Another proof of a Lions type existence result

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Abstract. This paper concerns a nonlinear elliptic equation involving a critical Sobolev growth and a lower-order term. Under a Lions’s condition, we prove the existence of at least one positive solution. Our approach consists in constructing a relatively compact Palais–Smale sequence for the associated variational problem.

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1 Introduction and main results

We study the following nonlinear elliptic partial differential equation with zero Dirichlet boundary condition

\[-\Delta u = K(x)u^q + \mu u \quad \text{in } \Omega,\]
\[u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,\]

where \(\Omega \subset \mathbb{R}^n, n \geq 3\), is a bounded domain with a smooth boundary \(\partial \Omega\), \(K(x)\) is a continuous function in \(\bar{\Omega}\), \(q + 1 = \frac{2n}{n-2}\) is the critical exponent for the embedding \(H_0^1(\Omega)\) into \(L^{q+1}(\Omega)\) and \(0 < \mu \leq \mu_1(\Omega)\), where \(\mu_1(\Omega)\) denotes the first eigenvalue of \((-\Delta)\) in \(H_0^1(\Omega)\).

One motivation to study this equation comes from its resemblance to the well known scalar curvature problem on an \(n\)-dimensional closed Riemannian manifold \((M^n, g_0), n \geq 3\), which consists to find a new metric \(g\) conformally equivalent to \(g_0\) with prescribed scalar curvature \(K(x)\) on \(M^n\); see, e.g., [2].

Before setting forth the main existence result, let us introduce some notations. Let \(\langle , \rangle\) denotes the scalar product defined on \(H_0^1(\Omega)\) by

\[\langle u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v\]

and let \(\| \cdot \|\) denotes its associated norm. Let \(K_\infty, S\) and \(L_{K,\mu}\) denote the following constants

\[K_\infty := \sup_{\bar{\Omega}}(K), \quad S := \inf\{\|u\|^2, \quad u \in H_0^1(\Omega) \text{ and } \|u\|_{q+1} = 1\},\]
\[L_{K,\mu} := \inf\{\|u\|^2 - \mu\|u\|^2_2, \quad u \in H_0^1(\Omega) \text{ and } J(u) = 1\},\]

(1.2)
where \( J(u) := \int_\Omega K(x)|u(x)|^{q+1} \, dx \) and \( \|u\|_p^p = \int_\Omega |u(x)|^p \, dx \), for any \( p > 1 \). \( S \) is known as the best Sobolev constant.

For our present problem, we read the Lions type theorem as follows:

**Theorem 1.1**  Let \( n \geq 3 \). Assume that \( K_\infty > 0 \) and that \( 0 < \mu < \mu_1(\Omega) \). If

\[
L_{K,\mu} < \frac{1}{(K_\infty)^{\frac{n-2}{n}}} S,
\]

then the problem (1.1) has a solution \( u \) satisfying

\[
[J(u)]^{\frac{n-2}{n}} L_{K,\mu} \leq \|u\|^2 - \frac{\mu}{2} \|u\|_2^2 < \frac{1}{(K_\infty)^{\frac{n-2}{n}}} S[J(u)]^{\frac{n-2}{n}}.
\]

In [9], Lions introduced a concentration-compactness method, which enabled him, from others, to study the loss of compactness related to the constrained minimization problem

\[
\inf\{\|u\|^2 - \frac{\mu}{2} \|u\|_2^2, \quad u \in H^1_0(\Omega) \text{ and } J(u) = 1\}. \quad (1.4)
\]

In [10, Corollary 4.1], the author proved that the hypotheses of Theorem 1.1 are sufficient to ensure that any minimizing sequence of this problem is relatively compact, and then a solution to problem (1.1) is regained at the level set \( \varpi := (1/n)L_{K,\mu}^{n/2} \) for the functional \( I_{K,\mu} \) defined in (1.5) below. See the proof of [9, Theorem I.2] for more details.

The problem (1.1) enjoys a variational structure. Indeed solution of (1.1) corresponds to positive critical point of the functional \( I_{K,\mu} \) defined on \( H^1_0(\Omega) \) by

\[
I_{K,\mu}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{q+1} \int_\Omega K|u|^{q+1} - \frac{\mu}{2} \int_\Omega u^2, \quad \forall \, u \in H^1_0(\Omega). \quad (1.5)
\]

To resolve (1.1) one can think to the Palais–Smale (P-S for short) condition for \( I_{K,\mu} \). Let \( \partial I_{K,\mu} \) denotes the gradient of \( I_{K,\mu} \).

