The Probabilistic Normal Epipolar Constraint for Frame-To-Frame Rotation Optimization under Uncertain Feature Positions

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Abstract

The estimation of the relative pose of two camera views is a fundamental problem in computer vision. Kneip et al. proposed to solve this problem by introducing the normal epipolar constraint (NEC). However, their approach does not take into account uncertainties, so that the accuracy of the estimated relative pose is highly dependent on accurate feature positions in the target frame. In this work, we introduce the probabilistic normal epipolar constraint (PNEC) that overcomes this limitation by accounting for anisotropic and inhomogeneous uncertainties in the feature positions. To this end, we propose a novel objective function, along with an efficient optimization scheme that effectively minimizes our objective while maintaining real-time performance. In experiments on synthetic data, we demonstrate that the novel PNEC yields more accurate rotation estimates than the original NEC and several popular relative rotation estimation algorithms. Furthermore, we integrate the proposed method into a state-of-the-art monocular rotation-only odometry system and achieve consistently improved results for the real-world KITTI dataset.

1. Introduction

Extracting the 3D geometry of a scene from images is a long-standing problem in computer vision and has numerous applications, including augmented and virtual reality, autonomous driving, or robots that can help with everyday life. One key component of many such approaches is the estimation of the relative pose between two viewpoints of a scene. For example, relative pose estimation is the foundation of geometric vision algorithms like structure from motion (SfM) or visual odometry (VO). Global SfM pipelines rely on accurate pairwise relative poses for use as fixed measurements in global motion averaging [23, 48]. In VO, relative pose estimation is used to construct a trajectory from a stream of images. Like for all odometry systems, small errors in the relative pose estimation lead to a drift in VO.

The most widely used concept for relative pose estimation is the essential matrix [43] in the calibrated case, or the fundamental matrix [25] in the general case. Respective approaches rely on correspondences between feature points, and are generally known to provide fast and accurate results [24]. However, approaches based on the essential matrix suffer from fundamental problems, with the most prominent being solution multiplicity [17, 25] and planar degeneracy [33]. To address such issues, often it is necessary to consider more involved solution strategies, which also lead to even more accurate relative poses as shown by...
Kneip et al. [33] and in this work.

To this end, Kneip et al. [34] proposed a constraint that avoids these problems. Their epipolar plane normal epipolar parallelity constraint (NEC) allows the estimation of the rotation independent of the translation. A later work by Kneip and Lyyen [33] provides a fast and reliable eigenvalue-based solver for the NEC, which allows for real-time relative pose estimation. This approach has been incorporated into rotation-only VO systems that estimate the rotation independent of the translation and has led to promising results [9, 39].

Yet, like many VO systems, neither of them considers the quality of the correspondences. After removing outliers from the feature matches, every match contributes equally to the final result. However, two-dimensional feature correspondences exhibit different error distributions depending on the content of the image and the specific method used to extract the correspondences, which can be seen in Fig. 2. A correspondence lying on an edge is accurately localized perpendicular to the edge and possesses higher position uncertainty parallel to it. This fine-grained information about the quality of the matches is completely ignored. It has been shown that considering the uncertainty is beneficial for fundamental matrix estimation [5]. While Kanazawa et al. [30] argue that the uncertainty needs to be sufficiently inhomogeneous to see the aforementioned benefit, our experiments show that the PNEC improves over the NEC even for homogeneous uncertainty due to the geometry of the problem.

The main objective of our work is to improve the accuracy of rotation estimation techniques. We achieve this based on the following technical contributions:

- We introduce the novel probabilistic normal epipolar constraint (PNEC), see Fig. 1, which for the first time makes it possible to incorporate uncertainty information into the normal epipolar constraint (NEC).
- We propose an efficient two-stage optimization strategy for the PNEC that achieves real-time performance.
- We analyse singularities in the PNEC energy function and address them with a simple regularization scheme.
- Experimentally, we compare our PNEC to several popular relative pose estimation algorithms, namely 8pt [24], 7pt [25], Stewenius 5pt [56], Nistér 5pt [51], and NEC [33], and demonstrate that our PNEC delivers more accurate rotation estimates. Moreover, we integrate our PNEC into a visual odometry system and achieve state-of-the-art results on real-world data.
- We publish the code for all experiments to facilitate future research.

