Game-theoretic characterization of the Gurarii space

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Abstract. We present a simple and natural infinite game building an increasing chain of finite-dimensional Banach spaces. We show that one of the players has a strategy with the property that, no matter how the other player plays, the completion of the union of the chain is linearly isometric to the Gurarii space.

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1. Introduction. We consider the following game. Namely, two players (called Eve and Odd) alternately choose finite-dimensional Banach spaces $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$, with no additional rules. The inclusion $E_n \subseteq E_{n+1}$ means that $E_n$ is a linear subspace of $E_{n+1}$ and the norm of $E_{n+1}$ restricted to $E_n$ coincides with that of $E_n$. For obvious reasons, Eve should start the game. The result is the completion of the chain $\bigcup_{n \in \mathbb{N}} E_n$. We shall denote this game by $BM(\mathcal{B})$. This is in fact a special case of an abstract Banach–Mazur game studied recently in [7]. In model theory, this is sometimes called the $\forall\exists$-game, see [5]. Main result:

Theorem 1. There exists a unique, up to linear isometries, separable Banach space $G$ such that Odd has a strategy $\Sigma$ in $BM(\mathcal{B})$ leading to $G$, namely, the completion of every chain resulting from a play of $BM(\mathcal{B})$ is linearly isometric to $G$, assuming Odd uses strategy $\Sigma$, and no matter how Eve plays.

Furthermore, $G$ is the Gurarii space.

The result above may serve as a strong argument that the Gurarii space (see the definition below) should be considered as one of the classical Banach spaces. Indeed, Theorem 1 is completely elementary and can even be presented with
no difficulties to undergraduate students who know the very basic concepts of Banach space theory.

It turns out that the Gurari˘ı space \( G \) (constructed by Gurari˘ı in 1966) is not so well known, even to people working in functional analysis. The reason might be that this is a Banach space constructed usually by some inductive set-theoretic arguments, without providing any concrete formula for the norm. Furthermore, the fact that \( G \) is actually unique up to linear isometries was proved by Lusky [9] only ten years after Gurari˘ı’s work [4]. An elementary proof of the uniqueness of \( G \) has been found recently by Kubiš and Solecki [8]. Theorem 1 offers an alternative argument, still using the crucial lemma from [8].

In fact, uniqueness of a space \( G \) satisfying the assertion of Theorem 1 is almost trivial: if there were two Banach spaces \( G_0, G_1 \) in Theorem 1, then we can play the game so that Odd uses his strategy leading to \( G_1 \), while after the first move Eve uses Odd’s strategy leading to \( G_2 \). Both players win, therefore \( G_1 \) is linearly isometric to \( G_2 \).

Below, after recalling the definition of the Gurari˘ı space, we show that it indeed satisfies the assertion of Theorem 1. Finally, we discuss other variants of the Banach–Mazur game, for example, playing with separable Banach spaces or with a fixed (rich enough) subclass of finite-dimensional spaces. Again, the Gurari˘ı space is the unique object for which Odd has a winning strategy.

2. Preliminaries. The Gurari˘ı space is the unique separable Banach space \( G \) satisfying the following condition:

\[ (G) \text{ For every } \varepsilon > 0, \text{ for every finite-dimensional normed spaces } A \subseteq B, \text{ every isometric embedding } e: A \rightarrow G \text{ has an extension } f: B \rightarrow G \text{ that is an } \varepsilon\text{-isometric embedding, namely,} \]
\[ (1 - \varepsilon)\|x\| \leq \|f(x)\| \leq (1 + \varepsilon)\|x\| \]

for every \( x \in B \).

As we have already mentioned, this space has been found by Gurari˘ı [4] in 1966, yet its uniqueness was proved only ten years later by Lusky [9] using rather advanced method of representing matrices. An elementary proof can be found in [8]. According to [3, Thm. 2.7], the Gurari˘ı space can be characterized by the following condition:

\[ (H) \text{ For every } \varepsilon > 0, \text{ for every finite-dimensional normed spaces } A \subseteq B, \text{ for every isometric embedding } e: A \rightarrow G, \text{ there exists an isometric embedding } f: B \rightarrow G \text{ such that } \|e - f \upharpoonright A\| < \varepsilon. \]

Actually, in the proof of equivalence \( (G) \iff (H) \), one has to use the crucial lemma from [8]:

**Lemma 1.** Let \( 0 < \varepsilon < 1 \), and let \( f: X \rightarrow Y \) be an \( \varepsilon\)‐isometric embedding between Banach spaces. Then there exists a norm on \( X \oplus Y \) such that, denoting by \( i: X \rightarrow X \oplus Y, j: Y \rightarrow X \oplus Y \) the canonical embeddings, it holds that \[ \|j \circ f - i\| \leq \varepsilon. \]
The proof given in [8] uses functionals, however there is a direct formula for the norm on \( X \oplus Y \) satisfying the assertion of Lemma 1:
\[
\| (x, y) \| = \inf \{ \| x_0 \| + \| y_0 \| + \varepsilon \| x_1 \| : x = x_0 + x_1, \ y = y_0 - f(x_1) \}.
\]
Easy computations showing that it works can be found in [2, p. 753]. In fact, [2] deals with \( p \)-Banach spaces; \( p = 1 \) is our case.

