Minimality, homogeneity and topological 0-1 laws for subspaces of a Banach space

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Abstract

If a Banach space is saturated with basic sequences whose linear span embeds into the linear span of any subsequence, then it contains a minimal subspace. It follows that any Banach space is either ergodic or contains a minimal subspace.

For a Banach space $X$ with an (unconditional) basis, topological 0-1 law type dichotomies are stated for block-subspaces of $X$ as well as for subspaces of $X$ with a successive FDD on its basis. A uniformity principle for properties of block-sequences, results about block-homogeneity, and a possible method to construct a Banach space with an unconditional basis, which has a complemented subspace without an unconditional basis, are deduced.

The starting point of this article is the solution to the Homogeneous Banach Space Problem given by W.T. Gowers [12] and R. Komorowski - N. Tomczak-Jaegermann [18]. A Banach space is said to be homogeneous if it is isomorphic to its infinite dimensional closed subspaces; these authors proved that a homogeneous Banach space must be isomorphic to $l_2$.

Gowers proved that any Banach space with a basis must either have a subspace with an unconditional basis or a hereditarily indecomposable subspace. By properties of hereditarily indecomposable Banach spaces, it follows that a homogeneous Banach space must have an unconditional basis (see e.g. [12] for details about this). Komorowski and Tomczak-Jaegermann proved that a Banach space

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with an unconditional basis must contain a copy of \( l_2 \) or a subspace with a successive finite-dimensional decomposition on the basis (2-dimensional if the space has finite cotype) which does not have an unconditional basis. It follows that a homogeneous Banach space must be isomorphic to \( l_2 \).

While Gowers’ dichotomy theorem is based on a general Ramsey type theorem for block-subspaces in a Banach space with a Schauder basis, the subspace with a finite-dimensional decomposition constructed in Komorowski and Tomczak-Jaegermann’s theorem can never be isomorphic to a block-subspace. If one restricts one’s attention to block-subspaces, the standard homogeneous examples become the sequence spaces \( c_0 \) and \( l_p, 1 \leq p < +\infty \), with their canonical bases; these spaces are well-known to be isomorphic to their block-subspaces. Furthermore there are classical theorems which characterize \( c_0 \) and \( l_p, 1 \leq p < +\infty \) only by means of their block-subspaces. An instance of this is Zippin’s theorem ([19] Theorem 2.a.9): a normalized basic sequence is perfectly homogeneous (i.e. equivalent to all its normalized block-sequences) if and only if it is equivalent to the canonical basis of \( c_0 \) or some \( l_p \). See also [19] Theorem 2.a.10.

So it is very natural to ask what can be said on the subject of (isomorphic) homogeneity restricted to block-subspaces of a given Banach space with a Schauder basis: if a Banach space \( X \) with a basis \( (e_n)_{n \in \mathbb{N}} \) is isomorphic to its block-subspaces, does it follow that \( X \) is isomorphic to \( c_0 \) or \( l_p, 1 \leq p < +\infty \)? Note that such a basis is not necessarily equivalent to the canonical basis of \( c_0 \) or some \( l_p \), take \( l_2 \) with a conditional basis.

In the other direction, if a Banach space is not homogeneous, then how many non-isomorphic subspaces must it contain? This question may be asked in the setting of the classification of analytic equivalence relations on Polish spaces by Borel reducibility. This area of research originated from the works of H. Friedman and L. Stanley [11] and independently from the works of L. A. Harrington, A. S. Kechris and A. Louveau [14], and may be thought of as an extension of the notion of cardinality in terms of complexity, when one compares equivalence relations.

If \( R \) (resp. \( S \)) is an equivalence relation on a Polish space \( E \) (resp. \( F \)), then it is said that \( (E, R) \) is Borel reducible to \( (F, S) \) if there exists a Borel map \( f : E \to F \) such that \( \forall x, y \in E, xRy \iff f(x)Sf(y) \). An important equivalence relation is the relation \( E_0 \): it is defined on \( 2^\omega \) by

\[
\alpha E_0 \beta \iff \exists m \in \mathbb{N} \forall n \geq m, \alpha(n) = \beta(n).
\]

The relation \( E_0 \) is a Borel equivalence relation with \( 2^\omega \) classes and which, furthermore, is not Borel reducible to equality on \( 2^\omega \), that is, there is no Borel
map $f$ from $2^\omega$ into $2^\omega$ (equivalently, into a Polish space), such that $\alpha E_0 \beta \iff f(\alpha) = f(\beta)$; such a relation is said to be non-smooth. In fact $E_0$ is the $\leq_B$ minimal non-smooth Borel equivalence relation [14].

There is a natural way to equip the set of subspaces of a Banach space $X$ with a Borel structure [1], and the relation of isomorphism is analytic in this setting. The relation $E_0$ appears to be a natural threshold for results about the relation of isomorphism between separable Banach spaces [8],[9],[10],[23]. A Banach space $X$ is said to be ergodic if $E_0$ is Borel reducible to isomorphism between subspaces of $X$; in particular, an ergodic Banach space has continuum many non-isomorphic subspaces, and isomorphism between its subspaces is non-smooth. The results in [1],[8],[9],[10],[23] suggest that every Banach space non-isomorphic to $l_2$ should be ergodic, and we also refer to these articles for an introduction to the classification of analytic equivalence relations on Polish spaces by Borel reducibility, and more specifically to the complexity of isomorphism between Banach spaces.

Restricting our attention to block-subspaces, the natural question becomes the following: if $X$ is a Banach space with a Schauder basis, is it true that either $X$ is isomorphic to its block-subspaces or $E_0$ is Borel reducible to isomorphism between the block-subspaces of $X$?

Let us provide some ground for this conjecture by noting that, if we replace isomorphism by equivalence of the corresponding basic sequences, it is completely solved by the positive by a result of [9] using the theorem of Zippin: if $X$ is a Banach space with a normalized basis $(e_n)_{n \in \mathbb{N}}$, then either $(e_n)_{n \in \mathbb{N}}$ is equivalent to the canonical basis of $c_0$ or $l_p$, $1 \leq p < +\infty$, or $E_0$ is Borel reducible to equivalence between normalized block-sequences of $X$.

This article is divided in three sections. The results and methods in the first two sections are mainly independant, although some notation defined in the first section might be used in the second section. We obtain partial answers to the above conjectures in various directions; our methods also provide results of combinatorial nature which are of independant interest in Banach space theory. The third section contains a refined version of a principle proved in the third section, with an application to the study of complemented subspaces of a Banach space with an unconditional basis.

A.M. Pelczar has proved that a Banach space which is saturated with subsymmetric sequences contains a minimal subspace [22]. Our main theorem in this
article (Theorem 1.1), proved in the first section, is the following. If a Banach space $X$ is saturated with basic sequences whose linear span embed in the linear span of any subsequence, then $X$ contains a minimal subspace. In particular, define a basic sequence to be isomorphically homogeneous if all subspaces spanned by subsequences are isomorphic; our result implies that a Banach space saturated with isomorphically homogeneous basic sequences contains a minimal subspace. This is the isomorphic counterpart of Pelczar’s result.

In combination with a result of C. Rosendal [23], it follows that if $X$ is a Banach space with a Schauder basis, then either $E_0$ is Borel reducible to isomorphism between block-subspaces of $X$, or $X$ contains a block-subspace which is block-minimal (i.e. embeds as a block-subspace of any of its block-subspaces), Corollary 1.13. This improves a result of [10] which states that a Banach space contains continuum many non-isomorphic subspaces or a minimal subspace.

The second topological 0-1 law (Theorem 8.47 in [16]) states that in a infinite product space of Polish spaces, a set with Baire Property which is a tail set (i.e. invariant by change of a finite number of coordinates), is either meager or comeager. In the second section, we study the set $bb_d(X)$ of “rational normalized block-sequences” of a Banach space $X$ with a Schauder basis, and a characterization of comeager sets in the natural topology on $bb_d(X)$ that was obtained in [10], to deduce a principle of topological 0-1 law for block-subspaces in $bb_d(X)$ (Theorem 2.2, Theorem 2.4).

We deduce from this principle a uniformity theorem (remark after Proposition 2.5), and an application to some problems related to the block-homogeneity property (Proposition 2.10).

In the third section, we prove a principle of 0-1 topological law in a Banach space $X$ with a Schauder basis, for subspaces with a successive finite dimensional decomposition on the basis (Proposition 3.2), again continuing on some work from [10]. We derive a possible application to a long-standing open question in Banach space theory: does a complemented subspace of a Banach space with an unconditional basis necessarily have an unconditional basis (Corollary 3.3)?

Let us fix some notation. Let $X$ be a Banach space with a Schauder basis $(e_n)_{n \in \mathbb{N}}$. If $(x_n)_{n \in J}$ is a finite or infinite block-sequence of $X$ then $[x_n]_{n \in \mathbb{N}}$ will stand for its closed linear span. We shall also use some standard notation about finitely supported vectors on $(e_n)_{n \in \mathbb{N}}$, for example, we shall write $x < y$ and say that $x$ and $y$ are successive when $\max(supp(x)) < \min(supp(y))$. The set of
normalized block-sequences in $X$ is denoted $bb(X)$. It is a Polish space when equipped with the product of the norm topology on $X$.

Let $Q(X)$ be the set of non-zero blocks of the basis (i.e. finitely supported vectors) which have rational coordinates on $(e_n)_{n \in \mathbb{N}}$ (or coordinates in $\mathbb{Q} + i\mathbb{Q}$ if we deal with a complex Banach space). We denote by $bb_Q(X)$ the set of block-bases of vectors in $Q(X)$, and by $\mathcal{G}_Q(X)$ the corresponding set of block-subspaces of $X$.

The notation $bb_\leq^\omega(X)$ (resp. $bb^\omega_Q(X)$) will be used for the set of finite (resp. length $n$) block-sequences with vectors in $Q(X)$; the set of finite block-subspaces generated by a block-sequence in $bb_\leq^\omega(X)$ will be denoted by $Fin^\omega_Q(X)$.

We shall consider $bb_Q(X)$ as a topological space, when equipped with the product of the discrete topology on $Q(X)$. As $Q(X)$ is countable, this turns $bb_Q(X)$ into a Polish space. Likewise, $Q(X)^\omega$ is a Polish space.

For a finite block sequence $\bar{x} = (x_1, \ldots, x_n) \in bb_\leq^\omega(X)$, we denote by $N_Q(\bar{x})$ the set of elements of $bb_Q(X)$ whose first $n$ vectors are $(x_1, \ldots, x_n)$; this is the basic open set associated to $\bar{x}$.

The set $[\omega]^\omega$ is the set of increasing sequences of integers, which we sometimes identify with infinite subsets of $\omega$. It is equipped with the product of the discrete topology on $\omega$. The set $[\omega]^{<\omega}$ is the set of finite increasing sequences of integers. If $a = (a_1, \ldots, a_k) \in [\omega]^{<\omega}$, then $[a]$ stands for the basic open set associated to $a$, that is the set of increasing sequences of integers of the form $(a_1, \ldots, a_k, a_{k+1}, \ldots)$. If $A \in [\omega]^\omega$, then $[A]^\omega$ is the set of increasing sequences of integers in $A$ (where $A$ is seen as a subset of $\omega$).