2. Related Work

The focus of this paper is the integration of feature position uncertainties into frame-to-frame rotation estimation and the application to visual odometry. Hence, we restrict our discussion of related work to relative pose estimation, uncertainty for feature correspondences, and visual odometry. For a broader overview we refer the reader to the excellent books by Szeliski [57] and by Hartley and Zisserman [25] and to more topic-specific overview papers [6, 59].

Relative Pose Estimation. Estimating the relative pose between two viewpoints is a long-standing problem in computer vision with the first known solution proposed in 1913 by Krupa [35]. Most methods either rely on previously computed feature correspondences (feature-based) or directly consider the intensity differences between the two images (direct). While direct methods have recently shown promising results [14, 15], they are currently limited to images that exhibit photo consistency and hence cannot be used for general problems, e.g. structure from motion. Feature-based methods are considerably more robust to viewpoint and appearance changes. Therefore, we use feature correspondences within this paper.

Given feature correspondences, many methods [36, 41, 43, 51, 55] estimate the essential matrix in the case of a calibrated camera, or the fundamental matrix in the general case. Nistér [51] proposes a minimal solution using polynomials and root bracketing, while the solver proposed by Longuet-Higgins [43] is linear and requires careful normalization for good performance [24]. Alternatively, the relative pose can be estimated directly using quaternions [16].

The essential matrix constraint deteriorates in zero-translation situations without noise, due to it being a zero matrix. Most essential-matrix-based algorithms estimate the correct motion only implicitly [33]. To address this problem, recent works have proposed algorithms that can estimate the rotation independent of the translation [34, 42]. Our work is based on the normal epipolar constraint (NEC) proposed by Kneip et al. [34] and the direct optimization scheme proposed in a follow-up paper [33]. Briales et al. [4] show how to obtain the global minimum for the NEC, however, their Shor relaxation is not applicable to our non-polynomial energy function.

Uncertainty for Feature Correspondences. Kanade-Lucas-Tomasi (KLT) tracks [45, 58] are widely used, and the position uncertainty has been extensively investigated [19, 53, 54, 65]. Based on the unscented transform [60], the position uncertainty has also been integrated directly into the KLT tracking [13]. Zeisl et al. [64] have shown a method to obtain anisotropic and inhomogeneous covariances for SIFT [44] and SURF [1] features.

The integration of the position uncertainty into the alignment problem has been studied from a statistical perspective [28, 29], in the photogrammetry community [47], as well as in the computer vision community [5, 30]. Brooks et al. [5] show that covariance information can be used beneficially if the estimated covariance is sufficiently accurate. Kanazawa et al. [30] question the practical use of covari-
the epipolar plane normal vectors. To address this we cannot simply discard feature points. To address this sufficiently constrain the 3D geometry \cite{14, 49} and thus that correspondences in all areas of the image are required of the position distributions. Moreover, it is well-known Fig. 2 clearly shows the anisotropy and inhomogeneity parallel to it, which is also known as the aperture problem. Feature correspondences are given by pairs of unit bearing vectors $f_i$ and $f'_i$ in the host frame ($O$) and target frame ($O'$), respectively. Each pair of bearing vectors spans an epipolar plane (yellow, orange, red), and has an associated normal vector $n_i$, given in Eq. 1. All epipolar planes intersect in the line defined by the translation $t$ (dashed line). The normal vectors span the epipolar normal plane (gray) that is orthogonal to $t$. For visual clarity we show only three feature correspondences.

problem, we propose the probabilistic normal epipolar constraint (PNEC), which is able to take into account the uncertainty of feature positions by associating an anisotropic covariance matrix to each feature point.

