On the dynamics of magnetorotational turbulent stresses

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ABSTRACT

The turbulent stresses that lead to angular momentum transport in accretion discs have often been treated as resulting from an isotropic effective viscosity, related to the pressure through the alpha parametrization of Shakura and Sunyaev. This simple approach may be adequate for the simplest aspects of accretion disc theory, and was necessitated historically by an incomplete understanding of the origin of the turbulence. More recently, Balbus and Hawley have shown that the magnetorotational instability provides a robust mechanism of generating turbulent Reynolds and Maxwell stresses in sufficiently ionized discs. The alpha viscosity model fails to describe numerous aspects of this process. The present paper introduces a new analytical model that aims to represent more faithfully the dynamics of magnetorotational turbulent stresses and bridge the gap between analytical studies and numerical simulations. Covariant evolutionary equations for the mean Reynolds and Maxwell tensors are presented, which correctly include the linear interaction with the mean flow. Non-linear and dissipative effects, in the absence of an imposed magnetic flux and in the limit of large Reynolds number and magnetic Reynolds number, are modelled through five non-linear terms that represent known physical processes and are strongly constrained by symmetry properties and dimensional considerations. The resulting model explains the development of statistically steady, anisotropic turbulent stresses in the shearing sheet, a local representation of a differentially rotating disc, in agreement with numerical simulations. It also predicts that purely hydrodynamic turbulence is not sustained in a flow that adequately satisfies Rayleigh’s stability criterion. The model is usually formally hyperbolic and therefore ‘causal’, and guarantees the realizability of the stress tensors. It should be particularly useful in understanding the dynamics of warped, eccentric and tidally distorted discs, non-Keplerian accretion flows close to black holes, and a variety of time-dependent accretion phenomena.

Key words: accretion, accretion discs – hydrodynamics – MHD – turbulence.

1 INTRODUCTION

The magnetorotational instability is one of the most important instabilities in astrophysical fluid dynamics. It applies to a differentially rotating, electrically conducting fluid in which the angular velocity decreases in magnitude away from the axis of rotation.1 In the presence of a weak magnetic field of arbitrary configuration, such a flow is subject to a dynamical instability, with a growth rate comparable to the shear rate of the flow (Velikhov 1959; Chandrasekhar 1960; Fricke 1969; Acheson 1978; Balbus & Hawley 1991, 1992; Papaloizou & Szuszkiewicz 1992; Balbus 1995; Foglizzo & Tagger 1995; Ogilvie & Pringle 1996; Terquem & Papaloizou 1996).

The principal application of the magnetorotational instability is to accretion discs, in which the profile of angular velocity is fixed by Kepler’s third law. The non-linear development of the instability leads to sustained magnetohydrodynamic (MHD) turbulence, which transports angular momentum outwards in a vain attempt to neutralize the destabilizing gradient of angular velocity (Hawley, Gammie & Balbus 1995; Brandenburg et al. 1995; Stone et al. 1996; Balbus & Hawley 1998). Owing to the generality of the conditions for instability, however, further applications exist to stellar interiors and other astrophysical objects, and possibly to geophysical situations. Laboratory experiments are also in progress (Ji, Goodman & Kageyama 2001).

In one sense, the magnetorotational instability is generally considered to have ‘solved’ the problem of the effective viscosity of...
accretion discs, through providing a robust means of angular momentum transport. However, the realization of the importance and ubiquity of the instability has, in some ways, made matters more difficult. A strict view might be taken that any calculation of an accretion disc must now involve a numerical simulation of three-dimensional MHD turbulence. These simulations are demanding to execute, complicated to analyse and still restricted in their scope and physical content.

To illustrate the limitations of the direct numerical approach, let us consider how the computational requirements scale with the parameters of the problem being studied. Most simulations conducted so far have been based on the shearing box (Hawley et al. 1995), which is a local representation of a differentially rotating disc. The simulated volume is comparable to $H^3$, where $H$ is the vertical scale-height or semithickness of the disc, and the models can be run for many times the dynamical (orbital) time-scale. Let us accept, for the purposes of this argument, that recent versions of such simulations (e.g. Miller & Stone 2000) have achieved an adequate fidelity to physical reality (although questions indeed remain concerning numerical resolution and convergence, vertical boundary conditions, thermal and radiative physics, etc.). How much more difficult would it be to simulate a global model of a realistic thin disc over many times the viscous time-scale, as would be required in a typical application? Suppose that the disc has a constant angular semithickness $H/r$ and is simulated in a conical wedge in spherical polar coordinates $(r, \theta, \phi)$ with logarithmic grid spacing in $r$ between $r_a$ and $r_{out}$, and uniform spacing in $\theta$ and $\phi$. To achieve a resolution comparable to that of a shearing box, the number of grid zones must increase by a factor of the order of $(r/H) \ln (r_{out}/r_{in})$ in $r$, and a factor of the order of $r/H$ in $\phi$. The global viscous time-scale exceeds the dynamical time-scale in the inner part of the disc (where the Courant condition on the time-step is most severe) by a factor of the order of $(1/\alpha) (r/H)^3 (r_{out}/r_{in})^{3/2}$, where $\alpha$ is the viscosity parameter of Shakura & Sunyaev (1973). Therefore the computational requirements increase by a factor of the order of

$$\frac{1}{\alpha} \left( \frac{r_{out}}{r_{in}} \right)^2 \left( \frac{r_{out}}{r_{in}} \right)^{3/2} \ln \left( \frac{r_{out}}{r_{in}} \right), \quad (1)$$

Reasonable estimates of this factor for discs around young stars, in cataclysmic variables and in X-ray binaries, respectively, are $10^{13}$, $10^{11}$ and $10^{13}$, or even larger. (These numbers are reduced if the inner part of the disc is truncated by a stellar magnetosphere.) For this reason, although some aspects of large-scale disc dynamics can now be addressed in direct numerical simulations (e.g. Hawley 2001), global studies of realistic thin discs are currently inaccessible.

How then are we to study situations such as the thermal–viscous instability of discs in cataclysmic variables, the dynamics of warped or eccentric discs, tidally distorted discs in binary stars and protoplanetary systems, non-Keplerian accretion flows close to black holes, or the propagation of waves in discs? The traditional approach, of course, has been to treat the turbulence as an effective viscosity given by the alpha parametrization of Shakura & Sunyaev (1973). Despite suspicions that this treatment is likely to be inaccurate in various respects, very few studies have aimed specifically at testing this issue (Abramowicz, Brandenburg & Lasota 1996; Torkelsson et al. 2000) and it is still unclear in what respects magnetorotational turbulence does or does not behave as an effective viscosity. In a standard accretion disc the rate-of-strain tensor has a dominant $r \phi$ component, and an isotropic viscous model predicts that the $r \phi$ component of the turbulent stress would similarly dominate. This is not true of magnetorotational turbulence, in which other stress components (e.g. $\phi \phi$) are larger. However, for the simplest problems such as the evolution of the surface density of a standard accretion disc, these additional stress components play essentially no role in the dynamics. The alpha model is probably perfectly adequate in these circumstances, where one dimensionless number, $\alpha$, is parametrizing a single physical quantity, the vertically integrated stress coefficient $T_{r \phi}$.3

In situations such as warped, eccentric, tidally distorted and non-Keplerian discs there is a large-scale, possibly slowly varying, velocity field that is different from that in a standard disc. From the perspective of a local observer, however, the flow resembles a shearing-box model, but with a non-standard (and possibly slowly varying) velocity gradient tensor. Stress components other than the $r \phi$ component can play a significant role in the dynamics of these systems, but whether the full turbulent stress tensor resembles a viscous stress in such situations is not understood. In principle, shearing-box studies could be used to formulate a more faithful analytical or semi-analytical model of the relevant properties of the turbulence, which could then be used in studies of the large-scale behaviour of accretion discs. This would bridge the gap between the local dynamics (on scales from the dissipation length up to $H$) and the large-scale dynamics, without requiring colossal advances in computation. The formation and analysis of such a model is also likely to result in a better physical understanding.

In view of the formidable complexity of turbulent flows, such a programme might appear overambitious. A complete theory of turbulence does not exist, because turbulent flows have an unlimited number of statistical properties that cannot be calculated from first principles. However, in the present context, our attention is restricted to the question of how the mean turbulent stress in a patch of the disc responds to changes in the large-scale velocity gradient. As will be made clear, certain linear aspects of this problem can be deduced directly from the MHD equations; other non-linear aspects cannot be treated rigorously and require a closure model that is, nevertheless, strongly constrained by symmetry properties and dimensional considerations. Although the model affords only an imperfect description of turbulence, its physically motivated nature and its connection to the MHD equations ensure that it performs more faithfully than the alpha viscosity model, and this is borne out by numerous test problems.

