A probabilistic algorithm for the secant defect of Grassmann varieties

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1 Abstract

In this paper we study the higher secant varieties of Grassmann varieties in relation to Waring’s problem for alternating tensors and to Alexander-Hirschowitz theorem. We show how to identify defective higher secant varieties of Grassmannians using a probabilistic method involving Terracini’s Lemma, and we describe an algorithm which can compute, by numerical methods, \( \dim(G(k, n)) \) for \( n \leq 14 \). Our main result is that, except for Grassmannians of lines, if \( n \leq 14 \) and \( k \leq \frac{n-1}{2} \) (if \( n = 14 \) we have studied the case \( k \leq 5 \)) there are only the four known defective cases: \( G(2, 6)^3 \), \( G(3, 7)^3 \), \( G(3, 7)^4 \) and \( G(2, 8)^4 \).

2 Introduction

Waring’s problem for alternating tensors can be expressed in the following form (see [II])

Given a vector space \( V \) of dimension \( n + 1 \) and an alternating tensor \( \omega \in \Lambda^{k+1} V \), what is the least integer \( s \) such that \( \omega \) can be written as the sum of \( s \) decomposable tensors of the form \( v_1 \wedge \ldots \wedge v_{k+1} \)?
This problem is still open and in this paper we will give some evidence for what we expect the correct answer to be.

In order to formulate our result, we will consider a vector space $V$ of dimension $n + 1$ defined over a field $K$ of characteristic zero, and the Grassmann variety $G(\mathbb{R}^{k+1}, V) = G(k, n)$, which parametrises the decomposable tensors in the projective space of $\bigwedge^{k+1} V$. As will be explained in the next section, the problem translates into finding the dimension of the $s$-secant variety $G(k, n)^s$ (see definition 3.2). The expected dimension of $G(k, n)^s$ is $\min\{\binom{n+1}{k+1} - 1, s(n - k)(k + 1) + s - 1\}$, otherwise $G(k, n)^s$ is called defective (see definition 3.3).

It is well known that the Grassmannians of lines $G(\mathbb{P}^1, \mathbb{P}^n)^s$ are defective until they fill the ambient space and a list of four defective $G(k, n)^s$ is given in [2].

We would like to know if there exist other defective varieties which are still unknown.

Computing the dimension of $G(k, n)^s$ is quite difficult, even with the aid of a symbolic computation package; indeed just after the defective examples of [2], the computer's memory reaches its limit with the usual elimination technique using Gröbner basis.

The main idea behind this paper is that one can compute $\dim(G(k, n)^s)$ by means of a probabilistic method, which consists in studying the span of the tangent spaces at $s$ chosen random points. The dimension of this span can be computed by numerical methods as the rank of a large matrix, and when this dimension coincides with that expected, we can be sure that $G(k, n)^s$ is not defective, indeed with another choice of points the dimension cannot be larger because of inequality (1).

This technique allows us to take the computations further and our main result is the following.

**Theorem 2.1.** If $n \leq 14$ and $k \leq \frac{n-1}{2}$ (if $n = 14$ we consider $k \leq 5$), $G(k, n)^s$ is defective only for

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1If we choose a basis for $V$ there is a natural 1-1 correspondence between the associated bases of $\bigwedge^{k+1} V$ and $\bigwedge^{n-k} V$, and between the varieties $G(k, n)$ and $G(n - k - 1, n)$. Thus we
• \( k = 1 \)
• \((k, n, s) = (2, 6, 3), \delta = 1\)
• \((k, n, s) = (3, 7, 3), \delta = 1\)
• \((k, n, s) = (3, 7, 4), \delta = 4\)
• \((k, n, s) = (2, 8, 4), \delta = 2\).

This theorem is equivalent to the following answer to Waring's problem:

**Corollary 2.2.** Given a finite dimension vector space \( V \) and an alternating tensor \( \omega \in \bigwedge^{k+1} V \), if \( n \leq 14 \) and \( k \leq \frac{n-1}{2} \) (if \( n = 14 \) we consider \( k \leq 5 \)), then \( \omega \) can be written as the sum of \( s \) decomposable tensors of the form \( v_1 \land \ldots \land v_{k+1} \), where:

\[
s = \left\lceil \frac{1}{(k+1)(n-k)+1} \right\rceil
\]

except for \((k, n, s) = (2, 6, 3), (3, 7, 3), (3, 7, 4), (2, 8, 4)\) and \( k = 1 \).

