Forcing geodesic number of a fuzzy graph

S Rehmani\textsuperscript{1,a} and M S Sunitha\textsuperscript{2,b}
\textsuperscript{1}Department of Mathematics, Sullamussalam Science College, Area code-673639, Kerala, India.
\textsuperscript{2}Department of Mathematics, National Institute of Technology Calicut, Kozhikode-673601, Kerala, India.
E-mail: \textsuperscript{a}sameeha.rehmani@gmail.com, \textsuperscript{b}sunitha@nitc.ac.in

Abstract. Chartrand and Zhang in 1999 introduced the concept of forcing geodetic number of crisp graphs and studied it for several classes of graphs. In this paper, this concept is extended to fuzzy graphs using geodesic distance and is called the forcing geodesic number. For a geodesic basis $S$ of a fuzzy graph $G : (V, \sigma, \mu)$, a subset $T$ of $S$ with the property that $S$ is the unique geodesic basis containing $T$ is called a forcing subset of $S$. The minimum cardinality of a forcing subset of $S$ is called the forcing geodesic number of $S$ in $G$ and is denoted by $g_{nf}(S)$. The forcing geodesic number of $G$, denoted by $g_{nf}(G)$, is defined as $g_{nf}(G) = \min\{g_{nf}(S)\}$ where the minimum is taken over all geodesic bases $S$ in $G$. A characterization of the forcing geodesic number depending on the geodesic bases in the fuzzy graph is identified. The forcing geodesic number of fuzzy trees and of complete fuzzy graphs is obtained. It is proved that if the geodesic number of a fuzzy graph is 2, then its forcing geodesic number is always less than 2.

1. Introduction

Zadeh in 1965 [1] developed the concept of fuzzy sets which gave a platform for describing the uncertainties prevailing in day-to-day life situations. The theory of fuzzy graphs was later on developed by Rosenfeld in the year 1975 [2] along with Yeh and Bang [3]. Rosenfeld also obtained the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness along with some of their properties [2] and the concept of fuzzy trees [4], automorphism of fuzzy graphs [5], fuzzy interval graphs [6], cycles and co-cycles of fuzzy graphs [7] etc has been established by several authors during the course of time. Fuzzy groups and the notion of a metric in fuzzy graphs was introduced by Bhattacharya [8]. The concept of strong arcs [9] was introduced by Bhutani and Rosenfeld in the year 2003. The definition of fuzzy end nodes and some of its properties were established by the same authors in [10]. The concept of geodesic distance was introduced by Bhutani and Rosenfeld in [11]. Using this geodesic distance, Suvarna and Sunitha in [12] introduced the concept of geodesic iteration number and geodesic number of a fuzzy graph and studied some of their properties. The same concepts using $\mu$-distance was introduced by Linda and Sunitha in [13].

In this paper, the concept of forcing geodesic number of a fuzzy graph is introduced. A characterization of the forcing geodesic number of a fuzzy graph depending on the geodesic bases present is identified. The forcing geodesic number of fuzzy trees and of complete fuzzy graphs are obtained and a necessary condition for the forcing geodesic number of a fuzzy graph to be less than 2 is established.
2. Preliminaries

A graph is a pair \((V, E)\), where \(V\) is a set and \(E\) is a relation on \(V\). The elements of \(V\) are thought of as vertices of the graph and the elements of \(E\) are thought of as the edges. Sometimes, there can be vagueness in the description of vertices or in its relationships or in both. In such cases, designing a fuzzy graph model becomes useful as it is more realistic in natural situations.

In this section, a brief summary of some basic definitions in fuzzy graph theory taken from [2, 9, 10, 11, 14, 15] are given.

A fuzzy graph \(G : (V, \sigma, \mu)\) is a non-empty set \(V\) together with a pair of functions \(\sigma : V \rightarrow [0, 1]\) and \(\mu : V \times V \rightarrow [0, 1]\) such that for all \(x, y \in V\), \(\mu(x, y) \leq \sigma(x) \land \sigma(y)\). We call \(\sigma\) the fuzzy vertex set of \(G\) and \(\mu\) the fuzzy edge set of \(G\), respectively.

