Higher symmetries can emerge at low energies in a topologically ordered state with no symmetry, when some topological excitations have very high energy scales while other topological excitations have low energies. The low energy properties of topological orders in this limit, with the emergent higher symmetries, may be described by higher symmetry protected topological order. This motivates us, as a simplest example, to study a lattice model of $\mathbb{Z}_n$-1-symmetry protected topological (1-SPT) states in 3+1D for even $n$. We write down an exactly solvable lattice model and study its boundary transformation. On the boundary, we show the existence of anyons with non-trivial self-statistics. For the $n = 2$ case, where the bulk classification is given by an integer $m \mod 4$, we show that the boundary can be gapped with double semion topological order for $m = 1$ and toric code for $m = 2$. The bulk ground state wavefunction amplitude is given in terms of the linking numbers of loops in the dual lattice. Our construction can be generalized to arbitrary 1-SPT protected by finite unitary symmetry.

I. INTRODUCTION

In the last few decades, there has been rapid progress in understanding “topological phases” of matter, which despite sharing the same symmetry, must undergo a phase transition to reach one phase from another. Some famous examples are the topological ordered states with no symmetry\textsuperscript{1,2} which have degenerate ground states on topological non-trivial closed manifolds, as well
as symmetry protected topological (SPT) states with symmetry\(^3\)–\(^6\), which does not have topological order and have a unique gapped ground state in closed manifolds.

A 3+1D topological order can have point-like and string-like topological excitations\(^7\)–\(^9\). For example, a 3+1D topological order described by \(Z_n\) gauge theory has \(Z_n\) charges (the point-like topological excitations) and \(Z_n\) flux-lines (the string-like topological excitations). If the \(Z_n\) charges have very large energy gap, then the theory for low energy \(Z_n\) flux-lines will have an emergent higher symmetry – \(a\) \(Z_n\) 1-symmetry\(^10\). In other words the low energy effective Hamiltonian is invariant under the symmetry transformations that act on all closed 2-dimension subspaces of the 3-dimensional space. Thus to understand the topological orders in such a limit, we can study Hamiltonians with a 1-symmetry. This motivates us to study 1-symmetry in this paper, such as the lattice Hamiltonian that realize 1-symmetry and the associated symmetry protected topological order, as well as their boundaries.

We will refer the transformations that act closed 2-dimension subspaces as the transformation membrane. If the 3d space have a boundary, the transformation membrane may intersect with the boundary. Such an intersection will be called transformation string.

A. Statement of results

In this paper, we will study lattice systems with higher symmetries\(^11\)–\(^24\). Like the usual symmetry (0-symmetry) that can have SPT order\(^3\)–\(^6\), higher symmetry can also have higher SPT order\(^10,19,21\). In this paper, we will concentrate on 3+1D systems with \(Z_n\) 1-symmetry and the associated associated 3+1D \(Z_n\) 1-SPT states. Those systems can appear as low energy effective theories for 3+1D \(Z_n\) topological order where the \(Z_n\) charges have a large energy gap.

The 3+1D \(Z_n\) 1-SPT states are known to have a \(Z_{2n}\) classification\(^25,26\), labeled by \(m \in Z_{2n}\). We study them in the Hamiltonian formalism and write down an exactly solvable bulk Hamiltonian, which has a compact expression when \(m\) is even.

The boundary of our system can also have the \(Z_n\) 1-symmetry, but such a \(Z_n\) 1-symmetry is anomalous\(^19,21,27\). We find that on the boundary, the transformation strings can carry non-trivial self-statistics, as a reflection of the anomaly. This predicts the gapped boundary of the 1-SPT to have emergent anyons. We also find that it is possible for its surface state to be a gapped topological ordered state. The topological ordered boundary state has degenerate ground states if the surface manifold has non-zero genus. These degenerate states exhibit the spontaneous breaking of 1-symmetry. We also give a geometric interpretation of the ground state wave function, by writing the wave function amplitude in terms of the linking numbers of loops in the dual lattice.

B. Notations and conventions

In some part of this paper, we will use the Lagrangian formalism to describe quantum lattice systems. This allows us to use extensively the notion of cochain, cocycle, and coboundary, as well as their higher cup product \(\langle a,b \rangle\) and Steenrod square \(Sq^k\), to construct exactly solvable Lagrangian that realize topological orders and (higher) SPT orders. The reason to use modern mathematical formalisms is that they allow us to see the features of topological order and (higher) SPT order easily and quickly.

But the modern mathematical formalisms are not widely used in condensed matter theory. So we provide a brief introduction in Appendix A. Also, the Lagrangian formalism does not give us a lattice Hamiltonian explicitly. So in this paper, we present a systematic and direct way to obtain a lattice Hamiltonian from the those exactly solvable Lagrangian.

We will abbreviate the cup product of cochains \(a \cup b\) as \(ab\) by dropping \(\cup\). We will use \(\triangleq\) to mean equal up to \(\pm\) to mean equal up to \(df\) \((i.e.\) up to a coboundary). We will use \((l,m)\) to denote the greatest common divisor of \(l\) and \(m\) \((0,m) \equiv m\). We will also use \([x]\) to denote the integer that is closest to \(x\). (If two integers have the same distance to \(x\), we will choose the smaller one, \(eg\). \([\frac{1}{2}] = 0\).)

In this paper, we will deal with \(Z_n\)-value quantities. We will denote them as:

\[
a^{\pm n} := a - n\lfloor \frac{a}{n}\rfloor,
\]

so the value of \(a^{\pm n}\) has a range from \([-\lfloor \frac{n-1}{2} \rfloor\) to \(\frac{n}{2} \rfloor\). We will sometimes lift a \(Z_n\)-value to \(Z\)-value, and when we do so we omit the superscript, \(eg\). \(a^{\pm n} \rightarrow a^{\pm} = a\), so we can make sense of expressions like \(a^{\pm n} + a^{\pm} \sqrt[2]{\frac{z}{n}}\), which means \(a^{\pm} + a^{\pm} \sqrt[2]{\frac{z}{n}}\). Since \((a + nu^{\pm})^{\pm n} = a^{\pm n}\), whenever we lift a \(Z_n\)-value to \(Z\)-value we need to take care whether the final result is independent of choice of lifting, \(i.e.\) choice of \(u^{\pm}\).

We will also use \(D\) to denote spacetime dimensions and \(d\) to denote space dimensions.

C. Overview of paper

The structure of the paper and a road map for reading is presented as follows.

In section II, we review some background information connecting the cohomology models we studied to the standard many-body theory. We explained what are those cohomology models, and some simple examples of those model that realize simple topological orders and (higher) SPT orders.

In section III we present an intuitive, informal argument for one of our major results, the self and mutual statistics of boundary transformation strings, without using the mathematical machinery of cochains and cocycles. The formal argument begins from section IV, where
we cite from the literature that the $\mathbb{Z}_n$-1 SPT has $\mathbb{Z}_{2n}$ classification from cohomology, such that each phase is labeled by $m \in \mathbb{Z}_{2n}$. We write down the exactly solvable Lagrangian, the expression for $\omega_4$ in (11). We also show that it changes by a boundary term under gauge transformation via (16): \[
abla \omega_4[\hat{B}^{Z_m} + da^{Z_m}] = \omega_4[\hat{B}^{Z_m}] + d\phi_3[\hat{B}^{Z_m}, a^{Z_m}]
\]
for some function $\phi_3$. This implies $\omega_4[\hat{B}^{Z_m} + da^{Z_m}]$ and $\omega_4[\hat{B}^{Z_m}]$ gives the same answer when summed over a closed manifold, which is expected from gauge invariance (1).

In section V we specialize to the case $\hat{B}^{Z_m} = 0$ and give the explicit form of $\phi_3[a^{Z_m}]$ in (28). In Appendix B and C, we argue that on a closed spatial manifold $\mathcal{M}^3$, $e^{2\pi i \int_{\mathcal{M}^3} \phi_3[a^{Z_m}]}$ is the amplitude of the ground state wavefunction. We achieve this by examining the time-evolution operator $e^{-\mathcal{T}H_{\infty}}$ whose matrix elements are given in (B3). We show that it is a projection operator (hence an infinite gap) and has trace 1 (hence a unique ground state). We further argue this transfer matrix can be decomposed into local commuting projection operators $P_{ij}$ (B6). We then build our exactly solvable Hamiltonian with a finite gap by summing over the $-P_{ij}$'s. We then verify that the ground state wavefunction is indeed given in terms of $\phi_3[a^{Z_m}]$.

To write down the exactly solvable Hamiltonian, we consider a particular triangulation of $\mathcal{M}^3$, given in Appendix D. We compute the explicit form for $P_{ij}$ for the even $m$ case in Appendix F and the odd $m$ case in Appendix G. Unfortunately, we are unable to further simplify the expression in the odd $m$ case. The results are summarized and presented in section V.

In section VI we consider the case when $\mathcal{M}^3$ has a boundary. We introduced the notion of a “boundary state” (36), which is obtained by fixing the degrees of freedom on the boundary and relaxing the bulk degrees of freedom to their ground state. As a result, the originally non-anomalous 1-symmetry transformation from the bulk now transform the boundary states with an additional phase $e^{2\pi i \int_{\mathcal{M}^3} \phi_3[a^{Z_m}, h^{Z_m}]}$. This phase captures the ’t Hooft anomaly of 1-symmetry in the boundary. Any boundary Hamiltonian must be symmetric under this anomalous 1-symmetry in order to cancel the ’t Hooft anomaly. We show that $\phi_2[a^{Z_m}]$ is related to the ground state wavefunction $\phi_3[a^{Z_m}]$ by (40):

\[
\phi_3[(a + dh)^{Z_m}] = \phi_3[a^{Z_m}] = -d\phi_2[a^{Z_m}, h^{Z_m}]
\]
which states that under the 1-symmetry, the ground state wavefunction changes by a boundary term. We write down the explicit form of $\phi_2[a^{Z_m}, h^{Z_m}]$ in (41). Using this explicit form, we are able to compute the self (49) and mutual (51) statistics of the transformation strings. Details of the computation are given in H. The boundary transformation strings may be interpreted as hopping operators for anyons residing on the end of the strings. This predicts the emergence of such anyon on the boundary theory and is the main result of the paper.

In section VII we test our prediction by writing down some gapped boundary Hamiltonians which obeys the anomalous 1-symmetry. We specialize to $n = 2$ and check the cases $m = 2$ and $m = 1$. We show that the gapped boundary is identical to the toric code model (for $m = 2$) and the double semion model (for $m = 1$). We verify in both cases that the boundary indeed contains an anyon with the predicted statistics. Details of the computation for the boundary Hamiltonian are given in I.

In section VIII we return to examine the ground state wavefunction. We present the geometric interpretation of the bulk wave function amplitude as a knot invariant (linking number) of loops dual to $da^{Z_m}$.

In section IX we extend our study to the case with a non-zero background gauge field. In the even $m$ case, we find a line charge with charge $-m$ is attached to the dual line of the background gauge field. Details are presented in Appendix B2 and C2.

In Appendix J we go deeper into the origin of the connections between $\omega_4$, $\phi_3$, $\phi_2$, and show that they are members of a series of algebraic objects $\phi_k$ which encodes the same cocycle $\omega_4$ at sub-manifolds of dimension $k$.

In Appendix K we present the result of generalizing the computation of boundary string statistics to other unitary groups.

II. A BRIEF REVIEW OF TOPOLOGICAL ORDER, SPT STATES, AND HIGHER SPT STATES

A large class of topological orders can be realized by exactly solvable Lagrangian model. To write down the Lagrangian model, we first triangulate the spacetime to obtain a spacetime lattice $\mathcal{M}^D$, whose vertices are labeled by $i, j, \cdots$. The physical degrees of freedom $B_{ij}$ live on the link $ij$, and takes value in a group $G$, i.e. $B_{ij} \in G$. In this paper, we always assume $G$ to be Abelian. The collection of those values $B_{ij}$ give us a field $B$ on spacetime, which, in this case, is also called a gauge configuration. A quantum system in Lagrangian formulation is described by a path integral with an action amplitude. For our model, the action amplitude assigns a $U(1)$ phase $e^{2\pi i S^{\text{top}}[\mathcal{M}^D, B]}$ to a gauge configuration $B$ on a $D$-dimensional spacetime lattice $\mathcal{M}^D$. The gauge field $B$ satisfies the “flatness condition” $d\mathcal{B} = 0$ which is enforced by an energy penalty term $U|d\mathcal{B}|^2$ in $U \to \infty$ limit. The model is exactly solvable if the $U(1)$ phase is a topological invariant, meaning it remains unchanged under “deformations” of the lattice $\mathcal{M}^D$ (change of triangulation), and is also invariant under gauge transformations $B \to B + da$, i.e. (in this paper we will assume the underlying group $G$ is an Abelian finite group.)

\[
S^{\text{top}}[\mathcal{M}^D, B + da] = S^{\text{top}}[\mathcal{M}^D, B], \quad a \in G
\]
Here $\equiv$ means equal up to 1. The partition function, after summing all the degrees of freedom (i.e. the $G$ values in all the links), is given by

$$ Z(M^D) = \sum_{\{\bar{B}\}} e^{2\pi i S^{\text{top}}[M^D, \bar{B}]} $$

Up to a volume term, the partition function $Z(M^D)$ is a topological invariant of manifold $M^D$, that characterize a topological order. When $S^{\text{top}}[M^D, \bar{B}] = 0$, our model realize a $G$ topological order described by a $G$-gauge theory. When $S^{\text{top}}[M^D, \bar{B}] \neq 0$, our exactly solvable model realizes a topological order described by a twisted $G$-gauge theory, which is also known as Dijkgraaf-Witten model.

The action amplitude $e^{2\pi i S^{\text{top}}[M^D, \bar{B}]}$ of the exactly solvable model can also be viewed as an SPT invariant that characterizes an SPT order protected by symmetry $G$, if we view $\bar{B}$ as the background gauge field $\hat{B}$ that describes the symmetry twist on the space-time $M^D$. Such a relation is also referred to as “ungauging” a topological order, which results in a SPT order. The SPT invariant $e^{2\pi i S^{\text{top}}[M^D, \bar{B}]}$ characterizes a large class of SPT orders.

To realize the SPT states characterized by the above SPT invariant, we write $\bar{B} = \hat{B} + da$, fix a background gauge configuration $\hat{B}$, and treat the different gauge transformations $a$ as distinct physical fields. The partition function, after summing all the degrees of freedom $a$, reproduces the SPT invariant, up to a space-time volume term:

$$ Z[M^D, \hat{B}] = \int \mathcal{D}a e^{2\pi i S^{\text{top}}[M^D, \hat{B} + da]} \sim e^{2\pi i S^{\text{top}}[M^D, \hat{B}]} $$

Note that the action is invariant under the symmetry $a \to a + \alpha$ for $\alpha$ satisfying $da = 0$. An SPT is trivial if $e^{2\pi i S^{\text{top}}[M^D, \hat{B}]} = 1$ for all closed manifolds and background gauge fields $\hat{B}$. SPTs also form an Abelian group under stacking. The topological action $S^{\text{top}}$ for the stacked SPT is the sum of the topological actions of its layers. The trivial SPT is the identity element under stacking and describes a direct product state.

“Group cohomology construction” is one way to write down $S^{\text{top}}[M^D, \hat{B}]$. In this construction, we assume that $S^{\text{top}}[M^D, \hat{B}]$ can be written as a sum over all the $D$-simplices $\Delta^D$:

$$ S^{\text{top}}[M^D, \hat{B}] = \int_{M^D} \omega_D[\hat{B}] = \sum_{\Delta^D} \omega_D[\hat{B}] $$

where $\omega_D[\hat{B}]$ assigns a number to each $D$-simplex. The requirement that $S^{\text{top}}[M^D, \hat{B}]$ is invariant under triangulation leads to the following constraint on $\omega_D[\hat{B}]$, known as the “cocycle condition”:

$$ d\omega_D[\hat{B}] \equiv 0, $$

whose solutions are called cocycles. (The left hand side is evaluated on a $D+1$-simplex and $d$ is called the coboundary operator analogous to the exterior derivative for differential forms. See Appendix A for further details.) Distinct solutions of the cocycle condition do not necessarily correspond to distinct topological phases, since two solutions $\omega_D, \omega'_D$ may give the same $S^{\text{top}}$ on closed manifolds if $\omega_D = \omega'_D + d\beta_D - d$ for some function $\beta_D$. Defining an equivalence relation $\omega_D \sim \omega'_D + d\beta_D - d$ on cocycles and solving for the equivalence classes of cocycles, the resulting algebraic object is known as a cohomology group, which also provide a way to classify SPTs.

In the traditional SPT, the gauge field $\hat{B}$ assigns a group element of $G$ to every 1-dimensional simplex (i.e. links), and are thus called 1-cochain. (A $G$-valued $m$-cochain is an assignment of a group element of $G$ to each $m$-simplex.) Gauge transformations are parameterized by a 0-cochain $a$ which assigns a group element to every 0-dimensional simplex (i.e. vertices). Symmetry is parameterized by 0-cochain $\alpha$. The condition $da = 0$ implies $\alpha$ is a constant function on every connected component. Physically this corresponds to a global symmetry acting on a connected component of the spatial slice. An example is the $\mathbb{Z}_2$-protected SPT in $D = 2 + 1$. The $\mathbb{Z}_2$ symmetric ground state wavefunction can be constructed as the superposition of domain walls in the $\mathbb{Z}_2$ symmetric breaking state, with $(-1)^\text{no. of domain walls}$ as its amplitude.

With the above description of usual SPT states, we can now derive higher SPTs. Higher SPT states, or “$k$-symmetry protected topological states” (k-SPT), is a generalization of traditional SPTs. They have symmetry acting on closed sub-lattices of codimension $k$. The 1-cochain (i.e. the vector field) $\bar{B}$ is promoted to $(k+1)$-cochain. The gauge transformation is now described by a $k$-cochain $a$: $\bar{B} \to \bar{B} + da$.

The path integral on spacetime lattice $M^D$ that realize a higher SPT state is given by

$$ Z[M^D, \hat{B}] = \sum_{\{a\}} e^{2\pi i \int_{M^D} \omega_D(\hat{B} + da)} $$

where the dynamical field $a$ is now a $k$-cochain (a field which takes values on the $k$ simplices), and $S^{\text{top}}[M^D, \bar{B}]$ is given by eqn. (2). In such a lattice model, the higher symmetry is generated by a $k$-cocycle $\alpha$:

$$ a \to a + \alpha. $$

We see that the symmetry acts on $k$-simplices where $\alpha \neq 0$. Such $k$-simplices are dual to a $(D-k)$-dimensional manifold $\hat{\alpha}$ on the dual lattice. The condition $da = 0$ implies $\hat{\alpha}$ has no boundary within the space-time manifold. $\hat{\alpha}$ may have a non-empty boundary if it intersects the boundary of the space-time manifold $\partial M^D$.

When $\omega_D = 0$, eqn. (4) describes a state with trivial $k$-SPT order. When $\omega_D$ is a non-trivial cocycle, eqn. (4)
realizes a state with a non-trivial $k$-SPT order. The traditional SPT corresponds to $k = 0$ case.

