ON THE RIGIDITY OF UNIFORM ROE ALGEBRAS OVER UNIFORMLY LOCALLY FINITE COARSE SPACES

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Abstract. Given a coarse space $(X, E)$, one can define a $C^*$-algebra $C^*_u(X)$ called the uniform Roe algebra of $(X, E)$. It has been proved by J. Špakula and R. Willett that if the uniform Roe algebras of two uniformly locally finite metric spaces with property A are isomorphic, then the metric spaces are coarsely equivalent to each other. In this paper, we look at the problem of generalizing this result for general coarse spaces and on weakening the hypothesis of the spaces having property A.

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1. Introduction

The concept of coarse spaces generalizes the idea of metric spaces and gives us the appropriate framework to study large-scale geometry. In a nutshell, a coarse space consists of a pair $(X, E)$, where $X$ is a set and $E$ is a family of subsets of $X \times X$ which measures ‘boundedness’ in $X$ (we refer the reader to Section 2 for precise definitions of the terminology used in this introduction). A (connected) coarse space $(X, E)$ happens to be metrizable exactly when its coarse structure is generated by $\aleph_0$ subsets.

Given a uniformly locally finite coarse space $(X, E)$, one can define a algebra $C^*_u[X]$ and a $C^*$-algebra $C^*_u(X)$, with $C^*_u[X]$ dense in $C^*_u(X)$, called the algebraic uniform Roe algebra of $(X, E)$ and uniform Roe algebra of $(X, E)$, respectively.

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The motivation to study this $C^*$-algebra comes from its intrinsic relation with the coarse Baum-Connes conjecture and, as a consequence, to the Novikov conjecture (see [Yu00]). More to the point, rigidity problems considered in this paper are directly concerned with the Baum–Connes conjecture (see the discussion in [ŠW13, Section 1.2]). Many coarse properties of $(X, E)$ reflect on $C^*$-algebraic properties of $C^*_u(X)$ and vice versa (see [LW17, Sak13], and [WZ10]). However, the rigidity question—whether the uniform Roe algebra completely determines the coarse structure of the coarse space—remains open.

An isomorphism $\Phi : C^*_u(X) \to C^*_u(Y)$ is spatially implemented if there exists a unitary $U : \ell_2(X) \to \ell_2(Y)$ such that $\Phi = \text{Ad}U$ (where $\text{Ad}U(T) = UTU^*$). In this case we say that $C^*_u(X)$ and $C^*_u(Y)$ are spatially isomorphic. If in addition there exists $\varepsilon > 0$ such that
\[
\inf_{x \in X} \sup_{y \in Y} |\langle U^* \delta_y, \delta_x \rangle| > \varepsilon \quad \text{and} \quad \inf_{y \in Y} \sup_{x \in X} |\langle U \delta_x, \delta_y \rangle| > \varepsilon
\]
then we say that $\Phi$ is a rigid isomorphism between $C^*_u(X)$ and $C^*_u(Y)$ and that $C^*_u(X)$ and $C^*_u(Y)$ are rigidly isomorphic. Analogous terminology applies to the case when $\Phi$ is an isomorphism between algebraic Roe algebras $C^*_u[X]$ and $C^*_u[Y]$.

We study the following relations between coarse spaces $(X, E)$ and $(Y, F)$ (see Section 2 for the definitions).

(I) $(X, E)$ and $(Y, F)$ are coarsely equivalent.

(II) $(X, E)$ and $(Y, F)$ are bijectively coarsely equivalent.

(III) $C^*_u[X]$ and $C^*_u[Y]$ are isomorphic.

(IV) There is an isomorphism $C^*_u(X) \to C^*_u(Y)$ taking $\ell_2(X)$ to $\ell_\infty(Y)$.

(V) $C^*_u(X)$ and $C^*_u(Y)$ are rigidly isomorphic.

(VI) $C^*_u(X)$ and $C^*_u(Y)$ are spatially isomorphic.

(VII) $C^*_u(X)$ and $C^*_u(Y)$ are isomorphic.

Notice that, although the implication (II) $\Rightarrow$ (I) is trivial, (I) in general does not imply any of the other properties (for example if $X$ and $Y$ are connected and finite coarse spaces of different cardinalities, or if $X$ and $Y$ are $\mathbb{R}$ and $\mathbb{Z}$ with their standard metrizable coarse structures). This paper revolves around the following question.

**Problem 1.1.** Let $(X, E)$ and $(Y, F)$ be uniformly locally finite coarse spaces. Do all properties above imply (I)? Moreover, do we have that

(II) $\iff$ (III) $\iff$ (IV) $\iff$ (V) $\iff$ (VI) $\iff$ (VII)?

The implications (V) $\Rightarrow$ (VI) $\Rightarrow$ (VII) and (III) $\Rightarrow$ (VII) are trivial and the implications (II) $\Rightarrow$ (III) and (II) $\Rightarrow$ (IV) are quite straightforward. We list below what is known regarding the remaining implications.

1. Properties (VII) and (VII) are equivalent. More precisely, every isomorphism as in (III) or (VII) is spatially implemented (see [SW13, Lemma 3.1, or Lemma 3.1 below).)
2. For uniformly locally finite metric spaces with property A, (VII) implies (II) ([SW13, Theorem 4.1]).
3. Properties (VII) and (V) are equivalent for uniformly locally finite metric spaces with property A. Moreover, every isomorphism between the algebras is a rigid isomorphism (SW13, Lemma 4.6).
(4) Properties (II) and (IV) are equivalent for uniformly locally finite metric spaces ([WW16], Corollary 1.16).

(5) Properties (II), (III), (IV), (V), (VI), and (VII) are all equivalent in the following situations.

(a) for uniformly locally finite metric spaces with finite decomposition complexity ([WW16], Corollary 1.16 and its proof),

(b) for countable locally finite groups with proper left-invariant metrics ([LL17, Theorem 1.1]; the standard Cayley graph metric is proper and left-invariant).

(6) Property (I) implies that $C^*_u(X)$ and $C^*_u(Y)$ are Morita equivalent (this was proved for metric spaces in [BNW07, Theorem 4], but the proof clearly translates to coarse spaces).

We proceed to describe our results after a definition and a short discussion.

An operator $T \in C^*_u(X)$ is called a ghost if $\langle T\delta_x, \delta_{x'} \rangle \to 0$, as $(x, x') \to \infty$ on $X \times X$ (see Definition 2.9 for details). A uniformly locally finite metric space $(X, d)$ has property A if and only if all ghost operators in $C^*_u(X)$ are compact (see Proposition 11.43 of [Roe03] and Theorem 1.3 of [RW14]). Since there exist uniformly locally finite metric spaces with non-compact ghost operators in which all ghost projections are compact (see [RW14], Theorem 1.4), the property of a uniformly locally finite metric space having only compact ghost projections is strictly weaker than property A. Also, it was proved in [Yu00], Theorem 2.7, that having property A implies coarse embeddability into a Hilbert space.

(7) Properties (II), (III), and (IV) are equivalent for all uniformly locally finite coarse spaces (Theorem 8.1).

(8) (V) implies (I) for uniformly locally finite metric spaces (Theorem 4.12).

(9) More generally, (V) implies (I) for uniformly locally finite coarse spaces whose coarse structures are generated by less than $2^{\aleph_0}$ subsets, if Martin’s Axiom holds (see Theorem 4.12 for a stronger result).

(10) (VII) implies (V) for uniformly locally finite metric spaces which coarsely embed into a Hilbert space (Theorem 7.4).

(11) (VII) implies (V) for uniformly locally finite metric spaces in which all the ghost projections are compact (Theorem 6.2). More precisely, if an isomorphism is not rigidly implemented, then at least one of the spaces has a Cartan masa which contains non-compact ghost projections (Theorem 6.1).

By (10) and (8), we have the following.

**Corollary 1.2.** Let $(X, d)$ and $(Y, \partial)$ be uniformly locally finite metric spaces which coarsely embed into a Hilbert space. If $C^*_u(X)$ and $C^*_u(Y)$ are isomorphic, then $(X, d)$ and $(Y, \partial)$ are coarsely equivalent.

By (11) and (8), we have the following.

**Corollary 1.3.** Let $(X, d)$ and $(Y, \partial)$ be uniformly locally finite metric spaces such that all the ghost projections in $C^*_u(X)$ and $C^*_u(Y)$ are compact. If $C^*_u(X)$ and $C^*_u(Y)$ are isomorphic, then $(X, d)$ and $(Y, \partial)$ are coarsely equivalent.

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1. Since we do not make use of it, we do not define finite decomposition complexity in these notes. We only point out that this property is stronger than property A for uniformly locally finite metric spaces and refer the reader to Section 2 of [GTY12] for a definition.

2. We take the advantage of this fact and use it as an excuse not to give the definition of Property A; let’s just say that A appears to stand for ‘amenability’. 
We do not know whether a Cartan masa can contain a non-compact ghost projection (see the conclusion of (11)); see however Example 6.3.

A uniformly locally finite metric space which coarsely embeds into a Hilbert space but which does not have property A was constructed in [AGŠ12, Theorem 1.1]. Although this space does not have property A, all ghost projections in $C^*_u(X)$ are compact (see [RW14, Theorem 1.4]). In particular, it follows from Theorem 1.1 of [Yu00] that this space satisfies the coarse Baum–Connes conjecture.

Let $(X, E)$ and $(Y, F)$ be metrizable uniformly locally finite coarse spaces. We prove that any isomorphism $C^*_u(Y) \to C^*_u(X)$ must satisfy a ‘coarse-like’ property. Vaguely speaking, given such an isomorphism, we show that for every $\varepsilon > 0$ there exists an assignment $F \in F \mapsto E_F \in E$ so that this isomorphism takes operators supported in $F$ to operators supported in $E_F$ up to an error of $\varepsilon$. We refer to Definition 4.3 and Theorem 4.4 for precise statements and generalization for non-metrizable spaces. In the algebraic case, the error may be taken to be zero and the metrizability assumption can be completely forgotten (see Theorem 5.4).

This paper is organized as follows. In Section 2, we present all the necessary definitions and background for these notes. Section 3 is dedicated to show that Lemma 3.1 of [ŠW13] can be generalized to non-connected coarse spaces. In Section 4, we prove the ‘coarse-like’ property mentioned above which gives us (8) and (9). We can also prove rigidity of uniform Roe algebras for a subclass of uniformly locally finite coarse spaces which we call spaces with small partitions. This is done in Section 5 and we refer the reader to Definition 5.1 and Theorem 5.5 for precise statements. In Section 6, we look at Cartan masas (Definition 2.7) and show (11) above. In Section 7, we show that rigidity holds for metric spaces which coarsely embed into a Hilbert space (10) above and, in Section 8, we deal with rigidity of algebraic uniform Roe algebras and prove (7). At last, since our proof of (9) uses Baire category methods, it is therefore amenable to an extension along the lines of using mild forcing axioms from set theory. This is discussed in the Appendix 9.

2. Background

For more on coarse spaces, we refer to [Roe03] and [Ros].

2.1. Coarse spaces. Let $X$ be a set. Given subsets $E, F \subset X \times X$, we define

$E^{-1} = \{(x, y) \in X \times X \mid (y, x) \in E\}$

and

$E \circ F = \{(x, y) \in X \times X \mid \exists z \in X \text{ with } (x, z) \in E \text{ and } (z, y) \in F\}.$

We say that $E$ is symmetric if $E = E^{-1}$. For each $n \in \mathbb{N}$, define $E^{(n)}$ recursively as follows. Let $E^{(1)} = E$ and $E^{(n+1)} = E \circ E^{(n)}$, for all $n \geq 1$.

Definition 2.1. Let $X$ be a set. A collection $\mathcal{E}$ of subsets of $X \times X$ is called a coarse structure if

(i) $\Delta_X := \{(x, x) \in X \times X \mid x \in X\} \in \mathcal{E}$,
(ii) $E \in \mathcal{E}$ and $F \subset E$ implies $F \in \mathcal{E}$,
(iii) $E \in \mathcal{E}$ implies $E^{-1} \in \mathcal{E}$,
(iv) $E, F \in \mathcal{E}$ implies $E \cup F \in \mathcal{E}$, and
(v) $E, F \in \mathcal{E}$ implies $E \circ F \in \mathcal{E}$.

The elements of $\mathcal{E}$ are called entourages and the pair $(X, \mathcal{E})$ is called a coarse space.
Let $X$ be a set, $\mathcal{E} \subset \mathcal{P}(X \times X)$ and $A \subset X$ be a subset. We define
$$\mathcal{E}_A = \{ E \cap A \times A \mid E \in \mathcal{E} \}.$$ If $\mathcal{E}$ is a coarse structure on $X$, then $\mathcal{E}_A$ defines a coarse structure on $A$. A coarse space $(X, \mathcal{E})$ is called connected if $\{(x, y)\} \in \mathcal{E}$, for all $x, y \in X$. Since $\{(x, y)\} \in \mathcal{E}$ defines an equivalence relation on $X$, we can always write $X = \bigsqcup_{j \in J} X_j$, where each $(X_j, \mathcal{E}_{X_j})$ is connected and $X_j \cap X_i = \emptyset$, for all $j \neq i$. The subsets $(X_j)_{j \in J}$ are called the connected components of $X$.

Given a set $X$ and a family of subsets $\{E_i\}_{i \in I} \subset \mathcal{P}(X \times X)$, the intersection of all the coarse structures on $X$ containing the family $\{E_i\}_{i \in I}$, say $\mathcal{E}$, is still a coarse structure and it is called the coarse structure generated by $\{E_i\}_{i \in I}$. The family $\{E_i\}_{i \in I}$ is called a set of generators of $\mathcal{E}$. We say that a coarse structure $\mathcal{E}$ on $X$ is countably generated if it is generated by a countable family of subsets of $X \times X$.

Let $(X, d)$ be a metric space. For each $r \geq 0$, let
$$E_r = \{(x, y) \in X \times X \mid d(x, y) \leq r\}.$$ We call the the coarse structure generated by $\{E_r\}_{r \geq 0}$ the bounded coarse structure of $(X, d)$ and we denote it by $\mathcal{E}_d$. Clearly, $\mathcal{E}_d$ is countably generated.

A coarse space $(X, \mathcal{E})$ is metrizable if there exists some metric $d$ on $X$ such that $\mathcal{E}$ is the bounded coarse structure of $(X, d)$. A connected coarse structure $(X, \mathcal{E})$ is metrizable if and only if $\mathcal{E}$ is countably generated (see [Roe03], Theorem 2.55).

For $E \subset X \times X$ and $x \in X$ let
$$E_x = \{ y \in X \mid (x, y) \in E \} \quad \text{and} \quad E^x = \{ y \in X \mid (y, x) \in E \}.$$ Define
\[ \sup_{x \in X} |E_x| < \infty, \]
for all $E \in \mathcal{E}$. If $(X, d)$ is a metric space and $\mathcal{E}_d$ is its bounded coarse structure, we say that $(X, d)$ is a uniformly locally finite metric space if $(X, \mathcal{E}_d)$ is a uniformly locally finite coarse space.

Remark 2.3. We notice that a uniformly locally finite metric space $(X, d)$ is usually called a metric space with bounded geometry in the literature. However, the common definition of bounded geometry for general coarse spaces (see [Roe03], Chapter 3, Section 3.1) is not the generalization we need for these notes. Precisely, what we need is the idea of uniformly discrete coarse spaces defined in [Roe03], Definition 3.24. Since the terminology uniformly locally finite is also used by other authors (e.g., [Sak12], [Sak13] and [Ros]), we chose to use this less common terminology (even for metric spaces).

Let $(X, \mathcal{E})$ be a coarse space and let $E \in \mathcal{E}$. A subset $X'$ of $X$ is called $E$-separated if $(x, y) \notin E$, for all distinct $x, y \in X'$. The following lemma is a reformulation of Lemma 2.7(a), of [STY02]. For the convenience of the reader, we include a proof.

Proposition 2.4. Let $(X, \mathcal{E})$ be a coarse structure. Let $E \in \mathcal{E}$ be such that $n := \sup_{x \in X} |E_x| < \infty$. Then there exists a partition
$$X = X_1 \sqcup \ldots \sqcup X_n,$$ such that $X_i$ is $E$-separated, for all $i \in \{1, \ldots, n\}$. In particular, if $(X, \mathcal{E})$ is a uniformly locally finite coarse space, then such partition exists for all $E \in \mathcal{E}$.
Proof. Fix $E \in \mathcal{E}$ as in the proposition. Without loss of generality, we assume that $E$ is symmetric and that $\Delta_X \subset E$. By Zorn’s lemma, we can pick a maximal $E$-separated subset $X_1 \subset X$. Assume $X_i$ has being defined, for $i \geq 1$. If

$$X \setminus \bigcup_{n=1}^i X_n \neq \emptyset,$$

let $X_{i+1} \subset X \setminus \bigcup_{n=1}^i X_n$ be a maximal $E$-separated subset. If the proposition does not hold, this defines a finite sequence $(X_i)_{i=1}^{n+1}$ of nonempty pairwise disjoint subsets such that, for all $i \leq n + 1$, $X_i \subset X \setminus \bigcup_{n=1}^i X_n$ and $X_i$ is a maximal $E$-separated subset of $X \setminus \bigcup_{n=1}^i X_n$. Pick $x \in X_{n+1}$. For each $i \leq n$, we can pick $x_i \in X_i$ such that $(x, x_i) \in E$. Therefore,

$$\{x\} \cup \{x_i \mid j \in \{1, \ldots, n\}\} \subset E_x.$$

This gives us that $|E_x| \geq n + 1$; contradiction.

Definition 2.5. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces. If $Z$ is any set, then maps $f : Z \to X$ and $g : Z \to Y$ are said to be close if

$$\{(f(x), f(x')) \in X \times X \mid x, x' \in X\} \in \mathcal{E}.$$

(i) A map $f : X \to Y$ is called coarse if for all $E \in \mathcal{E}$ there exists $F \in \mathcal{F}$ such that $(x, x') \in E$ implies $(f(x), f(x')) \in F$.

