TRISECTIONS OF 3-MANIFOLDS

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ABSTRACT. We define a trisection of a closed, orientable three dimensional manifold into three handlebodies, and a notion of stabilization for these trisections. Several examples of trisections are described in detail. We define the trisection genus \( t(M) \) of a 3-manifold, and relate it to the Heegaard genus \( g(M) \), showing that \( t(M) \leq g(M) \leq 2t(M) \). We show moreover that the bound \( g(M) \leq 2t(M) \) is tight. We define stabilizations of trisections and show that all trisections of a 3-manifold are stably equivalent, providing an analogue of the Reidemeister-Singer theorem for trisections. We conclude by showing that there exist complicated trisections of \( S^3 \).

1. Introduction

A Heegaard splitting of a closed, orientable 3-manifold can be thought of as a “bisection” of the 3-manifold into two handlebodies. Gay and Kirby \[5\] introduced trisections of smooth, orientable 4-manifolds to create an analogous construction in the higher dimension, defining the trisection genus of a 4-manifold and proving that all trisections of a 4-manifold are stably equivalent. In this paper we consider these ideas back in the third dimension. Decompositions of non-orientable 3-manifolds into three orientable handlebodies have been analyzed by Gomez-Larrañaaga, Heil, and Núñez\[8\]\[9\] who also defined the notion of the trigenus of a nonorientable 3-manifold. Gomez-Larrañaaga also investigated which orientable 3-manifolds can be decomposed into three tori \[7\]. Coffey and Rubinstein have looked at orientable 3-manifolds formed by gluing three handlebodies in a sufficiently complicated way \[4\]. In this paper we will look at decompositions of orientable three manifolds into three handlebodies with connected pairwise intersections. This condition allows us to draw connections with the field of Heegaard splittings.

We will firsta trisection of a closed, orientable 3-manifolds and a notion of stabilization on these trisections. We then investigate several examples in detail, showing some surprising trisections. We define the trisection genus of a 3-manifold, and relate it to the Heegaard genus of the manifold. We analyze the behavior of trisection genus under
connect sum, showing that if $M$ is the connect sum of two manifolds of Heegaard genus $g$, $M$ has trisection genus equal to half its Heegaard genus. We then prove the main theorem of the paper, showing that with one trivial exception, all trisections of a closed, orientable 3-manifold $M$ can be made equivalent by stabilization.

We begin with the definition of a trisection. Let $M$ be a closed, orientable 3-manifold.

**Definition 1.** A $(h_1, h_2, h_3; b)$-trisection of $M$ is a quadruple $(H_1, H_2, H_3; B)$ such that

- $M = H_1 \cup H_2 \cup H_3$
- $H_i$ is a handlebody of genus $h_i$
- Each $S_{ij} = H_i \cap H_j$ is a compact connected surface with boundary $K$
- $B = H_1 \cap H_2 \cap H_3$ is a $b$-component link

If $h_1 = h_2 = h_3 = h$, the trisection is *balanced*, and we call it an $(h; b)$-trisection. Otherwise, it is *unbalanced*. If $(H_1, H_2, H_3; B)$ is a balanced $(h; b)$-trisection, we define the *genus* of the trisection to be $h$. Note that, in contrast to trisections of 4-manifolds where the genus refers to the complexity of the triple intersection, here it refers to the genus of the handlebodies $H_1, H_2, H_3$. The simplest trisection is the trisection of $S^3$ into three balls, with each pair of balls intersecting in a disk. We refer to this as the *trivial* trisection of $S^3$. Many more examples of trisections will be covered in section 2.

**Definition 2.** Let $i, j, k$ be the indices 1,2,3 in any order. Suppose that $S_{jk}$ is not a disk, and let $\alpha$ be a nonseparating arc in $S_{jk}$. Define a new trisection $(H'_1, H'_2, H'_3; B')$ by

\[
\begin{align*}
H'_i &= \overline{H_i \cup N(\alpha)} \\
H'_j &= \overline{H_j - H_j \cap N(\alpha)} \\
H'_k &= \overline{H_k - H_k \cap N(\alpha)} \\
B' &= H'_i \cap H'_j \cap H'_k
\end{align*}
\]

This results in a new trisection of $M$ where $h_i$ is increased by 1, and $b$ is changed by $\pm 1$. We call this operation a *stabilization*. This operation depends on the choice of $\alpha$, so it is not generally unique even after fixing a choice of $i, j, k$.

Two trisections $(H_1, H_2, H_3; B)$ and $(H'_1, H'_2, H'_3; B')$ are *isotopic* if there is an isotopy of $M$ taking each $H_i$ to the corresponding $H'_i$ and taking $B$ to $B'$. We say that one trisection of $M$ is a *stabilization* of another if it can be obtained by some sequence of stabilizations, up to
isotopy. Notice that a relabelling of the handlebodies does not necessarily produce an isotopic trisection. So, for example, \((H_1, H_2, H_3; B)\) and \((H_2, H_1, H_3; B)\) may be distinct trisections. In some settings, the order of the handlebodies may be unimportant. In the examples in the following section, we provide one possible order of the handlebodies, and implicitly treat all reorderings as part of the same class of examples.

