ON THE GEOGRAPHY AND BOTANY OF IRREDUCIBLE 4-MANIFOLDS WITH ABELIAN FUNDAMENTAL GROUP

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The geography and botany of smooth/symplectic 4-manifolds with cyclic fundamental group are addressed. For all the possible lattice points which correspond to non-spin manifolds of negative signature and a given homeomorphism type, an irreducible symplectic manifold and an infinite family of pairwise non-diffeomorphic non-symplectic irreducible manifolds are manufactured. In the same fashion, a region of the plane for manifolds with non-negative signature is filled in.

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1. Introduction

Our understanding of simply connected smooth 4-manifolds has witnessed a drastic improvement in recent years. A quick description of the blueprint to the chain of fresh successes of 4-dimensional topologists can be achieved through (great) oversimplification, by attributing them to two factors: an increase on the repertoire of techniques that manufacture small symplectic 4-manifolds and a new perspective on the usage of already existing mechanisms. The idea of using symplectic sums (cf. [21]) of non-simply connected building blocks along genus 2 surfaces to kill fundamental groups in an efficient way was introduced in [1]. Its immediate outcome was the construction of an exotic symplectic $\mathbb{CP}^2 \# 5 \mathbb{CP}^2$ and, later on, the existence of an exotic symplectic $\mathbb{CP}^2 \# 3 \mathbb{CP}^2$ (cf. [3]) was put on display. Shortly after, Luttinger surgery ([36], [7]), was invited to the game in [10] and in [40]. The combinations of these techniques produced another exotic symplectic $\mathbb{CP}^2 \# 3 \mathbb{CP}^2$ in [10]. Several of these constructions trace their origins to [17], where symplectic sums of products of 2-manifolds and surgery along nullhomologous tori were employed to construct symplectic and non-symplectic exotic 4-manifolds.

Concerning the (lack of) uniqueness of smooth structures on irreducible 4-manifolds, the article [19] introduces a technique to produce infinite families of distinct smooth structures on many smooth 4-manifolds. The influx of these rather elegant geometric-topological manufacturing mechanisms went successfully well and several of the small simply connected 4-manifolds $\mathbb{CP}^2 \# k \mathbb{CP}^2$ ($k \leq 9$), which were the most challenging 4-manifolds in terms of exhibiting the existence of one exotic smooth structure, were shown to admit infinitely many exotic smooth structures. We refer the reader to the papers [39], [18], [3], [10], [11], [2], [19] and [4] for a concise presentation of these ideas and for the current state of affairs in the subject.

Another major success in the 4-dimensional story was the use of these brand new manufactured exotic manifolds to produce a myriad of irreducible 4-manifolds and, thus, fill out a huge part of the symplectic geography plane and its botany counterpart (cf. [22], [18]). The combination of these results with previous efforts ([6], [4], [38]) provides us with a fairly comprehensive understanding of the geography/botany problem for simply connected symplectic 4-manifolds of negative signature. Although the non-negative signature region in the geography/botany plane is still a challenge ([39], [41], [37], [42], [8]), these new techniques have also been useful in the study of such manifolds ([8], [5]).

In this paper the focus is switched into the non-simply connected realm. The utility of these new techniques is extended to the study of smooth 4-manifolds with the following choices of fundamental group:

- $\pi_1 = \mathbb{Z}$,
- $\pi_1 = \mathbb{Z}_p$ or
- $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_q$ if $\gcd(p,q) = 1$. 
The geography and botany of smooth/symplectic irreducible 4-manifolds with cyclic fundamental group are addressed. For all the possible lattice points which correspond to non-spin manifolds of negative signature and a given homeomorphism type, an irreducible symplectic manifold and an infinite family of pairwise non-diffeomorphic non-symplectic irreducible manifolds are manufactured. Such goal involves building the smallest known 4-manifolds with cyclic fundamental group which are known to admit an exotic smooth structure and use them to fill in regions of the plane. In the same fashion, we construct the smallest manifolds with non-negative signature known to possess more than one smooth structure and employ them to fill in the corresponding region in the planes. The corresponding coordinates are given within the results for the convenience of the reader. The tools in [19] help us conclude that the manufactured manifolds have infinitely many smooth structures.

The first examples of exotic 4-manifolds with cyclic fundamental group were constructed in [30], [31], [23], [29] [24] and [46]. Unlike the complex surfaces built in these articles, most of the manifolds constructed in this paper are not even symplectic. Efforts towards more general fundamental groups can be found in [9], [11] and in [12].

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2. Statements of Results

The results obtained in this paper are in two directions. First, several symplectic 4-manifolds with cyclic fundamental group and small Euler characteristic are constructed. The second is to fill out regions of the Geography/Botany plane of each fundamental group.

2.1. Notation. The following notation will be used to denote the manufactured symplectic manifolds:

\[ X_{\pi_1}^{\pi_1, b_2^{-}}. \]

The corresponding topological prototypes for which exotic smooth structures are constructed will be

- \( b_2^+ \mathbb{C}P^2 \# b_2^- \mathbb{C}P^2 \# L(p, 1) \times S^1 \) and
\[ b_2^+ \mathbb{CP}^2 \# b_2^- \mathbb{CP}^2 \# S^1 \times S^3. \]

For example, \( X_{1,3}^{2_2} \) denotes the symplectic manifold with finite cyclic fundamental group \( \mathbb{Z}_p \) and \( X_{2,4}^{2_2} \) stands for the one with infinite cyclic fundamental group, both have Euler characteristic \( e = 6 \) and signature \( \sigma = -2 \). For the topological prototypes for finite cyclic fundamental groups, we have the following. The piece \( L(p, 1) \times S^1 \) stands for the surgered product \( L(p, 1) \times S^1 \) of a Lens space with the circle; the surgery is performed along \( \{ pt \} \times S^1 \) to kill the loop corresponding to the generator of the infinite cyclic group factor so that all \( \pi_1 \) comes from the fundamental group of the lens space.

We point out that this notation gives away all the information needed to establish a homeomorphism. In the infinite cyclic fundamental group case, we recall that \( b_2(X) = e(X) \); in particular notice that \( e(S^1 \times S^3) = 0 \) since the Euler characteristic is multiplicative. In the finite cyclic case, \( e(L(p, 1) \times S^1) = 2 \). Thus, these manifolds share the same Euler characteristic \( e = 6 \) and signature \( \sigma = -2 \). Thus, \( X_{1,3}^{2_2} \) is an exotic \( \mathbb{CP}^2 \# 3 \mathbb{CP}^2 \# \tilde{L}(p, 1) \times S^1 \).

The following definition introduced by Akhmedov and Park in [5] will be used for practical reasons.

**Definition 1.** A smooth 4-manifold \( X \) has the infinite \( \infty \)-property if and only if there exists an irreducible symplectic 4-manifold and infinitely many pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds, all of them homeomorphic to \( X \).

### 2.2. Small in terms of the Euler Characteristic.

For the small manifolds manufactured we have

**Theorem 2.** Let \( k = 2, 3, 4, 5, 6, 7, 8 \). There exists an irreducible symplectic 4-manifold with \( e = k + 1 \), \( \sigma = 1 - k \) and cyclic fundamental group

\[ X_{2,k+1}^{2} \text{ and } X_{1,k}^{2}. \]

Furthermore, the manifolds \( 3 \mathbb{CP}^2 \# k \mathbb{CP}^2 \# L(p, 1) \times S^1 \) have the infinite \( \infty \)-property.

For a little bit bigger manifolds we have

**Theorem 3.** Let \( k \in \{ 5, 6, 7, 8, 9, \ldots, 12, 14, 16, 18 \} \). The manifolds

\[ 3 \mathbb{CP}^2 \# k \mathbb{CP}^2 \# L(p, 1) \times S^1, \]
\[ 4 \mathbb{CP}^2 \# (k + 1) \mathbb{CP}^2 \# S^1 \times S^3 \]

have the infinite \( \infty \)-property.

### 2.3. Regions in the Plane.

Expressed in terms of the Geography/Botany problems, we manufacture irreducible symplectic 4-manifolds with cyclic fundamental group and infinitely many pairwise non-diffeomorphic non-symplectic 4-manifolds
with cyclic fundamental groups that realize the coordinates

\[(e, \sigma) \text{ if } 2e + 3\sigma \geq 0, \ e + \sigma \equiv 0 \pmod{4} \text{ and } \sigma \leq -1\]
or equivalently

\[(c_1^2, \chi_h) \text{ if } 0 \leq c_1^2 \leq 8\chi - 1.\]

The following result is an extension of the combined efforts of \cite{6} and \cite{4}.

**Theorem 4.** For any pair \((c, \chi)\) of non-negative integers satisfying \(0 \leq c \leq 8\chi - 1\)

there exists an irreducible symplectic 4-manifold with finite cyclic fundamental group \(Y_{\Z_p} = X_{2\chi - 1, 10\chi - c - 1}\) and an irreducible symplectic 4-manifold with infinite cyclic fundamental group \(Y_{\Z} = X_{2\chi, 10\chi - c}\) both with odd intersection form and

\[c = c_1^2(Y) \text{ and } \chi = \chi_h(Y)\]

Hence \(Y_{\Z_p}\) and \(Y_{\Z}\) are homeomorphic (accordingly) but not diffeomorphic to

\[(2\chi - 1)\CP^2 \# (10\chi - c - 1)\CP^2 \# L(p, 1) \times S^1\]

and

\[2\chi\CP^2 \# (10\chi - c)\CP^2 \# S^1 \times S^3.\]

Besides the main result, one is able to fill in other regions. A sample of such results is given below.

From the constructions in \cite{4} a bigger region can be filled. Which overlaps with one included in the result above:

**Theorem 5.** There exists an irreducible symplectic 4-manifold and an infinite family of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds, all of which have cyclic fundamental group \(\pi_1\) which are homeomorphic to

\[2n\CP^2 \# (2n + 1)\CP^2 \# S^1 \times S^3 \text{ if } n \geq 2 \text{ and } \pi_1 = \Z \text{ or}\]

\[(2n - 1)\CP^2 \# 2n\CP^2 \# L(p, 1) \times S^1 \text{ if } n \geq 1 \text{ and } \pi_1 = \Z_p.\]

The following theorem provides a method to manufacture a myriad of examples from an initial one. We observe that one can direct results in \cite{6} and \cite{4} into the study of the geography problem for 4-manifolds with cyclic fundamental group.

**Theorem 6.** Let \(X\) be a symplectic 4-manifold that contains a symplectic torus \(T\) with self-intersection 0. Assume the homomorphism \(\pi_1(T) \to \pi_1(X)\) induced by inclusion is surjective. Then for any pair \((c, \chi)\) of non-negative integers satisfying

\[0 \leq c \leq 8\chi - 1\]

there exist a symplectic 4-manifold \(Y\) with cyclic fundamental group with

\[\chi_h(Y) = \chi_h(X) + \chi \text{ and}\]

\[c_1^2(Y) = c_1^2(X) + c.\]
Moreover, if $X$ is minimal then $Y$ minimal as well and it has an odd indefinite intersection form.

The following results fill in several more regions in the plane.

**Proposition 7.** For each $g \geq 0$ there exist an irreducible symplectic 4-manifold $X_{1+2g,3+2g}$ and an infinite family of pairwise non-diffeomorphic non-symplectic 4-manifolds $\{X_n\}$, all of them homeomorphic to

$$(1 + 2g)\mathbb{CP}^2 \# (3 + 2g)\mathbb{CP}^2 \# L(p,1) \times S^1.$$

Analogously, for each $g \geq 1$ there exist an irreducible symplectic 4-manifold $X_{2+2g,4+2g}$ and an infinite family of pairwise non-diffeomorphic non-symplectic 4-manifolds $\{X_n\}$, all of them homeomorphic to

$$(2 + 2g)\mathbb{CP}^2 \# (4 + 2g)\mathbb{CP}^2 \# S^1 \times S^1.$$

The following theorems extend some results in [37] and [19].

**Theorem 8.** For each integer $k$, $10 \leq k \leq 18$, there exists an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible 4-manifolds with the following characteristics.

- Only one member is symplectic,

- the characteristic numbers for all the members of the family can be chosen from the following three pairs: $\chi_h = 2$ and $c_1^2 = 19 - k$; $\chi_h = 3$ and $c_1^2 = 19 - k$ or $\chi_h = 3$ and $c_1^2 = 27 - k$.

- each member of the family contains a symplectic surface $\Sigma_2$ of genus 2 and self-intersection 0. The fundamental group of the complement of $\Sigma_2$ in each manifold is isomorphic to the fundamental group of the ambient manifold.

**Theorem 9.** For each integer $k$, $10 \leq k \leq 18$, there exists an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible 4-manifolds with the following characteristics.

- Only one member is symplectic,

- the characteristic numbers for all the members of the family can be chosen from the following two pairs: $\chi_h = 4$ and $c_1^2 = 33 - k$ or $\chi_h = 5$ and $c_1^2 = 41 - k$.

- each member of the family contains a symplectic torus $T$ of self-intersection 0. The fundamental group of the complement of $T$ in each manifold is isomorphic to the fundamental group of the ambient manifold.
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Corollary 10. Let $k$ be an integer such that $10 \leq k \leq 18$. The following 4-manifolds have the $\infty$-property:

- $4\mathbb{CP}^2 \# (1 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3$,
- $6\mathbb{CP}^2 \# (3 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3$, $5\mathbb{CP}^2 \# (2 + k)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1$,
- $6\mathbb{CP}^2 \# (5 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3$, $5\mathbb{CP}^2 \# (4 + k)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1$,
- $8\mathbb{CP}^2 \# (7 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3$, $7\mathbb{CP}^2 \# (6 + k)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1$,
- $10\mathbb{CP}^2 \# (9 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3$, $9\mathbb{CP}^2 \# (8 + k)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1$.

Theorem 11. For all integers $k \geq 45$, there exist an irreducible symplectic 4-manifold $X^2_{2k+1,2k+1}$ with $\pi_1 = \mathbb{Z}_p$ and an irreducible symplectic 4-manifold $X^2_{2k+2,2k+2}$ with $\pi_1 = \mathbb{Z}$. Both of them have Euler characteristic $4k + 4$ and signature $\sigma = 0$.

For all integers $q \geq 49$, there exist an irreducible symplectic 4-manifold $X^2_{2q-1,2q}$ with $\pi_1 = \mathbb{Z}_p$ and an irreducible symplectic 4-manifold $X^2_{2q,2q+1}$ with $\pi_1 = \mathbb{Z}$. Both of them have Euler characteristic $4q + 1$ and signature $\sigma = -1$.

Accordingly, the manifolds from this theorem are homeomorphic to

$$
(2k + 2)\mathbb{CP}^2 \# (2k + 2)\overline{\mathbb{CP}^2} \# S^1 \times S^3
$$

$$
(2k + 1)\mathbb{CP}^2 \# (2k + 1)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1
$$

for $k \geq 45$; for $q \geq 49$ they are homeomorphic to

$$
(2q)\mathbb{CP}^2 \# (2q + 1)\overline{\mathbb{CP}^2} \# S^1 \times S^3
$$

$$
(2q - 1)\mathbb{CP}^2 \# (2q)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1
$$

With the last result we start our enterprise into the non-negative signature region of the geography plane. Following an idea of Stipsicz [11] employed in [10] and using the recent efforts in [8], [27] and [5], the following points/regions in the plane non-negative signature are shown to be realized.

Theorem 12. There exists a closed minimal symplectic 4-manifold $X$ with cyclic $\pi_1(X)$ for the following choices of characteristic numbers:

- $e = 94$ and $\sigma = 2$ corresponding to $(c_1^2, \chi_h) = (194, 24)$,
- $e = 98$ and $\sigma = 2$ corresponding to $(c_1^2, \chi_h) = (202, 25)$ or
- $e = 100$ and $\sigma = 0$ corresponding to $(c_1^2, \chi_h) = (200, 25)$.

Theorem 13. There exists a closed minimal symplectic 4-manifold $X$ with cyclic $\pi_1(X)$ for the following choices of characteristic numbers:

- $e = 100$ and $\sigma = 4$ corresponding to $(c_1^2, \chi_h) = (212, 26)$,
- $e = 104$ and $\sigma = 4$ corresponding to $(c_1^2, \chi_h) = (220, 27)$ or
- $e = 106$ and $\sigma = 2$ corresponding to $(c_1^2, \chi_h) = (218, 27)$.
These manifolds are used to fill in the following regions:

- \((e, \sigma) = (2m + 2, 0)\) and \((c_1^2, \chi_h) = (4m + 4, 1/2(m + 1))\),
- \((e, \sigma) = (2m + 1, 1)\) and \((c_1^2, \chi_h) = (4m + 5, 1/2(m + 1))\) and
- \((e, \sigma) = (2m, 0)\) and \((c_1^2, \chi_h) = (4m + 6, 1/2(m + 1))\).

The following result states these regions in terms of the topological prototypes.

**Proposition 14.** Let \(m\) be an odd positive integer. If \(m \geq 49\), then

- \(m\text{CP}^2 \# m\text{CP}^2 \# \tilde{L}(p, 1) \times S^1\),
- \((m + 1)\text{CP}^2 \# (m + 1)\text{CP}^2 \# S^1 \times S^3\).
- \(m\text{CP}^2 \# (m - 1)\text{CP}^2 \# \tilde{L}(p, 1) \times S^1\),
- \((m + 1)\text{CP}^2 \# m\text{CP}^2 \# S^1 \times S^3\),

have the \(\infty\)-property.

If \(m \geq 47\), then

- \(m\text{CP}^2 \# (m - 2)\text{CP}^2 \# \tilde{L}(p, 1) \times S^1\) and
- \((m + 1)\text{CP}^2 \# (m - 1)\text{CP}^2 \# S^1 \times S^3\)

have the \(\infty\)-property.

In a similar manner, the regions

- \((e, \sigma) = (2m - 1, 3)\) and \((c_1^2, \chi_h) = (4m + 7, 1/2(m + 1))\) and
- \((e, \sigma) = (2m - 2, 0)\) and \((c_1^2, \chi_h) = (4m + 8, 1/2(m + 1))\).

are shown to be realized by an irreducible (symplectic) smooth 4-manifold in the following result.

**Proposition 15.** Let \(m\) be an odd positive integer. If \(m \geq 53\), then

- \(m\text{CP}^2 \# (m - 3)\text{CP}^2 \# \tilde{L}(p, 1) \times S^1\) and
- \((m + 1)\text{CP}^2 \# (m - 2)\text{CP}^2 \# S^1 \times S^3\)

have the \(\infty\)-property. If \(m \geq 51\), then

- \(m\text{CP}^2 \# (m - 4)\text{CP}^2 \# \tilde{L}(p, 1) \times S^1\) and
- \((m + 1)\text{CP}^2 \# (m - 3)\text{CP}^2 \# S^1 \times S^3\)

have the \(\infty\)-property.

**Proposition 16.** For each odd integer \(m \geq 1\) and \(10 \leq k \leq 18\), there exists an irreducible symplectic 4-manifold \(Y\) with cyclic fundamental group whose characteristic numbers can be chosen amongst the following options:
(1) $\chi(Y) = 25m^2 + 31m + 5$ and $c_1^2(Y) = 225m^2 + 248m + 35 - k$;

(2) $\chi(Y) = 25m^2 + 31m + 6$ and $c_1^2(Y) = 225m^2 + 248m + 43 - k$;

(3) $\chi(Y) = 25m^2 + 31m + 6$ and $c_1^2(Y) = 225m^2 + 248m + 41 - k$;

(4) $\chi(Y) = 25m^2 + 31m + 7$ and $c_1^2(Y) = 225m^2 + 248m + 49 - k$;

(5) $\chi(Y) = 25m^2 + 31m + 8$ and $c_1^2(Y) = 225m^2 + 248m + 57 - k$.

Moreover, the manifolds with the first three choices of coordinates contain a symplectic genus 2 surface $\Sigma$ of self-intersection zero; the manifolds from the last two choices contain a symplectic torus $T$ of self-intersection zero and $\pi_1(Y - \Sigma) = \pi_1(Y - T)$.

**Proposition 17.** Let $n \geq 2$. There exists a symplectic minimal 4-manifold with cyclic fundamental group whose characteristic numbers can be chosen amongst the following three choices:

- $e = 75n^2 + 256n + 130$ and $\sigma = 25n^2 - 68n - 78$; $(c_1^2, \chi_h) = (225n^2 + 298n + 26, 25n^2 + 94n + 13)$

- $e = 75n^2 + 256n + 134$ and $\sigma = 25n^2 - 68n - 78$; $(c_1^2, \chi_h) = (225n^2 + 298n + 30, 25n^2 + 94n + 14)$ or

- $e = 75n^2 + 256n + 136$ and $\sigma = 25n^2 - 68n - 80$; $(c_1^2, \chi_h) = (225n^2 + 298n + 32, 25n^2 + 94n + 14)$.

These manifolds have the $\infty$-property.

3. **Background Results on 4-Manifolds**

The corresponding topological prototypes used to determine the homeomorphism type of the manufactured manifolds will be a connected sum of $p\mathbb{CP}^2 \# q\mathbb{CP}^2$ with a non-simply connected manifold responsible for the fundamental group. For $\pi_1 = \mathbb{Z}$, we build exotica for $q\mathbb{CP}^2 \# p\mathbb{CP}^2 \# S^1 \times S^3$. For $\pi_1 = \mathbb{Z}_p$, then the prototype manifolds would be of the form $q\mathbb{CP}^2 \# p\mathbb{CP}^2 \# L(p, 1) \times S^1$. Here the piece $L(p, 1) \times S^1$ stands for the surgered product $L(p, 1) \times S^1$ of a Lens space with the circle; the surgery is performed along $\{pt\} \times S^1$ to kill the loop corresponding to the generator of the infinite cyclic group factor so that all $\pi_1$ comes from the fundamental group of the lens space.

