SU(2) Dirac-Yang-Mills quantum mechanics of spatially constant quark and gluon fields

H.-P. Pavel
Institut für Kernphysik, TU Darmstadt, Schloßgartenstr. 9, D-64289 Darmstadt, Germany∗
and
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia

Mai 3, 2011

Abstract

The quantum mechanics of spatially constant SU(2) Yang-Mills- and Dirac-fields minimally coupled to each other is investigated as the strong coupling limit of 2-color-QCD. Using a canonical transformation of the quark and gluon fields, which Abelianises the Gauss law constraints to be implemented, the corresponding unconstrained Hamiltonian and total angular momentum are derived. In the same way as this reduces the colored spin-1 gluons to unconstrained colorless spin-0 and spin-2 gluons, it reduces the colored spin-1/2 quarks to unconstrained colorless spin-0 and spin-1 quarks. These however continue to satisfy anti-commutation relations and hence the Pauli-exclusion principle. The obtained unconstrained Hamiltonian is then rewritten into a form, which separates the rotational from the scalar degrees of freedom. In this form the low-energy spectrum can be obtained with high accuracy. As an illustrative example, the spin-0 energy-spectrum of the quark-gluon system is calculated for massless quarks of one flavor. It is found, that only for the case of 4 reduced quarks (half-filling) satisfying the boundary condition of particle-antiparticle C-symmetry, states with energy lower than for the pure-gluon case are obtained. These are the ground state, with an energy about 20% lower than for the pure-gluon case and the formation of a quark condensate, and the sigma-antisigma excitation with an energy about a fifth of that of the first glueball excitation.

Keywords: Yang-Mills theory, fermions, low-energy QCD, gauge invariance, strong coupling, glueball spectrum

PACS numbers: 11.10.Ef, 11.15.Me, 11.15.Tk, 12.38.Aw, 02.20.Tw, 03.65.-w

1 Introduction

For a complete description of the physical properties of low-energy QCD, such as color confinement, chiral symmetry breaking, the formation of condensates and flux-tubes, and the spectra and strong interactions of hadrons, it might be advantageous if one could first reformulate QCD in terms of gauge invariant dynamical variables, before applying any approximation schemes (see e.g.[1]). Using a canonical transformation of the dynamical variables, which Abelianises the Non-Abelian Gauss-law constraints, such a reformulation has been achieved for pure SU(2) Yang-Mills theory on the classical [2, 3, 4] and on the quantum level [5]. The resulting unconstrained SU(2) Yang-Mills Hamiltonian admits a systematic strong-coupling expansion in powers of \( \lambda = g^{-2/3} \), equivalent to an expansion in the number of spatial derivatives. The leading order term in this expansion constitutes the unconstrained Hamiltonian of SU(2) Yang-Mills quantum mechanics of spatially constant gluon fields [6]-[12], for which the low-energy spectra can be calculated with high accuracy. Subject of the present work is its generalisation to the case of SU(2) Dirac-Yang-Mills quantum mechanics of quark and gluon fields\(^1\). First steps in this direction on the classical level have been done in [2].

∗email: hans-peter.pavel@physik.tu-darmstadt.de

\(^1\)This is different from the works [13, 14] where the effect of dynamical quarks, satisfying anti-periodic boundary conditions, on the spatially constant gluon fields in a small volume has been investigated. Here we study spatially constant quark and gluon fields in large volumes.
Hamiltonian obtained from (1) via Legendre-transformation reads
\[ L = \frac{1}{2} \left( A_{ai} - g \epsilon_{abc} A_{b0} A_{ci} \right)^2 - \frac{1}{2} B_{ai}^2 (A) + \frac{i}{2} \left( \psi^*_{\alpha\tau} \psi_{\alpha\tau} - \psi^*_{\alpha\tau} \psi_{\alpha\tau} \right) - g A_{a0} \rho_a (\psi) + g A_{ai} j_{ia} (\psi) - m \psi^* \psi , \]
with the magnetic field \( B_{ai} (A) = (1/2) g \epsilon_{abc} \epsilon_{ijk} A_{bj} A_{ck} \), the Lorentz scalar \( \overline{\psi} \psi = \psi^*_{\alpha\tau} \beta_{rs} \psi_{\alpha s} \), and the color-densities and currents (using the Pauli matrices \( \tau_a \) \( (a = 1, 2, 3) \))
\[ \rho_a (\psi) := \frac{1}{2} \psi^*_{\alpha\tau} (\tau_a)_{\alpha\beta} \psi_{\beta \tau} , \quad j_{ia} (\psi) := \frac{1}{2} \psi^*_{\alpha\tau} (\tau_a)_{\alpha\beta} (\omega_s)_{rs} \psi_{\beta s} . \]

For simplicity we limit ourselves here to one quark flavor. The local \( SU(2) \) gauge invariance of the original 2-color-QCD action reduces to the symmetry under the \( SU(2) \) transformations \( U (\omega (t)) \) of the fermion fields, local in time,
\[ \psi^\omega_{\alpha\tau} (t) = U (\omega (t))_{\alpha\beta} \psi_{\beta \tau} (t) , \]
and the corresponding \( SO(3) \) transformations \( O (\omega (t))_{ab} = \frac{1}{2} \text{Tr} \left( U^{-1} (\omega (t)) \tau_a U (\omega (t)) \tau_b \right) \) of the gauge fields,
\[ A^\omega_{a0} (t) = O (\omega (t))_{a0} A_{b0} (t) - \frac{1}{2 g} \epsilon_{abc} \left( O (\omega (t)) \hat{O} (\omega (t)) \right)_{bc} , \]
\[ A^\omega_{ai} (t) = O (\omega (t))_{ab} A_{bi} (t) . \]

The rotational invariance of the original Yang-Mills action reduces to the global spatial rotations
\[ A^X_i = A_{ai} R (\chi)_{ji} , \quad \psi^X_{\alpha\tau} = \Lambda (\chi)_{rs} \psi_{\alpha s} , \]
where the \( 4 \times 4 \) Dirac rotation matrix \( A (\chi) \) is related to \( R (\chi) \) via \( R (\chi)_{ij} = \frac{1}{2} \text{Tr} \left( \Lambda^{-1} (\chi) \gamma_i \Lambda (\chi) \gamma_j \right) \). The canonical Hamiltonian obtained from (1) via Legendre-transformation reads
\[ H_C = \frac{1}{2} \Pi_{ai} \Pi_{ai} + \frac{1}{2} B_{ai}^2 (A) + g A_{a0} (\epsilon_{abc} \Pi_{bi} \Pi_{ci} + \rho_a (\psi)) - g A_{ai} j_{ia} (\psi) + m \overline{\psi} \psi , \]
where \( \Pi_{ai} \) are the momenta canonical conjugate to the spatial components \( A_{ai} \).

In the constrained Hamiltonian formulation (see e.g. [1]) the time dependence of the gauge transformations (4) is exploited to put the Weyl gauge
\[ A_{a0} = 0 , \quad a = 1, 2, 3 , \]
on the remaining dynamical degrees of freedom \( A_{ai} , \Pi_{ai} , \psi_{\alpha\tau} \) and \( \psi^*_{\alpha\tau} \) are quantized in the Schrödinger functional approach by imposing the equal time commutation relations
\[ \Pi_{ai} \rightarrow -i \partial / \partial A_{ai} : \quad [\Pi_{ai} , A_{bj}] = -i \delta_{ab} \delta_{ij} , \]
and anti-commutation relations
\[ \psi^*_{\alpha\tau} \rightarrow \psi^\dagger_{\alpha\tau} : \quad \{ \psi_{\alpha\tau} , \psi^\dagger_{\beta s} \} = \delta_{\alpha\beta} \delta_{rs} , \quad \{ \psi_{\alpha\tau} , \psi_{\beta s} \} = 0 . \]
where the quark and gluon field operators commute
\[ [A_{ai} , \psi_{\alpha\tau}] = 0 , \quad [\Pi_{ai} , \psi_{\alpha\tau}] = 0 . \]
The physical states \( \Phi \) have to satisfy both the Schrödinger equation and the three Gauss law constraints
\[ H \Phi = \left[ \frac{1}{2} \left( \frac{\partial}{\partial A_{ai}} \right)^2 + \frac{1}{2} B_{ai}^2 (A) - A_{ai} j_{ia} (\psi) + m \overline{\psi} \psi \right] \Phi = E \Phi , \]
\[ G_a \Phi = \left[ -i \epsilon_{abc} A_{bij} \frac{\partial}{\partial A_{ci}} + \rho_a (\psi) \right] \Phi = 0 , \quad a = 1, 2, 3 . \]

Footnote: Everywhere in the paper we put the spatial volume \( V = 1 \). As result the coupling constant \( g \) becomes dimensionful with \( g^{2/3} \) having the dimension of energy. The volume dependence can be restored in the final results by replacing \( g^2 \) with \( g^{2/3} V \).
The $G_a$ are the generators of the residual time independent gauge transformations, satisfying $[G_a, H] = 0$ and $[G_a, G_b] = \imath \epsilon_{abc} G_c$. Furthermore $H$ commutes with the angular momentum operators

$$ J_i = -\imath \epsilon_{ijk} A_{aj} \frac{\partial}{\partial A_{ak}} + \Sigma_i(\psi) , \quad i = 1, 2, 3 , $$

with the quark-spin

$$ \Sigma_i(\psi) := \frac{i}{8} \epsilon_{ijk} \psi_{\alpha \beta}^{\dagger} [\gamma_j, \gamma_k]_{\alpha \beta} \psi_{\alpha \beta} . $$

The matrix element of an operator $O$ is given in the Cartesian form

$$ \langle \Phi | O | \Phi \rangle \propto \int dA \overline{d\psi} d\psi \, \Phi^*(A, \overline{\psi}, \psi) O \Phi(A, \overline{\psi}, \psi) . $$

For carrying out quantum mechanical calculations it is desirable to have a corresponding unconstrained Schrödinger equation and to find its eigenstates in an effective way with high accuracy at least for the lowest states.