By a chain of normed spaces we mean a sequence \( \{ E_n \}_{n \in \mathbb{N}} \) such that each \( E_n \) is a normed space, \( E_n \) is a linear subspace of \( E_{n+1} \), and the norm of \( E_{n+1} \) restricted to \( E_n \) coincides with that of \( E_n \) for every \( n \in \mathbb{N} \). All mappings in this note are assumed to be linear.

3. Proof of Theorem 1. Let us fix a separable Banach space \( G \) satisfying (H). We do not assume a priori that it is uniquely determined, therefore the arguments below will also show the uniqueness of \( G \). Odd’s strategy \( \Sigma \) in BM(\( \mathcal{B} \)) can be described as follows.

Fix a countable set \( \{ v_n \}_{n \in \mathbb{N}} \) linearly dense in \( G \). Let \( E_0 \) be the first move of Eve. Odd finds an isometric embedding \( f_0 : E_0 \to G \) and finds \( E_1 \supseteq E_0 \) together with an isometric embedding \( f_1 : E_1 \to G \) extending \( f_0 \) and such that \( v_0 \in f_1[E_1] \).

Suppose now that \( n = 2k > 0 \) and \( E_n \) was the last move of Eve. We assume that a linear isometric embedding \( f_{n-1} : E_{n-1} \to G \) has been fixed. Using (H) we choose a linear isometric embedding \( f_n : E_n \to G \) such that \( f_n \upharpoonright E_{n-1} \) is \( 2^{-k} \)-close to \( f_{n-1} \). Extend \( f_n \) to a linear isometric embedding \( f_{n+1} : E_{n+1} \to G \) so that \( E_{n+1} \supseteq E_n \) and \( f_n \upharpoonright [E_{n+1}] \) contains all the vectors \( v_0, \ldots, v_k \). The finite-dimensional space \( E_{n+1} \) is Odd’s move. This finishes the description of Odd’s strategy \( \Sigma \).

Let \( \{ E_n \}_{n \in \mathbb{N}} \) be the chain of finite-dimensional normed spaces resulting from a fixed play, when Odd was using strategy \( \Sigma \). In particular, Odd has recorded a sequence \( \{ f_n : E_n \to G \}_{n \in \mathbb{N}} \) of linear isometric embeddings such that \( f_{2n+1} \upharpoonright E_{2n-1} \) is \( 2^{-n} \)-close to \( f_{2n-1} \) for each \( n \in \mathbb{N} \). Let \( E_\infty = \bigcup_{n \in \mathbb{N}} E_n \). For each \( x \in E_\infty \) the sequence \( \{ f_n(x) \}_{n \in \mathbb{N}} \) is Cauchy, therefore we can set \( f_\infty(x) = \lim_{n \to \infty} f_n(x) \), thus defining a linear isometric embedding \( f_\infty : E_\infty \to G \). The assumption that \( f_{2n+1}[E_{2n+1}] \) contains all the vectors \( v_0, \ldots, v_n \) ensures that \( f_\infty[E_\infty] \) is dense in \( G \). Finally, \( f_\infty \) extends to a linear isometry from the completion of \( E_\infty \) onto \( G \). This completes the proof of Theorem 1.

4. Playing with a subclass of finite-dimensional spaces. It is natural to ask whether Theorem 1 remains true when the game is restricted to a rich enough subclass of finite-dimensional normed spaces. Of course, the minimal assumption on the class must be the existence of a chain whose completion is the Gurarii space. It turns out that this is sufficient.

Let \( \mathcal{F} \) be a class of finite-dimensional normed spaces, closed under isometries. Namely, if \( E \in \mathcal{F} \) and \( E' \) is linearly isometric to \( E \), then \( E' \in \mathcal{F} \). We say that \( \mathcal{F} \) is dominating (in the class of all finite-dimensional spaces) if for every \( E \in \mathcal{F} \), for every isometric embedding \( e : E \to X \) with \( X \) finite-dimensional, for every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-isometric embedding \( f : X \to F \) such that \( F \in \mathcal{F} \) and \( f \circ e \) is an isometric embedding. Note that, by condition (G), if
\{F_n\}_{n \in \mathbb{N}} is a chain of finite-dimensional subspaces of the Gurarii space whose union is dense, then the class \(F\) consisting of all spaces linearly isometric to some \(F_n\) is dominating.

