Cost and Routing of Continuous Variable Quantum Networks

Federico Centrone,1,2,* Frederic Grosshans,1,† and Valentina Parigi3,‡

1Sorbonne Université, CNRS, LIP6, 4 place Jussieu, F-75005 Paris, France
2Université de Paris, CNRS, IRIF, 8 Place Aurélie Nemours, 75013 Paris, France
3Laboratoire Kastler Brossel, Sorbonne Université, CNRS, ENS-Université PSL, Collège de France, 4 place Jussieu, F-75252 Paris, France

We study continuous-variable graph states as quantum communication networks. We explore graphs with regular and complex network shapes distributed among different agents and we report for their cost as a global measure of squeezing and number of squeezed modes that are necessary to build the network. We show that the trend of the squeezing cost presents a non-trivial scaling with the size of the network strictly dependent on its topology. We devise a routing protocol based on local quadrature measurements for reshaping the network in order to perform teleportation protocol between two arbitrary nodes of the networks. The Routing protocol, which is based on wire-shortening over parallel paths among the nodes, improves the final entanglement between the two nodes in a considerable amount of cases, and it is particularly efficient in running-time for complex sparse networks.

I. INTRODUCTION

Photonics quantum networks are essential resources for quantum communication and information protocols, they represent an essential part of the future quantum internet where quantum states of light will allow for efficient distribution and manipulation of information [1–3]. In this work we explore continuous-variable (CV) entangled states with regular and complex network topologies, that are distributed among different agents. Continuous-variable quantum states span on infinite dimensional Hilbert space so allowing for the encoding of larger amount of information when compared to Discrete Variables (DV). Moreover, generation and measurement of CV states requires only coherent control on classical laser source and weakly non-linear materials, along with coherent (homodyne) detection which, differently from photon counting detectors, can be highly efficient at room temperature and easily integrated in classical communication networks. However, CV encoding in quantum networking/routing protocols has not been extensively studied yet.

Our model aims at reproducing the existing photonic platforms with realistic experimental constraints, such as limited amount of squeezing, but without taking into account propagation losses [42]. At the same time we probe their capabilities while the scaling of the network increases beyond the capacities allowed by the state-of-the-art technology. In particular we explore the cost of different networks topologies in term of number of needed squeezers at fixed number of nodes in the networks and of the global amount of squeezing.

We then explore their potentialities to perform efficient quantum communications between two arbitrary nodes when assisted with a given class of Gaussian Local Operations and Classical Communication (GLOCC) by all the agents in the network. A typical approach of quantum networking and routing consists in distributing photonics states like single-photons, Bell pair or Gaussian state and then use synchronous local operations that build the wanted entanglement structure between the agent [2, 4–7]. We rather consider the case where a pre-existing CV multipartite entangled state is distributed among the players and then local operations reconfigure the entanglement connections, similarly to some protocol in the DV case [8]. The choice is motivated by the fact that multimode entangled states can be directly generated via optical platform [9–13] and their shape can be easily manipulated [14, 15]. The article is structured as it follows. In section II we discuss Gaussian quantum states, their generation by quadratic Hamiltonian and their decomposition. We then introduce the Gaussian quantum networks studied in this Article. In section III we revise a resource theory of squeezing and in section IV we adopt it in order to estimate the cost of expanding the network. Although in entangled qubits networks the resource usage is always proportional to the number of links, we show that in CV gaussian networks the trend of the squeezing cost is vastly richer, presenting non-trivial scaling with the size of the network strictly depending on its topology. We present as well a few instances of the full squeezing spectra — i.e. the needed amount of squeezers with the required squeezing values — of regular and complex networks, showing that some topologies are equivalent up to a linear optical transformation.

In section V we propose a CV architecture for the quantum internet based on the gaussian network previously described. We simulate quantum communication protocols through the network by letting the spatially separated agents present at each node perform a homodyne measurement on their optical mode and look for the optimal measurement strategy to maximize the negativity of the entangled pair shared by the two users who want communicate, Alice and Bob. We prove a compelling result that could have potential applications, notably that when multiple entangled paths connect Alice to Bob the optimal measurement strategy allows to increase the negativity in the final pair. This parallel enhancement of entanglement can be used to increase the quality of quantum communications in some selected network topologies.

Lastly, in section VII, we employ our previous find-
ings to implement an heuristic routing protocol for distributing and boosting the entanglement between two arbitrary clusters. The algorithm we provide, on the one hand, is much more efficient than directly checking all possible combinations of quadratures measurements and, on the other hand, it always provides higher negativity than the classical scheme, which is directly employing the shortest path between Alice and Bob and neglect the parallel channels.

II. ARBITRARY GAUSSIAN NETWORK

A. Gaussian quantum states

The generation of continuous variables multimode entangled states has been demonstrated in several optical setups. In such experiments we recover networks structures as naturally appearing entanglement correlations [11], reconfigurable Gaussian interactions [15], or imprinted cluster states [9, 13, 14, 16].

These quantum states produced via parametric processes and linear optical transformation are characterized by Gaussian statistical distribution of the quadratures of the involved optical modes [17]. The quadratures $\tilde{q}_j$ and $\tilde{p}_j$ of the $j$th mode are canonical conjugate variables, such that $[\tilde{q}_j, \tilde{p}_k] = i\hbar \delta_{j,k}$, associated to the quantum harmonic oscillator describing the light mode. In this work we adopt the following relation with creation and annihilation operators $\hat{a}^\dagger = (\tilde{q} - i\tilde{p})/\sqrt{2}$ and $\hat{a} = (\tilde{q} + i\tilde{p})/\sqrt{2}$, such that the variance of the vacuum mode are canonical conjugate variables.

The produced states can then be completely characterized by the first two moments of the quadratures $\bar{r} = \text{Tr}[\hat{r}]$ and $\sigma = \text{Tr}[\rho((\hat{r} - \bar{r}),(\hat{r} - \bar{r})^T)]$, where $\rho$ is the density matrix of the Gaussian state and $\bar{r} = (\bar{q} \ldots \bar{q}_N, \bar{p} \ldots \bar{p}_N)$ we follow here $qp$-ordering.

