STRONGLY GRADED GROUPOID AND DIRECTED GRAPH ALGEBRAS

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Abstract. We show the reduced $C^*$-algebra of a graded ample groupoid is a strongly graded $C^*$-algebra if and only if the corresponding Steinberg algebra is a strongly graded ring. We apply this result to get a theorem about the Leavitt path algebra and $C^*$-algebra of an arbitrary graph.

The prototypical example of a strongly $\mathbb{Z}$-graded ring is the ring $K[x,x^{-1}]$ of Laurent polynomials over a field $K$. The homogeneous components are the sets $K^{x_p}$, and for $m, n \in \mathbb{Z}$ such that $m+n = p$, every element $rx^p \in K^{x_p}$ can be realised as a product $(rx^m)(1x^n)$ of elements in the homogeneous components $K^{x_m}$ and $K^{x_n}$. The $C^*$-algebraic analogue of $K[x,x^{-1}]$ is the $C^*$-algebra $C(T)$ of continuous functions on the unit circle $T \subseteq \mathbb{C}$, which by the Stone–Weierstrass theorem contains the trigonometric polynomials as a dense subspace. This is the prototype of a strongly $\mathbb{Z}$-graded $C^*$-algebra with homogeneous subspaces $C(T)^n = \mathbb{C} z^n$.

To be more precise, suppose that $A$ is a $C^*$-algebra and $\Gamma$ is a group. Following Exel [9, Definition 16.2], we say that $A$ is a $\Gamma$-graded $C^*$-algebra if there are linearly independent closed subspaces $\{A_\gamma : \gamma \in \Gamma\}$ of $A$ such that for every $\alpha, \beta \in \Gamma$, we have

(a) $A_\alpha \cdot A_\beta := \text{span}\{ab : a \in A_\alpha, b \in A_\beta\} \subseteq A_{\alpha\beta}$,
(b) $A_\alpha^* := \{a^* : a \in A_\alpha\} \subseteq A_{\alpha^{-1}}$ and
(c) $\bigoplus_{\gamma \in \Gamma} A_\gamma$ is dense in $A$.

We say that $A$ is a strongly $\Gamma$-graded $C^*$-algebra if in addition $A_\alpha \cdot A_\beta$ is dense in $A_{\alpha\beta}$ for all $\alpha\beta \in \Gamma$.

Our work is inspired by a recent result of Clark, Hazrat and Rigby in [7] which identifies the groupoids for which the Steinberg algebra is a strongly graded ring. Given an ample groupoid $G$ with Hausdorff unit space and a continuous cocycle $c : G \to \Gamma$, $c$ induces a grading on

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the Steinberg $R$-algebra $A_R(G)$. The groupoid $G$ is strongly graded in the sense that $c^{-1}(\alpha)c^{-1}(\beta) = c^{-1}(\alpha\beta)$ for all $\alpha, \beta \in \Gamma$ if and only if $A_R(G)$ is a strongly $\Gamma$-graded ring [7, Theorem 3.11]. Strongly graded groupoids also appear in [4]. In Proposition 2 we show that $c$ also induces a $\Gamma$-grading on the reduced $C^*$-algebra $C_r^*(G)$. We then show in Theorem 3 that $C_r^*(G)$ is a strongly $\Gamma$-graded $C^*$-algebra if and only if the complex Steinberg algebra $A(G)$ is a strongly $\Gamma$-graded ring. Our main tool in the proof is the injective, continuous linear map $j$ from $C_r^*(G)$ into the space of functions from $G$ to $\mathbb{C}$ that are bounded with respect to the uniform norm from [14, Proposition 4.4.2].

The key examples for us are the $C^*$-algebras $C^*(E)$ of directed graphs $E$, for which the gauge action $\gamma : \mathbb{T} \to \text{Aut} C^*(E)$ gives a $\mathbb{Z}$-grading with

$$C^*(E)_n := \{ a \in C^*(E) : \gamma_z(a) = z^n a \text{ for all } z \in \mathbb{T} \}.$$

In [7], the authors identify the directed graphs $E$ for which the Leavitt path algebras $L_R(E)$ are strongly $\mathbb{Z}$-graded in [7, Theorem 4.2]. (See also [10, Theorem 1.3].) We use our Theorem 3 to show $L_C(E)$ is strongly $\mathbb{Z}$-graded if and only if $C^*(E)$ is strongly $\mathbb{Z}$-graded. Thus the graph condition of [7, Theorem 2.4] (which we will describe soon) also characterises when $C^*(E)$ is strongly $\mathbb{Z}$-graded.

