On the treatment of divergent integrals in perturbative quantum field theory

R. Trinchero

Instituto Balseiro, 8400 Bariloche, Argentina

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A simple integral that illustrates the concepts of regularization, subtraction, renormalization and renormalization group employed in perturbative quantum field theory (PQFT) is considered.

I. INTRODUCTION

The description of processes where particles can be created and annihilated, i.e. involving energies above the rest energy of the particles involved, is well done in many cases in terms of relativistic quantum field theories. The perturbative approximation to these quantum field theories provides excellent agreement with experiment in the description of fundamental interactions, such as for example the electroweak interactions [1]. Probability amplitudes for different processes in the perturbative approximation of quantum field theories are written in terms of Feynman diagrams. Feynman rules associate to each diagram a certain contribution. For diagrams which involve closed loops, the corresponding contribution is in general given in terms of integrals that are not well defined, they diverge. The renormalization program [2] in perturbative quantum field theory is the process that allows to obtain physical results from these ill defined integrals. This program is both one of the major and most historically controversial developments in the subject. It is a fundamental tool in getting physical results and at the same time involves a careful handling of ill defined integrals. Nowadays renormalization is an important tool for any field-theorist. From the point of view of learning renormalization the subject normally presents some difficulties. These difficulties are mostly related to the fact that in field theories describing nature the length and subtlety of the calculations involved obscure the conceptual ideas behind the renormalization procedure.

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The purpose of this paper is to provide a pedagogical example of how divergent integrals are dealt with in perturbative quantum field theory (PQFT). The interest is focused not in how these integrals arise but on what is done once you have them. As mentioned above in realistic examples of this procedure the main ideas are obscured by the length of the calculations involved. A example of these integrals is presented where the concepts of regularization, subtraction, renormalization and renormalization group are clearly illustrated even when the calculations involved are very simple. Furthermore it permits to see in this particular example different approaches to topics in this field, some of which are considered in this paper. To the knowledge of the author, such an example and its pedagogical treatment are not available in the literature.

The paper is organized as follows. Section II presents the example, analyses its superficial degree of divergence and introduces the concept of regularization. Section III shows how finite parts independent of the regularization employed can be obtained from the original integral. Section IV presents the renormalization procedure and section V studies its dependence on the subtraction point, leading to the concepts of renormalization group transformation and renormalization group equations.

II. THE EXAMPLE

In PQFT the contribution of Feynman diagrams involving loops leads to integrals of rational functions of the integration variables (internal momenta) that depend on other variables (external momenta) and other parameters (coupling constants, masses etc.). The momenta varying in some $\mathbb{R}^n$ where $n$ is the dimension of space-time. Next the following one-dimensional example is considered,

$$I_m(q) = \int_0^\infty dp \frac{1}{p + q + m^2}; \quad q, m \in \mathbb{R}, \quad q > 0$$  \hspace{1cm} (1)

here $p(q)$ play the role of internal (external) momenta and $m^2$ is a parameter. This integral is not well defined, it diverges. A way to measure its degree of divergence is provided but what is usually called superficial degree of divergence. This quantity measures the behaviour of the integrand in its upper limit of integration. In order to define it consider the following change of integration variable $p = \lambda p'$ in the integral $I_m(q)$,

$$I_m(q) = \int_0^\infty dp' \frac{1}{\lambda p' + q + m^2}$$
and consider of the limit of the new integrand for $\lambda \to \infty$. In this example it goes like $\lambda^0$, in general the exponent of $\lambda$ which is obtained in this way is what we refer to as superficial degree of divergence (SDD) of the integral and we denote by $\omega$. The integral of this limit is for this example,

$$I_{m}^{UV} = \int_{0}^{\infty} dp' \frac{1}{p'}$$

which is a logarithmically divergent integral. It is worth noting that the handling of divergent integrals exemplified bellow is applicable to integrands that for high integration momenta behave as an integer power of this momenta. For those integrands the SDD is an integer. In these cases if the integrand has no singular points in the interval of integration and has SDD $\omega < 0$ then the integral converges.

