A convergence result of the Lagrangian mean curvature flow

Mu-Tao Wang *

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email: mtwang@math.columbia.edu

Abstract

We prove the mean curvature flow of the graph of a symplectomorphism between Riemann surfaces converges smoothly as time approaches infinity.

1 Introduction

Let $\Sigma_1$ and $\Sigma_2$ be two homeomorphic compact Riemann surfaces without boundary. We assume $\Sigma_1$ and $\Sigma_2$ are both equipped with Riemannian metrics of the same constant curvature $c$, $c = -1, 0, \text{ or } 1$. Let $\omega_1$ and $\omega_2$ denote the volume or symplectic forms of $\Sigma_1$ and $\Sigma_2$, respectively. The Riemannian product space $\Sigma_1 \times \Sigma_2$ is denoted by $M$. We take $\omega' = \omega_1 - \omega_2$ to be the Kähler form of $M$ and $M$ becomes a Kähler-Einstein manifold with the Ricci form $\text{Ric} = c\omega'$. Let $\Sigma$ be the graph of a symplectomorphism $f : \Sigma_1 \to \Sigma_2$, i.e. $f^*\omega_2 = \omega_1$. $\Sigma$ can be considered as a Lagrangian submanifold with respect to the symplectic form $\omega'$.

The mean curvature flow deforms the initial surface $\Sigma^0 = \Sigma$ in the direction of its mean curvature vector. Denote by $\Sigma^t$ the time slice of the flow at $t$. That $\Sigma^t$ remains a Lagrangian submanifold follows from a result of Smoczyk [9]. The long-time existence and convergence problems of this flow were studied in [10] and [12].

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In [12], the author proved the long time existence of the flow and showed that $\Sigma^t$ for $t > 0$ remains the graph of a symplectomorphism $f_t$. When $c = 1$, the author proved the $C^\infty$ convergence as $t \to \infty$. However, only $C^0$ convergence was achieved in the case when $c = -1$ or 0.

Independently, in [10], Smoczyk studied the case when $c = -1$ or 0 assuming an extra angle condition. He discovered a curvature estimate and showed that the second fundamental form is uniformly bounded under this condition, and thus established the long time existence and $C^\infty$ convergence at infinity.

In view of the above results, it is interesting to see whether the $C^\infty$ convergence of the flow does require the angle condition. In this paper, we show this assumption is unnecessary.

**Theorem 1.1** Let $(\Sigma_1, \omega_1)$ and $(\Sigma_2, \omega_2)$ be two homeomorphic compact Riemann surface of the same constant curvature $c = -1, 0, \text{or } 1$. Suppose $\Sigma$ is the graph of a symplectomorphism $f : \Sigma_1 \to \Sigma_2$ as a Lagrangian submanifold of $M = (\Sigma_1 \times \Sigma_2, \omega_1 - \omega_2)$ and $\Sigma^t$ is the mean curvature flow with initial surface $\Sigma^0 = \Sigma$. Then $\Sigma^t$ remains the graph of a symplectomorphism $f_t$ along the mean curvature flow. The flow exists smoothly for all time and $\Sigma^t$ converges smoothly to a minimal Lagrangian submanifold as $t \to \infty$.

The long time existence part was already proved in [12]. The smooth convergence was established through a new integral estimate (Lemma 3.1) related to the second variation formula. This estimate is most useful when $c = -1$ or 0. We remark the existence of such minimal Lagrangian submanifold was proved using variational method by Schoen [7] (see also Lee [5]).

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## 2 Background material

First we recall some formulas from [12]. The restriction of the Kähler form $\omega'$ to $\Sigma'$ gives a time-dependent function $\eta = *\omega'$. Since $\Sigma'$ is Lagrangian, $*\omega' = 2 * \omega_1$. $*\omega_1$ is indeed the Jacobian of the projection $\pi_1$ from $M$ to $\Sigma_1$ when restricted to $\Sigma$ and $\eta > 0$ if and only if $\Sigma$ is locally a graph over $\Sigma_1$. $\eta$ satisfies the following evolution equation:

$$\frac{d}{dt} \eta = \Delta \eta + \eta[2|A|^2 - |H|^2] + c\eta(1 - \eta^2)$$

(2.1)
along the mean curvature flow.