### 3 Waring’s problem and some notations

Waring's polynomial problem has attracted considerable attention from geometers and algebraists throughout its long and absorbing history since it was first put forward in 1770. This problem is connected with crucial issues in both representation theory and coding theory.

It poses the following question: if \( f \) is a homogeneous polynomial of degree \( k \) in \( n \) variables, what is the least integer \( s \) such that \( f \) can be written as the sum of \( k \)-th-powers of \( s \) linear forms?

This formulation of Waring’s problem was solved in 1995 by J. Alexander and A. Hirschowitz, who produced a formula for finding the integer \( s \); nevertheless this formula has four well known exceptions.

will only consider the variety \( G(k, n) \) where \( k \leq \frac{n-1}{2} \).
Theorem 3.1. Let \( \text{char}(\mathbb{K}) = 0 \). A homogeneous polynomial \( f \in \mathbb{K}[X_0, \ldots, X_n] \) of degree \( k \) can be represented as the sum of \( s \) powers of linear forms

\[
f = L_1^k + \ldots + L_s^k,
\]

where \( s = \lceil \frac{1}{n+1} \binom{k+n}{n} \rceil \), except in the cases where \((k, n, s) = (4, 2, 5), (4, 3, 9), (4, 4, 14), (3, 4, 7)\) and \( k = 2 \).

This challenging result can also be expressed in geometrical terms, as we will explain.

First let us recall some fundamental definitions.

Let \( X \subseteq \mathbb{P}^N \) be an \( n \)-dimensional irreducible projective variety,

**Definition 3.2.** The \( s \)-secant variety is the closure of the union of all linear spaces spanned by \( s \) points of \( X \), which is expressed as follows:

\[
X^s = \bigcup_{x_1, \ldots, x_s \in X} < x_1, \ldots, x_s >.
\]

The choice of \( s \) points in \( X \) gives rise to \( sn \) free parameters. In addition, \( s \) points span a space of projective dimension \( s - 1 \) and \( X^s \) will be embedded in \( \mathbb{P}^N \). Consequently, we should expect the dimension of \( X^s \) to be given by \( \min\{N, sn + s - 1\} \): this is called the expected dimension for secant varieties.

The following estimate on the dimension of \( X^s \) is valid in general:

\[
dim(X^s) \leq \min\{N, sn + s - 1\}.
\]  

(1)

From our viewpoint, the cases where the strict inequality applies are the most interesting.

**Definition 3.3.** The secant variety \( X^s \) is called **defective** if

\[
dim(X^s) < \min\{N, sn + s - 1\}
\]

and the quantity \( \delta = \min\{N, sn + s - 1\} - \dim(X^s) \) is its defectiveness.
One well-known result that is useful in finding the dimensions of the multi-
secant varieties is Terracini's lemma.

**Lemma 3.4 ([4])**. Let \( x_1, \ldots, x_s \in X \) be generic points; let us refer to the
projectivised tangent spaces to \( X \) at these points as \( T_{x_1}X, \ldots, T_{x_s}X \), then

\[
\dim(X^s) = \dim(<T_{x_1}X, \ldots, T_{x_s}X>).
\]

Let us now take a homogeneous polynomial \( f(X_0, \ldots, X_n) \) of degree \( k \).
Asking whether \( f \) can be written as the sum of powers of degree \( k \) of \( s \) linear
forms \( L_1, \ldots, L_s \) is the same as asking whether \( f \) belongs to the \( s \)-secant variety
of the \( k \)th Veronese embedding of \( \mathbb{P}^n \), which we call \( V_{k,n+1} \). It is therefore
important to know the dimension of \( V_{k,n+1}^s \), and consequently the cases where
\( V_{k,n+1} \) is defective for \( s \)-secant varieties.
The result obtained by Alexander and Hirschowitz is extremely useful in our
case and translates geometrically as follows:

**Theorem 3.5. ([5])** The Veronese variety \( V_{k,n+1} \) is defective for \( s \)-secant varie-
ties only in the following cases:

\[
(k, n, s) = (4, 2, 5), (4, 3, 9), (4, 4, 14), (3, 4, 7)
\]

and \( k = 2 \).

We have therefore obtained a full classification of defective Veronese varieties.

In this paper we will analyse the problem of defectiveness with respect to another
important family of classical varieties, the Grassmannians,\(^2\) which are related
to exterior algebras.
If \( k \leq \dim(V) \) is a positive integer, we define the Grassmannian \( G(k, V) \) to be
the variety of projective subspaces of \( \mathbb{P}(V) \) of dimension \( k \). When \( V = \mathbb{K}^{n+1} \),
\( G(k, V) \) will be denoted by \( G(k, n) \).