We can now state the main theorem.

**Theorem 3.** Let \(M\) be a closed, orientable 3-manifold with two trisections. If \(M = S^3\), assume that neither of the two trisections is the trivial trisection into three balls. Then there exists a third trisection isotopic to a stabilization of each of the original two trisections.

In the following section, we will discuss several examples of trisections, and methods to obtain interesting trisections for many classes of manifolds. Section 3 we discuss how to get a balanced trisection from an unbalanced one. In section 4 we will define the trisection genus of a manifold and relate the trisection genus and Heegaard genus of \(M\). Section 5 will present the proof of Theorem 3. In the final section we will prove that there is no reasonable analogue of Waldhausen’s theorem for trisections.

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2. Examples

We begin with a few ways to get trisections of any 3-manifold, and then present some more interesting trisections of specific classes of 3-manifolds. The order of the handlebodies is unimportant for producing examples, so we will use whatever order is convenient, usually ordering from largest to smallest genus.

**Example 4.** Let \(M\) be any closed orientable 3-manifold, and let \(V \cup_{\Sigma} W\) be a genus \(g\) Heegaard splitting of \(M\). Let \(D\) be a disk in \(\Sigma\). Define \(H_3\) to be a regular neighborhood of \(D\), and let \(H_1 = V - V \cap H_3\), \(H_2 = W - W \cap H_3\). Then this defines a \((g, g, 0; 1)\)-trisection of \(M\). We can stabilize \(H_3\) \(g\) times to produce a balanced genus \(g\) trisection of \(M\). See Figure 2. The trivial \((0, 0, 0; 1)\)-trisection of \(S^3\) is a special case of this construction.
A (2, 2, 0, 1) trisection is constructed from a genus-2 trisection of $S^3$ by the construction of Example 4. A balanced trisection can be obtained by stabilizing $H_3$ twice.

**Example 5.** Suppose $(K, \phi)$ is an open book decomposition of $M$ with binding circle $K$ and $\phi : M - K \to S^1$. Then we can define

\begin{align*}
H_1 &= K \cup \phi^{-1}([0, 1/3]) \\
H_2 &= K \cup \phi^{-1}([1/3, 2/3]) \\
H_3 &= K \cup \phi^{-1}([2/3, 1])
\end{align*}

Since each $H_i$ is a thickening of a once punctured surface, each is indeed a handlebody. This gives a $(2g, 2g, 2g; 1)$-trisection of $M$ where $g$ is the genus of the fiber surface.

In fact, this is a special case of Example 4. If we set $V = K \cup \phi^{-1}([0, 1/2])$ and $W = K \cup \phi^{-1}([1/2, 1])$ we get a Heegaard splitting of $M$. Applying the technique of Example 4 gives a $(g, g, 0; 1)$-trisection which can be stabilized to the $(g; 1)$-trisection described in this example. See Figure 2.
It is known that any two open book decompositions of $S^3$ are related by plumbing and deplumbing Hopf bands [6]. Hopf plumbings gives a different notion of stabilization from that used here, but it is worth noting that plumbing and deplumbing of Hopf bands is also not a unique operation. Trisections of this form appear on the boundary of relative trisections of 4-manifolds as defined in [5] and [3].

**Figure 2.** We see a $(2, 2, 0; 1)$ trisection where the first two handlebodies are neighborhoods of Seifert surfaces of the trefoil knot. If we stabilize $H_3$ along the two red arcs, we get a trisection where each handlebody is a neighborhood of a Seifert surface, and the triple intersection curve $B$ is a trefoil.

**Example 6.** We can generalize the construction of Example 4 as follows. Let $V \cup_\Sigma W$ be a genus $g$ Heegaard splitting of $M$. Let $H_1 = W$. Choose some disk properly embedded in $V$ that cuts it into two handlebodies $H_2$ and $H_3$ of genus $h$ and $g - h$ respectively, where $0 \leq h \leq g$. Then this gives a $(g, h, g - h; 1)$-trisection. When $h = 0$ or $h = g$ this construction reduces to the construction of Example 4 possibly after relabelling the handlebodies.
Figure 3. Here we construct a \((4, 2, 2, 1)\)-trisection of \(S^3\) using the technique of Example 6. When \(H_2\) or \(H_3\) has genus 0 then we can relabel handlebodies and perform an isotopy to get the upper trisection in Figure 1. Although the disk \(H_2 \cap H_3\) here cuts the complement of \(H_1\) into two standard handlebodies, it is possible that \(H_2\) and \(H_3\) are knotted. An example of this is described in Example 7.

