AFTER RUNAWAY: THE TRANS-HILL STAGE OF PLANETESIMAL GROWTH

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ABSTRACT

When planetesimals begin to grow by coagulation, they first enter an epoch of runaway, during which the biggest bodies grow faster than all the others. The questions of how runaway ends and what comes next have not been answered satisfactorily. We show that runaway is followed by a new stage—the “trans-Hill stage”—that commences when the bodies that dominate viscous stirring (“big bodies”) become trans-Hill, i.e., when their Hill velocity matches the random speed of the small bodies they accrete. Subsequently, the small bodies’ random speed grows in lockstep with the big bodies’ sizes, such that the system remains in the trans-Hill state. Trans-Hill growth is crucial for determining the efficiency of growing big bodies, as well as their growth timescale and size spectrum. Trans-Hill growth has two sub-stages. In the earlier one, which occurs while the stirring bodies remain sufficiently small, the evolution is collisionless, i.e., collisional cooling among all bodies is irrelevant. The efficiency of forming big bodies in this collisionless sub-stage is very low, \( \sim 10^{-\alpha} \ll 1 \), where \( \alpha \sim 0.005(a/AU)^{-1} \) is the ratio between the physical size of a body and its Hill radius. Furthermore, the size spectrum is flat (equal mass per size decade, i.e., \( q = 4 \)). This collisionless trans-Hill solution explains results from previous coagulation simulations for both the Kuiper Belt and the asteroid belt. The second trans-Hill sub-stage commences once the stirring bodies grow big enough (>\( \alpha^{-1} \times \) the size of the accreted small bodies). After that time, collisional cooling among small bodies controls the evolution. The efficiency of forming big bodies rises and the size spectrum becomes more top heavy. Trans-Hill growth can terminate in one of two ways, depending on the sizes of the small bodies. First, mutual accretion of big bodies can become significant and conglomeration proceeds until half of the total mass is converted into big bodies. This mode of growth may explain the observed size distributions of small bodies in the solar system and is explored in our subsequent work. Second, if the big bodies’ orbits become separated by their Hill radius, oligarchy commences. This mode likely precedes the formation of fully fledged planets.

Key words: celestial mechanics – Kuiper Belt: general – minor planets, asteroids: general – planets and satellites: formation

Online-only material: color figures

1. INTRODUCTION

Our understanding of how planets form remains inadequate. This hinders attempts to explain the many recent discoveries of extrasolar planets, protoplanetary disks, and debris disks. Planet formation is often decomposed into a number of stages. In the first stage, planetesimals form out of dust embedded in protoplanetary disks. How that happens is highly uncertain because it depends on complicated physics such as how particles stick and how they interact with a turbulent gas disk (see Chiang & Youdin 2010 for a review). The initial planetesimals are often assumed to be kilometer-sized, but they could be much smaller or larger than that (e.g., Johansen et al. 2007). In the second stage, sometimes called coagulation, the planetesimals attract one another gravitationally, merge, and grow. This stage is more easily understood than the first, because the dominant physical process is simply gravity (see Goldreich et al. 2004b, hereafter GLS, for a review). Nonetheless, past studies have given different, sometimes even conflicting, views on this process. What happens after coagulation depends on local conditions. Just beyond the snow line, coagulation can produce cores that then accrete massive gaseous atmospheres to form gas giants (Pollack et al. 1996). In the terrestrial zone, coagulation produces dozens of sub-Earth-sized bodies that then undergo a velocity instability once they have accreted half the planetesimals, leading to an epoch of large-scale chaos and giant impacts (Chambers & Wetherill 1998; Goldreich et al. 2004a). In the asteroid belt, Kuiper Belt, and extrasolar debris disks, coagulation is thought to have been incomplete, perhaps because the coagulating planetesimals were excited by exterior planets before forming planets themselves or because the initial surface density was very low.

It is the second stage, coagulation, that is the topic of this paper. Our goal is to build a theory that explains the properties of bodies that ultimately form, such as their number, size distribution, formation timescale, and efficiency, where the efficiency is the fraction of mass in the original planetesimals that ends up in big bodies. This theory can then be compared against observations, especially of the asteroid and Kuiper Belts, whose large bodies are thought to be frozen remnants of the coagulation process.

Coagulation is usually studied with numerical simulations. Starting from an assumed initial state, the planetesimals begin to merge. The merging rate depends on gravitational focusing factors, which are functions of the relative speeds. The relative speeds are in turn affected by a variety of two-body processes such as viscous stirring, dynamical friction, and inelastic collisions. While there have been many such simulations, using a variety of techniques (e.g., Greenberg et al. 1978; Wetherill & Stewart 1989; Kokubo & Ida 1998; Weidenschilling et al. 1997; Kenyon & Luu 1999; Inaba et al. 2001; Morbidelli et al. 2009; Ormel et al. 2010b; Weidenschilling 2011; Schlichting & Sari 2011), the simulations are complicated and hence it is often difficult to disentangle the various effects in order to understand the results and be confident that they are correct. Moreover, the results from different groups do not always agree (e.g.,
Morbidelli et al. 2009; Weidenschilling 2011). Compounding the difficulty are a number of fundamental uncertainties—such as the unknown initial size and velocity distributions, how collisional fragmentation occurs, and when gaps are opened in the circumstellar disk. A theory for coagulation would be desirable to guide and interpret simulations and to determine how sensitive the results are to assumptions.

Early coagulation simulations led to the discovery of runaway growth, in which the size distribution quickly develops a tail extending to extremely large sizes (Safronov 1972; Greenberg et al. 1978; Wetherill & Stewart 1989). Runaway occurs because, with gravitational focusing included, the growth rate of bodies (d ln R/dt) can be an increasing function of their radius R (see also Section 3.1 below). This implies that the largest bodies continue to double to infinite size before smaller ones double a single time. The properties of runaway growth have been studied analytically (e.g., Lee 2000; Malyskin & Goodman 2001). A variety of theories have been proposed to explain how runaway growth ends and what comes next (e.g., Ida & Makino 1993; Kokubo & Ida 1998; Ormel et al. 2010b; Schlichting & Sari 2011). It is often thought that runaway accretion is followed by self-regulated oligarchic growth, during which each big body heats its own food. However, as we show in this paper, there is a critical intervening stage of growth that has hitherto been overlooked: the “trans-Hill stage.”

In the absence of an understanding of the trans-Hill stage, previous studies (e.g., GLS) could not explain two key results of conglomeration simulations: the size spectrum and the formation efficiency of large bodies. Here, we show that trans-Hill growth is a critical stage for determining these. Based on our analytical understanding, we provide simple scalings to explain the results of previous numerical simulations.

During trans-Hill growth, conglomeration can be categorized as collisionless or collisional, with very different outcomes for formation efficiency and size spectrum. In the collisionless case, small bodies rarely collide, while in the collisional case, the small bodies’ dispersion is reduced by frequent inelastic collisions. Collisionless growth has been often simulated before, but without a clear interpretation. Collisional growth, on the other hand, is just starting to be explored (Weidenschilling 2011; A. Shannon et al. 2012, in preparation). Our study here provides a unified framework for both regimes.

