ARITHMETIC EULER TOP

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Abstract. The theory of differential equations has an arithmetic analogue in which derivatives of functions are replaced by Fermat quotients of numbers. Many classical differential equations (Riccati, Weierstrass, Painlevé, etc.) were previously shown to possess arithmetic analogues. The paper introduces an arithmetic analogue of the Euler differential equations for the rigid body.

1. Introduction

The theory of differential equations has an arithmetic analogue in which derivatives of functions are replaced by Fermat quotients of numbers. This arithmetic analogue was introduced in [3]; for an exposition of part of the resulting theory we refer to [4]. The present paper morally fits into the theory developed in [3, 4]. However our paper is written so as to be entirely self-contained and, in particular, it is independent of [3, 4]; the few facts we need from [3, 4] will be quickly reviewed here.

Many remarkable classical differential equations have arithmetic analogues. Examples of such classical differential equations are: the Riccati equation [7], the Weierstrass equation [3], the Painlevé VI equation [3, 5], Schwarzian equations (satisfied by modular forms) [7], and linear differential equations corresponding to special connections in Riemannian geometry (such as Chern, Levi-Civita, etc.) [6, 7, 2, 8].

The purpose of the present paper is to develop an arithmetic analogue of the Euler differential equations for the rigid body (the Euler top). This is a system of 3 ordinary (non-linear) differential equations in 3 variables which is one of the simplest examples of algebraically completely integrable systems [11]. As such its flow on 3-space, referred to in what follows as the classical Euler flow, can be viewed as a derivation $\delta$ on the polynomial ring $\mathbb{C}[x_1, x_2, x_3]$ given by

$$\delta = (a_2 - a_3)x_2x_3 \frac{\partial}{\partial x_1} + (a_3 - a_1)x_3x_1 \frac{\partial}{\partial x_2} + (a_1 - a_2)x_1x_2 \frac{\partial}{\partial x_3},$$

where $a_1, a_2, a_3 \in \mathbb{C}$ are distinct complex numbers. This flow is trivially seen to have 2 independent prime integrals

$$H_1 = \sum_{i=1}^{3} a_i x_i^2, \quad H_2 = \sum_{i=1}^{3} x_i^2,$$

in the sense that

$$\delta H_1 = \delta H_2 = 0.$$

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For generic \( c = (c_1, c_2) \in \mathbb{C}^2 \), the loci \( E_c \) in 3-space, given by
\[
(1.4) \quad H_1 = c_1, \quad H_2 = c_2,
\]
are affine elliptic curves. We refer to \( E_c \) as the level sets of \( H_1, H_2 \). Then the classical Euler flow \( \delta \) is “linearized” when restricted to these level sets \( E_c \) in the sense that, if one denotes by \( \delta_c \) the action of \( \delta \) as Lie derivative on the 1-forms on \( E_c \) and if \( \omega_c \) is the “canonical” invariant 1-form on \( E_c \), then
\[
(1.5) \quad \delta_c \omega_c = 0.
\]

Here is our arithmetic analogue of the above.

Let \( A \) be a complete discrete valuation ring with maximal ideal generated by an odd prime \( p \) and residue field \( \mathbb{F} = A/pA \) algebraic over the prime field. Let \( a_1, a_2, a_3 \in A \) be distinct mod \( p \) and consider again \( H_1, H_2 \) as in 1.2. Recall from [3] that a \( p \)-derivation on a ring \( B \) (in which \( p \) is a non-zero divisor) is a map \( \delta : B \to B \) such that the map \( \phi : B \to B, \phi(b) := b^p + p\delta b \), is a ring homomorphism; \( \phi \) is then referred to as the Frobenius lift attached to \( \delta \). Of course, \( A \) has a unique \( p \)-derivation; any \( p \)-derivation on an \( A \)-algebra will be tacitly assumed compatible with the one on \( A \). We also denote by \( ^\wedge \) the \( p \)-adic completion of rings or schemes.

We will prove:

**Theorem 1.1.** There exists an open set \( X \) of the affine 3-space over \( A \), a \( p \)-derivation \( \delta \) on \( \mathcal{O}(\hat{X}) \), with attached Frobenius lift \( \phi \), and an invertible function \( \lambda \in \mathcal{O}(X)^\times \) such that:

1) The following equalities hold in \( \mathcal{O}(\hat{X}) \):
\[
\delta H_1 = \delta H_2 = 0;
\]

2) For any \( c \in A^2 \) in a suitable Zariski open set and satisfying \( \delta c = 0 \) we have
\[
\frac{\phi_c^*}{p} \omega_c \equiv \lambda(c) \cdot \omega_c \mod p,
\]
on \( \hat{E}_c \cap \hat{X} \), where \( E_c \) are the level sets of \( H_1, H_2 \), the map \( \phi_c \) is the restriction of \( \phi \) to \( \hat{E}_c \cap \hat{X} \), and \( \omega_c \) is the invariant 1-form on \( E_c \).

Conditions 1 and 2 in our theorem are of course to be viewed as analogues of conditions 1.3 and 1.5 respectively. For a more precise statement of the above theorem we refer to Theorem 6.1 in the text; this more precise statement will explicitly give \( X \) and \( \lambda \). As we shall see the function \( \lambda \) will be the “inverse of the Hasse invariant” for a family of plane quartics isogenous to the family of level sets of \( H_1, H_2 \).

A \( p \)-derivation \( \delta \) as in Theorem 1.1 will be referred to as an arithmetic Euler flow on \( \hat{X} \). Arithmetic Euler flows are, of course, not unique. Remarkably, it will turn out that the set of arithmetic Euler flows mod \( p \) is parameterized by the “prime integrals mod \( p^2 \)” for the classical Euler flow 1.1. Indeed formula 1.1 defines a derivation on the \( \mathbb{F} \)-vector space \( \mathcal{O}(\hat{X}) \), where \( \hat{X} := X \otimes \mathbb{F} \), and we have:

**Theorem 1.2.** Assume \( \mathbb{F} \) is infinite. Then the set of equivalences classes mod \( p \) of arithmetic Euler flows on \( \hat{X} \) is a principal homogeneous space for the \( \mathbb{F} \)-vector space of all elements in \( \mathcal{O}(\hat{X}) \) killed by the derivation 1.1.
Cf. Theorem 6.5 in the text for a more precise statement in which $X$ and $\lambda$ are, again, specified explicitly.

The principal homogeneous space in Theorem 1.2 is non-empty by Theorem 1.1.

It is reasonable to expect that, more generally, higher dimensional algebraically completely integrable systems might have similar arithmetic analogues; no straightforward generalization of our arguments here seems possible, however, at this point.

