Learning Distributions from their Samples under Communication Constraints

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Abstract—We consider the problem of learning high-dimensional, nonparametric and structured (e.g. Gaussian) distributions in distributed networks, where each node in the network observes an independent sample from the underlying distribution and can use $k$ bits to communicate its sample to a central processor. We consider three different models for communication. Under the independent model, each node communicates its sample to a central processor by independently encoding it into $k$ bits. Under the more general sequential or blackboard communication models, nodes can share information interactively but each node is restricted to write at most $k$ bits on the final transcript. We characterize the impact of the communication constraint $k$ on the minimax risk of estimating the underlying distribution under $\ell_2$ loss. We develop minimax lower bounds that apply in a unified way to many common statistical models and reveal that the impact of the communication constraint can be qualitatively different depending on the tail behavior of the distribution under constraint. We study three different instances of this estimation problem:

1) High-dimensional discrete distributions: in this case we assume that $P = (p_1, \ldots, p_d)$ is a discrete distribution with known support size $d$ and $\mathcal{P}$ denotes the probability simplex over $d$ elements. By “high-dimensional” we mean that the support size $d$ of the underlying distribution may be comparable to the sample size $n$.

2) Non-parametric distributions: in this case $X_1, X_2, \ldots, X_n \sim f$, where $f$ is a density that possesses some standard Hölder continuity property [18].

3) Structured distributions: in this case, we assume that we have some additional information regarding the structure of the underlying distribution or density. In particular, we assume that the underlying distribution or density can be parametrized such that

$$X_1, X_2, \ldots, X_n \sim P_\theta$$

where $\theta \in \Theta \subset \mathbb{R}^d$. In this case, estimating the underlying distribution amounts to estimating the parameters of this distribution and we are interested in the following parameter estimation problem under squared $\ell_2$ risk

$$\inf_{\Pi} \inf_{\theta, \theta' \in \Theta} \sup_{P} \mathbb{E}_P \|\hat{\theta} - \theta\|^2_2,$$

where $\hat{\theta}(\cdot)$ is an estimator of $\theta$.

Statistical estimation in distributed settings has gained increasing popularity over the recent years motivated by the fact that modern data sets are often distributed across multiple machines and processors, and bandwidth and energy limitations in networks and within multiprocessor systems often impose significant bottlenecks on the performance of algorithms. There are also an increasing number of applications in which data is

I. INTRODUCTION

Estimating a distribution from samples is a fundamental unsupervised learning problem that has been studied in statistics since the late nineteenth century [19]. Consider the following distribution estimation model

$$X_1, X_2, \ldots, X_n \sim P$$

where we would like to estimate the unknown distribution $P$ under $\ell_2$ loss. Unlike the traditional statistical setting where samples $X_1, \ldots, X_n$ are available to the estimator as they are, in this paper we consider a distributed setting where each observation $X_i$ is available at a different node and has to be communicated to a central processor by using $k$ bits. We consider three different communication protocols:

1) Independent communication protocol $\Pi_{\text{ind}}$: each node sends a $k$-bit string $M_i$ simultaneously (independent of the other nodes) to the central processor and the final transcript is $Y = (M_1, \ldots, M_n)$;

2) Sequential communication protocol $\Pi_{\text{seq}}$: the nodes sequentially send a $k$-bit string $M_i$, where the quantization of the sample $X_i$ can depend on the previous messages $M_1, \ldots, M_{i-1}$;

3) Blackboard communication protocol $\Pi_{\text{BB}}$ [17]: all nodes communicate via a publicly shown blackboard while the total number of bits each node can write in the final transcript $Y$ is limited by $k$. When one node writes a message (bit) on the blackboard, all other nodes can see the content of the message and depending on the written bit another node can take the turn to write a message on the blackboard.

Upon receiving the transcript $Y$, the central processor produces an estimate $\hat{P}$ of the distribution $P$ based on the transcript $Y$ and known protocol $\Pi$ which can be of type $\Pi_{\text{ind}}, \Pi_{\text{seq}}$, or $\Pi_{\text{BB}}$. Our goal is to jointly design the protocol $\Pi$ and the estimator $\hat{P}(\cdot)$ so as to minimize the worst case squared $\ell_2$ risk, i.e., to characterize

$$\inf_{\Pi} \inf_{P} \sup_{\mathcal{P}} \mathbb{E}_P \|\hat{P} - P\|^2_2,$$

where $\mathcal{P}$ denotes the class of distributions which $P$ belongs to. We study three different instances of this estimation problem:

1) High-dimensional discrete distributions: in this case we assume that $P = (p_1, \ldots, p_d)$ is a discrete distribution with known support size $d$ and $\mathcal{P}$ denotes the probability simplex over $d$ elements. By “high-dimensional” we mean that the support size $d$ of the underlying distribution may be comparable to the sample size $n$.

2) Non-parametric distributions: in this case $X_1, X_2, \ldots, X_n \sim f$, where $f$ is a density that possesses some standard Hölder continuity property [18].

3) Structured distributions: in this case, we assume that we have some additional information regarding the structure of the underlying distribution or density. In particular, we assume that the underlying distribution or density can be parametrized such that

$$X_1, X_2, \ldots, X_n \sim P_\theta$$

where $\theta \in \Theta \subset \mathbb{R}^d$. In this case, estimating the underlying distribution amounts to estimating the parameters of this distribution and we are interested in the following parameter estimation problem under squared $\ell_2$ risk

$$\inf_{\Pi} \inf_{\theta, \theta' \in \Theta} \sup_{P} \mathbb{E}_P \|\hat{\theta} - \theta\|^2_2,$$

where $\hat{\theta}(\cdot)$ is an estimator of $\theta$.
generated in a distributed manner and the data (or features of it) are communicated over bandwidth-limited links to central processors. In particular, recent works [6], [12], [13] focus on a special case of the distributed parameter estimation problem described above, when the underlying distribution is known to have Gaussian structure, i.e., \( P_\theta = N(\theta, \sigma^2 I_d) \) with \( \sigma^2 \) known and \( \theta \in \Theta \subseteq \mathbb{R}^d \), often called the Gaussian location model. On the other hand, [4] focuses on the first two problems described above, distributed estimation of high-dimensional and non-parametric distributions, under \( \ell^1 \) loss.

In this paper, we approach all of these problems in a unified way by developing a framework that characterizes the Fisher information for estimating an underlying unknown parameter \( \theta \in \mathbb{R}^d \) from a \( k \)-bit quantization of a sample \( X \sim P_\theta \). Equivalently, we ask the question: how can we best represent \( X \) with \( k \) bits so as to maximize the Fisher information it provides about \( \theta \)? This framework was first introduced by the authors in [1, 2]. There has been some previous work in analyzing Fisher information from a quantized scalar random variable such as [7], [9], [10], [11]. Different from these works, here we consider the arbitrary quantization of a random vector and are able to study the impact of the quantization rate \( k \) along with the dimension \( d \) of the underlying statistical model on the Fisher information. As an application of our framework, we use upper bounds on Fisher information to derive lower bounds on the minimax risk of the distributed estimation problems we discuss above and recover many of the existing results in the literature [6], [12], [13], [4], which are known to be (order-wise) tight. Our technique is significantly simpler and more transparent than those in [6], [12], [13], [4]. In particular, the approach in [3], [4] is to convert the estimation problem to a hypothesis testing problem and involves a highly technical analysis of the divergence between distributions of the transcript under different hypotheses. The current paper recovers the results in [3] (with the exception of their result on the sparse Gaussian location model which closes a logarithmic gap left open in [12]) via the simpler approach that builds on the analysis of Fisher information from quantized samples. We also extend the results of [3] to find minimax lower bounds for statistical models with sub-exponential score functions, which is the case, for example, when we are interested in estimating not the mean but the variance of a Gaussian distribution.

