Continuous Measurement of Spin Systems with Spatially-Distinguishable Particles

Ben Q. Baragiola, Bradley A. Chase, and JM Geremia

Department of Physics & Astronomy, The University of New Mexico, Albuquerque, New Mexico USA

(Dated: October 20, 2009)

Continuous measurement of hyperfine spin is typically performed by interacting an atomic sample with a probe laser and then detecting the forward-scattered field. In conventional treatments of free-space coupling, the atoms are described by their collective spin and the field is approximated explicitly as a single spatial mode. However, the size of the atomic sample in typical experiments is large compared to the laser wavelength, thus it is possible (at least in principle) to image the scattered laser field, making a single-mode approximation inadequate. The purpose of this letter is to illustrate that even if the forward-scattered field is focused onto a single detector, models based on a collective, single-mode approximation incorrectly predict even qualitative features of the measurement (please see Fig. 1).

Physical Model— Consider a system of $N$ spin-1/2 particles interacting with a linearly-polarized, far-detuned probe laser propagating along the laboratory $z$-axis. After adiabatic elimination of excited atomic states, the conditions of Fig. $1(a)$ result in the Faraday-effect interaction Hamiltonian $H_C = \hbar k J_z t$, where $J_z$ is the $z$-component of the time-dependent Stokes operator for the single-mode field and the $J_i = \sum_{n=1}^N j_i^{(n)}$ are collective atomic spin operators with $j_i^{(n)} = \sigma_i^{(n)} / 2$. Under the conditions of Fig. $1(b-c)$, however, adiabatic elimination of atomic excited states yields an interaction Hamiltonian of the form $H_S = \hbar k \sum_{n=1}^N j_x^{(n)} n_x t$, where $n_x$ is the Stokes operator of the field mode that interacts only with atom $n$. For simplicity, we assume that the atom-field coupling $k$ is identical for all atoms.

The dynamics of any operator $O_{AF} = \hat{X}_A \otimes \hat{Y}_F$ acting on both the atoms and field is given by the quantum flow $j_t(O_{AF}) = U_t^\dagger O_{AF} U_t$, where $U_t$ is the propagator for the joint system and can be determined from $H_C$ or $H_S$ using well-established procedures from quantum stochastic calculus. Continuous detection of the scattered laser field provides a measurement record $\mathcal{Y}_t$ comprised of the sequence of random measurement outcomes $\tilde{\mathcal{Y}}_i$ during $0 \leq t$. The conditional expectation value of any atomic operator $\hat{X}$ given the measurement data

