WALL-CROSSING MORPHISMS IN KHOVANOV-ROZANSKY HOMOLOGY

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Abstract. We define a wall-crossing morphism for Khovanov-Rozansky homology; that is, a map between the KR homology of knots related by a crossing change. Using this map, we extend KR homology to an invariant of singular knots categorifying the Vassiliev derivative of the HOMFLY polynomial, and of \( sl_n \) quantum invariants.

1. Introduction

In [Shi1], the first author outlined a program of classification of combinatorially defined homological theories, such as Khovanov and Ozsvath-Szabo’s categorifications of knot and 3-manifold invariants.

In this paper we construct the wall-crossing morphism for Khovanov-Rozansky (KR) homology which will allow us to put the KR theory into such a framework.

The theory that was outlined in [Shi1] combines results of V. Vassiliev, A. Hatcher and M. Khovanov and the resulting theory can be considered as a “categorification of Vassiliev theory” or a classification of categorifications of knot invariants. We defined Khovanov homology for singular knots, introduced the definition of a theory of finite type \( n \) and have shown that Khovanov homology restricted to the subcategories of knots of bounded crossing number have finite type [Shi2].

The main idea in [Shi1] was to consider a knot homology theory as a local system, or a constructible sheaf on the space of all objects (knots, including singular ones), extend this local system to the singular locus and introduce the analogue of the “Vassiliev derivative” for categorifications.

The Khovanov homology was just the first example of a theory satisfying our axioms and in the present paper we show that the analogous constructions can be carried out for Khovanov-Rozansky homology.

In [Vas], Vassiliev introduced finite type invariants by considering the space of all immersions of \( S^1 \) into \( \mathbb{R}^3 \) and relating the topology
of the singular locus to the topology of its complement via Alexander duality. He resolved and cooriented the discriminant (the space of immersions with self-intersections) and introduced a spectral sequence with a filtration, which suggested the simple geometrical and combinatorial definition of an invariant of finite type.

Let $\lambda$ be an arbitrary invariant of oriented knots in oriented space with values in an abelian group $A$. Extend $\lambda$ to be an invariant of 1-singular knots (knots that may have a single singularity which is a double point), using the formula which was interpreted by Birman and Lin as a “Vassiliev derivative”: if $K_0$ has a single double point and $K_+$ and $K_-$ are its “positive” and “negative” resolutions, then

\begin{equation}
\lambda(K_0) = \lambda(K_+) - \lambda(K_-)
\end{equation}

Furthermore, extend $\lambda$ to the set $\mathcal{K}^n$ of $n$-singular knots (knots with $n$ double points) by repeatedly using the relation (1).

**Definition 1.1.** We say that $\lambda$ is of type $n$ if its extension $\lambda|_{\mathcal{K}^{n+1}}$ to $(n+1)$-singular knots vanishes identically. We say that $\lambda$ is of finite type if it is of type $n$ for some $n$.

Given this formula, the definition of an invariant of finite type becomes similar to that of a polynomial: its $(n+1)$-st Vassiliev derivative is zero.

All known invariants are either of finite type, or are infinite linear combinations of those. For example, it was shown by Bar-Natan that the $n$th coefficient of the Conway polynomial is a Vassiliev invariant of order $n$.

To categorify the Vassiliev derivative for a link homology theory $\mathcal{H}$, where $\mathcal{H}(K)$ is a complex attached to a link $K$ (typically whose Euler characteristic is a known link invariant), we must define a **wall-crossing morphism** $\mathcal{W}_w : \mathcal{H}(K_-) \rightarrow \mathcal{H}(K_+)$ for each wall $w$ of codimension 1 in the discriminant.

We let $\mathcal{H}(K_0)$ be the cone of $\mathcal{W}_w$:

$$\mathcal{H}(K_0) \cong \mathcal{H}(K_+) \oplus \mathcal{H}(K_-)[1] \quad d_0 = \begin{bmatrix} -d_+ & \mathcal{W}_w \\ 0 & -d_- \end{bmatrix}$$

Assuming these satisfy the obvious commutativity relation $\mathcal{W}_{w_1}\mathcal{W}_{w_2} = \mathcal{W}_{w_2}\mathcal{W}_{w_1}$ for any two walls $w_1, w_2$ which meet, then $\mathcal{W}$ allows us to extend $\mathcal{H}$ to the whole discriminant, by simply taking successive cones, as described in Section 3.