The remainder of this paper is organized as follows. In Section 2, I introduce a conceptual model system for magnetorotational turbulence, and make some simple arguments concerning the saturation of the turbulence. I develop the equations for the mean Reynolds and Maxwell tensors as far as possible analytically in Section 3, and then discuss the requirements of a non-linear closure model. A specific model satisfying these principles is presented and explained physically in Section 4. The outcome of the model in the shearing sheet, under a variety of conditions, is investigated in Section 5. In Section 6 the effects of a compressible mean flow are incorporated and the total energy budget is explained. Some preliminary applications of the model are worked out in Section 7. In Section 8 a comparison is made with existing models in engineering and in astrophysics, and also with published numerical simulations. A summary follows in Section 9.

2 Nevertheless, considerable interest and debate remain regarding the operation of the magnetorotational instability, or alternative mechanisms of angular momentum transport, in weakly ionized discs, e.g. around young stars and in the quiescent state of dwarf novae.

3 It may not be adequate for predicting the vertical distribution of energy dissipation, and therefore the vertical structure of the disc.
2 A MINIMAL MODEL SYSTEM FOR MAGNETOROTATIONAL TURBULENCE

The magnetorotational instability has certain minimal requirements: rotation and shear (in the correct relative orientation), electrical conductivity and a seed magnetic field. Let us consider the simplest conceptual model system for studying magnetorotational turbulence: the incompressible shearing sheet (Goldreich & Lynden-Bell 1965). Let \((x, y, z)\) be Cartesian coordinates in a frame of reference rotating with uniform angular velocity \(\Omega = \Omega e_z\). An incompressible fluid of uniform density \(\rho\), kinematic viscosity \(\nu\) and magnetic diffusivity \(\eta\) has a uniformly shearing velocity field characterized by a uniform velocity gradient tensor \(\nabla \mathbf{u}\). In a standard disc this is of the form \(\mathbf{u} = -2\Omega xy\), where \(A\) is Oort’s first constant. The \(x, y\) and \(z\) directions correspond to the radial, azimuthal and vertical directions from the perspective of a corotating observer in a thin, differentially rotating disc. The flow is unbounded in the \((x, y)\) plane but of bounded vertical extent, with either ‘physical’ or periodic boundary conditions. The distance between the vertical boundaries defines a characteristic length-scale \(L = L_z\), which, in a real disc, is related to the vertical scaleheight or thickness.

As defined above, the system involves just three dimensionless parameters: the Rossby number \(Ro = A/\Omega\), the Reynolds number \(Re = L^2 \Omega / \nu\) and the magnetic Prandtl number \(Pm = \nu / \eta\). In a Keplerian disc, the Rossby number is fixed at \(Ro = 3/4\). The microscopic diffusivities \(\nu\) and \(\eta\) are included in this conceptual model only to regularize the system by allowing for dissipation and irreversibility. The Reynolds number may be considered to be arbitrarily large.

Apart from the assumption of incompressibility and the inclusion of microscopic diffusivities, this system is equivalent to the shearing box studied by Hawley et al. (1995) in the limit that the horizontal scales \(L_x\) and \(L_y\) of the box tend to infinity while retaining a finite vertical scale. Note that the limit \(L_x, L_y \gg L_z\) corresponds to the physical situation in a thin disc. It is natural to make the following assumptions, which are not contradicted by existing numerical simulations:

(i) the macroscopic statistical properties of the turbulence do not depend on the details of the dissipative mechanisms, and are therefore independent of \(Re\) and \(Pm\) in the limit \(Re \to \infty\);

(ii) the macroscopic statistical properties of the turbulence are bounded and well defined in the limit \(L_x, L_y \to \infty\).

The system is horizontally homogeneous, in the sense that any point in the \((xy)\)-plane is equivalent (modulo a Galilean boost). If periodic vertical boundary conditions are imposed, the system is also vertically homogeneous. It is then possible for the system to develop statistically steady and homogeneous (although anisotropic) turbulence.

If magnetorotational turbulence is initiated in this system, the properties of the saturated state are strongly constrained by dimensional considerations. The rms turbulent velocity fluctuation, \(\langle u^2 \rangle^{1/2}\), for example, must equal \(L \Omega\) times a dimensionless coefficient of the order of unity, if the assumed independence of \(Re\) and \(Pm\) is correct. Therefore the vertical scale of the system plays an essential role in the non-linear saturation of the turbulence, presumably by limiting the size of coherent structures (‘eddies’). In an unbounded system with \(L \to \infty\), the turbulence would presumably grow indefinitely without saturation. It is sometimes suggested that turbulent motions and magnetic fields in an accretion disc are limited by shock formation or magnetic buoyancy, but there is little evidence of this in numerical simulations of magnetorotational turbulence, and neither limiting mechanism operates in the incompressible shearing sheet.4

3 STRESS EQUATIONS AND CLOSURES

3.1 Basic equations

Consider an incompressible fluid of uniform density \(\rho\), kinematic viscosity \(\nu\) and magnetic diffusivity \(\eta\). In the Cartesian tensor notation, the equation of motion in a frame of reference rotating with uniform angular velocity \(\Omega\) is

\[
(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{\partial \Pi}{\rho} + \mathbf{b} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u},
\]

where

\[
b_i = (\mu_0 \rho)^{1/2} B_i
\]

is the Alfven velocity and

\[
\Pi = \frac{P}{\rho} + \frac{1}{2} \mathbf{b}^2 + \Phi
\]

is the modified pressure including the gas pressure, the magnetic pressure and the gravitational and centrifugal potentials. The induction equation is

\[
(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{b},
\]

and the velocity and magnetic fields satisfy the solenoidal conditions,

\[
\partial_i u_i = \partial_i b_i = 0.
\]

Now separate the varying quantities into mean and fluctuating parts, e.g.

\[
u_i = \bar{u}_i + u'_i,
\]

where \(\bar{u}_i = \langle u_i \rangle\) and \(\langle u'_i \rangle = 0\). The angle brackets denote a suitable averaging procedure such as a spatial, temporal or ensemble average. The mean forms of the above equations are then

\[
(\partial_t + \bar{u} \cdot \nabla) \bar{u}_i + 2\epsilon_{ijk} \Omega_j \bar{u}_k = -\partial_j \Pi + \bar{b}_j \partial_i \bar{b}_j + \nu \partial_{ij} \bar{u}_i + \partial_j (\bar{M}_{ij} - \bar{R}_{ij}),
\]

\[
(\partial_t + \bar{u} \cdot \nabla) \bar{b}_i = \bar{b}_j \partial_j \bar{u}_i + \eta \partial_{ij} \bar{b}_j + \partial_j \bar{F}_{ij},
\]

\[
\partial_i \bar{u}_i = \partial_i \bar{b}_i = 0,
\]

where

\[
\bar{M}_{ij} = \bar{b}_j b'_i,
\]

\[
\bar{R}_{ij} = u'_i u'_j,
\]

\[
\bar{F}_{ij} = u'_i b'_j - u'_j b'_i
\]

are the Maxwell, Reynolds and Faraday tensors. The problem at hand is how to determine the mean quantities \(\bar{M}_{ij}\), \(\bar{R}_{ij}\) and \(\bar{F}_{ij}\) and thereby close the system of equations.5

Unless the mean velocity field acts as a dynamo, the mean magnetic field can be sustained only by the mean Faraday tensor, which represents the turbulent electromotive force or ‘alpha effect’. For

4 Numerical simulations of an incompressible shearing box using a spectral method (unpublished work by the author) produce results similar to those of Hawley et al. (1995).

5 For the high Reynolds number situations considered in this paper, the diffusive terms involving \(\nu\) and \(\eta\) in the mean equations may be neglected.
simplicity, it will be assumed in the following that there is no mean magnetic field and no large-scale dynamo, i.e. $\tilde{b}_i = \tilde{F}_{ij} = 0$. The remaining problem is to determine the Maxwell and Reynolds stress tensors that appear in the mean equation of motion.

In the case $\tilde{b}_i = \tilde{F}_{ij} = 0$, the induction equation reads

$$ (\partial_t + \tilde{u}_l \partial_l) \tilde{b}'_i + u'_j \partial_j \tilde{b}'_i = b'_j \partial_j \tilde{u}_i + b'_j \partial_j u'_i + \eta \partial_j b'_j. \quad (14) $$

From this, an exact equation for the mean Maxwell stress can be obtained, in the form

$$ (\partial_t + \tilde{u}_l \partial_l) \tilde{M}_{ij} - \tilde{M}_{ik} \partial_k \tilde{u}_j - \tilde{M}_{jk} \partial_k \tilde{u}_i = \cdots, \quad (16) $$

retaining the exact form of the linear terms, and then investigate simple closure models for the non-linear terms on the right-hand side.

