Abstract. We use the Riccati equation method with other ones to establish new oscillation and interval oscillation criteria for linear matrix Hamiltonian systems. We investigate the oscillation problem for linear matrix Hamiltonian systems in a new direction, which is to break the positive definiteness condition, imposed on one of the coefficients of the system.

Key words: Riccati equation, oscillation, interval oscillation, conjoined (prepared, preferred) solutions, unitary transformation, comparison theorem.

1. Introduction. Let $A(t)$, $B(t)$, and $C(t)$, be complex valued continuous matrix functions on $[t_0, +\infty)$ and let $B(t)$ and $C(t)$ be Hermitian, i.e., $B(t) = B^*(t)$, $C(t) = C^*(t)$, $t \geq t_0$ (here and after * denotes the conjugation sign). Consider the linear matrix Hamiltonian system

\[
\begin{cases}
\Phi' = A(t)\Phi + B(t)\Psi, \\
\Psi' = C(t)\Phi - A^*(t)\Psi,
\end{cases}
\tag{1.1}
\]

By a solution of this system we mean an ordered pair $(\Phi(t), \Psi(t))$ of continuously differentiable matrix functions $\Phi(t)$ and $\Psi(t)$ of dimension $n \times n$ on $[t_0, +\infty)$, satisfying (1.1) on $[t_0, +\infty)$.

**Definition 1.1.** A solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called conjoined (or prepared, preferred) if $\Phi^*(t)\Psi(t) = \Psi^*(t)\Phi(t)$, $t \geq t_0$.

**Definition 1.2.** A conjoined solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called oscillatory if $\det \Phi(t)$ has arbitrary large zeroes.

**Definition 1.3** The system (1.1) is called oscillatory if its all conjoined solutions are oscillatory.

Let $[a, b] \subset [t_0, +\infty)$.

**Definition 1.4.** A conjoined solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called oscillatory on the interval $[a, b]$ if $\det \Phi(t)$ vanishes on $[a, b]$. 

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Definition 1.5 The system (1.1) is called oscillatory on the interval \([a, b]\), if its all conjoined solutions are oscillatory on \([a, b]\).

Study of the oscillatory behavior of the system (1.1) is an important problem of qualitative theory of differential equations and many works are devoted to it (see e.g., [1-12] and cited works therein). Usually the oscillation behavior of the system (1.1) is studied under the hypothesis that the matrix function \(B(t)\) is positive definite on \([t_0, +\infty)\), and this restriction is essential from the point of view of the using methods of investigations. Meanwhile in the applications "the nature" of the restriction on \(B(t)\) is that it must be non negative definite (The Legendre’s condition).

In [11] two oscillation criteria are obtained in a new direction which is to break the positive definiteness restriction imposed on \(B(t)\). In [11] the last restriction was replaced by the non negative definiteness condition with the condition of solvability of the linear matrix equation

\[
\sqrt{B(t)}X[A(t)\sqrt{B(t)} - \sqrt{B(t)}] = A(t)\sqrt{B(t)} - \sqrt{B(t)}, \quad t \geq t_0. \tag{1.2}
\]

Remark 1.1. Eq. (1.2) has always a solution on \([t_0, +\infty)\) when \(B(t)\) is invertible, in particular, when \(B(t)\) positive definite on \([t_0, +\infty)\) \((X = \sqrt{B(t)^{-1}} t \geq t_0)\). But it can also have a solution on \([t_0, +\infty)\) in some cases when \(B(t)\) is not positive definite but it is nonnegative definite (see [11]).

Another replacements of the mentioned above restriction are considered in [12], in which some new oscillation and interval oscillation criteria for the system (1.1) are obtained.

In this paper we continue the study of the oscillation problem of the system (1.1) in the mentioned above direction. The Riccati equation method used to obtain new oscillation and interval oscillation criteria. The unitary transformation approach allows to obtain oscillation and interval oscillation criteria without solvability condition, imposed on Eq. (1.2).

2. Main results. The non negative (positive) definiteness of any Hermitian matrix we denote by \(H \geq 0 (H > 0)\). Hereafter we will always assume that \(B(t) \geq 0, t \geq t_0\) (then \(\sqrt{B(t)}, t \geq t_0\) exists) and, when it is necessary, we will assume that \(\sqrt{B(t)}\) is continuously differentiable on \([t_0, +\infty)\) (or an interval \([a, b] \subset [t_0, +\infty)\)).