Parametric processes are described by quadratic Hamiltonians $\mathcal{H}_I = i\mathcal{H}\mathcal{H}^T$, whose dynamics is implemented on the quadratures by $S_H = e^{i\mathcal{H}t}$, as

$$r' = S_H r_0$$

where $\Omega = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ [43], $r_0$ are quadratures of the initial state and $r'$ are the quadratures of the final state.

Since any pure Gaussian state can be obtained by the application of a quadratic Hamiltonian $\mathcal{H}$ to the vacuum, the most general pure Gaussian state covariance matrix is given by applying $S_H$ by congruence to the vacuum covariance matrix $\sigma_0 = 1/2$:

$$\sigma = S_H \sigma_0 S_H^T = \frac{S_H S_H^T}{2}$$

Singular value decomposition allows one to write the symplectic transformation in the so called Bloch-Messiah decomposition [17] as a product of an orthogonal, a diagonal and an orthogonal matrices $S_H = O\Delta O'$, which can be interpreted as a basis rotation, a squeezing in the diagonal basis and another rotation. The mode-basis in which the covariance matrix is diagonal and each component is independently squeezed is named the supermode basis. In [11, 14] where the pump and the phase matching function can be described by a gaussian spectral profile, the supermode basis corresponds to Hermite-Gauss spectral modes. The squeezing values of $\Delta$ can be derived from the eigenvalues of the Hamiltonian $\mathcal{H}_I$, while the orthogonal matrix $O$ can be interpreted as a measurement basis change or, equivalently, as a passive linear optical transformation. The other orthogonal matrix $O'$ is simplified in the product $S_H S_H^T$ and can be disregarded:

$$\sigma = \frac{S_H S_H^T}{2} = \frac{1}{2} O \Delta O^T.$$ (3)

The diagonal matrix $\Delta$ contains the information on the minimum number of squeezed modes in the system and their value of squeezing, which will later be used in the chosen resource theory. If we consider a single mode field, the squeezing operation is defined as a Gaussian transformation that reduces the variance of $\hat{p}$ by a factor $10^{-s/10}$, where $s$, measured in dB throughout this article, is called squeezing factor. Squeezing is represented by the local symplectic matrix

$$S_{\text{sq}}(s) = \begin{pmatrix} 10^{s/20} & 0 \\ 0 & 10^{-s/20} \end{pmatrix}$$

The multimode $\Delta$ matrix can then be written as

$$\Delta = \text{diag}\{10^{s_1/20}, 10^{s_2/20}, \ldots 10^{s_N/20}, 10^{-s_1/20}, 10^{-s_2/20}, \ldots 10^{-s_N/20}\}$$

This formalism can be used to visualize and manipulate Gaussian quantum states, that are readily available in most well-equipped photonics laboratories and, although the number of modes and their connections is still in large part limited, many efforts are employed to improve the capacities of these systems.

Targeted Gaussian quantum states, including the quantum networks of the next section, can be generated via the two following strategies: i) by tailoring Hamiltonions $\mathcal{H}_I$ of multimode parametric processes in order to get the decomposition of eq. (3) corresponding to the desired covariance matrix; ii) by getting a number of single-mode squeezers equal to the number of elements with $s_j \neq 0$ of $\Delta$ in eq. (3) and producing the corresponding $s_j$ squeezed states, that are injected in a linear optic interferometer corresponding to the orthogonal matrix $O$ in eq. (3).

B. Graph states as quantum networks

The above formalism can be employed to describe Gaussian graph states, that can be used as CV quantum networks. We at first recall that a network is mathematically described by a graph $G(V,E)$, which is a set of vertices $V$ (or nodes) connected by a set of edges $E$. Labeling the nodes of the graph in some arbitrary order, we can define a symmetric adjacency matrix $A = A^T$ whose $(j,k)^{th}$ entry $A_{jk}$ is equal to the weight of the edge linking node $j$ to node $k$ (with no edge corresponding to a weight of 0). Typically, the adjacency matrix
is enough to completely characterize a graph, however we will see that in our case there are other degrees of freedom such as the squeezing of a node and its angle.

We can now describe the quantum networks we use in this work that are called graph- or cluster-states [44] [18–20]. Theoretically, they can be built by entangling a number of squeezed modes of light via CZ-gates, which is a Gaussian operation implementing a correlation of strength \( g \) between the \( \hat{q} \) and the \( \hat{p} \) of the two modes on which it acts. The corresponding symplectic matrix is

\[
S_{CZ}(g) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & g & 1 & 0 \\
g & 0 & 0 & 1
\end{pmatrix}
\]

The graph associated to the graph states identify edges as CZ-gates applied between nodes, that are the squeezed modes, weighted with \( g \).

In order to simplify the many degrees of freedom present in our networks, for the moment we shall assume that all the nodes will be squeezed in \( \hat{p} \) by \( s \) and all the edges have a correlation strength of \( g \). If we apply a CZ-gate network with adjacency matrix \( A \) to a multimode squeezed vacuum \( \sigma_s \), with squeezing factor \( s \) we obtain a Gaussian network with covariance matrix [21]

\[
\sigma = \begin{pmatrix}
\sigma_{qq} & \sigma_{qp} \\
\sigma_{pq} & \sigma_{pp}
\end{pmatrix} = \begin{pmatrix}
\mathbb{1} & 0 \\
0 & \mathbb{1}
\end{pmatrix} \sigma_s \begin{pmatrix}
\mathbb{1} & A \\
A & \mathbb{1}
\end{pmatrix} = \begin{pmatrix}
\mathbb{1} & 0 \\
0 & \mathbb{1}
\end{pmatrix} A \mathbb{1} \mathbb{1} \begin{pmatrix}
\mathbb{1} & 0 \\
0 & \mathbb{1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
R & 0 \\
0 & RA
\end{pmatrix} \begin{pmatrix}
R & 0 \\
0 & RA
\end{pmatrix} = \begin{pmatrix}
R^2 & 0 \\
0 & R^2 + 1
\end{pmatrix}
\]

(5)

Where \( R = 10^{s/10} \). The \( 2N \times 2N \) covariance matrix \( \sigma \) is divided in four \( N \times N \) blocks, where the blocks \( \sigma_{qq} \) and \( \sigma_{pp} \) represent the correlations among the different nodes’ \( \hat{q} \)- and \( \hat{p} \)-quadratures, respectively, whereas the blocks \( \sigma_{qp} \) and \( \sigma_{pq} \) describes the correlations between \( \hat{q} \) and \( \hat{p} \)-quadratures.

