Stable phase retrieval with low-redundancy frames

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Abstract We investigate the recovery of vectors from magnitudes of frame coefficients when the frames have a low redundancy, meaning a small number of frame vectors compared to the dimension of the Hilbert space. We first show that for complex vectors in $d$ dimensions, $4d - 4$ suitably chosen frame vectors are sufficient to uniquely determine each signal, up to an overall unimodular constant, from the magnitudes of its frame coefficients. Then we discuss the effect of noise and show that $8d - 4$ frame vectors provide a stable recovery if part of the frame coefficients is bounded away from zero. In this regime, perturbing the magnitudes of the frame coefficients by noise that is sufficiently small results in a recovery error that is at most proportional to the noise level.

Keywords Magnitude measurements · Trigonometric polynomials · Roots of complex polynomials · Newton’s identities

Mathematics Subject Classifications (2010) 15A29 · 42C15 · 30C15

1 Introduction

Phase retrieval is a topic that is currently extensively researched. Part of the effort is directed towards applications in X-ray crystallography, where the Fourier transform dictates the form of the measured quantities from which a signal is recovered [8, 14].
It is well known that the magnitudes of the Fourier transform need to be complemented by additional information about the signal to make recovery feasible [1], for example the magnitudes of the fractional Fourier transform [17]. It is then a challenge whether the additional measurements can be realized experimentally. Another main motivation for phase retrieval is quantum communication, where quantum states need to be estimated from the relative frequencies of outcomes occurring in quantum measurements [15]. In this paper, we investigate the abstract question of finding a small number of linear measurements such that their magnitudes characterize a vector in a finite dimensional Hilbert space, up to an overall unimodular constant. In addition, we wish to make the recovery procedure resilient against noise affecting the magnitude measurements. The central idea is that the redundancy inherent in the frame coefficients, resulting from the linear dependencies among the frame vectors, compensates the loss of information when passing from frame coefficients to their magnitudes. In fact, in frame theory the notion of redundancy is usually understood to be the number of frame vectors divided by the dimension of the Hilbert space. In this paper, we choose a small number of frame vectors in a specific way to recover vectors from magnitudes of their frame coefficients, up to a unimodular constant. We address the following main questions: How small can we choose the size of a frame and still characterize each vector uniquely? What conditions ensure that the vector can be recovered with a guaranteed accuracy if the measured magnitudes are affected by noise?

Several strategies have been applied to the problem of phase retrieval, for example the reformulation of recovery a vector \( x \) in terms of the rank-one hermitian \( x \otimes x^* \) [4–6]. This was solved with techniques from compressed sensing by rank minimization in an underdetermined system [9–13], or even without rank minimization [21]. However, this technique does not specify what precise level of redundancy is sufficient for recovery. Other recovery procedures embed in even higher dimensional spaces by taking more tensor powers of \( x \) with itself [3]. Constructions based on expander graphs and the polarization identity give us randomized constructions with an explicit bound on the redundancy [7]. However, this is still far from the necessary number of vectors derived from the theory of projective embeddings [16], see also [18–20].

The phase retrieval problem can be represented as finding a set of measurement vectors \( \{v_1, \ldots, v_m\} \) so that a vector \( x \) in a Hilbert space can be determined by the magnitude of inner products \( |\langle x, v_j \rangle| \). The recovery procedure outlined here assumes that the signal in a \( d \)-dimensional Hilbert space is realized as a complex polynomial of degree \( d - 1 \). The space of polynomials is used so that the analytic properties of this space (the Residue Theorem in particular) may be utilized. To obtain our measurements, we will use point evaluations of the polynomial. For any polynomial \( f(z) \) and any point \( \omega \), the evaluation \( f(\omega) = \sum_{k=0}^{d-1} c_k \omega^k \) is the inner product of \( f \) with the polynomial \( z \mapsto \sum_{k=0}^{d-1} \omega^k z^k \). Similarly, the evaluation of \( f'(\omega) \) is the inner product of \( f \) with the polynomial \( z \mapsto \sum_{k=1}^{d-1} k \omega^{k-1} z^k \). This means that these polynomials, or linear combinations of them, may be used as the measurement vectors for the recovery procedure. In the absence of noise, the recovery proceeds in several steps:

1. The measured quantities are \( \{|f(\omega_j)|^2, |f(\omega_j) + v_l f'(\omega_j)|^2 : 0 \leq j \leq 2n - 2, 0 \leq l \leq 2\} \) where \( \omega_j = e^{2\pi i j/(2n-1)} \) and \( v_l = e^{2\pi i l/3} \). Using the Dirichlet
kernel and the polarization identity, these are extrapolated to the values $|f(z)|^2$ and $f'(z)\overline{f(z)}$ for each $z$ in the unit circle.

2. From the values of these two functions on the unit circle we determine moments of the roots of $f$.

3. The moments determine the polynomial up to an overall unimodular constant.

We investigate how the presence of noise affects each of these steps, and show that under certain conditions, for all sufficiently small $\epsilon$, perturbing the measured quantities up to $\epsilon$ still gives an approximate recovery with an error of order $\epsilon$. This provable stability only extends up to a certain noise level. Numerical experiments show that the domain in which the linear error bound holds extends far beyond this level.

This paper is organized as follows: After fixing the notation, we show in Section 2 that a vector, up to a unimodular constant, is determined by a specific choice of $4d - 4$ measurements. Section 3 establishes criteria for stability of the recovery procedure, complemented with the results of a numerical simulation.

