Multipole analysis on stationary massive vector and symmetric tensor fields with irreducible Cartesian tensors

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ABSTRACT

The multipole expansions for massive vector and symmetric tensor fields in the region outside spatially compact stationary sources are obtained by using the symmetric and trace-free formalism in terms of the irreducible Cartesian tensors, and the closed-form expressions for the source multipole moments are provided. The expansions show a Yukawa-like dependence on the massive parameters of the fields, and the integrals of the stationary source multipole moments are all modulated by a common radial factor. For stationary massive vector field, there are two types of “magnetic” multipole moments, among which one is the generalization of that of the magnetostatic field, and another, being an additional set of multipole moments of the stationary massive vector field, can not be transformed away. As to the stationary massive symmetric tensor field, its multipole expansion is presented when the trace of its spatial part is specified, where besides the counterparts of the mass and spin multipole moments of massless symmetric tensor field, three additional sets of multipole moments also appear. The multipole expansions of the tensor field under two typical cases are discussed, where it is shown that if the spatial part of the tensor field is trace-free, the monopole and dipole moments in the corresponding expansion will vanish.

I. INTRODUCTION

A massive vector field is usually used to describe a massive particle with spin-1 [1, 2], and mathematically, the field equations of a massive vector field $A^\mu$ can be written as

\begin{align}
\Box A^\mu - m_A^2 A^\mu &= -4\pi j^\mu, \tag{1.1a} \\
\partial_\mu A^\mu &= 0, \tag{1.1b}
\end{align}

where $\Box := \eta^{\mu\nu} \partial_\mu \partial_\nu$ with $\eta^{\mu\nu}$ as the Minkowskian metric in a Minkowski spacetime and $\partial_\mu$ as the partial derivative with respect to the Minkowskian coordinate, $m_A$ is the massive parameter of the field, and $j^\mu$ is the external source. Although Eq. (1.1a) formally coincides with the Lorenz gauge condition in Maxwell’s electromagnetic theory, it is important to realize that due to the existence of $m_A$, Eq. (1.1b) arises dynamically, i.e. as consequence of the the continuity equation satisfied by $j^\mu$, and there is no longer the local gauge symmetry for $A^\mu$. As a contrast, a massive spin-2 particle could be depicted by a massive symmetric tensor field (MSTF) $h^{\mu\nu}$, and its field equations can be mathematically assumed to be

\begin{align}
\Box h^{\mu\nu} - m_h^2 h^{\mu\nu} &= -4\pi T^{\mu\nu}, \tag{1.2a} \\
\partial_\mu h^{\mu\nu} &= 0, \tag{1.2b}
\end{align}

where $m_h$ is the massive parameter of the field, $T^{\mu\nu}$ is the external symmetric source, and the constraint (1.2b) is compatible with the conservation of $T^{\mu\nu}$. The above two equations have a wide range of applications in the models of massive gravity [3–6]. Unlike the case in linearized General Relativity (GR), the term $m_h^2 h^{\mu\nu}$ in Eq. (1.2a) results in that $h^{\mu\nu}$ does not have the usual massless gravity gauge symmetry. Both of massive vector and symmetric tensor fields have important applications in theoretical physics [7–12], and therefore, it is very necessary to explore the solutions to their field equations.

One of the most significant ways to describe the external field of the source localized in a finite region of space is the multipole expansion. The symmetric and trace-free (STF) formalism in terms of the irreducible Cartesian tensors, developed by Thorne [13], Blanchet, and Damour, and Iyer [14–16], is one useful method with respect to the multipole expansion. In view of the important applications of the above massive vector and symmetric tensor fields in physics, the multipole analysis on them in the region outside spatially compact sources is very necessary. In Ref. [16], the relativistic multipole expansions for massless vector and symmetric tensor fields are obtained using the STF formalism, and the corresponding source multipole moments are derived. By following the same approach, we can also make a multipole analysis on the massive vector and symmetric tensor fields.

The starting point of this study is the multipole expansion for the massive scalar field, namely the Klein-Gordon field. When the source is time-independent, the field equation of the Klein-Gordon field reduces to the screened Poisson equation, and since the Green’s function of this equation is easier to handle, the derivation in such case
can be greatly simplified. In Ref. [17], the multipole expansion for the Klein-Gordon field with a spatially compact stationary source is derived, and the closed-form expressions of the source multipole moments are provided. In this paper, using the STF formalism, we will make use of this result to make a multipole analysis on the massive vector and symmetric tensor fields in the region outside spatially compact stationary sources.

The derivation can be performed by following the conventional method in Refs. [14, 16], and compared with the results for stationary massless vector and symmetric tensor fields, the multipole expansions for stationary massive fields show a Yukawa-like dependence on the massive parameters of the fields, and the integrals of the stationary source multipole moments are all modulated by a common radial factor. For stationary massive vector field \(A^\mu\), the multipole expansion of \(A^0\) field and the “electric” multipole moments are compatible with the multipole expansion of the scalar potential of the electrostatic field presented in Ref. [16], and it is shown that outside the source region, the stationary \(A^0\) field can be equivalently generated by the source built from \(\delta\)-function, which is referred to as the skeleton of the stationary \(A^0\) field. As a contrast, the multipole expansion of \(A^i\) field is different from that of the vector potential of the magnetostatic field because two sets of “magnetic” multipole moments appear in the expansion, where one of them is the generalization of that of the magnetostatic field, and another one, as an additional set of multipole moments of the stationary massive vector field, can not be transformed away because there is no the local gauge symmetry for \(A^i\). Moreover, it should be pointed out that for the massive vector field, both of the two types of “magnetic” monopole moments always vanish.

As to the stationary massive symmetric tensor field \(h^{\mu\nu}\), its multipole expansion is also presented. In the expansion, besides the counterparts of the mass and spin multipole moments of massless symmetric tensor field, there are three additional sets of multipole moments. Similarly, since \(h^{\mu\nu}\) has no the usual massless gravity gauge symmetry, these three additional sets of multipole moments can also not be transformed away. The derivation indicates that the multipole expansion for \(h^{ij}\) is dependent on its trace, and hence, for future application, the expression of \(h^{kk}\) also needs to be given. The results under the cases of \(h^{kk} = 0\) and \(h^{kk} = \delta_{00}\) are provided in the present paper. It is shown that when \(h^{kk} = 0\), the monopole and dipole moments in the expansion of \(h^{ij}\) vanish, which is compatible with the general form of the STF-tensor spherical harmonics expansion for a trace-free tensor field of “spin” 2 on the unit sphere centered at the coordinate origin [14].

The multipole expansion for stationary massive symmetric tensor field, as the external solution to Eqs. (1.2a) and (1.2b) for any spatially compact stationary source, describes the effects of the source at all orders, so it must have important applications in the models of massive gravity. In addition, it also plays an important role in the alternative theories of gravity. In Ref. [18], the metric for the external gravitational field of a spatially compact stationary source is discussed in \(F(X,Y,Z)\) gravity, a generic fourth-order theory of gravity, where \(X := R\) is the Ricci scalar, \(Y := R_{\mu\nu}R^{\mu\nu}\) is the quadratic contraction of two Ricci tensors, \(Z := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\) is the quadratic contraction of two Riemann tensors, and \(F\) is a general function of \(X, Y\), and \(Z\). In the derivation, the linearized gravitational field equations of \(F(X,Y,Z)\) gravity are transformed into d’Alembert equation and Klein-Gordon equations with external sources by imposing a new type of gauge condition. Specifically, it is shown that there exists a symmetric tensor \(P^{\mu\nu}\), constructed from the linearized Ricci tensor \(R^{\mu\nu(1)}\) and the linearized Ricci scalar \(X^{(1)}\), satisfying

\[
\Box P^{\mu\nu} - m_i^2 P^{\mu\nu} = -4\pi i S^{\mu\nu}, \tag{1.3a}
\]

\[
\partial_\mu P^{\mu\nu} = 0 \tag{1.3b}
\]

outside the source, where the massive parameter \(m_i\) is defined by the coefficients of \(X, Y\), and \(Z\) when \(F(X,Y,Z)\) is expressed as a power series, \(i\) is a constant, and the symmetric tensor \(S^{\mu\nu}\) is relevant to the energy-momentum tensor of the source living in a Minkowski spacetime. The tensor \(P^{\mu\nu}\) presents a massive propagation in \(F(X,Y,Z)\) gravity [19, 20], and one can obtain the metric for the gravitational field outside a spatially compact stationary source in \(F(X,Y,Z)\) gravity only after the solution to Eqs. (1.3a) and (1.3b) is given.

