AN OPERATOR THAT RELATES TO SEMI-MEANDER POLYNOMIALS
VIA A TWO-SIDED $q$-WICK FORMULA

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Abstract. We consider the sequence $(Q_n)_{n=1}^{\infty}$ of semi-meander polynomials which are used in the enumeration of semi-meandric systems (a family of diagrams related to the classical stamp-folding problem). We show that for a fixed $d \in \mathbb{N}$, $(Q_n(d))_{n=1}^{\infty}$ appears as sequence of moments for a compactly supported probability measure $\nu_d$ on $\mathbb{R}$. More generally, we consider a sequence of two-variable polynomials $(rQ_n)_{n=1}^{\infty}$ related to a natural concept of “self-intersecting semi-meandric system”, where the second variable of $rQ_n$ keeps track of the crossings of such a system; one has, in particular, that $Q_n(t) = rQ_n(t, 0)$. We prove that for fixed $d \in \mathbb{N}$ and $q \in (-1, 1)$, $(rQ_n(d, q))_{n=1}^{\infty}$ can be identified as sequence of moments for a compactly supported probability measure $\nu_{d,q}$ on $\mathbb{R}$. The measure $\nu_{d,q}$ is found as scalar spectral measure for an operator $T_{d,q}$ constructed by using left and right creation/annihilation operators on the $q$-Fock space over $\mathbb{C}^d$, a deformation of the full Fock space over $\mathbb{C}^d$ introduced by Bożejko and Speicher. The relevant calculations of moments for $T_{d,q}$ are made by using a two-sided version of a (previously studied in the one-sided case) “$q$-Wick formula”, which involves the number of crossings of a pair-partition.

1. Introduction

A meandric system of order $n$ is a picture obtained by independently drawing two non-crossing pair-partitions (a.k.a. “arch-diagrams”) of $\{1, \ldots, 2n\}$, one of them above and the other one below a horizontal line, as exemplified in Figure 1. The combined arches of the two non-crossing pair-partitions create a family of disjoint closed curves which wind up and down the horizontal line. If this family consists of only one curve going through all of $1, \ldots, 2n$, then the meandric system in question is called a meander.

Figure 1. Two meandric systems of order 4, where one of them (on the right) is a meander.

For every $n \in \mathbb{N}$, a meandric system of order $n$ has at least 1 and at most $n$ components; for $k \in \{1, \ldots, n\}$, we will denote by $m_{n}^{(k)}$ the number of such meandric systems which have

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Proposition 5.9). Thus the scalar spectral distribution of 
$X$ for every $n$ precisely the probability measure 
$(1.3) \mu (1.3)$ the positive element 
of polynomials $(\in n$ is by identifying them with a special class of meandric 
systems. More precisely, for every $n$ systems 
$\rho$ the horizontal line is constrained to equal 
$R$ can be described as follows. We start with a free family 
$a$ a scalar spectral measure for some bounded selfadjoint op erator in a 
distribution of variance $(1.4) \sigma (1.4)$ and let 
$R$ (1.4) $\rho$ the numerical sequence 
$P$ works only for integer values of 
t for $t$ $t$ can be identified as moment sequence for a compactly 
supported probability measure $\mu_t$ on $\mathbb{R}$. The typical way of finding the measure $\mu_t$ is as 
a scalar spectral measure for some bounded selfadjoint operator in a $C^*$-probability space. 
The largest range of $t$’s for which this can be done appears to be \{2 $\cos \frac{\pi}{n} \mid n \geq 3\} \cup [2, \infty)$, as found in [6, Section 3] by using an operator model which lives in a planar algebra. The 
considerations of the present paper bear an analogy with a simpler operator model, which 
works only for integer values of $t$, and was described in [16] in terms of a free semicircular 
system of operators, following the idea of a random matrix model from [9, 14]. More precisely: 
for $t = d \in \mathbb{N}$, the probability measure $\mu_d$ on $\mathbb{R}$ determined uniquely by the moment 
conditions 
$(1.2) \int_{\mathbb{R}} x^n d\mu_d(x) = P_n(d) = \sum_{k=1}^{n} m_n^{(k)} d^k, \forall n \in \mathbb{N}$ 
can be described as follows. We start with a free family $a_1, \ldots, a_d$ of selfadjoint elements 
in a $C^*$-probability space $(A, \varphi)$, such that every $a_i$ $(1 \leq i \leq d)$ has centred semicircular 
distribution of variance 1. We then consider the $C^*$-probability space $(A \otimes A, \varphi \otimes \varphi)$, and the 
positive element 
$(1.3) X_d = (a_1 \otimes a_1 + a_2 \otimes a_2 + \cdots + a_d \otimes a_d)^2 \in A \otimes A.$ 
For every $n \in \mathbb{N}$, the moment $(\varphi \otimes \varphi)(X_d^n)$ turns out to be equal to $P_n(d)$ (cf. [16], 
Proposition 5.9). Thus the scalar spectral distribution of $X_d$ with respect to $\varphi \otimes \varphi$ is 
precisely the probability measure $\mu_d$ from Equation (1.2).

In this paper we study the analogue of (1.2) for semi-meander polynomials, a sequence 
of polynomials $(Q_n)_{n=1}^{\infty}$ which are used in connection to the enumeration of semi-meandric 
systems. There are several equivalent descriptions for what is a semi-meandric system (see 
e.g. Section 2.2 of [9]). For our purposes, the most convenient way to look at these objects 
is by identifying them with a special class of meandric systems. More precisely, for every 
n \in \mathbb{N}$ let us consider the rainbow pair-partition 
$(1.4) \rho_{2n} := \{\{1, 2n\}, \{2, 2n - 1\}, \ldots, \{n, n + 1\}\}$, 
and let $\mathcal{R}_n$ denote the set of meandric systems of order $n$ for which the pair-partition under 
the horizontal line is constrained to equal $\rho_{2n}$. (For example, both the meandric systems 
shown in Figure 1 are from $\mathcal{R}_4$.) For every $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$ we will denote by $r_n^{(k)}$ 
the number of meandric systems in $\mathcal{R}_n$ which have exactly $k$ components. The polynomial 
$(1.5) Q_n(t) := \sum_{k=1}^{n} r_n^{(k)} t^k$
is called the $n$th semi-meander polynomial. (For instance $Q_1(t) = t, Q_2(t) = t + t^2$. A table with numerical data concerning the numbers $r_n^{(k)}$ can be found in [9, Section 2.3].)

On a historical note, we mention that the meandric systems in $\mathcal{R}_n$ truly deserve special attention, due to their connection to the old (but still open, to our knowledge) problem of enumerating the foldings of a strip of stamps, which can be tracked back to the treatise on number theory by Lucas [12]. More precisely: the foldings of a strip of $n$ stamps where stamp no.1 stays on top of the folding are in bijective correspondence with the connected meandric systems in $\mathcal{R}_n$, and are thus counted by the linear coefficient $r_n^{(1)}$ of the polynomial $Q_n$. As the topic of the present paper is not directly related to foldings, we will not elaborate on that but rather refer the reader to [9, Section 2] (see also Sections 2 and 5 of the survey paper [11]) for the details of this bijective correspondence.

In the present paper we show that, analogously to the discussion for meander polynomials: when we set $t = d \in \mathbb{N}$, $(Q_n(d))_{n=1}^\infty$ is the moment sequence of a compactly supported probability measure $\nu_d$ on $\mathbb{R}$ which is related to free probability. More precisely, we find $\nu_d$ as the distribution of a selfadjoint operator $T_d$ which arises naturally in the framework of two-faced free probability theory introduced by Voiculescu in [18] (a survey of this direction of research, and many references, can be found in the expository paper [19]). In order to describe $T_d$, we review a bit of terminology concerning creation and annihilation operators on the full Fock space $\mathcal{F}_d$ over $\mathbb{C}^d$: let $\varphi_{\text{vac}} : B(\mathcal{F}_d) \to \mathbb{C}$ denote the vacuum-state on $B(\mathcal{F}_d)$, and let $L_1, \ldots, L_d$ and $R_1, \ldots, R_d$ denote the left and respectively right creation operators on $\mathcal{F}_d$ associated to the vectors $e_1, \ldots, e_d$ from the standard orthonormal basis of $\mathbb{C}^d$. (These notations are reviewed in more detail in Section 3 below.)

**Proposition 1.1.** In the framework just described, we put

\begin{equation}
T_d := \sum_{i=1}^{d} (L_i + L_i^*)(R_i + R_i^*) \in B(\mathcal{F}_d).
\end{equation}

Then $T_d = T_d^*$, and has

\begin{equation}
\varphi_{\text{vac}}(T_d^n) = Q_n(d), \quad \forall n \in \mathbb{N}.
\end{equation}

**Corollary 1.2.** For every $d \in \mathbb{N}$, $(Q_n(d))_{n=1}^\infty$ appears as sequence of moments for a compactly supported probability measure $\nu_d$ on $\mathbb{R}$. The measure $\nu_d$ can be found as the distribution of the operator $T_d$ from Proposition 1.1 with respect to the vacuum-state on the full Fock space $\mathcal{F}_d$.

In the framework of Proposition 1.1, it can be easily verified that, for every $1 \leq i \leq d$, the operator $(L_i + L_i^*)(R_i + R_i^*)$ is selfadjoint and has Marchenko-Pastur distribution (the free analogue of the standard Poisson distribution) with respect to the vacuum-state. Thus the operator $T_d$ from (1.6) is the sum of $d$ elements with free Poisson distributions – however, these $d$ elements are not “independent” in the sense of some non-commutative probability theory (so it is not clear if moment-cumulant methods could be applied to calculate the distribution of $T_d$ by starting from the distribution of its $d$ summands).

We will obtain Proposition 1.1 as the special case of a more general result, where we look at a two-variable generalization of the semi-meander polynomials $Q_n$. Suppose that in the construction of a meandric system of order $n$ we actually allow the two pair-partitions drawn above and below the horizontal line to run in the full set $\mathcal{P}_2(2n)$ of all pair-partitions of $\{1, \ldots, 2n\}$ (that is, we give up the non-crossing requirement). We then get a larger collection
of pictures, which could be called *self-intersecting meandric systems*; some examples of such pictures are shown in Figure 2. A self-intersecting meandric system of order $n$ consists of a family of $k$ closed curves, with $1 \leq k \leq n$, and where now we may also have a number of *crossings* in between these curves (including the possibility that one of the curves crosses itself, or the possibility that two of the curves cross each other multiple times). The total number of crossings of a self-intersecting meandric system of order $n$ is at least 0 and at most $n^2 - n$, where the upper bound is found by noting that every crossing arises either above or below the horizontal line, and there can be at most $n$-choose-$2$ crossings of each of these two kinds. The usual “meandric systems” discussed above are retrieved, of course, as the self-intersecting meandric systems which have 0 crossings. (The formal definitions of the notions introduced in this paragraph appear in Section 3.4 below – including precise formulas for what we mean by “number of closed curves” and by “number of crossings” for a self-intersecting meandric system of order $n$.)

![Figure 2. Two self-intersecting meandric systems of order 4, one with 2 closed curves and 3 crossings, and the other with 1 closed curve and 3 crossings.](image)

**Notation 1.3.** (1) For every $n \in \mathbb{N}$ we let $\mathcal{R}_n$ denote the set of self-intersecting meandric systems of order $n$ for which the pair-partition under the horizontal line is constrained to be the rainbow pair-partition $\rho_{2n}$ of Equation (1.4). (For instance the second picture in Figure 2 shows a self-intersecting meandric system in $\mathcal{R}_4 \ \setminus \mathcal{R}_4$.)

(2) For every $n \in \mathbb{N}$ we consider the polynomial

$$ Q_n(t, u) := \sum_{k=1}^{n} \sum_{\ell=0}^{n^2-n} r_n^{(k,\ell)} t^k u^\ell, $$

where $r_n^{(k,\ell)}$ stands for the number of self-intersecting meandric systems in $\mathcal{R}_n$ which have exactly $k$ closed curves and $\ell$ crossings\(^\dagger\). (For instance $\tilde{Q}_1(t, u) = t$, $\tilde{Q}_2(t, u) = t(1+u)+t^2$. Clearly, for every $n \in \mathbb{N}$, the semi-meander polynomial $Q_n(t)$ can be retrieved as $\tilde{Q}_n(t, 0)$.)

Now, the framework of creation/annihilation operators on $T_d$ used in Proposition 1.1 can be viewed as the special case $q = 0$ of a $q$-deformation introduced by Bożejko and Speicher [2] which has been well-studied since the 1990’s. For every $q \in (-1,1)$ one has a $q$-Fock space $T_{dq}$ and one can consider the operators $L_{1,q}, \ldots, L_{d,q}$ and $R_{1,q}, \ldots, R_{d,q}$ of left and respectively right creation associated to the vectors $e_1, \ldots, e_d$ from the standard orthonormal basis of $\mathbb{C}^d$. (The precise definition of these operators is reviewed in Section 3.3 below.) The statement of Proposition 1.1 generalizes as follows.

\(^\dagger\)In the summations from Equation (1.8) one may actually restrict the range of $\ell$ to an upper bound of $(n^2 - n)/2$, since a system in $\mathcal{R}_n$ can only have crossings above the horizontal line.
Theorem 1.4. Let $d \in \mathbb{N}$ and $q \in (-1, 1)$, and consider the operator

$$
T_{d,q} := \sum_{i=1}^{d} (L_{i;q} + L_{i; q}^*) (R_{i; q} + R_{i; q}^*) \in B(T_{d,q}).
$$

Then $T_{d,q} = T_{d,q}^*$ and its moments with respect to the vacuum-state $\varphi_{\text{vac}}$ on the $q$-Fock space are

$$
\varphi_{\text{vac}}(T_{d,q}^n) = \tilde{Q}_n(d, q), \quad n \in \mathbb{N}.
$$

Corollary 1.5. For every $d \in \mathbb{N}$ and $q \in (-1, 1)$, $(\tilde{Q}_n(d, q))_{n=1}^{\infty}$ appears as sequence of moments for a compactly supported probability measure $\nu_{d,q}$ on $\mathbb{R}$. The measure $\nu_{d,q}$ can be found as the distribution of the operator $T_{d,q}$ from Theorem 1.4 with respect to the vacuum-state on the $q$-Fock space $T_{d,q}$.

