On exponential stability for linear discrete-time systems in Banach spaces

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In this paper we investigate four concepts of exponential stability for difference equations in Banach spaces. Characterizations of these concepts are given. They can be considered as variants for the discrete-time case of the classical results due to Barbashin [6] and Datko [5]. An illustrative example clarifies the relations between these concepts.

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1. Introduction

Let \(X\) be a real or complex Banach space, \(\mathcal{B}(X)\) be the Banach algebra of all bounded linear operators from \(X\) into itself, and \(X^*\) be the Banach space of all continuous linear functionals on \(X\) (the dual space of \(X\)).

The norms on \(X\), in \(\mathcal{B}(X)\) and occasionally in the dual space \(X^*\) are all denoted by \(\|\cdot\|\). Let \(\Delta\) be the set of all pairs \((m, n)\) of positive integers satisfying the inequality \(m \geq n\). We also denote by \(\mathcal{T}\) the set of all triplets \((m, n, p)\) of positive integers with \((m, n)\) and \((n, p)\) \(\in \Delta\).

We consider the linear discrete-time system

\[
x_{n+1} = A(n)x_n,
\]

where \(A : \mathbb{N} \to \mathcal{B}(X)\) is a given \(\mathcal{B}(X)\)-valued sequence.

For \((m, n) \in \Delta\) we define

\[
A^n_m = \begin{cases} 
A(m) \cdots A(n + 1), & m \geq n + 1 \\
I, & m = n
\end{cases}
\]

(\(I\) is the identity operator on \(X\)).

It is obvious that \(A^n_mA^p_n = A^n_m\) for all \((m, n, p) \in \mathcal{T}\).

In the theory of difference equations both in finite and infinite dimensional spaces, the concepts of exponential stability play a central role in the study of the asymptotical behaviors of solutions of discrete-time systems. In this sense we recall the classical monographs due to Agarwal [1], Elaydi [2], Gil [3], Lakshmikantham and Trigiante [4], where the stability properties of discrete-time systems are studied.
In this paper we consider four concepts of exponential stability for linear discrete-time systems: uniform exponential stability, nonuniform exponential stability, strong exponential stability and exponential stability.

Our main objective is to obtain appropriate versions of the well-known stability theorems due (in the continuous case) to Datko [5] and Barbashin [6]. An illustrative example clarifies the implications between these exponential stability concepts.

We remark that in comparison with the classical notion of uniform exponential stability, the concepts of strong exponential stability and exponential stability are much weaker behaviors.

A principal motivation for weakening the assumption of uniform exponential behavior is that from the point of view of ergodic theory, almost all variational equations in finite dimensional spaces have a nonuniform exponential behavior.

2. Uniform exponential stability

In this section we consider the well-known concept of uniform exponential stability of a linear discrete-time system (A) given by:

**Definition 1.** The linear discrete-time system (A) is said to be **uniformly exponentially stable** (and denoted as u.e.s.) if there are some constants \( N \geq 1 \) and \( \alpha > 0 \) such that
\[
\|A^n_m x\| \leq Ne^{-\alpha(m-n)}\|A^n_p x\|, \quad \text{for all } (m, n, p, x) \in T \times X.
\] (2)

**Remark 1.** The linear discrete-time system (A) is uniformly exponentially stable if and only if there are some constants \( N \geq 1 \) and \( \alpha > 0 \) such that
\[
\|A^n_m x\| \leq Ne^{-\alpha(m-n)}\|x\|, \quad \text{for all } (m, n, x) \in \Delta \times X.
\] (3)

A preliminary result for uniform exponential stability is given by:

**Proposition 1.** For every linear discrete-time system (A) the following statements are equivalent:

(i) (A) is uniformly exponentially stable;

(ii) there exist two constants \( N \geq 1 \) and \( a \in (0, 1) \) such that
\[
\|A^n_m x\| \leq Na^{m-n}\|A^n_p x\|, \quad \text{for all } (m, n, p, x) \in T \times X;
\] (4)

