TORSIONAL NODELESS VIBRATIONS OF A QUAKING NEUTRON STAR RESTORED BY THE COMBINED FORCES OF SHEAR ELASTIC AND MAGNETIC FIELD STRESSES

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ABSTRACT

Within the framework of Newtonian magneto-solid mechanics, relying on equations appropriate for a perfectly conducting elastic continuous medium threaded by a uniform magnetic field, the asteroseismic model of a neutron star undergoing axisymmetric global torsional nodeless vibrations under the combined action of Hooke’s elastic and Lorentz magnetic forces is considered with emphasis on a toroidal Alfvén mode of differentially rotational vibrations about the dipole magnetic-moment axis of the star. The obtained spectral equation for frequency is applied to ℓ-pole identification of quasi-periodic oscillations (QPOs) of the X-ray flux during the giant flares of SGR 1806-20 and SGR 1900+14. Our calculations suggest that detected QPOs can consistently be interpreted, within the framework of this model, as being produced by global torsional nodeless vibrations of a quaking magnetar if they are considered to be restored by the joint action of bulk forces of shear elastic and magnetic field stresses.

Key words: stars: individual (SGR 1900+14, SGR 1806-20) – stars: neutron – stars: oscillations

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1. INTRODUCTION

The recent detection of quasi-periodic oscillations (QPOs) on the light-curve tails of X-ray flaring SGR 1806-20 and SGR 1900+14 (Israel et al. 2005; Watts & Strohmayer 2006) that have been ascribed to torsional seismic vibrations of a quaking magnetar raises several questions of general interest for the asteroseismology of degenerate solid stars, namely, whether these differentially rotational oscillations are predominately of an elastic nature (that is, restored by Hooke’s force of shear solid-mechanical stresses) or should be thought of as the toroidal magnetic Alfvén mode of axisymmetric vibrations about the dipole magnetic moment axis of the star and restored by the Lorentz force of magnetic field stresses (Glampedakis et al. 2006). Also, it remains questionable whether these vibrations are of global character, that is, excited in the entire volume, or can be explained as locked in the peripheral finite-depth crustal region of a neutron star. These and related issues are currently the subject of intense theoretical investigations by classical theory methods of material continua (Piro 2005; Levin 2007; Lee 2008; Baturkov et al. 2007a, 2007b, 2008a) and by general relativity methods (Sotani et al. 2008; Samuelson & Andersson 2007).

In this paper, continuing the above investigation, we study an asteroseismic model of a neutron star with uniform internal and dipolar external magnetic field undergoing quake-induced global differentially rotational, torsional, vibrations about the dipole magnetic moment axis of the star under the joint action of the elastic Hooke’s and the magnetic Lorentz forces. In doing so, we confine our attention to the regime of an extremely long wavelength of differentially rotational fluctuations of material displacements about a motionless stationary state, which are insensitive to core–crust compositional stratification of the quaking neutron star. The characteristic feature of this regime is that conducting and highly robust to compressional distortions of solid-state material of both core and crust, coupled by Maxwell stresses of a uniform fossil magnetic field frozen in the star, sets in coherent axisymmetric differentially rotational vibrations with the nodeless toroidal field of material displacements identical to that for global torsional vibrations of a spherical mass of an elastic continuous medium capable of transmitting perturbation by transverse shear elastic waves that are generic to a solid state of condensed matter, not a liquid one. In the context of the above QPO problem, such vibrations have been analyzed in some detail in our recent works (Baturkov et al. 2007a, 2007b, 2008a), but the driving force was assumed to be only of elastic nature, that is, owing its origin to fluctuations in shear elastic stresses. The focus of this paper is placed, therefore, on the toroidal Alfvén vibrational mode in which the magnetic field and the field of material displacement undergo coupled fluctuations restored by the Lorentz force with Ampère’s form of the conduction current density. Before proceeding to the details of calculations it is worth noting that the liquid star model with an axisymmetric poloidal uniform magnetic field inside and dipolar field outside has been the subject of serious investigations in the past in the context of magnetic variables (e.g., Schwarzschild 1949; Ledoux & Walraven 1958). The situation may be quite different for ultra-strong internal magnetic fields that are frozen in the super dense matter of the end-products of stellar evolution, white dwarfs, and neutron stars.

