ON THE WEAK LIMITS OF SMOOTH MAPS FOR THE DIRICHLET ENERGY BETWEEN MANIFOLDS

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Abstract. We identify all the weak sequential limits of smooth maps in $W^{1,2}(M,N)$. In particular, this implies a necessary and sufficient topological condition for smooth maps to be weakly sequentially dense in $W^{1,2}(M,N)$.

1. Introduction

Assume $M$ and $N$ are smooth compact Riemannian manifolds without boundary and they are embedded into $\mathbb{R}^l$ and $\mathbb{R}^l$ respectively. The following spaces are of interest in the calculus of variations:

$$W^{1,2}(M,N) = \{u \in W^{1,2}(M,\mathbb{R}^l) : u(x) \in N \text{ a.e. } x \in M\},$$

$$H^{1,2}_W(M,N) = \{u \in W^{1,2}(M,N) : \text{there exists a sequence } u_i \in C^\infty(M,N) \text{ such that } u_i \rightharpoonup u \text{ in } W^{1,2}(M,N)\}.$$  

For a brief history and detailed references on the study of analytical and topological issues related to these spaces, one may refer to [2, 3, 7]. In particular, it follows from theorem 7.1 of [3] that a necessary condition for $H^{1,2}_W(M,N) = W^{1,2}(M,N)$ is that $M$ satisfies the 1-extension property with respect to $N$ (see section 2.2 of [3] for a definition). It was conjectured in section 7 of [3] that the 1-extension property is also sufficient for $H^{1,2}_W(M,N) = W^{1,2}(M,N)$. In [1, 7], it was shown that $H^{1,2}_W(M,N) = W^{1,2}(M,N)$ when $\pi_1(M) = 0$ or $\pi_1(N) = 0$. Note that if $\pi_1(M) = 0$ or $\pi_1(N) = 0$, then $M$ satisfies the 1-extension property with respect to $N$. In section 8 of [4], it was proved that the above conjecture is true under the additional assumption that $N$ satisfies the 2-vanishing condition. The main aim of the present article is to confirm the conjecture in its full generality. More precisely, we have

**Theorem 1.1.** Let $M^n$ and $N$ be smooth compact Riemannian manifolds without boundary ($n \geq 3$). Take a Lipschitz triangulation $h : K \to M$, then

$$H^{1,2}_W(M,N)$$

$$= \{u \in W^{1,2}(M,N) : u_\#(h) \text{ has a continuous extension to } M \text{ w.r.t. } N\}$$

$$= \{u \in W^{1,2}(M,N) : u \text{ may be connected to some smooth maps}\}.$$  

In addition, if $\alpha \in [M,N]$ satisfies $\alpha \circ h|_{K^1_1} = u_\#(h)$, then we may find a sequence of smooth maps $u_i \in C^\infty(M,N)$ such that $u_i \to u$ in $W^{1,2}(M,N)$, $[u_i] = \alpha$ and $du_i \to du$ a.e..
Here \( u_{\#} \) is the 1-homotopy class defined by White [8] (see also section 4 of [3]) and \([M, N]\) means all homotopy classes of maps from \( M \) to \( N \). It follows from Theorem 1.1 that

**Corollary 1.1.** Let \( M^n \) and \( N \) be smooth compact Riemannian manifolds without boundary and \( n \geq 3 \). Then smooth maps are weakly sequentially dense in \( W^{1,2}(M, N) \) if and only if \( M \) satisfies the 1-extension property with respect to \( N \).

For \( p \in [3, n - 1] \) being an natural number, it remains a challenging open problem to find out whether the weak sequential density of smooth maps in \( W^{1,p}(M, N) \) is equivalent to the condition that \( M \) satisfies the \( p - 1 \) extension property with respect to \( N \). This was verified to be true under further topological assumptions on \( N \) (see section 8 of [4]). However, even for \( W^{1,3}(S^1, S^2) \), it is still not known whether smooth maps are weakly sequentially dense. Some very interesting recent work on this space can be found in [5].