## 2 Unconstrained Dirac-Yang-Mills Hamiltonian

The local symmetry transformation (4) of the gauge potentials $A_{ai}$ prompts us with the set of coordinates in terms of which the separation of the gauge degrees of freedom occurs. This can be achieved [2] using the polar decomposition for arbitrary $3 \times 3$ quadratic matrices, and the new fermionic variables $\psi'_s$

$$ A_{ai}(q, S) = O_{ak}(q) S_{ki} , $$

$$ \psi'_\alpha(q, \psi') = U_{\alpha \beta}(q) \psi'_\beta $$

with the orthogonal matrix $O(q)$, parametrized by the three angles $q_i$, which is the adjoint representation of the unitary $2 \times 2$ matrix $U(q)$

$$ O_{ab}(q) = \frac{1}{2} \text{Tr} \left( U^{-1}(q) \tau_a U(q) \tau_b \right) . $$

and the positive definite, symmetric $3 \times 3$ matrix $S$. The decomposition (16) is unique and corresponds to the symmetric gauge [3, 4]

$$ \chi_i(A) = \epsilon_{ijk} A_{jk} = 0 . $$

Preserving the canonical commutators (8) and (10) one obtains the expressions for the old canonical momenta in terms of the new variables

$$ -i \frac{\partial}{\partial A_{ai}} = O_{ak} \left[ -i \frac{\partial}{\partial S_{ki}} + \epsilon_{kil} \gamma_{l}^{-1}(S) \left( -i \Omega_{sj}(q) \frac{\partial}{\partial q_j} + \rho_{s}(\psi') - i \epsilon_{smn} S_{bm} \frac{\partial}{\partial S_{mn}} \right) \right] $$

where $\Omega_{jm}(q) \equiv (1/2) \epsilon_{mkl} \left[ O^T(q) \partial O(q) / \partial q_j \right]_{kl}$. The Jacobian is $| \partial(A_{ai}) / \partial(q, S) | \propto \det \Omega(q) \prod_{i<j} (\phi_i + \phi_j)$, where $\phi_1, \phi_2, \phi_3$ are the eigenvalues of $S$. The variables $S$ and $\partial / \partial S$ make no contribution to the Gauss law operators

$$ G_a = -i O_{as}(q) \Omega_{sj}(q) \frac{\partial}{\partial q_j} . $$

Hence, assuming the invertibility of the matrix $\Omega$, the non-Abelian Gauss laws (12) can be replaced by an equivalent set of Abelian constraints

$$ G_a \Phi = 0 , \quad a = 1, 2, 3 \quad \Leftrightarrow \quad \frac{\partial \Phi}{\partial q_i} = 0 , \quad i = 1, 2, 3 . \quad (\text{Abelianisation}) $$

and the unconstrained Hamiltonian and total angular momentum of $SU(2)$ Dirac-Yang-Mills quantum mechanics read

$$ H = \frac{1}{2} \sum_{m,n} \left[ - \left( \frac{\partial}{\partial S_{mn}} \right)^2 - \left( \gamma_{mn}^{-1}(S) - \delta_{mn} \text{tr} (\gamma^{-1}(S)) \right) \frac{\partial}{\partial S_{mn}} + B_{mn}^2(S) \right] $$

$$ -g \psi'^\dagger \alpha_m S_{mn} \gamma_n \psi' + \frac{1}{2} \left( J_m - \imath K_l^Q \right) \gamma_{mn}^{-2}(S) \left( J_n - \imath K_l^Q \right) \right] + \frac{1}{2} m \overline{\psi'} \psi' . $$

$$ J_i = -2\imath \epsilon_{ijk} A_{aj} \frac{\partial}{\partial S_{ak}} + J_i^Q , \quad [J_i, H] = 0 , $$
in terms of the reduced variables, where \( \gamma_{ik}(S) := S_{ik} - \delta_{ik}\text{tr}S \), and \( J^Q_i \) and \( K^Q_i \) are the quark operators

\[
J^Q_i := \Sigma_i(\psi') + \rho_i(\psi') , \quad K^Q_i := -i\left(\Sigma_i(\psi') - \rho_i(\psi')\right) , \quad i = 1, 2, 3 ,
\]

satisfying

\[
[j^Q_i , j^Q_j] = i\epsilon_{ijk}j^Q_k , \quad [j^Q_i , k^Q_j] = i\epsilon_{ijk}k^Q_k , \quad [k^Q_i , k^Q_j] = -i\epsilon_{ijk}j^Q_k .
\]

The matrix element of a physical operator \( O \) is given by

\[
\langle \Phi'|O|\Phi \rangle \propto \int dS \ d\varphi d\psi' \left[ \prod_{i<j} (\phi_i + \phi_j) \right] \Phi^*(S, \varphi', \psi') O \Phi(S, \varphi', \psi') .
\]

The classical analog of the unconstrained Hamiltonian (23) in some equivalent form has already been obtained in [2], the expression (24) of the unconstrained total angular momentum, however, is new. Note that the extra factor of 2 in the gluonic part and the additional term \( \rho_i(\psi') \) in the quark part \( J^Q_i \) of the physical total spin (24), in comparison with the constrained form (13), originate from the anti-symmetric part of the momenta (20) and have important consequences.

The variables \( S \) transform under spatial rotations as symmetric tensor field,

\[
[j_i , S_{mn}] = i \left( \epsilon_{imj}S_{jn} + \epsilon_{inj}S_{mj} \right) , \quad i = 1, 2, 3 ,
\]

which can be decomposed into spin-0 and spin-2 components (using Clebsch-Gordan coefficients)

\[
S_{ik} = C^A_{11ik} S_A^{(2)} + \frac{1}{\sqrt{3}} \delta_{ik} S^{(0)}
\]

In the Weyl-representation, the quark-spin operators \( J^Q_i \) can be written

\[
J^Q_i =: \psi_L^t \tilde{S}^{(4)}_i \psi_L^t + \psi_R^t \tilde{S}^{(4)}_i \psi_L^t , \quad \tilde{S}^{(4)}_i = \frac{1}{2} \left( \tau_i \otimes 1_2 + 1_2 \otimes \tau_i \right) , \quad [\tilde{S}^{(4)}_i , \tilde{S}^{(4)}_j] = i\epsilon_{ijk} \tilde{S}^{(4)}_k ,
\]

with the three \( 4 \times 4 \) matrices \( \tilde{S}^{(4)}_i \) generate SO(3) spatial rotations and read explicitly

\[
\tilde{S}^{(4)}_1 = \frac{1}{2} \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix} , \quad \tilde{S}^{(4)}_2 = \frac{1}{2} \begin{pmatrix}
0 & -i & -i & 0 \\
i & 0 & 0 & -i \\
i & 0 & 0 & -i \\
0 & i & i & 0
\end{pmatrix} , \quad \tilde{S}^{(4)}_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} ,
\]

It follows that the combinations

\[
\psi^{(0)}_L := \frac{1}{\sqrt{2}} \left( -\psi'_L + \psi'_L \right) , \quad \left( \psi^{(1)}_L , \psi^{(1)}_L , \psi^{(1)}_L \right) := \left( -\frac{1}{\sqrt{2}} (\psi'_L - \psi'_L) , -\frac{i}{\sqrt{2}} (\psi'_L + \psi'_L) , \frac{1}{\sqrt{2}} (\psi'_L + \psi'_L) \right),
\]

of the reduced quark fields \( \psi_L' \), and analogously for the right-handed components, transform as spin-0 and spin-1 fields

\[
[j_i , \psi^{(0)}_{L(R)}] \equiv [J^Q_i , \psi^{(0)}_{L(R)}] = 0 , \quad [j_i , \psi^{(1)}_{L(R)}] \equiv [J^Q_i , \psi^{(1)}_{L(R)}] = i\epsilon_{ijk} \psi^{(1)}_{L(R)k} , \quad i = 1, 2, 3 .
\]

Hence implementation of the Gauss law constraints reduces the original spin-1/2 quark fields \( \psi \) to the unconstrained quark fields \( \psi' \) carrying integer spin-0 and spin-1, just as the original spin-1 gluon fields \( A \) are reduced to the corresponding unconstrained spin-0 and spin-2 gluon fields \( S \). It is however to notice that due to the unitarity of transformation (17), the unconstrained quarks continue to satisfy anti-commutation relations and hence the Pauli exclusion principle\(^3\).