Bear in mind that the CZ-gate operations that theoretically identify the edges of the networks are seldom realized in any laboratory being very challenging to accomplish. What is commonly done, as explained in the previous subsubsection, is the reduction of the covariance matrix of the graph state in (5) to the form of eq. (3), that is also a receipt for building the graph states from a certain number of squeezed modes (\( \Delta \)) and linear optics transformations (\( O \)).

In this paper we will focus on the squeezing cost of employing highly multimodes Gaussian systems as quantum networks and discuss some strategies to increase the quality of quantum communication on these networks in some selected (but realistically relevant) scenarios. Our platform offers a wide range of applications for simulating complex structured quantum systems or implementing quantum information protocols in realistic networks.

III. A RESOURCE THEORY OF SQUEEZING

The Gaussian bosonic states of subsection II A are of particular significance in the theory of continuous variable quantum information, in particular in their quantum optical implementations. They are in fact resources for measurement based quantum computing [18, 20], quantum simulations [15], multi-party quantum communication [14, 22], and quantum metrology [23, 24].

Being interested in the nature of the correlations between such states, the first moments become irrelevant. In any practical realization of a quantum communication protocol with Gaussian states, first moments do play a role, but these are normally managed in the post-processing and do not interfere with the dynamics of the second moments. We can thus assume that our quantum states are fully described by their covariance matrix.

We have seen in the previous section that the squeezing is the essential resource for building Gaussian entangled states. A natural question is thus: what is the squeezing cost of producing a quantum state?

A general resource theory for Gaussian states is provided in [25]. The specific case of squeezing is described in [26] where they show an operational squeezing measure for any symplectic transformations \( S \):

\[
F : \mathbb{R}^{2N \times 2N} \rightarrow \mathbb{R}, \quad F(S) = 20 \sum_{i=1}^{N} \log_{10}(\frac{s_i(S)}{s_i(S)}), \quad (6)
\]

where \( s_i \) are the decreasingly ordered singular values of \( S \), while \( \log_{10} \) and the factor 20 ensures that the outcome is measured in dB. Using eq. (3), we can then define a squeezing measure for covariance matrices

\[
G : \mathbb{R}^{2N \times 2N} \rightarrow \mathbb{R}, \quad G(\sigma) = \sum_{i=1}^{N} 10 \log_{10}(\frac{1}{s_i(\sigma)}) \quad (7)
\]

This definition can be generalized for arbitrary quantum states, but assumes this particularly simple form for pure Gaussian ones and it works for any number of modes. We will employ it to classify networks topologies, basing on how their squeezing cost scales with the dimension of the network. As we said previously, the implementation of the CZ-gate is not trivial, however in our analysis the actual transformations used to implement the state are irrelevant because the eq. (7) only depends on the final state to which it is applied and not on the single symplectic maps used to implement it.

IV. SQUEEZING COST FOR NETWORK GENERATION

In this section we apply the results presented above to various topologies of Gaussian networks, to study how their squeezing requirement scales with the size of the network depending on its own structure. As stated in the first section, a node in the network is a a pure continuous variables Gaussian state, that will be called a qumode, and is completely defined by its own covariance matrix.

A. Regular Networks

Let us first discuss some regular network structures. We shall consider the following topologies:
• The **linear graph** $L_N$, with $N$ nodes and $N - 1$ edges, is accomplished by connecting each node in series to the next.

![Linear Graph](image)

• The **ring graph** $R_N$, with $N$ nodes and $N$ edges, is a linear graph with a closed loop.

![Ring Graph](image)

• In the **star graph** topology $S_N$, with $N$ nodes and $N - 1$ edges, every peripheral node is linked to a central node called hub.

![Star Graph](image)

• The **diamond graph** $D_N$, with $N$ nodes and $2(N - 2)$ edges, has 2 hubs, each linked to all the $N - 2$ central nodes of the network.

![Diamond Graph](image)

• In the **complete (or fully connected) graph** $F_N$, with $N$ nodes and $\frac{N(N-1)}{2}$ edges, all nodes are interconnected.

![Complete Graph](image)

We use the linear graph as a benchmark to see how the squeezing cost scales with the number of nodes and links. In fact, single mode squeezing and the CZ-gate both require a fixed amount of squeezing to be implemented, so we would expect $G(\sigma)$ to scale linearly with the number of links and nodes. This is indeed the case of the graphs in Fig. 1. In figure (a), we create a multimode squeezed vacuum with no connections ($g = 0$). We can see how the effect of the squeezing $s$ on each node is that of shifting the average cost. In figure (b) we set the initial squeezing to be null ($s = 0$) and only apply the CZ-gates. It is shown that for large values of $N$ the average cost is constant, hence the total cost is linear in $N$. We can always set $s = 0$ which would only contribute as a constant shift and see how the effect of the connections influence the squeezing requirement for the network. Let us now see how the total squeezing cost $G(\sigma)$ scales with the number of nodes $N$ for each of the network topologies presented above.

![Diagram](image)

**FIG. 1:** Trend of the average squeezing cost for the linear graph $L_N$ up to $N = 100$: (a) we increase the squeezing $s$ keeping the correlations null $g = 0$ and (b) we increase the correlations $g$ keeping the squeezing null $s = 0$.

![Diagram](image)

**FIG. 2:** (a) Trend of the squeezing cost $G(\sigma)$ for the regular topologies with $s = 0$ and $g = 1$: linear $L_N$, ring $R_N$, star $S_N$, diamond $D_N$, full $F_N$ networks up to $N = 100$ nodes. (b) Detail of the diamond and star graphs. These networks do not scale linearly unlike the others.

In Fig. 2 we can see how the linear graph in blue and the ring graph in orange are superposed, sharing the same squeezing cost, as well as the star graph in green and the diamond in red. The latter two graphs present much less squeezing than the others and do not grow linearly with $N$.

