Abstract

Adversarial examples have been shown to be the severe threat to deep neural networks (DNNs). One of the most effective adversarial defense methods is adversarial training (AT) through minimizing the adversarial risk $R_{adv}$, which encourages both the benign example $x$ and its adversarially perturbed neighborhoods within the $\ell_p$-ball to be predicted as the ground-truth label. In this work, we propose a novel defense method, the robust training (RT), by jointly minimizing two separated risks ($R_{stand}$ and $R_{rob}$), which is with respect to the benign example and its neighborhoods respectively. The motivation is to explicitly and jointly enhance the accuracy and the adversarial robustness. We prove that $R_{adv}$ is upper-bounded by $R_{stand} + R_{rob}$, which implies that RT has similar effect as AT. Intuitively, minimizing the standard risk enforces the benign example to be correctly predicted, and the robust risk minimization encourages the predictions of the neighbor examples to be consistent with the prediction of the benign example. Besides, since $R_{rob}$ is independent of the ground-truth label, RT is naturally extended to the semi-supervised mode (i.e., SRT), to further enhance the adversarial robustness. Moreover, we extend the $\ell_p$-bounded neighborhood to a general case, which covers different types of perturbations, such as the pixel-wise (i.e., $x + \delta$) or the spatial perturbation (i.e., $Ax + b$). Extensive experiments on benchmark datasets not only verify the superiority of the proposed SRT method to state-of-the-art methods for defending pixel-wise or spatial perturbations separately, but also demonstrate its robustness to both perturbations simultaneously. The code for reproducing main results is available at https://github.com/THUYimingLi/Semi-supervised_Robust_Training.

1 Introduction

It has been shown that deep neural networks (DNNs) are vulnerable to adversarial examples [1,2,3]. Considering the wide application of DNNs in many mission-critical tasks (e.g., face recognition), it is urgent to develop effective defense methods for adversarial examples. One of the most promising defense methods is adversarial training (AT) [4,5], which minimizes the adversarial risk $R_{adv}$. It firstly defines the perturbed neighborhood set bounded by $\ell_p$-norm around each benign example $x$, and the perturbation is generated by some off-the-shelf adversarial attack methods, such as PGD [5] or FGSM [4]. $R_{adv}$ indicates the maximal loss value among the neighborhood samples and the benign example. It couples the classification risks on both the benign example $x$ and its surrounding $\ell_p$-bounded perturbed neighborhood examples. Minimizing $R_{adv}$ encourages both $x$ and its neighborhoods to be correctly predicted.

In this work, we propose to separate the classification risk on $x$ and its perturbed neighborhoods, respectively. We propose a novel robust training (RT) method by minimizing $R_{stand} + \lambda R_{rob}$. We...
prove that \( R_{adv} \) is upper-bounded by \( R_{stand} + R_{rob} \), which guarantees that RT has the similar effect as AT. In RT, \( R_{stand} \) indicates whether benign example \( x \) is correctly predicted, while the robust risk \( R_{rob} \) indicates the maximal value of the 0/1-loss that measures whether \( x \) and one of its perturbed neighborhoods are predicted as the same class, among all neighborhoods of \( x \). Minimizing \( R_{rob} \) encourages the predictions of the neighborhood examples to be consistent with the prediction of \( x \). Compared to AT, RT has two additional benefits. First, the balance between the classification accuracy on benign examples and the robustness to adversarial examples can be explicitly controlled by adjusting a trade-off hyper-parameter, i.e., \( \lambda \). Second, one important property of \( R_{rob} \) is that the ground-truth label of \( x \) is not involved, while only the prediction of \( x \) is adopted. It means that \( R_{rob} \) doesn’t require labeled training examples. Therefore, RT can be naturally extended to the semi-supervised mode, dubbed semi-supervised robust training (SRT), by incorporating massive unlabeled examples into \( R_{rob} \). The later experiments will show that the usage of the unlabeled examples in SRT could significantly enhance both the classification accuracy and the robustness to adversarial examples.

Moreover, we notice that most existing works on adversarial training adopt the \( \ell_p \)-bounded neighborhood when generating adversarial examples during the training. This neighborhood corresponds to the pixel-wise adversarial perturbations, i.e., \( x + \delta \). Meanwhile, many researches have found that there are also many other types of adversarial perturbations, such as the spatial perturbation or the functional perturbation. It has been observed [6] that the defense designed for pixel-wise perturbations doesn’t work for other types of perturbations. Is it possible to train a robust model to defend different types of adversarial perturbations simultaneously? To explore this problem, we extend the \( \ell_p \)-bounded neighborhood definition to a general case, since the perturbation type is closely related to the perturbed neighborhood. This new definition allows a general transformation-based perturbation and measures the distance between \( x \) and its perturbations using different metrics. This extension could be naturally embedded into SRT, as it only updates the form of neighborhood set. Consequently, the model trained using SRT with general perturbed neighborhood could simultaneously defend different types of perturbations.

The main contributions of this work are three-fold. 1) We propose a robust training method by jointly minimizing the standard risk and the robust risk, which is naturally extended the semi-supervised mode. 2) By generalizing the definition of the perturbation neighborhood to cover different types of perturbations, our robust training achieves the joint robustness to different perturbations, such as the pixel-wise and spatial perturbation. 3) Extensive experiments on benchmark datasets verify the superiority of the proposed SRT method to state-of-the-art adversarial training methods, as well as the robustness of SRT to pixel-wise and spatial perturbations simultaneously.

