Effective Scaling Approach to Frictionless Quantum Quenches in Trapped Bose Gases

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We work out the effective scaling approach to frictionless quantum quenches in a one-dimensional Bose gas trapped in a harmonic trap. The effective scaling approach produces an auxiliary equation for the scaling parameter interpolating between the noninteracting and the Thomas-Fermi limits. This allows us to implement a frictionless quench by engineering inversely the smooth trap frequency, as compared to the two-jump trajectory. Our result is beneficial to design the shortcut-to-adiabaticity expansion of trapped Bose gases for arbitrary values of interaction, and can be directly extended to the three-dimensional case.

I. INTRODUCTION

Bose-Einstein condensates (BECs) and their related phenomena - such as collective excitations, collapse, and nonlinear dynamics, to mention a few - have aroused great interest since their first experimental realization [1, 2]. From the theoretical point of view, weakly interacting BECs can be accurately described within the framework of the Gross-Pitaevskii (GP) theory, which provides a remarkable agreement with experimental observations. In most cases of classical hydrodynamics [3] or scaling transformations [4, 5], exact analytical solution can be found for the collective dynamics and free expansion of BECs in time-dependent harmonic traps, both in the noninteracting limit and the Thomas-Fermi (TF) regime [2]. In this vein, symmetries give birth to an intriguing property of self-similarity, which allows to utilize the scaling approach for describing the dynamics of ultracold atomic systems, for instance, the atomic gases in the non-interacting and the hydrodynamic regimes [6, 7]. Tonks-Girardeau (TG) gas of impenetrable bosons [8, 9], superfluid Fermi gas [7, 10], and thermal cloud [11] in different geometries.

Besides, an effective scaling approach has further been proposed as an approximate solution for the evolution of both bosonic and fermionic density distributions, for describing the collective dynamics of a trapped Bose gas [12], the expansion of Fermi gas [13, 14] and of quantum degenerate Bose-Fermi mixtures [15]. It consists in a self-similar evolution in the hydrodynamic regime to be satisfied on average by integrating over the spatial coordinates, reduce the complexity of numerical treatment. Recently, the accuracy of such effective approach in reproducing the exact solution of quasi one-dimensional (1D) and three-dimensional (3D) GP equations for arbitrary values of the interactions has been discussed in Refs. [16, 17]. Remarkably, it turns out that the space-averaged self-similarity can provide an accurate description in several situations [18].

In a slightly different but relevant topic, the concept of shortcuts to adiabaticity (STA), originally proposed for fast schemes reproducing or approaching slow adiabatic process [19, 20], have extended further the control paradigms for frictionless atomic cooling in a expanding harmonic trap [21–26]. In the context of inverse engineering, the scaling approach [21] and Lewis-Riesenfeld dynamical invariant [22] bring out the various forms of the Ermakov equation for the scaling parameter, capturing the character of the self-similar evolution. Along with it, the harmonic trap frequency is thus inversely engineered for the propose, by choosing an interpolation function of the scaling parameter with the appropriate boundary conditions. This strategy can be applicable to other ultracold atomic systems as a TG gas [27], an anisotropic gas containing quantum defects [28] and a Fermi gas [29]. However, tracking back to a BEC described by the GP equation in the mean-field approximation, one can realize that the original Ermakov equation obtained in non-interacting case needs to be modified in the TF limit or in the case of a time-dependent interaction [21, 26]. To remedy it, the variational approximation [30] (which is equivalent to moment method [31]), can be complemented by the concept of STA, for studying the dynamics of BECs [32–34], valid for the range from zero to small atomic interaction, with the implication on the quantum speed limits and quantum thermodynamics [35, 36]. As a matter of fact, the accuracy of the variational approximation depends on the presumed ansatz in terms of nonlinearity [33, 34]. Thus, the motivation of this work is to fill the gap in more general theory on STA design for 1D Bose gas with arbitrary interactions.

