ON ISOMORPHICALLY POLYHEDRAL $\mathcal{L}_\infty$-SPACES

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ABSTRACT. We show that there exist $\mathcal{L}_\infty$-subspaces of separable isomorphically polyhedral Lindenstrauss spaces that cannot be renormed to be a Lindenstrauss space.

1. ISOMORPHICALLY POLYHEDRAL SPACES

A Banach space is said to be polyhedral if the closed unit ball of every finite dimensional subspace is the closed convex hull of a finite number of points. Polyhedrality is a geometrical notion: $c_0$ is polyhedral while $c$ is not. It is also an hereditary notion: every subspace of a polyhedral space is polyhedral. The isomorphic notion associated with polyhedrality is: A Banach space is said to be isomorphically polyhedral if it admits a polyhedral renorming. The simplest examples of isomorphically polyhedral spaces are the $C(\alpha)$ spaces for $\alpha$ an ordinal, and their subspaces. In [5] we surveyed what is known about polyhedral $\mathcal{L}_\infty$-spaces, which can be summarized as follows:

(1) There are polyhedral spaces which are not $\mathcal{L}_\infty$: indeed, any non $\mathcal{L}_\infty$ subspace of $c_0(\Gamma)$ — recall from [11] that subspaces of $c_0(\Gamma)$ are $\mathcal{L}_\infty$-spaces if and only if they are isomorphic to $c_0(\Gamma)$.
(2) There are Lindenstrauss spaces not polyhedral: $C[0,1]$.
(3) A result of Fonf [8] asserts that preduals of $\ell_1$ are isomorphically polyhedral.
(4) Fonf informed us [9] that the result fails for $\ell_1(\Gamma)$: Kunen’s compact $\mathcal{K}$ provides, under CH, a scattered, non metrizable, compact so that $C(\mathcal{K})$ space has the rare property that every uncountable set of elements contains one that belongs to the closure of the convex hull of the others. And this property was used by Jiménez and Moreno [13] to show that every equivalent renorming of $C(\mathcal{K})$ has only a countable number of weak*-strongly exposed points. Thus, no equivalent renorming can be polyhedral (see [10]). At the same time $C(\mathcal{K})^* = \ell_1(\Gamma)$ since $\mathcal{K}$ is scattered.
(5) The trees $T$ for which $C(T)$ is isomorphically polyhedral are characterized in [10]. Thus, there are scattered compact $K$ (not depending on CH as it occurs with Kunen’s compact) such that $C(K)$ is not isomorphically polyhedral.

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Fonf [9] asked [5, Section 4, problem 5] whether isomorphically polyhedral $L_\infty$-spaces are isomorphically Lindenstrauss. The purpose of this note is to show that the answer is no.

2. Preliminaries

A Banach space $X$ is said to be a $L_\infty,\lambda$-space if every finite dimensional subspace $F$ of $X$ is contained in another finite dimensional subspace of $X$ whose Banach-Mazur distance to the corresponding space $\ell_\infty^n$ is at most $\lambda$. The space $X$ is said to be an $L_\infty$-space if it is an $L_\infty,\lambda$-space for some $\lambda$. The basic theory and examples of $L_\infty$-spaces can be found in [18, Chapter 5]. A Banach space $X$ is said to be a Lindenstrauss space if it is an isometric predual of some space $L_1(\mu)$. Lindenstrauss spaces correspond to $L_\infty,1+$-spaces. A Lindenstrauss space is an $L_\infty,1$-space if and only if it is polyhedral (i.e., the unit ball of every finite dimensional subspace is a polytope) [18, p.199].

A Banach space $X$ is said to have Pełczyński’s property $(V)$ if each operator defined on $X$ is either weakly compact or an isomorphism on a subspace isomorphic to $c_0$. Pełczyński shows in [19] that $C(K)$-spaces enjoy property $(V)$, and Johnson and Zippin [14] that Lindenstrauss spaces also have $(V)$.

