Research Article

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The group inverse of circulant matrices depending on four parameters

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Abstract: Explicit expressions for the coefficients of the group inverse of a circulant matrix depending on four complex parameters are analytically derived. The computation of the entries of the group inverse are now reduced to the evaluation of a polynomial. Moreover, our methodology applies to both the invertible and the singular case, the latter being computationally less expensive. The techniques we use are related to the solution of boundary value problems associated with second order linear difference equations.

Keywords: circulant matrix, group inverse, Chebyshev polynomials, difference equations

MSC: 15B05, 15A09, 11B83

1 Introduction and Preliminaries

The problem of solving a linear system with circulant coefficient matrices appears in many problems related to the periodicity of that problem. This kind of system occurs in many applications: time series analysis, image processing, spline approximation, difference solutions of partial differential equations or the finite difference method to approximate elliptic equations with periodic boundary conditions, see for instance [4].

Even if the problem of computing the inverse, or the group inverse, of a circulant matrix can be considered solved from a theoretical point of view, the computational cost of finding the solution is very high, even for low dimensions.

Different approaches to solve the problem have focused on special classes of circulant matrices. For instance, S. R. Searle in [12], provided a method for obtaining analytic expressions for the coefficients of the inverse matrix of a family of three-element circulant matrices; O. Rojo in [11] gives the solution of a linear system having a symmetric circulant tridiagonal matrix, and L. Fuyong, [8], obtained the elements of the inverse matrix as functions of zero points of the characteristic polynomial of the circulant matrix. With the application of the FFT, M. Chen [5] gave algorithms to solve circulant systems in $O(n \log_2 n)$ operations instead of the $O(n^2)$ arithmetic operations required; and the properly election of circulant preconditioners, see [4], reduced the problem from $O(n^2)$ to $O(n)$.

We study circulant matrices of type $\text{Circ}(a, b, c, \ldots, c, d)$ in full generality and we compute their group inverse. We give necessary and sufficient conditions for the invertibility of circulant matrices of type $\text{Circ}(a, b, c, \ldots, c, d)$ and we obtain analytical expressions for the coefficients of their inverse or group inverse. This means that, by just checking some relations between the four coefficients, we can explicitly com-

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pute the coefficients of the inverse. In particular, we obtain the group inverse of any circulant matrix of order four. The results here obtained encompass those in [2], when the corresponding matrix is non singular, and the ones in [3] where we obtained the group inverse of singular circulant matrices depending on three parameters.

Moreover, if the inverse or the group inverse of a circulant matrix can be easily computed, we can slightly modify it by introducing new parameters in such a way that the group inverse of the new matrix is still computable with a reasonable number of new operations. We are able to obtain the corresponding explicit expression according to the values of the coefficients of the matrix. For instance, Corollary 4.4 is a generalization of the result given in [11].

The paper is organized as follows. Section 2 is devoted to notation and some results, both old and new, on circulant matrices. In Section 3 we focus on second order difference equations and the tools to tackle the problem according to the values of the coefficients of the matrix. For instance, Corollary 4.4 is a generalization of the result given in [11].

The real part of the complex number \( z \) is denoted by \( \Re(z) \). Given \( z \in \mathbb{C} \), we define \( z^n \) as \( z^{-1} \) if \( z \neq 0 \) and 0 otherwise.

For any \( n \in \mathbb{N} \), let \( R_n \) be the multiplicative group of \( n \)-th roots of unity. Given \( \rho \in \mathbb{C}^* \) we define

\[
R_n(\rho) = \left\{ \rho r + (\rho r)^{-1} : r \in R_n \right\}
\]

and hence, \( R_n(-\rho) = -R_n(\rho) \). Moreover, when \( |\rho| = 1 \), then \( R_n(\rho) = \{ \Re(\rho) : r \in R_n \} \) and in particular, \( R_n(1) = \{ \cos \left( \frac{2\pi k}{n} \right) : k = 0, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \} \). Therefore, if we consider the function \( \varphi : \mathbb{C}^* \to \mathbb{C} \) defined as

\[
\varphi(w) = \frac{1}{2}(w + w^{-1}),
\]

then, \( R_n(\rho) = \varphi(\rho R_n) \). In particular, \( \varphi(i z) = \frac{z^2 - 1}{2z} \) for any \( z \in \mathbb{C}^* \), \( i z = \varphi(i(z + \sqrt{1 + z^2})) \) for any \( z \in \mathbb{C} \) and hence \( \varphi(i R^n) = i R \). In addition, \( \varphi(w) = \Re(w) \) when \( |w| = 1 \), \( \varphi(R^n \setminus \{ \pm 1 \}) = R \setminus [-1, 1] \) and \( [-1, 1] = \varphi(S^1) \), where \( S^1 \) is the unit circle.

**Lemma 1.1.** Given \( \rho \in \mathbb{C}^* \), the following statements hold:

(i) \( 1 \in R_n(\rho) \) iff \( \rho \in R_n \).
(ii) When \( \rho^2 \notin R_n \), then \( |R_n(\rho)| = n \) and moreover \( \pm 1 \notin R_n(\rho) \).
(iii) When \( \rho^2 \in R_n \), then \( |R_n(\rho)| = \left\lfloor \frac{n+1}{2} \right\rfloor \), except when \( n \) is even and \( \rho \notin R_n \) in which case we have that \( |R_n(\rho)| = \left\lfloor \frac{n+1}{2} \right\rfloor \).

We denote by \( \tau \) the permutation \((1, n, n-1, \ldots, 2)\) of \((1, \ldots, n)\) defined as

\[
\tau(j) = 1 + (n + 1 - j)(\text{mod } n), \quad j = 1, \ldots, n.
\]

Moreover, if \( v \in \mathbb{C}^n \) we define by \( v_\tau \) the vector \( v_\tau = (v_1, v_n, v_{n-1}, \ldots, v_3, v_2)^T \). In particular, \( 1_\tau = 1 \) and \( (v_\tau)_j = v \) for any \( v \in \mathbb{C}^n \), since \( \tau^{-1} = \tau \). In addition, \( \langle a, b_\tau \rangle = \langle a, b \rangle \) for any \( a, b \in \mathbb{C}^n \) and in particular, \( \langle a, \tau, 1 \rangle = \langle a, 1 \rangle \), for any \( a \in \mathbb{C}^n \).
2 Circulant matrices

We use the same notation as in [2, 3], see also [12]. A square matrix $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ is named circulant with parameters $a_1, \ldots, a_n \in \mathbb{C}$ if

$$A = \begin{bmatrix}
a_1 & a_2 & \cdots & a_n \\
a_n & a_1 & \cdots & a_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_2 & a_3 & \cdots & a_1
\end{bmatrix} \quad (3)$$

or equivalently, $a_{ij} = a_{1+(i-j) \bmod n}$.

Given $a = (a_1, \ldots, a_n)^\top \in \mathbb{C}^n$, we denote $\text{Circ}(a) = \text{Circ}(a_1, \ldots, a_n)$ as the circulant matrix with parameters $a_1, \ldots, a_n$. The vector $e \in \mathbb{C}^n$ is such that $e_1 = 1$ and $e_j = 0$, $j = 2, \ldots, n$. In addition, $N(a)$ and $R(a)$ denote the null space and the range of $\text{Circ}(a)$ respectively, whereas $P_a$ is the orthogonal projection onto $N(a)$. Therefore, given $v \in \mathbb{C}^n$, $P_a(v) \in N(a)$ is characterized by satisfying that $v - P_a(v) \in N(a)^\perp$.

One of the main problems in this setting is to determine the group inverse of a circulant matrix and, moreover, to know when the matrix is invertible. This problem has been widely studied in the literature and solved using $R_n$, see [9]. We next give a short account of these results, since it will be useful in the rest of this paper.

For any $z \in \mathbb{C}$, we consider the vector $f(z)$ and the matrix $J(z)$ defined respectively as

$$f(z) = (1, z, \ldots, z^{n-1})^\top \quad \text{and} \quad J(z) = \text{Circ}(f(z)). \quad (4)$$

Clearly $f(0) = e$ and $f(1) = 1$. Moreover, $\tilde{f}(z) = f(\bar{z})$ and when $r \in R_n$, then $f(r) = \tilde{f}(r)$. In addition, $\{\frac{1}{\sqrt{n}} f(r) : r \in R_n\}$ is an orthonormal basis of $\mathbb{C}^n$.

**Remark:** If $\omega = e^{\frac{2\pi i}{n}}$, then the matrix $F = \frac{1}{\sqrt{n}} \left[ f(1), f(\omega), \ldots, f(\omega^{n-1}) \right]$ is called Fourier matrix and for any $a \in \mathbb{C}^n$, $F a = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} a_{k+1} f(\omega^k)$ is the discrete Fourier transform of $a$.

The following lemma provides a necessary and sufficient condition for the invertibility of $\text{Circ}(a)$ and gives a formula for its group inverse, see [5, 6, 12].

**Lemma 2.1.** For any $a \in \mathbb{C}^n$, the following properties hold:

(i) $\text{Circ}(a)f(r) = \langle a, f(\bar{r}) \rangle f(r)$, for any $r \in R_n$ and hence

$$\det \text{Circ}(a) = \prod_{r \in R_n} \langle a, f(r) \rangle.$$

(ii) If $g(a) = \frac{1}{n} \sum_{r \in R_n} \langle a, f(r) \rangle^\# f(r)$, then

$$\text{Circ}(a)^\# = \text{Circ}(g(a)) = \frac{1}{n} \sum_{r \in R_n} \langle a, f(r) \rangle^\# J(r).$$

First, notice that property (i) implies that all circulant matrices of order $n$ are diagonalizable and have the same eigenvectors but different eigenvalues. In particular, any two circulant matrices commute each other and the conjugate transpose of a circulant matrix is also circulant. Moreover, any circulant matrix is normal, which in turns implies that it is range-hermitian with index 1. On the other hand, part (ii) in the above Lemma establishes that the problem of finding the group inverse of a circulant matrix is completely solved. Although its computation is not straightforward in most cases, even in low dimensions the formulae of Lemma 2.1 for $g(a)$ can involve a great number of computations, see Corollaries 3.5 and 3.6 in [3] for the cases $n = 3$ and $n = 4$ with $d = c$, respectively.
The complexity of the formula in Lemma 2.1 for the determination of the group inverse of a circulant matrix grows with the order of the matrix, so it is not useful at all from the computational point of view, even when the matrix is invertible. So, it is reasonable to focus on special classes of circulant matrices and/or to look for alternative methods to compute their group inverses. Many papers have considered this topic, specially for circulant matrices depending on few parameters. In many of these cases, the special structure of the matrix is highly used and leads to the use of alternative methods, such as solving linear difference equations, either directly, see for instance [2, 3, 12], or through special LU decompositions, see [5, 11].

In addition, if the group inverse of a given circulant matrix can be easily computed, we can perturb it by introducing new parameters in such a way that the group inverse of the perturbed matrix is still computable with few new operations.