(iii) there exist a constant \( N \geq 1 \) and a sequence of positive real numbers \( (a_n) \) with \( a_n \to 0 \) such that
\[
\|A^n_m x\| \leq Na^{m-n}\|A^n_p x\|, \quad \text{for all } (m, n, p, x) \in T \times X;
\] (5)

(iv) there exist a constant \( N \geq 1 \) and a sequence of positive real numbers \( (a_n) \) with \( a_n \to 0 \) such that
\[
\|A^n_m x\| \leq Na^{m-n}\|x\|, \quad \text{for all } (m, n, x) \in \Delta \times X.
\] (6)

**Proof.** The implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) are obvious.

(iv) \( \Rightarrow \) (i) We observe that from \( a_n \to 0 \) it follows that there exists a positive integer \( k \) with \( Na_k < 1 \). Then for all \( (m, n) \in \Delta \) there exist two positive integers \( r \) and \( s \) with \( 0 \leq r < k \) such that \( m-n = ks+r \). If we define \( \alpha = -\frac{\ln(Na_k)}{k} > 0 \) then:

(a) For the case \( s = 0 \) we obtain
\[
\|A^n_m x\| \leq Na_r \|x\| \leq Ne^{\alpha r}e^{-\alpha r} \|x\| \\
\leq Ne^{-\ln(Na_k)}e^{-\alpha(m-n)}\|x\| = Me^{-\alpha(m-n)}\|x\|
\]
for all \( (m, n, x) \in \Delta \times X \).

(b) If \( s \neq 0 \) we have
\[
\|A^n_m x\| \leq Na_r \|A^n_{m+s} x\| \leq \cdots \leq N\|A^n_{m+k} x\| \\
\leq Ne^{\alpha k}e^{-\alpha(m-n)}\|x\| = Me^{-\alpha(m-n)}\|x\|
\]
for all \( (m, n, x) \in \Delta \times X \). Thus, (A) is u.e.s., which ends the proof. \( \square \)

Other characterizations of the uniform exponential stability property are given by:

**Theorem 2.** For every linear discrete-time system (A) the following assertions are equivalent:

(i) (A) is uniformly exponentially stable;

(ii) there are some constants \( D \geq 1 \) and \( d > 0 \) such that
\[
\sum_{m=n}^{\infty} e^{d(m-n)}\|A^n_m x\| \leq D\|A^n_p x\|, \quad \text{for all } (n, p, x) \in \Delta \times X;
\] (7)
There exists a constant $D \geq 1$ such that
\[ \sum_{m=n}^{\infty} \|A_m^n x\| \leq D \|A_n^0 x\|, \quad \text{for all } (n, p, x) \in \Delta \times X; \]
\[ \sum_{m=n}^{\infty} \|A_m^n x\| \leq D \|x\|, \quad \text{for all } (n, x) \in \mathbb{N} \times X. \]

**Proof.** The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are immediate. The implication (iv) $\Rightarrow$ (i) has been proved in [7]. □

**Remark 2.** The preceding theorem can be considered as a discrete-time variant of the well-known theorem due to Datko [5]. Similar results are obtained in [8, 1, 9].

**Theorem 3.** The following statements are equivalent:
(i) $\mathbf{(A)}$ is uniformly exponentially stable;
(ii) there are some constants $B \geq 1$ and $b > 0$ such that
\[ \sum_{k=0}^{m} e^{b(m-k)} \|(A_m^k)^{x^*}\| \leq B \|x^*\|, \quad \text{for all } (m, x^*) \in \mathbb{N} \times X^*; \]
(iii) there exists a constant $B \geq 1$ such that
\[ \sum_{k=0}^{m} \|(A_m^k)^{x^*}\| \leq B \|x^*\|, \quad \text{for all } (m, x^*) \in \mathbb{N} \times X^*. \]