Our principal result in this paper is to define just such a wall crossing morphism for KR homology $\mathcal{KR}_N$ for any integer $N$, or HOMFLY homology, which we denote $\mathcal{KR}_\infty$. 
Theorem 1.1. For $N = 1, \ldots, \infty$ and for any wall $w$ in the discriminant, we have a map $\mathcal{W}_w : \mathcal{K}\mathcal{R}_N(K_-) \to \mathcal{K}\mathcal{R}_N(K_+)$. The wall-crossing maps for adjacent walls commute, so this defines an extension of $\mathcal{K}\mathcal{R}_N$ to the discriminant locus.

The Euler characteristic of this extension is the extension of the $\mathfrak{sl}_N$-quantum invariant (or HOMFLY polynomial) to the discriminant by Vassiliev derivative.

There is, of course, an obvious guess for this wall-crossing map when $N < \infty$: that defined on $\mathcal{K}\mathcal{R}_N$ by the unique cobordism of minimal genus between $K_+$ and $K_-$. Unfortunately, the definition of Khovanov and Rozansky [KR1] is only well defined up to constant factors. Since it very important that our wall-crossing morphisms commute (and not, say, anti-commute), we will need to construct them explicitly, which is, in fact, much simpler than attempting to compute the action of a cobordism. Unfortunately, our wall-crossing maps are also only defined up to scalar, but it is clear from the construction that the scalars can be chosen consistently so that wall-crossing maps commute. This approach has the further advantage of applying to the triply graded theory introduced in [KR2], where even projective functoriality has not been described.

It seems likely to the authors that this wall-crossing map has connections to the action of the braid cobordisms on the derived category of coherent sheaves on the cotangent bundle of the flag variety described by Khovanov and Thomas [KT], but at the moment it seems unclear how.

One family of algebraic invariants of singular knots has already been categorified (as part of the definition of Khovanov-Rozansky homology): the MOY invariants, which give one extension of the HOMFLY polynomial and $\mathfrak{sl}_n$ quantum invariants to the discriminant locus, one which is different from the Vassiliev derivative. While this categorification and our theory both give extensions of Khovanov-Rozansky homology to singular knots, they categorify different extensions of the HOMFLY polynomial to singular knots, and thus obviously differ.

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2. Khovanov-Rozansky Homology

Since it is somewhat involved, we will only sketch the definition of Khovanov-Rozansky homology here, leaving out details that will not be relevant for our argument. For a more complete definition, see the papers of Rasmussen [Ras], Webster [Web], Khovanov [Kho] or the original papers of Khovanov and Rozansky [KR1]. We will concentrate on reduced HOMFLY homology, though our results are equally valid for unreduced homology.

Reduced HOMFLY homology is defined as follows:

Using Vogel’s algorithm, write your knot $K$ as the closure of a braid $\sigma = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_m}^{\epsilon_m}$, where $\{\sigma_i\}$ are the standard generators of the braid group $B_n$. We let $\alpha(\sigma) = \frac{1}{2}\left(n - \sum_{j=1}^{m} \epsilon_{i_j}\right)$, that is, half the braid index minus half the writhe.

Let $S = \mathbb{Q}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n)$, equipped with the obvious $S_n$-action, and considered with the grading where $\deg(x_i) = 2$. If $s_i = (i, i+1)$, then $S^{s_i}$ is the subring generated by $x_j$ for $j \neq i, i+1$, and the symmetrized polynomials $x_i + x_{i+1}$ and $x_i x_{i+1}$.

We define $S_i = S \otimes_{S^{s_i}} S\{-1\}$ (where $\{a\}$ denotes degree shift by $a$). This is free of rank two for the left and right action of $S$, and is equipped with an obvious multiplication map $m_i : S_i \to S$. Less obviously, we have a map $\iota_i : S\{2\} \to S_i$, such that $\iota_i(1) = x_i \otimes 1 - 1 \otimes x_{i+1}$.

We can define a categorification $F$ of the braid group setting

$$F(\sigma_i) = \cdots \to 0 \to S\{2\} \to S_i\{1\} \to 0 \to \cdots$$

$$F(\sigma_i^{-1}) = \cdots \to 0 \to S_i\{-1\} \to S\{-2\} \to 0 \to \cdots$$

and extending to an arbitrary element of $B_n$ by the relation $F(\sigma \sigma') = F(\sigma) \otimes_R F(\sigma')$. By a theorem of Rouquier [Rou], this is well-defined up to homotopy equivalence of complexes.