In Ref. [18], since the multipole expansions of the time-space and spatial components of \(P^{\mu\nu}\) are not derived, the metric provided in Ref. [18] is not expanded in terms of the irreducible Cartesian tensors, which greatly limits the application of the result. Comparing Eqs. (1.3a) and (1.3b) with Eqs. (1.2a) and (1.2b), respectively, we find that by applying the result derived in the present paper, the complete multipole expansion for the stationary \(P^{\mu\nu}\) will be easily gained, and as a consequence, the metric, presented in the form of the multipole expansion, for the external gravitational field of a spatially compact stationary source will also be obtained in \(F(X,Y,Z)\) gravity. According to the result in this paper, besides the counterparts of the mass and spin multipole moments in the linearized GR, there should be three additional sets of multipole moments appearing in the expansion of the metric, and when the metric is applied to some specific phenomenon in practice, the effects of those terms associated with the additional moments can be analyzed. For instance, for a gyroscope moving around the source in geodesic motion, one is able to use the metric to derive its spin’s angular velocity of precession, and by following the conventional method in Ref. [21], the precessional angular velocity in GR would be corrected by those terms associated with the additional moments. Then, by comparing these results with the data of the gyroscopic experiment, e.g., Gravity Probe B (GP-B), the effects of the additional moments in \(F(X,Y,Z)\) gravity will be obtained. Similarly, the metric can also be applied to the
anomalous perihelion advance of Mercury, the gravitational redshift of light, and the light bending, etc. By comparing
the theoretical results with the experimental or observational data, the further effects of the additional moments in
$F(X, Y, Z)$ gravity will also be derived. It could be expected that in the future, more and more applications of the
results in the present paper will be found.

This paper is organized as follows. In Sec. II, the STF formalism and the multipole expansion for a stationary
Klein-Gordon field are briefly reviewed. In Sec. III, a multipole analysis on stationary massive symmetric
tensor field. In Sec. IV, the STF formalism is extended to deal with the stationary massive symmetric
Klein-Gordon field are briefly reviewed. In Sec. III, a multipole analysis on stationary massive vector field is made by
using the STF formalism. In Sec. V, the conclusions and the related discussions are presented. Throughout this paper, when the
notation is concerned, the Greek letters denote spacetime indices and range from 0 to 3, whereas the Latin letters
denote space indices and range from 1 to 3. The repeated indices within a term represent that the sum should be
taken over.

II. PRELIMINARY

A. Relevant notations and formulas in the STF formalism

The notation in this paper is the same as that in Refs. [17, 22–24]. In a Minkowski spacetime with signature
$(-, +, +, +)$, the STF part of a Cartesian tensor $A_{Ii} := A_{i_1i_2⋯i_l}$ is denoted by

$$\hat{A}_{Ii} := A_{(Ii)} = A_{(i_1i_2⋯i_l)} := \sum_{k=0}^{[l/2]} c_k \delta_{(i_1i_2⋯i_l)} \delta_{(2k+1)⋯i_l} S_{i_1i_2⋯i_l} a_1a_2⋯a_k,$$  \hspace{1cm} (2.1)

where $[l/2]$ representing the integer part of $l/2$, $\delta_{ij}$ denoting the Kronecker symbol,

$$c_k := (-1)^k \frac{(2l-2k-1)!!}{(2l-1)!!} \frac{l!!}{(2k)!(l-2k)!},$$ \hspace{1cm} (2.2)

and

$$S_{Ii} := A_{(Ii)} = A_{(i_1i_2⋯i_l)} := \frac{1}{l!} \sum_{\sigma} A_{i_{\sigma(1)}i_{\sigma(2)}⋯i_{\sigma(l)}}$$ \hspace{1cm} (2.3)

is its symmetric part with $\sigma$ running over all permutations of $(12⋯l)$. Denote $(x^\mu) = (ct, x_i)$ and $(ct, r, \theta, \varphi)$ as
Minkowskian coordinates and the corresponding spherical coordinates, and then, there are

$$x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta.$$ \hspace{1cm} (2.4)

Let $\partial_i := \partial/\partial x_i$ be the coordinate basis vectors, and the radial vector and the unit radial vector are $x = x_i \partial_i$ and $n = n_i \partial_i = (x_i/r) \partial_i$, respectively. With $x_i$ and $n_i$, the tensor products of $l$ radial and unit radial vectors are

$$X_{Ii} = X_{i_1i_2⋯i_l} := x_{i_1}x_{i_2}⋯x_{i_l},$$

$$N_{Ii} = N_{i_1i_2⋯i_l} := n_{i_1}n_{i_2}⋯n_{i_l},$$ \hspace{1cm} (2.5)

and they satisfy

$$X_{Ii} = r^l N_{Ii}.$$ \hspace{1cm} (2.7)

In the STF formalism, one important result is that any Cartesian tensor $T_{I\mu}$ can be decomposed into a finite sum
terms of the type $\gamma_{I\mu}^{I'} \hat{R}_{I\mu}$, where $\gamma_{I\mu}^{I'}$ is a tensor invariant under the group of proper rotations SO(3), and $\hat{R}_{I\mu}$ is an
irreducible STF $l$ tensor ($l \leq p$) [14, 16, 25, 26]. This assertion can directly be proven by induction if one uses the following formula:

$$A_{Ii} \hat{R}_{I\mu} = \hat{R}^{(+)\mu}_{Ii} + \frac{l}{l+1} \epsilon_{ai(i} \hat{R}^{(0)}_{I(i_1⋯i_l-1)a} + \frac{2l-1}{2l+1} \delta_{ai(i} \hat{R}^{(-\mu)}_{I(i_1⋯i_l-1)},$$ \hspace{1cm} (2.8)

where

$$\hat{R}^{(+)\mu}_{Ii} := A_{(i+1)i} \hat{T}_{i_1⋯i_l},$$ \hspace{1cm} (2.9a)

$$\hat{R}^{(0)}_{I\mu} := A_{a} \hat{T}_{a(i_1⋯i_l-1)ab},$$ \hspace{1cm} (2.9b)

$$\hat{R}^{(-\mu)}_{Ii} := A_{a} \hat{T}_{a(i_1⋯i_l)ab}. $$ \hspace{1cm} (2.9c)
with $\epsilon_{ijk}$ as the Levi-Civita symbol. One particular case of Eq. (2.8) is

$$n_i\hat{N}_l = \hat{N}_i l + \frac{l}{2l+1}\delta_{i(l}\hat{N}_{i_1 \cdots i_{l-1})}.$$  (2.10)

Another important result in the STF formalism is that any scalar function $f(\theta, \phi)$ on the unit sphere centered at the coordinate origin can be expanded in powers of the unit radial vector $n$ [13, 14], namely,

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \hat{F}_I \hat{N}_I.$$  (2.11)

Here, the STF tensor coefficients $\hat{F}_I$ are unique, and one is able to obtain

$$\hat{F}_I = \frac{(2l + 1)!!}{4\pi l!} \int d\Omega \hat{N}_I f(\theta, \phi)$$  (2.12)

by virtue of the equality

$$\int (\hat{A}_I \hat{N}_I)(\hat{B}_J \hat{N}_J) d\Omega = \frac{4\pi l!}{(2l + 1)!!} \hat{A}_I \hat{B}_I \delta_{lJ}$$  (2.13)

where $\hat{A}_I$ and $\hat{B}_J$ are any two STF tensors, and $d\Omega$ is the element of solid angle.

