Size effects in statistical fracture

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Abstract

We review statistical theories and numerical methods employed to consider the sample size dependence of the failure strength distribution of disordered materials. We first overview the analytical predictions of extreme value statistics and fibre bundle models and discuss their limitations. Next, we review energetic and geometric approaches to fracture size effects for specimens with a flaw. Finally, we overview the numerical simulations of lattice models and compare with theoretical models.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Understanding how materials break is a fundamental problem of science and engineering. The difficulties stem from the non-trivial dependence of the fracture strength on the characteristic length scales of the samples. This was already noted by Leonardo da Vinci, who measured the carrying-capacity of metal wires of varying lengths [1]. He observed that the longer the wire, the less weight it could sustain. The reason for this behaviour is rooted in the disorder present in the material. If a sample can be divided into non-interacting subvolumes, then the strength is dominated by the weakest one and its distribution could in principle be computed using extreme value statistics [2, 3]. Longer wires are more likely to possess weak parts and are thus bound to fail at smaller loads on average. The quantitative understanding of this statistical size effect is difficult and the low tails of the strength probability distribution are not easy to sample experimentally. In addition, microcracks interact by long-range stress fields. Hence, the assumption made to use extreme value statistics, that of independent subvolumes, is difficult to justify in practice. These problems are particularly compelling in quasi-brittle materials, such as concrete and many other composites, where sample failure is preceded by significant damage accumulation [4].

The typical materials science question is to study the size effect with a pre-existing flaw, a notch. Failure in quasi-brittle materials is then determined by the competition between deterministic effects, due to the stress enhancement created by notch, and the damage accumulation around the defect due to the stress concentration [5, 6]. The effect of disorder is then treated in an effective medium sense by defining a fracture process zone (FPZ) around the crack tip. Starting from these observations, several theoretical formulations based on linear elastic fracture mechanics (LEFM) have been proposed in the literature and compared with experiments [5–11].

The problem is how to extend the LEFM when disorder is strong and cannot be treated as a small perturbation on the homogeneous solution.

The statistical physics approach to fracture is based on simple lattice models, which allow for a relatively simple description of disorder and elasticity. The models are sometimes amenable to analytical solutions, and are usually simulated numerically. In the simplest approximation, elastic interactions are replaced by a load transfer rule which is applied when elements fail or get damaged. These ‘fibre bundle models’ can be solved exactly in some cases and can thus provide a useful guidance for the simulations of more realistic models in which the elastic medium is represented by a network of springs or beams. In this case, the local displacements can then be found by standard methods for solving coupled linear equations. Disorder is modelled, for instance, by imposing random failure thresholds on the springs or by removing a fraction of the links. The lattice is loaded imposing appropriate boundary conditions and the fracture process can be followed step by step, in a series of quasi-equilibria. The cornerstone in this respect has been for the last 20 years the random fuse model (RFM) [12], a lattice model of the fracture of solid materials.
in which as a further key simplification vectorial elasticity has been substituted with a scalar field.

In this paper, we review the statistical physics approaches used to understand size effects in fracture. We first discuss the traditional weakest-link approaches and the solution of fibre bundle models. Next, we consider size effects arising from energetic considerations when a single dominating crack is present, and when its growth may be of importance. Finally, we discuss the results of lattice models for fracture and compare the results obtained with the theoretical arguments discussed previously.

2. Size effect from extreme value statistics

2.1. The weakest-link model

The key theoretical concepts needed to understand the distribution of fracture strength and the associated size effects in heterogeneous media date back to the pioneering work of Gumbel [2], Pierce [13] and Weibull [3] on the statistics of extremes. The general idea stems from a weakest link argument: the failure strength of an extended object is ruled by its weakest local subvolume. For a disordered system, the larger the sample the easier it is to find a weak region.

The easiest way to illustrate this concept is based on a one-dimensional model. Consider a chain composed by \( N \) elastic links that can sustain at most a stress \( s_i \) without breaking. We assume that the threshold stresses \( s_i \) are independent random variables distributed according to a probability density function \( p(s_i) \) and a corresponding cumulative distribution \( P(s_i < \sigma) = P(\sigma) = \int_0^\sigma p(x) \, dx \). The chain will fail as soon as one of the links, the weakest one, fails. Hence, the probability that a chain fails at a stress larger than \( \sigma \) is equal to the probability that the minimum of \( s_i \) is larger than \( \sigma \), which is just the probability that all the links have thresholds larger than \( \sigma \). This condition can be expressed in mathematical terms for a chain of \( N \) links as

\[
1 - P_N(\sigma) = \prod_{i=1}^N P(s_i > \sigma) = (1 - P(\sigma))^N, \tag{1}
\]

where \( P_N(\sigma) \) is the probability that the failure stress is smaller than \( \sigma \). In the large \( N \) limit, equation (1) can be approximated as

\[
P_N(\sigma) \simeq 1 - \exp[-NP(\sigma)]. \tag{2}
\]

