Approximation theorems for Pascali systems

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ABSTRACT
Based on the Runge theorem for generalized analytic vectors proved by Goldschmidt in 1979, we provide a Mergelyan-type and a Carleman-type approximation theorems for solutions of Pascali systems.

1. Introduction

Given a domain $\Omega \subset \mathbb{C}$ the Pascali system on $\Omega$ is an elliptic system which can be written in the following normalized form:

$$w_\zeta + B_1 w + B_2 \overline{w} = 0,$$

where $w: \Omega \rightarrow \mathbb{C}^n$ is a complex vector function, while $B_1$ and $B_2$ are $n \times n$ matrix functions. In the scalar case $n = 1$, the fundamental theory for such systems was developed by Bers [1] and Vekua [2], thus Equation (1) is often called the Bers-Vekua equation. In the case $n \geq 2$, the pioneering work was done by Pascali [3]. Furthermore, solutions of such a system are a subclass of generalized analytic vectors, which correspond to elliptic systems with vanishing Beltrami coefficient [4–7]. We denote them by $\mathcal{O}_B(\Omega)$. In general, their regularity depends on the regularity of $B_1$ and $B_2$ (see the preliminary section), however, in our theorems, we work with smooth matrix functions and thus elements of the set $\mathcal{O}_B(\Omega)$ are smooth vector functions.

We shall focus on approximation theorems that originate from the classical complex function theory (see, e.g. [8]). We rely on a version of Runge approximation theorem for generalized analytic vectors, which was proved by Goldschmidt [6, Theorem 3.1], and the recent work on Pascali systems by Sukhov and Tumanov [9, § 3]. All our results are valid for $n \geq 1$. 

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We prove a Mergelyan-type approximation theorem for maps from compact admissible sets: Let $\Omega \subset \mathbb{C}$ be an open subset and let $S \subset \Omega$ be a compact subset. Recall that the set $S$ is Runge in $\Omega$ if $\Omega \setminus S$ has no relatively compact components. We call $S$ admissible if it is Runge in $\Omega$ and if it can be written as $S = K \cup E$, where $K$ is a finite union of pairwise disjoint compact domains in $\mathbb{C}$ with smooth boundary and $E = \overline{S \setminus K}$ is a union of finitely many pairwise disjoint smooth Jordan arcs and closed Jordan curves meeting $K$ only in their endpoints (if at all) and such that their intersection with the boundary $bK$ is transverse.

Our first result is the following Mergelyan-type theorem.

**Theorem 1.1:** Let $\Omega \subset \mathbb{C}$ be an open subset, let $S \subset \Omega$ be an admissible compact set, and let $B_1$ and $B_2$ be $n \times n$ matrix functions with coefficients in $C^\infty(\Omega)$. Given $\epsilon > 0$ and a smooth map $f : S \to \mathbb{C}^n$ whose restriction to the interior $\overset{\circ}{S}$ lies in $O_B(\overset{\circ}{S})$, there exists $w \in O_B(\Omega)$ such that

$$|f(\zeta) - w(\zeta)| < \epsilon$$

for all $\zeta \in S$.

A similar version of such an approximation theorem on admissible sets was recently obtained for conformal minimal immersions, null curves [10,11] and for holomorphic Legendrian curves [11,12].

Carleman’s approximation theorem asserts that given a continuous function $f : \mathbb{R} \to \mathbb{C}$ and a strictly positive function $\epsilon : \mathbb{R} \to (0, \infty)$ there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ such that $|f(\zeta) - g(\zeta)| < \epsilon(\zeta)$ for every $\zeta \in \mathbb{R}$. An analogue of this statement was recently proved for conformal minimal immersions [13]. We provide a version for solutions of Pascali systems.

