ON INEQUALITIES FOR A-NUMERICAL RADIUS OF OPERATORS

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Abstract. Let $A$ be a positive operator on a complex Hilbert space $\mathcal{H}$. We present inequalities concerning upper and lower bounds for $A$-numerical radius of operators, which improve on and generalize the existing ones, studied recently in [A. Zamani, A-Numerical radius inequalities for semi-Hilbertian space operators, Linear Algebra Appl. 578 (2019) 159-183]. We also obtain some inequalities for $B$-numerical radius of $2 \times 2$ operator matrices where $B$ is the $2 \times 2$ diagonal operator matrix whose diagonal entries are $A$. Further we obtain upper bounds for $A$-numerical radius for product of operators which improve on the existing bounds.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with usual inner product $\langle ., . \rangle$ and $\| . \|$ be the norm induced from $\langle ., . \rangle$. Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$. Throughout this article we assume $I$ and $O$ are identity operator and zero operator on $\mathcal{H}$, respectively. A self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and is called strictly positive if $\langle Ax, x \rangle > 0$ for all $(0 \neq) x \in \mathcal{H}$. For a positive (strictly positive) operator $A$ we write $A \geq 0$ $(A > 0)$. Let $B = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$. Then $B \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is positive or strictly positive if $A$ is positive or strictly positive respectively. Let us fix the alphabets $A$ and $B$ for positive operator on $\mathcal{H}$ and $\mathcal{H} \oplus \mathcal{H}$ respectively. Clearly $A$ induces a positive semidefinite sesquilinear form $\langle ., . \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined as $\langle x, y \rangle_A = \langle Ax, y \rangle$ for $x, y \in \mathcal{H}$. Let $\| . \|_A$ denote the semi-norm on $\mathcal{H}$ induced from the sesquilinear form $\langle ., . \rangle_A$, that is, $\| x \|_A = \sqrt{\langle x, x \rangle_A}$ for all $x \in \mathcal{H}$. It is easy to verify that $\| . \|_A$ is a norm if and only if $A$ is a strictly positive operator. Also $(\mathcal{H}, \| . \|_A)$ is complete space if and only if the range $\mathcal{R}(A)$ of $A$ is closed in $\mathcal{H}$. By $\overline{\mathcal{R}(T)}$ we denote the norm closure of $\mathcal{R}(T)$ in $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, $A$-operator semi-norm of $T$, denoted as $\| T \|_A$, is defined as

$$\| T \|_A = \sup_{x \in \overline{\mathcal{R}(A)}, x \neq 0} \frac{\| Tx \|_A}{\| x \|_A}.$$ 

Here we note that for a given $T \in \mathcal{B}(\mathcal{H})$, if there exists $c > 0$ such that $\| Tx \|_A \leq c\| x \|_A$ for all $x \in \overline{\mathcal{R}(A)}$ then $\| T \|_A < +\infty$. Again $A$-minimum modulus of $T$,
denoted as $m_A(T)$ (see [26]), is defined as
\[m_A(T) = \inf_{x \in \mathcal{R}(A), x \neq 0} \frac{\|Tx\|_A}{\|x\|_A}.
\]
We set $\mathcal{B}^A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \|T\|_A < +\infty\}$. It is easy to verify that $\mathcal{B}^A(\mathcal{H})$ is not generally a subalgebra of $\mathcal{B}(\mathcal{H})$ and $\|T\|_A = 0$ iff $AT^*A = 0$. For $T \in \mathcal{B}(\mathcal{H})$, an operator $R \in \mathcal{B}(\mathcal{H})$ is called an $A$-adjoint of $T$ if for every $x, y \in \mathcal{H}$ such that $\langle Tx, y \rangle_A = \langle x, R y \rangle_A$, that is, $AR = T^*A$ where $T^*$ is the adjoint of $T$. For any operator $T \in \mathcal{B}(\mathcal{H})$, $A$-adjoint operator of $T$ may or may not exist. In fact, an operator $T \in \mathcal{B}(\mathcal{H})$ may have one or more than one $A$-adjoint operators, also it may have none. By Douglas Theorem [12], we have that an operator $T \in \mathcal{B}(\mathcal{H})$ admits $A$-adjoint if
\[\mathcal{R}(T^*A) \subseteq \mathcal{R}(A).
\]
Now we consider an example that $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{C}^2$.

Then we see that $\mathcal{R}(T^*A) = \{(x, 0) : x \in \mathbb{C}\}$ and $\mathcal{R}(A) = \{(0, x) : x \in \mathbb{C}\}$. So by Douglas Theorem [12] we conclude that $T$ have no $A$-adjoint. Let $\mathcal{B}_A(\mathcal{H})$ be the collection of all operators in $\mathcal{B}^A(\mathcal{H})$ which admits $A$-adjoint. Note that $\mathcal{B}_A(\mathcal{H})$ is a sub-algebra of $\mathcal{B}(\mathcal{H})$ which is neither closed nor dense in $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$, $A$-adjoint operator of $T$ is written as $T^{\sharp A}$. It is well known that $T^{\sharp A} = A^1T^*A$ where $A^1$ is the Moore-Penrose inverse of $A$, (see [20]). It is useful that if $T \in \mathcal{B}_A(\mathcal{H})$ then $AT^{\sharp A} = T^*A$. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be $A$-self-adjoint operator if $AT$ is self-adjoint, that is, $AT = T^*A$ and it is called $A$-positive if $AT \geq 0$. For $A$-positive operator $T$ we have
\[\|T\|_A = \sup\{\langle Tx, x \rangle_A : x \in \mathcal{H}, \|x\|_A = 1\}.
\]
An operator $U \in \mathcal{B}_A(\mathcal{H})$ is said to be $A$-unitary if $U^{\sharp A}U = (U^{\sharp A})^{\sharp A}U^{\sharp A} = P_A$, $P_A$ is the orthogonal projection onto $\mathcal{R}(A)$. Here we note that if $T \in \mathcal{B}_A(\mathcal{H})$ then $T^{\sharp A} \in \mathcal{B}_A(\mathcal{H})$, $(T^{\sharp A})^{\sharp A} = P_ATP_A$. Also $T^{\sharp A}T$, $TT^{\sharp A}$ are $A$-self-adjoint and $A$-positive operators and so
\[\|T^{\sharp A}T\|_A = \|TT^{\sharp A}\|_A = \|T\|^2_A = \|T^{\sharp A}\|^2_A.
\]
Also for $T, S \in \mathcal{B}_A(\mathcal{H})$, $(TS)^{\sharp A} = S^{\sharp A}T^{\sharp A}$, $\|TS\|_A \leq \|T\|_A\|S\|_A$ and $\|Tx\|_A \leq \|T\|_A\|x\|_A$ for all $x \in \mathcal{H}$. For further details we refer the reader to [1, 2, 3]. For an operator $T \in \mathcal{B}_A(\mathcal{H})$, we write $Re_A(T) = \frac{1}{2}(T + T^{\sharp A})$ and $Im_A(T) = \frac{1}{2i}(T - T^{\sharp A})$. For $T \in \mathcal{B}_A(\mathcal{H})$, $A$-numerical radius of $T$, denoted as $w_A(T)$, is defined as
\[w_A(T) = \sup\{\langle Tx, x \rangle_A : x \in \mathcal{H}, \|x\|_A = 1\}, \text{ (see [4])}.
\]
Also, for $T \in \mathcal{B}_A(\mathcal{H})$, $A$-Crawford number of $T$, denoted as $c_A(T)$ (see [26]), is defined as
\[c_A(T) = \inf\{\langle Tx, x \rangle_A : x \in \mathcal{H}, \|x\|_A = 1\}.
\]
For $T \in \mathcal{B}_A(\mathcal{H})$, it is well-known that $A$-numerical radius of $T$ is equivalent to $A$-operator semi-norm of $T$, (see [25]), satisfying the following inequality:
\[\frac{1}{2}\|T\|_A \leq w_A(T) \leq \|T\|_A.
\]
Over the years many mathematicians have studied numerical radius inequalities in [5, 7, 8, 9, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24]. Recently, Zamani [25] have studied A-numerical radius and computed some inequalities for A-numerical radius. In this paper, we compute some inequalities for B-numerical radius of \(2 \times 2\) operator matrices which generalize and improve on the existing inequalities. Also we obtain some inequalities for A-numerical radius of operators in \(B\) which improve on the existing inequalities in [25]. Further we obtain A-numerical radius bounds for sum of product of operators in \(B\) which improve on the existing bounds.