The paper is organized as follows. Section 2 sets the geometric stage for the Euler equations (for both the classical and the arithmetic case). Section 3 reviews the classical Euler flow. Section 4 quickly reviews Fermat quotients ($p$-derivations) following [3, 4]. Section 5 reviews the Hasse invariant. In section 6 we state our main results which are Theorems 6.1 and 6.5. Section 7 will provide the proofs of our theorems.

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2. Basic geometric setting

In this section we introduce the geometric objects that will be relevant throughout the paper.

Let $A$ be either a field or a discrete valuation ring and assume 2 is invertible in $A$. We denote by $A^\times$ the group of invertible elements. Let $x_1, x_2, x_3$ and $z_1, z_2$ be variables. Let $a_1, a_2, a_3 \in A$ be such that

$$\text{(2.1)} (a_1 - a_2)(a_2 - a_3)(a_3 - a_1) \in A^\times.$$ 

We consider the quadratic forms $H_1, H_2 \in A[x_1, x_2, x_3]$,

$$\text{(2.2)} H_1 := \sum_{i=1}^{3} a_i x_i^2, \quad H_2 := \sum_{i=1}^{3} x_i^2.$$ 

Also we consider the affine spaces

$$\text{(2.3)} \mathbb{A}^2 = \text{Spec } A[z_1, z_2], \quad \mathbb{A}^3 = \text{Spec } A[x_1, x_2, x_3]$$

and the morphism

$$\text{(2.4)} \mathcal{H}: \mathbb{A}^3 \to \mathbb{A}^2$$

defined by

$$\text{(2.5)} z_1 \mapsto H_1, \quad z_2 \mapsto H_2.$$ 

For any $A$-point $c = (c_1, c_2) \in A^2 = \mathbb{A}^2(A)$ we denote by

$$\text{(2.6)} E_c = \text{Spec } A[x_1, x_2, x_3]/(H_1 - c_1, H_2 - c_2)$$

the fiber of $\mathcal{H}$ at $c$ and by $i_c: E_c \to \mathbb{A}^3$ the inclusion.

Consider the polynomial

$$\text{(2.7)} N(z_1, z_2) = \prod_{i=1}^{3} (z_1 - a_i z_2) \in A[z_1, z_2].$$
Indeed, if we consider the ideal $\mathcal{I} = (x_1 - 1, x_2 - 1) \subseteq \mathbb{A}^2$, equivalently that all $2 \times 2$ minors of the matrix
\[
\left( \begin{array}{cccc} a_1 & a_2 & a_3 & c_1 \\ 1 & 1 & 1 & c_2 \end{array} \right)
\]
are invertible. Then, for $E_{ij}^c$ the locus in $E_c$ where $x_ix_j$ is invertible, we have that
\[
E_c = E_{12}^c \cup E_{23}^c \cup E_{31}^c
\]
and $E_c$ is smooth over $A$. Moreover, $E_c$ comes equipped with a global 1-form given by
\[
\omega_c = i_c^* \frac{dx_1}{(a_2 - a_3)x_2x_3} = i_c^* \frac{dx_2}{(a_3 - a_1)x_3x_1} = i_c^* \frac{dx_3}{(a_1 - a_2)x_1x_2},
\]
which we will refer to as the canonical 1-form on $E_c$. If one considers the projective closure $\overline{E}_c$ of $E_c$ in the projective space
\[
\mathbb{P}^3 = \text{Proj} A[t_0, t_1, t_2, t_3], \quad x_i = t_i/t_0,
\]
then, by the arguments above applied to the other affine charts of $\mathbb{P}^3$ one gets that $\overline{E}_c$ is smooth and comes equipped with a 1-form whose restriction to $E_c$ is $\omega_c$.

Here are more facts, needed later, about the geometry of the situation. Consider two more indeterminates $x, y$, and consider the polynomial
\[
F = F(z_1, z_2, x) \in A[z_1, z_2][x]
\]
defined by
\[
F := ((a_2 - a_3)x^2 + z_1 - a_2z_2)((a_3 - a_1)x^2 - z_1 + a_1z_2).
\]
This is a weighted homogeneous polynomial of degree 4 (a quartic) in $z_1, z_2, x$ with respect to the weights $(2, 2, 1)$. For any $c = (c_1, c_2) \in A^2$ set
\[
E_c' := \text{Spec} A[x, y]/(y^2 - F(c_1, c_2, x)).
\]
Then we have a morphism
\[
\pi : E_c \to E_c',
\]
given by
\[
x \mapsto x_3, \quad y \mapsto (a_1 - a_2)x_1x_2.
\]
Indeed if we consider the ideal
\[
I = (H_1 - z_1, H_2 - z_2) \subseteq A[x_1, x_2, x_3][z_1, z_2]
\]
we have the following congruences:
\[
(a_1 - a_2)x_1^2 \equiv (a_2 - a_3)x_2^2 + z_1 - a_2z_2 \mod I
\]
and
\[
(a_1 - a_2)x_2^2 \equiv (a_3 - a_1)x_3^2 - z_1 + a_1z_2 \mod I
\]
hence
\[
(a_1 - a_2)^2x_1^2x_2^2 \equiv F(x_3, z_1, z_2) \mod I.
\]
Note that, under the assumption that $\mathcal{I}(c_1, c_2) \subseteq \mathbb{A}^2$, the discriminant of $F$ is in $A^2$ so $E_c'$ is smooth over $A$. Moreover,
\[
\pi^*(\frac{dx}{y}) = i_c^* \frac{dx_3}{(a_1 - a_2)x_1x_2} = \omega_c.
\]
For $A$ a perfect field and $c_1, c_2$ satisfying $N(c_1, c_2) \neq 0$ we have that $E'_c$ is a smooth plane curve whose smooth projective model $\overline{E}_c$ is an elliptic curve and hence is induced by an isogeny of elliptic curves,

$$\overline{E}_c \rightarrow \overline{E}_c'.$$

This isogeny has degree 2, in particular it is separable.

### 3. Classical Euler Flow

In this section we review the classical Euler equations for the rigid body (the Euler top).

With notation as in section 2 assume, in the present section only, that $A$ is an algebraically closed field of characteristic $\neq 2$ (e.g., the complex field, which is the classical setting for the theory). Consider the $A$-derivation $\delta$ on the polynomial ring $A[x_1, x_2, x_3]$ given by

$$\begin{align*}
\delta x_1 &= (a_2 - a_3)x_2x_3, \\
\delta x_2 &= (a_3 - a_1)x_3x_1, \\
\delta x_3 &= (a_1 - a_2)x_1x_2.
\end{align*}$$

We refer to the derivation $\delta$ as the classical Euler flow.