3.1. **Homeomorphism Criteria:** Case $\pi_1 = \mathbb{Z}_p$. For the finite cyclic fundamental group case, the classification result we will use is given in [25] in the shape of theorem C.
Theorem 18. (Hambleton, Kreck). Let $X$ be a smooth closed oriented 4-manifold with finite cyclic fundamental group. $X$ is classified up to homeomorphism by the fundamental group, the intersection form on $H_2(M;\mathbb{Z})/\text{Tors}$ and the $\omega_2$-type. Moreover, any isometry of the intersection form can be realized by a homeomorphism.

Since in this scenario we do have 2-torsion, one is to be careful about determining the parity of the intersection form and its $\omega_2$-type. The Enriques surfaces are an example of the sublety of the situation: their intersection form is even, but they are not spin manifolds. In this case, there are three $\omega_2$-types:

1. $\omega_2(\tilde{X}) \neq 0$,
2. $\omega_2(X) = 0$,
3. $\omega_2(\tilde{X}) = 0$, but $\omega_2(X) \neq 0$.

By using the well-know work of Donaldson and of Minkowski-Hasse on the classification of the intersection forms, the previous result can be stated in the following practical terms.

Theorem 19. A smooth, closed, oriented 4-manifold with finite cyclic fundamental and indefinite intersection form is classified up to homeomorphism by the fundamental group, the Betti numbers $b^+_2$ and $b^-_2$, the parity of the intersection form and the $\omega_2$-type.

However, do notice that for these manifolds, to know the invariants $b^+_2$ and $b^-_2$ is equivalent to knowing any other two numerical invariants, like $e$ or $\sigma$.

Moreover, most of the manufactured manifolds are non-spin, aka, type II does not occur. Deciding the $\omega_2$ type boils down on distinguishing if the universal cover is spin or not.

Remark 1. This criteria also applies to manifolds with $\pi_1 = \mathbb{Z}_n \oplus \mathbb{Z}_m$ if $\gcd(n,m) = 1$.

3.2. Homeomorphism Criteria: Case $\pi_1 = \mathbb{Z}$. For a huge region, the following result (found in [32]) settles the homeomorphism criteria.

Theorem 20. (Hambleton-Teichner). If $X$ is a closed oriented smooth 4-manifold with infinite cyclic fundamental group and satisfies the inequality

$$b_2(X) - |\sigma(X)| \geq 6,$$

then $X$ is homeomorphic to the connected sum of $S^1 \times S^3$ with a unique closed simply-connected 4-manifold. In particular, $X$ is determined up to homeomorphism by its second Betti number $b_2(X)$, its signature $\sigma(X)$ and its $\omega_2$-type. In particular, $X$ is either spin or non-spin depending on the parity of its intersection form.

However, in more generality we have

Theorem 21. Let $X$ be a closed orientable 4-manifold with infinite cyclic fundamental group and suppose the intersection form on $X$ is extended from the integers.
Then $X$ is homeomorphic to a connected sum of $S^1 \times S^3$ with a simply-connected 4-manifold.

At this point the condition of a manifold to have an intersection form which is extended from the integers is equivalent to its algebraic numbers complying with the inequality above. It has been conjectured by Hambleton-Teichner that all smoothable 4-manifolds can be topologically decomposed as a connected sum of a simply-connected 4-manifold and $S^1 \times S^3$. Because of the equivalence, this is the same as the indefiniteness inequality $b_2 \geq |\sigma|$ being all that is needed for the forms to be extended from $\mathbb{Z}$.

The following result of Hambleton-Kreck-Teichner ([27]) will help us conclude on the homeomorphism type of our manifolds.

**Lemma 22.** Let $M$ be a closed oriented smooth 4-manifold with solvable Baumslag-Solitar fundamental group. Then $M$ has $\omega_2$-type I) if and only if the equivariant intersection form of $M$ is odd.

This is particularly true in the absence of 2-torsion.

### 3.3. Raw Materials

The elements employed in our constructions are borrowed from the constructions of other authors ([4], [6], [9], [10], [11], [19]). In this section we quote the notions and results we used the most for the convenience of the reader.

The following definition was introduced in [9].

**Definition 23.** An ordered triple $(X, T_1, T_2)$ consisting of a symplectic 4-manifold $X$ and two disjointly embedded Lagrangian tori $T_1$ and $T_2$ is called a telescoping triple if

1. The tori $T_1$ and $T_2$ span a 2-dimensional subspace of $H_2(X; \mathbb{R})$.
2. $\pi_1(X) \cong \mathbb{Z}^2$ and the inclusion induces an isomorphism $\pi_1(X - (T_1 \cup T_2)) \to \pi_1(X)$. In particular, the meridians of the tori are trivial in $\pi_1(X - (T_1 \cup T_2)) \to \pi_1(X)$.
3. The image of the homomorphism induced by the corresponding inclusion $\pi_1(T_1) \to \pi_1(X)$ is a summand $\mathbb{Z} \subset \pi_1(X)$.
4. The homomorphism induced by inclusion $\pi_1(T_2) \to \pi_1(X)$ is an isomorphism.

The telescoping triple is called minimal if $X$ itself is minimal. Some words are in order. Notice the importance of the order of the tori. The meridians $\mu_{T_1}, \mu_{T_2}$ in $\pi_1(X - (T_1 \cup T_2)) \to \pi_1(X)$ are trivial and the relevant fundamental groups are abelian. The push off of an oriented loop $\gamma \subset T_i$ into $X - (T_1 \cup T_2)$ with respect to any (Lagrangian) framing of the normal bundle of $T_i$ represents a well defined element of $\pi_1(X - (T_1 \cup T_2))$ which is independent of the choices of framing and base-point.

The first condition assures us that the Lagrangian tori $T_1$ and $T_2$ are linearly independent in $H_2(X; \mathbb{R})$. This allows for the symplectic form on $X$ to be slightly perturbed so that one of the $T_i$ remains Lagrangian while the other becomes symplectic. It can also be perturbed in such way that both of them become symplectic.
If we were to consider a symplectic surface $F$ in $X$ disjoint from $T_1$ and $T_2$, the perturbed symplectic form can be chosen so that $F$ remains symplectic.

Removing a surface from a 4-manifold usually introduces new generators into the fundamental group of the resulting manifold. The second condition indicates that the meridians are nullhomotopic in the complement and, thus, the fundamental group of the manifold and the fundamental group of the complement of the tori in the manifold coincide.

Out of two telescoping triples, one is able to produce one as follows

**Proposition 24.** (cf. [6]) Let $(X, T_1, T_2)$ and $(X', T'_1, T'_2)$ be two telescoping triples. Then for an appropriate gluing map the triple

$$(X \#_{T_2} X', T_1, T'_2)$$

is again a telescoping triple.

The Euler characteristic and the signature of $X \#_{T_2} X'$ are given by $e(X) + e(X')$ and $\sigma(X) + \sigma(X')$.

By Usher’s theorem, if both $X$ and $X'$ are minimal the resulting telescoping triple will be minimal too.

For the production of the exotic manifolds with cyclic fundamental groups we have the following.

**Proposition 25.** Let $(X, T_1, T_2)$ be a telescoping triple. Let $l_{T_1}$ be a Lagrangian push off of a curve on $T_1$ and $m_{T_2}$ the Lagrangian push off of a curve on $T_2$ so that $l_{T_1}$ and $m_{T_2}$ generate $\pi_1(X)$.

The symplectic 4-manifold obtained by performing either $+1$ Luttinger surgery on $T_1$ along $l_{T_1}$ or $+1$ surgery on $T_2$ along $m_{T_2}$ has infinite cyclic fundamental group. By applying a $+1$ Luttinger surgery on $T_1$ along $l_{T_1}$ and a $+1/p$ Luttinger surgery on $T_2$ along $m_{T_2}$ a symplectic manifold with finite cyclic fundamental group is obtained.

**Proof.** We start with the infinite cyclic fundamental group case. Denote by $Y$ the manifold resulting from applying one of the two mentioned surgeries surgeries. For the sake of definiteness, say $T_1$ is the surgered torus and let $T_1 = T$ to simplify notation. A $(0, +1)$ surgery is applied. By definition, the meridians of a telescoping triple are trivial. Therefore, we have

$$\pi_1(Y) = \pi_1(X - T)/N(\mu_T m_T^p l_T^p) = \mathbb{Z} \oplus \mathbb{Z}/N(1m_T^p 1),$$

where $N(m^p)$ is the normal subgroup generated by $m$, which is $\mathbb{Z}$. Therefore, $\pi_1(Y) = \mathbb{Z}$ generated by $t_1$. This could be stated as surgery on $T_1$ along $l_{T_1}$ kills $t_2$ in the fundamental group.

If we apply a $(0, +1/p)$ surgery on $T_2$ along $m_{T_2}$, we have $\pi_1 = \mathbb{Z}/N(m_T^p) = \mathbb{Z}_p$.\hfill $\square$

**Remark 2.** The fundamental group calculations for the more general torus surgeries are analogous. To check the validity of the claims, it suffices to state
\[ \pi_1 = \pi_1(X - T)/N(\mu_T m_T^p t_T^p). \]

**Theorem 26.** (Baldridge-Kirk [11]) There exists a minimal symplectic 4-manifold \( B \) containing a pair of homologically essential Lagrangian tori \( T_1 \) and \( T_2 \) and a square zero symplectic genus 2 surface \( F \) so that these three surfaces are pairwise disjoint, \( e(B) = 6, \sigma(B) = -2 \) and

1. \( \pi_1(B - (F \cup T_1 \cup T_2)) = \mathbb{Z}t_1 \oplus \mathbb{Z}t_2 \), i.e., \( \mathbb{Z}^2 \) generated by \( t_1 \) and \( t_2 \).
2. The inclusion \( B - (F \cup T_1 \cup T_2) \subset B \) induces an isomorphism on fundamental groups. In particular the meridians \( \mu_F \), \( \mu_{T_1} \) and \( \mu_{T_2} \) all vanish in \( \pi_1(B - (F \cup T_1 \cup T_2)) \).
3. The Lagrangian push-offs \( m_{T_1}, l_{T_1} \) of \( \pi_1(T_1) \) are sent to \( 1 \) and \( t_2 \) respectively in the fundamental group of \( B - (F \cup T_1 \cup T_2) \).
4. The Lagrangian push-offs \( m_{T_2}, l_{T_2} \) of \( \pi_1(T_2) \) are sent to \( t_1 \) and \( t_2 \) respectively in the fundamental group of \( B - (F \cup T_1 \cup T_2) \).
5. The push-off \( F \subset B - (F \cup T_1 \cup T_2) \) takes the first three generators of a standard symplectic generating set \( \{a_1, b_1, a_2, b_2\} \) for \( \pi_1(F) \) to 1 and the last element to \( t_2 \).
6. There exists a symplectic torus \( H_1 \subset B \) which intersects \( F \) transversally once, which has square -1 and the homomorphism \( \pi_1(H_1) \rightarrow \pi_1(B) \) takes the first generator to 1 and the second to \( t_1 \). Moreover, \( H_1 \) is disjoint from \( T_1 \) and \( T_2 \).

We refer the reader to theorems 20 and 13 and to proposition 12 in [11] for the proof and for more details. The following two results can be found in [6].

**Corollary 27.** For each \( g \geq 0 \), there exists a minimal telescoping triple \((B_g, T_1, T_2)\) satisfying \( e(B_g) = 6 + 4g, \sigma(B_g) = -2 \) and containing a square -1 genus \( g + 1 \) surface disjoint from \( T_1 \cup T_2 \).

**Theorem 28.**

- There exists a minimal telescoping triple \((C, T_1, T_2)\) with \( e(C) = 8, \sigma(C) = -4 \). Moreover, \( C \) contains a square -1 torus disjoint from \( T_1 \cup T_2 \).

- There exists a minimal telescoping triple \((D, T_1, T_2)\) with \( e(D) = 10, \sigma(D) = -6 \). Moreover, \( D \) contains a square -1 torus disjoint from \( T_1 \cup T_2 \).

By a repeated use of Lemma 2 in [11] and Usher’s theorem one proves the following

**Proposition 29.** Let \( X \) be one of the manifolds \( B, B_g, C, D \) and \( T_1, T_2 \) the corresponding Lagrangian tori as described in the previous results, with Lagrangian push-offs \( m_{T_1} \) and \( l_{T_1} \) and trivial meridians. Then the symplectic 4-manifolds obtained from \( \pm 1 \) Luttinger surgery on one Lagrangian tori along (accordingly) \( m_{T_2} \) or \( l_{T_2} \) are all minimal. The symplectic 4-manifolds obtained from \( \pm 1 \) Luttinger surgery on one Lagrangian tori along (accordingly) \( m_{T_2} \) or \( l_{T_2} \) and \( \pm 1/p \) Luttinger surgery on the other tori along the proper push-off are all minimal.
We move on now to imitate the procedure of lemma 10 in [6] to produce a non-minimal telescoping triple out of $(B, T_1, T_2)$ that suits perfectly our purposes. The statement is

**Lemma 30.** The blow-up $A = B \# 16\mathbb{CP}^2$ contains a genus 18 surface $F_{18}$ with trivial normal bundle and two Lagrangian tori $T_1 \times T_2$ so that the surfaces $F_{18}, T_1, T_2$ are pairwise disjoint, $(A, F_{18})$ is relatively minimal and:

1. $\pi_1(A - (F_{18} \cup T_1 \cup T_2)) = \mathbb{Z}t_1 \oplus \mathbb{Z}t_2$.
2. The inclusion $A - (F_{18} \cup T_1 \cup T_2) \subset A$ induces an isomorphism on fundamental groups. In particular the meridians $\mu_{F_{18}}, \mu_{T_1}, \mu_{T_2}$ all vanish in $\pi_1(A - (F_{18} \cup T_1 \cup T_2))$.
3. The Lagrangian push-offs $m_{T_1}, m_{T_2}$ of $\pi_1(T_1)$ are sent to 1 and $t_2$ respectively in the fundamental group of $A - (F_{18} \cup T_1 \cup T_2)$.
4. The Lagrangian push-offs $m_{T_2}, m_{T_1}$ of $\pi_1(T_2)$ are sent to $t_1$ and $t_2$ respectively in the fundamental group of $A - (F_{18} \cup T_1 \cup T_2)$.
5. There is a standard symplectic generating set $\{a_1, b_1, a_2, b_2, \ldots, a_{18}, b_{18}\}$ for $\pi_1(F_{18})$ so that the push-off $F_{18} \subset A - (F_{18} \cup T_1 \cup T_2)$ takes $b_{17}$ to $t_2$ and $b_{18}$ to $t_1$, and all other generators to 1.

In particular, $(A, T_1, T_2)$ is a telescoping triple.

Needless to say, a basic element in these constructions is the computation of fundamental groups. Serious technical issues arise when dealing with fundamental groups and cut-and-paste constructions; keeping track of the base point through out the operations is crucial. For example, in order to be able to apply van-Kampen’s theorem, the base points must lie on the boundary and great care is required when one is performing fundamental group calculations. The reader is referred to [10], [11], [12] and [19] for more detailed description on this issue. The mechanisms employed in this paper are much softer though, since they depend heavily on those calculations performed in the papers cited before. We recall for the convenience of the reader two results due to Baldridge and Kirk which are very useful for our constructions.

Let $Z$ be the symplectic sum of $T^4 \# 2\mathbb{CP}^2$ and $T^2 \times \Sigma_2$ along the symplectic genus 2 surfaces $F_2 \subset T^4 \# 2\mathbb{CP}^2$ and $\Sigma_{h'} \subset T^2 \times \Sigma_2$ using the diffeomorphism $\phi : F_2 \to \Sigma_{h'}$ which induces an isomorphism on the fundamental groups which maps the generators $s_i, t_i$ of $\pi_1(F_2)$ to the generators $a_i, b_i$ of $\pi_1(\Sigma_{2})$ as follows:

$$(s_1, t_1, s_2, t_2) \mapsto (b_1^{-1}, b_1 a_1 b_1^{-1}, b_2^{-1}, b_2 a_2 b_2^{-1}).$$

Denote by $x, y$ the generators of $\pi_1(T^2)$. We have the following expression for the fundamental group that comes in handy when applying surgery to $Z$. We now quote theorem 11 of [11]

**Theorem 31.** (Baldridge-Kirk). Let $Z$ be as above. Let $R$ be the union of the six Lagrangian tori $T'_1, T'_2, T_1, T_2, T_3, T_4$ and a symplectic genus 2 surface $F$. Then the fundamental group of $Z - R$ is generated by the loops $x, y, a_1, b_1, a_2, b_2$. The relations
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1 = [b_1, b_2] = [a_1, b_2] = [b_1, a_1] = [b_2, a_2]

and

1 = [x, a_1] = [y, a_1] = [x, a_2] = [y, a_2]

hold in \( \pi_1(Z - R) \).

The meridian of the surface \( F \) is given by \( \mu_F = [x, y] \). For the six Lagrangian tori, the meridians and Lagrangian push offs (with appropriate paths to the base point) in \( \pi_1(Z - R) \) are given by:

- \( T_1' : \mu_1' = [a_2^{-1}, a_1^{-1}], m_1' = b_1^{-1}, l_1' = b_2^{-1} \).
- \( T_2' : \mu_2' = [b_1, a_2], m_2' = b_1 a_2 b_1^{-1}, l_2' = b_2^{-1} \).
- \( T_1 : \mu_1 = [b_1^{-1}, y^{-1}], m_1 = x, l_1 = a_1 \).
- \( T_2 : \mu_2 = [x^{-1}, b_1], m_2 = y, l_2 = a_1 \).
- \( T_3 : \mu_3 = [b_2^{-1}, y^{-1}], m_3 = x, l_3 = a_2 \).
- \( T_4 : \mu_4 = [x^{-1}, b_2], m_4 = y, l_4 = a_2 \).

In the same paper, Baldridge and Kirk went on to determine some basic properties of \( Z \). We quote a part of their proposition 12 in [11] that we will need.

**Proposition 32.** (Baldridge-Kirk). The first homology \( H_1(Z) = \mathbb{Z}^6 \) is generated by the loops \( x, y, a_1, b_1, a_2, b_2 \). The second homology is \( H_2(Z) = \mathbb{Z}^{16} \). There exist six disjoint tori \( R_1', R_2', R_1, R_2, R_3, R_4 \) and a genus 2 surface \( H_3 \) with trivial normal bundles in \( Z \) which are geometrically dual to the six Lagrangian tori \( T_1, T_2, T_1, T_2, T_3, T_4 \) and the surface \( F \), in the sense that \( T_i' \) intersects \( R_i' \) transversally in one point and similarly for \( T_i, R_i \) and \( F, H_3 \), and all other intersections are pairwise empty.

There are two disjoint tori \( H_1, H_2 \) with square -1 which intersect \( F \) transversally once; in particular the intersection form is odd. The sixteen surfaces \( T_1', R_1', T_1, R_1, F, H_1, H_2, H_3 \) generate \( H_2(Z) \).

The signature is \( \sigma(Z) = -2 \) and the Euler characteristic \( \epsilon(Z) = 6 \).

The symplectic manifold obtained by performing any six Luttinger surgeries on the \( T_i' \) and \( T_i \) Lagrangian tori satisfy \( \epsilon = 6 \) and \( \sigma = -2 \) as well and contains a symplectic genus 2 surface \( F \) with trivial normal bundle and six Lagrangian tori.

The following lemma (lemma 16 in [6]) is useful if one wishes to produce an exotic manifold with cyclic fundamental group out of a simply connected one at the cost of increasing the Euler characteristic by 4.

**Lemma 33.** There exists a minimal symplectic 4-manifold \( Z \) with \( \epsilon(Z) = 4 \) and \( \sigma(Z) = 0 \) which contains eight homologically essential Lagrangian tori \( S_1, \ldots, S_8 \).
(in fact each $S_i$ has a geometrically dual torus $S_i^d$ so that all other intersections are zero) so that $\pi_1(Z - \cup_i S_i)$ is generated by $x_1, y_1, x_2, y_2$ and $a_1, b_1, a_2, b_2$, and so that the meridians and Lagrangian push offs are given by

- $S_1: \mu_1 = [b_1^{-1}, y_1^{-1}], m_1 = x_1, t_1 = a_1$,
- $S_2: \mu_2 = [x_1^{-1}, b_1], m_2 = y_1, t_2 = b_1a_1b_1^{-1}$,
- $S_3: \mu_3 = [b_2^{-1}, y_1^{-1}], m_3 = x_1, t_3 = a_2$,
- $S_4: \mu_4 = [x_1^{-1}, b_2], m_4 = y_1, t_4 = b_2a_2b_2^{-1}$,
- $S_5: \mu_5 = [b_1a_1^{-1}b_2^{-1}, y_2^{-1}], m_5 = x_2, t_5 = b_1^{-1}$,
- $S_6: \mu_6 = [x_2^{-1}, b_1a_1b_1^{-1}], m_6 = y_2, t_6 = b_1a_1^{-1}b_1^{-1}$,
- $S_7: \mu_7 = [b_2a_2^{-1}b_2^{-1}, y_2^{-1}], m_7 = x_2, t_7 = b_2^{-1}$,
- $S_8: \mu_8 = [x_2^{-1}, b_2a_2b_2^{-1}], m_8 = y_2, t_8 = b_2a_2b_2^{-1}a_2^{-1}b_2^{-1}$.

Since the tori are assumed to be homologically essential, the symplectic form can be perturbed so that any of them become symplectic.

The following lemma and proposition solve several geography problems in the simply connected case

**Lemma 34.** (J. Park, [37]). For each integer $k$, $10 \leq k \leq 18$, there exists a simply connected, minimal, symplectic 4-manifold $Z$ with $\chi = 2$ and $c_2^Z = 19 - k$ which contains a symplectic genus 2 surface $\Sigma_2$ of square 0 and a symplectic torus $T$ of square 0, disjoint from $\Sigma_2$, in a fishtail neighborhood, and $\pi_1(Z - \Sigma_2) = \pi_1(Z - T) = 1$.

**Proposition 35.** (J. Park, [37]). For each odd integer $m \geq 1$ and $10 \leq k \leq 18$, there exists a simply connected, minimal symplectic 4-manifold $Y$, which contains a symplectic genus 2 surface $\Sigma_2$ of square 0 and a torus of square 0 disjoint from $\Sigma_2$, in a fishtail neighborhood which satisfies

$$\pi_1(Y - \Sigma_2) = \pi_1(Y - T) = 1.$$  

Furthermore, it has $\chi(Y) = m^2 + 31m + 5$ and $c_2^Y = 225m^2 + 248m + 35 - k$.