The anti-Hermitian quark operators \( K^Q_i \) can be written as

\[
K^Q_i := \psi_L^t \tilde{T}^{(4)}_i \psi_L + \psi_R^t \tilde{T}^{(4)}_i \psi_L , \quad \tilde{T}^{(4)}_i = -\frac{i}{2} (\tau_i \otimes 1_2 - 1_2 \otimes \tau_i) , \quad [\tilde{T}^{(4)}_i , \tilde{T}^{(4)}_j] = -i\epsilon_{ijk} \tilde{S}^{(4)}_k ,
\]

\(^3\)Note, that there is no spin-statistics theorem for non-local theories.
with the three $4 \times 4$ matrices $\tilde{T}_i^{(4)}$

$$
\begin{pmatrix}
0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1 \\
-1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & -1 & 0 \\
-1 & 0 & 0 & -1 \\
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

which generate boosts

$$
[K^Q_i, \psi_L^{(0)}j] = -i\delta_{ij}\psi_L^{(0)}j, \quad [K^Q_i, \psi_L^{(1)}j] = -i\delta_{ij}\psi_L^{(0)}j, \quad i = 1, 2, 3,
$$
on the left(right)-handed 4-vector $(\psi_L^{(0)}j, \psi_L^{(1)}j, \psi_R^{(0)}j, \psi_R^{(1)}j)$ of reduced quark fields. For example, we easily verify the Lorentz invariant

$$
[K^Q_i, \bar{\psi}\psi] = [K^Q_i, (\psi_L^{(0)}j\psi_L^{(1)}j + \psi_R^{(0)}j\psi_R^{(1)}j) + \sum_{i=1}^{3}(\psi_R^{(0)}j\psi_R^{(1)}j + \psi_R^{(0)}j\psi_R^{(1)}j)] = 0.
$$

3 Unconstrained Hamiltonian in terms of rotational and scalar degrees of freedom

3.1 Transformation to rotational and scalar degrees of freedom

A more transparent form for the unconstrained Dirac-Yang-Mills Hamiltonian (23), maximally separating the rotational from the rotation invariant degrees of freedom, can be obtained using the transformation properties (28) and (33) of the canonical fields $S$ and $\psi'$ under spatial rotations generated by the physical total spin (24). We limit ourselves in this work to the case of principle orbit configurations of non-coinciding eigenvalues $\phi_1, \phi_2, \phi_3 > 0$ of the positive definite symmetric matrix $S$, which without loss of generality can be taken as

$$
0 < \phi_1 < \phi_2 < \phi_3 < \infty,
$$

(not considering singular orbits where two or more eigenvalues coincide) and perform a principal-axes transformation

$$
S(\chi, \phi) = R(\chi) \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} R^T(\chi),
$$

with the $SO(3)$ matrix $R(\chi)$ parametrized by the three Euler angles $\chi \equiv (\alpha, \beta, \gamma)$,

$$
R(\chi) = \exp(-i\alpha\tilde{S}_z)\exp(-i\beta\tilde{S}_y)\exp(-i\gamma\tilde{S}_z),
$$

with the $SO(3)$ generators $(\tilde{S}_i)_{jk} = -i\epsilon_{ijk}$. On the fermion fields in Weyl representation, the transformation

$$
\psi'(x, \bar{u}_L, \bar{v}_R) = \begin{pmatrix} \psi_L(x, \bar{u}_L) \\ \psi_R(x, \bar{v}_R) \end{pmatrix} = \begin{pmatrix} \bar{U}(\chi)\bar{u}_L \\ \bar{U}(\chi)\bar{v}_R \end{pmatrix}
$$

is performed, with the $U(4)$ matrix ($\det \bar{U}(\chi) = -i$)

$$
\bar{U}(\chi) = \bar{R}(\chi)\bar{U}_0 = \begin{pmatrix} 0 & D_{11}^{(1)}(\chi) & D_{11}^{(1)}(\chi) & D_{10}^{(1)}(\chi) \\ -1/\sqrt{2} & D_{01}^{(1)}(\chi)/\sqrt{2} & D_{01}^{(1)}(\chi)/\sqrt{2} & D_{00}^{(1)}(\chi)/\sqrt{2} \\ 1/\sqrt{2} & D_{01}^{(1)}(\chi)/\sqrt{2} & D_{01}^{(1)}(\chi)/\sqrt{2} & D_{00}^{(1)}(\chi)/\sqrt{2} \\ 0 & D_{11}^{(1)}(\chi) & D_{11}^{(1)}(\chi) & D_{10}^{(1)}(\chi) \end{pmatrix},
$$

which is the product of the $SU(4)$ matrix of spatial rotations

$$
\bar{R}(\chi) = \exp(-i\alpha\tilde{S}_z^{(4)})\exp(-i\beta\tilde{S}_y^{(4)})\exp(-i\gamma\tilde{S}_z^{(4)}),
$$

5
with the $SO(3)$ generators $S_i^{(4)}$ defined in (31), and the special $U(4)$ matrix $(\det \tilde{U}(\chi) = -i)$

$$\tilde{U}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -i & 0 \end{pmatrix}. \quad (44)$$

Hence the unconstrained spin-0 and spin-2 gluon fields read (using Wigner D-functions)

$$S^{(0)} = (\phi_1 + \phi_2 + \phi_3) / \sqrt{3}, \quad S_A^{(2)} = \sqrt{\frac{2}{3}} \left( \phi_3 - \frac{1}{2} (\phi_1 + \phi_2) \right) D_{A0}(\chi) + \sqrt{\frac{3}{2}} (\phi_1 - \phi_2) D_{A2+}^{(2)}(\chi) \right), \quad (45)$$

in terms of the principal axes variables, and the unconstrained spin-0 and spin-1 quark fields

$$\psi^0_L = u^0_L, \quad \psi^1_L = R_{ij}(\chi) u^j_L, \quad \psi^0_R = \tilde{u}^0_R, \quad \psi^1_R = R_{ij}(\chi) \tilde{u}^j_R, \quad (46)$$

in terms of the intrinsic quark variables

$$\tilde{u}_L = (\tilde{u}^0_L, \tilde{u}^1_L, \tilde{u}^2_L, \tilde{u}^3_L), \quad \tilde{v}_R = (\tilde{v}^0_R, \tilde{v}^1_R, \tilde{v}^2_R, \tilde{v}^3_R), \quad (47)$$

satisfying anti-commutation relations

$$\{\tilde{u}_L^\mu, \tilde{u}_L^\nu\} = \delta^{\mu\nu}, \quad \{\tilde{v}_R^\mu, \tilde{v}_R^\nu\} = \delta^{\mu\nu}, \quad \{\tilde{u}_L^\mu, \tilde{v}_R^\nu\} = 0. \quad (48)$$

### 3.2 Unconstrained Hamiltonian in terms of scalar and rotational variables

The transformation (41)-(44) on the quark fields yields (see the Appendix for details)

$$J_i^Q = R_{ij}(\chi) \tilde{J}_j^Q, \quad K_i^Q = R_{ij}(\chi) \tilde{K}_j^Q, \quad (49)$$

with the operators

$$\tilde{J}_i^Q := -i \epsilon_{ijk} \left( \tilde{u}^{(j)\dagger} \tilde{u}^{(k)} + \tilde{v}^{(j)\dagger} \tilde{v}^{(k)} \right), \quad \tilde{K}_i^Q := i \left( \tilde{u}^{(0)\dagger} \tilde{u}^{(i)} + \tilde{u}^{(i)\dagger} \tilde{u}^{(0)} + \tilde{v}^{(0)\dagger} \tilde{v}^{(i)} + \tilde{v}^{(i)\dagger} \tilde{v}^{(0)} \right), \quad (50)$$

satisfying the same Lie-algebra (26) as the original $J_i^Q$ and $K_i^Q$. Furthermore, one finds (see App. for details) that the minimal coupling part of the Hamiltonian is diagonalised,

$$g \psi^a \alpha_i S_i^{(1)} 1 \over 2 \tau_j \psi^b = \frac{g}{2} (\phi_1 + \phi_2 + \phi_3) \left( \tilde{N}_L^{(0)} - \tilde{N}_R^{(0)} \right) + \frac{g}{2} \sum_{ij} \left( \phi_i - (\phi_j + \phi_k) \right) \left( \tilde{N}_L^{(i)} - \tilde{N}_R^{(i)} \right) \right), \quad (51)$$

with the quark-number operators

$$\tilde{N}_L^{(\mu)} = \tilde{u}_L^{(\mu)\dagger} \tilde{u}_L^{(\mu)}, \quad \tilde{N}_R^{(\mu)} = \tilde{v}_R^{(\mu)\dagger} \tilde{v}_R^{(\mu)}, \quad \mu = 0, 1, 2, 3. \quad (52)$$

The transformation $\tilde{U}$ in (42) therefore leads to a diagonalisation of the Dirac-Hamiltonian for zero-momentum quarks in the background of a zero-momentum symmetric tensor field $S$. The eigenvectors are the quark states $\tilde{u}$ and $\tilde{v}$ and the corresponding eigenvalues simple linear combinations of the eigenvalues of the reduced gluon field.