Remark 1.6. (1) The main tool for proving the moment formula (1.10) in Theorem 1.4 is a two-sided version of a $q$-Wick formula (previously studied in the one-sided case, cf. [3, 7]) which involves the number of crossings of a pair-partition. The two-sided $q$-Wick formula is discussed in Sections 3.2 and 4.1 below.

(2) The case $q = 0$. Clearly, Proposition 1.1 is the special case $q = 0$ of Theorem 1.4. The recent body of work on two-faced free probability allows for several short proofs of this special case. Indeed, Proposition 1.1 is about the $(2d)$-tuple of operators $(L_i + L_i^*)_i \cup (R_i + R_i^*)_i$, which is the prototypical example of bi-free Gaussian system appearing in the bi-free central limit theorem from [18]. The explicit formulas that one has for the joint moments of this $(2d)$-tuple of operators can also be read by using the bi-free cumulant theory developed in [4, 5], or its precursor focused on canonical $(2d)$-tuples from [13]. Yet another approach to Proposition 1.1 can be found by combining a matrix model for semi-meander polynomials proposed in [9] with the bi-free large $N$ limits discussed in [17]. We give more details on these alternative proofs in Section 4.3 below.

In connection to the case $q = 0$ we also mention that, among the various patterns that can be prescribed for the bottom part of a meandric system, the rainbow pair-partition $\rho_{2n}$ is believed to provide the case which is hardest to approach (as opposed, for instance, to the case when the bottom part of the meandric system is prescribed to be the interval pair-partition $\{\{1,2\}, \ldots, \{2n-1,2n\}\}$ – see Section 6.3 of [9], particularly the comment in the last paragraph of that section). In this light, it is quite nice that the semi-meander polynomials $Q_n$ can nevertheless be related to moments of operators in the two-sided framework. This seems to be caused by the fortunate circumstance that in the rectangular pictures which we use (following [4]) to depict pair-partitions of $\{1, \ldots, 2n\}$, an important “labels-to-heights” permutation of the $2n$ points in the picture converts $\rho_{2n}$ into $\{\{1,2\}, \ldots, \{2n-1,2n\}\}$ (see Notation 2.1.3 below, and its follow-up in the proof of Lemma 3.4.3).

(3) On the lines of Theorem 1.4, we observe that the enlarged framework of self-intersecting meandric systems also works well in connection to the meander polynomials $P_n(t)$ from Equation (1.1). More precisely, let us look at the two-variable generalization of these polynomials, defined as follows.

Notation 1.7. For every $n \in \mathbb{N}$, consider the polynomial

$$
\tilde{P}_n(t, u) := \sum_{k=1}^{n} \sum_{\ell=0}^{n^2-n} m_n^{(k, \ell)} t^k u^\ell,
$$
where $m_n^{(k,\ell)}$ stands for the number of self-intersecting meandric systems of order $n$ which have exactly $k$ closed curves and $\ell$ crossings. (For instance $\widehat{P}_1(t,u) = t$, $\widehat{P}_2(t,u) = t(2 + 4u) + t^2(2 + u^2)$. Clearly, for every $n \in \mathbb{N}$, the meander polynomial $P_n(t)$ can be retrieved as $\widehat{P}_n(t,0)$.)

It is natural to ask if some $q$-deformation of the operator $X_d$ from Equation (1.3) could allow us to infer that $(\widehat{P}_n(d,q))_{n=1}^{\infty}$ appears as sequence of moments for a probability measure $\mu_{d,q}$ on $\mathbb{R}$, for $d \in \mathbb{N}$ and $q \in (-1,1)$. There is in fact an obvious candidate for how to do the $q$-deformation of $X_d$: the semicircular elements $a_i$ appearing in (1.3) can be concretely realized as $L_i + L_i^* \in B(\mathcal{T}_d)$, and can then be $q$-deformed to $L_{i,q} + L_{i,q}^* \in B(\mathcal{T}_{d,q})$, $1 \leq i \leq d$. It is easy to see (by a straightforward adjustment of the argument shown in [16, Proposition 5.9] for the case $q = 0$) that this candidate of $q$-deformation does indeed the required job. That is, we have the following proposition.

**Proposition 1.8.** Let $d \in \mathbb{N}$ and $q \in (-1,1)$, and consider the selfadjoint operator

\[(1.12) \quad X_{d,q} := \left( \sum_{i=1}^{d} (L_{i,q} + L_{i,q}^*) \otimes (L_{i,q} + L_{i,q}^*) \right)^2 \in B(\mathcal{T}_{d,q}) \otimes B(\mathcal{T}_{d,q}). \]

The moments of $X_{d,q}$ with respect to the state $\varphi_{\text{vac}} \otimes \varphi_{\text{vac}}$ on $B(\mathcal{T}_{d,q}) \otimes B(\mathcal{T}_{d,q})$ are

\[(1.13) \quad (\varphi_{\text{vac}} \otimes \varphi_{\text{vac}})(X_{d,q}^n) = \widehat{P}_n(d,q), \quad n \in \mathbb{N}. \]

**Corollary 1.9.** For every $d \in \mathbb{N}$ and $q \in (-1,1)$, $(\widehat{P}_n(d,q))_{n=1}^{\infty}$ appears as sequence of moments for a compactly supported probability measure $\mu_{d,q}$ on $\mathbb{R}$. The measure $\mu_{d,q}$ can be found as the distribution of the operator $X_{d,q}$ from Proposition 1.8 with respect to the state $\varphi_{\text{vac}} \otimes \varphi_{\text{vac}}$ on $B(\mathcal{T}_{d,q}) \otimes B(\mathcal{T}_{d,q})$.

**Organization of the paper.** Besides the present introduction, the paper has three sections. Section 2 covers the combinatorics relevant for proving the two-sided $q$-Wick formula. In Section 3 we prove Theorem 1.4; a good part of the section is devoted to establishing the special case of the two-sided $q$-Wick formula which is used in the proof of the theorem. The final Section 4 discusses some miscellaneous remarks related to the results of the paper: in Section 4.1 we complete the discussion of the two-sided $q$-Wick formula, in Section 4.2 we prove Proposition 1.8, and in Section 4.3 we make some comments related to the special case $q = 0$.

2. Pair-partitions and strings of symbols from $\{1, \ast\}$

2.1. Pair-partitions.

**Definition 2.1.1.** Let $n$ be a positive integer.

1. We denote by $\mathcal{P}_2(2n)$ the set of all pair-partitions of $\{1, \ldots, 2n\}$. A $\pi \in \mathcal{P}_2(2n)$ is thus of the form $\pi = \{V_1, \ldots, V_n\}$, where the sets $V_1, \ldots, V_n$ (called pairs, or blocks of $\pi$) satisfy: $\cup_{i=1}^{n} V_i = \{1, \ldots, 2n\}$, $V_i \cap V_j = \emptyset$ for $i \neq j$, and $|V_1| = |V_2| = \cdots = |V_n| = 2$.

2. Let $\pi = \{V_1, \ldots, V_n\}$ be in $\mathcal{P}_2(2n)$. We will say that two distinct blocks $V_i, V_j$ of $\pi$ are crossing to mean that upon writing $V_i \cup V_j = \{a, b, c, d\}$ with $a < b < c < d$, one finds $a, c$ in one of the two blocks and $b, d$ in the other. The number of crossings of $\pi$ is defined as $\text{cr}(\pi) := |\{(i,j) \mid 1 \leq i < j \leq n, \text{ } V_i \text{ crosses } V_j\}|$. 

A pair-partition \( \pi \in \mathcal{P}_2(2n) \) is said to be non-crossing when it has \( \text{cr}(\pi) = 0 \). The collection of all non-crossing pair-partitions in \( \mathcal{P}_2(2n) \) will be denoted by \( \text{NC}_2(2n) \).

**Remark 2.1.2.** In the rather extensive literature pertaining to pair-partitions, one finds their pictures drawn either in “linear” representation (with \( 2n \) points labelled by \( 1, \ldots, 2n \) depicted along a line) or in “circular” representation (with \( 2n \) points depicted around a circle). In either representation, \( \text{cr}(\pi) \) appears as the number of intersections between the curves drawn in the picture in order to represent the pairs of \( \pi \).

In this paper we will use pictures drawn in circular representation but where, following [4], our “circle” will in fact be a rectangle, and the \( 2n \) points labelled by \( 1, \ldots, 2n \) will be depicted on the vertical sides of the rectangle: labels \( 1, \ldots, n \) on the left side of the rectangle (running downwards) and labels \( n + 1, \ldots, 2n \) on the right side of the rectangle (running upwards).

For a concrete example, in Figures 3 and 4 we consider the pair-partition
\[
\pi = \{\{1, 9\}, \{2, 7\}, \{3, 10\}, \{4, 5\}, \{6, 8\}\} \in \mathcal{P}_2(10),
\]
which has \( \text{cr}(\pi) = 3 \). Figure 3 shows the linear representation of this \( \pi \), and Figure 4 shows its “rectangular” representation.
rectangle, and we measure heights \textit{downwards} from there. Thus in Figure 4 we get, with “Ht” for height \(^2\) and by writing the height as a function of the corresponding label:

\[
\text{Ht}(1) = 1, \text{Ht}(2) = 3, \ldots, \text{Ht}(5) = 9 \text{ and } \text{Ht}(6) = 10, \text{Ht}(7) = 8, \ldots, \text{Ht}(10) = 2.
\]

In general, for the “height in terms of label” map we will use the following notation. (We mention that “\(\xi_n\)” is a special case of a rather established notation for a permutation “\(\xi\)”, which will be reviewed in Section 4.1 below, and plays an essential role in the combinatorics of two-faced free probability.)

\textbf{Notation 2.1.3.} For every \(n \in \mathbb{N}\), we will denote by \(\xi_n\) the permutation of \(\{1, \ldots, 2n\}\) defined (in the usual two-line notation for permutations) as

\[
(2.1) \quad \xi_n := \begin{pmatrix} 1 & 2 & \cdots & n & n + 1 & \cdots & 2n - 1 & 2n \\ 1 & 3 & \cdots & 2n - 1 & 2n & \cdots & 4 & 2 \end{pmatrix}.
\]

Thus \(\xi_n(k)\) is the height of the point with label \(k\) \((1 \leq k \leq 2n)\) in the rectangular representation of any \(\pi \in \mathcal{P}_2(2n)\).

We will also consider the natural action of \(\xi_n\) on \(\mathcal{P}_2(2n)\), defined by

\[
(\pi = \{V_1, \ldots, V_n\}) \Rightarrow (\xi_n \cdot \pi = \{\xi_n(V_1), \ldots, \xi_n(V_n)\}).
\]

It is relevant to observe that the rainbow pair-partition \(\rho_{2n}\) from Equation (1.4) of the Introduction is transformed by the action of \(\xi_n\) into an interval pair-partition,

\[
(2.2) \quad \xi_n \cdot \rho_{2n} = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}\} \in \mathcal{P}_2(2n).
\]

2.2. \textit{Tuples in }\{1, \ast\}\textit{ }^{2n}\textit{ and the map }\Phi_n : \mathcal{P}_2(2n) \rightarrow \mathcal{D}(1, \ast)(2n)\textit{.}

We now start to look at strings made with the symbols “1” and “\(\ast\)”. It will be convenient to view such a string (i.e. tuple \(\varepsilon \in \{1, \ast\}^m\) for some \(m \in \mathbb{N}\)) as a map \(\varepsilon : \{1, \ldots, m\} \rightarrow \{1, \ast\}\).

\textbf{Definition 2.2.1.} A \((2n)\)-tuple \(\varepsilon : \{1, \ldots, 2n\} \rightarrow \{1, \ast\}\) is said to have the \textit{Dyck property} when it satisfies the inequalities

\[
(2.3) \quad |\{1 \leq i \leq h \mid \varepsilon(i) = 1\}| \geq |\{1 \leq i \leq h \mid \varepsilon(i) = \ast\}|, \quad \forall 1 \leq h \leq 2n,
\]

with equality when \(h = 2n\), that is

\[
(2.4) \quad |\{1 \leq i \leq 2n \mid \varepsilon(i) = 1\}| = n = |\{1 \leq i \leq 2n \mid \varepsilon(i) = \ast\}|.
\]

The collection of all the \((2n)\)-tuples with the Dyck property will be denoted by \(\mathcal{D}(1, \ast)(2n)\).

\textbf{Remark 2.2.2.} The use of the term “Dyck property” in Definition 2.2.1 is justified by a connection to lattice paths. To every \(\varepsilon \in \{1, \ast\}^{2n}\) one can associate a path with \(2n\) steps in \(\mathbb{Z}^2\) which starts at \((0, 0)\) and proceeds according to the following rule:

\begin{itemize}
  \item for every \(1 \leq h \leq 2n\) such that \(\varepsilon(h) = 1\) we perform a step of \((1, 1)\) (North-East step);
  \item for every \(1 \leq h \leq 2n\) such that \(\varepsilon(h) = \ast\) we perform a step of \((1, -1)\) (South-East step).
\end{itemize}

That is, our path visits successively the lattice points \((0, 0), (1, p_1), (2, p_2), \ldots, (2n, p_{2n})\), where for \(1 \leq h \leq 2n\) we put \(p_h = |\{1 \leq i \leq h \mid \varepsilon(i) = 1\}| - |\{1 \leq i \leq h \mid \varepsilon(i) = \ast\}|\). Clearly, the conditions (2.3), (2.4) in Definition 2.2.1 have the meaning that our lattice path never goes under the horizontal axis of \(\mathbb{Z}^2\), and ends at the point \((2n, 0)\) on that axis. These conditions constitute precisely the definition of a \textit{Dyck path} in \(\mathbb{Z}^2\).

