NONCOMMUTATIVE GEOMETRY OF TWISTS

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Abstract. Let \( k \subset \mathbb{C} \) be a number field and \( V(k) \) a projective variety over \( k \). Denote by \( \mathcal{A}_V \) the Serre \( C^* \)-algebra of \( V(k) \). It is proved that the \( k \)-isomorphisms of the variety \( V(k) \) correspond to the isomorphisms of the algebra \( \mathcal{A}_V \), while the \( \mathbb{C} \)-isomorphisms of \( V(k) \) correspond to the Morita equivalences of \( \mathcal{A}_V \). The case of rational elliptic curves is considered in detail.

1. Introduction

Let \( V \) be a complex projective variety given by a homogeneous coordinate ring \( \mathcal{A} \). If \( k \subset \mathbb{C} \) is a subfield of complex numbers and \( V(k) \) is a variety over \( k \), then isomorphisms of \( V(k) \) over the field \( \mathbb{C} \) cannot be restricted to the field \( k \) in general. When such a restriction fails, the variety \( V'(k) \) is called a twist of \( V(k) \). In other words, the varieties \( V(k) \) and \( V'(k) \) are not isomorphic over \( k \), yet they become isomorphic over the algebraically closed field \( \mathbb{C} \). Since all \( \mathbb{C} \)-isomorphic varieties correspond to the same \( \mathcal{A} \), such an algebra cannot distinguish the twists of \( V(k) \).

However, the twists of \( V(k) \) can be described in terms of the Galois cohomology [Serre 1997] [5, p. 123]. The Serre \( C^* \)-algebra \( \mathcal{A}_V \) of \( V \) is defined as the norm closure of a self-adjoint representation of the twisted homogeneous coordinate ring of \( V \) by the bounded linear operators on a Hilbert space \( \mathcal{H} \). We refer the reader to [Stafford & van den Bergh 2001] [7] and [3, Section 5.3.1] for an introduction and details. Recall that a \( C^* \)-algebra \( A \) is said to be Morita equivalent (stably isomorphic) to \( A' \), if \( A \otimes \mathcal{K} \cong A' \otimes \mathcal{K} \), where \( \mathcal{K} \) is the \( C^* \)-algebra of all compact operators on \( \mathcal{H} \) and \( \cong \) is an isomorphism of the \( C^* \)-algebras [Blackadar 1986] [1, Section 13.7.1]. Notice that if \( A \cong A' \) are isomorphic \( C^* \)-algebras, then \( A \) is Morita equivalent to \( A' \). The correspondence \( V \mapsto \mathcal{A}_V \) is a functor, which maps isomorphic varieties \( V \cong V' \) to the Morita equivalent algebras \( \mathcal{A}_V \) and \( \mathcal{A}_{V'} \). In other words, the Serre \( C^* \)-algebra \( \mathcal{A}_V \) is an analog of the homogeneous coordinate ring of \( V \). Notice that the \( \mathcal{A}_V \) is no longer a commutative algebra.

It is proved in this note that the Serre \( C^* \)-algebras distinguish twists of the variety \( V(k) \). Such a property is a far cry from the behavior of commutative algebra \( \mathcal{A} \) with respect to the twists. Namely, if the variety \( V(k) \) is \( k \)-isomorphic to a variety \( V'(k) \), then the Serre \( C^* \)-algebra \( \mathcal{A}_V \) is isomorphic to the Serre \( C^* \)-algebra \( \mathcal{A}_{V'} \). If \( V(k) \) is \( \mathbb{C} \)-isomorphic to \( V'(k) \), then the algebra \( \mathcal{A}_V \) is Morita equivalent to the algebra \( \mathcal{A}_{V'} \), see corollary 1.2. To formalize our results, we need the following definitions.

Let \( A \) be a unital \( C^* \)-algebra and denote by \( V(A) \) the union of projections in all the \( n \times n \) matrix \( C^* \)-algebra with entries in \( A \) [Blackadar 1986] [1, Section 5].

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Recall that projections $p, q \in V(A)$ are called equivalent, if there exists a partial isometry $u$ such that $p = u^* u$ and $q = uu^*$. The corresponding equivalence class is denoted by $[p]$. The equivalence classes of the orthogonal projections can be made to a semigroup with addition defined by the formula $[p] + [q] = [p + q]$. The Grothendieck completion of the semigroup to an abelian group, $K_0(A)$, is called the $K_0$-group of the algebra $A$. The functor $A \to K_0(A)$ maps the category of unital $C^*$-algebras into the category of abelian groups, so that projections in the algebra $A$ correspond to a positive cone $K_0^+(A) \subset K_0(A)$ and the unit element $1 \in A$ corresponds to an order unit $u \in K_0^+(A)$. An isomorphism class of the ordered abelian group $(K_0(A), K_0^+(A))$ is known as a dimension group. The dimension group $(K_0(A), K_0^+(A), u)$ with a fixed order unit $u$ is called a scaled dimension group [Blackadar 1986] [1, Section 6].