The triply-graded HOMFLY homology can be constructed by applying the functor Hochschild homology $HH^*$ to $F(\sigma)$. For each $i$, we have a complex $\mathcal{F}^i(\sigma) = HH^i(F(\sigma))$ (with Hochschild homology applied termwise) of graded modules over $S$.

**Definition 2.1.** The HOMFLY homology of $\sigma$ is the doubly graded complex of $S$-modules whose graded pieces are $K\mathcal{R}^{i,j,k}(\sigma) = \mathcal{F}^{i-\alpha(\sigma)}(\sigma)_j$, that is, the elements of grade $j$ of the complex $HH^{i-\alpha(\sigma)}(F(\sigma))$. 
As Khovanov and Rozansky noted, the Euler characteristic of this complex is the HOMFLY polynomial of links (after a slightly odd change of variables). If $\sigma$ is a knot, then the homology of this complex is finite dimensional.

It is worth noting, we can extend the functor to singular braids in an obvious way by defining a Rouquier complex $F(\sigma_i)$ for a single intersection point between the $i$th and $(i+1)$st strand of a braid, and applying the scheme above to a braidlike projection of the singular knot. This produces a categorification of the MOY state-sum invariant of singular links. This will not necessarily have finite-dimensional homology, even for singular knots.

To obtain $\mathfrak{sl}_N$-homology instead of HOMFLY homology, we need only replace Hochschild homology by a slightly different functor.

The algebraic basis for the construction of this functor is the theory of matrix factorizations. Let $M$ be a $\mathbb{Z}$-graded module over a ring $S$.

**Definition 2.2.** A ($\mathbb{Z}$-graded) matrix factorization on $M$ with potential $\varphi \in S$ is a map $d = d_+ + d_- : M \to M$ with $d_\pm$ of graded degree $\pm 1$ such that $d^2 = \varphi$.

Though this is not the usual definition of a matrix factorization (where typically we only assume a $\mathbb{Z}/2$ grading), this richer structure is also useful from the perspective of knot theory.

While this may look like a daunting definition, we will only be interested in essentially a single example of a matrix factorization.

Fix an integer $N$, and index $1 \leq i \leq n$, and let $\varphi_i = x_i \otimes 1 - 1 \otimes x_i$ and $\psi_i = \frac{x_i^{N+1} \otimes 1 - 1 \otimes x_i}{x_i \otimes 1 - 1 \otimes x_i}$. Define the matrix factorization $Z_i$ over $S \otimes S$ to be rank 2, with one copy of $S \otimes S$ in degree 0 and one in degree 1 with $d_+$ and $d_-$ by

$$Z_i = S \otimes S \xrightarrow{\varphi_i} S \otimes S \xleftarrow{\psi_i}.$$ 

and let $Z = \bigotimes_{i=1}^n Z_i$ (where the tensor product is essentially that of complexes applied to both differentials). Note that the potential of $Z$ is $\sum_{i=1}^n x_i^{N+1} \otimes 1 - 1 \otimes X_i^{N+1}$.

If $M$ is an $S - S$ bimodule annihilated by $p \otimes 1 - 1 \otimes p$ for any symmetric polynomial $p \in S^\otimes n$, then $Z \otimes_{S \otimes S} M$ is matrix factorization of potential 0. That is, the total differential $d$ is an honest differential, so the total homology $\tilde{HH}_N(M) = H(Z \otimes_{S \otimes S} M, d)$ is well defined, and in fact carries a single grading, which is a linear combination of the polynomial and matrix factorization gradings on $Z \otimes_{S \otimes S} M$ in which $d$ is homogeneous.
Note that if we only consider $d_-$, we obtain a free resolution of $S$ as a bimodule over itself. Thus the homology of the complex $H^i(Z_- \otimes_{S \otimes S} M)$ is simply $HH^i(M)$ for any $S - S$-bimodule $M$. Thus, we can think of $\bar{HH}_N(M)$ as a sort of non-flat deformation of $HH^*(M)$.

**Definition 2.3.** The $sl_N$ Khovanov-Rozansky homology $KR_N(\bar{\sigma})$ of $\bar{\sigma}$ is the bigraded complex $\bar{HH}_N(F(\sigma))$. This again has finite dimensional homology.

In fact, Jacob Rasmussen has shown that the total dimension of this homology if bounded above by that of the HOMFLY homology [Ras].

