POPULATING STELLAR ORBITS INSIDE A ROTATING, GASEOUS BAR

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ABSTRACT

In an effort to understand better the formation and evolution of barred galaxies, we have examined the properties of equatorial orbits in the effective potential of one specific model of a rapidly rotating, steady state gas dynamical bar that has been constructed via a self-consistent hydrodynamical simulation. At a given value of the Jacobi constant, roughly half of all test particles (stars) that are injected into the equatorial plane of this potential follow quasi-ergodic orbits; most regular prograde orbits have an overall “bow tie” shape; and some trace out trajectories that resemble the \( x_4 \) family of regular, retrograde orbits. The bow tie orbits appear to be related to the \( 4/1 \) orbit family discussed by Contopoulos in 1988, but particles moving along a bow tie orbit pass very close to the center of the bar twice each orbit. Unlike the barlike configurations that previously have been constructed using dissipationless, \( N \)-body simulation techniques, the effective potential of our gas dynamical bar is very shallow and generally does not support the \( x_4 \) family of orbits. If primordial galaxies evolve to a rapidly rotating barlike configuration before a significant amount of star formation has taken place and then stars form from the gas that makes up the bar, the initial stellar distribution function should consist of orbits that are (1) supported by the gaseous barlike potential and (2) restricted to have initial conditions dictated by the gasdynamics of the bar. With this “restriction hypothesis” in mind, we propose that stellar dynamical systems that form from gaseous bars will have characteristics that differ significantly from systems that form from a bisymmetric instability in an initially axisymmetric stellar system. Since bow tie orbits are preferred over \( x_4 \) orbits, for example, such systems should have a more boxy or peanut shape when seen face-on; there will be a mechanism for funneling material more directly into the center of the galaxy; and, near the galaxy center, stars may appear to move along retrograde trajectories.

Subject headings: celestial mechanics, stellar dynamics — galaxies: kinematics and dynamics

1. INTRODUCTION

1.1. Background

Bars are present in at least half of all disk galaxies, including our own Milky Way (Gerhard 1999; Binney & Tremaine 1987, § 6.5). Accordingly, there have been many observational and theoretical studies of bar attributes over the last few decades. Rather than attempt to summarize this extensive literature here, we refer the reader to publications from two recent conferences on the topic of barred galaxies (Sandqvist & Lindblad 1996; Buta, Crocker, & Elmegreen 1996) and to the reviews by Athanassoula (1984), Binney & Tremaine (1987), Contopoulos & Grosbol (1989), and especially Sellwood & Wilkinson (1993). Following the lead of Contopoulos et al. (1989), we find it useful to group previously published theoretical studies of the structure and stability of barred galaxies into the following four broad categories:

1. Orbit calculations for various two-dimensional analytical bar potentials.
2. \( N \)-body simulations of stellar bar formation.
3. \( N \)-body simulations of interstellar cloud collisions in bars.
4. Studies of gas dynamical flows in externally prescribed “stellar” bar potentials.

Our present work does not fit naturally into any of these categories because it involves a detailed analysis of the properties of a self-gravitating gas dynamical, rather than stellar dynamical, barlike configuration. However, there are strong parallels between our study and the analysis presented by Sparke & Sellwood (1987, hereafter SS87) of a purely stellar dynamical bar, so we will begin by reminding the reader of the key features of this earlier work.

In an effort to ascertain not only what kinds of orbits are allowed in the potential well of rapidly rotating, barred galaxies but also which orbit families are likely to be populated by stars in such galaxies, SS87 combined the tools and analysis techniques that previously had been associated with the separate categories of investigation listed as items 1 and 2 above. First, they used a two-dimensional \( N \)-body code to construct a steady state model of a rapidly rotating, infinitesimally thin bar (initially including a small axisymmetric bulge component, a surrounding axisymmetric disk, and a “hot” component). Then, using a standard “shooting technique” along with surfaces of section and a characteristic diagram, they mapped out the properties of available (stable and unstable) orbits in the effective potential well of this numerically generated, two-dimensional steady state bar. Finally, they identified which of the numerous possible orbit families were actually being populated by particles in their \( N \)-body simulation. Among other things, the SS87 work provided strong evidence in support of a restriction hypothesis\(^1\) first alluded to by Teuben & Sanders (1985), namely, that a real barred galaxy contains stars that largely follow a favorable subset of all possible orbit families. More specifically, SS87 found that the stellar distribution function that was associated with their steady state bar, \( DF_{SS} \), was dominated by particles whose trajectories were associated

\( ^1 \) We have chosen to use the phrase “restriction hypothesis” here in an effort to encapsulate the essence of the first sentence of § 4.3 in SS87 as well as the similar implication that appears in Teuben & Sanders (1985). This phrase does not appear in either SS87 or Teuben & Sanders (1985).
with and generally trapped around the \( x_1 \) family of orbits as defined by Contopoulos & Papayannopoulos (1980).

This restriction hypothesis has received additional support from Pfenniger & Friedli (1991), who extended the SS87 analysis to fully three-dimensional-\( N \)-body models of steady state bars, as well as from the study of Berentzen et al. (1998), in which a small amount of gas (8% of the total galaxy mass) was included in a self-consistent fashion along with a three-dimensional \( N \)-body simulation. It is not obvious how a nonlinear dynamical simulation (and by inference a real galaxy) that starts from a nearly axisymmetric distribution function, \( DF_{\text{axisym}} \), is able to select preferentially this restricted set of orbits (primarily related to the \( x_1 \) family) while evolving to a barlike configuration, but the outcome makes sense. Indeed, other generally available orbits, such as orbits associated with the \( x_2 \) or \( x_4 \) families (Contopoulos & Papayannopoulos 1980), have trajectories that generally do not support the overall shape of the bar.

Here we perform an analysis that is very similar to the one presented by SS87, but for a two-dimensional equatorial slice of a three-dimensional steady state bar that has been constructed from a self-gravitating, homentropic, compressible gas cloud using a finite-difference hydrodynamic technique (Cazes & Tohline 2000). We have been motivated to conduct this analysis, in part, because the steady state gasdynamical bars described by Cazes & Tohline (2000) are the first detailed models of their kind and we were curious to know to what degree their global attributes resemble the properties of their \( N \)-body counterparts. By using a shooting technique to inject test particles into the potential well of one of the Cazes & Tohline (2000) bars and then following the motion of the particles through many orbital periods, we have been able to produce surface of section diagrams to facilitate such a comparison.

1.2. Relevance to the Formation of Galaxies

We also have been motivated to conduct this analysis in the context of studies of the formation of galaxies and the earliest generation of stars. Models of barred galaxy formation historically have assumed that (1) baryonic material (containing cold gas but no stars) first settles into a flat, rotationally supported axisymmetric disk; then (2) a system of stars forms from this disk with a distribution function, such as \( DF_{\text{axisym}} \), that is prescribed by the location and motion of the gas from which it formed, i.e., roughly circular orbits confined to a disk with little vertical thickness; then (3) after most of the gas has been converted into stars so that the system of stars becomes sufficiently self-gravitating, the stellar dynamical system deforms into a non-axisymmetric barlike configuration because it is too cold to remain axisymmetric. The models presented, for example, in SS87, Pfenniger & Friedli (1991), and Berentzen et al. (1998) begin at step 2 in this chronological sequence of events and then follow the subsequent evolution using entirely (or at least predominantly, in the case of Berentzen et al. 1998) \( N \)-body simulation techniques.

But another scenario bears consideration. Just as cold, axisymmetric stellar dynamical configurations are known to be dynamically unstable toward a bisymmetric instability if they are sufficiently self-gravitating, the same is true for fluid configurations. Based on the classical analytical studies by Riemann and others of rotating incompressible fluids, this has been known for over 100 years (see Chandrasekhar 1969 for a thorough overview). More recently, three-dimensional hydrodynamical techniques have been used to demonstrate that this same type of bisymmetric instability arises as well in differentially rotating, compressible fluids. The eigenfunction that naturally develops as a result of the instability has a barlike structure from which a loosely wound, two-armed spiral emerges (Tohline, Durisen, & McCollough 1985; Williams & Tohline 1987; Pickett, Durisen, & Davis 1996; Toman et al. 1998). That is to say, it exhibits a structure very similar to the one seen in numerous simulations of purely stellar dynamical systems that start from qualitatively similar initial axisymmetric states (Zang & Hohl 1978; Miller & Smith 1979; Sellwood 1980; see the above reviews for additional references). Also like their \( N \)-body counterparts, as the bisymmetric distortion reaches nonlinear amplitude in a gasdynamical system, a significant amount of angular momentum redistribution can take place in a relatively short amount of time, and some mass (the relatively high specific angular momentum material) is shed into a roughly axisymmetric equatorial disk (Durisen et al. 1986; Williams & Tohline 1988; Durisen, Yang, & Grabhorn 1989; Imamura, Durisen, & Pickett 2000). Then a central object containing a majority of the mass usually settles down into a barlike structure that exhibits significant internal streaming motions but is spinning with a coherent pattern speed about its shortest axis. For two specific models of this type, Cazes & Tohline (2000) have carefully mapped the internal structural and flow properties of this central barlike object and have demonstrated that it is dynamically robust; as viewed from a frame rotating with the system pattern speed, the bar appears to be steady state and dynamically stable.