The structure of this paper is as follows. In Section 2, we present the equations of motion. We re-write these in Appendix A in a form suitable for numerical integration or analytic solution. In Section 3, we review runaway growth and show that it inexorably transitions into trans-Hill growth. We then derive qualitatively the properties of trans-Hill growth. In Section 4, we present exact solutions of trans-Hill growth, using numerical integrations as well as analytic self-similar solutions (derived in Appendix B). Section 5 describes what comes after trans-Hill growth. Section 6 examines our assumptions and delineates their range of validity. Section 7 discusses applications of our theory to the asteroid and Kuiper Belts and makes comparisons with a number of earlier papers. Section 8 provides a summary.

Shannon et al. (2012, A. Shannon et al. 2013, in preparation) present particle-in-a-box simulations in both the collisionless and collisional regimes, without the restrictive assumptions made in the present paper. The results there confirm and refine those of this study. The collisional paper in particular focuses on the formation of the Cold Classical Kuiper Belt. It presents a new picture where Kuiper Belt objects (KBOs) form out of a very low-mass planetesimal disk (the “minimum-mass Kuiper Belt”), 100 times less massive than the minimum-mass solar nebula.

2. ASSUMPTIONS AND EQUATIONS OF MOTION

We examine first the interactions between two groups of bodies, big ones and small ones, before proceeding to consider a distribution of bodies. This “two-group” approximation has been described by GLS and we follow their notation. Big bodies have radius R and surface density Σ; small bodies have radius s, surface density σ, and velocity dispersion u. All bodies have the same bulk density, ρ ∼ 1 g cm⁻³. We make the following assumptions and then check for self-consistency in Section 6.

1. The small bodies’ random speed u satisfies

$$\alpha^{1/2} < u/v_H < \alpha^{-1/2},$$

where α is the ratio of all bodies’ physical radius to their Hill radius (R/R_H) and v_H is the big bodies’ Hill velocity. Explicitly,

$$\alpha \equiv \frac{R}{R_H} \sim \frac{R_\odot}{a} \ll 1,$$

where the expression R_⊙/a (the ratio of the Sun’s radius to the semi-major axis) follows from the fact that the Sun’s density is comparable to that of solid bodies. In the asteroid belt α ∼ 2 × 10⁻³, while in the Kuiper Belt α ∼ 10⁻⁴. In addition,

$$v_H \sim R \sqrt{G \rho a}.$$  (3)

2. The big bodies’ random speed v is sub-Hill (v < v_H).

3. Σ ∝ σ, and so σ is essentially the total surface density and is treated as a constant during the growth. More stringently, we assume that big bodies grow only by accreting small ones. This is certainly true at early times, when there are only a few big bodies.

4. The small bodies’ size s is a constant parameter. Small bodies do not grow and they also do not fragment in collisions.

5. More than a single big body dominates viscous stirring at a given distance from the star and hence the big body number distribution can be treated as a continuous function. This assumption is violated when oligarchy commences.

With the above assumptions, big bodies grow by accreting small ones at the rate (GLS)

$$\frac{dR}{dt} = \frac{\sigma \Omega}{\rho R} \alpha^{-1} \left( \frac{v_H}{u} \right)^2, \quad \text{if } u > v_H$$

$$= \frac{\sigma \Omega}{\rho R} \alpha^{-2} \left( \frac{v_H}{u} \right)^4, \quad \text{if } u < v_H,$$  (4)

where $\Omega$ is the orbital angular speed around the Sun.

For the evolution of u, small bodies are damped by inelastic collisions among themselves and are viscously stirred by the big bodies. The net rate is (GLS).

$$\frac{du}{dt} = -\frac{\sigma \Omega \Sigma}{\rho s} + \frac{\sigma \Omega \Sigma}{\rho R} \alpha^{-2} \left( \frac{v_H}{u} \right)^2, \quad \text{if } u > v_H$$

$$= \frac{\sigma \Omega}{\rho s} \alpha^{-2} \left( \frac{v_H}{u} \right)^4, \quad \text{if } u < v_H.$$  (5)
If the first term on the right-hand side is important, the system is collisional; otherwise, it is collisionless.

The above equations for two groups of bodies are easily extended to a continuous distribution of big bodies. We denote the cumulative number distribution of big bodies as \(N(>R)\). Because we neglect accretion of big bodies, the number of big bodies remains constant as they grow. In other words, \(N\) satisfies the continuity equation

\[
\frac{\partial N}{\partial t} + V \frac{\partial N}{\partial R} = 0,
\]

where \(V \equiv dR/dt\) is given in Equation (4). In place of Equation (5), we replace \(\Sigma \to d\Sigma\), then set

\[
\frac{d(\Sigma/\sigma)}{d \ln R} = R^3 \frac{d(N/\eta)}{d \ln R}
\]

and integrate over \(dR\). Here, \(\eta\) is a constant; we shall not need its value because it is only the ratio \(N/\eta\) that is dynamically significant.

Equations (4)–(7) are the equations of motion. We solve them both analytically (Appendix B) and numerically (see the method in Appendix A). However, before presenting the exact solutions, we first use simple analytical arguments to show that runaway growth inexorably converges toward “trans-Hill growth.” This realization gives rise to a number of our main results on the growth of big bodies.

3. FROM RUNAWAY TO TRANS-HILL GROWTH

3.1. Runaway

We summarize the traditional picture of how runaway growth proceeds (e.g., Ida & Makino 1993). It is typically assumed that bodies of some characteristic size (here, \(s\)) emerge from a dissipating protoplanetary disk. The value of \(s\) is highly uncertain, since it is not even understood how the bodies formed.

On the timescale that bodies of size \(s\) collide, \(\rho s/\sigma \Omega\), they stir each other up to their surface escape speed, \(u \sim s \sqrt{G \rho} \sim s^{3/2} v_H|_s\), (Equation (5)), where \(v_H|_s\) is the Hill velocity for bodies of size \(s\). On the same timescale, they grow in mass by accreting each other.2 The evolution of those bigger bodies (radius labeled by \(R\)) proceeds in the runaway regime. This is because the growth rate when \(u \sim v_H\) is an increasing function of \(R\), i.e., \(d \ln R/dt \propto R\) (Equations (3) and (4)). Thus, comparing two big bodies that differ in size by a factor \(\gtrsim 2\), the bigger one will double in size faster than the smaller one and then will double again even faster. It will ultimately reach infinite size—or violate one of the assumptions made—before the smaller one has doubled a single time.

3.2. From Runaway to Trans-Hill Growth

Runaway growth requires \(u \sim v_H\). Since \(v_H\) increases linearly with \(R\), when runaway bodies get big enough their Hill velocity can be sufficiently large such that \(u \sim v_H\). In that sub-Hill case, big bodies grow in the “neutral” regime (GLS), i.e., since the growth rate is independent of \(R\) (Equation (4)), the distribution function of big bodies \(N(> R)\) maintains its shape while moving to larger \(R\).