2. A-numerical radius inequalities for operators in \(B_A(H)\)

We begin this section with the following three results proved by Zamani [25].

**Lemma 2.1.** Let \(T \in B_A(H)\) be an A-self-adjoint operator. Then
\[
w_A(T) = \|T\|_A.
\]

**Lemma 2.2.** Let \(T \in B_A(H)\). For every \(\theta \in \mathbb{R}\),
\[
w_A(R e_A(e^{i\theta}T)) = \|R e_A(e^{i\theta}T)\|_A.
\]

**Lemma 2.3.** Let \(T \in B_A(H)\). Then
\[
w_A(T) = \sup_{\theta \in \mathbb{R}} \|R e_A(e^{i\theta}T)\|_A \quad \text{and} \quad w_A(T) = \sup_{\theta \in \mathbb{R}} \|I m_A(e^{i\theta}T)\|_A.
\]

Next we compute B-numerical radius for some \(2 \times 2\) operator matrices. First we note that the operator \(T = (T_{ij})_{2 \times 2}\) is in \(B_B(H \oplus H)\) if the operator \(T_{ij}\) (for \(i, j = 1, 2\)) are in \(B_A(H)\) and in this case (see [10, Lemma 3.1]) \(T_{ij}^B = (T_{ij}^A)_{2 \times 2}\). We now prove the following lemma.

**Lemma 2.4.** Let \(X, Y \in B_A(H)\). Then the following results hold:

(i) \(w_B\left(\begin{array}{cc} X & O \\ O & Y \end{array}\right) = \max\{w_A(X), w_A(Y)\}\).

(ii) If \(A > 0\) then \(w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right) = w_B\left(\begin{array}{cc} O & Y \\ X & O \end{array}\right)\).

(iii) If \(A > 0\) then for any \(\theta \in \mathbb{R}\), \(w_B\left(\begin{array}{cc} O & e^{i\theta}X \\ y & O \end{array}\right) = w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\).

(iv) If \(A > 0\) then \(w_B\left(\begin{array}{cc} X & Y \\ Y & X \end{array}\right) = \max\{w_A(X + Y), w_A(X - Y)\}\).

In particular, \(w_B\left(\begin{array}{cc} O & Y \\ Y & O \end{array}\right) = w_A(Y)\).

**Proof.** (i) Let \(T = \left(\begin{array}{cc} X & O \\ O & Y \end{array}\right)\) and \(u = (x, y) \in H \oplus H\) with \(\|u\|_B = 1\), i.e., \(\|x\|^2_A + \|y\|^2_A = 1\). Now,
\[
|\langle Tu, u \rangle_B| \leq |\langle Xx, x \rangle_A| + |\langle Yy, y \rangle_A| \\
\leq w_A(X)\|x\|^2_A + w_A(Y)\|y\|^2_A \\
\leq \max\{w_A(X), w_A(Y)\}.
\]
Taking supremum over \( \|u\|_B = 1 \), we get
\[
   w_B(T) \leq \max \{ w_A(X), w_A(Y) \}.
\]
Suppose \( u = (x, 0) \in \mathcal{H} \oplus \mathcal{H} \) where \( \|x\|_A = 1 \). Then
\[
   |\langle Tu, u \rangle_B| = |\langle AXx, x \rangle| = |\langle Xx, x \rangle_A|.
\]
Taking supremum over \( \|x\|_A = 1 \), we get
\[
   \sup_{\|x\|_A=1} |\langle Tu, u \rangle_B| = w_A(X)
\]
and so we have \( w_B(T) \geq w_A(X) \). Similarly, if we take \( v = (0, y) \in \mathcal{H} \oplus \mathcal{H} \) with \( \|y\|_A = 1 \) then we can show that \( w_B(T) \geq w_A(Y) \). Therefore, \( w_B(T) \geq \max \{ w_A(X), w_A(Y) \} \). This completes the proof of Lemma 2.4 (i).

(ii) The proof follows from the observation that \( w_B(U^{\sharp B}TU) = w_B(T) \) (see \cite[Lemma 3.8]{10}) if \( U \) is an \( B \)-unitary operator on \( \mathcal{H} \oplus \mathcal{H} \), here we take \( U = \left( \begin{array}{cc} O & I \\ I & O \end{array} \right) \).

(iii) As in (ii) we now take \( U = \left( \begin{array}{cc} I & O \\ O & e^{i\theta} I \end{array} \right) \).