\(^2\)For the problem of defectiveness of Segre Varieties and its connection with the rank of
tensors, see [6].
Since $\dim(G(k,n)) = (k+1)(n-k)$, the expected dimension for the secant varieties $G(k,n)^s$ is:

$$\min \left\{ \binom{n+1}{k+1} - 1, s(n-k)(k+1) + s - 1 \right\}$$

We also have the two following important theoretical results to draw on.

**Theorem 3.6.** $G(1,n)^s$ is defective for $s < \lfloor \frac{n}{2} \rfloor$.

**Theorem 3.7.** (2) Let $k \geq 2$. If $s(k+1) \leq n+1$, then $G(k,n)^s$ has the expected dimension.

### 4 A probabilistic algorithm and proof of Theorem 2.1

To tackle our problem, we initially used the *Macaulay 2* computation system (see [7]), which was designed to study problems of algebraic geometry and commutative algebra, to write an algorithm generating parametric equations for the Grassmannians we are studying (see [8]).

To calculate the dimensions of the multisecant varieties, we favoured a probabilistic approach involving Terracini’s lemma. We took $s$ random points in $G(k,n)$ and studied their tangent spaces and the space spanned by these tangent spaces. If we found the expected dimension, the result was clearly correct, but if this revealed defectiveness, more checks needed to be performed.

Using this approach we constructed an algorithm that turned our problem into the calculation of the rank of fairly large matrices with constant coefficients; to study $\dim(G(k,n)^s)$ we needed to know the rank of a matrix of order $s(1+(k+1)(n-k)) \times N$. This algorithm enabled us to compute the dimension of $G(k,n)^s$ when $n \leq 11$, $k \leq 4$ and $s \leq 4$, at which stage the computer’s memory was used up. It was therefore clear that symbolic computation was not the best tool for this type of task.

To further proceed with our study, we decided to employ the *Matlab* software system, which is a computation system designed for dealing with numerical computations involving very large matrices.
The new algorithm obtained confirms the validity of the probabilistic approach. It is based on theoretical observation that the tangent spaces can be computed without having to define equations for the Grassmannian.

Let us take a point \( P = v_0 \wedge \ldots \wedge v_k \in G = G(k, n) \); the Plucker coordinates of \( P \) are all the \((k + 1) \times (k + 1)\) minors of the matrix \( A \) of order \((k + 1) \times (n + 1)\), which has the vectors \( v_0, \ldots, v_k \in V \) for rows. It is easy to check, by the Leibniz rule, that the following is true:

**Lemma 4.1.** \( T_P(G) \) is the projective space associated with

\[
V \wedge v_1 \wedge \ldots \wedge v_k + v_0 \wedge V \wedge v_2 \wedge \ldots \wedge v_k + \ldots + v_0 \wedge \ldots \wedge v_{k-1} \wedge V = T_0 + \ldots + T_k.
\]

If \( A_{i,j} \) stands for the matrix obtained from \( A \) by replacing the \( i \)th row by the \( j \)th row of the identity matrix \( I \) of order \((n + 1) \times (n + 1)\), then every \( T_i \) is parametrised by a matrix \( M_i \) of order \((n + 1) \times N \) whose \( j \)th row \( m_j \) contains the minors of maximum order of the matrix \( A_{i,j} \).

\[
A_{i,1} = \begin{pmatrix}
v_0 \\
\vdots \\
v_{i-1} \\
1 & 0 & \ldots & 0 \\
v_{i+1} \\
\vdots \\
v_k
\end{pmatrix}, \ldots, \begin{pmatrix}
v_0 \\
\vdots \\
v_{i-1} \\
0 & \ldots & 0 & 1 \\
v_{i+1} \\
\vdots \\
v_k
\end{pmatrix} = A_{i,n+1}
\]

\[
M_i = \begin{pmatrix}
m_1 \\
\vdots \\
m_{n+1}
\end{pmatrix}
\]

Our algorithm is described below.

- **Input:** positive integers \( n_m \) and \( n_M \).
• Repeat on parameters \( n_m \leq n \leq n_M, 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \) and \( 2 \leq s \leq S = \left\lceil \frac{1}{(k+1)(n-k+1)} \right\rceil \) to study \( G^s \).

• Define the matrix \( TA \) that contains the actual dimensions and the defectiveness of \( G^s \).

