WELL-POSEDNESS FOR THE CAUCHY PROBLEM OF THE KLEIN-GORDON-ZAKHAROV SYSTEM IN FOUR AND MORE SPATIAL DIMENSIONS

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Abstract. We study the Cauchy problem of the Klein-Gordon-Zakharov system in spatial dimension \(d \geq 4\) with radial or non-radial initial datum \((u, \partial_t u, n, \partial_t n)|_{t=0} \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)\). The critical value of \(s\) is \(s_c = d/2 - 2\). By the radial Strichartz estimates and \(U^2, V^2\) type spaces, we prove that the small data global well-posedness and scattering hold at \(s = s_c\) in \(d \geq 4\) for radial initial datum. For non-radial initial datum, we prove that the local well-posedness hold at \(s = 1/4\) when \(d = 4\) and \(s = s_c + 1/(d + 1)\) when \(d \geq 5\).

1. Introduction. We consider the Cauchy problem of the Klein-Gordon-Zakharov system:

\[
\begin{cases}
(\partial_t^2 - \Delta + 1)u = -nu, & (t, x) \in [-T, T] \times \mathbb{R}^d, \\
(\partial_t^2 - c^2 \Delta)n = \Delta |u|^2, & (t, x) \in [-T, T] \times \mathbb{R}^d, \\
(u, \partial_t u, n, \partial_t n)|_{t=0} = (u_0, u_1, n_0, n_1) \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d),
\end{cases}
\]

where \(u, n\) are real valued functions, \(d \geq 4, c > 0\) and \(c \neq 1\). The physical model of (1.1) is the interaction of the Langmuir wave and the ion acoustic wave in a plasma. In the physical model, \(c\) satisfies \(0 < c < 1\). When \(d = 3\), Ozawa, Tsutaya and Tsutsumi [26] proved that (1.1) is globally well-posed in the energy space \(H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3)\). They applied the Fourier restriction norm method to obtain the local well-posedness. Then by the local well-posedness and the energy method, they obtained the global well-posedness. For \(d = 3\), Guo, Nakanishi and Wang [6] proved the scattering in the energy class with small, radial initial data. They applied the normal form reduction and the radial Strichartz estimates. If we transform \(u_\pm := \omega_1 u + i \partial_t u, n_\pm := n + i(c\omega)^{-1} \partial_t n, \omega_1 := (1 - \Delta)^{1/2}, \omega := (-\Delta)^{1/2}\),

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\[ \begin{cases} (i_\partial + c_\omega) u_{\pm} = \pm (1/4) (n_{\pm} + n_{-}) (\omega_1^{-1} u_{+} + \omega_1^{-1} u_{-}), & (t, x) \in [-T, T] \times \mathbb{R}^d, \\ (i_\partial + c_\omega) u_{\pm n} = \pm (4c) \omega_1^{-1} u_{+} + \omega_1^{-1} u_{-} |^2, & (t, x) \in [-T, T] \times \mathbb{R}^d, \\ (u_{+}, n_{\pm}) |_{t=0} = (u_{+0}, n_{\pm0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d). \end{cases} \]

(1.2)

Our main result is as follows.

**Theorem 1.1.** (i) Let \( d = 4 \). Then (1.2) is locally well-posed in \( H^{1/4}(\mathbb{R}^4) \times \dot{H}^{1/4}(\mathbb{R}^4) \).

(ii) Let \( d \geq 5 \) and \( s = (d^2 - 3d - 2)/2(d + 1) \). Then (1.2) is locally well-posed in \( H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \).

(iii) Let \( d \geq 4, s = s_c = d/2 - 2 \) and assume the initial data \( (u_{\pm0}, n_{\pm0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \) is small and radial. Then, (1.2) is globally well-posed in \( H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \).

**Corollary 1.2.** The solution obtained in Theorem 1.1 (iii) scatters as \( t \to \pm \infty \).

For more precise statement of Theorem 1.1 and Corollary 1.2, see Propositions 4.1, 4.2. By the Duhamel principle, we consider the integral equation (4.2) corresponding to (1.2). For the integral equation (4.2), the theorem is proved by the Banach fixed point theorem. The key is the bilinear estimates (Proposition 3.1). We apply \( U^2, V^2 \) type spaces, which are introduced by Koch-Tataru. These space works well when we prove well-posedness at the scaling critical space. The critical regularity of (1.2) is \( s_c = d/2 - 2 \). First, we consider the radial case. To prove the bilinear estimate for Theorem 1.1 (iii), we need to make use of \( \omega_1^{-1} \) in the nonlinearities. We observe the first equation of (1.2). We regard the nonlinearity as \( n_{\pm} (\omega_1^{-1} u_{\pm}) \). We have

\[ \mathcal{F}_{t,x} [n_{\pm} (\omega_1^{-1} u_{\pm})](\tau, \xi) = \int \int \mathcal{F}_{t,x} [n_{\pm}](\tau + \xi, \xi') \langle \xi - \xi' \rangle^{-1} \mathcal{F}_{t,x} [u_{\pm}](\tau - \tau', \xi - \xi') d\tau' d\xi'. \]

We consider the following cases. The case \( |\xi'| \leq |\xi - \xi'| \) and the case \( |\xi'| \gg |\xi - \xi'| \). For the case \( |\xi'| \leq |\xi - \xi'| \), we have \( |\xi - \xi'|^{-1} \gg |\xi'|^{-1/2} |\xi - \xi'|^{-1/2} \). Hence, when \( d \geq 5 \), we can obtain the bilinear estimates at the critical space only by applying the Strichartz estimates. For the case \( |\xi'| \gg |\xi - \xi'| \), we need to gain a half derivative. We can gain a half derivative when we apply \( U^2, V^2 \) type spaces and the following inequality.

\[ M' := \max \{|\tau' \pm c|\xi'|, |\tau - \tau' \pm (\xi - \xi')|, |\tau \pm (\xi)| \} \gg |\xi'|. \]  

(1.3)

Here, \( \tau' \pm c|\xi'| \) (resp. \( \tau - \tau' \pm (\xi - \xi') \), \( \tau \pm (\xi) \)) denote the symbol of the linear part for the wave equation (resp. Klein-Gordon equation). There are three cases in (1.3). The cases (a) \( M' = |\tau' \pm c|\xi'| \), (b) \( M' = |\tau - \tau' \pm (\xi - \xi')| \), (c) \( M' = |\tau \pm (\xi)| \). For the case (a), we apply (1.3) for \( n_{\pm} \) and apply the Strichartz estimates for \( \omega_1^{-1} u_{\pm} \). Then we can obtain the bilinear estimate at the critical space. The same result also holds for (c) by the duality argument. Whereas for (b), when we apply the radial Strichartz estimates for \( n_{\pm} \) and apply (1.3) for \( \omega_1^{-1} u_{\pm} \), we can obtain the bilinear estimate at the critical space. The radial Strichartz estimates hold for wider range of \( (q, r) \). For more precise statement, see Propositions 2.11, 2.12. The restriction for radial admissible pair is weaker, hence we can make use of \( \omega_1^{-1} \) by the Sobolev embedding, so we can obtain the bilinear estimate at the critical space. When \( d = 4 \), we apply the radial Strichartz estimates for the case \( |\xi'| \leq |\xi - \xi'| \) as well as the case \( |\xi'| \gg |\xi - \xi'| \) and (b). Next, we consider \( d = 4 \) and the non-radial
case. When $d \leq 4$, it seems difficult to obtain well-posedness at the critical space because of the Lorentz invariance. Therefore, we consider well-posedness at the Lorentz regularity $s_l = 1/4$, which is determined by the Lorentz invariance. Similar to Theorem 1.1 (iii), we apply $U^2, V^2$ type spaces and we obtain Theorem 1.1 (i).

Since $s_c \geq s_l$ when $d \geq 5$, we expect well-posedness at the critical space. However, it seems difficult to prove the bilinear estimate for the case $|\xi| \gg |\xi - \xi'|$ and (b) as mentioned above. As a result, we have to impose more regularity, that is, $s = (d^2 - 3d - 2)/(2(d + 1)) = s_c + 1/(d + 1)$.

In section 2, we prepare some notations and lemmas with respect to $U^p, V^p$, in section 3, we prove the bilinear estimates and in section 4, we prove the main result.

2. Notations and preliminary lemmas. In this section, we prepare some notations, propositions and notations to prove the main theorem. $A \lesssim B$ means that there exists $C > 0$ such that $A \leq CB$. Also, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. Let $u = u(t, x), \mathcal{F}_t u, \mathcal{F}_x u$ denote the Fourier transform of $u$ in time, space, respectively. $\mathcal{F}_{t,x} u = \hat{u}$ denotes the Fourier transform of $u$ in space and time. Let $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \cdots < t_K = \infty$ and let $\mathcal{Z}_0$ be the set of finite partitions $-\infty < t_0 < t_1 < \cdots < t_K < \infty$. Let $H$ be $L^2_x$ or $L^2_x = \{ f \in L^2_x | f$ is radial $\}$ in this section.

Definition 1. Let $1 \leq p < \infty$. For $\{ \lambda_k \}_{k=0}^K \in \mathcal{Z}$ and $\{ \phi_k \}_{k=0}^{K-1} \subset H$ with $\sum_{k=0}^{K-1} \| \phi_k \|_H = 1$, we call the function $a : \mathbb{R} \rightarrow H$ given by

$$a = \sum_{k=1}^K 1_{(t_{k-1}, t_k)} \phi_{k-1}$$

a $U^p$-atom. Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^\infty \lambda_j a_j \mid a_j : U^p$-atom, $\lambda_j \in \mathbb{C}$ such that $\sum_{j=1}^\infty |\lambda_j| < \infty \right\}$$

with norm

$$\| u \|_{U^p} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| \mid u = \sum_{j=1}^\infty \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j : U^p$-atom $\right\}.$$
Proposition 2.2. Let \(1 \leq p < q < \infty\).

(i) Let \(v : \mathbb{R} \to H\) be such that

\[
\|v\|_{V^q_p} := \sup_{\{t_k\}_{k=0}^K} \left( \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{U_H}^p \right)^{1/p}
\]

is finite. Then, it follows that \(v(t^+_0) := \lim_{t \to t_0^+} v(t)\) exists for all \(t_0 \in [-\infty, \infty)\) and \(v(t^-_0) := \lim_{t \to t_0^-} v(t)\) exists for all \(t_0 \in (-\infty, \infty)\) and moreover,

\[
\|v\|_{V^q_p} = \|v\|_{V^q_p}^\prime.
\]

(ii) We define the closed subspace \(V^p_\infty(V^q_p)\) of all right-continuous \(V^p\) functions \((V^q\) functions). The spaces \(V^p\), \(V^p_\infty\), \(V^q\) and \(V^q_\infty\) are Banach spaces.

(iii) The embeddings \(U^p \subset V^p_\infty \subset U^q\) are continuous.

(iv) The embeddings \(V^p \subset V^q\) and \(V^p_\infty \subset V^q_\infty\) are continuous.

The proof of Proposition 2.2 is in [8] (Proposition 2.4 and Corollary 2.6). Let \(\{\mathcal{F}_t\}^{\varphi_n}(x) \subset \mathcal{S}(\mathbb{R}^d)\) be the Littlewood-Paley decomposition with respect to \(x\), that is to say

\[
\varphi \geq 0,
\supp \varphi = \{\xi \mid 2^{-1} \leq |\xi| \leq 2\},
\varphi_n(\xi) := \varphi(2^{-n}\xi), \quad \sum_{n=\infty}^{\infty} \varphi_n(\xi) = 1 \quad (\xi \neq 0),
\psi(\xi) := 1 - \sum_{n=0}^{\infty} \varphi_n(\xi).
\]

Let \(N = 2^n \) \((n \in \mathbb{Z})\) be a dyadic number. \(P_N\) and \(P_{<1}\) denote

\[
\mathcal{F}_t[P_N f](\xi) := \varphi(\xi/N)\mathcal{F}_t[f](\xi) = \varphi_n(\xi)\mathcal{F}_t[f](\xi),
\mathcal{F}_t[P_{<1} f](\xi) := \psi(\xi)\mathcal{F}_t[f](\xi).
\]

Similarly, let \(\tilde{Q}_N\) be

\[
\mathcal{F}_t[\tilde{Q}_N g](\tau) := \phi(\tau/N)\mathcal{F}_t[g](\tau) = \phi_n(\tau)\mathcal{F}_t[g](\tau),
\]

where \(\{\mathcal{F}_t\}^{\varphi_n}(t) \subset \mathcal{S}(\mathbb{R})\) is the Littlewood-Paley decomposition with respect to \(t\).

Let \(W_{\pm c}(t) = \sum_{k=1}^N \mathcal{F}_t[K \pm (\Delta)^{1/2}] \to L^2_{\pm c}\) be the Klein-Gordon unitary operator such that \(\mathcal{F}_t[K \pm (\Delta)^{1/2}] \to L^2_{\pm c}\). Similarly, we define the wave unitary operator \(W_{\pm c}(t) = \sum_{k=1}^N \mathcal{F}_t[K \pm (\Delta)^{1/2}] \to L^2_{\pm c}\) such that \(\mathcal{F}_t[W_{\pm c}(t)u](\xi) = \exp[\pm i\xi t]\mathcal{F}_t[u](\xi)\).

Definition 3. We define

(i) \(U^p_{K_{\pm}} = K_{\pm} U^p\) with norm \(\|u\|_{U^p_{K_{\pm}}} = \|K_{\pm}(-\cdot)u\|_{U^p}\),
(ii) \(V^p_{K_{\pm}} = K_{\pm} V^p\) with norm \(\|u\|_{V^p_{K_{\pm}}} = \|K_{\pm}(-\cdot)u\|_{V^p}\).

For a dyadic number \(N, M\),

\[
Q_N := K_{\pm}(\cdot)\tilde{Q}_N K_{\pm}(\cdot), \quad Q_{\geq M} := \sum_{N \geq M} Q_N, \quad Q_{< M} := Id - Q_{\geq M}.
\]

Here summation over \(N\) means that summation over \(n \in \mathbb{Z}\). Similarly, we define \(U^p_{W_{\pm c}}, V^p_{W_{\pm c}}\).

Remark 2.1. For \(A = K_{\pm}\) or \(W_{\pm c}\),

\[
U^2_A \subset V^2_{\infty,rc,A} \subset L^\infty(\mathbb{R}; H)
\]
Definition 4. For the Klein-Gordon equation, we define \( Y_{K^\pm}^s \) (resp. \( Z_{K^\pm}^s \)) as the closure of all \( u \in C(\mathbb{R}; H^s_x(\mathbb{R}^d)) \cap \langle \nabla_x \rangle^{-s} U_{K^\pm}^2 \) with \( Y_{K^\pm}^s \) (resp. \( Z_{K^\pm}^s \)) norm, where
\[
\|u\|_{Y_{K^\pm}^s} := \|P_{< 1} u\|_{V_{K^\pm}^2} + \left( \sum_{N \geq 1} N^{2s} \|P_N u\|_{V_{K^\pm}^2}^2 \right)^{1/2},
\]
\[
\|u\|_{Z_{K^\pm}^s} := \|P_{< 1} u\|_{U_{K^\pm}^2} + \left( \sum_{N \geq 1} N^{2s} \|P_N u\|_{U_{K^\pm}^2}^2 \right)^{1/2}.
\]

For the wave equation, we define \( \check{Y}_{W^\pm}^s \) and \( \check{Z}_{W^\pm}^s \) as the closure of all \( n \in C(\mathbb{R}; H^s_x(\mathbb{R}^d)) \cap \langle \nabla_x \rangle^{-s} V_{W^\pm}^2 \) (resp. \( n \in C(\mathbb{R}; H^s_x(\mathbb{R}^d)) \cap \langle \nabla_x \rangle^{-s} U_{W^\pm}^2 \)) with \( \check{Y}_{W^\pm}^s \) (resp. \( \check{Z}_{W^\pm}^s \)) norm, where
\[
\|n\|_{\check{Y}_{W^\pm}^s} := \left( \sum_N N^{2s} \|P_N n\|_{V_{W^\pm}^2}^2 \right)^{1/2}, \quad \|n\|_{\check{Z}_{W^\pm}^s} := \left( \sum_N N^{2s} \|P_N n\|_{U_{W^\pm}^2}^2 \right)^{1/2}.
\]

Definition 5. For a Hilbert space \( H' \) and a Banach space \( X \subset C(\mathbb{R}; H') \), we define
\[
B_r(H') := \{ f \in H' \mid \|f\|_{H'} \leq r \},
\]
\[
X([0, T)) := \{ u \in C([0, T); H') \mid \exists \tilde{u} \in X, \tilde{u}(t) = u(t), t \in [0, T) \}
\]
endowed with the norm \( \|u\|_{X([0, T))} = \inf \{\|\tilde{u}\|_{X} \mid \tilde{u}(t) = u(t), t \in [0, T)\} \).

We denote the Duhamel term
\[
I_{T, K^\pm}(n, v) := \pm \int_0^t 1_{[0, T]}(t') K_{\pm}(t - t') n(t') (\omega_{1}^{-1} v(t')) dt',
\]
\[
I_{T, W^\pm}(n, v) := \pm \int_0^t 1_{[0, T]}(t') W_{\pm}(t - t') \omega((\omega_{1}^{-1} v(t')) (\omega_{1}^{-1} u(t'))) dt'
\]
for the Klein-Gordon equation and the wave equation respectively.

Lemma 2.3. Let \( c > 0, c \neq 1 \) and \( \tau_3 = \tau_1 - \tau_2, \xi_3 = \xi_1 - \xi_2 \). If \( |\xi_1| \gg |\xi_2| \) or \( |\xi_1| \ll |\xi_2| \), then it holds that
\[
\max \{ |\tau_1 \pm |\xi_1| |, |\tau_2 \pm |\xi_2| |, |\tau_3 \pm c|\xi_3| | \} \geq \max \{|\xi_1|, |\xi_2| \} . \tag{2.1}
\]

Proof. We only prove the case \( |\xi_1| \gg |\xi_2| \) since the case \( |\xi_1| \ll |\xi_2| \) is proved by the same manner.
\[
(\text{I.h.s.}) \geq |(\tau_1 \pm (1 + |\xi_1|)) - (\tau_2 \pm (1 + |\xi_2|)) - (\tau_3 \pm c|\xi_3|)| \geq (2.2)
\]
If \( 0 < c < 1 \), then we take \( \varepsilon_c \) such that \( 0 < \varepsilon_c < (1 - c)/(1 + c), |\xi_2| \leq \varepsilon_c |\xi_1| \). Then, the right hand side of (2.2) is bounded by
\[
(1 + |\xi_1|) - (1 + |\xi_2|) - c|\xi_1 - \xi_2| \geq |\xi_1| - \varepsilon_c|\xi_1| - c(1 + \varepsilon_c)|\xi_1| \geq |\xi_1|. \]

If \( c > 1 \), then we take \( \tilde{\varepsilon}_c \) such that \( 0 < \tilde{\varepsilon}_c < (c - 1)/(c + 3), |\xi_2| \leq \tilde{\varepsilon}_c |\xi_1|, |\xi_1| \geq 1/\tilde{\varepsilon}_c \). Then, the right hand side of (2.2) is bounded by
\[
c|\xi_1 - \xi_2| - (1 + |\xi_1|) \geq c(1 - \tilde{\varepsilon}_c)|\xi_1| - (1 + \tilde{\varepsilon}_c)|\xi_1| - 2\tilde{\varepsilon}_c |\xi_1| \geq |\xi_1|. \]

The following proposition is in [8] (Theorem 2.8 and Proposition 2.10).
Proposition 2.4. \( u \in V_2 \subset U_2 \) be absolutely continuous on compact intervals. Then, \( \|u\|_{U_2} = \sup_{v \in V_2, \|v\|_{U_2} = 1} \left| \int_{-\infty}^{\infty} \langle u'(t), v(t) \rangle_{L^2_t} dt \right| \).