The above Lagrangian is a realization of higher SPT states. In this paper, we show how to convert the above Lagrangian realization into a Hamiltonian realization. In the Hamiltonian formalism, a $k$-symmetry operator acts on codimension $k$ sub-lattices in the spatial manifold. For example in a 3 space dimensions, a 1-symmetry operator acts on closed membranes. These membranes may intersect the boundary as strings. We show in Section VI that the 1-symmetry membrane operators in the bulk corresponds to 1-symmetry string operators on the boundary.

The hallmark of non-trivial SPT is that its boundary cannot be gapped with a unique ground state on all manifolds. If it were the case, we could start from a trivial SPT, nucleate a small bubble of the non-trivial SPT, and expand the bubble to fill up the entire space. This would have provided a path connecting the trivial and the non-trivial SPTs without closing the energy gap, leading to a contradiction. Generically the boundary of non-trivial SPT is gapless, breaks symmetry spontaneously, or has topological order. The inability for the boundary to achieve a uniquely gapped state on all manifolds is encoded by the ’t Hooft anomaly of the $k$-symmetry on the boundary. Therefore studying such anomaly is a way to probe the non-trivial nature of the topological bulk.

III. INTUITIVE ARGUMENT FOR BOUNDARY TRANSFORMATION STRING STATISTICS

In this section we present an informal argument for the self and mutual statistics of the boundary strings.

The ’t Hooft anomaly of the boundary transformation of SPTs may be interpreted via symmetry fractionalization\(^\text{34,35}\): when the symmetry acts on the entire boundary manifold, and hence can be extended into the bulk, the group representation structure is preserved. But when we attempt to examine the symmetry acting only on a local patch of the boundary manifold, various group representation structures may be spoiled.

Take for example the non-trivial $G$-protected 1d 0-SPT\(^\text{36}\), for which the AKLT\(^\text{37}\) chain with $G = SO(3)$ is a well-known instance. The boundary of a 1d segment are its two endpoints, indexed by $L$ and $R$ respectively. When the bulk is gapped, the low energy effective theory is described in terms of its boundary degrees of freedom, and the Hilbert space may be expressed as a tensor product $\mathcal{H}_L \otimes \mathcal{H}_R$ of the local Hilbert spaces $\mathcal{H}_L$ and $\mathcal{H}_R$ for the two ends. For two group elements $g, h$ acting on the tensor product space, we have

$$\mathcal{R}_L(g)\mathcal{R}_L(h) = \mathcal{R}_L(gh),$$

which says $\mathcal{R}_L(g) = \mathcal{R}_L(g) \otimes \mathcal{R}_R(g)$ is a linear representation of $G$. This is because when the same $g \in G$ acts on both boundaries, it may be extended into a symmetry acting globally in the bulk, where the group is represented linearly. When localizing on the left end, $\mathcal{R}_L$ becomes a projective representations of $G$:

$$\mathcal{R}_L(g)\mathcal{R}_L(h) = \omega(g, h)\mathcal{R}_L(gh).$$

The ’t Hooft anomaly is expressed as the non-trivial phase $\omega(g, h)$, which spoils the linearity of the representation.

In the same spirit, for our case with $\mathbb{Z}_n$-1-SPT, we may expect ’t Hooft anomaly to appear as the spoiling of some group representation structure when localizing to a part of the boundary. If the boundary symmetry can be extended into the bulk, the group representation structure is expected to be preserved.

Consider the case where we have two 1-symmetries, $W_1$ and $W_2$, associated with group elements $q^n_1$ and $q^n_2$ respectively. They act on two contractible loops, as shown below:

Each loop can be extended into the bulk as a 1-symmetry acting on a hemisphere. In the bulk, the 1-symmetries commute. We therefore expect that on the boundary, the two loop operators also commute. This is represented by the diagrammatic equation:

There are two intersections of the loops. Motivated by the symmetry fractionalization picture, we may guess that when localizing to one of the intersections, the commutativity is spoiled by a $U(1)$ phase $e^{2\pi i \theta_{q_1,q_2}}$:

If we further assume that two parallel lines associated with group elements $p^n_1$ and $q^n_1$ could stack into a single line $(p_1 + q_1)^n$ without incurring any phase, we can deduce that $\theta_{q_1,q_2}$ is a linear function of $q_1$ by the following
The reason is as follows, imagine when the left hand side extends into the bulk as two hemispheres which intersect on a line. On the intersection line, the $1$-symmetry acts trivially and may be removed by reconnecting the membranes near the line. Thus we may reconnect the two intersecting hemispheres into two non-intersecting membranes which terminates on the surface as shown on the right hand side. This implies

$$
\theta_q + \tilde{\theta}_q = 0.
$$

Adding this equation to (8), we get $2\theta_q = \theta_{qq}$. Thus

$$
\theta_q = \frac{1}{2} \theta_{qq} = \frac{1}{2} m q^2.
$$

For some integer-valued function $f$. An argument similar to that to the linearity of $\theta_q q_{1,2}$ implies $\theta_q$ is proportional to $q^2$. So $f(q,m,n) \propto q^2$ and the coefficient of $q^2$ in $\theta_q$ is an integer. Let’s redefine $m$ to be this integer coefficient. Thus we have

$$
\theta_q = m \frac{q^2}{2n^2}.
$$

Upon $q \to q + n$, the above equation transform as

$$
\theta_q = \frac{m}{2n} (q^2 + 2qn + n^2) \implies \theta_q + \frac{nm}{2} \equiv \theta_q \pmod{1}
$$

so in order for $\theta_q$ to be invariant mod $1$, $nm$ must be even. In the case $n$ is even, $m$ can be chosen from $0, 1, \ldots, 2n - 1$. With $n$ odd, $m$ must be an even number chosen from $0, 2, 4, \ldots, 2n - 2$. Each choice gives a distinct set of $\theta_q$ and $\theta_q q_{1,2}$.

Assuming the set of $\theta_q$ and $\theta_q q_{1,2}$ is bijective to the set of ‘t Hooft anomalies, which is bijective to the set of $1$-SPT phases in the bulk, we would expect the bulk $Z_n$-$1$-SPT to have $Z_{2n}$ classification for even $n$, and $Z_n$ classification for odd $n$. This agrees with the classification results in Ref.25 derived from cohomology and Ref.26 from cobordism group.

We stress that the odd $n$ case and the even $n$ case differs only in that the odd $m$’s are forbidden for odd $n$. In fact, all the results in our paper for even $m$ also applies to odd $n$ case.

In the rest of the paper, we will re-derive the expressions for $\theta_q q_{1,2}$ and $\theta_q$ formally, using the language of group cohomology.

IV. A 3+1D MODEL TO REALIZE A $Z_n$-$1$-SPT PHASE FOR EVEN $n$

To construct lattice models with higher symmetries, it is convenient to do so in the spacetime Lagrangian.

A similar argument shows $\theta_{q_1 q_2}$ is linear in $q_2$. We conclude

$$
\theta_{q_1, q_2} \propto q_1 q_2
$$

Since $q_1^n$, $q_2^n$ are defined up to multiples of $n$, we expect $e^{2\pi i \theta_{q_1 q_2}}$ to be invariant under $q_1 \to q_1 + n$. Thus the coefficient should be a fraction $\frac{m}{n}$ for some integer $m$.

$$
\theta_{q_1, q_2} = \frac{m}{n} q_1 q_2
$$

(7)

When $q_1 = q_2 = q$, we may also entertain the possibility that at an intersection, the transformation string may “change track” and incur a $U(1)$ phase $e^{2\pi i \theta_q}$ or $e^{2\pi i \theta_{\bar{q}}}$, depending on the orientation of the crossing:

$$
e^{-2\pi i \theta_q} = e^{-2\pi i \bar{\theta}_q}
$$

Comparing to (6) with $q_1 = q_2 = q$, we observe that

$$
\theta_q - \bar{\theta}_q = \theta_{qq}.
$$

(8)

On the other hand, we also have the equality of these two diagrams:

$$
\theta_q q_{1,2} \in \mathbb{Z}
$$

Manipulations:

$$
e^{2\pi i (\theta_{q_1 q_2} + \theta_{q_1 q_2})} = e^{2\pi i \theta_{(q_1 + q_2) q_2}} = e^{2\pi i \theta_{q_1 (q_1 + q_2)}} = e^{2\pi i \theta_{(q_1 + q_2) q_1}} = e^{2\pi i \theta_{q_1 (q_1 + q_2)}}
$$

IV. A 3+1D MODEL TO REALIZE A $Z_n$-$1$-SPT PHASE FOR EVEN $n$

To construct lattice models with higher symmetries, it is convenient to do so in the spacetime Lagrangian...
formalism. We construct a spacetime lattice by first triangulating a $D$-dimensional spacetime manifold $M^D$. So a spacetime lattice is a $D$-complex $\mathcal{M}^D$ with vertices labeled by $i$, links labeled by $ij$, triangles labeled by $ijk$, etc (see Fig. 1). The $D$-complex $\mathcal{M}^D$ also has a dual complex denoted as $\mathcal{M}^D_d$. The vertices of $\mathcal{M}^D_d$ correspond to the $D$-cells in $\mathcal{M}^D_d$. The links of $\mathcal{M}^D_d$ correspond to the $(D-1)$-cells in $\mathcal{M}^D_d$, etc.

Our spacetime lattice model may have a field living on the vertices, $g_i$. Such a field is called a 0-cochain. The model may also have a field living on the links, $a_{ij}$. Such a field is called a 1-cochain, etc. To construct spacetime lattice models, in particular, the topological spacetime lattice models, we will use extensively the mathematical formalism of cochains, coboundaries, and cocycles (see Appendix A).

### A. The bulk exactly solvable Lagrangian

We consider a 3+1D bosonic model on a spacetime complex $\mathcal{M}^4$, with $Z_n$-valued dynamic field $a^n_{ij}$ on the links $ij$ of the complex $\mathcal{M}^4$. Here $n$ is even. We also have a $Z_n$-valued non-dynamical background field $\hat{B}^{Z_n}$ on the triangles $ijk$ of the complex $\mathcal{M}^4$. $\hat{B}^{Z_n}$ is a $Z_n$-valued 2-cocycle

$$d\hat{B}^{Z_n} \equiv 0.$$  \hspace{1cm} (9)

The path integral of our bosonic model is given by

$$Z = \sum_{\{a^n_{ij}\}} e^{\frac{2\pi i}{n} \int_{\mathcal{M}^4} \omega_4[B]}$$  \hspace{1cm} (10)

$$\omega_4[B] := \frac{m}{2n} \text{Sq}^2 B^{Z_n},$$  \hspace{1cm} (11)

$$B := \hat{B} + da.$$  \hspace{1cm} (12)

where $m, n$ are integers, $\sum_{\{a^n_{ij}\}}$ sums over $Z_n$-valued 1-cochains $a^n_{ij}$. We have lifted the $Z_n$-valued quantities $\hat{B}^{Z_n}$ and $a^{Z_n}$ to $Z$-valued quantities $\hat{B}$ and $a$. Also $\text{Sq}^2$ is the generalized Steenrod square defined by eqn. (A21). We will show that the above model realizes a $Z_n$-1-SPT phase.

Since $\omega_4[B] = \omega_4[B^{Z_n}]$ and $B^{Z_n}$ is invariant under the transformation

$$\hat{B} \rightarrow \hat{B} + nb^{Z_n}, \quad a \rightarrow a + nu^{Z_n},$$  \hspace{1cm} (13)

where $b^{Z_n}$ and $u^{Z_n}$ are any $Z$ valued 2-cochain and 1-cochain, the action amplitude in eqn. (10) is invariant, even when $\mathcal{M}^4$ has a boundary. The above result also implies that the model has a $Z_n$-1-symmetry generated by

$$a \rightarrow a + \alpha^{Z_n}, \quad da^{Z_n} \equiv 0,$$  \hspace{1cm} (14)

even when $\mathcal{M}^4$ has a boundary.

Also it can be checked that $e^{2\pi i \omega_4[B]}$ is a $U(1)$-valued cocycle: Using (A21), (A18) and $d\hat{B}^{Z_n} \equiv 0$ which follows from (9), and remembering that $n$ is even, we have

$$d\omega_4[B^{Z_n}] = \frac{m}{2n} \text{Sq}^2 B^{Z_n}.$$  \hspace{1cm} (15)

In eqn. (12), $\hat{B}^{Z_n}$ is the $Z_n$ background 2-connection to describe the twist of the $Z_n$-1-symmetry. The model has a $Z_n$ gauge symmetry:

$$a \rightarrow a + \alpha^{Z_n}, \quad \hat{B} \rightarrow \hat{B} - d\alpha^{Z_n}.$$  \hspace{1cm} (16)

Also, using $d\hat{B}^{Z_n} \equiv 0$, (A21) and (A18),

$$\frac{m}{2n} \text{Sq}^2 B^{Z_n} = \frac{m}{2n} \text{Sq}^2 \hat{B} + \frac{m}{2} d(\hat{B} \rightarrow \hat{B} - d\alpha^{Z_n}).$$  \hspace{1cm} (17)

$$\frac{m}{2n} \text{Sq}^2 \hat{B} + \frac{m}{2} d(\hat{B} \rightarrow \hat{B} - d\alpha^{Z_n}) = \frac{m}{2n} \text{Sq}^2 \hat{B} + \frac{m}{2} d(\hat{B} + da) = \frac{m}{2n} \text{Sq}^2 \hat{B} + \frac{m}{2n} \text{Sq}^2 (\hat{B} + da).$$  \hspace{1cm} (18)

FIG. 1. (Color online) The black lines describe a 2-dimensional spacetime complex $\mathcal{M}^2$. The red lines describe the dual complex $\mathcal{M}^2_d$. 

The bulk exactly solvable Lagrangian

- Consider a 3+1D bosonic model on a spacetime complex $\mathcal{M}^4$, with $Z_n$-valued dynamic field $a^n_{ij}$ on the links $ij$ of the complex $\mathcal{M}^4$.
- Here $n$ is even. We also have a $Z_n$-valued non-dynamical background field $\hat{B}^{Z_n}$ on the triangles $ijk$ of the complex $\mathcal{M}^4$.
- $\hat{B}^{Z_n}$ is a $Z_n$-valued 2-cocycle $d\hat{B}^{Z_n} \equiv 0$.
- The path integral of our bosonic model is given by

$$Z = \sum_{\{a^n_{ij}\}} e^{\frac{2\pi i}{n} \int_{\mathcal{M}^4} \omega_4[B]}$$

$$\omega_4[B] := \frac{m}{2n} \text{Sq}^2 B^{Z_n},$$

$$B := \hat{B} + da.$$  

where $m, n$ are integers, $\sum_{\{a^n_{ij}\}}$ sums over $Z_n$-valued 1-cochains $a^n_{ij}$.

- We will show that the above model realizes a $Z_n$-1-SPT phase.

- Since $\omega_4[B] = \omega_4[B^{Z_n}]$ and $B^{Z_n}$ is invariant under the transformation

$$\hat{B} \rightarrow \hat{B} + nb^{Z_n}, \quad a \rightarrow a + nu^{Z_n},$$

where $b^{Z_n}$ and $u^{Z_n}$ are any $Z$ valued 2-cochain and 1-cochain, the action amplitude in eqn. (10) is invariant, even when $\mathcal{M}^4$ has a boundary. The above result also implies that the model has a $Z_n$-1-symmetry generated by

$$a \rightarrow a + \alpha^{Z_n}, \quad da^{Z_n} \equiv 0,$$
In the last step we reused (16) \( \equiv (18) \) with \( B \) replaced by \( \hat{B} \). Therefore
\[
e^{2\pi i \int_{\mathcal{M}^4} \frac{4\pi}{n} \mathbb{S}^2 (d\Omega_{\mathcal{M}^4} \Omega_{\mathcal{M}^4} + \Omega_{\mathcal{M}^4} \gamma d\hat{B}^2 \Omega_{\mathcal{M}^4}} \]
(21)
for closed spacetime \( \mathcal{M}^4 \). This is expected from gauge invariance \( (1) \). The model is exactly solvable and gapped using \( (16) \),
\[
e^{2\pi i \int_{\mathcal{M}^4} \frac{4\pi}{n} \mathbb{S}^2 (\hat{B}^2 + da)^2 \}
(19)
which is equal to 1 for all closed 4-complex \( M \). Therefore
\[
e^{2\pi i \int_{\mathcal{M}^4} \frac{4\pi}{n} \mathbb{S}^2 (\hat{B}^2 + da)^2}
(20)
where \( N_i \) is the number of links in the spacetime complex \( \mathcal{M}^4 \). \( n^{N_1} \) is the so called the volume term that is linear in the spacetime volume. The topological partition function \( Z^{\text{top}} \) is given by removing the volume term: \( Z^{\text{top}}(\mathcal{M}^4) = Z(\mathcal{M}^4)/n^{N_1} \),
(22)
which is equal to 1 for all closed 4-complex \( \mathcal{M}^4 \). Thus the above model has no topological order. After we turn on the flat \( Z_n \)-2-connection \( \hat{B}^2 \), the topological partition function of the model \( (10) \) becomes
\[
e^{2\pi i \int_{\mathcal{M}^4} \frac{4\pi}{n} \mathbb{S}^2 (\hat{B}^2 + da)^2 \Omega_{\mathcal{M}^4} \Omega_{\mathcal{M}^4} + \Omega_{\mathcal{M}^4} \gamma d\hat{B}^2 \Omega_{\mathcal{M}^4}} \]
(24)
\[
e^{2\pi i \int_{\mathcal{M}^4} \frac{4\pi}{n} \mathbb{S}^2 (\hat{B}^2 + da)^2 \}
(25)
using \( (16) \) \( \equiv (17) \) and \( (18) \) \( \equiv (19) \) with \( B = 0 \),
\[
e^{2\pi i \int_{\mathcal{M}^4} \frac{4\pi}{n} \mathbb{S}^2 (d\Lambda_{\mathcal{M}^4} \Lambda_{\mathcal{M}^4} + \Lambda_{\mathcal{M}^4} \gamma d\hat{B}^2 \Omega_{\mathcal{M}^4}} \]
(26)
(28) and (29) are obtained from the previous line by writing out \( a \). By construction we have \( \phi_3[a] = \phi_3[a + n\hat{a}] = \phi_3[a - \hat{a}] \) for any \( \mathbb{Z} \)-valued 1-cochain \( u \). However, \( \xi_3 \) and \( \xi_2 \) do not enjoy this property. (See Appendix J for relationship between \( \omega_4 \) and \( \phi_3 \) in general.)

We will analyze the cases for even and odd \( m \) separately. For each case we write down the Hamiltonian
\[
H = - \sum_{ij} P_{ij},
\]
which is the sum over links \( ij \) of projections \( P_{ij} \), as described in Appendix B. We can compute \( P_{ij} \) by assuming a hypercubic lattice for the space-time \( \mathcal{M}^4 = \mathbb{R}^4 \) triangulated as in Appendix D. The Hilbert space is spanned by \( \{|a^{\mathcal{M}^4}\rangle\} \) for links \( ij \) in the 3D cubic lattice.

