Black holes, fast scrambling and the breakdown of the equivalence principle

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Black holes are conjectured to be the fastest quantum scramblers in nature, with the stretched horizon being the scrambling boundary. Under this assumption, we show that any infalling body must couple to virtually the entire black hole Hilbert space even prior to the Page time in order for there to be any hope of preserving the often cited claim of the equivalence principle that such bodies should experience ‘no drama’ as they pass a black hole’s horizon. Further, under the scrambling assumption, we recover the usual firewall result at the black hole’s Page time for an initially pure-state black hole without the need for any complexity or computational assumptions. For a black hole that is initially impure we find that the onset of the firewall is advanced to times prior to the standard Page time. Finally, if black holes really do efficiently scramble quantum information, this suggests that, in order to preserve this claim of the equivalence principle even prior to the onset of a full blown firewall, the quantum state of a black hole interior must be a Bose-Einstein condensate.

I. Introduction

Outside a black hole, the physics is largely well understood. For stationary observers, an outgoing flux of radiation is observed. At spatial infinity, this Hawking radiation has a temperature that scales as $T_H \propto O(1/M)$ for a black hole of mass $M$. According to the ‘membrane paradigm,’ a simple thermodynamic argument suggests that stationary observers closer to the black hole should see a blue-shifted flux of this radiation, reaching a universal temperature of roughly one Planck energy (taking the Boltzmann constant as unity) when the stationary observer is roughly one Planck length from the horizon. Similarly, a freely-falling observer, sees an outward flux of radiation for distances larger than $O(3M)$. However due to quantum field renormalization effects, this flux reverses for infalling observers nearer the horizon, exactly vanishing at the horizon itself. This latter result at the horizon is often interpreted as following from the equivalence principle.

However, what about the physics inside the horizon of a black hole? Naively, the equivalence principle should continue to hold and a small infalling observer should continue to notice nothing special until they are torn apart by gravitational stresses as they approach the singularity. On the contrary, calculations based on quantum models suggest that an infalling observer will observe high-energy quanta near the horizon of a black hole which is older than the Page time (an ‘old black hole’). This led to the proposal that the horizon of an old black hole should be replaced by a firewall. The original proof of the firewall paradox\textsuperscript{3} required the infalling observer to extract enough information from the already present outgoing Hawking radiation before reaching the horizon. However, calculations based on quantum computation show that the time to extract this information is generally longer than the lifetime of the black hole, which forms a potential loophole of the original firewall claim.\textsuperscript{7} Though the firewall phenomenon may also be argued for without this extraction assumption.\textsuperscript{8}

Quantum scrambling on the other hand, denotes the dispersion of local quantum information into its neighborhoods and finally the entire system. Within a scrambling time, a quantum state has a random unitary applied to it; and each subsequent scrambling time would lead to the application of a new randomly selected unitary. Thus, a pure quantum state would be mapped to a random pure state, which would be further mapped to a new random pure state with each additional scrambling time. Indeed, the application of this effect is widely studied in the literature on black hole dynamics. By assuming that the radiation from a black hole is always a subsystem of a random pure state, Page proved that the entropy of the radiation will first increase and then decrease.\textsuperscript{9} He assumed that a black hole is a fast scrambler, without providing a concrete realization of it. Following this, Hayden and Preskill, for the first time, explicitly proposed that black holes obey fast random unitary transformations, and showed that old black holes behave as information mirrors.\textsuperscript{10} In their work, Hayden and Preskill argued that the scrambling time of a black hole should scale as $O(\sqrt{S} \log S)$, where $S$ is the entropy of the black hole.\textsuperscript{11} Note that the Page time is $O(M^3)$, while the scrambling time is $O(M \log M)$, implying that for a large (astrophysical) black hole ($M \gg 1$ in appropriate units), the latter is far shorter than the former.

By analyzing the spread of the perturbation on the stretched horizon\textsuperscript{11} of the D0-brane black hole and the ADS black hole, Sekino and Susskind further showed that the scrambling time of a black hole should be $\frac{1}{T_H} \log S$ times a constant, where $T_H$ is the Hawking temperature of the black hole.\textsuperscript{12} This scrambling time approx-
approximately equals that proposed by Hayden and Preskill for black holes far from extremality, which are the ones relevant to astrophysical black holes in nature. To have a sense of this time scale, consider a solar-mass black hole, its scrambling time is only about $10^{-1}$ second, while its Page time is about $10^{72}$ seconds. Indeed, black holes are believed by some to be the fastest scramblers in nature.

Although the original proofs of a black hole firewall require a black hole older than the Page time, Almheiri et al. speculate that the firewall phenomenon might already exist after the very short scrambling time. However, Susskind disagreed with this conjecture and claims that they have mistaken the scrambled state to be the generic state. He argued that while a generic state describes a maximally mixed state which corresponds to infinite temperature, a scrambled pure state will still be a pure state globally. Therefore, according to Susskind, a scrambled pure-state black hole cannot have an infinite temperature as does the firewall phenomenon. In section II we explicitly resolve this controversy with an explicit model of scrambling on a 1+1-dimensional lattice quantum field.

In section III, we review some basic concepts about black holes and quantum fidelity and then in section IV, we show that any sufficiently small neighborhood of a scrambled black hole will be infinitesimally close to a maximally mixed quantum state. This implies that locally an infalling observer will experience a high temperature as they pass the horizon. We allow for arbitrary amounts of emitted radiation and a black hole which may be initially pure or mixed. Finally, in section V we summarize our conclusions.

II. Local temperature of a scrambled lattice quantum field

Here we study an explicit model of scrambling. Consider a quantized real massless scalar field on a spatial lattice in 1+1-dimensional Minkowski spacetime, with lattice spacing $\delta$. The Hamiltonian takes the form (see Appendix A)

$$H = \frac{1}{2\delta} \left( \vec{\pi}^T \cdot \vec{1}_N \cdot \vec{\pi} + \vec{\phi}^T \cdot V \cdot \vec{\phi} \right) ,$$

where the field $\phi_i$ at each lattice site $i$ allows us to form the $N$-dimensional vector $\vec{\phi} = (\phi_1, \cdots, \phi_N)$ with conjugate momentum $\vec{\pi} = (\pi_1, \cdots, \pi_N)$ satisfying $[\phi_i, \pi_{i'}] = i\delta_{i,i'}$, with natural units so that $\hbar = 1$. The interaction matrix $V$ here given by

$$V = \frac{1}{4} \begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & \ddots & 0 \\
0 & 0 & \ddots & \ddots & -1 \\
0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix} .$$

The ground state of this Hamiltonian is given by

$$\Psi = | N \rangle \exp \left( -\frac{1}{2} \vec{\phi}^T \cdot \sqrt{V} \cdot \vec{\phi} \right) ,$$

see Appendix A. We shall be considering a very much reduced space of random unitaries which correspond simply to random permutations among the lattice sites. This allows us to easily construct the ‘scrambled’ ground state by the replacement $\vec{\phi} \rightarrow P \vec{\phi}$ in Eq. (3), for a random permutation operator $P$.

![FIG. 1: Cumulative distribution for 100 sampled energies seen by an observer locally coupled the scrambled ground state of a massless scalar field at a single lattice point (solid) on a lattice with $N = 300$ sites. The fit shown is for $k_B T \delta = 0.0205$ against the cumulative Boltzmann distribution $1 - e^{-E/k_B T}$ (dashed).]

![FIG. 2: Mean temperature $k_B T$ seen by an observer locally coupled to the scrambled ground state of a massless scalar field at a single lattice point within a lattice of $N$ sites. This local temperature appears to asymptote to a value approaching the cutoff scale $O(1/\delta)$.]

Consider some local observer weakly coupled to our field at a single lattice site $i$. We can now ask what will be the expected energy seen by such a local observer when coupled to the scrambled ground state above the actual ground state. Although each permutation yields a single expected energy, we may consider the distribution of these energies across random permutations. Each successive scrambling time induces a new random permutation and consequently a new energy seen by our local observer. By fitting this distribution to the Boltzmann distribution we may extract a local temperature $T$ as a function of lattice size $N$ (see for example, Fig. 1). Numerical results of randomly scrambled lattices of up to $N = 300$ are shown in Fig. 2.

We find that as the number of lattice sites becomes large, the temperature seen by an observer locally coupled to our scrambled scalar field approaches the cutoff...
scale $O(1/\delta)$; or within an order-of-magnitude or two of that scale. Our simplified model of scrambling involves only the permutation of lattice sites, rather than a full blown random unitary applied to the Hilbert space of interest.