Corollary 2.5. Let \( A = K_{\pm} \) or \( W_{\pm} \) and \( u \in V_{2,A} \subset U_2 \) be absolutely continuous on compact intervals. Then,

\[
\|u\|_{U_{2,A}} = \sup_{v \in V_{2,A}, \|v\|_{U_{2,A}} = 1} \left| \int_{-\infty}^{\infty} \langle A(t)(A(-)u)'(t), v(t) \rangle_{L^2_t} dt \right|
\]

Lemma 2.6. Let \( M > 0 \) and \( Q \in \{Q_{<M}, Q_{\geq M}\} \). For \( 1 \leq p \leq \infty \) and \( f \in V_{K_{\pm}} \), it holds that

\[
\|Q(1_{[0,T]}f)\|_{L^p_t L^2_x} \lesssim T^{1/p} \|f\|_{V_{K_{\pm}}}. \tag{2.3}
\]

Proof. By scaling, we only need to prove (2.3) for \( M = 1 \). We will show (2.3) for \( Q = Q_{<1} \). Put \( g := K_{\pm}(-)f \). Then (2.3) is equivalent to

\[
\|Q_{<1}(1_{[0,T]}K_{\pm}(\cdot)g)\|_{L^p_t L^2_x} \lesssim T^{1/p} \|g\|_{V^2}. \tag{2.4}
\]

By the unitarity of \( K_{\pm} \), we have

\[
\|Q_{<1}(1_{[0,T]}K_{\pm}(\cdot)g)\|_{L^p_t L^2_x} = \left\| \sum_{N<1} K_{\pm}(\cdot) \tilde{Q}_N K_{\pm}(-)(1_{[0,T]}K_{\pm}(\cdot)g) \right\|_{L^p_t L^2_x}
\]

\[
= \left\| \sum_{N<1} \tilde{Q}_N(1_{[0,T]}g) \right\|_{L^p_t L^2_x}
\]

\[
= \|\tilde{Q}_{<1}(1_{[0,T]}g)\|_{L^p_t L^2_x}. \tag{2.5}
\]

By the definition of \( \tilde{Q}_{<1} \), there exists a Schwartz function \( \phi \), it holds that

\[
\tilde{Q}_{<1}h = \phi \ast_t h.
\]

Hence by the Young inequality and the Hölder inequality, we have

\[
\|\tilde{Q}_{<1}(1_{[0,T]}g)\|_{L^p_t L^2_x} \lesssim \|\phi\|_{L^1_t} \|1_{[0,T]}g\|_{L^{2}_t L^2_x}
\]

\[
\lesssim \|1_{[0,T]}\|_{L^p_t} \|g\|_{L^p_t L^2_x}
\]

\[
\lesssim T^{1/p} \|g\|_{V^2}. \tag{2.6}
\]

Collecting (2.5)–(2.6), we obtain (2.4). For the proof of (2.3) for \( Q = Q_{\geq 1} \), we use \( Q_{\geq 1} = Id - Q_{<1} \) and \( \|Id(1_{[0,T]}g)\|_{L^p_t L^2_x} \lesssim T^{1/p} \|g\|_{V^2} \). \( \square \)

Proposition 2.7. Let \( T_0 : H \times \ldots \times H \to L^1_{loc}(\mathbb{R}^d; \mathbb{C}) \) be a \( n \)-linear operator. Assume that for some \( 1 \leq p,q \leq \infty \), it holds that

\[
\|T_0(K_{\pm}(\cdot)\phi_1, \ldots, K_{\pm}(\cdot)\phi_n)\|_{L^p(R; L^q_x(\mathbb{R}^d))} \lesssim \prod_{i=1}^{n} \|\phi_i\|_H.
\]

Then, there exists \( T : U_{K_{\pm}}^p \times \ldots \times U_{K_{\pm}}^p \to L^p(R; L^q_x(\mathbb{R}^d)) \) satisfying

\[
\|T(u_1, \ldots, u_n)\|_{L^p(R; L^q_x(\mathbb{R}^d))} \lesssim \prod_{i=1}^{n} \|u_i\|_{U_{K_{\pm}}^p},
\]

such that \( T(u_1, \ldots, u_n)(t)(x) = T_0(u_1(t), \ldots, u_n(t))(x) \) a.e.

See Proposition 2.19 in [8] for the proof of the above proposition.
Proposition 2.8. Let $d \geq 3, 2 \leq r < \infty, 2/q = (d - 1)(1/2 - 1/r), (q, r) \neq (2, 2(d - 1)/(d - 3))$ and $s = 1/q - 1/r + 1/2$. Then it holds that
\[
\|W_{\pm}(t)f\|_{L_t^p W_x^{-s,r}((1^+)R^d)} \lesssim \|f\|_{L_x^q(\mathbb{R}^d)}.
\]

For the proof of Proposition 2.8, see [12], [4].

Proposition 2.9. Let $d \geq 3, 2 \leq r < \infty, 2/q = (d - 1)(1/2 - 1/r), (q, r) \neq (2, 2(d - 1)/(d - 3))$ and $s = 1/q - 1/r + 1/2$. Then, it holds that
\[
\|K_{\pm}(t)f\|_{L_t^p W_x^{-s,r}((1^+)R^d)} \lesssim \|f\|_{L_x^q(\mathbb{R}^d)}.
\]

For the proof of Proposition 2.9, see [22]. Combining Proposition 2.8, Proposition 2.9, Proposition 2.10, Proposition 2.12, Proposition 2.13 and Proposition 2.7, we have the following proposition.

Proposition 2.11. Let $d \geq 3$. Then, for all radial functions $f \in L_x^q(\mathbb{R}^d)$, it holds that
\[
\|W_{\pm}(t)P_Nf\|_{L_t^p L_x^q((1^+)R^d)} \lesssim N^{d(1/2 - 1/r) - 1/q}\|P_Nf\|_{L_x^q(\mathbb{R}^d)},
\]
if and only if
\[
(q, r) = (\infty, 2) \text{ or } 2 \leq q < \infty, \quad 1/q < (d - 1)(1/2 - 1/r).
\]  

See Theorem 1.5 (a) in [7] for the proof of Proposition 2.11.

Proposition 2.12. Let $d \geq 2$. If $2d/(d - 1) < q \leq \infty, N \geq 1$ and $f \in L_x^q(\mathbb{R}^d)$ is radial function, then it holds that
\[
\|K_{\pm}(t)P_Nf\|_{L_t^p L_x^q((1^+)R^d)} \lesssim N^{d(2/2 - (d+1)/q)}\|P_Nf\|_{L_x^q(\mathbb{R}^d)}.
\]

If $q \geq (4d + 2)/(2d - 1), N < 1$ and $f \in L_x^q(\mathbb{R}^d)$ is radial function, then it holds that
\[
\|K_{\pm}(t)P_Nf\|_{L_t^p L_x^q((1^+)R^d)} \lesssim N^{d(2/2 - (d+2)/q)}\|P_Nf\|_{L_x^q(\mathbb{R}^d)}.
\]

See (3.13) in [7] for the proof of Proposition 2.12.

Proposition 2.13. Let $d \geq 2, N \geq 1$ and $(q, r)$ satisfy $2 \leq r \leq 2(d + 1)/(d - 1), r \leq q, (1/2)(d - 1)(1/2 - 1/r) \leq 1/q < (d - 1)(1/2 - 1/r)$. Then, for all radial function $f \in L_x^q(\mathbb{R}^d)$, it holds that
\[
\|K_{\pm}(t)P_Nf\|_{L_t^p L_x^q((1^+)R^d)} \lesssim N^{d(1/2 - 1/r) - 1/q}\|P_Nf\|_{L_x^q(\mathbb{R}^d)}.
\]

Proof. When $q = r$, (2.11) follows from (2.9). Interpolating $L_{t,x}^q$ with $L_{t,x}^\infty L_x^2$, we obtain (2.21).

Proposition 2.14. (i) Let $d \geq 3, (q, r)$ satisfy (2.8) in Proposition 2.11 and $s = d(1/2 - 1/r) - 1/q$. If $p < q$, then for all spherically symmetric function $u$, it holds that
\[
\|u\|_{L_t^q L_x^{s,r}((1^+)R^d)} \lesssim \|u\|_{W_x^{p,q} W_{t,x}^{p,q}}.
\]

(ii) Let $d \geq 2$ and $(q, r)$ satisfy the condition in Proposition 2.13. If $p < q, N \geq 1$ and $s_1 = d(1/2 - 1/r) - 1/q$, then for all spherically symmetric function $u$, it holds that
\[
\|P_Nu\|_{L_t^q L_x^{s_1,r}((1^+)R^d)} \lesssim \|P_Nu\|_{W_x^{p,q} W_{t,x}^{p,q}}.
\]
(iii) Let $d \geq 2$. If $p < q, N < 1$ and $s_2 = d/2 - (d + 2)/q$, then for all spherically symmetric function $u$, it holds that

$$
\|P_N u\|_{L^q_x W^{s_2,q}_{x,0}([R^{1+d})} \lesssim \|P_N u\|_{V^{s_2}_{k,\pm}}. \tag{2.13}
$$

Combining Proposition 2.2, Proposition 2.11 and Proposition 2.7, we have Proposition 2.14 (i). Combining Proposition 2.2, Proposition 2.13 and Proposition 2.7, we have Proposition 2.14 (ii). Combining Proposition 2.2, (2.10) and Proposition 2.7, we have Proposition 2.14 (iii).

**Proposition 2.15.** (i) Let $T > 0$ and $u \in Y_{k,\pm}([0, T]), u(0) = 0$. Then, there exists $0 \leq T' \leq T$ such that $\|u\|_{Y_{k,\pm}([0, T'])} < \varepsilon$.

(ii) Let $T > 0$ and $n \in Y_{W,\pm}([0, T]), n(0) = 0$. Then, there exists $0 \leq T' \leq T$ such that $\|n\|_{Y_{W,\pm}([0, T'])} < \varepsilon$.

For the proofs of (i) and (ii), see Proposition 2.24 in [8].

**Lemma 2.16.** Let $a \geq 0$. Then for $A = K_{\pm}$ or $W_{\pm, c}$, it holds that

$$
\|\langle \nabla_x \rangle^a f\|_{V^a_{k,\pm}} \lesssim \|f\|_{Y^{a}_{k,\pm}}.
$$

**Proof.** We only prove for $A = K_{\pm}$ since we can prove similarly for $A = W_{\pm, c}$. By $L^2_x$ orthogonality, we have

$$
\|\langle \nabla_x \rangle^a f\|_{V^a_{k,\pm}} \lesssim \sup_{\{t_i\}_{i=0}^{1}} \sum_{i=1}^{1} \left[ \|P_{<1}(K_{\pm}(-t_i)f(t_i) - K_{\pm}(-t_{i-1})f(t_{i-1}))\|_{L^2_x}^2 + \sum_{N \geq 1} N^{2a} \|P_N(K_{\pm}(-t_i)f(t_i) - K_{\pm}(-t_{i-1})f(t_{i-1}))\|_{L^2_x}^2 \right]
$$

$$
\lesssim \sup_{\{t_i\}_{i=0}^{1}} \sum_{i=1}^{1} \left[ \|K_{\pm}(-t_i)P_{<1}f(t_i) - K_{\pm}(-t_{i-1})P_{<1}f(t_{i-1})\|_{L^2_x}^2 + \sum_{N \geq 1} N^{2a} \sup_{\{t_i\}_{i=0}^{1}} \sum_{i=1}^{1} \|K_{\pm}(-t_i)P_Nf(t_i) - K_{\pm}(-t_{i-1})P_Nf(t_{i-1})\|_{L^2_x}^2 \right]
$$

$$
\lesssim \|f\|_{Y^a_{k,\pm}}^2.
$$

$\Box$

**Remark 2.2.** Similarly, we see

$$
\|\langle \nabla_x \rangle^a f\|_{V^a_{k,\pm}} \lesssim \|f\|_{Y^a_{k,\pm}}.
$$

**Proposition 2.17.** It holds that

$$
\|Q_M u\|_{L^2_x(R^{1+d})} \lesssim M^{-1/2}\|u\|_{V^{s_2}_{k,\pm}}, \quad \|Q_{\geq M} u\|_{L^2_x(R^{1+d})} \lesssim M^{-1/2}\|u\|_{V^{s_2}_{k,\pm}}, \tag{2.14}
$$

$$
\|Q_{< M} u\|_{V^{s_2}_{k,\pm}} \lesssim \|u\|_{V^{s_2}_{k,\pm}}, \quad \|Q_{\geq M} u\|_{V^{s_2}_{k,\pm}} \lesssim \|u\|_{V^{s_2}_{k,\pm}},
$$

$$
\|Q_{< M} u\|_{U^{s_2}_{k,\pm}} \lesssim \|u\|_{U^{s_2}_{k,\pm}}, \quad \|Q_{\geq M} u\|_{U^{s_2}_{k,\pm}} \lesssim \|u\|_{U^{s_2}_{k,\pm}}.
$$

The same estimates hold by replacing the Klein-Gordon operator $K_{\pm}$ by the wave operator $W_{\pm, c}$.
Lemma 2.18. If \( f, g \) are measurable functions, then for \( Q \in \{ Q_{< M}, Q_{\geq M} \} \), it holds that
\[
\int_{\mathbb{R}^{1+d}} f(t, x)Qg(t, x)dxdt = \int_{\mathbb{R}^{1+d}} (Qf(t, x))g(t, x)dxdt.
\]

For the proof of Lemma 2.18, see [13].

Lemma 2.19. Let \( \tilde{u}_{N_1} := 1_{[0,T]} P_{N_1} u, \tilde{v}_{N_2} := 1_{[0,T]} P_{N_2} v, \tilde{n}_{N_3} := 1_{[0,T]} P_{N_3} n, Q_1, Q_2 \in \{ Q_{< M}, Q_{\geq M} \} \) be operator for the Klein-Gordon equation and \( Q_3 \in \{ Q_{< M}, Q_{\geq M} \} \) be operator for the wave equation. Let \( s' := (d^2 - 3d - 2)/(d + 1) \), \( s_c := d/2 - 2 \). Then the following estimates hold for sufficiently small \( T > 0 \) if \( \theta > 0 \), and hold for all \( 0 < T \leq \infty \) if \( \theta = 0 \) or spherically symmetric \( (u, v, n) \):

(i) If \( N_3 \lesssim N_2 \sim N_1 \), then
\[
|I_1| := \left| \int_{\mathbb{R}^{1+d}} (\omega^{-1}_{N_1})\tilde{u}_{N_1})\tilde{v}_{N_2})\tilde{n}_{N_3})dxdt \right| \lesssim T^9 N_3^{d/3} u_{N_1} \| v_{N_2} \| n_{N_3} \| v_{N_2} \| ,
\]

where \( \theta, s) = (1/4, 1/4) \) for \( d = 4 \) and \( \theta, s) = (0, s_c), (1/(d + 1), s') \) for \( d \geq 5 \).

Moreover, if \( (u, v, n) \) are spherically symmetric, then for \( d \geq 4 \),
\[
|I_1| \lesssim \langle N_2 \rangle^{(d-8)/3} N_3^{(d+4)/6} u_{N_1} \| v_{N_2} \| n_{N_3} \| v_{N_2} \| .
\]

(ii) It holds that
\[
|I_2| := \left| \int_{\mathbb{R}^{1+d}} \tilde{n}(\omega^{-1}_{N_1})\tilde{v}(P_{<1}u)dxdt \right| \lesssim T^9 \| n \| \| v \| \| P_{<1}u \| \| v \| \| ,
\]

where \( \theta, s) = (1, 1/4) \) for \( d = 4 \) and \( \theta, s) = (0, s_c), (1/(d + 1), s') \) for \( d \geq 5 \).

Moreover, if \( (u, v, n) \) are spherically symmetric, then for \( d = 4 \),
\[
|I_2| \lesssim \| n \| \| v \| \| P_{<1}u \| \| v \| .
\]

(iii) If \( N_1 \sim N_2 \), then
\[
|I_3| := \left| \int_{\mathbb{R}^{1+d}} \left( \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3})\tilde{v}_{N_3})\tilde{n}_{N_3})dxdt \right| \lesssim T^9 \| n \| \| v \| \| u \| \| v \| ,
\]

where \( \theta, s) = (1/4, 1/4) \) for \( d = 4 \) and \( \theta, s) = (0, s_c), (1/(d + 1), s') \) for \( d \geq 5 \).

