Noncommutative unification of general relativity with quantum mechanics and canonical gravity quantization

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Abstract

The groupoid approach to noncommutative unification of general relativity with quantum mechanics is compared with the canonical gravity quantization. It is shown that by restricting the corresponding noncommutative algebra to its (commutative) subalgebra, which determines the space-time slicing, an algebraic counterpart of superspace (space of 3-metrics) can be obtained. It turns out that when this space-time slicing emerges the universe is already in its commutative regime. We explore the consequences of this result.

1 Introduction

In recent years a new approach has appeared to the quantization of gravity, the one based on noncommutative geometry. The idea is to make space-time a noncommutative space (which is essentially nonlocal) with the hope that in this way at least some major obstacles to the gravity quantization could eventually be overcome. There are many attempts in this direction [1]. In [2] we have followed Connes [3, p. 99] who, in order to make a space $X$ noncommutative defines a noncommutative algebra not directly on $X$ but rather on a groupoid over $X$. This approach, which has been further
developed in the series of works 4, will be called a groupoid approach to the
unification of general relativity and quantum mechanics.

The aim of the present paper is to compare the groupoid approach with
the canonical gravity quantization 5, which can be thought of as a “refer-
ence point” for other methods of quantizing gravity. The groupoid approach
is “more radical” in the sense that in this approach the noncommutative
counterpart of the differential structure is quantized whereas in the canoni-
cal method three-metrics play the role of “quantization variables”. We show
that in spite of this difference the superspace formulation of general rela-
tivity (which could be regarded as a prerequisite of the canonical quanti-
zation) can be obtained from the groupoid approach if the corresponding
noncommutative algebra is restricted to its commutative subalgebra which
determines a suitable slicing of space-time. Consequently, in the groupoid
approach when the space-time slicing appears gravity is already in its “clas-
sical (non-quantum) regime”. However, this conclusion could follow from
a simplification inherent in our model, and could eventually be avoided if
one considers a more general module of the noncommutative counterpart of
vector fields (the module of derivations of a given algebra).

We organize our material in the following way. To make the paper
self-contained and to fix our notation, in Section 2, we give a summary of
the groupoid approach to noncommutative unification of general relativity
with quantum mechanics. In Section 3, we define the noncommutative alge-
braic counterpart of the standard concept of superspace (the space of three
metrics). The comparison of the canonical gravity quantization with the
groupoid approach is done in Section 4, and some conclusions and comments
are collected in Section 5.

2 Basic ideas of the model

The main idea of the groupoid approach to the unification of general relativ-
ity and quantum mechanics is to forget, in the very beginning, the concept of
space-time and start with the abstract space $G = E \times \Gamma$, where $E$ is the total
space of a principal fibre bundle, and $\Gamma$ its structural group such that the
orbits of the action of $\Gamma$ on $E$ form a smooth manifold $M$ interpreted as space-
time (this construction can eventually be generalized to the category of differ-
ential spaces of constant dimension, see [3]). We endow $G$ with the groupoid
structure. In the present paper, for the sake of concreteness, we shall assume
that $E$ is the total space of the frame bundle over a space-time manifold $M$,
and \( \Gamma \) the group \( \text{SO}(3,1) \). Of course, \( M = (G/\text{SO}(3,1))/\text{SO}(3,1) \). Then one defines the algebra as the (intrinsic) direct sum

\[
\mathcal{A} = \mathcal{A}_{\text{const}} \oplus C_c^\infty(G, \mathbb{C})
\]

