Moduli spaces of $G_2$ manifolds

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Abstract

This paper is a review of current developments in the study of moduli spaces of $G_2$ manifolds. $G_2$ manifolds are 7-dimensional manifolds with the exceptional holonomy group $G_2$. Although they are odd-dimensional, in many ways they can be considered as an analogue of Calabi-Yau manifolds in 7 dimensions. They play an important role in physics as natural candidates for supersymmetric vacuum solutions of $M$-theory compactifications. Despite the physical motivation, many of the results are of purely mathematical interest. Here we cover the basics of $G_2$ manifolds, local deformation theory of $G_2$ structures and the local geometry of the moduli spaces of $G_2$ structures.

1 Introduction

Ever since antiquity there has been a very close relationship between physics and geometry. Originally, in Timaeus, Plato related four of the five Platonic solids - tetrahedron, hexahedron, octahedron, icosahedron to the elements fire, earth, air and water, respectively, while the fifth solid, the dodecahedron was the quintessence of which the cosmos itself is made. Later, Isaac Newton’s Laws of Motion and Theory of Gravitation gave a precise mathematical framework in which the motion of objects can be calculated. However Albert Einsteins’s General Relativity made it very explicit that the physics of spacetime is determined by its geometry. More recently, this fundamental relationship has been taken to a new level with the development of String and M-theory. Over the past 25 years, superstring theory has emerged as a successful candidate for the role of a theory that would unify gravity with other interactions. It was later discovered that all five superstring theories can be obtained as special limits of a more general eleven-dimensional theory known as M-theory and moreover, the low energy limit of which is the eleven-dimensional supergravity \cite{40,42}. The complete formulation of M-theory is, however, not known yet.

One of the key features of String and M-theory is that these theories are formulated in ten- and eleven-dimensional spacetimes, respectively. One of the techniques to relate this to the visible four-dimensional world is to assume that the remaining six or seven dimensions are curled up as a small, compact, so-called internal space. This is known as compactification. Such a procedure also leads to a remarkable interrelationship between physics and geometry, since the effective physical content of the resulting four-dimensional theory is determined by the geometry of the internal space. Usually the full multidimensional spacetime is regarded as a direct product $M_4 \times X$, where $M_4$ is a 4-dimensional non-compact manifold with Lorentzian signature $(- + + +)$ and $X$ is a compact six or seven dimensional Riemannian manifold. In
general, the parameters that define the geometry of the internal space give rise to massless scalar fields known as moduli, and the properties of the moduli space are determined by the class of spaces used in the compactification.

The properties of the internal space in String and M-theory compactifications are governed by physical considerations. A key ingredient of these theories is supersymmetry [11]. Supersymmetry is a physical symmetry between particles the spin of which differs by $\frac{1}{2}$ - that is, between integer spin bosons and half-integer spin fermions. Mathematically, bosons are represented as functions or tensors and fermions as spinors. When looking for a supersymmetric vacuum for which the metric is the only non-zero field, that is a Ricci-flat solution that is invariant under supersymmetry transformations, it turns out that a necessary requirement is the existence of covariantly constant, or parallel, spinor. That is, there must exist a non-trivial spinor $\eta$ on the Riemannian manifold $X$ that satisfies

$$\nabla \eta = 0$$

where $\nabla$ is the relevant spinor covariant derivative [8]. This condition implies that $\eta$ is invariant under parallel transport.

Properties of parallel transport on a Riemannian manifold are closely related to the concept of holonomy. Consider a vector $v$ at some point $x$ on $X$. Using the natural Levi-Civita connection that comes from the Riemannian metric, we can parallel transport $v$ along paths in $X$. In particular, consider a closed contractible path $\gamma$ based at $x$. As shown in Figure 1 if we parallel transport $v$ along $\gamma$, then the new vector $v'$ which we get will necessarily have the same magnitude as the original vector $v$, but otherwise it does not have to be the same. This gives the notion of holonomy group. Below we give the precise definition.

**Definition 1** Let $(X, g)$ be a Riemannian manifold of dimension $n$ with metric $g$ and corresponding Levi-Civita connection $\nabla$, and fix point $x \in X$. Let $\gamma: [0, 1] \rightarrow X$ be a loop based at $x$, that is, a piecewise-smooth path such that $\gamma(0) = \gamma(1) = x$. The parallel transport map $P_\gamma: T_x X \rightarrow T_x X$ is then an invertible linear map which lies in $SO(n)$. Define the Riemannian holonomy group $\text{Hol}_x(X, g)$ of $\nabla$ based at $x$ to be

$$\text{Hol}_x(X, g) = \{ P_\gamma : \gamma \text{ is a loop based at } x \} \subset O(n)$$

![Figure 1: Parallel transport of a vector](image)
| Geometry         | Holonomy      | Dimension |
|------------------|---------------|-----------|
| Kähler           | $U(k)$        | $2k$      |
| Calabi-Yau       | $SU(k)$       | $2k$      |
| HyperKähler      | $Sp(k)$       | $4k$      |
| Exceptional      | $G_2$         | $7$       |
| Exceptional      | $Spin(7)$     | $8$       |

Figure 2: List of special holonomy groups

defined for vectors, but can then be naturally extended to other objects like tensors and spinors, with the holonomy group acting on these objects via relevant representations.

Now going back to the covariantly constant spinor $\eta$, (1.1) implies that $\eta$ is invariant under the action of the holonomy group. This shows that the spinor representation of $\text{Hol}(X, g)$ must contain the trivial representation. For $\text{Hol}(X, g) = \text{SO}(n)$, this is not possible since the spinor representation is reducible, so $\text{Hol}(X, g) \subset \text{SO}(n)$. Hence the condition (1.1) implies a reduced holonomy group. Thus, Ricci-flat special holonomy manifolds occur very naturally in string and M-theory.

As shown by Berger [9], the list of possible special holonomy groups is very limited. In particular, if $X$ is simply-connected, and neither locally a product nor symmetric, the only possibilities are given in Figure 2. In this list manifolds with holonomy $SU(k), Sp(k), G_2$ and $Spin(7)$ are Ricci-flat. Moreover, these groups are subgroups of $\text{SO}(n)$ and are simply-connected. This implies that manifolds with these holonomy groups always admit a spin structure ([23, Proposition 3.6,2]). These are also precisely the manifolds that admit a parallel spinor. Kähler manifolds only admit parallel projective spinors - a line subbundle of the spinor bundle. Thus, for a Ricci-flat supersymmetric vacuum in a 10-dimensional theory, $X$ has to be 6-dimensional in order to reduce to 4 dimensions, and hence necessarily a Calabi-Yau manifold. Similarly, for an 11-dimensional theory, 7-dimensional manifolds with $G_2$ holonomy arise naturally.

We have thus seen that even rather simple physical requirement restrict the geometry of the manifold $X$ to rather special classes. In particular, the study of Calabi-Yau manifolds has been crucial in the development of String Theory, and in fact some very important discoveries in the theory of Calabi-Yau manifolds have been made thanks to advances in the physics. One such major discovery is Mirror Symmetry [36, 26]. This symmetry first appeared in String Theory where evidence was found that conformal field theories (CFTs) related to compactifications on a Calabi-Yau manifold with Hodge numbers $(h_{1, 1}, h_{2, 1})$ are equivalent to CFTs on a Calabi-Yau manifold with Hodge numbers $(h_{2, 1}, h_{1, 1})$. Mirror symmetry is currently a powerful tool both for calculations in String Theory and in the study of the Calabi-Yau manifolds and their moduli spaces.

In mathematical literature $G_2$ holonomy first appeared in Berger’s list of special holonomy groups in 1955 [9]. In 1966 Bonan has shown that manifolds with $G_2$ holonomy are Ricci-flat. It was known from general theory that having a holonomy group $G$ is equivalent to having a torsion-free $G$-structure. So it was natural to study $G_2$ structures on manifolds to get a better understanding of $G_2$ holonomy. The different classes of $G_2$ structures have been explored by Fernández and Gray in their 1982 paper [17]. In particular they have shown that a torsion-free $G_2$ structure is equivalent to the $G_2$-invariant 3-form $\varphi$ being closed and co-closed.

It was not known whether the group $G_2$ (or indeed $Spin(7)$ for that matter) does actually appear as a non-symmetric holonomy group until in 1987 Bryant [12] proved the existence of metrics with $G_2$ and $Spin(7)$ holonomy. In a later paper, Bryant and Salamon [11] constructed complete metrics with $G_2$ holonomy. However the first compact examples of $G_2$ holonomy manifolds have been constructed by Joyce in 1996 [27]. These examples are based on quotients.
$T^\Gamma / T$ where $\Gamma$ is a finite group. Such quotient spaces usually exhibit singularities, and Joyce has shown that it is possible to resolve these singularities in such a way as to get a smooth, compact manifold with $G_2$ holonomy. Since then, a number of other types of constructions have been found, in particular the construction by Kovalev [32] where a compact $G_2$ manifold is obtained by gluing together two non-compact asymptotically cylindrical Riemannian manifolds with holonomy $SU(3)$.

In the $G_2$ holonomy compactification approach to M-theory, the physical content of the four-dimensional theory is given by the moduli of $G_2$ holonomy manifolds. Such a compactification of M-theory is in many ways analogous to Calabi-Yau compactifications in String Theory, where much progress has been made through the study of the Calabi-Yau moduli spaces. In particular, as it was shown in [14] and [35], the moduli space of complex structures and the complexified moduli space of Kähler structures are both in fact, Kähler manifolds. Moreover, both have a special geometry: that is, both have a line bundle whose first Chern class coincides with the Kähler class. However, until recently, the structure of the moduli space of $G_2$ holonomy manifolds has not been studied in that much detail. Generally, it turns out that the study of $G_2$ manifolds is quite difficult. Unlike the study of Calabi-Yau manifolds where the machinery of algebraic geometry has been used with great success, in the case of $G_2$ manifolds there is no analogue, so analytical rather than algebraic study is needed.

In this review, we aim to give an overview of what is currently known about $G_2$ moduli spaces and corresponding deformations of $G_2$ structures. We first give an introduction to the properties of the group $G_2$ - definitions and representations. Then we look at general properties of $G_2$ structures. Finally we move on to properties of $G_2$ moduli spaces.

2 The group $G_2$

2.1 Automorphisms of octonions

The group $G_2$ is the smallest of the 5 exceptional Lie groups, the others being $F_4$, $E_6$, $E_7$ and $E_8$. Surprisingly, all of these Lie groups are related to the octonions, but $G_2$ is especially close. So let us first give a few facts about the octonions. The eight-dimensional algebra of octonions, denoted by $O$, is the largest possible normed division algebra. The others of course are the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$. Following Baez [6], it turns out that division algebras can be defined using the notion of triality. Given three real vector spaces $U, V, W$, then a triality is a non-degenerate trilinear map $t: U \times V \times W \rightarrow \mathbb{R}$.

Non-degenerate here means that for any fixed non-zero elements of $U$ and $V$, the induced functional on $W$ is non-zero. Hence, $t$ also defines a bilinear map $m$

$$m: U \times V \rightarrow W^*.$$ 

For each fixed element of $U$, this map defines an isomorphism between $V$ and $W^*$, and for each fixed element of $V$, an isomorphism between $U$ and $W^*$. Hence these three spaces are isomorphic to each, and if we choose to identify non-zero elements $e_1 \in U$, $e_2 \in V$, and $e_1 e_2 \in W^*$, we can identify the spaces $U, V, W$ with each other, and we can say that $m$ now defines multiplication on $U$ with identity element $e = e_1 = e_2 = e_1 e_2$. Note that in particular, the existence of a non-degenerate trilinear map implies that the original vector spaces $U, V, W$ are all of the same dimension.

Due to the non-degeneracy of the original triality, multiplication by a fixed element is an isomorphism, so in fact, $U$ is a division algebra. Assuming further that $U, V, W$ are inner product
spaces, if the triality map satisfies
\[ |t(u, v, w)| \leq \|u\| \|v\| \|w\| \]
and is such that for all \( u, v \) there exists a non-zero \( w \) such that the bound is attained (and similarly for cyclic permutations for \( u, v, w \)) then we get a normed division algebra. The converse is also true - any division algebra defines a triality.

As discussed in detail by Baez [6], on \( \mathbb{R}^n \) it is possible to construct bilinear maps \( m_n \) involving the vector and spinor representations of \( \text{Spin}(n) \):
\[
m_n : V_n \times S_n^{\pm} \rightarrow S_n^{\mp} \quad \text{for } n = 0, 4 \mod 8 \quad (2.1aa)
m_n : V_n \times S_n \rightarrow S_n \quad \text{otherwise} \quad (2.1ab)
\]
where \( V_n \) is the vector representation of \( SO(n) \), \( S_n^{(\pm)} \) are the (left- and right-handed) spinor representations.

The spinor representations in (2.1) are self-dual, so in principle, by dualizing the maps in (2.1), we could obtain trilinear maps into \( \mathbb{R} \). However, in order to obtain trialties, these maps have to be non-degenerate, and hence the dimensions of the relevant representations must agree. This happens only for \( n = 1, 2, 4, 8 \), and each of these trialties gives a normed division algebra of the corresponding dimension:
\[
t_1 : V_1 \times S_1 \times S_1 \rightarrow \mathbb{R} \rightarrow \mathbb{R}
t_2 : V_2 \times S_2 \times S_2 \rightarrow \mathbb{R} \rightarrow \mathbb{C}
t_4 : V_4 \times S_4^{+} \times S_4^{-} \rightarrow \mathbb{R} \rightarrow \mathbb{H}
t_8 : V_8 \times S_8^{+} \times S_8^{-} \rightarrow \mathbb{R} \rightarrow \mathbb{O} \quad (2.2)
\]
This way, via the trialties we obtain all of the normed division algebras.