Example 7. We describe a specific instance of the construction of Example 6. Let \(K\) be some knot in \(S^3\). Set \(H_1 = N(K)\). Let \(D\) be a disk in \(\partial H_1\), and \(\alpha_1, \ldots, \alpha_m\) be a tunnel system for \(K\), with the endpoints of each \(\alpha_i\) lying in \(D\). Then we can set \(H_2 = N(D \cup \bigcup \alpha_i)\) and \(H_3 = M - H_1 \cup H_2\). This gives a \((1, m, m + 1; 1)\) trisection of \(S^3\). Here \(H_1 \cup H_2\) is a handlebody and \(H_1 \cap H_2\) a disk. See Figure 4 for an example.

All examples so far have been stabilizations of the construction in Example 6. To provide some different classes of examples, we present some trisections where all three handlebodies have genus lower than the Heegaard genus of the manifold.

Example 8. Suppose \(M\) is the connect sum of two 3-manifolds of Heegaard genus 1. That is, each connect summand is either \(S^1 \times S^2\) or a lens space. \(M\) has Heegaard genus 2 by [10], so we can fix a genus 2 Heegaard splitting \(X_1 \cup X_2\). We produce a \((1, 1, 1; 2)\)-trisection of \(M\). See Figure 5 for the case where \(M\) is the connect sum of two copies of \(S^1 \times S^2\). We describe this case in detail.

Let \(S\) be the reducing sphere splitting \(M\) into the two copies of \(S^1 \times S^2\). Begin by splitting \(X_1\) along the disk \(X_1 \cap S\), resulting in two genus 1-handlebodies \(H_1\) and \(H_2\). Then both \(\partial H_1\) and \(\partial H_2\) contain essential curves \(\alpha_1, \alpha_2\) respectively bounding disks in \(X_2\). Let \(\gamma\) be an arc connecting \(\alpha_1\) and \(\alpha_2\) such that the intersection of \(\gamma\) with \(\partial H_1 \cap \partial H_2\) is only a single point. Let \(\alpha\) now denote the result of performing a handle slide of \(\alpha_1\) across \(\alpha_2\) using the arc \(\gamma\). \(\alpha\) intersects both \(\partial H_1\) and
∂H₂ in a single arc, and bounds a disk D in X₂. Therefore, we can isotope H₂ to add a neighbourhood of D. We continue to call the result of this isotopy H₂. H₁ and H₂ now intersect in an annulus essential in both ∂H₁ and ∂H₂, and H₃ = M – (H₁ ∪ H₂) is a genus 1-handlebody. It follows that each intersection Sᵢⱼ is an annulus, and so all pairwise intersections are connected. We therefore have a (1, 1, 1; 2)-trisection as desired. An identical argument can be applied when one or both of the connect summands are replaced with lens spaces.

**Example 9.** Now consider a more general connect sum M = M₁#M₂ where both of M₁, M₂ have Heegaard genus g. M then has Heegaard genus 2g [10]. Fix genus g Heegaard splittings (X₁, X₂, Σ) and (X₁*, X₂*, Σ*) for M₁ and M₂ respectively. Let α₁, ···, αₙ (resp. β₁, ···, βₙ) be a collectively nonseparating set of g disjoint curves on Σ (resp. Σ*) such that each curve bounds a disk in X₂ (resp. X₂*). (X₁#X₁*, X₂#X₂*; Σ#Σ*) is a minimal genus Heegaard splitting for M. Let C be a curve splitting Σ#Σ* into the punctured copies of Σ and Σ*. We can apply a diffeomorphism to Σ#Σ* that fixes C and sends Σ to Σ and Σ* to Σ* in order to get a Heegaard diagram of the form shown in Figure [8]. For each i, let γᵢ denote the result of sliding αᵢ across βᵢ using connecting arcs intersecting C once as shown in Figure [8].
Let $D_1 \ldots D_g$ denote the meridian disks in $X_2\#X_2^*$ bounded by the $\gamma_i$.

We can now define a trisection. Let $H_1 = X_1$, so it is a genus $g$ handlebody. Define $H_2$ to be the union of $X_1^*$ and the collection of all $N(D_i)$. Removing these disks $N(D_i)$ from $X_2\#X_2^*$ leaves another genus $g$ handlebody, which we define to be $X_3$. $H_2$ as defined is isotopic to $X_1^*$, because we defined it by attaching disks that intersected $X_1^*$ in a single arc each. Thus, attaching the disk $D_i$ is equivalent to isotoping $X_2$ to extend from the arc $D_i \cap \Sigma^*$ across the disk $D_i$ to the arc $D_i \cap \Sigma$. We can also observe that each such attachment introduces a new curve component of $H_1 \cap H_2$, so the resulting trisection is a $(g, g, g; g + 1)$ trisection.