In this paper, we integrate the effective scaling approach into inverse engineering for frictionless quantum quenches in trapped Bose gas, for arbitrary values of the atomic interaction strength. Here we focus on the 1D GP equation, but the result can be extended to 3D case. By assuming the scaling solution in the hydrodynamic limit, we derive the Ermakov-like equation for the scaling parameter interpolating between non-interacting and the TF limit. With this, the frictionless quench is designed, and also compared to the free expansion and two-jump trajectory of STA. Finally, the numerical simulation is performed to check the stability of our method, and the energetic cost of STA is discussed as well.
II. EFFECTIVE SCALING APPROACH

We start by considering a quasi-1D BEC confined in cigar-shaped trap, characterized by a longitudinal frequency \( \omega_0 \) and a tight transverse frequency \( \omega_x \gg \omega_0 \). Therefore, the system can be effectively described by a wave function \( \psi(x, t) \), whose dynamics is governed by the following GP equation,

\[
\frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} \omega^2(x)^2 \psi + g|\psi|^2 \psi,
\]

that is written here in dimensionless form, for convenience. To this end, we have used \( l_0 = \sqrt{\hbar/m_0} \) as unit length \( (m \) being the particle mass), \( \hbar \omega_0 \) as unit energy, and \( \omega_0^{-1} \) as unit time. The interaction strength can be written in terms of the scattering length \( a_s \) as \( g = 2Na_s/l_0 \), with \( N \) being the number of atoms and with the total density being normalized to one.

In order to elaborate the effective scaling approach, we apply the Madelung transformation \( \psi = \sqrt{n(x,t)} e^{i\phi(x,t)} \), such that the Lagrangian of the system can be written as

\[
\mathcal{L} = -\left[ \frac{1}{2} (\psi \nabla \psi)^2 + \frac{1}{8} \left( \frac{n}{n} \right)^2 + V(x,t) + g n \right],
\]

where \( V(x,t) = \omega^2(x) x^2/2 \). The essence of the variational Lagrangian formalism is to minimize the action \( S = \int \mathcal{L} dx dt \) with respect to the parameters \( q_i = \{ n, \phi \} \), that is, \( \delta S / \delta q_i = 0 \). The latter corresponds to the Euler-Lagrangian equations \( \partial L / \partial \dot{q}_i - \partial \mathcal{L} / \partial q_i ) dx = 0 \), from which the hydrodynamic equations are obtained as [2]

\[
\frac{\partial n}{\partial t} + \frac{\partial (n \nabla \phi)}{\partial x} = 0, \tag{3}
\]

\[
\frac{\partial (\nabla \phi)}{\partial t} + \frac{\partial}{\partial x} \left( P(x,t) + \frac{1}{2} \sqrt{n} + V(x,t) + gn \right) = 0, \tag{4}
\]

where \( P(x,t) = -\langle \partial^2 \sqrt{n} / (2 \sqrt{n} \rangle \) is the so-called quantum pressure. By inserting the scaling solution \( n = n_0(x/a)/a \) and \( v = \dot{x}/a \) [2] into Eq. (4), one can obtain

\[
- \frac{1}{2} \frac{\dot{a}}{a^2} x^2 = P(x,t) + V(x,t) + gn - P(0,t) - gn(0,0). \tag{5}
\]

Multiplying the resulting equation by \( n_0(\xi) \) after rescaling the spatial as \( \xi = x/a \), and integrating over the coordinates, we get the following effective Ermakov-like equation [17]

\[
\ddot{a} + \omega^2(t)a = \frac{A}{a^3} + \frac{B}{a^2}, \tag{6}
\]

where \( A \) and \( B \) read

\[
A = \frac{P(0,0) - E_0^0}{E_v^0}, \quad B = \frac{gn(0,0) - E_{int}^0}{E_0^0}, \tag{7}
\]

with \( E_v^0 \equiv \langle 1/2 \rangle \int \partial_x n(0,x)^2 dx \) being the kinetic energy, \( E_v^0 \equiv \int V(x,0)n_0(x)dx \) the potential energy, and \( E_{int}^0 \equiv \langle g/2 \rangle \int n_0^2(x)dx \) the interaction energy, all evaluated at the initial time \( t = 0 \). Here \( n_0(x) = \bar{n}(x,0) \) represents the ground state of the system in the harmonic trap. As shown in Ref. [17], the parameters \( A \) and \( B \) satisfy the relation \( A + B = 1 \) for arbitrary (positive) interactions.