where \( \mathcal{A}_{\text{const}} = pr^*(C^\infty(M, \mathbb{C})) \), and \( C_c^\infty(G, \mathbb{C}) \) is the family of smooth compactly supported complex valued functions on \( G \). The multiplication in the algebra \( \mathcal{A} \) is defined in the following way: (1) if \( a, b \in C_c^\infty(G, \mathbb{C}) \), their multiplication is the convolution \( (a \ast b)(\gamma) = \int_{G_p} a(\gamma_1)b(\gamma_2) \), where \( \gamma = \gamma_1\gamma_2 \) with \( \gamma, \gamma_1, \gamma_2 \in G_p \), \( G_p \) being the fiber in \( G \) over \( p \in E \); integration is with respect to the Haar measure; (2) if \( a, b \in \mathcal{A}_{\text{const}} \) they are multiplied in the usual way, i.e., \( a \ast b = a \cdot b \); (3) if \( a \in \mathcal{A}_{\text{const}} \) and \( b \in C_c^\infty(G, \mathbb{C}) \), one sets \( (a \ast b)(\gamma) = (b \ast a)(\gamma) = k(p) \int_{G_p} b(\gamma_1^{-1}\gamma) \) where \( k(p) = \int_{G_p} a(\gamma_1) \). \( \mathcal{A} \) is evidently a noncommutative algebra. We also define the involution of \( a \in \mathcal{A} \) by \( a^*(\gamma) = \overline{a(\gamma^{-1})} \) where \( \gamma = (p, g) \), \( p \in E \), \( g \in \Gamma \).

Let us also define the subalgebra \( \mathcal{A}_{\text{proj}} = \pi_M^*C^\infty(M, \mathbb{C}) \subset \mathcal{A}_{\text{const}} \). It plays the important role in our model since by restricting the algebra \( \mathcal{A} \) to the subalgebra \( \mathcal{A}_{\text{proj}} \) we recover the space-time manifold of general relativity.

\[\text{footnote}{\text{One should notice that we have corrected the definition of the algebra } \mathcal{A} \text{ as compared with our previous works (see } [2, 4]). \text{This correction does not change our previous results.}}\]
Let us consider the set $\text{Der}\mathcal{A}$ of all derivations of the algebra $\mathcal{A}$. $\text{Der}\mathcal{A}$ is a $\mathcal{Z}(\mathcal{A})$-module, where $\mathcal{Z}(\mathcal{A})$ denotes the center of $\mathcal{A}$, and can be regarded as a noncommutative counterpart of vector fields. In the following, we shall consider a noncommutative differential geometry as defined by the $\mathcal{Z}(\mathcal{A})$-submodule $V$ of $\text{Der}\mathcal{A}$ such that $V = V_E \oplus V_\Gamma$ where $V_E$ and $V_\Gamma$ are derivations of $\mathcal{A}$ parallel to $E$ and $\Gamma$, respectively (this is only a simplifying assumption which in the general case should be relaxed).

First, we define a metric on the $\mathcal{Z}(\mathcal{A})$-submodule $V$ as a $\mathcal{Z}(\mathcal{A})$-bilinear non-degenerate symmetric mapping $g : V \times V \to \mathcal{A}$, and for our model we choose the following metric adapted to the product structure of $V$

$$g = pr_E^*g_E + pr_\Gamma^*g_\Gamma$$

where $g_E$ and $g_\Gamma$ are metrics on $E$ and $\Gamma$, respectively, and $pr_E$ and $pr_\Gamma$ are the obvious projections. It turns out that the “vertical component” $pr_\Gamma^*g_\Gamma$ of the metric $g$ is essentially unique (this is true for a broad class of derivation based noncommutative differential calculi, see [4]), whereas the “parallel component” $pr_E^*g_E$ of $g$ is a lifting of the Lorentz metric in space-time $M$ (see also [5]).

Now, with the help of the Koszul formula, we define the linear connection;
then the curvature and the usual Ricci operator \( R : V \to V \) which is the counterpart of the Ricci tensor with one index up and one index down (for details see \( [2] \)). In this way, we have all quantities needed to write the

noncommutative Einstein equation

\[
G = 0
\]

(2)

where \( G = R + 2\Lambda I \) with \( R \) being the Ricci operator, \( \Lambda \) a constant related to the usual cosmological constant, and \( I \) the identity operator. Because of the form of metric \( (1) \) \( G \) also assumes the form \( G_E + G_\Gamma \) (with obvious meaning of symbols).