(ii) A coarse map $f$ is a coarse embedding if for all $F \in \mathcal{F}$ there exists $E \in \mathcal{E}$ such that $(x, x') \notin E$ implies $(f(x), f(x')) \notin F$.

(iii) A coarse embedding $f$ is a coarse equivalence if there exists a coarse map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are close to $\text{Id}_X$ and $\text{Id}_Y$, respectively.

Equivalently, $f$ is a coarse equivalence if it is a coarse embedding and it is cobounded, i.e., there exists $F \in \mathcal{F}$ such that

$$Y = \{y \in Y \mid \exists x \in X, (f(x), y) \in F\}.$$

2.2. Uniform Roe algebras. We denote the algebra of bounded linear operators on a Hilbert space $H$ by $B(H)$.

Definition 2.6. Let $(X, \mathcal{E})$ be a coarse space. The algebraic uniform Roe algebra $C^*_u[X]$ is defined by setting

$$C^*_u[X, \mathcal{E}] = \left\{T \in B(\ell_2(X)) \mid \forall E \in \mathcal{E}, \forall (x, x') \notin E, \langle T\delta_x, \delta_{x'} \rangle = 0 \right\},$$

and the uniform Roe algebra $C^*_u(X, \mathcal{E})$ is defined as the norm closure of $C^*_u[X]$ in $B(\ell_2(X))$. Clearly, $C^*_u[X, \mathcal{E}]$ is a algebra and $C^*_u(X, \mathcal{E})$ is a $C^*$-algebra. We omit $\mathcal{E}$ whenever it is clear from the context and write $C^*_u[X]$ and $C^*_u(X)$ for $C^*[X, \mathcal{E}]$ and $C^*_u(X, \mathcal{E})$, respectively.

Since we will only work with uniformly locally finite coarse spaces $(X, \mathcal{E})$, it is worth noticing that, for any such $(X, \mathcal{E})$, the following holds. Let $T = (T_{xy})_{x,y \in X}$ be a family of complex numbers satisfying

(i) $\sup_{x,y \in X} |T_{xy}| < \infty$, and

(ii) there exists $E \in \mathcal{E}$ such that $T_{xy} = 0$ if $(x, y) \notin E$.

The family $T$ naturally induces a bounded operator on $\ell_2(X)$, which we still denote by $T$. The set of all such operators coincides with $C^*_u[X]$ and, for any such $T =$
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\[(T_{xy})_{x,y \in X}, \text{ it holds that}\]
\[\|T\| \leq \sup_{x \in X} |E^x| \cdot \sup\{|T_{xy}| \mid x, y \in X\},\]
for \(E \in \mathcal{E}\) as in (ii).

For a set \(X\) the Hilbert space \(\ell_2(X)\) has the standard orthonormal basis, \((\delta_x)_{x \in X}\).

The support of \(T \in B(\ell_2(X))\) is
\[
supp(T) = \{(x, x') \in X \times X \mid \langle T\delta_x, \delta_{x'} \rangle \neq 0\}.
\]

By identifying \(\ell_\infty(X)\) with the subspace of all \(T \in B(\ell_2(X))\) such that \(supp(T) \subseteq \Delta_X\) we have the inclusion \(\ell_\infty(X) \subset C^*_u(X)\).

The algebra \(\ell_\infty(X)\) is a maximal abelian subalgebra (or shortly, masa) with special properties, captured by the following definition ([WW16], Proposition 3.1).

**Definition 2.7.** A \(C^*\)-subalgebra \(B\) of a \(C^*\)-algebra \(A\) is a Cartan masa if

(i) \(B\) is a maximal abelian self-adjoint subalgebra (i.e., a masa) of \(A\),

(ii) \(B\) contains an approximate unit for \(A\),

(iii) the normalizer of \(B\) in \(A\) defined as
\[
N_A(B) = \{a \in A \mid aBa^* \cup a^*Ba \subset B\}
\]
generates \(A\) as a \(C^*\)-algebra, and

(iv) there is a faithful conditional expectation from \(A\) onto \(B\).

Given \(x, x' \in X\), we define an operator \(e_{xx'} \in B(\ell_2(X))\) by letting
\[e_{xx'}(\delta_z) = \langle \delta_z, \delta_{x'} \rangle \delta_x,\]
for all \(z \in Z\). For \(A \subset X\), let \(\chi_A = \sum_{x \in A} e_{xx}\). Clearly, \(\chi_A\) belongs to \(\ell_\infty(X)\).

Given \(T \in B(\ell_2(X))\) and \(x, x' \in X\), we let
\[T_{xx'} = e_{xx}Te_{x'x'}.
\]

So, if \(\{(x, x')\} \in \mathcal{E}\), we have that \(T_{xx'} \in C^*_u[X]\).

We state the following trivial lemma for future reference.

**Lemma 2.8.** Suppose \((X, \mathcal{E})\) is a uniformly locally finite coarse space and \((T_j)_{j \in J}\) is a uniformly bounded family of operators with disjoint supports such that \(E = \bigcup_{j \in J} supp(T_j)\) belongs to \(\mathcal{E}\). Then the series \(\sum_{j \in J} T_j\) converges in the strong operator topology to an element in \(C^*_u[X]\) with support contained in \(E\). \(\Box\)

We conclude this subsection with the definition of ghost operators in uniform Roe algebras.

**Definition 2.9.** Let \((X, \mathcal{E})\) be a uniformly locally finite coarse space. An operator \(T \in C^*_u(X)\) is a ghost if \(\langle T\delta_x, \delta_{x'} \rangle \to 0\) as \((x, x') \to \infty\) on \(X \times X\), i.e., if for all \(\varepsilon > 0\) there exists a finite set \(A \subset X\) such that
\[|\langle T\delta_x, \delta_{x'} \rangle| < \varepsilon,
\]
for all \(x, x' \in X \setminus A\).

The set of all ghost operators of a uniform Roe algebra forms an ideal which contains all compact operators. For more on the class of ghost operators, we refer to [CW04], [CW05] and [Roe03].
3. Spatially implemented isomorphisms

In this section, we show that a minor modification of the proof of [SW13, Lemma 3.1] gives that any isomorphism between uniform Roe algebras is spatially implemented, thus showing (1) from the introduction.

**Lemma 3.1.** Every isomorphism between uniform Roe algebras associated with coarse spaces is spatially implemented. Every isomorphism between algebraic uniform Roe algebras associated with coarse spaces is spatially implemented.

We only need to show that the assumptions that the spaces be connected and metrizable used in [SW13, Lemma 3.1] are not necessary for the conclusion. This requires only a little analysis of the center of an (algebraic) uniform Roe algebra.

Suppose \((X, ℰ)\) is a coarse space with the connected components \(X_i\), for \(i \in I\). The space \(ℓ_2(X)\) can be naturally identified with the Hilbert sum \(\bigoplus_{i \in I} ℓ_2(X_i)\). Denote the projection from \(ℓ_2(X)\) to \(ℓ_2(X_i)\) by \(P_i\). A corner of a \(C^*\)-algebra \(C\) is a subalgebra of the form \(PCP\) for some nonzero projection \(P\) in \(C\). A projection \(P\) in a \(C^*\)-algebra \(A\) is scalar if the corner \(PAP\) is isomorphic to \(C\). The center of an algebra \(A\) is denoted \(Z(A)\).

**Lemma 3.2.** Let \((X, ℰ)\) be a coarse space with the connected components \(X_i\), for \(i \in I\). Then, the projections \((P_i : ℓ_2(X) \to ℓ_2(X_i))_{i \in I}\) are the only scalar projections in \(Z(C_u^*(X))\). They are also the only scalar projections in \(Z(C_u^*(X))\).

**Proof.** If \(X\) is connected, then \(C_u^*[X]\) includes all finite rank operators and therefore \(Z(C_u^*[X]) = Z(C_u^*(X)) = \mathbb{C} \cdot 1\), so the statement follows. In the general case, via the identification from the previous paragraph, \(B(ℓ_2(X))\) is identified with \(\prod_{i \in I} B(ℓ_2(X_i))\). Note that \(P_iC_u^*(X)P_i\) is naturally isomorphic to \(C_u^*(X_i)\) and that each \(P_i\) clearly belongs to \(ℓ_∞(X)\cap Z(C_u^*(X))\). In order to prove that \(Z(C_u^*(X))\) has no other scalar projections, observe that on the level of the uniform Roe algebras we have the following inclusions

\[
\bigoplus_{i \in I} K(ℓ_2(X_i)) \subseteq C_u^*(X) \subseteq \prod_{i \in I} C_u^*(X_i).
\]

Therefore

\[
Z(C_u^*(X)) \subseteq \left\{ T \in \prod_{i \in I} C_u^*(X_i) \mid ST = TS \text{ for all } S \in \bigoplus_{i \in I} K(ℓ_2(X_i)) \right\}.
\]

The algebra on the right-hand side is the von Neumann algebra generated by orthogonal projections \(\{P_i \mid i \in I\}\), and the conclusion follows.

The proof in the case of the algebraic uniform Roe algebra is analogous. □

**Proof of Lemma 3.2.** Suppose \((X, ℰ)\) and \((Y, ℱ)\) are coarse spaces with connected components \(X_i\), for \(i \in I\), and \(Y_j\), for \(j \in J\), and that \(Φ : C_u^*(X) \to C_u^*(Y)\) is an isomorphism. Writing \(Q_j\) for the projection of \(ℓ_2(Y)\) to \(ℓ_2(Y_j)\), Lemma 3.1 implies that there exists a bijection \(f : I \to J\) such that \(Φ(P_i) = Q_{f(i)}\).

It remains to prove that the isomorphism between \(C_u^*(X_i)\) and \(C_u^*(Y_{f(i)})\) is implemented by a unitary for every \(i\).

Note that \(Φ\) sends scalar projections of \(C_u^*(X)\) to \(C_u^*(Y)\). Fix \(i\). The algebra of compact operators \(K(ℓ_2(X_i))\) is equal to the ideal of \(C_u^*(X_i)\) generated by its scalar projections, and the algebra of compact operators \(K(ℓ_2(Y_{f(i)}))\) is equal to the ideal of \(C_u^*(Y_{f(i)})\) generated by its scalar projections. Therefore \(Φ(K(ℓ_2(X_i))) = K(ℓ_2(Y_{f(i)}))\). Since an isomorphism between two algebras of compact operators is...
implemented by a unitary unique up to the multiplication by a scalar, a unitary $U_i : \ell_2(X_i) \to \ell_2(Y_{f(i)})$ implements the restriction of $\Phi$ to $C^*_u(X_i)$. This concludes the proof in the case of uniform Roe algebras.

An analogous argument shows the analogous statement holds for $C^*_u[X]$. □

**Corollary 3.3.** An isomorphism between uniform Roe algebras carries operators of rank $n$ to operators of rank $n$ for all $n \in \mathbb{N}$. It also carries operators with orthogonal images to operators with orthogonal images. □

**Corollary 3.4.** Let $(X, E)$ and $(Y, F)$ be coarse spaces with isomorphic uniform Roe algebras. Then $|X| = |Y|$.

*Proof.* This follows straightforwardly from Lemma 3.1, the fact that an unitary isomorphism $U : \ell_2(X) \to \ell_2(Y)$ must take an orthonormal basis of $\ell_2(X)$ to an orthonormal basis of $\ell_2(Y)$, and the fact that an orthonormal basis of $\ell_2(X)$ has cardinality $|X|$. □

4. Rigid isomorphism between uniform Roe algebras and Baire category

The goal of this section is to prove (8) and (9) (Theorem 4.12) from the introduction. The main step in the proof of this result is Theorem 4.4 below, which is a strengthening of Lemma 3.2 of [ŠW13] for coarse spaces.

Readers interested only in metric spaces will lose nothing by assuming all the spaces are metrizable throughout this section.

Other readers may want to consult [Kun11, Section III] for more details on infinitary combinatorics and Martin’s Axiom in particular. Define

$$D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$$

and endow $D$ with its usual metric. For a set $J$ consider the space $D^J$ with the product topology. This is a compact Hausdorff space. Let $\text{cov}(D^\kappa)$ denote the minimal cardinality of a family of nowhere dense subsets of $D^J$ that covers the space. The following lemma collects some well-known results.

**Lemma 4.1.** Suppose that $\kappa$ and $\mu$ are infinite cardinals.

(i) If $\text{cov}(D^\kappa) \leq \kappa$ then $\kappa$ is uncountable.

(ii) If $\kappa < \mu$ then $\text{cov}(D^\kappa) \geq \text{cov}(D^\mu)$.

(iii) If $\kappa \geq 2^{\aleph_0}$ then $\text{cov}(D^\kappa) \leq \kappa$.

(iv) Martin’s Axiom for $\kappa$ dense sets, $\text{MA}_\kappa$, implies that $\text{cov}(D^\mu) > \kappa$ for all $\mu$.

(v) It is consistent with ZFC that $\text{cov}(D^\mu) > \kappa$ for all $\kappa < 2^{\aleph_0}$ and all $\mu$.

*Proof.*

(i) is the Baire category theorem.

For (ii), note that if $D^\mu$ is homeomorphic to $D^\kappa \times D^\mu$, and that if $F$ is nowhere dense in $D^\kappa$ then $F \times D^\mu$ is nowhere dense in $\{ -1, 1 \}^\mu$.

For (iii), cover $D^\aleph_0$ by the singletons and apply (i).

The Tychonoff product of any family of separable spaces has the countable chain condition (ccc; see [Kun11, Definition III.2.1]) by [Kun11, Theorem III.2.8]. Therefore the space $D^\mu$ has the countable chain condition and (iv) is a consequence of

---

Footnotes:

3 This is because the Baire category theorem implies that all metric spaces are, in the terminology introduced below, small. It is hard to resist quoting from [She92]: “To these we have nothing to say at all, beyond a reasonable request that they refrain from using the countable additivity of Lebesgue measure.”
Theorem 4.4. Suppose that $\text{supp}(\varepsilon > 0, E \in \mathcal{E})$ is not small. In particular, a metric space with $|E|$ is not small. By Lemma 4.1(iv), if $|E| \geq 2^{\aleph_0}$ and $\text{cov}(\mathcal{D}) \leq \text{cov}(\mathcal{D}^{E=\omega})$, then $(X, \mathcal{E})$ is not small. In particular, a metric space with $|E| = 2^{\aleph_0}$ is not small.

Definition 4.3. Suppose $(X, \mathcal{E})$ is a coarse space, $\varepsilon > 0$, and $E \in \mathcal{E}$. An operator $T \in B(\ell_2(X))$ can be $\varepsilon$-approximated if there exists $S \in C_0^*(X)$ such that $\text{supp}(S) \subseteq E$ and $\|T - S\| \leq \varepsilon$. We say that $S$ is an $\varepsilon$-approximation to $T$.

Theorem 4.4. Suppose that $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are small and uniformly locally finite coarse spaces. Let $\Phi : C_0^*(Y) \to C_0^*(X)$ be a isomorphism. Then for every $F \in \mathcal{F}$ and $\varepsilon > 0$ there exists $E := E(F, \varepsilon) \in \mathcal{E}$ such that $\Phi(T)$ can be $\varepsilon$-approximated for all $T \in C_0^*(Y)$ with $\text{supp}(T) \subseteq F$.

The proof of Theorem 4.4 was inspired by the canonical Ramsey Theory (see e.g., [PV85]). For the convenience of the reader, we give a self-contained proof after a few elementary lemmas and some bad news.

Lemma 4.5. Suppose $(X, \mathcal{E})$ is a coarse space, $\varepsilon > 0$, and $E \in \mathcal{E}$. If operators $T_1$ and $T_2$ are $\varepsilon$-approximated, then the operator $T_1 + T_2$ can be $2\varepsilon$-approximated. □

Lemma 4.6. Suppose $(X, \mathcal{E})$ is a uniformly locally finite coarse space, $\varepsilon > 0$, and $E \in \mathcal{E}$. Let $T$ be an operator in $C_0^*(X)$ and $P$ be a finite rank projection in $\ell_\infty(X)$. The following holds.

(i) If $T$ is $\varepsilon$-approximated, so is $TP$.

(ii) If $TP$ is $(\varepsilon + \delta)$-approximated for all $\delta > 0$, then $TP$ is $\varepsilon$-approximated.

Proof. Since $\text{supp}(TP) \subseteq \text{supp}(T)$ and $\|P\| \leq 1$, (i) follows. For (ii), pick $S_n \in C_0^*(X)$ such that $\|TP - S_n\| \leq \varepsilon + 1/n$ and $\text{supp}(S_n) \subseteq E$, for all $n \in \mathbb{N}$. Let $X' = \{x \in X \mid P\delta_x \neq 0\}$ and $X'' = \{x \in X \mid (\exists x' \in X')(x', x) \in E\}$, and notice that both $X'$ and $X''$ are finite. Therefore, since $S_n P$ can be naturally identified with operators from $\ell_2(X')$ to $\ell_2(X'')$, by going to a subsequence, we can assume that $(S_n P)_{n \in \mathbb{N}}$ converges to some $S \in B(\ell_2(X))$ in norm. As $\text{supp}(S_n P) \subseteq E$, for all $n \in \mathbb{N}$, $\text{supp}(S) \subseteq E$. Clearly, $\|TP - S\| \leq \varepsilon$. □

Lemma 4.7. Suppose $(X, \mathcal{E})$ is a coarse space, $\varepsilon > 0$, and $E \in \mathcal{E}$. Also suppose that $(P_j)_{j \in J}$ is an increasing net of finite rank projections in $\ell_\infty(X)$ converging to 1 strongly. If $T \in B(\ell_2(X))$ cannot be $\varepsilon$-approximated, then $TP_\lambda$ cannot be $\varepsilon$-approximated for all large enough $\lambda$. □
Proof. Suppose otherwise. For every \( j \in J \) fix an \( \varepsilon \)-\( E \)-approximation \( S_j \) to \( TP_j \). Then \( \|S_j\| \leq \|T\| + \varepsilon \) for all \( j \in J \). Since the norm-bounded balls of \( B(\ell_2(X)) \) are compact in the weak operator topology, by going to a subnet if necessary we may assume that the \((S_j)_{j \in J}\) converges to some \( S \in B(\ell_2(X)) \) in the weak operator topology. Clearly, \( \supp(S) \subseteq \bigcup_{j \in J} \supp(S_j) \). So, \( S \in C^*_u(X) \) and, by our assumption, \( \|T - S\| > \varepsilon \). Choose unit vectors \( \xi \) and \( \eta \) in \( \ell_2(X) \) such that \( \|\langle (T - S)\xi, \eta \rangle\| > \varepsilon \).

