On the structure of the space of generalized connections

J. M. Velhinho

Departamento de Física, Universidade da Beira Interior
R. Marquês D’Ávila e Bolama, 6201-001 Covilhã, Portugal
jvelhi@dfisica.ubi.pt

Abstract

We give a modern account of the construction and structure of the space of generalized connections, an extension of the space of connections that plays a central role in loop quantum gravity.

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1 Introduction

Loop quantum gravity is an attempt to canonically quantize general relativity, starting from a $SU(2)$ gauge formulation of the classical theory, and using non-perturbative background independent methods.

The purpose of the present review is to give a modern and updated account of the structure of the so-called space of generalized (or distributional) connections $\bar{A}$, an extension of the classical space of smooth connections $A$, that plays a central role in loop quantum gravity. Since analogous spaces of generalized connections can be constructed for different compact gauge groups, this discussion may also be of interest for other (especially diffeomorphism invariant) models.

Although we will certainly not attempt to make a survey of loop quantum gravity\footnote{For reviews of loop quantum gravity, or of its by now numerous and very impressive achievements and applications, we recommend the excellent available works, ranging from quick surveys to very thorough and comprehensive treatments, by some of the most active researchers in the field [1, 2, 3, 4, 5, 6, 7, 8, 9].}, a minimal introduction to the general programme and its foundations is required, in order to appreciate the origins, place and role of the space of generalized connections. This is the purpose of the current section.

The starting point for loop quantum gravity is the $SU(2)$ version of Ashtekar’s canonical formulation of general relativity as a special kind of gauge theory [1, 10, 11]. The phase space $A \times E$ is made of canonically conjugate pairs $(A, E)$, where $A \in A$ are smooth connections on the spatial manifold $\Sigma$ and $E \in E$ are electric fields, i.e. $su(2)^*$-valued vector densities of weight one. As is well known, the theory has no Hamiltonian (in the usual sense), only constraints. Besides the Gauss constraint, that generates $SU(2)$ gauge transformations, there is the (spatial) diffeomorphism constraint, that generates spatial diffeomorphisms, and the Hamiltonian constraint, associated with reparametrizations of time.

Loop quantum gravity follows the Dirac method for the quantization of theories with constraints. In this method, one tries to impose the constraints after quantization of the unconstrained phase space. A kinematical algebra, i.e. a Poisson algebra of functions that separate points in the unconstrained phase space, must therefore be chosen, as the set of elementary variables to be quantized. The specific choice of variables is one of the most characteristic aspects of loop quantum gravity, and the one that gave it its name. It was introduced in quantum gravity by Jacobson, Rovelli and Smolin [12, 13].
In its modern formulation, the loop quantum gravity kinematical variables are: i) continuous complex functions of holonomies $A \mapsto h_c(A)$ of connections along certain curves $c$, as configuration variables, and ii) electric flux functions $E \mapsto E_{S,f}(E) := \int_S \star E_j f^j$ on surfaces $S$, where $(\star E_j)_{\mu\nu} = E_j^\alpha \epsilon_{\alpha\mu\nu}$ and $f$ are $su(2)$-valued functions, as momentum variables.

Besides the non-trivial fact that a well defined quantization of such an algebra was constructed [15, 16, 17, 18], these variables have in their favor the fact that they are well adapted to the constraints. Notice that, replacing the crucial role of the Hamiltonian in standard quantum field theory, the constraints are now the fundamental guidelines in the construction/selection of the quantization, as they must be implemented, either as self-adjoint operators or as unitary representations of the corresponding groups. It is therefore welcomed to start with classical variables that have a simple behaviour with respect to the constraints. In this respect, notice that under a $SU(2)$ gauge transformation $g$, the flux variables transform among themselves, $E_{S,f} \mapsto E_{S,g^{-1}f}$, and holonomies transform simply as $h_c \mapsto g(b)h_cg^{-1}(a)$, where $a$ and $b$ are the starting and end points of the curve $c$. Moreover, gauge invariant configuration functions are easily obtained, by considering closed curves (loops) $c$ and taking the trace. In what concerns diffeomorphisms, the important fact is that holonomies and flux variables are intrinsically, background independently, defined integrals, since, locally, connections $A$ are 1-forms and the objects $\star E_j f^j$ are 2-forms. Simple covariance properties then follow: under a spatial diffeomorphism $\varphi$ we have $h_c \mapsto h_{\varphi^{-1}c}$ and $E_{S,f} \mapsto E_{\varphi^{-1}S,\varphi^*f}$. As for the Hamiltonian constraint, early formal arguments [12, 13] suggested that a quantum version of it could be defined within this framework, and that certain "loop states" could be solutions of the "quantum Hamiltonian constraint".

The space of generalized connections $\bar{A}$ is an extension of $A$ determined by the above configuration variables, in the sense that those variables are quantized as functions on $\bar{A}$. Moreover, the flux variables are naturally realized as derivations of a certain algebra of functions in $\bar{A}$. Thus, $\bar{A}$ plays the role of a "universal quantum configuration space", i.e. the space where a Schrodinger-like $L^2$ representation (of the chosen variables) is naturally defined. Notice that this situation is reminiscent of the well known quantization of scalar fields, where a distributional extension of the space of classical variables would be considered.

There are, in fact, subtleties in the definition of the Poisson algebra generated by these functions, especially concerning the flux variables. A satisfactory answer to this issue was given in [14].
smooth fields is required in order that measures and corresponding $L^2$ spaces can be defined. In the present case, we are led to a compact Hausdorff space $\bar{A}$, and (regular Borel) measures are thus guaranteed to exist.

Before we go into any details, let us fix the particular framework considered in the present review.

There are several versions of the space of generalized connections, depending on the differentiability class of the curves $c$ used in the holonomies. We will assume that the spatial manifold $\Sigma$ is endowed with a (real) analytic structure and that the curves $c$ are piecewise analytic (analytic surfaces are accordingly used in the flux variables). While this may seem unnatural, there are reasons to believe that the analytic set-up is sufficiently general. As it avoids technical complications arising in more general settings, the analytic case is the most studied, and the one in which more rigorous results have been obtained. Nevertheless, important progress has been made in the case of piecewise smooth, or more general, curves [19, 20, 21, 22].

Notice also that we will describe the non gauge invariant space of generalized connections, as opposed to the gauge invariant space $A/G$ of generalized connections modulo gauge transformations, in which loop variables defined by closed curves are used. This was in fact the original approach [15, 16], but nowadays the non gauge invariant space $\bar{A}$ [23, 17] is typically preferred, leaving the solution of the Gauss constraint to a later stage. The relation between $\bar{A}$ and $A/G$, or how to solve the Gauss constraint in the $\bar{A}$ framework, is very well understood; the two approaches are seen to be fully equivalent.

Let us then see how the space of generalized connections appears in loop quantum gravity. The crucial fact is that the chosen set of configuration variables

$$\begin{align*}
A & \mapsto F(h_{c_1}(A), \ldots, h_{c_n}(A)), \quad F \in C((SU(2)^n), \\
& \text{is a unital } \star\text{-algebra of bounded functions, due to the compactness of} \\
& \text{the gauge group (this is called the algebra of cylindrical functions } Cyl(A)). \\
& \text{The natural sup norm } C^\star\text{-completion of this algebra, introduced by Ashtekar} \\
& \text{and Isham [15], is called the holonomy algebra } Cyl(A). \\
& \text{One then requires that the quantization of kinematical variables produces a representation of} \\
& \text{the unital commutative } C^\star\text{-algebra } Cyl(A). \\
& \text{By the Gelfand–Naimark fundamental characterization of commutative} \\
& C^\star\text{-algebras, we know that every commutative unital } C^\star\text{-algebra is (isomorphic to) the algebra } C(X) \\
& \text{of continuous complex functions on a unique (up to homeomorphism) compact Hausdorff space } X. \\
& \text{In the case of the holonomy algebra } Cyl(A), \text{the corresponding compact space is naturally realized}
\end{align*}$$


\begin{align*}
1 & \leftrightarrow F(h_{c_1}(A), \ldots, h_{c_n}(A)), \quad F \in C((SU(2)^n), \\
& \text{is a unital } \star\text{-algebra of bounded functions, due to the compactness of} \\
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\end{align*}
as the space of all morphisms from a certain groupoid of paths (equivalence classes of curves) to the gauge group. This is the space of generalized connections $\tilde{A}$.

The relevance of $\tilde{A}$ to the quantization process now becomes obvious. Again from general results, we know that: 

i) every representation of $C(\tilde{A})$ is a direct sum of cyclic representations; 

ii) every cyclic representation is a representation by multiplication operators on a Hilbert space $L^2(\tilde{A}, \mu)$, where $\mu$ is a measure on $\tilde{A}$. Thus, given the isomorphism $C_{\text{Y}}(\mathcal{A}) \cong C(\tilde{A})$, every quantization of the loop quantum gravity kinematical algebra decomposes into a direct sum

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i = \bigoplus_i L^2(\tilde{A}, \mu_i),$$

with $C_{\text{Y}}(\mathcal{A})$ being represented in each space $L^2(\tilde{A}, \mu_i)$ by multiplication operators:

$$(\hat{f}\psi)(\tilde{A}) = \hat{f}(\tilde{A})\psi(\tilde{A}),$$

where $\psi \in L^2(\tilde{A}, \mu_i)$, $f \in C_{\text{Y}}(\mathcal{A})$ and $\hat{f} \in C(\tilde{A})$ is the image of $f$.

