Analytical method and its convergence analysis based on homotopy analysis for the integral form of doubly singular boundary value problems

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Abstract
In this paper, we consider the nonlinear doubly singular boundary value problems
\[(p(x)y'(x))' + q(x)f(x, y(x)) = 0, \quad 0 < x < 1\]
with Dirichlet/Neumann boundary conditions at \(x = 0\) and Robin type boundary conditions at \(x = 1\). Due to the presence of singularity at \(x = 0\) as well as discontinuity of \(q(x)\) at \(x = 0\), these problems pose difficulties in obtaining their solutions. In this paper, a new formulation of the singular boundary value problems is presented. To overcome the singular behavior at the origin, with the help of Green’s function theory the problem is transformed into an equivalent Fredholm integral equation. Then the optimal homotopy analysis method is applied to solve integral form of problem. The optimal control-convergence parameter involved in the components of the series solution is obtained by minimizing the squared residual error equation. For speed up the calculations, the discrete averaged residual error is used to obtain optimal value of the adjustable parameter \(c_0\) to control the convergence of solution. The proposed method (a) avoids solving a sequence of transcendental equations for the undetermined coefficients (b) it is a general method (c) contains a parameter \(c_0\) to control the convergence of solution. Convergence analysis and error estimate of the proposed method are discussed. Accuracy, applicability and generality of the present method is examined by solving five singular problems.

Keyword: Optimal homotopy analysis method; Doubly singular boundary value problems; Green’s function; Lane-Emden equation; Fredholm integral equation; Approximations.

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1 Introduction

We consider the nonlinear doubly singular boundary value problems (DSBVPs) \[1–3\]

\[-(p(x)y'(x))' = q(x)f(x, y(x)), \quad 0 < x < 1, \tag{1.1}\]

\[y(0) = \delta_1, \quad \alpha_1 y(1) + \beta_1 y'(1) = \gamma_1, \tag{1.2}\]

or

\[\lim_{x \to 0} p(x)y'(x) = 0 \quad \text{(or } y'(0) = 0) \quad \alpha_2 y(1) + \beta_2 y'(1) = \gamma_2, \tag{1.3}\]

where \(\delta_1, \alpha_i, \beta_i\) and \(\gamma_i, i = 1, 2\) are any real constants. Here, \(p(0) = 0\) and \(q(x)\) may be discontinuous at \(x = 0\). Throughout this paper, the following conditions are assumed on \(p(x), q(x)\) and \(f(x, y)\):

\((C_1)\) \(p(x) \in C[0, 1] \cap C^1(0, 1), p(x) > 0, q(x) > 0 \in (0, 1],\)

\((C_2)\) \(\frac{1}{p(x)} \in L^1(0, 1)\) and \(\int_0^1 \frac{1}{p(x)} \int_x^1 q(s)ds dx < \infty, \) (BCs (1.2))

\((C_3)\) \(q(x) \in L^1(0, 1)\) and \(\int_0^1 \frac{1}{p(x)} \int_0^x q(s)ds dx < \infty, \) (for BCs (1.3))

\((C_4)\) \(f(x, y), f_y(x, y) \in C(\Omega)\) and \(f_y(x, y) \geq 0\) on \(\Omega\), where \(\Omega := \{(0, 1] \times \mathbb{R}\}.\)

Equation (1.1) with \(p(x) = 1, q(x) = x^{-\frac{1}{2}}\) and \(f = y^2\) is known as the Thomas-Fermi equations \([4, 5]\). The Lane-Emden is a special case of (1.1) with \(p(x) = q(x) = x^\alpha\) which has been used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents \([6]\). The Lane-Emden equation is a basic equation in the theory of stellar structure for a shape factor of \(\alpha = 2\), i.e., signifying spherical bodies. Equation (1.1) with \(p(x) = q(x) = x^2\) arises in oxygen diffusion in a spherical cell \([7, 8]\) with \(f = ny/y + k\), \(n > 0, k > 0\), and in modelling of heat conduction in human head \([9, 11]\) with \(f = \delta e^{-by}\), \(\theta > 0, \delta > 0\). Existence and uniqueness of doubly singular boundary value problems (1.1) with BCs. (1.2) and (1.3) can be found in \([1, 12, 14]\). In general, such singular problems are difficult to solve due to its singular behavior at \(x = 0\). There are several techniques to solve doubly singular boundary value problems (1.1) with BCs. (1.3) where \(p(x) = q(x) = x^\alpha\) for \(\alpha > 0\). The numerical study of doubly singular boundary value problems has been carried out for past couple of decades and still it is an active area of research to develop some better numerical schemes. So far various numerical methods such as the collocation methods \([15, 16]\), tangent chord method \([14]\), finite difference methods \([17, 19]\), spline finite difference methods \([20]\), B-Spline method \([21]\), spline method \([22]\), Chebyshev economization method \([23]\), Cubic
spline method \cite{24,26}, Adomian decomposition method (ADM) and modified ADM \cite{27-31}, ADM with Green’s function \cite{32-34}, variational iteration method (VIM) \cite{35,37}, the optimal modified VIM \cite{38}, homotopy analysis method \cite{39} and homotopy perturbation method \cite{40} and the references cited therein. Solving (1.1) using ADM or HAM is always a computationally involved task as it requires computation of unknown coefficients. In \cite{32-34}, the ADMGF was proposed to overcome the difficulties occurred in the ADM for (1.1). However, this method does not provide a mechanism to adjust and control the convergence region and rate of the series solutions.

In this paper, we use the OHAM to obtain approximate solutions of doubly singular problems (1.1). To overcome the singular behavior at the origin, the singular equation is transformed into an equivalent Fredholm integral equation and then the OHAM is applied to get approximate solutions. The most significant feature of the OHAM is the optimal control of the convergence of solutions by a convergence-control parameter which ensures a very fast convergence. In summary, the OHAM has the following advantages:

• Unlike HAM, the present approach does not require any additional computational work for unknown constants;
• Independent of small or large physical parameters;
• Guarantee of convergence;
• Flexibility on choice of base function and initial guess of solution;
• Useful analytic tool to investigate highly nonlinear problems with multiple solutions, singularity and perturbed.

