MIMO Radar Waveform Constraints for GMTI
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Abstract—Ground moving-target indication (GMTI) provides both an opportunity and challenge for coherent multiple-input multiple-output (MIMO) radar. MIMO techniques can improve a radar’s angle estimation and the minimum detectable velocity (MDV) for a target. However, the challenge of clutter mitigation places significant constraints on MIMO radar waveforms. In this paper, the loss of target return because of clutter mitigation (signal-to-noise ratio (SNR) loss) is the driving performance metric. The ideal, orthogonal repeated-pulse waveform is shown not to exist. Pulse-to-pulse time-varying waveforms, such as Doppler-division multiple access (DDMA), are shown to offer SNR loss performance approaching ideal MIMO systems.

Index Terms—Multiple-input multiple-output (MIMO) systems, pulse Doppler radar, radar clutter, radar theory.

I. INTRODUCTION

Despite a hundred-year history [1], radar research is still active. One active area of investigation is multiple-input multiple-output (MIMO) radar [2], [3]. The basic concept of MIMO radar is that there are multiple radiating and receiving sites [4]. The collected information is then processed together. Various researchers assume different types of radar concepts when using the nomenclature MIMO radar [2]. Broadly, MIMO radar can be broken into two classes of operation, based on whether or not the target response is coherent between various transmit and receive antenna pairs. Here, coherence indicates that the target has the same complex response up to some known relative delay caused by the antenna geometries. If the response is coherent, then this type of radar is sometimes denoted coherent MIMO radar [4]–[8]. Otherwise, it is denoted statistical MIMO radar [9]–[12]. The focus of this paper is coherent MIMO radar. A number of authors have discussed MIMO radar’s advantages in terms of angle estimation. Angle-estimation Cramer–Rao bounds for system or waveform optimization are discussed in [2] and [13]–[15]. Nonlocal extensions to angle estimation bounds are considered in [16] and [17]. Rather than discuss angle estimation, in the this paper, we focus on clutter mitigation performance.

In much of the literature, it is assumed that the waveforms radiated from each transmit phase center are orthogonal and have an idealized cross-waveform radar ambiguity functions. For real waveforms, this is not generally possible [18], [8]. Depending upon the application, orthogonal waveforms may or may not be acceptable. An extreme example is ground moving-target indication (GMTI) radar. Because of the clutter mitigation requirements of GMTI, this type of radar is particularly sensitive to waveform characteristics, requiring similar clutter response from each waveform [19], [8].

Typically, GMTI systems are side-looking radars mounted on airborne platforms that employ arrays of antennas enabling angle estimation. These radars transmit a sequence of pulses and coherently process this entire sequence of received signals. GMTI radars exploit the angle-Doppler frequency correlation to differentiate stationary clutter from moving targets. Commonly, displaced phase center aperture (DPCA) or space–time adaptive processing (STAP) [20]–[22] techniques are used to mitigate clutter. Improved clutter mitigation performance enables improved target detection, particularly at lower target velocities.

Implicit in the analysis in this paper is the notion that the scattering field is longer than the pulse. This is commonly true for GMTI systems. The results in this paper may be overly pessimistic for problems with a clutter scattering field of limited extent. Another assumption in this paper is that there is no scattering return between pulses. Because of the range to the fourth attenuation and other factors, there is little likelihood of confusion at the receiver between signal returns from the current pulse via nearby scatterers and signal returns from previous pulses via distant scatterers. This is often a good approximation, but this is not valid for all GMTI systems.

A. MIMO GMTI Radar

GMTI is an interesting application for MIMO radar. Target angle-estimation performance is dependent upon the antenna array size, and is therefore improved by MIMO’s larger virtual aperture. The minimum detectable velocity (MDV) of a target is improved by both the larger virtual aperture size and the longer coherent processing interval (CPI) used by MIMO GMTI radars [2].

Because the MIMO virtual array (constructed from the convolution of the positions of the transmit and receive phase centers) enables the use of sparse real arrays, a larger effective aperture can be employed. By design, the virtual array has a large, filled aperture with many more virtual degrees of freedom than the physical array. Clutter suppression techniques take advantage of angle-Doppler coupling of the stationary clutter. The larger MIMO virtual array increases the distance, in terms of angular resolution cells, between the clutter and the target within a given Doppler bin, making clutter suppression easier.

For a given CPI, a MIMO radar illuminates a larger region and integrates longer than the equivalent traditional focused beam radar. To cover the same search area, the focused beam of a traditional single-input multiple-output (SIMO) radar must
move from direction to direction over a number of CPIs. In this paper, it is assumed SIMO indicates a radar that has an array of transmit/receive phase centers. During a SIMO CPI, a single waveform up to some set of fixed phases is radiated from all elements of the transmit array for the entire CPI. The MIMO radar covers the same search region by staring continuously over a single, longer CPI. The illuminated region of the MIMO radar is typically determined by the pattern of the subarray or antenna associated with each phase center. The instantaneous target SNR for the SIMO system is larger because of the focused transmit beam, but this is compensated by the increased integration period of the MIMO system. Consequently, the integrated target SNR is the same for both approaches. For some applications, the longer integration can require increased computational sophistication because of the increased sensitivity to details of the target motion. While it is beyond the scope of this paper, it is possible for the MIMO radar to shape the illumination region, forming more complicated patterns than that determined by the subarray pattern, by controlling the structure of the transmit covariance matrix [17].

The minimum detectable velocity (MDV) of a target is also dependent upon the length of the CPI. The increase in CPI length of the MIMO radar reduces the size of a Doppler cell. The smaller Doppler cell both reduces the amount of clutter in a bin and increases the distance, in terms of Doppler resolution cells, from clutter to the target, making clutter suppression easier.

B. Waveform Dependent SNR Loss

In this paper, we use target signal-to-noise ratio loss (SNR loss) due to clutter suppression as our metric to assess performance. SNR loss for MIMO GMTI radar is discussed in [2], [8]. Because SNR loss is a function of the target velocity, SNR loss performance affects MDV. There are a number of fundamental questions associated with clutter mitigation performance as a function of waveform characteristics. This paper addresses some of those questions. To this end, notation and description of the MIMO GMTI response vector is introduced in Section II.

A common assumption in the MIMO radar literature is that of the orthogonal MIMO waveform. In principle, waveforms with disjoint frequency supports provide orthogonality at all delays. The adverse implications of these waveforms to clutter suppression are discussed in Section III. Range cells filled with scatterers cause random response fluctuations that vary with frequency. If the frequency supports are too dissimilar, then the clutter response becomes significantly different. As a consequence, the clutter rank in the MIMO covariance matrix for a given Doppler cell is increased with these spectrally disjoint waveforms. In fact, the effective nulling capability reverts to that based on the receiver phase centers alone and does not benefit from the virtual array. Ideally, the MIMO clutter covariance should be rank one. However, in Section III.C, we show the rank of the clutter grows to the number of transmitters.

In Section IV, relationships for SNR loss as a function of waveform characteristics are developed. It is common for GMTI systems to repeat the same waveform from pulse to pulse. A fundamental question is whether it is an advantage to change this mode of operation. The SNR loss relationship, developed in Section IV.D, allows for continuously changing waveforms. In Section IV.E, the SNR loss is specialized to the case of periodically repeating waveforms. In both cases, the SNR loss is shown to be consistent with the performance of idealized orthogonal waveforms. Such is not the case for waveforms that repeat in each pulse throughout a CPI.

Repeating waveforms cannot use the virtual array for nulling. In a Doppler cell, clutter nulling relies only on the receiver phase centers. Even so, the transmitter phase centers do play a role in clutter suppression. With a large number of transmitter phase centers and some array designs, it is possible to achieve MDV performance comparable to an ideal MIMO GMTI at a large cost in increased sidelobes or increased bandwidth. There may be niche applications where this trade-off makes sense, but this discussion is beyond the scope of this study.