The study of the structure of $\tilde{A}$ (and $A/G$) and of measures thereon was done in the first half of the 1990s [16, 23, 24, 25, 26, 17]. After the seminal work of Ashtekar and Isham [15], Ashtekar and Lewandowski introduced the analytic framework and succeeded in the construction of a very natural measure, distinguished by its simplicity and invariance properties [16]. This is the so-called Ashtekar–Lewandowski, or uniform, measure $\mu_0$. Although several other measures were constructed, it turns out that the uniform measure still stands as the only known measure that gives rise to a representation of the full kinematical algebra of holonomies and flux variables. In other words, the corresponding kinematical Hilbert space $\mathcal{H}_0 := L^2(\tilde{A}, \mu_0)$ supports the only known representation of the kinematical algebra of loop quantum gravity [18].

This $\mathcal{H}_0$ representation is cyclic with respect to $C(\tilde{A})$, of course, and is irreducible, as it should be, under the action of the kinematical algebra [27]. Moreover, the measure $\mu_0$ is gauge invariant and invariant under the action of analytic diffeomorphisms. This immediately leads to the required unitary representations of both groups. Thus, as far as the Gauss and diffeomorphism constraints are concerned, the $\mathcal{H}_0$ representation seems to qualify as an intermediate, or auxiliary, representation of kinematical variables and constraints, as required in the Dirac method. The above facts alone are sufficient to justify the absolutely central role of the $\mathcal{H}_0$ representation in
the canonical loop quantum gravity programme. The $\mathcal{H}_0$ representation is the kinematical representation used in loop quantum gravity; virtually all further developments are based upon it.

The seemingly unique status of the $\mathcal{H}_0$ representation was recently reinforced by a detailed analysis of the representation theory of the kinematical algebra $[28, 29, 30, 31]$. Although the uniqueness of the $\mathcal{H}_0$ representation was not fully established, it was shown $[30, 31]$ that an a priori large class of representations, that also support a unitary implementation of the group of analytic diffeomorphisms, contains in fact only reducible representations, and that every irreducible component is equivalent to the $\mathcal{H}_0$ representation.

Finally, for completeness, let us stress that the $\mathcal{H}_0$ representation is not, by far, the end of the quantization process. Important as it is, $\mathcal{H}_0$ is the starting point for the hardest and most interesting part of the quantization, and this is precisely the reason why it is so important that $\mathcal{H}_0$, and therefore $\bar{A}$, are well defined and well understood.

Once the constraints are represented, one must, of course, solve them. As already mentioned, the Gauss constraint is easily dealt with. It can be solved before or after solving the other constraints. If we choose to solve it prior to the other constraints, the solution is the (large) subspace of gauge invariant elements of $\mathcal{H}_0$.

On the contrary, already the diffeomorphism constraint cannot be solved within $\mathcal{H}_0$. In fact, the necessity of distributional, or generalized, solutions is typical of the Dirac method, when non-compact invariance groups, as the diffeomorphism group, are involved. Starting from $\mathcal{H}_0$, a complete space of solutions of the diffeomorphism constraint was constructed in a distributional extension of $\mathcal{H}_0$, and equipped with an inner product induced by that of $\mathcal{H}_0$ [18].

All these efforts still leave ahead far more challenging tasks, like the construction of the quantum Hamiltonian constraint and its space of solutions, recovery of (semi-)classical physics, construction of observables, or interpretational issues, just to mention a few. These subjects, and many more, are being actively pursued. Regarding e.g. the crucial and most difficult question of the Hamiltonian constraint, it is truly remarkable that a candidate quantum operator has been rigorously defined in $\mathcal{H}_0$ [7]. Although there are open questions regarding the physical correctness of this operator, the fact that all constraints can be implemented reinforces, again, the status of $\mathcal{H}_0$, and of the non-perturbative and background independent methods used in loop quantum gravity. More recently, a new proposal [32] suggests the possibility of defining the Hamiltonian constraint directly on the space
of solutions of the diffeomorphism constraint. If the expectations raised by that work are confirmed, then important technical as well as conceptual simplifications may occur, possibly leading to further and quicker progress.

Let us conclude with an overview of the present work. In section 2 we present the space of generalized connections \( \mathcal{A} \) and discuss its important projective structure. As first pointed out by Baez [33] and later on explicitly put forward in [34], the proper framework to express the algebraic properties of \( \mathcal{A} \) implicit in earlier formulations [23, 17] uses the language of category theory. The space \( \mathcal{A} \) is actually a space of functors, or morphisms. (The notions from category theory used in the present work are minimal and elementary – we review them in section 2.1.) The algebraic properties of \( \mathcal{A} \) reflect simply the algebraic properties of parallel transports, as functions of curves. These functions depend only on certain equivalence classes of curves, and this is precisely how the groupoid of paths \( \mathcal{P} \), discussed in section 2.2, emerges. As a bare set, \( \mathcal{A} \) is the set of functors from \( \mathcal{P} \) to the gauge group \( G \). This large set is actually a limit of a family of finite dimensional spaces, each of which is identified with some power \( G^n \) of the gauge group. It is by means of this characterization that \( \mathcal{A} \) is turned into an interesting and manageable space, with a rich structure [16, 26, 17, 25]. This so-called projective structure is discussed in some detail section 2.4, where we concentrate on topological aspects. Section 2.5 describes the “dual” inductive structure of the algebra of functions on \( \mathcal{A} \). In section 2.6 we show how gauge transformations and diffeomorphisms fit within the category formulation of \( \mathcal{A} \), and speculate on possible extensions of the diffeomorphism group suggested by this formulation.

After the description of \( \mathcal{A} \), we return in section 3 to its physical role as a possible ”quantum configuration space” for theories of connections. In section 3.1 we display \( \mathcal{A} \) as the compact space defined by the holonomy algebra. Section 3.2 is dedicated to the general structure of measures on \( \mathcal{A} \). Finally, we present the Ashtekar-Lewandowski measure \( \mu_0 \) and briefly discuss the most important properties of the corresponding representation.

2 The space of generalized connections \( \mathcal{A} \)

2.1 Elementary notions from category theory

A (small) category \( \mathcal{C} \) is formed by ”arrows” (or ”morphisms”) between ”objects”. The set of objects is denoted by \( \text{Obj} \mathcal{C} \) and the set of arrows by \( \text{Mor} \mathcal{C} \) (or simply by \( \mathcal{C} \)). For \( x, y \in \text{Obj} \mathcal{C} \), \( \text{Hom}_\mathcal{C} \left[ x, y \right] \) denotes the set of all arrows
from $x$ to $y$. The following two maps $r, s : \text{Mor} \mathcal{C} \to \text{Obj} \mathcal{C}$, called range and source, respectively, are naturally defined: $r(\gamma)$ is the object on which the arrow $\gamma$ ends, i.e. $r(\gamma) = x$ if $\gamma \in \text{Hom}_\mathcal{C}[\cdot, x]$; likewise, $s(\gamma)$ is the object on which $\gamma$ starts. The main characteristic of a category is the existence of an associative composition operation between compatible arrows, meaning that there are binary operations $\text{Hom}_\mathcal{C}[x,y] \times \text{Hom}_\mathcal{C}[y,z] \to \text{Hom}_\mathcal{C}[x,z]$, $(\gamma, \gamma') \mapsto \gamma' \gamma$ (4)

satisfying $\gamma''(\gamma' \gamma) = (\gamma'' \gamma')\gamma$. It is also required that for every object $x$ there exists a unique identity arrow $1_x \in \text{Hom}_\mathcal{C}[x,x]$, such that $1_x \gamma = \gamma$, $\forall \gamma \in \text{Hom}_\mathcal{C}[\cdot, x]$ and $\gamma 1_x = \gamma$, $\forall \gamma \in \text{Hom}_\mathcal{C}[x,\cdot]$. Naturally, an arrow $\gamma \in \text{Hom}_\mathcal{C}[x,y]$ is said to be invertible if there exists $\gamma^{-1} \in \text{Hom}_\mathcal{C}[y,x]$ such that $\gamma^{-1}\gamma = 1_x$ and $\gamma\gamma^{-1} = 1_y$.

Most important to our discussion is the notion of groupoid, which is simply a category in which every arrow is invertible.

Groups are a special class of groupoids, and therefore of categories. In this case the arrows are the elements of the group, composed through the group operation. As a category, a group has a single object, namely the group identity.

A map between categories that preserves the algebraic structure is called a functor. A functor $F : \mathcal{A} \to \mathcal{B}$ is made of two maps (usually denoted by the same symbol), between objects and between arrows, subjected to the conditions $F(1_x) = 1_{F(x)}$, $\gamma \in \text{Hom}_\mathcal{A}[x,y]$ implies $F(\gamma) \in \text{Hom}_\mathcal{B}[F(x), F(y)]$ and $F(\gamma' \gamma) = F(\gamma')F(\gamma)$.

