Fundamental energy cost for quantum measurement

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Measurements and feedback are essential in the control of any device operating at the quantum scale and exploiting the features of quantum physics. As the number of quantum components grows, it becomes imperative to consider the energetic expense of such elementary operations. Here we determine the fundamental energy requirements for physical implementations of any general quantum measurement. We show that the exact costs for projective measurements depend on the outcome probabilities only, causing more severe constraints on error correction and control protocols than previously known. In contrast, energy can be extracted from certain measurement processes even if their outcome is recorded. Our results constitute fundamental physical limitations against which to benchmark implementations of future quantum devices as they grow in complexity.

The ability to manipulate and measure individual quantum system1,2 enables ever more powerful devices that fully exploit the laws of the quantum world. This has facilitated the development of high-precision clocks2 and quantum simulators3 as well as the observation of fundamental decoherence processes4. Quantum measurements are crucial to the operation of scalable quantum computers5,6—for their final readout but importantly also for continual protection against external noise via error correction7,8—and quantum computation wholly based on measurements has been proposed9. The combination of quantum measurement with feedback is an essential primitive for all these applications, as it allows future actions to depend on past measurement outcomes. Thus, with quantum devices becoming increasingly complex, more measurements have to be performed and physical requirements such as the energy supply for implementing these elementary operations must be accounted for (Fig. 1). This is in parallel to the primitive of information erasure10 whose energetic expense will become a limiting technological factor within a few decades11,12 as the miniaturization of computers progresses13. The expense is needed for initializing a computer register and therefore accumulates when using a device repeatedly14 as is typical of measurement apparatuses. The physical ramifications of information erasure are summarized by Landauer’s Principle15, which demands $k_B T \ln 2$ of energy to be dissipated into a heat bath of temperature $T$ for the erasure of each bit of information.

So how much energy must be expended for measurements during quantum computing and error correction, or in central control protocols such as quantum Zeno stabilization16,17? We derive fundamental physical bounds on the energy cost of any general quantum measurement, and investigate the energetic requirements they place on real-world implementations of quantum devices.

Our results show that several tens of percent more energy than previously known must be expended to operate quantum devices robustly via active error correction schemes14,17. This will ultimately become a fundamental physical limitation to quantum computers unless the noise itself provides energy, in a way similar to the Landauer limit for classical computers13. More drastically, the energy cost of quantum Zeno control diverges in the limit of perfect stabilization. On the other hand, when the post-measurement state is irrelevant we devise a simple protocol that can extract useful energy from the measurement process, even if the outcome is recorded18. Our results are both more general and stronger than previous bounds19,20 as they apply to all measurements including inefficient ones21, and yield equality relations for the important case of projective measurements.

Let us consider the most general quantum measurement22, described by a collection of measurement operators $\{ M_i \}$, on a quantum system $S$. In order to allow for feedback in quantum applications23, it is not sufficient to track the average state during

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FIG. 1: Quantum measurement. Most quantum engineering protocols involve frequent measurements to maintain their stability or to control future actions. Whereas measurements are often considered as abstract primitives, we investigate their actual physical implementation and quantify the arising fundamental energy requirements $E_{\text{cost}}$.
the measurement, but instead the final state $\rho_{S,k}$, conditional on the outcome $k$. Subsequent feedback $\{V_k\}$ on $S$ is possible as different outcomes belong to orthogonal subspaces $H_k = Q_k H_M$ of the memory $M$. Before the device can be used again, step $R$ must reset $M$ to its proper initial state $\rho_M$, using a thermal resource $p_k$. Hamiltonians $H_M$ and $H_B$ determine the total energy cost $E_{\text{cost}} = \Delta E_M + \Delta E_R$ required to operate the device physically. Our main results (1) and (2) express this cost in terms of quantities that do not depend on such microscopic details of the device (blue and green parts), but are instead determined by system quantities alone (red), such as $\rho_S$ or the measurement specification $\{M_k\}$.

One may naively think that the energy cost of this measurement on any state $\rho_S$ equals the energy difference $\Delta E_S = \text{tr}[H_S(\rho_S' - \rho_S)]$ on $S$, where $\rho_S' = \sum_k p_k \rho_{S,k}'$ is the average post-measurement state and $H_S$ the Hamiltonian. This however neglects the measurement device $M$ needed in a concrete physical realization of the measurement step $\mathcal{M}$ to store the measurement outcome $k$ in a memory for readout and feedback; and also to allow for an implementation by physical, unitary or Hamiltonian, dynamics in the first place. Secondly, since in typical applications the measurement device will be used repeatedly, it has to be restored to its proper initial state $\rho_M$ during a resetting step $R$. The total energy cost to operate a measurement device consists of those in the measurement and resetting steps.

To satisfy the above readout requirement, the memory Hilbert space $H_M = \bigoplus_k H_k$ must be composed of subspaces $H_k$ with orthogonal projections $Q_k$ corresponding to the classical measurement outcomes. Step $\mathcal{M}$ is then fully microscopically described by specifying a unitary interaction $U_{SM}$ between $S$ and $M$ (Fig. 2): for any initial state $\rho_S$, we obtain outcome $k$ with probability $p_k = \text{tr}[U_S(\rho_S \otimes \rho_M) U_{SM}^\dagger]$, together with a conditional $SM$ state

$$\rho_{SM,k} = (\mathbb{1} \otimes Q_k) U_{SM}(\rho_S \otimes \rho_M) U_{SM}^\dagger (\mathbb{1} \otimes Q_k)/p_k.$$  

The full state after measurement is thus given by $\rho_{SM} = \sum_k p_k \rho_{SM,k}$ and makes it possible to perform unitary feedback operations $\{V_k\}$ via the overall unitary interaction $\sum_k V_k \otimes Q_k$ without disturbing $M$, meaning that $k$ is indeed stored classically on the memory. A quantum measurement $\{M_k\}$ has many physical implementations $(\rho_M, U_{SM}, \{Q_k\})$, so we will look for the least expensive in terms of energy. Regarding the non-unitary projection part $\{Q_k\}$ of step $\mathcal{M}$, we show that it can be implemented by a physical unitary operation involving an environment system $E$, and this at zero energy cost but not less (supplementary material).

The resetting step $R$ leaves $S$ untouched, but should restore the memory to its initial state. This is impossible by unitary evolutions on $M$ alone because it is in general necessary to change the eigenvalues. Hence, we supply the thermal state $\rho_B = e^{-H_B/k_B T}/\text{tr}[e^{-H_B/k_B T}]$ of a bath $B$ at some temperature $T$, as in the usual Landauer erasure process.\[10,25\] Step $R$ proceeds then by a unitary $U_{MB}$ (Fig. 2).

$$\text{tr}_B[U_{MB}(\rho_M' \otimes \rho_B) U_{MB}^\dagger] = \rho_M,$$

where $\rho_M' = \text{tr}_S[\rho_{SM}']$ is the marginal memory state after $\mathcal{M}$. Thermal states like $\rho_B$ are free resources at ambient temperature $T$, but we must keep track of the energy expended during step $R$, just as for step $\mathcal{M}$.

We now compute the actual energy cost $E_{\text{cost}} = \Delta E_M + \Delta E_R$ to operate the measurement device. Importantly, we express it in terms of system quantities like $\rho_S$, $p_k$, and $\rho_{S,k}'$ determined by the desired measurement $\{M_k\}$ already, and so obtain fundamental results that are independent of the concrete physical measurement implementation. Step $\mathcal{M}$ incurs an energy expense of $\Delta E_M = \text{tr}[H_M(\rho_M' - \rho_M)]$, whereas the cost $\Delta E_S$ on $S$ is already given in terms of system quantities and may change further due to feedback $\{V_k\}$. The former turns out to split into a sum of operationally meaningful quantities, $\Delta E_M = k_B T (\Delta S + \Delta Q + \mathcal{Z}) + \Delta F_M$ (supplementary text), where $\Delta S = S(\rho_S) - \sum_k p_k S(\rho_{S,k}')$ denotes the change in information about the system state, measured by the von Neumann entropy $S(\rho) = -\text{tr}[\rho \ln \rho]$, and $\Delta F_M = F_M(\rho_M') - F_M(\rho_M)$ the change in free energy $F_M(\rho) = \text{tr}[\rho H_M - k_B T S(\rho)]$, not requiring $\rho_M$ or $\rho_M'$ to be thermal.\[10,25\] The terms $\Delta Q$ and $\mathcal{Z}$ denote the entropy increase due to the projections $\{Q_k\}$ and the conditional $SM$ correlations, respectively. Both are non-negative, so as a consequence we can lower bound the cost of step $\mathcal{M}$, generalizing previous results\[23\] to measurements that are not necessarily efficient:

$$\Delta E_M \geq k_B T [S(\rho_S) - \sum_k p_k S(\rho_{S,k}')] + \Delta F_M.$$
Fig. 3: Extracting energy from measurement. A pure qubit $\rho_S = |\psi\rangle\langle\psi|$ is measured in the spin-$z$ basis by a device with bipartite memory $M = M_A M_B$ and initial state $\rho_M = |0\rangle\langle0|_{M_A} \otimes 1_{M_B}/2$, in one of two different ways. Either measurement implementation yields the same outcome distribution and enables feedback via $Q_k = |k\rangle\langle k|_{M_A} \otimes 1_{M_B}$ but counterintuitively, the inefficient one allows to extract useful energy from the operation, in contrast to previous results\cite{18,19} (a) The implemented measurement is efficient with $M_k = |k\rangle\langle k|$. Operating such a device always costs energy $E_{\text{proj}} \geq 0$ according to Eq. (2). (b) The measurement is given by $M_{ki} = |i\rangle\langle i|/\sqrt{2}$ and thus is inefficient. Intuitively, the device implements the same unitary interaction $U_{ZM_A M_B}$ as in (a) with an additional swap of the systems $S$ and $M_B$ before the projections $\{Q_k\}$. This modified projective measurement always outputs a fully mixed state $\rho_{S,k} = 1_S/2$ and yields energy $(-E_{\text{cost}}) \geq 0$ (supplementary text).