We recall that two basic sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are said to be equivalent if the map $T : [x_n]_{n \in \mathbb{N}} \to [y_n]_{n \in \mathbb{N}}$ defined by $T(x_n) = y_n$ for all $n \in \mathbb{N}$ is an isomorphism. For $C \geq 1$, they are $C$-equivalent if $\|T\|\|T^{-1}\| \leq C$. A basic sequence is said to be ($C$-)subsymmetric if it is ($C$-)equivalent to all its subsequences.

We shall sometimes use “standard perturbation arguments” without expliciting them. This expression will refer to one of the following well-known facts about block-subspaces of a Banach space $X$ with a Schauder basis. Any basic sequence (resp. block-basic sequence) in $X$ is an arbitrarily small perturbation of a basic sequence in $Q(X)^\omega$ (resp. block-basic sequence in $bb_Q(X)$), and in particular is $1 + \epsilon$-equivalent to it, for arbitrarily small $\epsilon > 0$. Any subspace of $X$ has a subspace which is an arbitrarily small perturbation of a block-subspace of $X$ (and in particular, with $1 + \epsilon$-equivalence of the corresponding bases, for arbitrarily small $\epsilon > 0$). If $X$ is reflexive, then any basic sequence in $X$ has a subsequence which is a perturbation of a block-sequence of $X$ (and in particular, is $1 + \epsilon$-equiv-
equivalent to it, for arbitrarily small \( \epsilon > 0 \).

We shall also use the fact that any Banach space contains a basic sequence.

1 Minimal subspaces, isomorphically homogeneous sequences, and reductions of \( E_0 \).

We recall different notions of minimality for Banach spaces. A Banach space \( X \) is said to be minimal if it embeds into any of its subspaces. If \( X \) has a basis \( (e_n)_{n \in \mathbb{N}} \), then it is said to be block-minimal if every block-subspace of \( X \) has a further block-subspace which is isomorphic to \( X \), and is said to be equivalence block-minimal if every block-sequence of \( (x_n)_{n \in \mathbb{N}} \) has a further block-sequence which is equivalent to \( (x_n)_{n \in \mathbb{N}} \).

The theorem of Pelczar \([22]\) states that a Banach space which is saturated with subsymmetric sequences must contain an equivalence block-minimal subspace with a basis. In this section we prove a version of her theorem for isomorphism (Theorem 1.1).

We recall that a basic sequence \( (x_n)_{n \in \mathbb{N}} \) in a Banach space is said to be isomorphically homogeneous if all subspaces spanned by subsequences of \( (x_n)_{n \in \mathbb{N}} \) are isomorphic. The relevant property for our theorem will be the following property, which is obviously weaker than the property of being isomorphically homogeneous: say that a basic sequence embeds (resp. \( C \)-embeds) into its subsequences if its linear span embeds (resp. \( C \)-embeds) into the linear span of any of its subsequences.

**Theorem 1.1** A Banach space which is saturated with basic sequences which embed into their subsequences contains a minimal subspace.

For \( N \in \mathbb{N} \) let \( d_c(N) \) denote an integer such that if \( X \) is a Banach space with a basis \( (e_n)_{n \in \mathbb{N}} \) with basis constant \( c \), and \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) are normalized block-basic sequences of \( X \) such that \( x_n = y_n \) for all \( n > N \), then \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) are \( d_c(N) \)-equivalent. We leave as an exercise to the reader to check that such an integer exists.

**Lemma 1.2** Let \( (x_n)_{n \in \mathbb{N}} \) be a basic sequence in a Banach space which embeds into its subsequences. Then there exists \( C \geq 1 \) and a subsequence of \( (x_n)_{n \in \mathbb{N}} \) which \( C \)-embeds into its subsequences.
Proof: Let \((x_n)_{n \in \mathbb{N}}\) be a basic sequence which embeds into its subsequences, and let \(c\) be its basis constant. It is clearly enough to find a subsequence \((y_n)_{n \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\) and \(C \geq 1\) such that \((x_n)_{n \in \mathbb{N}}\) \(-\text{embeds into any subsequence of } (y_n)_{n \in \mathbb{N}}\) (with the obvious definition).

Assuming the conclusion is false, we construct by induction a sequence of subsequences \((x^k_n)_{n \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\), such that for all \(k \in \mathbb{N}\), \((x^k_n)_{n \in \mathbb{N}}\) is a subsequence of \((x^{k-1}_n)_{n \in \mathbb{N}}\) such that \((x_n)_{n \in \mathbb{N}}\) does not \(kd_c(k)\)-embeds into \((x^k_n)_{n \in \mathbb{N}}\).

Let \((y_n)_{n \in \mathbb{N}}\) be the diagonal subsequence of \((x_n)_{n \in \mathbb{N}}\) defined by \(y_n = x^0_n\). Then \((x_n)_{n \in \mathbb{N}}\) does not \(kd_c(k)\)-embeds into \((x^k_1, \ldots, x^k_{k-1}, y_k, y_{k+1}, \ldots)\). So \((x_n)_{n \in \mathbb{N}}\) does not \(k\)-embeds in \((y_n)_{n \in \mathbb{N}}\). Now \(k\) was arbitrary, so this contradicts our hypothesis.

Lemma 1.3 Let \(X\) be a Banach space which is saturated with basic sequences which embed in their subsequences. Then there exists a subspace \(Y\) of \(X\) with a Schauder basis, and a constant \(C \geq 1\) such that every block-sequence of \(Y\) (resp. in \(bbQ(Y)\)) has a further block-sequence (resp. in \(bbQ(Y)\)) which \(C\)-embeds into its subsequences.

Proof: By properties of hereditarily indecomposable Banach spaces [15], a basic sequence which embeds into its subsequences cannot span a hereditarily indecomposable space. Thus \(X\) does not contain a hereditarily indecomposable subspace and by Gowers’ dichotomy theorem, we may assume \(X\) has an unconditional basis (let \(c\) be its basis constant). If \(c_0\) or \(l_1\) embeds into \(X\) then we are done, so by the classical theorem of James, we may assume \(X\) is reflexive. Thus by standard perturbation arguments, every normalized block-sequence in \(X\) has a further normalized block-sequence in \(X\) which embeds into its subsequences (here we also used the obvious fact that if a basic sequence \((x_n)_{n \in \mathbb{N}}\) embeds into its subsequences, then so does any subsequence of \((x_n)_{n \in \mathbb{N}}\).

Assuming the conclusion is false, we construct by induction a sequence of block-sequences \((x^k_n)_{n \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\), for \(k \in \mathbb{N}\), such that for all \(k \in \mathbb{N}\), \((x^k_n)_{n \in \mathbb{N}}\) is a block-sequence of \((x^{k-1}_n)_{n \in \mathbb{N}}\) such that no block-sequence of \((x^k_n)_{n \in \mathbb{N}}\) \(kd_c(k)^2\)-embeds into its subsequences.

Let \((y_n)_{n \in \mathbb{N}}\) be the diagonal block-sequence of \((x_n)_{n \in \mathbb{N}}\) defined by \(y_n = x^0_n\), and let \((z_n)_{n \in \mathbb{N}}\) be an arbitrary block-sequence of \((y_n)_{n \in \mathbb{N}}\).

Then \((x^k_1, \ldots, x^k_{k-1}, z_k, z_{k+1}, \ldots)\) is a block-sequence of \((x^k_n)_{n \in \mathbb{N}}\) and so, does not \(kd_c(k)^2\)-embed into its subsequences. So \((z_n)_{n \in \mathbb{N}}\) does not \(k\)-embeds into its subsequences - this is true as well of its subsequences. As \(k\) was arbitrary, we
deduce from Lemma 1.2 that \((z_n)_{n \in \mathbb{N}}\) does not embed into its subsequences. As \((z_n)_{n \in \mathbb{N}}\) was an arbitrary block-sequence of \((y_n)_{n \in \mathbb{N}}\), this contradicts our hypothesis.

By standard perturbation arguments, we deduce from this the stated result with block-sequences in \(bbQ(Y)\). □

Recall that \(Q(X)^\omega\) is equipped with the product of the discrete topology on \(Q(X)^\omega\) which turns it into a Polish space.

**Definition 1.4** Let \(X\) be a Banach space with a Schauder basis, and let \((x_n)_{n \in \mathbb{N}} \in Q(X)^\omega\). We shall say that \((x_n)_{n \in \mathbb{N}}\) continuously embeds (resp. \(C\)-continuously embeds) into its subsequences if there exists a continuous map \(\phi : [\omega]^{\omega} \to Q(X)^\omega\) such for all \(A \in [\omega]^{\omega}\), \(\phi(A)\) is a sequence of vectors in \([x_n]_{n \in A} \cap Q(X)\) which is equivalent (resp. \(C\)-equivalent) to \((x_n)_{n \in \mathbb{N}}\).

This definition depends on the Banach space \(X\) in which we pick the basic sequence \((x_n)_{n \in \mathbb{N}}\); this will not cause us any problem, as it will always be clear which is the underlying space \(X\).

The interest of this notion stems from the following lemma, which was essentially obtained by Rosendal as part of the proof of [23], Theorem 11. To prove it, we shall need the following fact, which is well-known to descriptive set theorists. The algebra \(\sigma(\Sigma^1_1)\) is the \(\sigma\)-algebra generated by analytic sets. For any \(\sigma(\Sigma^1_1)\)-measurable function from \([\omega]^{\omega}\) into a metric space, there exists \(B \in [\omega]^{\omega}\) such that the restriction of \(f\) to \([B]^{\omega}\) is continuous.

Indeed, by Silver’s Theorem [16] 21.9, any analytic set in \([\omega]^{\omega}\) is completely Ramsey, and so any \(\sigma(\Sigma^1_1)\) set in \([\omega]^{\omega}\) is (completely) Ramsey as well (use for example [16] Theorem 19.14). One concludes using the proof of [20] Theorem 9.10 which only uses the Ramsey-measurability of the function.

**Lemma 1.5** Let \(X\) be a Banach space with a Schauder basis, let \((x_n)_{n \in \mathbb{N}} \in bbQ(X)\) be a block-sequence which \(C\)-embeds into its subsequences, and let \(\epsilon\) be positive. Then some subsequence of \((x_n)_{n \in \mathbb{N}}\) \(C + \epsilon\)-continuously embeds into its subsequences.

**Proof:** By standard perturbation arguments, we may find for each \(A \in [\omega]^{\omega}\) a sequence \((y_n)_{n \in \mathbb{N}} \in Q(X)^\omega\) such that \(y_n \in [x_k]_{k \in A}\) for all \(n \in \mathbb{N}\), and such that the basic sequences \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) are \(C + \epsilon\)-equivalent. The set \(P \subset\)
$[\omega]^\omega \times \mathbb{Q}(X)^\omega$ of couples $(A, (y_n))$ with this property is Borel (even closed), so by the Jankov-von Neumann Uniformization Theorem (Theorem 18.1 in [16]), there exists a $C$-measurable selector $f : [\omega]^\omega \to \mathbb{Q}(X)^\omega$ for $P$. By the fact before this lemma, there exists $B \in [\omega]^\omega$ such that the restriction of $f$ to $[B]^\omega$ is continuous. Write $B = (b_k)_{k \in \mathbb{N}}$ where $(b_k)_{k \in \mathbb{N}}$ is increasing. By composing $f$ with the obviously continuous maps $\psi_B : [\omega]^\omega \to [B]^\omega$, defined by $\psi_B((n_k)_{k \in \mathbb{N}}) = (b_{n_k})_{k \in \mathbb{N}}$, and $\mu_B : \mathbb{Q}(X)^\omega \to \mathbb{Q}(X)^\omega$, defined by $\mu_B((y_n)_{n \in \mathbb{N}}) = (y_{b_n})_{n \in \mathbb{N}}$, we obtain a continuous map $\phi : [\omega]^\omega \to \mathbb{Q}(X)^\omega$ which indicates that $(x_n)_{n \in B} C + \epsilon$-continuously embeds into its subsequences.

