Analytic continuations of the Horn $H_1$ and $H_5$ functions

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Abstract

The analytic continuations (ACs) of the double variable Horn $H_1$ and $H_5$ functions have been derived for the first time using the automated symbolic Mathematica package Olsson.wl. The use of Pfaff-Euler transformations have been emphasised to derive AC to cover regions which are otherwise not possible. The corresponding region of convergence (ROC) is obtained using its companion package ROC2.wl. A Mathematica package HornH1H5.wl, containing all the derived ACs and the associated ROCs, along with a demonstration file of the same is made publicly available in this URL: https://github.com/souvik5151/Horn_H1_H5.git

1 Introduction

Feynman integrals are an integral part of the precision calculations in quantum field theory [1]. They occur in the higher-order terms of the perturbation theory. Apart from this, they are also interesting from a mathematical point of view. Interest in Feynman integrals has also been motivated by processes at the LHC. In this quest to obtain the analytic properties, it was observed that hypergeometric functions and Feynman integrals are intimately related [2]. There are various ways to obtain and explore these hypergeometric connections of Feynman integrals. A general technique to evaluate these Feynman integrals, thus giving rise to multi-variable hypergeometric functions (MHFs) is that of using the Mellin-Barnes (MB) representation. Some of the classic results using it are results for general $N$-point one loop integrals [3–5]. Such evaluations give rise to the $N$-variable hypergeometric function of the Horn type. On the same front recently developed method of Conic Hull Mellin-Barnes (CHMB) [6–9], a generalised technique for finding the series representation of $N$-fold MB integrals also give a solution of Feynman integrals in terms of MHFs. Another method which achieves the same, and can be shown to be equivalent to the MB method is that of Negative Dimensional Integration Method [10–16]. In some cases the hypergeometric representation of Feynman integrals can be obtained using the integration of the Feynman parametric integral [17–25] and without the use of any other specialised techniques. Recently functional relations were used to evaluate one-loop Feynman integrals (see [26] and references within) in terms of MHFs. Feynman integrals can also be realized as GKZ hypergeometric system [27–29]. These MHFs also appear in some physical processes too like Appell $F_3$ appears in the calculation of one loop four-photon scattering amplitude [30, 31]. Apart from this, the evaluation of the pentagon integral [32] in certain Regge limit gives rise to Appell $F_4$ and Kampé de Fériet functions. To give a few more examples, two-loop massive sunset diagram [33, 34], three-loop vacuum diagram [24], one loop two, three and four-point scalar functions [22, 35, 36] are also evaluated in terms of MHFs.

The theory of hypergeometric functions and their analytic continuations (ACs) in one and more than one variable is now a highly explored subject for over a century and is of fundamental importance in many branches of mathematical physics [37–43]. The one variable Gauss $\, _2F_1$ function has been generalized to two variables in [44], which are now known as Appell $F_1, F_2, F_3$ and $F_4$ functions. Another ten double variable hypergeometric functions $G_1, G_2, G_3$ and $H_1, H_2, \ldots, H_7$ are introduced by Horn [45]. These fourteen Appell-Horn functions form the set of distinct, second-order complete

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hypergeometric functions in two variables. More general higher-order functions, namely the Kampé de Fériet functions, frequently appear in the study of analytic continuations of these functions [38]. Using the detailed transformation theory of two variables hypergeometric functions it was found by Erdélyi [46] that all the second order, two variable hypergeometric functions, except $F_3, H_1$ and $H_5$ can be related to $F_2$ via the transformation formulae. The Appell $F_2$ has been studied by Olsson [47] and recently have also been studied in [48] where a large number of ACs and their numerical implementation in Mathematica has been developed. The Appell $F_1$ was treated by Exton [49], and these results were further modified in [50]. It was also pointed out by Exton that $H_1$ and $H_5$ should be investigated in a similar way [49]. A recent study for the determination of the solution of the PDEs of various Appell and Horn functions along with that of $H_1$ and $H_5$ have been done in [51]. Some properties of the $H_1$ and $H_5$ have recently been studied in [52–56].

In all of the above examples, both mathematical and physical it is thus important to obtain the ACs of the obtained MHFs. Mathematically such ACs are important to obtain the solutions of the associated system of partial differential equations around various singular points. In physical applications, they are important so as to evaluate these functions in various kinematical regions. In [46] it was pointed out that the well-known transformation formulae of the one variable $_pF_{p-1}$ functions can be used to find the ACs of the MHFs. The method has been used for the Appell $F_1$ [57], $F_4$ [49] and recently for $F_2$ [48]. In principle, such computations are possible and have been achieved by hand before, but as we discuss below in the upcoming section that the analysis for the cases of $H_1$ and $H_5$ is tedious and demands the use of more powerful computational tools. Specifically one of the important steps in such analysis, the ROC analysis is nearly impossible to do by hand. These occur due to the higher-order hypergeometric functions that appear in these studies.

With all the above motivation, in this work, we consider the non-trivial cases of Horn $H_1$ and $H_5$ series. Due to the difficulty in the computation of ACs and ROCs for these non-trivial cases, we use the automated Mathematica packages Olsson.wl and ROC2.wl[58], thus also aiming to test their efficacy. We aim to find the analytic continuations(ACs) of these so as to cover the whole real $x−y$ plane and their respective ROCs. We will also, tacitly exclude the exceptional values of the parameters that are values that make the gamma functions that appear in the AC to be not defined. While deriving the ACs, we observe that only the use of AC of $_pF_{p-1}(z)$ around $z = \infty$ and the AC of $2F_1(z)$ around $z = 1$ as has done in a similar previous analysis of $F_4$ and $F_2$, is no more adequate. To resolve this issue we use the Pfaff-Euler transformations (PETs) of the $2F_1$, wherever applicable, which allow us to cover the regions that are otherwise difficult to cover. For the case of $H_1$, we found that such an approach allows us to cover the whole real plane, apart from some boundaries and singular points. On the other hand, for the Horn $H_5$ function, there still remains a small finite size region where no AC is valid (see Figure 29). Since it is impossible to present all the results in the form of running text, we provide along with this manuscript a Mathematica [59] package and a demonstration file of the package. The package can be used to access the ACs and their corresponding ROCs as required by the user. However, we point out that special care has to be taken for the numerical evaluation as the ACs can have multivaluedness issues which are not discussed in the present paper. These non-trivial evaluations are thus a challenge to the package in terms of both evaluating the analytic continuations for these two cases as well as finding the region of convergence for various higher-order series that appear in these analytic continuations. These results can further be used as a benchmark for future numerical tests of the same.