Let \(F(t)\) be a matrix function of dimension \(n \times n\) on \([t_0, +\infty)\). Set:

\[
\begin{align*}
A_F(t) & \equiv F(t)\sqrt{B(t)} - \sqrt{B(t)} \equiv (a_{Fjk}(t))_{j,k=1}^n, \\
C_B(t) & \equiv \sqrt{B(t)}C(t)\sqrt{B(t)} = (c_{Bjk}(t))_{j,k=1}^n, \\
\theta_{Fj}(t) & \equiv c_{Bjj}(t) + \sum_{m=1}^{n} |a_{Fmj}(t)|^2, \quad j = 1, n, \quad t \geq t_0.
\end{align*}
\]
Theorem 2.1. Let the following conditions be satisfied.
1) Eq. (1.2) has a solution $F(t)$ on $[t_0, +\infty)$;
2) for some $j \in \{1, \ldots, n\}$ the scalar equation

$$\phi'' + 2\Re a_{Fj}(t)\phi' + \theta_{Fj}(t)\phi = 0, \quad t \geq t_0$$

(2.1)

is oscillatory.

Then the system (1.1) is also oscillatory.

□

Theorem 2.2. Let the following conditions be satisfied.
1') Eq. (1.2) has a solution $F(t)$ on $[a, b]$;
2') for some $j \in \{1, \ldots, n\}$ the scalar equation

$$\phi'' + 2\Re a_{Fj}(t)\phi' + \theta_{Fj}(t)\phi = 0, \quad t \in [a, b]$$

is oscillatory on $[a, b]$.

Then the system (1.1) is also oscillatory on $[a, b]$.

□

Remark 2.1. An explicit interval oscillation criterion for second order linear ordinary differential equations (therefore for Eq. (2.1)) id obtained in [13] (see [13], Theorem 3.2)

The next result is based on the use of an unitary transformation, which allows us to overcome the restriction of solvability of Eq. (1.2), presented in the conditions of Theorem 2.1.

Let $p_{jk}(t), \ j, k = 1, 2$ be real-valued locally integrable functions on $[t_0, +\infty)$. Consider the linear system of ordinary differential equations

$$\begin{cases}
\phi' = p_{11}(t)\phi + p_{12}(t)\psi, \\
\psi' = p_{21}(t)\phi + p_{22}(t)\psi, \quad t \geq t_0.
\end{cases}$$

(2.1)

Definition 2.1. A solution $(\phi(t), \psi(t))$ of the system (2.1) is called oscillatory if $\phi(t)$ has arbitrary large zeroes.

Definition 2.2. The system (2.1) is called oscillatory if its all solutions are oscillatory.

Definition 2.3. A solution $(\phi(t), \psi(t))$ of the system (2.1) is called oscillatory on the interval $[a, b]$, if $\phi(t)$ vanishes on $[a, b]$.

Definition 2.4. The system (2.1) is called oscillatory on the interval $[a, b]$ if its all solutions are oscillatory on $[a, b]$.

Let $U_B(t)$ be an unitary matrix function of dimension $n \times n$ on $[t_0, +\infty)$ such that

$$B(t) = U_B^*(t)B_0(t)U_B(t), \quad t \geq t_0.$$  

(2.2)
where \( B_0(t) \equiv \text{diag}\{b_1(t),...,b_n(t)\}, \ t \geq t_0 \) - is a diagonal matrix function on \([t_0, +\infty)\).

**Remark 2.2.** It is well known that for any Hermitian matrix \( H \) of dimension \( n \times n \) there exists an unitary matrix (transformation) \( U_H \) such that \( H = U_H^* \text{diag}\{h_1,...,h_n\} U_H \), where \( h_1,...,h_n \) are real numbers.