**Theorem 2.2.** Given \( a \in \mathbb{C}^n \) and \( \emptyset \neq S \subset R_n \), consider \( \hat{a} = a + \sum_{s \in S} b_s f(s) \), where \( b_s \in \mathbb{C} \) for any \( s \in S \). Then, it is satisfied that

\[
g(\hat{a}) = g(a) + \frac{1}{n} \sum_{s \in S} \left( (a, f(s)) + nb_s \right)^* - (a, f(s))^* f(s),
\]

or equivalently

\[
(\text{Circ}(a) + \sum_{s \in S} b_s J(s))^* = \text{Circ}(a)^* + \frac{1}{n} \sum_{s \in S} \left( (a, f(s)) + nb_s \right)^* - (a, f(s))^* f(s).
\]

Moreover,

\[
det \left( \text{Circ}(a) + \sum_{s \in S} b_s J(s) \right) = \prod_{r \in S} \left( (a, f(r)) + nb_r \right) \prod_{r \in R_n \setminus S} (a, f(r)).
\]

**Proof.** Since \( \{ \sqrt{n} f(r) \}_{r \in R_n} \) is an orthonormal system, given \( r \in R_n \), then \( \langle \hat{a}, f(r) \rangle = \langle a, f(r) \rangle + \sum_{s \in S} b_s \langle f(s), f(r) \rangle \) and hence, \( \langle \hat{a}, f(r) \rangle = \langle a, f(r) \rangle \) when \( r \notin S \), whereas \( \langle \hat{a}, f(r) \rangle = \langle a, f(r) \rangle + b_r \langle f(r), f(r) \rangle = \langle a, f(r) \rangle + nb_r \) when \( r \in S \). Applying Lemma 2.1 we get

\[
g(\hat{a}) = \frac{1}{n} \sum_{r \in R_n} \langle \hat{a}, f(r) \rangle^* f(r) - \frac{1}{n} \sum_{r \in R_n \setminus S} \langle a, f(r) \rangle^* f(r) = \frac{1}{n} \sum_{r \in R_n} \left( (a, f(r)) + nb_r \right)^* - (a, f(r))^* f(r).
\]

The formula for the determinant is a consequence of part (i) of Lemma 2.1. \( \square \)

Observe that, given \( r \in R_n \), we have

\[
\frac{1}{n} \left( \left( (a, f(r)) + nb_r \right)^* - (a, f(r))^* \right) = \begin{cases} \frac{b_r^*}{n^2}, & \text{if } \langle a, f(r) \rangle (a, f(r)) + nb_r = 0, \\ -\langle a, f(r) \rangle (a, f(r)) + nb_r, & \text{otherwise.} \end{cases}
\]

Of course, when \( |S| \) is big, close to \( n \), the computational complexity of the expression for \( g(\hat{a}) \) in the above result is similar to that in the general case. Therefore, in practice, we must achieve a compromise between the computation of \( g(a) \), that we assume to be easy and with low cost, and the number of perturbations of \( a \). For instance, if \( S = \{ s \} \), the perturbation only adds the computation of \( \left( (a, f(s)) + nb_r \right)^* - (a, f(s))^* f(s) \). In particular, when \( s = 1 \), there is nothing new to compute, since \( a, 1 = a_1 + \cdots + a_n \) consists only of sums, and hence we only need to add the value \( \frac{1}{n} \left( \left( a_1 + \cdots + a_n + nb_r \right)^* - (a_1 + \cdots + a_n)^* \right) \) to the entries of \( \text{Circ}(a)^* \).

The following results are straightforward consequences of Theorem 2.2 that are interesting of themselves.
Corollary 2.3. Given \( \{b_r\}_{r \in R_n} \subset \mathbb{C} \), then
\[
\left( \sum_{r \in R_n} b_r(r) \right)^\# = \frac{1}{n^2} \sum_{r \in R_n} b_r^\#(r).
\]

Corollary 2.4. Given \( a, b \in \mathbb{C} \) and \( r \in R_n \), then \( (a + b)(r) \) is singular iff \( a(a + nb) = 0 \) and moreover,
\[
(a + b)(r)^\# = a^\#1 + \frac{1}{n} \left( (a + nb)^\# - a^\# \right)(r).
\]

Sometimes, for a fixed vector \( a \), it happens that \( \text{Circ}(a)^\# \) or even \( a\text{Circ}(a)^\# + \sum_{s \in S} b_s(s) \) for some \( a, b_s \in \mathbb{C} \) determines a structured family of matrices depending on the parameters \( a, b_s \). So, we can use Theorem 2.2 to establish a sort of converse to compute the group inverse of this new class of matrices. We take into account that \( \langle g(a), f(r) \rangle = \langle a, f(r) \rangle^\# \) for any \( r \in R_n \).

Proposition 2.5. Given \( a \in \mathbb{C}^n \) and \( \emptyset \neq S \subset R_n \), consider \( A = a\text{Circ}(a)^\# + \sum_{s \in S} b_s(s) \), where \( a \in \mathbb{C} \) and \( b_s \in \mathbb{C} \) for any \( s \in S \). Then,
\[
\det A = a^{n-\#S} \prod_{r \in S} (a(f(r)))^\# + nb_r \prod_{r \in R_n \setminus S} (a(f(r)))^\#
\]
and moreover,
\[
A^\# = a^\#\text{Circ}(a) + \frac{1}{n} \sum_{s \in S} \left( (a(f(s)))^\# + nb_s \right)^\# - a^\# \langle a, f(s) \rangle J(s).
\]

Corollary 2.6. Given \( a \in \mathbb{C}^n \) consider \( A = a\text{Circ}(a)^\# + b \), where \( a, b \in \mathbb{C} \). Then,
\[
\det A = a^{n-1} (a_1 + \cdots + a_n)^\# + nb \prod_{r \in R_n \setminus \{1\}} (a(f(r))^\#
\]
and moreover,
\[
A^\# = a^\#\text{Circ}(a) + \frac{1}{n} \left( (a_1 + \cdots + a_n)^\# + nb \right)^\# - a^\# (a_1 + \cdots + a_n) J.
\]

Among all the possible generalized inverses of \( \text{Circ}(a) \) some of them are circulants, see [3, Proposition 2.5]. In particular, we are interested in the group inverse \( \text{Circ}(g(a)) \). The next result gives the algebraic characterization of the vector \( g(a) \).

Theorem 2.7. ([3, Theorem 2.7]) Given \( a \in \mathbb{C}^n \), there exists a unique \( g(a) \in N(a)^\perp \) such that \( \text{Circ}(g(a)) \) is the group inverse of \( \text{Circ}(a) \). Moreover, \( g(a) \) is characterized as the unique solution of the system \( \text{Circ}(a)z = e - Pa(e) \) belonging to \( N(a)^\perp \). In particular, \( \text{Circ}(a) \) is invertible iff the system \( \text{Circ}(a)z = e \) is compatible.

In [2], we computed the inverse matrix of some circulant matrices of order \( n \geq 3 \) with three real parameters at most and in [3] we also considered complex parameters and we determined the expression for their group inverse. Indeed, we reduced significantly the computational cost of applying Lemma 2.1, since the key point for finding the mentioned inverse matrix consists in solving the system that provides \( g(a)_r \) by means of a non-homogeneous first order linear difference equation.

We aim here to extend the methodology to the computation of the group inverse of the matrices \( \text{Circ}(a, b, c, \ldots, d) \). The main difficulty in this objective is that we need to solve non-homogeneous second order linear difference equations.

### 3 Second order difference equations and Chebyshev polynomials

Since the computation of the group inverse of the circulant matrices here considered involves the solution of second order linear difference equations with constant coefficients, we have enumerated some of the properties of this kind of difference equations, as well as a review of Chebyshev polynomials in Section 4.
Given \( z \in \mathbb{C}, \rho \in \mathbb{C}^* \) and \( w \in \mathbb{C}^n \), we are interested in finding \( h \in \mathbb{C}^n \) such that
\[
w_j = -\rho^2 h_{j-1} + 2zh_j - h_{j+1}, \quad j = 2, \ldots, n - 1.
\] (5)
In this case, \( h \) is named solution of the difference equation. Clearly, the choice of \( h_0, h_1 \in \mathbb{C} \) uniquely determines \( h_j, j = 2, \ldots, n \) and hence the solution \( h \).

In order to solve the second order difference equation (5), for any \( \rho, z \in \mathbb{C} \) where \( \rho \neq 0 \), we consider the vectors \( t(\rho, z) \) and \( u(\rho, z) \) in \( \mathbb{C}^n \) whose components are defined as
\[
t_j(\rho, z) = \rho^j T_j(z) \quad \text{and} \quad u_j(\rho, z) = \rho^{j-1} U_{j-1}(z), \quad j = 1, \ldots, n.
\] (6)
Where \( T_j(z) \) and \( U_{j-1}(z) \) stand for the first and second kind Chebyshev polynomials, see Appendix (13). In addition, given \( w \in \mathbb{C}^n \), we also consider the vector \( \Psi^w(\rho, z) \in \mathbb{C}^n \) whose components are
\[
\psi_j^w(\rho, z) = \sum_{s=1}^{j} \rho^{j-s} U_{j-s}(z)w_s, \quad j = 1, \ldots, n.
\] (7)
Notice that \( \psi_1^w(\rho, z) = 0 \) and moreover \( \psi_j^w(\rho, z) = \sum_{s=1}^{j-1} u_{j-s}(\rho, z)w_s \) for \( j = 2, \ldots, n \), since \( U_{-1} = 0 \). Observe that \( t(-\rho, z) = t(\rho, -z), u(-\rho, z) = u(\rho, -z), (t(-\rho, z), u(-\rho, z)) = (t(\rho, -z), u(\rho, z)) \) and also that \( \Psi^w(-\rho, -z) = \Psi^w(\rho, z) \) and \( \Psi^w(-\rho, z) = \Psi^w(\rho, -z) \) for any \( z, \rho \in \mathbb{C}, \rho \neq 0 \).

**Lemma 3.1.** ([1, Lemma 2.4] and [7, Theorem 4.3]) Given \( z \in \mathbb{C}, \rho \neq 0 \) and \( w \in \mathbb{C}^n \), then \( h \in \mathbb{C}^n \) satisfies
\[
-\rho^2 h_{j-1} + 2zh_j - h_{j+1} = w_j, \quad j = 2, \ldots, n - 1,
\]
iff there exist \( a, \beta \in \mathbb{C} \) such that \( h = at(\rho, z) + \beta u(\rho, z) - \Psi^w(\rho, z). \)

The characteristic polynomial for the Chebyshev recurrence \(-h_{j-1} + 2zh_j - h_{j+1} = 0\) suggests that Chebyshev polynomials must be related to the function \( \varphi \) defined in (1) via the Binet Formula. Specifically, the aimed relation is given by the following identities, see [10]: for any \( k \in \mathbb{Z} \). We have
\[
T_k\left(\frac{w + w^{-1}}{2}\right) = \frac{w^k + w^{-k}}{2}, \quad w \neq 0 \quad \text{and}
\]
\[
U_{k-1}\left(\frac{w + w^{-1}}{2}\right) = \frac{w^k - w^{-k}}{w - w^{-1}}, \quad w \neq \pm 1;
\]
that is, \( T_k(\varphi(w)) = \varphi(w^k) \) for \( w \neq 0 \), whereas \( U_{k-1}(\varphi(w)) = \frac{w^k - \varphi(w)}{w - \varphi(w)} \) for \( w \neq \pm 1 \). Moreover, since \( \lim_{w \to \pm 1} \frac{w^k - w^{-k}}{w - w^{-1}} = (\pm 1)k^{-1}k = U_{k-1}(\pm 1) \), the identity remains true for \( w = \pm 1 \).