**Definition 1.1**  Let \( c \in \mathbb{R} \).

1) Let \( (u_k)_k \) be a sequence in \( H^1_0(\Omega) \). We say that \( (u_k)_k \) is a P-S sequence at \( c \) for \( I_{K,\mu} \) if, up a subsequence, \( I_{K,\mu}(u_k) \to c \) and \( \partial I_{K,\mu}(u_k) \to 0 \) strongly in \( H^{-1}(\Omega) \).

2) We say that \( I_{K,\mu} \) satisfies the P-S condition at \( c \) if any P-S sequence at \( c \) is relatively compact.

As well-known in variational problems with critical exponent, concentration phenomena can occur and violate the P-S condition at some levels. Thus a local analysis becomes useful. In this direction, a mountain-pass procedure was introduced in [8]: In order to obtain a P-S sequence, the authors used, as a key tool, an Ambrosetti–Rabinowitz type result [8, Theorem 2.2]. For an adaptation of this procedure to the present problem, we refer the reader to [1].

Our aim in this work is to give another approach to prove such kind of existence result. The key idea in our arguments is inspired from [4]: By using the condition (1.3), we are able to consider a suitable flow line of a considerably simplified vector field and with some properties.
As a consequence, we construct a non-negative and bounded P-S sequence for $I_{K,\mu}$ under the threshold $c^\infty := \frac{\sqrt{2}}{\sqrt{n}}(K_{\infty})^{(n-2)/2}$. Finally, using the compactness result given in Proposition 2.1 below, we obtain a solution $u$ to problem (1.1) with

$$\varpi \leq I_{K,\mu}(u) < c^\infty.$$  

(1.6)

Our arguments enable us to regain the existence result [10, Corollary 4.1]. Namely, we have the following corollary:

**Corollary 1.1** Let $n \geq 3$. Assume that the hypotheses of Theorem 1.1 are satisfied. Then the problem (1.1) has a solution $u$ satisfying

$$I_{K,\mu}(u) = \varpi.$$  

(1.7)

**Remark 1.1** (1.6) and (1.7) enable us to ask: What about uniqueness of critical value or, more precisely, of the solution to (1.1) in the region $[\varpi, c^\infty]$?

As an immediate extension, we have the following existence result which concerns the case $\mu = \mu_1(\Omega)$. Let $e_1$ denotes the eigenfunction of $(-\Delta)$ corresponding to $\mu_1(\Omega)$ with $e_1 > 0$ and $\|e_1\| = 1$.

**Theorem 1.2** Let $n \geq 3$. Assume that the hypotheses of Theorem 1.1 are satisfied with $\mu = \mu_1(\Omega)$. If

$$\int_{\Omega} K(x)e_1^{q+1} < 0,$$

then the problem (1.1) has a solution $u$ satisfying $\varpi \leq I_{K,\mu_1(\Omega)}(u) < c^\infty$. Moreover, $u$ can be chosen such that $I_{K,\mu_1(\Omega)}(u) = \varpi$.

We finish this section by giving an example of function $K(x)$ dealing with the hypotheses of Theorems 1.1 and 1.2.

**Example 1.1** Let $y_0 \in \Omega$ and denote by $2d_0 := \text{dist}(y_0, \partial \Omega)$. We define a function $K : \overset{\circ}{\Omega} \rightarrow \mathbb{R}$ by

$$K(x) = -(1 - \theta(|x - y_0|)) + \varepsilon_0 \cdot \theta(|x - y_0|)(d_0^\beta - \eta|x - y_0|^\beta),$$

where $0 < \varepsilon_0 < d_0$, $0 < \eta \leq 1$ and $\beta \geq 2$ are three fixed constants and $\theta$ is a non-increasing cut-off function with $0 \leq \theta \leq 1$, $\theta(t) = 1$ if $0 \leq t \leq d_0 - \varepsilon_0$ and $\theta(t) = 0$ if $t \geq d_0$. A straightforward calculation shows that the function $K(x)$ satisfies

$$K_{\infty} = K(y_0) = \varepsilon_0 d_0^\beta > 0 \quad \text{and} \quad \int_{\Omega} K(x)e_1^{q+1} < 0 \quad \text{for} \ \varepsilon_0 \ \text{small enough}.$$ 