The equivalence of this definition to that originally given in [KR1] was communicated to the second author by Mikhail Khovanov, but to the best of our knowledge the first full proof appeared in [Web].

### 3. Categorified Vassiliev derivative.

In this section, we will discuss a general schema for Vassiliev theory of knot homology, as introduced in the introduction and [Shi1].

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{K}(\mathcal{A})$ be the category of complexes in $\mathcal{A}$ with morphisms considered up to homotopy. The category $\mathcal{K}(\mathcal{A})$ is not abelian; it no longer makes sense to consider the kernel or cokernel of a map. The closest notion we have is that of the cone of a map.

As usual, for a complex $X = (X^i, d_X)$, define the shift $X[j]$ of $X$ by

$$(X[j])^i = X^{i+j}, \quad d_X[j] = (-1)^j d_X$$

**Definition 3.1.** Let $f : X \to Y$ be a chain morphism. The cone of $f$ is the complex

$$C_f^i = X[1]^i \oplus Y^i \quad d_{C_f}(x^{i+1}, y^i) = (-d_X x^{i+1}, f(x^{i+1}) - d_Y y^i)$$

In $\mathcal{K}(\mathcal{A})$, the cone naturally fits into an exact triangle:

$$\xymatrix{ & C_f \ar[dl]_{[1]} \ar[dr] & \\
X \ar[rr]^w & & Y \ar[ll]^v \\
}$$

For our purposes, cones have two important properties. The first is a rather trivial observation. If we let $\chi(X)$ be the Euler characteristic of $X$ (a class in the Grothendieck group $K^0(\mathcal{A})$ then

**Proposition 3.1.** $\chi(C_f) = \chi(X) - \chi(Y)$
The second is the behavior of successive cones. Whenever we have a commuting square of chain maps

\[
\begin{array}{ccc}
X_- & \xrightarrow{\varphi_-} & X_+ \\
\downarrow{\psi} & & \downarrow{\psi} \\
X_+ & \xrightarrow{\varphi_+} & X_-
\end{array}
\]

then one has natural induced maps \(\varphi : C_\psi \to C_\varphi\) and \(\psi : C_\varphi \to C_\psi\). It is a simple exercise to show that \(C_\psi\) and \(C_\varphi\) are naturally isomorphic. More generally if we have a commuting hypercube of any dimension, we will get the same answer taking cones in any order. This iterated cone can be seen as the total complex of a bicomplex defined by our chain maps.

Consider a point of self intersection of the discriminant of codimension \(n\). There are \(2^n\) chambers adjacent to this point. Since the discriminant was resolved by Vassiliev [Vas], this point can be considered as a point of transversal self intersection of \(n\) hyperplanes in \(\mathbb{R}^n\), or an origin of the coordinate system of \(\mathbb{R}^n\).

For an invariant \(\lambda\) of knots valued in an abelian group, we can extend \(\lambda\) to the discriminant by the Vassiliev derivative. If \(K\) is a singular knot with \(n\) self-intersection points, then there are \(n\) codimension 1 “walls” of the discriminant intersecting transversely at \(n\), each being cooriented (having a “positive” and a “negative” side). Thus, any neighborhood of \(K\) is split into \(2^n\) chambers, one for each map \(\sigma : [1, \ldots, n] \to \{\pm 1\}\). Let \(K_\sigma\) be a representative for chamber corresponding to \(\sigma\), and let

\[
\lambda(K) = \sum_{\sigma} (-1)^\nu(\sigma) \lambda(K_\sigma)
\]

where \(\nu(\sigma) = \sum_{i=1}^n (1 + \sigma(i))/2\).

As usual in categorification, when we pass to a categorification, we would like to replace the sum above with a chain complex. The previous work of the first author [Shi1] suggests that to categorify a knot homology theory \(\mathcal{H}\), we should define **wall-crossing morphisms** \(W_w : \mathcal{H}(K_-) \to \mathcal{H}(K_+)\) where \(K_\pm\) are the knots adjacent to a wall \(w\), such that the diagram

\[
\begin{array}{ccc}
\mathcal{H}(K_-, -) & \xrightarrow{W_{w_1}} & \mathcal{H}(K_-, +) \\
\downarrow{W_{w_2}} & & \downarrow{W_{w_2}} \\
\mathcal{H}(K_+, -) & \xrightarrow{W_{w_1}} & \mathcal{H}(K_+, +)
\end{array}
\]
commutes, where $K_{\pm, \pm}$ are the knots adjacent to a generic element in
the intersection of two walls $w_1$ and $w_2$.

