Abstract—This paper studies the optimal output-feedback control of a linear time-invariant system where a stochastic event-based scheduler triggers the communication between the sensor and the controller. The primary goal of the use of this type of scheduling strategy is to provide significant reductions in the usage of the sensor-to-controller communication and, in turn, improve energy expenditure in the network. In this paper, we aim to design an admissible control policy, which is a function of the observed output, to minimize a quadratic cost function while employing a stochastic event-triggered scheduler that preserves the Gaussian property of the plant state and the estimation error. For the infinite horizon case, we present analytical expressions that quantify the trade-off between the communication cost and control performance of such event-triggered control systems. This trade-off is confirmed quantitatively via numerical examples.

I. INTRODUCTION

Over the past decade, distributed control and estimation over networks have been a major trend. Thanks to the forthcoming revolution of the Internet-of-Things (IoT) and resulting interconnection of smart technologies, the importance of decision making over communication networks grows ever larger in our modern society. These technological advances, however, bring new challenges regarding how to use the limited computation, communication, and energy resources efficiently. Consequently, event- and self-triggered algorithms have appeared as an alternative to traditional time-triggered algorithms in both estimation and control; see, e.g., [1].

A vast majority of the research in this area has mainly focused on proving the stability of the proposed control schemes, and demonstrating the effectiveness of such control systems, as compared to periodically sampled ones, through numerical simulations. However, an important stream of work in such schemes is analytically characterizing the trade-off between the control performance and communication rate achieved via these algorithms. Early works on event-triggered control, such as [2]–[5], provided expressions for event decisions but only for scalar systems. The authors of [6] later extended the work of [2] to a class of second-order systems. The work in [7] studies state estimation for multiple plants across a shared communication network, and quantified communication and estimation performance. Recently, the authors of [8] investigated the minimum-variance event-triggered output-feedback control problem; cf. [2]. They established a separation between the control strategy and the scheduling decision, and they also showed that scheduling decisions are determined by solving an optimal stopping problem. Our initial work in [9] considered a certain structure of controllers such as dead-beat controllers, and analytical expressions for the control performance and communication rate were obtained. Differing from [9], in the current work, we will focus on designing optimal controllers by establishing a separation between the controller and the scheduler.

Optimal event-triggered controller design requires the joint design of an optimal control law and an optimal event-based scheduler. The associated optimization problem becomes notoriously difficult [10].

Contributions.

In this paper, we consider optimal output-feedback control of a linear time-invariant system where a stochastic event-based triggering algorithm dictates the communication between the sensor and the controller. The proposed scheduler decides at each time step whether or not to transmit new state estimates from the sensor to the controller. The main contributions of this manuscript are as follows:

1) We develop a framework for quantifying the closed-loop control performance and the communication rate in the channel between the sensor and the controller.

2) We confirm that the certainty-equivalent controller is optimal under the scheduling rule based on estimation errors. Our previous work [9] used a transmission strategy based on the plant state, and employed a sequence of deadbeat control actions to establish a resetting property, but this was not optimal since the separation principle between control and scheduling does not hold.

3) We derive analytical expressions for the (average) communication rate and control performance. Our analysis relies on a Markov chain characterization of the evolution of the state prediction error (cf. [9]) where the states of this Markov chain describe the time elapsed since the last transmission.

4) Due to the use of the stochastic triggering rule, we can compute the conditional covariance of the comparison error (i.e., the difference between the state estimation error at the sensor and the state estimation error at the controller) in a closed-form. Consequently, it becomes almost effortless to compute the closed-loop control performance; cf. [9].

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We assume that the pairs \((A, B)\) and \((A, V^{1/2})\) are controllable while the pair \((A, C)\) is observable.

Sensor, pre-processor, and scheduler

Using a standard Kalman filter, the sensor locally computes minimum mean squared error (MMSE) estimates \(\hat{x}_{k|k}'\) of the plant state \(x_k\) based on the information available to the sensor at time \(k\), and transmits them to the controller. As noted in [17], sending local state estimates, in general, provides better performance than transmitting measurements. The sensor also employs a transmission scheduler, which decides whether or not to send the current state estimate to the controller at each time-step \(k \in \mathbb{N}_0\) as determined by

\[
\sigma_k = \begin{cases} 
1 & \text{if MMSE estimate } \hat{x}_{k|k}' \text{ is sent,} \\
0 & \text{otherwise.}
\end{cases}
\]

Assumption 1 The sensor \(S\) has precise knowledge of the control policy used to generate control actions, which are computed by the controller and applied by the actuator to the plant. Hence, the information set of the smart sensor \(S\) contains all controls used up to time \(k - 1\).

The information set available to the sensor at time \(k \in \mathbb{N}_0\) is:

\[
I_k = \{\sigma_0, \ldots, \sigma_{k-1}; y_0, \ldots, y_k; u_0, \ldots, u_{k-1}\}.
\]

The minimum mean squared error estimate \(\hat{x}_{k|k}'\) of the plant state \(x_k\) can be computed recursively starting from the initial condition \(\hat{x}_{0|0}' = \bar{x}_0\) and \(P_{0|0}' = X_0\) using a Kalman filter [18]. At this point, it is worth reviewing the fundamental equations underlying the Kalman filter algorithm. The algorithm consists of two steps:

- **Prediction step:** This step predicts the state, estimation error, and estimation error covariance at time \(k\) dependent on information at time \(k - 1\):

  \[
  \hat{x}_{k|k-1}' = A\hat{x}_{k-1|k-1}' + B u_{k-1} \\
  \tilde{x}_{k|k-1}' = A\hat{x}_{k-1|k-1}' + w_{k-1} \\
  P_{k|k-1}' = A P_{k-1|k-1}' A^\top + W.
  \]

- **Update step:** This step updates the state, estimation error, and estimation error covariance using a blend of the predicted state and the observation \(y_k\):

  \[
  \hat{x}_{k}' = \hat{x}_{k|k-1}' + K_k (y_k - C\hat{x}_{k|k-1}') \\
  \tilde{x}_{k}' = \hat{x}_{k}' + (I_n - K_k C) w_{k-1} - K_k v_k \\
  P_{k}' = (I_n - K_k C) P_{k|k-1}' (I_n - K_k C)^\top + N_k.
  \]

where the gain matrix is given by

\[
K_k \triangleq P_{k|k-1}' C^\top \left(C P_{k|k-1}' C^\top + V\right)^{-1}.
\]

It is worth noting that the estimation error at the sensor \(\tilde{x}_{k|k}'\) is Gaussian with zero-mean and co-variance \(P_{k|k}'\), which evolves according to the standard Riccati recursion [19, Chapter 9]. Since the pair \((A, C)\) is observable and the pair \((A, W^{1/2})\) is controllable, the matrices \(P_{k|k-1}'\) and \(K_k\) converge exponentially to steady state values \(P_{\infty}'\) and \(K_{\infty}\), respectively. Similarly, the matrix \(P_{k|k}'\) also converges to a steady state value, i.e., \(P_{\infty}' \triangleq \left(I_n - K_{\infty} C\right) P_{\infty}'\).

The scheduler and the sensor are collocated, and the scheduler has access to all available information at the sensor. Moreover, the scheduler employs an event-based triggering mechanism to decide if there is a need for transmission of an updated state estimate from the sensor. The scheduler monitors the difference between the current state estimate and the past estimate, and transmits a new state estimate if the difference exceeds a predefined threshold. This ensures that the state estimate is transmitted only when an update is necessary, thus reducing unnecessary transmission.
sensor to the controller. The occurrence of information transmission is defined as

\[ \sigma_k = \begin{cases} 1 & \text{if } \delta_k = 1 \text{ or } \tau_{k-1} = T, \\ 0 & \text{otherwise}, \end{cases} \quad (12) \]

where \( \delta_k \) is a (random) binary decision variable (which in this paper evolves according to \( \{\delta_k\}_{k \geq 0} \)), \( \tau_k \) is a non-negative integer variable introduced to describe the time elapsed since the last transmission, and \( T \) is a time-out interval. Such a time-out mechanism is critical in event-triggered control systems to guard against faulty components; see, e.g., [9].

To maintain the Gaussianity of the comparison error

\[ e_{k|k-1} \triangleq \hat{x}_{k|k-1} - x_{k|k-1}, \]

(note that \( \hat{x}_{k|k} \) is defined in [9] while \( \hat{x}_{k|k-1} \) is introduced in (17)) a variant of the stochastic triggering mechanism proposed in [12]-[14] is used. More specifically, the scheduler will decide to transmit a new sensor packet according to the following decision rule:

\[ \delta_k = \begin{cases} 0 & \text{with prob. } e^{-\lambda (e_{k|k-1}, s_{k|k-1})}, \\ 1 & \text{with prob. } 1 - e^{-\lambda (e_{k|k-1}, s_{k|k-1})}, \end{cases} \quad (13) \]

where the triggering parameter \( \lambda \) is a given positive scalar. As can be seen in [13], the probability of transmitting a new sensor packet (i.e., \( \sigma_k = 1 \)) converges to one as \( \lambda \) goes to infinity. In other words, for large values of \( \lambda \), the communication between the sensor and the controller is more likely to be triggered.

The integer-valued random process \( \{\tau_k\}_{k \geq 0} \) in (12) describes how many time instances ago the last transmission of a sensor packet occurred. Whenever a sensor packet is transmitted from the sensor to the controller, \( \tau_k \) is reset to zero. Thus, the evolution of the random process \( \{\tau_k\}_{k \geq 0} \) is defined by

\[ \tau_k = \begin{cases} 0 & \text{if } \delta_k = 1 \text{ or } \tau_{k-1} = T, \\ 1 + \tau_{k-1} & \text{otherwise}, \end{cases} \quad (14) \]

or equivalently,

\[ \tau_k = \begin{cases} 0 & \text{if } \sigma_k = 1, \\ 1 + \tau_{k-1} & \text{if } \sigma_k = 0, \end{cases} \quad (15) \]

where \( \tau_{-1} \triangleq 0 \). Notice that the number of time steps between two consecutive transmissions is bounded by the time-out interval \( T < \infty \). If the number of samples since the last transmission exceeds a time-out value of \( T \), the sensor will attempt to transmit new data to the controller even if the comparison error does not satisfy the triggering condition (13). Thus, a transmission (i.e., \( \sigma_k = 1 \)) will occur when either \( \delta_k = 1 \) or there is a time-out.

Remark 2 It is worth noting that, as can be seen in [15], the events \( \{\sigma_k = 1\} \) and \( \{\tau_k = 0\} \) are equivalent to each other.