Notation. Vectors are denoted by bold lowercase letters (e.g. $f$) and matrices by bold uppercase letters (e.g. $\Sigma$). The hat operator applied to a vector $u \in \mathbb{R}^3$ gives a skew-symmetric matrix $\hat{u} \in \mathbb{R}^{3 \times 3}$ that computes the cross product between two vectors, i.e. $u \times v = \hat{u}v$. The superscript $\top$ denotes the transpose. A rigid-body transformation is represented by a rotation matrix $R \in SO(3)$ and a unit length translation $t \in \mathbb{R}^3$ ($\|t\| = 1$ is imposed since the two-view problem is scale-invariant).

3.1. Background – NEC

In the following we summarize the main idea of the NEC proposed in \cite{34}. Given are a host frame and a target frame that observe at least five feature correspondences that are defined by pairs of unit bearing vectors $f_i$ and $f'_i$ in the host and target frame, respectively (see Fig. 3). A 3D point $x'$ in the target frame is transformed into the host frame by applying the relative rotation $R$ and translation $t$ s.t. $x = Rx' + t$. In the ideal, error-free case, a single feature correspondence, together with the two viewpoints, creates an epipolar plane, represented by its normal vector

$$n_i = f_i \times Rf'_i. \quad (1)$$

All normal vectors are orthogonal to the translation and they span the epipolar normal plane.

The rotation is estimated by enforcing the coplanarity of the normal vectors. The residual of the model is given by

$$\sum_{i=1}^{5} \left( n_i \cdot f'_i \right) = 0.$$
the normalized epipolar error
\[ e_i = |t^T n_i|, \] (2)
i.e. the Euclidean distance of a normal vector to the epipolar normal plane. An energy function
\[ E(R, t) = \sum_i e_i^2 = \sum_i |t^T (f_i \times R f'_i)|^2 \] (3)
is constructed from the residuals. For a more detailed derivation, we kindly refer the reader to the original paper [34] or the recent paper by Lee and Civera [38], which offers numerous geometric interpretations of the NEC.

### 3.2. Deriving the PNEC

The probabilistic normal epipolar constraint (PNEC) extends the NEC by incorporating uncertainty. To be more specific, the PNEC allows the use of the anisotropic and inhomogeneous nature of the uncertainty of the feature position in the energy function. The feature position error is considered in the target frame as shown in Fig. 1 and we assume that the position error follows a 2D Gaussian distribution in the image plane with a known covariance matrix \( \Sigma_{2D,i} \) per feature. In the supplementary we show how the covariance matrix can be extracted for KLT tracks from the KLT energy function using Laplace’s approximation [3].

Given the 2D covariance matrix of the feature position in the target frame \( \Sigma_{2D,i} \), we propagate it through the unprojection function using the unscented transform [60] in order to obtain the 3D covariance matrix \( \Sigma_i \) of the bearing vector \( f'_i \). Using the unscented transform ensures full-rank covariance matrices after the transform. We derive the details of the unscented transform in the supplementary material and show qualitative examples.

Propagating this distribution to the normalized epipolar error gives the probabilistic distribution of the residual. Due to the linearity of the transformations, the distribution of the residual is a univariate Gaussian distribution \( \mathcal{N}(0, \sigma_i^2) \), with variance
\[ \sigma_i^2(R, t) = t^T f_i \Sigma_i R^T f'_i t. \] (4)

We integrate this variance into the cost-function of the NEC so that the Euclidean distance becomes the Mahalanobis distance [8] and define the **PNEC Energy Function**
\[ E_P(R, t) = \sum_i \frac{e_i^2}{\sigma_i^2} = \sum_i \frac{|t^T (f_i \times R f'_i)|^2}{t^T f_i \Sigma_i R^T f'_i t}, \] (5)
which results in a weighted optimization problem. In the supplementary material we show a geometric interpretation of the above derivation.