On the other hand, Lions [10, Remark 4.7] showed that, for $n \geq 5$, the condition (1.3) is satisfied provided that $(n - 2)^2 \bar{c}_2 \Delta K(y_0)/(2nK(y_0)) > -\mu \bar{c}_3$, where $\bar{c}_2$ and $\bar{c}_3$ are two positive constants depending only on $n$. This means that

$$\eta \frac{(n - 2)^2 \bar{c}_2}{d_0^2} < \mu \bar{c}_3 \quad \text{for} \ \beta = 2 \quad \text{and} \ \mu > 0 \quad \text{for} \ \beta > 2.$$
The argument used in [10, Remark 4.7] is still valid to show that if \( n = 4 \), then (1.3) is satisfied for any \( 0 < \mu \leq \mu_1(\Omega) \). If \( n = 3 \), we estimate the quantity \( L_{K,\mu} \) by considering the test function \( u_\varepsilon \) defined by

\[
    u_\varepsilon(x) = \cos\left(\frac{\pi|x-y_0|}{4d_0}\right) \quad \text{on } B(y_0, 2d_0) \quad \text{and} \quad u_\varepsilon(x) = 0 \quad \text{on } \Omega \setminus B(y_0, 2d_0),
\]

where \( \varepsilon > 0 \) is a constant small enough, and we use similar computations as that given in the proof of [8, Lemma 1.3] in order to show that (1.3) is satisfied if \( \mu > \pi^2/16d_0^2 \). This last condition is significant if, for example, \( \Omega = B(y_0, 2d_0) \).

## 2 Proof of the results

**Proof of Theorem 1.1.** To prove Theorem 1.1, we need the following result:

**Proposition 2.1** Let \( K(x) \in C(\bar{\Omega}) \) satisfying \( K_\infty > 0 \) and let \( 0 < \mu \leq \mu_1(\Omega) \). Let \( c < c_\infty \) be a fixed constant. Then any non-negative and bounded P-S sequence for \( I_{K,\mu} \) at \( c \) is relatively compact.

The proof is an adaptation of the arguments used to prove [8, Lemma 1.2] and [5, Lemma 1]. We include it in the Appendix for the reader’s convenience.

Let, for any \( p \geq 1 \), \( M_p \) denotes the following open subset of \( H_0^1(\Omega) \):

\[
    M_p := \left\{ u \in H_0^1(\Omega) : \|u\| > (p + 1)^{-1} \quad \text{and} \quad \int_\Omega K u^{2n/(n-2)} > (p \cdot c_\infty)^{n/(n-2)} \left( \int_\Omega |\nabla u|^2 - \mu \int_\Omega u^2 \right)^{n/(n-2)} \right\},
\]

where \( c_\infty := (n \cdot c_\infty)^{2/n} \). The fact that \( K_\infty > 0 \) and the condition (1.3) assert that \( M_p \) is non-empty. We define the functional \( J_{K,\mu} : M_5 \to \mathbb{R} \) by

\[
    J_{K,\mu}(u) = \int_\Omega |\nabla u|^2 - \mu \int_\Omega u^2 \left( \int_\Omega K |u|^{2n/(n-2)} \right)^{n/(n-2)}. \tag{2.1}
\]

Let \( \Sigma := \{ u \in H_0^1(\Omega) : \|u\| = 1 \} \) and let \( \bar{u}_0 \in (M_1 \cap \Sigma) \) be fixed with \( \bar{u}_0 \geq 0 \). Let \( \tau \) be a smooth non-negative cut-off function such that \( \tau = 1 \) in \( M_2 \) and that \( \tau = 0 \) in \( H_0^1(\Omega) \setminus M_4 \). Finally, consider the following Cauchy problem

\[
    \frac{\partial \eta}{\partial s}(s) = W(\eta(s)), \quad \eta(0) = \bar{u}_0 \in M_1, \tag{2.2}
\]

where \( W(u) \) denotes the following locally Lipschitz vector field defined by

\[
    W(u) = \begin{cases} 
        \tau(u) [-\partial J_{K,\mu}(u)] & \text{if } u \in M_4, \\
        0 & \text{if } u \in H_0^1(\Omega) \setminus M_4.
    \end{cases}
\]

Let $[0, T)$ denote the positive maximal interval defining the solution $\eta(s)$ of (2.2). We will prove some facts satisfied by the flow line $\eta(s)$. We claim that

$$T = +\infty \quad \text{and} \quad \eta(s) \in M_1, \quad \forall \ s \geq 0. \quad (2.3)$$