**Example 10.** Let $\Sigma$ be a closed orientable genus-$g$ surface. Let $M$ be a surface bundle $\Sigma \times [0, 1]/(x, 0) \sim (\phi(x), 1)$. $M$ has a genus $2g + 1$ Heegaard splitting. If the surface bundle is a product bundle $\Sigma \times S^1$ or if the translation distance of the monodromy map is sufficiently high.
relative to the genus of $\Sigma$, then it is known that the genus $2g + 1$ Heegaard splitting is minimal \cite{Brendle02} \cite{Brendle10}. We produce a $(2g, g + 1, g + 1; b)$ trisection of $M$, where $b$ is either 1 or 3 depending on whether $g$ is even or odd.

Case 1. $g$ is even.

See Figure 7. There is a curve $C \subset \Sigma$ cutting $\Sigma$ into two punctured genus $g/2$ surfaces. Let $\alpha$ be a path in $\Sigma$ such that $\alpha(0)$ lies on $\phi(C)$ and $\alpha(1)$ lies on $C$. Then the path $P = \{\alpha(2t) \times t : 0 \leq t \leq 1/2\}$ is transverse to the fibers, so $\Sigma \times [0, 1/2] - N(P)$ is homeomorphic to a thickened punctured genus $g$ surface, and is therefore a genus $2g$ handlebody. Let $H_1$ be this handlebody. Now, $C \times [1/2, 1]$ cuts $\Sigma \times [1/2, 1]$ into two genus $g$ handlebodies $H_2, H_3$. Split the tube $N(P)$ into two halves as in Figure 7 and assign half to $H_2$ and half to $H_3$ so that they become genus $g + 1$ handlebodies. Note that performing twists to $N(P)$ will possibly produce non-isotopic trisections. In the resulting trisection, $H_2 \cap H_3$ is a punctured torus, and each of $H_1 \cap H_2$ and $H_1 \cap H_3$ is the union of two punctured genus $g$ surfaces connected by a band.
Case 2. \( g \) is odd.

The idea is approximately the same. Instead of \( C \) we choose two curves \( C_1, C_2 \) cutting \( \Sigma \) into two twice punctured genus \((g-1)/2\) surfaces. Choose the path \( \alpha \) to connect a point on \( C_1 \) to a point on \( C_2 \). Everything else goes through as before, and we end up with a \((2g, g, g; 3)\) trisection. \( H_2 \cap H_3 \) is now a thrice punctured planar surface, and each of \( H_1 \cap H_2 \) and \( H_1 \cap H_3 \) is a thrice punctured genus \( g-1 \) surface.

3. Balancing Trisections

We prove that an unbalanced trisection can be turned into a balanced trisection without increasing the genus of the largest handlebody.
Proposition 11. Let \((H_1, H_2, H_3; B)\) be an \((h_1, h_2, h_3; b)\) trisection of \(M\). Then there is a balanced \((h'; b')\)-trisection of \(M\) that is a stabilization of \((H_1, H_2, H_3; B)\), where \(h' = \max(h_1, h_2, h_3)\). Additionally, we can ensure that \(b' \leq \max(b, 2)\). That is, either \(b' \leq b\) or \(b' = 2\) and \(b = 1\).

Proof. Choose \(i, j, k\) to be a permutation of 1, 2, 3 such that \(h_i \geq h_j \geq h_k\). If \(h_i = h_j = h_k\) then we are done. Otherwise, we know that \(h_i > h_k\). Now, if \(S_{ij}\) were a disk, then \(H_i \cup H_j\) would be a genus \(h_i + h_j\) handlebody with complement \(H_k\). This would give a Heegaard splitting and would imply that \(h_i + h_j = h_k\), which contradicts \(h_i > h_k\). Therefore \(S_{ij}\) is not a disk. Hence there exists some nonseparating arc \(\alpha\) properly embedded in \(S_{ij}\). Moreover, unless \(b = 1\), we can choose \(\alpha\) to have its endpoints lie on two distinct components of \(B\). Performing a stabilization with this choice of \(\alpha\) gives a \((h_i, h_j, h_k + 1; b')\)-trisection, where \(b'\) is either \(b - 1\) or \(2\). This operation does not increase the genus of any handlebody beyond \(h_i\), and only increases \(b\) if \(b = 1\). Therefore, we can repeat the operation until we get a balanced trisection of genus \(h_i\). □

We investigate the surfaces \(S_{ij}\). Again, let \(i, j, k\) be some permutation of 1, 2, 3. We can compute the genus of the handlebody \(H_i\) from \(b\) and the genera \(g(S_{ij})\) and \(g(S_{ik})\) of \(S_{ij}\) and \(S_{ik}\) by the formula \(h_i = g(S_{ij}) + g(S_{ik}) + b - 1\). In a balanced trisection, \(h_1 = h_2 = h_3\). Comparing the formula for \(h_1\) and \(h_2\) we see that \(g(S_{13}) = g(S_{23})\). Similarly we can compare the formulas for \(h_2\) and \(h_3\) to see that \(g(S_{12}) = g(S_{13})\). Therefore, in a balanced trisection, all \(g(S_{ij})\) are the same, and are equal to \(\frac{h + 1 - b}{2}\). Since this must be an integer, we also get the following:

Remark 12. In a balanced \((h, b)\)-trisection, \(b\) and \(h\) must have opposite parities.

