DIFFERENTIAL AS A HARMONIC MORPHISM WITH RESPECT TO CHEEGER–GROMOLL TYPE METRICS

W. KOZLOWSKI AND K. NIEDZIAŁOMSKI

ABSTRACT. We investigate horizontal conformality of a differential of a map between Riemannian manifolds where the tangent bundles are equipped with Cheeger–Gromoll type metrics. As a corollary, we characterize the differential of a map as a harmonic morphism.

1. INTRODUCTION AND PRELIMINARY RESULTS

1.1. Introduction. In [2], M. Benyounes, E. Loubeau and C. M. Wood introduced a new class of Riemannian metrics in the tangent bundle over a Riemannian manifold. These metrics \( h_{p,q} \), depending on two constants \( p, q \), generalize Sasaki and Cheeger–Gromoll metrics. Wider class of this type of metrics was investigated by M. I. Munteanu [8] and by the authors [6].

Now, there are several articles concerning geometry of a tangent bundle equipped with Cheeger–Gromoll type metric. For example, M. Benyounes, E. Loubeau and C. M. Wood [3] considered \( h_{p,q} \) metrics in the context of harmonic maps, whereas the first named author and Sz. M. Walczak [7] in the context of Riemannian submersions and Gromov–Hausdorff topology.

In [6], the authors studied the conformality of a differential \( \Phi = \varphi_* : (TM, \tilde{h}) \to (TN, h) \) of a map \( \varphi : M \to N \) between Riemannian manifolds, where \( \tilde{h} = h_{p,q,\alpha} \) and \( h = h_{r,s,\beta} \) are metrics of Cheeger–Gromoll type. In this paper, we continue considerations concerning the differential of a map. We give necessary and sufficient conditions for a differential to be horizontally conformal and as a corollary we characterize the differential as a harmonic morphisms.

The idea is the following. The vertical part of Cheeger–Gromoll type metrics is nonlinear with respect to the base point, excluding Sasaki metric. Hence, conformal change of a metric on \( M \) does not effect conformal change of Cheeger–Gromoll type metric. Scaling the base point, the condition of horizontal conformality reduces to vanishing of a certain polynomial, which
gives restrictions to coefficients $p, q, r, s$ and $\alpha, \beta$. That is why, for horizontal conformatility of $\Phi$ there are not many natural choices of $\tilde{h}$ and $h$ (Theorem 2.1), whereas in the context of a harmonic morphism the only possibility is Sasaki metric (Theorem 2.2).

1.2. Preliminary results. Let $(M, g)$ be a Riemannian manifold and $\pi : TM \to M$ its tangent bundle. The Levi-Civita connection and the projection $\pi$ gives the natural splitting $TTM = H \oplus V$ of the second tangent bundle $\pi_* : TTM \to TM$, where the vertical distribution $V$ is the kernel of $\pi_*$ and the horizontal distribution $H$ is the kernel of the conection map $K$. If $X, \xi \in T_\xi M$ then there is a unique vertical vector $X^v_\xi$ and a unique horizontal vector $X^h_\xi$ in $T_\xi TM$ such that $\pi^* X^h_\xi = X$ and $KX^v_\xi = X$. Moreover, any $A \in T_\xi TTM$ has a unique decomposition into horizontal and vertical part, $A = \nabla A + H A$. For more details on decomposition of the tangent bundle see [5].

Let $p, q, \alpha$ be constants, $q$ non–negative, $\alpha$ positive. Define $(p, q, \alpha)$-metric $h = h_{p,q,\alpha}$ on $TM$ as follows. For every $A, B \in T_x M$, $h(A, B) = g(\pi_* A, \pi_* B) + \omega_\alpha(\xi)^p (g(KA, KB) + qg(KA, \xi)g(KB, \xi))$ where $\omega_\alpha(\xi) = (1 + \alpha g(\xi, \xi))^{-1}$. The Riemannian metric $h_{p,q,\alpha}$ is a generalization of the metric considered in [2, 3] and is a special case of a metric considered in [8]. In particular, $h_{0,0,\alpha}$ (or $h_{p,0,0}$) is Sasaki metric, $h_{1,1,1}$ Cheeger-Gromoll metric.

We will often write $\langle \cdot, \cdot \rangle_M$ for $g$ and $\langle \cdot, \cdot \rangle_{TM}$ for $h = h_{p,q,\alpha}$. The length of a vector will be denoted by $| \cdot |_M$ and $| \cdot |_{TM}$, respectively.

Consider now a smooth map $\varphi : M \to N$ between Riemannian manifolds $M$ and $N$. Let $\varphi^{-1}TN \to M$ be a pull–back bundle. There is a unique connection $\nabla^\varphi$ in this bundle characterised by the property [1]

$$\nabla^\varphi_X (Y \circ \varphi) = \nabla^N_{\varphi_* X} Y, \quad X \in T_x M, Y \in \Gamma(TN).$$

Then we easily obtain

$$\nabla^\varphi_X \varphi_* Y - \nabla^\varphi_Y \varphi_* X = \varphi_* [X, Y], \quad X, Y \in \Gamma(TM).$$

The second fundamental form of $\varphi$ is $B = \nabla \varphi_*$,

$$B(X, Y) = \nabla^\varphi_X \varphi_* Y - \varphi_* (\nabla^N_X Y), \quad X, Y \in \Gamma(TM).$$