2 Injectivity of the magnitude map

**Definition 2.1** The space of complex polynomials of degree at most $n$ is denoted as $P_n$. It is equipped with the inner product induced by the Lebesgue measure on the unit circle, so $p, q \in P_n$ have the inner product

$$\langle p, q \rangle = \int_{[0,2\pi]} p(e^{it})\overline{q(e^{it})}dt$$

where the overbar denotes complex conjugation. The space of trigonometric polynomials of degree at most $n$, henceforth called $T_n$, is understood to consist of all linear combinations of complex polynomials in $P_n$ and of their complex conjugates.

Thus, $P_n$ is the subspace of analytic functions in $T_n$. On the other hand, the map $A : f \mapsto |f|^2$ takes $f \in P_n$ to a trigonometric polynomial in $T_n$. The first question we wish to resolve is at how many points $A(f)$ needs to be evaluated in order to determine $\{\lambda f : |\lambda| = 1\}$. The second problem is that of noisy recovery. If the measured quantities are perturbed, is it possible to estimate the set accurately?

The following theorem shows that to determine a polynomial of degree at most $d - 1$ up to a unimodular constant, is enough to know its magnitude at $4d - 4$ points in the complex plane. An essential ingredient is a result of Philippe Jaming’s [17], here the special case for polynomials.

**Lemma 2.2** Let $d \in \mathbb{N}$. If $g \in P_{d-1}$ then it is determined up to a unimodular constant by the values of $|g|^2$ on two lines $L$ and $L_\alpha$ that intersect in an angle $\alpha \in \mathbb{R} \setminus \pi \mathbb{Q}$.

**Proof** Without loss of generality we take $L = \mathbb{R}$ and $L_\alpha = e^{i\alpha}\mathbb{R}$. Then by the positivity of $|g|^2$ on $\mathbb{R}$ and by the Schwarz reflection principle, the boundary values
of $|g|^2$ on $\mathbb{R}$ extend uniquely to a polynomial whose roots come in pairs related by complex conjugation. More precisely, if the degree of $g$ is $n$, then

$$g(z) = C \prod_{j=1}^{n} (z - z_j)$$

and the polynomial $p$ is

$$p(z) = |C|^2 \left( \prod_{j=1}^{n} (z - z_j) \right) \prod_{j=1}^{n} (z - \overline{z_j}).$$

Thus, the roots of $p$ complement those of $g$ with their complex conjugates.

On the other hand, the same applies to the analytic continuation $p_\alpha$ of $|g|^2$ from $\mathbb{L}_\alpha$ to the complex plane and reflections of its roots about $\mathbb{L}_\alpha$. We conclude that the roots of $g$ are contained in the intersection of the roots of $p$ and $p_\alpha$. This motivates a selection principle for the roots of $g$: We pick the intersection of the sets of roots obtained from the two extensions.

If $g$ were not determined by this, then it would need to have a root in the symmetric difference of the roots from the two extensions $p$ and $p_\alpha$. However, it is straightforward to verify that the symmetric difference is invariant under reflecting first about $\mathbb{L}$ and then about $\mathbb{L}_\alpha$. This composition of the two reflections is an irrational rotation, and so if the symmetric difference is non-empty, it gives a dense set of roots in a circle, which means $g = 0$.

**Theorem 2.3** Let $g(z) = \sum_{j=0}^{d-1} c_j z^j$, let $\mathbb{S}$ be the unit circle as before and $\mathbb{S}_\alpha = \phi_\alpha(\mathbb{R}) \cup \{1\}$ with $\alpha \in \mathbb{R} \setminus \pi \mathbb{Q}$ and $\phi_\alpha(z) = e^{\alpha z - \alpha} e^{\alpha z - 1}$, $\omega = e^{2\pi i/(2d-1)}$. Then sampling $\{|g(z(\alpha)_j)|^2\}$ on $2d - 1$ equidistantly spaced points $\{z(\alpha)_j\}_{j=0}^{2d-2}$ of $\mathbb{S}_\alpha$ and $\{|g(\omega^j)|^2\}_{j=2}^{2d-2}$, determines $g$ uniquely, up to an overall unimodular factor.

**Proof** We proceed in several steps:

Step 1. Given $f \in T_{d-1}$, $\omega = e^{2\pi i/(2d-1)}$ and the normalized Dirichlet kernel $D_{d-1}(z) = \frac{1}{2d-1} \sum_{i=-d+1}^{d-1} z^i$, then $f(z) = \sum_{j=0}^{2d-2} f(\omega^j) D((z - a) \omega^{-j})$. By substitution, for $a \in \mathbb{C}$, $r > 0$, $f(z) = \sum_{j=0}^{2d-2} f(a + r \omega^j) D(z \omega^{-j}/r)$. This means from the values at $2d - 1$ equidistant points on a circle we can interpolate any trigonometric polynomial of degree at most $d - 1$. Consequently, the magnitudes of $|g(z)|^2$ on $2d - 1$ equidistant points on $\mathbb{S}_\alpha$ determine $|g(z)|^2$ on the entire circle. Because the circles $\mathbb{S}$ and $\mathbb{S}_\alpha$ intersect in $0$ and $\omega_1$, this also determines the magnitudes of $g$ at two of the sample points on $\mathbb{S}$. Once the additional magnitudes $\{|g(\omega^j)|^2\}_{j=2}^{2d-2}$ are obtained, the Dirichlet kernel determines the magnitude of $|g|^2$ on all points in $\mathbb{S} \cup \mathbb{S}_\alpha$.