#### 4. Optimization

To optimize the PNEC energy function Eq. 5, we propose a two-stage optimization scheme consisting of an alternating iterative optimization and a joint refinement. We propose this two-stage approach since the eigenvalue-based optimization of the NEC [33] cannot be naively applied to our derived PNEC energy function, which we show in Sec. 4.1. The whole PNEC optimization is given in Alg. 1 and we detail the first stage in Sec. 4.2 & Sec. 4.3, and the second stage in Sec. 4.4.

#### 4.1. Background – Optimizing the NEC

Following [33], the NEC energy function Eq. 3 is rewritten as \( E(R, t) = t^T M(R) t \) using the (symmetric and positive-semidefinite) Gramian matrix
\[ M(R) = \sum_i (f_i \times R f'_i)(f_i \times R f'_i)^T. \] (6)

Because the energy is a quadratic form in the unit vector \( t \), the optimization over the translation \( t \) can be carried out analytically, i.e.
\[ \min_{R \in SO(3)} t^T M(R) t = \min_{R \in SO(3)} \lambda_{\text{min}}(M(R)). \] (7)
The eigenvector corresponding to the smallest eigenvalue \( \lambda_{\text{min}} \) of \( M(R) \) minimizes the Rayleigh quotient \( t^T M(R) t \) over all unit-length vectors \( t \). The constructed sub-problem is then optimized over the rotation \( R \) using the Levenberg-Marquardt algorithm [40, 46], whereas the translation \( t \) is obtained by solving an eigenvalue problem.

#### 4.2. Optimizing the PNEC - Translation

The PNEC energy function Eq. 5 is the sum of generalized Rayleigh quotients (GRQs) in the translation \( t \), and thus the optimum is not simply given by an eigenvalue as for the NEC. Optimizing the sum of GRQs over
the unit sphere has recently been studied in the context of data science and wireless communications [2, 66, 67], and it has been shown by Zhang et al. [67] that the self-consistent-field (SCF) algorithm [26] outperforms generic manifold optimization methods.

Since the sum of GRQs can exhibit many local minima [2], and thus the SCF iteration is not guaranteed to converge to a global optimum, we propose a simple, yet effective globalization strategy. To this end, we make use of the intrinsic low dimension of the unit sphere in $\mathbb{R}^3$ by sampling evenly distributed initial points $t_k$ efficiently using the Fibonacci lattice [21]. We then pick the point with the lowest objective function and apply the SCF iteration for $N$ steps. Due to the inherent parallelism, the resulting optimization procedure can be implemented efficiently. We present the effectiveness of the globalization strategy, as well as technical details for the SCF iteration, in the supplementary material.

4.3. Optimizing the PNEC - Rotation

Kneip and Lynen [33] have shown how to optimize Eq. 7 efficiently using the Levenberg-Marquardt algorithm with the rotation parametrized based on the Cayley transformation [7]. To account for the weights in the PNEC energy function Eq. 5, we employ an optimization scheme similar to the popular iteratively reweighted least squares (IRLS) algorithm [37]. Specifically, given a previous estimate of the rotation and translation $(R_p, t_p)$, we compute fixed weights $\tilde{\sigma}_i = \sigma_i(R_p, t_p)$ for all $i$, and define the weighted matrix

$$M_P(R, \{\tilde{\sigma}_i\}) = \sum_i \frac{(f_i \times Rf'_i)(f_i \times Rf'_i)^\top}{\tilde{\sigma}_i}$$

(8)

that depends only on the rotation $R$. The rotation is obtained by finding $R$ such that the smallest eigenvalue of $M_P(R, \{\tilde{\sigma}_i\})$ is minimal. After doing so based on the optimizer of Kneip and Lynen [33], the weights $\{\tilde{\sigma}_i\}$ are updated with new $R$, $t$.

