COULOMB BRANCHES OF NONCOTANGENT TYPE

ALEXANDER BRAVERMAN, GURBIR DHILLON, MICHAEL FINKELBERG, SAM RASKIN, AND ROMAN TRAVKIN, WITH APPENDICES BY GURBIR DHILLON AND THEO JOHNSON-FREYD

Abstract. We propose a construction of the Coulomb branch of a 3d $N = 4$ gauge theory corresponding to a choice of a connected reductive group $G$ and a symplectic finite-dimensional representation $M$ of $G$, satisfying certain anomaly cancellation condition. This extends the construction of [BFN1] (where it was assumed that $M = N \oplus N^*$ for some representation $N$ of $G$). Our construction goes through certain “universal” ring object in the twisted derived Satake category of the symplectic group $Sp(2n)$. The construction of this object uses a categorical version of the Weil representation; we also compute the image of this object under the (twisted) derived Satake equivalence and show that it can be obtained from the theta-sheaf $[Ly, LL]$ on $Bun_{Sp(2n)}(P^1)$ via certain Radon transform. We also discuss applications of our construction to a potential mathematical construction of $S$-duality for super-symmetric boundary conditions in 4-dimensional gauge theory and to (some extension of) the conjectures of Ben-Zvi, Sakellaridis and Venkatesh.

Contents

1. Introduction 2
1.1. Symplectic duality 2
1.2. 3-dimensional $N = 4$ quantum field theories 3
1.3. Coulomb branches via ring objects in the derived Satake category 4
1.4. Ring objects for general $M$ and twisted Satake category 4
1.5. The universal twisted ring object 5
1.6. Idea of the construction 6
1.7. $S$-duality and Ben-Zvi-Sakellaridis-Venkatesh conjectures 7
1.8. The universal ring object under Satake equivalence 10
1.9. Acknowledgments 11
2. Setup and notation 11
2.1. Affine Grassmannians 11
2.2. D-modules 12
2.3. Weyl algebra 13
2.4. Twisted derived Satake 13
3. The universal ring object 14
3.1. The internal Hom construction 14
1. Introduction

1.1. Symplectic duality. Let $X$ be an algebraic variety over $\mathbb{C}$. We say that $X$ is singular symplectic (or $X$ has symplectic singularities) if

(1) $X$ is a normal Poisson variety;

(2) There exists a smooth dense open subset $U$ of $X$ on which the Poisson structure comes from a symplectic structure. We shall denote by $\omega$ the corresponding symplectic form.

(3) There exists a resolution of singularities $\pi: \tilde{X} \to X$ such that $\pi^*\omega$ has no poles on $\tilde{X}$.

We say that $X$ is a conical symplectic singularity if in addition to (1)-(3) above one has a $\mathbb{C}^*$-action on $X$ which acts on $\omega$ with some positive weight and which contracts all of $X$ to one point.

A symplectic resolution $\pi: \tilde{X} \to X$ is a proper and birational morphism $\pi$ such that $\pi^*\omega$ extends to a symplectic form on $\tilde{X}$. Here is one example. Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$ and let $\mathcal{N}_0 \subset \mathfrak{g}^*$ be its nilpotent cone. Let $\mathcal{B}$
COULOMB BRANCHES OF NONCOTANGENT TYPE

denote the flag variety of $\mathfrak{g}$. Then the Springer map $\pi : T^*\mathcal{B} \to N_\mathfrak{g}$ is proper and birational, so if we let $X = N_\mathfrak{g}, \check{X} = T^*\mathcal{B}$ we get a symplectic resolution.

The idea of symplectic duality is this: often conical symplectic singularities come in “dual” pairs $(X, X^*)$ (the assignment $X \to X^*$ is by no means a functor; we just have a lot of interesting examples of dual pairs). What does it mean that $X$ and $X^*$ are dual? This is in general not easy to tell, but many geometric questions about $X$ should be equivalent to some other geometric questions about $X^*$. For example, we should have $\dim H^\bullet(\check{X}, \mathbb{C}) = \dim H^\bullet(\check{X}^*, \mathbb{C})$ (but these spaces are not supposed to be canonically isomorphic). We refer the reader to [BPW], [BLPW] for more details. There should be a lot of other connections between $X$ and $X^*$ which will take much longer to describe; we refer the reader to loc.cit. for the description of these properties as well as for examples.

1.2. 3-dimensional $N = 4$ quantum field theories. One source of dual pairs $(X, X^*)$ comes from quantum field theory in the following way. Physicists have a notion of 3-dimensional $N = 4$ super-symmetric quantum field theory. Any such theory $\mathcal{T}$ is supposed to have a well-defined moduli space of vacua $\mathcal{M}(\mathcal{T})$. This space is complicated, but it should have two special pieces called the Higgs and the Coulomb branch; we shall denote these by $\mathcal{M}_H(\mathcal{T})$ and $\mathcal{M}_C(\mathcal{T})$. They are supposed to be (singular) symplectic complex algebraic varieties (in fact, they don’t even have to be algebraic but for simplicity we shall only consider examples when they are).

Let $G$ be a complex reductive algebraic group and let $M$ be a symplectic vector space with a Hamiltonian action of $G$. Then to the pair $(G, M)$ one is supposed to associate a theory $\mathcal{T}(G, M)$ provided that $M$ satisfies certain anomaly cancellation condition, which can be formulated as follows. The representation $M$ defines a homomorphism $G \to \text{Sp}(M)$ and thus a homomorphism $\pi_4(G) \to \pi_4(\text{Sp}(M)) = \mathbb{Z}/2\mathbb{Z}$. The anomaly cancellation condition is the condition that this homomorphism is trivial. Without going to further details at the moment we would like to emphasize the following:

1) Any $M$ of the form $T^*N = N \oplus N^*$ where $N$ is some representation $G$ satisfies this condition.

2) The anomaly cancellation condition is a “$\mathbb{Z}/2\mathbb{Z}$-condition” (later on we are going to formulate it more algebraically).

Assume that we are given $M$ as above for which the anomaly cancellation condition is satisfied. Then the theory $\mathcal{T}(G, M)$ is called gauge theory with gauge group $G$ and matter $M$. Its Higgs branch is expected to be equal to $M \sslash G$: the Hamiltonian reduction of $M$ with respect to $G$. In particular, all Nakajima quiver varieties arise in this way (the corresponding theories are called quiver gauge theories).
The corresponding Coulomb branches are much trickier to define. Physicists had some expectations about those but no rigorous definition in general (only some examples). The idea is that at least in the conical case the pair \((M_H(T), M_C(T))\) should produce an example of a dual symplectic pair. A mathematical approach to the definition of Coulomb branches was proposed in [N]. A rigorous definition of the Coulomb branches \(M_C(G, M)\) is given in [BFN1] under the assumption that \(M = T^*N = N \oplus N^*\) for some representation \(N\) of \(G\).

The varieties \(M_C(G, M)\) are normal, affine, Poisson, generically symplectic and satisfy the monopole formula. We expect that they are singular symplectic, but we can not prove this in general, cf. [We]. The main ingredient in the definition is the geometry of the affine Grassmannian \(\text{Gr}_G\) of \(G\). In [BFN1, BFN2, BFN3] these varieties are computed in many cases (in particular, in the case of so called quiver gauge theories — it turns out that one can associate a pair \((G, N)\) to any framed quiver). The quantizations of these varieties are also studied, as well as their (Poisson) deformations and (partial) resolutions.

1.3. Coulomb branches via ring objects in the derived Satake category.
Let \(\mathcal{K} = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[t]\). The affine Grassmannian ind-scheme \(\text{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}\) is the moduli space of \(G\)-bundles on the formal disc equipped with a trivialization on the punctured formal disc. One can consider the derived Satake category \(D_{G_{\mathcal{O}}}(\text{Gr}_G)\). This is a monoidal category which is monoidally equivalent to \(D^{G^\vee}(\text{Sym}^\bullet(\mathfrak{g}^\vee[-2]))\): the derived category of dg-modules over \(\text{Sym}^\bullet(\mathfrak{g}^\vee[-2])\) endowed with a compatible action of \(G^\vee\) (the monoidal structure on this category is just given by tensor product over \(\text{Sym}^\bullet(\mathfrak{g}^\vee[-2])\)); we shall denote the corresponding functor from \(D_{G_{\mathcal{O}}}(\text{Gr}_G)\) to \(D^{G^\vee}(\text{Sym}^\bullet(\mathfrak{g}^\vee[-2]))\) by \(\Phi_G\). In [BFN3] we have attached to any \(N\) as above a certain ring object \(A_{G, M}\) in \(D_{G_{\mathcal{O}}}(\text{Gr}_G)\) (here as before we set \(M = T^*N\) such that the algebra of functions on \(M_C(G, M)\) is equal to \(H^{\bullet}_{G_{\mathcal{O}}}(\text{Gr}_G, A_{G, M})\) (this cohomology has an algebra structure coming from the fact that \(A_{G, M}\) is a ring object).

1.4. Ring objects for general \(M\) and twisted Satake category. One of the main goals of this paper is to construct the ring object \(A_{G, M}\) for arbitrary symplectic representation \(M\) satisfying the anomaly cancellation condition. In fact, we can construct the ring object \(A_{G, M}\) for any symplectic \(M\) but instead of

---

1In addition to the Coulomb branch \(M_C(G, M)\), in [BFN1, Remark 3.14] the authors define the \(K\)-theoretic Coulomb branch \(M^K_C(G, M)\) under the same assumption (physically, it should correspond to the Coulomb branch of the corresponding 4d gauge theory of \(\mathbb{R}^3 \times S^1\)). We would like to emphasize that at this point we are not able to extend this construction to arbitrary symplectic \(M\) with anomaly cancellation condition.

2In fact we are going to work with a certain renormalized version of it, cf. §2.1.

3Another construction of the Coulomb branch of a 3d \(N = 4\) gauge theory in the noncotangent case was proposed by C. Teleman [T].
being an object of the derived Satake category $D_{G^0}(\text{Gr}_G)$ it will be an object of a certain twisted version of it. More precisely, the representation $M$ defines certain determinant line bundle $D_M$ on $\text{Gr}_G$ which is equipped with certain multiplicative structure; we shall denote by $D^0_M$ the total space of this bundle without the zero section. The line bundle $D_M$ is also $G_0$-equivariant. In particular, for any $\tau \in \mathbb{C}$ one can consider the category $D^{G^0}_\tau(\text{Gr}_G)$ of $G_0$-equivariant sheaves on $D^0_M$ which are $\mathbb{C}^\times$-monodromic with monodromy $q = e^{2\pi i \tau}$. This category is again monoidal (because of the above multiplicative structure on $D_M$). If $\tau$ is a rational number and $D^0_M$ exists as a multiplicative line bundle on $\text{Gr}_G$, the twisted category $D^{G^0}_\tau(\text{Gr}_G)$ is naturally equivalent to $D_{G^0}(\text{Gr}_G)$ (as a monoidal category).

In this paper we shall construct a ring object $A_{G,M} \in D_{G^0}^{-1/2}(\text{Gr}_G)$. It turns out (see Proposition 4.1.1) that the anomaly cancellation condition is equivalent to the existence of a multiplicative square root of $D_M$. So, we can construct the ring object $A_{G,M}$ but it will be untwisted only if the anomaly cancellation condition is satisfied. In particular, we can take its $G_0$-equivariant cohomology (and thus define the algebra of functions on the corresponding Coulomb branch) only under the anomaly cancellation assumption.

1.5. The universal twisted ring object. In fact in order to construct the ring object $A_{G,M}$ for any $G$ and $M$ it is enough to do it when $G = \text{Sp}(2n)$ and $M = \mathbb{C}^{2n}$ is its tautological representation. The reason is as follows. Assume first that $M = T^*N$ and let $i: G' \to G$ be a homomorphism of connected reductive groups. It induces a morphism $\tilde{i}: \text{Gr}_{G'} \to \text{Gr}_G$, and it follows from the construction of [BFN3] that $A_{G',M} = \tilde{i}^!A_{G,M}$. Assuming that the same is true for arbitrary $M$ and since the symplectic representation $G \to \text{Sp}(M)$ we see that the case $G = \text{Sp}(M)$ is universal in the sense that the object $A_{G,M}$ in general should just be equal to the $!$-pullback of $A_{\text{Sp}(M),M}$.

In this paper we do the following:

1) We construct the object $A_{G,M}$ (as was explained above it is enough to do it in the case $G = \text{Sp}(M)$).

2) We check that when $M = T^*N$ for some representation $N$ of $G$, this construction coincides with the one of [BFN3].

3) In the case when $G = \text{Sp}(M)$ we compute the image of $A_{G,M}$ under the twisted version of the derived geometric Satake equivalence (see §1.8 below). To do that we express $A_{G,M}$ as a Radon transform of a certain theta-sheaf [Ly, LL] for the curve $\mathbb{P}^1$ (the necessary facts and definitions about the Radon transform are reviewed in Appendix A). The idea that $A_{G,M}$ should be related to the theta-sheaf also belongs to V. Drinfeld.

---

4This was first observed by V. Drinfeld.
1.6. **Idea of the construction.** Let us briefly explain the idea of the construction of $A_{G,M}$. Let $\mathcal{C}$ be a (dg) category endowed with a strong action of an algebraic group $H$ (e.g. one can take $\mathcal{C}$ to be the (dg-model of the) derived category of $D$-modules on a scheme $X$ endowed with an action of $H$). Let $F$ be an object of $\mathcal{C}$ which is equivariant under some closed subgroup $L$ of $H$. Then one can canonically attach to $F$ a ring object $A_F \in D\text{-mod}_{L,H/L}$ (the $L$-equivariant derived category of $D$-modules on $H/L$; this category is endowed with a natural monoidal structure with respect to convolution). This object has the property that its $!$-restriction to any $h \in H$ is equal to $R\text{Hom}(F,F^h)$.

Here is a variant of this construction. Assume that $H$ is endowed with a central extension $1 \rightarrow \mathbb{C}^\times \rightarrow \tilde{H} \rightarrow H \rightarrow 1$, which splits over $L$. Then for any $\kappa \in \mathbb{C}$ it makes sense to talk about an action $H$ on $\mathcal{C}$ of level $\kappa$. Then in the same way as above we can define $A_\sigma \in D\text{-mod}_{L,\kappa}(H/L)$ (the $L$-equivariant derived category of $D$-modules on $H/L$; this category is endowed with a natural monoidal structure with respect to convolution). This object has the property that its $!$-restriction to any $h \in H$ is equal to $R\text{Hom}(\mathcal{F},\mathcal{F}^h)$.