Set $\bar{s} := \sup \{0 \leq s < T : \eta(t) \in M_2, \ \forall \ 0 \leq t \leq s\}$. The continuity of the function $s \mapsto \eta(s)$ implies that $\bar{s} > 0$. Thus we get

$$W(\eta(s)) = -\partial J_{K,\mu}(\eta(s)), \quad \forall \ 0 \leq s < \bar{s}. \quad (2.4)$$

This, together with (2.2), implies that $J_{K,\mu}(\eta(s)) \leq J_{K,\mu}(\bar{u}_0) < c_\infty(K_\infty)$ for any $0 \leq s < \bar{s}$. Thus we obtain

$$\eta(s) \in M_1, \quad \forall \ 0 \leq s < \bar{s}. \quad (2.5)$$

Using, again, the continuity of the function $s \mapsto \eta(s)$, the definition of $\bar{s}$, (2.4) and the fact that $M_1 \subset M_2$ we derive that $\bar{s} = T$. In particular, (2.2) becomes

$$\frac{\partial \eta}{\partial s}(s) = -\partial J_{K,\mu}(\eta(s)), \quad \forall \ T > s \geq 0, \quad (2.5)$$

$$\eta(0) = \bar{u}_0 \in M_1. \quad (2.6)$$

Using the fact that the functional $J_{K,\mu}$ is homogenous we derive from (2.5) that

$$\langle -\partial J_{K,\mu}(\eta(s)), \eta(s) \rangle = 0, \quad \forall \ T > s \geq 0. \quad (2.7)$$

Combining (2.5)–(2.7) we get

$$\|\eta(s)\| = \|\bar{u}_0\| = 1, \quad \forall \ T > s \geq 0. \quad (2.8)$$

On the other hand, by using the fact that $\mu < \mu_1(\Omega)$ we derive the existence of a constant $c_0 > 0$ such that

$$\int_\Omega |\nabla u|^2 - \mu \int_\Omega u^2 \geq c_0, \quad \forall \ u \in \Sigma. \quad (2.9)$$

The expression of $-\partial J_{K,\mu}$ and (2.9) imply that $\|\partial J_{K,\mu}\|$ is bounded on $M_4 \cap \Sigma$. Thus $W$ is bounded on $\Sigma$. In particular, we derive from (2.8) that

$$T = +\infty. \quad (2.10)$$

This finishes the proof of the claim (2.3). On the other hand, (1.2), (2.1), (2.3), (2.9), together with Sobolev’s inequality and the fact that $\sup_\Omega(K) > 0$, imply that

$$J_{K,\mu}(\eta(s)) \geq L_{K,\mu} > 0, \quad \forall \ s \geq 0. \quad (2.10)$$

(2.3), (2.10) and the fact that $J_{K,\mu}(\eta(s))$ is a non-increasing function imply that

$$L_{K,\mu} \leq \lim_{s \to +\infty} J_{K,\mu}(\eta(s)) = c < c_\infty. \quad (2.11)$$

Combining (2.5) and (2.11) we obtain

$$\int_0^{+\infty} \|\partial J_{K,\mu}(\eta(s))\|^2 \, ds < +\infty.$$
In particular, we derive the existence of a sequence \((s_k), s_k \to +\infty\), such that
\[
\partial J_{K,\mu}(\eta(s_k)) \to 0 \quad \text{strongly in } H^{-1}(\Omega).
\] (2.12)
(In fact, by using similar arguments as that given in the proof of [3, Lemma A1] we can prove that \(\lim_{s \to +\infty} \|\partial J_{K,\mu}(\eta(s))\| = 0\). Finally, up to minor modifications as that given in [?] we can suppose that
\[
\eta(s_k) \geq 0, \quad \forall \ k \geq 0,
\] (2.13)
(more details will be given in a next new version).