We note that our local observer here is an inertial observer, as our analysis is within 1+1-dimensional Minkowski spacetime. Therefore despite the equivalence principle, unitary scrambling of the degrees-of-freedom within some spatial patch will produce a noticeable (pre-principle, unitary scrambling of the degrees-of-freedom). The quantum state of this subsystem is thereby unitary, its state after a unitary transformation can be written as

$$\rho_0 \rightarrow U \rho_0 U^\dagger \equiv \rho_U$$

where $U$ is a unitary operator, i.e., $U^\dagger = U^{-1}$.

Next, consider a small particle (compared to the black hole) falling into this black hole. It passes the horizon and interacts with a small neighborhood surrounding it within the black hole. The quantum state of this neighborhood, which is a tiny subsystem of the black hole, may be obtained by tracing out the other degrees of the black hole. If we assume that the dimension of this subsystem is $n$, its state may be expressed as $\text{tr}_n(\rho_U)$, where $\hat{n}$ represents the degrees-of-freedom of the state complementary (i.e., orthogonal) to degrees-of-freedom contained within this tiny subsystem.

Since we assume that a black hole is a fast scrambler, we shall assume that the particle arrives at the black hole horizon after the scrambling time, which is believed to be quite small for any reasonably sized black hole.13 At this stage, the state of the black hole may be calculated as the average over all the unitary transformations. Under these conditions, we will prove that the local quantum state 'observed' by the particle (i.e., with which it directly interacts) is almost a maximally mixed state, hence having an almost infinite temperature. To show this, we will calculate the fidelity $F$ between $\text{tr}_n(\rho_U)$ and $\frac{\hat{n}}{n}$ after the scrambling time. In this paper, we define the fidelity of two states characterized by the density matrices $\rho$ and $\sigma$ as $F(\rho, \sigma) = \sqrt{\text{tr} \rho \sigma}$. It can be shown that $F(\rho, \sigma) = F(\sigma, \rho)$. It can also be shown that the fidelity of two states $\rho$ and $\sigma$ satisfies the inequalities13

$$1 - \frac{1}{2} \| \rho - \sigma \|_1 \leq F(\rho, \sigma) \leq 1 - \frac{1}{4} \| \rho - \sigma \|_1^2,$$  

where the Schatten p-norm of $A$ is defined as $\| A \|_p = (\text{tr}(A^A)^{p/2})^{1/2}$. Since $\rho - \sigma$ is Hermitian, the Cauchy-Schwarz inequality implies13

$$\| \rho - \sigma \|_1 \leq \| \rho \|_2 \times \| \rho - \sigma \|_2,$$  

where $\| \cdot \|_2$ has the same dimensionality as $\rho$ and $\sigma$. Applying this Cauchy-Schwarz inequality and $\| \rho - \sigma \|_1 \leq \| \rho - \sigma \|_2$ to Eq. (5) yields

$$1 - \frac{1}{2} \| \rho - \sigma \|_2 \leq F(\rho, \sigma) \leq 1 - \frac{1}{4} \| \rho - \sigma \|_2^2. \quad (6)$$

Since we would like to study the fidelity between $\text{tr}_n(\rho_U)$ and $\frac{\hat{n}}{n}$ after scrambling, we need to insert these two matrices into Eq. (6). Under the assumption of fast scrambling, the mean fidelity is averaged over all unitary operators of the state of the black hole. We obtain the following two relations

$$F(\text{tr}_n(\rho_U), \frac{\hat{n}}{n}) \leq 1 - \frac{1}{4} \int_U \| \text{tr}_n(\rho_U) - \frac{\hat{n}}{n} \|_2^2 dU, \quad (7)$$

and

$$F(\text{tr}_n(\rho_U), \frac{\hat{n}}{n}) \geq 1 - \frac{1}{2} \int_U \| \text{tr}_n(\rho_U) - \frac{\hat{n}}{n} \|_2 dU = 1 - \frac{1}{2} \sqrt{n} \int_U \| \text{tr}_n(\rho_U) - \frac{\hat{n}}{n} \|_2 dU. \quad (8)$$

IV. Temperatures of different black holes

Since the scrambling time is very short, when we consider a newly formed black hole in a pure quantum state, yet after its scrambling, we may ignore the effect of any radiation. For this scenario, the fidelity relation in Eqs. (7) and (8) may be simplified to (see Appendix C)

$$1 - \frac{n}{2\sqrt{N}} \leq F(\text{tr}_n(\rho_U), \frac{\hat{n}}{n}) \leq 1 - \frac{n^2 - 1}{4(N + 1)n}. \quad (9)$$

Recall that for a stellar-mass black hole $N \sim \exp(10^{80})$. Therefore, the upper bound in Eq. (9) is not very interesting since it just means the fidelity is slightly less than one. However, the lower bound is important since it determines how close the local subsystem encountered by an infalling particle is to a maximally mixed state. Since a maximally mixed quantum state corresponds to an infinity high temperature (see Appendix D for a detailed analysis), this means that the infalling object will experience a very high temperature.

To have some idea about the exact value of this temperature, we now try to write out the expression of this quantum state. The Fidelity $F(\rho, \sigma)$ of a pair of states $\rho$ and $\sigma$ is the maximum overlap over all purifications $|\psi_\rho\rangle$ and $|\psi_\sigma\rangle$, respectively, of these states, i.e.,

$$F \equiv F(\rho, \sigma) = \max_{\psi_\rho, \psi_\sigma} \langle \psi_\rho | \psi_\sigma \rangle. \quad (10)$$

Consequently, there exist purifications $|\psi_\rho\rangle$ and $|\psi_\sigma\rangle$ which satisfy13

$$|\psi_\rho\rangle = F|\psi_\sigma\rangle + \sqrt{1 - F^2} |\psi_\sigma\rangle, \quad (11)$$
where \( |\psi\rangle \) is some quantum state orthogonal to \( |\psi\rangle \). Taking the partial trace of this pure-state representation yields
\[
\rho = F^2 \sigma + O(\sqrt{1 - F^2}), \quad \text{for} \quad 1 - F^2 \ll 1. \quad (12)
\]
Applying this result to Eq. (9), we find that typically
\[
\text{tr}_n(\rho_U) = (1 - \epsilon) \frac{\sqrt{n}}{n} + O(\sqrt{\epsilon}),
\]
for \( \epsilon \equiv 1 - F^2 \leq \frac{n}{\sqrt{N}} \ll 1. \quad (13)
\]
Thus, for example, for a stellar-mass black hole this reduced state of the local neighborhood that our particle interacts with will be in a quantum state exceedingly close to a completely mixed state — a state with infinite temperature — provided only that \( n \ll \sqrt{N} \). (see Appendix D for an analysis for how one can estimate the temperature more precisely if we knew the Hamiltonian describing the black hole system.)

Now we consider the above black hole that has radiated a non-negligible amount of itself away. If we assume the dimension of the radiation is \( R \), then the dimension of the remaining black hole will be \( N_R = N/R \). The quantum state of the remaining black hole may be represented by \( \text{tr}_R(\rho_U) \). For this scenario, the fidelity relations in Eqs. (7) and (8) equal (see Appendix E)
\[
1 - \frac{n}{2\sqrt{N}} \leq F \leq 1 - \frac{n^2 - 1}{4(N + 1)n}. \quad (14)
\]
Note that Eq. (14) is identical to Eq. (9). So again the quantum state of any local neighborhood encountered by an infalling object typically has the form
\[
\text{tr}_n(U_B \text{tr}_R(\rho_U) U_B^\dagger) = (1 - \epsilon) \frac{\sqrt{n}}{n} + O(\sqrt{\epsilon}), \quad (15)
\]
where \( 0 \leq \epsilon \leq \frac{n}{\sqrt{N}} \ll 1 \). Therefore, infalling objects will observe a very high temperature (see Appendix D for how to estimate the temperature). We emphasize that this result does not require the black hole to radiate half of itself away, which means that this phenomenon may occur much earlier than the Page time.

In the above analysis we have assumed that the initial state of the black hole is a pure state. However, it may seem natural that the quantum state of an astrophysical black hole does not begin as a pure state. Therefore, now we consider a black hole that begins from a generic quantum state \( \rho_0 \) with Hilbert space dimension \( N \). If it is a newly formed black hole with negligible radiations, the fidelity relations in Eqs. (7) and (8) will become (see Appendix F)
\[
1 - \frac{n}{2N} \sqrt{N \text{tr}(\rho_0^2) - 1} \leq F \leq 1 - \frac{(N \text{tr}(\rho_0^2) - 1)(n^2 - 1)}{4(N^2 - 1)n}. \quad (16)
\]
However, identical bounds are found when radiation is allowed for (see Appendix G). Therefore Eq. (16) represents the generic result, with \( N \) the dimensionality of the original black hole, or equivalently the product of the dimensionality of the current state of the black hole and that of the radiation.

