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Lifan Guan and Ronggang Shi

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Abstract

For a one-parameter subgroup action on a finite-volume homogeneous space, we consider the set of points admitting divergent-on-average trajectories. We show that the Hausdorff dimension of this set is strictly less than the manifold dimension of the homogeneous space. As a corollary we know that the Hausdorff dimension of the set of points admitting divergent trajectories is not full, which proves a conjecture of Cheung [Hausdorff dimension of the set of singular pairs, Ann. of Math. (2) 173 (2011), 127–167].

1. Introduction

Let $G$ be a connected Lie group, $\Gamma$ be a lattice of $G$ and $F = \{f_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of $G$. A discrete subgroup $\Gamma \leq G$ is called a lattice if there exists a finite $G$-invariant measure on the homogeneous space $G/\Gamma$. The action of $F$ on the homogeneous space $G/\Gamma$ by left translation defines a flow. In this paper we consider the dynamics of the semiflow given by the action of $F^+ \overset{\text{def}}{=} \{f_t : t \geq 0\}$. For $x \in G/\Gamma$ we say the trajectory $F^+x \overset{\text{def}}{=} \{f_t x : t \geq 0\}$ is divergent if $f_t x$ leaves any fixed compact subset of $G/\Gamma$ provided $t$ is sufficiently large. We say $F^+ x$ is divergent on average if for any characteristic function $1_K$ of a compact subset $K$ of $G/\Gamma$ one has

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T 1_K(f_t x) \, dt = 0.$$ 

Clearly, if the trajectory $F^+ x$ is divergent, then it is divergent on average. The aim of this paper is to understand the set of divergent points

$$\mathcal{D}(F^+, G/\Gamma) \overset{\text{def}}{=} \{x \in G/\Gamma : F^+ x \text{ is divergent}\},$$

and the set of divergent on average points

$$\mathcal{D}(F^+, G/\Gamma) \overset{\text{def}}{=} \{x \in G/\Gamma : F^+ x \text{ is divergent on average}\},$$

in terms of their Hausdorff dimensions. Here the Hausdorff dimension is defined by attaching $G/\Gamma$ with a Riemannian metric. It is well known that different choices of Riemannian metrics will not affect the Hausdorff dimension of subsets of $G/\Gamma$. Indeed, a specific Riemannian metric will be used later for the sake of convenience.

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According to the work of Margulis [Mar75] and Dani [Dan84, Dan86], if $F$ is Ad-unipotent then the space $G/\Gamma$ admits no divergent on average trajectories of $F^+$. In other words, the set $\mathcal{D}(F^+, G/\Gamma)$, hence the set $\mathcal{D}(F^+, G/\Gamma)$, is empty. On the other hand, the set $\mathcal{D}(F^+, G/\Gamma)$ can be complicated when $F$ is Ad-diagonalizable. For example, it was proved by Cheung in [Che11] that the Hausdorff dimension of $\mathcal{D}'(F^+, SL_3(\mathbb{R})/SL_3(\mathbb{Z}))$ with $F = \{\text{diag}(e^t, e^t, e^{-2t}) : t \in \mathbb{R}\}$ is equal to $7\frac{1}{4}$. Based on his results, Cheung raised the following conjecture in [Che11].

**Conjecture 1.1.** Let $\Gamma$ be a lattice of a connected Lie group $G$ and let $F = \{f_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of $G$. Then the Hausdorff dimension of $\mathcal{D}'(F^+, G/\Gamma)$ is strictly less than the manifold dimension of $G/\Gamma$.

Since Ad-unipotent flows admit no divergent trajectories, the conjecture is actually about nonquasiunipotent flows, i.e., the flows $F$ such that $\text{Ad}(F)$ has a nontrivial diagonal part (see Lemma 1.3 for the related concepts). It is proved by Kleinbock and Margulis [KM96] that for all nonquasiunipotent flows, the set of points that have bounded trajectories have full Hausdorff dimension. Divergent trajectories can be thought of as the dual to bounded trajectories, and Conjecture 1.1 asks if these two dynamical properties always have opposite characterizations.

The conjecture is known to be true in the following cases where $G$ is a semisimple Lie group without compact factors and $F$ is Ad-diagonalizable:

(i) $G$ is of rank one [Dan85];
(ii) $G = \prod_{i=1}^k \text{SO}(n, 1)$ where $n \geq 2$, $\Gamma = \prod_{i=1}^k \Gamma_i$ where each $\Gamma_i$ is a lattice in $\text{SO}(n, 1)$ and $F \leq G$ is the diagonal embedding of an $\mathbb{R}$-split torus of $\text{SO}(n, 1)$ [Che07, Yan19];
(iii) $G/\Gamma = \text{SL}_{m+n}(\mathbb{R})/\text{SL}_{m+n}(\mathbb{Z})$ and $F = F_{n,m} = \{\text{diag}(e^{nt}, \ldots, e^{nt}, e^{-mt}, \ldots, e^{-mt}) : t \in \mathbb{R}\}$ where $m, n \geq 1$ [Che11, CC16, KKLM17].

Indeed, for all the cases listed above, the Hausdorff dimensions of the corresponding $\mathcal{D}'(F^+, G/\Gamma)$ have been determined.

There is evidence that a stronger version of this conjecture is true. It was proved by Einsiedler and Kadyrov in [EK12] that the Hausdorff dimension of $\mathcal{D}(F^+, SL_3(\mathbb{R})/SL_3(\mathbb{Z}))$ is at most $7\frac{1}{2}$ when $F = F_{1,2}$ as in (iii). Using the contraction property of the height function introduced in [EMM98], it was proved by Kadyrov, Kleinbock, Lindenstrauss and Margulis in [KKLM17] that for any $m, n \geq 1$, the Hausdorff dimension of $\mathcal{D}(F^+, SL_{m+n}(\mathbb{R})/SL_{m+n}(\mathbb{Z}))$ is at most $\dim G - mn/(m + n)$ when $F = F_{n,m}$ as in (iii). See also [ELMV12, Kad12, KP17, LSST19, DFSU17, Wei04] for related results.

Now we state the main result of this paper, from which Cheung’s conjecture follows.

**Theorem 1.2.** Let $\Gamma$ be a lattice of a connected Lie group $G$ and let $F = \{f_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of $G$. Then the Hausdorff dimension of $\mathcal{D}(F^+, G/\Gamma)$ is strictly less than the manifold dimension of $G/\Gamma$.

We will reduce the proof of Theorem 1.2 to the special case where $G$ is a semisimple linear group. Recall that a connected semisimple Lie group $G$ contained in $\text{SL}_k(\mathbb{R})$ has a natural structure of real algebraic group. So terminologies of algebraic groups have natural meanings for $G$ and are independent of the embeddings of $G$ into $\text{SL}_k(\mathbb{R})$. In particular, the one-parameter group $F$ has the following real Jordan decomposition which is a special case of [Bor91, Theorem 4.4].
Lemma 1.3. Let $G \leq \text{SL}_k(\mathbb{R})$ be a connected semisimple Lie group. For any one-parameter subgroup $F = \{f_t : t \in \mathbb{R}\}$, there are uniquely determined one-parameter subgroups

$$K_F = \{k_t : t \in \mathbb{R}\}, \quad A_F = \{a_t : t \in \mathbb{R}\} \quad \text{and} \quad U_F = \{u_t : t \in \mathbb{R}\}$$

with the following properties:

- $f_t = k_t a_t u_t$;
- $K_F$ is bounded, $A_F$ is $\mathbb{R}$-diagonalizable and $U_F$ is unipotent;
- all the elements of $K_F, A_F$ and $U_F$ commute with each other.

The subgroups $K_F, A_F$ and $U_F$ are called compact, diagonal and unipotent parts of $F$, respectively. In §2 we will reduce the proof of Theorem 1.2 to its following special case which contains the main unknown situations.

Theorem 1.4. Let $G \leq \text{SL}_k(\mathbb{R})$ be a connected center-free semisimple Lie group without compact factors. Let $F = \{f_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of $G$ such that the compact part $K_F$ is trivial but the diagonal part $A_F$ is nontrivial. We assume the following hold:

- $G = \prod_{i=1}^m G_i$ is a direct product of connected normal subgroups $G_i$;
- $\Gamma = \prod_{i=1}^m \Gamma_i$ where each $\Gamma_i$ is a nonuniform irreducible lattice of $G_i$;
- the group $A_F$ has nontrivial projection to each $G_i$.

Then the Hausdorff dimension of $\mathcal{D}(F^+, G/\Gamma)$ is strictly less than the manifold dimension of $G/\Gamma$.

The proof of Theorem 1.4 is from §3 till the end of the paper. Indeed, during the proof we will give an algorithm to calculate explicitly the upper bound of the Hausdorff dimension in the setting of Theorem 1.4. Roughly speaking, the dimension gap is given by an explicitly calculable contraction rate of a height function defined in §4. We believe that an optimal contraction rate makes it possible to give a sharp upper bound. The upper bound we give is not sharp in general, since we do not try to calculate the optimal contraction rate.

The basic strategy for proving Theorem 1.4 is the same as in [KKLM17]. Indeed, when $F$ is diagonalizable, i.e. $F = A_F$, Theorem 1.2 can be established using the strategy developed in [KKLM17] and the contraction property of the height function proved in [Shi14]. The Eskin–Margulis height function (abbreviated as EM height function) is introduced in [EM04] and serves as our main tool. But when $F$ has nontrivial unipotent parts, essential new ideas are needed. Indeed, when $U_F$ is nontrivial, we can no longer reduce the study of $\mathcal{D}(F^+, G/\Gamma)$ to its intersections with unstable manifolds $G^+ x$ (see §3 for the precise definition of $G^+$) as in [KKLM17] due to the existence of nontrivial $U_F$. Namely, we need to consider the intersection of $\mathcal{D}(F^+, G/\Gamma)$ with $G^c G^+ x$ where $G^c$ is the connected component of the centralizer of $A_F$.

