A zig-zag index

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Abstract: We study the large $N$ superconformal index of quiver gauge theories describing the worldvolume of D3 branes probing toric Calabi Yau singularities. The index has been previously noticed to factorize over the set of the extremal BPS mesonic operators of the gauge theory. We review this factorization and reformulate it in terms of zig-zag paths in the dimer model associated to the quiver. By using this reformulation, we argue that the factorization is valid not only for the exact $R$-charge but for every set of $R_{\text{trial}}$ respecting the marginality constraints. Moreover, we show the factorization of the index also in theories with orbifold singularities, previously not investigated. We conclude by providing an expression for the index in terms of the toric data of the dual geometry.
1 Introduction

The superconformal index (SCI) of four dimensional superconformal field theories \([1, 2]\) is the supersymmetric partition function of the theory defined on the euclidean space \(S^3 \times S^1\). Alternatively, it can be defined as a weighted (over the fermion number) sum
of the states of the theory, where the contribution of the long multiplets vanishes. The index counts the short BPS multiplets and it is invariant under marginal deformations of the theory. It has been extensively studied in the recent years, especially to check field theory dualities and the AdS/CFT correspondence [1–8].

There are many prescriptions for obtaining the functional form of the index [1–3, 9–12]. In the large $N$ limit, the computation of the index simplifies and in some cases it can be carried over with matrix model techniques.

In this paper, we focus on a large class of superconformal gauge theories, namely the quiver gauge theories arising as the world volume of D3 branes probing a toric CY$_3$ singularity. It has been shown that the large $N$ index for such theories can be computed, matches with the dual description, and that it usually factorizes on a specific subset of operators, the so called extremal BPS mesons, corresponding to the edges of the dual cone of the toric fan.

This factorization was first observed in [7] for the SCI of of the $Y^{pq}$ families [13] of quiver gauge theories. By fixing the value of the superconformal $R$-charge imposed by $a$-maximization the authors computed the index in the $Y^{p0}$ and $Y^{pp}$ theories and guessed a general behavior for the $Y^{pq}$ case. A proof for the conjecture was later provided in [8], where the authors explained the factorization of the index from the properties of the toric geometry, for the case of smooth CY$_3$’s.

In this paper we show that the factorization property of the SCI for toric quiver gauge theories is more general. First, we observe that the index factorizes without fixing the exact superconformal $R$-charge, but just by requiring that the NSVZ beta functions vanish and the superpotential is marginal $^1$. Second, we show that the factorization holds also in gauge theories dual to geometries with additional singularities.

For this purpose, we reformulate the factorization of the SCI on extremal BPS mesons as a factorization of the SCI over a set of paths in the brane tiling. These paths are called zig-zag paths because they turn maximally left (right) at the black (white) nodes of the bipartite tiling. We conjecture a general factorized formula for the SCI in terms of the zig-zag paths, as a function of a trial $R$-charge. This expression continues to be well defined in the case of quiver gauge theories dual to geometries with orbifold singularities.

We check the validity of our formula and the factorization of the SCI index over the zig-zag paths in various examples, including infinite families of orbifold singularities. Moreover, we verify the invariance of our formula under Seiberg duality. As a byproduct, the factorization over the zig-zag path allows us to express the SCI directly

$^1$With a slight abuse of notation we keep on referring to this supersymmetric partition function on $S^3 \times S^1$ as the superconformal index also in this case.
in terms of the CY geometry and the toric data.

The paper is organized as follows. In section 2 we review the relevant aspects of D3 branes at toric CY$_3$ singularities and of the large $N$ calculation of the superconformal index. In section 3 we explain the factorization of the index over the extremal BPS mesons as discovered in [8]. In 3.2 we give the prescription to relate the $R$-charges of the extremal BPS mesons to the ones of the zig-zag paths and we re-formulate the factorization in terms of these paths. In section 4 we study the factorization over the zig-zag paths, in some simple examples, for general values of the trial $R$-charges that satisfy the constraints imposed by marginality. In section 5 we prove the factorization in the infinite families of $L_{aba}$ non-chiral singularities. In section 6 we show that our formula is preserved by Seiberg duality. In section 7 we show the role of the global, non anomalous and non $R$-symmetries in the factorization. In section 8 we translate the index from the zig-zag paths to their geometric counterpart. We conclude in 9 with some open problems. In appendix A we compare the zig-zag factorization with the one discovered in [7] for the whole $Y_{pq}$ family.

2 Review: SCI and toric quivers

2.1 D3 branes on toric CY$_3$

In this section we review some aspects of the world-volume theory describing D3 branes probing a toric CY$_3$ singularity, that will be useful for the rest of the paper (see [14] and references therein for a comprehensive review).

We start by the definition of a quiver gauge theory. A quiver is a graph made of vertices with directed edges connecting them. The vertices represent the $SU(N)$ gauge groups and the edges represent bifundamental or adjoint matter fields. The direction of the arrow of an edge is associated to the representation of the corresponding matter field under the gauge groups.

Since we study SCFTs there are two classes of constraints imposed by superconformality, both associated to the vanishing of the beta functions.

The first constraint comes from requiring the vanishing of the NSVZ beta function for each gauge group. This corresponds to the requirement of the existence of a non anomalous $R$-symmetry in the SCFT and hence becomes a constraint on the $R$-charges. At the $k$-th node of the quiver we have

$$\sum_{i=1}^{n_k} (r_i - 1) + 2 = 0 \quad (2.1)$$

where the sum is over all the $n_k$ bifundamentals charged under the $k$-th gauge group.

The second constraint comes from imposing the marginality of the superpotential terms.
These two constraints restrict the possible $R$-charge assignments of the superconformal field theory to a subset named $R_{\text{trial}}$. The extra freedom is fixed through $a$-maximization [15], that gives eventually the exact $R$-charge. In the following we refer to the case where $R$ is exact as the on-shell case, while the case obtained by just imposing the marginality constraints is referred as the off-shell case.

Note that in general the superpotential cannot be read from the quiver, but in the case of toric CY it is possible thanks to the notion of planar quiver. Toric quiver gauge theories have the property that each field appears linearly in the superpotential and in precisely two terms with opposite signs. It can be shown that we can exploit this structure of superpotential terms to transmute the quiver into a planar quiver embedded in $T^2$. The planar quiver is thus a periodic quiver built from the original one by separating all the possible multiple arrows connecting the nodes such that corresponding to each superpotential term there is a plaquette whose boundaries are given by the arrows, the bifundamental fields appearing in that superpotential term. Plaquettes representing superpotential terms with a common bifundamental are glued together along the corresponding edge. The sign of a superpotential term corresponds to orientation of its plaquette.