At time instances when \( \sigma_k = 1 \), the sensor transmits its local state \( \hat{x}_{k|k} \) to the controller. As a result, the information set available to the controller at time \( k \in \mathbb{N}_0 \) (and before deciding upon \( u_k \)) can be defined as:

\[ \mathcal{I}_k \triangleq \{\sigma_0, \cdots, \sigma_k; \sigma_0 \hat{x}_{0|0}, \cdots, \sigma_k \hat{x}_{k|k}; u_0, \cdots, u_{k-1}\}. \]

Under the event-based scheduling mechanism, (12) - (14), the controller runs an MMSE estimator to compute estimates of the plant state \( x_k \) as follows:

\[ \hat{x}_{k|k} \triangleq \mathbb{E}[x_k | \mathcal{I}_k] = \begin{cases} \hat{x}_{k|k} & \text{if } \sigma_k = 1, \\ \hat{x}_{k|k-1} & \text{otherwise}, \end{cases} \quad (16) \]

\[ \hat{x}_{k|k-1} \triangleq \mathbb{E}[x_k | \mathcal{I}_{k-1}] = A \hat{x}_{k-1|k-1} + Bu_{k-1}, \quad (17) \]

where \( \hat{x}_{k|k-1} \) is the optimal estimate at the controller if the sensor did not transmit any information at time-step \( k \in \mathbb{N}_0 \). Note that the optimality of this estimator can be shown by using a similar argument to that provided in [14, Lemma 4].

Assumption 3 In addition to computing \( \hat{x}_{k|k} \), the sensor operates another estimator, which mimics the one at the controller, since transmission decisions rely on both \( \hat{x}_{k|k} \) and \( \hat{x}_{k|k-1} \); see [13]. This can be done provided we make the following assumption.

Assumption 4 Both the smart sensor \( S \) and the controller \( C \) know the plant model \( G \) (but not realizations of the noise processes).

Controller design and performance criterion. We aim at finding the control strategies \( u_k \), as a function of the admissible information set \( \mathcal{I}_k \), to minimize a quadratic cost function of the form

\[ J_N = \mathbb{E} \left[ x_N^T Q_f x_N + \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) \right], \quad (18) \]

where \( Q, Q_f \in \mathbb{S}_+^{n_x} \) and \( R \in \mathbb{S}_+^{n_u} \). At time instances when \( \sigma_k = 1 \) (i.e., the controller has received sensor packets), the controller uses the state estimate \( \hat{x}_{k|k} \) which is transmitted by the sensor. However, at time instances when \( \sigma_k = 0 \), the controller uses the outcome of the estimator at the controller side. As is well-known in related situations (see e.g., [10]), if the transmission decision \( \sigma_k \) is independent of the control strategy \( u_k \), then the certainty equivalent controller is optimal. In Section III, we will confirm that the certainty equivalent controller is optimal under the event-based scheduler proposed above.

III. MAIN RESULTS

We wish to quantify the communication rate and control performance of the feedback control system described by (11) and (2), where the event-based triggering mechanism (13) determines the communication between the sensor and the controller. We will first demonstrate that the time elapsed between two consecutive transmissions can be regarded as a discrete-time, finite state, time-homogeneous Markov chain. Then, using an ergodicity property, we will provide an analytical formula for the communication rate between the sensor and the controller. Subsequently, we will show that the certainty equivalent controller is still optimal with the event-triggering rule (13). Lastly, we will compute the control performance analytically for the infinite horizon case.

Assumption 5 In the rest of this paper, we will assume that the local Kalman filter at the sensor runs in steady state.

We first define the state prediction error at the controller

\[ \hat{x}_{k|k-1} \triangleq x_k - \hat{x}_{k|k-1}, \quad (19) \]

which evolves as

\[ \hat{x}_{k+1|k} \triangleq \begin{cases} A \hat{x}_{k|k} + w_k & \text{if } \sigma_k = 1, \\ A \hat{x}_{k|k-1} + w_k & \text{if } \sigma_k = 0. \end{cases} \quad (20) \]

Then, we define the state estimation error at the controller

\[ \hat{x}_{k|k} \triangleq \hat{x}_k - \hat{x}_{k|k}, \quad (21) \]

which evolves as

\[ \hat{x}_{k|k} = \begin{cases} A \hat{x}_{k|k-1} + w_k & \text{if } \sigma_k = 1, \\ A \hat{x}_{k-1|k-1} + w_{k-1} & \text{if } \sigma_k = 0. \end{cases} \quad (22) \]

Define also the comparison errors:

\[ e_{k|k-1} \triangleq \hat{x}_{k|k-1} - \hat{x}_{k|k-1} = \hat{x}_{k|k-1} - \hat{x}_{k|k}, \quad (23) \]

\[ e_{k|k} \triangleq \hat{x}_{k|k} - \hat{x}_{k|k} = \hat{x}_{k|k} - \hat{x}_{k|k}. \quad (24) \]
Whenever a transmission occurs \( i.e., \tau_k = 0 \), the state estimation error \( \tilde{x}_{k|k} \) at the controller is equal to \( \tilde{x}_{k|k} \), since the most recent sensor packet is available at the controller. It is then possible to write the stochastic recurrence equations (23) and (24) as

\[
e_{k+1|k} = \begin{cases} 
\eta_k & \text{if } \tau_k = 0, \\
e_{k|k-1} + \eta_k & \text{if } \tau_k \neq 0,
\end{cases}
\]

and

\[
e_{k|k} = \begin{cases} 
0 & \text{if } \tau_k = 0, \\
e_{k|k-1} + \eta_k & \text{if } \tau_k \neq 0,
\end{cases}
\]

where \( \eta_k \equiv K_{\infty}(C(\tilde{x}_{k|k} + \omega_k) + \nu_{k+1}) \). Notice that the comparison errors \( e_{k|k-1} \) and \( e_{k|k} \) propagate according to a linear system with open-loop dynamics \( A \), driven by the process \( \eta_k \).

