On products of skew rotations.

M.D. Arnold \(^*\)\(^\dagger\) E.I. Dinaburg \(^*\)\(^\ddagger\) G.B. Dobrushina \(^*\)
S.A. Pirogov \(^*\) A.N. Rybko \(^*\)

October 14, 2011

Abstract

Let \(\{S^1_t\}, ..., \{S^n_t\}\) be the one-parametric groups of shifts along the orbits of Hamiltonian systems generated by time-independent Hamiltonians \(H_1, ..., H_n\) with one degree of freedom. In some problems of population genetics there appear planar transformations having the form \(S^{h_n}_n \circ \cdots \circ S^{h_1}_1\) under some conditions on Hamiltonians \(H_1, ..., H_n\). In this paper we study asymptotical properties of trajectories of such transformations. We show that under classical non-degeneracy condition on Hamiltonians trajectories stay in the invariant annuli for generic combination of lengths \(h_1, ..., h_n\) while for the special case \(h_1 + \cdots + h_n = 0\) there exists trajectory escaping to infinity.

Mathematics Subject Classification: 37J40, 37J15, 37M05;
Keywords: KAM–Theory, Hamiltonian Systems

1 Introduction.

Let \(H_1(p, q), H_2(p, q)\) be two time-independent Hamiltonians with one degree of freedom. Typical examples can be \(H_1 = \sqrt{(p - p_1)^2 + (q - q_1)^2}\) and \(H_2 = \sqrt{(p - p_2)^2 + (q - q_2)^2}\) where \(p_1, q_1, p_2, q_2\) are some constants. Sometimes instead of simplectic coordinates \((p, q)\) we shall write \((x, y)\) or in complex notation \(z = x + iy\). Denote by \(\{S^1_t\}, \{S^2_t\}\) the one-parametric groups of shifts along the trajectories of the first and second system. Ya.G. Sinai formulated a general question about asymptotic properties of transformations \(T^{(h_1, h_2)} = S^{h_2}_2 \circ S^{h_1}_1\) where \(h_1, h_2\) are fixed numbers.

Similar transformations appear in some problems of population genetics (see [2]). In the example described above map \(T^{(h_1, h_2)}\) moves each point \((p, q)\) along the circle with center \((p_1, q_1)\) for the arc length \(h_1\) in positive direction and then moves its image along the circle centred at

\(^*\)Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute), Bolshoi Karetny per. 19, Moscow, 127994, Russia

\(^\dagger\)International Institute of Earthquake Prediction Theory and Mathematical Geophysics of the Russian Academy of Sciences, Profsoyuznaya str., 84/32, Moscow, 117997, Russia

\(^\ddagger\)Schmidt Institute of Physics of the Earth of the Russian Academy of Sciences, B. Gruzinskaya str., 10, Moscow, 123995, Russia
Let in the previous example $p_1 = q_1 = 0$. Then $H_1 = \sqrt{p^2 + q^2}$ and the corresponding Hamiltonian flow in usual polar coordinates $\varphi = \arg(z)$, $\rho = |z|$ has the form

$$
\begin{cases}
\varphi_t = \varphi_0 + \frac{t}{\rho} \\
\rho_t = \rho_0
\end{cases}
$$

(1)

For any $t$ this map is simplectic i.e. leaves invariant the element of area $dx \wedge dy = \rho d\rho \wedge d\varphi$.

**Definition 1** For a given point $F$ on the plane and usual polar coordinates $(\rho, \varphi)$ with the center $F$ introduce the inverse polar coordinates with the center $F$: $(r = \frac{1}{\rho}, \varphi)$. We call the map $T$

$$
\begin{cases}
\varphi_1 = \varphi + hr \\
r_1 = r
\end{cases}
$$

(2)

the skew rotation with the center $F$.

**Definition 2** The map $T_h$ is called the perturbed skew rotation if it is invertible, simplectic and has asymptotics

$$
\begin{cases}
\varphi_1 = \varphi + hr + O(r^2) \\
r_1 = r + O(r^3)
\end{cases}
$$

(3)

as $r \to 0$.

Note that in this definition $(\varphi, r) \in [0, 2\pi) \times \mathbb{R}_+$ are not necessarily the inverse polar coordinates but any "polar" coordinate system in the neighbourhood of the infinity point $r = 0$. If the correction terms $O(r^2)$, $O(r^3)$ in right hand side of (3) are absent we call the map $T_h$ the generalized skew rotation.