Step 2. Using the Cayley map $z \mapsto \frac{1+z}{1-z}$ and the associated polynomial automorphism

$$Wg(z) = (1+z)^{k-1} g\left( \frac{1-z}{1+z} \right)$$
we map both $S$ and $S_\alpha$ to lines $L$ and $L_\alpha$, $g$ to a polynomial $Wg$, and $|g|^2$ to a trigonometric polynomial $|Wg|^2$. By conformality, the angle between the lines at $\frac{1+\omega}{1-\omega}$ is the same as the angle between the circles. However, the tangent vector $\phi'_\alpha(0) = -(1+\omega)e^{i\alpha}$ has an irrational argument, whereas $i\omega$ does not, so the two circles intersect with an angle that is an irrational multiple of $\pi$. By the conformality of the Cayley map, the same is true for the intersection of $L$ and $L_\alpha$.

Step 3. Next, we use P. Jaming’s argument to show that magnitudes of $Wg$ on the two lines uniquely determines $Wg$, up to a unimodular multiplicative constant [17]. Applying the inverse of the map $W$ yields that the magnitudes of $g$ on the two circles $S$ and $S_\alpha$ determine $g$, up to a unimodular multiplicative constant.

3 Stable recovery in the presence of noise

The recovery procedure we outline below heavily relies on the analyticity properties of the function space. Although an absolute phase can not be measured, we have access to a relative phase such as $f'(z)/f(z)$ for some $z \in \mathbb{C}$. If these values were known on the entire unit circle $S = \{z \in \mathbb{C} : |z| = 1\}$ then we could recover $f$, up to an overall multiplicative constant from contour integrals and Cauchy formulas as outlined further below.

We first examine how such integrals are affected by perturbed measurements. Unless noted otherwise, for any continuous $f : \mathbb{C} \to \mathbb{C}$, $\|f\|_\infty = \max_{z \in S} |f(z)|$.

**Lemma 3.1** Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function, and let $p_1 : S \to \mathbb{C}$ and $p_2 : S \to \mathbb{R}$ with $\|p_1\|_\infty < \epsilon$ and $\|p_2\|_\infty < \epsilon$. If there exists a $\delta > 0$ such that $(|f|^2 + p_2)(z) > \delta$ for all $z \in S$, then

$$\left\| \frac{f'\overline{f} + p_1}{|f|^2 + p_2} - \frac{f'\overline{f}}{|f|^2} \right\|_\infty < \frac{\epsilon}{\delta} \left(1 + \left\| \frac{f'}{f} \right\|_\infty \right).$$

**Proof** The Minkowski inequality gives

$$\left\| \frac{f'\overline{f} + p_1}{|f|^2 + p_2} - \frac{f'\overline{f}}{|f|^2} \right\|_\infty = \left\| \frac{|f|^2 p_1 - f'\overline{f} p_2}{(|f|^2 + p_2)|f|^2} \right\|_\infty \leq \left\| \frac{|f|^2 p_1}{(|f|^2 + p_2)|f|^2} \right\|_\infty + \left\| \frac{f'\overline{f} p_2}{(|f|^2 + p_2)|f|^2} \right\|_\infty = \left\| \frac{p_1}{|f|^2 + p_2} \right\|_\infty + \left\| \frac{p_2}{|f|^2 + p_2} \frac{f'\overline{f}}{|f|^2} \right\|_\infty < \frac{\epsilon}{\delta} \|f\|_\infty + \frac{\epsilon}{\delta} \left(1 + \left\| \frac{f'}{f} \right\|_\infty \right).$$

\(\square\)
Intrinsically the recovery procedure is linked to the moments of the roots of the polynomial inside the unit disk. We study how Newton’s identities are affected when the moments are perturbed.

**Lemma 3.2** Let $f$ be a complex polynomial with $N_0$ roots $\{z_j\}_{j=1}^{N_0}$ in the open unit disk. Let $f_i(z) = \sum_{k=0}^{N_0} b_k z^k = \prod_{j=1}^{N_0} (z - z_j)$ define the monic factor of $f$ whose roots are precisely the roots of $f$ that are inside the unit disk. Given the perturbed moments $\{\tilde{\mu}_k\}_{k=1}^{N_0}$ of the roots such that $|\tilde{\mu}_k - \mu_k| < \gamma$ for some $0 \leq \gamma \leq 1$ and $\mu_k = \sum_{j=1}^{N_0} z_j^k$ for all $k \in \{1, 2, \ldots, N_0\}$ then there exists $C$ which only depends on $N_0$ such that $\{\tilde{\mu}_k\}_{k=1}^{N_0}$ uniquely determine coefficients $\{\tilde{b}_k\}$ with $|\tilde{b}_k - b_k| \leq C\gamma$ for all $k \in \{1, 2, \ldots, N_0\}$.