We assume that \(V\) is finite and non-empty, \(\mu\) is reflexive (i.e., \(\mu(x, x) = \sigma(x), \forall x\)) and symmetric (i.e., \(\mu(x, y) = \mu(y, x), \forall (x, y)\)). Also we denote the underlying crisp graph by \(G^* : (\sigma^*, \mu^*)\) where \(\sigma^* = \{x \in V/|\sigma(x) > 0\}\) and \(\mu^* = \{(x, y) \in V \times V/\mu(x, y) > 0\}\). Here we assume \(\sigma^* = V\).

The fuzzy graph \(H : (V, \tau, \nu)\) is called a partial fuzzy subgraph of \(G : (V, \sigma, \mu)\) if \(\tau \subseteq \sigma\) and \(\nu \subseteq \mu\). Similarly, the fuzzy graph \(H : (P, \tau, \nu)\) is called a fuzzy subgraph of \(G : (V, \sigma, \mu)\) induced by \(P\) if \(P \subseteq V\), \(\tau(x) = \sigma(x)\) for all \(x \in P\) and \(\nu(x, y) = \mu(x, y)\) for all \(x, y \in P\). A fuzzy subgraph \(H : (P, \tau, \nu)\) of a fuzzy graph \(G : (V, \sigma, \mu)\) is in fact a special case of a partial fuzzy subgraph obtained as follows.

\[
\tau(x) = \begin{cases} 
\sigma(x) & \text{if } x \in P \\
0 & \text{if } x \notin V - P
\end{cases}
\]

\[
\nu(x, y) = \begin{cases} 
\mu(x, y) & \text{if } (x, y) \in P \times P \\
0 & \text{if } (x, y) \in V \times V - P \times P
\end{cases}
\]

A fuzzy subgraph \(H : (V, \tau, \nu)\) of \(G : (V, \sigma, \mu)\) is said to span \(G\) if \(\tau = \sigma\). In this case, we call \(H : (V, \tau, \nu)\) a spanning fuzzy subgraph of \(G : (V, \sigma, \mu)\). A fuzzy graph \(G : (V, \sigma, \mu)\) is called trivial if \(|\sigma^*| = 1\). Otherwise it is called non-trivial. Also a fuzzy graph \(G : (V, \sigma, \mu)\) is a complete fuzzy graph if \(\mu(x, y) = \sigma(x) \land \sigma(y)\) \(\forall x, y \in \sigma^*\).

A sequence of distinct nodes \(u_0, u_1, ..., u_n\) such that \(\mu(u_i, u_{i-1}) > 0\), \(i = 1, 2, 3, ..., n\) is called a path \(P_n\) of length \(n\). An arc \((x, y)\) of \(G : (V, \sigma, \mu)\) with least non-zero membership value \(\mu(x, y)\) is a weakest arc of \(G\). The degree of membership of a weakest arc in the path is defined as the strength of the path. The strength of connectedness between two nodes \(u\) and \(v\) in \(G : (V, \sigma, \mu)\) is the maximum of the strengths of all paths between \(u\) and \(v\) and is denoted by \(CONN_G(u, v)\). The fuzzy graph \(G : (V, \sigma, \mu)\) is said to be connected if \(CONN_G(u, v) > 0\) for every \(u, v \in \sigma^*\).

An arc \((u, v)\) of a fuzzy graph \(G\) is called strong if its weight is at least as great as the strength of connectedness of its end nodes \(u, v\) when the arc \((u, v)\) is deleted and a \(u - v\) path \(P\) is called a strong path if \(P\) contains only strong arcs. A connected fuzzy graph \(G : (V, \sigma, \mu)\) is called a fuzzy tree if it has a spanning fuzzy subgraph \(F : (V, \sigma, \nu)\), which is a tree such that for all arcs \((u, v)\) not in \(F\), \(CONN_F(u, v) > \mu(u, v)\).