Under some broad assumptions on the tail of \( P(\sigma) \), it can be shown that \( P_N(\sigma) \) converges to a stable asymptotic form (i.e. a distribution with a shape that does not depend on \( N \)). In more technical terms a stable distribution is invariant under equation (1) provided the variable is linearly transformed according to \( \sigma_N = a_N \sigma + b_N \). For large \( N \), the statistics is dominated by the low value tail of the distribution \( P(\sigma) \), if we assume it to scale close to \( \sigma = 0 \) as \( P(\sigma) \simeq (\sigma/\sigma_0)^\mu \), where \( \sigma_0 \) is a characteristic stress. Under this assumption, we obtain

\[
P_N(\sigma) \simeq 1 - \exp[-N(\sigma/\sigma_0)^\mu], \tag{3}
\]

which is invariant under equation (1) if we transform the variable \( \sigma \) with \( a_N = N^{-1/\mu} \) and \( b_N = 0 \).

Equation (3) is the celebrated Weibull distribution [3], predicting that the average strength of a chain composed by \( N \) links decreases as

\[
\langle \sigma_N \rangle \propto N^{-1/\mu}. \tag{4}
\]

The Weibull distribution represents still today the main tool used to analyse failure statistics in various materials, although the validity of its underlying assumptions is in general difficult to demonstrate. Real samples cannot generally be schematized as a chain of independent elements with random failure thresholds. In many cases, such as in quasibrittle materials, the sample does not even fracture at once but sustains a considerable amount of damage before failure. Furthermore, long-range elastic interactions could correlate different regions of the sample invalidating the assumptions used to derive the Weibull statistics. That the interactions and microcrack growth would be irrelevant at sufficiently large scale, leading to the recovery of the Weibull distribution, has never been proven rigorously. Indeed, fracture strength data obtained through numerical simulation of discrete lattice models such as the RFM, random spring model and random beam models do not conform to the Weibull distribution (see figure 2(a)) for all the system sizes; instead, Weibull appears to be an approximation over a limited range of system sizes and probability level.

2.2. The largest crack model

It would be desirable to relate the failure statistics to some geometrical characteristics of the microstructure of a material, going beyond a simple description of a sample as a collection of regions with different random strengths. An attempt in this direction was proposed by Freudenthal [14] and then rederived later by various authors for different models [15–17]. The idea is to schematize the microstructure as a set of independent linear cracks of length \( a_i \), distributed according to given probability density function \( p(a_i) \). If the cracks are sufficiently diluted, we can treat them as completely isolated and assess their stability by the simple energetic argument due to Griffith. The idea is establish the conditions under which crack growth becomes energetically favourable. Considering for simplicity a two-dimensional geometry, the elastic energy released by a crack of length \( a \) is given by

\[
\mathcal{E}_{el} = -\frac{\pi \sigma^2 a^2}{2E}. \tag{5}
\]

where \( \sigma \) is the applied stress and \( E \) is the Young modulus. Forming a crack involves the creation of a crack surface with an energy cost, which in the ideal case would be related to the rupture of atomic bonds,

\[
\mathcal{E}_{surf} = 2aG_f, \tag{6}
\]

where \( G_f \) is the fracture toughness. The crack becomes unstable when the total energy decreases

\[
\frac{d\mathcal{E}}{da} = -\pi \sigma^2 a/E + 2G_f < 0, \tag{7}
\]
which implies that a crack of length \( a \) becomes unstable when the stress is equal to

\[
\sigma_c = \sqrt{\frac{2EGf}{\pi a}}. \tag{8}
\]

If we have a collection of \( N \) independent random cracks in a volume \( V \), we can assume that the sample will fail when the least stable one will become unstable. The problem thus reduces to finding the distribution of the largest crack. As discussed in section 2.1, the cumulative distribution is given by

\[
P_N(a) = 1 - \exp[-\rho V P(a)], \tag{9}
\]

where \( \rho \equiv N/V \) is the crack density and \( P(a) \) is the cumulative distribution associated with the crack density distribution \( p(a) \). The asymptotic behaviour of equation (9) is ruled by the large-value tail of \( P(a) \). It can be shown that if the tail decays faster than an exponential, equation (9) converges to the Gumbel distribution [2] for large \( N \)

\[
P_N(a) = 1 - \exp[-\rho V \exp(-a/\sigma_c)], \tag{10}
\]

where \( \sigma_c \) is the characteristic scale of the crack length distribution. Combining equations (10) and (8), one can derive the strength distribution as

\[
P_N(\sigma) = 1 - \exp\left[-\rho V \exp\left(-\frac{\sigma_0^2}{\sigma}\right)^2\right], \tag{11}
\]

where \( \sigma_0^2 \equiv 2EGf/\pi a_c \). Equation (11) predicts a logarithmic size effect for the average strength:

\[
(\sigma_N) \simeq \frac{A}{B + C \log(V)}. \tag{12}
\]