In neutron stars, self-gravity is counterbalanced by the degeneracy pressure of relativistic electrons in the crust, whose
conducting matter is capable of sustaining persistent current-carrying flows, and by degeneracy pressure of nonrelativistic neutrons in the cores of neutron stars whose material can be in the state of paramagnetic magnetization caused by Pauli’s mechanism of field-induced alignment of spin magnetic moment of neutrons along the frozen-in star fossil magnetic field (e.g., Bastrukov et al. 2002a, 2002b). Of course, there is no compelling evidence that this is the case and there is certainly no general agreement on how the magnetic fields of pulsars and magnetars are produced. Over the years, a model of a neutron star with a uniform internal and dipolar external field, pictured in Figure 1, has been invoked to discuss the evolution of magnetic dipole fields of isolated radio pulsars (Flowers & Ruderman 1977), whose surface dipolar magnetic fields are found to be highly stable to spontaneous decay (Bhattacharya & van den Heuvel 1991; Chanmugam 1994), as well as the origin and evolution of ultra-strong magnetic fields of magnetars (Braithwaite & Spruit 2006; Geppert & Rheinhardt 2006; Spruit 2008). Therefore the study of axisymmetric torsional vibrations within the framework of a neutron star model with uniform internal magnetic field seems amply justified. This model, which is interesting in its own right, describes the situation when a physically meaningful analytical solution of the eigenfrequency problem can be found, and discloses mathematical difficulties one must confront when computing the frequency of Alfvén oscillations in neutron stars (Flowers & Ruderman 1977; Bhattacharya & van den Heuvel 1991; Chanmugam 1994; Spruit 2008), that is, they have been inherited from massive MS progenitors and amplified in the processes of implosive gravitational collapse proceeding under control of the magnetic-flux conservation. Also it is commonly agreed that the external magnetic field of pulsars and magnetars is of dipolar symmetry, but the geometrical shape of the fossil magnetic field frozen in the super dense motionless matter of the star remains fairly uncertain. The subject of the present paper is the formulation of a solid-mechanical variational scheme of computing the frequency of Alfvén oscillations in neutron stars an with an arbitrary configuration of the frozen-in fossil magnetostatic field.

The equation of magneto-solid mechanics describing coupled oscillations of material displacements $\mathbf{u}$ and magnetic field $\delta \mathbf{B}$ about the motionless stationary state of perfectly conducting, elastically deformable neutron star matter in the presence of the frozen-in fossil magnetic field is written as follows:$^5$

$$\delta \mathbf{E} = \frac{1}{c} \mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{B})$$

$$\delta \mathbf{E} = -\frac{1}{c} (\mathbf{\nabla} \times \mathbf{B})$$

This link between fluctuating electric field $\delta \mathbf{E}$ and constant magnetic field $\mathbf{B}$ frozen in the conducting flow follows from Ohm’s law $\delta j / \sigma = (1/c) (\mathbf{\nabla} \times \mathbf{B})$ for the extremely large, effectively infinite, electrical conductivity, $\sigma \rightarrow \infty$ (the case of the medium designated as a perfect conductor). Taking into account that $\mathbf{\nabla} \times \mathbf{u} = 0$ and eliminating the time derivative from that obtained in the above equation of the field-flow coupling $\delta \mathbf{B} = \mathbf{\nabla} \times (\delta \mathbf{\nabla} \times \mathbf{B})$, one arrives at the last equation in the line (1).

2. MAGNETIC ALFVÉN MODES OF GLOBAL NODELESS VIBRATIONS ABOUT THE AXIS OF THE MAGNETIC DIPOLE MOMENT OF A NEUTRON STAR

It is common knowledge today that, shortly before the discovery of pulsars, it had been realized that neutron stars must come into existence as compact sources of super strong dipolar magnetic fields operating as a chief promoter of their electromagnetic activity. In particular, it has been suggested that in view of the absence in neutron stars of a thermonuclear source of electromagnetic emission from the star surface, they could manifest their presence in the universe by the conversion of energy of hydromagnetic Alfvén vibrations in highly incompressible stellar matter into the energy of electromagnetic waves propagating out into the space (Hoyle et al. 1964). In this work we investigate this proposition in the context of the current development of neutron star asteroseismology by proceeding from equations of magneto-solid mechanics underlying the study of noncompressional Alfvén magnetic oscillations in quaking neutron stars. Such an approach implies that neutron star material, regarded as a perfectly conducting medium threaded by a magnetic field frozen in the star on the final stage of gravitational collapse of its massive main sequence (MS) progenitor, possesses mechanical properties of super-dense strained-solid highly robust to compressional distortions, rather than a flowing liquid, as is the case of MS stars whose theoretical asteroseismology relies on equations of fluid mechanics and magnetohydrodynamics. To this end, it is worth emphasizing that, contrary to the MS stars whose magnetic fields are generated by a self-exciting dynamo process in a peripheral convective zone with the energy supply of this process coming from the central thermonuclear reactive zone (e.g., Parker 1979), in neutron stars there are no internal energy sources to drive relative motions of extremely dense conducting material that is under immense gravitational pressure; consequently the mobility of neutron star matter is heavily suppressed. The common belief today is that magnetic fields of neutron stars are fossils (e.g., Flowers & Ruderman 1977; Bhattacharya & van den Heuvel 1991; Chanmugam 1994; Spruit 2008), that is, they have been inherited from massive MS progenitors and amplified in the processes of implosive gravitational collapse proceeding under control of the magnetic-flux conservation. Also it is commonly agreed that the external magnetic field of pulsars and magnetars is of dipolar symmetry, but the geometrical shape of the fossil magnetic field frozen in the super dense motionless matter of the star remains fairly uncertain. The subject of the present paper is the formulation of a solid-mechanical variational scheme of computing the frequency of Alfvén oscillations in neutron stars an with an arbitrary configuration of the frozen-in fossil magnetostatic field.

