Asymptotics of spacing distributions at the hard edge for $\beta$-ensembles

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Abstract

In a previous work [J. Math. Phys. 35 (1994), 2539–2551], generalized hypergeometric functions have been used to give a rigorous derivation of the large $s$ asymptotic form of the general $\beta > 0$ gap probability $E_{\beta}^{\text{hard}}(0; (0, s); \beta a/2)$, provided both $\beta a/2 \in \mathbb{Z}\geq 0$ and $2/\beta \in \mathbb{Z}^+$. It is shown how the details of this method can be extended to remove the requirement that $2/\beta \in \mathbb{Z}^+$. Furthermore, a large deviation formula for the gap probability $E_{\beta}(n; (0, x); ME_{\beta,N}(\lambda^{\beta/2}e^{\beta N\lambda/2}))$ is deduced by writing it in terms of the characteristic function of a certain linear statistic. By scaling $x = s/(4N)^2$ and taking $N \to \infty$, this is shown to reproduce a recent conjectured formula for $E_{\beta}^{\text{hard}}(0; (0, s); \beta a/2)$, $\beta a/2 \in \mathbb{Z}_{\geq 0}$, and moreover to give a prediction without the latter restriction. This extended formula, which for the constant term involves the Barnes double gamma function, is shown to satisfy an asymptotic functional equation relating the gap probability with parameters $(\beta, n, a)$, to a gap probability with parameters $(4/\beta, n', a')$, where $n' = \beta(n + 1)/2 - 1$, $a' = (a - 2)/2 + 2$.

1 Introduction

The topic of eigenvalue spacing distributions within random matrix theory is of interest both for its practical utility in comparisons with experimental data, and for its rich mathematical content leading to many explicit functional forms. Moreover, the function theory associated with the latter is intimately related to integrable systems theory, and this in turn offers a number of powerful asymptotic methods to further quantify the spacing distributions. We refer to [15] for survey articles relating to explicit functional forms, and to [23] for a recent review on the associated asymptotics.

Spacing distributions depend on a symmetry parameter $\beta$, and the region of the eigenvalue spectrum being scaled. In relation to $\beta$, according to Dyson’s three fold way [10], random matrix ensembles corresponding to quantum systems with a time reversal symmetry $T$ such that $T^2 = 1$ must have a probability density function invariant under conjugation by orthogonal matrices. If the quantum system does not have a time reversal symmetry,
or if the time reversal symmetry is such that \( T^2 = -1 \), the probability density function must be invariant under conjugation by unitary matrices, or by unitary symplectic matrices respectively. The three cases are labelled by the parameter \( \beta \), with \( \beta = 1 \) for orthogonal symmetry, \( \beta = 2 \) for unitary symmetry and \( \beta = 4 \) for unitary symplectic symmetry. One reason for this labelling is that the joint probability density function of \( 2 \times 2 \) matrices with these symmetries is generically proportional to \(|\lambda_1 - \lambda_2|^\beta\) for near degenerate levels.

Regarding the region of the eigenvalue spectrum being scaled, depending on the details of the ensemble under consideration the eigenvalue density \( \rho(\lambda) \) can exhibit various asymptotic forms. Our interest is when

\[
\rho(\lambda) = 0 \quad \text{for} \quad \lambda < 0, \quad \rho(\lambda) \sim \frac{1}{2\pi \lambda^{1/2}} \quad \text{for} \quad \lambda \to \infty. \quad \text{(1.1)}
\]

This corresponds to the so-called hard edge scaled state. As a concrete example of a random matrix ensemble which exhibits this behaviour, consider an \( n \times p \) \((n \geq p)\) real Gaussian matrix \( X \), each entry independently distributed as a standard normal, and form the corresponding covariance matrix \( X^\dagger X \). The joint eigenvalue probability density function is then proportional to

\[
\prod_{l=1}^{p} \lambda_l^{a \beta / 2} e^{-\beta \lambda_l / 2} \prod_{1 \leq j < k \leq p} |\lambda_k - \lambda_j|^\beta, \quad \lambda_l > 0, \quad \text{(1.2)}
\]

with \( a = n - p + 2 / \beta - 1 \) and \( \beta = 1 \). The same construction, but with \( X \) a standard complex Gaussian matrix gives (1.2) in the case \( \beta = 2 \). The probability density function (1.2) for general parameters will be referred to as \( \text{ME}_{\beta,N}(\lambda^{a \beta / 2} e^{-\beta \lambda / 2}) \). For general \( \beta > 0 \), fixed \( a \geq 0 \), and with \( \lambda_l \mapsto \lambda_l / 4p \), \((l = 1, \ldots, p)\), \( p \to \infty \), the eigenvalues in the neighbourhood of \( \lambda = 0 \) have spacing of order unity, and moreover exhibit the asymptotic behaviour (1.1).

We therefore see that the hard edge scaled state is specified by the symmetry parameter \( \beta \), and the power law exponent \( a \beta / 2 \) (which represents the microscopic repulsion from the origin) in (1.2). The notation \( E_{\beta}^{\text{hard}}(n; (0, s); a \beta / 2) \) will be used to refer to the probability that in the hard edge state so specified, there are exactly \( n \) eigenvalues in the interval \((0, s)\). Similarly, the notation \( E_{\beta}(n; (0, s); \text{ME}_{\beta,N}(\lambda^{a \beta / 2} e^{-\beta N \lambda / 2})) \) will be used for the probability that in the ensemble specified by (1.2), there are \( n \) eigenvalues in \((0, s)\).

The most prominent application in physics of the hard edge state is to the study of eigenvalues in lattice QCD. This comes about by the use of the matrix structure

\[
H = \begin{bmatrix}
O_{n \times n} & X \\
X^\dagger & O_{p \times p}
\end{bmatrix} \quad \text{(1.3)}
\]

to model the effective Hamiltonian for given topological anomaly \( \nu = n - p \), and with elements of \( X \) independent standard Gaussian which are real \((\beta = 1)\), complex \((\beta = 2)\) or real quaternion \((\beta = 4)\). The square of the non-zero eigenvalues of (1.3) are equal to the eigenvalues of (1.2), when the latter corresponds to the eigenvalue PDF of (1.2).
Significantly, the exact distribution of the $k$-th smallest eigenvalue, $p^{\text{hard}}(k; (0, s); \beta a/2)$ say, which is related to $\{E_\beta^{\text{hard}}(l; (0, s); \beta a/2)\}$ by

$$p^{\text{hard}}(k; (0, s); \beta a/2) = -\frac{d}{ds} \sum_{l=0}^{k} E_\beta^{\text{hard}}(l; (0, s); \beta a/2),$$

can be compared against data from lattice QCD simulations [12].

Very recently, the explicit $s \to \infty$ asymptotic form of $E_\beta^{\text{hard}}(n; (0, s); \beta a/2)$ for general $\beta > 0$, but requiring $\beta a/2 \in \mathbb{Z}_0$, has been derived up to an error term which goes to zero [23]. The derivation relies on an unproved conjecture relating to the asymptotics of a certain generalized hypergeometric function. Such multivariable special functions, introduced into random matrix theory by Constantine and Muirhead in the case $\beta = 1$ in the 60’s and 70’s (see [30]), are finding their way into a number of recent works relating to asymptotics of eigenvalue distributions [7, 35, 28, 18, 8, 19]. The result of [23] also assumes knowledge of the corresponding asymptotics of $E_\beta^{\text{hard}}(0; (0, s); \beta a/2)$. The latter was first studied using generalized hypergeometric functions in [14], and the result

$$E_\beta^{\text{hard}}(0; (0, s); \beta a/2) \sim e^{-\beta s/8 + \beta a s^{1/2}/2} \left( \frac{1}{s} \right)^{\alpha(\beta a/2+1)/4 - \beta a/4} \tau_{\beta a/2, \beta} \left( 1 + O\left( \frac{1}{s^{1/2}} \right) \right), \quad (1.4)$$

where

$$\tau_{\beta a/2, \beta} = 2^{(1-\beta/2)a} \left( \frac{1}{2\pi} \right)^{\beta a/2} \prod_{j=1}^{\beta a/2} \Gamma(2j/\beta), \quad (1.5)$$

was rigorously established for $\beta a/2 \in \mathbb{Z}_0$, $2/\beta \in \mathbb{Z}^+$. Soon after Dyson’s log-Coulomb gas method was used to derive (but not prove) this same asymptotic form, up to the explicit form of the constant, for general parameters [5]. In the case $\beta = 2$, (1.4) has been proved for general $\beta a/2 > -1$ in [8], and for $|\beta a/2| < 1$ in [11]. Very recently [33], stochastic differential equation methods based on tridiagonal matrices realizing the eigenvalue PDF (1.2) have been used to prove (1.4) for general $\beta > 0$, $\beta a/2 > -1$, up to the explicit form of the constant.