Hereafter we will assume that \( U_B(t) \) is continuously differentiable on \([t_0, +\infty)\) and \( B_0(t) \) is continuous on \([t_0, +\infty)\). Set:

\[
A^0_B(t) \equiv U_B(t)[A(t)U^*_B(t) - \{U^*_B(t)\}'] = (a^0_{jk}(t))_{j,k=1}^n,
\]

\[
C^0_B(t) \equiv U_B(t)C(t)U^*_B(t) = (c^0_{jk}(t))_{j,k=1}^n,
\]

\[
\left[ \frac{|a^0_{mj}(t)|^2}{b_m(t)} \right]_0 \equiv \begin{cases} \frac{|a^0_{mj}(t)|^2}{b_m(t)}, & \text{if } b_m(t) \neq 0, \\ 0, & \text{if } b_m(t) = 0, \end{cases} \quad m = \overline{1,n},
\]

\[
\chi_j(t) \equiv c^0_{jj}(t) + \sum_{m=1 \atop m \neq j}^n \left[ \frac{|a^0_{mj}(t)|^2}{b_m(t)} \right]_0, \quad j = \overline{1,n}, \quad t \geq t_0.
\]

**Theorem 2.3.** Let the following conditions be satisfied:

3) \( b_m(t) \geq 0, \ m = \overline{1,n}, \ t \geq t_0; \)

4) for some \( j \in \{1,...,n\} \) the function \( \chi_j(t) \) is continuous on \([t_0, +\infty)\) and the scalar system

\[
\begin{cases}
\phi' = 2 \Re a^0_{jj}(t)\phi + b_j(t)\psi, \\
\psi' = \chi_j(t)\phi, \quad t \geq t_0
\end{cases}
\]

is oscillatory.

Then the system (1.1) is also oscillatory.

\[\square\]

**Remark 2.3.** Explicit oscillatory criteria for the system (2.1) (therefore for the system (2.3)) are obtained in [14].

**Corollary 2.1.** Let the following conditions be satisfied

5) \( B(t) = \text{diag}\{b_1(t),...,b_n(t)\}, \ b_m(t) \geq 0, \ m = \overline{1,n}, \ t \geq t_0, \)

6) for some \( j \in \{1,...,n\} \)

\[
\int_{t_0}^{+\infty} b_j(\tau)d\tau = - \int_{t_0}^{+\infty} c_{jj}(\tau)d\tau = +\infty.
\]
Then the system
\[
\begin{cases}
\Phi' = B(t)\Psi, \\
\Psi' = C(t)\Phi,
\end{cases}
\]  
(2.4)

is oscillatory.

**Remark 2.4.** Corollary 2.1 is a generalization of Leighton’s oscillation criterion (see [15, Theorem 2.24]).

**Theorem 2.4.** Let the following conditions be satisfied:

3') \(b_m(t) \geq 0, \ m = 1, n, \ t \in [a, b];\)

4') for some \(j \in \{1, \ldots, n\}\) the function \(\chi_j(t)\) is continuous on \([a, b]\) and the scalar system
\[
\begin{cases}
\phi' = 2\Re a_{jj}^0(t)\phi + b_j(t)\psi, \\
\psi' = \chi_j(t)\phi, \quad t \in [a, b]
\end{cases}
\]
is oscillatory on \([a, b].\)

Then the system (2.4) is also oscillatory on \([a, b].\)

\[\square\]

**Corollary 2.2.** Let the following conditions be satisfied

5') \(B(t) = \text{diag}\{b_1(t), \ldots, b_n(t)\}, \ b_m(t) \geq 0, \ m = 1, n, \ t \in [a, b],\)

6') for some \(j \in \{1, \ldots, n\}\)
\[
\int_a^b \min[b_j(t), -c_{jj}(t)]dt \geq \pi.
\]

Then the system (2.4) is oscillatory on the interval \([a, b].\)

3. **Proof of the main results.** Let \(f_k(t), g_k(t)\) and \(h_k(t), \ k = 1, 2\) be real-valued continuous functions on \([t_0, +\infty).\) Consider the scalar Riccati equations
\[
y' + f_k(t)y^2 + g_k(t)y + h_k(t) = 0, \ t \geq t_0, \ k = 1, 2 
\]  
(3.1k)

and the differential inequalities
\[
y' + f_k(t)y^2 + g_k(t)y + h_k(t) = 0, \ t \geq t_0, \ k = 1, 2. 
\]  
(3.2k)

**Remark 3.1.** Every solution of Eq. (3.1k) on \([t_1, t_2)\) \((t_0 \leq t_1 < t_2 \leq +\infty)\) is also a solution of the inequality (3.2k), \( k = 1, 2.\)
Remark 3.2. If \( f_k(t) \geq 0, \ t \geq t_0, \) then every solution of the linear equation
\[
\zeta' + g_k(t)\zeta + h_k(t) = 0, \quad t \geq t_0
\]
is also a solution of the inequality (3.2\(_k\)), \( k = 1, 2.\)

The following comparison theorem plays a crucial role in the proof of the main results.