Figure 1. Result of a numerical integration of the collisionless equations of motion (Equations (4)–(7) with \(s = \infty\) and \(\alpha = 10^{-3}\)). The leftmost panels (top and bottom) show the initial conditions, where the bottom panel is the big body distribution function and the top panel is the stirring rate per \(\ln R\) (the right-hand side of Equation (5) with \(\Sigma \to d\Sigma/d \ln R\)). The red cross marks \(R_{\text{run}}\), defined to be the maximum of the curve in the top panel. The blue circle marks \(R_{\text{run}} \equiv u/\sqrt{G \rho a}\). Initially, \(u\) is the escape speed from bodies with \(R = 1\) (i.e., \(R_{\text{run}} = a^{-1/2} = 100\). The panels at subsequent times \((\tau = 10, 500, 10^4, \text{with } \tau \text{ being the scaled time; } \tau \equiv t \sigma \Omega/(\rho a)\); see Equation (A1)) show that the system evolves from \(R_{\text{run}} \ll R_{\text{run}}\) to the trans-Hill state \(R_{\text{run}} \sim R_{\text{run}}\) and thereafter remains trans-Hill.

(A color version of this figure is available in the online journal.)

In general, some big bodies grow in the runaway regime and others grow in the neutral regime, depending on their radius:

- runaway growth, if \(R < R_{\text{run}}\)
- neutral growth, if \(R > R_{\text{run}}\)

where the trans-Hill radius is

\[
R_{\text{run}} \equiv \frac{u}{\sqrt{G \rho a}},
\]

which is the radius of the big bodies that have Hill velocities equal to the small bodies’ speed \(v_H|_{\text{run}} = u\).

As we now show, the big bodies’ distribution function is always driven to the trans-Hill state, defined as \(R_{\text{stir}} \sim R_{\text{run}}\), where \(R_{\text{stir}}\) is the radius of the big bodies that dominate viscous stirring. Subsequently, the small-body velocity dispersion and the size of the bodies that dominate stirring grow in lockstep, maintaining \(R_{\text{stir}} \sim R_{\text{run}}\). Our argument proceeds by considering the two initial situations, \(R_{\text{stir}} \ll R_{\text{run}}\) or \(R_{\text{stir}} \gg R_{\text{run}}\), in turn.

1. If initially \(R_{\text{stir}} \ll R_{\text{run}}\) (Figure 1), as is relevant for the early phases of planetesimal growth, the small bodies are super-Hill with respect to the stirrers. All big bodies with \(R > R_{\text{stir}}\) repeatedly double in size before the stirring bodies \(R_{\text{stir}}\) have doubled once.3 This runaway produces new bodies that

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2 It is possible that some bodies grow to very large sizes before \(u\) is stirred up to the small bodies’ escape speed, depending on the details of how the small bodies formed. We discuss the effect of this on trans-Hill growth below.

3 Even though very large bodies, with radii above \(R_{\text{run}}\), grow in the neutral regime, their doubling rate also exceeds that of the stirring bodies.
and both increase with time, following (Equation (4)). The Astrophysical Journal

\[ \Sigma \] 

tion is driven toward the trans-Hill state than they do, i.e., the distribution becomes frozen in time at small

grow significantly because those with \( R > R_{\text{stir}} \) double faster than they do, i.e., the distribution becomes frozen in time at small \( R \). Furthermore, bodies with \( R > R_{\text{stir}} \) follow neutral growth and their distribution maintains its shape. Throughout, \( R_{\text{trans}} \sim R_{\text{stir}} \) and both increase with time, following (Equation (4))

\[ R_{\text{stir}} \sim R_{\text{trans}}. \]  

or equivalently

\[ u \sim v_H |_{R_{\text{trans}}}. \] 

and that after that the big bodies continue to grow in this state. We call this new stage of growth trans-Hill growth.

During trans-Hill growth, all bodies with \( R < R_{\text{stir}} \) do not grow significantly because those with \( R > R_{\text{stir}} \) double faster than they do, i.e., the distribution becomes frozen in time at small \( R \). Furthermore, bodies with \( R > R_{\text{stir}} \) follow neutral growth and their distribution maintains its shape. Throughout, \( R_{\text{trans}} \sim R_{\text{stir}} \) and both increase with time, following (Equation (4))

\[ R_{\text{stir}} \sim \frac{\sigma \Omega}{\rho \alpha t}. \] 

We may also evaluate the mass fraction in bodies that dominate stirring, \( \Sigma_{\text{stir}}/\sigma \), as a function of \( R_{\text{stir}} \). We find,

dropping the “stir” subscripts,

\[ \frac{\Sigma_{\text{stir}}}{\sigma} \sim \alpha, \]  
collisionless \( (R < s/\alpha) \)

\[ \frac{\Sigma_{\text{stir}}}{\sigma} \sim \alpha^2 R/s, \]  
collisional \( (R > s/\alpha) \),

(14)

where the collisionless expression above follows from setting \( d \ln u/dt \sim d \ln R_{\text{stir}}/dt \) and dropping the collisional term in Equation (5) and the collisional expression follows from dropping the left-hand side of Equation (5). The transition from collisionless to collisional trans-Hill evolution occurs once \( R_{\text{stir}} \) exceeds a critical size

\[ R_{\text{stir}} > R_{\text{coll}} = s/\alpha, \]  

because beyond this size, small bodies collisionally cool faster than the growth of trans-Hill bodies. This transition occurs at \( t \sim \rho s/(\sigma \Omega) \), i.e., the small bodies’ collision time.

Following conventional practice, we define the power-law index \( q \) of the differential number distribution

\[ \frac{dN}{dR} \propto R^{-q} \]  

\[ \propto \Sigma(>R) R^{-4}, \]  

in which case we have

\[ q = \begin{cases} 4, & \text{collisionless} \ (R < s/\alpha) \\ 3, & \text{collisional} \ (R > s/\alpha). \end{cases} \]  

(17)

4. EXACT SOLUTIONS

We present exact solutions of the equations of motion. We first summarize the analytical self-similar solutions to these equations (derived in Appendix B) and then integrate the equations numerically for a number of cases—both when the initial conditions are as given by the self-similar solution and for more realistic initial conditions.

4.1. Analytical Self-similar Solutions

It is shown in the appendices that the equations of motion (Equations (4)–(7) or, equivalently, Equations (A1)–(A4)) admit self-similar solutions when the radii \( R \) are scaled relative to \( R_{\text{trans}} \) and when the stirring is either in the collisionless or collisional regime. In either case, there is a single free parameter, the power-law exponent for bodies that grow neutrally, i.e., the value of \( \gamma \) such that \( N \propto R^{-\gamma} \) at \( R > R_{\text{trans}} \). The value of \( \gamma \) remains frozen in time by the nature of neutral growth, as long as the assumptions of Section 2 remain valid. However, the value of \( \gamma \) in real disks is difficult to ascertain from first principles and likely depends both on how the bodies form and on the early stages of coagulation. Fortunately, the solutions are quite insensitive to \( \gamma \) as long as \( \gamma \gg 3 \), or equivalently as long as stirring is dominated by bodies at finite sizes, not by those with \( R = \infty \).