Since we are proving a $C^*$-algebraic result, we use the convention that the spanning elements \{\alpha\beta\} of $L_C(E)$ are determined by pairs $\alpha, \beta \in E^*$ satisfying $s(\alpha) = s(\beta)$. In other words, we use the conventions of [13] rather than those of [11]. With these conventions, a directed graph $E$ has property (Y) if, for every infinite path $x \in E^\infty$ and every $k \in \mathbb{N}$, there are an initial segment $\alpha$ of $x$ and a finite path $\beta \in E^*$ such that $s(\alpha) = s(\beta)$ and $|\beta| - |\alpha| = k$, that is, $\beta$ has $k$ more edges than $\alpha$. The reason for the name is the picture:

![Graph Diagram]

We can state our main graph algebra result as follows.
Theorem 1. Suppose that $E$ is a directed graph. Then the graph algebra $C^*(E)$ is strongly $\mathbb{Z}$-graded if and only if $E$ is row-finite, has no sources, and has property (Y).

As a corollary, we get that $C^*(E)$ is a strongly $\mathbb{Z}$-graded $C^*$-algebra if and only if $L_C(E)$ is strongly $\mathbb{Z}$-graded ring. We conclude with a brief analysis of $(L(E))_0$, the core of a Leavitt path algebra which is dense inside of the core of $C^*(E)$. We show that every element of the core of a Leavitt path algebra is contained in a finite-dimensional subalgebra. Previously this was known for row-finite graphs, see for example the last line of the proof of [2, Theorem 5.3].

1. Strongly graded Steinberg algebras and groupoid $C^*$-algebras

We begin this section with some definitions. Suppose $R$ is a ring. If $A$ and $B$ are subsets of $R$, then we write $AB$ for the set of all finite sums of elements of the form $ab$ where $a \in A$ and $b \in B$. Let $\Gamma$ be a group. We say that $R$ is a $\Gamma$-graded ring if there are subgroups $\{R_\gamma : \gamma \in \Gamma\}$ in $R$ such that for every $\alpha, \beta \in \Gamma$:

(a) $R_\alpha R_\beta \subseteq R_{\alpha \beta}$ and
(b) $\bigoplus_{\gamma \in \Gamma} R_\gamma = R$.

Further, we say $R$ is a strongly $\Gamma$-graded ring if the containment in (a) above is in fact an equality.

We say a groupoid $G$ is ample if it is a topological groupoid and has a basis of compact open bisections. When $G$ is an ample groupoid such that $G^{(0)}$ is Hausdorff, the complex Steinberg algebra of $G$ is the space

$$A(G) := \text{span}\{1_B : B \text{ is a compact open bisection}\}$$

where $1_X$ denotes the characteristic function from $G$ to $\mathbb{C}$ of $X$. Addition and scalar multiplication are defined pointwise, and the convolution product is such that

$$1_B 1_D = 1_{BD}$$

for compact open bisections $B$ and $D$. If $c : G \to \Gamma$ is a continuous cocycle into a discrete group $\Gamma$, then the Steinberg algebra $A(G)$ is a $\Gamma$-graded ring such that for $\gamma \in \Gamma$

$$A(G)_\gamma := \{f \in A(G) : \text{supp} f \subseteq G_\gamma\}$$

where $G_\gamma := c^{-1}(\gamma)$. Thus

$$A(G) = \bigoplus_{\gamma \in \Gamma} A(G)_\gamma.$$ 

(See [8, Lemma 3.11].)
The next proposition shows that this structure also gives a grading on the groupoid $C^*$-algebra. For this proposition (and also the following theorem) we use the injective continuous linear map
\[ j : C^*_r(G) \to B(G), \]
where $B(G)$ denotes the normed vector space of bounded functions from $G$ to $\mathbb{C}$ with respect to $\| \cdot \|_{\infty}$. This map restricts to the identity on $A(G)$. (See, for example, [5, page 3680].)