With this in mind, we suggest that a reasonable alternative to the standard model of barred galaxy formation is one in which the initially axisymmetric, cold gaseous disk evolves to a barlike structure before significant star formation has taken place, then the stars form from the gas that makes up the gaseous bar. Via this scenario, the system of stars that emerges from the gas will have an initial distribution function, \( DF_{\text{bar}} \), that is quite different from the ones (such as \( DF_{\text{axisym}} \)) that have been adopted in various \( N \)-body simulations to model the formation of barred galaxies. By examining the types of particle orbits that are supported by the effective potential of one of the steady state bars described by Cazes & Tohline (2000), we will be in a position to determine better what would be the initial distribution function for a system of stars that forms directly from a gaseous bar. In making this determination we will introduce a new restriction hypothesis that is distinctively different from the one discussed above in the context of the Teuben & Sanders (1985) and SS87 papers. Specifically, after determining via a shooting technique what orbits are allowed in the potential well of the gaseous bar, we propose that the realistic stellar distribution function will not contain all such orbits but, rather, must be restricted to the subset of those orbits that are consistent with the positions and the velocities of the gas from which the stars would form. We will illustrate to what degree this restriction hypothesis places interesting constraints on the resulting stellar distribution function.

Because it appears that the baryonic component of many, if not most, barred galaxies is dominated by stars at the present epoch, at some point in time most of the gas in the bar must be converted into stars before this alternative model of barred galaxy formation can be considered plaus-
ible. It would therefore be interesting to know whether or not a stellar dynamical system with a distribution function similar to $DF_{\text{bar}}$ can, by itself, produce a self-consistent bar that mimics the original gaseous bar configuration. That is, can the transformation from a barred galaxy that is predominately gaseous into one that is predominantly stellar be a relatively smooth one that essentially preserves the system’s overall geometric shape (and its basic morphological features), or must the system relax to an entirely different structure after the majority of its mass has been converted into stars? Directly related to this question is the recent study (Lütticke, Dettmar, & Pohlen 2000) that finds that the fraction of box- and peanut-shaped bulges in edge-on galaxies is roughly the same as the fraction of strongly barred face-on galaxies (see also Bureau & Freeman 1999). The implication is that the bar and peanut bulges are barred galaxies seen in profile. In the context of the transformation between gaseous and stellar systems, this suggests that the relaxation of the emerging stellar system should not radically alter the morphology of the initial gaseous bar (if this alternative bar formation scheme is to remain plausible). An answer to this question would also be useful in the context of efforts to understand the evolutionary connection between galaxies at the present epoch and galaxies at high redshift that are now being directly imaged (Lilly et al. 1998; Driver et al. 1998; Simard et al. 1999 and references therein) and would be particularly relevant to observations that are designed to ascertain how barred galaxies have evolved (Abraham et al. 1999; Bunker 1999; Eskridge et al. 2000). Furthermore, it would be useful to know whether or not the final stellar dynamical configuration that is created via this alternative galaxy evolution scenario can in any way be differentiated from the stellar bars that are produced via $N$-body techniques directly from $DF_{\text{axisym}}$. It will be necessary to have an answer to this question before we will be able to distinguish critically between the standard scenario of barred galaxy formation and the alternative one being discussed here. We will not attempt to address these follow-up questions in the present work, although we expect to do so in the future.

In what follows (§ 2) we briefly review how Cazes & Tohline (2000) constructed two steady state gasdynamical bars, summarize the structural and internal flow properties of the system (their model B) that we have selected to analyze in detail, and then briefly describe the shooting technique that we have used to probe the properties of test particle orbits in the effective potential well of this rapidly rotating, gasdynamical bar. In § 3 we use surface of section diagrams and sample particle orbit trajectories to illustrate the types of stellar orbits that can, in principle, be supported inside this bar. (We do not consider orbits outside the bar.) In an effort to understand better the origin of key orbit asymmetries, in § 4 we develop and investigate orbits in analytical potential functions that mimic the numerically created bar potential. In § 5 we apply our new restriction hypothesis in order to ascertain into which of the allowed particle orbits stars would actually be injected if they formed from the gas that makes up the bar. The results of this study are summarized in § 6.

2. INITIAL CONDITIONS AND TOOLS

2.1. The Cazes Bar

Self-gravitating, triaxial configurations that are either stationary in inertial space or spinning about their shortest axis are of broad astrophysical interest. Aside from their relevance to the global properties of spiral and elliptical galaxies, spinning triaxial configurations are thought to be a stage through which dense cores of molecular clouds must evolve in order to produce binary stars (Lebovitz 1987; Cazes & Tohline 2000). Such configurations can also arise in the context of the late stages of stellar evolution (Lai, Rasio, & Shapiro 1993; New, Centrella, & Tohline 2000). In recent years interest in triaxial compact stellar objects has been renewed because they are potentially detectable sources for the gravitational wave detectors that are being constructed worldwide.

Our theoretical understanding of such structures has grown out of the general class of incompressible, ellipsoidal figures of equilibrium originally identified over 100 years ago by Maclaurin, Jacobi, Dedekind, and Riemann and recently studied in detail by Chandrasekhar (1969). The Riemann $S$-type ellipsoids, in particular, are an extremely useful family of equilibrium fluid configurations because they have analytically prescriptible properties that span a broad range of geometric parameters. Unfortunately, Riemann ellipsoids are not completely satisfactory models of galaxies, protostellar clouds, or compact stellar objects because they are uniform density configurations with very simple internal flows, whereas most astrophysically interesting systems are centrally condensed objects that exhibit a wide assortment of different angular momentum profiles.

In an effort to study the rotational fission instability in more realistic models of protostellar gas clouds, Cazes (1999) has recently utilized numerical hydrodynamic techniques to construct two different steady state models of rapidly rotating, triaxial gas clouds having a compressible (specifically, an $n = 3/2$ polytropic) equation of state. These models have been described in detail by Cazes & Tohline (2000). As far as we have been able to ascertain, these are the only fully self-consistent models of self-gravitating, compressible gas bars with nontrivial internal flows that have been presented or discussed in the literature. Because these models provide structures that are more realistic than Riemann ellipsoids, we have decided to examine the properties of one of them, specifically the one referred to as “model B” in Cazes & Tohline (2000), here in the context of the formation and evolution of barred galaxies. Hereafter we will refer to this model as the “Cazes bar.”

The gasdynamical simulation described by Cazes & Tohline (2000) that ultimately produced the Cazes bar began from a rotationally flattened, axisymmetric, $n = 3/2$ polytropic gas cloud that was in equilibrium and dynamically stable against axisymmetric disturbances. The initial model was constructed with an angular velocity profile such that, in equatorial projection, the model had uniform vortensity, where vortensity is defined as the ratio of vorticity to surface density. The model had a ratio of rotational to gravitational potential energy $T/|W| = 0.282$ and therefore was sufficiently rapidly rotating that it was unstable toward the development of a bisymmetric, non-axisymmetric distortion. Although primarily barlike in structure, the eigenfunction of the unstable bisymmetric mode had a slight, loosely wound, two-armed spiral character. Some redistribution of angular momentum occurred via gravitational torques as the mode grew to nonlinear amplitude. After approximately 30 dynamical times, the system settled down into a new, dynamically stable, spinning...
barlike structure containing 98% of the initial cloud mass and 95% of the cloud's original total angular momentum. At this point in the system's evolution, Cazes reconfigured the hydrodynamical code so that the evolution could be continued in a frame of reference that was rotating at a constant angular frequency, the pattern frequency of the bar, and then he followed the system's evolution through an additional 30 dynamical times. This extended evolution showed that, to a high degree of accuracy, the Cazes bar had settled into a steady state configuration and was dynamically stable.