The exact self-similar solutions (displayed in Equations (B8) and (B12) and graphed in Figures 2 and 3) confirm what was qualitatively derived in Section 3. In particular, from the

4 Because the spectrum is frozen at small \( R \), Equation (14) gives not only the temporal evolution of \( \Sigma_{\text{stir}} \), but also the frozen spectrum at \( R < R_{\text{stir}} \)—that is why we drop the “stir” subscript.
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**Figure 2.** Collisionless evolution of Equations (4)–(7), initialized with the self-similar size distribution at scaled time $\tau = 10 \Omega_\odot / (\rho_\odot) = 1$, where $R$ is measured in units of the initial value of $R_{\text{trans}}$. Top panel: the cumulative number distribution of big bodies (normalized by constant $\eta$) is plotted at three subsequent times. The numerical and analytic self-similar solutions agree. The initial parameters are $\alpha = 10^{-4}$ and $\gamma = 7$, i.e., $N \propto R^{-3}$ at large $R$. Bottom panel: the differential mass distribution, derived from the top panel via Equation (7).

**Figure 3.** Similar to Figure 2, but showing the collisional case with parameters $s = 0.1\alpha$ (in units where the initial $R_{\text{trans}} = 1$) and, as before, $\alpha = 10^{-4}$ and $\gamma = 7$. The size spectrum is $dN/dR \propto R^{-3}$ at sizes below $R_{\text{trans}}$ and the efficiency of forming large bodies is much higher than the collisionless case. The top panel shows that our numeric and analytic integrations of Equations (4)–(7) agree well. Note, however, that these equations neglect accretion of big bodies, which, with the chosen parameters, would become important for $\tau > 10^6$. (A color version of this figure is available in the online journal.)

**Figure 4.** Two integrations were initialized with nearly all bodies at the same size, $R = 1$ (the black profile marked $\tau = 0$). Numerical integrations of the equations of motion are plotted at scaled time $\tau = 2.5 \times 10^5$. For the red (high $u$) curve, the initial $u$ is the surface escape speed from bodies with $R = 1$ (i.e., $R_{\text{trans}} = 1/\sqrt{\sigma} = 100$); for the blue curve, it is 10 times smaller ($R_{\text{trans}} = 10$). Both integrations begin in the runaway stage and converge onto the collisional self-similar solution (Equation (19), dotted line) when $R \gg s/\sigma$ (Equation (14)). The low $u$ case follows the collisionless solution (Equation (18), dashed line) for some time before collisional cooling becomes important. This explains the flat mass spectrum from $R = 10^5$–$10^6$. The high $u$ case takes longer to reach the self-similar solution because its initial conditions differ more from the trans-Hill state; in fact, it effectively skips the collisionless solution entirely.

(A color version of this figure is available in the online journal.)

**Equation (14):**

$$\frac{d}{d \ln R} \frac{\Sigma}{\sigma} = -3g_{\text{coll}}(\gamma)\alpha, \quad \text{collisionless} \quad (18)$$

$$\frac{d}{d \ln R} \frac{\Sigma}{\sigma} = -2g_{\text{coll}}(\gamma)\alpha^2 \frac{R}{\sigma}, \quad \text{collisional.} \quad (19)$$

The order-unity functions $g_{\text{coll}}(\gamma)$ and $g_{\text{coll}}(\gamma)$ are defined in Appendix B. They asymptote to $g_{\text{coll}}(\gamma \to \infty) = g_{\text{col}}(\gamma \to \infty) = 10$.

**4.2. Numerical Solutions**

Figure 2 shows a numerical integration of the equations of motion in the collisionless case ($s = \infty$), initialized with the collisionless self-similar solution (Equation (B8)) at scaled time $\tau = 1$. It is apparent from the figure that the evolution remains self-similar at all times and agrees with the analytic expression (Equation (B8)). The mass distribution per logarithmic bin is indeed constant, in agreement with Equation (17).

Figure 3 shows an integration for the collisional case. It is initialized with Equation (B12) at $\tau = 1$, has $s = 0.1\alpha \times R_{\text{trans}}$, and other parameters as before. The numerical integration agrees with the analytic expression and shows that the mass distribution is top heavy with $dN/dR \propto R^{-3}$ (Equation (17)).

In Figure 4, we experiment with a more commonly adopted initial spectrum—one that is more strongly peaked at a single size ($R = 1$). We perform two integrations, both of which have $\alpha = 10^{-4}$, $s = 1$ (i.e., $s$ is the same as the location of the peak in the initial size distribution), and initially $N \propto R^{-7}$ at
$R > 1$. The two integrations differ in the initial $u$: one has an initial velocity equal to the escape speed of bodies with $R = 1$ and the other has $u$ 10 times lower. Over time, both converge to the collisional self-similar solution. The low $u$ case undergoes a temporary phase of collisionless trans-Hill growth, when the big bodies have yet to grow to $R \geq s/\alpha$. In contrast, the high $u$ case takes longer to reach the trans-Hill solution because its initial condition is more discrepant from trans-Hill. In fact, this case completely skips the collisionless trans-Hill regime. We have also experimented with different values for the initial power-law slope of $N$. The main resulting difference is the value of $R$ at which the collisionless trans-Hill solution commences. For example, when $N \propto R^{-5}$ at $R > 1$ (rather that $R^{-7}$ as in Figure 4), then the high $u$ solution follows the collisionless theory from $R = 10^{2.5} - 10^4$.

In summary, as long as the evolution is collisionless, the efficiency of forming big bodies is small, with $\Sigma/\sigma \approx 10\alpha$, independent of $R$. But once the evolution becomes collisional ($R_{\text{stir}} > s/\alpha$), the efficiency grows toward unity as $R_{\text{stir}}$ increases.

5. AFTER TRANS-HILL GROWTH

5.1. Growth by Accreting Big Bodies: Equal Accretion

One of the ways that trans-Hill growth can end is when accretion of big bodies becomes important, violating assumption 3 in Section 2. To establish when that occurs, we calculate the accretion rate of big bodies, which requires knowing the big bodies’ random speed $v$. Balancing dynamical friction damping due to small bodies with viscous stirring due to other big bodies yields (GLS)

$$\frac{1}{v} \frac{dv}{dt} \sim \frac{\Omega}{\rho R} \alpha^{-2} \left(-\sigma + \frac{\Sigma v_H}{v}\right) \sim 0.$$ (20)

