Constructing $p,n$-forms from $p$-forms via the Hodge star operator and the exterior derivative

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Abstract

In this paper, we aim to explore the properties and applications on the operators consisting of the Hodge star operator together with the exterior derivative, whose action on an arbitrary $p$-form field in $n$-dimensional spacetimes makes its form degree remain invariant. Such operations are able to generate a variety of $p$-forms with the even-order derivatives of the $p$-form. To do this, we first investigate the properties of the operators, such as the Laplace-de Rham operator, the codifferential and their combinations, as well as the applications of the operators in the construction of conserved currents. On basis of two general $p$-forms, then we construct a general $n$-form with higher-order derivatives. Finally, we propose that such an $n$-form could be applied to define a generalized Lagrangian with respect to a $p$-form field according to the fact that it incudes the ordinary Lagrangians for the $p$-form and scalar fields as special cases.

$Keywords$: $p$-form; Hodge star; Laplace-de Rham operator; Lagrangian for $p$-form.
1 Introduction

Differential forms are a powerful tool developed to deal with the calculus in differential geometry and tensor analysis. Their applications in mathematics and physics have brought about the increasingly widespread attention \[1, 2, 3, 4\]. Particularly, in multifarious branches of theoretical physics, such as general relativity, supergravity, (super)string theories, M-theory and so on, antisymmetric tensor fields are essential ingredients of these theories, while each of such fields is naturally in correspondence to a certain $p$-form (with the form degree $p = 0, 1, 2, \cdots$). As a consequence, the introduction of $p$-forms is able to offer great conveniences for the manipulation of antisymmetric tensor fields, even of various quantities that just contain antisymmetric parts. For instance, in some cases, if the Lagrangian of an $n$-dimensional theory is put into an $n$-form, the analysis on the symmetry of the theory becomes more convenient.

As is known, apart from the wedge product, both the Hodge star operator and the exterior derivative are thought of as two important and fundamental tools for manipulating differential forms. Their combinations can generate various useful non-zero operations \[5, 6, 7, 8\], such as the well-known Laplace-de Rham operator, codifferential (divergence operator) and d’Alembertian operation. Furthermore, if letting an arbitrary combined operation act upon any $p$-form field in $n$-dimensional (pseudo-)Riemannian manifold, one observes that the operator is able to generate $l$-forms only with six types of form degrees, that is, $l \in \{p, q, p \pm 1, q \pm 1\}$, where and in what follows $q = n - p$ \[5\]. However, in the present work, of particular interest are operators that mix the Hodge star operation with the exterior derivative and can generate $p$-forms from a $p$-form because of the significance and relevance of these fields to general relativity and gauge theory. To move on, it is of great necessity to find all the probable structures of such operators first, as well as to exploit their main properties then. It will be demonstrated below that the degree-preserving operations can be expressed through the Laplace-de Rham operator together with the combinations of the codifferential and the exterior derivative.

As a matter of fact, working with the action of the combined operators preserving the form degree upon a certain $p$-form field, one can observe that all the newly-generated $p$-forms are just the even-order derivatives of this field. Furthermore, these higher-order derivative $p$-forms, as well as their Hodge dualities, enable us to construct $n$-forms involving higher derivatives of fields. As a significant application, in terms of the fact that the ordinary
Lagrangians for the $p$-form gauge fields and scalars can be included as special cases of those $n$-forms, they may be adopted as appropriate candidates for Lagrangians with respect to $p$-form fields. If so, such a proposition would provide a novel understanding towards the $p$-form gauge theories. For the sake of clarifying this, we recognize that at least the structure of the $n$-forms, together with their main characters, should be illustrated at the mathematical level. This is just our main motivation.

The remaining part of the current paper is organized as follows. In section 2, in terms of the action upon arbitrary $p$-form fields, we plan to carry out detailed investigations on the operators generating $p$-forms. We shall pay special attention to the Laplace-de Rham operator, together with the codifferential. In section 3, a general $n$-form, as well as its equivalent, will be constructed on basis of two general $p$-forms. In addition, their properties will be analyzed in detail. In section 4, inspired with the forms of the ordinary Lagrangians for $p$-form fields and scalars, we are going to put forward that the $n$-forms could play the role of the generalized Lagrangians with respect to $p$-forms. Then the equations of motion for the fields will be derived. The last section is our conclusions. For convenience to the reader, we summarize our notations and conventions in appendix A.

## 2 The operators preserving the form degree

In this section, through the combination of the Hodge star operation $\star$ and the exterior derivative $d$, we shall present a general operator that can generate differential forms of degree $p$ from an arbitrary $p$-form field $F$, which is expressed as $F = (p!)^{-1} F_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ (From here on, $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ will always refer to the abbreviation for $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ and $F_{\mu_1 \cdots \mu_p}$ denotes a totally anti-symmetric rank-$p$ tensor). Further to illustrate such an operator, the Laplace-de Rham operator $\Delta$, which could be expressed by means of the d’Alembertian (wave) operator $\Box$ as well as the codifferential $\delta$, will be discussed in detail.

Without loss of generality, in an $n$-dimensional spacetime, which is a pseudo-Riemannian manifold $M$ endowed with the metric $g_{\mu\nu}$ of a Lorentzian signature $(-, +, \cdots, +)$, we introduce Hodge star that is also referred to as the Hodge duality operator as the map from $p$-forms to $q$-forms ($q = n - p$), that is, $\Box$ as well as the codifferential $\delta$, will be discussed in detail.

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\[ \star : \Omega^p(M) \to \Omega^q(M), \]  

(2.1)

where $\Omega^p(M)$ stands for the space of $p$-forms on $M$. More specifically, by means of its action
on the $p$-form $\mathbf{F} \in \Omega^p(M)$, we obtain the following $q$-form
\[
\star \mathbf{F} = \frac{1}{p!q!} F^{\nu_1 \cdots \nu_p} \epsilon_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_q} d\nu_1 \cdots \mu_q.
\] (2.2)

Here the completely anti-symmetric rank-$n$ Levi-Civita tensor $\epsilon_{\nu_1 \cdots \nu_n}$ is defined through $\epsilon_{01 \cdots (n-1)} = \sqrt{-g}$. Hence the Hodge duality of $\star \mathbf{F}$ is read off as $\star \star \mathbf{F} = (-1)^{pq+1} \mathbf{F}$.

Apart from the ordinary Hodge star operation, its various generalizations, as well as their applications in physics, have been investigated in many works (see [9, 10, 11, 12, 13, 14], for example). What is more, the usual exterior derivative of the $p$-form is presented by
\[
d \mathbf{F} = (p!)^{-1} \partial_\sigma F_{\nu_1 \cdots \nu_p} d\nu_1 \cdots \nu_p d\sigma_{\mu_1 \cdots \nu_q},
\]
directly leading to a significant property of the exterior derivative that it gives zero when applied twice in succession to an arbitrary differential form, i.e., it fulfills the identity $d^2 = 0$.

As what has been demonstrated in [5], both the operators $O_1$ and $O_2$, defined in terms of the Hodge star operation together with the exterior derivative and expressed as
\[
O_1 = \star d \star d, \quad O_2 = d \star \star,
\] (2.3)
respectively, can preserve the degree of the $p$-form field $\mathbf{F}$. It is easy to check that $O_1 O_2 = 0 = O_2 O_1$ with the help of the identity $d^2 = 0$. See [5, 6, 7, 8] for more information of the properties about $O_1$ and $O_2$. Particularly, the works [6, 7, 8] have explicitly exploited the properties of the operator algebra generated by Hodge star and the exterior derivative.

More generally, through the linear combination of the operators $O_1^j$ and $O_2^k$, we further propose an operator $O_{jk}$ that is presented by
\[
O_{jk}(\alpha_{1j}, \alpha_{2k}) = \alpha_{1j} O_1^j + \alpha_{2k} O_2^k,
\] (2.4)
where both $\alpha_{1j}$ and $\alpha_{2k}$ are constant parameters (it is also allowed that they are functions of spacetime coordinates). It is straightforward to verify that $O_{jk}$ still guarantees the degree of an arbitrary $p$-form field to remain unchanged, namely, $O_{jk} : \Omega^p(M) \to \Omega^p(M)$, as well as $O_{jk}^n = \alpha_{1j}^n O_1^m + \alpha_{2k}^n O_2^m$. Making use of the operator $O_{jk}^n$ to act on $\mathbf{F}$, we are able to obtain a variety of $p$-form fields $O_{jk}^n \mathbf{F}$ that are the derivatives of order $\text{Max}\{2m_j, 2m_k\}$ of $\mathbf{F}$.