**Proposition 2.** Let $G$ be an ample groupoid such that $G^0$ is Hausdorff and let $c : G \to \Gamma$ be a continuous cocycle into a discrete group $\Gamma$. Then $C^*_r(G)$ is a $\Gamma$-graded $C^*$-algebra such that for each $\gamma \in \Gamma$ we have
\[ C^*_r(G)_\gamma := \overline{A(G)_\gamma}. \]

**Proof.** Fix $\alpha, \beta \in \Gamma$. Because $A(G)_\alpha \cdot A(G)_\beta$ is contained in the closed set $C^*_r(G)_{\alpha\beta}$,
\[ C^*_r(G)_\alpha \cdot C^*_r(G)_\beta \subseteq C^*_r(G)_{\alpha\beta} \]
by the continuity of multiplication. Similarly
\[ C^*_r(G)^*_\gamma \subseteq C^*_r(G)^{-1}_\gamma \]
by the continuity of the involution $\ast$. Next we verify that each $C^*_r(G)_\gamma$ is a linearly independent subspace. To do this, we claim that
\[ j(C^*_r(G)_\gamma) \subseteq B(G_\gamma). \]
We have that $j$ is the identity map on $A(G)_\gamma \subseteq B(G_\gamma)$. It is straightforward to check that $B(G_\gamma)$ is closed in $B(G)$ with respect to $\| \cdot \|_{\infty}$ and hence the claim follows from the continuity $j$.

Now consider a finite linear combination $\sum a_i = 0$ where each $a_i \in C^*_r(G)_{\gamma_i}$ and the $\gamma_i$ are distinct elements of $\Gamma$. By linearity
\[ 0 = j(\sum a_i) = \sum j(a_i). \]
Since the supports of $j(a_i)$ are all disjoint, $j(a_i) = 0$ for every $i$. Thus $a_i = 0$ because $j$ is an injective linear map.

Finally notice
\[ A(G) = \bigoplus_{\gamma \in \Gamma} A(G)_\gamma \subseteq \bigoplus_{\gamma \in \Gamma} C^*_r(G)_\gamma \subseteq C^*_r(G). \]
Thus since $A(G)$ is dense in $C^*_r(G)$ by [17, Proposition 6.7], we have that
\[ \bigoplus_{\gamma \in \Gamma} C^*_r(G)_\gamma \text{ is dense in } C^*_r(G). \]
\[ \square \]
**Theorem 3.** Let $G$ be an ample groupoid such that $G^{(0)}$ is Hausdorff and let $c : G \to \Gamma$ be a continuous cocycle into a discrete group $\Gamma$. The following are equivalent:

(a) $G$ is strongly $\Gamma$-graded groupoid in the sense that $G_\alpha G_\beta = G_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$;

(b) $A(G)$ is a strongly $\Gamma$-graded ring;

(c) $C^*_r(G)$ is a strongly $\Gamma$-graded $C^*$-algebra.

**Remark 4.** Using [7, Theorem 3.11], we could replace item (b) in Theorem 3 with the equivalent statement: $A_R(G)$ is a strongly $\Gamma$-graded ring for any commutative ring $R$ with identity.

The equivalence of (a) and (b) is established in [7, Theorem 3.11]. We establish the equivalence of (b) and (c). That (c) implies (b) is straightforward: Take $\alpha, \beta \in \Gamma$. Since $C^*_r(G)_{\alpha\beta}$ is closed and $C^*_r(G)$ is $\Gamma$-graded, $C^*_r(G)_{\alpha} \cdot C^*_r(G)_{\beta}$ trivially has closure in $C^*_r(G)_{\alpha\beta}$. To see that this closure is all of $C^*_r(G)_{\alpha\beta}$, take $a \in C^*_r(G)_{\alpha\beta}$. Since $C^*_r(G)_{\alpha\beta} = A(G)_{\alpha\beta}$, there exist $f_n \in A(G)_{\alpha\beta}$ such that $f_n \to a$ in $C^*_r(G)$. Since $A(G)$ is strongly graded, we have

$$f_n \in A(G)_{\alpha} \cdot A(G)_{\beta} \subseteq C^*_r(G)_{\alpha} \cdot C^*_r(G)_{\beta}$$

for each $n$ and hence the limit $a$ belongs to $C^*_r(G)_{\alpha\beta}$.

