CRITICAL 2D ZAKHAROV-KUZNETSOV EQUATION
POSED ON A HALF-STRIP

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Abstract. An initial-boundary value problem for the critical
generalized 2D Zakharov-Kuznetsov equation posed on the right half-
strip is considered. Existence, uniqueness and exponential decay
rate of global regular solutions for small initial data are established.

1. Introduction

We are concerned with an initial-boundary value problem (IBVP)
for the critical Zakharov-Kuznetsov (ZK) equation posed on the right
half-strip
\[ u_t + u^2 u_x + u_{xxx} + u_{xyy} = 0 \]  (1.1)
which is a two-dimensional analog of the generalized Korteweg-de Vries
(KdV) equation
\[ u_t + u^k u_x + u_{xxx} = 0 \]  (1.2)
with plasma physics applications that has been intensively studied
last years.

Equations (1.1) and (1.2) are typical examples of so-called dispersive
equations attracting considerable attention of both pure and applied
mathematicians. The KdV equation is more studied in this context.
The theory of the initial-value problem (IVP henceforth) for (1.2) is
considerably advanced today.

Although dispersive equations were deduced for the whole real line,
necessity to calculate numerically the Cauchy problem approximating
the real line by finite intervals implies to study initial-boundary study
value problems posed on bounded and unbounded intervals. What concerns (1.2) with \( k > 1, \ l = 1 \), called generalized
KdV equations, the Cauchy problem was studied in and later
in, where it has been established that for \( k = 4 \) (the
critical case) the problem is well-posed for small initial data, whereas
for arbitrary initial data solutions may blow-up in a finite time. The
generalized Korteweg-de Vries equation was studied for understanding
the interaction between the dispersive term and the nonlinearity in
the context of the theory of nonlinear dispersive evolution equations
[12, 13, 15]. In [29], the initial-boundary value problem for the gen-
eralized KdV equation with an internal damping posed on a bounded
interval was studied in the critical case; exponential decay of weak sol-
solutions for small initial data has been established. In [2], decay of weak
solutions in the case \( l = 2, k = 2 \) has been established.

Recently, due to physics and numerics needs, publications on initial-
boundary value problems in both bounded and unbounded domains
for dispersive equations have been appeared [23, 24, 27, 28, 33, 34].
In particular, it has been discovered that the KdV equation posed on
a bounded interval possesses an implicit internal dissipation. This al-
lowed to prove the exponential decay rate of small solutions for (1.2)
with \( k = 1 \) posed on bounded intervals without adding any artificial
damping term [4]. Similar results were proved for a wide class of dis-
persive equations of any odd order with one space variable [7, 23, 24].

The interest on dispersive equations became to extend their study
for multi-dimensional models such as Kadomtsev-Petviashvili (KP) and
ZK equations. We call (1.1) a critical ZK equation by analogy with the
critical KdV equation (1.2) for \( k = 4 \). It means that we could not prove
the existence and uniqueness of global regular solutions without small-
ness restrictions for initial data similarly to the critical case for the KdV
equation [8, 10, 11, 24, 31, 32]. As far as the ZK equation is concern
ed, the results on both IVP and IBVP can be found in [5, 6, 8, 27, 28, 30].
Our work has been inspired by [29] where critical KdV equation with
internal damping posed on a bounded interval was considered and ex-
ponential decay of weak solutions has been established. We must note
that solvability of initial-boundary value problems in classes of global
regular solutions for the regular case of the 2D ZK equation \((uu_x)\) has
been established in [4, 6, 18, 19, 22, 26, 30, 36, 37] for arbitrary smooth
initial data. On the other hand, for the 3D ZK equation, the convective
term \( uu_x \), which is regular for the 2D ZK equation, corresponds to a
critical case. It means that to prove the existence and uniqueness of
global regular solutions one must put restrictions of small initial data
[20, 21, 25].

The main goal of our work is to prove for small initial data the
existence and uniqueness of global-in-time regular solutions for (1.1)
posed on bounded rectangles and the exponential decay rate of these
solutions.
The paper is outlined as follows: Section I is the Introduction. Section 2 contains formulation of the problem and auxiliaries. In Section 3, Galerkin’s approximations are used to prove the existence and exponential decay of strong solutions. In Section 4, regularity of strong solutions their uniqueness and decay are established.

2. Problem and preliminaries

Let \((x, y) \equiv (x_1, x_2) \in \Omega\) and \(\Omega\) be a domain in \(\mathbb{R}^2\). We use the usual notations of Sobolev spaces \(W^{k,p}\), \(L^p\) and \(H^k\) and the following notations for the norms [1]:

\[
\| f \|_{L^p(\Omega)} = \int_{\Omega} |f| \, d\Omega, \quad \| f \|_{W^{k,p}(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \| D^\alpha f \|_{L^p(\Omega)}, \quad p \in (1, +\infty),
\]

\[
\| f \|_{L^\infty(\Omega)} = \text{ess sup}_{\Omega} |f(x, y)|.
\]

Let

\[
\mathcal{D}(\Omega) = \{ f \in C^\infty(\Omega); \text{suppf is a compact set of } \Omega, \}
\]

\[
\mathbb{R}^+ = \{ t \in \mathbb{R}, \ t > 0. \}
\]

The closure of \(\mathcal{D}(\Omega)\) in \(W^{k,p}(\Omega)\) is denoted by \(W^{k,p}_0(\Omega), \ H^k_0(\Omega)\) when \(p = 2\).