Moreover, it is possible to define a set of paths on the planar quiver called zigzag paths. They are loops on the torus defining the planar quiver. These loops are composed by the arrows. These arrows are chosen such that if a path turns mostly left at one node it turns mostly right at the next one. This notion is not illuminating on the quiver but it becomes more important in the description of the moduli space on the dual graph, called the bipartite tiling or the dimer model.

The dimer model is built from the planar quiver by reversing the role of the faces and of the vertices. The superpotential terms become the vertices of the tiling, and the orientation is absorbed in the color (black or white), i.e. the tiling is bipartite. The edges are mapped to dual edges, and the orientation is lost (all the information is in the vertices). The faces represent the gauge groups.

The zig-zag paths are oriented closed loops on the tiling with non trivial homology along the $T^2$. Every node of the tiling is surrounded by a closed loop made out of the zig-zag paths, and the orientation of the loops determines the color of the vertices, consistently with the bipartite structure of the tiling.

On the bipartite tiling there are sets of edges, called perfect matchings, that connect black and white nodes, such that every node is covered by exactly one edge. As already mentioned, the tiling is defined on the torus, that possesses two winding cycles $\gamma_\omega$ and $\gamma_z$. An intersection number with the homology classes $(1, 0)$ and $(0, 1)$ of the two winding cycles is associated to each perfect matching.

A monomial in $z^m \omega^n$ is associated to each perfect matching, where $m$ and $n$ rep-
resent the intersection number of the perfect matching with the cycles $\gamma_\omega$ and $\gamma_z$. A polynomial that counts the perfect matchings in the brane tiling is obtained by summing over these monomials.

The convex hull of the exponents of this polynomial is a polyhedral on $\mathbb{Z}^2$, the toric diagram. This rational polyhedral encodes the informations of the moduli space of the D3 probing the toric CY$_3$.

2.2 Large N index in toric quivers

The superconformal index for a four dimensional $\mathcal{N} = 1$ field theory is defined as

$$I = Tr(-1)^F e^{-\beta \Xi} t^{R_{-2J_3}} y^{2J_3} \prod \mu_i^{q_i}$$

(2.2)

where $\Xi = \{Q_1, Q_1^\dagger\}$ represents the superconformal algebra on $S^3 \times S^1$. The index gets contributions only from the states with $\Xi = 0$ and hence it is independent from $\beta$.

The chemical potentials $t$, $y$ and $\mu$ are associated to the abelian symmetries of the theory that commute with $Q_1$ and their charges are the exponents, $R$ is the $R$-symmetry, $J_3$ and $\bar{J}_3$ are the Cartan of the $SU(2)_L \times SU(2)_R \in SO(4,2)$ and $q_i$ are the charges of the flavor symmetries. The single particle index receives contributions from both the chiral and the vector multiplet. In the first case we have

$$I_{s.p}(\phi) = \frac{t^{r_\phi}}{(1 - ty)(1 - t/y)} \quad , \quad I_{s.p}(\psi^\dagger) = -\frac{t^{2-r_\phi}}{(1 - ty)(1 - t/y)}$$

(2.3)

where both $\phi$ and $\psi$ belong to the chiral multiplet $\Phi$. The contribution of the vector multiplet is

$$I_{s.p}(V) = \frac{2t^2 - t(1 + 1/y)}{(1 - ty)(1 - t/y)}$$

(2.4)

In the case of quiver gauge theories there are only two possible representations, bifundamental and adjoint. A bifundamental superfield $X_{ij}$ contains a scalar in the fundamental for the $i$-th group and in the antifundamental for the $j$-th group. The fermion $\psi^\dagger$ is in the opposite representation.

The single particle index $I(t, y, \chi)$ associated to the quiver is the sum of the contributions of the vector multiplets and the bifundamental multiplets in the quiver. At each node $i$ there is a contribution $I_{s.p.}(V_i)\chi_i^{\text{adj}}$, where $\chi_i^{\text{adj}}$ is the character of the adjoint representation of the $i$-th gauge group. For every bifundamental $\Phi_{ij}$ there is a contribution

$$I_{i,j}(t, y, \chi) = I_{s.p.}(\phi_{ij})\chi_i\bar{\chi}_j + I_{s.p.}(\psi_{ji}^\dagger)\bar{\chi}_i\chi_j$$

(2.5)

where the $\chi_i$ and $\bar{\chi}_i$ are the characters of the fundamental and antifundamental representation associated to the $SU(N_i)$-th gauge group. If the matter field is a bifundamental the product $\chi_i\bar{\chi}_j$ in (2.5) must be substituted with $\chi_i^{\text{adj}}$. 

\[ -5 - \]
The single trace index is obtained by taking the plethystic exponential \([16]\). In order to single out contributions from gauge-invariant states, we also need to integrate over the gauge measure. In formulae

\[
I_{\text{m.t.}}(x) = \int \prod_{i=1}^{G} [d\alpha_i] PE[I(t, y, \chi(\alpha_i))]
\]

(2.6)

where the \(\alpha_i\) are the Cartan of the \(i\)-th gauge group. By taking the large \(N\) limit this becomes a Gaussian integral and the index is

\[
I_{\text{m.t.}}(t, y) = \prod_k \frac{e^{\frac{1}{2} \text{Tr} i(t^k, y^k)}}{\text{det}(1 - i(t^k, y^k))}
\]

(2.7)

where

\[
1 - i(t, y) = \frac{1 - m(t) + t^2 m^T (t^{-1}) - t^2}{(1 - ty)(1 - t/y)} \equiv \frac{M(t)}{(1 - ty)(1 - t/y)} \quad \text{with} \quad m_{ij}(t) = \sum_{e:i \to j} t^{R(e)}
\]

(2.8)

The matrix \(m(t)\) represents the adjacency matrix weighted by the \(R\)-charge. For every edge \(e\), connecting the \(i\)-th node to the \(j\)-th one in the quiver, the matrix picks up a contribution \(t^{R(e)}\). The index can be further simplified and it becomes

\[
I_{\text{s.t.}}(t, y) = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \text{det} M(t^k) - \text{Tr} \left( \frac{m(t) - t^2 m(t^{-1})}{(1 - ty)(1 - t/y)} \right)
\]

(2.9)

where \(\varphi\) is the Euler-phi function. Observe that the second term in (2.9) vanishes in absence of adjoint matter because \(m(t)\) becomes traceless.

3 Factorization of the SCI

3.1 SCI over the extremal BPS mesons

The factorization of the index was first observed in \([7]\) and then proven in \([8]\) for toric \(\text{CY}_3\) without additional singularities away from the tip of the cone.