In general, suppose we have a triality \( t : U_1 \times U_2 \times U_3 \rightarrow \mathbb{R} \). Then to define a normed division algebra from \( t \), we fix two vectors in the two of the three spaces. Hence the automorphism of the division algebra is the subgroup of the automorphism group of the triality that fixes these two vectors. For \( t_8 \) the automorphism group of the triality turns out to be \( \text{Spin}(8) \), while \( G_2 \) is defined as the automorphism group of the corresponding octonion algebra. Thus we have

**Definition 2** The group \( G_2 \) is the automorphism group of the octonion algebra.

Since \( G_2 \) is the automorphism group of octonions, it is the subgroup of \( \text{Spin}(8) \) (the automorphism group of the triality \( t_8 \)) that preserves unit vectors in \( V_8 \) and \( S_8^{+} \). As explained by Baez in [6], the subgroup of \( \text{Spin}(8) \) that fixes a unit vector in \( V_8 \) is \( \text{Spin}(7) \). Moreover, if the representation \( S_8^{+} \) is restricted to \( \text{Spin}(7) \), we get the spinor representation \( S_7 \). Therefore, \( G_2 \) is the subgroup of \( \text{Spin}(7) \) that fixes a unit vector in \( S_7 \). In this representation, \( \text{Spin}(7) \) acts transitively on the unit sphere \( S^7 \), so we have
\[
\text{Spin}(7) / G_2 = S^7. \quad (2.3)
\]
Hence we have the following result.

**Proposition 3** The group \( G_2 \) has dimension 14.

**Proof.** From (2.3),
\[
\dim G_2 = \dim (\text{Spin}(7)) - \dim S^7 = 21 - 7 = 14.
\]
\[ \blacksquare \]
The automorphism group fixes the identity, so in fact $G_2$ acts non-trivially on octonions that are orthogonal to the identity - the imaginary octonions, denoted by $\text{Im}(\mathbb{O})$ and thus we get a natural 7-dimensional representation of $G_2$. A closer look at this representation reveals another description of $G_2$. Using octonion multiplication, we can define a cross product on $\text{Im}(\mathbb{O})$ by

$$a \times b = \text{Im}(ab) = \frac{1}{2}(ab - ba). \quad (2.4)$$

But $G_2$ preserves octonion multiplication, hence any element of $G_2$ preserves the 7-dimensional cross product. Alternatively, (2.4) can be written as

$$a \times b = ab + \langle a, b \rangle \quad (2.5)$$

where $\langle, \rangle$ is the octonionic inner product, in general defined by

$$\langle a, b \rangle = \frac{1}{2}(a^*b + ba^*).$$

Also, it can be shown that

$$\langle a, b \rangle = -\frac{1}{6} \text{Tr}(a \times (b \times \cdot)) \quad (2.6)$$

Therefore, from (2.5), multiplication of imaginary octonions can be defined in terms of the cross product, hence any transformation preserving the cross product preserves multiplication on $\text{Im}(\mathbb{O})$, and is thus in $G_2$. So, $G_2$ is precisely the group that preserves the 7-dimensional cross product.

Moreover, from the cross product we can form a “scalar triple product” on $\text{Im}(\mathbb{O})$ given by

$$\varphi_0(a, b, c) = \langle a, b \times c \rangle = \langle a, bc \rangle. \quad (2.7)$$

This defines $\varphi_0$ as an anti-symmetric trilinear functional - that is, a 3-form on $\mathbb{R}^7$. Equivalently, for a basis $e_i$ of $\text{Im}(\mathbb{O})$,

$$e_i \times e_j = \varphi_0^{k} _{ij} e_k. \quad (2.8)$$

So in this description, the components of $\varphi_0$ are essentially the structure constants of the algebra of imaginary octonions.

A well-known way to encode the multiplication rules for the octonions is the Fano plane [6]. It is shown in Figure 3. In the diagram, the vertices $e_1, ..., e_7$ are the seven square roots of $-1$. Multiplication follows along the six straight lines (sides of the triangle and the altitudes) and along the central circle in the direction of the arrows. So if $e_i, e_j, e_k$ are in this order on a straight line, then $e_i e_j = e_k$ and $e_j e_i = -e_k$.

However, from (2.8) we see that $\varphi_0$ encodes precisely the same information as the Fano plane. Suppose $x^1, ..., x^7$ are coordinates on $\mathbb{R}^7$ and let $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$; then just reading off from the Fano plane, $\varphi_0$ can be written as

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \quad (2.9)$$

Note that in order to keep the same convention for $\varphi_0$ as Joyce [28], in the Fano plane we have a different numbering for the octonions compared to Baez [6].

With this choice of coordinates, the inner product on $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ is given by the standard Euclidean metric

$$g_0 = (dx^{1})^2 + ... + (dx^{7})^2. \quad (2.10)$$

As seen from (2.6), $G_2$ preserves the inner product on $\text{Im}(\mathbb{O})$, so it clearly preserves $g_0$ and is hence a subgroup of $SO(7)$.

Since $\varphi_0$ defines the 7-dimensional cross product, and $G_2$ is the symmetry group of this cross product, $G_2$ is the stabilizer of $\varphi_0$ in $GL(7, \mathbb{R})$. So we can state:
Theorem 4 (Bryant, [12]) The subgroup of $GL(7, \mathbb{R})$ that preserves the 3-form $\varphi_0$ is $G_2$. From the metric $g_0$ we can define the Hodge star $\ast_0$ on $\mathbb{R}^7$, and using this, the dual 4-form $\psi_0 = \ast_0 \varphi_0$ which is given by

$$\psi_0 = e^{4567} + e^{2367} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \tag{2.11}$$

This is a key property of $G_2$ and as such this is often taken as the definition of the group $G_2$, in particular in [28]. As we have seen, $G_2$ preserves both $\varphi_0$ and $g_0$, so it also preserves $\psi_0$. In particular, $\varphi_0$ and $\psi_0$ give alternate descriptions of the trivial 1-dimensional representation of $G_2$.

It also turns out that $\psi_0$ is closely related to the associator on $\text{Im} (\mathbb{O})$. As the octonions are non-associative, we can define a non-trivial associator map

$$[\cdot, \cdot, \cdot] : \text{Im} (\mathbb{O}) \times \text{Im} (\mathbb{O}) \times \text{Im} (\mathbb{O}) \rightarrow \text{Im} (\mathbb{O})$$
given by

$$[a, b, c] = a (bc) - (ab) c. \tag{2.12}$$

Just as $\varphi_0$ is defined as a dualization of the cross product using the inner product to obtain the map

$$\varphi_0 : \text{Im} (\mathbb{O}) \times \text{Im} (\mathbb{O}) \times \text{Im} (\mathbb{O}) \rightarrow \mathbb{R}$$
so it turns out that up to a constant multiple the map

$$\psi_0 : \text{Im} (\mathbb{O}) \times \text{Im} (\mathbb{O}) \times \text{Im} (\mathbb{O}) \times \text{Im} (\mathbb{O}) \rightarrow \mathbb{R}$$
is a dualization of the associator, given by

$$\psi_0 (a, b, c, d) = \frac{1}{2} \langle [a, b, c], d \rangle. \tag{2.13}$$

It is possible to show that $\varphi_0$ and $\psi_0$ satisfy various contraction identities. In particular, from [13] [20] [30], we have
Proposition 5 The 3-form $\varphi_0$ and the corresponding 4-form $\psi_0$ satisfy the following identities:

\[
\begin{align*}
\varphi_{0abc} \varphi_{0mn} &= g_{0am}g_{0bn} - g_{0an}g_{0bm} + \psi_{0abmn} \quad (2.14a) \\
\psi_{0abc} \psi_{0mnq} &= 3 \left( g_{0a[m} \varphi_{0np]b} - g_{0b[m} \varphi_{0np)a} \right) \quad (2.14b) \\
\psi_{0abcd} \psi_{0}^{mnpq} &= 24 \delta_a^m \delta_b^n \delta_c^p \delta_d^q + 72 \psi_{0[ab} [mnp] \delta_c^p \delta_d^q - 16 \varphi_{0[abc} \varphi_{0}^{[mnp] \delta_d^q} \quad (2.14c)
\end{align*}
\]

where $[m \, n \, p]$ denotes antisymmetrization of indices and $\delta_a^b$ is the Kronecker delta, with $\delta_a^a = 1$ if $a = b$ and 0 otherwise.

The above identities can be of course further contracted - the details can be found in [20, 30]. These identities and their contractions are crucial whenever any calculations involving $\varphi_0$ and $\psi_0$ have to be done. In particular, these are very useful when studying $G_2$ manifolds.

2.2 Representations of $G_2$

As we will see in section 3, a crucial role in the study of $G_2$ structures is played by the representations of $G_2$. Since $G_2$ is a subgroup of $SO(7)$, it has a fundamental vector representation on $\mathbb{R}^7$. In the study of $G_2$ manifolds, it is very important to understand the representations of $G_2$ on $p$-forms. So let us consider first the representations of $G_2$ on antisymmetric tensors in $\mathbb{R}^7$. For brevity let $V = \mathbb{R}^7$. Following Bryant [13], we first look at the the Lie algebra $\mathfrak{so}(7)$, which is the space of antisymmetric 7 × 7 matrices on $V$. For a vector $\omega \in V$, define the map

\[ \rho_\varphi : V \to \mathfrak{so}(7) \text{ given by } \rho_\varphi(\omega) = \omega \varphi_0 \]

which is clearly injective. Conversely, define the map

\[ \tau_\varphi : \mathfrak{so}(7) \to V \text{ given by } \tau_\varphi(\alpha_{ab})^c = \frac{1}{6} \varphi_0^{c(ab} \alpha_{ab} \]

From (2.15), we get that

\[ \tau_\varphi(\rho_\varphi(\omega)) = \omega, \]

so that $\tau_\varphi$ is a partial inverse of $\rho_\varphi$. Thus we get a decomposition

\[ \mathfrak{so}(7) = \ker \tau_\varphi \oplus \rho_\varphi(V) \]

where $\dim \rho_\varphi(V) = 7$ and $\dim \ker \tau_\varphi = 14$. It turns out that $\ker \tau_\varphi$ is in fact a Lie algebra with respect to the matrix commutator. This is the Lie algebra bracket on $\mathfrak{so}(7)$ and satisfies the Jacobi identity. It is hence only necessary to show that for $\alpha, \beta \in \ker \tau_\varphi$, we have $[\alpha, \beta] \in \ker \tau_\varphi$. This is an exercise in applying the contractions for $\varphi$. Thus we get a 14-dimensional Lie subalgebra of $\mathfrak{so}(7)$. However, this is precisely the Lie algebra $\mathfrak{g}_2$ [30], that is

\[ \mathfrak{g}_2 = \ker \tau_\varphi = \left\{ \alpha \in \mathfrak{so}(7) : \varphi_{0abc} \alpha^{bc} = 0 \right\}. \]

This further implies that we get the following decomposition of $\mathfrak{so}(7)$:

\[ \mathfrak{so}(7) = \mathfrak{g}_2 \oplus \rho_\varphi(V). \]

The group $G_2$ acts via the adjoint representation on the 14-dimensional vector space $\mathfrak{g}_2$ and via the fundamental vector representation on the 7-dimensional space $\rho_\varphi(V)$. This is a $G_2$-invariant irreducible decomposition of $\mathfrak{so}(7)$ into the representations 7 and 14. Hence we get the following result:
Theorem 6 (Bryant, [12]) The space $\Lambda^2$ of 2-forms on $V$ decomposes as

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14},$$

with the components $\Lambda^2_7$ and $\Lambda^2_{14}$ given by:

$$\Lambda^2_7 = \{ \omega \cdot \varphi : \omega \text{ a vector} \}$$

$$\Lambda^2_{14} = \left\{ \alpha = \frac{1}{2} \alpha_{ab} e^a \wedge e^b : (\alpha_{ab}) \in \mathfrak{g}_2 \right\}$$

An alternative, but fully equivalent, description of $\Lambda^2_7$ and $\Lambda^2_{14}$ presents them as eigenspaces of the operator

$$T\psi : \Lambda^2 \longrightarrow \Lambda^2$$

given by $T\psi(\alpha_{ab}) = \psi_{0abcd} \alpha^{cd}$

With this description, we have [30]:

$$\Lambda^2_7 = \left\{ \alpha \in \Lambda^2 : T\psi \alpha = 4\alpha \right\}$$

$$\Lambda^2_{14} = \left\{ \alpha \in \Lambda^2 : T\psi \alpha = -2\alpha \right\}.$$  

Correspondingly, the description of the 7 and 14 pieces of $\Lambda^5$ is obtained from (2.21a) and (2.21b) via Hodge duality.

