Rotation-induced Asymmetry of Far-field Emission from Optical Microcavities

Li Ge,1,2† Raktim Sarma,3 and Hui Cao3†

1Department of Engineering Science and Physics, College of Staten Island, CUNY, Staten Island, NY 10314, USA
2The Graduate Center, CUNY, New York, NY 10016, USA
3Department of Applied Physics, Yale University, New Haven, CT 06520-8482, USA

(Dated: April 22, 2014)

We study rotation-induced asymmetry of far-field emission from optical microcavities, based on which a new scheme of rotation detection may be developed. It is free from the “dead zone” caused by the frequency splitting of standing-wave resonances at rest, in contrast to the Sagnac effect. A coupled-mode theory is employed to provide a quantitative explanation and guidance on the optimization of the far-field sensitivity to rotation. We estimate that a $10^4$ enhancement of the minimal detectable speed can be achieved by measuring the far-field asymmetry, instead of the Sagnac effect, in microcavities 5 microns in radius and with distinct emission directions for clockwise and counterclockwise waves.

Optical microcavities have found a wide range of applications from coherent light sources in integrated photonic circuits to cavity quantum electrodynamics, single-photon emitters, and biochemical sensors [1, 2]. Recently they have also been proposed as a platform for rotation detection [3–5], replacing their tabletop counterparts in optical gyroscopes for reduced system size and weight [6–13]. An optical gyroscope utilizes the Sagnac effect [14–18], with rotation (see Appendix A). Therefore, we need to obtain directional emission so that we can detect the rotation speed increases, either the CW or CCW wave in a resonance gradually become the dominant component, and consequently the far-field intensity pattern changes appreciably if the CW and CCW waves have very different output directionality.

One well-studied non-rotating ARC with directional emission is the limaçon cavity [22], defined by $\rho(\theta) = R(1 + \epsilon \cos \theta)$ in the polar coordinates, where $R$ is the average radius and $\epsilon$ is the deformation parameter. The main emission direction of both CW and CCW waves is in the forward direction ($\theta = 0$) for a wide range of $\epsilon$, but for a transverse magnetic (TM) mode the CW and CCW waves also have a significant peak in $\theta \in [120^\circ, 150^\circ]$ and $[210^\circ, 240^\circ]$ respectively, located symmetrically above the horizontal axis [see Fig. 1(a)]. The CW and CCW waves couple to form non-degenerate standing-wave resonances when the cavity does not rotate. Their frequency splitting barely changes at low rotation speed until a critical value is reached [3]. Such a “dead zone” limits the minimal speed in rotation sensing in microcavities using the Sagnac effect. On the other hand, a gradual change of the weights of the CW and CCW waves in a resonance due to rotation leads to an asymmetry of the far-field intensity pattern. We find that this asymmetry increases linearly at low rotation speed, which is then free from the “dead zone” that plagues the Sagnac effect.

In this Letter we propose to use the asymmetry of the far-field emission pattern of deformed microcavity lasers as a measurable signature of rotation, which shows surprisingly high sensitivity. In a perfectly circular cavity, the output directionality of a resonance remains isotropic with rotation (see Appendix A). Therefore, we need to employ asymmetric resonant cavities (ARCs) [20, 21] to obtain directional emission so that we can detect the change in output directionality by rotation. As the rotation speed increases, either the CW or CCW wave in a resonance gradually become the dominant component.

* Electronic address: li.ge@csi.cuny.edu
† Electronic address: hui.cao@yale.edu
Below we focus on the TM polarized resonances in two-dimensional (2D) microcavities without loss of generality. Their electric field is in the cavity plane, and their magnetic field, represented by $\psi(\vec{r})$, is perpendicular to the cavity plane. To the leading order of the rotation speed $\Omega$, the resonances are determined by the modified Helmholtz equation [3]

$$
\left[ \nabla^2 + n(\vec{r})^2k^2 + 2ik\frac{\Omega}{c} \frac{\partial}{\partial \theta} \right] \psi(\vec{r}) = 0, \quad (1)
$$

where $n(\vec{r})$ is the refractive index, $k$ is the complex frequency of mode $\psi(\vec{r})$ in the unit of inverse length, and $\theta$ is the azimuthal angle. We have assumed that the angular velocity is perpendicular to the cavity plane and that $\Omega > 0$ indicates a CCW rotation.