4. Trisections, Heegaard Splittings, and Trisection Genus

Just as the Heegaard genus \(g(M)\) of a 3-manifold is defined as the smallest \(g\) for which \(M\) has a genus \(g\) Heegaard splitting, we can define the trisection genus \(t(M)\) to be the smallest \(t\) for which \(M\) has a balanced trisection of genus \(t\). Here we state some facts about trisection genus, and about how trisections relate to Heegaard splittings.

Proposition 13. If \(M\) is a closed orientable 3-manifold, its Heegaard genus \(g(M)\) and trisection genus \(t(M)\) are related by

\[
t(M) \leq g(M) \leq 2t(M)
\]
Proof. First, note that by combining the construction of Example 4 or 6 with Proposition 11, whenever \( M \) has a Heegaard splitting of genus \( g \) we can also construct balanced trisections of genus \( g \). It follows that \( t(M) \leq g(M) \). We can also get a Heegaard splitting from a trisection \((H_1, H_2, H_3; B)\) as follows. First choose one of the three handlebodies \( H_i \), and let \( j, k \) be the indices not chosen. Choose a maximal set of nonseparating arcs in \( S_{jk} \), and stabilize \( H_i \) along each of these arcs in turn to get a new trisection \((H'_1, H'_2, H'_3; B')\). In this trisection, \( S'_{jk} \) is now a disk, since if it were not then there would be some nonseparating arc in it, contradicting the maximality of our choice of arcs. Therefore, \( H'_j \cup H'_k \) is a handlebody. If we started with a balanced trisection of genus \( h \) then \( h \) stabilizations were required to make \( S'_{jk} \) a disk, so \( g(H'_i) = 2h \). It follows that \((H'_i, H'_j \cup H'_k; \partial H'_i)\) is a genus \( 2h \) Heegaard splitting. Applying this construction to a minimal genus balanced trisection of \( M \), we conclude that \( g(M) \leq 2t(M) \). □

Since the construction used in the previous proposition is quite useful, we set it aside as a definition.

Definition 14. Suppose \((H_1, H_2, H_3; B)\) is a trisection. If we stabilize \( H_i \) along a maximal set of arcs in \( H_{jk} \), then we get a trisection \((H'_1, H'_2, H'_3; B')\) where \( H_j \cup H_k \) is a handlebody, so \((H_i, H_j \cup H_k; \partial H_i)\) is a Heegaard splitting. We call this the Heegaard splitting built from the trisection \((H_1, H_2, H_3; B)\) by stabilizing \( H_i \). If we do not care which \( i \) was chosen, we just say that it is a Heegaard splitting built from the trisection.

Remark 15. For a given trisection \((H_1, H_2, H_3; B)\), we do not know that, for example, the Heegaard splitting built by stabilizing \( H_1 \) and the Heegaard splitting built by stabilizing \( H_2 \) are isotopic. However, after fixing a choice of handlebody \( H_i \) we are stabilizing along a maximal set of arcs in \( S_{jk} \). Any two such maximal system of arcs in \( S_{jk} \) are slide equivalent, so any two choices of arc systems will result in isotopic Heegaard splittings. It follows that there are at most three isotopy classes of Heegaard splittings that can be built from a given trisection, one for each choice of handlebody \( H_i \).

It is natural to ask how strict these inequalities are. We have already shown that the inequality \( g(M) \leq 2t(M) \) is the best general bound possible; Example 9 demonstrates that if \( M = M_1 \# M_2 \) is a connect sum with \( g(M_1) = g(M_2) \) then \( t(M) = g(M) \). However, it is known that \( g(M) = 2g(M_1) = 2t(\text{[10]}) \). This gives us
Proposition 16. Suppose both $M_1$ and $M_2$ are closed orientable 3-manifolds with Heegaard genus $g$. Let $M = M_1 \# M_2$. Then $M$ has Heegaard genus $2g$ and trisection genus $g$.

Corollary 17. For each integer $t \geq 0$, there exists a 3-manifold with trisection genus $t$ and Heegaard genus $2t$.

We can also ask whether for every $t \geq 0$ there exists a 3-manifold $M$ such that both the trisection and Heegaard genus are equal to $t$. $S^3$ satisfies this for $t = 0$, and any Lens space satisfies it for $t = 1$. The fact that there exist examples for $t = 2$ follows from the following proposition relating Heegaard splittings built from trisections to Hempel distance [11].

Proposition 18. Suppose $(H_1, H_2, H_3; B)$ is a trisection of $M$ such that no $H_i$ has genus $g(H_i) = 0$. Then any Heegaard splitting built from $(H_1, H_2, H_3; B)$ has distance at most 2.