**Theorem 3.1 [16, Theorem 3.1].** Let Eq. (3.1\(_2\)) have a real valued solution \( y_2(t) \) on \([t_0, \tau_0)\) \((t_0 < \tau_0 \leq +\infty)\) and let the following conditions be satisfied: \( f_1(t) \geq 0 \) and \( \int_{t_0}^{\infty} \exp \left\{ \int_{t_0}^{t} [f(s)(\eta_1(s) + \eta_2(s)) + g(s)] ds \right\} \left\{ (f_1(\tau) - f(\tau))y_2^2(\tau) + (g_1(\tau) - g(\tau))y_2(\tau) + h_1(\tau) - h(\tau) \right\} d\tau \geq 0, \ t \in [t_0, \tau_0) \) where \( \eta_1(t) \) and \( \eta_2(t) \) are solutions of the inequalities (3.2\(_1\)) and (3.2\(_2\)) respectively on \([t_0, \tau_0)\) such that \( \eta_j(t_0) \geq y_2(t_0), \ j = 1, 2.\) Then for every \( \gamma_0 \geq y_2(t_0) \) Eq. (3.1\(_1\)) has a solution \( y_1(t) \) on \([t_0, \tau_0), \) satisfying the condition \( y_1(t_0) = \gamma_0.\)

**Remark 3.3.** One can easily verify, that in the case \( \tau_0 < +\infty \) Theorem 3.1 remains valid if we replace \([t_0, \tau_0)\) by \([t_0, \tau_0]\) in it.

Set \( E(t) \equiv p_{11}(t) - p_{22}(t), \ t \geq t_0. \)

**Theorem 3.2 [14, Theorem 2.4].** Let the following conditions be satisfied:
\[
p_{12}(t) \geq 0, \ t \geq t_0; \int_{t_0}^{\infty} p_{12}(t) \exp \left\{ - \int_{t_0}^{t} E(\tau) d\tau \right\} = - \int_{t_0}^{\infty} p_{21}(t) \exp \left\{ \int_{t_0}^{t} E(\tau) d\tau \right\} dt = +\infty.
\]
Then the system (2.1) is oscillatory. \( \square \)

**Theorem 3.3 [14, Theorem 2.3].** Let the following conditions be satisfied:
\[
p_{12}(t) \geq 0, \ t \in [a; b]; \int_{a}^{b} \min \left[ p_{12}(t) \exp \left\{ - \int_{a}^{t} E(\tau) d\tau \right\}, -p_{21}(t) \exp \left\{ \int_{a}^{t} E(\tau) d\tau \right\} \right] dt \geq \pi.
\]
Then the system (2.1) is oscillatory on \([a; b).\) \( \square \)

Consider the scalar Riccati equation
\[
y' + p_{12}(t)y^2 + (p_{11}(t) - p_{22}(t))y - p_{21}(t) = 0, \quad t \geq t_0.
\]
The solutions \( y(t) \) of this equation, existing on some interval \([t_1, t_2)\) \((t_0 \leq t_1 < t_2 \leq +\infty)\) are connected with solutions \((\phi(t), \psi(t))\) of the system (2.1) by relations (see [16])
\[
\phi(t) = \phi(t_1) \exp \left\{ \int_{t_1}^{t} [p_{12}(\tau)y(\tau) + a_{11}(\tau)] d\tau \right\}, \ \phi_1(t_1) \neq 0, \ \psi(t) = y(t)\phi(t), \quad (3.3)
\]
\( t \in [t_1, t_2). \)
Let \( p(t) \) and \( q(t) \) be real-valued locally integrable functions on \([t_0, +\infty)\). Consider the second order linear ordinary differential equation

\[
\phi'' + p(t)\phi' + q(t)\phi = 0, \quad t \geq t_0
\]  