(iv) Let \( U = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} I & O \\ -I & I \end{array} \right) \) and \( T = \left( \begin{array}{cc} X & Y \\ Y & X \end{array} \right) \). Then an easy calculation we have
\[
   U^{\sharp B}TU = \left( \begin{array}{cc} X - Y & O \\ O & X + Y \end{array} \right).
\]
Using Lemma 2.4 (i) and \( w_B(U^{\sharp B}TU) = w_B(T) \) we get
\[
   w_B(T) = \max \{ w_A(X + Y), w_A(X - Y) \}.
\]
Taking \( X = O \) we get
\[
   w_B \left( \begin{array}{cc} O & Y \\ Y & O \end{array} \right) = w_A(Y).
\]
This completes the proof of Lemma 2.4 (iv).

Next we prove the following important lemma for \( A \)-positive operators.

**Lemma 2.5.** Let \( X, Y \in \mathcal{B}_A(\mathcal{H}) \) be \( A \)-positive. If \( X - Y \) is \( A \)-positive then
\[
   \|X\|_A \geq \|Y\|_A.
\]

**Proof.** From the definition of \( A \)-positive operator we have, for all \( x \in \mathcal{H} \)
\[
   \langle (X - Y)x, x \rangle_A \geq 0 \Rightarrow \langle Xx, x \rangle_A \geq \langle Yx, x \rangle_A \Rightarrow w_A(X) \geq \langle Yx, x \rangle_A.
\]
Taking supremum over \( \|x\|_A = 1 \), we get
\[
   w_A(X) \geq w_A(Y).
\]
Since \( X, Y \) are \( A \)-self-adjoint operators, so \( \|X\|_A \geq \|Y\|_A \).
Theorem 2.6. Let $X, Y \in \mathcal{B}_A(\mathcal{H})$. Then

$$w_B^2\left( \begin{array}{c} O \\ Y \\ O \end{array} \right) X \geq \frac{1}{4} \max \left\{ \| XX^*A + Y^2A^*Y \|_A, \| XX^*A^*X + YY^*A^*Y \|_A \right\}$$
$$w_B^2\left( \begin{array}{c} O \\ Y \\ O \end{array} \right) X \leq \frac{1}{2} \max \left\{ \| XX^*A + Y^2A^*Y \|_A, \| XX^*A^*X + YY^*A^*Y \|_A \right\}.$$

Proof. Let $T = \left( \begin{array}{c} O \\ Y \\ O \end{array} \right) X$, $H_\theta = \text{Re}_A(e^{i\theta}T)$ and $K_\theta = \text{Im}_A(e^{i\theta}T)$. Then from an easy calculation we have,

$$H_\theta^2 + K_\theta^2 = \frac{1}{2} \left( \begin{array}{c} M \\ O \\ N \end{array} \right)$$
where $M = XX^*A + Y^2A^*Y$, $N = XX^*A^*X + YY^*A^*Y$.

Taking norm on both sides and then using Lemma 2.3, we get

$$\frac{1}{2} \left\| \left( \begin{array}{c} M \\ O \\ N \end{array} \right) \right\|_B = \| H_\theta^2 + K_\theta^2 \|_B \leq \| H_\theta \|_B^2 + \| K_\theta \|_B^2 \leq 2w_B^2(T).$$

Therefore we get,

$$\frac{1}{2} \max \left\{ \| M \|_A, \| N \|_A \right\} \leq 2w_B^2(T).$$

This completes the proof of the first inequality.

Again, from $H_\theta^2 + K_\theta^2 = \frac{1}{2} \left( \begin{array}{c} M \\ O \\ N \end{array} \right)$ we have, $H_\theta^2 - \frac{1}{2} \left( \begin{array}{c} M \\ O \\ N \end{array} \right) = -K_\theta^2 \leq 0$.

Therefore, $H_\theta^2 \leq \frac{1}{2} \left( \begin{array}{c} M \\ O \\ N \end{array} \right)$. Using Lemma 2.5, we get

$$\| H_\theta \|_B^2 \leq \frac{1}{2} \left\| \left( \begin{array}{c} M \\ O \\ N \end{array} \right) \right\|_B = \frac{1}{2} \max \left\{ \| M \|_A, \| N \|_A \right\}.$$

Taking supremum over $\theta \in \mathbb{R}$, we get

$$w_B^2(T) \leq \frac{1}{2} \max \left\{ \| M \|_A, \| N \|_A \right\}.$$

This completes the proof of the second inequality of the theorem. \qed

Next we state the corollary, the proof of which follows easily by considering $X = Y = T$ and $A > 0$ in Theorem 2.6.

Corollary 2.7. Let $T \in \mathcal{B}_A(\mathcal{H})$ and $A > 0$. Then

$$\frac{1}{4} \| TT^*A + T^2A^*T \|_A \leq w_A^2(T) \leq \frac{1}{2} \| TT^*A^* + T^2A^*T \|_A.$$

Remark 2.8. (i) Kittaneh [18, Th. 1] proved that if $T \in \mathcal{B}(\mathcal{H})$ then

$$\frac{1}{4} \| TT^* + T^*T \| \leq w_2(T) \leq \frac{1}{2} \| TT^* + T^*T \|,$$

which follows easily from Corollary 2.7 by taking $A = I$.

(ii) Zamani [25, Th. 2.10] proved that

$$w_A^2(T) \leq \frac{1}{2} \| TT^*A^* + T^2A^*T \|_A.$$
which clearly follows from the inequality obtained in Corollary 2.7.

Next we prove the following theorem.

**Theorem 2.9.** Let $X, Y \in \mathcal{B}_A(\mathcal{H})$. Then

\[
w_B^2 \left( \begin{array}{cc} O & X \\ Y & O \end{array} \right) \geq \frac{1}{16} \max \left\{ \|P\|_A, \|Q\|_A \right\}
\]

and

\[
w_B^4 \left( \begin{array}{cc} O & X \\ Y & O \end{array} \right) \leq \frac{1}{8} \max \left\{ \|XX^{\sharp_A} + YY^{\sharp_A}\|_A^2 + 4w^2_A(XY), \right\}
\]

\[
\|XX^{\sharp_A} + YY^{\sharp_A}\|_A^2 + 4w^2_A(YX) \right\},
\]

where $P = (XX^{\sharp_A} + YY^{\sharp_A})^2 + 4(Re_A(XY))^2$, $Q = (XX^{\sharp_A} + YY^{\sharp_A})^2 + 4(Re_A(YX))^2$.

