The universal Banach space with a $K$-suppression unconditional basis

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Abstract

Using the technique of Fraïssé theory, for every constant $K \geq 1$ we construct a universal object in the class of Banach spaces with normalized $K$-suppression unconditional Schauder bases.

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1 Introduction

A Banach space $X$ is complementably universal for a given class of Banach spaces if $X$ belongs to this class and every space from the class is isomorphic to a complemented subspace of $X$.

In 1969 Pełczyński [12] constructed a complementably universal Banach space for the class of Banach spaces with a Schauder basis. In 1971 Kadec [7] constructed a complementably universal Banach space for the class of spaces with the bounded approximation property (BAP). In the same year Pełczyński [10] showed that every Banach space with BAP is complemented in a space with a basis. Pełczyński and Wojtaszczyk [13] constructed in 1971 a universal Banach space for the class of spaces with a finite-dimensional decomposition. Applying Pełczyński’s decomposition argument [11], one immediately concludes that all three universal spaces are isomorphic. It is worth mentioning a negative result of Johnson and Szankowski [6] saying that no separable Banach space can be complementably universal for the class of all separable Banach spaces. In [4] the second author constructed an isometric version of the Kadec-Pełczyński-Wojtaszczyk space. The universal Banach space from [4] was constructed using the general categorical technique of Fraïssé limits [8]. This method was also applied by Kubiś and Solecki in [9] for constructing the Gurarii space [5], which possesses the property of extension of almost isometries, which implies the universality property that is stronger than the standard universality property of the Banach spaces $\ell_\infty$ or $C[0,1]$.

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In this paper we apply the categorical method of Fraïssé limits for constructing a universal space in the class of Banach spaces with a normalized $K$-suppression unconditional Schauder basis. The universal space constructed by this method has a nice property of extension of almost isometries, which is better than just the standard universality, established in the papers of Pelczynski [12] and Schechtman [13] (who gave a short alternative construction of universal space for class of Banach spaces with an unconditional bases).

2 Preliminaries

All Banach spaces considered in this paper are separable and over the field $\mathbb{R}$ of real numbers.

2.1 Definitions

Let $X$ be a Banach space with a Schauder basis $(e_n)_{n=1}^{\infty}$ and let $(e^*_n)_{n=1}^{\infty}$ be the corresponding sequence of coordinate functionals. The basis $(e_n)_{n=1}^{\infty}$ is called $K$-suppression for a real constant $K$ if for every finite subset $F \subset \mathbb{N}$ the projector $pr_F : X \to X$, $pr_F : x \mapsto \sum_{n \in F} e^*_n(x) \cdot e_n$, has norm $\|pr_F\| \leq K$. It is well-known [1] 3.1.5 that each $K$-suppression Schauder basis $(e_n)_{n=1}^{\infty}$ is unconditional. So for any $x \in X$ and any permutation $\pi$ of $\mathbb{N}$ the series $\sum_{n=1}^{\infty} e^*_\pi(n)(x) \cdot e_{\pi(n)}$ converges to $x$. This means that we can forget about the ordering and think of a $K$-suppression basis of a Banach space as a subset $B \subset X$ such that for some bijection $e : \mathbb{N} \to B$ the sequence $(e(n))_{n=1}^{\infty}$ is a $K$-suppression Schauder basis for $X$.

More precisely, by a normalized $K$-suppression basis for a Banach space $X$ we shall understand a subset $B \subset X$ for which there exists a family $\{e^*_b\}_{b \in B} \subset X$ of continuous functionals such that

- $\|b\| = 1 = e^*_b(b)$ for any $b \in B$;
- $e^*_b(b') = 0$ for every $b \in B$ and $b' \in B \setminus \{b\}$;
- $x = \sum_{b \in B} e^*_b(x) \cdot b$ for every $x \in X$;
- for any finite subset $F \subset B$ the projector $pr_F : X \to X$, $pr_F : x \mapsto \sum_{b \in F} e^*_b(x) \cdot b$, has norm $\|pr_F\| \leq K$.

The equality $x = \sum_{b \in B} e^*_b(x) \cdot b$ in the third item means that for every $\varepsilon > 0$ there exists a finite subset $F \subset B$ such that $\|x - \sum_{b \in E} e^*_b(x) \cdot b\| < \varepsilon$ for every finite subset $E \subset B$ containing $F$.