**Theorem 1.2:** Let $B_1$ and $B_2$ be $n \times n$ matrix functions with coefficients in $C^\infty(\mathbb{C})$. Given a continuous map $f : \mathbb{R} \to \mathbb{C}^n$ and a strictly positive function $\epsilon : \mathbb{R} \to (0, \infty)$ there exists $w \in O_B(\mathbb{C})$ such that

$$|f(\zeta) - w(\zeta)| < \epsilon(\zeta)$$

for all $\zeta \in \mathbb{R}$.

### 2. Preliminaries

Let $p > 2$ and assume that the coefficients of $B_1$ and $B_2$ belong to the Lebesgue class $L^p_{\text{loc}}(\Omega)$. Then one can consider weak solutions of (1) defined as vector functions $w \in L^q_{\text{loc}}(\Omega, n)$, $\frac{1}{p} + \frac{1}{q} = 1$, satisfying the condition

$$\int\int_{\Omega} (w^T \cdot \varphi_{\zeta} - (B_1 w + B_2 \overline{w})^T \cdot \varphi) \, d\zeta \, d\overline{\zeta} = 0$$

for all smooth vector functions $\varphi \in C_0^\infty(\Omega, n)$ with compact support (note that here the minus sign arises from the integration by parts). We denote by $O_B^*_{\text{loc}}(\Omega) \subset L^q_{\text{loc}}(\Omega, n)$ the subset of all such solutions. However, using the standard bootstrapping arguments, it turns out that $O_B^*_{\text{loc}}(\Omega)$ indeed lies in the Sobolev class $W^{1,p}_{\text{loc}}(\Omega, n)$. That is, the $L^p_{\text{loc}}$-regularity of $B_1$ and $B_2$ implies the $W^{1,p}_{\text{loc}}$-regularity of $w \in O_B^*_{\text{loc}}(\Omega)$. Similarly, we can take $k \in \mathbb{N}_0$ and $0 < \alpha < 1$ and introduce the Hölder spaces $C^{k,\alpha}(\Omega)$ and $C^{k,\alpha}(\Omega, n)$. Again, the ellipticity of the Pascali system implies that if the coefficients of $B_1$ and $B_2$ belong to $C^{k,\alpha}(\Omega)$, the solutions...
of (1) which are at least $L^q_{\text{loc}}$-regular automatically belong to the class $C^{k+1,\alpha}(\Omega, n)$. As indicated in the introduction, we denote the set of all differentiable solutions of (1) by $\mathcal{O}_B(\Omega)$ and remark that in our main theorems the coefficients of $B_1$ and $B_2$ are assumed to be $C^\infty$-regular, hence $\mathcal{O}_B(\Omega) \subset C^\infty(\Omega, n)$. For more details on the regularity and bootstrapping methods, see, e.g. [2].

From now on, we denote by

$$\bar{\partial}_B(w) = w_\xi + B_1w + B_2\bar{w},$$

where $w_\xi$ means either the usual derivative with respect to the conjugate complex variable or its weak analogue (it will be clear from the context). Let $\mathcal{D} \subset \mathbb{C}$ be an open disc and let the coefficients of $B_1$ and $B_2$ belong to the class $C^{k,\alpha}(\mathcal{D})$ for some $k \in \mathbb{N}_0$ and $0 < \alpha < 1$. Then $\bar{\partial}_B$ can be treated as a linear operator mapping from $C^{k+1,\alpha}(\mathcal{D}, n)$ into the space $C^{k,\alpha}(\mathcal{D}, n)$. For $n = 1$, this operator is known to be surjective. Indeed, let $T_D$ be the classical Cauchy-Green operator for $\mathcal{D}$. Then, we can introduce a bijective integral operator $P_D : C^{k+1,\alpha}(\mathcal{D}, n) \to C^{k+1,\alpha}(\mathcal{D}, n)$ given by

$$P_D(w) = w + T_D(B_1w + B_2\bar{w}).$$

It turns out that $P_D^{-1} \circ T_D$ is a bounded right inverse for $\bar{\partial}_B$. For $n \geq 2$, an analogue of this fact was proved only recently. In particular, in the higher-dimensional case $P_D$ may have a non-trivial kernel and has to be slightly perturbed before it can be inverted [9, Theorem 3.1, Corollary 3.6].