An AF-algebra $B$ (Approximately Finite $C^*$-algebra) is the norm closure of an ascending sequence of the finite-dimensional $C^*$-algebras $M_n(C)$, where $M_n(C)$ is the $C^*$-algebra of the $n \times n$ matrices with entries in $C$ [Blackadar 1986] [1, Section 7.1]. The scaled dimension group $(K_0(B), K_0^+(B), u)$ is an isomorphism invariant of the algebra $B$. In contrast, the dimension group $(K_0(B), K_0^+(B))$ is an invariant of the Morita equivalence of the AF-algebra $B$ [Blackadar 1986] [1, Section 7.3]. The Serre $C^*$-algebra $A_V$ is not an AF-algebra, but there exists a dense embedding $A_V \to B$, where $B$ is an AF-algebra such that $(K_0(A_V), K_0^+(A_V)) \cong (K_0(B), K_0^+(B))$ [4, Lemma 3.1].

By $H^1(Gal(C|k), \text{Aut}_{C^*}^{ab}(V))$ we understand the first Galois cohomology group of the extension $k \subset C$, where $\text{Aut}_{C^*}^{ab}(V)$ is the maximal abelian subgroup of the group of $C$-automorphisms of the variety $V(k)$ [Serre 1997] [5, p. 123]. The $H^1(Gal(C|k), \text{Aut}_{C^*}^{ab}(V))$ is a dimension group of stationary type, see lemma 3.1. Our main results can be formulated as follows.

**Theorem 1.1.** $H^1(Gal(C|k), \text{Aut}_{C^*}^{ab}(V)) \cong (K_0(A_V), K_0^+(A_V)), \text{ where } \cong \text{ is an order-isomorphism of the dimension groups.}$

**Corollary 1.2.** The $V(k)$ and $V'(k)$ are $k$-isomorphic if and only if the Serre $C^*$-algebras $A_V \cong A_{V'}$ are isomorphic. The $V(k)$ and $V'(k)$ are $C$-isomorphic if and only if the $C^*$-algebras $A_V \otimes K \cong A_{V'} \otimes K$ are isomorphic.

The article is organized as follows. In Section 2 we briefly review the Serre $C^*$-algebras and the Galois cohomology. Theorem 1.1 and corollary 1.2 are proved in Section 3. An illustration of corollary 1.2 can be found in Section 4.

## 2. Preliminaries

In this section we briefly review the Galois cohomology and the Serre $C^*$-algebras. We refer the reader to [Serre 1997] [5, Chapter I, §5] and [3, Section 5.3.1] for a detailed account.

### 2.1. Serre $C^*$-algebras.

Let $V$ be an $n$-dimensional complex projective variety endowed with an automorphism $\sigma : V \to V$ and denote by $B(V, L, \sigma)$ its twisted homogeneous coordinate ring [Stafford & van den Bergh 2001] [7]. Let $R$ be a commutative graded ring, such that $V = Spec\,(R)$. Denote by $R[t, t^{-1}; \sigma]$ the ring of skew Laurent polynomials defined by the commutation relation $b^\sigma t = tb$ for all $b \in R$, where $b^\sigma$ is the image of $b$ under automorphism $\sigma$. It is known, that $R[t, t^{-1}; \sigma] \cong B(V, L, \sigma)$. 
Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For a ring of skew Laurent polynomials $R[t, t^{-1}; \sigma]$, consider a homomorphism:

$$\rho : R[t, t^{-1}; \sigma] \rightarrow \mathcal{B}(\mathcal{H}). \quad (2.1)$$

Recall that $\mathcal{B}(\mathcal{H})$ is endowed with a $*$-involution; the involution comes from the scalar product on the Hilbert space $\mathcal{H}$. We shall call representation $(2.1)$ *-coherent, if (i) $\rho(t)$ and $\rho(t^{-1})$ are unitary operators, such that $\rho^*(t) = \rho(t^{-1})$ and (ii) for all $b \in R$ it holds $(\rho^*(b))^{\sigma(\rho)} = \rho^*(b^{\sigma})$, where $\sigma(\rho)$ is an automorphism of $\rho(R)$ induced by $\sigma$. Whenever $B = R[t, t^{-1}; \sigma]$ admits a $*$-coherent representation, $\rho(B)$ is a $*$-algebra. The norm closure of $\rho(B)$ is a $C^*$-algebra denoted by $\mathfrak{A}_V$. We refer to the $\mathfrak{A}_V$ as the Serre $C^*$-algebra of variety $V$.