For any $c = (c_1, c_2) \in A^2$ with $N(c_1, c_2) \neq 0$ denote by $\delta_c$ the derivation on $\mathcal{O}(E_c) = A[x_1, x_2, x_3]/(H_1 - c_1, H_2 - c_2)$ induced by the derivation $\delta$ on $A[x_1, x_2, x_3]$.

Here are a couple of trivial facts about the derivation $3.1$ that we collect in the form of the Theorem below; in this Theorem $\langle \; , \; \rangle$ stands for the pairing between derivations and 1-forms.

**Theorem 3.1.**

1) The following equalities hold in $A[x_1, x_2, x_3]$:

$$\delta H_1 = \delta H_2 = 0.$$

2) For any $c = (c_1, c_2) \in A^2$ with $N(c_1, c_2) \neq 0$ the derivation $\delta_c$ on $\mathcal{O}(E_c)$ and the canonical 1-form $\omega_c$ on $E_c$ are dual to each other in the sense that

$$\langle \delta_c, \omega_c \rangle = 1.$$

**Remark 3.2.**

1) Condition 1 above is interpreted as saying that $H_1$ and $H_2$ are prime integrals for $\delta$.

2) Condition 2 implies that the derivation $\delta_c$ on $E_c$ extends to a derivation of the structure sheaf of $\overline{E}_c$; morally this says that $\delta_c$ is linearized on $E_c$.

3) Let $\delta_c$ be any $A$-derivation on $E_c$ and let us still denote by $\delta_c$ the action of the Lie derivative along $\delta_c$ on 1-forms on $E_c$. Then we have

$$\langle \delta_c, \omega_c \rangle \in A \iff \delta_c \omega_c = 0.$$

This follows from Cartan’s formula for the Lie derivative (because 2-forms on curves vanish) or, more directly, one can argue as follows. Set $\omega_c = f dx_3$ and $\delta_c = g \frac{\partial}{\partial x_3}$.
So $\langle \delta_c, \omega_c \rangle = fg$. Now,

\[
\delta_c \omega_c = \delta_c (f dx_3) \\
= (\delta_c f) dx_3 + f d(\delta_c x_3) \\
= g \cdot \frac{\partial f}{\partial x_3} \cdot dx_3 + f \cdot dg \\
= gdf + f dg \\
= d(fg),
\]

which ends the proof of 3.2. In particular condition 2 in Theorem 3.1 implies (and “up to a scalar” is implied by) $\delta_c \omega_c = 0$. The condition $\langle \delta_c, \omega_c \rangle = 1$ does not have a direct analogue in the arithmetic case; but, as we shall see, the condition $\delta_c \omega_c = 0$ will.

4) Assertions 1 and 2 in Theorem 3.1 make the classical Euler flow fit into the paradigm of algebraic complete integrability [1].

4. Review of the arithmetic setting

Following [3, 4] we review, in this section, some of the basic definitions needed in the arithmetic setting.

Let $p$ be an odd prime. The sign $\wedge$ will always mean $p$-adic completion of rings or schemes. So for a ring $B$ we have

\[
\hat{B} = \lim \leftarrow B/p^n B
\]

while, for an affine scheme $X = \text{Spec} B$, $\hat{X}$ is the formal scheme [9],

\[
\hat{X} = \text{Spf} \hat{B}.
\]

Recall that a Frobenius lift on a ring $B$ is a ring homomorphism $\phi = \phi^B : B \to B$ whose reduction mod $p$ is the $p$-power Frobenius on $B/pB$. More generally, for a $B$-algebra $C$, a Frobenius lift from $B$ to $C$ is a ring homomorphism $B \to C$ whose reduction mod $p$ is the structure map of the algebra composed with the $p$-power Frobenius. Given a ring $B$, in which $p$ is a non-zero divisor, a $p$-derivation on $B$ is a map of sets $\delta = \delta^B : B \to B$ such that the map $\phi = \phi^B : B \to B$ defined by

\[
\phi(b) = b^p + p\delta b
\]

is a ring homomorphism; such a $\phi^B$ is, of course, a Frobenius lift and we say that $\delta^B$ and $\phi^B$ are attached to each other. More generally we may consider $p$-derivations from any ring $B$ into a $B$-algebra $C$ in which $p$ is a non-zero divisor; they are maps of sets such that [11] is a ring homomorphism. We shall still say that $\delta$ and $\phi$ are attached to each other. If $X$ is a scheme and $\delta$ is a $p$-derivation on $\mathcal{O}(\hat{X})$ (with $p$ a non-zero divisor in the latter) we also say that $\delta = \delta^X$ is a $p$-derivation on $\hat{X}$; morally $\delta^X$ should be viewed as an arithmetic analogue of a flow on $X$. The same convention will be in place for Frobenius lifts on formal schemes. The requirement above about $p$ being a nonzero divisor can be relaxed; cf. [3, 4]. This relaxation is necessary for the general theory but not for what we will need in the present paper.
Form now on, in this paper, \( A \) will denote a complete discrete valuation ring with maximal ideal generated by an odd prime \( p \) such that the residue field \( \mathbb{F} := A/pA \) is algebraic over the prime field \( \mathbb{F}_p \).

Such an \( A \) has a unique Frobenius lift and hence a unique \( p \)-derivation. From now all \( p \)-derivations, respectively Frobenius lifts, on \( A \)-algebras will be assumed compatible with the \( p \)-derivation, respectively the Frobenius lift, on \( A \).

5. Hasse invariant

In this section we review some well known facts and trivial observations about the Hasse invariant.

**Definition 5.1.** Let \( B \) be any ring and let \( F \in B[x] \) be a polynomial in one variable \( x \). Define the Hasse invariant of \( F \) to be the coefficient \( A_{p-1}(F) \in B \) of \( x^{p-1} \) in the polynomial \( F^{p-1} \).

Let \( C = Spec \mathbb{F}_p[x,y]/(y^2 - F(x)) \). We have the following well known facts:

**Lemma 5.2.** Assume \( B = \mathbb{Z}_p \) is the ring of \( p \)-adic integers and \( F \in \mathbb{Z}_p[x] \) and set \( N_p = |C(\mathbb{F}_p)| \). Then the following hold:

1) If \( \deg(F) = 3 \) then
\[
A_{p-1}(F) \equiv -N_p \mod p.
\]

2) If \( \deg(F) = 4 \) and \( F(x) = ax^4 + (\text{lower terms}) \) then
\[
A_{p-1}(F) \equiv -N_p - \left( \frac{a}{p} \right) \mod p,
\]

where \( \left( \frac{a}{p} \right) \) is the Legendre symbol.