The scheme of this paper is as follows: In section 2 we have first analyzed $H_1$, starting with the derivation of one of its simple AC for the illustration. All the other ACs are derived using the package Olsson.wl that follows the same methodology. Similarly, we have discussed $H_5$ in section 3. In all these sections, we have discussed ways along with a road map to derive enough ACs that cover the whole real plane (boundaries excluded). We have also illustrated the importance and uses of the Pfaff-Euler transformations of $2F_1$ hypergeometric function during the process.

## 2 The Horn $H_1$ series

The Horn $H_1$ series is defined as [38, 45]

$$H_1(a, b, c, d, x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_{m+n}(c)_n x^m y^n}{m!n!(d)_m} \tag{1}$$
with the ROC : \(2\sqrt{|x||y|} + |y| < 1 \land |y| < 1 \land |x| < 1\), which for real \(x\) and \(y\) is shown in Figure 1.

It satisfies the following system of partial differential equations

\[
x(1-x)r - y^2t + [d - (a + b + 1)x]p - (a - b - 1)yx - abz = 0
\]
\[
-y(1+y)t - x(1-y)s + [a - 1 - (b + c + 1)y]q - cxp - cbz = 0
\]

where the symbols are defined as follows

\[
\begin{align*}
z &= H_1, & p &= \frac{\partial z}{\partial x}, & q &= \frac{\partial z}{\partial y} \\
r &= \frac{\partial^2 z}{\partial x^2}, & t &= \frac{\partial^2 z}{\partial y^2}, & s &= \frac{\partial^2 z}{\partial x \partial y}
\end{align*}
\]

The singular curves of the above equation are as follows, see Figure 2.

\[
x = 0, y = 0, x = 1, y = -1, -4xy + y^2 + 2y + 1 = 0
\]

In the following section, we obtain the various ACs of the \(H_1\) series by using the ACs of \(_2F_1\). These ACs ensure that we cover the whole real \(xy\) plane with the exception of certain singular lines, in other words, we can find the value of \(H_1\) outside its ROC given as in Fig.1. We also make use of the Pfaff-Euler transformations of \(_2F_1\) in the intermediate steps and show how to obtain the various other ACs that could not be or are at times hard to obtain using just the ACs of \(_2F_1\).

### 2.1 An Illustrative example

As an illustrative example of the procedure of finding the ACs, by deriving one simple AC of \(H_1\). We will derive the AC around \((1, 0)\).

We take Eq. (1) and sum over \(m, n\),

\[
H_1(a, b, c, d, x, y) = \sum_{m, n=0}^{\infty} \frac{(-1)^n y^m (b)_n (c)_n}{n!(1-a)_n} _2F_1(b + n, a - n; d; x)
\]

Now using the AC of \(_2F_1(x)\) around \(x = 1\) Eq. (46) we find two series,

\[
H_{(1,0)}^{(1)}(a, b, c, d, x, y) = \frac{\Gamma(d)\Gamma(-a - b + d)}{\Gamma(d - a)\Gamma(d - b)} \sum_{m, n=0}^{\infty} \frac{y^m (b)_n (c)_n (b - d + 1)_n}{n!(1-a)_n(d - a)_n}
\]
\[
\times _2F_1(a - n, b + n; a + b - d + 1; 1 - x)
\]
\[
= \frac{\Gamma(d)\Gamma(-a - b + d)}{\Gamma(d - a)\Gamma(d - b)} \sum_{m, n} \frac{(c)_n (a)_m (b - d + 1)_n (b)_m (1 - x)^m (-y)^n}{m!n!(d - a)_n(a + b - d + 1)_m}
\]

\(3\)
with ROC : $2\sqrt{|1-x||-y| + |-y| < 1 \land |1-x| < 1 \land |-y| < 1$ and,

$$H^{(2)}_{(1,0)}(a, b, c, d, x, y) = (1-x)^{-a-b+d} \frac{\Gamma(d)\Gamma(a+b-d)}{\Gamma(a)\Gamma(b)} \sum_{m,n=0}^{\infty} \frac{y^n(c)n}{n!}$$

$$\times \, _2F_1(-b+d-n, -a+d+n; -a-b+d+1; 1-x)$$

$$= (1-x)^{-a-b+d} \frac{\Gamma(d)\Gamma(a+b-d)}{\Gamma(a)\Gamma(b)}$$

$$\sum_{m,n=0}^{\infty} \frac{(c)n(b-d+1)n(d-a)m+n(d-b)m+n}{m!n!(d-a)n(-a-b+d+1)m}(1-x)^m(-y)^n$$

(6)

So we get the following AC around (1,0)

$$H_1(a, b, c, d, x, y) = \frac{\Gamma(d)\Gamma(-a-b+d)}{\Gamma(d-a)\Gamma(-d-b)} \sum_{m,n=0}^{\infty} \frac{(c)n(a)m-n(b-d+1)n(d-a)m+n}{m!n!(d-a)n(a+b-d+1)m}(1-x)^m(-y)^n$$

(7)

with ROC : $2\sqrt{|1-x||-y| + |-y| < 1 \land |1-x| < 1 \land |-y| < 1$ as shown in Figure.3

Figure 3: ROC of Eq. (7)

We observe from Fig.3 that around the singular point (1,0) one AC is enough to cover the space around it. However, there might arise a situation where this is not possible. For example for the other singular point (0, −1), where we need two ACs around that singular point.

The following diagram illustrates the possible chain of the process that can be used to derive both the ACs around the singular point (0, −1)

First AC

Second AC

(0, −1)

(0, −1)

(0, 0)

Using the AC of $\, _pF_{p-1}(z)$ around $z = \infty$, we have the following two possible situations
Taking into account all these possibilities we can have in general the following possibilities for $H_1$.

Figure 4: All the possibilities of ACs for $H_1$. The thick lines indicate the one that has been derived and the dashed lines denote which are not. The region that can be covered using the ACs denoted by dashed lines can be covered using the ACs denoted by solid line. The derived ACs are part of the package and the AC $T_i$ in the above graph is the $i$-th AC in the package.