By a $K$-based Banach space we shall understand a pair $(X, B_X)$ consisting of a Banach space $X$ and a normalized $K$-suppression basis $B_X$ for $X$. A $K$-based Banach space $(X, B_X)$ is a subspace of a $K$-based Banach space $(Y, B_Y)$ if $X \subseteq Y$ and $B_X = X \cap B_Y$.

A finite dimensional $K$-based Banach $(X, B_X)$ is rational if its unit ball is a convex polyhedron spanned by finitely many vector whose coordinates in the basis $B_X$ are rational.

2.2 Categories

Let $\mathfrak{A}$ be a category. For two objects $A, B$ of the category $\mathfrak{A}$, by $\mathfrak{A}(A, B)$ we will denote the set of all $\mathfrak{A}$-morphisms from $A$ to $B$. A subcategory of $\mathfrak{A}$ is a category $\mathfrak{L}$ such that each object of $\mathfrak{L}$ is an object of $\mathfrak{A}$ and each arrow of $\mathfrak{L}$ is an arrow of $\mathfrak{A}$.

A category $\mathfrak{L}$ is cofinal in $\mathfrak{A}$ if for every object $A$ of $\mathfrak{A}$ there exists an object $B$ of $\mathfrak{L}$ such that the set $\mathfrak{A}(A, B)$ is nonempty. A category $\mathfrak{A}$ has the amalgamation property if for every objects
In this section we prove that the category property.

Let us show that the basis \( B \) of the coordinate functionals of the basis \( \mathbf{z} \) on the other hand, for every \( k \in \mathbb{R} \) and morphisms \( f' \in \mathfrak{R}(B, D) \), \( g' \in \mathfrak{R}(C, D) \) such that \( f' \circ j = g' \circ j \)

In this paper we shall work in the category \( \mathfrak{R} \), whose objects are \( K \)-based Banach spaces for some fixed real constant \( K \geq 1 \). For two \( K \)-based Banach spaces \( (X, B_X), (Y, B_Y) \), a morphism of category \( \mathfrak{R} \) is a linear continuous operator \( T : X \to Y \) such that \( T(B_X) \subseteq B_Y \). A morphism \( T : X \to Y \) of the category \( \mathfrak{R} \) is called an isometry if \( \|T(x)\|_Y = \|x\|_X \) for any \( x \in X \).

### 2.3 Amalgamation

In this section we prove that the category \( \mathfrak{R} \) of \( K \)-based Banach spaces has the amalgamation property.

**Lemma 1. (Amalgamation Lemma)** Let \( X, Y, Z \) be \( K \)-based Banach spaces and \( j : Z \to X \), \( i : Z \to Y \) be isometries. Then there exist a \( K \)-based Banach space \( W \) and isometries \( j' : Y \to W \) and \( i' : X \to W \) such that the diagram

\[
\begin{array}{ccc}
Y & \overset{j'}{\longrightarrow} & W \\
\downarrow{i'} & & \downarrow{i} \\
Z & \overset{j}{\longrightarrow} & X
\end{array}
\]

is commutative.

Moreover, if the \( K \)-based Banach spaces \( X, Y, Z \) are finite-dimensional (rational), then so is the \( K \)-based Banach space \( W \).

**Proof.** Without loss of generality we may assume that \( Z = X \cap Y \), \( B_Z = B_X \cap B_Y \) and the isometries \( i, j \) are identity inclusions. Here \( B_X, B_Y, B_Z \) are the normalized \( K \)-suppression bases of the \( K \)-based Banach spaces \( X, Y, Z \). It follows from \( B_Z = B_X \cap B_Y \) that the coordinate functionals of the bases \( B_X \) and \( B_Y \) agree on the intersection \( Z = X \cap Y \).

Consider the direct sum \( X \oplus Y \) of the Banach space \( X, Y \) endowed with the norm \( \|(x, y)\| = \|x\|_X + \|y\|_Y \). Let \( W = (X \oplus Y)/\Delta \) be the quotient by the subspace \( \Delta = \{(z, -z) : z \in Z\} \). We define linear operators \( i' : X \to W \) and \( j' : Y \to W \) by \( i'(x) = (x, 0) + \Delta \) and \( j'(y) = (0, y) + \Delta \).

