Transporting a prediction model for use in a new target population

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Abstract

We consider methods for transporting a prediction model and assessing its performance for use in a new target population, when outcome and covariate information for model development is available from a simple random sample from the source population, but only covariate information is available on a simple random sample from the target population. We discuss how to tailor the prediction model for use in the target population, how to assess model performance in the target population (e.g., by estimating the target population mean squared error), and how to perform model and tuning parameter selection in the context of the target population. We provide identifiability results for the target population mean squared error of a potentially misspecified prediction model under a sampling design where the source study and the target population samples are obtained separately. We also introduce the concept of prediction error modifiers that can be used to reason about the need for tailoring measures of model performance to the target population and provide an illustration of the methods using simulated data.

Keywords: transportability, generalizability, model performance, prediction error modifier, covariate-shift, domain adaptation


1 Introduction

Users of prediction models usually want to obtain model-derived predictions in a specific target population. For example, a healthcare system may want to deploy a clinical risk prediction model [1] to identify high-risk individuals among all patients receiving care. Prediction models are often built using information from large prospective epidemiological cohort studies, confirmatory randomized trials [2], or observational studies in routinely collected healthcare data (e.g., electronic health records) [3]. Typically, the observations in the source study that provides data for developing the prediction model are not randomly sampled from the target population of interest. As a result, the source study population, from which the source study data is derived, typically has a different data distribution compared to the target population. A model developed using the source study data may not apply to the target population and the performance of the model in the source study data may not reflect performance in the target population.

Consider a setup where outcome and covariate information are available from a source study (i.e., a sample from the source population) and only covariate information is available from a sample of the target population. For example, covariate information from the target population may be available from routinely collected data, but outcome information may be unavailable either because it is not collected (e.g., when outcome ascertainment requires specialized assessments) or because it is not readily usable for modeling (e.g., when the number of outcome events is small due to inadequate followup duration). Developing and assessing the performance of a prediction model for the target population is not possible using standard methods because of the lack of sufficient outcome information; using data from the source population can be an attractive alternative. Yet, as noted above, directly applying a prediction model developed in the source study to the target population, or treating model performance measures (e.g., mean squared prediction error) estimated in the source study as reflective of performance in the target population may be inappropriate when the data distribution of the source population differs from the data distribution of the target population.

Tailoring a prediction model for use in a target population other than the one used for model development and assessing the performance of the model in that target population has received
attention in the computer science literature under the names of covariate shift, domain adaptation, or as a special case of transfer learning [4–12]. In epidemiology, however, the transportability of prediction models has been mostly treated heuristically and commonly used methods do not have well-understood statistical behavior. The related problem of generalizability or transportability of treatment effects to a target population has received more attention [13–16], but there are important differences between transportability of treatment effects and prediction models in terms of the parameters being estimated and the methods used for estimation.

Here, we examine the conditions that allow transporting prediction models from the source study to the target population. We discuss the implications of these conditions both for tailoring the models for use in the target population and for assessing model performance in that context. We show that many popular measures of model performance (e.g., the mean squared prediction error) are identifiable and can be estimated using covariate and outcome data from the source study and just covariate data from a simple random sample of the target population, without the strong assumption that the prediction model is correctly specified. We discuss the relevance of our results when using modern model-building approaches such as cross-validation-based model selection. We introduce the concept of prediction error modifiers, which is useful for reasoning about transportability of measures of model performance to the target population. Last, we illustrate the methods using simulated data.

2 Data structure, identifiability conditions, and sampling design

Let \( Y \) be the outcome of interest and \( X \) a covariate vector. We assume that outcome and covariate information is obtained from a simple random sample from the population underlying the source study, \( \{(X_i, Y_i) : i = 1, \ldots, n_{\text{source}}\} \). Furthermore, covariate information is obtained from a simple random sample from the target population, \( \{X_i : i = 1, \ldots, n_{\text{target}}\} \); no outcome information is available from the target population. Let \( S \) be an indicator for the population from which data are obtained, with \( S = 1 \) for the source population and \( S = 0 \) for the target population and denote \( n = n_{\text{source}} + n_{\text{target}} \) as the sample size of the composite dataset consisting of the data from the source study and sample of the target population. We assume that this composite dataset is randomly
split into a training set and a test set. The training set is used to build a prediction model for the expectation of the outcome conditional on covariates, $E[Y|X, S = 1]$, and then, the test set is used to evaluate model performance. We use the notation $g_{\beta}(X)$ to denote the model, indexed by the parameter $\beta$ and the notation $g_{\beta}(X)$ to denote the model with estimated parameter $\hat{\beta}$. We use $f(\cdot)$ to denote generic densities.