(3.4)

and the corresponding scalar Riccati one

\[
y' + y^2 + p(t)y + q(t) = 0, \quad t \geq t_0.
\]

(3.5)

Since Eq. (3.4) is equivalent to the system

\[
\begin{align*}
\phi' &= \psi, \\
\psi' &= -q(t)\phi - p(t)\psi, \quad t \geq t_0
\end{align*}
\]

by (3.3) we have that the solutions \( y(t) \) of Eq. (3.5), existing on an interval \([t_1, t_2]\), are connected with solutions \( \phi(t) \) of Eq. (3.4) by relations

\[
\phi(t) = \phi(t_1) \exp \left\{ \int_{t_1}^{t} [y(\tau) + p(\tau)] \, d\tau \right\}, \quad \phi(t_1) \neq 0, \quad t \in [t_1, t_2).
\]

(3.6)

Consider the matrix Riccati equation

\[
Z' + ZB(t)Z + A^*(t)Z +ZA(t) - C(t) = 0, \quad t \geq t_0.
\]

(3.7)

It is not difficult to verify that the solutions \( Z(t) \) of this equation, existing on an interval \([t_1, t_2) \quad (t_0 \leq t_1 < t_2 \leq +\infty)\) are connected with solutions \( \Phi(t), \Psi(t) \) of the system (1.1) by the relations

\[
\Phi'(t) = [A(t) + B(t)Z(t)]\Phi(t), \quad \Phi(t_1) \neq 0, \quad \Psi(t) = Z(t)\Phi(t), \quad t \in [t_1, t_2).
\]

(3.8)

### 3.1. Proof of Theorem 2.1.

Suppose the system (1.1) is not oscillatory. Then there exists a conjoined solution \((\Phi(t), \Psi(t))\) of that system such that \( \det \Phi(t) \neq 0, \quad t \geq t_1 \)

for some \( t_1 \geq t_0 \). By (3.8)(3.9) from here it follows that \( Z(t) \equiv \Psi(t)\Phi^{-1}(t), \quad t \geq t_1 \) is a Hermitian solution of Eq. (3.7)(3.8) on \([t_1, +\infty), \) i. e., \( Z^*(t) = Z(t) \) and

\[
Z'(t) + Z(t)B(t)Z(t) + A^*(t)Z(t) + Z(t)A(t) - C(t) = 0, \quad t \geq t_1.
\]

Multiply both sides of this equality at left and at right by \( \sqrt{B(t)}, \quad t \geq t_1 \). Taking into account the equality

\[
\sqrt{B(t)}Z'(t)\sqrt{B(t)} = [\sqrt{B(t)}Z(t)\sqrt{B(t)}]' - \sqrt{B(t)}Z(t)\sqrt{B(t)} - \sqrt{B(t)}Z(t)\sqrt{B(t)}, \quad t \geq t_1
\]
and the condition 1) of the theorem we obtain

\[ V'(t) + V^2(t) + A_F^*(t)V(t) + V(t)A_F(t) - C_B(t) = 0, \quad t \geq t_1, \quad (3.9) \]

where \( V(t) \equiv \sqrt{B(t)}Z(t)\sqrt{B(t)}, \quad t \geq t_1. \) Denote by \([M]_{jk}\) the \(jk\)-th entry of any square matrix \(M\) \((j, k = 1, n)\). Set: \([V(t)]_{jk} \equiv v_{jk}(t), \quad t \geq t_1, \quad k = 1, n. \) Since \( V(t) \) is a Hermitian matrix function on \([t_1, +\infty)\) it is not difficult to verify that

\[ [V^2(t)]_{11} = v_{11}^2(t) + |v_{12}(t)|^2 + ... + |v_{1n}(t)|^2, \]

\[ [V^2(t)]_{22} = |v_{21}(t)|^2 + v_{22}^2(t) + ... + |v_{2n}(t)|^2, \]

\[ ... \]

\[ [V^2(t)]_{nn} = |v_{n1}(t)|^2 + |v_{n2}(t)|^2 + ... + v_{nn}^2(t) \]

\[ [V(t)A_F(t)]_{jj} = \sum_{m=1}^{n} v_{jm}(t)a_{F_{mj}}, \quad [A_F^*(t)V(t)]_{jj} = \sum_{m=1}^{n} \overline{v_{jm}(t)} a_{F_{mj}}, \quad t \geq t_1. \]