Given \( \rho, r \in \mathbb{C}^* \), we have the following identities for any \( k \in \mathbb{N} \)
\[
\rho^k T_k(\varphi(\rho r)) = \frac{r^k}{2}((\rho r)^{2k} + 1) \quad \text{and} \quad \rho^{k-1} U_{k-1}(\varphi(\rho r)) = \frac{r^{1-k}}{(\rho^2 r^2 - 1)}((\rho r)^{2k} - 1).
\]
(9)
As we can see below, the zeroes of Chebyshev polynomials \( T_n(z) - q \), where \( q \in \mathbb{C} \), play a fundamental role in the computation of the group inverse of the circulant matrices here considered. Since \( q = \varphi(w) \) for some \( w \in \mathbb{C}^* \), we can take \( \rho = w^{\frac{1}{2}} \) and then \( q = T_n(\varphi(\rho)) \).

**Lemma 3.2.** Given \( \rho \in \mathbb{C}^* \) and \( n \in \mathbb{N} \), the following properties hold:
(i) \( R_n(\rho) \) is the set of roots of the polynomial \( T_n(z) - T_n(\varphi(\rho)) \). Moreover, when \( \rho^2 \notin R_n \) all roots are simple, whereas when \( \rho^2 \in R_n \) all roots in \( R_n(\rho) \) \( \{\pm 1\} \) are double.
(ii) If \( \rho^2 \in R_n \) and \( z \in R_n(\rho) \), then \( T_{m+\ell n}(z) = \rho^{-\ell n} T_m(z) \) for any \( \ell, m \in \mathbb{Z} \) and hence, \( T_n(z) = \rho^{-n} \).
(iii) If \( \rho^2 \in R_n \) and \( z \in R_n(\rho) \) \( \{\pm 1\} \), then \( U_{m+\ell n}(z) = \rho^{-\ell n} U_m(z) \) for any \( \ell, m \in \mathbb{Z} \) and hence, \( U_{n-1}(z) = 0 \).
Lemma 3.3. Given \( n \in \mathbb{N}, \rho \in \mathbb{C} \) such that \( |\rho| = 1 \) and \( x \in \mathbb{R} \), we have that
\[
\langle t(\rho, x), u(\rho, x) \rangle = \frac{\rho}{2} U_{n-1}(x) U_n(x), \quad ||t(\rho, x)||^2 = \frac{1}{4} \left( U_{2n}(x) + 2n - 1 \right), \quad \text{and}
\]
\[
||u(\rho, x)||^2 = \frac{1}{4(x^2 - 1)} \left( U_{2n}(x) - 2n - 1 \right). \text{In particular,}
\]
\[
\langle t(\rho, \pm 1), u(\rho, \pm 1) \rangle = (\pm 1) \frac{\rho}{2} n(n+1), \quad ||t(\rho, \pm 1)||^2 = n, \quad \text{and}
\]
\[
||u(\rho, \pm 1)||^2 = \frac{n}{4(n+1)(2n+1)}, \text{whereas if } \rho^2 \in R_n, \text{ then}
\]
\[
\langle t(\rho, \varphi(\rho)), u(\rho, \varphi(\rho)) \rangle = 0, \quad ||t(\rho, \varphi(\rho))||^2 = \frac{n}{2}, \quad \text{and}
\]
\[
||u(\rho, \varphi(\rho))||^2 = \frac{n}{2(1 - \varphi(\rho)^2)}, \text{for any } r \in R_n \setminus \{ \tilde{\rho} \}, \text{ since } U_{n-1}(\varphi(\rho)) = 0 \text{ and } U_{2n}(\varphi(\rho)) = 1. \text{ In addition, if } \rho \in R_n, \text{ then}
\]
\[
\langle t(\rho, 1), f(\rho) \rangle = np \quad \text{and} \quad \langle u(\rho, 1), f(\rho) \rangle = \frac{n}{2}(n+1),
\]
whereas if \(-\rho \in R_n, \text{ then}
\]
\[
\langle t(\rho, -1), f(-\rho) \rangle = -np \quad \text{and} \quad \langle u(\rho, -1), f(-\rho) \rangle = \frac{n}{2}(n+1).
\]

We remark that when \( \rho^2 \in R_n \) and \( r \in R_n \setminus \{ \tilde{\rho} \} \) then \( \varphi(\rho r) \in \mathbb{R} \) and \(-1 < \varphi(\rho r) < 1\).

Lemma 3.4. Given \( n \in \mathbb{N} \) and \( \rho \in \mathbb{C}^* \) such that \( \rho^2 \notin R_n \), for any \( r \in R_n \) we have
\[
\langle t(\rho, \varphi(\rho)), f(\tilde{r}) \rangle = \frac{\rho^{n+1}}{2} U_{n-1}(\varphi(\rho)) + \frac{n}{2r},
\]
\[
\langle u(\rho, \varphi(\rho)), f(\tilde{r}) \rangle = -\frac{r^{n+1}}{(\rho^2 r^2 - 1)} U_{n-1}(\varphi(\rho)) - \frac{n}{(\rho^2 r^2 - 1)}.
\]

We end this section with new identities about inner products, whose proofs newly use (15), (16), (17) and in addition that \( \sum_{j=1}^n \sum_{i=1}^j a_{ij} = \sum_{i=1}^n \sum_{j=i}^n a_{ij} \).

Lemma 3.5. Consider \( w \in \mathbb{C}^n \). Then, when \( \rho^2 \notin R_n \) and \( r \in R_n \) we have
\[
\langle \Psi^w(\rho, \varphi(\rho)), f(\tilde{r}) \rangle = \frac{1}{\rho^2 r^2 - 1} \left( \sum_{s=1}^n \rho^{n-s} U_{n-s}(\varphi(\rho)) + sr^2 \right) w_s - \frac{n(n+1)}{\rho^2 r^2 - 1} \langle w, f(\tilde{r}) \rangle.
\]

Moreover, if \( \rho \in R_n \), then
\[
\langle \Psi^w(\rho, 1), f(\rho) \rangle = \frac{n(n+1)}{2\rho} \langle w, f(\rho) \rangle - \frac{1}{2} \sum_{s=1}^n \rho^{-s}(2n+1-s)w_s,
\]
if \(-\rho \in R_n, \text{ then}
\]
\[
\langle \Psi^w(\rho, -1), f(-\rho) \rangle = \frac{-n(n+1)}{2\rho} \langle w, f(-\rho) \rangle - \frac{1}{2} \sum_{s=1}^n (-\rho)^{-s}(2n+1-s)w_s,
\]
whereas when \( \rho^2 \in R_n \), for any \( r \in R_n \setminus \{ \tilde{\rho} \},
\]
\[
\langle \Psi^w(\rho, \varphi(\rho)), t(\rho, \varphi(\rho)) \rangle = \frac{1}{2} \sum_{s=1}^n \rho^{-(1+s)} w_s U_{n-1}(\varphi(\rho)) - \frac{(n+1)}{2\rho^{n+1}} \langle w, u(\rho, \varphi(\rho)) \rangle
\]
\[
+ \frac{1}{2}(\varphi(\rho)^2 - 1) \left( np \langle w, t(\rho, \varphi(\rho)) \rangle + \varphi(\rho) \langle w, u(\rho, \varphi(\rho)) \rangle \right).
\]
4 The Group Inverse of Matrices $\text{Circ}(a, b, c, \ldots, c, d)$

Our aim in this section is the computation of $\text{Circ}(a, b, c, \ldots, c, d)^\#$ where $a, b, c, d \in \mathbb{C}$, which requires matrices of order $n \geq 4$. Actually, for $n = 4$, we obtain the group inverse of any circulant matrix. Moreover, we also get explicit expressions for the coefficients of the group inverse of any symmetric circulant matrix of this type, that corresponds to taking $d = b$. Defining $a = (a, b, c, \ldots, c, d)^\top$, since $(a, 1) = a + b + d + (n - 3)c$ and 

$$(a, f(r)) = a + b\bar{r} + c(r^2 + \cdots + r^{n-2}) + d\bar{r}^{n-1} = a - c + \bar{r}(b - c) + r(d - c), \text{ for } r \in \mathbb{R}_n \setminus \{1\},$$

from Lemma 2.1 we get that

$$\text{det} \text{Circ}(a, b, c, \ldots, c, d) = (a + b + d + (n - 3)c) \prod_{r \in \mathbb{R}_n \setminus \{1\}} (a - c + \bar{r}(b - c) + r(d - c))$$

and moreover, if

$$g(a, b, c, d) = \frac{1}{n} (a + b + d + (n - 3)c)^\# 1 + \frac{1}{n} \sum_{r \in \mathbb{R}_n \setminus \{1\}} (a - c + \bar{r}(b - c) + r(d - c))^\# f(r),$$

then $\text{Circ}(a, b, c, \ldots, c, d)^\# = \text{Circ}(g(a, b, c, d))$ or, equivalently,

$$\text{Circ}(a, b, c, \ldots, c, d)^\# = \frac{1}{n} (a + b + d + (n - 3)c)^\# 1 + \frac{1}{n} \sum_{r \in \mathbb{R}_n \setminus \{1\}} (a - c + \bar{r}(b - c) + r(d - c))^\# g(r).$$

Except for very elemental cases, it is very difficult to reduce the expression for $g(a, b, c, d)$ into a closed and manageable formula. For instance, this can be easily done when $b = d = c$, since $\sum f(r) = n e$ and hence, $\text{det} \text{Circ}(a, c, \ldots, c) = (a - c)^{n-1} (a + (n - 1)c)$ and moreover,

$$g(a, c, c, c) = (a - c)^\# e + \frac{1}{n} ((a + (n - 1)c)^\# - (a - c)^\# 1).$$

However, in the remaining cases, the computation of $g(a, b, c, d)$ is not so easy. For instance, reference [3] is entirely devoted to obtaining an explicit formula for $g(a, b, c, c)$.

To achieve the objective, we use the same strategy as in our previous works on the topic, [2, 3]. The main idea is to interpret $a$ as a simple perturbation of a vector associated with a circulant matrix whose group inverse is known. So, the aimed result will come from applying Theorem 2.2. In fact, we can obtain the above result on $g(a, c, c, c)$ from Corollary 2.4, taking into account that $(a, c, \ldots, c) = (a - c)e + c1$.

Since $a = (a - c, b - c, 0, \ldots, 0, d - c)^\top + c1$, from Theorem 2.2 we know that

$$\text{Circ}(a, b, c, \ldots, c, d)^\# = \text{Circ}(a - c, b - c, 0, \ldots, 0, d - c)^\#$$

$$\quad + \frac{1}{n} \left((a + b + d + (n - 3)c)^\# - (a + b + d - 3c)^\#\right),$$

and hence our aim can be reduced to the computation of

$$k(a, b, c, d) = g(a - c, b - c, 0, d - c).$$

The solution of this problem when $d = c$ is given at [3, Theorem 3.4]). In addition, the case $b = c$ is equivalent to the previous one, since $g(a, c, c, d) = g_d(a, d, c, c)$.