A similar approach applied to the fluctuating part of the equation of motion leads to the exact equation

$$ (\partial_t + \tilde{u}_l \partial_l) \tilde{R}_{ij} + \tilde{R}_{ik} \partial_k \tilde{u}_j + \tilde{R}_{jk} \partial_k \tilde{u}_i + 2\epsilon_{ijk} \Omega_k \tilde{R}_{ij} + 2\epsilon_{ijk} \Omega_k \tilde{R}_{jk} = - \langle u'_j \partial_j \Pi' \rangle + \epsilon_{ijk} \partial_k (M_{jk} - R_{jk}) + \epsilon_{ijk} \partial_k (M_{jk} - R_{jk}) \quad (17) $$

for the mean Reynolds tensor. The terms on the right-hand side require a closure model, but the form of the linear terms,

$$ (\partial_t + \tilde{u}_l \partial_l) \tilde{R}_{ij} + \tilde{R}_{ik} \partial_k \tilde{u}_j + \tilde{R}_{jk} \partial_k \tilde{u}_i + 2\epsilon_{ijk} \Omega_k \tilde{R}_{ij} + 2\epsilon_{ijk} \Omega_k \tilde{R}_{jk} = \cdots, \quad (18) $$

may be retained. Note that the Maxwell and Reynolds tensors interact with the mean velocity gradient in different ways, and only the Reynolds tensor is affected by rotation.

### 3.2 The pressure–strain correlation

In homogeneous turbulence, the term involving $\Pi'$ on the right-hand side of equation (17) has the alternative form

$$ \langle \Pi' (\partial_j u'_i + \partial_i u'_j) \rangle, \quad (19) $$

and is known as the pressure–strain correlation. The identification of suitable closures for this term, even in purely hydrodynamic turbulence, has been a matter of considerable controversy (e.g. Speziale 1991), leading to highly elaborate, but still imperfect, models (e.g. Sjögren & Johansson 2000). An expression for $\Pi'$ can be obtained by taking the divergence of the fluctuating part of the equation of motion. If the spectrum of the turbulence is known, part of the pressure–strain correlation can then be expressed in terms of $\tilde{R}_{ij}$, while part is undoubtedly non-linear. However, as the spectrum itself is determined through non-linear dynamics, one may take the view that the entire pressure–strain correlation is a non-linear term for which a non-deductive closure must be proposed.

In the following, the fluctuating quantities will not be referred to, and the bars on $\tilde{u}_i$, $\tilde{M}_{ij}$ and $\tilde{R}_{ij}$ will be omitted.

### 3.3 Linear dynamics in the shearing sheet

In the shearing sheet the angular velocity is $\Omega = \Omega e_z$, and the only non-vanishing component of the mean velocity gradient is $\partial_y u_z = -2A$. The linearized equations for the Maxwell and Reynolds stresses then have the form

$$ \partial_t M_{xx} = 0, \quad \partial_t M_{xy} + 2AM_{x1} = 0, \quad \partial_t M_{xz} = 0, \quad \partial_t M_{yz} + 4AM_{y1} = 0, \quad \partial_t M_{zy} + 2AM_{z1} = 0, \quad \partial_t M_{zz} = 0, $$

$$ \partial_t R_{xx} - 4\Omega R_{xy} = 0, \quad \partial_t R_{xy} + 2(\Omega - A)R_{xx} - 2\Omega R_{yx} = 0, \quad \partial_t R_{xz} = 0, \quad \partial_t R_{yz} + 4(\Omega - A)R_{yy} = 0, \quad \partial_t R_{zz} = 2(\Omega - A)R_{zz} = 0. $$

The Maxwell and Reynolds tensors are decoupled. The general solution for the magnetic component is

$$ M_{xx} = M_{x1,0}, \quad M_{xy} = M_{x1,0} - 2M_{z1,0}A, \quad M_{xz} = M_{x1,0}, \quad M_{yz} = M_{y1,0} - 4M_{z1,0}A + 4M_{z1,0}A'^2, \quad M_{zy} = M_{z1,0} - 2M_{z1,0}A, \quad M_{zz} = M_{z1,0}, $$

and allows for algebraic growth through the shearing of magnetic field lines. The solution for the kinetic component depends on the Rayleigh discriminant, or squared epicyclic frequency,

$$ \kappa^2 = \Omega(\Omega - A). \quad (20) $$

determined through non-linear dynamics, one may take the view that the entire pressure–strain correlation is a non-linear term for which a non-deductive closure must be proposed. The system in the absence of rotation, the system is Rayleigh-neutral and the solution for $R_{ij}$ allows for algebraic growth (it resembles the solution for $M_{ij}$, but with the sign of $A$ reversed). When $\kappa^2 > 0$, the solution is oscillatory, as can be seen by reducing the problem to equations of the form

$$ \partial_t R_{yy} + 4\kappa^2 R_{yy} = 0, \quad \partial_t R_{xz} + 2\kappa^2 R_{zz} = 0, \quad \partial_t R_{yz} = 0. $$

So $R_{yy}$ and $R_{xz}$ oscillate at twice the epicyclic frequency, $R_{xz}$ and $R_{yz}$ at the epicyclic frequency, and $R_{zz}$ is constant. On the other
3.4 Requirements of a non-linear closure model

Whenever the linear dynamics indicates unbounded growth or undamped oscillations, the non-linear terms will determine the eventual outcome. The approach adopted in this paper is to explore the consequences of simple closure models of the form

\[ D^{(1)} R_{ij} = F^{(1)}_{ij}(R_{ij}, M_{ij}, \ldots), \]
\[ D^{(2)} M_{ij} = F^{(2)}_{ij}(R_{ij}, M_{ij}, \ldots), \]

where the operators \( D \) are defined by the left-hand sides of equations (18) and (16) respectively, and the quantities \( F_{ij} \) are non-linear tensorial functions of their arguments. The dots represent the parameters of the problem, on which the functions \( F_{ij} \) may depend.

Such a model ought to satisfy the following fundamental principles.

(i) There should be no ‘source terms’. \( R_{ij} = M_{ij} = 0 \) should always be a solution, although it may be unstable.

(ii) An unmagnetized state, \( M_{ij} = 0 \), should always be a solution even when \( R_{ij} \neq 0 \), although it may be unstable to dynamo action.

(iii) The non-linear terms on the right-hand side should be manifestly covariant, as the left-hand sides are.

(iv) The positive semidefinite nature of the tensors \( R_{ij} \) and \( M_{ij} \), implicit in their definition, should be preserved by the model.\(^6\) This ensures not only that the turbulent kinetic and magnetic energy densities remain non-negative, but also that the stress tensors can be realized by genuine velocity and magnetic fields.

The following two additional requirements appear plausible, although they cannot be regarded as fundamental principles.

(v) The non-linear terms should not refer to the angular velocity \( \Omega \) or the velocity gradient \( \nabla \mathbf{u} \), because their effects are fully represented in the linear terms.

(vi) The non-linear terms should not refer to the microscopic diffusion coefficients, because one has assumed an asymptotic independence of \( \text{Re} \) and \( \text{Pm} \) in the limit \( \text{Re} \to \infty \).

The non-linear terms are also strongly constrained by dimensional considerations. Both \( R_{ij} \) and \( M_{ij} \), as defined in this incompressible system, have the dimensions of velocity-squared. A quantity with the dimensions of \( R_{ij} \) cannot be formed from \( R_{ij} \) and \( M_{ij} \) alone. The only other physical quantity that can appear in the functions \( F_{ij} \), according to the above assumptions, is the length-scale \( L \). In that case the non-linear terms must have the form

\[ F_{ij}^{(n)} = L^{-3} G_{ij}^{(n)}(R_{ij}, M_{ij}), \]

where the quantities \( G_{ij} \) are homogeneous functions of degree 3/2.

\(^6\) A real symmetric \( n \times n \) matrix (or second-rank tensor) \( A_{ij} \) is said to be positive semidefinite if \( A_{ij}X_iX_j \geq 0 \) for all real \( n \)-component vectors \( X_i \), or, equivalently, if all the eigenvalues of \( A_{ij} \) are non-negative. This condition is relevant because \( R_{ij}X_iX_j = (u_i'u_iX_iX_j) = (\mathbf{u}' \cdot \mathbf{X}^2) \geq 0 \), and similarly for \( M_{ij} \). Note that off-diagonal components such as \( R_{ij} \) can have either sign.

4 A SIMPLE MODEL AND ITS PROPERTIES

4.1 Statement of the model

A simple model based on the above principles is as follows:

\[ \partial_t R_{ij} + u_i \partial_j R_{ij} + R_{ik} \partial_k u_j + R_{jk} \partial_k u_i + 2 \epsilon_{ijk} \Omega_k R_{lj} + 2 \epsilon_{ijk} \Omega_k R_{lj} = -C_1 L^{-1/2} R_{ij} - C_2 L^{-1} R_{ij}^{1/2} \left( R_{ij} - \frac{1}{2} R \delta_{ij} \right) \]
\[ + C_3 L^{-1} M_{ij}^{1/2} M_{ij} - C_4 L^{-1} R^{1/2} M_{ij}, \]

(23)

\[ \partial_t R_{ij} + u_i \partial_j R_{ij} - M_{ik} \partial_k u_j - M_{jk} \partial_k u_i = C_1 L^{-1} R_{ij}^{1/2} M_{ij} - (C_3 + C_5) L^{-1} M_{ij}^{1/2} M_{ij}, \]

(24)

where \( R = R_{ij} \) and \( M = M_{ij} \) are the traces of the Reynolds and Maxwell tensors, and \( C_1, \ldots, C_5 \) are positive dimensionless coefficients of a universal nature.