Let $\partial_{i(l} = \partial_{i_1i_2\cdots i_l} := \partial_{i_1}\partial_{i_2}\cdots \partial_{i_l}$, there is

$$\hat{\partial}_{l} = \sum_{k=0}^{\lfloor \frac{l}{2} \rfloor} c_k \delta_{(i_1i_2 \cdots i_{l-k-1}i_{l-k+1} \cdots i_l)} (\nabla^2)^{k},$$  (2.14)

and related formulas of direct use in later sections are

$$\hat{\partial}_{l} \left(\frac{F(r)}{r}\right) = \hat{N}_l \sum_{k=0}^{l} \frac{(l+k)!}{(-2)^k k!(l-k)!} \frac{\partial_{(l-k}^{l-k} F(r)}{r^{k+1}},$$  (2.15)

$$\partial_{(i(l} := \partial_{i} \hat{\partial}_{l} = \partial_{i} \hat{N}_l + \frac{l}{2l+1} \delta_{(i(l} \delta_{i_1 \cdots i_l)} \nabla^2,$$  (2.16)

where $\nabla^2 = \partial_a \partial_a$ is the Laplace operator, $\partial_{(l-k}^{l-k}$ is the $(l-k)$-th derivative with respect to $r$, and Eq. (2.16) can be derived by following the proof of Eq. (2.10).

**B. Multipole expansion for a stationary Klein-Gordon field [17]**

In this subsection, we will briefly review the multipole expansion for a stationary Klein-Gordon field. Mathematically, the field equation of a Klein-Gordon field $V(t, x)$ with an external source $S(t, x)$ is [27]

$$\square V - m^2 V = -4\pi S$$  (2.17)

with $m$ as the massive parameter of the field. For a stationary source $S(x)$ that is spatially compact, the field equation of $V(x)$ reduces to the screened Poisson equation:

$$\nabla^2 V - m^2 V = -4\pi S.$$  (2.18)

The Green’s function of this equation is

$$G(x; x') = \frac{e^{-m|x-x'|}}{4\pi|x-x'|},$$  (2.19)

which satisfies

$$(\nabla^2 - m^2)G(x; x') = -\delta^3(x - x')$$  (2.20)
As a consequence, it is understood that when the Klein-Gordon field reduces to D’Alembert equation, namely
and thus, from Eqs. (2.24) and (2.27), we get

where \( \delta(z) \) is the Dirac delta function. Then, the solution to Eq. (2.18) is

\[
V(x) = 4\pi \int G(x; x') S(x') d^3 x'.
\]

As in Ref. [17], the Green’s function \( G(x; x') \) can be rewritten as

\[
G(x; x') = \sum_{l=0}^{\infty} \frac{(2l+1)!!}{4\pi l!} m_i l((mr^-)_l) \hat{N}(\theta, \phi) \hat{N}(\theta', \phi'),
\]

where \((\theta, \phi)\) and \((\theta', \phi')\) are the angle coordinates of \( x \) and \( x' \), respectively, \( r^- \) represents the lesser of \( r = |x| \) and \( r' = |x'| \), and \( r^+ \) the greater. Functions

\[
i_l(z) := \sqrt{\frac{\pi}{2z}} I_{l+\frac{1}{2}}(z), \quad k_l(z) := \sqrt{\frac{2}{\pi z}} K_{l+\frac{1}{2}}(z)
\]

are the spherical modified Bessel functions of \( l \)-order [28], and \( I_{l+1/2}(z) \), \( K_{l+1/2}(z) \) are the modified Bessel functions of \((l+1/2)\)-order. Insert Eq. (2.22) into Eq. (2.21), and with the aid of equalities [28]

\[
i_l(z) = z^l \left( \frac{d}{dz^l} \right) \left( \frac{\sinh z}{z} \right), \quad k_l(z) = \frac{e^{-z}}{z} \sum_{k=0}^{l} \frac{(l+k)!}{k!(l-k)!} \left( \frac{1}{2z^k} \right)
\]

and Eq. (2.15), one could obtain the multipole expansion of \( V(x) \) outside the source region (namely \( r = r^+ \) and \( r' = r^- \)),

\[
V(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{F}_l \partial_l \left( \frac{e^{-mr}}{r} \right),
\]

where the stationary source multipole moments \( \hat{F}_l \) are expressed as

\[
\hat{F}_l = \int \hat{X}'_l \delta_l(mr') S(x') d^3 x'
\]

with \( X'_l = X'_{l_1 l_2 \ldots l_i} := x'_{l_1} x'_{l_2} \cdots x'_{l_i} \) and

\[
\delta_l(z) := (2l+1)!! \left( \frac{d}{dz^l} \right) \left( \frac{\sinh z}{z} \right).
\]

The expansion (2.25) explicitly depends on the Yukawa “potential” \( e^{-mr}/r \), where \( m \) is related to the mass of the quanta of Klein-Gordon field, and in subsequent sections, we will see that such Yukawa-like dependence on the massive parameter of the field is a salient feature of the multipole expansions for massive fields. In addition, the expressions of the stationary source multipole moments \( \hat{F}_l \) display that their integrals are all modulated by a common radial factor \( \delta_l(mr') \), which is another salient feature of the multipole expansions for massive fields. Now, let’s discuss the property of this radial factor \( \delta_l(mr') \). For function \( i_l(z) \), the following formula holds [28]:

\[
\lim_{z \to 0} \frac{i_l(z)}{z^l} = \frac{1}{(2l+1)!!}
\]

and thus, from Eqs. (2.24) and (2.27), we get

\[
\lim_{m \to 0} \delta_l(mr') = 1.
\]

As a consequence, it is understood that when the Klein-Gordon field reduces to D’Alembert equation, namely \( m = 0 \), there are

\[
V(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{F}_l \partial_l \left( \frac{1}{r} \right),
\]

\[
\hat{F}_l = \int \hat{X}'_l S(x') d^3 x'.
\]

This is the multipole expansion for the stationary massless scalar field \( V(x) \), which is identical to the result presented in Ref. [16]. Obviously, the Yukawa-like dependence in the multipole expansion for Klein-Gordon field has reduces to the Coulomb-like dependence, which reflects the fact that when \( m = 0 \), the mass of the quanta of the scalar field vanishes.
III. A MULTIPOLe ANALYSIS ON STATIONARY MASSIVE VECTOR FIELD

As noted previously, the field equations of a massive vector field $A^\mu$ are (1.1a) and (1.1b), where $A^\mu$ has no the local gauge symmetry. For a spatially compact stationary source $j^\mu(x)$, the field equations of $A^\mu$ reduce to

$$\nabla^2 A^\mu - m_A^2 A^\mu = -4\pi j^\mu,$$

$$\partial_i A^i = 0.$$  \hspace{1cm} (3.1a, 3.1b)

Each component of $A^\mu$ satisfies the screened Poisson equation, and then from Eqs. (2.25)—(2.27), we have

$$A^0 = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{Q}_{I_l} \partial_{I_l} \left( e^{\frac{-m_A r'}{r}} \right),$$

$$A^i = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{F}^{(I_l)} \partial_{I_l} \left( e^{\frac{-m_A r'}{r}} \right),$$

where

$$\hat{Q}_{I_l} := \hat{F}^{0}_{(I_l)} = \int \hat{X}' I_l, \delta_i (m_A r') j^0 (x') d^3 x',$$

$$\hat{F}^{(I_l)} = \int \hat{X}' I_l, \delta_i (m_A r') j^i (x') d^3 x'.$$

For simplicity, define

$$U_{i(I_l)} := \frac{(-1)^l}{l!} \hat{F}^{i}_{(I_l)},$$

and then, there is

$$A^i = \sum_{l=0}^{\infty} U_{i(I_l)} \partial_{I_l} \left( e^{\frac{-m_A r'}{r}} \right).$$