What is more, without the consideration of the operations $(\star \star)^k \in \{-1, 1\}$, the combined

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1We only focus on the combined operators depending on the Hodge star operator and the exterior derivative in this paper, while Eq. (2.13) shows that the degree-preserving d’Alembertian $\Box$ is also dependent of the curvature tensors $R_{\mu \nu \rho \sigma}$ and $R_{\mu \nu}$ in curved spacetimes. As a result, the d’Alembertian operator does not enter into the definition of $O_{jk}$. 
operation $\Sigma_{j,k}O_{jk}$ could be viewed as the most universal operator constructed from the combination of the Hodge duality operation and the exterior derivative, which takes an arbitrary $p$-form back into a certain $p$-form.

Subsequently, substituting $\alpha_{11} = (-1)^{np+1}$ and $\alpha_{21} = (-1)^n\alpha_{11}$ into the combined operator $O_{11}(\alpha_{11}, \alpha_{21})$, one can construct the well-known Laplace-de Rham operator $\Delta$ (it is also called as Laplacian or Laplace-de Rham operator in literature) from its action on the $p$-form field $F$, written as \[ \Delta = (-1)^{np+1}O_1 + (-1)^{np+n+1}O_2 \] (2.5)

Here the duality of the operation combining the exterior derivative and Hodge star $\hat{\delta} = (-1)^{np+1} \ast d \ast$ represents the well-known codifferential, or coderivative, or co-exterior derivative [1][2][4], which fulfills $\hat{\delta}^2 = 0$, $\hat{\delta}d = (-1)^{np+1}O_1$ (or $d \ast d = (-1)^p \ast \hat{\delta}d$) and $d\hat{\delta} = (-1)^{np+n+1}O_2$, while we adopt $\hat{\delta}$ instead of the conventional $\delta$ to denote the codifferential since we prefer to reserve the latter for the variational symbol. The last equality of Eq. (2.5) apparently demonstrates that the Laplace-de Rham operator is just the anticommutator of the codifferential and the exterior derivative, namely, $\Delta = \{\hat{\delta}, d\}$. By making use of the equalities $\hat{\delta}^2 = 0$ and $d^2 = 0$, we obtain the important properties associated with the three de Rham cohomological operators $(d, \hat{\delta}, \Delta)$ (they are essential ingredients involving in the famous Hodge decomposition theorem [1][4]), including $\Delta^k = (\hat{\delta}d)^k + (d\hat{\delta})^k$, together with the commutation relations $[\hat{\delta}, d] \neq 0$, $[d, \Delta^k] = 0$ and $[\hat{\delta}, \Delta^k] = 0$. Moreover, the combination of the coderivative with the Hodge star operation gives rise to

\[ \ast \hat{\delta} = (-1)^{p+1}d \ast, \quad \ast \hat{\delta}d \ast = (-1)^{pq+1}d \hat{\delta}, \]
\[ \hat{\delta} \ast = (-1)^p \ast d, \quad \ast d \hat{\delta} \ast = (-1)^{pq+1}d \hat{\delta} \] (2.6)

due to their respective actions on arbitrary $p$-forms. More properties and applications on the two operators $\Delta$ and $\hat{\delta}$ could be found in the works [6][15][16].

By contrast with the exterior derivative, which increases the degree of a differential form by one unit, the codifferential decreases that of a form by one. Specifically, its operation on the $p$-form field $F$ sends this one to the $(p-1)$-form

\[ (\hat{\delta}F)_{\mu_2 \cdots \mu_p} = \nabla^{\mu_1} F_{\mu_1 \cdots \mu_p} = (\text{div}_g F)_{\mu_2 \cdots \mu_p}, \] (2.7)
that is to say, \( \hat{\delta} \) is consistent with the usual divergence operator \( \text{div}_g \). Furthermore, if the \( p \)-form \( \mathbf{F} \) is an exact form, demanding that \( \mathbf{F} = d\mathbf{A} \), where the field \( \mathbf{A} \in \Omega^{p-1}(M) \) is a \((p-1)\)-form, the coderivative of \( \mathbf{F} \) becomes

\[
\nabla^\mu_{\mu_1 \cdots \mu_p} F_{\mu_1 \cdots \mu_p} = (\delta d\mathbf{A})_{\mu_2 \cdots \mu_p} - (d\hat{\delta}\mathbf{A})_{\mu_2 \cdots \mu_p},
\]

(2.8)
in which, after some algebraic manipulations, the component expressions for \( \hat{\delta}d\mathbf{A} \) and \( d\hat{\delta}\mathbf{A} \) are given by

\[
(\hat{\delta}d\mathbf{A})_{\mu_2 \cdots \mu_p} = p\nabla^\sigma \nabla_{[\sigma} A_{\mu_2 \cdots \mu_p]} = \Box A_{\mu_2 \cdots \mu_p} - (p-1)\nabla^\sigma \nabla_{[\mu_2 A_{|\sigma|\mu_3 \cdots \mu_p]},
\]

(2.9)
respectively. Here and henceforth, the covariant d'Alembertian operator with respect to the metric tensor \( g_{\mu\nu} \) is presented by \( \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \). As a convention, a pair of square brackets on \( p \) indices refer to anti-symmetrization over those indices with the common factor of \((p!)^{-1}\), while the horizontal bars around an index denote that this one remains out of the anti-symmetrization.

Let us make some discussions on the applications of Eq. (2.9). In the case where \( \mathbf{A} \) is an arbitrary scalar \((0\text{-form}) \phi(x)\), it yields \( \hat{\delta}d\phi = \Box \phi \) and \( d\hat{\delta}\phi = 0 \). On the other hand, for the case where \( \mathbf{A} \) is a vector field, Eq. (2.9) gives rise to the result \((O_1 \mathbf{A})^\mu = 2(-1)^n \nabla_\nu \nabla^{[\mu} A^{\nu]}\), ensuring that the conserved current \( J^\mu_K = 2\nabla_\nu \nabla^{[\mu} A^{\nu]} \) involved in the well-known Komar integral \([17]\) can be alternatively expressed as \( J_K = -\hat{\delta}d\xi \) (its Hodge duality gives rise to the usual \((n-1)\)-form current). Here it is worthwhile to note that it is unnecessary to restrict \( \xi^\mu \) to a Killing vector and it can be arbitrary, so \( J_K \) covers the generalized Komar current with respect to an almost-Killing vector presented in \([18, 19]\) as a special case (see the quite recent work \([20]\) for some properties of the conserved current associated with almost-Killing vectors). One is able to check that the current naturally yields the divergence-free equation \( \hat{\delta}J_K = 0 \). Furthermore, we put forward a more general conserved current associated with the arbitrary vector \( \xi^\mu \), taking the form

\[
J_{(1)} = \sum_{i=1}^2 \chi_i (\delta d)^i \xi.
\]

(2.10)
Here and in what follows, \( \chi_i \)'s denote arbitrary constant parameters. Hence the 2-form potential \( K_{(2)} \) corresponding to \( J_{(1)} \) could be expressed as \( K_{(2)} = -\sum_{i=1}^2 \chi_i (d\hat{\delta})^{i-1} d\xi \) on
basis of the relation $J_{(1)} = -\hat{\delta}K_{(2)}$. In general, based upon an arbitrary $p$-form field $F$, the action of the operator $O_1 = (-1)^{np+1}\hat{\delta}d$ on it renders the possibility to construct an anti-symmetric $p$-index tensor

$$J_{(p)} = \sum_{i=1}^{\chi_i} O_i^1 F,$$  

which apparently obeys the constraint of covariant divergencelessness $\hat{\delta}J_{(p)} = 0$. Through a replacement of $O_1$ with $O_2$ in $J_{(p)}$, one obtains a general closed $p$-form $J_{(p)}(O_1 \to O_2)$.