Therefore, $v/v_H \sim \Sigma/\sigma \ll 1$. The growth rate by accreting big bodies with $v < v_H$ is (Rafikov 2003a, GLS)

$$\frac{1}{R} \frac{dR}{dt}_{\text{big}} \sim \frac{\Sigma \Omega}{\rho R} \alpha^{-3/2},$$ (21)

which is larger than the usual (isotropic) sub-Hill accretion formula (i.e., Equation (4)) because big bodies lie in a zero-inclination disk. Comparing growth by accreting big bodies with that by accreting small bodies,

$$\frac{d \ln R/dt}_{\text{big}} \sim \frac{\Sigma}{\sigma} \alpha^{-1/2}$$ (22)

$$\sim \left\{\begin{array}{ll}
\alpha^{1/2}, & \text{collisionless} \\
\alpha^{-3/2} R_{\text{stir}}/s, & \text{collisional}
\end{array}\right.$$ (23)

using the trans-Hill expressions. We conclude that in the collisionless regime it is always safe to ignore accretion by big bodies, in disagreement with (Schlichting & Sari 2011; see also Section 7). However, in the collisional regime, big body accretion becomes important once $R_{\text{stir}}$ exceeds a critical value

$$R_{\text{stir}} \gtrsim R_{\text{stir}}^{(b, \text{accrete})} \approx \alpha^{-3/2}$$

$$\approx 2800 \text{ km} \left(\frac{s}{1 \text{ km}}\right) \left(\frac{a}{1 \text{ AU}}\right)^{3/2}. (24)$$

This occurs at time $t \sim \alpha^{-1/2} \rho s/(\sigma \Omega)$ and at that time the fraction of mass in big bodies (i.e., the efficiency) is $\Sigma/\sigma \sim \alpha^{1/2} \sim 7\% (a/1 \text{ AU})^{-1/2}$.

As shown in A. Shannon et al. (2013, in preparation), after collisional trans-Hill accretion ends, an epoch of “equal accretion” begins, during which big bodies grow by accreting comparable mass in big and in small bodies. Equal accretion terminates when half of the mass has been converted into big bodies, $\Sigma \sim \sigma$.

The scenario outlined above is applicable as long as oligarchy has not yet begun. We discuss oligarchy next.

5.2. Oligarchic Growth

As accretion proceeds, the number of stirring bodies decreases and hence neighboring stirrers become increasingly separated. Eventually, they become so separated that each small body is predominantly stirred by, and accreted onto, a single big body. When that happens, assumption 5 is violated (Section 2) and oligarchy commences (Kokubo & Ida 1998, GLS). In oligarchy, the nature of growth is modified. Instead of a continuous size spectrum, the largest body in each radial annulus separates from the size spectrum of smaller bodies. The velocity dispersion $u$ is no longer trans-Hill.

The value of $R_{\text{stir}}$ at which oligarchy starts depends on the size spectrum, for which we now have a simple model. For trans-Hill velocity dispersion, oligarchy begins when the separation between adjacent big bodies ($\Delta a \sim \rho R^3/(\alpha \Sigma)$) exceeds their Hill radius ($R_H \sim R/\alpha$). If we define the oligarchy parameter

$$\text{OP} \equiv \frac{\Delta a}{R_H} = \frac{\rho \alpha R^2}{\Sigma a},$$ (25)

then oligarchy begins when OP $\sim 1$. Inserting the efficiency of big body formation (Equation (14)) into the above equation, we find that trans-Hill growth transitions to oligarchy once $R_{\text{stir}}$ exceeds

$$R_{\text{stir}}^{(\text{oligarchy})} \approx \left\{\begin{array}{ll}
150 \text{ km} \cdot \sigma_{16}^{1/2} \left(\frac{a}{1 \text{ AU}}\right)^{1/2} & \text{collisionless} \\
(\frac{\Sigma}{\sigma}) \frac{\sigma_{16}}{\rho} \approx 120 \text{ km} \cdot \sigma_{16} \left(\frac{a}{1 \text{ AU}}\right)^{-1} & \text{collisional}
\end{array}\right.$$ (26)

where $\sigma_{16} \equiv \sigma/(16 \text{ g cm}^{-2})$. For the MMSN density profile, $\sigma_{16} \approx (a/1 \text{ AU})^{-3/2}$ with an enhancement beyond the snow line by a factor of $\sim 5$. The collisionless expression above applies as long $R_{\text{stir}}^{(\text{oligarchy})} < R_{\text{stir}} \approx s/\alpha$, otherwise the collisional expression applies. However, we caution that depending on parameter values the equal accretion stage discussed in Section 5.1 may begin earlier than oligarchy, which would lead to a different transition criterion.

Once oligarchy takes hold, neighboring regions evolve independently under the stirring of their respective oligarchs, as discussed in Sections 9–10 of GLS. Neighboring oligarchs likely converge in size, their battling leading to scattering or merging. The evolution depends to a large degree on the fate of the small bodies. At late times, small bodies collide at such high speeds that they almost certainly fragment. As the small bodies grind down each other, the evolution can become increasingly collisional. The small bodies may cool and the oligarchs may carve gaps around themselves, effectively sabotaging their accretion (Rafikov 2003b; Levison et al. 2010). We defer considerations of these dynamics to future work.

If the oligarchs eventually reach the isolation mass, they may undergo orbital instability and experience giant impacts. The
timescale to form planets of size $R$ is

$$t_{\text{impact}} \approx \frac{\rho R}{\Sigma \Omega} \sim 10^8 \left( \frac{R}{R_{\oplus}} \right) \sigma_{16}^{-1} \left( \frac{a}{1 \, \text{AU}} \right)^{3/2} \, \text{yr},$$

(27)

which is very long in the outer solar system.

Our criterion for oligarchy (OP $\sim 1$) differs from that of Ormel et al. (2010a). They stipulate, based on empirical evidence from Monte Carlo simulations, that oligarchy begins when the stirring rate by one single large body equals its growth rate and that, in turn, equals the small-body collision rate. Their definition of oligarchy differs from ours: their oligarchy begins when the ratio of small-body random velocity to Hill velocity of the biggest body reaches a minimum. In fact, their oligarchy is similar to (but still somewhat different from) the trans-Hill phase. To some extent, this is a matter of how one defines oligarchy. But irrespective of definitions, there should be a transition from trans-Hill to oligarchic behavior when $\Delta a \sim R_H$.

6. EXAMINING ASSUMPTIONS

We examine the assumptions made in deriving the properties of trans-Hill growth (Section 2).

1. The requirement that $u$ satisfies Equation (1) is equivalent to insisting that we only consider big bodies with $\alpha^{1/2} < R/R_{\text{trans}} < \alpha^{-1/2}$. Bodies that violate these restrictions grow more slowly than Equation (4) predicts (GLS), but this has little effect on trans-Hill growth.

2. The assumption that $v/v_H < 1$ is confirmed by Equation (20).

3. We ignored growth by accretion of big bodies. When that assumption is violated, equal accretion begins (Section 5.1).

4. We assumed that $s$ is constant. In truth, at late times $u$ can become sufficiently large that collisions between small bodies fragment them. We do not treat this in detail because the physics of fragmentation is complex. We note, however, that since bodies of smaller sizes are typically more resistant to fragmentation (for $s \ll 1 \, \text{km}$), the result of fragmentation is likely that $s$ is a decreasing function of the stirring bodies’ sizes. This will alter the spectrum of trans-Hill accretion (Equation (14)) in that one should replace the $s$ in that equation with the function $s(R_{\text{eq}})$. A similar remark applies to other mechanisms for damping $u$, such as gas drag. Since $s$ only appears in the equations of motion through its damping effect on $u$, gas drag may also be modeled by adopting a much smaller effective $s$.