Now, to prove Theorem 1.1 we need to construct a sequence \((u_k)\) satisfying the hypotheses of Proposition 2.1. For this, we set, for any \(k \geq 0\),
\[
u_k := \beta_k \cdot \eta(s_k),
\] (2.14)
where \(\beta_k := J_{K,\mu}^{n/4}(\eta(s_k)) / (\int_{\Omega} \|
abla \eta(s_k)\|^2 - \mu \int_{\Omega} |\eta(s_k)|^2)^{1/2}\). Denoting
\[
\beta_k := 2 J_{K,\mu}^{(4-n)/4}(\eta(s_k)) / (\int_{\Omega} \|
abla \eta(s_k)\|^2 - \mu \int_{\Omega} |\eta(s_k)|^2)^{1/2}.
\] A direct calculation shows that
\[
\partial J_{K,\mu}(\eta(s_k)) = \beta_k \cdot \partial I_{K,\mu}(u_k) \quad \text{and} \quad I_{K,\mu}(u_k) = J_{K,\mu}^{n/4}(\eta(s_k)), \quad \forall \ k \geq 0.
\] (2.15)
On the other hand, we derive from (2.3) and (2.8)–(2.10) the existence of two constants \(\tilde{c}_2, \tilde{c}_3 > 0\) such that
\[
\tilde{c}_2 \leq \beta_k, \quad \beta_k \leq \tilde{c}_3, \quad \forall \ k \geq 0.
\] (2.16)
Combining (2.3) and (2.11)–(2.16) we get
\[
(u_k) \text{ is a non-negative and bounded sequence in } H^1_0(\Omega),
\]
\[
\nu \leq \lim_{k \to +\infty} I_{K,\mu}(u_k) = \frac{1}{n} \tilde{c}_2 < c^\infty,
\]
\[
\partial I_{K,\mu}(u_k) \to 0 \quad \text{strongly in } H^{-1}(\Omega).
\] (2.17)
These mean that the sequence \((u_k)\) satisfies the hypotheses of Proposition 2.1 and then, up to a subsequence, \((u_k)\) converges strongly in \(H^1_0(\Omega)\) to a critical point \(u\) of \(I_{K,\mu}\) with \(u \geq 0\), \(\|u\| \neq 0\) and \(\nu \leq I_{K,\mu}(u) < c^\infty\). It follows from the regularity theory for this kind of equation (1.1); see, e.g., [8, Lemma 1.5] and [6, Chapter 9], that \(u \in C^2(\Omega)\). Therefore the strong maximum principle shows that \(u > 0\). This finishes the proof of Theorem 1.1.

**Proof of Corollary 1.1** Let \((w_k)\) be a non-negative minimizing sequence of the problem (1.4) satisfying
\[
\nu \leq \frac{1}{n} (\|w_k\|^2 - \mu \|w_k\|^2)^{1/2} < c^\infty, \quad \forall \ k.
\] (2.18)
From (2.18) we can repeat the proof of Theorem 1.1 by using \(w_k/\|w_k\|\) instead of \(\bar{u}_0\). Therefore, from (2.14), (2.16) and (2.17) we obtain a sequence \((u_k)\) of solutions for the problem (1.1) satisfying
\[
\varpi \leq I_{K,\mu}(u_k) = \frac{1}{n}c_2 \leq \frac{1}{n}\bar{I}_{K,\mu}(w_k) = \frac{1}{n}(\|w_k\|^2 - \mu\|w_k\|_2^2)^{\frac{1}{2}},
\]
\[
c_2 \leq \|u_k\| \leq c_3.
\]
Thus \((u_k)\) is a positive and bounded P-S sequence for \(I_{K,\mu}\) at \(\varpi\). This, together with Proposition 2.1 and the rest of the proof of Theorem 1.1, implies that the problem (1.1) has a solution at \(\varpi\) for \(I_{K,\mu}\). This finishes the proof of Corollary 1.1.

**Proof of Theorem 1.2.** To get the claims of Theorem 1.2, it is sufficient to prove the next lemma:

**Lemma 2.1** If
\[
\int_{\Omega} K(x)e_1^{q+1} < 0,
\]
then there exists a constant \(c > 0\) such that, for any \(u \in H_0^1(\Omega)\) satisfying \(\|u\| = 1\) and \(\int_{\Omega} K(x)|u|^{q+1} \geq 0\), we have
\[
\|u\|^2 - \mu_1(\Omega)\|u\|_2^2 \geq c.
\]

Indeed, by using Lemma 2.1 instead of the condition \(\mu < \mu_1(\Omega)\), (2.9) remains valid in \(\Sigma \cap \{u \in H_0^1(\Omega) : \int_{\Omega} K(x)|u|^{q+1} \geq 0\}\) with a uniform constant \(c_0\). Now, we follow the proof of Theorem 1.1 and Corollary 1.1 step by step in order to prove that (1.1) has a solution \(u\) with \(\varpi \leq I_{K,\mu_1(\Omega)}(u) < c\) and \(I_{K,\mu_1(\Omega)}(u) = \varpi\), respectively.