Two nodes \(u\) and \(v\) in \(G\) are neighbors (adjacent) if \(\mu(u, v) > 0\) and \(v\) is called a strong neighbor of \(u\) if the arc \((u, v)\) is strong. A node \(v\) is called a fuzzy end node of \(G\) if it has at most one strong neighbor in \(G\).

A strong path \(P\) from \(u\) to \(v\) is called geodesic if there is no shorter strong path from \(u\) to \(v\) and the length of a \(u - v\) geodesic is the geodesic distance from \(u\) to \(v\) denoted by \(d_g(u, v)\). The geodesic eccentricity \((g\)-eccentricity\) \(e_g(u)\) of a node \(u\) in a connected fuzzy graph \(G : (V, \sigma, \mu)\) is given by \(e_g(u) = \max_{x \in V} d_g(u, v)\). A node \(u\) with maximum \(e_g(u)\) is called a \(g\)-peripheral node or diametral node. The \(g\)-diameter of \(G\), \(d_g(G) = \max \{ e_g(v) : v \in V \}\) and the \(g\)-radius of \(G\), \(r_g(G) = \min \{ e_g(v) : v \in V \}\).

The following definitions and results have been taken from [11, 12].
Definition 2.1. Let $S$ be a set of nodes of a connected fuzzy graph $G$. The **geodesic closure** $(S)$ of $S$ is the set of all nodes in $S$ together with the nodes that lie on geodesics between nodes of $S$.

$S$ is said to be **geodesic cover** (**geodesic set**) of $G$ if $(S) = V(G)$ and any cover of $G$ with minimum number of nodes is called a **geodesic basis** of $G$. Order of a geodesic basis is the number of nodes in it.

Definition 2.2. The **geodesic number** of a fuzzy graph $G : (V, \sigma, \mu)$ is the order of a geodesic basis of $G$ and is denoted by $\text{gn}(G)$.

**Proposition 2.3.** A fuzzy tree has a unique geodesic basis consisting of its fuzzy end nodes.

**Proposition 2.4.** For a complete fuzzy graph $G$ on $n$ nodes, $\text{gn}(G) = n$.

**Proposition 2.5.** For a connected fuzzy graph $G$, $\text{gn}(G) = 2$ if and only if there exist $g$-peripheral nodes $u$ and $v$ such that every node of $G$ is on a diametral path joining $u$ and $v$.

**Example 2.6.** Consider the fuzzy graph $G : (V, \sigma, \mu)$ given in Figure 1.

![Figure 1. A fuzzy graph.](image)

Note that here the arc $(u, y)$ is the **weakest arc** of $G$. Now, consider two nodes $w$ and $x$ of $G$. The **paths** joining $w$ to $x$ are $P_1 : w - x, P_2 : w - z - x, P_3 : w - v - z - x, P_4 : w - z - y - x, P_5 : w - v - z - y - x, P_6 : w - v - y - x, P_7 : w - v - u - y - x, P_8 : w - z - u - y - x, P_9 : w - v - u - z - x, P_{10} : w - z - v - u - y - x, P_{11} : w - v - u - y - z - x, P_{12} : w - v - z - u - y - x, P_{13} : w - v - u - z - y - x, P_{14} : w - v - y - z - x$ and $P_{15} : w - z - v - y - x$
The strengths of these paths are listed in Table 1 as follows.

| Path $P_i$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | $P_7$ | $P_8$ | $P_9$ | $P_{10}$ | $P_{11}$ | $P_{12}$ | $P_{13}$ | $P_{14}$ | $P_{15}$ |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|--------|--------|--------|--------|--------|
| Strength of $P_i$ | 0.5   | 0.6   | 0.6   | 0.6   | 0.6   | 0.3   | 0.3   | 0.3   | 0.3   | 0.4    | 0.3    | 0.4    | 0.6    | 0.6    | 0.6    |

The strength of connectedness between the nodes $w$ and $x$ is given by $\text{CONN}_G(w, x) = \max\{0.5, 0.6, 0.3, 0.4\} = 0.6$.