Fig. 3 shows the average cost for each node and each edge respectively. It results that the constant behaviour of the linear and ring topologies is rather an exception and that in general the squeezing required to generate a determined Gaussian network is not simply proportional to the number of nodes or edges but sublinear. The intrinsic connection between the squeezing of a Gaussian network and its topology was already put in evidence by Gu et al. [20], by proving a relation between the squeezing required to produce a CV graph state and the singular value decomposition of the associated adjacency matrix. As we said, the linear and ring graphs have a constant average cost per edge with the growth of the network, whereas the complete graph seems to be the
one with lowest average cost per edge, having also the highest degree of edges.

An objection one could make at this point is that our cost functional \( G(\sigma) \) does not fully characterize our Gaussian networks. In fact, two states that have the same squeezing cost are not necessarily equivalent up to an orthogonal transformation. For example, a 10dB single mode squeezed vacuum and a two mode squeezed vacuum with 5dB of squeezing each would have the same cost but cannot be transformed into each other using only passive optics. As a matter of fact, resource theories can seldom give a complete view of the problem in exam, notably in the light of an experimental implementation.

Although the squeezing cost is indeed the most insightful figure of merit to investigate what happens as the size of the Gaussian networks grows, the most complete picture of number of the amount of needed experimental resources is given by the decomposition defined in eq. (3), which gives us the minimum number of squeezed modes and their squeezing value.

In Fig. 4 we then show the distribution of the logs of the singular values of the covariance matrix of the regular topologies studied above, for \( N = 100 \).

Notice that the diamond and the star graphs only have two intrinsic squeezed modes. In the specific, the number of squeezers in all the networks grows linearly with \( N \), except the star and diamond, that are built by squeezing only two modes and mixing them with \( N - 2 \) vacuum modes with passive optics interferometry. A straight consequence, is that these two very different types of networks are completely equivalent up to an orthogonal transformation, which means that they can be exactly reshaped one into the other using linear optics [45]. Interestingly, the first mode of the \( F \) network has the majority of squeezing, while the rest is shared equally among all other nodes.

B. Complex Networks

So far we described graphs that are built through a deterministic algorithm, though we can also construct a graph based on statistical models [27, 28]. This is the difference between regular and random networks. An exemplary standard for random networks is the Erdős–Rényi model \( G_{ER}(N, p) \), in which each pair of the \( N \) nodes have a probability \( p \) to be linked; the network thus has \( \binom{N}{2}p \) edges on average [29].

Most of the network properties observed in nature, however, simply cannot be described by regular or random graphs. For this reason, a youthful branch of scientific research is committed to the study of complex networks. In the field of network theory, complex networks are a type of graph with non trivial topological features, that are shared by neither regular nor random graphs, but are rather akin to networks modeling real systems [30].

An important class of complex networks is characterized by the small world property. These networks exhibit the peculiarity of having a low average path length, which is the mean distance between two arbitrary nodes, and a high clustering, which is a measure of the degree to which nodes in a graph tend to cluster together. The emblematic network presenting this feature is the Watts–Strogatz model \( G_{WS}(N, Q, \beta) \) [31]. In this model, we first construct a graph with \( N \) nodes and \( \frac{NQ}{2} \) edges where each node has exactly \( Q \) neighbors, then with probability \( \beta \) we rewire each edge with another node chosen uniformly at random while avoiding self loops and link duplications.

The second relevant class of complex networks present the typical aspect of being scale-free and having long-tailed structures. Scale-free networks show a power law in the degree distribution \( P(k) \propto k^{-\gamma} \) for some \( \gamma > 0 \), which is self-similar at all values of \( k \) in the tail of the distribution, unlike the ER and WS models that go to zero very quickly and have no tails. This fractal like attribute is well modeled by the Barabasi–Albert model \( G_{BA}(N, K) \), which can also reproduce growth and preferential attachment in networks [32]. The graph is built sequentially by adding one node at the time and wiring it to \( K \) other nodes with a probability that is proportional to the number of links that the target node already has. While adding one node at a time, the first \( K \) nodes initially will not be linked to anything, hence this graph will result having \( (N - K)K \) edges, mostly connected to a great hub. This type of graph is the canonical example to reproduce some properties of the World Wide Web. A particular case of the Barabasi–Albert networks is the Tree-graph \( G_T(N) = G_{BA}(N, 1) \).

Another notable class of complex networks is constituted by technological networks, artificial networks designed typically for distribution of some merchandise or resource, such as electricity or information. The most famous example in this category is the Internet Autonomous System (AS) \( G_{AS}(N) \), the physical global computer data network. In order to study this topology we will base on the work put forward in ref. [33].

All the complex networks mentioned so far are meant to emulate man-made structures, however complex topologies appear in nature in the most surprising ways. There is a large variety of biological networks, among these we will consider specifically the protein–protein interaction network model \( G_{PP}(N, \sigma) \) developed in ref. [34].

In Fig. 6 we report the trend of the total squeezing...
cost, average cost per node and average cost per edge as a function of the number of nodes for each of the above complex topologies. From the plots we notice that the most expensive growth belongs to the ER topology, while the AS seems to be the cheapest, which is a relevant quality in prospect of an actual implementation of the quantum internet. Another interesting feature, that did not emerge for regular networks, is the fact that for the ER the average cost of squeezing per node increases with the size of the graph. It is in part surprising that the complete graph behaves so differently from the ER graph. In fact, even though they are topologically very different, the scaling of the number of edges is similar so one could have expected a similar trend. On the other hand, the cost per node in the BA, WS and PP is approximately constant whereas it slowly decreases for the AS topology. Moreover, we observe that the average price per edge is decreasing for all but the WS topology. The fact that a high connectivity does not imply a high resource usage is a particularly inviting property of Gaussian networks, especially for their applications in quantum communications.

In Fig. 4 we show the squeezing cost distribution for the various topologies of complex networks by showing the squeezing cost of all the principal modes. We evince that the AS is not only the cheapest, but is also the one that has the least number of squeezed modes, turning to a tremendous advantage for experimental applications. This feature is further emphasized in Fig. 8, in which we plot the trend of the number of squeezed modes necessary to build the network as a function of \( N \). In this plot we see that the ER, BA and WS networks have a similar trend, unlike the PP and AS.