For the reverse implication, we use the following lemma.

**Lemma 5.** Let $G$ be an ample groupoid such that $G^{(0)}$ is Hausdorff and let $c : G \to \Gamma$ be a continuous cocycle into a discrete group $\Gamma$. Then for $\alpha, \beta \in \Gamma$ we have

$$A(G)_\alpha \cdot A(G)_\beta \subseteq \{ f \in A(G) : f(x) \neq 0 \implies x \in G_\alpha G_\beta \}.$$

**Proof.** Observe that $A(G)_\alpha \cdot A(G)_\beta$ is equal to

$$\text{span}\{1_{BD} : B \subseteq G_\alpha, D \subseteq G_\beta \text{ are compact open bisections}\}$$

(see [7, bottom of page 53]). Since the support of a sum of functions is contained in the union of the supports of the summands, the containment follows.

**Proof of Theorem 3** To complete the proof, suppose $A(G)$ is not strongly graded. Then $G$ is not strongly graded by [7, Theorem 3.11], that is, by (a) $\implies$ (b) of Theorem 3. Then there exists $\gamma \in \Gamma$ and $x \in G_e \setminus G_\gamma G_{\gamma^{-1}}$.
by [7, Lemma 3.1]. Since $c$ is continuous, $G_e$ is open so we can find $B \subseteq G_e$ a compact open bisection containing $x$. Since $1_B(x) = 1,$

$$1_B \in A(G)_e \setminus A(G)_\gamma A(G)_{\gamma^{-1}}$$

by Lemma 5. We show

$$1_B \notin A(G)_\gamma A(G)_{\gamma^{-1}}$$

using an adaptation of the argument in [6, Proposition 3.3(i)]. By way of contradiction, suppose there exists a sequence $(f_n) \subseteq A(G)_\gamma \cdot A(G)_{\gamma^{-1}}$ that converges to $1_B$ in $C^*_r(G)$. Again we apply the injective continuous linear map $j : C^*_r(G) \to B(G)$. Since $j$ restricts to the identity map on $A(G)$ we have $j(f_n) = f_n \to j(1_B) = 1_B$ in $\| \cdot \|_{\infty}$. Recall that $1_B(x) \neq 0$. However, $x \notin G_\gamma G_{\gamma^{-1}}$ so $f_n(x) = 0$ for all $n \in \mathbb{N}$ by Lemma 5, which is a contradiction. Therefore

$$f \notin A(G)_\gamma A(G)_{\gamma^{-1}}$$

and $C^*_\nu(G)$ is not a strongly $\Gamma$-graded $C^*$-algebra. \hfill $\Box$

2. LEAVITT PATH ALGEBRAS

We want to connect properties of $C^*(E)$ and properties of $L_C(E)$, and hence we want to be able to view $L_C(E)$ as a subalgebra of $C^*(E)$ using the homomorphism $\iota$ which takes a spanning element $\alpha \beta^*$ to the element $s_\alpha s_\beta^*$ of $C^*(E)$. That $\iota$ is an injection was established (for example) in [3, §1.3], but with an annoying hypothesis of “no sources”. So we pause to prove the following more general result.

**Proposition 6.** Suppose that $E$ is a directed graph. Then the homomorphism $\iota : L_C(E) \to C^*(E)$ is an isomorphism of $L_C(E)$ onto the $\ast$-subalgebra

$$A := \text{span} \{ s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } s(\alpha) = s(\beta) \} \text{ of } C^*(E).$$

**Proof.** We want to apply the Graded Uniqueness Theorem for Leavitt path algebras [11, Theorem 2.2.15], but $C^*(E)$ is not $\mathbb{Z}$-graded in the algebraic sense, and hence that result does not apply to $\iota : L_C(E) \to C^*(E)$. However, the $\ast$-subalgebra $A$ is $\mathbb{Z}$-graded, with

$$A_n = \text{span} \{ s_\alpha s_\beta^* : s(\alpha) = s(\beta) \text{ and } |\alpha| - |\beta| = n \}.$$ 

Since we can always find Cuntz-Krieger $E$-families $\{S, P\}$ with $P_v \neq 0$ for every $v \in E^0$ (see the top of [13, page 8]), every $p_v$ is non-zero. Thus [11, Theorem 2.2.15] implies that $\iota$ is an injection of $L_C(E)$ into
A; since every $s_\alpha s_\beta^*$ belongs to the range of $\iota$, it is an isomorphism, as claimed.