**Remark.** As it is known (see for example [3]) the Hamiltonian system on the plane with closed trajectories can be written in the canonical form

$$
\begin{cases}
\dot{\varphi} = \omega(I) \\
\dot{I} = 0
\end{cases}
$$

(4)

for particular variables $(\varphi, I) \in [0, 2\pi) \times \mathbb{R}_+$ which are called action-angle coordinates. Function $\omega$ is called frequency or angular velocity. The action variable $I$ represents up to the numerical factor the area surrounded by the level curve of Hamiltonian. Let the frequency $\omega(I)$ monotonously tends to 0 as $I \to \infty$. So we can use $(\omega, \varphi)$ as a "polar" coordinates in the neighbourhood of infinity. In this coordinates the shift along the orbits of the flow (4) is the generalized skew rotation.

Let $T_1$ and $T_2$ are perturbed skew rotations in two different coordinate systems $(r, \varphi)$ and $(\tilde{r}, \tilde{\varphi})$ in the neighbourhood of infinity. What can one say about the existence of closed invariant curves for the map $T = T_1 \cdot T_2$?
Figure 1: Product of two skew rotations around the points \( F_1 \) and \( F_2 \) with \( h_1, h_2 > 0 \).

As we shall show the qualitative behaviour of the product of several skew rotations is similar to the behaviour of the product of usual rotations of the plane. Consider \( N \) motions of the plane given in complex notation as \( z \mapsto a_i z + b_i, |a_i| = 1 \). The product of these motions is

\[
z \mapsto az + b \quad a = \prod_{i=1}^{N} a_i
\]

If \( a \neq 1 \) then this map is a rotation and so the trajectories of this map lie on the circles with the center \( z = -\frac{b}{a} \) in the fixed point of this rotation. We generalize this result to the skew rotations and perturbed skew rotations. Our result shows that the typical trajectories lie on the closed invariant curves. The rest trajectories are contained in the annuli surrounded by these curves. This follows from the topological lemma.

**Lemma 1** Let \( A \) be the annulus bounded by two closed curves \( \gamma_0 \) and \( \gamma_1 \). Let \( T \) be the homeomorphism of the domain \( B \supset A \). If \( T(\gamma_i) = \gamma_i, \ i = 0, 1 \) then \( T(A) = A \).

We use KAM-theory technique to solve the described problem. There is a theorem formulated in other notations in [1, §34].

**Theorem 1** Let \( T \) be a real analytic invertible map defined in some "polar" coordinates \((\varphi, r) \in [0, 2\pi) \times \mathbb{R}_+\) as

\[
\begin{cases}
\varphi_1 = \varphi + hr^{2\ell} + O(r^{2\ell+1}), & h \neq 0 \\
r_1 = r + O(r^{2\ell+2})
\end{cases}
\]

for some \( \ell \in \frac{1}{2}\mathbb{N} \). Suppose \( T \) satisfies the intersection property: the image of any closed curve surrounding the point \( r = 0 \) in some its neighbourhood intersects original curve.

Then \( T \) has an invariant curve surrounding stable point \( r = 0 \) in any neighbourhood of this point.

The measure \( d\varphi dr \) of all points that do not belong to the closed invariant curves of \( T \) is \( o(\varepsilon) \) in the \( \varepsilon \)-neighbourhood of stable point \( r = 0 \).
Theorem 1 with $2\ell = 1$ can be applied to the perturbed skew rotations due to the following lemma.

**Lemma 2** Any perturbed skew rotation $T$ satisfies the intersection property.

From the Lemma 2 and Theorem 1 it follows that the perturbed skew rotation has infinitely many closed invariant curves in any neighbourhood of infinity. By Lemma 1 it means that for the perturbed skew rotation the trajectory of any point is contained in the invariant annulus.

In the next section we present the proof of the following theorem.

**Theorem 2** Let $R_{j,h_j}$ denote the skew rotations with centres $F_j$, $j = 1, \ldots, N$. Then for any $h_j$ such that $\sum_{j=1}^N h_j \neq 0$ trajectories of the map $T = \prod_{j=1}^N R_{j,h_j}$ are bounded. Actually they are contained in the invariant annuli as it is mentioned above.

Case $\sum_{j=1}^N h_j = 0$ is considered in section 3 (for $N = 2$). In section 4 we present natural generalisations of Theorem 2. In section 5 we analyse the non-smooth case. At last, in section 6 we present the results of numerical computations.