The paper is written as follows. In Section 2, we will present some technical preparations. This was proved by Pakzad and Riviere in [7]. The following lemma is a rough version of Luckhaus’s lemma [6]. For reader’s convenience, we will use those notations and concepts in section 2, 3 and 4 of [3]. The following local result, which was proved by Pakzad and Riviere in [7], plays an important role in our discussion.

**Theorem 2.1 ([7]).** Let \( N \) be a smooth compact Riemannian manifold. Assume \( n \geq 3 \), \( B_1 = B_1^1 \), \( f \in W^{1,2}(\partial B_1, N) \cap C(\partial B_1, N) \), \( f \sim \text{const} \), \( u \in W^{1,2}(B_1, N) \), \( u|_{\partial B_1} = f \), then there exists a sequence \( u_i \in W^{1,2}(B_1, N) \cap C(B_1, N) \) such that \( u_i|_{\partial B_1} = f \), \( u_i \rightarrow u \) in \( W^{1,2}(B_1, N) \) and \( du_i \rightarrow du \) a.e. Moreover, if \( v \in W^{1,2}(B_2 \setminus B_1, N) \cap C(B_2, B_1, N) \) satisfies \( v|_{\partial B_1} = f \) and \( v|_{\partial B_2} \equiv \text{const} \), then we may estimate

\[
\int_{B_1} |du_i|^2 dH^n \leq c(n, N) \left( \int_{B_1} |du|^2 dH^n + \int_{B_2 \setminus B_1} |dv|^2 dH^n \right).
\]

For convenience, we will use those notations and concepts in sections 2, 3 and 4 of [3]. The following lemma is a rough version of Luckhaus’s lemma [6]. For reader’s convenience, we sketch a proof of this simpler version using results from section 3 of [3].

**Lemma 2.1.** Assume \( M^n \) and \( N \) are smooth compact Riemannian manifolds without boundary. Let \( \varepsilon > 0 \), \( 0 < \delta < 1 \), \( A > 0 \), then there exists an \( \varepsilon = \varepsilon(e, \delta, A, M, N) > 0 \) such that for any \( u, v \in W^{1,2}(M, N) \) with \( |du|_{L^2(M)} \leq A \) and \( |u - v|_{L^2(M)} \leq \varepsilon \), we may find a \( w \in W^{1,2}(M \times (0, \delta), N) \) such that, in the trace sense \( w(x, 0) = u(x) \), \( w(x, \delta) = v(x) \) a.e. \( x \in M \) and

\[
|du|_{L^2(M \times (0, \delta))} \leq c(M) \sqrt{\left( |du|_{L^2(M)} + |dv|_{L^2(M)} + \varepsilon \right)}.
\]

**Proof.** Let \( \varepsilon_M > 0 \) be a small positive number such that

\[
V_{2\varepsilon_M}(M) = \{ x \in \mathbb{R}^l : d(x, M) < 2\varepsilon_M \}
\]

is a tubular neighborhood of \( M \). Let \( \pi_M : V_{2\varepsilon_M}(M) \rightarrow M \) be the nearest point projection. Similarly we have \( \varepsilon_N, V_{2\varepsilon_N}(N) \) and \( \pi_N \) for \( N \). Choose a Lipschitz
cubeulation $h : K \to M$. We may assume each cell in $K$ is a cube of unit size. For $\xi \in B^d_{\epsilon_M}, x \in |K|$, let $h_\xi(x) = \pi_M(h(x) + \xi)$. Assume $\epsilon_M$ is small enough such that all $h_\xi$’s are bi-Lipschitz maps. Set $m = \left[\frac{1}{2}\right] + 1$, using $[0, 1] = \cup_{i = 1}^m \left[\frac{i - 1}{m}, \frac{i}{m}\right]$, we may divide each $k$-cube in $K$ into $m^k$ small cubes. In particular, we get a subdivision of $K$, called $K_m$. It follows from section 3 of [3] that for a.e. $\xi \in B^d_{\epsilon_M}$, $u \circ h_\xi, v \circ h_\xi \in W^{1,2}(K_m, N)$. Applying the estimates in section 3 of [3] to each unit size $k$-cube in $|K_m|$, we get