We assume that the following identifiability conditions, which are fairly standard in the covariate-shift literature [12,17], hold:

A1. Conditional independence of the outcome $Y$ and the data source $S$. For every $x$ with positive density in the target population, $f(X = x, S = 0) > 0$,

$$f(Y|X = x, S = 1) = f(Y|X = x, S = 0).$$

Informally, this condition means that the “true” relationship between $Y$ and $X$ is the same in the source population and the target population and it implies that the conditional expectation of $Y$ given $X$ is the same in the two populations, $E[Y|X, S = 1] = E[Y|X, S = 0]$. If the outcome is binary, condition A1 can be written as $\Pr[Y = 1|X = x, S = 1] = \Pr[Y = 1|X = x, S = 0]$.

A2. Positivity. For every $x$ such that $f(X = x, S = 0) > 0$, $\Pr[S = 1|X = x] > 0$, for every $x$ such that $f(X = x, S = 0) > 0$. Informally, this condition means that every covariate pattern in the target population can occur in the source study sample, as sample size goes to infinity.

We will now discuss implications of these conditions depending on how the estimated model $g_{\beta}(X)$ behaves in the target population: in Section 3, we discuss how the model $g_{\beta}(X)$ can be estimated such that $g_{\beta}(X)$ is tailored for use in the target population; in Section 4, we discuss how to assess the model’s performance in the target population.
3 Tailoring the model to the target population

Recall that $g_\beta(X)$ is a model for $E[Y|X,S = 1]$. Let $\Theta$ be the sample space of the parameter $\beta$. We shall say that the model is correctly specified if there exists a $\beta_0 \in \Theta$ such that $g_{\beta_0}(X) = E[Y|X,S = 1]$. Under a correctly specified model, we can construct a model-based estimator $g_{\hat{\beta}}(X)$ that consistently estimates $E[Y|X,S = 1]$.

Correctly specified model: Suppose that the model estimated using the training set from the source study is correctly specified. Under condition A1, a correctly specified model for the population underlying the source study is also correctly specified for the target population and a consistent estimator for $E[Y|X,S = 1]$ is also consistent for $E[Y|X,S = 0]$ (because the two expectations are equal when condition A1 holds). Moreover, if the model for the conditional expectation is parametric and the parameter $\beta$ is estimated using maximum likelihood methods, then the maximum likelihood estimator $\hat{\beta}$ estimated using only the source study data (assigning weight equal to 1 to each observation from the source study training set) is optimal in terms of having the smallest asymptotic variance [18,19]. This is expected because, under random sampling, the covariates $X$ are ancillary for the parameter $\beta$ and factor out of the likelihood.

Misspecified model: Now, suppose, as is more likely to be the case in practice, that the model that is estimated using data from the source study is misspecified. In that case, theoretical work on the behavior of weighted maximum likelihood estimators for $\beta$ under covariate shift [19] shows that the maximum likelihood estimator estimated using source study data only is no longer optimal (defined in terms of minimizing the Kullback-Leibler divergence between the estimated and true conditional density of the outcome given covariates). The minimum Kullback-Leibler divergence is obtained by upweighting contributions to the likelihood from observations that have covariate patterns that are more likely to come from the target population. More specifically, the optimal source study training set weights for the weighted maximum likelihood estimator are obtained if the weights are set to the ratio of the densities in the target and source study population, that is $f(X|S = 0)/f(X|S = 1)$. The ratio of the densities in the target and source study population
can be used for general M-estimators \cite{20}, but for the rest of this section we focus on maximum likelihood estimators.

The development in reference \cite{19} assumes that the density ratio is known which is rarely the case and implementation usually relies on estimating the density ratio using the observed data. As the density ratio is, up to a proportionality constant, equal to the inverse of the odds of participation in the source study,

\[
\frac{f(X|S = 0)}{f(X|S = 1)} \propto \frac{Pr[S = 0|X]}{Pr[S = 1|X]},
\]

we can use the inverse odds weights instead of the density ratio to obtain an optimal estimator of the model, tailored for use in the target population. The estimation of density ratios using the training set is challenging even with moderately high-dimensional covariate vectors, but there is large literature on practical methods for estimating conditional probabilities with high-dimensional data \cite{21}. Thus, a reasonable approach for tailoring a potentially misspecified prediction model for use in the target population would be to (1) estimate the probability of participation in the source study, using data from the source study and target population in the training set; (2) use the estimated probabilities to construct inverse odds of participation weights for observations from the source study in the training set (observations from the target population in the training set do not have outcome data and are assigned weight equal to 0); and (3) estimate the prediction model using all observations from the source study in the training set by applying the weights from the previous step.