To find the resonances in a rotating ARC and their far-field intensity patterns, one can use the finite-difference time-domain method adapted to the rotating frame [19]. Here we employ a more effective and grid-free approach, the scattering matrix method. It applies generally to a concave cavity with a uniform refractive index and a mirror symmetry of the cavity, the wave functions of the resonances are centered about $|\alpha|$ and $-|\alpha|$. Their electric field is in the cavity plane, and their magnitude is perpendicular to the cavity plane and that $\Omega > 0$ indicates a CCW rotation.

As the cavity rotates, the initial balance between the waves, respectively [Fig. 1(a)]. The intensity of the main wave function of a resonance is decomposed in the angular momentum basis, i.e. $\psi(\vec{r}) = \sum_m A_m(r)e^{im\theta}$, where

$$
A_m(r) = \begin{cases} 
\alpha_m H^+_m(\tilde{k}_m r) + \beta_m H^-_m(\tilde{k}_m r), & r < \rho(\theta), \\
\gamma_m H^+_m(\tilde{k}_m r), & r > \rho(\theta), 
\end{cases} \quad (2)
$$

and $H^\pm$ are the Hankel functions of the first (outgoing) and second (incoming) kind. Compared with the non-rotating cavities [23, 24], the difference lies in the $m$-dependent frequencies $\tilde{k}_m \equiv [(nk)^2 - 2mk\Omega/R]^{1/2}$ and $\tilde{k}_m \equiv [k^2 - 2mk\Omega/R]^{1/2}$, where $\Omega = R\Omega/c$ is the dimensionless rotation speed. The details of the scattering matrix formulation are given in Appendix [3].

We first apply the method above to a limaçon and comparison with the Sagnac effect. The cavity deformation is $\epsilon = 0.41$ and the refractive index is $n = 3$. A resonance at $k_0R \approx 33.78$ and symmetric about the horizontal axis at rest is considered. (a) Far-field intensity patterns in the polar coordinates of CW (solid thick line) and CCW (thick dashed) waves at rest and their superposition (thin solid) in the symmetric resonance. (b) Far-field intensity pattern $I(\theta)$ calculated at the normalized rotation speed $\tilde{\Omega} = 10^{-8}$ (thick line), $10^{-9}$ (medium line), $10^{-10}$ (thin line). The maximum intensity is normalized to unity for each curve. (c) Numerical data (dots) of far-field asymmetry $\chi$ displaying no “dead zone” at low rotation speed. The solid line is a linear fit. (d) Calculation by the coupled-mode theory (solid lines) agrees quantitatively with the numerical simulation of $\xi(\tilde{\Omega})$ (dots) - the ratio of the amplitude of $\psi^*$ and $\psi$ in this resonance, and the Sagnac effect - normalized frequency splitting $R\epsilon[\Delta kR]$ (diamonds). The latter has a “dead zone” below $\tilde{\Omega}_c \approx 1.07 \times 10^{-9}$.

