Injection and nucleation of topological defects in the quench dynamics of the Frenkel-Kontorova model

Oksana Chelpanova¹, Shane P. Kelly¹, Giovanna Morigi², Ferdinand Schmidt-Kaler¹ and Jamir Marino¹

¹ Institut für Physik, Johannes Gutenberg-Universität Mainz, D-55099 Mainz, Deutschland
² Theoretische Physik, Universität des Saarlandes, D-66123 Saarbrücken, Deutschland

Abstract – Topological defects have strong impact on both elastic and inelastic properties of materials. In this article, we investigate the possibility to controllably inject topological defects in quantum simulators of solid state lattice structures. We investigate the quench dynamics of a Frenkel-Kontorova chain, which is used to model discommensurations of particles in cold atoms and trapped ionic crystals. The interplay between an external periodic potential and the inter-particle interaction makes lattice discommensurations, the topological defects of the model, energetically favorable and can tune a commensurate-incommensurate structural transition. Our key finding is that a quench from the commensurate to incommensurate phase causes a controllable injection of topological defects at periodic time intervals. We employ this mechanism to generate quantum states which are a superposition of lattice structures with and without topological defects. We conclude by presenting concrete perspectives for the observation and control of topological defects in trapped ion experiments.

Introduction. – The study of topological defects, such as solitons, is interdisciplinary, permeating different areas of physics such as cold atoms [14–17], spintronics [5–7], and nano-friction in material science [8–13]. They are also at the core of a variety of technological applications encompassing quantum computing [14–17] and atomic sensors [18–19]. In material science, they arise as lattice defects in solid state crystals and are particularly important in determining the mechanical properties such as rigidity [21].

In this article, we investigate the possibility to controllably inject and eject topological defects in quantum simulators of solid state lattice structures, with the perspective of performing similar nano-scale control of individual defects in real materials. To pursue this goal, we consider the quench dynamics of the Frenkel-Kontorova model [22–23], originally introduced to understand the macroscopic and static properties of dislocations in crystalline materials. This model can be simulated in a variety of quantum platforms such as ultra cold atoms [24], and trapped ion crystals [10, 25, 26], unlike in materials, microscopic control over these systems is currently possible, allowing for experiments that can perform repeatable quenches which are sensitive to the quantum nature of the constituents particles.

The topological defects in this model are lattice discommensurations induced by the interplay between an external periodic potential and inter-particle interaction. By tuning the external potential, the lattice can undergo a structural first-order transition from a (topologically trivial) commensurate state, in which particles stay at the minima of an underlying periodic potential, to an incommensurate state, where particles are dislocated from their natural equilibria, and the local discommensuration of particles can be mapped to a kink in a Frenkel-Kontorova model.

Our main result is that these topological defects can be injected from the boundaries of the system by performing quenches which originate from the commensurate phase. This mechanism occurs in a controlled fashion with atomic discommensurations entering the chain at discrete intervals of time and building a metastable state characterized by a temporal staircase of kinks, with life-time scaling linearly with the chain size. We also explore the im-
pact of non-equilibrium fluctuations at the commensuratemiscommensurate transition, and show that they can mold novel states in the quantum Frenkel-Kontorova model. By quenching the system close to the boundary of the commensurate-miscommensurate transition, quantum fluctuations lead to statistical nucleation of kinks, which drive the chain in a quantum superposition of configurations with and without topological defects.

We focus throughout the paper on parameters regimes proper of trapped ion simulators, and we discuss in the concluding sections how to observe these phenomena in low dimensional ultracold atomic wires, arguing for broad applicability of our results to other quantum simulators.

Model. – We consider \( N \) atoms forming a one-dimensional chain with inter-particle spacing \( a_0 \). The system is subject to a periodic external lattice potential of period \( a_s \). The physics of the discommensuration between the atoms spacing and the lattice periodicity can be captured by the Frenkel-Kontorova model \cite{22,23,28,30}:

\[
H = \sum_n \frac{p_n^2}{2} + m_K^2 (1 + \cos \phi_n) + \sum_{r=\pm 1} \frac{(\phi_{n+r} - \phi_n - 2\pi r \delta)^2}{2}
\]

where \( p_n = \dot{\phi}_n \). Here the first term describes the kinetic energy of the atoms, the second one depicts the inter-particle interaction. Here \( \phi_n/(2\pi) = x_n/a_s - n \) denotes a classical field, characterizing the displacement of the \( n \)-th particle position \( x_n \) from the coordinate of the \( n \)-th minima of the lattice potential \((n = 1, \ldots, N)\). The dimensionless parameter \( m_K^2 \) captures the amplitude of the periodic lattice potential and it plays the role of the kink mass in the following. The ‘misfit parameter’ \( \delta = (a_0 - a_s)/a_s \) quantifies instead the degree of discommensuration between \( a_0 \) and \( a_s \).