Since \( 1/N \leq \text{tr}(\rho_0^2) < 1 \) for any impure quantum state, the lower bound to the fidelity for a black hole originating from a non-pure state is larger than that for the pure-state scenarios studied above. Consequently, the reduced state of a neighborhood within an initially impure black hole will be even closer to being a maximally mixed state than for an initially pure state black hole.

Again, the local quantum state with which an infalling object interacts may be written as typically given by
\[
\text{tr}_n(U \rho_0 U^\dagger) = (1 - \epsilon) \frac{\sqrt{n}}{n} + O(\sqrt{\epsilon}), \quad (17)
\]
where \( 0 \leq \epsilon \leq \frac{n}{N} \sqrt{N \text{tr}(\rho_0^2) - 1} \leq n/\sqrt{N} \ll 1 \). Therefore an infalling observer will typically experience even higher temperatures (see Appendix D) as impurity of the quantum state of the initial black hole is increased.

\section*{V. Discussion}

Black holes are conjectured to be fast scramblers; possibly even the fastest scramblers in the universe.\cite{10} The scrambling process itself is conceived of as a random unitary operation on the black hole interior Hilbert space. Across the black hole’s entire lifetime during any time \( O(M \log M) \), for a black hole of mass \( M \), a random unitary operates on the black hole’s interior Hilbert space. This time is called the scrambling time.

For simplicity, let us suppose the black hole interior is initially pure and let us ignore any evaporation process or new material being added to the black hole. The question arises as to whether the behavior of the quantum state of the black hole should be treated as an ensemble average over the random unitaries\cite{11} or as a single (though randomly selected) pure state\cite{12}. In the former case, the reduced state of a sufficiently small subsystem would appear to be the generic maximally mixed state corresponding to an infinite temperature.\cite{13} In the latter case, it has been argued that a random pure state is still pure and therefore has zero associated temperature.\cite{14}

We have addressed this controversy head on in section II. There we study an explicit model of a 1+1-dimensional quantum field on a lattice undergoing random lattice site permutations (a highly restricted class of random unitary operations on the Hilbert space of the quantum field). We compute the energy of an observer weakly coupled to a single lattice site for a quantum field initially in the ground (vacuum) state. Each random unitary yields a distinct energy above the ground state for this observer. After each additional scrambling time a new random unitary will cause our local observer to experience a new local energy. As the number of scrambled lattice sites increases, the distribution of energies experienced by our local observer is found to be well approximated by a Boltzmannian distribution. At each point in
time the global quantum state is pure, nevertheless, local behavior is correctly described by the ensemble statistics of the random unitaries (in this case permutations of lattice sites). The specific temperature found in this analysis seen by an inertial (freely falling) observer is found to approach the natural cutoff scale of that model, but depends in detail on the choice of random operations, and the underlying Hamiltonian.

To overcome these model-dependent limitations, we consider a more general, information theoretic, approach in section IV (with the methods used given in section III). This allows us to take into account both the inclusion of black hole evaporation but also an initial black hole state which may be anywhere from completely pure to maximally mixed. We find that if an observer is only coupled to a sufficiently small ‘neighborhood’ of the entire interior Hilbert space they will experience a maximally mixed state to a close approximation and therefore a near infinite temperature. We find that a neighborhood is sufficiently small to achieve this high-temperature behavior provided only that it’s Hilbert space dimensionality, \( n \), satisfies

\[
n \leq \varepsilon \sqrt{N}, \tag{18}
\]

where \( N \) is the dimensionality of the newly formed pure state black hole. (A looser bound is found when the initial back hole is impure which only strengthens our discussion below, see section IV for details.) The prefactor satisfies \( \varepsilon \ll 1 \); for simplicity we take \( \varepsilon = 2^{-10} \approx 10^{-3} \), though using values of \( 2^{-100} \) or \( 2^{-1000} \) makes only trivial changes to our discussion below.

In particular, rather than quantifying the size of the neighborhood in terms of its dimensionality, it is more physically intuitive to quantify it in terms of qubits. Note that we are not saying that any part of the black hole is actually made up of two-level systems, only that the number of two-level systems that could be supported by its dimensionality is a more familiar quantity — analogous to entropy the number of qubits is additive. So the total dimensionality is a more familiar quantity — analogous to values of \( 2^{\# \text{qubits}} \) for a stellar mass black hole.

Consider an initially pure-state black hole that has partially evaporated leaving a fraction, \( f \), of the qubits in the current state of the black hole, which therefore has

\[
\#_{\text{BH}} = f \#_{\text{newly-formed-black-hole}}, \tag{20}
\]

qubits remaining (or equivalently, it’s area has shrunk to this fraction of its original size). Then Eq. (19) becomes

\[
\#_{\text{neighborhood}} \leq \frac{1}{2f} \#_{\text{BH}} - 10. \tag{21}
\]

Thus, as \( f \to \frac{1}{2} \), (i.e., as the evaporation approaches the Page time), any infalling body must simultaneously couple to virtually the entire black hole interior to have any hopes of experiencing no drama has it passes the horizon. From the Page time onwards, even this is not sufficient and we recover the usual firewall result, though without the need for any decoding or complexity assumption.

In order to preserve the claim of ‘no drama’ for scrambling black holes even prior to the onset of a full blown firewall. Any infalling body must couple to virtually the entire interior Hilbert space of the black hole. Further, it must do so in a uniform manner without random phases appearing in the coupling whatever the direction of the infalling body. We can envisage only one scenario where this is possible: that the quantum state of the black hole interior is actually described by a Bose-Einstein condensate, so that all interior degrees-of-freedom correspond to excitations of a single (or small number of) Bose-Einstein condensate modes. However, we note, that even with this radical assumption, the onset of the firewall is not affected.

If the black hole’s quantum state is initially partially mixed the constraints on satisfying the equivalence principle become even more extreme. It is convenient to define the log-purity of the initial black hole as

\[
\ell \equiv -\log_2 (\text{tr}(\rho_0^2) - 1/N) \geq 0, \tag{22}
\]

where \( \rho_0 \) is the initial black hole’s density matrix and \( N \) its dimensionality. Black hole scrambling therefore implies that a fully developed firewall will be present once the black hole’s area has shrunk to the fraction

\[
f = \frac{1}{2 - \ell/\#_{\text{BH}}} \geq \frac{1}{2}, \tag{23}
\]

of its original size. In other words, for a black hole with an initially partially mixed quantum state, the firewall becomes fully developed prior to the Page time.
Appendix A

1. Multivariate Gaussian integral

Before calculating the energy difference between the ground state and the scrambled ground state, we first review some results about the multivariate Gaussian integral that we will use later.

For single variable case, we know that
\[
\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}.
\]

Then, the multivariate Gaussian integral with linear term may be calculated as
\[
\int e^{-\frac{1}{2}x^T A x + J^T \vec{x}} d^n \vec{x} = \sqrt{(2\pi)^n} \det A. \quad (26)
\]

where the diagonal matrix allows us to treat this integral as multiple single-variable Gaussian integral times each other.

Then we try to derive another useful result about the multivariate Gaussian integral.

\[
\int x_i x_j B_{ij} e^{-\frac{1}{2} x^T A x + J^T \vec{x}} d^n \vec{x} = B_{ij} \left( \int e^{-\frac{1}{2} x^T A x} d^n \vec{x} \right) \delta_{ij} = B_{ij} \sqrt{(2\pi)^n} \det A. \quad (27)
\]

If we assume \( \vec{J} = 0 \) in Eq. (27), then we will obtain
\[
\int e^{-\frac{1}{2} x^T A x} d^n \vec{x} = \sqrt{\frac{(2\pi)^n}{\det A}} \operatorname{tr}(BA^{-1}). \quad (28)
\]

2. Massless scalar field based on the lattice representation

The Lagrangian of the massless scalar field in a \((n+1)\)-dimensional Minkowski space can be written as
\[
L = \int d^n x d\tau \mathcal{L} = \int d^n x d\tau \left( \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) = \frac{1}{2} \int \left( \partial_\mu \phi \partial_\mu \phi - \sum_{i=1}^{n} \partial_{x_i} \phi \partial_{x_i} \phi \right) d^n x d\tau, \quad (29)
\]

where \( \eta^{\mu\nu} \) is the Minkowski metric. If we define the momentum as \( \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} = \dot{\phi} \), the Hamiltonian of the massless scalar field at a given time equals
\[
H = \int \left( \dot{\phi} \pi - \mathcal{L} \right) d^n x = \frac{1}{2} \int \left( \pi^2 + \sum_{A=1}^{n} (\partial_{x_i} \phi)^2 \right) d^n x. \quad (30)
\]