These developments motivate us to further consider the asymptotics of the hard edge gap probability $E_\beta^{\text{hard}}(n; (0, s); \beta a/2)$, both for $n = 0$ and general $n \in \mathbb{Z}^+$. In Section 2 we summarize previous work on this problem, as has been deduced using the method of generalized hypergeometric functions. In particular, the theory leading to (1.4) is revised. The aim of Section 3 is to further develop this theory, so that the form (1.4) can be rigorously established for $\beta a/2 \in \mathbb{Z}_0$ and general $\beta > 0$. In Section 4 we make use of a Gaussian fluctuation formula for linear statistics to determine a large deviation formula for $E_\beta^{\text{hard}}(n; (0, x); ME_{\beta,N}(\lambda^{a\beta/2} e^{\beta N x/2}))$, first in the case $n = 0$, then for general $n$. Substituting $x = s/(4N)^2$, then taking $N \to \infty$, these large deviation formulas are shown to scale to the asymptotic formulas for $E_\beta^{\text{hard}}(n; (0, s); \beta a/2)$ of Section 2, and furthermore give formulas which hold for general $\beta a/2$ and $\beta > 0$. This latter point requires a suitable continuation of the product in (1.5) beyond positive integer values of its upper terminal, which is achieved by making use of the Barnes double gamma function. As a check on these formulas,
an asymptotic functional equation known from [16], which relates the asymptotics of the spacing for a particular \((\beta, n, a)\) to those for parameters \((4/\beta, \beta(n+1)/2-1, \beta(a-2)/2+2)\), is shown to hold true.

## 2 Generalized hypergeometric function expressions

It turns out that for \(\beta a/2 \in \mathbb{Z}^+\), the gap probability \(E_\beta(n; (0, s); \text{ME}_{\beta,N}(\lambda a^\beta e^{\beta N\lambda/2}))\) can be recognized as an integral representation of certain hypergeometric functions based on Jack polynomials. Moreover, the hard edge scaling limit corresponds to a confluence limit of the hypergeometric function in question, giving back another hypergeometric function for which there is also a known integral representation, now as a \(\beta a/2\)-dimensional integral and further requiring \(2/\beta \in \mathbb{Z}^+\) [14].

The class of generalized hypergeometric functions in question can be defined by the series

\[
pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; x_1, \ldots, x_m) := \sum_{\kappa} \frac{1}{|\kappa|!} \frac{[a_1]_{\kappa}^{(a)} \cdots [a_p]_{\kappa}^{(a)}}{[b_1]_{\kappa}^{(a)} \cdots [b_q]_{\kappa}^{(a)}} C_{\kappa}^{(a)}(x_1, \ldots, x_m). \tag{2.1}
\]

Here the sum is over all partitions \(\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_m \geq 0\) of non-negative integers, \(|\kappa| := \sum_{j=1}^m \kappa_j\), and the generalized Pochhammer symbol \([a]_{\kappa}^{(a)}\) is defined by

\[
[a]_{\kappa}^{(a)} = \prod_{j=1}^m \left(a - \frac{1}{\alpha}(j-1)\right)_{\kappa_j}, \quad (a)_k = a(a+1) \cdots (a+k-1). \tag{2.2}
\]

The function \(C_{\kappa}^{(a)}(x_1, \ldots, x_m)\) is proportional to the Jack symmetric polynomial (see e.g. [17, §12.6]), and as such is a homogeneous symmetric polynomial of degree \(|\kappa|\). For \(m = 1\), \(C_{\kappa}^{(a)} = x^{\kappa_1}\) and (2.1) reduces to the classical definition of \(pFq\) in one variable. Like \(pFq\) in one variable, \(pFq^{(a)}\) exhibits the confluence property

\[
\lim_{a_p \to \infty} pFq^{(a)}(a_1, \ldots, a_p; b_1, \ldots, b_q; x_1/a_p, \ldots, x_m/a_p) = p-1Fq^{(a)}(a_1, \ldots, a_{p-1}; b_1, \ldots, b_q; x_1, \ldots, x_m). \tag{2.3}
\]

In some cases there are explicit integral formulas for the generalized hypergeometric functions. One example is

\[
1F1^{(\beta/2)}(-N; a+2m/\beta; t_1, \ldots, t_m) = \frac{1}{C_{\beta a/2+m,\beta,m}} \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \prod_{j=1}^N \left(x_j^{\beta/2} e^{-\beta x_j/2} \prod_{k=1}^m (x_j - t_k)\right) \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta} \tag{2.4}
\]

where \(C_{\beta a/2+m,\beta,m}\) is equal to the integral with \(t_k = 0\), \((k = 1, \ldots, m)\). This integral formula, obtained explicitly in [13], follows as a corollary of a similar integral formula for \(2F1^{(\beta/2)}\) due to Kaneko [25].
On the other hand the gap probability \( E_\beta(0; (0, s); ME_{\beta,N}(\lambda^{a/2} e^{-\beta \lambda/2})) \) is obtained from (1.2) by integrating each \( \lambda_l \) over \((0, s)\). Changing variables \( \lambda_l \mapsto \lambda_l + s \) allows this gap probability to be recognized in terms of the integral in (2.4), provided \( \beta a/2 \in \mathbb{Z}_{\geq 0} \), thus showing that in this circumstance

\[
E_\beta(0; (0, s); ME_{\beta,N}(\lambda^{a/2} e^{-\beta \lambda/2})) = e^{-\beta N s^2/2} F_1^{(\beta/2)}(-N; a; (-s)^{\beta a/2}).
\tag{2.5}
\]

Here, in the argument of \( F_1^{(\beta/2)} \), the notation \((u)^r\) means \( u \) repeated \( r \) times. Similarly, since by definition

\[
E_\beta(n; (0, s); ME_{\beta,N+n}(\lambda^{a/2} e^{\beta N \lambda/2})) = \frac{(N)_n}{n!} \frac{C_{\beta a/2, \beta, N}}{C_{\beta a/2, \beta, N+n}} \int_0^n dy_1 \cdots \int_0^n dy_n \left( \prod_{l=1}^n y_l^{a/2} e^{-\beta y_l/2} \right) \prod_{1 \leq j < l \leq n} |y_j - y_l|^\beta
\times \int_s^\infty dx_1 \cdots \int_s^\infty dx_N \left( \prod_{j=1}^N (x_j^{a/2} e^{-\beta x_j/2} \prod_{l'=1}^n |x_l - y_{l'}|^{\beta}) \right) \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta},
\tag{2.6}
\]

we see from (2.4) that for \( \beta a/2 \in \mathbb{Z}_{\geq 0}, \beta \in \mathbb{Z}^+ \).

\[
E_\beta(n; (0, s); ME_{\beta,N+n}(\lambda^{a/2} e^{-\beta \lambda/2})) = \frac{(N)_n}{n!} \frac{C_{\beta a/2, \beta, N}}{C_{\beta a/2, \beta, N+n}} e^{-\beta s(N+k)/2} \int_0^n dy_1 \cdots \int_0^n dy_n \left( \prod_{l=1}^n (s - y_l)^{a/2} e^{\beta y_l/2} \right) \prod_{1 \leq j < l \leq n} |y_j - y_l|^\beta
\times F_1^{(\beta/2)}(-N; a + 2n; (-s)^{a/2}; (-y_1)^a, \ldots, (-y_n)^a).
\tag{2.7}
\]

To proceed from (2.5) and (2.7) to the computation of \( E_\beta^{\text{hard}}(0; (0, s); \beta a/2) \) requires that

\[
s \mapsto s/4N, \quad N \to \infty.
\tag{2.8}
\]

Using the confluence (2.3) it is a simple exercise to deduce from (2.5) that for \( \beta a/2 \in \mathbb{Z}_{\geq 0} \)

\[
E_\beta^{\text{hard}}(0; (0, s); \beta a/2) = e^{-\beta s/8} F_0^{(\beta/2)}(a; (s/4)^{\beta a/2}).
\tag{2.9}
\]

Similarly, using (2.3) and (2.7), together with knowledge of the explicit gamma function functional form of \( C_{a,\beta,N} \) [17, Prop. 4.7.3] one has that for \( \beta a/2 \in \mathbb{Z}_{\geq 0}, \beta \in \mathbb{Z}^+ \)

\[
E_\beta^{\text{hard}}(n; (0, s); \beta a/2) = A_{\beta}(n, a)s^{n+(\beta/2)n(a+1)} e^{-\beta s/8}
\times \int_0^1 dy_1 \cdots \int_0^1 dy_n \prod_{j=1}^n (1 - y_j)^{\beta a/2} \prod_{1 \leq j < k \leq n} |y_k - y_j|^\beta
\times F_0^{(\beta/2)}(a + 2n; (s/4)^{\beta a/2}; (sy_1/4)\beta, \ldots, (sy_n/4)\beta),
\tag{2.10}
\]