Consider a toric \(\text{CY}_3\) cone probed by a D3 brane. This cone is described by the fan \(\mathcal{C}\), a convex polyhedral cone in \(R^3\). The BPS mesons (their vev), up to F-term equivalences, are in 1-1 correspondence with the semigroup of integer points in \(\mathcal{C}^*\), the dual cone of \(\mathcal{C}\). The three integer numbers defining the points in the dual cone (and equivalently the BPS mesons) are the three \(U(1)\) isometries of the \(\text{CY}_3\) or equivalently the mesonic symmetries of the field theory \((U(1)^2_F \times U(1)_R)\). The points in the dual
cone can be divided in points on the edges, on the faces and on the internal of the cone itself.

After this geometrical digression we can now report the result of [8] on the factorization of the index. It states that the determinant \( \det(M(t)) \) factorizes over the extremal BPS mesons [17] that are described by the edges of the dual cone \( C^* \)

\[
\det(M(t)) = \prod_{i \in E_M} \left( 1 - t^{r_i} \mu_1^{F_i} \mu_2^{\tilde{F}_i} \right) \tag{3.1}
\]

where \( E_M \) refers to the edges of the dual cone or equivalently to the extremal BPS mesons. The charges appearing in [7, 8] are the exact \( R \)-charge of the SCFT and the two \( U(1)_F \).

There are some interesting questions following from the factorization. The first regards the exactness of the \( R \)-charge. One may wonder if the exact \( R \)-charge is a necessary condition for the factorization of the index, or if it possible to relax this assumption, just by imposing the marginality constraints (vanishing of the beta functions), corresponding to the \textit{off-shell} \( R_{\text{trial}} \) case defined above.

A second question regards theories with extra singularities far from the tip of the cone. These theories are characterized by having extra points on the edges of the toric diagram. In the dual cone these points are not associated to any edge but they live on the faces. These theories have not been investigated in [8] and one may wonder how the factorization formula is modified in these cases.

### 3.2 Extremal BPS mesons and zig-zag paths

In this section we study the two problems discussed above by using the brane tiling instead of the dual cone. By starting from the observation that both the extremal BPS mesons and the zig-zag paths are in 1-1 correspondence with the primitive vectors of the toric diagram we give a prescription to extract the charges of the extremal BPS mesons from the charges of the zig-zag paths. This allows us to define a factorization formula for the SCI in terms of the zig-zag paths.

The BPS mesons, not necessarily extremal, are represented on the tiling as string of operators built by connecting a face with its image by a path. These paths have to cross the edges of the tiling by leaving the nodes of the same color on the same side. A BPS meson is the product of the edges crossed by such paths. Products of operators with the same homology and the same \( R \)-charge are \( F \)-term equivalent. There is a set of these BPS mesons that have maximal \( U(1) \)-charge (up to a sign) for a given \( R \)-charge. These are the extremal BPS mesons, corresponding to the edges of the dual cone [17].
They can be built (up to degenerations) from the zig-zag paths. First we associate an orientation to every zig-zag path such that they leave a black node on the right and a white node on the left. For every black $n$-valent node the $i$-th zig-zag path crosses two edges. The $i$-th extremal BPS meson is obtained by associating the other $n-2$ edges at every black node crossed by the $i$-th zig-zag path.

For example in the figure 1 we highlight in red the three zig-zag paths of $\mathbb{C}^3/\mathbb{Z}_3$ and in green the three extremal BPS mesons. From this definition we obtain a general formula relating the $R$-charges of the extremal BPS mesons and the $R$-charges of the zig-zag paths.

At each $n$-valent black node the condition of marginality of the superpotential implies that

$$\sum_{j=1}^{n} r_j = 2$$  \hspace{1cm} (3.2)

where $r_j$ are the charges of the fields related to the edges connected with the black node that we are considering.

Let us suppose that the first two ($j = 1, 2$) are in the zig-zag paths and the others in the extremal BPS meson. By using the previous relation we have that

$$r_3 + \cdots + r_n = 2 - r_1 - r_2 = (1 - r_1) + (1 - r_2)$$  \hspace{1cm} (3.3)

and we have expressed the $R$-charges of the fields forming the extremal BPS meson in terms of the $R$-charges of the edges belonging to the zig-zag path.

\footnote{The same correspondence can be obtained by using the white nodes crossed by the $i$-th zig-zag path}
By summing over all the black nodes crossed by the zig-zag path we obtain the $R$-charge of the extremal BPS meson associated to the $i$-th zig-zag path (denoted with $Z_i$)

$$R_{BPS_i} = \sum_{k \in \{Z_i\}} (1 - r_k^{(i)})$$

(3.4)

where $k$ runs over the set of edges $\{Z_i\}$ belonging to the $i$-th zig-zag path, and $r_k^{(i)}$ is the $R$-charge of the $k$-th field in the $i$-th zig-zag path.

By using the relation between the $R$-charges of the extremal BPS mesons and of the zig-zag paths the determinant $\det M(t)$ factorizes over the zig-zag paths as

$$\det M = \prod_{i=1}^{Z} (1 - t \sum_{j \in \{Z_i\}} (1 - r_j^{(i)}))$$

(3.5)

where $Z$ is the number of zig-zag paths, and $\{Z_i\}$ and $r_j^{(i)}$ are defined as above.

We conjecture (3.5) to be valid also off-shell and in the singular cases. In the rest of the paper we study the validity of this formula with many examples and checks.

4 Examples

In this section we study the two simplest examples of quiver gauge theories described by a bipartite graph and associated to a toric CY$_3$ singularity. They are the $\mathcal{N} = 4$ SYM and the conifold.

In both cases we explicitly show how the Gaussian integral obtained in the large $N$ limit factorizes over the zig-zag paths off-shell.

4.1 N=4

We start by considering the $\mathcal{N} = 4$ SYM. We study this theory as an $\mathcal{N} = 1$ theory. In $\mathcal{N} = 1$ notations there is an $SU(N)$ gauge group and three adjoint fields, that we call $X_1$, $X_2$ and $X_3$. The interaction is $W = X_1 [X_2, X_3]$ which imposes $r_{X_1} + r_{X_2} + r_{X_3} = 2$. The three zig-zag paths correspond to the three products of fields

$$zz_1 = X_1 X_2 \quad zz_2 = X_2 X_3 \quad zz_3 = X_3 X_1$$

(4.1)

In this theory the determinant at large $N$ (3.1) is given by

$$\det(M(t)) = 1 - t^2 + \sum_{i=1}^{3} t^{r_i} + \sum_{i=1}^{3} t^{2-r_i}$$

(4.2)

\[\text{In the following we set } \mu_1 = \mu_2 = 1, \text{ at the end of the paper we will show how to insert these symmetries back in the index.}\]
We now show that this determinant factorize in a product over the zig-zag path as claimed in (3.5), by manipulating each term in expression (4.2).