To apply scrambling to such a scalar field, we assume the coordinates \( \{x^A\} \) as a (discontinuous) lattice labeled by \( \{x^A_i\} \). Then, we may define
\[
\int_{x^A_i}^{x^A_i+1} \cdots \int_{x^B_j}^{x^B_j+1} \phi(x^A)d^n x = \phi(x^A) \delta^n = \delta_{i\cdots j} \delta^n, \quad (31)
\]

where \( \delta = x^A_{i+1} - x^A_i = \cdots = x^B_j - x^B_{j-1} \) is the lattice spacing. For simplify, we may write \( \int_{x^A_i}^{x^A_i+1} \cdots \int_{x^B_j}^{x^B_j+1} \) as \( \int_{\Delta} \). Similarly, the momentum may be written as
\[
\int_{\Delta} \pi(x^A) d^n x = \pi(x^A) \delta^n = \pi_{i\cdots j} \delta^b. \quad (32)
\]

With these constructions, the canonical commutation relation may be calculated as
\[
\left[ \int_{\Delta} \phi(x^A) d^n x, \int_{\Delta} \pi(x^A) d^n x' \right] = \delta^{ab} \left[ \phi_{i\cdots j}, \pi_{i'\cdots j'} \right]. \quad (33)
\]

On the other hand, the canonical commutation relation also equals
\[
\left[ \int_{\Delta} \phi(x^A) d^n x, \int_{\Delta} \pi(x^A) d^n x' \right] = \int_{\Delta} d^n x d^n x' \left[ \phi(x^A), \pi(x^A) \right] = i \delta_{i\cdots j} \cdots \delta_{i'\cdots j'} \delta^n. \quad (34)
\]

Comparing Eqs. (33) and (34) yields
\[
[\phi_{i\cdots j}, \pi_{i'\cdots j'}] = i \delta_{i\cdots j} \cdots \delta_{i'\cdots j'}, \quad (35)
\]
providing \(a + b = n\). Inserting \(a + b = n\) into Eqs. (31) and (32) yields
\[
\phi(x^A) = \delta^{a-n}\phi_{i \cdots j} \quad \text{and} \quad \pi(x^A) = \delta^{-a}\pi_{i \cdots j}.
\]

(36)

With the above results, the Hamiltonian on the lattice may be written as
\[
H = \frac{1}{2} \int \left( \pi^2 + \sum_{A=1}^{n} (\partial_x \phi_A)^2 \right) d^nx
\]
\[
= \frac{1}{2} \delta \sum_{i \cdots j} \left( \delta^{a-n} \pi^2_{i \cdots j} + \frac{\delta^2(a-n)}{4\delta^2}(\phi_{i+1 \cdots j} \phi_{i-1 \cdots j})^2 \cdots \right).
\]

(37)

Since each term in Eq. (37) should have the same power of \(\delta\), we have \(-2a = 2a - 2n - 2 \Rightarrow a = \frac{n+1}{2}, b = \frac{n-1}{2}\). Therefore, the Hamiltonian becomes
\[
H = \frac{1}{2\delta} \sum_{i \cdots j} \left( \pi^2_{i \cdots j} + \frac{1}{4}(\phi_{i+1 \cdots j} \phi_{i-1 \cdots j})^2 \cdots \right).
\]

(38)

In this work, we consider \(n = 1\) as an example, and study of higher dimensions should be similar. We then have
\[
H = \frac{1}{2\delta} \sum_{i=1}^{N} \left( \pi^2_i + \frac{1}{4}(\phi_{i+1} - \phi_{i-1})^2 \right)
\]
\[
= \frac{1}{2\delta} \sum_{i=1}^{N} \left( \pi^T \phi^i + V_{ij} \phi_i \phi_j \right)
\]
\[
= \frac{1}{2\delta} \left( \pi^T \Pi_N + \phi^T \cdot V \cdot \phi \right),
\]

(39)

where
\[
\bar{\phi} = (\phi_1, \cdots, \phi_N),
\]

(40)

\[
V_{ij} = \frac{1}{4} \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix},
\]

(41)

and \(\Pi_N\) is the \((N \times N)\) identity matrix.

3. Vacuum state, scrambled vacuum state and their energy

The ground state of the above scalar field in the lattice representation is written as a Gaussian as follows
\[
\Psi = \mathcal{N} e^{-\frac{1}{2} \phi^T \cdot V \cdot \phi},
\]

(42)

where \(Q\) is a \((N \times N)\) matrix to be determined below. Recall that the canonical commutation relation \([x_i, \pi_j] = i\delta_{ij}\) implies \(\pi_j = -i\frac{\partial}{\partial x_j}\). Therefore, Eq. (39) can be written as
\[
H = \frac{1}{2\delta} \left( -\delta_{ij} \frac{\partial}{\partial \phi_i} \frac{\partial}{\partial \phi_j} + \phi^T \cdot V \cdot \phi \right).
\]

(43)

Applying this Hamiltonian operator to the ground state yields
\[
H \Psi = \mathcal{N} \left( -\delta_{ij} \frac{\partial}{\partial \phi_i} \frac{\partial}{\partial \phi_j} + \phi^T \cdot V \cdot \phi \right) e^{-\frac{1}{2} \bar{\phi}^T \cdot Q \cdot \bar{\phi}}
\]
\[
= \mathcal{N} \left( Q_{ii} - \delta_{ij} Q_{ik} \phi_k Q_{jk} \phi_i + \phi^T \cdot V \cdot \phi \right) e^{-\frac{1}{2} \bar{\phi}^T \cdot Q \cdot \bar{\phi}}
\]
\[
= \frac{1}{2\delta} (\text{tr} Q + \phi^T \cdot (V - Q^2) \cdot \phi) \Psi,
\]

(44)

where a similar technique as in Eq. (27) has been used in going from the first to the second line.

Now since Eq. (42) is an eigenstate of \(H\), it follows from Eq. (44) that \(Q = \sqrt{V}\). Thus, the ground state simply becomes
\[
\Psi = \mathcal{N} e^{-\frac{1}{2} \bar{\phi}^T \cdot \sqrt{V} \cdot \bar{\phi}},
\]

(45)

with the corresponding ground state energy
\[
E_0 = \frac{1}{2\delta} \text{tr} \sqrt{V}.
\]

(46)

To normalize the ground state, we require \(\int \Psi^2 d^N \phi = 1\), i.e.,
\[
1 = \mathcal{N}^2 \int e^{-\frac{1}{2} \bar{\phi}^T \cdot 2\sqrt{V} \cdot \bar{\phi}} d^N \phi = \mathcal{N}^2 \sqrt{\frac{(2\pi)^N}{\det (2\sqrt{V})}}
\]
\[
= \mathcal{N}^2 \sqrt{\frac{\pi^N}{\det \sqrt{V}}},
\]

(47)

where we have used Eq. (26). This implies
\[
\mathcal{N} = \left( \det V \right)^{\frac{1}{2}} \frac{1}{\pi^{\frac{N}{2}}}. \]

(48)

Next, if we randomly permute (scramble) the scalar field \(\phi_i\) at the lattice points, such that \(\bar{\phi} \rightarrow P \bar{\phi}\), where \(P\) is the permutation operator, and repeat the above analysis, we obtain similar results. Namely, the scrambled Hamiltonian becomes
\[
H = \frac{1}{2\delta} \left( \pi^T \Pi_N \phi^i \cdot K \cdot \phi^j \right),
\]

(49)

where \(K = PV^TP\) is the new potential matrix in the randomly permuted \(\bar{\phi}\). The scrambled ground state is now
\[
\Psi = \left( \det K \right)^{\frac{1}{2}} \frac{1}{\pi^{\frac{N}{2}}} e^{-\frac{1}{2} \bar{\phi}^T \cdot \sqrt{K} \cdot \bar{\phi}},
\]

(50)

with the ground state energy
\[
E_0' = \text{tr} \sqrt{K} = \frac{1}{2\delta} \text{tr} \sqrt{P \sqrt{V} \sqrt{P} \sqrt{V} \sqrt{P}} = \frac{1}{2\delta} \text{tr} \sqrt{V}. \]

(51)
which is identical to Eq. (46). In fact, scrambling the Hamiltonian and ground state simultaneously is akin to using a different coordinate system to describe the same physics, such that the energy does not change. Therefore to measure the scrambling effect, we need to calculate the mean value of the original Hamiltonian Eq. (39) in the scrambled ground state Eq. (50):

\[ E = \langle H \rangle = \int (\Psi^* H \Psi) d^N \phi \]

\[ = \frac{N^2}{2\delta} \int \left( \pi^T \pi + \phi^T V \phi \right) e^{-\frac{1}{2} \phi^T \kappa \phi} d^N \phi \]

\[ = \frac{N^2}{2\delta} \int \left( \text{tr} \sqrt{K} + \phi^T (V - K) \phi \right) e^{-\frac{1}{2} \phi^T \kappa \phi} d^N \phi \]

\[ = \frac{1}{2\delta} \left( \text{tr} \sqrt{K} + \text{tr} [(V - K)(2\sqrt{K})^{-1}] \right) \]

\[ = \frac{1}{2\delta} \left( \text{tr} \sqrt{V} + \frac{1}{2} \text{tr} [(V - K)(\sqrt{K})^{-1}] \right) \] (52)

where Eq. (28) has been used in moving from the third to the fourth line.