Let us now look at 3-forms in more detail. Consider $\text{Sym}^2 (V^*)$ - the space of symmetric 2-tensors on $V$, and define a map

$$i_\varphi : \text{Sym}^2 (V^*) \longrightarrow \Lambda^3$$

given by $i_\varphi (h)_{abc} = h_{[a} \varphi_{b]c}d.$

We can decompose $\text{Sym}^2 (V^*) = \mathbb{R} g_0 \oplus \text{Sym}^2_0 (V^*)$ where $\mathbb{R} g_0$ is the set of symmetric tensors proportional to the metric $g_0$ and $\text{Sym}^2_0 (V^*)$ is the set of traceless symmetric tensors. This is a $G_2$-invariant irreducible decomposition of $\text{Sym}^2 (V^*)$ into 1-dimensional and 27-dimensional representations. We clearly have

$$i_\varphi (g_0)_{abc} = \varphi_{0abc},$$

so the map $i_\varphi$ is also $G_2$-invariant and is injective on each summand of this decomposition. Looking at the first summand, we get that $i_\varphi (\mathbb{R} g_0) = \Lambda^3_1$ - the one-dimensional singlet representation of $G_2$. Now look at the second summand and consider $i_\varphi (\text{Sym}^2_0 (V^*))$. This is 27-dimensional and irreducible, so it gives a 27-dimensional representation of $G_2$ on 3-forms:

$$i_\varphi (\text{Sym}^2_0 (V^*)) = \Lambda^3_{27} (V^*).$$

Now, $\Lambda^3$ is 35-dimensional, and we have accounted for $1+27 = 28$ dimensions. Thus we still have 7 dimensions left unaccounted for in $\Lambda^3$. So let us extend the map $i_\varphi$ to $\Lambda^2$ - the antisymmetric 2-tensors on $\mathbb{R}^7$. Suppose $\beta \in \Lambda^2_7$. Then $\beta = \omega \cdot \varphi_0$, for some vector $\omega \in V$ so

$$i_\varphi (\beta)_{abc} = \varphi^d_0 [a|c| \varphi_{0bc}]d \omega^e = \psi_{0abcd} \omega^d$$

where we have used (2.24). This defines a $G_2$-invariant map from $V$ to $\Lambda^3$ and hence gives $\Lambda^3_7$.

So overall we thus have a decomposition of 3-forms into irreducible representations of $G_2$:

Theorem 7 (Bryant, [13]) The space $\Lambda^3$ of 3-forms on $V$ decomposes as

$$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$$

where

$$\Lambda^3_1 = \{ \chi \in \Lambda^3 : \chi_{abc} = f \varphi_{0abc} \text{ for scalar } f \}$$

$$\Lambda^3_7 = \{ \omega \cdot \varphi_0 : \omega \text{ a vector} \}.$$  

$$\Lambda^3_{27} = \{ \chi \in \Lambda^3 : \chi_{abc} = h^d_{[a} \varphi_{0bc]d} \text{ for } h_{ab} \text{ traceless, symmetric} \}.$$
From the identities for contraction of $\varphi_0$ and $\psi_0$, it is possible to see that an equivalent description of $\Lambda_2^3$ is

$$\Lambda_2^3 = \{ \chi \in \Lambda^3 : \chi \wedge \varphi_0 = 0 \text{ and } \chi \wedge \psi_0 = 0 \}.$$  

A similar decomposition of 4-forms is again obtained via Hodge duality.

Suppose we have $\chi \in \Lambda^3$, then define $\pi_1$, $\pi_7$ and $\pi_{27}$ to be projections of $\chi$ onto $\Lambda^3_1$, $\Lambda^3_7$ and $\Lambda^3_{27}$, respectively. Using contraction identities for $\varphi$ and $\psi$, we get the following relations [20]:

**Proposition 8** Given a 3-form $\chi \in \Lambda^3$, the projections of $\chi$ onto the components (2.26) of $\Lambda^3$ are given by:

$$\pi_1(\chi) = \frac{1}{7} (\chi, \varphi_0) \text{ with } |\pi_1(\chi)|^2 = \frac{7}{9} a^2$$  

$$\pi_7(\chi) = \omega \psi_0 \text{ where } \omega^a = \frac{1}{24} \chi_{mnp} \psi_0^{mnpa} \text{ with } |\pi_7(\chi)|^2 = \frac{4}{9} |\omega|^2$$

$$\pi_{27}(\chi) = i_\varphi (h) \text{ where } h_{ab} = -\frac{3}{4} \chi_{mnp} \varphi_{ab} \text{ with } |\pi_{27}(\chi)|^2 = \frac{2}{9} |h|^2.$$  

Here $\{a b\}$ denotes the traceless symmetric part.

Note that similar projections can be defined for 4-forms as well.

### 3 $G_2$ structures

#### 3.1 Definition

As we shall see, the notion of holonomy is closely related to $G$-structures on manifolds. Let us give the necessary definitions

**Definition 9** Let $X$ be a manifold of dimension $n$. Suppose $TX$ is the tangent bundle over $X$. Define the manifold $F$ by

$$F = \{ (x, e_1, ..., e_n) : x \in X \text{ and } (e_1, ..., e_n) \text{ is a basis for } T_xX \}.$$  

This then has a projection $\pi : (x, e_1, ..., e_n) \mapsto x$ onto $X$ and a natural left action by $GL(n, \mathbb{R})$ on the fibres. $F$ is thus a principal bundle over $X$ with fibre $GL(n, \mathbb{R})$, called the frame bundle of $X$.

**Definition 10** Let $X$ be a manifold of dimension $n$. Let $G$ be a Lie subgroup of $GL(n, \mathbb{R})$. Then a $G$-structure on $X$ is a principal subbundle $P$ of $F$ with fibre $G$.

The framework of $G$-structures is very powerful, and a number of geometrical structures can be reformulated in this language. In particular, a Riemannian metric on a manifold is equivalent to an $O(n)$ structure. We are in particular interested in torsion-free $G$-structures. A $G$-structure is torsion-free if and only if there exists a compatible torsion-free connection on $TM$. A connection $\nabla$ on $TM$ is equivalent to a connection $D$ on the frame bundle $F$, and we say $\nabla$ is compatible with the $G$-structure $P$ if $D$ reduces to a connection on $P$. For example, given a Riemannian metric, a unique torsion-free Levi-Civita connection can always be defined, hence all $O(n)$ structures are torsion-free. On a complex manifold with complex dimension, an integrable complex structure is equivalent to a torsion-free $GL(m, \mathbb{C})$ structure. A Kähler structure is then equivalent to a torsion-free $U(m)$-structure. From [28] we have a key result that relates torsion-free structures and holonomy:
Proposition 11 Let \((X, g)\) be a Riemannian manifold of dimension \(n\), with \(O(n)\)-structure \(P\) corresponding to \(g\). Let \(G\) be a Lie subgroup of \(O(n)\). Then \(\text{Hol}(g) \subseteq G\) if and only if \(X\) admits a torsion-free \(G\)-structure \(Q\) that is a subbundle of \(P\).

As Proposition 11 shows, the study of Riemannian holonomy is equivalent to studying torsion-free \(G\)-structures. Hence in order to study \(G_2\) holonomy manifolds we will first consider \(G_2\) structures.

Now suppose \(X\) is a smooth, oriented 7-dimensional manifold. Following Joyce [28], define a 3-form \(\varphi\) to be positive if locally we can choose a frame such that \(\varphi\) is written in the form \(3.1\) - that is for every \(p \in X\) there is an oriented isomorphism \(q_p\) between \(T_pX\) and \(\mathbb{R}^7\) such that \(\varphi|_p = \varphi_0\). For each \(p \in X\) define \(\mathcal{P}^3_pX\) to be set of such 3-forms. To each positive \(\varphi\) we can associate a metric \(g\) and a Hodge dual \(*\varphi\) which are identified with \(g_0\) and \(\psi_0\) under the \(q_p\) and the associated metric is written \(3.1\).

Since \(\varphi_0\) is preserved by \(G_2\) and \(GL(7, \mathbb{R})_+\) acts transitively on \(\mathcal{P}^3_pX\), it follows that

\[
\mathcal{P}^3_pX \cong GL(7, \mathbb{R})_+ / G_2
\]

and hence \(\dim \mathcal{P}^3_pX = \dim GL(7, \mathbb{R})_+ - \dim G_2 = 49 - 14 = 35\). This is equal to the dimension of \(\Lambda^3T_p^*X\), hence \(\mathcal{P}^3_pX\) is an open subset of \(\Lambda^3T_p^*X\). Moreover if we consider the bundle \(\mathcal{P}^3X\) over \(X\) with fibre \(\mathcal{P}^3_pX\), it will be an open subbundle of \(\Lambda^3T^*X\).

Given a positive 3-form \(\varphi\) on \(X\), consider at each point \(p\) the set \(Q_p\) of isomorphisms \(q_p\) between \(T_pX\) and \(\mathbb{R}^7\) such that \(\varphi|_p = \varphi_0\). It is then easy to see that \(Q_p \cong G_2\) and that the bundle \(Q\) over \(X\) with fibre \(Q_p\) is in fact a principal subbundle of the frame bundle \(F\). So in fact, \(Q\) is a \(G_2\) structure. The converse is also true - given an oriented \(G_2\) structure \(Q\), we can uniquely define a positive 3-form \(\varphi\) and associated metric \(g\) and 4-form \(\psi\) that correspond to \(\varphi_0, g_0\) and \(\psi_0\) respectively. We thus have a key result:

Theorem 12 (Joyce, [28]) Let \(X\) be an oriented 7-dimensional manifold. There exists a \(1\)-\(1\) correspondence between positive 3-forms on \(X\) and oriented \(G_2\)-structures \(Q\) on \(X\). Moreover, to each positive 3-form \(\varphi\) we can associate a Riemannian metric \(g\) and a corresponding 4-form \(*\varphi\varphi = \psi\) such that \(\varphi|_p = \varphi_0\). Under the isomorphism \(q_p : T_pX \rightarrow \mathbb{R}^7\), these quantities are identified with \(\varphi_0, g_0\) and \(\psi_0\) respectively.

So given a positive 3-form \(\varphi\) on \(X\), it is possible to define a metric \(g\) associated to \(\varphi\). This metric then defines the Hodge star, which we denote by \(*\varphi\) to emphasize the dependence on \(\varphi\). Given the Hodge star, we can in turn define the 4-form \(\psi = \varphi^\ast\varphi\). Thus in fact both the metric \(g\) and the 4-form \(\psi\) are functions of \(\varphi\). By definition, at point \(p \in X\) there is an isomorphism that identifies \(\varphi\) with \(\varphi_0\), \(\psi\) with \(\psi_0\) and \(g\) with \(g_0\). Therefore, properties of \(\varphi_0\) and \(\psi_0\) such as the contraction identities \(2.14\) that we encountered in Section 2.1 also hold for the differential forms \(\varphi\) and \(\psi\).

In general, any \(G\)-structure on a manifold \(X\) induces a splitting of bundles of \(p\)-forms into subbundles corresponding to irreducible representations of \(G\). The same is of course true for \(G_2\) structures. The decomposition of \(p\)-forms on \(\mathbb{R}^7\) carries over to any manifold with a \(G_2\) structure, so from the previous section we have the following decomposition of the spaces of \(p\)-forms \(\Lambda^p\):

\[
\begin{align*}
\Lambda^1 & = \Lambda_7^1, \\
\Lambda^2 & = \Lambda_7^2 \oplus \Lambda_{14}^2, \\
\Lambda^3 & = \Lambda_7^3 \oplus \Lambda_7^5 \oplus \Lambda_{27}^3, \\
\Lambda^4 & = \Lambda_7^4 \oplus \Lambda_{14}^4 \oplus \Lambda_{27}^4, \\
\Lambda^5 & = \Lambda_7^5 \oplus \Lambda_{14}^5, \\
\Lambda^6 & = \Lambda_7^6.
\end{align*}
\]

(3.1a) (3.1b) (3.1c) (3.1d) (3.1e) (3.1f)
Here each \( \Lambda^p_k \) corresponds to the \( k \)-dimensional irreducible representation of \( G_2 \). Moreover, for each \( k \) and \( p \), \( \Lambda^p_k \) and \( \Lambda^{7-p}_k \) are isomorphic to each other via Hodge duality, and also \( \Lambda^p_7 \) are isomorphic to each other for \( n = 1, 2, \ldots, 6 \).

Define the standard inner product on \( \Lambda^p \), so that for \( p \)-forms \( \alpha \) and \( \beta \),

\[
\langle \alpha, \beta \rangle = \frac{1}{p!} \alpha_{a_1 \ldots a_p} \beta^{a_1 \ldots a_p}.
\]

This is related to the Hodge star, since

\[
\lambda \wedge \ast \beta = \langle \lambda, \beta \rangle \text{ vol}
\]

where \( \text{vol} \) is the invariant volume form given locally by

\[
\text{vol} = \sqrt{\det g} dx^1 \wedge \ldots \wedge dx^7.
\]

Then the decompositions (3.1) are orthogonal with respect to (3.2). Note that \( \langle \varphi, \varphi \rangle = 7 \), so in fact we have

\[
V = \frac{1}{7} \int \varphi \wedge \ast \varphi.
\]

We know that the metric \( g \) is defined by the 3-form \( \varphi \) and we can use some of the results from Section 2.1 to find a direct relationship between the two quantities.

