Efficient Search of First-Order Nash Equilibria in Nonconvex-Concave Smooth Min-Max Problems

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Abstract
We propose an efficient algorithm for finding first-order Nash equilibria in smooth min-max problems of the form \( \min_{x \in X} \max_{y \in Y} F(x, y) \), where the objective function is nonconvex with respect to \( x \) and concave with respect to \( y \), and the set \( Y \) is convex, compact, and projection-friendly. The goal is to reach an \((\varepsilon_x, \varepsilon_y)\)-first-order Nash equilibrium point, as measured by the norm of the corresponding (proximal) gradient component. The proposed approach is fairly simple: essentially, we perform approximate proximal point iterations on the primal function, with inexact oracle provided by Nesterov’s algorithm run on the regularized function \( F(x_t, \cdot) \) with \( O(\varepsilon_y) \) regularization term, where \( x_t \) is the current primal iterate. The resulting iteration complexity is \( O(\varepsilon_x^{-2} \varepsilon_y^{-1/2}) \) up to a logarithmic factor. In particular, in the regime \( \varepsilon_y = O(\varepsilon_x^2) \), our algorithm gives \( O(\varepsilon_x^{-3}) \) complexity for finding \( \varepsilon_x \)-stationary point of the canonical Moreau envelope of the primal function. Moreover, when the function \( F(x, y) \) is strongly concave in \( y \), the complexity of our algorithm improves to \( O(\varepsilon_x^{-2} \kappa_y^{1/2}) \), up to logarithmic factors, with \( \kappa_y \) being the condition number of the dual function for the canonical Moreau envelope. In both cases, the proposed algorithm outperforms or matches the performance of several recently proposed schemes while, arguably, being more transparent and easier to implement.

1 Problem setup
In this note, we study the following min-max problem:

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} F(x, y). \tag{1}
\]

Here, \( \mathcal{X} \) is a Euclidean space; \( \mathcal{Y} \) is a “projection-friendly” convex body (i.e., convex and compact set with non-empty interior) inscribed into a ball with radius \( R_y < \infty \) in the Euclidean space \( \mathcal{Y} \); finally, \( F : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) is concave in \( y \) for all \( x \in \mathcal{X} \), and has Lipschitz gradient; precisely, the bounds

\[
\|\nabla_x F(x', y) - \nabla_x F(x, y)\| \leq L_{xx}\|x' - x\|, \\
\|\nabla_y F(x, y') - \nabla_y F(x, y)\| \leq L_{yy}\|y' - y\|, \\
\|\nabla_x F(x, y') - \nabla_x F(x, y)\| \leq L_{xy}\|y' - y\|, \\
\|\nabla_y F(x', y) - \nabla_y F(x, y)\| \leq L_{xy}\|x' - x\|. \tag{2}
\]

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hold uniformly over \(x, x' \in X\) and \(y, y' \in Y\) with Lipschitz constants \(L_{xx}, L_{yy}, L_{xy}\). In what follows, \(\|\cdot\|\) and \(\langle \cdot, \cdot \rangle\) denote the standard Euclidean norm and inner product (regardless of the space), and \([\nabla_x F(x, y), \nabla_y F(x, y)]\) refer to the components of the full gradient \(\nabla F(x, y)\). Rather than solving the problem (1) itself, we focus on the much easier task of finding approximate first-order Nash equilibria as defined below.

**Definition 1.** A point \((\hat{x}, \hat{y}) \in X \times Y\) is called \((\varepsilon_x, \varepsilon_y)\)-approximate first-order Nash equilibrium \(((\varepsilon_x, \varepsilon_y)\text{-FNE})\) of the problem (1) if the following holds:

\[
\|\nabla_x F(\hat{x}, \hat{y})\| \leq \varepsilon_x, \\
L_{yy} \left\| \hat{y} - \text{proj}_Y \left( \hat{y} + \frac{1}{L_{yy}} \nabla_y F(\hat{x}, \hat{y}) \right) \right\| \leq \varepsilon_y, \tag{3}
\]

where \(\text{proj}_Y(\cdot)\) is the operator of Euclidean projection onto the set \(Y\).

Note that the second inequality in Eq. (3) bounds the norm of the proximal gradient (see, e.g., [Nes13a]) corresponding to the projected gradient ascent step from \(\hat{y}\) with natural stepsizes \(1/L_{xx}\) (resp., \(1/L_{yy}\)), and reduces to \(\|\nabla_y F(\hat{x}, \hat{y})\| \leq \varepsilon_y\) if the updated point remains in the feasible set.

The goal of this note is to provide an efficient algorithm for finding \((\varepsilon_x, \varepsilon_y)\text{-FNE}\) given access to the full gradient \(\nabla F(x, y)\) at a point. As it is common in the literature on first-order algorithms, we assume that the feasible set \(Y\) is “projection-friendly”, i.e. solving the optimization problem \(\min_{y' \in Y} \|y - y'\|^2\) for any \(y \in Y\) is computationally cheap; thus, the natural notion of efficiency is simply the number of oracle calls.

Besides the accuracies \(\varepsilon_x, \varepsilon_y\), Lipschitz constants \(L_{xx}, L_{yy}, L_{xy}\), and the “radius” \(R_y\) of \(Y\), we need a parameter quantifying the “size” of the primal problem – that of minimizing the primal function

\[
\varphi(x) := \max_{y \in Y} F(x, y). \tag{4}
\]

Since the primal variable is unconstrained, the natural choice of such parameter is the primal gap \(\Delta := \varphi(x_0) - \min x \varphi(x)\), where \(x_0\) is the initial iterate. To give a concise and intuitive statement our main result, it is helpful to define the “coupling-adjusted” counterpart of \(L_{yy}\),

\[
L_{yy}^+ := L_{yy} + \frac{L_{xy}^2}{L_{xx}}, \tag{5}
\]

as well as the unit-free quantities

\[
T_x := \frac{L_{xx} \Delta}{\varepsilon_x^2}, \quad T_y := \sqrt{\frac{L_{yy}^+ R_y}{\varepsilon_y}}. \tag{6}
\]

Upon consulting the literature ([CDHS17, Nes12]), one recognizes \(T_x\) as the iteration complexity of finding \(\varepsilon_x\)-stationary point in the class of non-convex unconstrained minimization problems with \(L_{xx}\)-smooth objective and initial suboptimality gap \(\Delta\). On the other hand, \(T_y\) is the corresponding complexity bound in the class of smooth constrained maximization problems with concave and \(L_{yy}^+\)-smooth objective and \(R_y\)-bounded feasible set. We can now state our main result.

\footnotetext{1}{Note that we do not assume that \(x\) belongs to a compact set, hence one may not have an exact stationary point.}
Theorem 1.1 (Abridged formulation of Theorem 3.1). There exists an algorithm that, given the data $(\varepsilon_x, \varepsilon_y, L_{xx}, L_{xy}, L_{yy}, R_y, \Delta)$, finds $(\varepsilon_x, \varepsilon_y)$-FNE of the problem (1) in
\[ \widetilde{O}(T_xT_y) \] computations of $\nabla F(x, y)$ and projections onto $Y$, where $\widetilde{O}(\cdot)$ hides logarithmic factors in $T_x$ and $T_y$.

Discussion. We see that Eq. (7) can be interpreted as the product of the “primal” complexity of finding an $\varepsilon_x$-FNE of $\varphi(x)$, and the “dual” complexity of finding $\varepsilon_y$-FNE for any $x \in X$ by Nesterov’s regularization technique [Nes12]. Thus, under the condition $\varepsilon_y = O(\varepsilon_x^2)$ -- more precisely when $T_x = O(T_y^2)$ -- our result recovers the recent $\widetilde{O}(\varepsilon_x^{-3})$ guarantee of [TJNO19] for finding $\varepsilon_x$-stationary point in the Moreau envelope regime using [LJJ19, Proposition 5.2].

We note that, to the best of our understanding, our algorithm (and thus Theorem 1.1) can be extended to constrained minimization in $x$, non-Euclidean prox-functions, and composite objective. As our focus in this note is a simple presentation of our approach, we avoid such extensions.

Finally, we would like to mention that, while finalizing this manuscript, we were notified of the concurrent work [LJJ+20] where an algorithm with similar iteration complexity is proposed for nonconvex-concave min-max problems.