We now start the proof of Theorem 1.1. So we consider a Banach space $X$ which is saturated with basic sequences which embed into their subsequences and wish to find a minimal subspace in $X$.

By Lemma 1.3 and Lemma 1.5, we may assume that $X$ is a Banach space with a Schauder basis and that there exists $C \geq 1$ such that every block-sequence in $bb_\mathbb{Q}(X)$ has a further block-sequence in $bb_\mathbb{Q}(X)$ which $C$-continuously embeds into its subsequences.

For the rest of the proof $X$ and $C \geq 1$ are fixed with this property. Recall that the set of block-subspaces of $X$ which are generated by block-sequences in $bb_\mathbb{Q}(X)$ is denoted by $G_\mathbb{Q}(X)$; the set of finite block-subspaces which are generated by block-sequences in $bb^\omega_\mathbb{Q}(X)$ is denoted by $Fin_\mathbb{Q}(X)$. If $n \in \mathbb{N}$ and $F \in Fin_\mathbb{Q}(X)$ we write $n \leq F$ to mean that $n \leq \min(\text{supp}(x))$ for all $x \in F$.

We first express the notion of continuous embedding in terms of a game. For $L = [l_n]_{n \in \mathbb{N}}$ with $(l_n)_{n \in \mathbb{N}} \in bb_\mathbb{Q}(X)$, we define a game $H_L$ as follows. A $k$-th move for Player 1 is some $n_k \in \mathbb{N}$. A $k$-th move for Player 2 is some $(F_k, y_k) \in Fin_\mathbb{Q}(X) \times \mathbb{Q}(X)$, with $n_k \leq F_k \subset L$ and $y_k \in \bigcup_{j=1}^k F_j$.

Player 2 wins the game $H_L$ if $(y_n)_{n \in \mathbb{N}}$ is $C$-equivalent to $(x_n)_{n \in \mathbb{N}}$.

We claim the following:

**Lemma 1.6** Let $X$ be a Banach space with a Schauder basis, and $(l_n)_{n \in \mathbb{N}} \in bb_\mathbb{Q}(X)$ be a block-sequence which $C$-continuously embeds into its subsequences. Let $L = [l_n]_{n \in \mathbb{N}}$. Then Player 2 has a winning strategy in the game $H_L$.

**Proof:** Let $\phi$ be the continuous map in Definition 1.4. We describe a winning strategy for Player 2 by induction.
We assume that Player 1’s moves were \((n_i)_{i \leq k-1}\) and that the \(k-1\) first moves prescribed by the winning strategy for Player 2 were \((F_i, y_i)_{i \leq k-1}\), with \(F_i\) of the form \([l_{n_i}, \ldots, l_{m_i}]\), \(n_i \leq m_i\), for all \(i \leq k - 1\); letting \(a_{k-1} = [n_1, m_1] \cup \ldots \cup [n_{k-1}, m_{k-1}] \in [\omega]^\omega\), we also assume that \(\phi([a_{k-1}]) \subset N_G(y_1, \ldots, y_{k-1})\). We now describe the \(k\)-th move of the winning strategy for Player 2.

Let \(n_k\) be a \(k\)-th move for Player 1. We may clearly assume that \(n_k > m_{k-1}\). Let \(A_k = \cup_{i \leq k-1} [n_i, m_i] \cup [n_k, +\infty) \in [\omega]^\omega\). The sequence \(\phi(A_k)\) is of the form \((y_1, \ldots, y_{k-1}, y_k, z_{k+1}, \ldots)\) for some \(y_k, z_{k+1}, \ldots\) in \(Q(X)\). By continuity of \(\phi\) in \(A_k\) there exists \(m_k > n_k\) such that, if \(a_k = [n_1, m_1] \cup \ldots \cup [n_k, m_k] \in [\omega]^\omega\), then \(\phi([a_k]) \subset N_Q(y_1, \ldots, y_k)\). We may assume that \(\max(\text{supp}(x_{m_k})) \geq \max(\text{supp}(y_k))\); so as \(y_k \in [x_i]_{i \in A}\), we have that \(y_k \in \bigoplus_{j=1}^k [x_i]_{i \in [n_j, m_j]}\). So \((F_k, y_k) = ([l_{n_k}, \ldots, l_{m_k}], y_k)\) is an admissible \(k\)-th move for Player 2 for which the induction hypotheses are satisfied.

Repeating this by induction we obtain a sequence \((y_n)_{n \in \mathbb{N}}\) which is equal to \(\phi(A)\), where \(A = \cup_{k \in \mathbb{N}} [n_k, m_k]\), and so which is, in particular, \(C\)-equivalent to \((x_n)_{n \in \mathbb{N}}\).

\[\square\]

**Definition 1.7** Given \(L, M\) two block-subspaces in \(G_Q(X)\), define the game \(G_{L,M}\) as follows. A \(k\)-th move for Player 1 is some \((x_k, n_k) \in Q(X) \times \mathbb{N}\), with \(x_k \in L\), and \(x_k > x_{k-1}\) if \(k \geq 2\). A \(k\)-th move for Player 2 is some \((F_k, y_k) \in F\in Q(X) \times Q(X)\) with \(n_k < F_k \subset M\) and \(y_k \in F_1 \oplus \ldots \oplus F_k\) for all \(k \in \mathbb{N}\).

\[G_{L,M}^{\Lambda} : \quad \begin{array}{c} x_1, n_1 & x_2, n_2 & \ldots \\ I & \end{array} \]

\[G_{L,M}^{\Xi} : \quad \begin{array}{c} F_1, y_1 & F_2, y_2 & \ldots \\ II & \end{array} \]

Player 2 wins \(G_{L,M}\) if \((y_n)_{n \in \mathbb{N}}\) is \(C\)-equivalent to \((x_n)_{n \in \mathbb{N}}\).

The following easy fact will be needed in the next lemma: if \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) are \(C\)-equivalent basic sequences, then for any scalar sequence \((\lambda_i)_{i \in \mathbb{N}}\) and sequence \((I_n)_{n \in \mathbb{N}}\) of successive subsets of \(\mathbb{N}\) such that \(\{i \in I_n : \lambda_i \neq 0\} \neq \emptyset, \forall n \in \mathbb{N}\), the basic sequences \((\sum_{i \in I_n} \lambda_i x_i)_{n \in \mathbb{N}}\) and \((\sum_{i \in I_n} \lambda_i y_i)_{n \in \mathbb{N}}\) are \(C\)-equivalent as well.

**Lemma 1.8** Assume \((l_n)_{n \in \mathbb{N}}\) is a block-sequence in \(bb_Q(X)\) which \(C\)-continuously embeds into its subsequences, and let \(L = [l_n, n \in \mathbb{N}]\). Then Player 2 has a winning strategy in the game \(G_{L,L}\).
Proof: Assume Player 1’ first move was \((x_1, n_1)\); write \(x_1 = \sum_{j \leq k_1} \lambda_j l_j\). Letting in the game \(H_L\) Player 1 play the integer \(n_1\), \(k_1\) times, the winning strategy of Lemma 1.6 provides moves \((F_1^1, z_1), \ldots, (F_1^{k_1}, z_{k_1})\) for Player 2 in that game. We let \(y_1 = \sum_{j \leq k_1} \lambda_j z_j\), and \(F_1 = \sum_{j=1}^{k_1} F_1^j\). In particular, \(n_1 \leq F_1 \subseteq L \) and \(y_1 \in F_1\).

We describe the choice of \(F_p\) and \(y_p\) at step \(p\). Assuming Player 1’ \(p\)-th move was \((x_p, n_p)\); we write \(x_p = \sum_{k_p-1 < j \leq k_p} \lambda_j l_j\). Letting in the game \(H_L\) Player 1 play \(k_p - k_p-1\) times the integer \(n_p\), the winning strategy of Lemma 1.6 provides moves \((F_p^{k_p-1+1}, z_{k_p-1+1}), \ldots, (F_p^{k_p}, z_{k_p})\) for Player 2 in that game. We let \(y_p = \sum_{k_p-1 < j \leq k_p} \lambda_j z_j\), and \(F_p = \sum_{k_p-1 < j \leq k_p} F_p^j\). In particular, \(n_p \leq F_p \subseteq L \) and \(y_p \in \sum_{j=1}^{k_p} F_j\).

Finally by construction, \((z_n)_{n \in \mathbb{N}}\) is \(C\)-equivalent to \((l_n)_{n \in \mathbb{N}}\). It follows that \((y_p)_{p \in \mathbb{N}}\) is \(C\)-equivalent to \((x_p)_{p \in \mathbb{N}}\). \(\square\)

The non-trivial Lemma 1.8 will serve as the first step of a final induction which is on the model of the demonstration of Pelczar in [22] (note that there, the first step of the induction was straightforward). The rest of our reasoning in this section will now be along the lines of her work, with the difference that we chose to express the reasoning in terms of games instead of using trees, and that we needed the moves of Player 2 to include the choice of finite dimensional subspaces \(F_n\’s\) in which to pick the vectors \(y_n\’s\). This is due to the fact that the basic sequence which witnesses the embedding of \(X\) into a given subspace generated by a subsequence is not necessarily successive on the basis of \(X\).

Let \(L, M\) be block-subspaces in \(\mathcal{G}_Q(X)\). Let \(a \in bb^{<\omega}_Q(X)\) and \(b \in (\text{Fin}_Q(X) \times Q(X))^<\omega\) be such that \(|a| = |b|\) or \(|a| = |b| + 1\) (here \(|x|\) denotes as usual the length of the finite sequence \(x\)). Such a couple \((a, b)\) will be called a state of the game \(G_{L,M}\) and the set of states will be written \(St(X)\). It is important to note that \(St(X)\) is countable. The empty sequence in \(bb^{<\omega}_Q(X)\) (resp. \((\text{Fin}_Q(X) \times Q(X))^{<\omega}\) will be denoted by \(\emptyset\).

We define \(G_{L,M}(a, b)\) intuitively as “the game \(G_{L,M}\) starting from the state \((a, b)\)”’. More precisely, if \(|a| = |b|\), then write \(a = (a_1, \ldots, a_p)\) and \(b = (b_1, \ldots, b_p)\), with \(b_i = (B_i, \beta_i)\) for \(i \leq p\).

A \(k\)-th move for Player 1 is \((x_k, n_k) \in Q(X) \times \mathbb{N}\), with \(x_k \in L, x_1 > a_p\) if \(k = 1\) and \(a \neq \emptyset\), and \(x_k > x_{k-1}\) if \(k \geq 2\). A \(k\)-th move for Player 2 is \((F_k, y_k) \in \text{Fin}_Q(X) \times Q(X)\) with \(n_k \leq F_k \subseteq M\) and \(y_k \in B_1 \oplus \ldots \oplus B_p \oplus F_1 \oplus \ldots \oplus F_k\) for all \(k\).
Lemma 1.9

Define the following order relation on $G_LM(a, b)$, see also the proof by B. Maurey of Gowers’ dichotomy theorem [21]. We define the following order relation on $G_LM(a, b)$, see also the proof by B. Maurey of Gowers’ dichotomy theorem [21].