3) If \( F \) has distinct roots in the algebraic closure \( \overline{\mathbb{F}}_p \) and if \( \overline{C} \) is the smooth projective model of \( C \), then \( \overline{C} \) is an elliptic curve and satisfies
\[
|C(\mathbb{F}_p)| \equiv 1 - A_{p-1}(F) \mod p.
\]

**Proof.** Assertion 1 is proved in [10], p. 141.

Assertion 2 can be proved similarly.

Let us prove assertion 3. If \( \deg(F) = 3 \) then \( \overline{C} \) is a cubic in \( \mathbb{P}^2 \) and the complement of \( C \) in \( \overline{C} \) consists of one \( \mathbb{F}_p \)-rational point and assertion 3 follows. If \( \deg(F) = 4 \) then, by [10], p. 27, \( \overline{C} \) can be embedded in \( \mathbb{P}^3 \) such that \( \overline{C} \) minus an appropriate hyperplane is \( \mathbb{F}_p \)-isomorphic to \( C \) and the intersection of \( \overline{C} \) with the hyperplane consists of 2 points \((0 : 0 : \pm\sqrt{a} : 1)\). So these 2 extra points are \( \mathbb{F}_p \)-rational if and only if \( \left( \frac{a}{p} \right) = 1 \) and, again, assertion 3 follows.

**Corollary 5.3.** Let \( A = \mathbb{Z}_p \), let \( c = (c_1, c_2) \in A^2 \) be such that \( N(c_1, c_2) \not\equiv 0 \mod p \) (where \( N \) is as in [27]), let \( E_c \) be as in [26], let \( F \) be as in [24] and let \( A_{p-1} = A_{p-1}(F) \) be the Hasse invariant of \( F \). Then
\[
A_{p-1}(c_1, c_2) \equiv 0 \mod p
\]
if and only if \( E_c \) has supersingular reduction mod \( p \).
Proof. Recall from \[2.18\] that there is a separable isogeny between the smooth projective models of the reductions mod $p$ of $E_c$ and $E'_c$; so one is supersingular if and only if the other one is. But by assertion 3 in Lemma \[5.2\] the smooth projective model of the reduction mod $p$ of $E'_c$ is supersingular if and only if $A_{p-1}(c_1, c_2) \equiv 0 \mod p$, hence the same is true about $E_c$. \hfill \Box

**Lemma 5.4.** Let $F \in A[z_1, z_2][x]$ be the quartic polynomial in \[2.11\] and consider its Hasse invariant $A_{p-1} := A_{p-1}(F) \in A[z_1, z_2]$. Then $A_{p-1}$ is homogeneous in $z_1, z_2$ of degree $\frac{p-1}{2}$ and

\[ A_{p-1} \not\equiv 0 \mod p. \tag{5.1} \]

Proof. Since the polynomial $F \in A[z_1, z_2][x]$ in \[2.11\] is weighted homogeneous of degree $4$ in $z_1, z_2, x$ with respect to the weights $(2, 2, 1)$ we have that $F^{\frac{p-1}{2}}$ is weighted homogeneous of degree $2p - 2$ with respect to the same weights. Hence $A_{p-1}(F) \in A[z_1, z_2]$ is weighted homogeneous of degree $p - 1$ in $z_1, z_2$ with respect to the weights $(2, 2)$, equivalently homogeneous of degree $\frac{p-1}{2}$ in $z_1, z_2$. To check \[6.1\] set $t_1 = z_1 - a_2 z_2, t_2 = -z_1 + a_1 z_2$. Since

\[
\begin{bmatrix}
0 & -a_2 \\
-1 & a_1
\end{bmatrix} \not\equiv 0 \mod p,
\]

we have $A[z_1, z_2] = A[t_1, t_2]$ and

\[ F = ((a_2 - a_3)x^2 + t_1)((a_2 - a_3)x^2 + t_2). \]

Then we have

\[ A_{p-1} = (a_2 - a_3) \frac{p-1}{2} t_2^\frac{p-1}{2} + \rho, \]

with $\rho \in A[t_1, t_2]$ a polynomial of degree $< \frac{p-1}{2}$ in the variable $t_2$ and \[5.1\] follows. \hfill \Box

### 6. Statement of the main results

In this section we introduce some more notation and state our main results. Recall our standing assumption that $A$ a discrete valuation ring with maximal ideal generated by $p$ and residue field algebraic over $\mathbb{F}_p$. We also consider the objects

\[ H_1, H_2 \in A[x_1, x_2, x_3], \]
\[ F \in A[z_1, z_2][x], \]
\[ N \in A[z_1, z_2], \]
\[ \mathcal{E}_c, \omega_c, \]

introduced in section \[2\]. Moreover we let

\[ A_{p-1} = A_{p-1}(F), \quad A_{p-1} = A_{p-1}(z_1, z_2) \in A[z_1, z_2] \]

be the Hasse invariant of $F$ introduced in section 5 and we set

\[ Q = x_1 x_2 \cdot N(H_1, H_2) \cdot A_{p-1}(H_1, H_2) \in A[x_1, x_2, x_3] \tag{6.1} \]

and

\[ X = \text{Spec } A[x_1, x_2, x_3][1/Q]. \tag{6.2} \]

In particular $Q \not\equiv 0 \mod p$ in $A[x_1, x_2, x_3]$ by Lemma \[5.4\].

Assume in addition that $c = (c_1, c_2) \in A^2$ satisfies

\[ \delta c_1 = \delta c_2 = 0 \quad \text{and} \quad N(c_1, c_2) \cdot A_{p-1}(c_1, c_2) \in A^\times \tag{6.3} \]
and let $\delta^X$ be any $p$-derivation on $\hat{X}$ satisfying

$$\delta^X H_1 = \delta^X H_2 = 0.$$  

Since $\phi^X(H_i) = H_i^p$ and $\phi(c) = c^p$, the Frobenius lift $\phi^X$ on $O(\hat{X})$ induces a Frobenius lift $\phi_c := \phi^E_0$ on $\hat{E}_c^0$ where $E_c^0$ is the open set of

$$E_c = \text{Spec } (A[x_1, x_2, x_3]/(H_1 - c_1, H_2 - c_2))$$

given by

$$E_c^0 := E_c \cap X$$

$$= \text{Spec } (A[x_1, x_2, x_3][1/Q]/(H_1 - c_1, H_2 - c_2))$$

$$= \text{Spec } (A[x_1, x_2, x_3][1/x_1x_2]/(H_1 - c_1, H_2 - c_2)).$$

We refer to $\phi_c$ as the Frobenius lift on $\hat{E}_c^0$ attached to $\delta^X$.