Step $R$ incurs costs on $M$ similar to those above and on $B$ as in Landauer erasure\cite{11} totalling $\Delta E_R = -\Delta F_M + \Delta F_B + k_B T I_{MB}$ with a correlation term $I_{MB}$ (supplementary text). The last two contributions are non-negative as $p_B$ was thermal, and one can actually engineer the bath Hamiltonian $H_B$ and interaction $U_{MB}$ such that both vanish\cite{12}; i.e. $\Delta E_R = -\Delta F_M$; we will assume this henceforth, noting that even otherwise all our results remain valid lower bounds. We thus obtain our first main result, a constraint on the expense $E_{\text{cost}}$ of any physical implementation of the measurement $\{M_{ki}\}$ on a state $\rho_S$ in terms of system quantities:

$$E_{\text{cost}} \geq k_B T [S(\rho_S) - \sum_k p_k S(\rho_{S,k}')] \quad (1)$$

This shows that measurements necessarily consume energy whenever they lower the system entropy on average. Inefficient ones\cite{21} may however increase the entropy: for $\{M_{ki}\}$ with inefficiency $I$, i.e. index range $i = 1,\ldots,I$, we merely have $E_{\text{cost}} \geq -k_B T \ln I$ (supplementary text). Indeed, Fig. 3 shows how an amount $(-E_{\text{cost}}) \geq 0$ of useful energy can be extracted from measuring a system, even when the measurement outcome is correctly stored. This result is especially surprising as it contrasts previous statements that were restricted\cite{11,12} to $I = 1$ or addressed the unselective case\cite{13}.

Our second main result concerns the specific case of projective measurement\cite{22}, which constitute the textbook examples of quantum measurements and are of principal importance for applications, as exemplified below. These are efficient measurements $\{M_k\}$ with projection operators $M_k = M_k^\dagger = M_k^2$. Their rigid structure fixes all entropic terms in $\Delta E_M$ above and we obtain an exact equality (supplementary text):

$$E_{\text{proj}} = k_B T \sum_k p_k \ln 1/p_k \quad (2)$$

The energy cost of projective measurements therefore depends on the outcome probabilities $p_k = \text{tr}[\rho_S M_k]$ only, the last sum being simply their Shannon entropy $H(\{p_k\})$. We now use this result to investigate the energetic cost of important quantum protocols that involve measurements, illustrating its power in applications.

As a first application, let us start with quantum Zeno stabilization\cite{23}, a paradigmatic and ubiquitous quantum control protocol\cite{24}. The task is to stabilize a qubit in a pure state $|0\rangle$ against its free Hamiltonian time evolution when $|0\rangle$ is not an eigenstate, such as for $H_S = E X$ with the Pauli operator $\sigma_X$ and energies $\pm E$. The protocol applies the projective measurement $\{M_0 = |0\rangle\langle0|, M_1 = |1\rangle\langle1|\}$, with $M_0$ the projector onto the desired state, at $N$ regular time intervals $\delta t = t/N$ over the time span
suitable feedback operations directly limits the achievable accuracy. Such drastic en-
ter exclusive quantum error correction (QEC) schemes, which in this context since syndrome measurements with feed-
ter encoding the noise, the heart of QEC consists in performing repeated C code measurements of the noise, the heart of QEC consists in performing repeated measurements on the system to ensure high fidelity. The energy expense of the whole protocol consists of the costs $E_{\text{proj}}^\text{(n)} \simeq k_B T \varepsilon_n \ln 1/\varepsilon_n$ for all Zeno measurements $n = 1, \ldots, N$, where we have evaluated our sharp result Eq. (2) to leading order. The total energy required to achieve high target fidelity $F$ is then (supplementary text)

$$E_{\text{Zeno}} \simeq k_B T \frac{(Et/h)^2}{2} \ln \frac{1}{1-F}. \tag{3}$$

This expense diverges as $F$ approaches 1, so any restriction on the energy available for the stabilization scheme directly limits the achievable accuracy. Such drastic energy demands apply to Zeno schemes for dragging or holonomic computation as well.

Measurement and feedback are essential also for stabilizer quantum error correction (QEC) schemes, which allow quantum computations to reach capabilities beyond classical computers even in the presence of noise. After encoding the logical qubits $L$ redundantly into a QEC code $C$, which is simply a physical system subject to noise, the heart of QEC consists in performing repeated measurements of the error syndrome $s$ on $C$ followed by suitable feedback operations $V_s$. When these control operations are performed frequently enough, reliable computation is possible on noisy hardware by the threshold theorem. Energetic considerations are paramount in this context since syndrome measurements with feedback must be performed many times and on many qubits for a scalable setup. As a paradigmatic example we examine the 5-qubit code $C_5$, which encodes a single logical qubit $|\psi\rangle \in C_2 \subset C_5$ and whose syndrome measurement $\{P_s\}_{s=0}^{15}$ consists of two-dimensional projectors. We take each of the five physical qubits to be subject to excitation loss by amplitude damping noise $N_\gamma$ at strength $\gamma \in [0,1]$. The costs for each QEC step come from implementing the measurement $\{P_s\}$ on the noisy state $\rho_{S,\gamma} = N_\gamma^{\otimes 5}(|\psi\rangle\langle\psi|)$.

The minimum energy expense $E_{\text{min}}^{\text{sep}}$ according to Eq. (2) is shown in Fig. 4 (red curve), increasing from 0 in the noiseless case to $4k_B T \ln 2$ as $\gamma \to 1$. Much more energy must therefore be expended than the previously known bound $E_{\text{LU}}^{\text{SU}}$ predicts (green curve), which for example vanishes at $\gamma = 1$. A naive application of Landauer’s bound to the code system $S = C_5$ itself would yield an even weaker bound $E_{C_5} \geq k_B T [S(\rho_{S,\gamma}) - S(\rho_{S,0})] = E_{\text{LU}}^{\text{min}}$, negative at large noise $\gamma$ (black curve). The comparison illustrates the strength of result (2) obtained by an exact treatment of the measurement device; even in the sub-threshold regime $\gamma \lesssim 0.05$ where effectively noiseless quantum computing is possible, the improvement amounts to 15%. Whereas the fundamental energy cost is $E_{\text{min}} = k_B T H(\{p_s\}_{s=0}^{15})$, practical QEC will exploit the stabilizer structure to obtain syndrome $s = (s^1, s^2, s^3, s^4)$ by four commuting measurements, each with two outcomes $s^j = \pm 1$ only. When these four devices operate independently the costs total $E_{C_5}^{\text{sep}} = k_B T \sum_s H(\{p_s\}_s) \geq E_{C_5}^{\text{min}}$ (blue curve), showing that additional expenses can arise in simpler measurement implementations that disregard correlations (supplementary text).

The general results (1) and (2) determine the energetic cost for quantum measurements, imposing significant limitations on the performance of quantum error correction and control. Remarkably, our results establish a novel link between the viability of quantum technologies and the existence of uncertainty relations: Whenever incompatible measurements have to be performed, as in quantum state tomography or quantum Monte Carlo sampling, entropic uncertainty relations yield strictly positive energy bounds via Eq. (2), independent of the input state. Given finite energy supply, uncertainty relations therefore place fundamental limitations on tomographic accuracy or sample quality.

Our energy results are in fact statements about thermodynamic work as we accounted for all energetic contributions while employing unitary actions. It is then surprising that, in contrast to previous findings, we can extract useful work from measurement, while still respecting the Second Law of Thermodynamics. Our study paves the way for investigations into the energy costs of further elementary operations in the quantum sciences or engineering. Particularly interesting venues include single-shot scenarios and processes operating on unknown input states.
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Supplementary Material

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Appendix A: The measurement model

A quantum measurement on a system $S$ with Hilbert space $\mathcal{H}_S$ is mathematically described by a quantum instrument, i.e. a set of completely positive maps $\{T_k\}_{k=1,...,K}$ on $\mathcal{B}(\mathcal{H}_S)$ satisfying $\sum_k T_k^* T_k = \mathbb{I}_S$, where $k$ corresponds to the measurement outcome and $T_k^*$ denotes the adjoint of $T_k$. The action of the map $T_k$ on a state $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ can always be written in terms of Kraus operators $M_k$, i.e. $T_k(\rho_S) = \sum_{i=1}^{J(k)} M_{ki} \rho_S M_{ki}^*$, where $J(k)$ is the Kraus rank of $T_k$. A measurement is called efficient or pure if $J(k) = 1$ for all $k$. A measurement that is not efficient is called inefficient.$^{[2][3]}

A quantum instrument characterises both the probability $p_k = \text{tr} [T_k(\rho_S)]$ to obtain outcome $k$ and the corresponding post-measurement state $\rho_{S,k} = T_k(\rho_S) / p_k$. In contrast, if the post-measurement state can be disregarded and only the outcome probabilities are of interest, it is sufficient to consider a POVM (positive operator valued measure) defined by positive operators $\{E_k\}_{k=1,...,K}$ satisfying $\sum_k E_k = \mathbb{I}_S$. The probability to obtain outcome $k$ is then given by $p_k = \text{tr} [E_k \rho_S]$. Any quantum instrument $\{T_k\}_k$ determines a POVM by $E_k = T_k^* (\mathbb{I}_S) = \sum_i M_{ki}^* M_{ki}$. In the following we will always consider quantum instruments in order to be able to describe applications such as quantum error correction, where the post-measurement state cannot be disregarded. Only in the energy extraction example (Appendix F) we employ the idea of POVMs to investigate the implications of disregarding the post-measurement state.

While a quantum instrument accurately describes the measurement as an abstract process on the measured system, we are considering physical implementations of an instrument that incorporate all relevant systems that are involved in the measurement process. In particular, the measurement outcome $k$ has to be stored in degrees of freedom of a physical system $M$. We model this register by a quantum system with Hilbert space $\mathcal{H}_M = \bigoplus_{k=1}^{K} \mathcal{H}_k$ and Hamiltonian $H_M = \bigoplus_{k=1}^{K} H_k$, which naturally captures all the important properties one generally demands from a classical memory.$^{[23]}$ We consider a state $\rho_{M,k} \in \mathcal{H}_M$ to store the measurement outcome $k$ if it has support only on the subspace $\mathcal{H}_k$ corresponding to $k$. In this case, projection operators $\{Q_k\}_{k=1,...,K}$ which project onto the respective subspaces $\mathcal{H}_k$, i.e. satisfying $\sum_k Q_k = \mathbb{I}_M$ with $Q_k^2 = Q_k = Q_k^*$ for all $k$ and $Q_k \psi_k = \psi_k$ for all $\psi_k \in \mathcal{H}_k$, can be applied to read out the measurement outcome from the register.
More formally, an implementation of a quantum measurement is a tuple $(\rho_M, U_{SM}, \{Q_k\})$ determining the initial state $\rho_M$ of the memory register, the unitary dynamics $U_{SM}$ that describes the interaction between measured system and register, and the projections $\{Q_k\}$ on $M$ with which the outcome $k$ can be read out from the register after measurement. To any such tuple $(\rho_M, U_{SM}, \{Q_k\})$ we associate a measurement step $M$, i.e. the channel that takes as input an arbitrary initial state $\rho_S$ of $S$ and outputs the post-measurement state

$$\rho'_{SM,k} = (\mathbb{1} \otimes Q_k) U_{SM}(\rho_S \otimes \rho_M) U_{SM}^\dagger (\mathbb{1} \otimes Q_k)/p_k$$

on $S$ and $M$ with probability

$$p_k = \text{tr}[(\mathbb{1} \otimes Q_k) U_{SM}(\rho_S \otimes \rho_M) U_{SM}^\dagger],$$

for each $k = 1, ..., K$. We say that a tuple $(\rho_M, U_{SM}, \{Q_k\})$ is an implementation of a given measurement $\{T_k\}$ if the associated measurement step outputs the correct post-measurement states on the measured system, $\text{tr}_M[\rho'_{SM,k}] = \rho'_{S,k} = T_k(\rho_S)/p_k$, with correct probability $p_k = \text{tr}[T_k(\rho_S)]$ for all possible input states $\rho_S$. The measurement step $M$ therefore outputs the state

$$\rho'_M = \sum_k p_k \rho'_{SM,k} = \sum_k (\mathbb{1} \otimes Q_k) U_{SM}(\rho_S \otimes \rho_M) U_{SM}^\dagger (\mathbb{1} \otimes Q_k)$$

on $S$ and $M$, which correctly stores the outcome $k$ on $M$, since $\rho'_{M,k} = \text{tr}_S[\rho'_{SM,k}]$ has by construction only support on $\mathcal{H}_k$.