2 Description of the method

2.1 The equivalent integral form of (1.1) and (1.2)

Let us consider the homogeneous version of the problem (1.1) with (1.2) as

\[
\begin{align*}
-(p(x)g(x))' &= 0, \quad x \in (0, 1), \\
g(0) &= \delta_1, \quad \alpha_1 g(1) + \beta_1 g'(1) = \gamma_1.
\end{align*}
\]  

(2.1)

Its solution is given by

\[
g(x) = \delta_1 + \frac{(\gamma_1 - \delta_1 \alpha_1)}{\mu} h(x),
\]  

(2.2)

where

\[
\mu = \alpha_1 h(1) + \beta_1 h'(1), \quad h(x) = \int_0^x \frac{dx}{p(x)}, \quad h(1) = \int_0^1 \frac{dx}{p(x)} \quad \text{and} \quad h'(1) = \frac{1}{p(1)}.
\]
Integrating (1.1) twice w.r.t $x$ first from $x$ to 1 and then from 0 to $x$ and changing the order of integration, and applying the BCs $y(0) = 0$, $\alpha_1 y(1) + \beta_1 y'(1) = 0$, we obtain

$$y(x) = -\frac{1}{\mu} \int_0^1 \alpha_1 h(x)h(s)q(s)f(s,y(s))ds + \int_0^x h(s)q(s)f(s,y(s))ds + \int_x^1 h(x)q(s)f(s,y(s))ds.$$  

Splitting the first integral into two parts from 0 to $x$ and $x$ to 1, we get

$$y(x) = -\frac{1}{\mu} \int_0^x \alpha_1 h(x)h(s)q(s)f(s,y(s))ds - \frac{1}{\mu} \int_x^1 \alpha_1 h(x)h(s)q(s)f(s,y(s))ds$$

$$+ \int_0^x h(s)q(s)f(s,y(s))ds + \int_x^1 h(x)q(s)f(s,y(s))ds.$$  

Combining the first and last, and second and third integrals, we obtain

$$y(x) = \int_0^x h(s) \left[ 1 - \frac{\alpha_1 h(x)}{\mu} \right] q(s)f(s,y(s))ds + \int_x^1 h(x) \left[ 1 - \frac{\alpha_1 h(s)}{\mu} \right] q(s)f(s,y(s))ds.$$  

(2.3)

Combining (2.2) and (2.3), we get Fredholm integral form of doubly singular boundary value problems (1.1) and (1.2) as

$$y(x) = g(x) + \int_0^1 G(x,s)q(s)f(s,y(s))ds,$$  

(2.4)

where $g(x)$ and $G(x,s)$ are given by

$$g(x) = \delta_1 + \frac{1}{\mu} (\gamma_1 - \delta_1 \alpha_1)h(x),$$  

(2.5)

$$G(x,s) = \begin{cases} 
    h(x) \left[ 1 - \frac{\alpha_1 h(s)}{\mu} \right], & 0 \leq x \leq s, \\
    h(s) \left[ 1 - \frac{\alpha_1 h(x)}{\mu} \right], & s \leq x \leq 1. 
\end{cases}$$  

(2.6)

### 2.2 The equivalent integral form of (1.1) and (1.3)

We again consider the homogeneous version of the problem (1.1) and (1.3) as

$$-(p(x)g'(x))' = 0, \quad x \in (0,1),$$

$$\lim_{x \to 0^+} p(x)g'(x) = 0, \quad \alpha_2 g(1) + \beta_2 g'(1) = \gamma_2.$$  

(2.7)
The unique solution of (2.7) is given by
\[ g(x) = \frac{\gamma_2}{\alpha_2}. \] (2.8)

Integrating (1.1) w.r.t. \( x \) first from 0 to \( x \) and then from \( x \) to 1, then changing the order of integration, and applying the BCs \( \lim_{x \to 0^+} p(x)y'(x) = 0, \alpha_2y(1) + \beta_2y'(1) = 0 \), we get
\[ y(x) = \int_0^1 \frac{\beta_2}{\alpha_2 p(1)} q(s) f(\xi, y(s)) ds + \int_0^1 \left[ \int_s^x \frac{1}{p(x)} q(s) f(s, y(s)) ds - \int_0^x \left[ \int_s^x \frac{1}{p(x)} q(s) f(s, y(s)) ds \right] ds \right]. \]

Splitting the first and second integrals into two parts from 0 to \( x \) and \( x \) to 1, we get
\[ y(x) = \int_0^1 \frac{\beta_2}{\alpha_2 p(1)} q(s) f(s, y(s)) ds + \int_x^1 \left[ \int_s^x \frac{1}{p(x)} q(s) f(s, y(s)) ds \right] ds - \int_0^x \left[ \int_s^x \frac{1}{p(x)} q(s) f(s, y(s)) ds \right] ds, \quad s > 0. \]

By combining the integrals of same limits, we obtain
\[ y(x) = \int_0^1 \left[ \int_s^x \frac{1}{p(x)} - \int_s^x \frac{1}{p(x)} + \frac{\beta_2}{\alpha_2 p(1)} \right] q(s) f(s, y(s)) ds + \int_x^1 \left[ \int_s^x \frac{1}{p(x)} + \frac{\beta_2}{\alpha_2 p(1)} \right] q(s) f(s, y(s)) ds. \] (2.9)

Combining (2.8) and (3.11), we get Fredholm integral form of doubly singular boundary value problem (1.1) and (1.3) as
\[ y(x) = g(x) + \int_0^1 G(x, s) q(s) f(s, y(s)) ds. \] (2.10)

where \( g(x) \) and \( G(x, s) \) are given by
\[ g(x) = \frac{\gamma_2}{\alpha_2}, \] (2.11)
\[ G(x, s) = \begin{cases} \int_s^x \frac{1}{p(x)} + \frac{\beta_2}{\alpha_2 p(1)}, & 0 < x \leq s, \\ \int_s^x \frac{1}{p(x)} - \int_s^x \frac{1}{p(x)} + \frac{\beta_2}{\alpha_2 p(1)}, & s \leq x \leq 1. \end{cases} \] (2.12)

or
\[ G(x, s) = \begin{cases} h(1) - h(s) + \frac{\beta_2}{\alpha_2} h'(1), & 0 < x \leq s, \\ h(1) - h(x) + \frac{\beta_2}{\alpha_2} h'(1), & s \leq x \leq 1. \end{cases} \] (2.13)
2.3 Analytical method based on homotopy analysis