Let \(L, B\) be finite positive numbers. Define

\[
D = \{ (x, y) \in \mathbb{R}^2 : x \in (0, L), \ y \in (0, B) \}, \quad Q = D \times \mathbb{R}^+; \quad \gamma = \partial D \text{ is a boundary of } D.
\]

Consider the following IBVP:

\[
Au \equiv u_t + u^2 u_x + u_{xxx} + u_{xyy} = 0, \quad \text{in } Q; \quad (2.1)
\]

\[
u_{\gamma \times t} = 0, \quad t > 0; \quad (2.2)
\]

\[
u_x(L, y, t) = 0, \quad y \in (0, B), \ t > 0; \quad (2.3)
\]

\[
u(x, y, 0) = \nu_0(x, y), \quad (x, y) \in D, \quad (2.4)
\]

where \(\nu_0 : D \to \mathbb{R}\) is a given function.

Hereafter subscripts \(u_x, u_{xx}, \text{etc.}\) denote the partial derivatives, as well as \(\partial_x\) or \(\partial^2_{xx}\) when it is convenient. Operators \(\nabla\) and \(\Delta\) are the gradient and Laplacian acting over \(D\). By \((\cdot, \cdot)\) and \(\| \cdot \|\) we denote the inner product and the norm in \(L^2(D)\), and \(\| \cdot \|_{H^k(D)}\) stands for the norm in \(L^2\)-based Sobolev spaces.

We will need the following result [16, 17].
Lemma 2.1. Let $u \in H^1(D)$ and $\gamma$ be the boundary of $D$. If $u|_\gamma = 0$, then
\[\|u\|_{L^q(D)} \leq \beta \|\nabla u\|^\theta \|u\|^{1-\theta}.\] (2.5)

We will use frequently the following inequalities:
\[\|u\|_{L^4(D)} \leq 2^{1/2} \|\nabla u\|^{1/2} \|u\|^{1/2}, \quad \|u\|_{L^8(D)} \leq 4^{3/4} \|\nabla u\|^{3/4} \|u\|^{1/4}.\]

If $u|_\gamma \neq 0$, then
\[\|u\|_{L^q(D)} \leq C_D \|u\|_{H^1(D)} \|u\|^{1-\theta},\] (2.6)

where $\theta = 2(\frac{1}{2} - \frac{1}{q})$.

Lemma 2.2. Let $v \in H^1_0(0, L)$. Then
\[\|v_x\|^2 \geq \frac{\pi^2}{L^2} \|v\|^2.\] (2.7)

Proof. The proof is based on the Steklov inequality [35]: let $v(t) \in H^1_0(0, \pi)$, then by the Fourier series $\int_0^\pi v^2(t) dt \geq \int_0^\pi v^2(t) dt$. Inequality (2.7) follows by a simple scaling. \qed

Proposition 2.1. Let for a.e. fixed $t$ $u(x, y, t) \in H^1_0(D)$ and $u_{xy}(x, y, t) \in L^2(D)$. Then
\[\sup_{(x, y) \in D} u^2(x, y, t) \leq 2 \left[ \|u\|_{H^1_0(D)}^2(t) + \|u_{xy}\|_{L^2(D)}^2(t) \right] \leq 2 \|u\|_{H^1_0(D)}^2(t) + 2 \|u\|_{H^1_0(D)}^2(t).\] (2.8)

Proof. For a fixed $x \in (0, L)$ and for any $y \in (0, B)$, it holds
\[u^2(x, y, t) = \int_0^y \partial_s u^2(x, s, t) ds \leq \int_0^B u^2(x, y, t) dy + \int_0^B u_y^2(x, y, t) dy \equiv \rho^2(x, t).\]

On the other hand,
\[\sup_{(x, y) \in D} u^2 \leq \sup_{x \in (0, L)} \rho^2(x) = \sup_{x \in (0, L)} \left| \int_0^x \partial_s \rho^2(s) ds \right| \leq 2 \int_0^L \int_0^B (u^2 + u_x^2 + u_y^2 + u_{xy}^2) dx dy \leq 2 \|u\|_{H^2(D)}^2 \cap H^1_0(D).\]

The proof of Proposition 2.1 is complete. \qed
3. Existence theorem

Define the space $W(D)$ with the norm
\[ \|u\|_{W(D)} = \|u\|_{H^2(D) \cap H^1_0(D)} + \|\Delta u_x\|. \]

**Theorem 3.1.** Given $u_0 \in W(D)$ and $D$ such that $u_0|_\gamma = u_0|_{x=L} = 0$ and
\[ \|u_0\| < \min\left(\frac{1}{2}, m\right), \] (3.1)

where
\[ m < \left(\frac{\pi^2}{4(1 + L)^2} \left(\frac{1}{L^2} + \frac{1}{B^2}\right) \left[5 \times 2^7(1 + L)\|u_t(0)\| + 5^2 \times 2^{15}(1 + L)^6\|u_t(0)\|\right]^{-1}\right)^{1/3}, \]
\[ \|u_t(0)\| \leq \|\Delta u_{0x}\| + \|u_{0x}^2\| \leq C\left(\|u_0\|_{H^2(D) \cap H^1_0(D)}\right). \]