Finally, the momenta canonically conjugate to the $S^{(0)}$ and $S_A^{(2)}$ are

$$-i \frac{\partial}{\partial S^{(0)}} = -i \left( \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2} + \frac{\partial}{\partial \phi_3} \right) / \sqrt{3}, \quad (53)$$

$$-i \frac{\partial}{\partial S_A^{(2)}} = \sqrt{\frac{2}{3}} \left[ -i \left( \frac{\partial}{\partial \phi_3} - \frac{1}{2} \left( \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2} \right) \right) D_{A0}^{(2)}(\chi) - \sqrt{\frac{3}{2}} i \left( \frac{\partial}{\partial \phi_1} - \frac{\partial}{\partial \phi_2} \right) D_{A2+}^{(2)}(\chi) \right]$$

$$+ \frac{1}{\sqrt{2}} \left[ D_{A1+}^{(2)}(\chi) \frac{\xi_G^{(2)}}{\phi_2 - \phi_3} + D_{A1-}^{(2)}(\chi) \frac{\xi_G^{(2)}}{\phi_3 - \phi_1} + D_{A2-}^{(2)}(\chi) \frac{\xi_G^{(2)}}{\phi_1 - \phi_2} \right], \quad (54)$$


\[ \xi^G_i := \xi_i - J^Q_i, \quad i = 1, 2, 3, \quad [\xi^G_i, \xi^G_j] = -i\epsilon_{ijk}\xi^G_k, \] 

where

\[ \xi_i := -iM^{-1}_{ij}\partial_{\chi_i}, \quad M_{ij} := -\frac{1}{2}\epsilon_{jst}(R^T\partial_{\chi_j})_{st}, \quad [\xi_i, \xi_j] = -i\epsilon_{ijk}\xi_k. \] 

For the case of Euler angles \( \chi = (\alpha, \beta, \gamma) \) we have

\[ M^{-1} = \begin{pmatrix} \sin \gamma & -\cos \gamma/\sin \beta & \cos \gamma \cot \beta \\ \cos \gamma & \sin \gamma/\sin \beta & -\sin \gamma \cot \beta \\ 0 & 0 & 1 \end{pmatrix}. \]

Hence we obtain

\[ \frac{1}{2}\sum_{m,n} \left[ -\left( \frac{\partial}{\partial S_{mn}} \right)^2 - \left[ \gamma^{-1}_{mn}(S) - \delta_{mn}\text{tr}(\gamma^{-1}(S)) \right] \frac{\partial}{\partial S_{mn}} + B^2_{mn}(S) \right] = \frac{1}{2}\sum_{ijk} \left[ -\left( \frac{\partial}{\partial \phi_i} \right)^2 - \frac{2}{\phi_i^2 - \phi_j^2} (\phi_i \partial_{\phi_i} - \phi_j \partial_{\phi_j}) + \frac{1}{2} \left( \frac{\xi^G_i}{\phi_i - \phi_j} \right)^2 + g^2 \phi_i^2 \phi_j^2 \right] \]

and

\[ J^G_i := -2i\epsilon_{ijk}S_{aj}\frac{\partial}{\partial S_{ak}} = R_{ij}(\chi) \xi^G_i. \]

Altogether, after rescaling the fields \( \phi_i \rightarrow g^{-1/3}\phi_i \) \( i = 1, 2, 3 \), and then reinstalling \( g^2 \rightarrow g^2/V \), we obtain the unconstrained Hamiltonian of \( SU(2) \) Dirac-Yang-Mills quantum mechanics in the final form

\[ H = \frac{g^{2/3}}{V^{1/3}} \left[ \mathcal{H}^G + \mathcal{H}^D + \mathcal{H}^C \right] + H_m, \]

with

\[ \mathcal{H}^G := \frac{1}{2}\sum_{ijk} \left( -\frac{\partial^2}{\partial \phi_i^2} - \frac{2}{\phi_i^2 - \phi_j^2} (\phi_i \partial_{\phi_i} - \phi_j \partial_{\phi_j}) + (\xi_i - J^Q_i)^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + \phi_j^2 \phi_k^2 \right), \]

\[ \mathcal{H}^D := \frac{1}{2}(\phi_1 + \phi_2 + \phi_3)(\vec{N}^{(i)}_L - \vec{N}^{(i)}_R) + \frac{1}{2}\sum_{ijk} (\phi_i - (\phi_j + \phi_k))(\vec{N}^{(i)}_L - \vec{N}^{(i)}_R), \]

\[ \mathcal{H}^C := -\frac{1}{4}\sum_{ijk} (\xi_i - \vec{J}^Q_i)^2 - (\xi_i - i\vec{K}^Q_i)^2 \frac{1}{(\phi_i + \phi_k)^2}, \]

the quark-mass term

\[ H_m := -\frac{1}{2}m \left[ \sum_{i=1}^{3} \vec{u}^{(i)}_L \vec{v}^{(i)}_R + \sum_{i=1}^{3} \sum_{j=1}^{3} \vec{u}^{(i)}_L \vec{v}^{(i)}_R \right] + h.c. \]

and the total angular momentum

\[ J_i = R_{ij}(\chi) \left( \xi^G_j + \vec{J}_j^Q \right) = R_{ij}(\chi) \xi_j. \]

Since the Jacobian of (39) is \( |\partial S/\partial (\alpha, \beta, \gamma, \phi)| \propto \sin \beta \prod_{i<j} (\phi_i - \phi_j) \), the matrix elements of an operator \( O \) are given as

\[ \langle \Phi | O | \Phi \rangle \propto \int d\vec{u}_L d\vec{v}_L d\vec{u}_R d\vec{v}_R \int d\alpha \sin \beta \partial_\beta d\gamma \prod_{0<\phi_i<\phi_2<\phi_3}^{\text{cyclic}} \int d\phi_i \left( \phi_j^2 - \phi_k^2 \right) \Phi^* O \Phi. \]

The representation of the unconstrained Hamiltonian (59)-(63) and total angular momentum (64) of \( SU(2) \) Dirac-Yang-Mills quantum mechanics of spatially constant quark and gluon fields, is the main result of the present work and is new to the best of my knowledge. In order to keep the formulas simple, only the 1-flavor case has been shown in this work. The generalization of the Hamiltonian (59)-(63) to flavor-numbers larger than one is trivial: The operators \( \vec{N}^{(i)}_{L(R)}, \vec{J}^Q_i, \vec{K}^Q_i \) and \( H_m \) become sums over different flavors.
3.3 Symmetries of the Hamiltonian

Due to \([J_i, \xi_j] = 0\), the angular momenta \(J_i\) commute with the Hamiltonian, and the eigenstates of \(H\) can be characterized by the quantum numbers \(J\) and \(M\) of total spin of the quark-gluon system. Furthermore \(H\) is invariant under cyclic permutations \(\sigma_{123}\) of the three indices 1, 2, 3, parity transformations \(P\)

\[
P : \quad (\phi_i \rightarrow -\phi_i \quad (i = 1, 2, 3)) \quad \& \quad (\tilde{u}^{(\mu)}_L \leftrightarrow \tilde{v}^{(\mu)}_R) \quad (\mu = 0, 1, 2, 3) ,
\]

time inversion \(T\) (anti-unitary)

\[
T : \quad (\phi_i \rightarrow -\phi_i \quad (i = 1, 2, 3)) \quad \& \quad (|0\rangle \rightarrow |8\rangle) \quad \& \quad \tilde{u}^{(\mu)}_L \rightarrow \tilde{v}^{(\mu)}_R, \tilde{v}^{(\mu)}_R \rightarrow -\tilde{u}^{(\mu)}_L \quad (\mu = 0, 1, 2, 3) ,
\]

where \(|0\rangle\) and \(|8\rangle\) denote the energy degenerate and complex conjugate\(^4\) quark states with no and with all levels filled, and charge conjugation \(C\)

\[
C : \quad (|0\rangle \rightarrow |8\rangle) \quad \& \quad \tilde{u}^{(0)}_L \rightarrow \tilde{v}^{(0)}_R, \tilde{v}^{(0)}_R \rightarrow -\tilde{u}^{(0)}_L \quad \& \quad \tilde{u}^{(i)}_L \rightarrow -\tilde{v}^{(i)}_R, \tilde{v}^{(i)}_R \rightarrow \tilde{u}^{(i)}_L \quad (i = 1, 2, 3) ,
\]

such that

\[
[H, J_i] = 0 , \quad [H, \sigma_{123}] = 0 , \quad [H, P] = 0 , \quad [H, T] = 0 , \quad [H, C] = 0 .
\]