A. Organization of the Paper

In the next section, we introduce the problem of characterizing Fisher information from a quantized sample. We present a geometric characterization for this problem and use it to derive two upper bounds on Fisher information as a function of the quantization rate. We evaluate these upper bounds for common statistical models. In Section III we formulate the problem of distributed learning of distributions under communication constraints with independent, sequential and blackboard communication protocols. We use the upper bounds on Fisher information from Section III to derive lower bounds on the minimax risk of distributed estimation of discrete and structured distributions. There we also provide a more detailed comparison of our results with those in the literature. Finally, in Section IV we discuss extending these results to non-parametric density estimation.

II. Fisher Information from a Quantized Sample

Let \( P_\theta \) be a family of probability measures on \( \mathcal{X} \) parameterized by \( \theta \in \Theta \subseteq \mathbb{R}^d \). Suppose each \( P_\theta \) is dominated by some base measure \( \nu \) and that each \( P_\theta \) has density \( f(x|\theta) \) with respect to \( \nu \). Let \( X \in \mathcal{X} \) be a single sample drawn from \( f(x|\theta) \). Any (potentially randomized) \( k \)-bit quantization strategy for \( X \) can be expressed in terms of the conditional probabilities

\[
b_m(x) = p(m|x) \quad \text{for} \quad m \in [1 : 2^k], \quad x \in \mathcal{X}.
\]

We assume that there is a well-defined joint probability distribution with density

\[
f(x, m|\theta) = f(x|\theta)p(m|x)
\]

and that \( p(m|x) \) is a regular conditional probability. For a given \( \Theta \subseteq \mathbb{R}^d \) and quantization strategy, denote the likelihood that the quantization \( M \) takes a specific value \( m \) by \( p(m|\theta) \). Let

\[
S_\theta(m) = (S_{\theta_1}(m), \ldots, S_{\theta_d}(m)) = \left( \frac{\partial}{\partial \theta_1} \log p(m|\theta), \ldots, \frac{\partial}{\partial \theta_d} \log p(m|\theta) \right)
\]

be the score of this likelihood. In an abuse of notation, we will also denote the score of the likelihood \( f(x|\theta) \) by

\[
S_\theta(x) = (S_{\theta_1}(x), \ldots, S_{\theta_d}(x)) = \left( \frac{\partial}{\partial \theta_1} \log f(x|\theta), \ldots, \frac{\partial}{\partial \theta_d} \log f(x|\theta) \right).
\]

The Fisher information matrix for estimating \( \theta \) from \( M \) is

\[
I_M(\theta) = \mathbb{E}[S_\theta(M)^T S_\theta(M)]
\]

and likewise the Fisher information matrix for estimating \( \theta \) from an \( X \) is

\[
I_X(\theta) = \mathbb{E}[S_{\theta}(X)^T S_{\theta}(X)].
\]

We will assume throughout that \( f(x|\theta) \) satisfies the following regularity conditions:

1. For each \( j \) and fixed \( \theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_d \), the function \( \sqrt{f(x|\theta)} \), thought of as a function of \( \theta_j \), is continuously differentiable with respect to \( \theta_j \) at \( \nu \)-almost every \( x \in \mathcal{X} \).
2. For each \( j \) and all \( \theta \), the expected value \( [I_X(\theta)]_{jj} = \mathbb{E}[S_{\theta_j}(X)^2] \) exists and is continuous in \( \theta_j \).
These two conditions justify interchanging differentiation and integration as in
\[
\frac{\partial}{\partial \theta_j} p(m|\theta) = \frac{\partial}{\partial \theta_j} \int f(x|\theta)p(m|x)d\nu(x)
= \int \frac{\partial}{\partial \theta_j} f(x|\theta)p(m|x)d\nu(x)
\]
for each \(j\), and they also ensure that \(p(m|\theta)\) is continuously differentiable with respect to each \(\theta_j\) (see Lemma 1, Section 26 in [14]). We will also assume, without loss of generality, that \(f(x|\theta) > 0\). For each fixed \(\theta\), this can be done by restricting the domain \(\mathcal{X}\) to only include those \(x\) values such that \(f(x|\theta) > 0\) when taking an expectation, or equivalently, by defining \(S_\theta(x) = 0\) whenever \(f(x|\theta) = 0\). In the same way we will assume that \(p(m|\theta) > 0\).

Our first two lemmas establish a geometric interpretation of the trace \(Tr(I_M(\theta))\). The first lemma is a slight variant of Theorems 1 and 2 from [5], and our debt to that work is clear.

**Lemma 1.** The \((i,i)\)-th entry of the Fisher information matrix \(I_M(\theta)\) is
\[
[I_M(\theta)]_{i,i} = \mathbb{E} \left[ \mathbb{E}[S_\theta(X)|M]^2 \right].
\]
The inner conditional expectation is with respect to the distribution \(f(x|\theta)\), while the outer expectation is over the conditioning random variable \(M\).

**Proof.** By Lemma 1,
\[
\sum_{i=1}^{d} [I_M(\theta)]_{i,i} = \sum_{i=1}^{d} \mathbb{E} \left[ \mathbb{E}[S_\theta(X)|M]^2 \right]
= \mathbb{E} \left[ \sum_{i=1}^{d} \mathbb{E}[S_\theta(X)|M]^2 \right]
= \mathbb{E} \left[ \mathbb{E}[S_\theta(X)|M]^2 \right]
= \sum_{m} p(m|\theta) \|\mathbb{E}[S_\theta(X)|m]\|^2.
\]

In order to get some geometric intuition for the quantity \(1\), consider a special case where the quantization is deterministic and the score function \(S_\theta(x)\) is a bijection. In this case, the quantization map partitions the space \(\mathcal{X}\) into disjoint quantization bins, and this induces a corresponding partitioning of the score functions values \(S_\theta(x)\). Each vector \(\mathbb{E}[S_\theta(X)|m]\) is then the centroid of the set of \(S_\theta(x)\) values corresponding to quantization bin \(m\) (with respect to the induced probability distribution on \(S_\theta(X)\)). Lemma 2 shows that \(Tr(I_M(\theta))\) is equal to the average magnitude squared of these centroid vectors.

**A. Upper Bounds on \(Tr(I_M(\theta))\)**

In this section, we give two upper bounds on \(Tr(I_M(\theta))\). The proofs appear in Appendix A. The first theorem upper bounds \(Tr(I_M(\theta))\) in terms of the variance of \(S_\theta(X)\) when projected onto any unit vector.

**Theorem 1.** If for any \(\theta \in \Theta\) and any unit vector \(u \in \mathbb{R}^d\),
\[
\text{var}(\langle u, S_\theta(X) \rangle) \leq I_0,
\]
then
\[
Tr(I_M(\theta)) \leq \min\{Tr(I_X(\theta)), 2^k I_0\}.
\]

The upper bound \(Tr(I_M(\theta)) \leq Tr(I_X(\theta))\) follows easily from the data processing inequality for Fisher information [11]. The theorem shows that when \(I_0\) is finite, \(Tr(I_M(\theta))\) can increase at most exponentially in \(k\).

Recall that for \(p \geq 1\), the \(\Psi_p\) Orlicz norm of a random variable \(X\) is defined as
\[
\|X\|_{\Psi_p} = \inf\{K \in (0, \infty) \mid \mathbb{E}[\Psi_p(\|X\|/K)] \leq 1\}
\]
where
\[
\Psi_p(x) = \exp(x^p) - 1.
\]

A random variable with finite \(p = 1\) Orlicz norm is sub-exponential, while a random variable with finite \(p = 2\) Orlicz norm is sub-Gaussian [15]. Our second theorem shows that when the \(\Psi_p\) Orlicz norm of the projection of \(S_\theta(X)\) onto any unit vector is bounded by some finite constant, \(Tr(I_M(\theta))\) can increase at most like \(k^{\frac{p}{2}}\).