A. Even \( m \) case

When \( m \) is even, \( (26) \) and \( (28) \) are simplified considerably. The result is
\[
\omega_4[a] = \frac{m}{2n} \mathbb{S}^2 (d\Lambda_{\mathcal{M}^4} \Lambda_{\mathcal{M}^4} + \Lambda_{\mathcal{M}^4} \gamma d\hat{B}^2 \Omega_{\mathcal{M}^4}} \]
(30)
\[
\phi_3[a] = \frac{m}{2n} a d\Lambda_{\mathcal{M}^4} \Lambda_{\mathcal{M}^4} \gamma d\hat{B}^2 \Omega_{\mathcal{M}^4}} \]
(31)
We will also triangulate \( \mathbb{R}^3 \) as described in Appendix D. The variables in our lattice model lives on links. There are three types of links: 1-diagonal, 2-diagonal or 3-diagonal. A link \( ij \) is defined to be \( k \)-diagonal if the displacement vector from \( i \) to \( j \) differs by \( k \) distinct link vectors \( \{x_1, x_2, x_3\} \). In the even \( m \) case, as shown in Appendix F, the 2-diagonal and 3-diagonal links form product states and can be ignored.

For the 1-diagonal links, the topological action
\[
Z^{\text{top}} = \frac{1}{n^{N_1}} \sum_{\{a^{\mathcal{M}^4}\}} e^{2\pi i \int_{\mathcal{M}^4} \frac{4\pi}{n} \mathbb{S}^2 (d\Lambda_{\mathcal{M}^4} \Lambda_{\mathcal{M}^4} + \Lambda_{\mathcal{M}^4} \gamma d\hat{B}^2 \Omega_{\mathcal{M}^4}} \]
(32)
leads to mutually commuting projections (F2)

\[ P_{ij} = \frac{1}{n} \sum_{k=0}^{n} \hat{X}_{ij} e^{2\pi i \frac{k}{n} [F_{\beta\gamma} (\vec{r}_{ij} + \frac{\vec{r}_{ij}}{2}) + F_{\beta\gamma} (\vec{r}_{ij} - \frac{\vec{r}_{ij}}{2})]} \]  

where the sum is carried over 1-diagonal links \( ij = (i, i+1) \), \( \hat{X}_{ij} [a^{Z_n}_{ij}] = |(aij + 1)Z_n \rangle \), \( \vec{r}_{ij} = \vec{r} + \frac{\vec{r}_{ij}}{2} \) is the mid-point of the link, \( \frac{\vec{r}_{ij}}{2} = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \) and \( F_{\beta\gamma}(\vec{r}) \) reads off the “flux” through the square centered at \( \vec{r} \), or more specifically

\[ F_{\beta\gamma}(\vec{r}) := \langle \partial a^{Z_n}(\vec{0}, \vec{r}_1, \vec{r}_2) \rangle = \langle (a^{Z_n}) - \langle (a^{Z_n}) \rangle \rangle (34) \]

where \( \langle a^{Z_n} \rangle \) is a shorthand for \( \langle a^{Z_n}(a, b, c) \rangle \). The Hamiltonian is illustrated in Fig. 2

**B. General m case**

In appendix G we show for general m the corresponding projections are given by (G1):

\[ P_{ij} = \frac{1}{n} \sum_{k=0}^{n} \hat{X}_{ij} e^{2\pi i f_{\beta\gamma} \delta _k \phi_3 [a^{Z_n}]} \]

Here \( \delta _k \phi_3 [a^{Z_n}] \) is the change in \( \phi_3 [a^{Z_n}] \) when a single link \( ij \) changes as \( a^{Z_n}_{ij} \rightarrow (aij + k)Z_n \). Under our triangulation, it is evaluated for 1-, 2-, 3- diagonal links in Appendix G.

**VI. GROUND STATE WAVEFUNCTIONS AND BOUNDARY TRANSFORMATIONS**

By Appendix C, the ground state wavefunction in closed space 3-manifold \( M^3 \) is given by

\[ |\psi_0 \rangle = \sum_{\{a^{Z_n}\}} e^{2\pi i f_{\beta\gamma} \phi_3 [a^{Z_n}]} |\{a^{Z_n}\} \rangle \]  

(35)

For physical interpretation of these wavefunctions, see Section VIII.

**A. Boundary States and their 1-symmetry transformations**

Suppose we are interested in space 3-manifold which has a boundary. We may write down a “boundary state” by separating \( \{a^{Z_n}\} = \{a^{Z_n}_{bulk}, a^{Z_n}_{\partial}\} \) into boundary and bulk links, fixing the values of \( a^{Z_n}_{\partial} \) at the boundary in (35) and only sum over links \( a^{Z_n}_{bulk} \) inside the bulk.

\[ |\{a^{Z_n}_{\partial}\} \rangle _\partial := \sum_{\{a^{Z_n}_{bulk}\}} e^{2\pi i f_{\beta\gamma} \phi_3 [a^{Z_n}_{bulk}, a^{Z_n}_{\partial}]} |\{a^{Z_n}_{bulk}, a^{Z_n}_{\partial}\} \rangle \]  

(36)

Consider a 1-symmetry transformation

\[ |a^{Z_n}\rangle \rightarrow |a'^{Z_n}\rangle = |(a + \alpha)Z_n\rangle \],

where \( a^{Z_n} \) is a \( Z_n \)-valued 1-cochain. We have

\[ |\{a^{Z_n}_{\partial}\} \rangle _\partial \rightarrow \sum_{\{a^{Z_n}_{bulk}\}} e^{2\pi i f_{\beta\gamma} \phi_3 [a^{Z_n}]} |\{a + \alpha\}Z_n\rangle \]  

(37)

with

\[ \delta _\alpha \phi [a] := \phi [a + \alpha] - \phi [a] \]

for any function \( \phi [a] \).

For 1-symmetry, we have \( d a^{Z_n} = 0 \), then

\[ -\delta _\alpha \phi_3 [a^{Z_n}] = \frac{m}{2n} a^{Z_n} da^{Z_n} + \frac{m}{2} (\alpha a^{Z_n} + a^{Z_n} \frac{d a^{Z_n}}{n}) \]  

\[ = \frac{m}{2} a^{Z_n} \frac{d a^{Z_n}}{n} + \delta _\alpha (a^{Z_n} \frac{d a^{Z_n}}{n}) \]

\[ = \frac{1}{2} d (\frac{m}{2} \frac{d a^{Z_n}}{n} + \frac{m}{2} a^{Z_n} \frac{d a^{Z_n}}{n}) \]

(39)

Assuming \( \alpha = d h^{Z_n} \) for a \( Z_n \) valued 0-cochain \( h^{Z_n} \), then the last term can be made into a total derivative:

\[ \frac{m}{2} a^{Z_n} \frac{d a^{Z_n}}{n} = \frac{m}{2} \left( \alpha \frac{d a^{Z_n}}{n} + d a^{Z_n} \frac{d a^{Z_n}}{n} \right) \]

\[ = \frac{m}{2} dh^{Z_n} \frac{d h^{Z_n}}{n} \]

\[ \delta _\alpha \phi_3 [a^{Z_n}] = d \phi_2 [a, \alpha] \]  

(40)
\[
\phi_2[a, h] := \frac{m}{2n} a^Z_n a^Z_n + \frac{m}{2} a^Z_n \sum_i \left( \frac{d\alpha^Z_n}{n} \right) \\
+ \delta_{n, a} \xi_2[a^Z_n] + \frac{m}{2} \left( h^Z_n \left[ \frac{dh^Z_n}{n} \right] \right) \tag{41}
\]

\[
= \frac{m}{2n} a^Z_n a^Z_n + \frac{m}{2} \left( \alpha^Z_n \sum_i \frac{d\alpha^Z_n}{n} + \delta_{n, a} \xi_2[a] \right) \\
+ \left[ \frac{m}{2} \left( \alpha^Z_n \sum_i \frac{d\alpha^Z_n}{n} + \delta_{n, a} \xi_2[a] \right) \right] + \frac{m}{2} \left( h^Z_n \left[ \frac{dh^Z_n}{n} \right] \right) \\
\geq \frac{m}{2n} \left( \frac{dh^Z_n}{n} \right) + \alpha^Z_n \sum_i \frac{d\alpha^Z_n}{n} + \delta_{n, a} \xi_2[a, h] \tag{42}
\]

By construction (41) we have

\[
\phi_2[a, h] = \phi_2[a^Z_n, h^Z_n]. \tag{43}
\]

(See Appendix J for relationship between \(\omega_4\), \(\phi_3\) and \(\phi_2\) in general.)

We also see that \(\int_{\partial M^3} \delta_n \phi_3[a^Z_n] \) is independent of \(a^Z_{bulk} \) or \(a^Z_{bulk} \), so we may take it out of the sum in the last line of (38) and write:

\[
\begin{align*}
|\{a^Z_n\}|_{\partial M^3} &\to e^{-2\pi i \int_{\partial M^3} \delta_n \phi_3[a^Z_n]} |\{a^Z_n\}|_{\partial M^3} \\
&= e^{2\pi i \int_{\partial M^3} \phi_2[a^Z_n, h^Z_n]} |\{a^Z_n\}|_{\partial M^3}.
\end{align*}
\tag{44}
\]

In the even case, (41) simplifies to

\[
\phi_2[a, h] = \frac{m}{2n} \alpha^Z_n a^Z_n,
\]

so the non-on-site phase for the anomalous 1-symmetry is

\[
\int_{\partial M^3} \phi_2[a^Z_n, h^Z_n] = \frac{m}{2n} \int_{\partial M^3} \alpha^Z_n a^Z_n \\
= \frac{m}{2n} \int_{\partial M^3} a^Z_n. \tag{45}
\]

Here \(\cap\) is the cap product\(^{40}\), which takes as input a q-cochain \(\phi\) and n-chain \((0 \to n)\), and outputs a \((n - q)\)-chain given by:

\[
(0 \to n) \cap \phi_q := \langle \phi_q, (0 \to q) \rangle (q \to n). \tag{46}
\]

In the more general case, by (41) and (29), the non-on-site phase is:

\[
\begin{align*}
\int_{\partial M^3} \phi_2[a^Z_n, h^Z_n] &\leq \int_{\partial M^3} \frac{m}{2n} \alpha^Z_n a^Z_n \\
+ \frac{m}{2} \left( a^Z_n \sum_i \frac{d\alpha^Z_n}{n} + \alpha^Z_n \right) \sum_i \frac{d\alpha^Z_n}{n} \\
+ \left[ a^Z_n + \alpha^Z_n \right] + \frac{m}{2} \left( h^Z_n \left[ \frac{dh^Z_n}{n} \right] \right) \tag{47}
\end{align*}
\]

where \(\alpha^Z_n = (dh^Z_n)^Z_n\).

**B. Boundary transformation strings**

On the boundary \(\partial M^3\), the 1-cocycle \(\alpha^Z_n\) is Poincaré dual to closed loops \(\partial M^3 \cap \alpha^Z_n\). These loops are the boundary of a 2-manifold \(-M^3 \cap \alpha^Z_n\) in the bulk. While the 1-symmetry is on-site in the bulk, it is non-on-site on the boundary, accompanied by the phase \(\int_{\partial M^3} \phi_2\). Since the bulk is a non-trivial SPT with 1-symmetry, we expect that its boundary cannot be uniquely gapped without breaking the 1-symmetry.

The 1-symmetry acts on the boundary as string operators. These string operators can be thought of as hopping operators for some emergent flux anyons. We measure the statistics of these anyons in the following subsection.

**C. Self and mutual statistics of boundary transformation strings**

We triangulate the 2-dimensional boundary \(\partial M^3\) as shown in Fig. 3. We only focus on a yellow central square, whose links are labeled as \(a^Z_n, i = 0, 1, 2, 3, 4\). We define string operators: \(W_i^q, i = 1, 2, 3, 4\), to be the hopping operator depicted in the bottom of Fig. 3.

Each string operator \(W_i^q\) is represented by an oriented red line in the figure. The red line intersects links in the lattice (colored in gray). Every lattice link intersecting the red string is being updated as in (37) with \(\alpha = dh^Z_n\). \(h^Z_n = q\) in the pink shaded region and \(h^Z_n = 0\) in the other unshaded regions. The operator \(W_i^q\) acts on the boundary Hilbert space as described in (44), with \(\phi_2\) given by (45) or (47).

1. **Self-statistics**

To compute the self statistics for anyon with flux \(q\), we compare the result of hopping an anyon from bottom to top, then another anyon from left to right, versus the result of hopping an anyon from bottom to right, and another anyon from left to top.\(^{13}\) As shown in Fig. 4a, the resulting positions of the two final anyons are exchanged in the two processes. More explicitly the self-statistic is given by \(\theta_q\), where

\[
W_i^q \circ W_j^q = e^{2\pi i \theta_q W_i^q \circ W_j^q}. \tag{48}
\]

Using (47) to compute the actions of \(W_i^q\), the result is (derivation details in Appendix H)

\[
\theta_q = \frac{q^2 m}{2n}, \tag{49}
\]

which is consistent with Ref. 38: The 3+1D bulk state that we have constructed is a \(Z_n\)-1-SPT state labeled by \(m \in \{0, 1, \cdots, 2n - 1\}\), protected by an on-site (anomaly-free) \(Z_n\)-1-symmetry. Its boundary has an anomalous (non-on-site) \(Z_n\)-1-symmetry generated by closed string
operators (see eqn. (45) or eqn. (47)). The corresponding open string operators will create topological excitations on the boundary. The anomaly of the boundary 1-symmetry is encoded in the fractional statistics of those topological excitations. For instance if $n = 2$, $q = 1$, then for $m = 2$, the anyon is an emergent fermion. For $m = 1$ the anyon is an emergent semion.

2. Mutual-statistics

Similarly to compute the mutual statistics for two anyons with flux $q_1$ and $q_2$, we compare the result of hopping a flux $q_1$ anyon from left to right, then the flux $q_2$ anyon from bottom to top, versus the result of doing the two processes in a different order, as illustrated in Fig. 4b. The mutual-statistic is given by $\theta_{q_1 q_2}$, where (derivation details in Appendix H)

$$W_{1}^{q_1} \circ W_{2}^{q_2} = e^{2\pi i \theta_{q_1 q_2}} W_{2}^{q_2} \circ W_{1}^{q_1},$$

and the result is

$$\theta_{q_1 q_2} = q_1 q_2 \frac{m}{n}. \quad (51)$$

VII. GAPPED SYMMETRIC BOUNDARIES

In this section we attempt to write down boundary Hamiltonians which are symmetric under the non-onsite transformation (44), and contain emergent anyons with self-statistics predicted by (49). We will show that it is possible to gap out the boundary by realizing a topological order, which in the $(n,m) = (2,1)$ case is the double semion (DS) topological order, which contains an emergent semion. In the $(n,m) = (2,2)$ case the toric code is realized on the boundary, which contains an emergent fermion. The degenerate ground states for these systems on a manifold with non-trivial cycles spontaneously break the 1-symmetry.

An easy way to see this is as follows. From $\omega_4 = $
\[ d\phi_3[a], \text{ if } M^4 \text{ has a boundary,} \]
\[ Z_{\text{top}}^0 = \frac{1}{n^{N_{\text{d}}}} \sum_{\{a_{\alpha}^2\}} \epsilon^{2\pi i} f_{\alpha M^4} \omega_4[da^{2n}] \]
\[ = \frac{1}{n^{N_{\text{d}}}} \sum_{\{a_{\alpha}^2\}} \epsilon^{2\pi i} f_{\alpha M^4} \phi_3[a^{2n}], \quad (52) \]
where \( N_{\text{d}} \) is the number of links in the space-time triangulation of the boundary.

If we impose the constraint \( da^{2n} = 0 \) by hand (the constraint doesn’t violate 1-symmetry since it is invariant under (14)), then from the expression for \( \phi_3 \) (28), we have
\[ \phi_3[a^{2n}]|_{da^{2n} = 0} = \frac{m}{2n} a^{2n} da^{2n} = \frac{m}{2} a^2 \frac{\partial^2}{\partial a^2}, \]
where in the last step we specialized to the case \( n = 2 \). This can be recognized as the Lagrangian for the surface topological order. To recast it into a more familiar form, we have
\[ \frac{\partial a^{2n}}{2} = \beta_2 a^{2n} \bar{z}_0 = S_0^1(a^2) = a^2 a^{2n}, \]
where \( \beta_2 \) is the Bockstein homomorphism and the second equality follows from (A33), and the third equality is by definition of Steenrod square (A19) and \( da^{2n} \cong 0 \). So, (52) becomes
\[ Z_{\text{top}}^0|_{da^{2n} = 0} = \frac{1}{n^{N_{\text{d}}}} \sum_{\{da^{2n}\}} \epsilon^{2\pi i} f_{\alpha M^4} \frac{\partial a^{2n}}{\partial a^{2n}} \]
\[ = \frac{1}{n^{N_{\text{d}}}} \sum_{\{da^{2n}\}} \epsilon^{2\pi i} f_{\alpha M^4} \frac{\partial a^{2n} a^{2n}}{\partial a^{2n}}, \]
which for \( m = 1 \) is (up to a volume term) the partition function for double semion topological order (see for instance Ref. 41). For \( m = 2 \) the Lagrangian \( \hat{Z} \equiv 0 \) and describes the \( Z_2 \) gauge theory, i.e. toric code.

A. Engineering boundary gapped Hamiltonian

Alternatively, we can explicitly engineer a gapped Hamiltonian consisting of mutually commuting terms on the boundary Hilbert space respecting the anomalous 1-symmetry and realizing the DS topological order.

The following boundary Hamiltonian is proposed:
\[ H_\theta = - \sum_i H_{s,i} - \sum_\Delta H_{p,\Delta} \quad (53) \]
\[ H_{s,i} := W_{\hat{O}, i}, \quad \]
\[ H_{p,\Delta} := \delta_{(da, \Delta)^{2n}, 0}. \]
Here \( i \) is summed over all sites and \( \Delta \) is summed over all 2-simplices \( \text{(i.e. triangles)} \) in the boundary. \( \delta \) is the Kronecker delta function. \( \langle da, \Delta \rangle \) is evaluating the 2-cochain \( da \) on the 2-simplex \( \Delta \). Hence \( H_{p,\Delta} \) enforces the “no flux” constraint \( da \cong 0 \) on every 2-simplices. \( W_{\hat{O}, i} \) is the 1-symmetry operator corresponding to a tiny loop surrounding site \( i \) (see Fig. 5).

1. \( H_\theta \) is exactly solvable and has 1-symmetry

To show that \( H_\theta \) consists of mutually commuting terms, which also commutes with the boundary 1-symmetry operators, it suffices to check the following commutators vanishes:
\[ [H_{p,\Delta}, W(h^{2n})] = 0 \quad (54) \]
\[ [W_{\hat{O}, i}, W(h^{2n})] = 0 \quad (55) \]
for any \( Z_2 \)-valued 0-cochain \( h^{2n} \), where \( W(h^{2n}) \) denotes a 1-symmetry operator parameterized by \( h^{2n} \), whose action is described by (44) with \( a_{\alpha}^{2n} = (a_\alpha + \delta h)^{2n} \).