**Proof.** Let $T = \left( \begin{array}{cc} O & X \\ Y & O \end{array} \right)$, $H_\theta = Re_A(e^{i\theta}T)$ and $K_\theta = Im_A(e^{i\theta}T)$. Then we get,

\[
H_\theta^4 + K_\theta^4 = \frac{1}{8} \left( \begin{array}{cc} P_0 & O \\ O & Q_0 \end{array} \right),
\]

where $P_0 = (XX^{\sharp_A} + YY^{\sharp_A})^2 + 4(Re_A(XY))^2$, $Q_0 = (XX^{\sharp_A} + YY^{\sharp_A})^2 + 4(Re_A(YX))^2$. Taking norm on both sides and using Lemma 2.3, we get

\[
\frac{1}{8} \left\| \left( \begin{array}{cc} P_0 & O \\ O & Q_0 \end{array} \right) \right\|_B = \|H_\theta^4 + K_\theta^4\|_B \leq \|H_\theta\|_B^4 + \|K_\theta\|_B^4 \leq 2w_B^4(T).
\]

Therefore we get,

\[
\frac{1}{8} \max \left\{ \|P_0\|_A, \|Q_0\|_A \right\} \leq 2w_B^4(T).
\]

This holds for all $\theta \in \mathbb{R}$, so taking $\theta = 0$ we get,

\[
\frac{1}{8} \max \left\{ \|P\|_A, \|Q\|_A \right\} \leq 2w_B^4(T).
\]

This completes the proof of the first inequality of the theorem.

Again, from $H_\theta^4 + K_\theta^4 = \frac{1}{8} \left( \begin{array}{cc} P_0 & O \\ O & Q_0 \end{array} \right)$ we have, $H_\theta^4 - \frac{1}{8} \left( \begin{array}{cc} P_0 & O \\ O & Q_0 \end{array} \right) = -K_\theta^4 \leq 0$.

Therefore, $H_\theta^4 \leq \frac{1}{8} \left( \begin{array}{cc} P_0 & O \\ O & Q_0 \end{array} \right)$.

Using Lemma 2.5, we get

\[
\|H_\theta\|_B^4 \leq \frac{1}{8} \left\| \left( \begin{array}{cc} P_0 & O \\ O & Q_0 \end{array} \right) \right\|_B = \frac{1}{8} \max \left\{ \|P_0\|_A, \|Q_0\|_A \right\}.
\]

Therefore using Lemma 2.3, we get

\[
\|H_\theta\|_B^4 \leq \frac{1}{8} \max \left\{ \|XX^{\sharp_A} + YY^{\sharp_A}\|_A^2 + 4w^2_A(XY), \|XX^{\sharp_A} + YY^{\sharp_A}\|_A^2 + 4w^2_A(YX) \right\}.
\]

Taking supremum over $\theta \in \mathbb{R}$ and using Lemma 2.3, we get

\[
w_B^4(T) \leq \frac{1}{8} \max \left\{ \|XX^{\sharp_A} + YY^{\sharp_A}\|_A^2 + 4w^2_A(XY), \|XX^{\sharp_A} + YY^{\sharp_A}\|_A^2 + 4w^2_A(YX) \right\}.
\]

This completes the proof of the second inequality of the theorem. \(\square\)

Now, taking $X = Y = T$ (say) and $A > 0$ in the above Theorem 2.9, we get the following inequality.
Corollary 2.10. Let $T \in B_A(H)$ where $A > 0$. Then
\[
\frac{1}{16} \|(TT^*A + T^*AT)^2 + 4(Re_A(T^2))^2\|_A \leq w_A^4(T)
\]
\[
\leq \frac{1}{8} \|TT^*A + T^*AT\|_A^2 + \frac{1}{2} w_A^2(T^2).
\]

Remark 2.11. (i) In [5, Th. 2.11] we proved that if $T \in B(H)$ then
\[
\frac{1}{16} \|TT^* + T^*T\|_H^2 + \frac{1}{4} m ((Re(T^2))^2) \leq w^4(T)
\]
\[
\leq \frac{1}{8} \|TT^* + T^*T\|_H^2 + \frac{1}{2} w^2(T^2),
\]
which follows easily from Corollary 2.10 by taking $A = I$.

(ii) Zamani [25, Th. 2.10] proved that
\[
w_A^2(T) \leq \frac{1}{2} \|TT^*A + T^*AT\|_A.
\]

Since $w_A(T^2) \leq w_A^2(T)$ (see [19, Prop. 3.10]), so $w_A(T^2) \leq \frac{1}{2} \|TT^*A + T^*AT\|_A$.

Therefore, the right hand inequality obtained in Corollary 2.10 improves on the inequality obtained by Zamani [25, Th. 2.10].

We next prove the following theorem.

Theorem 2.12. Let $T \in B_A(H)$ where $A > 0$. Then
\[
w_A^4(T) \leq \frac{1}{4} w_A^2(T^2) + \frac{1}{8} w_A(T^2P + PT^2) + \frac{1}{16} \|P\|_A^2,
\]
where $P = T^*A + TT^*A$.

Proof. From Lemma 2.3, we have $w_A(T) = \sup_{\theta \in \mathbb{R}} \|H_\theta\|_A$ where $H_\theta = Re_A(e^{i\theta}T)$.

Then
\[
H_\theta = \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*)
\]
\[
\Rightarrow 4H_\theta^2 = e^{2i\theta}T^2 + e^{-2i\theta}T^*A^2 + P
\]
\[
\Rightarrow 16H_\theta^4 = (e^{2i\theta}T^2 + e^{-2i\theta}T^*A^2 + P)(e^{2i\theta}T^2 + e^{-2i\theta}T^*A^2 + P)
\]
\[
= (e^{2i\theta}T^2 + e^{-2i\theta}T^*A^2)^2 + (e^{2i\theta}T^2 + e^{-2i\theta}T^*A^2)^2 + P^2
\]
\[
\Rightarrow \|H_\theta^4\|_A \leq \frac{1}{4} \|Re_A(e^{2i\theta}T^2)\|_A^2 + \frac{1}{8} \|Re_A(e^{2i\theta}(T^2P + PT^2))\|_A + \frac{1}{16} \|P\|_A^2
\]
\[
\leq \frac{1}{4} w_A^2(T^2) + \frac{1}{8} w_A(T^2P + PT^2) + \frac{1}{16} \|P\|_A^2.
\]