With the help of the operators $\Delta$ and $\hat{\delta}$, we are able to recast the general operator $O_{jk}$ given by Eq. (2.4) into

$$O_{jk}(\alpha_1 j, \alpha_2 k) = \alpha_1 j \mathbb{P}^j - \alpha_2 k \mathbb{P}^k + \alpha_2 k \Delta^k,$$  

where $\mathbb{P} = \hat{\delta}d = (-1)^{np+1}O_1$ for convenience, whose operation on a differential form has been given by Eq. (2.10). Within the above equation, if both $\alpha_1 j$ and $\alpha_2 k$ are allowed to be the functions of the spacetime coordinates, the properties $\mathbb{P}^j \phi = \Box^j \phi$ and $\Delta^k \phi = \mathbb{P}^k \phi$ are useful to the computation of $O_{jk}^m$. For example, when $m = 2$, we have $O_{jk}^2 = \alpha_1^2 \mathbb{P}^{2j} + \alpha_2^2 \Delta^2 - \mathbb{P}^2 k + \alpha_1 j \Upsilon_{jk}$ with $\Upsilon_{jk}$ presented by $\Upsilon_{jk} = (\Box^j \alpha_1 j) \mathbb{P}^j + (\Box^k \alpha_2 k) (\Delta^k - \mathbb{P}^k)$. Moreover, Eq. (2.12) implies that it is completely possible to utilize the Laplace-de Rham operator $\Delta$ to manipulate differential forms so as to assist with a new perspective upon $O_{jk}$. To demonstrate this, we address ourselves to such a problem. After letting $\Delta$ operate on an arbitrary $p$-form $F$, we arrive at the so-called Weitzenböck identity

$$\Delta F = \Box F + \Omega(F),$$  

$$\Box F = \frac{1}{p!} \Box F_{\mu_1 \ldots \mu_p} dx^\mu_1 \ldots dx^\mu_p,$$  

in which the $p$-form $\Omega(F)$, arising from the property of the non-commutativity of the co-variant derivative associated with the (pseudo-)Riemannian geometry, takes the form

$$\Omega_{\mu_1 \ldots \mu_p} = -p R_{\rho \sigma \mu_1 \mu_2 \ldots \mu_p}^\rho F_{\sigma |\mu_2 \ldots \mu_p |} + \frac{p(p-1)}{2} R_{\rho \sigma |\mu_1 \mu_2 |} F_{|\rho \sigma | \mu_3 \ldots \mu_p |}.$$

It should be pointed out that $\Omega = 0$ if $F$ is a 0-form (scalar). Throughout this work $R_{\rho \sigma \mu \nu}$ and $R_{\rho \sigma}$ stand for the standard Riemann curvature tensor and the Ricci tensor of metric respectively. Specifically, the former is defined through $(\nabla_\nu \nabla_\sigma - \nabla_\sigma \nabla_\nu)V_\mu = R_{\rho \sigma \mu \nu} V_\nu$ for an arbitrary vector $V_\nu$ [2, 3]. Eq. (2.13) indicates that the operation $\Delta$ could be generally expressed in terms of the d’Alembertian operator and the curvature tensors. As a result,
we obtain the commutation relations

\[
\begin{align*}
[\delta d, \square] F &= \Omega(\delta d F) - \delta d \Omega(F), \\
[d \delta, \square] F &= \Omega(d \delta F) - d \delta \Omega(F), \\
[\Delta, \square] F &= \Omega(\square F) - \square \Omega(F). 
\end{align*}
\] (2.15)

In particular, when the spacetime is Minkowskian, according to the vanishing of the Riemann curvature tensor in such a spacetime, namely, \(R_{\rho\sigma\mu\nu} = 0\), the operator \(\Delta\) is identified with the standard wave operator \(\square_\eta = \partial^\mu \partial_\mu\).

In the remainder of the present section, to provide a deeper understanding about the operator \(\Delta\), instead of the general situations, we take into account of its applications in several concrete types of fields.

First, when \(F = f(x) \epsilon\) is an arbitrary \(n\)-form, we obtain \(\Delta F = \epsilon \square f\). In parallel, the action of \(\Delta^k\) on the scalar field \(\phi(x)\) gives rise to \(\Delta^k \phi(x) = \square^k \phi(x)\), from which one can get the massless wave equation relevant to the scalar field \(\square \phi(x) = 0\).

Second, if it is supposed that the \((p - 1)\)-form \(A\) satisfies the restriction \(\mathbb{P} A = 0\), we have the identity \(\square A + \Omega(A) - d \delta A = 0\). Performing \(\Delta\) on the closed \(p\)-form \(F = dA\) further yields

\[
\Delta F = (-1)^{np+n+1} O_2 dA = 0. 
\] (2.16)

This means that \(F\) is a harmonic \(p\)-form. Hence Eq. (2.13) gives the wave equation of the closed \(p\)-form

\[
\square F = - \Omega(F). 
\] (2.17)

For example, when \(F\) is a closed and co-closed 2-form \(F(2) = F_{\mu\nu} dx^{\mu\nu}/2\), that is to say, \(dF(2) = 0\) together with \(\delta F(2) = 0\), we obtain the wave equation for such a field in the tensor form [2]

\[
\square F_{\mu\nu} + R^\sigma_{\mu\nu} F_{\nu\sigma} - R^\sigma_{\nu\sigma} F_{\mu\sigma} + R^\rho_\mu F_{\rho\sigma} = 0. 
\] (2.18)

What is more, making use of Eq. (2.13), we are able to reexpress the well-known Proca equation \(\delta d A_{(1)} = m^2 A_{(1)}\), which describes the co-closed vector field \(A_{(1)} = A_\mu dx^\mu\) with mass \(m\), into the form

\[
\square A_\mu - (R^\nu_\mu + m^2 \delta^\nu_\mu) A_\nu = 0. 
\] (2.19)

Specially, when the Ricci tensor \(R_{\mu\nu} = \lambda g_{\mu\nu}\), Eq. (2.19) becomes \(\square A_{(1)} = (\lambda + m^2) A_{(1)}\).
Third, with the help of the first equation of Eq. (2.9) and Eq. (2.13), one deduces the following commutation relationship between the exterior derivative and the covariant d’Alembertian operator:

\[ [d, \Box] A = \Omega(dA) + \frac{1}{(p-2)!} \nabla_{\mu_1} \nabla^\rho \nabla_{\mu_2} A_{\rho \mu_3} \cdots \mu_p \, dx^{\mu_1 \cdots \mu_p}. \] (2.20)

For instance, when the \((p-1)\)-form \(A\) is a scalar \(\phi\), this equation leads to

\[ [\nabla_\mu, \Box^m] \phi = -R^\nu_\mu \sum_{k=0}^{m-1} \Box^k \nabla_\nu \Box^{m-k-1} \phi \]

\[ - \sum_{k=1}^{m-1} (\Box^k R^\nu_\mu) \nabla_\nu \Box^{m-k-1} \phi, \] (2.21)

where the arbitrary integer \(m \geq 1\). Apart from Eq. (2.20), the commutation relation between the codifferential \(\hat{\delta}\) and the d’Alembertian operator \(\Box\) is

\[ [\hat{\delta}, \Box^m] F = \Omega(\hat{\delta}F) - \hat{\delta} \Omega(F) \] (2.22)

on basis of their action on an arbitrary \(p\)-form \(F\). Particularly, for an arbitrary vector \(V^\mu\), \(\Omega(\hat{\delta}V) = 0\). Thus we have

\[ [\hat{\delta}, \Box^m] V = - \sum_{k=0}^{m-1} \Box^k \hat{\delta} \Omega(\Box^{m-k-1} V). \] (2.23)

Obviously, if the spacetime is Ricci-flat, \([\hat{\delta}, \Box^m] V = 0\). When \(m = 1\), the above equation becomes

\[ [\hat{\delta}, \Box] V = \frac{1}{2} V^\mu \nabla_\mu R + R^\mu_\nu \nabla(\nu V^\nu). \] (2.24)

If \(V^\mu\) is a Killing vector \((\nabla(\nu V^\nu) = 0)\), we obtain \(V^\mu \nabla_\mu R = 0\) since \(2\Box V = \hat{\delta} dV\) and \(\hat{\delta} V = 0\). That is to say, the Lie derivative of the Ricci scalar along a Killing vector field disappears.