To show (54), notice that
\[ W(h^{2n})^{-1} H_{p,\Delta} W(h^{2n}) \]
\[ = W(h^{2n})^{-1} \delta_{(da, \Delta)^{2n}, 0} W(h^{2n}) \]
\[ = \delta_{(d(a + dh), \Delta)^{2n}, 0} = \delta_{(da, \Delta)^{2n}, 0} = H_{p,\Delta}, \]
where we used the fact that the non-onsite phases from \( W(h^{2n}) \) and \( W(h^{2n})^{-1} \) cancels, since \( \delta_{(da, \Delta)^{2n}, 0} \) does not change the value of \( a^{2n} \) in the ket.

To show (55), notice that for any two \( Z_2 \)-valued 0-cochain \( h_{1}^{2n} \) and \( h_{2}^{2n} \), we have
\[ W(h_{1}^{2n})^{-1} W(h_{2}^{2n}) W(h_{1}^{2n}) \]
\[ \quad = \exp \left[ 2\pi i \int_{\partial M^3} \phi_2((a + dh_1)^{2n}, h_{1}^{2n}) - \phi_2(a^{2n}, h_{2}^{2n}) \right] \]
\[ + \phi_2((a + dh_2)^{2n}, h_{2}^{2n}) + \phi_2(a^{2n}, h_{1}^{2n}) \]
\[ \quad = \exp \left[ 2\pi i \int_{\partial M^3} \phi_3(a^{2n}, h_{1}^{2n}, h_{2}^{2n}) \right] \]
\[ \quad = \exp \left[ 2\pi i \int_{\partial M^3} \delta_{(da, \Delta)^{2n}, 0} \phi_3(a^{2n}, h_{1}^{2n}, h_{2}^{2n}) \right] \]
\[ \quad = \exp \left[ 2\pi i \int_{\partial M^3} \phi_3(a^{2n}, h_{1}^{2n}, h_{2}^{2n}) \right] \]
where we applied (43) and (42) in the last step to show that the integrand in the exponent is a total derivative:
\[ \phi_2((a + dh_1)^{2n}, h_{2}^{2n}) - \phi_2(a^{2n}, h_{2}^{2n}) - (h_1 \leftrightarrow h_2) \]
\[ = \phi_2(a + dh_1, h_2) - \phi_2(a, h_2) - (h_1 \leftrightarrow h_2) \]
\[ = \frac{m}{2n} dh_1(a + dh_1) + \delta_{dh_1, \xi_2}(a + dh_1) + d\xi_1(a + dh_1, h_2) \]
\[ - \frac{m}{2n} dh_2(a + dh_2) - \delta_{dh_2, \xi_2}(a - d\xi_1(a, h_2) - (h_1 \leftrightarrow h_2) \]
\[ = \frac{m}{2n} dh_1 h_2 + \delta_{dh_1, \xi_1}(a, h_2) - (h_1 \leftrightarrow h_2) \]
\[ = \phi_3(a^{2n}, h_{1}^{2n}, h_{2}^{2n}) \]
where
\[ \phi_3(a, h_1, h_2) := \frac{m}{2n} h_{2}^{2n} dh_{1}^{2n} + \xi_1[a^{2n} + dh_{1}^{2n}, h_{2}^{2n}] \]
\[ - \xi_1[a^{2n}, h_{2}^{2n}] - (h_1 \leftrightarrow h_2) \quad (57) \]
\[ = \frac{m}{2n} h_2 dh_1 + \delta_{dh_1, \xi_1}(a, h_2) + d\xi_0[h_1, h_2] \]
\[ - (h_1 \leftrightarrow h_2) \]
\[ \xi_0[h_1, h_2] := \frac{m}{2n} \left( \frac{h_2}{n} - h_1 \right) + h_1 \frac{h_2}{n} \].
where

\[ h(z_n) = 1. \]

In general, for any \( W \) in general.

For the case of our interest (55), we may take

\[ h_{a} = h_{i} \]

where \( W_{O} = W(h_{i}^{Z_{n}}) \) as depicted in Fig. 5, and \( h_{i}^{Z_{n}} = h_{i}^{Z_{n}} \)

to be an arbitrary 1-symmetry. Alternatively we can evaluate (56) by integrating the exponent over a patch covering the region where \( h_{O}^{Z_{n}} \neq 0 \) and use

\[ \phi_{1}[a^{Z_{n}}, h_{i}^{Z_{n}} = 0, h_{i}^{Z_{n}}] = 0. \]

2. Topological ordered surface states for \( n = 2 \)

We can specialize to the case \( (n, m) = (2, 1) \) and evaluate \( W_{O} \).

Assuming “no flux” constraint is enforced, we have (see Appendix I for details)

\[
W_{O} = \prod_{(j', j)} (-1)^{a_{ij}a_{ij'}} \left| \{(a_{ij}, 1)^{Z_{n}}, a_{ij'}^{Z_{i}} \} \right|,
\]

where \( j, j' \in \{1, \ldots, 6\} \) are neighboring sites of \( i \) (see Fig. 5). The product is taken over six links with neighboring \( j, j' \). The resulting \( H_{D} \) gives rise to DS topological order.

For the \( m = 2 \) case, we have

\[
W_{O} = \prod_{(j', j)} (-1)^{a_{ij}a_{ij'}} \left| \{(a_{ij}, 1)^{Z_{n}}, a_{ij'}^{Z_{i}} \} \right|.
\]

So \( h_{a,i} \) is the usual star term and \( h_{a,i} \) is the usual plaque term for the toric code model. Thus \( H_{D} \) gives rise to the toric code topological order.

3. Connection to Works of Wan and Wang

A general theory of gapped symmetric boundaries of higher SPT is presented in Section III of Ref. 42, which is a generalization of Ref. 41. It was then applied in Section IX of Ref. 43, and Section 8 of Ref. 44, which also contains a lattice Hamiltonian description for a 4+1D bulk/3+1D boundary.

We give a rough review of their result in the following. In general, a 1-SPT protected by 1-form finite symmetry \( \Pi_{2} \) (which is Abelian) and 0-form finite symmetry \( G \), may be associated with a “2-group” \( \mathcal{G} \), such that its classifying space \( BG = B(G, \Pi_{2}) \) has \( \sigma_{1} = G, \sigma_{2} = \Pi_{2} \) and \( \pi_{0} = \sigma_{k,2} = 0 \). In addition, \( \mathcal{G} \) also contains the data of \( \omega_{2} : G \to \text{Aut}(\Pi_{2}) \) and \( n_{2} \in H^{3}(BG, \Pi_{2}^{2}) \) describing the interplay between \( G \) and \( \Pi_{2} \). A space-time field configuration is a map \( \phi : M^{D} \to BG \), and the cocycle \( \omega_{D} \) describing the 1-SPT is the pullback: \( \omega_{D} = \phi^{*} \omega_{D} \) for a topological term \( \omega_{D} \), which can be an element of \( H^{D}(BG, U(1)) \), or a bordism invariant in general. Section III of Ref. 42 claimed that the gapped boundary of 1-SPT corresponds to a fibration:

\[ Bk \to BH \to BG \]

such that the topological invariant \( \omega_{D} \) in \( BG \) is pulled back to a trivial topological invariant in \( BH \). Here \( H \) is a 2-group, viewed as an extension of \( G \). \( Bk \) is the total space of a fibration \( B^{2}K[1] \to Bk \to B_{0}[1] \), where \( K[0], K[1] \) are some 0-form and 1-form symmetries respectively. For a finite group \( G, B^{2}G = K(G, 2) \) is an Eilenberg-MacLane space for which the only non-trivial homotopy group is \( \pi_{2} = G \). To be precise, the topological invariants of the classifying spaces \( BG, BH \) are bordism invariants of the bordism groups \( \Omega_{D} \), \( \Omega_{D}^{H}(BG), \Omega_{D}^{H}(BH) \), computed with respect to an “S-structure”, \( S_{G,H} = SO/O/Spin/Pin^{\dagger} \), corresponding to unitary bosonic SPT/time-reversal invariant bosonic SPT/unitary fermionic SPT/time-reversal invariant fermionic SPT with \( T^{2} = (\mp)^{F} \), respectively.

In this framework, our \( \mathcal{Z}_{2} \)-1-SPT has \( D = 4 \), \( (G, \Pi_{2}) = (0, Z) \) and \( S_{G} = SO \). Gapping out its \( 2+1D \) boundary for \( m = 2 \) with toric code topological order corresponds to the fibration:

\[ BZ_{2} \to BSpin(4) \times B^{2}Z_{2} \to BSO(4) \times B^{2}Z_{2} \]

where the pullback of \( \omega_{4} \) is trivial because of a relation between \( Z_{2} \)-valued 2-cocycle \( B^{2}Z_{2} \) and the Stiefel-Whitney classes \( w_{1}, w_{2} \) (which is derived using Wu formula, eq of Appendix D.5 of Ref. 38):

\[ S_{G}^{2}(B^{2}Z_{2}) = (w_{1}^{2} + w_{2})B^{2}Z_{2}, \]

where \( w_{1}, w_{2} \) vanishes when pulled back to \( BSpin(4) \times B^{2}Z_{2} \), which is a spin manifold. The emergent fermion is due to the emergent spin structure.

VIII. GEOMETRIC INTERPRETATION OF GROUND STATE WAVEFUNCTION

In this section we attempt to provide an intuitive interpretation of the ground state wavefunction (35) on a closed 3-manifold.
Recall from (35) and (28), the ground state wavefunction is
\[
|\psi_0\rangle = \sum_{\{a^2_n\}} e^{2\pi i \int_{M^3} \phi_3[a^2_n]} |\{a^2_n\}\rangle
\]
\[
\phi_3[a] = \frac{m}{2n} a da + \frac{m}{2} da \sim \left[ \frac{da}{n} \right] + d\xi_2[a].
\]
In a closed 3-manifold $M^3$, we can ignore the $d\xi_2$ term. In the dual manifold $\tilde{M}^3$, $a$ is dual to 2-chains $\tilde{a}$, and $da$ is dual to $\partial a$, which is a 1-cycle.

If we focus on the term $\frac{m}{2n} a da$, which only depends on 1-diagonal links, we can imagine the dual 2-chains and 1-cycles as living on the dual faces and links of a simple cubic lattice. Geometrically, $\frac{m}{2n} a da$ is contributed from the intersections of $\tilde{a}$ and $\partial a'$, which is $\partial a$ displaced by the framing vector $-\frac{1}{2} = (-1/2, -1/2, -1/2)$.

\[
\int_{M^3} \frac{m}{2n} a da = \sum_{p \in \tilde{\partial} a \cap \partial a'} \frac{m}{2n} q_{a,p} q_{\partial a',p},
\]
where $q_{a,p}, q_{\partial a',p} \in Z$ denote the integer coefficients of the 2-chain $\tilde{a}$ and 1-cycle $\partial a'$ at the intersection point $p$.

If the 1-cycle $\partial a$ can be resolved into non-intersecting loops $K_i$, then $\tilde{a}$ are the Seifert surfaces $S_i$ for these loops. A Seifert surface of loop $K_i$ is an oriented surface with $K_i$ as its boundary. It is known that the signed intersection number between $K_i$ and a Seifert surface of $K'_j$ is the sum of signed crossings between $K_i$ and $K'_j$ (viewed from the $-\frac{1}{2}$ direction), which is the linking number $Lk(K_i, K'_j)^{47}$. Thus
\[
\int_{M^3} \frac{m}{2n} a da = \sum_{i,j} \sum_{S_i \cap K'_j} \frac{m}{2n} q_i q_j
\]
\[
= \frac{m}{2n} \sum_{i,j} q_i q_j Lk(K_i, K'_j)
\]
\[
= \frac{m}{2n} \sum_i q_i^2 w(K_i) + \frac{m}{n} \sum_{i<j} q_i q_j Lk(K_i, K_j),
\]
where $w(K_i) = Lk(K_i, K'_j)$ is the self-linking number of $K_i$, and for $i \neq j$, $Lk(K_i, K'_j) = Lk(K_i, K_j)$ is the linking number between $K_i$ and $K_j$. $q_i \in Z$ denote the “strength” of each loop $K_i$. Note the result (59) is invariant (mod 1) under $q_i \rightarrow q_i + m u_i$ for any integers \{u_i\} for general $m$. For example in Figure 6(a), we see that for an unknot with self linking number +1 carrying flux $da = q$, $\int_{M^3} \frac{m}{2n} a da = \frac{m}{2n} q^2$. In Figure 6(b), for the Hopf link with linking number 1, with two loops carrying flux $da = q_1$ and $q_2$, we have $\int_{M^3} \frac{m}{2n} a da = \frac{m}{2n} q_1 q_2$. This could be regarded as an alternative way to derive the self-statistics (49) and mutual-statistics (51) of the boundary transition strings, from the 3d bulk space perspective instead of the 2+1d boundary spacetime perspective.

However, when multiple lines intersect at a point, we need to carefully resolve the 1-cycle $\partial a$ into non-intersecting loops. We will consider the even $m$ case and the odd $m$ case separately.

**A. Even $m$ case**

In the case of even $m$, (28) is
\[
\phi_3[a] = \frac{m}{2n} a da.
\]

Each lattice point of the dual cubic lattice has six connecting dual links. Given a dual cycle configuration $\partial a$, we project these six links onto the plane perpendicular to the $-\frac{1}{2}$ framing vector. Then we resolve the intersection into disjoint loops with “no-crossing” resolution: requiring that no crossing occurs in this intersection. An example is shown in Fig. 7.

Since all the crossings are contributed away from intersections, the wavefunction amplitude (59) is
\[
\int_{M^3} \phi_3[a] = \frac{m}{2n} \sum_i q_i^2 w(K_i) + \frac{m}{n} \sum_{i<j} q_i q_j Lk(K_i, K_j),
\]
with $K_i$ obtained from $\partial a$ by “no-crossing” resolution at each vertex.

**B. Odd $m$ case**

In the case of odd $m$, (28) is
\[
\phi_3[a] = \frac{m}{2n} a da + \frac{m}{2} da \sim \left[ \frac{da}{n} \right].
\]

As explained previously,
\[
\int_{M^3} \frac{m}{2n} a da = \frac{m}{2n} \sum \left( \text{signed crossings away from intersections} \right).
\]
In the following we will also interpret the second term as $\frac{m}{2n} \times$ the sum of signed crossings under a “quotient-remainder” resolution at intersections.

Since the term $\frac{m}{2} da \sim \lfloor \frac{da}{n} \rfloor$ depends on 2- and 3-diagonal links, we need to use the full triangulation described in Appendix D with six tetrahedrons per unit cubic cell. The dual lattice is a cubic lattice with six sites forming a hexagon in each unit cell, depicted in Fig. 8(a).

The “quotient-remainder” resolution is the following: write $da = (da)^Z_n + n \lfloor \frac{da}{n} \rfloor$. These two terms are called the “remainder” and “quotient” respectively. The 1-cycles dual to $da$ live on the links of the dual lattice. We split each link in the dual lattice into two channels: the “remainder” channel dual to $(da)^Z_n$, and the “quotient” channel dual to $n \lfloor \frac{da}{n} \rfloor$. They are depicted as black and red links respectively in Fig. 8(b). If we detach the “quotient” intersections from the “remainder” intersections by displacing them slightly towards the center of each cube, then $da \sim n \lfloor \frac{da}{n} \rfloor$ is the sum of signed crossings (mod 2n) between the “remainder” channels and the “quotient” channels, viewed from $\frac{1}{2}$, as depicted in Fig. 8(b). All other intersections (black and red dots in Fig. 8(b)) are resolved with the “no-crossing” resolution.

Thus $\frac{m}{2} da \sim \lfloor \frac{da}{n} \rfloor$ is $\frac{m}{2n} \times$ sum of signed crossings between quotient channels and remainder channels at a vertex. As before, the sum of signed crossings is the sum of linking numbers between resolved loops. Hence the wavefunction amplitude (59) is

$$\int_{M^3} \phi_3[a] = \frac{m}{2n} \sum_i q_i^2 w(K_i) + \frac{m}{n} \sum_{i<j} q_i q_j Lk(K_i, K_j).$$

with $K_i$ obtained from $\partial a$ by “quotient-remainder” resolution at each vertex.

Since the term $\frac{m}{2} da \sim \lfloor \frac{da}{n} \rfloor$ is necessary to ensure that $\phi_3[a]$ is invariant mod 1 under $a \rightarrow a + nu$ for $Z$-valued 1-chain $u$. Indeed under $a \rightarrow a + nu$, all changes occur only in the quotient channel $n \lfloor \frac{da}{n} \rfloor \rightarrow n \lfloor \frac{da}{n} \rfloor + nu$. The change to $\phi_3[a]$ mod 1 is $\frac{1}{2} \times$ the sum of signed crossings between the dual of $(da)^Z_n$ in remainder channel and the dual of $du$ in the quotient channel. Since both of them are closed loops living in separate channels, the total number of signed crossing is even. Hence $\phi_3[a]$ is invariant mod 1.