### 2.2 The Pfaff-Euler transformations

In the intermediate steps of the analytic continuation procedure, it is sometimes possible to get $\mathbf{2F}_1$ hypergeometric function when carrying out one of the summations. Instead of using the usual AC of $\mathbf{2F}_1(z)$ around $z = 1$ or $z = \infty$, it is also possible to use the PETs, Eq. (45), in the intermediate steps. This process can also be done using Olsson.wl. Using this we get the following transformations for $H_1$.

We first take Eq. (1). Summing over $m$, we find,

$$H_1(a, b, c, d, x, y) = \sum_{m,n=0}^{\infty} \frac{(-1)^n y^n (b)_n (c)_n}{n! (1-a)_n} \mathbf{2F}_1(b+n, a-n; d; x)$$

Now using the PETs of $\mathbf{2F}_1$, we find,

$$H_1(a, b, c, d, x, y) = (1-x)^{-b} \sum_{m,n=0}^{\infty} \frac{(c)_n (b)_{m+n}(d-a)_{m+n}}{m!n!(1-a)_n(d-a)_n} \left( \frac{x}{x-1} \right)^m \left( \frac{y}{x-1} \right)^n$$

with ROC: $\sqrt{\left| \frac{y}{x-1} \right|} + \sqrt{\left| \frac{x}{x-1} \right|} < 1$
\[ H_1(a, b, c, d, x, y) = (1 - x)^{-a} \sum_{m, n=0}^{\infty} \frac{(b)_{n}(c)_{n}(a)_{m-n}(b - d + 1)_n(d - b)_{m-n}}{m!n!(d)_m} \left( \frac{x}{x-1} \right)^m (y(x-1))^n \] (10)

with ROC : \[ \sqrt{\frac{x}{x-1}} - \frac{1}{\sqrt{|y(x-1)|}} < -1 \land |y(x-1)| < 1 \land \left| \frac{x}{x-1} \right| < 1 \]

\[ H_1(a, b, c, d, x, y) = (1 - x)^{-a-b+d} \sum_{m, n=0}^{\infty} \frac{(b)_{n}(c)_{n}(b - d + 1)_n(d - a)_{m+n}(d - b)_{m-n}}{m!n!(1-a)_{n}(d)_{m}(d - a)_n} x^m y^n \] (11)

with the ROC same as \( H_1 : 2\sqrt{|x||y|} + |y| < 1 \land |y| < 1 \land |x| < 1 \)

Similarly, summing over \( n \), we find,

\[ H_1(a, b, c, d, x, y) = \sum_{m, n=0}^{\infty} \frac{x^m(a)_{m}(b)_m}{m!(d)_m} \, _2F_1(c, b + m; -a - m + 1; -y) \] (12)

Using PETs of \( _2F_1 \) we find,

\[ H_1(a, b, c, d, x, y) = (y + 1)^{-b} \sum_{m, n=0}^{\infty} \frac{(a + c)_{m}(b)_{m+n}(-a - c + 1)_{n-m}}{m!n!(d)_{m}(1-a)_{n-m}} \left( \frac{x}{y+1} \right)^m \left( \frac{y}{y+1} \right)^n \] (13)
with ROC $\frac{x}{y+1} + \frac{y}{y+1} < 1$

$$H_1(a, b, c, d, x, y) = (y + 1)^{-c} \sum_{m,n=0}^{\infty} \frac{(b)_m(c)_n(a + b)_{2m}(-a - b + 1)_{2m}}{m!n!(1-a)^n_m}(-x)^m \left(\frac{y}{y+1}\right)^n$$ (14)

with ROC : $\frac{1}{|x|} - 4 \left|\frac{y}{y+1}\right|^2 > 4 \left|\frac{y}{y+1}\right| \land |x| < 1 \land \left|\frac{y}{y+1}\right| < 1$

$$H_1(a, b, c, d, x, y) = (y + 1)^{-a-b-c+1} \sum_{m,n=0}^{\infty} \frac{(b)_m(a + b)_{2m}(a + c)_n(-a - b + 1)_{2m}(-a - c + 1)_{2m}}{m!n!(1-a)^n_m}(-x)^m \left(\frac{y}{y+1}\right)^n$$ (15)

with ROC $\frac{1}{\sqrt{|y+1|^2}} - |y| > 1 \land \left|\frac{x}{|y+1|^2}\right| < 1 \land |y| < 1$

Figure 9: The ROC of Eq. (14)  
Figure 10: The ROC of Eq. (15)

It is important to note that though the PETs have been used with the original definition of $H_1$ in the above case, one can also use them in the intermediate steps if summation over one of the indices gives $\sum F_1$. AC number 22 (in the package) of $H_1$ is derived using this strategy. One can also then use the whole procedure of finding the AC on the above-derived transformations of $H_1$ and obtain more ACs of $H_1$. We derive ACs number 20 and 21 using this idea.

3 The Horn function $H_5$

The Horn function $H_5$ is defined as [37, 38, 45],

$$H_5 := H_5(a, b, c, x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_{n-m}}{(c)_n} x^m y^n \frac{x^n}{m!n!}$$ (16)

The defining ROC is given by [38]

$$|x| < \frac{1}{4} \land |y| < 1$$

$$|y| < \left\{ \begin{array}{ll} \min \left(\frac{(1-12|x|)^{3/2}+36|x|+1}{54|x|}, \frac{(1-12|x|)^{3/2}+36|x|+1}{54|x|}, \frac{(12|x|+1)^{3/2}-36|x|+1}{54|x|}, \frac{(12|x|+1)^{3/2}-36|x|+1}{54|x|}\right) \sqrt{|x|} < \frac{1}{2\sqrt{3}} \right. \right\}$$ (17)
The ROC above is shown in Figure 11 for real values of $x$ and $y$.