• Define the function \( ed(k, n, s) \) that calculates the expected dimension of \( G^s \) and the matrix \( E \) of the expected dimensions.

• Choose \( s \) random points \( P_1, \ldots, P_s \) in \( G \).
  
  – Take a matrix \( B \) of order \( s(k + 1) \times (n + 1) \) with random rational coefficients in the interval \([-L, L]\).
  
  – Extract \( s \) submatrices \( A \) of order \((k + 1) \times (n + 1)\) from \( B \).

• Repeat for \( 1 \leq h \leq s \) and study \( TP_h G = T_1 + \ldots + T_{k+1} \).
  
  – Repeat for \( 1 \leq i \leq k + 1 \).
  
  – For every \( 1 \leq j \leq n + 1 \) calculate the minors \((k + 1) \times (k + 1)\) of \( A_{i,j} \), computed from \( A \) as in (2), and call the row of these minors \( m_j \).
  
  – Construct the matrix \( M_i \) with rows \( m_j \).

• Parametrise \( TP_1 G + \ldots + TP_s G \).
  
  – Concatenate \( M_1, \ldots, M_{k+1} \) vertically to obtain matrix \( M \).

• Determine the value of the projective dimension of \( G^s \).
  
  – Calculate the rank of \( M \), then subtract 1.
  
  – Define row \( \text{dim} \) of \( TA \) of actual dimensions and row \( \text{dif} = E - \text{dim} \) of defectiveness.

• Output: matrix \( TA \).

This is the text of the algorithm.
L=100
nm=3
nM=14
f=inline('floor((n-1)/2)')
F=[]
for u=nm:nM
F=[F 2*f(u)]
end
N=inline('factorial(n+1)/(factorial(k+1)*factorial(n-k))-1')
S=inline('ceil((N+1)/((n-k)*(k+1)+1))')
Smax=S(N(f(nM),nM),f(nM),nM)
TA=[]
for n=nm:nM
T=zeros(2*f(n),Smax+1)
   for k=1:f(n)
      E=[]
      T(2*k-1,1:2)=[k n]
      I=eye(n+1)
      v=nchoosek(1:n+1,k+1)
      l=size(v,1)
      dim=[]
      for s=2:S(N(k,n),k,n)
         M=[]
         ed=inline('min(N,s*(n-k)*(k+1)+s-1)')
         E=[E ed(N(k,n),k,n,s)]
         B=rand(k+1,(n+1)*s)
         B=(B-0.5)*2*L
         for h=1:s
            A=B(:,(h-1)*(n+1)+1:h*(n+1))
            for i=1:k+1
               r=A(i,:)
               for j=1:n+1
                  A(i,:)=I(j,:)
               end
               m=[]
               for w=1:l
                  D(w)=det(A(:,v(w,:)))
                  m=[m D(w)]
               end
               M=[M;m]
            end
            A(i,:)=r
         end
      end
      dim=[dim rank(M)-1]
      dif=E-dim
   end
end
$T(2k-1, s+1) = \text{dim}(1, s-1)$
$T(2k, s+1) = \text{dif}(1, s-1)$
end
end
TA=[TA; T]
end

TA

If $s = S = \left\lceil \frac{1}{(k+1)(n-k)+T(k+1)} \right\rceil$ and $G(k, n)^s$ is not defective, then the variety fills the ambient space $\mathbb{P}^N = \mathbb{P}^{(n+1)}_{(k+1)}$; if it is defective, then we find $\text{dif} > 0$ and it can happen that $\text{dim}(G(k, n)^S) < N$, so that we will have to calculate $\text{dim}(G(k, n)^s)$ for $s > S$.