Forth, for another interesting case of acting \(\Delta\) on an arbitrary Killing vector \(\xi^\mu\), by virtue of its null divergence \(\hat{\delta} \xi = 0\), we obtain \((d\hat{\delta})^k \xi = 0\). This implies that

\[ \Box^k \xi = \Delta^k \xi \] (2.25)

always holds true for an arbitrary nonnegative integer \(k\). As a result, application of the first equation in Eq. (2.9) and Eq. (2.13) to the computation of the above equation with \(k = 1\)
leads to the property $\square \xi = \Omega(\xi)$ or $\square \xi^\mu = -R^\mu_{\sigma\gamma} \xi^\sigma$ for Killing vectors. According to such a property, we make use of the equation \( (\Delta F_{(1)})^\mu = \square F^\mu_{(1)} - R^\mu_{\sigma\gamma} F^{\sigma\gamma}_{(1)} \) repeatedly to obtain

\[
(\Delta^1 \xi)^\mu = -2\xi^{\sigma_1} R^\mu_{\sigma_1},
\]

\[
(\Delta^2 \xi)^\mu = -2\xi^{\sigma_2} (\square R^\mu_{\sigma_2} - 2R^\mu_{\sigma_1} R^\sigma_{\sigma_1}),
\]

\[
(\Delta^3 \xi)^\mu = -2\xi^{\sigma_3} [\square R^\mu_{\sigma_3} - 3\square (R^\mu_{\sigma_2} R^\sigma_{\sigma_2}) + 4R^\mu_{\sigma_1} R^\sigma_{\sigma_2} R^\rho_{\sigma_3}].
\]  \(2.26\)

On basis of Eq. \(2.26\), one can go on doing so to get \( (\Delta^4 \xi), (\Delta^5 \xi), \cdots \). As a matter of fact, \(\Delta^k \xi\) must be of the general form \( (\Delta^k \xi)^\mu = -2\xi^{\sigma_k} X^\mu_{(k)\nu} \) where \(X^\mu_{(k)\nu}\) consists of the terms made up of \(R^\rho_{\sigma\gamma}, \square R^\rho_{\sigma\gamma}, \cdots, \square^{k-1} R^\rho_{\sigma\gamma}\). Therefore, if \(\Delta^k \xi = 0\) holds for arbitrary Killing vectors, it is demanded that \(X^\mu_{(k)\nu} = 0\), which enables one to construct equations containing higher-order derivative terms of curvature tensors. For example, in the \(k = 2\) case, we get the equation \(X^\mu_{(2)\nu} = \square R^\mu_{\sigma\nu} - 2R^\mu_{\sigma\gamma} R^\gamma_{\nu} = 0\) \([5]\). To the contrary, if the spacetime is Ricci-flat, namely, \(R_{\mu\nu} = 0\), we deduce that \(\Delta^k \xi = 0\) holds true for all Killing vectors. So we arrive at the conclusion that any Killing vector in Ricci-flat spacetime is harmonic.

Finally, let us summarize the main novel results obtained in this section. First, we give a generic degree-preserving combined operator \(O_{jk}\). Second, we propose a generic conserved current \(J^{(1)}\) associated with an arbitrary 1-form and a covariant divergence-free \(p\)-form \(J^{(p)}\). Third, we make use of differential forms to derive the wave equation \(2.17\) for a harmonic \(p\)-form. Fourth, we obtain the commutation relations given in Eqs. \(2.15\), \(2.21\) and \(2.23\). Fifth, the action of the operator \(\Delta^k\) on an arbitrary Killing vector is explicitly analysed.

### 3 The construction of \(n\)-forms and their properties

Within the present section, we shall utilize the operators \(O_{jk}(\alpha_{1j}, \alpha_{2k})\) in Eq. \(2.4\) to construct \(n\)-forms \(L_{\hat{m}\tilde{n}}(F, H)\) in terms of the \(p\)-form fields \(F\) and \(H\), as well as their equivalents. To understand those, we are going to investigate several special cases where the operators are specifically \(\Delta\) and \(\mathbb{P}\). Table 2 in Appendix \(A\) summarizes all the \(n\)-forms.

Let us start with the action of the operators

\[
\hat{O} = O_{ji}(\alpha_{1j}, \alpha_{2i}), \quad \tilde{O} = O_{st}(\beta_{1s}, \beta_{2t}),
\]  \(3.1\)

on the \(p\)-form fields \(F\) and \(H\). Due to the degree-preserving property of the operator \(O_{jk}\), we are able to obtain \(p\)-form fields \(\hat{O}^i F\) \((i = 0, 1, \cdots, \hat{m})\) and \(\tilde{O}^k H\) \((k = 0, 1, \cdots, \tilde{n})\), where
\( \hat{m} \) and \( \tilde{n} \) represent arbitrary non-negative integers. Consequently, the combination of these fields allows us to construct two new \( p \)-form fields \( \hat{F}_m, \hat{H}_n \in \Omega^p(M) \). Both of them are read off as

\[
\hat{F}_m = \sum_{i=0}^{\hat{m}} \gamma_i \hat{O}_i F, \quad \hat{H}_n = \sum_{k=0}^{\tilde{n}} \lambda_k \hat{O}_k H,
\]

respectively, where \( \gamma_i \)'s together with \( \lambda_k \)'s are coupling constants, and it is set that both the operators \( \hat{O}^0 \) and \( \tilde{O}^0 \) are taken to be the identity operation \( 1 \). Based upon the above two \( p \)-form fields \( \hat{F}_m \) and \( \hat{H}_n \), a general \( n \)-form \( L_{\hat{m}\tilde{n}}(F, H) \) is further defined through

\[
L_{\hat{m}\tilde{n}} = \hat{F}_m \wedge \star \hat{H}_n = \sum_{i=0}^{\hat{m}} \sum_{k=0}^{\tilde{n}} \gamma_i \lambda_k U^{ik},
\]

\( U^{ik} = \hat{O}^i F \wedge \star \hat{O}^k H. \)

(3.3)

The above \( L_{\hat{m}\tilde{n}} \) is one of our main desired results in the present work. It is worth noting that the motivation to adopt \( U^{ik} \) in the construction of the \( n \)-form \( L_{\hat{m}\tilde{n}}(F, H) \) mainly stems from the fact that \( U^{ik} \) covers certain Lagrangians associated with gauge fields. Apart from \( U^{ik} \), one may wonder whether \( \hat{O}_i F \wedge \tilde{O}_k H \) can be adopted to construct the \( n \)-form if \( H \) is a \( q \)-form. The answer is yes. This is attributed to the fact that \( \hat{O}_i F \wedge \tilde{O}_k H = (-1)^{pq+1} \hat{O}_i F \wedge \star \hat{O}_k (\beta_{2t}, \beta_{1s}) \tilde{H} \), where the \( p \)-form \( \tilde{H} = \star H \) is the Hodge duality of \( H \). Obviously, \( \hat{O}_i F \wedge \tilde{O}_k H \) can be recast into the form \( (-1)^{pq+1} \hat{O}_i F \wedge \star \hat{O}_k \tilde{H} \) if the operator \( \hat{O} \) is redefined as \( \hat{O} = \hat{O}_t (\beta_{2t}, \beta_{1s}) \). To this point, \( \hat{O}_i F \wedge \tilde{O}_k H \) is equivalent to \( U^{ik} \). What is more, the \( n \)-form \( U^{ik} \) can be alternatively defined via \( U^{ik} \rightarrow \tilde{U}^{ik} \), where \( \tilde{U}^{ik} = \hat{O}_k H \wedge \star \hat{O}_i F \).

In accordance with \( L_{\hat{m}\tilde{n}} = L_{\tilde{m}\hat{n}} \epsilon \), one obtains

\[
L_{\hat{m}\tilde{n}} = \sum_{i=0}^{\hat{m}} \sum_{k=0}^{\tilde{n}} \gamma_i \lambda_k U^{ik}.
\]

(3.4)

Here the scalar (or inner) product \( U^{ik} \) between two differential \( p \)-forms \( \hat{O}_i F \) and \( \tilde{O}_k H \) is defined through the contraction of their components, namely,

\[
U^{ik} = \langle \hat{O}_i F \cdot \tilde{O}_k H \rangle = \frac{1}{p!} (\hat{O}_i F)_{\mu_1 \cdots \mu_p} (\tilde{O}_k H)_{\mu_1 \cdots \mu_p}.
\]

(3.5)

Obviously, \( U^{ik} = U^{ki} \) if \( \hat{O} = \tilde{O} \) and \( H = F \).