Going back to our example $I_{m}(q)$, it is noted that if it would be well defined it would have the symmetry property,

$$I_{m}(q) = I_{-m}(q).$$  \hspace{1cm} (2)

Clearly, speaking about the symmetry of an object that is not well defined is at least also not well defined. However symmetry is a fundamental concept in physics, and in many cases it works as a guide in looking for a way to make sense of the mathematical expressions that appear in the description of physical theories. The treatment of divergent integrals that appear in perturbative quantum field theory is not an exception. As an example, the conservation of the electromagnetic current in quantum electrodynamics (QED) is considered as a fundamental physical requirement and the whole treatment of divergent integrals in QED aims to maintain this symmetry.

III. REGULARIZATION

The process regularization of a integral consist in considering a parameter dependent family of integrals which,

1. Is well defined for a continuous open interval of the parameter.

2. The original integral correspond to a limiting point of this interval (which will be denoted as cut-off limit).
For the example of the previous section the following regularization is considered,

\[ I_\Lambda^m(q) = \int_0^\Lambda dp \frac{1}{p + q + m^2} , \]  

which leads to,

\[ I_\Lambda^m(q) = [ \ln(p + q + m^2) ]_0^\Lambda \]

\[ = \ln \left[ \frac{\Lambda + q + m^2}{q + m^2} \right] . \]  

(3)

The original integral corresponds to the limit \( \Lambda \to \infty \), which in accordance with the value \( \omega = 0 \), shows a logarithmic divergence. This regularization preserves the symmetry (2), i.e.,

\[ I_\Lambda^m(q) = I_{-\Lambda}^m(q) \]

such regularizations are called invariant.

Another regularization is provided by,

\[ I_M^m(q) = \int_0^\infty dp \left\{ \frac{1}{p + q + m^2} - \frac{1}{p + q + mM} \right\} \]

\[ = \left[ \ln \frac{p + q + m^2}{p + q + mM} \right]_0^\infty \]

\[ = - \ln \left[ \frac{q + m^2}{q + mM} \right] = \ln \left[ \frac{q + mM}{q + m^2} \right] . \]  

(5)

The original integral corresponds to the limit \( M \to \infty \), which in accordance with the value \( \omega = 0 \), shows a logarithmic divergence. This regularization does not preserve the symmetry (2), i.e.,

\[ I_M^m(q) \neq I_{-\Lambda}^m(q) \]

IV. SUBTRACTION

Is it possible to separate a finite contribution from the regularized integral in a way that does not depend on the regularization chosen? It is clear that there are many ways to write the regularized integral as the sum of two terms, one of which tends to a finite limit in the cut-off limit and the other diverges in this limit. For example,

\[ I_\Lambda^m(q) = I_{\Lambda,F}^m(q) + I_{\Lambda,D}^m(q) , \]  

(6)
with
\[ I_{m,F}^{\Lambda}(q) = -\ln[q + m^2] \]
\[ I_{m,D}^{\Lambda}(q) = \ln[\Lambda + q + m^2] \, , \] (7)
is a possibility, but it is also possible to take,
\[ I_{m,F}^{\Lambda}(q) = -\ln\frac{q + m^2}{m^2 + f(q)} \]
\[ I_{m,D}^{\Lambda}(q) = \ln\frac{\Lambda + q + m^2}{m^2 + f(q)} \, , \] (8)
for the moment there is no criteria to perform this separations.

In order to proceed derivatives of the integrand in \( \Pi \) will be considered. Let \( \mathcal{F}(p, q) \) denote this integrand, i.e.,
\[ \mathcal{F}(p, q) = \frac{1}{p + q + m^2} \]
Then the integral of the derivative of the integrand respect to the external momenta \( q \) converges, indeed,
\[ I_{m}^{(1)}(q) \equiv \int_0^\infty dp \frac{\partial \mathcal{F}(p, q)}{\partial q} \]
\[ = \int_0^\infty dp \frac{-1}{(p + q + m^2)^2} \]
\[ = \left. \frac{1}{p + q + m^2} \right|_0^\infty = \frac{1}{q + m^2} \, . \] (9)
where the first equality is just the definition of \( I_{m}^{(1)}(q) \). It is clear that since \( I_{m}^{(1)}(q) \) is a convergent integral, then any regularization of this integral would give in the cut-off limit the same result obtained in \( \Pi \). Next consider the Taylor expansion of the integrand \( F(p, q) \) around \( q = 0 \), i.e.,
\[ \mathcal{F}(p, q) = \mathcal{F}(p, 0) + \left[ \frac{d\mathcal{F}(p, q)}{dq} \right]_{q=0}\, q + \ldots + \left[ \frac{d^n\mathcal{F}(p, q)}{dq^n} \right]_{q=0} \, q^n + \ldots \, . \] (10)
The only term in this expansion whose integral in \( p \) does not converge is the first one. The identification,
\[ I_{m}^{(1)}(q) = \frac{dI_{m}(q)}{dq} \] (11)
makes no sense because \( I_{m}(q) \) is not well defined. However the quantities,
\[ I_{m}^{(n)}(q) = \frac{d^nI_{m}^{\Lambda}(q)}{dq^n} \] (12)
\[ I_m^{(n)M}(q) = \frac{d^n I_m^M(q)}{dq^n}. \]  