By (1.1), we get that $B$ is symmetric and hence tensorial in both variables. If $B = 0$ we say that $\varphi$ is totally geodesic. We will need the following lemma.
Lemma 1.1. For $X, \xi \in T_xM$

\begin{align}
\varphi_{**}X_\xi^v &= (\varphi_*X)_{\varphi_*\xi}^v, \\
\varphi_{**}X_\xi^h &= (\varphi_*X)_{\varphi_*\xi}^h + (B(X,\xi))_{\varphi_*\xi}^v.
\end{align}

Proof. Follows by the definition of a connection map and vertical and horizontal distributions. Details are left to the reader. \qed

Let $\varphi : (M,\langle \cdot, \cdot \rangle_M) \to (N,\langle \cdot, \cdot \rangle_N)$ be a smooth map between Riemannian manifolds, $\dim M > \dim N$. Let $\mathcal{V}^\varphi = \ker\varphi_*$ be the vertical distribution and $\mathcal{H}^\varphi = (\mathcal{V}^\varphi)^\perp$, the orthogonal complement of $\mathcal{H}^\varphi$ with respect to $\langle \cdot, \cdot \rangle_M$, the horozontal distribution, $TM = \mathcal{V}^\varphi \oplus \mathcal{H}^\varphi$. Each $X \in T_xM$ has therefore a unique decomposition

$$X = X^\top + X^\perp$$

into vertical and horizontal part.

We say that $\varphi$ is \textit{horizontally conformal} if for every $x \in M$ either $\varphi_{*x} = 0$ or $\varphi_{*x} : \mathcal{H}^\varphi_x \to T_{\varphi(x)}N$ is surjective and

$$\langle \varphi_{*x}X, \varphi_{*x}Y \rangle_N = \lambda(x)\langle X, Y \rangle_M, \quad X, Y \in \mathcal{H}^\varphi_x,$$

where $\lambda(x)$ is positive. We call $\lambda$ the \textit{dilatation} of $\varphi$. Let $C_\varphi$ denote the set of \textit{critical points} i.e., points $x \in M$ such that $\varphi_{*x} = 0$. One can easily prove

Lemma 1.2. For $\Phi = \varphi_* : TM \to TN$ we have $\pi(C_\Phi) \subset C_\varphi$.

Moreover, by Lemma 1.1 we have

Lemma 1.3. If a map $\varphi : M \to N$ is a submersion, i.e., surjective map of maximal rank, then so is $\Phi = \varphi_* : TM \to TN$.

Remark 1.4. By Lemmas 1.2 and 1.3 if $\varphi : M \to N$ and $\Phi = \varphi_* : TM \to TN$ are horizontally conformal, then the differential of horizontally conformal submersion $\varphi : M \setminus C_\varphi \to N$ is a horizontaly conformal submersion $\Phi : TM \setminus \pi^{-1}(C_\varphi) \to TN$. Hence, throughout the paper, without loss of generality, we may assume that horizontaly conformal map $\varphi$ is a submersion i.e. the set of critical points is empty.

Theorem 1.5. Assume $\varphi : M \to N$ is a submersion. Let $\Phi = \varphi_* : TM \to TN$ and equip tangent bundles $TM$ and $TN$ with Cheeger–Gromoll type metrics $h_{p,q,\alpha}$ and $h_{r,s,\beta}$, respectively. Then, with respect to submersion $\Phi$, the second tangent bundle splits into orthogonal sum $T_\xi TM = \mathcal{V}^\Phi_\xi \oplus \mathcal{H}^\Phi_\xi$, where

\begin{align}
\mathcal{V}^\Phi_\xi &= \text{Span}\{\eta^h_\xi + (\nabla^M_\xi \eta)^e_\xi \mid \eta \in \Gamma(\mathcal{V}^\varphi)\},
\end{align}
\begin{align*}
\mathcal{H}_\xi^\phi &= \text{Span}\{ X^\nu - q\omega_q(\xi)\langle X, \xi \rangle_{\xi}^\nu + \omega_\alpha(\xi)\eta_p(\nabla^M X)_{\xi}^h | X \in \Gamma(\mathcal{H}^\phi) \}.
\end{align*}

**Proof.** Let \( \varphi : (M, g_M) \rightarrow (N, g_N) \) be a submersion. Fix \( \xi \in T_xM, \ x \in M \). Denote the right hand side of (1.5) by \( V \) and the right hand side of (1.6) by \( \mathbb{H} \). By Lemma 1.3 for \( \eta \in \Gamma(\mathcal{V}^\varphi) \)

\[
\Phi_*\eta^h + (\nabla^M_M \eta)_{\xi}^0 = \langle \varphi_* \eta, \xi \rangle_{\varphi_*\xi} + \langle \varphi, \nabla^M_M \eta \rangle_{\varphi_*\xi, \xi} = 0.
\]

Hence, \( V \subseteq \mathcal{V}^\varphi_\xi \). Let \( f \) be a smooth function such that \( \xi f = 1 \) and \( f(x) = 0 \). Then

\[
\mathcal{V}^\varphi_\xi \ni (f \eta)^h + (\nabla^M_M f \eta)_{\xi}^0 = \eta^\nu_{\xi}.
\]