**Proof** If we knew the values of $\mu_k$ we could recover the actual coefficients using Newton’s identities, which give the recurrence relation $b_{N_0-k} = -\frac{1}{k} \sum_{l=1}^{k} \mu_l b_{N_0-k+l}$ for all $k$ from 1 to $N_0$. Instead we use our approximations $\tilde{\mu}_k$ to find approximated coefficients $\tilde{b}_k$ using the recurrence relation

$$\tilde{b}_{N_0-k} = -\frac{1}{k} \sum_{l=1}^{k} \tilde{\mu}_l \tilde{b}_{N_0-k+l}$$

with $\tilde{b}_{N_0} = 1$. We inductively show that $|\tilde{b}_k - b_k|$ is $O(\gamma)$. For the base case, by assumption

$$|\tilde{b}_{N_0-1} - b_{N_0-1}| = |\tilde{\mu}_1 b_{N_0} - \mu_1 b_{N_0}| = |\tilde{\mu}_1 - \mu_1| < \gamma .$$

For the inductive step we note that

$$|\mu_l| = |\sum_{j=1}^{N_0} z_j^l| \leq \sum_{j=1}^{N_0} |z_j^l| \leq N_0 ,$$

and if $S_k$ is the set of all combinations of $k$ roots of $f(z)$ inside the unit disk, then

$$|b_{N_0-k}| = \left| \sum_{S \in S_k} \prod_{z_j \in S} z_j \right| \leq \sum_{S \in S_k} \prod_{z_j \in S} |z_j| \leq \sum_{S \in S_k} 1 = \binom{N_0}{k} .$$
Thus, with the inductive assumption that for all \( j < k \), there exists a constant \( C_j \) such that 
\[
\left| \tilde{b}_{N_0-j} - b_{N_0-j} \right| < C_j \gamma,
\]
we have
\[
\left| \tilde{b}_{N_0-k} - b_{N_0-k} \right| = \frac{1}{k} \left| \sum_{l=1}^{k} \tilde{\mu}_l \tilde{b}_{N_0-k+l} - \sum_{l=1}^{k} \mu_l b_{N_0-k+l} \right|
\leq \frac{1}{k} \sum_{l=1}^{k} \left| \tilde{\mu}_l \tilde{b}_{N_0-k+l} - \mu_l b_{N_0-k+l} \right|
= \frac{1}{k} \sum_{l=1}^{k} \left( |\tilde{\mu}_l| \left| \tilde{b}_{N_0-k+l} - \mu_l b_{N_0-k+l} \right| + |\tilde{\mu}_l - \mu_l| \right)
\leq \frac{1}{k} \sum_{l=1}^{k} \left( \gamma \alpha - \gamma \alpha \right)
\leq \frac{1}{k} \sum_{l=1}^{k} \left( \frac{N_0 + 1}{k} - \frac{N_0}{k} - \frac{1}{k} \right)
\leq \frac{1}{k} \sum_{l=1}^{k} \left( \frac{N_0 + 1}{k} \right)
\leq \frac{1}{k} \sum_{l=1}^{k} \left( \frac{N_0 + 1}{k} \right)
\leq \frac{1}{k} \left( \frac{N_0 + 1}{k} \right)
\]
thus \( C_k = (1/k) \sum_{i=1}^{k} \left( (N_0 + 1)C_{k-i} + \left( \frac{N_0}{k-i} \right) \right) \) suffices.

Next, we show how the coefficients of the monic polynomial factor containing the roots on the inside of the disk can be estimated from perturbed moments.

**Theorem 3.3** Let \( f(z) = \sum_{k=0}^{N} a_k z^k \), with fixed positive constant \( m \) such that \( 0 < m \leq |f(z)| \) for all \( z \) on the unit circle \( S \) and let \( M' = \|f'\|_\infty \). Let \( N_0 \) be the number of roots of \( f \) inside the unit circle. Let
\[
\alpha = \frac{1}{1 + 2 \left( 1 + \frac{M'}{m} \right)},
\]
and \( \varepsilon > 0 \) with \( \varepsilon < \alpha m^2 \), \( p_1 : S \to \mathbb{C} \) and \( p_2 : S \to \mathbb{R} \) with \( \|p_1\|_\infty < \varepsilon \), \( \|p_2\|_\infty < \varepsilon \), then
\[
\tilde{\mu}_k = \frac{1}{2\pi i} \oint_S z^k f'(z) \overline{f(z)} + p_1(z) \overline{f(z)} + p_2(z) \overline{f(z)} d\zeta
\]
for \( k \in \{1, 2, \ldots, N_0\} \) observes
\[ |\mu_k - \tilde{\mu}_k| \leq \frac{\epsilon}{(1 - \alpha)m^2} \left( 1 + \frac{M'}{m} \right) \equiv \gamma \]

and if \( \gamma \leq 1 \) which only depends on \( N_0 \) such that \( f_i(z) = \sum_{k=0}^{N_0} b_k z^k = \prod_{j=1}^{N_0} (z - z_j) \), the monic factor of \( f \) whose roots are precisely the roots of \( f \) that inside the unit disk, has approximate coefficients \( \{\tilde{b}_k\} \) with
\[ |\tilde{b}_k - b_k| \leq C \frac{\epsilon}{(1 - \alpha)m^2} \left( 1 + \frac{M'}{m} \right). \]

**Proof** Note that
\[ \forall z \in S, \ (|f|^2 + p_2)(z) \geq m^2 + p_2(z) \geq m^2 - \epsilon > (1 - \alpha)m^2 \]
so by the first lemma we have
\[ \left\| \frac{f^'f + p_1}{|f|^2 + p_2} - \frac{f^'f}{|f|^2} \right\|_{\infty} < \frac{\epsilon}{(1 - \alpha)m^2} \left( 1 + \left\| \frac{f'}{f} \right\|_{\infty} \right) \leq \frac{\epsilon}{(1 - \alpha)m^2} \left( 1 + \frac{M'}{m} \right) \]