Therefore, the energy difference before and after scrambling equals

\[ \Delta E = E - E_0 = \frac{1}{4\delta} \text{tr} [(V - K)(\sqrt{K})^{-1}] , \] (53)

where we have used Eqs. (46) and (52).

It follows from Eq. (53) that we can numerically simulate the change \( \Delta E \) with respect to the number of contiguous sites being scrambled, see Fig. 1 in the manuscript.

Appendix B

Let us first review a result of the Schur-Weyl duality. Suppose we have a Hermitian \( X \), then Schur-Weyl duality implies that\(^{19}\)

\[ \int_{U(A)} \left( U^\dagger \otimes U^\dagger \right) X (U \otimes U) dU = \alpha_+ \Pi_+^A + \alpha_- \Pi_-^A \] (54)

where

\[ \alpha_\pm = \frac{\text{tr} \left( X \Pi_\pm^A \right)}{\text{rank} (\Pi_\pm^A)} \text{ and } \Pi_\pm^A = \frac{1}{2} (1_{A,A} \pm \text{SWAP}_{A,A}) . \] (55)

Here \( \text{SWAP}_{A,B} \) represents the SWAP operator between the system \( A \) and \( B \).

Now suppose that \( A = A_1 \otimes A_2 \), which is what we will use in this work, then we have

\[ \text{rank} \Pi_\pm^A = \text{rank} \frac{1}{2} (1_{A,A} \pm \text{SWAP}_{A,A}) \]

\[ = \frac{1}{2} (d_{A_1}^2 + d_{A_2}^2) \]

\[ = \frac{1}{2} (d_{A_1} d_{A_2})^2 + d_{A_1} d_{A_2} , \] (56)

where \( d_X \) equals the dimension of the state \( X \). If we suppose \( X \) equals \( \text{SWAP}_{A_2,A_2} \), then \( \text{tr} \left( X \Pi_\pm^A \right) \) in Eq. (55) may be calculated as

\[ \text{tr} \left( \Pi_\pm^A \text{SWAP}_{A_2,A_2} \right) = \frac{1}{2} \text{tr} \left( [1_{A,A} \pm \text{SWAP}_{A_1,A_1} \otimes \text{SWAP}_{A_2,A_2}] \text{SWAP}_{A_2,A_2} \right) \]

\[ = \frac{1}{2} \left( \text{tr} \left( 1_{A,A} \otimes \text{SWAP}_{A_2,A_2} \right) \pm \text{tr} \left( \text{SWAP}_{A_1,A_1} \otimes 1_{A_2,A_2} \right) \right) \]

\[ = \frac{1}{2} (d_{A_1} d_{A_2} + d_{A_1} d_{A_2}) . \] (57)

Inserting Eqs. (55), (56) and (57) into Eq. (54) yields

\[ \int_{U(A)} \left( U^\dagger \otimes U^\dagger \right) X (U \otimes U) dU = \frac{d_{A_1}^2 d_{A_2}^2 + d_{A_1} d_{A_2}^2 - d_{A_1} d_{A_2}}{(d_{A_1} d_{A_2})^2 + d_{A_1} d_{A_2}} \Pi_+^A + \frac{d_{A_1}^2 d_{A_2} - d_{A_1} d_{A_2}^2}{(d_{A_1} d_{A_2})^2 - d_{A_1} d_{A_2}} \Pi_-^A \]

\[ = \frac{d_{A_1} + d_{A_2}}{d_{A_1} - d_{A_2}} \Pi_+^A + \frac{d_{A_1} - d_{A_2}}{d_{A_1} + d_{A_2}} \Pi_-^A \]

\[ = \frac{d_{A_1} + d_{A_2}}{d_{A_1}} \Pi_+^A + \frac{d_{A_1} - d_{A_2}}{d_{A_2}} \Pi_-^A \] (58)

Appendix C

Since the scrambling time is very short, a newly formed black hole in a pure state, yet after its scrambling, should have negligible radiations. For such a black hole, the two norm \( ||\text{tr}_n(\rho_U) - \frac{1}{d} ||_2^2 \) that appeared in the fidelity relation
Eqs. 7 and 8 in the manuscript may be calculated as

\[
\int_U \left\| \operatorname{tr}_n(\rho_U) - \frac{1}{n} \right\|^2 dU = \int_U \operatorname{tr}_n \left[ \operatorname{tr}_n(\rho_U) \operatorname{tr}_n(\rho_U) - \frac{2}{n} \operatorname{tr}_n(\rho_U) + \frac{1}{n^2} \right] dU
\]

\[
= \int_U \operatorname{tr}_n \left[ \operatorname{tr}_n(\rho_U) \operatorname{tr}_n(\rho_U) \right] dU - \frac{1}{n}
\]

\[
= \int_U \operatorname{tr}_n \left[ \operatorname{tr}_n(\rho_U) \operatorname{tr}_n(\rho_U) \right] dU - \frac{1}{n}
\]

\[
= \int_U \operatorname{tr}_n \left[ \operatorname{tr}_n(\rho_U) \operatorname{tr}_n(\rho_U) \right] dU - \frac{1}{n}
\]

\[
= \int_U \operatorname{tr} \left[ (U \otimes U)(\rho \otimes \rho)(U^\dagger \otimes U^\dagger)(\operatorname{SWAP}_{n,n} \otimes \operatorname{SWAP}_{n,n}) \right] dU - \frac{1}{n},
\]

where \(\|A\|^2 = \operatorname{tr}(A^\dagger A)\) is applied to the first step, and \(\operatorname{tr}(AB) = \operatorname{tr}(A \otimes B \operatorname{SWAP}_{A,B})\) is applied in moving from the second to the third line. Here \(\operatorname{SWAP}_{A,B}\) is the SWAP operator between the quantum subsystems \(A\) and \(B\). Applying the Schur-Weyl duality in Appendix B to Eq. \((59)\) then yields

\[
\int_U \left\| \operatorname{tr}_n(\rho_U) - \frac{1}{n} \right\|^2 dU = \frac{1}{2} \operatorname{tr} \left[ (\rho \otimes \rho) \left( \frac{N + n}{N+1} (1_{N,N} + \operatorname{SWAP}_{N,N}) + \frac{N - n}{N-1} (\mathbb{I}_{N,N} - \operatorname{SWAP}_{N,N}) \right) \right] - \frac{1}{N}
\]

\[
= \int_U \operatorname{tr} \left[ (\rho \otimes \rho) \left( \frac{N^2 - n}{N^2 - 1} 1_{N,N} + \frac{Nn - N}{N^2 - 1} \operatorname{SWAP}_{N,N} \right) \right] - \frac{1}{N}
\]

\[
= \frac{N^2 - n}{N^2 - 1} \operatorname{tr} \rho^2 + \frac{Nn - N}{N^2 - 1} \operatorname{tr} \rho^2 - \frac{1}{n} = \frac{N^2 - n}{N^2 - 1} + \frac{Nn - N}{N^2 - 1} - \frac{1}{n}
\]

\[
= \frac{Nn^2 - N - n^2 + 1}{(N^2 - 1)n} = \frac{n^2 - 1}{(N+1)n} = \frac{1}{N+1} \left( n - \frac{1}{n} \right).
\]

(60)

where \(\operatorname{tr} \rho = \operatorname{tr} \rho^2 = 1\) is used in the third line because the entire black hole is a pure quantum state. Since \(N \geq n \geq 1\), Eq. \((60)\) may be further simplified as

\[
\int_U \left\| \operatorname{tr}_n(\rho_U) - \frac{1}{n} \right\|^2 dU < \frac{n}{N}.
\]

(61)

Then the fidelity relation in Eqs. 7 and 8 may be written

\[
1 - \frac{n}{2\sqrt{N}} \leq F \left( \operatorname{tr}_n(\rho_U), \frac{1}{n} \right) \leq 1 - \frac{n^2 - 1}{4(N+1)n}.
\]

(62)

Appendix D

In statistical mechanics, a Boltzmann distribution describes the probability distribution of a system for different microstates \(i\) with respect to the state’s energy \(E_i\) and temperature of the system \(T\). The distribution may be written

\[
p_i = \frac{1}{Z} e^{-E_i/(k_B T)},
\]

(63)

where function \(Z\) is used to normalized the distribution and hence equals the sum of \(e^{-E_i/(k_B T)}\) for all different \(E_i\).

