Results on exact controllability of second-order semilinear control system in Hilbert spaces

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Abstract

In our manuscript, we extend the controllability outcomes given by Bashirov (Math. Methods Appl. Sci. 44(9):7455–7462, 2021) for a family of second-order semilinear control system by formulating a sequence of piecewise controls. This approach does not involve large estimations which are required to apply fixed point theorems. Therefore, we avoid the use of fixed point theory and the contraction mapping principle. We establish that a second-order semilinear system drives any starting position to the required final position from the domain of the system. To achieve the required results, we suppose that the linear system is exactly controllable at every non-initial time period, the norm of the inverse of the controllability Grammian operator increases as the time approaches zero with the slower rate in comparison to the reciprocal of the square function, and the nonlinear term is bounded. Finally, an example has been presented to validate the results.

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1 Introduction

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time is known or postulated. This is illustrated in classical mechanics where the motion of a body is described by its position and velocity as time varies. For the studies related to the existence of solution for integer- and fractional-order systems, one can refer to [2–30]. The concept of controllability is one of the underlying ideas in mathematical control theory. Controllability analysis is used in several real-life problems which include, but are not limited to, rocket launching problems for satellite and aircraft control, missiles and anti-missiles problems in defense, regulating inflation rate in the economy, controlling sugar level in the blood, etc. A systematic study of controllability was initiated by Kalman [31] in 1963 when the theory of controllability for time-invariant and time-varying control systems in state-space form was developed.

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Several engineering and scientific problems can be expressed by infinite-dimensional differential equations. Therefore, it becomes necessary to discuss the controllability results for infinite-dimensional systems. The controllability problems for finite-dimensional nonlinear systems have been immensely analyzed by several authors. Many authors have advanced the idea of controllability from finite-dimensional systems to infinite-dimensional systems and determined appropriate requirements for the controllability of nonlinear systems. Different techniques have been practiced for the examination of controllability including fixed point theorems [32–34]. For more problems on controllability and recent progresses on fractional calculus and its applications, please refer to [35–47].

Second-order differential equations represent abstract mathematical interpretations of several partial differential equations which occur in many applications related to the oscillation of fastened bars, the transverse motion of an extensible beam, and several other real-world physical phenomena. Hence it becomes really important to determine the controllability outcomes for this kind of system. Controllability discussions for various second-order nonlinear systems have been widely studied by several authors [33, 48–58]. Fixed point theorems have been utilized greatly in determining the existence and controllability results for different first-order and second-order systems which involve large estimations on system constants see [33, 59, 60]. Recently Bashirov [1] obtained the exact controllability results for first-order semilinear systems using a new technique that is based on the piecewise formulation of driving controls and without using fixed point theory. Earlier the same approach was applied in terms of approximate controllability [61]. Motivated by Bashirov [1], we extend the controllability results for second-order semilinear systems excluding the use of fixed point theory. It is centered on a piecewise formulation of steering controls and does not involve large estimations which are required to apply fixed point theorems. By constructing a piecewise sequence of controls, we determine that a second-order semilinear system is exactly controllable to the domain of the system operator, that is, it drives the system from any starting position to the required final position from the domain of the system. To achieve the required results, we suppose the subsequent requirements:

(a) The corresponding linear system is exactly controllable at every non-initial time period.

(b) The norm of the inverse of the controllability Grammian operator increases as the time approaches zero with the slower rate in comparison to the reciprocal of the square function.

(c) The nonlinear term is bounded.

Let us consider $Z = L_2[0, b; X]$ and $Y = L_2[0, b; U]$ as the function spaces defined on $J = [0, b], 0 \leq b < \infty$, where $X$ and $U$ are two Hilbert spaces. Consider the following second-order semilinear control system:

$$\begin{cases}
p''(t) = Ap(t) + Bv(t) + r(t, p(t)), & \text{for } t \in (0, b], \\
p(0) = p_0, \\
p'(0) = q_0, 
\end{cases} \quad (1.1)$$

where

1. $p(t)$ represents the state having values in Hilbert space $X$.
2. Control function $v$ is defined from $[0, b] \to U$. 

...
3 $B$ is a bounded and linear operator from $U$ into $X$.
4 The function $r : [0, b] \times X \rightarrow X$ is a purely nonlinear function which produces nonlinearity in the system.
5 $A : \text{dom}(A) \subseteq X \rightarrow X$ is linear, closed where $\text{dom}(A)$ is a dense subset of $X$.