Here is a variant of this construction. Assume that $H$ is endowed with a central extension $1 \rightarrow \mathbb{C}^\times \rightarrow \tilde{H} \rightarrow H \rightarrow 1$, which splits over $L$. Then for any $\kappa \in \mathbb{C}$ it makes sense to talk about an action $H$ on $\mathcal{C}$ of level $\kappa$. Then in the same way as above we can define $A_\sigma \in D\text{-mod}_{L,\kappa}(H/L)$ (the $L$-equivariant derived category of $D$-modules on $H/L$; this category is endowed with a natural monoidal structure with respect to convolution). This object has the property that its $!$-restriction to any $h \in H$ is equal to $R\text{Hom}(\mathcal{F},\mathcal{F}^h)$.

The relationship between the two is that $T(G,M)$ is obtained from $B(G,M)$ by pairing with the Dirichlet boundary condition for Yang-Mills; this implies that $T(G,M)$ has $G$-flavor symmetry (it comes from the corresponding symmetry of the Dirichlet boundary condition). Our constructions yield algebraic data attached to $A$-twists of the resulting physical theories. The category $W\text{-mod}$ is the category of line operators of (the $A$-twist of) $T(G,M)$, and the $G_\kappa$-action on $W\text{-mod}$ expresses the $G$-flavor symmetry of $T(G,M)$. More details about the connection between our language and the physics language can be found in [HR].

Remark 1.6.1. Here we make a remark about a connection between the above construction and some physics terminology. Suppose $M$ is a symplectic representation of $G$ and suppose the anomaly cancellation holds. In this case, physicists would say that there are two (closely related) structures attached to this data:

a) a 3d $N = 4$ theory $T(G,M)$ such that $T(G,M)$ has what physicists call $G$-flavor symmetry. In this case one can **gauge this symmetry** to get a new 3d $N = 4$ theory; this new theory is the theory $T(G,M)$ discussed in §1.2;

b) a supersymmetric boundary condition $B(G,M)$ for 4d $N = 4$ Yang-Mills.

The relationship between the two is that $T(G,M)$ is obtained from $B(G,M)$ by pairing with the Dirichlet boundary condition for Yang-Mills; this implies that $T(G,M)$ has $G$-flavor symmetry (it comes from the corresponding symmetry of the Dirichlet boundary condition). Our constructions yield algebraic data attached to $A$-twists of the resulting physical theories. The category $W\text{-mod}$ is the category of line operators of (the $A$-twist of) $T(G,M)$, and the $G_\kappa$-action on $W\text{-mod}$ expresses the $G$-flavor symmetry of $T(G,M)$. More details about the connection between our language and the physics language can be found in [HR].

---

Footnote 5: This action should be thought of as a categorical analog of the Weil representation, cf. [LL].
1.7. S-duality and Ben-Zvi-Sakellaridis-Venkatesh conjectures. This sub-
section is somewhat digressive from the point of view of the main body of this
paper. We include it here for completeness and in order to indicate some future
research directions.

1.7.1. S-duality for boundary conditions. The papers [GW1, GW2] developed the
theory of super-symmetric boundary conditions in 4d gauge theories; it follows
from loc.cit. that in addition to symplectic duality one should expect some kind
of S-duality for affine symplectic varieties M endowed with a Hamiltonian action
of G (here we no longer assume that M is a vector space) and with a C\(^\times\)-action
for which the symplectic form has degree 2 — again, satisfying some kind of
anomaly cancellation condition (we don’t know how to formulate it precisely, but
when M is a symplectic vector space with a linear action of G, it should be the
same condition as before; also, this condition should automatically be satisfied
when M = T\(^*\)N where N is a smooth affine G-variety). The S-dual of M is
another affine variety M\(^\vee\) endowed with a Hamiltonian action of the Langlands
dual group G\(^\vee\). In fact, this kind of duality is not expected to be well-defined for
arbitrary M — only in some “nice” cases, which we don’t know how to describe
mathematically. Physically, it is explained in loc.cit. that to any M as above
one can attach a super-symmetric boundary condition in the corresponding 4-
dimensional gauge theory; S-duality is supposed to be a well-defined operations on
such boundary conditions, but since not all super-symmetric boundary conditions
come from M as above, it follows that M\(^\vee\) will be well-defined only if we are
sufficiently lucky. It should also be noted that in general one should definitely
consider singular symplectic varieties. On the other hand, below we describe a
rather general construction and some expected properties of it. Let us also note
that more generally, when the anomaly cancellation condition is not satisfied,
one should expect a duality between varieties M and M\(^\vee\) endowed with some
additional “twisting data”.

1.7.2. The Whittaker reduction. Before we discuss a somewhat general approach
to the construction of the S-duality, let us give some explicit examples as well
as some properties of S-duality. First we need to recall the notion of Whittaker
reduction.

Let M be any Hamiltonian G-variety (i.e. M is a Poisson variety with a Hamil-
tonian G-action). Let \( \mu: M \to g^* \) be the corresponding moment map. Let also
\( U \subset G \) be a maximal unipotent subgroup of G and let \( \psi: U \to G_\alpha \) be a generic
homomorphism. Then we set Whit\(_G\)(M) to be the Hamiltonian reduction of M
with respect to \( (U, \psi) \). In other words, let us view \( \psi \) as an element of \( u^* \) (here
u is the Lie algebra of U) and let \( g^*_\psi \) be the pre-image of \( \psi \) under the natural
projection \( g^* \to u^* \). Then

\[
\text{Whit}_G(M) = (\mu^{-1}(g^*_\psi))/U.
\]
It is well-known (cf. [K]) that the action of $U$ on $g^*$ is free, so it is also free on $\mu^{-1}(g^*)$. However, in principle $\mu^{-1}(g^*)$ might be a dg-scheme. For simplicity we shall usually assume that it is not the case (for this it is enough to assume that $\mu$ is flat over the regular part of $g^*$).

More generally, we can talk about the Whittaker reduction of any $G$-equivariant $\text{Sym}(g)$-module. The connection between the Whittaker reduction and the derived Satake isomorphism is this: it is shown in [BeF] that for any $F \in D_{G}(\text{Gr}_{G})$ we have

$$H^*_{G^\circ}(\text{Gr}_{G}, F) = \text{Whit}_{G^\circ}(\Phi(F)).$$

1.7.3. Some expected properties of $S$-duality. Here are some purely mathematical properties that are expected to be satisfied by the $S$-dual variety $M^\vee$ (when it is well-defined):

1) Assume that $M$ is a point. Then $M^\vee = \text{Whit}_{G^\vee}(T^*G^\vee)$ (note that $T^*G^\vee$ is endowed with two commuting $G^\vee$-actions, so after we take the Whittaker reduction with respect to one of them, the 2nd one remains).

2) Let $H$ be a connected reductive group and set $G = H \times H$. Let $M = T^*H$ (with natural $G$-action). Then we should have $M^\vee = T^*H^\vee$.

3) Assume that $M$ is a linear symplectic representation of $G$ satisfying the anomaly cancellation condition. Then one should have

$$M_C(G, M) = \text{Whit}_{G^\vee}(M^\vee).$$

4) We expect that $(M^\vee)^\vee = M$ whenever it makes sense.

1.7.4. Construction of $M^\vee$ in the cotangent case. Here is a construction in the case when $M = T^*N$ where $N$ is a smooth affine $G$-variety. The construction of the ring object $A_{G,M}$ from [BFN3] makes sense verbatim in this case (in [BFN3] $N$ was a vector space but it is not important for the construction). Let us consider $\Phi_G(A_{G,M})$. This is a commutative ring object of the derived category of $G^\vee$-equivariant dg-modules over $\text{Sym}^*(g^\vee[-2])$. Passing to its cohomology $H^*(\Phi_G(A_{G,M}))$ we just get a graded commutative algebra over $\text{Sym}^*(g^\vee[-2])$. Assuming that it has no cohomology in odd degrees, we can pass to its spectrum $M^\vee$. This is an affine scheme with an action of $G^\vee$ which is endowed with a compatible map to $(g^\vee)^*$. In fact, the object $A_{G,M}$ is naturally equivariant with respect to the $C^\infty$-action which rescales $t \in \mathcal{K}$ (this action is usually called “loop rotation”). It is not difficult to see that (in the same way as in [BFN3]) this defines a natural non-commutative deformation of the ring $H^*(\Phi_G(A_{G,M}))$, and it particular, we get a Poisson structure on $M^\vee$. This Poisson structure is easily seen to be generically symplectic and the above map to $(g^\vee)^*$ is the moment map for the $G^\vee$-action and this Poisson structure. The grading on the ring

$^6$In all the interesting cases we know the algebra $\Phi_G(A_{G,M})$ is formal, so we do not loose any information after passing to cohomology.
COULOMB BRANCHES OF NONCOTANGENT TYPE

\(H^\bullet(\Phi_G(A_{G,M}))\) defines a \(\mathbb{C}^\times\)-action on \(M^\vee\) with respect to which the symplectic form has degree 2 (more precisely, we must divide the homological grading by 2: we can do that since we are assuming that we only have cohomology in even degrees).

It is easy to see that the above definition satisfies properties 1-3) of §1.7.3. Namely, 1) is proved in [BeF], 2) essentially follows from the construction of the derived Satake equivalence, and 3) immediately follows from (1.7.1). On the other hand, property 4) does not hold in this generality — it fails already when \(G\) is trivial; in general it is hard to formulate since typically even if \(M = T^*N\) with smooth \(N\), the variety \(M^\vee\) will be singular; also if it is smooth it might not be isomorphic to a cotangent of anything. But even when it is, the involutivity of the duality is far from obvious. Again, we believe that in some “nice” cases the equality \((M^\vee)^\vee = M\) makes sense and it is true (we do not know how to say what “nice” means, but some examples are discussed below).

One can construct a natural functor from \(D_{G_0}(N_K)\) to \(D^{G^\vee}(\Phi_G(A_{G,M}))\). Assuming formality of the ring \(\Phi_G(A_{G,M})\) we can just think about the latter category as the derived category of \(G^\vee\)-equivariant dg-modules over the coordinate ring \(\mathbb{C}[M^\vee]\), when the latter is regarded as a dg-algebra with trivial differential and grading given by the above \(\mathbb{C}^\times\)-action. Ben-Zvi, Sakellaridis and Venkatesh conjectured that when \(N\) is a spherical variety for \(G\) (i.e. when it has an open orbit with respect to a Borel subgroup of \(G\)), this functor is an equivalence. In fact, in this formulation the above conjecture is not very hard – the real content of the conjecture (which we are not going to describe here) is hidden in the explicit (essentially combinatorial) calculation of \(M^\vee\) when \(M = T^*N\), where \(N\) is a smooth spherical \(G\)-variety (this is done in [BZSV]; also, under some assumptions the conjecture of [BZSV] should hold for singular spherical \(N\), but in this case it is much harder to formulate).

1.7.5. An example. Here is another example. Let \(G = GL(N) \times GL(N-1)\) and let \(M = T^*GL(N)\) where the action of \(G\) comes from the action of \(GL(N)\) on itself by left multiplication and from the action of \(GL(N-1)\) by right multiplication via the standard embedding \(GL(N-1) \hookrightarrow GL(N)\). In this case \(N = GL(N)\) is a spherical \(G\)-variety. Then it is essentially proved in [BFGT] that \(M^\vee = T^*Hom(\mathbb{C}^N, \mathbb{C}^{N-1})\) and the Ben-Zvi-Sakellaridis-Venkatesh conjecture holds. It is, however, not clear how to deduce from this that \((M^\vee)^\vee = M\). A construction of the isomorphism \((T^*Hom(\mathbb{C}^N, \mathbb{C}^{N-1}))^\vee \simeq T^*GL(N)\) is going to appear in a forthcoming paper of T.-H. Chen and J. Wang.

1.7.6. S-duality outside of the cotangent type (linear case). In all of the above examples we only worked with cases when \(M = T^*N\) for some smooth affine \(G\)-variety \(N\). However, the main construction of this paper allows us to extend it to the case when \(M\) is an arbitrary symplectic representation of \(G\) satisfying the
anomaly cancellation condition.\footnote{One can also talk about $S$-duality for twisted objects, but we will not discuss it here.} Namely, as before we just let $M^\vee$ be the spectrum of $H^\bullet(\Phi_G(A_{G,M}))$ (also as before let us assume that there is no cohomology in odd degrees).

The following example is similar to the one of §1.7.5. Let $N$ be a positive integer. Let $G = \text{Sp}(2N) \times \text{SO}(2N)$. Let also $M$ be the bi-fundamental representation of $G$ (i.e. $M = \mathbb{C}^{2N} \otimes \mathbb{C}^{2N}$ with the natural action of $G$). Then $G^\vee = \text{SO}(2N+1) \times \text{SO}(2N)$, and we conjecture that $M^\vee = T^\ast \text{SO}(2N+1)$ (with the action of $G^\vee = \text{SO}(2N+1) \times \text{SO}(2N)$ defined similarly to the example in §1.7.5). Note that if $N > 2$ then $M$ is an irreducible representation of $G$, so it cannot be written as $T^\ast N$ for another representation $N$. On the other hand, $M^\vee$ is manifestly written as a cotangent bundle to $N^\vee = \text{SO}(2N+1)$ and the fact that $(M^\vee)^\vee = M$ (together with the corresponding special case of the Ben-Zvi-Sakellaridis-Venkatesh conjecture) is proved in [BFT]. However, we do not know at the moment how to prove that $M^\vee = T^\ast \text{SO}(2N+1)$ (but at least the main construction of this paper allows us to formulate this statement).

Here is a variant of this example. Let $G = \text{SO}(2N) \times \text{Sp}(2N-2)$ (here we assume that $N > 1$ and let $M$ be again its bi-fundamental representation. Then $G^\vee = \text{SO}(2N) \times \text{SO}(2N-1)$, and we expect that $M^\vee = T^\ast \text{SO}(2N)$ (the action of $G^\vee = \text{SO}(2N) \times \text{SO}(2N-1)$ is again defined similarly to the example in §1.7.5).

1.8. The universal ring object under Satake equivalence. Finally, we are able to describe the image of the universal ring object under the twisted Satake equivalence (answering a question of V. Drinfeld). First, it turns out that for $G = \text{Sp}(M)$, $g = \mathfrak{sp}(M)$, there is a monoidal equivalence $\Phi_G: D_{1/2}^G(Gr_G) \sim D^G(\text{Sym}^\bullet(\mathfrak{g}[-2]))$ [DLYZ]. Second, $\Phi_G(A_{G,M}) \cong \mathbb{C}[\text{Whit}_G(T^\ast G)]$ (Whittaker reduction of the shifted cotangent bundle of $G$ with respect to the left action. The cohomological grading arises from the one on $\mathbb{C}[T^\ast G] = \mathbb{C}[G] \otimes \text{Sym}^\bullet(g)$, where the generators in $g$ are assigned degree 2, while $\mathbb{C}[G]$ is assigned degree 0).