Notice that $0 < \eta \leq 1$. By the equation of $\eta$ and the comparison theorem for parabolic equations, we get

$$\eta(x, t) \geq \frac{\alpha e^{ct}}{\sqrt{1 + \alpha^2 e^{2ct}}}$$

(2.2)

where $\alpha > 0$ is given by $\frac{\alpha}{\sqrt{1 + \alpha^2}} = \min_{\Sigma_0} \eta$. Therefore $\eta(x, t)$ converges uniformly to 1 when $c = 1$ and is nondecreasing when $c = 0$. In any case, $\eta$ has a positive lower bound at any finite time and thus $\Sigma_t$ remains the graph of a symplectomorphism.

Using the fact that the second fundamental form for Lagrangian submanifold is a fully symmetric three tensor, one derives

$$|H|^2 \leq \frac{4}{3} |A|^2.$$ 

Plug this into (2.1) and we obtain

$$\frac{d}{dt} \eta \geq \Delta \eta + \frac{2}{3} |A|^2 \eta + c\eta(1 - \eta^2) \quad (2.3)$$

In [12], we apply blow-up analysis to this equation to show there exists a weak blow-up limit with vanishing $\int |A|^2$. This together with the lower bound of $\eta$ shows the limit is a flat space and White’s regularity theorem [13] implies the blow-up center is a regular point. This proves the long-time existence of the flow.

### 3 A monotonicity lemma

In this section, we derive a new monotonicity formula. First $|H|^2$ satisfies the following evolution equation:

$$(\frac{d}{dt} - \Delta)|H|^2 = -2|\nabla H|^2 + 2 \sum_{ij}(\sum_k H_k h_{kij})^2 + c(2 - \eta^2)|H|^2 \quad (3.1)$$

where the symmetric three-tensor $h_{ijk}$ is the second fundamental form and $H_k = h_{ikk}$, the trace of the second fundamental form, is the component of the mean curvature vector after identifying the tangent bundle and the normal bundle through $J$. 

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We remark that both equations (2.1) and (3.1) are derived in Lemma 5.3 of [10] where \( p \) in [10] and \( \eta \) are related by \( \eta^2 = \frac{4}{p} \) and \( S = 2c \).

We claim the following differential inequality is true:

**Lemma 3.1**

\[
\frac{d}{dt} \int_{\Sigma_t} \frac{|H|^2}{\eta} \leq c \int_{\Sigma_t} \frac{|H|^2}{\eta}
\]

**Proof.**

The proof is a direct computation by combining equations (2.1) and (3.1).

We compute

\[
\frac{d}{dt} \frac{|H|^2}{\eta} = \frac{\eta \Delta |H|^2 - |H|^2 \Delta \eta}{\eta^2} - 2 \frac{\nabla |H|^2}{\eta} + 2 \sum_{ij} \left( \sum_k H_k h_{kij} \right)^2 - 2 |H|^2 |A|^2 + |H|^4 + c \frac{|H|^2}{\eta}.
\]

Now

\[
\Delta \frac{|H|^2}{\eta} = \frac{\eta \Delta |H|^2 - |H|^2 \Delta \eta}{\eta^2} - 2 \eta \nabla \eta (\eta \nabla |H|^2 - |H|^2 \nabla \eta).
\]

We plug this into the previous equation and obtain

\[
\frac{d}{dt} \frac{|H|^2}{\eta} = \frac{\Delta |H|^2}{\eta} + \frac{2 \eta \nabla \eta (\eta \nabla |H|^2 - |H|^2 \nabla \eta)}{\eta^2} - 2 \frac{\nabla |H|^2}{\eta} + 2 \sum_{ij} \left( \sum_k H_k h_{kij} \right)^2 - 2 |H|^2 |A|^2 + |H|^4 + c \frac{|H|^2}{\eta}.
\]

Rearranging terms, we arrive at

\[
\frac{d}{dt} \frac{|H|^2}{\eta} = \frac{\Delta |H|^2}{\eta} + \frac{4 \eta |H| \nabla \eta \cdot \nabla |H| - 2 \nabla \eta^2 |H|^2 - 2 \eta^2 \nabla H|^2}{\eta^3} + 2 \sum_{ij} \left( \sum_k H_k h_{kij} \right)^2 - 2 |H|^2 |A|^2 + |H|^4 + c \frac{|H|^2}{\eta}.
\]
Integrate this identity and we have

\[ \frac{d}{dt} \int_{\Sigma_t} \frac{|H|^2}{\eta} = \int_{\Sigma_t} \frac{4\eta |H| \nabla \eta \cdot \nabla |H| - 2|\nabla \eta|^2 |H|^2 - 2\eta^2 \nabla |H|^2}{\eta^3} \]

\[ + \int_{\Sigma_t} \frac{2 \sum_{ij} (\sum_k H_k h_{kij})^2 - 2|H|^2 |A|^2}{\eta} + c \int_{\Sigma_t} \frac{|H|^2}{\eta}. \]