The Cartesian tensor $U_{i(I_l)}$ is reducible, and as stated in Eq. (2.8), it may be decomposed into three irreducible pieces denoted by

$$\hat{R}^{(+)}_{I_{l+1}} = \hat{U}_{I_{l+1}},$$

$$\hat{R}^{(0)}_{I_l} = U_{pq(i_1 \cdots i_{l-1})} \epsilon_{pq},$$

$$\hat{R}^{(-)}_{I_{l-1}} = U_{aaI_{l-1}},$$

so that

$$U_{i(I_l)} = \hat{R}^{(+)}_{I_{l+1}} + \frac{l}{l+1} \epsilon_{ax(i_1 \cdots i_{l+1})} a + \frac{2l - 1}{l+1} \delta_{(i_1} \hat{R}^{(-)}_{i_{l+1} \cdots i_{l-1})}. \hspace{1cm} (3.7)$$

From Eqs. (2.19) and (2.20), we get the identity

$$\left( \nabla^2 - m_A^2 \right) \left( \frac{e^{\frac{-m_A r'}{r}}}{r} \right) = -4\pi \delta^3(x),$$

and then, by substituting the decomposition (3.7) in the expansion (3.5) outside the source region, we finally derive, after suitable changes of the summation index,

$$A^i = \sum_{l=0}^{\infty} \hat{B}_{I_l} \partial_{I_l} \left( \frac{e^{\frac{-m_A r'}{r}}}{r} \right) + \sum_{l=1}^{\infty} \hat{C}_{I_{l-1}} \partial_{I_{l-1}} \left( \frac{e^{\frac{-m_A r'}{r}}}{r} \right) + \sum_{l=1}^{\infty} \epsilon_{abI_{l-1}} \partial_{aI_{l-1}} \left( \frac{e^{\frac{-m_A r'}{r}}}{r} \right)$$

with

$$\hat{B}_{I_l} := \frac{2l+1}{2l+3} \hat{R}_{I_l}^{(-)};$$

$$\hat{C}_{I_l} := \hat{R}_{I_l}^{(+)} - \frac{m_A^2 l}{2l+3} \hat{R}_{I_l}^{(-)};$$

$$\hat{D}_{I_l} := \frac{l}{l+1} \hat{R}_{I_l}^{(0)}.$$  \hspace{1cm} (3.10a, 3.10b, 3.10c)
Next, we will consider Eq. (3.1b). Plugging the expansion (3.9) into it, one can directly obtain

\[ \partial_i A^i = m_A^2 \hat{B} e^{-m_A r} + \sum_{l=1}^{\infty} \left( m_A^2 \hat{B} + \hat{C}_l \right) \partial_i \left( \frac{e^{-m_A r}}{r} \right) = 0, \]  

(3.11)

from which, by virtue of Eqs. (2.13) and (2.15), one can further obtain

\[ \hat{B} = 0, \quad m_A^2 \hat{B}_l + \hat{C}_l = 0, \quad l \geq 1. \]  

(3.12)

With these conditions, the expansion of \( A^i \) reduces to

\[ A^i = \sum_{l=1}^{\infty} \epsilon_{iab} \hat{M}_{ab,l-1} \partial_{bl-1} \left( \frac{e^{-m_A r}}{r} \right) + \sum_{l=1}^{\infty} \left[ \hat{B}_l \partial_{(l)} \left( \frac{e^{-m_A r}}{r} \right) - \frac{m_A^2}{2l+1} \hat{B}_{l-1} \partial_{l-1} \left( \frac{e^{-m_A r}}{r} \right) \right], \]  

(3.13)

where from Eqs. (3.4), (3.6b), (3.6c), (3.10a), and (3.10c), the multipole moments \( \hat{M}_l \) and \( \hat{B}_l \) are

\[ \hat{M}_l := -\hat{D}_l = (-1)^l \frac{l}{l!} \int F_{q(i_1 \cdots i_l-1)}^{p} \partial_{(l)}, \quad l \geq 1, \]  

(3.14a)

\[ \hat{B}_l = \frac{(-1)^{l+1} 2l+1}{(l+1)!} \int F_{a(l)}^{p}, \quad l \geq 1. \]  

(3.14b)

For later convenience, let us replace the above \( \hat{M}_l \) and \( \hat{B}_l \) by \((-1)^l l! \hat{M}_l \) and \((-1)^{(l+1)} (l+1)! \hat{B}_l \), respectively, and then, there are

\[ A^i = \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \epsilon_{iab} \hat{M}_{ab,l-1} \partial_{bl-1} \left( \frac{e^{-m_A r}}{r} \right) + \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{(l+1)!} \left[ \hat{B}_l \partial_{(l)} \left( \frac{e^{-m_A r}}{r} \right) - \frac{m_A^2}{2l+1} \hat{B}_{l-1} \partial_{l-1} \left( \frac{e^{-m_A r}}{r} \right) \right], \]  

(3.15)

where in the above derivation, the equality

\[ \hat{B}_l \partial_{(l)} \left( \frac{e^{-m_A r}}{r} \right) = \hat{B}_l \partial_{l-1} \left( \frac{e^{-m_A r}}{r} \right) + \frac{m_A^2 l}{2l+1} \hat{B}_{l-1} \partial_{(l)} \left( \frac{e^{-m_A r}}{r} \right) \]  

(3.16)

has been used, and it can be derived from Eq. (2.16). After inserting Eq. (3.3b) into Eqs. (3.14a) and (3.14b), we acquire the final expressions of the source multipole moments

\[ \hat{M}_l = -\frac{l}{l+1} \int \hat{X} q_{i1 \cdots i_l-1}^{\gamma(p)} \partial_{(l)} (m_A r') j^p (r') d^3 r', \quad l \geq 1, \]  

(3.17a)

\[ \hat{B}_l = \frac{2l+1}{2l+3} \int \hat{X} a_{i1}^{\gamma} \partial_{(l)} (m_A r') j^0 (r') d^3 r', \quad l \geq 1. \]  

(3.17b)

The expansions (3.2a) and (3.15) show that the stationary \( A^\mu \) field in the region exterior to the source can be expressed in terms of three infinite sets of STF multipole moments: \( \hat{Q}_{l}, \hat{M}_l, \) and \( \hat{B}_{l} \), where besides the “electric” multipole moments \( \hat{Q}_l \), there are two types of “magnetic” multipole moments, namely, \( \hat{M}_l \) and \( \hat{B}_l \). From Eqs. (3.2a) and (3.3a), one can easily verify that the expansion of \( A^\mu \) and the “electric” multipole moments \( \hat{Q}_l \) are compatible with the multipole expansion of the scalar potential of electrostatic field presented in Ref. [16]. For \( l = 0 \), according to Eq. (2.27), there are

\[ A^0 = \frac{\hat{Q}}{r} e^{-m_A r}, \]  

(3.18a)

\[ \hat{Q} = \int \delta_0 (m_A r') j^0 (x') d^3 x' = \int \frac{\sinh (m_A r')}{m_A r'} j^0 (x') d^3 x'. \]  

(3.18b)

This result implies that because of the existence of the radial factor \( \delta_0 (m_A r') \) in the integrand, \( \hat{Q} \) is not equal to the total charge of the source, which is different from the case of electrostatic field [29]. Substituting Eq. (3.2a) back in Eq. (3.1a) and using Eq. (3.8), the multipole approximation of \( j^0 (x) \) is derived,

\[ j^0 (x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{Q}_l \delta^3 (x), \]  

(3.19)
which means that outside the source region, the stationary $A^0$ field can be equivalently generated by the above source built from $\delta$-function. Thus, Eq. (3.19) could be referred to as the skeleton of the stationary $A^0$ field [30, 31].