Since \( \lim_{j \in J} P_j \xi = \xi \) in norm, we have \( \lim_{j \in J} \|\langle (TP_j - S_j)\xi, \eta \rangle\| > \varepsilon \) contradicting the assumption that \( S_j \) is an \( \varepsilon \)-\( E \)-approximation of \( TP_j \).

\( \square \)

Lemma 4.8. Suppose \((X, \mathcal{E})\) is a coarse space and \( K \subseteq C^*_u(X) \) is compact in the norm topology. Then for every \( \varepsilon > 0 \) there exists \( E \in \mathcal{E} \) such that every \( T \in K \) can be \( \varepsilon \)-\( E \)-approximated.

Proof. If \( K \) is finite then this is true because \( \mathcal{E} \) is directed. Fix a finite \( \varepsilon/2 \)-net \( K_0 \) in \( K \) and find \( E \) such that every \( T \in K_0 \) can be \( \varepsilon/2 \)-\( E \)-approximated.

In what follows, it will be convenient to write \( \bar{\lambda} \) for \((\lambda_j)_{j \in J} \in \mathbb{D}^J\).

Lemma 4.9. Suppose \((X, \mathcal{E})\) is a small and uniformly locally finite coarse space. Suppose \((T_j)_{j \in J} \in \mathcal{C}^*_u(X) \) is a family of finite rank operators in \( C^*_u(X) \) such that for every \( \bar{\lambda} \in \mathbb{D}^J \) the series \( \sum_{j \in J} \lambda_j T_j \) strongly converges to an operator \( T_{\bar{\lambda}} \in C^*_u(X) \). Then for every \( \varepsilon > 0 \) there exists \( E \in \mathcal{E} \) such that \( T_{\bar{\lambda}} \) can be \( \varepsilon \)-\( E \)-approximated for all \( \bar{\lambda} \in \mathbb{D}^J \).

Proof. Without loss of generality, assume that \( X \) is infinite and that \( T_j \neq 0 \) for all \( j \). Hence, it follows that \( |J| \leq |X| \). Otherwise, by a counting argument there is a pair \((x, x')\) in \( X^2 \) such that the set \( \{j \in J \mid T_{xx'} \neq 0\} \) is of cardinality \(|J|\). Since \( X \) is infinite, \( J \) is uncountable and therefore for some \( \varepsilon > 0 \) the set \( \{j \in J \mid |T_{xx'}| \geq \varepsilon\} \) is uncountable. This clearly contradicts the assumption that \( \sum_{j \in J} \lambda_j T_j \) converges strongly to some operator \( T \).

For each finite \( I \subseteq J \), write

\[
Z_I = \{\bar{\lambda} \in \mathbb{D}^J \mid \forall j \in I, \lambda_j = 0\} \text{ and } Y_I = \{\bar{\lambda} \in \mathbb{D}^J \mid \forall j \notin I, \lambda_j = 0\}.
\]

Assume that the conclusion of the lemma fails. Then the following holds.

(7) \((\exists \varepsilon > 0)(\forall E \in \mathcal{E})(\exists \bar{\lambda} \in \mathbb{D}^J)T_{\bar{\lambda}} \) is not \( \varepsilon \)-\( E \)-approximated.

This implies a stronger condition. In what follows, the symbol \( \forall \) abbreviates ‘for all finite’.

(8) \((\exists \varepsilon' > 0)(\forall E \in \mathcal{E})(\forall_{\text{finite } I \subset J})(\exists \bar{\lambda} \in Z_I)T_{\bar{\lambda}} \) is not \( \varepsilon' \)-\( E \)-approximated.

To prove this, suppose \( \text{(8)} \) fails for \( \varepsilon' > 0 \) and fix \( E \in \mathcal{E} \) and a finite \( I \subset J \) such that for all \( \bar{\lambda} \in Z_I \) the operator \( T_{\bar{\lambda}} \) can be \( \varepsilon' \)-\( E \)-approximated. For each \( \bar{s} \in Y_I \) the operator \( T_{\bar{s}} \) belongs to \( C^*_u(X) \). Since the function \( \bar{s} \in Y_I \mapsto T_{\bar{s}} \in C^*_u(X) \) is norm-continuous and \( Y_I \) is, being homeomorphic to \( \mathbb{D}^I \), compact, by Lemma 4.8 there exists \( E' \) such that \( T_{\bar{s}} \) can be \( \varepsilon' \)-\( E' \)-approximated, for all \( \bar{s} \in Y_I \). We may assume \( E \subset E' \). For every \( \bar{\lambda} \in \mathbb{D}^J \) the operator \( T_{\bar{\lambda}} \) can be written as a sum of an operator indexed in \( Y_I \) and one indexed in \( Z_I \). By Lemma 4.8 it can be \( 2\varepsilon' \)-\( E' \)-approximated, and therefore \( \text{(7)} \) fails for \( \varepsilon = 2 \varepsilon' \). As \( \varepsilon' \) is arbitrary, this contradicts \( \text{(7)} \).

Fix \( \varepsilon = \varepsilon'/2 \), where \( \varepsilon' \) is given by \( \text{(8)} \).

Claim 1. For each \( E \in \mathcal{E} \), the subset

\[
U_E = \left\{\bar{\lambda} \in \mathbb{D}^J \mid T_{\bar{\lambda}} \text{ is } \varepsilon \text{-}E\text{-approximated}\right\}
\]
is closed and has empty interior.

Proof. Suppose that $U_E$ is not closed and pick $\bar{\lambda} \in U_E^c \setminus \text{Int}(U_E^0)$. Since $T_\lambda$ is not $\varepsilon$-approximated, by Lemma 4.7 there exists a finite rank projection $P \in \ell_\infty(X)$ so that $T_\lambda P$ is not $\varepsilon$-approximated. Fix $\delta > 0$. Since $\sum_{j \in J} \theta_j T_j$ strongly converges to a bounded linear operator for every $\bar{\theta} \in D^J$, there exists a finite $I \subset J$ such that $\|T_{\bar{\theta}} P\| < \delta$, for all $\bar{\theta} \in Z_I$. Let $\bar{\lambda} = \bar{\lambda}_I + \bar{\lambda}_\infty$, for some $\bar{\lambda}_I \in Y_I$ and $\bar{\lambda}_\infty \in Z_I$. As $\bar{\lambda} \notin \text{Int}(U_E^0)$, there exists $\bar{\theta}_I \in Y_I$ and $\bar{\theta}_\infty \in Z_I$ such that $\|T_{\bar{\lambda}_I} - T_{\bar{\theta}_I}\| \leq \delta$ and such that $T_{\bar{\theta}_I} + T_{\bar{\theta}_\infty}$ is $\varepsilon$-approximated. Since $T_{\bar{\lambda}_I} P = (T_{\bar{\theta}_I} + T_{\bar{\theta}_\infty}) P + T_{\bar{\lambda}_I} P - T_{\bar{\theta}_I} P - T_{\bar{\theta}_\infty} P + T_{\bar{\lambda}_\infty} P$, Lemma 4.5 and Lemma 4.6(i) imply that $T_{\bar{\lambda}} P$ is $(\varepsilon + 3\delta)$-approximated. As $\delta$ is arbitrary, Lemma 4.6(ii) implies that $T_{\bar{\lambda}} P$ is $\varepsilon$-approximated; contradiction.

In order to notice that $U_E$ has empty interior, let $\bar{\lambda} \in D^J$ and fix a finite $I \subset J$. Let $\bar{\lambda}_I$ be defined as in the previous paragraph and pick $E' \in \mathcal{E}$ such that $T_{\bar{\lambda}_I}$ is $\varepsilon$-approximated. Without loss of generality, $E \subset E'$. By our choice of $\varepsilon$, there exists $\theta \in Z_I$ so that $T_{\bar{\theta}}$ cannot be $2\varepsilon$-approximated. Hence, by Lemma 4.5 $T_{\bar{\lambda}_I} + T_{\bar{\theta}}$ is not $\varepsilon$-approximated. Since $E \subset E'$, this implies $\lambda_I + \theta \notin U_E$. \qed

Let $\mathcal{U}$ be a set of generators of $\mathcal{E}$ of cardinality $\max\{|\mathcal{E}|_{\min}, |E|_{\min}\}$. Without loss of generality, assume that every element of $\mathcal{E}$ is contained in some element of $\mathcal{U}$. It follows that

$$D^J = \bigcup_{E \in \mathcal{U}} U_E.$$  

Since $(|\mathcal{E}|_{\min}, |X|)$ is small and $|J| \leq |X|$, it follows that $\text{cov}(D^J) > \max\{|\mathcal{E}|_{\min}, |E|_{\min}\}$. On the other hand, since $U_E$ is nowhere dense for every $E \in \mathcal{U}$, (4.1) implies that $\text{cov}(D^J) \leq \max\{|\mathcal{E}|_{\min}, |E|_{\min}\}$; contradiction. \quad \Box

Proof of Theorem 4.4 Fix $\varepsilon > 0$ and $F \in \mathcal{F}$. By Lemma 3.1 $\Phi$ is implemented by a unitary operator and therefore continuous with respect to the strong operator topology. Let $(T_j)_{j \in J}$ be the family of all $\Phi(e_{y'y'})$, for $(y,y') \in F$. This family has cardinality at most $|Y|$, and therefore, by Corollary 3.2 less than $|X|$. Since $(X, \mathcal{E})$ is small cardinal, Lemma 4.9 implies that there exists $E \in \mathcal{E}$ such that $T_{\bar{\lambda}}$ can be $\varepsilon$-approximated for all $\bar{\lambda} \in D^J$, as required. \quad \Box

It would be desirable to have $E$ as in the conclusion of Theorem 4.4 depend on $F$ only, instead of both $F$ and $\varepsilon$. Alas, in general this is not true; see Example 6.3.

Lemma 4.10 below plays the role of Lemma 4.5 of \cite{SW13} in our proof. This will be used later to show that the the maps we obtain between $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are coarse. Before stating and proving it, we prove a technical result which will be essential throughout this paper.

Lemma 4.10. Suppose $(X, \mathcal{E})$ is a uniformly locally finite coarse space and $E \in \mathcal{E}$. Let $\mathcal{F}$ be a directed set and let $(x^F_1)_{F \in \mathcal{F}}$ and $(x^F_2)_{F \in \mathcal{F}}$ be nets in $X$ such that $(x^F_1, x^F_2) \in E$, for all $F \in \mathcal{F}$. Then there exist subsets $I$ and $J$ of $\mathcal{F}$ and $\varphi : I \rightarrow J$ such that $I$ is cofinal and the following holds:

(i) $x^F_1 \neq x^{F'}_1$ and $x^F_2 \neq x^{F'}_2$, for all distinct $F$ and $F'$ in $J$, and
(ii) $x^F_1 = x^{\varphi(F)}_1$ and $x^F_2 = x^{\varphi(F)}_2$, for all $F \in I$.  

Proof. By Proposition 2.4 there exists a partition
\[ X = X_1 \sqcup \ldots \sqcup X_k, \]
such that \((x_1, x_2) \notin E^{(3)}\), for all \(i \leq k\) and all distinct \(x_1, x_2 \in X_i\). For each \(i \leq k\), let
\[ I_i = \{ F \in \mathcal{F} \mid x_1^F \in X_i \}. \]
Since \(I_i\) is directed, there exists \(i \leq k\) such that \(I_i\) is cofinal in \(I\). Let \(I = I_i\). Fix \(F\) and \(F'\) in \(I\) such that \(x_1^F \neq x_2^{F'}\). Then \((x_1^F, x_2^{F'}) \notin E^{(3)}\) and since \((x_1^F, x_2^G) \in E\) for all \(G \in J\), we conclude that \((x_1^F, x_2^{F'}) \notin E\), and in particular \(x_1^F \neq x_2^{F'}\).

By an analogous argument and going to a cofinal subset of \(I\) if necessary, we may assume that \(x_1^F \neq x_2^{F'}\) implies \(x_1^F \neq x_2^{F'}\) for all \(F\) and \(F'\) in \(I\).

To recap, we may assume the following:
\[(4.2) \quad x_1^F = x_1^{F'} \quad \text{if and only if} \quad x_2^F = x_2^{F'}, \quad \text{for all} \quad F, F' \in I.\]

Let \(\tilde{X} = \{ x_1^F \mid F \in I \}\). Fix an assignment \(x \in \tilde{X} \rightarrow F_x \in I\) such that \(x_1^{F_x} = x\), for all \(x \in \tilde{X}\). Let \(J = \{ F_x \in I \mid x \in \tilde{X} \}\) be the image of this assignment. Notice that \(x_1^F \neq x_2^{F'}\), for all distinct \(F, F' \in J\). So, by the backwards implication of (4.2), we have that \(x_1^F \neq x_2^{F'}\), for all distinct \(F, F' \in J\).

Define \(\varphi : I \rightarrow J\) by letting \(\varphi(F) = F_x\), for all \(F \in I\). By the definition of \(\varphi\) and the forward implication in (4.2) we have that
\[ x_1^F = x_1^{\varphi(F)} \quad \text{and} \quad x_2^F = x_2^{\varphi(F)}, \quad \text{for all} \quad F \in I. \]

This completes the proof of the lemma. \(\square\)

If the spaces \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) are metrizable, the assumption on \(\mathcal{E}\) and \(\mathcal{F}\) in the following lemma follows from the Baire category theorem.

Lemma 4.11. Suppose that \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) are small and uniformly locally finite coarse spaces. Let \(U : \ell_2(X) \rightarrow \ell_2(Y)\) be a unitary operator which spatially implements a isomorphism \(C^*_u(X) \rightarrow C^*_u(Y)\). Then for all \(E \in \mathcal{E}\) and all \(\delta > 0\) the following set belongs to \(\mathcal{F}\):
\[ F_{E, \delta} := \{(y_1, y_2) \in Y^2 \mid (\exists (x_1, x_2) \in E)(|\langle U \delta_{x_1}, \delta_{y_1} \rangle| \geq \delta \land |\langle U \delta_{x_2}, \delta_{y_2} \rangle| \geq \delta)\}. \]

Proof. Suppose otherwise. Then there exist \(E \in \mathcal{E}\) and \(\delta > 0\) such that \(F_{E, \delta} \notin \mathcal{F}\). Therefore for every \(F \in \mathcal{F}\) there exist \((x_1^F, x_2^F) \in E\) and \((y_1^F, y_2^F) \in Y^2\) such that \(|\langle U \delta_{x_1^F}, \delta_{y_1^F} \rangle| \geq \delta\), \(|\langle U \delta_{x_2^F}, \delta_{y_2^F} \rangle| \geq \delta\), and \((y_1^F, y_2^F) \notin F\).

Order \(\mathcal{F}\) by the inclusion; it is a directed set. By Lemma 4.10 there exist a cofinal \(I \subset \mathcal{F}\), \(J \subset \mathcal{F}\), and a map \(\varphi : I \rightarrow J\) such that
\begin{enumerate}
  \item \(x_1^F \neq x_2^{F'}\) and \(x_2^F \neq x_2^{F'}\), for all distinct \(F, F' \in I\), and
  \item \(x_1^F = x_1^{\varphi(F)}\) and \(x_2^F = x_2^{\varphi(F)}\), for all \(F \in I\).
\end{enumerate}

Fix \(\lambda \in \mathbb{D}^J\). Since \((x_1^F, x_2^F) \in E\) for all \(F \in I\) and \(\sup_{x \in X} (|E_x|, |E|^x)\) is finite, \(\Phi\) implies that the sum \(\sum_{F \in J} \lambda_F e^F e^{F'}\) converges in the strong operator topology to an operator in \(B(\ell_2(X))\). Since its support is included in \(E\), this operator belongs to \(C^*_u(X)\).

For each \(F \in \mathcal{F}\), let \(e(F) = e^F e^{F'}\). Hence, as \(\Phi\) is continuous in the strong operator topology, the sum
\[ \sum_{F \in J} \lambda_F \Phi(e(F)) \]
converges strongly to an operator in \(C^*_u(Y)\), for every \(\lambda \in \mathbb{B}^j\). By Theorem 1.3 there exists \(F_1 \in \mathcal{F}\) such that
\[
\|\chi_A \Phi(e(F)) \chi_B\| < \delta^2,
\]
for all \(F \in J\) and all \(A, B \subset Y\) which are \(F_1\)-separated. Since \(I\) is cofinal in \(\mathcal{F}\), we can pick \(F_1 \in I\). For now on, set
\[
a(x) = x_1^{F_1}, \quad b(x) = x_2^{F_1}, \quad a(y) = y_1^{F_1}, \quad \text{and} \quad b(y) = y_2^{F_1}
\]
and notice that \(\{a(y)\}\) and \(\{b(y)\}\) are \(F_1\)-separated. By (ii), it follows that
\[
\|\chi(a(y)) \Phi(e(F_1)) \chi(b(y))\| = \|\chi(a(y)) \Phi(e(F_1)) \chi(b(y))\| < \delta^2.
\]
Using \(\langle U^* \delta_{b(y)}, \delta_{b(x)}\rangle \delta_{a(x)} = e(F_1) U^* \delta_{b(y)}\), we have the following
\[
\delta^2 \leq |\langle \delta_{a(y)}, U \delta_{a(x)} \rangle \langle U \delta_{b(x)}, \delta_{b(y)} \rangle| = |\langle U e(F_1) U^* \delta_{b(y)}, \delta_{a(x)} \rangle| = |\langle U e(F_1) U^* \delta_{b(y)}, \delta_{a(x)} \rangle| = \|\chi(a(y)) \Phi(e(F_1)) \chi(b(y))\|:
\]
contradiction. \(\square\)

We are ready to prove (8) and (9) from the introduction.