If we combine the surjectivity of $\bar{\partial}_B$ and the extension theory for Hölder spaces, we obtain the following theorem concerning the approximation of vector functions with small $\bar{\partial}_B$-derivatives.

**Theorem 2.1**: Let $\mathcal{D} \subset \mathbb{C}$ be an open disc and assume that the coefficients of $B_1$ and $B_2$ belong to the class $C^{k,\alpha}(\mathcal{D})$. Given $\epsilon > 0$ there exists a constant $\delta > 0$ such that given any smoothly bounded domain $\Omega \Subset \mathcal{D}$ and $g \in C^{k+1,\alpha}(\overline{\Omega})$ with $\| \bar{\partial}_B(g) \|_{C^{k,\alpha}(\Omega)} < \delta$ there exists $w \in \mathcal{O}_B(\Omega)$ such that

$$\| w - g \|_{C^{k,\alpha}(\Omega)} < \epsilon.$$  

**Proof**: Note that $g$ is assumed to be regular up to the boundary of $\Omega$. Hence, we can introduce extension operators $E_{k,\alpha} : C^{k,\alpha}(\overline{\Omega}, n) \to C^{k,\alpha}(\mathbb{C}, n)$. It is crucial that, by [14, Theorem 4, p.177], they can be chosen so that their norms are bounded by constants that are independent of $\overline{\Omega}$. In particular, there exist a constant $C_{k,\alpha} \geq 1$ such that

$$\| E_{k,\alpha}(\bar{\partial}_B(g)) \|_{C^{k,\alpha}(\mathcal{D})} \leq C_{k,\alpha} \| \bar{\partial}_B(g) \|_{C^{k,\alpha}(\overline{\Omega})}$$

for any corresponding domain $\Omega \Subset \mathcal{D}$. 

Since the operator \( \tilde{\partial}_B : C^{k+1,\alpha}(D, n) \to C^{k,\alpha}(D, n) \) is surjective, there exist a solution \( u \in C^{k+1,\alpha}(D) \) of the non-homogeneous equation

\[
\tilde{\partial}_B(u) = E_{k,\alpha}(\tilde{\partial}_B(g)).
\]

We restrict it to \( \Omega \) and define \( w = g - u|_\Omega \). This is the map we seek since \( w \in \mathcal{O}_B(\Omega) \) and

\[
\|w - g\|_{C^{k,\alpha}(\Omega)} < C \cdot \delta,
\]

where the constant \( C > 0 \) arises as a product of \( C_{k,\alpha} \) and the operator norm of the bounded right inverse of \( \tilde{\partial}_B \) on \( D \).

We conclude this preliminary section by presenting Runge-type theorem for Pascali systems provided by Goldschmidt [6]. For the convenience of the reader, we include its proof. The proof relies on the Riesz representation theory for Lebesgue spaces; hence, it is essential to work with weak solutions of (1). However, as explained above, higher regularity of \( B_1 \) and \( B_2 \) automatically improves the regularity of solutions.

**Theorem 2.2:** Let \( U \subset \mathbb{C} \) be a domain and assume that the coefficients of \( B_1 \) and \( B_2 \) belong to the class \( L^p_{\text{loc}}(U, n) \), \( p > 2 \). Let \( K \subset U \) be a compact Runge subset. Given \( \epsilon > 0 \) and \( f \in \mathcal{O}^*_B(\Omega) \) for some neighbourhood \( \Omega \) of \( K \), there exists \( w \in \mathcal{O}^*_B(U) \) such that

\[
\|f - w\|_{L^\infty(K)} < \epsilon.
\]

**Proof:** Fix \( \epsilon > 0 \) and choose \( f \in \mathcal{O}^*_B(\Omega) \) as in the theorem. If necessary, we may shrink the neighbourhood \( \Omega \) so that we still have \( K \subset \Omega \) and that \( U \setminus \Omega \) admits no relatively compact connected components in \( U \).