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\(^2\)We thank the anonymous referee for putting forward this question.
Next, for the purpose of simplicity, we attempt to construct an alternative but equivalent formulation of the $n$-form $L_{\hat{m}\tilde{n}}(F, H)$. As a warmup, we take into consideration of an $n$-form $dB \wedge \star H$. Here and in what follows, $B$ denotes an arbitrary $(p - 1)$-form $B$. With the help of the codifferential, the $n$-form $dB \wedge \star H$ can be expressed as

$$dB \wedge \star H = -B \wedge \star \delta H + d(B \wedge \star H),$$

or equivalently

$$\langle dB \cdot H \rangle = -\langle B \cdot \delta H \rangle + \frac{1}{(p - 1)!} \nabla^{\mu_1} \cdots \nabla^{\mu_p} (H_{\mu_1} \cdots \mu_p B^{\mu_2 \cdots \mu_p}).$$

(3.7)

From Eq. (3.6), one observes that $dB \wedge \star H$ differs from $-B \wedge \star \delta H$ only by an exact form or a total derivative term. As a consequence of Eq. (3.7), one gets

$$\hat{\delta} dF \wedge \star H = F \wedge \star \hat{\delta} dH + d\Theta_1,$$

$$d \hat{\delta} F \wedge \star H = F \wedge \star d \hat{\delta} H + d\Theta_2,$$

(3.8)

with the boundary terms $\Theta_1, \Theta_2 \in \Omega^{n-1}(M)$ given by

$$\Theta_1 = H \wedge \star dF - F \wedge \star dH,$$

$$\Theta_2 = \hat{\delta} F \wedge \star H - \hat{\delta} H \wedge \star F,$$

(3.9)

respectively. According to Eq. (3.8), it is easy to check that both the $n$-forms $\hat{\delta} dF \wedge \star \hat{\delta} H$ and $d \hat{\delta} F \wedge \star d \hat{\delta} H$ are exact forms by virtue of $d \star \hat{\delta} = 0$. Furthermore, computations based upon Eq. (3.8) give rise to

$$\Delta F \wedge \star H = -dF \wedge \star dH - \delta F \wedge \star \delta H$$

$$+ d(H \wedge \star dF + \hat{\delta} F \wedge \star H)$$

$$= F \wedge \star \Delta H + d(\Theta_1 + \Theta_2),$$

(3.10)

as well as

$$\langle O_a^i F \cdot O_b^k H \rangle = \delta_{ab} \langle F \cdot O_a^{i+k} H \rangle + \nabla^\mu (\bullet)_{\mu},$$

(3.11)

where $a, b = 1, 2$ and $(\bullet)_{\mu}$ stands for the total derivative term. Eq. (3.10) implies that $\Delta F \wedge \star H$ must be an exact form provided that $H$ is a harmonic $p$-form.

Subsequently, with the help of Eq. (3.11), the $n$-form $L_{\hat{m}\tilde{n}}(F, H)$ can be recast into the following equivalent form:

$$L_{\hat{m}\tilde{n}} = \tilde{L}_{\hat{m}\tilde{n}} + d\Theta_{(n-1)},$$

(3.12)
in which the \((n-1)\)-form \(\Theta_{(n-1)}\) represents some boundary term while the \(n\)-form \(\mathbf{L}_{\tilde{m}\tilde{n}}(\mathbf{F}, \mathbf{H})\) is given by

\[
\mathbf{L}_{\tilde{m}\tilde{n}} = \sum_{i=0}^{\tilde{m}} \rho_i \mathbf{U}_\Delta^i + \sum_{k=0}^{\tilde{n}} \sigma_k \mathbf{U}_P^k
\]

(3.13)

with the \(n\)-forms \(\mathbf{U}_\Delta^i\) and \(\mathbf{U}_P^k\) defined through

\[
\mathbf{U}_\Delta^i = \mathbf{F} \wedge \star \Delta^i \mathbf{H},
\]

\[
\mathbf{U}_P^k = \mathbf{F} \wedge \star P^k \mathbf{H},
\]

(3.14)

respectively, where \(\rho_i\)'s and \(\sigma_k\)'s are constant parameters. Eq. (3.12) shows that \(\mathbf{L}_{\tilde{m}\tilde{n}}\) is determined by \(\mathbf{L}_{\tilde{m}\tilde{n}}\) up to a surface term only. As a consequence, ignoring the contribution from such a term, one can utilize \(\mathbf{L}_{\tilde{m}\tilde{n}}\) as an alternative but equivalent form of \(\mathbf{L}_{\tilde{m}\tilde{n}}\). Apparently the former has the great advantage of simplicity. What is more, when \(\mathbf{F} = \mathbf{H}\), under the transformation \(\mathbf{F} \rightarrow \mathbf{F} + d \mathbf{Y}\), where \(\mathbf{Y}\) is an arbitrary \((p-1)\)-form, one observes that the \(n\)-forms \(\mathbf{U}_\Delta^i(\mathbf{F}, \mathbf{F})\) and \(\mathbf{U}_P^k(\mathbf{F}, \mathbf{F})\) transform as

\[
\mathbf{U}_\Delta^i \rightarrow \mathbf{U}_\Delta^i + (2\mathbf{F} + d \mathbf{Y}) \wedge \star d \mathbf{P}^i \mathbf{Y} + d(\bullet),
\]

\[
\mathbf{U}_P^k \rightarrow \mathbf{U}_P^k + d(\bullet),
\]

(3.15)

respectively. If further provided that \(\mathbf{P} \mathbf{Y} = 0\), Eq. (3.15) leads to that \(\mathbf{L}_{\tilde{m}\tilde{n}}(\mathbf{F}, \mathbf{F})\) behaves like \(\mathbf{L}_{\tilde{m}\tilde{n}} \rightarrow \mathbf{L}_{\tilde{m}\tilde{n}} + d(\bullet)\) under the aforementioned transformation.

Finally, in order to illustrate the structure of the \(n\)-form \(\mathbf{L}_{\tilde{m}\tilde{n}}(\mathbf{F}, \mathbf{H})\), we move on to take into account three typical cases where the operators \(\hat{\mathbf{O}}\) and \(\tilde{\mathbf{O}}\) take the specific values \(\Delta\) and \(\mathbf{P}\). First, let \(\hat{\mathbf{O}} = \Delta = \tilde{\mathbf{O}}\). In such a case, \(\mathbf{U}^{ik}\) is denoted by the notation \(\mathbf{U}_\Delta^{ik}\), taking the form

\[
\mathbf{U}_\Delta^{ik} = \Delta^i \mathbf{F} \wedge \star \Delta^k \mathbf{H} = \mathbf{U}_\Delta^{i+k} + d \Theta_\Delta^{ik},
\]

(3.16)

with the \(n\)-form \(\mathbf{U}_\Delta^{i+k} = \mathbf{F} \wedge \star \Delta^{i+k} \mathbf{H}\) and the surface term \(\Theta_\Delta^{ik}\) defined through

\[
\Theta_\Delta^{ik} = \sum_{j=1}^{i} \left( \hat{\Theta}_\Delta^{ik,j} + \tilde{\Theta}_\Delta^{ik,j} \right),
\]

(3.17)

where the \((n-1)\)-forms \(\hat{\Theta}_\Delta^{ik,j}\) and \(\tilde{\Theta}_\Delta^{ik,j}\) \((1 \leq j \leq i)\), derived according to Eq. (3.15) or
are presented by

$$
\Theta_{\Delta}^{ik,j} = + \Delta^{k+j-1}H \wedge \ast d\Delta^{i-j}F
- \Delta^{i-j}F \wedge \ast d\Delta^{k+j-1}H,
$$

$$
\Theta_{\Delta}^{ik,j} = + \delta \Delta^{i-j}F \wedge \ast \Delta^{k+j-1}H
- \delta \Delta^{k+j-1}H \wedge \ast \Delta^{i-j}F,
$$

(3.18)

respectively. Neglecting surface terms, one straightforwardly finds that \(L_{\tilde{m}\tilde{n}}\) is equivalently described by

$$
\tilde{L}_{\Delta}^{\tilde{m}} = \sum_{i=0}^{\tilde{m}} \rho_i \tilde{U}^i_{\Delta}.
$$

(3.19)