Moreover, if \( (u, v, n) \) are spherically symmetric, then for \( d = 4 \),
\[
|I_3| \lesssim \| n \| \| v \| \| u \| \| v \| .
\]

(iv) If \( N_1 \sim N_3, N_1 \geq 1, M = \varepsilon N_1 \) and \( \varepsilon > 0 \) is sufficiently small, then
\[
|I_4| \lesssim T^9 \| n \| \| v \| \| u \| ,
\]

where \( \theta, s) = (1/4, 1/4) \) for \( d = 4 \), \( i = 4, 5, 6 \) and \( \theta, s) = (0, s_c) \) for \( d \geq 4, i = 4, 6 \) and \( \theta, s) = (1/(d + 1), s') \) for \( d \geq 5, i = 4, 5, 6 \). Moreover, if \( (u, v, n) \) are spherically symmetric, then for \( d \geq 4 \), it holds that
\[
|I_4| \lesssim \| n \| \| v \| \| u \| .
\]
Proof. We show (i) first. For $f \in V^d_2$, $A \in \{K_\pm, W_{\pm c}\}$, we see

$$
\|1_{[0,T]}f\|_{V^d_2} \lesssim \|f\|_{V^d_2}.
$$

(2.15)

First, we show it for $d = 4$. We apply the Hölder inequality, Proposition 2.10, (2.15) and $N_3 \lesssim N_1 \sim N_2$, then we have

$$
|I_1| \lesssim \|\omega^{-1} \tilde{u}_{N_3}\|_{L^4_{t,x}} \|\omega^{-1} \tilde{v}_{N_2}\|_{L^{4/3}_{t,x}} \|\omega \tilde{w}_{N_3}\|_{L^{4/3}_{t,x}}
$$

$$
\lesssim T^{1/4} \|\langle \nabla_x \rangle \omega^{-1} \tilde{u}_{N_3}\|_{V^2_{K_\pm}} \|\langle \nabla_x \rangle \omega^{-1} \tilde{v}_{N_2}\|_{V^2_{K_\pm}} \|\tilde{w}_{N_3}\|_{V^2_{W \pm c}}.
$$

(2.19)

(2.20)

For $d \geq 5$, we apply the Hölder inequality to have

$$
|I_1| \lesssim \|\omega^{-1} \tilde{u}_{N_3}\|_{L^{2(d+1)/(d-1)}_{t,x}} \|\omega^{-1} \tilde{v}_{N_2}\|_{L^{2(d+1)/(d-1)}_{t,x}} \|\omega \tilde{w}_{N_3}\|_{L^{2(d+1)/2}_{t,x}}.
$$

(2.16)

We apply Proposition 2.10, (2.15) and the Sobolev inequality, then we have

$$
\|\tilde{f}_N\|_{L^{2(d+1)/(d-1)}_{t,x}} \lesssim \langle N \rangle^{-1/2} \|f_N\|_{V^2_{K_\pm}},
$$

(2.17)

$$
\|\tilde{w}_{N_3}\|_{L^{2(d+1)/2}_{t,x}} \lesssim \|\langle \nabla_x \rangle^{d-5/2} \langle \omega \rangle^{1/2} \|\omega \tilde{w}_{N_3}\|_{L^{2(d+1)/2}_{t,x}}.
$$

(2.18)

Collecting (2.16), (2.17), (2.19) and $N_3 \lesssim N_1 \sim N_2$, we obtain

$$
|I_1| \lesssim N_3^{\varepsilon} \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}} \|n_{N_3}\|_{V^2_{W \pm c}}.
$$

In (2.16), if we apply the Hölder inequality, the Sobolev inequality and Proposition 2.10, then we have

$$
\|\omega \tilde{w}_{N_3}\|_{L^{2(d+1)/2}_{t,x}} \lesssim \|1_{[0,T]}\|_{L^{d+1}_{t}} \|\omega \tilde{w}_{N_3}\|_{L^{d+1}_{t,x}}
$$

$$
\lesssim T^{1/(d+1)} \|\langle \nabla_x \rangle^{d(d-4d-1)/2(d^2-1)} \omega \tilde{w}_{N_3}\|_{L^{d+1}_{t,x}}
$$

$$
\lesssim T^{1/(d+1)} \|\omega \tilde{w}_{N_3}\|_{V^2_{W \pm c}}.
$$

(2.21)
Next, we prove it for $d \geq 4$ and spherically symmetric functions $(u, v, n)$. We apply the Hölder inequality to have
\[
|I_1| \lesssim \| \omega^{-1}_1 \tilde{u}_N_1 \|_{L^2_{t,x}} \| \omega^{-1}_1 \tilde{v}_N_2 \|_{L^2_{t,x}} \| \omega \tilde{n}_N_3 \|_{L^1_{t,x}}.
\] (2.22)

We apply Proposition 2.14, (2.15) and $N_3 \lesssim N_2 \sim N_1$, then we have
\[
\| \omega^{-1}_1 \tilde{u}_N_1 \|_{L^2_{t,x}} \lesssim \langle N_1 \rangle^{(d-2)/6-1} \| u_{N_1} \|_{V^2_{\tilde{k},+}} \lesssim \langle N_2 \rangle^{(d-8)/6} \| u_{N_1} \|_{V^2_{\tilde{k},+}},
\] (2.23)
\[
\| \omega^{-1}_1 \tilde{v}_N_2 \|_{L^2_{t,x}} \lesssim \langle N_2 \rangle^{(d-8)/6} \| v_{N_2} \|_{V^2_{\tilde{k},-}},
\] (2.24)
\[
\| \omega \tilde{n}_N_3 \|_{L^1_{t,x}} \lesssim \| |\nabla|^2_{\tilde{k},+} \omega \tilde{n}_N_3 \|_{V^2_{\tilde{k},+}} \lesssim \langle N_3 \rangle^{(d+4)/6} \| n_{N_3} \|_{V^3_{\tilde{k},+}}.
\] (2.25)

From (2.22)–(2.25), we obtain
\[
|I_1| \lesssim \langle N_2 \rangle^{(d-8)/3} \langle N_3 \rangle^{(d+4)/6} \| u_{N_1} \|_{V^2_{\tilde{k},+}} \| v_{N_2} \|_{V^2_{\tilde{k},+}} \| n_{N_3} \|_{V^3_{\tilde{k},+}}.
\]

Next, we prove $(ii)$. For $d = 4$, we apply the Hölder inequality, the Sobolev inequality, Remark 2.1, (2.15), Remark 2.2, discarding $\omega^{-1}_1$ and Lemma 2.16, we obtain
\[
|I_2| \lesssim \| n \|_{L^2_{t,x}} \| \tilde{u} \|_{L^4_{t,x} L^{16/7}_{x}} \| \omega^{-1}_1 \tilde{v} \|_{L^4_{t,x} L^{16/7}_{x}} \| P_{< \tilde{k}} \tilde{u} \|_{L^4_{t,x} L^{4}_{x}}
\lesssim T \| \langle \nabla \rangle \tilde{u} \|_{L^4_{t,x} L^{4}_{x}} \| \langle \nabla \rangle \tilde{v} \|_{L^4_{t,x} L^{4}_{x}} \| \langle \nabla \rangle \tilde{v} \|_{L^4_{t,x} L^{4}_{x}} \| P_{< \tilde{k}} \tilde{u} \|_{L^4_{t,x} L^{4}_{x}}
\lesssim T \| n \|_{Y^1_{W^{\tilde{k},+}}} \| v \|_{Y^1_{W^{\tilde{k},+}}} \| P_{< \tilde{k}} \tilde{u} \|_{V^2_{\tilde{k},-}}
\lesssim T \| n \|_{Y^1_{W^{\tilde{k},+}}} \| v \|_{Y^1_{W^{\tilde{k},+}}} \| P_{< \tilde{k}} \tilde{u} \|_{V^2_{\tilde{k},-}}.
\] (2.26)

From Proposition 2.10, (2.18), Remark 2.2 and Lemma 2.16, we obtain
\[
\| \tilde{u} \|_{L^{(d+1)/2}_{t,x}} \lesssim \| n \|_{Y^1_{W^{\tilde{k},+}}},
\] (2.27)
\[
\| \omega^{-1}_1 \tilde{v} \|_{L^{(d+1)/(d-1)}_{t,x}} \lesssim \| \langle \nabla \rangle^{-1/2} \tilde{v} \|_{V^2_{\tilde{k},-}} \lesssim \| \langle \nabla \rangle \tilde{v} \|_{V^2_{\tilde{k},-}} \lesssim \| v \|_{Y^1_{W^{\tilde{k},+}}},
\] (2.28)
\[
\| P_{< \tilde{k}} \tilde{u} \|_{L^{(d+1)/(d-1)}_{t,x}} \lesssim \| \langle \nabla \rangle^{1/2} \tilde{u} \|_{V^2_{\tilde{k},-}} \lesssim \| P_{< \tilde{k}} \tilde{u} \|_{V^2_{\tilde{k},-}}.
\] (2.29)

Collecting (2.26)–(2.29), we obtain
\[
|I_2| \lesssim \| n \|_{Y^1_{W^{\tilde{k},+}}} \| v \|_{Y^1_{W^{\tilde{k},+}}} \| P_{< \tilde{k}} \tilde{u} \|_{V^2_{\tilde{k},-}}.
\]

Also for $d \geq 5$, from (2.20), Remark 2.2, (2.26), (2.28) and (2.29), we have
\[
|I_2| \lesssim T^{1/(d+1)} \| n \|_{Y^1_{W^{\tilde{k},+}}} \| v \|_{Y^1_{W^{\tilde{k},+}}} \| P_{< \tilde{k}} \tilde{u} \|_{V^2_{\tilde{k},-}}.
\]

We prove it for $d = 4$ and spherically symmetric functions $(u, v, n)$. Due to the operator $P_{< \tilde{k}}$.
\[
|I_2| \lesssim \left| \int_{\mathbb{R}^{1+d}} \left( \sum_{N_3 \lesssim 1} \tilde{n}_N_3 \right) \left( \sum_{N_2 < 1} \omega^{-1}_1 \tilde{v}_N_2 \right) (P_{< \tilde{k}} \tilde{u}) dx dt \right|
+ \sum_{N_2 \geq 1} \left| \int_{\mathbb{R}^{1+d}} \left( \sum_{N_3 \lesssim N_2} \tilde{n}_N_3 \right) (\omega^{-1}_1 \tilde{v}_N_2) (P_{< \tilde{k}} \tilde{u}) dx dt \right|
=: I_{2,1} + I_{2,2}.
First, we estimate $I_{2.2}$. We apply the Hölder inequality to have

$$|I_{2.2}| \lesssim \sum_{N_2 \geq 1} \left( \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \right) L_{t,x}^3 \left\| \omega_1^{-1} \tilde{v}_{N_2} \right\| L_{t,x}^1 \left\| P_{< 1} \tilde{u} \right\| L_{t,x}^3. \quad (2.30)$$

By Proposition 2.14, (2.15), $N_3 \lesssim N_2 \sim N_1$ and the Cauchy-Schwarz inequality, then we have

$$\left\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \right\| L_{t,x}^3 \lesssim \left\| \left\| \nabla_x |1/3 \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \right\| v_{K_\pm}^2 \right\| \lesssim \left( \sum_{N_3 \lesssim N_2} N_3^{2/3} \right)^{1/2} \left( \sum_{N_3 \lesssim N_2} \|n_{N_3}\| v_{K_\pm}^2 \right)^{1/2} \lesssim N_2^{1/3} \|n\|_{Y_{W_\pm}^K}. \quad (2.31)$$

From (2.23) and (2.24), we see

$$\left\| P_{< 1} \tilde{u} \right\| L_{t,x}^1 \lesssim \left\| P_{< 1} u \right\| v_{K_\pm}^1, \quad \left\| \omega_1^{-1} \tilde{v}_{N_2} \right\| L_{t,x}^3 \lesssim \left\{ N_2 \right\}^{-2/3} \left\| v_{N_2} \right\| v_{K_\pm}^2. \quad (2.32)$$

Collecting (2.30)–(2.32), $N_2 \geq 1$ and applying the Cauchy-Schwarz inequality, we obtain

$$|I_{2.2}| \lesssim \sum_{N_2 \geq 1} \left( \sum_{N_3 \lesssim N_2} N_3^{-1/3} \|n\|_{Y_{W_\pm}^K} \|v_{N_2}\| v_{K_\pm}^2 \right) \left\| P_{< 1} u \right\| v_{K_\pm}^1 \lesssim \left\| n \right\|_{Y_{W_\pm}^K} \left\| v \right\| v_{K_\pm}^2 \left\| P_{< 1} u \right\| v_{K_\pm}^1. \quad (2.33)$$

Next, we estimate $I_{2.1}$, By the Hölder inequality, we have

$$|I_{2.1}| \lesssim \left\| \sum_{N_3 \lesssim 1} \tilde{n}_{N_3} \right\| L_{t,x}^1 \left\| \sum_{N_2 < 1} \omega_1^{-1} \tilde{v}_{N_2} \right\| L_{t,x}^3 \left\| P_{< 1} \tilde{u} \right\| L_{t,x}^3. \quad (2.34)$$

From (2.31), (2.32) and discarding $\omega_1^{-1}$, we see

$$\left\| \sum_{N_3 \lesssim 1} \tilde{n}_{N_3} \right\| L_{t,x}^3 \lesssim \left\| n \right\|_{Y_{W_\pm}^K}, \quad \left\| \sum_{N_2 < 1} \omega_1^{-1} \tilde{v}_{N_2} \right\| L_{t,x}^3 \lesssim \left\| P_{< 1} v \right\| v_{K_\pm}^2. \quad (2.35)$$

Collecting (2.32), (2.34) and (2.35), we have

$$|I_{2.1}| \lesssim \left\| n \right\|_{Y_{W_\pm}^K} \left\| P_{< 1} v \right\| v_{K_\pm}^2 \left\| P_{< 1} u \right\| v_{K_\pm}^2 \lesssim \left\| n \right\|_{Y_{W_\pm}^K} \left\| v \right\| v_{K_\pm}^2 \left\| P_{< 1} u \right\| v_{K_\pm}^2. \quad (2.36)$$

From (2.33) and (2.36), we obtain $|I_2| \lesssim \left\| n \right\|_{Y_{W_\pm}^K} \left\| v \right\| v_{K_\pm}^2 \left\| P_{< 1} u \right\| v_{K_\pm}^2$. We prove (iii) for $d = 4$ below. We apply the Hölder inequality, Proposition 2.10 and (2.15), then we have

$$|I_3| \lesssim \left\| 1_{[0,T]} \right\| L_{t,x}^4 \left\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \right\| L_{t,x}^{20/3} \left\| \omega_1^{-1} \tilde{v}_{N_2} \right\| L_{t,x}^{20/3} \left\| \tilde{n}_{N_3} \right\| L_{t,x}^{20/3} \left\| u_{N_1} \right\| L_{t,x}^{20/3} \lesssim T^{1/4} \left\| \nabla_x |3/4 \sum_{N_3 \lesssim N_2} n_{N_3} \right\| v_{K_\pm}^2 \left\{ N_2 \right\}^{1/4} \left\| v_{N_2} \right\| v_{K_\pm}^2 \left\{ N_1 \right\}^{1/4} \left\| u_{N_1} \right\| v_{K_\pm}^2. \quad (2.37)$$
By the Cauchy-Schwarz inequality, we have
\[ \left\| \nabla_x \sum_{N_3 \leq N_2} n_{N_3} \right\|_{V^3_{W \pm \varepsilon}} \lesssim \sum_{N_3 \leq N_2} N_3^{3/2} \left\| n_{N_3} \right\|_{V^2_{W \pm \varepsilon}} \]
\[ \lesssim \left( \sum_{N_3 \leq N_2} N_3 \right)^{1/2} \left( \sum_{N_3 \leq N_2} N_3^{3/2} \left\| n_{N_3} \right\|_{V^2_{W \pm \varepsilon}}^2 \right)^{1/2} \]
\[ \lesssim N_2^{1/2} \left\| n \right\|_{Y^1_{W \pm \varepsilon}}. \quad (2.38) \]

Collecting (2.37), (2.38) and \( N_1 \sim N_2 \), we obtain
\[ |I_3| \lesssim T^{1/4} \left\| n \right\|_{Y^1_{W \pm \varepsilon}} \left\| v_{N_2} \right\|_{V^2_{K \pm}} \left\| u_{N_1} \right\|_{V^2_{K \pm}}. \]

We prove it for \( d \geq 5 \). We apply the Hölder inequality to have
\[ |I_3| \lesssim \sum_{N_3 \leq N_2} \tilde{n}_{N_3} \left\| \omega^{-1} \bar{u}_{N_2} \right\|_{L^{2(d+1)/(d-1)}_{t,x}} \left\| \tilde{u}_{N_1} \right\|_{L^{2(d+1)/(d-1)}_{t,x}}. \quad (2.39) \]

Similar to (2.18), the Sobolev inequality and Proposition 2.10, we have
\[ \left\| \sum_{N_3 \leq N_2} \tilde{n}_{N_3} \right\|_{L^{2(d+1)/2}_{t,x}} \lesssim \left\| \nabla_x \right\|^{2 \kappa_c}_{V^3_{W \pm \varepsilon}} \sum_{N_3 \leq N_2} \tilde{n}_{N_3} \right\|_{V^3_{W \pm \varepsilon}}. \quad (2.40) \]

By the \( L^2_x \) orthogonality, we obtain
\[ \left\| \nabla_x \right\|^{2 \kappa_c}_{V^3_{W \pm \varepsilon}} \sum_{N_3 \leq N_2} \tilde{n}_{N_3} \right\|_{V^3_{W \pm \varepsilon}}^2 \lesssim \sup_{\{t_i\}, \tilde{e}_i \in \mathbb{Z}} \sum_{i=1}^I \sum_{N} N^{2 \kappa_c} \left\| P_N \left\{ W_{\pm \varepsilon}(-t_i) \left( \sum_{N_3 \leq N_2} \tilde{n}_{N_3}(t_i) \right) \right. \right. \]
\[ \left. \left. \left. - W_{\pm \varepsilon}(-t_{i-1}) \left( \sum_{N_3 \leq N_2} \tilde{n}_{N_3}(t_{i-1}) \right) \right) \right\|^2_{L^2_{t,x}} \right. \]
\[ \lesssim \left\| \tilde{n} \right\|^2_{Y^{3 \kappa_c}_{W \pm \varepsilon}}. \quad (2.41) \]

Since \( P_N \tilde{n}_{N_3} = 0 \) if \( N_3 > 2N \) or \( N_3 < N/2 \) and \( P_N \) is projection, the right-hand side is bounded by
\[ \sup_{\{t_i\}, \tilde{e}_i \in \mathbb{Z}} \sum_{i=1}^I \sum_{N} N^{2 \kappa_c} \left\| W_{\pm \varepsilon}(-t_i) P_N \tilde{n}(t_i) - W_{\pm \varepsilon}(-t_{i-1}) P_N \tilde{n}(t_{i-1}) \right\|^2_{L^2_t} \]
\[ \lesssim \left\| \tilde{n} \right\|^2_{Y^{3 \kappa_c}_{W \pm \varepsilon}}. \quad (2.42) \]

Hence, from (2.39)–(2.42), (2.17) and \( N_1 \sim N_2 \), we have
\[ |I_3| \lesssim \left\| n \right\|_{Y^{3 \kappa_c}_{W \pm \varepsilon}} \left( \langle N_2 \rangle^{-1/2} \left\| v_{N_2} \right\|_{V^2_{K \pm}} \left( \langle N_1 \rangle \right)^{1/2} \left\| u_{N_1} \right\|_{V^2_{K \pm}} \right. \]
\[ \lesssim \left\| n \right\|_{Y^{3 \kappa_c}_{W \pm \varepsilon}} \left\| v_{N_2} \right\|_{V^2_{K \pm}} \left\| u_{N_1} \right\|_{V^2_{K \pm}}. \]

