A Vertex Correction in the Gap Equation for the High Temperature Superconductors

Yunkyu Bang
Depart. of Physics, Chonnam National University, Kwangju 500-757, Korea

We show that the Migdal theorem is obviously violated in the high Tc cuprates and the full vertex correction should be included, in particular, in the gap equation, in order to be consistent with the anomalously strong inelastic scattering in the region of the “hot spots”, which is observed from the various normal state experiments. The full vertex correction is obtained non-pertubatively by utilizing the generalized Ward identity, which is shown to hold in the important scattering channel of the pairing interaction in the high Tc cuprates. As a result, we find a strong enhancement of Tc from the vertex correction despite of the strong pair breaking effect due to the inelastic scattering.

74, 74.20, 74.20.Fg, 74.20.Mn

Since the discovery of the high Tc superconductors, there has been substantial progress in our understanding for the large part of normal state anomalies, if not its microscopic origin. The marginal Fermi liquid (MFL) ansatz\(^1\), for example, captures remarkably simple essence to understand the various normal state anomalies in a unified manner. The single and essential ingredient of the MFL phenomenology is the assumption of the anomalous scattering kernel only with temperature scale. Although the microscopic mechanism of this scattering is yet controversial, it is generally agreed on its existence and the essential role of it for the high temperature superconductivity (HTSC) phenomena. More recently, however, it becomes clear that the MFL ansatz is rather too simple to explain more problematic normal state anomalies, in particular, such as Hall resistance, c-axis charge dynamics, etc. The angle resolved photo emission (ARPES) experiment\(^2\), for example, reveals strong anisotropy of the scattering rate along the Fermi surface, conveniently termed as the “hot spot” region and the “cold spot” region according to strong or weak scattering rates in the different sections of the Fermi surface.

While it is natural to think that such an anomalous inelastic scattering, which is responsible for the normal state anomalies, should also play a role in the superconducting pairing mechanism, there has been no satisfactory attempt to incorporate it in the pairing mechanism beyond the leading order. Up to now, the superconducting pairing mechanisms for the high Tc cuprates can be largely grouped into two: non-Fermi liquid theories and Fermi liquid based theories. The former is typically represented by the interlayer pair tunneling mechanism\(^3\) and the latter one has more variety such as the antiferromagnetic paramagnon theory\(^4\), Van Hove singularity, or more exotic theory like various preformed boson scenarios\(^5\). Each theory has various degree of success to account the known data and we are not in the position to judge of them. In this paper, we would like to address the importance of the vertex correction when we apply the conventional BCS-Eliashberg pairing approach for HTSC.

It has been pointed out by many authors the inadequacy of the blind application of the Migdal theorem\(^6\) in HTSC. At the qualitative level, there are already quite simple reasons not to trust the Migdal theorem in HTSC: (1) \(w_D/E_F\) is not so small (\(w_D\) is the characteristic frequency of any mediating boson); (2) the Fermi surface anisotropy (certain degree of nesting or Van Hove singularity) poses a potential danger to invalidate the phase space constraint in the Migdal theorem argument. Nevertheless, the complexity of higher order vertex diagrams prohibits systematic progress along this line; at most it can only indicate the danger of vertex correction by calculating the first order correction.

In this paper, we took rather simple short cut by observing two experimental findings: (1) HTSC is a d-wave pairing state mediated by strongly peaked potential in the momentum space, which dominantly mediates scattering from one “hot spot” region to the other “hot spot” region in the Brillion zone\(^6\); (2) the electrons in the “hot spot” region has a singular self-energy correction, destroying almost its quasi-particle nature at all. From the above two observations, we show that we are exactly opposite limit, for the important pairing channel, from which the Migdal theorem is valid. In this opposite limit - we conveniently call it the “Ward identity limit” - we can easily read off the exact vertex from the generalized Ward identity\(^8\) given a set of reasonable assumptions. As a demonstration, we performed a numerical calculation to solve the model Tc equation for a d-wave state with a full vertex correction. We find a strong enhancement of Tc from the singular vertex correction despite of the strong pair breaking effect of the self-energy correction.