High-level description of the idea. In a nutshell, our algorithm consists in successively solving the strongly-convex-strongly-concave saddle-point problem
\[ \min_{x \in X} \max_{y \in Y} \left\{ F^\text{reg}_t(x, y) := F^\text{reg}(x, y) + L_{xx}\|x - x_{t-1}\|^2 \right\}, \] where $x_{t-1}$ is the previous primal iterate, and $F^\text{reg}(x, y) := F(x, y) - \lambda_y \|y - \bar{y}\|^2$ is the objective regularized with a small $O(\varepsilon_y)$ term à-la Nesterov [Nes12]. By the first-order optimality conditions, Eq. (8) amounts to the proximal-point type updates run on the regularized primal function $\varphi^\text{reg}(x) = \max_{y \in Y} F^\text{reg}(x, y)$, ensuring that
\[ \varphi^\text{reg}(x_t) + \frac{1}{4L_{xx}}\|\nabla_x F(x_t, y_t)\|^2 \leq \varphi^\text{reg}(x_{t-1}), \] where $y_t := \arg\max_{y \in Y} F^\text{reg}(x_t, y)$. The key novelty here is that, while the function $\varphi^\text{reg}$ is essentially non-smooth (with gradient only $O(1/\varepsilon_y)$-Lipschitz by Danskin’s theorem), one can still perform proximal point iterations with stepsize corresponding to the smoothness of $F(\cdot, y)$, using the readily available “max-oracle” for $F^\text{reg}(x_t, \cdot)$ in the form of Nesterov’s accelerated algorithm.

Application of this algorithm at each step gives the $\widetilde{O}(T_y)$ complexity factor, and ensures that $(x_t, y_t)$ remains $O(\varepsilon_y)$-stationary in $y$. On the other hand, repeating Eq. (9) for $T_x$ iterations ensures that at least one of the search points satisfies $\|\nabla_x F(x_t, y_t)\| \leq \varepsilon_x$, similarly to the classical argument for the proximal point algorithm.

We refer the reader to Sec. 3.1–3.2 for a more detailed presentation of our approach.

Outline. In Sec. 2 we present the “building blocks” for our algorithm. We first recap Nesterov’s algorithm for smooth convex optimization, and then show how to use it for the approximation of the proximal point operator. Next, in Sec. 3 we outline our approach, present the resulting algorithm, and prove the full version of Theorem 1.1. Finally, we give a brief overview of related work in Sec. 4.
2 Building blocks

Preliminaries. For convex set $Z$ in Euclidean space $Z$ and $z, \zeta \in Z$, we define the prox-mapping

$$\text{prox}_{z,Z}(\zeta) := \text{argmin}_{z' \in Z} \langle \zeta, z' \rangle + \frac{1}{2} \|z' - z\|^2.$$ (10)

In what follows, we assume this operator is computationally cheap. Note that in the unconstrained case with $Z = Z$, one has $\text{prox}_{z,Z}(\zeta) = z - \zeta$. Note also that, using Eq. (10), we can rewrite the second part of Eq. (3) as

$$\|\hat{y} - \text{prox}_{\hat{y},Y} \left( -\nabla_y F(\hat{x}, \hat{y}) \frac{L_{yy}}{L_{yy}} \right)\| \leq \frac{\varepsilon_y}{L_{yy}}.$$ (11)

Following [DGN14], we use the notion of inexact first-order oracle for a smooth convex function.²

Definition 2 ($\delta$-inexact oracle). Let $f : Z \to \mathbb{R}$ be convex and have $L$-Lipschitz gradient. The pair $[\tilde{f}(\cdot), \nabla f(\cdot)]$ is inexact oracle for $f(\cdot)$ with accuracy $\delta \geq 0$ if for any pair of points $z, z' \in Z$ one has

$$0 \leq f(z') - f(z) - \langle \nabla f(z), z' - z \rangle \leq \frac{L}{2} \|z' - z\|^2 + \delta.$$ (11)

Next we present Nesterov’s fast gradient method (FGM) for smooth convex optimization with inexact oracle (see [DGN14]) and a simple restart scheme for it. We use them in two scenarios:

(a) unconstrained minimization of a strongly convex and smooth function on $X$ with exact oracle;

(b) constrained maximization of a strongly concave and smooth function on $Y$ with $\delta$-inexact oracle.

2.1 Fast gradient method with inexact oracle

Assume we are given initial point $z_0 \in Z$, target number of iterations $T$, stepsize $\gamma > 0$, and access to a $\delta$-inexact (or, possibly, exact) oracle for function $f : Z \to \mathbb{R}$ which satisfies the requirements in Definition 2. We will use a variant of fast gradient method with inexact oracle due to [DGN14], given below as Algorithm 1, that performs $T$ iterations and outputs approximate minimizer $z_T$ of $f$; each of these iterations reduces to a single call of $\tilde{\nabla} f(\cdot)$, two prox-mapping computations, and a few entrywise vector operations. Note that the inexact oracle $\tilde{\nabla} f(\cdot)$ is passed as an input parameter (i.e., “function handle”); this means that such an oracle must be implemented as an external procedure.

Assume $\tilde{\nabla} f(\cdot)$ has small enough error $\delta$ in the sense of Definition 2. The work [DGN14] proves the standard $O(T^{-2})$ convergence of FGM is preserved in this case. Let us rephrase their result.

Theorem 2.1 ([DGN14, Thm. 5 and Eq. (42)]). Assume Algorithm 1 is run with $\gamma = 1/L$ and $\delta$-inexact oracle of $f$ that is $L$-smooth, convex, and minimized at $z^*$ such that $\|z_0 - z^*\| \leq R$, then

$$f(z_T) - f(z^*) \leq \frac{4LR^2}{T^2} + 2\delta T.$$ (12)

As a result, one has

$$f(z_T) - f(z^*) \leq \frac{5LR^2}{T^2} \text{ whenever } \delta \leq \delta_T := \frac{LR^2}{2T^3}.$$ (12)

²Unlike [DGN14], we do not include $L$ into the definition of inexact oracle, as in our situation this is unnecessary.
When $f$ is also $\lambda$-strongly convex, Eq. (12) results in the bound on the distance to $z^*$:

$$\|z_T - z^*\|^2 \leq \frac{10\kappa R^2}{T^2},$$

(13)

where $\kappa = L/\lambda$ is the condition number. That is, we are guaranteed to get twice closer to the optimum after $T = O(\sqrt{\kappa})$ iterations. Following [Nes13b], we exploit this fact to obtain linear convergence via the simple restart scheme given in Algorithm 2, and derive the following result.

**Corollary 2.1.** Given $\varepsilon > 0$, assume Algorithm 2 is run with $\gamma = 1/L$, parameters $T, S$ satisfying

$$T \geq \sqrt{40\kappa}, \quad S \geq \log_2 \left(\frac{3LR}{\varepsilon}\right),$$

(14)

and $\delta \leq \delta_T$, cf. (12). Then the final iterate $z^S$ satisfies

$$\|z^S - z^*\| \leq \frac{\varepsilon}{3L}, \quad L \left\| z^S - \text{prox}_{z^S, Z} \left( \frac{\nabla f(z^S)}{L} \right) \right\| \leq \varepsilon, \quad f(z^S) - f(z^*) \leq \frac{\varepsilon^2}{18L}.$$ 

(15)

**Proof.** By Eq. (13), with $T \geq \sqrt{40\kappa}$ iterations in $s$-th epoch we ensure $\|z^s - z^*\| \leq \frac{1}{2}\|z^{s-1} - z^*\|$. Hence, after $S \geq \log_2(3LR/\varepsilon)$ epochs, $z^S$ satisfies the first bound in Eq. (15). Similarly, we have

$$f(z^s) - f(z^*) \leq \frac{10\kappa [f(z^{s-1}) - f(z^*)]}{T^2} \leq \frac{1}{4} [f(z^{s-1}) - f(z^*)] = 4^{1-s}[f(z^1) - f(z^*)] \leq \frac{20LR^2}{4^sT^2} \leq \frac{LR^2}{2^{2s+1}},$$

where in the end we used $\kappa \geq 1$. Plugging in $2^{2s+1} = 18L^2R^2/\varepsilon^2$, we verify the last part of Eq. (15). Finally, for the second part of Eq. (15), by the first-order optimality conditions for Eq. (10) we have

$$z^* = \text{proj}_Z \left( z^* - \nabla f(z^*)/L \right).$$

Let $\hat{z}^S := \text{prox}_{z^S, Z}(\nabla f(z^S)/L) = \text{proj}_Z \left( z^S - \nabla f(z^S)/L \right)$. The triangle inequality then implies

$$\|z^S - \hat{z}^S\| \leq \|z^S - z^*\| + \|z^* - \hat{z}^S\| \leq 2\|z^S - z^*\| + \frac{1}{L}\|\nabla f(z^*) - \nabla f(z^S)\| \leq 2\|z^S - z^*\| = \frac{\varepsilon}{L},$$

where we used non-expansiveness of projection and the Lipschitzness of $\nabla f(\cdot)$.