Similar to (2.20), (2.41) and (2.42), then we have
\[ \left\| \sum_{N_3 \leq N_2} \tilde{n}_{N_3} \right\|_{L^{2(d+1)/2}_{t,x}} \lesssim T^{1/(d+1)} \left\| n \right\|_{Y^{3 \kappa_c}_{W \pm \varepsilon}}. \quad (2.43) \]

Collecting (2.17), (2.39) and (2.43), we obtain
\[ |I_3| \lesssim T^{1/(d+1)} \left\| n \right\|_{Y^{3 \kappa_c}_{W \pm \varepsilon}} \left\| v_{N_2} \right\|_{V^2_{K \pm}} \left\| u_{N_1} \right\|_{V^2_{K \pm}}. \]
When \( d = 4 \) and \((u, v, n)\) are spherically symmetric functions, we apply the Hölder inequality to have
\[
|I_3| \lesssim \left\| \sum_{N_1 \leq N_2} \tilde{u}_{N_1} \right\|_{L^3_t L^{13/2}_x} \|\omega^{-1}_1 \tilde{v}_{N_2}\|_{L^3_t L^{13/2}_x} \|\tilde{u}_{N_1}\|_{L^3_t L^{13/2}_x}.
\]
(2.44)

From (2.23), (2.24), we see
\[
\|\tilde{u}_{N_1}\|_{L^3_t L^{13/2}_x} \lesssim \langle N_2 \rangle^{-1/3} \|u_{N_1}\|_{V^2_{K_\pm}}, \quad \|\omega^{-1}_1 \tilde{v}_{N_2}\|_{L^3_t L^{13/2}_x} \lesssim \langle N_2 \rangle^{-2/3} \|v_{N_2}\|_{V^2_{K_\pm}}.
\]
(2.45)
Collecting (2.31), (2.44) and (2.45), we obtain
\[
|I_3| \lesssim \|n\|_{Y^{1/4}_{w,\pm}} \|v_{N_2}\|_{V^2_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}.
\]

We prove (iv). The estimate for \(I_4\) is obtained by the same manner as the estimate for \(I_3\), so we only estimate \(I_4, I_5\). First, we estimate \(I_4\) for \(d = 4\). We apply the Hölder inequality, Proposition 2.17, the Sobolev inequality, Lemma 2.6, Proposition 2.10 and (2.15) to have
\[
|I_4| \lesssim \|Q \geq M \tilde{n}_{N_1}\|_{L^3_t L^{13/2}_x} \left\| \sum_{N_2 \ll N_1} Q_2 \omega^{-1}_1 \tilde{v}_{N_2} \right\|_{L^3_t L^{13/2}_x} \|Q_1 \tilde{u}_{N_1}\|_{L^3_t L^{13/2}_x}
\]
\[
\lesssim N_1^{-1/2} \|\tilde{n}_{N_1}\|_{V^2_{K_\pm}} \langle \nabla_x \rangle^{5/4} \left\| \sum_{N_2 \ll N_1} Q_2 \omega^{-1}_1 \tilde{v}_{N_2} \right\|_{L^3_t L^2_x} \langle \nabla_x \rangle^{1/12} \|Q_1 \tilde{u}_{N_1}\|_{L^3_t L^2_x}
\]
\[
\lesssim N_1^{-1/2} \|\tilde{n}_{N_1}\|_{V^2_{K_\pm}} T^{1/4} \langle \nabla_x \rangle^{5/4} \left\| \sum_{N_2 \ll N_1} \omega^{-1}_1 \tilde{v}_{N_2} \right\|_{L^2_x} \langle N_1 \rangle^{1/2} \|u_{N_1}\|_{V^2_{K_\pm}}.
\]
(2.46)

Similar to Lemma 2.16, (2.41) and (2.42), then we have
\[
\left\| \langle \nabla_x \rangle^{1/4} \sum_{N_2 \ll N_1} \tilde{v}_{N_2} \right\|_{V^2_{K_\pm}}^2
\]
\[
\lesssim \sup_{\{t_i\}_{i=0}^I} \sum_{i=1}^I \left\| P_{<1}(K_{\pm}(-t_i) \left( \sum_{N_2 \ll N_1} \tilde{v}_{N_2}(t_i) \right) - K_{\pm}(-t_{i-1}) \left( \sum_{N_2 \ll N_1} \tilde{v}_{N_2}(t_{i-1}) \right) \right\|_{L^2_x}^2
\]
\[
+ \sum_{N \geq 1} N^{1/2} \left\| P_N \left( K_{\pm}(-t_i) \left( \sum_{N_2 \ll N_1} \tilde{v}_{N_2}(t_i) \right) - K_{\pm}(-t_{i-1}) \left( \sum_{N_2 \ll N_1} \tilde{v}_{N_2}(t_{i-1}) \right) \right) \right\|_{L^2_x}^2
\]
\[
\lesssim \sup_{\{t_i\}_{i=0}^I} \sum_{i=1}^I \left\| K_{\pm}(-t_i) P_{<1} \tilde{v}(t_i) - K_{\pm}(-t_{i-1}) P_{<1} \tilde{v}(t_{i-1}) \right\|_{L^2_x}^2
\]
\[
+ \sum_{N \geq 1} N^{1/2} \sup_{\{t_i\}_{i=0}^I} \sum_{i=1}^I \left\| K_{\pm}(-t_i) P_N \tilde{v}(t_i) - K_{\pm}(-t_{i-1}) P_N \tilde{v}(t_{i-1}) \right\|_{L^2_x}^2
\]
\[
\lesssim \|v\|_{H^{1/2}_{K_\pm}}^2.
\]
(2.47)

From (2.46), (2.47) and \(\langle N_1 \rangle \sim N_1 \gg 1\), we obtain
\[
|I_4| \lesssim T^{1/4} \|n_{N_1}\|_{V^2_{K_\pm}} \|v\|_{V^{1/4}_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}.
\]

Next, we prove it for \(d \geq 4\) and non-radial case. We apply the Hölder inequality to have
\[
|I_4| \lesssim \|Q \geq M \tilde{n}_{N_1}\|_{L^{2+\epsilon}_t L^{13/2}_x} \left\| \sum_{N_2 \ll N_1} Q_2 \omega^{-1}_1 \tilde{v}_{N_2} \right\|_{L^{2+\epsilon}_t L^{13/2}_x} \|Q_1 \tilde{u}_{N_1}\|_{L^{2+\epsilon}_t L^{13/2}_x}.
\]
(2.48)
By Proposition 2.17, (2.17) and (2.15), we have
\[ \| Q_{\geq M} \tilde{\eta} N_3 \|_{L_{t,x}^2} \lesssim N_1^{-1/2} \| n N_3 \|_{V_{K,\pm}^{2}} , \]  
(2.49)
\[ \| Q_1 \tilde{u}_N \|_{L_{t,x}^{2(d+1)/(d-1)}} \lesssim \langle N_1 \rangle^{1/2} \| u N_1 \|_{V_{K,\pm}^{2}} . \]  
(2.50)

We apply the Sobolev inequality, Proposition 2.10, Proposition 2.17 and (2.15), we have
\[ \left\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_N \right\|_{L_{t,x}^{d+1}} \lesssim \left\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)} \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_N \right\|_{L_{t,x}^{d+1} L_{x}^{2(d^2-1)/(d^2-5)}} \]
\[ \lesssim \left\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)+1/(d-1)-1} \sum_{N_2 \ll N_1} \tilde{v}_N \right\|_{V_{K,\pm}^{2}} . \]  
(2.51)

Similar to (2.47), we have
\[ \left\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)+1/(d-1)-1} \sum_{N_2 \ll N_1} \tilde{v}_N \right\|_{V_{K,\pm}^{2}} \lesssim \| v \|_{Y_{K,\pm}^{\infty}} . \]  
(2.52)

Collecting (2.48)–(2.52) and $N_1 \gg 1$, we obtain
\[ |I_4| \lesssim \| n N_3 \|_{V_{W_{\infty,\pm}^{2}}} \| v \|_{Y_{K,\pm}^{\infty}} \| u N_1 \|_{V_{K,\pm}^{2}} . \]

For $d \geq 5$ and non-radial case, in (2.48), if we apply the Sobolev inequality, Proposition 2.17 and Lemma 2.6, then we have
\[ \left\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_N \right\|_{L_{t,x}^{d+1}} \lesssim \left\| \langle \nabla_x \rangle^{d(d-1)/2(d+1)} \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_N \right\|_{L_{t,x}^{d+1} L_{x}^{2}} \]
\[ \lesssim T^{1/(d+1)} \left\| \langle \nabla_x \rangle^{(d^2-3d-2)/2(d+1)} \sum_{N_2 \ll N_1} \tilde{v}_N \right\|_{V_{K,\pm}^{2}} . \]  
(2.53)

Similar to (2.47), we have
\[ \left\| \langle \nabla_x \rangle^{(d^2-3d-2)/2(d+1)} \sum_{N_2 \ll N_1} \tilde{v}_N \right\|_{V_{K,\pm}^{2}} \lesssim \| v \|_{Y_{K,\pm}^{\infty}} . \]  
(2.54)

Collecting (2.48)–(2.50), (2.53), (2.54) and $N_1 \gg 1$, we obtain
\[ |I_4| \lesssim T^{1/(d+1)} \| n N_3 \|_{V_{W_{\infty,\pm}^{2}}} \| v \|_{Y_{K,\pm}^{\infty}} \| u N_1 \|_{V_{K,\pm}^{2}} . \]

Next, we prove $I_5$. When $d = 4$, by the H"older inequality, the Sobolev inequality, Lemma 2.6, Proposition 2.10, (2.15), Proposition 2.17, $N_1 \sim N_3$ and (2.47), we have
\[ |I_5| \lesssim \| Q_3 \tilde{n} N_3 \|_{L_{t,x}^{2(d+1)/(d-1)}} \left\| \sum_{N_2 \ll N_1} Q \geq M \omega_1^{-1} \tilde{v}_N \right\|_{L_{t,x}^{2} L_{x}^{2(d+1)/(d-1)}} \| Q_1 \tilde{u}_N \|_{L_{t,x}^{2(d+1)} L_{x}^{2}} \]
\[ \lesssim \| \nabla_x \|_{L_{t,x}^{1/2}} \| Q_3 \tilde{n} N_3 \|_{L_{t,x}^{2} L_{x}^{2}} \left\| \langle \nabla_x \rangle^{5/4} \sum_{N_2 \ll N_1} Q \geq M \omega_1^{-1} \tilde{v}_N \right\|_{L_{t,x}^{2}} T^{1/4} \| u N_1 \|_{V_{K,\pm}^{2}} \]
\[ \lesssim N_3^{1/2} \| n N_3 \|_{V_{W_{\infty,\pm}^{2}}} N_1^{-1/2} \left\| \langle \nabla_x \rangle^{5/4} \sum_{N_2 \ll N_1} \omega_1^{-1} \tilde{v}_N \right\|_{V_{K,\pm}^{2}} T^{1/4} \| u N_1 \|_{V_{K,\pm}^{2}} \]
\[ \lesssim T^{1/4} \| n N_3 \|_{V_{W_{\infty,\pm}^{2}}} \| v \| \| u N_1 \|_{V_{K,\pm}^{2}} . \]

For $d \geq 5$, by the H"older inequality, we have
\[ |I_5| \lesssim \| Q_3 \tilde{n} N_3 \|_{r_{t,x}^{2(d+1)/(d-1)}} \left\| \sum_{N_2 \ll N_1} Q \geq M \omega_1^{-1} \tilde{v}_N \right\|_{L_{t,x}^{2} L_{x}^{2(d+1)}} \| Q_1 \tilde{u}_N \|_{L_{t,x}^{2(d+1)} L_{x}^{2}} . \]  
(2.55)
Similar to (2.50), $N_3 \sim N_1$ and Lemma 2.6, we have

$$\|Q_3\tilde{n}_{N_3}\|_{L_{x,t}^{2(d+1)/(d-1)}} \lesssim \langle N_1 \rangle^{1/2} \|n_{N_3}\|_{V_{t,x}^{2,\pm}} \tag{2.56}$$

$$\|Q_1\tilde{u}_{N_1}\|_{L_{x,t}^{2}L_{x}^{2}} \lesssim T^{1/(d+1)} \|u_{N_1}\|_{V_{t,x}^{2,\pm}} \tag{2.57}$$

We apply the Sobolev inequality, Proposition 2.17, (2.15) and (2.54), we have

$$\|\sum_{N_2 \ll N_1} Q_{\geq M\omega_1^{-1}} \tilde{u}_{N_2}\|_{L_{x,t}^{2}L_{x}^{2}} \lesssim \|\langle \nabla_x \rangle^{d(d-1)/2(d+1)} \sum_{N_2 \ll N_1} Q_{\geq M\omega_1^{-1}} \tilde{v}_{N_2}\|_{L_{x,t}^{2}}$$

$$\lesssim N_1^{-1/2} \|\langle \nabla_x \rangle^{(d^2-3d-2)/2(d+1)} \sum_{N_2 \ll N_1} \tilde{v}_{N_2}\|_{V_{t,x}^{2,\pm}}$$

$$\lesssim N_1^{-1/2} \|v\|_{Y_{t,x}^{2,\pm}} \tag{2.58}$$

Collecting (2.55)–(2.58) and $N_1 \gg 1$, we obtain

$$|I_3| \lesssim T^{1/(d+1)} \|n_{N_3}\|_{V_{t,x}^{2,\pm}} \|v\|_{Y_{t,x}^{2,\pm}} \|u_{N_1}\|_{V_{t,x}^{2,\pm}} \cdot$$

Finally, we prove it for spherically symmetric functions $(u, v, n)$ and $d \geq 4$. By the Hölder inequality, Proposition 2.14, (2.15), the Sobolev inequality, $1 \ll N_1 \sim N_3$, Proposition 2.17 and (2.52), we have

$$|I_5| \lesssim \|Q_3\tilde{n}_{N_3}\|_{L_{x,t}^{2}L_{x}^{2^{d+1}/(d-1)}} \sum_{N_2 \ll N_1} Q_{\geq M\omega_1^{-1}} \tilde{v}_{N_2}\|_{L_{x,t}^{2}L_{x}^{2}} \|Q_1\tilde{u}_{N_1}\|_{L_{x,t}^{2}L_{x}^{2^{d+1}/(d-1)}}$$

$$\lesssim N_3^{1/4} \|n_{N_3}\|_{V_{t,x}^{2,\pm}} \|\langle \nabla_x \rangle^{(d-2)/2} \sum_{N_2 \ll N_1} Q_{\geq M\omega_1^{-1}} \tilde{v}_{N_2}\|_{L_{x,t}^{2}} \|n_{N_1}\|_{V_{t,x}^{2,\pm}} \|u_{N_1}\|_{V_{t,x}^{2,\pm}}$$

$$\lesssim N_3^{1/2} \|n_{N_3}\|_{V_{t,x}^{2,\pm}} \|\langle \nabla_x \rangle^{(d-2)/2} \sum_{N_2 \ll N_1} \omega_1^{-1} \tilde{v}_{N_2}\|_{V_{t,x}^{2,\pm}} \|u_{N_1}\|_{V_{t,x}^{2,\pm}}$$

$$\lesssim \|n_{N_3}\|_{V_{t,x}^{2,\pm}} \|v\|_{Y_{t,x}^{2,\pm}} \|u_{N_1}\|_{V_{t,x}^{2,\pm}} \cdot$$

\[\square\]

3. Bilinear estimates.

**Proposition 3.1.** (i) Let $(\theta, s) = (1/4, 1/4)$ for $d = 4$ and $(\theta, s) = (1/(d+1), (d^2 - 3d - 2)/2(d+1))$ for $d \geq 5$. For any $0 < T < 1$, it holds that

$$\|I_{T, K_{\pm}}(n, v)\|_{Z_{K_{\pm}}^{2}} \lesssim T^{\theta} \|n\|_{Y_{t,x}^{2,\pm}} \|v\|_{Y_{t,x}^{2,\pm}} \cdot \tag{3.1}$$

$$\|I_{T, W_{t,x}}(u, v)\|_{Z_{W_{t,x}}^{2}} \lesssim T^{\theta} \|u\|_{Y_{t,x}^{2,\pm}} \|v\|_{Y_{t,x}^{2,\pm}} \cdot \tag{3.2}$$

(ii) We assume that $(u, v, n)$ are spherically symmetric functions. Then for $d \geq 4$ and for all $0 < T < \infty$, (3.1), (3.2) also holds with $(\theta, s) = (0, d/2 - 2)$.

