Hyperbolic topology and bounded locally homeomorphic quasiregular mappings in 3-space

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(Presented by V. Ya. Gutlyanski˘ı)

The paper dedicated to the memory of my colleague and friend, brilliant mathematician Bogdan Bojarski

Abstract. We use our new type of bounded locally homeomorphic quasiregular mappings in the unit 3-ball to address long standing problems for such mappings, including the Vuorinen injectivity problem. The construction of such mappings comes from our construction of non-trivial compact 4-dimensional cobordisms $M$ with symmetric boundary components and whose interiors have complete 4-dimensional real hyperbolic structures. Such bounded locally homeomorphic quasiregular mappings are defined in the unit 3-ball $B^3 \subset \mathbb{R}^3$ as mappings equivariant with the standard conformal action of uniform hyperbolic lattices $\Gamma \subset \text{Isom} H^3$ in the unit 3-ball and with its discrete representation $G = \rho(\Gamma) \subset \text{Isom} H^4$. Here, $G$ is the fundamental group of our non-trivial hyperbolic 4-cobordism $M = (H^4 \cup \Omega(G))/G$, and the kernel of the homomorphism $\rho: \Gamma \to G$ is a free group $F_3$ on three generators.

Keywords. Bounded quasiregular mappings, Vuorinen problem, local homeomorphisms, hyperbolic group action, hyperbolic manifolds, cobordisms, group homomorphism, deformations of geometric structures.

1. Introduction

The theory of quasiregular mappings, initiated by the works of M. A. Lavrentiev and later by Reshetnyak and Martio, Rickman, and Väisälä, shows that they form (from the geometric function theoretic point of view) the correct generalization of the class of analytic functions to higher dimensions. In particular, Reshetnyak proved that non-constant quasiregular mappings are (generalized) branched covers, that is, continuous, discrete, and open mappings and, hence, local homeomorphisms modulo an exceptional set of (topological) codimension at least two, and that they preserve sets of measure zero. For the theory of quasiregular mappings, see [20, 21] and [26].

Here, we address some properties of bounded quasiregular mappings $f : B^3 \to \mathbb{R}^3$ in the unit ball $B^3$ and well-known problems on such quasiregular mappings. These results will be heavily based on our recent construction [9] (Theorem 4.1) of surjective locally homomorphic quasiregular mappings $F : S^3 \setminus S_\ast \to S^3$ with topological barriers at points of a dense subset $S_\ast \subset S^2 \subset S^3$. Due to its importance for the understanding of results of this paper, in the Appendix, we will give some details of this construction based on properties of non-trivial compact 4-dimensional cobordisms $M^4$ with symmetric boundary components – see [7–9]. The interiors of these 4-cobordisms have complete 4-dimensional real hyperbolic structures and are universally covered by the real hyperbolic space $H^4$, while the boundary components of $M^4$ have (symmetric) 3-dimensional conformally flat structures obtained by deformations of the same hyperbolic 3-manifold whose fundamental group $\Gamma$ is a uniform lattice in $\text{Isom} H^3$. Such conformal deformations of hyperbolic manifolds are well understood after...
their discovery in [3], see [6]. Nevertheless, till recently, such “symmetric” hyperbolic 4-cobordisms were unknown despite our well-known constructions of non-trivial hyperbolic homology 4-cobordisms with very asymmetric boundary components – see [10] and [4–6].

The above subset $S_*$ of the boundary sphere $S^2 = \partial B^3$ is a countable $\Gamma$-orbit of a Cantor subset with Hausdorff dimension $\ln 5 / \ln 6 \approx 0.89822444$ (where a uniform hyperbolic lattice $\Gamma$ conformally acts in the unit ball $B^3$). All its points are essential singularities of the bounded locally homeomorphic quasiregular mapping $f : B^3 \to \mathbb{R}^3$ defined as the restriction to the unit ball $B^3$ of our quasiregular mapping $F : S^3 \setminus S_* \to S^3$. This bounded quasiregular mapping $f : B^3 \to \mathbb{R}^3$ has no radial limits at all points $x \in S_* \subset S^2 = \partial B^3$ and gives an advance to Fatou’s well-known problem on the correct analog for higher-dimensional quasiregular mappings of Fatou’s theorem [13] on radial limits of a bounded analytic function of the unit disk. There are several results concerning radial limits of mappings of the unit ball. The most recent progress is due to Kai Rajala who proved, in particular, that radial limits exist for infinitely many points of the unit sphere, see [19] and references therein for some earlier results in this direction.

Another application of our construction [9] (Theorem 4.1) of surjective locally homeomorphic quasiregular mappings $F : S^3 \setminus S_* \to S^3$ is to a well-known open problem on injectivity of quasiregular mappings in space formulated by Matti Vuorinen in the 1970–1980s (see Vuorinen [24–25], [26] (p. 193, Problem 4) and Problem 7.66 in Hayman’s list of problems [11, 16]). The problem asks whether a proper quasiregular mapping $f$ in the unit ball $B^n, n \geq 3$, with a compact branching set $B_f \subset B^n$ is injective. It is false, when $n = 2$. The conjecture is known to be true in the special case $f(B^n) = B^n, n \geq 3$ – see [24]. In Section 2, we show that the quasiregular mapping $f : B^3 \to \mathbb{R}^3$ defined as the restriction to the unit ball $B^3$ of our quasiregular mapping $F : S^3 \setminus S_* \to S^3$ from [9] (Theorem 4.1) is essentially a counter-example to this conjecture. This mapping $f$ is essentially proper in the sense that any compact $C \subset f(B^3)$ has a compact subset $C' \subset B^3$ such that $f(C') = C$. This mapping $f$ is bounded locally homeomorphic, but not injective quasiregular mapping in the unit ball. Its restriction to a round ball $B_r \subset B^3$ of radius $r < 1$ arbitrarily close to one gives (after re-scaling) a proper bounded quasiregular mapping of the round ball $B^3$ serving as a counter-example to this Vuorinen conjecture (Theorem 2.2).

The last task of this paper is to investigate the asymptotic behavior of bounded locally homeomorphic quasiregular mappings in the unit ball in smaller balls $B^n(r) \subset B^n$ of radius $r$ close to one. There is an open Vuorinen conjecture that, in dimension $n \geq 3$, it is not possible that, for $y \in f(B^n)$ and all $r \in (1/2, 1)$, the cardinality

$$\text{card}(B^n(r) \cap (f^{-1}(y))) > \frac{1}{(1 - r)^{n-1}}. \quad (1.1)$$

In Section 3, we show that this question is closely related to the growth function of the kernel (a free group of rank 3) $F_3 \subset \Gamma$ of the homomorphism $\rho : \Gamma \to G$ of our hyperbolic lattice $\Gamma \subset \text{Isom} H^3$ to the constructed discrete group $G \subset \text{Isom} H^4$ – see [9]. We conclude that our bounded locally homeomorphic quasiregular mappings in the unit ball $B^3$ satisfy this conjecture.