Using the method of [KKLM17] and the EM height function, it is not hard to give a nontrivial upper bound for the dimension of $\mathcal{D}(F^+, G/\Gamma) \cap z G^+ x$ for each fixed $z \in G^c$. But the Hausdorff dimension of a set can be full even if the dimension of every slice is uniformly bounded by a nontrivial constant. So the main difficulty lies in the fact that there are no clear relations among those $\mathcal{D}(F^+, G/\Gamma) \cap z G^+ x$ with different $z$. We manage to deal with this issue by showing that for any $\delta$ sufficiently small the intersection $\mathcal{D}(F^+, G/\Gamma) \cap B_\delta z B_1^+ x$ can be covered by $\delta^{-(m-\epsilon)}$ balls of radius $\delta$, where $m$ is the manifold dimension of $G^+$ and $\epsilon > 0$. Here $B_\delta^c$ and $B_1^+$ denote the metric ball of radius $\delta$ and 1 centered at $1_G$ in $G^c$ and $G^+$, respectively. This argument is carried out in the last two sections. Sections 3 and 4 are devoted to prove a uniform contraction property for a family of one-parameter subgroups with respect to the EM height function that are needed in the last two sections.
2. Proof of Theorem 1.2

In this section we prove Theorem 1.2 assuming Theorem 1.4. Let $G, \Gamma, F$ be as in Theorem 1.2. We choose and fix a Euclidean norm $\| \cdot \|$ on the Lie algebra $g$ of $G$, which induces a right invariant Riemannian metric $\dist(\cdot, \cdot)$ on $G$. Moreover, this metric naturally induces a metric on $G/\Gamma$, also denoted by ‘dist’, as follows:

$$\dist(g\Gamma, h\Gamma) = \inf_{\gamma \in \Gamma} \dist(g\gamma, h) \quad \text{where } g, h \in G.$$  

Let $r$ be the maximal amenable ideal of the Lie algebra $g$ of $G$, i.e., the largest ideal whose analytic subgroup is amenable. The adjoint action of $G$ on $s = g/r$ defines a homomorphism $\pi : G \to \text{Aut}(s)$. Let $S$ be the connected component of $\text{Aut}(s)$. It follows from the Levi decomposition of $G$ that $\pi(G) = S$ and $S$ is a center-free semisimple Lie group without compact factors. It is known that $\Gamma \cap \text{Ker}(\pi)$ is a cocompact lattice in $\text{Ker}(\pi)$ and $\pi(\Gamma)$ is a lattice in $S$, see e.g. [BQ12, Lemma 6.1]. Therefore, the induced map $\overline{\pi} : G/\Gamma \to S/\pi(\Gamma)$ is proper, i.e., the preimage of a compact subset is still compact. Thus, a trajectory $F^+x$ is divergent on average if and only if the trajectory $\overline{\pi}(F)\overline{\pi}(x)$ is divergent on average, and consequently

$$\overline{\pi}(\mathcal{D}(F^+, G/\Gamma)) = \mathcal{D}(\pi(F^+), S/\pi(\Gamma)).$$  

(2.1)

Let $\varphi : s \to g$ be an embedding of Lie algebras such that $d\pi \circ \varphi$ is the identity map. It follows from (2.1) that for any $x \in G/\Gamma$ and any $v \in s$

$$\exp(v)\overline{\pi}(x) \in \mathcal{D}(\pi(F^+), S/\pi(\Gamma))$$

if and only if

$$\exp(\varphi(v))\exp(v')x \in \mathcal{D}(F^+, G/\Gamma) \quad \text{for all } v' \in r.$$

For any $x \in G/\Gamma$, there is a neighborhood $U_x$ of $x$ of the form $U_x = \exp(V_x)\exp(\varphi(W_x))x$, where $V_x$ (respectively $W_x$) is an open neighborhood of 0 in $s$ (respectively $r$) on which the exponential map is a bi-Lipschitz diffeomorphism onto its image. Here the metrics on $s$ and $r$ are given by Euclidean structures induced from that of $g$. Note that $\mathcal{D}(F^+, G/\Gamma) \cap U_x$ is equal to

$$\{\exp(v)\exp(w)x : v \in V_x, \exp(v)\varphi(x) \in \mathcal{D}(\pi(F^+), S/\pi(\Gamma)), w \in W_x\}.$$

Since the Hausdorff dimension is preserved by bi-Lipschitz maps and the induced metrics on homogeneous spaces are locally given by the metrics on Lie groups, we have

$$\dim \mathcal{D}(F^+, G/\Gamma) \cap U_x = \dim \{v \in V_x : \exp(v)\varphi(x) \in \mathcal{D}(\pi(F^+), S/\pi(\Gamma))\} \times W_x$$

$$\leq \dim \{v \in V_x : \exp(v)\varphi(x) \in \mathcal{D}(\pi(F^+), S/\pi(\Gamma))\} + \dim r$$

$$= \dim \mathcal{D}(\pi(F^+), S/\pi(\Gamma)) \cap \exp(V_x)\varphi(x) + \dim r$$

$$\leq \dim \mathcal{D}(\pi(F^+), S/\pi(\Gamma)) + \dim r,$$

(2.2)

where Marstrand’s product theorem (see, e.g. [BP17, Theorem 3.2.1]) is used to get the first inequality. Assuming that $\dim \mathcal{D}(\pi(F^+), S/\pi(\Gamma)) < \dim s$, then in view of (2.2) we have

$$\dim \mathcal{D}(F^+, G/\Gamma) = \sup_{x \in G/\Gamma} \dim \mathcal{D}(F^+, G/\Gamma) \cap U_x$$

$$\leq \dim \mathcal{D}(\pi(F^+), S/\pi(\Gamma)) + \dim r$$

$$< \dim s + \dim r = \dim G.$$

Hence to prove Theorem 1.2 it suffices to give a nontrivial upper bound of the Hausdorff dimension of $\mathcal{D}(\pi(F^+), S/\pi(\Gamma))$. We summarize what we have obtained as follows.
Lemma 2.1. Theorem 1.2 is equivalent to its special case where $G$ is a center-free semisimple Lie group without compact factors.

Proof of Theorem 1.2 modulo Theorem 1.4. According to Lemma 2.1, it suffices to prove the theorem under the additional assumption that $G$ is a center-free semisimple Lie group without compact factors. Under this assumption, the adjoint representation $\text{Ad} : G \to \text{SL}(g)$ is a closed embedding. According to the real Jordan decomposition in Lemma 1.3, the compact part $K_F$ does not affect the divergence on average property of the trajectories. So we assume without loss of generality that $K_F$ is trivial.

There exist finitely many connected semisimple subgroups $G_i$ such that $G = \prod_i G_i$ and $\Gamma_i \overset{\text{def}}{=} \Gamma \cap G_i$ is an irreducible lattice of $G_i$ for each $i$. It follows that $\prod_i \Gamma_i$ is a finite-index subgroup of $\Gamma$ and the natural quotient map $\Gamma \to G/\prod_i \Gamma_i \to G/\Gamma$ is proper. So we assume moreover that $\Gamma = \prod_i \Gamma_i$.

Denote by $\pi_j$ the projection of $G$ to $G/G_j = \prod_{i \neq j} \Gamma_i$ and denote by $\overline{\pi}_j$ the induced map from $G/\Gamma$ to $\pi_j(G)/\pi_j(\Gamma) = \prod_{i \neq j} \Gamma_i$. Here if $G = G_j$ we interpret $\prod_{i \neq j} \Gamma_i$ as a trivial group and $\prod_{i \neq j} \Gamma_i/\Gamma_i$ as a single point set. If $G_j/\Gamma_j$ is compact or the projection of $A_F$ to $G_j$ is trivial, then

$$\mathcal{D}(F^+, G/\Gamma) = \overline{\pi}_j^{-1}\left(\mathcal{D}\left(\pi_j(F^+, \prod_{i \neq j} \Gamma_i)\right)\right).$$

So either $\mathcal{D}(F^+, G/\Gamma)$ is an empty set or we finally can reduce the problem to the setting of Theorem 1.4 where each $\Gamma_i$ is a nonuniform lattice and the projection of $A_F$ to each $G_i$ is nontrivial. This completes the proof. \qed

3. Preliminary on linear representations

From this section, we start the proof of Theorem 1.4. At the beginning of each section we will set up some notation that will be used later. Let $G$ and $F$ be as in Theorem 1.4. Let $A_F = \{a_t : t \in \mathbb{R}\}$ and $U_F = \{u_t : t \in \mathbb{R}\}$ be the diagonal and the unipotent parts of $F$, respectively. Let $H$ be the unique connected normal subgroup of $G$ such that $A_F \leq H$ and the projection of $A_F$ to each simple factor of $H$ is nontrivial. Since $A_F$ is nontrivial and $G$ is center-free, $H$ is the (nontrivial) product of some simple factors of $G$. Hence $H$ is a semisimple Lie group without compact factors. Let $S$ be the product of simple factors of $G$ not contained in $H$. Then $S$ is a semisimple and normal subgroup of $G$ that commutes with $H$. Moreover, $G = HS$ and $H \cap S = 1_G$, where $1_G$ is the neutral element of $G$.

In this section we will prove a couple of auxiliary results for a finite-dimensional linear representation $\rho : G \to \text{GL}(V)$ on a (nontrivial real) normed vector space $V$. These results will be used in the next section to prove a uniform contracting property of the EM height function. We will use $\| \cdot \|$ to denote the norm on $V$.

