Pure-Exploration for Infinite-Armed Bandits with General Arm Reservoirs

Maryam Aziz
College of Computer and Information Sciences
Northeastern University
Boston, MA 02115, USA

Kevin Jamieson
Paul G. Allen School of Computer Science & Engineering
University of Washington
Seattle, WA 98195, USA

Javed Aslam
College of Computer and Information Sciences
Northeastern University
Boston, MA 02115, USA

Abstract
This paper considers a multi-armed bandit game where the number of arms is much larger
than the maximum budget and is effectively infinite. We characterize necessary and suffi-
cient conditions on the total budget for an algorithm to return an $\epsilon$-good arm with prob-
ability at least $1 - \delta$. In such situations, the sample complexity depends on $\epsilon, \delta$ and the
so-called reservoir distribution $\nu$ from which the means of the arms are drawn iid. While
a substantial literature has developed around analyzing specific cases of $\nu$ such as the beta
distribution, our analysis makes no assumption about the form of $\nu$. Our algorithm is
based on successive halving with the surprising exception that arms start to be discarded
after just a single pull, requiring an analysis that goes beyond concentration alone. The
provable correctness of this algorithm also provides an explanation for the empirical obser-
vation that the most aggressive bracket of the Hyperband algorithm of Li et al. (2017) for
hyperparameter tuning is almost always best.

Keywords: Infinite-armed bandit, pure-exploration, best-arm identification

1. Introduction
Consider a multi-armed bandit problem with $n$ arms where the $j$th pull from the $i$th arm
emits an independent random variable $X_{i,j} \in [0,1]$ with $\mu_i := E[X_{i,j}]$. Given $\epsilon, \delta \in (0,1)$,
how many total pulls must an algorithm make in order to return an arm $\hat{i} \in \{1, \ldots, n\}$ with
a small\footnote{While non-standard in stochastic bandits, seeking small means significantly simplifies notation in the
infinite-armed bandit setting; we translate all prior results to this equivalent perspective.} mean that satisfies $\mu_{\hat{i}} \leq \min_j \mu_j + \epsilon$ with probability at least $1 - \delta$? Much effort
has gone into answering this and closely related questions resulting in a rich collection of
algorithms. But each algorithm starts the same: Pull each arm $i \in \{1, \ldots, n\}$ once.
Pure-exploration Infinite-armed bandit game

**Input** $\epsilon, \delta \in (0, 1)$ and reservoir distribution $\nu_0$

**Initialize** Draw $\{\mu_i\}_i \sim \nu_0$ and set $N_i(t) := \sum_{s=1}^{t} 1\{I_s = i\}$ for all $t \in \mathbb{N}$ for $t = 1, 2, \ldots$

Player chooses $I_t \in \mathbb{N}$

Nature reveals $X_{I_t, N_I(t)}(t) \in \mathbb{R}$ where $E[X_{I_t, N_I(t)}(t)|I_t] = \mu_t$

Player recommends $J_t \in \mathbb{N}$

In this work we are interested in problems where the number of arms $n$ is so large that it is dwarfed by any available budget of total pulls. Necessarily, we are interested in problems where the budget necessary to identify an $\epsilon$-good arm among the $n$ arms with probability $1 - \delta$ is independent of $n$. Such cases arise when the proportion of $\epsilon$-good arms is independent of $n$ (e.g. $\frac{1}{n} \sum_{i=1}^{n} 1\{\mu_i \leq \min_j \mu_j + \epsilon\} \geq \epsilon^2$).

Consider a concrete example of the New Yorker caption contest dataset, where captions are voted on to find the funniest one (see Section 4). The bold blue line is the best-arm identification algorithm lil’UCB (Jamieson et al., 2014) executed on all 3,795 arms, lil’UCB-$X$ for $X$ in $\{10, 100, 1000, 10000\}$ is lil’UCB run on $X$ arms randomly drawn with replacement from the 3,795 arms, and ISHA is the proposed algorithm of this work. Each algorithm outputs the empirical best arm at any given total budget of pulls, breaking ties randomly. We observe that one can identify a “pretty good” arm faster when the number of drawn arms $X$ is small, but as a consequence this small set of arms will not have an arm very close to the best possible arm. We also observe that ISHA appears to naturally navigate this tradeoff.

When $n \gg B$ we can treat $n$ as effectively infinite and the difference between sampling an arm with or without replacement is indistinguishable. Towards this end, we define the infinite-armed bandit problem.

1.1 The Pure-exploration Infinite-armed Bandit Problem

Let $\nu_0$ be a fixed but arbitrary cumulative distribution function over $\mathbb{R}$ such that if $\mu \sim \nu_0$ then $\mathbb{P}(\mu \leq x) = \nu_0(x)$. In the finite-armed bandit case like the example of the previous section, one would take $\nu_0(x) = \frac{1}{n} \sum_{i=1}^{n} 1\{\mu_i \leq x\}$. Without loss of generality $\{\mu_i\}_{i=1}^{\infty}$ are drawn iid from $\nu_0$ before the start of the game and identified by their index, and the player has no prior knowledge of $\nu_0$. Consider the pure-exploration infinite-armed bandit game.

**Goal:** For a fixed reservoir distribution $\nu_0$ with $\mu_* = \inf\{x : x \in \text{support}(\nu_0)\}$ and $\epsilon \in (0, 1)$, how big must $\tau \in \mathbb{N}$ be to ensure that $\mu_{J_\tau} \leq \mu_* + \epsilon$ with high probability?

Said another way, minimize *simple regret* (Bubeck et al., 2009; Carpentier and Valko, 2015) in high probability, which implies a bound on $\mathbb{E}[\mu_{J_\tau}] - \mu_*$ (see Remark 2).
1.2 Prior work

The main objective of this work is pure-exploration where different arms are sampled different numbers of times with the goal of choosing $J_t$ such that the simple regret $\mu_{J_t} - \mu^* \leq \epsilon$ for as small $t$ as possible. Contrast this with exploration-vs-exploitation where the objective is to pull different arms to minimize the cumulative regret of all the plays of the arms pulled: $\sum_{s=1}^{t} \mu_{I_s} - \mu^*$. In pure-exploration the player is only evaluated on the mean $\mu_{J_t}$ of the recommended arm at time $t$; in exploration-vs-exploitation the player is evaluated on all the arms played $\{\mu_{I_s}\}_{s=1}^{t}$ up to time $t$. The infinite-armed case has also been studied in both the explore-vs-exploit and pure-exploration settings, which we briefly review.

1.2.1 Explore-vs-Exploit: Minimizing cumulative regret

While research on the finite-armed bandit problem for explore-vs-exploit is quite mature (Bubeck et al., 2012), many open problems still remain for the infinite-armed setting. To the best of our knowledge, a form of the infinite armed bandit problem was first proposed in Berry et al. (1997) which studies the particular case when observations are Bernoulli and $\nu_0$ is the uniform distribution over a known interval $[a,b] \subseteq [0,1]$, but also considers asymptotic upper bounds for their novel algorithm for a more general class of distributions $\nu_0$. This work inspired a number of followup works including Teytaud et al. (2007) that extended the algorithm of Berry et al. (1997) to settings where the time horizon of the algorithm was unknown in advance. These algorithms worked on the principle of flipping a coin until $m$ failures are observed at which time it would discard the current coin and sample a new one from $\nu_0$. Bonald and Proutiere (2013) studied a related algorithm for Bernoulli observations where $\nu_0$ is a beta distribution:

$$\nu(\mu) = \int_{\theta=0}^{\theta=\mu} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \theta^{\alpha_1-1}(1-\theta)^{\alpha_2-1} d\theta$$

with known parameters $\alpha_1, \alpha_2$; lower bounds are proven for any algorithm in this setting. Note that (1) with $\alpha_1 = \alpha_2 = 1$ is just the uniform distribution.

While these previous algorithms assumed that $\nu_0$ was equal to some known parameterization of a beta distribution on a known support, Wang et al. (2009) relaxed these conditions to simply assume there exists (known) constants $c, C, \beta, \epsilon_0$ such that

$$c\epsilon^\beta \leq \nu_0(\mu^* + \epsilon) \leq C\epsilon^\beta \quad \forall \epsilon \leq \epsilon_0.$$  

Clearly, for sufficiently small $\epsilon_0$ the beta distribution of (1) satisfies (2) with $\mu^* = 0$ and $\beta = \alpha_1$. In this more general setting Wang et al. (2009) proposed an algorithm with cumulative regret guarantees that only needed to know $\beta$, not the support or $\mu^*$. Chan and Hu (2018) recently proved lower bounds and proposed an algorithm based on confidence intervals. To our knowledge, there exists no algorithm in the regret setting that provably adapts to general, unknown reservoir distributions $\nu_0$ with near-optimal cumulative regret.

A related problem is when each arm’s reward distribution is a single-point distribution, or deterministic, but unknown until it is played. In this setting David and Shimkin (2014, 2015) studied reservoir distributions with conditions similar to (2).

Quantiles are a convenient object in infinite armed bandits since one can very accurately determine how many arms must be sampled to obtain at least one in the $q$th quantile without
knowing anything about $\nu_0$. In the quantile-regret minimization setting where $\mu_q$ denotes the $q$th percentile of $\nu_0$ for any $q \in (0, 1)$, Chaudhuri and Kalyanakrishnan provide an algorithm that obtains sub-linear regret with respect to $\mu_q$ (instead of $\mu^*$) for arbitrary reservoir distributions $\nu_0$.