Then for all finite positive $B$, $L$ there exists a unique regular solution to (2.1)-(2.4) such that
\[ u \in L^\infty(\mathbb{R}^+, H^2(D)) \cap L^2(\mathbb{R}^+, H^3(D)); \]
\[ \Delta u_x \in L^\infty(\mathbb{R}^+, L^2(D)) \cap L^2(\mathbb{R}^+, H^1(D)); \]
\[ u_t \in L^\infty(\mathbb{R}^+, L^2(D)) \cap L^2(\mathbb{R}^+, H^1(D)) \]
and
\[ \|u\|_{H^2(D)}^2(t) + \|\Delta u_x\|^2(t) + \|u_t\|^2(t) \leq C(\|u_0\|_{W(D)})e^{(-\chi t)}, \quad t > 0, \] (3.2)

where \[ \chi = \frac{\pi^2}{2(1 + L)} \left[\frac{5}{L^2} + \frac{1}{B^2}\right]; \]
\[ \int_{\mathbb{R}^+} \left\{\|u\|_{H^2(D)}^2(t) + \|\Delta u_x\|^2_{H^1(D)}(t) + \|u_x(0, y, t)\|^2_{H^2(0,B)}\right\} dt \leq C(\|u_0\|_{W(D)}, L, B). \] (3.3)

To prove this theorem, we will use the Faedo-Galerkin approximations. Let $w_j(y)$ be orthonormal in $L^2(D)$ eigenfunctions to the following Dirichlet Problem:
\[ w_{jyy} + \lambda_j w_j = 0, \quad y \in (0, B); \quad w_j(0) = w_j(L) = 0; \quad j \in \mathbb{N}. \] (3.4)

Define approximate solutions of (2.1)-(2.4) in the form:
\[ u^N(x, y, t) = \sum_{j=1}^{N} g_j^N(x, t) w_j(y) \] (3.5)
and \( g_j^N(x, t) \) are solutions to the following Korteweg-de Vries system:

\[
\begin{align*}
g_j^N + g_j^{Nxxx} - \lambda_j g_j^N + \int_0^B |u^N|^2 u_j^N w(y) dy &= 0, \\
g_j^N(0, t) &= g_j^N(L, t) = g_j^{Nxx}(L, t) = 0; \quad t > 0, \\
g_j^N(x, 0) &= (u_{0N}, w_j), \quad x \in (0, L),
\end{align*}
\]

where \( u_{0N} = \sum_{i=1}^N \alpha_{iN} w_i \to u_0 \) in \( W(D) \cap H_0^1(D), \quad j = 1, ..., N. \)

Since each regularized KdV equation from (3.6) is not critical, it is known [23, 24, 36, 37] that there exists a unique regular solution of (3.5)-(3.8) at least locally in \( t. \)

Our goal is to obtain global in \( t \) a priori estimates for the \( u^N \) independent of \( t \) and \( N, \) then to pass the limit as \( N \) tends to \( \infty \) getting a solution to (2.1)-(2.4).

**Lemma 3.1.** Under the conditions of Theorem 3.1, the following independent of \( N \) and \( t \) estimates hold:

\[
u^N \text{ is bounded in } L^\infty(\mathbb{R}^+; L^2(D)) \cap L^2(\mathbb{R}^+; H^1(D)) \tag{3.9}\]

and

\[
\|u^N\|^2(t) \leq ((1 + x)^{1/2}, u^N)(t) \leq ((1 + x), u_0^2) e^{-\chi t}
\]

\[
\leq \frac{1 + L}{4} e^{-\chi t}, \quad \text{where } \chi = \frac{\pi^2}{2(1 + L)} \left[ \frac{5}{L^2} + \frac{1}{B^2} \right]. \tag{3.10}\]

**Proof.** **Estimate I.** Multiply (3.6) by \( g_j^N, \) sum up over \( j = 1, ..., N \) and integrate over \( \Omega \times (0, t) \) to obtain

\[
\|u^N\|^2(t) + \int_0^t \int_0^B (u_j^N)^2(0, y, \tau) dy d\tau
\]

\[
= \|u_0^N\|^2 \leq \|u_0\|^2, \quad t > 0. \tag{3.11}\]

**Estimate II.** Write the inner product

\[
2 \left( Au^N, (1 + x)u^N \right)(t) = 0,
\]

dropping the index \( N, \) in the form:

\[
\frac{d}{dt} ((1 + x), u^2)(t) + \int_0^B u_x^2(0, y, t) dy + 3\|u_x\|^2(t) + \|u_y\|^2(t)
\]

\[
= \frac{1}{2} \int_D u^4 dx dy.
\]
Taking into account (2.5) and (3.11), we obtain

\[
\frac{1}{2} \int_D u^4 \, dx \, dy \leq \frac{1}{2}\|u\|_{L^4(D)}^4(t) \leq 2\|\nabla u\|^2(t)\|u_0\|^2(t).
\]

This implies

\[
\frac{d}{dt}((1 + x), u^2)(t) + \frac{1}{2}\|\nabla u\|^2(t) + \left(\frac{1}{2} - 2\|u_0\|^2\right)\|\nabla u\|^2(t) + 2\|u_x\|^2(t) + \int_0^B u_x^2(0, y, t) \, dy \leq 0.
\] (3.12)

Making use of (3.1) and Lemma 2.2, we get

\[
\frac{d}{dt}((1 + x), u^2)(t) + \frac{\pi^2}{2(1 + L)} \left[ \frac{5}{L^2} + \frac{1}{B^2} \right]((1 + x), u^2)(t) \leq 0.
\]