In Section V, examples of performance for various waveform concepts are compared. We show that near ideal MIMO GMTI MDV performance occurs when the MIMO waveforms are varied pulse to pulse. There are a variety of different types of variations that all perform well. We call out two here because they are easy to describe. The first MIMO variation uses random waveforms pulse to pulse. It does not achieve the best possible MDV in all cases but does exhibit clutter nulling using the virtual array. The second variation applies an independent pulse-to-pulse (slow time) modulation for each transmitter, which we denote Doppler domain, multiple access (DDMA) [8]. A related concept is discussed in [23]. These waveforms can achieve almost the same performance as ideal MIMO waveforms. Periodic repetition of the MIMO waveforms, modulated pulse to pulse, allows Doppler filtering to localize the interfering clutter and the target, reducing the degrees of freedom required for adaptation. This result does not seem possible with random waveforms for which adaptation and training occur in the full transmitter-by-receiver-by-pulse dimensionality. It is worth noting that pulse-to-pulse modulated waveforms have repeated clutter notches unlike their SIMO counterparts. Random waveforms do not suffer this problem.

II. MIMO GMTI RESPONSE VECTORS AND THE VIRTUAL ARRAY

A. Virtual Array

We consider pulse Doppler waveforms consisting of a sequence of phase coherent pulses. We assume bounded delays in the radar returns to the extent that there is no overlap between the returns of different pulses. An array is used for transmission and reception. Multiple transmitters allow the use of different simultaneous waveforms that may be either fixed or varying pulse to pulse.

We utilize a narrowband model for the array response of the radar waveforms considered here. In more explicit terms, the array response is modeled in terms of differential phase due to delay at the RF carrier $\omega$. In particular, the transmitter and receiver phase centers are not widely separated in terms of inverse bandwidths.

Throughout the paper, we employ the following notation. Tensors and vectors are represented by bold lowercase variables.
Operators and matrices are represented by bold or calligraphic uppercase variables. Sets are indicated by uppercase variables or calligraphic uppercase variables (from context there should be little chance of confusion). The Hermitian conjugate is denoted by $\dagger$. The Kronecker product is denoted by $\otimes$. The Hadamard (or element-by-element) product is denoted by $\boxtimes$. The L2-norm of a vector is indicated by $\| \cdot \|$. The absolute value of a scalar is indicated by $| \cdot |$. The expected value of an expression is indicated by $E[\cdot]$. The number of members in a set is denoted $|\cdot|$. The identity matrix of size $n$ is indicated $I_n$. The speed of light is denoted $c$.

Assume that the antenna array is one-dimensional along the $x$-axis with virtual phase centers located on a subset of a Nyquist (half-wavelength) grid. Assume, in addition, that a virtual pulse rate is coordinated with platform motion (the distance moved by the radar between virtual samples is one-half the grid spacing) so that physical pulses are also associated with a subset of grid elements. More explicitly, if $\ell_{\Delta}$ denotes the time between virtual (i.e., potential) pulses, then we require $\ell_{\Delta} = \lambda = 1$. Platform motion is aligned with the $x$-axis with constant velocity $v_p$. Collecting separable measurements (as if this were possible) between all transmitter and receiver pairs, a normalized array response to a point scatterer can be summarized in the form $a_R \otimes a_T \otimes a_D^{-1}$, where

\[
\begin{align*}
(A_R(\mu))_m &= e^{im\mu} & m &\in M_R \\
(A_T(\mu))_m &= e^{im\mu} & m &\in M_T \\
(A_D(\mu, \rho))_m &= e^{im(\mu+\rho)} & m &\in M_D
\end{align*}
\]

and where $\mu \stackrel{df}{=} \pi u_x u_y$ is the x-component of the direction-of-arrival vector $u$, and $\rho$ represents target Doppler shift in terms of phase change per virtual sample. The array response is written as a tensor product, but can be interpreted as a vector. This vector represents the differential phases between virtual phase centers due to differential delays at the RF carrier. For example, at time $m_{\Delta}$,

\[
\frac{\omega}{c} (2\pi m_{\Delta} v_p \cdot u) = m \mu.
\] (1)

$M_R$, $M_T$, and $M_D$ are each subsets of integers that index the phase centers used for transmitter, receiver, and pulse. The location of the scatterer is parameterized by the normalized spatial frequency $\mu$, $-\pi \leq \mu \leq \pi$. It is important to note that spatial and temporal samples can be sparse on the half-wavelength grid. However, the combined sampled may fill in all or some of the gaps. Although the sampling framework utilized here is built around a half-wavelength lattice of virtual, omnidirectional elements, similar reasoning would allow us to incorporate element patterns by grouping elements in subarrays and working with a down-sampled lattice.

The subsets of integers $M_R$, $M_T$, and $M_D$ form the set of triples given by $C = M_R \times M_T \times M_D$. We let

\[
S \stackrel{df}{=} M_R + M_T + M_D = \{ m + j + n : (m, j, n) \in C \}. \tag{2}
\]

In some cases, $S$ represents an interval of integers

\[
S = \{ k : k_L \leq k \leq k_U \}. \tag{3}
\]

We say that $M_T$, $M_R$, and $M_D$ satisfying (3) form a Nyquist virtual array. Note that this is an extension of the spatial virtual array concept to include the synthetic aperture created by sensor motion.

Define the vector $v$ with components $v_{ijk} = e^{ik\mu}$, with $k \in S$.

Next, define the matrix $A_C : R^S \rightarrow R^C$ so that, when $\rho = 0$,

\[
(a_R \otimes a_T \otimes a_D)_{ijk} = (A_C v)_{ijk} = v_{i+j+k}.
\] (4)

Given a Nyquist virtual array, the domain of $A_C$ has dimension $k_T - k_L + 1$ while the range space (not the range of the matrix $A_C$) always has at least as large a dimension as the domain.

A few properties of the operator $A_C$ are useful below. First, note that

\[
w^T A_C v = \sum_{(i,j,k) \in C} w_{ijk} v_{i+j+k} = \sum_{k \in S} \left( \sum_{(i,j,k) \in C \cap i+j+k=\ell} w_{ijk} \right) u_{\ell}
\]

so that $A_C^T$ is determined by

\[
(A_C^T w)_I = \sum_{(i,j,k) \in C} w_{ijk}. \tag{5}
\]

From this relation, we have

\[
(A_C^T A_C v)_I = \sum_{i+j+k=\ell} v_{i+j+k} = p(I) u_{\ell}
\] (6)

where $p(I) \stackrel{df}{=} \#\{ (i,j,k) \in C \text{ with } i+j+k = \ell \}$. In the same vein,

\[
(A_C (A_C^T A_C)^{-1} A_C^T w)_{ijk} = p^{-1}(i+j+k) \sum_{(i,m,n) \in C \cap i+m+n = i+j+k} w_{imm}.
\] (7)

The operator $P_{A_C} = A_C (A_C^T A_C)^{-1} A_C^T$ projects onto the range of $A_C$. For use below, let $n_R = \#M_R$, $n_T = \#M_T$, and $n_D = \#M_D$.