Another important notion is that of a natural transformation between functors. For two functors $S, T : \mathcal{A} \to \mathcal{B}$, a natural transformation $\tau : S \to T$ is a map from $\text{Obj} \mathcal{A}$ to $\text{Mor} \mathcal{B}$, assigning to each $x \in \text{Obj} \mathcal{A}$ an arrow $\tau(x) \in \text{Hom}_\mathcal{B}[S(x), T(x)]$ such that $T(\gamma)\tau(s(\gamma)) = \tau(r(\gamma))S(\gamma)$, $\forall \gamma \in \text{Mor} \mathcal{A}$. If, in particular, $\mathcal{B}$ is a group, one sees that natural transformations give us a representation of the product group $\times_{x \in \text{Obj} \mathcal{A}} \mathcal{B}$, acting on the set of all functors $F : \mathcal{A} \to \mathcal{B}$, by

$$F \mapsto \tau F : \ \tau F(\gamma) = \tau(r(\gamma))F(\gamma)\tau(s(\gamma))^{-1}. \quad (5)$$

2.2 The groupoid of paths $\mathcal{P}$

Let $\Sigma$ be an analytic, connected, orientable and paracompact $d$-dimensional manifold. Let us consider the set $\mathcal{C}$ of all continuous, oriented and piecewise
analytic parametrized curves in $\Sigma$, i.e. maps

$$c : [0, t_1] \cup \ldots \cup [t_{n-1}, 1] \to \Sigma$$

which are continuous in all the domain $[0,1]$, analytic in the closed intervals $[t_k, t_{k+1}]$ and such that the images $c([t_k, t_{k+1}])$ of the open intervals $]t_k, t_{k+1}[$ are submanifolds embedded in $\Sigma$. In what follows, $\sigma(c) := c([0,1])$ denotes the image of the interval $[0,1]$. The maps $s$ (source) and $r$ (range) are defined, respectively, by $s(c) = c(0)$, $r(c) = c(1)$.

Given two curves $c_1, c_2 \in \mathcal{C}$ such that $s(c_2) = r(c_1)$, let $c_2c_1 \in \mathcal{C}$ denote the natural composition given by

$$(c_2c_1)(t) = \begin{cases} 
  c_1(2t), & \text{for } t \in [0,1/2] \\
  c_2(2t-1), & \text{for } t \in [1/2,1].
\end{cases}$$

This composition defines a binary operation in a well defined subset of $\mathcal{C} \times \mathcal{C}$. Consider also the operation $c \mapsto c^{-1}$ given by $c^{-1}(t) = c(1 - t)$. Notice that this composition of parametrized curves is not truly associative, since the curves $(c_3c_2)c_1$ and $c_3(c_2c_1)$ are related by a reparametrization, i.e. by an orientation preserving piecewise analytic diffeomorphism $[0,1] \to [0,1]$. Similarly, the curve $c^{-1}$ is not the inverse of the curve $c$. We will refer to compositions of the form $c^{-1}c$ as retracings.

**Definition 1** Two curves $c, c' \in \mathcal{C}$ are said to be equivalent if

(i) $s(c) = s(c')$, $r(c) = r(c')$;

(ii) $c$ and $c'$ coincide up to a finite number of retracings and a reparametrization.

We will denote the set of all above defined equivalence classes by $\mathcal{P}$. It is clear by (i) that the maps $s$ and $r$ are well defined in $\mathcal{P}$. The image $\sigma$ can still be defined for special elements called edges. By edges we mean elements $e \in \mathcal{P}$ which are equivalence classes of analytic (in all domain) curves $c : [0,1] \to \Sigma$. It is clear that the images $c_1([0,1])$ and $c_2([0,1])$ corresponding to two equivalent analytic curves coincide, and therefore we define $\sigma(e)$ as being $\sigma(c)$, where $c$ is any analytic curve in the class of the edge $e$.

We discuss next the natural groupoid structure on the set $\mathcal{P}$. We will refer to generic elements of $\mathcal{P}$ as paths $p$, the symbol $e$ being reserved for edges.
The composition of paths is defined by the composition of elements of \( C \): if \( p, p' \in \mathcal{P} \) are such that \( r(p) = s(p') \), one defines \( p'p \) as the equivalence class of \( c'c \), where \( c \) (resp. \( c' \)) belongs to the class \( p \) (\( p' \)). The independence of this composition with respect to the choice of representatives follows from condition \((ii)\) above. The composition in \( \mathcal{P} \) is now associative, since \((c_3c_2)c_1 \) and \( c_3(c_2c_1) \) belong to the same equivalence class.

The points of \( \Sigma \) play the role of objects in this context. Points are in 1-1 correspondence with identity paths: given \( x \in \Sigma \) the corresponding identity \( 1_x \in \mathcal{P} \) is the equivalence class of \( c^{-1}c \), with \( c \in \mathcal{C} \) such that \( s(c) = x \). If \( p \) is the class of \( c \) then \( p^{-1} \) is the class of \( c^{-1} \). It is clear that \( p^{-1}p = 1_{s(p)} \) and \( pp^{-1} = 1_{r(p)} \).

One, therefore, has a well defined groupoid, whose set of objects is \( \Sigma \) and whose set of arrows is \( \mathcal{P} \). As usual, we will use the same notation, \( \mathcal{P} \), both for the set of arrows and for the groupoid. Notice that every element \( p \in \mathcal{P} \) can be obtained as a composition of edges. Therefore, the groupoid \( \mathcal{P} \) is generated by the set of edges, although it is not freely generated, since composition of edges may produce new edges.

### 2.3 The set of functors \( \text{Hom}[\mathcal{P}, G] \)

Let \( G \) be a (finite dimensional) connected and compact Lie group.

**Definition 2** \( \text{Hom}[\mathcal{P}, G] \) is the set of all functors from the groupoid \( \mathcal{P} \) to the group \( G \), i.e. is the set of all maps \( \bar{A} : \mathcal{P} \to G \) such that \( \bar{A}(p'p) = \bar{A}(p') \bar{A}(p) \) and \( \bar{A}(p^{-1}) = \bar{A}(p)^{-1} \).

To be consistent with the literature, we will refer to elements of \( \text{Hom}[\mathcal{P}, G] \) not as functors, but as morphisms, or as generalized (or distributional) connections, for reasons to be discussed next.

Let us show that the space \( \mathcal{A} \) of smooth \( G \)-connections on any given principal \( G \)-bundle over \( \Sigma \) is realized as a subspace of \( \text{Hom}[\mathcal{P}, G] \). We think of this bundle as being associated to a classical field theory of connections, and so we will also refer to \( \mathcal{A} \) as the classical configuration space. We will assume that a fixed trivialization of the bundle has been chosen. Connections can then be identified with local connection potentials.

The space of connections \( \mathcal{A} \) is injectively mapped into \( \text{Hom}[\mathcal{P}, G] \) through the use of the parallel transport, or holonomy, functions. The holonomy defined by a connection \( A \in \mathcal{A} \) and a curve \( c \in \mathcal{C} \) is denoted by \( h_c(A) \). Using the fixed trivialization, one can assume that holonomies \( h_c(A) \) take values on the group \( G \). The following properties of holonomies are seen to hold: i)
$h_c(A)$ is invariant under reparametrizations of $c$; ii) $h_{c_1c_2}(A) = h_{c_1}(A)h_{c_2}(A)$ and iii) $h_{c^{-1}}(A) = h_c(A)^{-1}$. One thus conclude that $h_c(A)$ depends only on the equivalence class of $c$, i.e., for fixed $A \in \mathcal{A}$, $h_c(A)$ defines a function on $\mathcal{P}$, with values on $G$. This function is, moreover, a morphism, by ii). Summarizing, we have a map $\mathcal{A} \rightarrow \text{Hom}[\mathcal{P},G]$:

$$A \mapsto \bar{A}_A, \quad \bar{A}_A(p) := h_p(A), \quad \forall p \in \mathcal{P}. \quad (6)$$

That this map is injective is guaranteed by the crucial fact that the set of holonomy functions $\{h_c, c \in \mathcal{C}\}$ separates points in $\mathcal{A}$ [35], i.e. given $A \neq A'$ one can find $c \in \mathcal{C}$ such that $h_c(A) \neq h_c(A')$, which is, of course, the expression of injectivity. The classical space $\mathcal{A}$ can then be seen as a subspace of $\text{Hom}[\mathcal{P},G]$.

The set $\text{Hom}[\mathcal{P},G]$ is, however, larger than the classical space $\mathcal{A}$. To begin with, depending on $\Sigma$ and $G$, different bundles may exist, and $\text{Hom}[\mathcal{P},G]$ contains the space of connections of all these bundles. Moreover, elements of $\text{Hom}[\mathcal{P},G]$ that do not correspond to any smooth connection do exist (see e.g. [16] for examples).

2.4 Projective structure and topology

Although $\text{Hom}[\mathcal{P},G]$ is a very large space, it is a well defined limit — a projective limit — of a family of finite dimensional spaces. Each space of this so-called projective family is identified, as a manifold [16, 26, 17], with some power of the Lie group $G$. This projective structure is critically important. It gives us a good understanding of $\text{Hom}[\mathcal{P},G]$, allowing e.g. to equip it with a rich variety of structures, from topology and measures [16, 26] to differential calculus [17]. In a precise sense, the projective structure reduces the task of dealing with a large infinite dimensional space to a problem in finite dimensions plus certain consistency conditions. Projective methods, and their inductive "dual" counterparts, are therefore basic tools of the present approach, from its foundations to current research.

We present next the projective structure of $\text{Hom}[\mathcal{P},G]$. In particular, this structure gives rise to a natural topology on $\text{Hom}[\mathcal{P},G]$. The compact space thus obtained is the space of generalized connections.