The overall geometric shape of the Cazes bar is illustrated well by a map of isodensity contours in the equatorial plane of the bar, as displayed here in Figure 1a. As detailed in the last column of Table 3 of Cazes & Tohline (2000; see also the bottom panels of their Figs. 8 and 9), the bar extends along the major (x) axis to a dimensionless\(^2\) radius of \(x_{\text{max}} = 1.07\), has an intermediate (y)–to–major (x) axis ratio of approximately 0.52, and possesses two shallow off-axis density maxima at \(|x| = 0.31\). The two spiral “kinks” that are immediately apparent in the second and fourth quadrants of the isodensity contours of Figure 1a identify the location of the two weak standing shocks that accompany the bar's internal flow, as described more fully below.

\(^2\) As discussed in § 3.1 of Cazes & Tohline (2000), the hydrodynamical simulation that created the “model B” Cazes bar was performed using a set of dimensionless units so that the model could be straightforwardly scaled to a variety of different types of astrophysically interesting systems. A so-called polytropic system of units was adopted in which \(M = G = K = 1\), where \(G\) is the gravitational constant, \(K\) is the polytropic constant in the \((n = 3/2)\) polytropic equation of state, and \(M_{\infty}\) is the total mass of the initial, axisymmetric, equilibrium configuration from which the Cazes bar formed. As is tabulated in Table 3 of Cazes & Tohline (2000), in these units, our Cazes bar has a mass \(M = 0.958\), a total angular momentum \(J = 0.941\), a semimajor axis length \(R_{\text{max}} = 8.47\), a pattern frequency \(\Omega = 0.522\), and a maximum density \(\rho_{\text{max}} = 6.69 \times 10^{-3}\). We note that all of the figures in this manuscript show lengths that have been additionally scaled to the equatorial radius \(R_{\text{eq}} = 7.95\) of the initial axisymmetric model from which the “model B” Cazes bar was formed; hence, \(x_{\text{max}} \equiv R_{\text{eq}}/R_{\text{max}} = 1.07\). The Appendix in Williams & Tohline (1987), for example, shows in detail how any physical variable can be converted from this “polytropic” system of units to more familiar dimensional units. By way of illustration, when the Cazes bar is scaled to \(M_{\infty} = 10^{10} M_{\odot}\) and \(x_{\text{max}} = 2\) kpc, it has a pattern period \(P_{\text{rot}} = 2\pi/\Omega \approx 1 \times 10^7\) yr and a maximum density of \(\approx 3 \times 10^{-22}\) g cm\(^{-3}\) (see also Cazes 1999).

As discussed by Cazes & Tohline (2000), the bar is spinning about its shortest (z) axis in a counterclockwise direction with respect to Figure 1a, with a well-defined pattern frequency, \(\dot{\Omega} = 0.522\), and exhibits a global ratio of rotational to gravitational potential energy, \(T/|W| = 0.235\). The bar appears to be spinning as a solid object, but in reality it is not. Instead, as viewed from a frame spinning with the bar's pattern frequency, each Lagrangian fluid element in the bar moves along a well-defined streamline in a periodic, prograde orbit (counterclockwise in Fig. 1a) with a frequency that varies with position along the streamline. The nested fluid streamlines (see the bottom panel of Fig. 9 in Cazes & Tohline 2000) do not cross one another, but streamlines associated with the lowest density (outermost) regions of the bar contain a pair of standing shock fronts.

The velocity of fluid elements that follow these outermost streamlines becomes supersonic (in the frame rotating with the bar) as they “fall” along the length of the bar; then, with the aid of the shock, the flow becomes subsonic in order to bend around the end of the bar. The two standing shocks are evidenced by the kinks in the isodensity contours displayed in Figure 1a (see also the related violin Mach surface in the bottom panel of Fig. 8 in Cazes & Tohline 2000). Moving radially outward along the shock, the flow exhibits Mach numbers that vary smoothly from 1.0 to roughly 2.0 (see the discussion in § 5.2 for more details). Hence, along its entire length, the standing shock is relatively weak.

Figure 1b shows equipotential contours of the effective potential,

\[
\Phi_{\text{eff}}(x, y) = \Phi(x, y) - \frac{1}{2}\Omega^2(x^2 + y^2),
\]

that is generated in the equatorial plane by the rotating Cazes bar. Hereafter we will refer to the numerically determined effective potential of the Cazes bar as \(\Phi_{\text{CB}}\). Notice that, as with simpler models of rotating bars or oval distort-

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[Fig. 1. Representative diagrams of the Cazes bar potential-density pair. (a) Equatorial plane isodensity contours of the “model B” Cazes bar. (b) Equipotential contours in the equatorial plane of the Cazes bar effective potential, \(\Phi_{\text{CB}}\). The dimensionless length scales, x and y, that are used here, as well as in all other figures throughout this paper, are defined in terms of “polytropic” units, as described in footnote 2. Asterisks mark the positions of the L1 and L2 Lagrangian points. In addition, note in (a) the presence of off–axis density maxima and two spiral “kinks” indicating the presence of shocks.]
tions (e.g., Binney & Tremaine 1987, § 3.3.2), $\Phi_{\text{CB}}$ displays four prominent extrema outside of the central, elongated potential well. Two relative maxima appear above and below the bar (these are associated with the traditional L4 and L5 Lagrangian points), and two saddle points (associated with the L1 and L2 Lagrangian points) are marked by asterisks to the left and right of the bar. The L1 and L2 points are located at a dimensionless distance $R_{L1, L2} = 1.36$ from the origin and, for all practical purposes, define the maximum extent of the bar along the major axis.

The solid curves in Figures 2a and 2b show the quantitative variation in $\Phi_{\text{CB}}$ along the major and intermediate axes, respectively, of the bar. Along the intermediate axis, for example, the effective potential varies from a value $\Phi_{\text{min}} = -1.018$ at $y = 0$ to a value associated with the L4 and L5 maxima of $\Phi_{L4, L5} = -0.503$. Along the major axis the effective potential climbs to a somewhat lower value, $\Phi_{L1, L2} = -0.603$, before dropping again at positions $|x| > R_{L2}$. As Figure 1a illustrates, the Cazes bar has two mild off-axis density maxima. These density maxima help support a corresponding pair of slightly off-axis minima in the effective potential. The minima are not immediately evident from the contour levels used in Figure 1b, but they can be seen in Figure 2a. We note that the equipotential contours do not trace out simple quadratic surfaces (they have, instead, an overall “peanut” shape) and the contours exhibit a slight spiral twist. As we replace the numerically generated $\Phi_{\text{CB}}$ with an analytical “fit” (see § 4), we will attempt to mimic these characteristic features.

2.2. Numerical Techniques and Analysis

Although the Cazes & Tohline (2000) simulations were fully three-dimensional, our investigation will be restricted to an analysis of two-dimensional orbits that reside in the equatorial plane of the Cazes bar. (We hope to extend this analysis to three-dimensional orbits in the future.) We will rely heavily upon surface of section diagrams to characterize the properties of stellar orbits that are allowed in the Cazes bar potential and to provide a means by which the general properties of this potential can be compared with other analytical and N-body potentials that have been examined previously in connection with studies of barred galaxies.

From the appearance of the surfaces of section, we can discern information about the various orbits that are supported by the potential. For example, if the surface of section for a given particle orbit consists of points that form a so-called invariant curve, then that orbit has two isolating integrals of motion and is called regular. Orbits that respect only one integral of motion (the particle’s specific effective energy; see eq. [2]) create a set of apparently disorganized points throughout the allowed phase space and hence do not produce invariant curves. These are referred to as irregular, or ergodic, orbits. There also may exist a class of orbits that are quasi-ergodic. The surfaces of section for these orbits seem to be intermediate between regular and irregular. While a quasi-ergodic orbit does not form an invariant curve surface of section, neither will it fill the entire phase space. There are two classes of quasi-ergodic orbits, stochastic and semistochastic (Goodman & Schwarzschild 1981). Stochastic orbit surfaces of section exclude regions that regular orbits would occupy but, given enough time, will otherwise fill the energetically allowed phase space. Semistochastic orbits also avoid regions occupied by regular orbits but are further constrained not to fill all of the energetically allowed phase space.

