Asymptotically tight worst case complexity bounds for initial-value problems with nonadaptive information

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Abstract

It is known that, for systems of initial-value problems, algorithms using adaptive information perform much better in the worst case setting than the algorithms using nonadaptive information. In the latter case, lower and upper complexity bounds significantly depend on the number of equations. However, in contrast with adaptive information, existing lower and upper complexity bounds for nonadaptive information are not asymptotically tight. In this paper, we close the gap in the complexity exponents, showing asymptotically matching bounds for nonadaptive standard information, as well as for a more general class of nonadaptive linear information.

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1 Introduction

We aim at closing a gap between upper and lower worst case complexity bounds for initial-value problems with nonadaptive information. A motivation comes from a discussion on this subject that we had with Stefan Heinrich in 2016. We deal with the solution of systems

\[
z'(t) = f(z(t)), \quad t \in [a, b], \quad z(a) = \eta,
\]

where \(a < b\), \(f : \mathbb{R}^d \to \mathbb{R}^d\) is a \(C^r\) function, \(d \geq 1\), and \(\eta \in \mathbb{R}^d\). A class of functions \(f\), denoted by \(F_{r,d}\), is given by (2). For \(\varepsilon > 0\), we measure the difficulty of the problem by the minimal cost of an algorithm, based on some information, that gives us an \(\varepsilon\)-approximation to the solution (the \(\varepsilon\)-complexity of the problem). If adaptive information is allowed, then the \(\varepsilon\)-complexity is denoted by \(\text{comp}(\varepsilon, F_{r,d})\). The notation \(\text{comp}^\text{nonad}(\varepsilon, F_{r,d})\), where the superscript is added, means that we restrict ourselves to the class of nonadaptive information. For precise definitions of basic notions, the reader is referred to the next section. Our aim is to establish the asymptotics of \(\text{comp}^\text{nonad}(\varepsilon, F_{r,d})\) as \(\varepsilon \to 0\) for nonadaptive information, as function of the regularity \(r\) and the dimension \(d\).

A question about potential advantages of adaptive over nonadaptive algorithms for solving various problems is an important issue in numerical analysis. Many different points of view cause some discussions and sometimes misunderstandings among numerical analysts in that respect. From practical point of view, adaption is claimed to be definitely better, which is supported by results of numerical experiments, see e.g. [2], [6], [7] and many other papers. A closer look however shows that advantages of adaption depend very much on the problem itself and the class of problem instances being solved. It is not a purpose of this paper to discuss the adaption/nonadaptation issue in details – to have a flavor of it, one can consult the monograph [9], or recent papers [1], [5], [8].

In what follows, for a positive function \(\gamma = \gamma(\varepsilon)\), the asymptotic expressions \(O(\gamma(\varepsilon))\), \(\Omega(\gamma(\varepsilon))\) and \(\Theta(\gamma(\varepsilon))\) will always be meant as \(\varepsilon \to 0\). It is known for many years that for problem (1) adaptive information is much more efficient in the worst case setting than nonadaptive one. It was shown for adaptive information that the \(\varepsilon\)-complexity of (1) is, (see [3]):

\[
\begin{align*}
\text{comp}(\varepsilon, F_{r,d}) &= \Theta\left(\frac{1}{\varepsilon^{1/r}}\right), & \text{for the class of standard adaptive information}, \\
\text{comp}(\varepsilon, F_{r,d}) &= \Theta\left(\frac{1}{\varepsilon^{1/(r+1)}}\right), & \text{for the class of linear adaptive information}.
\end{align*}
\]

In both cases of standard and linear information, the complexity bounds are asymptotical-
cally tight, and the asymptotics is independent of $d$.

In the nonadaptive case, the existing complexity bounds are not tight. In [4], we considered the class $F_{r,d}$ with $M = (0,1)^d$ and $D = 1$, see [2]. It was shown (translating the results from non-autonomous problems in [4] to the autonomous ones [1]) that

(a) $\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = \Omega \left( \frac{1}{\varepsilon} \frac{d}{d/(r+1)} \right)$, for the class of all linear nonadaptive information,
(b) $\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = O \left( \frac{1}{\varepsilon} \frac{d}{d/r} \right)$, for the class of standard nonadaptive information.

The influence of the dimension $d$ in the nonadaptive case is very significant, which indicates that the problem [1] is not well suited for nonadaptive solution. The complexity radically increases (asymptotically) with $d$.

The bounds (a) and (b) do not match, so that the asymptotics of the $\varepsilon$-complexity of the nonadaptive solution of [1] is not known. In this paper, we close the gap between lower and upper bounds in some important cases. We show that for $d \geq 2$

$$\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = \Omega \left( \frac{1}{\varepsilon} \frac{d}{d/(r+1)} \right),$$

for the class of all linear nonadaptive information.

For $d > r + 1$ this is an improvement over the lower bound (a). It shows in particular that it is not possible in general, as one may expect, to achieve the complexity proportional to $(1/\varepsilon)^{d/(r+1)}$ by allowing nonadaptive linear (nonstandard) information. Our main result, contained in Theorem 1 and next extended in Theorem 2, states that

$$\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = \Omega \left( \frac{1}{\varepsilon} \frac{d-1}{d/r} \right),$$

for a class of linear nonadaptive information

that includes any standard information.

This improves the lower bound (a), and matches the upper bound (b). The question about the asymptotics of the $\varepsilon$-complexity for the class of all linear nonadaptive information is still open. It is thought to be $\Theta \left( \frac{1}{\varepsilon} \frac{d}{d/r} \right)$ as $\varepsilon \to 0$, the same as for standard information.