Given \( w \in L^q_{\text{loc}}(U, n), \frac{1}{p} + \frac{1}{q} = 1 \), we define

\[
\tilde{\partial}_B^*(w) = w_\zeta - B_1^T w - B_2^T \bar{w}.
\]

Furthermore, let \( \varphi \in C^\infty_0(U, n) \). We have

\[
\varphi^T \cdot \tilde{\partial}_B(w) + w^T \cdot \tilde{\partial}_B^*(\varphi) = (\varphi^T \cdot w_\zeta + w^T \cdot \varphi_\zeta) + (\varphi^T \cdot B_2 \bar{w} - \varphi^T \cdot \bar{B}_2 w).
\]

Hence, applying the integration by parts, we get

\[
\int_U (\varphi^T \cdot \tilde{\partial}_B(w) + w^T \cdot \tilde{\partial}_B^*(\varphi)) \, d\zeta \, d\bar{\zeta} = \int_U (\varphi^T \cdot B_2 \bar{w} - \varphi^T \cdot \bar{B}_2 w) \, d\zeta \, d\bar{\zeta}.
\]

This last integral is imaginary, thus we have

\[
\text{Re} \left( \int_U \varphi^T \cdot \tilde{\partial}_B(w) + w^T \cdot \tilde{\partial}_B^*(\varphi) \, d\zeta \, d\bar{\zeta} \right) = 0 \quad \forall \varphi \in C^\infty_0(U, n).
\]

This implies that for \( w \in L^q_{\text{loc}}(U, n) \), we have

\[
w \in \mathcal{O}^*_B(U) \iff \text{Re} \left( \int_U w^T \cdot \tilde{\partial}_B^*(\varphi) \, d\zeta \, d\bar{\zeta} \right) = 0 \quad \forall \varphi \in C^\infty_0(U, n),
\]

where the reverse implication follows from the fact that if \( \varphi \in C^\infty_0(U, n) \), then we also have \( \bar{\varphi} \in C^\infty_0(U, n) \).
Given \( w \in \mathcal{O}_B^*(U) \) its restriction \( w|_\Omega \) belongs to \( \mathcal{O}_B^*(\Omega) \). We denote by \( \mathcal{O}_B^*(U)|_{\Omega} \) the set of all such restrictions. We would like to prove that the closure of this set in the topology of the space \( L^q_{\text{loc}}(\Omega, n) \) is equal to space \( \mathcal{O}_B^*(\Omega) \). If this holds, the desired conclusion follows from the bootstrapping argument. That is, there exists an extension \( w \) of \( f \) which is \( L^q_{\text{loc}} \)-close to \( f \) on \( \Omega \) and therefore \( W^{1,p} \)-close to \( f \) on some smaller neighbourhood \( \Omega' \) of \( K \) with smooth boundary. By the Sobolev embedding theorem such a function is \( L^{\infty} \)-close to \( f \) on \( \Omega' \) and therefore \( L^{\infty} \)-close to \( f \) on \( K \).

Assume the contrary. Then, by Hahn–Banach theorem, there is a bounded linear functional \( \mu : L^q_{\text{loc}}(\Omega, n) \rightarrow \mathbb{C} \) that vanishes on \( \mathcal{O}_B^*(U)|_{\Omega} \) and an element \( w_1 \in \mathcal{O}_B^*(\Omega) \setminus \mathcal{O}_B^*(U)|_{\Omega} \) such that \( \mu(w_1) \neq 0 \). In particular, we may assume that \( \text{Re}(\mu(w_1)) \neq 0 \) (otherwise, we take \( \mu \)). Furthermore, we can represent \( \mu \) by some \( g \in L^p(\Omega, n) \) such that \( \text{supp} \ g \subseteq \Omega \). Let us extend \( g \) to \( U \setminus \Omega \) by setting \( g = 0 \). We have

\[
\mu(w) = \iint_{\Omega} w^T \cdot g \, d\zeta \, d\bar{\zeta} = \iint_{U} w^T \cdot g \, d\zeta \, d\bar{\zeta}, \quad w \in L^q_{\text{loc}}(\Omega, n).
\]