1.2.2 Pure exploration: Simple regret, fixed budget, fixed confidence

The infinite-armed bandit setting for pure-exploration is also well-studied. The most-biased coin problem is a particular instance where

$$
\nu_0(\mu) = \int_{\theta=0}^{\mu} \pi \delta_\rho(\theta) + (1 - \pi) \delta_{\mu + \epsilon}(\theta) \, d\theta
$$

(3)

where $\delta_x(\theta)$ is a Dirac-delta at $x$ and parameters $\rho, \pi, \epsilon \in (0, 1)$ are known (Chandrasekaran and Karp, 2014; Malloy et al., 2012) and unknown (Jamieson et al., 2016). This parameterization is thought to be difficult because there is no incremental improvement towards the optimal arm over time: the optimal arm has either been identified or it has not. Quantile problems have also been studied in the pure-exploration setting, such as identifying an arm $\epsilon$-close to $\mu_q$ (Chaudhuri and Kalyanakrishnan, 2017; Aziz et al., 2018; Ren et al., 2018).

Carpentier and Valko (2015) proposed an algorithm specifically for reservoir distributions parameterized as (2) known as SIRI. Remarkably, they show that they can adapt to unknown parameters of this parametric model achieving a simple regret guarantee of

$$
\mathcal{O}\left(\max\left(t^{-1/2}, t^{-1/3}\text{polylog}(t)\right)\right)
$$

with high probability for their algorithm; they also provide nearly matching lower bounds on simpler regret for the $\beta$-parameterization of (2).

Li et al. (2017) proposed the Hyperband algorithm which to our knowledge is the only algorithm to obtain simple regret guarantees for general, unknown reservoir distributions (i.e., without a known parameterization like (1)-(3) of any kind). For any $\epsilon > 0$ and reservoir distribution $\nu_0$, they show that the simple regret of Hyperband is bounded by $\epsilon$ with high probability once the budget $t$ exceeds

$$
\epsilon^{-2} + \frac{1}{\nu_0(\mu^*) + \epsilon} \int_{\mu^* + \epsilon}^{\infty} \frac{1}{(\mu - \mu^*)^2} d\nu_0(\mu)
$$

(4)

pulls (up to poly-logarithmic factors). This result matches all known pure-exploration upper bounds, even those algorithms designed for specific reservoir distributions, up to poly-logarithmic factors. For any given value of $n \in \mathbb{N}$, Hyperband is nothing more than running $\log_2(n)$ copies of the Successive Halving algorithm of Karnin et al. (2013) each with a budget of $n$ and $2^k$ arms drawn from $\nu_0$ for $k = 1, 2, \ldots, \log_2(n)$; the whole procedure uses $n \log_2(n)$ total samples. While Hyperband is state-of-the-art, hedging over these $\log_2(n)$ copies of Successive Halving is inelegant, and empirically it was almost always observed that the most aggressive bracket with the most arms worked best.

1.3 Main Contributions

In this work we show that running just one version of Successive Halving, named ISHA, with $n$ arms and a budget of $n \log_2(n)$ pulls—where arms start being discarded after being pulled just once—achieves the same theoretical sample complexity guarantees of Hyperband up to poly-logarithmic factors, but the algorithm is considerably simpler and performs better.
empirically. That is, for any reservoir distribution \( \nu \) and \( \epsilon, \delta \), for sufficiently large \( n \) this procedure returns an \( \epsilon \)-good arm with probability at least \( 1 - \delta \). Remark 2 also characterizes the expected simple regret \( \mathbb{E}[J_T] - \mu_* \). Our analysis accounts for arms being discarded far before concentration of measure kicks in, requiring a novel analysis that may inspire future works. In exhaustive experimental studies, we show that our proposed algorithm is not only superior on most reservoir distributions (including those derived from finite-armed problems) but also against algorithms that were designed specifically for the reservoirs we evaluated them on, like parameterizations (1)-(3).

Our second contribution is an information theoretic lower bound for the infinite-armed bandit problem. Specifically, for any reservoir distribution \( \nu \) and any fixed \( \epsilon, \delta \in (0, 1) \) we prove a lower bound on the expected number of samples any algorithm must make in order to identify an \( \epsilon \)-good arm with probability at least \( 1 - \delta \) that depends on \( \nu, \epsilon, \delta \). The upper and lower bounds match the expression of (4) up to polynomial factors of \( \log(1/\delta) \) and other logarithmic factors.

2. Successive Halving for infinite-armed bandits

The Successive Halving algorithm of Karnin et al. (2013) is presented in Figure 2: our proposed algorithm, ISHA, sets \( T = \lceil n \log_2(n) \rceil \) for any \( n \in \mathbb{N} \). In what follows of this paper, any reference to ISHA means as taking the particular parameterization of \( T = \lceil n \log_2(n) \rceil \) for any \( n \in \mathbb{N} \). In words, our proposed algorithm is simple: for some \( n \in \mathbb{N} \), the algorithm draws \( n \) arms without replacement, pulls each arm once, discards the worst half, and on each successive round pulls the surviving arms twice as many times as the previous round before discarding the worst half. The whole process takes \( n \) pulls per round for \( \log_2(n) \) rounds for a total of \( n \log_2(n) \) total pulls.

```
Input: Budget T, number of arms n
Initialization: Draw n arms and add them to S_0
For k = 0, 1, \ldots, \lceil \log_2(n) \rceil - 1
    Pull each arm i \in S_k for t_k = \left\lfloor \frac{T}{|S_k| \log_2(n)} \right\rfloor times
    times and compute empirical means \( \hat{\mu}_{i,k} \)
    Set S_{k+1} to be \lfloor |S_k|/2 \rfloor arms with the lowest empirical means \( \hat{\mu}_{i,k} \)
Return Single arm in S_{\lceil \log_2(n) \rceil}
```

Figure 2: Successive Halving algorithm. The algorithm we propose for infinite-armed bandits is to use \( T = \lceil n \log_2(n) \rceil \) for any value of \( n \in \mathbb{N} \). For anytime, double \( n \) and repeat.

The main dilemma of choosing \( X \) for the lil’UCB-X strategies in the introduction of this paper, and more generally all infinite-armed bandit problems, is determining whether it is better to draw more arms from a reservoir distribution over arms in the hope of getting an arm with better mean reward, or to spend the remaining arm pull budget on identifying a “good” arm from among already drawn arms. ISHA navigates this dilemma by focusing not on individual arms but on populations of arms, where at each round the “fitter” (i.e. lower empirical mean) arms are more likely to survive and so from round to round the population as a whole “evolves,” in the sense that while any individual “good” arm might get unlucky
and be removed from the population, overall the average expected reward of the surviving arms tends to improve.

2.1 Analysis

Our main result relies on two assumptions about the distribution $\phi(\cdot; \mu)$ of observations from an arm given a drawn mean $\mu$. We make no assumptions about the shape or regularity of the reservoir $\nu_0$.

**Assumption 1** For all $m \in \mathbb{N}$ and $t \in \mathbb{R}$

$$
P\left(\frac{1}{m} \sum_{j=1}^{m} X_j \leq t \middle| X_j \sim \phi(\cdot; x)\right) \geq P\left(\frac{1}{m} \sum_{j=1}^{m} Y_j \leq t \middle| Y_j \sim \phi(\cdot; y)\right) \iff x \leq y.
$$

Assumption 1 states that the distribution with the smaller mean is more likely to have a smaller empirical mean. This holds, for example, if $\phi(\cdot, \mu)$ is Bernoulli($\mu$) or Gaussian observations with known variance $\mathcal{N}(\mu, R)$. A consequence of Assumption 1 is that

$$
P(\mu_i \in S_{k+1} \mid \mu_i = x, \mu_i \in S_k) \geq P(\mu_i \in S_{k+1} \mid \mu_i = y, \mu_i \in S_k) \quad \forall x \leq y.
$$

Assumption 1 often holds for the class of single-parameter exponential families that are well-studied in the multi-armed bandit literature (Audibert and Bubeck, 2010). Our second assumption is standard and allows us to rely on the concentration of measure phenomenon.

**Assumption 2** If $X \sim \phi(\cdot; \mu)$ then $X - \mu$ is an i.i.d. mean-zero, $R$-sub-Gaussian random variable such that $\mathbb{E}[\exp(\lambda(X - \mu))]) \leq \exp(\lambda^2 R/2)$ for any $\lambda > 0$.

Assumption 2 is quite benign: if $X \sim \mathcal{N}(\mu, \sigma^2)$ then $R = \sigma^2$.

**Theorem 1** Fix $\delta \in (0, 1)$. Under Assumptions 1 and 2, for any $\epsilon > 0$ define

$$
z_{n, \epsilon} := \frac{\log(2 \log_2(n)/\delta)}{\nu_0(\mu_\star + \epsilon)} \max \left\{1, 64R \log(4n \log_2(n)/\delta) \sup_{x \geq \mu_\star + \epsilon} (x - \mu_\star)^{-1/2} \nu_0(x)\right\}
$$

$$
\leq \log(2 \log_2(n)/\delta) \max \left\{\frac{1}{\nu_0(\mu_\star + \epsilon)}, 64R \log(4n \log_2(n)/\delta) \left(\epsilon^{-2} + \frac{1}{\nu_0(\mu_\star + \epsilon)} \int_{x=\mu_\star + \epsilon}^{\infty} \frac{1}{(x - \mu_\star)^2} d\nu_0(x)\right)\right\}.
$$

If Successive Halving of Figure 2 is run with $n$ arms and $T = \lceil n \log_2(n) \rceil$ total pulls where $n \geq z_{n, \epsilon}$ then with probability at least $1 - \delta$ the single arm returned is no greater than $\mu_\star + \epsilon$.

**Remark 2** Note that if the support of $\phi$ is bounded in $[0, 1]$ then taking $\delta = \epsilon$ in the above theorem implies $\mathbb{E}[|\mu_T| - \mu_\star] \leq 2\epsilon$ whenever $T = \lceil n \log_2(n) \rceil$ and $n \geq z_{n, \epsilon}$.