**Proposition 13** Given a positive 3-form \( \varphi \) on a 7-manifold \( X \), the associated metric \( g \) is given by

\[
g_{ab} = (\det s)^{-\frac{1}{2}} s_{ab}.
\]

with

\[
s_{ab} = \frac{1}{144} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \tilde{\varepsilon}^{mnpqrst}
\]

where \( \tilde{\varepsilon}^{mnpqrst} \) is the alternating symbol with \( \tilde{\varepsilon}^{12 \ldots 7} = +1 \). Alternatively, for \( u, v \) vector fields on \( X \),

\[
\langle u, v \rangle \text{ vol} = \frac{1}{6} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi
\]

where \( \lrcorner \) denotes interior multiplication:

\[
(u \lrcorner \varphi)_{bc} = u^a \varphi_{abc}.
\]

**Proof.** Consider the quantity \( P_{ab} \) given by

\[
P_{ab} = \varphi_{amn} \varphi_{bpq} \psi^{mnpq}
\]

Using identities (2.11) to contract \( \varphi \) and \( \psi \), this gives

\[
P_{ab} = 24 g_{ab}.
\]

Expanding \( \psi^{mnpq} \) in terms of \( \varphi \) and the Levi-Civita tensor we get

\[
P_{ab} = \frac{1}{6} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \varepsilon^{mnpqrst}.
\]

If we write \( \varepsilon^{mnpqrst} \) for the alternating symbol with \( \varepsilon^{12 \ldots 7} = +1 \), then we get

\[
g_{ab} \sqrt{\det g} = \frac{1}{144} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \varepsilon^{mnpqrst}.
\]
Figure 4: Notation that is used by different authors

Alternatively, let $u$ and $v$ be vector fields on $X$. Then

$$\langle u, v \rangle \sqrt{\det g} = \frac{1}{144} (u^a \varphi_{amn}) (v^a \varphi_{bpq}) \varphi_{rst} \hat{\varepsilon}^{mnpqrst}.$$ 

Hence we get (3.8). Now define

$$s_{ab} = \frac{1}{144} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \hat{\varepsilon}^{mnpqrst}$$

so that then, after taking the determinant of (3.9) we get (3.6).

Thus we see that even though given the 3-form $\varphi$ we can define the metric $g$, this relationship is rather complicated and non-linear. In particular, this also shows that $\psi = \ast \varphi$ depends on $\varphi$ in an even more non-trivial fashion, since the Hodge star depends itself on the metric.

Here we need to say a few words about the notation used for the $G_2$ 3-form $\varphi$ and the associated 4-form $\psi$. The notation that we use here is due to Karigiannis - where the Hodge dual of $\varphi$ is denoted by $\psi$ and was first introduced in [29]. In Figure 4 we summarize the different notations used by other authors: where $e^{ijk} = e^i \wedge e^j \wedge e^k$ and $e^{ijkl} = e^i \wedge e^j \wedge e^k \wedge e^l$ for basis covectors $e^i$.

### 3.2 Torsion-free structures

The definition of a $G_2$ structure only defines the algebraic properties of $\varphi$, and in general does not address the analytical properties of $\varphi$. Using the associated metric $g$ we can define the Levi-Civita connection $\nabla$ on $X$. Then it is natural to ask what are the properties of $\nabla \varphi$. This quantity is known as the torsion of the $G_2$ structure. Originally the torsion of $G_2$ structures was studied by Fernández and Gray [17], and their analysis revealed that there are in fact a total of 16 torsion classes of $G_2$ structures. Later on, Karigiannis reproduced their results using simple computational arguments [30].

Following [30], consider the 3-form $\nabla_X \varphi$ for some vector field $X$. We know that 3-forms split as $\Lambda^3_1 \oplus \Lambda^3_2 \oplus \Lambda^3_7$, so consider the projections $\pi_1, \pi_7$ and $\pi_2$ of $\nabla_X \varphi$ onto these components. Using (2.28), we have

$$\pi_1 (\nabla_X \varphi) = a \varphi$$

where

$$a = X^a (\nabla_a \varphi_{bcd}) \varphi^{bcd} = X^a \nabla_a \left( \varphi_{bcd} \varphi^{bcd} \right) - \varphi_{bcd} X^a \nabla_a \varphi^{bcd}$$

$$= -X^a (\nabla_a \varphi_{bcd}) \varphi^{bcd}$$

$$= 0.$$
Hence we see that the $\Lambda_3^1$ component vanishes. Similarly, for $\Lambda_3^2$, we have
\[
\pi_{27} (\nabla_X \varphi) = i_\varphi (h)
\]
where
\[
\begin{align*}
    h_{ab} &= \frac{3}{4} \left( X^c \nabla_c \varphi_{mn(a)} \right) \varphi_{b}^{\ mn} = \frac{3}{4} X^c \nabla_c \left( \varphi_{mn(a)} \varphi_{b}^{\ mn} \right) - \frac{3}{4} \varphi_{mn(a)} X^c \nabla_c \varphi_{b}^{\ mn} \\
    &= -\frac{3}{4} \left( X^c \nabla_c \varphi_{mn(a)} \right) \varphi_{b}^{\ mn} \\
    &= 0.
\end{align*}
\]
Here we have used the fact that $\varphi_{mna} \varphi_{b}^{\ mn} = 6g_{ab}$, the traceless part of which vanishes. Therefore, the $\Lambda_3^2$ part of $\nabla_X \varphi$ also vanishes. Now consider the $\Lambda_3^7$ component. In this case,
\[
\pi_7 (\nabla_X \varphi) = \omega \cdot \psi
\]
where
\[
\omega^a = -\frac{1}{24} X^c \left( \nabla_c \varphi_{mnp} \right) \psi^{mnpa} = \frac{1}{24} X^a \left( \nabla_a \psi_bce \right) \varphi_{bcd}.
\]
This quantity does not vanish in general, so we can conclude that
\[
\nabla_X \varphi \in \Lambda_3^7
\]
and thus overall,
\[
\nabla \varphi \in W = \Lambda_1^7 \otimes \Lambda_3^7.
\]  
(3.11)

Further classification of torsion classes depends on the decomposition of $W$ into components according to irreducible representations of $G_2$. Given (3.11), we can write
\[
\nabla_a \varphi_{bcd} = T_a^\ c \psi_{bcd}
\]  
(3.12)
where $T_{ab}$ is the full torsion tensor. This 2-tensor fully defines $\nabla \varphi$ since pointwise, it has 49 components and the space $W$ is also 49-dimensional (pointwise). In general we can split $T_{ab}$ as
\[
T = \tau_1 g + \tau_7 + \tau_{14} + \tau_{27}
\]  
(3.13)
where $\tau_i$ is a function, and gives the $1$ component of $T$, $\tau_7 \in \Lambda_3^2$ and hence gives the $7$ component, $\tau_{14} \in \Lambda_2^{14}$ gives the $14$ component and $\tau_{27}$ is traceless symmetric, giving the $27$ component. Note that the normalization of these components is different from [30]. Hence we can split $W$ as
\[
W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}.
\]  
(3.14)

The 16 torsion classes arise as the subsets of $W$ which $\nabla \varphi$ belongs to. Moreover, as shown in [30], the torsion components $\tau_i$ relate directly to the expression for $d \varphi$ and $d \psi$. In fact, in our notation,
\[
\begin{align*}
    d \varphi &= 4\tau_1 \psi + 3\tau_7 \wedge \varphi - *\tau_{27} \\
    d \psi &= 4\tau_7 \wedge \psi - 2 * \tau_{14}.
\end{align*}
\]  
(3.15a)
(3.15b)

Now suppose $d \varphi = d \psi = 0$. Then this means that all four torsion components vanish and hence $T = 0$, and as a consequence $\nabla \varphi = 0$. The converse is trivially true, since $d$ and $d*$ can both be expressed in terms of the covariant derivative. This result is due to Fernández and Gray [17]. If we add the fact that $Hol (g)$ is a subgroup of $G$ if and only if $X$ admits a torsion-free $G$ structure from Proposition [11] then we get the following important result.
Theorem 14 (Joyce, [28, Prop. 10.1.3]) Let $X$ be a 7-manifold with a $G_2$ structure defined by the 3-form $\varphi$ and equipped with the associated Riemannian metric $g$. Then the following are equivalent:

1. The $G_2$-structure is torsion-free
2. $\text{Hol}(g) \subseteq G_2$ and $\varphi$ is the induced 3-form
3. $\nabla \varphi = 0$ on $X$ where $\nabla$ is the Levi-Civita connection of $g$
4. $d \varphi = d \psi = 0$ where $\psi = * \varphi$ with the Hodge star defined by $g$

Different torsion classes of the $G_2$ structure also restrict the curvature of the manifold. Consider the curvature tensor $R_{abcd}$. Then for fixed $a, b$, we have

$$(R_{ab})_{cd} \in \Lambda^2,$$

so we can decompose it as

$$(R_{ab})_{cd} = (\pi_7 R_{ab})_{cd} + (\pi_{14} R_{ab})_{cd}. \quad (3.16)$$

Following Karigiannis [30], consider the operator $T_\psi$ acting on $R_{abcd}$. Then we have

$$
g^{ad} T_\psi R_{abcd} = R_{abef} \psi^{ef}_{cd} g^{ad}$$
$$= - (R_{beaf} + R_{eabf}) \psi^{ef}_{cd} g^{ad}$$
$$= - R_{beaf} \psi^{ef}_{c} a f + R_{fbae} \psi^{ef}_{c}$$
$$= - 2 g^{ad} T_\psi R_{abcd}$$
$$= 0$$

where we have used the cyclic identity for $R_{abef}$. Hence, from (2.23) we get

$$Ric_{bd} = 3 (\pi_7 R_{ab})_{cd} g^{ac} = \frac{3}{2} (\pi_{14} R_{ab})_{cd} g^{ac} \quad (3.17)$$

where $Ric_{bd}$ is the Ricci tensor. However, in general, by the Ambrose-Singer holonomy theorem [5], if $\text{Hol}(g) \subseteq G$, then $R_{abcd} \in \text{Sym}^2 (g)$ where $g$ is the Lie algebra of $G$. Therefore, in the $G_2$ case, if the $G_2$ structure is torsion-free and hence $\text{Hol}(g) \subseteq G_2$, then $R_{abcd} \in \text{Sym}^2 (g_2)$. This however implies that in (3.16), the $\pi_7$ component vanishes, and thus from (3.17), we have the following result:

Theorem 15 (Bonan, [10]) Let $X$ be a Riemannian 7-manifold with metric $g$. If $\text{Hol}(g) \subseteq G_2$, then $X$ is Ricci-flat.

In fact, this result can also be derived without invoking the general Ambrose-Singer theorem. In [30], Karigiannis expressed the $\Lambda^2_7$ component of the curvature tensor in terms of the torsion tensor $T_{ab}$, so that when the torsion vanishes, the curvature tensor is fully contained in $\Lambda^2_{14}$, thus directly confirming the Ambrose-Singer theorem in the $G_2$ case. The original proof of Theorem 15 due to Bonan [10] relied on the fact that the Lie algebra structure of $g_2$ imposes strong conditions on the Riemann tensor, and that these imply that the Ricci tensor cannot be non-vanishing.
Given a compact manifold with a torsion-free $G_2$ structure, the decompositions (3.1) carry over to de Rham cohomology [28], so that we have

\begin{align*}
H^1(X, \mathbb{R}) &= H^1_7 \\
H^2(X, \mathbb{R}) &= H^2_7 \oplus H^2_{14} \\
H^3(X, \mathbb{R}) &= H^3_7 \oplus H^3_{14} \oplus H^3_{27} \\
H^4(X, \mathbb{R}) &= H^4_7 \oplus H^4_{14} \oplus H^4_{27} \\
H^5(X, \mathbb{R}) &= H^5_7 \oplus H^5_{14} \\
H^6(X, \mathbb{R}) &= H^6_7 
\end{align*}

(3.18)

Define the refined Betti numbers $b^k_p = \dim (H^p_k)$. Clearly, $b^k_3 = b^k_4 = 1$ and we also have $b^1_k = b^7_k$ for $k = 1, \ldots, 6$. Moreover, it turns out that if $\text{Hol}(X, g) = G_2$ then $b^1 = 0$. Therefore, in this case the $H^7_k$ component vanishes in (3.18). It can be easily shown that on a Ricci-flat manifold, any harmonic 1-form must be parallel. However this happens if and only if $\text{Hol}(g)$ has an invariant 1-form. However the only $G_2$-invariant forms are $\varphi$ and $\psi$. Therefore there are no non-trivial harmonic 1-forms when $\text{Hol}(g) = G_2$ and thus $b^1 = 0$.

An example of a construction of a manifold with a torsion-free $G_2$ structure is to consider $X = Y \times S^1$ where $Y$ is a Calabi-Yau 3-fold. Define the metric and a 3-form on $X$ as

\begin{align*}
g_X &= d\theta^2 \times g_Y \\
\varphi &= d\theta \wedge \omega + \text{Re} \Omega
\end{align*}

(3.19)

(3.20)

where $\theta$ is the coordinate on $S^1$. This then defines a torsion-free $G_2$ structure, with

\begin{equation}
\ast \varphi = \frac{1}{2} \omega \wedge \omega - d\theta \wedge \text{Im} \Omega.
\end{equation}

(3.21)

However, the holonomy of $X$ in this case is $SU(3) \subset G_2$. From the K"unneth formula we get the following relations between the refined Betti numbers of $X$ and the Hodge numbers of $Y$

\begin{align*}
b^k_7 &= 1 \quad \text{for} \quad k = 1, \ldots, 6 \\
b^k_{14} &= h^{1,1} - 1 \quad \text{for} \quad k = 2, 5 \\
b^k_{27} &= h^{1,1} + 2h^{2,1} \quad \text{for} \quad k = 3, 4.
\end{align*}

(3.22)

In [27] and [28], Joyce describes a possible construction of a smooth manifold with holonomy equal to $G_2$ from a Calabi-Yau manifold $Y$. So suppose $Y$ is a Calabi-Yau 3-fold as above. Then suppose $\sigma : Y \rightarrow Y$ is an antiholomorphic isometric involution on $Y$, that is, $\chi$ preserves the metric on $Y$ and satisfies

\begin{align*}
\sigma^2 &= 1 \\
\sigma^* (\omega) &= -\omega \\
\sigma^* (\Omega) &= \Omega.
\end{align*}

(3.22)

Such an involution $\sigma$ is known as a real structure on $Y$. Define now a quotient given by

\begin{equation}
Z = (Y \times S^1) / \hat{\sigma}
\end{equation}

(3.23)

where $\hat{\sigma} : Y \times S^1 \rightarrow Y \times S^1$ is defined by $\hat{\sigma}(y, \theta) = (\sigma(y), -\theta)$. The 3-form $\varphi$ defined on $Y \times S^1$ by (3.20) is invariant under the action of $\hat{\sigma}$ and hence provides $Z$ with a $G_2$ structure. Similarly, the dual 4-form $\ast \varphi$ given by (3.21) is also invariant. Generically, the action of $\sigma$ on $Y$ will have a non-empty fixed point set $N$, which is in fact a special Lagrangian submanifold on $Y$ [28]. This
gives rise to orbifold singularities on $Z$. The singular set is two copies of $Z$. It is conjectured that it is possible to resolve each singular point using an ALE 4-manifold with holonomy $SU (2)$ in order to obtain a smooth manifold with holonomy $G_2$, however the precise details of the resolution of these singularities are not known yet. We will therefore consider only free-acting involutions, that is those without fixed points.