Particularly, one obtains \(U_{\Delta}^{ik} = \langle \Box^i F \cdot \Box^k H \rangle\) when the spacetime is Minkowskian. With imposition of the condition that \(F = dA\), Eq. \(2.16\) shows that performing \(\Delta^i\) upon \(F\) yields \(\Delta^i F = (-1)^{i+p-i+n+1}O_2^j dA\), demonstrating that the operator \(O_2\) acts on \(F\) only. Therefore, due to Eq. \(3.6\), the \(n\)-form \(U_{\Delta}^{ik}\) may be expressed as

$$
U_{\Delta}^{ik} = - \Delta^i A \wedge \ast \mathbb{P}^{k} \hat{\delta} H + d(\Delta^i A \wedge \ast \Delta^k H),
$$

(3.20)

or equivalently

$$
U_{\Delta}^{ik} = - \langle \Delta^i A \cdot \mathbb{P}^{k} \hat{\delta} H \rangle + \nabla^{\mu_1} B_{\mu_1}^{ik},
$$

\(B_{\mu_1}^{ik} = \frac{1}{(p-1)!}(\Delta^k H)_{\mu_1}^{\mu_2 \cdots \mu_p}(\Delta^i A)^{\mu_2 \cdots \mu_p}.\)

(3.21)

For concreteness, considering as a simple example the \(i, k = 0\) and \(H = F\) case of Eq. \(3.20\), we obtain

$$
U_{\Delta}^{00}(F, F) = \frac{1}{p!} F_{\mu_1 \cdots \mu_p} F^{\mu_1 \cdots \mu_p}
= - \langle A \cdot \mathbb{P} A \rangle + \nabla^{\mu_1} B_{\mu_1}^{00},
$$

(3.22)

which implies that our familiar \(n\)-form \(F \wedge \ast F\) can also be expressed with respect to the operators \(O_1\) and \(O_2\). Second, in analogy to the above-mentioned case, we take into account of replacing both the operators \(\hat{O}\) and \(\tilde{O}\) in Eq. \(3.3\) by the operator \(\mathbb{P}\). In this case, the \(n\)-form \(U_{\Delta}^{ik}\) in Eq. \(3.3\) is of the form

$$
U_{\mathbb{P}}^{ik} = \mathbb{P}^{i} F \wedge \ast \mathbb{P}^{k} H
= \tilde{U}_{\mathbb{P}}^{i+k} + d\Theta_{\mathbb{P}}^{ik},
$$

$$
\Theta_{\mathbb{P}}^{ik} = \sum_{j=1}^{i} \tilde{\Theta}_{\Delta}^{ik,j} (\Delta \rightarrow \mathbb{P}),
$$

(3.23)
where the \( n \)-form \( \tilde{U}_{\Delta}^{i+k} \) can be read off from Eq. (3.14), that is, \( \tilde{U}_{\Delta}^{i+k} = F \wedge \star \delta P^{i+k} H \). To obtain the surface term \( \Theta_{\Delta}^{ik} \), Eq. (3.8) has been used. Here note that \( \Theta_{\Delta}^{ik} \) together with \( \Theta_{\Delta} \) could be used to express the surface term \( \Theta_{(n-1)} \) given by Eq. (3.12). Third, in the case where \( \tilde{O} = \Delta \) and \( \tilde{O} = \mathbb{P} \) (or \( \tilde{O} = \mathbb{P}, \tilde{O} = \Delta \)), we have \( U^{ik} = U^{ik}_{\Delta,\mathbb{P}} \) (or \( U^{ik} = U^{ik}_{\mathbb{P},\Delta} \)). Both of them are written as

\[
U_{\Delta,\mathbb{P}}^{ik} = \Delta^{i} F \wedge \star \delta P^{k} H ,
\]

\[
U_{\mathbb{P},\Delta}^{ik} = \mathbb{P}^{i} F \wedge \star \delta \Delta^{k} H .
\] (3.24)

In comparison with Eq. (3.23), both the \( n \)-forms \( U_{\Delta,\mathbb{P}}^{ik} \) and \( U_{\mathbb{P},\Delta}^{ik} \) differ from \( U_{\mathbb{P}}^{ik} \) only by an exact form, that is,

\[
U_{\Delta,\mathbb{P}}^{ik} = U_{\mathbb{P}}^{ik} + d(\mathbb{P}^{i-1} \delta F \wedge \star \delta P^{k} H) ,
\]

\[
U_{\mathbb{P},\Delta}^{ik} = U_{\mathbb{P}}^{ik} + d(\mathbb{P}^{k-1} \delta H \wedge \star \delta F) .
\] (3.25)

This means that \( U_{\Delta,\mathbb{P}}^{ik} \) and \( U_{\mathbb{P},\Delta}^{ik} \) could be derived from \( U_{\mathbb{P}}^{ik} \), verifying the fact that the \( n \)-form \( \tilde{L}_{\tilde{m}\tilde{n}}(F, H) \) given by Eq. (3.13) is only dependent on \( \tilde{U}_{\mathbb{i}}^{i} \) and \( \tilde{U}_{\mathbb{P}}^{i} \). For Eqs. (3.23) and (3.24), obviously, if it is assumed that \( F = dA \) and \( H = dB \) hold, the \( n \)-forms \( U_{\mathbb{P}}^{ik}, U_{\Delta,\mathbb{P}}^{ik} \) and \( U_{\mathbb{P},\Delta}^{ik} \) disappear. What is more, comparing Eq. (3.20) with Eq. (3.24), we establish the following expression:

\[
U_{\Delta}^{ik}(F, H) = -U_{\Delta,\mathbb{P}}^{ik+1}(A, B) + d(\Delta^{i} A \wedge \star \delta P^{k} B)
\]

\[= -U_{\mathbb{P}}^{ik+1}(A, B) + d\Theta_{AB}^{ik} ,
\] (3.26)

with the surface term \( \Theta_{AB}^{ik} = \Delta^{i} A \wedge \star dP^{k} B - \mathbb{P}^{i-1} \Delta A \wedge \star \delta \mathbb{P}^{k+1} B \). In light of Eqs. (3.25) and (3.26), regardless of the total derivative term, we come to the conclusion that \( U_{\Delta}^{ik}, U_{\mathbb{P}}^{ik+1}, U_{\Delta,\mathbb{P}}^{ik+1} \) and \( U_{\mathbb{P},\Delta}^{ik+1} \) are naturally equivalent with each other.

With the \( n \)-form fields \( U_{\Delta}^{ik}, U_{\mathbb{P}}^{ik}, U_{\Delta,\mathbb{P}}^{ik} \) and \( U_{\mathbb{P},\Delta}^{ik} \) in hand, actually, it is completely feasible to express the general \( n \)-form \( U^{ik} \) associated with the operators \( \tilde{O} \) and \( \tilde{O} \) through those fields. This is a direct consequence of the fact that the operators \( O_{1}^{k} = (-1)^{npk+k} P^{k} \) and \( O_{2}^{k} = (-1)^{npk+nk+k}(\Delta^{k} - P^{k}) \). Therefore, in order to get the \( n \)-form \( L_{\tilde{m}\tilde{n}} \), one merely needs to carry out computations for \( U_{\Delta}^{ik}, U_{\mathbb{P}}^{ik}, U_{\Delta,\mathbb{P}}^{ik} \) and \( U_{\mathbb{P},\Delta}^{ik} \) alternatively, while the latter two could be derived from \( U_{\mathbb{P}}^{ik} \) in virtue of Eq. (3.25). What is more, under the condition that the contributions from the surface terms could be neglected, the equivalence between \( L_{\tilde{m}\tilde{n}} \) and \( \tilde{L}_{\tilde{m}\tilde{n}} \) guarantees that it is only necessary to evaluate \( \tilde{U}_{\Delta}^{i} \) and \( \tilde{U}_{\mathbb{P}}^{k} \). If so, the calculations are simplified notably.
4 A potential application to the construction of an extended Lagrangian associated to a \( p \)-form

In this section, as an application of the aforementioned \( n \)-form \( L_{\hat{m}\hat{n}}(F, H) \), we shall propose a Lagrangian associated with an arbitrary \( p \)-form field \( A_p \). For simplicity, we will only focus on the detailed analysis about its equivalent \( \tilde{L}_{\hat{m}\hat{n}}(F, H) \) given by Eq. (3.13) instead of \( L_{\hat{m}\hat{n}} \). More specifically, it is merely required to take into consideration of the \( n \)-forms \( \tilde{U}_\Delta \) and \( \tilde{U}_\delta^k \). According to such a Lagrangian, we are going to derive the equations of motion with respect to the fields.