5
where
\[
A_\beta(n, a) = \frac{1}{2^{2n}n!} \left( \frac{\beta}{2} \right)^n \frac{\beta^{n(a+n-1)\beta}}{4^n \prod_{j=0}^{2n-1} \Gamma(a\beta/2 + 1 + j\beta/2)}. \tag{2.11}
\]

Next, we want to revise how (2.9) and also (2.10) can be used for purposes of computing the \( s \to \infty \) asymptotics. In relation to (2.9), this follows from a \( \beta a/2 \)-dimensional integral formula for \( {}^1{}_{\beta}F_1^{(\beta/2)} \) in [23, Conj. 8]. In addition to the requirement \( \beta a/2 \in \mathbb{Z}_{\geq 0} \), the integral formula also requires that \( 2/\beta \in \mathbb{Z}^+ \). Under these assumptions, we then have [14]

\[
E_{\beta}^{\text{hard}}(0; (0, s); \beta a/2) = B_{a, \beta} e^{-\beta s/8} \left( \frac{1}{s} \right)^{(1+2/\beta)\beta a/2} \left( \frac{1}{2\pi} \right)^{a\beta/2} \times \int_{[-\pi, \pi]^{\beta a/2}} e^{\beta a/2 \cos \theta_1 e^{i(1-1/\beta)\theta_1}} \prod_{1 \leq j < k \leq \beta a/2} \left| e^{i\theta_1} - e^{i\theta_j} \right|^{1/\beta} d\theta_1 \cdots d\theta_{\beta a/2}, \tag{2.12}
\]

where
\[
B_{a, \beta} = \prod_{j=1}^{a\beta/2} \Gamma(1 + 2/j/\beta) \Gamma(2j/\beta). \tag{2.13}
\]

It was by using Laplace’s asymptotic method to this integral that (1.4) was derived. In [14] the asymptotic form (1.4) was conjectured to hold true for general \( \beta > 0 \), and general \( \beta a/2 \geq -1 \) (the latter subject to an appropriate interpretation of the product in (1.5), identified in [23] as relating to the Barnes double gamma function; see (4.13) below).

For the generalized hypergeometric function appearing in (2.10) there is no explicit integral form analogous to (2.12). Nonetheless a conjectured asymptotic form is available, which states that for \( s \to \infty \) and \( y_1, \ldots, y_n \approx 1 \) [23, eq. (3.45)],

\[
\begin{align*}
\phantom{=} & {}^1{}_{\beta}F_1^{(\beta/2)}(., c; (s/4)^{\beta a/2}, (sy_1/4)^\beta, \ldots, (sy_n/4)^\beta) \\
= & {}^1{}_{\beta}F_1^{(\beta/2)}(.; c; (s/4)^{\beta(a+2n)/2}) e^{\beta/\sqrt{s} \sum_{j=1}^{n}(1-y_j)^2} \left( 1 + O\left( \frac{1}{s^{1/2}} \right) \right), \tag{2.14}
\end{align*}
\]

and this used in (2.10) gives a conjecture for the \( s \to \infty \) asymptotics of \( E_{\beta}^{\text{hard}}(n; (0, s); \beta a/2) \) [23, Conj. 8].

**Conjecture 1.** For \( s \to \infty \) we have

\[
\frac{E_{\beta}^{\text{hard}}(n; (0, s); \beta a/2)}{E_{\beta}^{\text{hard}}(0; (0, s); \beta a/2)} = \tau_{\beta a/2, \beta}(n) \exp \left( -\beta \left\{ -\sqrt{s}n + \left( \frac{n^2}{2} + \frac{na}{2} \right) \log s^{1/2} \right\} \right) \left( 1 + O\left( \frac{1}{s^{1/2}} \right) \right), \tag{2.15}
\]

where in the case \( \beta n \in \mathbb{Z}_{\geq 0} \)

\[
\tau_{\beta a/2, \beta}(n) = \frac{2^{-(a+n)n \beta} n^{n(a+n-1)\beta/2}}{n!} \prod_{j=1}^{\beta n} \Gamma(a + 2j/\beta) \prod_{j=0}^{n-1} \Gamma(1 + (j + 1)\beta/2) \prod_{j=n}^{n-1} \Gamma(1 + (j + a)\beta/2). \tag{2.16}
\]
3 Integral formula for $E^\text{hard}_\beta(0; (0, s); \beta a/2)$ with $\beta a/2 \in \mathbb{Z}^+, \beta > 0$

We would like to generalize (2.12) so that the restriction $2/\beta \in \mathbb{Z}^+$ can be removed. For this we return to the finite $N$ gap probability formula (2.5), which has no such restriction. Thus with $\beta a/2 \in \mathbb{Z}_{\geq 0}$, but $\beta > 0$ general, we can use a $\beta a/2$-dimensional integral form of $1F_1^{(\beta/2)}(-N; a; (-s)^{\beta a/2})$, deduced in turn from an integral formula for $2F_1^{(\beta/2)}(-N, b; c; (s)^{\beta a/2})$ [17] Exercises 13.1 Q4(i) to deduce from (2.5) the following formula.

**Proposition 1.** We have

$$E_\beta(0; (0, s); \text{ME}_{\beta, N}(\lambda^{\alpha/2} e^{\beta N \lambda/2})) = e^{-\beta N s/2} \frac{1}{M_{\beta a/2}(N, -1 + 2/\beta, 2/\beta)} \times \int_{-1/2}^{1/2} dx_1 \cdots \int_{-1/2}^{1/2} dx_{\beta a/2} \left( \prod_{i=1}^{\beta a/2} e^{2\pi i x_i (-1 + 2/\beta)} (1 + e^{-2\pi i x_i})^{(1 + 2/\beta + N)} e^{s e^{2\pi i x_i}} \right) \times \prod_{1 \leq j < k \leq \beta a/2} |e^{2\pi i x_k} - e^{2\pi i x_j}|^{4/\beta},$$

(3.1)

where $M_{\alpha}(a, b, c)$ denotes the Morris integral [17] (Eq. 4.4)], which is evaluated as a product of gamma functions.

**Proof.** We indeed proceed as in [17] Exercises 13.1 Q4(i). Thus we begin with the formula [17] Eq. (13.11)]

$$\frac{1}{M_{\alpha}(a, b, c)} \int_{-1/2}^{1/2} dx_1 \cdots \int_{-1/2}^{1/2} dx_n \prod_{l=1}^{n} e^{\pi i x_l (a-b)} |1 + e^{2\pi i x_l}|^{a+b} (1 + t e^{2\pi i x_l})^{-r} \times \prod_{1 \leq j < k \leq n} |e^{2\pi i x_k} - e^{2\pi i x_j}|^{2/\alpha} = 2F_1^{(\alpha)} \left( r, -b; \frac{1}{\alpha} (n-1) + a + 1; (t)^{\alpha} \right).$$

(3.2)

Replacing $t$ by $t/r$ and taking $r \to \infty$ gives

$$\frac{1}{M_{\alpha}(a, b, c)} \int_{-1/2}^{1/2} dx_1 \cdots \int_{-1/2}^{1/2} dx_n \prod_{l=1}^{n} e^{\pi i x_l (a-b)} |1 + e^{2\pi i x_l}|^{a+b} e^{-te^{2\pi i x_l}} \prod_{1 \leq j < k \leq n} |e^{2\pi i x_k} - e^{2\pi i x_j}|^{2/\alpha} = 1F_1^{(\alpha)} (-b; a + 1 + (n-1)/\alpha; (t)^{\alpha}),$$

where on the RHS use has been made of (2.3). Recalling (2.5), we obtain a $\beta a/2$-dimensional integral formula for $E_\beta(0; (0, s); \text{ME}_{\beta, N}(\lambda^{\alpha/2} e^{\beta N \lambda/2}))$ by setting $n = \beta a/2, \alpha = \beta/2, t = -s, b = N$ and $a = 2/\beta - 1$. Simple manipulation of $|1 + e^{2\pi i x_l}|^{a+b}$ then gives (3.1). \qed
Essentially the same integral formula (3.1), but restricted to \(2/\beta \in \mathbb{Z}^+\), was used in [14] to compute the scaled limit (2.8) and thus derive (2.12). The utility of the assumption \(2/\beta \in \mathbb{Z}^+\) is that the integrand can readily be rewritten to be analytic in

\[ z_k := e^{2\pi i \theta_k}, \quad (k = 1, \ldots, \beta a/2) \]  

(3.3)

except for a pole at the origin, thus allowing for the integration domain \(|z_k| = 1, (k = 1, \ldots, \beta a/2)\), to be deformed. Deformation of the contours played a crucial role in the computing of the scaled limit (2.8). We will now show that it is possible to deform the contours, and so compute (2.8), without the need to assume \(2/\beta \in \mathbb{Z}^+\).