We briefly reexamine the Migdal theorem. Fig.1.a shows the typical vertex correction diagram of the first order. The typically interesting parameter regime of the momenta is that \(\vec{q},\vec{p},\vec{k}\sim\vec{k}_F\) and \(q_0,p_0,k_0\sim w_D\); here and afterward \(w_D\) is not necessarily the Debye frequency, but some characteristic frequency of any mediating boson. For the Migdal limit, i.e. \(\vec{q}\cdot\vec{v}_F>q_0(\sim w_D)\), the correction is of the order \(O(w_D/E_F)\) either due to the energy denominator in the electron propagators or due to the phase space constraint. Here the important observation is that \(\vec{q}\cdot\vec{v}_F\) should be understood as \(\epsilon(k+q)−\epsilon(k)\) in more general expression and particularly for the tight binding band. For the other limit, i.e. \(\vec{q}\cdot\vec{v}_F<q_0\), which we call the “Ward identity limit”, the correction...
is $O(1)$ as found by Migdal and also more clearly demonstrated by Engelsberg and Schrieffer using the generalized Ward identity. However, the above conclusion is true only under the following set of assumptions: (1) electron-phonon vertex, $g(\vec{k}, \vec{k}')$, has no strong momentum dependence – in practice, assumed to be constant in most of analysis; (2) similarly, the phonon dispersion, $w_q$, is assumed to be isotropic; namely, it doesn’t have any special directionality; (3) the Fermi surface doesn’t have a strong anisotropy so that it doesn’t introduce any special scattering phase space constraint; a pathological case is one dimensional system, for example; (4) in particular, for superconducting instability, the typical scattering process is involved with large momentum exchange ($|q| \sim O(|k_F|)$); for a contrasting example, the dynamic polarizability, $P(q, w)$, of electron gas for $|q|/p_F \to 0$ limit should have an important vertex correction as a trivial violation of the Migdal theorem.

Now let us examine the situation of the high Tc cuprates. First of all, $w_D/E_F$ is not that small since now the relevant $w_F$ is most probably electronic origin and also $E_F$ is renormalized to a smaller value. However, it is just one general fact to invalidate the Migdal theorem and it is not our main concern in this paper. There are two important observations just specific to the high Tc cuprates which break the Migdal theorem. First, the hole-doped high Tc cuprates is known to be a $d$-wave superconductor by now and accordingly the pairing potential is a fluctuation mediated potential which is strongly peaked for $\vec{k} - \vec{k}' = \vec{Q}$, where $\vec{Q} \simeq (\pm \pi, \pm \pi)$. Second, the Fermi surface is anisotropic; in particular, the important parts of the Fermi surface with the maximum gap opened are connected by $\vec{Q}$ (see Fig.1.b). Combining these two facts violates the Migdal assumptions (1)-(3) above. In particular, the scattering phase space constraint - the relative probability for satisfying $|q| \sim |k_F|$ is $O(w_D/E_F)$ with the Migdal assumptions - doesn’t hold anymore. Moreover, the last condition for superconductivity of large momentum exchange doesn’t help, too, because the superficial Migdal condition, i.e., $q \cdot \vec{v}_p > w_D$ with large $q$ for a parabolic band becomes $\epsilon(\vec{k} + \vec{q}) - \epsilon(\vec{k}) < w_D$ for the high Tc superconductors when both $\vec{k} + \vec{q}$ and $\vec{k}$ belong to the hot spot region. To summarize, in the high Tc cuprates, because of the strongly momentum dependent pairing potential, $V(k, k')$, inducing a $d$-wave pairing and the anisotropic Fermi surface from the tight binding nature, the important scattering process for the superconducting pairing is not in the Migdal regime ($\epsilon(\vec{k} + \vec{q}) - \epsilon(\vec{k}) > w_D$) but in the Ward identity regime ($\epsilon(\vec{k} + \vec{q}) - \epsilon(\vec{k}) < w_D$). In the following, we show that we can indeed extract the full vertex correction of this scattering channel ($\vec{k}, \vec{k} + \vec{q} \in \text{“hot spot” region}$) from the generalized Ward identity.

Engelsberg and Schrieffer derived the following general identity from the particle number conservation condition.

$$q_0 \Gamma^0 (k; k + q) - \vec{q} \cdot \overline{\Gamma}(k; k + q) = G^{-1}(k + q) - G^{-1}(k)$$