Before we prove the theorem, we need some notation and technical lemmas. Define

$$
\nu_k(x) = \mathbb{E}\left[\frac{1}{|S_k|} \sum_{i \in S_k} 1\{\mu_i \leq x\}\right] = P(\mu_i \leq x \mid \mu_i \in S_k)
$$

$$
\begin{align*}
&\mathbb{E}\left[\frac{1}{|S_k|} \sum_{i \in S_k} 1\{\mu_i \leq x\}\right] = P(\mu_i \leq x \mid \mu_i \in S_k) \\
&\leq \sum_{i \in S_k} P(\mu_i \leq x | \mu_i \in S_k) = \sum_{i \in S_k} P(\mu_i \leq x | \mu_i \in S_k) = \sum_{i \in S_k} P(\mu_i \leq x | \mu_i \in S_k) = \sum_{i \in S_k} P(\mu_i \leq x | \mu_i \in S_k).
\end{align*}
$$
Then

**Lemma 4**

For any $\ell \geq 0,1,\ldots,\log_2(n)-1$. Assume Assumption 1 holds. Fix some $\ell \geq 0,1,\ldots,\log_2(n)-1$. The next lemma refines the previous one, stating sufficient conditions for improvement.

**Lemma 5**

Assume Assumption 1 holds. For any $\ell \geq 0,1,\ldots,\log_2(n)-1$ if $n \geq \tilde{\xi}_{n,\ell} := \frac{\log(2\log_2(n)/\delta)}{2\nu(\mu_\ast + \Delta_\ell)}$ then

$$
\mathbb{P}
\left(\min_{i \in S_\ell} \mu_i > \mu_\ast + \Delta_\ell\right) \leq \frac{\delta}{2}.
$$

Moreover, $\mathbb{P}\left(\bigcup_{\ell=0}^{\log_2(n)-1} \left\{ \min_{i \in S_\ell} \mu_i > \mu_\ast + \Delta_\ell \right\} \right) \leq \frac{\delta}{2}$ whenever $n \geq \max_{\ell=0,1,\ldots,\log_2(n)-1} \tilde{\xi}_{n,\ell}$.

If $n \geq \max_{\ell=0,1,\ldots,\log_2(n)-1} \frac{\log(2\log_2(n)/\delta)}{2\nu(\mu_\ast + \Delta_\ell)}$, then combining the two above lemmas would give the desired result of Theorem 1. However, $\nu_k$ is not a natural quantity to reason about since it depends on the behavior of the algorithm. The next two lemmas characterize the behavior of this evolving distribution. The proof of the following lemma applies Bayes rule and exploits Assumption 1. It comes to the intuitive conclusion that the distribution of the true means gets no worse by thresholding the empirical means at the median (Appendix A.3).

**Lemma 6**

Assume Assumption 1 holds. Fix some $k \in \mathbb{N}$. Define the event

$$
\mathcal{E}_k = \{\min_{i \in S_k} \mu_i - \mu_\ast \leq \Delta_k, \max_{i \in S_k} |\hat{\mu}_{i,k} - \mu_i| \leq \Delta_k/2\}.
$$

Then

$$
\nu_{k+1}(\mu_\ast + \epsilon) \geq 1_{\mathcal{E}_k} \min\{1, 2\nu_k(\mu_\ast) + \nu_k(\mu_\ast + \epsilon)\} \quad \forall \epsilon \leq \Delta_k.
$$

**Proof**

Assume $\mathcal{E}_k$. We consider two exhaustive cases:

**Case 1:** $\max_{i \in S_{k+1}} \hat{\mu}_{i,k} \leq \mu_\ast + \Delta_k + (\Delta_k/2)$. Here

$$
\max_{i \in S_{k+1}} \mu_i \leq \max_{i \in S_{k+1}} \hat{\mu}_{i,k} + \Delta_k/2 \leq \mu_\ast + 2\Delta_k
$$
which means that no $2\Delta_k$-bad arms make it into $S_{k+1}$ and $\nu_{k+1}(\mu_* + 2\Delta_k) = 1$. Thus, applying the second result of Lemma 5 twice with $x = \mu_* + \epsilon$ and $y = \mu_* + 2\Delta_k$, we have $\nu_{k+1}(\mu_* + \epsilon) \geq \nu_k(\mu_* + \epsilon) \geq \nu_0(\mu_* + \epsilon) = 1$ for all $\epsilon < \Delta_k$.

**Case 2:** $\mu_* + \Delta_k + (\Delta_k/2) < \max_{i \in S_{k+1}} \hat{\mu}_{i,k}$ In this case all $\Delta_k$-good arms in $S_k$ are guaranteed to be in $S_{k+1}$. Thus $\nu_{k+1}(\mu_* + \Delta_k) = \min\{1, 2\nu_k(\mu_* + \Delta_k)\}$ by the first result of Lemma 5 with $x = \mu_* + \Delta_k$.

We are now ready to prove Theorem 1.

**Proof** Let $z_{n,\Delta_k}$ be $z_{n,\epsilon}$ from the theorem statement with $\epsilon = \Delta_k$. Define

$$
z_n := \max_{k=0,1,...,\log_2(n)-1} z_{n,\Delta_k} = z_{n,\Delta_{\log_2(n)-1}} \quad \text{and} \quad \xi_{n,k} := \frac{\log(2 \log_2(n)/\delta)}{2^{-k}\nu_k(\mu_* + \Delta_k)}
$$

where $\xi_{n,k}$ comes from Lemma 4. Note that $n \geq z_n$ by assumption. The theorem claims that with probability at least $1 - \delta$ the returned arm has a mean no greater than $\mu_* + \Delta_{\log_2(n)-1}$ which is implied if $E_{\log_2(n)-1}$ holds whenever $n \geq z_n = z_{n,\Delta_{\log_2(n)-1}}$ by simply making the substitution $\epsilon := \Delta_{\log_2(n)-1}$. Thus, our goal is to show that $E_{\log_2(n)-1}$ holds by proving the stronger statement that $\bigcap_{k=0}^{\log_2(n)-1} E_k$ holds.

By immediate application of Lemma 3 and Lemma 4, for all $k = 0, 1, \ldots, \log_2(n) - 1$ simultaneously

$$
\{n \geq \xi_{n,k}\} \implies \{\min_{i \in S_k} \mu_i \leq \mu_* + \Delta_k\} \cap \{\max_{i \in S_k} |\hat{\mu}_{i,k} - \mu_i| \leq \Delta_k/2\} = E_k
$$

(6)

with probability at least $1 - \delta$. In what follows, we make the implication of (6) holds for all $k$, since these occur with probability at least $1 - \delta$.

We will prove the theorem by induction starting with the base case

$$
\{n \geq z_n\} \implies \{n \geq z_{n,\Delta_0}\} \implies \{n \geq \frac{\log(2 \log_2(n)/\delta)}{\nu_0(\mu_* + \Delta_0)}\} \equiv \{n \geq \xi_{n,0}\} \implies E_0.
$$

Fix any $k \in \{1, \ldots, \log_2(n) - 1\}$. Note $\nu_k(\mu_* + \Delta_k) = 1$ implies $\min_{i \in S_k} \mu_i - \mu_* \leq \Delta_k$ so

$$
\{\nu_k(\mu_* + \Delta_k) = 1\} \cap \{\max_{i \in S_k} |\hat{\mu}_{i,k} - \mu_i| \leq \Delta_k/2\} \implies E_k.
$$

So assume $\nu_k(\mu_* + \Delta_k) < 1$. On $\left\{\bigcap_{\ell=0}^{k-1} E_{\ell}\right\}$ we have

$$
\nu_k(\mu_* + \Delta_k) \geq \min\{1, 2\nu_{k-1}(\mu_* + \Delta_k), \frac{\nu_0(\mu_* + \Delta_k)}{\nu_0(\mu_* + 2\Delta_{k-1})}\} \geq \min\{1, 2 \min\{1, 2\nu_{k-2}(\mu_* + \Delta_k), \frac{\nu_0(\mu_* + \Delta_k)}{\nu_0(\mu_* + 2\Delta_{k-2})}, \frac{\nu_0(\mu_* + \Delta_k)}{\nu_0(\mu_* + 2\Delta_{k-1})}\}\} \geq \min\{1, 2^k \nu_0(\mu_* + \Delta_k), \min_{\ell=0}^{k-1} 2^{k-1-\ell} \frac{\nu_0(\mu_* + \Delta_k)}{\nu_0(\mu_* + 2\Delta_{k-\ell})}\} = \min\{1, 2^k \nu_0(\mu_* + \Delta_k), \min_{\ell=0}^{k-1} \frac{1}{2^{\ell+1} \nu_0(\mu_* + 2\Delta_{k-\ell})}\} = 2^k \nu_0(\mu_* + \Delta_k) \max\{1, \min_{\ell=0}^{k-1} \frac{1}{2^{\ell+1} \nu_0(\mu_* + 2\Delta_{k-\ell})}\}
$$
where the last line uses \( \nu_k(\mu_\ast + \Delta_k) < 1 \) and moves the min to the denominator. Now

\[
\max_{\ell=0,...,k-1} 2^{\ell+1} \nu_0(\mu_\ast + 2\Delta) = \max_{\ell=0,...,k-1} \frac{16R \log(n \log_2(n))2^{-\ell+2}/\delta)}{\Delta^-2\nu_0(\mu_\ast + 2\Delta)} \\
\leq 64R \log(4n \log_2(n)/\delta) \max_{\ell=0,...,k-1} (2\Delta)^{-2} \nu_0(\mu_\ast + 2\Delta) \\
\leq 64R \log(4n \log_2(n)/\delta) \sup_{x \geq \mu_\ast + \Delta_k} (x - \mu_\ast)^{-2} \nu_0(x) \\
\leq 64R \log(4n \log_2(n)/\delta) \sup_{x \geq \mu_\ast + \Delta_k} (x - \mu_\ast)^{-2} \nu_0(x),
\]

\[
\max\{1, \max_{\ell=0,...,k-1} 2^{\ell+1} \nu_0(\mu_\ast + 2\Delta)\} \leq \max\{1, 64R \log(4n \log_2(n)/\delta) \sup_{x \geq \mu_\ast + \Delta_k} (x - \mu_\ast)^{-2} \nu_0(x)\} \\
= \frac{\nu_0(\mu_\ast + \Delta_k)}{\log(2 \log_2(n)/\delta)} z_n, \Delta_k.
\]