From now on, we assume that $(c - b)(c - d) \neq 0$. Therefore, given $a, b, c, d \in \mathbb{C}$ under the above condition, the identity

$$(a - c, b - c, 0, \ldots, 0, d - c)^\top = (c - b)\left(\frac{a - c}{c - b}, -1, 0, \ldots, 0, -\frac{c - d}{c - b}\right)^\top$$

motivates us to define the values $\rho \in \mathbb{C}^*$ and $z \in \mathbb{C}$ as

$$\rho = \sqrt{\frac{c - d}{c - b}} \quad \text{and} \quad z = \frac{a - c}{2\sqrt{(c - b)(c - d)}}$$

(11)
which implies that
\[(a - c, b - c, 0, \ldots, 0, d - c)^\top = (c - b)(2zp, -1, 0, \ldots, 0, -\rho^2)^\top\]
and hence,
\[k(a, b, c, d) = (c - b)^{-1}k(2zp, -1, 0, -\rho^2),\]
or equivalently,
\[\text{Circ}(a - c, b - c, 0, \ldots, d - c)^\# = (c - b)^{-1}\text{Circ}(2zp, -1, 0, \ldots, 0, -\rho^2)^\#;\]
that is, we can reduce our problem to compute the group inverse of the circulant matrices of the form \(\text{Circ}(2zp, -1, 0, \ldots, 0, -\rho^2)\), for any \(z \in \mathbb{C}\) and \(\rho \in \mathbb{C}^\ast\). Moreover, since
\[a - c + r(b - c) + r(d - c) = (c - b)(2zp - \bar{r} - \rho^2) = 2\rho(c - b)(z - \varphi(\rho))\]
we also conclude that
\[\det \text{Circ}(a, b, c, \ldots, d, c) = (2\rho(c - b))^{n-1}(a + b + d + (n - 3)c) \prod_{r \in \mathbb{R}_e \setminus \{1\}} (z - \varphi(\rho)).\]
In addition, if we denote by \(N(\rho, z)\) the nullity of \(\text{Circ}(2zp, -1, 0, \ldots, 0, -\rho^2)\) and by \(P_{\rho,z}\) the projection on it, from Theorem 2.7 we know that it is satisfied that \(k(2zp, -1, 0, -\rho^2) = h_r\), where \(h\) is the unique solution of the system \(\text{Circ}(2zp, -1, 0, \ldots, 0, -\rho^2)h = e - P_{\rho,z}(e)\) belonging to \(N(\rho, z)^\perp\).
As we mention above, our techniques come from difference equations theory. Observe that, given \(w \in \mathbb{C}^n\), \(h \in \mathbb{C}^n\) satisfies \(\text{Circ}(2zp, -1, 0, \ldots, 0, -\rho^2)h = w\) iff
\[2zph_1 - h_2 - \rho^2h_n = w_1,\]
\[-\rho^2h_{j-1} + 2zph_j - h_{j+1} = w_j, \quad j = 2, \ldots, n - 1,\]
\[-h_1 - \rho^2h_{n-1} + 2zph_n = w_n\]
and hence, \(h\) must be a solution of the second order linear difference equation (5). From Lemma 3.1 this happens iff \(h = at(\rho, z) + \beta u(\rho, z) - \Psi^w(\rho, z)\), where \(a, \beta \in \mathbb{C}\) must satisfy
\[w_1 = 2zph_1 - h_2 - \rho^2h_n = \]
\[= a\rho^2 \left(1 - \rho^nT_n(z)\right) - \beta \rho^{n+1}U_{n-1}(z) + w_1 + \rho^2\psi_n^w(\rho, z),\]
\[w_n = -h_1 - \rho^2h_{n-1} + 2zph_n = \]
\[= a\rho \left(\rho^nT_{n+1}(z)-z\right) + \beta (\rho^nU_{n}(z)-1) - \sum_{s=1}^{n-1} U_{n-s}(z)\rho^{n-s}w_s;\]
that is, \(M(\rho, z)(\alpha, \beta)^\top = (\psi_n^w(\rho, z), \theta_n^w(\rho, z))^\top\), where
\[M(\rho, z) = \begin{bmatrix} \rho^nT_n(z) - 1 & \rho^{n-1}U_{n-1}(z) \\ \rho(\rho^nT_{n+1}(z)-z) & \rho^nU_{n}(z)-1 \end{bmatrix} \text{ and } \theta_n^w(\rho, z) = \sum_{s=1}^{n} \rho^{n-s}U_{n-s}(z)w_s.\]

Therefore, the circulant matrix \(\text{Circ}(2zp, -1, 0, \ldots, 0, -\rho^2)\) is singular iff \(\det M(\rho, z) = 0\) and moreover
\[N(\rho, z) = \left\{ at(\rho, z) + \beta u(\rho, z) : M(\rho, z)(\alpha, \beta)^\top = (0, 0)^\top \right\}.\]
In particular, \(\dim N(\rho, z) = 2 - \text{rank } M(\rho, z)\). On the other hand, since
\[\det M(\rho, z) = \rho^{2n} \left(T_n(z)U_n(z) - T_{n+1}(z)U_{n-1}(z)\right) - \rho^n \left(T_n(z) + U_n(z) - zU_{n-1}(z)\right) + 1,\]
from the first identity in (15) we have that
\[ T_{n-1}(z) U_{n-1}(z) = \frac{1}{2} (U_{2n}(z) - 1), \]
and using from (16) that \( U_n(z) - z U_{n-1}(z) = T_n(z), \) we obtain that
\[ \det M(\rho, z) = 2\rho^n \left( T_n(\varphi(\rho)) - T_n(z) \right) \]
and hence, \( \text{Circ}(2z\rho, -1, 0, \ldots, 0, -\rho^2) \) is singular iff \( z \in R_n(\rho). \)

We remark that we could achieve the same conclusion computing directly the determinant of the circulant matrix \( \text{Circ}(2z\rho, -1, 0, \ldots, 0, -\rho^2) \): Since for any \( r \in R_n, \)
\[ \langle (2z\rho, -1, 0, \ldots, 0, -\rho^2), f(r) \rangle = 2z\rho - r - \rho^2 r^{n-1} = 2z\rho - r - \rho^2 r^{n-1} = 2\rho \left( z - \varphi(\rho r) \right), \]
from Lemma 2.1, we obtain that
\[ \det \text{Circ}(2z\rho, -1, 0, \ldots, 0, -\rho^2) = 2^n \rho^n \prod_{r \in R_n} (z - \varphi(\rho r)) = 2^n \rho^n \prod_{w \in R_n(\rho)} (z - w). \]

In addition, from identities (9), for any \( r \in R_n, \) we have that
\[ M(\rho, \varphi(\rho r)) = \frac{\rho^{2n} - 1}{2(\rho^2 r^2 - 1)} \begin{bmatrix} \rho^2 r^2 - 1 & 2r \\ r\rho^2 (\rho^2 r^2 - 1) & 2\rho^2 r^2 \end{bmatrix} \]
\[ = \frac{\rho^n - 1}{2r} U_{n-1}(\varphi(\rho r)) \begin{bmatrix} \rho^2 r^2 - 1 & 2r \\ r\rho^2 (\rho^2 r^2 - 1) & 2\rho^2 r^2 \end{bmatrix}. \]

When \( \rho^2 \notin R_n \) we have that rank \( M(\rho, \varphi(\rho r)) = 1 \) for any \( r \in R_n \) and moreover
\[ N(\rho, \varphi(\rho r)) = \text{span} \left\{ 2rt(\rho, \varphi(\rho r)) - (\rho^2 r^2 - 1)u(\rho, \varphi(\rho r)) \right\} = \text{span} \{ f(r) \}. \]

When \( \rho^2 \in R_n \) and \( r \in R_n \setminus \{ t\rho \}, \) then \( M(\rho, \varphi(\rho r)) = 0 \) and hence
\[ N(\rho, \varphi(\rho r)) = \text{span} \{ t(\rho, \varphi(\rho r)), u(\rho, \varphi(\rho r)) \}. \]

When \( \rho \in R_n \) and \( r = \rho, \) then \( M(\rho, \varphi(\rho r)) = M(\rho, 1) = \rho^{-1} \begin{bmatrix} 0 & 1 \\ 0 & \rho \end{bmatrix} \) and hence we have that \( N(\rho, \varphi(\rho r)) = N(\rho, 1) = \text{span} \{ t(\rho, 1) \} = \text{span} \{ f(\rho) \}. \)

When \( -\rho \in R_n \) and \( r = -\rho, \) then \( M(\rho, \varphi(\rho r)) = M(\rho, -1) = \rho^{-1} \begin{bmatrix} 0 & -1 \\ 0 & \rho \end{bmatrix} \) and hence we have
\[ N(\rho, \varphi(\rho r)) = N(\rho, -1) = \text{span} \{ t(\rho, -1) \} = \text{span} \{ f(-\rho) \}. \]

We are ready to compute the vector \( k(a, b, c, d) \) thus obtaining \( \text{Circ}(a - c, b - c, 0, \ldots, 0, d - c)^{\#} \) and hence, \( \text{Circ}(a, b, c, \ldots, c, d)^{\#} \). Previously, taking into account the definitions (11); that is, \( \rho = \sqrt{\frac{c-d}{c-b}} \) and \( z = \frac{a-c}{2\sqrt{(c-b)(c-d)}} = \frac{a-c}{\varphi(c-b)} \), we translate to the parameters \( a, b, c \) and \( d \) the conditions to have \( z \in R_n(\rho) \) and/or \( \rho^2 \notin R_n, \)
(i) \( z \in R_n(\rho) \) iff there exists \( r \in R_n \) such that \( a = c + (c-d)r + (c-b)r \) and then \( z = \varphi(\rho r) \).
(ii) \( \rho^2 \notin R_n \) iff \( (d-c)^n \neq (b-c)^n \).
(iii) \( \rho^2 \in R_n \) \& \( z \in R_n(\rho) \) iff there exists \( r, s \in R_n \) such that \( a = c + (c-d)r + (c-b)s \) or equivalently \( a = c + (c-b)(sr + r) = c + (c-b)\sqrt{3}(r\sqrt{5} + \frac{1}{r\sqrt{5}}) = c + 2\sqrt{(r\sqrt{5})\sqrt{5}(c-b)} \) and \( d = c + (b-c)s \). Moreover, if \( \vartheta(r\sqrt{5}) = \cos(\theta), \) where \( 0 \leq \theta < 2\pi, \) then \( z = \cos(\theta) \) and hence, \( z \neq \pm 1 \) iff \( \theta \notin 0, \pi. \) On the other hand, \( \vartheta(r\sqrt{5}) = \pm 1 \) if \( r\sqrt{5} = \pm 1 \) or equivalently \( s = \rho^2 \).