## 2 Related Work

**Supervised Adversarial Defense.** DNNs are known to be vulnerable to different types of well-designed small adversarial perturbation, such as pixel-wise perturbation [4][5][6] and spatial perturbation [7][8][9][10]. Those attacks usually based on the relation between the prediction and the ground-truth label of the sample, i.e. they are in a supervised manner. Toward the adversarial robustness against those attacks, several supervised adversarial defense methods were proposed. These methods can be roughly divided into three main categories, including adversarial training based defense [4][5][6][11], detection based defense [12][13][14], and reconstruction-based defense [15][16][17]. Adversarial training based defense is currently the most mainstream research direction, which improves the adversarial robustness via adding various adversarially manipulated samples during the training process. For example, standard adversarial training improved the model robustness through training on adversarial examples generated by FGSM [4] and PGD [5], and the trade-off inspired adversarial defense (TRADES) generated manipulated samples by further considering the trade-off between robustness and accuracy [11]. Although many defense methods have been proposed, there is still a long way to go for solving the adversarial vulnerability problem of DNNs.

Recently, some attempts have been proposed to defend multiple perturbations together. In [6], they observed that pixel-wise defense techniques have limited benefit to the spatial robustness. This problem is further investigated in [18], where the author proved that there exists a trade-off in robustness to different types of attacks in a natural and simple statistical setting. In other words, it is likely that there is no defense that could reach best robustness under every attacks. How to defend against different types of attacks simultaneously is still an important open question.
3 Proposed Method

3.1 Preliminaries

We denote the classifier as \( f_w : \mathcal{X} \rightarrow \{0, 1\}^{|\mathcal{Y}|} \), with \( \mathcal{X} \subset \mathbb{R}^d \) being the instance space and \( \mathcal{Y} = \{1, 2, \ldots, K\} \) being the label space. \( f(x) \) indicates the posterior vector with respect to \( K \) classes, and \( C(x) = \text{arg} \max f_w(x) \) denotes the predicted label. The labeled dataset is denoted as \( \mathcal{D}_L = \{(x_i, y_i) | i = 1, \ldots, N_l\} \), where \( (x_i, y_i) \) is independent and identically sampled from an unknown latent distribution \( \mathcal{P}_{\mathcal{X} \times \mathcal{Y}} \). \( \mathcal{D}'_L = \{x(x, y) \in \mathcal{D}_L\} \) indicates the instance set of \( \mathcal{D}_L \). \( \mathcal{D}_U = \{x_i | i = 1, \ldots, N_u\} \) denotes the unlabeled dataset.

Definition 1. The \( c \)-bounded transformation-based neighborhood set of the benign example \( x \) is defined as follows:

\[
\mathcal{N}_{c,T}(x) = \{T(x; \theta) | \text{dist}(T(x; \theta), x) \leq c\},
\]

where \( T(\cdot; \theta) \) indicates a parametric transformation, and \( \text{dist}(\cdot, \cdot) \) is a given distance metric corresponding to \( T(\cdot; \theta) \). Non-negative hyper-parameter \( c \) denotes the maximum perturbation size.

Remark. \( \mathcal{N}_{c,T} \) can reduce to some widely used neighborhood sets through different specifications of \( T(\cdot; \theta) \) and \( \text{dist}(\cdot, \cdot) \). For example, if we set \( T(x; \theta) = x + \theta \) and \( \text{dist}(T(x; \theta), x) = \|T(x; \theta) - x\|_\infty \), then \( \mathcal{N}_{c,T} \) becomes the \( \ell_\infty \)-bounded neighborhood set used in the pixel-wise adversarial attack and defense [25][5][11]. If we set \( T(x; \theta) = [\cos \theta, -\sin \theta; \sin \theta, \cos \theta]x \) and \( \text{dist}(T(x; \theta), x) = \theta \), then \( \mathcal{N}_{c,T} \) indicates the rotation-based neighborhood set used in the spatial adversarial attack and defense [6][10]. This flexibility is important for developing a method to defend multiple types of adversarial perturbations, which will be shown later.

Based on the general transformation-based neighborhood defined above, we can extend the traditional adversarial risk and the robust risk to the form as follows:

Definition 2 (Standard, Adversarial, and Robust Risk).

- The standard risk \( R_{\text{stand}} \) measures whether the prediction of \( x \) (i.e., \( C(x) \)), is same with its ground-truth label \( y \). Its definition with respect to a labeled dataset \( \mathcal{D} \) is formulated as

\[
R_{\text{stand}}(\mathcal{D}) = \mathbb{E}_{(x,y) \sim \mathcal{P}_{\mathcal{D}}} [\mathbb{I}\{C(x) \neq y\}],
\]

where \( \mathcal{P}_{\mathcal{D}} \) indicates the distribution behind \( \mathcal{D} \). \( \mathbb{I}(a) \) denotes the indicator function: \( \mathbb{I}(a) = 1 \) if \( a \) is true, otherwise \( \mathbb{I}(a) = 0 \).

- The \( \mathcal{N}_{c,T} \)-based adversarial risk with respect to \( \mathcal{D} \) is defined as

\[
R_{\text{adv}}(\mathcal{D}) = \mathbb{E}_{(x,y) \sim \mathcal{P}_{\mathcal{D}}} \left[\max_{x' \in \mathcal{N}_{c,T}(x)} \mathbb{I}\{C(x') \neq y\}\right].
\]

- The \( \mathcal{N}_{c,T} \)-based robust risk with respect to \( \mathcal{D} \) is defined as

\[
R_{\text{rob}}(\mathcal{D}) = \mathbb{E}_{x \sim \mathcal{P}_{\mathcal{D}'}} \left[\max_{x' \in \mathcal{N}_{c,T}(x)} \mathbb{I}\{C(x') \neq C(x)\}\right].
\]

Remark. \( R_{\text{adv}} \) considers the classification accuracies on both the benign example \( x \) and its adversarial perturbations within \( \mathcal{N}_{c,T}(x) \). \( R_{\text{stand}} \) is only related to the classification accuracy on \( x \). \( R_{\text{rob}} \) doesn’t involve the ground-truth label \( y \), and it only corresponds to the consistency between the prediction of \( x \) and its adversarial perturbations, i.e., robustness. Their relations are shown in Lemma 4. Due to the space limit, the proof of Lemma 4 will be presented in the Appendix.
Lemma 1. Standard Risk, adversarial risk and robust risk of a sample $x$ are correlated. Specifically,
\[ R_{adv}(x) = R_{stand}(x) + (1 - R_{stand}(x)) R_{rob}(x). \] (5)