**Lemma 6** \( \{\eta_k\}_{k \geq 0} \) is a sequence of pairwise independent Gaussian random vectors such that \( \eta_k \sim N(0, \Pi_n) \) with \( \Pi_n \equiv K_{\infty}C\Sigma_{w_{\infty}}K_{\infty}^T \).

**Remark 7** If the sensor has perfect state measurements \( i.e., y_k = x_k \), then \( \eta_k \) is equal to \( \omega_k \).

**Definition 8 (Cumulative error)** We shall characterize the cumulative comparison error \( i.e., \text{the error that occurs in estimation at the controller over time due to intermittent transmissions} \) via

\[
e_k(i) \equiv \sum_{j=0}^{i} A^j \eta_{k-j}.
\]

Using Definition 8 the stochastic recurrence equations (25) and (26) can be then re-written as

\[
e_{k+1|k} = e_k(\tau_k),
\]

\[
e_{k|k} = \begin{cases} 
0 & \text{if } \tau_k = 0, \\
e_{k|k-1} & \text{if } \tau_k \neq 0.
\end{cases}
\]

**Lemma 9 (Markov process)** The random process \( \{\tau_k\}_{k \geq 0} \) is an ergodic, time-homogeneous Markov chain with a finite state space \( B \equiv \{0, 1, \cdots, T\} \). Thus, it has a unique invariant distribution \( \pi \equiv [\pi(0) \pi(1) \cdots \pi(T)] \in \mathbb{R}^{1 \times T+1} \) such that \( \sum_{i \in B} \pi(i) = 1 \) and \( \pi(i) > 0 \) for all \( i \in B \).

**Lemma 10 (Augmented cumulative error vector)** Consider \( \hat{e}_k(i) \equiv \left[ e_k(0) \ e_{k+1}(1) \cdots e_{k+i}(i) \right]^T \) with \( e_k(i) \) as in (27). Then, \( \hat{e}_k(i) \) is a random vector having a multivariate normal distribution with zero-mean and co-variance:

\[
\Sigma_e (i) \equiv \begin{bmatrix} 
\Pi_n & 0 & \cdots & 0 \\
0 & \Pi_n A^T & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Pi_n A^T \\
\end{bmatrix}
\]

for any \( i \in \{0, 1, \cdots, T - 1\} \).

The next lemma computes the transition probabilities of the Markov chain defined in Lemma 9.

**Lemma 11 (Transition probabilities)** The transition matrix of the Markov chain \( \{\tau_k\}_{k \geq 0} \) is given by

\[
P_\tau = \begin{bmatrix} 
1 - p_{00} & 1 - p_{00} & 0 & \cdots & 0 \\
p_{10} & 1 - p_{10} & 0 & \cdots & 0 \\
0 & \vdots & \ddots & \vdots & \vdots \\
0 & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 - p_{T-1,0} \\
0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix},
\]

where the non-zero transition probabilities are computed as

\[
p_{ij} = \begin{cases} 
1 - \frac{1}{1 + \frac{1}{\Pi_n + 2\Lambda_s(0)} + \frac{1}{\Pi_n + 2\Lambda_s(i)}} & \text{if } i = 0, j = 0, \\
1 - \frac{1}{\Pi_n + 2\Lambda_s(i + 1)} & \text{if } i \in \{1, \cdots, T - 1\}, j = 0, \\
1 - \frac{1}{\Pi_n + 2\Lambda_s(i + 1)} & \text{if } i \in \{0, \cdots, T - 1\}, j = i + 1, \\
1 & \text{if } i = T, j = 0.
\end{cases}
\]

The visit of the Markov chain \( \{\tau_k\}_{k \geq 0} \) to the state 0 is analogous to a transmission \( i.e., \sigma_k = 1 \) of the estimate of the plant state \( x_k \) from the sensor to the controller. Using Remark 2 and the ergodic theorem for Markov chains [20, Theorem 5.3], we have:

\[
\sigma = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \sigma_k \iff \pi(0) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{I}(\tau_k = 0),
\]

where \( \pi(0) \) is the empirical frequency of transmissions. With the transition probabilities of this Markov chain, we can give an explicit characterization of the average communication rate of the event-triggered control system:

**Theorem 12 (Communication rate)** The average communication rate between the sensor and the controller under the stochastic event-based triggering mechanism, proposed in [12 13 14] for a fixed \( \lambda > 0 \) is given by

\[
\bar{\sigma} = \frac{1}{1 + \sum_{n=1}^{\infty} \Pi_n (1 - p_{n0})},
\]

**Remark 13** Note that, as \( \lambda \) goes to zero, the communication between the sensor and the controller becomes periodic.

The next theorem describes the optimal control law for the event-triggered control system at hand.