2.4.1 The set of labels

We start by introducing the directed set used as the set of labels of the projective family.
Definition 3 A finite set \( \{e_1, \ldots, e_n\} \) of edges is said to be independent if the edges \( e_i \) can intersect each other only at the points \( s(e_i) \) or \( r(e_i) \), \( i = 1, \ldots, n \).

The edges in an independent set are, in particular, algebraically independent, i.e. it is not possible to produce identity paths by (non-trivial) compositions of the edges and their inverses.

Let us consider subgroupoids of \( \mathcal{P} \) generated by independent sets of edges \( \{e_1, \ldots, e_n\} \). Recall that the subgroupoid generated by \( \{e_1, \ldots, e_n\} \) is the smallest subgroupoid containing all the edges \( e_i \), or explicitly, the subgroupoid whose objects are all the points \( s(e_i) \) and \( r(e_i) \) and whose arrows are all possible compositions of edges \( e_i \) and their inverses. Groupoids of this type are freely generated, given the algebraic independence of the edges.

Definition 4 The set of subgroupoids of \( \mathcal{P} \) that are generated by independent sets of edges is called the set \( \mathcal{L} \) of tame subgroupoids.

Clearly, the sets \( \{e_1, \ldots, e_n\} \) and \( \{e_1^{\epsilon_1}, \ldots, e_n^{\epsilon_n}\} \), where \( \epsilon_j = \pm 1 \) (i.e. \( e_j^{\epsilon_j} = e_j \) or \( e_j^{-1}\epsilon_j \)), generate the same subgroupoid, and this is the only ambiguity in the choice of the set of generators of a given groupoid \( L \in \mathcal{L} \). Thus, a groupoid \( L \in \mathcal{L} \) is uniquely defined by a set \( \{\sigma(e_1), \ldots, \sigma(e_n)\} \) of images corresponding to a set of independent edges. Notice that the union of the images \( \sigma(e_i) \) is a graph in the manifold \( \Sigma \), and therefore tame subgroupoids are uniquely associated to analytic graphs.

Let us consider in the set \( \mathcal{L} \) the partial-order relation defined by inclusion, i.e. given \( L, L' \in \mathcal{L} \), we will say that \( L' \geq L \) if and only if \( L \) is a subgroupoid of \( L' \). Recall that \( L \) is said to be a subgroupoid of \( L' \) if and only if all objects of \( L \) are objects of \( L' \) and for any pair of objects \( x, y \) of \( L \) every arrow from \( x \) to \( y \) is an arrow of \( L' \). It is not difficult to see that \( \mathcal{L} \) is a directed set with respect to the latter partial-order, meaning that for any given \( L \) and \( L' \) in \( \mathcal{L} \) there exists \( L'' \in \mathcal{L} \) such that \( L'' \geq L \) and \( L'' \geq L' \). Let us remark that this is a point where analyticity is very important, allowing to show e.g. that for every finitely generated subgroupoid \( \Gamma \subset \mathcal{P} \) there is an element \( L \in \mathcal{L} \) such that \( \Gamma \) is a subgroupoid of \( L \) [16].

2.4.2 Projective family

We introduce the projective family, induce a compact Hausdorff topology on each of its members, and show that consistent projections exist. Important
results concerning the projective limit of such so-called compact Hausdorff families were given in [26].

**Definition 5** For every $L \in \mathcal{L}$, let $\mathcal{A}_L := \text{Hom}[L,G]$ denote the set of all morphisms from the subgroupoid $L$ to the group $G$.

Let $\{e_1, \ldots, e_n\}$ be a set of generators of $L \in \mathcal{L}$. Since the morphisms $L \to G$ are uniquely determined by the images of the generators, one gets a bijection

$$\rho_{e_1, \ldots, e_n} : \mathcal{A}_L \to G^n,$$

given by

$$\rho_{e_1, \ldots, e_n}(A) \mapsto (A(e_1), \ldots, A(e_n)) \in G^n.$$  \hspace{1cm} (7)

Thus, every set $\mathcal{A}_L$ is in 1-1 correspondence with some $G^n$. Through this identification, every $\mathcal{A}_L$ becomes a compact Hausdorff space. Notice that the topology induced in $\mathcal{A}_L$ is independent of the choice of the generators, since maps of the form

$$(g_1, \ldots, g_n) \mapsto (g_{\epsilon_1 k_1}^{\epsilon_{k_1}}, \ldots, g_{\epsilon_n k_n}^{\epsilon_{k_n}}),$$  \hspace{1cm} (8)

where $(k_1, \ldots, k_n)$ is a permutation of $(1, \ldots, n)$ and $\epsilon_{k_i} = \pm 1$, are homeomorphisms $G^n \to G^n$.

We will show next that the family of compact spaces $\mathcal{A}_L$, $L \in \mathcal{L}$, is a compact Hausdorff projective family, meaning that given $L, L' \in \mathcal{L}$ such that $L' \geq L$ there exists a surjective and continuous projection $p_{L,L'} : \mathcal{A}_{L'} \to \mathcal{A}_L$ such that

$$p_{L,L''} = p_{L,L'} \circ p_{L',L''}, \quad \forall L'' \geq L' \geq L.$$  \hspace{1cm} (9)

For $L' \geq L$, the required projection

$$p_{L,L'} : \mathcal{A}_{L'} \to \mathcal{A}_L$$  \hspace{1cm} (10)

is naturally defined to be the map that sends each element of $\mathcal{A}_{L'}$ to its restriction to $L$. It is clear that (9) is satisfied. Let us show that the maps $p_{L,L'}$ are surjective and continuous. Let $\{e_1, \ldots, e_n\}$ be generators of $L$ and $\{e'_1, \ldots, e'_m\}$ be generators of $L' \geq L$. Let us consider the decomposition of the edges $e_i$ in terms of the edges $e'_j$:

$$e_i = \prod_j (e'_{r_{ij}})^{\epsilon_{ij}}, \quad i = 1, \ldots, n,$$  \hspace{1cm} (11)

where $r_{ij}$ and $\epsilon_{ij}$ take values in the sets $\{1, \ldots, m\}$ and $\{1, -1\}$, respectively. An arbitrary element of $\mathcal{A}_L$ is identified by the images $(h_1, \ldots, h_n) \in G^n$ of
the ordered set of generators \((e_1, \ldots, e_n)\). The map \(p_{L,L'}\) will, therefore, be surjective if and only if there are \((g_1, \ldots, g_m) \in G^m\) such that
\[
h_i = \prod_j g_{r_{ij}}^{e_{ij}}, \quad \forall i.
\] (12)

These conditions can indeed be satisfied, since they are independent. In fact, since the edges \(\{e_1, \ldots, e_n\}\) are independent, a given edge \(e'_k\) can appear at most once (in the form \(e'_k\) or \(e_k^{-1}\)) in the decomposition (11) of a given \(e_i\).

As for continuity, notice that, through the identifications (7), the maps \(p_{L,L'}\) correspond to projections \(\pi_{n,m} : G^m \to G^n,\)
\[
G^m \ni (g_1, \ldots, g_m) \mapsto \left(\prod_j g_{r_{ij}}^{e_{ij}}, \ldots, \prod_j g_{r_{nj}}^{e_{nj}}\right) \in G^n,
\] (13)
which are continuous.

2.4.3 Projective limit

The general notion of projective limit (see e.g [36]) applies in particular to the current situation.

**Definition 6** The projective limit of the family \(\{A_L, p_{L,L'}\}_{L,L' \in \mathcal{L}}\) is the subset \(A_\infty\) of the cartesian product \(\times_{L \in \mathcal{L}} A_L\) of those elements \((A_L)_{L \in \mathcal{L}}\) that satisfy the consistency conditions
\[
p_{L,L'} A_{L'} = A_L \quad \forall L' \geq L.
\] (14)

It is a simple, yet illustrative, exercise to show that \(\text{Hom} [\mathcal{P}, G]\) is in 1-1 correspondence with the projective limit \(A_\infty\). Let us consider the map
\[
\Phi : \text{Hom} [\mathcal{P}, G] \to A_\infty, \quad \hat{A} \mapsto (\hat{A}_L)_{L \in \mathcal{L}},
\] (15)
where \(\hat{A}_L\) is the restriction of \(\hat{A}\) to the tame subgroupoid \(L\). It is obvious that \((\hat{A}_L)_{L \in \mathcal{L}}\) belongs to \(A_\infty\), i.e. satisfies (14). To prove that \(\Phi\) is a bijection one just needs to remember that every path \(p \in \mathcal{P}\) belongs to some subgroupoid \(L(p) \in \mathcal{L}\), and therefore \(\hat{A}(p) = \hat{A}_L(p), \forall L \geq L(p)\). If \(\Phi(\hat{A}) = \Phi(\hat{A}')\), i.e. if \(\hat{A}_L = \hat{A}'_L \forall L\), we immediately get \(\hat{A}(p) = \hat{A}'(p) \forall p\), i.e. \(\hat{A} = \hat{A}'\), thus proving injectivity. Suppose now that we are given any \((A_L)_{L \in \mathcal{L}} \in A_\infty\). We construct its inverse image by \(\hat{A}(p) := A_L(p)\), where \(L\) is any tame subgroupoid such that \(p \in L\). This is well defined, since given two such \(L\) and \(L'\) one can always find \(L'' \geq L, L'\), ensuring that \(A_L(p) = p_{L,L''}A_{L''}(p)\) coincides with \(A_{L'}(p) = p_{L',L''}A_{L''}(p)\). As \(\hat{A}\) thus defined is obviously a morphism, surjectivity is proven.
Proposition 1 The map $\Phi$ (15) is a bijection.