2.2. Galois cohomology. Let $G$ be a group. The set $A$ is called a $G$-set, if $G$ acts on $A$ on the left continuously. If $A$ is a group and $G$ acts on $A$ by the group morphisms, then $A$ is called a $G$-group. In particular, if $A$ is abelian, one gets a $G$-module.

If $A$ is a $G$-group, then a 1-cocycle of $G$ in $A$ is a map $s \mapsto a_s$ of $G$ to $A$ which is continuous and such that $a_{st} = a_s a_t$ for all $s, t \in G$. The set of all 1-cocycles is denoted by $Z^1(G, A)$. Two cocycles $a$ and $a'$ are said to be cohomologous, if there exists $b \in A$ such that $a'_s = b^{-1} a_s b$. The quotient of $Z^1(G, A)$ by this equivalence relation is called the first cohomology set and is denoted by $H^1(G, A)$. The class of the unit cocycle is a distinguished element $1$ in the $H^1(G, A)$. Notice that in general there is no composition law on the set $H^1(G, A)$. If $A$ is an abelian group, the set $H^1(G, A)$ is a cohomology group.

If $G$ is a profinite group, then

$$H^1(G, A) = \lim_{\longrightarrow} H^1(G/U, A^U), \quad (2.2)$$

where $U$ runs through the set of open normal subgroups of $G$ and $A^U$ is a subset of $A$ fixed under action of $U$. The maps $H^1(G/U, A^U) \rightarrow H^1(G, A)$ are injective.

Let $k$ be a number field and $k$ the algebraic closure of $k$. Denote by $Gal(k/k)$ the profinite Galois group of $k$. Let $V(k)$ be a projective variety over $k$ and $Aut V(k)$ the group of the $k$-automorphisms of $V(k)$.

**Lemma 2.1.** [Serre 1997] [5, p. 124] There exists a bijective correspondence between the twists of $V(k)$ and the set $H^1(Gal(k/k), Aut V(k))$.

3. Proofs

3.1. Proof of theorem 1.1. We shall split the proof in a series of lemmas.

**Lemma 3.1.** The $H^1(Gal(C|k), Aut_{C}^{ab}(V))$ is a stationary dimension group.

**Proof.** It is known that the Galois group $Gal(C|k)$ of the field extension $k \subset C$ is a profinite group. We denote by $U_j$ an infinite ascending sequence of the open normal subgroups of $Gal(C|k)$. In other words,

$$Gal(C|k) = \lim_{\longrightarrow} U_j. \quad (3.1)$$

The corresponding Galois cohomology $(2.2)$ can be written in the form

$$H^1(Gal(C|k), Aut_{C}^{ab}(V)) = \lim_{\longrightarrow} H^1 \left( Gal(C|k)/U_j, (Aut_{C}^{ab}(V))^U_j \right). \quad (3.2)$$
The dimension group is called stationary dimension group, i.e. $n \in \mathbb{Z}$ the shift $j$-dimension group.

Remark 3.2. All the Galois cohomology in formulas (3.2) and (3.3) are abelian groups $\mathbb{Z}^{n_j}$, because the coefficient group $\text{Aut}_{\mathbb{C}^b(V)}^{ab}(V)$ is taken to be abelian.

Recall that a dimension group is an ordered abelian group which is the inductive limit of the sequence of abelian groups

$$\mathbb{Z}^{n_1} \xrightarrow{\varphi_1} \mathbb{Z}^{n_2} \xrightarrow{\varphi_2} \mathbb{Z}^{n_3} \xrightarrow{\varphi_3} \ldots$$

for some positive integers $n_j$ and some positive group homomorphisms $\varphi_j$, where $\mathbb{Z}^n$ is given the usual ordering

$$(\mathbb{Z}^n)^+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n : x_j \geq 0\}.$$  

The dimension group is called stationary, if $n_j = n = \text{Const}$ and $\varphi_j = \varphi = \text{Const}$.