**IX. NON-ZERO BACKGROUND GAUGE FIELD**

We may also extend our derivations to the case where the background gauge field $\hat{B}$ in (12) is non-zero. By keeping track of the coboundary terms in (16), it can be shown that (26),(27),(28),(29) become

$$\omega_4[\hat{B} + da]^Z_n = \frac{m}{2n} \xi q^2 \hat{B}^Z_n + d\phi_3[a, \hat{B}]$$

$$= \omega_4[\hat{B}^Z_n] + d\phi_3[a, \hat{B}]$$

where

$$\phi_3[a, \hat{B}] := \frac{m}{2n} \left( a^n da^n + a^n \hat{B}^Z_n + \hat{B}^Z_n a^n \right) + \xi_3[a^Z_n, \hat{B}^Z_n].$$

(60)
\[ \xi_3[a, \hat{B}] := \frac{m}{2n} \left( \hat{B} + da \right) \left\{ \frac{\hat{B} + da}{n} \right\} + a \left\{ \frac{a}{n} \right\} \]

\[ \xi_2[a, \hat{B}] := \frac{m}{2n} \left( \frac{a}{n} \right) + \frac{d}{n} \left\{ \frac{a}{n} \right\} + a \left\{ \frac{\hat{B}}{n} \right\} + \hat{B} \left\{ \frac{a}{n} \right\} \]

and (39), (40), (41), (42) become

\[ -\delta \phi_3[a, \hat{B}^2] = m \left( \frac{a - a^2}{2n} \right) \]

\[ + m \left( \frac{a}{n} + \delta \xi_2[a, \hat{B}] + \hat{B} \left\{ \frac{a}{n} \right\} \right) \]

\[ - m \left( \frac{a}{n} \right) + \frac{m}{2} \left\{ \frac{a}{n} \right\} = d \phi_2[a, h, \hat{B}] \]

Where

\[ \phi_2[a, h, \hat{B}] := m \left( \frac{a}{n} - \frac{a^2}{2n} \right) \]

\[ + m \left( \frac{a^2}{n} + \frac{d}{2} \frac{a}{n} + \delta \xi_2[a, \hat{B}] + \hat{B} \left\{ \frac{a}{n} \right\} \right) \]

\[ - m \frac{a}{n} + m \frac{d}{n} \left\{ \frac{a}{n} \right\} \]

\[ + m \frac{d}{2n} \left( dh - dh \right) \]

\[ \xi_1[a, h] := \frac{m}{2n} \left( \frac{d}{n} \right) \]

In Appendix B 2, we generalize the construction of an exactly soluble Hamiltonian with an unique ground state to the case of non-zero \( \hat{B} \). Also in Appendix C 2 we generalize the expression of the ground state wavefunction (C3) in terms of \( \phi_3 \):

\[ |\psi_0[\hat{B}]\rangle = \frac{1}{N^2} \sum_{\alpha} e^{2\pi i f M_3 \phi_3[a, \hat{B}] - \phi_3[0, \hat{B}]} |\alpha\rangle \]

In the following we consider the case \( m \) is even, where (60), (62) simplifies to

\[ \phi_3[a, \hat{B}] \]

\[ \phi_2[a, h, \hat{B}] \]

\[ \phi_1[a, h] \]

A. Exactly Solvable Hamiltonian

The bulk Hamiltonian is given by

\[ H = - \sum P_{ij}[\hat{B}] \]

By using (C4) to write down matrix elements of \( P_{ij}[\hat{B}] \), it can be shown that \( P_{ij}[\hat{B}] \) are the same as (33),

\[ P_{ij}[\hat{B}] = \frac{1}{n} \sum_{k=0}^{n} \delta_{ij} e^{2\pi i \frac{m}{n} a \hat{B} + \hat{B} a} \]

except that in the definition (34) of the flux \( F \), \( da \) is replaced by \( \hat{B} + da \):

\[ F_{\beta\gamma}(\hat{r}) := ((\hat{B} + da)^2, \hat{a} \hat{B} + \hat{B} + \hat{B} a) \]

\[ \alpha \leftrightarrow \gamma \]

(66)

B. Geometric interpretation of wavefunction

In (64), the background gauge field \( \hat{B} \) is coupled to \( a \) through the extra terms

\[ \frac{m}{2n} (a \hat{B} + \hat{B} a) \]

Geometrically, in 3d space, the 2-cocycle gauge field \( \hat{B} \) is dual to a 1d line \( \hat{B} \). We may shift these lines in the \( \pm \hat{B} \) directions to obtain \( \hat{B} \). Then the extra terms (67) contribute a phase \( e^{2\pi i \frac{m}{n}} \) to every signed intersections between \( \hat{B} \) and \( \hat{a} \) (Recall \( \hat{a} \) is the surface dual to \( a \)).

For simplicity let’s pretend \( \hat{a} \) will not fluctuate too wildly near the intersection, and so \( B_{\pm} \) gives the same number of signed interactions as \( \hat{B} \), then the extra terms contribute a phase \( e^{2\pi i \frac{m}{n}} \) for every such intersection.

\[ \int_{M^3} \frac{m}{2n} (a \hat{B} + \hat{B} a) \approx \frac{m}{n} \sum \left( \text{signed intersections} \right) \]

\[ \text{between } \hat{B} \text{ and } \hat{a} \]

We may interpret such phase as a charge attachment to \( \hat{B} \). In 1-SPT, charged objects are 1-dimensional: a charge \( k \) line pick up a phase \( e^{2\pi i \frac{m}{n}} \) for every intersection with a unit-shift (i.e. acting by the generator of \( \mathbb{Z}_n \)) 1-symmetry membrane operator. Charge lines live on the original lattice.

Thus in a \( \mathbb{Z}_n \)-SPT labeled by \( m \), consider the 1-symmetry transformation \( |\alpha\rangle \rightarrow |\{ \alpha + \alpha \} \rangle \rightarrow |\{ \alpha \} \rangle \), where \( \alpha \) is a unit-shift acting on a membrane intersecting \( \hat{B} \) once. We have as in (38)

\[ |\psi_0[\hat{B}]\rangle = \frac{1}{N^{2}} \sum_{\alpha} e^{2\pi i f M_3 \phi_3[a, \hat{B}] - \phi_3[0, \hat{B}]} |\alpha\rangle \]

\[ \alpha \rightarrow \{ \alpha + \alpha \} \rightarrow |\alpha \rangle \}

\[ e^{2\pi i f M_3 \phi_3[a', \hat{B}] - \phi_3[0, \hat{B}] - \delta \phi_3[a, \hat{B}]}} |\{ \alpha \} \rangle \]

using (61) and assuming \( \partial M^3 = \emptyset \). Hence the ground state wavefunction picks up a phase \( e^{-2\pi i \frac{m}{n}} \) due to the background gauge field. Thus the dual of the background gauge fields \( \hat{B} \) (with unit gauge strength) is attached a charge \(-m \text{ line}(\text{located at } \hat{B}_{\pm}, \text{ to be exact}).\)

C. Boundary perspective

We can alternatively consider the effect of background gauge field from a boundary perspective. Consider the
ary string operators create topological excitations at the closed string operators. We showed that those bound-

ary theory in 2+1-dimensions. The effective boundary state wavefunction. We also studied the effective bound-

ary theory, e.g., the background gauge field. \[ \hat{X} \] is denoted by a cross \( \times \). From the boundary, the endpoint of \( \hat{B} \) (depicted as a black dot) is enclosed in a region \( h^2 n = 1 \) (shaded in pink), where \( \alpha = dh \), and \( h = 0 \) outside the pink region. The line \( B \) with unit gauge strength acquires a phase \( e^{-2\pi i Z/\alpha} \) under a unit shift 1-symmetry \( Z \).

\[ \int_{\partial M^3} \phi_2[a_\alpha, h, \hat{B}] - \int_{\partial M^3} \phi_2[a_\alpha, h, 0] \]

\[ = \int_{\partial M^3} -\frac{m}{n} \hat{B} h = -\frac{m}{n} \]

contributes to the boundary transformation (44), compared to the case without background gauge fields. The boundary state hence acquires a phase \( e^{-2\pi i Z/\alpha} \) due to the background gauge field. i.e. the endpoint of \( B \) has charge \(-m\) under the boundary 1-symmetry. We observe that the above boundary argument extends to the odd \( m \) case as well. (68) still holds by inspecting (62) and again assuming \( dh = 0 \) near the end point where \( \hat{B} \neq 0 \). So we expect the same charge attachment also occurs for odd \( m \).

**X. CONCLUSIONS**

In this paper we studied the \( \mathbb{Z}_n \)-1-symmetry protected topological states in 3+1-dimensions, which is labeled by \( m \in \{0, 1, \cdots, 2n-1\} \). The \( \mathbb{Z}_n \)-1-symmetry is generated by closed membrane operators. We presented an exactly solvable Hamiltonian which commutes with the closed membrane operators, and wrote down the ground state wavefunction. We also studied the effective boundary theory in 2+1-dimensions. The effective boundary theory has an anomalous \( \mathbb{Z}_n \)-1-symmetry generated by closed string operators. We showed that those boundary string operators create topological excitations at the string ends, which may have non-trivial self-statistics. In particular for the \( n = 2 \) case, they have self-semantic (for \( m = 1 \)) or fermionic statistics (for \( m = 2 \)). In these cases we can gap out the boundary with an engineered boundary Hamiltonian with the anomalous \( \mathbb{Z}_n \)-1-symmetry, which gives the same ground state as the toric code model (for \( m = 2 \)) and double-semantic model (for \( m = 1 \)) on the boundary. We interpreted the wavefunction amplitudes of the bulk grounds states as linking numbers of strings in the dual lattice. Finally we extend to the case of non-zero background gauge field and find the lines dual to the background gauge field is attached with line charge \(-m\).

In the future, we would like to study the nature of the gapless boundary states. It is also interesting to see whether other knot invariants can be derived from the wavefunction amplitude for other 1-SPT’s.

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**Appendix A: Space-time complex, cochains, and cocycles**

In this paper, we consider models defined on a space-time lattice. A spacetime lattice is a triangulation of the \( D \)-dimensional spacetime \( M^D \), which is denoted by \( \mathcal{M}^D \). We will also call the triangulation \( \mathcal{M}^D \) as a spacetime complex, which is formed by simplices – the vertices, links, triangles, etc. We will use \( i, j, \cdots \) to label vertices of the spacetime complex. The links of the complex (the 1-simplices) will be labeled by \( (i, j), (j, k), \cdots \). Similarly, the triangles of the complex (the 2-simplices) will be labeled by \( (i, j, k), (j, k, l), \cdots \).

In order to define a generic lattice theory on the spacetime complex \( \mathcal{M}^D \) using local Lagrangian term on each simplex, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure.\(^5,48,49\) A branching structure is a choice of orientation of each link in the \( D \)-dimensional complex so that there is no oriented loop on any triangle (see Fig. 10).

The branching structure induces a local order of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming links, and the second vertex is the vertex with only one incoming link, etc. So the simplex in Fig. 10a has the following vertex ordering: 0, 1, 2, 3.

The branching structure also gives the simplex (and its sub-simplices) a canonical orientation. Fig. 10 illustrates two 3-simplices with opposite canonical orientations compared with the 3-dimension space in which they are embedded. The blue arrows indicate the canonical orientations of the 2-simplices. The black arrows indicate the
canonical orientations of the 1-simplices.

Given an Abelian group $(\mathbb{M}, +)$, an $n$-cochain $f_n$ is an assignment of values in $\mathbb{M}$ to each $n$-simplex, for example a value $f_{n;i,j,\ldots,k} \in \mathbb{M}$ is assigned to $n$-simplex $(i, j, \ldots, k)$. So a cochain $f_n$ can be viewed as a bosonic field on the spacetime lattice. $\mathbb{M}$ can also be viewed a $\mathbb{Z}$-module (i.e. a vector space with integer coefficient) that also allows scaling by an integer:

$$x + y = z, \quad x * y = z, \quad m x = y, \quad x, y, z \in \mathbb{M}, \quad m \in \mathbb{Z}.$$  
(A1)

The direct sum of two modules $\mathbb{M}_1 \oplus \mathbb{M}_2$ (as vector spaces) is equal to the direct product of the two modules (as sets):

$$\mathbb{M}_1 \oplus \mathbb{M}_2 \equiv \text{set} \{ \mathbb{M}_1 \times \mathbb{M}_2 \}.$$  
(A2)

We like to remark that a simplex $(i, j, \ldots, k)$ can have two different orientations. We can use $(i, j, \ldots, k)$ and $(j, i, \ldots, k) = -(i, j, \ldots, k)$ to denote the same simplex with opposite orientations. The value $f_{n;i,j,\ldots,k}$ assigned to the simplex with opposite orientations should differ by a sign: $f_{n;i,j,\ldots,k} = -f_{n;j,i,\ldots,k}$. So to be more precise $f_n$ is a linear map $f_n: n$-simplex $\to \mathbb{M}$. We can denote the linear map as $\langle f_n, n$-simplex $\rangle$, or

$$\langle f_n, (i_0 i_1 i_2 \ldots i_{n+1}) \rangle = f_{n;i_0 i_1 i_2 \ldots i_{n+1}} \in \mathbb{M}.$$  
(A3)

More generally, a cochain $f_n$ is a linear map of $n$-chains:

$$f_n: n$-chains $\to \mathbb{M},$$  
(A4)

or (see Fig. 11)

$$\langle f_n, n$-chain $\rangle \in \mathbb{M},$$  
(A5)

where a chain is a composition of simplices. For example, a 2-chain can be a 2-simplex: $(i, j, k)$, a sum of two 2-simplices: $(i, j, k) + (j, k, l)$, a more general composition of 2-simplices: $(i, j, k) - 2(j, k, l)$, etc. The map $f_n$ is linear respect to such a composition. For example, if a chain is $m$ copies of a simplex, then its assigned value will be $m$ times that of the simplex. $m = -1$ correspond to an opposite orientation.

We will use $C^n(\mathcal{M}^D; \mathbb{M})$ to denote the set of all $n$-cochains on $\mathcal{M}^D$. $C^n(\mathcal{M}^D; \mathbb{M})$ can also be viewed as a set all $\mathbb{M}$-valued fields (or paths) on $\mathcal{M}^D$. Note that $C^n(\mathcal{M}^D; \mathbb{M})$ is an Abelian group under the $+$-operation.

The total spacetime lattice $\mathcal{M}^D$ correspond to a $D$-chain. We will use the same $\mathcal{M}^D$ to denote it. Viewing $f_D$ as a linear map of $D$-chains, we can define an “integral” over $\mathcal{M}^D$:

$$\int_{\mathcal{M}^D} f_D \equiv \langle f_D, \mathcal{M}^D \rangle = \sum_{(i_0, i_1, \ldots, i_D)} s_{i_0 i_1 \ldots i_D}(f_D)_{i_0, i_1, \ldots, i_D}.$$  
(A6)

Here $s_{i_0 i_1 \ldots i_D} = \pm 1$, such that a $D$-simplex in the $D$-chain $\mathcal{M}^D$ is given by $s_{i_0 i_1 \ldots i_D}(i_0, i_1, \ldots, i_D)$.

We can define a derivative operator $\mathcal{D}$ acting on an $n$-cochain $f_n$, which give us an $(n + 1)$-cochain (see Fig. 11):

$$\langle df_n, (i_0 i_1 i_2 \ldots i_{n+1}) \rangle = \sum_{m=0}^{n+1} (-)^m (f_n, (i_0 i_1 i_2 \ldots \hat{i}_m \ldots i_{n+1}))$$  
(A7)

where $i_0 i_1 i_2 \ldots \hat{i}_m \ldots i_{n+1}$ is the sequence $i_0 i_1 i_2 \ldots i_{n+1}$ with $i_m$ removed, and $i_0, i_1, i_2 \ldots i_{n+1}$ are the ordered vertices of the $(n + 1)$-simplex $(i_0 i_1 i_2 \ldots i_{n+1})$.

A cochain $f_n \in C^n(\mathcal{M}^D; \mathbb{M})$ is called a cocycle if $df_n = 0$. The set of cocycles is denoted by $Z^n(\mathcal{M}^D; \mathbb{M})$. A cochain $f_n$ is called a coboundary if there exist a cochain $f_{n-1}$ such that $d f_{n-1} = f_n$. The set of coboundaries is denoted by $B^n(\mathcal{M}^D; \mathbb{M})$. Both $Z^n(\mathcal{M}^D; \mathbb{M})$ and
$B^n(M^D; \mathfrak{m})$ are Abelian groups as well. Since $d^2 = 0$, a coboundary is always a cocycle: $B^n(M^D; \mathfrak{m}) \subset Z^n(M^D; \mathfrak{m})$. We may view two cocycles differ by a coboundary as equivalent. The equivalence classes of cocycles, $[f_n]$, form the so called cohomology group denoted by

$$H^n(M^D; \mathfrak{m}) = Z^n(M^D; \mathfrak{m})/B^n(M^D; \mathfrak{m}),$$

(A8)

$H^n(M^D; \mathfrak{m})$, as a group quotient of $Z^n(M^D; \mathfrak{m})$ by $B^n(M^D; \mathfrak{m})$, is also an Abelian group.

For the $\mathbb{Z}$-valued cocycle $x_n$, $dx_n \equiv 0$. Thus

$$\beta_N x_n \equiv \frac{1}{N} dx_n \quad \text{(A9)}$$

is a $\mathbb{Z}$-valued cocycle. Here $\beta_N$ is Bockstein homomorphism.

We notice the above definition for cochains still makes sense if we have a non-Abelian group $(G, \cdot)$ instead of an Abelian group $(\mathfrak{m}, +)$, however the differential $d$ defined by eqn. (A7) will not satisfy $d^2 = 1$, except for the first two $d$’s. That is, one may still make sense of 0-cocycle and 1-cocycle, but no more further naively by formula eqn. (A7). For us, we only use non-Abelian 1-cocycle in this article. Thus it is ok. Non-Abelian cohomology is then thoroughly studied in mathematics motivating concepts such as gerbes to enter.

From two cochains $f_m$ and $h_n$, we can construct a third cochain $p_{m+n}$ via the cup product (see Fig. 12):

$$p_{m+n} = f_m \cup h_n,$$

$$\langle p_{m+n}, (0 \to m + n) \rangle = \langle f_m, (0 \to m) \rangle \times \langle h_n, (m \to m + n) \rangle, \quad \text{(A10)}$$

where $i \to j$ is the consecutive sequence from $i$ to $j$:

$$i \to j \equiv i, i+1, \cdots, j-1, j. \quad \text{(A11)}$$

Note that the above definition applies to cochains with global.