(1)

where $k, q$ stands for the four momenta, i.e., $k = (k_0, \vec{k})$, and $\Gamma^0$ and $\Gamma$ are scalar and vector vertices, respectively. $G(k) = (k_0 - \epsilon(k) - \Sigma(k))^{-1}$ is the full electron green function with a self-energy. The above equation is exact for any $k$ and $q$ and the famous “Ward identity” is derived by taking $\vec{q} = 0$ and $q_0 \to 0$ limit, so that $\Gamma^0(k; k) = 1 - \partial \Sigma(k)/\partial k_0$. The other limit, $q_0 = 0$ and $\vec{q} \to 0$, can be used to obtain the vector vertex $\overline{\Gamma}(k; k)$. Since we assume a $d$-wave pairing in the high Tc superconductors and the maximum gap is formed around the “hot spots”, the important part of the pair potential is the process which scatters from one “hot spot” region to the other “hot spot” region. Consequently, our interest is the scalar vertex, $\Gamma^0(\vec{k}, k_0; \vec{q} = \vec{Q}, q_0 < w_D)$, $\vec{k}$ belongs to “hot spot” region. In general, there is no simple solution for this unless we have a simple form of $\Sigma(k)$. Theoretical [11] and experimental [2] studies show that the self-energy correction around the “hot spots” is singular in frequency but its momentum dependence is negligible. Therefore, we can separate the scalar and vector vertices to a good degree of approximation as follows, for the interesting region of $\vec{q} \sim \vec{Q}$, $q_0 < w_D$ and $\vec{k}$ in “hot spot” region.

$$q_0 \Gamma^0(\vec{k}, k_0; \vec{Q}, q_0) \simeq$$

$$q_0 - [(\Sigma(\vec{k} + \vec{Q}, k_0 + q_0) - \Sigma(\vec{k}, k_0),)]$$

$$\vec{Q} \cdot \overline{\Gamma}(\vec{k}, k_0; \vec{k} + \vec{Q}, k_0 + q_0) \simeq \epsilon(\vec{k} + \vec{Q}) - \epsilon(\vec{k})$$

(3)

where $\epsilon(\vec{k})$ is the bare band dispersion. As argued above, we assumed that $\Sigma(\vec{k} + \vec{Q}, k_0) \simeq \Sigma(\vec{k}, k_0)$ when both $\vec{k} + \vec{Q}$ and $\vec{k}$ belong to the hot spot region for the above separation.

Now to be specific, we approximate the self-energy for the hot spot region as the marginal Fermi liquid type as follows.

$$\text{Im}\Sigma(w) = \alpha |w| \text{ for } |w| < w_c$$

(4)

where $w_c$ is the high energy cutoff, and $\alpha$ is about 0.6 from experiment [13] but it can be treated as a parameter for our purpose. In reality, the self-energy correction can be even more singular in the hot spot region [14], or maybe just Fermi liquid type ($\Sigma'' \sim w^2$) for very low frequency limit. In any case, our main conclusion doesn’t change, i.e. large vertex correction from the anomalous scattering in the hot spot region. Here the marginal Fermi liquid type self-energy assumption is just for demonstration purpose; nevertheless we think it is still reasonable assumption in view of experiments. Now
the full self-energy in the Matsubara frequency is written as $\Sigma(w_n) = -iw_n\frac{\alpha}{\pi} \log \frac{w_n^2 + w^2}{w_n}$. And the corresponding scalar vertex is written as

$$\Gamma^0(\vec{k}, w_n; \vec{q} \sim \vec{Q}, \Omega_n) = 1 - [\Sigma(w_n + \Omega_n) - \Sigma(w_n)]/i\Omega_n,$$

where $w_n = \pi k_B T (2n + 1)$ and $\Omega_n = \pi k_B T (2n)$ are the fermionic and bosonic Matsubara frequencies, respectively. Since we are interested in $\Omega_n < w_D$ regime, we take the $\Omega_n \to 0$ limit from the above equation. Now the full vertex for $\vec{q} \sim \vec{Q}, \vec{k}$ in “hot spot” region in the static limit is the following.

$$\Gamma^0(w_n) = \left[ 1 + \frac{\alpha}{\pi} \ln \frac{w_n^2 + w^2}{w_n^2} \right]$$

This vertex shows the ln$T$ divergence when $w_n \to 0$ limit due to the singular self-energy [14]. The first term of the Eq.(6) is nothing but $Z_n$ in Matsubara formalism; $Z_n$ is defined as $iw_n - \Sigma(w_n) = iw_n Z_n$. We notice that the singular suppression of the effective coupling constant due to $Z_n$ as $g_{eff} = g/Z_n$ can be almost recovered now as $g_{eff} = g \Gamma^0_n/Z_n$ by the corresponding vertex correction reminiscent of the original Ward identity in QED. With this vertex we solve the model $T_c$ equation for a d-wave state. The $T_c$ equation is written as

$$\Delta(k) = -k_B T \sum_{w_n} \sum_{k'} \frac{V(\vec{k}, \vec{k}') \Gamma^0(w_n) \Phi(\vec{k}, \vec{k}')} {Z_n^2 w_n^2 + \xi_{k'}^2} \Delta(k').$$