Note that for any instrument $\{T_k\}$ there exists an implementation $(\rho_M, U_{SM}, \{Q_k\})$. Conversely, any $(\rho_M, U_{SM}, \{Q_k\})$ is an implementation of some instrument $\{T_k\}$. In this sense the above operational measurement model does not place any restrictions on the set of measurements described.

After the measurement, the final state $\rho'_M = \sum_k p_k \rho'_{M,k}$ of the register stores the information of the measurement outcome $k$. This information has to be erased by resetting the register to its initial state $\rho_M$ before the same implementation of the measurement can be used another time. This process is called the resetting step $R$ and employs an additional quantum system, called thermal bath $B$, with Hamiltonian $H_B$ initially in a thermal state

$$\rho_B = \exp(-\beta H_B)/Z_B$$

at inverse temperature $\beta = \frac{1}{k_B T}$, where $Z_B = \text{tr}[\exp(-\beta H_B)]$ is the partition function. To achieve erasure, the register unitarily interacts with the thermal bath such that its state $\rho'_M$ after $M$ evolves back to the initial state

$$\text{tr}_B[U_{MB}(\rho'_M \otimes \rho_B) F_B] = \rho_M.$$  

This process is typically known as Landauer erasure.$^{[25]}$ Note that this process demands additional resources: Thermal states are needed since unitary dynamics on $M$ alone cannot alter the rank or spectrum of the state. In this framework we consider thermal states a free resource as they can easily be obtained by weakly coupling quantum systems to thermal baths at the desired ambient temperature $T$. Still, the energy cost of the resetting step, specifically to implement the unitary $U_{MB}$, needs to be accounted for. The overall energy expense needed to run the measurement device is therefore the sum of the cost of the measurement step $M$ and the cost of the resetting step $R$.

Appendix B: Relating the cost $\Delta E_M$ to operational quantities

Here we prove that the energy cost of implementing the measurement step $M$ (see Eq. (A1)),

$$\Delta E_M = \text{tr}[H_M(\rho'_M - \rho_M)],$$

splits into a sum of operational quantities as mentioned in the main text. More concretely, we denote by $\Delta S = S(\rho_S) - \sum_k p_k S(\rho'_{S,k})$ the average change in state information about the system where $S(\rho) = -\text{tr}[\rho \ln \rho]$ is the von-Neumann entropy. Moreover, $\Delta F_M = F(\rho'_M) - F(\rho_M)$ denotes the difference between free energies $F(\rho) = \text{tr}[\rho H] - S(\rho)/\beta$ of the memory before and after the measurement and $I = \sum_k p_k I(S:M|k)$ is the average amount of correlations built up between $S$ and $M$ as measured by the mutual information $I(S:M|k) := S(\rho'_{S,k}) + S(\rho'_{M,k}) - S(\rho'_{SM,k})$. Finally, we denote by $\Delta Q = S(\rho'_{SM}) - S(\rho_{SM})$, where $\rho_{SM} = \rho_S \otimes \rho_M$, the total entropy increase during the measurement step induced by the projections $\{Q_k\}$.

Using this notation we show the following theorem:
**Theorem 1.** Let $\mathcal{H}_S$ and $\mathcal{H}_M$ be finite-dimensional Hilbert spaces. Let $\rho_{SM} = \rho_S \otimes \rho_M$ be a quantum state on $\mathcal{H}_S \otimes \mathcal{H}_M$ and $H_M$ be a Hamiltonian on $\mathcal{H}_M$. For $\rho_M' = \text{tr}_S[\rho_{SM}']$, where $\rho_{SM}'$ is the state after a measurement step $M$ as in Appendix A, the energy cost $\Delta E_M = \text{tr}[H_M(\rho_M' - \rho_M)]$ satisfies

$$\beta \Delta E_M = \Delta S + I + Q \ .$$

**Proof.** Note that the post-measurement states $\rho_{SM,k}'$ (and hence, also the states $\rho_{SM,k}'$) are mutually orthogonal due to the projection operators $\{Q_k\}$. Denoting the Shannon entropy of the probability distribution $p_k$ by $H(\{p_k\}) = -\sum_k p_k \ln p_k$, we then find that the total entropy increase is

$$\Delta Q = S(\rho_{SM}') - S(\rho_{SM}) = H(\{p_k\}) + \sum_k p_k S(\rho_{SM,k}') - (S(\rho_S) + S(\rho_M)) = H(\{p_k\}) + \sum_k p_k (S(\rho_{S,k}') + S(\rho_{M,k}') - I(S : M|k)) - (S(\rho_S) + S(\rho_M)) = S(\rho_M') - S(\rho_M) + \sum_k p_k S(\rho_{S,k}') - S(\rho_S) - \sum_k p_k I(S : M|k) = S(\rho_M') - S(\rho_M) - \Delta S - I \ ,$$

where we used the additivity of the von-Neumann entropy under tensor products, i.e. $S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$ for all states $\rho$ and $\sigma$.

The energy cost of the measurement step is therefore given by, using (B2) in the last step,

$$\beta \Delta E_M = \beta \text{tr}[H_M(\rho_M' - \rho_M)] = \beta F(\rho_M') + S(\rho_M) - (\beta F(\rho_M) + S(\rho_M)) = \beta \Delta F_M + \Delta S + I + \Delta Q \ .$$

Note that Eq. (B1) is an exact equality, but contains quantities such as $I$ and $\Delta Q$ that are often hard to control as they require precise knowledge over the internal state $\rho_M$ of the measurement device. However, both $I$ and $\Delta Q$ are non-negative: $I$ inherits this property from the non-negativity of the mutual information $I(S : M|k)$, whereas $\Delta Q$ is non-negative because the measurement step $M$ corresponds to a unital measurement channel (see Eq. (A1)). Hence (B1) immediately implies the following inequality from the main text,

$$\Delta E_M \geq k_B T \left( S(\rho_S) - \sum_k p_k S(\rho_{S,k}') \right) + \Delta F_M \ .$$

**Appendix C: Computing the cost $\Delta E_R$ of the resetting step**

Similar to the previous section, the cost $\Delta E_R$ of the resetting step $R$ (see Eq. (A2)) can be expressed as a sum of operational quantities as mentioned in the main text. Here we present a proof of this statement.

We employ the same notation as above. Additionally we denote by $\Delta F_B = F(\rho_B') - F(\rho_B)$ the free energy increase in the thermal bath $B$ with $\rho_B' = \text{tr}_M[\rho_{SM}']$ where $\rho_{SM}' = U_{MB}(\rho_S' \otimes \rho_B')U_{MB}^\dagger$ is the joint state of $MB$ after the resetting step. We also introduce the mutual information term $I_{MB} = I(M : B)_{\rho_{SM}'}$ which measures the amount of correlations built up between $M$ and $B$ during step $R$.

**Theorem 2.** Let $\mathcal{H}_M$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces. Let $\rho_{MB}' = \rho_M' \otimes \rho_B'$ be a quantum state on $\mathcal{H}_M \otimes \mathcal{H}_B$ with $\rho_B' = e^{-\beta H_B} / \text{tr}[e^{-\beta H_B}]$ thermal and $H_M$ and $H_B$ be Hamiltonians on $\mathcal{H}_M$ and $\mathcal{H}_B$, respectively. Consider the resetting step $R$ as in Appendix A and denote by $\rho_{MB}'$ the final state of the process. Then the energy cost $\Delta E_R = \text{tr}[H_{MB}(\rho_{MB}' - \rho_{MB})]$ of the resetting step satisfies

$$\beta \Delta E_R = -\beta \Delta F_M + \beta \Delta F_B + I_{MB} \ .$$

Moreover, $\Delta F_B$ and $I_{MB}$ are both non-negative, such that

$$\Delta E_R \geq -\Delta F_M \ .$$
Proof. Note that the differing signs of the free energy terms appearing in \((C1)\) are just due to our notation, where \(\Delta F_M\) is defined as for the measurement step by \(\Delta F_M = F(\rho_M) - F(\rho_M) = -\left(F(\rho_M) - F(\rho_M)\right)\), where \(\rho_M = \text{tr}_B[\rho_{MB}] = \rho_M\), whereas \(\Delta F_B = F(\rho_B') - F(\rho_B')\). To show \((C1)\) we compute

\[
\Delta E_R = \text{tr}[H_{MB}(\rho_{MB}' - \rho_{MB})] \\
= \text{tr}[H_{MB}(\rho_{MB}' - \rho_{MB})] - \frac{1}{\beta}S(\rho_{MB}') + \frac{1}{\beta}S(\rho_{MB}) \\
= \text{tr}[H_{MB}(\rho_{MB}' - \rho_{MB})] - \frac{1}{\beta}\left[S(\rho_{MB}) + S(\rho_{MB}') - I(M : B)\right] \\
= F(\rho_M) + F(\rho_B') - F(\rho_M') - F(\rho_B') + \frac{1}{\beta}I_{MB} \\
= -\Delta F_M + \Delta F_B + \frac{1}{\beta}I_{MB},
\]

where we used in \((C3)\) that the unitary resetting step \((A2)\) does not change the entropies, \(S(\rho_{MB}') = S(\rho_{MB})\).