Player 2 wins $G_LM(a, b)$ if the sequence $(β_1, ..., β_p, y_1, y_2, ...)$ is $C$-equivalent to the sequence $(a_1, ..., a_p, x_1, x_2, ...)$. Player 2 wins $G_LM(a, b)$ if the sequence $(β_1, ..., β_p, y_1, y_2, ...)$ is $C$-equivalent to the sequence $(a_1, ..., a_p, x_1, x_2, ...)$. 

Now if $|a| = |b| + 1$, then write $a = (a_1, ..., a_{p+1})$ and $b = (b_1, ..., b_p)$, with $b_i = (B_i, β_i)$ for $i ≤ p$.

A first move for Player 1 is $n_1 ∈ N$. A first move for Player 2 is $(F_1, y_1) ∈ Fin_Q(X) × Q(X)$ with $n_1 ≤ F_1 ⊂ M$ and $y_1 ∈ B_1 ⊕ ... ⊕ B_p ⊕ F_1$.

For $k ≥ 2$, a $k$-th move for Player 1 is $(x_k, n_k) ∈ Q(X) × N$, with $x_k ∈ L$, $x_2 > a_{p+1}$ if $k = 2$, and $x_k > x_{k−1}$ if $k > 2$; a $k$-th move for Player 2 is $(F_k, y_k) ∈ Fin_Q(X) × Q(X)$ with $n_k ≤ F_k ⊂ M$ and $y_k ∈ B_1 ⊕ ... ⊕ B_p ⊕ F_1 ⊕ ... ⊕ F_k$.

Player 2 wins $G_LM(a, b)$ if the sequence $(β_1, ..., β_p, y_1, y_2, ...)$ is $C$-equivalent to the sequence $(a_1, ..., a_p, x_1, x_2, ...)$. 

We shall use the following classical stabilization process, called ”zawada” in [22], see also the proof by B. Maurey of Gowers’ dichotomy theorem [21]. We define the following order relation on $G_Q(X)$: for $M, N ∈ G_Q(X)$, with $M = [m_i]_{i ∈ N}, (m_i)_{i ∈ N} ∈ bb_q(X)$, write $M C^* N$ if there exists $p ∈ N$ such that $m_i ∈ N$ for all $i ≥ p$.

Let $τ$ be a mapping defined on $G_Q(X)$ with values in the set $2^Σ$ of subsets of some countable set $Σ$. Assume the map $τ$ is monotonous with respect to $C^*$ on $G_Q(X)$ and to inclusion on $2^Σ$. Then by [22] Lemma 2.1, there exists a block-subspace $M ∈ G_Q(X)$ which is stabilizing for $τ$, i.e. $τ(N) = τ(M)$ for every $N ⊂^* M$.

We now define a map $τ : G_Q(X) → 2^{St(X)}$ by $(a, b) ∈ τ(M)$ iff there exists $L ⊂^* M$ such that Player 2 has a winning strategy for the game $G_{LM}(a, b)$.

**Lemma 1.9** Let $M’$ and $M$ be in $G_Q(X)$. If $M’ ⊂^* M$ then $τ(M’) ⊂ τ(M)$.
Proof: Let $M' \subset^* M$, let $(a, b) \in \tau(M')$, and let $L \subset^* M'$ be such that Player 2 has a winning strategy in $G_{L, M'}(a, b)$. Let $m$ be an integer such that for any $x \in \mathbb{Q}(X)$, $x \in M'$ and $\text{min}(\text{supp}(x)) \geq m$ implies $x \in M$. We describe a winning strategy for Player 2 in the game $G_{L, M}(a, b)$: assume Player 1’s $p$-th move was $(n_p, x_p)$ (or just $n_1$ if it was the first move and $|a| = |b| + 1$), without loss of generality $n_p \geq m$. Let $(F_p, y_p)$ be the move prescribed by the winning strategy for Player 2 in $G_{L, M'}(a, b)$. Then $F_p \geq n_p \geq m$ and $F_p \subset M'$ so $F_p \subset M$. The other conditions are satisfied to ensure that we have described the $p$-th move of a winning strategy for Player 2 in the game $G_{L, M}(a, b)$. It remains to note that $L \subset^* M$ as well to conclude that $(a, b) \in \tau(M)$. □

By the stabilization lemma, there exists a block-subspace $M_0 \in \mathcal{G}_\mathbb{Q}(X)$ such that for any $M \subset^* M_0$, $\tau(M) = \tau(M_0)$.

For $L, M \in \mathcal{G}_\mathbb{Q}(X)$ we shall write $L =^* M$ if $L \subset^* M$ and $M \subset^* L$.

We now define a map $\rho : \mathcal{G}_\mathbb{Q}(X) \rightarrow 2^{\text{St}(X)}$ by $(a, b) \in \rho(M)$ iff there exists $L =^* M$ such that Player 2 has a winning strategy for the game $G_{L, M_0}(a, b)$.

Lemma 1.10 Let $M'$ and $M$ be in $\mathcal{G}_\mathbb{Q}(X)$. If $M' \subset^* M$ then $\rho(M') \supset \rho(M)$.

Proof: Let $M' \subset^* M$, let $(a, b) \in \rho(M)$, and let $L =^* M$ be such that Player 2 has a winning strategy in $G_{L, M_0}(a, b)$. Define $L' = M' \cap L$. As $L' \subset L$, it follows immediately that Player 2 has a winning strategy in the game $G_{L', M_0}(a, b)$. It is also clear that $L' =^* M'$ so $(a, b) \in \rho(M')$. □

So there exists a block-subspace $M_{00} \in \mathcal{G}_\mathbb{Q}(X)$ of $M_0$ which is stabilizing for $\rho$, i.e. for any $M \subset^* M_0$, $\rho(M) = \rho(M_{00})$.

Lemma 1.11 $\rho(M_{00}) = \tau(M_{00}) = \tau(M_0)$.

Proof: First it is obvious by definition of $M_0$ that $\tau(M_{00}) = \tau(M_0)$.

Let $(a, b) \in \rho(M_{00})$. There exists $L =^* M_{00}$ such that Player 2 has a winning strategy in $G_{L, M_0}(a, b)$; as $M \subset^* M_0$, this implies that $(a, b) \in \tau(M_0)$.

Let $(a, b) \in \tau(M_{00})$. There exists $L \subset^* M_{00}$ such that Player 2 has a winning strategy in $G_{L, M_{00}}(a, b)$. As $M_{00} \subset M_0$, this is a winning strategy for $G_{L, M_0}(a, b)$ as well. This implies that $(a, b) \in \rho(L)$ and by the stablization property for $\rho$, $(a, b) \in \rho(M_{00})$. □
We now turn to the concluding part of the proof of Theorem 1.1. By our assumption about $X$ just before Definition 1.7, there exists a block-sequence $(l_n)_{n \in \mathbb{N}}$ of $\text{bb}_Q(X)$ which is contained in $M_{00}$, and $C$-continuously embeds into its subspaces, and without loss of generality assume that $L_0 := [l_n, n \in \mathbb{N}] = M_{00}$. We fix an arbitrary block-subspace $M$ of $L_0$ generated by a block-sequence in $\text{bb}_Q(X)$ and we shall prove that $L_0$ embeds into $M$. By standard perturbation arguments this implies that $L_0$ is minimal.

We construct by induction a subsequence $(a_n)_{n \in \mathbb{N}}$ of $(l_n)_{n \in \mathbb{N}}$, a sequence $b_n = (F_n, y_n) \in (\text{Fin}_Q(X) \times Q(X))^\omega$ such that $F_n \subset M$ and $y_n \in F_1 \oplus \ldots F_n$ for all $n \in \mathbb{N}$, and such that $((a_n)_{n \leq p}, (F_n, y_n)_{n \leq p}) \in \rho(L_0)$ for all $p \in \mathbb{N}$.

By Lemma 1.8, Player 2 has a winning strategy in $G_{L_0 L_0}$, and so in particular $(\emptyset, \emptyset) \in \rho(L_0)$ (recall that $\emptyset$ denotes the empty sequence in the sets corresponding to the first and second coordinates). This takes care of the first step of the induction.

Assume $(a, b) = ((a_n)_{n \leq p-1}, (F_n, y_n)_{n \leq p-1})$ is a state such that $(a_n)_{n \leq p-1}$ is a finite subsequence of $(l_n)_{n \in \mathbb{N}}$, such that $F_n \subset M$ and $y_n \in F_1 \oplus \ldots F_n$ for all $n \leq p-1$, and such that $(a, b) \in \rho(L_0)$.

As $(a, b)$ belongs to $\rho(L_0)$, there exists $L =^* L_0$ such that Player 2 has a winning strategy in the game $G_{L_0, L_0}(a, b)$. In particular $L_0 =^* L$ so we may choose $m_p$ large enough so that $l_{m_p} > a_{p-1}$ and $l_{m_p} \in L$; we let Player 1 play $a_p = l_{m_p}$. Player 2 has a winning strategy in the game $G_{L_0, L_0}(a', b')$, where $a' = (a_n)_{n \leq p}$. In other words, $(a', b)$ belongs to $\rho(L_0)$. Now $\rho(L_0) = \tau(M)$, so there exists $L =^* M$ such that Player 2 has a winning strategy in the game $G_{L, M}(a', b)$. Let Player 1 play any integer $n_p$, and $(F_p, y_p)$ with $F_p \subset M$ and $y_p \in F_1 \oplus \ldots \oplus F_p$ be a move for Player 2 prescribed by that winning strategy in response to $n_p$. Once again, Player 2 has a winning strategy in $G_{L, M}(a', b')$, with $b' = (F_n, y_n)_{n \leq p}$; i.e. $(a', b') \in \tau(M) = \rho(L_0)$.

To conclude, note that $(a_n, b_n)_{n \leq p} \in \rho(L_0)$ implies in particular that $(a_n)_{n \leq p}$ and $(y_n)_{n \leq p}$ are $C$-equivalent, and this is true for any $p \in \mathbb{N}$, so $(a_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are $C$-equivalent. So $[a_n]_{n \in \mathbb{N}}$ embeds into $M$. Now $(a_n)_{n \in \mathbb{N}}$ is a subsequence of $(l_n)_{n \in \mathbb{N}}$ so by our hypothesis, $L_0$ $C$-embeds into $[a_n]_{n \in \mathbb{N}}$ and thus $C^2$-embeds in $M$, and this concludes the proof of Theorem 1.1.

D. Kutzarova drew our attention to the dual $T^*$ of Tsirelson’s space; it is minimal [6], but contains no block-minimal block-subspace (use e.g. [6] Proposition 2.4 and Corollary 7.b.3 in their $T^*$ versions, with Remark 1 after [6] Proposition 1.16). So Theorem 1.1 applies to situations where the Theorem of Pelczar does
not. On the other hand, we do have:

**Corollary 1.12** A Banach space with a Schauder basis which is saturated with isomorphically homogeneous basic sequences contains a block-minimal block-subspace.

