1. Introduction: conjugated differential operators and their properties.

Let us consider the following functional space \( \mathcal{H} := L_1(l^2_{(t,y)}; H) \), where \( H = L_2(\mathbb{R}_x; \mathbb{C}^N) \), \( N \in \mathbb{Z}_+ \), in which the next matrix-differential expressions

\[
\frac{\partial}{\partial y} - L := \frac{\partial}{\partial y} - \sum_{i=0}^{n(L)} a_i(x; y, t) \frac{\partial^i}{\partial x^i} := \mathcal{L},
\]

\[
\frac{\partial}{\partial t} - M := \frac{\partial}{\partial t} - \sum_{j=0}^{n(M)} b_j(x; y, t) \frac{\partial^j}{\partial x^j} := \mathcal{M},
\]

are defined. Here \( l^2_{(t,y)} = l^2 := [0, Y] \times [0, T] \subseteq \mathbb{R}^2_+ \), matrices \( a_i, b_j \in C^1(l^2_{(t,y)}; S(\mathbb{R}; \text{End} \mathbb{C}^N)) \), \( i = 1, n(L), j = 1, n(M) \), where \( S(\mathbb{R}; \text{End} \mathbb{C}^N) \) is the space of matrix-valued Schwartz class coefficient functions, and \( n(M), n(L) \in \mathbb{Z}_+ \) are fixed orders. Let \( \mathcal{H}^* := L_1(l^2_{(t,y)}; H^*) \) be the corresponding conjugated to \( \mathcal{H} \) space.

Let us define on the space \( \mathcal{H}^* \times \mathcal{H} \) an ordinary semi-linear scalar form according to the rule: for any pair \( (\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H} \)

\[
(\varphi, \psi) := \int_{l^2_{(t,y)}} dt dy \int_{\mathbb{R}} dx <\varphi, \psi> = \int_{l^2_{(t,y)}} dt dy \int_{\mathbb{R}} dx (\bar{\varphi}^T \psi),
\]

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where $\langle \cdot , \cdot \rangle$ is the standard semi-linear scalar form on $\mathbb{C}^N$, the bar $\bar{\cdot}$ means the usual complex conjugation and $\bar{\cdot}^T$ means the standard matrix transposition. Concerning the scalar form $(1.2)$ let us study a problem of existence the corresponding to $(1.1)$ conjugated matrix differential operators in the space $H^*$. Differential expressions $L, M : H \to H$ are defined on the domain $\text{Dom}(L, M) \subset H$, being dense in $H$. Then, by definition, the conjugated operators $L^*, M^* : H^* \to H^*$ exist and for all $\varphi, \psi \in W^1_1(\llbracket l_t, y \rrbracket; \text{Dom}(L, M))$ the following equalities

\begin{equation}
(\mathcal{L}^* \varphi, \psi) = (\varphi, \mathcal{L} \psi), \quad (\mathcal{M}^* \varphi, \psi) = (\varphi, \mathcal{M} \psi)
\end{equation}

obviously hold. Then one can consider the corresponding to $(1.3)$ relationships being analogs of the classical Lagrange identities for the operator $\mathcal{L}$

\begin{equation}
\langle \mathcal{L}^* \varphi, \psi \rangle = -\varphi, \mathcal{L} \psi \rangle = -\frac{\partial}{\partial x} Z_{(L)}[\varphi, \psi] + \frac{\partial}{\partial t}(\bar{\varphi}^T \psi)
\end{equation}

and for the operator $\mathcal{M}$

\begin{equation}
\langle \mathcal{M}^* \varphi, \psi \rangle = -\varphi, \mathcal{M} \psi \rangle = -\frac{\partial}{\partial x} Z_{(M)}[\varphi, \psi] - \frac{\partial}{\partial y}(\bar{\varphi}^T \psi),
\end{equation}

where $Z_{(L)}[\varphi, \psi]$ and $Z_{(M)}[\varphi, \psi]$ are some semi-linear forms on $H^* \times H$. From $(1.4)$ and $(1.5)$ one sees that the conjugated operators $L^* : H^* \to H^*$ and $M^* : H^* \to H^*$ are defined if there exists a matrix $\Omega \in C^1(\mathbb{R}^2 \times l^2; \mathbb{C})$ satisfying the expressions

\begin{equation}
\bar{\varphi}^T \psi := \partial \Omega / \partial x, \quad Z_{(L)}[\varphi, \psi] = \partial \Omega / \partial t, \quad Z_{(M)}[\varphi, \psi] = \partial \Omega / \partial y,
\end{equation}

together with conditions

\begin{equation}
\partial \Omega / \partial y, \partial \Omega / \partial t \in W^1_1(\llbracket l_t, y \rrbracket; H).
\end{equation}

By means of integration $(1.4)$ and $(1.5)$ correspondingly with respect to the measures $dt$ and $dx$ we find that a function $\Omega \in L^1(\mathbb{R}^1 \times l^2; \mathbb{C})$, called a Delsarte transmutation generator, exists if there holds the following condition: the differential form

\begin{equation}
Z^{(1)}[\varphi, \psi] := Z_{(L)}[\varphi, \psi]dy + Z_{(M)}[\varphi, \psi]dt + \bar{\varphi}^T \psi dx = d\Omega[\varphi, \psi],
\end{equation}

is exact, that is one can write down the following relationship

\begin{equation}
\Omega[\varphi, \psi] = \Omega_0[\varphi, \psi] + \int_{S(P; P_0)} Z^{(1)}[\varphi, \psi],
\end{equation}