Next we present the main result; that is, the computation of \( k(a, b, c, d) \) when \( (c-b)(c-d) \neq 0, \) that together with the results in [3], covers the computation of the group inverse of all matrices of the type \( \text{Circ}(a, b, c, \ldots, c, d). \)
Theorem 4.1. Given $a, b, c, d \in \mathbb{C}$ such that $(c - b)(c - d) \neq 0$, for any $j = 1, \ldots, n$ the $j$-th component of $k(a, b, c, d)$ is given by:

(i) If $a \neq c + (c - b)\bar{r} + (c - d)r$ for any $r \in R_n$, then

$$k_j(a, b, c, d) = \frac{\alpha}{2\sqrt{(c - b)(c - d)}}\frac{(c - b)^{-\frac{j}{2}}U_{n-j}^{-1}((a - c)/(2\sqrt{(c - b)(c - d)}))}{2} + \frac{\beta}{2\sqrt{(c - b)(c - d)}}T_n^{-1}\frac{(a - c)/(2\sqrt{(c - b)(c - d)})}{2}.$$ 

(ii) If there exists $r \in R_n$ such that $a = c + (c - b)\bar{r} + (c - d)r$ and $\frac{c - b}{c - d} \notin R_n$, then

$$k_j(a, b, c, d) = \frac{(c - b)^{\frac{j}{2}}(c - d)^{\frac{j}{2}}}{(c - d)^n - (c - b)^n} \left( \frac{\alpha}{2\sqrt{(c - b)(c - d)}}U_{n-j}^{-1}\frac{(a - c)/(2\sqrt{(c - b)(c - d)})}{2} + \frac{\beta}{2\sqrt{(c - b)(c - d)}}T_n^{-1}\frac{(a - c)/(2\sqrt{(c - b)(c - d)})}{2} \right).$$ 

(iii) If there exists $r \in R_n$ such that $a = c + (c - b)\bar{r} + (c - d)r$ and $\frac{c - b}{c - d} \in R_n \setminus \{r^2\}$, then

$$k_j(a, b, c, d) = \frac{(c - b)^{\frac{j}{2}}}{(c - d)^n - (c - b)^n} \left( \frac{(c - d)^n}{(c - d)^n - (c - b)^n} \right) + \frac{\beta}{2\sqrt{(c - b)(c - d)}}T_n^{-1}\frac{(a - c)/(2\sqrt{(c - b)(c - d)})}{2} + \frac{\beta}{2\sqrt{(c - b)(c - d)}}T_n^{-1}\frac{(a - c)/(2\sqrt{(c - b)(c - d)})}{2} \right).$$ 

(iv) If there exists $r \in R_n$ such that $a = c + (c - b)\bar{r} + (c - d)r$ and $\frac{c - b}{c - d} = r^2$, then

$$k_j(a, b, c, d) = \frac{\alpha}{12n(c - b)} \left( \frac{(n + 1 - 2j)(n + 1)(n + 1 - j)}{4(c - b)^2} \right).$$ 

Proof. Since $k(a, b, c, d) = (c - b)^{-1}k(2zp, -1, 0, -\rho^2)$, where $\rho = \sqrt{\frac{c - d}{c - b}}$ and $z = \frac{a - c}{2\sqrt{(c - b)(c - d)}}$, we need to obtain $k(2zp, -1, 0, -\rho^2)$ for any $(p, z) \in \mathbb{C}^* \times \mathbb{C}$. Moreover, if $w = p - P_{\rho, z}(e)$, then $h_r = k(2zp, -1, 0, -\rho^2)$, where $h_r$ is the unique solution of the system $\text{Circ}(2zp, -1, 0, \ldots, 0, -\rho^2)h = w$ belonging to $N(p, z)$. Therefore, $h = at(p, z) + \beta u(p, z) - \Psi_n(p, z)$, where $a$ and $\beta$ are uniquely determined by both the system $M(p, z)(a, b) = (\psi_n^0(p, z), \theta_n^0(p, z))^\top$ and the condition $h \in N(p, z)$. To imposing this last constraint in each case, we will apply the corresponding equality in Lemmas 3.3, 3.4 and 3.5. We remark that the system $M(p, z)(a, b) = (\psi_n^0(p, z), \theta_n^0(p, z))^\top$ is always compatible, since $w \in N(p, z)$. Once the vector $h$ is obtained, the components of $k$ are given by $k_1 = h_1$ and $k_j = h_{n+1-j}$, $j = 2, \ldots, n$. In all cases, we also show that the formula for the components of $h$ satisfies that $h_{n+1} = h_1$, so we can obtain all components of $k$ from a unique expression.

(i) The hypothesis means that $z \notin R_n(p)$, in which case $M(p, z)$ is invertible, so $N(p, z) = \{0\}$, $w = e$ and we need to find the unique solution of the system

$$M(p, z)(a, b) = \begin{pmatrix} \psi_n^0(p, z), \theta_n^0(p, z) \end{pmatrix}^\top = \begin{pmatrix} \rho^n - 2U_{n-2}(z), \rho^{n-1}U_{n-1}(z) \end{pmatrix}^\top.$$
Using identities (15) and (16), we have

\[
\begin{pmatrix}
\alpha \\ \beta
\end{pmatrix} = \frac{1}{2\rho^2\left(T_n(\varphi(\rho)) - T_n(z)\right)} \begin{pmatrix}
\rho^n U_n(z) - 1 & -\rho^{n-1} U_{n-1}(z) \\
\rho z - \rho^{n+1} T_{n-1}(z) & \rho^n T_n(z) - 1
\end{pmatrix} \begin{pmatrix}
\rho^{n+2} U_{n-2}(z) \\
\rho^n U_{n-2}(z)
\end{pmatrix}
\]

which implies that

\[
h = \frac{1}{2\rho^2\left(T_n(\varphi(\rho)) - T_n(z)\right)} \left( (\rho^n + U_{n-2}(z)) \varphi(\rho) + \rho(T_{n-1}(z) - \rho^n z) u(\rho, z) \right) - \Psi(\rho, z),
\]

where \(\psi_j(\rho, z) = \rho^{-j} U_{j-2}(z), j = 1, \ldots, n\). Therefore, the components of \(h\) are given by

\[
h_j = \frac{\rho^n (\rho^n + U_{n-2}(z)) T_j(z) + \rho(T_{n-1}(z) - \rho^n z) U_{j-1}(z)}{2\rho^2\left(T_n(\varphi(\rho)) - T_n(z)\right)} - \frac{2\rho^i\left(T_n(z) - T_n(\varphi(\rho))\right) U_j(z)}{2\rho^2\left(T_n(z) - T_n(\varphi(\rho))\right)} - \frac{\rho^{n+j} T_j(z) - z U_{j-1}(z)}{2\rho^2\left(T_n(z) - T_n(\varphi(\rho))\right)} + \frac{\rho^j \left(U_{n-2}(z) T_j(z) + T_{n-1}(z) U_{j-1}(z)\right)}{2\rho^n\left(T_n(z) - T_n(\varphi(\rho))\right)}
\]

for \(j = 1, \ldots, n\) and clearly it is satisfied that \(h_{n+1} = \frac{\rho^{n-1} U_{n-1}(z)}{2\rho^2\left(T_n(z) - T_n(\varphi(\rho))\right)} = h_1\).

(ii) The hypotheses mean that \(z = \varphi(\rho)\), so \(z \in R_\rho(\rho)\), but \(\rho^2 \notin R_n\). Therefore, \(M(\rho, z)\) is singular and \(N(\rho, z) = \text{span} \{ f(z) \}\), which implies that \(w = e - P_{\rho,z}(e) = e - \frac{1}{n} f(z)\). Moreover, \(h = \alpha \varphi(\rho) z + \beta\mathbf{u}(\rho, z) - \Psi^n(\rho, z)\), where \(\alpha\) and \(\beta\) satisfy

\[
(\rho^2 r^2 - 1)\alpha + 2n = \frac{2\theta^n(\rho, z)}{\rho^n U_{n-1}(z)}
\]

and also

\[
2\rho^n \theta^n(\rho, z) + 2r \sum_{s=1}^n s r^2 w_s = \alpha (\rho^2 r^2 - 1) \left( \rho^{n+1} U_{n-1}(z) + n \right) + 2r \beta \left( \rho^n U_{n-1}(z) - n \right);
\]

that is,

\[
(\rho^2 r^2 - 1)\alpha - 2n = \frac{2r n}{n} \sum_{s=1}^n s r^2 w_s = -\frac{(n-1)r^2}{n}.
\]

On the other hand,

\[
\theta^n(\rho, z) = \rho^{n-1} U_{n-1}(z) - \frac{1}{n} \sum_{s=1}^n \rho^{n-s} U_{n-s}(z) r^{1-s}
\]

\[
= \rho^{n-1} U_{n-1}(z) \left( 1 - \frac{\rho^2 r^2}{n(\rho^2 r^2 - 1)} \right) + \frac{r}{\rho^2 r^2 - 1}.
\]
So if we define \( K = \frac{2\theta^w(\rho, z)}{\rho^{n+1} U_{n-1}(z)} \), then \( K = \frac{2}{\rho^2} \left( \frac{\rho^{2n}}{\rho^{2n} - 1} - \frac{\rho^2 r^2}{n(\rho^2 r^2 - 1)} \right) \) and we have that
\[
\alpha = \frac{1}{2n(\rho^2 r^2 - 1)} (nK - (n-1)r^2), \quad \beta = \frac{1}{4nr} (nK + (n-1)r^2),
\]
which implies
\[
h = \frac{K}{4r(\rho^2 r^2 - 1)} (2rt(\rho, z) + (\rho^2 r^2 - 1)u(\rho, z)) - \frac{(n-1)r}{4n(\rho^2 r^2 - 1)} (2rt(\rho, z) - (\rho^2 r^2 - 1)u(\rho, z) - \Psi^w(\rho, z)) = \frac{K}{2r} u(\rho, z) + \frac{(nK - (n-1)r^2)}{2nr(\rho^2 r^2 - 1)} f(\bar{r}) - \Psi^w(\rho, z).
\]
Since the components of \( \Psi^w(\rho, z) \) are given by
\[
\psi^w(\rho, z) = \rho^{j-2} U_{j-2}(z) - \frac{\rho^2 r^2}{n(\rho^2 r^2 - 1)} \sum_{s=1}^{j} ((\rho r)^{2(j-s)} - 1) = \rho^{j-2} U_{j-2}(z) - \frac{\rho^2 r^2}{n(\rho^2 r^2 - 1)} (\rho^j - 1 - \rho^{-j}) = \rho^{j-2} U_{j-2}(z) \left( 1 - \frac{\rho^2 r^2}{n(\rho^2 r^2 - 1)} \right) + \frac{(j-1)r^2 - j}{n(\rho^2 r^2 - 1)}, \quad j = 1, \ldots, n
\]
and taking into account that \( \rho^{j-1} U_{j-1}(z) = \rho^j U_{j-1}(z) + r^{j-1} \) for any \( j \in \mathbb{Z} \), we obtain the components of \( h \)
\[
h_j = \rho^{j-2} U_{j-2}(z) \left( \frac{Kp^2}{2} - 1 + \frac{\rho^2 r^2}{n(\rho^2 r^2 - 1)} \right) + \frac{nKp^2 - (n-3+2j)}{2n(\rho^2 r^2 - 1)} r^{j-2} = \frac{\rho^{j-2}}{\rho^{2n} - 1} U_{j-2}(z) + \frac{r^{j-2}}{2n(\rho^2 r^2 - 1)} \left( \frac{2n}{\rho^{2n} - 1} - \frac{2\rho^2 r^2}{(\rho^2 r^2 - 1) - n + 1} \right),
\]
j = 1, \ldots, n. Notice that
\[
h_{n+1} = \frac{\rho^{n-1}}{\rho^{2n} - 1} U_{n-1}(z) + \frac{r}{2n(\rho^2 r^2 - 1)} \left( \frac{2n}{\rho^{2n} - 1} - \frac{2\rho^2 r^2}{(\rho^2 r^2 - 1) - n + 1} \right) = \frac{r}{2n(\rho^2 r^2 - 1)} \left( \frac{2n}{\rho^{2n} - 1} - \frac{2\rho^2 r^2}{\rho^2 r^2 - 1} + n + 1 \right) = h_1.
\]