2.3. Interacting cracks: fibre bundle models

The main shortcoming of extreme value statistics is the fact that local failure stresses are considered independent. This is difficult to justify in practice since cracks induce long-range interactions that may correlate stresses in different regions of the sample. A simple way to analyse the role of interaction in the fracture of disordered media is represented by the study of fibre bundle models [13, 18]. These models consist of a set of brittle fibres, with random failure thresholds, loaded in parallel. When a fibre breaks its load is redistributed to the other fibres according to a prescribed rule. The simplest possibility is the case of an equal load sharing (ELS), in which each intact fibre carries the same fraction of the load. This case represents a sort of mean-field approximation and allows for a complete analytic treatment [18–24]. At the other extreme lies the local load sharing model (LLS) where the load of a failed fibre is redistributed to the intact neighbouring fibres [25–31]. In between these two limits, one can consider global load sharing (GLS) where the load is redistributed in a region of finite size [32–34].

We consider first the case of ELS fibre bundles, in which \( N \) fibres of unitary Young modulus \( E = 1 \) are subject to an uniaxial load \( F \). Each fibre \( i \) obeys linear elasticity up to a critical load \( x_i \), which is randomly distributed according to a distribution \( p(x) \). When the load on a fibre exceeds \( x_i \), the fibre is removed. Due to the ELS rule, when \( n \) fibres are present each of them carries a load \( F_i = F/n \) and consequently a strain \( \epsilon = F/n \). The constitutive law for ELS fibre bundles can easily be obtained from a self-consistent argument. At a given load \( F \), the number of intact fibres is given by

\[
n = N \left(1 - \int_0^{F/n} p(x) \, dx\right). \tag{13}
\]

Rewriting equation (13) as a function of the strain, we obtain the constitutive law

\[
\frac{F}{N} = \epsilon \left(1 - P(\epsilon)\right), \tag{14}
\]

where \( P(\epsilon) \) is the cumulative distribution obtained from \( p(x) \). For simplicity we rewrite equation (14) as \( f = \epsilon (1 - P(\epsilon)) \), where \( f \equiv F/N \) is the load per fibre. Failure corresponds to the maximum \( \epsilon_c \) of the right-hand side, after that there is no solution for \( \epsilon(f) \). Expanding close to the maximum we obtain \( f \simeq f_c + B(\epsilon - \epsilon_c)^2 \), which implies that the average rate of bond failures increases very rapidly before fracture:

\[
\frac{df}{df} \sim (f_c - f)^{-1/2}. \tag{15}
\]

In contrast to the case of extreme type models, the strength of a fibre bundle with ELS does not vanish in the large \( N \) limit. In particular, for any threshold distribution such that \( 1 - p(x) \) goes to zero faster than \( 1/x \) for \( x \to \infty \), the strength distribution is Gaussian with average \( f_c = \epsilon_c (1 - P(\epsilon_c)) \) and standard deviation \( \sigma_c = \epsilon_c \sqrt{P(\epsilon_c)(1 - P(\epsilon_c))}/\sqrt{N} [18] \).

A typical LLS model considers a one-dimensional series of fibres loaded in parallel with random breaking thresholds from a distribution \( p(x) \). When the load on a fibre exceeds the threshold its load is redistributed to the neighbouring intact fibres. Thus the load on a fibre is given by \( f_i = f(1 + k/2) \), where \( k \) is the number of failed fibres that are nearest neighbours of the fibre \( i \) and \( f = F/N \) is the external load [25]. Even for this apparently simple one-dimensional model a closed form solution is not available, but several results are known from numerical simulations, exact enumeration and approximate analytical methods [23, 25–30, 32–34]. In contrast to the ELS model, the LLS fibre bundles normally exhibit non-trivial size effects as could be anticipated from general consideration of extreme value statistics. In particular, in the limit of large \( N \), it has been shown that the strength distribution follows the form

\[
W(f) = 1 - (1 - C(f))^N \simeq 1 - \exp(-NC(f)), \tag{16}
\]

where \( C(f) \) is a characteristic function, close to the Weibull form, but difficult to determine exactly [25]. The existence of a limit distribution has been recently proved under very generic conditions for the disorder distribution in [31], but a numerical estimate of its form typically requires extremely large samples. From equation (16) follows that the average bundle strength decreases with \( N \) as

\[
f_c \sim 1/\log(N), \tag{17}
\]

so that an infinitely large bundle would have zero strength.
3. Size effects from energy and geometry