Let $\alpha : A \rightarrow Z$ and $\beta : B \rightarrow Z$ be operators acting between Banach spaces. the pull-back space $PB$ is defined as $PB = PB(\alpha, \beta) = \{(a, b) \in A \oplus B : \alpha(a) = \beta(b)\}$. It has the property of yielding a commutative diagram

$$
\begin{array}{ccc}
P B & \xrightarrow{\beta} & A \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
B & \xrightarrow{\beta} & Z
\end{array}
$$

in which the arrows after primes are the restriction of the projections onto the corresponding factor. Needless to say (1) is minimally commutative in the sense that if the operators $"\beta : C \rightarrow A$ and $"\alpha : C \rightarrow B$ satisfy $\alpha \circ "\beta = \beta \circ "\alpha$, then there is a unique operator $\gamma : C \rightarrow PB$ such that $"\beta = \beta \gamma$ and $"\beta = \beta \gamma$. Clearly, $\gamma(c) = ("\beta(c),"\alpha(c))$ and $\|\gamma\| \leq \max\{|"\beta|, |"\alpha|\}$. Quite clearly $\alpha$ is onto if $\alpha$ is. As a consequence of this, if one has an exact sequence

$$
\begin{array}{cccc}
0 & \xrightarrow{i} & Y & \xrightarrow{\pi} & X & \xrightarrow{\pi} & Z & \xrightarrow{\pi} & 0
\end{array}
$$

and an operator $u : A \rightarrow Z$ then one can form the pull-back diagram of the couple $(\pi, u)$:

$$
\begin{array}{ccc}
0 & \xrightarrow{i} & Y & \xrightarrow{\pi} & X & \xrightarrow{\pi} & Z & \xrightarrow{\pi} & 0 \\
& \uparrow{u} & & \uparrow{u} & & \uparrow{u} & & \uparrow{u} & & \uparrow{u} \\
PB & \xrightarrow{\pi} & A
\end{array}
$$
Recalling that $\pi$ is onto and taking $j(y) = (0, \psi(y))$, it is easily seen that the following diagram is commutative:

$$
\begin{array}{cccccc}
0 & \longrightarrow & Y & \overset{j}{\longrightarrow} & X & \overset{\pi}{\longrightarrow} & Z & \longrightarrow & 0 \\
\| & \| & \| & \| & \| & \| & \| & \| & \\
0 & \longrightarrow & Y & \overset{j}{\longrightarrow} & \text{PB} & \overset{\pi}{\longrightarrow} & A & \longrightarrow & 0 \\
\end{array}
$$

Thus, the lower sequence is exact, and we shall refer to it as the pull-back sequence.

The well-known (see e.g., [2]) splitting criterion is: the pull-back sequence splits if and only if $u$ lifts to $X$; i.e., there is an operator $U : A \rightarrow X$ such that $\pi U = u$.

3. An isomorphically polyhedral $L_\infty$-space that is not Lindenstrauss

**Theorem 1.** There is a separable isomorphically polyhedral $L_\infty$ space that is not isomorphically Lindenstrauss. Moreover, it is a subspace of an isomorphically polyhedral Lindenstrauss space.

**Proof.** We need to recall from [1] the existence of nontrivial exact sequences

$$
0 \longrightarrow C(\omega_0) \longrightarrow \Omega \overset{q}{\longrightarrow} c_0 \longrightarrow 0
$$

in which the quotient map $q$ is strictly singular. This fact makes $\Omega$ fail Pełczyński’s property (V). Since Lindenstrauss spaces share with $C(K)$-spaces Pełczyński’s property (V), the space $\Omega$ is not isomorphic to a Lindenstrauss space. Of course it is an $L_\infty$-space since this is a 3-space property. Thus, our purpose is to show that there is an $\Omega$ as above that is isomorphically polyhedral.

We recall from [1] Section 3 the parameter $\rho_N(c_0)$, defined as the least constant such that if $T : c_0 \rightarrow \ell_\infty(\omega_N)$ is a bounded linear operator such that $\text{dist}(Tx, C(\omega_N)) \leq \|x\|$ for all $x \in c_0$ then there is a linear map $L : c_0 \rightarrow C(\omega_N)$ with $\|T - L\| \leq \rho_N(c_0)$. Theorems 3.1 and Lemma 3.2 in [1] show that $\lim \rho_N(c_0) = +\infty$. Now we need a specific choice for each $N$: this is provided by [1] Prop. 4.6: there is a bounded operator $T_N : c_0 \rightarrow \ell_\infty(\omega_N)$ so that $\text{dist}(T_N x, C(\omega_N)) \leq \|x\|$ for all $x \in c_0$ but such that if $E \subset c_0$ is a subspace of $c_0$ almost isometric to $c_0$ then $\rho_N(c_0) \leq 2\|T_N - L\|$ for any linear map $L : c_0 \rightarrow C(\omega_N)$.