3.2 Robust Training

Based on the risks defined in Definition 2, we propose a novel robust training (RT) method, as follows
\[ \min_w R_{stand}(D_L) + \lambda \cdot R_{rob}(D_L), \] (6)
where $\lambda > 0$ is a trade-off parameter. The relationship between RT and the standard adversarial training (AT) is analyzed in Theorem 1 (whose proof is in the Appendix). RT is minimizing the upper bound of $R_{adv}$, if we set $\lambda = 1$. It tells that RT could have the similar effect with AT. However, due to the separation of the classification accuracy and the robustness in RT, there are two important advantages compared to AT. 1) $\lambda$ can explicitly control the trade-off between the classification accuracy to benign examples and the robustness to adversarial examples, which will be verified in later experiments. $\lambda$ can be adjusted according to the user’s demand. 2) What is more important, since $R_{adv}$ doesn’t utilize the ground-truth labels, it can be defined with respect to any unlabeled example. Thus, it is natural to extend RT to the semi-supervised mode, as shown in Section 3.3.

Theorem 1. The relationship between the standard adversarial training and the robust training is as follows
\[ \min_w R_{adv}(D) \leq \min_w \{ R_{stand}(D) + R_{rob}(D) \}. \] (7)

3.3 Semi-supervised Robust Training

As stated above, $R_{rob}$ could be defined with respect to any unlabeled data. Thus, we propose the semi-supervised robust training (SRT), as follows
\[ \min_w R_{stand}(D_L) + \lambda \cdot R_{rob}(D_L \cup D_U). \] (8)
Note that RT (i.e., Eq. (6)) is a special case of Eq. (8), in the case that $D_U = \emptyset$.

Due to the non-differentiability of the indicator function $\mathbb{I}()$ used in $R_{stand}$ and $R_{rob}$, we resort to its approximated loss function, such as the cross-entropy. Besides, we replace the expectation with the sample mean with respect to $D$. Then, the approximated objective function is formulated as follows:
\[ \min_w \frac{1}{|D_L|} \sum_{(x,y) \in D_L} L(f_w(x), y) + \frac{\lambda}{|D_L \cup D_U|} \sum_{x \in D_L \cup D_U} \max_{x' \in \mathcal{N}_{x,T}(x)} L(f_w(x'), C(x)), \] (9)
where $L()$ indicates the cross-entropy function.

Similar to the optimization for adversarial training, the minimax problem can be solved by alternatively solving the inner-maximization and the outer-minimization sub-problem, as follows

- **Inner-maximization**: given $w$, $\forall x \in D_L \cup D_U$, we derive an adversarial example $x'$ by
\[ x' \leftarrow \arg \max_{x' \in \mathcal{N}_{x,T}(x)} L(f_w(x'), C(x)). \] (10)

- **Outer-minimization**: given all generated $x'$, the parameter $w$ is updated as follows:
\[ w \leftarrow \arg \min_w \frac{1}{|D_L|} \sum_{(x,y) \in D_L} L(f_w(x), y) + \frac{\lambda}{|D_L \cup D_U|} \sum_{x \in D_L \cup D_U} L(f_w(x'), C(x)). \] (11)

We notice that the name “robust training” has been used in some other works, such as in [26, 27, 28, 29], etc. However, their objective functions are totally different with ours. For example, the "robust training" in [29] indicates a robust hinge loss in the object function. Here we re-define "robust training" based on the robust risk.
4 Experiments

Unless otherwise specified, the training set is divided into two parts, one of which contains 10000 samples serves as the labeled dataset, and the remaining part is used as the unlabeled dataset in all experiments. In addition, in the semi-supervised setting, we use batch sizes proportional to the dataset size, \( i.e. \) for a total batch size \( m \), the number of labeled samples and unlabeled samples within a batch is \( m \cdot \frac{|D_u|}{|D_l|+|D_u|} \) and \( m \cdot \frac{|D_l|}{|D_l|+|D_u|} \), respectively. We conduct experiments in the CIFAR-10 [30] and MNIST [31] dataset. In addition, we use adversarial accuracy, which is defined as the accuracy on adversarial examples, to evaluate the adversarial robustness.

4.1 Comparison between RT and SRT

**Settings.** We conduct experiments in the CIFAR-10 dataset. In the pixel-wise experiments, we use the wide residual network WRN-34-10 [32], and set the perturbation size \( \epsilon_1 = 0.031 \). To evaluate the adversarial robustness, we apply PGD-20 with step size 0.003 and FGSM under perturbation size 0.031, as suggested in [11]. In the spatial adversarial settings, we use ResNet [33] with 4 residual groups with filter size [16, 16, 32, 64] and 5 residual units each. We perform GridAdv and RandAdv attack [6] to evaluate the spatial adversarial robustness. The attack parameters are chosen through random sampling (RandAdv) or grid search (GridAdv) with both rotations and translations. The maximum rotational perturbation is set as 30\(^\circ\), and we consider translations of at most 3 pixels. Other detailed settings, and additional results in the MNIST dataset are shown in the Appendix.

**Results.** As shown in Table 1, SRT is superior to RT under the same conditions, no matter in spatial scenario or pixel-wise scenario. In particular, SRT is much more robust than RT under stronger attack (PGD and GridAdv). For example, the adversarial accuracy of SRT is over more than 5\% compared to RT under PGD attack in most cases. This improvement is even more significant under GridAdv, which is at least 17\% (more than 20\% in most cases). In addition, the clean accuracy of SRT is also higher than the one of RT in all cases. As such, we will only use SRT in the following comparisons.

In particular, an interesting phenomenon is that the clean accuracy does not decrease as the hyper-parameter \( \lambda \) increases in the spatial adversarial settings. In other words, there is no trade-off between the spatial adversarial robustness and the standard prediction performance. In contrast, the inherent trade-off between clean accuracy and adversarial accuracy was verified theoretically and empirically in the pixel-wise adversarial settings [34]. This interesting contradiction was discussed in [10], which is out the scope of this paper.