Note that under the non-twisted Satake equivalence $\Phi_{G^\vee}: D_{G^\vee}^{G^\vee}(Gr_{G^\vee}) \sim D^G(\text{Sym}^\bullet(g[-2]))$, we have $\Phi_{G^\vee}(\omega_{Gr_{G^\vee}}) \cong \Phi_G(A_{G,M})$. This answer to Drinfeld’s question was proposed by D. Gaïtto.

Also, if we consider $G^\vee \cong \text{SO}(M')$ for a $2n+1$-dimensional vector space $M'$ equipped with a nondegenerate symmetric bilinear form, then $M \otimes M'$ carries a natural symplectic form and a natural action of $G \times G^\vee$. We have an isomorphism $\Phi_G(A_{G,M}) \cong \mathbb{C}[\text{Whit}_{G^\vee}(M \otimes M')]$ (with residual action of $G$. The cohomological grading arises from the one on $\text{Sym}^\bullet(M \otimes M')$ where all the generators are assigned degree 1).

Similarly, in the universal cotangent case, when $G = \text{GL}(N)$ for an $n$-dimensional vector space $N$, and $G^\vee \cong \text{GL}(N')$ for another $n$-dimensional vector space $N'$, we have the untwisted Satake equivalence...
\[ \Phi_G: D_{G_0}(Gr_G) \sim \to D^G(\text{Sym}^\bullet(\text{gl}(N)[-2])). \] Now \( \text{Hom}(N, N') \oplus \text{Hom}(N', N) \) carries a natural symplectic form and a natural action of \( G \times G^\vee \). We have an isomorphism \( \Phi_G(A_G, N) \cong C[\text{Whit}_{G^\vee} (\text{Hom}(N, N') \oplus \text{Hom}(N', N))] \) (with residual action of \( G \)). The cohomological grading arises from the one on \( \text{Sym}^\bullet(\text{Hom}(N, N') \oplus \text{Hom}(N', N)) \) where all the generators are assigned degree 1).

1.9. **Acknowledgments.** We are deeply grateful to D. Ben-Zvi, R. Bezrukavnikov, V. Drinfeld, P. Etingof, B. Feigin, D. Gaiotto, D. Gaitsgory, A. Hanany, T. Johnson-Freyd, S. Lysenko, H. Nakajima, Y. Sakellaridis, A. Venkatesh, J. Wang, E. Witten, P. Yoo and Z. Yun for many helpful and inspiring discussions. M.F. and S.R. thank the 4th Nisyros Conference on Automorphic Representations and Related Topics held in July 2019 for stimulating much of this work.

A.B. was partially supported by NSERC. G.D. was supported by an NSF Postdoctoral Fellowship under grant No. 2103387. M.F. was partially funded within the framework of the HSE University Basic Research Program and the Russian Academic Excellence Project ‘5-100’. S.R. was supported by NSF grant DMS-2101984.

2. Setup and notation

2.1. **Affine Grassmannians.** Let \( M \) be a \( 2n \)-dimensional complex vector space equipped with a symplectic form \( \langle , \rangle \). Its automorphism group is \( G = \text{Sp}(M) \).

Let \( K = C((t)) \supset O = C[[t]] \). The affine Grassmannian ind-scheme \( Gr_G = G_K/G_0 \) is the moduli space of \( G \)-bundles on the formal disc equipped with a trivialization on the punctured formal disc. The Kashiwara affine Grassmannian infinite type scheme \( \text{Gr}_G = G_K/G_{C[t^{-1}]} \) is the moduli space of \( G \)-bundles on \( \mathbb{P}^1 \) equipped with a trivialization in the formal neighbourhood of \( 0 \in \mathbb{P}^1 \).

The determinant line bundles over \( Gr_G \) and \( \text{Gr}_G \) are denoted by \( D \). The \( \mu_2 \)-gerbe of square roots of \( D \) over \( Gr_G \) (resp. \( \text{Gr}_G \)) is denoted \( \bar{Gr}_G \) (resp. \( \widetilde{Gr}_G \)).

The action of \( G_K \) on \( Gr_G \) and \( \text{Gr}_G \) lifts to the action of the metaplectic group-stack \( \widetilde{G}_K \) on \( \bar{Gr}_G \) and \( \widetilde{Gr}_G \). We have a splitting \( G_0 \hookrightarrow \widetilde{G}_K \).

In what follows we only consider the genuine constructible sheaves on \( \bar{Gr}_G \) and \( \widetilde{Gr}_G \), such that \(-1 \in \mu_2 \) acts on them as \(-1 \). We consider a dg-enhancement \( D^b_{G_0}(\bar{Gr}_G) \) of the (genuine) bounded equivariant constructible derived category. We denote by \( D_{G_0}(\widetilde{Gr}_G) \) the renormalized equivariant derived category defined as in [AGa, §12.2.3]. We also consider the category \( D_{G_0}(\widetilde{Gr}_G) \) defined as in [ArG, §3.4.1] (the inverse limit over the \( G_0 \)-stable open subgerbes of \( \widetilde{Gr}_G \), cf. §A.4). It contains the IC-sheaves of the \( G_0 \)-orbits closures.
An open sub-gerbe $\mathcal{T} \hookrightarrow \widetilde{\text{Gr}}_G \times \widetilde{\text{Gr}}_G$ is formed by all the pairs of transversal compact and discrete Lagrangian subspaces in $\mathcal{M}_X$. We denote by

$$\widetilde{\text{Gr}}_G \xleftarrow{p} \mathcal{T} \xrightarrow{q} \widetilde{\text{Gr}}_G$$

the natural projections. The Radon Transform is (cf. §A.5, where its $D$-module version is denoted $\text{RT}_!^{-1}$)

$$\text{RT} := p_* q^! : D_{G_o}(\widetilde{\text{Gr}}_G)_! \rightarrow D_{G_o}(\widetilde{\text{Gr}}_G)_!.$$ (2.1.1)

The Theta-sheaf $\Theta \in D_{G_o}(\widetilde{\text{Gr}}_G)_!$ introduced in [Ly] is the direct sum of IC-sheaves of two $G_o$-orbits in $\text{Gr}_G$: $\Theta_g$ of the open orbit, and $\Theta_s$ of the codimension 1 orbit.

### 2.2. D-modules.

The dg-category of $G_o$-equivariant $D$-modules on $\text{Gr}_G$ (resp. on $\text{Gr}_G$) twisted by the inverse square root $\mathcal{D}^{-1/2}$ is denoted $\text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G)$ (resp. $\text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G)_!$). More precisely, by $\text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G)$ we mean the renormalized equivariant category defined as in [AGa, §12.2.3], and $\text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G)_!$ is defined in §A.4. We have the Riemann–Hilbert equivalences

$$\text{RH} : \text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G)_! \sim \rightarrow D_{G_o}(\widetilde{\text{Gr}}_G)_!,$$

We denote $\text{RH}^{-1}(\Theta)$ by $\Theta \in \text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G)_!$, a direct sum of two irreducible $D$-modules, $\Theta_g$ with the full support, and $\Theta_s$ supported at the Schubert divisor.

The (derived) global sections $\Gamma(\text{Gr}_G, \Theta_g)$ and $\Gamma(\text{Gr}_G, \Theta_s)$ are irreducible $G_o$-integrable $\mathfrak{g}_{\text{aff}}$-modules of central charge $-1/2$, namely $L^0_{-1/2}$ and $L^\omega_{-1/2}$ [KT, Theorem 4.8.1]. Here $\mathfrak{g} = \mathfrak{sp}(M)$, and the highest component of $L^0_{-1/2}$ (resp. $L^\omega_{-1/2}$) with respect to $\mathfrak{g}_0$ is the trivial (resp. defining) representation of $\mathfrak{g}$.

The (derived) global sections functors

$$\Gamma : \text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G) \rightarrow \text{Rep}_{-1/2}^{G_o}(\mathfrak{g}_{\text{aff}}), \quad \Gamma : \text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G)_! \rightarrow \text{Rep}_{-1/2}^{G_o}(\mathfrak{g}_{\text{aff}})$$

($G_o$-integrable $\mathfrak{g}_{\text{aff}}$-modules with central charge $-1/2$) admit the left adjoints (see §§A.7,A.4)

$$\text{Loc} : \text{Rep}_{-1/2}^{G_o}(\mathfrak{g}_{\text{aff}}) \rightarrow \text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G), \quad \text{Loc} : \text{Rep}_{-1/2}^{G_o}(\mathfrak{g}_{\text{aff}}) \rightarrow \text{D-mod}^{G_o}_{-1/2}(\text{Gr}_G)_!.$$ (5.3)

According to [KT, Theorem 4.8.1(iv)], we have $\tau_{\geq 0} \text{Loc}(L^0_{-1/2} \oplus L^\omega_{-1/2}) = \Theta$ (the top cohomology in the natural $t$-structure).

---

8For a finite dimensional counterpart of this statement (about global sections of irreducible equivariant $D$-modules on the Lagrangian Grassmannian of $\mathfrak{g}$), see §5.3.
2.3. Weyl algebra. The symplectic form on $\mathbf{M}$ extends to the same named $\mathbb{C}$-valued symplectic form on $\mathbf{M}_{\mathcal{X}}$: $\langle f, g \rangle = \text{Res}(f, g) dt$. We denote by $\mathcal{W}$ the completion of the Weyl algebra of $(\mathbf{M}_{\mathcal{X}}, \langle , \rangle)$ with respect to the left ideals generated by the compact subspaces of $\mathbf{M}_{\mathcal{X}}$. It has an irreducible representation $\mathbb{C}[\mathbf{M}_O]$. Also, there is a homomorphism of Lie algebras $\mathfrak{g}_{\text{aff}} \rightarrow \text{Lie } \mathcal{W}$, see e.g. [FF]. According to [FF, rows 3,4 of Table XII at page 168], the restriction of $\mathbb{C}[\mathbf{M}_O]$ to $\mathfrak{g}_{\text{aff}}$ is $L^0_{-1/2} \oplus L^1_{-1/2}$ (even and odd functions, respectively).\footnote{9}

We consider the dg-category $\mathcal{W}$-mod of discrete $\mathcal{W}$-modules. More concretely, we identify $\mathcal{W}$ with the ring of differential operators on a Lagrangian discrete lattice $L \subset \mathbf{M}_{\mathcal{X}}$, e.g. $L = t^{-1} \mathbf{M}_{[t^{-1}]}$. Then $\mathcal{W}$-mod is the inverse limit of $\text{D-mod}(V)$ over finite dimensional subspaces $V \subset L$ with respect to the functors $i^V_{\leftarrow V'}$. Equivalently, $\mathcal{W}$-mod is the colimit of $\text{D-mod}(V)$ with respect to the functors $i^V_{\rightarrow V'}$. There is a twisted action $\text{D-mod}_{-1/2}(G_{\mathcal{X}}) \circ \mathcal{W}$-mod that gives rise to an action $\text{D-mod}_{-1/2}^{G_{\mathcal{X}}}(\text{Gr}_G) \circ \left( \mathcal{W} \text{-mod}^{G_{\mathcal{X}}} \right)$, see [R, §10].

2.4. Twisted derived Satake. One of the main results of [DLYZ] is a construction of a monoidal equivalence $\Phi: \tilde{D}_{G_{\mathcal{O}}}^{b}(\text{Gr}_G) \rightarrow D_{G_{\mathcal{O}}}^{G}(\text{Sym}^{\bullet}(\mathfrak{g}[-2]))$ (dg-category of perfect complexes of dg-modules over the dg-algebra $\text{Sym}^{\bullet}(\mathfrak{g}[-2])$ equipped with a trivial differential). It extends to a monoidal equivalence of Ind-completions $\Phi: D_{G_{\mathcal{O}}}^{G}(\text{Gr}_G) \rightarrow D^{G}(\text{Sym}^{\bullet}(\mathfrak{g}[-2]))$.

Here is one of the key properties of the twisted derived Satake equivalence $\Phi$. We choose a pair of opposite maximal unipotent subgroups $U_G, U_G^{-} \subset G$, their regular characters $\psi, \psi^{-}$, and denote by $\kappa: D^{G}(\text{Sym}^{\bullet}(\mathfrak{g}[-2])) \rightarrow D(\mathbb{C}[\Xi_{\mathfrak{g}}])$ the functor of Kostant-Whittaker reduction with respect to $(U_G^{-}, \psi^{-})$ (see e.g. [BeF, §2]). Here $\Xi_{\mathfrak{g}}$ with grading disregarded is the tangent bundle $T\Sigma_{\mathfrak{g}}$ of the Kostant slice $\Sigma_{\mathfrak{g}} \subset \mathfrak{g}^{\ast}$. Let us write $\kappa$ for the Ad-invariant bilinear form on $\mathfrak{g}$, i.e., level, corresponding to our central charge of $-1/2$. Explicitly, if we write $\kappa_b$ for the basic level giving the short coroots of $\mathfrak{g}$ squared length two, and $\kappa_c$ for the critical level, then $\kappa$ is defined by

$$\kappa = -1/2 \cdot \kappa_b - \kappa_c.$$  

If we consider the Langlands dual Lie algebra $\mathfrak{g}' \simeq \mathfrak{so}_{2n+1}$, the form $\kappa$ gives rise to identifications $\Sigma_{\mathfrak{g}} \simeq \Sigma_{\mathfrak{g}'}$ and $\Xi_{\mathfrak{g}} \simeq \Xi_{\mathfrak{g}'}$. Also, we have a canonical isomorphism $H^{\bullet}_{G_{\mathcal{O}}}^{G}(\text{Gr}_G) \simeq \mathbb{C}[\Xi_{\mathfrak{g}'}] \simeq \mathbb{C}[\Xi_{\mathfrak{g}}]$. This is a theorem of V. Ginzburg [G] (for a published account see e.g. [BeF, Theorem 1]).

Now given $\mathcal{F} \in D_{G_{\mathcal{O}}}^{b}(\text{Gr})$ we consider the tensor product $\mathcal{F} \otimes \text{RT}(\Theta)$ (notation of §2.1). Since the monodromies of the factors cancel out, it canonically descends.
to an object of $D_{G_0}(\text{Gr}_G)$. The aforementioned key property is a canonical isomorphism

\begin{equation}
(2.4.1) \quad H^*_G(\text{Gr}_G, \mathcal{F} \otimes \text{RT}(\Theta)) \cong \mathcal{H}(\mathcal{F})
\end{equation}

of $H^*_G(\text{Gr}_G) \cong \mathbb{C}[\Xi]$-modules.

3. The universal ring object

3.1. The internal Hom construction. To introduce the universal ring object and show its relation to the $\Theta$-sheaf, we recall the following general construction of internal Hom objects.