Proof. Suppose without loss of generalization that we build a Heegaard stabilization by stabilizing $H_1$. Let the trisection achieved by stabilization be $(H_1', H_2', H_3'; B)$ so that $(H_1', H_2' \cup H_3'; \partial H_1')$ is the Heegaard splitting. In order to demonstrate the distance bound we find a sequence of 3 curves $\alpha, \beta, \gamma$ on $\partial H_1'$ such that $\alpha$ bounds a disk in $H_1'$ and $\gamma$ bounds a disk in $H_2' \cup H_3'$. See Figure 8 for a picture of the case where each handlebody is genus 1, which easily generalizes to higher genus.

Let $\alpha$ be a loop enclosing a cocore of one of the stabilizations that took $H_1$ to $H_1'$. Let $\beta$ be some curve in $H_1 \cap H_3 \subset \partial H_1$, and note that the stabilization occurred away from $\beta$, so we can treat $\beta$ as also lying in $H_1' \cap H_3'$.

Now to find $\gamma$, first choose any meridian disk $D$ of $H_2'$. Since $H_2' \cap H_3'$ is a disk, we can isotope $D$ so that $\partial D$ lies in $H_1' \cap H_2'$. Set $\gamma$ to be $\partial D$. By construction $\alpha \subset H_1' - H_1 \cap H_3$, $\beta \subset H_1' \cap H_3'$, and $\gamma \subset H_1' \cap H_2'$. Thus, $\alpha \cap \beta$ is empty, as is $\beta \cap \gamma$. Moreover, $\alpha$ bounds a disk in $H_1'$ and $\gamma$ bounds a disk in the complement of $H_1'$, so this is indeed a distance 2 path.

Corollary 19. There exist 3-manifolds with both Heegaard genus and trisection genus equal to 2.

Proof. Suppose $M$ is a 3-manifold with a Heegaard splitting of genus $g = 2$ and distance at least 5. It is known that such manifolds exist by [11]. It is known that when a Heegaard surface has distance $d > 2g$, it represents the unique minimal genus Heegaard splitting [14]. It follows that $M$ has no other genus 2 Heegaard splittings, and hence has no
Figure 8. Here we see a Heegaard splitting built from a balanced genus 1 trisection. $H_1$ is the yellow torus on the left, and $H_2$ the union of the blue torus, the blue band, and the blue disk attached in some way along the boundary. $H'_1$ is obtained from $H_1$ by attaching an arc that intersects a meridian disk of the blue torus once. Therefore, $H'_1$ is isotopic to the union of $H_1$ and the blue torus. $\alpha$ is then a meridian curve of the blue torus, $\beta$ the red curve on the left, and $\gamma$ the attaching curve for the blue disk.

genus 2 Heegaard splitting of distance $\leq 2$. $M$ does have a trisection of genus 2 by Example 4. If $M$ had a genus 1 trisection, it would have a distance 2 Heegaard splitting by Proposition 18 which would be a contradiction. Therefore, both the trisection genus and Heegaard genus of $M$ must be equal to 2.

This corollary can also be derived using the classification of genus 1 trisections by Gomez-Larrañaga [7], since any 3-manifold not in his list that has a genus 2 Heegaard splitting necessarily also has trisection genus 2.

It would be interesting to know whether there exist higher genus examples with trisection genus equal to their Heegaard genus.

5. The Stabilization Theorem

Before proving the theorem, we need one more definition.

Definition 20. Suppose $(H_1, H_2, H_3; B)$ is a trisection of $M$ such that $(H_1, H_2 \cup H_3; \partial H_1)$ is a Heegaard splitting. Suppose moreover that there exists a disk $D$ properly embedded in $H_1$ such that $\partial D$ consists
of a nonseparating arc in $S_{12}$ and a nonseparating arc in $S_{13}$. We can stabilize $H_2$ and then $H_1$ as in Figure 9. We call this operation a fake Heegaard stabilization. The effect of this operation is to perform a “standard” stabilization between $H_1$ and $H_2$, as in Figure 10.

Remark 21. A disk $D$ as required in the above definition always exists if we have just stabilized $H_1$. Indeed, stabilizing changes $H_1$ by attaching a 1-handle to it, and a core disk of this one handle will satisfy the requirements. Once we have found such a disk, we can use parallel copies of it to perform an arbitrary number of fake Heegaard stabilizations.

Now we provide the proof of Theorem 3, which we restate here for convenience.

**Theorem 3.** Let $M$ be a closed, orientable 3-manifold with two trisections. If $M = S^3$, assume that neither of the two trisections is the trivial trisection into three balls. Then there exists a third trisection isotopic to a stabilization of each of the original two trisections.