*Proof:* Let $X$ have a Schauder basis and be saturated with isomorphically homogeneous basic sequences. By the beginning of the proof of Lemma 1.3 we may assume $X$ is reflexive. By Theorem 1.1 there exists a minimal subspace $Y$ in $X$, which is a block-subspace if you wish; passing to a further block-subspace assume furthermore that $Y$ has an isomorphically homogeneous basis. Take any block-subspace $Z$ of $Y = [y_n]_{n \in \mathbb{N}}$, then $Y$ embeds into $Z$. By reflexivity and standard perturbation results, some subsequence of $(y_n)_{n \in \mathbb{N}}$ spans a subspace which embeds as a block-subspace of $Z$. As $(y_n)_{n \in \mathbb{N}}$ is isomorphically homogeneous, this means that $Y$ embeds as a block-subspace of $Z$. □

We recall that a Banach space is said to be ergodic if the relation $E_0$ is Borel reducible to the relation of isomorphism between its subspaces.

**Corollary 1.13** A Banach space is either ergodic or contains a minimal subspace.

*Proof:* We prove the stronger result that if $X$ is a Banach space with a Schauder basis, then either $E_0$ is Borel reducible to isomorphism between block-subspaces of $X$ or $X$ contains a block-minimal block-subspace.

Assume $E_0$ is not Borel reducible to isomorphism between block-subspaces of $X$. By [23, Theorem 19], any block-sequence in $X$ has an isomorphically homogeneous subsequence. In particular $X$ is saturated with isomorphically homogeneous sequences, so apply Corollary 1.12. □

**Corollary 1.14** A Banach space $X$ contains a minimal subspace or the relation $E_0$ is Borel reducible to the relation of biembeddability between subspaces of $X$.

*Proof:* Note that the relation $\sim_{\text{emb}}$ of biembeddability between subspaces of $X$ is analytic. By [23, Theorem 15], if $E_0$ is not Borel reducible to biembeddability between subspaces of $X$, then every basic sequence in $X$ has a subsequence $(x_n)_{n \in \mathbb{N}}$ which is homogeneous for the relation between subsequences corresponding to $\sim_{\text{emb}}$, that is, for any subsequence $(x_n)_{n \in I}$ of $(x_n)_{n \in \mathbb{N}}$, $[x_n]_{n \in I} \sim_{\text{emb}} [x_n]_{n \in \mathbb{N}}$. 15
This means that \((x_n)_{n \in \mathbb{N}}\) embeds into its subsequences. So \(X\) is saturated with basic sequences which embed into their subsequences. \(\square\)

We conclude this section with a remark about the proof of Theorem 1.1. The sequences \((m_p)_{p \in \mathbb{N}} \in [\omega]^{\omega}\) (associated to a subsequence of \((l_n)_{n \in \mathbb{N}}\)) and \(b_p = (F_p, y_p) \in (\text{Fin}_Q(X) \times \mathbb{Q}(X))^{\omega}\) (with \((y_p)_{p \in \mathbb{N}}\) \(C\)-equivalent to \((l_m)_{p \in \mathbb{N}}\)) in our final induction may clearly be chosen with \(F_p \subset M_p\) for all \(p\), for an arbitrary sequence \((M_p)_{p \in \mathbb{N}}\) of block-subspaces of \(L_0\). Also, \((l_n)_{n \in \mathbb{N}}\) \(C\)-continuously embeds into its subsequences, i.e. there is a continuous map \(f : [\omega]^{\omega} \rightarrow \text{bb}_Q(X)\) such that \(f(A)\) is \(C\)-equivalent to \((l_n)_{n \in \mathbb{N}}\) for all \(A \in \text{bb}_Q(X)\).

By combining these two facts, it is easy to see that Player 2 has a winning strategy to produce a sequence \((y_n)_{n \in \mathbb{N}}\) which is \(C^2\)-equivalent to \((l_n)_{n \in \mathbb{N}}\), in a ”modified” Gowers’ game, where a \(p\)-th move for Player 1 is a block-subspace \(Y_p \in \mathbb{G}_Q(X)\), with \(Y_p \subset L_0\), and a \(p\)-th move for Player 2 is a couple \((F_p, y_p) \in (\text{Fin}_Q(X) \times \mathbb{Q}(X))^{\omega}\) with \(F_p \subset Y_p\) and \(y_p \in F_1 \oplus \ldots \oplus F_p\).

This is an instance of a result with a Gowers’ type game where Player 2 is allowed to play sequences of vectors which are not necessarily block-basic sequences.

## 2 Topological 0-1 laws for block-sequences.

In this section \(X\) denotes a Banach space with a normalized basis \((e_n)_{n \in \mathbb{N}}\). It will be necessary to restrict our attention to normalized block-bases in \(X\) to use compactness properties. We denote by \(\text{bb}(X)\) the set of normalized block-bases on \(X\). Let \(Q(X)\) be the set of normalized blocks of the basis that are a multiple of some block with rational coordinates (or coordinates in \(\mathbb{Q} + i\mathbb{Q}\) in the complex case). We denote by \(\text{bb}_d(X)\) the set of block-bases of vectors in \(Q(X)\) (here ”\(d\)” stands for ”discrete”, this notation was introduced in [9]). We consider \(\text{bb}_d(X)\) as a topological space, equipped with the product topology of the discrete topology on \(Q(X)\), which turns it into a Polish space.

The notation \(\text{bb}_d^{\leq \omega}(X)\) will denote the set of finite block-sequences with blocks in \(Q(X)\). For a finite block sequence \(\tilde{x} = (x_1, \ldots, x_n) \in \text{bb}_d^{\leq \omega}(X)\), we denote by \(N(\tilde{x})\) the set of elements of \(\text{bb}_d(X)\) whose first \(n\) vectors are \((x_1, \ldots, x_n)\); this is the basic open set associated to \(\tilde{x}\).

If \(s\) is a finite block-basis and \(y\) is a finite or infinite block-basis supported after \(s\), denote by \(s^{-}\) \(y\) the concatenation of \(s\) and \(y\). The notation \(x = (x_n)_{n \in \mathbb{N}}\) will be reserved to denote an infinite block-sequence, and \(\bar{x}\) will denote its closed
linear span: $\tilde{x}$ will denote a finite block-sequence, and $|\tilde{x}|$ its length as a sequence, $\text{supp}(\tilde{x})$ the union of the supports of the terms of $\tilde{x}$. For two finite block-sequences $\tilde{x} = (x_1, \ldots, x_n)$ and $\tilde{y} = (y_1, \ldots, y_m)$, write $\tilde{x} < \tilde{y}$ to mean that they are successive, i.e. $x_n < y_1$. For a sequence of successive finite block-sequences $(\tilde{x}_i)_{i \in I}$, we denote the concatenation of the block-sequences by $\tilde{x}_1 \cdots \tilde{x}_n$ if the sequence is finite with $I = \{1, \ldots, n\}$, or $\tilde{x}_1 \tilde{x}_2 \cdots$ if it is infinite, and we denote by $\text{supp}(\tilde{x}_i, i \in I)$ the support of the concatenation, by $[\tilde{x}_i]_{i \in I}$ the closed linear span of the concatenation.

2.1 A principle of topological 0-1 law for block-sequences.

We recall a characterization of comeager subsets of $bb_d(X)$ which was proved in [10]. If $A$ is a subset of $bb_d(X)$ and $\Delta = (\delta_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers, we denote by $A_\Delta$ the $\Delta$-expansion of $A$ in $bb_d(X)$, that is $x = (x_n) \in A_\Delta$ iff there exists $y = (y_n) \in A$ such that $\|y_n - x_n\| \leq \delta_n$, $\forall n \in \mathbb{N}$. Such an $y$ will be called a $\Delta$-perturbation of $x$. Given a finite block-sequence $\tilde{x} = (x_1, \ldots, x_n)$, we say that a (finite or infinite) block-sequence $(y_i)_{i \in \mathbb{N}}$ passes through $\tilde{x}$ if there exists some integer $m$ such that $\forall 1 \leq i \leq n, y_{m+i} = x_i$.

**Proposition 2.1** (V. Ferenczi, C. Rosendal [10]) Let $X$ be a Banach space with a Schauder basis. Let $A$ be comeager in $bb_d(X)$. Then for all $\Delta > 0$, there exists a sequence $(\tilde{a}_n)_{n \in \mathbb{N}} \in (bb_{d_{\Delta}}^\omega(X))^\omega$ of successive finite block-sequences such that any block-sequence of $bb_d(X)$ passing through infinitely many of the $\tilde{a}_n$’s is in $A_\Delta$.

As was noted in [10], the property in the conclusion of this proposition is essentially (i.e. up to perturbation) a characterization of comeager sets in $bb_d(X)$. Indeed, it easily implies that $A_\Delta$ is comeager.

Let $A$ have the Baire Property, that is, there exists an open set $U$ such that $A \setminus U$ and $U \setminus A$ are meager. Then either $A$ is meager, or $A$ is comeager in $N(\tilde{x}_0)$ for some finite block-sequence $\tilde{x}_0 \in bb_{d_{\Delta}}^\omega(X)$. This fact is to be combined with Proposition 2.1 For example, $A$ is comeager in $N(\tilde{x}_0)$ will imply that for all $\Delta > 0$, there exists a sequence of successive finite block-sequences $(\tilde{a}_n)_{n \in \mathbb{N}}$ such that any element of $bb_d(X)$ passing through $\tilde{x}_0$ and infinitely many of the $\tilde{a}_n$’s is in $A_\Delta$. This result will take more interest if one assumes a few natural additional properties for the set $A$. 

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In the following we identify a block-sequence \((x_k)_{k\in K}\) indexed on some infinite subset \(K = \{k_1, k_2, \ldots\}\) of \(\mathbb{N}\) (where \((k_n)_{n\in\mathbb{N}}\) is increasing), with the block-sequence \((x_{k_n})_{n\in\mathbb{N}}\) indexed on \(\mathbb{N}\); so given an infinite block-sequence, we may always chose the more convenient way to index it. We do the similar identification for finite block-sequences.

For \((x_n)_{n\in\mathbb{N}}\) a block-sequence and \(n_0 \in \mathbb{N}\), we call \(n_0\)-modification of \((x_n)_{n\in\mathbb{N}}\) a block-sequence \((y_n)_{n\in\mathbb{N}}\) such that \(x_n = y_n\) for all \(n > n_0\). An \(n_0\)-modification of \((x_n)_{n\in\mathbb{N}}\) for some \(n_0\) will be called a finite modification of \((x_n)_{n\in\mathbb{N}}\). For a block-sequence \((x_n)_{n\in\mathbb{N}}\) of \(X\), a couple \(((x_n)_{n\in I}, (x_n)_{n\in J})\) of block-sequences associated to a partition of \(\mathbb{N}\) in two infinite sets \(I\) and \(J\) will be called a partition of \((x_n)_{n\in\mathbb{N}}\).

Related to the notion of support is the useful notion of range: the range \(\text{ran}(x_0)\) of \(x_0 \in X\) is the smallest interval of integers containing the support of \(x_0\). If \(x = (x_n)_{n\in I}\) is a finite or infinite block-sequence, \(\text{ran}(x)\) will denote the union \(\bigcup_{n\in I} \text{ran}(x_n)\). When \(x = (x_n)_{n\in I}, y = (y_n)_{n\in J}\) are finite or infinite block-sequences whose ranges are disjoint, we call concatenation of \(x\) and \(y\) the unique (up to the choice of \(K\)) block-sequence \(z = (z_n)_{n\in K}\) such that \(\{z_n, n \in K\} = \{x_n, n \in I\} \cup \{y_n, n \in J\}\).

We are now ready to state our principle of topological 0-1 law for block-sequences.