As is well-known in the framework of gauge theory, for the Abelian \((p-1)\)-form gauge field \( A \) with the field strength \( F = dA \), one popular form of its Lagrangian can be presented by means of

\[
L_{00} = \gamma_0 \lambda_0 F \wedge \star F,
\]

which can be viewed as a generalization of the ordinary Lagrangian describing Maxwell’s theory of electromagnetism in four-dimensional Minkowski spacetime and with the electromagnetic four-potential \( A_\mu \). Further regardless of the surface term in Eq. (3.22), which makes no contribution to the equation of motion for the gauge field, one is able to reformulate the Lagrangian (4.1) as

\[
\begin{align*}
L_{00} &= -\gamma_0 \lambda_0 A \wedge (\star \overline{P} A), \\
L_{00} &= -\gamma_0 \lambda_0 A \wedge (\star \Delta A).
\end{align*}
\]

In order to get the second equation in Eq. (4.2), we have imposed the constraint that the field \( A \) satisfies the Lorentz gauge condition that \( A \) is co-closed, namely, \((\hat{\delta} A)_{\mu_3 \cdots \mu_p} = \nabla^\nu A_{\mu_3 \cdots \mu_p} = 0\), or we apply Eq. (3.10) to modify the Lagrangian (4.1) as the one

\[
\tilde{L}_{00} = \gamma_0 \lambda_0 \left( dA \wedge \star dA + \hat{\delta} A \wedge \star \hat{\delta} A \right),
\]

which covers the gauge-fixed Lagrangians within the Minkowskian spacetime manifold in [13]. Obviously, Eq. (4.2) demonstrates that the usual Lagrangian for the \((p-1)\)-form \( A \) can be described by the \( n \)-form \( \tilde{L}_{\hat{m}\hat{n}}(A, A) \) as well. In comparison with Eq. (4.1), as we shall see later, it is of great convenience to derive the equations of motion for the field in terms of the form of the Lagrangian given by Eq. (4.2). What is more, for the well-known Lagrangian \( L(\phi) \) with respect to the scalar field \( \phi \), usually given by \( L(\phi) = \sqrt{-g} \nabla^\mu \phi \nabla_\mu \phi \),
it could be equivalently of the form

$$L(\phi) = -U^0_\phi(\phi, \phi)$$

$$= -\phi(\star \mathcal{P} \phi) = -\phi(\star \Delta \phi)$$

(4.4)

without consideration of the contribution from the boundary term. That is to say, in analogy to the Lagrangian of the field $A$, the one for the scalar field might also be regarded as a special case of $L_{\bar{m} \bar{n}}$.

As a consequence, motivated by Eqs. (4.1), (4.2) and (4.4), here we propose that the $n$-form $L_{\bar{m} \bar{n}}(\mathcal{P}, H)$ with $\mathcal{P} = A_{(p)} = H$ could be a generalized Lagrangian associated with the $p$-form field $A_{(p)}$ from the mathematical point of view. In comparison, $L_{\bar{m} \bar{n}}(A_{(p)}, A_{(p)})$ makes such extensions $\Delta \rightarrow \Delta^i$ and $\mathcal{P} \rightarrow \mathcal{P}^k$ to the Lagrarians (4.2) and (4.4). Since the boundary $\Theta_{(p-1)}$ in Eq. (3.12) makes no contribution to the equations of motion for the fields, it is completely advisable for us to adopt the more convenient $n$-form $\tilde{L}_{\bar{m} \bar{n}}(A_{(p)}, A_{(p)})$ presented by Eq. (3.13) rather than $L_{\bar{m} \bar{n}}(A_{(p)}, A_{(p)})$ as the form of the Lagrangian.

In the remainder of this section, on basis of the Lagrangian $\tilde{L}_{\bar{m} \bar{n}}(A_{(p)}, A_{(p)})$, we take into account of the derivation for the Euler-Lagrange equation of motion associated with the $p$-form $A_{(p)}$. To do this, on one hand, via varying the $n$-form $\tilde{U}^i_\Delta(A_{(p)}, A_{(p)})$ with respect to $A_{(p)}$, we obtain

$$\delta \tilde{U}^i_\Delta = 2\delta A_{(p)} \wedge \star \Delta^i A_{(p)} + \sum_{j=1}^i d\Psi^i_\Delta,$$

(4.5)

where the surface terms $\Psi^i_\Delta$ are given by

$$\Psi^i_\Delta = +\Delta^i \delta A_{(p)} \wedge \star \Delta^i \delta A_{(p)}$$

$$-\Delta^i \delta A_{(p)} \wedge \star \Delta^i \delta A_{(p)}$$

$$+\delta \Delta^i \delta A_{(p)} \wedge \star \Delta^i \delta A_{(p)}$$

$$-\delta \Delta^i \delta A_{(p)} \wedge \star \Delta^i \delta A_{(p)}.$$

(4.6)

If $\delta \tilde{U}^i_\Delta = 0$, we get the equations of motion $\Delta^i A_{(p)} = 0$, demonstrating that a simple solution is the closed and co-closed form $A_{(p)}$ satisfying $dA_{(p)} = 0 = \delta A_{(p)}$. For instance, when $i = 1$, we have the field equation for the 2-form $A_{(2)}$ with the help of Eq. (2.13), that is,

$$\Box A_{\mu \nu} = R^\rho_{\mu \nu} A_{\sigma \nu} - R^\rho_{\nu \sigma} A_{\mu \rho} - R^m_{\mu \nu \rho} A_{\rho \sigma},$$

(4.7)
which could be interpreted as the field equation associated with the Kalb-Ramond Lagrangian $L_{KR} = dA(2) \wedge \star dA(2)/2$ \[21\] under the divergence-free gauge condition $\delta A(2) = 0$, and when $i = 2$, the equation of motion for the 1-form $A(1)$ is read off as
\[
\Box^2 A_\mu = 2R_\sigma^\mu \Box A_\sigma + A_\sigma \Box R_\mu^\sigma - R_\mu^\sigma R_\rho^\sigma A_\sigma.
\] (4.8)

On the other hand, in an analogous way to the above-mentioned analysis on $\delta U^i_\Delta$, let us deal with the variation of the $n$-form $\tilde{U}^k_F(A(p), A(p))$ with respect to the field $A(p)$. This gives rise to
\[
\delta \tilde{U}^k_F = 2\delta A(p) \wedge \star \tilde{\mathbb{P}}^k A(p) + \sum_{j=1}^k d\Psi^{kj}_F,
\] (4.9)
with the boundary terms $\Psi^{kj}_F$ defined through
\[
\Psi^{kj}_F = +\tilde{\mathbb{P}}^{j-1} A(p) \wedge \star \tilde{\mathbb{P}}^{k-j} \delta A(p) - \tilde{\mathbb{P}}^{k-j} \delta A(p) \wedge \star \tilde{\mathbb{P}}^{j-1} A(p).
\] (4.10)

Similarly, $\delta \tilde{U}^k_F = 0$ yields the equations of motion $\mathbb{P}^k A(p) = 0$, according to which one gains $\mathbb{P}^{p-1} \delta F_{(p+1)} = 0$ and $\Delta^k F_{(p+1)} = 0$, where the closed $(p + 1)$-form $F_{(p+1)} = dA(p)$. Letting us specialize to the case where $k = 1$ and $A(p)$ is the aforementioned $(p - 1)$-form $A$, we observe that
\[
(\mathbb{P}A)_{\mu_2 \cdots \mu_p} = \nabla^{\mu_1} F_{\mu_1 \cdots \mu_p} = 0,
\] (4.11)
which is just the field equation relevant for the Lagrangian (4.1). Here $F = dA$ as before. In the assumption that the Lorentz gauge condition $\delta A = 0$ holds, Eq. (4.11) further transforms to $\Delta A = 0$ or $\Box A_\mu = R_\sigma^\mu A_\sigma$.

