SOLITONS WITH INTEGER FERMION NUMBER

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Abstract

Necessary and sufficient conditions are found for any object in 3+1 dimensions to have integer rather than fractional fermion number. Nontrivial examples include the Jackiw-Rebbi monopole and the already well studied Su-Schrieffer-Heeger soliton, both displaying integer multiples of elementary charges in combinations that normally are forbidden.
Jackiw and Rebbi in ‘Solitons with fermion number $1/2$’\(^1\) introduced the concept of fractional charge. The aim of the following analysis is to define the limits of the fractional charge domain for objects in $3 + 1$ dimensions.

Fractional charge only is remarkable if it is an eigenvalue of the corresponding quantum observable, rather than just an expectation value – it must be sharp to be significant. The conditions for a localized charge to be sharp have been investigated for systems in $1 + 1$ dimensions,\(^2\) where the definition requires a spatially smoothed weighting of the corresponding charge density operator. In higher dimensions temporal smoothing or a frequency cutoff also is required.\(^3\) This means that without an energy threshold for charge-carrying excitations it may, (sometimes even must\(^4\)) be impossible to define a sharp charge, since the frequency cutoff does not preclude long-wavelength fluctuations in the charge density.

JR\(^1\) considered two examples of a quantum Fermi field coupled to a specific classical, static Bose field configuration (a soliton). In both cases the Dirac equation for the Fermi field includes a single mode with zero frequency, and the rest of the spectrum is completely symmetrical between positive and negative frequencies. Consequently, the state in which the zero frequency mode is occupied and that in which it is not ought to be charge conjugate to each other. Since they differ by one unit in fermion number $F$, they should be characterized by $F = \pm 1/2$. For the first example, where the Fermi field is Yukawa coupled to a sine-Gordon soliton in $1 + 1$ dimensions, the treatment of Fermi and Bose fields was made symmetrical by bosonization of the Fermi field. This allows back-reaction of the fermion on the boson degrees of freedom, and confirms the suggestion that the object carries half-integer $F$.\(^5\)

Su, Schrieffer, and Heeger\(^6\) independently discovered a system with fermion zero modes leading to JR oscillators, except that in their model (of polyacetylene) there are two species of fermion, electrons with spin up and electrons with spin down. This fermion doubling replaces fractional charge with charge \textit{dissociation} – an SSH soliton may exist in any
of four nearly degenerate states, characterized by $F = \pm 1$, spin $S = 0$ (spin singlet), or $F = 0, S = 1/2$ (spin doublet). The two charges (electron number and spin) which characterize a free electron have been ‘torn apart’ so that soliton states carry one or the other but not both.

The concept of fractional charge has blossomed since JR. A soliton which lacks charge conjugation symmetry may possess fractional charges other than $1/2$, whether rational\(^7\) or irrational.\(^8,9\) Perhaps the most striking experimentally studied example came from defining\(^10\) a charge somewhat different from the localized charges mentioned above, the ‘capacitor charge’ of a weak link in a conducting circuit, $\tilde{Q} = CV$. Here $V$ is the electric potential across the link, and $C$ is the classical electric capacitance of the link. If $\tilde{Q}(t)$ is sharp then electrons should jump across the link at sharply defined times separated by the unit $e/I$, with $e$ the electron charge and $I$ the mean current, a prediction\(^10\) verified by experiment.\(^11\) Provided a suitable frequency cutoff is introduced, $\tilde{Q}$ indeed should be sharp, even if the conductor is a normal metal which has zero energy threshold for charge-carrying excitations. This is possible because charge density fluctuations which produce a significant shift in $\tilde{Q}$ must be localized near the link, and do have a nontrivial energy threshold.

Since at least the potentiality for fractional charge is by now seen as ubiquitous, it becomes interesting to ask when is it not found, i.e., what are the conditions for a soliton to carry charges or quantum numbers with integer rather than fractional values? Let us begin to attack this question by recalling the

**Theorem of Jackiw and Schrieffer:**\(^12\) A soliton whose spectrum is invariant under a charge conjugation symmetry $C$ which reverses the sign of $F$ may have integer $F$, or half-integer $F$, but no other fractional value is allowed.