Furthermore, \(H\) commutes with the total number \(N\) of quarks

\[
[H, N] = 0 , \quad N := N_L + N_R , \quad N_{L(R)} := \sum_{\mu=0,1,2,3} \tilde{N}_L(R) .
\]

The eigenstates of \(H\) can therefore be characterized by the total number \(N = 0, 1, 2, .., 8\) of quarks. Due to the charge-conjugation symmetry \(C\) in (68), the states with quark numbers \(N\) and \(8 - N\) are degenerate in energy.

It is important to notice, that, in contrast to the parts \(\mathcal{H}^G\) in (60) and \(\mathcal{H}^C\) in (62), which both are invariant under the transformations \(\phi_i \rightarrow -\phi_i\) and \(L \leftrightarrow R\) separately, the part \(\mathcal{H}^D\) in (61) is invariant under the combination \(P\) in (66). This will be seen to be crucial for the lowering of the ground state energy of the quark-gluon system in comparison to the pure-gluon case and for the appearance of a quark-condensate.

3.4 Boundary conditions

Note that for the pure-gluon case, \(H\) reduces to

\[
\text{For } N = 0 : \quad H|_{N=0} = \mathcal{H}^G_{J^Q=0} .
\]

Its eigenstates

\[
\mathcal{H}^G_{J^Q=0}|\phi^{(J)\pm}_{n,M} = \epsilon^{(J)\pm}_{n} |\phi^{(J)\pm}_{n,M} ,
\]

can be chosen to have definite angular momentum and parity quantum numbers,

\[
|\phi^{(J)\pm}_{n,M} = \sum_{M'} \phi^{(J,M')\pm}_{n,M'} |\phi_{1,2,3} J M M' \rangle , \quad |J M M' \rangle = i J \sqrt{J+1} 2 \pi D^{(J)}_{M M'}(\chi) ,
\]

and are known with high accuracy [12]. The requirement of Hermiticity of \(\mathcal{H}^G_{J^Q=0}\) in the region bounded by the three boundary planes \(\phi_1 = 0, \phi_1 = \phi_2, \phi_2 = \phi_3\) and at positive infinity, leads to the conditions

\[
\partial_{\phi_1} \phi^{(J,M')}_n |_{\phi_1 = 0} = 0 , \quad \partial_{\phi_2} \phi^{(J,M')}_n |_{\phi_2 = \phi_1} = \text{finite} , \quad \phi^{(J,M')}_n |_{\phi_3 = \phi_2} = \text{finite} .
\]

Finally, normalisability of the wave functions requires that the wave functions vanish sufficiently fast at infinity.

\(^4\)The conditions \(\tilde{u}^{(\mu)}_L|0\rangle = 0, \tilde{v}^{(\mu)}_R|0\rangle = 0\) transform under \(T\) into \(\tilde{u}^{(\mu)}_L|0\rangle^* = 0, \tilde{v}^{(\mu)}_R|0\rangle^* = 0\), from which \(|0\rangle^* = |8\rangle\) follows.
In the general case of non-vanishing quark-number, one can put the quark boundary condition (possible only for the case of half-filling $N = 4$), to be eigenstates under the $C$ symmetry (68)\(^5\)

$$\{ N_L^{(\mu)} \} |\Psi\rangle = \left( 1 - N_R^{(\mu)} \right) |\Psi\rangle , \quad \forall \mu = 0, 1, 2, 3 , \quad (76)$$

invariant under all symmetries (69) and (70) of the Hamiltonian (59).

From all quark states $\Psi$ satisfying the boundary condition (76) one can build the quark spin and parity eigenstates

$$(J^Q)^2 |\Psi^{(j)\pm}_M\rangle = J(J + 1) |\Psi^{(j)\pm}_M\rangle , \quad J_z^Q |\Psi^{(j)\pm}_M\rangle = M |\Psi^{(j)\pm}_M\rangle , \quad P |\Psi^{(j)\pm}_M\rangle = \pm |\Psi^{(j)\pm}_M\rangle , \quad (77)$$

and diagonalize $\mathcal{H}$ in the basis

$$|J^{P_1}_n \otimes J^{P_2}_M \rangle = \delta_{P_1, P_2} - P \sum_{M_1, M_2} C_{J_1 M_1 J_2 M_2}^J \Phi_{n, M_1}^{(j_1)P_1} \otimes \Phi_{M_2}^{(j_2)P_2} \rangle . \quad (78)$$

### 4 Calculation of the spin-0 energy spectrum

For total spin-0 ($J^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 = 0$) and for massless quarks the Hamiltonian (59) reduces to

$$H_0 = \frac{g^2}{\sqrt{13}} \left[ \mathcal{H}_{\xi=0}^G + \mathcal{H}_D + \mathcal{H}_{\xi=0}^C \right] \quad (79)$$

In addition to the above stated symmetries the $m = 0$ Hamiltonian (79) and the boundary conditions (76) are invariant under independent global $U(1)$ phase rotations of the left- and the right-handed quarks, and the total numbers $N_L(R)$ of left-(right-) handed quarks are good quantum numbers.

Let us first consider 4-quark states with 2 left-handed and 2 right-handed quarks with the boundary condition (76), satisfied by the 6 states

$$|\Psi^{(\pm)}_i\rangle := \frac{1}{\sqrt{2}} \left( u^{(i)}_L \bar{u}^{(i)}_R \mp u^{(i)}_L \bar{v}^{(i)}_R \right) |0\rangle \pm \frac{1}{\sqrt{2}} \left( u^{(i)}_L \bar{v}^{(i)}_R \mp u^{(i)}_L \bar{v}^{(i)}_R \right) |0\rangle \quad i, j, k = 1, 2, 3 \quad (80)$$

which are $\pm$ parity-eigenstates, and construct the $(J^Q)^2 = (\bar{J}^Q)^2$ eigenstates

$$|\Psi^{(0)}\rangle := \left( |\Psi^+_1\rangle + |\Psi^+_2\rangle + |\Psi^+_3\rangle \right) / \sqrt{3} , \quad |\Psi_A^{(2)}\rangle := D_{A0}^{(2)}(\chi) |\Psi^{(2,0)}\rangle + D_{A2}^{(2)}(\chi) |\Psi^{(2,2)}\rangle , \quad (81)$$

with

$$|\Psi^{(2,0)}\rangle := \sqrt{\frac{2}{3}} \left( |\Psi^+_1\rangle - \frac{1}{2} \left( |\Psi^+_1\rangle + |\Psi^+_2\rangle \right) \right) , \quad |\Psi^{(2,2)}\rangle := \frac{1}{\sqrt{2}} \left( |\Psi^+_1\rangle - |\Psi^+_2\rangle \right) . \quad (82)$$

Furthermore, let

$$|\Phi^{(0)}_{n,\pm}\rangle = \Phi^{(0)}_{n,\pm}(\phi) |000\rangle , \quad n = 0, 1, 2, 3, \ldots , \quad |\Phi^{(2)}_{m,\pm}\rangle = \Phi^{(2)}_{m,\pm}(\phi) |2A0\rangle + \Phi^{(2,2)}_{m,\pm}(\phi) |2A2\rangle , \quad m = 0, 1, 2, 3, \ldots \quad (83)$$

be the complete set of $\pm$-parity spin-0 and spin-2 eigenstates [12] of the pure-gluon part $\mathcal{H}_R^{G_{J=0}}$ of the Hamiltonian (59). Since the action of the $(\bar{J}^Q)^2$ on the $|\Psi^{(0)}\rangle$, $|\Psi^{(2,0)}\rangle$ and $|\Psi^{(2,2)}\rangle$, is the same as that of the $\xi_i^2$ on the Wigner D-functions $D_{00}^{(2)} = 1, D_{A0}^{(2)}$ and $D_{A2}^{(2)}$, respectively, the positive-parity spin-0 combinations

$$|0^{\pm}_{n,\pm} \otimes 0^{\pm}_{n,\pm}\rangle + = \Phi^{(0)}_{n,\pm}(\phi) |\Psi^{(0)}\rangle \quad n = 0, 1, 2, 3, \ldots$$

$$|2^{\pm}_{m,\pm} \otimes 2^{\pm}_{m,\pm}\rangle + = \Phi^{(2,0)}_{m,\pm}(\phi) |\Psi^{(2,0)}\rangle + \Phi^{(2,2)}_{m,\pm}(\phi) |\Psi^{(2,2)}\rangle \quad m = 0, 1, 2, 3, \ldots \quad (84)$$