The set \( \text{ker}G = \text{ker}G_E \oplus \text{ker}G_\Gamma \) is a \( \mathbb{Z}(A) \)-submodule of \( V \) and represents a solution of eq. \( (2) \). Because of the uniqueness of the metric \( pr^*_\Gamma g_\Gamma \) the equation \( G_\Gamma = 0 \) should be solved for derivations \( v \in \text{ker}G_\Gamma \subset V_\Gamma \). The equation \( G_E = 0 \), as a “lifting” of the usual Einstein’s equation should be solved for the metric. All derivations \( v \in V_E \) satisfy it automatically (and all derivations \( v \in V_\Gamma \) satisfy it trivially, see \( [8] \)).

Let us consider the representation of the algebra \( A \) in the Hilbert space \( \mathcal{H} = L^2(G_q), \pi_q : A \to \mathcal{B}(\mathcal{H}) \), where \( \mathcal{B}(\mathcal{H}) \) denotes an algebra of bounded
operators on $\mathcal{H}$ and $G_q$ is the fiber of $G$ over $q \in E$, given by the formula

$$(\pi_q(a)\psi)(\gamma) = \int_{G_q} a(\gamma_1)\psi(\gamma_1^{-1}\gamma),$$

with $\gamma = \gamma_1 \circ \gamma_2$, $\gamma, \gamma_1, \gamma_2 \in G_q$, $q \in E$; $\psi \in \mathcal{H}$, $a \in \mathcal{A}$. The integral is taken with respect to the Haar measure. The completion of $\mathcal{A}$ with respect to the norm

$$\| a \| = \sup_{q \in E} \| \pi_q(a) \|$$

is a $C^*$-algebra (see [3, p. 102]). We shall denote this algebra by $\mathcal{E}$.

We assume (as a separate axiom) that the dynamics of a quantum gravitational system is described by the following equation

$$i\hbar \pi_q(v(a)) = [F_v, \pi_q(a)]$$

for every $q \in E$, where $v \in \ker G$, and $(F_v)_{v \in \ker G}$ is a one-parameter family of operators $F_v \in \text{End}\mathcal{H}$ with $\mathcal{H} = L^2(G_q)$ such that

$$F_{\lambda_1 v_1 + \lambda_2 v_2} = \lambda_1 F_{v_1} + \lambda_2 F_{v_2}$$

for $v_1, v_2 \in \ker G$, $\lambda_1, \lambda_2 \in \mathbb{C}$. We shall also assume that $[F_v, \pi_q(a)]$ is a bounded operator.

The fact that $v \in \ker G$ makes of eqs. (2) and (4) a “noncommutative dynamical system”. We could also say that noncommutative Einstein equa-
tion (2) plays the role of a “boundary condition” for quantum dynamical equation (1). To solve this system means to find the set

$$\mathcal{E}_G = \{ a \in \mathcal{E} : i\hbar \pi_q(v(a)) = [F_v, \pi_q(a)], \forall v \in \ker G \}. $$

It can be easily verified that it is a subalgebra of $\mathcal{E}$.

Let $\bar{\mathcal{E}}_G$ be the smallest closed involutive subalgebra of the algebra $\mathcal{E}$ containing $\mathcal{E}_G$. $\bar{\mathcal{E}}_G$ is said to be generated by $\mathcal{E}_G$. Since $\mathcal{E}$ is a $C^*$-algebra and every closed involutive subalgebra of a $C^*$-algebra is a $C^*$-algebra (see [10, Sec. 1.3.3]), $\bar{\mathcal{E}}_G$ is also a $C^*$-algebra; it will be called Einstein $C^*$-algebra or simply Einstein algebra, and the pair $(\bar{\mathcal{E}}_G, \ker G)$ – Einstein differential algebra.