3.4. Minimality/Irreducibility. The following results allow us to conclude on the irreducibility of the manufactured manifolds.

**Theorem 36.** (Hamilton and Kotschick, [28]). Minimal symplectic 4-manifolds with residually finite fundamental groups are irreducible.

Free groups and finite cyclic groups are a well-known example of residually finite fundamental groups. In particular, the results tell us that the only property we should worry about is minimality. For such purpose we will make use of the following

**Theorem 37.** (Usher, [45]). Let $X = Y \#_{\Sigma \cong \Sigma} Y'$ be the symplectic sum where the surfaces have genus greater than zero.
If either $Y - \Sigma$ or $Y' - \Sigma'$ contains an embedded symplectic sphere of square $-1$, then $X$ is not minimal.

If one of the summands, say $Y$ for definiteness, admits the structure of an $S^2$-bundle over a surface of genus $g$ such that $\Sigma$ is a section of this $S^2$-bundle, then $X$ is minimal if and only if $Y'$ is minimal.

In all other cases, $X$ is minimal.

Thus, to assure that the manufactured manifolds are minimal it suffices to exclude the first two cases. For such purpose, by taking a look at the building blocks of the symplectic sums, it is usual to blow-up points to obtain the symplectic surface of self-intersection 0 used for the construction. The exceptional spheres introduced by the blow-up process are the only $-1$ spheres. They are the only "threats" for our manifolds not being minimal. To assure that the first scenario of Usher’s theorem is not possible, we need to check that every exceptional sphere does indeed intersect transversally at one point the surface.

When working on a symplectic context, there is another useful method to eliminate the first two cases of Usher’s theorem. The result appears as Corollary 3 in [35], here we stated as a theorem due to its role.

**Theorem 38. (Li).** Let $X$ be a symplectic 4-manifold which is not rational or ruled. Then every smoothly embedded $-1$ sphere is homologous to a symplectic $-1$ curve up to sign. If $X$ is the blow-up of a minimal symplectic 4-manifold with $E_1, \ldots, E_n$ represented by exceptional curves, then the $E_i$ are the only classes represented by a smoothly embedded $-1$ sphere. Therefore, any orientation preserving diffeomorphism maps $E_i$ to some $E_j$ up to sign.

### 4. Strategy

The blueprint to the manufacturing process of symplectic irreducible 4-manifolds with cyclic fundamental group has two paths. Let us start by explaining the first one: when one is aiming at building a simply connected minimal symplectic 4-manifold using Luttinger surgeries, the process can be interrupted before applying the last $\pm 1$ Luttinger surgery. The fundamental group of the resulting manifold will be infinite cyclic. We can then go ahead and apply a $\pm 1/p$ Luttinger surgery on the Lagrangian tori which is still left unused and produce a manifold with finite cyclic fundamental group. This procedure was suggested for some constructions given in [12], [11], [6] and [4].

The second road consists of starting with a simply connected irreducible symplectic 4-manifold $X$ which contains a symplectic torus or a symplectic surface of genus 2, both of self-intersection zero and both having simply connected complement inside $X$. We can build the proper symplectic sum with one of the raw materials presented in section 3 along the corresponding symplectic surface. Then, we apply Luttinger surgeries to manipulate the manifolds as we need to obtain a cyclic fundamental group. This second road, of course, might have a toll on the Euler characteristic of the resulting manifold by increasing it accordingly.
Since the manufacturing process is strongly related for both types of cyclic groups, our proofs will carry on both cases at the same time. The required torus surgeries to produced an infinite family of pairwise non-diffeomorphic, non-symplectic manifolds are applied accordingly.

The manifolds constructed in the mentioned papers deserve a closer look for our purposes. We start by sketching the other authors’ constructions and use them to exemplify the first part of the recipe that we just outlined above.

**Example 39.** \((c_1^2, \chi_h) = (8m - 1, m)\) for \(\pi_1 = \mathbb{Z}_2\). In [4] (on page 8 remark 6) a smooth symplectic minimal 4-manifold \(S\) with \(\pi_1(X) = \mathbb{Z}_2\) satisfying
\[
\begin{align*}
e(X) &= 4m + 1, \\
\sigma(X) &= -1
\end{align*}
\]
is constructed from the fiber sum \(Y_n(M) \#_q(T^4 \# \mathbb{CP}^2)\) and an infinite family of non-symplectic 4-manifolds with \(\pi_1 = \mathbb{Z}_2\). We call this family \(\{X_n\}\). Let us see that their construction fill in the points of the form \((c_1^2, \chi_h) = (8m - 1, m)\) in the Geography plane for \(\pi_1 = \mathbb{Z}_2\).

We need to establish the topological prototype.

**Lemma 40.** Let \(m \geq 2\). There exists a smooth symplectic minimal 4-manifold \(X_{2m-1,2m}^{\mathbb{Z}_2}\) with \(\pi_1(X_{2m-1,2m}^{\mathbb{Z}_2}) = \mathbb{Z}_2\) and an infinite family \(\{X_n\}\) of non-symplectic manifolds satisfying
\[
\begin{align*}
e(X) &= 4m + 1 \text{ and} \\
\sigma(X) &= -1
\end{align*}
\]
For a given \(m\), the members of \(\{X_n\}\) and \(S := X_{2m-1,2m}^{\mathbb{Z}_2}\) are pair-wise homeomorphic to
\[
(2m - 1)\mathbb{CP}^2 \# 2m\mathbb{CP}^2 \# \widetilde{L(2,1)} \times S^1.
\]

**Proof.** Let us start by showing that they all have \(\omega_2\) - type I). The homological information implies that they have an odd intersection form, thus they can not be spin and \(\omega_2\) -type II) is ruled out. The universal cover of \(X_{2m-1,2m}^{\mathbb{Z}_2}\) has \(e = 8m + 2\) and \(\sigma = -2\), thus it is not spin and type III) is ruled out too. Therefore, all the \(X_{2m-1,2m}^{\mathbb{Z}_2}\) are of \(\omega_2\) -type I). For a fixed \(m\), Hambleton-Kreck’s theorem implies that they are pairwise homeomorphic. 

This allows one to conclude on the geography for this specific fundamental group. Notice that the procedure does construct a exotic smooth structure for \((2m - 1)\mathbb{CP}^2 \# 2m\mathbb{CP}^2 \# \widetilde{L(2,1)} \times S^1\). However, the focus problem is to find infinitely many exotic smooth structures.

**Remark 3.** The No-2-Torsion Hypothesis. One might be able to argue without any reference to the SW invariants that any given symplectic irreducible manifold constructed is an exotic copy of its corresponding topological prototype. However, in order to establish the existence of infinitely many exotic smooth structures one
does need these invariants. As it is explained in [19], the Morgan-Mrowka-Szabo formula [33] is employed to distinguish the Seiberg-Witten invariants and, by doing so, conclude that the members of the infinite family \( \{X_n\} \) of irreducible manifolds which were obtained by torus surgeries are pairwise non-diffeomorphic (see [19], the remark preceding corollary 14 in [11] and [2]).

This involves a one-to-one correspondence between the set of spin\(^C\) structures on the manifold and the characteristic elements of \( H^2 \). Our constructions build exotic manifolds for every single finite cyclic fundamental group. To establish the \( \infty \)-property on the manufactured manifolds, we assume that their fundamental groups do lack 2-torsion.

**Example 41.** \((c_1^2, \chi_h) = (4, 1)\). Baldridge and Kirk (cf. [10]) built a manifold \( P \) with Euler characteristic \( e = 8 \) and signature \( \sigma = -4 \) in theorem 7 of [11]. This manifold is \( X^2_{5,6} \) under our notation. By looking at the proof of the quoted theorem, one sees that the torus the Lagrangian torus \( T^2 \) has trivial meridian in \( P - T^2 \) and it is left unisurgered. One can then apply a \(+1/p\) - Luttinger surgery to produce \( X^p_{1,5} \) which is an exotic

\[
\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2 \# \widetilde{L(p, 1)} \times S^1.
\]

To prove this, it is just missing to observe that they share the same homeomorphism type. This follows from Hambleton-Kreck’s criteria by assuming \( 4p \not\equiv 0 \) mod 16.

**Example 42.** \((c_1^2, \chi_h) = (6, 1)\). In [6], \( X^2_{2,4} \) was already built out of the telescoping triple \((B, T_1, T_2)\) by only performing a single Luttinger surgery on \( T_2 \) along \( m_{T_2} \) (the authors call it \( \tilde{B} \) in that paper). By performing a \(+1/p\) Luttinger surgery on \( \tilde{B} \), we obtain \( X^p_{2,3} \). This way one has produced an exotic \( \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \widetilde{L(p, 1)} \times S^1 \) with the assumption \( 2p \not\equiv 0 \) mod 16.

These last two points belong to small 4-manifolds. The method to fill the region \((c_1^2, \chi_h) = (5 + 8k, 1 + k)\) (for \( k \geq 2 \)) for cyclic fundamental groups was suggested in [6]. Remark 1 of the same paper also suggests a recipe to obtain an analogous version of theorem 15 for manifolds with cyclic fundamental group. We sketch their recipe and point out the places where the needed chances are made in order to obtain the desired fundamental groups.

They start by defining two minimal simply connected symplectic 4-manifolds:

\[
X_- := X^1_{3,5} \quad \text{and} \quad X_+ := X^1_{1,3}
\]

with \( e(X_-) = 10, \sigma(X_-) = -2 \) and \( e(X_+) = 6, \sigma(X_+) = -2 \).

Each of these manifolds contain a symplectic surface \( F \) of genus 2 and trivial normal bundle, as well as a symplectic torus \( H_1 \) of square \(-1\). Out of these submanifolds a symplectic genus 3 surface \( F_3 \) of square 1 is obtained by symplectically resolving the union \( H_1 \cup F \). One gets rid of the self-intersection in the sense that
one considers now the proper transform $\tilde{F}_3$ of $F_3$ in $\tilde{X}_\pm$: the blow up $X_\pm$ at a point on $F_3$ provides us with a symplectic surface $\tilde{F}_3$ of genus three and self-intersection 0. The minimality of $X_\pm$ assures that the meridian of $\tilde{F}_3$ intersects the exceptional sphere, then $\pi_1(\tilde{X}_\pm - \tilde{F}_3) = 1$.

Consider now the product $F_3 \times G$ of a genus 3 surface with a genus $g$ surface and its product symplectic form. This is the step on the manufacturing process where manipulates the the fundamental groups to obtain the desired manifold. For our purposes, we will only perform $2g - 1$ Luttinger surgeries on the following $2g - 1$ disjoint Lagrangian tori along the corresponding curves

$$Y_1 \times A_j \text{ along } l_{Y_1 \times A_j} = a_j \text{ and }$$
$$Y_2 \times B_j \text{ along } l_{Y_2 \times B_j} = b_j.$$  

where $j = 1, \ldots, g$ by leaving (say for definiteness purposes) $Y_2 \times B_g$ alone, i.e., not performing this surgery. By doing so, one obtains a manifold $Z_g$. The fundamental group of $Z_g$ is given by theorem 1 of [6] to be the group generated by the $6 + 2g$ loops $x_1, y_1, x_2, y_2, x_3, y_3$ (from the $\pi_1(F_3)$) and $a_1, b_1, \ldots, a_g, b_g$ (from $\pi_1(G)$) and the relations:

$$[x_1, b_j] = a_j \text{ for } j = 1, \ldots, g \text{ and }$$
$$[x_2, a_j] = b_j \text{ for } j = 1, \ldots, g - 1.$$  

Build the symplectic sum $\tilde{Q}_{\pm, g} := \tilde{X}_\pm \#_{F_3 = F} Z_g$. Its fundamental group is infinite cyclic: notice that $\pi_1(\tilde{X} - \tilde{F}_3) = 1$, so this block kills the generators $x_i$ and $y_i$ during the symplectic sum. The relations from the Luttinger surgeries kill $a_j$ and $b_j$ except for $b_g$. Therefore, $\pi_1(\tilde{Q}_{\pm, g}) = <b_g>$.  

One can now perform a $+1/p$-Luttinger surgery along the remaining $Y_2 \times B_g$ along $l_{Y_2 \times B_g}$ to produce a manifold $Q_{\pm, g}$ with fundamental group $\mathbb{Z}_p$.

Since $\tilde{X}_\pm$ is relatively minimal by Li’s theorem, the only hypothesis needed to apply Usher’s theorem and conclude that $Q_{\pm, g}$ is minimal as well is $g \geq 1$. One then can go on and compute

$$e(Q_{-, g}) = 11 + 8g, \quad \sigma(Q_{-, g}) = -3$$
$$e(Q_{+, g}) = 7 + 8g, \quad \sigma(Q_{+, g}) = -3.$$  

If $k$ is even, rename $X_{1+2k, 4+2k} = Q_{+, k/2}$; if $k$ is odd, set $X_{1+2k, 4+2k} = Q_{-, (k-1)/2}$.

This procedure manufactures the manifolds of the result we now state.

**Theorem 43.** For each integer $k \geq 2$, there exists an irreducible symplectic 4-manifold $X^2_{1+2k, 4+2k}$ with $e(X_{1+2k, 4+2k}) = 7 + 4k$, $\sigma(X_{1+2k, 4+2k}) = -3$ and finite cyclic fundamental group which is homeomorphic to

$$(1 + 2k)\mathbb{CP}^2 \# (4 + 2k)\bar{\mathbb{CP}}^2 \# L(p, 1) \times S^1.$$

There exists an irreducible symplectic 4-manifold $X^2_{2+2k, 5+2k}$ homeomorphic to

$$(2 + 2k)\mathbb{CP}^2 \# (5 + 2k)\bar{\mathbb{CP}}^2 \# S^3 \times S^1.$$
The homeomorphism claims are a straightforward consequence of Hambleton-Kreck and Hambleton-Teichner’s results.

The following manifolds can be constructed out of the process to build exotic \(CP^2 \# 0CP^2\) carried out in \cite{4}.

**Example 44.** \((c^2, \chi_h) = (3, 1)\); in \cite{4} the following symplectic sum was used

\[
(T^4 \# CP^2) \#_{\Sigma_2} (T^2 \times S^2 \# 4CP^2)
\]

and applied two surgeries on the 4-torus blown up once to obtain the mentioned simply connected exotic manifold. Do notice that the symplectic sum kills already two generators without any help from the surgeries. Now, we want to skip a surgery to obtain a manifold with an infinite cyclic fundamental group. We proceed to exhibit the fundamental group calculations of the process.

Let \(\alpha_1, \alpha_2, \alpha_3, \alpha_4\) be the generators of the fundamental group of the 4-torus blown up once. Then, they all satisfy \([\alpha_i, \alpha_j]\) = 1. Let \(a'_i, b'_i\) \(i = 1, 2\) be the generators of the genus 2 surface \(\Sigma'_2 \subset T^4 \# CP^2\). Assume that the inclusion induces a homomorphism on the fundamental groups that map the generators as follows:

\[
a'_1 \mapsto \alpha_1, b'_1 \mapsto \alpha_2, a'_2 \mapsto \alpha_3^2, b_2 \mapsto \alpha_4.
\]

We will apply one Luttinger surgery on this block. From our first example above, we have learned that the generator \(\alpha_3\) is to be killed, otherwise one obtains a finite cyclic fundamental group of order 2. Thus, we will apply \((\alpha'_2 \times \alpha'_3, \alpha'_4, -p)\) and introduce the relation \(\alpha_3 = [\alpha_1^{-1}, \alpha_4]\) to kill \(\alpha_3\). The surgery \((\alpha''_2 \times \alpha'_4, \alpha'_1, -m/r)\) used in in \cite{4} to produce a simply connected manifolds will not be applied. Denote by \(S\) the surgered manifold.

Consider now the other building block. Let \(c, d\) be the generators of \(\pi_1(S^2 \times T^2 \# 4CP^2)\) satisfying \([c, d]\) = 1 and \(a_i, b_i\) the generators of \(\Sigma_2\). Assume the inclusion \(\Sigma'_2 \subset S^2 \times T^2 \# 4CP^2\) induces a map on the fundamental groups that map the generators as follows:

\[
a_1 \mapsto c, b_1 \mapsto d, a_2 \mapsto c^{-1}, b_2 \mapsto d^{-1}.
\]

We remark that both genus 2 surfaces intersect an exceptional sphere inside the corresponding block and thus both meridians are nullhomotopic. Assume the orientation reversing diffeomorphism \(\partial(\text{nbh}(\Sigma'_2)) \to \partial(\text{nbh}(\Sigma_2))\) induces a homomorphism on the fundamental groups which maps the generators of \(\pi_1\) as follows:

\[
a'_i \mapsto a_i, b'_i \mapsto b_i\text{ for }i = 1, 2.
\]

We build the symplectic sum \(S = Y \# \Sigma'_2 = \Sigma_2 T^2 \times S^2 \# 4CP^2\). The presentation of \(\pi_1(S)\) is

\[
< \alpha_1, \alpha_2, \alpha_3, \alpha_4, c, d | \alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}], [\alpha_1, \alpha_2] = [\alpha_1, \alpha_3] = [\alpha_2, \alpha_3] = [\alpha_2, \alpha_4], [c, d], c = a_1, d = a_2, a_2^2 = c^{-1}, a_4 = d^{-1} >.
\]

So we have \(\alpha_4 = \alpha_2^{-1}\) and substituting it in \(\alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}]\) implies that \(\alpha_3 = 1 = c\) since \(\alpha_1\) and \(\alpha_2\) commute. This establishes that the only surviving
generator is $\alpha_4 = d^{-1}$. Rename $S = X^2_{Z_7}$.

One can then apply $(\alpha_2' \times \alpha_4', \alpha_4', -1/p)$ Luttinger surgery on the unused Lagrangian torus to obtain $X^2_{Z_{1,6}}$.

Since $X^2_{1,6}$ has an odd intersection form and its universal cover has signature $\sigma = -5p$, it follows by Hamilton-Kreck’s criteria that it is homeomorphic to $\mathbb{CP}^2 \# 6\mathbb{CP}^2 \# L(p,1) \times S^1$. However, its minimality implies that they are not diffeomorphic.

**Remark 4.** The last Luttinger surgery applied in our constructions kills a loop carrying a generator of the fundamental group. At the cost of leaving the setting of symplectic manifolds, one could apply a more general torus surgery instead. The resulting core torus from the surgery is nullhomologous in the manufactured manifold. It serves as a dial to change the smooth structure at will ([20], [19]). One can then proceed to use the Morgan-Mrowka-Szabo formula to prove that the irreducible members of the infinite family produced by the torus surgery are pairwise non-diffeomorphic. We refer the reader to [20], [19] and [33] for details. A concise explanation is given in the remark above corollary 14 in [11].

On the fundamental group calculations related to the infinite family of exotic manifolds homeomorphic to some topological prototype. For an exotic manifold $X$, the fundamental group of $X$ differs only from the one of $X$ by replacing a single relations of the form $b = [a^{-1}, d]$ by $b = [a^{-1}, d]^n$. Thus the only thing needed is to check that raising the power of the commutator in such relation does not affect the fundamental group calculations (see [2] for more details).

Besides the examples mentioned above, other possible choices of Luttinger surgeries to skip in order to obtain 4-manifolds with infinite cyclic fundamental group are possible. One could then apply a $\pm 1/p$ Luttinger surgery to obtain 4-manifolds with finite cyclic fundamental group. We prove the following result in order to exemplify the situation. Instead of skipping $(a_2' \times c_4', c_4', +1)$ in the $Y_n(m)$ summand of $X_n(m)$, we will skip $(a_2' \times c_2', a_2', -1)$ to obtain the needed 4-manifolds with infinite cyclic fundamental group. We then proceed to apply $(a_2' \times c_2', a_2', -1/p)$ to conclude the finite cyclic fundamental group case.

**Theorem 45.** Let $n \geq 3$. The following 4-manifolds have the $\infty$-property.

- $(2n - 1)\mathbb{CP}^2 \# 2n\mathbb{CP}^2 \# L(p,1) \times S^1$
- $2n\mathbb{CP}^2 \# (2n + 1)\mathbb{CP}^2 \# S^1 \times S^3$.

**Proof.** Let $Z'$ be the irreducible symplectic 4-manifold constructed in [4]. It contains a genus 2 symplectic surface $\Sigma_2$ of self-intersection 0 and $\pi_1(Z' - \Sigma_2)$ is a quotient of the group $< \alpha_1, \alpha_2, \alpha_3, \alpha_4 | \alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}], [\alpha_1, \alpha_3] = 1, [\alpha_2, \alpha_3] = [\alpha_2, \alpha_4] = 1 >$. 


In section 2 of [4], an infinite family of pairwise non-diffeomorphic irreducible 4-manifolds which has the same cohomology ring as \((2n - 3)\) is constructed by applying \(2n + 3\) Luttinger surgeries and a single \(m\) torus surgery on the product \(\Sigma_2 \times \Sigma_n\) of a genus 2 surface with a genus \(n\) surface. Let \(Y_n(m)'\) be the 4-manifold obtained by applying only \(2n + 2\) Luttinger surgeries and a \(m\) torus surgery. Using the notation of [19] and [4], we choose not to apply \((a_2' \times c_2', a_2', -1)\); this means that in \(\pi_1(Y_n(m)')\) all the fundamental group relations given in [4] for \(Y_n(m)\) still hold except for \([b_2^{-1}, d_2^{-1}] = a_2\). There is a genus 2 symplectic surface \(\Sigma_2 \subset Y_n(m)'\) of self-intersection 0.