The cup product has the following property

$$d(h_n \cup f_m) = (dh_n) \cup f_m + (-)^n h_n \cup (df_m) \quad \text{(A12)}$$

for cochains with global or local values. We see that $h_n \cup f_m$ is a cocycle if both $f_m$ and $h_n$ are cocycles. If both $f_m$ and $h_n$ are cocycles, then $f_m \cup h_n$ is a coboundary if one of $f_m$ and $h_n$ is a coboundary. So the cup product is also an operation on cohomology groups $\cup: H^m(M^D; \mathfrak{m}) \times H^n(M^D; \mathfrak{m}) \to H^{m+n}(M^D; \mathfrak{m})$. The cup product of two cocycles has the following property (see Fig. 12)

$$f_m \cup h_n = (-)^{mn} h_n \cup f_m + \text{coboundary} \quad \text{(A13)}$$

We can also define higher cup product $f_m \cup h_n$ which gives rise to a $(m+n-k)$-cochain:

$$\langle f_m \cup h_n, (0, 1, \cdots, m + n - k) \rangle = \sum_{0 \leq i_0 < \cdots < i_{k} \leq n+m-k} (-)^{p} \langle f_{i_0} \cup \cdots \cup f_{i_{k}}, (0 \to i_0, i_1 \to i_2, \cdots) \rangle \times \langle h_{n}, (i_0 \to i_1, i_2 \to i_3, \cdots) \rangle, \quad \text{(A14)}$$

and $f_m \cup h_n = 0$ for $k < 0$ or for $k > m$ or $n$. Here $i \to j$ is the sequence $i, i+1, \cdots, j-1, j$, and $p$ is the number of permutations to bring the sequence

$$0 \to i_0, i_1 \to i_2, \cdots; i_0 + 1 \to i_1 - 1, i_2 + 1 \to i_3 - 1, \cdots \quad \text{(A15)}$$

to the sequence

$$0 \to m + n - k. \quad \text{(A16)}$$

For example

$$\langle f_m \cup h_n, (0 \to m + n - 1) \rangle = \sum_{i=0}^{m-1} (-)^{(m-i)(n+1)} \times \langle f_m, (0 \to i, i + n \to m + n - 1) \rangle \langle h_n, (i \to i + n) \rangle, \quad \text{(A17)}$$

We can see that $\cup = \circ$. Unlike cup product at $k = 0$, the higher cup product of two cocycles may not be a cocycle. For cochains $f_m, h_n$, we have

$$d(f_m \cup h_n) = df_m \cup h_n + (-)^m f_m \cup dh_n \quad \text{(A18)}$$

Let $f_m$ and $h_n$ be cocycles and $c_l$ be a chain, from eqn. (A18) we can obtain

$$d(f_m \cup h_n) = (-)^{m+n-k} f_m \cup h_n + (-)^{mn+n+m} h_n \cup f_m, \quad \text{(A19)}$$

From eqn. (A19), we see that, for $\mathbb{Z}_2$-valued cocycles $z_n$,

$$\text{Sq}^{-k}(z_n) \equiv z_n \cup z_n \quad \text{(A20)}$$

is always a cocycle. Here $\text{Sq}$ is called the Steenrod square. More generally $h_n \cup h_n$ is a cocycle if $n+k$ is odd and $h_n$ is a cocycle. Usually, the Steenrod square is defined only for $\mathbb{Z}_2$-valued cocycles or cohomology classes. Here, we like to define a generalized Steenrod square for $\mathfrak{m}$-valued cochains $c_n$:

$$\text{Sq}^{-k} c_n \equiv c_n \cup c_n + c_{n+k} \cup dc_n. \quad \text{(A21)}$$

From eqn. (A19), we see that

$$d\text{Sq}^k c_n = d(c_n \cup c_n + c_{n} \cup dc_n) \quad \text{(A22)}$$
\[ \text{Notice that (see eqn. (A18))} \]
\[ \text{we see that} \]
\[ \text{Sq}^k c_n \overset{d}{\Rightarrow} \text{Sq}^k d c_n. \]
\[ \text{(A23)} \]

Next, let us consider the action of \( \text{Sq}^k \) on the sum of two \( \mathbb{H} \)-valued cochains \( c_n \) and \( c'_n \):
\[ \text{Sq}^k (c_n + c'_n) = \text{Sq}^k c_n + \text{Sq}^k c'_n + \]
\[ c_n \overset{n-k}{\Rightarrow} c_n + c_n \overset{n-k}{\Rightarrow} c_n + c_n \overset{n-k+1}{\Rightarrow} d c' + c' n \overset{n-k+1}{\Rightarrow} d c_n \]
\[ = \text{Sq}^k c_n + \text{Sq}^k c'_n + [1 + (-1)^k] c_n \overset{n-k}{\Rightarrow} c'_n \]
\[ - (-)^{n-k} [(-)^{n-k} c'_n \overset{n-k}{\Rightarrow} c_n + (-)^n c_n \overset{n-k}{\Rightarrow} c'_n] \]
\[ + c_n \overset{n-k+1}{\Rightarrow} d c' + c' n \overset{n-k+1}{\Rightarrow} d c_n. \]
\[ \text{(A24)} \]

Notice that (see eqn. (A18))
\[ - (-)^{n-k} c'_n \overset{n-k}{\Rightarrow} c_n + (-)^n c_n \overset{n-k}{\Rightarrow} c'_n \]
\[ = d(c'_n \overset{n-k+1}{\Rightarrow} c_n) - d(c'_n \overset{n-k+1}{\Rightarrow} c_n) + (-)^n c'_n \overset{n-k+1}{\Rightarrow} d c_n, \]
\[ \text{(A25)} \]

we see that
\[ \text{Sq}^k (c_n + c'_n) = \text{Sq}^k c_n + \text{Sq}^k c'_n + [1 + (-1)^k] c_n \overset{n-k}{\Rightarrow} c'_n + (-)^{n-k} [d(c'_n \overset{n-k+1}{\Rightarrow} c_n) \]
\[ - d(c'_n \overset{n-k+1}{\Rightarrow} c_n) + (-)^k d(c'_n \overset{n-k+1}{\Rightarrow} c_n) \]
\[ = \text{Sq}^k c_n + \text{Sq}^k c'_n + [1 + (-1)^k] c_n \overset{n-k}{\Rightarrow} c'_n \]
\[ + [1 + (-1)^k] c'_n \overset{n-k}{\Rightarrow} c_n - (-)^{n-k} d(c'_n \overset{n-k+1}{\Rightarrow} c_n) \]
\[ - [(-)^{n-k+1} d(c'_n \overset{n-k+1}{\Rightarrow} c_n) - c_n \overset{n-k+1}{\Rightarrow} c'_n - c_n \overset{n-k+1}{\Rightarrow} d c'_n]. \]
\[ \text{(A26)} \]

Notice that (see eqn. (A18))
\[ - (-)^{n-k+1} d c'_n \overset{n-k+1}{\Rightarrow} c_n - c_n \overset{n-k+1}{\Rightarrow} d c'_n \]
\[ = d( d c'_n \overset{n-k+1}{\Rightarrow} c_n) + (-)^n d c'_n \overset{n-k+1}{\Rightarrow} d c_n, \]
\[ \text{(A27)} \]

we find
\[ \text{Sq}^k (c_n + c'_n) = \text{Sq}^k c_n + \text{Sq}^k c'_n + [1 + (-1)^k] c_n \overset{n-k}{\Rightarrow} c'_n \]
\[ + [1 + (-1)^k] c'_n \overset{n-k+1}{\Rightarrow} d c_n - (-)^{n-k} d(c'_n \overset{n-k+1}{\Rightarrow} c_n) \]
\[ - d(c'_n \overset{n-k+1}{\Rightarrow} c_n) - (-)^n d c'_n \overset{n-k+1}{\Rightarrow} d c_n \]
\[ = \text{Sq}^k c_n + \text{Sq}^k c'_n - (-)^n d c'_n \overset{n-k+1}{\Rightarrow} d c_n \]
\[ + [1 + (-1)^k] c_n \overset{n-k}{\Rightarrow} c'_n + c'_n \overset{n-k+1}{\Rightarrow} d c_n \]
\[ - (-)^{n-k} d(c'_n \overset{n-k+1}{\Rightarrow} c_n) - d(d c'_n \overset{n-k+1}{\Rightarrow} c_n). \]
\[ \text{(A28)} \]

We see that, if one of the \( c_n \) and \( c'_n \) is a cocycle,
\[ \text{Sq}^k (c_n + c'_n) \overset{d}{\Rightarrow} \text{Sq}^k c_n + \text{Sq}^k c'_n. \]
\[ \text{(A29)} \]

We also see that
\[ \text{Sq}^k (c_n + df_{n-1}) \]
\[ = \text{Sq}^k c_n + \text{Sq}^k df_{n-1} + [1 + (-1)^k] df_{n-1} \overset{n-k}{\Rightarrow} c_n \]
\[ - (-)^{n-k} d(c_n \overset{n-k}{\Rightarrow} df_{n-1}) - d(d c_n \overset{n-k+1}{\Rightarrow} df_{n-1}) \]
\[ = \text{Sq}^k c_n + [1 + (-1)^k][df_{n-1} \overset{n-k}{\Rightarrow} c_n + (-)^n \text{Sq}^{k+1} f_{n-1}] \]
\[ + d[\text{Sq}^k f_{n-1} - (-)^{n-k} c_n \overset{n-k}{\Rightarrow} df_{n-1} - d c_n \overset{n-k+1}{\Rightarrow} df_{n-1}]. \]

Using eqn. (A28), we can also obtain the following result if \( dc_n = \text{even} \)
\[ \text{Sq}^k (c_n + 2 c'_n) \]
\[ \overset{d}{\Rightarrow} \text{Sq}^k c_n + 2 d(c_n \overset{n-k+1}{\Rightarrow} c'_n) + 2 d c_n \overset{n-k+1}{\Rightarrow} c'_n \]
\[ \text{(A31)} \]

As another application, we note that, for a \( \mathbb{Q} \)-valued cochain \( m_d \) and using eqn. (A18),
\[ \text{Sq}^1 (m_d) = m_d \overset{d}{\Rightarrow} m_d + m_d \overset{d}{\Rightarrow} dm_d \]
\[ = \frac{1}{2} (-)^d [d(m_d \overset{d}{\Rightarrow} m_d) - dm_d \overset{d}{\Rightarrow} m_d] + \frac{1}{2} m_d \overset{d}{\Rightarrow} dm_d \]
\[ = (-)^d \beta_2 (m_d \overset{d}{\Rightarrow} m_d) - (-)^d \beta_2 m_d \overset{d}{\Rightarrow} m_d + m_d \overset{d}{\Rightarrow} \beta_2 m_d \]
\[ = (-)^d \beta_2 \text{Sq}^0 m_d - 2(-)^d \beta_2 m_d \overset{d}{\Rightarrow} m_d \]
\[ \text{(A32)} \]

This way, we obtain a relation between Steenrod square and Bockstein homomorphism, when \( m_d \) is a \( \mathbb{Z}_2 \)-valued cochain
\[ \text{Sq}^1 (m_d) \overset{2}{\Rightarrow} \beta_2 m_d. \]
\[ \text{(A33)} \]

where we have used \( \text{Sq}^0 m_d = m_d \) for \( \mathbb{Z}_2 \)-valued cochain. For a \( k \)-cochain \( a_k \), \( k = \text{odd} \) we find that
\[ \text{Sq}^k a_k = a_k a_k + a_k \overset{1}{\Rightarrow} da_k \]
\[ \text{(A34)} \]
\[ = \frac{1}{2} [da_k \overset{1}{\Rightarrow} a_k - a_k \overset{1}{\Rightarrow} da_k - d(a_k \overset{1}{\Rightarrow} a_k)] + a_k \overset{1}{\Rightarrow} da_k \]
\[ = \frac{1}{2} [da_k \overset{2}{\Rightarrow} da_k - d(a_k \overset{2}{\Rightarrow} da_k)] - \frac{1}{2} d(a_k \overset{1}{\Rightarrow} a_k) \]
\[ = \frac{1}{4} d(a_k \overset{3}{\Rightarrow} da_k) - \frac{1}{2} d(a_k \overset{1}{\Rightarrow} a_k + da_k \overset{2}{\Rightarrow} a_k) \]

Thus \( \text{Sq}^k a_k \) is always a \( \mathbb{Q} \)-valued coboundary, when \( k \) is odd.

Appendix B: Procedure for deriving Hamiltonian from topological partition function

We briefly review the procedure for writing down local commuting projection Hamiltonians from the topological action. The reader may refer to Ref.5,61 for details.
1. Zero background gauge field case

Suppose $\mathcal{M}^4 = \mathcal{M}^3 \times I$ for some closed 3-manifold $\mathcal{M}^3$ and $I$ is an interval parameterized by $t \in [0, T]$, to be regarded as the time direction. The space-time has boundaries at $t = 0, T$, where the field configurations are given by $\{a_0\}$ and $\{a_T\}$. The transfer matrix is given by

$$\langle \{a_T\} | e^{-T\hat{H}_\infty} | \{a_0\} \rangle = Z_{\mathcal{M}^4 \times I}^{\text{top}} [\{a_T\}, \{a_0\}] \quad \text{(B1)}$$

$$Z_{\mathcal{M}^4 \times I}^{\text{top}} [\{a_T\}, \{a_0\}] = \frac{1}{n^{N_{l,\text{int}} + (N_{l,0} + N_{l,T})/2}} \sum_{\{a_{\text{int}}\}} e^{2\pi i \int_{\mathcal{M}^3 \times I} \omega_4}, \quad \text{(B2)}$$

where $\int_{\mathcal{M}^3 \times I} \omega_4$ is evaluated with link configurations at its boundaries fixed to be $\{a_0\}, \{a_T\}$. Links not living on the boundary are called internal links. Their configuration is given by $\{a_{\text{int}}\}$. $N_{l,0}$, $N_{l,T}$ and $N_{l,\text{int}}$ are the number of links at the two boundaries and in the space-time bulk respectively. In the following we assume the two boundaries have the same triangulation so $N_{l,0} = N_{l,T} = N_{l,\mathcal{M}^3}$.

We may represent the transfer matrix diagrammatically as a spacetime cylinder

$$e^{-T\hat{H}_\infty} = \begin{array}{c}
\{a_T\} \\
\{a_0\}
\end{array}$$

where the top and bottom ellipses represent the spatial closed manifold $\mathcal{M}^3$ at $t = T, 0$ respectively. They are the boundaries of the space-time cylinder and are drawn as bold lines. Note that although $\mathcal{M}^3$ is a three-dimensional manifold, we draw it as a one-dimensional ellipse.

Recall from Ref.\textsuperscript{5,51} that under a local spacetime re-triangulation, the topological action $\int_{\mathcal{M}^3} \omega_4$ changes by $d\omega_4$. Hence the cocycle condition $d\omega_4 \equiv 0$ implies the action is invariant under re-triangulation mod 1. Moreover, during a re-triangulation, the boundary degrees of freedom cannot change, thus we can only conclude that the value of $\int_{\mathcal{M}^3 \times I} \omega_4$ is independent of triangulations of the internal bulk, but it could depend on the boundary triangulation. Furthermore, $\int_{\mathcal{M}^3 \times I} \omega_4$ is independent of the values of $a_{\text{int}}$. This is because during a re-triangulation, the internal link values are forgotten, which can be illustrated with the re-triangulation of a square:

$$\begin{array}{c}
a_{\text{int}} \\
0
\end{array}$$

Thus $Z_{\mathcal{M}^4 \times I}^{\text{top}} [\{a_T\}, \{a_0\}]$ is independent of both of the triangulation and field configuration of the internal bulk and only depends on the configuration at its boundaries.

We can show that the transfer matrix is a projection with a computation:

$$\langle \{a_{2T}\} | e^{-T\hat{H}_\infty} e^{-T\hat{H}_\infty} | \{a_0\} \rangle$$

$$= \sum \left\{ Z_{\mathcal{M}^3 \times [0, T]}^{\text{top}} [\{a_{2T}\}, \{a_T\}] Z_{\mathcal{M}^3 \times [T, 2T]}^{\text{top}} [\{a_T\}, \{a_0\}] \right\}$$

$$= \sum \left\{ \frac{1}{n^{N_{l,\text{int}} + 2N_{l,\mathcal{M}^3}}} e^{2\pi i \int_{\mathcal{M}^3 \times [0, T]} \omega_4 + \int_{\mathcal{M}^3 \times [T, 2T]} \omega_4} \right\}$$

$$= \frac{1}{n^{N_{l,\text{int}} + 2N_{l,\mathcal{M}^3}}} \sum_{\{a_{\text{int}}\}} e^{2\pi i \int_{\mathcal{M}^3 \times [0, 2T]} \omega_4}$$

$$= \langle \{a_{2T}\} | e^{-T\hat{H}_\infty} | \{a_0\} \rangle$$

where the label $\text{int}$ includes all the links not on the slices $t = 0, T, 2T$ and the label $\text{int}'$ includes all the links not on the slices $t = 0, 2T$. This computation can be expressed diagrammatically as

Since the eigenvalues of a projection is 1 or 0, correspondingly $\hat{H}_\infty$ has eigenvalues 0 or $\infty$, i.e. an infinite energy gap.

Moreover, the transfer matrix has trace 1. This is because $\text{Tr}[e^{-T\hat{H}_\infty}]$ is evaluated by identifying the top and bottom link configurations of the cylinder and summing over them. With the two ends identified, $\mathcal{M}^3 \times I = \mathcal{M}^3 \times S^1$ becomes a closed manifold. As we showed in (22), on a closed manifold without any background gauge fields, $\int_{\mathcal{M}^3} \omega_4 \equiv 0$. Thus we have

$$\text{Tr}[e^{-T\hat{H}_\infty}] = \frac{1}{n^{N_{l,\text{int}} + N_{l,\mathcal{M}^3}}} \sum_{\{a_{\text{int}}, a_0\}} 1 = 1 \quad \text{(B4)}$$

Diagrammatically, this is expressed as

$$\begin{array}{c}
\{a_{2T}\} \\
\{a_T\} \\
\{a_0\}
\end{array}\begin{array}{c}
\sum_{\{a_T\}} \\
\{a_{2T}\} = \\
\{a_T\}
\end{array}$$

$$\begin{array}{c}
\{a_{2T}\} \\
\{a_T\} \\
\{a_0\}
\end{array}\bigg\} = 1$$

hence the ground state of $\hat{H}_\infty$ is unique.

Although the transfer matrix is a non-local operator, it can be decomposed into a product of local operators. Suppose we evaluate $\int_{\mathcal{M}^3 \times I} \omega_4$ with a triangulation of the internal space-time, such that it consists of $N_{l,\mathcal{M}^3} + 1$ infinitesimal spatial slices, each slice having the same triangulation of the spatial slices at $t = 0, T$. Between
two adjacent slices, only a single link $ij$ is updated from $a_{0,ij}$ to $a_{T,ij}$, while all other links remains the same. We have

$$e^{-T\hat{H}\infty} = \prod_{ij} P_{ij}$$

$$\langle\{a_T\}|P_{ij}|\{a_0\}\rangle = n^{N_1 \times M^3 - 1} \prod_{i'j' \neq ij} \delta_{a_{0,i',j'},a_{T,i',j'}} \times Z_{\pi_M}^{\text{top}}\{\{a_T\},\{a_0\}\}$$

(B5)

$$P_{ij} \to \prod_{ij} P_{ij}$$

In diagrams, this means

In (B6), it is not very clear that $P_{ij}$ is a local operator. The locality of $P_{ij}$ can be seen by examining the diagrammatic expression for $P_{ij}$,

$$P_{ij} = \prod_{ij} P_{ij}$$

where double slash indicates the region in which field configurations on the top needs to be identified with that on the bottom. On the right hand side we see that $P_{ij}$ is associated with $M^3 \times S^1$ with a slit at the link $ij$. This means that in $M^3$, the links far away from $ij$ become internal links in the non-zero matrix elements of $P_{ij}$, and hence the non-zero matrix elements of $P_{ij}$ are independent of the value of links far away from $ij$. Thus $P_{ij}$ is a local operator.

Using the same arguments as before, it can be shown that $P_{ij}$ is a projection operator with trace $n^{N_1 \times M^3 - 1}$. So each projection by $P_{ij}$ reduces the dimension of the ground state Hilbert space by a factor or $n$. Furthermore, in the following we will show that any two such operators $P_{ij}, P_{kl}$ commute. The two orderings $P_{ij}P_{kl}$ or $P_{kl}P_{ij}$ corresponds to triangulations shown below

$$P_{ij}P_{kl} = \prod_{ij} P_{ij} \quad P_{kl}P_{ij} = \prod_{ij} P_{ij}$$

It is readily seen that the two diagrams only differs for the internal links. Thus $P_{ij}P_{kl} = P_{kl}P_{ij}$.

We note that the computation for $P_{ij}$ can be further simplified by setting $a_{\text{int}} = 0$.

$$\langle\{a_T\}|P_{ij}|\{a_0\}\rangle = \prod_{i'j' \neq ij} \delta_{a_{0,i',j'},a_{T,i',j'}} \frac{1}{n} e^{2\pi i \int_{M^3 \times I} \omega_{\text{int}}}|_{a_{\text{int}} = 0}$$

(B7)

The ground state of $\hat{H}\infty$ satisfies $P_{ij}|\psi_0\rangle = |\psi_0\rangle$. We can construct a Hamiltonian with finite gap but the same ground state as $\hat{H}\infty$ by defining

$$\hat{H} = - \sum_{ij} P_{ij}$$

(B8)

2. Non-zero background gauge field case

Suppose we are given a background gauge field on the spatial manifold $M^3$. In order to define the transfer matrix, we need to specify the background gauge field $\hat{B}$ on the spacetime $M^3 \times I$. We propose that $\hat{B}$ should be static, meaning that it should be invariant under time translation, i.e. $\hat{B}$ is the same on every spatial slice. This is sensible because a non-static background gauge field actually correspond to the insertion of a 1-symmetry operator into the transfer matrix.