Obviously, there are two types of “magnetic” multipole moments, so the multipole expansion for $A^i$ is different from that of the vector potential of the magnetostatic field. With Eqs. (3.15) and (3.16), the expansion of $A^i$ can be rewritten as

$$A^i = \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \epsilon_{iab} \hat{M}_{al - i} \hat{\partial}_{b l - 1} \left( \frac{e^{-m_A r}}{r} \right) + \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{(l+1)!} \left[ \hat{B}_{li} \hat{\partial}_{l i} \left( \frac{e^{-m_A r}}{r} \right) - m_A^2 \hat{B}_{l - 1 i} \hat{\partial}_{l - 1} \left( \frac{e^{-m_A r}}{r} \right) \right],$$

(3.20)

and when the massive vector field reduces to electromagnetic field, namely $m_A = 0$, the second term in the above expansion can be transformed away by using the local gauge transformation because it has reduced to the gradient of a scalar field. But for the massive vector field $A^i$, there is no longer the local gauge symmetry, and therefore, the second term in Eq. (3.20) does contribute to the multipole expansion of $A^i$. From this analysis, it is understood that “magnetic” multipole moments $\hat{M}_{li}$ are the generalization of those of the magnetostatic field, which can also be directly seen from the following argument: For $l = 1$,

$$\hat{M}_l = \frac{1}{2} \int \delta_l(m_A r') \epsilon_{ipq} x'_p j^q(x') d^3 x',$$

(3.21)

is able to reduce to the magnetic dipole moment of the magnetostatic field when $m_A = 0$. In addition, what should be pointed out is that for the massive vector field, both of the two types of “magnetic” monopole moments always vanish, which may reflect the nature of the vector field.

IV. A MULTIPOLe ANALYSIS ON STATIONARY MASSIVE SYMMETRIC TENSOR FIELD

Mathematically, the field equations of a massive symmetric tensor field $h^{\mu\nu}$ could be written as (1.2a) and (1.2b), where $h^{\mu\nu}$ has no the usual massless gravity gauge symmetry [3-6]. For a spatially compact stationary source $T^{\mu\nu}(x)$, the field equations of $h^{\mu\nu}$ reduce to

$$\nabla^2 h^{\mu\nu} - m_h^2 h^{\mu\nu} = -4\pi T^{\mu\nu},$$

(4.1a)

$$\partial_{\mu} h^{\mu\nu} = 0,$$

(4.1b)

where each component of $h^{\mu\nu}$ satisfies the screened Poisson equation. Thus, as in the previous section, we have the following multipole expansions for $h^{\mu\nu}$ in the region exterior to the source,

$$h^{00}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{M}^{(l)}_{l i} \hat{\partial}_{l i} \left( \frac{e^{-m_h r}}{r} \right),$$

(4.2a)

$$h^{0i}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{F}^{0i}_{l i} \hat{\partial}_{l i} \left( \frac{e^{-m_h r}}{r} \right),$$

(4.2b)

$$h^{ij}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{F}^{ij}_{l i} \hat{\partial}_{l i} \left( \frac{e^{-m_h r}}{r} \right)$$

(4.2c)

with

$$\hat{M}^{(l)}_{l i} := \frac{\hat{F}^{00}}{\hat{F}_{l i}} = \int \hat{X}'_{i l} \delta_l(m_h r') T^{00}(x') d^3 x',$$

(4.3a)

$$\hat{F}^{0i}_{l i} = \int \hat{X}'_{i l} \delta_l(m_h r') T^{0i}(x') d^3 x',$$

(4.3b)

$$\hat{F}^{ij}_{l i} = \int \hat{X}'_{i l} \delta_l(m_h r') T^{ij}(x') d^3 x'.$$

(4.3c)

For $h^{0i}$, Eqs. (4.2b), (4.3b), and (4.1b) are analogous to Eqs. (3.2b), (3.3b), and (3.1b), respectively, and then, by following the manipulations in Sec. III, one can directly obtain

$$h^{0i} = \sum_{l=1}^{\infty} \frac{(-1)^l}{(l+1)!} \epsilon_{iab} \hat{S}_{a l - i} \hat{\partial}_{b l - 1} \left( \frac{e^{-m_h r}}{r} \right) + \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{(l+1)!} \left[ \hat{B}_{li} \hat{\partial}_{l i} \left( \frac{e^{-m_A r}}{r} \right) - \frac{m_A^2 (l+1)}{2l+1} \hat{B}_{l - 1 i} \hat{\partial}_{l - 1} \left( \frac{e^{-m_A r}}{r} \right) \right].$$

(4.4)
with

$$
\hat{S}_{li} = - \int X'_{ij(i, \ldots, i_{l+1})} T^{00}(x') d^3x', \quad l \geq 1,
$$

$$
\hat{B}_{li}^{(h)} = \frac{2l+1}{2l+3} \int X'_{a(l+1)} T^{0a}(x') d^3x', \quad l \geq 1.
$$

(4.5a, 4.5b)

For $h^{ij}$, define

$$
U_{ij(l)} := \frac{(-1)^l}{l!} \mathcal{F}^{ij}_{(l)},
$$

(4.6)

and then, there is

$$
h^{ij} = \sum_{l=0}^{\infty} U_{ij(l)} \partial_l \left( \frac{e^{-m_hr}}{r} \right).
$$

(4.7)

Similarly to $U_{i(l)}$ in Sec. III, one could employ Eq. (2.8) to gain the decomposition of $U_{ij(l)}$,

$$
U_{ij(l)} = \hat{R}_{i(l)}^{(+)} + \frac{l}{l+1} \hat{R}_{i(l)}^{(0)} + \frac{2l+1}{2l+3} \hat{R}_{i(l)}^{(-)} + \delta_{ii, l+1} \delta_{ij, l1}.
$$

(4.8a)

$$
\hat{R}_{i(l)}^{(+)} = U_{i(l+1)} =: V^{(+)_{i(l+1)}},
$$

(4.8b)

$$
\hat{R}_{i(l)}^{(0)} = U_{i(l+1)} =: V^{(0)_{i(l+1)}},
$$

(4.8c)

$$
\hat{R}_{i(l)}^{(-)} = U_{i(l+1)} =: V^{(-)_{i(l+1)}},
$$

(4.8d)

where the symbol $|i|$ in $\hat{R}_{i(l+1)}^{(+)}$, $\hat{R}_{i(l+1)}^{(0)}$, and $\hat{R}_{i(l+1)}^{(-)}$ represents that it is not STF index. Since Cartesian tensors $V^{(+)}_{i(l+1)}$, $V^{(0)}_{i(l+1)}$, and $V^{(-)}_{i(l+1)}$ are still reducible, Eq. (2.8) needs to be used again to obtain their decompositions:

$$
V^{(+)}_{i(l+1)} = \hat{R}_{i(l+1)}^{(+)} + \frac{l}{l+1} \epsilon_{ai} \hat{R}_{i(l+1)}^{(0)} + \frac{2l+1}{2l+3} \delta_{ii, l+1} \hat{R}_{i(l+1)}^{(-)},
$$

(4.9a)

$$
\hat{R}_{i(l+1)}^{(+)} = V^{(+)}_{i(l+1)},
$$

(4.9b)

$$
\hat{R}_{i(l+1)}^{(0)} = V^{(0)}_{i(l+1)},
$$

(4.9c)

$$
\hat{R}_{i(l+1)}^{(-)} = V^{(-)}_{i(l+1)},
$$

(4.9d)