where $S(P; P_0) \subset \mathbb{R} \times l^2$ is some smooth curve connecting a running point $P(x; y, t) \in \mathbb{R} \times l^2$ with a fixed point $P(x_0; y_0, t_0) \in \mathbb{R} \times l^2$, a function $\Omega_0[\varphi, \psi]$ is a semilinear form on $H^* \times H$ constant with respect to variables $(x; y, t) \in \mathbb{R} \times l^2$. It is clear that conditions $(1.7)$ for the mapping $(1.3)$ are certain restrictions concerning $(x; t, y)$–parametric dependence of functions $(\varphi, \psi) \in H^*_0 \times H_0$. Let $H^*_0 \subset H_0$ be a closed subspace of pairs of functions $(\varphi, \psi) \in H^* \times H$ where $H^*_0 \times H_0$. are the correspondingly Hilbert-Schmidt rigged spaces $L^1(\llbracket l_t, y \rrbracket; H^* \times H_0) \times L^1(\llbracket l_t, y \rrbracket; H_0)$. Consider the expression $(1.5)$ for $\varphi \in H^*_0 \subset H^*$ and $\psi \in H_0 \subset H$ satisfying the conditions $(1.7)$. It is enough to assume that $\mathcal{L} \psi = 0$ and $\mathcal{M} \psi = 0$ for $\psi \in H_0$ oraz
\( \mathcal{L}^* \varphi = 0 \) and \( \mathcal{M}^* \varphi = 0 \) for all \( \varphi \in \mathcal{H}_0^* \), where

\[
\mathcal{H}_0^* := \{ \psi(\lambda; \xi) \in \mathcal{H}_-^* : \mathcal{L} \psi(\lambda; \xi) = 0, \quad \mathcal{M}^* \psi(\lambda; \xi) = 0, \quad \psi(\lambda; \xi)|_{x=0^+ y=0^+} = 0 \}
\]

\[
= \{ \psi_\lambda \in \mathcal{H}_-^*, \quad L \psi_\lambda = \lambda \psi_\lambda, \quad \psi(\xi)|_{x=x_0} = 0, \quad (\lambda; \xi) \in \Sigma \}
\]

\[
= \sigma(L, M) \cap \sigma(L^*, M^*) \times \Sigma_{\sigma},
\]

(1.10) \( \mathcal{H}_0^* := \{ \varphi(\lambda; \xi) \in \mathcal{H}_-^* : \mathcal{L}^* \varphi(\lambda; \xi) = 0, \quad \mathcal{M}^* \varphi(\lambda; \xi) = 0, \quad \varphi(\lambda; \xi)|_{x=0^+ y=0^+} = 0 \}
\]

\[
= \varphi_\lambda \in \mathcal{H}^*_-, \quad M \varphi_\lambda = \tilde{\lambda}, \quad \varphi_\lambda(\xi)|_{x=x_0} = 0, \quad (\lambda; \xi) \in \Sigma \]

\[
= \sigma(L, M) \cap \sigma(L^*, M^*) \times \Sigma_{\sigma}
\]

for some fixed point \( x_0 \in \mathbb{R} \) and a "spectral" set \( \Sigma = \sigma(L, M) \cap \sigma(L^*, M^*) \times \Sigma_{\sigma} \subseteq \mathbb{C}^p, \sigma(L, M) \subseteq \mathbb{C} \) is a combined spectrum of operators \( L \) and \( M \) and \( \sigma(L^*, M^*) \subseteq \mathbb{C} \) is that for the adjoint pair \( L^* \) and \( M^* \) in \( H \). Thereby one can formulate the following proposition.

**Proposition 1.1.** Consider a pair of functions \( (\varphi, \psi) \in \mathcal{H}_0^* \times \mathcal{H}_0 \); then there exists a semilinear form \( \tilde{\Omega} : \mathcal{H}_0^* \times \mathcal{H}_0 \to \mathbb{C} \), such that relationships (1.8) and (1.9) hold.

The proposition above makes it possible to study now local properties of operators \( \mathcal{L} \) and \( \mathcal{M} \) in \( \mathcal{H} \) with respect to some their dependence on parameter variables \( (y, t) \in I^2 \). To study this dependence let us proceed to its corresponding analysis in the next Chapter.

2. Structure of Delsarte-Darboux transformations

Consider now another pair of operators \( \tilde{\mathcal{L}} \) and \( \tilde{\mathcal{M}} : \mathcal{H} \to \mathcal{H} \) for which there exist the corresponding conjugated operators \( \tilde{\mathcal{L}}^* \) and \( \tilde{\mathcal{M}}^* : \mathcal{H}^* \to \mathcal{H}^* \). Making use of the Proposition 1 we can easily find another Delsarte transmutation generator \( \tilde{\Omega} \in C^1(\mathbb{R} \times I^2; \mathbb{C}) \) being a semilinear form on suitable pairs of functions \( (\tilde{\varphi}, \tilde{\psi}) \in \mathcal{H}_0^* \times \mathcal{H}_0 \) for which there exist the corresponding semi-linear forms \( \tilde{Z}_{(\tilde{\mathcal{L}})}[\tilde{\varphi}, \tilde{\psi}], \quad \tilde{Z}_{(\tilde{\mathcal{M}})}[\tilde{\varphi}, \tilde{\psi}] \) satisfying the conditions like (1.3), (1.5), (1.6) and (1.7):

\[
(2.1) \quad \tilde{\varphi}^T \tilde{\psi} := \partial \tilde{\Omega}/\partial x, \quad \tilde{Z}_{(\tilde{\mathcal{L}})}[\tilde{\varphi}, \tilde{\psi}] = \partial \tilde{\Omega}/\partial t, \quad \tilde{Z}_{(\tilde{\mathcal{M}})}[\tilde{\varphi}, \tilde{\psi}] = \partial \tilde{\Omega}/\partial y,
\]

with conditions

\[
(2.2) \quad \partial \tilde{\Omega}/\partial y, \quad \partial \tilde{\Omega}/\partial t \in W^1_{\mathcal{L}}(L^2(I^2, \mathcal{H})).
\]