Therefore, we see that \( \dim V = 2(m-n) \), where \( m = \dim M \) and \( n = \dim N \). Thus \( V = \mathbb{H} \), so (1.5) holds. Now, let \( \eta \in \Gamma(\mathcal{V}^\varphi) \) and \( X \in \Gamma(\mathcal{H}^\varphi) \). For \( A = \eta^h_{\xi} + (\nabla^M_M \eta)_{\xi}^0 \) and \( B = X^\nu - q\omega_q(\xi)\langle X, \xi \rangle_{\xi}^\nu + \omega_\alpha(\xi)\eta_p(\nabla^M X)_{\xi}^h \)

\[
(A, B)_{T_M} = \omega_\alpha(\xi)\langle X, \nabla^M_M \eta \rangle_M + q\langle X, \xi \rangle_M \langle \xi, \nabla^M_M \eta \rangle_M
\]

\[
- q\omega_q(\xi)\langle X, \xi \rangle_M (\nabla^M_M \eta, \xi)_{\xi}^0
\]

\[
- q^2\omega_q(\xi)\langle X, \xi \rangle_M (\nabla^M_M \eta, \xi)_{\xi}^0 (1 - \omega_q(\xi) - q\omega_q(\xi)\langle \xi |^2_M)
\]

\[
= q\omega_\alpha(\xi)\eta_p(X, \xi) (\nabla^M_M \eta, \xi)_{\xi}^0
\]

Thus \( \mathbb{H} \) is orthogonal to \( \mathcal{V}^\varphi_\xi \). Again, as above, taking a function \( f \) such that \( \xi f = 1 \) and \( f(x) = 0 \), we get \( X^h_{\xi} \in \mathbb{H} \) for \( X \in \Gamma(\mathcal{H}^\varphi) \). Hence, \( \dim \mathbb{H} = 2m \). Therefore, \( \mathbb{H} = \mathcal{H}^\phi_\xi \) and (1.6) holds.

The proof of Theorem 1.3 also implies

**Corollary 1.6.** Assume \( \varphi : M \rightarrow N \) is a submersion. Let \( \Phi = \varphi_* : TM \rightarrow TN \) and equip tangent bundles \( TM \) and \( TN \) with Cheeger–Gromoll type metrics \( h_{p,q,\alpha} \) and \( h_{r,s,\beta} \), respectively. Then

\begin{align*}
\eta^\nu_{\xi} &\in \mathcal{V}^\varphi_\xi \quad \text{for } \eta \in \mathcal{V}^\varphi, \\
X^h_{\xi} &\in \mathcal{H}^\varphi_\xi \quad \text{for } X \in \mathcal{H}^\varphi.
\end{align*}

Let \( \varphi : M \rightarrow N \), be a Riemannian submersion, \( \Phi = \varphi_* : TM \rightarrow TN \). A vector field \( \hat{Z} \in \Gamma(\mathcal{H}^\varphi) \) is called basic if there is a vector field \( Z \in \Gamma(TN) \) such that \( \varphi_* \hat{Z} = Z \circ \varphi \). The correspondence \( Z \mapsto \hat{Z} \) is one–to–one, since \( \varphi_* : \mathcal{H}^\varphi_\xi \rightarrow T_{\varphi(x)}N \) is an isomorphism. Let \( T : \Gamma(\mathcal{H}^\varphi) \times \Gamma(\mathcal{H}^\varphi) \rightarrow \Gamma(\mathcal{V}^\varphi) \) be an integrability tensor of \( \varphi \),

\[
T(X, Y) = \frac{1}{2}[X, Y]^T.
\]
Let $\nabla^M$ and $\nabla^N$ be the Levi–Civita conection of $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_N$, respectively. Then one can prove that for basic vector fields $\hat{Z}, \hat{W}$ and vertical vector field $\xi$

\[
\langle (\nabla^M_\xi \hat{Z})^\bot, \hat{W} \rangle_M = \langle \xi, T(\hat{Z}, \hat{W}) \rangle_M.
\]

(1.9)

For more details on Riemannian submersions see [4, Chapter 9].

Assume now, $\varphi : (M, \langle \cdot, \cdot \rangle_M) \to (N, \langle \cdot, \cdot \rangle_N)$ is a horizontally conformal map with a dilatation $\lambda$. If we put $\langle \cdot, \cdot \rangle_\lambda = \lambda \langle \cdot, \cdot \rangle_M$, then $\varphi : (M, \langle \cdot, \cdot \rangle_\lambda) \to (N, \langle \cdot, \cdot \rangle_N)$ is a Riemannian submersion. Clearly, horizontal and vertical distributions with respect to $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_\lambda$ coincide. The Levi–Civita connections $\nabla^M$ and $\nabla^\lambda$ of $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_\lambda$ satisfy

\[
\nabla^\lambda_X Y = \nabla^M_X Y + S(X, Y),
\]

where $S$ is a symmetric tensor field given by

\[
S(X, Y) = \frac{1}{2\lambda}((X\lambda)Y + (Y\lambda)X - \langle X, Y \rangle_M \text{grad } \lambda).
\]

Let $\Phi = \varphi_* : TM \to TN$ and equip $TM$ and $TN$ with Cheeger–Gromoll type metrics $h_{p,q}$ and $h_{r,s}$, respectively. For simplicity, put

\[
P(X, \xi) = \sum_{i=1}^n \langle \xi, T(X, e_i) \rangle_M \varphi_* e_i, \quad \xi \in T_x M, \quad X \in \mathcal{H}_x^\varphi,
\]

where $e_1, \ldots, e_n$ is an orthonormal basis of $\mathcal{H}_x^\varphi$.