Now that we have characterized the cost of implementing Gaussian quantum networks, we will describe how to use them as a substrate to perform quantum communications.

V. QUANTUM TELEPORTATION GAUSSIAN NETWORKS

Quantum entanglement is a paramount resource for quantum information purposes. In particular, bipartite entanglement represents the fundamental requirement that a shared quantum channel should have in order to enable a truly quantum teleportation. In the framework of Quantum Communications, the networks previously described can be seen as distributed Gaussian quantum teleportation networks [35], where each pair of nodes can employ the pre-established quantum correlations together with Local Operations and Classical Communications LOCC to teleport a Gaussian quantum state from one node to the other.

In a naive strategy, the teleportation between two arbitrary nodes can be implemented simply by ignoring all the other nodes and exploiting the residual bipartite entanglement together with classical communications. This strategy is a direct extension of the standard teleportation protocol from two to more stations and is called non-assisted protocol [36].

Another set of strategies is based upon a cooperative behavior, where all the other nodes assist the teleportation between the chosen pair (Alice and Bob) by means of LOCC. In fact, if the external nodes perform suitable local measurements and then classically communicate their outcomes to Bob, the latter can use this additional classical information to improve the process via modified conditional displacements. These strategies are called assisted protocols and are the ones that determine what we call networking/routing protocol in this Article.

According to Gu et al. [20] quadrature measurement on a mode of a Gaussian network like the ones we considered so far can be described by two simple rules:

- **Vertex Removal**: a \( \hat{q} \)-measurement on a qumode removes it from the network, along with all the edges that connect it.

- **Wire Shortening**: a \( \hat{p} \)-measurement on a qumode is just a \( \hat{q} \)-measurement after a Fourier Transform, which corresponds to a phase rotation of \( \frac{\pi}{2} \): \( S_F = S_R(\theta = \frac{\pi}{2}) \). The node will thus be re-
FIG. 5: Some complex networks and their degree distribution. In the distributions of the BA, AS and PP the y-axis is in log-scale.

FIG. 6: (a) Trend of the squeezing cost for complex topologies: Erdős–Rényi $\mathcal{G}_{ER}$, Barabasi–Albert $\mathcal{G}_{BA}$, Watts–Strogatz $\mathcal{G}_{WS}$, AS internet $\mathcal{G}_{AS}$ and Protein–Protein interaction $\mathcal{G}_{PP}$ networks up to $N = 500$ nodes. (b) Average squeezing per node, (c) average squeezing per edge.

FIG. 7: Squeezing cost distribution for complex topologies: Erdős–Rényi $\mathcal{G}_{ER}$, Barabasi–Albert $\mathcal{G}_{BA}$, Watts–Strogatz $\mathcal{G}_{WS}$, AS internet $\mathcal{G}_{AS}$ and Protein–Protein interaction $\mathcal{G}_{PP}$ networks in the $N = 100$ supermodes. The trend of the number of modes for each of these networks is shown in Fig. 8.
moved but the phase shift will induce correlations between the neighbouring edges. Thus, measurements in the momentum basis allow us to effectively “shorten” linear graph states.

If two nodes $A$ and $B$ need to teleport a quantum state, they can be helped by the other nodes in the network who will perform these operations in order to increase the strength of the entanglement in the final pair. A typical measure of entanglement is the negativity

$$N = -2 \log_2 \nu_-$$  \hspace{1cm} (8)

Where $\nu_-$ is the smallest symplectic eigenvalue of the partially transposed covariance matrix of the pair. Partial transposition is a necessary operation for the PPT criterion [37] and is easily implemented in gaussian states by changing the sign of the momentum of one of the two subsystems. The negativity is simply connected to another measure of entanglement, which is the fidelity of teleporting a coherent state through that quantum channel

$$F_{\text{coh}} = \frac{1}{1 + \nu_-}$$ \hspace{1cm} (9)

Simple classical communication attains at most $F_{\text{coh}} = \frac{1}{2}$ so a bipartite system presents truly quantum correlations only if $F_{\text{coh}} > \frac{1}{2}$, or equivalently $\nu_- < 1$ and $N > 0$.

The symplectic eigenvalues $\nu_{\pm}$ of a two-mode system can be computed through the invariants of the covariance matrix [38]. More specifically, we can define the seralian $\Delta = \det \sigma_A + \det \sigma_B + 2 \det \sigma_{AB}$, where $\sigma_A$ and $\sigma_B$ are the local covariance matrices of the single-mode sub-systems $A$ and $B$, and $\sigma_{AB}$ represents their correlations. From this we can compute the symplectic eigenvalues as:

$$\nu_{\pm}^2 = \frac{\Delta \pm \sqrt{\Delta^2 - 4 \det \sigma}}{2}$$ \hspace{1cm} (10)

In Fig. 9 we compare the effect of different regular topologies of quantum networks with the purpose of distributing entanglement between two of the furthest nodes inside the network. The simplest cases are the star and complete networks. In the first case the best assisted strategy is to let everyone perform a $\hat{q}$-measurement on their node except the central one who will make a wire shortening to link the final pair. In the complete network $A$ and $B$ are already linked by an edge so it is sufficient to measure the position in all the other qumodes (notice that this strategy outperforms the non-assisted protocol). In both these cases the entanglement is constant with the number of nodes in the network as we would expect, and the wire shortening of the central node in the star graph decreases the negativity with respect to the complete graph [39]. In the linear graph all the nodes have to wire shorten from $A$ to $B$. Here the negativity quickly decreases with the number of nodes. The decrease of entanglement with the wire shortening seem to be typical in all configurations except the diamond graph, where all the central nodes are $\hat{p}$-measured.

### VI. MULTI-PATH ENTANGLEMENT

The behaviour of the diamond graph is quite counter-intuitive and might be expected to increase the fidelity of quantum communications. It can be shown (proof in Appendix B) that the lowest symplectic eigenvalue for this system goes like

$$\left(\nu_{-}(\mathcal{D}_N)\right)^2 = \frac{1}{1 + 2NR^2}$$ \hspace{1cm} (11)

Where $R = 10^{7/10}$ is the inverse of the squeezing in $\hat{p}$, with squeezing factor $s$ in $dB$. Hence, the two modes become perfectly correlated in the limit of either infinite squeezing, infinite strength CZ-gate or infinite parallel measurements on $\hat{p}$.