FIG. 1: (Color online) Evolution of the asymmetry in far-field emission from a limaçon and comparison with the Sagnac effect. The cavity deformation is $\epsilon = 0.41$ and the refractive index is $n = 3$. A resonance at $k_0R \approx 33.78$ and symmetric about the horizontal axis at rest is considered. (a) Far-field intensity patterns in the polar coordinates of CW (solid thick line) and CCW (thick dashed) waves at rest and their superposition (thin solid) in the symmetric resonance. (b) Far-field intensity pattern $I(\theta)$ calculated at the normalized rotation speed $\tilde{\Omega} = 10^{-8}$ (thick line), $10^{-9}$ (medium line), $10^{-10}$ (thin line). The maximum intensity is normalized to unity for each curve. (c) Numerical data (dots) of far-field asymmetry $\chi$ displaying no “dead zone” at low rotation speed. The solid line is a linear fit. (d) Calculation by the coupled-mode theory (solid lines) agrees quantitatively with the numerical simulation of $\xi(\tilde{\Omega})$ (dots) - the ratio of the amplitude of $\psi^*$ and $\psi$ in this resonance, and the Sagnac effect - normalized frequency splitting $R\epsilon[\Delta kR]$ (diamonds). The latter has a “dead zone” below $\tilde{\Omega}_c \approx 1.07 \times 10^{-9}$.

CW and CCW waves is broken, similar to the finding in closed billiards [3]. In this case the weights of the CW waves become larger than their CCW counterparts. As a result, the intensity peak at $\theta_{cw}$ grows with respect to the one at $\theta_{ccw}$, as well as to the main one at $\theta = 0$ [see Fig. 1(b)]. The opposite takes place in the corresponding anti-symmetric resonance $\psi^-$ of the same $|m_0|$, with the CCW waves becoming the prevailing component and the intensity peak at $\theta_{ccw}$ increased. Our discussion below is based on the far-field intensity pattern of $\psi^+$. Utilizing the far-field difference of the CW and CCW waves, the asymmetry of the far-field intensity pattern
Applying Appendix C, we find two solutions that correspond to
\( \Delta \theta \) where \( \Delta \theta \) is the detection range of each peak and taken to
be 15°. It does not display a “dead zone” at low rotation speed: \( \chi \) increases linearly with \( \Omega \) [Fig. 1(c)], which is in stark contrast with the Sagnac effect; the latter barely changes until the rotation speed is higher than a critical value \( \Omega_c \equiv R\Omega_c/c \sim 10^{-9} \) [Fig. 2(d)]. If we assume that the far-field asymmetry can be measured experimentally down to \( 10^{-4} \), then the minimal detectable speed is about \( \Omega \sim 10^{-13} \), which is \( 10^3 \) times lower that the onset of the Sagnac effect at \( \Omega_c \). \( \chi \) starts to saturate only at very high speed in the log-log plot, when \( \Omega > \Omega_c \) and the CW waves dominate over the CCW waves. This does not imply that \( \Omega_c \) is the upper operation limit of our approach, as the change of the far-field asymmetry in the linear scale continues to be appreciable in the whole range of rotation speed shown in Fig. 1(c).

To understand why the far-field asymmetry does not display a “dead zone” at low rotation speed while the Sagnac effect does, we first note that the increase of the asymmetry can be viewed as a result of the mixing of the anti-symmetric resonance \( \psi^- \) with the symmetric one. Below we employ a coupled-mode theory to capture this behavior, which is similar to that given in Refs. [3, 4, 26]. It applies both in and beyond the “dead zone” and takes into account the phase of the coupling constant. Since \( \psi^+ \) and \( \psi^- \) are quasi-degenerate, their mutual coupling is much stronger than that with any resonance farther away in frequency [27, 28]. Therefore, it is a good approximation to write their wave function as \( \psi(\Omega) \approx (a^+ + a^-)(\Omega)\psi^+ + (a^+ - a^-)(\Omega)\psi^- \) when discussing how they evolve towards the CW and CCW resonances with rotation. By substituting \( \psi_m(\Omega) \) in Eq. (1) by this expansion and solving the resulting matrix equation for \( (a^+, a^-) \) (see Appendix C), we find two solutions that correspond to the CW- and CCW-prevalent resonances at \( \Omega \neq 0 \). The mixing ratio \( \xi(\Omega) \equiv a^- / a^+ \) in the initial \( \psi^+ \) resonance is given by