**Proof:** For an isolated soliton, the only way $F$ (assumed conserved) can change is by scattering processes in which the number of fermions incident differs from the number emerging. This means that two allowed values of $F$ must differ by an integer. $C$ symmetry
implies that for every allowed $F$, $-F$ also is allowed. Therefore the difference $2F$ must be an integer, and $F$ must be either an integer or a half-integer. If one allowed $F$ is an integer (half-integer), so must be all the others, since they differ by integers. QED

This theorem shows that to answer our question we need find only what conditions must supplement charge conjugation symmetry in order to exclude half-integer eigenvalues. It also suggests a strategy for determining those conditions. If absorbing a fermion changes other conserved charges of the soliton as well as $F$, then extra consistency requirements may follow. Therefore let us proceed by attending to other charges carried by fermions. The problem naturally divides solitons into two classes, (A) magnetic monopoles (interacting with fermions whose electric charge has the minimum magnitude $e$ allowed by the Dirac quantization condition\textsuperscript{13}), and (B) all others.

The reason for this division is that according to the spin-statistics connection a fermion in $3+1$ dimensions must carry half-integer spin. However, for a minimal electric charge in the field of a magnetic monopole there is an extra electromagnetic angular momentum which also is half-integer. Consequently, it is possible for the monopole to absorb a fermion with the appropriate magnitude of electric charge without absorbing angular momentum. For any other kind of soliton, or for a monopole interacting with fermions whose electric charges are even multiples of the minimum Dirac unit, absorbing a fermion requires the absorption of a half-integer unit of angular momentum. Therefore spin is a suitable candidate for the extra quantum number associated with depositing a fermion on a soliton for all cases in category B. For category A, the only apparent (and inevitably open) option for the extra quantum number is electric charge. Let us now analyze cases A and B separately.

**Integer $F$ Theorem A**: If the electromagnetic vacuum angle $\theta$ vanishes, then a magnetic monopole symmetric under fermion conjugation $C$ must carry integer $F$.

**Proof**: Let us begin by making explicit several assumptions. Take $C$ to reverse electric charge $Q$ as well as $F$, since the proof becomes trivial otherwise. Restrict attention only
to fermions with $Q$ (measured in units of the elementary fermion electric charge $e$) an odd integer, since the alternative will be covered under Theorem B, to follow. Treat both $Q$ and $F$ as conserved, localizable, sharp quantum observables. By the method of the Jackiw-Schrieffer theorem, the only possible values of $Q$ and $F$ are integers or half-integers, and furthermore both must be in the same class. This follows because adding one fermion changes both $Q$ and $F$ by an odd integer. Thus, if $F$ is a half-integer, so is $Q$. The proof is reduced to showing that $Q$ must be an integer.

To establish this, let us recall the basis for the Dirac quantization condition. Dirac’s discovery of the condition exploited the gauge invariance of electromagnetism, and could be phrased by saying that the monopole is described formally as one end of an infinitely thin magnet. In order that this magnetic line, or Dirac string, be unobservable, the Aharonov-Bohm phase for diffraction of electrically charged particles on either side of the line should be an integer multiple of $2\pi$, and this implies the quantization condition

$$qg = N\hbar c/2 ,$$

(1)

where $q$ is the electric charge, $g$ is the magnetic pole strength, and $N$ is an integer.

A second way to obtain the same condition is by insisting on proper quantization of the total angular momentum of a charge-pole system. At the simplest level, the argument is that the Thomson electromagnetic angular momentum along the line from charge to pole should be quantized. This suggests a possible escape from Dirac’s condition for the case of dyons carrying both electric and magnetic charge: The generalized condition

$$q_1g_2 - q_2g_1 = N\hbar c/2,$$

(2)

where the subscripts refer to the charges on the respective particles, has as a possible solution that each dyon carries a fractional electric charge proportional with a universal ratio to its magnetic charge. This makes the Thomson angular momentum of a dyon pair vanish identically, so that its quantization gives no further constraint.

However, if the introduction of these fractional charges is represented by a term in
the action density proportional to $K_\mu A_\mu$, where $K$ is the monopole current and $A$ is the usual electromagnetic four-potential, then the $q_i$ still must obey Dirac’s original condition. The reason is seen most easily in terms of Dirac strings. The fact that there is no velocity-dependent ‘magnetic’ force between two such dyons means that in any simply connected region of the relative coordinate the vector potential may be written as a pure gradient. Still we need to check for an observable Aharonov-Bohm phase when the relative motion corresponds to diffraction around what is now an endless Dirac string. To avoid a nontrivial phase, Dirac’s condition Eq. 1 is required.\textsuperscript{18}

The fact that the straightforward or obvious formulation of possible fractional dyon charges does not work should be no surprise, since one is giving mathematical expression to the placement of charges ‘by hand’ on each monopole. It seems natural that Dirac’s condition forbidding fractional charges at arbitrary locations should apply also to the limiting case when their locations are made to coincide with those of monopoles. Nevertheless, there is a unique, consistent way to introduce fractional charge, and that is by modifying electrodynamics to include a nontrivial vacuum angle $\theta$.\textsuperscript{8,19} Such an angle is believed to be a possible result of instanton tunneling effects, and is represented in the action density by a term proportional to $\theta E \cdot B$.\textsuperscript{20}