Let, for each $N$, a linear continuous operator $T_N : c_0 \rightarrow \ell_\infty(\omega_N)$ as above. We form the twisted sum space

$$
C(\omega_N) \oplus_{T_N} c_0 = \left( C(\omega_N) \times c_0, \| \cdot \|_{T_N} \right)
$$

endowed with the norm $\| (h, x) \|_{T_N} = \max \{ \|h - T_N x\|, \|x\| \}$. This yields an exact sequence

$$
0 \longrightarrow C(\omega_N) \overset{i_N}{\longrightarrow} C(\omega_N) \oplus_{T_N} c_0 \overset{q_N}{\longrightarrow} c_0 \longrightarrow 0
$$
with embedding $i_N(f) = (f,0)$ and quotient map $q_N(f,x) = x$. The identity map $id : C(\omega^N) \oplus_{T_N} c_0 \to C(\omega^N) \oplus_{\infty} c_0$ is an isomorphism since
\[ \|T_N\|^{-1}\|(f,x)\|_{T_N} \leq \|(f,x)\|_{\infty} \leq \|T_N\|\|(f,x)\|_{T_N} \]
and therefore the space $C(\omega^N) \oplus_{T_N} c_0$ is isomorphically polyhedral. We need now to use the main result in [7] asserting that in a separable isomorphically polyhedral space every norm can be approximated by a polyhedral norm. Let $\| \cdot \|_{P_N}$ be a polyhedral norm in $C(\omega^N) \oplus_{T_N} c_0$ that is 2-equivalent to $\| \cdot \|_{T_N}$.

The sequence (3) splits, but the norm of the projection goes to infinity with $N$: Indeed, if $P : C(\omega^N) \oplus_{T_N} c_0 \to C(\omega^N)$ is a linear continuous projection then $P$ has to have the form $P(f,x) = (f-Lx,0)$, where $L : c_0 \to C(\omega^N)$ is a certain linear map. Thus, if $x \in c_0$ is a norm one element, one gets $P(T_N x, x) = (T_N x - Lx,0)$ and thus $\|T_N x - Lx\| \leq \|P\|\|x\|$, hence $\|T_N - L\| \leq \|P\|$. The choice of $T_N$ forces $\lim_{N \to \infty} \inf \|P\| = +\infty$. Therefore, the $c_0$-sum
\[ 0 \longrightarrow c_0(C(\omega^N)) \longrightarrow c_0(C(\omega^N) \oplus_{P_N} c_0) \xrightarrow{(q_N)} c_0(c_0) \longrightarrow 0 \]
cannot split. The space $c_0(C(\omega^N) \oplus_{P_N} c_0)$ is isomorphically polyhedral as any $c_0$-sum of polyhedral spaces [12]. We now define a suitable operator $\Delta$ so that when making the pull-back diagram
\[ \begin{array}{cccccc}
0 & \longrightarrow & c_0(C(\omega^N)) & \longrightarrow & c_0(C(\omega^N) \oplus_{P_N} c_0) & \xrightarrow{(q_N)} & c_0(c_0) & \longrightarrow & 0 \\
\|\| & \|\| & \|\| & \|\| & \|\| & \|\| & \|\| & \|\| & \|\|
\end{array} \]
\[ \begin{array}{cccccc}
0 & \longrightarrow & c_0(C(\omega^N)) & \longrightarrow & \Omega & \xrightarrow{\delta} & \Omega & \xrightarrow{\Delta} & c_0 & \longrightarrow & 0 \\
0 & \longrightarrow & c_0(C(\omega^N)) & \longrightarrow & \Omega & \xrightarrow{q} & c_0 & \longrightarrow & 0 \\
\end{array} \]
the map $q$ is strictly singular. That prevents $\Omega$ from being Lindenstrauss under any equivalent renorming.

Pick as $\Delta$ the diagonal operator $c_0 \to c_0(c_0)$ induced by the scalar sequence $(\rho_N(c_0)^{-1/2}) \in c_0$; i.e.,
\[ \Delta(x) = (\rho_N(c_0)^{-1/2} x)_N. \]

Assume that $q$ is not strictly singular. Then, there is a subspace $E$ of $c_0$ and a linear bounded map $V : E \to \Omega$ so that $qV = \Delta|E$. By the $c_0$ saturation and the distortion properties of $c_0$, there is no loss of generality assuming that $E$ is an almost isometric copy of $c_0$. By the commutativity of the diagram $(q_N)\delta V = \Delta|E$, which in particular means that $q_N\delta V(e) = \rho_N(c_0)^{-1/2}e$ for all $e \in E$. This means that the map $\delta V$ has on $E$ the form $(L_N e, \rho_N(c_0)^{-1/2}e)_N$ where $L_N : E \to C(\omega^N)$ is a linear map; by continuity, there is a constant $M$ so that $\|(L_N e, \rho_N(c_0)^{-1/2}e)\| \leq M\|e\|$, which means
\[ \|L_N e - T_N \rho_N(c_0)^{-1/2}e\| \leq M\|e\|. \]
and thus
\[ \|\rho_N(c_0)^{1/2}L_N - T_N\| \leq M\rho_N(c_0)^{1/2}. \]
This contradicts the fact that \( E = c_0 \), the definition of \( \rho_N(c_0) \) and the choice of \( T_N \).