In the previous section, we discussed the statistical size effect, which is caused by the randomness in the material strength. In this section, we discuss two alternate relevant ideas, namely, the energetic size effect and the geometric size effect on the failure strength of quasi-brittle materials. The energetic size effect is based on the Griffith’s criterion of energy balance for stable crack propagation. Intuitively, the energetic size effect arises due to stress redistribution and the associated stored energy release as a large FPZ develops ahead of the crack tips. The size of the FPZ introduces an additional length scale into the problem and its influence on the fracture strength is the energetic size effect in quasi-brittle materials. This idea does not derive from statistical fracture, but turns out to be quite useful in understanding the results of the next section.

On the other hand, the geometric size effect is a consequence of fractal geometry of crack surfaces. The morphology of crack surfaces in disordered media can then be described by a self-affine transformation, which implies a length-scale dependence of the surface energy of a crack. This effect is supposed to introduce a geometric size effect on the fracture strength.

3.1. Energetic size effect

Classical continuum theories of elasticity, plasticity and damage mechanics do not possess a characteristic material length scale \( \ell \). Consequently, the nominal strength of geometrically similar structures is independent structure size, i.e. there is no size effect. Alternatively, the presence of a characteristic length scale introduces a size effect on the nominal strength of the material.

Recall the basic idea of Griffith’s criterion, that during quasi-static crack propagation the available energy must be equal to the energy required to create new crack surfaces. This is precisely the expression derived in equation (8) for brittle materials under plane stress criterion. For quasi-brittle materials, i.e. when the FPZ is comparable to or larger than the structural size, Griffith’s criterion can be generalized using an effective crack size concept that accounts for the influence of the FPZ size on nominal strength.

Following the classical continuum fracture mechanics treatment, a generic expression for the stress intensity factor around a crack of size \( a \) may be expressed as

\[
K_I = \sigma \sqrt{\pi a} \kappa(\alpha),
\]

where \( \kappa(\alpha) \) denotes a dimensionless function, \( \alpha = a/L \) denotes the relative crack size with \( a = a_0 + \Delta a \) such that \( a_0 \) is the initial crack size and \( \Delta a \) is the incremental crack size and \( L \) is a characteristic system size. Equation (18) represents an extension of the classical stress intensity factor around a central notch of size \( 2a \) in an infinite panel (\( K_I = \sigma \sqrt{\pi a} \) with \( \kappa(\alpha) = 1 \)) to finite system sizes and various boundary conditions. The influence of characteristic system size \( L \) on stress intensity factor can be seen clearly by rewriting equation (18) as

\[
K_I = \sigma \sqrt{L} \phi(\alpha),
\]

where \( \phi(\alpha) = \sqrt{\pi a} \kappa(\alpha) \). Using Irwin’s relation, the generic expression for elastic energy release rate \( G \) can be written as

\[
G = \frac{K_I^2}{E} = \frac{\sigma^2 L}{E} g(\alpha),
\]

where \( g(\alpha) = \phi^2(\alpha) \). For quasi-static crack growth, Griffith’s criterion states that

\[
G = \mathcal{R},
\]

where \( \mathcal{R} \) represents the crack resistance curve (R-curve: \( \mathcal{R}(\Delta a) \)), which is expressed more generically as

\[
\mathcal{R} = G_I \psi(\alpha).
\]
which implies

$$\frac{\partial G}{\partial \eta} = \frac{\partial R}{\partial \eta}.$$  \hspace{1cm} (27)

Dividing the above equation by $G = R$ and simplifying the result, we have

$$\frac{1}{g(\alpha_0, \eta, \vartheta)} \frac{\partial g(\alpha_0, \eta, \vartheta)}{\partial \eta} = \frac{1}{\psi(\alpha_0, \eta, \vartheta)} \frac{\partial \psi(\alpha_0, \eta, \vartheta)}{\partial \eta}.$$  \hspace{1cm} (28)

From equation (28), one can derive an implicit expression for the critical crack as

$$\eta = \eta_c(\alpha_0, \vartheta).$$  \hspace{1cm} (29)

Substituting equation (29) into Griffith’s fracture criterion $G = R$ and simplifying the result, we get

$$\sigma_c = \frac{\sqrt{E G_f}}{\sqrt{L h(\alpha_0, \vartheta)}}.$$  \hspace{1cm} (30)

where

$$h(\alpha_0, \vartheta) = \frac{g(\alpha_0, \eta_c(\alpha_0, \vartheta), \vartheta)}{\psi(\alpha_0, \eta_c(\alpha_0, \vartheta), \vartheta)}.$$  \hspace{1cm} (31)