In the following subsections the physical origin and implications of the various terms in the model will be made clear.

4.2 Hydrodynamic decay and return to isotropy

For freely decaying hydrodynamic turbulence in the absence of rotation and a mean flow, one might start with a model of the form

\[ \partial_t R_{ij} = -C_1 L^{-1/2} R_{ij}, \]

(25)

The turbulent kinetic energy density \( \frac{1}{2} R \) then decays monotonically and non-linearly according to

\[ \partial_t R = -C_1 L^{-1} R^{1/2}. \]

(26)

The functional form of the right-hand side is dictated by dimensional considerations, as discussed above, and is also suggested by the form of the triple correlations that appear in the exact equation (17). The predicted decay of energy, \( R \propto t^{-2} \) as \( t \to \infty \), is faster than the decay rate \( R \propto t^{-1} \) observed in turbulence generated by the passage of a stream of fluid through a grid (Batchelor 1953). The reason for this is that the length-scale of the energy-containing eddies is constrained by the constant \( L \) in the present model, while it expands continuously during the decay of grid-generated turbulence.

It is well known that freely decaying, anisotropic hydrodynamic turbulence also exhibits a tendency to return to isotropy. This suggests the addition of a traceless term that redistributes energy among the components of \( R_{ij} \), i.e.

\[ \partial_t R_{ij} = -C_1 L^{-1} R_{ij}^{1/2} R_{ij} - C_2 L^{-1} R_{ij}^{1/2} \left( R_{ij} - \frac{1}{2} R \delta_{ij} \right). \]

(27)

In this model, if initially \( R > 0 \) but \( R_{xx} = 0 \), for example, then \( R_{xx} \) will first grow and then decay.

4.3 Turbulent Lorentz forces and small-scale dynamo action

If the fluid is initially at rest and a random magnetic field is introduced, the Lorentz forces will immediately drive turbulent motions. This is the origin of the term \( C_3 \), which appears as a source for \( R_{ij} \) and a sink for \( M_{ij} \). Note that a random magnetic field that has a dominant \( x \) component, say, drives motions predominantly in the \( x \) direction through the \( \mathbf{B} \cdot \nabla \mathbf{B} \) force.\(^7\) For this reason the source for \( R_{ij} \) has the tensorial form of \( M_{ij} \).

\(^7\) This may seem paradoxical as the total Lorentz force is orthogonal to \( \mathbf{B} \). However, only the solenoidal part of the Lorentz force drives motion in an incompressible fluid, or in a compressible fluid under Boussinesq conditions. The remaining gradient part is compensated by a pressure gradient.
The term $C_4$ is a source for $M_\parallel$ and a sink for $R_\parallel$. It can be considered as the effect of small-scale dynamo action. Any turbulent motion acts as a small-scale dynamo and causes a growth of turbulent magnetic energy, at least when the field is weak. This occurs through the $B \cdot \nabla u$ term in the induction equation, and therefore the source for $M_\parallel$ has the tensorial form of $R_{ij}$.

4.4 Energetics, equipartition and realizability

The evolution of the turbulent kinetic and magnetic energy densities, $\frac{1}{2} R$ and $\frac{1}{2} M$, is governed by the traces of the equations for $R_{ij}$ and $M_{ij}$. Thus

$$\begin{align*}
(\partial_t + u_i \partial_i) R &= -2 R_{ij} \partial_j u_i - C_1 L^{-1} R^{3/2} + C_2 L^{-1} M^{1/2} \\
&\quad - C_4 L^{-1} R^{1/2} M, \quad (28)
\end{align*}$$

$$\begin{align*}
(\partial_t + u_i \partial_i) M &= 2 M_{ij} \partial_j u_i + C_3 L^{-1} R^{1/2} M - (C_1 + C_4) L^{-1} M^{1/2}.
\end{align*} \tag{29}$$

The total energy density satisfies

$$\begin{align*}
(\partial_t + u_i \partial_i) \left( \frac{1}{2} R + \frac{1}{2} M \right) &= (M_{ij} - R_{ij}) \partial_j u_i - C_1 L^{-1} \frac{1}{2} R^{3/2} - C_3 L^{-1} \frac{1}{2} M^{1/2}.
\end{align*} \tag{30}$$

The first term on the right-hand side represents the extraction of shear energy from the mean flow by the total turbulent stress $M_{ij} - R_{ij}$. The other two terms, $C_1$ and $C_4$, represent the turbulent dissipation of kinetic and magnetic energy, and can be thought of as operating through the formation of vortex sheets and current sheets, respectively.

Terms $C_3$ and $C_4$ transfer energy between kinetic and magnetic components. The net rate of transfer from kinetic to magnetic energy is

$$\begin{align*}
(\partial_t + u_i \partial_i) \left( C_3 R^{1/2} - C_4 M^{1/2} \right) R^{1/2} \frac{1}{2} M
\end{align*} \tag{31}$$

and is positive or negative according to whether the kinetic or magnetic energy dominates. There is therefore a tendency towards ‘equipartition’ in the ratio $M/R = (C_4/C_3)^2$.

An important requirement of the model is that the kinetic and magnetic energy densities should remain non-negative, not least because square roots appear in the model. In fact there is a stronger requirement, mentioned above, because the Reynolds and Maxwell tensors are by definition positive semidefinite tensors. In the Appendix it is shown that the positive semidefinite nature of $R_{ij}$ and $M_{ij}$ is preserved by the model, guaranteeing that they are realizable by genuine velocity and magnetic fields.

5 NON-LINEAR OUTCOME IN THE SHEARING SHEET

In this section, I examine the predictions of the model for the incompressible shearing sheet. The angular velocity and velocity gradient are fixed by the parameters of the shearing sheet. The Reynolds and Maxwell tensors may be assumed to be independent of position, corresponding to homogeneous turbulence. The model then reduces to a non-linear dynamical system describing the purely temporal evolution of the stress tensors. The non-trivial fixed points of the dynamical system represent possible states of statistically steady turbulence.

In numerical calculations I adopt the fiducial parameters $C_1 = C_2 = C_3 = C_4 = C_5 = 1$, although other values have been experimented with. The fact that this simple choice gives reasonable results indicates that no fine tuning is required for the model to behave physically. Ideally the parameters should be calibrated by comparison with suitable numerical simulations.

5.1 Freely decaying turbulence without rotation or shear

With $\Omega = A = 0$ there is nothing to initiate or sustain turbulence. Any turbulence introduced into the system decays, tending to isotropy as it does so. For freely decaying isotropic turbulence ($R_{ij} = \frac{1}{3} R \delta_{ij}$, etc.), the model states that

$$\begin{align*}
\dot{R} &= -C_1 L^{-1} R^{3/2} + C_2 L^{-1} M^{3/2} - C_3 L^{-1} R^{1/2} M, \\
\dot{M} &= C_4 L^{-1} M^{1/2} - (C_1 + C_3) L^{-1} M^{1/2},
\end{align*} \tag{32}$$

where the dot denotes differentiation with respect to time. A typical phase portrait is shown in Fig. 1. All trajectories tend towards the only fixed point, $R = M = 0$. During the decay there is a tendency towards equipartition.

5.2 Non-rotating shear flow without a magnetic field

The next simplest case to be examined is that of a non-rotating shear flow (plane Couette flow, $\Omega = 0$) with $M_\parallel = 0$ throughout. This case has been studied in the laboratory (with rigid boundaries) and in numerical simulations with the periodic boundary conditions of the shearing box (Pumir 1996). At high Reynolds number, sustained hydrodynamic turbulence develops. The closure model ought to describe this behaviour.

As described in Section 3.3, in the absence of rotation the system is Rayleigh-neutral and the linear dynamics of $R_{ij}$ allows for algebraic growth. The outcome is determined by the non-linear terms. The system has only two fixed points, one of which is the algebraically unstable trivial solution $R_{ij} = 0$. The non-trivial fixed point represents a state of steady hydrodynamic turbulence and is given by
Magnetorotational turbulent stresses

A typical result of numerical integration of the time-dependent equations is shown in Fig. 2. The integration is started close to the unstable fixed point $R_0 = 0$ by introducing a small positive value of $R_0$, only. Initially algebraic growth leads to a rapid approach to the stable fixed point.

5.3 Non-rotating shear flow with a magnetic field

The dynamics changes when a magnetic field is introduced, because a turbulent flow acts as a small-scale dynamo. The non-trivial fixed point representing a state of steady hydrodynamic turbulence is unstable to a magnetic perturbation. A new, stable fixed point appears, corresponding to a state of steady MHD turbulence.