The integral equations (2.4) or (3.12) may be written in the operator equation form

\[ T[y(x)] = y(x) - g(x) - \int_{0}^{1} G(x, s) q(s) f(s, y(s)) ds = 0, \quad (2.14) \]

where \( g(x) \) and \( G(x, s) \) are given by (2.5) or (3.13), respectively. Basic idea of homotopy analysis method for solving different scientific models can be found in [41–46] and optimal homotopy asymptotic method in [47–49]. According to homotopy analysis method, using \( r \in [0, 1] \) as an embedding parameter, the general zero-order deformation equation is constructed as

\[ (1 - q)[\phi(x; r) - y_0(x)] = r c_0 T[\phi(x; r)], \quad (2.15) \]

where \( y_0(x) \) denotes an initial guess for the exact solution \( y(x) \), \( c_0 \neq 0 \) is convergence-controller parameter, \( \phi(x; r) \) is an unknown function and \( N[\phi(x; r)] \) is given by

\[ N[\phi(x; r)] = \phi(x; r) - g(x) - \int_{0}^{1} G(x, s) q(s) f(s, \phi(s; r)) ds = 0. \quad (2.16) \]

When \( r = 0 \), the zero-order deformation \( 2.15 \) becomes \( \phi(x; 0) = y_0(x) \), and when \( r = 1 \), it leads to \( T[\phi(x; 1)] = 0 \), which is exactly the same as the original problem (2.14) provided that \( \phi(x; 1) = y(x) \). Expanding the function \( \phi(x; r) \) in a Taylor series with respect to the parameter \( r \), we obtain

\[ \phi(x; r) = y_0(x) + \sum_{k=1}^{\infty} y_k(x) r^k, \quad (2.17) \]

where \( y_k(x) \) is given by

\[ y_k(x) = \frac{1}{k!} \frac{\partial^k \phi(x; r)}{\partial r^k} \bigg|_{r=0}. \quad (2.18) \]

If the convergence controller parameter \( c_0 \neq 0 \) is chosen properly, the series (2.17) converges for \( r = 1 \) and it becomes

\[ \phi(x; 1) \equiv y(x) = y_0(x) + \sum_{k=1}^{\infty} y_k(x), \quad (2.19) \]

which will be one of solutions of the problem (2.14).

Defining the vector \( \vec{y}_k = \{y_0(x), y_1(x), \ldots, y_k(x)\} \) and differentiating (2.15), \( k \) times with respect to the parameter \( r \), dividing it by \( k! \), setting subsequently \( r = 0 \), we obtain the \( k \)th-order deformation equation as

\[ y_k(x) - \chi_k y_{k-1}(x) = c_0 R_k(\vec{y}_{k-1}, x), \quad (2.20) \]
where \( \chi_k \) is given by
\[
\chi_k = \begin{cases} 
0, & k = 0, 1 \\
1, & k \geq 2
\end{cases} \tag{2.21}
\]
and
\[
R_k(\overrightarrow{y}_{k-1}, x) = \frac{1}{(k-1)!} \left[ \frac{\partial^{k-1}}{\partial r^{k-1}} T \left( \sum_{j=0}^{\infty} y_j r^j \right) \right]_{r=0}
\]
\[
= y_{k-1}(x) - (1 - \chi_k) g(x) - \int_0^1 G(x, s) q(s) D_{k-1}[f(\phi)] ds \tag{2.22}
\]
where \( D_{k-1}[f(\phi)] \) is the \((k-1)\)th-order homotopy-derivative operator \([50]\) given by
\[
D_{k-1}[f(\phi)] = \frac{1}{(k-1)!} \left[ \frac{\partial^{k-1}}{\partial r^{k-1}} f \left( x, \sum_{j=0}^{\infty} y_j r^j \right) \right] \bigg|_{r=0} . \tag{2.23}
\]
Using (2.20) and (2.22), the \(k\)th-order deformation equation is simplified as
\[
y_k(x) - \chi_k y_{k-1}(x) = c_0 \left[ y_{k-1}(x) - (1 - \chi_k) g(x) - \int_0^1 G(x, s) q(s) D_{k-1}[f(\phi)] ds \right] . \tag{2.24}
\]
Using (2.24) with an initial guess \( y_0(x) = g(x) \), the solution components \( y_k(x) \) are obtained as
\[
y_1(x) = c_0 \left\{ y_0(x) - g(x) - \int_0^1 G(x, s) q(s) D_0[f(\phi)] ds \right\} \\
y_2(x) = (1 + c_0) y_1(x) - c_0 \left\{ \int_0^1 G(x, s) q(s) D_1[f(\phi)] ds \right\} \\
\vdots \\
y_k(x) = (1 + c_0) y_{k-1}(x) - c_0 \left\{ \int_0^1 G(x, s) q(s) D_{k-1}[f(\phi)] ds \right\} \quad k \geq 3 \tag{2.25}
\]
The \(M\)th-order approximate solution of the problem (2.14) is defined as
\[
\phi_M(x, c_0) = y_0(x) + \sum_{k=1}^{M} y_k(x, c_0) . \tag{2.26}
\]
Appropriate selection of the convergence control parameter \( c_0 \) has a big influence on the convergence region of series (2.19) and on the convergence rate as well \([50, 51]\). One of the methods for selecting the value of convergence control parameter is the so-called \( c_0 \)-curve and the horizontal line may be considered as the valid interval for \( c_0 \) \([42, 52]\). This method.
enables to determine the effective region of the convergence control parameter, however it does not give the possibility to determine the value ensuring the fastest convergence \[50\]. Another way to find the optimal value of the convergence-control parameter \(c_0\) is obtained by minimizing the squared residual of governing equation

\[ E_M(c_0) = \int_0^1 \left( T[\phi_M(x, c_0)] \right)^2 \, dx. \] (2.27)

The squared residual error defined by (2.27) is a kind of measurement of the accuracy of the Mth-order approximation. However, the exact squared residual error is expensive to calculate when \(M\) is large. For speed up the calculations Liao \[50, 53\] suggested to replace the integral in formula (2.27) by its approximate value obtained by applying the quadrature rules. So, we approximate \(E_M\) by the discrete averaged residual error defined by

\[ E_M(c_0) \approx \frac{1}{n} \sum_{j=1}^{n} \left( T[\phi_M(x_j, c_0)] \right)^2, \] (2.28)

where \(0 = x_1 < x_2 < \ldots x_{j-1} < x_j < \ldots < x_n = 1\) with nodal points \(x_j = jh, h = x_j - x_{j-1}, j = 1, 2, \ldots, n\). Since \(E_M(c_0)\) dependent upon \(c_0\), the optimal value is obtained by solving \(\frac{dE_M}{dc_0} = 0\), the effective region of the convergence control parameter is usually defined as \(R_{c_0} = \{c_0 : \lim_{M \to \infty} E_M(c_0) = 0\}\) and optimal value will satisfy \(E_M(\hat{c}_0) < E_M(c_0)\). Having computed the optimal value \(\hat{c}_0\) and substituting in (2.26), the approximate solution will be obtained.