5 We recall that the equation for $\delta \mathbf{B}$ in line (1) follows from Maxwell equation for Faraday’s induction $\delta \mathbf{B} = -c \mathbf{\nabla} \times \delta \mathbf{E}$ where

$\delta \mathbf{E} = -(1/c) \delta (\mathbf{\nabla} \times \mathbf{B})$. This link between fluctuating electric field $\delta \mathbf{E}$ and constant magnetic field $\mathbf{B}$ frozen in the conducting flow follows from Ohm’s law $\delta j / \sigma = (1/c) (\mathbf{\nabla} \times \mathbf{B})$ corrected for the extremely large, effectively infinite, electrical conductivity, $\sigma \rightarrow \infty$ (the case of the medium designated as a perfect conductor). Taking into account that $\mathbf{\nabla} \times \mathbf{u} = 0$ and eliminating the time derivative from that obtained in the above equation of the field-flow coupling $\delta \mathbf{B} = \mathbf{\nabla} \times (\delta \mathbf{\nabla} \times \mathbf{B})$, one arrives at the last equation in the line (1).
\[ \rho \ddot{u} = \frac{1}{c} [\delta j \times B], \quad \delta j = \frac{c}{4\pi} [\nabla \times \delta B], \quad \delta B = \nabla \times [u \times B], \]  
(1)

\[ \rho \ddot{u} = \frac{1}{4\pi} [\nabla \times (\nabla \times [u \times B]) \times B], \quad \nabla \cdot u = 0, \quad \nabla \cdot \delta B = 0. \]  
(2)

Scalar multiplication of Equation (2) by \( \dot{u} \) and integration over the star volume leads to the equation of energy balance

\[ \frac{\partial}{\partial t} \int \frac{\rho \dot{u}^2}{2} \, dV = \frac{1}{4\pi} \int \left[ \nabla \times (\nabla \times [u \times B]) \times B \right] \cdot \dot{u} \, dV. \]  
(3)

This equation provides a basis for computing the frequency spectrum of nodeless vibrations by Rayleigh’s energy variational method that has been used with success for analysis of shear nodeless vibrations driven solely by Hooke’s elastic force (Bastrukov et al. 2007a, 2007b). The key idea of this method is to use the following separable representation of the field of material displacements:

\[ u(r, t) = a(r) \alpha(t). \]  
(4)

In a solid star undergoing global torsional nodeless oscillations about the polar axis, the quadrupole and octupole overtones which are pictured in Figure 1, is given by (Bastrukov et al. 2007b)

\[ \dot{u}(r, t) = [\omega(r, t) \times r], \quad \omega = A_i \nabla r^2 P_\ell (\cos \theta) \dot{\alpha}(t), \]  
(5)

so that instantaneous displacements can be written as

\[ a_r = A_r \nabla \times [\chi(r)], \quad \chi(r) = r^\ell P_\ell (\cos \theta); \]  

\[ a_{\theta} = 0, \quad a_{\phi} = A_r r^\ell (1 - \cos^2 \theta)^{1/2}\frac{dP_\ell (\cos \theta)}{d\cos \theta}, \]  
(6)

where the arbitrary constant \( A_r \) eliminated from the boundary condition (Bastrukov et al. 2007b)

\[ \dot{u}(r, t) = [\omega(r, t) \times r] \bigg|_{r=R} = [\Omega \times r], \quad \Omega = A_i \nabla \dot{P}_\ell (\cos \theta) \dot{\alpha} (t) \to A_i = [R^{\ell e^{-1}}]^{-1}. \]  
(7)