$$
\begin{align*}
\int_{B^d_{\epsilon_M}} d\mathcal{H}^l(\xi) \int_{|K_m|} \left|d \left(u \circ h_\xi|_{K_m}\right)\right|^2 \, d\mathcal{H}^k \leq c(M) \delta^{k-n} \|du\|_{L^2(M)}^2, \\
\int_{B^d_{\epsilon_M}} d\mathcal{H}^l(\xi) \int_{|K_m|} \left|d \left(v \circ h_\xi|_{K_m}\right)\right|^2 \, d\mathcal{H}^k \leq c(M) \delta^{k-n} \|dv\|_{L^2(M)}^2,
\end{align*}
$$

and

$$
\begin{align*}
\left(\int_{B^d_{\epsilon_M}} \left|u \circ h_\xi - v \circ h_\xi\right|^2_{L^\infty(|K_m|)} \, d\mathcal{H}^l(\xi)\right)^{\frac{1}{2}} \\
\leq c(\delta, M) \left(\|d(u - v)\|_{L^2(M)}^2 \|u - v\|_{L^2(M)}^2 + \|u - v\|_{L^2(M)}^2\right) \\
\leq c(\delta, A, M) \epsilon_N^{\frac{1}{2}}.
\end{align*}
$$

By the mean value inequality, we may find a $\xi \in B^d_{\epsilon_M}$ such that $u \circ h_\xi, v \circ h_\xi \in W^{1,2}(K_m, N)$,

$$
\left|u \circ h_\xi - v \circ h_\xi\right|_{L^\infty(|K_m|)} \leq c(\delta, A, M) \epsilon_N^{\frac{1}{2}} < \epsilon
$$

when $\epsilon$ is small enough, and

$$
\begin{align*}
\int_{|K_m|} \left[d\left(u \circ h_\xi|_{K_m}\right)\right]^2 + d\left(v \circ h_\xi|_{K_m}\right)^2 \, d\mathcal{H}^k \\
\leq c(M) \delta^{k-n} \left(\|du\|_{L^2(M)}^2 + \|dv\|_{L^2(M)}^2\right)
\end{align*}
$$

for $1 \leq k \leq n$. Fix a $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta_{(-\infty, \frac{1}{2})} = 1$ and $\eta_{(\frac{1}{2}, \infty)} = 0$. Letting $f = u \circ h_\xi, g = v \circ h_\xi$, we will define $\phi : |K| \times [0, \delta] \to N$ inductively. First set $\phi(x, 0) = f(x)$ and $\phi(x, \delta) = g(x)$ for $x \in |K|$. For $\Delta \in K_m \setminus K_m^0$, on $\Delta \times [0, \delta]$, we let

$$
\phi(x, t) = \pi_N \left(\eta_{\left(\frac{t}{\delta}\right)} f(x) + \left(1 - \eta_{\left(\frac{t}{\delta}\right)}\right) g(x)\right), \quad x \in \Delta, 0 \leq t \leq \delta.
$$

For $\Delta \in K_m^2 \setminus K_m^1$, let $y_\Delta$ be the center of $\Delta$, and define $\phi$ on $\Delta \times [0, \delta]$ as the homogeneous degree zero extension of $\phi|_{\partial \Delta \times [0, \delta]}$ with respect to $(y_\Delta, \delta)$. Next we handle each 3-cube, 4-cube, $\cdots$, $n$-cube in a similar way. Calculations show that

$$
\begin{align*}
\int_{|K| \times [0, \delta]} |d\phi|^2 \, d\mathcal{H}^{n+1} \\
\leq c(n) \sum_{k=1}^n \delta^{n+1-k} \int_{|K_m|} \left[d\left(u \circ h_\xi|_{K_m}\right)\right]^2 + \left[d\left(v \circ h_\xi|_{K_m}\right)\right]^2 \, d\mathcal{H}^k + c(\delta, A, M) \epsilon_N^{\frac{1}{2}} \\
\leq c(M) \delta \left(\|du\|_{L^2(M)}^2 + \|dv\|_{L^2(M)}^2 + \epsilon^2\right)
\end{align*}
$$
when \( \varepsilon \) is small enough. Finally \( w : M \times [0, \delta] \to N \), defined by \( w(x,t) = \phi \left( h^{-1}_\xi(x) \right) \), is the needed map. \( \square \)

