Comment on “Temporal scaling at Feigenbaum point and nonextensive thermodynamics” by P. Grassberger

Recently, P. Grassberger focused on some relevant aspects of nonextensive statistical mechanics, based on the entropy $S_q = k \sum_i q \log^q p_i$. His Letter contains, side by side, correct results (most of them long known), and severely incorrect statements. The whole appears to reflect a somewhat abridged understanding of the theory that the author criticizes. The present Comment addresses some of the latter.

1. To start with, it is by now widely known that the $q \neq 1$ theory applies to macroscopic states that exhibit some kind of (quasi-) stationarity or metastability, and by no means applies to thermal equilibrium, which is definitively well described by standard statistical mechanics. This point and others are clarified in [3], skipped in Ref. 6 of [1].

2. For the Shannon and related entropic forms, it is reserved in [1] expressions such as “the only rationally justifiable ansatz” and “the only consistent probabilistic measure of information”. Regrettfully, on one hand this ignores the words of Shannon himself (“This theorem and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to lend a certain plausibility to some of our later definitions. The real justification of these definitions, however, will reside in their implications.” [4]). On the other hand, it also ignores relevant theorems and analytical properties that $q$-generalize those which currently characterize the Shannon entropy $S_1$. Let us mention a few of them.

(i) The Santos theorem, which $q$-generalizes the Shannon theorem: If an entropy $S$ is a continuous function of $\{p_i\}$, and monotonically increases with $W$ in the case of equal probabilities (i.e., $p_i = 1/W$), and satisfies $S(A \oplus B)/k = S(A)/k + S(B)/k + (1-q)S(A|B)/k^2$ for all independent subsystems $A$ and $B$, then $S$ is a conventional positive constant; $S(A) = \log^q 2^k$.

(ii) The Abe theorem, which $q$-generalizes the Khinchin theorem: If an entropy $S$ is a continuous function of $\{p_i\}$, and monotonically increases with $W$ in the case of equal probabilities (i.e., $p_i = 1/W$), and satisfies $S(A \oplus B)/k = S(A)/k + S(B)/k + (1-q)S(A|B)/k^2$ for arbitrary subsystems $A$ and $B$, then $S$ is a conventional positive constant; $S(A|B)$ is the conditional entropy, and

(iii) $S_q(\{p_i\})$ is concave for all $q > 0$ (Ref. 2 of [1]).

(iv) $S_q(\{p_i\})$ is Lesche-stable (also referred to as experimentally robust) [3].

(v) $S_q$ can be extensive: $q = 1$ for independent (or almost independent) subsystems, and $q \neq 1$ for some (strictly or asymptotically) scale-invariant systems [8]. (Moreover, this point is related to a possible generalization of the standard Central Limit Theorem [4]).

(vi) $S_q(t)$ leads, for a special value of $q$ and appropriate limits (including $t \to \infty$), a finite entropy production per unit time, where $S_q(t)$ denotes an ensemble-based entropy (defined in detail below, in [10], and elsewhere). This entropy production is a concept close (though different) to the so called Kolmogorov-Sinai entropy. Recent illustrations include those analytically discussed in Ref. 36 of [1] and in [11].

It is quite remarkable the fact that $S_q$ and $S_1$ share so many basic properties. It is worthy to mention, for instance, that Renyi entropy – certainly useful in the geometric characterization of chaotic dynamical systems – violates all the above properties (iii)-(vi) to be more precise, it satisfies (v) for all $q$ if the subsystems are (quasi-) independent, but it violates it for generic globally correlated subsystems.

3. We read in the Abstract of [1] that “there is no generalized Pesin identity for this system”. Let us revisit this point. On one hand, at the edge of chaos of one-dimensional unimodal dissipative maps such as the $z$-logistic one, it has been conjectured in Ref. 8 of [1], and analytically proved in Ref. 19 of [1] (and proved once again in [1], Eq. (3)), that the (upper bound of the) sensitivity to initial conditions $\xi = \lim_{\Delta t \to 0} \Delta x(t)/\Delta x(0)$ is given by $\xi = \lambda_{\text{sen}} \xi^q$, where $\lambda_{\text{sen}}$ generalizes the usual Lyapunov exponent (here recovered as the $q_{\text{sen}} = 1$ particular instance); $\xi$ stands for sensitivity. (Notice, by the way, that $\xi$ is the exact solution of $d\xi/dt = \lambda_{\text{sen}} \xi^q$. Many more details can be found in Ref. 36 of [1] and in [11]). On the other hand, we can partition the phase space $x \in [-1,1]$ of the system $x_{t+1} = 1 - a|x|^2$ in $W$ little cells (denoted by $i = 1, 2, ..., W$). Within one of these cells we can put $M$ initial conditions. We can then run the map for all these points and define the set of probabilities $p_i(t) = M_i(t)/M$ (with $\sum_{i=1}^W M_i(t) = M$). With these probabilities we can calculate $S_q(t)$, and focus on the supremum of such function over all partitions and initial cells. It follows that an unique value of $q$ exists, precisely $q_{\text{sen}}$, such that $K_q = \lim_{t \to \infty} \lim_{W \to \infty} \lim_{M \to \infty} S_q(t)/t$ is finite ($K_q$ vanishes for $q > q_{\text{sen}}$, and diverges for $q < q_{\text{sen}}$). It has been conjectured in Ref. 8 of [1], and analytically proved in Ref. 36 of [1], that $K_{q_{\text{sen}}} = \lambda_{q_{\text{sen}}}$. The particular case $q_{\text{sen}} = 1$ yields $K_1 = t$. This relation is not the same as Pesin identity, which uses, not the present $K_1(t)$, but rather the Kolmogorov-Sinai entropy rate. It is, however, totally similar, which makes that it is sometimes referred in the literature as Pesin-like, or simply Pesin identity.