On inserting Equation (4) in Equation (3), this latter equation is reduced to an equation for \( \alpha(t) \) having the form of equation of harmonic vibrations

\[ \frac{d^2 \alpha}{dt^2} = \omega^2 \alpha, \quad \omega^2 = \frac{\mathcal{M} \dot{\alpha}^2}{2 + K_m \alpha^2} \to \mathcal{M} \dot{\alpha} + K_m \alpha = 0, \]  
(8)

where the integral parameters of inertia \( \mathcal{M} \) and stiffness \( K_m \) of magnetic Alfvén vibrations are given by

\[ \mathcal{M} = \int \rho a^2 \, dV, \quad K_m = \frac{1}{4\pi} \int [\nabla \times [\nabla \times [a \times B]] \times B] \cdot a \, dV = \frac{1}{4\pi} \int (B \cdot \nabla) a^2 \, dV. \]  
(9)

In these last equations, the density \( \rho \) and the frozen-in star magnetostatic fossil magnetic field \( B \) are considered to be intrinsic characteristics of the equilibrium state of the star matter and the input parameters of the method in question. The mass parameter \( \mathcal{M} \) is the positively defined quantity, whereas the sign of the stiffness \( K_m \) depends on the specific form of the magnetostatic fossil magnetic field \( B \) frozen in the star. Thus, geometric configuration of the internal field is crucial to the question of whether quake-induced shear perturbation resulting in fluctuations of differentially rotational displacements (Equation 10) are developed as an oscillation mode, \( \omega^2 > 0 \), or a relaxation mode, \( \omega^2 < 0 \); within the above expounded method this issue, which has been a subject of controversy (Levin 2007; Glampedakis et al. 2006; Watts & Strohmayer 2007), cannot be resolved without computing \( \mathcal{K} \) for each imaginable form of \( B \).

The large-scale external magnetic fields of radiative magnetospheres of pulsars and magnetars are commonly thought of as produced by the magnetic dipole moment of the underlying neutron star and it is generally believed that the internal magnetic field of the neutron star has a strong poloidal component. With this in mind and for reasons of computational feasibility, in the remainder of this section we examine the above variational approach by considering an admittedly idealized neutron star model with uniform poloidal internal magnetostatic magnetic field \( B \), directed along the polar axis \( z \), undergoing global torsional oscillations about the dipolar magnetic moment axis which are insensitive to the above compositional core–crust stratification of the neutron star. The explicit form of spherical components of such a field inside the star is given by

\[ B_r = B_\zeta, \quad B_\theta = -B (1 - \zeta^2)^{1/2}, \quad B_\phi = 0, \]  
(10)

and the components of the dipolar configuration outside the star are

\[ B_r = B \left( \frac{R}{r} \right)^3 \zeta, \quad B_\theta = -\frac{B}{2} \left( \frac{R}{r} \right)^3 (1 - \zeta^2)^{1/2}, \quad B_\phi = 0 \]  
(11)

where \( R \) is the star radius. Computation of integrals for \( \mathcal{M} \) and \( K_m \) yields (Bastrukov et al. 1997)

\[ \mathcal{M}(\alpha_\ell) = 4\pi \rho A_i^2 R^{2e+3} \frac{\ell (\ell + 1)}{(2\ell + 1)(2\ell + 3)}, \]  
(12)

\[ K_m (\alpha_\ell) = B^2 A_i^2 R^{2e+1} \frac{\ell (\ell + 1)(\ell^2 - 1)}{(2\ell + 1)(2\ell - 1)}, \]  
(13)

and for the angular frequency of toroidal magnetic Alfvén mode of global torsional nodeless oscillations we obtain

\[ \omega_\ell (\alpha_\ell) = \omega_A \left[ (\ell^2 - 1) \frac{2\ell + 3}{2\ell - 1} \right]^{1/2}, \]  
(14)

where \( M \) and \( R \) stand for the mass and the radius of the star, respectively. It may be worth noting that the normal component of the magnetic field under consideration is continuous on the surface while the tangential component remains discontinuous and, thus, admits, in accord with standard boundary condition of electrodynamics, the surface current, provided that on the surface there is an excess of likely-charged particles distributed with the surface charge density \( \sigma \). The considered magnetic field inside the star is identical to that produced by a uniformly charged spherical shell of radius \( R \) set in the rotation about the polar axis with constant angular velocity \( \Omega \): the absolute
value of this field is given by $B = (2/3)σΩR$ (Griffiths 1981). So, the above inferences regarding frequency of nodeless Alfvén oscillations in the star volume are not affected when electrodynamic conditions inside the neutron star model under consideration are compatible with those for the latter case of the uniformly charged spherical shell.