Manifolds defined by (3.23) with a freely acting involution were called barely $G_2$ manifolds by Harvey and Moore in [24]. The cohomology of barely $G_2$ manifolds is expressed in terms of the cohomology of the underlying Calabi-Yau manifold $Y$:

$$H^2 (Z) = H^2 (Y)^+ \quad (3.24a)$$
$$H^3 (Z) = H^2 (Y)^- \oplus H^3 (Y)^+ \quad (3.24b)$$

Here the superscripts $\pm$ refer to the $\pm$ eigenspaces of $\sigma^*$. Thus $H^2 (Y)^+$ refers to two-forms on $Y$ which are invariant under the action of involution $\sigma$ and correspondingly $H^2 (Y)^-$ refers to two-forms which are odd under $\sigma$. Wedging an odd two-form on $Y$ with $d\theta$ gives an invariant 3-form on $Y \times S^1$, and hence these forms, together with the invariant 3-forms $H^3 (Y)^+$ on $Y$, give the three-forms on the quotient space $Z$. Also note that $H^1 (Z)$ vanishes, since the 1-form on $S^1$ is odd under $\hat{\sigma}$. Now, given a 3-form on $Y$, its real part will be invariant under $\sigma$, hence $H^3 (Y)^+$ is essentially the real part of $H^3 (Y)$. Therefore the Betti numbers of $Z$ in terms of Hodge numbers of $Y$ are

$$b^1 = 0 \quad (3.25a)$$
$$b^2 = h_{1,1}^+ \quad (3.25b)$$
$$b^3 = h_{1,1}^- + h_{2,1} + 1 \quad (3.25c)$$

A class of barely $G_2$ manifolds that are constructed from complete intersection Calabi-Yau manifolds has recently been considered in [19], where the Betti numbers of all such manifolds have been calculated explicitly.

Note that barely $G_2$ manifolds have holonomy $SU (3) \ltimes \mathbb{Z}_2$ while the first Betti number still vanishes. This shows that vanishing first Betti number is not a necessary and sufficient condition for $\text{Hol} (g) = G_2$. In fact, as shown by Joyce in [27], $\text{Hol} (g) = G_2$ if and only if the fundamental group $\pi_1 (X)$ is finite.

Let us briefly describe Joyce’s construction of compact torsion-free manifolds with $\text{Hol} (g) = G_2$. Here we follow [28]. On $T^7$ we can define a flat $G_2$ structure $(\varphi_0, g_0)$, similarly as on $\mathbb{R}^7$. Now suppose that $\Gamma$ is a finite group acting on $T^7$ that preserves the $G_2$ structure. Then we can define the orbifold $T^7/\Gamma$. The key to resolving the orbifold singularities is to consider appropriate Quasi Asymptotically Locally Euclidean (QALE) $G_2$ manifolds. These are 7-manifolds with a torsion-free $G_2$ structure that is asymptotic to the $G_2$ structure on $\mathbb{R}^7/G$ where $G$ is a finite subgroup of $G_2$. The orbifold $T^7/\Gamma$ is then resolved to obtain a smooth compact manifold. However on the resolution, the resulting $G_2$-structure is not necessarily torsion-free, so it is shown that it can be deformed to a torsion-free $G_2$ structure $(\varphi, g)$. Further, the fundamental group is calculated, and if it is finite, then $\text{Hol} (g) = G_2$. Using this method, Joyce found 252 topologically distinct $G_2$ holonomy manifolds with unique pairs of Betti numbers $(b^2, b^3)$.

4 Moduli space

4.1 Deformations of $G_2$ structures

One of the interesting directions in the study of $G_2$ holonomy manifolds is the structure of the moduli space. Essentially, the idea is to consider the space of all torsion-free $G_2$ structures
modulo diffeomorphisms on a manifold with fixed topology. The moduli space itself has an interesting geometry that may give further information about $G_2$ manifolds.

Currently, we can only say something about the very local structure of the $G_2$ moduli space. For this, we take a fixed $G_2$ structure and deform it slightly. The space of these deformations is the local moduli space. To study it, we thus need to understand the deformations of $G_2$ structures. Although, we are mostly interested in deformations of torsion-free $G_2$ structures, many of the results are valid for any $G_2$ structures.

Our aim is to consider infinitesimal deformations of $\varphi$ of the form

$$\varphi \rightarrow \varphi + \varepsilon \chi$$

for some 3-form $\chi$. As we already know, the $G_2$ structure on $X$ and the corresponding metric $g$ are all determined by the invariant 3-form $\varphi$. Hence, deformations of $\varphi$ will induce deformations of the metric. These deformations of metric will then also affect the deformation of $\psi = \star \varphi$. Theoretically, “large” deformations could also be considered, and in fact, as we shall see below in some cases closed expressions can be obtained for large deformations. However in that case, it is difficult to determine the resulting torsion class of the new $G_2$ structure [29]. In order for the deformed $\varphi$ to define a new $G_2$ structure, the new $\varphi$ must also be a positive form (as per the definition of a $G_2$ structure). However it is known [28] that the bundle of positive 3-forms on $X$ is an open subbundle of $\Lambda^3 T^* X$, so we can always find $\varepsilon$ small enough in order for the deformed $\varphi$ to be positive.

Using the decomposition of 3-forms (3.1c), we can split $\chi$ into $\Lambda_1^3$, $\Lambda_7^3$ and $\Lambda_{27}^3$ parts, and at first let us consider each one separately. As shown by Karigiannis in [29], metric deformations can be made explicit when the 3-form deformations are either in $\Lambda_1^3$ or $\Lambda_7^3$. Let us first review some of these results. First suppose

$$\tilde{\varphi} = f \varphi$$

We will also use the notation $\tilde{\psi} = \star \tilde{\varphi}$ where $\star$ is the Hodge star derived from the metric $\tilde{g}$ corresponding to $\tilde{\varphi}$. Then from (3.9) we get

$$\tilde{g}_{ab} \sqrt{|\det \tilde{g}|} = \frac{1}{144} \tilde{\varphi}_{ammn} \tilde{\varphi}_{bpqlrs} \tilde{\varphi}^{mnpqrst} \epsilon_{mnpqrst}$$

and hence

$$\tilde{g}_{ab} = f^2 g_{ab}$$

After taking the determinant on both sides, we obtain

$$\det \tilde{g} = f^{14} \det g.$$ (4.4)

Substituting (4.4) into (4.3), we finally get

$$\tilde{g}_{ab} = f^2 g_{ab}.$$ (4.5)

So, a scaling of $\varphi$ gives a conformal transformation of the metric. Hence deformations of $\varphi$ in the direction $\Lambda_1^3$ also give infinitesimal conformal transformation. Suppose $f = 1 + \varepsilon a$, then to third order in $\varepsilon$, we can write

$$\tilde{\psi} = \left(1 + \frac{4}{3} a \varepsilon + \frac{2}{9} a^2 \varepsilon^2 - \frac{4}{81} a^3 \varepsilon^3 + O(\varepsilon^4)\right) \psi.$$ (4.7)

Given a torsion-free $G_2$ structure, $d\varphi = d\psi = 0$, so if we want the deformed structure to be also torsion-free, $f$ must be constant.
Now, suppose in general that $\tilde{\varphi} = \varphi + \varepsilon \chi$ for some $\chi \in \Lambda^3$. Then using (3.8) for the definition of the metric associated with $\tilde{\varphi}$, after some manipulations, we get:

$$\tilde{\langle} u,v \tilde{\rangle} \tilde{\text{vol}} = \frac{1}{6} (u \wedge (v \wedge \varphi) + \varepsilon \left[ (u \wedge \chi) \wedge * (v \wedge \varphi) + (v \wedge \chi) \wedge * (u \wedge \varphi) \right] + \frac{1}{2} \varepsilon^2 (u \wedge \chi) \wedge (v \wedge \chi) \wedge \varphi$$

$$+ \frac{1}{6} \varepsilon^3 (u \wedge \chi) \wedge (v \wedge \chi) \wedge \chi. \quad (4.8)$$

Rewriting (4.8) in local coordinates, we get

$$\tilde{g}_{ab} \sqrt{\det \tilde{g}} = g_{ab} + \frac{1}{2} \varepsilon \chi^{mn} (a \varphi_b)^{mn} + \frac{1}{8} \varepsilon^2 \chi_{amn} \chi_{bpq} \psi^{mpnq} + \frac{1}{24} \varepsilon^3 \chi_{amn} \chi_{bpq} (\ast \chi)^{mpnq} \quad (4.9)$$

Now suppose the deformation is in the $\Lambda^3_{37}$ direction. This implies that

$$\chi = \omega \wedge \psi \quad (4.10)$$

for some vector field $\omega$. Look at the first order term in (4.9). From (2.28) we see that this is essentially a projection onto $\Lambda^3_{37} \oplus \Lambda^3_{27}$ - the traceless part gives the $\Lambda^3_{37}$ component and the trace gives the $\Lambda^3_1$ component. Hence this term vanishes for $\chi \in \Lambda^3_{37}$. For the third order term, it is more convenient to study at it in (4.8). By looking at

$$\omega \wedge ((u \wedge \omega \wedge \psi) \wedge (v \wedge \omega \wedge \psi) \wedge \psi) = 0$$

we immediately see that the third order term vanishes. So now we are left with

$$\tilde{g}_{ab} \sqrt{\det \tilde{g}} = \left( g_{ab} + \frac{1}{8} \varepsilon^2 \omega^d \psi_{camn} \psi_{dpq} \psi^{mpnq} \right) \sqrt{\det g}$$

$$= \left( g_{ab} \left( 1 + \varepsilon^2 |\omega|^2 \right) - \varepsilon^2 \omega_a \omega_b \right) \sqrt{\det g} \quad (4.11)$$

where we have used a contraction identity for $\psi$ twice. Taking the determinant of (4.11) gives

$$\sqrt{\det \tilde{g}} = \left( 1 + \varepsilon^2 |\omega|^2 \right)^{\frac{1}{2}} \sqrt{\det g}. \quad (4.12)$$

Eventually we have the following result:

**Theorem 16 (Karigiannis, [29])** Given a deformation of a $G_2$ structure (4.1) with $\chi = \omega \wedge \psi \in \Lambda^3_1$, then the new metric $\tilde{g}_{ab}$ is given by

$$\tilde{g}_{ab} = \left( 1 + \varepsilon^2 |\omega|^2 \right)^{-\frac{1}{2}} \left( g_{ab} \left( 1 + \varepsilon^2 |\omega|^2 \right) - \varepsilon^2 \omega_a \omega_b \right) \quad (4.13)$$

and the deformed 4-form $\tilde{\psi}$ is given by

$$\tilde{\psi} = \left( 1 + \varepsilon^2 |\omega|^2 \right)^{-\frac{1}{4}} \left( \psi + * \varepsilon (\omega \wedge \psi) + \varepsilon^2 \omega \wedge * (\omega \wedge \varphi) \right). \quad (4.14)$$

One of the key reasons why it is possible to get these closed form expressions for modified $g$ and $\psi$ is because as shown by Karigiannis in [29], the determinant of (4.11) can be calculated
in a closed form. Notice that to first order in $\varepsilon$, both $\sqrt{\det g}$ and $g_{ab}$ remain unchanged under this deformation. Now let us examine the last term in (4.14) in more detail. Firstly, we have
\[
\omega \land (\omega \lrcorner \varphi) = (\omega \land (\omega \lrcorner \varphi))
\]
and
\[
\left(\omega \land (\omega \lrcorner \varphi)\right)_{mnp} = 3\omega_{[m|\omega|^2]np} = 3i_{\varphi}(\omega \circ \omega) (4.15)
\]
where $(\omega \circ \omega)_{ab} = \omega_a \omega_b$. Therefore, in (4.14), this term gives $\Lambda^4_1$ and $\Lambda^4_2$ components. So, can write (4.14) as
\[
\tilde{\psi} = \left(1 + \varepsilon^2 |\omega|^2\right)^{-\frac{1}{4}} \left(1 + \frac{3}{7} \varepsilon^2 |\omega|^2\right) \psi + \varepsilon \left(\omega \lrcorner \varphi + \varphi \lrcorner \varphi\right). (4.16)
\]
Here $(\omega \circ \omega)_0$ denotes the traceless part of $\omega \circ \omega$, so that $i_{\varphi}(\omega \circ \omega)_0) \in \Lambda^3_{27}$ and thus, in (4.16), the components in different representations are now explicitly shown.