For $\lambda \in \mathbb{R}$, we denote the $\lambda$-Lyapunov subspace of $A_F$ by

$$V^\lambda = \{v \in V : \rho(a_t)v = e^{\lambda t}v\}.$$

Recall that if $V^\lambda \neq \{0\}$, then $\lambda$ is a called an Lyapunov exponent of $(\rho, V)$. Since $U_F$ commutes with $A_F$, every Lyapunov subspace $V^\lambda$ is $U_F$-invariant. As $A_F$ is $\mathbb{R}$-diagonalizable, the space $V$ can be decomposed as $V^+ \oplus V^0 \oplus V^-$ where

$$V^+ = \bigoplus_{\lambda > 0} V^\lambda \quad \text{and} \quad V^- = \bigoplus_{\lambda < 0} V^\lambda.$$
Now we consider the adjoint representation of $G$ on the Lie algebra $\mathfrak{g}$ of $G$. It is easily checked that $\mathfrak{g}^+, \mathfrak{g}^{-}$ and $\mathfrak{g}^c \overset{\text{def}}{=} \mathfrak{g}^0$ are subalgebras of $\mathfrak{g}$. The connected subgroup $G^+$ (respectively $G^-$) with Lie algebras $\mathfrak{g}^+$ (respectively $\mathfrak{g}^-$) is called unstable (respectively stable) horospherical subgroup of $a_1$. We denote the connected component of the centralizer of $a_1$ in $G$ by $G^0$ whose Lie algebra is $\mathfrak{g}^c$. Let $d, d^c, d^-$ be the manifold dimensions of $G^+, G^c$ and $G^-$, respectively. It follows from the nontriviality of $A_F$ that $d > 0$.

For $r \geq 0$ we let $B_r^G = \{ h \in G : \text{dist}(h, 1_G) < r \}$, $B_r^\pm = \{ h \in G^\pm : \text{dist}(h, 1_G) < r \}$ and $B_r^c = \{ h \in G^c : \text{dist}(h, 1_G) < r \}$. By rescaling the Riemannian metric if necessary, we may assume that:

(i) the product map $B_1^1 \times B_1^c \times B_1^\pm \to G$ is a bi-Lipschitz diffeomorphism onto its image;

(ii) the logarithm map is well defined on $B_1^G$ and is a bi-Lipschitz diffeomorphism onto its image.

According to (i), it is safe to identify the direct product $B_1^1 \times B_1^c \times B_1^\pm$ with its image $B_1^1 B_1^c B_1^\pm \subset G$ and we will mainly use the latter notation for sake of convenience. The same statement as (ii) also holds for $B_1^\pm$ and $B_1^c$.

We fix a Haar measure $\mu$ on $G^+$ normalized with $\mu(B_1^+) = 1$. Since the metric ‘dist’ is right invariant, any open ball of radius $r$ in $G^+$ has the form $B_r^+ h (h \in G^+)$ and there exists $C_0 \geq 1$ such that

$$C_0^{-1} r^d \leq \mu(B_r^+ h) = \mu(B_r^+) \leq C_0 r^d \quad \text{for all } 0 \leq r \leq 1.$$  \hfill (3.1)

For $g, h \in G$ we let $g^h = h^{-1} gh$. For $z \in G^c$, let $F_z = \{ f^z_t : t \in \mathbb{R} \}$ and $F^+_z = \{ f^z_t : t \geq 0 \}$. Note that $f^z_t = a_t u^z_t$ and

$$\{ u^z_t : z \in B_1^c \} \text{ is relatively compact.} \quad \hfill (3.2)$$

**Lemma 3.1.** Let $\rho : G \to \text{GL}(V)$ be a representation on a finite-dimensional normed vector space $V$. Let $\lambda$ be a Lyapunov exponent of $(\rho, V)$. For any $\delta > 0$, there exists $T_\delta > 0$ such that, for all $t \geq T_\delta$, $z \in B_1^c$ and unit vector $v \in V^\lambda$ we have

$$e^{(\lambda-\delta)t} \leq \|\rho(f^z_t)v\| \leq e^{(\lambda+\delta)t}. \quad \hfill (3.3)$$

**Proof.** For all $v \in V^\lambda$ with $\|v\| = 1$ we have $\|\rho(a_t)v\| = e^{\lambda t}$. On the other hand, in view of (3.2), there exist $C > 0$ and $n \in \mathbb{N}$ (in this paper $\mathbb{N} = \{1, 2, 3, \ldots\}$) such that

$$\|\rho(a_t^z)v\| \leq C(\|t\| + 1)^n$$

for all $z \in B_1^c$ and $t \in \mathbb{R}$. Therefore, for any unit vector $v \in V^\lambda$, $z \in B_1^c$ and sufficiently large $t$,

$$\|\rho(f^z_t)v\| \geq \|\rho(u^z_t)^{-1}\|\rho(a_t)v\| \geq C^{-1}(\|t\| + 1)^{-n} e^{\lambda t} \geq e^{(\lambda-\delta)t},$$

$$\|\rho(f^z_t)v\| \leq \|\rho(u^z_t)^{-1}\|\|\rho(a_t)v\| \leq C(\|t\| + 1)^n e^{\lambda t} \leq e^{(\lambda+\delta)t}. \quad \square$$

Since $z \in G^c$, it follows from the above discussion that the adjoint action of $f^z_t$ preserves each Lyapunov subspaces, and hence it preserves $\mathfrak{g}^+, \mathfrak{g}^-$ and $\mathfrak{g}^c$. Consequently, the conjugate action of $f^z_t$ preserves $G^+, G^-, G^c$. Moreover, we have the following.

**Lemma 3.2.** Let $\lambda$ be the top Lyapunov exponent of $A_F$ in the representation $(\text{Ad}, \mathfrak{g})$, i.e. $\lambda = \max\{ r \in \mathbb{R} : \mathfrak{g}^c \neq \{0\} \}$. Then for any $\delta > 0$, there exists $T_\delta > 0$ such that, for all $t \geq T_\delta$ and $z \in B_1^c$, we have

$$B^+_{e^{-(\lambda+\delta)t} r} \subset f^z_{-r} B^+_r f^z_t \subset B^+_{e^{-\delta t} r} \quad \text{for all } r \leq 1,$$

$$B^c_{e^{-\delta t} r} \subset f^z_{-r} B^c_r f^z_t \subset B^c_{e^{\lambda t} r} \quad \text{for all } r \leq e^{-\delta t}. \quad \hfill (3.4)$$

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Lemma 3.3. An open dense subset of $G$.

Proof. We only give detailed proofs for the second inclusion of (3.5). The proofs of the remaining inclusions are similar, and hence are omitted.

By assumption, the logarithm map from $B_1^G$ to $g$ is a bi-Lipschitz diffeomorphism onto its image. Hence there exists $R \geq 1$ such that, for any $h_1, h_2 \in B_1^G$, we have

$$R^{-1} \text{dist}(\log h_1, \log h_2) \leq \text{dist}(h_1, h_2) \leq R \text{dist}(\log h_1, \log h_2).$$

By applying Lemma 3.1 to the adjoint representation of $G$, we know that there exists $T_{\delta/2} > 2\log R$ such that, for all $t \geq T_{\delta/2}$, $v \in g^c$ and $z \in B_1^c$, we have

$$\|\text{Ad}(f_{-t}^z)v\| \leq e^{\delta t/2}\|v\|.$$

By taking $T_\delta' \geq T_{\delta/2}$ large enough, we may assume that $\text{Ad}(f_{-t}^z)v \in log B_1^c$ for all $t \geq T_\delta'$ and $v \in log B_1^c$ where $r \leq e^{-\delta t}$. Thus for $r \leq e^{-\delta t}$ and $h \in B_1^c$,

$$\text{dist}(f_{-t}^z h f_t^z, 1_G) = \text{dist}(f_{-t}^z \exp(\log h) f_t^z, 1_G) = \text{dist}(\exp(\text{Ad}(f_{-t}^z) \log h), 1_G)$$

$$\leq R \text{dist}(\text{Ad}(f_{-t}^z) \log h, 0) \leq R e^{\delta t/2} \text{dist}(\log h, 0) \leq R^2 e^{\delta t/2} \text{dist}(h, 1_G) < e^{\delta t} r.$$

This completes the proof of the second inclusion in (3.5). \qed

From now on till the end of this section, we assume that $\rho : G \to GL(V)$ is a representation on a finite-dimensional normed vector space $V$ which has no nonzero $H$-invariant vectors. As any two norms on $V$ are equivalent, we also assume without loss of generality that the norm is Euclidean.

Recall that a nonzero $H$-invariant subspace $V'$ of $V$ is said to be $H$-irreducible if $V'$ contains no $H$-invariant subspaces besides $\{0\}$ and itself. The complete reducibility of representations of $H$ implies that there exists a unique decomposition (called $H$-isotropic decomposition)

$$V = V_1 \oplus \cdots \oplus V_m$$

(3.6)

such that irreducible sub-representations of $H$ in the same $V_i$ are isomorphic but irreducible sub-representations in different $V_i$ are nonisomorphic. Since $S$ commutes with $H$, each $V_i$ is $S$-invariant, and hence $G$-invariant. Each $V_i$ is called an $H$-isotropic subspace of $V$.