Recall that \( w \in \mathcal{O}_B^*(U) \) precisely when

\[
\text{Re} \left( \iint_{U} w^T \cdot \overline{\sigma}_B^*(\varphi) \, d\zeta \, d\bar{\zeta} \right) = 0 \quad \forall \varphi \in C_0^\infty(U, n).
\]

Hence the map \( g \in L^p(U, n) \) belongs to the closure of the set \( \overline{\sigma}_B^*(C_0^\infty(U, n)) \) in the topology of the space \( L^p(U, n) \). Therefore, there exists a sequence \( \{ \varphi_m \} \subseteq C_0^\infty(U, n) \) with \( \text{supp} \ \sigma_B(\varphi_m) \subseteq \Omega \) and such that

\[
\lim_{m \to \infty} \overline{\sigma}_B^*(\varphi_m) = g.
\]

We claim that \( \varphi_m \in C_0^\infty(\Omega, n) \). That is, if the supports of \( \varphi_m \) are compact in \( U \), they must be compact in \( \Omega \) as well. Indeed, note that \( \sigma_B(\varphi_m) = 0 \) on \( U \setminus \Omega \). Moreover, \( \text{supp} \ \varphi_m \subseteq U \) implies that \( \varphi_m \) vanishes on some open set. However, by the Similarity principle for Pascali systems [15], every solution of (1) can be represented as a product of an invertible matrix function and a holomorphic vector function. This together with the fact that \( U \setminus \Omega \) admits no relatively compact components implies that \( \varphi_m = 0 \) in \( U \setminus \Omega \).

For \( w_1 \in \mathcal{O}_B^*(\Omega) \setminus \mathcal{O}_B^*(U)|_{\Omega} \) chosen above, we have

\[
\lim_{m \to \infty} \text{Re} \left( \iint_{\Omega} w_1^T \cdot \overline{\sigma}_B^*(\varphi_m) \, d\zeta \, d\bar{\zeta} \right) = \text{Re} \left( \iint_{\Omega} w_1^T \cdot g \, d\zeta \, d\bar{\zeta} \right) \neq 0.
\]

But then there is \( l \in \mathbb{N} \) such that

\[
\text{Re} \left( \iint_{\Omega} w_1^T \cdot \overline{\sigma}_B^*(\varphi_l) \, d\zeta \, d\bar{\zeta} \right) \neq 0.
\]

By an analogue of (2) this contradicts the fact that \( w_1 \in \mathcal{O}_B^*(\Omega) \).

\[\blacksquare\]

### 3. Proofs of Theorems 1.1 and 1.2

We begin by proving a local Mergelyan-type lemma for bounded domains.
Lemma 3.1: Let \( K \subset \mathbb{C} \) be a smoothly bounded compact domain. Given \( \epsilon > 0 \) and a smooth map \( f : K \to \mathbb{C}^n \) whose restriction to the interior \( \hat{K} \) is in \( \mathcal{O}_B(\hat{K}) \) there exist an open neighbourhood \( U \) of \( K \) and \( w \in \mathcal{O}_B(U) \) such that
\[
||f - w||_{L^\infty(K)} < \epsilon.
\]