Finally, in Remark 1 we point out how the proof of Theorem 1 can be modified to get the complexity lower bound for non-autonomous systems. In the lower bound of Theorem 1, one needs to replace $d$ by $d + 1$ in the exponent.

The paper is organized as follows. In Section 2 basic notation is established and known results are recalled. Section 3 is devoted to the proof of the main result in Theorem 1, and to its extension in Theorem 2. In the Appendix we give auxiliary constructions and bounds.
2 Preliminaries

2.1 Basic definitions

Let \( r \in \mathbb{N} \ (r \geq 1) \), \( D > 0 \) and \( M \) be a nonempty open subset of \( \mathbb{R}^d \). We consider the class of functions \( f \) with \( r \) continuous bounded derivatives

\[
F_{r,d} = \{ f : \mathbb{R}^d \to \mathbb{R}^d : f \in C^r(\mathbb{R}^d), f(y) = 0 \text{ for } y \notin M, |D^i f^j(y)| \leq D, y \in \mathbb{R}^d, i = 0, 1, \ldots, r, j = 1, 2, \ldots, d \},
\]

where \( D^i f^j \) denotes any partial derivative of order \( i \) of \( j \)th component of \( f \), \( f = [f^1, f^2, \ldots, f^d]^T \).

In particular, \( f \) is a Lipschitz function in \( \mathbb{R}^d \), with the Lipschitz constant denoted by \( L = L(D) \). We assume that \( \eta \in M \), which is the only case of interest (otherwise, \( z(t) \equiv \eta, t \in [a, b] \), for any \( f \in F_{r,d} \)). We shall use in what follows the maximum norm \( \| \cdot \| \) in \( \mathbb{R}^d \).

Let \( n \in \mathbb{N} \). The function \( f \) is accessible through information given by a vector

\[
N_n(f) = [L_1(f), L_2(f), \ldots, L_n(f)]^T,
\]

where \( L_j \) are linear functionals on \( C^r(\mathbb{R}^d) \). We will be interested in the power of nonadaptive information, which is defined by functionals \( L_j \) selected simultaneously in advance, before computation starts. Otherwise, if successive functionals are selected depending on previously computed values, the information is called adaptive. Most often, the functionals are defined by the values of \( f \) or its partial derivatives evaluated at certain points,

\[
L_j(f) = D^{k_j} f^{i_j}(y_j), \quad j = 1, 2, \ldots, n,
\]

where \( D^{k_j} \) is some partial derivative of order \( k_j \), \( 0 \leq k_j \leq r \), \( 1 \leq i_j \leq d \), and \( y_j \in \mathbb{R}^d \).

Such information is called standard. It is nonadaptive if the points \( y_j \), as well as \( k_j \) and \( i_j \), \( j = 1, 2, \ldots, n \), are given in advance.

By an algorithm we mean any function \( \Phi_n \) that computes an approximation \( l : [a, b] \to \mathbb{R}^d \) to the solution \( z \), based on \( N_n(f) \), \( l(t) = \Phi_n(N_n(f))(t) \). The worst case error of \( \Phi_n \) with information \( N_n \) in the class \( F_{r,d} \) is defined by

\[
e(\Phi_n, N_n, F_{r,d}) = \sup_{f \in F_{r,d}} \sup_{t \in [a, b]} \| z(t) - l(t) \|.
\]

For \( \varepsilon > 0 \), by the \( \varepsilon \)-(information) complexity of the problem (1) we mean the minimal number of functionals sufficient for approximating \( z \) with error at most \( \varepsilon \), that is,

\[
\text{comp}(\varepsilon, F_{r,d}) = \min \{ n \geq 1 : \text{there exist } N_n \text{ and } \Phi_n \text{ such that } e(\Phi_n, N_n, F_{r,d}) \leq \varepsilon \},
\]
with $\min \emptyset = +\infty$. The complexity obviously depends on the class of admitted information operators. If we restrict ourselves to the class of nonadaptive information, then the $\varepsilon$-complexity will be denoted by $\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d})$. An interesting question, in particular, is whether adaptation has advantages over nonadaptation.

2.2 Known results on adaptation versus nonadaptation for IVPs

We briefly recall what is known about the potential of nonadaptation for (1). It has been shown in [3] and [4] that the following holds. A straightforward modification is only needed in the nonadaptive case with respect to [4], where nonautonomous problems with nonadaptive information were considered.

**Theorem A ([3], [4])** For the class of standard adaptive information
\[
\text{comp}(\varepsilon, F_{r,d}) = \Theta \left( (1/\varepsilon)^{1/r} \right).
\]

For the class of linear adaptive information
\[
\text{comp}(\varepsilon, F_{r,d}) = \Theta \left( (1/\varepsilon)^{1/(r+1)} \right).
\]

For the class of linear nonadaptive information, for the function class $F_{r,d}$ with $M = (0,1)^d$ and $D = 1$, we have
\[
\Omega \left( (1/\varepsilon)^{d/(r+1)} \right) = \text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = O \left( (1/\varepsilon)^{d/r} \right).
\]

The upper bound in (9) is achieved by standard information.

The bounds show a substantial advantage of adaptive over nonadaptive information for the problem (1): the number of equations $d$ significantly increases the complexity, if we restrict ourselves to nonadaptive information.

Because of the gap between lower and upper bounds in (9), the asymptotics of the complexity in the nonadaptive case as $\varepsilon \to 0$ is not known (for both standard and linear information). We shall remove the gap in the next section for standard information, as well as for a class of linear information.