By means analysing conditions (2.1) we can similarly as before state that a function \( \tilde{\Omega} \in C^1(\mathbb{R} \times I^2; \mathbb{C}) \) exists if the differential form

\[
(2.3) \quad \tilde{Z}^{(1)}[\tilde{\varphi}, \tilde{\psi}] := \tilde{Z}_{(\tilde{\mathcal{L}})}[\tilde{\varphi}, \tilde{\psi}]dy + \tilde{Z}_{(\tilde{\mathcal{M}})}[\tilde{\varphi}, \tilde{\psi}]dt + \tilde{\varphi}^T \tilde{\psi}dx = d\tilde{\Omega}[\tilde{\varphi}, \tilde{\psi}],
\]

is exact, that is one can write down the corresponding to (2.2) relationship for \( \tilde{\Omega}[\tilde{\varphi}, \tilde{\psi}] := \tilde{\Omega}(\lambda; \xi; \mu; \eta) \) as follows:

\[
(2.4) \quad \tilde{\Omega}(\lambda; \xi; \mu; \eta) = \tilde{\Omega}_0(\lambda; \xi; \mu; \eta) + \int_{\hat{S}(p, \hat{p}_0)} \tilde{Z}^{(1)}[\tilde{\varphi}(\lambda; \xi), \tilde{\psi}(\mu; \eta)]
\]
for all pairs $(\lambda; \xi)$ and $(\mu; \eta) \in \Sigma$, where by definition,

\begin{equation}
\tilde{H}_0 := \{ \tilde{\psi}(\lambda; \xi) \in \mathcal{H}_*^\prime : \tilde{\psi}(\lambda; \xi) = 0, \tilde{\mathcal{M}} \tilde{\psi}(\lambda; \xi) = 0, \tilde{\psi}(\lambda; \xi)|_{y=0^+} = 0, \tilde{\psi}_{\lambda}(\lambda; \xi) \in \mathcal{H}_-, \tilde{\mathcal{L}} \tilde{\psi}_{\lambda} = \lambda \tilde{\psi}_{\lambda}, \tilde{\psi}(\lambda; \xi)|_{x=\tilde{x}_0} = 0, \}
\end{equation}

\begin{equation}
\tilde{H}_0^\circ := \{ \tilde{\varphi}(\lambda; \eta) \in \mathcal{H}_*^\prime : \tilde{\mathcal{L}}^* \tilde{\varphi}(\lambda; \eta) = 0, \tilde{\mathcal{M}}^* \tilde{\varphi}(\lambda; \eta) = 0, \tilde{\varphi}_{\lambda}(\lambda; \eta) \in \mathcal{H}_-, \tilde{\varphi}(\lambda; \eta)|_{x=\tilde{x}_0} = 0, \}
\end{equation}

for some fixed points $\tilde{x}_0 \in \mathbb{R}$ and the "spectral" parameter set $\Sigma := \sigma(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}) \times \Sigma_\sigma \subset \mathbb{C}^p$. Assume now that there exists an isomorphic bounded mapping $\Omega : \mathcal{H}_0 \subset \tilde{\mathcal{H}}_0$, parametrized by pairs $(\varphi(\lambda; \xi), \tilde{\varphi}(\mu; \eta)) \in \mathcal{H}_0 \times \mathcal{H}_0^\circ$ and $(\psi(\lambda; \xi), \psi(\mu; \eta)) \in \mathcal{H}_0 \times \mathcal{H}_0^\circ$, $(\lambda; \xi)$ and $(\mu; \eta) \in \Sigma$, which we will define as

\begin{equation}
\Omega : \psi(\lambda; \xi) \mapsto \tilde{\psi}(\lambda; \xi) := \psi(\lambda; \xi) \cdot \Omega^{-1} \Omega_0,
\end{equation}

where one supposes that an expression $\Omega^{-1} : L_2^\rho(\Sigma; \mathbb{C}) \rightarrow L_2^\rho(\Sigma; \mathbb{C})$ denotes the inverse operator for the corresponding kernel $\Omega(\lambda; \xi|\mu; \eta) \in L_2^\rho(\Sigma; \mathbb{C}) \times L_2^\rho(\Sigma; \mathbb{C})$, where $\rho$ is some finite Borel measure on the Borel subsets of a "spectral" parameter set $\Sigma$. The corresponding isomorphic bounded mapping between subspaces $\mathcal{H}_0^\circ$ and $\mathcal{H}_0^\circ$, i.e. $\tilde{\Omega}^\circ : \mathcal{H}_0^\circ \Rightarrow \mathcal{H}_0^\circ$ such that for any pair $(\varphi(\lambda; \xi), \tilde{\varphi}(\mu; \eta)) \in \mathcal{H}_0^\circ \times \mathcal{H}_0^\circ$

\begin{equation}
\tilde{\Omega}^\circ : \varphi(\lambda; \xi) \mapsto \tilde{\varphi}(\lambda; \xi) = \varphi(\lambda; \xi) \cdot \Omega^\circ, \Omega^{-1} \Omega_0,
\end{equation}

where, by definition, the kernel $\Omega^\circ(\lambda; \xi) := \tilde{\Omega}^\circ(\lambda; \xi) \in L_2^\rho(\Sigma; \mathbb{C}) \times L_2^\rho(\Sigma; \mathbb{C})$, $(\lambda; \xi) \in \Sigma$. It is easy to see now that the following proposition holds.

**Proposition 2.1.** The constructed above pair of mappings $(\Omega^\circ, \Omega)$ is consistent, i.e. there exists such a kernel $\Omega(\lambda; \xi) \in L_2^\rho(\Sigma; \mathbb{C}) \times L_2^\rho(\Sigma; \mathbb{C})$, that conditions (2.6) and (2.7) hold.