1.1. Acknowledgments

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2. Not injective bounded quasiregular mappings

Here, we apply our construction [9] (see Appendix: Theorems 4.1 and 4.5) of bounded locally homeomorphic quasiregular mapping \( F : B^3 \to \mathbb{R}^3 \) to solve the Vuorinen open problem on injectivity of quasiregular mappings in 3-dimensional space. This well-known problem was formulated by Matti Vuorinen in the 1970–1980s as a result of the investigations of quasiregular mappings in space (see Vuorinen [24–25], [26] (p. 193, Problem 4) and Problem 7.66 in Hayman’s list of problems [11, 16]).

The problem asks whether a proper quasiregular mapping \( f \) in the unit ball \( B^n \), \( n \geq 3 \), with a compact branching set \( B_f \subset B^n \) is injective. The mapping \( f(z) = z^2 \), where \( z \in B^2 \), shows that the conjecture is false, when \( n = 2 \). The conjecture is known to be true in the special case \( f(B^n) = B^n \), \( n \geq 3 \) – see [24]. Here we give a counter-example to this conjecture for \( n = 3 \).

**Proposition 2.1.** Let the uniform hyperbolic lattice \( \Gamma \subset \text{Isom} H^3 \) and its discrete representation \( \rho : \Gamma \to G \subset \text{Isom} H^3 \) with the kernel as a free subgroup \( F_3 \subset \Gamma \) be as in Theorem 4.1. Then the bounded locally homeomorphic quasiregular mapping \( F : B^3 \to \mathbb{R}^3 \) constructed in Theorem 4.5 as a \( \Gamma \)-equivariant mapping in (4.3) is an essentially proper bounded quasiregular mapping in the unit 3-ball \( B^3 \) which is locally homeomorphic (\( B_F = \emptyset \)), but not injective.

**Proof.** The discrete group \( G = \rho(\Gamma) \subset \text{Isom} H^4 \cong \text{Mob}(3) \) has its invariant bounded connected component \( \Omega_1 \subset \Omega(G) \subset S^3 \), where its fundamental polyhedron \( P_1 \) is quasiconformally homeomorphic to the convex hyperbolic polyhedron \( P_0 \) fundamental for our hyperbolic lattice \( \Gamma \subset \text{Isom} H^3 \) conformally acting in the unit ball \( B^3(0,1) \), \( \phi_1^{-1} : P_0 \to P_1 \). This homeomorphism \( \phi_1^{-1} \) from (4.2) maps polyhedral sides of \( P_0 \) to the corresponding sides of the polyhedron \( P_1 \) and preserves all dihedral angles.

Our bounded locally homeomorphic quasiregular mapping \( F : B^3 \to \Omega_1 \subset \mathbb{R}^3 \) defined in (4.3) as the equivariant extension of this homeomorphism \( \phi_1^{-1} \) maps the tessellation of \( B^3 \) by compact \( \Gamma \)-images of \( P_0 \) to the tessellation of \( \Omega_1 \) by compact \( G \)-images of \( P_1 \). This shows that, for any compact subset \( C \subset \Omega_1 = F(B^3) \) (covered by finitely many polyhedra \( g(P_1), g \in G \)), there is a compact subset \( C' \subset B^3 \) (covered by finitely many corresponding polyhedra \( \gamma(P_1), \gamma \in \Gamma \)) such that \( F(C') = C \).

On the other hand, this locally homeomorphic quasiregular mapping \( F \) is not injective in \( B^3 \). In fact, for any element \( \gamma \neq 1 \) in the kernel \( F_3 \subset \Gamma \) of the homomorphism \( \rho : \Gamma \to G \), the image \( F(P_0) = P_1 \) of the fundamental polyhedron \( P_0 \) of the lattice \( \Gamma \) is the same as the image \( F(\gamma(P_0)) \) of the translated polyhedron \( \gamma(P_0) \).

One may restrict our not injective essentially proper bounded quasiregular mapping \( F \) in the unit 3-ball \( B^3 \) from Proposition 2.1 to a round ball \( B_r \subset B^3 \) of radius \( r < 1 \) arbitrarily close to one. The composition of this restriction \( F_r : B_r \to \mathbb{R}^3 \) with the stretching of \( B_r \) to the unit ball \( B^3 \), i.e., the mapping

\[
 f : B^3 \to \mathbb{R}^3, \quad f(x) = F_r(rx),
\]

is a proper bounded locally homeomorphic quasiregular mapping of the unit ball \( B^3 \). For any point \( y \in f(B^3) \), the number \( N_y \) of its preimages in \( B^3 \), i.e., the cardinality of the set \( \{ x \in B^3 : f(x) = y \} \), is finite. In fact, due to Lemma 9.22 in [26], this number \( N_y \) is independent of \( y \in f(B^3) \) in the image set and is set as \( N_f \). The number \( N_f \) of such preimages of \( y \in f(B^3) \) is determined by the number of images \( \gamma(P_{ker}) \), \( \gamma \in F_3 \subset \Gamma \), of a fundamental polyhedron \( P_{ker} \subset H^3 \) in our round ball \( B_r \subset B^3 \) of radius \( r < 1 \) defining the mapping \( f \) in (2.1). Here, \( F_3 \) is the free subgroup \( F_3 \subset \Gamma \) in the uniform hyperbolic lattice \( \Gamma \subset \text{Isom} H^3 \) (the kernel of the discrete representation \( \rho : \Gamma \to G \subset \text{Isom} H^4 \) from Theorems 2.1 and 4.1), and \( P_{ker} \subset H^3 \) is its fundamental polyhedron in the hyperbolic space \( H^3 \).
Making the radius \( r < 1 \) sufficiently close to 1 (i.e., changing our quasiregular mapping \( f : B^3 \to \mathbb{R}^3 \)), one can make the number \( N_f \) arbitrarily large. This proves the following (the Vuorinen conjecture' counter-example):

**Theorem 2.2.** There are proper bounded quasiregular mappings \( f : B^3 \to \mathbb{R}^3 \) without branching sets \( (B_f = \emptyset) \) which are locally homeomorphic, but not injective. Their preimages \( \{ x \in B^3 : f(x) = y \} \) are finite and can be made arbitrarily large.

3. Asymptotics of bounded quasiregular mappings in the unit ball and growth in free groups

Here, we investigate the asymptotic behavior of bounded locally homeomorphic quasiregular mappings \( f \) in the unit ball. The question is how many preimages of a point \( y \in f(B^n) \) do we have in smaller balls \( B^n(r) \subset B^n \) of radius \( r \) close to one. There is an open Vuorinen conjecture that, in dimension \( n \geq 3 \), it is not possible that, for \( y \in f(B^n) \) and all \( r \in (1/2, 1) \), the cardinality of such preimage in \( B^n(r) \) is bigger than \((1 - r)^{1-n} - \text{see (1.1)}\).