**Theorem 14 (Optimal control law)** Consider the system (1) and (2), and the problem of minimizing the cost function (18) under the event-based triggering mechanism (12 13 14) for a fixed \( \lambda > 0 \). Then, there exists a unique admissible optimal control policy

\[
u_k = -L_k \mathbb{E}[x_k | T_k] = -L_k \tilde{x}_{k|k},
\]

where

\[
L_k = (B^T S_{k+1} B + R)^{-1} B^T S_{k+1} A,
\]

\[
S_k = A^T S_{k+1} A + Q
\]

with initial values \( S_N = Q_f \). The minimum value of the cost function is obtained as

\[
J_N = \mathbb{E}_0^T S_0 \mathbb{E}_0 + \sum_{k=0}^{N-1} \mathbb{E}[S_{k+1} W] + \sum_{k=0}^{N-1} \mathbb{E}[E_k^T M_k E_k] + \sum_{k=0}^{N-1} \mathbb{E}[E_k^T M_k E_k].
\]

where \( M_k \equiv L_k^T (B^T S_{k+1} B + R) L_k \).

**Remark 15** Our result should be viewed in the light of the limited information available to the controller. At every time step \( k \in \mathbb{N}_0 \), the controller computes an optimal control input based on the information set \( T_k^+ \). Our result is akin to the one derived in [19], however here we can also provide the closed-form expression of the control cost for the infinite horizon case (see Theorem 12).
Lemma 16 (Gaussianity-preservation) The conditional random variable, \(e_{k|\tau_k = i}\), has a Gaussian distribution with zero-mean and co-variance:

\[
\Sigma_e(0) = 0_n ,
\]

\[
\Sigma_e(i) = \frac{1}{2\lambda} I_n - \frac{1}{4\lambda^2} \left( A \Sigma_e(i - 1) A^T + \Pi_n + \frac{1}{2\lambda} I_n \right)^{-1} .
\]

Using the previous theorems, we have the following result to calculate the average control performance measured by a linear-quadratic function.

Theorem 17 (Infinite horizon control performance) Suppose the pairs \((A, B)\) and \((A, W^{1/2})\) are controllable, and the pairs \((A, C)\) and \((A, Q^{1/2})\) are observable. Moreover, suppose that \(\lambda > 0\). Then, we have the following:

(a) The infinite horizon optimal controller gain is constant:

\[
L_\infty \triangleq \lim_{k \to \infty} L_k = (B^T S_\infty B + R)^{-1} B^T S_\infty A .
\]

(b) The matrices \(S_\infty\) and \(P_\infty\) are the positive definite solutions of the following algebraic Riccati equations:

\[
S_\infty \triangleq A^T S_\infty A + Q - A^T S_\infty B (B^T S_\infty B + R)^{-1} B^T S_\infty A ,
\]

\[
P_\infty \triangleq A P_\infty A^T + W - A P_\infty C^T (C P_\infty C^T + V)^{-1} C P_\infty A^T .
\]

(c) The expected minimum cost converges to the following value:

\[
J_\infty \triangleq \lim_{N \to \infty} \frac{1}{N} J_N = \text{Tr} (S_\infty W) + \text{Tr} (F_\infty M_\infty)
\]

\[
+ \sum_{i=1}^{T} \pi(i) \text{Tr} (M_\infty \Sigma_e(i)) ,
\]

where \(F_\infty \triangleq (I_n - K_\infty C) P_\infty , M_\infty \triangleq L_\infty^T (B^T S_\infty B + R) L_\infty ,\) and \(\pi = [\pi(0) \pi(1) \cdots \pi(T)]\) satisfies \(\pi = \pi P_\infty .\)

IV. NUMERICAL EXAMPLE

In this section, numerical simulations are provided to assess the performance of the stochastic event-triggering algorithm proposed in Section III and verify the theoretical results presented in Section III. To this end, the system parameters are chosen as follows:

\[
A = \begin{bmatrix} 1.2 & 1 \\ 0 & 0.9 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} , \quad V = 1 ,
\]

\[
X_0 = W = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} , \quad Q = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix} , \quad R = 1 .
\]

The matrix \(A\) has one stable (i.e., 0.9) and one unstable eigenvalue (i.e., 1.2). The time-out interval is set to \(T = 50\). Notice that the pairs \((A, B)\) and \((A, W^{1/2})\) are controllable, the pairs \((A, C)\) and \((A, Q^{1/2})\) are observable, and \(R > 0\), as required by the assumptions of the theorems presented in Section III.

For various values of \(\lambda\) ranging from 0.01 to 100, we evaluate the communication rate and the control performance as predicted by Theorems 12 and 17 respectively. We compare the analytic results to Monte Carlo simulations of the closed-loop system. For each value of \(\lambda\), we conduct 25,000 Monte Carlo simulations for the horizon length of 10,000 samples, and obtain the mean communication rate and the control performance. The comparison is shown in Fig. 2 for the communication rate and the control performance. It can be seen that the analytic results match the Monte Carlo simulations very closely.

![Figure 2. A comparison of transmission rate (resp. control performance) derived from the analytic expression (31) (resp. (39)) and Monte Carlo simulations.](image)

![Figure 3. The trade-off between the communication rate and the control performance (the scheduling parameter \(\lambda\) is illustrated by gray scale).](image)

We can also obtain results on when changing the scheduling parameter \(\lambda\) has the most effect as demonstrated in Fig. 3. There are two extreme cases: 1) as \(\lambda\) goes to infinity, the communication rate becomes one, and the control performance converges to \(\text{Tr} (S_\infty W) + \text{Tr} (F_\infty M_\infty) \approx 53.23\); 2) as \(\lambda\) goes to zero, the transmission rate converges to zero, and the control performance becomes unbounded. We observe, for instance, that changing the scheduling parameter \(\lambda\) from one to infinity has minimal effect on the control performance, but nearly doubles the communication frequency. As can be seen in Fig. 2 by setting \(\lambda = 1\), we can reduce the communication between the sensor and the controller by almost 40\%, while only slightly sacrificing the control performance of the closed-loop system.