By tuning the amplitude and misfit parameter, \( m_K \) and \( \delta \), of the periodic lattice potential, the system at equilibrium undergoes a structural phase transition between two different configurations for \( \phi_n \). The equilibrium phase diagram is sketched in Fig. 1(a). For \( \delta < \delta_c \approx 2m_K/\pi^2 \) a strong lattice potential pins the particles to the potential minima, forming a commensurate phase \((\phi_n = \pi)\), see Fig. 1(b, d). Instead, for values of \( \delta \geq \delta_c \) atoms reside in a new incommensurate configuration associated with kinks in \( \phi_n \) placed at the location of the particles’ displacements, cf. Fig. 1(b, c). The kinks (anti-kinks) correspond to a local distribution of holes (excess particles) in the chain \cite{28,30,32}, and are represented by a ‘step’ in the displacement field, \( \phi_n \), equal to \( 2\pi(-2\pi) \). The number of kinks \( Q = (\phi_N - \phi_0)/(2\pi) \) is a parameter that allows us to distinguish between the commensurate and the incommensurate phases \cite{33}. This quantity estimates the number of ‘steps’ in \( \phi_n \), i.e., jumps in the value of the field \( \phi \) from \( \pi + 2\pi m \) to \( \pi + 2\pi (m+1) \), where \( m \) is an integer number. For instance, the incommensurate configuration in Fig. 1(b, c) has three ‘steps’ in \( \phi_n \), which corresponds to \( Q = 3 \). For large values of the misfit parameter \( \delta \), the number of kinks, \( Q \), is linearly proportional to \( \delta \). Accordingly, \( Q \) is directly related to the spacing between kinks, \( l_s \propto 1/\delta \).

In the following we investigate the dynamical injection of kicks and use the width of the kick and the excitation spectrum as key parameters to interpret. The width of the kick, \( l_m \), is equal to the distance over which \( \phi \) increases by \( 2\pi \), and it is roughly proportional to the inverse of the kick mass, \( l_m \propto 1/m_K \) \cite{28,30,32,33}. For large \( m_K \), the shape of the kick is sharp, since any broadening of the incommensurate region requires additional energy cost due to the lattice potential. For small \( m_K \) the lattice potential is almost negligible and even a single kink appears wide.

Following \cite{33,34}, the excitation spectrum of the model (1) in the two phases can be extracted by expanding the equations of motion for \( \phi_n \) around the classical solution \( \phi_n^{(0)} \) in powers of small fluctuations \( \eta_n \). The corresponding equations of motion for the linear displacements, \( \eta_n \), in momentum space read:

\[
\omega_q^2 \eta_q = -m_K^2 \cos(\phi_n^{(0)}) \eta_q + c^2 q^2 \eta_q + \ldots,
\]

where \( c \) is the speed of sound, which is equal to one for the
model \[Q\]. The spectrum in the incommensurate phase consists of two branches, cf. Fig. 1(e). The first \(Q\) modes correspond to gapless collective excitations, phonons, describing the propagation of kinks through the lattice at the speed of sound \(c\) (blue line). The remaining \(N - Q\) modes form an optical branch (red line) and they describe small harmonic displacements of the atoms around the potential minima: they are parabolically dispersing excitations with a gap set by \(m_K\). Within the commensurate phase, all the \(N\) modes are optical.

**Controlled Injection of Defects.** – We now show how to inject, and also eject, topological defects by keeping \(m_K\) fixed, and making an abrupt change of \(\delta_i\) from \(\delta_i\) to \(\delta_f\), which in terms of experimental parameters corresponds to the sudden change of the inter-particle spacing \(a_0\). We start by solving numerically the equations of motion that govern the dynamics of the topological defects in model \[Q\], which read

\[
\begin{align*}
\ddot{\phi}_1 &= m_K^2 \sin \phi_1 + (\phi_2 - \phi_1 - 2\pi \delta) \\
\ddot{\phi}_{n=2,N-2} &= m_K^2 \sin \phi_n + \\
&+ (\phi_{n+1} - \phi_n - 2\pi \delta) + (\phi_{n-1} - \phi_n + 2\pi \delta) \\
\ddot{\phi}_N &= m_K^2 \sin \phi_N + (\phi_{N-1} - \phi_N + 2\pi \delta).
\end{align*}
\]