**Lemma 1.7.** Let $X \in \mathcal{H}_x^\varphi$, $\xi \in T_x M$, $x \in M$. Then

\[
\Phi_* (X^\xi)_h = (\varphi_* X)^{\varphi_* \xi}_h + (\varphi_* S(X, \xi))^{\varphi_* \xi} + P(X, \xi)^{\varphi_* \xi}.
\]

(1.11)

**Proof.** We may assume that $X^\xi_h = (\gamma^h)'(0)$, where $\gamma$ is a curve on $M$ such that $\gamma(0) = x$, $\gamma(t) \in \mathcal{H}_x^\varphi$ and $\dot{\gamma}(0) = X$, and $\gamma^h$ is a horizontal lift of $\gamma$ to $TM$ such that $\gamma^h(0) = \xi$. Put $\eta = \gamma^h$ for simplicity. Then

\[
\nabla^N_{\varphi_* \gamma} \varphi_* \eta = \nabla^N_{\varphi_* \gamma} \varphi_* \eta^\bot = \varphi_* (\nabla^{\lambda}_\gamma \eta^\bot)
\]

and by (1.10) and the fact that $\eta$ is parallel with respect to $\nabla^M$,\n
\[
\nabla^{\lambda}_\gamma \eta^\bot = S(\dot{\gamma}, \eta).
\]

Therefore,

\[
\nabla^{\lambda}_\gamma \eta^\bot = S(\dot{\gamma}, \eta) - \nabla^{\lambda}_\gamma \eta^T.
\]

Next, for a vector field $Y$ on $N$, by (1.9)

\[
\langle \varphi_* (\nabla^{\lambda}_\gamma \eta^T), Y \rangle_N = \lambda (\nabla^{\lambda}_\gamma \eta^T, \dot{Y})_M = \lambda (\nabla^{\lambda}_{\eta^T} \dot{\gamma}, \dot{Y})_M = -\lambda (\eta^T, T(\dot{\gamma}, \dot{Y}))_M.
\]

Thus

\[
\varphi_* (\nabla^{\lambda}_\gamma \eta^T) = \frac{1}{\lambda} \sum_i \langle \varphi_* (\nabla^{\lambda}_\gamma \eta^T), \varphi_* e_i \rangle_N \varphi_* e_i = -\lambda \sum_i \langle \eta^T, T(\dot{\gamma}, e_i) \rangle_M \varphi_* e_i.
\]
Since
\[ K(\Phi_*X^h_\xi) = \nabla^N_{\varphi_*X}\varphi_\eta \]
\[ = \varphi_*(\nabla^2_{\varphi_*\eta}^T) \]
\[ = \varphi_*S(X,\eta) - \varphi_*(\nabla^2_{\varphi_*\eta}) \]
\[ = \varphi_*S(X,\eta) + \sum_i \langle \eta^T, T(X,e_i) \rangle_M \varphi_\eta e_i \]
\[ = \varphi_*S(X,\xi) + \sum_i \langle \xi, T(X,e_i) \rangle_M \varphi_\eta e_i \]
and
\[ \pi_* (\Phi_*X^h_\xi) = \varphi_*(\pi_*X^h_\xi) = \varphi_*X, \]
the equality (1.11) holds.

By Lemma 1.7 we have (see also [1, Lemma 4.5.1]).

**Corollary 1.8.**

1. Let \( X, Y \in H^{\varphi}_\xi, \xi \in T_xM \). Then
   \[ B(X,\xi) = \varphi_*(S(X,\xi)) + P(X,\xi), \]
   \[ \langle B(X,\xi), \varphi_*Y \rangle_N = \lambda(\langle S(X,\xi), Y \rangle_M + \langle \xi, T(X,Y) \rangle_M). \]
2. Let \( X \in H^{\varphi}_\xi, \xi \in V^{\varphi}_\xi \). Then
   \[ \langle \varphi_*(S(X,\xi)), P(X,\xi) \rangle_N = 0, \]
   \[ |B(X,\xi)|^2_N = |(\varphi_*(S(X,\xi)))|^2_N + |P(X,\xi)|^2_N. \]
3. \( B(X,\xi) = 0 \) for all \( X \in \Gamma(H^{\varphi}_\xi), \xi \in \Gamma(TM) \) if and only if \( T = 0 \) and \( S = 0 \).