We shall define a group homomorphism $\varphi_j : H^1\left(\text{Gal}(\mathbb{C}|k)/U_j, (\text{Aut}_{\mathbb{C}^b(V)}^{ab})^{U_j}\right) \to H^1\left(\text{Gal}(\mathbb{C}|k)/U_{j+1}, (\text{Aut}_{\mathbb{C}^b(V)}^{ab})^{U_{j+1}}\right)$ from a commutative diagram in Figure 1, where the injective homomorphisms $\alpha_j$ are defined by the formula (3.3). Note that $H^1\left(\text{Gal}(\mathbb{C}|k)/U_j, (\text{Aut}_{\mathbb{C}^b(V)}^{ab})^{U_j}\right)$ is isomorphic to $\mathbb{Z}^{n_j}$ for an integer $n_j \geq 0$ and $\varphi_j$ is a positive homomorphism after a proper basis in $\mathbb{Z}^n$ is fixed. Comparing formulas (3.2) and (3.4), we conclude that the cohomology group $H^1(\text{Gal}(\mathbb{C}|k), \text{Aut}_{\mathbb{C}^b(V)})$ is a dimension group.

Let us show that the dimension group $H^1(\text{Gal}(\mathbb{C}|k), \text{Aut}_{\mathbb{C}^b(V)})$ is a stationary dimension group, i.e. $n_j = n = \text{Const}$ and $\varphi_j = \varphi = \text{Const}$. Indeed, notice that the shift $j \mapsto j + 1$ in the RHS of formula (3.1) corresponds to an automorphism of the group $\text{Gal}(\mathbb{C}|k)$. (Namely, the inductive limits $\lim U_j$ and $\lim U_{j+1}$ generate isomorphic profinite groups $\text{Gal}(\mathbb{C}|k)$.) Such an automorphism gives rise to the shift automorphism of the Galois cohomology (3.2) and the corresponding dimension group (3.4). But the dimension group admits the shift automorphism if and only if it is a stationary dimension group; this follows from [Blackadar 1986] [1, Theorem 7.3.2].

**Lemma 3.3.** $H^1(\text{Gal}(\mathbb{C}|k), \text{Aut}_{\mathbb{C}^b(V)}) \cong (K_0(\mathcal{A}_V), K^+_0(\mathcal{A}_V))$. 

![Figure 1. The group homomorphism $\varphi_j$.](image)
Proof. Recall that there exists a dense embedding $\mathcal{A}_V \hookrightarrow \mathcal{B}$, where $\mathcal{B}$ is an AF-algebra such that $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V)) \cong (K_0(\mathcal{B}), K_0^+(\mathcal{B}))$ [4, Lemma 3.1]. Moreover, the $(K_0(\mathcal{B}), K_0^+(\mathcal{B}))$ is a stationary dimension group, ibid.

On the other hand, it is known that the Galois cohomology $H^1(Gal(C|k), Aut_C^{ab}(V))$ is a functor from the category of projective varieties $V(k)$ to a category of abelian groups [Serre 1997] [5]. We shall denote by $\mathcal{B}$ an AF-algebra such that

$$(K_0(\mathcal{B}), K_0^+(\mathcal{B})) \cong H^1(Gal(C|k), Aut_C^{ab}(V)).$$  \hspace{2cm} \text{(3.6)}$$

Since the Galois cohomology is the functor, we conclude that the AF-algebra $\mathcal{B}$ is a coordinate ring of the variety $V(k)$. But any such ring must be Morita equivalent to the Serre $C^*$-algebra $\mathcal{A}_V$. In other words,

$$(K_0(\mathcal{B}), K_0^+(\mathcal{B})) \cong (K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V)).$$  \hspace{2cm} \text{(3.7)}$$

The conclusion of lemma 3.3 follows from the formulas (3.6) and (3.7). \hfill \Box

Theorem 1.1 follows from lemma 3.3.

3.2. Proof of corollary 1.2.

Lemma 3.4. There exits a bijective correspondence between the elements of the abelian group $H^1(Gal(C|k), Aut_C^{ab}(V))$ and a subset of the set of twists of the variety $V(k)$.