for any integer \( n > 0 \) are well defined and satisfy,

\[ \lim_{\Lambda \to \infty} I_m^{(n)\Lambda}(q) = \lim_{M \to \infty} I_m^{(n)M}(q), \]

which simply reflects the independence on the regularization of convergent integrals. Therefore the quantity,

\[ \tilde{I}_m(q) = \lim_{\Lambda \to \infty} [I_m^{\Lambda}(q) - I_m^{\Lambda}(0)] \]

\[ = \lim_{\Lambda \to \infty} \left\{ \ln \left[ \frac{\Lambda + q + m^2}{q + m^2} \right] - \ln \left[ \frac{\Lambda + m^2}{m^2} \right] \right\} \]

\[ = \ln \left[ \frac{m^2}{q + m^2} \right] \]

is finite and independent of the regularization employed, indeed,

\[ \tilde{I}_m(q) = \lim_{M \to \infty} [I_m^M(q) - I_m^M(0)] \]

\[ = \lim_{M \to \infty} \left\{ \ln \left[ \frac{q + mM}{q + m^2} \right] - \ln \left[ \frac{mM}{m} \right] \right\} \]

\[ = \ln \left[ \frac{m^2}{q + m^2} \right]. \]

This construction of finite parts is very similar to what in the mathematical literature is called Hadamard’s finite parts. See for example ref.[6] for an account of this subject.

Next a series of remarks on the above construction are given,

1. What would happen is the SDD \( \omega \) would equal some positive integer \( k \)? For this type of integrals taking the derivative of the integrand respect to the external momenta reduces its SDD \( \omega \) by 1. Therefore the \( k + 1 \) derivative of the integrand respect to the external momenta would give a finite result when integrated. Thus subtracting to the regularized integral the integral of the first \( k \) terms of its Taylor expansion as in (10) leads to a finite quantity independent of the regularization employed. More precisely if \( K(q) \) denotes the integral with SSD \( \omega = k \) and \( K^\Lambda(q) \) denotes the corresponding regularized integral, then the quantity,

\[ \tilde{K}(q) = \lim_{\Lambda \to \infty} \left[ K^\Lambda(q) - \left( K^\Lambda(0) + \frac{dK^\Lambda(q)}{dq}|_{q=0} q + \cdots + \frac{d^kK^\Lambda(q)}{dq^k}|_{q=0} q^k \right) \right] \]

is finite and independent of the regularization employed.
2. The quantity between parenthesis in the last equation is called the subtraction. It is very important to note that the subtraction is a polynomial in the external momenta $q$. This implies that the Fourier transform of the subtraction, i.e. its expression in coordinate space, is a sum of derivatives of the delta function. Although not shown in this work, the important point is that this same terms in coordinate space would arise from adding to the Lagrangian of the field theory a certain number of local terms in the fields and its derivatives. This local terms are called counterterms and the coefficients in front of them should be of higher order in the expansion parameter and dependent on the regularization. Indeed temporal ordered products of fields, from which these integrals arise, are just not well defined for coincident space-time arguments, so the freedom of including counterterms can be thought as arising from this indefiniton.