For very large system sizes $L$, $\vartheta$ approaches zero asymptotically, i.e. $\vartheta \to 0$. Hence, $h(\alpha_0, \vartheta)$ can be approximated as

$$h(\alpha_0, \vartheta) \approx h_0(\alpha_0) + h_1 \vartheta,$$  \hspace{1cm} (32)

where

$$h_0 \equiv h(\alpha_0, 0),$$  \hspace{1cm} (33)

$$h_1 \equiv \frac{\partial h(\alpha_0, \vartheta)}{\partial \vartheta}|_{\vartheta=0}.$$  \hspace{1cm} (34)

Substituting equation (32) into equation (30), we have

$$\sigma_c = \frac{\sqrt{E G_f}}{\sqrt{L h_0 + \vartheta h_1}}.$$  \hspace{1cm} (35)

For quasi-brittle materials with an initial notch and constant R-curve, we have $h_0 = g(\alpha_0) \sim \alpha_0 = a_0/L$. Denoting $\xi = \ell h_1$, equation (35) can be specialized to the initially notched samples as

$$\sigma_c = \frac{K_c}{\sqrt{a_0 + \xi}}.$$  \hspace{1cm} (36)

Note that this argument states that $\xi$ depends on the partial derivative of the ratio of $G$ and $R$ on $\vartheta$, so it is related to the relative variation of how $G$ and $R$ vary with the characteristic length scales of material.

### 3.2. Geometric size effect

In this section, we discuss the implication of roughness of crack surfaces on the LEFM and therewith on the scaling of strength. For many years now, it has been hoped that there exists a simple relation between material toughness and the self-affine exponent of rough cracks, although such a hope appears too optimistic now (see [35] for controversies related to this topic). An exception is the indirect fact that explaining crack roughness using the theory of depinning of elastic manifolds implies that the critical stress intensity factor can be expressed in terms of a few relevant parameters such as crack front elasticity and the strength of the disorder, albeit the precise value depends on the geometry (planar cracks, notches or cracks in two or three dimensions, etc) [36–39]. There is, however, no size effect except for the one that arises from the finite crack length and the resulting correction to the critical stress intensity factor, in analogy with finite size corrections observed in critical phenomena [37, 40]. It is important to note that the roughness and strength may be directly coupled as the crack growth resistance and the roughness exponent (fractal dimension) follow from the same process. In what follows, the empirical observations of strength are explained *a posteriori* using as input a given roughness exponent for the crack.

In particular, roughness of crack surfaces influences both (i) the stress concentration around the crack [41–43] and (ii) the energy required to create crack opening through its dependence on the actual crack area [44]. Experiments on several materials under different loading conditions suggest that the crack surfaces exhibit self-affine scaling, which implies that if the in-plane length scales of a fracture surface are scaled by a factor $\lambda$, then the out of plane length scales (height) of the fracture surface scales by $\lambda^5$, where $\xi$ is the roughness exponent. This also implies that the actual (curvilinear) length of the crack is a non-trivial function of length scale. Consequently, an infinitesimal crack opening will now cost more surface energy than in the usual case, and thus the Griffith’s criterion is modified in a way to account for the increased energy requirement to propagate cracks, and possibly to account for the modified stress fields and hence the elastic energy due to a rough crack.

Following [41–43], the asymptotic near-tip stress fields around a rough crack may be expressed as

$$\sigma_{ij} = K_f r^{-\beta} f(\theta),$$  \hspace{1cm} (37)

where $r$ is the radial distance from the crack tip, $f(\theta)$ denotes the functional dependence of stress concentration around crack tip and $\beta = 1 - 1/2 \xi$ for $0.5 \leq \xi \leq 1$ and $\beta = 0$ otherwise. Equation (37) implies that the stress concentration around a rough crack can be expressed as (see [43]).

$$K_f \sim \sigma a^\beta \chi(\beta),$$  \hspace{1cm} (38)

where $\chi(\beta)$ is a dimensionless function and $a$ refers to the in-plane projected crack length. The influence of characteristic system size $L$ on stress intensity factor is incorporated into equation (38) as

$$K_f = \sigma L^\beta \phi(\alpha),$$  \hspace{1cm} (39)

$$G = \frac{\sigma^2 L^2}{E} g(\alpha, \eta, \vartheta).$$  \hspace{1cm} (40)