**Proof.** (1.12) follows by (1.11) and (1.14). For \( X, Y \in H^{\varphi}_\xi \) and \( \xi \in T_xM \) we have
\[ \langle B(X,\xi), \varphi_*Y \rangle_N = \langle \varphi_*(S(X,\xi)), \varphi_*Y \rangle_N + \sum_i \langle \xi, T(X,e_i) \rangle_M \langle \varphi_*e_i, \varphi_*Y \rangle_N \]
\[ = \lambda(\langle S(X,\xi), Y \rangle_M + \langle \xi, T(X,Y) \rangle_M, \]
which proves (1.13). Now, let \( X \in H^{\varphi}_\xi \) and \( \xi \in V^{\varphi}_\xi \). Then
\[ \varphi_*(S(X,\xi)) = \frac{1}{2\lambda}(\xi\lambda)\varphi_*X. \]
Thus
\[ \langle \varphi_*(S(X,\xi)), P(X,\xi) \rangle_N = \frac{1}{2\lambda}(\xi\lambda) \sum_i \langle \xi, T(X,e_i) \rangle_M \langle \varphi_*X, \varphi_*e_i \rangle_N \]
\[ = \frac{1}{2} \langle \xi, T(X,X) \rangle_M = 0, \]
since \( T \) is skew–symmetric. Hence (1.14) holds, which implies (1.15). (3) is a consequence of (1.12) and (1.14). □
Let $X \in \mathcal{H}_x^\varphi$ and $\xi \in T_x M$. Decompose $S(X, \xi)_x^\nu$ into horizontal and vertical part with respect to $V^\Phi$ and $H^\Phi$,

$$S(X, \xi)_x^\nu = S^\top_\Phi (X, \xi) + S^\bot_\Phi (X, \xi).$$

Equip tangent bundles $TM$ and $TN$ with Cheeger–Gromoll type metrics $h_{p,q,\alpha}$ and $h_{r,s,\beta}$, respectively.

**Lemma 1.9.** Assume $\varphi$ and $\Phi$ are both horizontally conformal with dilatations $\lambda$ and $\Lambda$, respectively. Let $X \in \mathcal{H}_x^\varphi$, $\xi \in T_x M$. Then

\begin{align}
(1.16) \quad \Lambda(\xi)|X|^2_M &= \lambda(x)|X|^2_M + |B(X, \xi)_x^\nu|_{TN}^2, \\
(1.17) \quad \Lambda(\xi)|X|^2_M + \Lambda(\xi)|S^\bot_\Phi (X, \xi)|^2_{TN} &= \lambda(x)|X|^2_M + |P(X, \xi)_x^\nu|_{TN}^2.
\end{align}

**Proof.** (1.16) follows directly by (1.4) (and Corollary 1.6). To prove (1.17), define $V = X_h^h - S(X, \xi)_x^\nu$. Then

$$V + S^\top_\Phi (X, \xi) = X_h^h - S^\bot_\Phi (X, \xi) \in H^\Phi.$$

Therefore,

$$|\Phi_*(X_h^h - S^\bot_\Phi (X, \xi))|_{TN}^2 = \Lambda(\xi)|X_h^h - S^\bot_\Phi (X, \xi)|_{TN}^2 = \Lambda(\xi)|X|^2_M + \Lambda(\xi)|S^\bot_\Phi (X, \xi)|_{TN}^2,$$

since $\langle X_h^h, S^\bot_\Phi (X, \xi) \rangle_{TM} = \langle X_h^h, S(X, \xi)_x^\nu \rangle = 0$, and, by (1.11),

$$|\Phi_*(V + S^\top_\Phi (X, \xi))|_{TN}^2 = |\Phi_* V|_{TN}^2 = |(\phi_*(X))^h_h + P(X, \xi)_x^\nu|_{TN}^2 = \lambda(x)|X|^2_M + |P(X, \xi)_x^\nu|_{TN}^2.$$

Hence (1.17) holds. $\square$

**Lemma 1.10.** If $\Phi$ and $\varphi$ are both horizontally conformal, then $\lambda$ is constant.

**Proof.** Let $X \in \mathcal{H}_x^\varphi$ and $\xi \in \mathcal{V}_x^\varphi$. Comparing (1.16) and (1.17) and using (1.15) we get

$$|S^\bot_\Phi (X, \xi)|_{TN}^2 + |(\varphi_*(S(X, \xi)))_x^\nu|_{TN}^2 = 0,$$

which implies, $\varphi_*(S(X, \xi)) = 0$, so $\xi \lambda = 0$. It follows that grad $\lambda$ is horizontal.

Assume now, $\xi \in \mathcal{H}_x^\varphi$. Then $P(X, \xi) = 0$. Again, comparing (1.16) and (1.17) we get

$$0 = \Lambda(\xi)|S^\bot_\Phi (X, \xi)|_{TN}^2 + |B(X, \xi)_x^\nu|_{TN}^2.$$

Thus

$$B(X, \xi) = 0 \quad \text{and} \quad S^\bot_\Phi (X, \xi) = 0.$$
Hence, as before, \( \varphi_*S(X, \xi) = 0 \). Since \( \text{grad} \lambda \) is horizontal, we may put \( X = \xi = \text{grad} \lambda \). Then
\[
0 = \varphi_*S(\text{grad} \lambda, \text{grad} \lambda) = -\frac{1}{2\lambda} |\text{grad} \lambda|^2_M \varphi_*(\text{grad} \lambda).
\]
Therefore, \( \text{grad} \lambda \) is vertical. Finally, \( \lambda \) is constant. \( \square \)

**Lemma 1.11.** Assume \( \dim N \geq 2 \). Let \( \Phi \) be a horizontally conformal map with dilatation \( \Lambda \). Then \( \varphi \) is horizontally conformal with constant dilatation \( \lambda = \Lambda(0) \). Moreover, \( \Lambda \) is constant and the horizontal distribution \( \mathcal{H}^\varphi \) is integrable.