With these results, substitute Eq. (4.8a) in the expansion (4.7) and use the identity

$$
(\nabla^2 - m_r^2) \left( \frac{e^{-m_hr}}{r} \right) = -4\pi \delta^3(x)
$$

(4.10)
outside the source region, we could derive, after suitable changes of the summation index,

\[ h^{ij} = \sum_{l=0}^{\infty} \hat{E}_l \partial_j h_i \left( \frac{e^{-m_h r}}{r} \right) + \sum_{l=0}^{\infty} \hat{F}_l \delta_{ij} \partial_l h_i \left( \frac{e^{-m_h r}}{r} \right) + \sum_{l=1}^{\infty} \hat{G}_{l-1 \ i \ j} \partial_{l-1} \left( \frac{e^{-m_h r}}{r} \right) \]

\[ + \sum_{l=1}^{\infty} \hat{H}_{bll-1} \epsilon_{ab(i \ j)} \partial_{l-1} \left( \frac{e^{-m_h r}}{r} \right) + \sum_{l=2}^{\infty} \hat{I}_{j \ i \ l-2} \partial_{l-2} \left( \frac{e^{-m_h r}}{r} \right) + \sum_{l=2}^{\infty} \epsilon_{ab(i \ j)} \hat{b}_{l-2} \partial_{a l-2} \left( \frac{e^{-m_h r}}{r} \right) \]  

(4.11)

with

\[ \hat{E}_l := \frac{2l + 1}{2l + 5} \hat{R}_l^{(--)}, \]  

(4.12a)

\[ \hat{F}_l := \frac{2l + 1}{(2l + 3)(l + 1)} \hat{R}_l^{(+--)} - \frac{l^2}{(l + 1)^2} \hat{R}_l^{(00)} - \frac{(2l + 1)m_h^2}{(2l + 5)(2l + 3)} \hat{R}_l^{(--)}, \]  

(4.12b)

\[ \hat{G}_l := \frac{(2l - 1)l}{(2l + 1)(l + 1)} \hat{R}_l^{(+--)} + \frac{l(2l - 1)}{(l + 1)^2} \hat{R}_l^{(00)} + \frac{2l - 1}{2l + 1} \hat{R}_l^{(---)} - \frac{2l(2l + 1)m_h^2}{(2l + 5)(2l + 3)} \hat{R}_l^{(--)}, \]  

(4.12c)

\[ \hat{H}_l := \frac{2l + 1}{(2l + 3)(l + 1)} \hat{R}_l^{(--)}, \]  

(4.12d)

\[ \hat{I}_l := \frac{l(l - 1)m_h^2}{(2l + 1)(l + 1)^2} \hat{R}_l^{(--)}, \]  

(4.12e)

\[ \hat{J}_l := \frac{l - 1 + \hat{R}_l^{(++)}}{l + 1} \hat{R}_l^{(00)} - \frac{l(l - 1)m_h^2}{(2l + 1)(l + 1)} \hat{R}_l^{(--)}. \]  

(4.12f)

In what follows, we will consider equation \( \partial_i h^{ij} = 0 \). Inserting the expansion (4.11) into it provides

\[ \left( \frac{m_h^2 \hat{G}_j}{2} + \frac{m_h^2}{3} \left( \frac{m_h^2 \hat{E}_j + \hat{F}_j + \frac{\hat{G}_j}{2}}{m_h^2 \hat{E}_j + \hat{F}_j + \frac{\hat{G}_j}{2}} \right) \right) e^{-m_h r} \left( \frac{m_h^2 \hat{E} + \hat{F}}{r} \right) \partial_j e^{-m_h r} \left( \frac{m_h^2 \hat{E} + \hat{F}}{r} \right) + m_h^2 \hat{H}_b \partial_a \left( e^{-m_h r} \right) \]

\[ + \left( \frac{m_h^2 \hat{G}_j}{2} + \hat{I}_j \right) \partial_l e^{-m_h r} \left( \frac{m_h^2 \hat{E} + \hat{F}}{r} \right) + \frac{2m_h^2}{5} \left( \frac{m_h^2 \hat{E}_j + \hat{F}_j + \frac{\hat{G}_j}{2}}{m_h^2 \hat{E} + \hat{F} + \frac{\hat{G}_j}{2}} \right) \partial_l e^{-m_h r} \]

\[ + \sum_{l=2}^{\infty} \left( \frac{m_h^2 \hat{E}_{l-1} + \hat{F}_{l-1} + \frac{\hat{G}_{l-1}}{2}}{m_h^2 \hat{E} + \hat{F} + \frac{\hat{G}_j}{2}} \right) \partial_{l-1} e^{-m_h r} \left( \frac{m_h^2 \hat{E}_{l-1} + \hat{F}_{l-1} + \frac{\hat{G}_{l-1}}{2}}{r} \right) + \sum_{l=2}^{\infty} \frac{m_h^2}{2l + 1} \left( \frac{m_h^2 \hat{E}_{l-1} + \hat{F}_{l-1} + \frac{\hat{G}_{l-1}}{2}}{r} \right) \partial_{l-2} e^{-m_h r} \left( \frac{m_h^2 \hat{E}_{l-1} + \hat{F}_{l-1} + \frac{\hat{G}_{l-1}}{2}}{r} \right) \]

\[ + \sum_{l=2}^{\infty} \frac{1}{l} \left( \frac{m_h^2 \hat{H}_{bll-1} + \hat{J}_{bll-1}}{r} \right) \epsilon_{ab(i \ j)} \partial_{l-1} \left( \frac{e^{-m_h r}}{r} \right) + \sum_{l=2}^{\infty} \left( \frac{m_h^2 \hat{G}_l}{2} + \hat{I}_l \right) \partial_{l-2} \left( \frac{e^{-m_h r}}{r} \right) = 0, \]  

(4.13)

and then, by means of Eqs. (2.13) and (2.15), one is able to acquire the conditions,

\[ m_h^2 \hat{E} + \hat{F} = 0, \quad \hat{G}_j = 0, \quad \hat{H}_j = 0, \]  

(4.14a)

\[ m_h^2 \hat{E}_l + \hat{F}_l + \frac{\hat{G}_l}{2} = 0, \quad l \geq 1, \]  

(4.14b)

\[ m_h^2 \hat{G}_l + \hat{I}_l = 0, \quad l \geq 2, \]  

(4.14c)

\[ m_h^2 \hat{H}_l + \hat{J}_l = 0, \quad l \geq 2. \]  

(4.14d)

In many circumstances, the trace of \( h^{ij} \) is also specified, and here, we assume that

\[ h^{kk} = \sum_{l=0}^{\infty} \hat{A}_l \partial_l \left( \frac{e^{-m_h r}}{r} \right) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{A}_l \partial_l \left( \frac{e^{-m_h r}}{r} \right), \]  

(4.15)

Under this case, from the expansion (4.13) and (4.14a), additional conditions are provided,

\[ \begin{cases} 
  m_h^2 \hat{E} + 3 \hat{F} = \hat{A}, \\
  m_h^2 \hat{E}_j + 3 \hat{F}_j = \hat{A}_j, \\
  m_h^2 \hat{E}_l + 3 \hat{F}_l + \frac{\hat{G}_l}{2} = \hat{A}_l, \quad l \geq 2.
\end{cases} \]  