**Theorem 2.** If for any \(\theta \in \Theta\), some \(p \geq 1\), and any unit vector \(u \in \mathbb{R}^d\),
\[
\|\langle u, S_\theta(X) \rangle\|_{\Psi_p} \leq N,
\]
then
\[
Tr(I_M(\theta)) \leq \min\{Tr(I_X(\theta)), 2^k I_0\}.
\]
\[
\text{Tr}(I_M(\theta)) \leq \min \{ \text{Tr}(I_X(\theta)), C N^2 k^2 \}
\]
where \( C = \frac{8}{\sigma^2} + 4 \).

B. Applications to Common Statistical Models

We next apply the above results to common statistical models. We will see that neither of these bounds is strictly stronger than the other and depending on the statistical model, one may yield a tighter bound than the other. The proofs of Corollaries 1 through 4 appear in Appendix B. In the next section, we show that Corollaries 3 and 4 yield tight results for the minimax risk of the corresponding distributed estimation problems.

**Corollary 1** (Gaussian location model). Consider the Gaussian location model \( X \sim \mathcal{N}(\theta, \sigma^2 I_d) \) where we are trying to estimate the mean \( \theta \) of a d-dimensional Gaussian random vector with fixed covariance \( \sigma^2 I_d \). In this case,

\[
\text{Tr}(I_M(\theta)) \leq \min \left\{ \frac{d}{\sigma^2}, C \frac{k}{\sigma^2} \right\}
\]

where

\[
C = \frac{8}{3} \left( \frac{8}{\epsilon^2} + 4 \right).
\]

The above corollary follows by showing that the score function associated with this model has finite \( p = 2 \) Orlicz norm and applying Theorem 2.

**Corollary 2** (variance of a Gaussian). Suppose \( X = (X_1, \ldots, X_d) \sim \mathcal{N}(0, \text{diag}(\theta_1, \ldots, \theta_d)) \) and \( \Theta \subseteq [\sigma_{\text{min}}^2, \sigma_{\text{max}}^2] \) with \( \sigma_{\text{max}} > \sigma_{\text{min}} > 0 \). In this case,

\[
\text{Tr}(I_M(\theta)) \leq \min \left\{ \frac{d}{2\sigma_{\text{min}}^2}, C \left( \frac{k}{\sigma_{\text{min}}^2} \right)^2 \right\}
\]

where \( C = \frac{4(\log 4 + 2(2 + \sqrt{2}))^2}{(\log 2)^2} \left( \frac{8}{\sigma^2} + 4 \right) \).

Corollary 2 similarly follows by showing that the score function associated with this model has finite \( p = 1 \) Orlicz norm and applying Theorem 2.

**Corollary 3** (distribution estimation). Suppose that \( X = \{1, \ldots, d+1\} \) and that

\[
f(x|\theta) = \theta_x.
\]

Let \( \theta_1, \ldots, \theta_d \) be the free parameters of interest and suppose they can vary from \( \frac{1}{d^2} \leq \theta_i \leq \frac{1}{d^2} \). In this case,

\[
\text{Tr}(I_M(\theta)) \leq 6 \min \{ d^2, d^2 k \}.
\]

Corollary 3 is a consequence of Theorem 1 along with characterizing the variance to the score function associated with this model.

**Corollary 4** (product Bernoulli model). Consider \( X = (X_1, \ldots, X_d) \sim \prod_{i=1}^d \text{Bern}(\theta_i) \). If \( \Theta = [1/2 - \epsilon, 1/2 + \epsilon]^d \) for some \( 0 < \epsilon < 1/2 \), i.e. the model is relatively dense, then

\[
\text{Tr}(I_M(\theta)) \leq C \min \{ d, k \}
\]

for some constant \( C \) that depends only on \( \epsilon \). If \( \Theta = [1/2 - \epsilon, 1/2 + \epsilon]^d \), i.e. the model is sparse, then

\[
\text{Tr}(I_M(\theta)) \leq \frac{2d}{\epsilon} \min \{ d, 2k \}.
\]

With the product Bernoulli model,

\[
S_{\theta_i}(x) = \begin{cases} \frac{1}{\alpha_i}, & x_i = 1 \\ \frac{1}{1-\alpha_i}, & x_i = 0 \end{cases}
\]

so that when \( \Theta = [1/2 - \epsilon, 1/2 + \epsilon]^d \), \( \text{var}(\langle u, S_\theta(X) \rangle) \) and \( \|\langle u, S_\theta(X) \rangle\|_2 \) are both \( \Theta(1) \). In this case, Theorem 1 gives

\[
\text{Tr}(I_M(\theta)) = O(2^k)
\]

while Theorem 2 gives

\[
\text{Tr}(I_M(\theta)) = O(k).
\]

In this situation Theorem 2 gives the better bound. On the other hand, if \( \Theta = [1/2 - \epsilon, 1/2 + \epsilon]^d \), then \( \text{var}(\langle u, S_\theta(X) \rangle) = \Theta(d) \) and \( \|\langle u, S_\theta(X) \rangle\|_2 \) is \( \Theta(d^2) \). In this case Theorem 1 gives

\[
\text{Tr}(I_M(\theta)) = O(d2^k)
\]

while Theorem 2 gives

\[
\text{Tr}(I_M(\theta)) = O(d^2k).
\]

In the sparse case \( \text{Tr}(I_X(\theta)) = \Theta(d^2) \), so only the bound from Theorem 1 is nontrivial. It is interesting that Theorem 2 is able to use the sub-Gaussian structure in the first case to yield a better bound – but in the second case, when the tail of the score function is essentially not sub-Gaussian, Theorem 1 yields the better bound.

III. DISTRIBUTED PARAMETER ESTIMATION

In this section, we focus on distributed estimation of parameters of an underlying distribution under communication constraints. Estimation of discrete distributions occurs as a special case of this problem. In the next section, we extend the discussion to distributed estimation of non-parametric distributions. The main technical exercise involves the application of Theorems 1 and 2 to statistical estimation with multiple quantized samples where the quantization of different samples can be independent or dependent as dictated by the communication protocol.

A. Problem Formulation

Let

\[
X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} P_\theta
\]

where \( \theta \in \Theta \subset \mathbb{R}^d \). We consider three different models for communicating each of these samples with \( k \) bits to a central processor that is interested to estimate the underlying parameter \( \theta \):

1. Independent communication protocol \( \Pi_{\text{ind}} \): under this model, each sample is independently quantized to \( k \) bits and communicated. Formally, each sample \( X_i \), for \( i = 1, \ldots, n \), is encoded to a \( k \)-bit string \( M_i \) by a
possibly randomized quantization strategy, denoted by $q_i(x_i) : \mathcal{X} \rightarrow [1 : 2^k]$, which can be expressed in terms of the conditional probabilities

$$p(m_i|x_i) \quad \text{for } m_i \in [1 : 2^k], \, x_i \in \mathcal{X}.$$  

(2) Sequential communication protocol $\Pi_{\text{Sec}}$: under this model, we assume that samples are communicated sequentially by broadcasting the communication to all nodes in the system including the central processor. Therefore, the quantization of the sample $X_i$ can depend on the previously transmitted quantized samples $M_1, \ldots, M_{i-1}$ corresponding to samples $X_1, \ldots, X_{i-1}$ respectively. Formally, each sample $X_i$, for $i = 1, \ldots, n$, is encoded to a $k$-bit string $M_i$ by a set of possibly randomized quantization strategies $\{q_{m_1,\ldots,m_{i-1}}(x_i) : \mathcal{X} \rightarrow [1 : 2^k] : m_1, \ldots, m_{i-1} \in [1 : 2^k]\}$, where each strategy $q_{m_1,\ldots,m_{i-1}}(x_i)$ can be expressed in terms of the conditional probabilities

$$p(m_i|x_i;m_1,\ldots,m_{i-1}) \quad \text{for } m_i \in [1 : 2^k] \text{ and } x_i \in \mathcal{X}.$$  