We henceforth identify $\text{Hom} [\mathcal{P}, G]$ with the projective limit $\mathcal{A}_\infty$, by means of the bijection $\Phi$ (15). In particular, the projections

$$p_L : \text{Hom} [\mathcal{P}, G] \to A_L$$

$$\bar{A} \mapsto \bar{A}|_L, L \in \mathcal{L}$$

(16)

correspond to the maps $\mathcal{A}_\infty \ni (A_L)_{L \in \mathcal{L}} \mapsto A_L \in \mathcal{A}_L$. It is clear that the following consistency conditions

$$p_L = p_{L,L'} \circ p_{L'}, \forall L' \geq L,$$

(17)

corresponding to (14), are satisfied.

2.4.4 Natural topology

Given the special nature of our particular projective family $\{A_L\}_{L \in \mathcal{L}}$, two important results follow. First, the projections $p_L$ above are guaranteed to be surjective$^3$ [16, 26]. Second, the projective limit is naturally a compact Hausdorff space [25, 26]. The projective limit topology in $\text{Hom} [\mathcal{P}, G]$ is the weakest topology such that all the projections $p_L$ (16) are continuous. We sketch below a simplified proof of the fact that the thus obtained topological space is compact Hausdorff. We use arguments that are adapted from those given in [25].

Let us start by giving an equivalent description of the topology. Notice that the projections (16) are continuous if and only if the maps

$$\text{Hom} [\mathcal{P}, G] \to G, \bar{A} \mapsto \bar{A}(e),$$

(18)

are continuous for every edge $e$, since every $A_L$ is homeomorphic to some $G^n$ and the topology on $G^n$ is generated by $G$-open sets. Since continuity of all maps (18) implies continuity of the maps

$$\pi_p : \text{Hom} [\mathcal{P}, G] \to G, \bar{A} \mapsto \bar{A}(p),$$

(19)

for all $p \in \mathcal{P}$, one can also characterize the topology on $\text{Hom} [\mathcal{P}, G]$ as the weakest topology such that all maps $\pi_p$ (19) are continuous. Consider now the set of all maps from $\mathcal{P}$ to $G$, identified with the product space $\times_{p \in \mathcal{P}} G$. The product space is compact Hausdorff with the Tychonov topology. It is $^3$A stronger result in fact holds, see lemma 1 in section 3.1.
clear that the topology defined by the maps (19) coincides with the subspace topology on \( \text{Hom} [\mathcal{P}, G] \) as a subset of \( \times_{p \in \mathcal{P}} G \). The Hausdorff property of \( \text{Hom} [\mathcal{P}, G] \) is inherited from \( \times_{p \in \mathcal{P}} G \), and from the fact that \( \text{Hom} [\mathcal{P}, G] \) contains only morphisms follows easily that it is closed, therefore compact.

**Definition 7** The space \( \text{Hom} [\mathcal{P}, G] \), equipped with the weakest topology such that all maps (16) (or equivalently (19)) are continuous, is called the space of generalized connections. This compact Hausdorff space will hereafter be denoted by \( \bar{A} \).

### 2.5 Algebra of functions and inductive structure

The projective characterization of the compact space \( \bar{A} \) has as a counterpart the inductive characterization of the corresponding \( C^* \)-algebra of continuous complex functions \( C(\bar{A}) \).

Let us consider the family of \( C^* \)-algebras \( \{ C(\mathcal{A}_L) \}_{L \in L} \), where \( C(\mathcal{A}_L) \) is the algebra of continuous complex functions on \( \mathcal{A}_L \). The pull-back \( p^*_L, L' \) of the projections \( p_L, L' \) (10) define injective \( C^* \)-morphisms

\[
p^*_L : C(\mathcal{A}_L) \to C(\mathcal{A}_L'), \quad L' \geq L, \tag{20}
\]

that satisfy the consistency conditions, following from (9),

\[
p^*_{L,L''} = p^*_{L,L'} \circ p^*_{L', L''}, \quad \forall L'' \geq L' \geq L. \tag{21}
\]

Such a structure is called an inductive family. Turning to \( \bar{A} \), the pull-back \( p^*_L \) of the projections \( p_L \) (16) define injective \( C^* \)-morphisms

\[
p^*_L : C(\mathcal{A}_L) \to C(\bar{A}) \tag{22}
\]

satisfying consistency conditions following from (17):

\[
p^*_L = p^*_L \circ p^*_{L', L''}, \quad \forall L' \geq L. \tag{23}
\]

Injectivity of \( p^*_L, L' \) and \( p^*_L \) follows from surjectivity of \( p_L, L' \) and \( p_L \), respectively. Likewise, we see that \( p^*_L \) and \( p^*_{L, L'} \) are isometries, i.e.

\[
\sup_{\bar{A}} |p^*_L f| = \sup_{\mathcal{A}_L} |p^*_L f| = \sup_{\mathcal{A}_L} |f|, \quad \forall f \in C(\mathcal{A}_L). \tag{24}
\]

Let us consider the set \( \bigcup_{L \in L} p^*_L C(\mathcal{A}_L) \) of all continuous functions in \( \bar{A} \) that are obtained by pull-back. It is obvious that this set is closed under complex conjugation. It also follows easily from (23) (and the fact that \( L \) is directed) that the above set is an algebra under pointwise multiplication. It is therefore a \( \star \)-subalgebra of \( C(\bar{A}) \).
Definition 8 The $\star$-algebra $\text{Cyl}(\bar{A}) := \bigcup_{L \in L} p^*_L C(A_L)$ is called the algebra of cylindrical functions.

Recalling the identification (7) of every $A_L$ with some $G^n$, it becomes clear that every cylindrical function, i.e. every element of Cyl($\bar{A}$), can be written in the form

$$\bar{A} \ni \bar{A} \mapsto f(\bar{A}) = F(\bar{A}(e_1), \ldots, \bar{A}(e_n)),$$

where $\{e_1, \ldots, e_n\}$ is a set of independent edges and $F : G^n \to \mathbb{C}$ is a continuous function. Notice that we can equally replace independent edges by arbitrary paths $\{p_1, \ldots, p_n\}$ in (25), as paths can always be decomposed using independent edges, and therefore $F(\bar{A}(p_1), \ldots, \bar{A}(p_n))$ can be written as $F'(\bar{A}(e_1), \ldots, \bar{A}(e_n))$, where $F'$ is again continuous.

Proposition 2 The algebra Cyl($\bar{A}$) of cylindrical functions on $\bar{A}$ is dense on the algebra $C(\bar{A})$ of continuous complex functions.

This result follows from the Stone-Weierstrass theorem, since Cyl($\bar{A}$) is a $\star$-algebra, contains the identity function and clearly separates points in $\bar{A}$, as the functions $\bar{A} \ni \bar{A} \mapsto \bar{A}(e) \in G$ separate points, when all edges $e$ are taken into account.

This latter result, together with (20–23), establishes that the $C^*$-algebra $C(\bar{A})$ is (isomorphic to) the so-called inductive limit of the family of $C^*$-algebras $\{C(A_L)\}_{L \in L}$ (see e.g. [37]).

Let us mention that, besides Cyl($\bar{A}$), a whole (decreasing) sequence of $\star$-algebras Cyl$^k(\bar{A})$, $k = \{1, 2, \ldots\} \cup \{\infty\}$, can be defined, exactly as Cyl($\bar{A}$), but considering only $C^k(A_L)$ functions, i.e. $C^k(G^n)$ functions $F$ in (25). Proposition 2 still holds for all these subalgebras, as already $C^\infty$ functions separate points in $G$. Differential calculus is naturally introduced in $\bar{A}$ by using derivations of these algebras of differentiable functions. In particular, vector fields in $\bar{A}$ can be defined by certain consistent families of vector fields in the finite dimensional spaces $A_L$ [17].

2.6 Generalized gauge transformations and diffeomorphisms

Two distinct groups act naturally and continuously on $\bar{A}$. One of these is the group of natural transformations (see section 2.1) of the set of functors Hom[$\mathcal{P}, G] \equiv \bar{A}$. This group is well understood, and is commonly accepted as the natural generalization of the group $\mathcal{G} = C^\infty(\Sigma, G)$ of smooth local gauge transformations to the present quantum context. The second group of interest is the group Aut($\mathcal{P}$) of automorphisms of the groupoid $\mathcal{P}$. It contains
the group $\text{Diff}^\omega(\Sigma)$ of analytic diffeomorphisms as a subgroup. Although extensions of $\text{Diff}^\omega(\Sigma)$ are welcome, the group $\text{Aut}(\mathcal{P})$ has not been studied yet.

We begin by discussing the group of natural transformations of $\tilde{\mathcal{A}}$. In this case, natural transformations form the group, hereafter denoted by $\tilde{\mathcal{G}}$, of all maps $g : \Sigma \to G$, under pointwise multiplication. The action of $\tilde{\mathcal{G}}$ on $\tilde{\mathcal{A}}$ can be written as $\tilde{\mathcal{A}} \times \tilde{\mathcal{G}} \ni (\tilde{\mathcal{A}}, g) \mapsto \tilde{\mathcal{A}}_g$ such that

$$\tilde{\mathcal{A}}_g(p) = g(r(p))\tilde{\mathcal{A}}(p)g(s(p))^{-1}. \quad (26)$$

This action is readily seen to be continuous. In fact, since the topology on $\tilde{\mathcal{A}}$ is the weakest such that all maps $\pi_p$ (19) are continuous, one can conclude that a map $\varphi : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ is continuous if and only if the maps $\pi_p \circ \varphi$ are continuous $\forall p$, which is obviously the case for elements of $\tilde{\mathcal{G}}$.