4.4. Optimizing the PNEC - Joint Refinement

After the first stage we improve the result using joint refinement. Specifically, we use a least-squares optimization strategy, which is effective for finding a local optimum of the energy function given a good starting point [52]. For the PNEC we optimize over

$$E_P(R, t) = \sum_i \left(\frac{t^\top (f_i \times Rf'_i)}{\sqrt{t^\top f_i R^\top 2 \Sigma R f_i^\top t}}\right)^2,$$

(9)

the least-squares formulation of the constraint. The Levenberg-Marquardt algorithm optimizes the objective function in the rotation $R$ and translation $t$ simultaneously and uses the solution of the first stage as the starting value. Because $R$ is a rotation matrix, we use manifold optimization [27] to optimize over the special orthogonal group $SO(3)$. For the translation $t$ we use spherical coordinates with the radius fixed to one in order to ensure that $|t| = 1$ holds. We would like to highlight that this joint refinement is different from bundle adjustment. Most notably, it does not need to calculate the 3D position of the features.

4.5. Singularities of the PNEC

The PNEC energy function Eq. 5 has a singularity if the translation $t$ is parallel to a bearing vector $f_i$ because the variance $\sigma_i^2$ vanishes due to $f_i t = f_i \times t = 0$. On the other hand, the numerator involves the same term and thus the energy function is bounded and possesses a finite discontinuity, as illustrated in Fig. 4. In the supplementary material we present the derivation of the directional limit of the energy function.

While the discontinuity is finite and less problematic than an infinite discontinuity, it still poses challenges. First, in contrast to the function values, the derivatives of the energy function are not bounded, which is problematic for the joint refinement. Second, the matrix $M_P$, unlike the energy function, includes $f_i t$ only in its denominator not the numerator. Hence $M_P$ tends to infinity for $t \to f_i$. To address these issues, we consider a variance of the form $\sigma_i^2 = \sigma^2_1 + c$ with regularization constant $c > 0$. Fig. 4 shows the effect that the regularizer has on the energy function.

5. Evaluation

We evaluate the performance of the PNEC and compare it to the original NEC on simulated data as well as in a visual odometry setting on real world data. On the simulated data, the proposed PNEC achieves better results than the

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Figure 4. Visualization of the NEC, the PNEC, and the PNEC with the regularization proposed in Sec. 4.5. The plot shows $t$ in a neighborhood of $f$ (in polar coordinates), where the center of the circle corresponds to $t = f$. For $t = f$, the PNEC shows a finite discontinuity, for which the limit depends on the direction. Our regularization eliminates this singularity while also maintaining the overall shape of the energy function.
Figure 5. Experiments for omnidirectional cameras. Results are averaged over 10 000 random instantiations for anisotropic inhomogeneous noise over different noise intensities. Our PNEC consistently leads to smaller errors compared to the NEC [33] for all noise levels. This holds for rotation and translation estimates in the general case in Fig. 5a and Fig. 5b, respectively, as well as the rotation in the zero-translation case in Fig. 5c.

Figure 6. Experiments for pinhole cameras. Results are averaged over 10 000 random instantiations for anisotropic inhomogeneous noise over different noise intensities. As for an omnidirectional camera in Fig. 5, our PNEC consistently leads to smaller errors compared to the NEC [33]. For a pinhole camera the average errors are higher for both methods in comparison to an omnidirectional camera. The experiments show that our PNEC is viable for the two most common camera types.

5.1. Frame-to-Frame Simulation

With the simulated experiments we evaluate the performance of the PNEC in a frame-to-frame setting. The experiments consist of randomly generated problems of two frames with known correspondences. We use

\[ e_{\text{rot}} := \angle(R^T \hat{R}), \quad \text{and} \]

\[ e_t := \arccos(t^T \hat{t}) \]

as error metrics between the ground truth \( R, t \) and the estimated values \( \hat{R}, \hat{t} \), where \( \angle(\cdot) \) returns the angle of the rotation matrix.