**Proof.** (i) $\Rightarrow$ (ii) Using **Definition 1** we have that
\[ \sum_{k=0}^{m} e^{b(m-k)} \|(A_m^k)^{x^*}\| \leq \sum_{k=0}^{m} e^{b(m-k)} Ne^{-\alpha(m-k)} \|x^*\| \leq \frac{Ne^{\alpha}}{e^{b\alpha} - e^{b\alpha}} \|x^*\| \]
for any $b \in (0, \alpha)$ and all $(m, x^*) \in \mathbb{N} \times X^*$.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) Let $(m, n, x^*) \in \Delta \times X^*$. By our hypothesis for $k \in \{n, n+1, \ldots, m\}$ and all $y^* \in X^*$ we have that
\[ \|(A_m^k)^{y^*}\| \leq B \|y^*\|. \]
If we consider $y^* = (A_m^n)^{x^*}$ we obtain
\[ \|(A_m^n)^{x^*}\| \leq \|(A_m^n)^{x^*}\| \leq B \|(A_m^n)^{x^*}\| \leq B \|(A_m^n)^{x^*}\|. \]

Hence, $(m - n + 1) \|(A_m^n)^{x^*}\| = \sum_{k=n}^{m} \|(A_m^n)^{x^*}\| \leq B \sum_{k=n}^{m} \|(A_m^n)^{x^*}\| \leq B \sum_{k=0}^{m} \|(A_m^n)^{x^*}\| \]
\[ \leq B \sum_{k=0}^{m} \|(A_m^n)^{x^*}\| \leq B^2 \|x^*\|. \]

Thus, we can conclude that Eq. (A) is u.e.s. □

**Remark 3.** The previous theorem is an analog for the linear discrete-time systems of a well-known theorem proved in the continuous case by Barbashin [6, Theorem 5.1, p. 169].

### 3. Nonuniform exponential stability

**Definition 2.** The linear discrete-time system $\mathbf{(A)}$ is said to be nonuniformly exponentially stable (and denoted as n.e.s.) if there exists a constant $\alpha > 0$ and a nondecreasing function $N : \mathbb{R}_+ \to [1, \infty)$ such that
\[ \|A_m^n x\| \leq N(n)e^{-\alpha(m-n)} \|A_n^p x\|, \quad \text{for all } (m, n, p, x) \in T \times X. \]

**Remark 4.** The linear discrete-time system $\mathbf{(A)}$ is n.e.s. if and only if there exist a constant $\alpha > 0$ and a nondecreasing function $N : \mathbb{R}_+ \to [1, \infty)$ such that
\[ \|A_m^n x\| \leq N(n)e^{-\alpha(m-n)} \|x\|, \quad \text{for all } (m, n, x) \in \Delta \times X. \]

It is obvious that if $\mathbf{(A)}$ is u.e.s. then it is n.e.s. The following example shows that the converse implication is not valid.
Example 1. Let \( (A) \) be the linear discrete-time system given by

\[
A(n) = ca_n I, \quad \text{where } a_n = \begin{cases} 
  e^{-n} & \text{if } n = 2k \\
  e^{n+1} & \text{if } n = 2k + 1 
\end{cases}
\]

with \( c \in (0, \frac{1}{e}) \). Let \( (m, n, x) \in \Delta \times X \). According to (1) we have that

\[
A^{m}_n x = \begin{cases} 
  e^{m-n} a_{mn} x & \text{if } m > n \\
  x & \text{if } m = n, 
\end{cases}
\]

where

\[
a_{mn} = \begin{cases} 
  1 & \text{if } m = 2q \text{ and } n = 2p \\
  e^{-n} & \text{if } m = 2q \text{ and } n = 2p + 1 \\
  e^{n+1} & \text{if } m = 2q + 1 \text{ and } n = 2p \\
  e^{m-n} & \text{if } m = 2q + 1 \text{ and } n = 2p + 1.
\end{cases}
\]

Firstly, we observe that if we suppose that Eq. \( (A) \) is u.e.s. then there exist some constants \( N \geq 1 \) and \( \alpha > 0 \) such that

\[
\|A^m_n x\| \leq Ne^{-\alpha(m-n)}\|x\|, \quad \text{for all } (m, n, x) \in \Delta \times X.
\]

In particular, for \( m = 2q + 1 \) and \( n = 2q \) it results that \( e^{\alpha + 2q + 2} \leq N \), for all positive integers \( q \), which is a contradiction. Hence, \( (A) \) is not u.e.s. for every \( c > 0 \).