**Proposition 2.** Let \(C\) be the contour which starts at the origin, runs along the negative real axis in the bottom half plane to \(z = -1 - 0i\), then along a counter clockwise circle to \(z = -1 + 0i\), and finally back along the negative real axis in the upper half plane. Let \(B_{a,\beta}\) be specified by (2.13), and assume \(\beta a/2 \in \mathbb{Z} \geq 0\). We have

\[
E_{\beta}^{\text{hard}}(0; (0, s); \beta a/2) = B_{a,\beta} e^{-\beta s/8} \left( \frac{1}{s} \right)^{(1+2/\beta)\beta a/2} \left( \frac{1}{2\pi} \right)^{a\beta/2} \times \int_{C_{\beta a/2}} e^{s^{1/2}(z_j+1/z_j)/2} z_{j}^{-1+2/\beta} \prod_{1 \leq j < k \leq \beta a/2} \frac{(z_k - z_j)(1/z_k - 1/z_j)^{2/\beta} dz_j}{2\pi i z_{j} 2\pi i z_{\beta a/2}}.
\]

(3.4)

Proof. We begin by noting that the integrand in (3.1) is symmetric in the variables \(\{x_j\}\), so we are free to choose the ordering

\[-1/2 \leq x_1 < x_2 < \cdots < x_{\beta a/2} < 1/2,\]

(3.5)

provided the integral is multiplied by \((\beta a/2)!\). But with this ordering

\[
\prod_{1 \leq j < k \leq \beta a/2} |e^{2\pi i x_k} - e^{2\pi i x_j}|^{1/\beta} = \prod_{1 \leq j < k \leq \beta a/2} \left( 2 \sin \pi(x_k - x_j) \right)^{4/\beta},
\]

(3.6)

which is an analytic function of each of the variables (3.3) in the corresponding complex planes cut along the negative real axis. The factors \(\prod_{l=1}^{\beta a/2} e^{2\pi i z_l (-1+2/\beta)(1+e^{-2\pi i z_l})^{-1+2/\beta+N}}\) share this property, which thus becomes a property of the integrand.

Let \(C_{N,t}\) denote the contour which starts at \(z = -1\), runs along the negative real axis in the bottom half plane to \(z = -N/\sqrt{t} - 0i\), then along a counter clockwise circle to \(z = -N/\sqrt{t} + 0i\), and finally back along the negative real axis in the upper half plane to \(z = -1 + 0i\). The analyticity properties of the integrand just discussed allow us to deform the original unit circle contours in the variables (3.3) to the contours \(C_{N,t}\), provided an ordering equivalent to (3.5) is adopted. Furthermore, in the variables (3.3), the integrand
and measure in (3.1) reads

$$
\prod_{j=1}^{\beta a/2} e^{s_1/2(z_j+1/z_j)/2} z_j^{-1+2/\beta} \left( 1 + \frac{1}{z_j} \right)^{-1+2/\beta + N/2} \prod_{1 \leq j < k \leq \beta a/2} \left( \frac{z_k - z_j}{z_k + 1/z_j} \right)^{2/\beta} \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_{\beta a/2}}{2\pi i z_{\beta a/2}}. \tag{3.7}
$$

This being symmetric in \( \{z_j\} \) we can ignore the ordering constraint, provided we remove the earlier introduced factor of \((\beta a/2)!\).

Now change variables \( z_j \mapsto (N/\sqrt{s}) z_j \). We see that in the scaling limit (2.8), (3.7) becomes proportional to \((N/\sqrt{s})^{2}\beta a/2\) times the integrand in (3.4), thus accounting for all \( s \)-dependent terms in the latter. The computation of the proportionality relies on also computing the asymptotics of the Morris integral in (3.1). But this is the very same task as already carried out in [14], and the factors as presented in (3.4) result. Finally, we note that with this change of variables and in the \( N \to \infty \) limit each contour \( C_{N,t} \) becomes the contour \( C \), with the integral being well defined thereon.

We remark that in the case \( 2/\beta \in \mathbb{Z}^+ \), the integrand no longer has a branch cut along the negative real axis. The contributions to the integral of the portions of \( C \) running along the latter therefore cancel, and after parameterizing the remaining unit circle, we reclaim (2.12).

The computation of the large \( s \) asymptotic form of (3.4) is essentially the same as done in [14] for (2.12). The reason is that method of stationary phase tells us that the maximum contribution to the integrand then comes from the neighbourhood of \( z_j = 1 \). Thus the portion of \( C \) running along the negative real axis plays no role in this limit. The working of [14] is therefore justified without the restriction \( 2/\beta \in \mathbb{Z}^+ \), and so we obtain (1.4), proved now for \( \beta a/2 \in \mathbb{Z}_0^+ \), and general \( \beta > 0 \).

### 4 Gaussian fluctuation formulas

The first aim of this section is to derive the large \( N \) asymptotic form of

$$
E_\beta(0; (0, \tilde{s}); \text{ME}_{\beta,N}^\epsilon(\lambda a^{\beta/2} e^{-2\beta N \lambda})).
$$

We will see that by scaling the corresponding asymptotic formula according to

$$
\tilde{s} \mapsto s/(4N)^2, \quad N \to \infty, \tag{4.1}
$$

we can reclaim (1.4). We will demonstrate too that an analogous procedure can be applied to compute the asymptotics of the scaling limit for there being \( n \) eigenvalues at \( \{y_j\}_{j=1,\ldots,n} \) in \((0, s)\) in the ensemble \( \text{ME}_{\beta,N+n}^\epsilon(\lambda a^{\beta/2} e^{-2\beta N \lambda}) \).
4.1 Strategy

Our method relies on reformulating the problem as one of computing the asymptotic form of a moment of the characteristic polynomial for $\text{ME}_{\beta,N}(e^{-2\beta N\lambda})$. In two recent works by the author [21, 20], the spectral densities for the Gaussian, Laguerre and Jacobi $\beta$-ensembles have been similarly formulated, and the corresponding asymptotics computed by recognizing that such averages can be interpreted as the characteristic function for the linear statistic $V(x) = \sum_{l=1}^{N} \log |x - \lambda_l|$. The significance of this is that for $\text{ME}_{\beta,N}(e^{-\beta N\lambda})$, which is the scaled Laguerre ensemble with $a = 0$, eigenvalue density $\rho_{(1),N}(t)$ supported to leading order on $(0,1)$, it is a known theorem [3] that

$$\langle N \prod_{l=1}^{N} (x + \lambda_l)^c \rangle_{\text{ME}_{\beta,N}(e^{-2\beta N\lambda})} \sim_{N \to \infty} e^{c\mu_N(v)} e^{(c\sigma(v))^2/2},$$

where, with $v(t) := \log(x + t), x > 0$

$$\mu_N(v) = \int_0^1 \rho_{(1),N}(t) v(t) dt \quad \text{(4.3)}$$

$$\sigma(v)^2 = \frac{1}{\beta \pi^2} \int_0^1 dt_1 \int_0^1 \frac{v(t_1)}{((1-t_1)t_1)^{1/2}} \int_0^1 dt_2 \frac{v'(t_2)}{t_2-t_1} \frac{((1-t_2)t_2)^{1/2}}{t_2-t_1}$$

$$= \frac{1}{2\beta} \sum_{k=1}^{\infty} k a_k^2, \quad a_k = \frac{2}{\pi} \int_0^\pi \left( \frac{1}{2} + \frac{1}{2} \cos \theta \right) \cos k\theta d\theta,$$

and the notation $\sim_{N \to \infty}$ means that in the $N \to \infty$ limit the ratio of the LHS and RHS tends to unity. In words, (4.3) says the characteristic function for the linear statistic exhibits Gaussian fluctuations, with an explicit mean (which is of order $N$), and an explicit variance (which is of order unity).

We will now show how this formalism applies to computing the sought large $N$ asymptotic forms.

4.2 Large $N$ form of $E_\beta(0; (0, \tilde{s}); \text{ME}_{\beta,N}(\lambda^{\alpha/2} e^{-\beta/2}))$

Our first task is to show that $E_\beta(0; (0, \tilde{s}); \text{ME}_{\beta,N}(\lambda^{\alpha/2} e^{-\beta/2}))$ can be expressed in terms of the average on the LHS of (4.2).