(7)

Some remarks are in order about the above $T_c$ equation. First of all, the above $T_c$ equation is the static limit of the Eliashberg equation, so that it would be the BCS equation were it not for the self-energy correction, $Z_n$, and the vertex correction, $\Gamma^0(w_n)$ of static limit. We could have solved a full Eliashberg equation with a given dynamic pair potential and it would be no problem to include the dynamic vertex correction, $\Gamma^0(w_n; \Omega_n)$, from Eq.(5). We think that qualitative results would not be changed. Second, the vertex correction we put in the $T_c$ equation is valid only when both incoming momentum, $\vec{k}$, and outgoing momentum, $\vec{k}' = \vec{k} + \vec{Q} + \Delta k$, belong to the hot spot region. In order to simulate this hot spot scattering channel constraint, we introduced a function $\Phi(\vec{k}, \vec{k}')$, which is just 1 only when both $\vec{k}$ and $\vec{k}'$ belong to the hot spot region connected by $\vec{Q}$; otherwise it replaces $\Gamma^0(w_n)$ with the bare vertex, $\Gamma^0 = 1$. It means that we do not consider any vertex correction for other scattering channels other than this special scattering channel. Some justifications for this are: (1) for other scattering channels, simply the vertex correction should not be as singular as for the hot spot scattering channel; (2) even if there should be some vertex correction, its effect in the $T_c$ equation is suppressed by the pair potential, $V(\vec{k}, \vec{k}')$, since it is peaked only for the hot spot scattering channel.

For simplicity of numerical calculation, we assume $V(\vec{k}, \vec{k}') = V|\sin(\phi - \phi')|$ and $\Delta(\vec{k}) = \Delta_{max} \cos(2\phi)$ in 2-dimensional momentum space, a circular Fermi surface, and $\phi$ is the angle along the Fermi surface. For all calculations, we take $w_c = 0.5eV$, $w_D = 0.3eV$ ($w_D$ enters as the BCS cutoff in the momentum summation) and the coupling constant parameter $\lambda = V\mathcal{N}(0)$ is taken to be 1.5 ($\mathcal{N}(0)$ is the density of states at the Fermi level). This choice of parameters is only for exemplary purpose. Also for the hot spot scattering channel constraint function, $\Phi(\vec{k}, \vec{k}')$, we define the the hot spot region by the angle $\theta_{hot}$ as indicated in Fig.1b; now $\Phi(\phi, \phi') = 1$ only when $\phi, \phi' \in $ “hot spot regions”, otherwise it will set $\Gamma^0(w_n) = 1$.

In Fig2, we show $T_c$ as a function of the strength of the self-energy correction, $\alpha$, for different values of $\theta_{hot}$. For example, $\theta_{hot} = \pi/2$ is the case that the whole Fermi surface is treated as the hot spot region. As expected, the $\theta_{hot} = \pi/2$ case (open square) shows an extreme slow-down of the suppression rate of $T_c$ because of the over-imposed vertex correction; the solid square indicates the result with no vertex correction at all but only with the self-energy correction, showing rapid suppression of $T_c$ with increasing $\alpha$. In between these two curves, the results with $\theta_{hot} = \pi/4$ and $\pi/8$ are shown and these should be the more realistic cases. In short, Fig.2 shows the dramatic effect of the vertex correction for $T_c$ even with a very narrow area of the hot spot region (see $\theta_{hot} = \pi/8$ case). When considering $\alpha \approx 0.6$ for YBCO ($T_c \sim 90$ K) [13] without the vertex correction at all (solid square), $T_c \approx 24$ K; only; the set of parameters (the dimensionless coupling constant, $\lambda = 1.5$, and a characteristic energy of mediating boson, $w_D = 0.3eV$ ) is already quite favorable choice for the pairing. Only with $\theta_{hot} = \pi/8$, $T_c$ is enhanced more than by a factor of 3 ($T_c \sim 85$ K) for the same $\alpha = 0.6$. The message is that in order to achieve $T_c \sim 100$ K of a d-wave state in HTSC, with strong inelastic scattering but without including the corresponding vertex correction, we are forced to choose rather unrealistic parameters in the conventional pairing model.