Proof: Let \( D \subset \mathbb{C} \) be an open disc containing \( K \). We apply the operator \( E_{1,\alpha} \) from the proof of Theorem 2.1 and extend \( f \) into \( \bar{f} = E_{1,\alpha}(f) \in C^{1,\alpha}(D, n) \). Let \( \{\Omega_m\} \) be a sequence of smoothly bounded domains that are compactly contained in \( D \) and shrink towards \( K \). We denote by \( f_m \) the restrictions of \( \bar{f} \) to \( \Omega_m \). Since \( f \) is smooth on \( K \) and \( f|\hat{K} \in \mathcal{O}_B(\hat{K}) \) we have
\[
\lim_{m \to \infty} ||\partial_B(f_m)||_{C^{0,\alpha}(\Omega_m)} = 0.
\]
Together with Theorem 2.1 this yields the desired conclusion. That is, we can set \( U = \Omega_m \) for some large \( m \in \mathbb{N} \).

Lemma 3.2: Let \( \gamma : [a, b] \to \mathbb{C} \) be a parametrization of a smooth Jordan arc or a closed Jordan curve. Given a smooth map \( f : \gamma([a, b]) \to \mathbb{C}^n \) there exist an open neighbourhood \( U \) of \( \gamma([a, b]) \) and a smooth map \( F \) on \( U \) which solves the Pascali system \((1)\) on \( \gamma([a, b]) \) and satisfies \( F|\gamma([a, b]) \equiv f \).

Proof: We only consider Jordan arcs, the proof for closed Jordan curves proceeds similarly. We can extend \( \gamma \) smoothly to \([a - \eta, b + \eta]\) for some \( \eta > 0 \). For any \( x \in [a - \eta, b + \eta] \), there are a neighbourhood \( U_x \) of \( \gamma(x) \) in \( \mathbb{C} \) and a smooth map \( f_x : U_x \to \mathbb{C}^n \), which agrees with \( f \) on \( U_x \cap \gamma([a, b]) \) and solves \( (1) \) on \( U_x \cap \gamma((a - \eta, b + \eta)) \). Indeed, locally near \( \gamma(x) \), the set \( \gamma((a - \eta, b + \eta)) \) is a graph above one of the coordinate axes. Without loss of generality, let us assume that this is the \( x \)-axis. Then there are a neighbourhood \( U_x = I_x \times J_x \) of \( \gamma(x) \) in \( \mathbb{C} \) and a map \( \psi : I_x \to I_x \) such that
\[
U_x \cap \gamma((a - \eta, b + \eta)) = \{(t + i\psi(t) : t \in I_x)\}.
\]
An extension of \( f \) to \( U_x \) which solves \( (1) \) on \( U_x \cap \gamma((a - \eta, b + \eta)) \) can be found in the form \( f_x(t + is) = f(t + i\psi(t)) + f_1(t)(s - \psi(t)) \), where \( f_1 \) is an appropriate function defined on \( I_x \).

Let \( U = \bigcup_{x \in \gamma([a, b])} U_x \) and let \( \{\varphi_x\} \) be a smooth partition of unity subordinate to the covering \( \{U_x\}_{x \in \gamma([a, b])} \). We define the map \( F \) on \( U \) by
\[
F = \sum_x \varphi_x f_x.
\]
Then, we have \( F|\gamma([a, b]) \equiv f \) and since \( f_x \) solves \( (1) \) on \( U_x \cap \gamma([a, b]) \), we obtain the following equality on \( \gamma([a, b]) \):
\[
\bar{\partial}_B(F) = \sum_x (\varphi_x) \bar{\partial}_B(f_x) + \sum_x \varphi_x \bar{\partial}_B(f_x) = \sum_x (\varphi_x) \bar{\partial}_B(f_x) = f \left( \sum_x \varphi_x \right) = 0.
\]
That is, \( F \) solves the Pascali system \((1)\) on \( \gamma([a, b]) \).
Proof of Theorem 1.1: Fix $\epsilon > 0$ and recall that $S = K \cup E$, where $K$ is a finite union of pairwise disjoint compact domains in $\mathbb{C}$ with smooth boundary and $E = S \setminus K$ is a union of finitely many pairwise disjoint smooth Jordan arcs and closed Jordan curves. By Lemma 3.1 there exist an open neighbourhood $U$ of $K$ and $g \in \mathcal{O}_B(U)$ such that