The linear system corresponding to (1.1) with state vector $q(t)$ and control $v$ is defined by

\[
\begin{cases}
q''(t) = Aq(t) + Bv(t), & \text{for } t \in (0, b], \\
q(0) = p_0, \\
q'(0) = q_0.
\end{cases}
\] (1.2)

The article is structured in the subsequent manner:
1 Sect. 2 presents a few basic results related to control theory and second-order systems.
2 Sect. 3 provides the assumptions which are required to obtain the controllability results.
3 Sect. 4 discusses the controllability results using the new technique.
4 Sect. 5 presents an example to verify the established outcomes.

2 Auxiliary results
Here, we will review fundamental theories and a few definitions which would be helpful for further discussions.

Definition 2.1 [62] A one-parameter family $\{C(t), t \in \mathbb{R}\}$ of bounded linear operators mapping the Hilbert space $X$ into itself is called a strongly continuous cosine family if and only if
1 $C(0) = I$;
2 $C(s + t) + C(s - t) = 2C(s)C(t)$;
3 $C(t)x$ is continuous in $t$ on $\mathbb{R}$ for each fixed $x \in X$.

If $\{C(t), t \in \mathbb{R}\}$ is a strongly continuous cosine family in $X$, then $\{S(t), t \in \mathbb{R}\}$ is a one-parameter family of operators in $X$ defined by

\[S(t) = \int_0^t C(s) \, ds, \quad t \in \mathbb{R}.
\]

The infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathbb{R}\}$ is the operator $A : X \rightarrow X$ defined by

\[Ax = \frac{d^2}{dt^2} C(0)x.
\]

The domain of operator $A$ is defined as

\[\text{dom}(A) = \{x \in X : C(t)x \text{ is a twice continuously differentiable function of } t\}.
\]

These cosine and sine families defined above and generator $A$ fulfill the following properties.
Lemma 2.2 ([28]) Suppose that \( A \) is the infinitesimal generator of a cosine family of operators \( \{C(t) : t \in \mathbb{R}\} \). Then the following hold:

1. There exist \( M' \geq 1 \) and \( \omega \geq 0 \) such that \( \|C(t)\| \leq M' e^{\omega t} \), and hence \( \|S(t)\| \leq M' e^{\omega t} \).
2. \( \hat{A} f = \int_{0}^{1} S(u)x \, du = [C(r) - C(s)]x \) for all \( 0 \leq s \leq r < \infty \).
3. There exists \( N' \geq 1 \) such that \( \|S(s) - S(r)\| \leq N' \left| \int_{r}^{s} e^{\omega s} \, ds \right| \) for all \( 0 \leq s \leq r < \infty \).

The uniform boundedness principle together with (1) implies that both \( \{C(t) : t \in J\} \) and \( \{S(t) : t \in J\} \) are uniformly bounded and \( M = M' e^{\omega |b|} \).

Proposition 2.3 ([62]) Let \( \{C(t), t \in \mathbb{R}\} \) be a strongly continuous cosine family in \( \mathbb{X} \) with infinitesimal generator \( A \). The following are true:

1. \( S(0) = 0 \).
2. \( C(t) = C(-t) \) and \( S(t) = -S(-t) \) for all \( t \in \mathbb{R} \).
3. If \( x \in E \), then \( S(t)x, C(t)x \in \text{dom}(A) \) and \( \frac{d}{dt} S(t)x = C(t)x \), and \( \frac{d}{dt} C(t)x = AS(t)x \), where \( E = \{x : C(t)x \text{ is once continuously differentiable function of } t\} \).
4. If \( x \in \text{dom}(A) \), then \( S(t)x \in \text{dom}(A) \) and \( AS(t)x = S(t)Ax \).
5. If \( x \in \text{dom}(A) \), then \( C(t)x \in \text{dom}(A) \) and \( AC(t)x = AC(t)x = C(t)Ax \).
6. If \( x \in E \), then \( \lim_{t \to 0} AS(t)x = 0 \).

Proposition 2.4 ([62]) Let \( \{C(t), t \in \mathbb{R}\} \) be a strongly continuous cosine family in \( \mathbb{X} \). The operator \( \hat{A} : \mathbb{X} \to \mathbb{X} \) defined by

\[
\hat{A}x = \lim_{t \to 0} \frac{C(2t)x - x}{2t^2},
\]

with domain \( x \in \mathbb{X} \), for which this limit exists, is the infinitesimal generator of the cosine family \( \{C(t), t \in \mathbb{R}\} \).