Now, the idea is to perform quantization with the help the usual $C^*$-algebraic method (see, for instance, [11], [12, chapter 9]) with the Einstein algebra $\bar{\mathcal{E}}_G$ as our basic $C^*$-algebra. According to this method, a quantum gravitational system is represented by $\bar{\mathcal{E}}_G$, and its observables by Hermitian elements of $\bar{\mathcal{E}}_G$. If $a$ is a Hermitian element of $\bar{\mathcal{E}}_G$, and $\phi$ a state on $\bar{\mathcal{E}}_G$ then $\phi(a)$ is the expectation value of the observable $a$ when the system is in the state $\phi$.

It can be shown that this gravity quantization scheme correctly repro-
duces the usual general relativity (on space-time) and quantum mechanics (in the Heisenberg picture) when the algebra $\mathcal{A}$ is restricted to its center $\mathcal{Z}(\mathcal{A})$ (or to some subset of $\mathcal{Z}(\mathcal{A})$) (see [2, 3]).

3 Algebraic version of superspace

First, let us recall the well known construction. Let $\text{Riem}(S)$ denote the space of all Riemannian metrics on a 3-manifold $S$, and let $\text{Diff}(S)$ be the group of all orientation preserving diffeomorphisms of $S$. For simplicity, we assume that $S$ is closed (e. g., compact and without boundary). We have the action of $\text{Diff}(S)$ on $\text{Riem}(S)$

$$\text{Diff}(S) \times \text{Riem}(S) \to \text{Riem}(S)$$

given by

$$(f, h) \mapsto f^* h.$$ 

The quotient space $\mathcal{S}(S) = \frac{\text{Riem}(S)}{\text{Diff}(S)}$ is called superspace. Its global properties were studied by Fischer [13] (see also [14]).

In a particular coordinate system any metric $h \in \text{Riem}(S)$ can be represented as a covariant metric tensor $h_{ij}(x)$ or as a contravariant metric tensor
$h^{ij}(x)$, $x \in S$. Then, as shown by DeWitt [15], there exists a metric on $S(S)$, called the \textit{Wheeler-DeWitt metric}, which assumes the form

$$G_{ijkl} = \frac{1}{2} h^{-1/2}(h_{ik}k_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}).$$ \hfill (5)

It has the signature $(-++++)$ for each point of the 3-geometry.

Let us now consider a slicing $(S_t)_{t \in T}$ of $M$ such that $S_t$ is diffeomorphic to $S$ for each $t \in T$. Let further $\mathcal{A}_S \subset \mathcal{A}$ be the subalgebra of functions which are constant on $pr^{-1}(S_t)_{t \in T}$, where $pr = pr_M \circ pr_E$ with $pr_M : E \to M$ being the canonical projection, and let us denote by $V_S$ the set of all derivations of $\mathcal{A}$ which are invariant with respect to $\mathcal{A}_S$, i. e., such that $V_S(\mathcal{A}_S) \subseteq \mathcal{A}_S$. Evidently, we have $V_S \subseteq V_E$. Let us notice that the subalgebra $\mathcal{A}_{proj}$ can be equivalently defined in another way; namely as consisting of functions of $\mathcal{A}$ which are constant on the equivalence classes of fibres $G_p = pr^{-1}(x)$, $pr_M(p) = x \in M$. Two fibres $G_p$ and $G_q$, $p, q \in E$ are equivalent if there is $g \in \Gamma$ such that $q = pg$. Now, it can be easily seen that $\mathcal{A}_S \subseteq \mathcal{A}_{proj} \subseteq \mathcal{Z}(\mathcal{A})$. Indeed, $pr^{-1}(x) \subseteq pr^{-1}(S)$ for every $x \in S$. Consequently, the differential algebra $(\mathcal{A}_S, V_S)$ is commutative. We denote the set of all metrics in the module $V_S$ by $\text{Riem}(\mathcal{A}_S)$. As an analogue of $\text{Diff}(S)$ we should
take the set $\text{Iso}_A$ of all isomorphisms of $A_S$ into itself. We have the action