**Theorem 2.2** (Topological 0-1 law for block-sequences) Let \(X\) be a Banach space with a Schauder basis. Assume \(A \subset \text{bd}(X)\) has the Baire Property and is invariant by finite modifications. Then \(A\) is either meager or comeager in \(\text{bd}(X)\).

This is a corollary of the following quantified version:

**Proposition 2.3** Let \(X\) be a Banach space with a Schauder basis. Let \((A_N)_{N\in\mathbb{N}}\) be an increasing sequence of subsets of \(\text{bd}(X)\) with the Baire Property, and let \(A = \bigcup_{N\in\mathbb{N}} A_N\). Assume that for any \(N \in \mathbb{N}\) and \(n_0 \in \mathbb{N}\), there exists \(K(N, n_0) \in \mathbb{N}\) such that whenever \((x_n)_{n\in\mathbb{N}}\) belongs to \(A_N\), then any \(n_0\)-modification of \((x_n)_{n\in\mathbb{N}}\) belongs to \(A_{K(N, n_0)}\).

Then either \(A\) is meager in \(\text{bd}(X)\), either there exists \(K \in \mathbb{N}\) such that \(A_K\) is comeager in \(\text{bd}(X)\).

**Proof:** We assume \(A\) is non-meager, then for some \(N \in \mathbb{N}\), \(A_N\) is non-meager. We reproduce a proof of [10]. As \(A_N\) has the Baire property, it is comeager in
some basic open set $U$, of the form $N(\tilde{x})$, for some finite block-sequence $\tilde{x} \in bb_d^{\leq\omega}(X)$.

We now prove that $A_K$ is comeager in $bb_d(X)$ for $K = Nc(2\max(supp(\tilde{x})))$ (for $n \in \mathbb{N}$, $c(n)$ denotes an integer such for any Banach space $X$, any $n$-codimensional subspaces of $X$ are $c(n)$-isomorphic, see e.g. [9]). So let us assume $V = N(\tilde{y})$ is some basic open set in $bb_d(X)$ such that $A_K$ is meager in $V$. We may assume that $|\tilde{y}| > |\tilde{x}|$ and write $\tilde{y} = \tilde{x}^\sim\tilde{z}$ with $\tilde{x} < \tilde{z}$ and $|\tilde{x}'| \leq \max(supp(\tilde{x}))$. Choose $\tilde{u}$ and $\tilde{v}$ in $bb_d^{\leq\omega}(X)$ such that $\tilde{u}, \tilde{v} > \tilde{z}$, $|\tilde{u}| = |\tilde{x}'|$ and $|\tilde{v}| = |\tilde{x}|$, and such that $\max(supp(\tilde{u})) = \max(supp(\tilde{v}))$. Let $U'$ be the basic open set $N(\tilde{x}^\sim\tilde{z}\sim\tilde{u})$ and let $V'$ be the basic open set $N(\tilde{x}'\sim\tilde{z}\sim\tilde{v})$. Again $A_N$ is comeager in $U'$ while $A_K$ is meager in $V'$.

Now let $T$ be the canonical map from $U'$ to $V'$. For all $u$ in $U'$, $T(u)$ differs from at most $|\tilde{x}| + \max(supp(\tilde{x})) \leq 2\max(supp(\tilde{x}))$ vectors from $u$, so $|T(u)|$ is $c(2\max(supp(\tilde{x})))$ isomorphic to $|u|$. Since $K = Nc(2\max(supp(\tilde{x})))$ it follows that $A_K$ is comeager in $V' \subset V$. The contradiction follows by choice of $V$. $\square$

Proposition 2.1 characterizes meager and comeager sets in the conclusion of Theorem 2.2. This leads us to the following theorem.

**Theorem 2.4** Let $X$ be a Banach space with a Schauder basis. Let $A$ be a subset of $bb_d(X)$ with the Baire Property, which is stable by $\Delta$-perturbations for some $\Delta > 0$, by finite modifications, and by taking subsequences. Assume that any sequence $(\tilde{x}_n)_{n \in \mathbb{N}} \in (bb_d^{\leq\omega}(X))^\omega$ of successive finite block-sequences admits a subsequence $(\tilde{x}_{n_k})_{k \in \mathbb{N}}$ such that the block-sequence $\tilde{x}_{n_1}^\sim\tilde{x}_{n_2}^\sim\ldots$ belongs to $A$.

Then every block-sequence in $bb_d(X)$ admits a partition in a couple of elements of $A$.

If furthermore, the set $A$ is stable by concatenation of pairs of block-sequences, then $bb_d(X) = A$.

Once again this is a corollary of a quantified version:

**Proposition 2.5** Let $X$ be a Banach space with a Schauder basis. Let $(A_N)_{N \in \mathbb{N}}$ be an increasing sequence of subsets of $bb_d(X)$ with the Baire Property, such that:

(a) there exists $\Delta > 0$ such that for any $N \in \mathbb{N}$, there exists $K_1(N) \in \mathbb{N}$ such that $(A_N)_\Delta \subset A_{K_1(N)}$.

(b) for any $N \in \mathbb{N}$ and $n_0 \in \mathbb{N}$, there exists $K_2(N, n_0) \in \mathbb{N}$ such that whenever $(x_n)_{n \in \mathbb{N}}$ belongs to $A_N$, then any $n_0$-modification of $(x_n)_{n \in \mathbb{N}}$ belongs to $A_{K_2(N, n_0)}$. 

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(c) for any \( N \in \mathbb{N} \), there exists \( K_3(N) \in \mathbb{N} \) such that whenever \( (x_n)_{n \in \mathbb{N}} \) belongs to \( A_N \) then any subsequence of \( (x_n)_{n \in \mathbb{N}} \) belongs to \( A_{K_3(N)} \).

Let \( A = \bigcup_{N \in \mathbb{N}} A_N \). Assume that any sequence \( (\tilde{x}_n)_{n \in \mathbb{N}} \in (bb_d^{\omega}(X))^{\omega} \) of successive finite block-sequences admits a subsequence \( (\tilde{x}_{nk})_{k \in \mathbb{N}} \) such that the block-sequence \( \tilde{x}_{n_1} \tilde{x}_{n_2} \ldots \) belongs to \( A \).

Then there exists \( N \in \mathbb{N} \) such that every block-sequence in \( bb_d(X) \) has a partition in two elements of \( A_N \). If furthermore,

(d) for any \( N \in \mathbb{N} \), there exists \( K_4(N) \in \mathbb{N} \) such that any concatenation of a couple of block-sequences in \( A^2_N \) belongs to \( A_{K_4(N)} \),

then \( bb_d(X) = A_N \) for some \( N \in \mathbb{N} \).

Proof: The part which is a consequence of (d) is obvious once we prove the first part of the proposition. We note that by Proposition 2.3 or \( A \) is meager, or \( A_N \) is comeager for some \( N \in \mathbb{N} \). By (a), there is some \( \Delta > 0 \) such that \( A = A_{\Delta} \). It follows that \( A_{\Delta} \cap A^C = \emptyset \), that is \( (A^C)_{\Delta} \cap A = \emptyset \).

If \( A \) is meager, Proposition 2.1 gives us a sequence of successive finite block-sequences \( (\tilde{x}_n)_{n \in \mathbb{N}} \) such that, in particular, \( \tilde{x}_{n_1} \tilde{x}_{n_2} \ldots \) is in \( (A^C)_{\Delta} \) for every subsequence \( (\tilde{x}_{nk})_{k \in \mathbb{N}} \), so for no subsequence \( (\tilde{x}_{nk})_{k \in \mathbb{N}}, \tilde{x}_{n_1} \tilde{x}_{n_2} \ldots \) is in \( A \).

So \( A_N \) is comeager for some \( N \in \mathbb{N} \). Applying Proposition 2.1 and up to modifying \( N \), let \( (\tilde{a}_n)_{n \in \mathbb{N}} \) be a sequence of successive block-sequences such that every block-sequence passing through infinitely many of the \( \tilde{a}_n \)’s is in \( A_N \).

Let now \( (x_n)_{n \in \mathbb{N}} \) be an arbitrary block-sequence in \( bb_d(X) \). We note that we may find a partition of \( (x_n)_{n \in \mathbb{N}} \) in two subsequences \( (x_n)_{n \in I} \) and \( (x_n)_{n \in J} \), and a subsequence \( (\tilde{a}_{nk})_{k \in \mathbb{N}} \) of \( (\tilde{a}_n)_{n \in \mathbb{N}} \) such that \( (x_n)_{n \in I} \) and \( (\tilde{a}_{nk})_{k \in \mathbb{N}} \) have disjoint ranges (let \( (i_n)_{n \in \mathbb{N}} \) denote their concatenation) and such that \( (x_n)_{n \in J} \) and \( (\tilde{a}_{nk+1})_{k \in \mathbb{N}} \) have disjoint ranges (let \( (j_n)_{n \in \mathbb{N}} \) denote their concatenation).

Now \( (i_n)_{n \in \mathbb{N}} \) belongs to \( A_N \), so by (c), for some \( N' \in \mathbb{N} \), \( (x_n)_{n \in I} \) belongs to \( A_{N'} \), and likewise \( (x_n)_{n \in J} \) belongs to \( A_{N'} \). \( \square \)

In particular we deduce a uniformity principle from Proposition 2.5. Under its hypotheses, and if every block-sequence of \( bb_d(X) \) is in \( A \), then there exists \( N \in \mathbb{N} \) such that every block-sequence of \( bb_d(X) \) is in \( A_N \). This method was first used in [10] to study the property of complementable embeddability ([10] Proposition 17).
Before passing to applications, we note that there is a case when Proposition 2.5 is not so interesting. It is when the set $A$ is $F_\sigma$ (in which case we may assume $A$ is the union of an increasing sequence of closed sets $(A_n)_{n \in \mathbb{N}}$). In that case, there is a much more direct proof of it, which does not use the subsequence hypothesis (c) nor the concatenation hypothesis (d). A typical instance of this situation is when $A$ is the set of block-sequences in $bb_d(X)$ which are equivalent to a given basic sequence.

**Remark 2.6** Let $X$ be a Banach space with a Schauder basis. Let $(A_n)_{n \in \mathbb{N}}$ be an increasing sequence of closed subsets of $bb_d(X)$ and let $A = \bigcup_{n \in \mathbb{N}} A_n$. Assume hypotheses (a) and (b) from Proposition 2.5 are satisfied. If any sequence $(\tilde{x}_n)_{n \in \mathbb{N}} \in (bb_d^{<\omega}(X))^\omega$ of successive finite block-sequences admits a subsequence $(\tilde{x}_{n_k})_{k \in \mathbb{N}}$ such that the block-sequence $\tilde{x}_{n_1}\tilde{x}_{n_2}\ldots$ belongs to $A$, then $bb_d(X) = A_N$ for some $N \in \mathbb{N}$.

**Proof:** Assume the conclusion does not hold. Note that for every $k \in \mathbb{N}$, every finite block-sequence $\tilde{x}_0 \in bb_d^{<\omega}(X)$, there exists a finite block-sequence $\tilde{x} \in bb_d^{<\omega}(X)$ such that $\tilde{x}_0^\sim \tilde{x}$ is not extendable in an element of $A_k$. Otherwise, by closedness of $A_k$, we would have that $\tilde{x}_0^\sim x \in A_k$ for all $x \in bb_d(X)$ supported after $\tilde{x}_0$, and using (b) we would deduce that every $x \in bb_d(X)$ belongs to $A_K$ for some $K$ depending on $k$ and the length of the support of $\tilde{x}_0$.