Expression (26) is a generalization of the action of smooth gauge transformations on the set of parallel transport functions $h_p(A)$ for smooth connections. It is therefore natural to accept $\tilde{\mathcal{G}}$ as the generalized group of gauge transformations on $\tilde{\mathcal{A}}$. This extension of the gauge group is in fact required: since $\tilde{\mathcal{A}}$ now contains arbitrary, e.g. non continuous, morphisms from $\mathcal{P}$ to $G$, to mod out only by smooth gauge transformations would leave spurious degrees of freedom untouched. This is seen most explicitly from the fact that $\tilde{\mathcal{A}}$ is actually homeomorphic to $\tilde{\mathcal{A}}/\tilde{\mathcal{G}} \times \tilde{\mathcal{G}}$, where the gauge invariant space $\tilde{\mathcal{A}}/\tilde{\mathcal{G}}$ is the original space of generalized connections modulo gauge transformations $[16, 17, 23, 25, 26, 34]$, and $\tilde{\mathcal{G}}$ is equipped with the Tychonov topology. A homeomorphism relating $\tilde{\mathcal{A}}/\tilde{\mathcal{G}}$ and the quotient space $\tilde{\mathcal{A}}/\tilde{\mathcal{G}}$ then follows, establishing the equivalence between the current approach and the original manifestly gauge invariant one.

Let us now turn to the group $\text{Aut}(\mathcal{P})$ of automorphisms of the groupoid $\mathcal{P}$. By definition, an automorphism of $\mathcal{P}$ is an invertible functor from $\mathcal{P}$ to itself. An element $F \in \text{Aut}(\mathcal{P})$ is therefore described by a bijection of $\Sigma$ and a composition preserving bijection on the set of paths, such that $F(1_x) = 1_{F(x)}$, $\forall x \in \Sigma$. The action of $\text{Aut}(\mathcal{P})$ on $\tilde{\mathcal{A}}$ is given by

$$\tilde{\mathcal{A}} \mapsto F\tilde{\mathcal{A}} : F\tilde{\mathcal{A}}(p) = \tilde{\mathcal{A}}(F^{-1}p), \forall p \in \mathcal{P}, F \in \text{Aut}(\mathcal{P}). \quad (27)$$

The continuity of this action is clear, as $\pi_p \circ F = \pi_{F^{-1}p}$. The group $\text{Aut}(\mathcal{P})$ contains as a subgroup the natural representation of the group $\text{Diff}^\omega(\Sigma)$ of analytic diffeomorphisms of $\Sigma$, whose action on curves factors through the equivalence relation that defines $\mathcal{P}$. 
The group $\text{Aut}(\mathcal{P})$ therefore emerges, in the current quantum context, as the largest natural extension of $\text{Diff}^\omega(\Sigma)$. Extensions of $\text{Diff}^\omega(\Sigma)$ are welcome. In fact, the very formalism seems to require some sort of extension of $\text{Diff}^\omega(\Sigma)$, in the spirit of the extension $\tilde{\mathcal{G}}$ of $\mathcal{G}$. Going a little outside the scope of the present work, let us point out that the diffeomorphism invariant Hilbert space [18] one obtains from the kinematical Hilbert space $\mathcal{H}_0$ (see sections 1 and 3.3) is non-separable, when only $\text{Diff}^\omega(\Sigma)$ is taken into account. (The invariant Hilbert space is, essentially, obtained by considering $\text{Diff}^\omega(\Sigma)$-orbits in $\mathcal{H}_0$, which are naturally realized as linear functionals over $\text{Cyl}^\infty(\mathcal{A})$.) Since a separable Hilbert space, although not absolutely required, is expected at this level, one is led to suspect that the group $\text{Diff}^\omega(\Sigma)$ is too small.

One possible way out of this situation is to accept certain non-smooth transformations of $\Sigma$ as gauge [38, 32, 39]. For instance, it was shown in [38] that the inclusion of piecewise analytic transformations is sufficient to achieve separability. While these transformations are not fully motivated from the classical perspective, it appears that the quantum enlargement $\tilde{\mathcal{A}}$ of $\mathcal{A}$ introduces additional spurious degrees of freedom that are no longer gauged away by the action of classical smooth transformations. Furthermore, the replacement of smooth transformations by a more general, perhaps combinatorial, group is not unlikely to occur in the final theory of quantum gravity, as smoothness itself is expected to dissolve at the Planck scale, thus being a meaningful concept at the semi-classical regime only (see e.g. [7, 5] and references therein for arguments along these lines).

The suggested requirement for a quantum enlargement of not only $\text{Diff}^\omega(\Sigma)$ but of $\text{Diff}^\infty(\Sigma)$ itself also removes some of the motivation for working with smooth curves and smooth diffeomorphisms, giving support to the idea that the differentiability class one starts with might be irrelevant [38, 7].

If these ideas turn out fruitful, then $\text{Aut}(\mathcal{P})$ appears, in the current context, as the largest natural group where to look for a quantum version of the diffeomorphism group. A natural candidate is e.g. the subgroup of $\text{Aut}(\mathcal{P})$ of those elements that are induced by homeomorphisms of $\Sigma$.

It must be strongly stressed, however, that there are other options to deal with the non-separability issue [7, 32, 40], and that possible extensions of the diffeomorphism group must be analyzed in depth and treated with great care, as they potentially have a large impact on the quantum theory.
3 Representations of the holonomy algebra

3.1 $\tilde{A}$ as the spectrum of the holonomy algebra

We have thus far shown that $\tilde{A}$ is a compact Hausdorff extension of the classical configuration space $A$, but its exact relation with $A$ and how it allows a quantization of a particular algebra of classical configuration functions was not yet established. We will do that next, by showing that $\text{Cyl}(\tilde{A})$ is naturally isomorphic, as a normed $\star$-algebra, to the classical configuration algebra $\text{Cyl}(A)$ introduced in section 1. (This ultimately justifies the slight misuse of language in calling both $\text{Cyl}(A)$ and $\text{Cyl}(\tilde{A})$ the algebra of cylindrical functions. Moreover, it is sometimes useful, and common practice, to actually identify $\text{Cyl}(A)$ with $\text{Cyl}(\tilde{A})$, and the Ashtekar-Isham holonomy algebra with $C(\tilde{A})$.)

Let us define $\text{Cyl}(A)$ more precisely.

**Definition 9** The algebra $\text{Cyl}(A)$ is the $\star$-algebra of all functions $f : A \to \mathbb{C}$ of the form

$$A \ni A \mapsto f(A) = F(h_{e_1}(A), \ldots, h_{e_n}(A)), \quad (28)$$

where $\{e_1, \ldots, e_n\}$ is a set of independent edges and $F \in C(G^n)$. $\text{Cyl}(A)$ is a normed $\star$-algebra with respect to the supremum norm.

It is clear that $\text{Cyl}(A)$ separates points in $A$, again by the crucial fact, mentioned in section 2.3, that holonomies separate points. Alternatively, notice that $\text{Cyl}(A)$ can be described as the restriction of $\text{Cyl}(\tilde{A})$ to the faithful image of $A \subset \tilde{A}$. The algebra $\text{Cyl}(A)$ is therefore a viable set of classical configuration functions on which to base the quantization process. (This is, of course, supplemented with the equally complete set of momentum variables $E_{S,f}$, see section 1, so that coordinates in phase space can be defined.)

**Definition 10** The holonomy algebra $\overline{\text{Cyl}}(A)$ is the $C^*$-completion of the normed $\star$-algebra $\text{Cyl}(A)$.

Let us see that $\overline{\text{Cyl}}(A)$ is isomorphic to $C(\tilde{A})$. This follows from denseness of $\text{Cyl}(\tilde{A})$ (proposition 2) and the natural identification between $\text{Cyl}(\tilde{A})$ and $\text{Cyl}(A)$. The 1-1 correspondence between $\text{Cyl}(\tilde{A})$ and $\text{Cyl}(A)$ is obvious. The isomorphism of normed algebras is ensured by the following non-trivial fact [16]:
Lemma 1. The maps

\[ A \ni A \mapsto (h(e_1, A), \ldots, h(e_n, A)) \in G^n \]

are surjective, for every set \( \{e_1, \ldots, e_n\} \) of independent edges.

By lemma 1, the supremum of \( |F(\bar{A}(e_1), \ldots, \bar{A}(e_n))| \) is already attained in \( A \subset \bar{A} \), and therefore the norm of \( F(\bar{A}(e_1), \ldots, \bar{A}(e_n)) \in \text{Cyl}(\bar{A}) \) equals the norm of \( F(h(e_1, A), \ldots, h(e_n, A)) \in \text{Cyl}(A) \).

Proposition 3. The \( C^\star \)-algebra \( C(\bar{A}) \) is isomorphic to \( \overline{\text{Cyl}(A)} \).