If \( c \in (0, \frac{1}{e}) \) then for \( \alpha = \ln \frac{1}{c e} > 0 \) and \( N(n) = e^{n+1} \) we have that

\[
\|A^m_n x\| \leq N(n)e^{-\alpha(m-n)}\|x\|, \quad \text{for all } (m, n, x) \in \Delta \times X,
\]

which shows that \( (A) \) is n.e.s.

Theorem 4. The linear discrete-time system \( (A) \) is nonuniformly exponentially stable if and only if there exists a constant \( d > 0 \) and a nondecreasing function \( N : \mathbb{R}_+ \rightarrow [1, \infty) \) such that

\[
\sum_{m=n}^{\infty} e^{d(m-n)}\|A^m_n x\| \leq N(n)\|A^p_n x\|, \quad \text{for all } (m, n, p, x) \in T \times X.
\]

Proof. Necessity. It is a simple verification.

Sufficiency. The inequality (16) implies that \( e^{d(m-n)}\|A^m_n x\| \leq N(n)\|x\|, \) for all \( (m, n, x) \in \Delta \times X \). By Remark 4 it results that \( (A) \) is n.e.s. \( \Box \)

Theorem 5. If there is a constant \( b > 0 \) and a nondecreasing function \( N : \mathbb{R}_+ \rightarrow [1, \infty) \) such that

\[
\sum_{k=n}^{m} e^{b(m-k)}\|(A^k_m)x^+\| \leq N(n)\|x^+\|, \quad \text{for all } (m, n, x^+) \in \Delta \times X^+.
\]

then the linear discrete-time system \( (A) \) is nonuniformly exponentially stable.

Proof. By (17) we have that \( \|A^p_n \| \leq N(n)e^{-\beta(m-n)}, \) for all \( (m, n) \in \Delta \), which implies that \( (A) \) is n.e.s. \( \Box \)

4. Exponential stability

In this section we study a particular concept of nonuniform exponential stability, studied by Barreira and Valls (see for example [10,11]).

Definition 3. The linear discrete-time system \( (A) \) is said to be exponentially stable (and denoted as e.s.) if there are some constants \( N \geq 1 \), \( \alpha > 0 \) and \( \beta \geq 0 \) such that

\[
\|A^p_n \| \leq Ne^{-\alpha(m-n)}e^{\beta n}\|A^p_n x\|, \quad \text{for all } (m, n, p, x) \in T \times X.
\]

Remark 5. The linear discrete-time system \( (A) \) is exponentially stable if and only if there are some constants \( N \geq 1 \), \( \alpha > 0 \) and \( \beta \geq 0 \) such that

\[
\|A^p_n \| \leq Ne^{-\alpha(m-n)}e^{\beta n}\|x\|, \quad \text{for all } (m, n, x) \in \Delta \times X.
\]
Proposition 6. The following statements are equivalent:

(i) \((A)\) is exponentially stable.
(ii) There exist some constants \(N \geq 1, \; \nu > 0\) and \(\beta \in [0, \nu)\) such that
\[
\|A^n_{m}x\| \leq Ne^{-\nu(m-n)}e^{\beta m}\|A^n_{p}x\|, \quad \text{for all } (m, n, p, x) \in T \times X.
\] (20)
(iii) There exist some constants \(N \geq 1, \; \nu > 0\) and \(\delta > 0\) with \(\delta \leq \nu\) such that
\[
\|A^n_{m}x\| \leq Ne^{-\delta m}e^{\nu n}\|A^n_{p}x\|, \quad \text{for all } (m, n, p, x) \in T \times X.
\] (21)

Proof. (i) \(\Rightarrow\) (ii) It is a simple verification for \(\beta \geq 0\) and \(\nu = \alpha + \beta\), with \(\alpha\) and \(\beta\) given by Definition 3.