In conclusion, we show that the Migdal theorem is maximally violated in HTSC, not only because of $w_D/E_F \sim O(1)$ but more importantly because of the strongly momentum dependent pairing potential, $V(\vec{k}, \vec{k}')$, and the existence of hot spot region in the anisotropic Fermi surface, which are dominantly scattered by the pairing potential in the main pairing channel of a d-wave state. Then, we show that we are in the “Ward identity limit” which allows us to extract the full vertex correction from the self-energy. Assuming a phenomenological self-energy $(\Sigma'' = \alpha |v|)$ and taking into account the corresponding vertex correction around the “hot spots”, we solved the model $T_c$ equation for a d-wave pairing including both the self-energy correction and the vertex correction. The results show the dramatic enhancing effect.
of Tc by the vertex correction despite of the strong pair breaking self-energy correction. Considering the problem of choosing realistic values of parameters for a d-wave pairing model, our result enforces the essential role of the singular vertex correction, at least for a conventional BCS-Eliashberg type pairing scenario. Amusing observation is that the strong inelastic scattering in the high Tc cuprates, which is the key entity responsible for the various anomalous normal state properties [1], turns out to be not so much destructive for superconductivity in the end, in contrast to the conventional superconductivity, thanks to the singular vertex correction. The origin of it is the strong anisotropies in the pairing potential $V(k, k')$, the Fermi surface, and the order parameters $\Delta(k)$, which are all related, though. Therefore, in HTSC the strong correlations not only in frequency domain but also in spatial domain play important roles on their own to make all these unusual materials.

We acknowledge the financial support of the Korea Research Foundation, 1996, the Korean Science and Engineering Foundation (KOSEF) Grant No. 961-0208-048-1, 976-0200-006-2, and the KOSEF through the SRC program of SNU-CTP.

---

1 C. M. Varma et. al., Phys. Rev. Lett. 63, 1996 (1989).
2 Z. X. Shen and D. Dessau, Physics Reports 253, 1 (1995).
3 J. M. Wheatley, T. C. Hsu, and P. W. Anderson, Phys. Rev. B 37, 5897 (1988); S. Chakravarty et. al., Science 261, 337 (1993).
4 D. Pines, Physica C 235-240, 113 (1994), and see more references therein.
5 A. S. Alexandrov et. al., Phys. Rev. Lett. 72, 1734 (1993); V. J. Emery and S. A. Kivelson, Phys. Rev. Lett. 74, 3253 (1995).
6 A. B. Migdal, Sov. Phys. JETP, 34, 996 (1958).
7 M. H. Sharifzadeh Amin and P.C.E. Stamp, Phys. Rev. Lett. 77, 3017 (1996); A. Chubukov et. al., Phys. Rev. B 56, 7789 (1997).
8 S. Engelsberg and J. R. Schrieffer, Phys. Rev. 131, 993 (1963).
9 E. Daggotto et. al., Phys. Rev. Lett. 73, 728 (1994).
10 This vertex correction in this scattering channel has been already studied by other authors, for example, in Ref[7]. The crucial difference between the previous works and the present one is that the previous ones are pertuative approach (only to the first order) while the present work is a non-pertuative approach. Nevertheless, the result of the present paper is qualitatively consistent with the pertuative one (in particular, A. Chubukov et al. in Ref.[7])
11 R. Hlubina and T. M. Rice, Phys. Rev. B 51, 9253 (1995); B. P. Stoijkovic and D. Pines, Phys. Rev. Lett. 76, 811 (1996); Phys. Rev. B 56, 11931 (1997); A. V. Chubukov, Cond-mat/9709221 9806200.
12 The hot spot region is defined as $\Delta k \xi < 1$. $\Delta k$ is the distance of $k$ from the nearest “hot spot” along the Fermi surface. In optimally doped YBCO sample, $\xi \sim 2$ above $T_c$, therefore the hot spot region can have substantial fraction along the FS (see, e.g., B. P. Stoijkovic and D. Pine in Ref.[11])
13 Z. Schlesinger et al., Phys. Rev. Lett. 65, 801 (1990).
14 If a Fermi liquid like self-energy, say, $\Sigma'' = \tilde{\alpha} w^2$ with $\tilde{\alpha} \sim O(1/E_F)$, the vertex correction is still $O(1)$ as $w \to 0$.
FIG. 2. Tc vs $\alpha$, the pair breaking parameter defined in $\Sigma'' = \alpha|w|$; solid square is the one without vertex correction, and open upper triangle, open circle, and open square are the ones with more vertex correction (wider area of the hot spot region) in increasing order.