The term $t^2$ generically corresponds to $t^{2n_G}$, where $n_G$ is the number of gauge groups in the quiver, and it can be re-written from the relation in the dimer as

$$n_{\text{Faces}} + n_{\text{Points}} - n_{\text{Edges}} = 0 \rightarrow 2n_{\text{fields}} - 2n_W = 2n_G \quad (4.3)$$

By imposing the superpotential constraint we have

$$\sum_{i=1}^{Z} \sum_{j \in \{Z_i\}} (1 - r_j^{(i)}) = 2n_G \quad (4.4)$$

In this case we have $t^2 \rightarrow t^{6-2(r_1+r_2+r_3)}$.

The term $\sum t^{r_i}$ can be re-written by using the constraints from the superpotential and it becomes $\sum_{i<j} t^{2-r_i-r_j}$. In the same way the last term becomes

$$2 - r_i = r_j + r_k = (2 - r_i - r_k) + (2 - r_j - r_i) \quad (4.5)$$

By putting everything together the final formula is

$$\det(M(t)) = (1 - t^{2-r_1})(1 - t^{2-r_2})(1 - t^{2-r_3}) \quad (4.6)$$

which corresponds to the expression (3.5), factorized over the three zig-zag paths.

### 4.2 Conifold

The second example is the worldvolume theory of a stack of $N$ D3 branes probing the conifold. This is represented by a quiver gauge theory with two gauge groups $SU(N)_1 \times SU(N)_2$ and two pairs of bifundamental-antibifundamental $(a_i, b_i)$ connecting them. The superpotential is $W = \epsilon_{ij} \epsilon_{kl} a_i b_k a_j b_k$ that imposes

$$r_{a_1} + r_{a_2} + r_{b_1} + r_{b_2} = 2 \quad (4.7)$$

At large $N$ the determinant of $M(t)$ is

$$1 - \sum_{i,j} t^{r_{a_i}+r_{b_j}} + 2t^2 + \sum_{i \neq j} \left( t^{2-a_i+a_j} + t^{2-b_i+b_j} \right) - \sum_{i,j} t^{4-a_i-b_j} + t^4 \quad (4.8)$$

we can reorganize the sum as a sum over the zig-zag paths as follows. There are four zig-zag paths parameterized by

$$\begin{align*}
zz_1 = a_1 b_1, & \quad zz_2 = a_2 b_1, & \quad zz_3 = a_1 b_2, & \quad zz_3 = a_2 b_2
\end{align*} \quad (4.9)$$
We keep fixed the first term in the sum (4.8). The second one becomes

$$\sum_{i,j} t^{r_{a_i} + r_{b_j}} \rightarrow \sum_{i,j} t^{2r_{a_j} - r_{b_i}} = \sum_{i=1}^{Z_t} t^{\sum_{j} (1-r^{(j)}_i)}$$

(4.10)

the third and the fourth terms can be written together and thanks to the relation (4.7) we have

$$2t^2 + \sum_{i \neq j} (t^{2a_i + a_j} + t^{2b_i + b_j}) \rightarrow \sum_{i=1}^{Z_t} \sum_{j=i+1}^{Z_t} t^{\sum_{k \in \{Z_i\}} (1-r^{(k)}_i) + \sum_{l \in \{Z_j\}} (1-r^{(l)}_j)}$$

(4.11)

Also in the fifth term of (4.8) we can insert the relation (4.7) and obtain

$$\sum_{i,j} t^{4-a_i-b_j} \rightarrow \sum_{i=1}^{Z_t} \sum_{j=i+1}^{Z_t} \sum_{k=j+1}^{Z_t} t^{\sum_{l \in \{Z_i\}} (1-r^{(l)}_i) + \sum_{m \in \{Z_j\}} (1-r^{(m)}_j) + \sum_{n \in \{Z_k\}} (1-r^{(n)}_k)}$$

(4.12)

The last term is obtained as already explained in the $N = 4$ case. Finally, by collecting all the terms, we have

$$\det(M(t)) = (1 - t^{2r_{a_1} - r_{b_1}})(1 - t^{2r_{a_1} - r_{b_2}})(1 - t^{2r_{a_2} - r_{b_1}})(1 - t^{2r_{a_2} - r_{b_2}})$$

(4.13)

5 The singular cases

The second result that we argue in this paper is that the determinant of the matrix $M(t)$ arising in the large $N$ calculation of the superconformal index (see formula (2.9)) factorizes over the zig-zag paths also in the case where new singularities arise far from the tip of the CY cone.

For example in the $L_{pqr}$ families [17–19] there are many examples corresponding to orbifolds. Inside these classes of orbifolds there are two infinite families, $L^{aaa}$ and $L^{aba}$, associated to non-chiral theories that can be studied in a unified way. In this section we show that $\det(M(t))$ factorizes in both these cases over the zig-zag paths. Moreover we study a non chiral case, $L^{264}$ corresponding to the $L^{a,b,\overleftarrow{a}}$ singular family, and observe the factorization.

5.1 The $L^{aaa}$ family

In this section we compute the large $N$ index for an infinite class of theories, the $L^{aaa}$ theories. These theories are vector like theories with a bifundamental and an antibifundamental connecting the $i$-th node and the $i + 1$-th one. We start by studying
Figure 2. Tiling and zig-zag paths for a generic $L^{aaa}$ model. We grouped the zig-zag paths with homology $(\pm 1, 0)$ with the green color while the blue ones have homology $(0, \pm 1)$. We distinguished the sign by specifying the orientation with.

![Figure 2](image)

Figure 3. Trial $R$-charge assignation for a generic $L^{aaa}$ model.

the phase without any adjoint matter field. Subsequently we show that the factorization of the index over the zig-zag paths is maintained even in phases that contain the adjoint fields.

By looking at the tiling there are four classes of zig-zag paths. The first two classes have homology $(1, 0)$ and $(-1, 0)$ respectively and contain $2a$ fields. By imposing the constraints imposed by the marginality we have two possible charge assignations, as in figure 3. The two zig-zag paths both contribute to the index with a factor $(1-t^a)$. There are also other $a$ zig-zag paths with homology $(0, 1)$ and $a$ with homology $(0, -1)$. The first class contains only fields with charge $r$, and every zig-zag of this kind contributes with a factor $(1-t^{2r})$. In the second case the charge is $1-r$ and the contribution is $(1-t^{2r})$. The final contribution to the index is