One difficulty with the above procedure is that usually individuals in the source study and the sample of the target population are sampled separately with sampling fractions from the corresponding underlying populations that are not known by the investigators and unlikely to be equal. When the sampling fractions are unequal, as is likely to be the case in practice, the probabilities \(Pr[S = 0|X]\) and \(Pr[S = 1|X]\) in the inverse odds weights are not identifiable from the data (an analogous problem occurs in the literature on generalizing or transporting causal effects \cite{22, 23}). Further complicating the inverse odds weights estimation, we propose to estimate the inverse odds weights for model fitting using only the training data, to ensure independence between the training and the test set. As a result, the training set data used to estimate the inverse odds weights are
a simple random sample of the available data with a known sampling probability equal to the proportion of observations that are in the training set. Under this sampling design, $\Pr[S = 1|X]$, $\Pr[S = 0|X]$, and the inverse odds weights are not identifiable (without knowledge of the sampling fractions). Nevertheless, we now argue that the inverse odds weights are still identifiable up to an unknown proportionality constant.

Let $D_{\text{train}}$ be an indicator if data from an observation is in the training set and used to estimate the inverse odds weights. The sampling design assumes that $\Pr[D_{\text{train}} = 1|X, S = 1] = a$ for some potentially unknown constant $a > 0$; and $\Pr[D_{\text{train}} = 1|X, S = 0] = b$ for some potentially unknown constant $b > 0$. Using this notation, $\Pr[S = 1|X, D_{\text{train}} = 1]$ is identifiable but $\Pr[S = 1|X]$ is not; furthermore, in Appendix A.1, we show that

$$\frac{\Pr[S = 0|X]}{\Pr[S = 1|X]} \propto \frac{\Pr[S = 0|X, D_{\text{train}} = 1]}{\Pr[S = 1|X, D_{\text{train}} = 1]}.$$  

This result implies that we can use the identifiable odds weights $\frac{\Pr[S = 0|X, D_{\text{train}} = 1]}{\Pr[S = 1|X, D_{\text{train}} = 1]}$ instead of the unidentifiable odds weights $\frac{\Pr[S = 0|X]}{\Pr[S = 1|X]}$, when estimating $\beta$ with the weighted maximum likelihood estimator.

4 Assessing model performance in the target population

We now turn out attention to assessing model performance in the target population. For concreteness, we focus on model assessment using the squared error loss function and on identifying and estimating its expectation, that is, the mean squared error (MSE), in the target population. The squared error loss $(Y - g_\beta(X))^2$ quantifies the discrepancy between the (observable) outcome $Y$ and the model-derived prediction $g_\beta(X)$ in terms of the square of their difference. The MSE in the target population is then defined as

$$\psi_\beta = E[(Y - g_\beta(X))^2|S = 0].$$
In the main text of this paper, we focus on the MSE because it is a commonly used measure of model performance. Our results, however, readily extend to other measures of performance: similar to the MSE, many measures of performance, such as the expected absolute error and the Brier score, are expectations of loss functions over the data distribution in the population where the model will be applied. In Appendix A.1, we provide identifiability results for general loss function-based measures of model performance.

4.1 Prediction error modifiers

To help explain why model performance measures need to be modified for use in the target population, we introduce the term “prediction error modifier” for any covariate that is associated with prediction error (as assessed with some specific measure of model performance, for a given model). When the distribution of prediction error modifiers differs between the source study and target population, measures of model performance in the source study data do not apply to the target population, in the sense that the performance of the model in the source study may be very different (either better or worse) compared to performance of the same model in the target population. Large differences in performance measures between the source and target population can occur even if the true outcome model in the two populations is the same (i.e., even if condition A1 holds) because most common measures of model performance average (marginalize) prediction errors over the data distribution of the target population, and the data distribution of the target population can be different from the distribution in the source population (e.g., the distribution of covariates $X$ may be different).

Figure 1 shows an example of a prediction error modifier that is differently distributed between the source study and target population resulting in a MSE in the target population that is higher than the MSE in the source population. The middle panel in Figure 1 shows the inverse odds weights as a function of $X$ and the bottom panel shows the squared errors as a function of $X$. Because both the squared errors and the inverse odds weights (and therefore the probability of being from the target population) increase as $X$ increases, the target population MSE (which is equal to the expectation of the squared errors) is larger than the source study MSE. Hence, directly
using the source study MSE in the context of the target population would lead to over-optimism about model performance.