A typical result of numerical integration of the time-dependent equations is shown in Fig. 3. The integration is started close to the trivial solution $R_0 = M_0 = 0$ by introducing small positive values of $R_0$ and $M_0$, only. The system approaches the stable fixed point representing magnetized turbulence.

5.4 Rotating shear flow without a magnetic field

If one seeks a non-trivial fixed point representing a state of steady hydrodynamic turbulence in rotating plane Couette flow, one obtains the formal solution

\[ R_{xx} = \left( \frac{3\sigma_1 + C_2}{C_1 + C_2} \right)^{1/3} R, \]
\[ R_{yy} = \left[ \frac{3(1 - \sigma_1)C_1 + C_2}{C_1 + C_2} \right]^{1/3} R, \]
\[ R_{zz} = \left( \frac{C_2}{C_1 + C_2} \right)^{1/3} R, \]
\[ R_{xy} = C_1 L^2 A^2, \]
\[ R_{xz} = R_{yz} = 0, \]

with

\[ R = \left( \frac{C_2}{C_1(C_1 + C_2)^2} \right)^{1/3} L^2 A^2. \]  

The existence of this non-trivial solution depends on both the shear and the return-to-isotropy coefficient $C_2$. The fixed point is stable for all parameter values and is universally attracting.

A typical result of numerical integration of the time-dependent equations is shown in Fig. 2. The integration is started close to the unstable fixed point $R_0 = 0$ by introducing a small positive value of $R_0$, only. Initially algebraic growth leads to a rapid approach to the stable fixed point.

Figure 2. Evolution of the Reynolds tensor in plane Couette flow, starting from a small hydrodynamic perturbation without a magnetic field. Fiducial parameters $C_1 = C_2 = 1$ are adopted. The solution tends towards the stable fixed point representing a state of statistically steady hydrodynamic turbulence. The components $R_{xz}$ and $R_{yz}$ are decoupled from the others, and tend to zero.

\[ R_{xx} = R_{zz} = \left( \frac{C_2}{C_1 + C_2} \right)^{1/3} R, \]
\[ R_{yy} = \left( \frac{3C_1 + C_2}{C_1 + C_2} \right)^{1/3} R, \]
\[ R_{xy} = \left( \frac{C_1}{4LA} \right)^{1/3} R, \]
\[ R_{xz} = R_{yz} = 0, \]

with

\[ R = \left( \frac{C_2}{C_1(C_1 + C_2)^2} \right)^{1/3} L^2 A^2. \]  

Figure 3. Evolution of the Reynolds tensor (left) and Maxwell tensor (right) in plane Couette flow, starting from small hydrodynamic and magnetic perturbations. Fiducial parameters $C_1 = C_2 = C_3 = C_4 = C_5 = 1$ are adopted. The solution tends towards the stable fixed point representing a state of statistically steady MHD turbulence. The solution approached in Fig. 2 is no longer attracting. The components $R_{xz}, R_{yz}, M_{xz}$ and $M_{yz}$ are decoupled from the others, and tend to zero.
Figure 4. Stability diagram for rotating plane Couette flow in the absence of a magnetic field. The two parameters of the system are the inverse Rossby number $\Omega/A$ and the ratio $C_2/C_1$, which measures the tendency to return to isotropy. When the system is Rayleigh-unstable ($0 < \Omega/A < 1$), or when the isotropizing tendency is sufficiently large, a stable state of steady hydrodynamic turbulence is permitted. Otherwise the system exhibits decaying epicyclic oscillations when perturbed. The Keplerian case $\Omega/A = 4/3$ is shown with a dotted line. Steady turbulence is permitted only if $C_2/C_1 > 8/3$.

\[
R = \left[ \frac{-6\alpha^{-1}(\Omega^{-1} - 1)C_1 + C_2}{C_1(C_1 + C_2)^2} \right]^2 L^2 A^2, \tag{36}
\]

which is the generalization of equation (34) for the non-rotating case. This solution is only meaningful, however, if $\rho_0$ is positive semidefinite, and this requires

\[
\frac{C_2}{C_1} > 6\alpha^{-1}(\Omega^{-1} - 1), \tag{37}
\]

as illustrated in Fig. 4. (In the case of equality, the solution is trivial.) Where it exists, this solution appears to be stable.

For the Keplerian case $\alpha = 3/4$ relevant to a standard accretion disc, steady hydrodynamic turbulence is possible only when $C_2/C_1 > 8/3$. Although this is possible in principle, such a value appears improbable a priori. When $C_2/C_1 = 1$, for example, steady turbulence is possible for $-0.145 < \Omega^{-1} < 1.145$. For rotation laws of the form $\Omega \propto r^{-q}$, this requires $q > 1.746$.

Numerical integration of the time-dependent system confirms that, when the system is perturbed from $\rho_0 = 0$, it tends towards the stable solution where this exists. Otherwise it exhibits decaying epicyclic oscillations about $\rho_0 = 0$.

5.5 Rotating shear flow with a magnetic field

The introduction of a magnetic perturbation allows steady MHD turbulence to develop in a rotating shear flow, even when Rayleigh’s stability criterion is amply satisfied, as in the Keplerian case $\alpha = 3/4$ (Fig. 5). This occurs through a non-linear magnetorotational instability (non-linear because there is no imposed magnetic flux) and the magnetic field is sustained through a non-linear dynamo process. The model therefore reproduces in broad terms the finding of numerical studies of the non-linear evolution of the magnetorotational instability (e.g. Hawley et al. 1995).

6 THE GOVERNING EQUATIONS IN COMPRESSIBLE FLOWS

The assumption of an incompressible fluid is useful in reducing the problem to a minimal, although still formidable, complexity. There are two reasons why the model may not be suitable for immediate application to a compressible fluid. First, the mean density may be non-uniform, especially if the mean velocity field has a non-zero divergence. Secondly, the turbulence may be essentially transonic, changing the character of the motions and leading to greatly enhanced dissipation through shock formation. The second case is beyond the scope of this paper and is unlikely to be important for the magnetorotational instability, at least in the part of an accretion disc where most of the mass resides. The first case, however, is important in a number of applications and can be treated in a simple way.
In a compressible fluid it is more appropriate to define the mean Reynolds and Maxwell tensors as
\[ R_{ij} = \langle \rho u'_i u'_j \rangle \]
and
\[ M_{ij} = \left( \frac{\mathbf{B}'_i \mathbf{B}'_j}{\mu_0} \right), \]
which have the dimensions of stress and differ from the earlier definitions by a factor of the density. It is convenient not to include the magnetic pressure perturbation in \( M_{ij} \), but to regard the mean magnetic stress as \( M_{ij} - \frac{1}{2} \mathbf{B} \mathbf{B} \).

The first issue to consider is whether a divergence of the mean velocity field, \( \partial_i u_i \), should affect the evolution of \( R_{ij} \) or \( M_{ij} \). This velocity divergence does not appear in the equation of motion, but does appear in the equation of mass conservation and the induction equation in the forms \( \partial_i \rho = -\partial_i \rho_0 \) and \( \partial_i B_i = -\partial_j B_j \partial_i u_j \) respectively. This motivates the addition of terms in the forms \( \partial_i R_{ij} = -\partial_i \rho \partial_i u_k \) and \( \partial_i M_{ij} = -2 \partial_i \rho_0 \partial_i u_j \) respectively.

The second issue concerns the definition of the vertical length-scale \( L \) in a stratified disc, and the effect of the stratification on the vertical profile of the stress. In the incompressible system the turbulence is homogeneous and the stress independent of z. Numerical simulations of stratified, isothermal accretion discs (Miller & Stone 2000) suggest that the stress is also stratified, being roughly proportional to the density (or pressure), and in many applications it is analytically convenient to assume that the stress scales with \( \rho \) or \( p \), and in many applications isothermal accretion discs (Miller & Stone 2000) indicate that the stress scales with \( \rho \) or \( p \).

Provided that the gravitational potential \( \Phi \) is independent of \( t \). Here \( e \) is the specific internal energy, such that \( de = T \, ds - p \, d(\rho^{-1}) \). The existence of this conservation law, with positive definite turbulent energy densities and heating rates, implies a certain self-consistency in the equations of the model. The terms that were added in passing to the compressible model are required to have the form that they do in order that energy be conserved.

It is worth remarking that when \( C_1 = \cdots = C_5 = 0 \) and \( R_{ij} = 0 \), these equations reduce exactly to those of ideal MHD if one identifies \( M_{ij} \) as the Maxwell tensor \( B_i B_j / \mu_0 \) of an arbitrary large-scale magnetic field \( B \) advected by the mean flow \( u \). The magnetic terms in the equation of motion are precisely the Lorentz force of such a magnetic field, and the equation for the Maxwell tensor is equivalent to the induction equation of ideal MHD. Furthermore, the magnetic energy density, \( \frac{1}{2} M = \frac{1}{2} B^2 / \mu_0 \), and the Poynting flux, \( M_{it} - M_{it} u_t = (B^2 u_i - B_i B_j \partial_j u_l) / \mu_0 = (E \times B)_l / \mu_0 \), can both be identified in the energy equation.