The Horn $H_5$ function satisfies the following set of PDEs

$$
\begin{align*}
  x(1 + 4x)r + (4x - 1)ys + y^2t + ((4a + 6)x - b + 1)p + 2(a + 1)yq + a(1 + a)z &= 0 \\
  (y - 1)yt + xys - 2x^2r - (x(a - 2b + 2))p + (y(a + b + 1) - c)q + abz &= 0
\end{align*}
$$

where like before the symbols $p, q, r, t$ and $s$ are given below

$$
\begin{align*}
  z &= H_5, \\
  p &= \frac{\partial z}{\partial x}, \\
  q &= \frac{\partial z}{\partial y}, \\
  r &= \frac{\partial^2 z}{\partial x^2}, \\
  t &= \frac{\partial^2 z}{\partial y^2}, \\
  s &= \frac{\partial^2 z}{\partial x\partial y}
\end{align*}
$$

It is to be noted that the PDE of $H_5$ given in the book [37] is not correct. This observation is also noted in [51]. It is worth mentioning that the singular locus of the $H_5$ can be found by using the theory of D-modules and Gröbner basis (see [60] for an application to Lauricella $F_C$). Some properties of the Horn $H_5$ function are studied in [53].

The singular curves for the Horn $H_5$ are as follows

$$
\begin{align*}
  x = 0, y = 0, x = -1/4, y = 1, 1 + 8x + 16x^2 - y - 36xy + 27xy^2 = 0
\end{align*}
$$

These are shown in Figure 12.

In the following section, we find the ACs of the function using Olsson’s method. All the ACs are derived using the package Olsson.wl [58]. The ROCs of the ACs are obtained using the companion package ROC2.wl. The ROCs that are shown below are plotted for real values of $x$ and $y$.

To proceed, we take the summation over the index $m$ and observe that the Gauss $2F_1$ appears inside the summand,

$$
H_5(a, b, c, x, y) = \sum_{n=0}^{\infty} \frac{y^n(a)_n(b)_n}{n!(c)_n} 2F_1 \left( \frac{a}{2} + \frac{n}{2}, \frac{a}{2} + \frac{n}{2} + \frac{1}{2}; -b - n + 1; -4x \right)
$$

Similarly taking the summation over the index $n$, we find

$$
H_5(a, b, c, x, y) = \sum_{m=0}^{\infty} \frac{(-1)^m x^m(a)_{2m}}{m!(1 - b)_m} 2F_1(b - m, a + 2m; c; y)
$$

The above two expressions (i.e. Eq. (20) and Eq. (21)) are the starting points for our analysis of the $H_5$ function. Applying the well known linear transformation formulae of Gauss $2F_1$ listed in the Section 6.1, the ACs of $H_5$ can be found for general values of its Pochhammer parameters.
Let us take Eq. (21). Applying Eq. (46), we find the AC of $H_5$ around the point $(0, 1)$,

$$H_5 = \frac{\Gamma(c)\Gamma(-a - b + c)}{\Gamma(c - a)\Gamma(c - b)} \sum_{m,n=0}^{\infty} \frac{(a - c + 1)2m(a)2m+n(b)_{n-m}}{(c - b)m(a + b - c + 1)_{m+n}} (-x)^m(1 - y)^n$$

$$+ (1 - y)^{-a - b + c} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} \sum_{m,n=0}^{\infty} \frac{(a - c + 1)2m(c - a)_{n-2m}(c - b)_{m+n}}{m!n!(c - b)m(-a - b + c + 1)_{n-m}} \left(-\frac{x}{1 - y}\right)^m(1 - y)^n$$

(22)

The first series of the above expression (Eq. (22)) is convergent in

$$|x| < \frac{1}{16} \land |1 - y| < 1 \land |1 - y| < \frac{(1 - 12|x|)^{3/2} - 18|x| + 1}{54|x|}$$

whereas the second series is valid for

$$\left|\frac{x}{y - 1}\right| < 1 \land |1 - y| < 1 \land$$

$$|1 - y| < \left\{ \min \left( \frac{1}{27} \left( -\frac{(4|x|+3|y-1|)^{3/2}}{\sqrt[3]{|y-1|}} \right) - \frac{8|x|}{|y-1|} + 9 \right), \frac{1}{27} \left( \frac{(4|x|+3|y-1|)^{3/2}}{\sqrt[3]{|y-1|}} \right) - \frac{8|x|}{|y-1|} - 9 \right) \right\} < 3$$

(23)

Hence the AC (Eq. (22)) is valid in the common ROCs of the two series. We plot the ROCs of the individual series in Figure 13 and 14.

Figure 13: The ROC of the first series of Eq. (22)  
Figure 14: The ROC of the second series of Eq. (22)

We observe that the AC of $H_5$ in Eq. (22) does not cover the whole space around the singular point $(0, 1)$. Thus we can find another AC that will cover the space that remains uncovered. To find such AC we take the second series of Eq. (22) (let us denote it by $S_2$) and take the summation over $m$ explicitly

$$S_2 = (1 - y)^{-a - b + c} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} (1 - y)^n(c - a)_n(c - b)_n$$

$$\times \quad {}_4F_3\left(\begin{array}{c} \frac{a}{2} + 1, \frac{a}{2} + 1, \frac{a}{2} + 1, \frac{a}{2} + 1, a + b - c - n, -b + c + n; c - b, \frac{a}{2} - n + 1, 1; \frac{a}{2} - c - n + 1; x \end{array}\right)$$

(24)

Now using the AC of ${}_4F_3(\ldots; z)$ at $z = \infty$ (Eq. (49)), we find that only one series is non-vanishing. Denoting it as $S'_2$

$$S'_2 = (1 - y)^{-a - b + c} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}$$

(25)
the process of evaluation. We plot the ROCs of both the ACs around the point (0, 0) and the ROC of it is plotted in Figure 15. This procedure can be described by a directed graph as the same point by transforming the second series of Eq. (22). The resultant AC is given in Eq. (27) around (0, 0) whose ROC is given in Eq. (17) and plotted in Figure 11. Taking summation over the index \(H\) defining ROC of \(H\) the Gauss 2 \(F_1\) where

\[
\Gamma(c)\Gamma(a - b + c)\sum_{m,n=0}^{\infty} \frac{(a - c + 1)_{2m}(a)_{2m+n}(b)_{n-m}(-x)^m(1-y)^n}{m!n!} m!n! (a + 2b - 2c + 1)_{m-n}(a + 2b - c)_{2(m-n)} (1 - \frac{y}{x})^n (-x)^m
\]

whose ROC is given by

\[
|r| < 1 \land |s| < \min(z_1, z_2, z_3) \land |s| < \frac{1}{16}
\]

where \(r = \frac{1-y}{x}\), \(s = -x\) and

\[
\begin{align*}
z_1 &= \text{second root of } 27z^4|r|^2 - 2z^2(9|r| + 8) - |r| - 1 = 0 \\
z_2 &= \text{second root of } 27z^4|r|^2 + 2z^2(9|r| - 8) + |r| - 1 = 0 \\
z_3 &= \text{second root of } 27z^4|r|^2 + 2z^2(9|r| + 8) - |r| - 1 = 0
\end{align*}
\]

This ROC is plotted in Figure 15.