Let $\mathcal{C}$ be a module category over D-mod$_{-1/2}(\text{Gr}_X)$. Given a subgroup $H$ of $G_X$ and an $H$-equivariant object $\xi$ of $\mathcal{C}$, convolution with it yields a D-mod$_{-1/2}(\text{Gr}_X)$-equivariant functor (D-mod$_{-1/2}(\text{Gr}_X)_H \to \mathcal{C}$, and upon restriction to spherical vectors a D-mod$_{-1/2}(\text{Gr}_G)\text{Go}^0$-equivariant functor D-mod$_{-1/2}(\text{Gr}_G)\text{Go}^0 \to (\mathcal{C})\text{Go}^0$. If both $\mathcal{C}$ and (D-mod$_{-1/2}(\text{Gr}_G)\text{Go}^0$ are dualizable as abstract dg-categories, we obtain the dual D-mod$_{-1/2}(\text{Gr}_G)\text{Go}^0$-equivariant functor

$$(\mathcal{C}^\vee)^{\text{Go}^0} \to \text{D-mod}_{-1/2}(\text{Gr}_X/H)^{\text{Go}^0}, \, \zeta \mapsto \mathcal{H}(\xi, \zeta).$$

We apply this as follows. First, taking $\mathcal{C} = W\text{-mod}$, $H = G_0$, and $\xi = \mathbb{C}[\mathcal{M}_0]$, we obtain a functor

$F \colon (W\text{-mod})^{\text{Go}^0} \to \text{D-mod}_{-1/2}(\text{Gr}_G)^{\text{Go}^0}$, $M \mapsto \mathcal{H}(\mathbb{C}[\mathcal{M}_0], M)$.

Setting $M = \mathbb{C}[\mathcal{M}_0]$, we obtain the internal Hom ring object

$\mathcal{R} := \mathcal{H}(\mathbb{C}[\mathcal{M}_0], \mathbb{C}[\mathcal{M}_0]) \in \text{D-mod}_{-1/2}(\text{Gr}_G)^{\text{Go}^0}$.

Second, taking $\mathcal{C} = W\text{-mod}$, $H = G_{C[t^{-1}]}$, and $\xi = \mathcal{C}^{t^{-1}[\mathcal{M}_{C[t^{-1}]^{-1}}]}$, i.e., the colimit of the dualizing sheaves $\omega_V$ over finite dimensional subspaces $V \subset t^{-1}[\mathcal{M}_{C[t^{-1}]^{-1}}]$, we obtain a functor

$F \colon (W\text{-mod})^{G_{C[t^{-1}]}} \to \text{D-mod}_{-1/2}(\text{Gr}_G)^{G_{C[t^{-1}]}}$, $M \mapsto \mathcal{H}(\mathcal{C}^{t^{-1}[\mathcal{M}_{C[t^{-1}]^{-1}]}, M)$.

**Lemma 3.1.1.** We have a canonical isomorphism $F(\mathbb{C}[\mathcal{M}_0]) \cong \Theta$.

**Proof.** We have $\mathbb{C}[\mathcal{M}_0] = W/(W\cdot \mathcal{M}_0)$. We denote $F(\mathbb{C}[\mathcal{M}_0])$ by $\mathcal{F}$ for short. For a Lagrangian discrete lattice $L$ representing a point of $\text{Gr}_G$, the fiber $\mathcal{F}_L$ of $\mathcal{F}$ at $L$ is $W/(W\cdot \mathcal{M}_0 + L\cdot W)$. According to [La, §2], the fiber $\Theta_L$ is $W/(W\cdot \mathcal{M}_0 + L\cdot W)$ as well.

For the reader’s convenience, let us briefly sketch a proof of the latter isomorphism. First, we consider the finite dimensional counterpart $S = S_g \oplus S_s$ of $\Theta$ as in §5.3. For a Lagrangian subspace $L \subset M$ representing a point of $L\text{Gr}_M$, the fiber $S_L$ of $S$ at $L$ is $W_M/(W_M \cdot N + L \cdot W_M)$ (notation of §5.3). This follows from the De Rham counterpart of the integral presentation [Ly, Proposition 5] of $S$. 
Second, representing $M_X$ as an ind-pro-limit of a growing family of finite dimensional symplectic spaces $M'$, we can construct the Theta $D$-module $\Theta_{Sato}$ on the co-Sato Lagrangian Grassmannian $Gr_{Sato}$ of Lagrangian discrete lattices in $M_X$ as a certain limit of baby Theta $D$-modules $M'\hat{S}$ on $LGr_M$, see [LL, §6.5]. The similar formula for the fibers of $\Theta_{Sato}$ follows. Finally, we have an embedding $Gr_G \hookrightarrow Gr_{Sato}$, and $\Theta$ is the pullback of $\Theta_{Sato}$ by [LL, Theorem 3]. Hence the desired formula for the fibers of $\Theta$. □

3.2. Radon transform. Recall the Radon transform (2.1.1). We keep the same notation for its $D$-module version $RT$: $D-mod_{-1/2}(Gr_G)^{G_0} \rightarrow D-mod_{-1/2}(Gr_G)^{G_0}$. See the Appendix starting from §A.5, where it is denoted $RT^{-1}$.

Proposition 3.2.1. We have an isomorphism $R \simeq RT\Theta$.

Proof. By Lemma 3.1.1, it suffices to show that the composition

$$(W-mod)^{G_0} \xrightarrow{F} D-mod_{-1/2}(Gr_G)^{G_0} \xrightarrow{RT} D-mod_{-1/2}(Gr_G)^{G_0}$$

is $D-mod_{-1/2}(Gr_G)^{G_0}$-equivariantly equivalent to $F$. By dualizing the appearing functors, we equivalently must show that the composition

$$D-mod_{-1/2}(Gr_G)^{G_0} \xrightarrow{RT^*} D-mod_{-1/2}(Gr_G)^{G_0} \xrightarrow{F^*} (W-mod)^{G_0}$$

sends the delta function at the origin $\delta_e$ to $\mathbb{C}[M_0]$.

To show this, writing $\text{Av}_{G_0}$ for the partially defined left adjoint to the forgetful functor $(W-mod)^{G_0} \rightarrow W-mod$, we have the following.

Lemma 3.2.2. The category $(W-mod)^{G_0}$ is compactly generated by a single object $\text{Av}_{G_0}(\mathbb{C}[M_0])$.

Proof. We have an equivalence $(W-mod)^{G_0} \simeq D-mod(\text{Heis})^{G_0 \times M_0 \times G_a \chi}$, where Heis is the Heisenberg central extension of $M_X$ with $G_a$ (canonically split after restriction to $M_0$), and $\chi$ is the character of $G_0 \times M_0 \times G_a$ obtained by composition of projection to $G_a$ and exponentiating. Indeed, the $W$-module $\mathbb{C}[M_0]$ is strongly $(G_0 \times M_0 \times G_a, \chi)$-equivariant, and so gives rise to a functor from $D-mod(\text{Heis})^{G_0 \times M_0 \times G_a \chi}$ to $(W-mod)^{G_0}$ that is the desired equivalence.

Now $\chi$ is non-trivial on the stabilizer of any point $m \in \text{Heis} \setminus (M_0 \times G_a)$. Indeed, given a vector $m \in M_X$ with nontrivial polar part, we can find $g \in G_0$ such that $gm = m + m'$, where $m' \in M_0$ has nonzero $\text{Res}(m, m')_X$. So $\chi|_{\text{Stab}(m)}$ is nontrivial.

Hence any object of $D-mod(\text{Heis})^{G_0 \times M_0 \times G_a \chi}$ must be supported on $M_0 \times G_a$. This yields an equivalence $(W-mod)^{G_0} \simeq D-mod(\text{pt}/G_0)$, which exchanges $\mathbb{C}[M_0]$ with the dualizing sheaf. Moreover, if we write $\langle \mathbb{C}[M_0] \rangle$ for the full subcategory of $W$-mod compactly generated by $\mathbb{C}[M_0]$, this exchanges the forgetful
functor

$$(W\text{-mod})^{G_o} \to \langle C[M_0] \rangle \simeq \text{Vect}$$

with the functor of $!$-pullback to the point

$$D\text{-mod}(pt/G_o) \to D\text{-mod}(pt) \simeq \text{Vect}.$$ 

The claim of the lemma now follows from the analogous fact for D-modules on $pt/G_o$, see for example [DG, §7.2.2].

We are now ready to calculate $F^\circ \circ RT^\circ(\delta_e)$. First, if we write $j_* \in D\text{-mod}_{-1/2}(\text{Gr}_G)^{G_o}$ for the $*$-extension of the constant D-module on the big cell, unwinding definitions we have that

$$F^\circ \circ RT^\circ(\delta_e) \simeq F^\circ(j_*) \simeq j_* \ast \omega_{t^{-1}M_{c[t]^{-1}}}.$$ 

To identify this with $C[M_0]$, by the proof of Lemma 3.2.2, particularly the exhibited equivalence $(W\text{-mod})^{G_o} \simeq D\text{-mod}(pt/G_o)$, we must show that $\text{Hom}_{(W\text{-mod})^{G_o}}(Av^G_o(C[M_0]), j_* \ast \omega_{t^{-1}M_{c[t]^{-1}}})$ is the trivial line $C$, placed in cohomological degree zero.

To see this, note that $j_*$ identifies with the relative $*$-averaging $(W\text{-mod})^G \to (W\text{-mod})^{G_o}$, and that, by the pronipotence of the kernel of $G_o \to G$ and the $G_o$-equivariance of $C[M_0]$, one has a canonical equivalence $\text{Av}^G_o(C[M_0]) \simeq \text{Av}^G_o(C[M_0])$. Therefore, we may compute

$$\text{Hom}_{(W\text{-mod})^{G_o}}(Av^G_o(C[M_0]), j_* \ast \omega_{t^{-1}M_{c[t]^{-1}}})$$

$$\simeq \text{Hom}_{(W\text{-mod})^{G_o}}(Av^G_o(C[M_0]), \omega_{t^{-1}M_{c[t]^{-1}}})$$

$$\simeq \text{Hom}_{D\text{-mod}}(C[M_0], \omega_{t^{-1}M_{c[t]^{-1}}})$$

$$\simeq \text{Hom}_{D\text{-mod}}(M_{t^{-1}c[t]^{-1}})(\delta_0, \omega_{t^{-1}M_{c[t]^{-1}}}) \simeq C,$$

as desired.

**Corollary 3.2.3.** We have an isomorphism $\Gamma(\mathcal{R}) \simeq C[M_0]$.

*Proof.* Recall that $\Gamma(\Theta) \simeq C[M_0]$ and apply Proposition A.7.1. \hfill \Box

### 3.3. Computation of $\text{RHR}$ under the twisted derived Satake

Recall the notation of §2.4. We consider an object $C[G] \otimes \text{Sym}^\bullet(g[-2]) \in D^G(\text{Sym}^\bullet(g[-2]))$. In fact, $C[G] \otimes \text{Sym}^\bullet(g[-2])$ has two such structures: with respect to the left (resp. right) $G$-action and the left (resp. right) comoment morphism. We consider the hamiltonian reduction with respect to the right $U_G$-action $(C[G] \otimes \text{Sym}^\bullet(g[-2]))/(U_G, \psi_C)$. This reduction has the residual left structure of a monoidal object of $D^G(\text{Sym}^\bullet(g[-2]))$. We will denote this object by $\mathfrak{R}$.

**Theorem 3.3.1.** We have an isomorphism $\Phi\text{RHR} \simeq \mathfrak{R}$.
Proof. Recall that derived Satake exchanges Verdier duality on the automorphic side with the composition on the spectral side of the Chevalley involution $C$ of $G$ and the standard duality of $D^G_{\text{perf}}(\text{Sym}^\bullet(\mathfrak{g}[-2]))$ sending a perfect complex to its dual (see [BeF, Lemma 14] and [DLYZ]). Equivalently, it exchanges the perfect pairing of dg-categories
\[
D_{G^0}(\widetilde{\text{Gr}}_G) \otimes D_{G^0}(\widetilde{\text{Gr}}_G) \to \text{Vect}, \quad \mathcal{F} \boxtimes \mathcal{G} \mapsto H^*_{G^0}(\text{Gr}_G, \mathcal{F} \boxtimes \mathcal{G}),
\]
which is continuous due to our renormalization, with the perfect pairing
\[
D^G(\text{Sym}^\bullet(\mathfrak{g}[-2])) \otimes D^G(\text{Sym}^\bullet(\mathfrak{g}[-2])) \to \text{Vect}, \quad F \boxtimes G \mapsto \mathcal{C}(M) \otimes_{\text{Sym}^\bullet(\mathfrak{g}[-2])} N.
\]
To prove the theorem it is enough to show that derived Satake interchanges the functor $D_{G^0}(\widetilde{\text{Gr}}_G) \to \text{Vect}$ given by pairing with $\text{RH}$ and the functor $D^G(\text{Sym}^\bullet(\mathfrak{g}[-2])) \to \text{Vect}$ given by pairing with $\mathcal{R}$. However, note the latter computes the underlying vector space of the Kostant–Whittaker reduction $\kappa$, cf. §2.4. Applying the Riemann–Hilbert correspondence to the statement of Proposition 3.2.1, we obtain that $\text{RH} \simeq \text{RT}(\Theta)$, and hence we are done by (2.4.1).

4. COULOMB BRANCHES OF NONCOTANGENT TYPE

4.1. Anomaly cancellation. A symplectic representation $M$ of a reductive group $G$, i.e. a homomorphism $G \to \text{Sp}(M) = G$ gives rise to a morphism $s: \text{Gr}_G \to \text{Gr}_G$. The pullback $s^*\mathcal{D}$ of the determinant line bundle of $\text{Gr}_G$ is a multiplicative line bundle $L$ on $\text{Gr}_G$ (i.e. its pullback $m^*L$ to the convolution diagram $\text{Gr}_G \times \text{Gr}_G \to \text{Gr}_G$ is isomorphic to $L \boxtimes L$, and this isomorphism satisfies a natural cocycle condition). It is well known that the multiplicative line bundles on $\text{Gr}_G$ are in natural bijection with the invariant (with respect to the Weyl group of $G$) integral bilinear forms on the coweight lattice $X_*(G)$ assuming even values on all the coroots. The bilinear form $B$ corresponding to $L$ is nothing but the pullback of the trace form on $\mathfrak{g} = \mathfrak{sp}(M)$. In case $B/2$ is still an integral bilinear form assuming even values on all the coroots, there exists a multiplicative line bundle $\sqrt{L}$. We choose such a square root, and the pullback of the gerbe $\widetilde{\text{Gr}}_G$ trivializes. Hence the pullback $A_{G,M} := s^*\text{RH}$ can be viewed as a ring object of $D_{\text{G}^0}(\text{Gr}_G)$ (no twisting).