V. CONCLUSIONS AND DISCUSSIONS

This paper has focused on the optimal control of a linear stochastic system, where a stochastic event-based scheduling mechanism governs the communication between the sensor and the controller. The scheduler is co-located at the sensor and employs a local Kalman filter. Based on the prediction error, the scheduler decides whether or not to send a new state estimate to the controller. The use of
this transmission strategy reduces the communication burden in the channel. We showed that, in this setup, the optimal controller is the certainty-equivalent controller since the measurement quality is not affected by the control policy. We also provided analytical expressions to quantify the trade-off between the communication rate and the control performance.

VI. APPENDIX: PROOFS

Proof of Lemma 9 By Assumption 5 the Kalman filter has reached its steady state. Consequently, the Kalman gain $K_k$ and the error covariance matrices, $P_{k-1}$ and $P_k$, become constant, i.e., $K_{∞}, P_{∞}$, and $P_{∞} = (I_n - K_{∞} C)P_{∞}$, respectively. Let us define the following random process:

$$η_k = K_{∞}CAx_{k|k} + K_{∞}Cw_k + K_{∞}v_{k+1}.$$  

Since $x_{k|k}, w_k$ and $v_{k+1}$ are mutually independent Gaussian vectors with zero-mean and co-variances $P_{∞}$, $W$, and $V$, respectively. $η_k$ is Gaussian with zero-mean and co-variance:

$$\Pi_η = E[η_kη_k^T] = E[(K_{∞}CAx_{k|k} + K_{∞}Cw_k + K_{∞}v_{k+1})^T](K_{∞}CAx_{k|k} + K_{∞}Cw_k + K_{∞}v_{k+1})$$

$$= K_{∞}CA(CA^T + W)CT + V)K_{∞}^T$$

$$= K_{∞}CA(CA^T + W) + V)K_{∞}^T$$

where (a) is derived by writing $P_{∞} = (I_{n} - K_{∞} C)P_{∞}$ while (b) is obtained by replacing $K_{∞}$ with $(CP_{∞}CT + V)^{-1}CP_{∞}$. Since $\{η_k\}_{k ≥ 0}$ are Gaussian random vectors, pair-wise independence is equivalent to

$$E[η_kη_l^T] = Ω, \ 0 ≤ k < l < ∞ .$$

For $k < l$, we have:

$$E[η_kη_l^T] = E[(K_{∞}CAx_{k|k} + K_{∞}Cw_k + K_{∞}v_{k+1})^T](K_{∞}CAx_{l|l} + K_{∞}Cw_l + K_{∞}v_{l+1})$$

$$= K_{∞}CAK_{∞}CT$$

where (a) holds since $w_k$ and $v_{k+1}$ are independent of $x_{k|k}$, $w_k$ and $v_{k+1}$; (b) is obtained by replacing $x_{k|k}$ iteratively from $l$ to $k$ and using the fact that $w_k$ and $v_{k+1}$ are independent of $\{w_{k+2}, \ldots, w_{k+1}\}$ and $\{v_{k+2}, \ldots, v_{k+1}\}$; (c) is obtained by writing $P_{∞} = (A^T + W); and (d)$ follows from [40]. This concludes the proof.

Proof of Lemma 10 For simplicity, we will use a slight abuse of notation and write $e_{k+1} = e_{k+1|k}$. We begin by proving that the process $\{η_k\}_{k ≥ 0}$ is a Markov chain. Using the total law of probabilities and the fact that $e_{k+1} \in R^n$, we have:

$$P(η_{k+1} | η_k, η_{k-1}, \ldots, η_0)$$

$$= \int_{R^n} P(η_{k+1} | e_{k+1}, η_k, η_{k-1}, \ldots, η_0) P(e_{k+1} | η_k, η_{k-1}, \ldots, η_0) dη_{k+1}$$

$$= \int_{R^n} \prod_{i=0}^k P(e_{k+1} | η_i, η_{i-1}, \ldots, η_0) dη_{k+1}$$

$$= P(η_{k+1} | η_k) \cdot P(e_{k+1} | η_k) \cdot P(η_k)$$

where (a) and (c) come from the definition of conditional probability, and (b) holds since $e_{k+1}$ depends stochastically only on $η_k$ as described in [29], and $η_{k+1}$ depends on $e_{k+1}$ and $η_k$ as described in [4]. Bear in mind that knowing $η_k = j$ implies knowing $η_k = j - 1, \ldots, η_{k-j} = 0$. Consequently, the process $\{η_k\}_{k ≥ 0}$ is a Markov chain.

We now show the ergodicity of this Markov chain. Since the Markov chain $\{η_k\}_{k ≥ 0}$, depicted in Fig. 4, has positive transition probabilities for any $λ > 0$, the chain is evidently irreducible. The chain is also aperiodic because the state 0 has a non-zero probability of being reached for any $λ > 0$. By [23] Theorem 3.3, this irreducible chain with finite state space $B$ is positive recurrent. Since the process $\{η_k\}_{k ≥ 0}$ is irreducible, aperiodic and positive recurrent, it is also ergodic. As the process $\{η_k\}_{k ≥ 0}$ is an irreducible aperiodic Markov chain with finite many states, it has a unique invariant distribution $π$ such that $πP_λ = π$ and $π\Pi + 1 = 1$; see [21] Corollary 2.11. This concludes the proof.