As we will show, for our bounded quasiregular mappings in the unit ball \( B^3 \), \( F : B^3 \to \mathbb{R}^3 \), constructed in Theorem 4.5 is closely related to the growth function of the kernel \( F_3 \subset \Gamma \) of the homomorphism \( \rho : \Gamma \to G \) of our uniform hyperbolic lattice \( \Gamma \subset \text{Isom} H^3 \) to the constructed discrete group \( G \subset \text{Isom} H^4 \) – see Proposition 4.3 and Lemma 4.4 in Appendix. Here, \( F_3 \) is a free group on 3 generators.

For free groups \( F_m \) on \( m \) generators, one can use well-known facts about their growth functions, cf. [14]. The growth function \( \gamma_{G, \Sigma} \) of a group \((G, \Sigma)\) with a generating set \( \Sigma \) counts the number of elements in \( G \) whose length (in the word metric) is at most a natural number \( k \):

\[
\gamma_{G, \Sigma}(k) = \text{card}\{ g \in G : |g|_\Sigma \leq k \}.
\]

**Lemma 3.1.** A free group \( F_m \) on \( m \) generators (for any free system \( \Sigma \) of generators) has \( 2m(2m - 1)^{k-1} \) elements of length \( k \), and its growth function:

\[
\gamma_{F_m}(k) = 1 + \frac{m}{m-1}((2m - 1)^k - 1).
\]

**Proof.** Clearly, in a free group \( F_m \) on \( m \) generators, we have the number of elements with length \( i \) equals to \( c_i = \text{card}\{ g \in F_m : |g| = i \} = 2m(2m - 1)^i \) Therefore, the growth function \( \gamma_{F_m}(k) = 1 + 2m + 2m(2m - 1) + \ldots + 2m(2m - 1)^{k-1} \). This gives the value \( \gamma_{F_m}(k) \) in the lemma. \( \square \)

In the embedded Cayley graph \( \varphi(K(\Gamma, \Sigma)) \subset B^3 \) (i.e. the graph that is dual to the tessellation of \( B^3 \) by convex hyperbolic polyhedra \( \gamma(P_0), \gamma \in \Gamma \)), we consider its subgraph (a tree) corresponding to our free group \( F_3 \subset \Gamma \) on 3 generators (the kernel of the homomorphism \( \rho \)). The embedding \( \varphi \) of the Cayley graph \( K(\Gamma, \Sigma) \) is a \( \Gamma \)-equivariant proper embedding. It is a pseudoisometry, i.e., for the word metric \((*,*)\) on \( K(\Gamma, \Sigma) \) and the hyperbolic metric \( d \) in the unit ball \( B^3 \), there are positive constants \( K \) and \( K' \) such that \( (a,b)/K \leq d(\varphi(a), \varphi(b)) \leq K \cdot (a,b) \) for all \( a,b \in K(\Gamma, \Sigma) \) satisfying one of the following two conditions: either \( (a,b) \geq K' \) or \( d(\varphi(a), \varphi(b)) \geq K' \).

Let \( D \) be the maximum of the hyperbolic length of generators of the kernel \( F_3 \subset \Gamma \). All vertices of our tree subgraph corresponding to elements in \( F_3 \) of length at most \( k \) are in the hyperbolic ball centered at \( 0 \in B^3 \) with radius \( R = Dk \). This hyperbolic ball corresponds to the Euclidean ball \( B^3(0, r) \subset B^2(0, 1) \) of the radius \( r = (e^R - 1)/(e^R + 1) \).
Multiplying (1.1) by \((1 - r)^{n-1}\), we see that we need to estimate the asymptotics of

\[ (1 - r)^{n-1} \text{card}(B^n(r) \cap (F^{-1}(y))) \] \( (3.3) \)

for arbitrarily small \(\epsilon = (1 - r)\) or for arbitrarily large \(\lambda = \ln((2/(1 - r)) - 1)\).

In the case of our free group \(F_3\) on 3 generators (the kernel of the homomorphism \(\rho\)), Lemma 3.1 shows that the growth function \(\gamma_{\rho_{F_3}}(k) = 1 + 3(5^k - 1)/2\). This reduces the asymptotics of (3.3) to the asymptotics of \(3(5^\lambda D - 1)/e^{2\lambda}\) for arbitrarily large \(\lambda\). Since the last expression tends to 0, when \(\lambda\) tends to \(\infty\), we conclude that our bounded locally homeomorphic quasiregular mappings \(F: B^3 \to \Omega_1 \subset \mathbb{R}^3\) in the unit ball \(B^3\) satisfy the Vuorinen conjecture (1.1).

**Remark 3.2.** There is an important observation. If, in our analysis of the asymptotics of (3.3) (and in our construction of groups \(\Gamma\) and \(G\)), the kernel of the corresponding homomorphism \(\rho: \Gamma \to G \subset \text{Isom } H^4\) were a free subgroup \(F_m\) on a big number \(m\) of generators, then our last expression would tend to \(\infty\), when \(\lambda\) tends to \(\infty\). This would provide a way to constructing a similar bounded locally homeomorphic quasiregular mapping in the unit ball giving a possible counter-example to (1.1).

### 4. Appendix: Hyperbolic 4-cobordisms and deformations of hyperbolic structures

For the convenience of readers, we now provide essential details of our construction [9] of \(\text{ locally homeomorphic quasiregular surjective mappings } F: S^3 \setminus S_\epsilon \to S^3\) based on the properties of non-trivial “symmetric” hyperbolic 4-cobordisms \(M^4 = (H^1 \cup \Omega(G))/G\) constructed in [7]. Properties of the fundamental group \(\pi_1(M^4) \cong G \subset \text{Isom } H^4\) of such “symmetric” hyperbolic 4-cobordisms \(M^4 = H^4/G\) acting discretely in the hyperbolic 4-space \(H^4\) and in the discontinuity set \( \Omega(G) \subset \partial H^4 = S^3\) are very essential for our construction of the quasiregular mapping \(F\).