4.2 Assessing model performance in the target population

To assess model performance in the context of the target population we need to account for differences in the data distribution between the source population and the target population. To ensure independence between the data used for model development and the data used for assessment of model performance, we only use the source study and target test data to estimate \( \hat{\psi}_B \) and we define \( D_{\text{test}} \) as an indicator if an observation is in the source study or target test data. The key complications when estimating \( \hat{\psi}_B \) are the separate sampling with an unknown probability from the target and source population and that, in our setup, outcome information is unavailable from the target population. The following proposition shows that, under the setup described in Sections 2 and 3 and conditions A1 and A2, \( \hat{\psi}_B \) is identifiable.

**Proposition 1.** Under conditions A1 and A2 and when the source study and target data are obtained by separate simple random sampling of the corresponding underlying populations, with potentially unknown sampling probabilities, then the target population MSE, \( \hat{\psi}_B \), is identifiable as

\[
\psi_B = \mathbb{E}[\mathbb{E}[(Y - g_B(X))^2|X, S = 1, D_{\text{test}} = 1]|S = 0, D_{\text{test}} = 1]; \tag{2}
\]

or, using an inverse odds weighting representation,

\[
\psi_B = \frac{1}{\Pr[S = 0|D_{\text{test}} = 1]} \mathbb{E} \left[ \frac{I(S = 1) \Pr[S = 0|X, D_{\text{test}} = 1]}{\Pr[S = 1|X, D_{\text{test}} = 1]} (Y - g_B(X))^2 \bigg| D_{\text{test}} = 1 \right]. \tag{3}
\]

All quantities in expressions (2) and (3) condition on \( D_{\text{test}} = 1 \) and can therefore be calculated using the available test data. In Appendix A.1, we prove the identifiability of a broad class of loss function-based measures of model performance, including the MSE as a special case. An important feature of this result is that it does not require the prediction model to be correctly specified (i.e., we do not assume that \( g_B(X) \) converges to the true conditional expectation \( \mathbb{E}[Y|X, S = 1] \)). This implies that model performance measures in the target population are identifiable, both
for misspecified and correctly specified models. Informally, our identifiability results require the existence of a common underlying model for the source and study population (condition A1), but they do not require the (much less plausible) assumption that investigators can correctly specify that model.

So far we have focused on the scenario where the prediction model is built using the training data and evaluated using the test data, and where both the data from the source study and from the sample of the target population are split into a test and a training set. An alternative scenario is when there is an established external model (already developed using a different data source) that we want to assess target population model performance for (e.g., the Framingham risk score or the Gail model). In that case, no data from the source study or the sample of the target population need to be used for model development and all available data can be used to evaluate model performance and treated as a part of the “test set.” Thus, identifiability of the MSE when the model is externally developed is also addressed by the identifiability results in Proposition 1.

5 Tailoring models for the target population when model-building requires model and tuning parameter selection

Up to now we have proceeded as if the source study data in the training set is used to estimate parameters of a prespecified model, without employing any form of model selection (e.g., variable choice or other specification search) or tuning parameter selection. Yet, when developing prediction models, analysts often select between multiple different models and modern statistical learning algorithms usually have one or more tuning parameters. Importantly, data-driven methods for model and tuning parameter selection, such as cross-validation-based procedures, rely on optimizing some measure of model performance (e.g., MSE).

Consider the example of tuning parameter selection using $K$-fold cross-validation. We split the data into $K$ mutually exclusive folds and for each value of the tuning parameter we build the model with the selected tuning parameter value on $K - 1$ of the folds and estimate a measure of model performance on the fold that is not used for model building. This process is repeated
where each of the $K$ folds is left out of the model building process, resulting in $K$ estimates of model performance. The final cross-validated estimator of model performance associated with the tuning parameter value is the average of the $K$ estimators. The cross-validated value of the tuning parameter is selected as the value of the tuning parameter that optimizes the cross-validated estimator of model performance.

Clearly, data-driven model and tuning parameter selection relies on estimating measures of model performance. Furthermore, tailoring the cross-validated model for use in the target population and tuning parameter selection to improve model performance for use in the target population require incorporating the results from two preceding sections to account for differences in the distribution of covariates between the source study and target population. Specifically, when prediction error modifiers are deferentially distributed in the source and the target population, measures of model performance estimated using the source study data are biased estimators of model performance in the target population. Failing to adjust for this bias (e.g., by using inverse odds weighting) when performing cross-validation can lead to sub-optimal model or tuning parameter selection in the context of the target population. For example, a linear regression model might be preferred in the source study but a non-linear model in the target population, or the optimal penalization parameter in penalized regression might differ between the source and the target population. Note in passing that, when using weighting methods to adjust for the bias, to guarantee independence between the $K-1$ folds used for model building and the left out fold, the odds weights need to be re-estimated using only data from the $K-1$ folds.