This gives

\[
\|u^N\|^2(t) \leq ((1 + x)^{1/2}, u^N)^2(t) \leq ((1 + x), u_0^2)e^{(-\chi t)} 
\]

\[
\leq \frac{1 + L}{4} e^{(-\chi t)}.
\] (3.13)

Returning to (3.12), we obtain

\[
((1 + x), |u^N|^2)(t) + \int_0^t \int_0^B |u_x^N|^2(0, y, \tau) \, dy \, d\tau + \frac{1}{2} \int_0^t \|\nabla u^N\|^2(\tau) \, d\tau 
\]

\[
\leq ((1 + x), u_0^2), \ t > 0.
\] (3.14)

Moreover, we can rewrite (3.12) as

\[
4\|u_x^N\|^2(t) + \|\nabla u^N\|^2(t) + \int_0^B |u_x^N|^2(0, y, t) \, dy \leq 4|((1 + x)u, u^N)(t)| 
\]

\[
\leq 4\|(1 + x)^{1/2}u^N\|(t)\|(1 + x)^{1/2}u^N\|(t).
\] (3.15)

The proof of Lemma 3.1 is complete. □

**Lemma 3.2.** Under the conditions of Theorem 3.1, the following inequalities are true:

\[
((1 + x), |u^N|^2)(t) \leq C_0 e^{(-\chi t)},
\] (3.16)

\[
((1 + x), |u^N|^2)(t) + \int_0^t \int_0^B (\partial_x^2 u^N)^2(0, y, \tau) \, dy \, d\tau 
\]

\[
+ \frac{1}{2} \int_0^t \|\nabla \partial_x u^N\|^2(\tau) \, d\tau \leq C_0, \ t > 0.
\] (3.17)
It follows from (2.1) that
\[
C_0 = (1 + x, |u_t^N|^2)(0) \leq (1 + L)\|u_t^N(0)\|^2 \\
\leq (1 + L)\left[\|\Delta u_0\| + \|u_0^2u_{0x}\|^2\right].
\] (3.18)

**Proof.** Making use of (2.8) for \(t = 0\), we get
\[
sup_D u_t^2(x, y) = sup_D u^2(x, y, 0) \leq 2(\|u_0\|^2_{L^2(D)} + \|u_{0xy}\|^2) \equiv C_s. \quad (3.19)
\]

Substituting (3.19) into (3.18), we find
\[
(1 + x, |u_t^N|^2)(0) \leq C_0(L, C_s, \|u_0\|_{W(D)}). \quad (3.20)
\]

**Estimate III**

Dropping the index \(N\), write the inner product
\[
2 \left((1 + x)u_t^N, \partial_t (Au^N)\right)(t) = 0
\]
as
\[
\frac{d}{dt} \left((1 + x)u_t^2\right)(t) + \int_B u_{xt}^2(0, y, t) dy + 3\|u_{xt}\|^2(t) + \|u_{xt}\|^2(t)
\]
\[
= 2 \left((1 + x)u^2u_t, u_{xt}\right)(t) + 2(u^2, u_t^2)(t). \quad (3.21)
\]

Making use of Lemmas 2.1, 2.2, (3.15) and taking into account the first inequality of (3.1), we estimate
\[
I_1 = 2 \left((1 + x)u^2u_t, u_{xt}\right)(t) \leq 2(1 + L)\|u_{xt}\|\|u_t\|^2(t)L^1(D)\|u_t\|_{L^1(D)}
\]
\[
\leq 2^{3/2}(1 + L)\|u_{xt}\|\|u_0\|^{1/2}\|\nabla u\|^{3/2}(t)\|u_t\|^{1/2}(t)\|\nabla u_t\|^{1/2}(t)
\]
\[
\leq 2^{3/2}(1 + L)\|\nabla u_t\|^{3/2}(t)\|u_0\|^{1/2}\|\nabla u\|^{3/2}(t)\|u_t\|^{1/2}(t)
\]
\[
\leq 3\delta \|\nabla u_t\|^2(t) + \frac{216(1 + L)^4}{\delta^3}\|u_t\|^2(t)\|\nabla u\|^6(t)\|u_t\|^2(t)
\]
\[
\leq 3\delta \|\nabla u_t\|^2(t) + \frac{222(1 + L)^{11/2}}{\delta^3}\|u_0\|^{5/(1 + x)}\|u_t^2\|^{3/2}(t)\|u_t\|^2(t).
\]

Here and henceforth \(\delta\) is an arbitrary positive number. Similarly,
\[
I_2 = 2(u^2, u_t^2)(t) \leq 2\|u\|^2(t)L^1(D)\|u_t^2(t)L^1(D)
\]
\[
\leq 2\|u\|^2(t)\|\nabla u\|(t)\|u_t\|(t)\|\nabla u_t\|(t)
\]
\[
\leq \frac{\delta}{2}\|\nabla u_t\|^2(t) + \frac{2^4\delta}{\delta^3}\|u_t^2(t)\|\|\nabla u\|^2(t)\|u_t\|^2(t)
\]
\[
\leq \frac{\delta}{2}\|\nabla u_t\|^2(t) + \frac{2^7\delta}{\delta^3}(1 + L)^{1/2}\|u_0\|^{2/2}(1 + x)^{1/2}\|u_t\|(t)\|u_t\|^2(t).
\]
Substituting $I_1, I_2$ into (3.21), making use of Lemma 2.2, taking into account the first inequality of (3.1) and setting $\delta = \frac{1}{5}$, we come to the inequality