### 3 Convergence analysis

In this section, we establish the convergence of method defined in (2.25) the solution of equivalent integral form (2.14) of doubly singular boundary value problems (1.1) - (1.3). Let \(X = (C[0, 1], \|y\|)\) be a Banach space with \(\|y\| = \max_{x \in [0, 1]} |y(x)|, y \in X\).

**Theorem 3.1.** Let \(0 < \delta < 1\) and the solution components \(y_0(x), y_1(x), y_2(x), \ldots\) obtained by (2.25) satisfy the following condition:

\[ \exists \ k_0 \in \mathbb{N} \ \forall \ k \geq k_0 : \|y_{k+1}\| \leq \delta \|y_k\|, \] (3.1)

then the series solution \(\sum_{k=0}^{\infty} y_k(x)\) is convergent.

**Proof.** Define the sequence \(\{\phi_n\}_{n=0}^{\infty}\) as,

\[
\begin{align*}
\phi_0 &= y_0(x) \\
\phi_1 &= y_0(x) + y_1(x) \\
\phi_2 &= y_0(x) + y_1(x) + y_2(x) \\
& \quad \vdots \\
\phi_n &= y_0(x) + y_1(x) + y_2(x) + \cdots + y_n(x)
\end{align*}
\] (3.2)
and we show that is a Cauchy sequence in the Banach space $X$. For this purpose, consider
\[ \| \phi_{n+1} - \phi_n \| = \| y_{n+1} \| \leq \delta \| y_n \| \leq \delta^2 \| y_{n-1} \| \leq \cdots \leq \delta^{n-k_0+1} \| y_{k_0} \|. \]
For every $n, m \in \mathbb{N}$, $n \geq m > k_0$, we have
\[ \| \phi_n - \phi_m \| = \| (\phi_n - \phi_{n-1}) + (\phi_{n-1} - \phi_{n-2}) + \cdots + (\phi_{m+1} - \phi_m) \| \]
\[ \leq \| \phi_n - \phi_{n-1} \| + \| \phi_{n-1} - \phi_{n-2} \| + \cdots + \| \phi_{m+1} - \phi_m \| \]
\[ \leq (\delta^{n-k_0} + \delta^{n-k_0-1} + \cdots + \delta^{m-k_0+1}) \| y_{k_0} \| \]
\[ = \frac{1 - \delta^{n-m}}{1 - \delta} \delta^{m-k_0+1} \| y_{k_0} \|. \] (3.3)
and since $0 < \delta < 1$ so it follows that
\[ \lim_{n,m \to \infty} \| \phi_n - \phi_m \| = 0. \] (3.4)
Therefore, \( \{ \phi_n \}_{n=0}^{\infty} \) is a Cauchy sequence in the Banach space $X$ and it implies that the series solution defined in (2.26) converges. This completes the proof of Theorem 3.1.

**Theorem 3.2.** Assume that the series solution \( \sum_{k=0}^{\infty} y_k(x) \) defined in (2.26), is convergent to the solution $y(x)$. If the truncated series \( \phi_M(x, c_0) = \sum_{m=0}^{M} y_m(x, c_0) \) is used as an approximation to the solution $y(x)$ of the problem (2.14), then the maximum absolute truncated error is estimated as
\[ |y(x) - \phi_M(x, c_0)| \leq \frac{1}{1 - \delta} \delta^{M-k_0+1} \| y_{k_0} \|. \] (3.5)
**Proof.** From Theorem 3.1 following inequality (3.3), we have
\[ \| \phi_n - \phi_M(x, c_0) \| \leq \frac{1 - \delta^{n-M}}{1 - \delta} \delta^{M-k_0+1} \| y_{k_0} \|, \]
for $n \geq M$. Now, as $n \to \infty$ then $\phi_n \to y$ and $\delta^{n-M} \to 0$. So,
\[ \| y(x) - \phi_M(x, c_0) \| \leq \frac{1}{1 - \delta} \delta^{M-k_0+1} \| y_{k_0} \|. \] (3.6)
Theorems 3.1 and 3.2 together confirm that the convergence of series solution (2.26).

Now, we discuss about the uniqueness of the solution of problem (2.14). The operator equation form of (2.14) is written as
\[ y(x) = g(x) + \int_{0}^{1} G(x, s)q(s)f(s, y(s))ds. \] (3.7)
**Theorem 3.3.** Let $0 < \delta < 1$ and suppose there exists a constant $l > 0$ such that
\[ |f(x, z_1) - f(x, z_2)| \leq L|z_1 - z_2| \quad \forall (x, z_1), (x, z_2) \in \Omega. \] (3.8)
Then there exists one, and only one, solution $y(x)$ of equation (3.7) in $X$. 

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Proof. Let us assume that there are two solutions \( z_1(x), z_1(x) \in X \) of the problem (3.7)

\[
\|z_1 - z_2\| = \max_{x \in [0,1]} |z_1(x) - z_2(x)| = \max_{x \in [0,1]} \left| \int_0^1 G(x, s)q(s)\left(f(s, z_1(s)) - f(s, z_2(s))\right)ds \right|
\]

\[
\leq \max_{s \in [0,1]} |f(s, z_1(s)) - f(s, z_2(s))| \left( \max_{x \in [0,1]} \int_0^1 |G(x, s)q(s)ds| \right)
\]

using (3.8), the above inequality reduce to

\[
\|z_1 - z_2\| \leq LM \max_{s \in [0,1]} |z_1(s) - z_2(s)| = \delta \|z_1 - z_2\|,
\]

setting \( \delta = LM \) and \( M := \max_{x \in [0,1]} \int_0^1 |G(x, s)q(s)ds| \), we obtain

\[
\|z_1 - z_2\| \leq \delta \|z_1 - z_2\|,
\]

since \( 0 < \delta < 1 \), the equality \( z_1 = z_2 \) must hold. This means equation (3.7) has a unique solution in \( X \).

In the following theorem we show that the series defined in (2.26) is convergent, where the solution components \( y_k(x) \) are obtained from (2.25), then it must be a solution of the integral of (3.7).

**Theorem 3.4.** Assume that the series solution \( \sum_{k=0}^{\infty} y_k(x) \) defined in (2.26), is convergent to the solution \( y(x) \) then it must be a solution of the integral of (3.7).