Proposition 4.1.1. The bilinear form $B$ is divisible by 2 (and $B/2$ assumes even values on all the coroots) iff the induced morphism $\pi_4G \to \pi_4G = \mathbb{Z}/2\mathbb{Z}$ is trivial.

For a proof, see Appendix B.

Remark 4.1.2. The second condition of the proposition is the anomaly cancellation condition of [Wi].
In case the anomaly cancellation condition holds true, we can consider the ring \( \mathcal{A}(G,M) := H^*_{G_0}(Gr_G, \mathcal{A}_{G,M}) \). Since the universal ring object \( RH \) is commutative (by explicit calculation of Theorem 3.3.1), the ring object \( \mathcal{A}_{G,M} \) is commutative as well. Hence the ring \( \mathcal{A}(G,M) \) is also commutative, and the Coulomb branch \( \mathcal{M}_C(G,M) \) is defined as \( \text{Spec} \mathcal{A}(G,M) \).

4.2. Cotangent type. Assume that a symplectic representation \( M \) of a reductive group \( G \) splits as \( M = N \oplus N^* \) for some \( G \)-module \( N \). Then the anomaly cancellation condition holds true, and we obtain a ring object \( \mathcal{A}_{G,N} \in D_{G_0}(Gr_G) \). On the other hand, a ring object \( \mathcal{A}_{G,M} := \pi_* \omega_{\mathbb{R}}[-2 \dim N_0] \in D_{G_0}(Gr_G) \) is defined in [BFN3, 2(ii)], such that \( \mathcal{A}(G,N) = H^*_{G_0}(Gr_G, \mathcal{A}_{G,N}) \) (the ring of functions on the Coulomb branch of cotangent type).

**Lemma 4.2.1.** We have an isomorphism of ring objects \( \mathcal{A}_{G,N} \cong \mathcal{A}_{G,M} \).

**Proof.** The monoidal category \( \text{D-mod}(Gr_G)^{G_0} \) acts on \( (\mathcal{W}\text{-mod})^{G_0} \cong \text{D-mod}(N_{\mathbb{R}})^{G_0} \), and \( \mathcal{A}_{G,M}^{DR} := \text{Hom}(\delta_{N_0}, \delta_{N_0}). \) By definition, it represents the functor \( \text{D-mod}(Gr_G)^{G_0} \ni \mathcal{G} \mapsto \text{Hom}_{\text{D-mod}(N_{\mathbb{R}})^{G_0}}(\mathcal{G} \star \delta_{N_0}, \delta_{N_0}) \). Now \( \mathcal{A}_{G,M} \in D_{G_0}(Gr_G) \) is the image of \( \mathcal{A}_{G,M}^{DR} \in \text{D-mod}(Gr_G)^{G_0} \) under the Riemann–Hilbert correspondence.

More generally, given a group \( H \) acting on a variety \( X \) we denote by

\[
H \xrightarrow{pr_H} H \times X \xrightarrow{a_{pr_X}} X
\]

the natural projections and the action morphism. The monoidal derived constructible category \( \mathcal{D}(H) \) (with respect to convolution) acts on \( \mathcal{D}(X) \) (by convolution), and given \( \mathcal{F} \in \mathcal{D}(X) \), the internal Hom object \( \mathcal{H}\text{om}(\mathcal{F}, \mathcal{F}) \in \mathcal{D}(H) \) is given explicitly by \( \mathcal{H}\text{om}(\mathcal{F}, \mathcal{F}) = \text{pr}_H^* \mathcal{H}\text{om}(\text{pr}_X^* \mathcal{F}, a^* \mathcal{F}) \), where \( \mathcal{H}\text{om}(\mathcal{X}, \mathcal{Y}) = \mathcal{D}\mathcal{X} \otimes \mathcal{Y} \).

Now let \( Y \subset X \) be a smooth subvariety, and \( \mathcal{F} = \underline{\mathcal{O}}_Y \). Set \( Z := \{(h, y) \in H \times Y : hy \in Y\} \subset H \times X \).

Then \( \mathcal{H}\text{om}(\mathcal{F}, \mathcal{F}) = \text{pr}_H^* \omega_Z[-2 \dim Y] \).

Similar statement applies to the situation when \( H \) comes with a closed subgroup \( A \) such that \( Y \) is \( A \)-invariant, and we consider the action of \( \mathcal{D}(A \setminus H/A) \) on \( \mathcal{D}(X)^A \).

Applying this to \( H = G_{\mathbb{R}}, A = G_0, X = N_{\mathbb{R}}, Y = N_0 \) we obtain the desired isomorphism \( \mathcal{A}_{G,M} \cong \mathcal{A}_{G,N} := \pi_* \omega_{\mathbb{R}}[-2 \dim N_0] \in D_{G_0}(Gr_G) \) (see [BFN1, 2(ii)] for the meaning of the cohomological shift \( \omega_{\mathbb{R}}[-2 \dim N_0] \)).

4.3. Finite generation.

**Lemma 4.3.1.** \( \mathcal{A}(G,M) \) is a finitely generated integral domain.

**Proof.** We essentially repeat the argument of [BFN1, 6(iii)]. We choose a Cartan torus \( T \subset G \), restrict our symplectic representation \( M \) from \( G \) to \( T \), and consider the corresponding ring \( \mathcal{A}(T,M) \). Note that the \( T \)-module \( M \) is automatically of
cotangent type, i.e. $M \simeq N \oplus N^*$ for a $T$-module $N$. In notation of [BFN1, 3(iv)], we have $A(T, M) = A(T, N)$. Similarly to [BFN1, Lemma 5.17], we obtain an injective homomorphism $A(T, M) \hookrightarrow A(G, M) \otimes H^*_G(pt) H^*_T(pt)$.

Since $Gr_G$ is the union of its spherical Schubert subvarieties, we obtain a filtration by support on $A(G, M)$ (and the induced filtration on $A(T, M)$) numbered by the cone $X^*_+(G)$ of dominant coweights of $G$. For $\lambda \in X^*_+(G)$ let $i_{\lambda}!$ denote the locally closed embedding $Gr_G^\lambda \hookrightarrow Gr_G$. The key observation is that $i_{\lambda}! A(G, M)$ is a trivial one-dimensional local system on $Gr_G^\lambda$ (shifted to some cohomological degree determined by the monopole formula). It gives rise to an element $\mathcal{R}_{\lambda} \in \text{gr} A(G, M)$ (in the cotangent case this element was the fundamental class of the preimage of $Gr_G^\lambda$ in the variety of triples, hence the notation).

Now the proof of [BFN1, Proposition 6.2, Proposition 6.8] goes through word for word in our situation and establishes the desired finite generation. □

4.4. Normality.

Lemma 4.4.1. $A(G, M)$ is integrally closed.

Proof. Again we repeat the argument of [BFN1, 6(v)] with minor modifications. It reduces to an explicit calculation of $A(G, M)$ for $G = SL(2)$ or $G = PGL(2)$ as in [BFN1, Lemma 6.9]. Now any symplectic representation of $PGL(2)$ is of cotangent type (since any irreducible representation is odd-dimensional), so $A(PGL(2), M)$ is already computed in [BFN1, Lemma 6.9(2)]. For $SL(2)$, a representation $M = \oplus_{k \in \mathbb{N}} V^k \otimes M^k$ (where $V^k$ is an irreducible $SL(2)$-module of dimension $k + 1$, and $M^k$ is a multiplicity space) is symplectic iff $\dim M^k$ is even for $k$ even. Furthermore, it is easy to see that the anomaly cancellation condition is that the sum $\sum_{\ell \in \mathbb{N}} \dim M^{k+1}$ must be even. Equivalently, if for a weight $\chi \in X^*(SL(2)) = \mathbb{Z}$ we denote by $m_\chi$ the dimension of the $\chi$-weight space of $M$, then $N := \sum_{\chi \in \mathbb{Z}} |\chi|m_\chi/4$ must be integral.

Then the same argument as in the proof of [BFN1, Lemma 6.9(1)] identifies $A(SL(2), M)$ as an algebra with 3 generators $\delta, \xi, \eta$ and a single relation $\xi^2 = \delta\eta^2 - \delta^N - 1$ if $N > 0$, and $\xi^2 = \delta\eta^2 + \eta$ if $N = 0$. In particular, it is always integrally closed. □

5. Odds and ends

5.1. An orthosymplectic construction of $\mathfrak{K}$. The invariants $\text{Sym}^*(\mathfrak{g}[-2])^G$ form a free graded commutative algebra $\mathbb{C}[\Sigma^*_g]$ with generators in degrees $4, 8, \ldots, 4n$ (functions on a graded version of Kostant slice). Recall the ring object $\mathfrak{K}$ of $D^G(\text{Sym}^*(\mathfrak{g}[-2]))$ introduced in §3.3. It is well known that $\mathfrak{K} \simeq \mathbb{C}[G \times \Sigma^*_g]$, where $G$ acts in the RHS via $g \cdot (g', \sigma) = (gg', \sigma)$, and the morphism $G \times \Sigma^*_g \to \mathfrak{g}^*[2]$ is $(g, \sigma) \mapsto \text{Ad}_g \sigma$. 
We consider a locally closed subvariety \( Y \subset M' \times \text{Hom}(M',M)[1] \) formed by the pairs \((v, A)\) such that \(v\) is a cyclic vector for \(C := AA\) satisfying the orthogonality relations \((v, C^k v) = 0\) for any \(k < 2n\) (note that for odd \(k\) this orthogonality relation is automatically satisfied), and \((v, C^{2n} v) = 1\).

Clearly, \(Y\) is equipped with the action of \(G' \times G = \text{SO}(M') \times \text{Sp}(M)\) and with a morphism \(\pi: Y \to g[2] \cong g^*[2]\), \((v, A) \mapsto AA\). Hence the categorical quotient \(Y//G'\) carries the residual action of \(G\) and is equipped with the residual morphism \(\overline{\pi}: Y//G' \to g^*[2]\).

One can easily construct an isomorphism \(\text{Hom}(M, M')[1]// (U_{G'}, \psi_{g'}) \simeq Y//G'\). We will construct an isomorphism \(\mathbb{C}[Y]\)^{G'} \simeq \mathfrak{R}. More precisely, we will construct an isomorphism \(\mathbb{C}[Y]\)^{G'} \simeq \mathbb{C}[G \times \Sigma^*] \) with gradings disregarded, and it will be immediate to check that it respects the gradings (along with the \(G\)-action and the comoment morphism).

We consider a locally closed subvariety \(X \subset M \times g\) formed by the pairs \((u, x)\) such that \(u\) is a cyclic vector of \(x\) satisfying the orthogonality relations \(\langle u, x^k u \rangle = 0\) for any \(k < 2n - 1\) (note that for even \(k\) this orthogonality relation is automatically satisfied), and \(\langle u, x^{2n-1} u \rangle = 1\).

\(^{10}\) So strictly speaking we should consider the generators in \(\text{Hom}(M, M')[1]\) as having odd parity.
We have an isomorphism $\eta: X \xrightarrow{\sim} G \times \Sigma_0$ defined as follows. The second factor of $\eta(u, x)$ is the image of $x$ in $g \parallel G \cong g^* \parallel G = \Sigma_0$. The first factor of $\eta(u, x)$ is the symplectic $2n \times 2n$-matrix with columns $C_0, C_1, \ldots, C_{2n-1}$ defined as follows. First, we set $C_k = x^k u$ for $k = 0, \ldots, n$. Second, we set $C_{n+1} = (-1)^n (x^{n+1} u - \langle u, x^{2n+1} u \rangle x^{n-1} u)$ to make sure $\langle C_{n-2}, C_{n+1} \rangle = 1$ and $\langle C_n, C_{n+1} \rangle = 0$. Third, we define $C_{n+2}$ as $(-1)^{n-1} x^{n+2} u$ plus an appropriate linear combination of $x^n u$ and $x^{n-2} u$ to make sure that $\langle C_{n-3}, C_{n+2} \rangle = 1$, and $C_{n+2}$ is orthogonal to all the other previous columns. Then we continue to apply this ‘Gram-Schmidt orthogonalization process’ to $x^{n+3} u, \ldots, x^{2n-1} u$ in order to obtain the desired columns $C_{n+3}, \ldots, C_{2n-1}$.

Now we consider a morphism $\xi: Y \rightarrow X$, $(v, A) \mapsto (u = Av, x = AA^t)$. It factors through $Y \rightarrow Y \parallel G' \xrightarrow{\sim} X$, and it follows from the first fundamental theorem of the invariant theory for $SO(M)$ that $\xi$ is an isomorphism, cf. [BFT, proof of Lemma 2.8.1.(a)].

5.2. The universal ring object of cotangent type. We choose a pair of transversal Lagrangian subspaces $M = N \oplus N^\ast$. They give rise to a (Siegel) Levi subgroup $G = GL(N) \subset G = Sp(M)$. The corresponding embedding of the affine Grassmannians $Gr_G \hookrightarrow Gr_G$ is denoted by $s$. The pullback $s^* \mathcal{D}$ of the determinant line bundle of $Gr_G$ is the square of the determinant line bundle of $Gr_G$. Hence the pullback of the gerbe $\tilde{Gr}_G$ trivializes, and the pullback $R := s^* \mathcal{R} \mathcal{R}$ can be viewed as an object of $D_{C_0}(Gr_G)$ (no twisting). It is nothing but the ring object considered in [BFN3]: the direct image of the dualizing sheaf of the variety of triples associated to the representation $N$ of $G$ in [BFN1].

We will compute the image of $R$ under the derived Satake equivalence $\Phi: D_{C_0}(Gr_G) \xrightarrow{\sim} D^G(\text{Sym}^\ast(gl(N)[-2]))$. To this end, similarly to §5.1, we introduce another copy $N'$ of an $n$-dimensional complex vector space, and consider the moment map

$q: \text{Hom}(N', N) \times \text{Hom}(N, N') \rightarrow \text{gl}(N) \times \text{gl}(N') \cong \text{gl}(N)^\ast \times \text{gl}(N')^\ast,$

$(A, B) \mapsto (AB, BA),$

(we use the trace form to identify $\text{gl}(N)$ (resp. $\text{gl}(N')$) with its dual), and the natural action $GL(N') \times GL(N) \subset \text{Hom}(N', N) \times \text{Hom}(N, N')$. We choose a maximal unipotent subgroup $U \subset GL(N')$ and a regular character $\psi$ of its Lie algebra. The Hamiltonian reduction $\mathbb{C}[\text{Hom}(N', N) \times \text{Hom}(N, N')] \parallel (U, \psi)$ carries the residual action of $GL(N)$ and comoment morphism from $\text{Sym}(\text{gl}(N))$.