Suppose \( U_{ad} = L_{2}[0, b; \mathbb{U}] \), which is the set of admissible controls. We define the mild solution of the given semi linear system (1.1) and its corresponding linear system as follows.

Definition 2.5 The mild solution of system (1.1) is defined by a function \( p(\cdot) \in \mathbb{X} \) which satisfies the following integral equation:

\[
\begin{cases}
p(t) = C(t)p_0 + S(t)q_0 + \int_{0}^{t} S(t-s)[Bv(s) + r(s, p(s))] \, ds, & t \in [0, b], \\
p(0) = p_0, \\
p'(0) = q_0,
\end{cases}
\]

and the mild solution of the corresponding linear system (1.2) is described by the following integral equation:

\[
\begin{cases}
q(t) = C(t)p_0 + S(t)q_0 + \int_{0}^{t} S(t-s)Bv(s) \, ds, & t \in [0, b], \\
q(0) = p_0, \\
q'(0) = q_0.
\end{cases}
\]

Definition 2.6 ([1]) System (1.1) is said to be approximately controllable in the time interval \([0, b]\) if, for the given starting position \( (p_0, q_0) \in \mathbb{X} \) and the required final position
\((p_F, q_F) \in \mathbb{X}\) and \(\epsilon > 0\), there exists a control function \(v \in U_{ad}\) such that the solution of (1.1) satisfies
\[
\|p(b) - p_F\| < \epsilon, \quad \|p'(b) - q_F\| < \epsilon,
\]
where \(p(b)\) is the state value of system (1.1) at time \(t = b\). If \(p(b) = p_F\), then the system is said to be exactly controllable. The system is said to be exactly controllable to \(\text{dom}(A)\) on \([0, b]\) if, for the given starting position \((p_0, q_0) \in \mathbb{X}\) and the required final position \((p_F, q_F) \in \text{dom}(A)\), there exists a control function \(v \in U_{ad}\) such that the solution of the system satisfies \(p(b) = p_F\), \(p'(b) = q_F\).

**Remark 2.7** Note that the exact controllability to \(\text{dom}(A)\) lies in between the exact and approximate controllability. Therefore, it is a weaker concept than the exact controllability. In real life applications, sometimes we are more concerned with attaining the points from \(\mathbb{X} \setminus \text{dom}(A)\) as well, then it can be considered as an additional capability of the system.

### 3 Assumptions

Let us introduce the controllability Grammian operator \(W\) associated with linear system (1.2) by
\[
W(t) = \int_0^t S(s)BB^*S^*(s)\, ds, \quad 0 \leq t \leq b,
\]
where \(S^*(s)\) denotes the adjoint of \(S(s)\).

**Theorem 3.1** The corresponding linear system (1.2) is exactly controllable on the interval \([h, b]\) iff \(W(b - h)\) is coercive. The control which drives the system from the starting position \((q(h), q'(h)) \in \mathbb{X}\) to the final position \((p_F, q_F) \in \mathbb{X}\) is given by
\[
v(t) = B^*S^*(b - t)(W(b - h))^{-1}(p_F - C(b - h)q(h) - S(b - h)q'(h)) \quad (h \leq t \leq b). \tag{3.1}
\]

**Proof** The result can be seen in [59]. Moreover, it can be easily verified by substituting the above defined control in the mild solution of the corresponding linear system that it transfers \((q(h), q'(h))\) to \((p_F, q_F)\) on the interval \([h, b]\).

**Remark 3.2** The coercivity of \(W(t)\) indicates that \((W(t))^{-1}\) is a bounded linear operator. We say that \(W(t)\) is coercive if there exists \(\gamma > 0\) such that \((W(t)x, x) \geq \gamma \|x\|^2\) for all \(x \in \mathbb{X}\). Here \(W(0) = 0\) and, therefore, it fails to be coercive. But it may be coercive for \(0 < t \leq b\). Therefore, the above result holds on \([h, b]\).