3 Asymptotically tight complexity bounds

3.1 Standard information

In the following theorem we prove new lower complexity bound improving that in (9) for standard information.
Theorem 1  For the class of standard nonadaptive information
\[
\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = \Omega \left( (1/\varepsilon)^{d/r} \right).
\] (10)

Proof  In view of (7), it is enough to consider \( d \geq 2 \). It suffices to show that there exists a positive constant \( C_3 \) such that for sufficiently large \( n \), for any standard nonadaptive information \( N_n \) and any algorithm \( \Phi_n \), we have
\[
e(\Phi_n, N_n, F_{r,d}) \geq C_3 n^{-r/d}.
\] (11)

It is easy to see that for any \( f_1, f_2 \in F_{r,d} \) such that \( N_n(f_1) = N_n(f_2) \), for the respective solutions \( z_1 \) and \( z_2 \) of (11) it holds
\[
e(\Phi_n, N_n, F_{r,d}) \geq (1/2) \sup_{t \in [a,b]} \| z_1(t) - z_2(t) \|.
\] (12)

We shall now construct functions \( f_1 \) and \( f_2 \) to get the lower bound in (12) as large as possible. Let \( M_1 \) be an open set containing \( \eta \), whose closure is contained in \( M \). We define \( f_1 \) such that
\[
f_1(y) = \begin{cases} 
\vec{\alpha} & \text{for } y \in M_1, \\
0 & \text{for } y \notin M.
\end{cases}
\] (13)

Here \( \vec{\alpha} \) is a vector in \( \mathbb{R}^d \) with \( \| \vec{\alpha} \| = \Delta > 0 \) (the vector with components \( [\alpha^1, \ldots, \alpha^d]^T \) may be identified with a point with the same components). The number \( \Delta \) is sufficiently small to assure that:
1. \( f_1 \) can be extended to \( M \setminus M_1 \) so that \( f_1 \in F_{r,d} \),
2. \( \Delta \leq D/2 \),
3. \( B_\Delta = \{ y : \| y - \eta \| \leq \Delta(b - a) \} \subset M_1 \), which assures that the solution \( z_1 \) of (1) for \( f_1 \) is given by
\[
z_1(t) = \vec{\alpha}(t - a) + \eta, \quad t \in [a,b].
\]

The direction of \( \vec{\alpha} \) will be chosen later on.

The function \( f_2 \) will be given as \( f_2 = f_1 + H \), where \( N_n(H) = 0 \). We now construct \( H \).
Let \( 0 < T \leq \Delta(b - a) \). Consider a hypercube \( K \) contained in \( B_\Delta \),
\[
K = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]
\] (14)
such that \( [b_1, b_2, \ldots, b_d]^T = \eta \), and the length of the edges is \( T, b_j - a_j = T, j = 1, 2, \ldots, d \).
Let \( m = \lceil n^{1/d} \rceil \) and \( p \in \mathbb{N} \). We divide the edges \( [a_j, b_j], j = 1, 2, \ldots, d - 1 \) into \( pm \)
intervals using equidistant points $t_k^i = a_j + kT/(pm)$, $k = 0, 1, \ldots, pm$. The hypercube in the $(d - 1)$-dimensional hyperplane

$$\{ y = [y^1, y^2, \ldots, y^d]^T : y^d = a_d \}$$

given by

$$K_1 = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_{d-1}, b_{d-1}] \times [a_d, a_d]$$

is thus divided into $(pm)^{d-1}$ smaller hypercubes

$$K_{k_1, k_2, \ldots, k_{d-1}} = [t_{k_1}^1, t_{k_1+1}^1] \times [t_{k_2}^2, t_{k_2+1}^2] \times \ldots \times [t_{k_{d-1}}^{d-1}, t_{k_{d-1}+1}^{d-1}] \times [a_d, a_d],$$

$k_j = 0, 1, \ldots, pm - 1$.

Let $c_{k_1, k_2, \ldots, k_{d-1}} \in \mathbb{R}^d$ be the center of $K_{k_1, k_2, \ldots, k_{d-1}}$. The direction of $\vec{\alpha}$ in (13) will be selected from among $(pm)^{d-1}$ directions of the vectors

$$\vec{\alpha}_{k_1, k_2, \ldots, k_{d-1}} = c_{k_1, k_2, \ldots, k_{d-1}} - \eta, \quad ||\vec{\alpha}_{k_1, k_2, \ldots, k_{d-1}}|| = T.$$  

We now associate with each $\vec{\alpha}_{k_1, k_2, \ldots, k_{d-1}}$ a paralelepiped $P_{k_1, k_2, \ldots, k_{d-1}} \subset \mathbb{R}^d$ as follows.

Let $C$ be a cone with vertex $\eta$ defined as a convex hull of $K_{k_1, k_2, \ldots, k_{d-1}}$ and $\eta$. Denote by $\bar{K}_{k_1, k_2, \ldots, k_{d-1}}$ the intersection of $C$ with the $(d - 1)$-dimensional hyperplane $y^d = (b_d + a_d)/2$.

We define

$$P_{k_1, k_2, \ldots, k_{d-1}} = \{ y \in \mathbb{R}^d : y = \bar{y} + (1/2)\vec{\alpha}_{k_1, k_2, \ldots, k_{d-1}} l, \text{ for some } \bar{y} \in \bar{K}_{k_1, k_2, \ldots, k_{d-1}} \text{ and } l \in [0, 1] \}. \quad (15)$$

Note that the parallelepipeds $P_{k_1, k_2, \ldots, k_{d-1}}$ are contained in $B_\Delta$, and have disjoint interiors for different vectors $(k_1, k_2, \ldots, k_{d-1})$. We finally divide each $P_{k_1, k_2, \ldots, k_{d-1}}$ into $pm$ smaller parallelepipeds (cells) with disjoint interiors

$$P^0_{k_1, k_2, \ldots, k_{d-1}}, P^1_{k_1, k_2, \ldots, k_{d-1}}, \ldots, P^{pm-1}_{k_1, k_2, \ldots, k_{d-1}},$$

by setting

$$P^j_{k_1, k_2, \ldots, k_{d-1}} = \{ y \in P_{k_1, k_2, \ldots, k_{d-1}} : l \in [j/(pm), (j + 1)/(pm)] \},$$

$j = 0, 1, \ldots, pm - 1$. The total number of cells is $(pm)^d$.