B. Radar Response With Repeated Waveforms

If $s(t)$ denotes a vector of MIMO waveforms, the narrowband model of the return from a scatterer at angle $\mu$, delay $\tau$, and Doppler $\rho$ can be written

\[
[s(t - \tau) a_T(\mu)] [a_R(\mu) \otimes a_D(\mu, \rho)]
\] (8)

where the tensor products represent phase at the center frequency. It is convenient notionally to conjugate the waveforms on transmission. The differential phases in $a_D$ represent the effects of Doppler, including sensor motion, on the returns. In this simple model, the same waveforms are used for each pulse.
A more general model is introduced in Section IV. We denote by \( R(\tau) \) the waveform correlation matrix

\[
R(\tau) \overset{\text{def}}{=} \int s(t+\tau)s^\dagger(t)\,dt. \tag{9}
\]

Pulse compression correlates a response such as (8) with each of the transmitted waveforms, yielding

\[
R(\tau) a_T(\mu) \otimes a_R(\mu) \otimes a_D(\mu, \rho). \tag{10}
\]

It is often convenient to process radar returns from multiple pulses by using a Fourier transform in the pulse domain to localize returns to a single Doppler cell. The Doppler cell containing the moving target at a fixed angle \( \rho_T \) is denoted \( (\mu_T, \rho_T) \). In the same Doppler cell, we can find clutter returns (from motionless scatterers) with parameters \( (\mu_c, 0) \), given \( \mu_c = \mu_T + \rho \). Additional suppression of the clutter can occur in this fixed cell. The radar response in the Doppler cell becomes

\[
R(\tau) a_T(\mu) \otimes a_R(\mu) \tag{11}
\]

with \( \mu = \mu_T \) or \( \mu = \mu_c \). The target is received at a fixed delay that we can take as \( \tau = 0 \). The clutter returns occur over a large delay extent and are uncorrelated from delay to delay. If we assume equal return strengths from each delay, the covariance of clutter in the Doppler cell is proportional to

\[
\left( \int R(\tau) a_T(\mu) a_T^\dagger(\mu) R(\tau) d\tau \right) \otimes a_R(\mu) a_R^\dagger(\mu). \tag{12}
\]

In the idealized version of MIMO GMTI, the waveform cross-correlations are assumed to be expressed as \( R(\tau) = r(\tau) I_{nt} \) for some scalar autocorrelation function \( r(\tau) \). Although such cross-correlation functions are achievable, the obvious realizations suffer from significant problems, as we show in Section III.

### III. ORTHOGONAL MIMO WAVEFORMS

We briefly discuss several orthogonal MIMO waveforms and show that they fail to provide the full benefits of MIMO, at least in terms of clutter nulling. It should be noted that the orthogonal waveforms considered maintain orthogonality at all delays and are repeated pulse to pulse. In other words, the common waveform autocorrelation matrix \( R(\tau) \) is diagonal. It is easy to construct such waveforms to a high degree of approximation, but these constructions are shown to possess serious limitations.

#### A. Shifted Bands

Orthogonal MIMO waveforms can be constructed from a band-limited base waveform that is shifted in frequency by a multiple of the base waveform’s bandwidth. Each transmitter uses a different shift so that the transmitted waveforms have no frequency overlap. The cost of this procedure is additional bandwidth without any improvement in range resolution.

The problem with this approach for constructing MIMO waveforms arises from the decorrelation of clutter between different frequencies. Consider the return from a range cell at radian frequency \( \omega \)

\[
\Sigma(\omega) \overset{\text{def}}{=} \int_L \sigma(\tau)e^{j2\pi\omega\tau} \, d\tau \tag{13}
\]

where \( \sigma(\tau) \) is a random variable with mean zero and complex variance

\[
E[\sigma(r_1)\overline{\sigma(r_2)}] = \delta(r_1 - r_2). \tag{14}
\]

The variable \( \sigma(\tau) \) represents the random phase response associated with scatterers at different ranges within the range cell that is informally denoted by \( L \). For two frequencies, the cross-correlation of the cell responses is expressed by

\[
\gamma(\omega_1, \omega_2) \overset{\text{def}}{=} E[\Sigma(\omega_1)\overline{\Sigma(\omega_2)}] = \int_{-L/2}^{L/2} e^{j2\pi(\omega_2 - \omega_1)\tau} \, d\tau
\]

\[
= L \text{sinc} \left( \frac{\omega_1 - \omega_2}{2B} \right)
\]

\[
= L \text{sinc} \left( \frac{\omega_1 - \omega_2}{c} \right)
\]

where \( c \) is the speed of light, \( B \) is the waveform’s bandwidth, and the range resolution is \( c/2B \). We have assumed that the range cell is centered at 0. Other center points simply introduce a unit magnitude complex gain factor into the correlation. The null depth associated with a correlation coefficient \( \gamma(\omega_1, \omega_2) \) can be expressed by

\[
1 - \frac{\gamma^2(\omega_1, \omega_2)}{\gamma^2(\omega_1, \omega_1)\gamma^2(\omega_2, \omega_2)} = 1 - \text{sinc}^2 \left( \frac{\omega_1 - \omega_2}{2B} \right) \tag{15}
\]

Good null depth requires \( \gamma(\omega) \approx 1 \), which cannot occur if the frequency shift is on the order of the waveform bandwidth given disjoint MIMO waveforms.

#### B. Interleaved Frequencies

One way around the problem associated with shifted bands involves the use of a picket-fence base waveform. Small shifts in the waveform can be made orthogonal to each other. Thus, orthogonal MIMO waveforms are created without suffering significant loss in null depth. The minimal shift is set by the inverse pulse length, which determines the frequency resolution of the waveform. As an example, the argument of (15) can be expressed by \( \pi/TB \), where \( T \) is the pulse length. Even modest time-bandwidth products provide more than adequate null depth. However, because of the unusual spectral structure of this waveform, the autocorrelation function contains a large set of near-range ambiguities, complicating target range estimation.

#### C. Clutter Rank

There is another problem associated with disjoint waveform spectra. Note that the autocorrelation matrix \( R(\tau) \) can be expressed in the form

\[
R(\tau) = \int \hat{s}(\omega)\overline{\hat{s}(\omega)}e^{j\omega\tau} \frac{d\omega}{2\pi}. \tag{16}
\]
Thus, any diagonal autocorrelation matrix \( \mathbf{R}(\tau) \) must involve waveforms with essentially disjoint spectra. In fact, if, for \( j \neq k \)

\[
0 = R_{jk}(\tau) = \int \hat{s}_j(\omega)\hat{s}_k(\omega)e^{j\omega\tau} \frac{d\omega}{2\pi}
\]  

(17)

for all \( \tau \), then \( \hat{s}_j(\omega) \) and \( \hat{s}_k(\omega) \) can be jointly nonzero only on a set of measure zero. It is interesting to observe that an ideal orthogonal MIMO waveform of the form \( \mathbf{R}(\tau) = \tau(\tau)\mathbf{I}_{\text{rs}} \) does not exist since, for \( j \neq k \), \( R_{jj} = R_{kk} \) implies \( |\hat{s}_j(\omega)| = |\hat{s}_k(\omega)| \) almost everywhere and yet \( \hat{s}_j(\omega) \) and \( \hat{s}_k(\omega) \) have essentially disjoint supports, meaning that they must both be zero almost everywhere.

Define

\[
\Omega_j \triangleq \{ \omega : |\hat{s}_j(\omega)| \neq 0 \}.
\]

(18)

Up to a set of measure zero, the \( \{\Omega_j\} \) are assumed disjoint. The clutter covariance in a Doppler cell associated with a moving target is expressed in (12), but, by Parseval

\[
\int \mathbf{R}(\tau)\mathbf{a}_R(\tau)^\dagger d\tau = \int |\mathbf{a}_R(\tau)|^2|\hat{s}(\omega)|^2(\omega)\frac{d\omega}{2\pi}.
\]

(19)

Thus, using the notation \( \mathbf{e}_j \) to denote the vector with all zeros except a single 1 in the \( j \)th entry

\[
\int \mathbf{R}(\tau)\mathbf{a}_R(\tau)^\dagger d\tau = \sum_j \left( \int_{\Omega_j} |\mathbf{a}_R(\tau)|^2|\hat{s}_j(\omega)|^2 T(\omega) \frac{d\omega}{2\pi} \right) \mathbf{e}_j\mathbf{e}_j^\dagger
\]

which is a diagonal matrix. Given nonzero waveforms from each transmitter and, as is typical, a transmitter vector \( \mathbf{a}_R \) that has all nonzero components, the rank of the clutter covariance in the Doppler cell becomes \( m_{\text{cl}} \). Clutter suppression is associated solely with the rank-one receiver factor \( \mathbf{a}_R(\tau) \) of the Kronecker covariance (12) since the other factor is full rank. An ideal orthogonal MIMO waveform would not increase clutter rank. In fact, substituting \( \mathbf{R}(\tau) = \tau(\tau)\mathbf{I}_{\text{rs}} \) in (12) results in a rank-one covariance proportional to \( \mathbf{a}_R(\tau)\mathbf{a}_R(\tau)^\dagger \). We will see that it is possible to obtain rank-one clutter in a Doppler cell with MIMO waveforms in a different manner.