**Theorem 4.12.** Suppose that \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) are uniformly locally finite coarse spaces which are also small. If \(C^*_u(X)\) and \(C^*_u(Y)\) are rigidly isomorphic, then \(X\) and \(Y\) are coarsely equivalent.

**Proof.** Let \(U : \ell_2(X) \rightarrow \ell_2(Y)\) be a unitary operator which spatially implements a rigid isomorphism between \(C^*_u(X)\) and \(C^*_u(Y)\). Therefore, there exist \(\delta > 0\), \(f : X \rightarrow Y\) and \(g : Y \rightarrow X\) such that
\[
|\langle U \delta_x, \delta_f(y) \rangle| \geq \delta \quad \text{and} \quad |\langle U^* \delta_y, \delta_{g(y)} \rangle| \geq \delta,
\]
for all \(x \in X\) and all \(y \in Y\). Lemma 4.11 implies that \(f\) and \(g\) are coarse maps. Therefore, we only need to verify that \(g \circ f\) and \(f \circ g\) are close to \(\text{Id}_X\) and \(\text{Id}_Y\), respectively. By our choice of \(f\) and \(g\), it follows that
\[
|\langle U \delta_{g(y)}, \delta_{f(g(y))} \rangle| \geq \delta \quad \text{and} \quad |\langle U^* \delta_{g(y)}, \delta_y \rangle| = |\langle \delta_{g(y)}, U^* \delta_y \rangle| \geq \delta,
\]
for all \(y \in Y\). Let \(F = F_{\Delta_X, \delta}\) be as in the conclusion of Lemma 4.11 Then, since \((g(y), g(y)) \in \Delta_X\), we have that
\[
(y, f(g(y))) \in F,
\]
for all \(y \in Y\). This shows that \(f \circ g\) is close to \(\text{Id}_Y\). Similar arguments show that \(g \circ f\) is close to \(\text{Id}_X\). \(\square\)

We end this section with a metamathematical remark.

**Corollary 4.13.** (MA\(\kappa\)) Suppose that \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) are \(\kappa\)-generated uniformly locally finite coarse spaces. If \(C^*_u(X)\) and \(C^*_u(Y)\) are rigidly isomorphic, then \(X\) and \(Y\) are coarsely equivalent. \(\square\)

## 5. Coarse spaces with a small partition

In this section, we give a large class of examples of non-metrizable uniformly locally finite coarse spaces such that a rigid isomorphism between their uniform Roe algebras implies coarse equivalence (i.e., (Y) implies (I)). We prove this by
studying coarse spaces with a partition that ‘behave like a metric space’. As before, the reader not interested in set theory may replace the words ‘small’ in the definition below by ‘metrizable’.

**Definition 5.1.** Let \((X, \mathcal{E})\) be a coarse space. We say that \((X, \mathcal{E})\) has a small partition if there exists a partition \(X = \bigsqcup_{i \in I} X_i\) so that, letting
\[
\mathcal{E}_\Delta := \left\{ \bigsqcup_{i \in I} (X_i \times X_i) \cap E \mid E \in \mathcal{E} \right\},
\]
the coarse space \((X, \mathcal{E}_\Delta)\) is small and \(\mathcal{E}\) is generated by \(\mathcal{E}_\Delta \cup \{E \in \mathcal{E} \mid |E| < \infty\}\).

Notice that if \((X, \mathcal{E})\) is small, then \((X, \mathcal{E})\) trivially admits a small partition. We refer the reader to Example 5.6 for more interesting examples of uniformly locally finite coarse spaces which have a small partition.

We need some simple lemmas (which are also used in Section 7). Given a partition \(X = \bigsqcup_{i \in I} X_i\) we identify the product \(\prod_{i \in I} B(\ell_2(X_i))\) with a subalgebra of \(B(\ell_2(X))\).

**Lemma 5.2.** Consider a partition \(X = \bigsqcup_{i \in I} X_i\). If \(Q \in \prod_{i \in I} B(\ell_2(X_i))\) and \(R \in B(\ell_2(X))\) are such that \(\chi_{X_i} R \chi_{X_i} = 0\), for all \(i \in I\) then \(\|Q\| \leq \|Q + R\|\).

**Proof.** For each \(i \in I\), identify \(\ell_2(X_i)\) with a subspace of \(\ell_2(X)\) in the natural way. Then \((B_2(X_i))\) denotes the unit closed ball of \(\ell_2(X_i)\), for \(i \in I\)
\[
\|Q\| = \sup_{i \in I} \sup_{x \in B_2(X_i)} \|\chi_{X_i} Q \chi_{X_i}(x)\| = \sup_{i \in I} \sup_{x \in B_2(X_i)} \|\chi_{X_i}(Q + R) \chi_{X_i}(x)\|
\]
and we conclude that \(\|Q\| \leq \|Q + R\|\). \(\Box\)

**Lemma 5.3.** Let \((X, \mathcal{E})\) be a uniformly locally finite coarse space and \(X' \subset X\). Fix a partition \(X' = \bigsqcup_{i \in I} X_i\) and consider the coarse structure
\[
\mathcal{E}' := \left\{ \bigsqcup_{j \in J} (X_j \times X_j) \cap E \mid E \in \mathcal{E} \right\}
\]
on \(X'\). Then \(C_u^*(X) \cap \prod_{i \in I} B(\ell_2(X_i)) = C_u^*(X', \mathcal{E}')\).

**Proof.** The inclusion \(C_u^*(X) \cap \prod_{i \in I} B(\ell_2(X_i)) \supseteq C_u^*(X', \mathcal{E}')\) is clear. For the other inclusion, let \(Q\) be an operator in \(C_u^*(X) \cap \prod_{i \in I} B(\ell_2(X_i))\). To prove \(Q \in C_u^*(X', \mathcal{E}')\), pick sequences \((Q_n)_{n \in \mathbb{N}}\) and \((R_n)_{n \in \mathbb{N}}\) in \(C_u^*[X]\) such that
\[
\begin{align*}
(i) & \quad Q_n \in C_u^*[X', \mathcal{E}'], \text{ for all } n \in \mathbb{N}, \\
(ii) & \quad \chi_{X_i} R_n \chi_{X_i} = 0, \text{ for all } i \in I \text{ and all } n \in \mathbb{N}, \text{ and} \\
(iii) & \quad \lim_n (Q_n + R_n) = Q.
\end{align*}
\]
By Lemma 5.2 we have that
\[
\|Q - Q_n\| \leq \|Q - (Q_n + R_n)\|.
\]
So, \(\lim_n Q_n = Q\) and we conclude that \(Q \in C_u^*(X', \mathcal{E}')\). \(\Box\)

A version of Lemma 5.3 holds for coarse spaces admitting small partitions.

**Lemma 5.4.** Suppose that \((X, \mathcal{E})\) is a uniformly locally finite coarse space admitting a small partition. Suppose that \((T_j)_{j \in J}\) is a family of finite rank operators in \(C_u^*(X)\) such that \(\sum_{j \in J} \lambda_j T_j\) converges strongly to an operator \(T_\lambda \in C_u^*(X)\), for
every $\lambda \in \mathbb{D}^J$. Then, for every $\varepsilon > 0$, there exists $E \in \mathcal{E}$ such that
\[ \|\chi_{\{x\}}T_j\chi_{\{x'\}}\| < \varepsilon, \]
for all $j \in J$ and all $x, x' \in X$ with $(x, x') \notin E$.

**Proof.** Let $X = \bigsqcup_{i \in I} X_i$ be a small partition of $(X, \mathcal{E})$ and let $\mathcal{E}_\Delta$ be the coarse structure in Definition 5.1 related to this given partition of $X$. Let $P_j$ denote the projection of $\ell_2(X)$ onto $\ell_2(X_j)$, let $P : B(\ell_2(X)) \to \prod_{j \in J} B(\ell_2(X_j))$ be the conditional expectation,
\[ P(T) = \sum_{j \in J} P_j TP_j, \]
and set $R = 1 - P$. Lemma 5.2 implies that $\mathbb{P}(C_\Delta^*(X)) \subset C_n^*(X)$. Since $\sum_{j \in J} \lambda_j T_j$ converges strongly in $C_n^*(X)$ for all $\lambda \in \mathbb{D}^J$, so do $\sum_{j \in J} \lambda_j PT_j$ and $\sum_{j \in J} \lambda_j RT_j$. Clearly, $PT_{\lambda} = \sum_{j \in J} \lambda_j PT_j$ and $RT_{\lambda} = \sum_{j \in J} \lambda_j RT_j$, for all $\lambda \in \mathbb{D}^J$.

**Claim 1.** For all $\varepsilon > 0$, there exists $E \in \mathcal{E}$ such that $\|\chi_{\{x\}}RT_j\chi_{\{x'\}}\| < \varepsilon$, for all $j \in J$ and all $x, x' \in X$ with $(x, x') \notin E$.

**Proof.** Suppose not. Then there exists $\varepsilon > 0$, $(j^E)_{E \in \mathcal{E}}$ in $J$ and $(x^E_1, x^E_2)_{E \in \mathcal{E}}$ in $X \times X$ with $(x^E_1, x^E_2) \notin E$ and $\|\chi_{\{x^E_1\}}RT_j\chi_{\{x^E_2\}}\| \geq \varepsilon$, for all $E \in \mathcal{E}$. In particular, $\{(x^E_1, x^E_2)\} \in \mathcal{E}$ and $(x^E_1, x^E_2) \notin \bigsqcup_{i \in I} X_i \times X_i$, for all $E \in \mathcal{E}$. An easy induction produces a sequence $(j^n)_{n \in \mathbb{N}}$ in $J$ and a sequence $(x^n_1, x^n_2)_{n \in \mathbb{N}}$ in $X \times X$ of distinct elements such that $\|\chi_{\{x^n_1\}}RT_j^n\chi_{\{x^n_2\}}\| \geq \varepsilon$ and $(x^n_1, x^n_2) \notin \bigsqcup_{i \in I} X_i \times X_i$, for all $n \in \mathbb{N}$. Since $T_j$ has finite rank for all $j \in J$, without loss of generality, assume that $(j^n)_{n \in \mathbb{N}}$ is a sequence of distinct elements of $J$.

Going to a subsequence of $(j^n)_{n \in \mathbb{N}}$ if necessary, we can pick a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $\{-1, 1\}$ such that
\begin{enumerate}[label=(i), ref=(i), start=1]
\item $\left\|\chi_{\{x^n_1\}} \left( \sum_{i=1}^n \lambda_i RT^n_{j^i} \right) \chi_{\{x^n_2\}} \right\| \geq \varepsilon$, for all $n \in \mathbb{N}$, and
\item $\sum_{i>n} \left\|\chi_{\{x^n_1\}}RT^n_{j^i}\chi_{\{x^n_2\}}\right\| < \varepsilon/2$, for all $n \in \mathbb{N}$.
\end{enumerate}
Therefore,
\[ \left\|\chi_{\{x^n_1\}} \left( \sum_{i=1}^\infty \lambda_i RT^n_{j^i} \right) \chi_{\{x^n_2\}} \right\| \geq \frac{\varepsilon}{2}, \]
for all $n \in \mathbb{N}$. Since $\mathcal{E}$ is generated by $\mathcal{E}_\Delta \cup \{ E \in \mathcal{E} \mid |E| < \infty \}$, there exists $S_1, S_2 \subset C_n^*(X)$ with $\text{supp}(S_1) \in \mathcal{E}_\Delta$ and $\text{supp}(S_2)$ finite such that
\[ \left\| \sum_{i=1}^\infty \lambda_i RT^n_{j^i} - (S_1 + S_2) \right\| < \frac{\varepsilon}{2}. \]
Since $\text{supp}(S_2)$ is finite, fix $n \in \mathbb{N}$ so that $(x^n_1, x^n_2) \notin \text{supp}(S_2)$. Then
\[ \left\|\chi_{\{x^n_1\}} \left( \sum_{i=1}^\infty \lambda_i RT^n_{j^i} - (S_1 + S_2) \right) \chi_{\{x^n_2\}} \right\| = \left\|\chi_{\{x^n_1\}} \left( \sum_{i=1}^\infty \lambda_i RT^n_{j^i} \right) \chi_{\{x^n_2\}} \right\| \geq \frac{\varepsilon}{2}, \]
contradiction. \qed

Fix $\varepsilon > 0$ and let $E \in \mathcal{E}$ be given by Claim 1 for $\varepsilon/2$. By Lemma 5.3, $\mathbb{P}(C_\Delta^*(X)) \subset C_n^*(X, \mathcal{E}_\Delta)$. Hence, by Lemma 4.3, going to a larger $E \in \mathcal{E}$ if necessary, assume that $PT_j$ is $\varepsilon/2-E$-approximable, for all $j \in J$. This implies that
\[ \|\chi_{\{x\}}T_j\chi_{\{x'\}}\| \leq \|\chi_{\{x\}}PT_j\chi_{\{x'\}}\| + \|\chi_{\{x\}}RT_j\chi_{\{x'\}}\| \leq \varepsilon, \]
for all $j \in J$ and all $x, x' \in X$ with $(x, x') \notin E$. \qed
Theorem 5.5. Suppose \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) are uniformly locally finite coarse spaces admitting small partitions. If \(C^*_u(X)\) and \(C^*_u(Y)\) are rigidly isomorphic, then \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) are coarsely equivalent.

Proof. Fix \(\delta > 0\), \(f : X \to Y\) and \(g : Y \to X\) such that
\[
|U_{\delta x}, \delta f(x)| \geq \delta \quad \text{and} \quad |U^*_{\delta y}, \delta g(y)| \geq \delta,
\]
for all \(x \in X\) and all \(y \in Y\). Using Lemma 6.3 one can easily check that Lemma 4.11 applies to \((X, \mathcal{E})\) and \((Y, \mathcal{F})\). So both \(f\) and \(g\) are coarse maps. Proceeding analogously to the proof of Theorem 4.12, we obtain that \(g \circ f\) and \(f \circ g\) are close to \(\text{Id}_X\) and \(\text{Id}_Y\), respectively. \(\square\)

Example 5.6. We now give a natural way to construct examples of uniformly locally finite coarse spaces with property A which are non-metrizable but have a small partition. In particular, those spaces satisfy the conditions in Theorem 5.5.

Let \((X, \mathcal{E})\) be a metrizable coarse space, \(J\) be an index set, \(X_j = X\), and \(\mathcal{E}_j = \mathcal{E}\), for all \(j \in J\). Let \(X = \bigsqcup_{j \in J} X_j\), and for each \(j \in J\) identify \(X_j\) with a subset of \(X\) and \(E \in \mathcal{E}_j\) with a subset of \(X \times X\) in the natural way. Let \((E_n)_{n \in \mathbb{N}}\) be a sequence generating \(\mathcal{E}\) and for each \(j \in J\) let \((E_{n,j})_{n \in \mathbb{N}}\) be thought as a sequence of subsets of \(X_j \times X_j\). Define

(i) \(\mathcal{E}_1 = \{\bigcup_{j \in J} E_{n,j} \mid n \in \mathbb{N}\}\), and
(ii) \(\mathcal{E}_2 = \{E \subseteq X \times X \mid |E| < \infty\}\).

Let \(\mathcal{E}\) be the coarse structure on \(X\) generated by \(\mathcal{E}_1\) and \(\mathcal{E}_2\). The space \((X, \mathcal{E})\) is uniformly locally finite and, if \(J\) is uncountable, it is non-metrizable. It is clear that \((X, \mathcal{E})\) has a small partition. Also, one can easily check that the asymptotic dimension of \((X, \mathcal{E})\), i.e., \(\text{asdim}(X)\), is the same as the asymptotic dimension of \((X, \mathcal{E})\) (see [Roe03], Chapter 9, for definition of asymptotic dimension). Hence, if \(\text{asdim}(X) < \infty\), \((X, \mathcal{E})\) has property A (by [Roe03], Section 11.5).

6. Isomorphism between uniform Roe algebras and Cartan masas

The goal of this section is to prove (11) (Theorem 6.1 and Theorem 6.2) from the introduction. Only a definition stands between us and this proof. A Cartan masa \(A\) in \(C^*_u(X)\) is ghostly if it contains non-compact ghost projections \(Q_j\), for \(j \in J\), which are orthogonal and \(\sum_{j \in J} Q_j\) converges strongly to the identity of \(C^*_u(X)\).

The following is a strengthening of Lemma 4.6 of [SW13], where an analogous result was proved using the stronger property A.

Theorem 6.1. Suppose that \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) are metrizable uniformly locally finite coarse spaces and there exists an isomorphism \(\Phi : C^*_u(X) \to C^*_u(Y)\) which is not rigidly implemented. Then at least one of the following applies.

(i) \(\Phi|\ell_\infty(X)\) is a ghostly Cartan masa in \(C^*_u(Y)\).
(ii) \(\Phi^{-1}|\ell_\infty(Y)\) is a ghostly Cartan masa in \(C^*_u(X)\).

Dropping the assumption that the spaces be metric, we obtain a slightly weaker conclusion which is still incompatible with property A.