![Fig. 2.—Comparison between $\Phi_{\text{CB}}$ (solid curves) and our analytically specified, untwisted, rotating effective potential (dashed curves) as described in § 4.1. (a) Comparison along the major ($x$) axis. (b) Comparison along the intermediate ($y$) axis. Note the presence of two shallow, off-axis minima in (a).](image-url)
We will generally present and discuss the behavior of groups of particle orbits that all have the same Jacobi constant, or specific effective energy,

$$\epsilon_j \equiv E - \Omega \cdot L$$

$$= \frac{1}{2}(p_x^2 + p_y^2) - \Omega(xp_y - yp_x) + \Phi$$

$$= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \Phi_{\text{eff}},$$  \hspace{1cm} (2)

where the canonical momenta are

$$p_x = \dot{x} - \Omega y$$

and

$$p_y = \dot{y} + \Omega x,$$

$E$ is the particle’s total energy per unit mass, $\Omega$ is the angular velocity of the bar, $L$ is the angular momentum of the particle in the rotating frame, and $\Phi_{\text{eff}}$ is defined as in equation (1).

In order to follow individual particle orbits inside $\Phi_{\text{CB}}$, we will use a simple Verlet integration scheme (Verlet 1967). The advantage of using a Verlet algorithm is that there is good long-term conservation of the Hamiltonian. For the orbits studied in this paper, the Hamiltonian of individual orbits is conserved to better than 0.5% and the average or orbits studied in this paper, the Hamiltonian of individual good long-term conservation of the Hamiltonian. For the advantage of using a Verlet algorithm is that there is a simple Verlet integration scheme (Verlet 1967).

The fractional difference is on the order of $10^{-5}$. The second difficulty is that particle initial positions are chosen to be on grid lines, but as the orbit integration progresses, each particle position moves continuously. When the particle’s position does not fall precisely on the intersection of two grid lines, we evaluate both components of the acceleration at the four grid points that surround the particle’s position, then linearly interpolate these to the particle’s position. As a check on the scope of this problem, we performed an integration with the nonrotating, analytical potential studied by Binney & Spergel (1982). Our resulting orbits and surfaces of section were satisfactorily similar to their published results. From this we concluded that the griding of the potential would not wreak havoc with the integrations.

There is some ambiguity in the literature over the question of which surfaces of section to use when characterizing a two-dimensional potential, $(x, p_x)$ or $(y, p_y)$. For example, Binney (1982) used $(x, p_x)$, while SS87 and Teuben & Sanders (1985) examined $(y, p_y)$. In order to gleam as much information as possible about the orbits in $\Phi_{\text{CB}}$, we have decided to look at both the $(x, p_x)$ and $(y, p_y)$ surfaces of section. We will base our primary categorizing criteria on the $(x, p_x)$ surfaces of section, but, as is illustrated below, the $(y, p_y)$ surfaces of section also can convey some important information, so we use them accordingly. Each $(x, p_x)$ surface of section is obtained by plotting the $x$-component of the position and canonical momentum every time the particle crosses the $x$-axis with $p_x > 0$. Alternatively, by plotting the $y$-component of the position and canonical momentum each time the particle crosses the $y$-axis with $p_y < 0$, a $(y, p_y)$ surface of section is created.

The shooting technique that we have used to investigate orbits in the Cazes bar potential is as follows. First, a value of $\epsilon_j$ is chosen. Second, an initial position $x_1$ is selected along the major axis of the bar. At this position, we start with $x_1 = 0.0$. From $\epsilon_j, x_0, y_0$ and $|\dot{x}_0|$, the corresponding value of $|\dot{y}_0|$ is uniquely determined. Then a particle trajectory is integrated with each of the four combinations of these initial velocity components: $(\dot{x}_0, \dot{y}_0)$, $(-\dot{x}_0, \dot{y}_0)$, $(\dot{x}_0, -\dot{y}_0)$, and $(-\dot{x}_0, -\dot{y}_0)$. During each orbit integration, the points for the $(x, p_x)$ and $(y, p_y)$ surfaces of section are calculated. Without changing the initial position, another value of $|\dot{x}_0|$ is chosen; we typically proceed in steps of 0.2. A new $|\dot{y}_0|$ is thereby determined, and the integration is repeated for each of the four velocity combinations. This cycle continues until the maximum allowed value of $x_1$ has been reached for that initial position. At that point, the initial position is changed and the entire procedure is repeated. On our 800 × 800 grid, we began this entire cycle by selecting the value of $x_1$ corresponding to the 20th zone from the center, then moved out along the axis in steps of 20 grid zones until the energetically limiting position along the major axis was reached. As a result, we examined roughly 200 unique orbits for each selected value of $\epsilon_j$. Finally, we emphasize that, throughout our presentation, equatorial plane coordinate axes are always oriented such that the $x$-axis coincides with the major axis of the bar, as in Figure 1.

3. ANALYSIS OF THE CAZES BAR POTENTIAL

3.1. Composite Surfaces of Section

Figure 3a shows a composite $(x, p_x)$ surface of section diagram for six separate regular orbits that arise in the Cazes bar potential when the Jacobi constant $\epsilon_j = -0.75$. (This value of the Jacobi constant has been selected for illustrative purposes only; it has no special significance, other than it lies between $\Phi_{\text{min}}$ and $\Phi_{\text{1,1,1,1,1,1}}$.) The contour that corresponds to this value of $\epsilon_j$ is identified by the dot-dashed line in Figure 6a. Additionally, the surface of section diagrams in §§ 3 and 4 also display zero-velocity curves, i.e., the locus of points in the surface of section diagram at which the potential equals the value of $\epsilon_j$. Figure 3a contains the following:

1. One elongated region (crosses) confined to a narrow, short segment of the negative $x$-axis.
2. Five disconnected regions (squares) that lie mostly at negative values of $x$ and surround the narrow elongated region.
3. Four disconnected regions (diamonds) having $|p_x|$ values that are generally larger than that of the five regions marked by squares.
4. Three islands identified by two separate but nested surfaces of section (asterisks and triangles).
Fig. 3. 

(a) $(x, p_x)$ composite surface of section diagram for six selected regular orbits with $\epsilon = -0.75$ that are supported by $\Phi_{cb}$. The dotted line surrounding the invariant curves is the zero-velocity curve. 

(b) $(y, p_y)$ composite surface of section diagram for the same six orbits represented in (a). As in (a), the dotted line is the zero-velocity curve.
5. A set of three curves (plus signs) that appear to define a boundary between the three islands and the region of the diagram occupied by the other surfaces of section.

Figure 3b is a \((y, p_y)\) composite surface of section that complements the \((x, p_x)\) surface of section shown in Figure 3a. The orbits that create each of the surfaces of section in Figure 3b are marked by the corresponding symbols in Figure 3a. For example, the orbit that forms the smallest three-island surface of section in Figure 3a (triangles) creates the skewed ellipse that is centered on \(y \approx -0.05\) in Figure 3b. Note that the majority of points that make up the \((y, p_y)\) surfaces of section fall at negative values of \(y\), suggesting that the orbits from which they are derived are retrograde. As we shall show, the surface of section marked by crosses is derived from an orbit that appears to be related to the \(x_4\) family of retrograde orbits. The surfaces of section marked by squares and diamonds belong to retrograde orbits with higher order resonances. However, the orbits associated with the three islands in Figure 3a are, in fact, prograde.

One striking feature of all of the surfaces of section that make up Figures 3a and 3b is the lack of symmetry. In nonrotating, bisymmetric potentials, surfaces of section show reflection symmetry about both the \(x = 0\) \((y = 0)\) axis and the \(p_x = 0\) \((p_y = 0)\) axis. A variety of such symmetric surfaces of section may be found in Binney & Tremaine (1987), § 3.3. Rotating potentials lose the reflection symmetry about the \(x = 0\) \((y = 0)\) axis but generally retain it across the \(p_x = 0\) \((p_y = 0)\) axis. Some examples of surfaces of section with this symmetry intact may be seen in SS87 as well as in Teuben & Sanders (1985). Figures 3a and 3b exhibit the expected rotational based asymmetry with respect to the \(x = 0\) \((y = 0)\) axis, but they also display a slight asymmetry about the \(p_x = 0\) \((p_y = 0)\) axis. As we attempt to develop an analytical approximation to \(Q_{CB}\) in § 4, we will strive to reproduce the primary features seen in Figures 3a and 3b, including this asymmetry.