5. We ignored oligarchy. The transition to oligarchic stage is described in Section 5.2.

7. APPLICATIONS AND COMPARISONS

7.1. The Asteroid Belt

Here, we compare our results with published coagulation simulations for the asteroid belt. At a distance of 2 AU, $\alpha \approx 2.5 \times 10^{-3}$, so the transition from collisionless to collisional evolution occurs when large bodies grow beyond $s/\alpha \sim 400 \, \text{km}(s/1 \, \text{km})$.

The uppermost end of the asteroid mass distribution is roughly flat, i.e., $d\Sigma/d\ln R \approx \text{const}$ for $100 \, \text{km} \lesssim R \lesssim 300 \, \text{km}$. But for $R \lesssim 100 \, \text{km}$, $\Sigma$ falls off with decreasing $R$ (Jedicke et al. 2002; Bottke et al. 2005; taking a constant albedo of 0.04). Both Morbidelli et al. (2009) and Weidenschilling (2011) attempt to reproduce the observed distribution, including the 100 km “bump,” by running particle-in-a-box coagulation simulations that are initialized with single-sized small planetesimals. However, the two reach opposite conclusions. Morbidelli et al. (2009), with small planetesimal sizes ranging from $s \approx 0.6$ to 6 km, fail to produce the 100 km bump: the final distributions they find are roughly flat ($q \approx 4$) for a range in $R$ that extends an order of magnitude below 100 km. However, Weidenschilling succeeds when using $s \approx 0.1 \, \text{km}$. He attributes the difference to the erroneous neglect of sub-Hill accretion by Morbidelli et al. In the following, we focus on explaining Weidenschilling’s results.

Weidenschilling’s bump at 100 km appears to be due to the transition from runaway to collisional trans-Hill growth. These simulations resemble the red curve in Figure 4—with a transition directly from runaway to collisional trans-Hill—because the initial velocity is the escape speed from seeds. Like the red curve that shows a transition around $R \sim s/\alpha$, the final size distribution in Weidenschilling (2011) also shows a bump around 50 km (his Figure 8), produced when the small body dispersion goes from super-Hill (runaway) to trans-Hill. The value of $\Sigma/\sigma$ at the location of his bump and the slope to the right of the bump are all roughly consistent with what we derive in this paper.

If the observed 100 km bump in the asteroid belt is indeed produced by the transition from runaway to trans-Hill growth, that would imply that the initial mass of small planetesimals, relative to that in current big bodies, was at least $\sigma/\Sigma \sim 1/(10\alpha) \sim 50$ times greater (Equation (18))—or even greater if some asteroids were dynamically ejected (Morbidelli et al. 2009). It would also imply that the initial small bodies had radii $\sim \alpha \times 100 \, \text{km} \sim 0.2 \, \text{km}$. One interesting implication of such a scenario is that the vast majority of these small bodies could not be dynamically ejected, because if they were that would correspondingly reduce the number of big bodies too. Instead, they had to be ground down to dust that was then eliminated by radiation forces. Whether this can occur has yet to be examined carefully. However, we note that by the time the asteroids grew to $\sim 100 \, \text{km}$, the small bodies would have begun to collide with one another with collision speeds $v_H/R=100 \, \text{km} \sim 5 \, \text{m} \, \text{s}^{-1}$. Such speeds might have been sufficient to grind all the planetesimals and eliminate them, preventing further growth of asteroids.

7.2. The Kuiper Belt

With a value of $\alpha \approx 10^{-4}$ for the Kuiper Belt, the transition to trans-Hill collisional growth occurs at $10^7 \, \text{km}(s/1 \, \text{km})$. So, with the typical choice $s \approx 1 \, \text{km}$ and the largest bodies observed at 1000 km, there is little wonder that simulations to date have not probed the collisional regime (but see A. Shannon et al. 2013, in preparation).

Kenyon & Luu (1999) perform particle-in-a-box simulations for the formation of KBOs. Starting from bodies of size $s \approx 80 \, \text{m}$, their simulations produce big bodies with a flat mass distribution ($q = 4$) for $3 \, \text{km} \lesssim R \lesssim 1000 \, \text{km}$ (see, e.g., their Figure 8). This is roughly consistent with the observed mass distribution in the Kuiper Belt at large sizes (e.g., Fraser et al. 2010) and hence may be taken as evidence that the formation

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5 An additional difference between the simulations of Morbidelli et al. (2009) and Weidenschilling (2011) is that the former initialize velocities to be the Hill velocity of the initial bodies, whereas the latter initialize them to be the escape speed. This might tend to flatten out the post-runaway bump in the simulations of Morbidelli et al., similar to how, in Figure 4, the blue curve is flatter than the red.

6 The initial small-body radii would be less than 0.2 km by a factor of $\sim 5$ if the MMSN budget of gas was still present when the bump was formed (e.g., GLS).
of KBOs has been solved. However, recent data have cast doubt on such a simple picture (Shankman et al. 2013).

Our theory explains some aspects of Kenyon & Luu’s simulations. The growth they witness should be in the collisionless regime, in which case we would predict that the mass spectrum should be flat (Equation (14)), as indeed they observe. However, the efficiency that they find is $\sim 1\%$, whereas we predict $\sim 10\% \sim 0.1\%$. In Shannon et al. (2013), we repeat their simulations and find an efficiency of $\sim 0.1\%$. As we suggest there, the discrepancy could be due to the fact the big bodies in their simulations have too high a velocity dispersion.

Schlichting & Sari (2011) consider collisionless coagulation both analytically and with numerical simulations. They argue that big bodies should grow equally by accreting big and small bodies (“equal accretion”). As they show, the mass spectrum implied by equal accretion is flat, consistent with the trans-Hill collisionless solution—and with full coagulation simulations.7

However, the amplitude of the mass spectrum in equal accretion is $\Sigma/\sigma \sim a^{3/2}$ (Schlichting & Sari 2011), whereas in trans-Hill it is $\Sigma/\sigma \sim a$ (Equation (14)). Since the latter is smaller than the former, the trans-Hill solution will begin before equal accretion can. Once trans-Hill begins, the system remains trans-Hill with $\Sigma/\sigma \sim a$, preventing equal accretion from occurring. Nonetheless, the amplitudes predicted by the two theories differ by a small amount that, moreover, could be affected by order-unity numbers that both we and Schlichting & Sari (2011) neglect. In Shannon et al. (2013), we show with numerical simulations that account for order-unity coefficients that it is indeed trans-Hill rather than equal accretion that sets the size spectrum in the collisionless regime.

A troubling concern with low-efficiency collisionless growth of KBOs is that 99.9% of the small bodies must be ground down to dust and then blown out by radiation pressure. With so much mass in dust, the disk can become optically thick, thwarting blow-out. This concern has not been adequately treated in the literature. But it is alleviated if the initial planetesimal size is significantly smaller than 1 km. The formation efficiency can then be significantly boosted by collisional growth (A. Shannon et al. 2013, in preparation).