It is worth noting that SIMO waveforms with \( \mathbf{R}(\tau) \triangleq \tau(\tau)\mathbf{a}_R(\tau)\mathbf{a}_R(\tau)^\dagger \) would provide a rank-one clutter covariance proportional to \( \mathbf{a}_R(\tau)\mathbf{a}_R(\tau)^\dagger \), but this covariance would support nulling only on the basis of the receiver response vector \( \mathbf{a}_R \) since the transmitter vector \( \mathbf{a}_T \) is fixed for all Doppler cells: a change in the target Doppler would change \( \mu_c \) and hence \( \mathbf{a}_T(\mu_c) \) and \( \mathbf{a}_R(\mu_c) \), but not \( \mathbf{a}_R \).

IV. PULSE-TO-PULSE WAVEFORM VARIATION

A. Clutter Covariance

Thus far we have assumed that all MIMO waveforms are repeated pulse to pulse. There is good reason to believe that pulse-to-pulse waveform variations allow additional clutter nulling involving the receiver phase centers; these phase centers do not play a role in nulling when waveforms are repeated. We will demonstrate clutter nulling theoretically through an approximate expression for SNR loss and through numerical evaluations of some interesting cases.

Below, subscripts will range in the sets \( \{ \text{MR}, \text{MT}, \text{MD} \} \) introduced in Section II. In most cases, we do not indicate these restrictions in the notation, except for emphasis. The expressions of SNR loss are general, but some additional assumptions are used to obtain approximate expressions that demonstrate nulling. These assumptions require \( M_D \) to equal an interval of integers and \( M_T \) to equal an interval on a sublattice of the integer lattice (i.e., multiples of a fixed integer). Approximate results are asymptotic as the intervals grow in length.

For waveforms \( \mathbf{s}_k(t) \) that vary with pulse index \( k \), let

\[
\mathbf{g}_k(\tau, t, \mu) = \sigma(\tau)(\mathbf{s}_k(t-\tau)^\dagger\mathbf{a}_R(\mu)\mathbf{a}_D(\mu))\mathbf{a}_R(\alpha)
\]

denote the clutter response from the \( k \)th pulse due to a point scatterer at delay \( \tau \) with scattering amplitude (redefined as a function of time) \( \sigma(\tau) \). The response incorporates beamforming at the transmitter through the inner product with \( \mathbf{a}_R \), phase response between pulses at the center frequency through the \( k \)th component of \( \mathbf{a}_D \), and phase response between receiver phase centers through \( \mathbf{a}_R \). It is convenient to use the conjugation of \( \mathbf{s}(t) \) as the transmitted waveform. Then the total clutter response at angle \( \mu \) becomes

\[
\mathbf{h}(t, \mu) = \int \mathbf{g}(\tau, t, \mu) d\tau.
\]

(21)

We assume that the complex scattering amplitude \( \sigma(t) \) has zero mean and covariance \( \mathbb{E} [\sigma(t_1)\sigma(t_2)] = p_c(\tau_1 - \tau_2) \) with clutter scaling \( p_c \). Then temporal cross-correlations of clutter from angle \( \mu \) can be expressed as

\[
\mathbb{E} [\mathbf{h}(t_1, \mu)\mathbf{h}^\dagger(t_2, \mu)] = \mathbb{E} [(\mathbf{a}_D(\mu, 0)\mathbf{a}_D^\dagger(\mu, 0))\mathbb{I}(t_1 - t_2) \otimes \mathbf{a}_R(\mu)\mathbf{a}_R^\dagger(\mu)]
\]

(22)

where

\[
K_{\text{cl}}(t) \triangleq p_c \int |s_\ell(t - \tau)|\mathbf{a}_R(\mu)\mathbf{a}_R^\dagger(\mu)s_\ell(\tau) \frac{d\tau}{2\pi}.
\]

(23)

Consider a delay vector \( \mathbf{q}(t) \) that can be used to form the processed output \( \mathbf{h}(t, \mu)\mathbf{q}(t) \). The quadratic form based on (22) can be written

\[
\int \mathbf{q}_\ell(t) [(\mathbf{a}_D(\mu, 0)\mathbf{a}_D^\dagger(\mu, 0))\mathbb{I}(t_1 - t_2) \otimes \mathbf{a}_R(\mu)\mathbf{a}_R^\dagger(\mu)] \mathbf{q}(t) \frac{d\omega}{2\pi}.
\]

The Fourier transform block diagonalizes the quadratic form. Group together the Fourier-transformed waveforms for each pulse in an array

\[
\mathbf{S}(\omega) \triangleq [\hat{s}_1(\omega), \ldots, \hat{s}_{\text{rs}}(\omega)].
\]

(24)

Then

\[
\hat{\mathbf{K}}(\omega) = p_c \mathbf{S}^\dagger(\omega)\mathbf{a}_R(\mu)\mathbf{a}_D^\dagger(\mu)\mathbf{S}(\omega).
\]

(25)

Hence, the block-diagonalized quadratic form utilizes the kernel

\[
\mathbf{R}(\mu) = p_c\{[(\hat{\mathbf{S}}(\omega)\mathbf{a}_T(\mu)\hat{\mathbf{S}}^\dagger(\omega)\mathbf{a}_T(\mu)]\mathbb{I}(\mathbf{a}_D\mathbf{a}_D^\dagger) \otimes \mathbf{a}_R\mathbf{a}_R^\dagger.
\]

(26)
Recall
\[(\mathbf{a}_R \otimes \mathbf{a}_T \otimes \mathbf{a}_D)_{ijk} = (\mathbf{A}_c \mathbf{v})_{ijk} = v_{i+j+k}. \quad (27)\]

If \(\hat{s}_k(\omega)\) denotes the \(j\)th component of \(\hat{s}_k(\omega)\), we can expand (26) to get
\[
p_k \sum_{k_1,k_2} (\hat{s}_{k_1}(\omega)(\mathbf{a}_T)_{k_1})(\hat{s}_{k_2}(\omega)(\mathbf{a}_T)_{k_2})
\cdot (\mathbf{a}_D(\mu,0)_{m_1} \mathbf{a}_R(\mu)_{j_1})_{k_1} (~\mathbf{a}_R^T(\mu))_{j_2} \]
\[= p_k \sum_{k_1,k_2} \hat{s}_{k_1}(\omega) \hat{s}_{k_2}(\omega) \cdot (\mathbf{A}_c \mathbf{v})_{j_1,k_1} (\mathbf{A}_c \mathbf{v})_{j_2,k_2}. \]

Integration over \(\mu\) and using \(\mathbf{C} \overset{\text{def}}{=} \int \mathbf{v}(\mu)^\dagger(\mu)\, dm(\mu)\) yields
\[
p_k \sum_{k_1,k_2} (\mathbf{A}_c \mathbf{C}^\dagger \mathbf{v})_{j_1,k_1} (\mathbf{A}_c \mathbf{v})_{j_2,k_2} \hat{s}_{k_1}(\omega) \hat{s}_{k_2}(\omega). \quad (28)\]

Define the matrix \(\mathbf{B}_C\) so that
\[
(\mathbf{B}_C(\omega)\mathbf{v})_{jl} = \sum_{k \in M_T} \hat{s}_{k_1}(\omega)(\mathbf{A}_c \mathbf{v})_{j,k_{1,l}}. \quad (29)\]

Then, absorbing the clutter power \(p_k\) into the covariance \(\mathbf{C}\), the clutter covariance at frequency \(\omega\) can be expressed by
\[
\mathbf{B}_C(\omega)\mathbf{C} \mathbf{B}_C^\dagger(\omega). \quad (30)\]

In the special case that \(\mathbf{C} = \mathbf{v}(\mu)\mathbf{v}(\mu)^\dagger\), we have a rank-one covariance that is proportional to the outer product of the vector \(\mathbf{B}_C(\omega)\mathbf{v}(\mu)\) with itself. This vector represents the array response of a stationary target at angle \(\mu/\pi\).