From the crack resistance point of view, it is expected that the roughness of a self-affine crack leads to R-curve behaviour. Specifically, for a self-affine crack, the crack surface energy scales as

$$\mathcal{E}_{surf} = 2\gamma a^{1/\xi},$$  \hspace{1cm} (41)
where $\gamma$ denotes the specific energy per unit of fractal measure and $1/\zeta$ denotes the fractal dimension of the crack. Hence, the crack resistance is given by

$$G_t = \frac{2\gamma}{\zeta} a^{1/\zeta - 1}. \quad (42)$$

We also note that a size effect law based on anomalous scaling of crack surfaces has been proposed in [9, 10]. In the case of anomalous scaling of crack surfaces (see equation (43) below), the extra scale also enters the scaling argument, further changing the amount of surface energy needed for infinitesimal crack advancement. This implies a non-trivial ‘R-curve’, or crack resistance, since the energy consumed via forming new crack surface depends not only on the full set of exponents but again on the distance propagated. An expression can be written down using solely geometric arguments and the exponents, such as

$$\Delta h(\ell, x) \simeq A \left\{ \begin{array}{ll}
\ell^{\zeta_0} \xi(x) x^{\zeta_0} & \text{if } \ell \ll \xi(x), \\
\xi(x) & \text{if } \ell \gg \xi(x),
\end{array} \right. \quad (43)$$

where $\Delta h(\ell, x)$ denotes the height fluctuations of fracture surfaces estimated over a window of size $\ell$ along the $x$-axis and at a distance $x$ from the initial notch. Also, $\xi(x) = B x^{1/2}$ is a cross-over length along the $x$-axis below which the fracture surface is self-affine with local roughness exponent $\zeta_0$ and $z$ is referred to as the dynamic exponent. This cross-over length depends on the distance $x$ to the initial notch. For empirical reasons, one needs to note that there should be also a lower cut-off above which the self-affinity can be observed [45], and that the argumentation should change if the crack geometry is very branched [46].

There are two different regimes: for $\ell \gg \xi(x)$, $\Delta h(\ell, x) \sim x^{\zeta_0}$ and for $\ell \ll \xi(x)$, it is characterized by the exponent $(\zeta - \zeta_0)/z$. Since the effective area is (from which surface energy follows)

$$G_t \delta A_p = 2\gamma \delta A_c, \quad (44)$$

where $\delta A_p$ is the real area increment and $\delta A_c$ its projection on the fracture mean plane. More et al make now the simplest first order approximation, that equation (44) can be approximated as $G_t = 2\gamma s(x)/L$, where $s(x)$ is the length of the crack profile that can be estimated by covering the profile path with segments of length $\delta$ whose horizontal projection on the $x$-axis is $\xi_0$. The elastic energy released is not taken to be affected by the self-affinity of the crack. Consequently, we have $s(x) = (L/\xi_0)\delta$, which can be approximated as

$$s(x) \simeq L \left\{ \begin{array}{ll}
1 + \left( \frac{A(B x^{1/2} x^{\zeta_0})}{\xi_0^{\zeta_0}} \right)^{2} & \text{if } x \ll x_{sat}, \\
1 + \left( \frac{AL^{\zeta_0}}{\xi_0^{\zeta_0}} \right)^{2} & \text{if } x \gg x_{sat}. \end{array} \right. \quad (45)$$

Substituting equation (45) into $G_t = 2\gamma s(x)/L$, we get an expression for crack resistance as

$$G_t(\Delta a \ll \Delta a_{sat}) \simeq 2\gamma \sqrt{1 + \left( \frac{AB^{\zeta_0}}{\xi_0^{\zeta_0}} \right)^{2} \Delta a^{2(\zeta_0 - 1)/\zeta}} \Delta a^{2(\zeta_0 - 1)/\zeta}. \quad (46)$$

within the region of crack growth ($\Delta a \ll \Delta a_{sat} = x_{sat}$), and when $\Delta a \gg \Delta a_{sat}$, the crack growth resistance becomes

$$G_t(\Delta a \gg \Delta a_{sat}) \simeq 2\gamma \left( 1 + \left( \frac{A}{I_0^{1-\zeta_0}} \right)^{2} \Delta a^{2(\zeta_0 - 1)/\zeta}. \quad (47)$$

4. Size effects from disordered fracture models

Simulations of statistical fracture provide a tool to analyse the validity of various scenarios for the size dependence of strength. The major tool for this has been the RFM [12, 40], where one approximates continuum LEFM with a spatial discretization (lattice) and a scalar analogy (voltages) for the displacements. The material response is formulated in terms of ‘fuse elements’ which have a conductivity $\Sigma_0$ and a failure threshold $i_c$. Usually the ‘elasticity’ or the fuse conductances is kept constant, while various threshold distributions $p(i_c)$ are employed. Until now almost all the effort has been spent on two-dimensional systems for the sake of numerical convenience. The discussion of established results can be divided into two topics, studies of the weakest-link effect and analyses of the size effect for notched specimens.