Theorem 6.2. Suppose that \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) are uniformly locally finite coarse spaces and there exists an isomorphism \(\Phi : C^*_u(X) \to C^*_u(Y)\) which is not rigidly implemented. Then at least one of the following applies.
Proof of Theorem 6.2. Define sets
\[ C_n \text{ in } C_u^*(X) \text{ for all } n. \]
This shows that \( C \) between \( n \) induces a rigid isomorphism. Hence, for every \( n \geq 1 \), the set
\[ X_n := \{ x \in X \mid \max_{y \in Y} |\langle U \delta_x, \delta_y \rangle| \geq 2^{-n} \}. \]
is a proper subset of \( X \).

Claim 1. There is a partition \( X = \bigcup_m Z_m \) such that each \( Z_m \) is infinite but \( Z_m \cap \langle X_{m+1} \setminus X_n \rangle \) has at most one element for all \( m \) and all \( n \).

Proof. Since \( \bigcup_{n \in \mathbb{N}} X_n = X \), for every \( n \in \mathbb{N} \) the set \( X \setminus X_n \) is infinite for all \( n \). Since \( X \) is countable, we can find a partition \( X = \bigcup_{m \in \mathbb{N}} Z_m \) such that \( Z'_m \cap X_n \) is finite for all \( m \) and \( n \). Every \( Z'_m \) can now be partitioned into sets as required. □

Fix \( m \in \mathbb{N} \). Let \( (x_n)_{n \in \mathbb{N}} \) be an enumeration of \( Z_m \) such that \( x_n \in X \setminus X_n \) for all \( n \). Then \( \| U e_{x_n} U^* \delta_y \| < 2^{-n} \), for all \( n \in \mathbb{N} \) and all \( y \in Y \). So, \( \sum_{n \in \mathbb{N}} e_{x_n} U^* \delta_y \) converges in the strong operator topology to an operator in \( C_u^*(X) \).

Claim 2. The projection \( P_m = \sum_{n \in \mathbb{N}} U e_{x_n} U^* \) is a ghost.

Proof. Let \( \varepsilon > 0 \). Pick \( n_0 \in \mathbb{N} \) such that \( 2^{-n_0} < \varepsilon/2 \). Since \( U e_{x_n} U^* \) is compact (it has rank 1), for all \( n \in \mathbb{N} \), we can pick a finite \( A \subset X \) such that
\[ |\langle U e_{x_n} U^*, \delta_y, \delta_y' \rangle| < \frac{\varepsilon}{2n_0}, \]
for all \( n \in \{1, \ldots, n_0\} \) and all \( y, y' \in Y \setminus A \). Then, for all \( y, y' \in Y \setminus A \), we have that
\[ |\langle P_m \delta_y, \delta_y' \rangle| \leq \sum_{n=1}^{n_0} |\langle U e_{x_n} U^* \delta_y, \delta_y' \rangle| + \sum_{n>n_0} |\langle U e_{x_n} U^* \delta_y, \delta_y' \rangle| \]
\[ < \frac{\varepsilon}{2} + \sum_{n>n_0} \| U e_{x_n} U^* \delta_y \| \]
\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]
This shows that \( P_m \) is a ghost. □

The projections \( P_m \), for \( m \in \mathbb{N} \), are orthogonal, non-compact, ghost projections in \( A := \text{Ad} U(\ell_\infty(X)) \) and \( \sum_{m \in \mathbb{N}} P_m \) is the identity in \( C_u^*(X) \). Since \( \ell_\infty(X) \) is a Cartan masa in \( C_u^*(X) \), \( A \) is a Cartan masa in \( C_u^*(Y) \). Therefore, \( A \) is a ghostly Cartan masa in \( C_u^*(Y) \).

Proof of Theorem 6.3. Define sets \( X_n \), for \( n \in \mathbb{N} \), as in the proof of Theorem 6.2. Then \( X \setminus X_n \) is infinite for all \( n \), and we can choose \( Z \subseteq X \) such that \( Z \cap X_{n+1} \setminus
$X_n$ has at most one element for all $n$. The proof of Claim 2 shows that $P_n = \sum_{n \in \mathbb{N}} U_n e_{x,x_n} U^*$ is a ghost projection. It belongs to the Cartan masa $U\ell_\infty(X)U^*$ and it is clearly not compact.

The following is as close as we could get to the conclusion of Theorem 6.1. It also serves as a basis for a limiting example for possible improvements of Theorem 4.4 (Example 6.4) promised earlier.

**Example 6.3.** There exist uniformly locally finite metric spaces $(X, d)$ and $(Y, \partial)$ such that $C^*_u(Y)$ is isomorphic to a corner $PC^*_u(X)P$ where $P$ is a noncompact ghost projection in $C^*_u(X)$.

The metric space $(X, d)$ is the counterexample to the coarse Baum–Connes conjecture given in HLS02 on pages 348–349, a copy of which we assume that the reader has handy. Therefore $X = \bigsqcup_{n \in \mathbb{N}} X_n$, each $X_n$ is finite, and the distance between $x \in X_n$ and $x' \in X_m$ is (for definiteness) $m + n$ if $m \neq n$. Each $X_n$ is the set of vertices of a finite graph $G_n$ with the shortest path distance. The graphs $G_n$ were chosen to be a sequence of $k$-$\varepsilon$-expander graphs for a fixed $k$ and $\varepsilon > 0$. Let $\Delta_n$ be the Laplacian of $G_n$; this is the operator on $\ell_2(X_n)$ given by the matrix

$$kI - A,$$

where $A$ is the incidence matrix of $G_n$ and $I$ is the identity. Then $\Delta_n$ is positive and $\|\Delta_n\| \leq k$ for all $n$. Therefore the direct sum $\Delta$ of all $\Delta_n$ is positive and belongs to $B(\ell_2(X))$. By the definition, $\text{supp}(\Delta) \subseteq \{(x, x') : d(x, x') \leq 1\}$ and therefore $\Delta \in C^*_u(X)$.

Let $P$ denote the projection to $\ker(\Delta)$. Since the lowest nonzero eigenvalue of $\Delta$ is bounded away from zero, the restriction of the operator $T := 1 - \Delta\|\Delta\|^{-1}$ to $\ker(\Delta)^\perp$ has norm strictly less than 1. Therefore the sequence $(T^n)_{n \in \mathbb{N}}$ converges to $P$ in norm, and $P \in C^*_u(X)$. The range of $P_{X_n}$ consists of constant functions in $\ell_2(X_n)$. Thus (writing $m_n = |X_n|$) $P_{X_n}$ is given by the $m_n \times m_n$ matrix all of whose entries are equal to $m_n^{-1/2}$. Since $|m_n| \to \infty$, $P$ is a ghost.

Let $\xi_n$ be a unit vector in the range of $P_{X_n}$. Let $Y$ be the metric space with domain $\mathbb{N}$ and the metric $\partial(m, n) = m + n$. Then $U : \ell_2(Y) \to \ell_2(X)$ defined by $U(\delta_m) = \xi_m$ implements a isomorphism between $C^*_u(Y)$ and $PC^*_u(X)P$.

The following example shows that Theorem 4.4 cannot be improved by stating that for $\Phi : C^*_u(Y) \to C^*_u(X)$ the uniformity $E \in \mathcal{E}$ depends only on $F \in \mathcal{F}$ even when $E = \Delta_Y$.

**Example 6.4.** There exist metrizable uniformly locally finite coarse spaces $(X, \mathcal{E})$ and $(Y, \mathcal{F})$, a *-homomorphism $\Phi : C^*_u(Y) \to C^*_u(X)$, such that for every $E \in \mathcal{E}$ there exist $\varepsilon > 0$ for which $\Phi(1)$ cannot be $\varepsilon$-$E$-approximated.

Let $(X, \mathcal{E})$, $(Y, \mathcal{F})$, $P$ and $\Phi : C^*_u(Y) \to PC^*_u(X)P$ be as in Example 6.3. We have that $\Phi(1)$ is a ghost projection in $C^*_u(X)$. Given $E \in \mathcal{E}$. The fact that the *-homomorphism $\Phi$ is not unital is easily remedied: add a single point $y$ to $Y$ and send $e_{yy}$ to $1 - P$ (the complement of the ghost projection in $C^*_u(X)$).

We do not know whether if $\Phi$ as in Theorem 4.4 and Example 6.4 is assumed to be a isomorphism one can deduce that it has a stronger coarse-like property.

**Corollary 6.5.** Suppose $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are uniformly locally finite coarse spaces admitting small partitions. Suppose that all ghost projections in $C^*_u(X)$ and $C^*_u(Y)$
are compact. If \( C^*_n(X) \) and \( C^*_n(Y) \) are isomorphic, then \((X,\mathcal{E})\) and \((Y,\mathcal{F})\) are coarsely equivalent.

**Proof.** This follows straightforwardly from Theorem 6.2 and Corollary 5.5. \(\square\)

7. **Rigidity for spaces which coarsely embed into Hilbert space**

In order to prove (10) (Theorem 7.4 below), we need to introduce the notion of the Rips complex and coarse connected components.

**Definition 7.1.** Let \((X,\mathcal{E})\) be a coarse space and let \(E \in \mathcal{E}\) be symmetric and such that \(\Delta_X \subseteq E\). Let

\[
P_E(X) = \{ A \subset X \mid (x,x') \in E, \ \forall x,x' \in A \}.
\]

We call \(P_E(X)\) the **Rips complex of \((X,\mathcal{E})\) over \(E\)**. We define an equivalence relation \(\sim_E\) on \(P_E(X)\) by setting \(A \sim_E A'\) if there exist \(x_1,\ldots,x_n \in X\) such that \(x_1 \in A\), \(x_n \in A'\), and \((x_m,x_{m+1}) \in E\), for all \(m \in \{1,\ldots,n-1\}\).

**Definition 7.2.** Let \((X,\mathcal{E})\) be a uniformly locally finite coarse space. We say that \((X,\mathcal{E})\) has an **infinite coarse component** if there exists a symmetric \(E \in \mathcal{E}\), with \(\Delta_X \subseteq E\), such that the Rips complex \(P_E(X)\) has an infinite \(\sim_E\)-equivalence class. Otherwise, we say that \((X,\mathcal{E})\) has **only finite coarse components**.

Let us present the prototypical example of a coarse (metric) space with only finite coarse connected components. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of finite metric spaces which are uniformly locally finite, uniformly in the index \(n\). We define a metric space \((X,d)\), called the **coarse disjoint union of \((X_n)_{n}\)**, by letting \(X = \bigcup_{n \in \mathbb{N}} X_n\) and picking a metric \(d\) on \(X\) which is given by the metric on each component and such that \(d(X_n,X_m) \to \infty\) as \(n + m \to \infty\). Any such metric is unique up to coarse equivalence. It is straightforward to check that \(X\) has only finite coarse components.

The following lemma is proved by going to a library.

**Lemma 7.3.** Let \((X,d)\) be a uniformly locally finite metric space with only finite coarse components which coarsely embeds into a Hilbert space. Then all ghost projections in \(C^*_u(X)\) are compact.

**Proof.** A metric \(d\) is uniformly discrete if \(\inf_{x \neq x'} d(x,x') > 0\). It is clear that every coarse metric space \((X,d)\) carries a uniformly discrete metric \(\partial\) compatible with its coarse structure (let \(\partial(x,x') = d(x,x') + 1\) if \(x \neq x'\)).

A uniformly locally finite metric space which coarsely embeds into a Hilbert space must satisfy the coarse Baum–Connes conjecture by [Y00], Theorem 1.1. This implies that every ghost projection in \(C^*_u(X)\) is compact (this is the second sentence of Proposition 35 of [PS14]; note that \(\mu_0\) should be \(\mu\)). \(\square\)

We are now ready to prove (10) from the introduction.

**Theorem 7.4.** If \((X,\mathcal{E})\) and \((Y,\mathcal{F})\) are uniformly locally finite coarse spaces such that both \((X,\mathcal{E})\) and \((Y,\mathcal{F})\) coarsely embed into a Hilbert space then every isomorphism between \(C^*_u(X)\) and \(C^*_u(Y)\) is rigidly implemented.

**Proof.** Since \((X,\mathcal{E})\) and \((Y,\mathcal{F})\) coarsely embed into a Hilbert space, both \((X,\mathcal{E})\) and \((Y,\mathcal{F})\) are metrizable. Let \(d\) and \(\partial\) be metrics on \(X\) and \(Y\), respectively, such that \(\mathcal{E}_d = \mathcal{E}\) and \(\mathcal{E}_\partial = \mathcal{F}\).
Let $U : \ell_2(X) \to \ell_2(Y)$ be a unitary spatially implementing a isomorphism between $C_\ell^*(X)$ and $C_\ell^*(Y)$. By symmetry, we only need to show that there exist $\delta > 0$ and $f : X \to Y$ such that $|\langle U\delta_x, f(x) \rangle| \geq \delta$, for all $x \in X$. Assuming this is not the case, we obtain a sequence $(x_n)_{n \in \mathbb{N}}$ of distinct points in $X$ such that $\|Ue_{x_n,x_n}U^*\delta_y\| < 2^{-n}$, for all $n \in \mathbb{N}$ and all $y \in Y$. For each $n \in \mathbb{N}$, set

$$S_n = Ue_{x_n,x_n}U^*.$$ 

So, $S_n$ is a projection on the 1-dimensional subspace generated by $\zeta_n := U\delta_{x_n}$, i.e., $S_n = \langle \cdot, \zeta_n \rangle \zeta_n$, for all $n \in \mathbb{N}$. Since $(x_n)_{n \in \mathbb{N}}$ is a sequence of distinct points, $\sum_{n \in \mathbb{N}} S_n$ converges in the strong operator topology to an operator in $C_\ell^*(Y)$, for all $I \subseteq \mathbb{N}$.

Let us pick a sequence of finite subsets $(Y_k)_{k \in \mathbb{N}}$ of $Y$ and a subsequence $(\zeta_{n_k})_{k \in \mathbb{N}}$ such that

1. $Y_k \cap Y_\ell = \emptyset$, for all $k \neq \ell$ in $\mathbb{N}$,
2. $\partial(Y_k, Y_\ell) \to \infty$, as $k + \ell \to \infty$, and
3. $\|\zeta_{n_k} - \chi_{Y_k} \zeta_{n_k}\| < 2^{-k}$, for all $k \in \mathbb{N}$.

We proceed by induction on $k \in \mathbb{N}$. Let $Y_1 \subset Y$ be a finite subset such that $\|\zeta_1 - \chi_{Y_1} \zeta_1\| < 2^{-1}$ and set $n_1 = 1$. Let $k \geq 2$ and assume that $Y_j$ and $n_j$ have been defined, for all $j \leq k - 1$. Fix $y_0 \in Y$ and pick $r > 0$ such that

$$\partial\left(\bigcup_{j=1}^{k-1} Y_j, B_r(y_0)\right) > k.$$ 

Let $Y' = B_r(y_0)$ (the complement of the $r$-ball centered at $y_0$). Since $(\zeta_n)_{n \in \mathbb{N}}$ is an orthonormal sequence, $(\zeta_n)_{n \in \mathbb{N}}$ is weak null. Therefore, since $B_r(y_0)$ is finite, we can pick $n_k \in \mathbb{N}$ such that $\|\zeta_{n_k} - \chi_{Y'} \zeta_{n_k}\| < 2^{-k-1}$. Pick a finite $Y_k \subset Y'$ such that $\|\chi_{Y'} \zeta_{n_k} - \chi_{Y_k} \zeta_{n_k}\| < 2^{-k-1}$. The sequences $(Y_k)_{k \in \mathbb{N}}$ and $(\zeta_{n_k})_{k \in \mathbb{N}}$ have the desired properties.

For each $k \in \mathbb{N}$, let $\xi_k = \chi_{Y_k} \zeta_{n_k}/\|\chi_{Y_k} \zeta_{n_k}\|$ and let $P_k \in B(\ell_2(Y_k))$ be the 1-dimensional projection on the subspace generated by $\xi_k$. So, $P_k = \langle \cdot, \xi_k \rangle \xi_k$. We clearly have that

$$\|S_{n_k} - P_k\| < 2^{-k+2} \quad \text{and} \quad P_k = \chi_{Y_k} P_k \chi_{Y_k},$$ 

for all $k \in \mathbb{N}$. In particular, $P_k \in B(\ell_2(Y_k))$ and $\|P_k \delta_y\| < 2^{-k+3}$, for all $k \in \mathbb{N}$ and all $y \in Y$.

Since $Y_k$ is finite, $\text{supp}(P_k)$ is finite and $P_k \in C_\ell^*[Y_k]$. Therefore, $S_{n_k} - P_k \in C_\ell^*[Y_k]$, for all $k \in \mathbb{N}$. Since

$$\sum_{k \in \mathbb{N}} \|S_{n_k} - P_k\| < \infty,$$

it follows that $\sum_{k \in \mathbb{N}} (S_{n_k} - P_k)$ converges in norm to an operator in $C_\ell^*[Y]$. Hence, as $\sum_{k \in \mathbb{N}} S_{n_k} \in C_\ell^*[Y]$, we have that $\sum_{k \in \mathbb{N}} P_k \in C_\ell^*[Y]$.

Let $P = \sum_{k \in \mathbb{N}} P_k$. By (i) above, and since $P_k \in B(\ell_2(Y_k))$ and $\xi_k \in \ell_2(Y_k)$, we have that $P \xi_k = \xi_k$, for all $k \in \mathbb{N}$. Let $\hat{Y} = \bigcup_{k \in \mathbb{N}} Y_k$. Therefore, since

$$\hat{P} \in \prod_{k \in \mathbb{N}} B(\ell_2(Y_k)),$$

Lemma 5.3 implies that $\hat{P} \in C_\ell^*[\hat{Y}]$.

**Claim 1.** The projection $\hat{P}$ is a ghost in $C_\ell^*[\hat{Y}]$. 
Proof. Since \( \|P_k\delta_x\| \leq 2^{-k+3} \), for all \( k \in \mathbb{N} \) and all \( y \in Y \), proceeding as in the proof of Theorem 5.1, we obtain that \( \tilde{P} \) is a ghost. Since \( P_k \) is a projection, for all \( k \in \mathbb{N} \), and since \( P_kP_\ell = 0 \), for all \( k \neq \ell \), we have that \( \tilde{P} \) is a projection.