We start with our construction [7] of such discrete group \(G \subset \text{Isom } H^4\) and the corresponding discrete representation \(\rho: \Gamma \to G\) of a uniform hyperbolic lattice \(\Gamma \subset \text{Isom } H^3\). These discrete groups \(G\) and \(\Gamma\) produce a non-trivial (not a product) hyperbolic 4-cobordism \(M^4 = (H^1 \cup \Omega(G))/G\) whose boundary components \(N_1\) and \(N_2\) are topologically and geometrically symmetric to each other. These \(N_1\) and \(N_2\) are covered by two \(G\)-invariant connected components \(\Omega_1\) and \(\Omega_2\) of the discontinuity set \(\Omega(G) \subset S^3\), \(\Omega(G) = \Omega_1 \cup \Omega_2\). The conformal action of \(G = \rho(\Gamma)\) in these components \(\Omega_1\) and \(\Omega_2\) is symmetric and has contractible fundamental polyhedra \(P_1 \subset \Omega_1\) and \(P_2 \subset \Omega_2\) of the same combinatorial type allowing one to realize them as a compact polyhedron \(P_0\) in the hyperbolic 3-space, i.e., the dihedral angle data of these polyhedra satisfy Andreev’s conditions [1]. Nevertheless, this geometric symmetry of boundary components of our hyperbolic 4-cobordism \(M^4(G)\) does not make the group \(G = \pi_1(M^4)\) quasi-Fuchsian, and our 4-cobordism \(M^4\) is non-trivial.

Here, a Fuchsian group \(\Gamma \subset \text{Isom } H^3 \subset \text{Isom } H^4\) conformally acts in the 3-sphere \(S^3 = \partial H^4\) and preserves a round ball \(B^3 \subset S^3\), where it acts as a cocompact discrete group of isometries of \(H^3\). Due to the Sullivan structural stability (see [22] for \(n = 2\) and [4], Theorem 7.2), the space of quasi-Fuchsian representations of a hyperbolic lattice \(\Gamma \subset \text{Isom } H^3\) into \(\text{Isom } H^4\) is an open connected component of the Teichmüller space of \(H^3/\Gamma\) or the variety of conjugacy classes of discrete representations \(\rho: \Gamma \to \text{Isom } H^4\). Points in this (quasi-Fuchsian) component of the variety correspond to trivial hyperbolic 4-cobordisms \(M(G)\), where the discontinuity set \(\Omega(G) = \Omega_1 \cup \Omega_2 \subset S^3 = \partial H^4\) is the union of two topological 3-balls \(\Omega_i, i = 1, 2\), and \(M(G)\) is homeomorphic to the product of \(N_1\) and the closed interval \([0, 1]\).

We may consider hyperbolic 4-cobordisms \(M(\rho(\Gamma))\) corresponding to uniform hyperbolic lattices \(\Gamma \subset \text{Isom } H^3\) generated by reflections. Natural inclusions of these lattices into \(\text{Isom } H^4\) act at infinity.
\( \partial H^4 = S^3 \) as Fuchsian groups \( \Gamma \subset \text{M"ob}(3) \) preserving a round ball \( B^3 \subset S^3 \). In this case, our construction of the mentioned discrete groups \( \Gamma \) and \( G = \rho(\Gamma) \) results in the following (see [7]):

**Theorem 4.1.** There exists a discrete Möbius group \( G \subset \text{M"ob}(3) \) on the 3-sphere \( S^3 \) generated by finitely many reflections such that:

1. Its discontinuity set \( \Omega(G) \) is the union of two invariant components \( \Omega_1, \Omega_2 \);

2. Its fundamental polyhedron \( P \subset S^3 \) has two contractible components \( P_i \subset \Omega_i, \ i = 1, 2 \), having the same combinatorial type (of a compact hyperbolic polyhedron \( R_0 \subset H^3 \));

3. For the uniform hyperbolic lattice \( \Gamma \subset \text{Isom} H^3 \) generated by reflections in sides of the hyperbolic polyhedron \( P_0 \subset H^3 \) and acting on the sphere \( S^3 = \partial H^4 \) as a discrete Fuchsian group \( i(\Gamma) \subset \text{Isom} H^4 = \text{M"ob}(3) \) preserving a round ball \( B^3 \) (where \( i : \text{Isom} H^3 \subset \text{Isom} H^4 \) is the natural inclusion), the group \( G \) is its image under a homomorphism \( \rho : \Gamma \to G \), but it is not quasiconformally (topologically) conjugate in \( S^3 \) to \( i(\Gamma) \).

**Construction:** We define a finite collection \( \Sigma \) of reflecting 2-spheres \( S_i \subset S^3, \ 1 \leq i \leq N \). As the first three spheres \( S_1, S_2, \) and \( S_3 \), we take the coordinate planes \( \{ x \in \mathbb{R}^3 : x_j = 0 \} \), and \( S_4 = S^2(0, R) \) is the round sphere of some radius \( R > 0 \) centered at the origin. The value of the radius \( R \) will be determined later. Let \( B = \bigcup_{1 \leq i \leq 4} B_i \) be the union of the closed balls bounded by these four spheres, and let \( \partial B \) be its boundary (a topological 2-sphere) having four vertices which are the intersection points of four triples of our spheres. We consider a simple closed loop \( \alpha \subset \partial B \) which does not contain any of our vertices and symmetrically separates two pairs of these vertices from each other, as the white loop does on the tennis ball. This loop \( \alpha \) can be considered as the boundary of a topological 2-disk \( \sigma \) embedded in the complement \( D = S^3 \setminus B \) of our four balls. Our geometric construction needs a detailed description of such a 2-disk \( \sigma \) and its boundary loop \( \alpha = \partial \sigma \) shown in Fig. 1.

The desired disk \( \sigma \subset D = S^3 \setminus B \) can be described as the boundary in the domain \( D \) of the union of a finite chain of adjacent blocks \( Q_j \) (regular cubes) with disjoint interiors whose centers lie on the coordinate planes \( S_1 \) and \( S_2 \) and whose sides are parallel to the coordinate planes. This chain starts from the unit cube whose center lies on the second coordinate axis, in \( e_2 \cdot \mathbb{R}_+ \subset S_1 \cap S_3 \). Then our chain goes up through small adjacent cubes centered in the coordinate plane \( S_1 \), changes its direction at some point to the horizontal one toward the third coordinate axis, where it turns its horizontal direction by a right angle again (along the coordinate plane \( S_2 \)), goes toward the vertical line passing through the second unit cube centered in \( e_1 \cdot \mathbb{R}_+ \subset S_2 \cap S_3 \), then goes down along that vertical line and finally ends at that second unit cube, see Fig. 1. We will define the size of small cubes \( Q_i \) in our block chain and the distance of the centers of two unit cubes to the origin in the next step of our construction.