**Proof.** We denote $\tilde{u}_{N_1} := 1_{[0, T]} P_{N_1} u$, $\tilde{v}_{N_2} := 1_{[0, T]} P_{N_2} v$, $\tilde{n}_{N_3} := 1_{[0, T]} P_{N_3} n$. To prove (3.1), we need to estimate the following.

$$\|I_{T, K_{\pm}}(n, v)\|_{Z_{K_{\pm}}^{2}} \lesssim \sum_{i=0}^{3} J_i \cdot$$
where

\[ J_0 := \left\| \int_0^t 1_{[0,T]}(t') K_\pm(t - t') P_{<1}(\tilde{n}(\omega_1^{-1}\tilde{v})))(t') dt' \right\|_{U^2_{K_\pm}}. \]

\[ J_1 := \sum_{N_1 \geq 1} N_1^{2s} \left\| \int_0^t 1_{[0,T]}(t') K_\pm(t - t') \sum_{N_2 \ll N_1} \sum_{N_3 \ll N_2} P_{N_1}(\tilde{n}(\omega_1^{-1}\tilde{v}_{N_2}))(t') dt' \right\|_{U^2_{K_\pm}}. \]

\[ J_2 := \sum_{N_1 \geq 1} N_1^{2s} \left\| \int_0^t 1_{[0,T]}(t') K_\pm(t - t') \sum_{N_2 \ll N_1} \sum_{N_3 \ll N_1} P_{N_1}(\tilde{n}(\omega_1^{-1}\tilde{v}_{N_2}))(t') dt' \right\|_{U^2_{K_\pm}}. \]

\[ J_3 := \sum_{N_1 \geq 1} N_1^{2s} \left\| \int_0^t 1_{[0,T]}(t') K_\pm(t - t') \sum_{N_2 \gg N_1} \sum_{N_3 \ll N_2} P_{N_1}(\tilde{n}(\omega_1^{-1}\tilde{v}_{N_2}))(t') dt' \right\|_{U^2_{K_\pm}}. \]

By Corollary 2.5, we have

\[ J_0^{1/2} \lesssim \sup_{\|u\|_{U^2_{K_\pm}}} \left| \int_{\mathbb{R}^{1+d}} \tilde{n}(\omega_1^{-1}\tilde{v})(P_{<1}\tilde{u}) dx dt \right|. \tag{3.3} \]

We apply Corollary 2.5 to have

\[ J_1 \lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{U^2_{K_\pm}}} \left| \int_{\mathbb{R}^{1+d}} \tilde{n}(\omega_1^{-1}\tilde{v})_{N_1} \tilde{u}_{N_1} dx dt \right|^2. \tag{3.4} \]

For the estimate of \( J_2 \), we take \( M = \varepsilon N_1 \) for sufficiently small \( \varepsilon > 0 \). Then, from Lemma 2.3, we have

\[ Q_{<M}(Q_{<M}\tilde{n}_{N_1})(Q_{<M}\omega_1^{-1}\tilde{v}_{N_2}) \]

\[ = Q_{<M} \left[ \mathcal{F}^{-1} \left( \int_{\tau_1 = \tau_2 + \tau_3, \xi_1 = \xi_2 + \xi_3} (Q_{<M}\tilde{n}_{N_1})(\tau_2, \xi_2)(Q_{<M}\omega_1^{-1}\tilde{v}_{N_2})(\tau_2, \xi_2) \right) \right] = 0 \]

when \( N_1 \gg (N_2) \). Therefore,

\[ \tilde{n}_{N_1}(\omega_1^{-1}\tilde{v}_{N_2}) = \sum_{i=1}^{3} F_i, \]

where \( Q_1, Q_2 \in \{Q_{<M}, Q_{\leq M}\} \) be operator for the Klein-Gordon equation, \( Q_3 \in \{Q_{<M}, Q_{\geq M}\} \) be operator for the wave equation and

\[ F_1 := Q_{<M}(Q_{<M}\tilde{n}_{N_1})(Q_{2\omega_1^{-1}}\tilde{v}_{N_2}), \quad F_2 := Q_{<M}(Q_{3\tilde{n}_{N_1}})(Q_{2\omega_1^{-1}}\tilde{v}_{N_2}), \]

\[ F_3 := Q_{>M}(Q_{3\tilde{n}_{N_1}})(Q_{2\omega_1^{-1}}\tilde{v}_{N_2}). \]

For the estimate of \( F_1 \), we apply Corollary 2.5 and Lemma 2.18 to have

\[ \sum_{N_1 \geq 1} N_1^{2s} \left\| \int_0^t 1_{[0,T]}(t') K_\pm(t - t') \sum_{N_2 \ll N_1} \sum_{N_3 \ll N_1} P_{N_1} F_1(t') dt' \right\|_{U^2_{K_\pm}} \]

\[ \lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{U^2_{K_\pm}}} \left| \int_{\mathbb{R}^{1+d}} (Q_{<M}\tilde{n}_{N_1})(Q_{2\omega_1^{-1}}\tilde{v}_{N_2})(Q_{1\tilde{u}_{N_1}}) dx dt \right|^2. \tag{3.5} \]
For the proof of \((3.2)\), by Corollary 2.5, we only need to estimate
\[
\sum_{N_1 \geq 1} N_1^{2s} \left\| \int_0^t 1_{[0,T]}(t') K_\pm(t - t') \sum_{N_2 \ll N_1, N_3 \sim N_1} P_{N_1} F_2(t') dt' \right\|^2_{U^2_{K_\pm}} 
\lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{v_2^{K_\pm}} = 1} \left| \sum_{N_2 \ll N_1, N_3 \sim N_1} \int_{\mathbb{R}^{1+d}} (Q_3 \tilde{n}_{N_3})(Q_{\geq M}^{-1} \tilde{v}_{N_2})(Q_{\geq M}^{-1} \tilde{u}_{N_1}) dx \right|^2.
\]  
(3.6)

For the estimate of \(F_3\), we apply Corollary 2.5 and Lemma 2.18 to have
\[
\sum_{N_1 \geq 1} N_1^{2s} \left\| \int_0^t 1_{[0,T]}(t') K_\pm(t - t') \sum_{N_2 \ll N_1, N_3 \sim N_1} P_{N_1} F_3(t') dt' \right\|^2_{U^2_{K_\pm}} 
\lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{v_2^{K_\pm}} = 1} \left| \sum_{N_2 \ll N_1, N_3 \sim N_1} \int_{\mathbb{R}^{1+d}} (Q_3 \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2})(Q_{\geq M}^{-1} \tilde{u}_{N_1}) dx \right|^2.
\]  
(3.7)

By Corollary 2.5 and the triangle inequality to have
\[
J_3 \lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{v_2^{K_\pm}} = 1} \left| \sum_{N_2 \gg N_1, N_3 \sim N_2} \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \right|^2 
\lesssim \sum_{N_1 \geq 1} N_1^{2s} \left( \sum_{N_2 \gg N_1, N_3 \sim N_2} \sup_{\|u\|_{v_2^{K_\pm}} = 1} \left| \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \right| \right)^2. 
\]  
(3.8)

For the proof of \((3.2)\), by Corollary 2.5, we only need to estimate \(K_i\) \((i = 1, 2, 3)\):
\[
K_1 := \sum_{N_1 \leq 1} N_1^{2s} \sup_{\|u\|_{v_2^{K_\pm}} = 1} \left| \sum_{N_2 \ll N_1, N_3 \leq N_1} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1})(\omega_1^{-1} \tilde{v}_{N_2}) (\overline{\omega n_{N_3}}) dx dt \right|^2,
\]
\[
K_2 := \sum_{N_1} N_1^{2s} \sup_{\|u\|_{v_2^{K_\pm}} = 1} \left| \sum_{N_2 \ll N_1, N_3 \ll N_1} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1})(\omega_1^{-1} \tilde{v}_{N_2}) (\overline{\omega n_{N_3}}) dx dt \right|^2.
\]
\[
K_3 := \sum_{N_1} N_1^{2s} \sup_{\|u\|_{v_2^{K_\pm}} = 1} \left| \sum_{N_2 \gg N_1, N_3 \sim N_2} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1})(\omega_1^{-1} \tilde{v}_{N_2}) (\overline{\omega n_{N_3}}) dx dt \right|^2.
\]

First, we estimate \(K_1\). Put \(K_1 = K_{1,1} + K_{1,2}\) where
\[
K_{1,1} := \sum_{N_1 \leq 1} N_1^{2s} \sup_{\|u\|_{v_2^{K_\pm}} = 1} \left| \sum_{N_2 \ll N_1, N_3 \leq N_1} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1})(\omega_1^{-1} \tilde{v}_{N_2}) (\overline{\omega n_{N_3}}) dx dt \right|^2, 
\]
\[
K_{1,2} := \sum_{N_1 \gg 1} N_1^{2s} \sup_{\|u\|_{v_2^{K_\pm}} = 1} \left| \sum_{N_2 \ll N_1, N_3 \ll N_1} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1})(\omega_1^{-1} \tilde{v}_{N_2}) (\overline{\omega n_{N_3}}) dx dt \right|^2.
\]  
(3.9)
For the estimate for $K_{1,2}$, we take $M = \varepsilon N_2$ for sufficiently small $\varepsilon > 0$. Then, from Lemma 2.3, we have

$$
Q_{<M\omega_1^{-1}}((Q_{<M\omega_1^{-1}} \tilde{v}_{N_2})(Q_{<M\omega} \tilde{n}_{N_3}))
= Q_{<M\omega_1^{-1}} \left[ F^{-1} \left( \int_{\tau_1 = \tau_2 + \tau_3, \xi_1 = \xi_2 + \xi_3} (Q_{<M\omega_1^{-1}} \tilde{v}_{N_2})(\tau_2, \xi_2)(Q_{<M\omega} \tilde{n}_{N_3})(\tau_3, \xi_3) \right) \right]
= 0
$$

when $N_2 \gg (N_1)$. Therefore,

$$(\omega_1^{-1} \tilde{v}_{N_2})(\omega \tilde{n}_{N_3}) = \sum_{i=1}^{3} G_i,
$$

where $Q_1, Q_2 \in \{Q_{<M}, Q_{\geq M}\}$ be operator for the Klein-Gordon equation, $Q_3 \in \{Q_{<M}, Q_{\geq M}\}$ be operator for the wave equation and

$$
G_1 := Q_{\geq M}((Q_{<M\omega_1^{-1}} \tilde{v}_{N_2})(Q_{3\omega} \tilde{n}_{N_3})),
G_2 := Q_1((Q_{\geq M\omega_1^{-1}} \tilde{v}_{N_2})(Q_{3\omega} \tilde{n}_{N_3})),
G_3 := Q_1((Q_{<M\omega_1^{-1}} \tilde{v}_{N_2})(Q_{\geq M\omega} \tilde{n}_{N_3})).
$$

Hence, it follows that

$$
K_{1,2} \leq K_{1,2,1} + K_{1,2,2} + K_{1,2,3}
$$

where

$$
K_{1,2,1} := \sum_{N_3 \gg 1} N_3^{2s} \sup_{\|n\|_{V_{x,t}^{2s}}^2} \left| \int_{\tau_1 = \tau_2 + \tau_3, \xi_1 = \xi_2 + \xi_3} (\omega_1^{-1} \tilde{u}_{N_1})(G_1) dxdt \right|^2,
$$

$$
K_{1,2,2} := \sum_{N_3 \gg 1} N_3^{2s} \sup_{\|n\|_{V_{x,t}^{1s}}^2} \left| \int_{\tau_1 = \tau_2 + \tau_3, \xi_1 = \xi_2 + \xi_3} (\omega_1^{-1} \tilde{u}_{N_1})(G_2) dxdt \right|^2,
$$

$$
K_{1,2,3} := \sum_{N_3 \gg 1} N_3^{2s} \sup_{\|n\|_{V_{x,t}^{1s}}^2} \left| \int_{\tau_1 = \tau_2 + \tau_3, \xi_1 = \xi_2 + \xi_3} (\omega_1^{-1} \tilde{u}_{N_1})(G_3) dxdt \right|^2.
$$

By Lemma 2.18, we have

$$
K_{1,2,1} \lesssim \sum_{N_3 \gg 1} N_3^{2s} \sup_{\|n\|_{V_{x,t}^{2s}}^2} \left| \int_{\tau_1 = \tau_2 + \tau_3, \xi_1 = \xi_2 + \xi_3} (Q_{\geq M\omega_1^{-1}} \tilde{u}_{N_1})(Q_{2\omega_1^{-1}} \tilde{v}_{N_2}) \right|^2,
$$

$$
K_{1,2,2} \lesssim \sum_{N_3 \gg 1} N_3^{2s} \sup_{\|n\|_{V_{x,t}^{1s}}^2} \left| \int_{\tau_1 = \tau_2 + \tau_3, \xi_1 = \xi_2 + \xi_3} (Q_{1\omega_1^{-1}} \tilde{u}_{N_1})(Q_{\geq M\omega_1^{-1}} \tilde{v}_{N_2}) \right|^2,
$$

$$
K_{1,2,3} \lesssim \sum_{N_3 \gg 1} N_3^{2s} \sup_{\|n\|_{V_{x,t}^{1s}}^2} \left| \int_{\tau_1 = \tau_2 + \tau_3, \xi_1 = \xi_2 + \xi_3} (Q_{1\omega_1^{-1}} \tilde{u}_{N_1})(Q_{2\omega_1^{-1}} \tilde{v}_{N_2}) \right|^2.
$$

By symmetry, the estimate for $K_2$ is obtained by the same manner as the estimate for $K_1$. Hence, we omit the estimate for $K_2$. We estimate $K_3$. By the triangle
inequality, we have

\[
J_0^{1/2} \lesssim \sum_{N_2} \sum_{N_1 \sim N_2} \left\{ \sum_{N_3 \lesssim N_2} N_3^{2s} \right\}
\sup_{\|n\|_{V^2_{K \pm}}} \left| \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1})(\omega_1^{-1} \tilde{v}_{N_2})(\omega \tilde{n}_{N_3}) dx dt \right|^{2/1}.
\]

(3.16)

First, we prove (3.1) for \( d = 4, s = 1/4 \). From (3.3), Lemma 2.19 (ii) and \( \|P_1 u\|_{V^2_{K \pm}} \lesssim \|u\|_{Y^1_{K \pm}} \), we obtain

\[
J_0^{1/2} \lesssim T\|n\|_{Y^{1/4}_{W_{\pm}}} \|v\|_{Y^{1/4}_{K \pm}}.
\]

By (3.4), \( N_1 \sim N_2 \), Lemma 2.19 (iii) and \( \|u_{N_1}\|_{V^2_{K \pm}} \lesssim \|u\|_{V^2_{K \pm}} \), we have

\[
J_1 \lesssim \sum_{N_3 \gtrsim 1} N_2^{1/2} T^{1/2} \|n\|_{Y^{1/4}_{V_{K \pm}}} \|v_{N_2}\|_{V^2_{K \pm}} \lesssim T^{1/2} \|n\|_{Y^{1/4}_{V_{K \pm}}} \|v\|_{Y^{1/4}_{K \pm}}.
\]

(3.17)

From Lemma 2.19 (iv), \( N_3 \sim N_1 \geq 1 \) and \( \|u_{N_1}\|_{V^2_{K \pm}} \lesssim \|u\|_{V^2_{K \pm}} \), the right-hand side of (3.5) is bounded by

\[
T^{1/2} \sum_{N_3 \gtrsim 1} N_3^{1/2} \|n_{N_3}\|_{V^2_{K \pm}} \|v\|_{Y^{1/4}_{V_{K \pm}}} \lesssim T^{1/2} \|n\|_{Y^{1/4}_{V_{K \pm}}} \|v\|_{Y^{1/4}_{K \pm}}.
\]

(3.18)

We apply Lemma 2.19 (iv), \( N_3 \sim N_1 \geq 1 \) and \( \|u_{N_1}\|_{V^2_{K \pm}} \lesssim \|u\|_{V^2_{K \pm}} \), then the right-hand side of (3.6) is bounded by

\[
T^{1/2} \sum_{N_3 \gtrsim 1} N_3^{1/2} \|n_{N_3}\|_{V^2_{K \pm}} \|v\|_{Y^{1/4}_{V_{K \pm}}} \lesssim T^{1/2} \|n\|_{Y^{1/4}_{V_{K \pm}}} \|v\|_{Y^{1/4}_{K \pm}}.
\]

(3.19)

Collecting (3.17), (3.18) and (3.19), we obtain

\[
J_2 \lesssim T^{1/2} \|n\|_{Y^{1/4}_{V_{K \pm}}} \|v\|_{Y^{1/4}_{K \pm}}.
\]

(3.20)

In the same manner as (2.37), we have

\[
\int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}) \tilde{u}_{N_1} dx dt \lesssim T^{1/4} N_3^{1/4} \|n_{N_3}\|_{V^2_{K \pm}} \|v_{N_2}\|_{V^2_{K \pm}} \|u_{N_1}\|_{V^2_{K \pm}}.
\]

(3.20)

From (3.20), the right-hand side of (3.8) is bounded by

\[
\sum_{N_1 \gtrsim 1} \left( \sum_{N_3 \gtrsim N_1} \sum_{N_2 \lesssim N_3} N_1^{1/4} T^{1/4} N_3^{1/4} \|n_{N_3}\|_{V^2_{K \pm}} \|v_{N_2}\|_{V^2_{K \pm}} \right)^2.
\]

Hence, \( \| \cdot \|_{T^{1/4}} \lesssim \| \cdot \|_{T^{1/4}} \) and the Cauchy-Schwarz inequality to have

\[
J_3^{1/2} \lesssim \sum_{N_2 \gtrsim 1} \sum_{N_3 \gtrsim N_2} \left( \sum_{N_1 \lesssim N_2} N_1^{1/2} T^{1/2} N_3^{1/2} \|n_{N_3}\|_{V^2_{K \pm}} \|v_{N_2}\|_{V^2_{K \pm}} \right)^{1/2}
\]

\[
\lesssim T^{1/4} \sum_{N_2 \gtrsim 1} \sum_{N_3 \gtrsim N_2} N_2^{1/4} N_3^{1/4} \|n_{N_3}\|_{V^2_{K \pm}} \|v_{N_2}\|_{V^2_{K \pm}}
\]

\[
\lesssim T^{1/4} \|n\|_{Y^{1/4}_{W_{\pm}}} \|v\|_{Y^{1/4}_{K \pm}}.
\]
We prove (3.2) for \( d = 4, s = 1/4 \). By the same manner as the estimate for Lemma 2.19 (iv), we find

\[
\left| \int_{\mathbb{R}^{1+4}} \left( \sum_{N_1 \ll N_3} \omega_1^{-1} \bar{u}_{N_1} \right) \left( \omega_1^{-1} \tilde{v}_{N_2} \right) \left( \omega \tilde{m}_{N_3} \right) dx dt \right|
\]

\[
\lesssim \left\| 1_{[0,T]} \right\| L_t^2 \left\| \sum_{N_1 \ll N_3} \omega_1^{-1} \bar{u}_{N_1} \right\|_{L_t^{20/3} L_x^{12/5}} \left\| \omega_1^{-1} \tilde{v}_{N_2} \right\|_{L_t^{20/3} L_x^{12/5}} \left\| \omega \tilde{m}_{N_3} \right\|_{L_t^{20/3} L_x^{12/5}}
\]

\[
\lesssim T^{1/4} \left\| \left( \nabla_x \right)^{1/4} \sum_{N_1 \ll N_3} \omega_1^{-1} \bar{u}_{N_1} \right\|_{\mathbb{H}_N^2} \left\| \left( \nabla_x \right)^{1/4} \omega_1^{-1} \tilde{v}_{N_2} \right\|_{\mathbb{H}_N^2} \left\| \left( \nabla_x \right)^{1/4} \omega \tilde{m}_{N_3} \right\|_{W_{2,c}^{3/4}}
\]

Hence, from (3.13) and (3.22), we have

\[
K_{1,1} \lesssim \sum_{N_2 \leq 1} N_2^{1/2} \left( T^{1/4} N_2^{3/4} \left\| u \right\|_{Y_{K^\pm}^{1/4}} \left\| v_{K^\pm} \right\| \right)^2 \lesssim T^{1/2} \left\| u \right\|_{Y_{K^\pm}^{1/4}}^2 v_{K^\pm}^2.
\]

By the same manner as the estimate for Lemma 2.19 (iv), we have

\[
K_{1,2} \lesssim \sum_{N_2 \gg 1} N_2^{1/2} \left( T^{1/4} N_2^{3/4} \left\| u \right\|_{Y_{K^\pm}^{1/4}} \left\| v_{K^\pm} \right\| \right)^2 \lesssim T^{1/2} \left\| u \right\|_{Y_{K^\pm}^{1/4}}^2 v_{K^\pm}^2.
\]

Hence, from (3.13), we have

\[
K_{1,2,1} \lesssim \sum_{N_2 \gg 1} N_2^{1/2} \left( T^{1/4} \left\| u \right\|_{Y_{K^\pm}^{1/4}} \left\| v_{K^\pm} \right\| \right)^2 \lesssim T^{1/2} \left\| u \right\|_{Y_{K^\pm}^{1/4}}^2 v_{K^\pm}^2.
\]

By the same manner as the estimate for Lemma 2.19 (iv), we have

\[
K_{1,2,2} \lesssim \sum_{N_2 \gg 1} N_2 \left( T^{1/4} \left\| u \right\|_{Y_{K^\pm}^{1/4}} \left\| v_{K^\pm} \right\| \right)^2 \lesssim T^{1/2} \left\| u \right\|_{Y_{K^\pm}^{1/4}}^2 v_{K^\pm}^2.
\]

Hence, from (3.13), we have

\[
K_{1,2,3} \lesssim \sum_{N_2 \gg 1} N_2 \left( T^{1/4} \left\| u \right\|_{Y_{K^\pm}^{1/4}} \left\| v_{K^\pm} \right\| \right)^2 \lesssim T^{1/2} \left\| u \right\|_{Y_{K^\pm}^{1/4}}^2 v_{K^\pm}^2.
\]
Hence, $K_{1,2} \lesssim T^{1/2} \| u \|_{Y^1/4}^2 \| v \|_{Y^1/4}^2$. Therefore, we obtain $K_1 \lesssim T^{1/2} \| u \|_{Y^1/4}^2 \| v \|_{Y^1/4}^2$.