3. The relation between the calculation of a given integral in two regularizations is considered. If the result for the integral is known in one regularization then it suffices to calculate the subtraction for the other regularization in order to know the whole expression in this last regularization. This is shown for the example of section III. Let $I_{m}^{\text{Reg}}(q)$ and $I_{m}^{\text{Reg}'}(q)$ denote the integral in two regularizations, then,

$$I_{m}^{\text{Reg}}(q) = \tilde{I}_{m}(q) + I_{m}^{\text{Reg}}(0),$$

and,

$$I_{m}^{\text{Reg}'}(q) = \tilde{I}_{m}(q) + I_{m}^{\text{Reg}'}(0).$$

thus,

$$I_{m}^{\text{Reg}'}(q) = I_{m}^{\text{Reg}}(q) - I_{m}^{\text{Reg}}(0) + I_{m}^{\text{Reg}'}(0).$$

4. What can be said about the behaviour of the subtractions under the transformation $m \rightarrow -m$? From eqs. (15) it follows that for the first regularization considered the subtraction is invariant, however from (16) it is clear that this is not the case for the second regularization. The general result is that subtractions are invariant if and only if the corresponding regularizations respect the symmetry. At the level of the Lagrangian this implies that counterterms are invariant under the symmetry if and only if the associated regularization is invariant.

5. The above arguments suggest another way of getting a regularization independent finite part of the integral. The original integral can be replaced by one in which the
first term in the Taylor’s expansion around \( q = 0 \) of the integrand is subtracted to the integrand, i.e. replacing,

\[
I_m(q) = \int_0^\infty dp \frac{1}{p + q + m^2}
\]

by,

\[
\tilde{I}_m(q) = \int_0^\infty dp \left[ \frac{1}{p + q + m^2} - \frac{1}{p + m^2} \right].
\] (17)

from which it is readily obtained,

\[
\tilde{I}_m(q) = \left[ \ln \frac{p + q + m^2}{p + m^2} \right]_0^\infty = \ln \left[ \frac{m^2}{q + m^2} \right].
\] (18)

which of course coincides with the result in (15) and (16). This procedure allows to subtract without previously employing a regularization, it was proposed in ref.[7].

V. RENORMALIZATION

As was shown in the previous section by means of the subtraction procedure it is possible to separate in a regularization independent way a finite part from the original divergent integral. However this does not solve the problem of making physical sense of a theory were such divergent integrals appear. In order to address this point it is necessary to know how does the divergent integral enters in the physical theory. Two cases will be considered here, the ones of masses and coupling constants in quantum field theory (QFT), which correspond in general to additive and multiplicative renormalization. In the first case the integral appears in a physical quantity \( E \) in the form,

\[
E = \mu_0 + I_m(q),
\] (19)

where \( \mu_0 \) is a parameter of the theory. In order to rewrite (19) in terms of the finite part \( \tilde{I}_m(q) \) the following identity is considered,

\[
E = \mu_0 + \left[ I_m^{\text{Reg}}(q) - I_m^{\text{Reg}}(0) \right] + I_m^{\text{Reg}}(0),
\] (20)

defining the renormalized parameter \( \mu_R \) by,

\[
\mu_R = \mu_0 + I_m^{\text{Reg}}(0)
\] (21)
and taking the cut-off limit leads to,

$$E = \mu_R + \tilde{I}_m(q).$$  \hfill (22)

the parameter $\mu_R$ corresponds to a measurable physical quantity (the mass for example) and therefore is given a finite value. Therefore \[21\] implies that the original parameter $\mu_0$ appearing in the theory should depend on the regularization parameter such as to cancel the contribution of $I_{m}^{Reg}(0)$. This is the key idea in renormalization, bare quantities such as $\mu_0$ are not observable, it does not matter what is their value. In other words, perturbation theory is just an approximation, the unperturbed part of the Lagrangian of the field theory under consideration has no physical reality, interactions can not be switched off in nature they are always switched on.

For the case of coupling constant renormalization the integral appears in a physical quantity $E'$ as follows,

$$E' = g_0 \left(1 + g_0 I_m(q)\right).$$  \hfill (23)

in terms of the finite part of $I_m(q)$ the last equation can be written as,

$$E' = g_0 \left\{1 + g_0 \left[I_{m}^{Reg}(q) - I_{m}^{Reg}(0)\right] + g_0 I_{m}^{Reg}(0)\right\}.$$  \hfill (24)

defining the renormalized parameter $g_R$ by,

$$g_R = g_0 \left[1 + g_0 I_{m}^{Reg}(0)\right]$$  \hfill (25)

leads to,

$$E' = g_R + g_0^2 \tilde{I}_m(q).$$  \hfill (26)

Inverting \[25\] and expanding for $g_R \ll 1$, implies,

$$E' = g_R \left[1 + g_R \tilde{I}_m(q)\right].$$  \hfill (27)

This last expansion for $g_R \ll 1$ is taken because the perturbative expansion parameters are by definition the couplings constants. It is important to note that the cut-off limit should be taken after the renormalization.