Remarkably, the Ermakov-like Eq. (6) permits to describe the dynamics of the system in terms of an effective self-similar evolution, for arbitrary interactions. Note that this equation is more accurate than the one obtained by applying a Gaussian ansatz for arbitrary interactions, \( a(t) = 1/a^3 + g/\sqrt{2 \pi a^2} \) [30, 33]. Obviously, Eq. (6) reproduces the exact scaling in the non-interacting \( (g = 0) \) and Thomas-Fermi limits \( (g \gg 1) \). Namely, for \( g = 0 \) the above Eq. (6) corresponds to the original Ermakov equation,

\[
\ddot{a} + \omega^2(t)a = 1/a^3, \tag{8}
\]

with \( n_0(x) = e^{-x^2/2 \pi a^2} \) and \( A = 1, \quad B = 0 \). This is consistent with the results derived from variational control [30, 33] and also from the scaling approach and Lewis-Riesenfeld dynamical invariant [21, 22]. In the opposite TF regime, the ground state density is \( n_0 = [\mu - \omega(0)x^2/2]/\mu \), with \( \mu = E_{int}^0 + E_{kl}^0 \) being the chemical potential (the kinetic energy can be safely neglected in this limit [2]). Then, inserting \( n_0 \) into Eq. (7) yields \( A = 0, \quad B = 1 \).

III. SHORTCUTS TO ADIABATICITY

In this section, we use the Ermakov-like equation (6) to construct a STA protocol for \( \omega(t) \), for achieving a frictionless quench from the initial trap frequency \( \omega_0 \equiv \omega(0) \) (with \( \omega(0) = 1 \) fixed by our notation choice) to a final value \( \omega_f \equiv \omega(t_f) \) within a short time \( t_f \), with \( \gamma \equiv \sqrt{\omega_0/\omega_f} > 1 \). That is, in a finite-time non-adiabatic expansion, the trap frequency is changed to some lower final value, while keeping the populations of the initial and final levels invariant, thus without generating friction and heating. The frictionless cooling of ultracold atoms trapped in time-dependent harmonic traps was originally investigated in two noninteracting and TF limits [21, 22]. Later, the variational approximation was used to design the same process in the weakly interacting regime [33]. Here, we propose a general approach based on the formulation discussed in previous section, for arbitrary values of the interactions, ranging from the non-interacting to the TF regime.

To this end, we recast Eq. (6) in the form of the perturbative Kepler problem in classical mechanics, in the presence of the effective potential

\[
\mathcal{U}(a) = \frac{A}{2a^2} + \frac{B}{a} + \frac{1}{2} \omega^2(t)a^2, \tag{10}
\]

for a fictitious particle with unit mass, satisfying the Newton equation \( \ddot{a} = -\partial \mathcal{U}(a)/\partial a \), derived from Eq. (6). The total energy of the particle reads

\[
E(a) = \frac{\dot{a}^2}{2} + \frac{A}{2a^2} + \frac{B}{a} + \frac{1}{2} \omega^2(t)a^2. \tag{11}
\]
The conditions for an adiabatic evolution are $\dot{a} = 0$ and $\partial U(a)/\partial a = 0 \ [32]$, yielding
\begin{equation}
\dot{a}^2 \omega^2(t) - aB = A. \tag{12}
\end{equation}