To determine the main result, we make the following assumptions on the controllability Grammian operator \(W\) and the nonlinear function \(r(t, p)\):

1. \(W(t)\) is coercive for all \(0 < t \leq b\).
2. There exists some \(N > 0\) such that
\[
t^{1+\alpha}\|(W(t))^{-1}\| \leq N \quad \text{for all} \ 0 < t \leq b, \ 0 < \alpha < 1.
\]
That is, \( \| (W(t))^{-1} \| \to \infty \) as \( t \to 0^+ \) with the slower rate in comparison to the reciprocal of the square function as

\[
\| (W(t))^{-1} \| \leq \frac{N}{t^{1+\alpha}} < \frac{N}{t^2},
\]

for small values of \( t \).

(III) The nonlinear function \( r \) is Lebesgue measurable in \( t \).

(IV) \( r \) is Lipschitz continuous in \( p \).

(V) \( r \) is bounded in \([0, b] \times X\), i.e., there exists \( K > 0 \) such that

\[
\| r(t,p) \| \leq K \quad \text{for all} \ (t,p) \in [0, b] \times X.
\]

4 Results on controllability

In this section, we primarily focus on the study of exact controllability of the assumed system.

**Theorem 4.1** System (1.1) is exactly controllable to \( \text{dom}(A) \) on the interval \([0, b]\) for every \( b > 0 \) provided assumptions (I)–(V) hold.

**Proof:** We construct a piecewise sequence of driving controls to formulate the required control function \( v \) which drives the given system from the starting position \((p_0, q_0) \in X\) to the final position \((p_F, q_F) \in \text{dom}(A)\) in the following manner.

For this, consider the sequence \( \{h_n\} \) which is defined by \( h_n = \frac{b}{2^n} \) for \( n = 1, 2, \ldots \). We have \( \sum_{n=1}^{\infty} h_n = b \). For the sake of simplicity, let us take \( h_0 = 0 \) and

\[
b_0 = h_0, \quad b_1 = h_0 + h_1, \quad \ldots, \quad b_n = \sum_{k=0}^{n} h_k, \quad \ldots
\]

Then \( \lim_{n \to \infty} b_n = \sum_{k=0}^{\infty} h_k = b \).

Using Theorem 3.1, the corresponding linear system (1.2) is exactly controllable on \([b_0, b_1]\) along with the control

\[
v_1(\varrho) = B^* S^* (b_1 - \varrho) (W(h_1))^{-1} (C(h_1)(p_F - p_0) + S(h_1)(q_F - q_0)), \quad b_0 \leq \varrho \leq b_1,
\]

which steers the initial state \( p_0 \) to \( C(h_1)p_F + S(h_1)q_F \).

That is,

\[
C(h_1)p_F + S(h_1)q_F = C(h_1)p_0 + S(h_1)q_0 + \int_{b_0}^{b_1} S(b_1 - s)Bv_1(s) \, ds.
\]

Define \( v \) on \([b_0, b_1]\) by letting \( v(\varrho) = v_1(\varrho) \). Then, from (2.1), we obtain

\[
p(b_1) = C(h_1)p_F + S(h_1)q_F + \int_{b_0}^{b_1} S(b_1 - s)r(s, p(s)) \, ds.
\]

For brevity, let \( p(b_1) = p_1 \). Next, consider (1.2) on \([b_1, b_2]\). By Theorem 3.1, the control

\[
v_2(\varrho) = B^* S^* (b_2 - \varrho) (W(h_2))^{-1} (C(h_2)(p_F - p_1) + S(h_1)(q_F - p'(b_1))), \quad b_1 \leq \varrho \leq b_2,
\]
steers $p_1$ to $C(h_2)p_F + S(h_2)q_F$. Writing $p'(b_1) = q_1$, then the control $u_2(\varrho)$ can be written as

$$v_2(\varrho) = B^*S^*(b_2 - \varrho)(W(h_2))^{-1}(C(h_2)(p_F - p_1) + S(h_2)(q_F - q_1)), \quad b_1 \leq \varrho \leq b_2.$$ 

That is,

$$C(h_2)p_F + S(h_2)q_F = C(h_2)p_1 + S(h_2)q_1 + \int_{b_1}^{b_2} S(b_2 - s)Bv_2(s) \, ds.$$ 

Define $v$ on $(b_1, b_2]$ by letting $v(\varrho) = v_2(\varrho)$. Then, from (2.1), we obtain

$$p(b_2) = C(h_2)p_F + S(h_2)q_F + \int_{b_1}^{b_2} S(b_2 - s)r(s, p(s)) \, ds.$$ 

For the sake of convenience, let $p(b_2) = p_2$. Progressing in this fashion, we acquire a sequence of driving controls

$$v_n(\varrho) = B^*S^*(b_n - \varrho)(W(h_n))^{-1}(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})), \quad b_{n-1} \leq \varrho \leq b_n,$$ 

(4.1)

where $q_{n-1} = p'(b_{n-1})$.