Let us consider one of our cubes \( Q_i \), i.e., a block of our chain, and let \( f \) be its square side having a nontrivial intersection with our 2-disk \( \sigma \subset D \). For that side \( f \), we consider spheres \( S_j \) centered at its vertices and having a radius such that each two spheres centered at the ends of an edge of \( f \) intersect each other at an angle of \( \pi/3 \). In particular, for the unit cubes, such spheres have radius \( \sqrt{3}/3 \). From such defined spheres, we select those spheres that have centers in our domain \( D \) and then include them in the collection \( \Sigma \) of reflecting spheres. Now, we define the distance of the centers of our big (unit) cubes to the origin. It is determined by the condition that the sphere \( S_4 = S^2(0, R) \) is orthogonal to the sphere \( S_j \in \Sigma \) centered at the vertex of such a cube closest to the origin. As in Fig. 2, let \( f \) be a square side of one of our cubic blocks \( Q_i \) having a nontrivial intersection \( f_\sigma = f \cap \sigma \) with our 2-disk \( \sigma \subset D \). We consider a ring of four spheres \( S_i \) whose centers are interior points of \( f \) which lie outside of
the four previously defined spheres $S_j$ centered at vertices of $f$ and such that each sphere $S_i$ intersects two adjacent spheres $S_{i-1}$ and $S_{i+1}$ (we enumerate spheres $S_i$ mod 4) at an angle of $\pi/3$. In addition, these spheres $S_i$ are orthogonal to the previously defined ring of bigger spheres $S_j$, see Fig. 2. From such defined spheres $S_i$, we select those spheres that have nontrivial intersections with our domain $D$ outside the previously defined spheres $S_j$ and then include them in the collection $\Sigma$ of reflecting spheres. If our side $f$ is not the top side of one of the two unit cubes, we add another sphere $S_k \in \Sigma$. It is centered at the center of this side $f$ and is orthogonal to the four previously defined spheres $S_i$ with centers in $f$, see Fig. 2.

Now, let $f$ be the top side of one of the two unit cubes of our chain. Then, as before, we consider another ring of four spheres $S_k$. Their centers are interior points of $f$, lie outside of the four previously defined spheres $S_i$ closer to the center of $f$, and are such that each sphere $S_k$ intersects two adjacent spheres $S_{k-1}$ and $S_{k+1}$ (we enumerate spheres $S_k$ mod 4) at an angle of $\pi/3$. In addition, these new four spheres $S_k$ are orthogonal to the previously defined ring of bigger spheres $S_i$, see Fig. 2. We note that the centers of these four new spheres $S_k$ are vertices of a small square $f_s \subset f$ whose edges are parallel to the edges of $f$, see Fig. 2. We set this square $f_s$ as the bottom side of the small cubic box adjacent to the unit one. This finishes our definition of the family of twelve round spheres whose interiors cover the square ring $f \setminus f_s$ on the top side of one of the two unit cubes in our cube chain and tells us which two spheres among the four new defined spheres $S_k$ were already included in the collection $\Sigma$ of reflecting spheres (as the spheres $S_j \in \Sigma$ associated to small cubes in the first step).

This also defines the size of small cubes in our block chain. Now, we can vary the remaining free parameter $R$ (which is the radius of the sphere $S_4 \in \Sigma$) in order to make two horizontal rows of small blocks with centers in $S_1$ and $S_2$, correspondingly, to share a common cubic block centered at a point.
in $e_3 \cdot \mathbb{R}_+ \subset S_1 \cap S_2$, see Fig. 1.

We can use the collection $\Sigma$ of reflecting spheres $S_i$ to define a discrete reflection group $G = G_{\Sigma} \subset \text{M"ob}(3)$. Important properties of $\Sigma$ are: (1) the closure of the disk $\sigma \subset D$ is covered by balls $B_j$; (2) any two spheres $S_j, S_j' \in \Sigma$ either are disjoint or intersect at an angle of $\pi/2$ or $\pi/3$; (3) the complement of all balls $B_j, 1 \leq j \leq N$, is the union of two disjoint contractible polyhedra $P_1$ and $P_2$ of the same combinatorial type and equal corresponding dihedral angles. So, the discontinuity set $\Omega(G) \subset S^3$ of $G$ consists of two invariant connected components $\Omega_1$ and $\Omega_2$ which are the unions of the $G$-orbits of $\bar{P}_1$ and $\bar{P}_2$, and this defines a Heegaard splitting of the 3-sphere $S^3$ (see [9]):

**Lemma 4.2.** The splitting of the discontinuity set $\Omega \subset S^3$ of our discrete reflection group $G = G_{\Sigma} \subset \text{M"ob}(3)$ into $G$-invariant components $\Omega_1$ and $\Omega_2$ defines a Heegaard splitting of the 3-sphere $S^3$ of infinite genus with ergodic word hyperbolic group $G$ action on the separating boundary $\Lambda(G)$.

To finish our construction in Theorem 4.1, we notice that the combinatorial type (with magnitudes of dihedral angles) of the bounded component $P_1$ of the fundamental polyhedron $P \subset S^3$ coincides with the combinatorial type of its unbounded component $P_2$. Applying Andreev’s theorem on 3-dimensional hyperbolic polyhedra [1], one can see that there exists a compact hyperbolic polyhedron $P_0 \subset H^3$ of the same combinatorial type with the same dihedral angles ($\pi/2$ or $\pi/3$). So one can consider a uniform hyperbolic lattice $\Gamma \subset \text{Isom} H^3$ generated by reflections in sides of the hyperbolic polyhedron $P_0$. This hyperbolic lattice $\Gamma$ acts in the sphere $S^3$ as a discrete co-compact Fuchsian group $i(\Gamma) \subset \text{Isom} H^4 = \text{M"ob}(3)$ (i.e., as the group $i(\Gamma) \subset \text{Isom} H^4$, where $i : \text{Isom} H^3 \subset \text{Isom} H^4$ is the natural inclusion) preserving a round ball $B^3$ and having its boundary sphere $S^2 = \partial B^3$ as the limit set. Obviously, there is no self-homeomorphism of the sphere $S^3$ conjugating the action of the groups $G$ and $i(\Gamma)$, because the limit set $\Lambda(G)$ is not a topological 2-sphere. So, the constructed group $G$ is not a quasi-Fuchsian group.

One can construct a natural homomorphism $\rho : \Gamma \to G, \rho \in R_3(\Gamma)$, between these two Gromov hyperbolic groups $\Gamma \subset \text{Isom} H^3$ and $G \subset \text{Isom} H^4$ defined by the correspondence between sides of the hyperbolic polyhedron $P_0 \subset H^3$ and reflecting spheres $S_i$ in the collection $\Sigma$ bounding the fundamental polyhedra $P_1$ and $P_2$. Then we have
Proposition 4.3. The homomorphism \( \rho \in \mathcal{R}_3(\Gamma) , \rho : \Gamma \to G \), in Theorem 4.1 is not an isomorphism. Its kernel \( \ker(\rho) = \rho^{-1}(e_G) \) is a free rank 3 subgroup \( F_3 < \Gamma \).