Proof. Since \( \sum_{k=0}^{\infty} y_k(x) \) is convergent, then

\[
\lim_{n \to \infty} y_n(x) = 0, \quad \forall \ x \in [0,1].
\]

(3.9)

By summing up the left hand-side of (2.20), we get

\[
\sum_{k=1}^{n} \left[y_k(x) - \chi_k y_{k-1}(x)\right] = y_1(x) + \ldots + (y_n(x) - y_{n-1}(x)) = y_n(x).
\]

(3.10)

Letting \( n \to \infty \) and using (3.9), equation (2.8) reduces to

\[
\sum_{k=1}^{\infty} \left[y_k(x) - \chi_k y_{k-1}(x)\right] = \lim_{n \to \infty} y_n(x) = 0.
\]

(3.11)

Using (3.11) and right hand-side of the relation (2.20), we obtain

\[
\sum_{k=1}^{\infty} c_0 R_k(\tilde{f}_{k-1}, x) = \sum_{k=1}^{\infty} \left[y_k(x) - \chi_k y_{k-1}(x)\right] = 0.
\]

(3.12)
Since \( c_0 \neq 0 \), then equation (3.12) reduces to
\[
\sum_{k=1}^{\infty} R_k \left( \vec{y}_{k-1}, x \right) = 0.
\]
(3.13)

Using (3.13) and (2.22), we have
\[
0 = \sum_{k=1}^{\infty} R_k \left( \vec{y}_{k-1}, x \right) = \sum_{m=1}^{\infty} \left[ y_{k-1}(x) - g(x) - \int_{0}^{1} G(x, s) q(s) \mathcal{D}_{k-1}[f(s, \phi)] ds \right]
\]
\[
= \sum_{k=1}^{\infty} y_{k-1}(x) - g(x) - \int_{0}^{1} G(x, s) q(s) \sum_{k=1}^{\infty} \mathcal{D}_{k-1}[f(s, \phi)] ds,
\]

since \( \sum_{k=0}^{\infty} y_k(x) \) converges to \( y(x) \), then \( \sum_{k=0}^{\infty} \mathcal{D}_{k-1}[f(x, \phi)] \) converges to \( f(x, y) \) \([54]\),
\[
y(x) = g(x) + \int_{0}^{1} G(x, s) q(s) f(s, y(s)) ds.
\]

Hence, \( y(x) \) is the exact solution of integral equation (3.7).

4 Numerical results

To examine the accuracy and applicability of the OHAM, we consider five examples of singular boundary value problem. All of the computations have been performed using MATHEMATICA. Here, \( y(x) \), \( \phi_M(x) \) and \( \psi_M(x) = \phi_M(x, -1) \) denote the exact, OHAM, and ADMGF solutions, respectively.

Problem 4.1. Doubly Singular Boundary Value Problem \([32, 34]\)

Consider nonlinear doubly singular boundary value problems
\[
\begin{align*}
(x^\alpha y'(x))' &= x^{\alpha+\beta-2} \left[ \beta \left( \beta x^\beta e^{2y} - e^y(\alpha + \beta - 1) \right) \right], \quad 0 < x < 1, \\
y(0) &= \ln \left( \frac{1}{4} \right), \quad y(1) = \ln \left( \frac{1}{5} \right), \quad 0 < \alpha < 1, \quad \beta > 0.
\end{align*}
\]

Here, \( p(x) = x^\alpha \), \( q(x) = x^{\alpha+\beta-2} \), \( f(x, y) = \left[ \beta \left( \beta x^\beta e^{2y} - e^y(\alpha + \beta - 1) \right) \right] \) with \( \delta_1 = \ln \left( \frac{1}{4} \right), \alpha_1 = 1, \beta_1 = 0 \) and \( \gamma_1 = \ln \left( \frac{1}{5} \right) \). Its exact solution is \( y(x) = \ln \left( \frac{1}{4+x^\beta} \right) \). Applying OHAM \([2.24]\) with an initial guess \( y_0(x) = \ln \left( \frac{1}{4} \right) \), we get the approximate solution \( \phi_M(x, c_0) \). Using the formula (2.28), we obtain optimal values, \( \tilde{c}_0 = [-0.970001; -0.970011] \) (for \( \alpha = 0.5, \beta = 1 \)) with \( M = 5, 10 \), respectively. We define absolute error as
\[
E^M_\alpha = |y(x) - \phi_M(x)| \quad \text{and} \quad e^M_\alpha = |y(x) - \psi_M(x)|, \quad x \in [0, 1].
\]

The numerical results of absolute errors and approximate solutions are shown in Table 1. One can see that OHAM method provides better results compared with ADMGF method.
Consider nonlinear singular boundary value problems
\[
\begin{align*}
(x^2y'(x))' &= \sigma^2 x^2 y^n(x), \quad 0 < x < 1, \\
y'(0) &= 0, \quad y(1) = 1,
\end{align*}
\]
Here, \( p(x) = q(x) = x^2 \), \( f(x, y) = \sigma^2 y^n(x) \) with \( \alpha_2 = 1 \), \( \beta_2 = 0 \) and \( \gamma_2 = 1 \). Applying OHAM (2.24) with an initial guess \( y_0(x) = 1 \), we get the approximate solution \( \phi_M(x, c_0) \).

Using the formula (2.28), we obtain optimal values, \( \hat{c}_0 = [-0.8929193; -0.8712345] \), (for \( n = 1.5, \sigma = 1 \)), \( \hat{c}_0 = [-0.6890655; -0.6666666] \) (for \( n = 2, \sigma = 1.5 \)) and \( \hat{c}_0 = [-0.5723102; -0.4809289] \) (for \( n = 2, \sigma = 2 \)) with iterations \( M = 5, 10 \), respectively. Since exact solution is not known so we define the absolute residual error as
\[
E_{res}^M(x) = |(x^2\phi'_M(x))' - \sigma^2 x^2 \phi^n_M(x)|, \quad e_{res}^M(x) = |(x^2\psi'_M(x))' - \sigma^2 x^2 \psi^n_M(x)|.
\]
The numerical results of the absolute residual errors and the approximate solutions are shown in Tables 2,3 and 4. One can observe that the residual error not converging to zero with the increase in \( \sigma \) and \( n \) by ADMGF technique whereas the proposed method OHAM gives stable solution and converges to exact solution.