\[
\xi(\Omega)^2 \approx \frac{D - \sqrt{D^2 + (2g^2/c^2) \Sigma^2 \Omega^2}}{D + \sqrt{D^2 + (2g^2/c^2) \Sigma^2 \Omega^2}},
\]

where \( D \equiv k_0^2 + k_0^{-2}, \Sigma \equiv k_0^2 + k_0^{-2}, \) and \( k_0^\pm \) are the two resonant frequencies at rest, \( g \equiv 2\sqrt{-G_{\pm}G_{\mp}/n^2} \) is the dimensionless coupling constant between the standing-wave resonances \( \psi^- \) and \( \psi^+ \), where \( G_{\pm} = \int_{\text{cavity}} \psi^+ \psi^- \partial_\theta \psi^- d\vec{r} \) and \( G_{\mp} \) is defined similarly. We emphasize that \( g \) should be differentiated from the coupling of CW and CCW waves in the non-rotating cavity. It can be shown that \( G_{\pm} \approx -G_{\mp} \) in a cavity slightly deformed from a circular disk, and \( g \) is approximately real and positive as a result. The mixing ratio in the initial \( \psi^- \) resonance is given by the inverse of Eq. (4).

Due to the resonance splitting at rest, \( D \neq 0 \) in a limacon cavity, \( \psi^+ \) and \( \psi^- \) gradually evolve towards the CW and CCW resonances as \( \Omega \) increases, and \( \xi(\Omega) \to -1 \). It is important to note that in this process the \( \Omega \)-dependence of \( \xi \) is linear inside the “dead zone,” as can be seen from

\[
\xi(\Omega) \approx \pm i \frac{\Omega}{\sqrt{2\Omega_c}}
\]

for \( \Omega \ll \Omega_c \equiv c|k_0^+ - k_0^-|/g \). This leads to the linear dependence of \( \chi \) on \( \Omega \) we have seen in Fig. 1(c). On the other hand, the difference of the two resonances is given by

\[
\Delta k(\Omega) = \left( (\Delta k_0)^2 + \left( \frac{g}{c} \Omega \right)^2 \right)^{1/2},
\]

where \( \Delta k_0 = k_0^+ - k_0^- \). It shows a “dead zone” for \( \Omega \lesssim \Omega_c \): the leading \( \Omega \)-dependence of \( \Delta k(\Omega) \) is quadratic at low rotation speed, and the sensitivity is reduced by a factor of \( \Omega/2\Omega_c \) when compared with the initially degenerate case \( (\Delta k_0 = 0) \). Far beyond the “dead zone,” \( \Delta k(\Omega) \) approaches its asymptote \( g\Omega/c \), which is the same as the initially degenerate case.

To check the validity of the coupled-mode theory, we compare the values of \( \xi(\Omega) \) and \( \Delta k(\Omega) \) given by Eqs. (4) and (6) with the numerical result from the scattering matrix approach. Good agreement is found for the resonances with \( |m_0| = 101 \) discussed above, with no free parameters [Fig. 1(d)]; the only inputs of the coupled-mode theory are \( g = 21.45 - 0.004 i \) and \( \Delta k_0 R = (2.29 + 0.90i) \times 10^{-8} \) obtained from the scattering matrix method, or equivalently, \( \Omega_c = 1.07 \times 10^{-9} \).