Take the fiber sum

\[ S_n(m) = Y_n(m)' \#_4 Z' \]

using a diffeomorphism \(\Phi : \partial(N_{\Sigma_2}) \to \partial(N_{\Sigma'_2})\). Notice that \(S_n(m)\) is symplectic if \(m = 1\). Let \(a'_i, b'_i\) be the standard generators of \(\pi_1(\Sigma'_2)\) and \(a_i, b_i\) be the generators of \(\pi_1(\Sigma_2)\); thus, the fundamental group of \(Y_n(m)'\) is generated by \(a_1, b_1, a_2, b_2, c_1, d_1, \ldots, c_n, d_n\). Assume that \(\Phi_*\) maps the generators of \(\pi_1\) as follows:

\[ a_i \mapsto a'_i, b_i \mapsto b'_i. \]

The group \(\pi_1(S_n(m))\) is a quotient of the group \(\pi(Y_n(m) - N_{\Sigma_2}) \pi_1(Z' - N_{\Sigma'_2})/ < a_1 = a_1, b_1 = a_2, b_2 = a_4, \mu(\Sigma_2) = \mu(\Sigma'_2) >\). Notice that the existence of a -1 sphere in \(Z'\) which intersects the genus 2 surface allows us to build a null-homotopy for the meridian of \(\Sigma_2\). We proceed to show that all but one generators in \(\pi_1(S_n(m))\) are trivial.

Using the relations given in section 2 of [AP] we have that \(a_1 = [b_1^{-1}, d_1^{-1}] = [b_1^{-1}, [c_1^{-1}, b_2]^{-1}] = [b_1^{-1}, [b_2, c_1^{-1}]]\). Moreover, from the fundamental group of the building block \(Z'\) we know that \([a_2, a_4] = 1\) and since \(b_2\) is identified with \(a_2\) and \(b_4\) with \(a_4\), we have that \([b_1, b_2] = 1\). Since \(b_1\) commutes with \(c_1\), then \(a_1 = 1\).

Once we have killed \(a_1\), one can get rid of \(b_1, b_2, c_1, c_2, d_1\) and \(d_2\) by using the first 7 surgeries in (4) of section 2 in [4], and we conclude that \(c_n = 1 = d_n\) for \(n \geq 3\) by using the last 2\((n - 2)\) Luttinger surgeries of (4). This implies that \(\alpha_1 = 1 = \alpha_2 = \alpha_4\) and \(\alpha_3 = 1 = [a_1^{-1}, b_2^{-1}] = 1\). Thus, the infinite family of irreducible pairwise non-diffeomorphic 4-manifolds have \(\pi_1(S_n(m)) = < a_2 > = \mathbb{Z}\). Notice that the Lagrangian torus \(a_2 \times c_2\) is still unused. We can now go and apply \((a_2' \times c_2', a_2', -1)p\) Luttinger surgery to \(S_n(m)\) and obtain an infinite family \(\{X_n(m)\}\) of pairwise non-diffeomorphic irreducible 4-manifolds with \(\pi_1 = \mathbb{Z}_p\); \(X_n(1)\) is symplectic.

We can compute

\[ e(S_n(m)) = e(X_n(m)) = e(Y_n(m)') + e(Z') - 2e(\Sigma_2) = 4n + 1, \]

\[ \sigma(S_n(m)) = \sigma(X_n(m)) = \sigma(Y_n(m)') + \sigma(Z') = -1. \]

From Hambleton-Teichner result, we conclude that \(S_n(m)\) is homeomorphic to \(2n\mathbb{CP}^2 \# (2n + 1)\mathbb{CP}^2 \# S^1 \times S^3\) when \(n \geq 2\). By Kreck-Hambleton’s criteria, we conclude that \(S_n(m)\) is homeomorphic to \((2n - 1)\mathbb{CP}^2 \# 2n\mathbb{CP}^2 \# L(p, 1) \times S^1\) when
We point out some other irreducible symplectic 4-manifolds with cyclic fundamental group whose manufacturing procedure is verbatim to the proof of the last theorem. Other possible choices of Luttinger surgeries to skip to produce only the infinite cyclic fundamental groups were outlined in [4].

Example 46. 

- \((c, \chi) = (7, 1)\): In the process to produce an exotic copy of \(\mathbb{CP}^2\) blown up twice \([4]\), if one chooses to not perform a Luttinger surgery one obtains \(X_{2,3}^2\). This was already observed in the same paper. One can then go ahead and used the "skipped" surgery and apply \((a_2' \times c', c', +1/p)\) to \(X_{2,3}^2\) in order to obtain \(X_{1,2}^p\).

Thus, out of this process we obtain an irreducible symplectic 4-manifolds with Euler characteristic \(e = 5\) and signature \(\sigma = -1\); we are able to conclude that the manifold \(\mathbb{CP}^2 \# 2\mathbb{CP}^2 \# L(p, 1) \times S^1\) has the \(\infty\)-property. More details are given in section 5.

- \((c, \chi) = (5, 1)\): By proceeding similarly, in the process to build an exotic \(\mathbb{CP}^2 \# 4\overline{\mathbb{CP}^2}\), if we skip the surgery that kills the last generator, one obtains \(X_{2,5}^2\) and by applying \((a_2'' \times \alpha'_4, \alpha'_4, -1/p)\) one obtains \(X_{1,4}^p\). In particular, \(\mathbb{CP}^2 \# 4\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1\) has the \(\infty\)-property.

- \((c, \chi) = (8 \chi - 1, \chi)\): This region has already been filled in theorem 45 above. We use it now to exemplify how the choice of Luttinger surgeries is not unique. In \([4]\), a different Luttinger surgery was chosen to be skipped in order to produce manifolds with infinite cyclic fundamental group. In that paper, \(X_{2\chi, 2\chi+1}^p\) was built by not applying the surgery \((a_2' \times c'_1, c'_1, +1)\).

One could go on then and apply \((a_2' \times c'_1, c'_1, +1/p)\) to obtain \(X_{2\chi, 2\chi+1}^{2p}\). These choices are different from ours.

- \((c, \chi) = (13, 2)\): by skipping \((a_2' \times c', c', +1/p)\) one obtains \(X_{2,7}^{2p}\). This manifold is non-spin, it has characteristic numbers \(e = 11\) and \(\sigma = -3\). The theorem of Hambleton-Teichner imply that it is homeomorphic to

\[4\overline{\mathbb{CP}^2} \# 7\overline{\mathbb{CP}^2} \# S^1 \times S^3.\]

We proceed to apply the surgery \((a_2' \times c', c', +1/p)\) on the unused Lagrangian tori with the given Lagrangian framing in order to obtain \(X_{3,6}^{2p}\). This manifold is non-spin as well. Its universal cover has signature \(\sigma = -3p\), which by Rohlin’s theorem, implies it is non-spin either. Hambleton-Kreck’s result says that \(X_{3,6}^{2p}\) is homeomorphic to

\[3\overline{\mathbb{CP}^2} \# 6\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1.\]
(c, \chi) = (11, 2): by skipping a surgery one obtains \( X^Z_{4,9} \) and by applying \((a''_2 \times \alpha_4', -1/p)\) one obtains \( X^Z_{3,8} \).

• (c, \chi) = (9, 2): \( X^Z_{4,11} \) and by applying \((a''_2 \times \alpha_4', -1/p)\) one obtains \( X^Z_{3,10} \).

The choice of Luttinger surgery to skip is not unique. In the next section we exemplify this phenomena. It is unknown if the resulting manifolds, independently of the chosen Luttinger surgeries, are diffeomorphic.

Other more general results of Baldridge and Kirk are also worth mentioning. Their efforts go into deriving an upper bound for the minimum of the Euler characteristic and signature in terms of a presentation for \( \pi_1 \). In [9] for example, theorem 6 of that paper provides an exotic symplectic irreducible version of \( 4\mathbb{CP}^2 \# 20\mathbb{CP}^2 \# S^3 \times S^3 \) and one of \( 5\mathbb{CP}^2 \# 29\mathbb{CP}^2 \# L(p,1) \times S^1 \). Theorem 24 of the same authors in [11] produces an exotic \( 6\mathbb{CP}^2 \# 10\mathbb{CP}^2 \# S^1 \times S^3 \) and an exotic \( 7\mathbb{CP}^2 \# 13\mathbb{CP}^2 \# L(p,1) \times S^1 \), both of them irreducible and symplectic. Recently, Paul Kirk [?] communicated to the author that Jonathan Yazinski has improved the generic bound for arbitrary fundamental groups. As a corollary to his result, one obtains an exotic \( 6\mathbb{CP}^2 \# 8\mathbb{CP}^2 \# S^1 \times S^3 \), an exotic \( 7\mathbb{CP}^2 \# 9\mathbb{CP}^2 \# L(p,1) \times S^1 \). All of them are irreducible and symplectic.

4.1. Manufacturing Manifolds via Telescoping Triples. In this section, the telescoping triples \((X, T_1, T_2)\) built in [6] will be employed on the manufacturing procedure of the exotic manifolds. The fundamental group of the manifold \( X \) is \( \mathbb{Z} t_1 \oplus \mathbb{Z} t_2 \). One is able to think as the Lagrangian push-off \( m_{T_2} \) being responsible for the \( \mathbb{Z} t_1 \) factor and the Lagrangian push-off \( l_{T_1} \) responsible for the \( \mathbb{Z} t_2 \) factor. To produce an infinite family \( \{X_n\} \) of irreducible 4-manifolds with infinite cyclic fundamental group, it suffices to apply a single torus surgery: either \((T_1, l_{T_1}, +n/1)\) and obtain \( \pi_1 = \mathbb{Z} t_1 \) or One could apply \((T_2, m_{T_2}, +n/1)\) as well and get \( \pi_1 = \mathbb{Z} t_2 \). The family produced in both cases as a unique symplectic member for \( n = 1 \).

In order to obtain 4-manifolds with finite cyclic fundamental group of order \( p \), one does need to apply two surgeries. Start by applying \((T_1, l_{T_1}, +1/p)\) and then \((T_2, m_{T_2}, +n/1)\). The first surgery is a Luttinger surgery and it provides us with a manifold with fundamental group \( \mathbb{Z} t_1 \oplus \mathbb{Z} p t_2 \). The second surgery is a general torus surgery. It has two duties: kill the \( \mathbb{Z} t_1 \) factor in \( \pi_1 \) and produce an infinite family of \( \{X_n\} \) of irreducible 4-manifolds where \( X_1 \) is the only member having a symplectic structure.

One obtains several organical results by applying the previous recipe to the telescoping triples. By using corollary 9 in [6], one obtains that all manifolds in the points \((c^2, \chi_h) = (6+8g, 1+g)\) (for a non-negative integer \( g \)) of the plane have the \( \infty \)-property. In this case the manifolds \( X^Z_{1+2g,3+2g} \) and the infinite family \( \{X_n\} \) come out of the manifold \( B_g \).
Proposition 47. For each \( g \geq 0 \) there exists a minimal symplectic 4-manifold \( X_{1+2g,3+2g}^Z \) and an infinite family of pairwise non-diffeomorphic non-symplectic 4-manifolds \( \{X_n\} \), all of them homeomorphic to
\[
(1 + 2g)\mathbb{CP}^2 \# (3 + 2g)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1.
\]

We need to take care of the topological prototype. The two surgeries already provided us with the dial to change the smooth structure at will.

Proof. From the characteristic numbers of these manifolds we get \( b_2^+ = 1 + 2g \) and \( b_2^- = 3 + 2g \). They all have an odd intersection form. We claim they are of \( \omega_2 \)-type I). Their universal cover is not spin by Rokhlin’s theorem (this argument leaves out the \( n = 0 \pmod{8} \) cases). Thus Hambleton-Kreck’s criteria say that they are homeomorphic to the chosen topological prototype.

\[ \square \]

Similarly we have

Proposition 48. For each \( g \geq 1 \) there exists a minimal symplectic 4-manifold \( X_{2+2g,4+2g}^Z \) and an infinite family of pairwise non-diffeomorphic non-symplectic 4-manifolds \( \{X_n\} \), all of them homeomorphic to
\[
(2 + 2g)\mathbb{CP}^2 \# (4 + 2g)\overline{\mathbb{CP}^2} \# S^3 \times S^1.
\]

We build the following manifold in order to illustrate the procedure that will allow us to fill in larger regions in the following section.

Proposition 49. There exists a minimal symplectic manifold \( X_{3,5}^Z \) homeomorphic to
\[
3\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1
\]

and an irreducible symplectic manifold \( X_{4,6}^Z \) homeomorphic to
\[
4\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2} \# S^3 \times S^1.
\]

We take the manifold \( \hat{B} \) described in page 18 of [6]. It contains a symplectic torus with Lagrangian push-offs \( mT_1 = 1 \) and \( lT_1 = t_2 \) and \( \pi_1(\hat{B} - (F \cup F \cup H_1)) = \mathbb{Z} t_2 \). All our computations on the effect of the surgeries on the fundamental group are taken from lemma 16 and the discussion that follows in [6]. We apply the surgeries 3, 4, 1, 7 and 6 exactly as they did to obtain a manifold whose fundamental group consists of only two generators.

Proof. Consider the symplectic sum \( S := \hat{B} \# T_1 = S_8 \) along the gluing maps that
\[ m_8 \mapsto lT_1, \ l_8 \mapsto m_8^{-1}. \]

The characteristic numbers are \( e = 7 \) and \( \sigma = -2 \).

The Seifert-Van Kampen theorem says that the fundamental group is generated by \( t_2, x_1, y_1, x_2, a_1, b_1, a_2, b_2 \). Let us start by getting rid of 7 generators. More precisely, we describe the manufacturing process of an intermediate manifold whose
fundamental group is generated by $a_1$ and $b_1$.

The fact that the meridian $m_5$ is trivial in $\pi_1(B - (F \cup F \cup H_1))$ implies that the meridian $\mu_8$ of the tori $S_8$ employed in the symplectic sum is trivial in $\pi_1(S - F)$. By our choice of gluing map, $l_8$ is trivial as well and $m_8$ is expressed as $t_2$ in the generators. Moreover, we have that $l_8$ is conjugate of $b_2^{-1}$ which implies that $b_2$ was killed during the symplectic sum as well. Since $m_8 = y_2$, then $y_2 = t_2$. At this point we have unveiled that actually the numbers of generators of the fundamental group can be lowered immediately to 7. Now the surgeries come into play.

By applying $(S_4, m_3, +1)$, we introduce the relation $x_1^{-1} = [b_2^{-1}, y_1^{-1}]$ and, since $b_2 = 1$, it kills $x_1 = 1$. This enables the use of the surgeries $S_2$, $S_4$ in the sense that $\mu_2$ and $\mu_4$ contain the trivial element $x_1 = 1$. Perform $(S_2, m_2, +1)$ to kill $y_1$ and $(S_4, l_4, +1)$ to kill $a_2$. This enables the use of $S_1$ since $\mu_1 = 1$: we mention this surgery, because we will use it to obtain our desired fundamental group. But now we keep on killing the remaining undesired generators.

By applying $(S_7, m_7, +1)$, the relation $x_2^{-1} = [b_2 a_2^{-1} b_2^{-1}, y_2^{-1}]$ is introduced; it kills $x_2 = 1$ and $\mu_6 = 1$. Now, surgerying with $(S_6, m_6, +1, m_7, +1)$ takes down $y_2 = 1$ and $\mu_5 = 1$. Let $M$ be the manifold obtained at this point.

As it was mentioned before, we use the surgery $(S_1, l_1, +p/1)$ on $M$ to obtain the relation $a_1^p = 1$. At this point we point out the fact that $a_1$ and $b_1$ commute since $1 = [a_1, b_1][a_2, b_2] = [a_1, b_1]$. Finally, we apply a more general $(S_5, l_5, +n/1)$ torus surgery to produce an infinite family $\{X_n\}$. The manifold $X_{3,5}$ would be the member $X_1$ of this bunch.

Once again their minimality follows from Usher’s theorem and their irreducibility from Hamilton-Kotschik’s result.

All the members of the family produced have odd intersection form, thus they are all either of $\omega_2$ - type I) or type III). The corresponding universal coverings will have $\sigma = -2p$: Hambleton-Kreck’s criteria imply that they are homeomorphic (except for the cases $p \equiv 0 \mod 8$).

It is worth noticing that we went to all this trouble with the surgeries to address the botany problem for $(c_1, \chi_b) = (14, 2)$ as well. One could also use the exact same surgeries in [I] and then twist the last surviving generator with the corresponding surgery.

The telescoping triples of theorem 11 and theorem 12 produce $X_{2,6}^{Z_p}$, $X_{2,8}^{Z_p}$, $X_{1,5}^{Z_p}$ and $X_{1,7}^{Z_p}$. This addresses the points $(c^2, \chi) = (8, 1)$ and $(-2, 1)$.

**Proposition 50.** Let $k \in \{5, 7\}$. The manifolds $\mathbb{C}P^2 \# k \mathbb{C}P^2 \# L(p, 1) \times S^1$ and have the $\infty$-property.

4.2. More Examples. In [II], a minimal symplectic 4-manifold $X_1$ with fundamental group $\mathbb{Z}$ was constructed. This manifold provides a smaller substitute for
$E(1)$ to be used in symplectic sums when only one generator is desired to be killed. By gluing either $X_1$ or $E(1)$ with the manifolds from the telescoping triples given in [6] along $T_1$ or along $T_2$ accordingly, one is able to fill in the following points for $q = 1, 2, 3, 4, 5, 6$

$$(c_1^2, \chi_h) = (2q, 2)$$

and the regions

$$(c_1^2, \chi_h) = (12 + 8g, 2 + g) \text{ and } (c_1^2, \chi_h) = (6 + 8g, 2) \text{ for } g \geq 1.$$  

To make it visually clear, the constructions are collected into the next table. The columns are arranged as follows. The first one indicates the corresponding symplectic sum with either $X_1$ or $E(1)$. The second and third columns display the manifold of infinite cyclic fundamental group that we obtain right out of the symplectic sum and its finite cyclic fundamental group brother one obtains after applying $+1/p$ Luttinger surgery on $T_2$. The chosen notation immediately gives away the Euler characteristic and signature of the manifolds. The last column indicates the coordinates on the plane.

**Table 1. Putting the pieces together**

| Symplectic Sum | $\pi_1 = \mathbb{Z}$ | $\pi_1 = \mathbb{Z}_p$ | $(c_1^2, \chi_h)$ |
|---------------|---------------------|---------------------|-------------------|
| $D\#_{T_1=T} X_1$ | $X_{4,12}^Z$ | $X_{4,11}^{Z_p}$ | $(8, 2)$ |
| $C\#_{T_1=T} X_1$ | $X_{4,10}^Z$ | $X_{3,9}^{Z_p}$ | $(10, 2)$ |
| $B\#_{T_1=T} X_1$ | $X_{4,8}^Z$ | $X_{3,7}^{Z_p}$ | $(12, 2)$ |
| $B_g\#_{T_1=T} X_1$ | $X_{4,8+2g,8+2g}^Z$ | $X_{3,7+2g,7+2g}^{Z_p}$ | $(12 + 8g, 2 + g)$ |
| $D\#_{T_1=T} E(1)$ | $X_{4,18}^Z$ | $X_{3,17}^{Z_p}$ | $(2, 2)$ |
| $C\#_{T_1=T} E(1)$ | $X_{4,16}^Z$ | $X_{3,15}^{Z_p}$ | $(4, 2)$ |
| $B\#_{T_1=T} E(1)$ | $X_{4,14}^Z$ | $X_{3,13}^{Z_p}$ | $(6, 2)$ |
| $B_g\#_{T_1=T} E(1)$ | $X_{4,14+2g,14+2g}^Z$ | $X_{3,13+2g,13+2g}^{Z_p}$ | $(6 + 8g, 2)$ |

The gathering of these constructions yields

**Proposition 51.** Let $k \in \{5, 6, 7, 8, 9, \ldots, 12, 14, 16, 18\}$. The manifolds

- $3\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1$
- $4\mathbb{CP}^2 \# (k + 1)\overline{\mathbb{CP}^2} \# S^1 \times S^3$

have the $\infty$-property.

Filling in the points $(c_1^2, \chi_h) = (19 - k, 2)$. In the same spirit, the points with shape

**Lemma 52.** Let $g \geq 1$. The manifolds

- $(3 + 2g)\mathbb{CP}^2 \# (7 + 2g)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1$
- $(4 + 2g)\mathbb{CP}^2 \# (8 + 2g)\overline{\mathbb{CP}^2} \# S^1 \times S^3$

have the $\infty$-property.

Thus, the points $(c_1^2, \chi_h) = (12 + 8g, 2 + g)$ are filled in.
For the remaining part of the section, we will take the second path we mentioned at the beginning to produce more manifolds. This extends the efforts set on the simply connected case into cyclic fundamental groups. We remind the reader that the second road has a starting point a simply connected irreducible symplectic manifold $X$ which contains a symplectic torus of self-intersection 0 (or a symplectic surface of self-intersection 0 in general) with a and simply connected complement inside $X$. One builds the symplectic sum with a minimal symplectic 4-manifold with non-trivial fundamental group and apply Luttinger surgeries to it in order to obtain a a manifold with cyclic fundamental group.

**Proposition 53.** Let $Y$ be a minimal symplectic 4-manifold which contains a symplectic torus $T$ of self intersection 0. Assume $\pi_1(Y) = 1 = \pi_1(Y - T)$. Then there exists an infinite family of pairwise non-diffeomorphic irreducible 4-manifolds $\{X_n\}$ which only has one symplectic member. Moreover, all of its members can be chosen to have as characteristic numbers one of the following three choices:

1. $e = e(Y)$ and $\sigma = \sigma(Y)$;
2. $e = e(Y) + 4$ and $\sigma = \sigma(Y)$;
3. $e = e(Y) + 6$ and $\sigma = \sigma(Y) = -2$.

The next procedure will be followed to obtain the claimed manifolds. First, build the symplectic sum of $Y$ and a minimal non-simply connected 4-manifold $\tilde{X}$. The manifold $\tilde{X}$ must have the required characteristic numbers, enough Lagrangian tori with geometrically dual tori to surger that are disjoint from the surface involved in the symplectic sum and the map $\pi_1(T) \to \pi_1(\tilde{X})$ must not be surjective. By modifying our chosen $\tilde{X}$, one obtains the possible characteristic numbers.

The symplectic sum kills some of the generators. To obtain our cyclic groups, the rest of the generators will be dealt with by the surgeries. The existence of the infinity family will follow from applying a general torus surgery at the corresponding point and corollary 3 in [19].