**Problem 4.3. Distribution of Heat Sources in the Human Head** [3,11]

Consider singular boundary value problems [33]
\[
\begin{align*}
-(x^2y'(x))' &= \delta x^2 e^{-y(x)} \quad 0 < x < 1, \\
y'(0) &= 0, \quad \alpha_2 y(1) + \beta_2 y'(1) = \gamma_2,
\end{align*}
\]
Here, \( p(x) = q(x) = x^2 \), \( f(x, y) = \delta e^{-y(x)} \) and \( \delta = 1 \). Applying the OHAM (2.24) with an initial guess \( y_0(x) = 0 \), we get the approximate solution \( \phi_M(x, c_0) \).

Using the formula (2.28), we obtain optimal values, \( \hat{c}_0 = [-0.6842013; -0.666463] \) (for \( \alpha_2 = \beta_2 = 1, \gamma_2 = 0 \)) and \( \hat{c}_0 = [-0.7759493; -0.7701234] \) (for \( \alpha_2 = 2, \beta_2 = 1, \gamma_2 = 0 \)) with iterations \( M = 5 \).
Table 2 Numerical results of residual error and solutions of Problem 4.2 for \( n = 1.5, \sigma = 1 \)

| \( x \) | \( e_{\text{res}}^{5} \) | \( E_{\text{res}}^{5} \) | \( e_{\text{res}}^{10} \) | \( E_{\text{res}}^{10} \) | \( \psi_{10} \) | \( \phi_{10} \) |
|---|---|---|---|---|---|---|
| 0.1 | 5.20E-04 | 7.56E-06 | 3.75E-07 | 1.22E-13 | 0.859202 | 0.859202 |
| 0.2 | 4.88E-04 | 7.15E-06 | 3.50E-07 | 4.96E-12 | 0.863188 | 0.863188 |
| 0.3 | 4.38E-04 | 6.52E-06 | 3.10E-07 | 1.34E-11 | 0.879303 | 0.879303 |
| 0.4 | 3.74E-04 | 5.72E-06 | 2.61E-07 | 2.56E-11 | 0.879303 | 0.879303 |
| 0.5 | 3.02E-04 | 4.79E-06 | 2.07E-07 | 4.03E-11 | 0.891566 | 0.891566 |
| 0.6 | 2.27E-04 | 3.66E-06 | 1.53E-07 | 9.18E-11 | 0.906766 | 0.906766 |
| 0.7 | 1.56E-04 | 2.08E-06 | 1.03E-07 | 4.09E-11 | 0.925033 | 0.925033 |
| 0.8 | 9.25E-05 | 5.69E-07 | 5.95E-08 | 4.75E-11 | 0.946527 | 0.946527 |
| 0.9 | 3.98E-05 | 5.62E-06 | 2.50E-08 | 2.56E-11 | 0.971441 | 0.971441 |

Table 3 Results of residual error and solutions of Problem 4.2 for \( n = 2, \sigma = 1.5 \)

| \( x \) | \( e_{\text{res}}^{5} \) | \( E_{\text{res}}^{5} \) | \( e_{\text{res}}^{10} \) | \( E_{\text{res}}^{10} \) | \( \psi_{10} \) | \( \phi_{10} \) |
|---|---|---|---|---|---|---|
| 0.1 | 3.69E-01 | 1.21E-03 | 1.34E-01 | 9.57E-08 | 0.759370 | 0.750609 |
| 0.2 | 3.46E-01 | 1.14E-03 | 1.25E-01 | 1.88E-07 | 0.765211 | 0.756965 |
| 0.3 | 3.10E-01 | 1.03E-03 | 1.10E-01 | 3.33E-07 | 0.775144 | 0.767704 |
| 0.4 | 2.64E-01 | 8.79E-04 | 9.13E-02 | 5.10E-07 | 0.789464 | 0.783048 |
| 0.5 | 2.13E-01 | 6.84E-04 | 7.16E-02 | 6.56E-07 | 0.808584 | 0.803324 |
| 0.6 | 1.60E-01 | 4.01E-04 | 5.24E-02 | 9.30E-07 | 0.833035 | 0.828980 |
| 0.7 | 1.10E-01 | 7.74E-05 | 3.50E-02 | 1.88E-07 | 0.863480 | 0.860603 |
| 0.8 | 6.53E-02 | 9.87E-04 | 2.03E-02 | 3.12E-06 | 0.900738 | 0.898953 |
| 0.9 | 2.85E-02 | 2.82E-03 | 8.71E-03 | 1.21E-05 | 0.945823 | 0.945003 |

Table 4 Results of residual error and solutions of Problem 4.2 for \( n = 2, \sigma = 2 \)

| \( x \) | \( e_{\text{res}}^{5} \) | \( E_{\text{res}}^{5} \) | \( e_{\text{res}}^{10} \) | \( E_{\text{res}}^{10} \) | \( \psi_{10} \) | \( \phi_{10} \) |
|---|---|---|---|---|---|---|
| 0.1 | 0.083 | 1.63E-04 | 1.140 | 9.69E-07 | 4.265570 | 0.641604 |
| 0.2 | 0.316 | 6.14E-04 | 4.160 | 4.10E-06 | 4.060708 | 0.649873 |
| 0.3 | 0.650 | 1.23E-03 | 7.990 | 9.92E-06 | 3.741832 | 0.663942 |
| 0.4 | 1.013 | 1.87E-03 | 11.37 | 1.15E-05 | 3.440492 | 0.684262 |
| 0.5 | 1.317 | 2.30E-03 | 13.26 | 1.26E-05 | 2.887879 | 0.715090 |
| 0.6 | 1.482 | 2.17E-03 | 13.32 | 1.83E-05 | 2.424517 | 0.746636 |
| 0.7 | 1.444 | 6.53E-04 | 11.30 | 6.65E-05 | 1.981583 | 0.790944 |
| 0.8 | 1.176 | 4.45E-03 | 8.071 | 4.25E-04 | 1.585305 | 0.846200 |
| 0.9 | 0.684 | 1.89E-02 | 4.130 | 1.67E-04 | 1.254310 | 0.914796 |
and $M = 10$, respectively. Since exact solution is not known so we define the absolute residual errors as

$$E^M_{\text{res}}(x) = \left|(x^2 \phi'_M(x))' + \delta x^2 e^{-\phi_M(x)}\right|, \quad e^M_{\text{res}}(x) = \left|(x^2 \psi'_M(x))' + \delta x^2 e^{-\psi_M(x)}\right|.$$  

The numerical results are given in Tables 5 and 6. We observe that that the residual error not converging to zero by ADMGF method whereas the OHAM gives stable solution and converges to exact solution.