**Lemma 2.2.** Assume \( N \) is a smooth compact Riemannian manifold, \( n \geq 2 \), \( B_1 = B_1^n \), \( u, v \in W^{1,2}(B_1, N) \) such that \( u|_{\partial B_1} = v|_{\partial B_1} \). Define \( w : B_1 \times (0,1) \to N \) by

\[
w(x,t) = \begin{cases} u \left( \frac{x}{|x|^2} \right), & x \in B_1 \setminus B_1; \\ v \left( \frac{x}{|x|^2} \right), & x \in B_1 \setminus B_1; \\ \end{cases}
\]

then \( w \in W^{1,2}(B_1 \times (0,1), N) \) and

\[
|dw|_{L^2(B_1 \times (0,1))} \leq c(n) \left( |du|_{L^2(B_1)} + |dv|_{L^2(B_1)} \right).
\]

**Proof.** Note that

\[
|dw(x,t)| \leq \begin{cases} |du(x)|, & t < |x|; \\ c(n) \left| du \left( \frac{x}{|x|^2} \right) \right| \frac{t^2}{|x|}, & t^2 < |x| < t; \\ c(n) \left| dv \left( \frac{x}{|x|^2} \right) \right| \frac{1}{t^2}, & |x| < t^2.
\end{cases}
\]

Hence

\[
\int_{0 < t < 1} \int_{t^2 < |x| < t} |dw(x,t)|^2 \, dH^{n+1}(x,t)
\]

\[
\leq c(n) \int_0^1 dt \int_{t^2}^{t} \int_{\partial B_r} \left| du \left( \frac{y^2}{y^2} \right) \right| \frac{r^2}{r^4} \, dH^{n-1}(x)
\]

\[
= c(n) \int_0^1 dt \int_{t}^{1} ds \int_{\partial B_s} \left| du(y) \right|^2 \, dH^{n-1}(y)
\]

\[
\leq c(n) |du|_{L^2(B_1)}^2;
\]

and

\[
\int_{0 < t < 1} \int_{|x| < t^2} |dw(x,t)|^2 \, dH^{n+1}(x,t)
\]

\[
\leq c(n) \int_0^1 dt \int_{B_{1/2}} \left| dv \left( \frac{x}{t^2} \right) \right| \frac{1}{t^4} \, dH^n(x)
\]

\[
\leq c(n) |dv|_{L^2(B_1)}^2.
\]

The lemma follows. \( \square \)

### 3. Identifying weak limits of smooth maps

In this section we shall prove Theorem 1.1 and Corollary 1.1.

**Proof of Theorem 1.1.** Let \( h : K \to M \) be a Lipschitz cubeulation. We may assume each cell in \( K \) is a cube of unit size. Let \( \varepsilon_M > 0 \) be a small number such that

\[
V_{2\varepsilon_M}(M) = \{ x \in \mathbb{R}^l : d(x, N) < 2\varepsilon_M \}
\]

is a tubular neighborhood of \( M \). Denote \( \pi_M : V_{2\varepsilon_M}(M) \to M \) as the nearest point projection. For \( \xi \in B_1^l \), we let \( h_\xi(x) = \pi_M \left( h(x) + \xi \right) \) for \( x \in |K| \), the polytope of \( K \). We may assume \( \varepsilon_M \) is small enough such that all \( h_\xi \) are bi-Lipschitz maps. Replacing \( h \) by \( h_\xi \) when necessary, we may assume \( f = u \circ h \in W^{1,2}(K, N) \).
Then we may find a $g \in C (|K|, N) \cap W^{1,2} (K, N)$ such that $[g \circ h^{-1}]=\alpha$ and $g|_{|K^1|}=f|_{|K_1|}$ (see the proof of theorem 5.5 and theorem 6.1 in [4]). For each cell $\Delta \in K$, let $y_\Delta$ be the center of $\Delta$. For $x \in \Delta$, let $|x|_\Delta$ be the Minkowski norm with respect to $y_\Delta$, that is

$$|x|_\Delta = \inf \left\{ t > 0 : y_\Delta + \frac{x-y_\Delta}{t} \in \Delta \right\}.$$  