4.2 Spatial Adversarial Defense

**Settings.** We select Worst-of-k [6] and K-L divergence based regularization (KLR) [10] as the baseline methods in the following evaluations. Those methods reach the state-of-the-art results in the realm of spatial adversarial defense. Besides, we also provide the standard training model (Standard) and the pixel-wise adversarial training (AT) [5] as other important baselines for comparison. In SRT, the we set \( \lambda = 0.2 \) in both CIFAR-10 and MNIST according to previous experiments, and the hyper-parameters of baseline methods are set according to their paper. Other detailed settings are shown in the Appendix.
Table 3: Adversarial accuracy (%) of different models under spatial attacks in CIFAR-10 and MNIST dataset. The attack parameters are chosen through random sampling (RandAdv) or grid search (GridAdv) with rotations and translations considered both together and separately (“RandAdv.R” and “GridAdv.R” for rotations, and “RandAdv.T” and “GridAdv.T” for transformations).

| Defense | Clean | RandAdv | GridAdv | RandAdv.T | GridAdv.T | RandAdv.R | GridAdv.R |
|---------|-------|---------|---------|-----------|-----------|-----------|-----------|
| CIFAR-10 |       |         |         |           |           |           |           |
| Standard | 80.63 | 8.82    | 0.09    | 33.61     | 19.67     | 19.55     | 10.73     |
| AT       | 65.59 | 4.92    | 0.22    | 16.23     | 7.49      | 14.47     | 8.37      |
| Worst-of-k | 82.02 | 70.92   | 54.80   | 75.49     | 69.45     | 73.18     | 68.22     |
| KLR      | 85.40 | 72.77   | 56.28   | 77.43     | 72.71     | 74.80     | 71.04     |
| SRT      | 88.87 | 78.99   | 64.83   | 82.16     | 78.47     | 80.84     | 77.24     |
| MNIST    |       |         |         |           |           |           |           |
| Standard | 97.19 | 14.01   | 0.00    | 55.91     | 5.12      | 65.06     | 51.32     |
| AT       | 97.96 | 29.84   | 0.01    | 51.83     | 10.72     | 71.66     | 57.71     |
| Worst-of-k | 98.05 | 94.77   | 84.64   | 96.07     | 93.99     | 95.70     | 94.24     |
| KLR      | 98.43 | 95.26   | 86.08   | 96.63     | 95.07     | 95.92     | 94.48     |
| SRT      | 98.64 | 97.02   | 92.12   | 97.68     | 96.85     | 97.33     | 96.54     |

Table 4: Adversarial accuracy (%) of different models under pixel-wise attacks.

| Defense | Clean | FGSM | PGD | MI-FGSM | JSMA | C&W | Point-wise Attack | DDNA |
|---------|-------|------|-----|---------|------|-----|--------------------|------|
| CIFAR-10 |       |      |     |         |      |     |                    |      |
| Standard | 88.43 | 7.26 | 0   | 0       | 9.98 | 0.14 | 2.06               | 0.01 |
| AT       | 77.17 | 50.35| 40.2| 39.79   | 11.90| 36.09| 12.32              | 22.37|
| TRADES   | 76.23 | 51.98| 40.2| 39.79   | 11.90| 36.09| 12.32              | 22.37|
| UAT      | 78.45 | 56.35| 44.86|44.56   | 13.23| 40.03| 12.62              | 24.08|
| RST      | 79.99 | 59.55| 48.38|47.97   | 13.78| 42.99| 13.42              | 27.73|
| SRT      | 78.46 | 59.34| 48.66|48.24   | 16.99| 43.33| 18.80              | 27.71|
| MNIST    |       |      |     |         |      |     |                    |      |
| Standard | 99.01 | 41.06| 2.87| 4.37    | 10.18| 0.01| 0.04               | 16.44|
| AT       | 98.99 | 96.61| 94.69|93.57   | 20.99| 94.67| 2.47               | 95.35|
| TRADES   | 98.99 | 96.92| 95.12|93.98   | 18.48| 92.37| 2.08               | 94.12|
| UAT      | 99.16 | 97.51| 96.14|95.65   | 23.79| 96.16| 5.52               | 96.39|
| RST      | 98.83 | 97.21| 95.47|95.05   | 26.63| 95.34| 3.30               | 94.25|
| SRT      | 99.28 | 97.77| 96.60|95.79   | 26.79| 96.19| 5.81               | 96.10|

Results. As shown in Table 3, SRT significantly exceeds all existing methods. For example, SRT achieves more than 5% improvement under any attacks in the CIFAR-10 dataset, compare with the current state-of-the-art method, the KLR. In addition, similar to the phenomenon in Section 4.1, the clean accuracy of SRT is also higher compared to all existing methods. This improvement of clean accuracy is presumably because the spatial adversarial defense serve as an effective data augmentation approach. Note that the pixel-wise adversarial defense has almost no benefit to the spatial adversarial robustness. In other words, pixel-wise adversarial robustness does not necessarily mean general adversarial robustness. Therefore, developing a general adversarial framework toward general adversarial robustness is of great significance.

4.3 Pixel-wise Adversarial Defense

Settings. We compare SRT with PGD-based adversarial training (AT) [5], trade-off inspired adversarial defense (TRADES) [11], unsupervised adversarial training (UAT) [34] and robust self-training (RST) [23]. TRADES is the state-of-the-art supervised defense, while UAT and RST are the most advanced semi-supervised adversarial defenses. In SRT, we set $\lambda = 1$ in both CIFAR-10 and MNIST dataset. To evaluate the robustness, we apply FGSM, PGD, momentum iterative attack (MI-FGSM) [35] under $\ell_\infty$ norm. We also apply some other attack methods, including jacobian saliency map attack (JSMA) [25], Carlini & Wagner attack (C&W) [36], point-wise attack [37], and decoupled direction and norm attack (DDNA) [38] for the evaluation.