**Proof.** Let \( X, Y \in \mathcal{H}^\varphi_x, 0 = 0_x \in T_xM \). Then by Corollary 1.6, \( X_0^h, Y_0^h \in \mathcal{H}^\Phi_0 \). By (1.4), \( \Phi_*(X_0^h) = (\varphi_*X)_0^h \) and \( \Phi_*(Y_0^h) = (\varphi_*Y)_0^h \). Thus
\[
\Lambda(0)(X, Y)_M = \langle \Phi_*X_0^h, \Phi_*Y_0^h \rangle_{TN} = \langle \varphi_*X, \varphi_*Y \rangle_N.
\]
Hence \( \varphi \) is conformal with dilatation \( \lambda(x) = \Lambda(0_x) \). By Lemma 1.10 \( \lambda \) is constant. Thus \( S = 0 \).

Let \( X, Y \in \Gamma(\mathcal{H}^\varphi), \xi \in \mathcal{V}^\varphi \). Then \( X_\xi^\varphi + \omega_\alpha(\xi)^p(\nabla_\xi X)_\xi^h, Y_\xi^h \in \mathcal{H}^\Phi_\xi \) by Theorem 1.5 and Corollary 1.6. Moreover, by Corollary 1.8
\[
\langle B(Y, \xi, \varphi_*X) \rangle_N = \lambda \langle \xi, T(Y, X) \rangle_M.
\]
Hence, by horizontal conformality of \( \Phi \) and (1.9) we get
\[
(\Lambda(\xi) - \lambda - \lambda(1 + \alpha|\xi|^2)^p)\langle \xi, T(Y, X) \rangle_M = \langle B(\nabla_\xi^M X, \xi), B(Y, \xi) \rangle_N.
\]
Put \( A(\xi) = \Lambda(\xi) - \lambda \). Then by (1.16), \( A(t\xi) = t^2 A(\xi) \). Hence, replacing \( \xi \) by \( t\xi \) and assuming for simplicity \( |\xi|_M = 1 \), we get
\[
t^2(\Lambda(\xi)\langle \xi, T(Y, X) \rangle_M - \langle B(\nabla_\xi^M X, \xi), B(Y, \xi) \rangle_M) = \lambda(1 + \alpha t^2)^p \langle \xi, T(Y, X) \rangle_M.
\]
Therefore, \( \langle \xi, T(Y, X) \rangle_M = 0 \) and, since \( \xi \in \Gamma(\mathcal{V}^\varphi) \) was taken arbitrary, \( T = 0 \). This implies, together with the fact that \( S = 0 \) and Corollary 1.8 that
\[
B(X, \xi) = 0 \quad \text{for} \quad X \in \mathcal{H}^\varphi \quad \text{and} \quad \xi \in TM.
\]
Hence, by (1.16), \( \Lambda(\xi) = \lambda \) for any \( \xi \). \( \square \)

### 2. Main results

**2.1. Differential as a horizontally conformal map.** Consider a smooth map \( \varphi : (M, \langle \cdot, \cdot \rangle_M) \to (N, \langle \cdot, \cdot \rangle_N), \dim M > \dim N \geq 2 \), between Riemannian manifolds. Let \( \Phi = \varphi_* : (TM, \tilde{h}) \to (TN, h) \), where \( \tilde{h} = h_{p,q,\alpha} \) and \( h = h_{r,s,\beta} \) are Cheeger–Gromoll type metrics.
Theorem 2.1. \( \Phi \) is horizontally conformal if and only if \( \varphi \) is totally geodesic and horizontally conformal with constant dilatation \( \lambda \), \( p \alpha = r \beta = 0 \) and \( q = \lambda s \). Then, the dilatation \( \Lambda \) of \( \Phi \) is constant and equal to \( \lambda \), horizontal distribution \( \mathcal{H}^\varphi \) is integrable and vertical distribution \( \mathcal{V}^\varphi \) is totally geodesic.

Proof. Assume \( \Phi \) is horizontally conformal. Then by Lemma 1.11, \( \varphi \) is horizontally conformal with constant dilatation \( \lambda \), the dilatation \( \Lambda \) of \( \Phi \) is constant and equal to \( \lambda \) and by Corollary 1.8, \( B(X, \xi) = 0 \) for \( X \in \mathcal{H}^\varphi \) and \( \xi \in TM \).