$$\det M(t) = (1-t^a)^2(1-t^{2-2r})^a(1-t^{2r})^a$$

(5.1)
We now give a proof of our claimed factorization. We start by writing the matrix
\[
M(t) = \begin{pmatrix}
a_1 & b_1 & 0 & \ldots & \ldots & c_a \\
b_1 & a_2 & c_1 & \ldots & \ldots & 0 \\
0 & c_1 & 0 & \ldots & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & c_{a-1} & 0 \\
c_a & 0 & 0 & \ldots & b_a & a_{2a}
\end{pmatrix}
\] (5.2)
where
\[
a_i = (1 - t^2), \quad b_{2i} = c_{2i} = (t^{r+1} - t^{1-r}), \quad b_{2i+1} = c_{2i+1} = (t^{2-r} - t^r) \quad (5.3)
\]
Since (5.2) is a circulant matrix the determinant can be easily computed. Actually here we use a more complicate technique, more useful for the $L^{aba}$ case. The determinant of (5.2) can be written in an equivalent way by the formula
\[
det M(t) = Tr \prod_{i=1}^{2a} L_i - 2 \prod_{i=1}^{a} b_i c_i, \quad L_{2j} = \begin{pmatrix} a_j & -b_{j-1}^2 \\ 1 & 0 \end{pmatrix}, \quad L_{2j+1} = \begin{pmatrix} a_j & -c_{j-1}^2 \\ 1 & 0 \end{pmatrix}
\] (5.4)
The trace is easily computed by defining $F = L_i L_{i+1}$ and by observing that
\[
Tr F^a = Tr \prod_{i=1}^{2a} L_i \quad (5.5)
\]
The trace is computed from the eigenvalues of $F$. We have
\[
Tr F^a = Tr \left( \begin{pmatrix} \lambda_1^a & 0 \\ 0 & \lambda_2^a \end{pmatrix} \right) = (1 + t^{2a})(1 - t^{2-2r})^a(1 - t^{2r})^a \quad (5.6)
\]
By adding the extra contribution
\[
\prod_{i=1}^{a} b_i c_i = t^a(1 - t^{2-2r})^a(1 - t^{2r})^a \quad (5.7)
\]
the expected factorization is obtained.

It is interesting to observe the behavior of the index under Seiberg duality. As we will show later the factorization of the determinant is not affected by the duality. Here the problem is that a duality on the $n$-th node adds two extra adjoints on the $n \pm 1$-th nodes. But as we already observed in section 2 the extra adjoints must be subtracted in the computation of the index.
While the $\mathcal{N} = 1$ vector multiplet usually cancels the $y$ dependence of the index, the presence of the extra adjoints fields reintroduces this and in principle one may expect that the index does not match among different phases. However, this extra contribution is

$$\frac{1}{(1-ty)(1-t/y)} \left( t^{2r} - t^{2(1-r)} + t^{2(1-r)} - t^{2-2(1-r)} \right)$$

and it vanishes in the dual phase.

### 5.2 The $L^{aba}$ family

In this section we generalize the case of the $L^{aaa}$ theories studied above to the whole $L^{aba}$ family. In this case the contributions from the extra adjoint matter fields has to be subtracted, and the index is $y$ dependent. Nevertheless the determinant of the matrix $M$ still factorizes over the zig-zag paths. By parameterizing the fields as in figure 4 there are four classes of zig-zag paths:

- $a$ paths formed by the pairs of fields $X_{i,i+1}$ and $X_{i+1,i}$ with charge $r$. They contribute to the index as $(1 - t^{2(1-r)})^a$.

- $b$ paths formed by the pairs of fields $X_{i,i+1}$ and $X_{i+1,i}$ with charge $1 - r$. They contribute to the index as $(1 - t^{2r})^b$.

- One path formed by all the adjoints and all the fields $X_{i,i+1}$. It contributes to the index as $(1 - t^{ar+b(1-r)})$.

- One path formed by all the adjoints and all the fields $X_{i+1,i}$. It contributes to the index as $(1 - t^{ar+b(1-r)})$.
With the parameterization of the charges in figure 4 the matrix $M$ is

$$
M = \begin{pmatrix}
  a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & c_b \\
  b_1 & a_2 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & c_1 & \ldots & \ldots & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & \ldots & b_a & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & b_a & a_{2a-1} & c_a & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & c_a & d_1 & c_{a+1} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & c_{a+1} & d_2 & c_{a+2} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & c_{a+2} & \ldots & \ldots & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & c_{b-1} & c_b \\
  c_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{b-1} & d_{b-a}
\end{pmatrix}
$$

where

$$
\begin{align*}
a_i &= 1 - t^2 \\
c_i &= t^{i-1} - t^{1-r} \\
a_{2a-1} &= 1 - t^2 \\
b_i &= t^{2-r} - t^r \\
d_i &= 1 - t^2 - t^{2r} + t^{2(1-r)}
\end{align*}
$$

As before the determinant of this matrix can be obtained by defining the two-dimensional $L$ matrices (5.4). The determinant becomes

$$
\det M = \text{Tr} \prod_{i=1}^{a+b} L_i + 2(-1)^{b+1} \prod_{i=1}^{a} b_i \prod_{j=1}^{b} c_j
$$

The first trace can be evaluated by redefining the matrices $L_iL_{i+1} = K$ for $i = 1, \ldots, 2a - 1$ and $L_j = J$ for $j = 2a + 1 \ldots, a + b$. The trace becomes $\text{Tr} K^a J^{b-a}$ where

$$
K^a = \frac{\left(1 - t^{2(1-r)}\right)^{a-1} (1 - t^{2r})^{a-1}}{t^{2r}} \left(\frac{1 - t^{2r}}{1 - t^{2a}}\right) \left(\frac{t^{2r} - t^{2(1-r)}}{1 - t^{2(1-r)}}\right)
$$

$$
J^{b-a} = \frac{(1 - t^{2r})^{-a+b-1}}{t^{2r} - t^2} \left(\frac{t^{2r} - t^{4r}}{t^{2r}}\right) \left(\frac{1 - t^{2(1-r)(-a+b+1)}}{1 - t^{2(1-r)(-a+b-1)}}\right)
$$

After plugging (5.12) in (5.11) we have

$$
\det M(t) = (1 - t^{2-2r})^a (1 - t^{2r})^b (1 - t^{ar+b(1-r)})^2
$$

that coincides with the formula computed from the zig-zag paths.
5.3 A chiral orbifold