Let $\lambda_i$ be the top Lyapunov exponent of $A_F$ in $(\rho, V_i)$, i.e.,

$$\lambda_i = \max\{\lambda \in \mathbb{R} : V_i^\lambda \neq \{0\}\}.$$

Since the projection of $A_F$ to each simple factor of $H$ is nontrivial, every $\lambda_i$ is positive. Let $\lambda$ be the minimum of top Lyapunov exponents in all the $V_i$, i.e.,

$$\lambda = \min\{\lambda_i : 1 \leq i \leq m\} > 0.$$

(3.7)

Let $\pi_i : V_i \to V_i^{\lambda_i}$ be the $A_F$-equivariant projection.

Lemma 3.3. For all $v \in V_i \setminus \{0\}$, the map

$$\varphi_v : G^+ \to \mathbb{R} \quad \text{where} \quad \varphi_v(h) = \|\pi_i(\rho(h)v)\|^2$$

is not identically zero.

Proof. Suppose $\varphi_v$ is identically zero. Then $\rho(G^+)v \subset V_i'$ where $V_i' \subset V_i$ is the $A_F$-invariant complimentary subspace of $V_i^{\lambda_i}$. This implies that $\rho(G^- G^+ G^+)v \subset V_i'$. Since $G^- G^+ G^+$ contains an open dense subset of $G$, see e.g. [MT94, Proposition 2.7], we moreover have that $\rho(G) v \subset V_i'$. This is impossible since the intersection of $V_i^{\lambda_i}$ with each $H$-invariant subspace of $V_i$ is nonzero. This contradiction completes the proof. \qed
Lemma 3.4. For all \( v \in V_i \setminus \{0\} \) and \( r \geq 0 \), let
\[
E(v,r) = \{ h \in B^+_1 : \| \pi_i(\rho(h)v) \| \leq r \}.
\]
Then there exists \( \theta_i > 0 \) such that
\[
C_i \overset{\text{def}}{=} \sup_{\| v \|=1, v \in V_i} \sup_{r>0} r^{-\theta_i} \mu(E(v,r)) < \infty. \tag{3.9}
\]
In particular, \( \mu(E(v,0)) = 0 \).

Proof. Since \( G^+ \) is a unipotent group, it is simply connected and by [CG90, Theorem 1.2.10(a)] there is an isomorphism of affine varieties \( \mathbb{R}^d \to G^+ \) such that the Lebesgue measure of \( \mathbb{R}^d \) corresponds to the Haar measure \( \mu \). During the proof, we will identify the group \( G^+ \) with \( \mathbb{R}^d \) for convenience.

By Lemma 3.3, for every nonzero \( v \in V_i \) the map \( \varphi_v \) in (3.8) is a nonzero polynomial map. So \( \varphi_v|_{B^+_1} \) is nonzero. Note that the degrees of \( \varphi_v \) (\( v \in V_i \)) are uniformly bounded from above. Therefore, the \((C,\alpha)\)-good property of polynomials in [BKM01, §3] implies that there exist positive constants \( C \) and \( \alpha \) such that
\[
\mu(E(v,r)) \leq C \left( \frac{r^2}{\sup_{h \in B^+_1} \varphi_v(h)} \right)^\alpha \tag{3.10}
\]
for all nonzero \( v \in V_i \). Since the set of unit vectors of \( V_i \) is compact,
\[
\inf_{\| v \|=1, v \in V_i} \sup_{h \in B^+_1} \varphi_v(h) > 0. \tag{3.11}
\]
So (3.9) follows from (3.10) and (3.11) by taking \( \theta_i = 2\alpha \). \( \square \)

Remark 3.5. According to [BKM01, Lemma 3.2] we have \( \alpha = 1/dl \) where \( d \) is the manifold dimension of \( G^+ \) and \( l \) is a uniform upper bound of the degree of \( \varphi_v \) (\( v \in V_i \)). So the constant \( \theta_i \) can be calculated explicitly.

Lemma 3.6. Let \( \theta_0 = \min_{1 \leq i \leq m} \theta_i \) where \( \theta_i > 0 \) so that Lemma 3.4 holds and let \( \lambda \) be as in (3.7). Then for any \( 0 < \delta < \theta < \theta_0 \), there exists \( T_{\theta,\delta} > 0 \) such that, for all \( t \geq T_{\theta,\delta} \), \( z \in B^+_1 \) and \( v \in V \) with \( \| v \| = 1 \), we have
\[
\int_{B^+_1} \| \rho(f_t^zh)v \|^{-\theta} \, d\mu(h) \leq e^{-(\theta-\delta)\lambda t}. \tag{3.12}
\]

Proof. Without loss of generality, we assume further that the Euclidean norm \( \| \cdot \| \) on \( V \) satisfies the following properties:

- the Lyapunov subspaces of \( A_F \) are orthogonal to each other;
- the \( H \)-isotropic subspaces \( V_i \) (\( 1 \leq i \leq m \)) are orthogonal to each other.

Let
\[
R_i = \sup_{v \in V_i, \| v \|=1} \| \pi_i(\rho(h)v) \| \quad \text{and} \quad R = \max\{ R_i : 1 \leq i \leq m \}. \tag{3.13}
\]
Let \( C = \max\{ C_i : 1 \leq i \leq m \} \) where \( C_i \) is given in (3.9). Let \( \theta' = \max\{ \theta_i : 1 \leq i \leq m \} \).

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According to Lemma 3.1, there exists $T''_{\eta, \delta} > 0$ such that (3.3) holds for any $t \geq T''_{\eta, \delta}$, any nonzero $v \in V^\lambda_i$ (1 \leq i \leq m) and any $z \in B_1^v$, i.e.,
\[
\|\rho(f_t^z)v\|^{-\theta} \leq e^{-(1-\delta/2)\theta \lambda t} \|v\|^{-\theta} \leq e^{-(\theta-\delta/2)\lambda t} \|v\|^{-\theta}.
\]
This inequality and the assumption of the norm implies that for all nonzero $v \in V_i$ and $t \geq T''_{\eta, \delta}$
\[
\|\rho(f_t^z h)v\|^{-\theta} \leq e^{-(\theta-\delta/2)\lambda t} \|\pi_i(\rho(h)v)\|^{-\theta},
\]
where $\frac{1}{\delta}$ is interpreted as $\infty$. Let $T_{\eta, \delta} > T''_{\eta, \delta}$ be a large enough real number so that $t \geq T_{\eta, \delta}$ implies
\[
\frac{(2m)^{\theta} CR^{\theta'-\theta}}{1 - 2^{\theta'-\theta_0}} e^{-(\theta-\delta/2)\lambda t} \leq e^{-(\theta-\delta)\lambda t}.
\]
Let $v$ be a unit vector of $V$. We write $v = v_1 + \cdots + v_m$ where $v_i \in V_i$. Since we assume different $V_i$ are orthogonal to each other, there exists an integer $i \in [1, m]$ such that $m \|v_i\| \geq \|v\| = 1$.

There is a disjoint union decomposition of $B_1^v$ as
\[
E(v_i, 0) \cup \left( \bigcup_{n \geq 0} E^+(v_i, 2^{-n}R_i) \right),
\]
where
\[
E^+(v_i, 2^{-n}R_i) = E(v_i, 2^{-n}R_i) \setminus E(v_i, 2^{-n-1}R_i).
\]
Since $\mu(E(v_i, 0)) = 0$, for any $z \in B_1^c$ and $t \geq T_{\eta, \delta}$ we have
\[
\int_{B_1^v} \|\rho(f_t^z h)v\|^{-\theta} d\mu(h) \leq \sum_{n=0}^{\infty} \int_{E^+(v_i, 2^{-n}R_i)} \|\rho(f_t^z h)v_i\|^{-\theta} d\mu(h)
\]
(by (3.14))
\[
\leq e^{-(\theta-\delta/2)\lambda t} \sum_{n=0}^{\infty} \int_{E^+(v_i, 2^{-n}R_i)} \|\pi_i(\rho(h)v_i)\|^{-\theta} d\mu(h)
\]
(by (3.9))
\[
\leq e^{-(\theta-\delta/2)\lambda t} \sum_{n=0}^{\infty} C_12^\theta (2^{-n}R_i)^{\theta_0} \|v_i\|^{-\theta_i}
\]
\[
\leq \frac{m^2 \theta' CR^{\theta'-\theta}}{1 - 2^{\theta'-\theta_0}} e^{-(\theta-\delta/2)\lambda t}
\]
(by (3.15))
\[
\leq e^{-(\theta-\delta)\lambda t}.
\]
\[\square\]

4. Eskin–Margulis height function

Let the notation be as in Theorem 1.4. In this section, we will establish a uniform contraction property of the EM height function on $G/\Gamma$ with respect to a family of one-parameter subgroups $F_z$ (z $\in B_1^v$).

Recall that $G/\Gamma = \prod_{i=1}^m G_i/\Gamma_i$ where each $G_i/\Gamma_i$ is a noncompact irreducible quotient of a semisimple Lie group without compact factors. Since we assume the projection of $A_F$ to each $G_i$ is nontrivial, we have $H = \prod_{i=1}^m H_i$, where each $H_i = G_i \cap H$ is a nontrivial connected normal subgroup of $G_i$.

Let us recall the definition of the EM height function from [EM04]. The EM height function is constructed on each $G_i/\Gamma_i$ using a finite set $\Delta_i$ of $\Gamma_i$-rational parabolic subgroups of $G_i$. Recall that a parabolic subgroup $P$ of $G_i$ is $\Gamma_i$-rational if the unipotent radical of $P$ intersects $\Gamma_i$ in

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a lattice. If the rank of $G_i$ is bigger than one, then Margulis' arithmeticity theorem implies that there is a $\mathbb{Q}$-structure on $G_i$ such that $\Gamma_i$ is commensurable with $G_i(\mathbb{Z})$. In this case the set $\Delta_i$ consists of standard $\mathbb{Q}$-rational maximal parabolic subgroups of $G_i$ with respect to a fixed $\mathbb{Q}$-split torus and fixed positive roots. So the irreducibility of $\Gamma_i$ implies that no conjugates of $H_i$ is contained in any $P \in \Delta_i$. The same conclusion holds in the case where $G_i$ has rank one. The reason is that in this case $H_i = G_i$ and $\Delta_i = \{ P \}$ where $P$ is a maximal parabolic subgroup defined over $\mathbb{R}$.