Such static background gauge field $\hat{B}$ on $M^3 \times I$ can be constructed from a given flat $\hat{B}$ on $M^3$ as follows. We triangulate $M \times I$ such that any 2-cell $(ijk)$ in $M^3 \times I$, when projected onto $M^3$, is either also a 2-cell $(i0j0k0)$ in $M^3$, or a lower dimensional cell. Then we define

$$\langle\hat{B}, (ijk)\rangle := \begin{cases} \langle\hat{B}, (i0j0k0)\rangle & \text{if } (ijk) \text{ projects to a 2-cell} \\ 0 & \text{else} \end{cases}$$
it can be checked $\hat{B} = 0$.

We then construct the transfer matrix with such static background gauge field. Diagrammatically, the transfer matrix is represented as follows:

$$e^{-T H_{\infty}[B]} = \begin{array}{c}{a_T} \\ {a_0} \end{array} \begin{array}{c}{a_T} \\ {a_0} \end{array}$$

where the wiggly vertical line represents the static $\hat{B}$. We may repeat the same analysis as in the previous subsection, except that we include a wiggly vertical line in the diagrams. For example, in showing the transfer matrix is a projection, we have

$$\sum_{\{a_T\}} \begin{array}{c}{a_{2T}} \\ {a_T} \\ {a_0} \end{array} \begin{array}{c}{a_{2T}} \\ {a_T} \\ {a_0} \end{array}$$

We need to be slightly careful about generalizing the argument that trace of transfer matrix is 1. Recall from the previous section at (B4), we used the fact that on a closed manifold $M^4 = M^3 \times S^1$, we have

$$\int_{M^4} \omega_4[da] \overset{!}{=} \int_{M^4} \omega_4[0] = 0,$$

which is due to gauge invariance of the topological action (21). In the present case we have

$$\int_{M^4} \omega_4[\hat{B} + da] \overset{!}{=} \int_{M^4} \omega_4[\hat{B}]$$

To complete the argument, note that a “static” background gauge field on $M^3 \times S^1$ may be extended into a higher dimensional manifold $M^3 \times D^2$, where $\partial D^2 = S^1$ with the same construction as before. Thus

$$\int_{M^4} \omega_4[\hat{B}] = \int_{\partial(M^3 \times D^2)} \omega_4[\hat{B}]$$

$$= \int_{M^3 \times D^2} d\omega_4[\hat{B}] \overset{!}{=} 0.$$ (B9)

using Stoke’s theorem and the cocycle condition.

Therefore we have

$$\text{Tr} \left( e^{-T H_{\infty}[\hat{B}]} \right) = \begin{array}{c}{a_T} \\ {a_0} \end{array} = 1$$

and the ground state is unique.

All the arguments in the previous section will follow through for the present case. We can construct commuting projections $P_{ij}[\hat{B}]$ which differs from the zero-gauge projections only when $ij$ is near the non-zero $\hat{B}$. Its corresponding diagram is

$$P_{ij}[\hat{B}] = \begin{array}{c}{a_{T,ij}} \\ {a_{0,ij}} \end{array}$$

Appendix C: Ground state wavefunction

1. Zero background gauge field case

Suppose $\omega_4 = d\phi_3$ for some 3-cochain $\phi_3$ (which may not have 1-symmetry, so this does not mean $\omega_4$ is a coboundary with 1-symmetry), then $\phi_3$ can be interpreted as the phase of a ground state wavefunction. Define $|\psi_0\rangle = \frac{1}{N_\psi} \sum_{\{a\}} e^{2\pi i \int_{M^3} \phi_3[a]} |\{a\}\rangle$ with normalization $N_\psi = \sqrt{n^{M_3}}$. Suppose the spatial manifold $M^3 = \partial M^4_0$ is the boundary of some manifold $M^3_0$ (such $M^4_0$ exists for any closed, oriented 3-manifold $\mathbb{M}^3$). Then the amplitude is

$$\int_{M^3} \phi_3[a] = \int_{\partial M^4_0} \phi_3[a] = \int_{M^4_0} d\phi_3[a] = \int_{M^3_0} \omega_4[a].$$

So $|\psi_0\rangle$ may be represented diagrammatically as

$$|\psi_0\rangle = \begin{array}{c}{a_0} \\ {a_T} \end{array} M^4_0$$

We check that $|\psi_0\rangle$ survives the $P_{ij}$ projection:

$$\langle \{a_T\} | P_{ij} | \psi_0 \rangle$$

$$= \frac{1}{N_\psi n} \sum_{\{a_0\}} \prod_{i,j \neq ij} \delta_{a_0,i',j',a_T,i',j'} e^{2\pi i [\int_{M^3 \times I} \omega_4 + \int_{M^3} \phi_3[a_0]]}$$

$$= \frac{1}{N_\psi n} \sum_{\{a_0\}} \prod_{i,j \neq ij} \delta_{a_0,i',j',a_T,i',j'} e^{2\pi i \int_{M^3} \phi_3[a_T]}$$

$$= \frac{1}{N_\psi} e^{2\pi i \int_{M^3} \phi_3[a_T]} = \langle \{a_T\} | \psi_0 \rangle,$$ (C1)

where in the second step we used Stoke’s theorem $\int_{M^3 \times I} \omega_4 = \int_{M^3} \phi_3[a_T]$. The same result can also be de-
rived diagrammatically as follows:

\[
\sum_{\{a_0\}} \mathcal{M}_0^4 \{a_T\} = \{a_T\} \quad \sum_{\{a_0\}} \mathcal{M}_0^4 \{a_0\} = \{a_0\}
\]

Therefore, the transfer matrix is

\[e^{-T \hat{H}_\infty} = |\psi_0\rangle\langle\psi_0|,\]

represented diagrammatically by

and the local projections \(P_{ij}\) can be expressed in terms of \(\phi_3\) as

\[\langle\{a_T\}|P_{ij}|\{a_0\}\rangle = \prod_{i'j' \neq ij} \delta_{a_0,i',a_T,j'} \delta_{a_T,i,j} \frac{1}{n} e^{2\pi i \int_{\mathcal{M}^3} (\phi_3[a_T] - \phi_3[a_0])}\] (C2)

which is

\[
\sum_{\{a_0\}} \mathcal{M}_0^4 \{a_T\} = \{a_T\} \quad \sum_{\{a_0\}} \mathcal{M}_0^4 \{a_0\} = \{a_0\}
\]

2. Non-zero background gauge field case

Suppose \(\omega_B[\hat{B} + da] = \omega_B[\hat{B}] + d\phi_3[a, \hat{B}]\). While it is still true that \(\mathcal{M}_0^4 = \partial \mathcal{M}_0^4\) for some manifold \(\mathcal{M}_0^4\), there may be obstructions in \(\mathcal{M}_0^4\) that forbids the extension of the background gauge field into \(\mathcal{M}_0^4\), while respecting the flatness constraint \(d\hat{B} = 0\).

So we will instead take \(\mathcal{M}_0^4 = \mathcal{M}^3 \times I\), where \(I = [-1, 0]\) is an interval. The boundary now have two components \(\partial \mathcal{M}_0^4 = \mathcal{M}^3 \times \{0\} \bigcup \mathcal{M}^3 \times \{-1\}\). We take the first component to be the original spatial manifold and extend the field configurations such that on the other end \(\mathcal{M}^3 \times \{-1\}\), we fix \(a = 0\). The background gauge field is extended to be “static” as in the previous section.

We define

\[|\psi_0[\hat{B}]\rangle = \frac{1}{N\psi} \sum_{\{a\}} e^{2\pi i \int_{\mathcal{M}^3} \phi_3[a, \hat{B}] - \phi_3[0, \hat{B}]} |\{a\}\rangle.\] (C3)

Thus we have

\[
\int_{\mathcal{M}_0^4} \phi_3[a, \hat{B}] - \phi_3[0, \hat{B}] = \int_{\partial \mathcal{M}_0^4} \phi_3[a, \hat{B}]
\]

\[
= \int_{\mathcal{M}_0^4} d\phi_3[a, \hat{B}] = \int_{\mathcal{M}_0^4} \omega_4[\hat{B} + da] - \int_{\mathcal{M}_0^4} \omega_4[\hat{B}]
\]

\[
= \int_{\mathcal{M}_0^4} \omega_4[\hat{B} + da].
\]

where in the last step the term \(\int_{\mathcal{M}_0^4} \omega_4[\hat{B}] = 0\) because its field configuration at \(\mathcal{M}_0^4 \times \{0\}\) and \(\mathcal{M}_0^4 \times \{-1\}\) are the same and the two ends can be glued together to form a closed manifold. The same arguments used in (B9) can be applied.

In diagram, this means

\[|\psi_0[\hat{B}]\rangle = \begin{cases} \{a\} & \{a\} \\ \{0\} & \{0\} \end{cases}\]

and it is the ground state for the projections \(P_{ij}[\hat{B}]\):

\[
\sum_{\{a_0\}} \mathcal{M}_0^4 \{a_T\} = \{a_T\} \quad \sum_{\{a_0\}} \mathcal{M}_0^4 \{a_0\} = \{a_0\}
\]

and the matrix elements of \(P_{ij}[\hat{B}]\) can be expressed in terms of \(\phi_3[a, \hat{B}]\):

\[
\langle\{a_T\}|P_{ij}[\hat{B}]|\{a_0\}\rangle = \prod_{i'j' \neq ij} \delta_{a_0,i',a_T,j'} \delta_{a_T,i,j} \frac{1}{n} e^{2\pi i \int_{\mathcal{M}^3} (\phi_3[a_T, \hat{B}] - \phi_3[a_0, \hat{B}])}\] (C4)

Appendix D: Triangulation of hypercubic lattice

\(\mathbb{R}^d\) may be triangulated by first admitting a hypercubic lattice, and triangulating each hypercube \(I^d = \{x_1, \ldots, x_d : 1 \geq x_i \geq 0 \forall i\}\) into \(d!\) simplices \(\Delta_p\) labeled by \(p\) in the permutation group \(S_d\):

\[\Delta_p = \{1 \geq x_{p(1)} \geq \cdots \geq x_{p(d)} \geq 0\}\]
The vertices and branching structure for each $\Delta_p$ are given by

$$\tilde{\theta} = (0, \ldots, 0)$$

$$\rightarrow \tilde{\rho}(1)$$

$$\rightarrow \tilde{\rho}(1) + \tilde{\rho}(2)$$

$$\ldots$$

$$\rightarrow \tilde{\rho}(1) + \cdots + \tilde{\rho}(d) = (1, \ldots, 1) = \tilde{1},$$

where $\tilde{v}$ is the unit vector in the $x_i$ direction. The orientation of $\Delta_p$ is given by $\sigma(p) = e^{\rho(1)\cdots\rho(d)}$.

**Appendix E: Evaluation of $\int_{t^4}(b^Z)^2$ in a hypercube**

Let $b^Z$ be a 2-cocycle. Under the triangulation in Appendix D for $d = 4$, we have

$$\int_{t^4}(b^Z)^2 = \langle (b^Z)^2, \sum_p \sigma(p) \Delta_p \rangle = \langle (b^Z)^2, e^{\mu\nu\rho\sigma} \Delta_{(\mu\nu\rho\sigma)} \rangle$$

$$= e^{\mu\nu\rho\sigma} \langle b^Z, (\tilde{0}, \tilde{\mu}, \tilde{\mu} + \tilde{\nu}) \rangle \langle b^Z, (\tilde{\mu} + \tilde{\nu}, \tilde{\mu} + \tilde{\nu} + \tilde{\rho}, \tilde{1}) \rangle$$

$$+ \langle b^Z, (\tilde{\mu} + \tilde{\nu}, \tilde{\mu} + \tilde{\nu} + \tilde{\rho}, \tilde{1}) \rangle (\mu \leftrightarrow \nu)$$

$$= \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(\tilde{\mu} + \tilde{\nu} + \frac{\tilde{\rho}}{2}) F_{\rho\sigma}(\tilde{\mu} + \tilde{\nu} + \frac{\tilde{\rho}}{2} + \frac{\tilde{\sigma}}{2})$$

where

$$F_{\mu\nu}(\tilde{r}) := \langle (b^Z, (\tilde{0}, \tilde{\mu} + \tilde{\nu} + \tilde{\rho}), (\tilde{0}, \tilde{\mu} + \tilde{\nu}) \rangle \frac{\tilde{\rho}}{2} - \frac{\tilde{\sigma}}{2} - (\mu \leftrightarrow \nu) \rangle.$$

**Appendix F: Evaluation of $P_{ij}$ in the $m=$even case**

In this section we follow the procedure described in Appendix B and write down the projections $P_{ij}$ in the $m=$even case for the topological action (32) with $M^3 = \mathbb{R}^3$. The matrix elements are given by (B7) and (30):

$$\langle (a^Z_T)_{ij} | P_{ij} | (a^Z_0)_j \rangle = \prod_{i' \neq i} \delta_{a_{i'i}^Z, a_{i'i}'^Z} \times \langle 0 \rangle^2 \frac{1}{n} e^{2\pi i \int_{\mathbb{R}^3} \frac{\alpha_m^a}{\alpha_m^a} (da^Z),$$

where

$$\int_{\mathbb{R}^3 \times I} \alpha_m^a (da^Z) = \int_{\mathbb{R}^3} da^Z da^Z$$

$$= \int_{\mathbb{R}^3} da^Z da^Z = \int_{\mathbb{R}^3} da^Z da^Z_{ij}$$

$$= \int_{\mathbb{R}^3} da^Z da^Z + a^Z_0 da^Z + a^Z_0 da^Z = \delta \left( \int_{\mathbb{R}^3} da^Z da^Z \right)$$

summed only over 1-diagonal links $ij = (\tilde{0}, \tilde{\mu} + \tilde{\nu})$, $\tilde{X}_{ij}$ increments $a^Z_{ij}$ by 1. It can be checked that $[P_{ij}, P_{i'j'}] = 0$ for distinct 1-diagonal links $ij$ and $i'j'$.

$$\frac{1}{4} \int_{\mathbb{R}^3} \delta a^Z da^Z + da^Z da^Z = \frac{1}{4} \int_{\mathbb{R}^3} \delta a^Z da^Z + da^Z da^Z,$$

where we have defined $\delta(x) := x_0^T$. In the last step we used integration by part and the fact that $\delta a^Z da^Z = 0$ because it is impossible for both factors of the cup product to be non-zero, since $\delta a^Z$ is non-zero for only one link $ij$. So Using the triangulation of Appendix D for $D = 3$ space, we have

$$\int_{\mathbb{R}^3} \delta a^Z da^Z + da^Z da^Z = \int_{\mathbb{R}^3} \delta a^Z da^Z + da^Z da^Z = \frac{1}{4} \int_{\mathbb{R}^3} \delta a^Z da^Z + da^Z da^Z.$$
Appendix G: Evaluation of $P_{ij}$ for general $m$

As in the $m$=even case, the matrix elements of the projections $P_{ij}$ are given by (B7):

$$
\langle \{ a_{ij}^n \} | P_{ij} | \{ a_0^n \} \rangle = \prod_{i, j \neq i, j'} \delta_{n_{i,j'}} \delta_{n_{i,j}} \int_{\theta} \frac{1}{n} e^{2\pi i \sum_{k=0}^{n-1} \delta_{k,n} a_{ij}^n + \delta_{k,n} a_{ij'}^n},
$$

where the exponent is

$$
\int_{\theta} \frac{m}{2n} \delta_{k,n} a_{ij}^n = \int_{\theta} \frac{m}{2n} \delta_{k,n} a_{ij' n}^n = \int_{\theta} \frac{m}{2n} \delta_{k,n} a_{ij'}^n.
$$

where $\phi_3$ is given in (28). Again $\delta a_{ij}^n \delta a_{ij'}^n = 0$ since we only change by one link.

$$
\delta_{ij}(a_{ij}^n) = \frac{m}{2n} \left( \delta a_{ij}^n \delta a_{ij}^n + \frac{m}{2} \right) \left( \delta a_{ij}^n \delta a_{ij}^n \right) + \frac{1}{n} \delta_{k,n} a_{ij}^n + \frac{1}{n} \delta_{k,n} a_{ij'}^n.
$$

Where we integrated by part in the last step and used $\delta a_{ij}^n \delta a_{ij}^n = 0$. Evaluating on the $d = 3$ lattice triangulation described in Appendix D, we have

$$
\delta_{ij} = \frac{m}{2n} \left( \delta a_{ij}^n \delta a_{ij}^n + \delta a_{ij}^n \delta a_{ij}^n \right) + \frac{1}{n} \delta_{k,n} a_{ij}^n + \frac{1}{n} \delta_{k,n} a_{ij'}^n.
$$

The projections can be written as

$$
P_{ij} = \frac{1}{n} \sum_{k=0}^{n} \tilde{X}_{ij}(a_{ij}^n + k)^n,
$$

where $\delta_k a_{ij}^n = (a_{ij}^n + k)^n$ is non-zero for only one link $ij$. There are three cases to consider: $ij$ can be 1-, 2- or 3-diagonal, as defined in subsection V A of the main text. For the 3-diagonal links $ij = (\vec{n}, \vec{n} + \vec{1})$,

$$
\int_{\theta} \delta_{ij} a_{ij}^n = \sum_{\alpha, \beta, \gamma} e^{\alpha \beta \gamma} a_{ij}^n
$$

where $(\vec{a}, \vec{b}, \vec{c})_n$ is a shorthand for $(\vec{a} + \vec{n}, \vec{b} + \vec{n}, \vec{c} + \vec{n})$.

We see that the 3-diagonal link $ij$ is coupled to $[a_{ij}^n]$ on twelve triangles making up the six faces of the cube whose diagonal is $ij$.

For the 2-diagonal links $ij = (\vec{n}, \vec{n} + \vec{2})$,

$$
\int_{\theta} \delta_{ij} a_{ij}^n = \sum_{\alpha, \beta, \gamma} e^{\alpha \beta \gamma} a_{ij}^n
$$

where $(\vec{a}, \vec{b}, \vec{c})_n$ is a shorthand for $(\vec{a} + \vec{n}, \vec{b} + \vec{n}, \vec{c} + \vec{n})$.

For instance, if $\alpha, \beta = 1, 2$, the link $ij$ is involved as $\delta_{ij} a_{ij}^n$ in two triangles making up the square in $x_1-x_2$ plane enclosing $ij$. Each of the triangles is coupled to $a_{ij}^n$ on two other faces in the $(x_1 + x_2)-x_3$ plane. All four triangles intersect at $ij$.