6 Illustration

The identifiability result in expression (3) suggests the following inverse odds weighting estimator $[17,19]$ for the target population MSE:

$$
\hat{\psi}_\beta = \frac{1}{\sum_{i=1}^{n} I(S_i = 0, D_{\text{test}} = 1)} \sum_{i=1}^{n} I(S_i = 1, D_{\text{test}} = 1) \hat{o}(X_i) \left(Y_i - g_{\hat{\beta}}(X_i)\right)^2.
$$

(4)
Here, \( \hat{o}(X) \) is an estimator for the inverse odds weights \( \frac{\Pr[S=0|X,D_{\text{test}}=1]}{\Pr[S=1|X,D_{\text{test}}=1]} \) estimated using the test set.

We performed simulations to illustrate (i) the effect on model performance of using inverse odds of participation weights for model building, with correctly or incorrectly specified prediction models; (ii) that the naive (unweighted) MSE estimator that uses only the source study data is biased for the target population MSE in the presence of prediction error modifiers; and (iii) that the inverse odds weighting estimator can adjust for the bias. We simulated the outcome using the linear model \( Y = 1 + X + 0.5X^2 + \varepsilon \), where \( \varepsilon \sim \mathcal{N}(0, X) \) and \( X \sim \text{Uniform}(0, 10) \). Under this model, the errors are heteroscedastic because the error variance directly depends on the covariate \( X \). We simulated participation in the source study using a logistic regression model \( \log \left( \frac{\Pr[S=1|X]}{1-\Pr[S=1|X]} \right) = 1.5 - 0.3X \).

We set the total sample size was set to 1000 and the source study and the target population data were randomly split in a 1:1 ratio into a training set used to implement the prediction model and a test set used to estimate target population MSE.

This is a setting where the target population MSE is larger than the source study population MSE and both conditions A1 and A2 are satisfied. We considered two prediction models, a correctly specified linear regression model that included main effects of \( X \) and \( X^2 \) and an incorrectly specified linear regression model that only included the main effect of \( X \) (i.e., it failed to include the main effect of \( X^2 \)). We also considered two approaches for estimating each posited prediction model: ordinary least squares regression (unweighted, OLS) and weighted least squares regression (WLS) where the weights were equal to the inverse of estimated odds of participation in the source study training set. We estimated the inverse odds of participation in the training set \( \Pr[S = 0|X, D_{\text{train}} = 1]/\Pr[S = 1|X, D_{\text{train}} = 1] \) using a correctly specified logistic regression model for the probability \( \Pr[S = 1|X, D_{\text{train}} = 1] \). Figure 2 shows an example of data simulated using this setup and the relationship between the correct model and the weighted and unweighted misspecified models. For the inverse odds weighting estimator \( \hat{\psi}\hat{\beta} \), we estimated the odds weights \( \hat{o}(X) \) in the test set by fitting a correctly specified logistic regression model for \( \Pr[S = 1|X, D_{\text{test}} = 1] \) using the test set data.

The results from 10,000 simulations are presented in Table 1. For both OLS and WLS estimation
of the prediction model, correct model specification resulted in smaller target population and source population MSE than incorrect specification. When comparing the performance of OLS and WLS estimation of the prediction model in the target population OLS performed slightly better than WLS when the model was correctly specified (MSE of 45.8 vs. 46.2). When the prediction model was incorrectly specified, the OLS performed worse than WLS (MSE of 66.3 vs. 58.0). The last column in the Table shows that the inverse odds weighting MSE estimator $\hat{\psi}_\beta$ was unbiased for the target population MSE for all combinations of model specifications and use of weights. In all scenarios of this simulation, the source population MSE estimator was substantially lower than the target population MSE. Hence, using the estimated source study MSE as an estimator for the target population MSE would lead to substantial underestimation of the MSE (i.e., showing model performance to be better than it is in the context of the target population). In contrast, the inverse odds weighting estimator would give a more accurate assessment of model performance in the target population.

7 Discussion

We considered transporting prediction models to a different population than was used for original model development, when outcome and covariate data are available on a simple random sample from the source population and covariate information is available on a simple random sample from the target population. We described the adjustments needed when the covariate distribution differs between the source and target population and provided identification results under a sampling design where the source study and the target population sample are obtained separately with unknown sampling probabilities. We discussed how to tailor the prediction model to the target population and how to calculate measures of model performance in the context of the target population, without requiring the prediction model to be correctly specified. We also examined tailoring data-driven model and tuning parameter selection to the target population. The key insight is that most measures of model performance average over the covariate distribution and, as a result, estimators of these measures obtained in data from the source population will typically be biased for the corresponding measures in the target population, when the covariate distribution differs
between the two populations. This relates to the problem of generalizability or transportability of treatment effects [15, 24], because treatment effect estimators for this problem are also averaged over the covariate distribution of the target population.