We use $|\nabla |H|| \leq |\nabla H|$ in the first summand on the right hand side and complete the square

\[ \frac{4\eta |H| \nabla \eta \cdot \nabla |H| - 2|\nabla \eta|^2 |H|^2 - 2\eta^2 \nabla |H|^2}{\eta^3} = -2\frac{\nabla \eta |H| - \eta \nabla |H|}{\eta^3}. \]

At last, we apply Cauchy-Schwarz inequality to the second summand and the differential inequality is proved.

\[ \square \]

4 Proof of the theorem

The smooth convergence in the case when $c = 1$, i.e. when $\Sigma_1$ and $\Sigma_2$ are both standard $S^2$, was proved in [12].

We prove the $C^\infty$ convergence in the case $c = 0$ and $c = -1$ in the following. By the general convergence theorem of Simon [8], it suffices to show $|A|^2$ is bounded independent of time.

In the case when $c = 0$, by (2.2), $\eta$ has a positive lower bound. We have

\[ \int_{\Sigma_t} |H|^2 \leq \int_{\Sigma_t} \frac{|H|^2}{\eta} \leq K_1 \int_{\Sigma_t} |H|^2 \]

for some constant $K_1$.

Since $\int_0^\infty \int_{\Sigma_t} |H|^2 < \infty$, there exists a subsequence $t_i$ such that $\int_{\Sigma_{t_i}} |H|^2 \to 0$ and thus $\int_{\Sigma_{t_i}} \frac{|H|^2}{\eta} \to 0$ as well. Because $\int_{\Sigma_t} \frac{|H|^2}{\eta}$ is non-increasing, this implies $\int_{\Sigma_t} \frac{|H|^2}{\eta} \to 0$ for the continuous parameter $t$ as it approaches $\infty$. Together with (4.1) this implies $\int_{\Sigma_t} |H|^2 \to 0$ as $t \to \infty$. By the Gauss formula, $\int_{\Sigma_t} |A|^2 = \int_{\Sigma_t} |H|^2 \to 0$. The $\epsilon$ regularity theorem in [4] (see also [2]) implies $\sup_{\Sigma_t} |A|^2$ is uniformly bounded.

In the case when $c = -1$, we have
\[
\frac{d}{dt} \int_{\Sigma_t} \frac{|H|^2}{\eta} \leq -\int_{\Sigma_t} \frac{|H|^2}{\eta}
\]

or

\[
\int_{\Sigma_t} \frac{|H|^2}{\eta} \leq K_2 e^{-t}
\]

for some constant \( K_2 \).

Since \( \eta \leq 1 \), we have

\[
\int_{\Sigma_t} |H|^2 \leq K_2 e^{-t}.
\]

This implies \( \Sigma_t \to \Sigma_\infty \) in Radon measure. Indeed, for any function \( \phi \) on \( M \) with compact support, it is easy to see

\[
\frac{d}{dt} \int_{\Sigma_t} \phi = \int_{\Sigma_t} \nabla^M \phi \cdot H + \int_{\Sigma_t} \phi |H|^2,
\]

and thus

\[
\int_{\Sigma_t} \phi \to \int_{\Sigma_\infty} \phi.
\]

exponentially. Also the limit measure \( \Sigma_\infty \) is unique.

The argument in [12] shows the limit \( \Sigma_\infty \) is smooth. It seems one can adapt the proof of the local regularity theorem of Ecker (Theorem 5.3) [2] or the original local regularity theorem of Brakke [1] to get the uniform bound on second fundamental form. This does require versions of these theorem in a general ambient Riemannian manifold.

We circumvent this step by quoting a theorem in minimal surfaces. Suppose the second fundamental form is unbounded. The blow-up procedure in Proposition 3.1 of [12] produces a limiting flow that exists on \(( -\infty, \infty )\). The flow has uniformly bounded second fundamental form \( A(x, t) \) and \( |A|(0, 0) = 1 \). It is not hard to see each slice is the graph of an area-preserving map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). Since \( \int_{\Sigma_t} |H|^2 \leq K_2 e^{-t} \), the limiting flow will satisfies

\[
\int |H|^2 \equiv 0
\]

Therefore, we obtain a minimal area-preserving map. A result of Ni [6] generalizing Schoen’s theorem [7] shows this is a linear diffeomorphism. This contradicts to the fact that \( |A|(0, 0) = 1 \).
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