Now we consider $\mathbb{C}[\text{Hom}(N', N) \times \text{Hom}(N, N')]$ as a dg-algebra with trivial differential and with cohomological grading such that all the generators in $\text{Hom}(N', N)^\ast \oplus \text{Hom}(N, N')^\ast$ have degree 1. We will denote this algebra by
The commutant morphism is a homomorphism of dg-algebras

\[ q^*: \text{Sym}^* (\mathfrak{gl}(N)[-2] \oplus \mathfrak{gl}(N')[-2]) \to \mathbb{C} [\text{Hom}(N', N)[1] \times \text{Hom}(N, N')[1]], \]

and \( \mathbb{C} [\text{Hom}(N', N)[1] \times \text{Hom}(N, N')[1]] \cong (U, \psi) \) is a ring object of \( D^G (\text{Sym}^* (\mathfrak{gl}(N)[-2])) \).

**Proposition 5.2.1.** We have an isomorphism

\[ \Phi R \simeq \mathbb{C} [\text{Hom}(N', N)[1] \times \text{Hom}(N, N')[1]] \cong (U, \psi). \]

**Proof.** We consider an open subvariety \( Z \subset N' \times \text{Hom}(N', N)[1] \times \text{Hom}(N, N')[1] \) formed by the triples \((v, A, B)\) such that \( v \) is a cyclic vector for \( BA \circ N' \). It is equipped with a morphism \( \varpi: Z \to \mathfrak{gl}(N)[2] \cong \mathfrak{gl}(N)^* [2], (v, A, B) \mapsto AB \), and a natural action of \( \text{GL}(N') \times \text{GL}(N) \). Hence the categorical quotient \( Z/\text{GL}(N') \) carries the residual action of \( \text{GL}(N) \) and is equipped with the residual morphism \( \varpi: Z/\text{GL}(N') \to \mathfrak{gl}(N)^* [2] \).

One can easily construct an isomorphism

\[ (\text{Hom}(N', N)[1] \times \text{Hom}(N, N')[1]) \cong Z//\text{GL}(N'). \]

It remains to construct an isomorphism \( \Phi R \simeq \mathbb{C} [Z]^{\text{GL}(N')} \) compatible with the commutant morphisms from \( \text{Sym}^* (\mathfrak{gl}(N)[-2]) \) and with the actions of \( \text{GL}(N) \).

The desired isomorphism is a corollary of [BFGT, Theorem 3.6.1]. Indeed, in notation of [BFGT, §3.2, §3.10], we have \( R = u_0^*(E_0 \oplus \omega_{\text{Gr}^0_{\text{GL}(N)} \times N_0} \oplus E_0) \) by comparison of definitions (say \( E_0 \) stands for the constant sheaf on \( \text{Gr}^0_{\text{GL}(N)} \times N_0 \), see [BFGT, §3.9], while \( \omega \) stands for the dualizing sheaf). So we have to compute this triple convolution in terms of the mirabolic Satake equivalence. The corresponding convolution on the coherent side is defined in [BFGT, §§3.4,3.5]. The convolution of 3 objects is computed via the double cyclic quiver \( \tilde{A}_3 \) on 4 vertices, cf. [BFGT, (3.4.1)]. The result of this computation is nothing but \( \mathbb{C} [Z]^{\text{GL}(N')} \). \( \square \)

### 5.3. Baby version

Let \( P \subset G \) stand for the stabilizer of the Lagrangian subspace \( N \subset M \) (Siegel parabolic). Let \( P' \subset P \) stand for the derived subgroup. We consider the Lagrangian Grassmannian \( \text{LGr}_M = G/P \). The \( \mu_2 \)-gerbe of square roots of the ample determinant line bundle \( D \) over \( \text{LGr}_M \) is denoted \( \tilde{\text{LGr}}_M \). The group \( P' \) acts on \( \tilde{\text{LGr}}_M \). We consider the derived constructible category \( D^b_{P'}(\tilde{\text{LGr}}_M) \) of genuine sheaves on \( \text{LGr}_M \) (such that \(-1 \in \mu_2 \) acts by \(-1 \)). An open sub-gerbe \( \mathcal{F} \hookrightarrow \text{LGr}_M \times \text{LGr}_M \) is formed by all the pairs of transversal Lagrangian subspaces in \( M \). We denote by \( \text{LGr}_M \overset{\rho_1}{\leftarrow} \mathcal{F} \overset{\rho_2}{\rightarrow} \text{LGr}_M \) the two projections, and we define the Radon Transform \( RT := \rho_2 \circ \rho_1^{-1}: D^b_{P'}(\text{LGr}_M) \to D^b_{P'}(\text{LGr}_M) \). Finally, we

\[ 11 \] So strictly speaking we should consider the generators in \( \text{Hom}(N', N)^* \oplus \text{Hom}(N, N')^* \) as having odd parity.
consider the $P'$-equivariant derived category $\text{D-mod}^{P'}_{1/2}(\text{LGr}_M)$ of $D$-modules on $\text{LGr}_M$ twisted by the negative square root of the determinant line bundle $\mathcal{D}$. We have the Riemann–Hilbert equivalence $\text{RH}: \text{D-mod}^{P'}_{1/2}(\text{LGr}_M) \xrightarrow{\sim} \mathcal{D}b_{P'}(\tilde{\text{LGr}}_M)$.

The Weyl algebra of the symplectic space $M$ is denoted by $W_M$. The homomorphism $g = \text{sp}(M) \to \text{Lie}W_M$ (oscillator representation) goes back to [S], see [H, §2] and [La, §1.1]. The restriction of the $W_M$-module $\mathbb{C}[N]$ to $g$ is a direct sum of two irreducible modules $L_{\lambda_g} \oplus L_{\lambda_s}$ (even and odd functions). Here in the standard orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$ of a Cartan Lie subalgebra of Lie $P$ we have

$$\lambda_g = -\frac{1}{2} \sum_{i=1}^n \varepsilon_i,$$

and

$$\lambda_s = \lambda_g - \varepsilon_n.$$

The baby version $S$ of $\Theta$-sheaf, introduced in [Ly, Definition 2] and studied in [LL, §2], is the direct sum of IC-sheaves of two $P$-orbits in $\tilde{\text{LGr}}_M$: $S_g$ of the open orbit, and $S_s$ of the codimension 1 orbit. We have irreducible twisted $D$-modules $S_g = \tau_{\geq 0}\text{Loc}L^{\lambda_g}$, $S_s = \tau_{\geq 0}\text{Loc}L^{\lambda_s}$, and $\text{RH}(S_g) = S_g$, $\text{RH}(S_s) = S_s$.

Finally, $\text{RT}(S)$ is isomorphic to $S$ up to a shift. More precisely, we have $\text{RT}(S_g) \simeq S_g[n^2 + 2]$, and $\text{RT}(S_s) \simeq S_g[n^2]$ for $n$ odd, while for $n$ even we have $\text{RT}(S_s) \simeq S_s[n^2 + 2]$ and $\text{RT}(S_g) \simeq S_s[n^2]$. This follows e.g. from [LY, Theorem 10.7].

**Appendix A. Localization and the Radon transform**

By Gurbir Dhillon

**A.1. Lie groups and algebras.** Let $G$ be an almost simple, simply connected, group and $\mathfrak{g}$ its Lie algebra.\footnote{The results discussed below straightforwardly generalize to any connected reductive group $G$.} Let $\kappa$ be a level, i.e. an Ad-invariant bilinear form on $\mathfrak{g}$, and consider the associated affine Lie algebra

$$0 \to \mathbb{C} \cdot 1 \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0.$$

**A.2. Levels.** Let us write $\kappa_c$ for the critical level, i.e., minus one half times the Killing form. We recall that a level $\kappa$ is called positive if

$$\kappa \notin \kappa_c + \mathbb{Q}^{\geq 0} \cdot \kappa_c.$$

Similarly, a level $\kappa$ is called negative if

$$\kappa \notin \kappa_c - \mathbb{Q}^{\geq 0} \cdot \kappa_c.$$

Note that, in this convention, an irrational multiple of the critical level is considered both positive and negative.
A.3. **Localization on the thin Grassmannian.** For any level $\kappa$, one has a $\text{D-mod}_\kappa(G_K)$-equivariant functor of global sections

$$\Gamma_\kappa : \text{D-mod}_\kappa(Gr_G) \to \hat{\mathfrak{g}}_\kappa \text{-mod}.$$ 

It is the unique equivariant functor sending the delta D-module at the trivial coset $\delta_e$ to the vacuum module, i.e., the parabolically induced module

$$\mathbb{V}_\kappa := \text{pind}^{\hat{\mathfrak{g}}}_0 \mathbb{C}.$$ 

The functor admits a right adjoint. Moreover, after passing to spherical vectors, it also admits a left adjoint. That is, one has an adjunction

$$\text{Loc}_\kappa : \hat{\mathfrak{g}}_\kappa \text{-mod} \rightleftarrows \text{D-mod}_\kappa(Gr_G)^{\ast} : \Gamma_\kappa.$$ 

A.4. **Localization on the thick Grassmannian.** Let us denote the usual and dual categories of D-modules on the thick Grassmannian by

$$\text{D-mod}_\kappa(Gr_G)^{!} \quad \text{and} \quad \text{D-mod}_\kappa(Gr_G)^{\ast}.$$ 

By definition, if we let $U_i$ range through the quasicompact open subschemes of $G_G$, we have

$$\text{D-mod}_\kappa(Gr_G)^{!} \simeq \lim_{\leftarrow \frac{i}{i}} \text{D-mod}_\kappa(U_i) \quad \text{and} \quad \text{D-mod}_\kappa(Gr_G)^{\ast} \simeq \lim_{\rightarrow \frac{i}{i}} \text{D-mod}_\kappa(U_i),$$

where the transition maps are given by $!$-restriction and $\ast$-pushforward, respectively.

Following Arkhipov–Gaitsgory [ArG], one has $\text{D-mod}_\kappa(G_K)$-equivariant localization and global sections functors

(A.4.1) $$\text{Loc}_\kappa : \hat{\mathfrak{g}}_\kappa \text{-mod} \to \text{D-mod}_\kappa(Gr_G)^{!} \quad \text{and} \quad \Gamma_\kappa : \text{D-mod}_\kappa(Gr_G)^{\ast} \to \hat{\mathfrak{g}}_\kappa \text{-mod}.$$ 

Upon passing to spherical vectors, one has the following adjunctions, which are sensitive to the sign of the level. If $\kappa$ is positive, $\text{Loc}_\kappa$ admits a right adjoint of (smooth) global sections

(A.4.2) $$\text{Loc}_\kappa : \hat{\mathfrak{g}}_\kappa \text{-mod}^{G_0} \rightleftarrows \text{D-mod}_\kappa(Gr_G)^{G_0} : \Gamma_\kappa.$$ 

Similarly, if $\kappa$ is negative, $\Gamma_\kappa$ admits a left adjoint

(A.4.3) $$\text{Loc}_\kappa : \hat{\mathfrak{g}}_\kappa \text{-mod}^{G_0} \rightleftarrows \text{D-mod}_\kappa(Gr_G)^{G_0} : \Gamma_\kappa.$$ 

We emphasize that the sources of the functors denoted $\Gamma_\kappa$ in (A.4.1) and (A.4.2) are distinct, as are the sources of the functors denoted $\text{Loc}_\kappa$ in (A.4.1) and (A.4.3).
A.5. **Radon Transform.** For any level $\kappa$, consider the Radon transform functors

\[
\text{RT}_! : \text{D-mod}_{\kappa}(\text{Gr}_G) \to \text{D-mod}_{\kappa}(\text{Gr}_G)_! \quad \text{and} \\
\text{RT}_* : \text{D-mod}_{\kappa}(\text{Gr}_G) \to \text{D-mod}_{\kappa}(\text{Gr}_G)_* .
\]

These are by definition $\text{D-mod}_{\kappa}(G_K)$-equivariant, and are characterized by sending $\delta_e$ to the $!$- and $*$-extensions of the constant intersection cohomology D-module

\[
\mathbb{C}[G_O \cdot G_{C[t^{-1}]} / G_{C[t^{-1}]}],
\]

respectively. In what follows, we denote these objects by $j_!$ and $j_*$, respectively.

It is standard that $\text{RT}_!$ and $\text{RT}_*$ induce equivalences on spherical vectors, and in particular are fully faithful embeddings.

A.6. **Global sections and the Radon transform: negative level.** We now turn to the relationship between the global sections functors on the thin and thick Grassmannians and the Radon transform. We begin with the case of $\kappa$ negative.

**Proposition A.6.1.** Suppose $\kappa$ is negative. Then the functor of global sections on the thin Grassmannian

\[
\Gamma_\kappa : \text{D-mod}_{\kappa}(\text{Gr}_G) \to \widehat{\mathfrak{g}}_\kappa\text{-mod}
\]

is canonically $\text{D-mod}_{\kappa}(G_X)$-equivariantly equivalent to the composition

\[
\text{D-mod}_{\kappa}(\text{Gr}_G) \xrightarrow{\text{RT}_*} \text{D-mod}_{\kappa}(\text{Gr}_G)_* \xrightarrow{\Gamma_*} \widehat{\mathfrak{g}}_\kappa\text{-mod}. 
\]

**Proof.** It is enough to show that the composition (A.6.2) sends $\delta_e$ to the vacuum module $\mathbb{V}_\kappa$. Unwinding definitions, we have

\[
\Gamma_\kappa \circ \text{RT}_*(\delta_e) \simeq \Gamma_\kappa(j_*) \simeq \mathbb{C}[G_O \cdot G_{C[t^{-1}]} / G_{C[t^{-1}]}],
\]

i.e., $\delta_e$ is sent to the algebra of functions on the big cell. The function which is identically one on the cell yields, by its $G_O$ invariance, a canonical map of $\widehat{\mathfrak{g}}_\kappa$-modules

\[
\mathbb{V}_\kappa \to \mathbb{C}[G_O \cdot G_{C[t^{-1}]} / G_{C[t^{-1}]}].
\]

It is straightforward to see that the characters of the two appearing modules coincide. Moreover, by our assumption on $\kappa$, $\mathbb{V}_\kappa$ is irreducible, hence the map is an isomorphism, as desired. \qed

**Remark A.6.2.** The functions on the big cell, at any level, are canonically isomorphic to the contragredient dual of the vacuum. In particular, at a positive rational level $\kappa$, the assertion of Proposition A.6.1 is false. We will meet its corrected variant in Proposition A.7.1 below.