Next we analyze how to optimize the far-field asymmetry for rotation sensitivity. First of all, the CW and CCW waves must have very different emission directionality, so that the far-field asymmetry changes significantly as a function of the rotation speed. Once this condition is satisfied, we note that the linear increase of the far-field asymmetry at low rotation speed is due to the relation \( \xi \). Thus we need to reduce \( \Omega_c \) or, equivalently, reduce the resonance splitting \( |\Delta k_0| \) at rest and increase the coupling constant \( g \) between \( \psi^+ \) and \( \psi^- \). From the definition of \( G_{\pm}, G_{\mp} \), we know that they are roughly proportional to \( |m_0| \), which is approximately \( n k_0^\pm R \) for whispering-gallery resonances if \( |m_0| \gg 1 \). Therefore, \( g \) is proportional to \( k_0 R \) to a good approximation, which is expected as it is the coefficient in the asymptote of the Sagnac effect. Thus it scales linear with the cavity size and the operational frequency. On the other hand, \( \Delta k_0 \) can be reduced by using microcavities of higher symmetry groups, such as the \( D_3 \) cavity with \( \rho(\theta) = R(1 + \cos 3\theta) \). Ideally \( \Delta k_0 \) can be entirely eliminated for resonances in the \( D_3 \) cavity; if they are not simultaneously the eigenfunctions of the parity about the horizontal axis and \( 2\pi/3 \) rotation, or equivalently, if their angular momenta are not integer
times of 3. In practice, there is always inherent surface roughness introduced unintentionally during the fabrication process, which breaks the exact $D_3$ symmetry and lift the degeneracy of the CW and CCW waves at rest slightly. In the scattering matrix approach, the limited precision in carrying out the boundary integral (see Appendix B) can also be regarded as one type of surface roughness. Nevertheless, for the ideally degenerate high-$Q$ resonances near $nk_0^2 R \sim 100$ in a $D_3$ cavity of $\epsilon = 0.025$ and $n = 3$, we find that they all have a $|\Delta k_0 R|$ below our numerical accuracy ($\sim 10^{-13}$). Furthermore, from the linear behavior of $\Delta k(\Omega)$ shown in Fig. 2(b), we know that for $\Omega > 10^{-14}$ the frequency splitting is already in the asymptotic region of the Sagnac effect. Thus the critical speed $\Omega_c$ is below $10^{-14}$.

If we take this upper bound as a practically realizable value of $\Omega_c$, (or equivalently $\Delta k_0 R \sim 10^{-15}$ given $g \sim 10$), Eq. (5) then shows that the sensitivity of the mixing ratio $\xi(\Omega)$ in this $D_3$ cavity is about $10^5$ times higher than the limaçon cavity of the same average radius studied above. We note that the CW and CCW waves in the triangular resonance shown in Fig. 2(a) have distinct far-field emission directions. By constructing the far-field asymmetry using $\xi(\Omega)$ given by the coupled-mode theory, the coupling constant $g = 11.06$ from the scattering matrix method, and assuming $\Omega_c = 10^{-14}$, we find that a rotation sensitivity about $\Omega \sim 10^{-18}$ ($\Omega \sim 10 \text{ deg/h}$) may be achieved when the far-field asymmetry can be measured down to $\sim 10^{-4}$ experimentally [Fig. 2(c)], in a microcavity just $5 \mu m$ in radius. We note that this minimal speed is much lower than $\Omega_c$ (by a factor of $10^6$), the onset speed of the Sagnac effect in the same microcavity. The whispering-gallery resonances shown in Fig. 2(b) have directional emission and a similar sensitivity in this estimation (see Appendix D).

In summary, we have proposed to measure the far-field asymmetry as a promising approach for rotation sensing in deformed microcavities, which is free from the “dead zone” that plagues the Sagnac effect. The minimal detectable rotation speed is estimated to be $10^4$ smaller than the onset speed of the Sagnac effect in the same microcavity, which holds in both the limaçon and $D_3$ cavities discussed above. The sensitivity can be further increased by using a larger cavity, to which the critical speed $\Omega_c$ is inversely proportional via the coupling constant $g$. As mentioned previously, we have based our estimates on the far-field asymmetry of the $\psi^\pm$ resonance. For the initially anti-symmetric resonance $\psi^-$, its far-field intensity changes in opposite to that of $\psi^+$ with rotation and thus counteracts the change of the total far-field asymmetry if both resonances are excited. However, the cancelation is only partial unless they have equal intensities and they are phase incoherent or their relative phase is integer times of $\pi/4$ at one detection direction. We believe that nonlinear effects such as spatial hole burning in a homogeneously broaden gain medium will lead to different lasing intensities of these resonances, which will only lead to a fractional reduction of the far-field asymmetry and the rotation sensitivity estimated above. Detailed study of the nonlinear effects will be included in a further work.