Let \( \mu_k = \sum_{j=1}^{N_0} z_j^k \), the \( k \)th moment of the inner roots of \( f \), then the residue theorem gives us that for any integer \( k \in [0, N] \)
\[ \mu_k = \frac{1}{2\pi i} \oint_S z^k \frac{f'(z)f(z)}{|f(z)|^2} dz \]

If we let
\[ \tilde{\mu}_k = \frac{1}{2\pi i} \oint_S z^k \frac{f'(z)f(z) + p_1(z)}{|f(z)|^2 + p_2(z)} d\zeta = \mu_k + \frac{1}{2\pi i} \oint_S z^k \left( \frac{f'(z)f(z) + p_1(z)}{|f(z)|^2 + p_2(z)} - \frac{f'(z)f(z)}{|f(z)|^2} \right) d\zeta \]
then
\[ |\tilde{\mu}_k - \mu_k| = \frac{1}{2\pi} \left| \oint_S z^k \left( \frac{f'(z)f(z) + p_1(z)}{|f(z)|^2 + p_2(z)} - \frac{f'(z)f(z)}{|f(z)|^2} \right) d\zeta \right| \]
\[ \leq \frac{1}{2\pi} \oint_S |z|^k \left| \frac{f'(z)f(z) + p_1(z)}{|f(z)|^2 + p_2(z)} - \frac{f'(z)f(z)}{|f(z)|^2} \right| d\zeta \]
\[ \leq \frac{1}{2\pi} \oint_S \left\| \frac{f^'f + p_1}{|f|^2 + p_2} - \frac{f^'f}{|f|^2} \right\|_{\infty} d\zeta \]
\[ \leq \frac{\epsilon}{(1 - \alpha)m^2} \left( 1 + \frac{M'}{m} \right). \]

Note that \( N_0 = \mu_0 \). Because \( \epsilon < \alpha m^2 \) we have that
\[ |\tilde{\mu}_0 - \mu_0| \leq \frac{\epsilon}{(1 - \alpha)m^2} \left( 1 + \frac{M'}{m} \right) \leq \frac{\alpha m^2}{(1 - \alpha)m^2} \left( 1 + \frac{M'}{m} \right) = \frac{1}{(\frac{1}{\alpha} - 1)} \left( 1 + \frac{M'}{m} \right) = \frac{1}{2} \]
so rounding \( \tilde{\mu}_0 \) gives us \( N_0 \). Thus by the second lemma, we can recover an approximation for \( f_i(z) \) with approximated coefficients \( \tilde{b}_k \) such that \( |\tilde{b}_k - b_k| \leq C\gamma \). Now re-expressing \( \gamma \) in terms of \( \epsilon \) gives the desired result. \( \square \)

**Corollary 3.4** If \( f \) satisfies the hypotheses of the preceding theorem, and in addition we have \( M = \| f \|_\infty, M' = \| f' \|_\infty \), and \( \epsilon < \frac{\tilde{\mu}m^2}{d} \) for \( d > N \) with

\[
\beta = \frac{1}{1 + 2 \left( 1 + \frac{(d-1)M+M'}{m} \right)}
\]

and if \( g(z) = z^{d-1}f\left(\frac{1}{z}\right) \), then using the perturbation with \( p_1 \) and \( p_2 \) as above, we can recover an approximation for \( g_i(z) = \sum_{k=0}^{N_0} b_k z^k \) (the monic factor of \( g(z) \) whose roots are precisely the roots of \( g(z) \) inside the unit disk) with approximated coefficients \( \tilde{b}_k \) such that \( |\tilde{b}_k - b_k| \) is \( O(\epsilon) \) as \( \epsilon \to 0 \).

**Proof** First, we note that \( \frac{1}{z} = \bar{z} \) on \( \mathbb{S} \). Thus, \( |g(z)| = |z^{d-1}f(\bar{z})| = |f(\bar{z})| \) on \( \mathbb{S} \). Then because we have \( m \leq |f(\bar{z})| \leq M \) on \( \mathbb{S} \), we also have \( m \leq |g(z)| \leq M \) on \( \mathbb{S} \). We also know that \( g'(z) = (d-1)\frac{1}{z}g(z) - z^{d-1-2}f'(\frac{1}{z}) \), so that \( |g'(z)| \leq (d-1)|g(z)| + |f'(\bar{z})| \leq (d-1)M + M' \) on \( \mathbb{S} \). Note that on \( \mathbb{S} \), \( |g(z)|^2 = |f(\bar{z})|^2 \), which has a perturbation of \( p_2(\bar{z}) \), and

\[
g'(z)g(z) = \left( (d-1)\frac{1}{z}g(z) - z^{d-1-2}f'(\frac{1}{z}) \right) \overline{g(z)}
\]

\[
= (d-1)\bar{z}|g(z)|^2 - z^{d-1-2}f'(\bar{z})z^{d-1}\overline{f(\bar{z})}
\]

\[
= (d-1)\bar{z}|f(\bar{z})|^2 - \bar{z}^2 f'(\bar{z}) \overline{f(\bar{z})}
\]

which has a perturbation of \( (d-1)\bar{z}p_2(\bar{z}) - \bar{z}^2p_1(\bar{z}) \). Note that both of these perturbations are bounded by \( d\epsilon \). Thus, \( g(z) \) is a complex polynomial that satisfies the requirements of the theorem, with \( N \) replaced by \( d \), \( M' \) replaced by \( (d-1)M + M' \), \( \epsilon \) replaced by \( d\epsilon \), and \( \alpha \) replaced by \( \beta \). Thus by the theorem, using the perturbed functions for \( f(z) \), we can recover an approximation for \( g_i(z) \) with approximated coefficients \( \tilde{b}_k \) such that \( |\tilde{b}_k - b_k| \) is \( O(d\epsilon) = O(\epsilon) \). \( \square \)

The objective of the following proposition is to control the error when the recovery of the polynomial factors from the inner and the outer roots are combined.