We apply Lemma 2.19 (i) and the Cauchy-Schwarz inequality, the right-hand side of (3.16) is bounded by

$$
\sum_{N_2} \sum_{N_1=1} \left\{ \sum_{N_1} N_1^{1/2} \left( T^{1/4} N_3^{1/4} \| u_{N_1} \|_{V^2_{K}}^2 \| v_{N_1} \|_{V^2_{K}}^2 \right)^2 \right\}^{1/2}
\lesssim T^{1/4} \sum_{N_2} \sum_{N_1=1} \left( N_2 \| u_{N_1} \|_{V^2_{K}}^2 \| v_{N_2} \|_{V^2_{K}}^2 \right)^{1/2}
\lesssim T^{1/4} \left( \sum_N N^{1/2} \| u_N \|_{V^2_{K}}^2 \right)^{1/2} \left( \sum_N N^{1/2} \| v_N \|_{V^2_{K}}^2 \right)^{1/2}.
$$

Since

$$
\sum_{N<1} N^{1/2} \| u_N \|_{V^2_{K}}^2 \lesssim \sum_{N<1} N^{1/2} \| P_{<1} u \|_{V^2_{K}}^2 \lesssim \| P_{<1} u \|_{V^2_{K}}^2,
$$

we obtain $K_3^{1/2} \lesssim T^{1/4} \| u \|_{Y^1/4}^2 \| v \|_{Y^1/4}^2$.

Next, we prove (3.1) for $d \geq 5$ and $s = s' = (d^2 - 3d - 2)/2(d + 1)$ by the same manner as the proof for $d = 4, s = 1/4$. From (3.3) and Lemma 2.19 (ii), we have

$$
J_0^{1/2} \lesssim T^{1/(d+1)} \| n \|_{Y^{s'}_{W,\pm}} \| v \|_{Y^{s'}_{K,\pm}}.
$$

By (3.4), $N_1 \sim N_2$, Lemma 2.19 (iii) and $\| u_{N_1} \|_{V^2_{K}}^2 \lesssim \| u \|_{V^2_{K}}^2$, we have

$$
J_1 \lesssim \sum_{N_2 \gtrsim 1} N_2^{2s'} T^{2/(d+1)} \| n \|_{Y^{s'}_{W,\pm}}^2 \| v_{N_2} \|_{V^2_{K}}^2 \lesssim T^{2/(d+1)} \| n \|_{Y^{s'}_{W,\pm}}^2 \| v \|_{V^{s'}_{K,\pm}}^2.
$$

From Lemma 2.19 (iv), $N_3 \sim N_1 \gtrsim 1$ and $\| u_{N_3} \|_{V^2_{K}}^2 \lesssim \| u \|_{V^2_{K}}^2$, the right-hand side of (3.5) is bounded by

$$
T^{2/(d+1)} \sum_{N_3 \gtrsim 1} N_3^{2s'} \| n_{N_3} \|_{V_{R,\pm}}^2 \| v \|_{Y^{s'}_{K,\pm}}^2 \lesssim T^{2/(d+1)} \| n \|_{Y^{s'}_{W,\pm}}^2 \| v \|_{V^{s'}_{K,\pm}}^2.
$$

From Lemma 2.19 (iv), $N_3 \sim N_1 \gtrsim 1$ and $\| u_{N_3} \|_{V^2_{K}}^2 \lesssim \| u \|_{V^2_{K}}^2$, the right-hand side of (3.6) is bounded by

$$
T^{2/(d+1)} \sum_{N_3 \gtrsim 1} N_3^{2s'} \| n_{N_3} \|_{V_{R,\pm}}^2 \| v \|_{Y^{s'}_{K,\pm}}^2 \lesssim T^{2/(d+1)} \| n \|_{Y^{s'}_{W,\pm}}^2 \| v \|_{V^{s'}_{K,\pm}}^2.
$$

From Lemma 2.19 (iv), $N_3 \sim N_1 \gtrsim 1$ and $\| u_{N_3} \|_{V^2_{K}}^2 \lesssim \| u \|_{V^2_{K}}^2$, the right-hand side of (3.7) is bounded by

$$
T^{2/(d+1)} \sum_{N_3 \gtrsim 1} N_3^{2s'} \| n_{N_3} \|_{V_{R,\pm}}^2 \| v \|_{Y^{s'}_{K,\pm}}^2 \lesssim T^{2/(d+1)} \| n \|_{Y^{s'}_{W,\pm}}^2 \| v \|_{V^{s'}_{K,\pm}}^2.
$$

Collecting (3.24)–(3.26), we obtain $J_2 \lesssim T^{2/(d+1)} \| n \|_{Y^{s'}_{W,\pm}}^2 \| v \|_{V^{s'}_{K,\pm}}^2$. By the same manner as the estimate for Lemma 2.19 (iii), we obtain

$$
\left| \int_{R^{1+d}} \hat{\tilde{n}}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \overline{u_{N_1}} \, dt \right| \lesssim T^{1/(d+1)} N_3^{s'} \| n_{N_3} \|_{V^2_{R,\pm}} \| v_{N_2} \|_{V^2_{K}} \| u_{N_1} \|_{V^2_{K}}^2.
$$

(3.27)
From (3.27), the right-hand side of (3.8) is bounded by
\[
\sum_{N_1 \geq 1} \left( \sum_{N_2 \gg N_1} \sum_{N_2 \sim N_2} N_1^{s'} T^{(d+1)} N_2^{s'} N_2 \|n_{N_1}\|_{V^2_{W_{L^2}}} \|\nu_{N_2}\|_{V_{K^\pm}}^2 \right)^2.
\]
Hence, \( \| \cdot \|_{L^2_t} \lesssim \| \cdot \|_{L^2_t} \) and the Cauchy-Schwarz inequality to have
\[
J_3^{1/2} \lesssim \sum_{N_2 \gg N_1 \sim N_2} \left( \sum_{N_1} N_1^{2s'} T^{2(d+1)} N_2^{2s'} \|n_{N_1}\|_{V^2_{W_{L^2}}} \|\nu_{N_2}\|_{V_{K^\pm}}^2 \right)^{1/2}
\]
\[
\lesssim T^{1/(d+1)} \sum_{N_2 \gg 1} N_2 \sum_{N_1} N_1^{2s'} \|n_{N_1}\|_{V^2_{W_{L^2}}} \|\nu_{N_2}\|_{V_{K^\pm}}^2
\]
\[
\lesssim T^{1/(d+1)} \|n\|_{V_{K^\pm}} \|v\|_{V_{K^\pm}}.
\]

We prove (3.2) for \( d \geq 5, s = s' = (d^2 - 3d - 2)/(d + 1) \) by the same manner as the proof for \( d = 4, s = 1/4 \). By the Hölder inequality to have
\[
\left| \int_{\mathbb{R}^1} \left( \sum_{N_1 \ll N_3} \omega^{-1}_1 \tilde{u}_{N_1} \right) \tilde{v}_{N_3} \tilde{\omega} \frac{dx}{dn} \right| dt \lesssim \left\| \sum_{N_1 \ll N_3} \omega^{-1}_1 \tilde{u}_{N_1} \right\|_{L^2_{x,t}} \left\| \omega^{-1}_1 \tilde{v}_{N_3} \right\|_{L^2_{x,t}} \left\| \omega \frac{dx}{dn} \right\|_{L^{1/2}_{x,t}}. \tag{3.28}
\]

By Proposition 2.10, discarding \( \omega^{-1}_1 \), by the same manner as (2.47) and \( s' > 1/2 \), we have
\[
\left\| \sum_{N_1 \ll N_3} \omega^{-1}_1 \tilde{u}_{N_1} \right\|_{L^2_{x,t}} \lesssim \left( \sum_{N_1 \ll N_3} \omega^{-1}_1 \tilde{u}_{N_1} \right) \|v\|_{V_{K^\pm}}^2
\]
\[
\lesssim \|u\|_{V_{K^\pm}}^2
\]
\[
\lesssim \|u\|_{V_{K^\pm}}. \tag{3.29}
\]

From (3.9), (3.28), (3.29), (2.17), (2.21) and \( N_2 \sim N_3 \lesssim 1 \), we obtain
\[
K_{1,1} \lesssim \sum_{N_2 \ll N_3} N_2^{s'} (T^{(d+1)} \langle N_2 \rangle^{-1/2} N_2^{s'+1} \|u\|_{Y_{K^\pm}} \|\nu_{N_2}\|_{V_{K^\pm}}^2)^2
\]
\[
\lesssim T^{2(d+1)} \|u\|_{V_{K^\pm}}^2 \sum_{N_2 \ll 1} N_2^{s'} \|v_{N_2}\|_{V_{K^\pm}}^2
\]
\[
\lesssim T^{2(d+1)} \|u\|_{V_{K^\pm}}^2 \|v\|_{V_{K^\pm}}. \tag{3.30}
\]

By the same manner as the estimate for Lemma 2.19 (iv), \( i = 5 \), we see
\[
\left| \int_{\mathbb{R}^1} \left( \sum_{N_1 \ll N_3} Q_{\geq M} \omega^{-1}_1 \tilde{u}_{N_1} \right) \tilde{v}_{N_3} \frac{dx}{dn} \right| dt \lesssim T^{1/(d+1)} \langle N_2 \rangle^{-1/2} \|u\|_{Y_{K^\pm}} \|\nu_{N_2}\|_{V_{K^\pm}} \|n_{N_3}\|_{V_{W_{L^2}}} \|v\|_{V_{K^\pm}}. \tag{3.30}
\]

From (3.13), (3.30), \( N_3 \gg 1 \) and \( N_2 \sim N_3 \), we have
\[
K_{1,2,1} \lesssim \sum_{N_2 \gg 1} N_2^{s'} (T^{(d+1)} \|u\|_{Y_{K^\pm}} \|v_{N_2}\|_{V_{K^\pm}}^2)^2 \lesssim T^{2(d+1)} \|u\|_{V_{K^\pm}}^2 \|v\|_{V_{K^\pm}}. \tag{3.30}
\]
By the same manner as the estimate for Lemma 2.19 (iv), \( i = 6, \) we see
\[
\left| \int_{\mathbb{R}^{d+1}} \left( \sum_{N_1 \leq N_3} Q_{1 \omega_1}^{-1} \tilde{u}_N \right) (Q_{\geq M \omega_1}^{-1} \tilde{v}_{N_2}) (Q_{\leq M \omega_1} N_3) \, dx \, dt \right| \\
\lesssim T^{1/(d+1)} (N_2)^{-1} \| u \|_{Y^{s'}_{K^+}} \| v_{N_2} \|_{V^2_{K^+}} \| n_{N_3} \|_{V^2_{\tilde{w}_{K^+}}}.
\]
(3.31)

From (3.14), (3.31), \( N_3 \gg 1 \) and \( N_2 \sim N_3, \) we have
\[
K_{1,2,2} \lesssim \sum_{N_2 \gg 1} N_2^{2s'} (T^{1/(d+1)} \| u \|_{Y^{s'}_{K^+}} \| v_{N_2} \|_{V^2_{K^+}})^2 \lesssim T^{2/(d+1)} \| u \|_{Y^{s'}_{K^+}} \| v \|_{Y^{s'}_{K^+}}.
\]

By the same manner as the estimate for Lemma 2.19 (iv), \( i = 4, \) we see
\[
\left| \int_{\mathbb{R}^{d+1}} \left( \sum_{N_1 \leq N_3} Q_{1 \omega_1}^{-1} \tilde{u}_N \right) (Q_{\geq M \omega_1}^{-1} \tilde{v}_{N_2}) (Q_{\leq M \omega_1} N_3) \, dx \, dt \right| \\
\lesssim T^{1/(d+1)} (N_2)^{-1} \| u \|_{Y^{s'}_{K^+}} \| v_{N_2} \|_{V^2_{K^+}} \| n_{N_3} \|_{V^2_{\tilde{w}_{K^+}}}.
\]
(3.32)

From (3.15), (3.32), \( N_3 \gg 1 \) and \( N_2 \sim N_3, \) we have
\[
K_{1,2,3} \lesssim \sum_{N_2 \gg 1} N_2^{2s'} (T^{1/(d+1)} \| u \|_{Y^{s'}_{K^+}} \| v_{N_2} \|_{V^2_{K^+}})^2 \lesssim T^{2/(d+1)} \| u \|_{Y^{s'}_{K^+}} \| v \|_{Y^{s'}_{K^+}}.
\]

We apply Lemma 2.19 (i) and the Cauchy-Schwarz inequality, the right-hand side of (3.16) is bounded by
\[
\sum_{N_2 \gg 1} \sum_{N_1 \sim N_2} \sum_{N_3} N_2^{2s'} (T^{1/(d+1)} N_3^{ \frac{1}{2}} \| u_{N_1} \|_{V^2_{K^+}} \| v_{N_2} \|_{V^2_{K^+}})^2 \lesssim T^{1/(d+1)} \sum_{N_2 \gg 1} \sum_{N_1 \sim N_2} \left( N_2^{4s'} \| u_{N_1} \|_{V^2_{K^+}} \| v_{N_2} \|_{V^2_{K^+}} \right)^{1/2}
\]
\[
\lesssim T^{1/(d+1)} \left( \sum_{N} N_2^{2s'} \| u_{N_1} \|_{V^2_{K^+}} \right)^{1/2} \left( \sum_{N} N_2^{2s'} \| v_{N_2} \|_{V^2_{K^+}} \right)^{1/2}.
\]

Since \( s' > 0, \) we have
\[
\sum_{N} N_2^{2s'} \| u_{N_1} \|_{V^2_{K^+}} \lesssim \sum_{N} N_2^{2s'} \| P_{< 1} u \|_{V^2_{K^+}} \lesssim \| P_{< 1} u \|_{V^2_{K^+}}.
\]

Thus, we obtain
\[
K_{1}^{1/2} \lesssim T^{1/(d+1)} \| u \|_{Y^{s'}_{K^+}} \| v \|_{Y^{s'}_{K^+}}.
\]

Finally, we prove (3.1) for \( d \geq 4, s = s_c = d/2 - 2 \) and spherically symmetric functions \((u, v, n)\) by the same manner as the proof of \( d = 4, s = 1/4. \) From (3.3) and Lemma 2.19 (ii), we obtain
\[
J_0^{1/2} \lesssim \| n \|_{Y^{s_c}_{W_{K^+}}} \| v \|_{Y^{s_c}_{K^+}}.
\]

By (3.4), \( N_1 \sim N_2, \) Lemma 2.19 (iii) and \( \| u_{N_1} \|_{V^2_{K^+}} \lesssim \| u \|_{V^2_{K^+}}, \) we have
\[
J_1 \lesssim \sum_{N_2 \gg 1} N_2^{2s_c} \| n \|_{Y^{s_c}_{W_{K^+}}} \| v_{N_2} \|_{V^2_{K^+}} \lesssim \| n \|_{Y^{s_c}_{W_{K^+}}} \| v \|_{Y^{s_c}_{K^+}}.
\]

From Lemma 2.19 (iv), \( N_3 \sim N_1 \geq 1 \) and \( \| u_{N_3} \|_{V^2_{K^+}} \lesssim \| u \|_{V^2_{K^+}}, \) the right-hand side of (3.5) is bounded by
\[
\sum_{N_3 \gg 1} N_3^{2s_c} \| n_{N_3} \|_{Y^{s_c}_{W_{K^+}}} \| v \|_{Y^{s_c}_{K^+}} \lesssim \| n \|_{Y^{s_c}_{W_{K^+}}} \| v \|_{Y^{s_c}_{K^+}}.
\]
From Lemma 2.19 (iv), \(N_3 \sim N_1 \geq 1\) and \(\|u_{N_1}\|_{V_{K_\pm}^2} \lesssim \|u\|_{V_{K_\pm}^2}\), the right-hand side of (3.6) is bounded by

\[
\sum_{N_3 \gtrsim 1} N_3^{2s_c} \|n_{N_3}\|_{V_{W_{K_\pm}^2}^2}^2 \|v\|_{Y_{K_\pm}^{s_c}}^2 \lesssim \|n\|_{V_{W_{K_\pm}^2}^2}^2 \|v\|_{Y_{K_\pm}^{s_c}}^2 .
\] (3.34)

From Lemma 2.19 (iv), \(N_3 \sim N_1 \geq 1\) and \(\|u_{N_1}\|_{V_{K_\pm}^2} \lesssim \|u\|_{V_{K_\pm}^2}\), the right-hand side of (3.7) is bounded by

\[
\sum_{N_3 \gtrsim 1} N_3^{2s_c} \|n_{N_3}\|_{V_{W_{K_\pm}^2}^2}^2 \|v\|_{Y_{K_\pm}^{s_c}}^2 \lesssim \|n\|_{V_{W_{K_\pm}^2}^2}^2 \|v\|_{Y_{K_\pm}^{s_c}}^2 .
\] (3.35)