(iii) The hypotheses mean that \( z = \varphi(\rho r) \) and \( z \neq \pm 1 \), so \( r \neq \pm \rho \) or equivalently, \( z \in R_\theta(\rho) \setminus \{\pm 1\} \), and \( \rho^2 \in R_n \). Moreover, \( z \in \mathbb{R} \), in fact \(-1 < z < 1\). Therefore, \( M(\rho, z) = 0 \) and hence \( N(\rho, z) = \text{span} \{ t(\rho, z), u(\rho, z) \} \), which implies \( w = e - P_{\rho, z}(e) = e - \frac{2}{n} (\rho z t(\rho, z) + (1-z^2)u(\rho, z)) \). Moreover, \( h = at(\rho, z) + \beta u(\rho, z) - \Psi^w(\rho, z) \) where
\[
\alpha = \frac{1}{n} \sum_{s=1}^{n} \rho^{-s} s w_s U_{s-1}(z), \quad \beta = \frac{1}{n} \sum_{s=1}^{n} \rho^{-s} s w_s T_s(z)
\]
and hence, from the first identity in (15),
\[
h_j = \frac{\rho^{j-1}}{n} \sum_{s=1}^{n} \rho^{-s} s w_s \left( T_j(z) U_{s-1}(z) - T_s(z) U_{j-1}(z) \right) - \sum_{s=1}^{j} \rho^{j-1-s} U_{j-1-s}(z) w_s = -\frac{1}{n} \sum_{s=1}^{n} \rho^{j-1-s} s U_{j-1-s}(z) w_s - \sum_{s=1}^{j} \rho^{j-1-s} U_{j-1-s}(z) w_s.
\]
On the other hand, the components of \( P_{\rho,z}(e) \) are
\[
P_{\rho,z}(e)_s = \frac{\rho^{s-1}}{n} \left( 2zT_s(z) - T_{s+1}(z) + T_{s-1}(z) \right) = \frac{2}{n} \rho^{s-1} T_{s-1}(z), \quad s = 1, \ldots, n
\]
which implies that the components of \( h \) are
\[
h_j = -\frac{(n+1)}{n} \rho^j T_j(z) + \frac{2\rho^j}{n} \sum_{s=1}^{n} U_{j-1-s}(z) T_{s-1}(z)
\]
\[
+ \frac{2\rho^j}{n} \sum_{s=1}^{j} U_{j-1-s}(z) T_{s-1}(z)
\]
\[
= \frac{(j - n - 1)}{2n} \rho^j T_j(z) + \frac{\rho^j}{n^2} \sum_{s=1}^{n} U_{j-2s}(z) + \frac{\rho^j}{n} \sum_{s=1}^{j} U_{j-2s}(z),
\]
\( j = 1, \ldots, n \), where we have newly applied the first identity in (15). Moreover, since \( z^2 \neq 1 \), we can use the first equality in (17) to obtain that for any \( j = 1, \ldots, n \),
\[
\sum_{s=1}^{j} U_{j-2s}(z) = \frac{1}{2(z^2 - 1)} \sum_{s=1}^{j} (T_{j-2(s-1)}(z) - T_{j-2s}(z))
\]
\[
= \frac{1}{2(z^2 - 1)} \left( \sum_{s=0}^{j-1} T_{j-2s}(z) - \sum_{s=1}^{j} T_{j-2s}(z) \right)
\]
\[
= \frac{1}{2(z^2 - 1)} (T_j(z) - T_{j-2s}(z)) = 0.
\]

On the other hand, taking into account that if \( k \in \mathbb{Z} \), then \( U_{m+kn}(\phi(pr)) = \rho^{-kn} U_m(\phi(pr)) \) and \( T_{m+kn}(\phi(pr)) = \rho^{-kn} T_m(\phi(pr)) \) for any \( m \in \mathbb{Z} \), for any \( j = 1, \ldots, n \) we also have
\[
\sum_{s=1}^{n} sU_{j-2s}(z) = \frac{1}{2(z^2 - 1)} \sum_{s=1}^{n} s(T_{j-2(s-1)}(z) - T_{j-2s}(z))
\]
\[
= \frac{1}{2(z^2 - 1)} \left( \sum_{s=0}^{n-1} (s + 1)T_{j-2s}(z) - \sum_{s=1}^{n} sT_{j-2s}(z) \right)
\]
\[
= \frac{1}{2(z^2 - 1)} \left( -nT_{j-2n}(z) + \sum_{s=0}^{n-1} T_{j-2s}(z) \right) = -\frac{n}{2(z^2 - 1)} T_j(z),
\]
since from the identity (16),
\[
\sum_{s=0}^{n-1} T_{j-2s}(z) = \frac{1}{2} \sum_{s=0}^{n-1} (U_{j-2s}(z) - U_{j-2(s+1)}(z)) = \frac{1}{2} (U_j(z) - U_{j-2n}(z)) = 0.
\]

Therefore, we have obtained that
\[
h_j = \frac{(j - n - 1)}{2n} \rho^j T_j(z) + \frac{\rho^j}{n^2} \sum_{s=1}^{n} U_{j-2s}(z)
\]
\[
+ \frac{\rho^j}{n} \sum_{s=1}^{j} U_{j-2s}(z),
\]
and moreover,
\[
h_{n+1} = \frac{\rho^{n-1}}{4n(1 - z^2)} \left( (n+1)T_{n-1}(z) - (n-1)T_{n+1}(z) \right)
\]
\[
= \frac{1}{4np(1 - z^2)} \left( (n+1)T_{n-1}(z) - (n-1)T_{n+1}(z) \right)
\]
\[
= \frac{1}{2np(1 - z^2)} T_1(z) = \frac{1}{4np(1 - z^2)} \left( (1 - n)T_{-1}(z) + (n+1)T_1(z) \right) = h_1.
\]
(iv) The hypotheses mean that either \( \rho = \bar{\rho} \) and hence \( z = 1 \) or \( \rho = -\bar{\rho} \) and hence \( z = -1 \).

If \( \rho = \bar{\rho} \) and \( z = 1 \), then \( N(\rho, 1) = \text{span}\{f(\rho)\} \), which implies that \( w = e - P_{\rho, z}(e) = e - \frac{1}{n} f(\rho) \) and that

\[
\psi_w^w(\rho, 1) = \sum_{s=1}^{j} \rho^{j-1-s}(j-s)w_s = \rho^{j-2}(j-1) - \frac{\rho^{j-2}}{n} \sum_{s=1}^{j} (j-s) = \frac{(j-1)(2n-j)}{2n} \rho^{j-2},
\]

\( j = 1, \ldots, n \). Moreover,

\[
h = at(\rho, 1) + \frac{\rho}{n} \psi_w^w(\rho, 1)u(\rho, 1) - \Psi^w(\rho, 1) = at(\rho, 1) + \left(\frac{n-1}{2n} \right) u(\rho, 1) - \Psi^w(\rho, 1),
\]

and hence, from Lemma 3.3, \( h \in N(\rho, 1) \) iff

\[
-\frac{n}{\rho} + \frac{1}{2n} \sum_{s=1}^{n} s(2n+1-s) = -\frac{n}{\rho} + \frac{(n-1)(n+1)}{6\rho} = \frac{(2n-1)(n-1)}{6\rho} = anp + \frac{n^2-1}{4n};
\]

that is, iff \( a = \frac{(n-1)(n-5)}{12n^2} \). Therefore, the components of \( h \) are given by

\[
h_j = \frac{(n-1)(n-5)}{12n^2} t(\rho, 1) + \frac{(n-1)}{2n} u(\rho, 1) - \frac{(j-1)(2n-j)}{2n} \rho^{j-2}
\]

\[
= \frac{\rho^{j-2}}{12n} \left( n^2 - 1 - 6(j-1)(n+1-j) \right), \quad j = 1, \ldots, n,
\]

and clearly it is satisfied that \( h_{n+1} = \frac{n^2 - 1}{12np} = h_1 \).

If \( \rho = -\bar{\rho} \) and \( z = -1 \), then \( N(\rho, -1) = \text{span}\{f(-\rho)\} \), which implies that \( w = e - P_{\rho, z}(e) = e - \frac{1}{n} f(-\rho) \) and that

\[
\psi_w^w(\rho, -1) = \sum_{s=1}^{j} (-\rho)^{j-1-s}(j-s)w_s = \frac{(j-1)(2n-j)}{2n} (-\rho)^{j-2}, \quad j = 1, \ldots, n.
\]

Moreover,

\[
h = at(\rho, -1) - \frac{\rho}{n} \psi_w^w(\rho, -1)u(\rho, -1) - \Psi^w(\rho, -1) = at(\rho, -1) - \frac{(n-1)}{2n} u(\rho, -1) - \Psi^w(\rho, -1),
\]

and hence, from Lemma 3.3, \( h \in N(\rho, 1) \) iff

\[
\frac{n}{\rho} - \frac{1}{2n} \sum_{s=1}^{n} s(2n+1-s) = \frac{(n-1)(n-1)}{6\rho} = -anp - \frac{(n^2-1)}{4n};
\]

that is, iff \( a = \frac{(n-1)(n-5)}{12n^2} \). Therefore, the components of \( h \) are given by

\[
h_j = \frac{(n-1)(n-5)}{12n^2} t(\rho, -1) - \frac{(n-1)}{2n} u(\rho, -1) - \frac{(j-1)(2n-j)}{2n} (-\rho)^{j-2}
\]

\[
= \frac{(-\rho)^{j-2}}{12n} \left( n^2 - 1 - 6(j-1)(n+1-j) \right), \quad j = 1, \ldots, n,
\]

and it is satisfied that \( h_{n+1} = \frac{n^2 - 1}{12np} = h_1 \). \( \Box \)

The case \( n = 4 \) deserves special attention, since it describes the group inverse of any circulant matrix with this order. In the next result, we include together those obtained in [3, Theorem 3A)] and Theorem 4.1.