Proofs of Lemmas 2 and 1 are given in Appendix.

**Acknowledgements** The authors thank Ya. G. Sinai for many fruitful discussions and for the substantial help in the writing of this article. Also useful remarks were made by M. Blank.

## 2 Proof of Theorem 2.

Since the product of two perturbed skew rotations in the same coordinate system is again a perturbed skew rotation to prove Theorem 2 it is sufficient to show that the skew rotation with one center is the perturbed skew rotation in the inverse polar coordinate system with any other center. We verify the last statement by direct calculation.

Without any loss of generality consider the case of the skew rotation with the center $F_0 = 0$ in the inverse polar coordinate system with the center $F_1 = 1$.

The skew rotation $R_{0,h_0}$ is the shift along trajectories of the system

\[
\begin{align*}
\dot{\varphi} &= r \\
\dot{r} &= 0
\end{align*}
\]

(6)

where $(\varphi, r)$ are inverse polar coordinates with the center $F_0$. In the complex notation $\varphi = \arg(z)$, $r = \frac{1}{|z|}$.

Inverse polar coordinates $(\tilde{\varphi}, \tilde{r})$ with the center $F_1$ can be expressed as

\[
\begin{align*}
\tilde{\varphi} &= \arg(z - 1) = \text{Im} \left( \ln(z - 1) \right) = \text{Im} \left( \ln z + \ln \left( 1 - \frac{1}{z} \right) \right) = \\
&= \text{Im} \left( \ln z - \frac{1}{z} + O(z^{-2}) \right) = \varphi + r \sin \varphi + O(r^2)
\end{align*}
\]

(7)
\[ \hat{r} = \frac{1}{|z-1|} = \exp \left( -\text{Re}(\ln(z-1)) \right) = \exp \left( -\text{Re}(\ln z) - \text{Re} \left( \ln \left( 1 - \frac{1}{z} \right) \right) \right) = \frac{1}{|z|} \left( 1 + \text{Re} \left( \frac{1}{z} \right) + O(z^{-2}) \right) = r(1 + r \cos \varphi) + O(r^3) \]

Thus system (6) in coordinates \((\hat{\varphi}, \hat{r})\) gets the form
\[
\begin{cases}
\dot{\hat{\varphi}} = r(1 + r \cos \varphi + O(r^2)) = \hat{r} + O(r^3) \\
\dot{\hat{r}} = (-r^2 \sin \varphi + O(r^3))r = O(r^3)
\end{cases}
\]

Since from (8) it follows \(\hat{r} \to 1\) as \(r \to 0\) we have \(O(r^3) = O(\hat{r}^3)\) in the neighbourhood of \(r = 0\). Since map \(R_{0,h_0}\) is the shift along the trajectories of (6) it is also a perturbed skew rotation in the coordinate system \((\hat{\varphi}, \hat{r})\).

3 The unbounded case.

Here we analyze the case of the product of two opposite skew rotations.

Remind the notion of the composition of two Hamiltonian systems with Hamiltonians \(H_1\) and \(H_2\) as a system corresponding to vector field which is equal to linear combination of vector fields given by \(H_j, j = 1, 2\). Such system also can be written in Hamiltonian form with Hamiltonian (see [3])
\[ H = h_1H_1 + h_2H_2 \quad (10) \]

For \(h_1 \neq -h_2\) level curves of Hamiltonian \(H\) are closed and are called Cartesian Ovals – algebraic curves which are defined by equation
\[ h_1|z - F_1| + h_2|z - F_2| = \text{const} \quad (11) \]

For particular case \(h_1 = h_2\) level curves of the Hamiltonian \(H\) is the family of confocal ellipses with foci \(F_1, F_2\). If \(h_1 = -h_2\) relation (11) defines the family of confocal hyperbolas with the same foci.

To this end Theorem 2 states that for \(h_1 + h_2 \neq 0\) orbits of the map \(T^{(h_1,h_2)}\) lie near the orbits of completely integrable Hamiltonian system with Hamiltonian (10). For the case \(h_1 = -h_2\) the map \(T^{(h,-h)}\) is an \(O(r^{-3})\) perturbation of identity map and KAM – theory methods are not applicable.