(ii) \(\Rightarrow\) (iii) We have that \(\|A^n_{m}x\| \leq Ne^{\nu n}e^{-(\nu-\delta)m}\|A^n_{p}x\|\). Hence, for \(\nu > 0\), and \(\delta = \nu - \beta\) we obtain relation (21).

(iii) \(\Rightarrow\) (i) Using (21) we obtain that \(\|A^n_{m}x\| \leq Ne^{-(\nu-\beta) m}\|A^n_{p}x\|\).

For \(\alpha = \delta\) and \(\beta = \nu - \delta\) we obtain that Eq. (A) is e.s. \(\square\)

Remark 6. It is obvious that u.e.s. \(\Rightarrow\) e.s. The following example shows that the converse implication is not valid.

Example 2. Let \((A)\) be the discrete-time system given by (14), with \(c > 0\). From (15) it results that if \(c \in (0, \frac{1}{e})\) then \((A)\) is e.s.

The converse implication is also true, because if we suppose that \((A)\) is e.s. and \(c \geq \frac{1}{e}\) then for \(m = 2q + 1, \; n = 1\) with \(q \in \mathbb{N}\) we obtain \(N \geq (e^{\alpha+1}c)^\delta e^{-\beta}\) which is false for \(q \rightarrow \infty\). In conclusion \((A)\) is e.s. if and only if \(c \in (0, \frac{1}{e})\).

Theorem 7. The linear discrete-time system \((A)\) is exponentially stable if and only if there exist some constants \(D \geq 1, \; d > 0\) and \(c \geq 0\) such that
\[
\sum_{m=0}^{\infty} e^{\delta(m-n)}\|A^m_{m}x\| \leq De^{cn}\|x\|, \quad \text{for all } (n, x) \in \mathbb{N} \times X.
\] (22)

Proof. Necessity. It is a simple verification for \(c = \beta, \; d \in (0, \alpha)\) and \(D = 1 + \frac{Ne^{\alpha\delta}}{e^{\alpha\delta} - d}\), with \(N, \alpha\) and \(\beta\) offered by Remark 5.

Sufficiency. The inequality (22) implies \(\|A^n_{m}x\| \leq De^{cn}e^{d(m-n)}\|x\|\), for all \((m, n, x) \in \Delta \times X\), which shows that \((A)\) is e.s. \(\square\)

Theorem 8. The linear discrete-time system \((A)\) is exponentially stable if and only if there exist some constants \(B \geq 1, \; b > 0\) and \(c \in [0, b)\) such that
\[
\sum_{k=0}^{m} e^{\delta(m-k)}\|(A^k_{m})^n x^n\| \leq Be^{cm}\|x^n\|, \quad \text{for all } (m, x^n) \in \mathbb{N} \times X^*.
\] (23)

Proof. Necessity. It is a simple verification for \(c = \beta, \; b \in (\beta, \alpha + \beta)\) and \(B = 1 + \frac{Ne^{\alpha\delta+b}}{e^{\alpha\delta+b} - e^{\beta b}}\), with \(N, \alpha\) and \(\beta\) given by Remark 5.