\[
\frac{d}{dt}((1 + x), u^2_t)(t) + \int_0^B u^2_{xt}(0, y, t) \, dy + 2\|u_{xt}\|_2^2(t) + \frac{1}{2}\|\nabla u_t\|_2^2(t)
\]

\[
+ \left[ \frac{1}{4(1 + L)} \pi^2 \left( \frac{1}{L^2} + \frac{1}{B^2} \right) - 5 \times 2^7(1 + L)^{1/2}\|u_0\|^3((1 + x), u^2_t)^{1/2}(t) \{1 + 5^2 \times 2^{15}(1 + L)^{5}\|u_0\|^2((1 + x), u^2_t)(t) \} \right]((1 + x), u^2_t)(t) \leq 0. \quad (3.22)
\]

Using (3.1), Lemma 2.2 and standard arguments, we obtain that

\[
\frac{1}{4(1 + L)} \pi^2 \left( \frac{1}{L^2} + \frac{1}{B^2} \right) - 5 \times 2^7(1 + L)^{1/2}\|u_0\|^3((1 + x), u^2_t)^{1/2}(t) \{1 + 5^2 \times 2^{15}(1 + L)^{5}\|u_0\|^2((1 + x), u^2_t)(t) \} > 0, \quad t > 0.
\]

Returning to (3.21) and using the Steklov inequalities (2.7), we can rewrite it as

\[
\frac{d}{dt}((1 + x), u^2_t)(t) + \int_0^B u^2_{xt}(0, y, t) \, dy
\]

\[
+ \frac{\pi^2}{2(1 + L)} \left[ \frac{5}{L^2} + \frac{1}{B^2} \right]((1 + x), u^2_t)(t) \leq 0. \quad (3.23)
\]

This implies

\[
((1 + x), u^2_t)(t) \leq ((1 + x), u^2_t)(0)e^{-\chi t} \leq C_0 e^{(-\chi t)}. \quad (3.24)
\]

Since (3.23) can be rewritten as

\[
\frac{d}{dt}((1 + x), u^2_t)(t) + \int_0^B u^2_{xt}(0, y, t) \, dy + \frac{1}{2}\|\nabla u_t\|_2^2(t) \leq 0,
\]

integrating it we get

\[
((1 + x), u^2_t)(t) + \int_0^t \int_0^B (\partial_{x\tau}^2 u)^2(0, y, \tau) \, dy \, d\tau + \frac{1}{2} \int_0^t \|\nabla \partial_{\tau} u\|^2(\tau) \, d\tau
\]

\[
\leq (1 + x), u^2_t)(0) = C_0. \quad (3.25)
\]

Inequalities (3.24), (3.25) complete the proof of Lemma 3.2. □

Returning to (3.15), we find

\[
\|\nabla u^N\|_2^2(t) \leq C e^{(-\chi t)}, \quad (3.26)
\]

where the constant $C$ does not depend on $N, t > 0$. 

Lemma 3.3. Under the conditions of Theorem 3.1, the following inequality holds:

\[ \| \nabla u_y \|^2(t) + \int_0^B |u_{xyy}(0, y, t)|^2 \, dy \leq C(L, \| u_0 \|, \| u_t(0) \| e^{(-\chi t)}). \]  

\((3.27)\)

Proof. Multiplying \( j \)-th equation of (3.6) by \( \lambda_j \), and summing up the results over \( J = 1, \ldots, N \), dropping the index \( N \), we transform the inner product

\[-2 ((1 + x) \partial^2_y u, Au)(t) = 0 \]

into the inequality

\[ 3 \| u_{xy} \|^2(t) + \| u_{yy} \|^2(t) + \int_0^B u^2_{xy}(0, y, t) \, dy + \]

\[ = \frac{2}{3} \left( (1 + x)(u^3_x, u_{yy})(t) + 2((1 + x)u_{yy}, u_t)(t) \right) \]

\[ = -\frac{2}{3} \left( (1 + x)(u^3_{yx}, u_y)(t) + 2((1 + x)u_{yy}, u_t)(t) \right) \]

\[ \leq \frac{2}{3} \left( u^3_y, u_y + (1 + x)u_{xy} \right) + 2(1 + L)\| u_t \| \| u_{yy} \|(t) \]

\[ \leq \delta \| \nabla u_y \|^2(t) + \frac{(1 + L)^2}{\delta} \| u_t \|^2(t) \]

\[ + 2(u^2, u^2_y(t) + 2(1 + x)u^2_y, u_{xy})(t). \]  

\((3.28)\)