**Step 1:** For every $\Delta \in K^2 \setminus K^1$, we may find a sequence $\phi_i \in C (\Delta, N) \cap W^{1,2} (\Delta, N)$ such that $\phi_i|_{\partial \Delta} = g|_{\partial \Delta}$, $\phi_i \rightarrow f|_{\Delta}$ in $W^{1,2} (\Delta, N)$ and $d\phi_i \rightarrow d (f|_{\Delta})$ a.e. (see lemma 4.4 in [3]). For $x \in \Delta$, let

$$f_i (x) = \begin{cases} \phi_i (x), & |x|_\Delta \geq \frac{1}{2}; \\ \phi_i \left( y_\Delta + \frac{1}{2} \frac{x-y_\Delta}{|x|_\Delta} \right), & \frac{1}{4} \leq |x|_\Delta \leq \frac{1}{2}; \\ g \left( y_\Delta + 2^{2j} (x-y_\Delta) \right), & |x|_\Delta \leq \frac{1}{2}. \end{cases}$$

It is clear that $f_i \rightharpoonup f|_{\Delta}$ in $W^{1,2} (\Delta, N)$, $df_i \rightarrow d (f|_{\Delta})$ a.e. on $\Delta$,

$$|df_i|_{L^2 (\Delta)} \leq c \cdot \left( |d\phi_i|_{L^2 (\Delta)} + |d (g|_{\Delta})|_{L^2 (\Delta)} \right) \leq c (f, g)$$

and $f_i \in C (|K^2|, N)$. In addition, if we define $h_{2,i} : \Delta \times [0, 1] \rightarrow N$ by

$$h_{2,i} (x, t) = \begin{cases} \phi_i (x), & |x|_\Delta \geq \frac{1}{2}; \\ \phi_i \left( y_\Delta + \frac{1}{2} \frac{x-y_\Delta}{|x|_\Delta} \right), & \frac{1}{8} + \frac{2^{j-1} t}{|x|_\Delta} \leq |x|_\Delta \leq \frac{1}{2} + \frac{2^{j-1} t}{|x|_\Delta}; \\ g \left( y_\Delta + \frac{x-y_\Delta}{\frac{1}{2} + \frac{2^{j-1} t}{|x|_\Delta}} \right), & |x|_\Delta \leq \left( \frac{1}{2} + \frac{2^{j-1} t}{|x|_\Delta} \right)^2. \end{cases}$$

Then by Lemma 2.2 we know $h_{2,i} \in W^{1,2} (\Delta \times [0, 1], N)$,

$$|dh_{2,i}|_{L^2 (\Delta \times [0, 1])} \leq c \cdot \left( |d\phi_i|_{L^2 (\Delta)} + |d (g|_{\Delta})|_{L^2 (\Delta)} \right) \leq c (f, g)$$

and $h_{2,i} \in C (|K^2| \times [0, 1], N)$.

**Step 2:** Assume for some $2 \leq k \leq n-1$, we have a sequence $f_i \in C (|K^k|, N) \cap W^{1,2} (K^k, N)$ and $h_{k,i} \in C (|K^k| \times [0, 1], N)$ such that for each $\Delta \in K^k$, $f_i \rightharpoonup f|_{\Delta}$ in $W^{1,2} (\Delta, N)$, $h_{k,i} \in W^{1,2} (\Delta \times [0, 1], N)$,

$$|d (f_i)|_{L^2 (\Delta \times [0, 1])} \leq c (f, g), \quad |dh_{k,i}|_{L^2 (\Delta \times [0, 1])} \leq c (f, g)$$

and $h_{k,i} (x, 0) = f_i (x)$, $h_{k,i} (x, 1) = g(x)$ for $x \in |K^k|$. Since for every $\Delta \in K^{k+1} \setminus K^k$, $f_i \rightharpoonup f|_{\partial \Delta}$ in $W^{1,2} (\partial \Delta, N)$, for fixed $j$ by Lemma 2.1 we may find a $n_j \geq j$ such that for each $\Delta \in K^{k+1} \setminus K^k$, there exists a $w_j \in W^{1,2} (\partial \Delta \times [0, 2^{-j}], N)$ with $w_j (x, 0) = f (x)$, $w_j (x, 1) = f_{n_j} (x)$ and