(4.16)
With the conditions (4.14a)—(4.14d) and (4.16), the expansion of \( h^{ij} \) reduces to

\[
h^{ij} = - \sum_{l=0}^{\infty} \left[ \frac{\hat{A}_{l}}{2m_{h}^{2}} + \frac{\hat{G}_{l}}{4m_{h}^{2}} \right] \partial_{ij}l_{l} \left( \frac{e^{-m_{h}r}}{r} \right) - \left( \frac{\hat{A}_{l}}{2} - \frac{\hat{G}_{l}}{4} \right) \delta_{ij} \partial_{l}l_{l} \left( \frac{e^{-m_{h}r}}{r} \right) \\
+ \sum_{l=2}^{\infty} \left[ \hat{G}_{l-1}l_{l-1} \left( \frac{e^{-m_{h}r}}{r} \right) - \frac{m_{h}^{2}}{2} \hat{G}_{l-2} \partial_{l}l_{l-2} \left( \frac{e^{-m_{h}r}}{r} \right) \right] \\
+ \sum_{l=2}^{\infty} \left[ \hat{H}_{l}l_{l-1} \epsilon_{ab(l)} \partial_{l}l_{l-1} \left( \frac{e^{-m_{h}r}}{r} \right) - m_{h}^{2} \epsilon_{ab}l_{l} \partial_{l-2}l_{l-2} \left( \frac{e^{-m_{h}r}}{r} \right) \right],
\]

where \( \hat{G} = 0 \) is presumed, and according to Eqs. (4.12a)—(4.12d), (4.8b)—(9.1), and (4.14b), the source multipole moments \( \hat{G}_{l} \) and \( \hat{H}_{l} \) can be given,

\[
\hat{G}_{l} = \frac{(-1)^{l}}{l!} \left( \frac{4(2l+1)m_{h}^{2}}{(2l+5)(l+2)(l+1)} + 2F_{(ab)}^{\alpha \mu} \right), \quad l \geq 2,
\]

\[
\hat{H}_{l} = \frac{(-1)^{l+1}}{(l+1)!} \left( \frac{2(2l+1)}{(2l+3)(l+2)}F_{apq}^{\alpha \mu \epsilon_{i_{l}} \epsilon_{j_{l}} \epsilon_{k_{l}} \epsilon_{l_{l}} \epsilon_{m_{l}} \epsilon_{n_{l}} \epsilon_{o_{l}} \epsilon_{p_{l}}} \right), \quad l \geq 2.
\]

As in Sec. III, replacing \( \hat{G}_{l} \) and \( \hat{H}_{l} \) by \((-1)^{l/l!} \hat{G}_{l} \) and \((-1)^{l+1}(l+1)! \hat{H}_{l} \), respectively, is convenient, and thus, \( h^{ij} \) is given by,

\[
h^{ij} = \frac{1}{3} \delta_{ij} \sum_{l=1}^{\infty} \left( \frac{(-1)^{l}}{l!} \hat{A}_{l}l_{l} \partial_{ij}l_{l} \left( \frac{e^{-m_{h}r}}{r} \right) - \sum_{l=0}^{\infty} \left( \frac{(-1)^{l}}{l!} \left( \frac{\hat{A}_{l}}{2m_{h}^{2}} - \frac{\hat{G}_{l}}{4m_{h}^{2}} \right) \right) \partial_{ij}l_{l} \left( \frac{e^{-m_{h}r}}{r} \right) \\
- \sum_{l=1}^{\infty} \left( \frac{(-1)^{l}}{l!} \left( \frac{2l+3}{2l+1} \hat{A}_{l-1}(\partial_{j}l_{l-1}) \left( \frac{e^{-m_{h}r}}{r} \right) + \frac{(l+2)}{2l+3} \hat{G}_{l-1}(\partial_{j}l_{l-1}) \left( \frac{e^{-m_{h}r}}{r} \right) \right) - \frac{m_{h}^{2}}{2(2l+1)(2l-1)} \hat{G}_{l-2} \partial_{l-2}l_{l-2} \left( \frac{e^{-m_{h}r}}{r} \right) \right) \\
+ \sum_{l=2}^{\infty} \left( \frac{(-1)^{l+1}}{(l+1)!} \hat{H}_{l}l_{l-1} \epsilon_{ab(l)} \partial_{l}l_{l-1} \left( \frac{e^{-m_{h}r}}{r} \right) - m_{h}^{2} \epsilon_{ab}l_{l} \partial_{l-2}l_{l-2} \left( \frac{e^{-m_{h}r}}{r} \right) \right),
\]

where in the above derivation, the equality

\[
\hat{E}_{l}l_{l} \partial_{ij}l_{l} \left( \frac{e^{-m_{h}r}}{r} \right) = \hat{E}_{l}l_{l} \partial_{ij}l_{l} \left( \frac{e^{-m_{h}r}}{r} \right) + \frac{m_{h}^{2}l}{2l+3} \delta_{ij} \hat{E}_{l}l_{l} \left( \frac{e^{-m_{h}r}}{r} \right) \\
+ \frac{2m_{h}^{2}l}{2l+3} \hat{E}_{l-1}(\partial_{j}l_{l-1}) \left( \frac{e^{-m_{h}r}}{r} \right) + \frac{m_{h}^{2}l(l-1)}{2l+3} \hat{E}_{l-2} \partial_{l-2}l_{l-2} \left( \frac{e^{-m_{h}r}}{r} \right)
\]

has been used, and it can be deduced by using Eq. (2.16) twice in succession. Finally, plugging Eq. (4.3c) into Eqs. (4.18a) and (4.18b), we acquire the closed-form expressions of the source multipole moments \( \hat{G}_{l} \) and \( \hat{H}_{l} \), namely,

\[
\hat{G}_{l} = \frac{4(2l+1)m_{h}^{2}}{(2l+5)(l+2)(l+1)} \int \hat{X}_{l}l_{l} \delta_{l+2}l_{l}(m_{h}r')T^{ab}(x')d^{3}x' + 2 \int \hat{X}_{l}l_{l} \delta_{l}(m_{h}r')T^{aa}(x')d^{3}x', \quad l \geq 2,
\]

\[
\hat{H}_{l} = \frac{2(2l+1)}{(2l+3)(l+2)} \int \hat{X}_{l}l_{l} \delta_{l+2}l_{l}(m_{h}r')T^{aa}(x')d^{3}x', \quad l \geq 2.
\]

From the expansions (4.2a), (4.4), and (4.19), it is easily seen that if the trace of the spatial part of the stationary massive symmetric tensor field \( h^{\mu \nu} \) is specified, the multipole expansions in the region exterior to the source depend on five infinite sets of STF multipole moments, namely \( M_{l}^{(h)} \), \( S_{l}^{(h)} \), \( B_{l}^{(h)} \), \( \hat{G}_{l} \), and \( \hat{H}_{l} \). When \( h^{\mu \nu} \) reduces to massless field, such as the gravitational field amplitude in the linearized GR [32], by applying the gauge transformation preserving the condition (1.2b) [16], one could verify that the above multipole moments \( B_{l}^{(h)} \), \( \hat{G}_{l} \), and \( \hat{H}_{l} \) can be transformed away. Therefore, \( M_{l}^{(h)} \), \( \hat{S}_{l} \) should be the counterparts of the mass and spin multipole moments for massless symmetric
tensor field, which may also be directly seen from the following facts:

\[ \hat{M}^{(h)} = \int T^{00}(x')\delta_0(m_hr')d^3x', \quad \text{for } l = 0, \tag{4.22a} \]

\[ \hat{\mathcal{S}}_l = \int \epsilon_{pqrs'}T^{0q}(x')\delta_1(m_hr')d^3x', \quad \text{for } l = 1 \tag{4.22b} \]

are able to reduce to the total mass and the angular momentum of the source when \( m_h = 0 \). As a consequence, \( \hat{M}^{(h)}_l \) and \( \hat{\mathcal{S}}_l \) in Eqs. (4.22a) and (4.4) could be referred to as the “mass” and “spin” multipole moments of the massive symmetric tensor field.