4.1. Statistical size effect

The major results on the strength of the RFM were obtained in the late 1980s by Duxbury and co-workers [17, 47]. The starting point is to consider weak disorder employing a failure threshold distribution composed by two delta-functions at $i_c = 0$ and $i_c = 1$ which naturally translates into ‘dilution disorder’ or the removal of a fraction $1 - p$ of the fuses at the beginning. In the diluted limit ($p \approx 1$), it is possible to formulate a non-rigorous theory of the size effect. First, the flaw size statistics indicates an exponentially decaying length distribution for the microcracks that are induced into the samples by the dilution disorder. Second, in practice the critical current $I_c$ essentially corresponds to the current $I_1$ at which the weakest fuse fails. This means that the case under study is very brittle, and the crack growth resistance is small. Given these two assumptions, Duxbury et al write down a modified extremal statistics theory of the type of equation (10), and compare it with the RFM simulations in two dimensions with small dilution. There are straightforward generalizations of the weak-disorder dilution theory towards $p \sim p_c$, the bond percolation of the RFM geometry (for square lattices, $p_c = 1/2$), since at the percolation point the outcome can be simply understood via the ‘red bond’ or singly connected bond arguments of percolation theory [47]. This limit is not relevant for most materials, but in general the defect size statistics cannot be assumed exponential, and often a noticeable damage accumulation takes place before failure.

To this end, we consider a variant of the RFM where $p(i_c)$ has a finite support which extends down to zero. In particular, we consider disorder with a cumulative distribution $P(i_c) = \Delta i_c^2$, $i_c \in [0, 1]$. The case $\Delta = 1$ corresponds to the uniform distribution, while different values of $\Delta$ allow to tune the strength of disorder. Since a fuse can fail even at low currents, with these kind of distributions there is a considerable damage
accumulation before failure. We illustrate this in figure 1 which shows the relation between the accumulated damage \(d\), defined as the cumulative fraction of broken bonds, and the failure stress \(\sigma_c\) on a sample-to-sample basis for three different disorder strengths \(\Delta\) and several lattice sizes \(L\).

Damage accumulation is responsible for a fundamental complication in the evaluation of size effects which should be relevant for real materials. To relate this to the discussion in section 2, the cumulative \(P(\sigma)\) in equation (9) becomes dependent on \(\sigma\). Figure 2 shows numerical results for the cumulative strength distribution \(P(\sigma_c,L)\) for the RFM. As demonstrated in the figure, the data are not described by a Weibull distribution, but neither the modified Gumbel distribution mentioned above captures the observed behaviour. It is interesting to note, nevertheless, that for larger \(L\) the double exponential plot seems to reveal the presence of an asymptotic scaling form.

A critical defect size, which when reached anywhere in the sample leads to catastrophic failure. From these models one could conclude that damage accumulation changes the asymptotic distribution even in the diluted limit and that the generalized Gumbel distribution does not apply [51].

**4.2. Strength of materials with flaws**

Another fundamental question is whether equation (8) from the Griffiths’ argument remains valid for realistic materials. A suitable generalization as discussed in the previous section is \(\sigma_c = K_c/\sqrt{\xi + a_0}\), where we have added a scale \(\xi\) and written \(K_c \sim \sqrt{E}\) for the fracture toughness (numerical factors relating, for example, plane stress or plane strain loading scenarios have been omitted). \(\xi\) plays here the role of incorporating a disorder effect: that cracks are masked when they become very small by fluctuations, and thus the strength saturates to \(K_c/\sqrt{\xi}\). As noted by Bazant (see, e.g. [8]) a natural reason for the length scale is the FPZ which is non-negligible in quasi-brittle media.

Extensive numerical simulations of the RFM and also other models—random spring model, random beam...
model—have allowed to understand the physics of size effect here and to elaborate on a scaling theory \[11, 52\]. Figure 3 shows an example of two-dimensional RFM data plotted using the inverse square of equation (36). For small flaws one sees a cross-over away from the behaviour predicted by the above scaling, and the inset illustrates how this depends on the specimen size, which is otherwise absent in principle from equation (36).

The dependence of the size effect on \(L\) and the disorder strength \(\Delta\) (in full generality, we take \(\Delta = 0\) to correspond to pure systems with no disorder) can be summarized with the equation (36).

\[ K_c^2 / \sigma_c^2 = \xi + a_0 f(\xi(a_0)), \]

where the scaling function \(f(y)\) follows

\[ f(y) \approx \begin{cases} 1 & \text{if } y \ll 1, \\ y & \text{if } y \gg 1. \end{cases} \]

The important point here is that equation (49) takes into account of the weak-link effects discussed above. The length scale \(a_c \approx (K_c(\Delta)/\sigma(L, \Delta))^2 - \xi(\Delta)\) indicates the cross-over between that regime and the LEFM one where equation (36) is valid.