For the 1-diagonal links $ij = (\vec{n}, \vec{n} + \vec{1})$,
\[ + d\bar{a}\Pi_n (\hat{\phi}, \hat{\beta}, \hat{\alpha})_{\bar{n} - \bar{\gamma}} d\left( \frac{d\bar{a}\Pi_n}{n} \right) (\hat{\phi}, \hat{\beta}, \hat{\alpha})_{\bar{n} - \bar{\gamma}} \]
\[ + d\bar{a}\Pi_n (\hat{\phi}, \hat{\beta}, \hat{\alpha})_{\bar{n} - \bar{\gamma}} d\left( \frac{d\bar{a}\Pi_n}{n} \right) (\hat{\phi}, \hat{\beta}, \hat{\alpha})_{\bar{n} - \bar{\gamma}} \]
\[ = \sum_{\beta, \gamma} \phi_{\beta, \gamma} \left\{ \frac{m}{2n} \delta \left( \frac{d\bar{a}\Pi_n}{n} \right) (\hat{\phi}, \hat{\beta}, \hat{\alpha})_{\bar{n}} \right. \]
\[ \times (d\bar{a}\Pi_n) \left[ (\hat{\phi}, \hat{\beta}, \hat{\alpha})_{\bar{n}} + (-\hat{\beta} - \hat{\gamma}, -\hat{\gamma}, \hat{\alpha})_{\bar{n}} \right] \]
\[ + \frac{m}{2} \left( d\left( \frac{d\bar{a}\Pi_n}{n} \right) (\hat{\phi}, \hat{\beta}, \hat{\alpha})_{\bar{n}} \right. \]
\[ \times (d\bar{a}\Pi_n) \left[ (\hat{\phi}, \hat{\beta}, \hat{\alpha})_{\bar{n}} + (-\hat{\beta} - \hat{\gamma}, -\hat{\gamma}, \hat{\alpha})_{\bar{n}} \right] \]
\[ + \left. \delta \left( \frac{d\bar{a}\Pi_n}{n} \right) \left[ (\hat{\phi}, \hat{\beta}, \hat{\alpha})_{\bar{n}} + (-\hat{\beta} - \hat{\gamma}, -\hat{\gamma}, \hat{\alpha})_{\bar{n}} \right] \right\} \}

In the case \( n = 2 \), \( \left| \frac{d\bar{a}\Pi_n}{2} \right| \geq \zeta_1 (a^{z = 2}) \).

**Appendix H: Calculation details for \( \theta_q, \theta_{q, q2} \)**

It turns out we only need to keep track of the two triangles and five links in the central square, shown in Fig. 3. This is slightly non-trivial, essentially due to \( \phi_2[a, h] = 0 \) when \( dh = 0 \). In this section we assume \( a_i = a_i^{z = n} \) and \( q = q^{z = n} \). Applying (44), we have

\[ W_0 (a^{z = n}) = \exp \left( \frac{2\pi i}{n} \{ \phi_2 (a^{z = n}, h(W_0^n)) - \phi_2 (a^{z = n}, h(W_0^n)) \} \right) \]
\[ \times \{ \{ [a + dh(W_0^n)]^{z = n} \} \}, \]

with \( h(W_0^n) \) depicted in the bottom of Fig. 3.

Evaluating \( \phi_2 \) using (47), we have

\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} (a - a_0) \left( \frac{a + q}{n} \right) \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} a \frac{q}{n} \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} (a - a_0) \left( \frac{a + q}{n} \right) \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} a \frac{q}{n} \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} (a - a_0) \left( \frac{a + q}{n} \right) \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} a \frac{q}{n} \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} (a - a_0) \left( \frac{a + q}{n} \right) \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} a \frac{q}{n} \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} (a - a_0) \left( \frac{a + q}{n} \right) \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} a \frac{q}{n} \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} (a - a_0) \left( \frac{a + q}{n} \right) \]
\[ \phi_2 (a, q, h(W_0^n)) = \frac{m}{2} a \frac{q}{n} \]

\[ + \theta_0 \frac{1}{2} \left( \phi_2 (a, q, h(W_0^n)) - \phi_2 (a, q, h(W_0^n)) \right) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]

So for self-statistics (48), after some algebra, we are left with

\[ \theta_q = \phi_2 (a, q, h(W_0^n)) - \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]

Whereas for mutual-statistics (50), we have

\[ \theta_{q, q2} = \frac{1}{2} \phi_2 (a, q, h(W_0^n)) - \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]
\[ + \phi_2 (a, q, h(W_0^n)) \]

**Appendix I: Evaluation of \( W_0 \), for \((n, m) = (2, 1)\)**

In this section we derive (58). We also assume \( a = a^{z = n} \) for all initial link values in this section. Restricting to \((n, m) = (2, 1)\) and enforcing “no flux” rule \( da = 0 \), (47) is

\[ \phi_2 [a, h^{Z = 2}] = \frac{1}{4} [a Z^{2} a + \frac{1}{2} (a - \frac{d\bar{a}^{Z = 2}}{2}) ] + [a + a^{Z = 2}] \frac{d\bar{a}^{Z = 2}}{2} \]

Applying (44), we have

\[ W_0 \phi_2 [a, h^{Z = 2}] = \frac{m}{2} a \frac{q}{n} \]

where

\[ \Phi[a] = \langle \phi_2 (1, 2, i) \rangle - \langle \phi_2 (2, 3, i) \rangle + \langle \phi_2 (i, 3, 4) \rangle \]
\[ - \langle \phi_2 (i, 4, 5) \rangle + \langle \phi_2 (6, 5, i) \rangle - \langle \phi_2 (1, 6, i) \rangle \]

Applying (11) for each 2-simplex in Fig. 5, we get

\[ \langle \phi_2 (1, 2, i) \rangle = \frac{1}{2} \left( a_2 + \frac{1}{2} \right) \]
\[ \langle \phi_2 (2, 3, i) \rangle = \frac{1}{2} \left( a_3 + \frac{1}{2} \right) \]
\[ \langle \phi_2 (i, 3, 4) \rangle = \frac{1}{2} \left( a_3 + \frac{1}{2} \right) \]
\[ \langle \phi_2 (i, 4, 5) \rangle = \frac{1}{2} \left( a_3 + \frac{1}{2} \right) \]
\[ \langle \phi_2 (i, 5, 6) \rangle = \frac{1}{2} \left( a_3 + \frac{1}{2} \right) \]
\[ \langle \phi_2 (6, 5, i) \rangle = \frac{1}{2} \left( a_3 + \frac{1}{2} \right) \]
\[ \langle \phi_2 (1, 6, i) \rangle = \frac{1}{2} \left( a_3 + \frac{1}{2} \right) \]


\[ \langle \phi_2, (i, 3, 4) \rangle = \frac{1}{4} a_{34} \]
\[ \langle \phi_2, (i, 5, 4) \rangle = \frac{1}{4} a_{34} \]
\[ \langle \phi_2, (6, i, 5) \rangle = \frac{1}{4} a_{55} + \frac{1}{2} (a_{65} + (a_{6i} + 1)(a_{i5} + 1)) \]
\[ \langle \phi_2, (1, 6, i) \rangle = \frac{1}{2} (a_{16} + a_{6i} + 1). \]

Note for \( a = a_Z^2 \) and \( a' = a'_{Z^2} \), we have \( |a + a'| = aa' \).

Also for any simplex \((i, j, k)\), the “no flux” constraint means
\[
 a_{ij} = (a_{ij} + a_{ik})^Z = a_{ij} + a_{ik} - 2\left[ \frac{a_{ij} + a_{ik}}{2} \right].
\]

After a bit of algebra, simplifying using the above identities, we finally arrive at
\[
 \Phi[a] = \frac{1}{2} \sum_{(j'j')} a_{ij}a_{ij'}. \]

1. DS projection Hamiltonian

For completeness, we supplement this section by briefly explaining the projection Hamiltonian for DS topological order from the action (up to a volume term)
\[
 Z_{DS} = \sum_{da, \alpha \in 0} e^{2\pi i f_{M3} \hat{s}_{aba}}.
\]

The construction was well-studied in the literature, see eg. Ref. 51. It is similar to that described in Appendix B, except that six links connecting to the same site is updated. We have
\[
 \hat{H} = -\sum_{i} P_i \int_{\Delta_i} \delta_{(da, \Delta), o} - \sum_{\Delta} \delta_{(da, \Delta), o},
\]
where \( \Delta \) is summed over all 2-simplices, \( \Delta_i \) are product over all 2-simplices having \( i \) as a vertex.
\[
 P_i \{(a_{ij}, a_{i'j'})\} = e^{2\pi i \Phi_{DS}[a]} \{(a_{ij} + 1)^Z, a_{i'j'}\},
\]
and \( \Phi_{DS}[a] \) is evaluating the cocycles on the six tetrahedrons involved when a site is updated. Using Fig. 5 and updating \( i \) to \( i' \) with \( i' \) out of paper, where \( a_{i'j'} = (a_{ij} + 1)^Z \) and \( a_{i'i'} = 1 \), the result is
\[
 \Phi_{DS}[a] = \frac{1}{2} [a_{12}a_{21} + a_{21}(a_{31} + 1) + (a_{31} + 1)a_{34}
 + (a_{51} + 1)a_{54} + a_{61}(a_{51} + 1) + a_{16}a_{61}] 
 + \frac{1}{2} \sum_{(j'j')} a_{ij}a_{ij'}. 
\]

We see it describes the same phase as \( H_\partial \) in (53).

### Appendix J: \( \omega_4, \phi_3 \) and \( \phi_2 \)

In the main text, we find that for \( Z_4 \)-1-SPT, the 4-cocycle \( \omega_4 \), the ground state wavefunction amplitude \( \phi_3 \), and the boundary transform anomalous phase \( \phi_2 \) are related via (26) and (40):
\[
 \omega_4[da^Z] = d\phi_3[a] \\
 - \delta_a \phi_3[a^Z] = d\phi_2[a, h].
\]

In general, given \( \omega_4 \) satisfying \( d\omega_4 = 0 \), we can define the 3-cochain \( \phi_0^3 \) as follows:
\[
 \langle \phi_0^3, (1234) \rangle := \langle \omega_4, (01234) \rangle,
\]
where we have introduced an extra “reference” vertex \( \partial \). A heuristic way to interpret \( \omega_4 \) is that it is located at \( t = -\infty \) whereas the other vertices \( i = 1, 2, 3, 4 \) are located at a spatial slice at \( t = 0 \). So \( a_{i'i'} \) are “spatial” links and \( a_{0i} \) are “temporal” links. We may choose the links \( a_{0i} = 0 \), \( i = 1, 2, 3, 4 \) as a convention. The dependence of \( \phi_0^3 \) on \( \partial \) is the choice of such convention. For arbitrary 4-chain (01234), we have
\[
 \langle d\phi_3^0, (01234) \rangle \\
 = \sum_{m=0}^{4} (-)^m \langle \phi_3^0, (0 \ldots \hat{m} \ldots 4) \rangle \\
 = \sum_{m=0}^{4} (-)^m \langle \omega_4, (0 \ldots \hat{m} \ldots 4) \rangle \\
 = \langle \omega_4, (01234) \rangle - \langle d\omega_4, (01234) \rangle \\
 = \langle \omega_4, (01234) \rangle.
\]

so \( \omega_4 = d\phi_3^0 \).

To generalize (40), note that if we have a 1-symmetry \( \alpha = dh \) only on the spatial links, then we can use the invariance of \( \omega_4 \) under space-time 1-symmetry to undo \( h \) from the spatial links and act \( (-h) \) on the temporal links instead, i.e.
\[
 \langle \phi_3^0[a + \alpha], (1234) \rangle = \langle \phi_3^0[a + dh], (1234) \rangle \\
 = \langle \omega_4[a + (dh)_{\text{spatial}}], (01234) \rangle \\
 = \langle \omega_4[a], (11234) \rangle \\
 = \langle \phi_3^0[a], (1234) \rangle.
\]

So \( \delta_{\alpha} \phi_3^0 = \phi_3^1 - \phi_3^0 \). Here \( (dh)_{\text{spatial}} \) means it only exists on spatial links \( a_{i'i'} \), and we have introduced a new vertex \( \hat{i} \) where \( a_{0i} := a_{0i} - h_i = -h_i \).
If we define
\[
\langle \phi_{2}^{01}, (234) \rangle := \langle \omega_{4}, (\hat{0} \hat{1} 234) \rangle,
\]
it can then be checked that for arbitrary 3-chain (1234), we have
\[
\langle d\phi_{2}^{01}, (1234) \rangle = \frac{4}{m} \sum_{m=1}^{4} (-m)^{m} \langle \omega_{4}, (\hat{0} \hat{1} m \ldots 4) \rangle = \frac{4}{m} \sum_{m=1}^{4} (-m)^{m} \langle \omega_{4}, (\hat{0} \hat{1} 11 \ldots m \ldots 4) \rangle
\]
\[
= \sum_{m=0}^{1} (-m) \langle \omega_{4}, (\hat{0} \hat{1} \hat{m} \ldots 1234) \rangle + \langle d\omega_{4}, (\hat{0} \hat{1} 1234) \rangle
\]
\[
= -\langle \omega_{4}, (11234) \rangle + \langle \omega_{4}, (\hat{0}1234) \rangle
\]
\[
= -\langle \phi_{1}^{01}, (1234) \rangle + \langle \phi_{1}^{01}, (1234) \rangle.
\]
So \( \delta_{\alpha} \phi_{0}^{01} = -d\phi_{2}^{01} \).

In general we may define
\[
\langle \phi_{k}^{0 \ldots (4-k-1)}, (01234) \rangle := \langle \omega_{k}, (\hat{0} \ldots (4-k-1)(4-k) \ldots 4) \rangle
\]
for \( k = 3, 2, 1, 0, -1 \). They represent the anomaly in the boundary transformation in \( k \)-dimensional sub-manifolds in the boundary. \( k = -1 \) means dimension 0 in the bulk. They satisfy
\[
\hat{d}\phi_{k} = (-k)^{k} d\phi_{k-1},
\]
where
\[
(\hat{d}\phi_{k})^{0 \ldots (4-k)} := \sum_{m=0}^{4-k} (-m)^{m} \phi_{k}^{0 \ldots \hat{m} \ldots (4-k)}.
\]

**Appendix K: Generalization of (49) and (51) to \( G \)-protected 1-SPT for finite unitary groups**

In general, we can carry through the calculations for self-statistics and mutual-statistics for transformation strings, for a \( G \)-protected 1-SPT in 3+1D as well, where \( G \) is any unitary group. Note \( G \) is Abelian since it is a 1-symmetry. In this section we will only present the final results.

Following similar strategies for deriving self- and mutual-statistics in the \( \mathbb{Z}_{n} \) case, it can be shown that for general unitary group \( G \), the self- and mutual-statistics of transformation strings are given by
\[
\theta_{q} = -\omega_{4}(-q, -q, 0, -q, 0, q) + \omega_{4}(-q, q, -q, -q, 0, 0) + \omega_{4}(0, 0, 0, -q, 0, -q, 0) - \omega_{4}(0, 0, -q, 0, -q, -q, 0)
\]
\[
\theta_{q_{1}q_{2}} = \left\{ \begin{array}{l}
[\omega_{4}(-q_{1}, 0, -q_{1}, q_{1}, 0, -q_{1} - q_{2}) \\
+ \omega_{4}(0, 0, -q_{1}, -q_{2}, -q_{1} - q_{2}, -q_{1}) \\
- \omega_{4}(q_{1}, 0, 0, -q_{1}, -q_{1} - q_{2}, -q_{2}) \\
- (q_{1} \to 0) - (q_{2} \to 0) \end{array} \right\} + (q_{1} \leftrightarrow q_{2}),
\]
where \( q, q_{1}, q_{2} \in G \) labels the group element associated with the transformation string. \( \omega_{4}[\mathcal{B}] = \omega_{4}(\mathcal{B}_{012}, \mathcal{B}_{013}, \mathcal{B}_{014}, \mathcal{B}_{023}, \mathcal{B}_{024}, \mathcal{B}_{034}) \) where \( d\mathcal{B} = 0 \). It can be checked (K1) and (K2) are topological invariants, namely, they are unchanged under \( \omega_{4} \to \omega_{4} + d\beta_{3} \) for any 1-symmetric 3-cochain \( \beta_{3} \).

We will check that (K1) and (K2) recovers (49) and (51) in the case \( G = \mathbb{Z}_{n} \). The \( \mathbb{Z}_{n} \) 4-cocycle (11) is
\[
\omega_{4}[\mathcal{B}] = \frac{m}{2n} \sum_{q}^{2} \mathbb{Z}^{q} = \frac{m}{2n} (\mathbb{B}^{Z}_{012} \mathbb{B}^{Z}_{023} + \mathbb{B}^{Z}_{013} \mathbb{B}^{Z}_{024} + \mathbb{B}^{Z}_{014} \mathbb{B}^{Z}_{034}),
\]
(K3)

where
\[
\mathbb{B}^{Z}_{ij} = (\mathbb{B}^{Z}_{0jk} - \mathbb{B}^{Z}_{0ik} + \mathbb{B}^{Z}_{0ij})^{Z} \quad \text{for } i \neq 0,
\]
so (K1) and (K2) are
\[
\theta_{q} = -\omega_{4}(-q, -q, 0, -q, 0, q) + \omega_{4}(-q, -q, -q, -q, 0, 0) + \omega_{4}(0, 0, 0, 0, 0, -q, 0) - \omega_{4}(0, 0, -q, 0, -q, -q, 0)
\]
\[
\theta_{q_{1}q_{2}} = \left\{ \begin{array}{l}
[\omega_{4}(-q_{1}, 0, -q_{1}, q_{1}, 0, -q_{1} - q_{2}) \\
+ \omega_{4}(0, 0, -q_{1}, -q_{2}, -q_{1} - q_{2}, -q_{1}) \\
- \omega_{4}(q_{1}, 0, 0, -q_{1}, -q_{1} - q_{2}, -q_{2}) \\
- (q_{1} \to 0) - (q_{2} \to 0) \end{array} \right\} + (q_{1} \leftrightarrow q_{2})
\]
\[
\theta_{q_{1}q_{2}} = \left\{ \begin{array}{l}
[\omega_{4}(-q_{1}, 0, -q_{1}, q_{1}, 0, -q_{1} - q_{2}) \\
+ \omega_{4}(0, 0, -q_{1}, -q_{2}, -q_{1} - q_{2}, -q_{1}) \\
- \omega_{4}(q_{1}, 0, 0, -q_{1}, -q_{1} - q_{2}, -q_{2}) \\
- (q_{1} \to 0) - (q_{2} \to 0) \end{array} \right\} + (q_{1} \leftrightarrow q_{2})
\]
\[
\theta_{q_{1}q_{2}} = \left\{ \begin{array}{l}
[\omega_{4}(-q_{1}, 0, -q_{1}, q_{1}, 0, -q_{1} - q_{2}) \\
+ \omega_{4}(0, 0, -q_{1}, -q_{2}, -q_{1} - q_{2}, -q_{1}) \\
- \omega_{4}(q_{1}, 0, 0, -q_{1}, -q_{1} - q_{2}, -q_{2}) \\
- (q_{1} \to 0) - (q_{2} \to 0) \end{array} \right\} + (q_{1} \leftrightarrow q_{2})
\]
Thus (49) and (51) are recovered.