**Proposition 3.5** If \( f \) satisfies the hypotheses of the preceding theorem, and in addition \( M = \| f \|_\infty, M' = \| f' \|_\infty \), and \( \epsilon < \frac{\tilde{\mu}m^2}{d} \) for \( d > N \) with

\[
\beta = \frac{1}{1 + 2 \left( 1 + \frac{(d-1)M+M'}{m} \right)}
\]

then using the perturbed functions for \( f(z) \), we can recover an approximation (up to a multiplicative constant) for \( f(z) = \sum_{k=0}^{d-1} c_k z^k \) with approximated coefficients \( \tilde{c}_k \) such that \( \max_k |\tilde{c}_k - c_k| \) is \( O(\epsilon) \).
Proof Let $N_i$ be the number of roots of $f(z)$ inside the unit disk, and let $N_o$ be the number of roots of $f(z)$ outside the unit disk. Note that $g_i(z)$ obtained in the previous corollary has roots \( \left( \frac{1}{z_j} \right)_{j=1}^{N_o} \) for all roots $z_j$ of $f(z)$ outside the unit disk, with $d - 1 - N$ additional roots at 0. Thus,

\[
g_i(z) = z^{d-1-N} \prod_{j=1}^{N_o} \left( z - \frac{1}{z_j} \right)
\]

Then if we let $f_o(z) = z^{d-1-N_i} g_i \left( \frac{1}{z} \right)$, we get

\[
f_o(z) = z^{d-1-N_i} z^{N-(d-1)} \prod_{j=1}^{N_o} \left( \frac{1}{z} - \frac{1}{z_j} \right) = z^{N-N_i-N_o} \prod_{j=1}^{N_o} \frac{1}{z_j} (z_j - z) = \prod_{j=1}^{N_o} \frac{(-1)^{N_o}}{z_j} (z - z_j)
\]

and so if \( \left( z'_j \right)_{j=1}^{N_i} \) are the roots of $f(z)$ inside the unit disk, by applying the theorem we get

\[
f_o(z) f_i(z) = \prod_{j=1}^{N_o} \frac{(-1)^{N_o}}{z_j} \left( z - z_j \right) \prod_{j=1}^{N_i} \left( z - z'_j \right)
\]

which is a constant multiple of $f(z)$. Thus, we just need an approximation for $f_o(z) f_i(z)$. In terms of the coefficients of $g_i(z)$, we have

\[
f_o(z) = z^{d-1-N_i} \sum_{k=0}^{N_0+d-1-N} b_k z^{d-1-N_i-k} = \sum_{k=0}^{d-1-N_i} b_{d-1-N_i-k} z^k
\]

Note that if $r$ is the multiplicity of the 0 root of $g_i(z)$, then for all $k$ from 0 to $r - 1$, $b_k = 0$. Thus, the coefficients for the $r$ highest degree terms of $f_o(z)$ are equal to 0, and $f_o(z)$ has degree $N_o$ as it should. Note that we can obtain $f_o(z)$ simply by reversing the order of the coefficients of $g_i(z)$, so our approximated coefficients $\tilde{b}_k$ for $f_o(z)$ are the same approximated coefficients that we obtained in the previous corollary, but in reverse order. Thus \( |\tilde{b}_k - b_k| = O(\epsilon) as \epsilon \to 0 \). In other words, there are numbers $C_{d,b_k}$ which do not depend on $\epsilon$, such that \( |\tilde{b}_k - b_k| \leq C_{d,b_k} \epsilon \). Also, from the theorem, we have that the approximated coefficients $\tilde{a}_j$ for $f_i(z) = \sum_{j=0}^{N_i} a_j z^j$ also have error that is $O(\epsilon)$ as $\epsilon \to 0$, so there are numbers $C_{d,a_j}$ which do not depend on $\epsilon$, such that \( |\tilde{a}_j - a_j| \leq C_{d,a_j} \epsilon \). Note that on $\mathbb{S}$

\[
|f_i(z)| = \prod_{j=1}^{N_i} |z - z'_j| \leq 2^{N_i}
\]

and

\[
|f_o(z)| = \left| z^{d-1-N_i} z^{N-(d-1)} \prod_{j=1}^{N_o} \left( \frac{1}{z} - \frac{1}{z_j} \right) \right| = \prod_{j=1}^{N_o} \left| \frac{z - \frac{1}{z_j}}{z_j} \right| \leq 2^{N_o}, z \in \mathbb{S}
\]