Taking supremum over $\theta \in \mathbb{R}$, we get,
\[
\Rightarrow w_A^4(T) \leq \frac{1}{4} w_A^2(T^2) + \frac{1}{8} w_A(T^2P + PT^2) + \frac{1}{16} \|P\|_A^2.
\]
Remark 2.13. Using the inequality in Corollary 3.3, it is easy to see that if $A > 0$ then $w_A(T^2P + PT^2) \leq 2w_A(T^2)||P||_A$. In case $A > 0$, we would like to remark that the inequality obtained in Theorem 2.12 improves on the inequality [25, Th. 2.11] obtained by Zamani. As for numerical example, if we consider $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on $\mathbb{C}^3$, then by simple computation we have

$$\frac{1}{4}w_A^3(T^2) + \frac{1}{8}w_A(T^2P + PT^2) + \frac{1}{16}||P||_A^2 = \frac{39}{16} < \frac{1}{16}(||P||_A + 2w_A(T^2))^2 = \frac{49}{16}.$$

Now we prove the following theorem.

Theorem 2.14. Let $T \in \mathcal{B}_A(\mathcal{H})$ where $A > 0$. Then

$$w_A^3(T) \leq \frac{1}{4}w_A(T^3) + \frac{1}{4}w_A(T^2T^3A + T^3A^2 + TT^3A).$$

Moreover if $T^2 = 0$ then $w_A(T) = \frac{1}{2}\sqrt{||TT^3A + T^3A||_A}$ and if $T^3 = 0$ then $w_A^3(T) = \frac{1}{4}w_A(T^2T^3A + T^3A^2 + TT^3A T).$

Proof. From Lemma 2.3, we have $w_A(T) = \sup_{\theta \in \mathbb{R}} ||H_\theta||_A$ where $H_\theta = Re_A(e^{i\theta}T).$

Then,

$$H_\theta = \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^3A)$$

$$\Rightarrow 4H_\theta^2 = e^{2i\theta}T^2 + e^{-2i\theta}T^3A^2 + T^3AT + TT^3A$$

$$\Rightarrow 8H_\theta^3 = (e^{2i\theta}T^2 + e^{-2i\theta}T^3A^2 + T^3AT + TT^3A)(e^{i\theta}T + e^{-i\theta}T^3A)$$

$$\Rightarrow H_\theta^3 = \frac{1}{4}Re_A(e^{3i\theta}T^3) + \frac{1}{4}Re_A(e^{i\theta}(T^2T^3A + T^3A^2T + TT^3A T))$$

$$\Rightarrow ||H_\theta^3||_A \leq \frac{1}{4}||Re_A(e^{3i\theta}T^3)||_A + \frac{1}{4}||Re_A(e^{i\theta}(T^2T^3A + T^3A^2T + TT^3A T))||_A$$

$$\leq \frac{1}{4}w_A(T^3) + \frac{1}{4}w_A(T^2T^3A + T^3A^2T + TT^3A T).$$

Taking supremum over $\theta \in \mathbb{R}$, we get the desired inequality.

If $T^2 = 0$, then $4H_\theta^2 = T^3AT + TT^3A$ and so $w_A(T) = \frac{1}{2}\sqrt{||TT^3A + T^3A||_A}.$

If $T^3 = 0$, then $H_\theta^3 = \frac{1}{4}Re_A(e^{i\theta}(T^2T^3A + T^3A^2T + TT^3A T))$ and so $w_A^3(T) = \frac{1}{4}w_A(T^2T^3A + T^3A^2T + TT^3A T).$ \hfill \Box

Remark 2.15. Here we would like to remark that the bound obtained in Theorem 2.14 improves on the existing upper bound in [25, Cor. 2.8] when $A > 0$. Note that if $T^2 = 0$ then $w_A(T) = \frac{1}{2}\sqrt{||TT^3A + T^3A||_A}.$ But converse is not true, that is, $w_A(T) = \frac{1}{2}\sqrt{||TT^3A + T^3A||_A}$ does not always imply $T^2 = O.$ As for example we consider $T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on $\mathbb{C}^3$. Then we see that

$$w_A(T) = \frac{1}{2}\sqrt{||TT^3A + T^3A||_A} = 1$$

but $T^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq O.$
Next we prove the following inequality.

**Theorem 2.16.** Let \( T \in \mathcal{B}_A(\mathcal{H}) \). Then for each \( r \geq 1 \),
\[
    w_A^{2r}(T) \leq \frac{1}{2} w_A^r(T^2) + \frac{1}{4} \left\| (T^2)^r + (TT^2)^r \right\|_A.
\]

**Proof.** From Lemma 2.3, we get \( w_A(T) = \sup_{\theta \in \mathbb{R}} \| H_\theta \|_A \) where \( H_\theta = Re_A(e^{i\theta}T) \). Now,
\[
    H_\theta = \frac{1}{2} (e^{i\theta}T + e^{-i\theta}T^\sharp_A)
\]
\[
\Rightarrow 4H_\theta^2 = e^{2i\theta}T^2 + e^{-2i\theta}T^\sharp_A^2 + T^\sharp_A T + T T^\sharp_A
\]
\[
\Rightarrow H_\theta^2 = \frac{1}{2} Re_A(e^{2i\theta}T^2) + \frac{1}{4} (T^\sharp_A T + T T^\sharp_A)
\]
\[
\Rightarrow \| H_\theta^2 \|_A \leq \frac{1}{2} \left\| Re_A(e^{2i\theta}T^2) \right\|_A + \frac{1}{4} \left\| T^\sharp_A T + T T^\sharp_A \right\|_A.
\]

For \( r \geq 1 \), \( t^r \) and \( t^\frac{1}{r} \) are convex and concave functions respectively and using that we get,
\[
\left\| H_\theta^2 \right\|_A^r \leq \left\{ \frac{1}{2} \left\| Re_A(e^{2i\theta}T^2) \right\|_A + \frac{1}{2} \left\| \frac{T^\sharp_A T + T T^\sharp_A}{2} \right\|_A \right\}^r
\]
\[
\leq \frac{1}{2} \left\| Re_A(e^{2i\theta}T^2) \right\|_A^r + \frac{1}{2} \left\| \frac{T^\sharp_A T + T T^\sharp_A}{2} \right\|_A^r
\]
\[
\leq \frac{1}{2} \left\| Re_A(e^{2i\theta}T^2) \right\|_A^r + \frac{1}{2} \left\| \left( \frac{T^\sharp_A T}{2} + \frac{T T^\sharp_A}{2} \right)^r \right\|_A
\]
\[
= \frac{1}{2} \left\| Re_A(e^{2i\theta}T^2) \right\|_A^r + \frac{1}{2} \left\| \left( \frac{T^\sharp_A T}{2} + \frac{T T^\sharp_A}{2} \right)^r \right\|_A
\]
\[
\leq \frac{1}{2} w_A^r(T^2) + \frac{1}{4} \left( (T^\sharp_A T)^r + (T T^\sharp_A)^r \right)_A.
\]