Claim 2. All ghost projections in \( C^*_u(\tilde{Y}) \) are compact.

Proof. Since \( Y \) coarsely embeds into a Hilbert space, so does \( \tilde{Y} \). By our discussion on finite coarse components preceding this lemma, it is clear that \( \tilde{Y}, d \) has only finite coarse components. Therefore, Lemma 7.3 implies that all ghost projections in \( C^*_u(\tilde{Y}) \) must be compact.

Claim 1 and Claim 2 together imply that \( \tilde{P} \) is compact, contradicting the facts that \( (\xi_k)_{k \in \mathbb{N}} \) is orthonormal and that \( \tilde{P} \xi_k = \xi_k \), for all \( k \in \mathbb{N} \).

8. Isomorphism between algebraic uniform Roe algebras

In this section, we prove rigidity of algebraic uniform Roe algebras for general uniformly locally finite coarse spaces (\( \mathcal{X} \) from the introduction).

Theorem 8.1. Let \((X, E)\) and \((Y, F)\) be uniformly locally finite coarse spaces. The following are equivalent.

(i) \((X, E)\) and \((Y, F)\) are bijectively coarsely equivalent.

(ii) \( C^*_u[X] \) and \( C^*_u[Y] \) are isomorphic.

(iii) There is an isomorphism \( \Phi : C^*_u(X) \to C^*_u(Y) \) so that \( \Phi(\ell_\infty(X)) \subset \ell_\infty(Y) \).

Although the implication \((i) \Rightarrow (ii)\) in Theorem 8.1 is quite straightforward and it has already been proved for metric spaces with bounded geometry ([WW16, Corollary 1.16]), for the convenience of the reader and for completeness, we present a proof of this result here.

Proof of \((i) \Rightarrow (ii)\) of Theorem 8.1. Let \( f : (X, E) \to (Y, F) \) be a bijective coarse equivalence. Define an operator \( U : \ell_2(X) \to \ell_2(Y) \) by letting \( U\delta_x = \delta_{f(x)} \), for each \( x \in X \). Since \( f \) is a bijection, it follows that \( U \) is a unitary isomorphism. Let \( E \in \mathcal{E} \). So there exists \( F \in \mathcal{F} \) such that \((x, x') \in E \) implies \((f(x), f(x')) \in F \). Let us show that \( \text{supp}(UTU^*) \subset F \), for all \( T \in C^*_u(X) \) with \( \text{supp}(T) \subset E \).

Notice that

\[
\langle UTU^*\delta_y, \delta_{y'} \rangle = \langle T\delta_{f^{-1}(y)}, \delta_{f^{-1}(y')} \rangle \neq 0
\]

if and only if \( (f^{-1}(y), f^{-1}(y')) \in \text{supp}(T) \). Say \((y, y') \notin F \). Then, by our choice of \( F \), \((f^{-1}(y), f^{-1}(y')) \notin E \). Therefore, as \( \text{supp}(T) \subset E \), this implies that \( \langle UTU^*\delta_y, \delta_{y'} \rangle = 0 \). This shows that \( \text{supp}(UTU^*) \subset F \).

Define an isomorphism \( \Phi : B(\ell_2(X)) \to B(\ell_2(Y)) \) by letting \( \Phi(T) = UTU^* \), for all \( T \in B(\ell_2(X)) \). The discussion above shows that \( \Phi(C^*_u[X]) \subset C^*_u[Y] \) and, by symmetry, it follows that \( \Phi^{-1}(C^*_u[Y]) \subset C^*_u[X] \). Therefore, \( \Phi \mid C^*_u[X] \) is an isomorphism between \( C^*_u[X] \) and \( C^*_u[Y] \).

We now turn to the proof of \((ii) \Rightarrow (iii)\) in Theorem 8.1. We show that if \( \Phi : C^*_u[X] \to C^*_u[Y] \) is an isomorphism, then \( \Phi \) satisfies the following ‘coarse-like’ property: there exists an assignment \( E \in \mathcal{E} \to F_E \in \mathcal{F} \) such that \( \text{supp}(T) \subset E \) implies \( \text{supp}(\Phi(T)) \subset F_E \), for all \( T \in C^*_u[X] \) (see Theorem 8.3 below).

Lemma 8.2. Let \((X, E)\) be a uniformly locally finite coarse space. Then \( \text{supp}(T) \) is finite, for every rank 1 operator \( T \in C^*_u[X] \).
Lemma 8.3. Let \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) be uniformly locally finite coarse spaces and assume that \(\Phi : C^*_u[X] \to C^*_u[Y]\) is an isomorphism. Let \(E \in \mathcal{E}\) and let \((T_j)_{j \in J}\) be a bounded family of rank 1 operators in \(C^*_u[X]\) with mutually orthogonal images and such that \(\text{supp}(T_j) \subset E\), for all \(j \in J\). Then,

\[
\bigcup_{j \in J} \text{supp}(\Phi(T_j)) \in \mathcal{F}.
\]

Proof. The proof consists of a series of claims. Throughout this proof, fix \(E \in \mathcal{E}\).

Claim 1. Suppose \((T_j)_{j \in \mathbb{N}}\) is a bounded family of rank 1 operators in \(C^*_u[X]\) such that \(\text{supp}(T_j) \subset E\), for all \(j \in \mathbb{N}\). Then, there exists an operator \(T \in C^*_u[X]\) such that \(\text{supp}(T) \subset E\) and

\[
\text{supp}(\Phi(T)) = \bigcup_{j \in \mathbb{N}} \text{supp}(\Phi(T_j)).
\]

In particular,

\[
\bigcup_{j \in \mathbb{N}} \text{supp}(\Phi(T_j)) \in \mathcal{F}.
\]

Proof. For each \(j \in \mathbb{N}\), let

\[
m_j := \inf \left\{ |\langle \Phi(T_j)\delta_y, \delta_{y'} \rangle| \mid (y, y') \in \text{supp}(\Phi(T_j)) \right\}.
\]

By Corollary 3.3 and Lemma 8.2, \(\text{supp}(\Phi(T_j))\) is finite, so \(m_j > 0\), for all \(j \in \mathbb{N}\). Let \(M = \sup_{j \in \mathbb{N}} \|T_j\|\). Define a sequence \((\lambda_j)_{j \in \mathbb{N}}\) of positive reals as follows. Let \(\lambda_1 = 1\) and assume that \(\lambda_{j-1}\) had been defined for \(j \geq 2\). Let \(\lambda_j\) be a positive real smaller than

\[
\min\{m_1, \ldots, m_{j-1}\} / 2^j M.
\]

Clearly, \(\sum_j |\langle \lambda_j T_j \delta_x, \delta_{x'} \rangle| \leq M\), for all \(x, x' \in X\). Also, \(\langle \lambda_j T_j \delta_x, \delta_{x'} \rangle = 0\), for all \((x, x') \notin E\). Hence, by the definition of \(C^*_u[X]\), it follows that \(T := \sum_{j \in \mathbb{N}} \lambda_j T_j \in C^*_u[X]\) is well-defined. So, \(\Phi(T) \in C^*_u[Y]\) is defined and \(\text{supp}(\Phi(T)) \in \mathcal{F}\). At last, notice that the sequence \((\lambda_j)_{j}\) was chosen such that

\[
\text{supp}(\Phi(T)) = \bigcup_{j \in \mathbb{N}} \text{supp}(\Phi(T_j)).
\]

This completes the proof.

Claim 2. Let \((T_j)_{j \in J}\) be an infinite family of nonzero operators in \(B(\ell_2(X))\) with mutually orthogonal images. There exists an infinite \(J' \subset J\) and a family \((x_j, x'_j)_{j \in J'}\) of pairwise distinct elements in \(X \times X\) such that \((x_j, x'_j) \in \text{supp}(T_j)\), for all \(j \in J'\).

Proof. Since the operators \((T_j)_{j \in J}\) have mutually orthogonal images, it is clear that, for all infinite \(J_0 \subset J\), there is no finite \(A \subset X \times X\) such that \(\text{supp}(T_j) \subset A\), for all \(j \in J_0\). We now construct \(J' \subset J\) and the required family \((x_j, x'_j)_{j \in J'}\) by induction. Pick any \(j_1 \in J\) and any \((x_{j_1}, x'_{j_1}) \in \text{supp}(T_{j_1})\). Let \(n \geq 1\) and assume
that \( j_k \) and \((x_{j_k}, x'_{j_k})\) have already been picked, for all \( k \in \{1, \ldots, n\} \). Let
\[
A = \bigcup_{k=1}^{n} \{(x_{j_k}, x'_{j_k})\} \quad \text{and} \quad J_0 = J \setminus \{j_1, \ldots, j_n\}.
\]

Since \( A \) is finite and \( J_0 \) is infinite, there exists \( j_{k+1} \in J_0 \) such that \( \text{supp}(T_j) \not\subset A \). Now pick \((x_{j_{k+1}}, x'_{j_{k+1}})\) ∈ \( \text{supp}(T_j) \setminus A \). This proves the claim. □

**Claim 3.** Let \( A \subset X \times X \) be finite and let \((T_j)_{j \in J}\) be an infinite family of rank 1 operators in \( B(\ell_2(X)) \) with mutually orthogonal images such that \( A \cap \text{supp}(T_j) \neq \emptyset \), for all \( j \in J \). Then, there exist \( x \in X \), an infinite \( J' \subset J \) and a family \((x_j)_{j \in J'}\) in \( X \) of pairwise distinct elements such that either
(i) \((x, x_j) \in \text{supp}(T_j)\), for all \( j \in J' \), or
(ii) \((x_j, x) \in \text{supp}(T_j)\), for all \( j \in J' \).

**Proof.** Let \( A \subset X \times X \) and \((T_j)_{j \in J}\) be as above. Since \( A \) is finite, a simple pigeonhole argument gives us \( x, x' \in X \) and an infinite \( J_0 \subset J \) such that \((x, x') \in \text{supp}(T_j)\), for all \( j \in J_0 \). By Claim 2, we can pick a countably infinite \( J_1 \subset J_0 \) and a family \((x_j, x'_{j})\) \( j \in J_1 \) of distinct elements such that \((x_j, x'_{j}) \in \text{supp}(T_j)\), for all \( j \in J_1 \). Since \( J_1 \) is infinite, there exists an infinite \( J' \subset J_1 \) such that either \((x_j)_{j \in J'}\) or \((x'_{j})_{j \in J'}\) is a sequence of pairwise distinct elements. Assume that \((x_j)_{j \in J'}\) is a sequence of pairwise distinct elements.

As the operators \((T_j)_{j \in J'}\) have rank 1, so do their adjoints \((T_j^*)_{j \in J'}\). Since \((x, x'), (x_j, x'_{j}) \in \text{supp}(T_j)\), for all \( j \in J' \), we have that \( T_j^*\delta_{x'} \neq 0 \) and \( T_j^*\delta_{x'_{j}} \neq 0 \), for all \( j \in J' \). Hence, for all \( j \in J' \), there exists \( \lambda_j \neq 0 \) such that \( T_j^*\delta_{x'} = \lambda_j T_j^*\delta_{x'_{j}} \).

It follows that
\[
\langle T_j\delta_{x_j}, \delta_{x'} \rangle = \langle \delta_{x_j}, T_j^*\delta_{x'} \rangle = \langle \delta_{x_j}, \lambda_j T_j^*\delta_{x'_{j}} \rangle = \lambda_j \langle T_j\delta_{x_j}, \delta_{x'_{j}} \rangle \neq 0,
\]
for all \( j \in J' \). So, \((x_j, x') \in \text{supp}(T_j)\), for all \( j \in J' \), and (ii) holds. If we had assumed that \((x'_{j})_{j \in J'}\) is a sequence of distinct elements, similar arguments would give us that (i) holds. □

**Claim 4.** Let \( A \subset X \times X \) be finite and let \((T_j)_{j \in J}\) be a family of rank 1 operators in \( C^*_u[X] \) with mutually orthogonal images and such that \( \text{supp}(T_j) \subset E \), for all \( j \in J \). Then,
\[
\left| \{j \in J \mid A \cap \text{supp}(T_j) \neq \emptyset \} \right| < \infty.
\]

**Proof.** Suppose the claim does not hold and let \( x \in X \), \( J' \subset J \) and \((x_j)_{j \in J'}\) be given by Claim 3. Without loss of generality, assume that \((x, x_j) \in \text{supp}(T_j)\), for all \( j \in J' \). Hence, as \( \text{supp}(T_j) \subset E \), for all \( j \in J' \), it follows that
\[
\{x_j \mid j \in J'\} \subset E_x.
\]

Since \((x_j)_{j \in J'}\) is an infinite sequence of distinct elements, this shows that \( |E_x| = \infty \), which contradicts the fact that \((X, E)\) is uniformly locally finite. □

**Claim 5.** Let \( B \subset Y \times Y \) be finite and let \((T_j)_{j \in J}\) be a bounded family of rank 1 operators in \( C^*_u[X] \) with mutually orthogonal images and such that \( \text{supp}(T_j) \subset E \), for all \( j \in J \). Then,
\[
\left| \{j \in J \mid B \cap \text{supp}(T_j) \neq \emptyset \} \right| < \infty.
\]
Proof. Assume the claim does not hold. By Corollary 3.3, \((\Phi(T_j))_{j \in J}\) is a bounded family of rank 1 operators in \(C_u[Y]\) with mutually orthogonal images. Let \(y \in Y\), \(J' \subset J\) and \((y_j)_{j \in J'}\) be given by Claim 3 applied to \((\Phi(T_j))_{j \in J}\). Without loss of generality, assume that \(J'\) is countable and that \((y, y_j) \in \text{supp}(\Phi(T_j))\), for all \(j \in J'\). Since \(J'\) is countable, Claim 1 gives us that
\[
\{(y, y_j) \mid j \in J'\} \subset \bigcup_{j \in J'} \text{supp}(\Phi(T_j)) \in \mathcal{F}.
\]
Since the elements \((y_j)_{j \in J'}\) are pairwise distinct and \((Y, \mathcal{F})\) is uniformly locally finite, this gives us a contradiction. \(\Box\)

We can finally finish the proof of the lemma. For now on, we fix a bounded family \((T_j)_{j \in J}\) of rank 1 operators in \(C_u[X]\) with mutually orthogonal images. Define an equivalence relation \(\sim\) on \(J\) as follows. First, define a relation (not necessarily an equivalence relation) \(\sim\) on \(J\) by saying that \(j \sim j'\) if either
\[
\text{supp}(T_j) \cap \text{supp}(T_{j'}) \neq \emptyset \text{ or } \text{supp}(\Phi(T_j)) \cap \text{supp}(\Phi(T_{j'})) \neq \emptyset.
\]
Then, say that \(j \sim j'\) if there exist \(n \in \mathbb{N}\) and \(j_1, \ldots, j_n \in J\), with \(j_1 = j\) and \(j_n = j'\), such that \(j_k \sim j_{k+1}\), for all \(k \in \{1, \ldots, n-1\}\). This defines a partition on \(J\), say \(J = \bigcup_{i \in I} J_i\), for some index set \(I\). By the definition of \(\sim\), we have that
\[
\text{supp}(T_j) \cap \text{supp}(T_{j'}) = \emptyset \text{ and } \text{supp}(\Phi(T_j)) \cap \text{supp}(\Phi(T_{j'})) = \emptyset,
\]
for all \(j, j' \in J\) such that \(j \not\sim j'\).

By Lemma 3.2, Claim 4 and Claim 5, \(J_i\) is countable, for all \(i \in I\). Therefore, by Claim 1, there exists an operator \(T^{(i)} \in C_u[X]\) such that \(\text{supp}(T^{(i)}) \subset \mathcal{E}\) and
\[
\text{supp}(\Phi(T^{(i)})) = \bigcup_{j \in J_i} \text{supp}(\Phi(T_j)).
\]
By multiplying \(T^{(i)}\) by an appropriate scalar if necessary, assume that \(|\langle T^{(i)} \delta_x, \delta_x' \rangle| \leq 1\), for all \(x, x' \in X\). Since \(\text{supp}(T_j) \cap \text{supp}(T_{j'}) = \emptyset\), for all \(j \not\sim j'\), it follows that \(\text{supp}(T^{(i)}) \cap \text{supp}(T^{(i')}) = \emptyset\), for all \(i \neq i'\). This shows that \(\sum_{i \in I} T^{(i)}\) is a well-defined element of \(C_u[X]\) with support contained in \(\mathcal{E}\).