7 APPLICATIONS AND IMPLICATIONS

7.1 Character of the equations

The governing equations set out in the previous section usually have the formal character of a hyperbolic system, indicating that information is propagated in a causal manner at finite speeds. To see this, consider the behaviour of infinitesimal wave-like perturbations of an arbitrary solution of the equations. Let the perturbations have the form of a rapidly varying phase factor, \( \exp \{ i \omega(x, t) \} \), multiplied by slowly varying functions of \( x \) and \( t \). The wave number \( k = \nabla \phi \) and frequency \( \omega = -\partial \phi \) are assumed to be such that \( |k|^{-1} \) and \( |\omega|^{-1} \) are small compared to the typical length-scales and time-scales of the system. Although the model is not necessarily valid in this regime, such an analysis serves to uncover the formal mathematical structure of the equations.

By linearizing equations (40)–(41) and eliminating perturbations in favour of the velocity perturbation \( u'_i \), one obtains the algebraic eigenvalue problem
\[ \rho (\omega - k_j u_j) \hat{u}'_i = \left( R_{jk} + M_{jk} \right) \hat{k}_k \hat{k}_l \hat{u}'_l + \left( \rho c_s^2 \delta_{ij} + 2 R_{ij} \right) \hat{u}'_j \]
\[ + \left( M_{jk} \hat{k}_j - M_{kl} \hat{k}_j \hat{k}_l - M_{jk} \hat{k}_k \hat{u}'_j \right) \]
for the local WKB modes of the system, where \( c_s^2 = \langle \partial^2 \rangle / \partial \rho \) is the square of the adiabatic sound speed. As in compressible MHD, the WKB dispersion relation has six frequency eigenvalues \( \omega \) for any choice of the real wavevector \( k \). If any of the eigenvalues has a non-zero imaginary part, the system experiences a local instability. Otherwise the eigenvalues are all real and the group velocities \( \partial \omega / \partial k \) are independent of \( k \), indicating that the wave propagation in this limit is anisotropic but non-dispersive. The system of equations is then formally hyperbolic.

A full investigation of the dispersion relation is difficult but the following observations may be made. First, when \( R_{ij} \neq 0 \), the dispersion relation is the standard one of compressible hydrodynamics, and information is propagated at the sound speed relative to the mean flow. Secondly, when \( R_{ij} = 0 \) but \( M_{ij} \neq 0 \), the system can be shown to be hyperbolic. The dispersion relation is related to that of compressible MHD. Thirdly, in an incompressible fluid, modes with \( k, u' \neq 0 \) are eliminated as their speed of propagation is infinite. The remaining part of the dispersion relation describes the propagation of transverse modes, similar to Alfvén waves, at finite speed. Finally, there are circumstances in which the system fails...
to be hyperbolic because the dispersion relation indicates a local instability, but this appears to be atypical.

7.2 Stratified shearing sheet

In a stratified shearing sheet, magnetorotational turbulence would develop according to these equations just as in an incompressible shearing sheet, except that all stress components would be proportional to \( p/\Omega^2 \) rather than \( L^2 \). The total shear stress \( M_{xy} - R_{xy} \) can be compared with that corresponding to an effective viscosity \( \mu = \alpha p/\Omega \). For the fiducial parameters in a Keplerian disc, this gives \( \alpha \approx 0.081 \). If all the parameters \( C_i \) are scaled by a constant factor, this value of \( \alpha \) scales as \( 1/C_i^2 \).

7.3 Non-Keplerian rotation

When the Rossby number \( R_0 = A/\Omega \) of the shearing sheet is varied in the range \( 0 < R_0 < 1 \) in which the system is Rayleigh-stable but magnetorotationally unstable, the total shear stress \( M_{xy} - R_{xy} \), in the steady turbulent state is not simply proportional to the shear rate \( 2A \), at fixed angular velocity. This is shown in Fig. 6, and demonstrates one aspect in which the present model differs significantly from a viscous representation of the turbulence.

7.4 Damping of the two-dimensional wave

The local dispersion relation for axisymmetric density waves in a two-dimensional disc model is

\[
\omega^2 = k^2 + v_z^2 k^2, \tag{47}
\]

where \( k \) is the radial wavenumber (Goldreich & Tremaine 1979). The same mode can be identified within the present model if one considers a three-dimensional compressible shearing sheet without vertical gravity and stratification. To avoid complications it is convenient to replace the thermal energy equation by the isothermal condition \( p = c_s^2 \rho \), \( c_s = \) constant. Equations (40)–(43), linearized around a basic state consisting of steady, homogeneous magnetorotational turbulence, then admit solutions proportional to \( \exp (kx - i \omega t) \), with no perturbations of \( (u_x, R_{xz}, R_{yz}, M_{xz}, M_{yz}) \). In the absence of turbulent stresses, the dispersion relation (47) is recovered. A numerical solution of the dispersion relation in the presence of turbulent stresses indicates that the two-dimensional wave is damped in a Keplerian disc when the fiducial parameters \( C_i = 1 \) are adopted (Fig. 7). The real part of the frequency is somewhat greater than predicted by equation (47), and agrees much better if \( v_1 \) is replaced by the ‘magnetosonic speed’ \( (c_s + M/\rho)^{1/2} \). Interestingly, the imaginary part corresponds quite closely with a viscous damping rate \( \nu k^2 \) if the effective viscosity is \( \nu = \alpha c_s^2/\Omega \) with \( \alpha \approx 0.06 \). This is remarkable when it is considered that there are no viscous or diffusive terms in the governing equations.

7.5 Damping of shearing epicyclic motions

When a stratified disc is warped, horizontal pressure gradients are introduced that drive epicyclic motions with an amplitude proportional to the distance \( z \) above the mid-plane (Papaloizou & Pringle 1983). The amplitude and phase of these oscillations are critical to the propagation and damping of the warp. As discussed by Torkelsson et al. (2000), the dynamics of these shearing epicyclic motions, and in particular their damping time-scale, can be studied within the shearing sheet.

Consider equations (40)–(43), linearized around a basic state consisting of steady, homogeneous magnetorotational turbulence, in which the perturbations depend only on \( z \) and \( t \). It is possible to arrange for all perturbations to vanish other than those of \( (u_x, R_{xz}, R_{yz}, M_{xz}, M_{yz}) \). Then

\[
\rho (\partial_t u_x' - 2\Omega u_z') = \partial_t (M_{xz}' - R_{xz}'), \tag{48}
\]

\[
\rho (\partial_t u_z' + 2(\Omega - A)u_x') = \partial_t (M_{xz}' - R_{xz}'), \tag{49}
\]

\[
\partial_z R_{xz}' - 2\Omega R_{yz}' + R_{xz} \partial_z u_x' = \Omega \rho^{-1/2} \left[ -(C_1 + C_2) R_{xz}' + C_1 M_{xz}' \right]
\]

\[
- C_3 R^{-1/2} M_{xz}' \right], \tag{50}
\]

\[
\partial_z R_{yz}' + 2(\Omega - A)R_{xz}' + R_{xz} \partial_z u_y' = \Omega \rho^{-1/2} \left[ -(C_1 + C_2) R_{yz}' + C_1 M_{yz}' \right]
\]

\[
- C_3 R^{-1/2} M_{yz}' \right], \tag{51}
\]

\[
\partial_z M_{xz}' - M_{xx} \partial_z u_x' = \Omega \rho^{-1/2} \left[ C_4 R_{xz}' - (C_3 + C_5) M_{xz}' \right], \tag{52}
\]

\[
\partial_z M_{yz}' + 2AM_{xz}' - M_{xx} \partial_z u_y' = \Omega \rho^{-1/2} \left[ C_4 R_{yz}' - (C_3 + C_5) M_{yz}' \right]. \tag{53}
\]

These equations admit solutions in which all perturbations are proportional to \( \exp (-i\omega t) \), while \( u_x' \) and \( u_y' \) are proportional to \( z \), and \( R_{xz}' \), \( R_{yz}' \), \( M_{xz}' \), and \( M_{yz}' \), are proportional to \( p \). The six eigenvalues \( \omega \), corresponding to three physically distinct modes, follow from an algebraic eigenvalue problem that is easily solved numerically. Two of these modes are strongly damped, while the other represents a slowly damped shearing epicyclic motion. For a Keplerian disc and fiducial parameters \( C_i = 1 \), the damping rate is 0.0254\( \Omega \), the same that would be obtained from an alpha viscosity if \( \alpha = 0.0254 \). This is smaller than the value of 0.081 required to account for the shear stress \( M_{xz} - R_{xz} \).