While these two models can be motivated by a distributed estimation scenario where the topology of the underlying network can dictate the type of the protocol (see Figure 1b) to be used, they can also model the quantization and storage of samples arriving sequentially at a single node. For example, consider a scenario where a continuous stream of samples is captured sequentially and each sample is stored in digital memory by using $k$ bits/sample. In the independent model, each sample would be quantized independently of the other samples (even though the quantization strategies for different samples can be different and jointly optimized ahead of time), while under the sequential model quantization of each sample $X_i$ would depend on the information $M_1, \ldots, M_{i-1}$ stored in the memory of the system at time $i$. This is illustrated in Figure 1a.

We finally introduce a third communication protocol that allows nodes to communicate their samples to the central processor in a fully interactive manner while still limiting the number of bits used per sample to $k$ bits. Under this model, each node can see the previously written bits on a public blackboard, and can use that information to determine its quantization strategy for subsequently transmitted bits. This is formally defined below.

(3) Blackboard communication protocol $\Pi_{\text{BB}}$: all nodes communicate via a publicly shown blackboard while the total number of bits each node can write in the final transcript $Y$ is limited by $k$ bits. When one node writes a message (bit) on the blackboard, all other nodes can see the content of the message. Formally, a blackboard communication protocol $\Pi_{\text{BB}}$ can be viewed as a binary tree (17), where each internal node $v$ of the tree is assigned a deterministic label $l_v \in [n]$ indicating the identity of the node to write the next bit on the blackboard if the protocol reaches tree node $v$; the left and right edges departing from $v$ correspond to the two possible values of this bit and are labeled by 0 and 1 respectively. Because all bits written on the blackboard up to the current time are observed by all nodes, the nodes can keep track of the progress of the protocol in the binary tree. The value of the bit written by node $l_v$ (when the protocol is at node $v$ of the binary tree) can depend on the sample $X_{l_v}$ observed by this node (and implicitly on all bits previously written on the blackboard encoded in the position of the node $v$ in the binary tree). Therefore, this bit can be represented by a function $b_v(x) = p_v(1|x) \in [0, 1]$, which we associate with the tree node $v$; node $l_v$ transmits 1 with probability $b_v(X_{l_v})$ and 0 with probability $1 - b_v(X_{l_v})$. Note that a proper labeling of the binary tree together with the collection of functions $\{b_v(\cdot)\}$ (where $v$ ranges over all internal tree nodes) completely characterizes all possible (possibly probabilistic) communication strategies for the nodes.

The $k$-bit communication constraint for each node can be viewed as a labeling constraint for the binary tree; for each $i \in [n]$, each possible path from the root node to a leaf node can visit exactly $k$ internal nodes with label $i$. In particular, the depth of the binary tree is $nk$ and there is one-to-one correspondence between all possible transcripts $y \in \{0, 1\}^{nk}$ and paths in the tree. Note that there is also one-to-one correspondence between $y \in \{0, 1\}^{nk}$ and the $k$-bit messages $m_1, \ldots, m_n$ transmitted by the $n$ nodes. In particular, the transcript $y \in \{0, 1\}^{nk}$ contains the same amount of information as $m_1, \ldots, m_n$, since given the transcript $y$ (and the protocol) one can infer $m_1, \ldots, m_n$ and vice versa (for this direction note that the protocol specifies the node to transmit first, so given $m_1, \ldots, m_n$ one can deduce the path followed in the protocol tree).
Under all three communication protocols above, the end goal is to produce an estimate \( \hat{\theta} \) of the underlying parameter \( \theta \) from the \( nk \) bit transcript \( Y \) or equivalently the collection of \( k \)-bit messages \( M_1, \ldots, M_n \) observed by the estimator. Note that the encoding strategies/protocols used in each case can be jointly optimized and agreed upon by all parties ahead of time. Formally, we are interested in the following parameter estimation problem under squared \( L^2 \) risk

\[
\inf_{\Pi} \inf_{\hat{\theta}} \sup_{\theta \in \Theta} \mathbb{E}_{\theta} \| \hat{\theta} - \theta \|_2^2,
\]

where \( \hat{\theta}(M_1, \ldots, M_n) \) is an estimator of \( \theta \) based on the quantized observations. Note that with an independent communication protocol, the messages \( M_1, \ldots, M_n \) are independent, while this is no longer the case with the sequential and blackboard protocols.

B. Main Results for Distributed Parameter Estimator

We next state our main results for distributed parameter estimation. We will show in the next subsection that the next two theorems can be applied to obtain tight lower bounds for distributed estimation under many common statistical models, including discrete distribution estimation and the Gaussian location model.

**Theorem 3.** Suppose \( \Theta = [-B, B]^d \). For any estimator \( \hat{\theta}(M_1, \ldots, M_n) \) and communication protocol \( \Pi \in \Pi_{\text{Ind}} \) or \( \Pi \in \Pi_{\text{Seq}} \), if \( S_0(X) \) satisfies the hypotheses in Theorem \ref{thm:seq} then

\[
\sup_{\theta \in \Theta} \mathbb{E}[\| \hat{\theta} - \theta \|_2^2] \geq \frac{d^2}{I_0 2kn + \frac{d\sigma^2}{\theta^2}},
\]

and if \( S_0(X) \) satisfies the hypotheses in Theorem \ref{thm:seq} then

\[
\sup_{\theta \in \Theta} \mathbb{E}[\| \hat{\theta} - \theta \|_2^2] \geq \frac{d^2}{C N^2 k^2 n + \frac{d\sigma^2}{\theta^2}},
\]

where \( C = \frac{8}{\sqrt{\pi}} + 4 \).

With the blackboard communication protocol we have the following slightly more restrictive theorem where we can prove the second bound only for the case \( p = 2 \).

**Theorem 4.** Suppose \( \Theta = [-B, B]^d \). For any estimator \( \hat{\theta}(Y) \) and communication protocol \( \Pi \in \Pi_{BB} \), if \( S_0(X) \) satisfies the hypotheses in Theorem \ref{thm:seq} then

\[
\sup_{\theta \in \Theta} \mathbb{E}[\| \hat{\theta} - \theta \|_2^2] \geq \frac{d^2}{I_0 2kn + \frac{d\sigma^2}{\theta^2}},
\]

and if \( S_0(X) \) satisfies the hypotheses in Theorem \ref{thm:seq} with \( p = 2 \) then

\[
\sup_{\theta \in \Theta} \mathbb{E}[\| \hat{\theta} - \theta \|_2^2] \geq \frac{d^2}{2N^2 kn + \frac{d\sigma^2}{\theta^2}}.
\]

We next show how Theorem \ref{thm:seq} by a rather straightforward application of the Van Trees Inequality combined with the conclusions of Theorems \ref{thm:ind} and \ref{thm:seq} The proof of Theorem \ref{thm:seq} is given in the Appendix \ref{app:seq}