For each $P_{i,j} \in \Delta_i$, there exists a representation $\rho_{i,j} : G_i \to \text{GL}(V_{i,j})$ on a normed vector space and a nonzero vector $w_{i,j} \in V_{i,j}$ such that the stabilizer of $\mathbb{R}w_{i,j}$ is $P_{i,j}$. In fact, the map $\rho_{i,j}$ is given by Chevalley's theorem and is defined over $\mathbb{Q}$ when $\Gamma$ is arithmetic. We consider $\rho_{i,j}$ as a representation of $G$ so that $\rho(G_s)$ is the identity linear map if $s \neq i$. Let $V^H_{i,j}$ be the $H$-invariant subspace of $V_{i,j}$ consisting of $H$-invariant vectors. Let $\pi_{i,j}$ be the projection of $V_{i,j}$ to the $H$-invariant subspace $V^H_{i,j}$ complementary to $V^H_{i,j}$. Since no conjugates of $H_i$ is contained in $P_{i,j}$ and $G_i = K_iP_{i,j}$ for some maximal compact subgroup $K_i$ of $G_i$, there exists $C \geq 1$ such that

$$\|v\| \leq C\|\pi_{i,j}(v)\|$$

for all $v \in \rho_{i,j}(G)w_{i,j}$. Note that $V^H_{i,j}$ is $G$-invariant and it has no nonzero $H$-invariant vectors. Therefore, Lemma 3.6 implies the following lemma which corresponds to Condition A in [EM04]. For simplicity, the set of indices $j$ such that $P_{i,j} \in \Delta_i$ is also denoted as $\Delta_i$. This will not cause any confusion.

**Lemma 4.1.** For any $1 \leq i \leq m$ and $j \in \Delta_i$, there exist positive constants $\theta_0^{i,j}$ and $\lambda^{i,j}$ such that, for any $0 < \delta < \theta < \theta_0^{i,j}$, any nonzero $v \in \rho_{i,j}(G)w_{i,j}$ and any $z \in B^i_c$, one has

$$\int_{B^i_1} \|\rho_{i,j}(f_t^zh)v\|^{-\theta_0^{i,j}} dh \leq e^{-\theta - \delta t\lambda^{i,j}} \|v\|^{-\theta}$$

(4.1)

provided $t \geq T_{\theta,\delta}^{i,j}$ where $T_{\theta,\delta}^{i,j} > 0$ is a constant depending on $\theta$ and $\delta$.

**Proof.** We assume without loss of generality that for all $V_{i,j}$ the norm $\| \cdot \|$ is Euclidean and $V^H_{i,j}$ and $V^V_{i,j}$ are orthogonal to each other. According to Lemma 3.6, for each representation $\rho_{i,j} |_{V^V_{i,j}}$, there exist positive constants $\theta_0^{i,j}$ and $\lambda^{i,j}$ with the following properties: for any $0 < \delta < \theta < \theta_0^{i,j}$ there exists $T_{\theta,\delta}^{i,j} > 0$ such that, for any $t \geq T_{\theta,\delta}^{i,j}$, $z \in B^i_c$ and any nonzero $v \in \rho_{i,j}(G)w_{i,j}$, one has

$$\int_{B^i_1} \|\rho_{i,j}(f_t^zh)v\|^{-\theta_0^{i,j}} dh \leq \int_{B^i_1} \|\rho_{i,j}(f_t^zh)\pi_{i,j}(v)\|^{-\theta_0^{i,j}} dh$$

$$\leq e^{-\theta - \delta t\lambda^{i,j}} \|\pi_{i,j}(v)\|^{-\theta}$$

$$\leq C^\theta e^{-\theta - \delta t\lambda^{i,j}} \|v\|^{-\theta}.$$

It is not hard to see from the above estimate that (4.1) holds for sufficiently large $t$. \qed

Besides $\rho_{i,j}$, the EM height function is constructed using positive constants $c_{i,j}$ and $q_{i,j}$ which are combinatorial data determined by the root system, see [EM04, (3.22), (3.28)]. Although only the product $c_{i,j}q_{i,j}$ is used in this paper, the constants $c_{i,j}$ and $q_{i,j}$ are given by different combinatorial data and we use both of them for the consistency with [EM04]. We assume the norm $\| \cdot \|$ on $V_{i,j}$ is $K_i$-invariant and let

$$u_{i,j}(g\Gamma) = \max_{\gamma \in \Gamma} \frac{1}{\|\rho_{i,j}(g\gamma)w_{i,j}\|^{1/c_{i,j}q_{i,j}}}.$$
where \( g \in G \). The maximum is used in the definition of \( u_{i,j} \) since supremum on the right-hand side is achieved for some \( \gamma \in \Gamma \). In the case where \( \Gamma \) is arithmetic this follows from \( \mathbb{Q} \)-rationality of \( \rho_{i,j} \). The rank one case is also well known, see e.g. [KW13, Proposition 3.1]. The main property of \( u_{i,j} \) proved in [EM04, §3.2] is summarized in the following lemma.

**Lemma 4.2** [EM04]. Let \( 1 \leq i \leq m \). There are combinatorial data \( \lambda_{i,j,k} > 0 \) where \( j, k \in \Delta_i \) such that the following hold:

(i) \( \sum_{k \neq j} \lambda_{i,j,k} < 1 \);

(ii) for any \( g \in G \), if \( u_{i,j}(g \Gamma) \) is achieved at \( \gamma \in \Gamma \), then for any \( t, \theta > 0 \) and \( z \in B^*_i \) there exist \( C', b' \geq 1 \) depending on \( t \) and \( \theta \) such that

\[
\int_{B^*_i} u^\theta_{i,j} (f^\gamma_i h \gamma g \Gamma) d\mu(h) \leq \int_{B^*_i} \frac{1}{\|\rho_{i,j}(f^\gamma_i h \gamma \Gamma)\|} d\mu(h) + C' \prod_{k \neq j} u^\lambda_{i,j,k}(x) + b'.
\]

The values \( \lambda_{i,j,k} \) are given in [EM04, (3.31)] and (i) is [EM04, (3.32)]. The statement of Lemma 4.2(ii) is summarized from the proof of [EM04, (3.35)] and (4.3) is the explicit version of [EM04, (3.35)]. We can make the values \( C' \) and \( b' \) independent of \( i \) since there are only finitely many \( i \). The dependence of \( C' \) and \( b' \) on \( \theta \) can be removed if \( \theta \) is bounded from above.

Let

\[
\theta_1 = \max \left\{ \theta > 0 : \frac{\theta}{c_{i,j} q_{ij}} \leq \theta_0^{i,j} \text{ for all } i, j \right\} \quad \text{and} \quad \alpha_1 = \min_{i,j} \left\{ \frac{\theta_1}{c_{i,j} q_{ij}} \right\},
\]

where \( \theta_0^{i,j} \) and \( \lambda^{i,j} \) are constants given by Lemma 4.1. We call \( \alpha_1 \) a contraction rate of the dynamical system \((G/\Gamma, F^+)\).

**Remark 4.3.** We will see in the following two sections that \( \alpha_1 \) plays an important role in bounding the Hausdorff dimension of \( \mathcal{D}(F^+, G/\Gamma) \). We believe that by obtaining the optimal \( \alpha_1 \), it would be possible to give a sharp upper bound on the dimension. By Remark 3.5, the constant \( \theta_0^{i,j} \) can be explicitly calculated, so are the constants \( \theta_1 \) and \( \alpha_1 \). Consequently, it will be clear from the proof in the next two sections that the upper bound of the dimension we obtain can also be explicitly calculated, although not optimal.

**Lemma 4.4.** For every \( \alpha < \alpha_1 \), there exist \( \theta = \theta(\alpha) \in (0, \theta_1) \) and \( T = T(\alpha) > 0 \) such that, for all \( t \geq T \) and \( \epsilon \) sufficiently small depending on \( t \), the EM height function

\[
u : G/\Gamma \to (0, \infty) \quad \text{defined by} \quad u(x) = \sum_{i \in [1, m], j \in \Delta_i} (\epsilon u_{i,j}(x))^\theta
\]

satisfies the following properties:

(i) \( u(x) \to \infty \) if and only if \( x \to \infty \) in \( G/\Gamma \);

(ii) for any compact subset \( K \) of \( G \), there exists \( C = C(K, t) > 1 \) such that \( u(hx) \leq Cu(x) \) for all \( h \in K \) and \( x \in G/\Gamma \);

(iii) there exists \( b = b(t) > 0 \) such that, for all \( z \in B^*_1 \) and \( x \in G/\Gamma \), one has

\[
\int_{B^*_1} u(f^\gamma_i h x) d\mu(h) < \frac{1}{2} e^{-\alpha t} u(x) + b;
\]
(iv) there exists $\ell = \ell(t) \geq 1$ such that if $u(x) \geq \ell$, then for all $z \in B_1^c$

$$
\int_{B_1^+} u(f_t^zhx) \, d\mu(h) < e^{-\alpha t} u(x).
$$

(4.7)

The proof of Lemma 4.4 is essentially the same as in [EM04, §3.2], with the exception of the following minor differences: (i) we do not assume that $\Gamma$ is irreducible; (ii) we want (4.6) and (4.7) holds uniformly in $z$; (iii) we want to make the contraction constant in (4.6) as explicit as possible. A sketch of proof will be given below to show how to adapt the proof given in [EM04, §3.2] to our settings.