Let us show \( i' \) and \( j' \) are isometries. Indeed, for every \( x \in X \)

\[
\|i'(x)\|_W = \text{dist}((x, 0), \Delta) \leq \|(x, 0)\| = \|x\|_X + \|0\|_Y = \|x\|_X.
\]

On the other hand, for every \( z \in Z \)

\[
\|(x, 0) - (z, -z)\| = \|(x - z, z)\| = \|x - z\|_X + \|z\|_Y = \|x - z\|_X + \|z\|_X \geq \|x - z + z\|_X = \|x\|_X
\]

and hence \( \|x\|_X \leq \inf_{z \in Z} \|(x, 0) - (z, -z)\| = \|i'(x)\|_W \). Therefore \( \|i'(x)\|_W = \|x\|_X \). Similarly, we can show that \( j' \) is an isometry.

We shall identify \( X \) and \( Y \) with their images \( i'(X) \) and \( j'(Y) \) in \( W \). In this case \( B_W = B_X \cup B_Y \) is a normalized Schauder basis for the Banach space \( W \). Let \( \{b_i\}_{i \in B_W} \subset W^* \) be the sequence of coordinate functionals of the basis \( B_W \).

Let us show that the basis \( B_W = B_X \cup B_Y \) is \( K \)-suppression. Given any finite subset \( D \) of \( B_W \) we should prove that the projector \( \text{pr}_D : W \to W \) has norm \( \|\text{pr}_D\| \leq K \).
Write \( D = D_{Z} \cup D_{X} \cup D_{Y} \), where \( D_{Z} = D \cap B_{Z} = D \cap B_{X} \cap B_{Y}, D_{X} = D \setminus B_{Y} \) and \( D_{Y} = D \setminus B_{X} \).

Taking into account that bases \( B_{X} \) and \( B_{Y} \) are \( K \)-suppression, for any \( w \in W \) we obtain:

\[
\| \operatorname{pr}_{D}(w) \|_{W} = \inf \{ \| x \|_{X} + \| y \|_{Y} : x \in X, \ y \in Y, \ x + y = \operatorname{pr}_{D}(w) \} = \\
= \inf \{ \| \operatorname{pr}_{D_{X}}(w) + z'\|_{X} + \| z'' + \operatorname{pr}_{D_{Y}}(w) \|_{Y} : z', z'' \in Z, \ z' + z'' = \operatorname{pr}_{D_{Z}}(w) \} \leq \\
\leq \inf \{ \| \operatorname{pr}_{D_{X}}(w) + z'\|_{X} + \| z'' + \operatorname{pr}_{D_{Y}}(w) \|_{Y} : z', z'' \in \operatorname{pr}_{D_{Z}}(Z), \ z' + z'' = \operatorname{pr}_{D_{Z}}(w) \} = \\
= \inf \{ \| \operatorname{pr}_{B_{X} \setminus B_{Z}}(w) + z'\|_{X} + \| z'' + \operatorname{pr}_{B_{Y} \setminus B_{X}}(w) \|_{Y} : z' + z'' = \operatorname{pr}_{B_{Z}}(w) \} = \\
= K \cdot \inf \{ \| x \|_{X} + \| y \|_{Y} : x + y = w \} = K \cdot \| w \|_{W}. 
\]

This completes the proof. \( \square \)

## 3 Rational universality

### Definition 1

A \( K \)-based Banach space \( X \) is called \textit{rationally universal} if each finite dimensional \( K \)-based subspace of \( X \) is rational and for any finite-dimensional rational \( K \)-based Banach space \( A \) and subspace \( A' \subset A \), any isometry \( f' : A' \to X \) can be extended to an isometry \( f : A \to X \).