Then, we define the time-averaged energy
\begin{equation}
E_t = \frac{1}{t_f} \int_0^{t_f} E(a) dt, \tag{13}
\end{equation}
that will be used for quantifying the energetic cost of STA, in the discussion below. The initial boundary conditions read [32]:
\begin{align}
&\ a(0) = a_i, \quad \dot{a}(0) = \ddot{a}(0) = 0, \tag{14} \\
&\ a(t_f) = a_i, \quad \dot{a}(t_f) = \ddot{a}(t_f) = 0, \tag{15}
\end{align}
where $a_i$ and $a_f$ are the unique positive real solutions of Eq. (12), at $t = 0$ and $t = t_f$. These boundary conditions (14) and (15) guarantee that the initial and final states are adiabatic correspondences for designing STA protocols. Having fixed the boundary conditions, the trajectory of $a(t)$ can be interpolated, by choosing a simple polynomial ansatz of the form
\begin{equation}
a(t) = a_i - 6(a_i - a_t) s^5 + 15(a_i - a_t) s^4 - 10(a_i - a_t) s^3, \tag{16}
\end{equation}
with $s = t/t_f$. Consequently, the trap frequency $\omega(t)$ is determined by Eq. (6). If an imaginary trap frequency is allowed, the harmonic trap inverts to a parabolic repeller, instead of a trap, such that $t_f$ may be formally made arbitrarily short. However, since in experimental implementations there are always imperfections and limitations related, e.g., to the trap anharmonicity and to the laser power, this poses a constraint of the amplitude of the frequency that can be physically achieved, namely $|\omega^2(t)| \leq \delta$, with $\delta$ being a real number [33, 37, 38].

In Fig. 1 we show the results for a STA protocol of a frictionless quench from $\omega_i = 1$ to $\omega_f = 0.1$, within a time $t_f = 1$. Panels (a) and (b) illustrate the the evolution of the trap frequency $\omega(t)$ and of the width $a(t)$ for different values of the interactions, $g = 0.01, 1, 100$. With the designed protocol, we use the split-operator method to propagate numerically the wave function to the final state $|\tilde{\psi}_f\rangle$, that is then compared to the ground state of the final trap $|\tilde{\psi}_f\rangle$, see Fig. 1(c) (the latter is computed by means of a standard by imaginary-time evolution). The corresponding fidelity, $F \equiv |\langle \tilde{\psi}_f | \tilde{\psi}_f \rangle |^2$, is shown in Fig. 2 as a function of the interaction strength $g$, for different values of the trap frequency. Notice that $F \equiv 1$ in both the non-interacting ($g \ll 1$) and TF limits ($g \gg 1$), where the scaling ansatz is an exact solution. Remarkably, the fidelity always stays very close to one even in the intermediate regime, thanks to the accuracy of the effective scaling approach [17]. There the deviations from $F \equiv 1$ are less than 1%, and show a weak dependence on the trap frequency, i.e., the fidelity decreases $\omega_f$ is decreased. This is due to the fact that the interaction becomes dominant when the trap frequency is negligible. Notice also that the behaviour of the fidelity presented here can be also intuitively understood in terms of the stability of a particle in the presence of the effective potential, see Eq. (10).

**IV. SUDDEN QUENCH**

For completeness, here we also consider the case of a sudden quench of the trap frequency:
\begin{equation}
\omega(t) = \begin{cases} \omega_i, & t = 0 \\
\omega_f, & 0 < t < t_f \end{cases}. \tag{17}
\end{equation}

In the non-interacting case, the conventional Ermakov equation (8) gives the analytical solution
\begin{equation}
a(t) = \sqrt{1 + (\omega_i^2 - \omega_f^2) \sin^2(\omega_f t)/\omega_f^2}, \tag{18}
\end{equation}
along with the re-
corresponds to a collective oscillation with period \( \tau \) and
the mass of the wave packet, calculated from numerical simulation, is denoted by symbol. Parameters: \( g = 0.01 \) (sold line; with red circle), \( g = 1 \) (dashed line, with black diamond), and \( g = 100 \) (dotted line, with red square), and the initial trap frequency is \( \omega_0 = 1 \).

with initial boundary conditions \( a(0) = 1 \), and \( \dot{a}(0) = 0 \), which corresponds to a collective oscillation with period \( \tau = \pi / \omega_f \). Therefore, in general the boundary conditions (14) and (15) cannot be not attained at \( t = t_f \), and this implies a heating/excitation of the system.

In the limit \( \omega_f \rightarrow 0 \), the case of a free expansion, Eq. (18) reduces to \( a(t) = (1 + \omega_f^2 t^2)^{1/2} \). In more general cases, analytical solutions for a sudden quenches are not available, but one can still solve Eq. (6) numerically. Fig. 3, we demonstrate that Eq. (6) is accurate enough to describe the dynamics in sudden quenches, when the values of nonlinear interaction is changed from zero to infinity.