Its proof (see [9], Prop.2.4) is based on the following statement (see [9], Lemma 2.5) in combinatorial group theory:

Lemma 4.4. Let \( A = \langle a_1, a_2 \mid a_1^2, a_2^2, (a_1a_2)^2 \rangle \cong B = \langle b_1, b_2 \mid b_1^2, b_2^2, (b_1b_2)^2 \rangle \cong C = \langle c_1, c_2 \mid c_1^2, c_2^2, \ (c_1c_2)^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), and let \( \varphi : A \ast B \to C \) be a homomorphism of the free product \( A \ast B \) into \( C \) such that \( \varphi(a_1) = \varphi(b_1) = c_1 \) and \( \varphi(a_2) = \varphi(b_2) = c_2 \). Then the kernel \( \ker(\varphi) = \varphi^{-1}(e_C) \) of \( \varphi \) is a free rank 3 subgroup \( F_3 < A \ast B \) generated by elements \( x = a_1b_1, y = a_2b_2 \) and \( z = a_1a_2b_2a_1 = a_1a_1 \).

4.1. Bending homeomorphisms between polyhedra

Here, we sketch our construction of a quasiconformal homeomorphism \( \phi_1 : P_1 \to P_0 \) between the fundamental polyhedron \( P_1 \subset \Omega_1 \subset \Omega(G) \subset S^3 \) for the group \( G \) action in \( \Omega_1 \) and the fundamental polyhedron \( P_0 \subset B^3 \) for the conformal action of our hyperbolic lattice \( \Gamma \subset \text{Isom} H^3 \) from Theorem 4.1. This mapping \( \phi_1 \) is a composition of finitely many elementary "bending homeomorphisms". It maps faces to faces and preserves the combinatorial structure of the polyhedra and their corresponding dihedral angles.

First, we observe that, to each cube \( Q_j, 1 \leq j \leq m \), used in the above construction of the group \( G \) (see Fig. 1), we may associate a round ball \( B_j \) centered at the center of the cube \( Q_j \) and such that its boundary sphere is orthogonal to the reflection spheres \( S_i \) from our generating family \( \Sigma \) whose centers are at vertices of the cube \( Q_j \). In particular, for the unit cubes \( Q_1 \) and \( Q_m \), the reflection spheres \( S_i \) centered at their vertices have radius \( \sqrt{3}/3 \), so the balls \( B_1 \) and \( B_m \) (whose boundary spheres are orthogonal to those corresponding reflection spheres \( S_i \)) should have radius \( \sqrt{5}/12 \). We add also another extra ball \( B^3(0, R) \) (which we consider as two balls \( B_0 \) and \( B_{m+1} \) whose boundary is the reflection sphere \( S^2(0, R) = S_4 \in \Sigma \) centered at the origin and orthogonal to the closest reflection spheres \( S_i \) centered at vertices of two unit cubes \( Q_1 \) and \( Q_m \). Our different enumeration of this ball will be used, when we consider different faces of our fundamental polyhedron \( P_1 \) lying on that reflection sphere \( S_4 \).

Now, for each cube \( Q_j, 1 \leq j \leq m \), we may associate a discrete subgroup \( G_j \subset G \subset \text{M"ob}(3) \cong \text{Isom} H^3 \) generated by reflections in the spheres \( S_i \in \Sigma \) associated to that cube \( Q_j \) - see our construction in Theorem 4.1. One may think about such a group \( G_j \) as a result of quasiconformal bending deformations (see [4], Chapter 5) of a discrete Möbius group preserving the round ball \( B_j \) associated to the cube \( Q_j \) (whose center coincides with the center of the cube \( Q_j \)). As the first step in such deformations, we define two quasiconformal "bending" self-homeomorphisms of \( S^3 \), \( f_1 \) and \( f_{m+1} \), preserving the balls \( B_1, \ldots, B_m \) and the set of their reflection spheres \( S_i, i \neq 4 \), and transferring \( \partial B_0 \) and \( \partial B_m + 1 \) into 2-spheres orthogonally intersecting \( \partial B_1 \) and \( \partial B_m \) along round circles \( b_1 \) and \( b_{m+1} \), respectively - see (3.1) and Fig. 6 in [9].

In the next steps in our bending deformations, for two adjacent cubes \( Q_{j-1} \) and \( Q_j \), let us denote, by \( G_{j-1,j} \subset G \), the subgroup generated by reflections with respect to the spheres \( S_i \subset \Sigma \) centered at common vertices of these cubes. This subgroup preserves the round circle \( b_j = b_{j-1,j} = \partial B_{j-1} \cap \partial B_j \). This shows that our group \( G \) is a result of the so-called "block-building construction" (see [4], Section 5.4) from the block groups \( G_j \) by sequential amalgamated products:

\[
G = G_1 *_{G_{1,2}} G_2 *_{G_{2,3}} \cdots *_{G_{j-2,j-1}} G_{j-1} *_{G_{j-1,j}} G_j *_{G_{j,j+1}} \cdots *_{G_{m-1,m}} G_m. \tag{4.1}
\]

The chain of these building balls \( \{B_j\}, 1 \leq j \leq m \), contains the bounded polyhedron \( P_1 \subset \Omega_1 \). For each pair \( B_{i-1} \) and \( B_i \) with the common boundary circle \( b_i = \partial B_{i-1} \cap \partial B_i, 1 \leq i \leq m \), we construct
a quasiconformal bending homeomorphism \( f_i \) that transfers \( B_i \cup B_{i-1} \) onto the ball \( B_i \) and which is conformal in dihedral \( \zeta \)-neighborhoods of the spherical disks \( \partial B_i \backslash \overline{B}_{i-1} \) and \( \partial B_{i-1} \backslash \overline{B}_{i} \) - see (3.3) and Fig. 7 in [9]. In each \( i \)-th step, \( 2 \leq i \leq m \), we reduce the number of balls \( B_i \) in our chain by one. The composition \( f_{m+1} f_i f_{i-1} \cdots f_2 f_1 \) transfers all spheres from \( \Sigma \) to spheres orthogonal to the boundary sphere of our last ball \( B_m \) which we renormalize as the unit ball \( B(0,1) \). We note that all intersection angles between these spheres do not change. We define our quasiconformal homeomorphism

\[
\phi_i : P_1 \to P_0
\]

(4.2)
as the restriction of the composition \( f_{m+1} f_m f_{m-1} \cdots f_2 f_1 \) of our bending homeomorphisms \( f_j \) on the fundamental polyhedron \( P_1 \subset \Omega_1 \).