We now apply in each $P^j_{k_1, k_2, \ldots, k_{d-1}}$ the construction described in point I of the Appendix to get bump functions $\hat{H}^j_{k_1, k_2, \ldots, k_{d-1}} : \mathbb{R}^d \to \mathbb{R}$ with supports $P^j_{k_1, k_2, \ldots, k_{d-1}}$. We use (40) with $P = P^j_{k_1, k_2, \ldots, k_{d-1}}$, $q_s - p_s = T/(2pm)$, $s = 1, 2, \ldots, d - 1$, and $N = T/(2pm)$.

The function $H : \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$H(y) = [\hat{H}(y), 0, \ldots, 0]^T, \quad y \in \mathbb{R}^d,$$  

$$7$$
any of the information points given by (3) and (4). It suffices to eliminate all cells containing in the interior \( \beta \) where the values \( \hat{H}(y) = \sum_{k_1, k_2, \ldots, k_{d-1}, j} \beta^j_{k_1, k_2, \ldots, k_{d-1}} \hat{H}^j_{k_1, k_2, \ldots, k_{d-1}}(y) \) (17)

with \( |\beta^j_{k_1, k_2, \ldots, k_{d-1}}| \leq 1 \). The support of the \( C^\infty \) function \( H \) is the sum of all parallelepipeds (cells) \( P^j_{k_1, k_2, \ldots, k_{d-1}} \). The number of parameters \( \beta^j_{k_1, k_2, \ldots, k_{d-1}} \) is \( (pm)^d \).

Coefficients \( \beta^j_{k_1, k_2, \ldots, k_{d-1}} \) are now selected to assure that \( N_n(\tilde{H}) = 0 \), where \( N_n \) is standard information given by (3) and (4). It suffices to eliminate all cells containing in the interior any of the information points \( y_j \), that is, to set all corresponding \( \beta \)'s to 0. The remaining \( \beta \)'s, whose number is at least \( (pm)^d - n \), are set to 1.

Since \( (pm)^d - n \geq (pm)^d - m^d = (p^d - 1)m^d \), one can observe for \( p \geq 2 \), that there must exist \( (k_1, k_2, \ldots, k_{d-1}) \) such that

\[
\beta^j_{k_1, k_2, \ldots, k_{d-1}} = 1 \text{ for at least } 1/2 \text{ of indices } j = 0, 1, \ldots, pm - 1. \tag{18}
\]

We now select \( \vec{\alpha} \) in (13) to be

\[
\vec{\alpha} = \frac{\vec{\alpha}_{k_1, k_2, \ldots, k_{d-1}}}{\|\vec{\alpha}_{k_1, k_2, \ldots, k_{d-1}}\|} \Delta, \tag{19}
\]

for the chosen indices \( k_1, k_2, \ldots, k_{d-1} \).

Summarizing the above construction, functions \( f_1 \) and \( f_2 = f_1 + H \) belong to \( F_{r,d} \), \( N_n(f_1) = N_n(f_2) \), and the solution for \( f_1 \) has the form \( z_1(t) = \vec{\alpha}(t - a) + \eta, \ t \in [a, b] \). To complete the proof, it remains to bound from below the distance \( \sup_{t \in [a,b]} \|z_1(t) - z_2(t)\| \). From (41),

\[
\sup_{t \in [a,b]} \|z_1(t) - z_2(t)\| \geq \frac{1}{1 + L(b - a)} \left\| \int_a^b H(\vec{\alpha}(\xi - a) + \eta) \, d\xi \right\|
\]

\[
= \frac{1}{1 + L(b - a)} \left| \int_a^b \hat{H}(\vec{\alpha}(\xi - a) + \eta) \, d\xi \right|. \tag{20}
\]

The values \( \hat{H}(\vec{\alpha}(\xi - a) + \eta) \) are 0 if the argument is outside the parallelepiped \( P_{k_1, k_2, \ldots, k_{d-1}} \).

The line \( \vec{\alpha}(\xi - a) + \eta \) intersects the hyperplanes \( y^d = (a_d + b_d)/2 \) and \( y^d = a_d \) for \( \xi = t_1 = a + T/(2\Delta) \) and \( \xi = t_2 = a + T/\Delta (\leq b) \), respectively. Hence,

\[
\int_a^b \hat{H}(\vec{\alpha}(\xi - a) + \eta) \, d\xi = \int_{t_1}^{t_2} \hat{H}(\vec{\alpha}(\xi - a) + \eta) \, d\xi
\]

\[
= \sum_{j=0}^{pm-1} \int_{\xi_j}^{\xi_{j+1}} \hat{H}(\vec{\alpha}(\xi - a) + \eta) \, d\xi, \tag{21}
\]
where \( \xi_j = t_1 + (t_2 - t_1)j/(pm) = a + T/(2\Delta) + jT/(2\Delta pm) \), \( j = 0, 1, \ldots, pm \).

Note that for \( \xi \in [\xi_j, \xi_{j+1}] \), and only for such, the argument \( \tilde{\alpha}(\xi - a) + \eta \) remains in \( P_{k_1, \ldots, k_{d-1}}^j \).