Hence we find the second AC of \(H_5\) around the point (0, 1) by combining the first series of Eq. (22) with \(S_2^{(1)}\) (i.e. Eq. (25))

\[
H_5 = \frac{\Gamma(c)\Gamma(-a - b + c)}{\Gamma(c - a)\Gamma(c - b)} \sum_{m,n=0}^{\infty} \frac{(a - c + 1)_{2m}(a)_{2m+n}(b)_{n-m}(-x)^m(1-y)^n}{m!n!} m!n! (a + 2b - 2c + 1)_{m-n}(a + 2b - c)_{2(m-n)} (1 - \frac{y}{x})^n (-x)^m
\]

The ROC of this AC is the same as Eq. (26), which is plotted in Figure 15.

Figure 15: The ROC of series \(S_2^{(1)}\) (i.e. Eq. (26)) is plotted for real values of \(x\) and \(y\).

Let us summarise what we have achieved for \(H_5\) so far. We started with the definition of the \(H_5\) whose ROC is given in Eq. (17) and plotted in Figure 11. Taking summation over the index \(n\) yields the Gauss 2 \(F_1\) in the summand. Then good use of the AC of 2 \(F_1\) Eq. (46) is used to find the AC of \(H_5\) at (0, 1). The ROC of that AC is plotted in Figure 14 for real values of \(x, y\). Since the AC of \(H_5\) around (0, 1) does not cover the whole space around that point, we have derived another AC around the same point by transforming the second series of Eq. (22). The resultant AC is given in Eq. (27) and the ROC of it is plotted in Figure 15. This procedure can be described by a directed graph as shown in Figure 16, where the vertices represent the singular points and the directed edges denotes the process of evaluation. We plot the ROCs of both the ACs around the point (0, 1) along with the defining ROC of \(H_5\) in Figure 17.

Figure 16: The procedure to find the ACs around (0, 1) is described as a directed graph.
The ROC of the first series is given by

\[ |y| < \frac{1}{27} \quad |x| < \frac{1}{y^2} < 27 \land \min \left( \frac{1}{32} \left( \frac{8}{y^2} - \frac{36}{|y|} + \frac{8}{|y|^2} + 27 \right), \frac{1}{|y|} \right) < 1 \]  

The ROC of the first series is given by

\[ |y| < \frac{1}{27} \land |x| < \frac{1}{y^2} \land |x/y|^2 < 27 \land \min \left( \frac{1}{32} \left( \frac{8}{y^2} - \frac{36}{|y|} + \frac{8}{|y|^2} + 27 \right), \frac{1}{|y|} \right) < 1 \]  

These are plotted in Figure 19 and 20 respectively. The ROC of the AC given in Eq. (28) is the intersection of the ROCs of both the series.

We observe that the second series can be transformed further to find the AC of \( H_5 \) around \((\infty, \infty)\). We find the AC around \((0,1)\) in a similar way to the second AC around \((0,1)\) (Eq. (27)). The procedure is demonstrated via a directed graph in Figure 18. To find the AC around \((0,1)\), we use Eq. (47) in Eq. (21) to find

\[ H_5 = \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(b) \Gamma(c - a)} \sum_{m,n=0}^\infty \frac{(a)_{2m+n}(a-c+1)_{2m+n}}{m!n!(a-b+1)_{3m+n}} \left( -\frac{x}{y^2} \right)^m \left( \frac{1}{y} \right)^n \]

We go on to derive the ACs of \( H_5 \) around \((0, \infty)\) and \((\infty, \infty)\). The procedure is demonstrated via a directed graph in Figure 18. To find the AC around \((0, \infty)\), we use Eq. (47) in Eq. (21) to find

\[ H_5 = (-y)^{-a} \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(b) \Gamma(c - a)} \sum_{m,n=0}^\infty \frac{(a)_{2m+n}(a-c+1)_{2m+n}}{m!n!(a-b+1)_{3m+n}} \left( -\frac{x}{y^2} \right)^m \left( \frac{1}{y} \right)^n \]

We observe that the second series can be transformed further to find the AC of \( H_5 \) around \((\infty, \infty)\). We find the AC around \((0, \infty)\) and \((\infty, \infty)\) are described as a directed graph.
We do not write the ROC of the above AC explicitly. It can be obtained from the ancillary file (See Section 6.3). The ROC is shown in Figure 21.

\[
\sum_{m,n=0}^{\infty} \frac{(\frac{3}{3} - \frac{b}{3} + \frac{1}{3})_m (\frac{3}{3} + \frac{3}{3} + \frac{1}{3})_m (\frac{3}{3} + \frac{3}{3} - c + \frac{1}{3})_m (\frac{1}{27xy})^m ((\frac{2}{3} - \frac{1}{3}y)^n}{m!n!(\frac{3}{3})_m (\frac{3}{3} - \frac{2}{3} + \frac{1}{3})_m (\frac{3}{3} + \frac{2}{3} + \frac{1}{3})_m (\frac{1}{27xy})^m (\frac{3}{3} + \frac{2}{3} - c + \frac{1}{3})_m (\frac{2}{3} - \frac{1}{3}y)^n}
\]

(31)

To summarize, we have found four ACs of \( H_5 \) around the singular points \((0,1), (0,\infty)\) and \((\infty,\infty)\). Two ACs are found around the neighbourhood of \((0,1)\). The ROCs of these four ACs are plotted in Figure 22. In an analogous way, starting from Eq. (20) one can find ACs around \((1,0)\), which can further be used to find ACs around \((\infty,0), (1,\infty)\) and \((\infty,\infty)\). The whole procedure of finding ACs of \( H_5 \) is demonstrated using the directed graph in Figure 23. However finding all the ACs according to the graph is quite a laborious task in practice, due to the lengthy expressions of the ACs, even when one uses the computer program \texttt{Olsson.wl}. On the other hand, one may use the PETs of the Gauss
\( _2F_1 \) Eq. (45) to find some ACs of \( H_5 \). We now show examples of how to find such ACs.