$$\|f - g\|_{L^\infty(K)} < \frac{\epsilon}{4}.$$ 

Since $f$ and $g$ are close on $K$ and continuous on $U \cap E$, and since Jordan curves from $E$ meet the boundary $bK$ transversally, there is an open neighbourhood $U_1$ of $K$, $K \subset U_1 \subset U$, such that

$$|f(\zeta) - g(\zeta)| < \frac{\epsilon}{2} \quad \text{for all } \zeta \in E \cap U_1.$$ 

We now glue $f$ and $g$ on $E \cap U_1$ by a smooth cut off function. That is, we choose a smoothly bounded neighbourhood $U_2$ of $K$, $K \subset U_2 \subset U_1$, and a smooth cut off function $\varphi: \mathbb{C} \to [0, 1]$ that satisfies $\varphi|_{U_2} \equiv 1$ and supp $\varphi \subset U_1$. We define a smooth map $h: U_2 \cup E \to \mathbb{C}^n$ by

$$h = \varphi g + (1 - \varphi)f.$$ 

Note that such a map satisfies $h \equiv g$ on $U_2$, $h \equiv f$ on $E \cap (\mathbb{C} \setminus \overline{U}_1)$, and

$$|f(\zeta) - h(\zeta)| < \frac{\epsilon}{2} \quad \text{for all } \zeta \in S.$$ 

We now use Lemma 3.2 to get an open neighbourhood $V$ of $E$ and a smooth map $F$ on $V$ which solves (1) on $E$ and satisfies $F|_E \equiv h$. We glue $F$ with $h$ in order to obtain a map that admits a vanishing $\partial_B$-derivative on $S$. Let $U_3$ be a smoothly bounded neighbourhood of $K$, $K \subset \overline{U}_3 \subset U_2$, and let $\varphi: \mathbb{C} \to [0, 1]$ be a smooth cut off function that satisfies $\varphi|_{U_3} \equiv 1$ and supp $\varphi \subset U_2$. Let a smooth map $H: U_3 \cup V \to \mathbb{C}^n$ be given by

$$H = \varphi h + (1 - \varphi)F.$$ 

We have

$$|H(\zeta) - f(\zeta)| < \frac{\epsilon}{2} \quad \text{for all } \zeta \in S.$$ 

On $E$ we have $h = F$, $\partial_B(h) = \partial_B(F) = 0$ and therefore

$$\partial_B(H) = \varphi(\zeta)(h - F) + \varphi \partial_B(h) + (1 - \varphi)\partial_B(F) = 0.$$ 

Furthermore, the fact that $H \equiv h \equiv g$ on $U_3$ implies that $H \in \mathcal{O}_B(U_3)$. In particular, for every $\zeta \in S$, we have $\partial_B(H)(\zeta) = 0$.

We now proceed as in the proof of Lemma 3.1 and set $\{\Omega_m\}$ to be a sequence of smoothly bounded domains that are compactly contained in $U_3 \cup V$ and shrink towards $S$. 

We denote by $H_m$ the restriction of $H$ to the set $\Omega_m$. We have
\[
\lim_{m \to \infty} \| \tilde{\delta}_B(H_m) \|_{C^0(\Omega_m)} = 0.
\]
Therefore, by Theorem 2.1, for $m \in \mathbb{N}$ that is large enough we can find $w_m \in \mathcal{O}_B(\Omega_m)$ such that
\[
\| w - H_m \|_{C^1(\Omega_m)} < \frac{\epsilon}{4}.
\]
Finally, by Runge theorem 2.2, there exists $w \in \mathcal{O}_B(\Omega)$ such that
\[
\| w - w_m \|_{L^\infty(S)} < \frac{\epsilon}{4}.
\]
This is the map we seek. \hfill \blacksquare

Finally, we prove the Carleman theorem using Mergelyan’s theorem for Pascali systems following the usual inductive construction [8].