Results. As shown in Table 4, compared to the supervised adversarial defense methods, semi-supervised defense approaches achieve significant improvement in both clean accuracy and adversarial robustness with additional unlabeled samples. Among three different semi-supervised defenses, SRT reaches the best performance. Specifically, in the CIFAR-10 and MNIST datasets, SRT has the best adversarial robustness in most cases. In the case where a few are not the best, SRT is still the sub-optimal. In addition, although UAT and RST perform similarly to and even exceed SRT in a few cases, their superiority is not consistent across different datasets. For example, RST has better performance in CIFAR-10, whereas UAT is better in MNIST.
4.4 Compound Adversarial Attack and Defense

Settings. Here we evaluate the defense to the attack consisting of multiple types of adversarial attacks, dubbed compound attack. We evaluate two types of compound attacks, including 1) combining GridAdv on the top of PGD (PGD+), and 2) combining PGD on the top of GridAdv (GridAdv+). To defend compound attacks, we provide two compound defenses, including 1) combining Worst-of-k on the top of AT (AT+), and 2) combining spatial SRT on the top of pixel-wise SRT (SRT+). Other detailed settings are shown in the Appendix.

Results. As shown in Figure 1 compared to the single type of attack, compound adversarial attacks (GridAdv+ and PGD+) have a much stronger threat under the same conditions. This phenomenon indicates that by simply combining different types of attacks, a powerful attack can be constructed, which poses a huge threat to DNNs. In particular, the spatial adversarial defenses have limited benefit to the pixel-wise defense, which can be observed by the subgraphs in third column. Especially in the CIFAR-10 dataset, the adversarial accuracy could quickly drop to zero under PGD attack with small perturbation fraction. This again confirms that a single type of defense may not have much effect on defending against another type of attack. Besides, the results also suggest that the model trained with the general perturbed neighborhood (i.e., AT+ and SRT+) could simultaneously defend different types of perturbations, i.e., the compound attack. Compared to single-type defense methods, compound defenses achieve significant improvements defending against compound attacks, as well good defending performance to single attacks. In addition, in the case of compound defense, our method (SRT+) is also better than the extension of previous methods (AT+).

4.5 The Effect of Unlabeled Data

In this section, we use PGD and GridAdv to evaluate the adversarial robustness under pixel-wise and spatial scenario respectively. Except for the number of unlabeled samples, all settings are the same as those used in Section 4.2-4.3.

Figure 2: Adversarial accuracy w.r.t the number of unlabeled data.

We notice that the name “compound attack” was used in the area of network security, indicating the combined attack of multiple cyberattacks.
(a) spatial scenario  

(b) pixel-wise scenario

Figure 3: Comparison between the decision surfaces of the SRT and those of the standard training model in the MNIST dataset. First row: the 3D decision surface and their corresponding 2D version of the standard training model. Second row: the decision surface of the model trained with SRT. In the 3D decision surfaces, the $X$ and $Y$ axis represent two different perturbation directions and its value indicates the perturbation size, while the $Z$-axis indicates the decision value. If and only if the decision value is positive, the prediction is correct. In those figures, the red area indicates the correctly classified region, while the blue area is the misclassified region.

As shown in Figure 2, the adversarial accuracy of models trained with SRT increases with the number of unlabeled samples in all settings. Moreover, leveraging large amounts of unlabeled examples, SRT achieves similar adversarial robustness to the supervised adversarial defense methods (AT and Worst-of-$k$) trained with all labeled samples in the original training set (i.e., 60000 samples in MNIST, and 50000 samples in CIFAR-10). In particular, in the setting of spatial attacks in MNIST, the adversarial accuracy of SRT-based model trained with 10000 labeled samples and 50000 unlabeled samples even exceeds that of Worst-of-$k$ with 60000 labeled samples. Besides, the adversarial accuracy still has an upward trend at the end of the curve (i.e., using all unlabeled data), which implies that the robustness can be further improved if additional unlabeled samples are utilized.

4.6 Verification via Decision Surface

In this section, we verify the effectiveness of SRT through visualizing of the geometry of the decision surface in MNIST dataset. The decision value $D$ of a sample $x$ is defined as $D(x) = p_y - \max_{i \neq y} p_i$, where $f_w(x) = [p_1; \cdots; p_K]$. We visualize the decision surface instead of the loss surface, since it can be further used to evaluate the prediction performance, where $D(x) > 0$ indicates that the prediction of $x$ is correct and vice versa.

As shown in Figure 3, the geometric property of decision surfaces of SRT-based model and that of the standard training model are significantly different. Compared to SRT, the decision surfaces of the standard training model have sharper peaks and larger slopes, which explains that the prediction of this model is vulnerable to small perturbation. In contrast, the surfaces of SRT are relatively flat and located on a plateau with positive decision confidence around the benign sample, though the decision surface in the spatial scenario is more rugged than that in the pixel-wise scenario. Consequently, the output of SRT still lies in the region of correct classification if perturbed with a certain range. It explains robustness of the SRT-based model.

5 Conclusion

In this paper, we propose a novel defense method, dubbed robust training (RT), by jointly minimizing the standard risk and robust risk. We prove that $R_{\text{adv}}$ is upper-bounded by $R_{\text{stand}} + R_{\text{rob}}$, which implies that RT has the similar effect as adversarial training. In addition, we extend RT to the semi-supervised mode (i.e., SRT) to further enhance the adversarial robustness, due to the fact that the robust risk is independent of the ground-truth label. Moreover, we extend the $f_p$-bounded neighborhood to a general case, which covers different types of perturbations. Consequently, the model trained using SRT with general perturbed neighborhood could simultaneously defend different types of perturbations. Extensive experiments verify the superiority of SRT for defending pixel-wise or spatial perturbations separately, as well as both perturbations simultaneously. Note that our extension of the perturbation neighborhood is general and not limited to the pixel-wise and spatial scenario. More types of transformations, such as blurring and color shifting, could be also covered. It will be explored in our future work.
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Appendix

A Proofs

Lemma 1. Standard Risk, adversarial risk and robust risk of a sample $x$ are correlated. Specifically,

$$R_{adv}(x) = R_{stand}(x) + (1 - R_{stand}(x)) R_{rob}(x).$$

Proof. According to the definition of $R_{adv}$ (see Definition 2 in the main manuscript), we have

$$R_{adv}(x) = \max_{x' \in N_{\delta,T}(x)} \mathbb{I}\{C(x') \neq y\}. \quad (13)$$

Since $\max_{x' \in N_{\delta,T}(x)} \mathbb{I}\{C(x') \neq y\} \in \{0, 1\}$, and its value is 1 iff $\exists x' \in N_{\delta,T}(x)$ s.t. $C(x') \neq y$.