Denote by \( \mathcal{F} \) the subcategory of \( \mathcal{K} \) whose objects are rational finite-dimensional \( K \)-based Banach spaces and morphisms are linear isometries of such spaces. Obviously, up to isomorphism the category \( \mathcal{F} \) contains countably many objects. By Lemma 1 the category \( \mathcal{F} \) has the amalgamation property. We now use the concepts from \[8\] for constructing a “generic” sequence in \( \mathcal{F} \). A sequence \( (X_{n})_{n \in \omega} \) of objects of the category \( \mathcal{F} \) is called a \textit{chain} if each space \( X_{n} \) is a subspace of the \( K \)-based Banach space \( X_{n+1} \).

### Definition 2

A chain of \( (U_{n})_{n \in \omega} \) of objects of the category \( \mathcal{F} \) is \textit{Fraïssé} if for any \( n \in \omega \), and any morphism \( f : U_{n} \to Y \) of \( \mathcal{F} \), there exist \( m > n \) and a morphism \( g : Y \to U_{m} \) of the category \( \mathcal{F} \) such that \( g \circ f : U_{n} \to U_{m} \) is the identity inclusion of \( U_{n} \) to \( U_{m} \).

The name “Fraïssé sequence”, as in \[8\], is motivated by the model-theoretic theory of Fraïssé limits developed by Roland Fraïssé \[3\]. One of the results in \[8\] is that every countably cofinal category with amalgamation has a Fraïssé sequence. Applying this general result to our category \( \mathcal{F} \) we get:

### Theorem 1 (\[8\])

\textit{The category} \( \mathcal{F} \) \textit{has a Fraïssé sequence.}

From now on, we fix a Fraïssé sequence \( (U_{n})_{n \in \omega} \) in \( \mathcal{F} \), which can be assumed to be a chain of finite-dimensional rational \( K \)-based Banach spaces. Let \( \mathbb{U} \) be the completion of the union \( \bigcup_{n \in \omega} U_{n} \) and \( B_{\mathbb{U}} = \bigcup_{n \in \omega} B_{U_{n}} \).

### Lemma 2

\( (\mathbb{U}, B_{\mathbb{U}}) \) \textit{is a} \( K \)-\textit{based Banach space.}

\textit{Proof}. We have to prove that \( B_{\mathbb{U}} = \bigcup_{n \in \omega} B_{U_{n}} \) is a normalized \( K \)-suppression Schauder basis for \( \mathbb{U} \). For each \( n \) the spaces \( U_{n} \) are \( K \)-based Banach spaces, so \( \| b \| = 1 \) for every \( b \in U_{n} \). This shows that \( B_{\mathbb{U}} \) is normalized. The fact that \( B_{\mathbb{U}} \) is a \( K \)-suppression unconditional Schauder basis follows from Lemma 6.2 and Fact 6.3 in \[2\]. \( \square \)

Definition 1 and the construction of the \( K \)-based Banach space \( \mathbb{U} \) implies the following theorem.
Theorem 2. The $K$-based Banach space $U$ is rationally universal.

Theorem 3. Any rationally universal spaces $X$, $Y$ are isometric.

Proof. Let $X$, $Y$ be the completions of unions $\bigcup_{n \in \omega} X_n$ and $\bigcup_{n \in \omega} Y_n$ of chains of finite dimensional $K$-based Banach spaces such that $X_0 = \{0\}$ and $Y_0 = \{0\}$. We define inductively sequences of linear operators $\{f_k\}_{k \in \omega}$, $\{g_k\}_{k \in \omega}$ and increasing number sequences $(n_k)$, $(m_k)$ such that the following conditions are satisfied for every $k \in \omega$:

1. $f_k : X_{n_{k-1}} \to Y_{m_k}$ and $g_k : Y_{m_k} \to X_{n_k}$ are isometric embeddings;

2. $f_{k+1} \circ g_k = \id \upharpoonright Y_{m_k}$ and $g_{k+1} \circ f_{k+1} = \id \upharpoonright X_{n_k}.$