To conclude the proof, the definition of pull-back space implies that \( \Omega \) is actually a subspace of \( c_0(C(\omega^N) \oplus P_N c_0) \oplus \infty c_0 \), hence isomorphically polyhedral. \( \square \)

Since \( c_0(C(\omega^N)) \cong C(\omega^N) \), the space \( \Omega \) above yields a twisted sum
\[
0 \longrightarrow C(\omega^\omega) \longrightarrow \Omega \overset{q}{\longrightarrow} c_0 \longrightarrow 0
\]
in which \( q \) is strictly singular. The dual sequence
\[
0 \longrightarrow \ell_1 \longrightarrow \Omega^* \longrightarrow \ell_1 \longrightarrow 0
\]
necessarily splits and thus \( \Omega^* \) can be renormed to be \( \ell_1 \), although \( \Omega \) cannot be endowed with an equivalent norm \( \| \cdot \| \) so that \( (\Omega, \| \cdot \|)^* = \ell_1 \). Moreover, \( \Omega \) is actually a subspace of the isomorphically polyhedral Lindenstrauss space \( c_0(C(\omega^N) \oplus P_N c_0) \oplus c_0 \).

4. An isomorphically polyhedral \( L_\infty \) space that is not a Lindenstrauss-Pelczyński space

We show now that one can produce an \( L_\infty \)-variation of \( \Omega \) still farther from Lindenstrauss spaces. Lazar [15] and Lindenstrauss [16] showed that Lindenstrauss polyhedral spaces \( X \) enjoy the property that compact \( X \)-valued operator admit equal norm extensions. In [3], the authors introduce the Lindenstrauss-Pelczyński spaces (in short \( \mathcal{L}\mathcal{P} \)-spaces) as those Banach spaces \( E \) such that all operators from subspaces of \( c_0 \) into \( E \) can be extended to \( c_0 \). The spaces are so named because Lindenstrauss and Pelczyński first proved in [17] that \( C(K) \)-spaces have this property. Lindenstrauss spaces have also the property (see [17] [6]) as well as \( L_\infty \)-spaces not containing \( c_0 \) [3] and, of course, all their complemented subspaces. The construction of the space \( \Omega \) above has been modified in [4] to show that for every subspace \( H \subset c_0 \) there is an exact sequence
\[
0 \longrightarrow C(\omega^\omega) \longrightarrow \Omega_H \longrightarrow c_0 \longrightarrow 0
\]
in which the space \( \Omega_H \) is not a Lindenstrauss-Pelczyński space [17]; more precisely, there is an operator \( H \rightarrow \Omega_H \) that cannot be extended to the whole \( c_0 \).

**Proposition 1.** There is an isomorphically polyhedral \( L_\infty \)-space that is not an \( \mathcal{L}\mathcal{P} \)-space.

**Proof.** Consider the exact sequence \( 0 \rightarrow C(\omega^\omega) \rightarrow \Omega \rightarrow c_0 \rightarrow 0 \) with strictly singular quotient constructed above. Since every quotient of \( c_0 \) is isomorphic to a subspace of \( c_0 \), we can consider that there is an embedding \( u_H : c_0/H \rightarrow c_0 \). The pull-back sequence \( 0 \rightarrow C(\omega^\omega) \rightarrow P_H \overset{p}{\rightarrow} c_0/H \rightarrow 0 \) also has strictly singular quotient map. We form the commutative diagram
to show, exactly as in [17] that $\Omega_H$ is not an $\mathcal{LP}$-space since $j$ cannot be extended to $c_0$ through $i$. The space $\Omega_H$ has been obtained from a pull-back diagram

\[ 0 \to c_0(C(\omega^N)) \to \Omega \to c_0 \to 0 \]

and thus it is a subspace of $\Omega \oplus c_0$, hence isomorphically polyhedral. \qed

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