Proof. A restriction of the coefficient group $Aut_C V(k)$ of the Galois cohomology $H^1(Gal(C|k), Aut_C V(k))$ to its maximal abelian subgroup $Aut_C^{ab} V(k))$ defines an inclusion of the sets

$$H^1(Gal(C|k), Aut_C^{ab}(V)) \subseteq H^1(Gal(C|k), Aut_C V(k)).$$  \hspace{2cm} \text{(3.8)}$$

In view of the Serre’s Lemma 2.1, one gets from the inclusion (3.8) a bijection between the elements of the abelian group $H^1(Gal(C|k), Aut_C^{ab}(V))$ and a subset of the set of twists of the variety $V(k)$. Lemma 3.4 is proved. \hfill \Box

Corollary 3.5. There exits a bijective correspondence between the elements of the dimension group $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V))$ and a subset of the set of twists of the variety $V(k)$.

Proof. This corollary is an implication of theorem 1.1 and lemma 3.4. \hfill \Box

Lemma 3.6. There exits a bijective correspondence between the set of all scaled dimension groups $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V), u)$ and a subset of the set of twists of the variety $V(k)$.

Proof. It is known, that each $u \in K_0(\mathcal{A}_V)$ can be taken for an order-unit of the scaled dimension group $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V), u)$. Moreover, the obtained scaled dimension groups $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V), u)$ are distinct for different elements $u \in K_0(\mathcal{A}_V)$ and any such group can be obtained in this way [Blackadar 1986] [1, Section 6.2]. Thus lemma 3.6 follows from the corollary 3.5. \hfill \Box

Lemma 3.7. There exits a bijective correspondence between the set of all Morita equivalent but pairwise non-isomorphic Serre $C^*$-algebras $\mathcal{A}_V$ and a subset of the set of twists of the variety $V(k)$. 

Proof. Recall that there exists a dense embedding \( \mathcal{A}_V \hookrightarrow \mathcal{B} \), where \( \mathcal{B} \) is an AF-algebra, such that
\[
(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V)) \cong (K_0(\mathcal{B}), K_0^+(\mathcal{B})).
\] (3.9)
It is known that the dimension group \( (K_0(\mathcal{B}), K_0^+(\mathcal{B})) \) is an invariant of the Morita equivalence of the AF-algebra \( \mathcal{B} \), while the scaled dimension group \( K_0(\mathcal{B}), K_0^+(\mathcal{B}), u \) is an invariant of the isomorphism of \( \mathcal{B} \), see e.g. [Blackadar 1986] [1, Theorem 7.3.2]. In view of (3.9), the same is true of the Serre C*-algebra \( \mathcal{A}_V \). Lemma 3.7 follows from the corollary 3.5 and lemma 3.6. □

Corollary 1.2 follows from lemma 3.7 and the definition of a twist.

4. RATIONAL ELLIPTIC CURVES

To illustrate corollary 1.2, we shall consider the case \( V(k) \cong \mathcal{E}(k) \), where \( \mathcal{E}(k) \) is a rational elliptic curve. We briefly review the related definition and facts.

4.1. Elliptic curves. By an elliptic curve we understand the subset of the complex projective plane of the form
\[
\mathcal{E}(k) = \{ (x, y, z) \in \mathbb{C}P^2 \mid y^2z = x^3 + Axz^2 + Bz^3 \},
\] (4.1)
where \( A, B \in k \) are some constants. Recall that the number \( j(\mathcal{E}) = \frac{1728(4A^3)}{4A^3 + 27B^2} \) is an invariant of the \( \mathbb{C} \)-isomorphisms of the elliptic curve \( \mathcal{E}(k) \). The twists \( \mathcal{E}_t(k) \) of \( \mathcal{E}(k) \) are given by the equations
\[
\begin{align*}
y^2z &= x^3 + t^2Axz^2 + t^3Bz^3, & \text{if } j(\mathcal{E}) \neq 0, 1728, \\
y^2z &= x^3 + tAxz^2, & \text{if } j(\mathcal{E}) = 1728, \\
y^2z &= x^3 + tBz^3, & \text{if } j(\mathcal{E}) = 0,
\end{align*}
\] (4.2)
where \( t \in k \), see e.g. [Silverman 1985] [6, Proposition 5.4]. It is easy to verify, that \( j(\mathcal{E}_t(k)) = j(\mathcal{E}(k)) \).