Therefore, we have

$$\mathbb{I}\{C(x') \neq y\} = \mathbb{I}\{\exists x' \in N_{\delta,T}(x) \text{ s.t. } C(x') \neq y\}. \quad (14)$$

Event $A$: $\exists x' \in N_{\delta,T}(x)$ s.t. $C(x') \neq y$ can be divided into two disjoint sub-events, as follows:

$$A_1 : (\exists x' \in N_{\delta,T}(x) \text{ s.t. } C(x') \neq y) \cap (C(x) = y),$$

$$A_2 : (\exists x' \in N_{\delta,T}(x) \text{ s.t. } C(x') \neq y) \cap (C(x) = y).$$

Let $B : \exists x' \in N_{\delta,T}(x) \text{ s.t. } C(x') \neq y, C : C(x) = y$, now we prove that $A_1$ is true if and only if event $\overline{C} : C(x) \neq y$ is true.

1) Suppose $\overline{C}$ is true, since $x \in N_{\delta,T}(x)$, we have

$$\exists x' \in N_{\delta,T}(x) \text{ s.t. } C(x') \neq y$$

holds, i.e., $B$ is true, therefore $A_1 = B \cap \overline{C}$ is true.

2) Suppose $\overline{C}$ is false, then $A_1 = B \cap \overline{C}$ is false.

3) Suppose $A_1 = B \cap \overline{C}$ is true, then $\overline{C}$ is true.

4) Suppose $A_1 = B \cap \overline{C}$ is false, we can prove that $\overline{C}$ is false by seeking the contradiction.

Thus, we have

$$\mathbb{I}\{A_1\} = \mathbb{I}\{C(x) \neq y\}. \quad (15)$$

Since $A_2 = B \cap C = (B \cap C) \cap C$ and

$$(B \cap C) : \exists x' \in N_{\delta,T}(x) \text{ s.t. } C(x') \neq C(x),$$

$$\mathbb{I}\{A_2\} = \mathbb{I}\{C\} \times \mathbb{I}\{B \cap C\} = \mathbb{I}\{C(x) = y\} \times \mathbb{I}\{\exists x' \in N_{\delta,T}(x) \text{ s.t. } C(x') \neq C(x)\}. \quad (16)$$

Therefore, combining Eqs. (15) and (16), we obtain

$$\mathbb{I}\{A_1\} = \mathbb{I}\{A_2\}$$

$$\mathbb{I}\{C(x) \neq y\} + \mathbb{I}\{C(x) = y\} \times \mathbb{I}\{\exists x' \in N_{\delta,T}(x) \text{ s.t. } C(x') \neq C(x)\}. \quad (17)$$

Similar to Eq. (14), we have

$$\mathbb{I}\{\exists x' \in N_{\delta,T}(x) \text{ s.t. } C(x') \neq C(x)\} = \max_{x' \in N_{\delta,T}(x)} \mathbb{I}\{C(x') \neq C(x)\}. \quad (18)$$

Combining Eqs. (17) and (18), we have

$$\mathbb{I}\{C(x) \neq y\} + \mathbb{I}\{C(x) = y\} \times \max_{x' \in N_{\delta,T}(x)} \mathbb{I}\{C(x') \neq C(x)\}. \quad (19)$$

According to the definitions of $R_{adv}$, $R_{rob}$ and $R_{stand}$,

$$\mathbb{I}\{A_1\} = R_{stand}(x) + (1 - R_{stand}(x)) R_{rob}(x). \quad (20)$$

$\square$
**Theorem 1.** The relationship between the standard adversarial training and the robust training is as follows
\[
\min_{\omega} R_{adv}(D) \leq \min_{\omega}\{ R_{stand}(D) + R_{rob}(D) \}.
\]

**Proof.** Since \( R_{stand}(x), R_{rob}(x) \in [0, 1] \), according to Lemma \[1\] it is easy to obtain that
\[
R_{adv}(x) \leq R_{stand}(x) + R_{rob}(x).
\]
By seeking the expectation on both sides of \((22)\), we obtain
\[
R_{adv}(D) \leq R_{stand}(D) + R_{rob}(D).
\]
Utilizing the fact that all three risks are non-negative, we further obtain
\[
\min_{\omega} R_{adv}(D) \leq \min_{\omega}\{ R_{stand}(D) + R_{rob}(D) \}.
\]
\[\Box\]

**B Settings for Comparison between RT and SRT**

**Training Setup.** In the spatial adversarial settings, we use a standard ResNet [33] with 4 residual groups with filter size [16, 16, 32, 64] and 5 residual units each. We set the learning rate \( \eta_2 = 0.1 \), batch size \( m_2 = 128 \), and run 80000 iterations. In the pixel-wise experiments, we use the wide residual network WRN-34-10 [32], and set the perturbation size \( \epsilon_1 = 0.031 \), perturbation step size \( \alpha_1 = 0.007 \), number of iterations \( K_1 = 10 \) (for the inner maximization sub-problem), learning rate \( \eta_1 = 0.1 \), batch size \( m_1 = 128 \), and run 100 epochs in the training dataset.