\section{Target-to-Clutter Ratio}

One consequence of this formulation of response vector and clutter covariance is an expression for the target-to-clutter ratio of a stationary target, given by
\[
\max_{\mathbf{w}} \frac{1}{2\pi} \int |\mathbf{w}(\omega)\mathbf{B}_C(\omega)\mathbf{v}(\mu)\, d\omega|^2. \quad (31)\]

Assuming that the integrals are restricted to a set \(\Omega\) of finite measure, the Schwarz inequality shows that the maximum becomes
\[
\frac{\text{vol}(\Omega)}{2\pi} \mathbf{v}(\mu)^\dagger \mathbf{C}^{-1} \mathbf{v}(\mu) \quad (32)\]

provided we can solve
\[
\mathbf{B}_C^\dagger(\omega)\mathbf{w}(\omega) = \mathbf{C}^{-1} \mathbf{v}(\mu). \quad (33)\]

Rather than pursue this analytically, we address loss in target-to-clutter ratio numerically in Section V using the objective function of (31) in comparison with optimal performance expressed by (32).

\section{Response Vectors With Doppler}

It is more useful to work with the response vectors whose components are expressed by (29). Making use of the definition of \(\mathbf{v}(\mu)\), we express the \((j,l)\)th component of the response vector through the definition
\[
\hat{x}_{jl}(\mu,0,\omega) \overset{\text{def}}{=} \sum_{k \in M_T} e^{i(j+k-l)\mu} \hat{s}_k(\omega) = e^{i(j+l)\mu} \mathbf{a}_T(\mu)^T \mathbf{a}_R(\mu). \quad (34)\]

The response associated with target motion involves a phase shift between pulses that can be represented by elaborating the definition to
\[
\hat{x}_{jl}(\mu,0,\omega) = e^{i\mu(\mu+\rho)\omega} \mathbf{a}_T(\mu)^T \mathbf{a}_R(\mu) \mathbf{S}(\omega) \mathbf{v}(\mu). \quad (35)\]

The doted subscripts represent all indices at their position. For example \(\hat{s}_{l_j}(\omega)\) represents the vector formed from the \(l_j\)th column of \(\mathbf{S}(\omega)\). The complex exponential in (35) provides an extra phase shift associated with pulse number.

We observe
\[
\mathbf{B}_C(\omega)\mathbf{v}(\mu) = \mathbf{x}(\mu,0,\omega). \quad (36)\]

By defining the phase response vector
\[
\mathbf{a}(\mu,\rho) \overset{\text{def}}{=} \mathbf{a}_R(\mu) \otimes \mathbf{a}_T(\mu) \otimes \mathbf{a}_D(\mu,\rho) \quad (37)\]

and defining a waveform factor
\[
\mathbf{E}(\omega,\mu) \overset{\text{def}}{=} e_\mu \otimes \hat{s}_l(\omega) \otimes e_l \quad (38)\]

we can connect the MIMO response vectors \(\mathbf{x}\) with the phase responses expressed in \(\mathbf{a}(\mu,\rho)\) through
\[
\mathbf{x}(\mu,\rho,\omega) = \mathbf{E}(\omega)^\dagger \mathbf{a}(\mu,\rho). \quad (39)\]

Inner and outer products of the MIMO response vectors can be written as
\[
\mathbf{x}^\dagger \mathbf{x} = (\mathbf{a}_R^\dagger(\mu_1) \mathbf{a}_R(\mu_2)) \cdot (\mathbf{a}_T^\dagger(\mu_1) \sum_{l \in M_T} e^{i\mu_2 - \mu_1 - \rho_1} \mathbf{S}(\omega) \mathbf{a}_R(\mu_2)) \]
and
\[
\mathbf{x}(\mu_1,\rho_1,\omega) \mathbf{x}(\mu_2,\rho_2,\omega) = e^{i(\mu_2 - \mu_1)} e^{i(\mu_1 + \rho_2)(\omega - t)} \cdot (\mathbf{a}_R^\dagger(\mu_1) \mathbf{S}(\omega) \mathbf{S}(\omega)^\dagger \mathbf{a}_T(\mu_2)). \quad (40)\]
D. SNR Loss With Pulse-to-Pulse Waveform Variation

1) General Expression: Given the measure \( m \) on the space of angles \( \mu \), we define the correlation matrix

\[
(R_X(m, \rho))(j,l) \overset{\text{def}}{=} \left( \mathbf{E} \left[ \int_{-\pi}^{\pi} x(\mu, \rho, \omega) \times x^*(\mu, \lambda, \omega) dm(\mu) \right] \right)(j,l) \text{ } . \tag{41}
\]

The grouped subscripts \((j,l)\) indicate that the pair of indices can be interpreted as the index into a vector given, for example, lexicographical ordering of the indices. The expectation anticipates the use of stochastic waveforms; it is not required for deterministic waveforms. In this notation, the SNR loss on a source at \((\mu, \rho)\) can be expressed as

\[
\frac{\int x^*(\mu, \rho, \omega)(x_{R,\rho,\omega} + R_X(m, 0))^{-1} x(\mu, \rho, \omega) d\omega}{\int x^*(\mu, \rho, \omega)x(\mu, \rho, \omega) d\omega} \tag{42}
\]

given white, block-identity noise covariance per radian. This expression is used in the examples of Section V.

2) Approximation for Stochastic Waveforms: Although (42) is used to evaluate performance in Section V, it is useful to have approximate expressions to gain an understanding of performance as an aid to waveform design. The approximation we use is justified by an heuristic argument involving, in part, a localization result sketched in Appendix A. The argument requires additional assumptions on the phase centers represented by \( M_T, M_R, \) and \( M_D \) and restrictions on the types of waveforms. Once the localization approximation is made, the SNR loss can be evaluated by techniques similar to those used in [24]. We sketch this derivation.

We make several assumptions below. One restricts the domains \( M_R \) and \( M_T \) to finite intervals of integers in the former case and intervals of multiples of a fixed integer \( n_T \) in the latter. In addition, \( M_D \) is taken to be an interval of integers. The integer intervals (except that of \( M_T \)) are also assumed to be symmetric about zero. We assume the waveforms are stochastic and are i.i.d.\(^2\) between frequencies \( \omega \), which play a trivial role here. In particular, to approximate (42), consider treating the waveforms as stochastic with zero mean and covariance

\[
\mathbf{E}[x(\mu, \rho, \omega)x^*(\mu, \rho, \omega)] = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\theta_1, \theta_2) e^{i(\omega-\omega_0)\theta_1} e^{i(t-t')\theta_2} d\theta_1 d\theta_2 (2\pi)^2 , \tag{43}
\]

based on the spectral density \( f(\theta_1, \theta_2) \).

\[
\frac{1}{n_T} \mathbf{E}[x(\mu, \rho, \omega)x^*(\mu, \rho, \omega)] = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|x(\mu, \rho, \omega)|^2}{n_T} f(\theta_1, \theta_2) e^{i(\omega-\omega_0)\theta_1} e^{i(t-t')\theta_2} d\theta_1 d\theta_2 (2\pi)^2 \rightarrow n_T^{-1} \sum_{k} \int_{-\pi}^{\pi} f(\mu - \gamma_k, \omega) e^{i(t-1)\theta} d\theta 2\pi \tag{44}
\]

where

\[
\chi(\theta) \overset{\text{def}}{=} \sum_{p \in M_T} e^{ip\theta} \tag{45}
\]

has the localization property (assuming that \( M_T \subset n_T \mathbb{Z} \) consists of consecutive multiples of \( n_T \) of increasing length \( n_T \))

\[
\frac{\left| \chi(\theta) \right|^2}{n_T} = \frac{2\pi}{n_T} \sum_{k} \delta(\theta - \gamma_k) \tag{46}
\]

and where \( \gamma_k \equiv 2\pi k/n_T, 0 \leq k < n_T \). Additions are interpreted modulo \( 2\pi \) so that sums lie in the interval \([-\pi, \pi] \). Thus, using (35) and (44)

\[
n_T^{-1}(R_X(m, \rho))(j,l) \rightarrow \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \hat{f}(\mu, \rho) e^{i\mu(j-j')} e^{i(t-t')} d\mu d\rho \rightarrow \frac{d\mu d\rho}{2\pi} \tag{47}
\]

where \( dm(\mu) \) represents integration over the clutter measure \( m \) and \( \hat{f}(\mu, \rho) = n_T^{-1} \sum_{k} \hat{f}(\mu - \gamma_k, \rho) \).