4.2. Bounded locally homeomorphic quasiregular mappings

Now, we define bounded quasiregular mappings \( F : B^3 \to \mathbb{R}^3 \) as in Theorem 4.1 in [9]:

**Theorem 4.5.** Let the uniform hyperbolic lattice \( \Gamma \subset \text{Isom} H^3 \) and its discrete representation \( \rho : \Gamma \to G \subset \text{Isom} H^4 \) with the kernel as a free subgroup \( F_3 \subset \Gamma \) be as in Theorem 4.1. Then there is a bounded locally homeomorphic quasiregular mapping \( F : B^3 \to \mathbb{R}^3 \) whose all singularities lie in an exceptional subset \( S_* \) of the unit sphere \( S^2 \subset \mathbb{R}^3 \) and form a \( \Gamma \)-orbit dense in \( S^2 \) of a Cantor subset with Hausdorff dimension \( \ln 5 / \ln 6 \approx 0.8982444 \). These (essential) singularities create a barrier for \( F \) in the sense that, at points \( x \in S_* \), the map \( F \) does not have radial limits.

**Construction:** We construct our quasiregular mapping \( F : B^3 \to \Omega_1 = F(B^3) \) in the unit ball \( B^3 \) by equivariant extension of the above quasiconformal homeomorphism \( \phi_i^{-1} : P_0 \to P_1 \) which maps polyhedral sides of \( P_0 \) to the corresponding sides of the polyhedron \( P_1 \) and preserves combinatorial structures of polyhedra as well as their dihedral angles:

\[
F(x) = \rho(\gamma) \circ \phi_i^{-1} \circ \gamma^{-1}(x) \quad \text{if} \ |x| < 1, \ x \in \gamma(P_0), \ \gamma \in \Gamma
\]

(4.3)
The tessellations of \( B^3 \) and \( \Omega_1 \) by corresponding \( \Gamma \)- and \( G \)-images of their fundamental polyhedra \( P_0 \) and \( P_1 \) are perfectly similar. This implies that our quasiregular mapping \( F \) defined by (4.3) is bounded and locally homeomorphic. It follows from Lemma 4.2 that the limit set \( \Lambda(G) \subset S^3 \) of the group \( G \subset \text{Möb}(3) \) defines a Heegaard splitting of infinite genus of the 3-sphere \( S^3 \) into two connected components \( \Omega_1 \) and \( \Omega_2 \) of the discontinuity set \( \Omega(G) \). The action of \( G \) on the limit set \( \Lambda(G) \) is an ergodic word hyperbolic action. For this ergodic action, the set of fixed points of loxodromic elements \( g \in G \) (conjugate to similarities in \( \mathbb{R}^3 \)) is dense in \( \Lambda(G) \). Preimages \( \gamma \in \Gamma \) of such loxodromic elements \( g \in G \) for our homomorphism \( \rho : \Gamma \to G \) are loxodromic elements in \( \Gamma \) with two fixed points \( p, q \in \Lambda(\Gamma) = S^2, p \neq q \). This and Tukia’s arguments of the group completion (see [23] and [4], Section 4.6) show that our mapping \( F \) can be continuously extended to the set of fixed points of such elements \( \gamma \in \Gamma \), \( F(\text{Fix}(\gamma)) = \text{Fix}(\rho(\gamma)) \). The sense of this continuous extension is that if \( \gamma \in \Gamma \) is a loxodromic preimage of a loxodromic element \( g \in G, \rho(\gamma) = g \), and if \( x \in S^3 \backslash S^2 \) tends to its fixed points \( p \) or \( q \) along the hyperbolic axis of \( \gamma \) (in \( B(0, 1) \) or in its complement \( B(0, 1) \)) (i.e. radially) then \( \lim_{|x| \to 1} F(x) \) exists and equals to the corresponding fixed point of the loxodromic element \( g = \rho(\gamma) \in G \). In that sense, one can say that the limit set \( \Lambda(G) \) (the common boundary of the connected components \( \Omega_1, \Omega_2 \subset \Omega(G) \)) is the \( F \)-image of points in the unit sphere \( S^2 \subset S^3 \). So, the mapping \( F \) is extended to a map onto the closure \( \overline{\Omega_1} = \Omega_1 \cup \Lambda(G) \subset \mathbb{R}^3 \).

Nevertheless, not all loxodromic elements \( \gamma \in \Gamma \) in the hyperbolic lattice \( \Gamma \subset \text{Isom} H^3 \) have their images \( \rho(\gamma) \in G \) as loxodromic elements. Proposition 4.3 shows that \( \ker \rho \cong F_3 \) is a free subgroup on
three generators in the lattice $\Gamma$, and all elements $\gamma \in F_3$ are loxodromic. Now, we look at radial limits $\lim_{x \rightarrow p} F(x)$, when $x$ radially tends to a fixed point $p \in S^2$ of this loxodromic element $\gamma \in F_3 \subseteq \Gamma$.

Let $K(\Gamma, \Sigma)$ be the Cayley graph for a group $\Gamma$ with a finite generating set $\Sigma$. Our lattice $\Gamma \subset \text{Isom} \, H^3$ has an embedding $\varphi$ of its Cayley graph $K(\Gamma, \Sigma)$ in the hyperbolic space $H^3 \cong B^3$. For a point $0 \in H^3$ not fixed by any $\gamma \in \Gamma \setminus \{1\}$, the vertices $\gamma \in K(\Gamma, \Sigma)$ are mapped to $\gamma(0)$, and the edges joining vertices $a, b \in K(\Gamma, \Sigma)$ are mapped to the hyperbolic geodesic segments $[a(0), b(0)]$. In other words, $\varphi(K(\Gamma, \Sigma))$ is dual to the tessellation of $H^3$ by polyhedra $\varphi(F_0), \gamma \in \Gamma$. The map $\varphi$ is a $\Gamma$-equivariant proper embedding: for any compact $C \subset H^3$, its preimage $\varphi^{-1}(\varphi(K(\Gamma, \Sigma)) \cap C)$ is compact. Moreover, for any convex cocompact group $\Gamma \subset \text{Isom} \, H^n$, this embedding $\varphi$ is a pseudoisometry (see [12] and [4], Theorem 4.35), i.e., for the word metric on $K(\Gamma, \Sigma)$ and the hyperbolic metric $d$, there are $K > 0$ and $K' > 0$ such that $|a, b|/K \leq d(\varphi(a), \varphi(b)) \leq K \cdot |a, b|$ for all $a, b \in K(\Gamma, \Sigma)$ such that either $|a, b| \geq K'$ or $d(\varphi(a), \varphi(b)) \geq K'$.

This implies (see [4], Theorem 4.38) that the limit set of any convex-cocompact group $\Gamma \subset \text{Möb}(n)$ can be identified with its group completion $\overline{\Gamma}, \overline{\Gamma} = \overline{K(\Gamma, \Sigma)} \setminus K(\Gamma, \Sigma)$. Namely, there exists a continuous $\Gamma$-equivariant bijection $\varphi_T : \overline{\Gamma} \rightarrow \Lambda(\Gamma)$.