In other words, \( \bar{A} \) is the compact Hausdorff space, whose existence and uniqueness (up to homeomorphism) is guaranteed by the Gelfand-Naimark theorem, on which the unital commutative \( C^\star \)-algebra \( \overline{\text{Cyl}(A)} \) is realized as an algebra of continuous functions. In the \( C^\star \)-algebraic language, \( \bar{A} \) is called the spectrum of \( \text{Cyl}(A) \). It follows from general topological arguments that \( \bar{A} \) is actually a compactification of \( A \), i.e. the image of \( A \subset \bar{A} \) is dense.

A quantization of the configuration algebra \( \text{Cyl}(A) \) is naturally defined to be a representation of the holonomy algebra. This is now seen to be the same as a representation of \( C(\bar{A}) \). The framework of representation theory of commutative unital \( C^\star \)-algebras therefore comes into play, with measures on \( \bar{A} \) (or, equivalently, states of the algebra \( C(\bar{A}) \)) playing a crucial role.

3.2 General structure of measures on \( \bar{A} \)

Since \( \bar{A} \) is a compact Hausdorff space, regular Borel measures are known to exist. In fact, normalized measures on \( \bar{A} \) are in 1-1 correspondence with states of the \( C^\star \)-algebra \( C(\bar{A}) \). (Recall that by a state it is meant a linear functional \( \omega : C(\bar{A}) \to \mathbb{C} \) that is positive, i.e. \( \omega(ff^*) \geq 0 \), \( \forall f \in C(\bar{A}) \), and normalized, i.e. \( \|\omega\| = \omega(1) = 1 \). Such linear functionals are necessarily continuous.) This follows from the Riez-Markov theorem:

Theorem 1. Let \( X \) be a compact Hausdorff space. For any state \( \omega \) of the \( C^\star \)-algebra \( C(X) \) there is a unique regular Borel probability measure \( \mu \) on \( X \) such that

\[ \omega(f) = \int_X f d\mu, \ \forall f \in C(X). \]  

(29)

Every (regular Borel probability) measure \( \mu \) on \( \bar{A} \) produces a cyclic representation \( \pi \) of \( C(\bar{A}) \), by multiplication operators on \( L^2(\bar{A}, \mu) \):

\[ (\pi(f)\psi)(\bar{A}) = f(\bar{A})\psi(\bar{A}), \ \psi \in L^2(\bar{A}, \mu), \ f \in C(\bar{A}), \]  

(30)
with cyclic vector $\Omega = 1$ (i.e. $\{\pi(f)\Omega, \ f \in C(\bar{A})\}$ is dense). Moreover, we know from general $C^*$-algebra results (see e.g. [41]) that every cyclic representation is (unitarily equivalent to) a representation of the above type (30), and that general, non-cyclic, representations are obtained as direct sums of such cyclic representations.

Measure theory on $\bar{A}$ is most conveniently addressed by combining the Riesz-Markov theorem with the projective-inductive structure.

**Definition 11** A family of measures $\{\mu_L\}_{L \in \mathcal{L}}$, where $\mu_L$ is a regular Borel probability measure on $A_L$, is said to be consistent if, $\forall L, L'$ such that $L' \geq L$, the measure $\mu_L$ coincides with the push-forward measure $(p_{L,L'})^*\mu_{L'}$, i.e. if

$$\int_{A_L} p_{L,L'}^*f \, d\mu_{L'} = \int_{A_L} f \, d\mu_L, \ \forall f \in C(A_L). \quad (31)$$

It is clear that a regular Borel probability measure $\mu$ on $\bar{A}$ determines a consistent family $\{\mu_L\}$ by

$$\int_{\bar{A}} f \, d\mu_L = \int_{\bar{A}} p_{L}^*f \, d\mu, \ \forall f \in C(A_L). \quad (32)$$

It turns out that the converse is also true [26]. To see this, let us consider a given consistent family $\{\mu_L\}_{L \in \mathcal{L}}$. To simplify formulae, we work with the corresponding family of states $\{\omega_L\}_{L \in \mathcal{L}}$, where $\omega_L(f) = \int_{A_L} f \, d\mu_L$. From $\{\omega_L\}$ one can define a linear functional $\omega : \text{Cyl}(\bar{A}) \to \mathbb{C}$ as follows. Let $F \in \text{Cyl}(\bar{A})$. By definition, there exists $L \in \mathcal{L}$ and $f \in C(A_L)$ such that $F = p_{L}^*f$. The functional $\omega$ is defined by:

$$\omega(F) = \omega(p_{L}^*f) = \omega_L(f). \quad (33)$$

Let us check that this is a well defined functional. Let $F = p_{L'}^*f'$ be another way of writing the cylindrical function $F$. By considering $L'' \geq L, L'$, one arrives at $F = p_{L''}^*p_{L,L''}^*f$ and $F = p_{L''}^*p_{L',L''}^*f'$, by (23). Since $p_{L''}^*$ is injective, we then get $p_{L,L''}^*f = p_{L',L''}^*f'$. The consistency property (31) of the family now ensures that

$$\omega_{L'}(f') = \omega_{L''}(p_{L',L''}^*f') = \omega_{L''}(p_{L,L''}^*f) = \omega_L(f). \quad (34)$$

The linear functional $\omega$ is clearly continuous, as

$$|\omega(p_{L}^*f)| = |\omega_L(f)| \leq \sup_{A_L} |f| = \sup_{\bar{A}} |p_{L}^*f|, \quad (35)$$
by (24). Since \( \text{Cyl}(\bar{A}) \) is dense, \( \omega \) uniquely defines a continuous linear functional \( \omega : C(\bar{A}) \to \mathbb{C} \). Moreover, since \( \|\omega_L\| = \omega(1) = 1 \) (and using again \( \sup_{\bar{A}} |p^*_L f| = \sup_{A_L} |f| \)), we obtain \( \|\omega\| = \omega(1) = 1 \). Standard \( C^* \)-algebra arguments show that the above conditions are sufficient for \( \omega \) to qualify as a state of the algebra \( C(\bar{A}) \). Finally, using again the Riez-Markov theorem, we obtain a measure satisfying (32). Thus:

**Proposition 4** Regular Borel probability measures on \( \bar{A} \) are in 1-1 correspondence with consistent families of measures. Explicitly, a consistent family \( \{\mu_L\} \) uniquely determines a measure \( \mu \) on \( \bar{A} \) satisfying condition (32).

A measure on \( \bar{A} \) is therefore equivalent to a consistent family of measures on the finite dimensional spaces \( A_L \). We then reencounter a familiar situation in quantum field theory, with the important difference that the finite dimensional configuration spaces \( A_L \) are now compact.

The projective-inductive structure is reflected in every Hilbert space \( \mathcal{H} = L^2(\bar{A}, \mu) \). Given a measure \( \mu \) with associated family \( \{\mu_L\} \), let \( \mathcal{H}_L \) denote the Hilbert space \( L^2(A_L, \mu_L) \). It is clear that the pull-backs \( p^*_{L,L'} \) (20) define injective inner product preserving maps \( p^*_{L,L'} : \mathcal{H}_L \to \mathcal{H}_{L'} \), satisfying the consistency conditions (21). Likewise, the pull-backs \( p^*_L \) (22) define linear transformations \( p^*_L : \mathcal{H}_L \to \mathcal{H} \), which are unitary when considered as maps

\[
p^*_L : \mathcal{H}_L \to p^*_L \mathcal{H}_L
\]

onto their images, the closed subspaces \( p^*_L \mathcal{H}_L \). Consistency conditions (23) and the denseness of the subspace \( \bigcup_{L \in \mathcal{L}} p^*_L \mathcal{H}_L \supset \text{Cyl}(\bar{A}) \) characterize \( \mathcal{H} \) as the so-called inductive limit of the inductive family of Hilbert spaces \( \{\mathcal{H}_L\}_{L \in \mathcal{L}} \). This structure is useful e.g. in the construction of operators in \( \mathcal{H} \), starting from consistent families of operators in the spaces \( \mathcal{H}_L \), and finds application in the quantization of momentum, and more general, operators [17, 7].

### 3.3 The Ashtekar-Lewandowski representation

As discussed in the introduction, the Ashtekar-Lewandowski, or uniform, measure \( \mu_0 \) is a central object in the canonical loop quantum gravity programme. Although several other diffeomorphism invariant (and non-invariant) measures on \( \bar{A} \) were constructed [23, 24, 26, 17], \( \mu_0 \) is the only known measure that supports a quantization of the flux variables \( E_{S,f} \). The uniform measure was introduced by Ashtekar and Lewandowski [16] in the \( \mathcal{A}/\mathcal{G} \) gauge
invariant context, and later on the original construction was generalized by Baez [23] to produce the uniform measure in \( \bar{A} \). In this section we present the uniform measure and briefly discuss its most important properties.