Table 5 Results of residual error and solutions of Problem 4.3 when $\alpha^2 = \beta^2 = 1$, $\gamma^2 = 0$

| $x$  | $e^5_{\text{res}}$ | $E^5_{\text{res}}$ | $e^{10}_{\text{res}}$ | $E^{10}_{\text{res}}$ | $\psi_{10}$ | $\phi_{10}$ |
|------|-------------------|--------------------|---------------------|---------------------|-------------|-------------|
| 0.1  | 1.18E-01          | 2.05E-04           | 8.05E-02            | 2.41E-06            | 0.3442719   | 0.3663613   |
| 0.2  | 1.15E-01          | 1.87E-04           | 7.87E-02            | 2.40E-06            | 0.3442719   | 0.3663613   |
| 0.3  | 1.12E-01          | 1.57E-04           | 7.58E-02            | 2.38E-06            | 0.3442719   | 0.3663613   |
| 0.4  | 1.07E-01          | 1.15E-04           | 7.19E-02            | 2.37E-06            | 0.3442719   | 0.3663613   |
| 0.5  | 1.01E-01          | 6.13E-05           | 6.73E-02            | 2.36E-06            | 0.3442719   | 0.3663613   |

Table 6 Results of residual error and solutions of Problem 4.3 when $\alpha^2 = 2$, $\beta^2 = 1$, $\gamma^2 = 0$

| $x$  | $e^5_{\text{res}}$ | $E^5_{\text{res}}$ | $e^{10}_{\text{res}}$ | $E^{10}_{\text{res}}$ | $\psi_{10}$ | $\phi_{10}$ |
|------|-------------------|--------------------|---------------------|---------------------|-------------|-------------|
| 0.1  | 1.35E-02          | 6.11E-05           | 8.11E-04            | 9.12E-08            | 0.2686241   | 0.2687568   |
| 0.2  | 1.31E-02          | 5.54E-05           | 7.77E-04            | 8.83E-08            | 0.2686241   | 0.2687568   |
| 0.3  | 1.24E-02          | 4.62E-05           | 7.25E-04            | 8.39E-08            | 0.2686241   | 0.2687568   |
| 0.4  | 1.14E-02          | 3.37E-05           | 6.58E-04            | 7.87E-08            | 0.2686241   | 0.2687568   |
| 0.5  | 1.03E-02          | 1.82E-05           | 5.80E-04            | 7.35E-08            | 0.2686241   | 0.2687568   |
| 0.6  | 9.08E-03          | 2.51E-07           | 4.98E-04            | 6.93E-08            | 0.2686241   | 0.2687568   |
| 0.7  | 7.77E-03          | 1.22E-07           | 4.16E-04            | 6.72E-08            | 0.2686241   | 0.2687568   |
| 0.8  | 6.46E-03          | 4.92E-05           | 3.38E-04            | 6.87E-08            | 0.2686241   | 0.2687568   |
| 0.9  | 5.21E-03          | 8.36E-05           | 2.68E-04            | 7.65E-08            | 0.2686241   | 0.2687568   |

Problem 4.4. Oxygen Diffusion in a Spherical Cell [7, 8, 56]

Consider the following nonlinear singular boundary value problem:

$$\begin{aligned}
(x^2 y'(x))' &= n x^2 \frac{y(x)}{y'(x) + k}, & 0 < x < 1 \\
y'(0) &= 0, & 5y(1) + y'(1) = 5.
\end{aligned}$$

Here, $p(x) = q(x) = x^2$, $f(x, y) = n \frac{y(x)}{y'(x) + k}$ with $n = 0.76129$, $k = 0.03119$, $\alpha^2 = \gamma^2 = 5$ and $\beta^2 = 1$ as in [33, 36]. Applying the OHAM (2.24) with an initial guess $u_0(x) = 1$, we get the approximation to solution $\phi_M(x, c_0)$. Using the formula (2.28), we obtain optimal
values \( \hat{c}_0 = [-1.045949; -1.010201] \) with \( M = 5, 10 \), respectively. Since exact solution is not known so we define the absolute residual error as

\[
E_{res}^M(x) = \left| x^2 \phi_M'(x) - x^2 n \frac{\phi_M'(x)}{\phi_M(x) + k} \right|, \quad E_{res}^M(x) = \left| x^2 \psi_M'(x) - x^2 n \frac{\psi_M'(x)}{\psi_M(x) + k} \right|.
\]

The numerical results are presented in Table 7. From the numerical results we observe that OHAM give slightly better results compared to ADMGF method.

| \( x \) | \( e_{res}^5 \) | \( E_{res}^5 \) | \( e_{res}^{10} \) | \( E_{res}^{10} \) | \( \psi_{10} \) | \( \phi_{10} \) |
|---|---|---|---|---|---|---|
| 0.1 | 2.80E-06 | 7.95E-07 | 1.95E-10 | 1.04E-10 | 0.829706092 | 0.829706092 |
| 0.2 | 2.49E-06 | 6.94E-07 | 1.50E-10 | 7.76E-11 | 0.833374734 | 0.833374734 |
| 0.3 | 2.03E-06 | 5.54E-07 | 9.42E-11 | 4.61E-11 | 0.839489914 | 0.839489914 |
| 0.4 | 1.50E-06 | 4.07E-07 | 4.56E-11 | 2.00E-11 | 0.848052785 | 0.848052785 |
| 0.5 | 9.93E-07 | 2.83E-07 | 1.45E-11 | 4.59E-12 | 0.859064927 | 0.859064927 |
| 0.6 | 5.66E-07 | 2.01E-07 | 6.60E-13 | 1.23E-12 | 0.872528320 | 0.872528320 |
| 0.7 | 2.63E-07 | 1.59E-07 | 2.52E-12 | 1.82E-12 | 0.888453060 | 0.888453060 |
| 0.8 | 8.70E-08 | 1.49E-07 | 1.77E-12 | 9.98E-13 | 0.906818548 | 0.906818548 |
| 0.9 | 1.13E-08 | 1.52E-07 | 7.45E-13 | 4.01E-13 | 0.927650988 | 0.927650988 |

**Problem 4.5. Perturbed Second Kind Lane-Emden Equation**

Consider the following perturbed singular boundary value problem:

\[
\begin{align*}
-(x^\alpha y'(x))' & = \delta \ x^\alpha \exp \left( \frac{y(x)}{1 + \epsilon y(x)} \right), \quad 0 < x < 1 \\
y'(0) & = 0, \quad 2 y(1) + y'(1) = 0.
\end{align*}
\]