(3)

Here the effect of the discommensuration between \(a_0\) and \(\alpha_s\) cancels for particles in the bulk, as it produces opposite contributions from interaction with right and left neighbours. However, the misfit parameter affects directly dynamics of the boundary particles, because the contribution from discommensuration is not compensated. In this way, after the quench of \(\delta\) only the edge is initially driven out of equilibrium such that the boundary acts as a reservoir for topological defects (see also \[35\]). Furthermore, due to their topological nature, once the defects enter the bulk, they cannot be destroyed \[35\].

In order to see this concretely, we numerically initialize a system in the equilibrium state with misfit parameter \(\delta_i\), and then numerically solve its evolution according to equation (3) with the misfit parameter \(\delta_f\). In Fig. 2(a) we plot a diagram of the possible dynamical responses in the system using the long-time averaged number of kinks \(Q\), where time averages are taken over several periods \(\propto 2\pi/m_K\). In the white region of Fig. 2(a) no kinks or anti-kinks are injected during dynamics. The red region in the phase diagram corresponds to kinks’ injection after quenching the misfit parameter above a dynamical critical threshold, \(\delta^c_f\). Finally, the blue region in Fig. 2(a) corresponds to quenches which decrease the total number of kinks. This occurs by injecting anti-kinks into the system. Injection of a single kink (an anti-kink) from the boundary is associated with the excitation of a collective sonic mode.

Fig. 2(b) shows how the number of kinks in the system increases following the quench from the commensurate to a target point of the incommensurate phase, marked with \(\star\) in Fig. 2(a). Dynamics start from \(Q = 0\) and exhibit a staircase structure in time due to the injection of kinks from the boundaries up to time \(t \propto N/c\) when finite-size effects become relevant. The duration of the various plateaus of the staircase is proportional to \(l_{\delta_f}/c\). Arranging kinks at \(r < l_\delta + l_m\) requires an additional energy cost. In order to show that, we calculate the energy of two kinks as a function of the relative distance between their centers \(r\), cf. Fig. 2(c). If the kinks overlap, their energy is higher than the energy of the configuration with two infinitely separated kinks, resulting in short-distance repulsion \[29\]. Thus, the average distance between kinks after the quench is fixed to approximately \(l_m + l_\delta\). As a result, the number of kinks saturates at long times to \(Q \approx 2N/(l_m + l_\delta)\), which is basically given by the total number of solitons fitting in the system, for a given size \(N\) (cf. inset in Fig. 2(b)). Note, that \(N/(l_m + l_\delta)\) kinks enter the system from each boundary. Indeed, up to times \(\tau\), systems with a different number of particles evolve with the same staircase temporal profile, since dynamics are

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**Fig. 2:** (a) Dynamical response of the system after a quench of the misfit parameter from \(\delta_i\) to \(\delta_f\), with \(m_K = 0.5\) (here \(N = 80\)). We plot the time-averaged number of kinks, \(Q - Q_{(t=0)}\). The solid lines mark the critical thresholds, \(\delta^c_f\), corresponding to the discrete injection of a certain number of (anti-)kinks. The dashed line corresponds to \(\delta_i = \delta_f\). The red region corresponds to the injection of kinks into the system, while the blue one pertains to anti-kinks. (b) Dynamics of the number of kinks in the system after the quench marked with a \(\star\) in panel (a). Here we fix \(\delta_i = 0\) and \(\delta_f = 0.16 > \delta^c_f\) and consider different system sizes \(N\) (marked with different colors). The plot illustrates periodic injection of kinks mirroring in a ‘step-like’ increase of \(Q\) in time. The time scale \(\tau \propto N/c\) denotes the saturation time of \(Q\) for \(N = 140\) ions after which finite-size effects take over. The maximal number of kinks in the system after the quench scales linearly with \(N\) (see inset). (c) Energy of two kinks as a function of the relative distance between their centers. The effective kinks repulsion takes place at distances \(r < l_\delta + l_m\). The energy scale in panel (c) is normalized by the energy of two infinitely separated kinks \(E_\infty\).
universally ruled by δ_t and m_K. Notice that the height of each ‘step’ in Fig. 3(b) is equal to two, since we simultaneously inject one kink from the left and another one from the right boundary, after the quench. Finally, at late times t > τ, the number of kinks, Q(t), oscillates around the steady state value, see Fig. 2. This behaviour results from both sonic and optical modes. The first one alters the value of charge as a result of scattering of the injected kinks against the boundaries. The second one describes individual oscillations of particles around potential minima.