The response vector \( x(\mu, \rho, \omega) \) can be written in terms of the vector

\[
\xi(\mu_1, \mu_2; \mu, \rho, \omega) \overset{\text{def}}{=} \sum_{j \in M_R, l \in M_D} (x(\mu, \rho, \omega))_{jl} e^{-ij\mu_1} e^{-il\rho_2} \tag{48}
\]

\[
= (\mathbf{R}_R(\mu_1) \mathbf{a}_R(\mu_2)) \cdot \sum_{l \in M_D} (\tilde{s}_l(\omega))^l \mathbf{a}_D(\mu_2) e^{i(l-\mu_1)\rho} \tag{49}
\]

The integrand \( \xi(\mu_1, \mu_2; \mu, \rho, \omega) \) is a random variable. The numerator of (42) can be expanded about \( \rho = 0 \). When the measure \( m(\rho) \) is large, the dominant low-order term is expressed by

\[
\int x^3(\mu_1, 0, \omega)(x_{R,\rho,\omega} + R_X(m, 0))^{-1} x(\mu, 0, \omega) d\omega \tag{50}
\]

where we use variable-subscripted expressions like \( x_p \), to denote differentiation with respect to the relevant variable. From (49), after integrating by parts, we have

\[
(x_{\rho}(\mu, 0, \omega))_{\mu} = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \rho_2} \xi(\mu_1, \mu_2; \mu, 0, \omega) e^{i\mu_1} e^{i\rho_2} \frac{d\mu_1 d\rho_2}{(2\pi)^2} .
\]

For large \( n_T, n_R, \) and \( n_D \), the \( \rho_2 \) derivative of the function \( \xi(\mu_1, \mu_2; \mu, 0, \omega) \) is localized about \( \mu_1 = \mu \). In fact, the derivat-
tive has zero mean with a covariance approximated, at large $n_T$, by
\[
    n_T^{-1} \mathbb{E} \left[ \left( \delta_{n_D}(\theta) \right) \right]^2 \approx \left| a_{\mathbf{R}}(\mu) \right|^2 \int_{\pi}^{\pi} \tilde{f}(\mu, \theta - \mu) \left| \delta_{n_D}(\theta) \right|^2 \frac{d\theta}{2\pi},
\]
where $\delta_{n_D}(\theta) \overset{\text{def}}{=} n_D^{-1/2} \sum_{\ell \in \mathbb{M}_D} \epsilon^{i\ell \theta}$. The covariance peaks at $\mu_1 = \mu$. Thus, using the result of Appendix A, we can localize the measure $m$ in (47) about $\mu$ without significantly changing the value of (50). This suggests that, for small $\rho$ and high clutter-to-noise ratio, the SNR loss expressed by (42) can be approximated by
\[
    \frac{\int \mathbf{x}^H(\mu, \rho, \omega) \mathbf{P} \mathbf{x}(\mu, \rho, \omega) d\omega}{\int \mathbf{x}^H(\mu, \rho, \omega) \mathbf{x}^H(\mu, \rho, \omega) d\omega}.
\]
Projection $\mathbf{P} \perp$ is made onto the complement of the span of $\mathbf{x}(\mu_0, \omega)$ and $\mathbf{x}_\omega(\mu_0, \omega)$. We take (51) as an ansatz for approximating SNR loss.

To evaluate (51), we assume that $f(\theta_1, \theta_2)$ in (43) does not depend on $\theta_2$. Assuming that, for fixed $\omega$, the $\mathbf{s}_G(\omega)$ are Gaussian, then the random outer products $\mathbf{s}_G(\omega) \mathbf{s}_G(\omega)^H$ are independent and have the same mean, which we denote $\Gamma$. An approximate expression for SNR loss similar to (51) is used in [24], namely (in our notation)
\[
    \frac{\mathbf{a}_G(\mu, \rho) \mathbf{P} \mathbf{a}_G(\mu, \rho) d\omega}{\mathbf{a}_G(\mu, \rho) \mathbf{a}_G(\mu, \rho) d\omega}.
\]
where $\mathbf{a}_G(\mu, \rho) \overset{\text{def}}{=} (\mathbf{I}_{n_R} \otimes \mathbf{\Gamma}_1^{-1/2} \otimes \mathbf{I}_{n_D}) \mathbf{a}(\mu, \rho)$. We show, with additional assumptions, that there is agreement between these two expressions in a limiting sense.

Straightforward calculations show that
\[
    \mathbb{E}[\mathbf{x}^H(\mu, \rho, \omega) \mathbf{x}(\mu, \rho, \omega)] = \mathbf{a}_G(\mu, \rho) \mathbf{a}_G(\mu, \rho).
\]
\[
    \mathbb{E}[\mathbf{x}^H(\mu, \rho, \omega) \mathbf{x}_\omega(\mu, \rho, \omega)] = \mathbf{a}_G(\mu, \rho) \mathbf{a}_G(\mu, \rho),
\]

In other words, all expected values of inner products based on $\mathbf{x}(\mu, \rho, \omega)$ with itself or one of its partial derivatives have the same value as the deterministic, corresponding inner products based on $\mathbf{a}(\mu, \rho)$. Since these inner products are required to evaluate (52) for values of $(\mu, \rho)$ near $(\mu_0, 0)$, the procedure used in [24] can be used to approximate the relationship between MDV of the source and loss in SNR due to clutter suppression in the regime of slow source motions. The resulting relationship would also apply to (51). However, this conclusion requires replacing inner products of vectors based on $\mathbf{x}$ with their expectations. This replacement can be justified as follows. The random components of all relevant inner products are based on the three random matrices $\mathbf{A}_G$, $\mathbf{B}_G$, and $\mathbf{C}_G$, where $\mathbf{A}_G$ consists of all $\mathbf{s}_G(\omega) \mathbf{s}_G(\omega)^H$, and $\mathbf{B}_G$ consists of all $\mathbf{s}_G(\omega) \mathbf{\epsilon}_G(\omega)$, and $\mathbf{C}_G$ consists of all $\mathbf{\epsilon}_G(\omega) \mathbf{s}_G(\omega)$. In the first and last cases, as long as $\mathbf{\Gamma} \geq \mathbf{\epsilon}_G$, the nonzero mean increases with a higher power of $n_D$ than the standard deviation, justifying the approximation in these cases. The second case is more problematic since its mean is zero. However, it is plausible to believe that the localization principle expressed by (51) also holds if we assume that $\mathbf{s}_G$ and $\mathbf{s}_G(\omega)$ are perfectly correlated. Then, since $M_D$ is an interval of integers symmetric about 0, we would have $\mathbf{\epsilon}_G \mathbf{s}_G(\omega) \mathbf{s}_G(\omega)^H \mathbf{\epsilon}_G$. A more formal justification of the localization principle, given the correlation just introduced, can be based on a modification of the argument in Appendix A.