To show the non-negativity of \(\Delta F_B\) we first express the free energy \(F(\rho)\) of any quantum state \(\rho\) w.r.t. a Hamiltonian \(H\) in terms of the relative entropy \(D(\rho||\rho_{\text{can}})\), where \(\rho_{\text{can}} = e^{-\beta H}/Z\) with \(Z = \text{tr}[e^{-\beta H}]\) is a thermal state w.r.t. the same Hamiltonian \(H\). We have

\[
F(\rho) = \text{tr}[H\rho] - \frac{1}{\beta}S(\rho) \\
= \text{tr}[H\rho] + \frac{1}{\beta}D(\rho||\rho_{\text{can}}) + \frac{1}{\beta}\text{tr}[\rho\ln \rho_{\text{can}}] \\
= \text{tr}[H\rho] + \frac{1}{\beta}D(\rho||\rho_{\text{can}}) + \frac{1}{\beta}\text{tr}[\rho(-\beta H - \ln Z)] \\
= -\frac{1}{\beta}\ln Z + \frac{1}{\beta}D(\rho||\rho_{\text{can}}). 
\]

Hence, we find that the difference of free energies in the thermal bath,

\[
\Delta F_B = F(\rho_B') - F(\rho_B') \\
= -\frac{1}{\beta}\ln Z + \frac{1}{\beta}D(\rho_B'||\rho_B') + \frac{1}{\beta}\ln Z - \frac{1}{\beta}D(\rho_B'||\rho_B') \\
= -\frac{1}{\beta}D(\rho_B'||\rho_B'),
\]

is, by Klein’s inequality\(^{22}\), indeed non-negative. \(\square\)

It has been shown that an optimal process, in the sense that exact equality in \((C2)\) holds, does in general not exist\(^{22}\). However, if the dimension of the thermal bath \(B\) is not restricted, one can approach the lower bound \(-\Delta F_M\) arbitrarily closely, e.g. by conducting a process that consists of multiple intermediate steps in which the memory gets temporarily thermalised\(^{23,24}\). Since we place no restriction on the Hilbert space dimension of \(B\) in our framework, we assume that the resetting step \(R\) is conducted in such a way that \((C2)\) is saturated as closely as desired, i.e. \(\Delta E_R = -\Delta F_M\). We emphasise that this assumption merely simplifies our results, but does not restrict their validity in a more general setting. Dropping this assumption will only increase the lower bounds by additional non-negative quantities.

Appendix D: Overall energy cost of a general quantum measurement

As described in the main text, the overall energy cost needed to implement a quantum measurement is the sum of the cost of step \(M\) and step \(R\),

\[
E_{\text{cost}} = \Delta E_M + \Delta E_R.
\]

Combining Theorem \([1]\) and the optimal implementation of the resetting step \(R\), \(\Delta E_R = -\Delta F_M\) (see Appendix \([C]\)), the overall energy cost is given by

\[
\beta E_{\text{cost}} = \Delta S + I + \Delta Q.
\]
As shown by the computation in Eq. (B2), we have 
\[ \Delta S + \mathcal{I} + \Delta Q = S(\rho'_M) - S(\rho_M), \]
so we find that the overall energy cost of a quantum measurement equals the entropy change in the memory
\[ \beta E_{\text{cost}} = S(\rho'_M) - S(\rho_M) \]  
(D1)
as predicted by Landauer’s principle\cite{10,25}. The equality (D1) will turn out to be useful to derive the equality results in the following sections, but requires knowledge about the internal state of the measurement device. In contrast, our first main result in the main text (1)
\[ E_{\text{cost}} \geq \Delta S = k_B T \left[ S(\rho_S) - \sum_k p_k S(\rho'_{S,k}) \right] \]  
(D2)
is independent of the specific measurement implementation \((U_{SM}, \rho_M, \{Q_k\})\).

\textbf{Appendix E: Lower bound in terms of inefficiency}

Here we prove the lower bound on the energy cost in terms of the inefficiency of the measurement presented in the main text. A measurement is said to be efficient if each measurement outcome \(k\) has just one corresponding measurement operator \(M_k\), i.e. the (unnormalized) post-measurement states on the measured system \(S\) all take the form \(T_k(\rho_S) = M_k \rho_S M_k^\dagger\) or, in other words, the Kraus rank of all maps \(T_k\) is one. Efficient measurements however do not describe all possible quantum measurements. Instead, as described in Appendix A, the most general form of measurement is obtained through inefficient measurements described as \(\rho_S \mapsto \rho'_S = \sum_{k,i} M_{ki} \rho_S M_{ki}^\dagger\). The index \(i\) ranges from 1 to the Kraus rank \(I(k)\) of the channel \(T_k\). We henceforth call the maximal Kraus rank of all elements \(T_k\) of a given quantum instrument the inefficiency \(I\) of the quantum instrument \(\{T_k\}\). Clearly, if \(I = 1\) we recover the case of efficient measurements.

For general measurements we prove the following theorem:

**Theorem 3.** Let \((U_{SM}, \rho_M, \{Q_k\})\) be an implementation of a quantum measurement with inefficiency \(I\). The energy cost of operating this device is then lower bounded as
\[ \beta E_{\text{cost}} \geq - \ln I. \]  
(E1)

We highlight two consequences of this theorem: First, if we can construct a measurement device for an inefficient measurement with \(I > 1\) that saturates (E1), then useful energy can be extracted from the device during this operation. Remarkably, as shown in Figure 3 in the main text, such devices exist. Further details on this example and how to construct such measurement devices will be presented in Appendix F. Second, such extraction of energy in a measurement is only possible for inefficient measurements. Efficient measurements can never yield energy\cite{19}, \(E_{\text{cost}} \geq 0\).

**Proof.** In order to prove inequality (E1) let us denote by \(p_k = \text{tr} \left[ \sum_i M_{ki} \rho_S M_{ki}^\dagger \right]\) the probability of receiving outcome \(k\) and by \(\rho'_{S,k} = \sum_i M_{ki} \rho_S M_{ki}^\dagger / p_k\) the corresponding post-measurement state on \(S\). Furthermore, define \(r_{ki} := \)
Additionally, we take projections \( Q \). The outcome probabilities are then given by \( \rho \). and we denote by \( \rho \) the initial state of the measured system (as depicted in Figure 3). The first example is a rank-1 projective measurement on a qubit system consisting of two qubits \( A \) and \( B \) with Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively, that starts in the state \( |\psi\rangle \). A specific measurement device \( \rho_S \) that implements this projective measurement is characterised as follows: We take a memory \( M \) of the form \( \rho = \rho_S |\psi\rangle \langle \psi| \). Then we have

\[
\sum_k p_k S(\rho_{S,k}) = \sum_k p_k S\left( \sum_i \frac{M_{ki} \rho_S M_{ki}^\dagger}{p_k} \right) = \sum_k p_k S\left( \sum_i \frac{r_{ki}}{p_k} M_{ki} \rho_S M_{ki}^\dagger \right) \leq \sum_k p_k \left[ H\left( \left\{ \frac{r_{ki}}{p_k} \right\} \right) + \sum_i \frac{r_{ki}}{p_k} S\left( \frac{M_{ki} \rho_S M_{ki}^\dagger}{r_{ki}} \right) \right] \leq \sum_k p_k \ln I + \sum_i \frac{r_{ki}}{p_k} S\left( \sqrt{\frac{\rho_S}{r_{ki}}} \sqrt{\rho_S} M_{ki} \sqrt{\rho_S} \right) = \ln I + \sum_i \frac{r_{ki}}{p_k} S\left( \sqrt{\frac{\rho_S}{r_{ki}}} \sqrt{\rho_S} M_{ki} \sqrt{\rho_S} \right) \leq \ln I + S\left( \sum_{k,i} \sqrt{\frac{\rho_S}{r_{ki}}} \sqrt{\rho_S} M_{ki} \sqrt{\rho_S} \right) = \ln I + S(\rho_S),
\]

which by (D2) proves the desired statement (E1). Inequalities (E2) and (E3) are obtained using the property of the von-Neumann entropy that for any convex combination of quantum states, \( \sum_j p_j |\psi_j\rangle \langle \psi_j| \), we have that (E2)

\[
\sum_j p_j S(\sigma_j) \leq S\left( \sum_j p_j |\psi_j\rangle \langle \psi_j| \right) = H\left( \left\{ p_j \right\} \right) + \sum_j p_j S(\sigma_j).
\]

Inequality (E3) is obtained using the following two statements: First, the Shannon entropy of any probability distribution with \( I \) elements is upper bounded by \( \ln I \) and, second, \( S(LL^\dagger) = S(L^\dagger L) \) for any linear operator \( L \) since \( LL^\dagger \) and \( L^\dagger L \) have the same non-vanishing eigenvalues.

\[\square\]

Appendix F: How to extract energy through measurement

In this section we present and discuss in detail two examples of a measurement on a qubit \( S \) and compute how much energy is extracted during each process (see Figure 3 in the main text for a brief summary). The first example will be an efficient measurement, for which we already know that no energy can be extracted (see Theorem 3). The second example will be a slight variation of the first: Although quite similar to the first measurement, the second is inefficient and allows to extract much energy during each process (see Figure 3 in the main text for a brief summary). The first example will be an efficient measurement, for which we already know that no energy can be extracted (see Theorem 3). The second example will be a slight variation of the first: Although quite similar to the first measurement, the second is inefficient and allows to extract much energy during each process (see Figure 3 in the main text for a brief summary). The final state of \( S \) and \( M \) is of the form

\[
\rho_{SM} = \sum_{k=0,1} \langle k| \rho_S |k\rangle \cdot |k\rangle_S \otimes \rho_{M,k}^k.
\]

where the states \( \rho_{M,k}^k \) have support on orthogonal subspaces such that the outcome value \( k \) is reliably stored on \( M \). The outcome probabilities are then given by \( p_k = \langle k| \rho_S |k\rangle \). A specific measurement device \( (\rho_M, U_{SM}, \{ Q_k \}) \) that implements this projective measurement is characterised as follows: We take a memory \( M \) consisting of two qubits \( M_A \) and \( M_B \) with Hilbert spaces \( \mathcal{H}_{M_A} \) and \( \mathcal{H}_{M_B} \), respectively, that starts in the state

\[
\rho_M = |0\rangle_{M_A} \langle 0| \otimes \frac{1}{2} \mathbb{I}_{M_B^2}.
\]

Additionally, we take projections \( Q_k = |k\rangle_{M_A} \langle k| \otimes \mathbb{I}_{M_B} \) and consider the following unitary interaction between system and memory

\[
U_{SM} = \left( |0\rangle_S \langle 0| \otimes |0\rangle_{M_A} \langle 0| + |1\rangle_S |1\rangle \otimes |1\rangle_{M_A} \langle 0| + \ldots \right) \otimes \mathbb{I}_{M_B}. \]
where the dots indicate that we are free to choose any unitary extension. Indeed, evaluating (A1) for this implementation \((U_{SM}, \rho_M, \{Q_k\})\) we find that this measurement device outputs the desired final state (F1) with 
\[
\rho'_{SM} = |k\rangle_M|k\rangle \otimes \frac{1}{2} \rho_M.
\]

In this paper, we provide various ways to calculate the energy cost of conducting this particular measurement: In the main text we claimed that the energy cost of any projective measurements is directly given by the Shannon entropy of the outcome probability distribution, \(H(\{p_k\})\) (see eq. (2)). While this claim is proven in Appendix G we can also verify this result for the example at hand using (D1). Indeed, since we specified the microscopic details of our showcase measurement device, we are able to compute the energy cost of this projective measurement
\[
\beta E_{\text{proj}} = \beta \Delta E_M + \beta \Delta E_R
= S(\rho_M') - S(\rho_M)
= (H(\{p_k\}) + \ln 2) - \ln 2
= H(\{p_k\}).
\]

Hence, indeed we find that for all initial states of \(S\) the energy cost in non-negative, \(E_{\text{proj}} \geq 0\), i.e. no energy can be extracted. To run the device at zero energy cost the measured system has to start in any of the states \(\{|k\rangle_S|k\rangle\}_{k=0,1}\).