$$\text{Iso}(A_S) \times \text{Riem}(A_S) \to \text{Riem}(A_S)$$

defined by

$$(f, h) \mapsto f^* h.$$ 

Any isomorphism $f : A_S \to A_S$ induces the mapping (which is also an isomorphism)

$$f^# : V_S \to V_S$$

by

$$f^#(v)(\alpha) = v(f^* \alpha) = v(\alpha \circ f)$$

where $v \in V_S$, $\alpha \in A$. Therefore, one has

$$(f^* h)(v_1, v_2) = h(f^# v_1, f^# v_2),$$

$v_1, v_2 \in V_S$, $h \in \text{Riem}_A$, and we can define the superspace associated with the algebra $A$ as

$$\mathcal{S}(A) := \frac{\text{Riem}(A_S)}{\text{Iso}(A_S)}.$$ 

We have the following conclusion: By restricting the algebra $A$ to its subalgebra $A_S$ and considering the set $\text{Riem}(A_S)$ of all Riemannian met-
rics in the $\mathcal{Z}(A)$-submodule $V_S$ one obtains the algebraic counterpart of the standard concept of superspace.

4 Noncommutative gravity and canonical quantization

We now briefly recollect the canonical method of quantizing gravity to compare it with our approach. Any space-time metric can be locally written in the form

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dt dx^i + h_{ij} dx^i dx^j,$$

where $h_{ij}, i, j = 1, 2, 3$ is the metric tensor on the spacelike hypersurface $S =$const, $N$ is called lapse function; it measures the proper time separation between hypersurfaces $t =$const. The so-called shift vector $N_i$ measures the deviation of curves $x^i =$const from the normal to $S$ (in the following we use units such that $c = \hbar = 1$). The extrinsic curvature of $S$ can be written as

$$K_{ij} = \frac{1}{2N} \left[ \frac{\partial h_{ij}}{\partial t} + 2N_{i|j} \right].$$
where the stroke \( "|" \) denotes covariant differentiation with respect to the 3-metric \( h_{ij} \). The momentum canonically conjugated to \( h_{ij} \) is given by

\[
\pi^{ij} = -h^{1/2}(K^{ij} - h^{ij}K),
\]

where \( K = K_i^i \).

The classical Hamiltonian is

\[
H = \int (NH_0 + N_iH^i)d^3x, \tag{7}
\]

where

\[
H_0 = G_{ijkl}\pi^{ij}\pi^{kl} - h^{1/2}(3R - 2\Lambda),
\]

\[
H^i = -2\pi^{ij}_j,
\]

with \( 3R \) being the scalar curvature of \( h_{ij} \) and \( \Lambda \) the cosmological constant.

By making the standard substitution: \( h_{ij} \mapsto h_{ij}, \pi^{ij} \mapsto -i\delta_{\delta h_{ij}} \) (\( \delta \) is the functional derivative) one obtains the counterpart of the Schrödinger equation

\[
\hat{H}\Psi = 0. \tag{8}
\]

The \( \hat{H}_0 \)-part of this equation

\[
- [G_{ikl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + h^{1/2}(3R - 2\Lambda)]\Psi[h_{ij}] = 0. \tag{9}
\]
is the celebrated *Wheeler-DeWitt equation*. This is the fundamental equation for the “wave function of the universe” $\Psi[h_{ij}]$ which is the functional of the 3-metric (we do not take into account any matter fields).

We should emphasize that in the Wheeler-DeWitt approach it is the 3-metric that is quantized (and the momentum canonically conjugated to it), whereas in our approach the “quantization variables” are elements of the Einstein $C^*$-algebra $\mathcal{E}_G$. However, we can ask the question: what would happen to the equations of our theory (eqs. (2) and (4)) if we restrict $\mathcal{E}_G$ to $(\mathcal{E}_G)_S$, i.e. if we go to the “superspace limit”?