Sketch of proof. It follows from the corresponding properties for each function $u_{i,j}$ proved in [EM04, §3.2] that the first two conclusions hold for any choice of $\theta$ and $\epsilon$. Note that (iv) is a direct corollary of (iii).

Now we prove (iii). We fix $\delta > 0$ sufficiently small such that

$$
\alpha + \delta + \frac{\delta \lambda_{i,j}}{c_{i,j} q_{i,j}} < \alpha_1 \quad \forall i,j.
$$

According to the definitions of $\theta_1$, $\alpha_1$ and the choice of $\delta$ above, there exists $\theta > 0$ such that

$$
\theta < \theta_1 \quad \text{and} \quad (\theta - \delta) \lambda_{i,j} \geq \alpha + \delta \quad \forall i,j.
$$

(4.8)

Let $\delta_{i,j} = \delta/c_{i,j} q_{i,j}$, $\theta_{i,j} = \theta/c_{i,j} q_{i,j}$, then according to Lemma 4.1 there exists $T^{i,j} > 0$ such that, for $t \geq T^{i,j}$, one has that (4.1) holds with $\delta = \delta_{i,j}$ and $\theta = \theta_{i,j}$. We will show that Lemma 4.4 holds for $T = \delta^{-1} \log 4 + \max_i T^{i,j}$.

Now let us fix $z \in B_1^c$ and $x = g \Gamma \in G/\Gamma, t \geq T$ and an index $i,j$. By (4.8), Lemmas 4.1 and 4.2 we have

$$
\int_{B_1^c} u_{i,j}^\theta(f_t^{i,j} x) \, d\mu(h) \leq e^{-(\alpha + \delta)t} u_{i,j}^\theta(x) + C' \prod_{k \neq j} u_{i,k}^\lambda_{i,j,k}(x) + b',
$$

(4.9)

where $C'$ and $b'$ are positive constants depending on $t$.

Now we fix $\epsilon > 0$ sufficiently small which will be specified later and define $u(x)$ as in (4.5). We can write (4.9) as

$$
\int_{B_1^c} e^\theta u_{i,j}^\theta(f_t^{i,j} x) \, d\mu(h) \leq e^{-(\alpha + \delta)t} e^\theta u_{i,j}^\theta(x) + \epsilon_{i,j} \prod_{k \neq j} e^{\lambda_{i,j,k}} u_{i,k}^\lambda_{i,j,k}(x) + e^\theta b',
$$

where $\epsilon_{i,j} = C' \epsilon^1 - \sum_{k \neq j} \lambda_{i,j,k}$. According to Lemma 4.2(1) and the Jensen’s inequality, we have

$$
\prod_{k \neq j} (e u_{i,k}^\lambda_{i,j,k})^{\lambda_{i,j,k}} \leq \prod_{k \neq j} (e u_{i,k}^\lambda_{i,j,k})^{\lambda_{i,j,k}} \leq 1 + \sum_{k \neq j} (e u_{i,k}^\lambda_{i,j,k})^{\lambda_{i,j,k}}.
$$

Hence we have

$$
\int_{B_1^c} e^\theta u_{i,j}^\theta(f_t^{i,j} x) \, d\mu(h) \leq e^{-(\alpha + \delta)t} e^{\theta} u_{i,j}^\theta(x) + \epsilon_{i,j} u(x) + b'',
$$

where $b''$ is some constant depending on $t$. Add the above inequalities up over all indices $i,j$, we have

$$
\int_{B_1^c} u(f_t^zhx) \, dh \leq e^{-(\alpha + \delta)t} u(x) + \left( \sum_{i,j} \epsilon_{i,j} \right) u(x) + b,
$$

(4.10)

where $b$ is a constant depending on $t$. In view of (4.10), for all $\epsilon$ sufficiently small so that $\sum_{i,j} \epsilon_{i,j} \leq e^{-(\alpha + \delta)t}$, one has (4.6) holds. \qed
5. Applications of the uniform contraction property

In this section we will introduce some auxiliary sets closely related to $\mathcal{D}(F^+, G/\Gamma)$ and study them using the uniform contraction property of the EM height function established in Lemma 4.4. To be specific, we will prove some covering results for these auxiliary sets in Proposition 5.1 and these covering results will play an important role in bounding the Hausdorff dimension of $\mathcal{D}(F^+, G/\Gamma)$.

Let $\alpha_1$ be a contraction rate of the dynamical system $(G/\Gamma, F^+)$ given by (4.4) and let $\lambda$ be the top Lyapunov exponent of $A_F$ in the representation $(\text{Ad}, g)$. We also fix an auxiliary $\delta \in (0, \lambda)$ which will eventually go to zero. Let $T_0 > 0$ be given by Lemma 3.2 so that for all $t \geq T_0$ we have (3.4) and (3.5) hold. We fix $\alpha < \alpha_1$,

$$t > \max\{T_0, (\alpha), \log 4/\alpha\delta, \log 2/\lambda\},$$

where $T(\alpha)$ is given by Lemma 4.4 and an EM height function $u : G/\Gamma \to (0, \infty)$ so that Lemma 4.4 holds. Let $\ell \geq 1$ so that (4.7) holds for all $z \in B^1_1$ if $u(x) \geq \ell$. By Lemma 4.4(ii), there exists $C = C(t) \geq 1$ such that

$$C^{-1}u(x) \leq u(f_s hx) \leq Cu(x) \quad \text{for all} \quad 0 \leq s \leq t, \quad h \in B^2_{G} \quad \text{and} \quad x \in G/\Gamma.$$  \hspace{1cm} (5.2)

Note that the logarithm map from the metric space $(B^+_1, \text{dist})$ to the Lie algebra $\mathfrak{g}^+$ is a bi-Lipschitz diffeomorphism to its image with respect to the induced Euclidean structure. Therefore $(B^+_1, \text{dist})$ is Besicovitch, see [MAT95], namely, for any subset $D$ of $B^+_1$ and a covering of $D$ by balls centered at $D$, there is a finite sub-covering such that each element of $D$ is covered by at most $E'$ times. Therefore, there exists $E \geq E'$ such that, for all $0 < r \leq 1$, the set $B^{1/2}_{1/2}$ can be covered by no more than $Er^{-d}$ open balls of radius $r$, where $d = \dim G^+$.

We use $|I|$ to denote the cardinality of a finite set $I$. The following is the main result of this section.

**PROPOSITION 5.1.** Let $x \in G/\Gamma$. There exists $0 < \sigma < 1$ and $E_0 \geq 1$ such that, for $z \in B^1_1$ and $N \in \mathbb{N}$, the set

$$\mathcal{D}_x(z, N, \sigma, s) \overset{\text{def}}{=} \{h \in B^1_{G} : \{1 \leq n \leq N : u(f_n^zhx) \geq s\} \geq \sigma N\}$$  \hspace{1cm} (5.3)

with $s = C^2\ell$ can be covered by no more than $E_0e^{(d\lambda - \alpha + \delta(d + \alpha))tN}$ open balls of radius $e^{-(\lambda + \delta)tN}$ in $B^1_1$.

The rest of this section is devoted to show that Proposition 5.1 holds for

$$\sigma = \frac{(1 - \delta/2) \alpha t + \log C}{\alpha t + \log C}.$$  \hspace{1cm} (5.4)

In the rest of this section we fix $z \in B^1_1$ and $N \in \mathbb{N}$. We begin with the following simple observation.

**LEMMA 5.2.** If $B \subset G^+$ is a ball of radius $e^{-(\lambda + \delta)tN}$ centered at $\mathcal{D}_x(z, N, \sigma, C^2\ell)$, then $B \subset \mathcal{D}_x(z, N, \sigma, C\ell)$.

**Proof.** Let $h_0$ be the center of $B$ and $h \in B$. It suffices to show that, for all $1 \leq n \leq N$, if $u(f_n^zhx) \geq C^2\ell$ then $u(f_n^zhx) \geq C\ell$. By (3.4) we have

$$\text{dist}(f_n^zh_0, f_n^zh) = \text{dist}(1_G, f_n^zh_0^{-1}f_n^zh) < 1.$$  

By (5.2)

$$u(f_n^zhx) = u(f_n^zh^{-1}f_{-nt} \cdot f_{nt}h_0x) \geq C^{-1}u(f_n^zh_0x) \geq C^{-1} \cdot C^2\ell = C\ell.$$  \hspace{1cm} $\square$
For a subset \( I \subset \{1, \ldots, N\} \), we let
\[
\mathcal{D}_x(z, I, C\ell) = \{ h \in B_{1/2}^+ : u(f^nzhx) \geq C\ell \text{ for all } n \in I \}.
\]

Since \( \mathcal{D}_x(z, N, \sigma, C\ell) = \bigcup_{|I| \geq \sigma N} \mathcal{D}_x(z, I, C\ell) \), one has
\[
\mu(\mathcal{D}_x(z, N, \sigma, C\ell)) \leq \sum_{|I| \geq \sigma N} \mu(\mathcal{D}_x(z, I, C\ell)). \tag{5.5}
\]

The following lemma will play an important role in the proof of Proposition 5.1.