Let \( X \in \Gamma(\mathcal{H}^\varphi) \) and \( \xi \in \Gamma(\mathcal{V}^\varphi) \). Then \( A = X^\varphi_\xi + \omega_\alpha(\xi)^p(\nabla^M X)^h_\xi \in \mathcal{H}^\varphi_\xi \) by Theorem 1.5. Since \( T = 0 \), by (1.9), \( \nabla^M X \in \mathcal{V}^\varphi \). Then \( |\Phi_* A|_{TN}^2 = \lambda |A|_{TM}^2 \) implies

\[
\lambda |X|_M^2 (1 + \alpha |\xi|_M^2)^p (1 - (1 + \alpha |\xi|_M^2)^p) + \lambda |\nabla^M X|_M^2 = |B(\nabla^M X, \xi)|_{TN}^2 + 2(1 + \alpha |\xi|_M^2)^p |B(\nabla^M X, \xi), \varphi_* X|_{TN}.
\]

Replacing \( \xi \) by \( t\xi \), assuming \( |\xi|_M^2 = 1 \) and computing the second derivative at \( t = 0 \), we get

\[-p\alpha \lambda |X|_M^2 + \lambda |\nabla^M X|_M^2 = 2|B(\nabla^M X, \xi), \varphi_* X|_{TN}.
\]

Since \( |B(\nabla^M X, \xi), \varphi_* X|_{TN} = \lambda |\nabla^M X|_M^2 \), we obtain \(-p\alpha |X|_M^2 = |\nabla^M X|_M^2 \).

Therefore, \( p\alpha = 0 \) and \( \nabla^M X = 0 \). This implies \( B = 0 \) globally, so \( \varphi \) is totally geodesic. In particular, horizontal distribution is integrable and vertical distribution is totally geodesic.

Moreover, for \( X \in \Gamma(\mathcal{H}^\varphi) \) basic and \( \xi \in \mathcal{H}^\varphi_\xi \) such that \( \xi \) and \( X_\xi \) are orthogonal, by Theorem 1.5 and the fact that \( \alpha p = 0 \) we have \( A = X^\varphi_\xi + (\nabla^M X)^h_\xi \in \mathcal{H}^\varphi_\xi \). Since \( \nabla^M X \in \mathcal{H}^\varphi_\xi \), the condition \( |\Phi_* A|_{TN}^2 = \lambda |A|_{TM}^2 \) simplifies to

\[
q^3 \omega_\varphi(\xi)^2 |\xi|_M^2 - 2q^2 \omega_\varphi(\xi) |\xi|_M^2 = \lambda s + \lambda s q^2 \omega_\varphi(\xi)^2 |\xi|_M^2 - 2\lambda s q \omega_\varphi(\xi) |\xi|_M^2.
\]

Hence \( q = \lambda s \).

Conversely, assume \( \varphi \) is totally geodesic and horizontally conformal with constant dilatation \( \lambda \), \( p\alpha = r \beta = 0 \) and \( q = \lambda s \). By Theorem 1.5, \( \Phi : TM \to TN \) is a submersion and vertical and horizontal distributions \( \mathcal{V}^\Phi \) and \( \mathcal{H}^\Phi \) are given by (1.5) and (1.6), respectively. It remains to show that
Φ* : HΦ → T(TN) is conformal, but this follows immediately by simple calculations.

2.2. Differential as a harmonic morphism. Let ϕ : (M, ⟨·, ·⟩_M) → (N, ⟨·, ·⟩_N). Consider notation from the first section. We say that ϕ is harmonic if its tension field τ(ϕ) = trB vanishes. If e₁, ..., eₙ is an orthonormal frame on M then

\[
\tau(\varphi) = \sum_i (\nabla_{e_i}^M \varphi^* e_i - \varphi^* (\nabla_{e_i}^M e_i)).
\]

A map ϕ : M → N is said to be a harmonic morphism if for every harmonic function f : G → ℝ defined on an open subset G of N with ϕ⁻¹(G) nonempty, the composition f ◦ ϕ is a harmonic map on ϕ⁻¹(G).

There is a useful characterisation of harmonic morphisms due to Fuglede and Ishihara. Namely, see [1], ϕ is a harmonic morphism if and only if it is horizontally conformal and harmonic. Moreover, if ϕ is horizontally conformal, then its tension field simplifies to [1]

\[
\tau(\varphi) = -\frac{n - 2}{2} \varphi^*(\text{grad}(\ln \lambda)) - (m - n) \varphi^*(\kappa_ϕ),
\]

where \(\kappa_ϕ = \frac{1}{m - n} \sum_i (\nabla_{e_i}^M e_i))^⊥\),

where \(e_1, \ldots, e_{m-n}\) is an orthonormal frame of \(V^ϕ\).

Equip tangent bundle \(TM\) with Cheeger–Gromoll type metric \(h_{p,q,α}\). Let \(ω_q(ξ) = (1 + q|ξ|^2_M)^{-1}\). The Levi–Civita connection \(∇_{T M}^\xi\) corresponding to \(h_{p,q,α}\) evaluated at \(ξ\) is given by [8]

\[
\begin{align*}
∇_{X^h}^T Y^h &= (∇_X^Y)^h - \frac{1}{2} (R(X, Y)ξ)^v, \\
∇_{X^h}^T Y^v &= (∇_X^Y)^v + \frac{1}{2} ω(ξ)^p(R(ξ, Y)X)^h, \\
∇_{X^v}^T Y^h &= \frac{1}{2} ω(ξ)^p(R(ξ, X)Y)^h, \\
∇_{X^v}^T Y^v &= -αpω(ξ)(⟨X, ξ⟩_M Y + ⟨Y, ξ⟩_M X)^v \\
&+ (αpω(ξ) + q)ω_q(ξ)⟨X, Y⟩_M ξ^v + αpqω(ξ)ω_q(ξ)⟨X, ξ⟩_M (Y, ξ)_M ξ^v.
\end{align*}
\]

Let \(ϕ : (M, ⟨·, ·⟩_M) → (N, ⟨·, ·⟩_N)\), \(\dim M > \dim N ≥ 2\), be a smooth map between Riemannian manifolds, \(Φ = ϕ_* : (TM, ˜h) → (TN, h)\), where \(˜h = h_{p,q,α}\) and \(h = h_{r,s,β}\) are Cheeger–Gromoll type metrics.