The generalization of Eq. \((63)\) to a quantum mechanics may be written

\[
\hat{\rho} = \frac{1}{Z} \exp \left( \frac{\hat{H}}{k_B T} \right),
\]

(64)

where \(\hat{\rho}\) is the density matrix of the quantum system, \(\hat{H}\) is the system’s Hamiltonian operator, and \(\exp\) represents an exponential function for quantum operator (or matrix). Here, the function \(Z\) is used to normalized the density matrix of the quantum state and equals the trace of \(\exp(-\hat{H}/(k_B T))\).
Now we prove that Eq. (64) is the quantum version of the Boltzmann distribution Eq. (63). Suppose the different basis state of the quantum system is labeled by $i$ and $\{|i\rangle\}$ is a complete basis, then Eq. (64) may be written

$$\hat{\rho} = \frac{1}{Z} \exp \left( -\frac{\hat{H}}{k_B T} \right) = \frac{1}{Z} \sum_{i,j} |i⟩⟨i| \exp \left( -\frac{\hat{H}}{k_B T} \right) |j⟩⟨j| = \sum_{i,j} p_{i,j} |i⟩⟨j|, \quad \text{where} \quad p_{i,j} = \frac{1}{Z} ⟨i| \exp \left( -\frac{\hat{H}}{k_B T} \right) |j⟩. \quad (65)$$

Here $p_{i,j}$ is the probability to find the quantum system in the quantum state $|i⟩⟨j|$ which has the similar physical interpretation as $p_i$ in the Boltzmann distribution Eq. (63).

If we choose the complete basis $\{|i⟩\}$ as a complete basis of the energy eigenstates, the $p_{i,j}$ in Eq. (65) may be further simplified

$$p_{i,j} = \frac{1}{Z} ⟨i| \exp \left( -\frac{\hat{H}}{k_B T} \right) |j⟩ = \frac{1}{Z} e^{-\frac{E_i}{k_B T}} = \frac{1}{Z} e^{-\frac{E_i}{k_B T}} \delta_{i,j}, \quad (66)$$

where $E_j$ is the eigenvalues $|j⟩$. Inserting Eq. (66) into Eq. (65) yields

$$\hat{\rho} = \sum_{i,j} \frac{1}{Z} e^{-\frac{E_i}{k_B T}} \delta_{i,j} |i⟩⟨j| = \sum_{i} \frac{1}{Z} e^{-\frac{E_i}{k_B T}} |i⟩⟨i|, \quad (67)$$

where $Z = \sum_{i=1}^{N} e^{-\frac{E_i}{k_B T}}$ and $N$ is total number of the basis eigenvectors. The density matrix is diagonal in this basis and each entries give the probability for the quantum system with a specific energy.

If we suppose the temperature of the system is high ($\frac{E_i}{k_B T}$ is small), then Eq. (67) may be approximated by

$$\hat{\rho} \approx \sum_{i} \frac{1}{N} e^{-\frac{E_i}{k_B T}} |i⟩⟨i| \approx \frac{1}{N} \sum_{i} \left( 1 - \frac{E_i}{k_B T} + \mathcal{O}\left(\frac{1}{k_B T}\right) \right) |i⟩⟨i| = \sum_{i} \frac{1}{N} \left( 1 + \frac{\langle E \rangle - E_i}{k_B T} + \mathcal{O}\left(\frac{1}{k_B T}\right) \right) |i⟩⟨i|, \quad (68)$$

where $\langle E \rangle = \frac{\sum_{i=1}^{N} E_i}{N}$. When the temperature is infinity high, the quantum state of the system Eq. (68) reduces to $\frac{1}{N} |i⟩⟨i|$. Thus a maximally mixed quantum state corresponds to an infinity high temperature for the quantum system.

### Appendix E

For a black hole begins from a pure state with non-negligible radiations, we have the following analysis. If we still assume the dimension of initial state of the black hole to be $N$, and assume the dimension of the radiation is $R$, then the dimension of the remaining black hole will be $N_B = N/R$. The quantum state of the remaining black hole may be represented by $\text{tr}_R(ρ_U)$. The key step in our calculation of the 2-norm is to take an average over all possible unitary operators, since black holes are fast scramblers. Now, with part of the initial state radiated outside the horizon, we also take an average over the unitary operators of the remaining black hole $U_B$. Note that $U_B$ is different from the $U$, and does not influence the radiation outside the horizon. With this physical picture, the fidelity relations Eqs. (7) and (8) in the manuscript may be written

$$1 - \frac{1}{2} \sqrt{n} \int_{U_B} \left\| \text{tr}_B \left( U_B \text{tr}_R(ρ_U) U_B^\dagger \right) - \frac{1}{n} I \right\|_2 dU_B \leq F \leq 1 - \frac{1}{4} \int_{U_B} \left\| \text{tr}_B \left( U_B \text{tr}_R(ρ_U) U_B^\dagger \right) - \frac{1}{n} I \right\|_2^2 dU_B. \quad (69)$$
Using similar analysis to that used in Appendix C, the 2-norm in Eq. \[69\] may be calculated as

\[ \int_{U_B} \left\| \text{tr}_n \left( U_B \text{tr}_R (\rho_U) U_B^\dagger \right) - \frac{A_n}{n} \right\|_2^2 dU_B \]

\[ = \int_{U_B} \left[ \text{tr}_n \left( U_B \text{tr}_R (\rho_U) U_B^\dagger \right) \text{tr}_n \left( U_B \text{tr}_R (\rho_U) U_B^\dagger \right) - \frac{2}{n} \text{tr}_n \left( U_B \text{tr}_R (\rho_U) U_B^\dagger \right) + \frac{A_n}{n^2} \right] dU_B \]

\[ = \int_{U_B} \left[ \text{tr}_n \left( U_B \text{tr}_R (\rho_U) U_B^\dagger \right) \text{tr}_n \left( U_B \text{tr}_R (\rho_U) U_B^\dagger \right) \right] dU_B - \frac{2}{n} \int_{U_B} \text{tr}_R \left( U_B \text{tr}_R (\rho_U) U_B^\dagger \right) dU_B + \frac{1}{n} \]

\[ = \int_{U_B} \left[ \text{tr}_n,\tilde{\text{tr}} \left( U_B \text{tr}_R (\rho_U) U_B^\dagger \right) \otimes \left( U_B \text{tr}_R (\rho_U) U_B^\dagger \right) \right] \text{SWAP}_n \tilde{n} \tilde{n} \right] dU_B - \frac{2}{n} \int_{U_B} \text{tr}_R \left( \text{tr}_R (\rho_U) \right) dU_B + \frac{1}{n} \]

\[ = \int_{U_B} \text{tr} \left( U_B \otimes U_B \right) \left( \text{tr}_R (\rho_U) \otimes \text{tr}_R (\rho_U) \right) \left( U_B^\dagger \otimes U_B^\dagger \right) \left( \text{SWAP}_n \tilde{n} \otimes 1,\tilde{n} \tilde{n} \right) \left( U_B \otimes U_B \right) dU_B - \frac{1}{n}, \quad (70) \]

where \( \|A\|_2^2 = \text{tr}(A^\dagger A) \) is applied to the first step, and \( \text{tr}(AB) = \text{tr}(A \otimes \text{SWAP}_{A,B}) \) is applied in moving from the third to the fourth line. Applying the Schur-Weyl duality (see Appendix B) to Eq. \[70\] yields

\[ = \frac{1}{2} \text{tr} \left[ \left( \text{tr}_R (\rho_U) \otimes \text{tr}_R (\rho_U) \right) \left( \frac{N_B}{N} + \frac{n}{N_B + 1} (1,\text{SWAP}_{N_B, N_B}) + \frac{N_B}{N_B - 1} (1,\text{SWAP}_{N_B, N_B}) \right) \right] - \frac{1}{n} \]

\[ = \text{tr} \left[ \left( \text{tr}_R (\rho_U) \otimes \text{tr}_R (\rho_U) \right) \left( \frac{(N_B)^2 - n}{N_B - 1} \left( 1,\text{SWAP}_{N_B, N_B} \right) \right) \right] - \frac{1}{n} \]

\[ = \frac{(N_B)^2 - n}{N_B - 1} \left( \text{tr} \left( \text{tr}_R (\rho_U) \right) \right)^2 + \frac{N_B - n}{N_B - 1} \text{tr} \left( \left( \text{tr}_R (\rho_U) \right) \right)^2 - \frac{1}{n} \]

\[ = \frac{(N_B)^2 - n}{N_B - 1} \left( \text{tr} \left( \rho_U \right) \right)^2 + \frac{N_B - n}{N_B - 1} \text{tr} \left( \left( \text{tr}_R (\rho_U) \right) \right)^2 - \frac{1}{n}. \quad (71) \]