By Lemma 3.1, \(\Phi : C_u[X] \to C_u[Y]\) is continuous in the strong operator topology. By Lemma 2.8, the series \(\sum_{i \in I} T^{(i)}\) is convergent in the strong operator topology, so
\[
\Phi\left(\sum_{i \in I} T^{(i)}\right) = \sum_{i \in I} \Phi(T^{(i)}).
\]
Since
\[
\text{supp}(\Phi(T_j)) \cap \text{supp}(\Phi(T_{j'})) = \emptyset,
\]
for all \(j \not\sim j'\), it follows that \(\text{supp}(\Phi(T^{(i)})) \cap \text{supp}(\Phi(T^{(i')}) = \emptyset\), for all \(i \neq i'\). Therefore,
\[
\text{supp}\left(\sum_{i \in I} \Phi(T^{(i)})\right) = \bigcup_{i \in I} \text{supp}(\Phi(T^{(i)})) = \bigcup_{j \in J} \text{supp}(\Phi(T_j)).
\]
As \(\text{supp}(\Phi(\sum_{i \in I} T^{(i)})) \in \mathcal{F}\), this completes the proof. \(\Box\)

We can now prove that a isomorphism \(\Phi : C_u[X] \to C_u[Y]\) between the algebraic uniform Roe algebras of uniformly locally finite coarse spaces must satisfy a ‘coarse-like’ property (cf. Theorem 6.3).
**Theorem 8.4.** Let \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) be uniformly locally finite coarse spaces and let \(\Phi : C^*_u[X] \to C^*_u[Y]\) be a isomorphism. For all \(E \in \mathcal{E}\), there exists \(F \in \mathcal{F}\) such that, for all \(T \in C^*_u[X]\),

\[
\text{supp}(T) \subset E \implies \text{supp}(\Phi(T)) \subset F.
\]

**Proof.** Suppose otherwise. Then, there exists \(E \in \mathcal{E}\) such that for all \(F \in \mathcal{F}\) there exists \(T^F \in C^*_u[X]\) with \(\text{supp}(T^F) \subset E\) and \(\text{supp}(\Phi(T^F)) \not\subset F\). Without loss of generality, assume that \(\Delta_X \subset E\). By multiplying \(T^F\) by an appropriate scalar if necessary, assume that \(\|T^F\| \leq 1\), for all \(F \in \mathcal{F}\). For each \(F \in \mathcal{F}\), pick \((y^F_1, y^F_2) \in \text{supp}(\Phi(T^F))\) such that \((y^F_1, y^F_2) \not\in F\). By Lemma 8.5, \(\Phi\) is continuous in the strong operator topology, therefore, using Lemma 2.8, it follows that

\[
\Phi(T^F) = \Phi \left( \sum_{(x_1, x_2) \in E} T^F_{x_1 x_2} \right) = \sum_{(x_1, x_2) \in E} \Phi(T^F_{x_1 x_2}).
\]

So, for each \(F \in \mathcal{F}\), there exists \((x^F_1, x^F_2) \in E\) such that \((y^F_1, y^F_2) \in \text{supp}(\Phi(T^F_{x^F_1 x^F_2}))\).

We make \(\mathcal{F}\) into a directed set by setting \(F_1 \preceq F_2\) if \(F_1 \subset F_2\). By Lemma 4.10 there exists a cofinal subset \(I\) of \(\mathcal{F}\), a subset \(J\) of \(I\), and a map \(\varphi : I \to J\) such that

(i) \(x^F_2 \neq x^F_2\), for all \(F \neq F'\) in \(J\), and

(ii) \(x^F_1 = x^F_1(\varphi(F))\) and \(x^F_2 = x^F_2(\varphi(F))\), for all \(F \in I\).

Notice that \(\|T^F_{x^F_1 x^F_2}\| \leq \|T^F\| \leq 1\). Hence, by Item (i), \((T^F_{x^F_1 x^F_2})_{F \in I}\) is a bounded family of rank 1 operators in \(C^*_u[X]\) with mutually orthogonal images. Therefore, by Lemma 8.3 it follows that

\[
F' := \bigcup_{F \in J} \text{supp} \left( \Phi(T^F_{x^F_1 x^F_2}) \right) \in \mathcal{F}.
\]

As \(I\) is cofinal in \(\mathcal{F}\), we can pick \(F \in I\) such that \(F' \subset F\). Fix such \(F \in I\). By our choice of \((y^F_1, y^F_2)\), we have that \((y^F_1, y^F_2) \not\in F\). On the other hand,

\[
(y^F_1, y^F_1) \in \text{supp} \left( \Phi(T^F_{x^F_1 x^F_2}) \right) = \text{supp} \left( \Phi(T^{\varphi(F)}_{x^F_1(\varphi(F)) x^F_2(\varphi(F))}) \right) \subset F',
\]

contradiction. \(\square\)

We need to introduce some notation which will be used in the following lemmas. Let \((X, \mathcal{E})\) and \((X, \mathcal{F})\) be uniformly locally finite coarse spaces and let \(U : \ell_2(X) \to \ell_2(Y)\) be a unitary isomorphism which spatially implements a isomorphism between \(C^*_u[X]\) and \(C^*_u[Y]\). For each \(x \in X\) and \(y \in Y\), define

\[
X_y = \{x \in X \mid e_{xx} U^* e_{yy} U \neq 0\}
\]

and

\[
Y_x = \{y \in Y \mid U e_{xx} U^* e_{yy} \neq 0\}.
\]

**Lemma 8.5.** Let \((X, \mathcal{E})\) and \((X, \mathcal{F})\) be uniformly locally finite coarse spaces and let \(U : \ell_2(X) \to \ell_2(Y)\) be a unitary isomorphism which spatially implements a isomorphism between \(C^*_u[X]\) and \(C^*_u[Y]\). The following holds.

(i) There exists \(E \in \mathcal{E}\) such that \((x, x') \in E\), for all \(y \in Y\) and all \(x, x' \in X_y\). In particular, \(\sup_{y \in Y} |X_y| \leq \sup_{x \in X} E_x\).

(ii) There exists \(F \in \mathcal{F}\) such that \((y, y') \in F\), for all \(x \in X\) and all \(y, y' \in Y_x\). In particular, \(\sup_{x \in X} |Y_x| \leq \sup_{y \in Y} F_y\).
Proof. By symmetry, we only need to show (i). Let $E$ be given by Theorem 8.4 applied to the isomorphism $T \in C^*_u[Y] \mapsto U^*TU \in C^*_u[X]$ and $\Delta_Y \in \mathcal{F}$. So, $\text{supp}(U^*e_{yy}U) \subset E$.

for all $y \in Y$. Without loss of generality, assume that $E$ is symmetric. Let $y \in Y$ and $x, x' \in X_y$. Since $e_{xx}U^*e_{yy}U \neq 0$, there exists $x'' \in X$ such that $e_{xx}U^*e_{yy}U \delta_{x''} \neq 0$. As $U^*e_{yy}U$ is a rank 1 operator and $e_{xx'}U^*e_{yy}U \neq 0$, we have that $e_{xx'}U^*e_{yy}U \delta_{x''} \neq 0$. In other words, this gives us that $\langle U^*e_{yy}U \delta_x, \delta_{x''} \rangle \neq 0$ and $\langle U^*e_{yy}U \delta_{x'}, \delta_{x''} \rangle \neq 0$.

Therefore, for each $(x, x'')$, $(x', x'') \in \text{supp}(U^*e_{yy}U) \subset E$. As $E$ is symmetric, this implies that $(x, x') \in E \circ E$ has the desired property. 

Lemma 8.6. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniformly locally finite coarse spaces and let $U : \ell_2(X) \to \ell_2(Y)$ be a unitary isomorphism which spatially implements a isomorphism between $C^*_u[X]$ and $C^*_u[Y]$. The following holds.

(i) $|\bigcup_{y \in B} X_y| \geq |B|$, for all $B \subset Y$.
(ii) $|\bigcup_{y \in B} Y_x| \geq |B|$, for all $B \subset X$.

Proof. By symmetry, we only prove (i). Let $B \subset X$ and let $X_B = \bigcup_{y \in B} X_y$. Since $\text{supp}(U^*e_{yy}U) \subset X_y \times X_y$, for all $y \in Y$, it follows that

$$\bigcup_{y \in B} \text{supp}(U^*e_{yy}U) \subset X_B \times X_B.$$ 

Therefore, for each $y \in B$, we can naturally identify $U^*e_{yy}U$ with an operator in $\ell_2(X_B)$. For each $y \in B$, let $x_y \in X_B$ be such that $U^*e_{yy}U \delta_{x_y} \neq 0$. Since the images of $U^*e_{yy}U$ and $U^*e_{y'y'}U$ are orthogonal for all $y \neq y'$ in $B$, we have that $(U^*e_{yy}U \delta_{x_y})_{y \in B}$ is an orthogonal family of nonzero vectors in $\ell_2(X_B)$. So $|B| \leq |X_B|$.

Lemma 8.7. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniformly locally finite coarse spaces and let $U : \ell_2(X) \to \ell_2(Y)$ be a unitary isomorphism which spatially implements a isomorphism between $C^*_u[X]$ and $C^*_u[Y]$. There exist injections $f : X \to Y$ and $g : Y \to X$ such that

(i) $f(x) \in Y_x$, for all $x \in X$, and
(ii) $g(y) \in X_y$, for all $y \in Y$.

Proof. By symmetry, we only prove (i). Define $\varphi : X \to \mathcal{P}(Y)$ by letting $\varphi(x) = Y_x$, for all $x \in X$. By Lemma 8.4, we know that $|B| \leq \left| \bigcup_{x \in B} Y_x \right| = \left| \bigcup_{x \in B} \varphi(x) \right|$, for all $B \subset X$. By Lemma 8.3(ii), we also have that $Y_x$ is finite, for all $x \in X$.

Therefore, by Hall’s marriage theorem (see [Hal67, Theorem 5.1.2]), there exists a map $\psi : \{Y_x | x \in X\} \to Y$ such that $\psi(Y_x) \in Y_x$, for all $x \in X$, and $\psi(Y_x) \neq \psi(Y_{x'})$, for all $x \neq x'$. Therefore, the map $f = \psi \circ \varphi$ is an injection such that $f(x) \in Y_x$, for all $x \in X$. 

Theorem 8.8. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniformly locally finite coarse spaces and suppose that $\Phi : C^*_u[X] \to C^*_u[Y]$ is a isomorphism. Then, there exists a unitary operator $V \in C^*_u[Y]$ such that $V\Phi(\ell_\infty(X))V^* = \ell_\infty$. 

□
Proof. Let \( f : X \to Y \) and \( g : Y \to X \) be the injections given by Lemma \([8.7]\). Using König’s proof of the Cantor-Schröder-Bernstein theorem to the injections \( f \) and \( g \), we obtain a bijection \( h : Y \to X \) such that, for all \( y \in Y \), we have that either \( h(y) = g(y) \) or \( y \in \text{Im}(f) \) and \( h(y) = f^{-1}(y) \).

Let \( F \in \mathcal{F} \) be given by Lemma \([8.5]\)ii). Without loss of generality, assume that \( F \) is symmetric. So, \((y, y') \in F\), for all \( x \in X \) and all \( y, y' \in Y_x \).

**Claim 1.** \( Y_h(y) \subset F_y \), for all \( y \in Y \).

**Proof.** Fix \( y \in Y \). Suppose \( y \in \text{Im}(f) \) and \( h(y) = f^{-1}(y) \). By the definition of \( f \), we have that \( y = f(h(y)) \in Y_h(y) \). Therefore, \((y, y') \in F\), for all \( y' \in Y_h(y) \). So, \( Y_h(y) \subset F_y \). Suppose now that \( h(y) = g(y) \). By the definition of \( g \), \( g(y) \in X_y \). Hence, by the definition of \( X_{g(y)} \), \( e_{g(y)g(y)}^e U^* e_y U \neq 0 \) and we must have that \( U e_{g(y)g(y)}^e U^* e_y \neq 0 \). Therefore, by the definition of \( Y_h(y) \), it follows that \( y \in Y_h(y) \) and we conclude that \( Y_h(y) \subset F_y \). This finishes the proof of the claim. \( \square \)

For each \( x \in X \), let \( \xi_x = U \delta_x \). So \((\xi_x)_{x \in X}\) is an orthonormal basis of \( \ell_2(Y) \). We define a unitary operator \( V \in B(\ell_2(Y)) \) by letting

\[ V\xi = \sum_{x \in X} (\xi, \xi_x) \delta_{h^{-1}(x)} \]

for all \( \xi \in \ell_2(Y) \).

**Claim 2.** \( \text{supp}(V) \subset F \). In particular, \( V \in C_u^*[Y] \).

**Proof.** Let \( y, y' \in Y \). Then, by the definition for \( V \), we have that

\[
|\langle V \delta_y, \delta_{y'} \rangle| = \left| \sum_{x \in X} \langle \delta_y, \xi_x \rangle \delta_{h^{-1}(x)} \delta_{y'} \right|
\]

\[
= |\langle \delta_y, \xi_{h(y')} \rangle|
\]

\[
= |\langle \delta_y, U \delta_{h(y')} \rangle|
\]

\[
= |\langle U^* \delta_{y'}, \delta_{h(y')} \rangle|
\]

\[
= \|e_{h(y')}^e h(y')^e U^* e_y\|
\]

\[
= \|U e_{h(y')}^e h(y')^e U^* e_y\|
\]

By Claim[1] \( Y_h(y') \subset F_y \). Hence, \((y', y) \not\in F\) implies \( y \not\in Y_h(y') \). By the definition of \( Y_h(y') \), \( y \not\in Y_h(y') \) implies \( U e_{h(y')}^e h(y')^e U^* e_y = 0 \). Since \( F \) is symmetric, we conclude that \((y, y') \not\in F\) implies \( \langle V \delta_y, \delta_{y'} \rangle = 0 \). Therefore, \( \text{supp}(V) \subset F \) and the claim is proven. \( \square \)

In order to finish the proof, notice that \( VU\ell_\infty(X)U^* V^* = \ell_\infty(X) \). Indeed, a simple computation gives us that

\[ V^* \xi = \sum_{x \in X} \langle \xi, \delta_{h^{-1}(x)} \rangle \xi_x \]

for all \( \xi \in \ell_2(Y) \). Hence, for any \( x \in X \), we have that

\[ VU e_x U^* V^* \xi = \langle \xi, \delta_{h^{-1}(x)} \rangle \delta_{h^{-1}(x)} e_{h^{-1}(x)} x \xi, \]

for all \( \xi \in \ell_2(Y) \). So, \( VU e_x U^* V^* = e_{h^{-1}(x) h^{-1}(x)} \). Since \( h \) is a bijection, we conclude that \( VU\ell_\infty(X)U^* V^* = \ell_\infty(X) \). This finishes the proof. \( \square \)

**Proof of (ii)⇒(iii) of Theorem \([8.7]\)** Let \( \Phi : C_u^*[X] \to C_u^*[Y] \) be a isomorphism and let \( V \) be given by Theorem \([8.8]\). Define a isomorphism \( \Psi : C_u^*[X] \to C_u^*[Y] \) by
Let $\Phi : C^*_u(X) \to C^*_u(Y)$ extend to a bijection $\bar{\Phi} : C^*_u(X) \to C^*_u(Y)$, so the proof is complete.

We now turn to the proof of (iii) $\Rightarrow$ (i) of Theorem 8.11. We show that a isomorphism $\Phi : C^*_u(X) \to C^*_u(Y)$ sending $\ell_\infty(X)$ to $\ell_\infty(Y)$ must satisfy the same `coarse-like' property of Theorem 8.3 above (see Theorem 8.11 below). In particular, we obtain that $\Phi(C^*_u[X]) \subset C^*_u[Y]$.

**Lemma 8.9.** Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniformly locally finite coarse spaces and consider a isomorphism $\Phi : C^*_u(X) \to C^*_u(Y)$ such that $\Phi(\ell_\infty(X)) \subset \ell_\infty(Y)$. There exists a bijection $f : X \to Y$ such that

(i) $\Phi(e_{xx}) = e_{f(x)f_z(z)}$, for all $x \in X$, and

(ii) for all $x, x' \in X$ there exists $\lambda_{x,x'} \in \mathbb{C}$, with $|\lambda_{x,x'}| = 1$, such that $\Phi(e_{x,x'}) = \lambda_{x,x'} e_{f(x)f'(x')}$. In particular $\Phi^{-1}(\ell_\infty(Y)) \subset \ell_\infty(X)$.

**Proof.** (i) Fix $x \in X$. As $e_{xx} \in \ell_\infty(X)$, we have that $\Phi(e_{xx}) \in \ell_\infty(Y)$. Since $e_{xx}$ has rank 1, so does $\Phi(e_{xx})$. Therefore, there exists $y_x \in Y$ and $\lambda_x \in \mathbb{C} \setminus \{0\}$ such that $\Phi(e_{xx}) = \lambda_x e_{y_x}$. Since $\sigma(e_{xx}) = (0, 1)$, it follows that $\sigma(\Phi(e_{xx})) = (0, 1)$ and we must have $\lambda_x = 1$.

Define $f(x) = y_x$, for all $x \in X$, and let us show that $f$ is a bijection. Say $x \neq x'$. Then $e_{xx} e_{x,x'} = 0$, so

$$e_{f(x)f_z(f_z)} e_{f(x)f'(x')} = \Phi(e_{xx} e_{x,x'}).$$

Hence, $f(x) \neq f(x')$ and $f$ is injective. Say $f$ is not surjective. Then, there exists $y \in Y$ such that $\Phi(e_{xx}) \neq e_{yy}$, for all $x \in X$. Then, $e_{yy} e_{f(x)f_z(x)} = 0$, for all $x \in X$. This implies that $\Phi^{-1}(e_{yy}) e_{xx} = 0$, for all $x \in X$, so $\Phi^{-1}(e_{yy}) = 0$. Since $\Phi^{-1}$ is an isomorphism, this gives us a contradiction. This shows that $f$ is a bijection.

(ii) Let $f : X \to Y$ be the bijection in Item (i). Fix $x, x' \in X$. Using Item (i) and that $e_{x,x'} = e_{x,x'}, e_{x,x} = e_{x,x} e_{x,x}$ and $e_{x,x'} = e_{x,x'} e_{x,x'}$, we have that

$$\langle \Phi(e_{x,x'} \delta_y \delta_{y'} \rangle = \langle \Phi(e_{x,x'}) e_{f(x)f'(x')} \delta_y \delta_{y'} \rangle$$

$$= (e_{f(x)f'(x')} \delta_y \Phi(e_{xx}) \delta_{y'})$$

$$= \langle e_{f(x)f'(x')} \delta_y \Phi(e_{xx}) \delta_{y'} \rangle.$$

Since $\Phi(e_{x,x'}) \neq 0$, it follows that $\langle \Phi(e_{x,x'}) \delta_y \delta_{y'} \rangle \neq 0$ if and only if $y = f(x)$ and $y' = f(x')$. So, $\Phi(e_{x,x'}) = \lambda_{x,x'} e_{f(x)f'(x)}$, for some $\lambda_{x,x'} \in \mathbb{C}$. At last, notice that $|\lambda_{x,x'}| = \| \Phi(e_{x,x'}) \|$.