Taking supremum over \( \theta \in \mathbb{R} \), we get
\[
w_A^{2r}(T) \leq \frac{1}{2} w_A^r(T^2) + \frac{1}{4} \left\| (T^\sharp_A T)^r + (T T^\sharp_A)^r \right\|_A.
\]

\[\square\]

**Remark 2.17.** Here we would like to remark that if we take \( r = 1 \) in the above Theorem 2.16, we get the inequality [25, Th. 2.11] proved by Zamani.

Now we obtain a lower bound for \( A \)-numerical radius.

**Theorem 2.18.** Let \( T \in \mathcal{B}_A(\mathcal{H}) \) where \( A > 0 \). Then
\[
w_A^2(T) \geq \frac{1}{4} C_A^2(T^2) + \frac{1}{8} C_A(T^2 P + PT^2) + \frac{1}{16} \| P \|_A^2,
\]
where \( P = T^\sharp_A T + T T^\sharp_A \), \( C_A(T) = \inf_{\| x \|_A = 1} \inf_{\phi \in \mathbb{R}} \| Re_A(e^{i\phi}T)x \|_A \).
Proof. We know that $w_A(T) = \sup_{\phi \in \mathbb{R}} \|H_\phi\|_A$ where $H_\phi = Re_A(e^{i\phi}T)$. Let $x$ be a unit vector in $H$ and $\theta$ be a real number such that

$$e^{2i\theta}\langle(T^2P + PT^2)x, x\rangle_A = |\langle(T^2P + PT^2)x, x\rangle_A|.$$

Then,

$$H_\theta = \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*_A)$$

$$\Rightarrow 4H_\theta^2 = e^{2i\theta}T^2 + e^{-2i\theta}T^*_A + P$$

$$\Rightarrow 16H_\theta^4 = \left(e^{2i\theta}T^2 + e^{-2i\theta}T^*_A + P\right)^2 \left(e^{2i\theta}T^2 + e^{-2i\theta}T^*_A + P\right)$$

$$= \left(e^{2i\theta}T^2 + e^{-2i\theta}T^*_A\right)^2 + \left(e^{2i\theta}T^2 + e^{-2i\theta}T^*_A\right)^2 + P$$

$$+ P\left(e^{2i\theta}T^2 + e^{-2i\theta}T^*_A\right) + P^2$$

$$= 4\left(Re_A(e^{2i\theta}T^2)\right)^2 + 2Re_A(e^{2i\theta}(T^2P + PT^2)) + P^2$$

$$\Rightarrow 16w_A^4(T) \geq \left|4\left(Re_A(e^{2i\theta}T^2)\right)^2 + 2Re_A(e^{2i\theta}(T^2P + PT^2)) + P^2\right|_A$$

$$\geq \left|\langle(4\left(Re_A(e^{2i\theta}T^2)\right)^2 + 2Re_A(e^{2i\theta}(T^2P + PT^2)) + P^2)x, x\rangle_A\right|$$

$$= \left|4\langle Re_A(e^{2i\theta}T^2)\rangle x, x\rangle_A + 2Re_A(e^{2i\theta}\langle(T^2P + PT^2)x, x\rangle_A) + \langle P^2x, x\rangle_A\right|$$

$$= 4\left|\langle Re_A(e^{2i\theta}T^2)\rangle x\right|_A^2 + 2\left|\langle(T^2P + PT^2)x, x\rangle_A\right| + \left\|P^2\right\|_A^2$$

$$\geq 4\left\|\langle Re_A(e^{2i\theta}T^2)\rangle x\right\|_A^2 + 2c_A(T^2P + PT^2) + \left\|Px\right\|_A^2$$

$$\Rightarrow 16w_A^4(T) \geq 4C_A^2(T^2) + 2c_A(T^2P + PT^2) + \sup_{\|x\|_A = 1} \|Px\|_A^2$$

$$= 4C_A^2(T^2) + 2c_A(T^2P + PT^2) + \left\|P\right\|_A^2$$

$$\Rightarrow w_A^4(T) \geq \frac{1}{4}C_A^2(T^2) + \frac{1}{8}c_A(T^2P + PT^2) + \frac{1}{16}\|P\|_A^2.$$

This completes the proof. □

Remark 2.19. It is clear that $\frac{1}{4}C_A^2(T^2) + \frac{1}{8}c_A(T^2P + PT^2) + \frac{1}{16}\|P\|_A^2 \geq \frac{1}{16}\|T^2A + TT^Ax\|_A^2 \geq \frac{1}{16}\|T\|_A^4$. So, if $A > 0$ then the inequality obtained in Theorem 2.18 is better than the first inequality in [25, Cor. 2.8], obtained by Zamani.

3. $A$-numerical radius inequalities for product of operators in $B_A(H)$

We begin this section with the following $A$-numerical radius inequality for sum of product of operators.