**Proof.** Assume we want to build a manifold with characteristic numbers as in 2). Consider the minimal symplectic manifold $Z$ of proposition in section 3. This manifold has $e = 4$ and $\sigma = 0$. Since $S_8 \subset Z$ is a homologically essential Lagrangian torus, the symplectic sum can be perturbed so that $S_8$ becomes symplectic, while all the other tori stay Lagrangian. Consider the symplectic sum $V$ of $Y$ and $Z$ along the tori $T$ and $S_8$. The manifold $V$ is minimal by Usher’s theorem. Its characteristic numbers are $e(V) = e(Y) + e(Z) = e(Y) + 4$ and $\sigma(V) = \Sigma(Y) + \sigma(Z) = \sigma(Y)$.

The fundamental group of $V$ is generated by $x_1, y_1, x_2, y_2, a_1, b_1, a_2, b_2, g_1, \ldots, g_n$. In this notation the $g_i$’s represent the meridian of the torus. Although we have an specific commutator representing the meridian, we will not use for what follows; thus our choice of notation. Furthermore, since $\pi_1(X - T) = 1$ and the meridians are identified, we have that the symplectic sum kills all the these $g_i$’s as well as two generators $y_2$ and $b_2$. In particular, the normal subgroup of $\pi_1(V)$ generated by
the meridian and the corresponding relations is trivial. We proceed to show how the needed generators are killed via surgeries.

We start by applying \((S_3, l_3, +1)\) which introduces the relation \([b_2^{-1}, y_1^{-1}] = a_2^{-1}\). Thus, \(a_2\) is killed. The relations introduced by the surgeries \((S_7, m_7, +1)\), \((S_6, l_6, +1)\), \((S_1, m_1, +1)\) and \((S_2, l_2, +n/1)\) kill \(x_2, b_1, x_1\) and \(a_1\) respectively. If one stops at this point, a manifold with infinite cyclic fundamental group generated by \(y_1\) is obtained.

In order to produce minimal symplectic manifolds with finite cyclic fundamental group, one applies \((S_4, m_4, +1/p)\). Now the generator \(y_1\) is subject to the relation \(y_1^p = 1\). Notice that all the surgered Lagrangian tori have geometrically dual tori. We can apply Fintushel-Park-Stern’s corollary to conclude that the manifolds in the family \(\{X_n\}\) are pairwise non-diffeomorphic. Hamilton-Kotschick’s result imply that the manifolds are irreducible.

The other two cases are similar. Assume we want to build a manifold with characteristic numbers as in 3). Consider the manifold \(B\) of section 3 built by Baldridge and Kirk. Perturb the symplectic sum so that \(T_1\) becomes symplectic while \(T_2\) remains Lagrangian. Build the symplectic sum of \(B\) and \(Y\) along \(T_1\) and \(T\). The symplectic sum has \(\pi_1 = \mathbb{Z}t_1, e = e(Y) + 6, \sigma = \sigma(Y) - 2\) and it is an irreducible symplectic manifold. By applying \((T_2, mT_2, +1/p)\), one obtains the finite cyclic fundamental group manifolds. To produce the manifolds with \(e = e(Y)\) and \(\sigma = \sigma(Y)\), one glues in a copy of \(T^4\) (see [10]).

One can alter the above procedure and glue along genus 2 surfaces instead of tori to obtain a similar proposition with the appropriate increase in the characteristic numbers.

**Proposition 54.** Let \(Y\) be a minimal symplectic 4-manifold which contains a symplectic surface \(\Sigma_2\) of genus 2 and self intersection 0. Assume \(\pi_1(Y) = 1 = \pi_1(Y - \Sigma_2)\). Then there exists an infinite family of pairwise non-diffeomorphic irreducible 4-manifolds \(\{X_n\}\) which only has one symplectic member. Moreover, all of its members can be chosen to have as characteristic numbers one of the following three choices:

1. \(e = e(Y) + 10\) and \(\sigma = \sigma(Y) - 2\);
2. \(e = e(Y) + 14\) and \(\sigma = \sigma(Y) - 2\);

**Proof.** The proof is similar to the last proposition. One glues \(B\) and \(Y\) along \(F\) and \(\Sigma_2\) to obtain a symplectic 4-manifold with \(\pi_1 = \mathbb{Z}t_1\); by applying \((T_2, mT_2, +1/p)\) one obtains 4-manifolds with \(\pi_1 = \mathbb{Z}p\). The corresponding torus surgery produces the infinite family. For the choice of manifold for 2), see lemma 17 in [6].
The utility of these two results can be noted right away since they imply that the following manifolds have the $\infty$-property.

- $(b_2^+(Y) + 1)\mathbb{C}P^2 \# (b_2^-(Y) + 1)\mathbb{C}P^2 \# S^1 \times S^3$,
- $(b_2^+(Y) + 2)\mathbb{C}P^2 \# (b_2^-(Y) + 2)\mathbb{C}P^2 \# L(p, 1) \times S^3$,
- $(b_2^+(Y) + 3)\mathbb{C}P^2 \# (b_2^-(Y) + 3)\mathbb{C}P^2 \# S^1 \times S^3$,
- $(b_2^+(Y) + 4)\mathbb{C}P^2 \# (b_2^-(Y) + 4)\mathbb{C}P^2 \# L(p, 1) \times S^3$,
- $(b_2^+(Y) + 5)\mathbb{C}P^2 \# (b_2^-(Y) + 5)\mathbb{C}P^2 \# S^1 \times S^3$,
- $(b_2^+(Y) + 6)\mathbb{C}P^2 \# (b_2^-(Y) + 6)\mathbb{C}P^2 \# L(p, 1) \times S^3$.

This procedure allows us to construct some more manifolds if we specify the chosen simply connected blocks used in the symplectic sums. If we combine the procedure with lemma 2.1 and/or proposition 2.1 in [37] accordingly, we fill out another wide region. If we build the symplectic sum along tori, we have:

**Theorem 55.** For each integer $k$, $10 \leq k \leq 18$, there exists an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible 4-manifolds with the following characteristics.

- Only one member is symplectic,
- the characteristic numbers for all the members of the family can be chosen from the following three pairs: $\chi_h = 2$ and $c_1^2 = 19 - k$; $\chi_h = 3$ and $c_1^2 = 19 - k$ or $\chi_h = 3$ and $c_1^2 = 27 - k$.
- each member of the family contains a symplectic surface $\Sigma_2$ of genus 2 and self-intersection 0. The fundamental group of the complement of $\Sigma_2$ in each manifold is isomorphic to the fundamental group of the ambient manifold.

If instead of using the tori found in the manifold built by J. Park, we used the symplectic genus 2 surfaces we obtain

**Theorem 56.** For each integer $k$, $10 \leq k \leq 18$, there exists an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible 4-manifolds with the following characteristics.

- Only one member is symplectic,
- the characteristic numbers for all the members of the family can be chosen from the following two pairs: $\chi_h = 4$ and $c_1^2 = 33 - k$ or $\chi_h = 5$ and
\[ a_1^2 = 41 - k. \]

- each member of the family contains a symplectic torus \( T \) of self-intersection \( 0 \). The fundamental group of the complement of \( T \) in each manifold is isomorphic to the fundamental group of the ambient manifold.

**Corollary 57.** Let \( k \) be an integer such that \( 10 \leq k \leq 18 \). The following 4-manifolds have the \( \infty \)-property:

- \( 4\mathbb{CP}^2 \#(1 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3, \)
- \( 6\mathbb{CP}^2 \#(3 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3, 5\mathbb{CP}^2 \#(2 + k)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1, \)
- \( 6\mathbb{CP}^2 \#(5 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3, 5\mathbb{CP}^2 \#(4 + k)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1, \)
- \( 8\mathbb{CP}^2 \#(7 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3, 7\mathbb{CP}^2 \#(6 + k)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1, \)
- \( 10\mathbb{CP}^2 \#(9 + k)\overline{\mathbb{CP}^2} \# S^1 \times S^3, 9\mathbb{CP}^2 \#(8 + k)\overline{\mathbb{CP}^2} \# \widetilde{L}(p, 1) \times S^1. \)

Other choices of simply connected 4-manifolds as building blocks found in the literature produce the following manifolds.

**Example 58.** Consider the symplectic manifold \( E'(k) = E(k)_{2,3} \) obtained from the elliptic surface \( E(k) \) by performing two log transforms of order 2 and 3. It contains a torus \( T \) with trivial normal bundle and \( \pi_1(E'(k) - T) = 1 \). By the procedure suggested in the previous proposition one is able to build two manifolds: one with \( \pi_1 = \mathbb{Z} \) and other with \( \pi_1 = \mathbb{Z}_p \), both with \( e = 12k \) and \( \sigma = -8k \).

One gets that the constructed manifolds have \( b_2^+ = 2k, b_2^- = 10k \) and odd intersection form. This amounts to say that the \( \infty \)-property is satisfied by the following manifolds

\[ 2k\mathbb{CP}^2 \# 10k\overline{\mathbb{CP}^2} \# S^1 \times S^3 \text{ and } (2k - 1)\mathbb{CP}^2 \# (10k - 1)\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1. \]

have the \( \infty \)-property. One still needs the \( k \geq 2 \) hypothesis on the infinite cyclic case.

Other choices of simply connected manifolds produce some other manifolds we will need later.

- Using the minimal manifold built by Gompf \( S_{1,1} \) \((21)\) with \( e = 23 \) and \( \sigma = -15 \), we conclude that

\[ 4\mathbb{CP}^2 \# 18\overline{\mathbb{CP}^2} \# S^1 \times S^3 \text{ and } 3\mathbb{CP}^2 \# 17\overline{\mathbb{CP}^2} \# L(p, 1) \times S^1 \]

have the \( \infty \)-property.
• Other manifolds built by Gompf in [21] that will come in handy later on are the following. Applying the proposition to the manifold $R_{2,1}$, which has $e = 21$ and $\sigma = -13$ we obtain that the manifolds

$$4\mathbb{CP}^2 \# 17\mathbb{CP}^2 \# S^1 \times S^3$$

and

$$3\mathbb{CP}^2 \# 16\mathbb{CP}^2 \# \tilde{L}(p, 1) \times S^1$$

have the $\infty$-property.

The same procedure for $R_{2,2}$ with $e = 19$ and $\sigma = -11$ leads to the conclusion that

$$4\mathbb{CP}^2 \# 15\mathbb{CP}^2 \# S^1 \times S^3$$

and

$$3\mathbb{CP}^2 \# 14\mathbb{CP}^2 \# \tilde{L}(p, 1) \times S^1$$

have the $\infty$-property as well.

• B. D. Park constructed a minimal simply connected symplectic 4-manifold with $e = 17$ and $\sigma = -9$ which contains a torus with the required characteristics (cf [13]). Out of his manifold one shows that the manifolds

$$4\mathbb{CP}^2 \# 13\mathbb{CP}^2 \# S^1 \times S^3$$

and

$$3\mathbb{CP}^2 \# 12\mathbb{CP}^2 \# \tilde{L}(p, 1) \times S^1$$

have the $\infty$-property.

5. Small Examples

5.1. Littlest Examples up to now. The construction of manifolds with cyclic fundamental group out of the process to manufacture the exotic $\mathbb{CP}^2 \# 2\mathbb{CP}^2$ was already observed by Akhmedov and Park in [4].

They used the symplectic sum along genus 2 symplectic surfaces with trivial normal bundles of a $\Sigma_2 \times T^2$ which was (Luttinger) surgered 3 times and a $T^4 \# 2\mathbb{CP}^2$ on which two surgeries were applied. We proceed to describe the fundamental group calculations needed for our purposes.

Let $\alpha_i$ ($i = 1, 2, 3, 4$) be the generators of $\pi_1(T^4 \# 2\mathbb{CP}^2)$ and $a'_i, b'_i$ be the standard generators of the genus 2 surface $\Sigma'_2 \subset T^4 \# 2\mathbb{CP}^2$. Assume the inclusion maps the generators as follows:

$$a'_1 \mapsto \alpha_1, b'_1 \mapsto \alpha_2, a'_2 \mapsto \alpha'_3, b'_2 \mapsto \alpha_4.$$

The pairs $\alpha_2 \times \alpha_3$ and $\alpha_1 \times \alpha_4$, $\alpha_1 \times \alpha_3$ and $\alpha_2 \times \alpha_4$ are geometrically dual Lagrangian tori pairwise disjoint from $\Sigma'_2$. Denote by $Z := Z''(1/q, m/r)$ ($q \geq 0$, $r \geq 0$, $m \geq 0$) the manifold resulting from applying the next two surgeries on $T^4 \# 2\mathbb{CP}^2$:

• $(\alpha'_2 \times \alpha'_3, \alpha'_3, -1/q)$ which replaces the commutator relation $[\alpha_1, \alpha_4] = 1$ by $\alpha_3 = [\alpha^{-1}_1, \alpha^{-1}_4]$.

• $(\alpha'_2 \times \alpha'_4, \alpha'_4, -m/r)$ which replaces the commutator relation $[\alpha_1, \alpha_3] = 1$ by $\alpha'_4 = [\alpha_1, \alpha^{-1}_3]^m$.

Thus, the relations $[\alpha_2, \alpha_3] = 1 = [\alpha_2, \alpha_4]$ still hold in the fundamental group of the surgered manifold. Notice that only $m = 1$ results in a symplectic manifold.
We proceed to describe the other building block.

Let \( a_1, b_1 \) (for \( i = 1, 2 \)) and \( c, d \) be the standard generators of \( \pi_1(\Sigma_2 \times T^2) = \pi_1(\Sigma_2) \times \pi_1(T^2) \) respectively. Apply 3 surgeries on \( \Sigma_2 \times T^2 \) and let \( Y_1(0, 1/q) := Y \) denote the resulting manifolds. The chosen surgeries to be applied and the relations they introduced into the fundamental group of the surgered manifold are

\[
\begin{align*}
(1) \quad (a_1' \times c', a_1', -1) & \quad [b_1^{-1}, d^{-1}] = a_1, \\
(2) \quad (b'_1 \times c'', b'_1, -1) & \quad [a_1^{-1}, d] = b_1, \\
(3) \quad (a''_2 \times d', d', -1/q) & \quad [c^{-1}, b_2] = d''.
\end{align*}
\]

The following relations hold in \( \pi_1(Y) \):

\[
\begin{align*}
[b_1^{-1}, d^{-1}] = a_1, & \quad [a_1^{-1}, d] = b_1, & \quad [c^{-1}, b_2] = d'', \\
[a_1, c] = 1, & \quad [b_1, c] = 1, & \quad [a_2, c] = 1, & \quad [a_2, d] = 1, \\
[a_1, b_1] & \quad [a_2, b_2] = 1, & \quad [c, d] = 1.
\end{align*}
\]

One can choose the base-point of \( \pi_1(Y) \) on the boundary of the tubular neighborhood of the genus 2 surface \( \Sigma_2^\prime \) along which the symplectic sum will take place, do that \( \pi_1(Y - \Sigma_2, z) \) is normally generated by \( a_i, b_i \) (\( i = 1, 2 \)) and \( c, d \). All the relations that were described above still hold, except for the commutator \([c, d] \) which is now the meridian of the surface \( \mu_{\Sigma_2} \).

Build the symplectic sum

\[
X_1(m) := Y \#_{\psi} Z''(1, m)
\]

using an orientation reversing diffeomorphism \( \psi \) between the boundaries of the tubular neighborhoods of the genus 2 surfaces. Among the manufactured manifolds, only \( S := X_1(1) \) admits a symplectic structure. The induced map \( \psi_* \) is required to map the generators of \( \pi_1 \) as follows:

\[
a_i \mapsto a_i', b_i \mapsto b_i', \quad i = 1, 2.
\]

We can use these relations to identify the generators of both building blocks: \( \alpha_2 \mapsto b_1, \alpha_4 \mapsto b_2, \alpha_3' \mapsto a_2 \) and \( \alpha_1 \mapsto a_1 \). By lemma 5 of [AP], one has that the fundamental group of \( S \) will be the quotient of \( \pi_1(Y - \Sigma_2^\prime) \ast \pi_1(Z - \Sigma_2) \) by the group generated by \(<a_1 = \alpha_1, b_1 \mapsto \alpha_2, b_2 \mapsto \alpha_4, \mu(\Sigma_2) = \mu(\Sigma_2') \ast \mu(\Sigma_2) \ast \mu(\Sigma_2) >\).

We proceed to see that we are able to kill all the generators of \( \pi_1(S) \) except for \( c \). Using the relations coming from the first two Luttinger surgeries on the \( \Sigma_2 \times T^2 \) piece we have:

\[
a_1 = [b_1^{-1}, d^{-1}] = [b_1^{-1}, [c^{-1}, b_2]^{-1}] = [b_1^{-1}, [b_2, c^{-1}]] \text{ and since } b_1 \text{ and } b_2 \text{ commute as well as } b_1 \text{ and } c, \text{ this implies that } a_1 = 1. \text{ Substituting this again in the relation from the second surgery, we have } b_1 = 1.
\]

From the way it was described above how the generators of the fundamental groups of the building blocks were related through the symplectic sum, this implies \( 1 = b_1 = \alpha_2 \) and \( 1 = a_1 = \alpha_3 \). Combining the last one with \( \alpha_3 = [\alpha_1, \alpha_4] \) and
\[ \alpha_4 = [\alpha_1, \alpha_3] \] we obtain \( 1 = \alpha_4 = b_2 \). This implies \( d = 1 \).

This in particular proves that the meridian \( \mu(\Sigma_2') = [\alpha_3, \alpha_4] \) is trivial. This implies that \( a_2 = \alpha_3^2 = 1 \). This implies that only \( c \) survives to the symplectic sum/surgery process. Rename \( S = X_{2,3}^z \).

We point out that although the process above was only described for the Luttinger surgeries, the relations for a torus surgery will also die the same way. For example, we proved that \( d^q = [c^{-1}, b_2] \) was trivial since \( b_2 \). Thus, by killing \( b_2 \) we can also see that \([c^{-1}, b_2]^m \) is trivial.

Furthermore, we could apply a \((a'_2 \times c', c', +1/p)\) surgery on \( S \) to obtain a manifold sharing the characteristic numbers of \( S \) but with finite cyclic fundamental group. Call such manifold \( X_{1,2}^{z_p} \). We can prove now the following:

**Proposition 59.** There exists an irreducible symplectic 4-manifold \( X_{2,3}^z \) with infinite cyclic fundamental group and an irreducible symplectic 4-manifold \( X_{1,2}^{z_p} \) which is homeomorphic to \( \mathbb{CP}^2 \# 2 \mathbb{CP}^2 \# L(p,1) \times S^1 \).

We note that the lack of a homeomorphism criteria for the infinite cyclic case is the only thing that prevents us to settling the point \((c_1^2, \chi_h) = (7,1)\) in the Geography/Botany plane for both fundamental groups.

**Proof.** We claim that the prototype manifold is

\[ \mathbb{CP}^2 \# 2 \mathbb{CP}^2 \# L(p,1) \times S^1. \]

Let us start by showing that all the manufactured manifolds with finite fundamental group have \( \omega_2 \) - type I). From the signature of the manifolds we conclude that their intersection forms are odd, so they can not be spin and \( \omega_2 \) -type II) is ruled out. The universal cover of \( X_{1,2}^z \) has \( e = 5p \) and \( \sigma = -p \), thus, as long as \( p \neq 0 \) mod 16, it is not spin and type III) is ruled out too. Therefore, all the \( X_{2m-1,2m}^z \) are of \( \omega_2 \) -type I). Hambleton-Kreck’s criteria proves the claim.

To build the next example we use theorem 11 and proposition 12 of [11]. In the previous sense, these manifolds come out of the process to find exotic \( \mathbb{CP}^2 \# 3 \mathbb{CP}^2 \)'s.

**Example 60.** The raw material is the symplectic sum \( Z \) of \( T^2 \times \Sigma_2 \) and \( T^4 \# 2 \mathbb{CP}^2 \) along a genus 2 surface. However our starting point is the construction of Baldridge and Kirk in [11]. We already have a minimal symplectic 4-manifold \( Z \) (we use their calculations and follow notation as of the Lagrangian tori, meridians, generators of the fundamental group, etc.).

**Proposition 61.** There exists irreducible symplectic 4-manifolds \( X_{2,4}^z \) and \( X_{1,3}^z \). The manifold \( \mathbb{CP}^2 \# 3 \mathbb{CP}^2 \# L(p,1) \times S^1 \) has the \( \infty \)-property.
First, we will surgery $Z$ to obtain the minimal symplectic $X^\mathbb{Z}_{2,4}$, $X^\mathbb{Z}_{1,3}$ and an infi nite family of non-symplectic pairwise non-diffeomorphic minimal 4-manifolds for each one.

Then we will prove that the manufactured manifolds with fi nite cyclic fundamental group have the claimed underlying topological prototype by establishing the existence of a homeomorphism using Hambleton-Kreck’s criteria. We follow Baldridge-Kirk’s notation.

Proof. The chosen surgeries and the relations they introduce into the fundamental group are

\begin{align*}
(1) & \quad (T'_1, m'_1, +1) - - - - - - - - - - - - - b_1 = [a_2^{-1}, a_1^{-1}], \\
(2) & \quad (T_1, l_1, -1) - - - - - - - - - - - - - a_1 = [b_1^{-1}, \gamma^{-1}], \\
(3) & \quad (T'_2, l'_2, +1) - - - - - - - - - - - - - b_2 = [b_1, a_2], \\
(4) & \quad (T_3, m_3, -1) - - - - - - - - - - - - - x^{-1} = [b_2^{-1}, \gamma^{-1}], \\
(5) & \quad (T_4, l_4, -1/p) - - - - - - - - - - - - - a_p^p = [x^{-1}, b_2], \\
(6) & \quad (T_2, m_2, -k/1) - - - - - - - - - - - - - y = [x^{-1}, b_1^k].
\end{align*}

The two relations introduced by the first two surgeries take down $a_1$ and $b_1$: $a_1 = [b_1^{-1}, \gamma^{-1}] = [[a_2^{-1}, a_1^{-1}], \gamma^{-1}] = 1$ and by theorem 11 in \[ y \text{ commutes with both } a_1 \text{'s. This results in } a_1 = 1, \text{ which implies } b_1 = 1. \text{ The relation introduced by the surgery on } T'_2 \text{ along } l'_2 \text{ and the fact that } b_1 = 1 \text{ kill } b_2. \]

The fourth surgery (along $T_3$) takes out $x^{-1}$ and the fifth surgery sets $a_p^p = 1$. We kill the last surviving generator by surgering $T_2$ along $m_2$. This establishes

$$
\pi_1 = \mathbb{Z}_p
$$

and $X^\mathbb{Z}_{1,3}$ has been produced. If we apply a $p = 1$ surgery instead and kill the other generator, we can obtain a manifold with $\pi_1 = \mathbb{Z}$, i.e., this different path manufactures $X^\mathbb{Z}_{2,4}$.