For correctly specified prediction models, exchangeability in mean over $S$, that is $E[Y|X, S = 1] = E[Y|X, S = 0]$, is sufficient for the parameter $\beta$ to be identifiable using data from the source study alone. Exchangeability in mean over $S$ is a weaker condition than condition A1; that is, condition A1 implies exchangeability in mean, but the converse is not true. Similarly, for transportability of treatment effects mean exchangeability within each treatment arm is sufficient [13,16]. On the contrary, exchangeability in mean is not sufficient for transportability of MSE. In Appendix B we give an example of a setting where exchangeability in mean holds but it is not sufficient to identify the target population MSE.

The current literature on transporting prediction models focuses on non-nested designs, that is a setting where the sample from the target population is separately obtained from the data from the source study population. An alternative design is where the source study is nested in a larger population that represents the target population [16, 22]. Examples of such nested designs are when the source study is embedded within a larger health care system or comprehensive cohort studies [25] where covariate information is collected on all individuals including those who refuse to participate in the source study. Although we focused on non-nested designs, the results hold with minor modifications for nested designs. In Appendix A we prove an identification result for nested designs that is analogous to proposition 1.

Some algorithms for building prediction models use measures of model performance to drive decisions within the algorithmic procedure itself [17]. For example, splitting decisions in the regression tree algorithm [26] are made by comparing estimates of MSE. Regression trees are building blocks for many popular machine learning algorithms such as random forests, bagging, and boosting. Using MSE estimates appropriate for the context of the target population is expected to improve target population performance of these algorithms.

To simplify notation, we have used the same set of covariates ($X$) to satisfy the conditional independence condition (A1) and to build the prediction model $g_\beta(X)$. In practice, the set of co-
variates used to build the prediction model might not suffice to satisfy the conditional independence condition. For example, popular cardiovascular risk prediction models [27, 28] are built using only a few covariates and the information contained in that set of covariates might not be enough to satisfy the conditional independence condition. With minor modifications, the developments in this manuscript allow for the set of covariates used to satisfy the conditional independence assumption to be larger than the set of covariates used in the prediction model.

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Figure 1: An example of a prediction error modifier (X). The top panel shows a scatter-plot of the data and the solid black line is the $\mathbb{E}[Y|X, S = 1]$. The middle panel shows the inverse odds weights (IOW) as a function of X and the bottom panel shows the mean squared error (MSE) as a function of X. Larger values of X have increased MSE and higher inverse odds weights (higher probability of being from the target population ($S = 0$)). Hence, X is a prediction error modifier that is differentially distributed between the source study and the target population. This leads to the source study population MSE (=0.47) being smaller than the target population MSE (=0.74).
Figure 2: An example of simulated data used for illustrating transportability of prediction models using the setup described in Section 6. The solid curve is $E[Y|X, S = 1]$, the dashed line is the value that the misspecified unweighted model is estimating, and the dotted line is the value that the misspecified weighted model is estimating. Compared to the unweighted model, the weighted model prioritizes higher values of $X$ as higher values of $X$ are associated with higher probability of being from the target population. This is seen in the figure as for high values of $X$ the weighted model is closer to the truth compared to the unweighted model but the opposite is true for smaller values of $X$. 
Table 1: Target population mean squared error (MSE), source study MSE estimator, and the estimator for the target population MSE that weights observations by the inverse odds of source study participation.

| Model specification, estimation approach | Target population MSE | Source Study MSE Estimator | Weighted MSE estimator |
|----------------------------------------|-----------------------|-----------------------------|------------------------|
| Correctly specified, OLS               | 45.8                  | 22.5                        | 45.8                   |
| Incorrectly specified, OLS             | 66.3                  | 34.5                        | 66.3                   |
| Correctly specified, WLS               | 46.2                  | 22.8                        | 46.2                   |
| Incorrectly specified, WLS             | 58.0                  | 43.6                        | 57.9                   |

Correctly specified and incorrectly specified refers to the specification of the posited prediction model. OLS = model estimation using ordinary least squares regression (unweighted); WLS = model estimation using weighted least squares regression with weights equal to the inverse of the odds of participation in the source study. Weighted MSE estimator results were obtained using the estimator in equation (4). Results were averaged over 10,000 simulations.
Appendix A  Proofs of key results

A.1  Identifiability for non-nested designs

Proof of Proposition 1:

We will prove the proposition for a general loss function $L(Y, g_{\hat{\beta}}(X))$. Many common performance measures, including the mean squared error, absolute error, and the Brier score, are special cases of expected loss functions. For the first representation we have

$$
\psi_{\hat{\beta}} = E[L(Y, g_{\hat{\beta}}(X))|S = 0]
$$

$$
= E \left[ E[L(Y, g_{\hat{\beta}}(X))|X, S = 0]|S = 0 \right]
$$

$$
= E \left[ \int L(y, g_{\hat{\beta}}(X))dF(y|X, S = 0)|S = 0 \right]
$$

$$
= E \left[ \int L(y, g_{\hat{\beta}}(X))dF(y|X, S = 1)|S = 0 \right]
$$

$$
= E \left[ E[L(Y, g_{\hat{\beta}}(X))|X, S = 1]|S = 0 \right],
$$

where the first equality follows from the definition of $\psi_{\hat{\beta}}$, the second from the law of iterated expectations, and the fourth from identifiability condition A1. All expectations conditional on $(X, S = 1)$ in the above formula are well defined by the positivity condition A2. Rewrite

$$
\psi_{\hat{\beta}} = E[E[L(Y, g_{\hat{\beta}}(X))|X, S = 1]|S = 0]
$$

$$
= \int E[L(Y - g_{\hat{\beta}}(X))|X = x, S = 1]dF(x|S = 0).
$$

The conditional expectation $E[L(Y, g_{\hat{\beta}}(X))|X = x, S = 1]$ can be estimated using the sampling design as data on a random sample of observations from $S = 1$ are available. As we assume a random sample from $S = 0$ is available the conditional distribution $F(x|S = 0)$ is also identifiable using the sampling scheme. More formally, the random sampling ensures that

$$
\psi_{\hat{\beta}} = E[E[L(Y, g_{\hat{\beta}}(X))|X, S = 1, D_{\text{test}} = 1]|S = 0, D_{\text{test}} = 1].
$$
For the inverse odds weighting representation

\[
\psi_\beta = \mathbb{E}\left[\mathbb{E}\left[ L(Y, g_\beta(X)) | X, S = 1, D_{\text{test}} = 1 \right] | S = 0, D_{\text{test}} = 1 \right]
\]

\[
= \frac{1}{\mathbb{P}(S = 0 | D_{\text{test}} = 1)} \mathbb{E}\left[ \mathbb{E}\left[ \frac{I(S = 1)}{\mathbb{P}(S = 1 | X, D_{\text{test}} = 1)} L(Y, g_\beta(X)) | X, D_{\text{test}} = 1 \right] | S = 0, D_{\text{test}} = 1 \right] D_{\text{test}} = 1
\]

\[
= \frac{1}{\mathbb{P}(S = 0 | D_{\text{test}} = 1)} \mathbb{E}\left[ \frac{I(S = 1)}{\mathbb{P}(S = 1 | X, D_{\text{test}} = 1)} L(Y, g_\beta(X)) | X, D_{\text{test}} = 1 \right] D_{\text{test}} = 1
\]

\[
= \frac{1}{\mathbb{P}(S = 0 | D_{\text{test}} = 1)} \mathbb{E}\left[ \frac{I(S = 1) \mathbb{P}(S = 0 | X, D_{\text{test}} = 1)}{\mathbb{P}(S = 1 | X, D_{\text{test}} = 1)} L(Y, g_\beta(X)) | X, D_{\text{test}} = 1 \right] D_{\text{test}} = 1
\]

For the fourth equality we have used that

\[
\mathbb{E}\left[ \frac{I(S = 1)}{\mathbb{P}(S = 1 | X, D_{\text{test}} = 1)} L(Y, g_\beta(X)) | X, D_{\text{test}} = 1 \right] D_{\text{test}} = 1
\]

\[
= \frac{1}{\mathbb{P}(S = 0 | D_{\text{test}} = 1)} \mathbb{E}\left[ \frac{I(S = 1) \mathbb{P}(S = 0 | X, D_{\text{test}} = 1)}{\mathbb{P}(S = 1 | X, D_{\text{test}} = 1)} L(Y, g_\beta(X)) | X, D_{\text{test}} = 1 \right] D_{\text{test}} = 1
\]

All of the quantities in

\[
\frac{1}{\mathbb{P}(S = 0 | D_{\text{test}} = 1)} \mathbb{E}\left[ \frac{I(S = 1) \mathbb{P}(S = 0 | X, D_{\text{test}} = 1)}{\mathbb{P}(S = 1 | X, D_{\text{test}} = 1)} L(Y, g_\beta(X)) | D_{\text{test}} = 1 \right]
\]

condition on \(D_{\text{test}} = 1\) and are therefore identifiable using the observed data.