The situation changes if we consider the following slight variation of the above setup, which is our second example. Assume a situation where we are only interested in the outcome probabilities \(p_k\) of our measurement and not in the final state of \(S\), i.e. we fix the POVM but before the quantum instrument. We can then construct a measurement device that in addition to the previous device performs, after \(U_{SM}\) but before the projections \(\{Q_k\}\), a swap operation \(U_{SM} \otimes M_k\) between \(S\) and \(M_k\) (see Figure 3). The unitary interaction between measured system and memory in this device is therefore simply given by \(U_{SM} \otimes M_k\). The post-measurement state then reads
\[
\rho'_{SM} = \frac{1}{2} \otimes \sum_{k=0,1} \langle k|\rho_S|k\rangle \cdot |k\rangle_M \rho_a \otimes |k\rangle_M \rho_b \langle k| .
\]

Note that the measurement device correctly outputs the outcome probabilities, i.e. the measurement outcomes can be read off from the memory via the projections \(Q_k\) with the correct probabilities \(p_k = \langle k|\rho_S|k\rangle\), and therefore still allows for conditioning on the outcome. However, in contrast to the device in the first example, it always leaves the measured system in the completely mixed state. Remarkably, operating this device allows for extracting energy and, by (D1), we can compute the energy costs for this modified measurement:
\[
\beta E_{\text{cost}} = \beta \Delta E_M + \beta \Delta E_R = H(\{p_k\}) - \ln 2 \leq 0.
\]

Hence, if the measured system starts in any of the states \(\{|k\rangle_S|k\rangle\}\), this measurement device outputs \(-E_{\text{cost}} = k_B T \ln 2\) of useful energy. The reason why this slight modification of the setup allows for extracting energy is that the additional swap process introduces inefficiency into the measurement: A measurement that always outputs states of the form \(\rho_{SM} = \frac{1}{\beta} \otimes \rho_M\) cannot have a one-to-one correspondence between measurement operator \(A_k\) and outcome \(k\). Indeed, our device implements the quantum instrument \(\{T_k(\rho_S) = \sum_{i=1}^2 \frac{1}{\beta} |i\rangle \langle i| \langle k| \rho_S |k\rangle \langle i|\}\) with inefficiency \(I = 2\) and hence saturates our inefficiency bound (E1).

### Appendix G: Energy cost of projective measurements

Projective measurements are the textbook examples of “standard” quantum measurements. They are described by projective measurement operators \(M_k = P_k\) with \(P_k^2 = P_k^\dagger = P_k\) and map the initial state \(\rho_S\) of the measured system to the post-measurement state \(\rho_{SM}' = P_k \rho_S P_k / p_k\) with probability \(p_k = \text{tr}(P_k \rho_S)\). In particular, projective measurements belong to the class of efficient measurements due to the one-to-one correspondence between measurement operator \(P_k\) and outcome \(k\). A measurement device that implements such a projective measurement \(\{P_k\}\) on \(S\) is described by a tuple \((U_{SM}, \rho_M, \{Q_k\})\) satisfying
\[
\text{tr}_M \left[(1 \otimes Q_k)U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger(1 \otimes Q_k)\right] = P_k \rho_S P_k \quad \forall \rho_S \forall k.
\]

Again we require this equality to hold for all states \(\rho_S\) on \(S\), otherwise the device does not perform the projective measurement on all possible input states.

We now prove that the energy cost of implementing such a measurement is exactly given by Eq. (2) in the main text and is hence necessarily non-negative.
Let \((U_{SM}, \rho_M, \{Q_k\})\) be an implementation of a projective quantum measurement \(\{P_k\}\), as prescribed by (G1). Then the energy cost of operating this device on an initial state \(\rho_S\) is given by

\[
\beta_{E_{\text{proj}}} = H(\{p_k\}),
\]

where \(H(\{p_k\}) = -\sum_k p_k \ln p_k\) is the Shannon entropy of the outcome probability distribution \(p_k = \text{tr}[P_k \rho_S]\).

The proof of this theorem is based on the following lemma:

**Lemma 5.** Let \(\{P_k\}\) be a projective measurement on some quantum system \(S\). Let a dilation of the “measurement channel” \(T_S(\rho_S) := \sum_k P_k \rho_S P_k\) be given by

\[
\sum_k P_k \rho_S P_k = \text{tr}_E \left[ U_{SE}(\rho_S \otimes \rho_E) U_{ESE}^\dagger \right] \quad \forall \rho_S,
\]

where \(\rho_E\) is an initial state of a quantum system \(E\) and \(U_{SE}\) is a unitary on \(S\) and \(E\).

Then there exist quantum states \(\sigma_{E,k}\) with \(S(\rho_E) = S(\sigma_{E,k})\) for all \(k\) such that, for all quantum states \(\rho_S\), the post-measurement state of \(E\), \(\rho_E' = \text{tr}_S \left[ U_{SE}(\rho_S \otimes \rho_E) U_{ESE}^\dagger \right] \), can be written as

\[
\rho_E' = \sum_k \text{tr}[P_k \rho_S] \sigma_{E,k} \quad \forall \rho_S.
\]

If, additionally, \(U_{SE}\) and \(\rho_E\) together with projections \(\{Q_k\}\) on \(E\) form an implementation \(\{U_{SE}, \rho_E, \{Q_k\}\}\) of the projective measurement \(\{P_k\}\) on \(S\), i.e. \(\sum_k Q_k = \mathbb{1}\) and

\[
P_k \rho_S P_k = \text{tr}_E \left[ (\mathbb{1} \otimes Q_k) U_{SE}(\rho_S \otimes \rho_E) U_{ESE}^\dagger (\mathbb{1} \otimes Q_k) \right] \quad \forall \rho_S \forall k,
\]

then \(\sigma_{E,k} = Q_k \sigma_{E,k} Q_k\) for all \(k\), i.e. the \(\sigma_{E,k}\) are mutually orthogonal.

**Proof.** (Lemma 5) The proof is based on the Stinespring dilation theorem, according to which we can always write the channel \(T_S(\rho_S)\) as a unitary \(U_{SA}\) acting on \(S\) and an ancilla \(A\) initially in a pure state \(|0\rangle_A\langle 0|\),

\[
T_S(\rho_S) = \text{tr}_A[U_{SA}(\rho_S \otimes |0\rangle_A\langle 0|) U_{SA}^\dagger].
\]

The minimal Stinespring dilation can be chosen to be any unitary extension \(U_{SA}\) of the operator \(\sum_k P_k \otimes |k\rangle_A\langle k|\), whose action is only defined on states of the form \(|\psi\rangle_S \otimes |0\rangle_A\), where the ancilla Hilbert space \(A\) is spanned by the orthonormal basis \(|k\rangle^A\). The corresponding complementary channel takes the form

\[
T_A(\rho_S) = \text{tr}_S[U_{SA}(\rho_S \otimes |0\rangle_A\langle 0|) U_{SA}^\dagger] = \text{tr}_S \left[ \sum_{k,k'} P_k \rho_S P_{k'} \otimes |k\rangle_A \langle k'| \right],
\]

\[
= \sum_k \text{tr}[P_k \rho_S] |k\rangle_A \langle k|.
\]

However, this channel is not the only possible complementary channel of \(T_S\). Using (G2), we find another complementary channel,

\[
T_{E\tilde{E}}(\rho_S) = \text{tr}_S[U_{SE\tilde{E}}(\rho_S \otimes \psi_{E\tilde{E}}) U_{SE\tilde{E}}^\dagger],
\]

where \(\tilde{E}\) is a purifying system of \(E\) such that the pure state \(\psi_{E\tilde{E}}\) satisfies \(\text{tr}_{\tilde{E}}[\psi_{E\tilde{E}}] = \rho_E\) and \(U_{SE\tilde{E}} := U_{SE} \otimes \mathbb{1}_{\tilde{E}}\). The Stinespring theorem states that these complementary channels are related by an isometry \(V : H_A \rightarrow H_E \otimes H_{\tilde{E}}\), i.e.

\[
T_{E\tilde{E}}(\rho_S) = VT_A(\rho_S)V^\dagger = \sum_k \text{tr}[P_k \rho_S] |\gamma_k\rangle_{E\tilde{E}} \langle \gamma_k|
\]

with \(|\gamma_k\rangle_{E\tilde{E}} := V|k\rangle_A\) again forming an orthonormal basis. Note that the complementary channel \(T_{E\tilde{E}}\) and the final state \(\rho_E'\) of \(E\) are, by construction, linked via the partial trace,

\[
\rho_E' = \text{tr}_S \left[ U_{SE}(\rho_S \otimes \rho_E) U_{SE}^\dagger \right] = \text{tr}_{\tilde{E}}[T_{E\tilde{E}}(\rho_S)].
\]
Hence, for every $\rho_S$, the final state on $E$ takes the form
\[ \rho'_E = \sum_k \text{tr}[P_k \rho_S] \sigma_{E,k} \quad \forall \rho_S, \] (G4)
where we define the states $\sigma_{E,k} = \text{tr}_E[V|k\rangle \langle k| V^\dagger]$, which are independent of $\rho_S$.

To show $S(\rho_E) = S(\sigma_{E,k})$, let now $\rho_S = \psi_k$ be a pure state supported on the subspace characterised by one $P_k$, i.e. $P_k \psi_k = \psi_k$. Then by (G4)
\[ \rho'_E = \text{tr}_S[U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger] = \sigma_{E,k} \] (G5)
and the final state on $S$ is pure,
\[ \rho'_S = \text{tr}_E[U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger] = \sum_{k'} P_{k'} \psi_k P_{k'} = \psi_k. \]
Hence there are no correlations between the marginals of the final $SE$ state, i.e. $U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger = \psi_k \otimes \sigma_{E,k}$. Since unitaries do not change the spectrum, we have $S(\rho_E) = S(\sigma_{E,k})$, which concludes the first part of the proof.