\footnote{Even though they are measured downwards, we will nevertheless refer to the distances measured from the top horizontal side of the rectangle by calling them “heights”.
}
Proposition and Notation 2.2.3. Let $n$ be a positive integer.

(1) Let $\pi$ be in $\mathcal{P}_2(2n)$. We consider the pair-partition $\mathcal{s}_n \cdot \pi$ (as in Notation 2.1.3), and we write it explicitly, $\mathcal{s}_n \cdot \pi = \{W_1, \ldots, W_n\}$. Let $\varepsilon : \{1, \ldots, 2n\} \to \{1, \ast\}$ be defined by

$$
\varepsilon(\min(W_1)) = \cdots = \varepsilon(\min(W_n)) = 1, \text{ and }
\varepsilon(\max(W_1)) = \cdots = \varepsilon(\max(W_n)) = \ast.
$$

Then $\varepsilon \in D_{(1,\ast)}(2n)$.

(2) We will denote by $\Phi_n : \mathcal{P}_2(2n) \to D_{(1,\ast)}(2n)$ the map which associates to every $\pi \in \mathcal{P}_2(2n)$ the tuple $\varepsilon \in D_{(1,\ast)}(2n)$ described in part (1) above.

Proof that $\varepsilon$ of (2.5) has the Dyck property. Fix an $h \in \{1, \ldots, 2n\}$ and observe that

$$
|\{1 \leq i \leq h \mid \varepsilon(i) = 1\}| = |\{1 \leq i \leq h \mid i \in \{\min(W_1), \ldots, \min(W_n)\}\}|
$$

Similarly, we see that $|\{1 \leq i \leq h \mid \varepsilon(i) = \ast\}| = |\{1 \leq m \leq n \mid \min(W_m) \leq h\}|$. Thus (2.3) amounts to $|\{1 \leq m \leq n \mid \min(W_m) \leq h\}| \geq |\{1 \leq m \leq n \mid \max(W_m) \leq h\}|$, and holds true due to the obvious implication "$(\max(W_m) \leq h) \Rightarrow (\min(W_m) \leq h)$". The equality (2.4) is also clear, from how $\varepsilon$ is defined in (2.5). \qed

Example 2.2.4. Consider again the pair-partition $\pi \in \mathcal{P}_2(10)$ depicted in Figure 4, and let us determine what is $\Phi_5(\varepsilon) \in D_{(1,\ast)}(10)$. We thus look at the heights of the 10 points marked around the rectangle, and: whenever two points at heights $i$ and $j$ are paired, with $i < j$, we assign $\varepsilon(i) = 1$ and $\varepsilon(j) = \ast$. Formally, we write:

$$
\pi = \{\{1, 9\}, \{2, 7\}, \{3, 10\}, \{4, 5\}, \{6, 8\}\}
$$

$$
\Rightarrow \mathcal{s}_{10}(\pi) = \{\{1, 4\}, \{2, 5\}, \{3, 8\}, \{6, 10\}, \{7, 9\}\}
$$

$$
\Rightarrow \varepsilon := \Phi_5(\pi) \text{ has } \varepsilon^{-1}(1) = \{1, 2, 3, 6, 7\} \text{ and } \varepsilon^{-1}(\ast) = \{4, 5, 8, 9, 10\}
$$

$$
\Rightarrow \Phi_5(\pi) \text{ is the tuple } \varepsilon = (1, 1, 1, \ast, \ast, 1, 1, \ast, \ast, \ast).
$$

Remark 2.2.5. For the subsequent discussion and figures, it will come in handy to give names to the actual points (geometric entities) marked on the boundary of the rectangle: we will denote them as $P_1, \ldots, P_{2n}$, in such a way that

$(\text{label of } P_k) = k \text{ and } (\text{height of } P_k) = \mathcal{s}_n(k), \text{ for } 1 \leq k \leq 2n.$

\begin{figure}[h]
\centering
\begin{tabular}{|c|c|}
\hline
P_1 & $\bigcirc$ \\
\hline
P_2 & $\bigcirc$ \\
\hline
P_3 & $\bigcirc$ \\
\hline
P_4 & $\bigcirc$ \\
\hline
P_5 & $\bigcirc$ \\
\hline
P_6 & $\bigcirc$ \\
\hline
P_7 & $\bigcirc$ \\
\hline
P_8 & $\bigcirc$ \\
\hline
P_9 & $\bigcirc$ \\
\hline
P_{10} & $\bigcirc$ \\
\hline
\end{tabular}
\caption{10 decorated points $P_1, \ldots, P_{10}$ in a rectangular picture.}
\end{figure}

In terms of these points $P_1, \ldots, P_{2n}$, the assignments of the form "$\varepsilon(i) = 1$" or "$\varepsilon(j) = \ast$" introduced in Notation 2.2.3 will be referred to by saying that the point at height $i$ is "1"-decorated and respectively that the point at height $j$ is "$\ast$"-decorated. For illustration, Figure...
5 shows the points $P_1, \ldots, P_{10}$ and their decorations, as they come out of Example 2.2.4 (e.g. $P_2$ has label 2, height 3, and is “1”-decorated, while $P_6$ has label 6, height 10, and is “*”-decorated).

2.3. Choice numbers and the enumeration of $\Phi_n^{-1}(\varepsilon)$. 

**Remark and Notation 2.3.1.** Let $n$ be a positive integer and let us fix a tuple $\varepsilon \in \mathcal{D}(1,*) (2n)$. For the proof of the two-sided $q$-Wick formula in the next section, it will be important to have a good description of the pre-image $\Phi_n^{-1}(\varepsilon) \subseteq \mathcal{P}_2 (2n)$, where $\Phi_n$ is the map introduced in Notation 2.2.3. To this end, we will make the following (ad-hoc) definition: for $3$ every $h \in \varepsilon^{-1}(\ast) \subseteq \{2, \ldots, 2n\}$, we call “choice number of $\varepsilon$ at $h$” the number

$$\delta \in \mathcal{D}(1,*)(2n).$$

By invoking the Dyck property satisfied by $\varepsilon$, we infer that $\delta \in \mathcal{P}_2 (2n)$. Indeed:

$$0 \leq \{1 \leq i \leq h \mid \varepsilon(i) = 1\} - \{1 \leq i \leq h \mid \varepsilon(i) = \ast\} = \{1 \leq i \leq h - 1 \mid \varepsilon(i) = 1\} - \{1 \leq i \leq h - 1 \mid \varepsilon(i) = \ast\} + 1 = \delta.$$ 

The rationale for the term “choice” used in (2.6) is that the partitions in $\Phi_n^{-1}(\varepsilon)$ are naturally parametrized by “tuples of choices” of the form

$$\gamma_h \in [1, \delta] \cap \mathbb{N}, \forall h \in \varepsilon^{-1}(\ast).$$

Note that, as an obvious consequence of (2.7), one has

$$|\Phi_n^{-1}(\varepsilon)| = \prod_{h \in \varepsilon^{-1}(\ast)} \delta.$$ 

The procedure of retrieving a pair-partition $\pi \in \mathcal{P}_2 (2n)$ from the information provided by $\varepsilon$ and a tuple of $\gamma_h$’s as in (2.7) is rather standard in the literature on lattice paths, with the slight difference in terminology that instead of $\varepsilon$ one usually refers to the corresponding Dyck path mentioned in Remark 2.2.2. More precisely: a tuple of $\gamma_h$’s as indicated in (2.7) can be viewed as an additional piece of structure imposed on the Dyck path corresponding to $\varepsilon$, and this yields the notion of “weighted Dyck path”, which is thoroughly studied e.g. in Section 5.2 of the monograph [10].

In order to explain how a tuple as in (2.7) parametrizes a pair-partition from $\Phi_n^{-1}(\varepsilon)$, we find it more illuminating to discuss a relevant concrete example, where the presentation can be illustrated with pictures.

**Example 2.3.2.** Suppose that $n = 5$ and that $\varepsilon = (1, 1, 1, \ast, \ast, 1, \ast, \ast, \ast, \ast) \in \mathcal{D}(1,*) (10)$. Thus $\varepsilon^{-1}(\ast)$ is $\{4, 5, 8, 9, 10\}$, and the corresponding choice-numbers are:

$$\delta(4) = 3 - 0 = 3, \delta(5) = 3 - 1 = 2, \delta(8) = 5 - 2 = 3,$$

$$\delta(9) = 5 - 3 = 2, \delta(10) = 5 - 4 = 1.$$ 

For this $\varepsilon$, we consider the set of pair-partitions $\Phi_n^{-1}(\varepsilon) \subseteq \mathcal{P}_2 (2n)$, and we will discuss its enumeration.

---

3 Note that $1 \notin \varepsilon^{-1}(\ast)$, due to the assumption that $\varepsilon$ has the Dyck property.
Let us draw points $P_1, \ldots, P_{10}$ around a rectangle, in the same way as in Figure 5, and for every $h \in \{1, \ldots, 10\}$ let us decorate the point at height $h$ by the symbol $\varepsilon(h)$ (this also is exactly as in Figure 5). The enumeration of $\Phi_5^{-1}(\varepsilon)$ then amounts to the enumeration of all the pair-partitions of $P_1, \ldots, P_{10}$ which have the following property: every chord of the pair-partition must connect a point decorated as "1" to a point decorated as "∗", where the "∗"-decorated endpoint has a bigger height than the "1"-decorated endpoint of the chord.

In order to draw a generic pair-partition of $P_1, \ldots, P_{10}$ which has the required property, we proceed in five steps, as follows: we visit the five "∗"-decorated points among $P_1, \ldots, P_{10}$, in increasing order of their heights, and for every such point $P_m$ we choose its pair out of the "1"-decorated points with heights smaller than the one of $P_m$, and which have not already been used in a preceding step.

To be specific: the first step of our construction is to visit the point at height 4, and to choose a pair for it; we have three possibilities for doing so (which corresponds to the fact that $\text{Choice}_4(4) = 3$), namely, we can choose any of the points at heights 1, 2 or 3. Let’s say that we choose the pair to be at height 3, as shown in Figure 6. The second step of our construction is then to go to the point at height 5, and choose a pair for that one; we have two possibilities for doing so, corresponding to the fact that $\text{Choice}_5(5) = 3 - 1 = 2$ (indeed, there are three points with decoration "1" and with heights < 5, but one of them is already engaged in a different pair of the construction). So we must pair the point at height 5 with one of the points at heights 1 or 2 – let’s say we choose the one at height 1.

Continuing in the same way, we will then visit the points at heights 8, 9 and 10, and choose pairs for them (in three possible ways for the point $P_7$ at height 8, then in two possible ways for the point $P_9$ at height 9, and in a unique way for $P_{10}$ at height 10). We hope this discussion convinces the reader that, in our running example, the set $\Phi_5^{-1}(\varepsilon)$ has $3 \cdot 2 \cdot 3 \cdot 2 \cdot 1 = 36$ pair-partitions, as claimed by the formula (2.8).

Let us also describe precisely what are the parameters $(\gamma_h)_{h \in \varepsilon^{-1}(\ast)}$ mentioned in (2.7) which govern our successive choices of pairs for the "∗"-decorated points in the picture. We will use the convention that: whenever we want to assign a pair to a "∗"-decorated point on the left side of the rectangle, the possible choices of pairs are considered in clockwise order; while for "∗"-decorated points which lie on the right side of the rectangle, the possible choices of pairs are considered in counterclockwise order (so we always follow the boundary of the rectangle in the direction going towards the top horizontal side).

For instance, the first step of the construction presented earlier in this example assigned a pair to the point at height 4; the choice of the pair was governed by a parameter $\gamma_4 \in \{1, 2, 3\}$,

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6.png}
\caption{First two steps in the construction of a pair-partition in $\Phi_5^{-1}(\varepsilon)$, for the $\varepsilon$ of Figure 5.}
\end{figure}
and our convention says that (for what was picked in Figure 6) we had \( \gamma_4 = 3 \). Then the second step of the construction was to choose a pair for the point at height 5, governed by a parameter \( \gamma_5 \in \{1, 2\} \), and our convention says that (for what was picked in Figure 6) we had \( \gamma_5 = 1 \). Figure 7 shows how the pair-partition \( \pi \in \Phi_5^{-1}(\varepsilon) \) looks like when the choices \( \gamma_4 = 3 \), \( \gamma_5 = 1 \) from Figure 6 are followed by \( \gamma_8 = \gamma_9 = 2 \) and \( \gamma_{10} = 1 \). (Or referring to Figure 4, which has the same \( \varepsilon \) as in this example: the sequence of choices for the pair-partition in that figure is \( \gamma_4 = \gamma_5 = \gamma_8 = 2 \), \( \gamma_9 = \gamma_{10} = 1 \).)

![Figure 7. The pair-partition in \( \Phi_5^{-1}(\varepsilon) \) which arises when we choose \( \gamma_4 = 3 \), \( \gamma_5 = 1 \), \( \gamma_8 = \gamma_9 = 2 \), \( \gamma_{10} = 1 \).](image)

The convention for how the parameters \( (\gamma_h)_{h \in \varepsilon^{-1}(\ast)} \) are used in the construction of a \( \pi \in \Phi_n^{-1}(\varepsilon) \) is useful because it generates a nice formula for number of crossings, as described in the next proposition.