Plugging this result back into the previous display and rearranging, we obtain

\[
\nu_k(\mu_\ast + \Delta_k) \geq \frac{2^k \log(2 \log_2(n)/\delta)}{z_n, \Delta_k} \equiv z_{n, \Delta_k} \geq \xi_{n, k}.
\]

Thus,

\[
\{n \geq z_n\} \cap \left\{ \bigcap_{\ell=0}^{k-1} \mathcal{E}_\ell \right\} \implies \{n \geq z_n\} \cap \{z_n, \Delta_k \geq \xi_{n, k}\} \cap \left\{ \bigcap_{\ell=0}^{k-1} \mathcal{E}_\ell \right\} \stackrel{\text{(6)}}{\implies} \{n \geq z_n\} \cap \left\{ \bigcap_{\ell=0}^{k-1} \mathcal{E}_\ell \right\}
\]

Because \( k \) was chosen arbitrarily, and because \( \mathcal{E}_0 \) holds, we have proven that \( \{n \geq z_n\} \implies \left\{ \bigcap_{\ell=0}^{\log_2(n)-1} \mathcal{E}_\ell \right\} \) with probability at least \( 1 - \delta \).

3. Lower bound

Fix any reservoir distribution \( \nu_0 \) and \( \epsilon, \delta \in (0, 1) \). Our upper bound of Theorem 1 states that if the proposed algorithm is provided a budget of \( \epsilon^{-2} + \frac{1}{\nu_0(\mu_\ast + \epsilon)} \int_{\mu_\ast + \epsilon}(\mu - \mu_\ast)^{-2}d\nu_0(\mu) \) pulls (up to poly-logarithmic factors), then the prescribed procedure outputs an \( \epsilon \)-good arm with probability at least \( 1 - \delta \). In this section, we argue that any algorithm that identifies an \( \epsilon \)-good arm with probability at least \( 1 - \delta \) must take nearly this many total pulls in expectation. We follow the lower bound technique of Malloy et al. (2012) beginning with a definition borrowed from Berry et al. (1997).

**Definition 7** A non-recalling strategy is one that always draws a new arm from \( \nu_0 \) when switching from the current arm and never pulls a previous arm again.

Implicit in Malloy et al. (2012) is the assumption that there exists a non-recalling strategy for every \( \nu_0, \epsilon, \delta \) that is near-optimal with respect to any strategy. Such an assumption is reasonable because observations from any particular arm are conditionally independent given the mean of the arm, and knowing the mean of one arm provides no information about the mean of another. Thus, the number of times any particular arm is pulled depends only
on the observations from that arm, and because the means of the arms are drawn iid from \( \nu \), each arm should be treated identically. Thus, the procedure will continue to discard arms until it finds one and commits to it for all time. Of course, any such non-recalling strategy would require precise knowledge of \( \nu_0 \) making it purely a thought experiment, but it is useful for a lower bound. Nevertheless, many algorithms for the regret setting of infinite-armed bandits make very strong assumptions and are non-recalling strategies (c.f., Berry et al. (1997); Bonald and Proutiere (2013); Chan and Hu (2018).

Define \( KL(\mu, \mu') = \int \phi(x; \mu) \log \left( \frac{\phi(x; \mu)}{\phi(x; \mu')} \right) dx \) where we assume \( KL(c, d) \geq KL(a, b) \) for all \([a, b] \subseteq [c, d] \). This is a common assumption and holds for families of distributions \( \phi(\cdot; \mu) \) parameterized by their mean (e.g., Bernoulli, Gaussian, Poisson) Kaufmann et al. (2016).

**Theorem 8** Fix a reservoir distribution \( \nu, \delta \in (0, 1/15) \), and \( \epsilon > 0 \) such that \( \nu(\mu_\ast + \epsilon) \leq 1/2 \). If at time \( \tau \in \mathbb{N} \) a non-recalling strategy outputs an arm \( \hat{i} \in \mathbb{N} \) that satisfies \( \mathbb{P}(\mu_\hat{i} \leq \mu_\ast + \epsilon) \geq 1 - \delta \), then

\[
\mathbb{E}[\tau] \geq (1 - \delta) \log \left( \frac{1 - \delta}{\epsilon d \nu(\mu_\ast + \epsilon)} \right)KL(\mu_\ast, \mu) - \frac{2}{\nu(\mu_\ast + \epsilon, \mu_\ast)} + \frac{3/8}{\nu(\mu_\ast + \epsilon)} \int_{\mu_\ast + \epsilon}^{\mu_\ast} KL(\mu, \mu_\ast) d\nu(\mu)
\]

for any \( \tilde{\mu} \) with \( \tilde{\nu} := \nu(\tilde{\mu}) - \nu(\mu_\ast + \epsilon) > \frac{\delta \nu(\mu_\ast + \epsilon)}{1 - \delta} \).

We related this lower bound to previously known upperbounds using Gaussian realizations (or Bernoulli’s near 1/2) where \( KL(\mu, \mu') \leq c(\mu - \mu')^2 \):

**Continuous as \( \mu \to \mu_\ast \):** Take \( \tilde{\kappa} = \sqrt{\delta} \). As \( \delta \to 0 \) we have \( \tilde{\mu} \to \mu_\ast + \epsilon \) to yield a sample complexity of \( \epsilon^{-2} \log(\frac{1}{\epsilon d \nu(\mu_\ast + \epsilon)}) + \frac{1}{\nu(\mu_\ast + \epsilon)} \int_{\mu_\ast + \epsilon}^{\mu_\ast} \frac{1}{(\mu - \mu_\ast)^2} d\nu(\mu) \).

**Polynomial-tail, Equation 2:** \( \nu(\mu_\ast + x) = \mathbb{P}(\mu \leq \mu_\ast + x) \propto x^\beta \). Take \( \tilde{\mu} = \mu_\ast + 2^{1/\beta} \epsilon \) so that \( \tilde{\kappa} = \nu(\mu_\ast + 2^{1/\beta} \epsilon) - \nu(\mu_\ast + \epsilon) \propto \epsilon^{\beta} \) yielding a sample complexity of \( \epsilon^{-2} \log(1/\delta) + \epsilon^{-\beta} \).

We will only sketch the proof of Theorem 2, leaving the technical details to the appendix.

Since each arm is treated identically, one realizes that such a procedure is performing a sequence of composite binary hypothesis tests where the test decides to keep sampling or not given the observations up to the current time. Let \( \mathbb{P}_\mu \) and \( \mathbb{E}_\mu \) be the probability law and the expectation of observations from an arm with mean \( \mu \). It will be also convenient to define \( \pi := \nu(\mu_\ast + \epsilon) \) and \( N_i \) be the random number of times the \( i \)th arm is pulled before it is either discarded (denoted by the event \( R_i^\epsilon \)) or declared as \( \epsilon \)-good (\( R_i \)). Note that \( R_i, R_1 \) as well as \( N_i, N_1 \) for all \( i \) are independent and identically distributed for any non-recalling algorithm by the iid nature of the draws from \( \nu_0 \). Define \( \alpha := \frac{1}{1 - \pi} \int_{\mu_\ast + \epsilon}^{\mu_\ast + \epsilon} \mathbb{P}_\mu(R_1) d\nu(\mu_1) \) and \( \beta := \frac{1}{\pi} \int_{\mu_\ast + \epsilon}^{\mu_\ast + \epsilon} \mathbb{P}_\mu(R_i) d\nu(\mu_1) \). Then

\[
\mathbb{E}[\tau] = \mathbb{E} \left[ \sum_{i \geq 1} N_i \right] = \mathbb{E}[N_1] + \mathbb{E} \left[ \sum_{i > 1} N_i | R_i^\epsilon \right] ((1 - \alpha)(1 - \pi) + \beta \pi)
\]

\[
= \mathbb{E}[N_1] + \mathbb{E} \left[ \sum_{i \geq 1} N_i \right] ((1 - \alpha)(1 - \pi) + \beta \pi)
\]

by the iid nature of \( \mu_i \sim \nu \) and thus memoryless property of the process. After rearranging,

\[
\mathbb{E} \left[ \sum_{i \geq 1} N_i \right] = \frac{\mathbb{E}[N_1]}{\alpha(1 - \pi) + (1 - \beta) \pi}.
\]
Lemma 9 Fix $\alpha, \beta \in (0, 1)$. For any $\kappa \in \left(\frac{\alpha(1-\pi)}{1-\beta}, 1\right)$

$$\mathbb{E}[N_1] \geq \frac{\pi d(1-\beta, \frac{\alpha(1-\pi)}{\kappa})}{KL(\mu_*, \bar{\mu})} + d(\frac{\alpha(1-\pi)}{\kappa}, 1-\beta) \left(\frac{-\kappa}{KL(\mu_* + \epsilon, \mu_*)} + \frac{1}{2} \int_{\mu_* + \epsilon}^{\mu_*} \frac{1}{KL(\mu, \mu_*)} d\nu(\mu)\right)$$

for any $\bar{\mu}$ satisfying $\bar{\mu} := \nu(\mu) - \nu(\mu_* + \epsilon) > \frac{\alpha(1-\pi)}{1-\beta}$.