In the unconstrained use case for Algorithm 2, the problem will be parametrized with the initial gap $\Delta f = f(z^0) - f(z^*)$ instead of $R$, and the exact oracle $\nabla f(\cdot)$ will be available (function value $f(\cdot)$ will not be used). Using that $LR^2 \leq 2\kappa \Delta f$ by strong convexity, we make the following observation.

**Remark 2.1.** When running Algorithm 2 with $\delta = 0$, the second condition in Eq. (14) replaced with

$$S \geq \frac{1}{2} \log_2 \left(\frac{18\kappa L \Delta f}{\varepsilon^2}\right),$$

and other parameters set as in the premise of Corollary 2.1, the guarantees of Eq. (15) remain valid.


2.2 Proximal point operator and its implementation

Our next goal is to provide a brief overview of the proximal point method, which forms the backbone of our approach, in the context of searching for stationary points. Then we show how its iteration – the proximal point operator – can be efficiently implemented with sufficient accuracy via Algorithm 2.

Given $x \in \mathcal{X}$ and $\phi: \mathcal{X} \to \mathbb{R}$ with $L$-Lipschitz gradient, without assuming convexity of $\phi$, the \textit{proximal point operator of} $\phi$ \textit{with stepsize} $0 < \gamma < 1/L$ \textit{at} $x$ is

$$
PP_{\gamma \phi}(x) := \arg\min_{x' \in \mathcal{X}} \phi(x') + \frac{1}{2\gamma} \|x' - x\|^2.
$$

(16)

Note that, by the first-order optimality conditions, $x^+ = PP_{\gamma \phi}(x)$ satisfies

$$
x^+ = x - \gamma \nabla \phi(x^+).
$$

(17)

For large stepsizes, computing $PP_{\gamma \phi}(x)$ at a point might be as hard as minimizing $\phi$. However, with large enough regularization, namely when $\gamma = c/L$ for $0 < c < 1/2$, the task becomes easy, since the objective in Eq. (16) is strongly-convex and well-conditioned with condition number

$$
\kappa = (1 + c)/(1 - c) \leq 3.
$$

(18)

On the other hand, with such stepsize the \textit{proximal point method}, as given by the iterate sequence

$$
x_t = PP_{\gamma \phi}(x_{t-1}),
$$

(19)

attains the optimal rate of minimizing gradient norm. Indeed, Eq. (16) with $\gamma = c/L$, when combined with the implicit update formula $x^+ = x - \gamma \nabla \phi(x^+)$, gives

$$
\phi(x^+) + \frac{c}{2L} \|\nabla \phi(x^+)\|^2 \leq \phi(x);
$$

(20)
iterating this $T$ times according to (19) results in
\[
\min_{t \in [T]} \| \nabla \phi(x_t) \| \leq \sqrt{\frac{1}{T} \sum_{t \in [T]} \| \nabla \phi(x_t) \|^2} = \sqrt{\frac{2L\Delta}{cT}},
\] (21)
where $\Delta = \phi(x_0) - \min_{x \in \mathcal{X}} \phi(x)$ is the initial minimization gap. This gives the iteration complexity
\[
T(\varepsilon) = O\left(\frac{L\Delta}{\varepsilon^2}\right)
\] (22)
of minimizing the gradient norm up to $\varepsilon$, matching the dimension-free lower bound [CDHS17]. Of course, this argument would be meaningless without robustness to errors in the computation of $\text{PP}_{\gamma\phi}(x)$. We verify such robustness by making two observations. First, if instead of exactly ensuring Eq. (20) we find a point $\bar{x}^+$ which satisfies, for some $C > 0$,
\[
\phi(\bar{x}^+) + \frac{c}{2L}(\| \nabla \phi(\bar{x}^+) \|^2 - C\varepsilon^2) \leq \phi(x),
\]
then the same argument will still result in Eq. (22) since the right-hand side of Eq. (21) will only receive $\sqrt{C}\varepsilon$ additive error. Second, such $\bar{x}^+$ can be provided by minimizing the regularized function
\[
\phi_{L,x}(\cdot) = \phi(\cdot) + L\| \cdot - x \|^2,
\] (23)
i.e., the objective of Eq. (16) with $\gamma = 1/(2L)$, up to $O(\varepsilon)$ error in the gradient norm. Similar conclusions hold in the general (constrained) case, with the gradient norm replaced by the norm of the proximal gradient with natural stepsize. Namely, we have the following result (the proof is conceptually simple but technical, and we defer it to Appendix A).

**Proposition 2.1.** Assume $x^+$ minimizes $\phi_{L,x}$ from Eq. (23), and one is given a point $\bar{x}^+$ such that
\[
\| \nabla \phi(\bar{x}^+) - \nabla \phi(x^+) \| \leq \frac{\varepsilon}{3} \quad \text{and} \quad \| \bar{x}^+ - x^+ \| \leq \frac{\varepsilon}{3L},
\] (24)
then
\[
\phi(\bar{x}^+) + \frac{1}{10L}(\| \nabla \phi(\bar{x}^+) \|^2 - 7\varepsilon^2) \leq \phi(x).
\] (25)

Finally, observe that the point $\bar{x}^+$ satisfying Eq. (24) can be obtained by running FGM with restarts (Algorithm 2) with a near-constant total number of oracle calls, since the condition number of the function minimized in Eq. (16) satisfies $\kappa \leq 3$ due to Eq. (18). Namely, combining Corollary 2.1, Remark 2.1, and Proposition 2.1, we obtain the following result.

**Proposition 2.2** (Implementation of proximal point operator via FGM). Given $x \in \mathcal{X}$, let $\phi : \mathcal{X} \to \mathbb{R}$ have $L$-Lipschitz gradient, and let $x^+ = \text{PP}_{\phi/(2L)}(x)$. Let $\bar{x}^+$ be the output of Algorithm 2 run on $\phi_{L,x}(\cdot)$, cf. Eq. (23), with exact oracle $\nabla \phi_{L,x}(\cdot)$, initial point $z^0 = x$, $Z = \mathcal{X}$, and parameters
\[
T = 11, \quad \gamma = \frac{1}{3L}, \quad \text{and} \quad S \geq \frac{1}{2} \log_2 \left(\frac{18L\Delta_{L,x}}{\varepsilon^2}\right),
\] (26)
where $\varepsilon > 0$ is the desired error, and $\Delta_{L,x} := \phi(x) - \min_{x^*} \phi_{L,x}(x^*)$. Then Eqs. (24)–(25) hold, and
\[
\phi_{L,x}(\bar{x}^+) - \phi_{L,x}(x^+) \leq \frac{\varepsilon^2}{6L}.
\] (27)
To illustrate this idea, let us first consider the idealized updates (where the first bound is due to Eq. (14)), guarantees that
\[ \|x^+ - x^+\| \leq \frac{3\varepsilon}{3(3L)} \leq \frac{\varepsilon}{3L}, \quad \|\nabla \phi(x^+) - \nabla \phi(x^+)\| \leq \frac{\varepsilon}{3}, \quad \phi_{L,x}(x^+) - \phi_{L,x}(x^+) \leq \frac{\varepsilon^2}{6L}, \]
where the first bound is due to Eq. (15), and the next two bounds are by smoothness of \( \phi \) and \( \phi_{L,x} \).

\[ \phi_{L,x}(\cdot) \text{ is } 3L\text{-smooth, has condition number } \kappa \leq 3, \text{ and its suboptimality gap at } x \text{ is } \Delta_{L,x}. \]
Hence, running Algorithm 2 with \( T = 11 > \sqrt{40\kappa} \) inner loop iterations, and the number of restarts
\[ S \geq \frac{1}{2} \log_2 \left( \frac{18L\Delta_{L,x}}{\varepsilon^2} \right) \geq \frac{1}{2} \log_2 \left( \frac{18\kappa(3L)\Delta_{L,x}}{(3\varepsilon)^2} \right), \]
cf. Eq. (14), guarantees that

3 Algorithm and main result

Before giving the outline of our approach, let us recap the problem formulation. We assume that \( F(x, y) \) is the objective of the nonconvex-concave min-max problem Eq. (1) with a “prox-friendly” dual feasible set \( Y \) contained in a ball with radius \( R_y \). Our goal is to find \( (\varepsilon_x, \varepsilon_y) \)-FNE \((\tilde{x}, \tilde{y})\) in a small number of the computations of \( \nabla F(x, y) \) and projections onto \( Y \). We assume to be given the initial point \( x_0 \in X \), and fix an arbitrary point \( \bar{y} \in Y \); note that \( \|y - \bar{y}\| \leq 2R_y \) for any \( y \in Y \).