Using this algorithm, at the stage $(n, k) = (6, 14)$ the computer’s memory was used up. Our results are summarised in the following tables.
| N | S | k | n | \( G^2 \) | \( G^3 \) | \( G^4 \) | \( G^5 \) | \( G^6 \) | \( G^7 \) | \( G^8 \) | \( G^9 \) | \( G^{10} \) | \( G^{11} \) | \( G^{12} \) | \( G^{13} \) | \( G^{14} \) | \( G^{15} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 2 | 1 | 3 | 5 |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 9 | 2 | 1 | 4 | 9 |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 14 | 2 | 1 | 5 | 13* | 14 |   |   |   |   |   |   |   |   |   |   |   |   |
| 19 | 2 | 2 | 5 | 19 |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 20 | 2 | 1 | 6 | 17* | 20 |   |   |   |   |   |   |   |   |   |   |   |   |
| 34 | 3 | 2 | 6 | 25 | 33* | 34 |   |   |   |   |   |   |   |   |   |   |   |
| 27 | 3 | 1 | 7 | 21* | 26* | 27 |   |   |   |   |   |   |   |   |   |   |   |
| 55 | 4 | 2 | 7 | 31 | 47 | 55 |   |   |   |   |   |   |   |   |   |   |   |
| 69 | 5 | 3 | 7 | 33 | 49* | 63* | 69 |   |   |   |   |   |   |   |   |   |   |
| 35 | 3 | 1 | 8 | 25* | 32* | 35 |   |   |   |   |   |   |   |   |   |   |   |
| 83 | 5 | 2 | 8 | 37 | 56 | 73* | 83 |   |   |   |   |   |   |   |   |   |   |
| 125 | 6 | 3 | 8 | 41 | 62 | 83 | 104 | 125 |   |   |   |   |   |   |   |   |   |
| 44 | 3 | 1 | 9 | 29* | 38* | 43* | 44 |   |   |   |   |   |   |   |   |   |   |
| 119 | 6 | 2 | 9 | 43 | 65 | 87 | 109 | 119 |   |   |   |   |   |   |   |   |   |
| 209 | 9 | 3 | 9 | 49 | 74 | 99 | 124 | 149 | 174 | 199 | 209 |   |   |   |   |   |   |
| 251 | 10 | 4 | 9 | 51 | 77 | 103 | 129 | 155 | 181 | 207 | 233 | 251 |   |   |   |   |   |
| 54 | 3 | 1 | 10 | 33* | 44* | 51* | 54 |   |   |   |   |   |   |   |   |   |   |
| 164 | 7 | 2 | 10 | 49 | 74 | 99 | 124 | 149 | 164 |   |   |   |   |   |   |   |   |
| 329 | 12 | 3 | 10 | 57 | 86 | 115 | 144 | 173 | 202 | 231 | 260 | 289 | 318 | 329 |   |   |
| 461 | 15 | 4 | 10 | 61 | 92 | 123 | 154 | 185 | 216 | 247 | 278 | 309 | 340 | 371 | 402 | 433 | 461 |
| 65 | 4 | 1 | 11 | 37* | 50* | 59* | 64* | 65 |   |   |   |   |   |   |   |   |   |
| 219 | 8 | 2 | 11 | 55 | 83 | 111 | 139 | 167 | 195 | 219 |   |   |   |   |   |   |   |
| 494 | 15 | 3 | 11 | 65 | 98 | 131 | 164 | 197 | 230 | 263 | 296 | 329 | 362 | 395 | 428 | 461 | 494 |
| 791 | 22 | 4 | 11 | 71 | 107 | 143 | 179 | 215 | 251 | 287 | 323 | 359 | 395 | 431 | 467 | 503 | 539 |
| 923 | 25 | 5 | 11 | 73 | 110 | 147 | 184 | 221 | 258 | 295 | 332 | 369 | 406 | 443 | 480 | 517 | 554 |
| 77 | 6 | 1 | 12 | 41* | 56* | 67* | 74* | 77 |   |   |   |   |   |   |   |   |   |
| 285 | 10 | 2 | 12 | 61 | 92 | 123 | 154 | 185 | 216 | 247 | 278 | 285 |   |   |   |   |
| 714 | 20 | 3 | 12 | 73 | 110 | 147 | 184 | 221 | 258 | 295 | 332 | 369 | 406 | 443 | 480 | 517 | 554 |
| 1286 | 32 | 4 | 12 | 81 | 122 | 163 | 204 | 245 | 286 | 327 | 368 | 409 | 450 | 491 | 532 | 573 | 614 |
| 1715 | 40 | 5 | 12 | 85 | 128 | 171 | 214 | 257 | 300 | 343 | 386 | 429 | 472 | 515 | 558 | 601 | 644 |