### 3.2. Individual Orbits

While the surface of section is a useful tool for categorizing orbits, the orbits themselves are of primary importance. We begin with a description of the regular orbits that have just been identified in connection with the Cazes bar potential. The frames in the left column of Figure 4 illustrate individual \((x, p_x)\) surfaces of section from the Figure 3a composite diagram, while the frames in the right column of Figure 4 illustrate the \(x-y\) orbital trajectories from which each corresponding surface of section was derived.

Figure 4f shows the nearly closed orbit that leads to the smallest, three-island surface of section (triangles) illustrated in Figure 3a. This orbit as well as each of the two closely related orbits depicted in Figures 4b and 4d have the shape of a bow tie. Hence, we will refer to the regions of the surface of section diagrams that are occupied by these orbits, the three islands in \((x, p_x)\) and the skewed ellipses near the origin in \((y, p_y)\), as the bow tie regions. Particles travel on bow tie orbits in a counterclockwise direction (i.e., the overall motion is prograde) and make four radial oscillations before completing one full orbit cycle. Hence, the orbits illustrated in Figures 4a–4f are almost certainly related to the 4/1 family of orbits discussed by Contopoulos (1988; see especially his Fig. 1a). However, during two of the radial oscillations in a bow tie orbit, the particle passes very close to and, indeed, around the origin in such a way that its direction of motion formally becomes retrograde. This is why the \((y, p_y)\) surface of section for these orbits generally resides at negative values of \(y\). We note as well that the bow tie orbits do not exhibit perfect reflection symmetry about the \(x\)-axis. For example, the top and bottom sections of the orbit shown in Figure 4d seem to be tilted with respect to the intermediate \((y)\) axis, and the bottom of the “\(V\)” shape that is formed on the top of the orbit shown in Figure 4b does not lie directly above the inverted “\(V\)” that is formed on the bottom of that orbit.

The relatively simple orbit shown in Figure 4l is a retrograde orbit. This is clear from the \((x, p_x)\) surface of section (Fig. 4k), which shows that each time a particle on this orbit crosses the \(x\)-axis with a positive \(p_x\) it is to the left of the origin (i.e., at negative \(x\)), as well as from the \((y, p_y)\) surface of section (Fig. 3b, crosses), which shows that each time the particle crosses the \(y\)-axis with a negative \(p_y\) it is below the origin (i.e., at negative \(y\)). This orbit is almost certainly a member of the \(x_4\) family of orbits, as defined by Contopoulos & Papayannopoulos (1980).

The regular orbits shown in Figures 4h and 4j are also largely retrograde. However, these orbits are much more complex than the one illustrated in Figure 4f. Using the terminology of Contopoulos (1988), Figure 4h displays a 5/1 orbit, that is, the orbit makes five radial oscillations for every complete orbit cycle. Similarly, Figure 4j displays a 6/1 orbit. It is easier to understand why these two nearly closed orbits display, respectively, four and five disconnected regions in the \((x, p_x)\) surface of section diagram if, rather than counting radial oscillations, we count how many \(y\) oscillations the orbit undergoes before completing one full (horizontal) excursion along the bar. In this sense, Figure 4h displays a 4:1 orbit while Figure 4j displays a 5:1 orbit, exactly matching the number of disconnected regions that arise in the surface of section diagram.

Like a number of other previously investigated, non-axisymmetric potentials, the Cazes bar potential supports a rich variety of regular orbits that have a recognizable \(n:m\) oscillatory pattern, in the sense just discussed. Particles following these trajectories complete \(n\) oscillations perpendicular to the major \((x)\) axis in the time that it takes them to complete \(m\) circuits along the major axis. In the case of a closed orbit in which the oscillations perpendicular to the major axis actually cross the major axis, such an orbit would be represented by \(n\) distinct points in an \((x, p_x)\) surface of section diagram. However, it is also possible that not every oscillation will cross the major axis. As an example of this variety, Figure 5a illustrates a nearly closed, regular 15:5 orbit. However, the \((x, p_x)\) surface of section diagram in Figure 5b displays only 13 islands. The difference between what is expected (15 islands) and what is observed (13 islands) is due to the interesting behavior of this particular orbit. When the two apparently straight sections of the orbit (at \(y \approx 0.2\) and \(y \approx -0.2\)) are closely scrutinized, they each definitely exhibit a small \(y\) oscillation. Since neither of these nearly horizontal segments crosses the \(x\)-axis, neither generates a corresponding island in Figure 5b.

The Cazes bar potential also allows quasi-ergodic orbits to develop. In fact, approximately 40% of the \(\approx 200\) orbits studied at this Jacobi constant are quasi-ergodic (as determined from surface of section diagrams). As mentioned
FIG. 4.—Plots of the six individual surfaces of section taken from the Fig. 3a composite diagram and their corresponding orbits. Surfaces of section are shown in the frames in the left column, orbits are on the right. The symbols used for each surface of section are the same as in Fig. 3a.
Fig. 5.—Plots illustrating the behavior of a 15:5 orbit that is supported in \( \Phi_{CB} \). (a) Orbit showing 15 vertical oscillations for every five horizontal oscillations. (b) Corresponding \((x, p_x)\) surface of section diagram.

Fig. 6.—(a) Quasi-ergodic orbit with \( \epsilon_L = -0.75 \) that is supported by \( \Phi_{CB} \) shown superimposed on equipotential contours of that potential. The dot-dashed contour drawn at \( \Phi_{CB} = -0.75 \) also serves as a boundary of the area inside which this orbit is confined. As in Fig. 1, asterisks mark the positions of the L1 and L2 Lagrangian points. (b) \((x, p_x)\) surface of section for this orbit.
Fig. 7.—Composite ($x, p_x$) surfaces of section for four different values of $\epsilon_J$. (a) $\epsilon_J = -0.96$; this value of the Jacobi constant traps particles near the bottom of the potential well. (b) $\epsilon_J = -0.85$. (c) $\epsilon_J = -0.75$. (d) $\epsilon_J = -0.63$; this value of the Jacobi constant allows particles to move throughout the entire bar. Note the presence of bow tie and 5:1 orbital surfaces of section in each frame. Also, $x_4$ orbits appear only in (c) and (d), while $x_1$ orbits appear only in (d) (see also Fig. 9).

Fig. 8.—Composite ($y, p_y$) surfaces of section for four different values of $\epsilon_J$. The energy levels are the same as in Fig. 7.
earlier, quasi-ergodic orbits wander through the bar without an overall shape. For this reason, they are difficult to discuss individually, but collectively they have characteristics of interest. These orbits cross the major axis of the bar many times as they move along the length of the bar. Most importantly, they support the shape of the bar. A sample quasi-ergodic orbit is shown superimposed on equipotential contours of the Cazes bar in Figure 6a. The corresponding surface of section is shown in Figure 6b.

3.3. Composite Surfaces of Section for Varying $\epsilon_j$

While the previous sections have dealt with orbits at a single energy, it is interesting to see phase-space structure at various energy (Jacobi constant) levels. Figure 7 contains composite $(x, p_x)$ surface of section diagrams for four separate values of $\epsilon_j$: (a) $\epsilon_j = -0.96$ (near the bottom of the potential well), (b) $\epsilon_j = -0.85$, (c) $\epsilon_j = -0.75$ (this is the same as in Fig. 3a), and (d) $\epsilon_j = -0.63$ (almost at the L1, L2 energy level). Figure 8 shows the corresponding $(y, p_y)$ surfaces of section. As before, quasi-ergodic surfaces of section are not shown but do exist at each of these energies. The most striking aspect of these diagrams is their similarity to one another. The bow tie and 5:1 orbits appear in all diagrams. One difference among these diagrams is the small loop that appears at $(x \approx -0.1, p_x \approx 0.0)$ in Figures 7c and 7d. As mentioned in § 3.2, these loops are created by $x_4$ orbits. More difficult to distinguish is the presence of an $x_1$ surface of section in Figure 7d. This surface of section is composed of two pieces that lie close to the zero-velocity curve (outer dotted line). A better view of the $x_1$ surface of section is shown in Figure 8d; it is the small loop located at $(y \approx 0.4, p_y = 0.0)$. The relationship between these $(x_1, x_4$, and bow tie) orbits and the energy range over which they exist is best illustrated by a characteristic diagram, as shown in Figure 9. This diagram displays the location at which each periodic orbit crosses the $y$-axis as a function of the energy (Jacobi constant in this case) of that orbit. Figure 9 demonstrates that the bow tie orbits are the dominant regular orbital family in the Cazes bar potential.