**Proof.** Indeed, by using expressions (2.3), (2.8), (2.6) and (2.7) we easily obtain that

\[
d\tilde{\Omega} = \Omega_0 \Omega^{-1} d\Omega \Omega^{-1} \Omega_0 = -d (\Omega_0 \Omega^{-1} \Omega_0),
\]

whence the mapping $\tilde{\Omega} = -\Omega_0 \Omega^{-1} \Omega_0$ and the condition $\tilde{\Omega}_0 = -\Omega_0$ holds.  

Since the functional spaces $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0$ are consistent, the expressions for $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{H}}_0^\prime$ and $\tilde{\mathcal{M}}^\prime$ prove to be differential too. The corresponding mappings $\Omega$ and $\tilde{\Omega}^\circ : \mathcal{H} \rightarrow \mathcal{H}$ are often called Delsarte-Darboux type transformations and were for the first time used by Darboux and in general form was studied by Delsarte and J. Lions.

Consider now the compatible pair of invertible Delsatre mappings $(\Omega, \Omega^\circ)$ from the closed functional spaces $\mathcal{H}_0 \times \mathcal{H}_0^\circ \subset \mathcal{H}_- \times \mathcal{H}_-^\prime$ to closed subspaces $\mathcal{H}_0 \times \mathcal{H}_0^\circ \subset \mathcal{H}_- \times \mathcal{H}_-^\prime$, reduced naturally upon $\mathcal{H} \times \mathcal{H}^\circ$. It means that the following diagram

\[
\begin{array}{c}
\mathcal{M}, \mathcal{L} \downarrow \mathcal{H} \quad \begin{array}{c}
\Omega \quad \Omega^\circ \\
\mathcal{H} \quad \mathcal{H}
\end{array} \downarrow \tilde{\mathcal{L}}, \tilde{\mathcal{M}}
\end{array}
\]
is commutative, and consequently the relationships \( \Omega \cdot L = \hat{L} \cdot \Omega \) and \( \Omega \cdot M = \hat{M} \cdot \Omega \) hold. These relationships connect evolution operators \( L \) and \( M \) in the whole space \( \mathcal{H} \) with the corresponding evolution operators \( \hat{L} \) and \( \hat{M} \).

In order to define an exact form of mappings \( \Omega : \mathcal{H} \to \mathcal{H} \) and \( \Omega^\ominus : \mathcal{H}^* \to \mathcal{H}^* \) we will make use of the mappings (2.6) and (2.7) on fixed elements \((\varphi(\lambda; \xi), \psi(\mu; \eta)) \in \mathcal{H}_0 \times \mathcal{H}_0, (\lambda; \xi), (\mu; \eta) \in \Sigma.. \) Namely, from (2.4) we get that

\[
\tilde{\psi}(\lambda; \xi) = \Omega(\psi(\lambda; \xi)) := \int \int \Omega^{-1}_x(\mu; \eta|\nu; \gamma) \Omega_x(\nu; \gamma|x; \lambda),
\]

(2.8) \[
\tilde{\varphi}(\lambda; \xi) = \Omega^\ominus(\varphi(\lambda; \xi)) := \int \int \Omega^\ominus^{-1}_x(\mu; \eta|\nu; \gamma) \Omega^\ominus_x(\nu; \gamma|x; \lambda).
\]

that makes it possible to define the operators \( \Omega : \mathcal{H} \to \mathcal{H} \) and \( \Omega^\ominus : \mathcal{H}^* \to \mathcal{H}^* \) as

\[
\Omega : = 1 - \int \int \Omega^{-1}_x(\mu; \eta|\nu; \gamma) \tilde{\psi}(\mu; \eta) \Omega^{-1}_x(\mu; \eta|\nu; \gamma) \times \int \int_{P_0} Z^{(m-1)}(\varphi(\nu; \gamma), (\cdot)) \]

(2.9) \[
\Omega^\ominus : = 1 - \int \int \Omega^\ominus^{-1}_x(\mu; \eta|\nu; \gamma) \tilde{\varphi}(\mu; \eta) \Omega^\ominus^{-1}_x(\mu; \eta|\nu; \gamma) \times \int \int_{P_0} Z^{(m-1)}(\varphi(\nu; \gamma), (\cdot)) \]

with \( \rho \) being as before some finite Borel measure on the Borel subsets of a "spectr-ral" parameter set \( \Sigma \).

Now based on expressions (2.9), one can easily find the "dressed" operators \( \hat{L}, \hat{M} : \mathcal{H} \to \mathcal{H} \), and thereby their coefficient matrix functions subject to the corresponding coefficients of operators \( L, M : \mathcal{H} \to \mathcal{H} \), which are also called the Darboux-Backlund transformations [7].

Note also that the compatibility condition for the dressed differential operators \( \hat{L}, \hat{M} \) is equivalent to some system of nonlinear evolution equations in partial derivatives and often this pair is called [8, 5, 7] a Lax type or a Zakharov-Shabat pair.