Making use of Lemma 2.1, we estimate

\[ I_1 \equiv 2(u^2, u^2_y)(t) \leq 2\| u \|_{L^4(D)}(t)\| u_y \|_{L^4(D)}(t) \]

\[ \leq 4\| u \| (t)\| \nabla u \| (t)C_D^2 \| u_y \| (t)\| \nabla u_y \| (t) \]

\[ \leq \delta \| \nabla u_y \|^2(t) + \frac{4C_D^4}{\delta} \| u \|^2(t)\| \nabla u \|^4(t), \]

\[ I_2 \equiv 2(1 + x)u^2_y, u_{xy})(t) \leq 2(1 + L)\| u_{xy} \| (t)\| u^2 \|_{L^4(D)}(t)\| u_y \|_{L^4(D)} \]

\[ \leq 2^{3/2}C_D^{1/2} (1 + L)\| u_{xy} \| (t)\| u_y \|^{1/2}(t)\| \nabla u_y \|^{1/2}(t)\| u \|_{L^8(D)}^2 \]

\[ \leq 2^{3/2}C_D^{1/2} (1 + L)\| \nabla u_y \|^{3/2}(t)\| u_0 \|^{1/2}\| \nabla u \|^{3/2}(t) \]

\[ \leq \frac{3\delta}{4} \| \nabla u_y \|^2(t) + \frac{2^{16}(1 + L)^4 C_D^2}{\delta^3} \| u_0 \|^2\| \nabla u \|^6(t). \]
Substituting $I_1, I_2$ into (3.28), we transform it into the following inequality:

$$
\frac{1}{2} \| \nabla u_y \|^2(t) + \left( \frac{1}{2} - \frac{11\delta}{4} \right) \| \nabla u_y \|^2(t) + \int_0^B u_{xy}^2(0, y, t) \, dy \\
\leq \frac{C}{\delta} \left[ (1 + L)^2 \| u_t \|^2(t) + \| u_0 \|^2 \| \nabla u \|^4(t) \right. \\
+ \frac{2^{16}}{\delta^2} (1 + L)^4 \| u_0 \|^2 \| \nabla u \|^6(t) \bigg].
$$

(3.29)

Taking $11\delta = 2$ and making use of (3.14), (3.15), we come to (3.27). The proof of Lemma 3.3 is complete. \(\Box\)

**Lemma 3.4.** Under the conditions of Theorem 3.1, the following inequality holds:

$$
\int_0^t \left[ \| \nabla u_{yy}^N \|^2(\tau) + \int_0^B |u_{xxyy}(0, y, \tau)|^2 \, dy \right] \, d\tau \\
\leq C(D, \| u_0 \|, \| u_t \|(0), \| u_{yy} \|(0)), \quad t > 0.
$$

(3.30)

**Proof.** Estimate IV

Multiply each of the $j$-th equation of (3.6) by $\lambda_j^2$, sum up over $j = 1, ..., N$ and, dropping the index $N$, write the scalar product

$$
2 \left( (1 + x)\partial_y^4 u, Au \right)(t) = 0
$$

in the form

$$
\frac{d}{dt} \left( (1 + x)u_{yy}^2 \right)(t) + 3\| \partial_y^2 u_x \|^2(t) + \| \partial_y^3 u \|^2(t) + \int_0^B u_{xxyy}^2(0, y, t) \, dy \\
= -\frac{2}{3} \left( (1 + x)u_{yy}, (u^3)_{yy} \right)(t).
$$

(3.31)

Denote

$$
I = -\frac{2}{3} \left( (1 + x)u_{yy}, (u^3)_{yy} \right)(t) = \frac{2}{3} \left( u_{yy}, (u^3)_{yy} \right)(t) \\
+ \frac{2}{3} \left( (1 + x)u_{xxyy}, (u^3)_{yy} \right)(t) \equiv I_1 + I_2,
$$

where

$$
I_1 = \frac{2}{3} \left( u_{yy}, (u^3)_{yy} \right)(t) = 4(uu_{yy}^2, u_{yy})(t) + 2(u^2, u_{yy}^2)(t) = I_{11} + I_{12}.
$$
By Proposition 2.1 and (3.13), (3.18), (3.26), (3.27),
\[
\sup_{(x,y)\in D, t>0} u^2(x,y,t) \leq 2 \left[ \|u\|_{H^1_0(D)}^2(t) + \|u_{xy}\|_{L^2(D)}^2(t) \right]
\leq Ce^{-\chi t}, \quad t > 0.
\] (3.32)

Then
\[
I_{11} = 2(u^2, u^2_{yy})(t) \leq 2 \sup_{(x,y)\in D} |u(x,y,t)|^2 \|u_{yy}\|^2(t)
\leq C \left( (1 + x), u^2_{yy}(t) \leq Ce^{-\chi t} \right)
\]
and
\[
I_{12} = 2(u_{yy}, u^2_{yy})(t) \leq 2 \sup_D |u(x,y,t)||u_{yy}||u_{yy}||L^4(D)(t)
\leq 2C^2 D \sup_D |u(x,y,t)||u_{yy}||u_{yy}||H^1(D)(t)
\leq Ce^{-\chi t}.
\]

Similarly,
\[
I_2 = \frac{2}{3}((1 + x)u_{xyy}, (u^3)_{yy})(t) = 2((1 + x)u_{xyy}, (u^2u_{yy})_y)(t)
\leq 4||(1 + x)u_{xyy}, u^2_{yy}(t)|| + 2||(1 + x)u_{xyy}, u_{yy}||^2(t)||
\leq 4||(1 + x)u_{xyy}, u^2_{yy}(t)|| + 2 \sup_D u^2(x,y,t)||(1 + x)u_{xyy}, u_{yy}(t)||
\leq 4(1 + L) \sup_D |u(x,y,t)||u_{xyy}||u_{yy}||^2(t)L^4(D)
+ 2(1 + L) \sup_D u^2(x,y,t)||u_{yy}||u_{xyy}||^2(t)||
\leq \delta \|u_{xyy}\|^2(t) + \frac{C(D)}{\delta} \left[ \sup \|u^2(x,y,t)||u_y||^2(t)||\nabla u_y||^2(t)
+ \sup_D u^4(x,y,t)||u_{yy}||^2(t) \right].
\]