The game BM(\(F\)) is defined precisely in the same way as BM(\(B\)), simply restricting the class of spaces to \(F\).

**Theorem 2.** Let \(F\) be a dominating class of finite-dimensional normed spaces. Then Odd has a strategy \(\Sigma\) in BM(\(F\)) leading to the Gurarii space \(G\). Namely, the completion of every chain resulting from a play of BM(\(F\)) is linearly isometric to \(G\) whenever Odd uses strategy \(\Sigma\).

**Proof.** The strategy is a suitable adaptation of the one from the proof of Theorem 1. Fix a linearly dense set \(\{v_n\}_{n \in \mathbb{N}}\) in \(G\) such that \(\|v_i\| = 1\) for \(i \in \mathbb{N}\). Suppose \(n = 2k \geq 0\) and \(E_n \in F\) was the last move of Eve. We assume that a linear isometric embedding \(f_{n-1}: E_{n-1} \to G\) has been defined, where \(f_{-1} = 0\) and \(E_{-1} = \{0\}\). Using (H) we choose an isometric embedding \(f_n: E_n \to G\) such that \(f_n \restriction E_{n-1}\) is \(2^{-k}\)-close to \(f_{n-1}\). Extend \(f_n\) to a linear isometric embedding \(g: X \to G\) so that \(X \supseteq E_n\) is finite-dimensional and \(\{v_0, \ldots, v_k\} \subseteq g[X]\). We need to “correct” \(X\) so that it becomes a member of \(F\). Using the fact that \(F\) is dominating, we find a \(2^{-(k+1)}\)-isometric embedding \(s: X \to F\) such that \(F \in F\) and \(s \restriction E_n\) is isometric. We may assume that \(X \subseteq F\) and \(s\) is the inclusion. We set \(E_{n+1} := F\). This finishes the description of Odd’s strategy, yet for the inductive arguments we still need to define the embedding \(f_{n+1}\).

Using Lemma 1, we find isometric embeddings \(i: X \to Z\), \(j: F \to Z\) such that \(Z\) is finite-dimensional and \(\|j \circ s - i\| \leq 2^{-(k+1)}\). Using (H), we find an isometric embedding \(h: Z \to G\) such that \(\|h \circ i - g\| \leq 2^{-(k+1)}\). We set \(f_{n+1} := h \circ j\).

Note that \(f_{n+1} \restriction X = h \circ j \restriction X = h \circ j \circ s\), therefore
\[
\|f_{n+1} \restriction X - g\| \leq \|h \circ j \circ s - h \circ i\| + \|h \circ i - g\| \leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.
\]
Thus \(\|f_{n+1} \restriction E_n - f_n\| \leq 2^{-k}\). Furthermore, if \(v_i = g(x_i)\), then \(\|f_{n+1}(x_i) - v_i\| = \|f_{n+1}(x_i) - g(x_i)\| \leq 2^{-k}\|x_i\| = 2^{-k}\), showing that \(\text{dist}(v_i, f_{n+1}[E_{n+1}]) \leq 2^{-k}\) for \(i \leq k\).

Let \(\{E_n\}_{n \in \mathbb{N}} \subseteq F\) be the chain resulting from a play when Odd was using the strategy described above. In particular, we have a sequence \(\{f_n: E_n \to G\}_{n \in \mathbb{N}}\) of linear isometric embeddings converging uniformly to an isometric embedding \(f_\infty: E_\infty \to G\), where \(E_\infty = \bigcup_{n \in \mathbb{N}} E_n\). Finally, \(E_\infty\) is dense in \(G\), because
\[
\lim_{n \to \infty} \text{dist}(v_i, f_n[E_n]) = 0
\]
for each \(i \in \mathbb{N}\). It follows that the unique extension of \(f_\infty\) to the completion of \(E_\infty\) is an isometry onto \(G\). This completes the proof. □

An immediate corollary to Theorem 2 is that if \(F\) is a dominating class of finite-dimensional normed spaces, then there exists a chain in \(F\) whose union is isometric to a dense subspace of the Gurarii space. Another corollary is the
known fact that the Gurari˘ı space contains a chain of finite-dimensional $\ell_\infty$-spaces with a dense union, as the class of all such spaces is easily seen to be dominating.