$$|dw_j|_{L^2 (\partial \Delta \times [0, 1])} \leq c (n_j) \frac{c (f, g)}{2^j} \left( |d (f|_{\partial \Delta})|_{L^2 (\partial \Delta)} + |df_{n_j}|_{L^2 (\partial \Delta)} + 1 \right) \leq \frac{c (f, g)}{2^j}.$$  

Without loss of generality, we may replace $f_i$ by $f_{n_i}$ and $h_{k,i}$ by $h_{k,n_i}$. Fix a $\Delta \in K^{k+1} \setminus K^k$. For $x \in \Delta$, let

$$\psi_i (x) = \begin{cases} f \left( y_\Delta + \frac{2^j (x-y_\Delta)}{2^{j-1}} \right), & |x|_\Delta \leq \frac{2^j-1}{2^{j-1}}; \\ w_j \left( y_\Delta + \frac{x-y_\Delta}{|x|_\Delta} \right), & \frac{2^j-1}{2^{j-1}} \leq |x|_\Delta \leq 1. \end{cases}$$
Then $\psi_i|_{K^k} = f_i$ and $\psi_i \rightharpoonup f_1$ in $W^{1,2}(\Delta, N)$ as $i \to \infty$ for each $\Delta \in K^{k+1}\setminus K^k$. By Theorem 2.1 and (3.3) (use $h_{k,i}$ and $g$ for the needed “$v$” in Theorem 2.1 one may refer to lemma 9.8 of [4]), for every $\Delta \in K^{k+1}\setminus K^k$, we may find $\phi_i \in C(\overline{\Delta}, N) \cap W^{1,2}(\Delta, N)$ such that $\phi_i|_{\partial \Delta} = f_i|_{\partial \Delta}$, $|\phi_i - \psi_i|_{L^2(\Delta)} \leq \frac{1}{2^i}$, $|d\phi_i|_{L^2(\Delta)} \leq c(f, g)$ and

$$
\int_M \frac{|d\phi_i - d\psi_i|}{1 + |d\phi_i - d\psi_i|} \, d\mathcal{H}^{k+1} \leq \frac{1}{2^i}.
$$

After passing to subsequence, we may assume $d\phi_i \rightharpoonup d(f|_\Delta)$ a.e. on $\Delta$. Fix a $\Delta \in K^{k+1}\setminus K^k$, for any $x \in \Delta$, define

$$
g_{k+1,i}(x) = \begin{cases} 
  h_{k,i} \left( y_{\Delta} + \frac{x - y_{\Delta}}{|x_{\Delta}|}, 1 + 2 \left( \frac{1}{2} - |x_{\Delta}| \right) \right), & \frac{1}{2} \leq |x_{\Delta}| \leq 1; \\
  g \left( y_{\Delta} + 2(x - y_{\Delta}) \right), & |x_{\Delta}| \geq \frac{1}{2}; \\
  \phi_i (x), & |x_{\Delta}| \leq \frac{1}{2},
\end{cases}
$$

$$
f_i(x) = \begin{cases} 
  \phi_i (x), & |x_{\Delta}| \geq \frac{1}{2}; \\
  g_{k+1,i} \left( y_{\Delta} + \frac{1}{2^{k+1} |x_{\Delta}|} \frac{x - y_{\Delta}}{|x_{\Delta}|} \right), & \frac{1}{2} \leq |x_{\Delta}| \leq \frac{1}{2^i}; \\
  g_{k+1,i} \left( y_{\Delta} + 2^{k+1} \left( x - y_{\Delta} \right) \right), & |x_{\Delta}| \leq \frac{1}{2},
\end{cases}
$$