Since the surgeries respect the Euler characteristic and the signature we have that $e = 6$ and $\sigma = -2$ and both have an odd intersection form.

Now we proceed on to seeing that we have chosen the correct topological prototype for the homeomorphism type. We have that $b_2^2 = 1$ and $b_2^3 = 3$. Since the intersection form of the manifold is odd, type II) is ruled out. We claim that the manifolds are of type I) indeed. To rule out type III), we observe that the universal cover has Euler characteristic $6p$ and signature $-2p$. For simplicity, assume $p \neq 0 \mod 8$, then by Rohlin’s theorem the universal cover will not be spin. Thus, these manifolds are of $\omega_2$-type I). The homeomorphism follows from the quoted result of Hambleton-Kreck. \hfill \Box

Remark 5. To cover the case when $p$ is a multiple of 8, one needs to go to the definition of type III) and check that the injectivity of the map

The last examples realize the pairs $(c^2_1, \chi) = (7, 1)$ and $(6, 1)$ in the Geography/Botany for irreducible (symplectic) 4-manifolds with cyclic $\pi_1$.

Proposition 62. The 4-manifolds
Remark 6. P. Kirk and S. Baldridge obtained a similar result for $\mathbb{CP}^2 \# 3 \mathbb{CP}^2 \# L(p,1) \times S^1$ (cf. [34]).

5.2. Small Examples with Odd Signature.

Proposition 63. There exist irreducible 4-manifolds $X_{6,11}^Z$, $X_{5,10}^Z$ and $X_{p,1}^Z$ with $\sigma = -5$ that are homeomorphic (respectively) to $6\mathbb{CP}^2 \# 11\mathbb{CP}^2 \# S^1 \times S^3$ and $5\mathbb{CP}^2 \# 10\mathbb{CP}^2 \# L(p,1) \times S^1$.

Proof. Consider the manifold $B$ of the telescoping triple from theorem 7 in [6]. It contains a genus 2 symplectic surface $F$ of square zero and a geometrically dual symplectic torus $H_1$ with square -1. One produces a genus 3 symplectic surface $F_3 \subset B$ of square 1 by symplectically resolving $F \cup H_1$.

We get rid of its self-intersection by the standard procedure. Blow up $B$ at a point on $F_3$ to obtain $\tilde{B}$ and consider the proper transform $\tilde{F}_3$ of $F$. So we have a square zero symplectic surface of genus 3 $\tilde{F}_3 \subset \tilde{B}$.

Li’s theorem tells us that $\tilde{F}_3$ intersects the exceptional sphere in $\tilde{B}$. Therefore, we can find a nullhomotopy for its meridian through the sphere and obtain an isomorphism

$$\pi_1(\tilde{B} - \tilde{F}) \to \pi_1(B).$$

Rename $\tilde{B} = A$ and $\tilde{F}_3 = F_3$. Lemma 10 in [6] says that $\pi_1(A - (F_3 \cup T_1 \cup T_2)) = \mathbb{Z}t_1 \oplus \mathbb{Z}t_2$.

Now produce a genus 3 surface inside $Y = T^2 \times F_2$ by taking the union of the geometrically dual symplectic surfaces $T^2 \times \{p\}$ and $\{q\} \times F_2$ and symplectically resolving it. Note that the inclusion induces a surjective homomorphism $\pi_1(F_3) \to \pi_1(Y)$.

We obtain a symplectic genus 3 surface $F_3$ with self-intersection 2. Get rid of the self-intersection as before to obtain $\tilde{F}_3 \subset \tilde{Y} = T^2 \times F_3 \# 2\mathbb{CP}^2$ as before.

Once again the meridian of $\tilde{F}_3$ is nullhomotopic. Consider the symplectic sum

$$S = A \# F_3 = \tilde{F}_3 \tilde{Y}.$$  

By lemma 5, we know that the generators $a_1, b_1, a_2, b_2, a_3, b_3$ of $\pi_1(F_3)$ are taken to 1 except for $b_2 \mapsto t_2$ and $b_3 \mapsto t_1$. One can use the Lagrangian push-offs suggested in the same lemma and apply the surgeries $(T_2, mT_2, +1)$ and $(T_1, tT_1, +1/p)$ to produce a manifold with $\pi_1 = \mathbb{Z}_p$. One would leave the last one out to obtain $\pi_1 = \mathbb{Z}$.
Usher’s theorem says that $S$ is a minimal manifold and its characteristic numbers can be computed to be $e(S) = 17$ and $\sigma(S) = -5$. The homeomorphism is settled by either Hambleton-Kreck’s theorem or by Hambleton-Teichner. Rename $S$ accordingly.

This settles the point $(c_1^2, \chi_h) = (19, 3)$. One can go ahead and play with the building blocks in the previous process to address the points $(17, 3)$ and $(15, 3)$.

**Proposition 64.** There exist irreducible 4-manifolds $X_{6,13}^Z$ and $X_{5,11}^Z$ with $e = 19$ and $\sigma = -7$ that are homeomorphic (respectively) to $6\mathbb{CP}^2 \# 13\mathbb{CP}^2 \# S^1 \times S^3$ and $5\mathbb{CP}^2 \# 11\mathbb{CP}^2 \# L(p, 1) \times S^1$.

For these manifolds consider the product of two tori $Z = T^2 \times T^2$ and build a genus 3 symplectic surfaces as follows. Take three distinct points $p_1, p_2, p_3 \in T^2$ to indicate the three symplectic surfaces $T^2 \times \{p_1\}, T^2 \times \{p_2\}$ and $\{p_3\} \times T^2$. By resolving symplectically their union, one obtains a genus 3 symplectic surface $F_3' \subset Z$ of square 4. The homomorphism $\pi_1(F_3') \to \pi_1(Z)$ induced by inclusion is surjective. One proceeds to blow up $Z$ at four points along $F_3'$ to obtain a surface with trivial normal bundle. The proper transform $F_3' \subset \tilde{Z} = Z \# 4\mathbb{CP}^2$ is such surface.

**Proposition 65.** There exist irreducible 4-manifolds $X_{6,15}^Z$ and $X_{5,14}^Z$ with $e = 21$ and $\sigma = -9$ that are homeomorphic (respectively) to $6\mathbb{CP}^2 \# 11\mathbb{CP}^2 \# S^1 \times S^3$ and $5\mathbb{CP}^2 \# 10\mathbb{CP}^2 \# L(p, 1) \times S^1$.

Consider $Z = T^2 \times S^2$ and choose three different points $p_1, p_2, p_3 \in S^2$ and $\{q\} \in T^2$. We could use them to point out the four symplectic surfaces $T^2 \times \{p_i\}$ $(i = 1, 2, 3)$ and $\{q\} \times S^2$. Consider their union and symplectically resolve it to obtain a genus 3 symplectic surfaces of square 6 $F_3' \subset Z$. Once again, the homomorphism induced by inclusion $\pi_1(F_3') \to \pi_1(Z)$ is surjective. Blow up $Z$ at six points along $F_3'$ and consider the proper transform $F_3' \subset \tilde{Z} = Z \# 6\mathbb{CP}^2$, which is now the genus 3 symplectic surface with trivial normal bundle needed to build the symplectic sum.

**Corollary 66.** The above manifolds have the $\infty$-property.

6. Region

6.1. Main Region. In this section we address the question of existence of an irreducible symplectic 4-manifold and infinitely many pairwise non-diffeomorphic non-symplectic 4-manifolds having finite cyclic fundamental group that realize the coordinates:

$$(e, \sigma) \text{ when } 2e + 3\sigma \geq 0, e + \sigma \equiv 0 \pmod{4} \text{ and } \sigma \leq -1.$$ 

In other terms, we wish to construct irreducible manifolds with finite cyclic fundamental group realizing all pairs of integers

$$(c_1^2, \chi_h) \text{ when } 0 \leq c_1^2 \leq 8\chi_h - 1.$$
The plan of attack to establish that these manifolds have the \(\infty\)-property is to generalize the main result of [6] (Theorem B and Theorem 22); this settles the region with signature at most \(-2\). Then, we fill in the gaps by generalizing the results contained in [4] to extend the region up to signature at most \(-1\).

We quote the arithmetic lemma 21 of [6] for reference. The purpose of this lemma is to produce the number of each model manifolds \(B < b, C, D, E'(k)\) which is needed to construct a 4-manifold with a specified signature and Euler characteristic.

**Lemma 67.** Given any pair of non-negative integers \((m, n)\) such that

\[
0 \leq m \leq 4n - 1
\]

there exist non-negative integers \(b, c, d, g\) and \(k\) so that

\[
m = d + 2c + 3b + 4g\quad\text{and}\quad n = b + c + d + g + k
\]

and so that \(b \geq 1\) if \(g > 0\).

For an algorithm on how to find such numbers, the reader is referred to the proof of the lemma in [6].

Note that under the chosen coordinates, a 4-manifold with \(c_1^2 = 8\chi + k\) has signature \(k\); therefore, the line \(c_1^2 = 8\chi - 1\) corresponds to manifolds with \(\sigma = -2\).

**Theorem 68.** For any pair \((c, \chi)\) of non-negative integers satisfying

\[
0 \leq c \leq 8\chi - 2
\]

there exists a minimal symplectic 4-manifold with finite cyclic fundamental group

\[
Y = X_{2\chi - 1, 10\chi - c - 1}
\]

with odd intersection form and

\[
c = c_1^2(Y)\quad\text{and}\quad \chi = \chi_k(Y)
\]

Hence \(Y\) is homeomorphic but not diffeomorphic to

\[
(2\chi - 1)\mathbb{CP}^2 \# (10\chi - c - 1)\mathbb{CP}^2 \# L(p, 1) \times S^1.
\]

**Proof.** The beginning of the proof consists of manufacturing the manifolds that realize all the pairs. This task is divided in two with respect to the parity of \(c\). Let us start by considering \(c\) to be even. Set \((m, n) = (1/2c, \chi)\).

By the quoted lemma, we have integers \(b, c, d, g\) and \(k\) so that

\[
m = d + 2c + 3b + 4g\quad\text{and}\quad n = b + c + d + g + k\quad\text{and}\quad g > 0\implies b \geq 1.
\]

The pairs are realized via symplectic sums where the raw materials are the manifolds \(B, B_g, C, D\) and \(E'(k)\) of [4]. The relation between the arithmetic setting on the characteristic numbers and the number of manifolds needed for the correct mix is

1. \(b\) copies of \(B\) if \(g = 0\) and \(b - 1\) copies of \(B_g\) when \(g \geq 1\),
2. \(c\) copies of \(C\), and
3. \(d\) copies of \(D\).

Each one of the manifolds \(B, C\) and \(D\) are taken out of a telescoping triple, thus they contain two essential Lagrangian tori. We will chain them together along these tori via symplectic sums to create a symplectic manifold \(Z\), which will be minimal.
by Usher’s theorem. Notice that proposition 3 of [6] assures us that at each step of the process, the result is a telescoping triple.

If \( g = 0 \), then
\[
Z := B \#_s \cdots B \#_s C \#_s \cdots C \#_s \#_s D \cdots \#_s D.
\]

The notation chosen in [6] indicates the symplectic sum along the appropriate tori. This might involve that the symplectic forms on the manifolds need to be perturbed so that a Lagrangian tori becomes symplectic. This is possible since the building blocks are all members of a telescoping triple. The unused Lagrangian tori will be relabelled \( T_1 \) and \( T_2 \) and we construct indeed a telescoping triple \((Z, T_1, T_2)\).

The characteristic numbers of \( Z \) are computed to be
\[
e(Z) = be(B) + ce(C) + de(D) = 6b + 8c + 10d \quad \text{and} \quad \sigma(B) = -2(b + 2c + 3d).
\]

If \( g \geq 1 \), we take
\[
Z := B_g \#_s B \#_s \cdots B \#_s C \#_s \cdots C \#_s \#_s D \cdots \#_s D
\]
with characteristic numbers
\[
e(Z) = 4g + 6b + 8c + 10d \quad \text{and} \quad \sigma(Z) = -2(b + 2c + 3d).
\]

At this point we point out that the building blocks \( B_g, B, C \) and \( D \) all contain a surface of odd square which is disjoint from the Lagrangian tori used to perform the symplectic sum and the following surgeries on \( Z \). Therefore, all the manifolds coming out of performing either symplectic sums with \( Z \) and a manifold with odd intersection form and/or surgeries on \( Z \) will have odd intersection forms.

We carry on with the process of realizing the the given pairs with irreducible manifolds with the desired fundamental group for. We divide the enterprise by cases.

Case \( k = 0 \): apply \((T_2, mT_2, +1/p)\) Luttinger surgery to \( Z \) to obtain an intermediate manifold \( Z_0 \) with \( \pi_1(Z_0) = \mathbb{Z}_p \oplus \mathbb{Z} \). Then apply \((T_1, lT_1, +n/1)\) torus surgery to kill the \( \mathbb{Z} \) factor on the fundamental group and produce an infinite family \( \{Y_n\} \) whose minimal members all have finite cyclic fundamental group and only \( X_1 \) has a symplectic structure.

Case: \( k \geq 1 \) and one of \( b, c, d \) is positive. Take the symplectic sum
\[
S := Z \#_{T_1} E'(k)
\]
of the manifold of the telescoping triple \((Z, T_1, T_2)\) with the symplectic manifold obtained from the elliptic surface of odd intersection form by applying log transformations \( E'(k) := E(k)_{2,3} \) along \( T_1 \) and a generic fiber of \( E'(k) \). We claim that \( \pi_1(S) = \mathbb{Z} \). Since \( \pi_1(E'(k)) = 1 = \pi_1(E'(k) - T) \) and \( Z \) is part of a telescoping triple, the Seifert- Van Kampen theorem shows that the symplectic sum only killed one generator of \( \pi_1(Z - T_1) \), thus the fundamental group of \( S \) is infinite cyclic. We can then apply a \( +1/p \) Luttinger surgery to obtain our desired symplectic minimal
manifold.

Case: $k \geq 1$ and all $b, c, d$ are zero. Consider the symplectic sum $S$ of the symplectic manifold $E'(k)$ and $T^4$ along a symplectic torus of self-intersection zero. This produces a minimal symplectic manifold with $\pi_1(S) = \mathbb{Z} \oplus \mathbb{Z}$ and the same characteristic numbers since $e(T^4) = 0 = \sigma(T^4)$. Notice that the four Lagrangian tori in $T^4$ can be pushed off and remain Lagrangian within the standard Weinstein neighborhoods while they lie in the complement of some small tubular neighborhoods of the two symplectic tori. One can now apply the usual procedure to obtain the desired fundamental group.

At this point, we would also like to mention that minimal elliptic surfaces $E(n,0)_{p,q}$ with $1 \leq p \leq q$ for which $\pi_1 = \mathbb{Z}_{\gcd(p,q)}$ have already been constructed (see theorem 8.3.12 in [22]).

This concludes part one of the proof.

Suppose that $c$ is odd and consider the region $7 \leq c \leq 8\chi - 11$. Let us reparametrize the region by setting $(c', \chi') = (c - 7, \chi - 2)$. Now the region looks like $0 \leq c' \leq 8\chi' - 2$ and $c'$ is even. Consider the manifold $Z$ of the telescoping triple constructed to realize the pair $(c', \chi')$. Perturb the symplectic form of $Z$ so that $T_1$ becomes symplectic while $T_2$ stays Lagrangian. This is possible since the Lagrangian tori are linearly independent in $H_2(Z; \mathbb{R})$ (cf. [21]).

Consider the simply connected, minimal, symplectic 4-manifold $S_{1,1}$ (lemma 5.5 in [21]) with $e = 23$ and $\sigma = -15$. It contains a symplectic torus $F_1$ with $\pi_1(S_{1,1} - F_1) = 1$. Construct the symplectic sum $S$ of $Z$ with $S_{1,1}$ along $T_1$ and $F_1$. Just like above, one concludes that $\pi_1(S) = \mathbb{Z}$. We apply $(T_2, lT_2, +1/p)$ Luttinger surgery on $S$ to produce a minimal symplectic manifold $X_{c, \chi}$ with $\pi_1 = \mathbb{Z}_p$. Since $c^2_1(S_{1,1}) = 1$ and $\chi_4(S_{1,1}) = 2$, we have that $X_{c, \chi}$ realizes the following coordinates in the geography plane:

$$(c_1^2, \chi_k) = (c, \chi).$$

Let us work now on the region $7 \leq c \leq 8\chi - 11$ while still assuming $c$ to be odd. The process is analogous to the previous paragraph, with only a small change on the ingredients of the construction. Now we re-parametrize by $(c', \chi') = (c - 7, \chi - 2)$; so one has $0 \leq c' \leq 8\chi' - 2$ and $c'$ even. Consider the manifold $Z$ constructed for the corresponding pair $(c', \chi')$.

Now consider the simply connected, minimal, symplectic 4-manifold $X_{1,12}^4$ build by D. Park in [13]. It has $e = 17$, $\sigma = -9$ and contains a symplectic torus $T_{2,4}$ with simply connected complement. Take the symplectic sum

$$S := Z \#_{T_1 = T_{2,4}} X_{3,12}^4$$

along $T_{2,4}$ and $T_1$ (and NOT $T_2$ like it was done in [3]). By using Seifert-van Kampen theorem we conclude that $\pi_1(S) = \mathbb{Z}$. By applying $(T_2, lT_2, +1/p)$ we obtain a minimal symplectic 4-manifold with $\pi_1 = \mathbb{Z}_p$ that realizes the pair $(c, \chi)$. 


This last assertion is true since \( c_1^2(X_{3,12}^1) = 7 \) and \( \chi(X_{3,12}^1) = 2 \).

In order to realize all pairs \((c, \chi)\) with \( c \) odd and within the region \( 21 \leq c \leq 8\chi - 5 \), one proceeds as above but instead of gluing in \( X_{3,12}^1 \), one uses the manifold \( P_{5,8} \) constructed in [6]. It has \( \pi_1 = \mathbb{Z}, e = 14 \) and \( \sigma = -3 \), aka, \( c_1^2 = 21 \) and \( \chi_h = 3 \).

The region \( 21 \leq c = 8\chi - 3 \) is expressed by the pairs \((c, \chi) = (5 + 8k, 1 + k)\) for any \( k \geq 2 \) and it was already covered using telescoping triples in a previous section.

Concerning the homeomorphism types of the constructed manifolds we mention the following. The manifolds constructed in all these regions have odd intersection forms and we know their \( b^+_2 \) and \( b^-_2 \) numbers. For the lines in the plane corresponding to odd signatures and those which are not multiples of 16, one concludes immediately that they all have type I) and that the homeomorphism type is as claimed.

The region corresponding to the manifolds \((2\chi - 1)\mathbb{CP}^2 \# 2\chi\mathbb{CP}^2 \# L(p, 1) \times S^1\) and \( 2\chi\mathbb{CP}^2 \# (2 + 1)\mathbb{CP}^2 \# S^1 \times S^3 \) was filled in the previous section. To conclude the proof, one needs to apply the proper homeomorphism criteria.

The procedure of the proof leaves out several points of the Geography plane. We point them out now and sketch how they are filled.

Remark 7. The previous procedure does not consider the following manifolds

- \((c_1^2, \chi) = (1, 1)\) corresponding to \( X_{2,9}^{\mathbb{Z}} \) and \( X_{1,8}^{\mathbb{Z}} \).

  These manifolds are obtained from the symplectic sum \((T^4 \# 3\mathbb{CP}^2) \# \Sigma_2(T^2 \times S^2 \# 4\mathbb{CP}^2)\) along genus 2 surfaces. The symplectic sum has \( \pi_1 = \mathbb{Z} \oplus \mathbb{Z} \) and the methods described in section 4 produce the manifolds with the given characteristics and their corresponding infinite families.

- \((c_1^2, \chi) = (3, 1)\) corresponding to \( X_{2,7}^{\mathbb{Z}} \) and \( X_{1,6}^{\mathbb{Z}} \).

  These manifolds were built in section 4.

- \((c_1^2, \chi) = (5, 1)\) corresponding to \( X_{2,5}^{\mathbb{Z}} \) and \( X_{1,4}^{\mathbb{Z}} \).

  The construction of the manifold \( X_{2,5}^{\mathbb{Z}} \) was sketched in [4]. Out of it we obtain \( X_{1,4}^{\mathbb{Z}} \) by performing \(+1/p\) Luttinger surgery.

- \((c_1^2, \chi) = (1, 2)\) corresponding to \( X_{4,19}^{\mathbb{Z}} \) and \( X_{3,18}^{\mathbb{Z}} \).

  These manifolds come out of the symplectic sum of the manifold \( S_{1,1} \) (lemma 5.5 in [Go]) constructed by Gompf and a copy of \( T^2 \times \Sigma_2 \) (see [6] for the fundamental group calculations). We then proceed to surger the symplectic sum.
To fill in the next point, one can build and surger the symplectic sum of the minimal symplectic 4-manifold homeomorphic to $3\mathbb{CP}^2 \# 12\mathbb{CP}^2$ containing a torus of self-intersection 0 and simply connected complement build by D. Park in [38].

- $(c_1^2, \chi) = (7, 2)$ corresponding to $X_{4,13}^Z$ and $X_{3,12}^Z$,

The following three points were filled in in the previous section.