Lemma 18 Suppose that $ζ_k^1, \ldots, ζ_k^i$ is a sample of $ζ_∞ \sim Uni(0,1)$. Define the following events:

$$E_i = \{δ_{k+1} = 0, \ldots, δ_{k+i} = 0\}$$

$$= \bigcap_{j=0}^{i-1} \{ζ_{k+j+1} ≤ e^{-λ(κ_{k+j}(j) + κ_{k+j}(j))}\}$$

for all $i \in \{1,2,\ldots,T\}$, with the convention that $E_0$ is a sure event. For any given $λ > 0$, the probability of these events $E_i$, for all $i \in \{1,2,\ldots,T\}$, can be computed as:

$$P(E_i) = \frac{1}{\sqrt{|I_{n+i} + 2λΣ_{i}(i-1)|}}$$

Proof of Lemma 19 Assume that $ζ_k^1, ζ_k^2, \ldots, ζ_k^i$ is a sample of $ζ_∞ \sim Uni(0,1)$. Since $e_{k+1|k} = e_{k+1(i-1)}$ when $η_k = i-1$, $\forall i \in \{1,2,\ldots,T\}$, the stochastic triggering rule [13] can be rewritten as

$$δ_{k+i} = \begin{cases} 0 & \text{if } ζ_{k+i} ≤ e^{-λ(κ_{k+i+1}(i-1) + κ_{k+i+1}(i-1))} \\ 1 & \text{otherwise} \end{cases}$$

For any given $λ > 0$, we compute:

$$P(E_i) = P(δ_{k+1} = 0, \ldots, δ_{k+i} = 0)$$
\[
P(\tau_{k+1} = 1 \mid \delta_k) = \frac{P(\delta_{k+1} = 1)}{P(\delta_{k+1} = 1)} = 1 - P(\delta_k = 0),
\]
where \((a)\) is true as \(\delta_k\) is independent of the random variable \(\eta_k\). For any \(i \in \{1, \ldots, T-1\}\), we derive:

\[
\begin{align*}
p_{00} &= P(\tau_{k+1} = 0 \mid \iota_k) \\
&= P(\delta_{k+1} = 0 \mid \delta_k = 0) \\
&= P(\delta_{k+1} = 0 \mid \delta_k = 0, \delta_{k-1} = 0) \\
&= \frac{P(\delta_{k+1} = 0, \delta_k = 0, \delta_{k-1} = 0)}{P(\delta_k = 0, \delta_{k-1} = 0)} = 1 - P(\delta_k = 0),
\end{align*}
\]

\[
\text{where (b) comes from the Markov property, (c) is the result of (14), and (d) holds since \(\delta_{k+1}\) is independent of the random variables \(\eta_k, \ldots, \eta_{k-1}\). Using the result from Lemma 18 we can straightforwardly compute the transition probabilities as given in the statement of the lemma.}
\]

**Proof of Theorem 7.** The proof of this theorem follows similar steps as in [22] pp. 98. **Proof of Theorem 7.** Since \(13\) is a fixed scheduling rule with a predefined, constant parameter (i.e., \(\delta > 0\)) and is a function of random variables \(\{x_0, u_0, \ldots, x_k, u_k, x_{k+1}, \ldots, x_{N-1}, u_{N-1}\}\), the transmission decisions \(\sigma_k\) (or consequently \(\tau_k\)) are independent of the control law \(u_k\); see [10] Lemma 1. Therefore, the separation principle holds.

The proof of this theorem employs a dynamic programming argument; see [22]. Define the optimal value function \(V_k(x_k)\) as follows:

\[
V_k(x_k) = \min_{u_{k-1}} \quad \mathbb{E}\left[ x_N^\top (A^\top S_{k+1} A + Q - L_0^\top (B^T S_{k+1} B + R) L_0) x_k + \sum_{t=k}^{N-1} x_t^\top Q x_t + u_t^\top R u_t \right].
\]

We claim that the solution of the functional equation (43) is a quadratic function of the form

\[
V_k(x_k) = \mathbb{E}\left[ x_k^\top S_k x_k \mid T_k \right] + s_k,
\]

where \(S_k\) is a non-negative definite matrix and \(s_k\) is a scalar. Indeed, this claim is clearly true for \(k = N\) with the choice of parameters \(S_N = Q\) and \(s_N = 0\). Suppose that the claim now holds for \(k+1\). The value function at time-step \(k\) is

\[
V_k(x_k) = \min_{u_k} \mathbb{E}\left[ x_k^\top Q x_k + u_k^T R u_k + V_{k+1}(x_{k+1}) \mid T_k \right]
\]

which is obtained by writing \(L_k \triangleq (B^T S_{k+1} B + R)^{-1} B^T S_{k+1} A\) and by replacing \(x_k\) with \(x_{k+1} = x_k - x_{k+1} - x_{k+1}\). Hence, the minimum is obtained for

\[
u_k = -L_k \mathbb{E}[x_k \mid T_k] = -L_k \bar{x}_k[i].
\]

Consequently, the claim provided in (44) is satisfied also for the time step \(k\) for all \(x_k\) if and only if

\[
S_k = A^T S_{k+1} A + Q - L_0^T (B^T S_{k+1} B + R) L_0 \\
k = \min_{u_k} \mathbb{E}[x_k \mid T_k]
\]

\[
\text{are satisfied. This concludes the proof.}
\]

**Proof of Theorem 7.** The proof of this lemma follows similar arguments to \(13\) Lemma 4), while also making use of the matrix inversion lemma.