**Lemma 5.3.** Suppose that \( I \subset \{1, \ldots, N\} \) and \( |I| \geq \sigma N \). Then
\[
\mu(\mathcal{D}_x(z, I, C\ell)) \leq C^2 u(x)e^{-(1-\delta/2)atN}. \tag{5.6}
\]

We fix \( I \) as in the statement of Lemma 5.3. Our strategy is to estimate the measure of \( \mathcal{D}_x(z, I, C\ell) \) by relating it to a subset coming from random walks on \( G/\Gamma \) with alphabet \( f^\ell_1B_1^+ \). Let \( p = \sup I \) and for \( 1 \leq k \leq p \) let
\[
Z_k = \{(h_1, \ldots, h_k) \in (B_1^+)^k : u(f^\ell_{n_1}h_n \cdots f^\ell_{n_k}h_1) \geq \ell \forall n \in (I \cap [1,k])\}.
\]

Define \( \eta : (B_1^+)^p \to G^c \) by
\[
\eta(h_1, \ldots, h_p) = \hat{h}_p \cdots \hat{h}_1 \quad \text{where } \hat{h}_n = f^\ell_{(n-1)}h \in \mathcal{D}_x(z, I, C\ell).
\]

By (3.4), we know that \( \hat{h}_n \in B_{1/4n-1}^+ \), and hence by the right invariance of \( \text{dist}(\cdot, \cdot) \), the image of \( \eta \) is contained in \( B_2^+ \). The following two lemmas are needed in the proof of Lemma 5.3.

**Lemma 5.4.** For all \( h \in \mathcal{D}_x(z, I, C\ell) \) one has \( \eta^{-1}(h) \subset Z_p \).

**Proof.** Suppose that \( \eta(h_1, \ldots, h_p) = h \) where \( h_i \in B_1^+ \). In view of (5.7), for \( 1 \leq n \leq p \) we have
\[
f^\ell_hz = f^\ell_{n}h \in \mathcal{D}_x(z, I, C\ell) \quad \text{and} \quad f^\ell_{n_1}h_1 \cdots f^\ell_{n_k}h_1 = f^\ell_{n_1}h_1 \cdots f^\ell_{n_k}h_1.
\]

Therefore,
\[
f^\ell_{n_1}h \cdot (f^\ell_{n_2}h_2 \cdots f^\ell_{n_k}h_k)^{-1} = \prod_{n < i < p} f^\ell_{n_i}h_i (f^\ell_{n_i})^{-1}
\]
which is contained in the image of \( \eta \) with \( p \) replaced by \( n - p \). Since the image of \( \eta \) is always contained in \( B_2^+ \) and the metric on \( G \) is right invariant, we have
\[
\text{dist}(f^\ell_{n_1}h, f^\ell_{n_2}h_2 \cdots f^\ell_{n_k}h_k) = \text{dist}(f^\ell_{n_1}h \cdot (f^\ell_{n_2}h_2 \cdots f^\ell_{n_k}h_k)^{-1}, 1_G) < 2.
\]

Therefore, by (5.2) we have for \( n \in I \)
\[
u(f^\ell_{n_1}h \cdots f^\ell_{n_k}h_1x) \geq C^{-1}u(f^nzhx) \geq \ell.
\]
So \( (h_1, \ldots, h_p) \in Z_p \) and the proof is complete. \( \square \)

Let \( \tilde{\mu}_n \) be the Radon measure on \( G^c \) defined by
\[
\int_{G^c} \varphi(h) \, d\tilde{\mu}_n(h) = \int_{B_1^+} \varphi(f^\ell_{n_1}h \cdots f^\ell_{n_k}h_1) \, dh \tag{5.8}
\]
for all \( \varphi \in C_c(G^c) \). For any positive integer \( n \) let \( \mu_n = \tilde{\mu}_n - \cdots - \tilde{\mu}_1 * \tilde{\mu}_0 \) be the measure on \( G^c \) defined by the \( n \) convolutions. Clearly, \( \mu_n \) is absolutely continuous with respect to \( \mu \), and \( \mu_p \) is the pushforward of \( (\mu|_{B_1^+})^p \) by the map \( \eta \). The following lemma shows that \( \mu_n \) has density bigger than or equal to 1 at every \( h \in B_{1/2}^+ \).
Lemma 5.5. For all \( n \leq N \) and \( h \in B_{1/2}^+ \) we have \( (d\mu_n/d\mu)(h) \geq 1 \).

Proof. The conclusion is clear if \( n = 1 \). Now we assume \( n > 1 \) and let

\[
\nu = \bar{\mu}_{n-1} \ast \bar{\mu}_{n-2} \ast \cdots \ast \bar{\mu}_1.
\]

It follows from (3.4) and (5.8) that for \( k > 0 \) the probability measure \( \bar{\mu}_k \) is supported on \( B_{1/4k}^+ \).

Since the metric on \( G^+ \) is right invariant, the measure \( \nu \) is supported on \( B_{1/2}^+ \). Suppose \( \nu = \varphi \, d\mu \), then \( \mu_n = \nu \ast \bar{\mu}_0 = \varphi \ast 1_{B_1^+} \, d\mu \). So for any \( h \in B_{1/2}^+ \), we have

\[
\varphi \ast 1_{B_1^+}(h) = \int_{G^+} \varphi(h_1) 1_{B_1^+}(h_1^{-1}h) \, d\mu(h_1) \geq \int_{B_{1/2}^+} \varphi(h_1) \, d\mu(h_1) = 1. \quad \Box
\]

Now we are ready to prove Lemma 5.3.

Proof of Lemma 5.3. By Lemmas 5.4 and 5.5,

\[
\mu(\mathcal{D}_x(z, I, C \ell)) \leq \mu_p(\mathcal{D}_x(z, I, C \ell)) \leq \mu^\otimes p(Z_p). \quad (5.9)
\]

So it suffices to estimate \( \mu^\otimes p(Z_p) \). For \( 1 \leq k \leq p \)

\[
s(k) = \int_{Z_k} u(f_i^k h_k \cdots f_i^1 h_1 x) \, d\mu^\otimes k(h_1, \ldots, h_k).
\]

Let

\[
s(p + 1) = \int_{Z_p} \left[ \int_{B_1^+} u(f_i^k h_k \cdots f_i^1 h_1 x) \, d\mu(h_{p+1}) \right] \, d\mu^\otimes (h_1, \ldots, h_p). \quad (5.10)
\]

Then, for every \( 1 < k \leq p + 1 \),

\[
s(k) \leq \int_{Z_{k-1}} \left[ \int_{B_1^+} u(f_i^k h_k f_i^{k-1} h_{k-1} \cdots f_i^1 h_1 x) \, d\mu(h_k) \right] \, d\mu^\otimes (h_1, \ldots, h_{k-1}).
\]

If \( k - 1 \in I \), then \( s(k) \leq e^{-\alpha t} s(k - 1) \) by (4.7). If \( k - 1 \notin I \), then by (5.2) we have \( s(k) \leq C s(k - 1) \). We apply this estimate to \( k = p + 1, p, \ldots, 2 \); then we have

\[
s(p + 1) \leq C^{(N-|I|)} e^{-|I|\alpha t} \int_{B_1^+} u(f_i h x) \, d\mu(h) \leq C^{1+\sigma N} e^{-\sigma \alpha t N} u(x),
\]

where in the second inequality we use the assumption \( |I| \geq \sigma N \) and (5.2).

According to (5.4), we have \( C^{\sigma N} e^{\sigma \alpha t N} = C^{N e^{(1-\delta/2)\alpha t N}} \), and hence

\[
s(p + 1) \leq C^{-N} e^{-(1-\delta/2)\alpha N C^{1+N} u(x)} = C e^{-(1-\delta/2)\alpha t N} u(x). \quad (5.11)
\]

On the other hand, in view of (5.10), (5.2) and the fact \( p = \sup I \) we have

\[
s(p + 1) \geq C^{-1} s(p) \geq C^{-1} \ell \cdot \mu^\otimes p(Z_p). \quad (5.12)
\]

Therefore, (5.6) follows from (5.9), (5.11) and (5.12) and the observation \( \ell \geq 1 \). \quad \Box
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Proof of Proposition 5.1. As before we fix \( z \) and \( N \) as in the statement. Let \( \sigma \) be as in (5.4). Since \((B_1^+, \text{dist})\) is Besicovitch, there exists a covering \( \mathcal{U} \) of \( \mathcal{D}_x(z, N, \sigma, C^2\ell) \) by open balls of radius \( e^{-(\lambda+\delta)tN} \) centered at \( \mathcal{D}_x(z, N, \sigma, C^2\ell) \) such that each element of \( \mathcal{D}_x(z, N, \sigma, C^2\ell) \) is covered by at most \( E \) elements. By Lemma 5.2, each \( B \in \mathcal{U} \) is contained in \( \mathcal{D}_x(z, N, \sigma, C^2\ell) \). So in view of (3.1)

\[
\mu(\mathcal{D}_x(z, N, \sigma, C\ell)) \geq \frac{|\mathcal{U}|}{E} \mu(B_{e^{-(\lambda+\delta)tN}}^+) \geq \frac{|\mathcal{U}|}{C_0E} e^{-(\lambda+\delta)dtN}.
\] (5.13)

On the other hand, there are \( 2^N \) subsets \( I \subset \{1, \ldots, N\} \). So by (5.5) and Lemma 5.3, we have

\[
\mu(\mathcal{D}_x(z, N, \sigma, C\ell)) \leq C^2 2^N e^{-(1-\delta/2)atN} u(x). \] (5.14)

Note that \( 2^N < e^{\delta atN/2} \) by (5.1). This together with (5.14) implies

\[
\mu(\mathcal{D}_x(z, N, \sigma, C\ell)) \leq C^2 e^{-(1-\delta)atN} u(x). \] (5.15)

By (5.13) and (5.15),

\[
|\mathcal{U}| \leq u(x)C_0C^2\ell e^{(d\lambda-\alpha(d+\alpha))tN}.
\]

The conclusion now follows by taking \( E_0 = u(x)C_0C^2\ell \). \( \square \)

6. Upper bound of Hausdorff dimension

In this section, we complete the proof of Theorem 1.4. We will use the same notation as in § 5 prior to Proposition 5.1. For \((z, h) \in B_1^cB_1^+, \ell > 0 \) and \( N \in \mathbb{N} \), let \( I_{N,\ell}(z, h) \) denote the set of \( n \in \{1, \ldots, N\} \) satisfying \( u(f_{ntzh}x) \geq \ell \). For \( x \in G/\Gamma \), let

\[
\mathcal{D}_x^0(F^+, N, \sigma, \ell) = \{(z, h) \in B_{1/2}^cB_1^+: |I_{N,\ell}(z, h)| \geq \sigma N\}. \] (6.1)

Lemma 6.1. Let \( x \in G/\Gamma \). Then there exist \( 0 \leq \alpha < 1 \) and \( E_2 \geq 1 \) such that, for any \( N \in \mathbb{N} \), the set \( \mathcal{D}_x^0(F^+, N, \sigma, C^4\ell) \) can be covered by no more than \( E_2e^{(d'\lambda+\alpha(d'+d+\alpha))tN} \) open balls of radius \( e^{-(\lambda+\delta)tN} \) in \( C^4G^+ \).