Making use of Proposition 2.1, (3.31) and Lemmas 3.2, 3.3, we get
\[
I_2 \leq \delta \|u_{xyy}\|^2(t) + \frac{C(L, \|u_0\|, \|u_t(0)\|)}{\delta} e^{-\chi t}.
\]

Taking \(\delta = \frac{1}{2}\) and substituting \(I_1, I_2\) into (3.25), we find that
\[
\frac{d}{dt} ((1 + x), u^2_{yy})(t) + \frac{1}{2} \|\nabla u_{yy}\|^2(t) + \int_0^B u^2_{xyy}(0, y, t)dy
\leq C(D, \|u_0\|, \|u_t(0)\|) e^{-\chi t}.
\]

Simple integration completes the proof of Lemma 3.4. \(\square\)
Taking into account (3.3), (3.4), write (3.6) in the form
\[
\int_0^B (u_{xxx}^N - \lambda_j u_j^N) w_j dy = - \int_0^B \left[ u_t^N + |u|^2 u_x^N \right] w_j dy.
\]

Multiplying it by \( g_{jxxx} - \lambda_j g_j x \), summing over \( j = 1, \ldots, N \) and integrating with respect to \( x \) over \((0, L)\), we obtain
\[
\| \Delta u_x^N \|^2(t) \leq \| u_t^N \|(t) \| \Delta u_x^N \|(t) + \| |u|^2 u_x^N \|(t) \| \Delta u_x^N \|(t)
\]
or
\[
\| \Delta u_x^N \|(t) \leq \| u_t^N \|(t) + \| |u|^2 u_x^N \|(t).
\]

Making use of (2.8), (3.24), we get
\[
\| \Delta u_x^N \|(t) \leq C e(-\chi t) \tag{3.33}
\]
where the constant \( C \) does not depend on \( N, t > 0 \).

### 3.1. Passage to the limit as \( N \to \infty \)

Since the constants in Lemmas 3.1-3.4 and (3.33) do not depend on \( N, t > 0 \), then making use of the standard arguments, see \[16, 37\], one may pass to the limit as \( N \to \infty \) in (3.6) to obtain for all \( \psi(x, y) \in L^2(D) \):
\[
\int_D [u_t + u^2 u_x + \Delta u_x] \psi \, dx \, dy = 0. \tag{3.34}
\]

Taking into account Lemmas 3.1-3.4, we establish the following result:

**Lemma 3.5.** Let all the conditions of Theorem 3.1 hold. Then there exists a strong solution \( u(x, y, t) \) to (2.1)-(2.4) such that
\[
\| u \|^2_{H^1(D)}(t) + \| \nabla u \|^2(t) + \| u_t \|^2(t) + \| \Delta u_x \|^2(t) + \| u_x(0, y, t) \|^2_{H^1_0(0, B)} \leq C e(-\chi t), \tag{3.35}
\]
\[
\int_0^t \left\{ \| \nabla u_{yy} \|^2(\tau) + \| \Delta u_x \|^2_{L^2(D)}(\tau) + \| u_t \|^2_{H^1(D)}(\tau)
\right\}
\]
\[
+ \| u_x(0, y, \tau) \|^2_{H^1_0(0, B)} d\tau \right\}
\]
\[
\leq C(D, \| u_0 \|, \| u_t \|(0), \| u_{yy} \|(0)), \ t > 0. \tag{3.36}
\]

### 4. More regularity

In order to complete the proof of the existence part of Theorem 3.1 it suffices to show that
\[
u \in L^\infty \left( \mathbb{R}^+; H^2(D) \right) \cap u \in L^2 \left( \mathbb{R}^+; H^3(D) \right),
\]
\[
u_t \in L^\infty \left( \mathbb{R}^+; L^2(D) \right) \cap u \in L^2 \left( \mathbb{R}^+; H^1(D) \right).
\]

These inclusions will be proved in the following lemmas.
Lemma 4.1. A strong solution from Lemma 3.5 satisfies the following inequality:

\[
\int_{\mathbb{R}^+} \left\{ \|u\|^2_{H^3(D)}(t) + \|\Delta u_x\|^2_{H^1(D)}(t) \right\} \, dt \\
\leq C(D, \|u_0\|, \|u_t\|(0), \|u_{yy}\|(0)) e^{(-\chi t)}. \tag{4.1}
\]

Proof. Taking into account (3.35), (3.36) and Proposition 2.1, we write (3.34) in the form

\[
\Delta u_x = -u_t - u^2 u_x \equiv f(x, y, t) \in L^\infty(\mathbb{R}^+; L^2(D)),
\]

\[
u_x(0, y, t) \equiv \phi(y, t) \in L^2(\mathbb{R}^+; H^2(0, B)) \cap L^\infty(\mathbb{R}^+; H^1_0(0, B)),
\]

\[
u_x(x, 0, t) = u_x(x, B, t) = u_x(L, y, t) = 0.
\]

Denote \(\Phi(x, y, t) = \phi(y, t)(1 - x/L)\). Obviously,

\[
\Phi \in L^2(\mathbb{R}^+; H^2(D))
\]