Let \(F_1 = (-1,0)\) and \(F_2 = (1,0)\) be two centres. Let \(R_{1,h}\) and \(R_{2,-h}\) be the skew rotations with centres \(F_1, F_2\) with opposite values of parameters. Then there is the following theorem.

**Theorem 3** For the map \(T = R_{1,h} \cdot R_{2,-h}\) there exists a trajectory escaping to infinity.

Informally Theorem 3 states that in the case \(h_1 = -h_2\) trajectories of the map \(T^{(h_1,h_2)}\) also follow trajectories of the Hamiltonian system with Hamiltonian (10).

**Proof.** Consider an arc \(z_0, z_1\) of length \(h\) with center at \(F_1\) from some point \(z_0 = x_0 + iy_0, \ x_0 > 0\) which intersects straight line \(x = 0\) in its middle point. Distance from \(z_1\) to the line \(x = 0\) equals \(-\text{Re}(z_1) = -x_1 = x_0\). After rotation \(R_2\) the distance from \(z_2 = R_2(z_1)\) to the axis will be greater then \(x_0\) since \(|z_1 - F_2| > |z_0 - F_2|\). Thus \(\text{Re}(z_2) > \text{Re}(z_0)\). Similarly, for
Figure 2: Linear combination of Hamiltonians $H_1$ and $H_2$. 

$z_3 = R_1(z_2)$ we get $\text{Re}(z_3) < \text{Re}(z_1)$. At each step the distance from the image of the point $z_0$ to the axis will be greater then that of the previous step. Incidentally, arc obtained on each iteration should intersect an axis since $|x_j| < h$. Thus because of convexity of the circle, sequence $\{y_j\}$—coordinates of the intersection of the axis and arc $z_j, z_{j+1}$ is increasing. Suppose, that this sequence has some finite limiting point $y$. Then for $(0, y)$ one gets $T(0, y) = (0, y)$ which is definitely impossible for any finite $y$.

Similar considerations can be done for any point $z \in \mathbb{C}$. Role of the axis $x = 0$ in this case will play some hyperbola with foci $F_j$. Theorem 3 is proven. 

Figure 3: Hyperbolic trajectories.
4 Generalisations.

Now let us turn to the general case of perturbed skew rotations given in different coordinate systems.

**Definition 3** We shall call two coordinate systems \((r, \varphi)\) and \((\tilde{r}, \tilde{\varphi})\) concordant in the neighbourhood of \(r = 0\) if

1. Point \(r = 0, \varphi \in [0, 2\pi)\) corresponds to \(\tilde{r} = 0\). So \(\frac{\partial \tilde{r}}{\partial \varphi} \bigg|_{r=0} = 0\).
2. Transform \((r, \varphi) \rightarrow (\tilde{r}, \tilde{\varphi})\) is real analytic invertible map.
3. This transform asymptotically does not change angles and radii, i.e. \(\frac{\partial \tilde{\varphi}}{\partial \varphi} \bigg|_{r=0} = 1\) and \(\frac{\partial \tilde{r}}{\partial r} \bigg|_{r=0} = 1\).

Obvious the concordance of coordinate systems is equivalence relation.

**Remark.** Inverse polar coordinate systems with different centres are concordant.

**Lemma 3** If coordinate systems \((r, \varphi)\) and \((\tilde{r}, \tilde{\varphi})\) are concordant in the neighbourhood of \(r = 0\) then \(\frac{\partial \tilde{r}}{\partial \varphi} \bigg|_{r=0} = O(r^2)\) for \(r \rightarrow 0\).

**Proof.** From the definition \(\frac{\partial \tilde{r}}{\partial \varphi} \bigg|_{r=0} = 0\) and \(\frac{\partial}{\partial r} \left( \frac{\partial \tilde{r}}{\partial \varphi} \right) \bigg|_{r=0} = \frac{\partial}{\partial \varphi} \left( \frac{\partial \tilde{r}}{\partial r} \right) \bigg|_{r=0} = 0\). ■

**Theorem 4** Let \((r, \varphi)\) and \((\tilde{r}, \tilde{\varphi})\) are two coordinate systems concordant in the neighbourhood of \(r = 0\). Let real analytic invertible map \(T\) satisfies conditions (3) in coordinates \((r, \varphi)\). Then \(T\) also satisfies conditions (3) in coordinates \((\tilde{r}, \tilde{\varphi})\).