Collecting (3.33)–(3.35), we have \(J_2 \lesssim \|n\|_{V_{W_{K_\pm}^2}^2}^2 \|v\|_{Y_{K_\pm}^{s_c}}^2\). By the same manner as the estimate for Lemma 2.19 (iii), we obtain

\[
\left| \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}) \overline{u_{N_1}} dx dt \right| \lesssim N_3^{s_c} \|n_{N_3}\|_{V_{W_{K_\pm}^2}^2} \|v_{N_2}\|_{Y_{K_\pm}^{s_c}} \|u_{N_1}\|_{V_{K_\pm}^2} .
\] (3.36)

From (3.36), the right-hand side of (3.8) is bounded by

\[
\sum_{N_1 \geq 1} \left( \sum_{N_2 \geq 1} \sum_{N_3 \sim N_2} N_1^{s_c} N_3^{s_c} \|n_{N_3}\|_{V_{W_{K_\pm}^2}^2} \|v_{N_2}\|_{Y_{K_\pm}^{s_c}} \right)^2 .
\]

Hence, when \(d > 4\), by \(s_c > 0\), \(\|\cdot\|_{L^1} \lesssim \|\cdot\|_{L^2}\) and the Cauchy-Schwarz inequality, we have

\[
J_3^{1/2} \lesssim \sum_{N_2 \geq 1} \sum_{N_3 \sim N_2} \left( \sum_{N_1 \ll N_2} N_1^{2s_c} N_3^{2s_c} \|n_{N_3}\|_{V_{W_{K_\pm}^2}^2} \|v_{N_2}\|_{V_{K_\pm}^2} \right)^{1/2} 
\lesssim \sum_{N_2 \geq 1} \sum_{N_3 \sim N_2} N_1^{s_c} N_3^{s_c} \|n_{N_3}\|_{V_{W_{K_\pm}^2}^2} \|v_{N_2}\|_{V_{K_\pm}^2} 
\lesssim \|n\|_{V_{W_{K_\pm}^2}^2} \|v\|_{Y_{K_\pm}^{s_c}} .
\]

When \(d = 4\), similar to (2.22)–(2.25) and \(1 \leq N_1 \ll N_2 \sim N_3\) to have

\[
\left| \int_{\mathbb{R}^{1+4}} \tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}) \overline{u_{N_1}} dx dt \right| \lesssim N_1^{1/3} N_2^{-1/3} \|n_{N_3}\|_{V_{W_{K_\pm}^2}^2} \|v_{N_2}\|_{V_{K_\pm}^2} \|u_{N_1}\|_{V_{K_\pm}^2} .
\] (3.37)

From (3.37), (3.8) and the Cauchy-Schwarz inequality to have

\[
J_3^{1/2} \lesssim \sum_{N_2 \geq 1} \sum_{N_3 \sim N_2} \left( \sum_{N_1 \ll N_2} N_1^{2/3} N_2^{-2/3} \|n_{N_3}\|_{V_{W_{K_\pm}^2}^2} \|v_{N_2}\|_{V_{K_\pm}^2} \right)^{1/2} 
\lesssim \sum_{N_2 \geq 1} \sum_{N_3 \sim N_2} \|n_{N_3}\|_{V_{W_{K_\pm}^2}^2} \|v_{N_2}\|_{V_{K_\pm}^2} 
\lesssim \|n\|_{V_{W_{K_\pm}^2}^2} \|v\|_{Y_{K_\pm}^{s_c}} .
\]

We prove (3.2) for \(d \geq 4, s = s_c = d/2 - 2\) and spherically symmetric functions \((u, v, n)\) by the same manner as the proof of \(d = 4, s = 1/4\). By the H"older inequality
to have
\[ \left\| \int_{\mathbb{R}^{1+d}} \left( \sum_{N_1 \ll N_3} \omega_1^{-1} \bar{u}_{N_1} \right) \left( \omega_2^{-1} \bar{v}_{N_2} \right) (\omega\bar{n}_{N_3}) dx dt \right\| \leq \left\| \sum_{N_1 \ll N_3} \omega_1^{-1} \bar{u}_{N_1} \right\|_{L^2_{t,x}} \left\| \omega_2^{-1} \bar{v}_{N_2} \right\|_{L^2_{t,x}} \left\| \omega\bar{n}_{N_3} \right\|_{L^2_{t,x}}. \tag{3.38} \]

By discarding \( \omega_1^{-1} \) and the same manner as \( (2.23) \), we find
\[ \left\| \sum_{N_1 \ll N_3} \omega_1^{-1} \bar{u}_{N_1} \right\|_{L^2_{t,x}} \lesssim \left\| (\nabla_x)^{(d-2)/2} P_{<1} \left( \sum_{N_1 \ll N_3} \bar{u}_{N_1} \right) \right\|_{V^2_{K_\pm}} \lesssim \left\| P_{<1} u \right\|_{V^2_{K_\pm}} \tag{3.39} \]

in the case \( N_1 \ll N_3 \lesssim 1 \). Collecting \( (3.9), (3.39), (3.39), (2.24), (2.25) \) and \( N_2 \sim N_3 \lesssim 1 \), we obtain
\[ K_{1,1} \lesssim \sum_{N_2 \ll 1} N_2^{2s_\epsilon} \left( \left\| P_{<1} u \right\|_{V^2_{K_\pm}} \right)^2 \lesssim \left\| P_{<1} u \right\|_{V^2_{K_\pm}}^2 \]

By the same manner as the estimate for Lemma 2.19 \((iv)\), \( i = 5 \), we obtain
\[ \left\| \int_{\mathbb{R}^{1+d}} \left( \sum_{N_1 \ll N_3} Q_{>M} \omega_1^{-1} \bar{u}_{N_1} \right) (Q_2 \omega_1^{-1} \bar{v}_{N_2}) (Q_3 \omega\bar{n}_{N_3}) dx dt \right\| \lesssim \left\langle N_2 \right\rangle N_3 \left\| u \right\|_{Y^1_{K_\pm}} \left\| v_{N_2} \right\|_{V^2_{K_\pm}} \left\| n_{N_3} \right\|_{V^2_{W^1_{K_\pm}}}. \tag{3.40} \]

From \( (3.13), (3.40) \), \( N_3 \gg 1 \) and \( N_2 \sim N_3 \), we have
\[ K_{1,2,1} \lesssim \sum_{N_2 \gg 1} N_2^{2s_\epsilon} \left( \left\| u \right\|_{Y^1_{K_\pm}} \left\| v_{N_2} \right\|_{V^2_{K_\pm}} \right)^2 \lesssim \left\| u \right\|_{Y^1_{K_\pm}}^2 \left\| v \right\|_{Y^1_{K_\pm}}^2. \]

By the same manner as the estimate for Lemma 2.19 \((iv)\), \( i = 6 \), we obtain
\[ \left\| \int_{\mathbb{R}^{1+d}} \left( \sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \bar{u}_{N_1} \right) (Q_{>M} \omega_1^{-1} \bar{v}_{N_2}) (Q_3 \omega\bar{n}_{N_3}) dx dt \right\| \lesssim \left\langle N_2 \right\rangle^{-1} N_3 \left\| u \right\|_{Y^1_{K_\pm}} \left\| v_{N_2} \right\|_{V^2_{K_\pm}} \left\| n_{N_3} \right\|_{V^2_{W^1_{K_\pm}}}. \tag{3.41} \]

From \( (3.14), (3.41) \), \( N_3 \gg 1 \) and \( N_2 \sim N_3 \), we have
\[ K_{1,2,2} \lesssim \sum_{N_2 \gg 1} N_2^{2s_\epsilon} \left( \left\| u \right\|_{Y^1_{K_\pm}} \left\| v_{N_2} \right\|_{V^2_{K_\pm}} \right)^2 \lesssim \left\| u \right\|_{Y^1_{K_\pm}}^2 \left\| v \right\|_{Y^1_{K_\pm}}^2. \]

By the same manner as the estimate for Lemma 2.19 \((iv)\), \( i = 4 \), we obtain
\[ \left\| \int_{\mathbb{R}^{1+d}} \left( \sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \bar{u}_{N_1} \right) (Q_2 \omega_1^{-1} \bar{v}_{N_2}) (Q_{>M} \omega\bar{n}_{N_3}) dx dt \right\| \lesssim \left\langle N_2 \right\rangle^{-1} N_3 \left\| u \right\|_{Y^1_{K_\pm}} \left\| v_{N_2} \right\|_{V^2_{K_\pm}} \left\| n_{N_3} \right\|_{V^2_{W^1_{K_\pm}}}. \tag{3.42} \]

From \( (3.15), (3.42) \), \( N_3 \gg 1 \) and \( N_2 \sim N_3 \), we have
\[ K_{1,2,3} \lesssim \sum_{N_2 \gg 1} N_2^{2s_\epsilon} \left( \left\| u \right\|_{Y^1_{K_\pm}} \left\| v_{N_2} \right\|_{V^2_{K_\pm}} \right)^2 \lesssim \left\| u \right\|_{Y^1_{K_\pm}}^2 \left\| v \right\|_{Y^1_{K_\pm}}^2. \]
From (3.16), Lemma 2.19 (i) and the Cauchy-Schwarz inequality, we have

\[ K_{3}^{1/2} \lesssim \sum_{N_{2}} \sum_{N_{1} \sim N_{2}} \left\{ \sum_{N_{3} \lesssim N_{2}} N_{3}^{-2/(3)} \langle N_{2} \rangle^{(d-8)/3} N_{3}^{(d+4)/6} \| u_{N_{1}} \|_{V_{K_{+}}^{2}} \| v_{N_{2}} \|_{V_{K_{-}}^{2}}^{2} \right\}^{1/2} \]

\[ \lesssim \sum_{N_{2}} \sum_{N_{1} \sim N_{2}} \left( \sum_{N_{3} \lesssim N_{2}} N_{3}^{4(d-2)/3} \langle N_{2} \rangle^{2(d-8)/3} \| u_{N_{1}} \|_{V_{K_{+}}^{2}}^{2} \| v_{N_{2}} \|_{V_{K_{-}}^{2}}^{2} \right)^{1/2} \]

\[ \lesssim \sum_{N_{2}} \sum_{N_{1} \sim N_{2}} N_{2}^{2(d-2)/3} \langle N_{2} \rangle^{(d-8)/3} \| u_{N_{1}} \|_{V_{K_{+}}^{2}} \| v_{N_{2}} \|_{V_{K_{-}}^{2}} \]

\[ \lesssim \left( \sum_{N} N^{2(d-2)/3} \langle N \rangle^{(d-8)/3} \| u_{N} \|_{V_{K_{+}}^{2}}^{2} \right)^{1/2} \]

\[ \left( \sum_{N} N^{2(d-2)/3} \langle N \rangle^{(d-8)/3} \| v_{N} \|_{V_{K_{-}}^{2}}^{2} \right)^{1/2} \] (3.43)

For \( N < 1 \), we have \( \langle N \rangle^{(d-8)/3} \lesssim 1 \). Hence we obtain

\[ \sum_{N < 1} N^{2(d-2)/3} \langle N \rangle^{(d-8)/3} \| u_{N} \|_{V_{K_{+}}^{2}}^{2} \lesssim \sum_{N < 1} N^{2(d-2)/3} \| P_{< 1} u \|_{V_{K_{+}}^{2}}^{2} \lesssim \| P_{< 1} u \|_{V_{K_{+}}^{2}}^{2} \]. (3.44)

For \( N \geq 1 \), we have \( \langle N \rangle \sim N \). Hence we obtain

\[ \sum_{N \geq 1} N^{2(d-2)/3} \langle N \rangle^{(d-8)/3} \| u_{N} \|_{V_{K_{+}}^{2}}^{2} \lesssim \sum_{N \geq 1} N^{d-4} \| u_{N} \|_{V_{K_{+}}^{2}}^{2} \]. (3.45)

From (3.44) and (3.45), we have

\[ \left( \sum_{N} N^{2(d-2)/3} \langle N \rangle^{(d-8)/3} \| u_{N} \|_{V_{K_{+}}^{2}}^{2} \right)^{1/2} \lesssim \| u \|_{Y_{K_{+}}^{\infty}} \]. (3.46)

Similarly, we have

\[ \left( \sum_{N} N^{2(d-2)/3} \langle N \rangle^{(d-8)/3} \| v_{N} \|_{V_{K_{-}}^{2}}^{2} \right)^{1/2} \lesssim \| v \|_{Y_{K_{-}}^{\infty}} \]. (3.47)

Collecting (3.43), (3.46) and (3.47), we obtain \( K_{3}^{1/2} \lesssim \| u \|_{Y_{K_{+}}^{\infty}} \| v \|_{Y_{K_{-}}^{\infty}} \). \( \Box \)

4. The proof of the main theorem. We define

\[ u_{\pm} := \omega_{1} u \pm i \partial_{t} u, \quad n_{\pm} := n \pm i (e \omega)^{-1} \partial_{t} n \]

where \( \omega_{1} := (1 - \Delta)^{1/2}, \omega := (-\Delta)^{1/2} \). Then the wave equation in (1.1) is rewritten into

\[
\begin{aligned}
\begin{cases}
 & i \partial_{t} u_{\pm} + \omega_{1} u_{\pm} = \pm (1/4)(n_{+} + n_{-})(\omega_{1}^{-1} u_{+} + \omega_{1}^{-1} u_{-}), & (t, x) \in [-T, T] \times \mathbb{R}^{d}, \\
 & i \partial_{t} n_{\pm} + e \omega n_{\pm} = \pm (4c)^{-1} \omega |\omega_{1}^{-1} u_{+} + \omega_{1}^{-1} u_{-}|^{2}, & (t, x) \in [-T, T] \times \mathbb{R}^{d}, \\
 & (u_{\pm}, n_{\pm})|_{t=0} = (u_{\pm 0}, n_{\pm 0}) \in H^{s}(\mathbb{R}^{d}) \times H^{s}(\mathbb{R}^{d}).
\end{cases}
\end{aligned}
\] (4.1)

Hence by the Duhamel principle, we consider the following integral equation corresponding to (4.1) on the time interval \([0, T]\) with \(0 < T \leq \infty\):

\[ u_{\pm} = \Phi_{1}(u_{\pm}, n_{+}, n_{-}), \quad n_{\pm} = \Phi_{2}(n_{\pm}, u_{+}, u_{-}), \] (4.2)
where
\[\Phi_1(u_\pm, n_\pm, n_-) := K_\pm(t)u_{\pm 0} \pm (1/4)\{\dot{I}_{T,K_\pm}(n_+, u_+)(t) + I_{T,K_\mp}(n_+, u_-)(t) + I_{T,K_\pm}(n_-, u_+)(t) + I_{T,K_\mp}(n_-, u_-)(t)\},\]
\[\Phi_2(n_\pm, u_+, u_-) := W_{\pm c}(t)n_{\pm 0} \pm (4c)^{-1}\{\dot{I}_{T,W_{\pm c}}(u_+, u_+)(t) + I_{T,W_{\pm c}}(u_+, u_-)(t) + I_{T,W_{\pm c}}(u_-, u_+)(t) + I_{T,W_{\pm c}}(u_-, u_-)(t)\}.

**Proposition 4.1.** (i) Let \(s = 1/4\) for \(d = 4\) or \(s = (d^2 - 3d - 2)/2(d + 1)\) for \(d \geq 5\). For any \(\delta > 0\) and for any initial data \((u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))\), there exists \(T > 0\) and a unique solution of (4.2) on \([0, T]\) such that
\[\Phi(t) \in Y_{K_\pm}^s([0, T]) \times \dot{Y}_{W_{\pm c}}^s([0, T]) \subset C([0, T]; H^s(\mathbb{R}^d)) \times C([0, T]; \dot{H}^s(\mathbb{R}^d)).\]
Moreover, let \(d \geq 4\), \(s = s_c = d/2 - 2\) and \(\delta > 0\) be sufficiently small. If \((u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))\) be radial, then for all \(0 < T < \infty\), there exists a unique spherically symmetric solution of (4.2) on \([0, T]\) such that
\[\Phi(t) \in C([0, T]; H^s(\mathbb{R}^d)) \times C([0, T]; \dot{H}^s(\mathbb{R}^d)).\]
(ii) The flow map obtained by (i):
\[B_\delta(H^s(\mathbb{R}^d)) \times B_\delta(\dot{H}^s(\mathbb{R}^d)) \ni (u_{\pm 0}, n_{\pm 0}) \mapsto (u_\pm, n_\pm) \in Y_{K_\pm}^s([0, T]) \times \dot{Y}_{W_{\pm c}}^s([0, T])\]
is Lipschitz continuous.

**Remark 4.1.** Due to the time reversibility of the Klein-Gordon-Zakharov equation, Proposition 4.1 also holds on corresponding time interval \([-T, 0]\).

**Remark 4.2.** By (i) in Proposition 4.1 and Remark 4.1, for any \(T > 0\), we have solutions to (4.2) \((u_(t), n_(t))\) on \([0, T]\) and \([-T, 0]\). If radial initial data \((u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))\), then we can take \(T\) arbitrary large and by uniqueness, spherically symmetric function \((u_\pm(t), n_\pm(t)) \in C((-\infty, \infty); H^s(\mathbb{R}^d)) \times C((-\infty, \infty); \dot{H}^s(\mathbb{R}^d))\) can be defined uniquely.

**Proposition 4.2.** Let the spherically symmetric solution \((u_\pm(t), n_\pm(t))\) to (4.2) on \((-\infty, \infty)\) obtained by Proposition 4.1, Remark 4.1 and Remark 4.2 with radial initial data \((u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))\). Then, there exist \((u_\pm, +\infty, n_\pm, +\infty)\) and \((u_\pm, -\infty, n_\pm, -\infty)\) in \(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)\) such that
\[\|u_\pm(t) - K_\pm(t)u_{\pm, +\infty}\|_{H^s_2(\mathbb{R}^d)} + \|n_\pm(t) - W_{\pm c}(t)n_{\pm, +\infty}\|_{\dot{H}^s_2(\mathbb{R}^d)} \to 0\]
as \(t \to +\infty\) and
\[\|u_\pm(t) - K_\pm(t)u_{\pm, -\infty}\|_{H^s_2(\mathbb{R}^d)} + \|n_\pm(t) - W_{\pm c}(t)n_{\pm, -\infty}\|_{\dot{H}^s_2(\mathbb{R}^d)} \to 0\]
as \(t \to -\infty\).