According to the fact that the Lagrangian $\tilde{L}_{\tilde{m}\tilde{n}}(A(p), A(p))$ is the linear combination of the $n$-forms $\tilde{U}^j_\Delta(A(p), A(p))$ and $\tilde{U}^k_F(A(p), A(p))$, we find that the variations of $\tilde{U}^j_\Delta$ and $\tilde{U}^k_F$ are sufficient for the derivation of the field equations related to the Lagrangian $\tilde{L}_{\tilde{m}\tilde{n}}(A(p), A(p))$. As a result, we make use of Eqs. (4.5) and (4.9) to vary $\tilde{L}_{\tilde{m}\tilde{n}}(A(p), A(p))$ with respect to the field $A(p)$ and obtain the equation of motion
\[
\sum_{i=0}^{\tilde{n}} \rho_i \Delta^i A(p) + \sum_{k=0}^{\tilde{n}} \sigma_k \tilde{\mathbb{P}}^k A(p) = 0.
\] (4.12)

Apparently, the left hand side of the above equation results from the linear combination of the $p$-forms $\Delta^i A(p)$ and $\tilde{\mathbb{P}}^k A(p)$. 

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Ultimately, we have to point out that it is of great importance to verify the stability with respect to the Lagrangian \( \tilde{L}_{\tilde{m}\tilde{n}}(A_{(p)}, A_{(p)}) \). In fact, since this Lagrangian includes terms with higher-order time derivatives, it generally encounters the ghost-like instability referred to as the Ostrogradsky instability, which is a linear instability existing in the Hamiltonian associated with a non-degenerate Lagrangian containing time derivative terms higher than the first order [22]. As is known, for the purpose of constructing well-behaved scalar-tensor theories involved in higher-order derivatives, several approaches have been proposed to avoid the Ostrogradsky instabilities of these theories (see works [23, 24, 25], for instance). Such methods maybe provide some insight into the avoidance of the linear instability of the Hamiltonian related to the Lagrangian \( \tilde{L}_{\tilde{m}\tilde{n}} \), which is left for future work.

5 Conclusions

In the present paper, we have systematically investigated the properties of the operators that are able to generate differential forms maintaining the invariance of the degree for an arbitrary \( p \)-form field, as well as their applications in constructions of \( n \)-forms and Lagrangians associated with \( p \)-form fields. More explicitly, through the linear combination of the operators \( O_1 \) and \( O_2 \), which are made up of the Hodge star operation and the exterior derivative, we have obtained the general operator \( O_{jk}(\alpha_{1j}, \alpha_{2k}) \) given by Eq. (2.4) or (2.12). With the help of \( O_1 \), a new conserved \( p \)-form \( J_{(p)} \) is presented in Eq. (2.11). In order to understand the operators preserving the form degree, we have given detailed analysis on the Laplace-de Rham operator \( \Delta \), as well as commutation relations between two operations and the action of \( \Delta^k \) on an arbitrary Killing vector. Particularly, the concrete relationship between the Laplace-de Rham operator and the d’Alembertian operator has been established through Eq. (2.13). Subsequently, in terms of the actions of the operator \( O_{jk} \) on the two \( p \)-forms \( F \) and \( H \), the general \( p \)-forms \( \tilde{F}_{\tilde{m}} \) and \( \tilde{H}_{\tilde{n}} \) have been presented via Eq. (3.2). Based upon them, the general \( n \)-form \( L_{\tilde{m}\tilde{n}}(F, H) \) in Eq. (3.3) has further been constructed, and its three special cases where the operators are specifically the ones \( \Delta \) and \( P \) have been discussed in detail. As a matter of fact, we have demonstrated that \( L_{\tilde{m}\tilde{n}} \) can be expressed as an alternative but equivalent \( n \)-form \( \tilde{L}_{\tilde{m}\tilde{n}}(F, H) \) given by Eq. (3.13) without the contribution from the total derivative term.

Finally, inspired by the forms (4.2) and (4.4) of the usual Lagrangians for the \( p \)-forms

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\footnote{We thank the anonymous referee for pointing out this.}
and scalar fields, we suggest that the $n$-form $L_{\tilde{m}\tilde{n}}(A_{(p)}, A_{(p)})$ or $\tilde{L}_{\tilde{m}\tilde{n}}(A_{(p)}, A_{(p)})$, including the higher-order derivative $p$-forms $\Delta^i A_{(p)}$ and $\mathbb{P}^k A_{(p)}$, could be thought of as a higher-order derivative generalization of the usual Lagrangian related to the $p$-form $A_{(p)}$ in the mathematical point of view. In terms of the Lagrangian, we have derived the equations of motion for the fields by varying the $n$-forms $\tilde{U}_\Delta^i (A_{(p)}, A_{(p)})$, $\tilde{U}_{\mathbb{P}}^k (A_{(p)}, A_{(p)})$ and $\tilde{L}_{\tilde{m}\tilde{n}}(A_{(p)}, A_{(p)})$ with respect to $A_{(p)}$. However, apart from the mathematical aspects of the extended Lagrangian $\tilde{L}_{\tilde{m}\tilde{n}}(A_{(p)}, A_{(p)})$, it is of great importance to seek the physical understandings behind the Lagrangian, for instance, the Ostrogradsky instability arising from the higher-order time derivatives in the Lagrangian. This remains to be investigated in the future research. Besides, the potential applications of the conserved $p$-form $J_{(p)}$ in theories involving higher derivatives of fields are deserved to be investigated in future.

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A Notations and conventions

Throughout this paper, the positive integer $n$ represents the dimension of the spacetime. The non-negative integers $p$ and $q = n - p$ stand for the form degree. As usual, the tensor (or spacetime coordinate) indices will be labeled by the Greek letters $\mu, \nu, \mu_1, \nu_1, \mu_2, \nu_2, \cdots, \rho, \sigma$. Each of them runs from 0 to $(n - 1)$. All the Latin indices $i, j, k, l, s, t, m, \tilde{m}$ and $\tilde{n}$ are non-negative integers, and they will be used to represent exponents of the operators and labels. The quantities $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \lambda, \chi, \rho_i$ and $\sigma_k$ denote arbitrary constant parameters. Like in [3], we use boldface letters to denote differential forms to avoid confusion with functions.

Table displays the definitions for the main operators appearing in this paper.
Table 1: Directory of operators

| Operator | Definition |
|----------|------------|
| ⋆ | Hodge star, given by Eq. (2.2) |
| d | exterior derivative |
| ˆδ | codifferential: $(-1)^{np+n+1}d*$ |
| ∆ | Laplace-de Rham: $\hat{\delta}d + d\hat{\delta}$ |
| □ | d'Alembertian: $g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ |
| P | $\hat{\delta}d$ |
| O₁ | $\star d \star d$ |
| O₂ | $d \star d\star$ |
| Oₖ | $\alpha_{1j}O^l_1 + \alpha_{2k}O^k_2$ |
| O | $\alpha_{1j}O^l_1 + \alpha_{2l}O^l_2$ |
| ˜O | $\beta_{1s}O^s_1 + \beta_{2t}O^t_2$ |

Through the action of the degree-preserving operators on the $p$-forms $F$ and $H$, we have obtained some novel $n$-forms, which are displayed by Table 2.

Table 2: Definitions for $n$-forms

| $n$-form | Expression |
|----------|------------|
| $U^{ik}$ | $\hat{O}^iF \wedge \star \hat{O}^kH$ |
| $\hat{U}^i_\Delta$ | $F \wedge \star \Delta^iH$ |
| $\hat{U}^k_p$ | $F \wedge \star \bar{P}^kH$ |
| $U^{ik}_\Delta$ | $\Delta^iF \wedge \star \Delta^kH$ |
| $U^{ik}_p$ | $\bar{P}^iF \wedge \star \bar{P}^kH$ |
| $U^{ik}_{\Delta,p}$ | $\Delta^iF \wedge \star \bar{P}^kH$ |
| $U^{ik}_{p,\Delta}$ | $\bar{P}^iF \wedge \star \Delta^kH$ |
| $L_{\hat{m}\hat{n}}$ | $\sum_{k=0}^{\hat{n}} \gamma_k \lambda_k U^{ik}$ |
| $\hat{L}_{\hat{m}\hat{n}}$ | $\sum_{i=0}^{\hat{m}} \rho_i \hat{U}^i_\Delta + \sum_{k=0}^{\hat{n}} \sigma_k \hat{U}^k_p$ |
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