**Proposition 2.3.3.** Let \( n \) be a positive integer and let \( \varepsilon \) be a tuple in \( \mathcal{D}(1, \varepsilon)(2n) \). Consider a tuple \( (\gamma_h)_{h \in \varepsilon^{-1}(\ast)} \) as discussed in Remark 2.3.1, and let \( \pi \in \Phi_n^{-1}(\varepsilon) \) be the pair-partition which corresponds to this \( (\gamma_h)_{h \in \varepsilon^{-1}(\ast)} \), in the way described in Example 2.3.2. Then one has

\[
\text{cr}(\pi) = \sum_{h \in \varepsilon^{-1}(\ast)} (\gamma_h - 1).
\]

**Proof.** Pick a \( k \in \varepsilon^{-1}(1) \), and consider the point at height \( k \) in the picture of \( \pi \). This point is paired with a point at height \( h \in \varepsilon^{-1}(\ast) \), where \( h > k \) (and where we recall that, in our pictures, heights are measured downwards from the top horizontal side of the rectangle). Say that the pairing of the points at heights \( h \) and \( k \) was done in the \( m \)-th step of the construction of \( \pi \), where \( 1 \leq m \leq n \). This means that there were \( m - 1 \) other \( \ast \)-decorated points, with heights \( h_1 < \cdots < h_{m-1} < h \), which were visited and paired before we arrived to visit and choose a pair for the point at height \( h \). For some of the \( i \)'s in \( \{1, \ldots, m-1\} \) it may have been the case that the point at height \( k \) was considered but then “skipped” \(^4\) in the process of choosing the pair for the point at height \( h_i \). We denote by “Skipped(\( k \))” the number of values of \( 1 \leq i \leq m - 1 \) for which this happened.

By starting from how the family of numbers \( \text{Skipped}(k) \) \( k \in \varepsilon^{-1}(1) \) was defined in the preceding paragraph, an elementary counting of crossings gives the formula

\[
\text{cr}(\pi) = \sum_{k \in \varepsilon^{-1}(1)} \text{Skipped}(k).
\]

\(^4\) This means that: (i) \( h_i > k \); (ii) the parameter \( \gamma_{h_i} \) was large enough so that, upon considering the possible choices of pairs for the point at height \( h_i \), we passed the point at height \( k \) before arriving at the choice dictated by \( \gamma_{h_i} \).
This is because (as seen by examining the various possible cases) every new chord which is drawn in our construction of \( \pi \) intersects precisely \( \text{Skipped}(k) \) of the previously drawn chords, where \( k \) is the height of the “1”-decorated endpoint of the new chord. When we sum over \( k \in \varepsilon^{-1}(1) \), we will thus consider all the intersections between chords in the drawing of \( \pi \), with every such intersection counted exactly once.

But on the other hand, the sum on the right-hand side of (2.10) just gives the total number of “skips” that were made during the construction of \( \pi \). The convention (explained at the end of Example 2.3.2) for how we choose a “1”-decorated point at every step of our construction makes clear that this total number of skips is equal to

\[
\sum_{h \in \varepsilon^{-1}(\ast)} (\gamma_h - 1)
\]

(where we now organize our counting according to the “∗”-decorated endpoints of the chords of \( \pi \)). By replacing the quantity (2.11) on the right-hand side of Equation (2.10), we obtain the formula for \( \text{cr}(\pi) \) stated in the proposition. ■

3. A two-sided \( q \)-Wick formula, and proof of Theorem 1.4

3.1. Review of \( q \)-creation and \( q \)-annihilation operators.

Throughout this subsection we fix a positive integer \( d \) and a real number \( q \in (-1, 1) \). We start from the finite dimensional Hilbert space \( \mathbb{C}^d \) and we consider the \( q \)-deformed Fock space over it, as defined by Bożejko and Speicher [2]. We will denote this \( q \)-deformed Fock space by \( \mathcal{T}_{dq} \). It is described as follows.

- First, for every \( n \in \mathbb{N} \) one considers the inner product \( \langle \cdot, \cdot \rangle_q \) on \( (\mathbb{C}^d)^{\oplus n} \) which is determined by the requirement that

\[
\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n \rangle_q = \sum_{\tau \in S_n} \langle v_1, w_{\tau(1)} \rangle \cdots \langle v_n, w_{\tau(n)} \rangle q^{\epsilon(\tau)},
\]

where \( S_n \) is the group of permutations of \( \{1, \ldots, n\} \) and where for \( \tau \in S_n \) we put \( \epsilon(\tau) = |\{(i, j) \mid 1 \leq i < j \leq n, \tau(i) > \tau(j)\}| \). The fact that Equation (3.1) defines indeed an inner product is proved in [2].

- \( \mathcal{T}_{dq} \) is then defined to be the Hilbert space

\[
\mathcal{T}_{dq} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} (\mathbb{C}^d)^{\oplus n} \quad \text{(orthogonal direct sum),}
\]

where every summand \( (\mathbb{C}^d)^{\oplus n} \) is considered with the inner product from (3.1). The number \( 1 \in \mathbb{C} \) in the first summand on the right-hand side of (3.2) is called the vacuum vector of \( \mathcal{T}_{dq} \), and will be denoted by \( \xi_{\text{vac}} \).

For every \( v \in \mathbb{C}^d \) one can verify that there exist operators \( L(v), R(v) \in B(\mathcal{T}_{dq}) \), called left \( q \)-creation operator and respectively right \( q \)-creation operator associated to \( v \), which are determined by the requirements that \( [L(v)](\xi_{\text{vac}}) = v = [R(v)](\xi_{\text{vac}}) \) and that

\[
\begin{cases}
[L(v)](v_1 \otimes \cdots \otimes v_n) = v \otimes v_1 \otimes \cdots \otimes v_n, & \forall n \in \mathbb{N} \text{ and } v_1, \ldots, v_n \in \mathbb{C}^d, \\
[R(v)](v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n \otimes v, & \forall n \in \mathbb{N} \text{ and } v_1, \ldots, v_n \in \mathbb{C}^d.
\end{cases}
\]

The adjoints of \( L(v) \) and \( R(v) \) are called left \( q \)-annihilation operator and respectively right \( q \)-annihilation operator associated to \( v \). They will be denoted as \( L^*(v) \) and \( R^*(v) \), and they
act as follows: 
\[ [L^*(v)](\xi_{\text{vac}}) = 0 = [R^*(v)](\xi_{\text{vac}}), \]
and
\[
\begin{cases}
[L^*(v)](v_1 \otimes \cdots \otimes v_n) = \sum_{k=1}^n q^{-k-1}(v_k, v)v_1 \otimes \cdots \otimes v_{k-1} \otimes v_{k+1} \otimes \cdots \otimes v_n, \\
[R^*(v)](v_1 \otimes \cdots \otimes v_n) = \sum_{k=1}^n q^{k-1}\langle v_{n-k+1}, v \rangle v_1 \otimes \cdots \otimes v_{n-k} \otimes v_{n-k+2} \otimes \cdots \otimes v_n,
\end{cases}
\]
for all \( n \in \mathbb{N} \) and \( v_1, \ldots, v_n \in \mathbb{C}^d \).

The left/right operators of \( q \)-creation and \( q \)-annihilation on \( \mathcal{T}_{d^2} \) are known to satisfy a number of commutation relations. Among these, it is of interest for the present paper to record the one stated in the following lemma (where we use the standard notation \( [X, Y] := XY - YX \) for \( X, Y \in B(\mathcal{T}_{d^2}) \)).

**Lemma 3.1.1.** For every \( v, w \in \mathbb{C}^d \) one has

\[ [L(v) + L^*(v), R(w) + R^*(w)] = (\langle w, v \rangle - \langle v, w \rangle) Q, \]
where \( Q \in B(\mathcal{T}_{d^2}) \) is the operator determined by the requirement that \( Q\xi_{\text{vac}} = \xi_{\text{vac}} \) and that

\[ Q(v_1 \otimes \cdots \otimes v_n) = q^n v_1 \otimes \cdots \otimes v_n, \quad \forall n \in \mathbb{N} \text{ and } v_1, \ldots, v_n \in \mathbb{C}^d. \]

**Proof.** By comparing how \( L^*(v)R(w) \) and \( R(w)L^*(v) \) act on tensors \( v_1 \otimes \cdots \otimes v_n \), one immediately finds that \( [L^*(v), R(w)] = \langle w, v \rangle Q \). A similar calculation leads to \( [L(v), R^*(w)] = -\langle v, w \rangle Q \). Since it is immediate that \( [L(v), R(w)] = [L^*(v), R^*(w)] = 0 \), the required formula (3.3) follows from the bilinearity of the commutator. \( \square \)

**Corollary 3.1.2.** Let \( v, w \in \mathbb{C}^d \) be such that \( \langle v, w \rangle \in \mathbb{R} \). Then \( L(v) + L^*(v) \) commutes with \( R(w) + R^*(w) \) and, consequently, the product \( (L(v) + L^*(v)) \cdot (R(w) + R^*(w)) \) is a selfadjoint operator. \( \square \)

We conclude this review with a notation that will be useful in the next subsection: a \( q \)-annihilation operator is a weighted sum of “annihilations at specified distance” (counting from the left or from the right, as needed), and we want to have names for the individual terms of this weighted sum.

**Notation and Remark 3.1.3.** Let \( \mathcal{T}^{(\text{alg})}_{d^2} \) be the dense subspace of \( \mathcal{T}_{d^2} \) which is obtained by only considering in Equation (3.2) the algebraic direct sum of the spaces \( (\mathbb{C}^d)^{\otimes n} \). For \( v \in \mathbb{C}^d \) and \( k \in \mathbb{N} \) we will denote by \( A_k(v) \) the linear operator on \( \mathcal{T}^{(\text{alg})}_{d^2} \) determined by the requirements that \( [A_k(v)](\xi_{\text{vac}}) = 0 \) and that for \( n \in \mathbb{N} \) and \( v_1, \ldots, v_n \in \mathbb{C}^d \) we have

\[ [A_k(v)](v_1 \otimes \cdots \otimes v_n) = \begin{cases} 0, & \text{if } n < k; \\
q^{k-1}\langle v_k, v \rangle v_1 \otimes \cdots \otimes v_{k-1} \otimes v_{k+1} \otimes \cdots \otimes v_n, & \text{if } n \geq k.
\end{cases} \]

The “(*)” included in the notation is merely a reminder that \( A_k(v) \) will be viewed as a piece of the annihilation operator \( L^*(v) \) (but we are not trying to identify \( A_k(v) \) as the adjoint of some other operator). It is clear that one has the formula

\[ [L^*(v)](v_1 \otimes \cdots \otimes v_n) = \sum_{k=1}^n [A_k(v)](v_1 \otimes \cdots \otimes v_n), \]
holding for every \( n \in \mathbb{N} \) and \( v, v_1, \ldots, v_n \in \mathbb{C}^d \).

For the sake of uniformity in notations, for every \( v \in \mathbb{C}^d \) we will also consider a linear operator \( A^{(1)}_1(v) \) on \( \mathcal{T}^{(\text{alg})}_{d^2} \), which is defined by

\[ [A^{(1)}_1(v)](\xi) = [L(v)](\xi), \quad \forall \xi \in \mathcal{T}^{(\text{alg})}_{d^2}. \]
The point of this extra notation is that we get to have a full collection of linear operators

\[ A_k^{(\theta)}(v), \text{ defined for } v \in \mathbb{C}^d, \theta \in \{1, *\} \text{ and certain } k \in \mathbb{N} \]
(where if \( \theta = * \) then \( k \) can take any value in \( \mathbb{N} \), but if \( \theta = 1 \) then we are forced to put \( k = 1 \)).

Symmetrically, we consider the collection of linear operators

\[ B_k^{(\theta)}(v), \text{ defined for } v \in \mathbb{C}^d, \theta \in \{1, *\} \text{ and certain } k \in \mathbb{N} \]
where for \( \theta = 1 \) we impose \( k = 1 \) and define \( B_1^{(1)}(v) \) to be the restriction of \( R(v) \) to \( T_{d,q}^{(\text{alg})} \),
while for \( \theta = * \) and arbitrary \( k \in \mathbb{N} \) we use the formula

\[
[B_k^{(*)}(v)](v_1 \otimes \cdots \otimes v_n) = \begin{cases} 
0, & \text{if } n < k; \\
q^{k-1}(v_{n-k+1}, v)v_1 \otimes \cdots \otimes v_{n-k} \otimes v_{n-k+2} \otimes \cdots \otimes v_n, & \text{if } n \geq k.
\end{cases}
\]

The operators \( B_k^{(*)}(v) \) are “pieces of the right \( q \)-annihilation operator associated to \( v \),” in the sense that one has

\[
[R^*(v)](v_1 \otimes \cdots \otimes v_n) = \sum_{k=1}^{n} [B_k^{(*)}(v)](v_1 \otimes \cdots \otimes v_n),
\]
holding for every \( n \in \mathbb{N} \) and \( v, v_1, \ldots, v_n \in \mathbb{C}^d \).

### 3.2. A two-sided \( q \)-Wick formula for an alternating left-right monomial.

In this subsection we fix the following data: a positive integer \( d \) and a real number \( q \in (-1, 1) \) (same as in subsection 3.1), and also:

- a positive integer \( n \);
- a tuple \( \varepsilon = (\varepsilon(1), \ldots, \varepsilon(2n)) \in D_{(1,*)}(2n) \) (with \( D_{(1,*)}(2n) \) as defined in Definition 2.2.1);
- a family of vectors \( u_1, \ldots, u_{2n} \in \mathbb{C}^d \).

We will use the notations related to \( q \)-creation and \( q \)-annihilation operators on the deformed Fock space \( T_{d,q} \) that were introduced in the preceding subsection, and the various notations and facts pertaining to \( \varepsilon \) that were discussed in Section 2.