A similar calculation to (7) reveals $\mathbb{P}(\text{error}) = \mathbb{P}(\bigcup_{i \geq 1} \{R_i, \mu_i > \mu_* + \epsilon\}) = \frac{\alpha(1-\pi)}{(1-\beta)\pi + \alpha(1-\pi)}$ and rearranging we observe that for some $\delta \in (0, 1)$

$$\mathbb{P}(\text{error}) = \frac{1}{1 + \frac{\pi(1-\beta)}{(1-\pi)\alpha}} \leq \delta \iff \frac{(1-\pi)\alpha}{\pi(1-\beta)} \leq \frac{\delta}{1-\delta}. \quad (8)$$

If $\kappa = 2\pi$ and $\mathbb{P}(\text{error}) \leq \delta$ then by the above implication, $\kappa = 2\pi > 2\frac{\pi \delta}{1-\beta} \geq 2\frac{(1-\pi)\alpha}{1-\beta} > \frac{(1-\pi)\alpha}{1-\beta}$ where the last strict inequality is precisely the condition on $\kappa$ for which Lemma 9 applies. Thus, we plug in the result of Lemma 9 with $\kappa = 2\pi$ into Equation 7 and simplify to obtain the theorem.

4. Empirical study

We evaluate ISHA against various baselines, using Bernoulli arms.

Datasets/reservoirs. We test on synthetic data, with arms drawn from various reservoirs. First, we use $Beta(1, 1)$, and $Beta(3, 1)$ reservoirs of Equation 1, both with support $[0, 1]$ and rescaled to have support in $[0.25, 0.75]$ to avoid arms with extreme means; these distributions correspond to $\beta = 1$ and $\beta = 3$ in Equation 2, respectively, regardless of scaling. We also use the reservoir described in Equation 3 composed of two spikes with gap $\epsilon$ and relative proportion $\pi$ such that $1/\pi \epsilon^2$ is a constant known to govern the sample complexity of this problem (Chandrasekaran and Karp, 2014; Malloy et al., 2012; Jamieson et al., 2016). In particular, the spikes are symmetric around $1/2$ and $(\pi, \epsilon) \in \{(10^{-3}, \sqrt{10^{-1}}), (10^{-2}, \sqrt{10^{-2}}), (10^{-1}, \sqrt{10^{-3}})\}$ so that $1/\pi \epsilon^2 = 10,000$.

We also test against a reservoir generated from data collected from the New Yorker Caption Contest dataset (Jamieson et al., 2015), using contest number 637 having 3,795 captions and 875,065 total votes uniformly at random distributed amongst the captions, resulting in about 231 votes per caption. For our experiment, arms are randomly drawn with replacement from the 3,795 arms, where the mean of each arm is taken to be the fraction of times “unfunny” was observed in the dataset, so small means are better. The arm reservoir CDF (Figure 9) can be found in Appendix D.

Algorithms. We compare against three main classes of algorithms: (1) pure exploration algorithms, (2) explore-vs-exploit algorithms, and (3) anytime algorithms. Detailed descriptions of these algorithms and their use in our empirical study are given in Appendix D. In brief, within the pure exploration family, we consider SIRI (Carpentier and Valko, 2015), lil’UCB (Jamieson et al., 2014), Hyperband (Li et al., 2017), and Successive Rejects (Audibert and Bubeck, 2010). We also devise a strong baseline for ISHA that we refer to as Chernoff. Chernoff has knowledge of $\mu_*$, and simply draws an arm from the reservoir, tests
it so long as its the empirical lower bound does not exceed $\mu_*$, discarding the arm and
drawing another if the arm is ever proven to be suboptimal. Within the explore-vs-exploit
family, we consider four infinite-armed bandit algorithms of Berry et al. (1997), CBT and
Empirical CBT (Chan and Hu, 2018), and fixed horizon Two Target (Bonald and Proutiere,
2013). We also consider an anytime version of ISHA: choose increasing dyadic numbers of
arms, $n = 2^i, i = 1, 2, \ldots$. For each value of $n$, sample $n$ new arms and run ISHA and save
the result as the best arm found so far. We compare this algorithm to Hyperband Anytime
and SIRI Anytime (using the budget doubling trick as proposed by the SIRI authors).

4.1 Experiments and Insights

Our proposed algorithm starts discarding arms after just a single pull, far before concentra-
tion of measure has kicked in. It is truly surprising that theoretical results can be obtained
for such a result, and consequently we begin with an empirical study of this phenomenon.
We evaluate ISHA against a variety of alternative approaches. Our exhaustive experiments
can be found in Appendix D, here we report only a representative sample.

**Successive Halving performance as a function of the number of arms for a fixed budget.** Figure 3a studies the tradeoff between the number of arms $n$ and budget $T$
for Successive Halving in terms of simple regret averaged over 500 replications. Let $n^*_T$ be
the maximum number of arms in Successive Halving, i.e. $T \geq n^*_T \log_2 n^*_T$. Each “sheet” of
the plot corresponds to a single value of budget $T$ for a number of arms $n = 2, 2^2, 2^3, \ldots, n^*_T$.
We observe that across a variety of reservoirs (see Appendix D) Successive Halving with
the maximum possible number of arms $n^*_T$ (ISHA) appears to perform better than or as
well as Successive Halving with any other value of $n$.

**Simple regret vs. Budget.** In the next several plots we compare the simple regret
of ISHA to that of various baselines. The results are averaged over 200 replications. In
Figures 3b and 3c we compare to state-of-the-art algorithms for infinitely-armed bandit
models. While different algorithms are most competitive on different reservoirs, ISHA and
Successive Rejects are the only algorithms that are consistently superior across all reservoirs.

Figure 3d highlights, as discussed in some detail in Section 1, the difficulty of choosing
the optimal number of arms for UCB-style algorithms, such as the lil’UCB.

Figure 3f compares mainly against exploration-vs-exploitation baselines. Recall that
many of these baseline algorithms are designed specifically for Beta reservoirs and assume
knowledge of of the reservoir. Even so, ISHA does as well or better.

Finally, in Figure 3e, we compare ISHA and ISHA Anytime to several other anytime
algorithms. ISHA Anytime easily outperforms its baselines, performing nearly as well as its
non-Anytime version.
Figure 3: A sampled set of results. See Appendix D for more.
References

Jean-Yves Audibert and Sébastien Bubeck. Best arm identification in multi-armed bandits. In COLT-23th Conference on Learning Theory-2010, pages 13–p, 2010.

Maryam Aziz, Jesse Anderton, Emilie Kaufmann, and Javed Aslam. Pure exploration in infinitely-armed bandit models with fixed-confidence. In ALT 2018-Algorithmic Learning Theory, 2018.

Donald A. Berry, Robert W. Chen, Alan Zame, David C. Heath, and Larry A. Shepp. Bandit problems with infinitely many arms. The Annals of Statistics, 25(5):2103–2116, 10 1997.

Thomas Bonald and Alexandre Proutiere. Two-target algorithms for infinite-armed bandits with bernoulli rewards. In Advances in Neural Information Processing Systems, pages 2184–2192, 2013.

Sébastien Bubeck, Rémi Munos, and Gilles Stoltz. Pure exploration in multi-armed bandits problems. In International conference on Algorithmic learning theory, pages 23–37. Springer, 2009.

Sébastien Bubeck, Nicolo Cesa-Bianchi, et al. Regret analysis of stochastic and nonstochastic multi-armed bandit problems. Foundations and Trends® in Machine Learning, 5(1):1–122, 2012.

Alexandra Carpentier and Michal Valko. Simple regret for infinitely many armed bandits. CoRR, abs/1505.04627, 2015.

Hock Peng Chan and Shouri Hu. Infinite arms bandit: Optimality via confidence bounds. CoRR, abs/1805.11793, 2018.

Karthekeyan Chandrasekaran and Richard Karp. Finding a most biased coin with fewest flips. In Conference on Learning Theory, pages 394–407, 2014.

Arghya Roy Chaudhuri and Shivaram Kalyanakrishnan. Quantile-regret minimisation in infinitely many-armed bandits.

Arghya Roy Chaudhuri and Shivaram Kalyanakrishnan. Pac identification of a bandit arm relative to a reward quantile. In AAAI, pages 1777–1783, 2017.

Thomas M. Cover and Joy A. Thomas. Elements of Information Theory (Wiley Series in Telecommunications and Signal Processing). Wiley-Interscience, 2006. ISBN 0471241954.

Yahel David and Nahum Shimkin. Infinitely many-armed bandits with unknown value distribution. In Joint European Conference on Machine Learning and Knowledge Discovery in Databases, pages 307–322. Springer, 2014.

Yahel David and Nahum Shimkin. Refined algorithms for infinitely many-armed bandits with deterministic rewards. In Joint European Conference on Machine Learning and Knowledge Discovery in Databases, pages 464–479. Springer, 2015.

Kevin G. Jamieson, Matthew Malloy, Robert D. Nowak, and Sébastien Bubeck. lil’ ucb : An optimal exploration algorithm for multi-armed bandits. In COLT, 2014.

Kevin G Jamieson, Lalit Jain, Chris Fernandez, Nicholas J. Glattard, and Rob Nowak. Next: A system for real-world development, evaluation, and application of active learning. In Advances in Neural Information Processing Systems 28, pages 2656–2664. 2015.

Kevin G Jamieson, Daniel Haas, and Benjamin Recht. The power of adaptivity in identifying statistical alternatives. In Advances in Neural Information Processing Systems, pages 775–783, 2016.
Zohar Karnin, Tomer Koren, and Oren Somekh. Almost optimal exploration in multi-armed bandits. In *Proceedings of the 30th International Conference on Machine Learning (ICML-13)*, volume 28, pages 1238–1246, May 2013.