For use in Section V, we write the SNR loss from [24] (after some translation and using the notation of Section II) as
\[
    1 - \frac{\left| a_{\mathbf{R}}(\mu) \mathbf{a}_G(\mu) \right|^2}{\left| a_{\mathbf{R}}(\mu) \mathbf{a}_G(\mu) \right|^2} = 1 - \frac{\left| \mathbf{\Gamma}_1^{-1/2} \mathbf{a}_G(\mu) \right|^2}{\left| \mathbf{\Gamma}_1^{-1/2} \mathbf{a}_G(\mu) \right|^2}.
\]

in the limit $n_D \rightarrow \infty$. Unlike in [24], the loss is referenced to the SNR achieved in the absence of clutter and not to the SNR achieved by SIMO beamforming on the moving target in the absence of clutter. Note that the loss expression separates target and clutter using both receiver and transmitter phase centers, unlike the case of repeated, pulse-compressed waveforms. In fact, (53) expresses loss as if the target response is $\mathbf{\Gamma}_1^{-1/2} \mathbf{a}_G(\mu) \otimes \mathbf{a}_G(\mu)$ and the clutter has a rank-one covariance based on the response vector $\mathbf{\Gamma}_1^{-1/2} \mathbf{a}_G(\mu) \otimes \mathbf{a}_G(\mu)$, in the limit of large clutter power. This model of SNR loss is similar to that in Section II-B provided $\mathbf{R}(\tau) = r(\tau) \mathbf{\Gamma}_1^{-1/2}$, in the notation of Section II-B. The difference in the results lies in the definition of both $\mathbf{R}(\tau)$ and $\mathbf{\Gamma}$ as different types of waveform correlation matrices. However, the response vectors $\mathbf{R}(\tau) \mathbf{a}_G \otimes \mathbf{a}_G$ of Section II-B are based on pulse compressing the radar returns and ignoring the resulting induced noise correlation. With noise whitening, the effective response vectors become $\mathbf{R}(\omega)^{1/2} \mathbf{a}_G \otimes \mathbf{a}_G$ in the frequency domain. Thus, for slow-moving targets and strong clutter, the performance of idealized MIMO GMTI waveforms with autocorrelation $\mathbf{R}(\tau) = r(\tau) \mathbf{I}_{n_R}$ can be realized with time-varying waveforms.

**E. Periodic Pulse Repetition**

Under the same conditions imposed in Section IV-D2 and with small modifications to the arguments, we can evaluate the performance of periodic waveforms by assuming $f(\theta_1, \theta_2)$ has the form $\phi(\omega) e^{i\phi(\omega)}$, which results in stochastic waveforms $\mathbf{s}_G(\omega)$ with pulse-periodicity $n_p$. With large $n_p$, we can utilize the approximate SNR loss given by (53). However, a more direct calculation is relevant.

There is a particularly simple interpretation of clutter nulling when deterministic waveforms are allowed to repeat periodically. Specifically, assume
\[
    \delta_{kl}(\omega) = \delta_{l'}(\omega) \text{ when } l = l' \mod n_p
\]
where $l \in M_D$ ranges over consecutive integers that we can assume begin with index 0. Then
\[
    x_{kl}(\mu, \rho, \omega) = e^{i(\mu+\rho) \omega n_p} x_{k}(\mu, \rho, \omega).
\]
With a large number of pulses and a fixed \( n_p \), we have, to a good approximation

\[
x^\dagger(\mu_2, 0, \omega) x(\mu_1, \rho, \omega)
\approx \frac{n_D}{n_p} \sum_{j=0}^{n_p} \sum_{0 \leq r < n_p} e^{i(\mu_1 - \mu_2 + \rho)r}
\times \bar{x}_r(\mu_1, \rho, \omega) \bar{x}_r(\mu_2, 0, \omega)
\]

(56)

as long as \( \mu_2 = \mu_1 + \rho + 2\pi k/n_p \), for integer \( k \); the inner product approximates zero otherwise. These inner products allow us to identify the closest points on the clutter ridge associated with Doppler \( \rho \) and target angle \( \mu_1 \). The SNR loss is expressed through these inner products via (51). By using (35), (56) becomes

\[
\begin{align*}
\mathbf{a}_R(\mu_2) & \cdot \sum_{0 \leq r < n_p} e^{i(\mu_1 - \mu_2 + \rho)r} \mathbf{a}_T^\dagger(\mu_2) \hat{s}_r(\omega) \\
& \times \hat{s}_r(\omega)^\dagger \mathbf{a}_T(\mu_1).
\end{align*}
\]

(57)

For the moment, assume \( \mu_c \equiv \mu_2 = \mu_1 + \rho \equiv \mu_t + \rho \). Then (57) becomes

\[
\begin{align*}
(\mathbf{a}_R(\mu_c) & \cdot (\mathbf{a}_T(\mu_c) \Delta \mathbf{a}_T(\mu_c))) \\
= (\Delta^{1/2}(\omega) \mathbf{a}_T(\mu_c) \otimes \mathbf{a}_R(\mu_c))^\dagger \\
& \cdot (\Delta^{1/2}(\omega) \mathbf{a}_T(\mu_c) \otimes \mathbf{a}_R(\mu_c)).
\end{align*}
\]

where

\[
\Delta(\omega) \equiv \sum_{0 \leq r < n_p} \hat{s}_r(\omega) \hat{s}_r(\omega)^\dagger.
\]

(58)

The waveform correlation matrix \( \Delta(\omega) \) plays the same role as the matrix \( \mathbf{G} \). It can assume some interesting special forms. For example, with orthogonal beams for \( \hat{s}_r(\omega) \), \( \Delta(\omega) \) becomes a multiple of the identity. The loss associated with these inner products takes the same form as that expressed by (53), which leverages both the transmitter and receiver arrays for clutter suppression. In the SIMO case, \( \Delta(\omega) \) would be proportional to a single steering vector. In this case, only the receiver phase centers separate target and clutter; the transmitters provide only differential gain, which is often of little benefit.

In the more general case when \( \mu_2 = \mu_1 + \rho + 2\pi k/n_p \), (57) becomes

\[
\begin{align*}
\mathbf{a}_R^\dagger(\mu_2) & \cdot \mathbf{a}_T^\dagger(\mu_2) \left( \sum_{0 \leq r < n_p} e^{-2\pi kr/n_p} \hat{s}_r(\omega) \hat{s}_r(\omega)^\dagger \right) \mathbf{a}_T(\mu_1).
\end{align*}
\]

The left factor is typically small unless the receiver array has ambiguities at the \( 2\pi k/n_p \) spacing.

V. EXAMPLES OF PERFORMANCE

A. Discussion

Three types of MIMO waveforms are explored in the following examples. For the purpose of comparison, one type is a conventional SIMO beamformer steered at the target. Below, these waveforms are called SIMO waveforms. A second waveform consists of periodic repetitions of several beamformers. The repetition period equals the number of transmitters. The first \( n_p \) pulses cycle through orthogonal beamformers, one for each pulse. These beamformers together span physical beamspace. This pattern of beams is repeated throughout the CPI. In analogy to multiaccess communication waveforms, we label these waveforms Doppler domain, multiple access (DDMA) [8]. The last example waveform utilizes randomly chosen beamformers for each pulse. These beamformers are chosen in a completely random fashion, without regard to their physical patterns. The term random waveforms is applied to this example. All waveforms utilize an ideal, flat-spectrum transmitted signal; performance is evaluated using the expressions in Section IV.

The next figures address two aspects of performance. One is expressed by SNR loss associated with clutter suppression as a function of target motion. Another is expressed by adaptive beam patterns.

The examples below consider a target at broadside (perpendicular to the collinear transmitter and receiver arrays) assuming various radial velocities, measured as a fraction of the platform velocity, which is aligned along the array axes. The number of pulses used equals 64; the number of transmitters and receivers equals 4. We vary the transmitter and receiver array geometries between filled and sparse arrays. The filled array is uniformly linear with half-wavelength spacing. The sparse array is also uniformly linear, but with four times the aperture. The receiver array is assumed filled and the transmitter array is assumed sparse. Thus, the spatial virtual array has Nyquist spacing. The pulse spacing is assumed to sample the clutter Doppler spread at the Nyquist rate.