**Proof of Theorem \ref{thm:seq}:** We are interested in the quantity

\[
I(M_1, \ldots, M_n)(\theta)
\]

under each model. We have

\[
Tr(I(M_1, \ldots, M_n)(\theta)) = \sum_{j=1}^d [I(M_1, \ldots, M_n)(\theta)]_{j,j}
\]

\[
= \sum_{i=1}^n \sum_{j=1}^d [I(M_i(M_1, \ldots, M_{i-1})(\theta))]_{j,j}
\]

\[
= \sum_{i=1}^n \sum_{m_1, \ldots, m_{i-1}} p(m_1, \ldots, m_{i-1})Tr(I(M_i(m_1, \ldots, m_{i-1})(\theta)))
\]

(4)

due to the chain-rule for Fisher information. Under the independent model,

\[
[I(M_i(m_1, \ldots, m_{i-1})(\theta))]_{j,j} = [I(M_i(\theta))]_{j,j}.
\]

Under the sequential model, conditioning on specific \( m_1, \ldots, m_{i-1} \) only effects the distribution \( p(m_i|\theta) \) by fixing the quantization strategy for \( X_i \). Formally, for the sequential model,

\[
\mathbb{P}(M_i = m_i|\theta; m_1, \ldots, m_{i-1}) = \mathbb{P}(q_{m_1, \ldots, m_{i-1}}(X_i) = m_i|\theta; m_1, \ldots, m_{i-1}) = \mathbb{P}(q_{m_1, \ldots, m_{i-1}}(X_i) = m_i|\theta),
\]

where the last step follows since \( X_1, \ldots, X_{i-1} \) is independent of \( X_i \) and therefore conditioning of \( m_1, \ldots, m_{i-1} \) does not change the distribution of \( X_i \). Since the bounds from Theorems \ref{thm:ind} and \ref{thm:seq} apply for any quantization strategy, they apply to each of the terms in (4), and the following statements hold under both quantization models:

(i) Under the hypotheses in Theorem \ref{thm:ind}

\[
Tr(I(M_1, \ldots, M_n)(\theta)) \leq nI_0 2^k.
\]

(ii) Under the hypotheses in Theorem \ref{thm:seq}

\[
Tr(I(M_1, \ldots, M_n)(\theta)) \leq nCN^2 k^{\frac{n}{2}}.
\]

Consider the squared error risk in estimating \( \theta \):

\[
\mathbb{E}[\| \hat{\theta} - \theta \|_2^2] = \sum_{i=1}^d [I(\theta_i - \hat{\theta}_i)^2].
\]

In order to lower bound this risk, we will use the van Trees inequality from \cite{vaneq}. Suppose we have a prior \( \mu_i \) for the parameter \( \theta_i \). For convenience denote \( M = (M_1, \ldots, M_n) \). The van Trees inequality for the component \( \theta_i \) gives

\[
\int_{-B}^B \mathbb{E}[(\hat{\theta}_i(M) - \theta_i)^2] \mu_i(\theta_i) d\theta_i 
\]

\[
\geq \frac{1}{\int_{-B}^B [I(M(\theta))]_{i,i} \mu_i(\theta_i) d\theta_i + I(\mu_i)}
\]

(5)

where \( I(\mu_i) = \int_{-B}^B \frac{\mu_i(\theta)^2}{\mu_i(\theta)} d\theta \) is the Fisher information from the prior. Note that the required regularity condition that
\[ \mathbb{E}[S_{\theta}(\lambda)] = 0 \] follows trivially since the expectation over \( \lambda \) is just a finite sum:

\[ \mathbb{E}[S_{\theta}(\lambda)] = \sum_{m} \frac{\partial}{\partial \theta} p(m|\theta) = \sum_{m} p(m|\theta) = 0. \]

The prior \( \mu_{\lambda} \) can be chosen to minimize this Fisher information and achieve \( I(\mu_{\lambda}) = \pi^{2}/B^{2} \). Let \( \mu(\theta) = \prod_{i} \mu_{\lambda}(\theta_i) \). By summing over each component,

\[ \int_{\Theta} \sum_{i=1}^{d} \mathbb{E}[(\theta_{i} - \hat{\theta}_{i})^{2}] \mu(\theta) d\theta \geq \sum_{i=1}^{d} \frac{1}{\mathbb{E}[I_{M}(\Theta)]_{i,i} \mu(\theta) d\theta + \frac{\pi^{2}}{B^{2}}} \]

\[ = d \sum_{i=1}^{d} \frac{1}{\mathbb{E}[I_{M}(\Theta)]_{i,i} \mu(\theta) d\theta + \frac{\pi^{2}}{B^{2}}} \geq d \sum_{i=1}^{d} \frac{1}{\mathbb{E}[I_{M}(\Theta)]_{i,i} \mu(\theta) d\theta + \frac{\pi^{2}}{B^{2}}} \geq d \sum_{i=1}^{d} \frac{1}{\mathbb{E}[I_{M}(\Theta)]_{i,i} \mu(\theta) d\theta + \frac{\pi^{2}}{B^{2}}} \]

\[ = \sum_{i=1}^{d} \mathbb{E}[I_{M}(\Theta)_{i,i} \mu(\theta) d\theta + \frac{\pi^{2}}{B^{2}}]. \]

Therefore,

\[ \sup_{\theta \in \Theta} \mathbb{E}[|\hat{\theta}(M) - \theta|^{2}] \geq d^{2} \sum_{i=1}^{d} \mathbb{E}[I_{M}(\Theta)_{i,i} \mu(\theta) d\theta + \frac{\pi^{2}}{B^{2}}]. \]

The inequality (7) follows from Jensen’s inequality via the convexity of \( x \rightarrow 1/x \) for \( x > 0 \), and the inequality (6) follows both from this convexity and (5). We could have equivalently used the multivariate version of the van Trees inequality (16) to arrive at the same result, but we have used the single-variable version in each coordinate instead in order to simplify the required regularity conditions.

Combining (8) with (i) and (ii) proves the theorem.

### C. Applications to Common Statistical Models

Using the bounds we developed in Section II-B, Theorems 3 and 4 give a lower bound on the minimax risk for the distribution estimation of \( \theta \) under common statistical models.

We summarize these results in the following corollaries:

**Corollary 5** (Gaussian location model). Let \( X \sim \mathcal{N}(\theta, \sigma^{2}I_{d}) \) with \( \Theta = [-B, B]^{d} \). For \( n \sigma^{2} \min\{k, d\} \geq d\sigma^{2} \), we have

\[ \sup_{\theta \in \Theta} \mathbb{E}[|\hat{\theta} - \theta|^{2}] \geq C \sigma^{2} \max \left\{ \frac{d^{2}}{n k}, \frac{d}{n} \right\} \]

for any communication protocol \( \Pi \) of type \( \Pi_{\text{Ind}}, \Pi_{\text{Seq}} \) or \( \Pi_{\text{BB}} \) and estimator \( \hat{\theta} \), where \( C > 0 \) is a universal constant independent of \( n, k, d, \sigma^{2}, B \).

Note that the condition \( n \sigma^{2} \min\{k, d\} \geq d\sigma^{2} \) in the above corollary is a weak condition that ensures that we can ignore the second term in the denominator of (5). For fixed \( B, \sigma \), this condition is weaker than just assuming that \( n \) is at least order \( d \), which is required for consistent estimation anyways. We will make similar assumptions in the subsequent corollaries.

The corollary recovers the results in [6], [13] (without logarithmic factors in the risk) and the corresponding result from [3] without the condition \( k \geq \log d \). A blackboard communication protocol that matches this result is given in [13].