Sufficiency. Since \(0 \leq c < b\), using inequality (23) for all \((m, n) \in \Delta\) we have that \(e^{\delta(m-n)}\|A^n_{m}x\| = e^{\beta(m-n)}\|(A^n_{m})^* x^n\| \leq Be^{cm}\), which proves that \((A)\) is e.s. \(\square\)

5. Strong exponential stability

A particular concept of exponential stability is defined by:

Definition 4. The linear discrete-time system \((A)\) is said to be strongly exponentially stable (and denoted as s.e.s.) if there are some constants \(N \geq 1, \; \alpha > 0\) and \(\beta \in [0, \alpha)\) such that
\[
\|A^p_{m}x\| \leq Ne^{-\alpha(m-n)}e^{\beta n}\|A^p_{n}x\|, \quad \text{for all } (m, n, p, x) \in T \times X.
\] (24)

Remark 7. The linear discrete-time system \((A)\) is strongly exponentially stable if and only if there are some constants \(N \geq 1, \; \alpha > 0\) and \(\beta \in [0, \alpha)\) such that
\[
\|A^n_{m}x\| \leq Ne^{-\alpha(m-n)}e^{\beta n}\|x\|, \quad \text{for all } (m, n, x) \in \Delta \times X.
\] (25)

Proposition 9. The linear discrete-time system \((A)\) is strongly exponentially stable if and only if there exist some constants \(N \geq 1, \; \alpha > 0\) and \(\nu > 0\) with \(\nu \geq \alpha\) such that
\[
\|A^n_{m}x\| \leq Ne^{-\alpha m}e^{\nu n}\|A^n_{p}x\|, \quad \text{for all } (m, n, p, x) \in T \times X.
\] (26)
Proof. Necessity. Using our hypothesis and Definition 4 we have that
\[ \|A_{m}^{n}x\| \leq Ne^{-(\alpha(n-m))}e^{\beta n}\|A_{n}^{m}\| = Ne^{-(\alpha(n-m))}e^{(\beta+\alpha)n}\|A_{m}^{n}x\| \]
and for \(\alpha > 0\) and \(\nu = \beta + \alpha\) we obtain relation (26).

Sufficiency. We have that \(\|A_{m}^{n}x\| \leq Ne^{-(\alpha(n-m))}e^{-(\alpha(m-n))}\|A_{m}^{n}x\|\). For \(\alpha > 0\) and \(\beta = \nu - \alpha\) we obtain that (A) is s.e.s. \(\square\)

Remark 8. It is obvious that s.e.s. \(\Rightarrow\) e.s. The following example shows that the concept of strong exponential stability is a distinct concept of exponential stability.

Example 3. Let (A) be the linear discrete-time system given by (14).
If \(c \in \left[\frac{1}{2}, \frac{1}{2}\right]\) then from the considerations given in Example 2 it results that (A) is e.s. If we suppose that (A) is s.e.s. then there are \(N \geq 1\), \(\alpha > 0\) and \(\beta \in [0, \alpha]\) such that \(\|A_{m}^{n}x\| \leq Ne^{\alpha(n-m)}\|x\|\), for all \((m, n, x) \in \Delta \times X\). Then for \(m = 2q + 1\) and \(n = 1\) we obtain \(N \geq (e^{\alpha+1})^{q}e^{-\beta}\) which is impossible. Hence, for \(c \geq \frac{1}{2}\), (A) is not s.e.s., which shows that if (A) is s.e.s. then \(c \in \left(0, \frac{1}{2}\right)\). The converse implication is also valid. Indeed, for \(c \in \left(0, \frac{1}{2}\right)\), \(\alpha = -\ln(ce)\), \(\beta = 1\) and \(N = e\) we have that \(\|A_{m}^{n}x\| \leq Ne^{\beta n}e^{-\alpha(m-n)}\|x\|\), for all \((m, n, x) \in \Delta \times X\), with \(\alpha > \beta \geq 1\). Thus, (A) is s.e.s. In conclusion, (A) is s.e.s.