For the second part of the proof, we assume that we additionally have an implementation of the projective measurement on $S$, i.e. (G3) is satisfied. Note that we can obtain (G2) by summing (G3) over $k$; hence, all statements within the first part of the proof remain valid for this second part of the proof. We can thus take $\psi_k$ to be a state in the support of $P_k$ as above to find by (G5) that
\[ \sigma_{E,k} = \text{tr}_S[U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger] \]
Our aim is to show that by requiring (G3) we have, for all $k$, that
\[ \sigma_{E,k} = Q_k \sigma_{E,k} Q_k. \] (G6)
To this end observe that the quantity $Q_k \sigma_{E,k} Q_k$ is a positive operator with unit trace for all $k$ since by (G3)
\[ \text{tr}_E[Q_k \text{tr}_S[U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger]Q_k] = \text{tr}_S[(1 \otimes Q_k) U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger (1 \otimes Q_k)] = \text{tr}_S[P_k \psi_k P_k] = \text{tr} \psi_k = 1. \]
But then we can compute
\[ 1 = \text{tr}[\sigma_{E,k}] = \text{tr}[(Q_k + (1 - Q_k)) \sigma_{E,k} (Q_k + (1 - Q_k))] = \text{tr}[Q_k \sigma_{E,k} Q_k] + \text{tr}[(1 - Q_k) \sigma_{E,k} (1 - Q_k)] = 1 + \text{tr}[(1 - Q_k) \sigma_{E,k} (1 - Q_k)] \]
to find that $(1 - Q_k) \sigma_{E,k} (1 - Q_k) = 0$ which implies (G6).

Let us finally prove Theorem 4.

**Proof. (Theorem 4)** To compute the energy cost $E_{\text{proj}} = \Delta E_M + \Delta E_R$ of a projective measurement, we use (D1) to simplify the problem to computing the entropy difference in the memory, $\beta E_{\text{proj}} = S(\rho'_M) - S(\rho_M)$. While this difference is hard to control for general measurements, Lemma 5 gives us enough information to compute it exactly in the case of projective measurements.

Recall that the state of the memory after the measurement is given by
\[ \rho'_M = \text{tr}_S \left[ \sum_k (1 \otimes Q_k) U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger (1 \otimes Q_k) \right] = \sum_k Q_k \tilde{\rho}_M Q_k, \]
where we introduced the quantum state $\tilde{\rho}_M := \text{tr}_S[U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger]$. Also note that $(U_{SM}, \rho_M, \{Q_k\})$ is, by assumption, an implementation of the projective measurement $\{P_k\}$, i.e. (G1) (resp. (G3) of Lemma 5) is satisfied. Hence, by Lemma 5 we know that the state $\tilde{\rho}_M$ takes the form
\[ \tilde{\rho}_M = \sum_k \text{tr}[P_k \rho_S] \sigma_{M,k}, \]
where the $\sigma_{M,k} = Q_k \sigma_{M,k} Q_k$ are mutually orthogonal and have entropy $S(\sigma_{M,k}) = S(\rho_M)$ for all $k$. The post-measurement state of the memory is therefore given by

$$
\rho'_M = \sum_k Q_k \rho_M Q_k = \sum_k \left( \sum_{k'} \text{tr}[P_{k'} \rho_S] \sigma_{M,k'} \right) Q_k = \sum_k \text{tr}[P_k \rho_S] \sigma_{M,k} = \sum_k p_k \sigma_{M,k}
$$

From this it follows that

$$
\beta E_{\text{proj}} = S(\rho'_M) - S(\rho_M) = \left( H(\{p_k\}) + \sum_k p_k S(\sigma_{M,k}) \right) - S(\rho_M)
$$

$$
= H(\{p_k\}).
$$

\appendix{H: Energy costs of quantum Zeno measurements}

Here we compute the energy cost of conducting a stabilisation scheme via Zeno measurements – a process typical for the field of quantum control. As in the main text we consider a quantum system $S$, initially in the pure state $\rho_S = \langle 0 | 0 \rangle$, with Hamiltonian $H_S = E \sigma_X$, where $\sigma_X$ is the Pauli operator and $\pm E$ are the two energy eigenvalues of $H_S$. Our goal is to study the energy cost of conducting a quantum Zeno stabilisation protocol that stabilises $S$ against the free Hamiltonian time evolution over the time span $t$ by applying projective measurements $\{M_0 = \langle 0 | 0 \rangle, M_1 = | 1 \rangle \langle 1 | \}$ at $N$ regular time intervals $\delta t = t/N$. Since these measurements are projective, we can, by Theorem 4, compute the energy cost of each measurement exactly.

Note that the protocol employs multiple iterations of the same projective measurement. In our framework this may be equivalently described either by considering a single measurement device that is used repeatedly or by considering multiple devices, each possibly a different implementation of that measurement. The energy costs of both approaches are the same as the cost is by Theorem 4 independent of the specific implementation.

We find the following theorem for the total energy cost of Zeno stabilisation:

\textbf{Theorem 6.} Consider a quantum Zeno stabilisation scheme as above. To achieve high target fidelity $F$, the energy cost of operating the devices that implement the projective measurements $\{M_0 = \langle 0 | 0 \rangle, M_1 = | 1 \rangle \langle 1 | \}$ is given by

$$
E_{\text{Zeno}} \simeq \frac{1}{2} k_B T \left( \frac{E \delta t}{\hbar} \right)^2 \ln \left[ \frac{4.5}{1 - F} \right].
$$

Hence, we find that the total energy required for stabilisation grows logarithmically in $1/(1 - F)$ for increasing target fidelity $F$. In the asymptotic limit as $F \to 1$ the energy cost is given by Eq. (3) in the main text. Limited energy supply thus constrains our ability to stabilise a quantum system via Zeno control.

\textbf{Proof.} Let us denote the state on $S$ after the $n$-th measurement by $\rho_S^{(n)} = (1 - \epsilon_n)\langle 0 | 0 \rangle + \epsilon_n | 1 \rangle \langle 1 |$. The probability that the process returns the wrong state $| 1 \rangle$ after $n$ steps is then given by $\epsilon_n$; the fidelity $F = F(\rho_S^{(N)}, \rho_S) = \langle 0 | \rho_S^{(N)} | 0 \rangle$ of the final state at the end of all $N$ steps is $F = 1 - \epsilon_N$. Between the measurements, the system undergoes free time evolution according to the unitary $U = \exp(-i \delta t H_S / \hbar)$ such that the probabilities after the $(n + 1)$-th measurement change to $\epsilon_{n+1} = \epsilon_n \cos(2E \delta t / \hbar) + (1 - \epsilon_n) \sin(2E \delta t / \hbar)^2$. Since $\epsilon_0 = 0$ by assumption, this recursion formula has the solution

$$
\epsilon_n = \frac{1}{2} (1 - \cos(2E \delta t / \hbar)^n) = n \left( \frac{E \delta t}{\hbar} \right)^2 + \mathcal{O}(\delta t^4).
$$

According to our result for projective measurements (Theorem 4), the $n$-th measurement consumes energy $\beta E_{\text{proj}} = H(\{\epsilon_n, 1 - \epsilon_n\})$ such that the total energy required is given by

$$
\beta E_{\text{Zeno}} = \sum_{n=1}^{N} H(\{\epsilon_n, 1 - \epsilon_n\}).
$$

We are interested in stabilisation schemes that yield high target fidelity $F$, which can be achieved by applying the measurements in shorter and shorter time scales, $\delta t = t/N \to 0$, or in other words by applying more measurements
$N \to \infty$ in constant time span $t$. In this limit the higher order terms $O(\delta t^4)$ of $\epsilon_n$ will not contribute to the energy cost of the measurements, so we set $\epsilon_n \simeq n \left( \frac{E_t}{\hbar} \right)^2$. We then have $F \simeq 1 - \frac{1}{N} \left( \frac{E_t}{\hbar} \right)^2$ and

$$\beta E_{\text{Zeno}} = - \sum_{n=1}^{N} \epsilon_n \ln \epsilon_n - \sum_{n=1}^{N} \left( 1 - \epsilon_n \right) \ln \left( 1 - \epsilon_n \right)$$

$$\simeq - \sum_{n=1}^{N} n \left( \frac{E_t}{\hbar N} \right)^2 \ln \left[ n \left( \frac{E_t}{\hbar N} \right)^2 \right] - \sum_{n=1}^{N} \left( 1 - n \left( \frac{E_t}{\hbar N} \right)^2 \right) \ln \left[ 1 - n \left( \frac{E_t}{\hbar N} \right)^2 \right]$$

$$= - \left( \frac{E_t}{\hbar} \right)^2 \left( \sum_{n=1}^{N} \frac{n}{N} \ln \frac{n}{N} + \sum_{n=1}^{N} \frac{n}{N^2} \ln \left[ \frac{1}{N} \left( \frac{E_t}{\hbar} \right)^2 \right] \right)$$

$$\simeq - \left( \frac{E_t}{\hbar} \right)^2 \left( \sum_{n=1}^{N} \frac{1}{N} \ln \frac{1}{N} + \frac{N(N+1)}{2N^2} \ln \left[ 1 - F \right] \right)$$

$$= - \sum_{n=1}^{N} \left( 1 - n \left( \frac{E_t}{\hbar N} \right)^2 \right) \ln \left[ 1 - n \left( \frac{E_t}{\hbar N} \right)^2 \right].$$

In the limit $N \to \infty$ we have $\sum_{n=1}^{N} \frac{1}{N} \ln \frac{1}{N} \simeq \int_{1}^{\infty} x \ln x dx = -1/4$ and $\ln \left[ 1 - n \left( \frac{E_t}{\hbar N} \right)^2 \right] \simeq -n \left( \frac{E_t}{\hbar N} \right)^2$ (again higher orders of the expansion do not contribute). Hence, we have

$$\beta E_{\text{Zeno}} \simeq \frac{1}{4} \left( \frac{E_t}{\hbar} \right)^2 - \frac{1}{2} \left( \frac{E_t}{\hbar} \right)^2 \ln \left[ 1 - F \right] + \frac{1}{2} \left( \frac{E_t}{\hbar} \right)^2$$

$$= \frac{1}{2} \left( \frac{E_t}{\hbar} \right)^2 \left( 3 \ln \left[ 1 - F \right] \right)$$

$$\simeq \frac{1}{2} \left( \frac{E_t}{\hbar} \right)^2 \ln \left[ \frac{4.5}{1 - F} \right].$$