**Omnidirectional Camera.** In this experiment we follow the experimental outline proposed by Kneip and Lynen [33] closely. We differ from the original experiments in the following ways: we only add noise to the points in the second frame; to compensate for the lack of noise in the first frame, we scale the standard deviation by a factor of 2; we recreate the experiment with different noise types based on the classification by Brooks et al. [5] and generate individual covariance matrices for each point. A detailed description of how the matrices are generated can be found in the supplementary material. To show the effectiveness of the PNEC still holds even for pure rotation, we repeat the experiment with the translational difference fixed to zero.

Fig. 5 shows the results for anisotropic inhomogeneous noise for both experiments. The PNEC achieves consistently better results for the rotation over all noise levels in both experiments.

**Pinhole Camera.** Since most cameras are modeled as pinhole cameras we also repeat the previous experiments for pinhole cameras. The generation of the frames stays the same. Points are sampled in viewing direction of the coordi-
Relative rotation estimation. To capture the effects of both features. Second, we replace the NEC with our PNEC for also used in [61] to extract feature keypoints instead of ORB estimation. Our approach differs from MRO in two ways.

5.2. Visual Odometry

Besides the simulated experiments, we also validate the PNEC on real world data, namely the highly popular KITTI odometry dataset [20]. We compare our results with the MRO algorithm by Chng et al. [9] that uses the optimization from [33]. For MRO and our algorithm we disable rotation averaging and loop closure to focus on local rotation estimation. Our approach differs from MRO in two ways.

First, we use the KLT-based tracking implementation also used in [61] to extract feature keypoints instead of ORB features. Second, we replace the NEC with our PNEC for relative rotation estimation. To capture the effects of both changes, we compare the rotation estimation of MRO, as reported in [9], KLT-NEC, using KLT tracks and the NEC, and KLT-PNEC, the proposed PNEC with KLT tracks. Both KLT-NEC and KLT-PNEC use the same KLT tracks for the relative rotation estimation.

The proposed PNEC can account for uncertainties in the feature correspondence positions that approximately follow a Gaussian distribution. To overcome outlier correspondences from failed KLT tracks, we use the same RANSAC [18] routine as the NEC for estimating the rotation in the first loop of Alg. 1.

Fig. 7 shows a trajectory generated from the rotation estimates of MRO and our approach. In Tab. 2 we compare the mean performance over 5 runs of all approaches in the rotation-only version of the Relative Pose Error (RPE) for n camera poses as defined in [9]. The RPE evaluates the root mean square error (RMSE) of rotational residuals over frame pairs. The residual for a “time-step” Δ is

$$E_i := \angle((R_i^\top R_{i+\Delta})^\top (R_i^\top R_{i+\Delta})).$$

(12)

The RMSE is calculated over m := n − Δ residuals

$$\text{RMSE}(\Delta) := \left(\frac{1}{m} \sum_{i=1}^{m} E_i^2\right)^{\frac{1}{2}}.$$  

(13)

For our evaluation we use

$$\text{RPE}_1 := \text{RMSE}(1),$$

and

$$\text{RPE}_n := \frac{1}{n} \sum_{\Delta=1}^{n} \text{RMSE}(\Delta),$$

(15)

to capture local frame-to-frame rotation error and long term drift, respectively.

The results show the following: With a single exception (seq. 01), using KLT tracks instead of ORB features is beneficial for relative rotation estimation with the NEC. PNEC

| Noise level [px] | w/T | w/O T | w/T | w/O T | w/T | w/O T |
|------------------|-----|-------|-----|-------|-----|-------|
| Metric [degree]  | e_{rot} & e_\text{t} & e_{rot} & e_\text{t} & e_{rot} & e_\text{t} & e_{rot} & e_\text{t} & e_{rot} & e_\text{t} |
| 7pt (25)/8pt (24) | 0.19 & 0.1 & 0.26 & 0.33 & 0.07 & 0.15 & 0.06 & 0.15 & 0.42 & 0.29 & 0.29 |
| 8pt (25)/9pt (24) | 0.11 & 0.1 & 0.15 & 0.21 & 0.07 & 0.15 & 0.06 & 0.15 & 0.42 & 0.29 & 0.29 |
| 9pt (25)/10pt (24) | 0.11 & 0.1 & 0.15 & 0.21 & 0.07 & 0.15 & 0.06 & 0.15 & 0.42 & 0.29 & 0.29 |
| 10pt (25)/11pt (24) | 0.11 & 0.1 & 0.15 & 0.21 & 0.07 & 0.15 & 0.06 & 0.15 & 0.42 & 0.29 & 0.29 |