This property of the Diamond network, however, is not easily generalized to all graphs that present parallel
connections and the quest for the optimal measurement strategy in order to improve the final entanglement is by no means trivial. This is the case, for example, of the $D$ graph shown in Fig. 10, generated by taking the diamond network and add a CZ-gate link between adjacent central nodes. We can see that for $N > 9$ measuring always $P$ in this network is not the optimal strategy, whereas a better strategy is to alternate a $P$ and $Q$ measurement in order to restore a (smaller) diamond network.

Another important figure of merit is the entanglement per squeezing cost, shown in Fig 11 (a). We see that the diamond is the only one that gives the a ratio of entanglement per cost of the network that becomes constant for large $N$. However, the linear graph is the one that links two nodes that are the furthest away from each other. Conversely, figure 11 (b) shows the negativity in the final pair divided by the number of modes in the initial state. Once again, the diamond structure is particularly convenient, yielding the highest negativity while keeping a constant number of independent squeezers.

In order to give a fair comparison between the capacity of the linear network to bridge distant nodes and that of the diamond to increase the final entanglement we need to generalize the diamond graph to a diamond chain graph, $DC_{K,N}$, where $K$ is the number of parallel branches linking the two hubs that want to perform quantum communications as in figure 12.

We can then compare the entanglement concentrated using multiple path strategies to link two nodes far away from each other. We can see in figure 13 that the presence of parallel links has indeed the desired effect, despite the quality of the final pair, which still decreases exponentially with the distance. On the other hand, notice that the parallel links can help concentrating more entanglement until the system reaches a plateau and even the additional channels will not allow to increase the negativity. Moreover, the quality-price of this networks, specifically the ratio between the entanglement of the pair after the protocol and the squeezing cost before the protocol, is maximized by the linear graph.

Another important class of networks, notably for measurement based quantum computation, is constituted by grid cluster states that belong to graph shapes that allow for universal quantum computation [40]. Similarly to the diamond network, the presence of ancillary nodes between the emitter and the receiver can improve the quality of the quantum link with respect to the linear network. This, however, is not a general rule and sometimes the presence of additional links can be detrimental. This is the case of the triangular lattice, generated from the square lattice by adding a link between the nodes in the diagonal. There are two ways of generating the triangular and only one of the two decreases effectively the distance between Alice and Bob $T$. In both cases the result is detrimental, however $T$ is slightly better than $\tilde{T}$, while the square lattice $Q$ seems to be the most
VII. ROUTING PROTOCOL

In this section we aim at employing the abstract notions on gaussian graphs developed so far for a specific application: the routing of entanglement. In this scenario, the highly multimode entangled gaussian state corresponds to a distributed teleportation network, described in the previous section, where each node of the network is supplied with a mode of an electromagnetic harmonic oscillator and is linked to some other nodes in different geographical locations through quadrature correlations, e.g. quantum entanglement. We remark that this type of communication quantum networks is inherently different from the typical qubit networks that are currently being deployed in different metropolitan areas [41]. In those cases, for example, each entanglement link is pairwise between two qubits and as a consequence each node of the network will have to receive, storage and measure as many quantum states as neighbors it has. Conversely, in a gaussian quantum network the same qumode can be entangled with an arbitrary number of other nodes. Moreover, the production of such states, their manipulation to increase the entanglement among two nodes and their measurement to perform quantum teleportation can be achieved deterministically, unlike the discrete variables case. Nonetheless, qubits networks have been extensively studied over the last years, whereas gaussian teleportation networks is a very recent emerging field. Our purpose is, thus, not to prove the superiority of the latter, but rather to explore its properties and the differences from the DV schemes in order to get the best of both worlds.

The results of the previous sections, highlighted some outstanding properties of gaussian networks. The most important is the parallel enhancement of entanglement in the diamond graph. If properly used, this feature can most certainly improve the routing of entanglement in regular and complex shaped network. On these grounds, the search for an optimal protocol that exploits all the qualities of these gaussian networks is very desirable yet arduous, and will be subject of future investigations. Alternatively, we present a naive entanglement routing protocol that takes into account some of these properties and we will apply it to complex topologies, to show that the enhancement of the entanglement with respect to the trivial protocol is, in principle, easily achievable. Imagine we have a distributed network of entangled harmonic oscillators, where each node is honest and can perform classical communication and local homodyne measurement, and we want to establish an entangled pair between two nodes, Alice and Bob, that want to teleport a quantum state or perform QKD. The trivial protocol would be to find the shortest path between them and measure in P all the qumodes along this path and in Q all the others. A careful look at the inner structure of the network, however, might help us increase the strength of the correlation. For example if at any point, two nodes on the path are linked by multiple parallel routes, we can measure these in P to exploit the parallel enhancement.

In order to show this in practice, we will test the performances of three different routing protocols (shown in Figure 15 on various complex networks with the purpose of establishing a highly entangled pair. We choose Alice to be one of the hubs of the graph and evaluate the efficiency of the protocol in delivering entanglement to all the other nodes. The quantum protocol that we propose to exploit the parallel enhancement of entanglement will be simply called Routing.

- **Routing**: it takes as input the target node, Bob, lists all the shortest paths connecting it to Alice and measures all the nodes that are not in these paths in the Q quadrature, so that they will not influence the protocol. Among the list of paths it checks one by one those to be measure in P in order to maximize the negativity \( N \) of the final pair, while the rest will be measured in Q.

In the Routing protocol, in principle, we could have considered as well parallel paths of longer lengths that might have contributed to improve the negativity. However, in practice had the only effect of slowing down the performances while not increasing the entanglement for all the cases we considered. The effect of the parallel paths can be appreciated when comparing the negativity produced by Routing with that produced by Shortest.

- **Shortest**: the difference of the latter is that it only
exploits one of the shortest parallel paths, directly measuring everything else in Q.