4. We also read in [1] that “In the first papers [...], it was supposed but never substantiated that the parameter $q$ of NET can be obtained, also for the Feigenbaum map, by some maximum entropy principle.” This is a
strange statement indeed. From the very first steps of the nonextensive theory, the entropic index $q$ was assumed to have some special value (obtained in fact a priori from microscopic dynamics, in agreement with Einstein’s and Cohen’s philosophies \[13\]). Under appropriate circumstances, extremization of $S_q$ is to be sought for characterizing stationary or stationary-like states (in close correspondence with thermal equilibrium of Hamiltonian systems, which are known to extremize $S_1$). Why the author of \[11\] appears to suggest that $q$ itself would be determined through some maximum entropy principle remains as some sort of mystery.

There are several other points in \[11\] that would deserve comments were it not space limitations. However, we have illustrated that, by its poor knowledge of the relevant literature, \[11\] can be regretfully misleading for the nonspecialized readers.

[1] P. Grassberger, Phys. Rev. Lett. 95, 140601 (2005).
[2] A mathematical expression very similar to $S_q$ was, to the best of my present knowledge, first introduced by J. Harvda and F. Charvat in 1967 and was further discussed by I. Vajda, Kybernetika 4, 105 (1968) [in Czech]. With a different prefactor, it was rediscovered by Z. Daroczy, Inf. and Control 16, 36 (1970), and further commented by A. Wehrl, Rev. Mod. Phys. 50, 221 (1978). They did so in the context of cybernetics and control theory but apparently never found any application. More historical details can be found in C. Tsallis, (i) Chaos, Solitons and Fractals 6, 539 (1995); (ii) in Nonextensive Statistical Mechanics and Its Applications, eds. S. Abe and Y. Okamoto, Lecture Notes in Physics (Springer-Verlag, Heidelberg, 2001); and (iii) Chaos, Solitons and Fractals 13, 371 (2002). In September 1985 in Mexico City, I independently conceived of $S_q$, this time as the basis of a possible generalization of Boltzmann-Gibbs statistical mechanics. It was published in Ref. 2 of \[11\]. Many entropic forms are related with $S_q$. A special mention is deserved by the Renyi entropy $S_q^{R} \equiv (\ln \sum_i p_i^q)/(1-q) = \ln[1+(1-q)S_q]/(1-q)$, and by the Landsberg-Vedral-Abe-Rajagopal entropy $S_q^{LVAR} \equiv S_q/\sum_{i=1}^{W} p_i^q = S_q/[1+(1-q)S_q]$. The Renyi entropy was, according to I. Csiszar (in Transactions of the Seventh Prague Conference on Information Theory, Statistical Decision Functions, Random Processes, and the European Meeting of Statisticians, 1974 (Reidel, Dordrecht, 1978), p. 73), first introduced by P.-M. Schutzenberger, Publ. Inst. Statist. Univ. Paris 3, 3 (1954), and then by A. Renyi, in Proceedings of the Fourth Berkeley Symposium, 1, 547 (University of California Press, Berkeley, Los Angeles, 1961) (see also A. Renyi, Probability theory (North-Holland, Amsterdam, 1970)). $S_q^{LVAR}$ was independently introduced by P.T. Landsberg and V. Vedral, Phys. Lett. A 247, 211 (1998), and by A.K. Rajagopal and S. Abe, Phys. Rev. Lett. 83, 1711 (1999). Both $S_q^{R}$ and $S_q^{LVAR}$ are monotonic functions of $S_q$; consequently, under identical constraints, they are all optimized by the same probability distribution. They differ however with regard to many other properties. A two-parameter entropic form was introduced by B.D. Sharma and D.P. Mittal, J. Math. Sci. 10, 28 (1975), which reproduces both $S_q$ and Renyi entropy as particular cases. This scheme has been recently enlarged by M. Masi, Phys. Lett. A 338, 217 (2005).
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[8] C. Tsallis, M. Gell-Mann and Y. Sato, Proc. Natl. Acad. Sci. USA 102, 15377 (2005).
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[10] M. Gell-Mann and C. Tsallis, eds., Nonextensive Entropy - Interdisciplinary Applications (Oxford University Press, New York, 2004).
[11] G. Casati, C. Tsallis and F. Baldovin, Europhys. Lett. 72, 355-361 (2005).
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[13] A. Einstein, Annalen der Physik 33, 1275 (1910); E.G.D. Cohen, Boltzmann Award Lecture at Statphys-Bangalore-2004, Pramana 64, 635 (2005).