$$\pi_t[\hat{X}] = \mathbb{E}[\hat{X} | \mathcal{Y}_t]$$

(1)
can be determined using the methods of quantum filter-
Single-Mode (Collective) Measurement Model— We first consider the system of indistinguishable particles interacting with a single field mode, as in Fig. (a). Under the interaction $\hat{H}_C = \hbar k \hat{J}_z \hat{s}_z$, the atom-field system evolves according to the unitary propagator \[ \hat{U}^C_t = \left[ \sqrt{\kappa} \hat{J}_z (d\hat{S}^\dagger_t - d\hat{S}_t) - \frac{1}{2} \kappa \hat{J}_z^2 dt \right] \hat{U}^C_0, \] where $\kappa = 2\pi |k(\omega_a)|^2$ is the weak-coupling interaction strength and $d\hat{S}_t$ and $d\hat{S}_t^\dagger$ are quantum Brownian motion operators satisfying the Itô rules $d\hat{S}d\hat{S}^\dagger = dt$, $d\hat{S}d\hat{S} = d\hat{S}^\dagger d\hat{S}^\dagger = 0$. Continuous measurement of the $\hat{s}_y$ Stokes operator of the forward scattered field generates a measurement current that satisfies \[ dY_t^C = 2\sqrt{\kappa} \hat{J}_z [\hat{J}_z] dt + d\hat{S}^\dagger_t + d\hat{S}_t. \] Equations (2-3) constitute a system-observation pair, from which the conditional atomic dynamics can be inferred via the collective quantum filtering equation \[ d\pi_t^C [\hat{X}] = \kappa \pi_t \hat{L}^C [\hat{J}_z] \hat{X} dt \] \[ + \sqrt{\kappa} \left( \pi_t [\hat{J}_z \hat{X} + \hat{X} \hat{J}_z] - 2\pi_t [\hat{J}_z] \pi_t [\hat{X}] \right) dW_t^C, \] where $dW_t^C = dY_t^C - 2\sqrt{\kappa} \pi_t [\hat{J}_z] dt$ is a classical innovations process and $\hat{L}^C [\hat{J}_z] \hat{X}$ is the collective Lindbladian operator \[ \hat{L}^C [\hat{J}_z] \hat{X} = \hat{J}_z \hat{X} \hat{J}_z - \frac{1}{2} \hat{J}_z^2 \hat{X} - \frac{1}{2} \hat{X} \hat{J}_z^2. \] Note that the single-mode field model described by Eqs. (4-5) leads to a measurement that can be formulated entirely in terms of collective angular momentum operators. Multi-Mode (Symmetric) Measurement Model— We next consider a system of particles interacting with a multi-mode field as in Fig. (c). Under the interaction $\hat{H}_S$, the atom-field system evolves according to \[ d\hat{U}_t^S = \left[ \sum_{n=1}^N \sqrt{\kappa} \hat{j}^{(n)}_z \left( d\hat{S}^{(n)}_t^\dagger - d\hat{S}^{(n)}_t \right) - \frac{\kappa}{2} \hat{j}^{(n)}_z^2 dt \right] \hat{U}_0^S, \] where $\kappa$ is again the weak-coupling interaction strength but the $d\hat{S}^{(n)}_t$ and $d\hat{S}^{(n)}_t^\dagger$ now satisfy $d\hat{S}^{(n)}_td\hat{S}^{(n)}_t^\dagger = dt \delta_{n,m}$ (with all other products equal to zero). Under the condition that the multi-mode field intensity is chosen to be the same as the single mode field intensity, to achieve the same single atom-field coupling strength $\kappa$, continuous measurement of the $\hat{s}_y = \sum_{n=1}^N \hat{s}^{(n)}_y$ Stokes operator for the total forward scattered field generates a measurement current that satisfies \[ dY_t^S = \frac{1}{\sqrt{N}} \sum_{n=1}^N 2\sqrt{\kappa} \hat{j}^{(n)}_z (dt + d\hat{S}^{(n)}_t + d\hat{S}^{(n)}_t^\dagger) \] This new system-observations pair Eqs. (6-7) produces the symmetric quantum filtering equation \[ d\pi_t^S [\hat{X}] = \kappa \pi_t \hat{L}^S [\hat{j}_z] \hat{X} dt \] \[ + \sqrt{\frac{\kappa}{N}} \left( \pi_t [\hat{j}_z \hat{X} + \hat{X} \hat{j}_z] - 2\pi_t [\hat{j}_z] \pi_t [\hat{X}] \right) dW_t^S, \] with the innovations process $dW_t^S = (dY_t^S - 2\sqrt{\kappa/N} \pi_t [\hat{j}_z] dt)$ and the symmetric Lindbladian \[ \hat{L}^S [\hat{j}_z] \hat{X} = \sum_{n=1}^N \hat{j}_z^{(n)} \hat{X} \hat{j}_z^{(n)} - \frac{1}{2} \left( \hat{j}_z^{(n)} \hat{j}_z^{(n)} \right)^2, \] which cannot be expressed using collective operators \[14\]. Squeezing and Anti-Squeezing— A comparison between the measurement models can be accomplished by analyzing the conditional dynamics of the collective spin operators $\hat{J}_z$. We begin with the observed component $\hat{J}_z$. From Eqs. (4) and (8), the conditional $\hat{J}_z$ expectation values evolve according to the filtering equations \[ d\pi_t^C [\hat{J}_z] = 2\sqrt{\kappa} \Delta^C \hat{j}_z \left( dY_t^C - 2\sqrt{\kappa} \pi_t [\hat{J}_z] dt \right) \] \[ d\pi_t^S [\hat{J}_z] = 2\sqrt{\frac{\kappa}{N}} \Delta^S \hat{j}_z \left( dY_t^S - 2\sqrt{\kappa/N} \pi_t [\hat{j}_z] dt \right). \] The two measurement equations are structurally similar; both are fundamentally noise-driven because $\hat{J}_z$ commutes with the Lindbladians in both Eqs. (4) and (8). However, the effective measurement strength in the multi-mode model is smaller by a factor of $1/\sqrt{N}$, which causes its variance $\Delta^2 [\hat{J}_z] = \pi_t [\hat{j}_z^2] - (\pi_t [\hat{J}_z])^2$ to decrease more slowly than for the single-mode measurement. Since the evolution of the conditional expectation values depends on the variances, the multi-mode measurement not only converges more slowly, but the conditional expectations $\pi_t^C [\hat{J}_z]$ and $\pi_t^S [\hat{J}_z]$ do not have the same value for the two filters. More significantly, structural differences between the models are manifest in the dynamics of operators such as $\hat{j}_z^2$ because the actions of the two Lindblad terms, \[ \hat{L}^C [\hat{j}_z] \hat{j}_z^2 = \hat{j}_z^2 - \hat{j}_z^2, \] \[ \hat{L}^S [\hat{j}_z] \hat{j}_z^2 = \frac{N}{4} - \hat{j}_z^2, \] are different and non-zero. For the single-mode measurement, this term is non-zero in expectation when taken with respect to an $x$-polarized spin coherent state; it is the term responsible for anti-squeezing in the collective spin component $\hat{J}_z$. Quite remarkably, however, the multi-mode Lindblad term vanishes in expectation for an $x$-polarized state, suggesting that the multi-mode measurement generates no collective $\hat{J}_z$ anti-squeezing! Figure 2 illustrates many of these differences for an initial $x$-polarized spin coherent state subjected to the two forms of continuous measurement. As shown by Fig. (a) and the inset plot, the single-mode collective
FIG. 2: Comparison of the single-mode (a) and multi-mode (b) measurement models, simulated for an ensemble of $N = 60$ spin-1/2 particles with $\kappa = 6$ beginning from an initial $x$-polarized coherent state. The single-mode measurement exhibits the expected spin-squeezing and anti-squeezing; however the multi-mode measurement does not. While there is uncertainty reduction in the collective spin $\hat{J}_z$ under the multi-mode model, the squeezing parameter (inset plot) does not decrease, nor is there any anti-squeezing; the standard deviation $\Delta_{\hat{J}_z}$ remains constant. For clarity, $\pi_{\hat{J}_z}$ and $\pi_{\hat{J}_y}$ are not plotted.