Emilie Kaufmann, Olivier Cappé, and Aurélien Garivier. On the complexity of best-arm identification in multi-armed bandit models. *The Journal of Machine Learning Research*, 17(1):1–42, 2016.

Lisha Li, Kevin G Jamieson, Giulia DeSalvo, Afshin Rostamizadeh, and Ameet Talwalkar. Hyperband: A novel bandit-based approach to hyperparameter optimization. *Journal of Machine Learning Research*, 18:185–1, 2017.

Matthew L Malloy, Gongguo Tang, and Robert D Nowak. Quickest search for a rare distribution. In *Information Sciences and Systems (CISS), 2012 46th Annual Conference on*, pages 1–6. IEEE, 2012.

Wenbo Ren, Jia Liu, and Ness Shroff. Exploring $k$ out of top $\rho$ fraction of arms in stochastic bandits. *arXiv preprint arXiv:1810.11857*, 2018.

David Siegmund. *Sequential Analysis*. 1985.

Olivier Teytaud, Sylvain Gelly, and Michele Sebag. Anytime many-armed bandits. In *CAP07*, 2007.

Yizao Wang, Jean yves Audibert, and Rémi Munos. Algorithms for infinitely many-armed bandits. In *Advances in Neural Information Processing Systems 21*, pages 1729–1736. 2009.
Appendix A. Technical Lemmas

A.1 Proof of Lemma 3

Lemma 10 Assume Assumption 2 holds. We have

\[ P \left( \bigcup_{\ell=0}^{\log_2(n)-1} \bigcup_{i \in S_{\ell}} \{|\hat{\mu}_{i,\ell} - \mu_i| \geq \Delta_\ell / 2\} \right) \leq \delta / 2 \]

Proof By Assumption 2 and a Chernoff bound, recalling that for the \( i \)th arm \( \hat{\mu}_{i,\ell} \) is empirical mean of \( 2^\ell \) iid draws from \( \phi(\mu_i) \), we have

\[ P(|\hat{\mu}_{i,\ell} - \mu_i| \geq \Delta_\ell / 2) \leq 2 \exp \left( -\frac{\Delta_\ell^2}{2|S_\ell|} \frac{2 R}{n \log_2(n) 2^{-\ell+2}} \right) = 2 \frac{\delta}{2^{\log_2(n) - 1}} \frac{\delta/2}{|S_\ell| \log_2(n)} \]

By a union bound and conditioning on the elements in \( S_\ell \)

\[ P \left( \bigcup_{\ell=0}^{\log_2(n)-1} \bigcup_{i \in S_{\ell}} \{|\hat{\mu}_{i,\ell} - \mu_i| \geq \Delta_\ell / 2\} \right) \leq \sum_{\ell=0}^{\log_2(n)-1} \mathbb{E} \left[ P \left( \bigcup_{i \in S_{\ell}} \{|\hat{\mu}_{i,\ell} - \mu_i| \geq \Delta_\ell / 2\} \right) \right] \leq \sum_{\ell=0}^{\log_2(n)-1} \mathbb{E} \left[ \sum_{i \in S_{\ell}} P(|\hat{\mu}_{i,\ell} - \mu_i| \geq \Delta_\ell / 2) \right] \leq \sum_{\ell=0}^{\log_2(n)-1} \mathbb{E} \left[ \sum_{i \in S_{\ell}} \frac{\delta}{|S_\ell| \log_2(n)} \right] \leq \delta / 2 \]

A.2 Proof of Lemma 4

Lemma 11 For any \( \ell = 0, 1, \ldots, \log_2(n) - 1 \) if \( n \geq \xi_{n,\ell} := \frac{\log(2 \log_2(n) / \delta)}{2^{\log_2(n) - 1} (\xi_\ell + \Delta_\ell)} \) then

\[ P \left( \min_{i \in S_{\ell}} \mu_i > \mu_* + \Delta_\ell \right) \leq \frac{\delta}{2 \log_2(n)} \]

Moreover, \( P \left( \bigcup_{\ell=0}^{\log_2(n)-1} \{\min_{i \in S_{\ell}} \mu_i > \mu_* + \Delta_\ell\} \right) \leq \delta / 2 \) whenever \( n \geq \max_{\ell=0,1,\ldots,\log_2(n)-1} \xi_{n,\ell} \).

Proof By a union bound we have

\[ P \left( \bigcup_{\ell=0}^{\log_2(n)-1} \{\min_{i \in S_{\ell}} \mu_i > \mu_* + \Delta_\ell\} \right) \leq \sum_{\ell=0}^{\log_2(n)-1} P(\min_{i \in S_{\ell}} \mu_i > \mu_* + \Delta_\ell) \]
so it suffices to show

\[
P(\min_{i \in S_t} \mu_i > \mu_\ast + \Delta_t) = E\left[ P\left( \min_{i \in S_t} \mu_i > \mu_\ast + \Delta_t \mid S_t \right) \right]
\]

\[
= P\left( \min_{i=1}^{\mid S_t \mid} \mu_i > \mu_\ast + \Delta_t \mid \{\mu_i\}_{i=1}^{\mid S_t \mid} \sim \nu_t \right)
\]

\[
= P\left( \mu_i > \mu_\ast + \Delta_t \mid \{\mu_i\}_{i=1}^{\mid S_t \mid} \sim \nu_t \right)
\]

\[
= (1 - \nu_t(\mu_\ast + \Delta_t))^{\mid S_t \mid}
\]

\[
\leq \exp(-n2^{-\ell} \nu_t(\mu_\ast + \Delta_t))
\]

\[
\leq \frac{\delta}{2 \log_2(n)}
\]

where the second-to-last line uses the identity $n2^{-\ell} = |S_t|$ and that $1 - x \leq e^{-x}$ for all $x \geq 0$, and the last line plugs in the assumed condition on $n$.

\[ \square \]

### A.3 Proof of Lemma 5

**Lemma 12** Assume Assumption 1 holds. For any $k$ and $x \in \mathbb{R}$ we have

\[
\nu_{k+1}(x) = 2P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k) \nu_k(x)
\]

\[
\geq \nu_k(x).
\]

Moreover, for any $k$ and $x < y$ we have $\frac{\nu_{k+1}(x)}{\nu_k(x)} \geq \frac{\nu_{k+1}(y)}{\nu_k(y)}$.

**Proof** By Bayes’ rule

\[
\nu_{k+1}(x) = P(\mu_i \leq x \mid \mu_i \in S_{k+1})
\]

\[
= P(\mu_i \leq x \mid \mu_i \in S_{k+1}, \mu_i \in S_k)
\]

\[
= \frac{P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k) P(\mu_i \leq x \mid \mu_i \in S_k)}{P(\mu_i \in S_{k+1} \mid \mu_i \in S_k)}
\]

\[
= \nu_k(x) \frac{P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k)}{P(\mu_i \in S_{k+1} \mid \mu_i \in S_k)}.
\]

By definition, $P(\mu_i \in S_{k+1} \mid \mu_i \in S_k) = \frac{1}{2}$. On the other hand, ignoring ties (which are broken randomly) we have $\mu_i \in S_{k+1} \iff \tilde{\mu}_{i,k} \leq \tau$. Thus, by assumption 1,

\[
P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k) \leq P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k).
\]

Applying the law of total probability to $P(\mu_i \in S_{k+1} \mid \mu_i \in S_k)$ we have

\[
\frac{P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k)}{P(\mu_i \in S_{k+1} \mid \mu_i \in S_k)}
\]

\[
= \frac{P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k)}{\nu_k(x) P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k) + (1 - \nu_k(x)) P(\mu_i \in S_{k+1} \mid \mu_i > x, \mu_i \in S_k)}
\]

\[
\geq \frac{\nu_k(x) P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k) + (1 - \nu_k(x)) P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k)}{\nu_k(x) P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k) + (1 - \nu_k(x)) P(\mu_i \in S_{k+1} \mid \mu_i \leq x, \mu_i \in S_k)}
\]

\[
= 1.
\]

\[ 17 \]
Appendix B. Proof of Lemma 9

Proof By a manipulation of Wald’s identity Siegmund (1985), if \( N \) is a stopping time with finite expectation at which time the procedure declares the arm as \( \epsilon \)-good or not when run on an arm with mean \( \mu \), we have for any \( \mu' \neq \mu \)

\[
\mathbb{E}_\mu[N] \geq \sup_\epsilon d(\mathbb{P}_\mu(E), \mathbb{P}_{\mu'}(E)) \frac{KL(\mu, \mu')}{K L(\mu, \mu')}
\]

where \( d(x, y) = x \log(e^x) + (1 - x) \log(e^{1-x}) \). Now

\[
\int_{\mu=\mu_*} \mathbb{E}_\mu[N] d\nu(\mu) = \int_{\mu=\mu_*}^{\mu_*+\epsilon} \mathbb{E}_\mu[N] d\nu(\mu) + \int_{\mu=\mu_*+\epsilon}^{\epsilon} \mathbb{E}_\mu[N] d\nu(\mu).
\]

Consider the decomposition \( \nu = \nu^a + \nu^s \) where \( \nu^a \) and \( \nu^s \) are the absolutely continuous and singular components, respectively, of \( \nu \) with respect to the Lebesgue measure. Let \( \mathcal{A}^0 = \{ \{a\} : a \in [\mu_* + \epsilon, \infty) \cap \text{support}(\nu^s), \frac{d\nu^a(a)}{dx} \geq \kappa \} \) where \( \frac{d\nu^a(a)}{dx} \) is the Radon-Nikodym derivative with respect to the Lebesgue measure. Note that \( \mathcal{A}^0 \) does not contain all of the singular components, just those with mass at least \( \kappa \). Let \( \mathcal{A}^\perp \) be a collection of disjoint sets that have empty intersection with \( \mathcal{A}^0 \) constructed by first covering \( [\mu_* + \epsilon, \infty) \cap \text{support}(\nu^s) \) with intervals such that \( A \) is an interval, \( \kappa \leq \nu(A - \mathcal{A}^\perp) \leq 2\kappa \), and then set \( A = A \cup \mathcal{A}^\perp \) such that \( A \) is an interval in all but a set of measure 0. Finally, define \( \mathcal{A} = \mathcal{A}^0 \cup \mathcal{A}^\perp \). Note that \( \min_{A \in \mathcal{A}} \nu(A) \geq \kappa \). Also note that for any \( A \in \mathcal{A} \) we have \( \sup_{x,y \in A} |x - y| > 0 \) if and only if \( A \in \mathcal{A}^\perp \) so that we also have \( \nu(A) \leq 2\kappa \).