Note by the way that, if $E_c^0$ is the open set of $E_c'$ given by

$$E_c^0 = \text{Spec } A[x, y, y^{-1}]/(y^2 - F(c_1, c_2, x))$$

then

$$E_c^0 = \pi^{-1}(E_c^0),$$

where $\pi : E_c \to E_c'$ is as in 2.13.

On the other hand, the global 1-form $\omega_c$ in 2.10 restricted to $E_c^0$ will be referred to as the canonical 1-form on $E_c^0$ and will still be denoted by $\omega_c$.

The following provides an arithmetic analogue of the classical Euler flow: assertions 1 and 2 below are arithmetic analogues of assertions 1 and 2 in Theorem 3.1 respectively.

Theorem 6.1. There exists a $p$-derivation $\delta^X$ on $\hat{X}$ such that:

1) The following equalities hold in $O(\hat{X})$:

$$\delta^X H_1 = \delta^X H_2 = 0;$$

2) For any point $c = (c_1, c_2) \in A^2$ with

$$\delta c_1 = \delta c_2 = 0, \text{ and } N(c_1, c_2) \cdot K(c_1, c_2) \not\equiv 0 \mod p,$$

the Frobenius lift $\phi_c$ on $\hat{E}_c^0$ attached to $\delta^X$ and the canonical 1-form $\omega_c$ on $E_c^0$ satisfy the following congruence on $\hat{E}_c^0$:

$$\frac{\phi_c^*}{p} \omega_c \equiv A_{p-1}(c_1, c_2)^{-1} \cdot \omega_c \mod p.$$

Remark 6.2. It would be interesting to see whether a strengthening of Theorem 6.1 holds in which the last congruence in assertion 2 is replaced by an equality and $A_{p-1}$ replaced by an appropriate $p$-adic modular form.

Definition 6.3. A $p$-derivation $\delta^X$ satisfying conditions 1 and 2 in Theorem 6.1 is called an arithmetic Euler flow (on $\hat{X}$, attached to $H_1, H_2$).

Remark 6.4. The proof will show that one can find an arithmetic Euler flow $\delta^X$ such that

$$\delta^X x_3 = \frac{V}{A_{p-1}(H_1, H_2)},$$
where $V \in A[x_1, x_2, x_3]$ is a homogeneous polynomial of degree $2p-1$. In particular, since $A_{p-1}(H_1, H_2)$ is a homogeneous polynomial of degree $p-1$ in $x_1, x_2, x_3$, we get that $\phi^X(x_3)$ is, again, a quotient of a homogeneous polynomial of degree $2p-1$ by the homogeneous polynomial $A_{p-1}(H_1, H_2)$. Note that, for such a $\phi^X$, the element $\phi^X(x_3) \in O(\hat{X})$ is actually in $O(X)$; as we will see this automatically implies that $\phi^X(x_1), \phi^X(x_2)$ are integral over $O(X)$, cf. Lemma 7.1.

The next result gives a parametrization of arithmetic Euler flows “mod $p$.” Remarkably the parameterization will involve the “prime integrals mod $p$” of the classical Euler flow. Indeed let us say that two $p$-derivations $\delta^X$ and $\tilde{\delta}^X$ on $\hat{X}$ are congruent mod $p$ if

$$\delta^X K \equiv \tilde{\delta}^X K \mod p$$

for all $K \in O(\hat{X})$; equivalently if the attached Frobenius lifts $\phi^X$ and $\tilde{\phi}^X$ on $\hat{X}$ are congruent mod $p^2$ in the sense that they induce the same endomorphism of

$$O(\hat{X})/p^2O(\hat{X}) = O(X)/p^2O(X).$$

Also let $\mathbb{F} = A/pA$ be the residue field of $A$ and let $\overline{X} = X \otimes_A \mathbb{F}$. More generally, in what follows, an overline will always denote a “class mod $p$.” Next denote by $\overline{\delta}^cl$ the derivation on $O(\overline{X})$ obtained by reducing mod $p$ the classical Euler flow, i.e.,

$$\overline{\delta}^cl_{x_1} = (a_2 - a_3)x_2x_3,$$
$$\overline{\delta}^cl_{x_2} = (a_3 - a_1)x_3x_1,$$
$$\overline{\delta}^cl_{x_3} = (a_1 - a_2)x_1x_2.$$

The elements $\overline{K} \in O(\overline{X})$ such that

$$\overline{\delta}^cl_{\overline{K}} = 0$$

can be referred to as the prime integrals mod $p$ of the classical Euler flow.

Then we have:

**Theorem 6.5.** Assume $\mathbb{F}$ is infinite. Then the quotient set

$$\frac{\{\text{arithmetic Euler flows on } \hat{X}\}}{\{\text{congruence mod } p\}}$$

is a principal homogeneous space for the $\mathbb{F}$-linear space

$$\{\overline{K} \in O(\overline{X}); \overline{\delta}^cl_{\overline{K}} = 0\}.$$ 

**Remark 6.6.** Recall that a set is a principal homogeneous space for a group if it has a simply transitive action of that group. A principal homogeneous space is either empty or in bijection with the group. The principal homogeneous space in Theorem 6.5 is non-empty due to Theorem 6.1.

**Remark 6.7.** The $\mathbb{F}$-linear space in Theorem 6.5 is, of course, rather large: it contains the $\mathbb{F}$-subalgebra of $O(\overline{X})$ generated by

$$\overline{H}_1, \overline{H}_2, \overline{A}_{p-1}(\overline{H}_1, \overline{H}_2)^{-1}, \overline{N}(\overline{H}_1, \overline{H}_2)^{-1},$$

and by all the $p$-powers of elements of $O(\overline{X})$. 
7. Proof of the main results

We first concentrate on proving our Theorem 6.1; we need a preparation.

For the discussion below $Q$ can be any element of $A[x_1, x_2, x_3]$ with $Q \not\equiv 0 \mod p$ (which is not necessarily as in 6.1) and we continue to denote by $X$ the scheme in 6.2. Consider the morphism

$$\mathcal{H} : X \to \mathbb{A}^3,$$

obtained from 2.4 by restriction, hence given by $z_i \mapsto H_i$, $i = 1, 2$, and continue to denote by

$$\hat{\mathcal{H}} : \hat{X} \to \hat{\mathbb{A}}^2$$

the induced morphism. We will also be considering the trivial Frobenius lift $\phi_{\mathbb{A}^2}^0$ on $\hat{\mathbb{A}}^2$ given by $\phi_{\mathbb{A}^2}^0(z_i) = z_i^p$. Its attached $p$-derivation satisfies $\delta_{\mathbb{A}^2}^0 z_i = 0$.