3.1 Conceptual method: primal-dual proximal point iteration

First, following [Nes12], we reduce the problem of finding \( (\varepsilon_x, \varepsilon_y) \)-FNE in Eq. (1) to the problem of finding approximate FNE of the regularized function
\[ F^{\text{reg}}(x, y) := F(x, y) - \frac{\varepsilon_y}{2R_y} \|y - \bar{y}\|^2. \] (28)
Essentially, one can easily show that any \( (\varepsilon_x, \varepsilon_y) \)-FNE of \( F \) is an \( (\varepsilon_x, 3\varepsilon_y) \)-FNE for \( F^{\text{reg}} \). Moreover, while both \( F \) and \( F^{\text{reg}} \) might have no stationary points, the function \( F^{\text{reg}}(x, \cdot) \) has a unique maximizer for any \( x \in X \) as it is \( \varepsilon_y/R_y \)-strongly concave. This strong concavity will help us to obtain faster algorithms for finding \( (\varepsilon_x, \varepsilon_y) \)-FNE when applying standard accelerated procedures.

Our approach consists of running a version of primal-dual proximal point method, choosing the next iterate \((x_t, y_t)\) to approximately solve the strongly-convex-strongly-concave saddle-point problem
\[ \min_{x \in X} \max_{y \in Y} \left\{ F_t^{\text{reg}}(x, y) := F^{\text{reg}}(x, y) + \frac{1}{2\gamma_x} \|x - x_{t-1}\|^2 \right\} \text{ with } \gamma_x = \frac{1}{2L_{xx}}. \] (29)
To illustrate this idea, let us first consider the idealized updates \((\tilde{x}_t, \tilde{y}_t)\) corresponding to the exact saddle point in Eq. (29), which clearly exists and is unique. By definition of saddle point, we have
\[ F_t^{\text{reg}}(\tilde{x}_t, \tilde{y}_{t+1}) \leq F_t^{\text{reg}}(\tilde{x}_t, \tilde{y}_t) \leq F_t^{\text{reg}}(\tilde{x}_{t-1}, \tilde{y}_t) \] (30)
By the definition of \( F_t^{\text{reg}} \), and replacing \( x_{t-1} \) in Eq. (29) with \( \tilde{x}_{t-1} \), the right-hand side gives
\[ F^{\text{reg}}(\tilde{x}_t, \tilde{y}_t) + L_{xx} \|\tilde{x}_t - \tilde{x}_{t-1}\|^2 \leq F^{\text{reg}}(\tilde{x}_{t-1}, \tilde{y}_t). \]
Using that \( \nabla_x F^\text{reg}(x, y) \equiv \nabla_x F(x, y) \), the primal first-order optimality condition for Eq. (29) reads

\[
\hat{x}_t = \hat{x}_{t-1} - \gamma_x \nabla_x F(\hat{x}_t, \hat{y}_t),
\]

which mimics Eq. (17); plugging this back into the previous inequality, we arrive at

\[
F^\text{reg}(\hat{x}_t, \hat{y}_t) + \frac{1}{4L_{xx}} \| \nabla_x F(\hat{x}_t, \hat{y}_t) \|^2 \leq F^\text{reg}(\hat{x}_{t-1}, \hat{y}_t).
\]

(31)

By the left-hand side of Eq. (30), and since \( F^\text{reg}(x, \hat{y}_t) - F^\text{reg}(x, \hat{y}_{t+1}) \equiv F^\text{reg}(x, \hat{y}_t) - F^\text{reg}(x, \hat{y}_{t+1}) \), we replace \( \hat{y}_t \) in the first summand with \( \hat{y}_{t+1} \), thus establishing a recursive relation similar to Eq. (20):

\[
F^\text{reg}(\hat{x}_t, \hat{y}_{t+1}) + \frac{1}{4L_{xx}} \| \nabla_x F(\hat{x}_t, \hat{y}_t) \|^2 \leq F^\text{reg}(\hat{x}_{t-1}, \hat{y}_t).
\]

(32)

Unlike the previous one, this relation can be iterated, and thus analyzed in the same manner as Eq. (20), giving the \( T_x \) complexity factor. Indeed, repeating this for \( t \in \mathbb{T} \), we get (cf. Eq. (21)):

\[
\min_{t \in \mathbb{T}} \| \nabla_x F(\hat{x}_t, \hat{y}_t) \| \leq \sqrt{\frac{1}{T} \sum_{t \in \mathbb{T}} \| \nabla_x F(\hat{x}_t, \hat{y}_t) \|^2} = \sqrt{\frac{4L_{xx}[F^\text{reg}(\hat{x}_0, \hat{y}_1) - F^\text{reg}(\hat{x}_T, \hat{y}_T)]}{T}}.
\]

Finally, we can relate \( F^\text{reg}(\hat{x}_0, \hat{y}_1) - F^\text{reg}(\hat{x}_T, \hat{y}_T) \) to the primal gap \( \Delta = \varphi(\hat{x}_0) - \min_{x \in \mathcal{X}} \varphi(x) \):

\[
F^\text{reg}(\hat{x}_0, \hat{y}_1) \leq F(\hat{x}_0, \hat{y}_1) \leq \max_{y \in \mathcal{Y}} F(\hat{x}_0, y) = \varphi(\hat{x}_0),
\]

\[
F^\text{reg}(\hat{x}_T, \hat{y}_T) = \max_{y' \in \mathcal{Y}} F^\text{reg}(\hat{x}_T, y') \geq \max_{y \in \mathcal{Y}} F(\hat{x}_T, y) - \frac{\varepsilon_y}{2R_y} \max_{y' \in \mathcal{Y}} \| y' - \hat{y} \|^2 \geq \min_{x \in \mathcal{X}} \varphi(x) - 2\varepsilon_y R_y.
\]

Neglecting the additive to \( \Delta \) term \( 2\varepsilon_y R_y \), we can guarantee that there exists \( \tau \in \mathbb{T} \) such that

\[
\| \nabla_x F(\hat{x}_\tau, \hat{y}_\tau) \| = O\left( \sqrt{\frac{L_{xx} \Delta}{T}} \right),
\]

which corresponds to \( O(T_x) \) iterations (Eq. (6)) to ensure \( \| \nabla_x F(\hat{x}_\tau, \hat{y}_\tau) \| \leq \varepsilon_x \). At the same time,

\[
\left\| \hat{y}_\tau - \text{prox}_{y, \mathcal{Y}} \left( -\frac{\nabla_y F(\hat{x}_\tau, \hat{y}_\tau)}{L_{yy}} \right) \right\| \leq \left\| \text{prox}_{y, \mathcal{Y}} \left( -\frac{\nabla_y F(\hat{x}_\tau, \hat{y}_\tau)}{L_{yy}} \right) - \text{prox}_{y, \mathcal{Y}} \left( -\frac{\nabla_y F(\hat{x}_\tau, \hat{y}_\tau)}{L_{yy}} \right) \right\| \leq \frac{1}{L_{yy}} \| \nabla_y F(\hat{x}_\tau, \hat{y}_\tau) - \nabla_y F^\text{reg}(\hat{x}_\tau, \hat{y}_\tau) \| \leq \frac{2\varepsilon_y}{L_{yy}},
\]

where the equality is by the optimality condition in Eq. (29); thus, \( (\hat{x}_\tau, \hat{y}_\tau) \) is an \((\varepsilon_x, O(\varepsilon_y))\)-FNE.