In Figure 2, frequencies (left) and periods (right) of the toroidal Alfvén mode are plotted as functions of multipole degree $ℓ$ for a solid star with parameters typical for pulsars and magnetars, clearly showing general trends of $ν$ (in Hz) and $P$ (in seconds) as a function of multipole degree of oscillations and their order of magnitude.

In the above computations we have used a toroidal (axial) vector field of differentially rotational nodeless material displacements, which is one of two fundamental solutions of the vector Laplace equation

$$\nabla^2 \mathbf{a}(\mathbf{r}) = 0, \quad \nabla \cdot \mathbf{a}(\mathbf{r}) = 0, \quad (15)$$

built on the fundamental solution of the scalar Laplace equation

$$\nabla^2 \chi(\mathbf{r}) = 0, \quad \chi(\mathbf{r}) = r^ℓ P_ℓ(\zeta), \quad \zeta = \cos θ. \quad (16)$$

The second fundamental solution is given by the even-parity poloidal (polar) vector field (Bastrukov et al. 2007b)

$$\mathbf{a}_p = \frac{A_p}{ℓ + 1} \nabla \times \nabla \times [\mathbf{r} \chi(\mathbf{r})] = A_p \nabla \chi(\mathbf{r}), \quad (17)$$

$$\chi(\mathbf{r}) = r^ℓ P_ℓ(\zeta),$$

which is irrotational: $\nabla \times \mathbf{a}_p = 0$. The integral parameters of inertia $\mathcal{M}$ and stiffness $K_m$ in the poloidal mode are given by

$$\mathcal{M}(a_p^2) = 4πρA_p^2 R^{2ℓ+1} \frac{ℓ}{2ℓ + 1}, \quad (18)$$

$$K_m(a_p^2) = B^2 A_p^2 R^{2ℓ-1} \frac{ℓ^2(ℓ - 1)}{2ℓ - 1},$$

and for the frequency spectrum of poloidal magnetic Alfvén mode we obtain

$$ω(a_p^2) = ω_A^2(ℓ - 1) = 2ℓ + 1$$

$$ω_A^2 = \frac{V_p^2}{R^4} = \frac{B^2}{4πρR^2} = \frac{B^2 R}{3M}, \quad (19)$$

The spectral formulae such as that obtained above for toroidal and poloidal Alfvén modes are central to theoretical astroseismology of pulsars and magnetars in the sense that they provide the basis for interpretation of observable QPOs as produced by quake-induced vibrations restored by Lorentz force.

We recall that the Lorentz force density, $\mathbf{f} = (1/c)[\mathbf{j} \times \mathbf{B}]$, with the Ampère’s current density, $\mathbf{j} = (c/4π)\nabla \times \mathbf{B}$, having the form

$$\mathbf{f} = \frac{1}{4π} [(\nabla \times \mathbf{B}) \times \mathbf{B}] = \frac{1}{4π} \left[ (\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2} \nabla \mathbf{B}^2 \right] \quad (20)$$

To the best of our knowledge, classification of Alfvén vibrational modes as poloidal and toroidal has been introduced by Chandrasekhar (1956) in the context of hydromagnetic oscillations of fluid sphere. General properties of poloidal and toroidal fields are extensively discussed in Chandrasekhar (1961) and Ferraro & Plumpton (1961).
can be represented in terms of Maxwell tensor of magnetic field stresses as (e.g., Mestel 1999)
\[
f_i = \nabla \cdot \mathbf{T}_{ik}, \quad \mathbf{T}_{ik} = \frac{1}{8\pi} \left[ \mathbf{B}_i \delta \mathbf{B}_k + \mathbf{B}_k \delta \mathbf{B}_i - (\mathbf{B}_j \delta \mathbf{B}_j) \delta_{ik} \right].
\] (21)
In extremely dense matter of solid stars at the final stage, the effects of magnetic buoyancy are heavily suppressed and, hence, in the motionless matter of the stationary state of the solid star with the above frozen-in uniform magnetic field, \( f_i = f_i^0(\mathbf{B} = \text{const}) = 0 \). However, the Lorentz force surely comes into play as a result of quake-induced perturbation which leads to coupled fluctuations of material displacements \( \mathbf{u} \) and magnetic field \( \delta \mathbf{B} \) in accordance with the equation for field-flow coupling
\[
\delta \mathbf{B}(\mathbf{r}, t) = \nabla \times [\mathbf{u} \times \mathbf{B}].
\]
With this in mind, the equation of dynamics (Equation (2)) can be represented in the equivalent tensor form, to wit, in terms of the tensor of fluctuating magnetic field stresses (e.g., Franco et al. 2000)
\[
\rho \ddot{\mathbf{u}}_i = \nabla \cdot \mathbf{T}_{ik}, \quad \mathbf{T}_{ik} = \frac{1}{4\pi} \left[ \mathbf{B}_i \delta \mathbf{B}_k + \mathbf{B}_k \delta \mathbf{B}_i - (\mathbf{B}_j \delta \mathbf{B}_j) \delta_{ik} \right],
\] (22)
\[
\delta \mathbf{B} = \nabla \times [\mathbf{u} \times \mathbf{B}],
\]
where \( \mathbf{B} \) is the stationary magnetic field given in advance. In the next section we show that the above obtained spectral Equation (13) for the toroidal Alfvén mode of global differentially rotational oscillations of the star matter about the homogeneous frozen-in-field star can be regained on the basis of this last representation of the Lorentz force density.