- $(c_1^2, \chi) = (15, 3)$ corresponding to $X_{6,15}^Z$ and $X_{5,14}^Z$,
- $(c_1^2, \chi) = (17, 3)$ corresponding to $X_{6,13}^Z$ and $X_{5,12}^Z$,
- $(c_1^2, \chi) = (19, 3)$ corresponding to $X_{6,11}^Z$ and $X_{5,10}^Z$.

A bunch of words are in order at this point. We begin to point out utility of the lemma, its proof and the theorem, by adapting the examples given in [6] to our purposes.

**Remark 8.**
- To construct the corresponding irreducible symplectic manifold for the point $(c_1^2, \chi) = (2, 2)$, we look into the lemma for the pair $(m, n) = (1, 2)$. It indicates that one must now take $b = 1, c = 0, d = 0, g = 9$ and $k = 1$. This results in the symplectic sum $D \# T_2 E'(1)$. By applying the needed surgery one obtains an irreducible smooth structure (which admits a symplectic structure as well) on the prototype $3\mathbb{CP}^2 \# 17\mathbb{CP}^2 \# L(p, 1) \times S^1$.

- We $(c_1^2, \chi) = (78, 11)$. In the terms of the numerical algorithm, this point would be expressed as the pair $(m, n) = (39, 11)$, which is satisfied by taking the integers $b = 1, c = 0, d = 0, g = 9$ and $k = 1$. This indicates that the produced manifold is $S = D \# T_2 E'(1)$. By performing a $+1/p$ Luttinger surgery on the manifold $S$ one obtains an irreducible symplectic manifold $X_{3,17}^Z$ which is homeomorphic, but not diffeomorphic to $21\mathbb{CP}^2 \# 31\mathbb{CP}^2 \# L(p, 1) \times S^1$.

- The integers produced are not unique. If one choses the integers $b = 2, c = 0, d = 1, g = 8$ and $k = 0$ yields a manifold $B_8 \# T_1 \# D$. By performing the corresponding Luttinger surgeries, one obtains a minimal symplectic manifold and an infinite family of pairwise non-diffeomorphic non-symplectic 4-manifolds homeomorphic to $21\mathbb{CP}^2 \# 31\mathbb{CP}^2 \# L(p, 1) \times S^1$.

An interesting question is if these minimal (symplectic) manifolds are different or not.

**Remark 9.** Infinite exotic smooth structures. Most of the manifolds constructed in this theorem contain a nullhomologous torus that can be used as a dial to change the smooth structure.
6.2. **Signature greater or equal than** $-1$. The following theorem (cf. [4], [6]) provides a method to manufacture a myriad of examples from an initial one. It is useful to study the geography problem for 4-manifolds with arbitrary fundamental group.

**Theorem 69.** Let $X$ be a symplectic 4-manifold that contains a symplectic torus $T$ with self-intersection 0. Assume the homomorphism $\pi_1(T) \to \pi_1(X)$ induced by inclusion is trivial. Then for any pair $(c, \chi)$ of non-negative integers satisfying

$$0 \leq c \leq 8\chi - 1$$

there exists a symplectic 4-manifolds $Y$ with $\pi_1(Y) = \pi_1(X)$,

$$c_2^1(Y) = c_2^1(X) + c.$$

Moreover, if $X$ is minimal then $Y$ are minimal as well and they both have an odd indefinite intersection form.

**Remark 10.** One can make a couple of changes on the hypothesis of the theorem to address the geography problem for specific fundamental groups. When one wishes to obtain manifolds with cyclic fundamental group, the following changes in the hypothesis of the theorem are helpful. If one asks for $\pi_1(X)$ to be cyclic and for the homomorphism $\pi_1(T) \to \pi_1(X)$ induced by inclusion to be surjective. More particularly, one can assume $\pi_1(X)$ to be a cyclic group and that the homomorphism induced by inclusion maps one generator of the fundamental group of the torus to the identity and the other to the generator of $\pi_1(X)$. These cases result in a manifold $Y$ with cyclic fundamental group and the same characteristic numbers as in the theorem.

As it was done in [6], one can use an idea of Stipsicz (cf. [41]) to fill in the following regions for $\sigma = 0$ and $\sigma = -1$.

**Theorem 70.** For all the integers $k \geq 45$, there exists a minimal symplectic 4-manifold $X_{2k+1,2k+1}$ with Euler characteristic $4k + 4$, signature $\sigma = 0$ and $\pi_1 = \mathbb{Z}_p$. For all the integers $q \geq 49$, there exists a minimal symplectic 4-manifold $X_{2q-1,2q}$ with Euler characteristic $4q + 1$, signature $\sigma = -1$ and $\pi_1 = \mathbb{Z}_p$.

This results fills in the points of the form $(c_2^1, \chi_h) = (8k+8, k+1)$ for $k \geq 45$ and $(8q - 1, q)$ for $q \geq 49$. The prototype manifolds for these guys are (accordingly):

$$(2k+1)\mathbb{C}P^2 \# (2k+1)\mathbb{C}P^2 \# L(p,1) \times S^1$$

and

$$(2q-1)\mathbb{C}P^2 \# (2q)\mathbb{C}P^2 \# L(p,1) \times S^1.$$

**Proof.** Consider the telescoping triple $(B, T_1, T_2)$. The manifold $B$ contains a symplectic surface $F$ of genus 2 and trivial normal bundle and a geometrically dual surface $G$ of genus 2 and trivial normal bundle as well. The union $F \cup G$ is disjoint from the Lagrangian tori $T_1 \cup T_2$. Perform $+1$ Luttinger surgery on $T_1$ along $l_{T_1}$ to...
kill \( t_2 \). Let \( R \) be the resulting minimal symplectic manifold. Proceed to perturb the symplectic form on \( R \) so that \( T_2 \) becomes symplectic. Concerning the fundamental group, we have \( \pi_1(R - T_2) = \pi_1(R) = \mathbb{Z}t_1 \) and the map induced by inclusion \( \pi_1(T_2) \to \pi_1(R) \) is surjective.

Consider the symplectic sum \( Y \) of the irreducible symplectic manifold homeomorphic to \( \mathbb{C}P^2 \# 3 \mathbb{C}P^2 \) with \( T^2 \times \Sigma \) along a genus two surface. The manifold \( Y \) has fundamental group \( \mathbb{Z} \oplus \mathbb{Z} \) and contains the Lagrangian tori \( T_1, T_2, T_3, T_4 \) (see Theorem 18, [10]); it has \( e = 10 \) and \( \sigma = -2 \). One obtains a symplectic manifold with infinite cyclic fundamental group by applying \(-1 \) Luttinger surgery on \( T_1 \) along \( m_1 \). One can then apply \(-1/p \) Luttinger surgery on \( T_2 \) along \( m_2 \) to obtain a manifold with finite cyclic fundamental group of order \( p \). Denote by \( X \) the minimal symplectic manifold with cyclic fundamental group. There are two symplectic tori \( T_3 \) and \( T_4 \) left unused. They have both trivial normal bundles and trivial meridians in \( X - (T_3 \cup T_4) \) so by a correct choice of gluing map for the symplectic sum, the embedding \( T_3 \to X \) is chosen so that \( \pi_1(T_3) \to \pi_1(X) \) maps one generator to the identity and the other to the generator of \( \pi_1(X) \).

We build the symplectic sum:

\[
Q = X \#_{T_3 = T_2} R.
\]

Notice that the surfaces \( F \) and \( G \) persist in \( Q \) as symplectic surfaces of square zero and geometrically dual. The fundamental group of \( Q \) is cyclic, it has a single generator and the relation it inherits from \( \pi_1(X) \). The characteristic numbers can be computed to be \( e(Q) = 16 \) and \( \sigma(Q) = -4 \); it follows from them that \( Q \) is not a rational nor a ruled surface. The symplectic torus \( T_4 \) coming from the \( X \) piece has the quality that its meridian is trivial. So, the inclusion \( Q - T_4 \subset Q \) induces an isomorphism on fundamental groups.

One can go through the same procedure using the telescoping triple \( A \) of the lemma located in the fourth section of this paper instead of \( B \) and build \( A \#_{T_3 = T_2} X \).

Let us consider the Lefschetz fibration \( H \to K \) over a surface \( K \) of genus 2 constructed in [41] (lemma 2.1). The characteristic numbers of the fibration are \( e = 75 \) and \( \sigma = 25 \). It has a symplectic section of square -1 and the fibers are genus 16 surfaces. This fibration will be used as a building block. To argue that it is minimal, we notice that \( H \) is an algebraic surface; the BMY inequality (see [22] for details) implies that it is holomorphicly minimal. By a result of Hamilton and Kotschick (see [?]) it is minimal from a symplectic point of view as well. Since it lies on the BMY line, \( H \) is not rational nor ruled.

We proceed to construct a genus 18 surface of self-intersection zero from the fiber and section of this Lefschetz fibration. Consider the union of a fiber and section.
We have a surface of genus and with double points. By resolving symplectically, we obtain a genus 18 symplectic surface of square $1 \Sigma' \subset H$. The exact sequence

$$\pi_1(\Sigma') \to \pi_1(H) \to \pi_1(K)$$

implies that the homomorphism $\pi_1(\Sigma') \to \pi_1(H)$ is surjective.

Now let us get rid of the $[\Sigma'] = 1$ point. Blow-up $H$ once along $\Sigma'$ and consider the proper transform $\tilde{\Sigma}' \subset \tilde{H} = H \# \mathbb{P}^2$. Since $H$ was neither taional nor ruled, Li’s theorem implies that every exceptional sphere in $\tilde{H}$ intersects $\tilde{\Sigma}'$. Because the meridian of the surface intersects the exceptional sphere, the necessary nullhomotopy can be built and we have that $\pi_1(\tilde{H} - \tilde{\Sigma}') \to \pi_1(\tilde{H})$ is an isomorphism and $\phi: \pi_1(\tilde{\Sigma}') \to \pi_1(\tilde{H})$ is surjective.

Consider the symplectic sum $S = \tilde{Q} \# \tilde{\Sigma} = \tilde{\Sigma}' \tilde{H}$.

The surjectivity of $\phi$ implies that the map $\pi_1(\tilde{Q}) \to \pi_1(S)$ is surjective too. We need to establish that this last homomorphism is actually an isomorphism. For this we observe that the following: let $b_i$ be a generator for $\pi_1(\tilde{H})$. If we consider the fiber of this element under $\phi^{-1}$ and compose it with the map $\pi_1(F_{18}) \to \pi_1(S)$ as it was indicated in the lemma, we see that $x_i$ is not trivial only in the cases when the inverse image gets mapped either to $b_{18}$. Therefore $\pi_1(S)$ is cyclic.

The characteristic numbers are $e(S) = 176$ and $\sigma(S) = 4$, i.e., $e_1^2(S) = 364$ and $\chi_h(S) = 45).$ Out of these numbers one can conclude (in the absence of 2-torsion, as usual), that these manifolds have odd intersection forms. Another way of noticing this fact is to observe that the manifolds used in the construction have a torus of self-intersection -1. Furthermore, notice that the torus $T_4$ has not been used yet and the map $\pi_1(T_4) \to \pi_1(S)$ induced by inclusion is trivial.

Now we apply theorem 23 of [6] and its extension in [4] to produce the minimal symplectic 4-manifolds with cyclic fundamental group and odd intersection form

$$X_{\mathbb{Z}_p}^{89+2\chi,85+10\chi+c}$$

with characteristic numbers $e_1^2 = 364 + c$ and $\chi_h = 45 + \chi$ for any $(c, \chi)$ in the region $0 \leq c \leq 8\chi - 1$ when $c$ is even. To be able to appreciate better the zero signature quality of the manifolds produced, substitute $c = 8\chi - 4$ for any $\chi \geq 1$:

$$X_{\mathbb{Z}_p}^{89+2\chi,89+2\chi}$$

To produce minimal symplectic 4-manifolds with signature -1 we proceed as follows. Apply Luttinger surgery on $B$ to kill one factor of the fundamental group. Call the resulting manifold $\tilde{B}$ ($\pi_1(B) = \mathbb{Z}$). Build the symplectic sum

$$W = \tilde{B} \#_{T_1 = T} P_{4+2k,4+2k}$$
with the manifold $P_{1+2k,4+2k}$ of remark 1 in [6]. The homomorphism $\pi_1(T) \to \pi_1(P_{1+2k,4+2k})$ is surjective, $\pi_1(P_{1+2k,4+2k} - T) \to \pi_1(P_{1+2k,4+2k})$ is an isomorphism and that $\pi_1(T) \to \pi_1(\tilde{B})$ has image a cyclic summand assure; the gluing map in the symplectic sum $W$ can be chosen in such manner that the map $\pi_1(T_2) \to \pi_1(W) = \mathbb{Z}$ sends one generator to the identity and the other to the generator of $\pi_1(W) = \mathbb{Z}$.

Construct now the symplectic sum $Z = W \#_{T_2 = T_0} S$, where $S$ is the manifold constructed above. Then $Z$ has cyclic fundamental group and by renaming $Z$ accordingly to the fundamental group we produce

$$X^Z_{93+2k,94+2k} \text{ and } X^Z_{94+2k,95+2k}.$$

Their characteristic numbers are $e = 189 + 4k$ and $\sigma = -1$ for any $k \geq 2$.

The results of Hambleton-Kreck and Hambleton-Teichner settle the homeomorphism type of these manifolds (except for $p = -16q$).

6.3. **Non-negative signature.** The last result in the previous section fills in a big region for manifolds with zero signature. In this section, we proceed to fill in regions of the plane which correspond to non-spin manifolds with both zero and positive signature. We address both infinite cyclic and finite cyclic fundamental groups in every result. Our main sources to do so are the results in [41], [8], [37] and [5].

Let us start by using proposition 8 in [8] to fill in the following regions.

**Proposition 71.** Let $n \geq 2$. There exists a symplectic minimal 4-manifold with cyclic fundamental group whose characteristic numbers can be chosen amongst the following three choices:

- $e = 75n^2 + 256n + 130$ and $\sigma = 25n^2 - 68n - 78$; $(c_1^2, \chi_h) = (225n^2 + 298n + 26, 25n^2 + 94n + 13)$
- $e = 75n^2 + 256n + 134$ and $\sigma = 25n^2 - 68n - 78$; $(c_1^2, \chi_h) = (225n^2 + 298n + 30, 25n^2 + 94n + 14)$ or
- $e = 75n^2 + 256n + 136$ and $\sigma = 25n^2 - 68n - 80$; $(c_1^2, \chi_h) = (225n^2 + 298n + 32, 25n^2 + 94n + 14)$.

**Proof.** The manifold $W(n)$ constructed in [8] has a symplectic torus $T_2$ with trivial normal bundle and $\pi_1(W(n) - T) = 1$. We build the symplectic sum of $W(n)$ and a manifold from proposition 50 above along the corresponding tori. The possible characteristic numbers come from the three choices given in proposition 50.

From this proposition one concludes that the manifolds...
have the $\infty$-property.

Now we combine the building blocks of given in our proposition of section 3 above and proposition 2.1 in [37] to obtain the following result.

**Proposition 72.** For each odd integer $m \geq 1$ and $10 \leq k \leq 18$, there exists an irreducible symplectic 4-manifold $Y$ with cyclic fundamental group whose characteristic numbers can be chosen amongst the following options:

1. $\chi(Y) = 25m^2 + 31m + 5$ and $c_1^2(Y) = 225m^2 + 248m + 35 - k$;
2. $\chi(Y) = 25m^2 + 31m + 6$ and $c_1^2(Y) = 225m^2 + 248m + 43 - k$;
3. $\chi(Y) = 25m^2 + 31m + 6$ and $c_1^2(Y) = 225m^2 + 248m + 41 - k$;
4. $\chi(Y) = 25m^2 + 31m + 7$ and $c_1^2(Y) = 225m^2 + 248m + 49 - k$;
5. $\chi(Y) = 25m^2 + 31m + 8$ and $c_1^2(Y) = 225m^2 + 248m + 57 - k$.

Moreover, the manifolds with the first three choices of coordinates contain a symplectic genus 2 surface $\Sigma$ of self-intersection zero; the manifolds from the last two choices contain a symplectic torus $T$ of self-intersection zero and $\pi_1(Y - \Sigma) = \pi_1(Y - T)$.

The characteristic numbers of proposition 2.1 [37] are:

$e = 74m^2 + 124m + 25 + k$

$\sigma = 25m^2 - 5 - k$.

The following manifolds have the $\infty$-property and contain a symplectic genus 2 surface of self-intersection 0.

- $(50n^2 + 94n + 26)\mathbb{C}P^2 \# (25n^2 + 162n + 104)\mathbb{C}P^2 \# S^1 \times S^3$, $(50n^2 + 94n + 25)\mathbb{C}P^2 \# (25n^2 + 162n + 103)\mathbb{C}P^2 \# L(p, 1) \times S^1$;
- $(50n^2 + 94n + 28)\mathbb{C}P^2 \# (25n^2 + 162n + 104 + q)\mathbb{C}P^2 \# S^1 \times S^3$, $(50n^2 + 94n + 27)\mathbb{C}P^2 \# (25n^2 + 162n + 103 + q)\mathbb{C}P^2 \# L(p, 1) \times S^1$ for $q \in \{2, 4\}$. 

- $(50m^2 + 62m + 10)\mathbb{C}P^2 \# (25m^2 + 62m + 15 + k)\mathbb{C}P^2 \# S^1 \times S^3$, $(50m^2 + 62m + 9)\mathbb{C}P^2 \# (25m^2 + 62m + 14 + k)\mathbb{C}P^2 \# L(p, 1) \times S^1$.
- $(50m^2 + 62m + 12)\mathbb{C}P^2 \# (25m^2 + 62m + 17 + k)\mathbb{C}P^2 \# S^1 \times S^3$, $(50m^2 + 62m + 11)\mathbb{C}P^2 \# (25m^2 + 62m + 16 + k)\mathbb{C}P^2 \# L(p, 1) \times S^1$.
- $(50m^2 + 62m + 12)\mathbb{C}P^2 \# (25m^2 + 62m + 19 + k)\mathbb{C}P^2 \# S^1 \times S^3$, $(50m^2 + 62m + 11)\mathbb{C}P^2 \# (25m^2 + 62m + 18 + k)\mathbb{C}P^2 \# L(p, 1) \times S^1$. 

Theorem 74. The following manifolds contain a symplectic torus of self-intersection 0.

\begin{itemize}
  \item $(50m^2 + 62m + 14)\mathbb{CP}^2 \# (25m^2 + 62m + 21 + k)\mathbb{CP}^2 \# S^1 \times S^3$; $(50m^2 + 62m + 13)\mathbb{CP}^2 \# (25m^2 + 62m + 20 + k)\mathbb{CP}^2 \# L(p, 1) \times S^1$.
  \item $(50m^2 + 62m + 16)\mathbb{CP}^2 \# (25m^2 + 62m + 23 + k)\mathbb{CP}^2 \# S^1 \times S^3$; $(50m^2 + 62m + 15)\mathbb{CP}^2 \# (25m^2 + 62m + 22 + k)\mathbb{CP}^2 \# L(p, 1) \times S^1$.
\end{itemize}

**Remark 11.** The main theorem of [8] produces irreducible symplectic exotic smooth structures on

$(50n^2 + 94n + 30)\mathbb{CP}^2 \# (25n^2 + 162n + 124)\mathbb{CP}^2 \# S^1 \times S^3$ and $(50n^2 + 94n + 32)\mathbb{CP}^2 \# (25n^2 + 162n + 134)\mathbb{CP}^2 \# L(p, 1) \times S^1$ and

Which for $n \geq 4$ produces 4-manifolds with positive signature.

Using their own improvements (theorem 14 in [8] or theorem 24 [8]) or the improvement of Yazinski [34]

By going through the proof of theorem 1 [8], one realizes that it is possible to construct smaller examples by changing the corresponding building block in the original paper by other (smaller) constructions that were obtained in obtained in [9], [11], [12] and [31]. The main result of [8] can be improved as the following theorem;

**Theorem 73.** (Baldridge-Kirk-Yasinski). Let $G$ have a presentation with $g$ generators and $r$ relations. For the choices

- $K_e = 12(g + r + 1), K_\sigma = -8(g + r + 1)$
- $K_e = 12 + 8(g + r), K_\sigma = -8 - 4(g + r)$
- $K_e = 10 + 6(g + r), K_\sigma = -2 - 2(g + r)$
- $K_e = 10 + 4(g + r), K_\sigma = -2$

and for each integer $n > 1$, there exists a symplectic 4-manifold $M(G, n)$ with fundamental group $G$, Euler characteristic

$e(M(G, n)) = 75n^2 + 256 + 130 + K_e$

and signature

$\sigma(M(G, n)) = 25n^2 - 68n - 78 + K_\sigma$.

We will build now several manifolds with the smallest Euler characteristic among the 4-manifolds which are currently known to possess more than one smooth structure.

**Theorem 74.** There exists a closed minimal symplectic 4-manifold $X$ with cyclic $\pi_1(X)$ for the following choices of characteristic numbers:

- $e = 94$ and $\sigma = 2$; $(e^2, \chi_h) = (194, 24)$.
- $e = 98$ and $\sigma = 2$; $(e^2, \chi_h) = (202, 25)$ or
- $e = 100$ and $\sigma = 0$; $(e^2, \chi_h) = (200, 25)$.

**Proof.** Theorem 4.1 in [4] builds a manifold $M$ with $e(M) = 94, \sigma(M) = 2$ which contains a symplectic torus $T$ with self-intersection and $\pi_1(M - T) = 1$. Build
the symplectic sum of this manifolds with one of the manifolds from proposition of section 3.3 above. The different choices of characteristic numbers correspond to the three different manifolds that can be involved in the symplectic sum. To check that the different symplectic sums have the other claimed properties is straight-forward.

\[\square\]

**Remark 12.** For convenience to the reader, we would suggest a different process to build a 4-manifold with cyclic $\pi_1$ and $e = 94 \sigma = 2$.

We can now build the symplectic sum of the the 4-manifold produced in theorem 4.2 in [4] with the building block of proposition from section 3.3 to obtain the following result.