So far, we assumed that we can compute the step (29) exactly and we analyzed the iteration complexity of the iterative procedure (29). Next we show how the “conceptual” updates can be implemented using Algorithm 2, leading to our suggested algorithm and the complexity estimate of Theorem 1.1.
3.2 Implementation of conceptual method

As in the case of the usual proximal point method, the update stemming from the auxiliary min-max problem in Eq. (29) cannot be performed exactly. To address this problem, we can extend the approach described in Sec. 2.2 and approximately solve the (primal) minimization problem in Eq. (29) up to $O(\varepsilon_\gamma)$ gradient accuracy (cf. Proposition 2.2) via Algorithm 2. However, this generates a new difficulty: the minimization objective in Eq. (29) stems from the nested maximization problem, hence neither it nor its gradient can be computed exactly. Instead, one must provide inexact oracle for this function through the following steps:

- Given the current primal iterate $x_{t-1}$, consider the minimization problem corresponding to the dual function of Eq. (33) evaluated at some fixed $y \in Y$:

$$
\psi_t(y) := \min_{x \in X} \left\{ F_t^{\text{reg}}(x, y) := F^{\text{reg}}(x, y) + L_{\infty} \|x - x_{t-1}\|^2 \right\}.
$$

Fixing $y \in Y$ and solving the above minimization problem, we obtain an approximation $\tilde{x}_t(y)$ of the exact minimizer $\bar{x}_t(y)$ by running Algorithm 2 with exact oracle $\nabla_x F(\cdot, y) + 2L_{\infty}(\cdot - x_{t-1})$. As $F(\cdot, y)$ is well-conditioned, it only takes a logarithmic number of oracle calls to ensure a very small (inversely polynomial in the problem parameters) error of approximating $\tilde{x}_t(y)$. On the other hand, Danskin’s theorem ([Dan66] and [NSLR19, Lem. 24]) implies that $\psi_t(y)$ has $O(L_{\gamma}^+)$-Lipschitz gradient given by

$$
\nabla \psi_t(y) \equiv \partial_y F_t^{\text{reg}}(\tilde{x}_t(y), y).
$$

Hence, $\tilde{x}_t(y)$ provides a $\delta$-inexact oracle

$$
\tilde{\psi}_t(y) := F_t^{\text{reg}}(\tilde{x}_t(y), y), \quad \nabla \tilde{\psi}_t(y) := \partial_y F_t^{\text{reg}}(\tilde{x}_t(y), y).
$$

for $\psi_t(y)$ with arbitrarily small error $\delta$. For convenience, we outline the subroutine that returns $\tilde{x}_t(y)$ and the approximate dual gradient $\nabla \tilde{\psi}_t(y) = \partial_y F_t^{\text{reg}}(\tilde{x}_t(y), y)$ in Algorithm 3.

- Now, observe that, by strong duality, we can switch min and max in Eq. (29), recasting it as

$$
y_t = \arg \max_{y \in Y} \psi_t(y), \quad x_t = \tilde{x}_t(y_t).
$$

Naturally, we replace those with the approximate updates given by

$$
y_t \approx \arg \max_{y \in Y} \tilde{\psi}_t(y), \quad x_t = \tilde{x}_t(y_t),
$$

maximizing $\tilde{\psi}_t(y)$ by running Algorithm 2 with inexact gradient $\nabla \tilde{\psi}_t(y)$ defined in Eq. (34), and without using $\tilde{\psi}_t(y)$. Since $\tilde{\psi}_t(y)$ is $L_{\gamma\gamma}^+$-smooth and $\varepsilon_y/R_y$-strongly convex, in $O(T)$ calls of the inexact oracle $\nabla \tilde{\psi}_t(\cdot)$ Algorithm 2 finds $O(\varepsilon_y)$-approximate maximizer $y_t$ of $\tilde{\psi}_t$, ensuring

$$
L_{\gamma\gamma}^+ \left\| y_t - \text{prox}_{y, Y} \left( -\frac{1}{L_{\gamma\gamma}^+} \nabla \psi_t(y_t) \right) \right\| \leq \varepsilon_y, \quad \max_{y \in Y} \psi_t(y) - \psi_t(y_t) \leq \frac{\varepsilon_y^2}{18L_{\gamma\gamma}^+}.
$$

Combining the first of these inequalities with Eq. (33), and recalling that $\tilde{x}_t(y_t) \approx \tilde{x}_t(y_t)$ with very high accuracy, we ensure that $(x_t, y_t)$ obtained via Eq. (36) is $O(\varepsilon_y)$-stationary in $y$ (in
Algorithm 4 First-Order Stationary Point Search in Nonconvex-Concave Smooth Min-Max Problem

Require: $\nabla F(\cdot, \cdot), Y, x_0, \bar{y} \in Y, \bar{T}_x, \bar{T}_y, S_y, \gamma_x, \gamma_y, \lambda_y, T^o, S^o$

1: for $t \in [\bar{T}_x]$ do
   \hspace{1em} $\triangleright$ Using Algorithms 2 and 3 as subroutines
2: \hspace{1em} $y_t = \text{RestartFGM}(\bar{y}, Y, \gamma_y, T_y, S_y, -\nabla \psi_t(\cdot))$
3: \hspace{1em} with $\nabla \psi_t(y)$ returned by $\text{RegDualSol}(y, x_{t-1}, \bar{y}, \gamma_x, \lambda_y, T^o, S^o)$
4: \hspace{1em} $x_t = \tilde{x}_t(y_t)$ returned by $\text{RegDualSol}(y_t, x_{t-1}, \bar{y}, \gamma_x, \lambda_y, T^o, S^o)$
5: \hspace{1em} end for
6: return $(x_{\tau}, y_{\tau})$ with $\tau \in \text{Argmin}_{t \in [\bar{T}_x]} \|\nabla_x F(x_t, y_t)\|$

the sense of Definition 1). As this must be repeated for $t \in [\bar{T}_x]$, we recover Eq. (7). the first term in Eq. (7). On the other hand, the second inequality leads to the extra $O(\varepsilon_y^2/L_{yy})$ error in the saddle point relation Eq. (30), whereas this error must be $O(\varepsilon_x^2/L_{xx})$ in order to preserve the argument in Sec. 3.1. This is easy to fix: it suffices to perform a logarithmic in $\bar{T}_x$ number of additional restarts when maximizing $\psi_t(y)$. Thus, the argument in Sec. 3.1 remains valid, and we find an $(\varepsilon_x, O(\varepsilon_y))$-FNE of Eq. (1) in $\widetilde{O}(T_x T_y)$ gradient computations.

We present the resulting algorithm, which is our main methodological contribution, as Algorithm 4.

3.3 Convergence guarantee for Algorithm 4

Next we state our main result, the full version of Theorem 1.1.

Theorem 3.1. Define

$$\lambda_y := \frac{\varepsilon_y}{R_y}, \quad \Theta := L_{yy} R_y^2, \quad \Theta^+ := L_{yy}^+ R_y^2,$$

$$\delta := \min \left\{ 8\varepsilon_y R_y, \frac{\Theta}{2 T_y}, \sqrt{\frac{\Delta (\Theta^+ - \Theta)}{T_x T_y^2}} \right\}. \quad (37)$$

Running Algorithm 4 with

$$\gamma_x = \frac{1}{2L_{xx}}, \quad \gamma_y = \frac{1}{L_{yy}^+ + \lambda_y}, \quad (38)$$

$$\bar{T}_x \geq \frac{10L_{xx}(\Delta + 2\varepsilon_y R_y)}{\varepsilon_x^2}, \quad \bar{T}_y \geq \sqrt{\frac{40(L_{yy}^+ + \lambda_y)}{\lambda_y}}, \quad S_y \geq 2 \max \left\{ \log_2 (T_y), \log_2 \left( \frac{\Theta^+}{\delta} \right) \right\}, \quad (39)$$

$$T^o = 11, \quad \text{and} \quad S^o \geq \frac{1}{2} \log_2 \left( (1278 \Delta + 36 \Theta^+ + 36 \varepsilon_y R_y) \left[ \frac{L_{xx}}{\varepsilon_x^2} + \frac{9 \Theta^+}{\delta^2} + \frac{1}{\delta} \right] \right) \quad (40)$$

outputs $(4\varepsilon_x, 4\varepsilon_y)$-FNE of Eq. (1) after

$$[T^o S^o S_y \bar{T}_x \bar{T}_y] \quad (41)$$

computations of the gradient $\nabla F(x, y)$ and twice larger number of projections onto $Y$. 

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Remark 3.1 (Nonconvex-strongly concave setup). As follows from Theorem 3.1, Algorithm 4 can also be used when the objective function $F(x, y)$ is known to be $\lambda_y$-strongly concave in $y$ with general $\lambda_y$, leading to the complexity estimate $\tilde{O}(T_y (\kappa_y^+)^{1/2})$, where $\kappa_y^+ = L_{xy}^+/\lambda_y$ is the condition number of the dual function in (1), matching the best known rate in this case. It suffices to run the algorithm with parameter values as prescribed in of Theorem 3.1, but fixing a prescribed value for $\lambda_y$.