Let \( E \) denote the event that the current arm is the declared as \( \epsilon \)-good. Given such a partition, note that

\[
\max_{A \in \mathcal{A}} \int_{\mu \in A} \frac{1}{\nu(A)} \mathbb{P}_\mu(E) d\nu(\mu) \leq \sum_{A \in \mathcal{A}} \frac{1}{\nu(A)} \int_{\mu \in A} \mathbb{P}_\mu(E) d\nu(\mu)
\]

\[
\leq \frac{1}{\kappa} \sum_{A \in \mathcal{A}} \int_{\mu \in A} \mathbb{P}_\mu(E) d\nu(\mu)
\]

\[
= \frac{1}{\kappa} \int_{\mu=\mu_*+\epsilon} \mathbb{P}_\mu(E) d\nu(\mu)
\]

\[
= \frac{\alpha(1 - \pi)}{\kappa}
\]

\[
< 1 - \beta
\]
where the last line holds by assumption. By the definition of \( \bar{\mu} \) in the statement, if \( \bar{A} = (\mu_* + \epsilon, \bar{\mu}] \) then \( \nu(\bar{A}) \geq \kappa \) so

\[
\int_{\mu=\mu_*}^{\mu_*+\epsilon} \mathbb{E}_\mu[N]d\nu(\mu) = \frac{1}{\nu(\bar{A})} \int_{\mu' \in \bar{A}} \frac{1}{\nu(\mu')} \int_{\mu=\mu_*}^{\mu_*+\epsilon} \mathbb{E}_\mu[N]d\nu(\mu)d\nu(\mu') \\
\overset{(i)}{\geq} \frac{1}{\nu(\bar{A})} \int_{\mu' \in \bar{A}} \frac{1}{\nu(\mu')} \int_{\mu=\mu_*}^{\mu_*+\epsilon} \frac{d(\mathbb{P}_\mu(E), \mathbb{P}_{\mu'}(E))}{KL(\mu, \mu')} d\nu(\mu)d\nu(\mu') \\
\overset{(ii)}{\geq} \frac{1}{KL(\mu_*, \bar{\mu})} \frac{1}{\nu(\bar{A})} \int_{\mu' \in \bar{A}} \frac{1}{\nu(\mu')} \int_{\mu=\mu_*}^{\mu_*+\epsilon} \frac{d(\mathbb{P}_\mu(E), \mathbb{P}_{\mu'}(E))}{KL(\mu, \mu')} d\nu(\mu)d\nu(\mu') \\
\overset{(iii)}{\geq} \frac{1}{KL(\mu_*, \bar{\mu})} \frac{1}{\nu(\bar{A})} \int_{\mu' \in \bar{A}} \frac{1}{\nu(\mu')} \left( \frac{1}{\nu(\mu)} \int_{\mu=\mu_*}^{\mu_*+\epsilon} \mathbb{E}_\mu(E) d\nu(\mu) \right) d\nu(\mu) \\
\overset{(iv)}{\geq} \pi d(1 - \beta, \frac{\alpha(1-\pi)}{\nu(\bar{A})}) \frac{1}{KL(\mu_*, \bar{\mu})}
\]

where (i) follows from Equation 9, (ii) uses the fact that \( KL(a, b) \leq KL(c, d) \) whenever \([a, b] \subseteq [c, d] \), (iii) uses the fact that binary KL divergence is convex Cover and Thomas (2006), and (iv) holds because \( \max_{A \in \mathcal{A}} \frac{1}{\nu(\bar{A})} \int_{\mu \in \bar{A}} \mathbb{P}_\mu(E) d\nu(\mu) \leq \frac{\alpha(1-\pi)}{\kappa} < 1 - \beta \) by assumption.

The second term follows analogously

\[
\int_{\mu=\mu_*+\epsilon}^{\infty} \mathbb{E}_\mu[N]d\nu(\mu) = \sum_{A \in \mathcal{A}} \nu(\bar{A}) \frac{1}{\nu(\bar{A})} \int_{\mu \in \bar{A}} \mathbb{E}_\mu[N]d\nu(\mu) \\
= \sum_{A \in \mathcal{A}} \nu(\bar{A}) \frac{1}{\nu(\bar{A})} \int_{\mu' = \mu_*=\mu_*}^{\mu_*+\epsilon} \mathbb{E}_\mu[N]d\nu(\mu)d\nu(\mu') \\
\overset{(i)}{\geq} \sum_{A \in \mathcal{A}} \nu(\bar{A}) \frac{1}{\nu(\bar{A})} \int_{\mu' = \mu_*}^{\mu_*+\epsilon} \frac{d(\mathbb{P}_\mu(E), \mathbb{P}_{\mu'}(E))}{KL(\mu, \mu')} d\nu(\mu)d\nu(\mu') \\
\overset{(ii)}{\geq} \sum_{A \in \mathcal{A}} \frac{\nu(\bar{A})}{\sup_{\mu \in \bar{A}} KL(\mu, \mu_*)} \frac{1}{\nu(\bar{A})} \int_{\mu' = \mu_*}^{\mu_*+\epsilon} \frac{d(\mathbb{P}_\mu(E), \mathbb{P}_{\mu'}(E))}{KL(\mu, \mu')} d\nu(\mu)d\nu(\mu') \\
\overset{(iii)}{\geq} \sum_{A \in \mathcal{A}} \frac{\nu(\bar{A})}{\sup_{\mu \in \bar{A}} KL(\mu, \mu_*)} d \left( \frac{1}{\nu(\bar{A})} \int_{\mu \in \bar{A}} \mathbb{P}_\mu(E) d\nu(\mu), \frac{1}{\nu(\bar{A})} \int_{\mu' \in \bar{A}} \mathbb{P}_{\mu'}(E) d\nu(\mu') \right) \\
\overset{(iv)}{\geq} d \left( \frac{\alpha(1-\pi)}{\kappa}, 1 - \beta \right) \sum_{A \in \mathcal{A}} \frac{\nu(\bar{A})}{\sup_{\mu \in \bar{A}} KL(\mu, \mu_*)}
\]

where (i) – (iv) follow for identical reasons as above.
Index the sets of $A^k$ into $A_1, A_2, \ldots, A_{|A^k|}$ where $\sup_{x \in A_k} x \leq \inf_{y \in A_{k+1}} y$ for all $k$. Recalling that $\sup_{x \in A} x = \inf_{x \in A} x$ for all $A \in A^\circ$ and $\kappa \leq \nu(A) \leq 2\kappa$ for all $A \in A^+$ we have

\[
\sum_{A \in A^+} \frac{\nu(A)}{\sup_{x \in A} KL(\mu, \mu_*)} = \sum_{A \in A^+} \frac{\nu(A)}{\sup_{x \in A} KL(\mu, \mu_*)} + \sum_{A \in A^+} \frac{\nu(A)}{\inf_{x \in A} KL(\mu, \mu_*)} \geq \sum_{k=1}^{|A^+|} \frac{\nu(A_k) / 2}{\sup_{x \in A_k} KL(\mu, \mu_*)} + \sum_{A \in A^+} \frac{\nu(A)}{\inf_{x \in A} KL(\mu, \mu_*)} \geq \sum_{k=1}^{|A^+|} \frac{\nu(A_k) / 2}{\sup_{x \in A_k} KL(\mu, \mu_*)} + \sum_{A \in A^+} \frac{\nu(A)}{\inf_{x \in A} KL(\mu, \mu_*)} = \sum_{k=2}^{|A^+|} \frac{\nu(A_k) / 2}{\sup_{x \in A_k} KL(\mu, \mu_*)} + \sum_{A \in A^+} \frac{\nu(A)}{\inf_{x \in A} KL(\mu, \mu_*)} \geq -\frac{\kappa}{KL(\mu_*, \epsilon, \mu_*)} + \frac{1}{2} \int_{\mu_* + \epsilon}^{\mu_*} KL(\mu, \mu_*) d\nu(\mu),
\]

so that

\[
\int_{\mu = \mu_* + \epsilon}^{\infty} E[\mu[N] d\nu(\mu) \geq \frac{\left(1 - \beta, \frac{\alpha(1 - \pi)}{\kappa} \right)}{\alpha(1 - \pi) + (1 - \beta)\pi} KL(\mu, \mu_*) + \frac{\kappa}{KL(\mu_* + \epsilon, \mu_*)} + \frac{1}{2} \int_{\mu_* + \epsilon}^{\mu_*} KL(\mu, \mu_*) d\nu(\mu).}
\]

\[\square\]

Appendix C. Proof of Theorem 2

**Proof** We plug in the result of Lemma 9 with $\kappa = 2\pi$ into Equation 7 to obtain

\[
E[\sum_{i \geq 1} N_i] \geq \frac{\pi}{\alpha(1 - \pi) + (1 - \beta)\pi} KL(\mu, \mu_*) + \frac{\kappa}{KL(\mu_* + \epsilon, \mu_*)} + \frac{1}{2} \int_{\mu_* + \epsilon}^{\mu_*} KL(\mu, \mu_*) d\nu(\mu).
\]

For the first term we apply the assumption $\frac{1 - \pi}{\pi(1 - \beta)} \leq \frac{\delta}{1 - \beta}$ to obtain

\[
\frac{\pi}{\alpha(1 - \pi) + (1 - \beta)\pi} d(1 - \beta, \frac{\alpha(1 - \pi)}{\kappa} (1 - \beta)) \geq \frac{d(1 - \beta, \frac{\delta(1 - \beta)}{\pi(1 - \beta)} (1 - \beta))}{(1 - \beta)/(1 - \delta)} = (1 - \beta) \log\left(\frac{1 - \beta}{\beta}\right) + \beta \log(\beta) - \beta \log(1 - \frac{\delta(1 - \beta)}{1 - \beta}) \geq (1 - \delta) \log\left(\frac{1 - \beta}{\beta}\right) + (1 - \delta) \frac{\beta}{1 - \beta} \log(\beta) \geq (1 - \delta) \log\left(\frac{1 - \beta}{\beta}\right)
\]

using the fact that $-\frac{\beta}{1 - \beta} \log(\beta) \in (0, 1)$ for $\beta \in (0, 1)$. 