The multipole expansion (4.19) of \( h^{ij} \) is dependent on the STF tensor \( \hat{A}_l \) which are defined in terms of \( h^{kk} \), and hence, when one intends to apply this result, the expression of \( h^{kk} \) needs to be given. The first case that one may encounter is that \( h^{kk} = 0 \). In this situation, from Eq. (4.15), there are \( \hat{A}_l = 0 \), and thus, Eq. (4.19) yields

\[
h^{ij}_{l=0} = \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \frac{\hat{G}_{l0}}{4m^2_h} \hat{A}_{ij}l_0 \left( \frac{e^{-m_hr}}{r} \right) - \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \frac{3(l+2)}{2(2l+3)} \frac{\hat{G}_{l-1}(i\partial_j)l_{1-1}}{m^2_h} \left( \frac{e^{-m_hr}}{r} \right) + \sum_{l=2}^{\infty} \frac{(-1)^l-1}{(l+1)!} \left[ \hat{H}_{b1-1} \epsilon_{ab(i\partial_j)al_{1-2}} \left( \frac{e^{-m_hr}}{r} \right) \right] \]

\[ + \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \frac{(9l^2 - l - 2)m^2_h}{4(2l+1)(2l-1)} \frac{\hat{G}_{i(l-2)j(l-2)}l_{1-2}}{m^2_h} \left( \frac{e^{-m_hr}}{r} \right) + \sum_{l=2}^{\infty} \frac{(-1)^l-1}{(l+1)!} \left[ \hat{H}_{b1-1} \epsilon_{ab(i\partial_j)al_{1-2}} \left( \frac{e^{-m_hr}}{r} \right) \right]. \tag{4.23} \]

Obviously, when \( h^{kk} = 0 \), the monopole and dipole moments of \( h^{ij} \) vanish, which is compatible with the general form of the STF-tensor spherical harmonics expansion for a trace-free tensor field of “spin” 2 on the unit sphere centered at the coordinate origin [14]. With Eq. (4.23), the expansion (4.19) can be rewritten as

\[
h^{ij} = h^{ij}_{l=0} + \frac{1}{3} \delta_{ij} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{A}_{ij}l_0 \left( \frac{e^{-m_hr}}{r} \right) - \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \frac{\hat{A}_{ij}l_0}{2m^2_h} \hat{A}_{ij}l_0 \left( \frac{e^{-m_hr}}{r} \right) + \sum_{l=2}^{\infty} \frac{(-1)^l}{l!(l+1)!} \hat{H}_{b1-1} \epsilon_{ab(i\partial_j)al_{1-2}} \left( \frac{e^{-m_hr}}{r} \right) \]

\[ - \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \frac{l}{2l+3} \hat{A}_{ij}l_{1-1} \left( \frac{e^{-m_hr}}{r} \right) - \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \frac{l(l-1)m^2_h}{2(2l+1)(2l-1)} \hat{A}_{ij}l_{1-2} \left( \frac{e^{-m_hr}}{r} \right). \tag{4.24} \]

which explicitly indicates that in general, the monopole and dipole moments of \( h^{ij} \) do not vanish, and they are only related to \( \hat{A} \) and \( \hat{A}_j \). Another case that one may encounter is that \( h^{kk} = h^{00} \) or \( h^{\mu \nu} = 0 \). From Eqs. (4.2a) and (4.15), this case implies \( \hat{A}_l = \hat{M}^{(h)}_l \), and by substituting it back in Eq. (4.24), the final expansion of \( h^{ij} \) in this situation can be derived.

The above results of multipole expansion for stationary massive symmetric tensor field present the external solution to Eqs. (4.1a) and (4.1b) for any spatially compact stationary source, so they must have important applications in the models of massive gravity. In addition, these results can also be applied to alternative theories of gravity. As stated in the introduction section, such a typical model is the \( F(X,Y,Z) \) gravity, a generic fourth-order theory of gravity. In this model, there is a massive propagation of the linearized Ricci tensor [19, 20], and since the equations satisfied by its components can be recast in the form of Eqs. (1.2a) and (1.2b) the results provided in this section may help to obtain the corresponding stationary solutions to these equations. It should be pointed out that the multipole expansion for \( h^{\mu \nu} \) actually presents the effects of the source at all orders, and in practical application, the results need to be truncated to the leading pole order or the next leading pole order so that the dominant effects of the source could be obtained.

V. SUMMARY AND CONCLUSION

In this paper, by following the method in Refs. [14, 16], we have shown how to utilize the STF formalism to make a multipole analysis on stationary massive vector and symmetric tensor fields in a unified and structurally transparent manner. Due to the reason that stationary massive vector and symmetric tensor fields have no the usual gauge symmetries of massless fields [1–6], their multipole expansions can be expressed in terms of more infinite sets of STF multipole moments. In addition, differently from the results for stationary massless fields, the multipole expansions for stationary massive fields show a Yukawa-like dependence on the massive parameters of the fields, and the integrals of the source multipole moments are all modulated by a common radial factor.
In the stationary massive vector field case, the expansion of $A^I$ field and the “electric” multipole moments $\hat{Q}_I$ are compatible with the multipole expansion of the scalar potential of the electrostatic field presented in Ref. [16]. But it should be pointed out that at the leading pole order, the existence of the radial factor $\delta_0(mAr')$ in the integrand of the monopole moment results in that $\hat{Q}$ is not equal to the total charge of the source, which is different from the case of the electrostatic field [29]. For $A^I$ field, two types of “magnetic” multipole moments, namely $M_I$, $B_I$, appear in its multipole expansion, where from the expression of $M_I$, we know that $M_I$ are the generalization of those of the magnetostatic field, and $B_I$, are additional multipole moments of the stationary massive vector field.

In the stationary massive symmetric tensor field case, the expansions of $h^{\mu\nu}$ depend on five infinite sets of STF multipole moments, namely $M_I^{(h)}$, $S_I^{(h)}$, $\hat{B}_I^{(h)}$, $\hat{G}_I$, $\hat{H}_I$, and the added efficiency of the STF technique has allowed us to obtain the closed-form expressions of these moments in terms of the energy-momentum tensor of the source. Among these moments, $M_I^{(h)}$, $S_I^{(h)}$, in the expansions of $h^{00}$ and $h^{0i}$ are the counterparts of the mass and spin multipole moments for massless symmetric tensor field, because at the leading pole order, $M^{(h)}$ and $S^I$ are able to reduce to the total mass and the angular momentum of the source when the mass parameter of the field vanishes. The trace of the spatial part $h^{ij}$ of the tensor field plays an important role in its expansion, and the expansions of $h^{ij}$ under the cases of $h^{k0} = 0$ and $h^{kk} = h^{00}$ are provided in the present paper. The result indicates that when $h^{kk} = 0$, the monopole and dipole moments in the expansion of $h^{ij}$ vanish [14].

In general, if a spatially compact source is in a static state, and namely, the elements of the source do not move in the spatial directions, the field generated by it is stationary. Thus, by making use of the results in the present paper, the effects of the scale and shape of the source can be analyzed when these results are applied to some specific phenomenon [24]. In addition, it should be pointed out that even if the source is not in a static configuration, its external field may still be stationary. Let us discuss a rigidly rotating source with an angular velocity that is independent of position within the source. If the rotation is steady in the sense that the angular velocity is independent of time, and the motions of the elements of the source are taken to be purely rotational, the system is taken to be stationary when viewed by a non-rotating observer. Under such a case, our results in this paper can also be applied to the external field of the source.

As far as we know, such results of multipole expansions for stationary massive vector and symmetric tensor fields have not been given before. Although these results in this paper are valid only for stationary sources, since they describe the effects of the source at all orders they must have important applications in the models of massive gravity and alternative theories of gravity. In the future, it could be expected that more and more applications of these results will be found. These results will be extended to non-stationary sources in our subsequent work so as to obtain the relativistic time-dependent multipole expansions for massive scalar, vector, and symmetric tensor fields. In order to complete the discussion, the force equation for a particle in a massive field also needs to be addressed so that one can discuss the motion of the particle. In general, since the action integral for the particle is a scalar field, its expression could be assumed to be the same as that when the field is massless. Thus, the force equation for the particle can be directly derived by virtue of the Euler-Lagrange equations. This derivation is trivial, and interested readers could consult relevant references.

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Data Availability Statement

No new data were created or analysed in this study.

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