| 90 | 4 | 1 | 13 | 45* | 62* | 75* | 84* | 90 |   |   |   |   |   |   |   |   |   |
| 363 | 11 | 2 | 13 | 67 | 101 | 135 | 169 | 203 | 237 | 271 | 305 | 339 | 363 |   |   |   |
| 1000 | 25 | 3 | 13 | 81 | 122 | 163 | 204 | 245 | 286 | 327 | 368 | 409 | 450 | 491 | 532 | 573 | 614 |
| 2001 | 44 | 4 | 13 | 91 | 137 | 183 | 229 | 275 | 321 | 367 | 413 | 459 | 505 | 551 | 597 | 643 | 689 |
| 3002 | 62 | 5 | 13 | 97 | 146 | 195 | 244 | 293 | 342 | 391 | 440 | 489 | 538 | 587 | 636 | 685 | 734 |
| 3431 | 69 | 6 | 13 | 99 | 149 | 199 | 249 | 299 | 349 | 399 | 449 | 499 | 549 | 599 | 649 | 699 | 749 |
| 104 | 4 | 1 | 14 | 49* | 68* | 83* | 94* | 101* | 104 |   |   |   |   |   |   |   |   |
| 454 | 13 | 2 | 14 | 73 | 110 | 147 | 184 | 221 | 258 | 295 | 332 | 369 | 406 | 443 | 480 | 517 | 554 |
| 1364 | 31 | 3 | 14 | 89 | 134 | 179 | 224 | 269 | 314 | 359 | 404 | 449 | 494 | 539 | 584 | 629 | 674 |
| 3002 | 59 | 4 | 14 | 101 | 152 | 203 | 254 | 305 | 356 | 407 | 458 | 509 | 560 | 611 | 662 | 713 | 764 |
| 5004 | 91 | 5 | 14 | 109 | 164 | 219 | 274 | 329 | 384 | 439 | 494 | 549 | 604 | 659 | 714 | 769 | 824 |
| N  | S  | k  | N  | S  | k  |
|----|----|----|----|----|----|
| 791 | 22 | 4  | 11 | 575 | 611 |
| 923 | 25 | 5  | 11 | 591 | 628 |
| 714 | 20 | 3  | 12 | 591 | 628 |
| 1286| 32 | 4  | 12 | 655 | 696 |
| 1715| 40 | 5  | 12 | 687 | 730 |
| 1000| 25 | 3  | 13 | 665 | 696 |
| 2001| 44 | 4  | 13 | 735 | 781 |
| 3002| 62 | 5  | 13 | 783 | 832 |
| 3431| 69 | 6  | 13 | 799 | 849 |
| 1364| 31 | 3  | 14 | 719 | 764 |
| 3002| 59 | 4  | 14 | 815 | 866 |
| 5004| 91 | 5  | 14 | 879 | 934 |
| 1286| 32 | 4  | 12 | 1188| 1229|
| 1715| 40 | 5  | 12 | 1246| 1289|
| 2001| 44 | 4  | 13 | 1333| 1379|
| 3002| 62 | 5  | 13 | 1420| 1469|
| 3431| 69 | 6  | 13 | 1449| 1499|
| 1364| 31 | 3  | 14 | 1304| 1349|
| 3002| 59 | 4  | 14 | 1478| 1529|
| 5004| 91 | 5  | 14 | 1594| 1649|
| 2001| 44 | 4  | 13 | 1885| 1931|
| 3002| 62 | 5  | 13 | 2008| 2057|
| 3431| 69 | 6  | 13 | 2049| 2099|
| 3002| 59 | 4  | 14 | 2090| 2141|
| 5004| 91 | 5  | 14 | 2254| 2309|
| 3002| 62 | 5  | 13 | 2596| 2645|
| 3431| 69 | 6  | 13 | 2649| 2699|
| 3002| 59 | 4  | 14 | 2972| 2973|
| 5004| 91 | 5  | 14 | 2914| 2969|
| 3431| 69 | 6  | 13 | 3249| 3299|
| 5004| 91 | 5  | 14 | 3574| 3629|
| 5004| 91 | 5  | 14 | 4234| 4289|

| N  | S  | k  |
|----|----|----|
| 5004| 91 | 5  |

| N  | S  | k  |
|----|----|----|
| 5004| 91 | 5  |

| N  | S  | k  |
|----|----|----|
| 5004| 91 | 5  |

| N  | S  | k  |
|----|----|----|
| 5004| 91 | 5  |

| N  | S  | k  |
|----|----|----|
| 5004| 91 | 5  |

| N  | S  | k  |
|----|----|----|
| 5004| 91 | 5  |

| N  | S  | k  |
|----|----|----|
| 5004| 91 | 5  |

| N  | S  | k  |
|----|----|----|
| 5004| 91 | 5  |
These results confirm what is known about the defectiveness of Grassmannians. Except for Grassmannians of lines, we have identified four defective varieties. We have therefore proved theorem 2.1.

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