In Figure 3 we plot the frequency and period of Alfvénic modes \( \omega(\ell, \ell') \) upon multipole degree \( \ell \) of nodeless oscillations of material displacements in a neutron star whose internal magnetic field might be approximated by axisymmetric poloidal field of uniform shape. The presented absolute values for the frequencies show that they fall in the realm of observable QPOs in flaring SGR 1806-20 and SGR 1900+14 and, thus, suggest that the Lorentz restoring force must be taken into account when studying global seismic vibrations of magnetars. Together with this, it is worth emphasizing that obtained one-parametric spectral formula for toroidal Alfvén vibration mode in the model under consideration does not match the general trends in observed QPOs frequency as a function of multipole degree of torsional vibrations (Watts & Strohmayer 2007) because the slope of the computed frequency \( \omega(\ell, \ell') \) as a function of \( \ell \) is different from the slope of the overall trends of the observed QPO frequencies. In the next section, we approach this problem by considering in some detail the global nodeless torsional oscillations of a neutron star model with the above uniform internal magnetic field under the action of the combined forces of shear elastic and magnetic field stresses.

3. TORSIONAL NODELESS VIBRATIONS UNDER THE JOINT ACTION OF BULK FORCES OF ELASTIC AND MAGNETIC FIELD STRESSES

The equation of Newtonian magneto-solid dynamics appropriate for noncompressional \( (\delta \rho = -\rho \nabla \cdot \mathbf{u} = 0) \) shear vibrations of an elastic medium of infinite electrical conductivity is
\[
\rho \ddot{\mathbf{u}}_i = \nabla \cdot \mathbf{T}_{ik} + \nabla \cdot \mathbf{B}_{ik} \tau_{ik},
\] (23)
The first term in the right part of Equation (23) is the bulk force of elastic stresses obeying Hooke’s law,
\[
\sigma_{ik} = 2\mu \mathbf{u}_{ik}, \quad \mathbf{u}_{ik} = \frac{1}{2} (\nabla_i \mathbf{u}_k + \nabla_k \mathbf{u}_i), \quad \mathbf{u}_{kk} = \nabla_i \mathbf{u}_i = 0,
\] (24)
where \( \mu \) is the shear modulus of stellar matter linearly relating quake-induced shear stresses \( \sigma_{ik} \) and resulting shear deformations or strains \( \mathbf{u}_{ik} \). The second term is the above-defined Lorentz force represented in terms of fluctuating magnetic field stress \( \tau_{ik} \). Similar equations have recently been considered (Piro 2005; Glampedakis et al. 2006; Lee 2008). The conservation of energy is controlled by the equation
\[
\frac{\partial}{\partial t} \int \rho \dot{\mathbf{u}}_i \dot{\mathbf{u}}_i \ d\mathcal{V} = -2 \int \left[ \sigma_{ik} + \tau_{ik} \right] \dot{\mathbf{u}}_i d\mathcal{V},
\] (25)
\[
\dot{\mathbf{u}}_i = \frac{1}{2} [\nabla_i \dot{\mathbf{u}}_k + \nabla_k \dot{\mathbf{u}}_i],
\]
which is obtained after scalar multiplication of Equation (23) by \( \dot{\mathbf{u}}_i \) and integration over the star volume. To compute the eigenfrequency of torsional nodeless oscillations we again take advantage of Rayleigh’s energy method. On inserting separable representation of the field of material displacement
\[
\mathbf{u}_i(\mathbf{r}, t) = a_i(\mathbf{r}) \omega(t)
\] (26)
in the equation of energy balance, we again obtain the equation of harmonic oscillations of temporal amplitude $\alpha(t)$:

$$\frac{dH}{dt} = 0, \quad H = \frac{M \alpha^2}{2} + \frac{K \alpha^2}{2}, \quad \rightarrow \quad M \ddot{\alpha}(t) + K \alpha(t) = 0, \quad K = K_e + K_m. \quad (27)$$