By taking the statement of Proposition A.6.1, passing to spherical invariants, and then left adjoints, we deduce the following.
Corollary A.6.3. Suppose $\kappa$ is negative. Then the localization functor on the thin Grassmannian

$$\text{Loc}_\kappa : \widehat{\mathfrak{g}}_\kappa \text{-mod}^{G_0} \to \text{D-mod}_\kappa(G_0 \setminus G \setminus G_0)$$

is canonically $\text{D-mod}_\kappa(G_0 \setminus G \setminus G_0)$-equivariantly equivalent to the composition

$$\widehat{\mathfrak{g}}_\kappa \text{-mod}^{G_0} \xrightarrow{\text{Loc}_\kappa} \text{D-mod}_\kappa(G_0)_* \xrightarrow{\text{RT}^{-1}} \text{D-mod}_\kappa(G_0).$$

A.7. Global sections and the Radon transform: positive level. Let us now turn to the case of $\kappa$ of positive level. As we will see momentarily, the analog of the approach we took at negative level requires knowing the global sections of a $!$-extension, and is therefore less immediate.

Proposition A.7.1. Suppose $\kappa$ is positive. Then the functor of global sections on the thin Grassmannian

$$\Gamma_\kappa : \text{D-mod}_\kappa(G_0) \to \widehat{\mathfrak{g}}_\kappa \text{-mod}$$

is canonically $\text{D-mod}_\kappa(G_0 \setminus G \setminus G_0)$-equivariantly equivalent to the composition

$$\text{D-mod}_\kappa(G_0) \xrightarrow{\text{RT}_!} \text{D-mod}_\kappa(G_0)_! \xrightarrow{\Gamma_\kappa} \widehat{\mathfrak{g}}_\kappa \text{-mod}.$$

Proof. It is enough to show the composition sends $\delta_e$ to the vacuum module $\mathbb{V}_\kappa$.

By definition, we have that

$$\Gamma_\kappa \circ \text{RT}_!(\delta_e) \simeq \Gamma_\kappa(j_!) \circ \text{RT}_!(\delta_e).$$

We will deduce the calculation of the latter global sections from the work of Kashiwara–Tanisaki on localization at positive level [KT].

To do so, fix a Borel subgroup $B^-$ of $G$. Write $I^-$ for the ‘thick Iwahori’ group ind-scheme associated to $B^-$, i.e., the preimage of $B^-$ under the map

$$G_{[t^{-1}]} \to G$$

given by evaluation at infinity. Write $\text{Fl}_G \simeq G_\kappa I^- \setminus I^-$ for the thick affine flag variety. Consider the functor of (smooth) global sections

$$\Gamma_\kappa(\text{Fl}_G, -) : \text{D-mod}_\kappa(\text{Fl}_G) \to \widehat{\mathfrak{g}}_\kappa \text{-mod},$$

which is denoted in loc.cit. by $\overline{\Gamma}$.

Fix another Borel subgroup $B$ of $G$ in general position with $B^-$. Write $I$ for the associated Iwahori group scheme, i.e., the preimage of $B$ under the map $G_0 \to G$ given by evaluation at zero.

Let us denote by $j_!$ the $!$-extension of the constant intersection cohomology D-module on the open orbit $I \cdot I^- / I^-$. On the other side of $\Gamma_\kappa$, let us denote the Verma module of highest weight zero for $\mathfrak{g}$ by $M_0$, and note the Verma module for $\widehat{\mathfrak{g}}_\kappa$ of highest weight zero is given by $\text{pind}_{\widehat{\mathfrak{g}}_\kappa}(M_0)$. 
Then, the desired result of Kashiwara–Tanisaki is the canonical equivalence
\[ \Gamma_\kappa(\text{Fl}_G, j!) \simeq \text{pind}_{\mathfrak{b}}^\kappa(M_0), \]
see [KT, Theorem 4.8.1(ii)].

We are ready to deduce the proposition. Consider the projection
\[ \pi : \text{Fl}_G \to \text{Gr}_G. \]
As both functors denoted by \( \Gamma_\kappa \) are the smooth vectors in the naive global sections, and \( \pi \) is a Zariski locally trivial fibration with fibre \( G/B \), we have that
\[ \Gamma_\kappa(j!) \simeq \Gamma_\kappa(\text{Fl}_G, \pi^!\ast(j!)), \]
where \( \pi^! \):= \( \pi^![-\dim G/B] \). If we write \( \text{Av}^I_{G_\circ} \) for the functor of relative !-averaging from \( I \)-invariants to \( G_\circ \)-invariants, note that
\[ \pi^!\ast(j!) \simeq \text{Av}^I_{G_\circ}(j!). \]

By the equivariance of the appearing functors, we then have
\[ \Gamma_\kappa(\text{Fl}_G, \pi^!\ast(j!)) \simeq \Gamma_\kappa(\text{Fl}_G, \text{Av}^I_{G_\circ}(j!)) \]
\[ \simeq \text{Av}^I_{G_\circ} \circ \Gamma_\kappa(\text{Fl}_G, j!) \simeq \text{Av}^I_{G_\circ} \circ \text{pind}_{\mathfrak{b}}^\kappa(M_0) \]
\[ \simeq \text{pind}_{\mathfrak{b}}^\kappa \circ \text{Av}^{B,G}_{G_\circ}(M_0) \simeq \text{pind}_{\mathfrak{b}}^\kappa(\mathbb{C}) \simeq \mathbb{V}_\kappa, \]
as desired.

\[ \square \]

**Corollary A.7.2.** Suppose \( \kappa \) is positive. Then the functor of localization on the thin Grassmannian
\[ \text{Loc}_\kappa : \widehat{\mathfrak{g}}_\kappa \text{-mod}^{G_\circ} \to \text{D-mod}_{\kappa}(\text{Gr}_G)^{G_\circ} \]
is canonically \( \text{D-mod}_{\kappa}(G_\circ \backslash G_\kappa \backslash G_\circ) \)-equivariantly equivalent to the composition
\[ \widehat{\mathfrak{g}}_\kappa \text{-mod}^{G_\circ} \xrightarrow{\text{Loc}_\kappa} \text{D-mod}_{\kappa}(\text{Gr}_G)^{G_\circ} \xrightarrow{\text{RT}^{-1}} \text{D-mod}_{\kappa}(\text{Gr}_G)^{G_\circ}. \]

**Remark A.7.3.** Analogs of the results of this appendix hold, *mutatis mutandis*, after replacing the thick and thin Grassmannians by any opposite thick and thin partial affine flag varieties, by similar arguments, as well as for monodromic D-modules on the enhanced thick and thin affine flag varieties. Similarly, one may replace \( G_\kappa \) by a quasi-split form.

With some care about hypotheses on twists, similar results hold for a symmetrizable Kac–Moody group, again by similar arguments. We leave the details to the interested reader.

\[ ^{13} \text{Strictly speaking, Kashiwara–Tanisaki discuss only the case of } \kappa \text{ positive rational, but their argument applies more generally to any positive } \kappa. \]
Appendix B. Topological vs. Algebraic Anomaly Cancellation Condition

By Theo Johnson-Freyd

The goal of this appendix is to prove Proposition 4.1.1.

B.1. Simply connected case. Let $G$ be a connected complex reductive group with classifying space $BG$, and let $\varrho: G \to \text{Sp}(2n, \mathbb{C})$ a symplectic representation of $G$. Recall that $H^4(B\text{Sp}(2n, \mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$ is generated by the universal (quaternionic first) Pontryagin class $q_1$. Thus $\varrho$ has a (quaternionic first) Pontryagin class $q_1(\varrho) = \varrho^*(q_1) \in H^4(BG, \mathbb{Z})$, equal (up to a sign convention) to the second Chern class of the underlying complex representation $\varrho: G \to \text{Sp}(2n, \mathbb{C}) \to \text{SL}(2n, \mathbb{C})$. Recall furthermore that $\pi_4\text{Sp}(2n, \mathbb{C}) \cong \pi_5B\text{Sp}(2n, \mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$.

Theorem B.1.1. If $q_1(\varrho)$ is even, i.e. divisible by 2 in $H^4(BG, \mathbb{Z})$, then $\varrho$ induces the zero map $\pi_5\varrho: \pi_5BG \to \pi_5B\text{Sp}(2n, \mathbb{C})$. If $G$ is simply connected, then the converse holds: if $\pi_5\varrho = 0$, then $q_1(\varrho)$ is even.

Theorem B.1.1 obviously depends only on the homotopy 5-type $\tau_{\leq 5}B\text{Sp}(2n, \mathbb{C})$ of $B\text{Sp}(2n, \mathbb{C})$. This homotopy 5-type is independent of $n$, and so we will henceforth call it simply $\tau_{\leq 5}B\text{Sp}$. We will prove Theorem B.1.1 for any map $\varrho: BG \to \tau_{\leq 5}B\text{Sp}$.

Remark B.1.2. To see that simple connectivity is a necessary condition, consider $\varrho: G = \mathbb{C}^\times \hookrightarrow \text{Sp}(2, \mathbb{C})$ a Cartan torus of $\text{Sp}(2, \mathbb{C})$. Then $q_1(\varrho)$ is a generator of $H^4(BG, \mathbb{Z})$.

Proposition B.1.3. If $G$ is connected and simply connected, then $H_5BG$ is trivial.

Proof. Recall that $\pi_2G = \pi_3BG$ vanishes and $\pi_4BG = H_4BG$ is a free abelian group.\textsuperscript{14} Recall furthermore that $H^\ast(BG, \mathbb{Q})$ is concentrated in even degrees.\textsuperscript{15} From the universal coefficient theorem, we find that $H_5BG$ is torsion.

Choose a Borel subgroup $B \subset G$, and consider the flag variety $X = G/B$. The homology of $X$ is very well understood. Indeed, $X$ has a Schubert decomposition into cells of even real dimension. In particular, the homology of the manifold $X$ is free abelian and concentrated in even degrees.

Consider the homological Serre spectral sequence for the fibre bundle $X \to BB \to BG$:

\[ E^2_{ij} := H_i(BG, H_jX) \Rightarrow H_{i+j}BB. \]

\textsuperscript{14}Indeed, $\pi_3BG$ vanishes for every Lie group, with no conditions, and $\pi_4BG$ is always free abelian. The Hurewicz map $\pi_4BG \to H_4BG$ is an isomorphism if $G$ is simply connected, in which case $H_4BG$ has rank equal to the number of simple factors of $G$.

\textsuperscript{15}$H^\ast(BG, \mathbb{Q})$ is a polynomial algebra on generators of degrees twice the exponents of $G$. 
The \( E^2 \) page vanishes whenever \( j \) is odd and also when \( 1 \leq i \leq 3 \). Since \( B \) is homotopy equivalent to a torus, \( H_\bullet BB \) is free abelian and concentrated in even degrees, and hence the \( E^\infty \) page vanishes when \( i + j \) is odd. It follows that there is an exact sequence

\[
0 \to H_5 BG \to H_4 X \to H_4 BB \to H_4 BG \to 0.
\]

But \( H_5 BG \) is torsion, whereas \( H_4 X \) is free abelian.

**Corollary B.1.4.** Let \( G \) be a connected complex reductive Lie group, not necessarily simply connected, and let \( Y \) be any topological space. Suppose given a map \( BG \to \tau_{\leq 4} Y \) which admits a lift to \( Y \). Then any two lifts \( BG \to Y \) induce the same map \( \pi_5 BG \to \pi_5 Y \).

**Proof.** The lifts of a map \( BG \to \tau_{\leq 4} Y \) along \( \tau_{\leq 5} Y \to \tau_{\leq 4} Y \), assuming there are any, form a torsor for \( H^0(BG, \pi_5 Y) \). Suppose two lifts differ by some class in \( H^0(BG, \pi_5 Y) \). Then their actions on \( \pi_5 BG \) differ by the image of that class along the Hurewicz map \( H^5(BG, \pi_5 Y) \to \text{Hom}(\pi_5 BG, \pi_5 Y) \) induced from \( \pi_5 BG \to H_5 BG \).

Let \( G^{sc} \) denote the simply connected cover of \( G \). Then \( \pi_5 BG^{sc} \to \pi_5 BG \) is an isomorphism, and so the Hurewicz map \( \pi_5 BG \to H_5 BG \) factors through \( H_5 BG^{sc} \to H_5 BG \). But \( H_5 BG^{sc} = 0 \) by Proposition B.1.3.

To complete the proof of Theorem B.1.1, we will need to know the space \( \tau_{\leq 5} BSp \). It has precisely two nontrivial homotopy groups: \( \pi_4 = \mathbb{Z} \) and \( \pi_5 = \mathbb{Z}/2\mathbb{Z} \). Thus we will know it completely if we know its Postnikov k-invariant. Recall that the Postnikov k-invariant of the extension \( K(\mathbb{Z}/2\mathbb{Z}, 5) \to \tau_{\leq 5} BSp \to K(\mathbb{Z}, 4) \) is some universal cohomology operation \( f: H^4(-, \mathbb{Z}) \to H^6(-, \mathbb{Z}/2\mathbb{Z}) \). A map \( X \to K(\mathbb{Z}, 4) \) is, up to homotopy, a class \( \alpha \in H^4(X, \mathbb{Z}) \), and it lifts along \( \tau_{\leq 5} BSp \to K(\mathbb{Z}, 4) \) if and only if \( f(\alpha) = 0 \in H^6(X, \mathbb{Z}/2\mathbb{Z}) \).

**Lemma B.1.5.** The Postnikov k-invariant of the \( \tau_{\leq 5} BSp \) is \( \text{Sq}^2 \circ \text{mod} \ 2 \): \( H^4(-, \mathbb{Z}) \to H^6(-, \mathbb{Z}/2\mathbb{Z}) \), where \( \text{mod} \ 2 : H^4(-, \mathbb{Z}) \to H^4(-, \mathbb{Z}/2\mathbb{Z}) \) is the corresponding map on coefficients, and \( \text{Sq}^2 \) is the second Steenrod square.

**Proof.** Bott periodicity identifies \( \tau_{\leq 5} BSp \) with the 4-fold suspension of the infinite loop space \( \tau_{\leq 1} ko \). Thus the statement in the Lemma follows from (and is equivalent to) the fact that the k-invariant (at the level of infinite loop spaces) connecting \( \pi_0 ko = \mathbb{Z} \) to \( \pi_1 ko = \mathbb{Z}/2\mathbb{Z} \) is \( \text{Sq}^2 \circ \text{mod} \ 2 \).

**Proof of Theorem B.1.1.** Fix \( q: BG \to \tau_{\leq 5} BSp \). The class \( q_1(q) \in H^4(BG, \mathbb{Z}) \) is nothing but the image of \( q \) along \( \tau_{\leq 5} BSp \to \tau_{\leq 4} BSp = K(\mathbb{Z}, 4) \), and note that \( q_1(q) \) factors through \( \tau_{\leq 4} BG \).