**Corollary 6** (variance of a Gaussian). Let \( X \sim \mathcal{N}(0, \text{diag}(\theta_{1}, \ldots, \theta_{d})) \) with \( \Theta = [\sigma^{2}_{\min}, \sigma^{2}_{\max}]^{d} \). For \( n \left( \frac{\sigma^{2}_{\max} - \sigma^{2}_{\min}}{2} \right)^{2} \min\{k, d\} \geq d\sigma^{2}_{\min} \), we have

\[ \sup_{\theta \in \Theta} \mathbb{E}[|\hat{\theta} - \theta|^{2}] \geq C \sigma^{2}_{\min} \max \left\{ \frac{d^{2}}{n k^{2}}, \frac{d}{n} \right\} \]

for any communication protocol \( \Pi \) of type \( \Pi_{\text{Ind}}, \Pi_{\text{Seq}} \) or \( \Pi_{\text{BB}} \) and estimator \( \hat{\theta} \), where \( C > 0 \) is a universal constant independent of \( n, k, d \).

The bound in this Corollary 6 is new, and it is an unknown whether or not it is order optimal.

**Corollary 7** (distribution estimation). Suppose that \( X = \{1, \ldots, d + 1\} \) and that \( f(x|\theta) = \theta_{x} \).

Let \( \Theta \) be the probability simplex with \( d + 1 \) variables. For \( n \min\{2^{k}, d\} \geq d^{2} \), we have

\[ \sup_{\theta \in \Theta} \mathbb{E}[|\hat{\theta} - \theta|^{2}] \geq C \max \left\{ \frac{d}{n 2^{k}}, \frac{1}{n} \right\} \]

for any communication protocol \( \Pi \) of type \( \Pi_{\text{Ind}}, \Pi_{\text{Seq}} \) or \( \Pi_{\text{BB}} \) and estimator \( \hat{\theta} \), where \( C > 0 \) is a universal constant independent of \( n, k, d \).

This result recovers the corresponding result in [3] and matches the upper bound from the achievable scheme developed in [4] (when the performance of the scheme is evaluated under \( \ell^{2} \) loss rather than \( \ell^{1} \)).

**Corollary 8** (product Bernoulli model). Suppose \( X = (X_{1}, \ldots, X_{d}) \sim \prod_{i=1}^{d} \text{Bern}(\theta_{i}) \). If \( \Theta = [0, 1]^{d} \), then for \( n \min\{2^{k}, d\} \geq d^{2} \), we have

\[ \sup_{\theta \in \Theta} \mathbb{E}[|\hat{\theta} - \theta|^{2}] \geq C \max \left\{ \frac{d^{2}}{n k}, \frac{1}{n} \right\} \]

for any communication protocol \( \Pi \) of type \( \Pi_{\text{Ind}}, \Pi_{\text{Seq}} \) or \( \Pi_{\text{BB}} \) and estimator \( \hat{\theta} \), where \( C > 0 \) is a universal constant independent of \( n, k, d \).

If \( \Theta = \{(\theta_{1}, \ldots, \theta_{d}) \in [0, 1]^{d} : \sum_{i} \theta_{i} = 1\} \), then for \( n \min\{2^{k}, d\} \geq d^{2} \), we get instead

\[ \sup_{\theta \in \Theta} \mathbb{E}[|\hat{\theta} - \theta|^{2}] \geq C \max \left\{ \frac{d}{n 2^{k}}, \frac{1}{n} \right\}. \]

The corollary recovers the corresponding result from [3] and matches the upper bound from the achievable scheme presented in the same paper.
IV. DISTRIBUTED ESTIMATION OF NON-PARAMETRIC DENSITIES

Finally, we turn to the case where the $X_i$ are drawn independently from some probability distribution on $[0, 1]$ with density $f$ with respect to the uniform measure. We will assume that $\hat{f}$ is (uniformly) Hölder continuous with exponent $s$ and constant $L$, i.e. that
\[
|f(x) - f(y)| \leq L|x - y|^s
\]
for all $x, y \in [0, 1]$. Let $\mathcal{H}_2'(0, 1)$ be the space of all such densities. We are interested in characterizing the minimax risk $\inf_{\Pi} \inf_{f} \sup_{f \in \mathcal{H}_2'(0, 1)} \mathbb{E}\|f - \hat{f}\|_2^2$

where the estimators $\hat{f}$ are functions of the transcript $Y$. We have the following theorem.

**Theorem 5.** For any blackboard protocol $\Pi \in \Pi_{BB}$ and estimator $\hat{f}(Y)$,
\[
\sup_{f \in \mathcal{H}_2'(0, 1)} \mathbb{E}\|f - \hat{f}\|_2^2 \geq c \max\left\{n^{\frac{s}{2k+1}}, (n2^k)^{\frac{s}{2k+1}}\right\}.
\]

Moreover, this rate is achieved by an independent protocol so that
\[
\inf_{\Pi} \inf_{f} \sup_{f \in \mathcal{H}_2'(0, 1)} \mathbb{E}\|f - \hat{f}\|_2^2 \leq C \max\left\{n^{\frac{s}{2k+1}}, (n2^k)^{\frac{s}{2k+1}}\right\}
\]

where $C$ is a constant that depends only on $s, L$.

**Proof.** We start with the lower bound. Fix a bandwidth $h = 1/d$ for some integer $d$, and consider a parametric subset of the densities in $\mathcal{H}_2'(0, 1)$ that are of the form
\[
f_{P}(x) = 1 + \sum_{i=1}^{d} \frac{p_i - h}{h} g \left(\frac{x - x_i}{h}\right)
\]
where $P = (p_1, \ldots, p_d)$, $x_i = (i-1)h$, and $g$ is a smooth bump function that vanishes outside $[0, 1]$ and has $\int g(x)dx = 1$. The function $f_P$ is in $\mathcal{H}_2'(0, 1)$ provided that $|p_i - h| \leq c_0 h^{s+1}$ for some constant $c_0$. Let
\[
\mathcal{P} = \left\{ P : \sum_{i=1}^{d} p_i = 1, p_i \geq 0, |p_i - h| \leq c_0 h^{s+1} \right\}.
\]

For $P \in \mathcal{P}$ and any estimator $\hat{f}$ define $\hat{P} = (\hat{p}_1, \ldots, \hat{p}_d)$ by $\hat{p}_i = \int_{x_{i-1}}^{x_i} \hat{f}(x)dx$. We have the following chain of (inequalities):
\[
\mathbb{E}\|f_P - \hat{f}\|_2^2 = \mathbb{E}\left[ \int_0^1 (f_P(x) - f(x))^2 dx \right]
\]
\[
= \mathbb{E}\left[ \sum_{i=1}^{d} \int_{x_{i-1}}^{x_i} (f_P(x) - \hat{f}(x))^2 dx \right]
\]
\[
\geq d \mathbb{E}\left[ \sum_{i=1}^{d} \left( \int_{x_{i-1}}^{x_i} (f_P(x) - \hat{f}(x))dx \right)^2 \right]
\]
\[
= d \mathbb{E}\|P - \hat{P}\|^2.
\]

By the proofs of Theorem 1 and Corollary 7, the quantity in (9) can be upper bounded provided that $h$ is not too small. In particular we can pick $h^{s+1} = (n \min\{2^k, d\})^{-\frac{1}{2}}$ so that
\[
\sup_{P \in \mathcal{P}} \mathbb{E}\|f_P - \hat{f}\|_2^2 \geq c \max\{n^{\frac{s}{2k+1}}, (n2^k)^{\frac{s}{2k+1}}\}
\]
as desired.