In some cases the two protocols do not give a substantial difference, either because there are not parallel routes or because these do not help increasing the entanglement, however in many instances the effects of parallel routing are significant. The last protocol we compare with is All P.

- **All P**: it measures all the terminal nodes with degree 1 in Q and the rest in P.

This protocol is less effective than the first two but is always the quickest, whereas **Routing** can be computationally very slow on regular networks, which are characterized by long distances and many parallel paths, but becomes very efficient on complex sparse networks.

One instance of this program is given in figure 16 that shows the negativity provided by the three different protocols for each node of a $G_{AS}(N = 1000)$ network. At the beginning of the protocol, we pick Alice as the node, or one of the nodes, with the highest degree. The nodes are then sorted by their distance from Alice and, for the same distance, by the number of all the shortest paths connecting them to Alice. Additionally, the grey column represents the ratio of parallel paths that were useful to increase the entanglement. Notice that nodes at distance 1 cannot show a difference between the Routing and the Shortest protocols, however many nodes at distance 2 present a greater negativity than those at shorter distance after the Routing. This non-trivial effect of improving a channel capacity at larger distances has no classical equivalent.

In Figure 17 we show the graph of the network, where the nodes are again sorted by distance and number of parallel paths and the size of each node is proportional to its degree. In this figure Alice is ’0′ and has a thick red contour. The node with highest negativity and all the paths that improved its entanglement are highlighted with red thick lines.

The same analysis was done in several networks with different sizes and topologies with very different results that we report in figures 18, 19, 20 and 21. A property that is not apparent in Fig. 17, is that the node with the highest enhancement of entanglement due to the multiple paths is not necessarily the one with the highest negativity in absolute. This is the case of the ER network of Fig. 18, in which the node with the highest entanglement, highlighted in green in the graph representation, is at distance 1 while the node with the highest difference in negativity between the **Routing** and the Shortest protocols, highlighted in red, is at distance 3. In this case, the structure of the subgraph used throughout the Routing protocol is not a diamond chain and the intercorrelations among the parallel branches have limited the increase of the entanglement, as for the $D_N$ network in Fig. 10. In any case, in this network the nodes at greater distances are the ones that are most affected by our protocol and, although in some cases many parallel paths have been disregarded, as shown by the height of the grey column, all the nodes at distance 4 received a substantial enhancement.

The results of the simulation on the BA topology of Fig. 19 is similar to the AS, although the first only reaches a distance of 3. The nodes with the highest absolute negativity and the highest negativity difference produced by the Routing protocol coincide and are at distance 2 from Alice, whereas this time the subgraph of is a diamond with no interconnections. Also in this case distance 2 is favorable to perform quantum communications.

The WS structure of Fig. 20, on the other hand, is the worst to apply the **Routing protocol**. Only a few nodes, in fact, were poorly enhanced and mostly at large distances, while the negativity averaged over all the nodes for **Routing and Shortest** is comparable. The node 44 at distance 3 is the one that received the greatest boost from our protocol, whereas node 1 (like all the other nodes at distance 1) has the highest negativity.

Finally, the biological network of Fig. 21 produced the most interesting results. Once again, many nodes at distance 2 end up having more negativity than those at distance 1, and at this distance the nodes with the same degree have the same negativity that decreases exponentially with their degree. The nodes with highest negativity and highest difference coincide with node 139, which is linked to Alice through 33 intermediate nodes, forming a diamond network with no interconnections.
FIG. 16: Negativity produced by the three different protocols applied to each node of the the $G_{AD}(N = 1000)$ network. The nodes are labeled in order of distance and of number of paths connecting to Alice. The blue, orange and green stems represent the negativity of the final pair after the Routing, Shortest and All P protocols respectively, while the dashed lines represent the mean value for all the nodes. The color of the marker indicates the distance of the node from A and the grey columns represent the ratio of paths that improved the entanglement in Routing.

FIG. 17: Scheme of the $G_{AD}(N = 1000)$ network on which we performed the protocol and subgraph of the paths connecting to the node with highest negativity. The nodes are set in circles according to their distance from Alice and their size is proportional to the degree.
VIII. CONCLUSION AND OUTLOOK

A. Conclusion

We have studied Gaussian multimode quantum networks with regular and complex topologies for quantum communication protocols. In particular: i) we have studied their cost in terms of squeezing and number of necessary squeezed modes to build the network; ii) we have established a multi-path routing protocol distributing entanglement between two arbitrary nodes. In details:

- We have shown that the cost of the networks is not always linear with the number of edges and nodes and there are particular (regular and complex) graph shapes that optimize the cost and the number of squeezers over number of nodes/edges in the networks. Among regular networks the diamond and the star graph need only two squeezed nodes to be built, independently from their number of nodes. Among the complex networks shapes, the Internet Autonomous System model is the most convenient in number of needed squeezed states.

- We have studied the assisted teleportation protocol in Gaussian entangled networks, where couple of nodes are assisted in the teleportation by local measurement in all the other nodes. This naturally defines a routing protocol in Gaussian networks. In particular we have considered $q$ and $p$ homodyne quadrature measurement that allow respectively for vertex-removal and wire-shortening.

- The routing is optimized by different measurement schemes in regular networks. In the linear and the diamond networks the best strategy consists in the wire shortening, but the diamond network shows the largest ratio in reached entanglement over cost.

- Inspired by the behaviour of the diamond network we have devised a routing protocol that exploits wire shortening in parallel paths and we have applied it to complex networks graphs. The protocol named Routing is compared with Shortest, where wire shortening is done only in the shortest path, and All $P$, which removes all the terminal nodes while it wire-shorten all the others. In most cases, the Routing improves the entanglement compared with Shortest. Also, in terms of computational complexity, the Routing is much slower than All $P$ in regular networks, where there are long distances between nodes and several parallel paths, but it is very efficient in complex sparse networks.