**Proof of Proposition 4.1.** First, we prove (i). By Proposition 2.10, there exists \(C > 0\) such that
\[\|K_\pm(t)u_{\pm 0}\|_{Y_{K_\pm}^s} \leq C\|u_{\pm 0}\|_{H^s}, \quad \|W_{\pm c}(t)n_{\pm 0}\|_{\dot{Y}_{W_{\pm c}}^s} \leq C\|n_{\pm 0}\|_{\dot{H}^s}.
\]
We denote time interval \(I := [0, T]\). If \((u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)), (u_\pm, n_\pm) \in B_r(Y_{K_\pm}^s(I) \times \dot{Y}_{W_{\pm c}}^s(I)),\) then by Proposition 3.1, for \((\theta, s) = (1/4, 1/4), d = 4\) or for
\((\theta, s) = (1/(d+1), (d^2 - 3d - 2)/(d+1)), d \geq 5\), it holds that

\[
\|\Phi_1(u_\pm, n_+, n_-)\|_{Y_{K_\pm}^s(t)} \\
\leq C\|u_{\pm 0}\|_{H^s} + (1/4)CT^\theta (\|n_+\|_{\dot{Y}_{W_{\pm}^s}(t)} + \|n_-\|_{\dot{Y}_{W_{\pm}^s}(t)}) \|u_+\|_{Y_{K_+}^s(t)} + \|u_-\|_{Y_{K_-}^s(t)} \\
+ n_+\|\dot{Y}_{W_{\pm}^s}^s(t)\|_{u_+}+ n_-\|\dot{Y}_{W_{\pm}^s}^s(t)\|_{u_-} \leq \Phi(t) \\
\leq C\delta + C\|T^\theta r^2, \\
\|\Phi_2(n_\pm, u_+, u_-)\|_{\dot{Y}_{W_{\pm}^s}(t)} \\
\leq C\|n_{\pm 0}\|_{H^s} + (CT^\theta/4c)(\|u_{\pm 0}\|_{Y_{K_\pm}^s(t)} + 2\|u_+\|_{Y_{K_+}^s(t)} + \|u_-\|_{Y_{K_-}^s(t)}) \\
\leq C\delta + C\|T^\theta r^2/c.
\]

We take \(r = 2C\delta\) and \(T > 0\) satisfying

\[
4CT^\theta r \leq \min\{1, c\}.
\] (4.3)

Then we have

\[
\|\Phi_1(u_\pm, n_+, n_-)\|_{Y_{K_\pm}^s(t)} \leq r, \quad \|\Phi_2(n_\pm, u_+, u_-)\|_{Y_{N_{\pm}^s}(t)} \leq r.
\]

Hence, \((\Phi_1, \Phi_2)\) is a map from \(B((Y_{K_\pm}^s([0, T]) \times \dot{Y}_{W_{\pm}^s}^s([0, T]))\) into itself. Similarly, we assume \((v_{\pm 0}, m_{\pm}) \in B((H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)), (v_{\pm}, m_{\pm}) \in B((Y_{K_\pm}^s(t) \times \dot{Y}_{W_{\pm}^s}^s(t)),\)

then it holds that

\[
\|\Phi_1(u_\pm, n_+, n_-) - \Phi_1(v_{\pm}, m_+, m_-)\|_{Y_{K_\pm}^s(t)} \\
\leq (1/4)(\|I_{T, K_\pm}(n_+, u_+)(t) - I_{T, K_\pm}(m_+, v_+)(t)\|_{Y_{K_\pm}^s(t)} \\
+ \|I_{T, K_\pm}(n_+, u_-)(t) - I_{T, K_\pm}(m_+, v_-)(t)\|_{Y_{K_\pm}^s(t)} \\
+ \|I_{T, K_\pm}(n_-, u_+)(t) - I_{T, K_\pm}(m_-, v_+)(t)\|_{Y_{K_\pm}^s(t)} \\
+ \|I_{T, K_\pm}(n_-, u_-)(t) - I_{T, K_\pm}(m_-, v_-)(t)\|_{Y_{K_\pm}^s(t)}).
\] (4.4)

By Proposition 3.1, we have

\[
\|I_{T, K_\pm}(n_+, u_+)(t) - I_{T, K_\pm}(m_+, v_+)(t)\|_{Y_{K_\pm}^s(t)} \\
\leq CT^\theta (\|n_+ - m_+\|_{\dot{Y}_{W_{\pm}^s}^s(t)} + \|u_+\|_{Y_{K_\pm}^s(t)} + \|m_+\|_{\dot{Y}_{W_{\pm}^s}^s(t)} + \|u_- - v_+\|_{Y_{K_\pm}^s(t)}).
\] (4.5)

Similarly, we have

\[
\|I_{T, K_\pm}(n_+, u_-)(t) - I_{T, K_\pm}(m_+, v_-)(t)\|_{Y_{K_\pm}^s(t)} \\
\leq CT^\theta (\|n_+ - m_+\|_{\dot{Y}_{W_{\pm}^s}^s(t)} + \|u_-\|_{Y_{K_\pm}^s(t)} + \|m_+\|_{\dot{Y}_{W_{\pm}^s}^s(t)} + \|u_- - v_-\|_{Y_{K_\pm}^s(t)}, (4.6)
\]

\[
\|I_{T, K_\pm}(n_-, u_+)(t) - I_{T, K_\pm}(m_-, v_+)(t)\|_{Y_{K_\pm}^s(t)} \\
\leq CT^\theta (\|n_- - m_-\|_{\dot{Y}_{W_{\pm}^s}^s(t)} + \|u_+\|_{Y_{K_\pm}^s(t)} + \|m_-\|_{\dot{Y}_{W_{\pm}^s}^s(t)} + \|u_+ - v_+\|_{Y_{K_\pm}^s(t)}, (4.7)
\]

\[
\|I_{T, K_\pm}(n_-, u_-)(t) - I_{T, K_\pm}(m_-, v_-)(t)\|_{Y_{K_\pm}^s(t)} \\
\leq CT^\theta (\|n_- - m_-\|_{\dot{Y}_{W_{\pm}^s}^s(t)} + \|u_-\|_{Y_{K_\pm}^s(t)} + \|m_-\|_{\dot{Y}_{W_{\pm}^s}^s(t)} + \|u_- - v_-\|_{Y_{K_\pm}^s(t)}).
\] (4.8)
Hence from \( \|u_\pm\|_{Y_{K_\pm}^q} \leq r \), \( \|m_\pm\|_{\bar{Y}_{W_{\pm}}^q} \leq r \), (4.4)–(4.8) and (4.3), we have
\[
\|\Phi_1(u_\pm,n_+,n_-) - \Phi_1(v_\pm,m_+,m_-)\|_{Y_{K_\pm}^q} \leq (1/8)(\|u_+ - v_+\|_{Y_{K_+}^q} + \|u_- - v_-\|_{Y_{K_-}^q}) + \|m_+ - m_-\|_{\bar{Y}_{W_{\pm}}^q}.
\]
(4.9)

Similarly, we have
\[
\|\Phi_2(n_\pm,u_+,u_-) - \Phi_2(m_\pm,v_+,v_-)\|_{\bar{Y}_{W_{\pm}}^q} = (4c)^{-1}(\|I_{T,W_{\pm}}(u_+,u_+)(t) - I_{T,W_{\pm}}(v_+,v_+)(t)\|_{\bar{Y}_{W_{\pm}}^q} + \|I_{T,W_{\pm}}(u_+,u_-)(t) - I_{T,W_{\pm}}(v_+,v_-)(t)\|_{\bar{Y}_{W_{\pm}}^q} + \|I_{T,W_{\pm}}(u_-,u_+)(t) - I_{T,W_{\pm}}(v_-,v_+)(t)\|_{\bar{Y}_{W_{\pm}}^q} + \|I_{T,W_{\pm}}(u_-,u_-)(t) - I_{T,W_{\pm}}(v_-,v_-)(t)\|_{\bar{Y}_{W_{\pm}}^q}).
\]
(4.10)

By Proposition 3.1, we have
\[
\|I_{T,W_{\pm}}(u_+,u_+)(t) - I_{T,W_{\pm}}(v_+,v_+)(t)\|_{\bar{Y}_{W_{\pm}}^q} \leq CT^\theta(\|u_+\|_{Y_{K_+}^q} + \|v_+\|_{Y_{K_+}^q})\|u_+ - v_+\|_{Y_{K_+}^q}. \quad (4.11)
\]

Similarly, we have
\[
\|I_{T,W_{\pm}}(u_+,u_-)(t) - I_{T,W_{\pm}}(v_+,v_-)(t)\|_{\bar{Y}_{W_{\pm}}^q} \leq CT^\theta(\|u_+\|_{Y_{K_+}^q})\|u_+ - v_+\|_{Y_{K_+}^q} + \|v_+\|_{Y_{K_+}^q} + \|v_-\|_{Y_{K_-}^q} + \|u_-\|_{Y_{K_-}^q}) \quad (4.12)
\]
\[
\|I_{T,W_{\pm}}(u_-,u_+)(t) - I_{T,W_{\pm}}(v_-,v_+)(t)\|_{\bar{Y}_{W_{\pm}}^q} \leq CT^\theta(\|u_-\|_{Y_{K_-}^q})\|u_- - v_-\|_{Y_{K_-}^q} + \|v_-\|_{Y_{K_-}^q} + \|v_+\|_{Y_{K_+}^q} + \|u_+\|_{Y_{K_+}^q}) \quad (4.13)
\]
\[
\|I_{T,W_{\pm}}(u_-,u_-)(t) - I_{T,W_{\pm}}(v_-,v_-)(t)\|_{\bar{Y}_{W_{\pm}}^q} \leq CT^\theta(\|u_-\|_{Y_{K_-}^q}) + \|v_-\|_{Y_{K_-}^q} + \|u_- - v_-\|_{Y_{K_-}^q} \quad (4.14)
\]

From \( \|u_\pm\|_{Y_{K_\pm}^q} \leq r \), \( \|v_\pm\|_{Y_{K_\pm}^q} \leq r \), (4.10)–(4.14) and (4.3), we obtain
\[
\|\Phi_2(n_\pm,u_+,u_-) - \Phi_2(m_\pm,v_+,v_-)\|_{\bar{Y}_{W_{\pm}}^q} \leq (1/4)(\|u_+ - v_+\|_{Y_{K_+}^q} + \|u_- - v_-\|_{Y_{K_-}^q}).
\]
(4.15)

Therefore, \((\Phi_1, \Phi_2)\) is a contraction mapping on \( B_r(Y_{K_\pm}^q([0,T]) \times \bar{Y}_{W_{\pm}}^q([0,T])) \). Hence, by the Banach fixed point theorem, we have a solution to (4.2) in it.

Next, we prove uniqueness. Let \((u_\pm(n), n_)\), \((v_\pm(m), m_)\) \( \in Y_{K_\pm}^q([0,T]) \times \bar{Y}_{W_{\pm}}^q([0,T]) \) are two solutions satisfying \((u_\pm(0), n_0) = (v_\pm(0), m_0)\). Moreover,
\[
T' := \sup\{0 \leq t \leq T : u_\pm(t) = v_\pm(t), n_\pm(t) = m_\pm(t)\} < T.
\]
By a translation in $t$, it suffices to consider $T' = 0$. Fix $0 < \tau \leq T$ sufficiently small. From (4.4)–(4.8) and Proposition 2.15, we obtain

$$
\|u_+ - u_+\|_{Y_{K^+}^s([0,\tau])} \leq (1/4)C^{\alpha}(\|u_+\|_{Y_{K^+}^s([0,\tau])} + \|u_-\|_{Y_{K^-}^s([0,\tau])})
\times \left((n_+ + m_+\|Y_{W^+}^s([0,\tau])\| + \|n_- - m_\|Y_{W^-}^s([0,\tau])\|)
+ \left(\|m_+\|Y_{K^+}^s([0,\tau]) + \|m_-\|Y_{K^-}^s([0,\tau])\right)\left(\|u_+ - v_+\|_{Y_{K^+}^s([0,\tau])} + \|u_- - v_-\|_{Y_{K^-}^s([0,\tau])}\right)\right)
\leq (1/8)(n_+ + m_+\|Y_{W^+}^s([0,\tau])\| + \|n_- - m_\|Y_{W^-}^s([0,\tau])\|)
+ \|u_+ - v_+\|_{Y_{K^+}^s([0,\tau])} + \|u_- - v_-\|_{Y_{K^-}^s([0,\tau])}). 
$$

From (4.16), we obtain

$$
\|u_+ - v_+\|_{Y_{K^+}^s([0,\tau])} \leq (1/7)(n_+ + m_+\|Y_{W^+}^s([0,\tau])\| + \|n_- - m_\|Y_{W^-}^s([0,\tau])\| + \|u_+ - v_+\|_{Y_{K^+}^s([0,\tau])}).
$$

Similarly, we have

$$
\|u_- - v_-\|_{Y_{K^-}^s([0,\tau])} \leq (1/7)(n_+ + m_+\|Y_{W^+}^s([0,\tau])\| + \|n_- - m_\|Y_{W^-}^s([0,\tau])\| + \|u_+ - v_+\|_{Y_{K^+}^s([0,\tau])}).
$$

From (4.10)–(4.14) and Proposition 2.15, we have

$$
\|n_\pm - m_\|_{Y_{W^\pm}^s([0,\tau])} \leq (1/4)(\|u_+ - v_+\|_{Y_{K^+}^s([0,\tau])} + \|u_- - v_-\|_{Y_{K^-}^s([0,\tau])}).
$$

Hence, collecting (4.17)–(4.19), we obtain

$$
u_\pm = v_\pm, \quad n_\pm = m_\pm
$$
on $[0, \tau]$. This contradicts the definition of $T'$.

If $(u_0, n_0) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$ is radial, $s = s_c = d/2 - 2$ with $d \geq 4$ and $(u_\pm, n_\pm) \in B_r(Y_{K^\pm}^s(I) \times \dot{Y}_{W^\pm}^s(I))$ is spherically symmetric, then by Proposition 3.1, we have

$$
\|\Phi_1(u_\pm, n_\pm, u_-)\|_{Y_{K^\pm}^s(I)} \leq C \delta + (1/4)C(\|n_\pm\|_{Y_{W^\pm}^s(I)} + \|n_\|_{Y_{K^\pm}^s(I)}\|u_\|_{Y_{W^\pm}^s(I)}\|n_-\|_{Y_{K^\pm}^s(I)}\|u_-\|_{Y_{W^\pm}^s(I)}).
$$

Taking $\delta = r^2$ and $r = \min\{1, c\}/(4C)$, then we have

$$
\|\Phi_1(u_\pm, n_\pm, u_-)\|_{Y_{K^\pm}^s(I)} \leq r, \quad \|\Phi_2(n_\pm, u_\pm, u_-)\|_{Y_{W^\pm}^s(I)} \leq r.
$$

Hence, $(\Phi_1, \Phi_2)$ is a map from $B_r(Y_{K^\pm}^s([0, T]) \times \dot{Y}_{W^\pm}^s([0, T]))$ into itself. If we also assume $(v_\pm, m_\pm) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$ is radial and $(v_\pm, m_\pm) \in B_r(Y_{K^\pm}^s(I) \times \dot{Y}_{W^\pm}^s(I))$.
Proof. There exists Proposition 4.2 in [13].

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Thus, \( (\Phi, \Phi_2) \) is a contraction mapping on \( B_r(Y_{K^+}^s([0, T]) \times \dot{Y}_{W_{2r}}^s([0, T])) \). Hence, by the Banach fixed point theorem, we have a solution to (4.2) in it. We assume that \( (u_{\pm}(0), n_{\pm}(0)) \), \( (v_{\pm}(0), m_{\pm}(0)) \) are both radial and \( s = s_c = d/2 - 2 \) with \( d \geq 4 \).

Let \( (u_{\pm}(0), n_{\pm}(0)) \), \( (v_{\pm}, m_{\pm}(0)) \) are two spherical symmetric solutions satisfying \( (u_{\pm}(0), n_{\pm}(0)) = (v_{\pm}(0), m_{\pm}(0)) \). Then by the same manner as the proof for non-radial initial data, the uniqueness of the solution \( (u_{\pm}, n_{\pm}) \) is showed. (ii) follows from the standard argument, so we omit the proof.

Finally, we prove Proposition 4.2. The proof is the same manner as the proof for Proposition 4.2 in [13].

Proof. There exists \( M > 0 \) such that for all \( 0 < T < \infty \),

\[
\|u_{\pm}\|_{Y_{K^+}^s([0, T])} + \|n_{\pm}\|_{\dot{Y}_{W_{2r}}^s([0, T])} < M,
\]

\[
\|u_{\pm}\|_{Y_{K^+}^s([-T, 0])} + \|n_{\pm}\|_{\dot{Y}_{W_{2r}}^s([-T, 0])} < M
\]

holds since \( r \) in the proof of Proposition 4.1 does not depend on \( T \). Take \( \{t_k\}_{k=0}^K \in \mathbb{Z}_0 \) and \( 0 < T < \infty \) such that \(-T < t_0, t_K < T\). By \( L^2_x \) orthogonality,

\[
\left( \sum_{k=1}^K \|\langle \nabla_x \rangle^s (K_{\pm}(-t_k)u_{\pm}(t_k) - K_{\pm}(-t_{k-1})u_{\pm}(t_{k-1}))\|_{L^2_x}^2 \right)^{1/2} \lesssim \|\langle \nabla_x \rangle^s u_{\pm}\|_{V_{K^+}^s([0, T])} + \|\langle \nabla_x \rangle^s u_{\pm}\|_{V_{K^+}^s([-T, 0])}
\]

\[
\lesssim \|u_{\pm}\|_{Y_{K^+}^s([0, T])} + \|u_{\pm}\|_{Y_{K^+}^s([-T, 0])} < 2M.
\]

Thus,

\[
\sup_{\{t_k\}_{k=0}^K} \left( \sum_{k=1}^K \|\langle \nabla_x \rangle^s K_{\pm}(-t_k)u_{\pm}(t_k) - \langle \nabla_x \rangle^s K_{\pm}(-t_{k-1})u_{\pm}(t_{k-1})\|_{L^2_x}^2 \right)^{1/2} < 2M.
\]

Hence, there exists \( f_{\pm} := \lim_{t \to \pm \infty} \langle \nabla_x \rangle^s K_{\pm}(-t)u_{\pm}(t) \in L^2_x(\mathbb{R}^d) \). Then put \( u_{\pm \infty} := \langle \nabla_x \rangle^{-s}f_{\pm} \), we obtain

\[
\|\langle \nabla_x \rangle^s K_{\pm}(-t)u_{\pm}(t) - f_{\pm}\|_{L^2_x} = \|u_{\pm}(t) - K_{\pm}(t)u_{\pm \infty}\|_{H^s_x} \to 0
\]

as \( t \to \pm \infty \). The scattering result for the wave equation is obtained similarly.

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