**Corollary 4.2.** Given \( a, b, c, d \in \mathbb{C} \) such that \((c - b)(c - d) \neq 0\), the matrix \( \text{Circ}(a, b, c, d) \) is invertible iff \( a \neq c \pm i(b-d) \), \(-c \pm (b+d) \) and moreover,

\[
\text{Circ}(a, b, c, d)^\# = \frac{1}{4} \left( (a + b + c + d)^\# - (a + b + d - 3c)^\# \right) + \text{Circ}(k(a, b, c, d)),
\]

where vector \( k(a, b, c, d) \) is given by:
(i) If $a \neq c \pm (2c - b - d)$, $c \pm i(b - d)$, then

$$k(a, b, c, d) = \begin{cases} \frac{(a - c)((a - c)^2 - 2(c - b)(c - d))}{((a - c)^2 - 2(c - b - d)^2)((a - c)^2 + (b - d)^2)} \\
\frac{(c - b)((a - c)^2 - (c - b)(c - d)) + (c - d)^3}{((a - c)^2 - 2(c - b - d)^2)((a - c)^2 + (b - d)^2)} \\
\frac{(a - c)((c - b)^2 + (c - d)^2)}{((a - c)^2 - 2(c - b - d)^2)((a - c)^2 + (b - d)^2)} \\
\frac{(c - d)((a - c)^2 - (c - b)(c - d)) + (c - b)^3}{((a - c)^2 - 2(c - b - d)^2)((a - c)^2 + (b - d)^2)} \
\end{cases}.$$ 

(ii) If $d \neq c \pm (c - b)$, $c \pm i(c - b)$, then

$$k(c \pm (2c - b - d), b, c, d) = \begin{cases} \frac{\pm(10c(b + d - c) - 3(b^2 + d^2) - 4bd)}{8(b + d - 2c)((c - d)^2 + (c - b)^2)} \\
\frac{2c^2 + (2b - 6d)c - b^2 + 3d^2}{8(b + d - 2c)((c - d)^2 + (c - b)^2)} \\
\frac{\pm(6c(c - b - d) + b^2 + 4bd + d^2)}{8(b + d - 2c)((c - d)^2 + (c - b)^2)} \\
\frac{2c^2 + (2d - 6b)c - d^2 + 3b^2}{8(b + d - 2c)((c - d)^2 + (c - b)^2)} \
\end{cases},$$ 

and

$$k(c \pm i(b - d), b, c, d) = \begin{cases} \frac{\pm i(2c(b + d - c) - 3(b^2 + d^2) + 4bd)}{8(b - d)((c - d)^2 + (c - b)^2)} \\
\frac{-\left(2c^2 + (2b - 6d)c - b^2 + 3d^2\right)}{8(b - d)((c - d)^2 + (c - b)^2)} \\
\frac{\pm i(2c(c - b - d) - b^2 + 4bd - d^2)}{8(b - d)((c - d)^2 + (c - b)^2)} \\
\frac{2c^2 + (2d - 6b)c - d^2 + 3b^2}{8(b - d)((c - d)^2 + (c - b)^2)} \
\end{cases}.$$ 

(iii) If $b \neq c$,
\[ k(c, b, c, c \pm (c-b)) = \frac{1}{4(c-b)} (0, \pm 1, 0, -1)^\top, \]
\[ k(c \pm (c-b)(1-i), b, c, c + i(c-b)) = \frac{1}{8(c-b)} (\pm (1 + i), -i, 0, -1)^\top, \]
\[ k(c \pm (c-b)(1-i), b, c, c - i(c-b)) = \frac{1}{4(c-b)} (\pm (1 + i), i, 0, -1)^\top, \]
\[ k(c \pm (c-b)(1+i), b, c, c + i(c-b)) = \frac{1}{4(c-b)} (\pm (1 - i), -i, 0, -1)^\top, \]
\[ k(c \pm (c-b)(1+i), b, c, c - i(c-b)) = \frac{1}{8(c-b)} (\pm (1 - i), i, 0, -1)^\top, \]
\[ k(c \pm 2(c-b), b, c, b) = \frac{1}{16(c-b)} (\pm 5, -1, \mp 3, -1)^\top, \]
\[ k(c \pm 2i(c-b), b, c, 2c-b) = \frac{1}{16(c-b)} (\mp 5, 1, \mp 3, -1)^\top. \]

**Proof.** First, we have that \( R_4 = \{\pm 1, \pm i\} \), and \( T_4(z) = 8z^4 - 8z^2 + 1 \), which implies that \( T_4(z) - T_4(w) = 8(z^2 - w^2)(z^2 + w^2 - 1) \).

On the other hand, when \( a = c \pm (2c-b-d) \), then \( c-b = -(c-d) \) iff \( a = c, b-d = i(c-d) \) iff \( a = c \pm (c-b)(1-i) \) and \( (c-b) = -(c-d) \) iff \( a = c \pm (c-b)(1+i) \), whereas when \( a = c \pm i(b-d) \), then \( c-b = -(c-d) \) iff \( a = c \pm i(b-d) \), whereas when \( a = c \pm z(b-d) \), then \( c-b = -(c-d) \) iff \( a = c \pm 2i(c-b) \). Moreover in all these cases \( b = c \) iff \( d = c \) and hence \( (c-b)(c-d) \neq 0 \) iff \( b \neq c \).

**Remark:** Notice that if we permit the equality \( (b-c)(c-d) = 0 \) in the identities (i) and (ii), then we recover the expressions obtained in [3]: For case (i), we obtain

\[
k(a, b, c, c) = \frac{1}{(a-c)^3 - (c-b)^3} \left( (a-c)^3, (a-c)^2(c-b), (a-c)(c-b)^2, (c-b)^3 \right)^\top, \]
\[
k(a, c, c, d) = \frac{1}{(a-c)^3 - (c-d)^3} \left( (a-c)^3, (a-c)(c-d)^2, (c-a)^2(c-d)^2 \right)^\top, \]

for case (ii)

\[
k(c \pm (2c-b-d), b, c, d) = \frac{1}{8} (c-b)^8 \left( \pm 3, 1, \mp 1, -3 \right)^\top + (c-d)^8 \left( \pm 3, -3, \mp 1, 1 \right)^\top. \]

wheras

\[
k(c \pm i(b-c), b, c, c) = \frac{1}{8(c-b)} (\pm 3i, -1, \pm i, -3)^\top \]
\[
k(c \pm i(c-d), b, c, d) = \frac{1}{8(c-d)} (\mp 3i, -3, \pm i, -1)^\top, \]

**Remark:** The application of Lemma 2.1 (ii) leads to the apparently manageable expression

\[
g(a, b, c, d) = \frac{1}{4} \left( (a+b+c+d)^8(1, 1, 1, 1)^\top + (a-b+c-d)^8(1, -1, 1, 1)^\top \right) \]
\[+ \frac{1}{4} \left( (a-c+i(d-b))^8(1, i, -1, -i)^\top + (a-c-i(d-b))^8(1, -i, -1, i)^\top \right) \]
\[+ \frac{1}{4} \left( (a+b+c+d)^8 - (a+b+d-3c)^8 \right) (1, 1, 1, 1)^\top + k(a, b, c, d) \]

where

\[
k(a, b, c, d) = \frac{1}{4} \left( (a+b+d-3c)^8(1, 1, 1, 1)^\top + (a+b+c-d)^8(1, -1, 1, 1)^\top \right) \]
\[+ \frac{1}{4} \left( (a-c+i(d-b))^8(1, i, -1, -i)^\top + (a-c-i(d-b))^8(1, -i, -1, i)^\top \right). \]
To change the above identity into an explicit formula, we need to analyze the cases when \( a \) is equal or not to any of the values \( c \pm (2c - b - d) \) and \( c \pm i(b - d) \). This is precisely the analysis contained in Corollary 4.2 and our methodology allows us to do the same in higher dimensions, where easy identities for \( g(a, b, c, d) \) as the previous one are no longer available.

The circulant matrix \( \text{Circ}(a, b, c, \ldots, c, d) \) is symmetric iff \( d = b \). We remark that the necessary and sufficient conditions for the invertibility of \( \text{Circ}(a, b, c, \ldots, c, b) \) when \( a, b, c \in \mathbb{R} \), together with the expression for its inverse, were already obtained in [2, Theorem 3.5]. Next, we describe the expression for the group inverse.

**Corollary 4.3.** Given \( a, b, c \in \mathbb{C} \), the matrix \( \text{Circ}(a, b, c, \ldots, c, b) \) is invertible iff

\[
(a + 2b + (n - 3)c) \prod_{j=1}^{[\frac{n-1}{2}]} \left( a - c + 2(b - c) \cos \left( \frac{2j\pi}{n} \right) \right) \neq 0.
\]

Moreover,

\[
\text{Circ}(a, b, c, \ldots, c, b)^{\#} = \text{Circ}(ka, b, c, \ldots, c, b)^{\#} + \frac{1}{n} \left( (a + 2b + (n - 3)c)^{\#} - (a + 2b - 3c)^{\#} \right)
\]

where for any \( j = 1, \ldots, n \) the \( j \)-th component of \( k(a, b, c, b) \) is given by:

(i) If \( a \neq c + 2(c - b) \cos \left( \frac{2k\pi}{n} \right) \), \( k = 0, \ldots, \lceil \frac{n-1}{2} \rceil \), then

\[
k_j(a, b, c, b) = \begin{cases} 
\frac{U_{n-j} \left( \frac{a - c}{2(c - b)} \right) + U_{j-2} \left( \frac{a - c}{2(c - b)} \right)}{2(c - b)} \left( T_n \left( \frac{a - c}{2(c - b)} \right) - 1 \right), & \text{if } b \neq c; \\
\frac{e_j}{a - c}, & \text{if } b = c.
\end{cases}
\]

(ii) If \( a = c + 2(c - b) \cos \left( \frac{2k\pi}{n} \right) \) for some \( k = 0, \ldots, \lfloor \frac{n-1}{2} \rfloor \), then

\[
k_j(a, b, c, b) = \begin{cases} 
\frac{(n^2 - 1 - 6j(n - j)(n + 1) - j)}{3(n + 3 - 2j)(2n - 2j - 1) \sin^2 \left( \frac{2\pi k}{n} \right)}, & k = 0, \ldots, \lfloor \frac{n-1}{2} \rfloor; \\
(-1)^j (n^2 - 1 - 6j(n - j)(n + 1 - j)), & n \text{ even} \quad \text{and } k = \lfloor \frac{n-1}{2} \rfloor.
\end{cases}
\]

In particular, we next analyze the case \( c = 0 \). Notice that the group inverse of \( \text{Circ}(a, b, 0, \ldots, 0, b) \) was obtained by O. Rojo in [11] under the hypothesis of strictly diagonally dominance, that is \( |a| > 2|b| \), which implies \( a \neq -2b \cos \left( \frac{2\pi k}{n} \right) \neq 0 \) for any \( k = 0, \ldots, \lfloor \frac{n-1}{2} \rfloor \). Here we obtain \( \text{Circ}(a, b, 0, \ldots, 0, b)^{\#} \) with no additional hypotheses, except \( b \neq 0 \), of course.