**Lemma 7.1.** Assume $Q$ is in the ideal $x_1x_2A[x_1, x_2, x_3]$. Let $F$ be the set of Frobenius lifts $\phi^X$ on $\hat{X}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\hat{X} & \xrightarrow{\phi^X} & \hat{X} \\
\mathcal{H} \downarrow & & \downarrow \mathcal{H} \\
\hat{\mathbb{A}}^2 & \xrightarrow{\phi_{\mathbb{A}^2}^0} & \hat{\mathbb{A}}^2
\end{array}
$$

(7.1)

Then the following hold:

1) The set $F$ is in bijection with the ring $\mathcal{O}(\hat{X})$; the bijection is given by

$$\phi^X \mapsto \Delta_3,$$

where $\phi^X(x_3) = x_3^p + p\Delta_3$.

2) If $\phi^X_3(x_3)$ (equivalently $\Delta_3$) is in $\mathcal{O}(X)$ then $\phi^X(x_1)$ and $\phi^X(x_2)$ are integral over $\mathcal{O}(X)$.

3) Two Frobenius lifts $\phi^X$ and $\tilde{\phi}^X$ in $F$ are congruent mod $p^2$ if and only if the corresponding elements $\Delta_3$ and $\tilde{\Delta}_3$ are congruent mod $p$ in the ring $\mathcal{O}(\hat{X})$.

**Remark 7.2.** Let $\delta^X$ be the $p$-derivation attached to $\phi^X$. Then the commutativity of diagram (7.1) is equivalent to the condition:

$$\delta^X H_i = 0, \quad i = 1, 2.$$

**Remark 7.3.** We will give two proofs of assertion 1 in Lemma 7.1. The first one is more conceptual but non-explicit. The second one is more computational but also more explicit, and leads to proofs of assertions 2 and 3 as well. Assertion 2 can be used to prove the existence of correspondence structures (in the sense of [8]) for our Frobenius lifts $\phi^X$ in case $\Delta_3$ is in $\mathcal{O}(X)$. However we will not be concerned here with these correspondence structures.

For the first proof of assertion 1 in Lemma 7.1 we need to recall the following:

**Lemma 7.4.** Let $X \to Y$ be an étale morphism of affine schemes of finite type over $A$. Then any $p$-derivation $\mathcal{O}(\hat{Y}) \to \mathcal{O}(\hat{X})$ lifts to a unique $p$-derivation $\mathcal{O}(\hat{Y}) \to \mathcal{O}(\hat{X})$.

**Proof.** Cf. [4], Lemma 3.14, p. 76.

**First proof of assertion 1 in Lemma 7.1.** Let $Y = \mathbb{A}^3 = \text{Spec} A[z_1, z_2, z_3]$ be an affine 3-space with coordinates $z_1, z_2, z_3$, and consider the morphism

$$\hat{\mathcal{H}} : X \to Y$$
given by
\[ z_1 \mapsto H_1, \ z_2 \mapsto H_2, \ z_3 \mapsto x_3. \]
Clearly \( \tilde{H} \) composed with the projection \( Y = \mathbb{A}^3 \to \mathbb{A}^2 \) onto the first 2 components equals \( H \). Let \( \Delta_3 \in (\hat{X}) \) be an arbitrary element; we shall construct a Frobenius lift \( \phi^X \in \mathcal{F} \) such that \( \phi^X(x_3) = x_3^p + p\Delta_3 \) and this will end the proof of assertion
1. Note that the Jacobian matrix of \( \tilde{H} \) equals
\[
\begin{pmatrix}
2a_1x_1 & 2a_2x_2 & 2a_3x_3 \\
2x_1 & 2x_2 & 2x_3 \\
0 & 0 & 1
\end{pmatrix}
\]
so \( \tilde{H} \) is \'{e}tale because \( x_1x_2 \) is invertible on \( X \). Consider the \( p \)-derivation \( \delta^YX : O(\hat{Y}) \to O(\hat{X}) \) defined by
\[
\delta^YX(z_1) = 0, \quad \delta^YX(z_2) = 0, \quad \delta^YX(z_3) = \Delta_3.
\]
By Lemma 7.1 \( \delta^YX \) extends to a unique \( p \)-derivation \( \delta^X : O(\hat{X}) \to O(\hat{X}) \), hence we will have
\[
\delta^X(H_1) = 0, \quad \delta^X(H_2) = 0, \quad \delta^X(x_3) = \Delta_3,
\]
so the Frobenius lift attached to \( \delta^X \) is in \( \mathcal{F} \) and satisfies the desired requirement. □

**Second proof of assertion 1 Lemma 7.1** Let \( \phi^X \) be a Frobenius lift on \( \hat{X} \) and let \( \Delta_i \in O(\hat{X}) \) be such that
\[
\Phi_i := x_i^p + p\Delta_i = \phi^X(x_i), \quad i = 1, 2, 3.
\]
For the sake of uniformity in notation write \( a_{1j} = a_j \) and \( a_{2j} = 1 \). Then the commutativity of the diagram 7.1 is equivalent to the condition
\[
(7.3) \quad \sum_{j=1}^3 \phi(a_{ij})\Phi_j^2 = \left( \sum_{j=1}^3 a_{ij}x_j^2 \right)^p, \quad i = 1, 2.
\]
We can rewrite the system 7.3 as
\[
(7.4) \quad \sum_{j=1}^2 \phi(a_{ij})\Phi_j^2 = \left( \sum_{j=1}^3 a_{ij}x_j^2 \right)^p - \phi(a_{i3})(x_3^p + p\Delta_3)^2, \quad i = 1, 2.
\]
Note that the right hand side of (7.4) is congruent mod \( p \) to
\[
\sum_{i=1}^2 a_{ij}^p x_j^{2p}.
\]
Fix an element \( \Delta_3 \) in \( O(\hat{X}) \) and let \( M \) be the matrix \( (a_{ij})_{i,j=1}^3 \). By Cramer’s rule the system (7.4), viewed as a linear system with unknowns \( \Phi_1^2, \Phi_2^2 \), has a unique solution given by
\[
\Phi_i^2 = \frac{D_i}{D}
\]
where \( D = \det(\phi(M)) \) and \( D_i \) is the determinant of the matrix obtained from the matrix \( \phi(M) \) by replacing the \( i \)-th column with the column vector appearing in the right hand side of the system (7.4). Now clearly
\[ D \equiv \det \begin{pmatrix} a_{11}^p & a_{12}^p \\ a_{21}^p & a_{22}^p \end{pmatrix} \mod p. \]