We start the inductive construction letting $n_0 = 0 = m_0$ and $f_0 : X_0 \to Y_0$, $g_0 : Y_0 \to X_0$ be the unique linear operators of trivial Banach spaces. To make an inductive step, assume that for some $k \in \omega$, the numbers $n_k$, $m_k$ and isometries $f_k : X_{n_{k-1}} \to Y_{m_k}$, $g_k : Y_{m_k} \to X_{n_k}$ have been constructed. Definition of rational universality of the sequence $(Y_m)_{m \in \omega}$ yields a number $m_k < n_{k-1}$ and an isometry $f_{k+1} : X_{n_k} \to Y_{m_{k+1}}$ such that $f_{k+1} \circ g_k$ coincides with the identity inclusion $Y_{m_k}$ in $Y_{m_{k+1}}$. Using the rational universality of the sequence $(X_n)_{n \in \omega}$ we can find a number $m_{k+1} > m_k$ and an isometry $g_{k+1} : Y_{m_{k+1}} \to X_{m_{k+1}}$ such that $g_{k+1} \circ f_{k+1}$ is the identity inclusion $X_{n_k}$ in $X_{n_{k+1}}$. This complete the inductive step.

After completing the inductive construction consider the isometries $f : \bigcup_{n \in \omega} X_n \to \bigcup_{m \in \omega} Y_m$ and $g : \bigcup_{m \in \omega} Y_m \to \bigcup_{n \in \omega} X_n$ such that for every $k \in \omega$ $f \upharpoonright X_{n_k} = f_{k+1}$ and $g \upharpoonright Y_{m_k} = g_k$.

By the uniform continuity, the isometries $f$, $g$ extend to isometries $\bar{f} : X \to Y$ and $\bar{g} : Y \to X$.

The condition (2) of the inductive construction implies that $\bar{f} \circ \bar{g} = \id_Y$ and $\bar{g} \circ \bar{f} = \id_X$, so $f$ and $g$ are isometric isomorphisms of the $K$-based Banach spaces $X$ and $Y$. \hfill \Box

4 Almost universality

A linear operator $f$ between Banach spaces $X$ and $Y$ is called an $\varepsilon$-isometry for a positive real number $\varepsilon$, if

$$(1 + \varepsilon)^{-1} \cdot \|x\|_X < \|f(x)\|_Y < (1 + \varepsilon) \cdot \|x\|_X$$

for every $x \in X \setminus \{0\}$.

Definition 3. A $K$-based Banach space $X$ called almost-universal if for any $\varepsilon > 0$ and finite dimensional $K$-based Banach space $A$, any $\varepsilon$-isometry $f' : A' \to X$ defined on a based subspace $A' \subseteq A$ can be extended to a $\varepsilon$-isometry $f : A \to X$.

Theorem 4. Any rational universal $K$-based Banach space $X$ is almost-universal.

Proof. We shall use the fact, that every finite dimensional Banach space can be approximated by a rational Banach space.

To prove that $X$ is almost-universal we take any $\varepsilon > 0$, any finite-dimensional $K$-based Banach spaces $A \subseteq A'$ and an $\varepsilon$-isometry $f : A \to X$. Consider the new norm $\| \cdot \|_1$ on $A$ defined by $\|a\|_1 = \|f(a)\|_X$ for $a \in A$. Since $X$ is rational and $K$-based, $\| \cdot \|_1$ is a rational norm on $A$ such that $\|pf(a)\|_1 \leq K \cdot \|a\|_1$ for every $a \in A$ and every subset $F \subseteq B_A$. Taking into account that $f$ is an $\varepsilon$-isometry, we conclude that $(1 + \varepsilon)^{-1}\|x\| < \|x\|_1 < (1 + \varepsilon)\|x\|$ for every $x \in A$ with $\|x\| = 1$. By the compactness of the unite sphere in $A$, there exists a positive $\delta < \varepsilon$ such
that \((1 + \delta)^{-1}\|x\| < \|x\|_1 < (1 + \delta)\|x\|\) for every \(x \in A\) with \(\|x\| = 1\). This inequality implies \(\frac{1}{1 + \delta}O \subset O_1 \subset (1 + \delta)O\), where \(O = \{x \in A : \|x\| \leq 1\}\) and \(O_1 = \{x \in A : \|x\|_1 \leq 1\}\) are the unit balls of \(A\) in the norms \(\|\cdot\|\) and \(\|\cdot\|_1\). Choose \(\delta'\) such that \(\delta < \delta' < \varepsilon\). Let \(O' = \{x \in A' : \|x\| \leq 1\}\). Choose a rational polyhedron \(O_2'\) in \(A'\) such that \(O_2' = -O_2'\) and \(\frac{1}{1 + \delta'}O' \subset O_2' \subset (1 + \delta)O'\). Next consider the convex hull \(O_2 = \text{conv}(P)\) of the set \(P = O_1 \cup O_2' \cup \bigcup_{F \subset B_{A'}} \frac{1}{K}\text{pr}_F(O_2')\). Taking into account that \(A'\) is a \(K\)-based Banach space, we conclude that