4. ANALYTICAL POTENTIALS

4.1. Rotating Bar

In an effort to understand better why the Cazes bar potential supports this particular variety of particle orbits, we have developed an analytical potential that shares its major structural features. The effective potential that we have developed empirically has the form

$$
\Phi_{\text{eff}}(x, y) = N \left\{ 1 - \left[ 1 + \left( \frac{x}{R_{L1,2}} \right)^2 + \left( \frac{y}{q R_{L1,2}} \right)^2 \right]^{-n/2} \right\} - \frac{1}{2} \Omega^2 (x^2 + y^2) + \Phi_{\text{min}},
$$

where $N$ is a normalization factor, $q$ determines the strength of the barlike distortion, $x$ and $n$ are exponents whose values are to be determined, and $R_{L1,2}$, $\Omega$, and $\Phi_{\text{min}}$ have the same definitions as in the Cazes bar potential. Note that unlike $\Phi_{CB}$, the $\Phi_{\text{eff}}$ given in equation (3) is fourfold symmetric, i.e., the potential looks the same under the transformations $x \rightarrow -x$ and $y \rightarrow -y$. The form of equation (3) is certainly not a unique way to model the Cazes bar potential. It has been adopted simply because it produces a potential sufficiently similar to the Cazes bar potential as well as supporting orbits like those discussed in § 3.2.

After setting $R_{L1,2} = 1.36$, $\Phi_{\text{min}} = -1.018$, and $\Omega = 0.522$, as in the Cazes bar, we have found that equation (3) is a good fit to $\Phi_{CB}$ if we select the following parameter values: $q = 0.8$, $N = 0.7$, $x = 4$, and $n = 8$. Figure 10 is a plot of the equipotential contours generated by equation (3) with this set of parameters, and the dashed curves in Figures 2a and 2b show the variation of this analytical potential along its major $(x)$ and minor $(y)$ axes. As Figure 2 demonstrates, along the principal axes of the bar, this potential matches

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**Figure 9.** Characteristic diagram for three families of orbits in the Cazes bar potential. The dot-dashed line at $-\epsilon_j = 0.603$ marks the value of the potential at the L1, L2 points. The plus signs represent the position along the $y$-axis where $\epsilon_j = \Phi_{CB}$. Bow tie orbits exist over the entire energy range that exists inside the Cazes bar. Near the bottom of the potential well ($-\epsilon_j \approx 0.85$), the periodic bow tie orbits become fully prograde. The $x_4$ and $x_1$ families exist over only a limited (higher energy) range.

**Figure 10.** Equipotential contours of the effective potential that is defined analytically by eq. (3). The degree to which this analytical function matches the numerically prescribed $\Phi_{CB}$ can be judged by comparing this figure to Fig. 1b. The dashed curves drawn in Figs. 2a and 2b show more quantitatively the behavior of this analytical function along its $x$ and $y$ principal axes, respectively, in comparison to the behavior of $\Phi_{CB}$.
Fig. 11—(a) $(x, p_x)$ composite surface of section diagram for five selected regular orbits with $\epsilon = -0.75$ that are supported by the rotating analytical potential described in § 4.1. As before, the dotted line surrounding the invariant curves denotes the zero-velocity boundary. This diagram should be compared with Fig. 3a. (b) $(y, p_y)$ composite surface of section diagram for the same orbits represented in (a). Again, the dotted line is the zero-velocity curve. This figure should be compared with Fig. 3b.
Fig. 12.—Plots of the five individual surfaces of section taken from the Fig. 11a composite diagram and their corresponding orbits. Surfaces of section are shown in the frames in the left column, orbits are on the right. A comparison between this figure and Fig. 4 illustrates the degree to which the analytically specified effective potential discussed in § 4.1 supports orbits that are like the orbits supported by $\Phi_{\text{eq}}$. 

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The analytical potential closely matches the Cazes bar potential. 

4.2. Twisted, Rotating Bar

In an effort to construct an analytical effective potential that supports orbits having all of the asymmetries seen in the Cazes bar orbits, we have added a slight spiral twist to the potential function given in equation (3). Specifically, our chosen rotating, twisted potential has the form

$$\Phi_{\text{eff}}(x', y') = \mathcal{N}\left\{1 - \left[1 + \left(\frac{x'}{R_c}\right)^2 + \left(\frac{y'}{qR_c}\right)^2\right]^{-n/2}\right\}$$

$$-\frac{1}{2} \Omega^2(x'^2 + y'^2) + \Phi_{\text{min}},$$

(4)

where

$$x' \equiv x \cos(a\sqrt{x^2 + y^2}) - y \sin(a\sqrt{x^2 + y^2})$$

and

$$y' \equiv x \sin(a\sqrt{x^2 + y^2}) + y \cos(a\sqrt{x^2 + y^2}).$$

For this potential, shown in Figure 13, $a = 0.1$; otherwise the values of the parameters are the same as for the rotating effective potential discussed in § 4.1.

The composite ($x, p_x$) surface of section for regular orbits with $\epsilon_j = -0.75$ that are supported by this twisted potential is shown in Figure 14a; Figure 14b shows the corresponding composite ($y, p_y$) surface of section. About 50% of the orbits studied in this potential are quasi-ergodic. Notice that the overall composite surface of section bears a strong resemblance to that of the untwisted potential. The major difference lies in the symmetry of the surface of section. The reflection symmetry about the $p_x = 0$ and $p_y = 0$ axis is now gone. An example of this feature is that the three islands (triangles and asterisks) that are positioned symmetrically in the Figure 11a surface of section diagram are twisted slightly from those positions in Figure 14a. The effect of the twisting of the potential on the orbits can be seen in Figures 15a–15j. The orbits in Figure 15 resemble those from § 3.2 even more closely than the orbits shown in Figure 12 in that there is now an asymmetry due to the twisting. The effective potential given by equation (4) appears to provide an excellent approximation to $\Phi_{\text{CB}}$.

5. RESTRICTION HYPOTHESIS

Up to this point, we have identified many stellar orbits that, in principle, could be supported by the Cazes bar potential. In the context of our restriction hypothesis (RH), we next ask, which of these orbits would be populated by stars that form from the gas and therefore have initial velocities determined by the gas in the bar?

5.1. Restriction Hypothesis Orbits

In order to maintain consistency between the discussion of our RH orbits and the previously discussed orbits, we want to focus on the orbits of stars that are created with a Jacobi constant $\epsilon_j = -0.75$. However, we must abandon the method of choosing initial conditions as outlined in § 2.2. From the gas motions that are an integral part of the Cazes bar structure, we now have specific values of the velocity associated with each coordinate position in the bar.
Fig. 14.—(a) $(x, p_x)$ composite surface of section diagram for five selected regular orbits with $\epsilon_j = -0.75$ that are supported by the rotating and twisted analytical potential described in § 4.2. As before, the dotted line marks the zero-velocity curve. This diagram should be compared with Fig. 3a. (b) $(y, p_y)$ composite surface of section diagram for the same orbits represented in (a). The dotted line is the zero-velocity curve. This figure should be compared with Fig. 3b.
Fig. 15.—Plots of the five individual surfaces of section taken from the Fig. 14a composite diagram and their corresponding orbits. Surfaces of section are shown in the frames in the left column, orbits are on the right. A careful comparison between these plots and the corresponding ones displayed in Fig. 12 shows that the slight spiral “twist” that has been added to eq. (3) in order to generate eq. (4) produces a “north-south” asymmetry like the one that arises in $\Phi_{eq}$ (see Fig. 4).
With this in mind, Figure 16 shows contours of constant $\epsilon_j$, where the known velocity of the gas has been used in the determination of $\epsilon_j$ at each $(x, y)$ location. The dot-dashed contour underlying the large assortment of symbols identifies at what locations in the bar stars could form with $\epsilon_j = -0.75$. Because these contours are dependent on the velocity field of the gas, the shocks mentioned in § 2.1 become noticeable in Figure 16, whereas they were not readily identifiable in our earlier plots of $\Phi_{CB}$ (Fig. 1a) or $\Phi_{eff}$ (Figs. 10 and 13).