Dynamics close to the boundary of the commensurate-incommensurate transition. — An exciting feature of cold atom and trapped ion platforms is their sensitivity to quantum fluctuations. Generally, the effect of quantum fluctuations within a given phase only quantitatively modify the dynamics, but the dynamics close to the boundary between two phases can have non-trivial effects such as destroying order, generating new dynamical phases, or provoking a quantum critical region in the system [42–51]. As such, we investigate the effect of quantum fluctuations for quenches close to the boundary between two phases can have non-trivial effects such as destroying order, generating new dynamical phases, or provoking a quantum critical region in the system [42–51]. As such, we investigate the effect of quantum fluctuations for quenches close to the boundary of the commensurate-incommensurate transition, and we introduce the quantum version of Hamiltonian (1):

\[ \hat{H} = \sum_n \frac{\hat{p}_n^2}{2} + m_K^2(1 + \cos \phi_n) + \sum_{\tau = \pm 1} \frac{(\phi_{n+\tau} - \phi_n - 2\pi \tau \delta)^2}{2} \]  

(4)

where we have promoted \( \hat{p}_n \) and \( \phi_n \) to the dimensionless quantum operators \( \hat{p}_n \) and \( \phi_n \) with canonical commutation relations [\( [\phi_n, \hat{p}_m] = i\hbar \delta_{n,m} \)]. Here we take \( h_{\text{eff}} \) as a dimensionless free parameter which controls the strength of quantum fluctuations. Below we give an expression for \( h_{\text{eff}} \) for the trapped ion implementation.

In the following, we take \( h_{\text{eff}} \ll 1 \) and investigate the impact of quantum fluctuations using the semi-classical approximation known as the Truncated Wigner Approximation (TWA) [52,54]. We consider a quench from the ground state of the Hamiltonian (1), that can be approximated [52] by a Gaussian wave function with the Wigner function equal to

\[ W(\hat{p}_n^0, \phi_n^0) = \prod_q \exp \left[ -\frac{1}{\sigma_q} |\hat{p}_q^0|^2 - \frac{1}{b_q} |\phi_q^0|^2 \right] , \]  

(5)

where \( \hat{p}_q^0 = \phi_q^0 - \langle \phi_q \rangle_0 \) and \( \phi_q^0 = \hat{p}_q^0 - \langle \hat{p}_q \rangle_0 \) are the phase space variables relative to the mean value of the operators in the initial state, while \( \sigma_q = \langle \hat{p}_q^2 \rangle_0 = h_{\text{eff}}/\omega_q \) and \( b_q = \langle \phi_q^2 \rangle_0 = h_{\text{eff}}/\omega_q \) encode the variance of quantum fluctuations in the initial state. Finally, the dispersion relation \( \omega_q \) is determined by Eq. (2). At zeroth order in \( h_{\text{eff}} \), the TWA approximation is simply the mean field dynamics discussed above where the system evolves by the classical equations of motion Eq. (3) with initial state taken as the mean field ground state. At first order in \( h_{\text{eff}} \), the dynamics of phase space variables are still described by Eq. (3) but the initial state is sampled from the positive Wigner distribution in Eq. (5). Observables at later times are calculated by averaging over the trajectories resulting from the initial quantum noise. Taking into account higher orders quantum corrections can be performed by including quantum jumps during dynamics in each trajectory, with each jump introducing \( O(h_{\text{eff}}^3) \) corrections [52,55].