**Theorem 2.2.** \(Φ\) is a harmonic morphism if and only if \(ϕ\) is a totally geodesic harmonic morphism, the dilatation of \(ϕ\) is constant and \(h, \tilde{h}\) are Sasaki metrics.
Proof. Assume $\Phi$ is a harmonic morphism. In particular, $\Phi$ is horizontally conformal. Hence, by Theorem 2.1, $\varphi$ is horizontally conformal, the dilatations of $\Phi$ and $\varphi$ are constant, $\Lambda = \lambda$, and $\varphi$ is totally geodesic. Therefore $\varphi$ is harmonic, so $\varphi$ is a harmonic morphism. Thus, by (2.2), $\kappa_\varphi = 0$, $\kappa_\Phi = 0$. Moreover, by Theorem 2.1 $p\alpha = r\beta = 0$ and $q = \lambda s$. Let $\xi \in \mathfrak{H}^\varphi$ and let $e_1, \ldots, e_{m-n}$ be an orthonormal frame for $\mathcal{V}_\xi^\varphi$, $m = \text{dim } M$, $n = \text{dim } N$. Since vertical distribution $\mathcal{V}_\xi^\varphi$ is, by Theorem 2.1 totally geodesic, by Theorem 1.5 and Corollary 1.6
\[
E_i = (e_i)_\xi^h, \quad F_i = (e_i)_{\xi}^v, \quad i = 1, \ldots, m - n,
\]
is an orthonormal frame for $\mathcal{V}_\xi^\varphi$. Then
\[
\nabla^T_M E_i = (\nabla^M e_i)_\xi^h, \quad \nabla^T_M F_i = q\omega_q(\xi)_{\xi}^v.
\]
Therefore,
\[
\Phi_*(\kappa_\Phi) = \frac{1}{2}(\varphi_*(\kappa_\varphi))_{\varphi_*\xi}^h + \frac{1}{2}q\omega_q(\xi)_{\varphi_*\xi}^v.
\]
Since $\Phi_*(\kappa_\Phi) = 0$ and $\varphi_*(\kappa_\varphi) = 0$, it follows that $q = 0$. Hence, $s = 0$ and $\hat{h}$, $h$ are Sasaki metrics.

Conversely, assume $\varphi$ is a totally geodesic harmonic morphism, the dilatation of $\varphi$ is constant and $\hat{h}$, $h$ are Sasaki metrics. Then, by Theorem 2.1 $\Phi$ is horizontally conformal with constant dilatation. Moreover, $\varphi_*(\kappa_\varphi) = 0$. Hence, similarly as above,
\[
\Phi_*(\kappa_\Phi) = \frac{1}{2}(\varphi_*(\kappa_\varphi))_{\hat{h}}^h = 0,
\]
and by (2.2), $\Phi$ is a harmonic morphism. □

Remark 2.3. Condition $p\alpha = 0$ for Cheeger–Gromoll metric $h_{p,q,\alpha}$ is equivalent to condition $p = 0$ for the metric $h_{p,q}$. It follows that in the case of horizontal conformality of a differential, and in consequence for a differential to be a harmonic morphism, it is sufficient to consider only $h_{p,q}$ metrics introduced in [3].

References

[1] P. Baird, J. C. Wood, Harmonic morphisms between Riemannian manifolds, Oxford University Press, Oxford 2003.
[2] M. Benyounes, E. Loubeau, C. M. Wood, Harmonic sections of Riemannian vector bundles, and metrics of Cheeger-Gromoll type, Differential Geom. Appl. 25 (2007), no. 3, 322–334.
[3] M. Benyounes, E. Loubeau, C. M. Wood, The geometry of generalised Cheeger-Gromoll metrics, preprint, arXiv:math/0703059.
[4] A.L. Besse, Einstein Manifolds, Springer-Verlag, Berlin, Heidelberg, 1987.
[5] P. Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew. Math. 210 (1962), 73–88.

[6] W. Kozłowski, K. Niedziałomski, *Conformality of a differential with respect to Cheeger–Gromoll type metrics*, preprint, arXiv:0809.4427.

[7] W. Kozłowski, Sz. M. Walczak, *Collapse of unit horizontal bundles equipped with a metric of Cheeger–Gromoll type*, Differential Geom. Appl. 27 (2009), no. 3, 378–383.

[8] M. I. Munteanu, *Cheeger–Gromoll type metrics on the tangent bundle*, Sci. Ann. Univ. Agric. Sci. Vet. Med. 49 (2006), no. 2, 257–268.

Department of Mathematics and Computer Science
University of Łódź
ul. Banacha 22, 90-238 Łódź
Poland
E-mail address: wojciech@math.uni.lodz.pl

Department of Mathematics and Computer Science
University of Łódź
ul. Banacha 22, 90-238 Łódź
Poland
E-mail address: kamiln@math.uni.lodz.pl