On the other hand,
\[ D_1 \equiv \det \left( \sum_{j=1}^{2} a_{1j}^p x_j^{2p} \right) = x_1^{2p} \cdot D \mod p, \]
and, similarly,
\[ D_2 \equiv x_2^{2p} \cdot D \mod p. \]

Hence we have
\[ \Phi_i^2 = x_i^{2p}(1 + pG_i), \quad i = 1, 2, \]
for some \( G_i \in \mathcal{O}(\tilde{X}). \) Hence the system \( (7.4) \) viewed as a non-linear system with unknowns \( \Phi_1, \Phi_2, \) has (under the restriction \( \Phi_1 \equiv x_1^p \mod p \)) a unique solution given by
\[ \Phi_i = x_i^p(1 + pG_i)^{1/2} \in \mathcal{O}(\tilde{X}), \]
where the \( 1/2 \) power is the principal root (the root congruent to 1 mod \( p \)). This ends our proof of assertion 1. \( \square \)

**Proof of assertions 2 and 3 in Lemma 7.1** To check assertion 2 note that if \( \Delta_3 \) in the second proof above is in \( \mathcal{O}(X) \) then so are \( D_1, D_2, \) and hence so are \( \Phi_1^2, \Phi_2^2; \) so \( \phi^X(x_1), \phi^X(x_2) \) are integral over \( \mathcal{O}(X). \)

The “only if” part of assertion 3 is clear. To check the “if” part assume \( \Delta_3 \equiv \tilde{\Delta}_3 \mod p \) for two Frobenius lifts \( \phi^X, \tilde{\phi}^X \in \mathcal{F}. \) Then, clearly \( \phi^X(x_3) \equiv \tilde{\phi}^X(x_3) \mod p^2 \) and, by the second proof above,
\[ p^2 \mid \Phi_1 - \Phi_1 = \tilde{\Phi}_1 - (\Phi_1 - \tilde{\Phi}_1)(\Phi_1 + \tilde{\Phi}_1), \]
where \( \tilde{\Phi}_1 = \tilde{\phi}^X(x_1). \) Since \( p \) is prime in \( \mathcal{O}(\tilde{X}) \) and \( p \) does not divide \( \Phi_1 + \tilde{\Phi}_1 \) (the latter being congruent to \( 2x_1^p \mod p \)) it follows that
\[ p^2 \mid \Phi_1 - \tilde{\Phi}_1 \]
in \( \mathcal{O}(\tilde{X}) \) so \( \phi^X(x_1) \equiv \tilde{\phi}^X(x_1) \mod p^2. \) Similarly \( \phi^X(x_2) \equiv \tilde{\phi}^X(x_2) \mod p^2 \) and we are done. \( \square \)

At this point we are ready to conclude the proof of our Theorem 6.1

**Proof of Theorem 6.1** Write
\[ A_{p-1}(z_1, z_2)^{-1} \cdot F(x, z_1, z_2) x^{p-1} = \sum_{i=0}^{2p-2} R_i(z_1, z_2) x^i, \]
with
\[ R_i(z_1, z_2) \in A_{p-1}(z_1, z_2)^{-1} \cdot A[z_1, z_2]. \]
Since \( R_{p-1} = 0 \) we have that
\[ S(z_1, z_2, x) := \sum_{i=0}^{2p-2} R_i(z_1, z_2) \cdot x^{i+1} \in A_{p-1}(z_1, z_2)^{-1} \cdot A[z_1, z_2][x]. \]
Define
\[ \Delta_3 := S(H_1, H_2, x_3) \in A_{p-1}(H_1, H_2)^{-1} \cdot A[x_1, x_2, x_3] \subset \mathcal{O}(X). \]
By the bijection \( \ref{bijection} \) in Lemma \( \ref{lemma} \) there is a \( p \)-derivation \( \delta^X \) on \( \hat{X} \) killing \( H_1 \) and \( H_2 \) whose Frobenius lift \( \phi^X \) satisfies
\[ \phi^X(x_3) = x_3^p + \Delta_3. \]
Let now \((c_1, c_2) \in A^2 \) satisfy the conditions in assertion 2 of Theorem \( \ref{theorem} \). We claim that the congruence in the conclusion of assertion 2 holds and this will end the proof. Indeed we have
\[ \frac{\phi^*_c}{p} \omega_c = \frac{\phi^*_c}{p} \left( \frac{dx_3}{(a_1 - a_2)x_1x_2} \right) = \frac{(d\Phi_{3c})/p}{\phi(a_1 - a_2)\Phi_{1c}.\Phi_{2c}}, \]
where, for \( i = 1, 2, 3 \), \( \Phi_{ic} = i_c^*\Phi_i \) is the restriction of \( \Phi_i = \phi^X(x_i) \) to \( E^0_c \). Now
\[ \Phi_{3c} = x_3^p + pi^*_c\Delta_3 \]
\[ = x_3^p + pS(c_1, c_2, x_3) \]
\[ = x_3^p + p\sum_{i=0}^{2p-2} R_i(c_1, c_2) \cdot x_i^{3i+1}, \]
where, in order to simplify the notation, we continued to write \( x_i \) in place of \( i^*_c x_i \). Hence, with the same abuse of notation,
\[ \frac{(d\Phi_{3c})/p}{p} = (x_3^{p-1} + \sum_{i=0}^{2p-2} R_i(c_1, c_2)x_i^3)dx_3 \]
\[ = A_{p-1}(c_1, c_2)^{-1} F(x_3, c_1, c_2)^{\frac{p-1}{2}} \cdot dx_3 \quad \text{(by \ref{8.6})} \]
\[ = A_{p-1}(c_1, c_2)^{-1} ((a_2 - a_3)x_3^3 + c_1 - a_2c_2)^{\frac{p-1}{2}} \times \]
\[ \times (a_3 - a_1)x_3^3 - c_1 + a_1c_2)^{\frac{p-1}{2}}dx_3 \]
\[ = A_{p-1}(c_1, c_2)^{-1}(a_1 - a_2)^{p-1}x_1^{p-1}x_2^{p-1}dx_3 \quad \text{(by \ref{9.10})}. \]
So we have
\[ \frac{\phi^*_c}{p} \omega_c \equiv A_{p-1}(c_1, c_2)^{-1} \cdot \frac{(a_1 - a_2)^{p-1}x_1^{p-1}x_2^{p-1}dx_3}{(a_1 - a_2)x_1x_2} \quad \text{mod } p \]
\[ \equiv A_{p-1}(c_1, c_2)^{-1} \cdot \frac{dx_3}{(a_1 - a_2)x_1x_2} \quad \text{mod } p \]
\[ = A_{p-1}(c_1, c_2)^{-1} \cdot \omega_c. \]
This ends our proof. \( \square \)