It follows easily that for any $k \in \mathbb{N}$, for any finite block-sequences $\tilde{x}_0, \ldots, \tilde{x}_n$ in $bb_d^{<\omega}(X)$, there exists $\tilde{x}$ such that $\tilde{x}_i^\sim \tilde{x}$ is extendable in an element of $A_k$ for no $i \leq n$.

Using this fact, we find $\tilde{a}_1 \in bb_d^{<\omega}(X)$ not extendable in $A_1$, and by induction, for any $n \in \mathbb{N}$, a finite block-sequence $\tilde{a}_n > \tilde{a}_{n-1}$ such that $\tilde{a}_{i_1} \ldots \tilde{a}_{i_p} \tilde{a}_n$ is extendable in $A_n$ for no finite sequence $i_1 < \cdots < i_p < n$.

Consider now the sequence $(\tilde{a}_n)_{n \in \mathbb{N}}$. By construction, for any subsequence $(\tilde{a}_{n_k})_{k \in \mathbb{N}}$, the block-sequence $\tilde{a}_{n_1}\tilde{a}_{n_2}\ldots$ is in $A_{n_k}$ for no $n_k, k \in \mathbb{N}$, so does not belong to $A$, a contradiction with the hypothesis. \qed

**2.2 Isomorphism between block-subspaces.**

Recall that two basic sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are said to be *permutatively equivalent* if there is a permutation $\sigma$ on $\mathbb{N}$ such that $(x_n)_{n \in \mathbb{N}}$ is equivalent to $(y_{\sigma(n)})_{n \in \mathbb{N}}$. By Zippin’s theorem and by a result of Bourgain, Casazza, Lindenstrauss and Tzafriri [2], the homogeneity problem is solved for equivalence or
permutative equivalence of normalized block-sequences, the only solutions being the canonical bases of $c_0$ or $l_p$, $1 \leq p < +\infty$. Unfortunately, the situation is not nearly as nice when we replace permutative equivalence by isomorphism. We shall say that a Banach space $X$ with a basis is block-homogeneous if $X$ is isomorphic to all its block-subspaces. We recall our conjecture:

**Question 2.7** Let $X$ be a Banach space with an (unconditional) Schauder basis $(e_n)_{n \in \mathbb{N}}$, which is block-homogeneous. Does it follow that $X$ is isomorphic to $c_0$ or $l_p$?

Note that an (even unconditional) basis $(e_n)_{n \in \mathbb{N}}$ of a block-homogeneous Banach space $X$ need not be equivalent to the canonical basis of $c_0$ or $l_p$: for $1 < p < +\infty$, $X = (\oplus_{n \in \mathbb{N}} l_2^n)_p$ (with the associated canonical basis) is isomorphic to $l_p$, and every block-subspace of $X$ is complemented in $X$ ([19], Proposition 2.a.12) and thus isomorphic to $l_p$ as well.

In some special cases however, results of uniqueness of unconditional basis will allow us to pass from isomorphism to permutative equivalence and use the previous results. We recall some definitions and results from [5]. A sequence space $X$ is said to be left (resp. right) dominant if there exists a constant $C \geq 1$ such that whenever $(u_i)_{i \leq n}$ and $(v_i)_{i \leq n}$ are finite block-sequences, with $\|u_i\| \geq \|v_i\|$ (resp. $\|u_i\| \leq \|v_i\|$) and $v_i > u_i$ for all $i \leq n$, then $\|\sum_{i=1}^{n} v_i\| \leq C \|\sum_{i=1}^{n} u_i\|$ (resp. $\|\sum_{i=1}^{n} u_i\| \leq C \|\sum_{i=1}^{n} v_i\|$). When $X$ is left or right dominant, then there exists exactly one $r = r(X)$ such that $l_r$ is finitely disjointly representable in $X$, and we call $r$ the index of $X$.

We refer to [19], [15] for the definition of and background about Banach lattices. If $X$ and $Y$ are Banach lattices, a bounded linear operator $V : X \to Y$ is called a lattice homomorphism if $V(x_1 \vee x_2) = Vx_1 \vee Vx_2$ for all $x_1, x_2 \in X$. Following [5], define a Banach lattice $X$ to be sufficiently lattice-euclidean if there exists $C \geq 1$ such that for all $n \in \mathbb{N}$, there exist operators $S : X \to l_2^n$ and $T : l_2^n \to X$ such that $ST = I_{l_2^n}$, $\|S\| \|T\| \leq C$ and such that $S$ is a lattice homomorphism. This is equivalent to saying that $l_2$ is finitely representable as a complemented sublattice of $X$. A Banach lattice which is not sufficiently lattice-euclidean is said to be anti-lattice euclidean.

For an unconditional basis $(x_n)_{n \in \mathbb{N}}$ of a Banach space (seen as a Banach lattice), being sufficiently lattice-euclidean is the same as having, for some $C \geq 1$ and every $n \in \mathbb{N}$, a $C$-complemented, $C$-isomorphic copy of $l_2^n$ whose basis is disjointly supported on $(x_n)_{n \in \mathbb{N}}$. 22
Proposition 2.8 Let $X$ be a Banach space with a normalized unconditional basis $(e_n)_{n \in \mathbb{N}}$ which is isomorphically homogeneous. Assume $(e_n)_{n \in \mathbb{N}}$ is right or left dominant with $r(X) \neq 2$ and that $(e_n)_{n \in \mathbb{N}}$ is equivalent to $(e_{2n})_{n \in \mathbb{N}}$. Then $(e_n)_{n \in \mathbb{N}}$ is equivalent to the canonical basis of $l_p$, $p \neq 2$ or $c_0$.

Proof: Let $(y_n)_{n \in \mathbb{N}}$ be any subsequence of $(e_n)_{n \in \mathbb{N}}$, and $Y = [y_n]_{n \in \mathbb{N}}$. The sequence $(y_n)_{n \in \mathbb{N}}$ is equivalent to an unconditional basis $(u_n)_{n \in \mathbb{N}}$ of $X$. It is enough to note now that the proof of [5], Theorem 5.7, is still valid as long as we prove that $(u_n)_{n \in \mathbb{N}}$ is anti-lattice euclidean. But this is clear because $r(Y) = r(X) \neq 2$. So $(y_n)_{n \in \mathbb{N}}$ must be permutatively equivalent to $(e_n)_{n \in \mathbb{N}}$.

It follows by [2] Proposition 6.2 that some subsequence $(v_n)_{n \in \mathbb{N}}$ of $(e_n)_{n \in \mathbb{N}}$ is subsymmetric. By [5], $X$ is asymptotically $c_0$ or $l_p$ for some $p \neq 2$, so $(v_n)_{n \in \mathbb{N}}$ is equivalent to the canonical basis of $c_0$ or $l_p$, $p \neq 2$ and $(e_n)_{n \in \mathbb{N}}$ as well. □

The right or left dominant hypothesis in Proposition 2.8 cannot be removed: the canonical basis $(e_n)_{n \in \mathbb{N}}$ of Schlumprecht’s space $S$ [24] is unconditional, subsymmetric, but $S$ does not even contain a copy of $c_0$ or $l_p$.

It is of interest to note that $S$ is however quite homogeneous in some sense: any constant coefficient block-subspace of $S$ is isomorphic to $S$ (see [17], Remark before Proposition 9, for the proof and [19] for the definition). So $S$ is an example of a non $c_0$ or $l_p$, yet ”constant coefficient block-homogeneous” sequence space. This contrasts with the Theorem of Zippin (resp. the Theorem of Bourgain, Casazza, Lindenstrauss, Tzafriri) for equivalence (resp. permutative equivalence) which can be proved using only constant coefficient block-sequences in $X$ [2].

The question of uniformity in the homogeneous Banach space problem was raised by Gowers [12]. Of course, since a homogeneous Banach space must be isomorphic to $l_2$, it is trivial that if $X$ is homogeneous, then there exists a constant $C \geq 1$ such that $X$ is $C$-isomorphic to any of its subspaces. However, there does not seem to be a direct proof of this fact. Note also that uniformity is the first step in the proof of the theorem of Zippin. So the following question is natural:

Question 2.9 Let $X$ be a Banach space with an (unconditional) basis $(e_n)_{n \in \mathbb{N}}$. Assume $X$ is block homogeneous. Does there exists $C \geq 1$ such that $X$ is $C$-block homogeneous?

By a $C$-block homogeneous Banach space with a basis, we mean a Banach space $C$-isomorphic to all its block-subspaces.
As a partial result, we may use the primeness of the spaces $c_0$ and $l_p$ to get a positive answer to Question 2.9 when $X$ is isomorphic to $l_p$ or $c_0$:

**Proposition 2.10** Let $p \geq 1$. Let $X$ be a Banach space with an unconditional basis. Assume any sequence $(\tilde{x}_n)_{n \in \mathbb{N}} \in (bb_d^{<\omega}(X))^\omega$ of successive finite block-sequences admits a subsequence $(\tilde{x}_{n_k})_{k \in \mathbb{N}}$ such that the block-subspace $[\tilde{x}_{n_1}, \tilde{x}_{n_2}, \ldots]$ is isomorphic to $l_p$. Then $X$ is isomorphic to $l_p$, and furthermore, there exists $C \geq 1$, such that all block-subspaces of $X$ are $C$-isomorphic to $l_p$. The similar result holds for $c_0$.

**Proof:** We may assume the unconditional basis of $X$ is 1-unconditional (then all canonical projections on subspaces spanned by subsequences are of norm 1). The set $A_N = \{(x_n)_{n \in \mathbb{N}} \in bb_d(X) : [x_n]_{n \in \mathbb{N}} \sim^N l_p \}$ is analytic and so has Baire Property (this is true of any isomorphism class in $bb_d(X)$, see [10] about this). We check the hypotheses of Proposition 2.5. Given $\epsilon > 0$, there exists $\Delta > 0$ such that the $\Delta$-perturbation of a block-sequence $(x_n)_{n \in \mathbb{N}}$ in $bb_d(X)$ spans a space which is $1 + \epsilon$ isomorphic to $[x_n]_{n \in \mathbb{N}}$, so (a) follows. (b) is true with $K(N, n_0) = NC(n_0)$ (here $c(n)$ is the previously used constant such that in any Banach space, any two subspaces of codimension $n$ are $c(n)$-isomorphic). If $[x_n]_{n \in \mathbb{N}}$ is $C$-isomorphic to $l_p$, and if $[x_{n_k}]_{k \in \mathbb{N}}$ is a subsequence of $(x_n)_{n \in \mathbb{N}}$, then as $[x_{n_k}]$ is 1-complemented in $[x_n]_{n \in \mathbb{N}}$, it is $C$-isomorphic to a $C$-complemented subspace of $l_p$, so is $K(C)$-isomorphic to $l_p$, for some constant $K(C)$. Finally it is easy to check that if $x, y$ in $bb_d(X)$ are disjointly supported, and $[x]$ and $[y]$ are $C$-isomorphic to $l_p$, then the concatenation of $x$ and $y$ will span a subspace which is $k(C)$ isomorphic to $l_p$, for some constant $k(C)$.

In particular, if $X$ has a block-homogeneous unconditional basis and is isomorphic to $l_p$ or $c_0$, then it is $C$-block-homogeneous for some $C \geq 1$.

To conclude this section, it is worth noting the form that our topological 0-1 law takes when $A$ is really an isomorphic property of the span of a block-sequence in $bb_d(X)$.