In order to investigate the behavior of an assortment of our RH orbits, we positioned 30 particles along the $\epsilon_j = -0.75$ contour, as shown in Figure 16, and assigned to each the velocity of the Cazes bar gas at that location. For discussion purposes, we divided these particles into two groups: one that begins on the positive side of the major axis and terminates where the $\epsilon_j = -0.75$ contour crosses the intermediate $(y)$ axis, and one that begins near the intermediate axis and ends where this energy contour crosses the negative side of the major axis. The composite $(x, p_x)$ surface of section diagrams that we derived from the first and second groups are shown in Figures 17a and 18a, respectively. Corresponding $(y, p_y)$ composite surface of section diagrams for the first and second groups are displayed in Figures 17b and 18b, respectively. The symbols marking the initial positions of the particles in Figure 16 are the same symbols used to make the corresponding surfaces of section in Figures 17 and 18.

By comparing these new figures with Figure 3, we are able to assess the general impact of our RH. One of the most striking differences between these figures results simply from the fact that we have elected to include the surfaces of section of quasi-ergodic orbits in Figures 17 and 18, whereas the equivalent orbits were not displayed in Figure 3. This is clear at a glance that the RH permits a mixture of regular and quasi-ergodic orbits to be populated. In particular, nine of the 30 starting positions shown in Figure 16, that is, 30% of our RH particles, produced quasi-ergodic orbits. It is also clear that the bow tie region of phase space is well populated within the constraints of our RH. The holes that appear at the centers of the bow tie orbit "islands" in Figures 17 and 18 falsely suggest that a strictly periodic bow tie orbit does not arise under our RH. Instead, these empty regions (and the analogous gaps that appear between some of the other regular orbit surfaces of section) arise because the spacing that we have chosen between initial particle positions in Figure 16 was relatively coarse. With a finer spacing, these regions would have been filled by regular bow tie orbits, and we probably would have identified two points, one for each group, on the precisely periodic bow tie orbit. Most significantly, the composite surface of section diagrams (Figs. 17a and 18a) that result from our RH present a large region of...
phase space that is completely unoccupied. This is the region that previously had been occupied by retrograde orbits. It is obvious, therefore, that under the constraints of our RH, no true retrograde orbits are produced. This is perhaps not surprising, given that all of the gas in the Cazes bar is moving along prograde streamlines.

It is informative to study the sequence of orbits that appears as one moves to different starting positions along the $\epsilon_j = -0.75$ contour of Figure 16, in a counterclockwise fashion starting from the position marked by the plus sign on the positive $x$-axis. This point on the major axis is the initial position for an orbit that is similar to that shown in Figure 4b. Moving along the contour of constant $\epsilon_j$, each successive initial position gives rise to bow tie orbits that are more and more closed. The fifth and sixth points (square and cross, respectively) have nearly identical orbits; they form the innermost three-island surfaces of section in Figure 17a and the corresponding innermost curves of the bow tie region in Figure 17b. A particle starting from a position somewhere between these two points would probably trace the periodic bow tie orbit. Proceeding toward the minor axis, the sequence reverses and the orbits become less closed. The twelfth point (cross) is the origin for an orbit that is basically the same as that for the first point. The last three initial positions in this group, up to and including the point on the minor axis, produce quasi-ergodic orbits. These are the orbits that, for example, create the swarm of points that surround the three bow tie region islands in Figure 17a.

We now discuss the second group of initial positions, whose $(x, p_x)$ surfaces of section are displayed in Figure 18a and $(y, p_y)$ surfaces of section are shown in Figure 18b. In the absence of the shock, it would be reasonable to assume that the progression seen in the first group would simply be reversed. However, whereas only three positions nearest the minor axis gave rise to quasi-ergodic orbits in the first group, five positions nearest the minor axis lead to quasi-ergodic orbits in group two. The sixth position (cross) produces an orbit similar to the one shown in Figure 4b and leads to a large three-island surface of section in Figure 18a. The next four points (ending with the triangle just after the shock) mirror the sequence in the first group by beginning bow tie orbits that become more and more closed. The

eleventh point in the second group (square) is the initial position for the most closed orbit in both groups. This orbit forms the smallest three-island surface of section in Figure 18a and the smallest curve in the bow tie region of Figure 18b. Continuing toward the major axis, the orbits become less closed. The point on the major axis (diamond) gives rise to an orbit that is the same as that orbit associated with the first point among the first group of points. Since the number of quasi-ergodic RH orbits is greater for the quadrant containing a shock, it seems that the presence of a shock (even one as mild as this) can influence orbital structure. We will examine the effects of shocks more directly in the next section.

Observations in the solar neighborhood indicate that stars are born with velocities that have some dispersion about the mean motion of the gas. With this in mind, we have examined the properties of stellar orbits that begin with the velocity of the gas plus a modest random component. The results vary little from what has already been presented here under the strict RH, even if a random velocity of up to 30% of the magnitude of the initial velocity is added. For larger (but $\leq 50\%$) random velocities, orbits that were bow tie shaped with no perturbation maintain their basic morphology, but unperturbed quasi-ergodic orbits can be "kicked" into regular orbits.

5.2. Orbits Originating near Shocks

If stars were to form from the gas with equal probability at all locations throughout the Cazes bar, then the distribution function of stars that would be created at a Jacobi constant $\epsilon_f = -0.75$ would contain a uniform mixture of all the orbits discussed in § 5.1. For example, most would be quasi-periodic and 30%, by number, would be quasi-ergodic. In real barred galaxies, however, one usually does not find that star formation occurs at a uniform rate throughout the entire volume of the bar. In particular, the star formation rate usually is higher in the vicinity of a shock (Binney & Merrifield 1998, § 5.1.8). Since the Cazes bar model contains shocks, this is an ideal opportunity to examine how such a process would impact the resulting distribution function of newly formed stars.

In order to model this scenario, we placed four groups of 15 particles in the vicinity of the shock structure that is
evident in the fourth quadrant of Figure 1a; see Figure 19 for details regarding the distribution of these particles. These particles were given the gas velocity corresponding to their initial positions, according to our RH. Note that, as described in § 2.1, the shock becomes stronger as the distance from the major axis increases, but along most of its length, the shock is relatively weak. More specifically, using the initial particle positions in Figure 19 as a guide, the diamond located at \( y \approx -0.31 \) identifies the contour level at which the shock front officially begins (Mach number 1.0), the flow reaches Mach 1.5 between the square and triangle at \( y \approx -0.45 \), and at the lowest density contour shown the Mach number is approximately 2.0. These particles no longer share a common value of \( \epsilon_J \). Hence, individual surfaces of section for regular orbits are likely to overlap, and it becomes much less useful to produce composite surface of section diagrams. For this reason, here we will discuss only individual surfaces of section.

From this entire group of 60 initial particle positions, we find that only the three particles in the second column and farthest from the \( x \)-axis (plus sign, asterisk, and diamond) follow quasi-ergodic orbits. All other particles follow quasi-periodic orbits. We focus, then, on this second column of particles. The particle that began farthest from the \( x \)-axis (Fig. 19, plus sign) created the \((x, p_x)\) surface of section shown in Figure 20a and the corresponding orbit shown in Figure 20b. Moving progressively closer to the \( x \)-axis, most of the particles trace orbits that have the general bow tie shape. For example, particles starting from the positions marked by the square (\( y_i \approx -0.43 \)) and the asterisk (\( y_i \approx -0.34 \)) generate the orbits shown in Figures 20d and 20f. The orbits shown in Figures 20h and 20j are followed by particles that are deep in the central region of the potential well (initial positions marked by the plus sign \([\epsilon_J = -0.899]\) and diamond \([\epsilon_J = -0.953]\) that appear closest to the major axis in Fig. 19). These orbits are quite thin and have a strong overall bar shape. (Note that these orbits do not appear to be thin in the figures because we have expanded the vertical axis in order to reveal more orbit details.)

We believe that the presence of the quasi-ergodic orbits is connected to the large velocities that are present in the gas that is located immediately before the shock. Since bow tie orbits have turning points in the vicinity of this shock, stars that are created with small \( x \)-velocities in this region (i.e., from the postshock gas) have a better chance to fall onto such an orbit than do stars that are created with a large \( x \)-velocity (i.e., from preshock gas). Basically, these high-velocity stars are shot through the region occupied by bow tie orbits and onto the only other available trajectories, that is, quasi-ergodic orbits. Hence, the presence of the shock does influence the trajectories onto which stars will be injected according to our RH, but in such a way that stars that form from the postshock gas are unlikely to end up on quasi-ergodic orbits.