Theorem 3.1. Let $P, Q, X, Y \in B_A(H)$ where $A > 0$. Then

$$w_A(PXQ^{\sharp_A} + QYP^{\sharp_A}) \leq 2\|P\|_A\|Q\|_A w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right).$$

In particular,

$$w_A(PXQ^{\sharp_A} + QXP^{\sharp_A}) \leq 2\|P\|_A\|Q\|_A w_A(X).$$
Proof. Let $C = \begin{pmatrix} P & Q \\ O & O \end{pmatrix}$ and $Z = \begin{pmatrix} O & X \\ Y & O \end{pmatrix}$. Then from an easy calculation we get,

$$\begin{pmatrix} P X Q^*A + Q Y P^*A \\ O \\ O \end{pmatrix}$$

Therefore,

$$w_A(P X Q^*A + Q Y P^*A) = w_B \begin{pmatrix} P X Q^*A + Q Y P^*A \\ O \\ O \end{pmatrix}$$

$$= w_B(C Z C^*B), \ \text{using Lemma 2.4 (i)}$$

$$\leq \|C\|_B^2 w_B(Z), \ \text{using \cite{25, Lemma 4.4}}$$

$$= \|P P^*A + Q Q^*A\|_A w_B(Z)$$

$$\leq (\|P\|_A^2 + \|Q\|_A^2) w_B(Z).$$

Replacing $P$ and $Q$ by $tP$ and $\frac{1}{t}Q$ respectively with $t > 0$ in this above inequality, we get

$$w_A(P X Q^*A + Q Y P^*A) \leq \left(\frac{t^4\|P\|_A^2 + \|Q\|_A^2}{t^2}\right) w_B(Z).$$

Note that

$$\min_{t>0} \frac{t^4\|P\|_A^2 + \|Q\|_A^2}{t^2} = 2\|P\|_A\|Q\|_A$$

and so

$$w_A(P X Q^*A + Q Y P^*A) \leq 2\|P\|_A\|Q\|_A w_B \begin{pmatrix} O & X \\ Y & O \end{pmatrix}.$$  

Replacing $Y$ by $-Y$ in the above inequality and using Lemma 2.4 (iii), we get

$$w_A(P X Q^*A - Q Y P^*A) \leq 2\|P\|_A\|Q\|_A w_B \begin{pmatrix} O & X \\ Y & O \end{pmatrix}.$$  

Taking $X = Y$ and using Lemma 2.4 (iv), we get

$$w_A(P X Q^*A \pm Q X P^*A) \leq 2\|P\|_A\|Q\|_A w_A(X).$$

This completes the proof of the theorem. \qed

Remark 3.2. Here we note that the inequality

$$w_A(P X Q^*A + Q Y P^*A) \leq 2\|P\|_A\|Q\|_A w_B \begin{pmatrix} O & X \\ Y & O \end{pmatrix}$$

in Theorem 3.1 holds also when $A \geq 0$.

Considering $X = Y = T$ (say), $P = I$ in Theorem 3.1, we get the following inequality.

Corollary 3.3. Let $T, Q \in \mathcal{B}_A(H)$ where $A > 0$. Then

$$w_A(T Q^*A \pm Q T) \leq 2w_A(T)\|Q\|_A.$$  

Next we prove the following lemma, the idea of which is based on the result \cite{6, Lemma 3} proved by Bernau and Smithes.
Lemma 3.4. Let $X, T, Y \in \mathcal{B}_A(\mathcal{H})$ where $A > 0$. Then, for all $x \in \mathcal{H}$

$$|\langle X^{2A}T Y x, x \rangle_A| + |\langle Y^{2A}TX x, x \rangle_A| \leq 2w_A(T) ||Xx||_A ||Yx||_A.$$ 

Proof. Let $x \in \mathcal{H}$ and $\theta, \phi$ be real numbers such that $e^{i\theta} \langle Y^{2A}TX x, x \rangle_A = |\langle Y^{2A}TX x, x \rangle_A|$, $e^{2i\theta} \langle e^{-i\phi}X^{2A}TY x, x \rangle_A = |\langle e^{-i\phi}X^{2A}TY x, x \rangle_A|$. Then for non-zero real number $\lambda$, we have

$$2e^{2i\theta} \langle TY x, e^{i\phi}X x \rangle_A + 2e^{i\phi} \langle TX x, Y x \rangle_A$$

$$= \langle e^{i\theta} \left( \lambda e^{i\theta}Yx + \frac{1}{\lambda} e^{i\phi}X x \right), \lambda e^{i\theta}Yx + \frac{1}{\lambda} e^{i\phi}X x \rangle_A$$

$$- \langle e^{i\theta} \left( \lambda e^{i\theta}Yx - \frac{1}{\lambda} e^{i\phi}X x \right), \lambda e^{i\theta}Yx - \frac{1}{\lambda} e^{i\phi}X x \rangle_A$$

$$\Rightarrow 2e^{2i\theta} \langle e^{-i\phi}X^{2A}TY x, x \rangle_A + 2e^{i\phi} \langle Y^{2A}TX x, x \rangle_A$$

$$= \langle e^{i\theta} \left( \lambda e^{i\theta}Yx + \frac{1}{\lambda} e^{i\phi}X x \right), \lambda e^{i\theta}Yx + \frac{1}{\lambda} e^{i\phi}X x \rangle_A$$

$$- \langle e^{i\theta} \left( \lambda e^{i\theta}Yx - \frac{1}{\lambda} e^{i\phi}X x \right), \lambda e^{i\theta}Yx - \frac{1}{\lambda} e^{i\phi}X x \rangle_A$$

$$\Rightarrow 2 |\langle X^{2A}T Y x, x \rangle_A| + 2 |\langle Y^{2A}TX x, x \rangle_A|$$

$$\leq \left| \langle e^{i\theta} \left( \lambda e^{i\theta}Yx + \frac{1}{\lambda} e^{i\phi}X x \right), \lambda e^{i\theta}Yx + \frac{1}{\lambda} e^{i\phi}X x \rangle_A \right|$$

$$+ \left| \langle e^{i\theta} \left( \lambda e^{i\theta}Yx - \frac{1}{\lambda} e^{i\phi}X x \right), \lambda e^{i\theta}Yx - \frac{1}{\lambda} e^{i\phi}X x \rangle_A \right|$$

$$\Rightarrow 2 |\langle X^{2A}T Y x, x \rangle_A| + 2 |\langle Y^{2A}TX x, x \rangle_A|$$

$$\leq w_A(T) \left( \| \lambda e^{i\theta}Yx + \frac{1}{\lambda} e^{i\phi}X x \|^2_A + \| \lambda e^{i\theta}Yx - \frac{1}{\lambda} e^{i\phi}X x \|^2_A \right)$$

$$\Rightarrow |\langle X^{2A}T Y x, x \rangle_A| + |\langle Y^{2A}TX x, x \rangle_A| \leq w_A(T) \left( \lambda^2 ||Yx||_A^2 + \frac{1}{\lambda^2} ||Xx||_A^2 \right).$$

This holds for all non-zero real $\lambda$. If $||Yx||_A \neq 0$, then we choose $\lambda^2 = \frac{||Xx||_A}{||Yx||_A}$. So, we get

$$|\langle X^{2A}T Y x, x \rangle_A| + |\langle Y^{2A}TX x, x \rangle_A| \leq 2w_A(T) ||Xx||_A ||Yx||_A.$$ 

Clearly this inequality also holds when $||Yx||_A = 0$, i.e., $Yx = 0$. This completes the proof of the lemma. □
Remark 3.5. In [11] we have already generalized the result obtained by Bernau and Smithes [6, Lemma 3] and proved some important numerical radius inequalities.