**Lemma 1.** Let $\mathcal{N}(\text{ME}_{\beta,N}(x^{\beta/2} e^{-\beta x/2}))$ denote the normalization required to make (1.2) a probability density function, and let $\tilde{s} = s/4N$. We have

$$E_\beta(0; (0, \tilde{s}); \text{ME}_{\beta,N}(\lambda^{\alpha/2} e^{-\beta/2})) = e^{-\beta N s/2} (4N)^{\alpha/2} \frac{\mathcal{N}(\text{ME}_{\beta,N}(e^{-\beta x/2}))}{\mathcal{N}(\text{ME}_{\beta,N}(x^{\beta/2} e^{-\beta x/2}))} \left\langle \prod_{l=1}^{N} (\tilde{s} + \lambda_l)^{\alpha/2} \right\rangle_{\text{ME}_{\beta,N}(e^{-\beta N\lambda})}. \quad (4.5)$$
Proof. A simple change of variables $\lambda_l \mapsto \lambda_l + \tilde{s}$ in the definition shows
\[
E_\beta(0; (0, s); ME_{\beta, N}(x^{\beta a/2} e^{-\beta x/2}))
= e^{-\beta N \tilde{s}/2} \frac{N(ME_{\beta, N}(x^{\beta a/2} e^{-\beta x/2}))}{N(ME_{\beta, N}(x^{\beta a/2} e^{-\beta x/2}))} \left\langle \prod_{l=1}^N (s + \lambda_l)^{\beta a/2} \right\rangle_{ME_{\beta, N}(e^{-\beta \lambda/2})}.
\]
We see from the definitions that with $\tilde{s} = s/4N$. The result \((4.5)\) now follows by noting that the change of variables $\lambda_l \mapsto 4N\lambda_l$ in the average on the RHS implies
\[
\left\langle \prod_{l=1}^N (s + \lambda_l)^{\beta a/2} \right\rangle_{ME_{\beta, N}(e^{-\beta \lambda/2})} = (4N)^N \left\langle \prod_{l=1}^N (\tilde{s} + \lambda_l)^{\beta a/2} \right\rangle_{ME_{\beta, N}(e^{-2\beta N \lambda})}. \]

We know from [22, eqns. (6.21) and (6.22)] and [21, Eq. (4.1) with $N = M$] that for $ME_{\beta, N}(e^{-2\beta N x})$ the density on $[0, 1]$ is such
\[
2t\rho_{(1)},N(t^2) = \frac{4N}{\pi}(1 - t^2)^{1/2} + \left(\frac{1}{2\beta} - \frac{1}{4}\right) \left(\delta(t - 1) - \delta(t)\right) + O\left(\frac{1}{N}\right). \tag{4.6}
\]
With this explicit form, we want to compute \((4.3)\). The following integral evaluation is required.

**Lemma 2.** Let $x > 0$. We have
\[
\frac{2}{\pi} \int_{-1}^{1} \log |ix + t| (1 - t^2)^{1/2} dt = \sqrt{x(x + 1)} - x - \log \left(2(\sqrt{x^2 + 1} - \sqrt{x})\right) - \frac{1}{2}. \tag{4.7}
\]

Proof. This can be deduced by an appropriate analytic continuation of [21, Eq. (3.2)].
\[\square\]

**Corollary 1.** We have
\[
\int_0^1 \log |\tilde{s} + t| \rho_{(1)},N(t) dt = 2N \left(\sqrt{\tilde{s}(\tilde{s} + 1) - \tilde{s}} - \log \left(2(\sqrt{\tilde{s}^2 + 1} - \sqrt{\tilde{s}})\right) - \frac{1}{2}\right) + \left(\frac{1}{2\beta} - \frac{1}{4}\right) \log \left|\frac{1 + \tilde{s}}{\tilde{s}}\right| \tag{4.8}
\]

Proof. This follows by first noting
\[
\int_0^1 \log |\tilde{s} + t| \rho_{(1)},N(t) dt = \int_{-1}^{1} \log |i\sqrt{\tilde{s}} + t|(2t\rho_{(1)},N(t^2)) dt,
\]
where on the RHS $2t\rho_{(1)},N(t)$ is given by the RHS of \((4.36)\), extended to be an even function by the addition of $\left(\delta(t + 1) - \delta(t)\right)$. Now \((4.8)\) can be read off from \((4.37)\). \[\square\]
For the variance as specified by (4.4), a straightforward modification of the working which gave [21, Eq. (4.4)] shows
\[
\sigma^2 = -\frac{2}{\beta} \log \left( \frac{\tilde{s}}{\tilde{s} + 1} \right)^{1/2} - \frac{4}{\beta} \log \left( 2((\sqrt{\tilde{s}^2 + 1}) - \tilde{s}) \right) - \frac{1}{2}.
\] (4.9)

Substituting (4.8) in (4.3) and (4.9) in (4.4), then substituting into (4.2) we read off the large \(N\) asymptotic form of the average on the RHS of (4.5).

Proposition 3. We have
\[
\langle \prod_{l=1}^{N} (\tilde{s} + \lambda)^{\beta a / 2} \rangle_{\text{ME}_{\beta,N}(e^{-2\beta N\lambda})} \sim \exp \left( N\beta a \left( \sqrt{\tilde{s}(\tilde{s} + 1)} - \tilde{s} - \log \left( 2((\sqrt{\tilde{s}^2 + 1}) - \sqrt{\tilde{s}}) \right) - \frac{1}{2} \right) + \frac{\beta a}{4} \left( \frac{1}{\beta} - \frac{1}{2} \right) \log \left| 1 + \frac{1}{\tilde{s}} \right| \right) \times \exp \left( -\frac{\beta a^2}{4} \log(\tilde{s}(\tilde{s} + 1))^{1/2} + \frac{\beta a^2}{2} \log \left( \frac{\tilde{s} + 1/2 + \tilde{s}^{1/2}}{2} \right) \right).
\] (4.10)

Recalling (4.5), it remains to compute the large \(N\) forms of the ratio of normalizations. Each is an example of a limiting form of the Selberg integral, and so is given by a product of gamma functions (see e.g. [17, Prop. 4.7.3]),
\[
\frac{\mathcal{N}(\text{ME}_{\beta,N}(e^{-\beta x/2}))}{\mathcal{N}(\text{ME}_{\beta,N}(x^{\beta a/2}e^{-\beta x/2}))} = (\beta/2)^{Na/2} \prod_{j=0}^{N-1} \frac{\Gamma(1 + j\beta/2)}{\Gamma(a\beta/2 + 1 + j\beta/2)}. \tag{4.11}
\]

In the case that \(a \in \mathbb{Z}_{\geq 0}\), simple manipulation of the product and use of Stirling’s formula shows that the large \(N\) form of (4.11) is
\[
N^{-aN^{\beta/2}}(N\beta/2)^{-\beta(a-1)/4} \pi N\beta^{-a/2}e^{aN^{\beta/2}} \prod_{j=0}^{a-1} \Gamma(1 + j\beta/2). \tag{4.12}
\]

To compute the large \(N\) form of (4.11) without the assumption \(a \in \mathbb{Z}_{\geq 0}\), we follow the lead of the recent works [4, 31] and introduce the Barnes double gamma function \(\Gamma_2\). This function is related to the usual gamma function through the two functional equations
\[
\frac{1}{\Gamma_2(z + 1; 1, \tau)} = \frac{\tau^{z/\tau - 1/2}}{\sqrt{2\pi}} \frac{\Gamma(z/\tau)}{\Gamma_2(z; 1, \tau)}, \quad \frac{1}{\Gamma_2(z + \tau; 1, \tau)} = \frac{1}{\sqrt{2\pi}} \frac{\Gamma(z)}{\Gamma_2(z; 1, \tau)}, \tag{4.13}
\]
and is normalized by requiring that \(\lim_{z \to 0} z \Gamma_2(z; 1, \tau) = 1\). A result of Shintani [34] gives that \(\Gamma_2(z; 1, \tau)\) can be written as an infinite product of gamma functions,
\[
\Gamma_2(z; 1, \tau) = (2\pi)^{z/2} \exp \left\{ \left( \frac{z - z^2}{2\tau} - \frac{z}{2} \right) \log \tau + \frac{(z^2 - z)\gamma}{2\tau} \right\} \times \Gamma(z) \prod_{n=1}^{\infty} \frac{\Gamma(z + n\tau)}{\Gamma(1 + n\tau)} \exp \left\{ \frac{z^2}{2n\tau} - (1 - z) \log(n\tau) \right\}. \tag{4.14}
\]
where \( \gamma \) denotes Euler’s constant. Now, as shown in [4], it is easy to check from the underlying recurrence that

\[
f_{\beta/2}(n+1) := \prod_{j=0}^{n} \Gamma(1 + \beta j/2)
\]

can be written in terms of the Barnes double gamma function according to

\[
f_{\beta/2}(n+1) = (2\pi)^{n/2} \tau^{-(n^2 - n(1-\tau))/2 \tau} \frac{\Gamma(n) \Gamma(1 + n/\tau)}{\Gamma_2(n; 1, \tau)}, \quad \tau := 2/\beta.
\] (4.15)