To first order, we thus have the deformations
\[
\tilde{\varphi} = \varphi + \varepsilon \left(\omega \lrcorner \varphi\right)
\]
\[
\tilde{\psi} = \psi + \varepsilon \left(\omega \lrcorner \varphi\right).
\]
If originally $d\varphi = d\psi = 0$, that is, the $G_2$ structure is torsion-free, then for the deformed structure to be torsion-free to first order we need
\[
d(\omega \lrcorner \psi) = d(\omega \lrcorner \varphi) = 0.
\]
By expanding $d(\omega \lrcorner \psi)$ in terms of the decomposition of $\Lambda^4$, and setting each term individually to 0, we find that the symmetric part of $\nabla \omega$ and the $\Lambda^2_7$ part of $d\omega^\flat$ must vanish. Furthermore, by expanding $\ast d (\omega \lrcorner \psi)$ in terms of the decomposition of $\Lambda^2$ we find that the $\Lambda^4_1$ part of $d\omega^\flat$ must also vanish. Hence we get that $\nabla \omega = 0$. If $Hol(g) = G_2$, then we know that in this case $\omega = 0$, so there are no interesting small $\Lambda^3_7$ deformations of manifolds with holonomy equal to $G_2$.

As we have seen above, in the cases when the deformations were in $\Lambda^3_7$ or $\Lambda^3_2$ directions, there were some simplifications, which make it possible to write down all results in a closed form. In the case of deformations in $\Lambda^3_2$, the only known way to get results for deformations of the metric and the 4-form $\psi$ is to consider the deformations order by order in $\varepsilon$. This analysis has been carried out in [20], and here we will review those results. So suppose we have a deformation
\[
\tilde{\varphi} = \varphi + \varepsilon \chi
\]
where $\chi \in \Lambda^3_{27}$. Now let us set up some notation. Define
\[
\tilde{s}_{ab} = \frac{1}{144} \frac{1}{\sqrt{\det g}} \tilde{\varphi}_{amn} \tilde{\varphi}_{bpq} \tilde{\varphi}_{rst} \tilde{\varepsilon}^{mnpqrst} (4.18)
\]
\[
\tilde{g}_{ab} = \frac{\det \tilde{g}}{\det g} \frac{\sqrt{\det g}}{\det \tilde{g}} (4.19)
\]
From (4.19), the untilded $s_{ab}$ is then just equal to $g_{ab}$. We can rewrite (4.19) as
\[
\tilde{g}_{ab} = \left(\frac{\det g}{\det \tilde{g}}\right) (g_{ab} + \delta s_{ab}) (4.20)
\]
where $\delta g_{ab}$ is the deformation of the metric and $\delta s_{ab}$ is the deformation of $s_{ab}$, which from (4.39) is given by

$$
\delta s_{ab} = \frac{1}{2} \varepsilon \chi_{mn(\varphi b)} mn + \frac{1}{8} \varepsilon^2 \chi_{amn} \chi_{bpq} \psi^{mnpq} + \frac{1}{24} \varepsilon^3 \chi_{amn} \chi_{bpq} (\ast \chi)^{mnpq}.
$$

(4.21)

Also introduce the following short-hand notation

$$
s_k = \text{Tr} \left( (\delta s)^k \right)
$$

(4.22)

where the trace is taken using the original metric $g$. From (4.21), note that since $\chi \in \Lambda^{3,3}_2$, when taking the trace the first order term vanishes, and hence $s_1$ is at least second-order in $\varepsilon$. Clearly, for $k > 1$, $s_k$ are at least of order $k$ in $\varepsilon$. Similarly as before, take the determinant of (4.18):

$$
\frac{(\det \tilde{g})^\frac{2}{7}}{\det g} = \frac{\det (g + \delta s)}{\det g}.
$$

(4.23)

Unlike in the case of $\Lambda_3^{2,3}$ deformations, we cannot compute $\det (g + \delta s)$ in closed form, so we have to calculate it order by order in $\varepsilon$. From the standard expansion of $\det (I + X)$, we find

$$
\frac{\det (g + \delta s)}{\det g} = 1 + s_1 + \frac{1}{2} (s_1^2 - s_2) + \frac{1}{6} (s_1^3 - 3s_1 s_2 + 2s_3) + O(\varepsilon^4)
$$

(4.24)

However, as we noted above, $s_1$ is second-order in $\varepsilon$, so this expression actually simplifies:

$$
\frac{\det (g + \delta s)}{\det g} = 1 + \left( s_1 - \frac{1}{2} s_2 \right) + \frac{1}{3} s_3 + O(\varepsilon^4).
$$

(4.25)

Raising this to the power of $-\frac{1}{7}$, and expanding again to fourth order in $\varepsilon$, we get

$$
\left( \frac{\det g}{\det \tilde{g}} \right)^\frac{2}{7} = 1 + \left( \frac{1}{18} s_2 - \frac{1}{9} s_3 \right) - \frac{1}{27} s_3 + O(\varepsilon^4).
$$

(4.26)

Using this and (4.20), we can immediately get the deformed metric, but the expressions using the current form of $\delta s_{ab}$ are not very useful. So far, the only property of $\Lambda_3^{3,3}$ that we have used is that it is orthogonal to $\varphi$, thus in fact, up to this point everything applies to $\Lambda_3^{3,3}$ as well. Now however, let $\chi$ be of the form

$$
\chi_{abc} = h_{[a}^{d} \varphi_{bc]}^d
$$

(4.27)

where $h_{ab}$ is traceless and symmetric, so that $\chi \in \Lambda_3^{3,3}$. Let us first introduce some further notation. Let $h_1, h_2, h_3, h_4$ be traceless, symmetric matrices, and introduce the following shorthand notation

$$
(\varphi h_1 h_2 \varphi)_{mn} = \varphi_{alm} h_1^{ad} h_2^{be} \varphi_{dcm}
$$

(4.28a)

$$
\varphi h_1 h_2 h_3 \varphi = \varphi_{ab} h_1^{ad} h_2^{be} h_3^{cf} \varphi_{def}
$$

(4.28b)

$$
(\psi h_1 h_2 h_3 \psi)_{mn} = \psi_{abm} \psi_{def} h_1^{ad} h_2^{be} h_3^{cf} h_{mn}^{ef}
$$

(4.28c)

$$
\psi h_1 h_2 h_3 \psi = \psi_{abcm} \psi_{def} h_1^{ad} h_2^{be} h_3^{cf} h_{mn}^{ef}
$$

(4.28d)

It is clear that all of these quantities are symmetric in the $h_i$ and moreover $(\varphi h_1 h_2 \varphi)_{mn}$ and $(\psi h_1 h_2 h_3 \psi)_{mn}$ are both symmetric in indices $m$ and $n$. Then, it can be shown that

$$
\chi_{(a[mn][\varphi b]}^{mn} = \frac{4}{3} h_{ab}
$$

$$
\chi_{amn} \chi_{bpq} \psi^{mnpq} = -\frac{4}{7} |\chi|^2 g_{ab} + \frac{16}{9} (h^2)_{(ab)} - \frac{4}{9} (\varphi h_1 \varphi)_{(ab)}
$$

$$
\chi_{amn} \chi_{bpq} \ast \chi^{mnpq} = -\frac{32}{189} \text{Tr} (h^3) g_{ab} - \frac{8}{9} (\varphi h^2 \varphi)_{(ab)}
$$
and symmetric. Then to first order the conditions for 
\( d\chi \) is thus equivalent to 
\[ \psi = \frac{1}{63} \varepsilon^2 \text{Tr}(h^2) + \frac{4}{961} \varepsilon^3 \text{Tr}(h^3) \]
and correspondingly, the deformed 4-form \( \tilde{\psi} \) is given by

\[ \tilde{\psi} = \psi - \varepsilon \ast \chi + \varepsilon^2 \left( -\frac{1}{189} \text{Tr}(h^2) \psi + \frac{1}{6} \ast \iota_{\phi}((\phi hh\varphi)_0) \right) + \varepsilon^3 \left( \frac{2}{1701} (\phi hh\varphi)_0 \psi - \frac{5}{108} \text{Tr}(h^2) \ast \chi + \frac{1}{18} \ast \iota_{\phi}((\psi hh\psi)_0) - \frac{1}{36} \ast \iota_{\phi}((\psi hh\psi)_0) + \frac{1}{324} \alpha \wedge \varphi \right) + O(\varepsilon^4) \]
where \((\phi hh\varphi)_0, h_3^3\) and \((\psi hh\psi)_0\) denote the traceless parts of \((\phi hh\varphi)_0, (h^3)_0\) and \((\psi hh\psi)_0\), respectively, and

\[ \alpha_a = \psi_{\alpha\mu
n p} \varphi_{\mu
n p} h^{\mu
n p} h^{\alpha\mu} h^{\alpha\nu} \]

In general if such a deformation is performed on a torsion-free G2 structure, then it is not known what conditions must \( h \) satisfy in order for the torsion class to be preserved. If we restrict our analysis only to first order deformations, then it is easier to see these conditions.

Suppose we have \( d\varphi = d\psi = 0 \) and we apply a deformation (4.1) with \( \chi = \iota_{\phi}(h) \) for traceless and symmetric. Then to first order the conditions for \( d\tilde{\varphi} = d\tilde{\psi} = 0 \) are

\[ d\chi = d \ast \chi = 0. \]

Hence the deformation must be a form that is closed and co-closed. For a compact manifold this is thus equivalent to \( \chi \) being harmonic. We can also find what this condition means in terms of \( h \). By decomposing \( d\chi \) into \( \Lambda_1^4, \Lambda_2^4 \) and \( \Lambda_{27}^4 \) components, we find that we must have

\[ \nabla_r h^r_a = 0 \]
\[ \nabla_m h_a (\nu^m a) c = 0 \]

where as before \( \{a \ b\} \) denotes the traceless symmetric part. Using this and (4.21), we can now express \( \delta s_{ab} \) in terms of \( h \):

\[ \delta s_{ab} = \frac{2}{3} \varepsilon h_{ab} + g_{ab} \left( -\frac{1}{63} \varepsilon^2 \text{Tr}(h^2) + \frac{4}{961} \varepsilon^3 \text{Tr}(h^3) \right) + \varepsilon^2 \left( \frac{2}{9} (h^2)_{(ab)} - \frac{1}{18} (\varphi hh\varphi)_{(ab)} \right) - \frac{\varepsilon^3}{27} (\varphi hh^2 \varphi)_{(ab)} \]

and hence

\[ s_1 = \text{Tr}(\delta s) = -\frac{1}{9} \varepsilon^2 \text{Tr}(h^2) + \frac{4}{81} \varepsilon^3 \text{Tr}(h^3) \]
\[ s_2 = \text{Tr}(\delta s^2) = \frac{4}{9} \varepsilon^2 \text{Tr}(h^2) + \varepsilon^3 \left( \frac{8}{27} \text{Tr}(h^3) - \frac{2}{27} (\varphi hh\varphi) \right) \]
\[ s_3 = \text{Tr}(\delta s^3) = \frac{8}{27} \varepsilon^3 \text{Tr}(h^3) \]

Substituting these expressions into (4.26) and (4.20), we can get the full expression for the deformed metric (up to third order in \( \varepsilon \)) and correspondingly the expression for the deformed 4-form \( \tilde{\psi} \):

**Theorem 17 (Grigorian and Yau, [20])**
Given a deformation of a G2 structure (4.7) with \( \chi_{abc} = h^d_{(a} \varphi_{bc)d} \in \Lambda_3^2 \), then the new metric \( \tilde{g}_{ab} \) is given to third order in \( \varepsilon \) by

\[ \tilde{g}_{ab} = \left( 1 + \frac{1}{18} \varepsilon^2 \text{Tr}(h^2) + \frac{1}{81} \varepsilon^3 \text{Tr}(h^3) - \frac{1}{243} \varepsilon^3 (\varphi hh\varphi) \right) g_{ab} + \frac{2}{3} \varepsilon h_{ab} \]

and correspondingly, the deformed 4-form \( \tilde{\psi} \) is given by

\[ \tilde{\psi} = \psi - \varepsilon \ast \chi + \varepsilon^2 \left( -\frac{1}{189} \text{Tr}(h^2) \psi + \frac{1}{6} \ast \iota_{\phi}((\phi hh\varphi)_0) \right) + \varepsilon^3 \left( \frac{2}{1701} (\phi hh\varphi)_0 \psi - \frac{5}{108} \text{Tr}(h^2) \ast \chi + \frac{1}{18} \ast \iota_{\phi}((\psi hh\psi)_0) + \frac{1}{324} \alpha \wedge \varphi \right) + O(\varepsilon^4) \]

where \((\phi hh\varphi)_0, h_3^3\) and \((\psi hh\psi)_0\) denote the traceless parts of \((\phi hh\varphi)_0, (h^3)_0\) and \((\psi hh\psi)_0\), respectively, and

\[ \alpha_a = \psi_{\alpha\mu
n p} \varphi_{\mu
n p} h^{\mu
n p} h^{\alpha\mu} h^{\alpha\nu} \]
Further, if we decompose \( *d* \chi \) into \( \Lambda^2_7 \) and \( \Lambda^2_{14} \) components, we again get (4.34a) and moreover get a new constraint

\[
\nabla_m h_{a[b} \varphi^{ma}_{\ c]} = 0 \quad (4.35a)
\]

Thus overall, for \( h \) traceless and symmetric, \( \chi = i \varphi (h) \) being closed and co-closed is equivalent to

\[
\nabla_r h^r_a = 0 \quad \text{and} \quad \nabla_m h_{ab} \varphi^{ma}_{\ c} = 0.
\]

On a compact manifold \( \chi \) being closed and co-closed is equivalent to \( \chi \) being harmonic. It also turns out [2] that, if \( \chi \) is defined as above, then

\[
\Delta \chi = 0 \iff \Delta_L h = 0
\]

where \( \Delta_L \) is the Lichnerowicz operator given by

\[
\Delta_L h_{ab} = \nabla^2 h_{ab} + 2 R_{acbd} h^{cd}.
\]

Therefore to preserve the torsion-free \( G_2 \) structure, we have to limit our attention to zero modes of the Lichnerowicz operator. Note that, to linear order, traceless deformations of the metric which preserve the Ricci tensor are also precisely the Lichnerowicz zero modes, and this is consistent with (4.31) where the linear term in the metric deformation is proportional to \( h \).