The size effect can thus be summarized with the aid of two parameters, \(\xi\) and \(K_c \equiv \sqrt{GE}\), where \(E\) is the elastic modulus and \(G\) the fracture energy. It has to be emphasized that all the three quantities of \(\xi\), \(E\) and \(G\) depend on the presence of disorder. Figure 4 illustrates the FPZ damage profile, which underlies \(\xi\). It measures the exponential decay of the damage from the crack tip, in a situation where the LEFM stress profile (and its correspondence in the RFM) is not seen in the damage due to screening. The stronger is the disorder, the larger is \(\xi\) as one would expect, indicating also a strength reduction. It would be very interesting to develop an analytical model of this screening effect. Figure 4 demonstrates, however, that there are strong sample-to-sample fluctuations: the exponential damage profile and its role in the average strength behaviour or size effect is to be understood as a statistical average.

It is interesting to note that these statistical models exhibit a non-trivial R-curve if one computes the crack extension using the changing scale of the damage cloud, \(\xi(\sigma)\). The strength prediction of equation (36) is then ‘self-consistent’. For those \(a_0\) for which it is valid, \(\xi\) is a constant to which value \(\xi(\sigma)\) grows as damage accumulates from zero. Likewise, \(K_c\) is to be measured at \(\sigma_c\) as well. There are no clear signs of any kind of geometrical size effects, since though the cracks formed are self-affine, the growth takes place after the stress maximum \[52\]. However, the damage cloud is truly two-dimensional and we thus note that the results differ from those obtained for uniaxial fibre composites \[53, 54\] which exhibit a scaling similar to equation (36) but with a logarithmic correction and which reproduce a R-curve in a similar fashion as the statistical models.

This brings us to the important issue of strength fluctuations in the presence of a dominating notch. In the previous sections we have discussed the important role of extremal statistics on average strength. Figure 5 illustrates the behaviour of \(P(\sigma_c, a_0)\). As \(a_0\) grows the average strength goes down, and the width of the distribution reaches a maximum at an intermediate value of \(a_0\). For small notches, \(a_0 < a_c\) the distribution naturally is close to the finite-size form for \(p(\sigma_c, a_0 = 0, L)\). For large values of \(a_0\) we see a narrowing of the distribution (in fact, \(\Delta\sigma_c/\sigma_c\) goes to zero with \(a_0\) increasing) and the distribution is well approximated in the central part by a Gaussian shape. This is interesting, since the Gaussian distribution is also obtained in GLS fibre bundles \[18\] and in chains-of-fibre-bundle models with LLS and heterogeneous fibre strengths \[50\]. However, even for the largest \(a_0 = 48\) the tails are broader than in the Gaussian case.
are obtained from 1000 to 4000 samples depending on $\Delta l$. For this reason it would be interesting to study these models with statistical fracture and other quasibrittle materials (Boca Raton, FL: CRC Press).

5. Conclusions

In this paper we have reviewed various statistical approaches to fracture size effect. We first discussed the Weibull theory and similar approaches based on the weakest-link concept. According to these extreme value statistics arguments, larger samples should be more likely to possess weak regions and are thus bound to fail at a smaller load. While this general statistical argument is generally believed to be correct, the presence of the stress enhancement around a crack and damage accumulation prior to failure make a quantitative theory for size effects in materials still elusive. In cases when a dominating crack, or a notch, is present one can rely on energetic arguments that link the crack geometry to the failure stress. Further complications arise when the dominating crack is not straight but has a self-affine geometry, as observed in experiments. These general statistical approaches have been tested using statistical lattice models for fracture, where a set of discrete elements with random failure thresholds are subjected to an increasing load. These models allow to study the cross-over between notch dominated size effects and statistically induced ones. Despite this, the role of damage accumulation for fracture size effects in unnotched samples still remains unclear. On the experimental side, large statistical sampling would be needed to reach firm conclusions about the asymptotic behaviour of materials strength. The above observations apply for quasi-brittle fracture first and foremost, and if viscoelastic or viscoplastic deformation is important our understanding is much less developed. These important questions remain open for future investigation, so that statistical fracture promise to pose interesting challenges for the years to come.

Figure 5. The distribution $\rho(\sigma_l, a_0)$ for a RFM with linear size $L = 192$ and intermediate disorder $\Delta = 0.6$. Three different $a_0$ are used: $L = 4, 16, 48$. The last dataset is compared with a Gaussian fit in the central part around $\sigma_l = \langle \sigma_l \rangle$. The numerical distribution are obtained from 1000 to 4000 samples depending on $\Delta$ and $a_0$. The fuse threshold distribution would be relevant \cite{48}, and for this reason it would be interesting to study these models with different thresholds distributions such as a Weibull distribution for $P(\sigma_l)$.

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