Conditional Quantum State Dynamics— The only structural difference in the two filters, Eqs. (4) and (8), resides in the form of their Lindbladians. Dynamics that can be expressed entirely in terms of collective spin operators, such as the single-mode Lindbladian Eq. (5), preserve states that are invariant under the permutation of particle labels. Provided that the initial state is permutation invariant, the dynamics are then confined to an $(N + 1)$-dimensional sub-Hilbert space corresponding to the maximum $\hat{J}_z$ eigenvalue $J_{\text{max}} = N/2$, the so-called symmetric group [17]. Since the multi-mode Lindbladian Eq. (6) is not generated by collective spin operators, it can be shown to couple the different total-$J$ irreducible representations (irreps) of the rotation group [14]. The dynamics are not restricted to the symmetric group, but rather the $O(N^2)$-dimensional sub-Hilbert space that is invariant across degenerate copies of the different total-$J$ irreps. Such states are called generalized collective states and are described by density operators of the form [14]

$$\hat{\rho}_C = \bigoplus J \hat{\rho}_J = \sum_{J,M,M'} \rho_{J,M,M'} |J,M\rangle \langle J,M'|,$$

i.e., the direct sum over irreps $\hat{\rho}_J$ corresponding to different total spin $J = 0, 1, 2, \ldots, N/2$. The form of the symmetric Lindbladian and the resulting coupling between total-$J$ irreps accounts for the decoherence due to ignoring the “which-mode” information discussed in Fig. [1]. Evolution under the multi-mode measurement does not preserve pure states (even in the
of focusing a multi-mode probe field onto a single detector does not produce a true collective spin measurement for dilute atomic samples. Previous theoretical treatments based on symmetric states fail to account for the decoherence that results from ignoring “which-particle” information that is available in principle for spatially-resolvable particles. While typical experiments will generally lie somewhere between the limiting cases of Figs. 1(a) and 1(c), one would still expect measurement models based entirely on symmetric states and collective spin operators \[ \rho_I \] to overestimate significantly the expected degree of squeezing and anti-squeezing. Such models are likely inadequate to describe dispersive measurements performed by coupling a large spin ensemble to a probe laser field in free space.

We thank Rob Cook for helpful discussions. This work was supported by the NSF under grants PHY-0639994 and PHY-0652877. Please visit \url{http://qmc.phys.unm.edu/} to download the simulation code used to generate our results as well as all data files used to generate the figures in this paper.

---

**FIG. 4:** Purity comparison of the steady states \( M = \{2, 1, 0\} \) for an ensemble of 4 particles with \( \kappa = 25 \), with respective values \( \{1, 1/4, 1/6\} \) according to \( 1/\alpha^M_N \cdot N = 4 \) was chosen for reasons of clarity.

---

of focusing a multi-mode probe field onto a single detector does not produce a true collective spin measurement for dilute atomic samples. Previous theoretical treatments based on symmetric states fail to account for the decoherence that results from ignoring “which-particle” information that is available in principle for spatially-resolvable particles. While typical experiments will generally lie somewhere between the limiting cases of Figs. 1(a) and 1(c), one would still expect measurement models based entirely on symmetric states and collective spin operators \[ \rho_I \] to overestimate significantly the expected degree of squeezing and anti-squeezing. Such models are likely inadequate to describe dispersive measurements performed by coupling a large spin ensemble to a probe laser field in free space.

We thank Rob Cook for helpful discussions. This work was supported by the NSF under grants PHY-0639994 and PHY-0652877. Please visit \url{http://qmc.phys.unm.edu/} to download the simulation code used to generate our results as well as all data files used to generate the figures in this paper.