**Data Preprocessing.** We conduct standard data augmentation techniques for benign images. Specifically, 4-pixel padding is used before performing random crops of size \( 32 \times 32 \), followed by a left-and-right random flipping with probability 0.5.

**Attack Setup.** We perform GridAdv and RandAdv attack [6] to evaluate the spatial adversarial robustness. The attack parameters are chosen through random sampling (RandAdv) or grid search (GridAdv) with both rotations and translations. Specifically, the maximum rotational perturbation is set as \( 30^\circ \), and we consider translations of at most 3 pixels. For the GridAdv, we consider 5 and 31 values equally spaced per direction for translation and rotation, respectively. For the RandAdv, we perform a uniformly random perturbation from the attack spaced used in GridAdv. These settings are based on those suggested in [6]. To evaluate the pixel-wise adversarial robustness, we apply PGD-20 with step size 0.003 and FGSM under perturbation size 0.031, as suggested in [11].

| Table 5: Comparison between SRT and RT under spatial attacks in MNIST dataset. | Table 6: Comparison between SRT and RT under pixel-wise attacks in MNIST dataset. |
|--------------------------------|----------------------------------|
| Standard | Clean | RandAdv | GridAdv | Clean | FGSM | PGD |
| RT (\( \lambda = 0.15 \)) | 98.74 | 92.88 | 72.85 | 99.02 | 93.80 | 86.12 |
| SRT (\( \lambda = 0.15 \)) | 98.61 | 96.55 | 91.52 | 98.61 | 96.55 | 91.52 |
| RT (\( \lambda = 0.20 \)) | 98.64 | 97.02 | 92.12 | 99.14 | 97.57 | 96.23 |
| SRT (\( \lambda = 0.20 \)) | 98.33 | 93.66 | 76.68 | 99.14 | 97.57 | 96.23 |
| RT (\( \lambda = 0.25 \)) | 98.29 | 93.91 | 78.00 | 99.06 | 97.78 | 96.92 |
| SRT (\( \lambda = 0.25 \)) | 98.63 | 96.91 | 91.70 | 99.34 | 98.47 | 97.81 |
| RT (\( \lambda = 0.30 \)) | 98.42 | 93.86 | 77.74 | 99.11 | 97.90 | 97.06 |
| SRT (\( \lambda = 0.30 \)) | 98.62 | 97.08 | 91.44 | 99.35 | 98.53 | 97.86 |

**C Comparison between SRT and RT in MNIST Dataset**

**Setup.** We use a simple CNN with four convolutional layers followed by three fully-connected layers [11] in both pixel-wise and spatial scenario. In spatial scenario, except for the model architecture, the other settings are the same as those stated in Section B. In the pixel-wise experiments, we set perturbation size as 0.1, perturbation step size as 0.01, number of iterations as 20, with learning rate \( \eta_3 = 0.01 \), batch size \( m_3 = 128 \), and run 50 epochs in the training dataset. We apply PGD-40 with step size 0.005 and FGSM under perturbation size 0.1, which are suggested in [11].
where

**Attack Setup**. There are mainly two different ways to construct a compound attack based on

we apply PGD-40 and MI-FGSM-40 with step size 0.01 and FGSM under perturbation size 0.3.

**Training Setup**

**Defense Setup**

A spatial SRT on the top of pixel-wise SRT (SRT+).

compound defenses, including 1) combining Worst-of-

first performing a pixel-wise attack and then a spatial attack. Specifically, we provide two different

the top of GridAdv (GridAdv+):

T

pound adversarial attacks, including 1) combining GridAdv on the top of PGD (PGD+):

translation, and maximum pixel-wise perturbation respectively. We evaluate two types of com-

user-defined maximum perturbation size, where

discussed in [18], and we discuss the second way in this paper. Let

1 in the CIFAR-10 dataset according to cross-validation.

Besides, we observe that when the regularization weight

in both CIFAR-10 and MNIST, and the hyper-parameter of baseline

1 in both CIFAR-10 and MNIST, and the hyper-parameter of baseline

in the CIFAR-10 dataset according to cross-validation.

**Robustness Evaluation Setup.** We apply FGSM, PGD, momentum iterative attack (MI-FGSM) 

[35] under \( \ell_\infty \) norm in the evaluation, since these methods are similar to the one (PGD) used in

solving the inner-maximization in SRT. We also apply some other attack methods, including jacobian

saliency map attack (JSMA) [25], Carlini & Wagner attack (C&W) [36], point-wise attack [37], and

decoupled direction and norm attack (DDNA) [38] for the evaluation. All attacks are implemented

based on Advertorch [39] and Foolbox [40]. In CIFAR-10, we set perturbation size \( \epsilon = 0.031 \) for

FGSM, PGD and MI-FGSM, the step size \( \alpha = 0.007 \), and the iteration step is set to 10. In MNIST,

we apply PGD-40 and MI-FGSM-40 with step size 0.01 and FGSM under perturbation size 0.3.

**Results.** Similar to the results in the CIFAR-10 dataset, SRT is superior to RT under the same

conditions in the MNIST dataset. SRT is also much more robust than RT under stronger attack (PGD

and GridAdv), especially in the spatial scenario.

**D Settings for Spatial Adversarial Defense**

**Setup.** The training and the attack setup follow the same settings used in Section [3] in SRT,

we set \( \lambda = 0.2 \) in both CIFAR-10 and MNIST according to previous experiments, and the

hyper-parameters of baseline methods are set according to their paper.

**E Settings for Pixel-wise Adversarial Defense**

**Training Setup.** The model architectures are the same as those used in Section [3]. To ensure that

all methods have converged, we train 120 epochs in this experiment, and the initial learning rate is

changed to 0.05. The number of iterations and the perturbation size is modified to 40 and 0.3 in

CIFAR-10 respectively, as suggested in [11]. Other settings are the same as those used in Section [3].

above. In SRT, we set \( \lambda = 1 \) in both CIFAR-10 and MNIST, and the hyper-parameter of baseline

methods are set according to their paper. In particular, there are two default settings discussed

in TRADES, we choose the one with better performance (1/\( \lambda = 1 \) in MNIST and 1/\( \lambda = 6 \) in

CIFAR-10). Besides, we observe that when the regularization weight \( \beta \) is set to 6, as suggested in the

paper, the UAT is hard to learn in the CIFAR-10 dataset. As such, we modify the hyper-parameter \( \beta \)
to 1 in the CIFAR-10 dataset according to cross-validation.