In reference to the data fixed above, we consider the product of operators

\[
M := R^\varepsilon(2n)(u_{2n}) \cdot L^\varepsilon(2n-1)(u_{2n-1}) \cdots R^\varepsilon(2)(u_2) \cdot L^\varepsilon(1)(u_1) \in B(T_{d,q}),
\]
and we will examine the action of \( M \) on the vacuum vector \( \xi_{\text{vac}} \in T_{d,q} \). It is quite easy (by keeping in mind how \( D_{(1,*)}(2n) \) is defined) to see that \( \xi_{\text{vac}} \) is an eigenvector for \( M \); the goal of the present subsection is to describe explicitly what is the corresponding eigenvalue. The eigenvalue will appear (cf. Proposition 3.2.6 below) as a sum over pair-partitions, giving a formula of “\( \text{Wick} \)” type.

The fact that \( M \) in (3.7) has alternating “\( L \)” and “\( R \)” factors is due to the intended use of \( M \), towards finding a relation to semi-meander polynomials. It is actually easy to extend the result of Proposition 3.2.6 to a two-sided \( q \)-Wick formula for general products of operators \( L^\theta(v) \) and \( R^\theta(v) \), with \( \theta \in \{1, *\} \) and \( v \in \mathbb{C}^d \) – see Section 4.1 below.

**Notation 3.2.1.** Recall that for every \( h \in \varepsilon^{-1}(*) \subseteq \{2, 3, \ldots, 2n\} \) we defined a choice number \( \text{Choice}_\varepsilon(h) \in \mathbb{N} \) (cf. Notation 2.3.1). We will denote

\[
\Gamma := \left\{ (\gamma_1, \ldots, \gamma_{2n}) \in \mathbb{N}^{2n} \right| \begin{array}{l}
\text{if } h \in \varepsilon^{-1}(1), \text{ then } \gamma_h = 1; \\
\text{if } h \in \varepsilon^{-1}(*) \text{, then } 1 \leq \gamma_h \leq \text{Choice}_\varepsilon(h)
\end{array} \right\}.
\]

Note that a \((2n)\)-tuple in \( \Gamma \) is nothing but a tuple of choices \((\gamma_h)_{h \in \varepsilon^{-1}(*)}\) as considered in (2.7) of Remark 2.3.1, where we “filled in some values of 1” by putting \( \gamma_k = 1 \) for \( k \in \varepsilon^{-1}(1) \).
Thus $\Gamma$ is in natural bijection with the set of pair-partitions $\Phi_n^{-1}(\varepsilon) \subseteq \mathcal{P}_2(2n)$, in the way discussed in Remark 2.3.1 and in Example 2.3.2.

The role of $\Gamma$ in calculations related to $M(\xi_{\text{vac}})$ shows up in the following lemma.

**Lemma 3.2.2.** One has

\begin{equation}
M(\xi_{\text{vac}}) =
\sum_{(\gamma_1, \ldots, \gamma_{2n}) \in \Gamma} \left[ B^{(2n)}_{\gamma_2}(u_{2n}) \cdot A^{(2n-1)}_{\gamma_2}(u_{2n-1}) \cdots B^{(2)}_{\gamma_2}(u_2) \cdot A^{(1)}_{\gamma_1}(u_1) \right](\xi_{\text{vac}}),
\end{equation}

where the operators of the form $A^{(\theta)}_k(v)$ and $B^{(\theta)}_k(v)$ are as in Notation 3.1.3.

**Proof.** Referring to the definition of $M$ in Equation (3.7), consider the vectors $\xi_0, \xi_1, \ldots, \xi_{2n} \in T^{(\text{alg})}_{d,q}$ obtained by putting $\xi_0 := \xi_{\text{vac}}$ and then

\begin{align}
\xi_1 &= [L^{(1)}(u_1)](\xi_0), \quad \xi_2 = [R^{(2)}(u_2)](\xi_1), \ldots, \\
\xi_{2n-1} &= [L^{(2n-1)}(u_{2n-1})](\xi_{2n-2}), \quad \xi_{2n} = [R^{(2n)}(u_{2n})](\xi_{2n-1}).
\end{align}

Clearly, $M(\xi_{\text{vac}}) = \xi_{2n}$.

For every $p \in \mathbb{N} \cup \{0\}$, let $\mathcal{V}_p$ denote the copy of $(\mathbb{C}^d) \otimes p$ which sits inside $T^{(\text{alg})}_{d,q}$. In view of how $q$-creation and $q$-annihilation operators act on a space $\mathcal{V}_p$ (by mapping it to $\mathcal{V}_{p+1}$ and respectively $\mathcal{V}_{p-1}$), it is immediate that the vectors introduced in (3.10) are such that $\xi_1 \in \mathcal{V}_{p_1}, \xi_2 \in \mathcal{V}_{p_2}, \ldots, \xi_{2n} \in \mathcal{V}_{p_{2n}}$, with

\begin{equation}
p_h := \left| \left\{ 1 \leq i \leq h \mid \varepsilon(i) = 1 \right\} \right| - \left| \left\{ 1 \leq i \leq h \mid \varepsilon(i) = * \right\} \right|, \quad \text{for } 1 \leq h \leq 2n.
\end{equation}

For every $h \in \{1, \ldots, 2n\}$ such that $\varepsilon(h) = 1$ we have

\begin{equation}
\xi_h = \begin{cases} 
[L(u_h)](\xi_{h-1}), & \text{if } h \text{ is odd} \\
[R(u_h)](\xi_{h-1}), & \text{if } h \text{ is even}
\end{cases} = \begin{cases} 
[A^{(1)}_1(u_h)](\xi_{h-1}), & \text{if } h \text{ is odd} \\
[B^{(1)}_1(u_h)](\xi_{h-1}), & \text{if } h \text{ is even}
\end{cases},
\end{equation}

On the other hand, for every $h \in \{1, \ldots, 2n\}$ such that $\varepsilon(h) = *$ we have

\begin{equation}
\xi_h = \begin{cases} 
[L^*(u_h)](\xi_{h-1}), & \text{if } h \text{ is odd} \\
[R^*(u_h)](\xi_{h-1}), & \text{if } h \text{ is even}
\end{cases} = \sum_{\gamma=1}^{p_{h-1}} [A^{(*)}_{\gamma}(u_h)](\xi_{h-1}), \quad \text{if } h \text{ is odd} \quad \sum_{\gamma=1}^{p_{h-1}} [B^{(*)}_{\gamma}(u_h)](\xi_{h-1}), \quad \text{if } h \text{ is even}.
\end{equation}

In (3.13) we took into account that the action of an operator $L^*(v)$ on a specified $\mathcal{V}_p$ can be replaced by the action of $\sum_{\gamma=1}^{p} A^{(*)}_{\gamma}(v)$ (cf. Equation (3.5) in Remark 3.1.3), with a similar statement holding for $R^*(v)$ and $\sum_{\gamma=1}^{p} B^{(*)}_{\gamma}(v)$.

In connection to Equation (3.13) we also make the remark that $p_{h-1} = \text{Choice}_e(h)$ (compare the definition of $p_{h-1}$ to Equation (2.6) in Remark 2.3.1); hence the sums indicated in (3.13) have precisely $\text{Choice}_e(h)$ terms.

When we write the recursion for the vectors $\xi_1, \xi_2, \ldots, \xi_{2n}$ by using the Equations (3.12) and (3.13), we get to have $\xi_{2n}$ written precisely as on the right-hand side of Equation (3.9). Since we know that $\xi_{2n} = M(\xi_{\text{vac}})$, this concludes the proof. $\blacksquare$

We now concentrate our attention on a tuple $(\gamma_1, \ldots, \gamma_{2n}) \in \Gamma$. We know that $(\gamma_h)_{h \in \varepsilon^{-1}(*)}$ is a tuple of choices which parametrizes a pair-partition $\pi \in \Phi_n^{-1}(\varepsilon) \subseteq \mathcal{P}_2(2n)$, where the map $\Phi_n : \mathcal{P}_2(2n) \to \mathcal{P}_{(1,*)}(2n)$ is as described in Notation 2.2.3, and where the procedure for finding $\pi$ is described in detail in Remark 2.3.1 and Example 2.3.2.
Lemma 3.2.3. With $\gamma \in \Gamma$ and $\pi \in \Phi_n^{-1}(\varepsilon)$ as above, we have
\begin{equation}
[B_{(\varepsilon(2n))}(u_{2n}) \cdot A_{(\varepsilon(2n-1))}(u_{2n-1}) \cdots B_{(\varepsilon(2))}(u_2) \cdot A_{(\varepsilon(1))}(u_1)](\xi_{\text{vac}}) = c \xi_{\text{vac}},
\end{equation}
where the scalar $c$ is described as follows:
\begin{equation}
c = \prod_{\{k,h\} \text{ pair in } S_n, \pi \text{ with } \varepsilon(h)=* \text{, } \varepsilon(k)=1} \left( \langle u_k, u_h \rangle \cdot q^{\gamma h-1} \right).
\end{equation}

Proof. Consider the vectors $\eta_0, \eta_1, \ldots, \eta_{2n} \in T^{(\text{alg})}_{d \cdot q}$ obtained by putting $\eta_0 := \xi_{\text{vac}}$ and then
\begin{equation}
\begin{cases}
\eta_1 = [A_{(\varepsilon(1))}(u_1)](\eta_0), & \eta_2 = [B_{(\varepsilon(2))}(u_2)](\eta_1), \ldots, \\
\eta_{2n-1} = [A_{(\varepsilon(2n-1))}(u_{2n-1})](\eta_{2n-2}), & \eta_{2n} = [B_{(\varepsilon(2n))}(u_{2n})](\eta_{2n-1}).
\end{cases}
\end{equation}
The vector on the left-hand side of Equation (3.14) is then $\eta_{2n}$.

We next observe that $\eta_1 \in V_{p_1}, \eta_2 \in V_{p_2}, \ldots, \eta_{2n} \in V_{p_{2n}}$, where $p_1, \ldots, p_{2n} \in \mathbb{N} \cup \{0\}$ are exactly as in Equation (3.11) in the proof of the preceding lemma, and the spaces “$V_p$” have the same meaning as in the said proof. The specifics of how the operators $A_{(\varepsilon)}(v)$ and $B_{(\varepsilon)}(v)$ act on tensors actually give us that every $\eta_h$ is of the form
\begin{equation}
\eta_h = c_h \zeta_h,
\end{equation}
where $c_h \in \mathbb{C}$ and $\zeta_h$ is a tensor product of $p_h$ vectors \footnote{It may happen (e.g. for $h = 2n$) that $p_h = 0$; in such a case, the vector $\zeta_h$ of Equation (3.17) is $\zeta_h = \xi_{\text{vac}}$.} picked (in some order) out of $u_1, \ldots, u_h$. We can, moreover, follow on how $c_h$ and $\zeta_h$ are obtained from $c_{h-1}$ and $\zeta_{h-1}$, depending on whether $\varepsilon(h)$ is a “1” or a “∗”:
\begin{itemize}
\item If $\varepsilon(h) = 1$ then $\zeta_h$ is obtained out of $\zeta_{h-1}$ by adding a component (either at the left end or at the right end of the tensor), and we have $c_h = c_{h-1}$.
\item If $\varepsilon(h) = *$ then $\zeta_h$ is obtained out of $\zeta_{h-1}$ by removing a component “$u_k$” (which could be anywhere in the tensor), and we have $c_h = \langle u_k, u_h \rangle \cdot q^{\gamma h-1} \cdot c_{h-1}$.
\end{itemize}

We leave it as an exercise to the reader to check that the above procedure of adding and removing components in the tensors $\zeta_h$ corresponds precisely to the procedure used in Section 2 in order to obtain the pair-partition $\pi \in \Phi_n^{-1}(\varepsilon)$ which is encoded by the choicenumbers $(\gamma_h)_{h \in \varepsilon^{-1}(*)}$. In particular, the scalar $c_{2n}$ comes out to be a product indexed by pairs of the partition $S_n \cdot \pi$, exactly in the way stated on the right-hand side of Equation (3.15). Since $\zeta_{2n} = \xi_{\text{vac}}$ and $\eta_{2n}$ is the vector on the left-hand side of (3.14), this concludes the proof. \hfill \Box

Example 3.2.4. For illustration, we show how the procedure explained in the proof of Lemma 3.2.3 works on a concrete example. Let us consider again the situation discussed in Example 2.3.2, where $n = 5$, $\varepsilon = (1, 1, 1, *, *, 1, 1, *, *, *, *) \in D_{(1, *)}(10)$, and the tuple $(\gamma_1, \ldots, \gamma_{10}) \in \Gamma$ used for Figure 7 is $(1, 1, 1, 3, 1, 1, 1, 2, 2, 1)$. Thus the vector considered in Lemma 3.2.3 is, in this concrete example:
\begin{equation}
[B_{(1)}(u_{10}) A_{(2)}(u_9) B_{(2)}(u_8) A_{(1)}(u_7) B_{(1)}(u_6) A_{(1)}(u_5)](\xi_{\text{vac}}) \times
\end{equation}
\begin{equation}
\times B_{(3)}(u_4) A_{(1)}(u_3) B_{(1)}(u_2) A_{(1)}(u_1))(\xi_{\text{vac}}).
\end{equation}
The last 3 of the 10 factors in the above product are creation operators, and they map \( \xi_{\text{vac}} \) to \( c_3 \zeta_3 \), where \( c_3 = 1 \) and \( \zeta_3 = u_3 \otimes u_1 \otimes u_2 \). Let us examine what happens when the next 2 factors of the product (both of them doing annihilation) are applied. We have:

\[
[B_3^*(u_4)](u_3 \otimes u_1 \otimes u_2) = \left( q^2 \langle u_3, u_4 \rangle \right) u_1 \otimes u_2,
\]
and then

\[
[A_1^*(u_5)](u_1 \otimes u_2) = \left( \langle u_1, u_5 \rangle \right) u_2;
\]

hence \( \zeta_5 = u_2 \) and \( c_5 = \langle u_1, u_5 \rangle \langle u_3, u_4 \rangle q^2 \). Observe that this corresponds exactly (in the respect of what points are being paired in the rectangular picture, and also in the respect of "how many points are skipped" at every step, and from what direction) to the construction shown in Figure 6. In particular, the equality \( \zeta_5 = u_2 \) corresponds to the fact that the point at height 2 is still not paired in Figure 6, while the other four points at heights \( \leq 5 \) are paired – height 3 paired with height 4, then height 1 paired with height 5, corresponding to the two inner products multiplied in the formula for \( c_5 \).