**Appendix I: Energy cost of quantum error correction**

In this section we investigate another application of our result on the energy cost of projective measurements (Theorem 4), namely computing the energy cost of conducting quantum error correcting protocols. In particular, we consider the 5-qubit code in which the state of a single logical qubit $|\psi\rangle = \alpha_0|0_L\rangle + \alpha_1|1_L\rangle$ in the code space $\mathbb{C}^2_5$, with $\alpha_0, \alpha_1 \in \mathbb{C}$ and $|\alpha_0|^2 + |\alpha_1|^2 = 1$, is encoded into the space $\mathbb{C}_5 \equiv (\mathbb{C}^2)^{\otimes 5}$ of five physical qubits by using the codewords

$$|0_L\rangle = \frac{1}{\sqrt{4}} \left[ |00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle + |00101\rangle - |11011\rangle - |00110\rangle - |11000\rangle - |11100\rangle - |01111\rangle - |01101\rangle - |01011\rangle - |01001\rangle - |00111\rangle - |00101\rangle - |00011\rangle - |00010\rangle - |00001\rangle - |00000\rangle \right],$$

$$|1_L\rangle = \frac{1}{\sqrt{4}} \left[ |11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle + |11010\rangle - |00110\rangle - |00011\rangle - |00100\rangle - |01100\rangle - |11110\rangle - |11100\rangle - |10111\rangle - |10101\rangle - |10011\rangle - |10010\rangle - |10001\rangle - |10000\rangle \right].$$

As in the main text we assume that each physical qubit is affected by the amplitude damping channel

$$N_\gamma (\rho) = J_1 \rho J_1^+ + J_2 \rho J_2^+.$$
with Kraus operators \( J_1 = \sqrt{\gamma}|0\rangle\langle 1| \) and \( J_2 = \sqrt{1 - \gamma^2}J_1 \), where \( \gamma \in [0,1] \) determines the noise strength (\( \gamma = 0 \) corresponding to the noiseless case). Note however that our formalism applies to arbitrary noise models.

The error-correcting protocol is then a feedback scheme that allows to approximately recover the logical state \(|\psi\rangle\) from the noisy state \( \rho_{S,\gamma} = N_{\gamma}^{\otimes 5}(|\psi\rangle\langle\psi|) \) by applying so-called syndrome measurements

\[
S^1 = X \otimes Z \otimes Z \otimes X \otimes I , \quad S^2 = I \otimes X \otimes Z \otimes Z \otimes X , \quad S^3 = X \otimes I \otimes X \otimes Z \otimes Z , \quad S^4 = Z \otimes X \otimes I \otimes X \otimes Z ,
\]

where \( X, Y, Z \) denote the Pauli operators and \( I \) is the identity matrix. Each syndrome measurement \( S^j \) has outcomes \( s^j \in \{-1,1\} \), occurring with probability \( p_{s^j} = \text{tr}[P_{s^j}^j \rho_{S,\gamma}] \), where \( P_{s^j}^j \) denotes the projector on the subspace corresponding to eigenvalue \( s^j \). Furthermore all syndrome measurements commute and are hence jointly measurable. The measurement operators of the joint measurement \( S \) are given by projections \( \{ P_s \}_{s=0}^{15} \) with \( P_s = P_s^{(1)} P_s^{(2)} P_s^{(3)} P_s^{(4)} \) and outcomes (“syndromes”) \( s \equiv (s^1, s^2, s^3, s^4) \) that occur with probability \( p_s = \text{tr}[P_s \rho_{S,\gamma}] \).

We call a specific realisation of the 5-qubit code a separate measurement scheme if it employs four devices, each implementing one of the syndrome measurements \( S^j \). If only a single device is employed that implements a joint measurement \( S \) we call it the joint measurement scheme. Note that, for both separate and joint measurement scheme, any single-qubit Pauli error is uniquely identified by one of the 16 possible syndromes \( s \equiv (s^1, s^2, s^3, s^4) \). Hence, such errors can be corrected with certainty by applying the conditional unitary

\[
V_{SM} = \sum_s V_s \otimes P_s ,
\]

which applies the unitary \( V_s \) on the measured system \( S \) if the syndrome (which is stored in the memory after the measurement and read out by projections \( P_s \)) is \( s \). More concretely, since all single qubit errors (\( X, Y \) or \( Z \)) square to identity, we simply apply, say, \( V_s = X \otimes I \otimes I \otimes I \otimes I \) if the syndrome \( s \) identified an \( X \)-error on the first physical qubit. In the following we denote the final state after applying the joint measurement \( S \) and corresponding feedback by

\[
\tilde{\rho}_{S,\gamma} = \sum_s V_s P_s \rho_{S,\gamma} P_s V_s^\dagger .
\]

Both the separate and the joint measurement schemes can be used for quantum error correction, but they come at different energy costs as may be verified by employing Theorem 4. The joint measurement scheme demands energy

\[
E_{C_{\text{sep}}} = k_B T H(\{p_{s^j}\}_{s^j=0}^{15}) ,
\]

which is always less or equal to the energy

\[
E_{C_{\text{sep}}} = k_B T H(\{p_{s^j}\}_{s^j=\pm 1})
\]

required to implement the four separate syndrome measurements. The reason for this is that a joint measurement scheme can exploit correlations in the measurement outcomes to reduce the cost of the resetting step \( R \). More concretely, for all joint probability distributions \( p(s \equiv (s^1, s^2, s^3, s^4)) \) of random variables \( S^1, S^2, S^3, S^4 \) the following relation holds:

\[
H(S^1 S^2 S^3 S^4) = H(S^1) + H(S^2 S^3 S^4|S^1) = H(S^1) + H(S^2|S^1) + H(S^3|S^1 S^2) + H(S^4|S^1 S^2 S^3) = \sum_{j=1}^{4} H(S^j) - I(S^1 : S^2) - I(S^1 S^2 : S^3) - I(S^1 S^2 S^3 : S^4) ,
\]

where \( H(S^1 S^2 S^3 S^4) := H(\{p(s \equiv (s^1, s^2, s^3, s^4))\}_{s}) \) is the Shannon entropy of the joint probability distribution, \( H(S^2|S^1) := H(S^1 S^2) - H(S^1) \) is the conditional entropy and \( I(S^1 : S^2) := H(S^1) - H(S^1 S^2) \) is the mutual information, which quantifies the correlations between the respective random variables and is always non-negative. Hence, only if the measurement outcomes are uncorrelated (i.e. all mutual information terms in \( I_{\text{sep}} \) vanish), we have that \( H(\{p_s\}_{s=0}^{15}) = \sum_{s=1}^{4} H(\{p_{s^j}\}_{s^j=\pm 1}) \) and the energy cost of the joint measurement equals the cost of all four
separate measurements. In Fig. 4, the difference between the blue and red curve is due to the mutual information terms of the syndrome bits on the specific noisy states,

$$E_{CS}^{\text{sep}} - E_{CS}^{\text{min}} = k_B T \left( I(S^1 : S^2) + I(S^1 S^2 : S^3) + I(S^1 S^2 S^3 : S^4) \right) .$$

Note that exactly determines the minimum energy requirements for quantum error correction and is hence a drastic improvement to all previous results, which only provide lower bounds: Evaluating the result obtained in ref. 19 which is generalised to all measurements by our first main result (D2), yields the best lower bound previously known

$$E_{CS}^{\text{SU}} := k_B T \left[ S(\rho_{S,\gamma}) - \sum_s p_s S(P_s \rho_{S,\gamma} P_s / p_s) \right] \leq E_{CS}^{\text{min}},$$

which itself is a slight improvement (due to concavity of the von Neumann entropy) upon the simple application of Landauer’s principle as in ref. 19, which is a slight improvement (due to concavity of the von Neumann entropy) upon the simple application of Landauer’s principle as in ref. 19.

Evaluating the result obtained in ref. 19, which itself is a slight improvement (due to concavity of the von Neumann entropy) upon the simple application of Landauer’s principle as in ref. 19.

$$E_{C5}^{\text{SU}} := k_B T \left[ S(\rho_{S,\gamma}) - \sum_s p_s S(P_s \rho_{S,\gamma} P_s / p_s) \right] \leq E_{C5}^{\text{min}},$$

which itself is a slight improvement (due to concavity of the von Neumann entropy) upon the simple application of Landauer’s principle as in ref. 19. However, we prove this statement. This result on the energy cost of dephasing operations may be of independent interest.

Recall from Appendix A that the Hamiltonian of the memory is given by

$$H_M = \bigoplus_{k=1}^K H_k,$$

where $H_k$ is a Hamiltonian on the respective subspace $\mathcal{H}_k$ with corresponding projection $Q_k$.

**Theorem 7.** Let $T_M(\sigma_M) = \sum_k Q_k \sigma_M Q_k$ be the dephasing operation on the memory $M$. Then for any dilation of $T_M$,

$$T_M(\sigma_M) = \text{tr}_E[U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger] \quad \forall \sigma_M,$$

where $\sigma_E$ is a thermal state on an environment $E$ with Hamiltonian $H_E$, the corresponding energetic cost

$$E_{\text{deph}} := \text{tr}[H_{ME}(U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger - \sigma_M \otimes \sigma_E)]$$

is non-negative,

$$E_{\text{deph}} \geq 0 .$$

Conversely, there exist $\sigma_E$ and $U_{ME}$ such that the energetic cost $E_{\text{deph}}$ of the corresponding dilation of $T_M(\sigma_M)$ is precisely zero.

**Proof.** Our goal is to quantify the energetic cost $E_{\text{deph}} = \text{tr}[H_{ME}(\sigma'_M - \sigma_{ME})]$ of implementing $T_M$ unitarily as in (J1), where $\sigma_{ME} := \sigma_M \otimes \sigma_E$ and $\sigma'_M := U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger$, denote the initial and final state of $M$ and $E$, respectively. Due to the direct sum structure of $H_M = \bigoplus_k H_k$ (see Appendix A) we know that $[Q_k, H_k] = 0$, which implies that the dephasing operation $T_M$ does not change the average energy on $M$. Hence, all energy expenses of this implementation are due to energy changes in the environment $E$.

$$E_{\text{deph}} = \text{tr}[H_{E}(\sigma'_E - \sigma_E)].$$

(J2)

The initial state of the environment is, by assumption, thermal, i.e. $\sigma_E = \exp(-\beta H_E)/Z_E$ with $Z_E$ the partition function. The final state of $E$ on the other hand can be characterised by applying Lemma 5 from which we know that there exist states $\sigma_{E,k}$ with $S(\sigma_{E,k}) = S(\sigma_E)$ for all $k$ such that

$$\sigma'_E = \sum_k \text{tr}[Q_k \sigma_M] \sigma_{E,k} \quad \forall \sigma_M .$$

(J3)
But then
\[ E_{\text{deph}} = \text{tr}[H_E(\sigma'_{E} - \sigma_E)] = \sum_k \text{tr}[Q_k \sigma_M] \text{tr}[H_E(\sigma_{E,k} - \sigma_E)] \geq 0 , \]
where the last inequality is due to the non-negativity of the relative entropy \( D \) which implies that the thermal state minimizes the average energy on an entropic orbit, \( 0 \leq \beta \text{tr}[H_E(\sigma_{E,k} - \sigma_E)] = D(\sigma_{E,k}, \sigma_E) \) for all \( k \).