We thank Takahisa Harayama and Jan Wiersig for helpful discussions. L.G. acknowledges PSC-CUNY 45 Research Award. R.S. and H.C. acknowledges NSF support under Grant No. ECCS-1128542.

**Appendix A: Resonances in a rotating circular cavity**

In a circular microcavity, a pair of CW and CCW modes, $\psi_m(\vec{r}) \propto e^{\pm im\theta}$ ($m$: angular momentum number), are degenerate when the cavity does not rotate. As a result, any mixture of them is still an eigenstate of the system, such as the standing waves proportional to $\sin(m\theta), \cos(m\theta)$. At a nonzero rotation speed $\Omega$, however, Eq. (1) in the main text permits only CW or CCW resonances with different frequencies (the same is true in cavities of high symmetry groups, such as the one with $D_3$ symmetry [4]). They can be found by solving the continuity equation of

\[
\frac{\partial \psi_m}{\partial t} = i\Delta_k \psi_m + \nabla \cdot (\mathbf{D} \nabla \psi_m) + \mathbf{E} \cdot \nabla \psi_m - i g \sum_{m'} \psi_{m'} \mathbf{R}_{m'm} \psi_m,
\]

where $\mathbf{R}_{m'm}$ is the matrix connecting modes $\psi_m$ and $\psi_{m'}$.
\( \psi_m(\vec{r}) \) and its radial derivative at \( r = R \), i.e.

\[
\tilde{k}_m \frac{J_m(\tilde{k}_m R)}{J_m(\tilde{k}_m R)} = \tilde{k}_m \frac{H^{+ \prime}_m(\tilde{k}_m R)}{H^+_m(\tilde{k}_m R)},
\]

(A1)

in which \( \tilde{k}_m \equiv (nk)^2 - 2mk\Omega/R \), \( \tilde{k}_m \equiv [k^2 - 2mk\Omega/R]^{1/2} \), and \( \Omega \equiv R\Omega/c \) is the dimensionless rotation speed defined in the main text. We note that the resulting \( \tilde{k}_m \neq k_{m,\pm} \), CW and CCW modes do not mix even though the angular momentum is still a good quantum number; the mixing of the CW and CCW waves of the same \( |m| \) found in Ref. \[26\] is not caused by rotation but rather the excitation method in the finite-difference-time-domain method.

Appendix B: Scattering matrix method for rotating ARCs

The (internal) scattering matrix \( S(k) \) used in the main text maps the internal incident waves on the cavity boundary \( (\alpha_m \text{ in Eq. (2) of the main text}) \) to the scattered waves inside \( (\beta_m) \). \( S(k) \) is found by solving the continuity conditions of the TM wave function and its radial derivative at the cavity boundary, which can be put into the following matrix form

\[
\mathcal{H}^+|\alpha\rangle + \mathcal{H}^-|\beta\rangle = \hat{H}^+|\gamma\rangle,
\]

(B1)

\[
\mathcal{D}^+|\alpha\rangle + \mathcal{D}^-|\beta\rangle = \tilde{D}^+|\gamma\rangle,
\]

(B2)

where

\[
\mathcal{H}_{lm}^\pm = \int_0^{2\pi} H_m^\mp(\tilde{k}_m \rho(\theta)) e^{i(m-\ell)\theta} d\theta,
\]

(B3)

\[
\mathcal{D}_{lm}^\pm = \int_0^{2\pi} \tilde{k}_m H_m^\mp(\tilde{k}_m \rho(\theta)) e^{i(m-\ell)\theta} d\theta,
\]