Suppose that \( q_1(q) \) is even. Then \( \text{Sq}^2(q_1(q) \mod 2) = 0 \), and so \( q_1(q): \tau_{\leq 4} BG \to \tau_{\leq 4} BSp \) lifts to a map \( \tau_{\leq 4} BG \to \tau_{\leq 5} BSp \). The composition \( BG \to \tau_{\leq 4} BG \to \tau_{\leq 5} BSp \) vanishes on \( \pi_5 BG \). This composition might not be equal to \( q \), but it and
\( q \) are both lifts of the same map \( \text{BG} \to \tau_{\leq 4}\text{BSp} \). And so by Corollary B.1.4 they have the same (trivial) value on \( \pi_3\text{BG} \).

Now suppose that \( G \) is connected and simply connected. Then \( \tau_{\leq 4}\text{BG} \cong K(A, 4) \) where \( A \) is a free abelian group, and \( H^4(\text{BG}, \mathbb{Z}/2\mathbb{Z}) = H^4(K(A, 4), \mathbb{Z}/2\mathbb{Z}) = \text{Hom}(A, \mathbb{Z}/2\mathbb{Z}) \). We claim that \( \text{Sq}^2 \colon H^4(K(A, 4), \mathbb{Z}/2\mathbb{Z}) \to H^6(K(A, 4), \mathbb{Z}/2\mathbb{Z}) \) is injective. Indeed, suppose that \( \alpha \neq 0 \in \text{Hom}(A, \mathbb{Z}/2\mathbb{Z}) \), and let \( a : \mathbb{Z} \to A \) be an element such that \( \alpha(a) \neq 0 \). By restricting along the corresponding map \( K(\mathbb{Z}, 4) \to K(A, 4) \), if suffices to prove the claim when \( A = \mathbb{Z} \) and \( \alpha \) is the map that reduces mod 2. There is a nonzero map \( \beta : K(\mathbb{Z}/2\mathbb{Z}, 3) \to K(\mathbb{Z}, 4) \), and the composition \( K(\mathbb{Z}/2\mathbb{Z}, 3) \xrightarrow{\beta} K(\mathbb{Z}, 4) \xrightarrow{\alpha} K(\mathbb{Z}/2\mathbb{Z}, 4) \) is the class \( \text{Sq}^1z \in H^4(K(\mathbb{Z}/2\mathbb{Z}, 3), \mathbb{Z}/2\mathbb{Z}) \), where \( z \in H^3(K(\mathbb{Z}/2\mathbb{Z}, 3), \mathbb{Z}/2\mathbb{Z}) \) generates \( H^3(K(\mathbb{Z}/2\mathbb{Z}, 3), \mathbb{Z}/2\mathbb{Z}) \) over the Steenrod algebra. Then \( \text{Sq}^2(\alpha)(\beta) = \text{Sq}^2\text{Sq}^1z \neq 0 \in H^6(K(\mathbb{Z}/2\mathbb{Z}, 3), \mathbb{Z}/2\mathbb{Z}) \). It follows that \( \text{Sq}^2(\alpha) \neq 0 \), proving the claim that \( \text{Sq}^2 : H^4(K(A, 4), \mathbb{Z}/2\mathbb{Z}) \to H^6(K(A, 4), \mathbb{Z}/2\mathbb{Z}) \) is injective.

Suppose that \( \pi_5\emptyset = 0 \). Then the map \( \tau_{\leq 5}\emptyset : \tau_{\leq 5}\text{BG} \to \tau_{\leq 5}\text{BSp} \) factors through the cofibre of the inclusion \( K(\pi_5\text{BG}, 5) \to \tau_{\leq 5}\text{BG} \). Note that this inclusion is the fibre of the map \( \tau_{\leq 5}\text{BG} \to \tau_{\leq 4}\text{BG} \). In general, given a fibre bundle of spaces \( F \to E \to B \), there is a canonical map \( \text{cofibre}(F \to E) \to B \), but it is not always an equivalence. However, assuming \( G \) is connected and simply connected, then \( \tau_{\leq 5}\emptyset : \tau_{\leq 5}\text{BG} \to \tau_{\leq 5}\text{BSp} \) is canonically a map of infinite loop spaces, and for infinite loop spaces, a fibre and cofibre sequences agree. In particular, if \( \pi_5\emptyset = 0 \) and \( G \) is connected and simply connected, then \( \tau_{\leq 5}\emptyset : \tau_{\leq 5}\text{BG} \to \tau_{\leq 5}\text{BSp} \) factors through \( \tau_{\leq 4}\text{BG} \).

But this means that \( q_1(\emptyset) : \tau_{\leq 4}\text{BG} \to \tau_{\leq 4}\text{BSp} \) does lift along \( \tau_{\leq 5}\text{BSp} \to \tau_{\leq 5}\text{BSp} \), and so \( \text{Sq}^2(q_1(\emptyset) \text{ mod } 2) = 0 \in H^6(\tau_{\leq 4}\text{BG}, \mathbb{Z}/2\mathbb{Z}) \). On the other hand, since \( G \) is connected and simply connected, \( \text{Sq}^2 : H^4(\tau_{\leq 4}\text{BG}, \mathbb{Z}/2\mathbb{Z}) \to H^6(\tau_{\leq 4}\text{BG}, \mathbb{Z}/2\mathbb{Z}) \) is injective. Thus \( q_1(\emptyset) \text{ mod } 2 = 0 \in H^4(\tau_{\leq 4}\text{BG}, \mathbb{Z}/2\mathbb{Z}) = H^4(\text{BG}, \mathbb{Z}/2\mathbb{Z}) \), or in other words \( q_1(\emptyset) \) is even. \( \square \)

### B.2. General case (proof of Proposition 4.1.1)

We choose a Cartan torus \( T \subset G \). The Weyl group of \((G, T)\) is denoted \( W \). If \( G \) is simply connected, then the coweight lattice \( X_c(T) \) coincides with the coroot lattice \( Q \). The cohomology group \( H^4(\text{BG}, \mathbb{Z}) \) is canonically identified with the group \( \text{Bil}(Q)^W \) of \( W \)-invariant integer-valued bilinear forms on \( Q \) such that \( B(\lambda, \lambda) \in 2\mathbb{Z} \) for any \( \lambda \in Q \) (invariant even bilinear forms). Let \( \text{Tr} : \mathfrak{sp}(2n, \mathbb{C}) \times \mathfrak{sp}(2n, \mathbb{C}) \to \mathbb{C} \) stand for the trace form of the defining representation of \( \text{Sp}(2n, \mathbb{C}) \). Given a representation

\footnote{In general, a space all of whose homotopy groups are in degrees \( (n, 2n) \) for some \( n \) is automatically an infinite loop space.}
\( \varrho \colon G \to \text{Sp}(2n, \mathbb{C}) \), we obtain a bilinear form \( \varrho^* \text{Tr} \in \text{Bil}(Q)^W \). According to Theorem B.1.1, the vanishing of \( \pi_4Q \) is equivalent to the divisibility \( \varrho^* \text{Tr} \in 2\text{Bil}(Q)^W \).

For arbitrary reductive \( G \) with a Cartan torus \( T \), we denote by \( \text{Bil}(X_*(T))^W \) the group of \( W \)-invariant integer-valued bilinear forms on \( X_*(T) \) such that \( B(\lambda, \lambda) \in 2\mathbb{Z} \) for any \( \lambda \) in the coroot sublattice \( Q \subset X_*(T) \). For a representation \( \varrho \colon G \to \text{Sp}(2n, \mathbb{C}) \) we have to check the equivalence of conditions \( \pi_4\varrho = 0 \) and \( \varrho^* \text{Tr} \in 2\text{Bil}(X_*(T))^W \).

First, if \( G = T \) is a torus, then \( \pi_4(T) = 0 \), and it is immediate to check that \( \varrho^* \text{Tr} \in 2\text{Bil}(X_*(T)) \). Hence the desired equivalence holds true for any symplectic representation of any group of the form \( G_{sc} \times T \).

Now for general \( G \), choose a finite cover \( \varpi \colon G' \times T \to G \), where \( G' \) is semisimple simply-connected. It remains to check that \( \varrho^* \text{Tr} \) is divisible by 2 iff \( (\varrho \circ \varpi)^* \text{Tr} \) is divisible by 2. This is clear since \( G \) and \( G' \times T \) share the same coroots, and the pullback of the trace form to the coweight lattice of any torus is always divisible by 2.

This completes the proof of Proposition 4.1.1.

References

[AGa] D. Arinkin, D. Gaitsgory, Singular support of coherent sheaves, and the geometric Langlands conjecture, Selecta Math. (N.S.) 21 (2015), 1–199.

[ArG] S. Arkhipov, D. Gaitsgory, Localization and the long intertwining operator for representations of affine Kac-Moody algebras, https://people.math.harvard.edu/~gaitsgde/GL/Arkh.pdf (2009).

[BZSV] D. Ben-Zvi, Y. Sakellaridis and A. Venkatesh, Duality in the relative Langlands program, in preparation.

[BeF] R. Bezrukavnikov, M. Finkelberg, Equivariant Satake category and Kostant-Whittaker reduction, Mosc. Math. J. 8 (2008), no. 1, 39–72.

[BPW] T. Braden, N. Proudfoot, B. Webster, Quantizations of conical symplectic resolutions I: local and global structure, Astérisque 384 (2016), 1–73.

[BLPW] T. Braden, A. Licata, N. Proudfoot, B. Webster, Quantizations of conical symplectic resolutions II: category \( \mathcal{O} \) and symplectic duality, Astérisque 384 (2016), 75–179.

[BFN1] A. Braverman, M. Finkelberg, H. Nakajima, Towards a mathematical definition of Coulomb branches of 3-dimensional \( \mathcal{N} = 4 \) gauge theories, II, Adv. Theor. Math. Phys. 22 (2018), no. 5, 1071–1147.

[BFN2] A. Braverman, M. Finkelberg, H. Nakajima, Coulomb branches of 3d \( \mathcal{N} = 4 \) quiver gauge theories and slices in the affine Grassmannian (with appendices by Alexander Braverman, Michael Finkelberg, Joel Kamnitzer, Ryosuke Kodera, Hiraku Nakajima, Ben Webster, and Alex Weekes), Adv. Theor. Math. Phys. 23 (2019), no. 1, 75–166.

[BFN3] A. Braverman, M. Finkelberg, H. Nakajima, Ring objects in the equivariant derived Satake category arising from Coulomb branches (with appendix by Gus Lonergan), Adv. Theor. Math. Phys. 23 (2019), no. 2, 253–344.

[BFGT] A. Braverman, M. Finkelberg, V. Ginzburg, R. Travkin, Mirabolic Satake equivalence and supergroups, Compos. Math. 157 (2021), no. 8, 1724–1765.
[BFT] A. Braverman, M. Finkelberg, R. Travkin, Orthosymplectic Satake equivalence, arXiv:1912.01930.
[DG] V. Drinfeld, D. Gaitsgory, On some finiteness questions for algebraic stacks, Geom. Funct. Anal. 23 (2013), no. 1, 149-294.
[DLYZ] G. Dhillon, Y.-W. Li, Z. Yun, X. Zhu, Endoscopy for affine Hecke categories, in preparation.
[FF] A. Feingold, I. Frenkel, Classical Affine Algebras, Adv. Math. 56 (1985), 117–172.
[GW1] D. Gaiotto and E. Witten, Supersymmetric Boundary Conditions in N = 4 Super Yang-Mills Theory, Journal of Statistical Physics, 135 (2009), 789–855.
[GW2] D. Gaiotto and E. Witten, S-duality of boundary conditions in N = 4 super Yang-Mills theory, Adv. Theor. Math. Phys. 13 (2009), no. 3, 721–896.
[GR] D. Gaitsgory and N. Rozenblyum, A study in derived algebraic geometry. Vol. I. Correspondences and duality, Mathematical Surveys and Monographs 221 (2017).
[G] V. Ginzburg, Perverse sheaves on a loop group and Langlands duality, arXiv:alg-geom/9511007.
[HR] J. Hilburn and S. Raskin, Tate’s thesis in the de Rham Setting, arXiv:2107.11325.
[H] R. Howe, Remarks on classical invariant theory, Trans. Amer. Math. Soc. 313 (1989), no. 2, 539–570. Erratum, Trans. Amer. Math. Soc. 318 (1990), no. 2, 823.
[KT] M. Kashiwara, T. Tanisaki, Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras. III. Positive rational case, Asian J. Math. 2 (1998), no. 4, 779–832.
[K] B. Kostant, Lie group representations on polynomial rings, American Journal of Math. 85 (1963), no. 3, 327–404.
[La] V. Lafforgue, Correspondance Theta pour les D-modules, unpublished manuscript.
[Ly] S. Lysenko, Moduli of metaplectic bundles on curves and theta-sheaves, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 3, 415–466.
[LL] V. Lafforgue, S. Lysenko, Geometric Weil representation: local field case, Compos. Math. 145 (2009), no. 1, 56–88.
[LY] G. Lusztig, Z. Yun, Endoscopy for Hecke categories, character sheaves and representations, Forum Math. Pi 8 (2020), e12, 93pp. Corrigendum, arXiv:1904.01176v3.
[N] H. Nakajima, Towards a mathematical definition of Coulomb branches of 3-dimensional N = 4 gauge theories, I, Adv. Theor. Math. Phys. 20 (2016), no. 3, 595–669.
[R] S. Raskin, Homological methods in semi-infinite contexts, https://web.ma.utexas.edu/users/sraskin/topalg.pdf (2019).
[S] D. Shale, Linear symmetries of free boson fields, Trans. Amer. Math. Soc. 103 (1962), 149–167.
[T] C. Teleman, Coulomb branches for quaternionic representations, https://math.berkeley.edu/~teleman/math/perimeter.pdf (2020).
[We] A. Weekes, Quiver gauge theories and symplectic singularities, arXiv:2005.01702.
[Wi] E. Witten, An SU(2) Anomaly, Phys. Lett. B 117 (1982), no. 5, 324–328.
Department of Mathematics, University of Toronto and Perimeter Institute of Theoretical Physics, Waterloo, Ontario, Canada, N2L 2Y5;
Skolkovo Institute of Science and Technology
Email address: braval@math.toronto.edu

Yale University, New Haven, CT 06511, USA
Email address: gurbir.dhillon@yale.edu

National Research University Higher School of Economics, Russian Federation, Department of Mathematics, 6 Usacheva st, 119048 Moscow;
Skolkovo Institute of Science and Technology;
Institute for the Information Transmission Problems
Email address: fnklberg@gmail.com

The University of Texas at Austin, Department of Mathematics, RLM 8.100, 2515 Speedway Stop C1200, Austin, TX 78712, USA
Email address: sraskin@math.utexas.edu

Skolkovo Institute of Science and Technology, Moscow, Russia
Email address: roman.travkin2012@gmail.com