When the arc $(w, x)$ is deleted, $\text{CONN}_{G-(w,x)}(w, x) = 0.6 > 0.5 = \mu(w, x)$. Therefore $(w, x)$ is not a strong arc. Note that all arcs of $G$ other than $(u, y)$ and $(w, x)$ are strong. Thus the strong paths joining $w$ to $x$ are $P_2, P_3, P_4, P_5, P_6, P_9, P_{13}, P_{14}$ and $P_{15}$.

Of these, the shortest strong path joining $w$ to $x$ is $P_2$ which is having length 2.

Therefore, $P_2$ is a $w - x$ geodesic and $d_g(w, x) = 2$.

Similarly we get $d_g(w, v) = 1, d_g(w, z) = 1, d_g(w, y) = 2, d_g(w, u) = 2$, and so on.

Now, $e_g(w) = \max\{d_g(w, v) : v \in V\} = \max\{1, 2\} = 2$.

Similarly we get $e_g(v) = 2, e_g(u) = 2, e_g(x) = 2, e_g(y) = 2$ and $e_g(z) = 1$.

Hence the $g$-diameter of $G$, $d_g(G) = \max\{e_g(v) : v \in V\} = \max\{1, 2\} = 2$ and the $g$-radius of $G$, $r_g(G) = \min\{e_g(v) : v \in V\} = \min\{1, 2\} = 1$.

Consider $S = \{u, y, w, x\}$. Note that the only geodesics joining $u$ to $y$ are $u - v - y$ and $u - z - y$. Thus $(S) = V(G)$ and so $S$ is a geodesic cover of $G$.

Since no proper subset of $V(G)$ having cardinality less than that of $S$ is a geodesic cover of $G$, $S$ is a geodesic basis of $G$ and hence the geodesic number $gn(G) = 4$.

3. Forcing geodesic number of a fuzzy graph

The concept of forcing geodetic number of crisp graphs was introduced by Gary Chartrand and Ping Zhang in 1999 [16]. In this section, the concept of forcing geodesic number of fuzzy graphs is defined as follows.

**Definition 3.1.** For a geodesic basis $S$ of a fuzzy graph $G : (V, \sigma, \mu)$, a subset $T$ of $S$ with the property that $S$ is the unique geodesic basis containing $T$ is called a forcing subset of $S$.

The minimum cardinality of a forcing subset of $S$ is called the **forcing geodesic number** of $S$ in $G$ and is denoted by $gn_f(S)$.

The forcing geodesic number of $G$, denoted by $gn_f(G)$, is defined as $gn_f(G) = \min\{gn_f(S)\}$ where the minimum is taken over all geodesic bases $S$ in $G$. 
Example 3.2. Consider the fuzzy graph $G$ given in Figure 2. Note that here the arcs $(w, y)$ and $(u, y)$ are not strong. The geodesic bases of $G$ are $S_1 = \{w, x, y\}$, $S_2 = \{v, x, z\}$, $S_3 = \{u, y, z\}$ and $S_4 = \{u, y, w\}$.

Since $S_2$ is the only geodesic basis containing $v$, it follows that $gn_f(S_2) = 1$.

No other node of $G$ belongs to only one geodesic basis, so $gn_f(S_i) \geq 2$ for $i = 1, 3, 4$. In particular, $gn_f(S_i) = 2$ for $i = 1, 3, 4$ since $T_1 = \{w, x\}$ or $\{x, y\}$, $T_3 = \{u, z\}$ or $\{y, z\}$ and $T_4 = \{u, w\}$ are the forcing subsets of minimum cardinality for $S_1$, $S_3$ and $S_4$ respectively. Hence $gn_f(G) = min\{1, 2\} = 1$.

Proposition 3.3. [17] For any non-trivial fuzzy graph $G$ on $n$ nodes, $2 \leq gn(G) \leq n$.