For the achievability side note that
\[
\mathbb{E}\|f - \hat{f}\|_2^2 \leq 2\mathbb{E}\|f - f_h\|_2^2 + 2\mathbb{E}\|f_h - \hat{f}\|_2^2
\]

and $f_h$ can be chosen to be a piece-wise constant function of the form
\[
f_h(x) = \sum_{i=1}^{d} \frac{P_i}{h} 1(x \in [x_i, x_{i+1}])
\]

that satisfies
\[
\|f - f_h\|_2^2 \leq C_0 h^{2s}
\]

for a constant $C_0$ that depends only on $L$. Choosing $\hat{f}(x) = \sum_{i=1}^{d} \frac{P_i}{h} 1(x \in [x_i, x_{i+1}])$

for some $\hat{P} = (\hat{p}_1, \ldots, \hat{p}_d)$ we get
\[
\mathbb{E}\|f - \hat{f}\|_2^2 \leq 2C_0 h^{2s} + 2d\mathbb{E}\|P - \hat{P}\|_2^2.
\]

By using the estimator $\hat{P}$ (along with the specific communication protocol) from the achievability scheme in 3 for discrete distribution estimation, we are left with
\[
\mathbb{E}\|f - \hat{f}\|_2^2 \leq 2C_0 h^{2s} + C_1 \frac{d^2}{n \min\{2^k, d\}}.
\]

Optimizing over $h$ by setting $h^{2(s+1)} = (n \min\{2^k, d\})^{-1}$ gives the final result.

\[\square\]

APPENDIX

A. Proofs of Theorems 1 and 2

Consider some $m$ and fix its likelihood $t = p(m|\theta)$. We will proceed by upper-bounding $\mathbb{E}[S_\theta(X)|m]$ from the right-hand side of (1). Note that
\[
\mathbb{E}[S_\theta(X)|m] = \mathbb{E}[S_\theta(X)b_m(X)]
\]

where $b_m(x) = t$ and $0 \leq b_m(x) \leq 1$ for all $x \in X$. We use $\langle \cdot, \cdot \rangle$ to denote the usual inner product.

For some fixed $m$ and $t = p(m|\theta)$, let $U$ be a $d$-by-$d$ orthogonal matrix with columns $u_1, u_2, \ldots, u_d$ and whose first column is given by
\[
u_1 = \frac{1}{\mathbb{E}[S_\theta(X)|m]} \mathbb{E}[S_\theta(X)|m].
\]

We have
\[
t\mathbb{E}[S_\theta(X)|m] = \int S_\theta(x)b_m(x)f(x|\theta)d\nu(x)
\]
\[
= \int \left( \sum_{i=1}^{d} u_i \langle u_i, S_\theta(x) \rangle \right) b_m(x)f(x|\theta)d\nu(x)
\]
\[
= \sum_{i=1}^{d} \left( \int \langle u_i, S_\theta(x) \rangle b_m(x)f(x|\theta)d\nu(x) \right) u_i
\]

and since $u_2, \ldots, u_d$ are all orthogonal to $E[S_0(X)|m]$, 

$$E[S_0(X)|m] = \frac{1}{t} \left( \int \langle u_1, S_0(x) \rangle b_m(x) f(x|\theta) dx \right) u_1 .$$

Therefore, 

$$||E[S_0(X)|m]|| = \frac{1}{t} E[(u_1, S_0(x)) b_m(x)] .$$

1) Proof of Theorem 7. To finish the proof of Theorem 1, note that the upper bound $Tr(I_M(\theta)) \leq Tr(I_X(\theta))$ follows easily from the data processing inequality for Fisher information [11]. Using (10) and the Cauchy-Schwarz inequality, 

$$t ||E[S_0(X)|m]||^2 = \frac{1}{t} \left( E[(u_1, S_0(X)) b_m(x)] \right)^2 
\leq \frac{1}{t} \left( E[(u_1, S_0(X))^2] E[b_m(x)^2] \right) 
= E[(u_1, S_0(X))^2] .$$

So if var$(u_1, S_0(X)) \leq I_0$, then because score functions have zero mean, 

$$t ||E[S_0(X)|m]||^2 \leq I_0 .$$

Therefore by Lemma 2, 

$$Tr(I_M(\theta)) \leq 2k I_0 .$$

2) Proof of Theorem 2. Turning to Theorem 2, we now assume that for some $p \geq 1$ and any unit vector $u \in \mathbb{R}^d$, the random vector $(u, S_0(X))$ has finite $\Psi_p$ norm less than or equal to $N$. For $p = 1$ or $p = 2$, this is the common assumption that $S_0(X)$ is sub-exponential or sub-Gaussian, respectively, as a vector.

In particular $(u_1, S_0(X))$ has $||\langle u_1, S_0(X) \rangle \psi_p \leq N$, and 

$$2 \geq E[\exp(|\langle u_1, S_0(X) \rangle/N)^p]) 
\geq E[b_m(X) \exp(|\langle u_1, S_0(X) \rangle/N)^p]) 
\geq \epsilon \exp(|\langle u_1, S_0(X) \rangle/N)^p)|m) 
\geq \epsilon \exp \left( \frac{1}{N} E[\langle u_1, S_0(X) \rangle|m] \right)^p 
\geq \epsilon \exp \left( \frac{1}{N} E[\langle u_1, S_0(X) \rangle|m] \right)^p ,$$

so that 

$$E[\langle u_1, S_0(X) \rangle|m] \leq N \left( \log \left( \frac{2}{\epsilon} \right) \right)^{1/p} .$$

Therefore by (10), 

$$||E[S_0(X)|m]|| \leq N \left( \log \left( \frac{2}{\epsilon} \right) \right)^{1/p} .$$

By Lemma 2 

$$Tr(I_M(\theta)) = \sum_m p(m|\theta) ||E[S_0(X)|m]||^2 ,$$

and therefore by (11) 

$$Tr(I_M(\theta)) \leq \sum_m N^2 p(m|\theta) \left( \log \left( \frac{2}{p(m|\theta)} \right) \right)^{2/p} .$$

To bound this expression we will use the following properties: 

(i) $x \mapsto (\log x)^{2/p}$ is concave for $0 \leq x \leq 2e^{p/2}$ 

(ii) $(\log x)^{2/p} \leq 2e^{-\frac{p}{2}x}$ for $0 \leq x \leq 1 

(iii) $\sum_m p(m|\theta) = 1$ 

There can be at most one $m$-value such that $p(m|\theta) \geq 2e^{p/2}$, so we’ll call such a term $m_0$, separate it out, and treat it separately.

$$\sum_m p(m|\theta) \left( \log \left( \frac{2}{p(m|\theta)} \right) \right)^{2/p} 
\leq 2e^{-\frac{p}{2} \left( \frac{2}{p} \right)^{2/p} \sum_m p(m|\theta) \left( \log \left( \frac{2}{p(m|\theta)} \right) \right)^{2/p} 
\leq 2e^{-\frac{p}{2} \left( \frac{2}{p} \right)^{2/p} + \sum_m p(m|\theta) \left( \log \left( \frac{2}{p(m|\theta)} \right) \right)^{2/p} 
\leq 2e^{-\frac{p}{2} \left( \frac{2}{p} \right)^{2/p} + \sum_m p(m|\theta) \left( \log \left( \frac{2}{p(m|\theta)} \right) \right)^{2/p} 
\leq \frac{8}{c^2} + (k + 1)^{2/p} .$$

In (12) we separate out the possible $m_0$ such that $p(m_0|\theta) \geq 2e^{p/2}$ and then use property (ii) to bound that specific term. Then (13) follows from Jensen’s inequality and the concavity described in property (i). Setting $C$ large enough so that (14) is less than or equal to $Ck^{2/p}$ completes the proof.