Consider now the structure of "dressed" operators

\[
\hat{L} = \Omega L \hat{\Omega}^{-1}, \quad \hat{M} = \Omega M \Omega^{-1}
\]

as elements of orbits of some Volterra group \( G_- \) [14, 15]. As one can see from (2.11), these operators lie correspondingly on orbits of elements \( L, M \in G^* \) with respect to the natural co-adjoint group action of the group of pseudo-differential operators \( G_- \), whose Lie co-algebra \( G^*_+ \) consists of Volterra type integral operators.
of the form \( l := \sum_{i=0}^{n(l)} a_i \partial^{-1} \hat{b}_i^T \), where \( n(l) \in \mathbb{Z}_+ \) is some finite number, i.e.

\[ (2.11) \]

\[ G^*_\rho = \{ l = \sum_{i=0}^{n(l)} a_i \partial^{-1} \hat{b}_i^T : a_i, \hat{b}_i \in C^1(I_{(\mathbb{R};E)}^N) \} \]

Let us show that these orbits leave the space \( G^*_\rho \) invariant, i.e. the "dressed" operators \( \tilde{\mathcal{L}} \) and \( \tilde{\mathcal{M}} : \mathcal{H} \to \mathcal{H} \) under transformation \( (2.14) \) persist to be differential with conservation of their orders. To do it let us consider an arbitrary pseudo-differential operator \( \mathcal{P} : \mathcal{H} \to \mathcal{H} \) and note that the following identity

\[ (2.12) \]

\[ \text{Tr}(\mathcal{P} f \partial^{-1} \hat{h}^T) := (\mathcal{P} f, \partial^{-1} \hat{h}^T)_\mathcal{G} = (h, \mathcal{P}_+ f)_H \]

holds, where, by definition, \((\cdot, \cdot)_{\mathcal{H}}\) denotes the scalar product in the Hilbert space \( H \),

\[ \text{Tr}(\cdot) := \int dx \text{res} \text{Sp}(\cdot), \]

and the operation \((\cdot)_+\) means the projection upon the differential part of a given pseudo-differential expression. Based on the relationship \( (2.12) \) it is easy to prove the following lemma.

**Lemma 2.2.** A pseudo-differential operator \( \mathcal{P} : \mathcal{H} \to \mathcal{H} \) is pure differential if and only if the following equality

\[ (2.13) \]

\[ (h, (\mathcal{P} \partial^i)_+ f)_H = (h, \mathcal{P}_+ \partial^i f)_H \]

holds with respect to the scalar product \((\cdot, \cdot)_{\mathcal{H}}\) in \( H \) for all \( i \in \mathbb{Z}_+ \) and any dense \( \mathcal{H}^* \times \mathcal{H} \) set of pairs \((f, h) \in \mathcal{H}^* \times \mathcal{H}\). That is the condition \( (2.13) \) is equivalent to equality \( \mathcal{P}_+ = \mathcal{P} \).

Making use of this Lemma in the case when \( \mathcal{P} := \mathcal{L} : \mathcal{H} \to \mathcal{H} \) and taking into consideration the condition \( (2.13) \), one gets that

\[ (h, (\hat{\mathcal{L}} \partial^i)_+ f) = (h, (\mathcal{L}(\partial^i \Omega^{-1} \partial^i)_+ f) \]

\[ = (h, \partial^i \partial^i f) - (h, \{[\mathcal{L}(\Omega^{-1} \Omega^{-1} \partial^i + \Omega^{-1} \mathcal{L} \Omega^{-1}) \partial^i \}_+ f) \]

\[ = (h, \partial^i \partial^i f) - \text{Tr} \{ (\mathcal{L}(\Omega^{-1} \partial^i + \Omega^{-1} \partial^i f \partial^{-1} \hat{h}^T) \}

\[ = (h, \partial^i \partial^i f) - \text{Tr} \{ (1 - \hat{\mathcal{L}} \partial^i \hat{\mathcal{L}} f \partial^{-1} \hat{h}^T) \}

\[ = (1 - \hat{\mathcal{L}} f \partial^{-1} \hat{h}^T) \}

\[ \equiv \text{Tr} \{ (\hat{\mathcal{L}} f \partial^{-1} \hat{h}^T) \} \]

\[ (2.14) \]

When deriving \( (2.14) \) we made use of the equalities \( \mathcal{L} \psi = 0, \mathcal{L}^* \varphi = 0 \) for any pair \((\varphi, \psi) \in \mathcal{H}_0^* \times \mathcal{H}_0 \) and the evident condition \( \mathcal{L}_+ = \mathcal{L} \). Thereby in accordance with lemma 1, the operator \( \hat{\mathcal{L}} : \mathcal{H} \to \mathcal{H} \) remains to be differential and, moreover, the order \( \text{ord} \hat{\mathcal{L}} = \text{ord} \mathcal{L} \) that follows from the definition \( (2.10) \). Similarly, the same proposition holds also for the "dressed" operator \( \hat{\mathcal{M}} : \mathcal{H} \to \mathcal{H} \), i.e. \( \mathcal{M}_+ = \mathcal{M} \) and \( \text{ord} \hat{\mathcal{M}} = \text{ord} \mathcal{M} \). As a conclusion from the results obtained above one can formulate the following proposition.
Proposition 2.3. The pair of "dressed" differential operators \( \tilde{L}, \tilde{M} : \mathcal{H} \to \mathcal{H} \) of the form (2.10), obtained as a result of the Delsarte-Darboux type transformation from a compatible Zakharov-Shabat commuting pair of differential operators \( L, M : \mathcal{H} \to \mathcal{H} \) in the form (1.1) persists to be a compatible pair of commuting differential operators in \( \mathcal{H} \) preserving their differential orders. The corresponding coefficient matrix functions of the Delsarte-Darboux transformed differential operators \( \tilde{L}, \tilde{M} : \mathcal{H} \to \mathcal{H} \) define a so-called Backlund-Darboux transformation for the coefficient matrix functions of the initially chosen compatible pair \( L, M : \mathcal{H} \to \mathcal{H} \) of differential operators.

From the practical point of view at proposition 3, it is clear that the Delsarte-Darboux transformations are especially useful for construction of a wide class of so-called soliton \([7, 10, 8, 9]\) and algebraic solutions to the corresponding system of nonlinear evolution differential equations, which is equivalent to the compatibility condition for the obtained pair of "dressed" operators (1.1). A great deal of papers is devoted (see, for example \([7, 9]\)) to such calculations, where particular solutions of solitons and other types were built for different evolution differential equations of mathematical physics.