After combining the above sequence of controls, we get the control function as follows:

$$v(\varrho) = \begin{cases} v_1(\varrho) & \text{if } b_0 \leq \varrho \leq b_1, \\ v_2(\varrho) & \text{if } b_1 < \varrho \leq b_2, \\ \vdots & \vdots \\ v_n(\varrho) & \text{if } b_{n-1} < \varrho \leq b_n, \\ \vdots & \vdots \end{cases}$$

and

$$p_n = p(b_n) = C(h_n)p_F + S(h_n)q_F + \int_{b_{n-1}}^{b_n} S(b_n - s)r(s, p(s)) \, ds.$$ 

(4.2)

Now by using the assumption (V), we get

$$\|p_n - p_F\| \leq \|C(h_n)p_F - p_F\| + \|S(h_n)q_F\| + \int_{b_{n-1}}^{b_n} \|S(b_n - s)\| \|r(s, p(s))\| \, ds$$

$$\leq \|C(h_n)p_F - p_F\| + \|S(h_n)q_F\| + KMh_n, \quad n = 1, 2, \ldots,$$ 

(4.3)

where $M = \sup_{[0,b]} \|C(\varrho)\|$ and $K = \sup_{[0,b] \times X} \|r(\varrho, p)\|$.

Since $C(\varrho)$ is strongly continuous, $S(0) = 0$ and $\lim_{n \to \infty} h_n = 0$.

Therefore, $\lim_{n \to \infty} p_n = p_F$. Also,

$$q_n = p'(b_n) = AS(h_n)p_F + C(h_n)q_F + \int_{b_{n-1}}^{b_n} C(b_n - s)r(s, p(s)) \, ds.$$
Thus, we have
\[
\|q_n - q_F\| \leq \|AS(h_n)p_F\| + \|C(h_n)q_F - q_F\| + \int_{b_{n-1}}^{b_n} \|C(b_n - s)\| \|r(s,p(s))\| \, ds \\
\leq \|AS(h_n)p_F\| + \|C(h_n)q_F - q_F\| + KMh_n, \quad n = 1, 2, \ldots \tag{4.4}
\]

Since \(C(q)\) is strongly continuous, \(S(0) = 0\) and \(\lim_{n \to \infty} h_n = 0\).

Therefore, \(\lim_{n \to \infty} q_n = q_F\).

Next we prove that \(v \in U_{ad}\). Since every \(v_n\) of \(v\) is continuous on the interval \((b_{n-1}, b_n]\) for \(n = 0, 1, 2, \ldots\), hence \(v\) is measurable. Also,
\[
\int_{b_{n-1}}^{b_n} \|v_\alpha(\varrho)\|^2 \, d\varrho \\
= \int_{b_{n-1}}^{b_n} \left\|B^*S^*(b_n - \varrho)(W(h_n))^{-1} \left(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})\right)\right\|^2 \, d\varrho \\
= \int_{b_{n-1}}^{b_n} \left\|(S(b_n - \varrho)B^*S^*(b_n - \varrho)(W(h_n))^{-1} \left(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})\right)\right\| \, d\varrho \\
= \left\|(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})\right)\right\| \, d\varrho \\
\leq M^2 \left\|(W(h_n))^{-1} \left(\|p_F - p_{n-1}\|^2 + \|q_F - q_{n-1}\|^2\right), \quad n = 1, 2, \ldots
\]

Therefore, by (4.3) and (4.4),
\[
\int_0^b \|v(\varrho)\|^2 \, d\varrho \\
= \sum_{n=0}^{\infty} \int_{b_n}^{b_{n+1}} \|v_{n+1}(\varrho)\|^2 \, d\varrho \\
\leq M^2 \sum_{n=0}^{\infty} \left\|(W(h_{n+1}))^{-1} \left(\|p_F - p_n\|^2 + \|q_F - q_n\|^2\right)\right\| \\
\leq M^2 \sum_{n=0}^{\infty} \left\|(W(h_{n+1}))^{-1} \left(\|C(h_n)p_F - p_F\| + \|S(h_n)q_F\| + KMh_n\right)\right\|^2 \\
+ \left(\|AS(h_n)p_F\| + \|C(h_n)q_F - q_F\| + KMh_n\right)^2.
\]