B. Outlook

The devised Routing protocol is very general so that it can be applied to arbitrary networks, and it is particularly efficient for sparse not regular networks. Our simple graph exploration approach would be improved in computational efficiency by real graph-based algorithms, especially if we allow for approximate solutions. Also it would be interesting to allow for non uniform distributions of squeezing $s$ and CZ gate strength $g$ or more general homodyne measurements, i.e. going beyond the two $p$ and $q$ cases and considering measurements along $q_\theta = \cos(\theta)q + \sin(\theta)p$. In addition, it could be interesting to examine a scenario in which the intermediate nodes are dishonest and do not cooperate to perform the routing. Moreover, in order to consider practical implementations, realistic parameters for losses and noise should be included in the model. Finally the routing protocol has been implemented to solve the particular task of creating a perfect EPR pair between two nodes, future protocols will consider general reshaping in arbitrary multiparty states.

IX. ACKNOWLEDGEMENTS

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FIG. 18: Negativity produced by the three different protocols applied to each node of the the $G_{ER}(N = 1000, p = 0.4)$ network.

FIG. 19: Negativity produced by the three different protocols applied to each node of the $G_{BA}(N = 1000, K = 4)$ network.

FIG. 20: Negativity produced by the three different protocols applied to each node of the $G_{WS}(N = 1000, Q = 4, \beta = 0.9)$ network.

FIG. 21: Negativity produced by the three different protocols applied to each node of the $G_{PP}(N = 1000, \sigma = 0.4)$ network.
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[42] This work is focused on the capabilities of pure CV quantum states to act as quantum networks and the resources needed for their generation. Of course propagation losses can be included in the model in future works. Also generation losses can be very low, so that the hypothesis of pure states is a realistic one, and propagation losses can be mitigated by considering local (short distances) networks.

[43] $\Omega$ is a $2N \times 2N$ skew-symmetric matrix associated to the $N$ dimensional Hilbert space allowing to write the commutation relation of the canonical variables as

$$[\hat{r}, \hat{p}] = i \Omega = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

[44] While usually the name cluster is used when the graph shape allows for universal quantum computing, in this work we will use the terms cluster state and graph state as synonyms.

[45] In general any CV graph can be reshaped in any other graph via a symplectic transformation, in this case the symplectic involves only linear optics without any supplementary squeezing.
Appendix A: Graphical Calculus

In ref. [19] it is provided a unified graphical calculus for all gaussian pure states that is particularly suited for describing highly multimode gaussian networks.

In this framework, a N mode gaussian state is completely described, up to displacements, by a $N \times N$ complex valued adjacency matrix:

$$ Z = V + iU $$

Where the real and imaginary part of $Z$, $V$ and $U$ respectively, are related to the covariance matrix through the following unique decomposition

$$ \sigma = \frac{1}{2} \begin{pmatrix} U^{-1} & U^{-1}V \\ VU^{-1} & U + VU^{-1}V \end{pmatrix} $$

Gaussian graph states have a particular simple graphical representation, being

$$ Z = A + iD $$

Where $A$ is the weighted adjacency matrix of the graph and $D$ is a diagonal matrix that represents momentum squeezing, i.e. for $D = 10^{-2} \mathbb{I}$ the momentum variance of all modes is reduced by 20 decibels.

All symplectic operations can be reproduced in this language, however, since we already know how to represent the resource graph states, we only need to implement the quadrature measurements in $\hat{x}$ and $\hat{p}$. We can express the state as

$$ Z = \begin{pmatrix} t & R^T \\ R & W \end{pmatrix} $$

Where $t$ is the target mode we want to measure, $W$ is the subgraph of the untouched modes and $R$ their correlations with the target mode. We have the following two rules:

- $Z \rightarrow Z_q = W$ after a $\hat{q}$ measurement.
- $Z \rightarrow Z_p = W - \frac{RRT}{t}$ after a $\hat{p}$ measurement.

Thus, for a measurement in $\hat{q}$ we remove the node and its link from the graph, whereas for a measurement in $\hat{p}$ we apply a $\pi/2$ phase rotation and then measure $\hat{q}$.

Appendix B: Parallel enhancement of entanglement

We can use the rules described in the previous sections to prove eq. (11), that expresses analytically the power of parallel enhancement of entanglement in the diamond network when measuring the central nodes in $\hat{p}$. Let us assume that the nodes $A$ and $B$ are squeezed by a factor $S_A$ and $S_B$ respectively, there are $N$ central nodes and the $k$th mode has squeezing $S_k$ and is correlated with $A$ and $B$ through a CZ-gate with strength $g_{Ak}$ and $g_{Bk}$. It can then be easily showed that the final pair will have a purely imaginary adjacency matrix of the form

$$ Z_{AB} = i \begin{pmatrix} \Sigma_A & \Gamma \\ \Gamma & \Sigma_B \end{pmatrix} $$

Where $\Sigma_A = S_A + \sum_k \frac{g_{Ak}^2}{S_k}$, $\Sigma_B = S_B + \sum_k \frac{g_{Bk}^2}{S_k}$ and $\Gamma = \sum_k \frac{g_{Ak} g_{Bk}}{S_k}$. These result can be derived by direct application of the rule for measuring $\hat{p}$ in the graphical calculus formalism, schematized in figure 22.

![Graphical representation of the diamond graph and its parallel enhancement of entanglement.](image)

FIG. 22: Graphical representation of the diamond graph and its parallel enhancement of entanglement.

Employing eq. (A1) and A2 and noticing that $V = 0$, we can reconstruct the covariance matrix of the final pair:

$$ \sigma_f = \begin{pmatrix} \Sigma_B - \Gamma^2 & 0 & 0 \\ -\Sigma_A \Sigma_B^{-1} \Gamma & \Sigma_A - \Gamma^2 \\ 0 & 0 & \Sigma_A \Gamma \end{pmatrix} $$

Notice that this state differs from a graph state up to a local phase.

By computing the seralian $\Delta$ of the partially transpose covariance matrix of the pair $\tilde{\sigma}_f$ and applying formula 10, we can derive the general lowest symplectic eigenvalue of the state

$$ \nu_f^2 = \frac{(\sqrt{\Sigma_A \Sigma_B} - \Gamma)^2}{\Sigma_A \Sigma_B - \Gamma^2} $$

Finally, if we assume that all the modes are equally squeezed in $\hat{p}$ of a factor $R^{-1} = 10^{25}$ and all the CZ-gate correlations have a strength $g$, we arrive to formula 11.