**Lemma 8.10.** Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniformly locally finite coarse spaces and consider a isomorphism $\Phi : C^*_u(X) \to C^*_u(Y)$ such that $\Phi(\ell_\infty(X)) \subset \ell_\infty(Y)$. Let $U : \ell_2(X) \to \ell_2(Y)$ be a unitary operator such that $\Phi(T) = UTU^*$, for all $T \in C^*_u(X)$. Let $f : X \to Y$ be the bijection given by Lemma 8.9. Then, for all $x \in X$, there exists $\lambda \in \mathbb{C}$, with $|\lambda| = 1$, such that $U(\delta_x) = \lambda \delta_f(x)$.

**Proof.** Fix $x \in X$. By Lemma 8.8, $U e_{xx} U^* = e_{f(x)f_z(x)}$, so $U \delta_x = e_{f(x)f_z(x)} U \delta_x$, for all $x \in X$. This gives us that $U \delta_x$ is a multiple of $\delta_f(x)$. Since $U$ is an isometry, it follows that $U(\delta_x) = \lambda \delta_f(x)$, for some $\lambda \in \mathbb{C}$, with $|\lambda| = 1$.

**Theorem 8.11.** Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniformly locally finite coarse spaces and let $\Phi : C^*_u(X) \to C^*_u(Y)$ be a isomorphism such that $\Phi(\ell_\infty(X)) \subset \ell_\infty(Y)$. For all
For each $F \in \mathcal{F}$, there exists $F' \in \mathcal{F}$ such that for all $T \in C_u(X)$, $\text{supp}(T) \subset E$ implies $\text{supp}(\Phi(T)) \subset F$. In particular $\Phi(C_u[X]) \subset C_u[Y]$.

Proof. Assume the conclusion of the theorem does not hold. Then, there exists $E \in \mathcal{E}$ such that for all symmetric entourage $F \in \mathcal{F}$ there exists $T^F \in C_u(X)$ with $\text{supp}(T^F) \subset E$ and $\text{supp}(\Phi(T^F)) \not\subset F$. We can assume that $\|T^F\| \leq 1$, for all $F \in \mathcal{F}$. For $F \in \mathcal{F}$, pick $(y^F_2, y^F_1) \in \text{supp}(\Phi(T^F))$ such that $(y^F_2, y^F_1) \not\in F$. By continuity of $\Phi$ in the strong operator topology, Lemma 2.8 and Lemma 3.1 imply that

$$\Phi(T^F) = \sum_{(x_1, x_2) \in E} T^F_{x_1 x_2} \sum_{(x_1, x_2) \in E} \Phi(T^F_{x_1 x_2}).$$

For each $F \in \mathcal{F}$, pick $(x^F_2, x^F_1) \in E$ such that $(y^F_2, y^F_1) \in \text{supp}(\Phi(T^F_{x^F_2, x^F_1}))$. Let $f : X \to Y$ be the bijection given in Lemma 8.9. By Lemma 8.9 $\text{supp}(\Phi(T^F_{x^F_2, x^F_1})) = \{(f(x^F_2), f(x^F_1)), y^F_1 = f(x^F_1)$ and $y^F_2 = f(x^F_2)$.

We make $\mathcal{F}$ into a directed set by setting $F_1 \leq F_2$ if $F_2 \subset F_1$. By Lemma 4.10, we can pick a cofinal subset $I$ of $\mathcal{F}$, a subset $J$ of $\mathcal{F}$ and a map $\varphi : I \to J$ such that

(i) $x^F_1 \neq x^{F'}_1$ and $x^F_2 \neq x^{F'}_2$, for all distinct $F, F' \in J$,
(ii) $x^F_1 = x^{\varphi(F)}_1$ and $x^F_2 = x^{\varphi(F)}_2$, for all $F \in I$.

By Item (i) and Lemma 2.8, the sum

$$\sum_{F \in J} e_{x^F_1 x^F_2}$$

converges in the strong operator topology to an operator in $C_u[Y]$. Let $(\lambda^F_{x^F_1 x^F_2})_{F \in J}$ be given by Lemma 8.9 (ii). Then, as $\Phi$ is continuous in the strong operator topology, the sum

$$\sum_{F \in J} \Phi(e_{x^F_1 x^F_2}) = \sum_{F \in J} \lambda^F_{x^F_1 x^F_2} e_{f(x^F_1), f(x^F_2)}$$

converges strongly to an operator $S$ in $C_u[Y]$. Pick $S' \in C_u[Y]$ such that $\|S - S'\| < 1$. In particular, $\text{supp}(S') \in \mathcal{F}$.

Claim 1. $(f(x^F_2), f(x^F_1)) \in \text{supp}(S')$, for all $F \in J$.

Proof. Notice that

$$|\langle S \delta_y, \delta_{y'} \rangle| = \begin{cases} 1, & \text{if } (y, y') = (f(x^F_2), f(x^F_1)) \text{ for some } F \in J, \\ 0, & \text{otherwise}. \end{cases}$$

Let $F \in \mathcal{F}$. Since $\|S(\delta_{f(x^F_2)}) - S'((\delta_{f(x^F_1)}))\| < 1$, we have that

$$|\langle S(\delta_{f(x^F_2)}) - S'(\delta_{f(x^F_1)}), \delta_{f(x^F_1)} \rangle| < 1.$$

This gives us that $(f(x^F_2), f(x^F_1)) \in \text{supp}(S')$, and the claim is proved.

Since $I$ is cofinal in $\mathcal{F}$, we can pick $F \in I$ such that $\text{supp}(S') \subset F$. Fix such $F \in I$. By hypothesis, $(f(x^F_2), f(x^F_1)) = (y^F_2, y^F_1) \notin F$.

Therefore, by Item (ii), we must have $(f(x^{\varphi(F)}_2), f(x^{\varphi(F)}_1)) \notin F$.

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Since $\varphi(F) \in J$ and $\text{supp}(S') \subseteq F$, the claim above gives us a contradiction. 

Proof of (iii)$\Rightarrow$(i) of Theorem 8.1

Let $f : X \to Y$ be the bijection given in Lemma 8.9. By Lemma 8.1 there exists a unitary isomorphism $U : \ell_2(X) \to \ell_2(Y)$ such that $\Phi(T) = UTU^*$, for all $T \in C^*_u[X]$. By Lemma 8.10 we have that $\langle U\delta_x, \delta_f(x) \rangle \neq 0$, for all $x \in X$. Let $g = f^{-1}$.

Claim 2. $f$ and $g$ are coarse maps.

Proof. By symmetry, we only need to show that $f$ is coarse. Let $E \in \mathcal{E}$ and let $F \in \mathcal{F}$ be given by Theorem 8.11. Without loss of generality, we can assume that $F$ is symmetric. For all $x, x' \in X$, we have that

$$\langle \Phi(e_{xx'}), \delta_f(x), \delta_f(x') \rangle = \langle Ue_{xx'}U^*\delta_f(x), \delta_f(x') \rangle = \langle e_{xx'}U^*\delta_f(x'), U^*\delta_f(x) \rangle = \langle \delta_x, U^*\delta_f(x') \rangle \delta_x, U^*\delta_f(x) \rangle = \langle \delta_x, U^*\delta_f(x) \rangle \delta_x', U^*\delta_f(x') \rangle$$

Therefore, by the definition of $f$, $\langle \Phi(e_{xx'}), \delta_f(x'), \delta_f(x) \rangle \neq 0$, for all $x, x' \in X$. Hence, if $(x, x') \in E$, this gives us that

$$(f(x'), f(x)) \in \text{supp}(\Phi(e_{xx'})) \subseteq F.$$ 

Since $F$ is symmetric, we are done.

Claim 3. $g \circ f$ and $f \circ g$ are close to $\text{Id}_X$ and $\text{Id}_Y$, respectively.

Proof. By symmetry, we only show that $f \circ g$ is close to $\text{Id}_Y$. Let $E \in \mathcal{E}$ be given by Theorem 8.11 applied to the isomorphism $\Phi^{-1} : C^*_u(Y) \to C^*_u(X)$ and the diagonal $\Delta_Y \subseteq Y \times Y$. By 8.1 above, we have that

$$\langle \Phi(e_{g(y)g(y)}), \delta_g, \delta_f(g(y)) \rangle = \langle U\delta_{g(y)}, \delta_f(g(y)) \rangle \langle U\delta_{g(y)}, \delta_y \rangle,$$

for all $y \in Y$. Therefore, by the definition of $f$ and $g$, we get that

$$\langle \Phi(e_{g(y)g(y)}), \delta_g, \delta_f(g(y)) \rangle = \langle U\delta_{g(y)}, \delta_f(g(y)) \rangle \langle \delta_y, U^*\delta_y \rangle \neq 0,$$

for all $y \in Y$. We conclude that

$$(y, f(g(y)) \in \text{supp}(\Phi(e_{g(y)g(y)})) \subseteq E,$$

for all $y \in Y$.

This finishes the proof.

9. Appendix: Generically absolute isomorphisms

In this appendix some familiarity with models of set theory and absoluteness is desirable; see for example Kun11 Section I.16 and Section II.5 or Jec03. Before defining the notion of a generically absolute isomorphism between uniform Roe algebras, we should point out that the absoluteness theorem for $\Pi^1_1$ statements (Jec03, Theorem 25.4) implies that every isomorphism between uniform Roe algebras associated to metric spaces is generically absolute. Therefore Theorem 9.7 below is a generalization of the instance of Theorem 4.12 for metric spaces.

Suppose that $M \subseteq N$ are two transitive models of a large enough fragment of ZFC with the same set of ordinals. (Because of metamathematical considerations
not directly relevant to our discussions, we cannot assume that ZFC is consistent and therefore have to work with a model of a large enough finite fragment of ZFC; see [Kun11 II] for an extensive discussion.) Furthermore suppose that $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are coarse spaces, $\Phi: C^*_u(X) \to C^*_u(Y)$ is an isomorphism and all those objects are in $M$.

Since $M \subseteq N$, all of these objects belong to $N$. However, they need not be objects of the required form. For example, in $N$ the set $\mathcal{E}$ is a collection of subsets of $X \times X$, and it satisfies (i), (iii), (iv), and (v) of Definition 2.1. (This is a consequence of the absoluteness of the notions involved in these axioms; see [Kun11 II.4].) However, $N$ may contain a subset of $X$ that does not belong to $M$, and in this case (ii) of Definition 2.1 will fail for $\mathcal{E}$. The way to remediate this issue is to take the coarse structure on $X$ generated by $\mathcal{E}$ in the model $N$.

We take (writing $\mathcal{P}(E) = \{E' \mid E' \subseteq E\}$)

$$\mathcal{E}^N = \bigcup_{E \in \mathcal{E}} \mathcal{P}(E)$$

as computed in $N$. Then we have that $(X, \mathcal{E}^N)$ is a coarse space in $N$\footnote{Purists may object our not distinguishing $X^M$ from $X^N$ and using $X$ to denote both sets; this is however the same set and we find writing $\ell_2(X)^N$ preferable to writing $\ell_2(X^N)$. The notation $K(\ell_2(X)^N)^N$ appears to be a necessary evil.}

We proceed to define interpretations of other relevant objects in the model $N$. For $\ell_2(X)^N$ we take the completion of $\ell_2(X)^M$; this space has $\delta_x$, for $x \in X$, as an orthonormal basis and it clearly agrees with $\ell_2(X)$ as computed in $N$. The coarse space $(Y, \mathcal{F}^N)$ and the Hilbert space $\ell_2(Y)^N$ are defined analogously. Then $U^M$ is a linear isometry between dense subspaces of $\ell_2(X)^N$ and $\ell_2(Y)^N$, and we let $U^N$ denote its continuous extension. This is a unitary.

It remains to see how the uniform Roe algebras $C^*_u(X)^N$ and $C^*_u(Y)^N$ relate to the uniform Roe algebras $C^*_u(X)^M$ and $C^*_u(Y)^M$. A minor inconvenience is caused by the following two facts. (We assume the ‘worst case scenario’ that $N$ contains a subset of $X$ which is not in $M$.)

1. The algebra $\ell_\infty(X)^N$ is equal the closure of $\ell_\infty(X)^M$ in the weak operator topology, and it is strictly larger than the closure of $\ell_\infty(X)^M$ in the norm topology.

2. The algebra $K(\ell_2(X)^N)^N$ is equal to the closure of $K(\ell_2(X)^M)^M$ in the norm topology. It is strictly smaller than the closure of $K(\ell_2(X)^M)^M$ in the weak operator topology.

Lemma 9.1 will provide us with a recipe for how to compute $C^*_u(X)^N$ directly from $(X, \mathcal{E}^M)$. (The reason for the absence of the superscripts $^M$ and $^N$ in the statement of Lemma 9.1 is that this lemma has nothing to do with models of fragments of ZFC.) The proof of the lemma is omitted, being an immediate consequence of the definition of $C^*_u(X)$.}

**Lemma 9.1.** Suppose that $(X, \mathcal{E})$ is a uniformly locally discrete coarse space. For $E \in \mathcal{E}$ the set

$$\mathcal{Z}_E = \{T \in B(\ell_2(X)) : \text{supp}(T) \subseteq E\}$$

is closed in the weak operator topology and contained in $C^*_u(X)$. If $\mathcal{E}_1 \subseteq \mathcal{E}$ is cofinal with respect to inclusion, then $C^*_u(X) = \bigcup_{E \in \mathcal{E}_1} \mathcal{Z}_E^\|\cdot\|$. \qed
Back to interpreting notions in models $M$ and $N$, we note that for $E \in \mathcal{E}^M$, the set $(\mathcal{Z}_E)^N$ is equal to the closure of $(\mathcal{Z}_E)^M$ in the weak operator topology. We can now define

$$C_u^*(X)^N = \bigcup_{E \in \mathcal{E}^M} (\mathcal{Z}_E)^N$$

where the norm closure is computed in $N$. Since $\mathcal{E}^M$ is cofinal in $\mathcal{E}^N$, Lemma 9.1 implies that $C_u^*(X)^N$ defined in this way coincides with the uniform Roe algebra of the coarse space $(X, \mathcal{E}^N)$ as computed in $N$.

Suppose that $\Phi: C_u^*(X) \to C_u^*(Y)$ is a isomorphism. Then $X, Y, C_u^*(X), C_u^*(Y)$, and $\Phi$ all belong to a large enough rank initial segment $R(\theta)$ (commonly denoted $V_\theta$; we use Kunen’s notation) of von Neumann’s cumulative universe for set theory. Let $M_0$ be a countable elementary submodel of $R(\theta)$ containing $X, Y, C_u^*(X), C_u^*(Y), \Phi$, and the unitary $U$ implementing $\Phi$, and let $M$ denote the Mostowski collapse of $M_0$. This is a countable transitive model isomorphic to $M_0$, and it contains copies $X^M, Y^M, C_u^*(X)^M, C_u^*(Y)^M, \Phi^M$, and $U^M$ of the above objects. (This time we write $X^M$ because $X$ may not belong to $M$.) By elementarity, $M$ will satisfy the assertion

$$(\forall T \in B(\ell_2(X))) T \in C_u^*(X) \Leftrightarrow UTU^* \in C_u^*(Y).$$

We proceed to describe how an extension $N$ of a model $M$ that will serve our purpose is obtained. Suppose that $J$ is a set in $M$. Observe that $\mathbb{D}^M$ is a countable dense subset of $\mathbb{D}$ and that $(\mathbb{D}^J)^M$ is a countable dense subset of $\mathbb{D}^J$. Therefore if $A \in M$ and $A \subseteq (\mathbb{D}^J)^M$, then $A$ is nowhere dense in $\mathbb{D}^J$ if and only if the assertion ‘$A$ is nowhere dense in $\mathbb{D}^J$’ holds in $M$.

By the Baire category theorem, we can choose $G \in \mathbb{D}^J$ such that $G$ does not belong to the closure of any nowhere dense subset of $\mathbb{D}^J$ that belongs to $M$. Such $G$ is said to be generic over $M$. Then $G$ is generic (in the technical sense from the theory of forcing) for the poset of all nonempty open subsets of $\mathbb{D}^J$ ordered by the inclusion. A transitive model $M[G]$ that contains $G$ and includes $M$ can be formed as in [Kun11, IV.2].

**Definition 9.2.** An isomorphism $\Phi: C_u^*(X) \to C_u^*(Y)$ implemented by a unitary $U$ is generically absolute if and only if for all $M$ and $M[G]$ as in the previous paragraph, $U$ implements an isomorphism between $C_u^*(X)$ and $C_u^*(Y)$.

A proof of the following lemma is now straightforward.

**Lemma 9.3.** A isomorphism $\Phi: C_u^*(X) \to C_u^*(Y)$ between uniform Roe algebras of coarse spaces is generically absolute if and only if it satisfies the conclusion of Lemma 9.1. More precisely, there is a function $f: \mathcal{E} \times \mathbb{N} \to \mathcal{F}$ such that for all $E \in \mathcal{E}$, every $n \geq 1$, and every $T \in C_u^*(X)$ such that $\text{supp}(T) \subseteq E$ there exist $S \in C_u^*(Y)$ such that $\text{supp}(S) \subseteq f(E, n)$ and $\|\Phi(T) - S\| < 1/n$. \qed

The proof of Theorem 9.12 in which the smallness assumption is replaced by the assumption that $\Phi$ be generically absolute gives the following.

**Theorem 9.4.** Suppose that $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are uniformly locally finite coarse spaces. If $C_u^*(X)$ and $C_u^*(Y)$ are rigidly isomorphic via a generically absolute isomorphism, then $X$ and $Y$ are coarsely equivalent. \qed

We do not know whether it is possible to construct an isomorphism between uniform Roe algebras that is not generically absolute.
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