Next we relate the grading spaces $C^*(E)_n$ for the $C^*$-algebra to those of the Leavitt path algebra. We define $\Phi_n : C^*(E) \to C^*(E)_n$ in terms of the gauge action $\gamma : \mathbb{T} \to \text{Aut} C^*(E)$ by

$$\Phi_n(a) = \int_{\mathbb{T}} w^{-n} \gamma_w(a) \, dw.$$ 

Left invariance of the Haar integral on $\mathbb{T}$ implies that $\Phi_n$ is a norm-decreasing $C^*(E)_n$-bilinear map of $C^*(E)$ onto $C^*(E)_n$ such that

$$\Phi_n(s_\mu s_\nu^*) = \begin{cases} s_\mu s_\nu^* & \text{if } |\mu| - |\nu| = n, \\ 0 & \text{otherwise.} \end{cases}$$

The map $\Phi_n \circ \iota$ is a bounded linear map of $L_C(E)$ onto a dense subspace of $C^*(E)_n$, and its restriction to $L_C(E)_n$ is an injection of $L_C(E)_n$ onto this dense subspace of $C^*(E)_n$. Hence, modulo the canonical injection $\iota$ of $L_C(E)$ in $C^*(E)$, $C^*(E)_n$ is the closure of $L_C(E)_n$. We suppress $\iota$, and write $C^*(E)_n = \overline{L_C(E)_n}$.

**Proposition 7.** Suppose that $E$ is a directed graph. Then $C^*(E)$ is a strongly $\mathbb{Z}$-graded $C^*$-algebra if and only if $L_C(E)$ is a strongly $\mathbb{Z}$-graded ring.

With the following lemma, the proof of Proposition 7 is an immediate corollary to Theorem 3.

**Lemma 8.** For any directed graph $E$, there is an ample Hausdorff amenable groupoid $G_E$ and an isomorphism $\pi : C^*(E) \to C^*(G_E)$ such that $\pi(C^*(E)_n) = C^*(G_E)_n$, and such that the restriction of $\pi$ to $L_C(E)$ is a $\mathbb{Z}$-graded (ring) isomorphism of $L_C(E)$ onto $A(G_E)$.

**Remark 9.** Because $G_E$ is amenable, the full and reduced $C^*$-algebras are equal so we drop the $r$ subscript from the $C^*_r(G_E)$ from now on.

**Proof.** Let $G_E$ be the boundary path groupoid of $E$ as defined by Paterson on page 653 of [12]. Then $G_E$ is ample and Hausdorff; see [15, Theorem 2.4] for a nice proof of this. The groupoid $G_E$ is amenable by [12, Theorem 4.2]. We view $C^*_r(G_E)$ as the universal groupoid $C^*$-algebra. In this sense, there is a representation $\pi_{\text{max}} : C_c(G_E) \to C^*(G_E)$ that is universal for representations of $C_c(G_E)$ and $\pi_{\text{max}}(C_c(G_E))$ is dense in $C^*(G_E)$. See [16, Theorem 3.2.2] for more details. Since we view $A(G_E) \subseteq C_c(G_E)$ as subsets of $C^*(G_E)$, we identify $C_c(G_E)$ with

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[1] Paterson calls $G_E$ the path groupoid.
$\pi_{\text{max}}(C_e(G_E))$ inside $C^*(G_E)$, which we can do because $\pi_{\text{max}}$ is injective by [16, Corollary 3.3.4].