Since \(C(h_0)p_F = p_F\) and \(KMh_0 = 0\), we obtain
\[
\int_0^b \|v(\varrho)\|^2 \, d\varrho \\
\leq M^2 \sum_{n=1}^{\infty} \left\|(W(h_{n+1}))^{-1} \left(\|C(h_n)p_F - p_F\| + \|S(h_n)q_F\| + KMh_n\right)\right\|^2 \\
\leq M^2 \sum_{n=1}^{\infty} \left(\|C(h_n)p_F - p_F\|^2 + \|S(h_n)q_F\|^2 + KMh_n\right)^2.
\]
+ M^2 \sum_{n=1}^{\infty} (W(h_{n+1}))^{-1} \left( \|AS(h_n)q_F - p_F\| + \|C(h_n)q_F - q_F\| + KMh_n \right)^2
\leq M^2 \sum_{n=1}^{\infty} (h_{n+1})^{1-\alpha} \left( W(h_{n+1}) \right)^{-1} \left( \frac{h_n}{h_{n+1}} \right)^2 \left( \frac{1}{(h_{n+1})^{\alpha-1}} \right) \left( \frac{\|C(h_n)q_F - p_F\|}{h_n} \right)
+ \frac{\|S(h_n)q_F\|}{h_n} + KM \right)^2
+ M^2 \sum_{n=1}^{\infty} (h_{n+1})^{1-\alpha} \left( W(h_{n+1}) \right)^{-1} \left( \frac{h_n}{h_{n+1}} \right)^2 \left( \frac{1}{(h_{n+1})^{\alpha-1}} \right) \left( \frac{\|AS(h_n)q_F\|}{h_n} \right)
+ \frac{h_n}{2} \left( \frac{2\|C(h_n)q_F - q_F\|}{h_n^2} \right) + KM \right)^2
= 4M^2 \sum_{n=1}^{\infty} (h_{n+1})^{1-\alpha} \left( W(h_{n+1}) \right)^{-1} \left( \frac{1}{(h_{n+1})^{\alpha-1}} \right) \left( \frac{h_n}{2} \right) \left( \frac{2\|C(h_n)q_F - p_F\|}{h_n^2} \right)
+ \frac{\|S(h_n)q_F\|}{h_n} + KM \right)^2
+ 4M^2 \sum_{n=1}^{\infty} (h_{n+1})^{1-\alpha} \left( W(h_{n+1}) \right)^{-1} \left( \frac{1}{(h_{n+1})^{\alpha-1}} \right) \left( \frac{\|S(h_n)Ap_F\|}{h_n} \right)
+ \frac{h_n}{2} \left( \frac{2\|C(h_n)q_F - q_F\|}{h_n^2} \right) + KM \right)^2,
\]

Since $\lim_{\theta \to 0} \frac{C(2\theta^p - p_F)}{2\theta^2} = \hat{A}p_F$, using proposition (2) results in

$$\lim_{n \to \infty} \frac{2\|C(h_n)q_F - q_F\|}{h_n^2} = \hat{A}q_F$$
and

$$\lim_{n \to \infty} \frac{2\|C(h_n)q_F - p_F\|}{h_n^2} = \hat{A}p_F,$$

which implies

$$\frac{2\|C(h_n)q_F - p_F\|}{h_n^2} \leq P$$
and

$$\frac{2\|C(h_n)q_F - q_F\|}{h_n^2} \leq P'$$

for some $P, P' > 0$.

Then, by using assumption (II),

$$\int_0^b \|v(q)\|^2 \, dq \leq 4M^2 N \sum_{n=1}^{\infty} \left( \frac{1}{(h_{n+1})^{\alpha-1}} \right) \left( \frac{Ph_n}{2} + C + KM \right)^2
+ 4M^2 N \sum_{n=1}^{\infty} \left( \frac{1}{(h_{n+1})^{\alpha-1}} \right) \left( \frac{L + Ph_n}{2} + KM \right)^2,$$

where $\|q_F\| \leq C$ and $\|Ap_F\| \leq L$ for some $C, L > 0$.