**Proof of Lemma 2.1** Arguing by contradiction, assuming that there exists a sequence \((u_k)\) in \(H_0^1(\Omega)\) satisfying
\[
\|u_k\| = 1 \quad \text{and} \quad \int_{\Omega} K(x)|u_k|^{q+1} \geq 0, \quad \forall \ k,
\]
\[
\lim_{k \to +\infty} \|u_k\|^2 - \mu_1(\Omega)\|u_k\|_2^2 = 0.
\]

Let, for every \(k\),
\[
u_k = \alpha_k e_1 + v_k
\]
be the decomposition of \(u_k\) in the Hilbert space \((L^2(\Omega), \|\|_2)\). By using the Hilbert basis \((e_s)_{s \geq 1}\) of \((L^2(\Omega), \|\|_2)\) defined by
\[
e_s \in H_0^1(\Omega) \quad \text{and} \quad -\Delta(e_s) = \mu_s(\Omega)e_s, \quad \forall \ s \geq 1,
\]
(see, e.g., [6, Theorem 9.31]), we derive that
\[
\|v_k\|^2 - \mu_1(\Omega)\|v_k\|_2^2 \geq \inf_{s \geq 2}(\frac{\mu_s(\Omega) - \mu_1(\Omega)}{\mu_s(\Omega)})\|v_k\|^2, \quad \forall \ k.
\]
Combining (2.22)–(2.24) we obtain, up to a subsequence,

\[
0 \leq \inf_{s \geq 2} \left( \frac{\mu_s(\Omega) - \mu_1(\Omega)}{\mu_s(\Omega)} \right) \lim_{k \to +\infty} \| v_k \|^2 \leq \lim_{k \to +\infty} \| v_k \|^2 - \mu_1(\Omega) \| v_k \|^2 = \lim_{k \to +\infty} \| u_k \|^2 - \mu_1(\Omega) \| u_k \|^2 = 0.
\]

This, together with the fact that \( \inf_{s \geq 2} \left[ (\mu_s(\Omega) - \mu_1(\Omega))/\mu_s(\Omega) \right] \neq 0 \), implies that

\[
\lim_{k \to +\infty} \| v_k \|^2 = 0.
\]

By combining (2.21), (2.23) and (2.25) we derive that

\[
u_k \to \pm e_1 \quad \text{strongly in} \quad H^1_0(\Omega).
\]

This, together with the continuity of the injection \( H^1_0(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega) \) and (2.21), implies that

\[
0 \leq \lim_{k \to +\infty} \int_{\Omega} K(x) | u_k |^{q+1} = \int_{\Omega} K(x) e_1^{q+1},
\]

which contradicts (2.19). Thus the claim (2.20) follows.

3 Appendix

Proof of Proposition 2.1. Let \((u_k)_k\) be a non-negative and bounded sequence in \( H^1_0(\Omega) \) satisfying

\[
I_{K,\mu}(u_k) \to c \quad \text{and} \quad \partial I_{K,\mu}(u_k) \to 0 \quad \text{in} \quad H^{-1}(\Omega) \quad \text{with} \quad c < c^\infty.
\]

Since \((u_k)_k\) is bounded in \( H^1_0(\Omega) \), then there exists \( u \in H^1_0(\Omega) \) such that, up to a subsequence still denoted by \((u_k)_k\),

\[
u_k \to u \quad \text{weakly in} \quad H^1_0(\Omega).
\]

Thus, due to the fact that the injection \( H^1_0(\Omega) \subset L^2(\Omega) \) is compact, we get, up to a subsequence,

\[
u_k \to u \quad \text{strongly in} \quad L^2(\Omega).
\]

In particular, we derive from (3.3) that, up to a subsequence,

\[
u_k \to u \quad \text{a.e. on} \quad \Omega.
\]

This, together with the fact that \((u_k)_k\) is bounded in \( L^{\frac{2n}{n-2}} \), implies that, passing to a further subsequence,

\[
u_k^{\frac{n+2}{n-2}} \to u^{\frac{n+2}{n-2}} \quad \text{weakly in} \quad L^{\frac{2n}{n+2}} \quad \text{(see, e.g., [6, Exercise 4.16])}.
\]