4.2. Noncommutative tori. A C*-algebra \( \mathcal{A}_\theta \) on two generators \( u \) and \( v \) satisfying the relation \( vu = e^{2\pi i \theta} uv \) for a constant \( \theta \in \mathbb{R} \) is called the noncommutative torus. The well known Rieffel’s Theorem [3, Theorem 1.1.2] says that the algebra \( \mathcal{A}_\theta \) is Morita equivalent to the algebra \( \mathcal{A}_{\theta'} \), if and only if,
\[
\theta' = \frac{a\theta + b}{c\theta + d} \text{ for a matrix } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).
\] (4.3)
In contrast, the algebra \( \mathcal{A}_\theta \) is isomorphic to the algebra \( \mathcal{A}_{\theta'} \) if and only if \( \theta' = \theta \). Notice that in terms of the continued fraction
\[
\theta = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} := [a_0, a_1, a_2, \ldots]
\] (4.4)
attached to the parameter \( \theta \), it means that the \( \mathcal{A}_{\theta'} \) is Morita equivalent to the \( \mathcal{A}_\theta \), if and only if, the continued fraction of \( \theta' \) coincides with such of \( \theta \) everywhere but a finite number of terms. In other words, an infinite tail of the corresponding continued fractions must be the same. Clearly, the \( \mathcal{A}_{\theta'} \) is isomorphic to the \( \mathcal{A}_\theta \), if and only if, the continued fraction of \( \theta' \) coincides with such of \( \theta \).
Remark 4.1. An infinite tail of continued fraction (4.4) is an invariant of the Morita equivalence of the algebra $\mathcal{A}_\theta$. In other words, such a tail is an analog of the $j$-invariant of an elliptic curve.

4.3. **Serre $C^*$-algebra of $\mathcal{E}(k)$**. It is known that the Serre $C^*$-algebra of an elliptic curve $\mathcal{E}(k)$ is isomorphic to the $\mathcal{A}_\theta$ [3, Theorem 1.3.1]. Moreover, if $k$ is a number field, then $\theta$ is a real quadratic number, i.e. the irrational root of a quadratic polynomial with integer coefficients, see [3, Theorem 1.4.1] for a special case. For such a number, the continued fraction (4.4) must be eventually periodic, i.e.

$$\theta = [a_0, \ldots, a_k; b_1, \ldots, b_n], \quad (4.5)$$

where $(b_1, \ldots, b_n)$ is the minimal period of the fraction.

**Corollary 4.2.** The period $(b_1, \ldots, b_n)$ of a continued fraction of $\theta$ corresponding to the twists $\mathcal{E}_t(k)$ of an elliptic curve $\mathcal{E}(k)$ does not depend on the value of $t \in k$.

**Proof.** Up to a cyclic permutation, the period $(b_1, \ldots, b_n)$ is the Morita invariant of the algebra $\mathcal{A}_\theta$, see remark 4.1. The corollary 4.2 follows from the corollary 1.2. \qed

Remark 4.3. Let $\mathbb{Z}$ be the ring of integers. Recall that the arithmetic geometry is focused on schemes over the spectrum $\text{Spec } \mathbb{Z}$ of the ring $\mathbb{Z}$ [Eisenbud & Harris 1999] [2, Chapter II, 4]. In particular, such a theory studies an analogy between $\text{Spec } \mathbb{Z}$ and complex projective plane $\mathbb{C}P^1$, and between the map $\text{Spec } O_k \to \text{Spec } \mathbb{Z}$ and a ramified covering $\mathcal{R} \to \mathbb{C}P^1$, where $O_k$ is the ring of integers of a number field $k$ and $\mathcal{R}$ is a Riemann surface [Eisenbud & Harris 1999] [2, p. 83]. Using corollary 1.2, the analogy can be proved as follows.

Let $R \cong \mathbb{Z}, O_k$ be a discrete commutative ring. Mimicking construction of Section 2.1, one gets a twisted homogeneous coordinate ring $R[t, t^{-1}; \sigma]$ of $R$. Without loss of generality, we can assume $R[t, t^{-1}; \sigma] \cong M_2(R)$, i.e. a ring of the two-by-two matrices over the ring $R$. The norm closure of a self-adjoint representation $\rho$ of the ring $R[t, t^{-1}; \sigma]$ given by the formula (2.1) is a $C^*$-algebra. We denote the $C^*$-algebra by $\mathcal{A}_R$.

Since $R$ is a discrete ring, the $\mathcal{A}_R$ cannot be a Serre $C^*$-algebra. However, corollary 1.2 says the $\mathcal{A}_R \otimes \mathcal{K}$ is such an algebra, since the tensor products correspond to taking the algebraic closure of the ring $R$. In other words,

$$\mathcal{A}_R \otimes \mathcal{K} \cong \mathcal{A}_V, \quad (4.6)$$

where $V$ is a complex projective variety. In particular, if $R \cong \mathbb{Z}$, then $V \cong \mathbb{C}P^1$ and if $R \cong O_k$, then $V \cong \mathcal{R}$.

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