Now using Lemma 3.4, we obtain the following inequalities involving A-numerical radius, A-Crawford number and A-operator norm.

**Theorem 3.6.** Let $X, T, Y \in \mathcal{B}_A(H)$ where $A > 0$. Then

$$c_A(X^{\sharp_A}TY) + w_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A,$$

$$w_A(X^{\sharp_A}TY) + c_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A.$$

**Proof.** Taking $\|x\|_A = 1$ in Lemma 3.4, we have

$$|\langle X^{\sharp_A}TY, x \rangle_A| + |\langle Y^{\sharp_A}TX, x \rangle_A| \leq 2w_A(T)\|X\|_A\|Y\|_A \Rightarrow c_A(X^{\sharp_A}TY) + |\langle Y^{\sharp_A}TX, x \rangle_A| \leq 2w_A(T)\|X\|_A\|Y\|_A.$$

Taking supremum over $\|x\|_A = 1$, we get

$$c_A(X^{\sharp_A}TY) + w_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A.$$

Again taking $\|x\|_A = 1$ in Lemma 3.4, we have

$$|\langle X^{\sharp_A}TY, x \rangle_A| + |\langle Y^{\sharp_A}TX, x \rangle_A| \leq 2w_A(T)\|X\|_A\|Y\|_A \Rightarrow |\langle X^{\sharp_A}TY, x \rangle_A| + c_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A.$$

Taking supremum over $\|x\|_A = 1$, we get

$$w_A(X^{\sharp_A}TY) + c_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A.$$

This completes the proof of the theorem.

Now taking $Y = I, T = X$ and $X = Y$ in the above Theorem 3.6, we get the following upper bounds for the numerical radius of product of two operators, which improve on the existing bounds.

**Corollary 3.7.** Let $X, Y \in \mathcal{B}_A(H)$ where $A > 0$. Then the following inequalities hold:

$$w_A(XY) \leq 2w_A(X)\|Y\|_A - c_A(Y^{\sharp_A}X),$$

$$w_A(XY) \leq 2w_A(Y)\|X\|_A - c_A(YX^{\sharp_A}).$$

**Remark 3.8.** For $A > 0$, it is clear that the inequalities obtained in Corollary 3.7 improve on the inequalities $w_A(XY) \leq 2w_A(X)\|Y\|_A$ and $w_A(XY) \leq 2w_A(Y)\|X\|_A$, (see [25, Th. 3.4]).

Finally using Lemma 3.4 we obtain new inequalities for B-numerical radius of $2 \times 2$ operator matrices with zero operators as main diagonal entries.
Theorem 3.9. Let $X, Y \in \mathcal{B}_A(\mathcal{H})$ where $A > 0$. Then the following inequalities hold:

(i) $\|X\|_A^2 + c_A(YX) \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|X\|_A$,

(ii) $m_A^2(X) + w_A(YX) \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|X\|_A$,

(iii) $\|Y\|_A^2 + c_A(YX) \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|Y\|_A$,

(iv) $m_A^2(Y) + w_A(YX) \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|Y\|_A$.

Proof. Taking $X = T$ and $Y = I$ in Lemma 3.4 we get,

$\|Tx\|_A^2 + \langle T^2x, x \rangle_A \leq 2w_A(T)\|Tx\|_A\|x\|_A$.

This also holds if we take $T = \left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)$ and $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\|_B = 1$, i.e., $\|x_1\|_A^2 + \|x_2\|_A^2 = 1$. Therefore we get,

$\|Xx_2\|_A^2 + \|Yx_1\|_A^2 + \langle XYx_1, x_1 \rangle_A + \langle YXx_2, x_2 \rangle_A \leq 2w_B(T)\left(\|Xx_2\|_A^2 + \|Yx_1\|_A^2\right)^{\frac{1}{2}}$.

Taking $x_1 = 0$, we get

$\|Xx_2\|_A^2 + \langle YXx_2, x_2 \rangle_A \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|Xx_2\|_A$,

$\Rightarrow \|Xx_2\|_A^2 + \langle YXx_2, x_2 \rangle_A \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|X\|_A$,

$\Rightarrow \|Xx_2\|_A^2 + c_A(YX) \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|X\|_A$.

Taking supremum over $\|x_2\|_A = 1$, we get the inequality (i), i.e.,

$\|X\|_A^2 + c_A(YX) \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|X\|_A$.

Again from the inequality

$\|Xx_2\|_A^2 + \langle YXx_2, x_2 \rangle_A \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|X\|_A$, we get

$m_A^2(X) + \langle YXx_2, x_2 \rangle_A \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|X\|_A$.

Taking supremum over $\|x_2\|_A = 1$, we get the inequality (ii), i.e.,

$m_A^2(X) + w_A(YX) \leq 2w_B\left(\begin{array}{cc} O & X \\ Y & O \end{array}\right)\|X\|_A$. 
Similarly taking $x_2 = 0$ and supremum over $\|x_1\|_A = 1$, we can prove the remaining inequalities.

Next taking $X = Y = T$ in Theorem 3.9 and using Lemma 2.4 (iv), we get the following lower bounds for $A$-numerical radius.

**Theorem 3.10.** Let $T \in \mathcal{B}_A(\mathcal{H})$ with $\|T\|_A \neq 0$ where $A > 0$. Then the following inequalities hold:

$$w_A(T) \geq \frac{\|T\|_A^2}{2} + \frac{c_A(T^2)}{2\|T\|_A},$$

$$w_A(T) \geq \frac{m_A^2(T)}{2\|T\|_A} + \frac{w_A(T^2)}{2\|T\|_A}.$$  

**Remark 3.11.** Here we note that the two inequalities obtain in Theorem 3.10 are incomparable. So, using these bounds we have a new lower bound

$$w_A(T) \geq \frac{1}{2\|T\|_A} \max \left\{ \|T\|_A^2 + c_A(T^2), m_A^2(T) + w_A(T^2) \right\},$$  

where $T \in \mathcal{B}_A(\mathcal{H})$ with $\|T\|_A \neq 0$. It is clear that this inequality improves on the first inequality in [25, Cor. 2.8].

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