Combining (3.1)–(3.3) and (3.5) we derive that

\[
\int_{\Omega} | \nabla u |^2 - \int_{\Omega} K u^{\frac{2n}{n+2}} - \mu \int_{\Omega} u^2 = 0.
\]
In particular, we obtain
\[ I_{K,\mu}(u) = \frac{1}{n} \left( \int_{\Omega} |\nabla u|^2 - \mu \int_{\Omega} u^2 \right) \geq 0. \] (3.7)

Let \( w_k := u_k - u \) for any \( k \). We deduce from the Brezis–Lieb’s result [7, Theorem 1] and (3.4) that
\[ \int_{\Omega} K(x)|u_k|^\frac{2n}{n-2} \, dx = \int_{\Omega} K(x)|w_k|^\frac{2n}{n-2} \, dx + \int_{\Omega} K(x)|u|^\frac{2n}{n-2} \, dx + o(1). \] (3.8)

By combining (3.1), (3.3), (3.6) and (3.8) we get
\[ o(1) = \langle \partial I_{K,\mu}(u_k), u_k \rangle = \int_{\Omega} |\nabla u_k|^2 - \int_{\Omega} K u_k^\frac{2n}{n-2} - \mu \int_{\Omega} u_k^2 = \int_{\Omega} |\nabla w_k|^2 - \int_{\Omega} K |w_k|^\frac{2n}{n-2} + o(1). \] (3.9)

This, together with (1.5), implies that
\[ I_{K,\mu}(w_k) = \frac{1}{n} \int_{\Omega} |\nabla w_k|^2 + o(1). \] (3.10)

On the other hand, by combining (1.5), (3.2), (3.3), (3.7) and (3.8) we get
\[ I_{K,\mu}(u_k) = I_{K,\mu}(w_k) + I_{K,\mu}(u) + o(1) \geq I_{K,\mu}(w_k) + o(1). \] (3.11)

We deduce from (3.1), (3.10) and (3.11) that, for \( \varepsilon > 0 \) a constant small enough and \( k_0 \) large enough,
\[ \int_{\Omega} |\nabla w_k|^2 \leq \frac{1}{(K_\infty)^\frac{n}{n-2}} (S - \varepsilon)^\frac{n}{2}, \quad \forall \ k \geq k_0. \] (3.12)

Finally, by using Sobolev’s inequality and the fact that \( K_\infty > 0 \) we obtain
\[ \int_{\Omega} K |w_k|^\frac{2n}{n-2} \leq K_\infty \cdot S^{-\frac{n}{n-2}} \left( \int_{\Omega} |\nabla w_k|^2 \right)^\frac{n}{n-2}, \quad \forall \ k. \] (3.13)

Combining (3.9), (3.12) and (3.13), we get
\[ \int_{\Omega} |\nabla w_k|^2 \leq \left( \frac{S - \varepsilon}{S} \right)^\frac{n}{n-2} \int_{\Omega} |\nabla w_k|^2 + o(1), \]
which implies that \( \lim_{k \to +\infty} \int_{\Omega} |\nabla w_k|^2 = 0 \), and then the claim of Proposition 2.1 follows.

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**References**

[1] S. Alama and G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities*, Calc. Var. Partial Differential Equations 1 (1993), no. 4, 439–475.
[2] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer-Verlag, Berlin, 1998.

[3] A. Bahri and J. M. Coron, *The scalar-curvature problem on the standard three-dimensional sphere*, J. Funct. Anal. 95 (1991), 106–172.

[4] Z. Boucheche, *Existence result for an elliptic equation involving critical exponent in three-dimensional domains*, Complex Var. Elliptic Equ. 64 (2019), no. 4, 649–675.

[5] H. Brezis, *Elliptic equations with limiting Sobolev exponents- the impact of topology*, Comm. Pure Appl. Math. 39 (1986), no. S1, S17–S39.

[6] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer-Verlag, New York, 2011.

[7] H. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88 (1983), no. 3, 486–490.

[8] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (1983), no. 4, 437–477.

[9] P. L. Lions, *The concentration compactness principle in the calculus of variations. The limit case, part 1*, Rev. Mat. Iberoam. 1 (1985), no. 1, 145–201.

[10] P. L. Lions, *The concentration compactness principle in the calculus of variations. The limit case, part 2*, Rev. Mat. Iberoam. 1 (1985), no. 2, 45–121.