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Thus, the max norms on $\mathbb{S}$ of $f_i(z)$ and $f_o(z)$ are bounded by $2^{N_i}$ and $2^{N_o}$ respectively. Because all norms on a finite dimensional space are equivalent, there exists a number $K_N$, which non-decreasingly depends only on $N$, such that for any complex polynomial $h(z) = \sum_{k=0}^{N} c_k z^k$ of degree less than or equal to $N$, we have $\max_{k \leq N} |c_k| \leq K_N \|h(z)\|_\infty$. Note that

$$f(z) = f_o(z) f_i(z) = \left( \sum_{k=0}^{d-1-N_i} b_{d-1-N_i-k} z^k \right) \left( \sum_{j=0}^{N_i} a_j z^j \right) = \sum_{n=0}^{d-1} \left( \sum_{k=0}^{n} b_{d-1-N_i-k} a_{n-k} \right) z^n$$

and thus, for the approximation $\tilde{f}(z) = \sum_{k=0}^{d-1} \tilde{c}_k z^k$ we have that for each $k$

$$\tilde{c}_k = \sum_{j=0}^{n} \tilde{b}_{d-1-N_i-j} \tilde{a}_{k-j}$$

Then for each $k$ we have

$$|\tilde{c}_k - c_k| = \sum_{j=0}^{n} |\tilde{b}_{d-1-N_i-j} \tilde{a}_{k-j} - b_{d-1-N_i-j} a_{k-j}|$$

$$\leq \sum_{j=0}^{n} \left| \tilde{b}_{d-1-N_i-j} \tilde{a}_{k-j} - b_{d-1-N_i-j} a_{k-j} \right|$$

$$\leq \sum_{j=0}^{n} \left( \left| \tilde{b}_{d-1-N_i-j} \tilde{a}_{k-j} - b_{d-1-N_i-j} a_{k-j} \right| + \left| \tilde{b}_{d-1-N_i-j} a_{k-j} - b_{d-1-N_i-j} a_{k-j} \right| \right)$$

$$= \sum_{j=0}^{n} \left( \left| \tilde{a}_{k-j} - a_{k-j} \right| + \left| \tilde{b}_{d-1-N_i-j} - b_{d-1-N_i-j} \right| \left| a_{k-j} \right| \right)$$

$$\leq \sum_{j=0}^{n} \left( K_{N_i} 2^{N_o} + C_d b_{d-1-N_i-j} \epsilon \right) C_d a_{k-j} \epsilon + C_d b_{d-1-N_i-j} \epsilon K_{N_i} 2^{N_i}$$

$$\leq K_d 2^{d} \sum_{j=0}^{n} \left( C_d a_{k-j} + C_d b_{d-1-N_i-j} \right) \epsilon + O(\epsilon^2)$$

Thus, since $\epsilon$ was bounded above, multiplying two polynomials, whose coefficients have an error that is $O(\epsilon)$ gives a polynomial whose coefficients have an error that is $O(\epsilon)$.

Finally, we obtain a finite number of measurements by discretizing and interpolating with the Dirichlet kernel.
Theorem 3.6 Let \( f(z) = \sum_{k=0}^{d-1} c_k z^k \) be a complex polynomial with degree at most \( d-1 \), with fixed positive constant \( m \) such that \( 0 < m \leq \| f(z) \| \) for all \( z \) on the unit circle \( \mathbb{S} \) and let \( M = \| f \|_\infty \) and \( M' = \| f' \|_\infty \). Let \( \omega = e^{ \frac{2 \pi i}{d} } \) and \( v = e^{ \frac{2 \pi i}{d} } \) be the \((2d-1)\)th and 3rd roots of unity, and let

\[
\beta = \frac{1}{1 + 2 \left( 1 + \frac{(d-1)M+M'}{m} \right)^2}
\]

Let \( \epsilon > 0 \) with \( \epsilon < \frac{\beta m^2}{(2d-1)d} \), and assume that we know \( 2d - 1 \) values \( \| f'(\omega^j) \|^2 + \epsilon_l, j \leq \epsilon \) with each \( \epsilon_l \leq \epsilon \). Then using only these values, we can recover an approximation (up to a multiplicative constant) for \( f(z) \) with approximated coefficients \( \tilde{c}_k \) such that \( |\tilde{c}_k - c_k| \) is \( O(\epsilon) \).

Proof If \( D_{d-1}(z) = \frac{1}{2d-1} \sum_{k=-d+1}^{d-1} z^k \) is the normalized Dirichlet kernel of degree \( d-1 \), then the set of functions \( \{ z \mapsto D_{d-1}(z\omega^l) \}_{l=0}^{2(d-1)} \) is orthogonal with respect to the \( L^2 \) norm on \( \mathbb{S} \), and in addition it provides interpolation identity for each trigonometric polynomial \( g(z) = \sum_{k=-d+1}^{d-1} c_k z^k \), \( g(z) = \sum_{l=0}^{2(d-1)} g(\omega^l) D_{d-1}(z\omega^l) \). If we let \( g_0(z) = |f(z)|^2 \) and \( g_j(z) = |f(z) + v^j f'(z)|^2 \) for \( j = 1, 2, 3 \), then because each of these is a trigonometric polynomial of degree at most \( d-1 \), we know that we have

\[
\sum_{l=0}^{2(d-1)} \left( g_j(\omega^l) + \epsilon_{l,j} \right) D_{d-1}(z\omega^{-l}) = \sum_{l=0}^{2(d-1)} g_j(\omega^l) D_{d-1}(z\omega^{-l}) + \sum_{l=0}^{2(d-1)} \epsilon_{l,j} D_{d-1}(z\omega^{-l})
\]

\[
= g_j(z) + \sum_{l=0}^{2(d-1)} \epsilon_{l,j} D_{d-1}(z\omega^{-l})
\]