Since $0 < \alpha < 1$, therefore the series

$$\sum_{n=1}^{\infty} \frac{1}{(h_{n+1})^{\alpha-1}},$$

$$\sum_{n=1}^{\infty} \frac{h_n}{(h_{n+1})^{\alpha-1}}$$
and

$$\sum_{n=1}^{\infty} \frac{h_n^2}{(h_{n+1})^{\alpha-1}}$$
are convergent.
This shows that \( v \in U_{ad} \). Therefore, \( p \) is continuous and
\[
\lim_{t \to b^-} p(\varrho) = \lim_{n \to \infty} p_n = p_F \quad \text{and} \quad \lim_{t \to b^-} p'(\varrho) = \lim_{n \to \infty} q_n = q_F.
\]

Thus, \( p(b) = p_F \) and \( p'(b) = q_F \).

5 Example

Consider the partial differential equation:
\[
\begin{cases}
\frac{\partial^2}{\partial t^2} u(\varrho, y) = u_{xx}(\varrho, y) + \gamma(\varrho, y) + \sigma(\varrho, u(\varrho, y)); \\
u(\varrho, 0) = u(\varrho, \pi) = 0 \quad \text{for} \ t > 0.
\end{cases}
\]

Let \( X = L^2[0, \pi] \) and \( \gamma : [0, b] \times (0, \pi) \to \mathbb{R} \) be a continuous control function in \( \varrho \). Define the operator \( A : D(A) \to X \) by
\[
A \eta = \eta''; \quad \eta \in D(A)
\]
with \( D(A) = \{ \eta \in X : \eta, \eta' \) are absolutely continuous \( \eta'' \in X, \eta(0) = \eta(\pi) = 0 \}. A \) is an infinitesimal generator of a strongly continuous cosine family \( C(\varrho) \) on \( X \). Moreover, the spectrum of \( A \) consists of eigenvalues \( -n^2 \) for \( n = 1, 2, 3, \ldots \), with the associated normalized eigenvectors \( \eta_n(s) = (2/\pi)^{1/2} \sin(ns) \). In particular,
\[
A \eta = \sum_{n=1}^{\infty} (-n^2)(\eta, \eta_n) \eta_n, \quad \eta \in D(A).
\]

The cosine function \( C(\varrho) \) and the sine function \( S(\varrho) \) are defined in the following way:
\[
\begin{align*}
C(\varrho) \eta &= \sum_{n=1}^{+\infty} \cos n\varrho(\eta, \eta_n) \eta_n, \quad \eta \in X, \\
S(\varrho) \eta &= \sum_{n=1}^{+\infty} \frac{1}{n} \sin n\varrho(\eta, \eta_n) \eta_n, \quad \eta \in X,
\end{align*}
\]
respectively. Define \( r : J \times X \to X \) by
\[
r(\varrho, u)(y) = \sigma(\varrho, u(y)); \quad u \in X, y \in [0, \pi].
\]

Let \( v : [0, b] \to \mathbb{U} \) be defined by
\[
v(\varrho)(y) = \gamma(\varrho, y); \quad y \in [0, \pi].
\]

Define the controllability operator in the following way:
\[
W(\varrho) = \int_0^\varrho S(s)B^* S^*(s) \, ds, \quad 0 \leq \varrho \leq b
\]
\[
= \int_0^\varrho S(s)S^*(s) \, ds, \quad \text{as } B = I.
\]
Let us choose the system operator $A$ in such a way that $W(\varrho)$ is coercive for all $0 < \varrho \leq b$ and there exists some $N \geq 0$ such that
\[ \varrho^{1+\alpha} \left\| (W(\varrho))^{-1} \right\| \leq N \quad \forall 0 < \varrho \leq b \text{ and } 0 \leq \alpha < 1. \]

Also, the nonlinear function $\sigma$ can be considered satisfying conditions (III)–(V).

The considered PDE (5.1) can be converted to (1.1). Therefore, system (5.1) is exactly controllable to $\text{dom}(A)$.

6 Conclusion
In the present manuscript, the exact controllability to $\text{dom}(A)$ for a second-order semilinear system has been discussed using a new technique which avoids fixed point theorems and does not involve large estimations on the system constants. The control function has been formulated by the piecewise construction of steering controls. These results can be further extended for systems with delay or deviated arguments with impulses and fractional-order systems.

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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