**Results.** Similar to the results in the CIFAR-10 dataset, SRT is superior to RT under the same

conditions in the MNIST dataset. SRT is also much more robust than RT under stronger attack (PGD

and GridAdv), especially in the spatial scenario.

**F Settings for Compound Adversarial Attack and Defense**

**Attack Setup.** There are mainly two different ways to construct a compound attack based on

combining different types of attacks. One way is to choose the best attack within many different

types of attacks, and the other is to conduct multiple attacks in sequence. The first way is

discussed in [18], and we discuss the second way in this paper. Let \( \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \) indicates

the user-defined maximum perturbation size, where \( \epsilon_1, \epsilon_2, \epsilon_3 \) is the maximum rotation, maximum

translation, and maximum pixel-wise perturbation respectively. We evaluate two types of compound

adversarial attacks, including 1) combining GridAdv on the top of PGD (PGD+): \( T_1(x) = A(\theta) \cdot (x + \arg \max_{r \in B_{\infty}(\epsilon)} \mathcal{L}(f(x + r), y)) + B \) and \( \text{dist}_1(T_1(x), x) = \theta, \| B \|_{\infty, \infty} \leq (\epsilon_1, \epsilon_2) \),

where \( A(\theta) = [\cos \theta, -\sin \theta; \sin \theta, \cos \theta] \) and \( B_{\infty}(\epsilon) = \{ x \| \| x \|_\infty \leq \epsilon \} \); and 2) combining PGD on the top of GridAdv (GridAdv+): \( T_2(x; r) = A^* x + B^* + r \) and \( \text{dist}_2(T_2(x), x) = \| r \|_\infty \leq \epsilon_3 \),

where

\[
(A^*, B^*) = \arg \max_{(A(\theta), B): \theta \leq \epsilon_1, \| B \|_{\infty, \infty} \leq \epsilon_2} \mathcal{L}(A(\theta) x + B, y).
\]

**Defense Setup.** We generate the perturbed samples needed for compound adversarial defense by

first performing a pixel-wise attack and then a spatial attack. Specifically, we provide two different

compound defenses, including 1) combining Worst-of-k on the top of AT (AT+), and 2) combining

spatial SRT on the top of pixel-wise SRT (SRT+).
Figure 4: The decision surfaces of SRT-based model and those of the standard training model in the CIFAR-10 dataset. In the 3D decision surfaces, the $X$ and $Y$ axis represent two different perturbation directions and its value indicates the perturbation size, while the $Z$-axis indicates the decision value. If and only if the decision value is positive, the prediction is correct. In those figures, the red area indicates the correctly classified region, while the blue area is the misclassified region.

Figure 5: The loss surfaces of SRT-based model and those of the standard training model in the MNIST dataset. In the 3D loss surface, the $X$ and $Y$ axis represent two different perturbation directions and its value indicates the perturbation size, while the $Z$-axis indicates the loss.

Figure 6: The loss surfaces of SRT-based model and those of the standard training model in the CIFAR-10 dataset. In the 3D loss surface, the $X$ and $Y$ axis represent two different perturbation directions and its value indicates the perturbation size, while the $Z$-axis indicates the loss.

**G Additional Decision and Loss Surfaces**

In this section, we present the visualization of decision surface in CIFAR-10 in Figure 4 and the visualizations of loss surfaces in both MNIST [31] and CIFAR-10 [30] database in Figures 5 and 6. As shown in Figure 4, the geometric properties of the decision surface in the CIFAR-10 dataset are very similar to those in MNIST. Specifically, compared with SRT, the decision surfaces of the standard training model have sharper peaks and larger slopes, which explains that its prediction is vulnerable to small perturbation. In contrast, the surfaces of SRT are flat and located on a plateau with positive decision value around the benign sample.
Moreover, as shown in Figures 5-6, the loss surfaces between SRT and those of the model with standard training also have different properties. Specifically, the loss surfaces of the standard training model have much higher peak and larger slopes, compared with the surfaces of SRT. Although the loss value is not directly related to the correctness of prediction compared with the decision value, the aforementioned difference still verifies the effectiveness of SRT.

Figure 7: The comparison between different types of compound adversarial defenses. The maximum perturbation size is the same as the one used in previous experiments. Perturbation fraction represents the ratio of the current perturbation size to the previous maximum perturbation size. **First row:** Results in CIFAR-10 dataset. **Second row:** Results in MNIST dataset.

### H Different Types of Compound Defenses

In this section, we compare two different types of the compound adversarial defense, including SRT+ and AT+ proposed in this paper, and those proposed in [18], dubbed AT (average) and AT (max). AT (average) and AT (max) respectively minimize the average error rate across perturbation types, or the error rate against an adversary that picks the worst perturbation type for each input. Specifically, for each training sample in the AT (average) and the AT (max), these methods build adversarial examples for all perturbation types and then train either on all examples or only the worst example. We conduct compound adversarial attacks, including GridAdv+ and PGD+, to evaluate the adversarial robustness of the compound adversarial defense methods. All training and evaluation settings are the same as those demonstrated in Section 4.4 of the main manuscript.

As shown in Figure 7, compared with the compound strategies proposed in [18], our methods reach better adversarial robustness under compound adversarial attacks. Moreover, in the case of compound defense, our method (SRT+) is also better than all other methods. Especially in the MNIST dataset, compared to the second best method, the AT+, our method achieves an increase of more than 7% in adversarial accuracy under PGD+ attack. This improvement is even more significant under GridAdv+ attack, which is more than 10%.