We now use (40). By construction, in any cell for which \( \beta_{k_1, k_2, \ldots, k_{d-1}}^j = 1 \), we have for \( m \geq T/(2p) \)

\[
\hat{H}(\tilde{\alpha}(\xi - a) + \eta) = \hat{C}(T/(2pm))^r \left( h(1/2) \right)^{d-1} h(\bar{l}/N), \quad \xi \in [\xi_j, \xi_{j+1}],
\]

where \( N = T/(2pm) \) and \( \bar{l}/N = (\xi - \xi_j)/(\xi_{j+1} - \xi_j) \). Hence, we have that

\[
\sum_{j=0}^{pm-1} \int_{\xi_j}^{\xi_{j+1}} \hat{H}(\tilde{\alpha}(\xi - a) + \eta) d\xi = \hat{C}(T/(2pm))^r \left( h(1/2) \right)^{d-1} \sum_{j-}^{\xi_{j+1}} \int_{\xi_j}^{\xi_{j+1}} h((\xi - \xi_j)/(\xi_{j+1} - \xi_j)) d\xi
\]

\[
= \hat{C}(T/(2pm))^r \left( h(1/2) \right)^{d-1} \int_0^1 h(x) dx \sum_{j-}^{\xi_{j+1}} (\xi_{j+1} - \xi_j),
\]

where the sum \( \sum_{j-} \) extends over all indices \( j \) for which \( \beta_{k_1, k_2, \ldots, k_{d-1}}^j = 1 \). By construction, there is at least \( pm/2 \) such indices \( j \), so that the sum \( \sum_{j-}(\xi_{j+1} - \xi_j) \) is a positive constant. This allows us to conclude that

\[
\left| \int_a^b \hat{H}(\tilde{\alpha}(\xi - a) + \eta) d\xi \right| = \Omega \left( m^{-r} \right).
\]

In view of (20) and the fact that \( m = \Theta(n^{1/d}) \), we get (21), which completes the proof. \( \blacksquare \)

### 3.2 Generalization to a class of linear information

Let a sequence \( \{k(n)\}_{n=1}^\infty \subset \mathbb{N} \) be such that there exist \( \alpha > 0 \) and \( n_0 \in \mathbb{N} \) such that

\[
k(n) \leq \alpha n^{1-1/d} \quad \text{for } n \geq n_0.
\]

We consider the following class of linear information operators

\[
N_n(f) = [\bar{L}_1(f), \ldots, \bar{L}_{n-k}(f), L_1(f), \ldots, L_k(f)]^T, \quad n \geq k + 1,
\]

where \( k = k(n) \), and \( \bar{L}_j \) are arbitrary standard information functionals defined by (4). The functionals \( L_j \) are arbitrary linear.

In the following theorem we first somewhat improve the lower bound in (9) of Theorem A in the class of all linear nonadaptive information. The bound (26) will yield that we cannot
in general reduce the exponent in the upper complexity bound from \(d/r\) to \(d/(r+1)\), when switching from nonadaptive standard to nonadaptive linear information (as one may expect). Next, more interesting, we generalize Theorem 1 to the class of information given by (25), by proving that the lower bound (10) still holds true in this case.

**Theorem 2**  
(i) For the class of linear nonadaptive information (3), we have

\[
\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = \Omega \left( \left( \frac{1}{\varepsilon} \right)^{\max\{(d-1)/r, d/(r+1)\}} \right) .
\]  
(26)

(ii) Let \(\{k(n)\}_{n=1}^{\infty}\) satisfy (24). For the class of nonadaptive information (25), we have

\[
\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = \Omega \left( \left( \frac{1}{\varepsilon} \right)^{d/r} \right) .
\]  
(27)

(The constant and the maximal value of \(\varepsilon\) in the ‘\(\Omega\)’ notation in (27) depend on \(\alpha\) and \(n_0\).)

**Proof of (i)** In view of Theorem A, it suffices to show that for \(d \geq 2\) we have

\[
\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = \Omega \left( \left( \frac{1}{\varepsilon} \right)^{(d-1)/r} \right) .
\]

We refer to the proof of Theorem 1. We now choose \(m = \lceil (n+1)^{(d-1)}/(d-1) \rceil\) and \(p = 1\), and construct the parallelepipeds \(P_{k_1, \ldots, k_{d-1}}\) as in (15). We do not divide them further into small cells, and define the function \(H\) by (16) with \(\hat{H}\) given by

\[
\hat{H}(y) = \sum_{k_1, k_2, \ldots, k_{d-1}} \beta_{k_1, k_2, \ldots, k_{d-1}} \hat{H}_{k_1, k_2, \ldots, k_{d-1}}(y)
\]  
(28)

with \(|\beta_{k_1, k_2, \ldots, k_{d-1}}| \leq 1\). Here the functions \(\hat{H}_{k_1, k_2, \ldots, k_{d-1}}\) are defined in a similar way as \(\hat{H}_{k_1, k_2, \ldots, k_{d-1}}\) in the proof of Theorem 1 (now with supports \(P_{k_1, \ldots, k_{d-1}}\)).

The number of unknowns \(\beta_{k_1, k_2, \ldots, k_{d-1}}\) is now \(m^{d-1} \geq n + 1\). The condition \(N_n(H) = 0\) leads to \(n\) linear homogenous equations, which have a solution with the maximum modulus coefficient \(\beta_{k_1, k_2, \ldots, k_{d-1}} = 1\) for some \(k_1, k_2, \ldots, k_{d-1}\). We use these indices in (19), and follow the further steps of the proof of Theorem 1. The sum of the integrals in (21) contains now only one element. We finally get that \(e(\Phi_n, N_n, F_{r,d}) = \Omega \left( m^{-r} \right)\) for any information \(N_n\) and algorithm \(\Phi_n\), which means that

\[
e(\Phi_n, N_n, F_{r,d}) = \Omega \left( n^{-r/(d-1)} \right), \quad n \to \infty.
\]  
(29)

This leads to the desired bound on the complexity.