There is a unique cobordism of genus 1 joining $K_-$ and $K_+$, and one
would hope to be able to define $\mathcal{W}_w$ as simply the functor $\mathcal{H}$ applied
to this cobordism. Obviously, if $\mathcal{H}$ is functorial on the nose, this will work,
but at the moment this approach is captive to the sign problems which
appear in many knot homology theories, including Khovanov-Rozansky
homology. In this sense, one can consider the existence of wall-crossing
morphisms as a weaker version of fixing the signs of functoriality.

Now, assume that we have constructed such morphisms $\mathcal{W}_w$ for all
walls in the discriminant. At each singular knot $K$, we can define $\mathcal{H}(K)$
as an iterated cone $C_{\mathcal{W}_{w_1}, \ldots, \mathcal{W}_{w_n}}$ of the wall-crossing maps corresponding
to walls $w_1, \ldots, w_n$ containing $K$.

As an alternative description, we can construct a complex $\mathcal{C}_K$ such
that

\[(\mathcal{C}_K)_i = \bigoplus_{\nu(\sigma)=i} \mathcal{H}(K_{\sigma})\]

and differentials given by appropriate sums of wall-crossing maps (as
usual, we will need to add signs to make sure we have a complex, but
this can be done by the standard conventions of supermathematics),
by simply collapsing the grading on the hypercube with each resolution
at a corner.

This is now a double-complex in the category of matrix factorizations
of potential 0. We can then extend $\mathcal{H}$ to the discriminant by taking
the total complex

$\mathcal{H}(K) = \text{Tot}(\mathcal{C}_K)$.

Proposition 3.1 implies

**Corollary 3.2.** The Euler characteristic of $\mathcal{H}(K)$ is the extension of
the knot invariant $\chi(\mathcal{H})$ to the discriminant, that is

$\chi(\mathcal{H}(K)) = (-1)^{\nu(\sigma)} \chi(K_{\sigma})$.

4. **Wall-Crossing Morphisms**

While the above discussion covered an essentially formal situation
which could apply to any categorification, we still need to define the
wall-crossing maps themselves, which will require us getting our hands
(a little) dirty.

Fix a wall $w$ and a generic singular knot $K$ in $w$ and let $K_{\pm}$ be the
knots on its positive and negative sides. Fix a braid-like projection of
$K$. In this projection, $K$ has a single self-intersection point, and a pro-
jection of $K_+$ (resp. $K_-$) is obtained by resolving this self-intersection
dures a canonical map $HH$ is a realizaion of the class $\phi$ point to a positive (resp. negative) crossing. Furthermore, $K$ is the closure of a singular braid $\sigma_i$. Let $\sigma_+, \sigma_-$ be the resolution of the self-intersection point of $\sigma_i$. We may assume (by moving the cutting point) that for some braid $\beta$, we have $\sigma_i = \sigma_i^+\beta, \sigma_+ = \sigma_i\beta$ and $\sigma_- = \sigma_i^{-1}\beta$. This in turn implies that

$$F(\sigma_+) \cong F(\sigma_i) \otimes_S F(\beta) \quad F(\sigma_-) \cong F(\sigma_i^{-1}) \otimes_S F(\beta)$$

**Theorem 4.1.** There exists a natural (up to scalar) chain map $W : \mathcal{KR}_N(K_-) \to \mathcal{KR}_N(K_+)$ for $N = 1, \ldots, \infty$. If $K$ is an $n$-singular link, there is consistent choice of scalars so that the induced cube of wall-crossing maps is commutative, and thus the iterated cone on this cube is independent of these scalars.

**Proof.** This map is induced by an element in $\text{Ext}^1(F(\sigma_-), F(\sigma_+))$, which, in turn, comes from one in $\varphi \in \text{Ext}^1(F(\sigma_i^{-1}), F(\sigma_i))$. This element can be described in several ways.

If $\tilde{S}$ is the quotient of $S \otimes S$ by the ideal generated by $x_j \otimes 1 - 1 \otimes x_j$ for $j \neq i, i + 1$, then we have a natural exact sequence of complexes:

$$\tilde{S} \xrightarrow{x_i \otimes 1 - 1 \otimes x_i} \tilde{S}\{-2\} \xrightarrow{1} S\{\{-2\} \xrightarrow{m_i} S\{-1\}$$

By simple degree considerations, this exact sequence induces an injection $\text{Hom}(\tilde{S}\{2\} \to \tilde{S}, F(\sigma_i)) \hookrightarrow \text{Ext}^1(F(\sigma_i^{-1}), F(\sigma_i))$. Since there is a unique (up to scalar) projection map $\tilde{S}\{2\} \to \tilde{S} \to F(\sigma_i)$, we can take $\varphi$ to be the image of this.