**Theorem 75.** There exists a closed minimal symplectic 4-manifold $X$ with cyclic $\pi_1(X)$ for the following choices of characteristic numbers:

- $e = 100$ and $\sigma = 4; (c_1^2, \chi_h) = (212, 26)$
- $e = 104$ and $\sigma = 4; (c_1^2, \chi_h) = (220, 27)$ or
- $e = 106$ and $\sigma = 2; (c_1^2, \chi_h) = (218, 27)$.

**Remark 13.** All the manifolds above contain a symplectic torus of self-intersection zero. Going through the proofs of theorems 4.1 and 4.2 in [4], one sees that the manifolds $M$ and $N$ are obtained by building the symplectic sum of $Y_n(1)$ and the total space $X_2$ of a genus 7 fibration over a surface of genus 2 ($n = 7$ to produce $M$ and $n = 9$ to produce $N$). In both cases, the building block $Y_n(1)$ has plenty of such tori. For example, $Y_9(1)$ contains 14 pairs of geometrically dual Lagrangian tori that are all disjoint from the genus 9 surface used to build the symplectic sum $N = Y_9(1) \# Y_9 X_2$. One could go ahead and use one of these 32 Lagrangian tori to obtain the claimed torus $T$ by perturbing the symplectic form on $N$ so that $T$ becomes symplectic.

Moreover, the homomorphism $\pi_1(T) \to \pi_1(S)$ (where $S$ is one of the manifolds from the last two theorems) factorizes through the respective $\pi_1(Y_n(1'))$. In particular, the images of the generators of $\pi_1(T)$ are trivial.

After using Hambleton-Teichner’s and Hambleton-Kreck’s homeomorphism criterias to the corresponding manifolds. one concludes that the following manifolds enjoy the $\infty$-property:

- $48\text{CP}^2 \# 46\text{CP}^2 \# S^1 \times S^3; 47\text{CP}^2 \# 45\text{CP}^2 \# L(p, 1) \times S^1$
- $49\text{CP}^2 \# 47\text{CP}^2 \# S^1 \times S^3; 48\text{CP}^2 \# 46\text{CP}^2 \# L(p, 1) \times S^1$
- $50\text{CP}^2 \# 50\text{CP}^2 \# S^1 \times S^3; 49\text{CP}^2 \# 49\text{CP}^2 \# L(p, 1) \times S^1$
- $52\text{CP}^2 \# 48\text{CP}^2 \# S^1 \times S^3; 41\text{CP}^2 \# 47\text{CP}^2 \# L(p, 1) \times S^1$
- $54\text{CP}^2 \# 50\text{CP}^2 \# S^1 \times S^3; 53\text{CP}^2 \# 49\text{CP}^2 \# L(p, 1) \times S^1$. 
Let Corollary 77. has an odd intersection form, then the corresponding $X$ intersection form. Moreover, if $X$ there exists a symplectic 4-manifolds $Y$ satisfying $\pi_1(\partial(N_T))$ contains a symplectic torus $T$ of self-intersection 0. Let $N_T$ be a tubular neighborhood of $T$ and $\partial(N_T)$ its boundary. Suppose that the homomorphism $\pi_1(\partial(N_T)) \to \pi_1(X - N_T)$ induced by the inclusion is trivial. Then for any pair of integers $(\chi, c)$ satisfying

$$\chi \geq 1 \text{ and } 0 \leq c \leq 8\chi,$$

there exists a symplectic 4-manifolds $Y$ with $\pi_1(Y) = \pi_1(X)$,

$$\chi_h(Y) = \chi(X) + \chi \text{ and } c_1(Y) = c_1(X) + c.$$ 

Moreover, if $X$ is minimal then $Y$ is minimal as well. If $c < 8\chi$ or if $c = 8\chi$ and $X$ has an odd intersection form, then the corresponding $Y$ has an odd indefinite intersection form.

We use this result now to produce 4-manifolds with signature $\sigma = 0, 1, 2$.

Corollary 77. Let $m$ be an odd positive integer. If $m \geq 49$, then

- $\text{54} \mathbb{CP}^2 \# 52 \mathbb{CP}^2 \# S^1 \times S^3$;
- $\text{53} \mathbb{CP}^2 \# 51 \mathbb{CP}^2 \# L(p, 1) \times S^1$.

We proceed to use these manifolds to fill in regions of the plane corresponding to non-negative signature. For such purposes the following result is very practical (cf. [5]).

Theorem 76. (Akhmedov-Park). Let $X$ be a closed symplectic 4-manifolds that contains a symplectic torus $T$ of self-intersection 0. Let $N_T$ be a tubular neighborhood of $T$ and $\partial(N_T)$ its boundary. Suppose that the homomorphism $\pi_1(\partial(N_T)) \to \pi_1(X - N_T)$ induced by the inclusion is trivial. Then for any pair of integers $(\chi, c)$ satisfying

$$\chi \geq 1 \text{ and } 0 \leq c \leq 8\chi,$$

there exists a symplectic 4-manifolds $Y$ with $\pi_1(Y) = \pi_1(X)$,

$$\chi_h(Y) = \chi(X) + \chi \text{ and } c_1(Y) = c_1(X) + c.$$ 

Moreover, if $X$ is minimal then $Y$ is minimal as well. If $c < 8\chi$ or if $c = 8\chi$ and $X$ has an odd indefinite intersection form, then the corresponding $Y$ has an odd indefinite intersection form.

We use this result now to produce 4-manifolds with signature $\sigma = 0, 1, 2$.

Corollary 77. Let $m$ be an odd positive integer. If $m \geq 49$, then

- $\text{54} \mathbb{CP}^2 \# 52 \mathbb{CP}^2 \# L(p, 1) \times S^1$,
- $\text{(m + 1)} \mathbb{CP}^2 \# \text{(m + 1)} \mathbb{CP}^2 \# S^1 \times S^3$ (with characteristic numbers $(e, \sigma) = (2m + 2, 0)$ and $(c_1^2, \chi_h) = (4m + 4, 1/2(m + 1))$.
- $\text{54} \mathbb{CP}^2 \# 51 \mathbb{CP}^2 \# L(p, 1) \times S^1$,
- $\text{(m + 1)} \mathbb{CP}^2 \# m \mathbb{CP}^2 \# S^1 \times S^3$ (with characteristic numbers $(e, \sigma) = (2m + 1, 1)$ and $(c_1^2, \chi_h) = (4m + 5, 1/2(m + 1))$.

have the $\infty$-property. If $m \geq 47$, then

- $\text{54} \mathbb{CP}^2 \# (m - 2) \mathbb{CP}^2 \# L(p, 1) \times S^1$ and
- $\text{(m + 1)} \mathbb{CP}^2 \# (m - 1) \mathbb{CP}^2 \# S^1 \times S^3$ (with characteristic numbers $(e, \sigma) = (2m, 0)$ and $(c_1, \chi_h) = (4m + 6, 1/2(m + 1))$.

have the $\infty$-property.

Proof. We already know that $\text{48} \mathbb{CP}^2 \# 46 \mathbb{CP}^2 \# S^1 \times S^3$ and $\text{47} \mathbb{CP}^2 \# 45 \mathbb{CP}^2 \# L(p, 1) \times S^1$ have the $\infty$-property. We apply Akhmedov-Park’s result to them. Since there is no margin for confusion, we deal with the infinite cyclic case and the finite cyclic case together. Let $X$ be either one of these two manifolds: $\chi_h(X) = 24$ and $c_1^2 = 194$. Then Akhmedov-Park’s result that there exists a minimal symplectic 4-manifold $Y$ with $\chi_h(Y) = \chi + 24$ and $c + 194$. By Hambleton-Teichner’s criteria in the infinite cyclic fundamental group and by Hambleton-Kreck’s criteria in the finite cyclic fundamental group case, such $Y$ is homeomorphic to
• if $\pi_1(Y) = \mathbb{Z}$: $(2\chi + 48)\mathbb{CP}^2 \#(10\chi - c + 46)\mathbb{CP}^2 \#S^1 \times S^3$ or
• if $\pi_1(X) = \mathbb{Z}_p$: $(2\chi + 47)\mathbb{CP}^2 \#(10\chi - c + 45)\mathbb{CP}^2 \#L(p, 1) \times S^1$.  

By setting the constants from Akhmedov-Park’s theorem to be $c = 8\chi - s$, where $s \in \{0, 1, 2\}$, we produce an irreducible symplectic 4-manifold $Y$ homeomorphic to

• if $\pi_1(Y) = \mathbb{Z}$: $(2\chi + 48)\mathbb{CP}^2 \#(2\chi + 46 + s)\mathbb{CP}^2 \#S^1 \times S^3$ or
• if $\pi_1(X) = \mathbb{Z}_p$: $(2\chi + 47)\mathbb{CP}^2 \#(2\chi + 45 + s)\mathbb{CP}^2 \#L(p, 1) \times S^1$.  

A torus surgery on a nullhomologous torus in $Y$ as explained in [FPS] produces infinite families of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds homeomorphic to $(2\chi + 48)\mathbb{CP}^2 \#(2\chi + 46 + s)\mathbb{CP}^2 \#S^1 \times S^3$ if $\pi_1(Y) = \mathbb{Z}$ or homeomorphic to $(2\chi + 47)\mathbb{CP}^2 \#(2\chi + 45 + s)\mathbb{CP}^2 \#L(p, 1) \times S^1$ if $\pi_1(Y) = \mathbb{Z}_p$.  

Now we address a large region of non-spin 4-manifolds with cyclic fundamental group and signature $\sigma = 3, 4$

**Corollary 78.** Let $m$ be an odd positive integer. If $m \geq 53$, then

- $m\mathbb{CP}^2 \#(m - 3)\mathbb{CP}^2 \#L(p, 1) \times S^1$ and
- $(m + 1)\mathbb{CP}^2 \#(m - 2)\mathbb{CP}^2 \#S^1 \times S^3$ (with characteristic numbers $(e, \sigma) = (2m - 1, 3)$ and $(c_1^2, \chi_h) = (4m + 7, 1/2(m + 1))$.

 have the $\infty$-property. If $m \geq 51$, then

- $m\mathbb{CP}^2 \#(m - 4)\mathbb{CP}^2 \#L(p, 1) \times S^1$ and
- $(m + 1)\mathbb{CP}^2 \#(m - 3)\mathbb{CP}^2 \#S^1 \times S^3$ (with characteristic numbers $(e, \sigma) = (2m - 2, 0)$ and $(c_1^2, \chi_h) = (4m + 8, 1/2(m + 1))$.

 have the $\infty$-property.

**Proof.** We already know that $52\mathbb{CP}^2 \#48\mathbb{CP}^2 \#S^1 \times S^3$ and $51\mathbb{CP}^2 \#47\mathbb{CP}^2 \#L(p, 1) \times S^1$ have the $\infty$-property. We apply Akhmedov-Park’s result to them. Since there is no margin for confusion, we deal with the infinite cyclic case and the finite cyclic case together. Let $X$ be either one of these two manifolds: $\chi_h(X) = 26$ and $c_1^2 = 212$. Then Akhmedov-Park’s result that there exists a minimal symplectic 4-manifold $Y$ with $\chi_h(Y) = \chi + 26$ and $c + 212$. By Hambleton-Teichner’s criteria in the infinite cyclic fundamental group and by Hambleton-Kreck’s criteria in the finite cyclic fundamental group case, such $Y$ is homeomorphic to

• if $\pi_1(Y) = \mathbb{Z}$: $(2\chi + 52)\mathbb{CP}^2 \#(10\chi - c + 48)\mathbb{CP}^2 \#S^1 \times S^3$ or
• if $\pi_1(X) = \mathbb{Z}_p$: $(2\chi + 51)\mathbb{CP}^2 \#(10\chi - c + 47)\mathbb{CP}^2 \#L(p, 1) \times S^1$.  

By setting the constants from Akhmedov-Park’s theorem to be $c = 8\chi - s$, where $s \in \{0, 1, 2\}$, we produce an irreducible symplectic 4-manifold $Y$ homeomorphic to

• if $\pi_1(Y) = \mathbb{Z}$: $(2\chi + 52)\mathbb{CP}^2 \#(2\chi + 48 + s)\mathbb{CP}^2 \#S^1 \times S^3$ or
• if $\pi_1(X) = \mathbb{Z}_p$: $(2\chi + 51)\mathbb{CP}^2 \#(2\chi + 47 + s)\mathbb{CP}^2 \#L(p, 1) \times S^1$.  

A torus surgery on a nullhomologous torus in $Y$ as explained in [19] produces infinite families of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds homeomorphic $(2\chi + 52)\mathbb{CP}^2 \# (2\chi + 48 + s)\mathbb{CP}^2 \# S^1 \times S^3$ if $\pi_1(Y) = \mathbb{Z}$ or homeomorphic to $(2\chi + 51)\mathbb{CP}^2 \# (2\chi + 47 + s)\mathbb{CP}^2 \# L(p,1) \times S^1$ if $\pi_1(Y) = \mathbb{Z}_p$.

Remark 14. We could have employed the other manifolds with bigger Euler characteristics of theorems 74 and 75 to fill in other regions of the geography/botany planes having a bigger manifold as a starting point. We do not know if the 4-manifolds located within the overlapping regions are diffeomorphic.

7. Work in Progress

7.1. Homeomorphism criteria for the smallest manifolds with $\pi_1 = \mathbb{Z}$. Efforts are being oriented towards bringing Hambleton-Teichner’s stability condition/inequality (at least) down to $b_2 - |\sigma| \geq 4$. Notice that this the only part keeping us from concluding that the manifolds of the form $X_{\mathbb{Z},k}$ have the $\infty$-property.

7.2. on the $\omega_2$-types. Some cases where left behind when determining the homeomorphism type for manifolds with finite cyclic group. In particular, the argument to see that the universal cover was not spin (Rohlin’s theorem) leaves out the universal covers whose signature is a multiple of 16. To conclude that the $\omega_2$ type III) does not occur in any case, a further examination is needed. Technical details are on their way.

7.3. Technical details on 2-torsion. Special care needs to be taken when one deals with the SW invariants of the infinite family of manifolds produced whose fundamental group have 2-torsion and which are candidates to be pairwise non-diffeomorphic. This part is under study.

7.4. Non-minimal examples. Build some other non-minimal examples which are not only blow-ups of the manifolds above. Use stable cohomotopy Seiberg-Witten Invariants to detect the exotic smooth structures.

7.5. Other manifolds with abelian $\pi_1$. The geography of irreducible symplectic 4-manifolds with other abelian, yet non-cyclic, fundamental groups is studied in other preprint (yet to appear).

References

[1] A. Akhmedov, Small Exotic 4-Manifolds, Algebraic & Geometric Topology 8 (2008), no.3, 1781-1794.
[2] A. Akhmedov, R. I. Baykur and B. D. Park, Constructing Infinitely Many Smooth Structures on Small 4-Manifolds, Journal of Topology, 2 (2008), 1-13.
[3] A. Akhmedov and B. D. Park, Exotic Smooth Structures on Small 4-Manifolds, Invent. Math., 173 (2008), no. 1, 209 - 223.
[4] A. Akhmedov and B. D. Park, Exotic Smooth Structures on Small 4-Manifolds with Odd Signatures, 2007. Preprint...
[5] A. Akhmedov and B. D. Park, New Symplectic 4-Manifolds with Non-negative Signature, Journal of Gokova Geometry Topology. Volume 2 (2008). 1 - 13.
[6] A. Akhmedov, S. Baldridge, R. I. Baykur, P. Kirk and B. D. Park, *Simply Connected Minimal Symplectic 4-Manifolds with Signature less than -1*, J. Eur. Math. Soc. (to appear), arXiv 0705.0778

[7] D. Auroux, S. K. Donaldson and L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, Math. Ann. 326 (2003) 185 - 203 MR1981618.

[8] S. Baldridge and P. Kirk, *On Symplectic 4-Manifolds With Prescribed Fundamental Group*, Comentari Math. Helv. 82 (2007).

[9] S. Baldridge and P. Kirk, *Symplectic 4-manifolds with arbitrary fundamental group near the Bogomolov-Miyaoka-Yau line*, Journal of Symplectic Geometry 4 (2006).

[10] S. Baldridge and P. Kirk, *Constructions of Small Symplectic 4-Manifolds using Luttinger Surgery*, Journal of Differential Geometry (to appear) arXiv 0703065.

[11] S. Baldridge and P. Kirk, *Luttinger Surgery and Interesting Symplectic 4-Manifolds with Small Euler Characteristic*, arXiv 0701400.

[12] S. Baldridge and P. Kirk, *A symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 3\mathbb{CP}^2$*, Geom. and Topology 12:2 (2008), 919 - 940.

[13] B. D. Park, *Constructing infinitely many smooth structures on $3\mathbb{CP}^2 \# n\mathbb{CP}^2$*, Math. Ann. 322 (2002), no.2, 267 - 278; and Erratum (to appear).

[14] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4. Springer-Verlag, Berlin, 1984, x + 304 pp.

[15] R. Fintushel and R. J. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish J. Math. 19 (1995), 145 - 157.

[16] R. Fintushel and R. J. Stern, *Knots, links and 4-manifolds*, Invent. Math. 134 (1998), no. 2, 363 - 400.

[17] R. Fintushel and R. J. Stern, *Families of simply connected 4-manifolds with the same Seiberg-Witten invariants*, Topology 43 (2004), 1449 - 1467.

[18] R. Fintushel and R. J. Stern, *Six Lectures on Four 4-Manifolds*, Park City Mathematics Institute Graduate Summer School on Low Dimensional Topology (2006).

[19] R. Fintushel, B. D. Park and R. J. Stern, *Reverse Engineering small 4-Manifolds*, Algebraic and Geom. Top. 7 (2007), 2103 - 2116.

[20] R. Fintushel and R. J. Stern, *Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$*, Journal of Topology, 1 (2008), 1-15.

[21] R. E. Gompf, *A New Construction of Symplectic Manifolds*. Ann. of Math. (2) 142 (1995), no. 3, 527 - 595.

[22] R. E. Gompf and A. I. Stipsicz, *4-Manifolds and Kirby Calculus*. Graduate Studies in Mathematics, 20. AMS, Providence, RI, 1999.

[23] I. Hambleton and M. Kreck, *Smooth structures on algebraic surfaces with cyclic fundamental group*, Invt. Math. 91 (1988), 53 - 59.

[24] I. Hambleton and M. Kreck, *Smooth structures on algebraic surfaces with finite fundamental group*, Invt. Math. 102 (1990), 109 - 114.

[25] I. Hambleton and M. Kreck, *Cancellation, Elliptic Surfaces and the Topology of Certain Four-Manifolds*, J. Reine Angew. Math., 444 (1993), 79-100.

[26] I. Hambleton and P. Teichner, *A Non-Extended Hermitian form over $\mathbb{Z}[2]$, Manuscripta Mathematica 94 (1997), 435 - 442.

[27] I. Hambleton, M. Kreck and P. Teichner, *Topological 4-Manifolds with Geometrically 2-dimensional Fundamental Groups*, 2009 Preprint.

[28] M. J. D. Hamilton and D. Kotschick, *Minimality and irreducibility of Symplectic Four-Manifolds*. Int. Math. Res. Not. vol. 2006, article ID 35032, 13pp.

[29] C. Okonek, *Fake Enrique Surfaces*, Topology 27 (1988), 415-427.

[30] M. übke and C. Okonek, *Differentiable structures on elliptic surfaces with cyclic fundamental group*, Comp. Math. 63 (1987), 217-22.

[31] F. Maier, *On the diffeomorphism type of elliptic surfaces with finite fundamental group*, Tulane University PhD Thesis (1987).

[32] V. Krushkal and R. Lee, *Surgery on Closed 4-Manifolds with Free Fundamental Group*, Math. Proc. Camb. Phil. Soc. (2002), 133, 305.

[33] J. Morgan, T. Mrowka and Z. Szabo, *Product formulas along $T^4$ for Seiberg-Witten invariants*. Math. Res. Lett. 2 (1997) 915-929.

[34] P. Kirk, *Private Communication*, 2009.
ON THE GEOGRAPHY AND BOTANY OF IRREDUCIBLE 4-MANIFOLDS WITH ABELIAN FUNDAMENTAL GROUP

[35] A. Liu, *Some new applications of wall crossing formula*, preprint.
[36] K. M. Luttinger, *Lagrangian Tori in $\mathbb{R}^4$*, J. Diff. Geom. 42 (1995) 220-228.
[37] J. Park, *The Geography of Symplectic 4-Manifolds with an Arbitrary Fundamental Group*. Proceedings of the AMS. Vol 135, No. 7 (2007) 2301-2307.
[38] B. D. Park and Z. Szabo. *The geography problem for irreducible spin four-manifolds*, Trans. Amer. Math. Soc. 352 (2000), 3639 - 3650.
[39] R. J. Stern, *Will we ever classify simply-connected smooth 4-manifolds*, Clay Mathematics Institute Proceedings 4 (2000), Floer Homology, Gauge Theory and Low Dimensional Topology, CMI/AMS Book Series, 225 - 240.
[40] R. J. Stern, Talk given at Tulane University, Topology of 4-Manifolds: A Conference in Honor of Ronald Fintushel’s 60th Birthday (2006).
[41] A. I. Stipsicz, *Simply-Connected symplectic 4-manifolds with positive signature*. Proceedings of the 6th Gokova Geometry-Topology Conference. Turkish J. Math. 23 (1999), no. 1, 145-150.
[42] A. I. Stipsicz, *A note on the geography of symplectic manifolds*, Turkish J. Math. 20 (1996), 135 - 139.
[43] A. I. Stipsicz, *The geography problem of 4-manifolds with various structures*, Preprint...
[44] A. I. Stipsicz and S. Szabo, *An exotic smooth structure on $\mathbb{CP}^2 \# 6 \mathbb{CP}^2$*, 2004. Preprint.
[45] M. Usher, *Minimality and Symplectic Sums*, Int. Math. Res. Not. vol. 2006, article ID 49857, 17 pp.
[46] S. Wang, *Smooth structures on complex surfaces with fundamental group $\mathbb{Z}_2$*, Proc. of the AMS vol. 125 no. 1 (1997), 287 - 292.

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