Further use of (4.13) shows that this simplifies to give

\[
f_{\beta/2}(n) = (2\pi)^{(n+1)/2} \tau^{-(n-1)n/2 - n/2} \frac{1}{\Gamma_2(n + \tau; 1, \tau)}.
\] (4.16)

But for \( a \in \mathbb{Z}_{\geq 0} \) we have that the product in (4.12) is equal to \( f_{\beta/2}(a) f_{\beta/2}(N)/f_{\beta/2}(N + a) \). This product is uniquely determined by its recurrence in \( N \), and its value 1 at \( N = 0 \), and thus the form implied by (4.15) persists without the assumption \( a \in \mathbb{Z}_{\geq 0} \). Moreover, the asymptotic form of \( \Gamma_2(z; 1, \tau) \) [32] then tells us that the large \( N \) form of \( f_{\beta/2}(N)/f_{\beta/2}(N + a) \) continues naturally off its \( a \in \mathbb{Z}_{\geq 0} \) form, and so can be read off (4.12). Thus the latter without the assumption \( a \in \mathbb{Z}_{\geq 0} \) reads

\[
N^{-aN\beta/2} (N\beta/2)^{-\beta(a-1)/4} (\pi N\beta)^{-a/2} e^{aN\beta/2} f_{\beta/2}(a),
\] (4.17)

where \( f_{\beta/2}(a) \) is specified by (4.16). Substituting (4.10) and (4.17) in (4.5) with \( c = \beta a/2 \) the sought large deviation formula results.

**Corollary 2.** Let \( \tilde{s} = s/(4N) \), and let \( f_{\beta/2}(a) \) be specified by (4.16). With \( \tilde{s} \) fixed and \( a \in \mathbb{Z}_{\geq 0} \) we have

\[
E_{\beta}(0; (0, s); ME_{\beta} \left( \lambda^{\beta a/2} e^{-\lambda/2} \right)) \sim
\]

\[
e^{-\beta^2 N^2 \tilde{s}} (N\beta/2)^{-\beta(a-1)/4} (\pi N\beta)^{-a/2} f_{\beta/2}(a)
\]

\[
\times \exp \left\{ N\beta a \left( \sqrt{\tilde{s}(\tilde{s} + 1)} - \tilde{s} + \log \left( \sqrt{\tilde{s}^2 + 1 + \sqrt{s}} \right) \right) + \frac{\beta a}{4} \left( \frac{1}{\beta} - \frac{1}{2} \right) \log \left| 1 + \frac{1}{\tilde{s}} \right| \right\}
\]

\[
\times \exp \left\{ - \frac{\beta a^2}{4} \log (\tilde{s}(\tilde{s} + 1))^{1/2} + \frac{\beta a^2}{2} \log \left( \frac{\tilde{s} + 1}{2} + \frac{\tilde{s}^{1/2}}{2} \right) \right\}.
\] (4.18)

An idea that goes back to Dyson [9] is to now scale the large deviation formula, which at the hard edge requires (2.8), to deduce the large \( s \) expansion of \( E_{\beta}^{\text{hard}}(0; (0, s); \beta a/2) \). A rigorous justification of this procedure, which plays an essential role in a recent study of asymptotics of the soft edge gap probability [2], requires that a uniform error bound be provided with the large deviation formula [27]. Unfortunately the present method does not provide us with an error estimate. Nonetheless we find that the hard edge scaling of (4.18) does reclaim (1.4).

**Proposition 4.** Setting \( \tilde{s} = s/(4N)^2 \), then taking \( N \to \infty \), the RHS of (4.18) is equal to

\[
A_{\alpha, \beta} \exp \left( - \frac{\beta s}{8} + \frac{\beta a}{2} \sqrt{\tilde{s}} - \frac{\beta}{4} a(a - 1) \log s^{1/2} - \frac{a}{4} \log s^{1/2} \right),
\] (4.19)
where
\[
A_{\beta,N} = (\beta/2)^{-\beta a(a-1)/4} (\pi \beta)^{-a/2} 2^{-\beta a/2 + a} f_{\beta/2}(a).
\] (4.20)

Moreover, in the case \(a\beta/2 \in \mathbb{Z}^+\) we have
\[
A_{\beta,N} = \tau_{\beta a/2,\beta},
\] (4.21)

where \(\tau_{\beta a/2,\beta}\) is given by (1.4), and thus (4.19) reclaims the large \(s\) asymptotic form of \(E^\text{hard}_\beta(0; (0, s); \beta a/2)\) as given by (1.4).

Proof. Obtaining (4.19) from (4.18) is an elementary calculation. The equality (4.21) can be established by verifying that both sides satisfy the same defining recurrence, making use of (4.13) on the LHS.

4.3 Probability density function for there being \(n\) eigenvalues at \(\{y_j\}_{j=1,\ldots,n}\) in \((0, s)\)

Let \(p(y_1, \ldots, y_n; (0, s); \text{ME}_{\beta,N+a}(x^{\beta a/2} e^{-\beta x/2})\) denote the probability density function that in the ensemble \(\text{ME}_{\beta,N+a}(x^{\beta a/2} e^{-\beta x/2})\) there are \(n\) eigenvalues at \(\{y_j\}_{j=1,\ldots,n}\) in \((0, s)\). The aim of this subsection is to compute the large \(N\) form of this probability density function, with
\[
\tilde{s}_0 = \frac{s}{4N}, \quad \tilde{s}_j = \frac{(s - y_j)}{4N} \quad (j = 1, \ldots, n)
\] (4.22)
fixed, and then the scaling limit
\[
\tilde{s}_0 = s_0/(4N)^2, \quad \tilde{s}_j = s_j/(4N)^2 \quad (j = 1, \ldots, n), \quad N \to \infty
\] (4.23)
of this hard edge form.

Lemma 3. In terms of the notation (4.22) we have
\[
p(y_1, \ldots, y_n; (0, s); \text{ME}_{\beta,N}(x^{\beta a/2} e^{-\beta x/2}))
\]
\[
= \frac{\mathcal{N}(\text{ME}_{\beta,N}(e^{-\beta x/2}))}{\mathcal{N}(\text{ME}_{\beta,N+a}(x^{\beta a/2} e^{-\beta x/2}))} \frac{(N)_n}{n!} (4N)^{\beta N(a/2+n)} e^{-\beta N s_0/2} \prod_{l=1}^{n} y_l^{\beta a/2}
\]
\[
\times e^{-\beta \sum_{j=1}^{s} y_j} \prod_{1 \leq j < k \leq n} |y_j - y_k|^\beta \left( \prod_{l=1}^{N} ((\tilde{s}_0 + x_l)^{\beta a/2}) \prod_{j=1}^{n} ((\tilde{s}_j + x_l)^\beta) \right)_{\text{ME}_{\beta,N}(e^{-2\beta N s})}
\] (4.24)

Proof. We essentially follow the strategy of the proof of Lemma 1.

Comparing the average in (4.24) with (2.4) we see that in the case \(\beta a/2 \in \mathbb{Z}_{\geq 0}, \beta \in \mathbb{Z}_{\geq 0}\), we have
\[
\frac{\mathcal{N}(\text{ME}_{\beta,N}(e^{-\beta x/2}))}{\mathcal{N}(\text{ME}_{\beta,N}(x^{(a+2n)\beta/2} e^{-\beta x/2}))} \left( \prod_{l=1}^{N} ((\tilde{s}_0 + x_l)^{\beta a/2}) \prod_{j=1}^{n} ((\tilde{s}_j + x_l)^\beta) \right)_{\text{ME}_{\beta,N}(e^{-2\beta N s})}
\]
\[
= (4N)^{-N\beta(a/2+n)} \Gamma_1^{(\beta/2)}(-N; a + 2n; (-4N\tilde{s}_0)^{\beta a/2}, (-4N\tilde{s}_1)^\beta, \ldots, (-4N\tilde{s}_n)^\beta).
\] (4.25)
Furthermore, we recognise the average as the form given for large $N$ by the RHS of (4.2) with $c = 1$ and

$$v(t) = \frac{\beta a}{2} \log(t + \tilde{s}_0) + \beta \sum_{j=1}^{n} \log(t + \tilde{s}_j).$$  \hspace{1cm} (4.26)$$

Thus our immediate task is to compute the mean and variance.