Let us compare what happens here to what happens on Calabi-Yau manifolds [14]. In that case, deformations of the metric \( \delta g_{mn} \) split into deformations of mixed type \( \delta g_{\mu\bar{\nu}} \) and deformations of pure type \( \delta g_{\mu\nu} \) and \( \delta g_{\bar{\mu}\bar{\nu}} \). From the mixed type deformations we can define a real \((1, 1)\)-form

\[
i \delta g_{\mu\bar{\nu}} dx^\mu \wedge dx^{\bar{\nu}} \quad (4.37)
\]

and given the holomorphic 3-form \( \Omega \), we can use the mixed type deformation to define a real \((2, 1)\)-form

\[
\Omega_{\kappa\lambda} \delta g_{\mu\bar{\nu}} dx^k \wedge dx^\lambda \wedge dx^{\bar{\mu}}. \quad (4.38)
\]

In order to preserve the Calabi-Yau structure, the metric deformation must preserve the vanishing Ricci curvature, and hence \( \delta g_{mn} \) must satisfy the Lichnerowicz equation:

\[
\Delta_L \delta g_{mn} = 0
\]

However, the Lichnerowicz equation for \( \delta g_{mn} \) becomes equivalent to both the \((1, 1)\)-form (4.37) and the \((2, 1)\)-form (4.38) being harmonic. Note that the definition (4.38) is very similar to \( \chi_{abc} = h_{[a} \varphi_{bc]} \) in the \( G_2 \) case with \( \varphi \) playing the role of \( \Omega \) and \( h \) the role of \( \delta g_{\mu\bar{\nu}} \).

### 4.2 Geometry of the moduli space

In the theory of Calabi-Yau moduli spaces, one of the key results is that the local moduli space of complex structure deformations is isomorphic to an open set in \( H^{m-1,1} (X) \) where \( X \) is a Calabi-Yau \( m \)-fold. Moreover, as it has been shown by Tian and Todorov [38, 39], any infinitesimal deformation can be in fact lifted to a full deformation. For the moduli spaces of \( G_2 \) manifolds however, we can only replicate the results about the local moduli space. First let us define the moduli space of torsion-free \( G_2 \) structures. Let \( \mathcal{X} \) be the set of of positive 3-forms \( \varphi \in \mathcal{P}^3 X \) such that \( d\varphi = d * \varphi = 0 \). Here we use \(*\) to emphasize that the Hodge star is defined using the \( G_2 \) holonomy metric that is defined by \( \varphi \) itself. Then \( \mathcal{X} \) gives the set of all 3-forms that correspond to oriented, torsion-free \( G_2 \) structures. However we do not want to distinguish between 3-forms that are related by a diffeomorphism. Hence, let \( \mathcal{D} \) be the group of all diffeomorphisms of \( X \) isotopic to the identity. This group then acts naturally on 3-forms. The moduli space of torsion-free \( G_2 \) structures is then defined as the quotient \( \mathcal{M} = \mathcal{X}/\mathcal{D} \). The key result by Joyce is that \( \mathcal{M} \) is locally diffeomorphic to an open set of \( H^3 (X, \mathbb{R}) \):
Theorem 18 (Joyce, [27]) Define a map \( \Xi : \mathcal{X} \rightarrow H^3(X, \mathbb{R}) \) by \( \Xi(\varphi) = [\varphi] \). Then \( \Xi \) is invariant under the action of \( \mathcal{D} \) on \( \mathcal{X} \). Moreover, \( \Xi \) induces a diffeomorphism between neighbourhoods of \( \varphi \mathcal{D} \in \mathcal{M} \) and \( [\varphi] \in H^3(X, \mathbb{R}) \).

Since the dimension of \( H^3(X, \mathbb{R}) \) is \( b^3(X) \), this result implies that \( \dim \mathcal{M} = b^3(X) \). The full proof of this result can be found either in [27] or [28]. This result covers the basic local properties of the \( G_2 \) moduli space, but we do not yet know anything about the global structure of \( \mathcal{M} \). So anything we can say about the moduli space only holds in a small neighbourhood.

Looking back at the study of Calabi-Yau moduli spaces, we know that the complex structure moduli space admits a Kähler structure, and the Kähler structure moduli space admits a Hessian structure [14]. It turns out that on the \( G_2 \) moduli space we can also define a Hessian structure.

First let us define the notion of a Hessian manifold [34]

**Definition 19** Let \( M \) be a smooth manifold and suppose \( D \) is a flat, torsion-free connection on \( M \). A Riemannian metric \( G \) on a flat manifold \( (M, D) \) is called Hessian if \( G \) can be locally expressed as

\[
G = D^2 H
\]

that is,

\[
G_{ij} = \frac{\partial^2 H}{\partial x^i \partial x^j}
\]

where \( \{x^1, ..., x^n\} \) is an affine coordinate system with respect to \( D \). Then \( H \) is called the Hessian potential.

Note that this is the closest analogue to a Kähler structure that can be defined on a real manifold. In fact, as shown by Shima [34], if we define a complex structure on the manifold \( TM \), then the straightforward extension of \( G \) onto \( TM \) is Kähler if and only if \( G \) is a Hessian metric on \( (M, D) \). Thus the complexification of a Hessian manifold is Kähler.

In the case of the \( G_2 \) moduli space \( \mathcal{M} \), we know that \( \mathcal{M} \) is locally diffeomorphic to an open set in \( H^3(X, \mathbb{R}) \). Suppose we choose a basis \([\varphi_0], ..., [\varphi_n]\) on \( H^3(X, \mathbb{R}) \) where \( n = b^3(X) - 1 \). Taking the unique harmonic representatives of the basis elements, we can expand \( \varphi \in \mathcal{M} \) as

\[
\varphi = \sum_{N=0}^{n} s^N \phi_N.
\]

Since \( H^3(X, \mathbb{R}) \) is a vector space, \( s_0, ..., s_n \) give an affine coordinate system, which in turn defines a flat connection \( D = d \) on \( \mathcal{M} \). It is trivial to check that this connection is well-defined [31].

In order to define a metric on \( \mathcal{M} \), we have to choose a Hessian potential function on \( \mathcal{M} \). The only natural function on \( \mathcal{M} \) is the volume function \( V(\varphi) \) given by (3.5):

\[
V(\varphi) = \frac{1}{7} \int_X \varphi \wedge \psi.
\]

Note that as before, \( \psi = *\varphi \) is itself a function of \( \varphi \). So we can consider \( V \) or some function of \( V \) as potential candidates for a Hessian potential. Let us calculate the Hessian of \( V \). Note that under a scaling \( s^M \rightarrow \lambda s^M \), \( \varphi \) scales as \( \varphi \rightarrow \lambda \varphi \) and from (4.6), \( *\varphi \) scales as \( *\varphi \rightarrow \lambda^{\frac{4}{3}} *\varphi \), and so \( V \) scales as

\[
V \rightarrow \lambda^{\frac{7}{3}} V.
\]

So \( V \) is homogeneous of order \( \frac{7}{3} \) in the \( s^M \), and hence

\[
s^M \frac{\partial V}{\partial s^M} = \frac{7}{3} V = \frac{1}{3} \int s^M \phi_M \wedge *\varphi
\]

24
and thus,
\[ \frac{\partial V}{\partial s^M} = \frac{1}{3} \int \phi_M \wedge \ast \varphi. \] (4.42)

Using our results on deformations of $G_2$ structures from Section 4.1, we can deduce that
\[ \partial_N (\ast \varphi) = \frac{4}{3} \ast \pi_1 (\phi_N) + \ast \pi_7 (\phi_N) - \ast \pi_{27} (\phi_N). \] (4.43)

Hence differentiating (4.42) again, we find that
\[ \frac{\partial V}{\partial s^M \partial s^N} = \frac{4}{9} \int \pi_1 (\varphi_M) \wedge \ast \pi_1 (\varphi_N) + \frac{1}{3} \int \pi_7 (\varphi_M) \wedge \ast \pi_7 (\varphi_N) \] 
\[ - \frac{1}{3} \int \pi_{27} (\varphi_M) \wedge \ast \pi_{27} (\varphi_N) \] (4.44)

Note that in the case when $b^1 (X) = 0$ (which in particular is true when $Hol (g) = G_2$), since $H^3_3 = H^1$, the $H^3_3$ component of $H^3 (X, \mathbb{R})$ is empty. Therefore, the second term in (4.44) vanishes, and we find that the signature of this metric is Lorentzian - $(1, b_3 - 1)$. Up to a constant factor, this definition of the moduli space metric has been been used in mathematical literature - in particular by Hitchin in [25] and Karigiannis and Leung in [31]. However in physics literature, in particular by Beasley and Witten in [7] and by Gutowski and Papadopoulos in [22], the potential $K$ given by
\[ K = -3 \log V \] (4.45)

has been used instead.

The motivation for using this modified potential is two-fold. Firstly, this is more in line with the logarithmic Kähler potentials on Calabi-Yau moduli spaces. Secondly, and perhaps most importantly is that the metric that arises from this potential appears as the target space metric of the effective theory in four dimensions when the action for the 11-dimensional supergravity is reduced to four dimensions on a $G_2$ manifold. We will hence define the moduli space metric $G_{MN}$ as
\[ G_{MN} = \frac{\partial^2 K}{\partial s^M \partial s^N}. \]

Using the definition of $K$ and (4.44), we get
\[ \frac{\partial^2 K}{\partial s^M \partial s^N} = \frac{1}{V} \left( \int \pi_1 (\varphi_M) \wedge \ast \pi_1 (\varphi_N) - \int \pi_7 (\varphi_M) \wedge \ast \pi_7 (\varphi_N) \right) \]
\[ + \int \pi_{27} (\varphi_M) \wedge \ast \pi_{27} (\varphi_N) \] (4.46)

In this case, if $b^1 (X) = 0$, we get
\[ G_{MN} = \frac{1}{V} \int_X \phi_M \wedge \ast \phi_N. \] (4.47)

This metric is then in fact Riemannian. In the physics setting, apart from the $G_2$ 3-form, there is another 3-form $C$ and when the 11-dimensional supergravity action is reduced to four dimensions, the parameters of $\varphi$ and $C$ naturally combine to give a complexification of the $G_2$ moduli space. The extension of the metric $G_{MN}$ to this complex space is then Kähler [7, 20, 22]. However since the metric on the complexified space does not depend on $C$, there is not much difference in treating the moduli space as a complexified Kähler manifold or a real Hessian manifold. Here we will treat $\mathcal{M}$ as a real Hessian manifold.
Now that we have fixed a metric on $\mathcal{M}$, we can proceed to various other geometrical quantities. For this we will need to use higher derivatives of $\psi$. In what follows we will assume that $b^1 (X) = 0$, so that there are no harmonic forms in $H^2_3$. Let us introduce local special coordinates on $\mathcal{M}$. Let $\phi_0 = a \varphi$ and $\phi_\mu \in \Lambda^3_3$ for $\mu = 1, ..., b^1_3$, so that $s^0$ defines directions parallel to $\varphi$ and $s^\mu$ define directions in $H^2_3$. Then, from the deformations of $\psi$ in Section 4.1, we can extract the higher derivatives of $\psi$ in these directions:

\[
\partial_0 \partial_0 \partial_0 \psi = \frac{4}{9} a^2 \psi \quad \partial_0 \partial_0 \partial_0 \psi = -\frac{8}{27} a^3 \psi \quad (4.48a)
\]

\[
\partial_0 \partial_\mu \psi = -\frac{1}{3} a * \phi_\mu \quad \partial_0 \partial_\mu \partial_\mu \psi = \frac{2}{9} a^2 * \phi_\mu \quad (4.48b)
\]

\[
\partial_\mu \partial_\nu \psi = -\frac{2}{189} \text{Tr} (h_\mu h_\nu) \psi + \frac{1}{3} * \varphi (\varphi h_\mu h_\nu \varphi)_0 \quad (4.48c)
\]

\[
\partial_0 \partial_\mu \partial_\nu \psi = \frac{4}{567} a \text{Tr} (h_\mu h_\nu) \psi - \frac{2}{9} a * \varphi (\varphi h_\mu h_\nu \varphi)_0 \quad (4.48d)
\]

\[
\partial_\mu \partial_\nu \partial_\kappa \psi = -\frac{5}{18} \text{Tr} (h_\mu h_\nu h_\kappa) \psi + \frac{1}{3} * \varphi (h_\mu h_\nu h_\kappa)_0 \quad (4.48e)
\]

where $h_{\mu, \nu}$ and $h_{\kappa}$ are traceless symmetric matrices corresponding to the 3-forms $\phi_{\mu, \nu}$ and $\phi_{\kappa}$, respectively. On a Hessian manifold, there is a natural symmetric 3-tensor given by the derivative of the metric, or equivalently the third derivative of the Hessian potential. We will denote this tensor $A_{MNP}$. By analogy with similar quantities on Calabi-Yau moduli spaces, this tensors is called the Yukawa coupling. Using these expressions, following [20] we can now write down all the components of $A_{MNR}$:

\[
A_{000} = -14 a^3 \quad (4.49a)
\]

\[
A_{00\mu} = 0 \quad (4.49b)
\]

\[
A_{0\mu\nu} = -\frac{2}{V} \int \phi_{\mu} \wedge * \phi_{\nu} = -2 a G_{\mu\nu} \quad (4.49c)
\]

\[
A_{\mu\nu\rho} = -\frac{2}{27 V} \int (\varphi h_{\mu} h_{\nu} h_{\rho} \varphi) dV \quad (4.49d)
\]