It should be remarked that there is an alternative way to renormalize the theory. This procedure is based on remark 2. of the last section. The idea is to add local counterterms
of higher order to the original field theory in order to cancel the divergent contributions. The addition of these counterterms being justified by the indefiniton of temporal ordered products for coincident spatial arguments\[^3\]. This scheme is more general than the previous one. This is so because it allows to renormalize even when the regularization employed is not invariant, you can not generate non-invariant counterterms by redefining the original parameters appearing in the theory. This is so because the original theory is invariant.

VI. SUBTRACTION POINT AND RENORMALIZATION GROUP

Why was the expansion in (10) done around \( q = 0 \) and not around other value \( q = q_s \)? In other words, is it possible to maintain consistency considering the following subtraction point \((q_s)\) dependent finite part,

\[
\tilde{I}_m(q, q_s) = \lim_{\Lambda \to \infty} \left[ I^{\text{Reg}^\Lambda}_m(q) - I^{\text{Reg}^\Lambda}_m(q_s) \right]?
\]

(28)

the one considered in (10) is,

\[
\tilde{I}_m(q) = \tilde{I}_m(q, 0).
\]

(29)

The difference between both finite parts is a finite renormalization, indeed,

\[
\tilde{I}_m(q, q_s) = \lim_{\Lambda \to \infty} \left\{ I^{\text{Reg}^\Lambda}_m(q) - I^{\text{Reg}^\Lambda}_m(0) + \left[ I^{\text{Reg}^\Lambda}_m(0) - I^{\text{Reg}^\Lambda}_m(q_s) \right] \right\}

= \tilde{I}_m(q, 0) + \Delta(q, q_s),
\]

(30)

where,

\[
\Delta(q, q_s) = \lim_{\Lambda \to \infty} \left[ I^{\text{Reg}^\Lambda}_m(0) - I^{\text{Reg}^\Lambda}_m(q_s) \right]

= \lim_{\Lambda \to \infty} \left\{ \ln \left[ \frac{\Lambda + m^2}{m^2} \right] - \ln \left[ \frac{\Lambda + q_s + m^2}{q_s + m^2} \right] \right\}

= \ln \left[ \frac{q_s + m^2}{m^2} \right]
\]

(31)

is a finite quantity. In other words, a change in the subtraction point just corresponds to the inclusion of finite renormalizations. Such operations are known as renormalization group transformations. The requirement that physical quantities do not depend on the choice of subtraction point, i.e,

\[
\frac{dE}{dq_s} = 0,
\]

(32)
leads to,

\[ \frac{d\mu}{dq_s} + \frac{d\tilde{I}_m(q, q_s)}{dq_s} = 0. \]  

which is usually called a renormalization group equation. Since the second term does not vanish, as follows from eqs. (30) and (31), then necessarily the parameter \( \mu_R \) should depend on the subtraction point \( q_s \). Defining,

\[ \mu_R(q_s) = \mu_0 + I_m^{\text{Reg}}(q_s) + \text{const.} = \mu_R(0) - \Delta(0, q_s) + \text{const.} \]  

eq. (33) holds, as follows from (30). Proceeding in a similar way for \( E' \), leads to,

\[ g_R(q_s) = g_0 \left[ 1 + g_0 I_m^{\text{Reg}}(q_s) \right] + \text{const.} = g_R(0)[1 - g_R(0) \Delta(0, q_s)] + \text{const.} . \]  

the physical interpretation of the quantities \( \mu_R(q_s) \) and \( g_R(q_s) \) is that they correspond to the value of masses and coupling constants for processes involving momenta of the order of \( q_s \). This dependence on the momentum scale is known as the "running" of coupling constants and masses.

[1] To get an idea of the phenomenological success of PQFT see for example, Ta-Pei Cheng and Ling-Fong Li, *Gauge Theory of elementary particle physics*, Oxford (1984)

[2] There are many books treating this subject. To get a complete list see http://books.google.com/books?q=renormalization&btnG=Search+Books References [3-5] is a partial list that makes contact with the approach exemplified in this paper.

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