Figure 9. We perform a fake Heegaard stabilization by stabilizing $H_2$ and then $H_1$. 

\[ (A) \quad (B) \quad (C) \quad (D) \]
Let \((H_1, H_2, H_3; B)\) and \((H^*_1, H^*_2, H^*_3; B^*)\) be two trisections of a closed orientable 3-manifold \(M\). To avoid excessive notation, we use the same notation for both a trisection and the stabilizations that we obtain from that trisection. The basic strategy is as follows:

1. Perform stabilizations until the quadruple \((h_1, h_2, h_3; b)\) is the same as \((h^*_1, h^*_2, h^*_3; b^*)\)
2. Stabilize \(H_1\) and \(H^*_1\) until \(H_2 \cup H_3\) and \(H^*_2 \cup H^*_3\) are handlebodies
3. Perform *fake Heegaard stabilizations* until the Heegaard splittings \((H_1, H_2 \cup H_3; \partial H_1)\) and \((H^*_1, H^*_2 \cup H^*_3; \partial H^*_1)\) are isotopic
4. Stabilize \(H_3\) and \(H^*_3\) until \(S_{12}\) and \(S^*_{12}\) are disks
5. Stabilize \(H_2\) and \(H^*_2\) until \(S_{13}\) and \(S^*_{13}\) are disks.

After these steps, we will show that the resulting trisections are isotopic. Since we will have started with two arbitrary trisections and stabilized both until we have isotopic trisections, the theorem follows.

*Proof of Theorem 3*

**Step 1.** First we stabilize so that the genera of the handlebodies in the two trisections are the same. By applying Proposition 11 we may assume both trisections are balanced. If \(b > 2\) we stabilize the first trisection along an arc connecting two components of \(B\), and then reapply Proposition 11. Do the same for the second trisection if \(b^* > 2\). Then we have an \((h; b)\) and an \((h^*; b^*)\) balanced trisection where both \(b, b^*\) are either 1 or 2. If \(h < h^*\), perform any stabilization on the first trisection, and then reapply Proposition 11 and repeat until \(h = h^*\). Do the same to the second trisection if \(h^* < h\). So we may assume
that $h = h^*$ and, by the proof of Proposition 11, $b$ and $b^*$ must still be $\leq 2$. By Remark 12, $b$ and $h$ must have opposite parities, so we see that $(h; b)$ is the same as $(h^*; b^*)$ as desired. In future steps we perform stabilizations equally to both trisections so as to retain the property that both trisections have the same tuple $(h_1, h_2, h_3; b)$.

**Step 2.** Choose a maximal nonseparating set of $h$ properly embedded arcs in $S_{23}$. Stabilizing along all arcs of this set results in a trisection where $S_{23}$ is a disk. This means the complement of $H_1$ is the union of two handlebodies $H_2 \cup H_3$ glued along a disk in their boundaries, so it, too, is a handlebody. Do the same thing to the other trisection. Since we began this step with a balanced trisection with $h > 0$, $S_{23}$ was not a disk, so at least one stabilization was required in this step. By Remark 21, both trisections now satisfy the necessary conditions to apply fake Heegaard stabilizations.

**Step 3.** We now have that $(H_1, H_2 \cup H_3; \partial H_1)$ and $(H_1^*, H_2^* \cup H_3^*; \partial H_1^*)$ are Heegaard splittings of $M$. By the Reidemeister-Singer theorem, there exists a common Heegaard stabilization of these two Heegaard splittings. The fake Heegaard stabilization operation affects the Heegaard splitting $(H_1, H_2 \cup H_3; \partial H_1)$ just as Heegaard stabilization does. Therefore, by repeatedly performing fake Heegaard stabilizations to both trisections we may assume that $(H_1, H_2 \cup H_3; \partial H_1)$ and $(H_1^*, H_2^* \cup H_3^*; \partial H_1^*)$ represent isotopic Heegaard splittings. In particular, $H_1$ and $H_1^*$ are isotopic in $M$.

**Step 4.** Choose a maximal set of nonseparating arcs properly embedded in $S_{12}$ and stabilize $H_3$ along all of them. Do the same for the other trisection. Since no stabilizations are performed on $H_1$ or $H_1^*$, $H_1$ and $H_1^*$ are still isotopic in $M$. After doing this, $S_{12}$ and $S_{12}^*$ are disks.

**Step 5.** Up to isotopy, we may now assume that $H_1 = H_1^*$ and $S_{12} = S_{12}^*$. Since $S_{12}$ is a disk, $H_1 \cup H_2$ can be obtained by attaching some 1-handles to $H_1$, with all attaching points occurring on $S_{12}$. We show that if we were allowed to slide the ends of these handles along loops in $\partial H_1$, we could arrange for them to be a set of small arcs each parallel rel $\partial$ into $S_{12}$. Note that sliding the ends around freely will not necessarily correspond to an isotopy of the trisection because the ends may need to slide across $S_{13}$ to trivialize the handles. See Figure 11.