3.4 Proof of Theorem 3.1

We use the notation introduced in Sec. 3.1–3.2, and refer to the arguments presented there.

1°. Consider first the “idealized” update from the primal iterate $x_{t-1}$:

$$y_t = \arg \max_{y \in Y} \psi_t(y), \quad x_t = \hat{x}_t(y_t).$$  \hfill (42)

Here, $\psi_t(y)$ and $\hat{x}_t(y)$ are defined as

$$\psi_t(y) := \min_{x \in X} F_t^\text{reg}(x, y) = F_t^\text{reg}(\hat{x}_t(y), y),$$

$$\hat{x}_t(y) := \arg \min_{x \in X} F_t^\text{reg}(x, y),$$ \hfill (43)

with

$$F_t^\text{reg}(x, y) := F(x, y) - \frac{\lambda_y}{2} \|y - \bar{y}\|^2,$$

$$F_t^\text{reg}(x, y) := F_t^\text{reg}(x, y) + L_{xx}\|x - x_{t-1}\|^2 = F_t(x, y) - \frac{\lambda_y}{2} \|y - \bar{y}\|^2,$$

Clearly, $\psi_t(y)$ is $\lambda_y$-strongly concave, with $\lambda_y = \varepsilon_y/R_y$. On the other hand, by Danskin’s theorem (see, e.g., [NSLR19, Lem. 24]), $\psi_t(y)$ is continuously differentiable with

$$\nabla \psi_t(y) = \partial_y F_t^\text{reg}(\hat{x}_t(y), y) = \partial_y F(\hat{x}_t(y), y) - \lambda_y(y - \bar{y}),$$ \hfill (44)

and $\nabla \psi_t(y)$ is $(L_{yy}^+ + \lambda_y)$-Lipschitz with $L_{xy}^+$ defined in Eq. (5).

2°. We now focus on the properties of the point $\hat{x}_t(y)$ returned when calling

$$\text{RegDualSol}(y, x_{t-1}, \bar{y}, \gamma_x, \lambda_y, T^o, S^o),$$

cf. line 3 of Algorithm 4, as well as the corresponding pair $[\tilde{\psi}_t(y), \tilde{\nabla} \psi_t(y)]$, cf. Eq. (34). Note that the function value $\tilde{\psi}_t(y)$ is never computed in Algorithm 4 and we only use it in the analysis. Inspecting the pseudocode of RegDualSol (Algorithm 3), we see that $\hat{x}_t(y)$ corresponds to the approximate minimizer of $F_t^\text{reg}(x, y)$ (thus also $F_t(x, y)$) in $x$, obtained by running restarted FGM (Algorithm 2) starting from $x_{t-1}$, with stepsize $\gamma = 1/(3L_{xx})$, $T^o = 11$ inner loop iterations, and the number of restarts $S^o$ given in Eq. (40). Observe that minimizing $F_t(\cdot, y)$ corresponds to computing the proximal operator $P_{\lambda_y} F_t(\cdot, y)(x_{t-1})$, where $F(\cdot, y)$ is $L_{xx}$-smooth. Hence, due to our choice of input parameters, the premise of Proposition 2.2 is satisfied; applying it with our choice of $S^o$ results in

$$\|\nabla_x F(\hat{x}_t(y), y) - \nabla_x F(\hat{x}_t(y), y)\| \leq \frac{\varepsilon_x}{3},$$ \hfill (45)

$$\|\hat{x}_t(y) - \tilde{x}_t(y)\| \leq \min \left\{ \frac{\varepsilon_x}{3L_{xx}}, \frac{\delta}{8L_{xy}R_y} \right\},$$ \hfill (46)

$$F_t^\text{reg}(\hat{x}_t(y), y) - F_t^\text{reg}(\hat{x}_t(y), y) \leq \min \left\{ \frac{\varepsilon_x^2}{6L_{xx}}, \frac{\delta}{2} \right\}. \hfill (47)$$
Here, Eq. (45) and the first terms in Eqs. (46)–(47) are due to the first of three terms in brackets under logarithm in Eq. (40), cf. Eq. (26), combined with a very crude uniform over $y \in Y$ bound

$$F_t^\reg(x_{t-1}, y) - \min_{x \in X} F_t^\reg(x, y) \leq 71 \Delta + 2 \Theta + 2 \varepsilon_y \mathcal{R}_y.$$  

(We defer the proof of Eq. (48) to appendix in order to streamline the presentation.) On the other hand, the second respective estimates in Eqs. (46)–(47) correspond to the two remaining terms in brackets under logarithm in Eq. (40), cf. Eq. (26). Now, Eqs. (45)–(47) have two immediate consequences.

- By Proposition 2.1 we get

$$F(\bar{x}_t(y), y) + \frac{1}{10 \mathcal{R}_{xx}} (\|\nabla_x F(\bar{x}_t(y), y)\|^2 - 7 \varepsilon_x^2) \leq F(x_{t-1}, y),$$

cf. Eq. (32). We will revisit this bound later on.

- As we verify in appendix, the second respective terms in the right-hand side of Eqs. (46)–(47) together ensure that the pair $[-\tilde{\psi}_t^\delta(y), -\tilde{\nabla} \psi_t(y)]$ with

$$\tilde{\psi}_t^\delta(y) := F_t^\reg(\bar{x}_t(y), y) + \frac{\delta}{4}, \quad \tilde{\nabla} \psi_t(y) = \partial_y F^\reg(\bar{x}_t(y), y).$$

amounts to a $\delta$-inexact first-order oracle for $-\psi_t(y)$ in accordance with Definition 2. In other words,

$$0 \leq -\psi_t(y') + \tilde{\psi}_t^\delta(y) + \langle \tilde{\nabla} \psi_t(y), y' - y \rangle \leq \frac{L^+_y + \lambda y}{2} \|y' - y\|^2 + \delta, \quad \forall y, y' \in Y,$$

where we used that $\psi_t$ is $(L^+_y + \lambda_y)$-smooth.

3°. Now consider the actual update performed in the for-loop of Algorithm 4. Namely, observe that

$$y_t \approx \arg \max_{y \in Y} \psi_t(y), \quad x_t = \bar{x}_t(y_t),$$

where the precise meaning of "$\approx$", cf. line 2, is that $y_t = \operatorname{RestartFGM}(\bar{y}, Y, \gamma_y, T_y, S_y, -\bar{\nabla} \psi_t(\cdot))$. In other words, $y_t$ is obtained by running Algorithm 2 with $\delta$-inexact gradient $\bar{\nabla} \psi_t(\cdot)$, starting from $\bar{y} \in Y$, with $T_y$ iterations in the inner calls of FGM and $S_y$ restarts, $T_y$ and $S_y$ being given in Eq. (39). Recall that $\psi_t(\cdot)$ is $(L^+_y + \lambda_y)$-smooth and $\lambda_y$-strongly convex with $\lambda_y = \varepsilon_y / \mathcal{R}_y$, and

$$\delta \leq \frac{\Theta}{2T_y} \leq \left( \frac{L^+_y + \lambda_y}{2} \right) \frac{R_y^2}{\mathcal{R}_{xx}^2},$$

cf. Eq. (37). Recalling our choice of $T_y$ and $S_y$ in Eq. (39), we can apply Corollary 2.1, and arrive at

$$\|y_t - y_t^*\| \leq \frac{\varepsilon_y}{3L^+_y},$$

$$\psi_t(y_t^*) - \psi_t(y_t) \leq \min \left\{ \frac{\varepsilon_y^2}{18 L^+_y}, \frac{\varepsilon_x^2}{L^+_y} \right\},$$

$$(L^+_y + \lambda_y) \left\| y_t - \operatorname{prox}_{y_t, Y} \left( -\frac{\tilde{\nabla} \psi_t(y_t)}{L^+_y + \lambda_y} \right) \right\| \leq \varepsilon_y,$$
where $y_t^*$ is the exact maximizer of $\psi_t$ (cf. Eq. (15)). Here we used the first lower bound in Eq. (39) for $S_y$ to obtain all estimates, except for the second estimate of $\psi_t(y_t^*) - \psi_t(y_t)$ for which we used the second lower bound for $S_y$, and substituted the third term in Eq. (37) for $\delta$. Moreover, it can be shown (see, e.g., [KSL14, Lemma 4.3]) that the norm of the proximal gradient monotonically decreases with stepsize, hence

$$L_{yy} \left\| y_t - \text{prox}_{y_t, Y} \left( - \nabla \psi_t(y_t) \right) \right\| \leq (L_{yy} + \lambda_y) \left\| y_t - \text{prox}_{y_t, Y} \left( - \nabla \psi_t(y_t) \right) \right\| \leq \varepsilon_y.$$  