7.3. Ormel et al. (2010b)

Ormel et al. (2010b) perform a comprehensive study of conglomeration, at semi-major axes ranging from 1 AU to 35 AU. Their numerical algorithm allows them to follow the evolution from runaway through oligarchy. We observe that there is a distinct trans-Hill phase in their simulations (e.g., Figures 8 and 11 of that paper). With their parameters of $s = 7.5$ km and $\sigma = 16$ g cm$^{-2}$, one expects the trans-Hill evolution (collisionless below $R_{\text{stir}} = 1500(a/1\text{AU})$ km) to lead to a characteristic size spectrum of $q = 4$ and a formation efficiency of $\Sigma/\sigma \sim 0.05(a/1\text{AU})$. These are indeed observed in their results.8 Above $R = 1500\text{km}(a/1\text{AU})$, the evolution is collisional. For some of their simulations (the ones at small $a$), oligarchy enters at around the same point. So, instead of observing a continuous mass spectrum of $q = 3$ (Equation (17)), they find a single body that grows to large sizes.

The reason both equal accretion and trans-Hill yield a flat mass spectrum in the collisionless regime is that, according to both, the ratio of growth by big-body accretion to that by small-body accretion is constant in time. But whereas that ratio is unity for equal accretion, it is $a^{1/2}$ for trans-Hill (Equation (23)).

Although Ormel et al. (2010b) fit their result with a power-law slope of $q = 5.5 (p = -2.5$ in their notation), it appears that the big bodies are better characterized by a flat mass distribution of $q = 4$ (see, e.g., their Figure 13).

8. SUMMARY

Runaway growth ends once the stirring bodies become trans-Hill ($v_{H\text{stir}} \sim u$ or, equivalently, $R_{\text{stir}} \sim u/\sqrt{\rho_\alpha a}$). Afterward, $R_{\text{stir}}$ and $u$ grow in unison and the stirring bodies remain trans-Hill. For $R_{\text{stir}} < s/\alpha$, trans-Hill growth is collisionless. The mass spectrum is flat ($q = 4$) and the efficiency low, $\sim 10\%$. However, for $R_{\text{stir}} > s/\alpha$, the evolution is collisional and the efficiency grows in proportion to $R_{\text{stir}}$, with a size spectrum that has $q = 3$. The time to reach the collisional transition is comparable to the small bodies’ collision time $(\rho s/\sqrt{\Omega})$. Our numerical simulations and self-similar calculations confirm these results, subject to the assumptions made in Section 2. The simulations also show that where the collisionless trans-Hill spectrum begins depends on the unknown initial conditions—especially the initial size spectrum and $u$, but the collisional trans-Hill spectrum invariably begins once $R_{\text{stir}} > s/\alpha$.

Having described trans-Hill growth, we briefly discussed what comes next when one or more of the assumptions in Section 2 breaks. Collisional trans-Hill growth ends once $R_{\text{stir}} \gtrsim sa^{3/2}$ (efficiency $\Sigma/\sigma \gtrsim \sqrt{\alpha}$), because once that happens mutual accretion of big bodies becomes important (Section 5). As we will show in a forthcoming paper (A. Shannon et al. 2013, in preparation), this regime is characterized by equal accretion of big and small bodies and can lead to order-unity efficiency. However, the mass spectrum below $R_{\text{stir}}$ is no longer frozen and hence if this regime is reached, it can wipe out the mass spectrum laid down during trans-Hill growth.

Trans-Hill growth can also be terminated when big bodies become separated by a Hill radius, in which case oligarchy sets in. Another complication is collisional fragmentation. We leave a more detailed consideration of these two effects to a future work. On the other hand, the complication of gas damping is easily incorporated by using an effective $s$.

Our theory explains results from previous studies of the asteroid and Kuiper Belt. Most simulations to date adopt parameters relevant for collisionless growth, typically with initial seeds $s \sim 1$ km. For the Kuiper Belt, this means the growth is collisionless at all times and that the efficiency is limited to $10\% \sim 0.1\%$. So, if the Kuiper Belt has been severely dynamically depleted, it would initially have had to contain multiple times the MMSN mass. Alternatively, the efficiency of formation can approach unity if the initial seed size is small. We explore how this may explain the formation of the Kuiper Belt in an upcoming publication.

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APPENDIX A

NUMERICAL SOLUTION OF THE EQUATIONS OF MOTION

In order to numerically integrate the equations of motion (Equations (4)–(7)), we first re-write the equations in terms of the trans-Hill radius $R_{\text{tran}} \equiv u/\sqrt{\rho_\alpha a}$ (Equation (10)) and the rescaled time:

$$\tau = \frac{\sigma \Omega t}{\rho_\alpha},$$

(A1)

which has units of length. Equation (4) becomes

$$\frac{dR}{d\tau} = \begin{cases} 
(R/R_{\text{tran}})^2, & \text{if } R < R_{\text{tran}}, \\
R/R_{\text{tran}}, & \text{if } R > R_{\text{tran}}, 
\end{cases}$$

(A2)
and Equation (5), with the replacement of Equation (7), becomes

\[
\frac{1}{R_{\text{tran}}} \frac{dR_{\text{tran}}}{d\tau} = -\frac{\alpha}{s} - \frac{1}{\alpha} \left( \int_{R<R_{\text{tran}}} R^6 \frac{dN}{\eta} \right) + \int_{R>R_{\text{tran}}} \frac{R^3}{R_{\text{tran}}^4} \frac{dN}{\eta} , \tag{A3}
\]

The continuity equation (Equation (6)) becomes

\[
\partial_t N + V \partial_R N = 0 , \tag{A4}
\]

where \( V \equiv dR/d\tau \) is given by Equation (A2).

Our numerical method is as follows. We choose an initial value for \( R_{\text{tran}} \) and an initial form for the function \( N(>R) \) on a logarithmic grid in \( R \), where the grid typically has 400 gridpoints per decade of \( R \). According to Equation (A4), the value of \( N \) remains constant at any \( R \) that satisfies Equation (A2). Therefore, for each initial \( N \) on the grid, we integrate its corresponding \( R \) according to Equation (A2). We simultaneously evolve \( R_{\text{tran}} \) according to Equation (A3).