Based on the above considerations, we can build a STA protocol with just two quenches, as proposed for the compression of solitons in nonlinear fibers [39]: an initial quench from \( \omega_i \) to \( \omega_c \) at \( t = 0 \), and a second one from \( \omega_c \) to \( \omega_f \), at the final time \( t_f \) such that

\[
    t_f = \pi / (2 \omega_c), \tag{19}
\]

with \( \omega_c = \sqrt{\omega_i \omega_f} \). Then, in the non-interacting case we have [see again Eqs. (17) and (18)]

\[
    a(t) = \sqrt{1 + (\omega_f^2 - \omega_c^2) \sin^2(\omega_c t) / \omega_c^2}, \tag{20}
\]

satisfying the above-mentioned conditions, \( a(0) = 1 \), \( a(t_f) = \gamma \), and \( \dot{a}(0) = \dot{a}(t_f) = 0 \) [33, 40]. Thus, Eqs. (19) and (20) provide a simple exact solution for the shortcut with just one intermediate frequency, the geometric mean of the initial and final frequencies. Importantly, the solution can be generalized for Eq. (6), without requiring a explicit form. The energy conservation in this perturbative Kepler problem implies \( U(a_i) = U(a_f) \), or, in an explicit form,

\[
    \omega_c = \sqrt{\frac{A}{a_i^2 a_f^2} + \frac{2B}{a_i a_f (a_i + a_f)}}. \tag{21}
\]

In this case, a simple expression for \( t_f \) is not available, but it can be written in the form of an integral:

\[
    t_f = \int_{a_i}^{a_f} \frac{da}{\sqrt{2[U(a_i) - U(a)]}}. \tag{22}
\]

where \( a_i \) and \( a_f \) are given by Eq. (12). Accordingly, the trajectory of \( a(t) \) can be obtained numerically from Eq. (6). The corresponding fidelity is plotted in Fig. 2 along with the results of the frictionless quench. This figure shows that the two-jump STA is less accurate than the STA by using smooth polynomial function, when the final frequency is smaller. Another important difference between the two approaches is the fact that in the STA protocol discussed in the previous section the final time is a free parameter, whereas in the present case it is fixed by the trap frequencies and the interaction strength, see Eq. (22). We should emphasize that the larger repulsive interaction somehow speed up the two-jump STA e.g. \( t_f = 4.9658 \) for \( g = 0.01 \) while \( t_f = 4.5380 \) for \( g = 100 \), when \( \omega_i = 1 \) and \( \omega_f = 0.1 \).

V. DISCUSSION

In the STA protocol, there is a trade-off between speed and cost in shortcuts to adiabaticity [41]. In principle, the transient energy excitation of the STA protocol, described by Eq. (11), is stipulated by the time-energy uncertainty, which implies an increase of the energy for shorter times. In detail,
VI. CONCLUSION

To conclude, we have employed the effective scaling approach to derive the Ermakov-like equation (6) for the scaling parameter interpolating between the noninteracting and the TF limits. By combining inverse engineering with the appropriate boundary conditions, this provides a general way to design accurate STA for 1D Bose gas, for arbitrary values of the atomic interaction strength. Here we have considered the case of a quasi-1D condensate confined in cigar-shaped trap, with a tight transverse frequency (that is, much larger than the longitudinal one). These results can be easily generalized to the 3D case in different interaction regimes, see Ref. [18].

In addition, the effective scaling approach is harnessed to design a STA protocol for a trapped 1D Bose gas in arbitrary repulsive interacting regime, where the previous methods, such as dynamical invariant or scaling approach, can not work successfully. We emphasize that the method is similar to, yet different from, the variational approximation [30, 33]. In the effective scaling approach, the scaling solution in the hydrodynamic regime is used as ansatz, but the Gaussian-shaped ground state in non-interacting limit is replaced as a presumed ansatz in variational approximation. In this sense, the effective scaling approach has more reasonable accuracy, when the atomic interaction with the arbitrary value is considered. Finally, some extensions are interesting for further exploration, for instance, the soliton dynamics by quenching the interactions of the BEC from repulsive to attractive [34, 43], or the expansion of a Bose gas in the crossover from TF to TG regimes [44].

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