**Proof of Theorem 7.** The proof of (a) and (b) can be found in [22]. We, here, focus on only the proof of (c). Let us define \(M_k \triangleq L_k (B^T S_{k+1} B + R) L_k\). As \(N \to \infty\), similar to [23], the expected minimum cost \(J^\infty\) can be written as

\[
J_\infty = \lim_{N \to \infty} \mathbb{E}\left[ \sum_{k=0}^{N-1} e_{k+1}^\top M_k e_k \right] = \mathbb{E}\left[ \sum_{k=0}^{N-1} e_{k+1}^\top M_k e_k \right] = \mathbb{E}\left[ \sum_{k=0}^{N-1} e_{k+1}^\top M_k e_k \right].
\]

The last term in \(J_\infty\) can be re-written as follows:

\[
\begin{align*}
\lim_{N \to \infty} \mathbb{E}\left[ \sum_{k=0}^{N-1} e_{k+1}^\top M_k e_k \right] &= \lim_{N \to \infty} \mathbb{E}\left[ \sum_{k=0}^{N-1} e_{k+1}^\top M_k e_k \right] \\
&= \lim_{N \to \infty} \sum_{k=0}^{N-1} \sum_{r=0}^{T} e_{k+1}^\top M_k e_k (\tau_k = i) \\
&= \lim_{N \to \infty} \sum_{k=0}^{N-1} \sum_{r=0}^{T} (M_k - M^\infty \Sigma_k (i)) P(\tau_k = i)
\end{align*}
\]

Since the pair \((A,B)\) is controllable and the pair \((A,Q^{1/2})\) is observable, there exists a steady state \(S_k \in \mathbb{S}_0^\infty\) for any initial matrix \(S_0 \in \mathbb{S}_0^\infty\). As a result, we have: \(\lim M_k = M^\infty\) (i.e., element-wise convergence). This implies that, for every \(\varepsilon > 0\), there exists \(N_\varepsilon\) such that, for all \(k > N_\varepsilon\),

\[
\left| \sum_{s=1}^{n} \sum_{r=1}^{m} (m_{sr} - m_{sr}^\infty) \right| \leq \frac{\varepsilon}{2},
\]

where \(m_{sr}^\infty\) is the \((s,r)\)-th entry of \(M_k\) and \(m_{sr}^\infty\) is the \((s,r)\)-th entry of \(M^\infty\). The first term of (45) can be upper-bounded as follows:

\[
\begin{align*}
\frac{1}{N} \sum_{k=0}^{N-1} \sum_{s=1}^{n} \sum_{r=1}^{m} (m_{sr} - m_{sr}^\infty) \max_{\iota_{s,r} \in \{1, \ldots, n\}} \{\sigma_{sr} (i)\},
\end{align*}
\]
As a result, the second term of (45) can be written as

\[ \text{have:} \]

\[ \varepsilon > 0. \text{ Choose } N_{\varepsilon} \text{ large enough such that} \]

\[ \frac{1}{N} \sum_{k=0}^{N-1} \left( \bar{m}_k - \bar{m}_{\infty} \right) \leq \frac{2}{N} \max_{k \in \{0, \ldots, N_{\varepsilon}-1\}} \left| \bar{m}_k - \bar{m}_{\infty} \right|, \]

hold for all \( N > N_{\varepsilon} \) and, if one chooses \( N \) satisfying

\[ N > \hat{N} \overset{\triangle}{=} \frac{2 N_{\varepsilon}}{\varepsilon} \max_{k \in \{0, \ldots, N_{\varepsilon}-1\}} \left| \bar{m}_k - \bar{m}_{\infty} \right|, \]

then the first term of (46) will be upper-bounded as follows:

\[ \frac{1}{N} \sum_{k=0}^{N-1} \left( \bar{m}_k - \bar{m}_{\infty} \right) \leq \frac{\varepsilon}{2}. \]

We now bound the second term of (46):

\[ \frac{1}{N} \sum_{k=N_{\varepsilon}}^{N} \left( \bar{m}_k - \bar{m}_{\infty} \right) \leq \frac{1}{N} \sum_{k=N_{\varepsilon}}^{N} \left| \bar{m}_k - \bar{m}_{\infty} \right| \]

\[ \leq \max_{k \in \{N_{\varepsilon}, \ldots, N-1\}} \left| \bar{m}_k - \bar{m}_{\infty} \right| \frac{N - N_{\varepsilon}}{N} \leq \frac{\varepsilon}{2}. \]

It follows that, for all \( N > \hat{N} \), the inequality (46) is bounded by \( \varepsilon \) (i.e., an arbitrarily chosen upper-bound). In other words, the first term of (45) converges to zero.

By the ergodic theorem [20] Theorem 5.3] for Markov chains, we have:

\[ \pi(i) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{I}(\tau_k=i), \]

which can be also represented, in the view of the bounded convergence theorem [24] pp. 138, as

\[ \pi(i) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\left[ \sum_{k=0}^{N-1} \mathbb{I}(\tau_k=i) \right]. \]

As a result, the second term of (45) can be written as

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \text{Tr}( \Sigma_{\infty} \Sigma^*(i) ) P(\tau_k=i) = \text{Tr}( \Sigma_{\infty} \Sigma^*(i) ) \pi(i). \]

This concludes our proof.