We now show the second part of Theorem 7 that is the existence of a thermal state \( \sigma_E \) and a unitary \( U_{ME} \) that implement the dephasing channel \( T_M \), i.e.
\[ \text{tr}_E[U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger] = \sum_k Q_k \sigma_M Q_k \quad \forall \sigma_M , \tag{J4} \]
at vanishing energy cost, \( E_{\text{deph}} = 0 \). Instead of just naming possible \( \sigma_E \) and \( U_{ME} \) to achieve the goal, we provide a characterisation of all unitaries \( U_{ME} \) which, given any fixed full rank state \( \sigma_E \), satisfy \( \text{(J4)} \). In the end, we will describe a simple explicit construction.

Consider a pure state \( \sigma_M = \psi = |\psi\rangle\langle\psi| \) on \( M \) in the support of a fixed projector \( Q_k \), i.e. \( Q_k \psi = \psi \). Then from \( \text{(J4)} \) we know that the marginal on \( M \),
\[ \text{tr}_E[U_{ME}(\psi \otimes \sigma_E)U_{ME}^\dagger] = \psi , \]
is pure, while by \( \text{(J3)} \) the marginal on \( E \) is
\[ \text{tr}_E[U_{ME}(\psi \otimes \sigma_E)U_{ME}^\dagger] = \sigma_{E,k} , \]
where \( \sigma_{E,k} \) is a state which is independent of \( \psi \). This implies
\[ U_{ME}(\psi \otimes \sigma_E)U_{ME}^\dagger = \psi \otimes \sigma_{E,k} . \tag{J5} \]
Let us now denote the spectral decomposition of \( \sigma_E \) by \( \sigma_E = \sum_j \lambda_j |\phi_j\rangle_E \langle \phi_j| \). We note that all eigenvalues \( \lambda_j \) are strictly positive since \( \sigma_E \) is assumed to have full rank. Substituting the decomposition of \( \sigma_E \) into \( \text{(J5)} \) we obtain
\[ \sum_j \lambda_j U_{ME}(\psi \otimes |\phi_j\rangle_E \langle \phi_j|)U_{ME}^\dagger = \psi \otimes \sigma_{E,k} . \tag{J6} \]
We therefore know that
\[ \text{tr}_E\left[U_{ME}(\psi \otimes |\phi_j\rangle_E \langle \phi_j|)U_{ME}^\dagger\right] = \psi \quad \forall j , \]
i.e. the marginal on \( M \) of each of the pure states \( U_{ME}(\psi \otimes |\phi_j\rangle_E \langle \phi_j|)U_{ME}^\dagger \) must be pure itself and identical regardless \( j \) – otherwise the marginal of the sum could not be pure as required by \( \text{(J6)} \). A unitary \( U_{ME} \) that satisfies \( \text{(J4)} \) with full rank states \( \sigma_E \) is therefore always of the form
\[ U_{ME}(|\psi\rangle_M \otimes |\phi_j\rangle_E) = |\psi\rangle_M \otimes V_\psi|\phi_j\rangle_E , \tag{J7} \]
with a unitary \( V_\psi \) depending on \( \psi \in Q_k \). Note however that the right-hand side of \( \text{(J7)} \) must be linear in \( \psi \) due to the linearity of the left-hand side. Hence, \( V_\psi \) can only depend on the label \( k \) of the subspace corresponding to \( Q_k \), so we write \( V_k = V_\psi \).

Up to now we have evaluated \( \text{(J4)} \) for pure states \( \sigma_M = \psi \) in the support of some \( Q_k \) only. To characterise the unitaries \( V_k \) we evaluate \( \text{(J4)} \) on all mixed initial states \( \sigma_M \) on \( M \) to find on one hand that
\[ \text{tr}_E[U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger] = \sum_k Q_k \sigma_M Q_k . \tag{J8} \]
On the other hand we can employ \( \text{(J7)} \) to compute
\[ U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger = U_{ME}\left( \sum_{ij} Q_i \sigma_M Q_j \otimes \sigma_E \right)U_{ME}^\dagger = \sum_{ij} Q_i \sigma_M Q_j \otimes V_i \sigma_E V_j^\dagger , \]
which together with \( (J8) \) yields
\[
\sum_{i,j} \langle i | \sigma_E V_j^\dagger \rangle Q_i \sigma_M Q_j = \sum_k Q_k \sigma_M Q_k \quad \forall \sigma_M .
\]
This holds if and only if the unitaries \( V_k \) satisfy
\[
\text{tr}[V_i \sigma_E V_j^\dagger] = \delta_{ij} \quad \forall i,j ,
\]
implying that the unitaries \( V_k \) must form an orthonormal unitary operator basis with respect to the modified scalar product \( (J9) \). Such orthonormal unitary operator bases only exist if the Hilbert space dimension \( d_E \) of the environmental system \( E \) is sufficiently large compared to the number \( K \) of possible outcomes \( k \), i.e. \( d_E \geq \sqrt{K} \). For further properties of unitary operator bases we refer to ref. \( \text{[S4]} \).

Hence, given a full rank state \( \sigma_E \) on \( E \), any unitary \( U_{ME} \) satisfying \( (J4) \) is of the form \( (J7) \) with unitaries \( V_k \) that meet the condition \( (J9) \).

To show that there exists an implementation of the dephasing channel \( (J4) \) with vanishing energy cost \( E_{\text{deph}} \), we may therefore choose the Hamiltonian of the environment \( E \) to be trivial, \( H_E = 0 \), implying that the initially thermal state of \( E \) is maximally mixed for all \( \beta \), \( \sigma_E = \mathbb{1}_E/d_E \), and that the energy cost is \( E_{\text{deph}} = 0 \) by \( (J2) \). The corresponding unitary \( U_{ME} \) that implements the dephasing channel is given by \( U_{ME} = \sum_k Q_k \otimes V_k \) with unitary operators \( V_k \) satisfying \( \text{tr}[V_i V_j^\dagger] = d_E \delta_{ij} \) for all \( i,j \), which can be easily checked by evaluating \( (J4) \)
\[
\text{tr}_E[U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger] = \text{tr}_E \left[ \left( \sum_j Q_j \otimes V_j \right) \left( \sigma_M \otimes \mathbb{1}_E/d_E \right) \left( \sum_k Q_k \otimes V_k^\dagger \right) \right]
= \frac{1}{d_E} \sum_{j,k} Q_j \sigma_M Q_k \text{tr}[V_j V_k^\dagger]
= \sum_k Q_k \sigma_M Q_k .
\]
The unitaries \( V_k \) can for example be chosen as distinct elements from the set of unitaries
\[
V_{l,m} = \sum_{r=0}^{d_E-1} e^{\frac{2\pi i}{d_E} lr} |l + r \rangle \langle r| , \quad l, m = 0, 1, ..., d_E - 1 ,
\]
where the addition in \( |l + r \rangle \) is taken modulo \( d_E \). These \( d_E^2 \) operators can be understood as a discrete version of the Heisenberg-Weyl operators and indeed satisfy, as one can easily compute,
\[
\text{tr}[V_{l,m} V_{s,t}^\dagger] = d_E \cdot \delta_{l,s} \delta_{m,t} \quad \forall l, m, s, t \in \{0, ..., d_E - 1\} .
\]

Appendix K: Work cost of quantum measurements and the 2nd law of thermodynamics

In this section we argue why our results on energy costs developed in this paper are, in fact, also statements about thermodynamic work.

More concretely, we argue that all energy costs of an implementation of a quantum measurement stem from unitary dynamics \( U \) only, so that thermodynamic work cost is given as the average energy change
\[
W = \text{tr}[H(U \rho U^\dagger - \rho)]
\]
of a system with Hamiltonian \( H \) initially in the state \( \rho \). Indeed, the energy expenses of the measurement step \( M \) are due to the unitary \( U_{SM} \) and not the projections \( \{Q_k\} \) as shown in Appendix \( \text{[J]} \) and the resetting step is unitary by construction. Hence, the overall work cost \( W_{\text{cost}} \) of conducting a general quantum measurement is exactly equal to the energy cost \( E_{\text{cost}} \) in our results.

As a consequence of this identification of energy and work, the energy extraction example in Appendix \( \text{[F]} \) illustrates a means to extract useful thermodynamic work by measurement. This may seem intriguing in the context of the 2nd
law of thermodynamics: Discussions on the net work gain in a whole cycle of a Szilard engine typically assume that no work can be extracted in the measurement itself and argue that all work gained in the extraction phase of the cycle is completely cancelled by the cost imposed by Landauer’s principle for resetting the memory that stores the measurement outcome.

We show however that our findings do not contradict the 2nd law of thermodynamics because of the following reasoning: The work cost of a quantum measurement is quantified by the energy expense needed to operate the measuring device and does not include the expense needed to induce the state change on the measured system $S$. When considering the overall work gain of a Szilard engine that employs a measurement device as described in our work extraction example (Appendix F), one needs to incorporate the cost of completing the thermodynamic cycle by restoring the initial pure state on $S$. This restoring step consumes all work gained during the measurement. Indeed, by explicitly incorporating the work cost $W_{\text{cost}} = \text{tr}[H_S(\rho'_S - \rho_S)]$ of the state change on $S$ to the overall work balance, one finds by Theorem 1 and Theorem 2

$$W_S + W_{\text{cost}} = \text{tr}[H_S(\rho'_S - \rho_S)] + E_{\text{cost}}$$
$$= \text{tr}[H_S(\rho'_S - \rho_S)] + k_B T(\Delta S + I + \Delta Q + I_{MB}) + \Delta F_B$$
$$\geq \text{tr}[H_S(\rho'_S - \rho_S)] + k_B T\Delta S$$
$$= \Delta F_S + k_B T(S(\rho'_S) - S(\rho_S)) + k_B T\Delta S$$
$$= \Delta F_S + k_B T \left( S(\rho'_S) - \sum_k p_k S(\rho'_{S,k}) \right)$$
$$\geq \Delta F_S,$$

(K1)

where $\Delta F_S = F_S(\rho'_S) - F_S(\rho_S)$ is the free energy change in the system during the measurement, which corresponds to the work cost of the aforementioned restoring step of the measured system $S$. The inequality (K1) follows from the concavity of the von-Neumann entropy (see (E5)).

This shows that the overall work expense in a full thermodynamic cycle that includes measurements is always non-negative and proves the validity of the 2nd law in our general setting.

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