**Lemma 4.** With $v(t)$ given by (4.20), and $\rho_{1,N}(t)$ as implied by (4.36) we have

$$\mu_{N}(v) = 2N\left(\frac{\beta a}{2}\right)\left(\sqrt{\tilde{s}_0(\tilde{s}_0 + 1)} - \tilde{s}_0 - \log\left(2\left(\sqrt{\tilde{s}_0^2 + 1} - \sqrt{\tilde{s}_0}\right)\right) - \frac{1}{2}\right)$$

$$+ 2N\beta \sum_{j=1}^{n} \left(\sqrt{\tilde{s}_j(\tilde{s}_j + 1)} - \tilde{s}_j - \log\left(2\left(\sqrt{\tilde{s}_j^2 + 1} - \sqrt{\tilde{s}_j}\right)\right) - \frac{1}{2}\right)$$

$$+ \left(\frac{1}{2\beta} - \frac{1}{4}\right) \left\{ \frac{\beta a}{2} \log \left|1 + \tilde{s}_0\tilde{s}_0\right| + \beta \sum_{j=1}^{n} \log \left|1 + \tilde{s}_j\tilde{s}_j\right| \right\}. \hspace{1cm} (4.27)$$

Proof. We proceed as in the proof of Corollary 1.

**Lemma 5.** Let $\nu_k := -(2\tilde{s}_k + 1) + 2(\tilde{s}^2_k + \tilde{s}_k)^{1/2}$. We have

$$(\sigma(v))^2 = 2\beta \left\{ -\left(\frac{a}{2}\right)^2 \log(1 - \nu_0^2) - \sum_{j=1}^{n} \log(1 - \nu_j^2) - a \sum_{j=1}^{n} \log(1 - \nu_0 \nu_j) - 2 \sum_{1 \leq j_1 < j_2 \leq n} \log(1 - \nu_{j_1} \nu_{j_2}) \right\}. \hspace{1cm} (4.28)$$

Proof. In the notation of (4.4) the first task is to compute

$$a_k = \frac{2}{\pi} \int_{0}^{\pi} \left\{ \frac{\beta a}{2} \log \left(1 + \frac{\cos \theta}{2\tilde{s}_0 + 1}\right) + \beta \sum_{j=1}^{n} \log \left(1 + \frac{\cos \theta}{2\tilde{s}_j + 1}\right) \right\} \cos k\theta d\theta.$$
Proposition 5. Let the large $N$ asymptotic form of the average (4.24), denoted by annotating the average by $\#$, be scaled according to (4.23). We have

$$(4N)^{N\beta(n/2+n)} \frac{N(ME_{\beta,N}(e^{-\beta N/2}))}{N(ME_{\beta,N}(x^{(a+2n)3/2}e^{-\beta x/2}))} \left( \prod_{l=1}^{N} \left( \tilde{s}_0 + x_l \right)^{\beta n/2} \prod_{j=1}^{n} \left( \tilde{s}_j + x_l \right)^{\beta} \right) \sim \tilde{A}_{a,n,\beta} e^{(\beta a/2)\sqrt{\pi \beta} \sum_{j=1}^{n} \sqrt{s_j}} \exp \left\{ -\frac{1}{2} \left( 1 - \frac{\beta}{2} \right) \left( a \log \frac{s_0}{4} + 2 \sum_{j=1}^{n} \log \frac{s_j}{4} \right) \right\} \times \exp \left( -\beta \left\{ \left( \frac{a}{2} \right)^2 \log \sqrt{s_0} + \sum_{j=1}^{n} \log \sqrt{s_j} + a \sum_{j=1}^{n} \log \left( \frac{\sqrt{s_0} + \sqrt{s_j}}{2} \right) \right\} \right) \times \exp \left( -2\beta \sum_{1 \leq j < j' \leq n} \log \left( \frac{\sqrt{s_{j'}} + \sqrt{s_j}}{2} \right) \right), \quad (4.29)$$

where, with $f_{\beta/2}$ specified by (4.17),

$$\tilde{A}_{a,n,\beta} = (\beta/2)^{-(\beta/4)(a+2n-1)(a+2n)}(\pi \beta)^{-\beta/2} f_{\beta/2}(a+2n-1). \quad (4.30)$$

According to (4.25) and (2.3), the scaled limit (4.23) of the LHS of (4.29) is equal to

$$0 F_{1}^{(\beta/2)}(\cdot; a + 2n; \left( \frac{s_0}{4} \right)^{\beta a/2}, \left( \frac{s_1}{4} \right)^{\beta}, \ldots, \left( \frac{s_n}{4} \right)^{\beta}). \quad (4.31)$$

Thus we obtain, as a conjecture, the corresponding large $\{s_j\}_{j=0,...,n}$ asymptotic form.

Conjecture 2. For large values of the arguments $\{s_j\}_{j=0,...,n}$ the generalized hypergeometric function (4.31) has the asymptotic form given by the RHS of (4.29), up to terms which vanish as the arguments approach infinity.

In the case $\beta = 4$ the asymptotic expansion of (4.31) can be derived from a matrix integral representation not available for general parameters $[29]$. The expression implied by the result of [29] is in precise agreement with the conjecture. Furthermore, this asymptotic form is also consistent with the conjecture (2.14).

In keeping with the origin of (4.31), and with the scaled variables as specified by (4.22) and (4.23), we can deduce from (4.24) that

$$s^n p(sy_1, \ldots, sy_n; (0, s); ME_{\beta,N+n}(x^{\beta a/2} e^{-\beta x/2}))$$

$$\sim A_\beta(n, a) s^{n+\beta(n+a-1)} e^{-\beta s/8} \prod_{j=1}^{n} (1 - y_j)^{\beta n/2} \prod_{1 \leq j < k \leq n} |y_k - y_j|^\beta$$

$$\times 0 F_{1}^{(\beta/2)}(\cdot; a + 2n; \left( \frac{s}{4} \right)^{\beta a/2}, \left( \frac{s(1 - y_1)}{4} \right)^{\beta}, \ldots, \left( \frac{s(1 - y_n)}{4} \right)^{\beta}).$$

But

$$E_\beta(n; (0, s); ME_{\beta,N+n}(\lambda^{\alpha/2} e^{-\beta N\lambda/2}))$$

$$= s^n \int_0^1 dy_1 \cdots \int_0^1 dy_n p(sy_1, \ldots, sy_n; (0, s); ME_{\beta,N+n}(x^{\beta a/2} e^{-\beta x/2}))$$

16
and thus, recalling the consistency of Conjecture 2 with (2.14), we reclaim (2.15), but now with requirement that $\beta n \in \mathbb{Z}_{\geq 0}$ relaxed by writing

$$\prod_{j=1}^{\beta n} \frac{(a + 2j/\beta)}{(2\pi)^{1/2}} = \frac{\Gamma(a + 2/\beta)}{\Gamma(2n + a + 2/\beta)} = \frac{\tau^{2n^2/\tau + 2na/\tau - n/\tau + n}}{(2\pi)^n} \frac{f_{1/\tau}(a)}{f_{1/\tau}(n + a)}, \quad \tau := 2/\beta. \tag{4.32}$$

A significant check on (2.15) with the substitution (4.32), supplemented by (1.4) and (2.14), is to verify that it satisfies the asymptotic functional equation [16]

$$E_{\beta}^{\text{hard}}(n; 0, s/\tilde{s}_\beta; \beta a/2) \sim \tilde{s}_{4/\beta} E_{4/\beta}^{\text{hard}}(\beta(n + 1)/2 - 1; (0, s/\tilde{s}_{4/\beta}; a - 2 + 4/\beta), \tag{4.33}$$

where $\tilde{s}_{4/\beta}(2/\beta)^2 = \tilde{s}_\beta$. Note that on the RHS the value for the number of particles in the gap is $\beta(n + 1)/2 - 1$ and thus not necessarily an integer. In addition to (4.32) we should also rewrite the remaining products in (2.16) according to

$$\prod_{j=1}^{n-1} \frac{\Gamma(1 + (j + 1)\beta/2)}{\Gamma(n + 1)(1 + (j + a)\beta/2)} = \frac{f_{\beta/2}(n + 1) f_{\beta/2}(n + a)}{f_{\beta/2}(2n + a)}, \tag{4.34}$$

where $f_{\beta/2}(n)$ is given by (4.16). Combining with (4.32) then gives

$$\tau_{\beta a/2, \beta}^{\text{hard}}(n) = 2^{-(a + n)2n/\tau \tau^{n(a + n)/\tau - n/\tau}} \frac{f_{1/\tau}(n + 1) f_{1/\tau}(n + a)}{n! (2\pi)^n f_{1/\tau}(a)} \frac{f_{1/\tau}(n + 2 - \tau) f_{1/\tau}(n + a)}{f_{1/\tau}(a)}, \quad \tau = 2/\beta, \tag{4.35}$$

where the second line follows from use of the definition (4.16) and the functional properties (4.13).