As before, our starting points are Eq. (20) and Eq. (21). Let us start with Eq. (20). Using the first PET (i.e. first equation of Eq. (45)) we find,

\[
H_5 = (4x + 1)^{-\frac{a}{2}} \sum_{m,n=0}^{\infty} \frac{(a)_{n} (\frac{a}{2})_{m+n} (\frac{1}{2}(-a - 2b + 1))_{m-2b} (\frac{1}{4}(1-b))_{m-n} (4x + 1)^{m} \left(-\frac{y}{\sqrt{4x+1}}\right)^{n}}{m! n! \left(\frac{1}{4}\right)^{n}}
\]

This AC is valid in

\[
\left\{ |r| < 1 \wedge |s| < \frac{1}{4} \wedge |s| < \left( \min (\Phi_1(r), \Phi_2(r), \Phi_3(r)) \bigg| \Phi_3(r) \right) \right\} \quad (33)
\]

where \( r = \frac{4x}{4x+1}, \ s = -\frac{y^{2}}{16x+4} \). The functions \( \Phi_1(r) \) are defined below

\[
\Phi_1(r) = -128|r|^3 + 96|r|^2 + 3|r| - 2\sqrt{1-|r|}\sqrt{-(4|r|-1)^{3}(8|r|+1) + 2}
\]

\[
\Phi_2(r) = -128|r|^3 + 96|r|^2 + 3|r| + 2(8|r| + 1)\sqrt{1-|r|}\sqrt{-(4|r|-1)^{3} + 2}
\]

\[
\Phi_3(r) = -2\sqrt{|r| + 1} - 8|r||4|r| + 1|^{3/2} + 128|r|^3 + 96|r|^2 - 3|r| + 2
\]

Similarly using other two PETs, we find two more ACs. These are written below

\[
H_5 = (4x + 1)^{\frac{1}{2}(-a-1)} \sum_{m,n=0}^{\infty} \frac{(a)_{n} (\frac{a+1}{2})_{m+n} (\frac{1}{2}(-b - b + 1))_{m-2b} (\frac{1}{4}(1-b))_{m-n} (4x + 1)^{m} \left(-\frac{y}{\sqrt{4x+1}}\right)^{n}}{m! n! \left(\frac{1}{4}\right)^{n}}
\]
We observe that the ROC of both the ACs Eq. (32) and Eq. (35) are the same. Hence the AC Eq. (35) is valid in the same ROC (i.e. Eq. (33)). The other AC can be found using the last PET (i.e. the last equation of Eq. (45))

\[
H_5 = (4x + 1)^{-a-b+\frac{1}{2}} \sum_{m,n=0}^{\infty} \frac{(a)_{n} \left(\frac{1}{2}(-a - 2b + 1)\right)_{m-a-1} \left(-\frac{a}{2} - b + 1\right)_{m-2a} (-4x)^{m} \left(-\frac{y}{4x+1}\right)^{n}}{m!n!(c)_{n}} \left(\frac{1}{2}(-a - 2b + 1)\right)_{-\frac{1}{2}(3n)} \left(-\frac{a}{2} - b + 1\right)_{-\frac{1}{2}(3n)} (1 - b)_{m-n}
\]

(36)

\[
|r| < 1 \land \sqrt{|s|} < \begin{cases} 
\text{min}(\Psi_1(r), \Psi_2(r), \Psi_3(r), \Psi_4(r)), & \sqrt{|r|} < \frac{1}{\sqrt{3}} \\
\text{True}, & \text{otherwise}
\end{cases} \land |s| < \frac{1}{4}
\]

(37)

where \( r = -4x, \ s = \frac{y^2}{4(4x+1)^2} \). The functions \( \Psi_i(r) \) are defined below

\[
\Psi_1(r) = \frac{-(1-3|r|)^{3/2} + 9r + 1}{27 \left(|r|^{3/2} + \sqrt{|r|} \right)^{2}}, \quad \Psi_3(r) = \frac{3|r| \left(\sqrt{3|r| + 1} - 3\right) + \sqrt{3|r| + 1} + 1}{27(|r| - 1)^2|r|}
\]

\[
\Psi_2(r) = \frac{(1-3|r|)^{3/2} + 9r + 1}{27 \left(|r|^{3/2} + \sqrt{|r|} \right)^{2}}, \quad \Psi_4(r) = \frac{\sqrt{3|r| + 1} + 3|r| \left(\sqrt{3|r| + 1} + 3\right) - 1}{27(|r| - 1)^2|r|}
\]

(38)

Starting with Eq. (21) and using three PETs we find three ACs below

\[
H_5 = (1 - y)^{-b} \sum_{m,n=0}^{\infty} \frac{(a)_{2m(a-c+1)}_{m-m} \left(b\right)_{n-m(c-a)}_{m-2m} \left(x - xy\right)^{m} \left(\frac{y}{y-1}\right)^{n}}{m!n!(c)_{n}} (x - xy)^{m} \left(\frac{y}{y-1}\right)^{n}
\]

(39)

with ROC

\[
4|x - xy| < 1 \land \left|\frac{y}{y-1}\right| < 1 \land
\]

\[
|x - xy| < \sqrt{\frac{y}{y-1}} + 8 \left|\frac{y}{y-1}\right|^{3/2} + \left|\frac{y}{y-1}\right|^2 - 20 \left|\frac{y}{y-1}\right| + 8 \sqrt{\left|\frac{y}{y-1}\right|} \left|\frac{y}{y-1}\right| + 8 - 8
\]

(40)
and

\[ H_5 = (1 - y)^{-a} \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(c-b)_{m+n}}{m!n!(1-b)_m(c-b)_m} \left( -\frac{x}{(y-1)^2} \right)^m \left( \frac{y}{y-1} \right)^n \]  

(41)

with ROC

\[
4 \left| \frac{x}{(y-1)^2} \right| < 1 \wedge 32 \left| \frac{x}{(y-1)^2} \right| + \sqrt{\left| \frac{y}{y-1} \right| (\left| \frac{y}{y-1} \right| + 8)^{3/2}} + \left| \frac{y}{y-1} \right|^2 < 20 \left| \frac{y}{y-1} \right| + 8 \wedge \left| \frac{y}{y-1} \right| < 1
\]

(42)

and the last one

\[ H_5 = (1 - y)^{-a-b+c} \sum_{m,n=0}^{\infty} \frac{(a)_{2m}(a-c+1)_{2m}(c-a)_{n-2m}(c-b)_{m+n}}{m!n!(1-b)_m(c-b)_m} \left( \frac{x}{y-1} \right)^m y^n \]  

(43)

with ROC

\[
4 \left| \frac{x}{y-1} \right| < 1 \wedge |y| < 1 \wedge \left( 4 \left| \frac{x}{y-1} \right| + 3 \right)^{3/2} + 8 \left| \frac{x}{y-1} \right| > 9(3|y| + 2)
\]

(44)

All the ROCs of the six ACs are shown in Figure 24, 25, 26, 27 and 28 for real values of \( x \) and \( y \).