---

**FIG. 4:** Purity comparison of the steady states \( M = \{2, 1, 0\} \) for an ensemble of 4 particles with \( \kappa = 25 \), with respective values \( \{1, 1/4, 1/6\} \) according to \( 1/\alpha^M_N \cdot N = 4 \) was chosen for reasons of clarity.

---

of focusing a multi-mode probe field onto a single detector does not produce a true collective spin measurement for dilute atomic samples. Previous theoretical treatments based on symmetric states fail to account for the decoherence that results from ignoring “which-particle” information that is available in principle for spatially-resolvable particles. While typical experiments will generally lie somewhere between the limiting cases of Figs. 1(a) and 1(c), one would still expect measurement models based entirely on symmetric states and collective spin operators \[ \rho_I \] to overestimate significantly the expected degree of squeezing and anti-squeezing. Such models are likely inadequate to describe dispersive measurements performed by coupling a large spin ensemble to a probe laser field in free space.

We thank Rob Cook for helpful discussions. This work was supported by the NSF under grants PHY-0639994 and PHY-0652877. Please visit \url{http://qmc.phys.unm.edu/} to download the simulation code used to generate our results as well as all data files used to generate the figures in this paper.

---

of focusing a multi-mode probe field onto a single detector does not produce a true collective spin measurement for dilute atomic samples. Previous theoretical treatments based on symmetric states fail to account for the decoherence that results from ignoring “which-particle” information that is available in principle for spatially-resolvable particles. While typical experiments will generally lie somewhere between the limiting cases of Figs. 1(a) and 1(c), one would still expect measurement models based entirely on symmetric states and collective spin operators \[ \rho_I \] to overestimate significantly the expected degree of squeezing and anti-squeezing. Such models are likely inadequate to describe dispersive measurements performed by coupling a large spin ensemble to a probe laser field in free space.

We thank Rob Cook for helpful discussions. This work was supported by the NSF under grants PHY-0639994 and PHY-0652877. Please visit \url{http://qmc.phys.unm.edu/} to download the simulation code used to generate our results as well as all data files used to generate the figures in this paper.

---

of focusing a multi-mode probe field onto a single detector does not produce a true collective spin measurement for dilute atomic samples. Previous theoretical treatments based on symmetric states fail to account for the decoherence that results from ignoring “which-particle” information that is available in principle for spatially-resolvable particles. While typical experiments will generally lie somewhere between the limiting cases of Figs. 1(a) and 1(c), one would still expect measurement models based entirely on symmetric states and collective spin operators \[ \rho_I \] to overestimate significantly the expected degree of squeezing and anti-squeezing. Such models are likely inadequate to describe dispersive measurements performed by coupling a large spin ensemble to a probe laser field in free space.

We thank RobCook for helpful discussions. This work was supported by the NSF under grants PHY-0639994 and PHY-0652877. Please visit \url{http://qmc.phys.unm.edu/} to download the simulation code used to generate our results as well as all data files used to generate the figures in this paper.

---

of focusing a multi-mode probe field onto a single detector does not produce a true collective spin measurement for dilute atomic samples. Previous theoretical treatments based on symmetric states fail to account for the decoherence that results from ignoring “which-particle” information that is available in principle for spatially-resolvable particles. While typical experiments will generally lie somewhere between the limiting cases of Figs. 1(a) and 1(c), one would still expect measurement models based entirely on symmetric states and collective spin operators \[ \rho_I \] to overestimate significantly the expected degree of squeezing and anti-squeezing. Such models are likely inadequate to describe dispersive measurements performed by coupling a large spin ensemble to a probe laser field in free space.

We thank Rob Cook for helpful discussions. This work was supported by the NSF under grants PHY-0639994 and PHY-0652877. Please visit \url{http://qmc.phys.unm.edu/} to download the simulation code used to generate our results as well as all data files used to generate the figures in this paper.

---

of focusing a multi-mode probe field onto a single detector does not produce a true collective spin measurement for dilute atomic samples. Previous theoretical treatments based on symmetric states fail to account for the decoherence that results from ignoring “which-particle” information that is available in principle for spatially-resolvable particles. While typical experiments will generally lie somewhere between the limiting cases of Figs. 1(a) and 1(c), one would still expect measurement models based entirely on symmetric states and collective spin operators \[ \rho_I \] to overestimate significantly the expected degree of squeezing and anti-squeezing. Such models are likely inadequate to describe dispersive measurements performed by coupling a large spin ensemble to a probe laser field in free space.

We thank Rob Cook for helpful discussions. This work was supported by the NSF under grants PHY-0639994 and PHY-0652877. Please visit \url{http://qmc.phys.unm.edu/} to download the simulation code used to generate our results as well as all data files used to generate the figures in this paper.