From here and from the equalities \( v_{jm}(t) = \overline{v_{mj}(t)}, \quad m = 1, n, \quad t \geq t_1 \) we obtain

\[ v_{jj}'(t) + v_{jj}^2(t) + 2\Re a_{F_{jj}}(t)v_{jj}(t) + \sum_{\substack{m=1 \\text{m} \neq j}}^{n} |v_{jm}(t) + \overline{a_{F_{mj}}(t)}|^2 - \theta_{F_j}(t) = 0, \quad t \geq t_1. \quad (3.10) \]

Consider the scalar Riccati equations

\[ y' + y^2 + 2\Re a_{F_{jj}}(t)y - \theta_{F_j}(t) = 0, \quad t \geq t_1, \quad (3.11) \]

\[ y' + y^2 + 2\Re a_{F_{jj}}(t)y - \theta_{F_j}(t) + \sum_{\substack{m=1 \\text{m} \neq j}}^{n} |v_{jm}(t) + \overline{a_{F_{mj}}(t)}|^2 = 0, \quad t \geq t_1. \quad (3.12) \]

By (3.10) \( v_{jj}(t) \) is a solution to the last equation on \([t_1, +\infty)\). Since

\[ \sum_{\substack{m=1 \\text{m} \neq j}}^{n} |v_{jm}(t) + \overline{a_{F_{mj}}(t)}|^2 \geq 0, \quad t \geq t_1, \]

using Theorem 3.1 to the pair of the equations (3.11) and (3.12) we conclude that Eq. (3.11) has a solution \( y_1(t) \) on \([t_1, +\infty)\). Then by (3.6)

\[ \phi_1(t) \equiv \exp \left\{ \int_{t_1}^{t} \left[ y_1(\tau) + \Re a_{F_{jj}}(t)y_1(\tau) \right] d\tau \right\}, \quad t \geq t_1 \]

is a solution of Eq. (2.1) on \([t_1, +\infty)\), which can be continued on \([t_0, +\infty)\) as a solution of Eq. (2.1). Since \( \phi_1(t) > 0, \quad t \geq t_1 \) Eq.
Remark 3.4. Theorem 2.2 can be proved by analogy of the proof of Theorem 2.1 by taking into account Remark 3.3.

3.2. Proof of Theorem 2.3. Suppose the system (1.1) is not oscillatory. Then there exists a conjoined solution \((\Phi(t), \Psi(t))\) of (1.1) such that \(\text{det} \Phi(t) \neq 0, \ t \geq t_1\) for some \(t_1 \geq t_0\). By virtue of (3.5) from here it follows that \(Z(t) \equiv \Psi(t)\Phi^{-1}(t), \ t \geq t_1\) is a Hermitian solution of Eq. (3.7) on \([t_1, +\infty)\), that is \(Z^*(t) = Z(t)\), \(t \geq t_1\) and

\[
Z'(t) + Z(t)B(t)Z(t) + A^*(t)Z(t) + Z(t)A(t) - C(t) = 0, \quad t \geq t_1
\]

Multiply both sides of the last equality at left by \(U_B^*(t)\) and at right by \(U_B(t)\). Taking into account (2.2) and the equality

\[
U_B(t)Z'(t)U_B^*(t) = [U_B(t)Z(t)U_B^*(t)]' - U_B'(t)Z(t)U_B^*(t) - U_B(t)Z'(t)[U_B^*(t)]', \quad t \geq t_1,
\]

we obtain

\[
V'(t) + V(t)B_0(t)V(t) + [A_B^0(t)]^*V(t) + V(t)A_B^0(t) - C_B^0(t) = 0, \quad t \geq t_1, \quad (3.13)
\]

where \(V(t) \equiv U_B(t)Z(t)U_B^*(t), \ t \geq t_1\). Let \(V(t) \equiv (v_{jk}(t))_{j,k=1}^n, \ t \geq t_1\). Since \(V(t)\) is a Hermitian matrix function it is not difficult to verify that

\[
[V(t)B_0(t)V(t)]_{11} = b_1(t)v_{11}^2(t) + b_2(t)|v_{12}(t)|^2 + \ldots + b_n(t)|v_{1n}(t)|^2,
\]