**Proof.** First consider

\[
\tilde{r}_1 - \tilde{r} = \frac{\partial \tilde{r}}{\partial r} (r_1 - r) + \frac{\partial \tilde{r}}{\partial \varphi} (\varphi_1 - \varphi)
\]

In the neighbourhood of \(r = 0\) function \(\frac{\partial \tilde{r}}{\partial r}\) is bounded and factor \((r_1 - r)\) is \(O(r^3)\) due to conditions (3). For the second term we get \(\frac{\partial \tilde{r}}{\partial \varphi} = O(r^2)\) from Lemma 3 and \((\varphi_1 - \varphi) = O(r)\) thanks to conditions (3). So we have

\[
\tilde{r}_1 - \tilde{r} = O(r^3) = O(\tilde{r}^3)
\]

since \(\tilde{r} \sim r\).

For the difference of angular coordinates we have

\[
\tilde{\varphi}_1 - \tilde{\varphi} = \frac{\partial \tilde{\varphi}}{\partial r} (r_1 - r) + \frac{\partial \tilde{\varphi}}{\partial \varphi} (\varphi_1 - \varphi)
\]

The first term is \(O(r^3)\) from the previous and the second term is

\[
\frac{\partial \tilde{\varphi}}{\partial \varphi} (\varphi_1 - \varphi) = (1 + O(r))(hr + O(r^2)) = hr + O(r^2) = h\tilde{r} + O(\tilde{r}^2)
\]

Theorem 4 is proven. ■

Now we can generalise Theorem 2.
Theorem 5 Let \((r, \varphi)\) and \((\tilde{r}, \tilde{\varphi})\) are two coordinate systems concordant in the neighbourhood of \(r = 0\). Let real analytic invertible map \(T_1\) is a perturbed skew rotation in coordinates \((r, \varphi)\), real analytic invertible map \(T_2\) is a perturbed skew rotation in coordinates \((\tilde{r}, \tilde{\varphi})\). Then the product 
\(T = T_1T_2\) is the perturbed skew rotation in coordinates \((r, \varphi)\) (or \((\tilde{r}, \tilde{\varphi})\) as well). If the sum of "angular coefficients" \(h_1 + h_2 \neq 0\) then the conclusion of Theorem 1 is valid for \(T\).

Proof follows from Lemma 2, Theorem 4 and Theorem 1.

Corollary 1 If there are two Hamiltonian systems with closed trajectories such that the functions \(\omega_1, \omega_2\) monotonously tend to 0 at infinity, coordinate systems \((\varphi_1, \omega_1)\) and \((\varphi_2, \omega_2)\) are concordant and \(h_1 + h_2 \neq 0\) then for the product of \(h_1\)– and \(h_2\)–shifts along trajectories of these Hamiltonian systems the conclusion of Theorem 1 is valid.

Corollary 2 Theorem 5 can be easily generalized to the case of any finite collection of invertible maps \(T_1, \ldots, T_N\).

5 Non-smooth case.

In this section we show that the smoothness of the system in previous consideration is essential. Now we orient the plane opposite to the usual way. So the positive rotation direction is clockwise. Let \(O_1 = (-\frac{1}{2}, 0)\) and \(O_2 = (\frac{1}{2}, 0)\). Remind the basic example of the product of two skew rotations with different centres. Transformation \(T\) moves each point \((x, y)\) along the level curve of the first Hamiltonian for the length \(h\) and then moves its image along the level curve of the second Hamiltonian for the length \(h\). In the basic example such level curves form two one-parametric families of concentric circles with centres \(O_1\) and \(O_2\). Consider the analogous system with two families of concentric squares, i.e. \(H_1 = |x - x_1| + |y - y_1|\) and \(H_2 = |x - x_2| + |y - y_2|\). For this system we show the coexistence of countably many periodic orbits and countably many trajectories escaping to infinity as \(O(\sqrt{t})\).

Let \(Q^{(j)}_R = \{(x, y): |x - x_j| + |y - y_j| = R\}, j = 1, 2\) be two families of concentric squares with centres \(O_1\) and \(O_2\) such that corresponding sides of the squares are parallel and \(O_1O_2\) belongs to the common diagonal of the squares. The trajectory consists of odd and even steps. For even steps the point is shifted along the side of the square of the first family for the length \(h\) and for odd steps — along the side of the square of the second family for the length \(h\) in the same direction. The plane is splitted onto three parts (see Fig. 5): half-planes \(x > \frac{1}{2}, x < -\frac{1}{2}\) and the strip \(|x| < \frac{1}{2}\). In the half-planes dynamics dictated by two families coincide while in the strip there exist a non-zero angle between isolines of first and second family.