3. General structure of Delsarte-Darboux transformations: a differential-geometric aspect

A preliminary analysis of the Delsarte-Darboux type transformation operators constructed above for differential operator expressions in the case of a single variable \( x \in \mathbb{R} \) shows that its form is rather restrictive concerning a class of possible transformations for operator differential expressions depending on two and more variables and admitting Lax type representations \([14, 15, 8, 13, 5]\). Therefore it is important to consider a nontrivial multi-dimensional generalization of the proposed above scheme for construction these Delsarte-Darboux type transformations. Below we will present some short sketch of such an approach to this problem based on the preliminary results obtained in \([20, 11, 19]\).

We consider as before a parametric functional space \( \mathcal{H} := L_1(t; H), l_t := [0,T] \in \mathbb{R}_+ \), where now \( H := L_2(\mathbb{R}^2; C^N) \), in which acts a (2+1)-dimensional differential operator expression \( L : \mathcal{H} \to \mathcal{H} \) of the form

\[
L = \partial_t - L(t; x, y) \partial^i\partial^j,
\]

with coefficients \( u_{ij} \in C^1(l; \mathcal{S}(\mathbb{R}^1; \text{End} C^N)), i, j = 1, n(L) \). Applying the same scheme as used above we find that for the expression (3.1) the standard identity

\[
\langle L^\ast \varphi, \psi \rangle - \langle \varphi, L \psi \rangle = \frac{\partial}{\partial t} (\varphi^T \psi)
\]

holds for all pairs \( (\varphi, \psi) \in D(L^\ast) \times D(L) \subset \mathcal{H}^* \times \mathcal{H} \), where \( Z^{(x)}[\varphi, \psi] \) and \( Z^{(y)}[\varphi, \psi] \) are some easily computable semi-linear forms on \( \mathcal{H}^* \times \mathcal{H} \). From (37) with respect...
to the oriented measure $dt \wedge dx \wedge dy$ one gets easily that
\[(<\mathcal{L}\varphi, \psi> - <\varphi, \mathcal{L}\psi>) dt \wedge dx \wedge dy = d(\bar{\varphi}^T \psi \wedge dx \wedge dy)\]
(3.3)

\[+Z^{(x)}[\varphi, \psi]dy \wedge dt - Z^{(y)}[\varphi, \psi]dx \wedge dt := dZ^{(2)}[\varphi, \psi],\]
where, by definition,
(3.4)

\[Z^{(2)}[\varphi, \psi] = \bar{\varphi}^T \psi dx \wedge dy + Z^{(x)}[\varphi, \psi] dy \wedge dt + Z^{(y)}[\varphi, \psi] dt \wedge dx\]
is a semilinear on $\mathcal{H}_+^* \times \mathcal{H}_-$ differential 2-form on $\mathbb{R}^2 \times l$. Therefore for all $t \in l$ and any $\lambda, \xi \in \mathcal{H}_0^+ \times \mathcal{H}_0 \subset \mathcal{H}_0 \subset \mathcal{H}_- \subset \mathcal{H}_+^* \times \mathcal{H}_-$, from a f closed subspace of the correspondingly Hilbert-Schmidt rigged \[\mathcal{H}_+^* \times \mathcal{H}_-\] parametric functional spaces $\mathcal{H}_+^* \times \mathcal{H}_-$ with $\Sigma \subset \mathbb{C}^p$ being some “spectral” parameter set, the expression on the right-hand side of relationship (3.3) can be made become identically zero if the conditions (3.5)

\[\mathcal{L}^* \varphi = 0, \quad \mathcal{L}\psi = 0\]
hold on $\mathcal{H}_0^* \times \mathcal{H}_0$. Thereby one can define the following closed dense subspaces $\mathcal{H}_0^* \subset \mathcal{H}_+^*$ and $\mathcal{H}_0 \subset \mathcal{H}_-$ similarly to (1.10) as
\[\mathcal{H}_0 : = \{\psi(\lambda; \xi) \in \mathcal{H}_- : \mathcal{L}\psi(\lambda; \xi) = 0, \quad \mathcal{M}^* (\lambda; \xi) = 0, \psi_{\lambda} \in \mathcal{H}_+^*, \quad L\psi_{\lambda} = \lambda \psi_{\lambda}, \quad \psi(\xi)|_t = 0, \quad t \in l, \quad \lambda, \xi \in \Sigma = \sigma(L, M) \cap \bar{\sigma}(L^*, M^*) \times \Sigma_\sigma\},\]
(3.6)

\[\mathcal{H}_0^* : = \{\varphi(\lambda; \xi) \in \mathcal{H}_+^* : \mathcal{L}^* \varphi(\lambda; \xi) = 0, \quad \mathcal{M}^* \varphi(\lambda; \xi) = 0, \varphi_{\lambda} \in \mathcal{H}_-^*, \quad L\varphi_{\lambda} = \lambda, \varphi(\xi)|_t = 0, \quad t \in l, \quad \lambda, \xi \in \Sigma = \sigma(L, M) \cap \bar{\sigma}(L^*, M^*) \times \Sigma_\sigma\},\]
where we imposed, correspondingly, on the spaces $\mathcal{H}_+^*$ and $\mathcal{H}_-$ some boundary conditions at $\Gamma \subset \mathbb{R}^2$, where $\Gamma$ is some one-dimensional smooth curve in $\mathbb{R}^2$. Now the differential 2-form (3.3) becomes closed, i.e. $dZ^{(2)}[\varphi, \psi] = 0$, that due to the Poincare lemma \[\mathcal{L}^2 \mathcal{L} = 0\] brings about the following equality
(3.7)

\[Z^{(2)}[\varphi(\lambda; \xi), \psi(\mu; \eta)] = d\Omega^{(1)}[\varphi(\lambda; \xi), \psi(\mu; \eta)]\]
for some differential 1-form $\Omega^{(1)}[\varphi(\lambda; \xi), \psi(\mu; \eta)]$ on the space $\mathbb{R}^3$ and all pairs $(\varphi(\xi), \psi(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0, \xi, \eta \in \Sigma$. Thus, the following proposition similar to one of \[\mathcal{L}^2 \mathcal{L} = 0\] holds.