**Proof of (ii)** Let \(d \geq 2\). We repeat the steps of the proof of Theorem 1 up to the definition of the function \(H\) in (16). A difference is in the definition of the coefficients.
In any such parallelepiped \( P_{n} \) cells is greater than \( s \). The number of remaining parallelepipeds is thus at least \( \beta \) corresponding of the information points that define the functionals \( \bar{P} \). All parallelepipeds \( P_{k_1, k_2, \ldots, k_{d-1}} \) (number these parallelepipeds is \( (pm)^{d-1} \)). To fulfill the standard information conditions \( \bar{L}_j(H) = 0 \), \( j = 1, 2, \ldots, n - k \), it suffices to eliminate cells which contain in the interior any of the information points that define the functionals \( \bar{L}_j \). This is done by setting the corresponding \( \beta \)'s to 0. The number of eliminated cells is at most \( n - k \). We also exclude all parallelepipeds \( P_{k_1, k_2, \ldots, k_{d-1}} \) which contain more than \( pm/2 \) eliminated cells. Denoting the number of these by \( x \), we observe that \( x \leq 2(n - k)/(pm) \). Indeed, the number of eliminated cells is at least \( pmx/2 \). If \( x > 2(n - k)/(pm) \), then the number of eliminated cells is greater than \( n - k \), which is a contradiction.

The number of remaining parallelepipeds is thus at least \( s = (pm)^{d-1} - 2(n - k)/(pm) \). In any such parallelepiped \( P_{k_1, k_2, \ldots, k_{d-1}} \), the number of coefficients \( \beta^j_{k_1, k_2, \ldots, k_{d-1}} = 0 \) is at most \( pm/2 \). We set the remaining \( \beta^j_{k_1, k_2, \ldots, k_{d-1}} \) to be the same for each \( j \), \( \beta^j_{k_1, k_2, \ldots, k_{d-1}} \equiv \beta_{k_1, k_2, \ldots, k_{d-1}} \). The number of unknown coefficients \( \beta_{k_1, k_2, \ldots, k_{d-1}} \) is thus at least \( s \).

We select them to satisfy the conditions \( L_j(H) = 0 \), \( j = 1, 2, \ldots, k \). This is the system of \( k \) linear homogenous equations with at least \( (pm)^{d-1} - 2(n - k)/(pm) \) unknowns. Since \( \{k(n)\}_{n=1}^{\infty} \) satisfies \( (24) \), we have for sufficiently large \( n (m) \) that

\[
k < (pm)^{d-1} - 2(n - k)/(pm).
\]

Indeed, the condition \( k \leq \alpha n^{1/d} \) for sufficiently large \( n \) yields that \( k \leq \alpha m^{d-1} \) for sufficiently large \( m \). A sufficient condition for \( (30) \) is

\[
\alpha m^{d-1} < \frac{(1/2)p^d - 1}{(1/2)pm - 1} m^d,
\]

which is fulfilled if

\[
\alpha \leq p^{d-1} - 2/p.
\]

The last condition holds true for sufficiently large (fixed) \( p \).

By \( (30) \), the system of equations has a solution with the maximum modulus component \( \beta_{k_1, k_2, \ldots, k_{d-1}} = 1 \) for some \( k_1, k_2, \ldots, k_{d-1} \). With this choice of \( k_1, k_2, \ldots, k_{d-1} \), we repeat, starting from \( (19) \), the remaining steps of the proof of Theorem 1. We use the fact that \( \beta^j_{k_1, k_2, \ldots, k_{d-1}} = 1 \) for at least \( pm/2 \) indices \( j \), while the remaining \( \beta^j_{k_1, k_2, \ldots, k_{d-1}} \) are 0.

Let \( d = 1 \). We show that for each \( \alpha \) there are \( C > 0 \) and \( \bar{n}_0 \) such that for any \( n \geq \bar{n}_0 \), any information \( N_n \) given by \( (25) \) with \( k \leq \alpha \), and any algorithm \( \Phi_n \) it holds

\[
e(\Phi_n, N_n, F_r, 1) \geq Cn^{-r}.
\]

(31)
We only sketch the proof, by showing how to define 'difficult' functions \( f_1 \) and \( f_2 \) in \( F_{r,1} \) with the same information, \( N_n(f_1) = N_n(f_2) \). Take an interval \([\eta, \eta + \delta]\), \( \delta > 0 \), contained in \( M \). Take \( f_1 \) such that \( f_1(y) \equiv \hat{\alpha} > 0, y \in [\eta, \eta + \delta] \). We choose \( \hat{\alpha} \) small enough to assure that: \( f_1 \) can be extended to \( \mathbb{R} \) so that the extension is in \( F_{r,1} \), \( \hat{\alpha} \leq D/2 \) and \( \hat{\alpha}(b-a) \leq \delta \).

Then the solution for \( f_1 \) in \([a, b]\) is given by \( z_1(t) = \eta + \hat{\alpha}(t-a), \ t \in [a, b] \).

We now divide \([\eta, \eta + \hat{\alpha}(b-a)]\) uniformly into \( k + 1 \) subintervals \( I_j, j = 1, 2, \ldots, k + 1 \). Next, each \( I_j \) is further divided into \( 2n \) equal subintervals \( I_j^l, l = 1, 2, \ldots, 2n \). We define function \( f_2 = f_1 + H \) (with the solution \( z_2 \)), where \( H \) is given as follows.

In each subinterval \( I_j^l \) containing in the interior an information point (see the functionals \( \bar{L}_p \)), we set \( H = 0 \). There is at most \( n - k \) such 'removed' subintervals \( I_j^l \). Hence, there is at most \( n - k \) removed (and at least \( n + k \) remaining) subintervals in each interval \( I_j \). The conditions \( \bar{L}_p(H) = 0, p = 1, 2, \ldots, n - k \) are automatically satisfied.