Figure 6. The non-linear relation between the total shear stress in steady magnetorotational turbulence and the shear rate, at fixed angular velocity. Fiducial parameters \( C_1 = C_2 = C_3 = C_4 = C_5 = 1 \) are adopted.
8 COMPARISON WITH PREVIOUS WORK

8.1 Comparison with analytical models by other authors

The historical development of closed model equations for the stress tensor in a turbulent fluid is described in the review article of Speziale (1991). Early studies were based on the concepts of eddy viscosity and mixing length introduced by Boussinesq and Prandtl. A systematic investigation of the equation for the mean Reynolds stress was initiated in the 1950s. Out of this arose (among others) the $K - \epsilon$ model, which is widely used in engineering applications despite its numerous deficiencies. More recent approaches have aimed at a greater fidelity to experimental and numerical results, typically by elaborating algebraic models of the pressure–strain correlation and trying to fix the coefficients therein.

The present work is related to these Reynolds-stress closure models in that I have worked with the exact equations for the mean Reynolds and Maxwell tensors, retaining the form of the linear terms and proposing closures for the non-linear terms. However, I have tried to take a fundamentally different approach to the non-linear effects by recognizing the essential role of the length-scale $L$ in the saturation of the turbulence and by identifying each non-linear term with a known physical process. In the absence of a magnetic stress, the equations adopted here for the Reynolds tensor are considerably simpler than those advocated by, for example, Speziale (1991), yet they do ensure realizability, allow for the return to isotropy, and may well give a more accurate representation of the linear and non-linear stability properties of rotating shear flows.

The application of Reynolds-stress closure models to accretion discs has been proposed in a series of papers by Kato (1994), Kato & Inagaki (1994) and Kato & Yoshizawa (1993, 1995, 1997). In some of these papers the Maxwell stress is also modelled. While these studies have a fair amount in common with the present approach, some important differences must be emphasized. In the absence of a magnetic stress, Kato & Yoshizawa (1997) have argued that steady hydrodynamic turbulence may be sustained in a Keplerian accretion disc. In fact, their closure model is based on one by Launder, Reece & Rodi (1975), which predicts stability for a Keplerian shear flow (see Speziale 1991); only by modifying one of the terms were Kato & Yoshizawa able to find a steady turbulent state. More importantly, the work of Kato and co-workers does not address the issue of the non-linear saturation of hydrodynamic or MHD turbulence. Although it predicts the relative magnitudes of the various components of the Reynolds (and Maxwell) tensors, it does not predict the magnitude of the turbulent energy in the saturated state.

Some further connections with the astrophysical literature can be made. Schramkowski et al. (1996) devised a kinematic mean-field theory for the evolution of the Maxwell tensor in a turbulent accretion disc, as an alternative to studying the mean magnetic field itself. Their theory is covariant and includes the interaction of the Maxwell tensor with the mean velocity field. In addition to the ‘alpha’ and ‘beta’ effects of standard mean-field electrodynamics, they emphasized the role of the ‘gamma’ effect by which a turbulent flow amplifies small-scale magnetic energy. This can be related to the $C_4$ term of the present model.

Soon after Balbus & Hawley (1991) first wrote on the magnetorotational instability in accretion discs, Tout & Pringle (1992) presented a simple model of a magnetic dynamo cycle that uses the differential rotation to generate azimuthal field from radial field, the Parker instability to convert azimuthal field into vertical field, and the magnetorotational instability to regenerate radial field from vertical field, while magnetic reconnection provides dissipation. In effect, they wrote down a non-linear dynamical system for the three field components, which could be thought of as either mean or rms values. The present model is ultimately based on a similar idea, although it treats the Maxwell stress covariantly as a single tensor object. The Parker instability (magnetic buoyancy) does not play a role here, and the magnetorotational instability is an outcome of the model rather than an ingredient.

8.2 Comparison with a viscoelastic model

In recent work on the dynamics of eccentric discs I questioned the use of a viscous model of the turbulent stress in an accretion disc (Ogilvie 2001). The assumption that the stress is linearly and...
instantaneously related to the rate of strain typically leads to an instability whereby an initially circular disc develops eccentricity on short radial length-scales (see also Lyubarskij, Postnov & Prokhorov 1994); in fact, this is essentially the same effect as the viscous overstability of axisymmetric waves discovered by Kato (1978). In order to take account of the non-zero relaxation time \( \tau \) of the turbulence, I proposed the use of a viscoelastic model for the Maxwell stress tensor,

\[
M_{ij} + \tau \delta M_{ij} = \mu (\partial_t u_i + \partial_j u_i) + \left( \mu_b - \frac{2}{3} \mu \right) (\partial_i u_j + \partial_j u_i),
\]

(54)

where \( \mu \) is the effective shear viscosity, \( \mu_b \) is the effective bulk viscosity, and \( \tau \) is a time-derivative operator such that \( \delta M_{ij} = \partial_j \mu_i \). This model is a compressible version of the upper-convected Maxwell fluid of non-Newtonian fluid dynamics, and is motivated by the fact that the Maxwell tensor satisfies the equation \( \delta M_{ij} = 0 \) exactly in ideal MHD. The analogy between viscoelasticity and MHD has been developed in some detail by Ogilvie & Proctor (2003), who show that the magnetorotational instability is also found in viscoelastic Couette flow. The introduction of the relaxation term, with \( \tau \) being comparable to the dynamical time-scale \( \Omega^{-1} \), has a profound influence on the dynamics of eccentric discs and tends to remove the viscous overstability.

The viscoelastic model was intended to introduce the effect of the relaxation time in the simplest possible way, and generalizes the conventional viscous model by the introduction of a single new parameter, the Weissenberg number \( \Omega \tau \). It is not a true magnetorotational model because it neglects the Reynolds stress and relies on an effective viscosity to generate a non-zero Maxwell stress. In contrast, the model proposed in the present paper does not require any artificial source terms because the stresses arise through an instability of the trivial state and through a cooperative non-linear interaction between the Reynolds and Maxwell stresses. The cost paid for this is the need for approximately twice as many terms, variables and parameters in the model. Nevertheless, the viscoelastic model can be compared closely with equation (43) if one identifies \( \tau^{-1} = (C_1 + C_2)\Omega x (M/p)^{1/2} \) and understands that the \( C_4 \) term, in acting as a source for \( M_{ij} \), is analogous to the effective viscosity.

8.3 Comparison with numerical simulations

The behaviour of the present model in the shearing sheet, as set out in the various parts of Section 5, agrees at least qualitatively with the results of existing numerical simulations. For a non-rotating shear flow without a magnetic field, one may compare with the simulations by Pumir (1996), which demonstrate the development of statistically steady, homogeneous and anisotropic turbulence. A detailed comparison with the mean stresses obtained in numerical simulations, preferably in the limit of horizontally extended shearing boxes \( L_x, L_y \gg L_z, \) would test the accuracy of the model and constrain the values of the parameters.

Rotating shear flows without a magnetic field have been studied numerically by Hawley, Balbus & Winters (1999). They investigated rotation laws of the form \( \Omega \propto r^{-q} \), finding that Keplerian shear flows \( (q = 3/2) \) are hydrodynamically stable while constant angular momentum shear flows \( (q = 2) \) are non-linearly unstable and become turbulent. They tried to locate the critical value of \( q \) at which hydrodynamic turbulence could just be sustained, and found it to be close to 1.95, although this may depend to some extent on the resolution of the numerical method and the initial conditions adopted. These results are in qualitative agreement with Fig. 4, which also suggests that hydrodynamic turbulence can be sustained only in flows that are Rayleigh-unstable or only just Rayleigh-stable.

The present model predicts the development of steady MHD turbulence in a Keplerian shear flow, much as seen in numerical simulations by Hawley et al. (1995), although it is more directly applicable to calculations without an imposed magnetic flux, such as those of Brandenburg et al. (1995) and Stone et al. (1996). The initial magnetic field in the latter simulations is well ordered, although of zero mean, and gives rise to an initial phase of exponential growth that is not captured in the model. Nevertheless, the properties of the saturated stresses appear to be in good qualitative agreement. A detailed numerical comparison would be inappropriate, because unfortunately there is no consensus as to the ‘correct’ absolute or relative magnitudes of the mean stress components in a Keplerian shear flow. For example, Brandenburg et al. (1995) and Stone et al. (1996), although adopting identical initial conditions and numerical resolutions, obtain widely differing quantitative results; in particular, they disagree on whether the turbulent kinetic energy or the turbulent magnetic energy is greater. It is unclear whether this discrepancy is attributable to the difference in boundary conditions or to a lack of numerical convergence.

Abramowicz et al. (1996) have simulated the magnetorotational instability in non-Keplerian shear flows, having in mind the application to the inner parts of accretion discs around black holes. They found a non-linear relation between the shear stress and the shear rate, at fixed angular velocity, in good agreement with Fig. 6.