We showed in [24] that (4.33) is satisfied up to the constant term in the corresponding asymptotic expansions of both sides. With $\tilde{s}_{4/\beta} = 1$, the constant term on the LHS of (4.33) is

$$\tau_{\beta a/2, \beta}^{\text{hard}}(n) \sim \tau_{\beta a/2, \beta}^{\text{hard}}(\beta/2)^{a(\beta a/2 + 1)/2 - \beta a/2 + (n^2 + na)\beta/2}, \tag{4.36}$$

while on the RHS the constant term is

$$\tau_{a - 2 + 4/\beta, 4/\beta}^{\text{hard}} \tau_{a - 2 + 4/\beta, 4/\beta}^{\text{hard}}(\beta(n + 1)/2 - 1). \tag{4.37}$$

Use of (4.35) and (4.21) shows that (4.36) is equal to

$$\frac{2^{-(a + n)2n/\tau - a/\tau + a}}{\Gamma((n + 1)/\tau)} \tau^{-1/2 \tau^{(n + 1)/\tau - 1/2}} \frac{f_{1/\tau}(n + 1 - \tau) f_{1/\tau}(n + a)}{f_{1/\tau}(a)}. \tag{4.38}$$
To verify the equality between (4.37) and (4.38), we require the fact that the Barnes double gamma function has the inversion property \[26]\]
\[
\Gamma_2(n; 1, \tau) = \tau^{-(1+n^2/2\tau)+n(1+\tau)/2\tau} \Gamma_2(n/\tau; 1, 1/\tau).
\]
(4.39)

Recalling (4.15) this implies
\[
f_{1/\tau}(a) = \frac{(2\pi)^{(a-1)/2} \tau^{(1-a)/2}}{(2\pi)^{(a-1)/2} \tau^{(a-1)/2}} f_\tau(a - 1) + 1).
\]
(4.40)

Making use of this in (4.37) shows the latter is equal to
\[
\frac{t-a'(a'-1)\tau/2-(a'/2+n')-n'(a'+n')t}{\Gamma(n'+1)(2\pi)^{n'+a'/2}} f_t(n' + 1) f_t(n' + a'),
\]
(4.41)

where \(a' = (a-2)t + 2\), \(n' = (n+1)t - 1\) and \(t = 1/\tau = \beta/2\). The expressions (4.35) and (4.21) reveal that this is precisely the RHS of (4.37), thus verifying that the functional equation (4.33) is a property of the asymptotic expansion of \(E_{\beta \text{hard}}(n; (0, s); \beta a/2)\) as implied by (1.4), (2.15), (4.35) and (4.21).

**Acknowledgement**

This work was supported by the Australian Research council.

**References**

[1] F. Bornemann, *On the numerical evaluation of distributions in random matrix theory: a review with an invitation to experimental mathematics*, Markov Processes Relat. Fields 16 (2010), 803–866.

[2] G. Borot, B. Eynard, S.N. Majumdar, and C. Nadal, *Large deviations of the maximal eigenvalue of random matrices*, J. Stat. Mech. 2011 (2011), P11024.

[3] G. Borot and A. Guionnet, *Asymptotic expansion of beta matrix models in the one-cut regime*, arXiv:1107.1167, 2011.

[4] A. Brini, M. Marino, and S. Stevan, *The uses of the refined matrix model recursion*, J. Math. Phys. 52 (2011), 052305(24pp).

[5] Y. Chen and S.M. Manning, *Asymptotic level spacing of the laguerre ensemble: a Coulomb fluid approach*, J. Phys. A 27 (1994), 3615–3620.

[6] P. Deift, A. Its, and J. Vasilevsk, *Asymptotics for a determinant with a confluent hypergeometric kernel*, Int. Math. Res. Not. 2011 (2011), 2117–2160.
[7] P. Desrosiers, *Duality in random matrix ensembles for all $\beta$*, Nucl. Phys. B 817 (2009), 224–251.

[8] P. Desrosiers and D.-Z. Liu, *Asymptotics for products of characteristic polynomials in classical $\beta$-ensembles*, arXiv:1112.1119.

[9] F.J. Dyson, *Statistical theory of energy levels of complex systems III*, J. Math. Phys. 3 (1962), 166–175.

[10] , *The three fold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics*, J. Math. Phys. 3 (1962), 1199–1215.

[11] T. Ehrhardt, *The asymptotics of a Bessel-kernel determinant which arises in Random Matrix Theory*, Advances Math. 225 (2010), 3088–3133.

[12] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, *Chiral properties of $su(3)$ sextet fermions*, JHEP 0911 (2009), 103.

[13] P.J. Forrester, *Exact integral formulas and asymptotics for the correlations in the $1/r^2$ quantum many body system*, Phys. Lett. A 179 (1993), 127–130.

[14] , *Exact results and universal asymptotics in the Laguerre random matrix ensemble*, J. Math. Phys. 35 (1994), 2539–2551.

[15] , *Spacing distributions in random matrix ensembles*, Recent perspectives in random matrix theory and number theory (F. Mezzadri and N.C. Snaith, eds.), London Mathematical Society Lecture Note Series, vol. 322, Cambridge University Press, Cambridge, 2005, pp. 279–308.

[16] , *A random matrix decimation procedure relating $\beta = 2/(r+1)$ to $\beta = 2(r+1)$*, Commun. Math. Phys. 285 (2009), 653–672.

[17] , *Log-gases and random matrices*, Princeton University Press, Princeton, NJ, 2010.

[18] , *Probabilities densities and distributions for spiked Wishart $\beta$-ensembles*, arXiv:1101.2261, 2011.

[19] , *Averaged characteristic polynomial for the Gaussian and chiral Gaussian ensembles with a source*, arXiv:1203.5838, 2012.

[20] , *Large deviation eigenvalue density for the soft edge Laguerre and Jacobi $\beta$-ensembles*, J. Phys. A 45 (2012), 145201(15pp).

[21] , *Spectral density asymptotics for Gaussian and Laguerre $\beta$-ensembles in the exponentially small region*, J. Phys. A 45 (2012), 075206(17pp).
[22] P.J. Forrester, N.E. Frankel, and T.M. Garoni, *Asymptotic form of the density profile for Gaussian and Laguerre random matrix ensembles with orthogonal and symplectic symmetry*, J. Math. Phys. 47 (2006), 023301.

[23] P.J. Forrester and M.J. Sorrell, *Asymptotics of spacing distributions 50 years later*, arXiv:1204.3225, 2012.

[24] P.J. Forrester and N.S. Witte, *Asymptotic forms for hard and soft edge general $\beta$ ensembles*, Nucl. Phys. B 859 (2012), 321–340.

[25] J. Kaneko, *Selberg integrals and hypergeometric functions associated with Jack polynomials*, SIAM J. Math Anal. 24 (1993), 1086–1110.

[26] K. Katayama and M. Ohtsuki, *On the multiple gamma-functions*, Tokyo J. Math. 21 (1998), 159–182.

[27] I. Krasovsky, *Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on the unit circle*, Int. Math. Res. Not. 2004 (2004), 1249–1272.

[28] M. Y. Mo, *The rank 1 real Wishart spiked model*, arXiv:1101.5144, 2011.

[29] R.J. Muirhead, *Latent roots and matrix variates: a review of some asymptotic results*, Ann. Stat. 6 (1978), 5–33.

[30] R.J. Muirhead, *Aspects of multivariate statistical theory*, Wiley, New York, 1982.

[31] D. Ostrovsky, *Selberg integral as a meromorphic functions*, to appear, IMRN.

[32] J.R. Quine, S.H. Heydari, and R.Y. Song, *Zeta regularized products*, Trans. Am. Math. Soc. 338 (1993), 213–231.

[33] J.A. Ramirez, B. Rider, and O. Zeitouni, *Hard edge tail asymptotics*, Elec. Comm. in Probab. 16 (2011), 741–752.

[34] T. Shintani, *A proof of the classical Kronecker limit formula*, Tokyo J. Math. 3 (1980), 191–199.

[35] D. Wang, *The largest eigenvalue of real symmetric, Hermitian and Hermitian self-dual random matrix models with rank one external source, part I*, J. Stat. Phys. 146 (2012), 719–761.