For the kernel subgroup $F_3 = \ker \rho \subset \Gamma \subset \text{Isom} \, H^3$ and for the above pseudoisometric embedding $\varphi$, we consider its Cayley subgraph in $\varphi(K(\Gamma, \Sigma)) \subset H^3$ which is a tree - see Fig. 5 in [9]. Since the limit set of $\ker \rho = F_3 \subset \Gamma$ corresponds to the “boundary at infinity” $\partial_{\infty} F_3$ of $F_3 \subset \Gamma$ (the group completion $\overline{F_3}$), it is a closed Cantor subset of the unit sphere $S^2 \subset \text{Hausdorff dimension ln} 5/\ln 6 \sim 0.8982244$. The $\Gamma$-orbit $\Gamma(\Lambda(F_3))$ of our Cantor set is a dense subset $S_* \subset S^2 = \Lambda(\Gamma)$ because of the density in the limit set $\Lambda(\Gamma)$ of the $\Gamma$-orbit of any limit point. In particular, we have such dense $\Gamma$-orbit $\Gamma(\{p, q\})$ of fixed points $p$ and $q$ of a loxodromic element $\gamma \in F_3 \subset \Gamma$ (the images of $p$ and $q$ are fixed points of $\Gamma$-conjugates of such loxodromic elements $\gamma \in F_3 \subset \Gamma$).

On the other hand, let $x \in l_\gamma$, where $l_\gamma$ is the hyperbolic axis in $B(0, 1)$ of an element $\gamma \in F_3 \subset \Gamma$. Denoting the translation distance of $\gamma$ by $d_\gamma$, we have that any segment $[x, \gamma(x)] \subset l_\gamma$ is mapped by our quasiregular mapping $F$ to a non-trivial closed loop $F([x, \gamma(x)]) \subset \Omega_1$, inside of a handle of the handlebody $\Omega_1$ (mutually linked with $\Omega_2$ - similar to the loops $\beta_1 \subset \Omega_1$ and $\beta_2 \subset \Omega_2$ constructed in the proof of Lemma 4.2). Therefore, when $x \in l_\gamma$ radially tends to a fixed point $p$ (in $\text{fix}(\gamma) \subset S^2$) of such element $\gamma$, its image $F(x)$ goes along the closed loop $F([x, \gamma(x)]) \subset \Omega_1$, because $F(\gamma(x)) = \rho(\gamma)(F(x)) = F(x)$. Immediately, this implies that the radial limit $\lim_{x \rightarrow p} F(x)$ does not exist. This shows that the fixed points of any element $\gamma \in F_3 \subset \Gamma$ (or its conjugates) are essential (topological) singularities of our quasiregular mapping $F$. So, our quasiregular mapping $F$ has no continuous extension to the subset $S_* \subset S^2$ which is a dense subset of the unit sphere $S^2 = \partial B^3 \subset S^3$.

REFERENCES

1. E. M. Andreev, “On convex polyhedra in Lobachevsky space,” Math. USSR Sbornik, 10, 413–440 (1970).

2. E. M. Andreev, “The intersections of the planes of faces of polyhedra with sharp angles,” Math. Notes, 8, 761–764 (1970).

3. B. Apanasov, “Nontriviality of Teichmüller space for Kleinian group in space” in: Riemann Surfaces and Related Topics, edited by I. Kra and B. Maskit, Princeton Univ. Press, Princeton, 1981, pp. 21–31.

4. B. Apanasov, Conformal Geometry of discrete Groups and Manifolds, W. de Gruyter, Berlin–New York, 2000.

5. B. Apanasov, “Quasisymmetric embeddings of a closed ball inextensible in neighborhoods of any boundary points,” Ann. Acad. Sci. Fenn., Ser. A I Math., 14, 243–255 (1989).
6. B. Apanasov, “Nonstandard uniformized conformal structures on hyperbolic manifolds,” *Invent. Math.*, **105**, 137–152 (1991).

7. B. Apanasov, “Hyperbolic 4-cobordisms and group homomorphisms with infinite kernel,” *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia*, **57**, 31–44 (2010).

8. B. Apanasov, “Group actions, Teichmüller spaces and cobordisms,” *Lobachevskii J. Math.*, **38**, 213–228 (2017).

9. B. Apanasov, “Topological barriers for locally homeomorphic quasiregular mappings in 3-space,” *Ann. Acad. Sci. Fenn. Math.*, **43**, 579–596 (2018).

10. B. N. Apanasov and A. V. Tetenov, “Nontrivial cobordisms with geometrically finite hyperbolic structures,” *J. of Diff. Geom.*, **28**, 407–422 (1988).

11. K. F. Barth, D. A. Brannan, and W. K. Hayman, “Research problems in complex analysis,” *Bull. London Math. Soc.*, **16**, No. 5, 490–517 (1984).

12. J. W. Cannon, “The combinatorial structure of cocompact discrete hyperbolic groups,” *Geom. Dedicata*, **16**, 123–148 (1984).

13. P. Fatou, “Séries trigonométriques et séries de Taylor,” *Acta Math.*, **30**, 335–400 (1906).

14. R. Grigorchuk, “Milnor’s problem on the growth of groups and its consequences,” in: *Frontiers in Complex Dynamics*, Princeton Univ. Press, Princeton, 2014, pp. 705–773.

15. M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, Boston, 1999.

16. W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, ArXiv: 1809.07200.

17. M. A. Lavrentiev, “On a class of continuous mappings,” *Mat. Sbornik*, **42**, 407–424 (1935).

18. O. Martio, S. Rickman, and J. Väisälä, “Distortion and singularities of quasiregular mappings,” *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **465**, 1–13 (1970).

19. K. Rajala, “Radial limits of quasiregular local homeomorphisms,” *Amer. J. Math.*, **130**, 269–289 (2008).

20. Y. G. Reshetnyak, *Space Mappings with Bounded Distortion*, Amer. Math. Soc., Providence, R.I., 1989.

21. S. Rickman, *Quasiregular Mappings*, Springer, Berlin, 1993.

22. D. Sullivan, “Quasiconformal homeomorphisms and dynamics, II: Structural stability implies hyperbolicity for Kleinian groups,” *Acta Math.*, **155**, 243–260 (1985).

23. P. Tukia, “On isomorphisms of geometrically Kleinian groups,” *Publ. Math. IHES*, **61**, 171–214 (1985).

24. M. Vuorinen, “Cluster sets and boundary behavior of quasiregular mappings,” *Math. Scand.*, **45**, No. 2, 267–281 (1979).

25. M. Vuorinen, “Queries No 249,” *Notic. Amer. Math. Soc.*, **28**, No. 7, 607 (1981).

26. M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Springer, Berlin, 1988.

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