The full Riemann curvature on a Hessian manifold is then defined by

\[
R^{MNPQ} = \frac{1}{4} (A^M_{QR} A^R_{NP} - A^M_{PR} A^R_{NQ}) \quad (4.50)
\]

Note that since the fourth derivative of $K$ is fully symmetric, the fourth derivative terms vanish here. However, we can also define the Hessian curvature tensor by

\[
Q_{KLMN} = \partial_M \partial_N \partial_L \partial_K K - A_{KMR} A^R_{LN} \quad (4.51)
\]

This tensor is the equivalent of the Kähler curvature, and carries more information than the actual Riemann tensor (4.50). The Riemann curvature tensor is obtained from $Q$ by

\[
R_{MNPQ} = \frac{1}{2} (Q_{MNPQ} - Q_{NMPQ}) \quad (4.52)
\]

From (4.48), we can calculate the fourth derivatives of $K$, and hence get all the components of $Q$:
Theorem 20 (Grigorian and Yau, [20]) The components of the Hessian curvature tensor $Q$ corresponding to the metric (4.47) on the local moduli space of torsion-free $G_2$ structures are given by:

\[
\begin{align*}
Q_{0000} &= 14a^4 \\
Q_{000\mu} &= 0 \\
Q_{00\mu\nu} &= 2a^2 G_{\mu\nu} \\
Q_{0\mu\nu\rho} &= -A_{\mu\nu\rho}a \\
Q_{\kappa\mu\nu\rho} &= \frac{1}{3} \left( G_{\mu\nu} G_{\kappa\rho} + G_{\mu\kappa} G_{\nu\rho} - \frac{5}{7} G_{\mu\rho} G_{\kappa\nu} \right) - G^{\tau\sigma} A_{\mu\tau\rho} A_{\kappa\nu\sigma} + \frac{1}{V} \int \left( -\frac{2}{27} \text{Tr} (h_\kappa h_\mu h_\nu h_\rho) + \frac{1}{27} \left( \psi h_\kappa h_\mu h_\nu h_\rho \psi \right) + \frac{5}{81} \text{Tr} (h_\kappa h_\mu) \text{Tr} (h_\nu h_\rho) \right) \text{vol}
\end{align*}
\]

Let us look in more detail at the expression for $A_{\mu\nu\rho}$. If we define $h^a_\mu = h^a_\mu dx^m$, then we get

\[
A_{\mu\nu\rho} = -\frac{4}{9V} \int \varphi_{abc} h^a_\mu \wedge h^b_\nu \wedge h^c_\rho \wedge \psi.
\]

Expressions for the $G_2$ Yukawa coupling has been derived by different authors - in particular by Lee and Leung, [33], de Boer, Naqvi and Shomer [16], and Karigiannis [30]. Similarly, we can rewrite (4.53e) as

\[
Q_{\kappa\mu\nu\rho} = \frac{1}{3} \left( G_{\mu\nu} G_{\kappa\rho} + G_{\mu\kappa} G_{\nu\rho} - \frac{5}{7} G_{\mu\rho} G_{\kappa\nu} \right) - G^{\tau\sigma} A_{\mu\tau\rho} A_{\kappa\nu\sigma} + \frac{8}{9V} \int \psi_{abc} h^a_\mu \wedge h^b_\nu \wedge h^c_\rho \wedge \varphi + \frac{1}{81} \int \left( 5 \text{Tr} (h_\kappa h_\mu) \text{Tr} (h_\nu h_\rho) - 6 \text{Tr} (h_\kappa h_\mu h_\nu h_\rho) \right) \text{vol}
\]

As we have mentioned previously, by complexifying the $G_2$ moduli space, it is possible to turn the Hessian structure into a Kähler structure. Similarly, the Hessian curvature $Q$ becomes Kähler curvature. On Calabi-Yau manifolds, the complex structure moduli space is naturally a complex manifold, and admits a Kähler structure, while the Kähler structure moduli space is naturally a Hessian manifold, but can be complexified to become Kähler itself. We compare the various quantities on $G_2$ moduli space and on the Calabi-Yau complex structure moduli space in Figure 5. We can see that there are a number of similarities. This leads to a speculation that perhaps the $G_2$ moduli space possesses more structures than it is currently known. One of the key features of Calabi-Yau moduli spaces is the special geometry, that is, both have a line bundle whose first Chern class coincides with the Kähler class [18, 35]. From physics point of view, special geometry relates to the effective theory having $N = 2$ supersymmetry. M-theory compactified on $G_2$ manifolds only gives $N = 1$ supersymmetry, so from this point of view it is perhaps unlikely that the (complexified) $G_2$ moduli space would admit precisely this structure. Moreover, it was shown by Alekseevsky and Cortés in [4] that a so-called special real structure on a Hessian manifold corresponds to special Kähler structure on the tangent bundle. A special real manifold is a Hessian manifold on which the cubic form $DG$ (with $D$ being the flat connection, and $G$ the Hessian metric) is parallel with respect to $D$. In our terms, this would mean that the derivative of the Yukawa coupling $A$ vanishes. This is a rather strong condition which is not necessarily fulfilled in our case. So perhaps instead there is some intermediate structure that could be defined on the $G_2$ moduli space or its complexification.
5 Concluding remarks

In this paper we have reviewed the developments in the study of $G_2$ moduli spaces. Currently only the local picture of the moduli space is known, so in the future it is natural to try and obtain at least some information on the global structure of the $G_2$ moduli space. On Calabi-Yau manifolds, the extension to the global moduli space was originally done by Tian and Todorov [38, 39]. We have seen that there are a number of similarities in the local structure of Calabi-Yau moduli spaces and $G_2$ moduli spaces, so it is feasible that it could also be possible to derive similar global properties of $G_2$ moduli spaces. However torsion-free $G_2$ structures are very non-linear in some aspects - in particular, the metric depends non-linearly on $\varphi$ and hence the differential equation $\nabla \varphi = 0$ for a torsion-free structure is also non-linear. Therefore, it is not clear how to extend infinitesimal deformations of a $G_2$ structure to large deformations, apart from considering deformations order by order. However even such expansions quickly get very complicated.

Another possible topic for study would be to further develop approaches to mirror symmetry on $G_2$ holonomy manifolds [21]. One possible direction for further research is to look at $G_2$ manifolds in a slightly different way. Suppose we have type IIA superstrings on a non-compact Calabi-Yau 3-fold with a special Lagrangian submanifold which is wrapped by a D6 brane which also fills $M_4$. Then, as explained in [3], from the M-theory perspective this looks like a $S^1$ bundle over the Calabi-Yau which is degenerate over the special Lagrangian submanifold, but this 7-manifold is still a $G_2$ manifold. The moduli space of this manifold will be then determined by the Calabi-Yau moduli and the special Lagrangian moduli. This possibly could provide more information about mirror symmetry on Calabi-Yau manifolds [37].

One more direction is to look at $G_2$ manifolds with singularities. So far in this work we have considered only smooth $G_2$ manifolds, however, from a physical point of view, $G_2$ manifolds with singularities are even more interesting, as they yield more realistic matter content [1]. Also, the moduli spaces which we studied are for manifolds with fixed topology. By allowing topological transitions through singularities [15], it may be possible to find some relations between the different moduli spaces. Understanding these questions would improve our grasp of both the geometry and physics of $G_2$ moduli spaces and the interplay between them.
References

[1] B. Acharya and E. Witten, *Chiral fermions from manifolds of G(2) holonomy*, hep-th/0109152.

[2] B. S. Acharya and S. Gukov, *M theory and Singularities of Exceptional Holonomy Manifolds*, Phys. Rept. 392 (2004) 121–189 hep-th/0409191.

[3] M. Aganagic, A. Klemm and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, Z. Naturforsch. A57 (2002) 1–28 hep-th/0105045.

[4] D. V. Alekseevsky and V. Cortes, *Geometric construction of the r-map: from affine special real to special Kähler manifolds*, 0811.1658.

[5] W. Ambrose and I. M. Singer, *A theorem on holonomy*, Trans. Am. Math. Soc. 75 (1953) 428–443.

[6] J. Baez, *The Octonions*, Bull. Amer. Math. Soc. (N.S.) 39 (2002) 145–205.

[7] C. Beasley and E. Witten, *A note on fluxes and superpotentials in M-theory compactifications on manifolds of G(2) holonomy*, JHEP 07 (2002) hep-th/0203061.

[8] K. Becker, M. Becker and J. H. Schwarz, *String theory and M-theory: A modern introduction*. Cambridge University Press, 2007.

[9] M. Berger, *Sur les groupes d’holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France 83 (1955).

[10] E. Bonan, *Sur les variétés riemanniennes à groupe d’holonomie g2 our spin(7)*, C. R. Acad. Sci. Paris 262 (1966) 127–129.

[11] R. Bryant and S. Salamon, *On construction of some complete metrics with exceptional holonomy*, Duke Math. J. 58 (1989) 829–850.

[12] R. L. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. (2) 126 (1987), no. 3 525–576.

[13] R. L. Bryant, *Some remarks on G_2-structures*, math/0305124.

[14] P. Candelas and X. de la Ossa, *Moduli space of Calabi-Yau manifolds*, Nucl. Phys. B355 (1991) 455–481.

[15] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, *M-theory conifolds*, Phys. Rev. Lett. 88 (2002) 121602 hep-th/0112098.

[16] J. de Boer, A. Naqvi and A. Shomer, *The topological G(2) string*, hep-th/0506211.

[17] M. Fernández and A. Gray, *Riemannian manifolds with structure group G_2*, Ann. Mat. Pura Appl. (4) 132 (1982) 19–45 (1983).

[18] D. S. Freed, *Special Kaehler manifolds*, Commun. Math. Phys. 203 (1999) 31–52 hep-th/9712042.

[19] S. Grigorian, *Betti numbers of a class of barely G2 manifolds*, 0909.4681.

[20] S. Grigorian and S.-T. Yau, *Local geometry of the G2 moduli space*, Commun. Math. Phys. 287 (2009) 459–488 0802.0723.
[21] S. Gukov, S.-T. Yau and E. Zaslow, Duality and fibrations on G(2) manifolds, hep-th/0203217.

[22] J. Gutowski and G. Papadopoulos, Moduli spaces and brane solitons for M theory compactifications on holonomy G(2) manifolds, Nucl. Phys. B615 (2001) 237–265 hep-th/0104105.

[23] F. R. Harvey, Spinors and Calibrations. Academic Press, 1990.

[24] J. A. Harvey and G. W. Moore, Superpotentials and membrane instantons, hep-th/9907028.

[25] N. J. Hitchin, The geometry of three-forms in six and seven dimensions, math/0010054.

[26] K. Hori et. al., Mirror symmetry. AMS - Providence, USA, 2003.

[27] D. D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2. I, II, J. Differential Geom. 43 (1996), no. 2 291–328, 329–375.

[28] D. D. Joyce, Compact manifolds with special holonomy. Oxford Mathematical Monographs. Oxford University Press, 2000.

[29] S. Karigiannis, Deformations of G_2 and Spin(7) Structures on Manifolds, Canadian Journal of Mathematics 57 (2005) 1012 math/0301218.

[30] S. Karigiannis, Geometric Flows on Manifolds with G_2 Structure, I, math/0702077.

[31] S. Karigiannis and N. C. Leung, Hodge theory for G2-manifolds: Intermediate Jacobians and Abel-Jacobi maps, 0709.2987.

[32] A. Kovalev, Twisted connected sums and special Riemannian holonomy, math/0012189.

[33] J.-H. Lee and N. C. Leung, Geometric structures on G(2) and Spin(7)-manifolds, math/0202045.

[34] H. Shima, The geometry of Hessian structures. World Scientific Publishing - Hackensack, NJ, 2007.

[35] A. Strominger, Special geometry, Commun. Math. Phys. 133 (1990) 163–180.

[36] A. Strominger, S.-T. Yau and E. Zaslow, Mirror symmetry is T-duality, Nucl. Phys. B479 (1996) 243–259 hep-th/9606040.

[37] A. Strominger, S.-T. Yau and E. Zaslow, Mirror symmetry is T-duality, Nucl. Phys. B479 (1996) 243–259 hep-th/9606040.

[38] G. Tian, Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric, in Mathematical aspects of string theory (San Diego, Calif., 1986), vol. 1 of Adv. Ser. Math. Phys., pp. 629–646. World Sci. Publishing, 1987.

[39] A. Todorov, The Weil-Petersson geometry of the moduli space of SU (n ≥ 3) (Calabi-Yau) manifolds I, Commun. Math. Phys. 126 (1989) 325–346.

[40] P. K. Townsend, The eleven-dimensional supermembrane revisited, Phys. Lett. B 350 (1995) 184–187 hep-th/9501068.

[41] P. C. West, Introduction to supersymmetry and supergravity. World Scientific Publishing - Singapore, 1990.
[42] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys. B* **443** (1995) 85–126 [hep-th/9503124].