\[
P \subset O_1 \cup \frac{1}{1 + \delta'}(O' \cup \bigcup_{F \subset B_{A'}} \frac{1}{K}\text{pr}_F(O')) = O_1 \cup \frac{1}{1 + \delta'}O' \subset (1 + \delta)O' \cup \frac{1}{1 + \delta'}O' = (1 + \delta)O'
\]

and hence

\[
\frac{1}{1 + \delta'}O' \subset O_2 \subset O_2 = \text{conv}(P) \subset (1 + \delta)O'.
\]

The convex symmetric set \(O_2\) determines a norm \(\|\cdot\|_2\) whose unite ball coincides with \(O_2\). Its is clear that the norm \(\|\cdot\|_2\) is rational and the base \(B_{A'}\) of the Banach space \((A', \|\cdot\|_2)\) is \(K\)-suppression. It remains to check that \(\|x\|_2 = \|x\|_1\) for each \(x \in A\), which is equivalent to equality \(O_2 \cap A = O_1\). The inclusion \(O_1 \subset O_2 \cap A\) is evident. To prove the reverse inclusion \(O_1 \supset O_2 \cap A\) observe that

\[
A \cap O_2 = A \cap \text{conv}(P) \subset A \cap \text{conv}(O_1 \cup \frac{1}{1 + \delta'}O') = A \cap \{tx_1 + (1 - t)x_2 : t \in [0, 1], \ x_1 \in O_1, x_2 \in \frac{1}{1 + \delta'}O'\} = \{tx_1 + (1 - t)x_2 : t \in [0, 1], \ x_1 \in O_1, x_2 \in \frac{1}{1 + \delta}(A \cap O')\} \subset \text{conv}(O_1 \cup O_1) = O_1.
\]

The inclusions \(\frac{1}{1 + \delta'}O' \subset O_2 \subset (1 + \delta)O'\) imply the strict inequality \((1 + \varepsilon)^{-1}\|x\| < \|x\|_2 < (1 + \varepsilon)\|x\|\) holding for all \(x \in A' \setminus \{0\}\). Using the rational universality of the \(K\)-based Banach space \(X\), find an isometric embedding \(f' : A' \to X\) from the rational \(K\)-based Banach space \((A', \|\cdot\|_2)\) such that \(f' : A = f\). The choice of the norm \(\|\cdot\|_2\) ensures that \(f'\) is an \(\varepsilon\)-isometry from the Banach space \(A'\), extending the \(\varepsilon\)-isometry \(f\). This completes the proof of the almost universality of \(X\).

\[\square\]

**Theorem 5.** Let \(\mathbb{U}\) and \(\mathbb{V}\) be almost-universal \(K\)-based Banach spaces and \(\varepsilon > 0\). Each \(\varepsilon\)-isometry \(f : X \to \mathbb{V}\) defined on a finite-dimensional based subspace \(X\) of the \(K\)-based Banach space \(\mathbb{U}\) can be extended to a bijective \(\varepsilon\)-isometry \(\tilde{f} : \mathbb{U} \to \mathbb{V}\).

**Proof.** Fix a positive real number \(\varepsilon\). Using the compactness of the unite sphere of the finite dimensional Banach space \(X\), we can find a positive \(\delta < \varepsilon\) such that \(f\) is a \(\delta\)-isometry. Write \(\mathbb{U}\) and \(\mathbb{V}\) as the completions of the unions \(\bigcup_{n \in \omega} X_n\) and \(\bigcup_{n \in \omega} Y_n\) of chains of finite dimensional \(K\)-based Banach spaces such that \(X_0 = X\) and \(Y_0 = f(X)\). We define inductively sequences of linear operators \(\{f_k\}_{k \in \omega}\) and increasing number sequences \((n_k), (m_k)\) such that \(m_0 = n_0 = 0\), \(f_0 = f\) and the following conditions are satisfied for every \(k \in \omega\):

1. \(f_k : X_{n_k - 1} \to Y_{m_k}\) and \(g_k : Y_{m_k} \to X_{n_k}\) are \(\delta\)-isometries;

2. \(f_{k+1} \circ g_k = \text{id} \mid Y_{m_k}\) and \(g_{k+1} \circ f_{k+1} = \text{id} \mid X_{n_k}\).