That this leads to fractional dyon charge can be seen directly from the equations of motion.\textsuperscript{21} That it avoids the breaking of gauge invariance inevitable with ‘hand placement’ has been shown to be due to the appearance of a second gauge-noninvariant term which exactly cancels the one coming from the $K_\mu A_\mu$ coupling.\textsuperscript{19} This result is easily understood, since the $\theta$ term in the action is manifestly gauge invariant without the necessity of an integration by parts, and therefore must produce only gauge invariant results. Provided only that there exists a gauge invariant, local, low-energy effective action describing charges and poles interacting with the electromagnetic field, the $\theta$ term is the unique mechanism for introducing fractional dyon charge, since it must include $K_\mu A_\mu$ and be gauge invariant. Any proposed alternate could be rewritten as the $\theta$ term plus another
gauge invariant piece not including $K_\mu A_\mu$ and therefore irrelevant to fractional charge.

We see that for vanishing $\theta$ there can be no fractional $Q$ on a monopole, but that means there can be no fractional $F$, and Theorem A is proven. Before going on to the much more straightforward Theorem B, let us pause to examine some related issues.

1) **Apparent confirmations of** $F = 1/2$ **for the JR monopole.** Following the suggestion of JR that the zero mode which they discovered for isospinor fermions coupled to the gauge and Higgs fields of an ’t Hooft-Polyakov monopole\(^ {22}\) implies $F = 1/2$, the same $F$ value was obtained by explicitly field-theoretic methods.\(^9\) However, the formalism behind this result was based on quantization of the Fermi fields only, with the Bose fields treated as adjustable classical backgrounds. Thus it amounts to a confirmation of the original JR calculation, but does not account for possible back-reaction of the fermion on the boson degrees of freedom.

The issue of back-reaction was attacked in an approach which bosonized the Fermi fields, introducing plausible boundary conditions for the bosonized fields at the core of the monopole.\(^ {23}\) This charge-conjugation-symmetric approach inevitably produced $Q = 1/2$ along with $F = 1/2$, and therefore included an implicit assumption of nontrivial vacuum angle. If we reject that assumption we must revise the boundary conditions, no matter how plausible they may appear, and are left with no evidence for fractional $F$.

2) **Counting states.** If JR’s disarmingly simple argument does not yield the required integer $F$, where is the the additional ‘duplexity’\(^ {24}\) which compensates or conceals the $\pm 1/2$ which they found? To give this the direct answer it deserves, let us repeat the JR analysis\(^1,12\) in algebraic form. Commuting with the monopole Hamiltonian are three operators, $\psi_0$ – the Hermitean part of the operator in the expansion of the Dirac field corresponding to the JR fermion zero mode, $CP$ – the relevant charge conjugation symmetry, and $F$. $F$ anticommutes with $CP$, and is raised or lowered one unit by $\psi_0$, which in turn commutes with $CP$. The minimum nontrivial representation of this algebra is two dimensional, and the basis states may be chosen as $|F = \pm 1/2 >$.  

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Now we are ready for the missing ingredient: In the classical limit electric charge $Q$ need not be sharp, but the charge signature operator $U = (-1)^Q$ still has definite, conserved eigenvalues. In the absence of $CP$-violating effects these eigenvalues are $U = \exp \left( 2\pi i T_3 \right)$. Here $T_3$ is the generator of rotations about the abelian or massless-photon direction in the $SO(3)$ gauge space, implying $\exp \left( 2\pi i T_3 \right) = \pm 1$. $U$ evidently commutes with $CP$ and $F$, but anticommutes with $\psi_0$. If the sector with $U = +$ contains a state with nonzero $F$, then the $CP$ conjugate state with $-F$ must differ by an even integer. Thus the minimum nontrivial $U = +$ multiplet must contain two states with $F = \pm 1$. Since this yields a two dimensional representation of $CP$, there must also be a two dimensional $F = 0$ representation for $U = -$, yielding a collection of four states as the smallest nontrivial multiplet. Although $Q$ is not sharp in this basis, the two states for $U = -$ may be chosen to carry the expectation values $<Q> = \pm 1$.

Of course it still is possible to choose two $CP$ conjugate states which differ in $F$ by one unit, and therefore have $<F> = \pm 1/2$. In other words, by ignoring $U$ we may repeat the original JR analysis with no error except that the charge conjugate states no longer are unique, and hence the fractional expectation values are not eigenvalues.