We conclude the analysis of the singular cases by studying a chiral orbifold of $L^{pqr}$. This model belongs to an infinite class of chiral orbifolds, $L^{a,b,c,d}$. We study a single case here, the $L^{264}$ theory, that is an orbifold of $L^{132}$. We show that $\det(M(t))$ factorizes over the zig-zag paths with an off shell $R_{\text{trial}}$. The tiling and the toric diagram are represented in (5) The matrix $M(t)$ is

$$M(t) = \begin{pmatrix}
1 - t^2 & -t^X_{1,2} - t^Y_{1,2} & t^X_{2,3} & 0 & 0 & -t^X_{1,6} & t^X_{1,7} & t^X_{1,8} \\
t^2 - r^X_{1,2} + t^2 - r^X_{1,2} & 1 - t^2 & -t^X_{2,3} & -t^X_{2,4} & t^2 - r^X_{3,2} & 0 & 0 & -t^X_{2,8} \\
t^X_{3,1} & t^X_{2,3} & 1 - t^2 & t^2 - r^X_{3,3} & -t^X_{3,5} & 0 & 0 & 0 \\
0 & t^X_{2,4} & -t^X_{3,4} & 1 - t^2 & -t^X_{4,5} & t^2 - r^X_{5,4} & 0 & 0 \\
0 & 0 & -t^X_{5,2} & t^2 - r^X_{5,3} & t^2 - r^X_{5,5} & 1 - t^2 & -t^X_{5,6} - t^X_{5,6} & t^2 - r^X_{5,6} \\
t^2 - r^X_{6,5} & 0 & 0 & -t^X_{6,4} & t^2 - r^X_{6,6} + t^2 - r^X_{6,6} & 1 - t^2 & -t^X_{6,7} & -t^X_{6,8} \\
-t^X_{7,1} & 0 & 0 & 0 & -t^X_{7,5} & t^X_{7,5} & 1 - t^2 & t^X_{7,6} \\
-t^X_{8,1} & t^2 - r^X_{2,8} & 0 & 0 & 0 & t^2 - r^X_{8,8} & -t^X_{8,7} & 1 - t^2
\end{pmatrix}
$$

(5.14)

The zig-zag paths are as

$$zz_1 = X_{1,2} X_{2,4} X_{4,5} X_{5,6} X_{6,7} X_{7,1}$$
$$zz_2 = X_{1,2} X_{1,6} X_{2,3} X_{2,8} X_{3,1} X_{4,5} X_{5,2} X_{6,4} X_{6,7} X_{7,5} X_{8,1} Y_{5,6}$$
$$zz_3 = X_{2,8} X_{3,1} X_{4,3} X_{5,6} X_{6,4} X_{7,5} X_{8,7} Y_{1,2}$$
$$zz_4 = X_{2,4} X_{3,5} X_{4,3} X_{5,2}$$
$$zz_5 = X_{1,6} X_{6,8} X_{7,1} X_{8,7}$$
$$zz_6 = X_{2,3} X_{3,5} X_{6,8} X_{8,1} Y_{1,2} Y_{5,6}$$

(5.15)
After imposing the NSVZ and the $W$ constraints we have

$$\det M(t) = \prod_{i=1}^{6} (1 - t^{\sum_{j \in \{z_i\}} (1 - r^{(i)}_j)})$$

(5.16)

### 6 Seiberg duality

In this section we study the invariance of the formula (3.5) under Seiberg duality. The duality on the dimer and on the zig-zag paths is shown in figure 6. The zig-zag paths involved in the duality are the four represented in the picture, the red (R), green (G), blue (B) and magenta (M). In the electric case the zig-zag paths that are involved in the duality are

$$zz_R = X_{AE} \quad X_{DA} \quad \tilde{zz}_R$$
$$zz_G = X_{DA} \quad X_{AC} \quad \tilde{zz}_G$$
$$zz_B = X_{AC} \quad X_{BA} \quad \tilde{zz}_B$$
$$zz_M = X_{BA} \quad X_{AE} \quad \tilde{zz}_M$$

(6.1)

where $\tilde{zz}_i$ is the part of the zig-zag part that does not transform under the duality. In the magnetic theory we have

$$zz'_R = Y_{DC} \quad Y_{CA} \quad Y_{AB} \quad Y_{BE} \quad \tilde{zz}_R$$
$$zz'_G = Y_{DE} \quad Y_{EA} \quad Y_{AB} \quad Y_{BC} \quad \tilde{zz}_G$$
$$zz'_B = Y_{BE} \quad Y_{EA} \quad Y_{AD} \quad Y_{DC} \quad \tilde{zz}_B$$
$$zz'_M = Y_{BC} \quad Y_{CA} \quad Y_{AD} \quad Y_{DE} \quad \tilde{zz}_M$$

(6.2)
The electric and magnetic $R$-charges are related by

\begin{align*}
    r^Y_{CA} &= 1 - r^X_{AC}, & r^Y_{DC} &= r^X_{DA} + r^X_{AC} \\
    r^Y_{AB} &= 1 - r^X_{BA}, & r^Y_{BE} &= r^X_{BA} + r^X_{AE} \\
    r^Y_{EA} &= 1 - r^X_{AE}, & r^Y_{BC} &= r^X_{BA} + r^X_{AC} \\
    r^Y_{AD} &= 1 - r^X_{DA}, & r^Y_{DE} &= r^X_{DA} + r^X_{AE}
\end{align*}

(6.3)

It is know easy to check that index calculated in the electric phase coincide with the one of the magnetic phase thanks to (6.3).

7 Global symmetries

In this section we show that the chemical potentials of the global symmetries preserve the factorization of the off-shell index over the zig-zag paths. There are two kind of global symmetries, baryonic and flavor symmetries. The first class of symmetries may be visualized as a sub set of the $U(1)$ symmetries inside the $U(N)$ at each node. The non anomalous baryonic symmetries are obtained from the trace anomaly $\text{Tr}SU(N)^2U(1)_{B_j}$. This can be visualized with the signed adjacency matrix. The kernel of this operator defines the combinations of baryonic symmetries that decouple in the IR or become anomalous. The zig-zag paths are uncharged under these symmetries, because they are gauge invariant paths, or equivalently they are closed on the quiver. This is consistent with the expectation that the baryonic symmetries do not contribute to the index. On the other hand the flavor symmetries are associated to the homologies of the paths in the tiling and they are expected to contribute. By assuming the factorization of the index over the zig-zag paths

\begin{equation}
    \det M(t) = \prod_{i=1}^Z (1 - t^{\sum_{j \in \{Z_i\}} (1 - r^{(i)}_j)})
\end{equation}

(7.1)

we now prove that

\begin{equation}
    \det M(t) = \prod_{z=1}^Z 1 - t^{\sum_{j \in \{Z_i\}} (1 - r^{(i)}_j) \frac{\mu_1}{\mu_2} - \sum_{j \in \{Z_i\}} F^{(i)}_j - \sum_{j \in \{Z_i\}} \tilde{F}^{(i)}_j}
\end{equation}

(7.2)

The index is a polynomial with three types of contributions $t^2$ and $t^{r_i}$ and $t^{2-r_i}$, where $r_i$ is the $R$-charge of the $i$-th scalar in the chiral multiplet. Every term in the polynomial is generically a set of disjoint closed loops in the quiver, a gauge invariant string of bosonic and fermionic fields. After adding the flavor symmetries the three possible
contributions change as

\[
\begin{align*}
    t^2 & \rightarrow t^2 \\
    t^{r_i} & \rightarrow t^{r_i} \mu_1^F \mu_2^\tilde{F} \\
    t^{2-r_i} & \rightarrow t^{2-r_i} \mu_1^{-F} \mu_2^{-\tilde{F}}
\end{align*}
\]  