To make the inductive step assume that for some \(k \in \omega\), the numbers \(n_k, m_k\) and \(\delta\)-isometries \(f_k : X_{n_k - 1} \to Y_{m_k}\), \(g_k : Y_{m_k} \to X_{n_k}\) have been constructed. Definition of almost-universality
of the sequence \((Y_m)_{m \in \omega}\) yields a number \(n_k > n_{k-1}\) and \(\delta\)-isometry \(f_{k+1} : X_{n_k} \to Y_{m_{k+1}}\) such that \(f_{k+1} \circ g_k = \text{id} \upharpoonright Y_{m_k}\). Using the almost-universality of the sequence \((X_n)_{n \in \omega}\), we can find a number \(m_{k+1} > m_k\) and a \(\delta\)-isometry \(g_{k+1} : Y_{m_{k+1}} \to X_{n_{k+1}}\) such that \(g_{k+1} \circ f_{k+1} = \text{id} \upharpoonright X_{n_k}\). This completes the inductive step.

After completing the inductive construction consider the \(\delta\)-isometries \(\tilde{f} : \bigcup_{n \in \omega} X_n \to \bigcup_{m \in \omega} Y_m\) and \(\tilde{g} : \bigcup_{m \in \omega} Y_m \to \bigcup_{n \in \omega} X_n\) such that for every \(k \in \omega\) \(\tilde{f} \upharpoonright X_{n_k} = f_{k+1}\) and \(\tilde{g} \upharpoonright Y_{m_k} = g_k\). The condition (2) of the inductive construction implies that \(\tilde{f} \circ \tilde{g}\) and \(\tilde{g} \circ \tilde{f}\) are the identity maps of \(\bigcup_{n \in \omega} X_n\) and \(\bigcup_{m \in \omega} Y_m\), respectively.

By the uniform continuity, the \(\delta\)-isometries \(\tilde{f}\), \(\tilde{g}\) extend to \(\varepsilon\)-isometries \(\bar{f} : U \to V\) and \(\bar{g} : V \to U\) such that \(\bar{f} \circ \bar{g} = \text{id}_V\) and \(\bar{g} \circ \bar{f} = \text{id}_U\).

\[\square\]

**Theorem 6.** For any \(\varepsilon > 0\), every \(K\)-based Banach space \(X\) can be \(\varepsilon\)-isometrically embedded into the almost-universal based Banach space \(U\).

**Proof.** Let \(X\) be a completion of the union \(\bigcup_{n \in \omega} X_n\) of a chain of finite dimensional \(K\)-based Banach spaces \(X_n\) such that \(X_0 = \{0\}\). Fix a positive real number \(\varepsilon\) and choose any \(\delta < \varepsilon\). We define inductively a sequence of \(\delta\)-isometries \((f_k : X_k \to U)_{k=0}^\infty\) such that \(f_k \upharpoonright X_{k-1} = f_{k-1}\) for every \(k > 0\).

We set \(f_0 = 0\). Suppose that for some \(k \in \omega\) a \(\delta\)-isometry \(f_k : X_k \to U\) has already been constructed. Using the definition of the almost-universality of the space \(U\), we can find a \(\delta\)-isometry \(f_{k+1} : X_{k+1} \to U\) such that \(f_{k+1} \upharpoonright X_k = f_k\). This completes the inductive step.

After completing the inductive construction consider the \(\delta\)-isometry \(f : \bigcup_{n=0}^\infty X_n \to U\) such that \(f \upharpoonright X_n = f_n\) for every \(n \in \omega\).

By the uniform continuity, the \(\delta\)-isometry \(f\) extends to an \(\varepsilon\)-isometry \(\bar{f} : X \to U\). \[\square\]
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