**APPENDIX B**

**SELF-SIMILAR SOLUTIONS**

If one considers either the collisionless or the collisional case, then there is only a single characteristic radius, \( R_{\text{tran}} \). Therefore, Equation (A4) may be written in self-similar form when appropriately scaled. We define the scaled independent variable to be the ratio of \( R \) to its characteristic value,

\[
x \equiv R/R_{\text{tran}}(\tau) , \tag{B1}
\]

and the scale-dependent variable to be

\[
y(x) \equiv R_{\text{tran}}^p N(R, \tau)/\eta , \tag{B2}
\]

with \( p \) to be determined below. In scaled variables, Equation (A4) becomes

\[
\frac{dR_{\text{tran}}}{d\tau} \left( x \frac{dy}{dx} + py \right) = V(x) \frac{dy}{dx} , \tag{B3}
\]

where

\[
V(x) = \begin{cases} 
   x^2 , & \text{if } x < 1 \\
   x , & \text{if } x > 1 
\end{cases} \tag{B4}
\]

Self-similarity demands

\[
\frac{dR_{\text{tran}}}{d\tau} = A \Rightarrow R_{\text{tran}} = A \tau , \tag{B5}
\]

where \( A \) is a constant to be determined. Substituting into Equation (B3), we may solve for \( y \) to obtain

\[
y(x) = y(1) \begin{cases} 
   x^{-p}(A-x)^p/(A-1)^p , & x < 1 \\
   x^{-p}/(A-1) , & x > 1 
\end{cases} \tag{B6}
\]

where we must have \( A > 1 \). Equation (A3) becomes

\[
\frac{A}{R_{\text{tran}}} = -\frac{\alpha}{s} - \frac{1}{\alpha} \rho \frac{2-p}{p} \left( \int_{x<1} x^6 dy + \int_{x>1} x^3 dy \right) . \tag{B7}
\]

It remains to determine the three constants \( \rho, A, \) and \( y(1) \). To do so, we consider the collisionless and collisional cases separately.

In the collisionless case, \( \alpha/s \ll A/R_{\text{tran}} \). This is equivalent to the condition \( t < \rho \sigma(\Omega) \), i.e., that the time elapsed is less than the collision time between small bodies. Equation (B7) demands that \( p = 3 \) since \( A \) is a constant in time, i.e., \( N/\eta = y(x)/R_{\text{tran}}^3 \). Note that since \( N \) is frozen (independent of time) at small \( x \) (i.e., for \( R \ll R_{\text{tran}} \)), this immediately implies that \( N \propto R^{-3} \) for \( R \ll R_{\text{tran}} \). Integrating Equation (B7) yields a relation between the remaining two constants, \( y(1) \) and \( A \). We write the resulting self-similar solution as follows:

\[
\frac{N(>R, t)}{\eta} = \frac{\alpha}{R_{\text{tran}}^3} g_{\text{nocoll}}(\gamma) f_3 \left( \frac{R}{R_{\text{tran}}}, \gamma \right) , \quad \text{collisionless} , \tag{B8}
\]

where

\[
R_{\text{tran}}(t) = \frac{\sigma \Omega}{\rho \alpha} \frac{t}{1 - 3/\gamma} \tag{B9}
\]

\[
g_{\text{nocoll}}(\gamma) = \frac{10\gamma^3}{\gamma^3 + 6\gamma^2 + 27\gamma + 108} \tag{B10}
\]

\[
f_3(x, \gamma) = \begin{cases} 
   x^{-p} \left( 1 - x \left( 1 - \frac{p}{\gamma} \right) \right)^p , & x < 1 \\
   x^{-\gamma} \left( \frac{p}{\gamma} \right)^p , & x > 1 
\end{cases} \tag{B11}
\]

In the collisional case (\( \alpha/s \gg A/R_{\text{tran}} \)), Equation (B7) implies that \( p = 2 \), which yields \( N \propto R^{-2} \) for \( R \ll R_{\text{tran}} \). Integrating Equation (B7) yields the self-similar solution

\[
\frac{N(>R, t)}{\eta} = \frac{1}{R_{\text{tran}}^2} \frac{\alpha^2}{s} g_{\text{coll}}(\gamma) f_2 \left( \frac{R}{R_{\text{tran}}}, \gamma \right) , \quad \text{collisional} , \tag{B12}
\]

where

\[
R_{\text{tran}} = \frac{\sigma \Omega}{\rho \alpha} \frac{t}{1 - 2/\gamma} \tag{B13}
\]

\[
g_{\text{coll}}(\gamma) = \frac{10\gamma(\gamma - 3)}{\gamma^2 + 5\gamma + 16} \tag{B14}
\]

and \( f_2(x, \gamma) \) is by Equation (B11).

**REFERENCES**

Bottke, W. F., Durda, D. D., Nesvorný, D., et al. 2005, Icar, 175, 111

Chambers, J. E., & Wetherill, G. W. 1998, Icar, 136, 304

Chiang, E., & Youdin, A. N. 2010, AREPS, 38, 493

Fraser, W. C., Brown, M. E., & Schwamb, M. E. 2010, Icar, 210, 944

Goldreich, P., Lithwick, Y., & Sari, R. 2004a, ApJ, 614, 497

Goldreich, P., Lithwick, Y., & Sari, R. 2004b, ARA&A, 42, 549 (GLS)

Greenberg, R., Hartmann, W. K., Chapman, C. R., & Wacker, J. F. 1978, Icar, 35, 1

Ida, S., & Makino, J. 1993, Icar, 106, 210

Inaba, S., Tanaka, H., Nakazawa, K., Wetherill, G. W., & Kokubo, E. 2001, Icar, 149, 235

Jedicke, R., Larsen, J., & Spahr, T. 2002, in Asteroids III, ed. W. F. Bottke, Jr., A. Cellino, P. Paolicchi, & R. P. Binzel (Tucson: AZ Univ. Arizona Press), 71

Johansen, A., Oishi, J. S., Low, M.-M. M., et al. 2007, Natur, 448, 1022

Kenyon, S. J., & Luu, J. X. 1999, AJ, 118, 1101

Kokubo, E., & Ida, S. 1998, Icar, 131, 171

Lee, M. H. 2000, Icar, 143, 74

Levison, H. F., Thommes, E., & Duncan, M. J. 2010, AJ, 139, 1297

Malyskin, L., & Goodman, J. 2001, Icar, 150, 314

Morbidelli, A., Bottke, W. F., Nesvorný, D., & Levison, H. F. 2009, Icar, 204, 558
Ormel, C. W., Dullemond, C. P., & Spaans, M. 2010a, ApJL, 714, L103
Ormel, C. W., Dullemond, C. P., & Spaans, M. 2010b, Icar, 210, 507
Pollack, J. B., Hubickyj, O., Bodenheimer, P., et al. 1996, Icar, 124, 62
Rafikov, R. R. 2003a, AJ, 126, 2529
Rafikov, R. R. 2003b, AJ, 125, 942
Safronov, V. S. 1972, Evolution of the Protoplanetary Cloud and Formation of the Earth and Planets (NASA-TTF-677; Jerusalem: Keter Publishing House)
Schlichting, H. E., & Sari, R. 2011, ApJ, 728, 68
Shankman, C., Gladman, B., Kaib, N., Kavelaars, J. J., & Petit, J.-M. 2013, ApJ, 764, L2
Shannon, A., Wu, Y., & Lithwick, Y. 2013, ApJ, submitted (arXiv:1303.3888)
Weidenschilling, S. J. 2011, Icar, 214, 671
Weidenschilling, S. J., Spaute, D., Davis, D. R., Marzari, F., & Ohtsuki, K. 1997, Icar, 128, 429
Wetherill, G. W., & Stewart, G. R. 1989, Icar, 77, 330