Alternatively, if we let $S' = \tilde{S}/(x_i \otimes 1 - 1 \otimes x_i)^2(x_i \otimes 1 - 1 \otimes x_{i+1})$, then the exact sequence

$$S_i\{1\} \xrightarrow{\varphi} S'\{-2\} \xrightarrow{1} S\{-2\} \xrightarrow{m_i} S_i\{-1\}$$

is a realaizao of the class $\varphi \in \text{Ext}^1(F(\sigma_i^{-1}), F(\sigma))$.

By standard homological algebra, any element $\psi \in \text{Ext}^1_k(M, N)$ induces a canonical map $HH^i(\psi) : HH^i(M) \to HH^{i+1}(N)$. Thus, the image of $\varphi$ in $\text{Ext}^1(F(\sigma_-), F(\sigma_+))$ induces a map $W : \mathcal{KR}_N(K_-) \to \mathcal{KR}_N(K_+)$, which is our wall-crossing map.

Note that if $K$ is an $n$-singular link, we can define the wall-crossing element of $\text{Ext}^1$ in a different tensor factor, so they will commute in
the Yoneda product, and thus induce commuting maps on $HH^i$. Even though $\varphi_\theta$ for each singular point $\theta$ is only defined up to a scalar, changing the $\text{Ext}^1$-term in the factor corresponding to $\theta$ by a scalar will change all the wall-crossing maps for that wall by the same scalar, so the iterated cone is still well-defined.

\[\square\]

Let $K$ be a singular link, and $\{K^\lambda\}$ be the collection of resolutions of $K$. We have a cube of wall-crossing maps whose vertices are the complexes $\mathcal{KR}_N(K^\lambda)$ as $\lambda$ ranges over resolutions. We denote the total complex of multi-complex by $\mathcal{VKR}_N(K)$, with the grading inherited from $\mathcal{KR}_N(K^\lambda)$ (or bigrading in the case of $\mathcal{KR}_\infty(K)$). The complex $\mathcal{VKR}_N(K)$ can also be realized as an iterated cone, over the wall-crossing maps, for the various walls which $K$ lies on.

**Definition/Theorem 4.2.** The homology of $\mathcal{VKR}_\infty(K)$ is a triply graded homology theory for singular links and its Euler characteristic is the Vassiliev derivative of the HOMFLY polynomial.

Similarly, the homology of $\mathcal{VKR}_N(K)$ for $N < \infty$ is a categorification of the Vassiliev derivative of the $\mathfrak{sl}_n$-quantum invariants.

**Proof.** The proof of invariance simply follows that of invariance of HOMFLY homology based on the Markov moves in [Web]. Markov I is clear, because $\varphi$ is depends on a single one of the tensor factors which are cyclically permuted by Markov I. For Markov II, we need only check that the inclusion of $\text{Hom}(\mathcal{F}_j(\sigma_1^{-1}), \mathcal{F}_j^{j+1}(\sigma_1))$ to $\text{Hom}(\mathcal{F}_j(\sigma_-), \mathcal{F}_j^{j+1}(\sigma_+))$ matches that after the stabilization under the isomorphisms that hold for all $\sigma \in B_n$. $\mathcal{F}_j(\sigma) \cong \mathcal{F}_j(\sigma_n)$ and $\mathcal{F}_j(\sigma) \cong \mathcal{F}_j^{1}(\sigma_n^{-1})$. Both these isomorphisms are induced by spectral sequences for the tensor product $F(\sigma) \otimes F(\sigma_n)$, and by the functoriality of these spectral sequences, the inclusions coincide. \[\square\]

**Proposition 4.3.** The homology of $\mathcal{VKR}(K)$ is finite dimensional for any singular knot $K$.

**Proof.** The iterated cone is the total chain complex of a double complex whose horizontal components are the complexes $\mathcal{KR}_N(K^\lambda)$, whose homology is finite dimensional (see, for example, [Ras, Proposition 7.1]). The spectral sequence of a double complex shows that the homology of the total complex is finite-dimensional as well. \[\square\]

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