\(^5\) Other possibilities, e.g. For $N = 2, 4, 6$ :

$$\tilde{N}_L^{(\mu)} |\Psi\rangle = \tilde{N}_R^{(\mu)} |\Psi\rangle , \quad \forall \mu = 0, 1, 2, 3 , \quad (75)$$

would restrict the quark states to positive-parity states and hence lead to a vanishing $\mathcal{H}_D$ in (61).
which are eigenstates of $C$, $P$ and $T$ symmetries (66)-(68), form a the complete set of eigenfunctions of $\mathcal{H}^G_{\xi=0}$,
\[
\mathcal{H}^G_{\xi=0} = \sum_n \left( c_n^{(0)} |0_n^+ \otimes 0^+\rangle_{(0)} + c_n^{(0)} |0_n^- \otimes 0^+\rangle_{(0)} + \epsilon_n^{(0)} |0_n^- \otimes 0^-\rangle_{(0)} + c_m^{(0)} |0_n^- \otimes 0^-\rangle_{(0)} \right) + \sum_m \left( c_m^{(2)} |2_m^+ \otimes 2^+\rangle_{(0)} + c_m^{(2)} |2_m^- \otimes 2^+\rangle_{(0)} + \epsilon_m^{(2)} |2_m^- \otimes 2^-\rangle_{(0)} + c_m^{(2)} |2_m^- \otimes 2^-\rangle_{(0)} \right). \tag{85}
\]
The interactions $\mathcal{H}^D$ and $\mathcal{H}^C_{\xi=0}$ can be written in the representation (84) as
\[
\mathcal{H}^D = 2 \sqrt{3} \sum_{n,n'} \langle \Phi_n^{(0)} | (|\phi_3\rangle |\Phi_n^{-}\rangle) \left( |0_{n'}^+ \otimes 0^+\rangle_{(0)} + |0_{n'}^- \otimes 0^-\rangle_{(0)} + h.c. \right) \]
\[
- \frac{2}{\sqrt{3}} \sum_{m,n} \left[ \langle \Phi_n^{(0)} | (|\phi_3\rangle |\Phi_n^{-}\rangle) \left( |0_{n}^+ \otimes 0^+\rangle_{(0)} + |2_{m}^- \otimes 2^-\rangle_{(0)} + h.c. \right) \right.
\]
\[
+ \langle \Phi_n^{-} | (|\phi_3\rangle |\Phi_n^{-}\rangle) \left( |0_{n}^+ \otimes 0^-\rangle_{(0)} + |2_{m}^+ \otimes 2^+\rangle_{(0)} + h.c. \right) \right] \]
\[
+ \frac{2}{\sqrt{3}} \sum_{m,m'} \left( \langle \Phi_n^{(0)} | (|\phi_3\rangle |\Phi_n^{-}\rangle) - \frac{\sqrt{7}}{2} \langle \Phi_n^{(0)} | (|\phi_3\rangle |\Phi_n^{-}\rangle) \left( |2_{m'}^+ \otimes 2^+\rangle_{(0)} + |2_{m}^- \otimes 2^-\rangle_{(0)} + h.c. \right)\right)(86)
\]
and
\[
\mathcal{H}^C_{\xi=0} = 2 \sqrt{3} \sum_{n,n'} \langle \Phi_n^{(0)} | (|\phi_3\rangle |\Phi_n^{-}\rangle) \left( |0_{n'}^+ \otimes 0^-\rangle_{(0)} + |0_{n'}^- \otimes 0^-\rangle_{(0)} \right)
\]
\[
+ \frac{1}{\sqrt{3}} \sum_{m,n} \langle \Phi_n^{(0)} | (|\phi_3\rangle |\Phi_n^{-}\rangle) \left( |0_{n}^+ \otimes 0^-\rangle_{(0)} + |2_{m}^- \otimes 2^-\rangle_{(0)} + h.c. \right) \right]
\]
\[
- \frac{1}{\sqrt{15}} \sum_{m,m'} \left( \langle \Phi_n^{(0)} | (|\phi_3\rangle |\Phi_n^{-}\rangle) \left( |0_{n}^+ \otimes 0^-\rangle_{(0)} + |2_{m}^- \otimes 2^-\rangle_{(0)} + h.c. \right) \right]
\]
\[
+ \sqrt{7} \langle \Phi_n^{(0)} | (|\phi_3\rangle |\Phi_n^{-}\rangle) \left( |0_{n}^+ \otimes 0^-\rangle_{(0)} + |2_{m}^- \otimes 2^-\rangle_{(0)} + h.c. \right) \right), \tag{87}
\]
with the abbreviations
\[
\langle \phi_3\rangle^{(0)} : = (\phi_1 + \phi_2 + \phi_3) / \sqrt{3} , \\
\langle \phi_3\rangle^{(2)} : = \sqrt{7} \left( \phi_3 - \frac{1}{2} (\phi_1 + \phi_2) \right) D^{(2)}_{10}(\chi) + \frac{1}{\sqrt{2}} (\phi_1 - \phi_2) D^{(2)}_{\chi 2}(\chi) .
\]

Truncating the space of states at 30 nodes and diagonalising the total $H_0$ one obtains the energy spectrum shown in the third column of Tab.1 and the second spectrum in Fig.1. The 3 numbers in brackets behind the energy values in the table give the contributions from the parts $\mathcal{H}^G_{\xi=0}$, $\mathcal{H}^D$, and $\mathcal{H}^C_{\xi=0}$ separately. The second column of Tab.1 and the first spectrum in Fig.1 show the energy levels for the corresponding pure-gluon case. Since the gluonic matrix elements are calculated with high accuracy, the errors are expected to be smaller than the last digits shown and lie inside the lines. The lowest state is (up to contributions $\leq 0.1$)$^6$
\[
|0_{QG(2L2R)}^{(0)}\rangle = \sum_{n,m} \left[ c_n^{(0)} |0_n^+ \otimes 0^+\rangle_{(0)} + c_n^{(0)} |0_n^- \otimes 0^-\rangle_{(0)} + \epsilon_n^{(0)} |0_n^- \otimes 0^-\rangle_{(0)} + c_m^{(2)} |2_m^+ \otimes 2^+\rangle_{(0)} + c_m^{(2)} |2_m^- \otimes 2^+\rangle_{(0)} \right]
\]
\[
= 0.80|0_n^+ \otimes 0^-\rangle_{(0)} + 0.38|2_m^+ \otimes 2^+\rangle_{(0)} - 0.28|0_n^- \otimes 0^-\rangle_{(0)} + 0.24|2_m^- \otimes 2^-\rangle_{(0)} + 0.19|0_n^+ \otimes 0^+\rangle_{(0)} + .. \, , \tag{88}
\]
and its energy $E_0^{(0)}_{QG(2L2R)} = 3.22 g^{2/3} / V^{1/3}$ about 20% lower than the lowest energy $E_{QG}^{(0)} = 4.117 g^{2/3} / V^{1/3}$ for the pure-gluon case. It is the lowest of all spin-0 states for different numbers of massless reduced quarks and therefore constitutes the ground state. Note that responsible for the lowering of the energy in the presence of quarks is $\mathcal{H}^D$, which leads to transitions between quark states with positive and negative parity, of course accompanied by a corresponding transition between quark states with opposite parity in order to be invariant

$^6$The coefficients in (88) satisfy $\sum_n \left[ (c_n^{(0)})^2 + (c_n^{(0)})^2 \right] + \sum_m \left[ (c_m^{(2)})^2 + (c_m^{(2)})^2 \right] = 1$. 