B. SNR Loss

Fig. 1 illustrates the performance of SIMO, DDMA, and random waveforms in terms of SNR loss as a function of target velocity. The clutter Doppler spread is sampled at the Nyquist rate. The clutter notch is significantly smaller for DDMA and random waveforms. The DDMA and random waveforms take advantage of the large, sparse transmitter aperture, which the SIMO waveforms cannot leverage. The pulse-to-pulse variation in orthogonal beamformers associated with DDMA results in additional Doppler ambiguities under target motion. For example, in this figure, only pulses 1, 5, 9, etc., illuminate the target. The other pulses are orthogonal physical beams. Thus, the target Doppler is undersampled in comparison to the SIMO case, which utilizes the same beamformer, pointed at the target, for each pulse. Of course, target velocities associated with the notch may be nonphysical, depending on the specific
Application. In addition, nonuniform spacing of the transmitter elements would ameliorate these ambiguities. Random waveforms do not suffer from additional Doppler ambiguities, but they have their own issues. Random waveforms do not accommodate localization in Doppler. Clutter and target are modulated spatially and energy is spread throughout Doppler. Thus, adaptation must occur in the full space-by-space-by-time dimensionality. SNR loss for these waveforms suffers slightly away from the clutter notch.

Not only is there potential undersampling of the target Doppler, associated with the notch in Fig. 1, but there is undersampling of the clutter Doppler spread for a similar reason. In this case, ambiguous clutter falls in the nulls of the receiver’s array pattern. For more general receiver apertures, it may be necessary to oversample the clutter Doppler spread (by increasing the pulse rate) in order to prevent ambiguous clutter from competing with the target and adversely effecting SNR loss.

The zero-target-Doppler target-to-clutter ratios of the above waveforms can be compared to that achieved ideally, as computed in Section IV-B. The numerical results indicate very little loss with all three waveforms.

C. Adapted Beam Patterns

The next set of plots shows the adapted beam patterns for SIMO, DDMA, and random waveforms. Fig. 2 shows the adapted pattern of a SIMO waveform in target angle-Doppler space. The beam is steered at the center of the crosshatches. The clutter is Nyquist sampled here and below. In this figure, the receiver array is filled and there is a single transmitter. The pattern response is concentrated in a single Doppler bin (diagonal here).

VI. CONCLUSION

In this paper, we discussed MIMO GMTI radar SNR loss performance for clutter mitigation as a function of waveform characteristics. We discussed the limitation of frequency shifting techniques for constructing waveforms. We developed SNR loss relationships that predict mitigation performance for waveforms that vary pulse to pulse. Pulse-to-pulse repeated
waveforms have comparatively poor clutter mitigation performance. Time varying waveforms, such as DDMA, can have relatively good performance, although they have the potential for Doppler ambiguities. Depending upon the system parameters, these ambiguities may or may not be physical. Using numerical examples, we have demonstrated pulse-to-pulse varying waveforms that enable MIMO GMTI low velocity performance that is significantly better than the corresponding SIMO system.

\section*{APPENDIX I}

\subsection*{A. Asymptotic Expressions for SNR Loss and Localization Properties}

In the Appendix, the notational conventions of the paper are violated in order avoid potential confusion with the notions of vector and operator, which are more fluid concepts in the reasoning here.

Let the infinite block matrix $\mathcal{D}$ have coordinates

\[ (\mathcal{D})_{j,l}(j',l') = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d(\mu, \mu') \times e^{i\mu(j-j')} e^{i\mu'(l-l')} \, \frac{d\mu d\mu'}{2\pi} . \]  

(59)

where $j$ and $l$ range over the integers. Denote by $\mathcal{D}^{(J,L)}$ the restriction of $\mathcal{D}$ to $|j| \leq J$ and $|l| \leq L$. Similarly, let the infinite blocked vector $\bm{X}$ have coordinates

\[ (\bm{X})_{j,l} = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x(\mu, \mu') \times e^{i\mu(j-j')} e^{i\mu'(l-l')} \, \frac{d\mu d\mu'}{2\pi} . \]  

(60)

Assuming $d(\mu, \mu')$ is bounded away from zero, we have the limiting behavior

\[ \lim_{J,L \to \infty} \left( \mathcal{D}^{(J,L)} \right)^{\dagger} \left( \mathcal{D}^{(J,L)} \right)^{-1} = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d^{-1}(\mu, \mu') x(\mu, \mu') \, \frac{d\mu d\mu'}{2\pi} . \]  

(61)

If $x(\mu, \mu')$ is concentrated in some subregion, the SNR loss expression of (61) shows that only the values of $d(\mu, \mu')$ in this subregion matter asymptotically.

We sketch a proof for a single variable $(d \equiv d(\mu))$; the two-variable case follows similarly. A different localization result occurs in [2].

Consider the Hilbert space $l_{2}$ consisting of sequences $\{x_{n}\}$ of complex numbers that have finite norm: $\sum_{k} |x_{k}|^{2} < \infty$. If $x(\theta)$ is an $L_{2}$ function on the circle $S^{1}$, represented by the interval $[-\pi, \pi]$, then the Fourier coefficients

\[ x_{n} \overset{\text{def}}{=} \int_{-\pi}^{\pi} x(\theta) e^{in\theta} \, \frac{d\theta}{2\pi} \]  

(62)

exist and form an element of $l_{2}$ with the same norm as $x(\theta)$. Thus, the correspondence $x(\theta) \rightarrow \{x_{n}\}$ can be expressed as a unitary operator $U$ from $l_{2}(S^{1})$ onto $l_{2}$. If $d(\theta)$ is continuous on $S^{1}$, the infinite matrix $D$ with elements

\[ D_{mn} = \int_{-\pi}^{\pi} d(\theta) e^{i(m-n)\theta} \, \frac{d\theta}{2\pi} \]  

(63)

is a bounded linear operator on $l_{2}$ and satisfies

\[ U^\dagger DU = M_{d} \]  

(64)

where the bounded linear operator $M_{d}$ acts on $l_{2}(S^{1})$ as $x(\theta) \rightarrow d(\theta) \cdot x(\theta)$. If $f, g \in l_{2}(S^{1})$, then $Uf, Ug \in l_{2}$ and

\[ \int_{-\pi}^{\pi} f(\theta) \overline{g(\theta)} \, \frac{d\theta}{2\pi} = (Ug)^{\dagger} Uf. \]  

(65)

Define $e_{k} = U e^{-ik\theta}$. This vector is the element of $l_{2}$ with 1 in the $k \text{th}$ component and zeros elsewhere. Define the infinite dimensional matrix $E = [e_{-L}, \ldots, e_{L}]$. For $y \in \mathbb{C}^{2L+1}$, we define the element of $l_{2}$: $\phi_{E}(y) = E \mathbf{y} = \sum_{|k| \leq L} y_{k} e_{k}$. Note that $y^{\dagger} \mathbf{x} = \phi_{E}(y)^{\dagger} \phi_{E}(\mathbf{x})$. We extend the definition of $\phi_{E}$ to matrices $G \in \mathbb{C}^{2L+1 \times 2L+1}$ so that

\[ \phi_{E}(G) = EGE^{\dagger} = \sum_{|j|,|l| \leq L} G_{j,k} e_{j} e_{k}^{\dagger}. \]  

(66)

It is easy to see that $\phi_{E}(G^{k}) = \phi_{E}(G^{k})^{k}$ for positive integer $k$. For any element $x \in l_{2}$, define the element $x_{E} \in \mathbb{C}^{2L+1}$ by $x_{E} = E^{\dagger} x$. Note that $\phi_{E}(x_{E}) = E E^{\dagger} x$. Denote by $D_{E}$ the $2L+1 \times 2L+1$ matrix $E^{\dagger} D E$. For any polynomial $p(x)$, we have, as $L \to \infty$, $p(\phi_{E}(D_{E})x_{E}) \to p(D)$ in the strong operator topology [25]. It follows that $x_{E}^{\dagger} f(D_{E}) x_{E} \to x^{\dagger} f(D) x$ for all such $f$. In particular, this holds for $f(z) = z^{-1}$.

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