(7.3)

By using the constraints from NSVZ and the superpotential we can convert the charge associated to a fermion \( \psi_{ij} \) in the charge associated to a product of bosons \( \prod \phi_\alpha \), where \( \alpha \in I \) is a set of pairs of labels that parameterizes the fields involved in this relation, we have

\[
    t^{2-r_{ij}} \mu_1^{-F_{ij}} \mu_2^{-\tilde{F}_{ij}} = \prod_{\alpha \in I} t^{r_\alpha} \mu_1^F \mu_2^{\tilde{F}_\alpha}
\]

(7.4)

We can also convert the terms in the diagonal entries of \( M(t) \), proportional to \( t^2 \) in \( t^{rW} \) or \( t^{rF} \), where the exponent is the sum of the charges of fields in a generic superpotential term or in a face in the tiling. Putting everything together we observe that before considering the flavor symmetries the index is a polynomial in \( P(t^{r_i}) \) where \( r_i \) represents the charge in the \( i \)-th scalar, while after we add these symmetries the index is a polynomial in the form \( P(t^{r_i} \mu_1^{F_i} \mu_2^{\tilde{F}_i}) \). This shows that the mesonic flavor symmetries preserve the factorization.

8 Geometric formulation

In this section we translate our formula of the index factorized over the zig-zag in terms of toric geometry. As a standard procedure a set of variables \( a_i \) is assigned to every external point of the toric diagram as in [20]. They are constrained by \( \sum a_i = 2 \), which in the geometry represents the superpotential constraint \( R(W) = 2 \). A variable \( b_i \) can be assigned to the primitive normals, that are \( 1-1 \) with the zig-zag paths, as

\[
b_i = \sum_{j=1}^{i} a_i
\]

(8.1)

such that \( b_d = 2 \) where \( d \) is the number of external point of the diagram. We give a pictorial representation of the toric diagram and the dual primitive vectors for dP1 in figure 7. On the tiling \( \pi b_i \) is the angle of intersection of the zig-zag paths with the rombhi edges in the isoradial embedding [21]. Every edges (fields) is crossed by two zig-zag paths and their \( R \) charges are defined as

\[
\left\{
\begin{array}{ll}
    R_{ij} = b_i - b_j & i < j \\
    R_{ij} = 2 - b_i + b_j & i > j
\end{array}
\right.
\]

(8.2)

\[4\]In this case we restrict to the case without points on the edges.
If more fields are crossed by the same pair of paths they have the same charge. Once we obtained the formula for the $R$-charges in terms of the geometry we can guess a formula that expresses the index in terms of the $b_i$ variables.

A geometric formula that reproduces the field theory index is

$$
\det M_{\text{geom}} = \prod_{i=1}^{d} (1 - t \sum_j |\omega_{ij}|(1-R_{ij}))
$$

(8.3)

This formula holds in the minimal phase, where the number of intersections between two zig-zag paths is fixed by

$$
\omega_{ij} = \langle \omega_i, \omega_j \rangle = \det \begin{pmatrix} p_i & q_i \\ p_j & q_j \end{pmatrix}
$$

(8.4)

where $\omega_i = (p_i, q_i)$ are the primitive normal vectors of the toric diagram. After Seiberg duality one can end up with non-minimal cases, where the number of intersections is just bounded from below by $\langle \omega_i, \omega_j \rangle$. In that case the formula is still valid because the extra intersections always come in pairs with an opposite orientation and they cancel in (8.4) [22].
As an example we study the dP$_1$ model. The quiver, the tiling and the toric diagram are shown in figure 8. First we write the index in terms of the zig-zag paths, and then we use the geometric formula and show that the two formulas agree. The superpotential is

$$W = \epsilon_{\alpha\beta} X_{23}^{(\alpha)} X_{34}^{(\beta)} X_{42} - \epsilon_{\alpha\beta} X_{12}^{(\alpha)} X_{23}^{(\alpha)} X_{34}^{(3)} X_{41}^{(\beta)} + \epsilon_{\alpha\beta} X_{34}^{(\alpha)} X_{41}^{(\beta)} X_{13}$$  \hspace{1cm} (8.5)$$

The four perfect matchings related to the external points of the toric diagram are

$$v_1 = (0, 1) \rightarrow X_{13} X_{24} X_{34}^{(3)}$$
$$v_2 = (-1, 0) \rightarrow X_{23}^{(1)} X_{34}^{(1)} X_{41}^{(1)}$$
$$v_3 = (0, -1) \rightarrow X_{12} X_{34}^{(1)} X_{34}^{(2)}$$
$$v_4 = (1, -1) \rightarrow X_{34}^{(2)} X_{23}^{(2)} X_{41}^{(2)}$$  \hspace{1cm} (8.6)$$

$$zz_1 = X_{13} X_{34}^{(1)} X_{42} X_{23}^{(1)} X_{34}^{(3)} X_{41}^{(1)}$$
$$zz_2 = X_{41}^{(1)} X_{12} X_{23}^{(1)} X_{34}^{(2)}$$
$$zz_3 = X_{41}^{(2)} X_{12} X_{23}^{(2)} X_{34}^{(1)}$$
$$zz_4 = X_{34}^{(2)} X_{13} X_{41}^{(2)} X_{34}^{(3)} X_{23}^{(2)} X_{42}^{(2)}$$
The index is computed from the matrix

\[
M(t) = \begin{pmatrix}
1 - t^2 & -t^r x_{12} & -t^r x_{13} & t^{2-r} x_{41}^{(1)} + t^{2-r} x_{41}^{(2)} \\
t^{2-r} x_{12} & 1 - t^2 & -t^r x_{13}^{(1)} & t^{2-r} x_{23}^{(1)} + t^{2-r} x_{23}^{(2)} \\
t^{2-r} x_{13} & t^{2-r} x_{13}^{(1)} \cdot t^{2-r} x_{13}^{(2)} & 1 - t^2 & t^{2-r} x_{34}^{(1)} - t^{2-r} x_{34}^{(2)} - t^{2-r} x_{34}^{(3)} \\
-t^r x_{41}^{(1)} - t^r x_{41}^{(2)} & -t^r x_{42} & t^{2-r} x_{34}^{(1)} + t^{2-r} x_{34}^{(2)} + t^{2-r} x_{34}^{(3)} & 1 - t^2 \\
\end{pmatrix}
\]  
(8.7)

The determinant of this matrix factorizes by imposing the marginality constraints and it is equivalent to

\[
(1 - t^{4-r_{zz1}})(1 - t^{6-r_{zz2}})(1 - t^{4-r_{zz3}})(1 - t^{6-r_{zz4}})
\]  
(8.8)