Proposition 3.4 gives a characterization of the forcing geodesic number of fuzzy graphs based on the geodesic bases. Using this result, forcing geodesic number of fuzzy trees and complete fuzzy graphs are obtained in Propositions 3.5 and 3.6. A necessary condition for the forcing geodesic number of a fuzzy graph to be less than 2 is established in Proposition 3.7.

Proposition 3.4. Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph. Then,

(i) $gn_f(G) = 0$ if and only if $G$ has a unique geodesic basis.

(ii) $gn_f(G) = gn(G)$ if and only if for all geodesic bases $S_i, i = 1, 2, 3, \ldots$ and for all $A \subset S_i$, there exists at least one geodesic basis $S_j, j \neq i$, such that $A \subset S_j$.

Proof. (i) Let $gn_f(G) = 0$. Then, by Definition 3.1, there exists some geodesic basis $S$ of $G$ such that $gn_f(S) = 0$ so that the empty set $\phi$ is the forcing subset of $S$ of minimum cardinality.

Since the empty set $\phi$ is a subset of every set, it follows that $S$ is the unique geodesic basis of $G$.

The converse is clearly true.

(ii) Let $gn_f(G) = gn(G)$. Then $gn_f(S) = gn(G) = |S|$ where $S$ is a geodesic basis of $G$.

Now by Proposition 3.3, since $gn(G) \geq 2$, we get $gn_f(G) \geq 2$. Hence, by part(1) of this Proposition, it is clear that $G$ has at least two geodesic bases (say) $S_i, i = 1, 2, 3, \ldots$.

Therefore $gn_f(S_i) = gn(G) = |S_i|$ for every geodesic basis $S_i$ of $G$.

Now let $A \subset S_i$ be such that $A \not\subset S_j$ for all $j \neq i$. Then by Definition 3.1, $A$ is a forcing subset of $S_i$ and so $gn_f(S_i) \leq |A| < |S_i|$ which is a contradiction. Thus $A \subset S_j, j \neq i$.

Conversely suppose that for all geodesic bases $S_i, i = 1, 2, 3, \ldots$, and for all $A \subset S_i$, there exists at least one geodesic basis $S_j, j \neq i$, such that $A \subset S_j$.

If possible suppose $gn_f(G) \neq gn(G) = |S|$ where $S$ is a geodesic basis of $G$. Then $gn_f(G) < |S|$, which implies that there exists a proper subset $A$ of $S$ such that $A$ is a
forcing subset of $S$. Then by Definition 3.1, since $A$ is a forcing subset of $S$, we get that $S$ is the unique geodesic basis containing $A$ which is a contradiction to our assumption.

Proposition 3.5. The forcing geodesic number of a fuzzy tree $G : (V, \sigma, \mu)$ is 0.

Proof. By Proposition 2.3, the fuzzy tree $G$ has a unique geodesic basis consisting of its fuzzy end nodes. So by proof of Part(1) of Proposition 3.4, $gn_f(G) = 0$.

Proposition 3.6. The forcing geodesic number of a complete fuzzy graph $G : (V, \sigma, \mu)$ on $n$ nodes is 0.

Proof. It follows from Proposition 2.4 that the geodesic number of a complete fuzzy graph $G$ on $n$ nodes is $n$. Therefore $V(G)$ is the unique geodesic basis of $G$ and so by proof of Part(1) of Proposition 3.4, we get $gn_f(G) = 0$.

Proposition 3.7. If $G : (V, \sigma, \mu)$ is a connected fuzzy graph with geodesic number $gn(G) = 2$, then its forcing geodesic number $gn_f(G) < 2$.

Proof. Let $gn(G) = 2$. Then there exists a set $S = \{u, v\}$ which is a geodesic basis of $G$. Also by Proposition 2.5, $d_g(u, v) = d_g(G)$, the diameter of $G$, and every node of $G$ lies on some $u - v$ geodesic in $G$.