Let us now revisit the quench of the misfit parameter within the commensurate phase, \( \delta_1 = 0 \rightarrow \delta_f \lesssim \delta_0 \). At the classical level this excites the displacement field \( \phi_n^{(0)} \) in an optical mode. At small times the dynamics of fluctuations are governed by \( \hat{h}_q \approx m_K^2 \cos (\phi^0(t)) \eta_0 - c^2 q^2 \eta_0 \) (cf.
with Eq. (2), with a periodically driven mass term set by the motion of $\phi_{n}^{(0)}(t)$. In cosmology [56] or cold atoms [57], a similar structure for dynamics of the low-energy modes results in an instability that can trigger non-linear effects. In our system the parametric amplification of low-energy modes can indeed cause the separation of quantum trajectories on times of order $\log m_{\text{eff}}$. Fig. 3(a) shows final configurations of the field for different samplings of the Wigner distribution associated to the pre-quench state. The yellow trajectories remain within the commensurate phase and contain no kinks, while the blue trajectories are the ones which have tunneled into the incommensurate region, and in fact they display kinks in the profile of the displacement field, $\phi_{n}$. Such a semi-classical ensemble represents a quantum state that is a superposition of an atomic lattice in the commensurate configuration and a configuration with a kink. We check that this result remains qualitatively unaltered after adding higher order quantum corrections in Fig. 3(b).

Classically, a quench below the dynamical critical threshold, $\delta_{c} = 0 \rightarrow \delta_{f} \ll \delta_{c}^{d}$, keeps the system within the commensurate phase since the injection of a single kink would require an additional energy cost $\Delta E = E(\delta^{d}_{f}) - E(\delta_{f})$. By adding quantum fluctuations, however, the system can tunnel through this barrier and nucleate a kink at the boundary. The fraction of trajectories that nucleated a kink after the quench of $\delta$ can be estimated via WKB approximation [58]. In this approximation, nucleation of a kink can be understood as tunneling of a quantum particle through the energy barrier separating a commensurate from an incommensurate configuration, see Fig. 2(c). The transition probability according to WKB reads

$$P = P_{0} \exp \left( - \alpha h_{\text{eff}}^{-1} \int_{\delta_{f}}^{\delta^{d}_{f}} \sqrt{E(\delta^{d}_{f}) - E(\delta)} d\delta \right),$$

which we fit in good agreement with numerical results in the inset of Fig. 3(a). Here the effective Planck constant $h_{\text{eff}}$ determines the width of the region in $\delta^{d}_{f} - \delta_{f}$ where nucleation of kinks can be observed, namely, $\sqrt{\Delta E(\delta^{d}_{f} - \delta_{f})} \lesssim h_{\text{eff}} \alpha^{-1}$. Quenches away from this region do not produce states with the above quantum superposition. Instead, quantum noise provokes dephasing among trajectories and can result in the broadening of the width of a single kink such that atoms will be harder to resolve.

Similar arguments hold for quenches deep into incommensurate region. As shown in the portrait of dynamical responses (Fig. 2(a)) there is a set of critical thresholds $\delta^{d}_{c}$ that correspond to the injection of multiple $(n = \pm 2, \pm 4, \ldots)$ solitons into the system. Therefore, by quenching $\delta$ in between $\delta^{d}_{c}$ and $\delta^{d}_{c+2}$ we can generate states which are in a superposition of $n$ and $n + 1$ kinks.

Notice that if the initial state was instead a Gibbs thermal state, transitions into the incommensurate state could also occur by thermal activation [59], and it would give rise qualitatively to the same results of Fig. 3. The major difference is that the system would be in a mixed state of crystal configurations with different values of $Q$ and with quantum coherence decreasing with increasing temperature.

**Trapped Ion Realization.** – We now discuss how the quench dynamics presented in the core of the paper can find application in trapped ion chain [10, 25–27]. An ionic crystal is formed by trapping ions in a 1D Paul trap whose strength sets the inter-particle distance $a_{0}$. The external lattice potential is provided by an optical lattice with period $a_{s}$, and amplitude, $\epsilon$, that sets the mass $m_{K} = \frac{(2\pi)^{2}(4\pi\epsilon a_{0}^{3})}{(4\pi^{2}a_{s}^{2})}$. Here $q$ is an electron charge and $\epsilon_{0}$ is a vacuum permittivity. The effective Planck constant takes the form $h_{\text{eff}} = h(2\pi)^{2}\sqrt{(4\pi\epsilon a_{0}^{3})(2\pi^{2}M)/a_{s}^{2}}$, where $M$ is the mass of a single $^{40}$Ca ion. In a trapped ion implementation the parameters of model (1) can be changed in the range $0 < m_{K} < 1$, $-0.5 \leq \delta \leq 0.5$ and $h_{\text{eff}} \leq 10^{-2}$. The discussion above holds except for an unscreened Coulomb interaction that creates a long range modification to the harmonic interaction between neighboring ions. In this section we explore how the long range interaction modifies the ability to controllably inject topological defects into the ionic lattice by considering the dynamics under the modified Hamiltonian

$$H = \sum_{n} \frac{P_{n}^{2}}{2} + m_{K}^{2}(1 + \cos \phi_{n}) + \sum_{r > 0} \left( \phi_{n+r} - \phi_{n} - 2\pi r \delta \right)^{2}/2|r|^{3},$$

where $r$ is an integer and runs over all neighbours.