Let \(h = a\sqrt{2}\). Shift along the square \(Q^{(j)}_t\) for the length \(h\) sufficiently far from the vertices of the square corresponds to the shift for the length \(a\) in each coordinate in positive or negative direction depending on the part of the square. Consider trajectory of the point \((-\frac{1}{2}, h_0)\).

Denote by \(h_n\) the ordinate of the point of \(n\)-th entry to the strip, by \(a_n\sqrt{2}\) length of the remaining part of the shift inside the strip and by \(a_n\) the number of the family modulo 2 (see Fig. 5). For the starting point corresponding triple is \((h_0, a, 0)\). Whenever \(|h_n| > a\) values \((h_{n+1}, a_{n+1}, a_{n+1})\) can be obtained by following recurrent relations.

\[\begin{align*}
\]
Denote by $\gamma_n$ the number of complete steps the trajectory spends in the strip after the $n$-th entry and by $\beta_n\sqrt{2}$ the initial part of the next step belonging to the strip. We have

$$
\gamma_n = \left\lfloor \frac{1 - a_n}{a} \right\rfloor \quad (12)
$$

$$
\beta_n = 1 - a_n - a\gamma_n = a \left( \frac{1 - a_n}{a} - \left\lfloor \frac{1 - a_n}{a} \right\rfloor \right) = a \left\{ \frac{1 - a_n}{a} \right\} \quad (13)
$$

where $[x]$ and $\{x\}$ denote integer and fractional part of $x$. For $h_{n+1}$ it immediately follows that

$$
h_{n+1} = - \left( h_n + (-1)^{\alpha_n} \left( a_n + (-1)^{\gamma_n+1} \beta_n - \frac{(1 - (-1)^{\gamma_n})a}{2} \right) \right) \quad (14)
$$

Compute the number of complete steps of the trajectory before the next entry to the strip. It is equal to $\left\lfloor \frac{2|h_{n+1}| + \beta_n - a}{a} \right\rfloor$. Thus we get

$$
\alpha_{n+1} \equiv \left( \alpha_n + \gamma_n + 1 + \left\lfloor \frac{2|h_{n+1}| + \beta_n}{a} \right\rfloor \right) \mod 2 \quad (15)
$$
and
\[ a_{n+1} = a \left( 1 - \left\{ \frac{2|h_n+1|+\beta_n}{a} \right\} \right) \] (16)

In order to simplify the analysis let us consider the case \( a = \frac{1}{m}, m \in \mathbb{N} \).

**Proposition 1** If \( h_0 = \ell a, \ell \in \frac{1}{2}\mathbb{N}, \ell \geq 1 \) and \( a_0 = a \) then \( h_n = \ell_n a, \ell_n \in \frac{1}{2}\mathbb{N}, \ell_n \geq 1 \) and \( a_n = a \) for all \( n \in \mathbb{N} \).

**Proof** goes by induction. From \( a_{n-1} = a \) and (13) it follows that \( \beta_{n-1} = 0 \). Thus \( h_n \) from (14) is equal \(-h_{n-1} \) or \(-h_{n-1} \pm a \) and so is again of the form \( \ell_n a \) where \( 2\ell_n \in \mathbb{N} \). Since \( \left\{ \frac{2|h_n|}{a} \right\} = 0 \) from (16) it follows \( a_n = a \). ■

Let \( a = \frac{1}{2m}, a_0 = a \) and \( h_0 \) is as above. Then from (12) it follows \( \gamma_n = 2m - 1 \) and \( h_{n+1} = -h_n \). So after 4(\( \ell + m \)) steps the point returns to the initial position.

For \( a = \frac{1}{2m}, a_0 = a \) it follows that \( \gamma_n = 2m - 2 \) and so \( h_{n+1} = -h_n - (-1)^{a_n}a \). So if \( h_0 = \ell a, \ell \in \mathbb{N} \) then \( h_{2n+2} = h_{2n} + 2a \) and the trajectory is expanding. If \( h_0 = \ell a, \ell \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N} \) then \( h_{2n+2} = h_{2n} \) and the trajectory is periodic with the period \( 4\ell + 2m - 1 \). More generally, if \( a = \frac{1}{k}, k \in \mathbb{N}, a_0 = a - \varepsilon, \varepsilon \in [0, a) \) then for \( h_0 = \ell a + \varepsilon, \ell \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N} \) the trajectory is again periodic.