Proof of expression 1:

Recall that the sampling design assumes that \(\mathbb{P}(D_{\text{train}} = 1 | X, S = 1) = a\) for some potentially unknown constant \(a > 0\) and \(\mathbb{P}(D_{\text{train}} = 1 | X, S = 0) = b\) for some potentially unknown constant.
\(b > 0\). Using that, we have
\[
\frac{\Pr[S = 0 | X, D_{train} = 1]}{\Pr[S = 1 | X, D_{train} = 1]} = \frac{\Pr[S = 0, D_{train} = 1 | X]}{\Pr[S = 1, D_{train} = 1 | X]} = \frac{\Pr[S = 0 | X]}{\Pr[S = 1 | X]} \times \frac{\Pr[D_{train} = 1 | X, S = 0]}{\Pr[D_{train} = 1 | X, S = 1]} = \frac{\Pr[S = 0 | X]}{\Pr[S = 1 | X]} \times \frac{b}{a} \propto \frac{\Pr[S = 0 | X]}{\Pr[S = 1 | X]}.
\]

A.2 Identifiability for nested designs

For nested designs the target parameter is
\[
\hat{\theta} = \mathbb{E} \left[ L(Y, g_{\hat{\beta}}(X)) \right].
\]

We introduce the following modified identifiability conditions:

B1. For every \(x\) such that \(f(X = x) > 0\) we have
\[
f(Y | X = x, S = 1) = f(Y | X = x).
\]

B2. We have \(\Pr[S = 1 | X = x] > 0\) for each \(x\) with \(f(x) > 0\).

**Proposition 2.** Under conditions B1 and B2, \(\hat{\theta}_{\beta}\) can be written as the observed data functional
\[
\hat{\theta}_{\beta} = \mathbb{E} \left[ \mathbb{E} \left[ L(Y, g_{\hat{\beta}}(X)) | X, S = 1 \right] \right].
\]

Or using the inverse probability weighting representation
\[
\hat{\theta}_{\beta} = \mathbb{E} \left[ \frac{I(S = 1)}{\Pr[S = 1 | X]} L(Y, g_{\hat{\beta}}(X)) \right].
\]
Proof of Proposition 2:

We have

\[
\hat{\theta}_\beta = E \left[ L(Y, g_\beta(X)) \right]
= E \left[ E \left[ L(Y, g_\beta(X)) \right| X \right]
= E \left[ E \left[ L(Y, g_\beta(X)) \right| X, S = 1 \right].
\]

For the inverse probability weighting representation

\[
\hat{\theta}_\beta = E \left[ E \left[ L(Y, g_\beta(X)) \right| X, S = 1 \right]
= E \left[ \frac{I(S = 1)}{Pr[S = 1|X]} L(Y, g_\beta(X)) \right| X \right]
= E \left[ \frac{I(S = 1)}{Pr[S = 1|X]} L(Y, g_\beta(X)) \right],
\]

which establishes the identifiability of \( \hat{\theta}_\beta \). \( \square \)
Appendix B  Example where mean exchangeability holds but inverse odds weighting estimators are expected to be biased

For correctly specified prediction models, exchangeability in mean over $S$, that is $\operatorname{E}[Y|X, S = 1] = \operatorname{E}[Y|X, S = 0]$, is sufficient for the parameter $\beta$ to be identifiable using data from the source study alone. Exchangeability in mean over $S$ is a weaker condition than condition A1; that is, condition A1 implies exchangeability in mean, but the converse is not true. Similarly, for transportability of treatment effects mean exchangeability within each treatment arm is sufficient [13, 16]. On the contrary, exchangeability in mean is not sufficient for transportability of MSE. This can be seen in Figure 3 where $\operatorname{E}[Y|X, S = 1] = \operatorname{E}[Y|X, S = 0]$ but $\operatorname{Var}[Y|X, S = 1] \neq \operatorname{Var}[Y|X, S = 0]$. As the conditional variance is different between the two populations, standardizing to the target population covariate distribution is not sufficient to transport the MSE to the target population.

![Figure 3: An example of a setting where condition A1 from Section 2 does not hold. Here, $\operatorname{E}[Y|X, S = 1] = \operatorname{E}[Y|X, S = 0]$ (the black line is the true conditional mean for both populations), but $\operatorname{Var}[Y|X, S = 1] < \operatorname{Var}[Y|X, S = 0]$ for all values of $X$. In this case, estimators of model performance measures that use weights equal to the inverse odds of participation in the source study (e.g., the MSE estimator in the main text of the paper) will be biased.](image-url)

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