5. Final remarks. Below we collect some comments around Theorem 1.

5.1. Universality. It has been noticed by Gurari˘ı that $G$ contains isometric copies of all separable Banach spaces. In fact, the space $G$ can be constructed in such a way that it contains any prescribed separable Banach space, e.g., the space $C([0,1])$, which is well known to be universal. The paper [8] contains a more direct and elementary proof of the isometric universality of $G$. The main result of this note offers yet another direct proof (cf. [7, Thm. 10]).

Namely, fix a separable Banach space $X$ and fix a chain $\{X_n\}_{n \in \mathbb{N}} \subseteq X$ of finite-dimensional spaces whose union is dense in $X$. We describe a strategy of Eve that leads to an isometric embedding of $X$ into $G$. Specifically, Eve starts with $E_0 := X_0$ and records the identity embedding $e_0: X_0 \to E_0$. Once Odd has chosen $E_n$ with $n = 2k + 1$, having recorded a linear isometric embedding $e_k: X_k \to E_{n-1}$, Eve finds $E_{n+1} \supseteq E_n$ such that there is a linear isometric embedding $e_{k+1}: X_{k+1} \to E_{n+1}$ extending $e_k$. This is her response to $E_n$. The only missing ingredient showing that such a strategy is possible is the amalgamation property of finite-dimensional normed spaces:

Lemma 2. Let $f: Z \to X$, $g: Z \to Y$ be linear isometric embeddings of Banach spaces. Then there are a Banach space $W$ and linear isometric embeddings $f': X \to W$, $g': Y \to W$ such that $f' \circ f = g' \circ g$. Furthermore, if $X, Y$ are finite-dimensional, then so is $W$.

The above lemma belongs to the folklore and can be found in several texts, e.g., [3] or [1].

In any case, when Eve uses the strategy described above and Odd uses a strategy leading to the Gurari˘ı space, Eve constructs a linear isometric embedding $e: X \to G$ such that $e \upharpoonright X_n = e_n$ for every $n \in \mathbb{N}$. This shows that $G$ is isometrically universal in the class of all separable Banach spaces.

5.2. Playing with separable spaces. It is natural to ask what happens if both players are allowed to choose infinite-dimensional separable Banach spaces. As it happens, in this case Odd has a very simple tactic (i.e., a strategy depending only on the last move of Eve) again leading to the Gurari˘ı space. This follows immediately from the following.

Proposition 1. ([3, Lemma 3.3]) Let $\{G_n\}_{n \in \mathbb{N}}$ be a chain of Banach spaces such that each $G_n$ is linearly isometric to the Gurari˘ı space. Then the completion of the union $\bigcup_{n \in \mathbb{N}} G_n$ is linearly isometric to the Gurari˘ı space.

Thus, knowing that $G$ contains isometric copies of all separable Banach spaces, Odd can always choose a space linearly isometric to $G$, so that the resulting chain consists of Gurari˘ı spaces.
5.3. Other variants of the game. It is evident that the Banach–Mazur game considered in this note can be played with other mathematical structures. The works [7] and [6] discuss this game in model theory, showing in particular that Odd has a winning strategy leading to the so-called Fraïssé limit of a class of structures (which exists under some natural assumptions). Another variant of this game appears when finite-dimensional normed spaces are replaced by finite metric spaces. Almost the same arguments as in the proof of Theorem 1 show that Odd has a strategy leading to the Urysohn space [10], the unique complete separable metric space $U$ satisfying the following condition:

(U) For every finite metric space $A \subseteq B$, every isometric embedding $e : A \to U$ can be extended to an isometric embedding $f : B \to U$.

It turns out that $U$ is uniquely determined by a weaker condition (analog of (H)) asserting that $f$ is $\varepsilon$-close to $e$ with arbitrarily small $\varepsilon > 0$, not necessarily extending $e$. An analog of Theorem 1 is rather obvious; the proof is practically the same as in the case of normed spaces, simply replacing all phrases “finite-dimensional” by “finite” and deleting all adjectives “linear”.

5.4. Strategies versus tactics. The proof of Theorem 1 (as well as of Theorem 2) actually gives a Markov strategy, that is, a strategy depending only on the step $n$ and the last move of Eve. When playing with separable spaces, Odd has a tactic, that is, a strategy depending on Eve’s last move only (such a strategy is also called stationary). We do not know whether Odd has a winning tactic in the Banach–Mazur game played with finite-dimensional normed spaces or finite metric spaces, where “winning” means obtaining the Gurarii space or the Urysohn space, respectively.

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