Using Eq. (44) and the Lipschitzness of $\nabla_y F(\cdot, y)$ and $\text{prox}_{y_t, Y} (\cdot)$, we see that $x_t = \bar{x}_t(y_t)$ satisfies

$$L_{yy} \left\| y_t - \text{prox}_{y_t, Y} \left( - \nabla_y F(x_t, y_t) \right) \right\| = L_{yy} \left\| y_t - \text{prox}_{y_t, Y} \left( - \frac{\partial_y F(\bar{x}_t(y_t), y_t)}{L_{yy}} \right) \right\| \leq L_{yy} \left\| y_t - \text{prox}_{y_t, Y} \left( - \frac{\partial_y F(\bar{x}_t(y_t), y_t)}{L_{yy}} \right) \right\| + \left\| \frac{\partial_y F(\bar{x}_t(y_t), y_t)}{L_{yy}} \right\| \leq L_{yy} \left\| y_t - \text{prox}_{y_t, Y} \left( - \nabla \psi_t(y_t) \right) \right\| + \frac{\varepsilon_y}{R} \| y_t - \bar{y} \| + L_{xy} \left\| \bar{x}_t(y_t) - \bar{x}_t(y_t) \right\| \\ \leq 3\varepsilon_y + \frac{\delta}{8R_y} \\ \leq 4\varepsilon_y,$$

where the penultimate transition uses Eq. (46), and the last transition uses our choice of $\delta$ in Eq. (37). Thus, the iterate $(x_t, y_t)$ is being kept $4\varepsilon_y$-stationary in $y$ at any iteration $t$.

4°. We now revisit Eq. (49). Applying it to $y = y_t$, we get

$$F_t^{\text{reg}}(x_t, y_t) + \frac{1}{10L_{xx}} \left( \| \nabla_y F(x_t, y_t) \|^2 - 7\varepsilon_y^2 \right) \leq F_t^{\text{reg}}(x_{t-1}, y_t),$$  

which mimics Eq. (31). Our goal, however, is to mimic Eq. (32), for which we must lower-bound, up to a small error, $F_t^{\text{reg}}(x_t, y_t)$ via $F_t^{\text{reg}}(x_t, y_{t+1})$, or, equivalently, $F_t^{\text{reg}}(x_t, y_t)$ via $F_t^{\text{reg}}(x_t, y_{t+1})$.

First,

$$F_t^{\text{reg}}(x_t, y_{t+1}) \leq \max_{y \in Y} F_t^{\text{reg}}(x_t, y) = \varphi_t(x_t),$$  

where $\varphi_t(x) := \max_{y \in Y} F_t^{\text{reg}}(x, y)$ is the primal function in the saddle-point problem Eq. (29). On the other hand, denoting $x_t^* = \bar{x}_t(y_t^*)$, so that $(x_t^*, y_t^*)$ is the unique saddle point in Eq. (29), we have

$$F_t^{\text{reg}}(x_t, y_t) = F_t^{\text{reg}}(\bar{x}_t(y_t), y_t) \geq F_t^{\text{reg}}(\bar{x}_t(y_t), y_t) = \psi_t(y_t) \quad (54) \geq \frac{\varepsilon_y^2}{L_{xx}T_y} \Rightarrow \psi_t(y_t^*) - \frac{\varepsilon_y^2}{L_{xx}T_y} = \varphi_t(x_t^*) - \frac{\varepsilon_y^2}{L_{xx}T_y} \quad (57).$$
It remains to compare \( \varphi_t(x_t) \) and \( \varphi_t(x_t^*) \). Combining \( F_t^{\text{reg}}(x_t^*, y_t^*) \geq F_t^{\text{reg}}(x_t^*, y_t) \) with the previous inequality, and observing that \( F_t^{\text{reg}}(\cdot, y_t) \) is \( L_{\infty} \)-strongly convex and minimized at \( \hat{x}_t(y_t) \), we obtain

\[
\| \hat{x}_t(y_t) - x_t^* \| \leq \frac{\sqrt{2\varepsilon_x}}{L_{\infty}T_y}.
\]

On the other hand, Eq. (46) applied to \( y = y_t \) gives

\[
\| x_t - \hat{x}_t(y_t) \| \leq \frac{\delta}{8L_{xy}R_y} \leq \frac{\Delta(\Theta^+ - \Theta)}{64L_{xy}^2 R_y^2 T_y^2 T_x} = \frac{\varepsilon_x}{8\sqrt{8L_{xy}T_y}},
\]

where we substituted the third term in Eq. (37) for \( \delta \). Combining these estimates, we get

\[
\| x_t - x_t^* \| \leq \frac{2\varepsilon_x}{L_{\infty}T_y}.
\]

Now, notice that \( \varphi_t \) is \( (3L_{\infty} + L_{xy}^2/\lambda_y) \)-smooth by Danskin’s theorem, and is minimized at \( x_t^* \). Thus,

\[
\varphi_t(x_t) - \varphi_t(x_t^*) \leq \frac{3(L_{\infty} + L_{xy}^2/\lambda_y)}{2}\| x_t - x_t^* \|^2 \leq \frac{6\varepsilon_x^2}{L_{\infty}} \left( 1 + \frac{L_{xy}^2}{\lambda_y L_{\infty} T_y^2} \right) \leq \frac{6\varepsilon_x^2}{L_{\infty}} \left( 1 + \frac{L_{yy}^2}{\lambda_y T_y^2} \right) \leq \frac{7\varepsilon_x^2}{L_{\infty}}.
\]

Returning to Eqs. (56) and (57), we arrive at

\[
F_{\text{reg}}^{\text{reg}}(x_t, y_t) \geq F_{\text{reg}}^{\text{reg}}(x_t, y_{t+1}) - \frac{8\varepsilon_x^2}{L_{\infty}}.
\]

Combining this with Eq. (55) we finally get an analogue of Eq. (32):

\[
F_{\text{reg}}^{\text{reg}}(x_t, y_{t+1}) + \frac{1}{10L_{\infty}} \| \nabla_x F(x_t, y_t) \|^2 - 15\varepsilon_x^2 \leq F_{\text{reg}}^{\text{reg}}(x_{t-1}, y_t).
\]

This inequality can be iterated to the next step, and we can proceed as outlined in Sec. 3.1:

\[
\min_{t \in [T_x]} \| \nabla_x F(x_t, y_t) \| \leq \sqrt{\frac{1}{T} \sum_{t \in [T_x]} \| \nabla_x F(x_t, y_t) \|^2} = \sqrt{\frac{10L_{\infty}[F_{\text{reg}}^{\text{reg}}(x_0, y_1) - F_{\text{reg}}^{\text{reg}}(x_{T_x}, y_{T_x})]}{T_x} + 15\varepsilon_x^2} \leq \sqrt{\frac{10L_{\infty}[\Delta + 2\varepsilon_y R_y]}{T_x} + 15\varepsilon_x^2} \leq 4\varepsilon_x.
\]

Combining this with the result of 3\(^o\), we have found an \( (4\varepsilon_x, 4\varepsilon_y) \)-FNE. On the other hand, we have performed \([T_0S\varepsilon_y T_x T_y]\) iterations of FGM (i.e., iterations in the for-loop of Algorithm 1) in total, with one computation of \( \nabla F \) and two projections on \( Y \) at each iteration.

\[\square\]

### 4 Related work

Our goal in this section is to briefly overview the recent stream of works on efficient algorithms for approximate FNE search in nonconvex-concave problems.
To the best of our knowledge, [NSLR19] was the first work providing non-asymptotic convergence rates for FNE search in general nonconvex-concave problems as in Eq. (1), without assuming special structure of the objective function. Their approach is to perform gradient descent directly on the primal function of the $O(\varepsilon_y)$-regularized version of the problem in Eq. (1), exploiting the fact that this function has $O(\varepsilon_y^{-1})$-Lipschitz gradient due to Danskin’s theorem. The resulting complexity is $O(\varepsilon_y^{-2} \varepsilon_y^{-3/2})$ in our notation, the extra $O(\varepsilon_y^{-1})$ factor stemming from the poor smoothness of the primal function, which restricts stepsize to be $O(\varepsilon_y)$.