Numerical experiments shows that for initial triples \((h_0, a, \alpha)\) with general \( h_0 \) and \( a \) the behaviour of the trajectory resembles random walk (see Fig. 8).

### 6 Numerical results.

First we present the results showing that for the case of two rotations reasoning from the section 3 actually takes place. Computer simulations shows that the trajectory of the map \( T^{(h_1, h_2)} \) at each step intersects one level curve of Hamiltonian (10).
Figure 6: Left: Trajectory of the point $z = 3 + 5i$ under the map $T^{(h_1, h_2)}$ for $h_1 = 2.5$, $h_2 = -3$ intersects Cartesian Oval $5|z - 1| - 6|z + 1| = \text{const}$ (red points correspond to steps $S_1^{h_1}$, blue points correspond to the steps $S_2^{h_2}$ and black dotted line depicts Cartesian Oval). Right: Trajectory of the point $z = 1.6 + 0i$ under the map $T^{(h_1, h_2)}$ for $h_1 = 2.5$, $h_2 = 0.25$ intersects Cartesian Oval $|z - 1| + 10|z + 1| = \text{const}$.

The map $T^{(h_1, h_2)}$ shows the typical for KAM-theory behaviour: the islands of stability surrounded by closed invariant curves.

Figure 7: Trajectories of the map $T^{(h_1, h_2)}$ for $h_1 = h_2 = 3.8$ and initial points $(0, 2.419)$ (large ovals) and $(0, 3)$ (islands surrounding the periodic points).

For the non-smooth case the dynamics looks like
Figure 8: Left: The points of the trajectory. Right: The distance of the trajectory $|T^n(x,y)|$ from 0 for the step $h = 2.43\sqrt{2}$ and randomly chosen initial point $(x_0,y_0)$: $|x_0| + |y_0| = 1$.

Appendix.

Proof of Lemma 2. The perturbed skew rotation $T$ is by definition an invertible area-preserving map. Consider any closed curve $\gamma$ surrounding points $z_1 = 0$ and $z_2 = T^{-1}(0)$. Then $T(\gamma)$ is a closed curve surrounding point $z_1 = 0$ and so, thanks to area-preserving property, $T(\gamma) \cap \gamma \neq \emptyset$. Since any closed curve surrounding the infinity in some its neighbourhood also surrounds points $z_1$ and $z_2$ the intersection property in the neighbourhood of infinity holds. ■

Proof of Lemma 1. Suppose that for some $x_0 \in A$ we have $T(x_0) \notin A$. Consider any curve $\gamma(t) \subset A$ having following properties

1. $\gamma(0) \in \gamma_0$, $\gamma(1) \in \gamma_1$
2. $\forall t \in (0,1): \gamma(t) \cap (\gamma_1 \cup \gamma_0) = \emptyset$
3. $\exists t_0 \in (0,1): \gamma(t_0) = x_0$

Since $T$ is a homeomorphism and $T(\gamma(t_0)) \notin A$ the image $T(\gamma(0,1))$ intersects the boundary $\gamma_1 \cup \gamma_0$ in some point $y_0 = T(\gamma(1))$

Then $\gamma(t_1) = T^{-1}(y_0) \in \gamma((0,1)) \cap (\gamma_1 \cup \gamma_0)$ which contradicts our choice of $\gamma$ (property 2). ■

References

[1] Siegel, C. L.; Moser, J. K. «Lectures on celestial mechanics.» Translated from the German by C. I. Kalme. Reprint of the 1971 translation. Classics in Mathematics. Springer-Verlag, Berlin, 1995. xii+290 pp. ISBN: 3-540-58656-3

[2] Y. Takuechi, N.H. Du, N.T. Hieu, K.Sato «Evolution of predator–prey systems described by Lotka–Volterra equation under random environment.» Journal of Mathematical Analysis and Applications, 323 (2006), 938-957
[3] V.I. Arnold «Mathematical Methods of Classical Mechanics» Series: Graduate Texts in Mathematics, Vol. 60 2nd ed. 1989, ISBN 978-0-387-96890-2

[4] M. Blank, T. Krüger, L. Pustylnikov «A KAM type theorem for systems with round-off errors.», arXiv:chao-dyn/9706005v1