The uniform measure \( \mu_0 \) is constructed from a consistent family of measures determined only by the Haar measure \( \mu_H \) on the group \( G \). It can be defined as follows. Let us consider the projective family \( \{A_L, \mu_L, L \} \) for \( L \in \mathcal{L} \). For \( L \in \mathcal{L} \), let \( (e_1, \ldots, e_n) \) be an ordered set of generators of the groupoid \( L \), and consider the associated homeomorphism \( \rho_{e_1, \ldots, e_n} : \mathcal{A}_L \to G^n \) (7). Let \( \mu_H^n \) be the Haar measure on \( G^n \) and let us denote by \( \mu_{0L} \) the measure in \( \mathcal{A}_L \) that is obtained by push-forward of \( \mu_H^n \) with respect to \( \rho_{e_1, \ldots, e_n}^{-1} \), i.e.,

\[
\mu_{0L} := (\rho_{e_1, \ldots, e_n}^{-1})^* \mu_H^n, \quad \forall f \in C(\mathcal{A}_L).
\]  

By construction, \( \mu_{0L} \) is a regular Borel probability measure on \( \mathcal{A}_L \). Since \( \mu_H^n \) is a product of \( n \) identical measures, it is obvious that \( \mu_{0L} \) is independent of the order of the generators \( \{e_1, \ldots, e_n\} \). Moreover, since the Haar measure \( \mu_H \) is invariant with respect to inversions \( g \mapsto g^{-1}, \quad g \in G \), it follows that \( \mu_{0L} \) is also independent of the choice of the set of generators of the groupoid \( L \).

The family of measures \( \{\mu_{0L}\} \) thus defined is consistent, also by the properties of the Haar measure. In fact, given \( L \) generated by \( \{e_1, \ldots, e_n\} \) and \( L' \) generated by \( \{e'_1, \ldots, e'_m\} \), with \( L' \geq L \), the consistency conditions (31) translate into

\[
\mu_H^n = (\pi_{n,m})_* \mu_H^m, \quad \text{or}
\]

\[
\int_{G^n} f(g_1, \ldots, g_n) d\mu_H^n = \int_{G^m} f(\pi_{n,m}(g_1, \ldots, g_m)) d\mu_H^m, \quad \forall f \in C(G^n),
\]

where \( \pi_{n,m} : G^n \to G^m \) is the projection (13). It is not difficult to verify that the conditions (39) are satisfied, taking into account the invariance properties of the Haar measure (see [16] for a complete proof).

The family \( \{\mu_{0L}\} \) defines, by proposition 4, a regular Borel probability measure on \( \bar{A} \). This is the uniform measure \( \mu_0 \).

The simplest integrable functions on \( \bar{A} \) are, of course, the cylindrical functions

\[
f(\bar{A}) = F(\bar{A}(e_1), \ldots, \bar{A}(e_n)),
\]
with \{e_1, \ldots, e_n\} a set of independent edges and \( F \in C(G^n) \). For such functions we have simply:

\[
\int_{\bar{A}} f(\bar{A})d\mu_0 = \int_{G^n} F(g_1, \ldots, g_n)d\mu^n_H. \tag{41}
\]

Let us denote by \( \mathcal{H}_0 := L^2(\bar{A}, \mu_0) \) the Hilbert space defined by \( \mu_0 \). Notice that \( \mathcal{H}_0 \) is a non-separable space. Notice also that an orthonormal basis of \( \mathcal{H}_0 \) is explicitly known. This is the important spin-network basis \([13, 33, 42]\). The Hilbert space \( \mathcal{H}_0 \) carries, of course, a cyclic representation of \( C(\bar{A}) \) by multiplication operators (30). We will refer to this representation simply as the \( \mathcal{H}_0 \) representation.

The most distinguished property of the \( \mathcal{H}_0 \) representation is that it supports a quantization of the loop quantum gravity kinematical algebra, introduced in section 1. Without going into any details (see e.g. [7]), this goes as follows. Instead of the full holonomy algebra, let us consider its subalgebra \( \text{Cyl}^\infty(\mathcal{A}) \), defined as \( \text{Cyl}^\infty(\bar{A}) \) (section 2.5), and obviously isomorphic to it. (This brings no loss of generality, as \( \text{Cyl}^\infty(\mathcal{A}) \) is dense. See moreover [30].)

It turns out that the flux variables \( E_{S,f} \) are naturally realized as derivations \( X_{S,f} \) of the algebra \( \text{Cyl}^\infty(\mathcal{A}) \), and that the Lie algebra – let us call it \( \text{ACZ} \), after Ashtekar, Corichi and Zapata – generated by \( \text{Cyl}^\infty(\mathcal{A}) \) and the set of derivations \( X_{S,f} \) is isomorphic to the Poisson algebra generated by the kinematical variables. Actually, the Lie algebra \( \text{ACZ} \) is the rigorous way to define the kinematical Poisson algebra, as Poisson brackets among kinematical variables are \textit{a priori} ill-defined, due to the particular smearing of connections and electric fields. It is only after proper regularization that one obtains a well defined Lie algebra, and this is the Ashtekar-Corichi-Zapata algebra \([14]\). Thus, a quantization of the kinematical algebra is defined to be a (Dirac) representation of the \( \text{ACZ} \) Lie algebra (meaning, of course, that we map real variables to self-adjoint operators and that a factor \( i \) is assumed in the commutators). It is clear that the assignments \( \text{Cyl}^\infty(\mathcal{A}) \ni f \mapsto f \) and \( X_{S,f} \mapsto iX_{S,f} \) formally satisfy the commutation relations, in any Hilbert space \( L^2(\bar{A}, \mu) \), where \( f \) is now seen as an element of \( \text{Cyl}^\infty(\bar{A}) \), and \( iX_{S,f} \) as linear operators densely defined on \( \text{Cyl}^\infty(\bar{A}) \). It turns out that for the \( \mu_0 \) measure, and only for that measure \([7, 30, 31]\), the linear operators \( iX_{S,f} \) are actually self-adjoint. Thus, the \( \mathcal{H}_0 \) representation extends to a quantization of the kinematical algebra \([18]\). This representation is irreducible \([27]\), and it is the only known irreducible representation of the loop quantum gravity kinematical algebra.

Let us see next that the measure \( \mu_0 \) is invariant under the induced action
on $\tilde{A}$ (27) of the group $\text{Diff}^\omega(\Sigma)$ of analytic diffeomorphisms of $\Sigma$. Notice that, by the Riez-Markov theorem and denseness of $\text{Cyl}^\infty(\tilde{A})$, it is sufficient to check invariance on cylindrical functions. This is easily confirmed as follows. It is clear that for every analytic diffeomorphism $\varphi \in \text{Diff}^\omega(\Sigma)$ and every set of independent edges $\{e_1, \ldots, e_n\}$, the diffeomorphic image $\{\varphi e_1, \ldots, \varphi e_n\}$ is again an independent set of edges. In other words, for every tame subgroupoid $L \in \mathcal{L}$, its diffeomorphic image $\varphi L$ is again a tame subgroupoid. Moreover, the spaces $A_L$ and $A_{\varphi L}$ are homeomorphic. Finally, both $A_L$ and $A_{\varphi L}$ are equipped with the same (Haar) measure, leading to invariance. Explicitly, for every cylindrical function $f$ (40) we obtain from (41):

$$\int_{\tilde{A}} F(\varphi^{-1} \tilde{A}(e_1), \ldots, \varphi^{-1} \tilde{A}(e_n))d\mu_0 = \int_{\tilde{A}} F(\tilde{A}(e_1), \ldots, \tilde{A}(e_n))d\mu_0 = \int_{\tilde{A}} F(\tilde{A}(e_1), \ldots, \tilde{A}(e_n))d\mu_0.$$ (42)

From invariance of the measure, a natural unitary representation $U_D$ of $\text{Diff}^\omega(\Sigma)$ in $H_0$ follows:

$$(U_D(\varphi)\psi)(\tilde{A}) = \psi(\varphi^{-1} \tilde{A}), \ \varphi \in \text{Diff}^\omega(\Sigma), \psi \in H_0.$$ (43)

We now turn to the action (26) of the (extended) gauge group $\hat{\mathcal{G}}$. Here we find an even simpler situation, as $\hat{\mathcal{G}}$ acts within each space $A_L$. Explicitly, for every $g \in \hat{\mathcal{G}}$ we have

$$\int_{\tilde{A}} F(\tilde{A}g(e_1), \ldots, \tilde{A}g(e_n))d\mu_0 = \int_{\tilde{A}} F(g_1 \tilde{A}(e_1)g_2, \ldots, g_n \tilde{A}(e_n))d\mu_0,$$ (44)

where $g_1 := g(r(e_i))$ and $g_2 := g((s(e_i))^{-1}$ are fixed elements of $G$, for each fixed set $\{e_1, \ldots, e_n\}$. From (41) and invariance properties of the Haar measure $\mu_H$ follows that $\mu_0$ is $\hat{\mathcal{G}}$-invariant. A representation of $\hat{\mathcal{G}}$ is therefore also obtained in $H_0$. In this case, the corresponding Gauss constraint is immediately solved by the gauge invariant subspace of $H_0$. This is a large closed subspace, that can be obtained by closure of the well understood subspace of gauge invariant cylindrical functions. Unfortunately, this is not the case for the diffeomorphism constraint, as the only $\text{Diff}^\omega(\Sigma)$-invariant elements of $H_0$ are the constant functions [43, 44]. As already mentioned, the solution space of the diffeomorphism constraint lies in the algebraic dual of the space $\text{Cyl}^\infty(\tilde{A})$ [18].
Acknowledgements

I thank Professor Gennadi Sardanashvily for the invitation to contribute to the special issue of *International Journal of Geometric Methods in Modern Physics* on "Advanced Geometric Techniques in Gauge Theory". I am most grateful to José Mourão for numerous discussions and suggestions. I also thank Thomas Thiemann for useful discussions. This work was supported in part by POCTI/33943/MAT/2000, CERN/P/FIS/43171/2001, POCTI/FNU/49529/2002 and POCTI/FP/FNU/50226/2003.

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