Subtler analyses of the problem have been provided in works [TJNO19, KM19], and, more recently, in the concurrent work [LJJ+20] of which we became aware upon finalizing this manuscript. While the underlying idea of solving the intermediate min-max problem Eq. (29) seems to be present in all these works, there are considerable differences between them. First, [TJNO19, KM19] focus on the problem of finding an $\varepsilon_x$-stationary point of the Moreau envelope of the primal function, reducing this problem to that of finding $(\varepsilon_x, \varepsilon_y)$-FNE of the associated min-max problem with $\varepsilon_y = O(\varepsilon_x^2)$. As a result, their complexity writes as $O(\varepsilon_x^{-3})$ for the Moreau envelope; nonetheless, we expect their results to be adaptable to the FNE search problem with complexity similar to ours. More crucially, the algorithms proposed in these works are somewhat less transparent than ours. In particular, [TJNO19] runs a mirror-prox type subroutine to approximate the proximal point step (while also using an FGM-type subroutine), and none of these works uses the readily available technical results such as [DGN14] on inexact-oracle FGM, and those of [LMH15, PLD+18] on the FGM-type implementation of the proximal operator. In contrast, our work simply puts together the existing available results in the literature to obtain the desired iteration complexity.

Although not directly relevant to this note, let us also mention some of the recent advances in distributed/multi-block non-convex min-max optimization [LMO+19, LTH19, LTHC94] as well as zero-th order methods methods [LLC+06, WBMR19].

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A Deferred proofs

A.1 Proof of Proposition 2.1

First recall (cf. Eq. (20)) that the actual minimizer \( x^+ \) of \( \phi_{L,x} \) satisfies

\[
\phi(x^+) + \frac{1}{4L} \| \nabla \phi(x^+) \|^2 \leq \phi(x) .
\]  

(58)

On the other hand, by the first part of Eq. (24) we have

\[
\| \nabla \phi(x^+) \| \leq \| \nabla \phi(x^+) \| + \varepsilon ,
\]  

(59)

whence, using also the second part of Eq. (24) and \( L \)-smoothness of \( \phi \),

\[
\phi(x^+) \leq \phi(x^+) + \langle \nabla \phi(x^+), x^+ - x^+ \rangle + \frac{L}{2} \| x^+ - x^+ \|^2 \\
\leq \phi(x^+) + \frac{\| \nabla \phi(x^+) \| \varepsilon}{3L} + \frac{\varepsilon^2}{18L} \\
= \phi(x^+) + \frac{1}{2L} \left[ \left( \| \nabla \phi(x^+) \| + \frac{\varepsilon}{3} \right)^2 - \| \nabla \phi(x^+) \|^2 \right] .
\]

Combining this with Eq. (59), for arbitrary \( c' > 0 \) we have

\[
\phi(x^+) + \frac{c'}{2L} \left( \| \nabla \phi(x^+) \|^2 - \varepsilon^2 \right) \\
\leq \phi(x^+) + \frac{1}{2L} \left[ \left( \| \nabla \phi(x^+) \| + \frac{\varepsilon}{3} \right)^2 - \| \nabla \phi(x^+) \|^2 + c' \left( \| \nabla \phi(x^+) \| + \frac{\varepsilon}{3} \right)^2 - c' \varepsilon^2 \right] \\
\leq \phi(x^+) + \frac{1}{2L} \left[ \frac{10}{9} \left( \| \nabla \phi(x^+) \|^2 + \varepsilon^2 \right) - \| \nabla \phi(x^+) \|^2 + \frac{10c'}{9} \left( \| \nabla \phi(x^+) \|^2 + \varepsilon^2 \right) - c' \varepsilon^2 \right] \\
= \phi(x^+) + \frac{1}{2L} \left[ \frac{1}{9} \| \nabla \phi(x^+) \|^2 + \frac{10 + c'}{9} \varepsilon^2 \right] ,
\]

where the second line is by \((a + \frac{1}{3}b)^2 \leq \frac{10}{9}(a^2 + b^2)\). Choosing \( c' = 1/5 \) and reusing (58), we arrive at

\[
\phi(x^+) + \frac{1}{10L} \left( \| \nabla \phi(x^+) \|^2 - \varepsilon^2 \right) \leq \phi(x) + \frac{17\varepsilon^2}{30L} \leq \phi(x) + \frac{3\varepsilon^2}{5L} ,
\]

whence (25) follows by rearranging the terms. \( \square \)

A.2 Verification of Eq. (51)

By concavity and \((L_{yy}^+ + \lambda_y)\)-smoothness of \( \psi_t \), one has

\[
0 \leq -\psi_t(y') + \psi_t(y) + \langle \nabla \psi_t(y), y' - y \rangle \leq \frac{L_{yy}^+ + \lambda_y}{2} \| y' - y \|^2 , \quad \forall y, y' \in Y .
\]

By Eqs. (43), (47) and (50),

\[
\frac{\delta}{4} \leq \overline{\psi_t}(y) - \psi_t(y) \leq \frac{3\delta}{4} , \quad \forall y \in Y .
\]
On the other hand, by the second part of Eq. (46),

$$\|\nabla \psi_t(y) - \nabla \psi_t(y)\| = \|\partial_y F(\tilde{x}_t(y), y) - \partial_y F(\tilde{x}_t(y), y)\| \leq L_{xy} \|\tilde{x}_t(y) - \tilde{x}_t(y)\| \leq \frac{\delta}{8R_y},$$

hence, since \(\|y' - y\| \leq 2R_y\) for any \(y', y \in Y\), we get

$$-\frac{\delta}{4} \leq (\nabla \psi_t(y) - \nabla \psi_t(y), y' - y) \leq \frac{\delta}{4}.$$

We obtain (51) by summing up the two-sided inequalities above. \(\square\)

A.3 Verification of Eq. (48)

Let \(\varphi_t(x) = \max_{y \in Y} F^t_{\text{reg}}(x, y)\) be the primal function of the saddle-point problem in Eq. (29). Then

$$\varphi_t(x) - 2L_{yy} R^2_y \leq F_t(x, y) \leq \varphi_t(x)$$

by bounding the variation of a smooth function \(F(x, \cdot)\) over \(y \in Y\), whence

$$F_t(x_{t-1}, y) - \min_{x \in X} F(x, y) \leq \varphi_t(x_{t-1}) - \min_x \varphi_t(x) + 2L_{yy} R^2_y$$

$$\leq \varphi_t(x_{t-1}) - \min_x \varphi(x) + 2L_{yy} R^2_y + 2\epsilon_y R_y,$$ (60)

where we used that \(F^t_{\text{reg}}(x, y) \geq F(x, y) - 2\epsilon_y R_y\). Thus, it only remains to prove that \(\varphi_t(x_{t-1})\) decreases in \(t\) up to a small error (since \(\varphi_t(x_0) \leq \varphi(x_0)\)). To this end, we proceed by induction.

The base is obvious: Eq. (48) is clearly satisfied when \(t = 1\), by observing that

$$\varphi_1(x) - 2\Theta \leq F^1_{\text{reg}}(x, y) \leq \varphi_1(x),$$

and \(\varphi_1(x_0) \geq \varphi(x_0) - 2\epsilon_y R_y\). Now, assume that Eq. (48) was satisfied at steps \(\tau \in [t - 1]\), so that our analysis of these steps was valid. Then, by part 5 of the proof of Theorem 3.1, at all these previous steps, including step \(t - 1\), the saddle-point problem in Eq. (29) has been solved up to the primal gap \(7\varepsilon_x^2 / L_{xx}\), i.e.,

$$\varphi_{\tau}(x_{\tau}) - \min_x \varphi_{\tau}(x) \leq \frac{7\varepsilon_x^2}{L_{xx}}, \quad \tau \in [t - 1].$$ (61)

On the other hand, \(\varphi_{\tau}(x_{\tau}) = \varphi_{\tau - 1}(x_{\tau - 1})\) by definition, cf. Eq. (29), and this holds for any \(\tau \in [T_x]\). Combining the two inequalities sequentially, we get

$$\varphi_t(x_{t-1}) \leq \varphi_{t-1}(x_{t-2}) + \frac{7\varepsilon_x^2}{L_{xx}} \leq \cdots \leq \varphi_1(x_0) + \frac{7(t - 1)\varepsilon_x^2}{L_{xx}} \leq \varphi(x_0) + \frac{7T_x \varepsilon_x^2}{L_{xx}} \leq \varphi(x_0) + 70\Delta.$$

Combining this with Eq. (60), we establish Eq. (48). \(\square\)