The patient reader can verify that when we apply to \( u_3 \) the next 3 factors from the product (3.18), we arrive to calculate

\[
[B_2^*(u_8)](u_7 \otimes u_2 \otimes u_6) = \left( q \langle u_2, u_8 \rangle \right) u_7 \otimes u_6,
\]

and this corresponds exactly to performing the next step (choosing a pair for the point at height 8) in the construction which led to Figure 7. Finally, upon applying the remaining 2 factors \( B_1^*(u_{10})A_2^*(u_9) \) to \( u_7 \otimes u_6 \) we arrive to the vector \( \eta_{10} = c_{10} \zeta_{10} \), where \( \zeta_{10} = \xi_{\text{vac}} \)

\[
c_{10} = \langle u_7, u_{10} \rangle \langle u_6, u_9 \rangle \langle u_2, u_8 \rangle \langle u_1, u_5 \rangle \langle u_3, u_4 \rangle q^4,
\]

exactly as claimed in Lemma 3.2.3.

**Remark 3.2.5.** In the framework of Lemma 3.2.3, one has

\[
\prod_{\{k,h\} \text{ pair in } \mathfrak{z}_n, \pi} q^{\gamma_{n}-1} = \prod_{h \in \varepsilon^{-1}(*)} q^{\gamma_{h}-1} = q^{c_{\pi}},
\]

where at the latter equality sign we invoked Proposition 2.3.3. Thus the scalar \( c \) appearing in Equation (3.15) of Lemma 3.2.3 can also be written as

\[
c = q^{c_{\pi}} \cdot \prod_{\{k,h\} \text{ pair in } \mathfrak{z}_n, \pi} \langle u_k, u_h \rangle \cdot \langle u_1, u_5 \rangle \langle u_3, u_4 \rangle q^4.
\]

The Wick type formula announced in the title of the subsection is then stated as follows.

**Proposition 3.2.6.** Let \( M \) be the operator defined in Equation (3.7). Then

\[
M(\xi_{\text{vac}}) = \sum_{\pi \in \Phi_n^{-1}(\varepsilon)} \left[ q^{c_{\pi}} \cdot \prod_{\{k,h\} \text{ pair in } \mathfrak{z}_n, \pi} \langle u_k, u_h \rangle \right] \cdot \xi_{\text{vac}}.
\]

**Proof.** This follows by combining Lemmas 3.2.2 and 3.2.3, where we use the natural bijection between \( \Gamma \) and \( \Phi_n^{-1}(\varepsilon) \) and the formula (3.19) noticed in Remark 3.2.5. \( \square \)
Remark 3.2.7. The $q$-Wick formula is usually stated as saying that $\varphi_{\operatorname{vac}}(M)$ is equal to the scalar which amplifies $\xi_{\operatorname{vac}}$ on the right-hand side of Equation (3.20).

We note here that, if we focus on $\varphi_{\operatorname{vac}}(M)$ rather than on the vector $M(\xi_{\operatorname{vac}})$, it is easy to extend the two-sided $q$-Wick formula to the framework where the tuple $\varepsilon$ would be picked in $\{1, \ast\}^{2n}$ without assuming (as we did throughout this subsection) that $\varepsilon \in D_{\{1, \ast\}}(2n)$. Indeed, if $\varepsilon$ was to be picked from $\{1, \ast\}^{2n} \setminus D_{\{1, \ast\}}(2n)$, then it is immediate that the vector $M(\xi_{\operatorname{vac}})$ would come out orthogonal to $\xi_{\operatorname{vac}}$, thus giving $\varphi_{\operatorname{vac}}(M) = 0$.

3.3. Alternating monomials in $L_{i,q}^0$ and $R_{i,q}^0$.

In this subsection we continue to keep fixed the $d \in \mathbb{N}$, $q \in (-1, 1)$, $n \in \mathbb{N}$ and $\varepsilon \in D_{\{1, \ast\}}(2n)$ from the preceding subsection, but not the vectors $u_1, \ldots, u_{2n} \in \mathbb{C}^d$ that were used there. We will do some further processing of the formula obtained in Proposition 3.2.6, in the special case when $u_1, \ldots, u_{2n}$ are picked from the standard orthonormal basis of $\mathbb{C}^d$.

So for $1 \leq i \leq d$, let $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)$ with the entry of 1 on position $i$, and let us put, consistently with the notations used in the Introduction,

$$L_{i,q} := L(e_i), \quad R_{i,q} := R(e_i).$$

Consider a $(2n)$-tuple $I : \{1, \ldots, 2n\} \to \{1, \ldots, d\}$ and let us examine the $q$-Wick formula from Proposition 3.2.6 in the case when $u_1 = e_{I(1)}, \ldots, u_{2n} = e_{I(2n)}$. Each of the inner products $\langle u_h, u_k \rangle$ appearing in the formula is equal to 0 or to 1, and asking that all these inner products are equal to 1 amounts to a compatibility condition between $I$ and $\pi$, which we record in the following (ad-hoc) definition.

Definition 3.3.1. We will say that a $(2n)$-tuple $I : \{1, \ldots, 2n\} \to \{1, \ldots, d\}$ and a pair-partition $\pi \in P_2(2n)$ are height-compatible to mean that the following condition is fulfilled:

$$I(h) = I(k), \quad \text{whenever } \{h, k\} \text{ is a pair of } (\varepsilon_n \cdot \pi).$$

Remark 3.3.2. Note that Definition 3.3.1 does not involve the tuple $\varepsilon \in D_{\{1, \ast\}}(2n)$, it just prescribes a relation between $I$ and $\pi$. If we fix a $\pi \in P_2(2n)$, then height-compatibility amounts to a system of $n$ equations that have to be satisfied by a $(2n)$-tuple $I$ (it is clear, in particular, that there are precisely $d^n$ tuples $I$ which are height-compatible to any given $\pi \in P_2(2n)$).

For illustration, say for instance that $n = 5$ and $\pi \in P_2(10)$ as is depicted in Figure 4. Then an $I : \{1, \ldots, 10\} \to \{1, \ldots, d\}$ is height-compatible with this $\pi$ if and only if it satisfies

$$I(1) = I(4), \quad I(2) = I(5), \quad I(3) = I(8), \quad I(6) = I(10), \quad I(7) = I(9).$$

When using the terminology from Definition 3.3.1, the special case of the two-sided $q$-Wick formula that we are interested in takes the following form.

Corollary 3.3.3. Consider the data $d \in \mathbb{N}$, $q \in (-1, 1)$, $n \in \mathbb{N}$ and $\varepsilon \in D_{\{1, \ast\}}(2n)$ fixed in this subsection. For every $(2n)$-tuple $I : \{1, \ldots, 2n\} \to \{1, \ldots, d\}$ we have

$$\varphi_{\operatorname{vac}}(Q_{2n}^{\varepsilon}L_{1(2n);q}^{\varepsilon(2n-1)} \cdots L_{1(2);q}^{\varepsilon(2)} L_{1(1);q}^{\varepsilon(1)}) = \sum_{\pi \in \Phi_n^1(\varepsilon), \text{height-compatible with } I} q^\epsilon(\pi).$$

(The index set in the summation on the right-hand side of (3.22) may be empty, in which case the corresponding sum is equal to 0.)

---

6 But things are not so nice if, vice-versa, an $I$ is given and we count the pair-partitions $\pi \in P_2(2n)$ which are height-compatible to it — the answer can vary between “0” (e.g. when $I(1) = I(2) = \cdots = I(2n-1) \neq I(2n)$) and “$(2n - 1)!$” (obtained when $I(1) = I(2) = \cdots = I(2n)$).
Proof. As an obvious consequence of Proposition 3.2.6 and of how the definition of height-compatibility was made, we get

$$\begin{align*}
(3.23) \quad [R_{1(2n);q}^{ε(2n)} I_{1(2n-1);q}^{ε(2n-1)} \cdots R_{1(2);q}^{ε(2)} L_{1(1);q}^{ε(1)}](ξ_{\text{vac}}) &= \sum_{π \in Φ_n^{-1}(ε)} q^{\text{cr}(π)} \cdot ξ_{\text{vac}},
\end{align*}$$

which in turn implies the formula stated in the corollary.

3.4. Proof of Theorem 1.4.
In this subsection we fix $d \in \mathbb{N}$ and $q \in (-1,1)$, and we will use the various notations pertaining to this $d$ and $q$ that were considered in Sections 3.1-3.3. Our goal is to prove Theorem 1.4 from the Introduction.

The theorem refers to the polynomials $\tilde{Q}_n(t,u)$, which were introduced in Notation 1.3. Since the definition of $\tilde{Q}_n$ relies on the concepts of “number of crossings” and “number of closed curves” for a self-intersecting meandric system, we start by indicating the precise formulas for these numbers. So let $n$ be a positive integer, let $π, σ \in \mathcal{P}(2n)$, and consider the self-intersecting meandric system which is obtained by drawing $π$ above a horizontal line and $σ$ under it (as in Figure 2 in the Introduction). Then:

• the number of crossings of the said self-intersecting meandric system is $\text{cr}(π) + \text{cr}(σ)$, and
• its number of closed curves is defined as $|π ∨ σ|$ (in words: the number of blocks of the join of $π$ and $σ$ in the lattice $\mathcal{P}(2n)$), where the join operation “$∨$” and a few other useful terms related to partitions are reviewed in the next definition.

Definition and Remark 3.4.1. Let $m$ be a positive integer.

(1) We denote by $\mathcal{P}(m)$ the set of all partitions of $\{1, \ldots, m\}$. A $π \in \mathcal{P}(m)$ is thus of the form $π = \{V_1, \ldots, V_k\}$, where the sets $V_1, \ldots, V_k$ (called blocks of $π$) are non-empty sets such that $\bigcup_{i=1}^k V_i = \{1, \ldots, m\}$ and $V_i \cap V_j = \emptyset$ for $i \neq j$.

(2) Let $π$ be in $\mathcal{P}(m)$. We will use the notation $|π|$ for the number of blocks of $π$. We will also use the notation “$\sim$” for the equivalence relation determined by $π$ on $\{1, \ldots, m\}$; that is, for $a, b \in \{1, \ldots, m\}$ we will write $a \sim b$ to mean that $a$ and $b$ belong to the same block of $π$.

(3) On $\mathcal{P}(m)$ we will use the partial order given by reverse refinement, where $π \leq π'$ means by definition that every block of $π'$ can be written as a union of blocks of $π$.

(4) Any two partitions $π, σ \in \mathcal{P}(m)$ have a least common upper bound, denoted as $π ∨ σ$, with respect to the reverse refinement order. The partition $π ∨ σ$ is called the join of $π$ and $σ$. It is easily verified that its blocks can be explicitly described as follows: two numbers $a, b \in \{1, \ldots, m\}$ belong to the same block of $π ∨ σ$ if and only if there exist $k \in \mathbb{N}$ and $a_0, a_1, \ldots, a_{2k} \in \{1, \ldots, m\}$ such that $a = a_0 \sim a_1 \sim a_2 \sim \cdots \sim a_{2k-1} \sim a_{2k} = b$.

(5) Let $s$ be a permutation of $\{1, \ldots, m\}$, and consider the natural action of $s$ on $\mathcal{P}(m)$ (if $π = \{V_1, \ldots, V_k\}$, then $s \cdot π = \{s(V_1), \ldots, s(V_k)\}$). This action respects the join operation, that is:

$$\begin{align*}
(3.24) \quad s \cdot (π ∨ σ) &= (s \cdot π) ∨ (s \cdot σ), \quad ∀ π, σ \in \mathcal{P}(m).
\end{align*}$$
Remark 3.4.2. Let \( n \) be in \( \mathbb{N} \) and consider the set of self-intersecting meandric systems \( \mathcal{R}_n \) from Notation 1.3. Every such system is parametrized by a \( \pi \in P_2(2n) \) (namely the \( \pi \) which appears, in the drawing of the system, above the horizontal line), and has:

- number of crossings equal to \( cr(\pi) \) (since the rainbow pair-partition has \( cr(\rho_{2n}) = 0 \));
- number of closed curves equal to \( |\pi \lor \rho_{2n}| \).

It is immediately seen that, upon plugging these numbers into the Equation (1.8) which was used to define the polynomial \( \tilde{Q}_n(t, u) \) in Notation 1.3, one comes to the formula

\[
(3.25) \quad \tilde{Q}_n(t, u) = \sum_{\pi \in P_2(2n)} t^{|\pi \lor \rho_{2n}|} u^{cr(\pi)}.
\]

We now proceed to the actual proof of Theorem 1.4. We will use the following lemma.

Lemma 3.4.3. Let \( n \) be in \( \mathbb{N} \) and let \( \pi \) be a pair-partition in \( P_2(2n) \). Consider the set of \( (2n) \)-tuples

\[
(3.26) \quad \mathcal{I}_\pi := \left\{ I : \{1, \ldots, 2n\} \rightarrow \{1, \ldots, d\} \mid I(2k - 1) = I(2k), \ \forall 1 \leq k \leq n, \text{ and } I \text{ is height-compatible with } \pi \right\},
\]

where the notion of “height-compatible” is as in Definition 3.3.1. Then \( |\mathcal{I}_\pi| = d^{|\pi \lor \rho_{2n}|} \).