The analytic form for the inertial $M$ and the stiffness $K_e$ of shear elastic oscillations derived in (Bastrukov et al. 2007a, 2007b) are given by

$$M = \int \rho a_i(r) a_i(r) dV, \quad K_e = \frac{1}{2} \int \mu [\nabla_k a_i + \nabla_l a_l] [\nabla_k a_i + \nabla_l a_l] dV, \quad \quad (28)$$

and for the stiffness of magnetic Alfvén shear oscillations we obtain

$$K_m = \frac{1}{8 \pi} \int \{ B_k [\nabla_j (B_j a_i - B_i a_j)] + B_i [\nabla_j (B_j a_i - B_i a_j)] \} (\nabla_k a_i + \nabla_l a_l) dV \quad = \frac{1}{4 \pi} \int \{ B_k \nabla_l a_i \} (\{ B_j \nabla_l a_j \}) dV. \quad (29)$$

It should be noted that this last equation can be derived from the standard MHD equations (Chandrasekhar 1961). Computation of integrals with $B_i$ given by Equation (3) and the toroidal field $a_i$ defined by Equation (9) yields

$$M_0(0 \ell) = 4 \pi \rho A_t R^{2 \ell+3} \frac{\ell(\ell+1)}{(2 \ell+1)(2 \ell+3)}, \quad (30)$$

$$K_e(0 \ell) = 4 \pi \mu A_t R^{2 \ell+1} \frac{\ell(\ell-1)}{(2 \ell+1)}, \quad (31)$$

$$K_m(0 \ell) = B^2 A_t R^{2 \ell+1} \frac{\ell(\ell+1)(\ell-1)}{(2 \ell+1)(2 \ell-1)}, \quad \quad (32)$$

and for the Hertz total frequency $v = \omega/2\pi$ (with $\omega = \sqrt{K/M}$) we obtain

$$v(0 \ell) = \left[ v_e^2 (\ell) + v_m^2 (\ell) \right]^{1/2}, \quad (32)$$

$$v_c(0 \ell) = v_e \left[ (2 \ell + 3)(\ell - 1) \right]^{1/2}, \quad (33)$$

In Figure 4, the fractional frequency of elastic $v_e(\ell)/v_c$ oscillations as a function of multipole degree $\ell$ is plotted in juxtaposition with fractional frequency $v_m(\ell)/v_A$ of magnetic Alfvén oscillations. One sees that the lowest overtones of both elastic and magnetic Alfvén modes are of quadrupole degree, $\ell = 2$. At $\ell = 1$, both parameters of elastic and magnetomechanical rigidity cancel, $K_e(\ell=1) = 0$ and $K_m(\ell=1) = 0$, and the mass parameter is equal to the moment of inertia of rigid sphere, $M = (2/5)MR^2$. It follows from the Hamiltonian that, in this dipole case, a star sets in rigid body rotation, rather than vibrations, about the axis of magnetic dipole moment.

The obtained spectral equation for total frequency can be conveniently represented in the following form

$$v(\ell) = v_e \left[ (2 \ell + 3)(\ell - 1) \right]^{1/2} \left[ 1 + \frac{v_m^2}{v_A^2} \right]^{1/2}, \quad (35)$$

where

$$v_e = \sqrt{\frac{\mu}{4 \pi^2 \rho R^2}}, \quad \beta = \frac{v_m^2}{v_A^2} = \frac{v_c^2}{v_e^2} = \frac{B^2}{4 \pi \mu}. \quad (36)$$