To assure that \( L_j(H) = 0, j = 1, 2, \ldots, k \), we construct in each of the remaining subintervals \( I_j^l \) a standard scalar (normalized) bump function \( h_j^l \) with support \( I_j^l \) (see the Appendix I), and define \( H(y) = \sum_{j=1}^{k+1} \beta_j \sum_l h_j^l(y) \) for \( y \in \mathbb{R} \). The second sum is taken over all \( l \) such that \( I_j^l \) has not been removed. The conditions \( L_j(H) = 0, j = 1, 2, \ldots, k \) are equivalent to \( k \) linear homogeneous equations with \( k + 1 \) unknowns \( \beta_j \). There exists a solution of the system with maximum modulus component \( \beta_j^* = 1 \). In the lower bound on \( \sup_{\xi \in [a, b]} |z_2(\xi) - z_1(\xi)| \) given in the Appendix II, we choose the interval \([x, t] = I_{j^*} \). Since the function \( H \) in the interval \( I_{j^*} \) is composed of at least \( n + k \) nonzero bump functions, the desired lower bound (31) follows.

By inspecting the proof, we see that (31) holds true as well for adaptive information \( N_n \).

We finally discuss the main result of this paper for non-autonomous systems.

**Remark 1** (*Theorem 1 for non-autonomous problems*) Consider a non-autonomous system
\[
z'(t) = f(t, z(t)), \ t \in [a, b], \ z(a) = \eta,
\]
where \( f : [a, b] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a function from the class
\[
F_{r,d}^{\text{non}} = \{ f : [a, b] \times \mathbb{R}^d \rightarrow \mathbb{R}^d : f \in C^r([a, b] \times \mathbb{R}^d), \ f(t, y) = 0 \text{ for } y \notin M, \ |D^i f^j(t, y)| \leq D, (t, y) \in [a, b] \times \mathbb{R}^d, i = 0, 1, \ldots, r, j = 1, 2, \ldots, d \}.
\]

We briefly sketch how the non-autonomous case can be covered by a similar analysis as in Theorem 1, by only pointing out differences in the proof. The corresponding function
to $f_1$ in (13) is now constructed so that

$$f_1(t, y) = \begin{cases} \bar{\alpha} & \text{for } t \in [a, b], \ y \in M_1, \\ 0 & \text{for } t \in [a, b], \ y \notin M, \end{cases}$$

(34)

where $\|\bar{\alpha}\| \leq \Delta$. The construction of the function $H : \mathbb{R}^{d+1} \to \mathbb{R}^d$ corresponding to that in (16) starts in the hypercube $K \subset \mathbb{R}^{d+1}$ contained in $[a, b] \times B_\Delta$ given by

$$K = [a, b] \times [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d],$$

(35)

where $B_\Delta$ is defined in point 3. in the proof of Theorem 1, and $a_s, b_s$ are given as in (14). Compared to the autonomous case, we have here an additional variable $t \in [a, b]$. The hypercube $K$ will contain all graphs of the solutions $[t, (t-a)\bar{\alpha}^T + \eta^T]^T$, $t \in [a, b]$, for functions $f_1$ (i.e., for vectors $\bar{\alpha}$) under consideration. Let $m = \lceil n^{1/(d+1)} \rceil$. We define the $d$-dimensional hypercube $K_1$ by

$$K_1 = [b, b] \times [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d].$$

We divide $K_1$ (uniformly) into $(pm)^d$ small hypercubes $K_{k_1,k_2,...,k_d}$ with centers $c_{k_1,k_2,...,k_d} = [b, \bar{c}_{k_1,k_2,...,k_d}]^T$, $\bar{c}_{k_1,k_2,...,k_d} \in \mathbb{R}^d$. The vector $\bar{\alpha}$ will be selected as

$$\bar{\alpha} = \bar{\alpha}_{k_1,k_2,...,k_d} = (\bar{c}_{k_1,k_2,...,k_d} - \eta) / (b - a),$$

for a proper choice of $k_1, k_2, \ldots, k_d$. Note that $\|\bar{\alpha}\| \leq \Delta$, and the graph $[t, (t-a)\bar{\alpha}^T + \eta^T]^T$ passes through $[a, \eta]^T$ for $t = a$, and $c_{k_1,k_2,...,k_d}$ for $t = b$.

For each $d$-tuple of the indices $k_1, k_2, \ldots, k_d$, we consider a cone $C \subset \mathbb{R}^{d+1}$ defined as the convex hull of $K_{k_1,k_2,...,k_d}$ and $[a, \eta]^T$. The intersection of $C$ with the $d$-dimensional hyperplane $t = (a + b)/2$ is denoted by $\hat{K}_{k_1,k_2,...,k_d}$. The remaining steps of the definition of the functions $\hat{H} : \mathbb{R}^{d+1} \to \mathbb{R}$, $H : \mathbb{R}^{d+1} \to \mathbb{R}^d$, and of the choice of indices $k_1, k_2, \ldots, k_d$ that define $\bar{\alpha}$ are similar to the steps described in (15)–(18) in the proof of Theorem 1, with $d := d + 1$. After replacing $\hat{H}(\bar{\alpha}(\xi - a) + \eta)$ in (20) by $\hat{H}(\xi, \bar{\alpha}(\xi - a) + \eta)$, the rest of the proof goes similarly as in the autonomous case with $d$ replaced by $d + 1$.