**Remark 7.5.** We claim that the arithmetic Euler flow \( \delta^X \) constructed in the proof above satisfies
\[ \delta^X x_3 = \frac{V}{A_{p-1}(H_1, H_2)} \]
with \( V \in A[x_1, x_2, x_3] \) homogeneous in \( x_1, x_2, x_3 \) of degree \( 2p - 1 \). To check this recall that the Hasse invariant \( A_{p-1} = A_{p-1}(F) \in A[z_1, z_2] \) is weighted homogeneous of degree \( p - 1 \) in \( z_1, z_2 \) with respect to the weights \((2, 2)\) so \( A_{p-1}(H_1, H_2) \in \)
Lemma 7.6. Let $\text{A}$ is a homogeneous polynomial in $x_1, x_2, x_3$ of degree $p - 1$. On the other hand the elements

$$A_{p-1}(z_1, z_2) \cdot R_i(z_1, z_2)$$

in our proof are weighted homogeneous polynomials of degree $2p - 2 - i$ in $z_1, z_2$, hence

$$A_{p-1} \cdot S(z_1, z_2, x)$$

is a weighted homogeneous polynomial of degree $2p - 1$ in $z_1, z_2, x$ with respect to the weights $(2, 2, 1)$, and hence

$$V(x_1, x_2, x_3) := A_{p-1}(H_1, H_2) \cdot \Delta_3,$$

is a homogeneous polynomial in $A[x_1, x_2, x_3]$ of degree $2p - 1$ in $x_1, x_2, x_3$.

We next concentrate on proving Theorem 6.5.

We need the following Lemma:

**Lemma 7.6.** Let $\overline{K} \in \mathcal{O}(\overline{X})$ and $c = (c_1, c_2) \in A^2$ with $\delta c_1 = \delta c_2 = 0$, $N(c_1, c_2) \not\equiv 0 \mod p$. Let $i_c : E_c^0 \to X$ be, as usual, the inclusion, let $\overline{i}_c : \overline{E}_c^0 \to \overline{X}$ be its reduction mod $p$, and consider the function $\overline{i}_c \overline{K} \in \mathcal{O}(\overline{E}_c^0)$. Let $\overline{\omega}_c$ be the 1-form on $\overline{E}_c^0$ obtained from $\omega_c$ by reduction mod $p$. Then we have the following equality of 1-forms on $\overline{E}_c^0$:

$$\overline{d}(\overline{i}_c \overline{K}) = \overline{i}_c(\overline{\delta} \overline{K}) \cdot \overline{\omega}_c.$$

**Proof.** We have the following computation (in which “$\sum_{(123)}$” means “sum obtained by cyclic permutations of the indices $1, 2, 3$”):

$$\overline{d}(\overline{i}_c \overline{K}) = \overline{\delta} \overline{i}_c (d\overline{K})$$

$$= \overline{\delta} \overline{i}_c (\partial \overline{K} / \partial x_1) dx_1 + \partial \overline{K} / \partial x_2 dx_2 + \partial \overline{K} / \partial x_3 dx_3$$

$$= \overline{\delta} \overline{i}_c (\sum_{(123)} (a_2 - a_3)x_2x_3 \partial \overline{K} / \partial x_1) dx_3$$

$$= \sum_{(123)} \overline{\delta} \overline{i}_c ((a_2 - a_3)x_2x_3 \partial \overline{K} / \partial x_1) \cdot \overline{i}_c (\delta \overline{K}) \cdot \overline{\omega}_c$$

$$\overline{d}(\overline{i}_c \overline{K}) = \overline{i}_c(\overline{\delta} \overline{K}) \cdot \overline{\omega}_c. \quad \Box$$

**Proof of Theorem 6.5.** By Lemma 7.1 the set of arithmetic Euler flows $\delta^X$ modulo $p$ is in a natural bijection with the set of all $\overline{\Delta}_3$ in $\mathcal{O}(\overline{X})$ satisfying the following equality of 1-forms

$$(7.7) \quad \frac{x_3^{p-1} dx_3 + \overline{d}(\overline{i}_c \overline{\Delta}_3)}{(a_1 - a_2)^c x_1 x_2 x_3^p} = A_{p-1}(c_1, c_2)^{-1} \frac{dx_3}{(a_1 - a_2)x_2x_3}$$

on $\overline{E}_c^0$ for all $c = (c_1, c_2) \in A^2$ with

$$(7.8) \quad \delta c_1 = \delta c_2 = 0, \quad N(c_1, c_2)A_{p-1}(c_1, c_2) \not\equiv 0 \mod p.$$
The bijection is given by the rule $\phi^X(x_3) = x_3^p + p\Delta_3$. So if $\tilde{\phi}^X = x_3^p + p\tilde{\Delta}_3$ is another arithmetic Euler flow and we set $K = \Delta_3 - \tilde{\Delta}_3 \in \mathcal{O}(X)$ then its reduction mod $p$, $\overline{K} \in \mathcal{O}(\overline{X})$, satisfies

$$d(\iota_c \overline{K}) = 0$$

for all $c$ that satisfy (7.8). By Lemma (7.6) we have that $\overline{K}$ satisfies

$$\iota_c^{-1}(\overline{\delta^{\mathcal{L}} K}) = 0$$

for all $c$ that satisfy (7.8). Now, for any point in $\mathbb{F}^2$ which is not a zero of $N \cdot A_{p-1}$, the Teichmüller lift $c$ of that point satisfies (7.8). Since $\mathbb{F}$ is infinite the set of points in $\mathbb{F}^2$ which are not zeros of $N \cdot A_{p-1}$ is Zariski dense in $\mathbb{A}_2^2$ hence the union of all curves $E^0_c$ with $c$ as in (7.8) is Zariski dense in $\mathbb{X}$. We conclude that $\overline{\delta^{\mathcal{L}} K} = 0$.

Conversely, if $\Delta_3$ satisfies (7.7) for all $c$ satisfying (7.8) and if $\overline{\delta^{\mathcal{L}} K} = 0$ then $\Delta_3 + K$ will still satisfy (7.7) for all $c$ satisfying (7.8). This shows that the $\mathbb{F}$-linear space in the statement of our theorem acts on the set of arithmetic Euler flows mod $p$; by the first part of the argument this action is transitive. Also the isotropy groups are clearly trivial. This ends our proof. □

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