$$
\overline{h}_{k+1,i}(x, t) = \begin{cases} 
  \phi_i \left( y_{\Delta} + \left( \frac{1}{t} + \frac{t - 1}{2} \right)^2 \frac{x - y_{\Delta}}{|x_{\Delta}|} \right), & \left( \frac{1}{t} + \frac{t - 1}{2} \right)^2 \leq |x_{\Delta}| \leq \frac{1}{2} + \frac{2(1-t)}{t}; \\
  g_{k+1,i} \left( y_{\Delta} + \left( \frac{1}{t} + \frac{t - 1}{2} \right)^2 \frac{x - y_{\Delta}}{|x_{\Delta}|} \right), & \frac{1}{2} + \frac{2(1-t)}{t} \leq |x_{\Delta}| \leq \frac{1}{2} + \frac{2(t-1)}{t};
\end{cases}
$$

$$
\underline{h}_{k+1,i}(x, t) = \begin{cases} 
  h_{k,i} \left( y_{\Delta} + \frac{x - y_{\Delta}}{|x_{\Delta}|}, 1 + 2 \left( \frac{1}{t^2} - |x_{\Delta}| \right) \right), & \frac{1}{t^2} \leq |x_{\Delta}| \leq 1; \\
  g \left( y_{\Delta} + \frac{2}{t^2} \left( x - y_{\Delta} \right) \right), & |x_{\Delta}| \leq \frac{1}{t^2},
\end{cases}
$$

and

$$
h_{k+1,i}(x, t) = \begin{cases} 
  \overline{h}_{k+1,i}(x, 2t), & 0 \leq t \leq \frac{1}{2}; \\
  \underline{h}_{k+1,i}(x, 2t - 1), & \frac{1}{2} \leq t \leq 1.
\end{cases}
$$

Simple calculations show that for any $\Delta \in K^{k+1}\setminus K^k$, $f_i \rightharpoonup f|_\Delta$ in $W^{1,2}(\Delta, N)$, $df_i \rightharpoonup d(f|_\Delta)$ a.e. on $\Delta$, $h_{k+1,i} \in W^{1,2}(\Delta \times [0, 1], N)$,

$$
|df_i|_{L^2(\Delta)} \leq c(f, g), \quad |dh_{k+1,i}|_{L^2(\Delta \times [0, 1])} \leq c(f, g)
$$
and $h_{k+1,i}(x, 0) = f_i(x)$, $h_{k+1,i}(x, 1) = g(x)$ as $x \in |K^{k+1}|$. Hence we finish when we reach $f_i \in C(|K|, N) \cap W^{1,2}(K, N)$ and $h_{k,i} \in C(|K| \times [0, 1], N)$. Let $v_i = f_i \circ h^{-1}$. Then it is clear that $v_i \in C(M, N) \cap W^{1,2}(M, N)$, $[v_i] = \alpha$, $|v_i - u|_{L^2(M)} \to 0$, $|dv_i|_{L^2(M)} \leq c(u, g)$ and $dv_i \to du$ a.e. on $M$. Hence, we may find $u_i \in C^\infty(M, N)$ such that $|v_i - u|_{L^2(M)} \to 0$, $|du_i|_{L^2(M)} \leq c(u, g)$, $[u_i] = \alpha$ and $du_i \to du$ a.e. on $M$. In particular, this shows

$$
H^{1,2}_{W}(M, N) \supset \{ u \in W^{1,2}(M, N) : u \#_2 (h) \text{ has a continuous extension to } M \text{ w.r.t. } N \}.
$$

The other direction of inclusion was proved in section 7 of [3]. To see

$$
H^{1,2}_{W}(M, N) = \{ u \in W^{1,2}(M, N) : u \text{ may be connected to some smooth maps} \}.
$$
we only need to use the above proved equality and proposition 5.2 of [3], which shows
\[ \{ u \in W^{1,2}(M, N) : u\#_2(h) \text{ has a continuous extension to } M \text{ w.r.t. } N \} \]
\[ = \{ u \in W^{1,2}(M, N) : u \text{ may be connected to some smooth maps} \}. \]

□

We remark that many constructions above are motivated from section 5 and section 6 of [4].

Proof of Corollary 1.1. This follows from Theorem 1.1 and corollary 5.4 of [3]. □

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