Consequently, in place of the bound in Theorem 1, we get for non-autonomous systems the (matching) bound

$$\text{comp}^{\text{nonad}}(\varepsilon, F_{r,d}) = \Omega \left( (1/\varepsilon)^{(d+1)/r} \right),$$

(36)
4 Appendix

I. Construction of bump functions on a parallelepiped

We show the construction of a $C^\infty$ bump function on a parallelepiped in $\mathbb{R}^d$, $d \geq 2$, which was used to define the function $H$ in [16]. The construction is, up to some details, standard. Let $h : \mathbb{R} \to \mathbb{R}$ be given by

$$h(x) = \begin{cases} \exp(1/x(x - 1)), & x \in (0, 1) \\ 0 & \text{otherwise} \end{cases}.$$  

We have that $h \in C^\infty(\mathbb{R})$, $h(x) > 0$, $x \in (0, 1)$, $\max_{x \in \mathbb{R}} h(x) = h(1/2) = c_0$, $\max_{x \in \mathbb{R}} |h^{(j)}(x)| = c_j$ for $j \geq 1$, and

$$\int_{\mathbb{R}} h(x) \, dx = \bar{c},$$  

where $c_j, \bar{c}$ are absolute positive constants.

Consider a parallelepiped $P$ in $\mathbb{R}^d$ defined as follows. Let $P_1 = [p_1, q_1] \times [p_2, q_2] \times \ldots \times [p_{d-1}, q_{d-1}]$, where $q_j - p_j > 0$. Let $\bar{\alpha} \in \mathbb{R}^d$ be such that $\cos(\bar{\alpha}, e_d) = \bar{\alpha}^T e_d/\|\bar{\alpha}\| \geq \bar{c} > 0$, where $e_d = [0, 0, \ldots, 1]^T$, and $\| \cdot \|_2$ is the Euclidean norm in $\mathbb{R}^d$. For $N > 0$, we set

$$P = \left\{ y \in \mathbb{R}^d : \text{there exist } \bar{y} \in P_1 \text{ and } \bar{l} \in [0, N] \text{ such that } y = [\bar{y}^T, 0]^T + \frac{\bar{\alpha}}{\|\bar{\alpha}\|} \bar{l} \right\}.$$  

We define a real-valued bump function $\hat{H}$ on $P$. A point $y \in \mathbb{R}^d = [y^1, y^2, \ldots, y^d]^T$ is uniquely defined by the pair $(\bar{y}, \bar{l})$ with $\bar{y} = [\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^{d-1}] \in \mathbb{R}^{d-1}$ and $\bar{l} \in \mathbb{R}$ ( $\bar{y}$ is a projection of $y$ on $\mathbb{R}^{d-1}$ along the direction $\bar{\alpha}$). We set for $\hat{C} > 0$

$$\hat{H}(y) = \hat{C} \left( \min_{1 \leq j \leq d-1} (q_j - p_j), N, 1 \right) \prod_{j=1}^{d-1} h \left( \frac{\bar{y}^j - p_j}{q_j - p_j} \right) h \left( \frac{\bar{l}}{N} \right).$$  

Since the change of the variables is given by

$$\bar{y} = y - \frac{\bar{\alpha}}{\|\bar{\alpha}\|} \cdot \frac{y^d}{\cos(\bar{\alpha}, e_d)} \quad \text{(the last component equal to 0 is omitted)}, \quad \bar{l} = \frac{y^d}{\cos(\bar{\alpha}, e_d)},$$

this describes a $C^\infty(\mathbb{R}^d)$ nonnegative mapping from $\mathbb{R}^d$ to $\mathbb{R}$ with support $P$. Furthermore, by selecting proper (sufficiently small, fixed) value of $\hat{C}$ independent of $p_j, q_j, N$, we assure that $\hat{H}(y) \leq D/2$ and $|D^k \hat{H}(y)| \leq D$ for any partial derivative of $\hat{H}$ of order $k$, $k = 1, 2, \ldots, r$, $y \in \mathbb{R}^d$.

In the proof of Theorem 1, we use [40] with $\bar{\alpha} = -\bar{\alpha}_{k_1, k_2, \ldots, k_{d-1}}$, and we have $\cos(\bar{\alpha}, e_d) \geq 1/\sqrt{d}$. 

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II. A lower bound on the distance between two solutions

We recall for completeness a standard lower bound that is used in this paper. Let \( z_j \) be the solution of (1) for a right-hand side \( f_j, j = 1, 2, f_2 = f_1 + H \). We have that for any \( a \leq x \leq t \leq b \)

\[
z_2(t) - z_1(t) - (z_2(x) - z_1(x)) = \int_x^t (f_2(z_2(\xi)) - f_2(z_1(\xi))) \, d\xi + \int_x^t (f_2(z_1(\xi)) - f_1(z_1(\xi))) \, d\xi.
\]

This gives

\[
2 \sup_{\xi \in [x,t]} \| z_2(\xi) - z_1(\xi) \| \geq \left\| \int_x^t H(z_1(\xi)) \, d\xi \right\| - \left\| \int_x^t (f_2(z_2(\xi) - f_2(z_1(\xi)) \, d\xi \right\|
\]

From the Lipschitz condition, we have that

\[
\sup_{\xi \in [x,t]} \| z_2(\xi) - z_1(\xi) \| \geq \frac{1}{2 + L(t - x)} \left\| \int_x^t H(z_1(\xi)) \, d\xi \right\|.
\] (41)

In particular, for any \( a \leq x \leq t \leq b \),

\[
\sup_{\xi \in [a,b]} \| z_2(\xi) - z_1(\xi) \| \geq \frac{1}{2 + L(t - x)} \left\| \int_x^t H(z_1(\xi)) \, d\xi \right\|. \] (42)

It is easy to see that for \( x = a \), the value \( 2 + L(t - a) \) in the denominator can be replaced by \( 1 + L(t - a) \).

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