**Proof of Theorem 1.2:** Without loss of generality, we may assume that the starting map $f$ is smooth (otherwise, work with its smooth approximation). For $m \in \mathbb{N}$, we define the sets
\[
S_m = \{ \zeta \in \mathbb{C}: |\zeta| \leq m \} \cup [-m - 2, m + 2],
\]
\[
\Omega_m = \{ \zeta \in \mathbb{C}: |\zeta| < m + \frac{1}{3} \},
\]
and a decreasing sequence
\[
\epsilon_m = \min\{ \epsilon(\zeta): |\zeta| \leq m + 2 \}.
\]
Let $f_0(\zeta) = f(\zeta)$ for $\zeta \in \mathbb{R}$. We construct inductively a sequence of smooth maps $f_m: \Omega_m \cup \mathbb{R} \to \mathbb{C}^n$, $m \geq 1$, which satisfies the following properties:

(i) $f_m \in \mathcal{O}_B(\Omega_m)$,
(ii) $f_m(\zeta) = f(\zeta)$ for $\zeta \in \mathbb{R}$ such that $|\zeta| \geq m + \frac{2}{3}$,
(iii) $|f_m(\zeta) - f_{m-1}(\zeta)| < \frac{\epsilon_{m-1}}{2m+1}$ for all $\zeta \in S_{m-1}$.

Let us fix $m \geq 1$ and assume that we have already constructed the map $f_{m-1}$. The set $S_{m-1}$ is admissible. Hence, by Theorem 1.1, there is $g_m \in \mathcal{O}_B(\mathbb{C})$ such that for every $\zeta \in S_{m-1}$ we have the following estimate:
\[
|g_m(\zeta) - f_{m-1}(\zeta)| < \frac{\epsilon_{m-1}}{2m+1}.
\]
Let $\varphi_m: \mathbb{C} \to [0, 1]$ be a smooth cut off function such that $\varphi_m \equiv 1$ on $\Omega_m$ and $\varphi_m(\zeta) = 0$ for $|\zeta| \geq m + \frac{2}{3}$. We define $f_m: \Omega_m \cup \mathbb{R} \to \mathbb{C}^n$ by setting
\[
f_m = \varphi_m g_m + (1 - \varphi_m) f_{m-1}.
\]
Note that such a function satisfies the properties (i)–(iii).
It follows from (iii) that for each \( m, k \geq 1 \) and \( \zeta \in S_m \) we have

\[
|f_{m+k}(\zeta) - f_m(\zeta)| \leq \sum_{i=1}^{k} |f_{m+i}(\zeta) - f_{m+i-1}(\zeta)| \leq \sum_{i=1}^{k} \frac{\varepsilon_{m+i-1}}{2^{m+i+1}} \leq \frac{\varepsilon_m}{2^{m+1}}. \tag{3}
\]

Therefore, the sequence \( f_m \) converges uniformly on compact sets in \( \mathbb{C} \). This, together with the fact that \( f_m \in \mathcal{O}_B(\Omega_m) \), implies the existence of its limit \( w \in \mathcal{O}_B(\mathbb{C}) \) (the uniform convergence implies that \( w \) is a weak solution of (1) on every compact subset of \( \mathbb{C} \)). Finally, by (ii) and (3) we have for every \( \zeta \), \( |\zeta| \in [m+1, m+2] \), the following estimate:

\[
|w(\zeta) - f(\zeta)| = |w(\zeta) - f_m(\zeta)| \leq \sum_{i=1}^{\infty} \frac{\varepsilon_{m+i-1}}{2^{m+i+1}} \leq \frac{\varepsilon_m}{2^{m+1}} < \varepsilon(\zeta).
\]

Hence \( w \) is the vector function we seek.

\[\blacksquare\]

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