Proof. The first requirement imposed on \( I \) on the right-hand side of (3.26) can be expressed as an inequality with respect to the reverse refinement order on \( P(2n) \), namely

\[ Ker(I) \geq \{ \{1, 2\}, \ldots, \{2n - 1, 2n\} \}, \]

where \( Ker(I) \) denotes the partition of \( \{1, \ldots, 2n\} \) into level-sets of \( I \) (two numbers \( a, b \in \{1, \ldots, 2n\} \) belong to the same block of \( Ker(I) \) if and only if \( I(a) = I(b) \)). Upon reviewing the definition of height-compatibility, we see that the second requirement imposed on \( I \) in (3.26) amounts to the inequality \( Ker(I) \geq \mathcal{s}_n \cdot \pi \). But then, in view of how the join operation is defined on \( P(2n) \), we find that the set \( \mathcal{I}_\pi \) defined in (3.26) can be written as

\[ \mathcal{I}_\pi = \left\{ I : \{1, \ldots, 2n\} \rightarrow \{1, \ldots, d\} \mid Ker(I) \geq \{ \{1, 2\}, \ldots, \{2n - 1, 2n\} \} \lor \mathcal{s}_n(\pi) \right\}. \]

This can be continued with

\[
= \left\{ I : \{1, \ldots, 2n\} \rightarrow \{1, \ldots, d\} \mid Ker(I) \geq (\mathcal{s}_n \cdot \rho_{2n}) \lor (\mathcal{s}_n \cdot \pi) \right\}
\]

(by Equation (2.2) in Notation 2.1.3)

\[
= \left\{ I : \{1, \ldots, 2n\} \rightarrow \{1, \ldots, d\} \mid Ker(I) \geq \mathcal{s}_n \cdot (\rho_{2n} \lor \pi) \right\} \quad \text{(by Eqn. (3.24))}
\]

In the latter formulation, it becomes clear that all the \( (2n) \)-tuples \( I \in \mathcal{I}_\pi \) are obtained, without repetitions, when we prescribe at will one value \( b \in \{1, \ldots, d\} \) for every block of \( \mathcal{s}_n(\rho_{2n} \lor \pi) \), and assign \( I \) to be identically equal to \( b \) on that block. This implies \( |\mathcal{I}_\pi| = d^{|\mathcal{s}_n(\rho_{2n} \lor \pi)|} \), and since \( |\mathcal{s}_n(\rho_{2n} \lor \pi)| = |\rho_{2n} \lor \pi| \), the required formula for \( |\mathcal{I}_\pi| \) follows. \( \blacksquare \)

Proof of Theorem 1.4. The operator \( T_{dq} \) is selfadjoint by Corollary 3.1.2. For the calculation of moments, we must prove (by taking into account the above formula Equation (3.25)) that

\[
(3.27) \quad \varphi_{vac}(T_{dq}) = \sum_{\pi \in P_2(2n)} d^{|\pi \lor \rho_{2n}|} q^{cr(\pi)}, \ \forall n \in \mathbb{N}.
\]

For the remaining part of the proof we fix an \( n \in \mathbb{N} \) for which we will verify that (3.27) holds.
By taking the power \(n\) of the sum which defines \(T_{d,q}^n\), we get

\[
T_{d,q}^n = \left( \sum_{i=1}^{d} (L_{i;q} + L_{i;q}^*) (R_{i;q} + R_{i;q}^*) \right)^n = \left( \sum_{i=1}^{d} (R_{i;q} + R_{i;q}^*) (L_{i;q} + L_{i;q}^*) \right)^n
\]

(3.28)

\[
= \sum_{I: \{1, \ldots, 2n\} \to \{1, \ldots, d\}} \sum_{\varepsilon: \{1, \ldots, 2n\} \to \{1, \ast\}} R_{I(2n);q}^{\varepsilon(2n)} L_{I(2n-1);q}^{\varepsilon(2n-1)} \cdots R_{I(2);q}^{\varepsilon(2)} L_{I(1);q}^{\varepsilon(1)}
\]

Upon applying \(\varphi_{\text{vac}}\) to (3.28) and then invoking an orthogonality observation analogous to the one in the last paragraph of Remark 3.2.7, we find that

\[
\varphi_{\text{vac}}(T_{d,q}^n) = \sum_{I: \{1, \ldots, 2n\} \to \{1, \ldots, d\}} \sum_{\varepsilon: \{1, \ldots, 2n\} \to \{1, \ast\}} \varphi_{\text{vac}} \left( R_{I(2n);q}^{\varepsilon(2n)} L_{I(2n-1);q}^{\varepsilon(2n-1)} \cdots R_{I(2);q}^{\varepsilon(2)} L_{I(1);q}^{\varepsilon(1)} \right).
\]

(3.29)

We next note that for any fixed \(I: \{1, \ldots, 2n\} \to \{1, \ldots, d\}\) we can write:

\[
\sum_{\varepsilon: \{1, \ldots, 2n\} \to \{1, \ast\}} \varphi_{\text{vac}} \left( R_{I(2n);q}^{\varepsilon(2n)} L_{I(2n-1);q}^{\varepsilon(2n-1)} \cdots R_{I(2);q}^{\varepsilon(2)} L_{I(1);q}^{\varepsilon(1)} \right) = \sum_{\varepsilon: \{1, \ldots, 2n\} \to \{1, \ast\}} \left[ \sum_{\pi \in \Phi_{n}^{-1}(\varepsilon), \text{height-compatible with } I} q^{\text{cr}(\pi)} \right] \quad \text{(by Corollary 3.3.3)}
\]

\[
= \sum_{\pi \in \mathcal{P}_2(2n), \text{height-compatible with } I} q^{\text{cr}(\pi)},
\]

where at the latter equality sign we took into account that \(\bigcup_{\varepsilon \in \mathcal{D}(1,*)} \Phi_{n}^{-1}(\varepsilon) = \mathcal{P}_2(2n)\), disjoint union.

When the result of the calculation from the preceding paragraph is plugged into Equation (3.29), we find that

\[
\varphi_{\text{vac}}(T_{d,q}^n) = \sum_{I: \{1, \ldots, 2n\} \to \{1, \ldots, d\}} \left[ \sum_{\pi \in \mathcal{P}_2(2n), \text{height-compatible with } I} q^{\text{cr}(\pi)} \right].
\]

(3.30)

Finally, we change the order of summation on the right-hand side of Equation (3.30). This leads to

\[
\varphi_{\text{vac}}(T_{d,q}^n) = \sum_{\pi \in \mathcal{P}_2(2n)} q^{\text{cr}(\pi)} \cdot |\mathcal{I}_\pi|,
\]

where the set \(\mathcal{I}_\pi \subseteq \{1, \ldots, d\}^{2n}\) is precisely the one considered in Lemma 3.4.3. The required formula for \(\varphi_{\text{vac}}(T_{d,q}^n)\) follows, since Lemma 3.4.3 has established that \(|\mathcal{I}_\pi| = d^{n \vee p_{2n}}\). \(\blacksquare\)

4. Some miscellaneous remarks

4.1. A more general two-sided \(q\)-Wick formula.

The derivation of two-sided \(q\)-Wick formula in Section 3.2 focused on the case needed in our main theorem, where the considered product of operators of the form \(L^q(u), R^q(u)\) has \(2n\) alternating factors "\(L\)" and "\(R\). We will outline here how this formula generalizes to products that are not necessarily alternating.
So let us consider again the data that was fixed throughout Section 3.2: \( d, n \in \mathbb{N} \), \( q \in (-1, 1) \), a tuple \( \varepsilon = (\varepsilon(1), \ldots, \varepsilon(2n)) \in \mathcal{D}_{(1, \nu)}(2n) \) and some vectors \( u_1, \ldots, u_{2n} \in \mathbb{C}^d \). In addition to that, we now fix a tuple

\[
\chi = (\chi(1), \ldots, \chi(2n)) \in \{\ell, r\}^{2n},
\]

and we look at the operator

\[
M := S_{\chi(2n)}^{\varepsilon}(u_{2n}) \cdots S_{\chi(1)}^{\varepsilon}(u_1) \in B(T_{d,q}),
\]

where for \( u \in T_{d,q} \) we denote (with "S" as a reminder of "shift operator"):

\[
L(u) =: S_\ell(u), \quad R(u) =: S_r(u).
\]

Same as in the special case discussed in Section 3.2 (which corresponds to putting \( \chi = (\ell, r, \ell, \ell, \ell, r, r, \ell, \ell) \)), we have that the vacuum-vector \( \xi_{\text{vac}} \) is an eigenvector for \( M \), and the Wick formula expresses the corresponding eigenvalue as a summation over \( \mathcal{P}_2(2n) \). When writing the derivation of this formula, it turns out that the only thing that needs to be changed (in the considerations leading to Proposition 3.2.6 from Section 3.2) is the definition of the "labels-to-heights" permutation \( s_n \) introduced in Notation 2.1.3. What is needed in the place of \( s_n \) is a permutation "\( s_\chi \)" (first noticed in [13], and then put to intensive use in [4, 5]) which plays an essential role in the combinatorics of two-faced free probability.

**Notation 4.1.1.** Let \( \chi \) be as in (4.1), and let us write explicitly:

\[
\begin{align*}
\chi^{-1}(\ell) &= \{i_1, \ldots, i_p\}, \quad \text{with } i_1 < \cdots < i_p, \text{ and} \\
\chi^{-1}(r) &= \{j_1, \ldots, j_q\}, \quad \text{with } j_1 < \cdots < j_q, \text{ where } q = n - p.
\end{align*}
\]

We denote by \( s_\chi \) the permutation of \( \{1, \ldots, 2n\} \) defined by

\[
s_\chi := \begin{pmatrix}
1 & 2 & \cdots & p & p+1 & \cdots & 2n-1 & 2n \\
i_1 & i_2 & \cdots & i_p & j_q & \cdots & j_2 & j_1
\end{pmatrix}.
\]

**Example 4.1.2.** Suppose that \( n = 5 \) and that \( \chi = (r, \ell, \ell, r, \ell, r, \ell, \ell) \in \{\ell, r\}^{10} \). Thus \( \chi^{-1}(\ell) = \{2, 3, 5, 6, 9, 10\}, \chi^{-1}(r) = \{1, 4, 7, 8\} \) and the labels-to-heights permutation \( s_\chi \) comes out as

\[
s_\chi := \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
2 & 3 & 5 & 6 & 9 & 10 & 8 & 7 & 4 & 1
\end{pmatrix}.
\]

When working with this \( \chi \), the pictures showing 10 points around a rectangle which were used for illustration in Section 2 must now be adjusted as shown in Figure 8.

![Figure 8](image-url)
Remark 4.1.3. Upon re-examining the Section 2, one sees that all the constructions and arguments presented there were based on the permutation \( s_0 \) introduced in Notation 2.1.3, and can be adjusted word-by-word to the framework where we use our arbitrary (but fixed) tuple \( \chi \in \{ \ell, r \}^{2n} \) by simply replacing everywhere "\( s_0 \)" by "\( \chi \)". First, we get a bijection
\[
\Phi_\chi : \mathcal{P}_2(2n) \to \mathcal{D}_{(1,*)}(2n),
\]
defined exactly as in Notation 2.2.3. Then the whole discussion from Section 2.3 goes through to the \( \chi \)-framework, and concludes with the analogue of Proposition 2.3.3, expressing the number of crossings of a partition \( \pi \in \Phi_\chi^{-1}(\varepsilon) \) in terms of the tuple of choice-numbers which encodes that partition. A concrete example of how this goes is shown in Figure 9.

![Figure 9](image)

**Figure 9.** Suppose that \( n = 5 \), \( \chi = (r, \ell, \ell, r, r, \ell, \ell, \ell, \ell, \ell) \), \( \varepsilon = (1, 1, 1, *, 1, *, 1, *, 1, *, *) \). The pair-partition \( \pi = \{(1,9),\{2,4\},\{3,6\},\{5,7\},\{8,10\}\} \in \Phi_\chi^{-1}(\varepsilon) \) is parametrized by the choices: \( \gamma_4 = 2 \), \( \gamma_6 = 2 \), \( \gamma_7 = 1 \), \( \gamma_9 = 2 \), \( \gamma_{10} = 1 \).

Next, it is straightforward to adjust to the \( \chi \)-framework the calculations shown in Sections 3.1 and 3.2, with the little nuisance that the \( \chi \)-version of the arguments in these sections must use a unified notation for the operators \( A_k^{(0)} \) and \( B_k^{(0)} \) from Notation 3.1.3. The end result of all this is a statement very similar to Proposition 3.2.6, as follows.

**Proposition 4.1.4.** Consider the data fixed at the beginning of the present subsection \( (d, n, q, \varepsilon, \chi) \) and the vectors \( u_1, \ldots, u_{2n} \in \mathbb{C}^d \), and let \( M \in B(\mathcal{T}_{d,q}) \) be the operator defined in Equation (4.2). Then
\[
(4.4) \quad M(\xi_{\text{vac}}) = \sum_{\pi \in \Phi_\chi^{-1}(\varepsilon)} \left[ q^{\text{cr}(\pi)} \cdot \prod_{\{k,h\} \text{ pair in } \xi_{\text{vac}}^\pi \text{ with } \varepsilon(k) = *, \varepsilon(h) = 1} \langle u_k, u_h \rangle \right] : \xi_{\text{vac}}. \]

Analogously to Remark 3.2.7, we note that \( \varphi_{\text{vac}}(M) \) equals the scalar that appeared on the right-hand side of (4.4), and moreover: if we focus on \( \varphi_{\text{vac}}(M) \) (rather than on the vector \( M(\xi_{\text{vac}}) \)), it is quite easy to extend the two-sided \( q \)-Wick formula to the case when \( \varepsilon \) would be picked in \( \{1,*\}^{2n} \setminus \mathcal{D}_{(1,*)}(2n) \), in which case we just get \( \varphi_{\text{vac}}(M) = 0 \).