Since $(\mathcal{E}_G)_S \subset Z(\mathcal{E}_G)$ eq. (4) reduces to the trivial identity $(0 \equiv 0)$ and hence it becomes insignificant. We are left with eq. (2) which, in this case, is reduced to the usual Einstein equations. In this way, gravity decouples from quantum mechanics. This is an important conclusion: if we go to the superspace limit quantum gravity effects become negligible. In this process, the slicing of space-time emerges, and consequently the concepts of time and instantaneous spaces become meaningful. This means that we are well beyond the Planck threshold in the non-quantum gravity regime (see [16] where the emergence of time from the noncommutative era has been studied).

As it is well known, the Wheeler-DeWitt equation corresponds to the sta-
tionary Schrödinger equation. Eq. (4) plays the similar role in our approach since, for weak gravitational fields it reduces to the Schrödinger equation (in the Heisenber picture of quantum mechanics). However, one should not forget that the Wheeler-DeWitt equation is the equation for three-metrics, whereas eq. (4) is the equation for elements of the algebra $E_G$.

5 Concluding remarks

We have demonstrated that if in the groupoid approach to the unification of general relativity and quantum mechanics, proposed in [4], the algebra $A = A_{proj} \oplus C_c^\infty(G, C)$ is restricted to its subalgebra $A_S$, consisting of functions constant on $pr^{-1}(S_t)_{t \in T}$, where $(S_t)_{t \in T}$ is a time slicing of space-time $M$, one obtains the superspace formulation of general relativity.

The important point is that our approach shows that at the level where time slicing of space-time appears, quantum gravity effects are already insignificant (i.e., gravity is too weak to exhibit quantum effects, see above Section 4). This seems reasonable since in the quantum gravity regime we would expect some kind of “foamy mixture” of space and time which is excluded by the well defined time slicing of space-time. This conclusion
could be the consequence of a simplifying assumption incorporated into our model, namely that our noncommutative differential algebra is based on the $\mathcal{Z}(\mathcal{A})$-submodule $V$ of $\text{Der}\mathcal{A}$ such that $V = V_E \oplus V_\Gamma$ where $V_E$ and $V_\Gamma$ are submodules of derivations parallel to $E$ and $\Gamma$, respectively. In this model “geometry parallel to $E$” is, in principle, responsible for gravity effects and “geometry parallel to $\Gamma$” is responsible for quantum effects. The fact that we have neglected “mixed terms” (those coming both from $V_E$ and $V_\Gamma$) means that in our model gravity is “weakly coupled” to quantum effects. Consequently, if we restrict the algebra $\mathcal{A}$ to its subalgebra $\mathcal{A}_S$ (this restricting essentially means that slicing of space-time enters the scene) all terms parallel to $\Gamma$ automatically are switched off. Such terms would be responsible for a “fluctuating slicing” of space-time which could be enough for an approximate validity of the canonical quantization of gravity. The decisive step in checking this hypothesis would be to construct a counterpart of our model based on a more general module of derivations.

The analogous situation occurs in the canonical quantization approach. One begins with the sliced classical space-time (with no quantum effects). Then one performs the canonical quantization, as the result of which 3-geometries begin to fluctuate, and the sliced regime of space-time becomes
As it is well known, when Einstein’s equations are formulated as a constrained Hamiltonian system, the Hamiltonian constraint and the equations of motion determine the evolution of three-metrics in superspace, and the momentum constraint implies that the Hamiltonian flow is orthogonal (in the Wheeler-DeWitt metric) to the orbits of the diffeomorphism group (although these two directions need not be disjoint [14]). Since in our algebraic approach the submodule $V_S$ corresponds to the family of vector fields on the superspace $S(A)$ the above mentioned regularities should be reflected in the structure of this submodule.

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