It worth noting that, the PETs of \( {}_2F_1 \) can be used in any intermediate step of the derivation, when the Gauss \( {}_2F_1 \) appears inside the summand, to find new ACs. We have applied it on the definition of the \( H_5 \) function only in this section to demonstrate the usefulness of the PETs of the Gauss \( {}_2F_1 \).

Even after analysing all the possible ways of applying the PETs in the intermediate steps of the derivation, we found no AC that is valid inside the small ‘white’ region as shown in Figure 29.

4 Summary

In this work, we have presented the various ACs and their corresponding ROCs of the Horn \( H_1 \) and \( H_5 \) functions. These results have been derived using the publicly available Mathematica packages Reson.wl and ROC2.wl. This ensures an easeful computation of the ACs and their ROCs which are otherwise prone to errors or difficult to compute by hand.

Our extensive work shows that the ACs of \( H_1 \) cover the whole real \( x - y \) plane, with the exception of a few boundaries and singular lines, whereas the ACs of \( H_5 \) can be used to cover everywhere, except for the small region shown in Figure 29. It is to be noted that in both cases we are not able to give
convergent expressions on the singular curves. It may be pointed out that the expressions given here have to be used carefully for numerical implementation due to the problem of multi-valuedness, which has not been discussed in this article. To this extent, it is the precise analogue of the work of Exton for Appell $F_4$ and Olsson for Appell $F_1$. All the ACs derived here are valid for generic values of Pochhammer parameters. If the ACs are required to be used for the exceptional parameters then one has to do the proper limiting procedure.

It is our view that investigations such as those reported here will go a long way in the advancements of the theory of hypergeometric functions, even for higher orders and higher numbers of variables, which in general are ubiquitous. Furthermore, the use of various symbolic computational packages used here makes them more significant for the analysis of complicated hypergeometric functions.

5 Acknowledgements

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6 Appendix

6.1 Appendix A

In this Appendix we list some formulae that are used in deriving the ACs of $H_1$ and $H_5$ functions.

- Euler transformation of $\, _2F_1$

\[
\begin{align*}
_2F_1(a, b, c; z) & = (1 - z)^{-a} _2F_1\left(a, c - b, c, \frac{z}{z - 1}\right) \\
& = (1 - z)^{-b} _2F_1\left(c - a, b, c, \frac{z}{z - 1}\right) \\
& = (1 - z)^{c - a - b} _2F_1(c - a, c - b, c, z) 
\end{align*}
\] (45)

- Analytic continuation of $\, _2F_1$ around $z = 1$

\[
\begin{align*}
_2F_1(a, b, c; z) & = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} _2F_1(a, b, a + b - c + 1; 1 - z) \\
& \quad + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}(1 - z)^{c - a - b} _2F_1(c - a, c - b, c - a - b + 1; 1 - z) 
\end{align*}
\] (46)
• Analytic continuation of $2F_1$ around $z = \infty$

$$2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} 2F_1(a, a-c+1, a-b+1; \frac{1}{z}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} 2F_1(b, b-c+1, b-a+1; \frac{1}{z}) \tag{47}$$

• Analytic continuation of $3F_2$ around $z = \infty$

$$3F_2\left(\begin{array}{c}
a_1, \ a_2, \ a_3 \\ b_1, \ b_2 \end{array}\right) = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(a_2-a_1)\Gamma(a_3-a_1)}{\Gamma(a_2)\Gamma(a_3)\Gamma(b_1-a_1)\Gamma(b_2-a_1)}(-z)^{-a_1} 3F_2\left(\begin{array}{c}
a_1, \ 1+a_1-b_1, \ 1+a_1-b_2 \\ 1+a_1-a_2, \ 1+a_1-a_3 \end{array}\frac{1}{z}\right) + \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(a_1-a_2)\Gamma(a_2-a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1-a_2)\Gamma(b_2-a_2)}(-z)^{-a_2} 3F_2\left(\begin{array}{c}
a_1, \ 1+a_2-b_1, \ 1+a_2-b_2 \\ 1+a_2-a_1, \ 1+a_2-a_3 \end{array}\frac{1}{z}\right)$$

$$+ \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(a_1-a_3)\Gamma(a_2-a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1-a_3)\Gamma(b_2-a_3)}(-z)^{-a_3} 3F_2\left(\begin{array}{c}
a_3, \ 1+a_3-b_1, \ 1+a_3-b_2 \\ 1+a_3-a_1, \ 1+a_3-a_2 \end{array}\frac{1}{z}\right) \tag{48}$$