As the inter-particle interaction increases, the form of a single kink modifies and it becomes less sharp on the tails [28, 60]. As a result, the effective interaction between kinks is also modified. Furthermore, in case of the long-range interaction all ions are sensitive to the presence of dislocations, and thus, injection of each kink into the system contributes immediately into dynamics of the ions in the middle of the chain. Nonetheless, as we show below,
our previous results remain qualitatively similar.

Similarly to the model with short range interactions, quenching the misfit parameter can drive commensurate-incommensurate transition in the long-range interacting model \(^7\). Fig. 1 shows the number of kinks in the system \(Q\) as a function of time in this case, and again shows kinks entering chain one by one from the boundaries in a periodic time intervals. Quantitatively, the duration of the plateau is shortened as the form of a single kink is broadened by the Coulomb repulsion. Furthermore, in the short-range interacting model injection of kinks takes place independently from left and right boundaries and it continues up to the time when first injected kink reaches the opposite boundary. However, in the long-range interacting model, dynamics of all atoms in the chain is immediately affected by the presence of the topological defects. Thus, the saturation time for a long-range interacting system, is approximately two times shorter than for its short-range interacting counterpart, \(\tau \approx N/2\epsilon\). It results from the fact that at this time \(N/2(l_m + l_s)\) kinks entered the chain from right boundary and \(N/2(l_m + l_s)\) kinks entered the chain from the left and the given density of kinks already minimizes the total energy of the system.

The role of Coulomb repulsion can be effectively suppressed by increasing \(m_K\), whose inverse sets its relative strength with respect to the lattice potential. In this case, a staircase structure in \(Q(t)\) can be observed more clearly. Even in a long-range interacting model the maximal number of kinks in the system scales linearly with the system size, sf. inset in Fig. 1 which is also the case for the nearest-neighbour interactions.

In order to observe these dynamics in experiments, it is necessary to resolve the commensurate and incommensurate configurations. One possibility would be to image the ions positions by applying non-destructive far-field measurements \({}^{61}\) using the diffractive pattern obtained from the scattered light. Alternatively, one could resort to AC-Stark shift methods \({}^{62}\) in order to distinguish the positions of the ions more accurately. The above results pertain to relevant parameters regime of a typical trapped ion experiment \({}^{10,25}\), and experimental details such as the effect of the external Paul trap, and an estimate of the temperature required to observe quantum suppositions of kinks in trapped ions experiments are elaborated in \({}^{65}\).

**Outlook.** — The effective long-wavelength description of Eq. (1), is a sine-Gordon model with \(m_K\) controlling the strength of the non-linearity (\(\sim m_K^2 \cos \phi\)), and the misfit parameter \(\delta\) coupling to the gradient of the field (\(\sim \delta \partial_x \phi\)). This model is known as Pokrovsky-Talapov field theory \({}^{33,66,67}\), and it finds numerous applications in ultra-cold gases \({}^{24,57,68,72}\), trapped ions applied to nano-friction \({}^{8,10}\), magnetism \({}^{29,30,73,74}\), and tangentially also strongly correlated systems \({}^{75,76}\). This naturally suggests broad applicability of our results to different platforms.

In addition to the trapped ion simulators discussed above, another example occurs in the cold atoms implementation of Ref. \({}^{24}\), where the Pokrovsky-Talapov model describes the dynamics of the phase difference between two Raman tunnel-coupled one-dimensional Bose quasi-condensates. The wave-vector difference of the Raman beams plays the role of the solitons chemical potential (\(\delta\) in the model of Eq. (1)). Quantum quench dynamics would therefore become accessible through interferometric measurements of the correlation functions of the phase difference \({}^{77,78}\).

These two examples represent just a few of the several interdisciplinary outreaches of our results which establish one more step towards merging the fields of topological defects and dynamical phase transitions in low-dimensional quantum simulators. In the far future one might imagine nano-scale control of topological defects in real materials by quenching high field optical potentials onto material crystals to controllably affect material properties. Experiments in trapped ion quantum simulators may offer the first steps in such a direction.

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