Let \( h_j(z) = \sum_{l=0}^{2(d-1)} \epsilon_{l,j} D_{d-1}(z\omega^{-l}) \) be the error obtained when using the Dirichlet kernel with the known given values to recover each \( g_j(z) \). Note that

\[
|h_j(z)| = \left| \sum_{l=0}^{2(d-1)} \epsilon_{l,j} D_{d-1}(z\omega^{-l}) \right| \leq \sum_{l=0}^{2(d-1)} \left| \epsilon_{l,j} D_{d-1}(z\omega^{-l}) \right| \leq (2d-1)\epsilon |D_{d-1}(z\omega^{-l})| = (2d-1)\epsilon
\]

Thus we have recovered approximations for the functions \( g_j(z) \) on \( \mathbb{S} \), including \( g_0(z) = |f(z)|^2 \) with perturbation that is less than \( (2d-1)\epsilon \). However, to use the previous corollary, we also need an approximation for \( f'(z) \). To obtain this, note that
\[
\frac{1}{3} \sum_{j=1}^{3} \nabla^j (g_j(z) + h_j(z)) = \frac{1}{3} \sum_{j=1}^{3} \nabla^j g_j(z) + \frac{1}{3} \sum_{j=1}^{3} \nabla^j h_j(z)
\]

\[
= \frac{1}{3} \sum_{j=1}^{3} \nabla^j \left| f(z) + v^j f'(z) \right|^2 + \frac{1}{3} \sum_{j=1}^{3} \nabla^j h_j(z)
\]

\[
= \frac{1}{3} \sum_{j=1}^{3} \left(\nabla^j |f(z)|^2 + \nabla^j f(z) f'(z) + \nabla^j |f'(z)|^2 \right) + \frac{1}{3} \sum_{j=1}^{3} \nabla^j h_j(z)
\]

\[
= f'(z) \overline{f(z)} + \frac{1}{3} \sum_{j=1}^{3} \nabla^j h_j(z)
\]

Note that \( \left| \frac{1}{3} \sum_{j=1}^{3} \nabla^j h_j(z) \right| \leq \frac{1}{3} \sum_{j=1}^{3} |h_j(z)| < \frac{bm^2}{d} \). Thus, we have recovered approximations for \( |f(z)|^2 \) and \( f'(z) \overline{f(z)} \) with error bounded by \( \frac{bm^2}{d} \). Then, by using the previous corollary with \((2d-1)\varepsilon\) in place of the \(\varepsilon\) in that corollary, we recover an approximation (up to a multiplicative constant) for \( f(z) \) with approximated coefficients \( \tilde{c}_k \) such that \( |\tilde{c}_k - c_k| \) is \( O(\varepsilon) \).

**Corollary 3.7** Let \( f \) be a complex polynomial of degree at most \( d - 1 \), let \( \omega \) and \( v \) and \( \varepsilon > 0 \) satisfy the conditions with \( m, M, M' \) and \( \beta \) be as in the preceding theorem, then \( \{ |f(\omega^l)|^2 + \varepsilon_{l,0} \}_{l=0}^{2(d-1)} \cup \{ |f(\omega^l) + v^j f'(\omega^l)|^2 + \varepsilon_{l,j} \}_{l=0}^{2(d-1)} \) \( j \) with each \( \varepsilon_{l,j} \leq \varepsilon \) determines an approximation of \( f \), up to a unimodular constant, with accuracy \( O(\varepsilon) \).

**Proof** The measured quantities \( \{ |f(\omega^l)|^2 + \varepsilon_{l,0} \}_{l=0}^{2(d-1)} \cup \{ |f(\omega^l) + v^j f'(\omega^l)|^2 + \varepsilon_{l,j} \}_{l=0}^{2(d-1)} \) with each \( \varepsilon_{l,j} \leq \varepsilon \) determines an approximation of \( f \), up to a unimodular constant, with accuracy \( O(\varepsilon) \).

Note that the lower bound \( m \) from the above theorems does not have a simple interpretation if we look at our vector in the space \( \mathbb{C}^n \). But we can note that as this bound shrinks, the radius of stability shrinks as well, to the point where stability is lost entirely if there is a root on the unit circle. Also note that the \( \frac{(d-1)M + M'}{m} \) term in...
the denominator of the radius of stability indicates that the radius of stability will be wider for a polynomial that is closer to constant on the unit circle.

To illustrate the linear stability results we applied the recovery algorithm to a random polynomial in dimension \(d = 10\), see Fig. 1. We generated the polynomial by selecting \(N = 9\) roots that are independent, identically normally distributed in their real and imaginary parts each with variance \(1/2\), and perturbed the measured quantities with noise that was uniformly distributed in an interval \([-\epsilon, \epsilon]\) before applying the recovery. For all sample polynomials, we obtained an empirical error bound on the difference in the coefficients of the recovered polynomial and the original polynomial, which grows proportionally with \(\epsilon\). Even when we perturb the values by 10 times the radius of stability \(\epsilon_0 = \frac{\beta m^2}{(2d-1)d^{d-1}}\) given by the theorem, the bound remains intact. This indicates that either the radius of stability used in our theorem could be improved, or that unstable behavior happens outside of this radius only for pathological examples.

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