While not directly useful for the present paper, we mention that a non-trivial analysis of \( M(\xi_{\text{vac}}) \) can actually be made for some choices of \( \varepsilon \in \{1,*\}^{2n} \setminus \mathcal{D}_{(1,*)}(2n) \), and also for some choices of \( \varepsilon \in \{1,*\}^m \) with \( m \) odd, by using "incomplete" pair-partitions of \( \{1, \ldots, m\} \). This kind of analysis was made in the one-sided \( q \)-Wick case in [3, Proposition 2.7] and [7, Section 3], and is pursued in depth (for several constructions of Wick products) in the recent paper [1].
4.2. Proof of Proposition 1.8. In this subsection we fix $d, n \in \mathbb{N}$ and $q \in (-1, 1)$, and we verify the moment formula stated in Proposition 1.8:

\[(\varphi_{\text{vac}} \otimes \varphi_{\text{vac}})(X_{d,q}^n) = \tilde{P}_n(d, q),\]

where $X_{d,q} := (\sum_{i=1}^d (L_{i,q} + L_{i,q}^*) \otimes (L_{i,q} + L_{i,q}^*))^2 \in B(\mathcal{T}_{d,q}) \otimes B(\mathcal{T}_{d,q})$, and the polynomial $\tilde{P}_n$ is as introduced in Notation 1.7. A discussion very similar to the one from Remark 3.4.2 shows that the Equation (1.11) used to define $\tilde{P}_n(t, u)$ can be re-written as

\[\tilde{P}_n(t, u) = \sum_{\pi, \sigma \in \mathcal{P}_2(2n)} q^{\varphi(\pi) + \varphi(\sigma)}.\]

In the calculations shown below it will be convenient to denote $L_{i,q} + L_{i,q}^* =: A_{i,q}, 1 \leq i \leq d$. The operators $A_{1,q}, \ldots, A_{d,q} \in B(\mathcal{T}_{d,q})$ go under the name of $q$-Gaussian random variables, and one has the following formula (cf. [2, Proposition 2]) to evaluate their joint moments of length $2n$ with respect to the vacuum-state: for every $I : \{1, \ldots, 2n\} \to \{1, \ldots, d\}$, one has

\[\varphi_{\text{vac}}(A_{I(2n);q} \cdots A_{I(1);q}) = \sum_{\pi \in \mathcal{P}_2(2n), \pi \leq \text{Ker}(I)} q^{\varphi(\pi)},\]

where Ker$(I)$ has the same meaning (partition of $\{1, \ldots, 2n\}$ into level-sets of $I$) as in the proof of Lemma 3.4.3, and the inequality $\pi \leq \text{Ker}(I)$ is with respect to the reverse refinement order on $\mathcal{P}(2n)$.

When on the left-hand side of (4.5) we replace $X_{d,q}$ from its definition and follow the underlying algebra, we find that

\[(\varphi_{\text{vac}} \otimes \varphi_{\text{vac}})(X_{d,q}^n) = \sum_{I: \{1, \ldots, 2n\} \to \{1, \ldots, d\}} \left(\varphi_{\text{vac}}(A_{I(2n);q} \cdots A_{I(1);q})\right)^2.\]

In view of (4.7), this can be continued with

\[= \sum_{I: \{1, \ldots, 2n\} \to \{1, \ldots, d\}} \sum_{\pi, \sigma \in \mathcal{P}_2(2n), \pi, \sigma \leq \text{Ker}(I)} q^{\varphi(\pi) + \varphi(\sigma)}.\]

The condition "$\pi, \sigma \leq \text{Ker}(I)$" is equivalent to $\pi \vee \sigma \leq \text{Ker}(I)$, hence when we change the order of summation in latter double sum we come to

\[= \sum_{\pi, \sigma \in \mathcal{P}_2(2n)} q^{\varphi(\pi) + \varphi(\sigma)} \left| \{I : \{1, \ldots, 2n\} \to \{1, \ldots, d\} \mid \text{Ker}(I) \geq \pi \vee \sigma\} \right|.\]

It is immediately seen that the set of $I$’s which has appeared in the summation (4.8) has cardinality $d^{\varphi(\pi \vee \sigma)}$. Hence this summation is nothing but the expression recorded in (4.6) for the value of $\tilde{P}_n(d, q)$, and this concludes the verification of the required formula (4.5).

4.3. Some remarks in the case $q = 0$.

In this subsection we fix $d \in \mathbb{N}$, we set $q = 0$, and we make some remarks around Proposition 1.1 (which is the special case $q = 0$ of our main result, Theorem 1.4). We will continue to use the framework built in Section 3, but, consistently with the statement of Proposition 1.1, we will omit the explicit occurrence of $q = 0$ in notations for operators – hence we will write \(L_i, R_i \in B(\mathcal{T}_{d})\) instead of \(L_{i,0}, R_{i,0} \in B(\mathcal{T}_{d,0})\), and such. It will be moreover convenient to denote

\[A_i := L_i + L_i^*, \quad B_i := R_i + R_i^*, \quad 1 \leq i \leq d.\]
Proposition 1.1 is thus concerned with the distribution of the operator

\begin{equation}
T_d := A_1 B_1 + \cdots + A_d B_d \in B(T_d).
\end{equation}

**Remark 4.3.1.** The \((2d)\)-tuple of operators \(A_1, \ldots, A_d, B_1, \ldots, B_d\) is the prototypical example of bi-free Gaussian system appearing in the bi-free central limit theorem from [18]. It can also be treated as a “\((2d)\)-tuple of canonical operators” in the sense of [13] (for which purpose one puts them in the form \(A_i = L_i^* (I + L_i^2 + \cdots + L_d^2)\) and \(B_i = R_i^* (I + R_i^2 + \cdots + R_d^2)\), \(1 \leq i \leq d\)). As a consequence, one has an explicit summation formula which describes the joint moments of this \((2d)\)-tuple, and which can be used in order to give a short proof of Proposition 1.1. The way to think of this summation formula is as a special case of “moment-cumulant machinery” (for a detailed presentation of the bi-free cumulant machinery we refer the reader to [4, 5], here we only state the formula relevant for Proposition 1.1.

First, a bit of notation: for every \(n \in \mathbb{N}\) let us put

\begin{equation}
\text{BNC}_2^{(alt)}(2n) := \{\pi_n, \pi \in \text{NC}_2(2n)\},
\end{equation}

where \(\pi_n\) is the permutation of \(\{1, \ldots, 2n\}\) introduced in Notation 2.1.3. The pair-partitions in \(\text{BNC}_2^{(alt)}(2n)\) are precisely those which are said to be bi-non-crossing with respect to the alternating \((2n)\)-tuple \((\ell, r, \ell, r, \ldots, \ell, r) \in \{\ell, r\}^{2n}\). They appear as indexing set in the following formula for alternating joint moments of \(A_1, \ldots, A_d, B_1, \ldots, B_d\): for every \(n \in \mathbb{N}\) and every \((2n)\)-tuple \(I: \{1, \ldots, 2n\} \to \{1, \ldots, d\}\) one has

\begin{equation}
\varphi_{\text{vac}}(A_{I(2n)} B_{I(2n-1)} \cdots A_{I(2)} B_{I(1)}) = \sum_{\pi \in \text{BNC}_2^{(alt)}(2n)} \text{term}(I, \pi),
\end{equation}

where for \(\pi \in \text{BNC}_2^{(alt)}(2n)\) we put

\[
\text{term}(I, \pi) := \begin{cases} 
1, & \text{if } \pi \leq \text{Ker}(I) \\
0, & \text{otherwise}.
\end{cases}
\]

**Outline 4.3.2.** *(Outline of proof of Proposition 1.1 via Equation (4.12).)*

We fix an \(n \in \mathbb{N}\) and we will verify that \(\varphi_{\text{vac}}(T_d^n) = Q_n(d)\), where \(Q_n\) is the \(n\)-th semi-meander polynomial from Equation (1.5). To this end we start in the same way as in the beginning of the proof of Theorem 1.4, and we derive a formula analogous to (3.29) from that proof, but where we now keep the \(A_i\)’s and \(B_i\)’s as they are, rather than breaking them in terms of \(L_i, L_i^*, R_i, R_i^*\). We get

\begin{equation}
\varphi_{\text{vac}}(T_d^n) = \sum_{I: \{1, \ldots, 2n\} \to \{1, \ldots, d\}} d^{|\pi \setminus \{\{1,2\}, \ldots, \{2n-1,2n\}\}|}.
\end{equation}

We next plug Equation (4.12) into (4.13), and change the order of summation in the ensuing double sum. By working a bit the inside summation over \(I\) (in a way similar to what we did in Section 3.4 for the proof of Theorem 1.4) we come to

\begin{equation}
\varphi_{\text{vac}}(T_d^n) = \sum_{\pi \in \text{BNC}_2^{(alt)}(2n)} d^{|\pi \setminus \{\{1,2\}, \ldots, \{2n-1,2n\}\}|}.
\end{equation}

Finally, in the summation on the right-hand side of (4.14) we perform the substitution \(\pi = \pi_n \cdot \tilde{\pi}\), with \(\tilde{\pi}\) running in the set \(\text{NC}_2(2n)\) of usual non-crossing pair-partitions. In connection to this substitution we also recall (cf. Equation (2.2) in Notation 2.1.3) that we can replace \(\{\{1,2\}, \ldots, \{2n-1,2n\}\}\) as \(\pi_n \cdot \rho_{2n}\), where \(\rho_{2n}\) is the rainbow pair-partition
from Equation (1.4). The exponent in the general term of the sum on the right-hand side of (4.14) thus becomes

$$|\pi \lor \{(1, 2), \ldots, (2n, 2n)\}| = |(\mathcal{E}_n \cdot \widehat{\pi}) \lor (\mathcal{E}_n \cdot \rho_{2n})|$$

$$= |(\mathcal{E}_n \cdot (\widehat{\pi} \lor \rho_{2n})| = |\widehat{\pi} \lor \rho_{2n}|.$$  

Hence the sum in (4.14) is turned by the substitution into $\sum_{\pi \in NC_2(2n)} d|\widehat{\pi} \lor \rho_{2n}|$, which is precisely the formula for $Q_n(d)$.

**Remark 4.3.3.** *(Approach via random matrix model.)* Yet another proof of Proposition 1.1 can be obtained by taking advantage of a random matrix model for semi-meander polynomials that was proposed in Section 5.4 of [9]. We outline here how this goes. For every $N \in \mathbb{N}$, let $G^{(N)}_1, \ldots, G^{(N)}_d$ be a $d$-tuple of independent Gaussian Hermitian random matrices (as reviewed for instance on pages 368-371 of Lecture 22 in [15]), and let $tr_N$ denote the normalized trace on complex $N \times N$ matrices. Equation (5.20) in Section 5.4 of [9] says that for every $n \in \mathbb{N}$ (and for the $d \in \mathbb{N}$ which is fixed in the present subsection) one has

$$Q_n(d) = \lim_{N \to \infty} \sum_{J: \{1, \ldots, n\} \to \{1, \ldots, d\}} (E \circ tr_N)(G^{(N)}_{J(1)} \cdots G^{(N)}_{J(n)} G^{(N)}_{J(1)} \cdots G^{(N)}_{J(1)}).$$

(4.15)  

Note that the the number of terms in the sum on the right-hand side of (4.15) is $d^n$, independent of $N$. It is natural to ask if it isn’t the case that each of these $d^n$ terms, taken separately, has its own limit for $N \to \infty$. The considerations from Section 4 of the paper [17] on bi-matrix models (specifically, Theorems 4.10 and 4.13 there) assure us that the separate limits of terms do indeed exist; and more precisely, for any fixed $J: \{1, \ldots, n\} \to \{1, \ldots, d\}$ one has

$$\lim_{N \to \infty} (E \circ tr_N)(G^{(N)}_{J(1)} \cdots G^{(N)}_{J(n)} G^{(N)}_{J(1)} \cdots G^{(N)}_{J(1)})$$

(4.16)  

$$= \varphi_{\text{vac}}(A_{J(1)} \cdots A_{J(n)} B_{J(1)} \cdots B_{J(n)}),$$

where $A_1, \ldots, A_d, B_1, \ldots, B_d$ are as in Equation (4.9). Since $A_i B_j = B_j A_i$ for all $1 \leq i, j \leq d$, the right-hand side of Equation (4.16) can also be written as

$$\varphi_{\text{vac}}(A_{J(1)} B_{J(1)} \cdots A_{J(n)} B_{J(n)}).$$

(4.17)  

Finally, summing in (4.17) over all tuples $J: \{1, \ldots, n\} \to \{1, \ldots, d\}$ leads to $\varphi_{\text{vac}}(\sum_{\{A_1 B_1 + \cdots + A_d B_d\}^n})$; hence Equation (4.15) implies the required formula (1.7) of Proposition 1.1.

**Remark 4.3.4.** It would be interesting to know what is the spectrum of the operator $T_d$, in particular if the spectrum is an interval. The spectral radius ($= \text{norm}$) of $T_d$ goes in a regime of “constant times d” (since it is clear that $\|T_d\| \leq 4d$, while on the other hand $\|T_d\| \geq \|T_d(\xi_{\text{vac}})\| = \sqrt{d + d^2}$). We note that, despite being a sum of $d$ operators with standard free Poisson distribution (where the said distribution is supported on the interval $[0, 4]$), the operator $T_d$ is not positive, for instance $\langle T_d(e_1 \otimes e_2 - e_2 \otimes e_1), e_1 \otimes e_2 - e_2 \otimes e_1 \rangle < 0$.

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