We now write the index from the geometric formula. The \((p, q)\) web is parameterized by the four vectors

\[
w_1 = (-1, 1) \quad w_2 = (-1, -1) \quad w_3 = (0, -1) \quad w_4 = (2, 1)
\]  
(8.9)

The \(R\)-charges of the fields intersecting on the zig-zag paths can be written in terms of \(b\) as

\[
R(1, 2) = 2(b_2 - b_1), \quad R(1, 3) = b_3 - b_1, \quad R(2, 3) = b_3 - b_2, \\
R(2, 4) = b_4 - b_2, \quad R(3, 4) = 2(b_4 - b_3), \quad R(4, 1) = 3(b_1 - b_4 + 2)
\]

In terms of the \(b\) variables the determinant is given by (8.3). We have

\[
det M(t) = (1 - t^{-3b_1 + b_2 + 2b_3})(1 - t^{b_1 + b_2 - 2b_4 + 4})(1 - t^{2b_1 - b_3 - b_4 + 4})(1 - t^{-2b_2 - b_3 + 3b_4})
\]  
(8.10)

The \(b\) are related to the \(a\) variables as \(b_i = \sum_{j=1}^{i} a_i\). By assigning the \(a_i\) variables to the external points we can calculate the \(R\)-charge of the fields in terms of the \(a_i\). We have

\[
\begin{array}{cccccccc}
X_{12} & X_{23}^{(1)} & X_{23}^{(2)} & X_{34}^{(1)} & X_{34}^{(2)} & X_{34}^{(3)} & X_{41}^{(1)} & X_{41}^{(2)} \\
/a_3 & a_2 & a_4 & a_2 + a_3 & a_3 + a_4 & a_1 & a_2 & a_4 \\
\end{array}
\]  
(8.11)

The expression in (8.10) coincides with (8.8) after substituting in the latter (8.11).

9 Conclusions

In this paper we observed that the superconformal index factorizes over a set of gauge invariant paths on the dimer, called zig-zag paths.
We showed that this factorization remains valid also for theories with orbifold singularities, and without fixing the exact $R$-charge but on a generic set of $R_{\text{trial}}$ satisfying the marginality constraints.

The zig-zag paths have an important role at geometrical level because they give a mirror dual interpretation of the tiling. Indeed, as observed in [23], the zig-zag paths are both $(p,q)$ winding cycles in the dimer and boundaries of the faces in the tiling of the Riemann surface associated to a punctured region. This allows a dual description in IIA in terms of mirror D6 branes. Our formulation in terms of the zig-zag paths may be interesting for a mirror interpretation of the index.

A different duality, called specular duality, has been recently discovered in [28]. This duality exchanges the tiling with its mirror dual, written in terms of the zig-zag paths. Since the zig-zag paths have a crucial role in the factorization of the index, it would be interesting to analyze the relation among the indices in specular dual phases, as done here for the case of the usual Seiberg duality.

Another interesting development regards the relation with the orientifolded theories. Indeed it is known that the orientifold action on the tiling corresponds to a fixed line or fixed point projection [29]. These projections are naturally extended to the zig-zag paths. It would be nice to study the relation between the zig-zag index and the orientifold in the tiling and in the geometry.

A further line of investigation concerns the bipartite field theories recently defined in [24–27]. Indeed, even if they are not usually conformal, the zig-zag paths are well defined on these theories. It would be interesting to understand if the formula we discussed in this paper has some field theoretical or geometrical interpretation in those cases.

Finally, as discussed in the text, the zig-zag path are in one to one correspondence with extremal BPS mesons. In [17] it has been shown that the extremal BPS mesons correspond to massless geodesics of semiclassical strings moving in the internal geometry. It would be intriguing to investigate possible connections between this hamiltonian system and the factorization of the superconformal index.

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Figure 9. Tiling for the $Y^{pq}$ theories. The different colors represent the fields $U$ (black), $V$ (blue), $Y$ (red) and $Z$ (green).

A $Y^{pq}$ theories

In [7] the on shell superconformal index has been computed for a generic $Y^{pq}$ theory [13], and the authors guessed a generic formula by looking at different cases. Here we show that by applying our formula in terms of the zig-zag paths we can match their result on shell, but off shell the factorization takes place over a different set of operators.

A $Y^{pq}$ theory is a quiver gauge theory with $2p$ gauge groups. In figure 9 we show the dimer and the four kind of fields distinguished by their representation under the global symmetries. From the figure one can extract the number of fields and their charges. They are given in the table

| Field   | Multiplicity | Charge  |
|---------|--------------|---------|
| $Z$-green | $p - q$       | $x$     |
| $Y$-red  | $p + q$       | $y$     |
| $V$-blue | $2q$          | $1 + \frac{1}{2}(x - y)$ |
| $U$-black| $2p$          | $1 - \frac{1}{2}(x + y)$ |

The charges $x$ and $y$ are determined by $a$-maximization.

\[
x = \frac{(-4p^2 - 2pq + 3q^2 + (2p + q)\sqrt{4p^2 - 3q^2})}{(3q^2)}
\]
\[
y = \frac{-4p^2 + 2pq + 3q^2 + (2p - q)\sqrt{4p^2 - 3q^2}}{3q^2}
\] (A.1)

There are four kind of zig-zag paths. Two of them involve all the $Z$ and $p(q) U(V)$ fields. The other zig-zag paths exchange $Z$ with $Y$. The contribution of these four
paths to the index are

\[
\sum_{j=1}^{Z_1} (1 - r_j^{(1)}) = \sum_{j=1}^{Z_2} (1 - r_j^{(2)}) = 2p - ((p - q)r_Z + qr_V + pr_U) = \frac{(p-q)(2-x)+(p+q)y}{2}
\]

\[
\sum_{j=1}^{Z_3} (1 - r_j^{(3)}) = \sum_{j=1}^{Z_4} (1 - r_j^{(4)}) = 2(p+q) - ((p+q)r_Y + qr_V + pr_U) = \frac{(p+q)(2-y)+(p-q)x}{2}
\]

By comparing the formula obtained in [7] with our formula we find that the two agree once the exact R-charge is imposed. If instead we just fix the constraints from the marginality of the couplings, i.e. we keep \(x\) and \(y\) as generic variables parameterizing a trial R-charge, we have

\[
\det(M(t)) = \prod_{i=1}^{4} \left( 1 - t^{\sum_{j=1}^{Z_i} (1-r_j^{(i)})} \right) \neq \left( 1 - t^{p(1+(x-y)/2)} \right)^2 \left( 1 - t^{p+1/2q(1-1/2(x+y))} \right)^2
\]

(A.2)

and the off-shell index still factorizes over the zig-zag paths.

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