In Figure 5, we plot the last two-parametric spectral equation for the total frequency $v(\ell)$ as a function of multipole degree $\ell$ of torsion nodeless vibrations computed with indicated values of parameters $v_e$ and $\beta$ carrying information about mechanical and electrodynamical properties of the neutron star matter, which are
adjusted so as to reproduce the observable frequency of QPOs (symbols) during the flare of SGR 1806-20 and SGR 1900+14 (the data are from Watts & Strohmayer 2007; Samuelsson & Andersson 2007). This figure demonstrates that the detected QPOs can be consistently explained from the viewpoint of the considered model as produced by global torsional nodeless vibrations when and only when combined forces of shear elastic and magnetic field stresses come into play in a coherent fashion. On the other hand, in our previous study reported in recent paper (Bastrukov et al. 2008a), it has been shown that this set of QPO data can be properly described, with the same degree of accuracy, on the basis of two-parametric spectral formula that has been derived on the basis of a two-component, core–crust model of quaking neutron star presuming that detected QPOs are produced by axisymmetric torsional nodeless seismic vibrations driven by a solely elastic restoring force and locked in the peripheral finite-depth seismogenic layer. The truth is, most probably, somewhere in between and in order to attain more definite conclusions, further investigations, both theoretical and observational, are needed. From a computational argument, the nodeless torsional oscillations entrapped in the neutron star crust as well as in the star models with a nonuniform axisymmetric internal magnetic field and a nonhomogeneous profile of shear modulus, require a more elaborate mathematical treatment. To keep our attention on the newly obtained results presented here (some of which are of interest, as is hoped, for general theoretical seismology; Lay & Wallace 1995; Aki & Richards 2002), we postpone a discussion of these latter cases to a forthcoming article.

A special comment should be made regarding the link between neutron stars and atomic nuclei which can be considered to be similar objects, as far as mechanical properties of degenerate nucleon material of normal nuclear density are concerned. It follows from the nuclear solid-globe model of giant resonances (within the framework of which these fundamental modes of nuclear excitations are properly described in terms of shear elastic vibrations of an ultra-small spherical piece of a nuclear Fermi-solid that is regarded as continuum matter) that the shear modulus is given by $\mu \sim 10^{33}$ dyn cm$^{-2}$ (e.g., Bastrukov et al. 2008b). Making use of this value of $\mu$ in the expression for parameter $\beta = [B^2/4\pi \mu] \sim 0.5$–0.6, which enters the above-derived spectral equation for QPO frequency, we can get an independent estimate for the intensity of the magnetic field frozen in the star. With the above values of $\beta$ and $\mu$ one finds that $B$ falls in the range $10^{15} < B < 10^{16}$ Gauss, that is, in the realm typical of magnetar magnetic fields. This latter inference demonstrates the theoretical potential of the asteroseismology, showing how physical interpretation of the QPOs as produced by quake-induced torsional vibrations of neutron star can be used to extract information about properties of the neutron star matter.

4. SUMMARY

There is a common belief today that gross features of the asteroseismology of pulsars and magnetars can be understood on the basis of a solid star model, presuming that quake-induced shear vibrations restored by bulk forces of intrinsic stresses of a different physical nature are governed and described by solid mechanics or elastodynamics (e.g., Hansen & van Horn 1979; McDermott et al. 1988; Bastrukov et al. 1999; Bastrukov et al. 2007a). This point of view is quite different from the theoretical approach to the asteroseismology of the MS stars at the base of which lies the liquid star model whose vibrations are treated within the framework of fluid-mechanical theory of continuous media, as in the case of helioseismology.

The main purpose of this work was to examine the magnetosolid mechanical variational method of the asteroseismology of neutron star by probing its interior with nonradial global differentially rotational, torsional vibrations with nodeless toroidal field of material displacements, which are insensitive to compositional stratification of the star matter. Bearing in mind that external magnetic fields of pulsars and magnetars are commonly thought to be produced by the magnetic dipole moment of underlying neutron star, we have considered a model of a neutron star with perhaps the simplest, from the viewpoint of computational feasibility, imaginable configuration of the magnetostatic fossil magnetic field, pictured in Figure 1. Proceeding from this admittedly idealized model, the two-parametric spectral equation for the frequency of global torsional vibrations has been derived in analytic form, showing that the larger the multipole degree $\ell$ of torsional nodeless vibrations, the higher the frequency $\omega_0(\ell)$. The application of the obtained two-parametric spectral formula to model analysis of QPOs during the flare of SGR 1806-20 and SGR 1900+14 shows that data on the QPO frequencies, with $\ell$ in the interval $2 \leq \ell \leq 20$, can be consistently interpreted as produced by global torsional nodeless vibrations restored by combined forces of shear elastic and magnetic field stresses. This inference is, of course, suggestive rather than conclusive, in view of the adopted, highly idealized configuration of a fossil magnetostatic field frozen in the star, and much remains to be done to be at all certain of interpretations suggested for QPOs in the X-ray flux during the giant flares of the above magnetars.

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