(B4)

and \( \hat{H}^+, \tilde{D}^+ \) are defined similarly with \( \tilde{k}_m \) substituted by \( k_m \). The apostrophe in Eq. (B4) represents the derivative of the Hankel functions. By eliminating \( \gamma_m \) from Eqs. (B1) and (B2), a matrix equation can be found in the form

\[
S(k)|\alpha\rangle = |\beta\rangle,
\]

which the complex quantity \( \phi \) is defined in such a way that \( |\beta\rangle = e^{i\phi} |\alpha\rangle \). The resonances \( k \) are then determined by the condition \( \phi = 0 \), so that \( |\alpha\rangle = |\beta\rangle \) and the wave function inside the cavity is given by the superposition of Bessel functions, i.e. \( J_m = (H^+_m + H^-_m)/2 \), which has a finite amplitude at the origin \( \vec{r} = 0 \). All other values of \( \phi \) lead to unphysical states with an infinite amplitude at \( \vec{r} = 0 \), because \( |H^+_m(r \to 0)| \to \infty \).

Appendix C: Coupled-mode theory

The coupled-mode theory for the Sagnac effect in a closed billiard was given in Refs. \[3, 4\], which applies only when the rotation speed is beyond the “dead zone.” In Ref. \[26\] the authors extended it into the “dead zone” but only for a ring laser. The coupled-mode theory presented in the main text shows that a similar approach can be applied to open cavities, both in and beyond the “dead zone,” and takes into account the phase of the coupling constant.

By substituting \( \psi(\Omega) \approx a^+ (\Omega) \psi^+ + a^- (\Omega) \psi^- \) to Eq. (1) of the main text, we find that the coefficients satisfy the following equation,

\[
\left( k^2 - k_0^2 \pm \frac{2ik\Omega}{cn^2} G_{+\pm} \right) \left( a^+ \right) = 0,
\]

(C1)

where \( k_0^+ \), \( k_0^- \) are the resonant frequencies at rest, \( G_{+\pm} = \int_{\text{cavity}} \psi^+ \psi^- d\vec{r} \), and \( G_{-\pm} \) is defined similarly. We note that \( G_{+\pm} \) and \( G_{-\pm} \), which would have appeared on the diagonal of the coupling matrix in Eq. (C1), vanish because their integrands are odd functions with respect to the horizontal axis. Likewise, \( \int_{\text{cavity}} \psi^+ \psi^- d\vec{r} \) vanishes even though resonances of an open cavity are not orthogonal or biorthogonal in general. We have used the normalization \( \int_{\text{cavity}} (\psi^+)^2 d\vec{r} = 1 \).

By solving Eq. (C1), we find the expression for the coupling ratio \( \xi(\Omega) \) and complex resonance splitting \( \Delta k(\Omega) \) given in the main text.

Appendix D: Near-field patterns

In Fig. 1(a) of the main text we have shown the near-field patterns of (a) the CW waves and (b) the CCW waves of the symmetric whispering-gallery resonance \( \psi^+ \) at \( \Omega = 0 \) in Fig. 1(a) of the main text.

![FIG. 3: False-color plot in the logarithmic scale showing the near-field intensity patterns of (a) the CW waves and (b) the CCW waves of the symmetric whispering-gallery resonance \( \psi^+ \) at \( \Omega = 0 \) in Fig. 1(a) of the main text.](image)
along side the triangular modes, and they have directional emission as well. This can be seen from Fig. 4(a) and (b), which show the near-field patterns of the CW and CCW waves of a symmetric whispering-gallery resonance $\psi^+$. The frequency difference of this resonance and its pairing anti-symmetric resonance is shown in Fig. 2(b) of the main text. Fig. 4(c) shows the estimate of the far-field asymmetry for the $\psi^+$ resonance, using $\xi(\theta)$ from the coupled-mode theory and the CW and CCW wave functions from the scattering matrix method. $\Omega_c = 10^{-14}$ is assumed, and $\theta_{cw} = 256^\circ$, $\theta_{ccw} = 106^\circ$, and $\Delta \theta = 15^\circ$ are used in calculating $\chi$. It has similar sensitivity to the triangular mode shown in Fig. 3 and discussed in Fig. 2(c) of the main text.