• Analytic continuation of $4F_3$ around $z = \infty$

$$4F_3\left(\begin{array}{c}
a_1, \ a_2, \ a_3, \ a_4 \\ b_1, \ b_2, \ b_3 \end{array}\right) = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(a_2-a_1)\Gamma(a_3-a_1)\Gamma(a_4-a_1)}{\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(b_1-a_1)\Gamma(b_2-a_1)\Gamma(b_3-a_1)}(-z)^{-a_1} 4F_3\left(\begin{array}{c}
a_1, \ 1+a_1-b_1, \ 1+a_1-b_2, \ 1+a_1-b_3 \\ 1+a_1-a_2, \ 1+a_1-a_3, \ 1+a_1-a_4 \end{array}\frac{1}{z}\right)$$

$$+ \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(a_3-a_2)\Gamma(a_4-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(b_2-a_2)\Gamma(b_3-a_2)}(-z)^{-a_2} 4F_3\left(\begin{array}{c}
a_2, \ 1+a_2-b_1, \ 1+a_2-b_2, \ 1+a_2-b_3 \\ 1+a_2-a_1, \ 1+a_2-a_3, \ 1+a_2-a_4 \end{array}\frac{1}{z}\right)$$

$$+ \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(a_1-a_3)\Gamma(a_2-a_3)\Gamma(a_4-a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(b_1-a_3)\Gamma(b_2-a_3)\Gamma(b_3-a_3)}(-z)^{-a_3} 4F_3\left(\begin{array}{c}
a_3, \ 1+a_3-b_1, \ 1+a_3-b_2, \ 1+a_3-b_3 \\ 1+a_3-a_1, \ 1+a_3-a_2, \ 1+a_3-a_4 \end{array}\frac{1}{z}\right) \tag{49}$$
\[
\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(a_1 - a_4)\Gamma(a_2 - a_4)\Gamma(a_3 - a_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(b_1 - a_4)\Gamma(b_2 - a_4)\Gamma(b_3 - a_4)}(-z)^{-a_4}
\]
\[
\quad F_3\left(\begin{array}{c}
a_4, 1 + a_4 - b_1, 1 + a_4 - b_2, 1 + a_4 - b_3 \\
1 + a_4 - a_1, 1 + a_4 - a_2, 1 + a_4 - a_3
\end{array} \mid \frac{1}{z}\right)
\]

(49)

- The analytic continuation around \( z = \infty \) of \( _pF_{p-1}(\ldots; z) \) function can be found using the \texttt{Olsson.wl} package.

### 6.2 Appendix- B

In this appendix we give an outline of the various methods that are used to find the ROC of the hypergeometric series [38]. The methods are applicable to series with more than two variables as well. But the discussion below is focused on the double variable hypergeometric series and are implemented in the companion package \texttt{ROC2.wl} of \texttt{Olsson.wl}.

1. **Cancellation of parameters**: Cancellation of parameters states that the region of convergence of a hypergeometric series is independent of the Pochhammer parameters. For example, consider the following Kampé de Fériet function

\[
S_1 = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}(d)_{m+n}}{(e)_{m} (f)_{n} m! n!} x^m y^n
\]

Since the ROC is independent of the Pochhammer parameters, we can choose \( d = g \) and the two Pochhammer symbols cancel and the series then effectively becomes similar to the Appell’s \( F_2 \) function and thus have the ROC: \( |x| + |y| < 1 \)

2. **Horn’s theorem**: Horn’s theorem provides a more general way to find the ROC of a given hypergeometric series for any number of variables. The automatized implementation of the Horn’s theorem for two variable hypergeometric series is performed in the package \texttt{ROC2.wl}. We will explain it’s working principle with an illustrative example below. Consider the following series

\[
S_2 = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}(d)_{2m+n}}{(d)_{2m+n} m! n!} X^m Y^n
\]

Writing the above series in a compact form

\[
S_2 = \sum_{m,n=0}^{\infty} A_{m,n} X^m Y^n
\]

We then evaluate following two ratios

\[
f(m, n) = \frac{A_{m+1,n}}{A_{m,n}} = \frac{(a + m + n)(b + m + n)(c + m)}{(d + 2m + n)(d + 1 + 2m + n)(1 + m)}
\]

\[
g(m, n) = \frac{A_{m,n+1}}{A_{m,n}} = \frac{(a + m + n)(b + m + n)}{(d + 2m + n)(1 + n)}
\]

Now we define two more functions as follows

\[
\Phi(\mu, \nu) = | \lim_{t \to \infty} f(\mu t, \nu t) |^{-1}
\]

\[
\Psi(\mu, \nu) = | \lim_{t \to \infty} g(\mu t, \nu t) |^{-1}
\]

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From this one can construct following two subsets of $\mathbb{R}_+^2$

$$C = \{(r, s) \mid 0 < r < \Phi(1, 0) \land 0 < s < \Psi(0, 1)\}$$

and

$$Z = \{(r, s) \mid \forall (m, n) \in \mathbb{R}_+^2 : 0 < r < \Phi(m, n) \lor 0 < s < \Psi(m, n)\}$$

The Horn’s theorem for two variable hypergeometric functions is then given as follows

**Theorem 1** *The union of $Z \cap C$ and its projections upon the co-ordinate axes is the representation in the absolute quadrant $\mathbb{R}_+^2$ of the region of convergence in $\mathbb{C}^2$ for the series $F$.***

For $S_2$ we then have

$$C = 0 < r < 4 \land 0 < s < 1$$

$$Z = r + s < 2s^{1/2}$$

Using Horn’s theorem the ROC of the series $S_2$ is then given by

$$|x| < 4 \land |y| < 1 \land |x| + |y| < 2|y|^{1/2}$$

### 6.3 Appendix C

In this appendix we briefly describe the *Mathematica* [59] package *HornH1H5.wl* for the ease of the reader. Since it is not possible to present all the ACs of the Horn $H_1$ and $H_5$ function due to their lengthy expressions so we provide this package as an ancillary file. The package can be downloaded from the link below.

[https://github.com/souvik5151/HornH1H5.git](https://github.com/souvik5151/HornH1H5.git)

Once the file is downloaded, it can be called, after setting the correct path, as

```mathematica
In[] := <<HornH1H5.wl
```

This package can be used to access all the ACs as well as their corresponding ROCs. We have included the two commands for each of the function $H_1$ and $H_5$ in the package. These are the following

1. **$H1\text{expose}$ and $H5\text{expose}$**: These commands take an integer (less than the number of ACs) and gives the output as a list of two elements containing the ROC and the expression of the AC, labelled by that number, of $H_1$ and $H_5$ respectively. There are 22 ACs of $H_1$ and 20 ACs of $H_5$.

2. **$H1\text{ROC}$ and $H5\text{ROC}$**: These commands plots the ROC of a particular AC for real values of $x$ and $y$, which is specified by a number given by the user, along with a user specified point $(x, y)$. This command can be used when the user wants to determine if a certain point lie inside the ROC of an AC or not.

More information about these commands are available to the user via the `Information` command of *Mathematica* after loading the package.

### 7 Conflict of Interest Statement

We have no conflicts of interest to disclose.

### 8 Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.
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