Then the function

\[
v = u_x - \Phi(x, y, t)
\]

solves in \(D\) the elliptic problem

\[
\Delta v = f(x, y, t) - \Phi_{yy}(x, y, t) \in L^2(R^+; L^2(0, D)), \quad v|_\gamma = 0
\]

which admits a unique solution \(v \in L^2(\mathbb{R}^+; H^2(D))\), see [16]. Consequently, \(u_x \in L^2(\mathbb{R}^+; H^2(D))\). Therefore (3.36) implies (4.1). This completes the proof of Lemma 4.1.. \(\square\)

Lemma 4.2. A strong solution given by Lemma 3.5 satisfies the following inequality:

\[
\|u\|^2_{H^2(D)}(t) + \|\Delta u_x\|^2(t) \\
\leq C(D, \|u_0\|, \|u_t\|(0), \|u_{yy}\|(0)) e^{(-\chi t)}, \quad t > 0. \tag{4.2}
\]

Proof. Making use of (3.35) and acting as by the proof of Lemma 4.1, we get

\[
\Delta u_x = -u_t - u^2 u_x \equiv f(x, y, t) \in L^\infty(\mathbb{R}^+; L^2(D)),
\]

\[
u_x(0, y, t) \equiv \phi(y, t) \in L^\infty(\mathbb{R}^+; H^1(0, B))
\]

\[
u_x(x, 0, t) = u_x(x, B, t) = u_x(L, y, t) = 0.
\]

Denote \(\Phi(x, y, t) = \phi(y, t)(1 - x/L)\). Obviously,

\[
\Phi \in L^\infty(\mathbb{R}^+; H^1(D))
\]

Then the function

\[
v = u_x - \Phi(x, y, t)
\]
solves in $D$ the elliptic problem
\[
\Delta v = f(x, y, t) - \Phi_{yy}(x, y, t) \in L^\infty \left( \mathbb{R}^+; H^{-1}(D) \right), \quad v|_\gamma = 0. \quad (4.3)
\]

By the elliptic equations theory \[16\], there exists a unique weak solution to (4.3),
\[
v \in L^\infty (R^+; H^1_0(D)).
\]

Consider the scalar product
\[
-(v, \Delta v)(t) = -(v, f - \Phi_{yy})(t) = -(v, f)(t) - (v_y, \Phi_y)(t)
\]
that can be rewritten in the form
\[
\|v_x\|^2(t) + \|v_y\|^2(t) \leq \frac{1}{2} [\|f\|^2(t) + \|v\|^2(t) + \|\Phi_y\|^2(t) + \|v_y\|^2(t)].
\]
This implies
\[
\|v\|^2_{H^1(D)}(t) \leq C[\|f\|^2(t) + \|\Phi_y\|^2(t)] \leq Ce^{(-\chi t)}. \quad (4.4)
\]
Hence, $u_x = v + \Phi \in L^\infty(\mathbb{R}^+; H^1(D))$ and since $u, u_y \in L^\infty(\mathbb{R}^+; H^1(D))$, then
\[
u \in L^\infty \left( \mathbb{R}^+; H^2(D) \cap H^1_0(D) \right).
\]
This completes the proof of Lemma 4.2. \hfill \Box

**Lemma 4.3.** The strong solution from Lemmas 3.5-3.7 is uniquely defined.

**Proof.** Let $u_1$ and $u_2$ be two distinct solutions to (2.1)-(2.4). Then $z = u_1 - u_2$ solves the following IBVP:

\[
Az \equiv z_t + z_x + \frac{1}{2}(u_1^3 - u_2^3)x + \Delta z_x = 0 \quad \text{in } Q \quad t > 0,
\]

\[
z(0, y, t) = z(L, y, t) = z_x(L, y, t) = z(x, 0, t) = z(x, B, t) = 0,
\]

\[
z(x, y, 0) = 0, \quad (x, y) \in D.
\]

From the scalar product
\[
2 (Az, (1 + x)z)(t) = 0,
\]
we infer
\[
\frac{d}{dt}((1 + x)z^2)(t) + 3\|z_x\|^2(t) + \|z_y\|^2(t) + \int_0^B z_x^2(0, y, t) dy
\]
\[
= -2((1 + x)(u_1^3 - u_2^3)x, z)(t) = 2((u_1^2 + u_1u_2 + u_2^2)z, z
\]
\[
+ (1 + x)z_x)(t) \leq 2\|z_x\|^2(t) + C(M)((1 + x), z^2)(t), \quad (4.8)
\]
where $M = \sup_D |u_1^2 + u_1 u_2 + u_2^2|(x, y, t)$. Due to Proposition 2.1 and (4.2), $M$ does not depend on $t > 0$. Hence (4.8) becomes
\[
\frac{d}{dt}((1 + x), z_2)(t) \leq C(M)((1 + x), z_2)(t).
\]
Since $z(x, y, 0) \equiv 0$, by the Gronwall lemma,
\[
\|z\|_2^2(t) \leq ((1 + x), z_2)(t) \equiv 0, \quad t > 0.
\]

The proof of Lemma 4.3 is complete. \hfill \Box

**Conclusions.** An initial-boundary value problem for the 2D critical generalized Zakharov-Kuznetsov equation posed on rectangles has been considered. Assuming small initial data, the existence of a regular global solution, uniqueness and exponential decay of $\|u\|_{H^2(D)}$ have been established.

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