[1] Optical Processes in Microcavities, edited by R. K. Chang and A. J. Campillo, Advanced Series in Applied Physics (World Scientific, Singapore, 1996).
[2] Optical Microcavities, edited by K. J. Vahala, Advanced Series in Applied Physics (World Scientific, Singapore, 2004).
[3] S. Sunada and T. Harayama, Phys. Rev. A 74, 021801(R) (2006).
[4] S. Sunada and T. Harayama, Opt. Express 15, 16245 (2007).
[5] J. Scheuer, Opt. Express 15, 15053 (2007).
[6] A. B. Matsko, A. A. Savchenkov, V. S. Ilchenko, and L. Maleki, Opt. Commun. 233, 107 (2004).
[7] B. Z. Steinberg, Phys. Rev. E 71, 056621 (2005).
[8] B. Z. Steinberg and A. Boag, J. Opt. Soc. Am. B 23, 1442 (2006).
[9] J. Scheuer and A. Yariv, Phys. Rev. Lett. 96, 053901 (2006).
[10] B. Z. Steinberg, J. Scheuer, and A. Boag, J. Opt. Soc. Am. B 24, 12161224 (2007).
[11] C. Peng, Z. Li, and A. Xu, Opt. Express 15, 3864 (2007).
[12] C. Sorrentino, J. R. E. Toland, and C. P. Search, Opt. Express 20, 354 (2012).
[13] R. Novitski, B. Z. Steinberg, and J. Scheuer, Phys. Rev. A 85, 023813 (2012).
[14] E. J. Post, Rev. Mod. Phys. 39, 475 (1967).
[15] W. W. Chow, J. Gea-Banacloche, L. M. Pedrotti, V. E. Sanders, W. Schleich, and M. O. Scully, Rev. Mod. Phys. 57, 61 (1985).
[16] F. Aronowitz, The Laser Gyro, in Laser Applications, edited by M. Ross (Academic, New York, 1971).
[17] C. Ciminelli, F. Dell’Olio, C. E. Campanella, and M. N. Armenise, Adv. Opt. Photon. 2, 370404 (2010).
[18] M. Terrel, M. J. F. Digonnet, and S. Fan, Laser Photon. Rev. 3, 452 (2009).
[19] R. Sarma and H. Cao, J. Opt. Soc. Am. B 29, 1648-1654 (2012).
[20] J. U. Noëckel and A. D. Stone, Nature 385, 45 (1997).
[21] C. Gmachl, F. Capasso, E. E. Narimanov, J. U. Noëckel, A. D. Stone, J. Faist, D. L. Sivco, and A. Y. Cho, Science 280, 15561564 (1998).
[22] J. Wiersig and M. Hentschel, Phys. Rev. Lett. 100, 033901 (2008).
[23] E. E. Narimanov, G. Hackenbroich, Ph. Jacquod, and A. D. Stone, Phys. Rev. Lett. 83, 4991-4994 (1999).
[24] H. E. Türeci, H. G. L. Schwefel, Ph. Jacquod, and A. D. Stone, Progress in Optics 47, 75 (2005).
[25] H. G. Schwefel, N. B. Rex, H. E. Türeci, R. K. Chang, A. D. Stone, T. Ben-Messaoud, and J. Zyss, JOSA B 21, 923034 (2004).
[26] S. Sunada, S. Tamura, K. Inagaki, and T. Harayama, Phys. Rev. A 78, 053822 (2008).
[27] L. Ge, Q. H. Song, B. Redding, and H. Cao, Phys. Rev. A 87, 023833 (2013).

[28] L. Ge, Q. Song, B. Redding, A. Eberspächer, J. Wiersig, and H. Cao, Phys. Rev. A 88, 043801 (2013).

[29] J. U. Nöckel, Ph.D. thesis, Yale University, 1997.