One can show that the collection

$$P_v := 1_{Z(v)}, \quad S_e := 1_{Z(e, s(e))}$$

for $v \in E^0$ and $e \in E^1$ is a Cuntz-Krieger $E$-family in $C^*(G_E)$. The universal property of $C^*(E)$ gives a homomorphism $\pi : C^*(E) \to C^*(G_E)$ such that

$$\pi(p_v) = P_v \quad \text{for } v \in E^0 \quad \text{and} \quad \pi(s_e) = S_e \quad \text{for } e \in E^1.$$

(See [13, page 42].) Notice that the restriction of $\pi$ to $L_C(E)$ is precisely the $\mathbb{Z}$-graded ring isomorphism from $L_C(E)$ onto $A(G_E)$ in [8, Example 3.2], and hence $\pi(L_C(E)_n) = A(G_E)_n$.

Paterson shows in in [12, Theorem 3.8] and [11, Theorem 3.1] that representations of $C^*(E)$ are in bijective correspondence with representations of $C_e(G_E)$. So the universal representation $p_v, s_e$ of $C^*(E)$ corresponds to a representation $\phi : C_e(G_E) \to C^*(E)$: indeed, by looking at the details of the proof of [12, Theorem 3.8], we see that $\phi(P_v) = p_v$ and $\phi(S_e) = s_e$ for $v \in E^0$ and $e \in E^1$. The universal property of $C^*(G_E)$ [16, Theorem 3.2.2] gives a homomorphism $\psi : C^*(G_E) \to C^*(E)$ such that $\psi = \phi$ on $C_e(G_E)$.

Notice that $\pi \circ \phi$ and $\phi \circ \pi$ restrict to identity maps on $A(G_E)$ and $L_C(E)$ respectively. We show that $\pi$ is a bijection by showing that $\pi \circ \psi$ and $\psi \circ \pi$ are the identity maps on $C^*(G_E)$ and $C^*(E)$ respectively. Fix $a \in C^*(G_e)$. Since $A(G_E)$ is dense in $C^*(G_E)$ by [17, Proposition 6.7], $a = \lim f_n$ where $f_n \in A(G_E)$. Then using that homomorphisms between $C^*$-algebras are automatically continuous, we have

$$\pi(\psi(a)) = \pi(\psi(\lim f_n)) = \lim \pi(\psi(f_n)) = \lim \pi(\phi(f_n)) = \lim f_n = a.$$

For the other direction, fix $b \in C^*(E)$. Then $b = \lim b_n$ where $b_n \in L_C(E)$, and recalling that $\pi$ takes elements of $L_C(E)$ to elements of $A(G_E)$, we deduce that

$$\psi(\pi(b)) = \psi(\lim(\psi(b_n))) = \lim(\psi(\pi(b_n))) = \lim(\phi(\pi(b_n))) = \lim b_n = b.$$

Thus $\pi$ is an isomorphism with inverse $\psi$. Continuity of $\pi$ and $\psi$ imply $\pi(C^*(E)_n) = C^*_r(G_E)_n$, and this completes the proof. \hfill $\square$

**Proof of Proposition 7.** Because the restricted map $\pi$ from Lemma 8 is a graded ring isomorphism, $L_C(E)$ is strongly $\mathbb{Z}$-graded if and only if $A(G_E)$ is strongly $\mathbb{Z}$-graded. Now Theorem 3 says this is equivalent to $C^*(G_E)$ being strongly $\mathbb{Z}$-graded. Since $\pi$ preserves the $C^*$-grading, the result follows. \hfill $\square$
Now we have the pieces needed to prove Theorem 1. Recall that [7, Theorem 4.2] says $L_C(E)$ is strongly $\mathbb{Z}$-graded if and only if $E$ is row-finite, has no sources and has property Y. Thus Theorem 1 follows from Proposition 7 above and [7, Theorem 4.2].

In strongly graded algebras, a lot of the structure of the algebra can be seen in the core, that is, in the homogeneous component of the identity $e$. Thus we conclude by establishing that the core of a Leavitt path algebra is build from finite dimensional subalgebras in the following way.

**Proposition 10.** Suppose that $E$ is a directed graph and $K$ is a field. Then every $a \in L_K(E)_0$ belongs to a finite-dimensional $*$-subalgebra of $L_K(E)_0$.