Table 1. Rotation and translation error for different algorithms. Results for omnidirectional and pinhole cameras for experiments with and without translation over different noise levels for anisotropic and inhomogenous noise. Errors are averaged over 10 000 random problems each with 10 points. For experiments without translation (W/O T) only e_{rot} is reported, due to e_\text{t} not being defined for zero translation. For all other algorithms apart from our PNEC we use the implementations from OpenGV [32]. (7pt) falls back to (8pt) for the non-minimal number of 10 correspondences. Our PNEC consistently achieves the best results, outperforming the NEC and several popular relative pose estimation algorithms. The last column gives the average error increase compared to the PNEC.
Figure 7. Qualitative trajectory comparison for KITTI seq. 08. The trajectory was generated with the estimated rotations of MRO [9] and PNEC, respectively, and are combined with the ground truth translations for visualization purposes. Relative rotations computed with the proposed PNEC lead to a significantly reduced drift.

| Seq. | MRO [9] RPE 1 | KLT-NEC RPE 1 | KLT-PNEC RPE 1 | KLT-PNEC (Ours) RPE 1 |
|------|---------------|---------------|-----------------|-----------------------|
| 00   | 0.360         | 8.670         | 0.125           | 5.922                 |
| 01*  | **0.290**     | **16.030**    | **0.093**       | **6.693**             |
| 02   | 0.290         | 16.030        | 0.122           | 9.687                 |
| 03   | 0.280         | 5.470         | 0.073           | 2.728                 |
| 04   | 0.040         | 1.080         | 0.041           | 0.619                 |
| 05   | 0.250         | 11.360        | 0.079           | 4.489                 |
| 06   | 0.180         | 4.720         | 0.073           | 3.162                 |
| 07   | 0.280         | 7.490         | 0.105           | 4.640                 |
| 08   | 0.270         | 9.210         | 0.070           | 5.523                 |
| 09   | 0.280         | 9.850         | 0.088           | 3.533                 |
| 10   | 0.380         | 13.250        | 0.073           | 3.959                 |

Table 3. Average frame processing time in milliseconds. For MRO, most of the time is needed for matching. KLT-NEC and KLT-PNEC (Ours) achieve real-time performance on KITTI.

### 6. Discussion and Future Work

While the proposed optimization scheme effectively optimizes the PNEC energy function, it relies on two consecutive stages, and is thus more involved than the optimization scheme proposed for the NEC [33]. Further and more detailed limitations of the proposed approach are given in the supplementary material. Nevertheless, we have shown in Sec. 5.3 that the proposed algorithm is real-time capable. As we explain in Sec. 4.2, the optimization over the translation alone is an actively studied problem for which no simple solution is known. Nevertheless, investigating improved optimization schemes for our PNEC energy function is a promising direction for future work. Recent works have shown that deep learning can boost the performance of visual odometry algorithms [22,62,63]. However, the focus of our work is on the correct modelling of the uncertainty for relative pose estimation, similar to [33,34].

### 7. Conclusion

This paper shows how to utilise 2D feature position uncertainties to obtain more accurate relative pose estimates from a pair of images. To this end, we introduce the probabilistic normal epipolar constraint (PNEC), and we propose an effective optimization scheme that runs in real-time. In synthetic experiments, the PNEC gives more accurate rotation estimates than the NEC and several popular relative rotation estimation algorithms for different noise levels and for the pure-rotation case. The results on KITTI show that the relative rotation estimation of the PNEC improves upon the NEC-based MRO, a state-of-the-art rotation-only VO system, and can be used e.g. for global initialization in SfM.

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