**Proposition 3.1.** If the differential 2-form (3.7) is closed for all pairs $(\varphi(\xi), \psi(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0, \xi, \eta \in \Sigma$, and vice versa if the 2-forms $Z^{(2)}[\varphi(\lambda; \xi), \psi(\mu; \eta)]$, $\xi, \eta \in \Sigma$, are closed, then the pair of conjugated differential operators $(L, L^*)$ is adjoint with respect to the scalar form on $\mathcal{H}_+^* \times \mathcal{H}_-$.

Applying now the Stokes theorem \[\mathcal{L}^2 \mathcal{L} = 0\] for a closed 2-form \[\mathcal{L}^2 \mathcal{L} = 0\] on $\mathbb{R}^2 \times l$, we obtain that
\[\int_{S^{(2)}(\sigma^{(1)}, \sigma_0^{(1)})} Z^{(2)}[\varphi(\lambda; \xi), \psi(\mu; \eta)] = \int_{\partial S^{(2)}(\sigma^{(1)}, \sigma_0^{(1)})} \Omega^{(1)}[\varphi(\lambda; \xi), \psi(\mu; \eta)]\]
for some piecewise imbedded smooth compact two-dimensional surface \( S^{(2)}(\sigma, \sigma_0) \subset \mathbb{R}^2 \times l \) with the boundary \( \partial S^{(2)}(\sigma^{(1)}, \sigma_0^{(1)}) = \sigma^{(1)} - \sigma_0^{(1)} \), where \( \sigma^{(1)}, \sigma_0^{(1)} \subset \mathbb{R}^2 \times l \) are some closed homological one-dimensional cycles without self-intersections parametrized correspondingly by running point \( P(x, y; t) \in \mathbb{R}^2 \times l \) and a fixed point \( P(x_0, y_0; t_0) \in \mathbb{R}^2 \times l \).

Making use of the surface integral (3.8) and assuming that the closed cycle \( \sigma_0^{(1)} \subset \mathbb{R}^2 \times l \) is fixed, one can define the following mappings for the corresponding Delsarte-Darboux transformations on pairs of functions \((\varphi, \psi) \in \mathcal{H}_0^* \times \mathcal{H}_0^*\):

\[
(3.8) \quad \tilde{\psi}(\lambda; \xi) = \text{\( \Omega(\psi(\lambda; \xi)): = \int_\Sigma \int_\Sigma d\rho(\mu; \eta)d\rho(\nu; \gamma)\psi(\mu; \eta) \times \Omega^{-1}(\mu; \eta|\nu; \gamma)\Omega_0(\nu; \gamma|\lambda; \xi), \)}
\]

\[
(3.9) \quad \tilde{\varphi}(\lambda; \xi) = \text{\( \Omega^*(\varphi(\lambda; \xi)): = \int_\Sigma \int_\Sigma d\rho(\mu; \eta)d\rho(\nu; \gamma)\varphi(\mu; \eta) \times \Omega^{*,-1}(\mu; \eta|\nu; \gamma)\Omega_0^*(\nu; \gamma|\lambda; \xi). \)}
\]

where the Delsarte transmutation generator expressions \( \Omega(\lambda; \xi|\mu; \eta) \) and \( \Omega^*(\lambda; \xi|\mu; \eta) \in L_2(\Sigma; \mathbb{C}) \times L_2(\Sigma; \mathbb{C}), \quad (\lambda, \xi), (\mu, \eta) \in \Sigma, \) are as before considered to be nondegenerate kernels from \( L_2(\Sigma; \mathbb{C}) \times L_2(\Sigma; \mathbb{C}). \) The following proposition concerning the pair of spaces \( \tilde{\mathcal{H}}_0 \supset \tilde{\psi} \) and \( \tilde{\mathcal{H}}_0 \supset \tilde{\varphi} \) holds.

**Proposition 3.2.** The pair of functional spaces \( \tilde{\mathcal{H}}_0 \) and \( \mathcal{H}_0 \) consisting correspondingly of functions \((\tilde{\varphi}, \tilde{\psi}) \in \mathcal{H}_0^* \times \mathcal{H}_0^*\) defined by the expressions (3.8), (3.9) can be equivalently characterized as follows:

\[
(3.10) \quad \tilde{\mathcal{H}}_0 : = \{ \tilde{\psi}(\lambda; \xi) \in \mathcal{H}_0^* : \tilde{\mathcal{L}}\tilde{\psi}(\lambda; \xi) = 0, \}
\]

\[
\tilde{\mathcal{M}}\tilde{\psi}(\lambda; \xi) = 0, \quad \tilde{\psi}(\lambda; \xi)|_{t_0} = \tilde{\psi}_\lambda \in \mathcal{H}_-^-, \quad \tilde{\mathcal{L}}\tilde{\psi}_\lambda = \lambda\tilde{\psi}_\lambda, \]

\[
\tilde{\varphi}(\lambda; \xi)|_{\tilde{F}} = 0, \quad (\lambda; \xi) \in \Sigma = \sigma(\tilde{L}, \tilde{M}) \cap \sigma(\tilde{L}^*, \tilde{M}^*) \times \Sigma_\sigma \}
\]

\[
\tilde{\mathcal{M}}^*\tilde{\varphi}(\lambda; \eta) = 0, \quad \tilde{\mathcal{L}}^*\tilde{\varphi}_\lambda = \lambda\tilde{\varphi}_\lambda, \quad \tilde{\varphi}(\lambda; \eta)|_{t_0} = \tilde{\varphi}_\lambda \in \mathcal{H}_0^*, \]

\[
\tilde{\varphi}(\lambda; \eta)|_{\tilde{F}} = 0, \quad (\lambda; \eta) \in \Sigma = \sigma(\tilde{L}, \tilde{M}) \cap \sigma(\tilde{L}^*, \tilde{M}^*) \times \Sigma_\sigma \}
\]

\[
(3.11) \quad \tilde{\mathcal{H}}_0^* : = 0
\]

for some piecewise smooth curve \( \tilde{\Gamma} \subset \mathbb{R}^2. \)