Here, \( p(x) = q(x) = x^\alpha, f(x, y) = \delta \exp \left( \frac{y(x)}{1 + \epsilon y(x)} \right) \) with \( \alpha_2 = 2, \beta_2 = 1 \) and \( \gamma_2 = 0 \) as in [38]. Applying the OHAM (2.24) with an initial guess \( u_0(x) = 0 \), we get the approximation \( \phi_M(x, c_0) \). Using the formula defined by (2.28), we obtain optimal values \( \hat{c}_0 = [-0.432512; -0.381111; -0.284943] \) (for \( \alpha = 1 \)) and \( \hat{c}_0 = [-0.608235; -0.471209; -0.381567] \) (for \( \alpha = 2 \)) with \( (\epsilon = 5, 10, 15), M = 10 \), respectively. Since exact solution is not known so we define the absolute residual errors as

\[
E_{res}^M(x) = \left| x^\alpha \phi_M'(x) + x^\alpha \delta \exp \left( \frac{\phi_M(x)}{1 + \epsilon \phi_M(x)} \right) \right|, \quad E_{res}^M(x) = \left| x^\alpha \psi_M'(x) + x^\alpha \delta \exp \left( \frac{\psi_M(x)}{1 + \epsilon \psi_M(x)} \right) \right|.
\]

In Tables 8 and 9, we consider the influence of \( \epsilon \) on the residual error for \((\epsilon = 5, 10, 15)\) with \( \alpha = 2 \) and \( M = 10 \). In each cases, we observe that the residual error not converging to zero with an increases in \( \epsilon \) by ADMGF technique whereas the OHAM gives stable solution and converges to exact solution.
Table 8 Numerical results of absolute residual error Problem 4.5 when \( \alpha = 1, \, \delta = 1 \)

| \( x \) | \( \epsilon = 5 \) | \( \epsilon = 10 \) | \( \epsilon = 15 \) |
|-------|------------------|------------------|------------------|
|       | \( e_{res}^{10} \) | \( E_{res}^{10} \) | \( e_{res}^{10} \) | \( E_{res}^{10} \) | \( e_{res}^{10} \) | \( E_{res}^{10} \) |
| 0.1   | 96.820 3.79E-04 | 1190.90 3.95E-05 | 173751.94 1.73E-03 |
| 0.2   | 200.650 4.41E-04 | 2334.05 2.93E-04 | 299967.18 1.11E-03 |
| 0.3   | 312.307 2.20E-05 | 3295.97 7.10E-04 | 350180.75 8.77E-03 |
| 0.4   | 423.542 1.03E-03 | 3858.23 8.85E-04 | 324708.56 1.11E-03 |
| 0.5   | 519.051 2.45E-03 | 3823.73 2.47E-04 | 248739.93 1.52E-02 |
| 0.6   | 582.390 4.01E-03 | 3159.91 1.51E-03 | 157883.64 2.08E-02 |
| 0.7   | 602.770 5.44E-03 | 2068.78 4.10E-03 | 081587.42 2.52E-02 |
| 0.8   | 579.289 6.62E-03 | 0914.41 6.73E-03 | 033136.89 2.90E-02 |
| 0.9   | 520.645 7.53E-03 | 0040.79 8.66E-03 | 010098.14 3.26E-02 |

Table 9 Numerical results of absolute residual error Problem 4.5 when \( \alpha = 2, \, \delta = 1 \)

| \( x \) | \( \epsilon = 5 \) | \( \epsilon = 10 \) | \( \epsilon = 15 \) |
|-------|------------------|------------------|------------------|
|       | \( e_{res}^{10} \) | \( E_{res}^{10} \) | \( e_{res}^{10} \) | \( E_{res}^{10} \) | \( e_{res}^{10} \) | \( E_{res}^{10} \) |
| 0.1   | 380.762 2.28E-03 | 1279.93 1.06E-03 | 29880.52 4.02E-03 |
| 0.2   | 332.801 1.83E-03 | 1129.84 4.31E-04 | 25360.17 6.66E-04 |
| 0.3   | 264.086 1.09E-03 | 0917.57 3.92E-04 | 19118.32 3.28E-03 |
| 0.4   | 187.844 1.19E-03 | 0686.12 1.18E-03 | 12594.80 6.41E-03 |
| 0.5   | 116.761 9.99E-04 | 0474.09 1.77E-03 | 07033.77 8.11E-03 |
| 0.6   | 059.804 2.13E-03 | 0305.58 2.12E-03 | 03128.11 8.61E-03 |
| 0.7   | 020.680 3.14E-03 | 0187.06 2.24E-03 | 00928.97 8.56E-03 |
| 0.8   | 001.774 3.97E-03 | 0111.44 2.21E-03 | 00018.88 8.44E-03 |
| 0.9   | 011.741 4.57E-03 | 0065.80 2.08E-03 | 00178.08 8.39E-03 |
5 Conclusions

In this paper, we have examined the doubly singular boundary value problems with Dirichlet/Neumann boundary conditions at $x = 0$ and Robin type boundary conditions at $x = 1$, arising in the reaction-diffusion process in a porous spherical catalyst [55], oxygen diffusion in a spherical cell [7], heat sources in the human head [11] and the perturbed second kind Lane-Emden equation is used in modelling a thermal explosion [57]. Due to the presence of singularity at $x = 0$ as well as discontinuity of $q(x)$ at $x = 0$, these problems pose difficulties in obtaining their solutions. In this paper, a new formulation of the singular boundary value problems has been presented. To overcome the singular behavior at the origin, with the help of Green’s function theory the problem has been transformed into an equivalent Fredholm integral equation. Then the optimal homotopy analysis method is applied to solve integral form of problem. The optimal control-convergence parameter involved in the components of the series solution has been obtained by minimizing the squared residual error equation. For speed up the calculations, the discrete averaged residual error has been used to obtain optimal value of the adjustable parameter $c_0$ to control the convergence of solution. Numerical results obtained by OHAM are better than the results obtained by the ADMGF [33] and are in good agreement with exact solutions, as shown in Tables 1-9. Unlike ADMGF [33], the OHAM always gives fast convergent series solution as shown in Tables. Convergence analysis and error estimate of the proposed method have been discussed. The proposed method has successfully applied to the perturbed second kind Lane-Emden Equation [57] whereas other method fails to give convergent series solution as shown in Tables 8 and 9.

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