10
under the total parity transformation $P$ in (66). At the same time, $\mathcal{H}^D$ is responsible for the formation of a quark condensate. The quark operator

$$O_{2L2R} := \sum_{i,j,k}^{\text{cyclic}} \left[ (u_L^{(0)} v_R^{(0)}) (u_L^{(i)} v_R^{(i)}) (\bar{v}_L^{(j)} u_R^{(j)}) (\bar{v}_L^{(k)} u_R^{(k)}) + h.c. \right]$$

(89)

which connects the parity-doublet partners in the quark wave function (80), has non-vanishing expectation values. In particular for the lowest state (88), we find the quark condensate according to the simple formula

$$\langle 0 | (\bar{\psi} \psi)^4 | 0 \rangle_{QG(2L2R)} = 24 \left( \sum_n \left[ (c_n^{(0)})^2 - (c_n^{(-)})^2 \right] + \sum_m \left[ (c_m^{(2)})^2 - (c_m^{(-)})^2 \right] \right) \simeq 16 ,$$

(90)

in addition to a gluon condensate (to be calculated) as in the pure-gluon case. Note however, that

$$\langle 0 | \bar{\psi} \psi | 0 \rangle_{QG(2L2R)} = 0.$$

For the case of 4-quark states with 3 left-handed and 1 right-handed quark or vice versa with the boundary condition (76), we have the 8 states\(^7\)

$$\tilde{\Psi}_0^\pm := \frac{1}{\sqrt{2}} \left( u_L^{(1)} \bar{u}_L^{(2)} \bar{u}_L^{(3)} v_R^{(0)} \right) \pm \frac{1}{\sqrt{2}} \left( \bar{v}_R^{(1)} \bar{v}_R^{(2)} \bar{v}_R^{(3)} v_R^{(0)} \right)$$

$$|\tilde{\Psi}_i^\pm \rangle := \frac{1}{\sqrt{2}} \left( u_L^{(j)} \bar{u}_L^{(k)} v_R^{(i)} \right) \pm \frac{1}{\sqrt{2}} \left( \bar{v}_R^{(j)} \bar{v}_R^{(k)} v_R^{(i)} \right), \quad i, j, k = 1, 2, 3 \ \text{cyc.}$$

(91)

A similar calculation as for the first case yields the energy spectrum shown in the fourth column of Tab.1 and the third spectrum of Fig.1. The energy $E_{0_{QG(3L1R/1L3R)}}^{(0)^+} = 3.42g^2/\sqrt{V^{1/3}}$ of its lowest state is also considerably lower than the ground state energy for the pure-gluon case but a little higher than for the first case of two right- and two left-handed quarks. It could be identified with the sigma-antisigma excitation in this massless 1-flavor 2-color investigation, its mass would be about one fifth of the first glueball excitation and therefore of the right order of magnitude.

Finally we mention, that for the case of 4 right-handed or 4 left-handed quarks with the boundary condition (76), we have the 2 states

$$|\tilde{\Psi}_0^\pm \rangle := \frac{1}{\sqrt{2}} \left( u_L^{(1)} \bar{u}_L^{(2)} \bar{u}_L^{(3)} \bar{v}_R^{(0)} \right) \pm \frac{1}{\sqrt{2}} \left( \bar{v}_R^{(1)} \bar{v}_R^{(2)} \bar{v}_R^{(3)} v_R^{(0)} \right),$$

(92)

leading to an energy spectrum which is the sum-set of the energy levels of the positive- and negative-parity pure-gluon cases, $\{E_{i_{QG(4L/4R)}}^{(0)^+}\} = \{E_{n_{G}}^{(0)^+}\} \cup \{E_{m_{G}}^{(0)^-}\}$, shown in the last column of Tab.1 and the last spectrum in Fig.1 (note that the first three and the 5th energy value coincide with $E_{i_{G}}^{(0)^+}, i = 0, 1, 2, 3$, whereas the 4th energy level is the lowest $E_{0_{G}}^{(0)^-}$ level).

All other number of particles and boundary conditions other than (76) do not lower the energy in comparison with the pure gluon case. Note, that if one imposed the boundary condition of Footnote 5 instead of (76) on the 4 quarks, $\mathcal{H}^D$ made no contribution and one would obtain a ground-state energy higher than that for the pure-gluon case. Also for boundary conditions which are rotationally invariant combinations of (76) and that in Footnote 5, putting one condition on the reduced spin-0 quarks and the other on the reduced spin-1 quarks, the energy is not lowered.

We have considered here the simplest case of one quark-flavor with mass $m = 0$. A non-vanishing quark-mass will lead to transitions between the states of the three massless cases discussed here. The generalization to $m > 0$, higher total spins and two or three flavors is straightforward and will be the subject of future work.

5 Summary and discussion

The quantum mechanics of spatially constant $SU(2)$ Yang-Mills fields minimally coupled to Dirac fields has been investigated as the strong coupling limit of 2-color QCD. Using the polar decomposition for the spatial

\(^7\)Due to the possibility of independent global $U(1)$ phase rotations of the left- and the right-handed quark fields for $m = 0$, each state is in fact a continuous family of energy-degenerate states. This is not the case for the states (80) of equal numbers of 2 left- and 2 right- handed quarks, in particular for the ground state (88), which are non-degenerate.
Table 1: The positive-parity spin-0 energy spectrum for the pure-gluon case (second column), for the quark-gluon case with 2 left- and 2 right-handed quarks (third column), with 3 left- and 1 right-handed quark and vice versa (fourth column), and with 4 left- or 4 right-handed quarks (fifth column). The 3 numbers in brackets in the third, fourth and fifth column refer to the contributions from the 3 parts $\mathcal{H}_{\xi=0}^G$, $\mathcal{H}^D$, and $\mathcal{H}_{\xi=0}^C$. The numerical errors are expected to be smaller than the last digit shown.

| $i$ | $E_{iG}^{(0)+}$ [$g^{2/3}/V^{1/3}$] | $E_{iQG(2L2R)}^{(0)+}$ [$g^{2/3}/V^{1/3}$] | $E_{iQG(3L1R/1L3R)}^{(0)+}$ [$g^{2/3}/V^{1/3}$] | $E_{iQG(4L4R)}^{(0)+}$ [$g^{2/3}/V^{1/3}$] |
|-----|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| 0   | 4.117                            | 3.22 ( 5.63, -2.43, 0.02)        | 3.42 ( 5.03, -1.82, 0.21)        | 4.117 (4.117, 0, 0)             |
| 1   | 6.386                            | 4.31 ( 7.43, -3.13, 0.01)        | 4.28 ( 5.03, -1.28, 0.53)        | 6.386 (6.386, 0, 0)             |
| 2   | 7.973                            | 5.30 ( 8.92, -3.62, 0.00)        | 5.15 ( 7.07, -2.04, 0.12)        | 7.973 (7.973, 0, 0)             |
| 3   | 9.204                            | 6.00 ( 7.93, -1.94, 0.01)        | 5.77 ( 7.10, -1.50, 0.17)        | 8.787 (8.787, 0, 0)             |
| 4   | ...                              | 6.64 (10.47, -3.84, 0.01)        | 6.53 ( 7.77, -1.94, 0.70)        | 9.204 (9.204, 0, 0)             |
| 5   | ...                              | 7.49 ( 8.68, -1.23, 0.04)        | 6.62 ( 8.66, -2.16, 0.12)        | ...                             |
| 6   | 7.74                             | 11.78 ( -4.06, 0.02)             | 7.32 ( 8.77, -1.85, 0.40)        | ...                             |
| 7   | 8.37                             | 10.48 ( -2.13, 0.02)             | 7.68 (10.06, -2.52, 0.14)        | ...                             |
| 8   | 9.14                             | 12.07 ( -2.95, 0.02)             | 8.11 ( 8.58, -0.87, 0.40)        | ...                             |
| 9   | 9.26                             | 11.14 ( -1.91, 0.03)             | 8.39 (10.78, -2.96, 0.57)        | ...                             |
| 10  | ...                              | ...                              | 8.73 (10.07, -1.62, 0.28)        | ...                             |
| 11  | 9.04                             | 9.46 ( -0.42, 0.00)              | 9.40 ( 9.46, -0.42, 0.00)        | ...                             |
| 12  | 9.31                             | 11.10 ( -1.84, 0.05)             | 9.31 (11.10, -1.84, 0.05)        | ...                             |

Figure 1: The energy spectrum of the first positive-parity spin-0 eigenstates for the pure-gluon case (first spectrum) and for the quark-gluon case with 2 left-handed and 2 right-handed quarks (second spectrum), with 3 left-handed and 1 right-handed quark and vice versa (third spectrum), and with 4 left-handed or 4 right-handed quarks (fourth spectrum). The numerical errors are expected to be inside the lines.
components of the original constrained gauge fields and a corresponding $SU(2)$ phase rotation of the constrained quark fields as in [2], the corresponding unconstrained Hamiltonian (23) and total spin operator (24) has been derived in the present work. The classical analog (in some equivalent form) of (23) has already been obtained in [2]. The expression for the total spin (24), in particular its quark part $J^Q_1$, is new in the present work and has important consequences.