Proof. Let \( 0 < \alpha < 1 \) and \( E_0 \geq 1 \) so that Proposition 5.1 holds. We will show that the lemma holds for this \( \sigma \) and some constant \( E_2 \) which will be specified later.

We fix \( N \in \mathbb{N} \). We claim that for any metric ball \( W = B_{e^{-(\lambda+\delta)tN}}^+z \subset B_1^c \) where \( z \in G^c \), we have

\[
(\mathcal{D}_x^0(F^+, N, \sigma, C^4\ell) \cap (WB_1^c)) \subset (W\mathcal{D}_x(z, N, \sigma, C^2\ell)).
\] (6.2)

Let \((z_1, h_1) \in WB_1^c\). Suppose that \( 1 \leq n \leq N \) and \( u(f_{ntzh}x) \geq C^4\ell \). In view of (5.2) we have

\[
u(f_{ntzh}x) = u(z_1f_{ntzh}x) \geq C^{-1}u(f_{ntzh}x) = C^{-1}u(f_{nt}(zz_1^{-1})f_{-nt}f_{ntzh}x).\] (6.3)

By the right invariance of the metric, \( \text{dist}(zz_1^{-1}, 1_G) = \text{dist}(z_1z_1^{-1}, 1_G) \leq e^{-(\lambda+\delta)tN} \). So (3.5) implies \( \text{dist}(f_{nt}(zz_1^{-1})f_{-nt}, 1_G) \leq e^{-\lambda N} \). This together with (6.3) and (5.2) imply

\[
u(f_{ntzh}x) \geq C^{-2}u(f_{ntzh}x) \geq C^2\ell.\]

In other words, we have proved that if \( n \in I_{N,\ell}(z_1, h_1) \), then \( u(f_{ntzh}x) \geq C^2\ell \). Therefore, if \((z_1, h_1) \) belongs to the left-hand side of (6.2), then it also belongs to the right-hand side.

Since \((B_1^c, \text{dist})\) is also Besicovitch, there exists \( E_1 \geq 1 \) such that, for all \( 0 < r \leq 1 \), \( B_{1/2}^c \) can be covered by no more than \( E_1r^{-\delta} \) open balls of radius \( r \) centered in \( B_{1/2}^c \). We fix a cover \( \mathcal{U}^c \)
of $B^c_{1/2}$ that consists of open balls of radius $e^{-(\lambda+\delta)Nt}$ with $|\Omega^c| \leq E_1 e^{e^{(\lambda+\delta)Nt}}$. We assume each element of $\Omega^c$ is centered in $B^c_{1/2}$ so that it is contained in $B^c_1$ by (5.1). Let $W_z \in \Omega^c$ be a ball centered at $z \in B^c_1$. Proposition 5.1 implies that there exists a covering $\Omega_z$ of $D_x(z, N, \sigma, C^2 \ell)$ by open balls of radius $e^{-(\lambda+\delta)Nt}$ such that

$$|\Omega_z| \leq E_0 e^{(d\lambda-\alpha_1+\delta d\lambda+\alpha_1)Nt}.$$  

In view of claim (6.2), the following class of sets

$$\{W_z B : W_z \in \Omega^c, B \in \Upsilon_z\} \quad (6.4)$$

forms an open cover of $D^0_x(F^+, G/\Gamma)$. It is easily checked that there exists $E_1' \geq 1$ not depending on $N$ such that each element $W_z B$ of (6.4) can be covered by $E_1'$ open balls of radius $e^{-(\lambda+\delta)Nt}$ in $G^c G^+$. Therefore the lemma holds with $E_2 = E_0 E_1 E_1'$.

**Theorem 6.2.** For any $x \in G/\Gamma$, the Hausdorff dimension of

$$D^0_x \overset{\text{def}}{=} \{ (z, h) \in B^c_{1/2} B^c_{1/2} : zhx \in D(F^+, G/\Gamma) \}$$

is at most $d^c + d - \alpha_1/\lambda$.

**Proof.** For each $\alpha < \alpha_1$ and $0 < \delta < 1$ we first choose $t > 0$, a height function $u$ and $\ell, C \geq 1$ so that Lemma 4.4, (3.4), (3.5), (5.2) and (5.1) hold. Then there exists $0 < \sigma < 1$ and $E_2 \geq 1$ so that Lemma 6.1 holds.

It follows from parts (i) and (ii) of Lemma 4.4 and the definition of $D(F^+, G/\Gamma)$ that

$$D^0_x \subset \bigcup_{M \geq 1} Q_M \quad \text{where} \quad Q_M = \bigcap_{N \geq M} D^0_N(F^+, N, \sigma, C^4 \ell).$$

Recall that for any metric space $S$, the Hausdorff dimension of $S$ is less than or equal to the lower box dimension, see e.g. [BP17, (1.2.3)]. So Lemma 6.1 implies

$$\dim_H Q_M \leq \liminf_{N \to \infty} \frac{[d^c \lambda + d \lambda - \alpha + \delta (d + d^c + \alpha)]tN + \log E_2}{\lambda N} = d^c + d - \frac{\alpha}{\lambda} + \delta \frac{d + d^c + \alpha}{\lambda}.$$ 

Therefore

$$\dim_H D^0_x \leq d^c + d - \frac{\alpha}{\lambda} + \delta \frac{d + d^c + \alpha}{\lambda}.$$ 

The conclusion follows by first letting $\delta \to 0$ and then letting $\alpha \to \alpha_1$.

**Lemma 6.3.** If $x \in D(F^+, G/\Gamma)$ and $h \in G^-$, then $hx \in D(F^+, G/\Gamma)$.

**Proof.** Note that, by Lemma 3.1,

$$\text{dist}(f_t hx, f_t x) \leq \text{dist}(f_t hf_{-t}, 1_G) \to 0$$

as $t \to \infty$. Therefore the lemma holds.
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Proof of Theorem 1.4. We will show that
\[ \dim H \mathcal{D}(F^+, G/\Gamma) \leq d^- + d^c + d - \frac{\alpha_1}{\lambda}. \]

In view of the local nature of the Hausdorff dimension and the definition of the metric on \( G/\Gamma \), it suffices to prove that for any \( x \in G/\Gamma \)
\[ \dim_H \{ g \in B_r^g : gx \in \mathcal{D}(F^+, G/\Gamma) \} \leq d^- + d^c + d - \frac{\alpha_1}{\lambda}, \]
where \( r > 0 \) so that \( B_r^g \subset B_{1/2} B_{1/2}^c B_1^+ \). Recall that we assume the logarithm map on \( B_1^c \) is bi-Lipschitz onto its image in \( g \) with respect to the metric given by the Euclidean structure. So Marstrand’s product theorem [BP17, Theorem 3.2.1] implies
\[ \dim B_1^{-} S \leq \dim S + d^- \]
for any \( S \subset B_{1/2}^c B_{1/2}^+ \). In particular, Lemma 6.3 implies
\[ \{ g \in B_r^g : gx \in \mathcal{D}(F^+, G/\Gamma) \} \subset B_1^{-} \mathcal{D}_x^0 \]
whose Hausdorff dimension is bounded from above by \( \dim_H \mathcal{D}_x^0 + d^- \). In view of Theorem 6.2, the Hausdorff dimension of \( \mathcal{D}(F^+, G/\Gamma) \) is at most \( d + d^c + d^- - \frac{\alpha_1}{\lambda} \), which is strictly less than the manifold dimension of \( G/\Gamma \).

Remark 6.4. It is worth mentioning that, if \( F = A_F \), then the contraction property of the Benoist–Quint height function proved in [Shi14] will allow us to prove a stronger result. Namely, we can get a nontrivial upper bound for the Hausdorff dimension of the intersection of \( \mathcal{D}(F^+, G/\Gamma) \) with orbits of the so-called \( (F^+, \Gamma) \)-expanding subgroups introduced in [KW13]. But unfortunately, we are not able to prove a uniform contracting property for the Benoist–Quint height function in some cases where the unipotent part of \( F \) is nontrivial.

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Lifan Guan  guanlifan@gmail.com
Department of Mathematics, University of York,
Heslington, York, YO10 5DD, UK

*Current address:* Mathematisches Institut, Georg-August Universität Göttingen,
Bunsenstrasse 3-5, D-37073 Göttingen, Germany

Ronggang Shi  ronggang@fudan.edu.cn
Shanghai Center for Mathematical Sciences, Fudan University,
Shanghai 200433, PR China

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