Based now on this Proposition, the mappings (3.10) can be extended naturally on the whole space \( \mathcal{H}_0^* \times \mathcal{H}_0^* \) by means of the just used before classical method of
variation of constants \cite{19} \cite{18} \cite{11} and give rise easily to the exact forms of the pair of Delsarte-Darboux mapping \((\Omega, \Omega^\otimes)\) upon the whole space \(\mathcal{H}^* \times \mathcal{H}\):

\[
\Omega : = 1 - \int d\rho(\mu; \eta) \int d\rho(\nu; \gamma) \hat{\psi}(\mu; \eta) \Omega^{-1}_0(\mu; \eta|\nu; \gamma)
\times \int_{S(2)(\sigma^{(1)}, \sigma_0^{(1)})} Z^{(m-1)}[\varphi(\nu; \gamma), (\cdot)]
\]

\[
(3.12) \quad \Omega^\otimes : = 1 - \int d\rho(\mu; \eta) \int d\rho(\nu; \gamma) \hat{\varphi}(\mu; \eta) \Omega^\otimes_0^{-1}(\mu; \eta|\nu; \gamma)
\times \int_{S(2)(\sigma^{(1)}, \sigma_0^{(1)})} Z^{(m-1)}[\tau(\cdot), \psi(\nu; \gamma)],
\]

defined for some imbedded into \(\mathbb{R}^2 \times \lambda\) piecewise smooth two-dimensional surface \(S(2)(\sigma^{(1)}, \sigma_0^{(1)}) \subset \mathbb{R}^2 \times \lambda\), spanned between two closed homological cycles \(\sigma^{(1)}\) and \(\sigma_0^{(1)} \subset \mathbb{R}^2 \times \lambda\) as its boundary, that is \(\partial S(2)(\sigma^{(1)}, \sigma_0^{(1)}) := \sigma^{(1)} - \sigma_0^{(1)}\). It is seen from \((3.12)\) that found above Delsarte transmutation operators \(\Omega : \mathcal{H} \to \mathcal{H}\) and \(\Omega^\otimes : \mathcal{H}^* \to \mathcal{H}^*\) are bounded of Volterra type integral operators, strongly depending on a measure \(\rho\) on the ”spectral” parameter space \(\Sigma\) and some piecewise smooth two-dimensional surface \(S(2)(\sigma^{(1)}, \sigma_0^{(1)})\) parametrized by a running point \(P(x, y; t) \in \mathbb{R}^2 \times \lambda\) and a fixed point \(P(x_0, y_0; t_0) \in \mathbb{R}^2 \times \lambda\).

Making now use of the bounded Delsarte-Darboux integral transformation operators \((3.12)\) of Volterra type, one can now as before to construct the corresponding Delsarte-Darboux transformed differential operator \(\hat{\mathcal{L}} : \mathcal{H} \to \mathcal{H}\) as follows:

\[
(3.13) \quad \hat{\mathcal{L}} = \mathcal{L} + [\Omega, \mathcal{L}] \Omega^{-1}.
\]

Since the expression \((3.13)\) contains the inverse integral operator \(\Omega^{-1} : \mathcal{H} \to \mathcal{H}\), it can be found from \((3.12)\) making use of the symmetry properties between closed subspaces \(\mathcal{H}_0^* \times \mathcal{H}_0\) and \(\mathcal{H}_0^* \times \mathcal{H}_0\):

\[
\Omega^{-1} : = 1 - \int d\rho(\mu; \eta) \int d\rho(\nu; \gamma) \psi(\mu; \eta) \tilde{\Omega}_0^{-1}(\mu; \eta|\nu; \gamma)
\times \int_{S(2)(\sigma^{(1)}, \sigma_0^{(1)})} Z[\hat{\varphi}(\nu; \gamma), (\cdot)]
\]

\[
(3.14) \quad \Omega^\otimes_{0, -1} : = 1 - \int d\rho(\mu; \eta) \int d\rho(\nu; \gamma) \varphi(\mu; \eta) \tilde{\Omega}^\otimes_0^{-1}(\mu; \eta|\nu; \gamma)
\times \int_{S(2)(\sigma^{(1)}, \sigma_0^{(1)})} Z^{(m-1)}[\tau(\cdot), \psi(\nu; \gamma)],
\]

for \((\hat{\varphi}(\lambda; \xi), \hat{\psi}(\mu; \eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0\), \((\lambda; \xi), (\mu; \eta) \in \Sigma\), satisfying the conditions \((3.10)\).

As a results of direct calculations in \((3.13)\) based on expressions \((3.14)\) one can find the corresponding Delsarte-Darboux transformed coefficient functions of the transformed operator \(\hat{\mathcal{L}} : \mathcal{H} \to \mathcal{H}\) parametrized by piecewise smooth closed one-dimensional homological cycles \(\sigma^{(1)}, \sigma_0^{(1)} \subset \mathbb{R}^2 \times \lambda\). We don’t present here these expressions in the general case of operator \((3.14)\) as they are too cumbersome for
writing down. Application of the constructions developed in the article we are going to deliver in detail in Part 2.

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