\[
[V(t)B_0(t)V(t)]_{22} = b_1(t)|v_{21}(t)|^2 + b_2(t)v_{22}^2(t) + \ldots + b_n(t)|v_{2n}(t)|^2,
\]

\[
[V(t)B_0(t)V(t)]_{nn} = b_1(t)|v_{n1}(t)|^2 + b_2(t)|v_{n2}(t)|^2 + \ldots + b_n(t)v_{nn}^2(t),
\]

\[
[V(t)A_B^0(t)]_{jj} = \sum_{m=1}^n v_{jm}(t)a_{mj}(t), \quad [(A_B^0(t))^*V(t)]_{jj} = \sum_{m=1}^n \overline{v_{jm}(t)}\overline{a_{mj}(t)}, \quad t \geq t_1.
\]

Taking into account the equalities \(v_{jm}(t) = v_{mj}(t)\), \(m = 1, n, \ t \geq t_1\) from here we obtain

\[
v_{jj}'(t) + b_j(t)v_{jj}^2(t) + 2\text{Re}\ a_{jj}(t)v_{jj}(t) + \sum_{m=1}^n b_m(t)v_{jm}(t) + \frac{a_{mj}(t)}{b_m(t)}t_0^2 - \chi_j(t) = 0, \quad (3.14)
\]
Consider the scalar Riccati equations

\begin{equation}
 y' + b_j(t)y^2 + 2\Re a_{jj}^0(t)y - \chi_j(t) = 0, \quad t \geq t_1, \quad (3.15)
\end{equation}

\begin{equation}
 y' + b_j(t)y^2 + 2\Re a_{jj}^0(t)y + \sum_{m=1}^{n} b_m(t) \left| v_{jm}(t) + \frac{a_{mj}^0(t)}{b_m(t)} \right|_0^2 - \chi_j(t) = 0, \quad t \geq t_1. \quad (3.16)
\end{equation}

By (3.14) $v_{jj}(t)$ is a solution to the last equation on $[t_1, +\infty)$. From the condition 4) of the theorem it follows that $\sum_{m=1, m \neq j}^{n} b_m(t) \left| v_{jm}(t) + \frac{a_{mj}^0(t)}{b_m(t)} \right|_0^2 \geq 0, \quad t \geq t_1$. Then using Theorem 3.1 to the pair of equations (3.15) and (3.16) we conclude that Eq. (3.15) has a solution $y(t)$ on $[t_1, +\infty)$. Hence in virtue of (3.1) the functions

\begin{align*}
 \phi(t) &\equiv \exp \left\{ \int_{t_1}^{t} [b_j(\tau)y(\tau) + 2\Re a_{jj}^0(\tau)]d\tau \right\}, \quad \psi(t) \equiv y(t)\phi(t), \quad t \geq t_1
\end{align*}

form a solution $(\phi(t), \psi(t))$ of the system (2.3) on $[t_1, +\infty)$, which can be continued on $[t_0, +\infty)$ as a solution of the system (2.3). Since, obviously, $\phi(t) > 0, \quad t \geq t_1$ the system (2.3) is not oscillatory, which contradicts the condition 4) of the theorem. The obtained contradiction completes the proof of the theorem.

**Remark 3.5.** Theorem 2.4 can be proved by analogy of the proof of Theorem 2.3 by taking into account Remark 3.3.

**3.3. Proof of Corollary 2.1.** Since according to the condition 5) $B(t)$ is a diagonal matrix, we can take the unitary transformation $U_B(t) \equiv I$. Then for the system (2.4) we will have $a_{jj}^0(t) \equiv 0, \quad \chi_j(t) = c_{jj}(t), \quad t \geq t_0$. Then by Theorem 2.3 from the condition 5) it follows that the system (2.4) is oscillatory provided the scalar system

\begin{align*}
 \phi' &= b_j(t)\psi, \\
 \psi' &= c_{jj}(t), \quad t \geq t_0
\end{align*}
is oscillatory. By Theorem 3.2 this condition holds provided the condition 6) is satisfied. The corollary is proved.

Corollary 2.2 can be proved by analogy of the proof of Corollary 2.1 using Theorem 3.3 instead of Theorem 3.2.

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