Firstly, such as the unconstrained gluon fields can be represented by spin-0 and spin-2 fields, the unconstrained quark fields are found to carry spin-0 and spin-1, but to continue to satisfy anti-commutation relations and hence the Pauli-exclusion principle. The states of $SU(2)$ Dirac-Yang-Mills quantum mechanics can therefore only have integer spin.

Secondly, the expression (30) for $J^Q_1$ in terms of $\hat{S}^{(4)}$ allows to determine the correct transformation properties of the unconstrained quark fields leading to the decomposition (41)-(44). Together with the well-known principle-axes representation (39) of the unconstrained gluon field $S$ it leads to a transparent form (59)-(63) of the unconstrained $SU(2)$-Dirac-Yang-Mills Hamiltonian, which separates the rotational from the scalar degrees of freedom and is new to the best of my knowledge. It generalizes the corresponding form of the pure-gluon Hamiltonian, known already for several decades, and allows to derive the energy spectrum of the Hamiltonian of $SU(2)$ Dirac-Yang-Mills quantum mechanics. In order to keep the formulas simple, only the 1-flavor case has been shown in this work, the generalization of the Hamiltonian (59)-(63) to two or three flavors is trivial.

As an illustrative example, the energy spectrum has been obtained here for the case of total spin-0 and for 4 massless quarks of one flavor (half-filling), imposing the boundary condition (76) of C-symmetry on the quark wave function. For the case of 2 left- and 2 right-handed quarks, the ground state energy is found to be lowered in comparison with the pure-gluon case by about 20%. Furthermore, the formation of a quark condensate appears, in addition to an expected gluon condensate (to be calculated). Responsible for these features is the part $H^D$, which is invariant under the combined transformation $\phi \rightarrow - \phi$ in the gluon sector and $L \leftrightarrow R$ in the quark sector, but not invariant under each of them separately. An energy slightly higher than the ground state, but still considerably lower than that for the pure gluon case, is obtained, if 3 left- and 1 right-handed quark and vice versa are considered. Its lowest state could be identified with the sigma-antisigma excitation in our investigation. The energy spectrum of the third case of 4 left- or 4 right-handed quarks, finally, is the sum-set of the positive- and negative-parity pure-gluon spectra. A non-vanishing quark-mass term will cause transitions between these three massless spectra. Furthermore, it turns out that the boundary condition (76) of C-symmetry is the only one that leads to energies lower than the pure-gluon ground state energy.

Finally, let us remark that, since the fields and states of $SU(2)$ Dirac-Yang-Mills quantum mechanics all carry integer spin, the expansion of the physical Hamiltonian in the number of spatial derivatives and coarse graining, developed in [5] for the case of $SU(2)$ Yang-Mills theory, can be straightforwardly generalized to the case of 2-color-QCD. One should note that the reduced quarks are not valence quarks, and that baryon number is expected to be related to topological quantum numbers of field configurations extending over several granulas. We conjecture here, that for the real case of QCD with three colors, the situation will be analogous to the 2-color case, except that baryons now carry half-integer spin and will appear naturally as skyrmions [15].

Acknowledgments

I would like to thank A. Dorokhov and J. Wambach for their interest and M. Buballa and L. v. Smekal for discussions on chiral symmetry. Financial support by the LOEWE-Program HIC for FAIR is gratefully acknowledged.

Appendix: Transformation of the quark operators to the intrinsic frame

The transformation (41) on the fermion fields is chosen such that $\hat{S}_i^{(4)}$ and $\hat{T}_i^{(4)}$ in (31) and (35) take the form \(^8\)

\[
\bar{U}(\chi) \hat{S}_i^{(4)} \bar{U}(\chi) = R_{ij}(\chi) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{S}_j & 0 \\ 0 & \hat{S}_j & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\(^8\) using the $SO(3)$ identities $\bar{R}(\chi) \hat{S}_i^{(4)} \bar{R}(\chi) = R_{ij}(\chi) \hat{S}_j^{(4)}$ and $\bar{R}(\chi) \hat{T}_i^{(4)} \bar{R}(\chi) = R_{ij}(\chi) \hat{T}_j^{(4)}$. 

$$\tilde{U}^\dagger(\chi) \tilde{T}_i^{(4)} \tilde{U}(\chi) = R_{ij}(\chi) \tilde{U}_0^\dagger \tilde{T}_j^{(4)} \tilde{U}_0 = R_{ij}(\chi) \begin{pmatrix} 0 & e_j \\ 0 & 0 \end{pmatrix}.$$  

with the 3 × 3 rotation matrices (50). Furthermore, using the operators (52) of reduced quarks.

$$J_i^Q = \psi_L^i \tilde{S}_i^{(4)} \psi_L + \psi_R^i \tilde{S}_i^{(4)} \psi_R = R_{ij}(\chi) J_j^Q,$$

$$K_i^Q = \psi_L^i \tilde{T}_i^{(4)} \psi_L + \psi_R^i \tilde{T}_i^{(4)} \psi_R = R_{ij}(\chi) K_j^Q,$$

with the operators (50). Furthermore, using

$$\tilde{U}^\dagger(\chi)(-S_{ij}(\chi, \phi) \tau_i \otimes \tau_j) \tilde{U}(\chi) = \tilde{U}_0^\dagger \begin{pmatrix} -\phi_3 & 0 & 0 & \phi_2 - \phi_1 \\ 0 & \phi_3 & -(\phi_1 + \phi_2) & 0 \\ 0 & 0 & \phi_3 & 0 \\ \phi_2 - \phi_1 & 0 & 0 & -\phi_3 \end{pmatrix} \tilde{U}_0 =$$

$$= \begin{pmatrix} \phi_1 + \phi_2 + \phi_3 & 0 & 0 & 0 \\ 0 & \phi_1 - (\phi_2 + \phi_3) & 0 & 0 \\ 0 & 0 & \phi_2 - (\phi_3 + \phi_1) & 0 \\ 0 & 0 & 0 & \phi_3 - (\phi_1 + \phi_2) \end{pmatrix}$$

one finds that the minimal coupling part of the Hamiltonian is diagonalised,

$$gU^{\dagger} \alpha_i S_{ij} \frac{1}{2} \tau_j \psi'^{\dagger} = \frac{g}{2} \begin{pmatrix} \psi_L^{i\dagger}, \psi_R^{i\dagger} \end{pmatrix} \begin{pmatrix} -S_{ij} \tau_i \otimes \tau_j & 0 \\ 0 & S_{ij} \tau_i \otimes \tau_j \end{pmatrix} \begin{pmatrix} \psi_L^i \\ \psi_R^i \end{pmatrix}$$

$$= \frac{g}{2} (\phi_1 + \phi_2 + \phi_3) \left( \tilde{N}_L^{(0)} - \tilde{N}_R^{(0)} \right) + \frac{g}{2} \sum_{ijk}^{\text{cyclic}} (\phi_i - (\phi_j + \phi_k)) \left( \tilde{N}_L^{(i)} - \tilde{N}_R^{(i)} \right).$$

with the number-operators (52) of reduced quarks.

References

[1] N.H. Christ and T.D. Lee, Phys. Rev. D 22 (1980) 939.
[2] S.A. Gogilidze, A.M. Khvedelidze, D. M. Mladenov and H.-P. Pavel, Phys. Rev. D 57 (1998) 7488.
[3] A.M. Khvedelidze and H.-P. Pavel, Phys. Rev. D 59 (1999) 105017.
[4] A.M. Khvedelidze, D. M. Mladenov, H.-P. Pavel, and G. Röpke, Phys. Rev. D 67 (2003) 105013.
[5] H.-P. Pavel, Phys. Lett. B 685 (2010) 353.
[6] M. Lüscher and G. Münster, Nucl. Phys. B232 (1984) 445.
[7] G. K. Savvidy, Phys. Lett. 159B (1985) 325.
[8] Yu. Simonov, Sov. J. Nucl. Phys. 41 (1985) 835.
[9] J. Koller and P. van Baal, Nucl. Phys. B273 (1986) 387; Nucl. Phys. B302 (1988) 1; P. van Baal and J. Koller, Ann. of Phys. (N.Y.) 174 (1987) 299.
[10] P. Weisz and V. Ziemann, Nucl. Phys. B284 (1987) 157.
[11] A. Khvedelidze and H.-P. Pavel, Phys. Lett. A 267 (2000) 96; A. Khvedelidze, H.-P. Pavel, and G. Röpke, Phys. Rev. D 61 (2000) 025017.
[12] H.-P. Pavel, Phys. Lett. B 648 (2007) 97.
[13] P. van Baal, Nucl. Phys. B307 (1988) 274.
[14] J. Kripfganz and C. Michael, Phys. Lett. 209B (1988) 77; Nucl. Phys. B314 (1989) 25.
[15] T.H.R. Skyrme, Proc. Royal Soc. A260 (1961) 127, Nucl. Phys. 31 (1962) 556.