Since the interior and radiations of the black hole are both random density matrices, the term \( \text{tr}((\text{tr}_R (\rho_U))^2) \) in Eq. \[71\] may be calculated

\[ \langle \text{tr}((\text{tr}_R (\rho_U))^2) \rangle = \int_U \text{tr}((\text{tr}_R (\rho_U))^2)dU = \int_U \text{tr}_U \left[ \text{tr}_R (\rho_U) \otimes \text{tr}_R (\rho_U) \text{SWAP}_{R,R} \right] dU \]

\[ = \int_U \text{tr} \left[ (\rho_U \otimes \rho_U) \text{SWAP}_{R,R} \right] dU = \int_U \text{tr} \left[ (U \otimes U)(\rho \otimes \rho)(U^\dagger \otimes U^\dagger) \text{SWAP}_{R,R} \right] dU \]

\[ = \text{tr} \left[ (\rho \otimes \rho) \int_U (U^\dagger \otimes U^\dagger) \text{SWAP}_{R,R} (U \otimes U) dU \right] \]

\[ = \frac{1}{2} \text{tr} \left[ (\rho \otimes \rho) \left( \frac{NR - N_B + R}{N^2 - 1} (1,\text{SWAP}_{N,N}) + \frac{N - R - R}{N - 1} (1,\text{SWAP}_{N,N}) \right) \right] \]

\[ = \text{tr} \left[ (\rho \otimes \rho) \left( \frac{NR - N_B}{N^2 - 1} (1,\text{SWAP}_{N,N}) + \frac{NNB - R}{N^2 - 1} (1,\text{SWAP}_{N,N}) \right) \right] \]

\[ = \frac{NR - N_B}{N^2 - 1} \langle \text{tr}(\rho)^2 \rangle + \frac{NNB - R}{N^2 - 1} \text{tr}(\rho^2) = \frac{NR - N_B}{N^2 - 1} + \frac{NNB - R}{N^2 - 1} \]

\[ = \frac{N_B + R}{N + 1}, \quad (72) \]

where the Schur-Weyl duality is used in moving from the third to the fourth line, \( R = N_B \) and \( N_B = N/R \) are used in moving from the fourth to the fifth line, and \( \langle \text{tr}(\rho)^2 \rangle = \text{tr}(\rho^2) = 1 \) is used in the sixth line.
Applying Eq. (72) and \((\text{tr}(\rho_U))^2 = 1\) to Eq. (71) yields

\[
\int_{U_B} \| \text{tr}_n \left( U_B \text{tr}_R(\rho_U) U_B^\dagger \right) - \frac{1_n}{n} \|_2^2 \, dU_B = \left( \frac{(N_B)^2 - n}{(N_B)^2 - 1} \right) \left( \frac{N_B n - N_B + R - \frac{1}{n}}{N + 1} \right) - \frac{1}{n} \]

\[
= \left( \frac{(N_B)^2 - n^2)(N + 1) + (N_B n^2 - N_B)(N_B + R) - (N_B^2 - 1)(N + 1)\right)\]

\[
= \frac{N_B^2 n^2 + N_B R n^2 - N n^2 - N_B^2 - N_B R + N - n^2 + 1}{(N_B)^2 - 1)(N + 1)n} = \frac{n^2 - 1}{(N + 1)n}, \tag{73}
\]

where \(N = N_B R\) is used in moving from the second to the third line.

Since \(N \geq n \geq 1\), Eq. (69) may be further simplified as

\[
\int_{U_B} \| \text{tr}_n \left( U_B \text{tr}_R(\rho_U) U_B^\dagger \right) - \frac{1_n}{n} \|_2^2 \, dU_B < \frac{n}{N}. \tag{74}
\]

Inserting Eqs. (73) and (74) into Eq. (69) yields

\[
1 - \frac{1}{2\sqrt{N}} \leq F \leq 1 - \frac{n^2 - 1}{4(N + 1)n}. \tag{75}
\]

**Appendix F**

Now we consider a black hole that begins from a generic quantum state \(\rho_0\), with Hilbert space dimension \(N\). The local subsystem observed by the infalling observer with dimension \(n\) may be written \(\text{tr}_n(\rho_U)\). The 2-norm in the fidelity relations Eqs. (7) and (8) in the manuscript may be calculated

\[
\int_U \| \text{tr}_n(\rho_U) - \frac{1_n}{n} \|_2^2 \, dU = \int_U \| \text{tr}_n(U \rho_0 U^\dagger) - \frac{1_n}{n} \|_2^2 \, dU
\]

\[
= \int_U \left[ \text{tr}_n(U \rho_0 U^\dagger) \text{tr}_n(\rho_U \rho_0 U^\dagger) - \frac{2}{n} \text{tr}_n(U \rho_0 U^\dagger) + \frac{1_n}{n^2} \right] \, dU
\]

\[
= \int_U \text{tr}_n \left[ \text{tr}_n(U \rho_0 U^\dagger) \right] \, dU - \frac{n}{n^2}
\]

\[
= \int_U \text{tr}_{n,n} \left[ \text{tr}_{n,n}(U \rho_0 U^\dagger) \otimes (U \rho_0 U^\dagger) \text{SWAP}_{n,n} \right] \, dU - \frac{1}{n}
\]

\[
= \int_U \left[ \text{tr}_{U} \left( (\rho_0 \otimes \rho_0) \left( U^\dagger \otimes U^\dagger \right) \left( \text{SWAP}_{n,n} \otimes 1_{n,n} \right) \right) \right] \, dU - \frac{1}{n}
\]

\[
= \text{tr} \left[ (\rho_0 \otimes \rho_0) \int_U \left( (U^\dagger \otimes U^\dagger) \left( \text{SWAP}_{n,n} \otimes 1_{n,n} \right) (U \otimes U) \right) \right] - \frac{1}{n}, \tag{76}
\]

where \(\text{tr}_n(U \rho_0 U^\dagger) = 1\) is applied to the second line. Applying the Schur-Weyl duality to Eq. (76) yields

\[
\int_U \| \text{tr}_n(\rho_U) - \frac{1_n}{n} \|_2^2 \, dU = \frac{1}{2} \text{tr} \left[ (\rho_0 \otimes \rho_0) \left( \begin{array}{c} \frac{N}{N + 1} (\mathbb{1}_{N,N} + \text{SWAP}_{N,N}) + \frac{N - n}{N - 1} (\mathbb{1}_{N,N} - \text{SWAP}_{N,N}) \end{array} \right) \right] - \frac{1}{n}
\]

\[
= \text{tr} \left[ (\rho_0 \otimes \rho_0) \left( \begin{array}{c} \frac{N^2 - n}{N^2 - 1} + \frac{Nn - n}{N^2 - 1} \text{SWAP}_{N,N} \end{array} \right) \right] - \frac{1}{n}
\]

\[
= \frac{N^2 - n}{N^2 - 1} \text{tr}(\rho_0)^2 + \frac{Nn - n}{N^2 - 1} \text{tr}(\rho_0^2) \right) - \frac{1}{n} = \frac{n^2 - n}{N^2 - 1} + \frac{Nn - n}{N^2 - 1} \text{tr}(\rho_0^2) - \frac{1}{n}
\]

\[
= \frac{Nn^2 - N}{(N^2 - 1)n} \text{tr}(\rho_0^2) - \frac{n^2 - 1}{(N^2 - 1)n} = \frac{(N n^2 - 1)(n^2 - 1)}{(N^2 - 1)n}, \tag{77}
\]
where \((\text{tr} \rho_0) = 1\) is used in the third line because the entire black hole is a pure quantum state.

Since \(1 < n \leq N\), Eq. (77) obeys

\[
\int_U \left\| \text{tr}_\bar{n}(\rho_U) - \frac{1}{n} \right\|_2^2 dU \leq \frac{(N \text{tr}(\rho_0^2) - 1)n}{N^2},
\]

(78)

where to obtain the inequality we have used the fact that \((x^2 - 1)/x^2\) is a monotonically increasing function for \(x > 0\).