20
For the second term we apply the assumption again to get

$$\frac{d}{\alpha(1-\pi) + (1-\beta)\pi} = \frac{d}{\alpha(1-\pi) + (1-\beta)\pi} \geq \frac{d}{1-\beta\pi/(1-\delta)} \geq \frac{1-\delta}{\pi} \log(\frac{\pi}{\kappa}) + \frac{1-\delta}{\pi} \left( 1 - \frac{\pi}{\kappa} (1-\beta) \right) \log(\frac{1}{1-\beta} + \log(\frac{1-\delta}{\pi}))$$

where the last lines use $\kappa = 2\pi$, $\log(1/\beta) \geq 1$ for all $\beta \in (0, 1)$, $\log(1-x) \geq -2x$ for $x \in (0, 1/2)$, and $\delta \in (0, 12)$. Thus, putting the pieces together we obtain

$$\mathbb{E} \sum_{i \geq 1} N_i \geq \frac{(1-\delta) \log((1-\delta)/\kappa)}{KL(\mu_*, \mu)} + \frac{1-\delta}{2} \log(\frac{1-\delta}{\pi}) \left( 1 - \frac{\kappa}{KL(\mu_*, \mu)} \right) + \frac{1}{2} \int_{\mu_* + \epsilon} \frac{1}{KL(\mu, \mu_*)} d\nu(\mu)$$

Finally, $1 \geq (1-\delta) \log((1-\delta)/\kappa) \geq 3/4$ for $\delta \leq 1/15$.

**Appendix D. Empirical study**

Here we present further results in addition to what we provided in Section 4. We start with describing in some detail the baselines we used and then present further experiments on various reservoirs.

**Pure exploration algorithms.** Our first batch of baselines are pure exploration algorithms.

We introduce the algorithm we call Chernoff, meant as a strong baseline for ISHA. This algorithm has knowledge of $\mu_*$. We define the algorithm in terms of the confidence lower bound $L_i(t) = \min\{q \in [0, \hat{\mu}_i] : N_i(t) KL(\hat{\mu}_i(t), q) \leq \log(1/\delta_i)\}$ where $t$ is the budget used so far, $N_i(t)$ is the number of times arm $i$ is pulled, and $\delta_i = \frac{1}{2N_i(t)}$ for $\delta = 0.1$ so that the overall error tolerance for an arm $i$ is $\delta$. This algorithm draws an arm from the reservoir and continues drawing rewards from it until $L_i(t) > \mu_*$. Once its confidence interval no longer contains $\mu_*$, the arm is discarded and a new arm is drawn. At time $T$ the algorithm stops and returns the arm that was sampled the most.

SIRI (Carpentier and Valko, 2015) is a recent UCB-style pure exploration algorithm in the fixed budget setting. It is based on a $\beta$-regularity assumption for the tail of the reservoir, and so makes use of additional information about the reservoir. We run this with $\delta = 0.1$.

lil’UCB (Jamieson et al., 2014), another UCB algorithm, is a fixed confidence pure exploration algorithm for finite bandits. While the original lil’UCB has its own stopping criterion, in our experiments it stops when it runs out of budget. For consistency, we use the $L_i(t)$ defined for Chernoff with the same $\delta$ value and schedule used therein.
We also run Hyperband (Li et al., 2017), described earlier. Its parameter $\eta$ decides which fraction of arms to discard in a given round of Successive Halving. We used $\eta = 2$ (discarding half the arms) in keeping with Successive Halving.

Successive Rejects (Audibert and Bubeck 2010), a fixed budget algorithm originally intended for finite bandits, is similar to Successive Halving. We run it with $n^*_T$ arms.

**Explore-vs-Exploit algorithms.** We use regret minimization infinite bandit algorithms as our next batch of baselines. Note that these algorithms are not natural competitors and do not have an arm recommendation strategy. At time $T$ we stop and return the arm that was sampled the most.

We experiment with four infinite bandit algorithms of Berry et al. (1997) designed for $Beta(1,1)$ with support on $[0,1]$: $f$-failure strategy (with $f = 1$) which samples an arm until $f$ failures are observed, $s$-run strategy (with $s = \sqrt{T}$) which samples each arm at most $s$ times until a failure is observed and then exploits the most successful arm until the budget is exhausted, $s$-run strategy (non-recall; $s = \sqrt{T}$) which samples from an arm until the first failure but exploits the arm until the budget is exhausted if $s$ successes are observed, and $m$-learning strategy (with $m = \log(T)\sqrt{T}$) which plays the $f$-failure strategy (with $f = 1$) for the first $m$ times (the arm at time $m$ is exploited until a failure is observed). From there the empirically-best arm is exploited.

CBT and Empirical CBT Chan and Hu (2018) are also used as baselines. CBT takes as input a target mean parameter $\mu_{target} = \sqrt{2}/T$ for reservoir $Beta(1,1)$, and $\mu_{target} = 4\sqrt{4}/T$ for $Beta(3,1)$ and thus assumes knowledge of the reservoir. Empirical CBT does not assume anything about the reservoir but instead estimates $\mu_{target}$.

Two Target (fixed horizon) Bonald and Proutiere (2013) uses two success target thresholds $l_1$ and $l_2$ as function of the reservoir parameters $\beta$ and $\alpha$, and budget $T$. Any arm which fails to have $l_1$ successes before its first failure is discarded, and any arm which has $l_2$ successes before its first $m$ failures is subsequently exploited until the budget is exhausted. We use $m = 3$ in our experiments.

**Anytime algorithms.** We also propose the following Anytime version of ISHA and compare it to other anytime algorithms. When an effectively unlimited budget is available, ISHA Anytime can be run as follows. Choose increasing dyadic numbers of arms, $n = 2^i, i = 1, 2, \ldots$. For each value of $n$, sample $n$ new arms and run ISHA and save the result as the best arm found so far. We compare this algorithm to Hyperband Anytime and SIRI Anytime (using the budget doubling trick as proposed by the SIRI authors).

**D.1 Experiments and Insights**

**Successive Halving performance as a function of the number of arms for a fixed budget.** The class of experiments in Figure 4 reports the average simple regret for ISHA (Successive Halving using the maximum number of arms s.t. $T \geq n^*_T \log_2(n^*_T)$) in comparison to various smaller choices of $n$. Across a variety of reservoirs, $n^*_T$ arms is always a good choice for Successive Halving in the infinite setting.

**Simple regret vs. Budget.** In the class of experiments in Figure 5 we compare the simple regret of ISHA to that of various pure exploration baselines.

Figure 6 highlights the difficulty of choosing the optimal number of arms for UCB-style algorithms, such as the lil’UCB.

Figure 7 compares mainly against various exploration-vs-exploitation baselines, some of which were specifically designed specifically for $Beta$ reservoirs.

Finally, in Figure 8, we compare ISHA and ISHA Anytime to several other anytime algorithms.
Figure 4: Impact of the number of arms for a fixed budget and reservoir.

(a) Beta(1, 1)

(b) Beta(1, 1) Scaled

(c) Beta(3, 1)

(d) Beta(3, 1) Scaled

(e) TwoSpike $\pi = 10^{-1}, \epsilon = \sqrt{10^{-1}}$

(f) Two Spike $\pi = 10^{-2}, \epsilon = \sqrt{10^{-2}}$

(g) Two Spike $\pi = 10^{-3}, \epsilon = \sqrt{10^{-1}}$

(h) New Yorker
Figure 5: Comparison to state-of-the-art pure exploration infinite bandit algorithms

(a) Beta(1, 1)  
(b) Beta(1, 1) Scaled  
(c) Beta(3, 1)  
(d) Beta(3, 1) Scaled  
(e) TwoSpike \( \pi = 10^{-1}, \epsilon = \sqrt{10^{-3}} \)  
(f) Two Spike \( \pi = 10^{-2}, \epsilon = \sqrt{10^{-2}} \)  
(g) Two Spike \( \pi = 10^{-3}, \epsilon = \sqrt{10^{-1}} \)  
(h) New Yorker
Figure 6: Comparison to lil’UCB

(a) Beta(1, 1)
(b) Beta(1, 1) Scaled
(c) Beta(3, 1)
(d) Beta(3, 1) Scaled

Figure 7: Comparison to state-of-the-art explore-vs-exploit infinite bandit algorithms

(a) Beta(1, 1)
(b) Beta(3, 1)
Figure 8: Anytime Performance

(a) \( \beta(1,1) \)

(b) \( \beta(1,1) \) Scaled

(c) \( \beta(3,1) \)

(d) \( \beta(3,1) \) Scaled
Figure 9: New Yorker CDF