Interestingly, the present model also allows for sustained MHD turbulence in the case of negative Rossby number, \( A/\Omega < 0 \), which corresponds to a situation in which the angular velocity increases outwards. This regime may appear astrophysically improbable but may be relevant to the boundary layer between a star and a disc. While boundary layers have been simulated in recent global studies (Armitage 2002; Steinacker & Papaloizou 2002), the case of negative Rossby number does not appear to have been explored in shearing-box simulations. There is no linear magnetorotational instability in this case [except when the Hall effect is important (cf. Balbus & Terquem 2001)], but the present model suggests that a turbulent MHD state may nevertheless exist and is accessed by a non-linear instability of the basic state. The energetic arguments presented by Balbus & Hawley (1998) appear to allow for this possibility, and it would be valuable to test this hypothesis in numerical simulations.

In one study, Hawley, Gammie & Balbus (1996) repeated one of their typical shearing-box simulations of magnetorotational turbulence, but omitted the Coriolis force, thereby converting the calculation into a local model of a non-rotating shear flow. They found that the magnetic energy was not sustained, although the turbulent kinetic energy continued to grow until the calculation was halted. This differs from the prediction of the present model in Section 5.3, that turbulent plane Couette flow acts as a small-scale dynamo and allows the magnetic energy to grow to a substantial fraction of equipartition with the turbulent kinetic energy, which itself reaches a saturated value. Indeed, the model is constructed on the basis that any three-dimensional turbulent flow at sufficiently high \( Rm \), by virtue of its stretching property, tends to amplify a weak, small-scale magnetic field introduced into it, at least until the Lorentz forces can modify the flow. In plane Couette flow, the shearing background provides a further means of amplifying the magnetic energy. The numerical result of Hawley et al. (1996), that the combination of turbulence and shear has the opposite characteristic, is therefore surprising. One possible explanation for the discrepancy between the numerical result and the prediction of the model is that \( Rm \) is too small in
the simulation to capture the small-scale dynamo. Alternatively, it may indicate a deficiency in the model. Further numerical studies may resolve this question.

Regrettably, there are no published numerical studies of the propagation and damping of the two-dimensional wave in turbulent discs, to compare with the predictions of Section 7.4. This problem is of some importance, especially in view of its application to tidally generated waves in binary stars and protoplanetary systems, and would make a further test of the present model.

However, Torkelsson et al. (2000) have already studied numerically the interaction of magnetorotational turbulence with the shear epicyclic motions found in warped discs (cf. Section 7.5). They found that the epicyclic motions are damped, but at a rate approximately one-half that expected from the action of an isotropic viscosity (the viscosity being measured from the total xy stress).

This is similar to the predictions of the present model as set out in Section 7.5.

9 CONCLUSION

The representation of the turbulent stresses in an accretion disc by the classical alpha viscosity model of Shakura & Sunyaev (1973) has been extremely successful in allowing the theory of accretion discs to develop even in the absence of a proper understanding of the turbulence, and in making possible a quantitative comparison between theory and observations. However, the magnetorotational instability is now widely accepted to be the origin of the turbulence in sufficiently ionized discs (e.g. Balbus & Hawley 1998), and the alpha viscosity model fails to describe numerous aspects of this process. In this paper, I have introduced a new analytical model that aims to represent more faithfully the dynamics of magnetorotational turbulent stresses and to bridge the gap between analytical studies and numerical simulations. Covariant evolutionary equations for the mean Reynolds and Maxwell tensors are presented, which correctly include the linear interaction with the mean flow. Non-linear and dissipative effects, in the absence of an imposed magnetic flux and in the limit of large Reynolds number and magnetic Reynolds number, are modelled through five non-linear terms that represent known physical processes and are strongly constrained by symmetry properties and dimensional considerations.

The cooperation of these linear and non-linear effects explains, without any fine tuning, the development of statistically steady, anisotropic turbulent stresses in the shearing sheet, a local representation of a differentially rotating disc, in agreement with numerical simulations. It reproduces other results seen in numerical studies: that purely hydrodynamic turbulence is not sustained in a flow that adequately satisfies Rayleigh’s stability criterion; that the shear stress in magnetorotational turbulence in non-Keplerian discs is non-linearly related to the shear rate, at fixed angular velocity; and that the shearing epicyclic motions found in warped discs are damped by the turbulence at a slower rate than predicted on the basis of an isotropic viscosity. It also makes predictions for future simulations, such as the decay of the two-dimensional wave and the possibility of sustained turbulence in situations where the angular velocity increases outwards. The model has good formal properties, being typically hyperbolic and therefore ‘causal’, guaranteeing the realizability of the stress tensors, and accounting satisfactorily for energy conservation.

Only the linear part of the model is derived directly from the MHD equations, and therefore the predictions of the model cannot be regarded as rigorous deductions. In particular, the non-linear hydrodynamic stability of circular Keplerian flow remains unproven.

Nevertheless, even without an accurate calibration of its parameters, the model almost certainly performs more faithfully than the alpha viscosity model in a variety of circumstances, and therefore represents an advance over previous work.

The general idea of seeking a model that connects the mean turbulent stress to changes in the mean velocity gradient can be related to one of the classical problems of continuum mechanics, which is to find the constitutive relation that characterizes a particular substance and allows the stress in a body to be related to its deformation history. From this perspective, the present model is reminiscent of certain classes of viscoelastic fluids. In fact, the coexistence of the Reynolds and Maxwell stresses, which are influenced by the mean flow in different ways, suggests a kind of composite viscoelastic material. The reason that a constitutive relation might be thought to exist at all for magnetorotational turbulence is that the physics of the magnetorotational instability is local, suggesting that the mean stress in a volume comparable to $H^3$ should indeed depend only on the recent deformation history of that parcel of fluid.

In recent years, numerical simulations of accretion flows subject to the magnetorotational instability have been taken beyond the confines of the shearing box to study the effects of cylindrical geometry, the flow across the marginally stable orbit around a black hole, or the boundary layer between a star and a disc (e.g. Armitage 1998; Armitage 2002; Hawley 2000; Hawley, Balbus & Stone 2001). These advances are to be welcomed, and represent significant computational achievements. However, I believe there is still, and will continue to exist, a need for a description such as the one developed in this paper. This is not just because global studies of realistic thin discs are well beyond the current capabilities of direct numerical simulations, but also because such a model will allow a variety of methods, analytical and less direct numerical ones, to continue to be used in studying accretion disc dynamics while taking seriously the nature of magnetorotational turbulence. It should be particularly useful in understanding the dynamics of warped, eccentric and tidally distorted discs, non-Keplerian accretion flows close to black holes, and a variety of time-dependent accretion phenomena.

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APPENDIX A: REALIZABILITY

The realizability constraint requires that the three eigenvalues of $R_{ij}$ and the three eigenvalues of $M_{ij}$ remain non-negative under the evolutionary model. A full investigation of this issue would require a consideration of a large number of degenerate cases. For simplicity, I demonstrate here only the weaker result that no one of the eigenvalues can become negative while the others remain positive.

It is therefore assumed that, in the initial condition, $R_{ij}$ and $M_{ij}$ are positive definite tensors. If either $R_{ij}$ or $M_{ij}$ were to develop a negative eigenvalue in the subsequent evolution, when the first such eigenvalue passed through zero the traces $R$ and $M$ (which are the sums of the eigenvalues) would still be positive. Therefore one may take $R, M > 0$ below, without loss of generality.

Define the quadratic forms $P = R_{ij} X_i X_j$ and $Q = M_{ij} Y_i Y_j$, where $X_i$ and $Y_i$ are differentiable vector fields advected according to the time-reversible equations

$$\left(\partial_t + u_j \partial_j\right) X_i - X_i \partial_t u_j + 2\epsilon_{ijkl} \Omega_k X_i = 0, \quad (A1)$$

$$\left(\partial_t + u_j \partial_j\right) Y_i + Y_i \partial_t u_j = 0. \quad (A2)$$

Then

$$\frac{D P}{D t} = - (C_1 + C_2) L^{-1} R^{1/2} P + \frac{1}{2} C_2 L^{-1} R^{3/2} X_i X_i$$

$$+ C_3 L^{-1} M^{1/2} M_{ij} X_i X_j - C_4 L^{-1} R^{-1/2} M P, \quad (A3)$$

$$\frac{D Q}{D t} = C_4 L^{-1} R^{-1/2} M R_{ij} Y_i Y_j - (C_1 + C_3) L^{-1} M^{1/2} Q, \quad (A4)$$

where $D / D t = \partial_t + u_j \partial_j$ is the Lagrangian time derivative.

In the initial condition, $P > 0$ everywhere for all $X_i$ and $Q > 0$ everywhere for all $Y_i$. According to equation (A3), $DP / D t > 0$ whenever $P = 0$ and $M_{ij}$ is positive definite, and therefore $P$ cannot become negative. According to equation (A4), $DQ / D t > 0$ whenever $Q = 0$ and $R_{ij}$ is positive definite, and therefore $Q$ cannot become negative.

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