To prove this result, we need to understand the finite-dimensional subalgebras of $L_K(E)_0$. We start by choosing an enumeration $\{e_j : j \in \mathbb{N}\}$ of the countable set $E^1$. For $J \in \mathbb{N}$, we write

$$E^k_J := \{ \alpha \in E^k : \alpha_i \in \{e_j : 1 \leq j \leq J\} \text{ for all } i \},$$

and $E^0_J := \{s(e_j) : 1 \leq j \leq J\}$. Notice that the sets $E^k_J$ are all finite, and $E^k_J \subseteq E^k_{J+1}$. We define

$$G_{k,J} := \text{span}\{\alpha \beta^* : \alpha, \beta \in E^k_J\}.$$

For $\beta, \gamma \in E^k_J$ we have $|\beta| = |\gamma| = k$, and hence we have

$$\beta^* \gamma = \begin{cases} s(\beta) = s(\gamma) & \text{if } \beta = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$G_{k,J}(v) := \text{span}\{\alpha \beta^* := \alpha, \beta \in E^k_Jv\}$$

is a (finite-dimensional) matrix algebra, and

$$v \neq w \implies G_{k,J}(v)G_{k,J}(w) = \{0\}.$$

Thus

$$G_{k,J} = \bigoplus_{v \in E^0_J} G_{k,J}(v)$$

is a finite direct sum of matrix algebras.

**Lemma 11.** Suppose that $k, l \in \mathbb{N}$ and $k > l$. Then for every $J \in \mathbb{N}$ and $v, w \in E^0_J$, we have

$$G_{k,J}(v)G_{l,J}(w) \subseteq G_{k,J}(v).$$
Proof. For $\alpha\beta^* \in G_{k,J}$ and $\gamma\delta^* \in G_{l,J}(w)$, we have $|\beta| = k > l = |\gamma|$, and hence

$$(\alpha\beta^*)(\gamma\delta^*) = \begin{cases} 
\alpha(\delta\beta')^* & \text{if } \beta = \gamma\beta' \\
0 & \text{otherwise.} 
\end{cases}$$

Since $s(\delta\beta') = s(\beta) = s(\alpha) = v$, we deduce that $(\alpha\beta^*)(\gamma\delta^*) \in G_{k,J}(v)$ (it could be 0, but that is in $G_{k,J}(v)$ too). A similar argument gives $(\gamma\delta^*)(\alpha\beta^*) \in G_{k,J}(v)$. \hfill $\square$

Corollary 12. For each $J \in \mathbb{N}$, the set

$$F_{k,J} := \text{span} \left( \bigcup \{ G_{l,J} : l \leq k \} \right)$$

is a finite-dimensional $\ast$-subalgebra of $L_K(E)_0$.

Proof. The set $F_{k,J}$ is a vector subspace of $L_K(E)_0$ and is closed under conjugation because each $G_{l,J}$ is. The lemma implies that it is closed under multiplication. It is finite-dimensional because there are only finitely many $G_{l,J}$ in play, and their union spans $F_{k,J}$. \hfill $\square$

Proof of Proposition 10. We suppose that $a \in L_K(E)_0$. Then there are a finite set $S$ of pairs $(\alpha, \beta) \in E^* \times E^*$ such that $s(\alpha) = s(\beta)$ and $|\alpha| = |\beta|$, and scalars $c_{\alpha,\beta}$ such that

$$a = \sum_{(\alpha,\beta) \in S} c_{\alpha,\beta} \alpha\beta^*.$$ 

Since the set $S$ is finite, the set of vertices

$$V := \{ v \in E^0 : v = s(\alpha_i) \text{ or } s(\beta_i) \text{ for some } (\alpha, \beta) \in S \text{ and } i \leq |\alpha| \}$$

is finite. Thus there exists $J$ such that $V \subseteq E^0_J$. Then with $k := \max\{|\alpha| : (\alpha, \beta) \in S\}$, we have

$$\alpha\beta^* \in G_{|\alpha|,J} \subseteq F_{k,J} \text{ for all } (\alpha, \beta) \in S.$$ 

Thus $a = \sum c_{\alpha,\beta} \alpha\beta^*$ belongs to the subspace $F_{k,J}$, which by Corollary 12 is a finite-dimensional $\ast$-subalgebra of $L_K(E)_0$. \hfill $\square$

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