Inserting Eqs. (77) and (78) into the fidelity relations Eqs. (7) and (8) yields

\[
1 - \frac{n}{2N} \sqrt{N \text{tr}(\rho_0^2) - 1} \leq F \leq 1 - \frac{(N \text{tr}(\rho_0^2) - 1)(n^2 - 1)}{4(N^2 - 1)n}.
\]

(79)

**Appendix G**

Similarly as Appendix E, we can also consider black holes originating from a generic black holes that has non-negligible radiations. If we still assume the initial dimension of the entire black hole state to be \(N\) and the dimension of the radiation is \(R\), then the dimension of the remaining black hole \(N_B\) will equal \(N/R\). The quantum state of the remaining black hole may be represented by \(\text{tr}_R(\rho_U)\). The key step in our calculation of the 2-norm is to take an average over all possible unitary operators, since black holes are fast scramblers. Now, with part of the initial state radiated outside the horizon, we first take an average over the unitary operators of the remaining black hole \(U_B\). Here \(U_B\) is different from the \(U\), and does not influence the radiation outside the horizon. With this physical picture, the fidelity will have a range of

\[
1 - \frac{1}{\sqrt{n}} \int_{U_B} \left\| \text{tr}_\bar{n}(U_B \text{tr}_R(\rho_U)U_B^\dagger) - \frac{1}{n} \right\|_2^2 dU_B \leq F \leq 1 - \frac{1}{4} \int_{U_B} \left\| \text{tr}_\bar{n}(U_B \text{tr}_R(\rho_U)U_B^\dagger) - \frac{1}{n} \right\|_2^2 dU_B.
\]

(80)

The 2-norm in Eq. (80) may be calculated

\[
\int_{U_B} \left\| \text{tr}_\bar{n}(U_B \text{tr}_R(\rho_U)U_B^\dagger) - \frac{1}{n} \right\|_2^2 dU_B \\
= \int_{U_B} \text{tr}_n \left[ \text{tr}_\bar{n}(U_B \text{tr}_R(\rho_U)U_B^\dagger) \text{tr}_\bar{n}(U_B \text{tr}_R(\rho_U)U_B^\dagger) - \frac{2}{n} \text{tr}_n(U_B \text{tr}_R(\rho_U)U_B^\dagger) \right] dU_B \\
= \int_{U_B} \text{tr}_n \left[ \text{tr}_\bar{n}(U_B \text{tr}_R(\rho_U)U_B^\dagger) \text{tr}_\bar{n}(U_B \text{tr}_R(\rho_U)U_B^\dagger) \right] dU_B - \frac{2}{n} \int_{U_B} \text{tr}_R(U_B \text{tr}_R(\rho_U)U_B^\dagger) dU_B + \frac{1}{n} \\
= \int_{U_B} \text{tr}_n,\bar{n} \left[ \text{tr}_\bar{n}(U_B \text{tr}_R(\rho_U)U_B^\dagger) \otimes (U_B \text{tr}_R(\rho_U)U_B^\dagger) \right] \text{SWAP}_{n,\bar{n}} dU_B - \frac{2}{n} \int_{U_B} \text{tr}_R(U_B \text{tr}_R(\rho_U)U_B^\dagger) dU_B + \frac{1}{n} \\
= \int_{U_B} \text{tr}_n \left[ (U_B \otimes U_B) \left( \text{tr}_R(\rho_U) \otimes \text{tr}_R(\rho_U) \right) \left( U_B^\dagger \otimes U_B^\dagger \right) \right] \left( \text{SWAP}_{n,\bar{n}} \otimes \text{SWAP}_{\bar{n},n} \right) dU_B - \frac{2}{n} \int_{U_B} \text{tr}_R(U_B \text{tr}_R(\rho_U)U_B^\dagger) dU_B + \frac{1}{n} \\
= \text{tr}_n \left[ \text{tr}_R(\rho_U) \otimes \text{tr}_R(\rho_U) \right] \int_{U_B} \left[ (U_B^\dagger \otimes U_B^\dagger) \left( \text{SWAP}_{n,\bar{n}} \otimes \text{SWAP}_{\bar{n},n} \right) (U_B \otimes U_B) \right] dU_B - \frac{1}{n},
\]

(81)

where \(\|A\|_2 = \text{tr}(A^\dagger A)\) is applied to the first step, and \(\text{tr}(AB) = \text{tr}(A \otimes B \text{SWAP}_{A,B})\) is applied in moving from the
third to the fourth line. Applying the Schur-Weyl duality in Appendix B to Eq. [81] yields

\[
\int_{U_B} \left\| \text{tr}_R \left( U_B \text{tr}_R (\rho_U) U_B^{-1} \right) - \frac{1}{n} \right\|_2^2 dU_B
\]

\[
= \frac{1}{2} \text{tr} \left[ \left( \text{tr}_R (\rho_U) \otimes \text{tr}_R (\rho_U) \right) \left( \frac{N_B}{N_B + 1} \mathbb{1}_{N_B} + \text{SWAP}_{N_B} + \frac{N_B - n}{N_B - 1} \mathbb{1}_{N_B} - \text{SWAP}_{N_B} \right) \right] - \frac{1}{n}
\]

\[
= \text{tr} \left[ \left( \text{tr}_R (\rho_U) \otimes \text{tr}_R (\rho_U) \right) \left( \frac{N_B}{N_B^2 - 1} \mathbb{1}_{N_B} + \frac{N_B - n}{N_B^2 - 1} \text{SWAP}_{N_B} \right) \right] - \frac{1}{n}
\]

\[
= \frac{(N_b)^2}{N_B^2} - \frac{n}{N_B^2} \left( \text{tr} (\rho_U)^2 \right)^2 + \frac{N_B n - n}{N_B^2 - 1} \text{tr} \left( (\text{tr}_R (\rho_U))^2 \right) - \frac{1}{n}
\]

where we have used \( (\text{tr}_R (\rho_U))^2 = 1 \) in the last step.

Since black holes are believed to be fast quantum scrambler, a black hole should already finish its quantum scrambling before it radiate too much state away. Thus, the term \( \text{tr}((\text{tr}_R (\rho_U))^2) \) in Eq. [82] may be calculated

\[
\langle \text{tr}((\text{tr}_R (\rho_U))^2) \rangle = \int_U \text{tr}((\text{tr}_R (\rho_U))^2)dU = \int_U \text{tr} \left[ \text{tr}_R (\rho_U) \otimes \text{tr}_R (\rho_U) \right] \text{SWAP}_{R,R} dU
\]

\[
= \int_U \text{tr} \left[ \rho_U \otimes \rho_U \text{SWAP}_{R,R} dU \right] = \int_U \text{tr} \left[ (U \otimes U)(\rho \otimes \rho)(U^\dagger \otimes U^\dagger) \right] \text{SWAP}_{R,R} dU
\]

\[
= \text{tr} \left[ (\rho \otimes \rho) \int_U (U^\dagger \otimes U^\dagger) \text{SWAP}_{R,R} (U \otimes U) dU \right]
\]

\[
= \frac{1}{2} \text{tr} \left[ (\rho \otimes \rho) \left( \frac{N/R + R}{N + 1} + \text{SWAP}_{N,N} + \frac{N/R - R}{N - 1} \text{SWAP}_{N,N} \right) \right]
\]

\[
= \text{tr} \left[ (\rho \otimes \rho) \left( \frac{NR - N_B}{N^2 - 1} \mathbb{1}_{N,R} + \frac{NN_B - R}{N^2 - 1} \text{SWAP}_{N,N} \right) \right]
\]

\[
= \frac{NR - N_B}{N^2 - 1} \text{tr}(\rho)^2 + \frac{NN_B - R}{N^2 - 1} \text{tr}(\rho^2)
\]

\[
= \frac{NR - N_B + (NN_B - R)\text{tr}(\rho^2)}{N^2 - 1},
\]

where the Schur-Weyl duality is used in moving from the third to the fourth line, \( \tilde{R} = N_B \) and \( N_B = N/R \) are used in moving from the fourth to the fifth line, and \( (\text{tr}(\rho))^2 = \text{tr}(\rho^2) = 1 \) is applied to the sixth line.
Applying Eq. (83) to Eq. (82) yields
\[ \int_{U_B} \left\| \rho_n \left( (U_B\rho U_B) - 1 \right) \right\|^2 \, dU_B = \frac{(N^2 - 1)n}{N^2 - 1} + \frac{(N^2 - 1)(N^2 - 1)n}{N^2 - 1} \]
\[ = (1 - n^2)(N^2 - 1)\frac{N^2(n^2 - 1) - N^2_B(n^2 - 1) + (N_B n^2 - N_B)(NN_B - R)(\rho^2)}{(N_B)^2 - 1) (N^2 - 1)n} \]
\[ = \frac{(n^2 - 1) + N(n^2 - 1)tr(\rho^2)}{(N^2 - 1)n} \]
\[ = \frac{(Ntr(\rho^2) - 1)(n^2 - 1)}{(N^2 - 1)n} \] (84)

where \( N = N_B R \) is used in moving from the third to the four line line. Since \( n \leq N \), Eq. (84) must satisfy
\[ \int_{U_B} \left\| \rho_n \left( (U_B\rho U_B) - 1 \right) \right\|^2 \, dU_B = \frac{(Ntr(\rho^2) - 1)(n^2 - 1)}{(N^2 - 1)n} \leq \frac{(Ntr(\rho^2) - 1)n}{N^2} \] (85)

Applying Eqs. (84) and (85) to Eq. (80) yields
\[ 1 - \frac{n}{2N} \sqrt{Ntr(\rho^2) - 1} \leq F \leq 1 - \frac{(Ntr(\rho^2) - 1)(n^2 - 1)}{4(N^2 - 1)n} \] (86)
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