To see that this is true, we define a Heegaard splitting of $H_2 \cup H_3$. Let $W$ be the union of $H_2$ with a regular neighborhood of $\partial(H_2 \cup H_3)$, so $V$ is a compression body. Let $W$ be the complement of $V$, so $W$ is a slightly shrunken version of $H_3$, and is a handlebody. Then $V \cup W$ is a Heegaard splitting of $H_2 \cup H_3$ as desired. We now use the fact that Heegaard splittings of handlebodies are standard, which follows from [2, 17].
and induction on genus. Since \( H_2 \cup H_3 \) is a handlebody, the Heegaard splitting must be a stabilization of the standard one. Therefore, \( W \) must topologically be the union of \( \partial(H_2 \cup H_3) \) and some number of trivial 1-handles. These 1-handles are therefore simultaneously parallel into \( \partial(H_2 \cup H_3) \).

![Diagram](image)

**Figure 11.** After step 4, \( H_2 \) looks like a set of 1-handles attached to the thickened disk \( N(S_{12}) \). The set of such 1-handles is simultaneously isotopic into \( \partial H_1 \), so it is possible to slide the ends around on \( \partial H_1 \) to get to the lower picture where the set of handles is parallel into \( S_{12} \). However, performing such slides might require sliding the ends of the handles across \( S_{13} \), which does not correspond to an isotopy of the trisection.

To allow us to slide the ends of these 1-handles freely, stabilize \( H_2 \) as much as possible until \( S_{13} \) is a disk. After these stabilizations, \( H_2 \) consists of \( N(\partial H_1 - \{\text{disk}\}) \cup \{\text{trivial 1-handles}\} \). If we have done the
same thing to the other trisection, we now know that there is an isotopy taking $H_1$ to $H_1^*$ and $H_2$ to $H_2^*$. Note that if we hadn’t performed step 1, it would be possible that $H_2$ had fewer or more of the trivial 1-handles than $H_2^*$. The isotopy must also necessarily take $H_3$ to $H_3^*$, so the trisections are in fact isotopic. Therefore, we have constructed a common stabilization of both initial trisections. This concludes the proof.

□

6. Trisections of $S^3$

Recall that any Heegaard splitting of $S^3$ is a stabilization of the standard splitting into two balls [17]. One might hope for a similar result for trisections. Since the genus 0 trisection cannot be stabilized, the simplest form of such a statement can be immediately ruled out. However, one might still hope that there exists some finite list of low genus trisections such that any trisection of $S^3$ is obtained by stabilizing something in the list. This turns out this too is impossible. Specifically,

**Proposition 22.** There exists an infinite class of $(1, 2, 2; 2)$ trisections of $S^3$ that are not stabilizations of any other trisection. Therefore, there is no finite list of trisections of $S^3$ that can be stabilized to cover all possible trisections of $S^3$.

**Proof.** First, we must investigate how to detect stabilized trisections. Since a stabilization is performed by adding a neighborhood of an arc in $S_{jk}$ to $H_i$, it follows that we can detect destabilizations as follows:

**Definition 23.** Suppose $M$ is a closed oriented 3-manifold with trisection $(H_1, H_2, H_3; B)$. A destabilizing disk $D$ is an essential nonseparating disk properly embedded in some $H_i$ such that $\partial D$ consists of a nonseparating arc of $S_{ij}$ and a nonseparating arc of $S_{ik}$.

If there exists a destabilizing disk $D$, we can pinch $H_j$ and $H_k$ together across $D$, performing a compression on $H_i$. Since $D$ is nonseparating, $H_i$ is still a handlebody, and $H_j$ and $H_k$ are unaffected topologically, so we still have a decomposition of $M$ into three handlebodies. Since the arcs of $\partial D$ in $S_{ij}$ and $S_{ik}$ were nonseparating, the two surfaces are still connected after the compression. $S_{jk}$ has been affected by attaching a band, so it too is still connected. Thus, all pairwise intersections are still connected. Therefore the result of this operation is indeed still a trisection, and we call this trisection a destabilization of the original trisection. This destabilization operation is the reverse of a stabilization.
Now we can demonstrate the class of examples. Koda and Ozawa describe a class of knots whose exteriors contain an incompressible, boundary incompressible twice punctured genus-1 surface $\Sigma$ cutting the exterior into two handlebodies $[12]$. This surface intersects $N(K)$ in two toroidal curves with nonzero rational slope on $\partial N(K)$. Let $H_1$ be $N(K)$ and let $H_2$ and $H_3$ be the two genus-2 handlebodies resulting from cutting the exterior of $K$ along $\Sigma$. Let $B$ be $\Sigma \cap H_1$. Then $(H_1, H_2, H_3; B)$ is a trisection of $S^3$. Since the components of $B$ are toroidal curves on $\partial H_1$, any disk properly embedded in $H_1$ must intersect each component of $B$ at least twice. Therefore, no such disk can be a destabilizing disk. Any destabilizing disk in $H_2$ or $H_3$ would contradict boundary incompressibility, so these also cannot exist. It follows that this $(1, 2, 2; 2)$-trisection is not a stabilization of any other trisection. Since the class of knots provided by Koda and Ozawa is infinite, the proposition follows.

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