Theorem 10. The linear discrete-time system (A) is strongly exponentially stable if and only if there exist some constants \(D \geq 1\), \(d > 0\) and \(c \geq 0\) with \(0 \leq c < d\) such that
\[ \sum_{m=n}^{\infty} e^{d(m-n)}\|A_{m}^{n}x\| \leq De^{cn}\|x\|, \text{ for all } (m, n) \in \mathbb{N} \times X. \]  
(27)

Proof. It results from the Definition 4 and the proof of Theorem 7. \(\square\)

Theorem 11. The linear discrete-time system (A) is strongly exponentially stable if and only if there exist some constants \(B \geq 1\), \(b > 0\) and \(c \geq 0\) with \(0 \leq 2c < b\) such that
\[ \sum_{k=0}^{m} e^{b(m-k)}\|(A_{m}^{k})x^{*}\| \leq Be^{cm}\|x^{*}\|, \text{ for all } (m, x^{*}) \in \mathbb{N} \times X^{*}. \]  
(28)

Proof. It results from Definition 4 and the proof of Theorem 8. \(\square\)

6. Conclusions
The paper considers four concepts of exponential stability for linear discrete-time systems in Banach spaces. We establish relations between these concepts. It is obvious that we have u.e.s. \(\Rightarrow\) s.e.s. \(\Rightarrow\) e.s. \(\Rightarrow\) n.e.s. and to emphasize the straightforwardness of the implication we have considered the linear discrete-time system (A) given by \(A(n) = ca_{n}I\), where \(c \geq 0\) and
\[ a_{n} = \begin{cases} e^{-n} & \text{if } n = 2k; \\ e^{n+1} & \text{if } n = 2k + 1. \end{cases} \]
Examples 1–3 show that:
(i) (A) is u.e.s. if and only if \(c = 0\);
(ii) (A) is n.e.s. if and only if \(c \in \left(0, \frac{1}{2}\right)\);
(iii) (A) is e.s. if and only if \(c \in \left(0, \frac{1}{2}\right)\);
(iv) (A) is s.e.s. if and only if \(c \in \left(0, \frac{1}{2}\right)\).

Theorems 2, 4, 7 and 10 give characterizations for the exponential stability concepts studied in this paper. They can be considered as variants for the discrete-time case of a well-known result due to Datko [5] in the continuous case.
The characterizations given by Theorems 3, 5, 8 and 11 can be considered as variants for the discrete-time case of a result due to Barbashin [6, Theorem 5.1, p. 169], in the continuous case.

7. Open problems
Our goal here is to present an interesting open problem. A characterization of Barbashin type for u.e.s. is given in [8]. This result implies:

Theorem 12. For every linear discrete-time system (A) the following statements are equivalent:
(i) (A) is uniformly exponentially stable;
(ii) there are some constants $B \geq 1$ and $b > 0$ such that
\[ \sum_{k=0}^{m} e^{b(m-k)} \| A_k^m \| \leq B, \quad \text{for all } m \in \mathbb{N}; \]  
\[ (29) \]
(iii) there exists a constant $B \geq 1$ such that
\[ \sum_{k=0}^{m} \| A_k^m \| \leq B, \quad \text{for all } m \in \mathbb{N}. \]
\[ (30) \]
Similar results hold for other concepts of exponential stability. We present as an open problem the following:

**Conjecture.** For every linear discrete-time system $(A)$ the following statements are equivalent:

(i) $(A)$ is uniformly exponentially stable;

(ii) there are some constants $B \geq 1$ and $b > 0$ such that
\[ \sum_{k=0}^{m} e^{b(m-k)} \| A_k^m x \| \leq B \| x \|, \quad \text{for all } (m, x) \in \mathbb{N} \times X; \]
\[ (31) \]
(iii) there exists a constant $B \geq 1$ such that
\[ \sum_{k=0}^{m} \| A_k^m x \| \leq B \| x \|, \quad \text{for all } (m, x) \in \mathbb{N} \times X. \]
\[ (32) \]

The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate. The problem is the proof of the implication $(iii) \Rightarrow (i)$. Of course, we are interested in obtaining similar results for n.e.s., e.s. and s.e.s. according to the previous condition.

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