**Theorem 2.11** Let $X$ be a Banach space with an unconditional basis $(e_n)_{n \in \mathbb{N}}$. Let $P$ be an isomorphic property of Banach spaces such that $A = \{(x_n)_{n \in \mathbb{N}} \in bb_d(X) : [x_n]_{n \in \mathbb{N}} has P \}$ has Baire Property, and which is stable by taking complemented subspaces.
Assume any sequence of successive finite block-sequences \((\tilde{x}_n)_{n \in \mathbb{N}}\) admits a subsequence \((\tilde{x}_{n_k})_{k \in \mathbb{N}}\) such that the block-subspace \([\tilde{x}_{n_1} \tilde{x}_{n_2} \ldots]\) satisfies \(P\).

Then every block-subspace of \(X\) is the sum of two disjointly supported block-subspaces satisfying \(P\).

Assume furthermore that any direct sum of two spaces with \(P\) satisfies \(P\), then every block-subspace in \(bb_d(X)\) satisfies \(P\).

3 Topological 0-1 law for subspaces with a successive finite dimensional decomposition.

We now turn to subspaces of a space \(X\) with a Schauder basis, which have a successive finite dimensional decomposition on the basis. There is a natural discretization of the set of such spaces, introduced in [10].

We say that two finite-dimensional subspaces \(F\) and \(G\) of \(X\) are successive, and write \(F < G\), if they are different from \(\{0\}\) and for any \(0 \neq x \in F, 0 \neq y \in G\), \(x\) and \(y\) are successive. A space with a successive finite dimensional decomposition (or successive FDD) in \(X\) is a subspace of \(X\) of the form \(\oplus_{k \in \mathbb{N}} F_k\), with successive, finite-dimensional subspaces \(F_k\). The associated sequence \((F_k)_{k \in \mathbb{N}}\) will be called a sequence of successive finite dimensional subspaces. Such a sequence passes through a finite sequence of successive finite dimensional subspaces \((A_i)_{1 \leq i \leq I}\) if there exists \(k\) such that \(F_{k+i} = A_i\) for all \(1 \leq i \leq I\). If the sequence \((A_i)\) is a length 1 sequence \((A)\), we shall just say that \((F_k)_{k \in \mathbb{N}}\) passes through \(A\).

We let \(fdd(X)\) be the set of infinite sequences of successive finite-dimensional subspaces, and \(fdd_d(X)\) be the Polish space of infinite sequences of successive finite-dimensional subspaces in \(Fin_{\mathbb{Q}}(X)\), equipped with the product of the discrete topology on \(Fin_{\mathbb{Q}}(X)\). The set of finite sequences of successive finite-dimensional subspaces in \(Fin_{\mathbb{Q}}(X)\) will be denoted by \(fdd_d^{< \omega}(X)\). \(F\) will denote a finite sequence of successive finite-dimensional subspaces, and \((F_n)_{n \in \mathbb{N}}\) an infinite sequence of such finite sequences. The usual notation about concatenation of finite sequences will be used. For \(S \in fdd_d(X)\), \([S]\) will denote the linear span of \(S\).

For \(E, F\) finite-dimensional spaces in \(X\), define the distance \(d(E, F)\) between \(E\) and \(F\) as the classical Hausdorff distance between the unit spheres of \(E\) and \(F\).

Let \(\Delta = (\delta_n)_{n \in \mathbb{N}} > 0\). Let \(A\) be a subset of \(fdd_d(X)\). The \(\Delta\)-expansion \(A_\Delta\) of \(A\) is the set of sequences of successive finite dimensional spaces \((F_k)_{k \in \mathbb{N}} \in fdd_d(X)\)
such that there exists \((E_k)_{k \in \mathbb{N}}\) in \(A\) with \(d(E_k, F_k) \leq \delta_k\) for all \(k \in \mathbb{N}\). The following theorem was essentially proved in [10].

**Theorem 3.1** Let \(X\) be a Banach space with a basis. If \(A\) is comeager in \(fdd_d(X)\), then for any \(\Delta > 0\), there exists a sequence \((\tilde{F}_n)_{n \in \mathbb{N}} \in (fdd_d(X))^{\omega}\) of successive finite sequences of successive finite dimensional subspaces, such that all elements of \(fdd_d(X)\) passing through infinitely many \(\tilde{F}_n\)'s are in \(A_\Delta\).

**Proof:** The proof is verbatim the same as in the case of block-sequences in [10] (this corresponds to Proposition 2.1 in this article), replacing blocks in \(Q(X)\) by finite-dimensional spaces in \(Fin_Q(X)\), and block-sequences in \(bb_d(X)\) by sequences of successive finite-dimensional subspaces in \(fdd_d(X)\). \(\Box\)

We shall use this theorem when \(A\) is in fact a property of \([x_n]_{n \in \mathbb{N}}\), in that case, each sequence \(\tilde{F}_n\) can be chosen to be of length 1, and the formulation becomes a little bit more tractable. It follows:

**Theorem 3.2** Let \(X\) be a Banach space with an unconditional basis. Let \(P\) be an isomorphic property of Banach spaces. Assume that the set \(\{(F_n)_{n \in \mathbb{N}} \in fdd(X) : [F_n]_{n \in \mathbb{N}} \text{ has } P\}\) has the Baire Property, and that \(P\) is stable by passing to complemented subspaces and by squaring. If every sequence in \(fdd(X)\) has a subsequence whose closed linear span satisfies \(P\), then all subspaces with a successive FDD in \(X\) satisfy \(P\).

**Proof:** Let \(A = \{(F_n)_{n \in \mathbb{N}} \in fdd(X) : [F_n]_{n \in \mathbb{N}} \text{ has } P\}\). For small enough \(\Delta > 0\), \(A_\Delta = A\) and \((A^C)_\Delta = A^C\). In Theorem 3.1 for sets corresponding to isomorphic properties (such as \(A\) or \(A^C\)), the sequence \(\tilde{F}_n\) may be chosen to be of length 1 for each \(n \in \mathbb{N}\). It follows from our hypotheses about \(P\) that \(A\) cannot be meager. For \(\tilde{E} = (E_1, \ldots, E_p) \in fdd_d^{\omega}(X)\), denote by \(N(\tilde{E})\) the set of sequences \((F_n)_{n \in \mathbb{N}} \in fdd_d(X)\) such that \(F_n = E_n\) for all \(n \leq p\). As \(A\) has the Baire Property, it is comeager in some open set \(N(\tilde{E})\), and without loss of generality \(\tilde{E}\) is a length 1 sequence \((E_1)\). We now prove that \(A\) is comeager in \(fdd_d(X)\).

Otherwise, \(A\) is meager in some open set \(N(\tilde{F})\), \(\tilde{F} \in fdd_d^{\omega}(X)\), and without loss of generality \(\tilde{F}\) is a length 1 sequence \((F_1)\). Now we may find \(E_2\) and \(F_2\) in \(Fin_Q(X)\), with \(E_1 < E_2\), \(\dim E_2 = \dim F_1\), \(F_1 < F_2\), \(\dim F_2 = \dim E_1\), and \(\max(\text{supp}(E_2)) = \max(\text{supp}(F_2))\). Let \(f\) be the canonical bijection between \(N((E_1, E_2))\) and \(N((F_1, F_2))\), defined by \(f((E_1, E_2)^c S) = (F_1, F_2)^c S\) for all.
$S \in \text{fdd}(X)$. It is routine to check that $f$ is an homeomorphism, and that for all $S \in N((E_1, E_2))$, $[f(S)]$ is isomorphic to $[S]$; in particular $S \in A$ if and only if $f(S) \in A$. It follows a contradiction with the fact that $A$ is meager in $N((F_1, F_2))$ and comeager in $N((E_1, E_2))$.

As $A$ is comeager, Theorem 3.1 applies. By properties of $P$, and because the basis of $X$ is assumed unconditional, $A$ is stable by taking subsequences and by concatenation of disjoint sequences. By the same method as in the end of the proof of Proposition 2.5 it follows that $A = \text{fdd}(X)$. □

One of the most important still open questions in Banach space theory is to know whether any complemented subspace of a Banach space with an unconditional basis must have an unconditional basis. A positive answer to this would have many consequences, for example concerning the Schroeder-Bernstein Property for Banach spaces (see e.g. [3] for a survey). The following corollary gives a direction for solving this question by the negative (here we use that ”spanning a subspace with an unconditional basis” is analytic and thus has the Baire Property in $\text{fdd}(X)$).

**Corollary 3.3** Let $X$ be a Banach space with an unconditional basis. Assume:

1. every sequence in $\text{fdd}(X)$ has a subsequence which spans a subspace with an unconditional basis,

2. there exists a sequence in $\text{fdd}(X)$ which spans a subspace without an unconditional basis.

Then there exists a subspace $F = \bigoplus_{n\in\mathbb{N}} F_n$ of $X$ with a successive FDD on the basis, which has an unconditional basis, and a subsequence $(G_k)_{k\in\mathbb{N}}$ of $(F_n)_{n\in\mathbb{N}}$ such that $G = \bigoplus_{k\in\mathbb{N}} G_k$, though complemented in $F$, does not have an unconditional basis.

We conclude by discussing some of the properties that a Banach space $X$ with (1) and (2) must have, supposing it to exist.

Recall that a Banach space $X$ is said to have Gordon-Lewis l.u.st. if there is a constant $C \geq 1$ such that for every finite dimensional subspace $E$ of $X$, there exists a finite dimensional space $F$ with a 1-unconditional basis, and maps $T : E \to F$, $U : F \to X$, such that $UT(x) = x$ for all $x \in E$ and such that $\|T\| \|U\| \leq C$. We note that having l.u.st. is an analytic property of Banach spaces, which is stable by passing to complemented subspaces and squaring. As (1) implies that every sequence in $\text{fdd}(X)$ has a subsequence which spans a
subspace with l.u.st., it follows from Theorem 3.2 that if \( X \) satisfies (1) and (2), then every subspace of \( X \) with a successive FDD must have l.u.st..

By the Theorem of Komorowski and Tomczak-Jaegermann [18], it follows that \( X \) must be \( l_2 \)-saturated. Also by [4] Theorem 3.8, every subspace of \( X \) with a uniform FDD on the basis must have an unconditional basis.

Another interesting fact is that the unconditional basis for \( \bigoplus F_n \) in the conclusion of Corollary 3.3 cannot be obtained in the obvious way, that is by constructing in each \( F_n \) a \( C \)-unconditional basis, and proving that the sequence which is the reunion of each basis is a \( K(C) \)-unconditional basis for \( \bigoplus F_n \), for some constant \( K(C) \). In that case, any subspace \( \bigoplus G_k \) associated to a subsequence \( (G_k)_{k \in \mathbb{N}} \) of \( (F_n)_{n \in \mathbb{N}} \) would inherit an unconditional basis (which is just a subsequence of the unconditional basis of \( \bigoplus F_n \)).

A natural candidate for \( X \) is the Orlicz sequence space \( l_F \) considered by P. Casazza and N.J. Kalton in [4]. It is reflexive, has cotype 2 and type \( 2 - \epsilon \) for any \( \epsilon > 0 \), and is \( l_2 \)-saturated. Among other interesting properties, every subspace of \( l_F \) with a uniform UFDD has an unconditional basis. We do not know whether \( l_F \) satisfies the hypotheses of Corollary 3.3.

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