Upon going to a basis with $Q$ sharp, which is allowed in the classical limit, and necessary if quantum corrections are to be included, we obtain as the natural structure for the multiplet of smallest mean-squared charge a quartet, one doublet with $F = \pm 1, Q = 0$, and one doublet with $F = 0, Q = \pm 1$. All this is in perfect analogy with the SSH soliton, so that the JR monopole actually is an example of charge dissociation rather than fractional charge.

3) Necessity of the $\theta = 0$ hypothesis? The assumption of vanishing vacuum angle is sufficient to prove Theorem A. It also is necessary, if the example of Dirac electrons interacting with a point Dirac monopole is considered. For that case a suitable $\theta$ can be found to make the monopole an ideal JR oscillator, whose degenerate ground states carry $F = Q = \pm 1/2$. For the nonsingular, and therefore perhaps more natural, 't
Hooft-Polyakov monopole, our question splits into two: Could the fermion-Higgs coupling somehow induce a nontrivial $\theta$? Even if there is such an angle, will the monopole acquire fractional $F$?

The first question describes a logical possibility, but I am unaware of any mechanism other than instanton tunneling which might produce nonzero $\theta$, so that the burden of proof should lie on proponents of such a notion. The answer to the second question seems more straightforward. Since $\theta$ is linked to $Q$, cranking $\theta$ from 0 to $\pi$ should take $Q$ through fractional values from 1 to 0, but should not influence $F$. Indeed, one can follow the spectrum of single-particle bound states associated with such a sequence. At $\theta = 0$ there are effectively two zero modes, one with positive and one with negative electric charge, yielding four nearly degenerate states with $F = \pm 1, Q = 0; F = 0, Q = \pm 1$. These are the states I have argued one should expect from the JR mechanism. Between $\theta = 0$ and $\theta = \pi/2$ the positive charge bound state rises from $E = 0$ to $E = mc^2$ (and then disappears into the continuum), while the negative charge state descends from $E = 0$ to $E = -mc^2$. At $\theta = \pi$ the monopole has a single ground state with $F = Q = 0$. Thus it appears that adjusting $\theta$ could undo the twice-doubled degeneracy associated with the JR mechanism in this setting, and could produce fractional $Q$, but it could not produce fractional $F$.

For all other solitons, including all condensed-matter defects, we have

**Integer $F$ Theorem B**: An object self-conjugate under a unitary charge conjugation symmetry $C$, with fermion number and spin both sharp, must carry integer $F$.

**Proof**: Again there are assumptions which should be made explicit. We are omitting the case of Theorem A, so that adding a fermion to the object does change its spin by a half-integer. $C$ is assumed to commute with rotations, so that the unitarity of $C$ implies its commutation with angular momentum and spin. Therefore charge conjugate states must have the same spin. This means that $2F$ must be an even integer, since otherwise the states would differ in spin by a half-integer. Hence $F$ is an integer – the proof is
complete. Let us now look for counterexamples if assumptions are violated.

1) **Antiunitarity.** If $C$ is antiunitary then it anticommutes with spin, and the above proof breaks down. A possible example is the SSH model with a very strong magnetic field along the polymer chain direction, so that the valence parallel-spin electron band is completely nonoverlapping with the antiparallel band. Then a soliton in a ‘quarter-filled’ background (i.e., the parallel band half-filled, the antiparallel unfilled) should have $S_z = F/2 = \pm 1/4$. Since rotational symmetry has been broken from $O(3)$ to $O(2)$, there is no intrinsic contradiction in a fractional eigenvalue for $S_z$. This is different from the monopole case discussed earlier – where $Q = 2T_3$ must be an integer – because there the isospin gauge symmetry (even though hidden by the Higgs mechanism) remains unbroken.

2) **Spin delocalization.** If spin is not localizable because of strong and long-range magnetic correlations, then adding a fermion need not change even the expectation value of spin – no spin is deposited locally. Consequently the difference in $F$ between charge conjugate states can be an odd integer, without any difference in the expectation value of spin. Precisely this may occur in certain organic charge-transfer salts, yielding $F = \pm 1/2, <S_z> = 0$, contrary to the previous claim of fractional spin for such a system.\(^{26}\)

We have seen that zero vacuum angle, localizability of $F$, $Q$ and $S$, and obedience to a unitary $C$ symmetry are the general necessary and sufficient conditions that a soliton in 3 + 1 dimensions carry integer $F$. The premier application and inspiration of this analysis is the subtle Jackiw-Rebbi monopole,\(^1\) which helped reveal a significant new phenomenon by evoking fractional charge even without possessing it.

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