**Corollary 4.4.** For \( a, b \in \mathbb{C}, b \neq 0 \), the circulant matrix \( \text{Circ}(a, b, 0, \ldots, 0, b) \) is invertible iff

\[
\prod_{k=0}^{[\frac{n-1}{2}]} \left( a + 2b \cos \left( \frac{2\pi k}{n} \right) \right) \neq 0
\]

and moreover, \( \text{Circ}(a, b, 0, \ldots, 0, b)^{\#} = \text{Circ}(ka, b, 0, b) \) where

\[
k_j(a, b, 0, b) =
\]
\[
\begin{aligned}
&\begin{cases}
\frac{1}{12bn} (6(j-1)(n+1-j)+1-n^2), & \text{if } a=-2b, \\
\frac{(-1)^j}{12bn} (6(j-1)(n+1-j)+1-n^2), & \text{if } n \text{ is even and } a=2b, \\
(n+1-2j) \cos \left( \frac{2nkj}{n} \right) - (n+3-2j) \cos \left( \frac{2nkj}{n} \right), & \text{if } a=-2b \cos \left( \frac{2nkj}{n} \right), \\
\frac{4nb \sin^2 \left( \frac{2nkj}{n} \right)}{2b \left( -1 \right)^n - T_n \left( \frac{n}{2b} \right)}, & \text{otherwise}. 
\end{cases}
\end{aligned}
\]

We can compare the obtained expression for \(k(2zp, -1, 0, -p^2)\) for any \(z \in \mathbb{C}\) and any \(p \in \mathbb{C}^*\), with the one given in part (ii) of Lemma 2.1. We thus generate some surprising identities that are new as the authors’ knowledge. It is also a good example of the complexity of the identities in part (ii) of Lemma 2.1.

**Corollary 4.5.** Given \(p \in \mathbb{C}^*\), for any \(j = 1, \ldots, n\), the following identities hold:

(i) If \(z \notin R_n(p)\), then
\[
\sum_{r \in R_n} \frac{r^j}{2zpr - p^2r^2 - 1} = \frac{n(U_{n-j}(z) + p^n U_{j-2}(z))}{2p \left( T_n(z) - T_n \left( \frac{p^2-1}{2p} \right) \right)}.
\]

(ii) If \(p^2 \notin R_n\), then for any \(s \in R_n\)
\[
\sum_{r \in R_n(s)} \frac{r^j}{(s-r)(p^2r^2 - s)} = \frac{\sqrt{2}}{2(p^2s^2 - 1)} \left( \frac{2np^{2(n+1)-j} s^{2(1-j)}}{p^{2n} - 1} - \frac{2p^2s^2}{p^{2n} - 1} - 2j - n - 1 \right).
\]

(iii) If \(p^2 \in R_n\), then for any \(s \in R_n \setminus \{z^p\}\),
\[
\sum_{r \in R_n(s)} \frac{r^j}{(s-r)(p^2r^2 - s)} = \frac{1}{4p^j \sin^2(\theta)} \left( (n+3-2j) \cos(j\theta) - (n+1-2j) \cos((j-2)\theta) \right),
\]

where \(\cos(\theta) = \Re(ps)\).

(iv) If \(p \in R_n\), then
\[
\sum_{r \in R_n(\{z\})} \frac{r^j}{(pr - 1)^2} = \frac{p^j}{12} \left( 6(j-1)(n+1-j) + 1 - n^2 \right).
\]

(v) If \(-p \in R_n\), then
\[
\sum_{r \in R_n(\{-p\})} \frac{r^j}{(pr + 1)^2} = \frac{(-p)^j}{12} \left( 6(j-1)(n+1-j) + 1 - n^2 \right).
\]

**Proof.** Theorem 4.1 and Lemma 2.1 imply that
\[
k(2zp, -1, 0, -p^2) = \frac{1}{2np} \sum_{r \in R_n} (z - \varphi(pr))^# f(r),
\]
that in turn, for any \(j = 1, \ldots, n\), implies that
\[
k_j(2zp, -1, 0, -p^2) = \frac{1}{n} \sum_{r \in R_n} r^j (2zpr - 1 - p^2r^2)^#.
\]
Conclusion

We have obtained the inverse and the group inverse of any circulant matrix depending on four parameters Circ(a, b, c, . . . , c, d) by solving the problem for matrices Circ(2zp, 1, 0, . . . , 0, −ρ²) where a, b, c, d, z, ρ ∈ C. The main tool has been to solve second order difference equations. As a consequence, we have the expression for the entries of the inverse, or the group inverse, of any circulant matrix of order four. In addition, we complete to the singular case, the study of the problem for matrices of type Circ(a, b, c, . . . , c, b) carried on by the authors in [2]. Moreover, we have removed the diagonal dominance hypothesis of some results in [11] and finally, some interesting yet surprising identities are derived.

Appendix

We enumerate some of the properties of second order linear difference equations with constant coefficients, see [7] and also [1] for the proofs. Given z ∈ C, ρ ∈ C* and w ∈ C^n, we are interested in finding h ∈ C^n such that

\[ w_j = -\rho^2 h_{j-1} + 2zph_j - h_{j+1}, \quad j = 2, \ldots, n - 1. \]  

(5)

In this case, h is named solution of the difference equation. Clearly, the choice of h₀, h₁ ∈ C uniquely determines h₁, j = 2, . . . , n and hence the solution h. When w = 0, Equation (5) is named homogeneous and then the set of its solutions is a (complex) 2-dimensional vector space.

Observe that Equation (5) encompasses all second order linear difference equations with constant coefficients, in the sense that, given \( \hat{z} \in C, \) a, b ∈ C* and \( \hat{w} \in C^n, \) then h ∈ C^n is a solution of the difference equation

\[ \hat{w}_j = -bh_{j-1} + 2\hat{z}h_j - ah_{j+1}, \quad j = 2, \ldots, n - 1, \]

iff it satisfies (5) where \( w = a^{-1} \hat{w}, z = (ab)^{−\frac{1}{2}} \hat{z} \) and \( \rho = (a^{-1}b)^{\frac{1}{2}}. \) Moreover, Equation (5) has interest of itself. For instance, in the case in which \( 2zp, \rho^2 \in Z, \) the homogeneous equation is related with many problems in Combinatorics and Enumerative Geometry.

The usual treatment of Equation (5) involves the so-called Binet Formula, that is based on the roots of the characteristic polynomial \( P(w) = -w^2 + 2zpw - \rho^2 \in C[w] \) and depends on the nature of such roots. However, it is more convenient to express the solutions of (5) in a more closed form that involves Chebyshev polynomials, see [7, Section 8] for a more complete explanation. This is the way followed in previous works by some of the authors, see [1, 2].

First, remember that a Chebyshev equation or Chebyshev recurrence corresponds to Equation (5) when ρ = 1. Our methodology is based on the simple fact that Equation (5), and so any second order linear equation, is equivalent to a Chebyshev equation: h ∈ C^n is a solution of Equation (5) iff h_j = ρ^k j, where k ∈ C^n is a solution of the Chebyshev equation −k_j + 2zk_j − k_{j+1} = ρ^{−(j−1)}w_j, j = 2, . . . , n − 1.

A Chebyshev sequence is a sequence of polynomials \( \{Q_n(z)\}_{n \in Z} \subset C[z] \) that satisfies the recurrence

\[ Q_{n+1}(z) = 2zQ_n(z) - Q_{n-1}(z), \quad \text{for each } n \in Z. \]

(13)

Notice that \( \{Q_n(z)\}_{n \in Z} \) is a Chebyshev sequence iff \( \{Q_{n+m}(z)\}_{n \in Z} \) is also a Chebyshev sequence for any m ∈ Z. In addition, recurrence (13) shows that any Chebyshev sequence is uniquely determined by the choice of polynomials of order zero and one, Q₀ and Q₁, respectively. In particular, the sequences \( \{T_n\}_{n=-\infty}^{\infty} \) and \( \{U_n\}_{n=-\infty}^{\infty} \) denote the first and second kind Chebyshev polynomials that are obtained when we choose \( T_0(z) = U_0(z) = 1, \) \( T_1(z) = z, \) \( U_1(z) = 2z. \)

The main role played by the first and second kind Chebyshev polynomials is shown by taking into account that if \( \{Q_n\}_{n=-\infty}^{\infty} \) is any Chebyshev sequence, there exist \( a, b \in C \) such that \( Q_n = aT_n + bU_{n-1}, \) for any \( n \in Z. \) Besides, it is easy to prove that for any \( n \in Z \) we have \( T_{2n}(0) = U_{2n}(0) = (-1)^n, \) \( T_{2n+1}(0) = U_{2n+1}(0) = 0, \)
In view of the importance of Chebyshev polynomials to solve Equation (5), we next enumerate other properties and relations between them that are well described in the literature, see for instance [10].

First, for any $k \in \mathbb{Z}$, we have that

$$T_k'(z) = kU_{k-1}(z), \quad z \in \mathbb{C} \text{ and } U_k'(z) = \frac{(n+1)T_{k+1}(z) - zU_k(z)}{z^2 - 1}, \quad z \in \mathbb{C} \setminus \{\pm 1\} \quad (14)$$

and moreover, $U_k'(1) = \frac{1}{2}k(k+1)(k+2)$ and $U_k'(-1) = \frac{(-1)^k}{2}k(k+1)(k+2)$. In addition, for any $z \in \mathbb{C}$ we have

$$2T_k(z)U_m(z) = U_{k+m}(z) + U_{m-k}(z) \quad \text{and} \quad (15)$$

$$2(z^2 - 1)U_k(z)U_m(z) = T_{k+m+2}(z) - T_{k-m}(z) \quad k, m \in \mathbb{Z}. \quad \text{In particular, taking into account that } U_0 = 1 \text{ and that } U_{-k} = -U_{k-2} \text{ from the above first identity, we also obtain that} \quad (16)$$

$$T_k(z) = \frac{1}{2}\left(U_k(z) - U_{k-2}(z)\right) = zU_{k-1}(z) - U_{k-2}(z) = U_k(z) - zU_{k-1}(z) \quad k \in \mathbb{N};$$

and from the second one,

$$T_k(z) = \frac{1}{2}\left(T_{k+2}(z) - T_k(z)\right) = zT_{k+1}(z) - T_k(z) \quad (17)$$

The roots of the Chebyshev polynomials $T_k$ and $U_k$ can be easily computed from the following identities:

$$T_m(\cos(\theta)) = \cos(m\theta) \quad \text{and} \quad U_m(\cos(\theta)) = \frac{\sin(m\theta)}{\sin(\theta)}, \quad m \in \mathbb{N}. \quad (18)$$

So, $T_n(z) = 0 \text{ iff } z = \cos \left( \frac{\pi k}{n} \right)(2k - 1), \quad k = 1, \ldots, n$, whereas $U_n(z) = 0 \text{ iff } z = \cos \left( \frac{\pi k}{n+1} \right), \quad k = 1, \ldots, n$. In this case, $U_{n+m}(z) = (-1)^kU_{m-1}(z)$ for any $m \in \mathbb{Z}$. Notice that all roots of $T_n$ and $U_n$ are simple.

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