6. DISCUSSION AND CONCLUSIONS

We have used a standard shooting technique to probe the structure of a rotating barlike potential, \( \Phi_{\text{CB}} \), that arises from the steady state gasdynamical bar that was constructed by Cazes (1999; see also Cazes & Tohline 2000 in a recent three-dimensional hydrodynamical simulation). This potential supports a roughly equal mixture of regular and quasi-ergodic orbits. Virtually all of the regular prograde orbits appear to belong to a single family that we have described as having a bow tie shape. These orbits are almost certainly related to the 4/1 family of orbits described by Contopoulos (1988) because particles on bow tie orbits make four radial oscillations for each complete azimuthal cycle, but they differ from the 4/1 orbit illustrated in, for example, Figure 1a of Contopoulos (1988) in that they pass very close to, and around, the center of the potential well twice each orbit cycle (see, e.g., Fig. 4f). The Cazes bar potential also supports a variety of regular retrograde orbits, including some that appear to be members of the \( x_4 \) orbit family.

Our analysis indicates that, over a large range of \( \epsilon_J, \Phi_{\text{CB}} \) does not support the family of \( x_1 \) orbits (see the characteristic diagram, Fig. 9). As illustrated in § 3, no such orbits were found with a Jacobi constant of \( \epsilon_J = -0.75 \), and after a careful probe at a number of other energy levels, we were only able to find a few \( x_1 \) orbits at energies close to \( \Phi_{0.2} \). This is perhaps the most striking difference between \( \Phi_{\text{CB}} \) and the potential wells that have been generated through self-consistent \( N \)-body simulations. \( N \)-body simulations tend to produce bars with stellar distribution functions, such as \( DF_{SS} \), that are dominated by \( x_1 \) orbits. We suspect that this is because the Cazes bar has a higher ratio of rotational to gravitational potential energy \( T/|W| \) than typical \( N \)-body bars and that, along its major axis, the Cazes bar potential is very shallow. In order to approx-
Fig. 20.—Plots of the $(x, p_x)$ surfaces of section and corresponding orbits produced in $\Phi_{cb}$ by five of the 15 particles whose initial positions are shown in Fig. 19. Surfaces of section are in the frames in the left column, orbits are on the right. In each case, the symbol used in the surface of section matches the symbol used to mark the corresponding particle’s initial position in Fig. 19. As discussed in § 5.2, the initial particle velocity is specified by the Cazes bar gas velocity at the particle’s initial position, as prescribed by our RH.
imate this behavior, we were driven to design an analytical effective potential that, while exhibiting a traditional quadratic dependence \([i.e., \text{changing as } (y/R_{1,2})^2]\) along the intermediate axis, changes as \((x/R_{1,2})^3\) from the center along the major axis.

We have considered the possibility that galaxies form central barlike structures while still in a predominantly gaseous state. Because it has been constructed in a self-consistent manner, the Cazes bar presents a reasonable representation of such a newly formed, gaseous galaxy configuration. If stars form from the gas in such a barred galaxy, our proposed RH illustrates the orbits into which the stars would be injected at the time of their formation. Our analysis indicates that the distribution function \(\text{DF}_{\text{bar}}\) of such a system of stars would contain no retrograde orbits, but it would consist of a reasonable mixture of quasi-ergodic orbits and regular prograde orbits predominately related to the bow tie \((4/1)\) orbit family. It is important to emphasize that these stellar orbits are distinctly different from the orbits that gas particles follow in the Cazes bar. Elements of gas are accelerated by local pressure gradients as well as by gradients in the underlying gravitational potential; also, unlike stellar orbits, gas particle orbits do not cross one another. As illustrated by Cazes & Tohline (2000), within the steady state Cazes bar the gas moves along closed streamlines that are approximately elliptical in shape. It is safe to say that no stars that form from such a gas flow will have similarly elliptical orbits. Searching many different initial conditions for particles in \(\Phi_{\text{CB}}\), we were unable to find any orbits that even approximated the gas streamlines.

There are two indications from our study that the presence of a shock front increases the ratio of quasi-ergodic to regular orbits. First, in the absence of a shock, in which case the potential would have exhibited a fourfold symmetry, we would have expected the ratio of quasi-ergodic to regular orbits to have been identical in the two samples of particles whose trajectories started from the positions shown in Figure 16. As discussed in § 5.1, however, we found that more of the test particles in the second group (with starting positions on or closer to the Cazes bar's second quadrant shock front) followed quasi-ergodic orbits. Then, in the case in which we purposely selected a group of starting positions along the fourth quadrant shock (see § 5.2), we found that particles starting from the highest velocity regions of the preshock gas landed in quasi-ergodic orbits.

There are several interesting points to be made about the bow tie orbit family and about stars that might be injected into bow tie orbits. Although bow tie orbits should certainly be classified as a prograde orbit family, stars that move along bow tie orbits will appear to be moving in a retrograde sense on the portions of their orbits that are nearest the center of the bar. Also, any star that moves along a bow tie orbit will (1) spend most of its time near the “four corners” of the orbit and (2) pass very close to the center of the potential well twice each orbit. When coupled with our discovery that a significant fraction of stars that form from gas in the Cazes bar will be injected into bow tie orbits, the first of these points leads us to suggest that a gaseous bar should produce a \(\text{DF}_{\text{bar}}\) that is rather boxy or peanut shaped. This is in contrast to distribution functions like \(\text{DF}_{\text{gs}}\) that are dominated by the \(x_1\) family of orbits and are therefore more elliptical in shape. The second of these points leads us to suggest that star formation in a primarily gaseous bar may provide a mechanism for funneling matter in toward the center of a galaxy in situations in which gas dissipation alone does not work efficiently. As noted by Norman & Silk (1983), triaxial potentials can provide a means of transporting stellar mass to a central black hole. Stars that travel close to a central black hole can become tidally disrupted, and the resulting gas can form an accretion disk that fuels an active galactic nucleus (AGN; see Evans & Kochanek 1989; Ho, Filippenko, & Sargent 1997). Admittedly, in our present model we have not examined to what extent a central point mass will scatter and thereby disrupt the regular bow tie orbit (Gerhard & Binney 1985). However, we find the existence of orbits that travel near the center of the potential over such a large range of energies \((-0.96 < \varepsilon_j < -0.63)\) intriguing. We hope to perform an investigation of this model of AGN fueling in the future.

We now consider whether a purely stellar dynamical bar could be created with a distribution function given by \(\text{DF}_{\text{bar}}\). That is to say, if a purely gaseous galaxy were to evolve initially into the form of a steady state Cazes bar and then slowly create stars from the gas, injecting them according to our RH into the orbits that make up \(\text{DF}_{\text{bar}}\), could a smooth evolutionary transition be made between the purely gaseous bar and one that is entirely made up of stars but that otherwise exactly resembles the Cazes bar? Using a technique similar to Schwarzschild’s method (Schwarzschild 1979) or that of Contopoulos & Grosbøl (1988), it is conceivable that the right combination of bow tie orbits and quasi-ergodic orbits could be assembled to produce a steady state stellar dynamical bar, and this configuration may even closely resemble the Cazes bar. (Given that we have found an analytical function \(\Phi_{\text{st}}\) that closely approximates \(\Phi_{\text{CB}}\), it should be relatively straightforward to conduct such a study.) However, it seems unlikely that a system of stars that forms according to our RH from the Cazes bar could lead to such a configuration because the specific distribution of gas in the Cazes bar is unlikely to produce the required proportion of bow tie and quasi-ergodic orbits. For example, if in order to create a steady state stellar bar one needs \(N_{\varepsilon_j}\) bow tie orbits with energy \(\varepsilon_j\), then there must be the right proportion of gas with energy \(\varepsilon_j\) at the proper positions to form stars for these orbits. With this additional constraint, it seems unlikely that there would be a clean transformation between a gaseous and a stellar system. We suspect, instead, that after more than half of the gas has been converted into stars, the entire configuration would dynamically relax to a new configuration that is dominated by the collective dynamics of the stars. Since such an evolution would begin from a relatively high \(T/|W|\) configuration that contains a large number of stars in bow tie orbits, it would be interesting to know whether this final state has a more boxy or peanut shape than the stellar dynamical configurations that have been created via \(N\)-body simulations from initially axisymmetric distribution functions. It may be necessary to answer this question before we are able to state with any certainty whether barred galaxies form from initially axisymmetric \(\text{DF}_{\text{axisym}}\) or nonaxisymmetric \(\text{DF}_{\text{bar}}\) stellar distributions.

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