Now we have to prove that $gn_f(G) < 2$. Note that it follows from Definition 3.1 that $gn_f(G) \leq gn(G)$. Suppose on the contrary that $gn_f(G) = 2 = gn(G)$. Then it follows from the proof of Part(2) of Proposition 3.4 that $S$ is not the unique geodesic basis containing the node $u$ (or $v$). Hence there exists some node $x \neq v$ (or $x \neq u$) such that $S' = \{u, x\}$ (or $S' = \{v, x\}$) is also a geodesic basis of $G$. Again since $gn(G) = 2$, we get by Proposition 2.5 that $d_g(u, x) = d_g(G)$ (or $d_g(v, x) = d_g(G)$).

Now since every node of $G$ lies on some $u - v$ geodesic in $G$, the node $x$ also lies on some $u - v$ geodesic in $G$ and so we get $d_g(u, x) < d_g(u, v) = d_g(G)$ (or $d_g(v, x) < d_g(u, v) = d_g(G)$), which is a contradiction.

4. Conclusion

In this paper, the concept of forcing geodesic number of a fuzzy graph is introduced along with a suitable example. The forcing geodesic number of fuzzy graphs is characterized depending on the geodesic bases present in the fuzzy graph. The forcing geodesic number of fuzzy tree and of complete fuzzy graph are identified. A necessary condition for the forcing geodesic number of a fuzzy graph to be less than 2 is established.

Acknowledgements

The first author is grateful to the University Grants Commission (UGC), New Delhi, India, for providing the financial assistance.

References

[1] Zadeh L A 1965 Fuzzy sets Information and Control 8 338–353
[2] Rosenfeld A 1975 Fuzzy graphs, In: L A Zadeh, K S Fu and M Shimura (Eds), Fuzzy Sets and their Applications (New York: Academic Press) pp 77–95
[3] Yeh R T and Bang S Y 1975 Fuzzy relations, fuzzy graphs and their application to clustering analysis, In: Fuzzy sets and their Application to Cognitive and Decision Processes L A Zadeh K S Fu and M Shimura Eds (New York: Academic Press) pp 125–49
[4] Mordeson J N and Yao Y Y 2002 Fuzzy cycles and fuzzy trees The Journal of Fuzzy Mathematics 10 189–202
[5] Bhutani K R 1989 On automorphisms of fuzzy graphs Pattern Recognition Letters 9 159–162
[6] Mordeson J N 1993 Fuzzy line graphs Pattern recognition Letters 14 381–384
[7] Mordeson J N and Nair P S 1996 Cycles and Cocycles of fuzzy graphs *Information Sciences* **90** 39–49
[8] Bhattacharya P 1987 Some Remarks on fuzzy graphs *Pattern Recognition Letters* **6** 297–302
[9] Bhutani K R and Rosenfeld A 2003 Strong arcs in fuzzy graphs *Information Sciences* **152** 319–322
[10] Bhutani K R and Rosenfeld A 2003 Fuzzy end nodes in fuzzy graphs *Information Sciences* **152** 323–326
[11] Bhutani K R and Rosenfeld A 2003 Geodesics in fuzzy graphs *Electronic Notes in Discrete Mathematics* **15** 49–52
[12] Suvarna N T and Sunitha M S 2015 *Convexity and Types of Arcs & Nodes in Fuzzy Graphs* (Scholar’s Press)
[13] Linda J P and Sunitha M S 2015 *Geodesic and Detour distances in Graphs and Fuzzy Graphs* (Scholars’ Press)
[14] Harary F 1969 *Graph Theory* (WesleyAddison: Addison)
[15] Moderson J N and Nair P S 2000 *Fuzzy Graphs and Fuzzy Hypergraphs* (Heidelberg:Physica-Verlag)
[16] Chartrand G and Zhang P 1999 The Forcing Geodetic Number of a graph *Discussiones Mathematicae Graph Theory* **19** 45–58
[17] Rehmani S and Sunitha M S 2017 Minimum Geodetic Fuzzy Subgraph *Electronic Notes in Discrete Mathematics* **63** 415–424