B. Proof of Corollaries 7 through 3

1) Proof of Corollary 7. The score function for the Gaussian location model is 

$$S_0(x) = \frac{1}{\sigma^2} (x - \theta)$$

so that $S_0(X) \sim N \left( 0, \frac{1}{\sigma^2} I_d \right)$. Therefore $(u, S_0(X))$ is a mean-zero Gaussian with variance $1/\sigma^2$ for any unit vector $u \in \mathbb{R}^d$. Hence, the $\Psi_2$ norm of the projected score function vector is 

$$||\langle u, S_0(X) \rangle||_{\psi_2} = \frac{1}{\sigma} \sqrt{\frac{8}{3}}$$

since it can be checked that 

$$\int \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} dx = 2$$

when $c = \frac{1}{\sigma} \sqrt{\frac{\pi}{2}}$. By Theorem 2, 

$$Tr(I_M(\theta)) \leq \min \left\{ \frac{d}{\sigma^2}, C, \frac{k}{\sigma^2} \right\}$$

where 

$$C = \left( \frac{8}{c^2} + 4 \right)^{\frac{8}{3}} .$$
2) Proof of Corollary \[2\] The components of the score function are
\[
S_\theta(x) = \frac{x^2}{2\theta^2} - \frac{1}{2\theta}.
\]
Therefore each independent component \(S_{\theta_i}(X)\) is a shifted, scaled version of a chi-squared distributed random variable with one degree of freedom. The \(\Psi_1\) norm of each component \(S_{\theta_i}(X)\) can be bounded by
\[
\|S_{\theta_i}(X)\|_{\Psi_1} \leq \frac{2}{\sigma_{\min}^2}.
\]  (15)
This follows since the pdf of \(Y = S_{\theta_i}(X)\) is
\[
f_Y(y) = \frac{2\theta}{\sqrt{2\pi}} \exp\left(-\left(\frac{1}{\theta} y + \frac{1}{2}\right)\right)
\]
for \(y \geq -\frac{1}{2\theta}\), and we have
\[
\int_{-\frac{1}{2\theta}}^{\infty} \frac{2\theta}{\sqrt{2\pi}} \exp\left(-\left(\frac{1}{\theta} y + \frac{1}{2}\right)\right) dy = \sqrt{\frac{2\pi}{\theta^2}}
\]
By picking \(K = \frac{2}{\theta}\) and using the identity
\[
\int_{0}^{\infty} \frac{1}{\sqrt{y}} e^{-xy} dy = \sqrt{\pi/c}
\]
for \(c > 0\), we get
\[
\mathbb{E}\left[\left|\frac{Y}{K}\right|\right] \leq \sqrt{2e}\frac{1}{\sqrt{\theta}} \leq 2.
\]
This proves (15). We next turn our attention to bounding
\[
\|\langle u, S_\theta(X)\rangle\|_{\Psi_1}
\]
for any unit vector \(u = (v_1, \ldots, v_d) \in \mathbb{R}^d\). To this end note that Markov’s inequality implies
\[
\mathbb{P}(|Y| \geq t) = \mathbb{P}\left(e^{\frac{|Y|}{K}} \geq e^t\right) \leq 2e^{-\frac{t}{\theta}}
\]  (16)
and we have the following bound on the moments of \(Y\):
\[
\mathbb{E}[|Y|^p] = \int_{0}^{\infty} \mathbb{P}(|Y| \geq t) t^{p-1} dt \leq \int_{0}^{\infty} 2e^{-\frac{t}{\theta}} t^{p-1} dt \leq 2p \int_{0}^{\infty} e^{-\frac{t}{\theta}} t^{p-1} dt = 2pK^p(p-1)!.
\]  (17)
This leads to a bound on the moment generating function for \(Y\) as follows:
\[
\mathbb{E}\left[e^{\frac{Y}{K}}\right] \leq 1 + \mathbb{E}\left[\frac{1}{p!} \left(\frac{v_i Y}{K}\right)^p\right] 
\]
\[
\leq 1 + \sum_{p=2}^{\infty} 2|v_i|^p - 1 + \frac{2\sigma_i^2}{1 - |v_i|} \leq e^{2\sigma_i^2}
\]  (18)
for \(|v_i| < 1\). Applying (18) to the moment generating function for \(\langle u, S_\theta(X)\rangle\) gives
\[
\mathbb{E}\left[\exp\sum_{i=1}^{d} \frac{v_i S_{\theta_i}(X)}{K}\right] = \exp\frac{\sum_{i=1}^{d} v_i S_{\theta_i}(X)}{K} \prod_{i \neq i_0} \frac{v_i S_{\theta_i}(X)}{K} \leq 2e^{2(2+\sqrt{2})}(20)
\]
and by exponentiating and using Jensen’s inequality,
\[
\mathbb{E}\left[\exp\sum_{i=1}^{d} \frac{v_i S_{\theta_i}(X)}{K}\right] \leq \left(\mathbb{E}\left[\exp\left|\sum_{i=1}^{d} \frac{v_i S_{\theta_i}(X)}{K}\right|\right]\right)^\alpha \leq (4e^{2(2+\sqrt{2})})^\alpha = 2
\]  (21)
where \(\alpha = \frac{\log 2}{\log 4 + 2(2+\sqrt{2})}\). We see that
\[
\|\langle u, S_\theta(X)\rangle\|_{\Psi_1} \leq \frac{2}{\sigma_{\min}^2\alpha}
\]
and using Theorem [2]
\[
Tr(I_M(\theta)) \leq \frac{4(\log 4 + 2(2+\sqrt{2})^2)\left(\frac{8}{e^2} + 4\left(\frac{k}{\sigma_{\min}^2}\right)^2\right)}{(\log 2)^2}.
\]
For the other bound note that \(\text{var}(S_{\theta_i}(X)) = \frac{1}{2\theta_i}\), so that
\[
Tr(I_X(\theta)) \leq \frac{d}{2\sigma_{\min}^4}.
\]
3) Proof of Corollary \[3\] Note that
\[
\theta_{d+1} = 1 - \sum_{i=1}^{d} \theta_i
\]
and
\[
S_{\theta_i}(x) = \begin{cases} \frac{1}{\theta_i}, & x = i \\ -\frac{1}{\theta_{i+1}}, & x = d + 1 \\ 0, & \text{otherwise} \end{cases}
\]
for \( i = 1, \ldots, d \). Recall that by assumption we are restricting our attention to \( \frac{1}{2d} \leq \theta_i \leq \frac{1}{2d} \) for \( i = 1, \ldots, d \). For any unit vector \( u = (v_1, \ldots, v_d) \),

\[
\text{var}((u, S_\theta(X))) = \frac{d+1}{\theta_x} \left( \sum_{i=1}^{d} v_i S_{\theta_i}(x) \right)^2 \\
= \theta_{d+1} \frac{1}{\theta_x^2} \left( \sum_{i=1}^{d} v_i^2 \right)^2 + \sum_{x=1}^{d} \theta_x \left( \sum_{i=1}^{d} v_i S_{\theta_i}(x) \right)^2 \\
\leq 2d + \sum_{x=1}^{d} \theta_x v_x^2 \frac{1}{\theta_x^2} \leq 6d.
\]

Finally by Theorem \( [1] \)

\[
Tr(I_M(\theta)) \leq 6 \min\{d^2, d^2 \kappa \}.
\]

4) Proof of Corollary \( [4] \) With the product Bernoulli model

\[
S_{\theta_i}(x) = \begin{cases} \frac{1}{\theta_i}, & x_i = 1 \\ \frac{1}{1-\theta_i}, & x_i = 0 \end{cases},
\]

so that in the case \( \Theta = [1/2 - \epsilon, 1/2 + \epsilon]^d \),

\[
\mathbb{E} \left[ \exp \left( \frac{S_{\theta_i}(X)}{K} \right)^2 \right] \leq \exp \left( \frac{1}{(\frac{1}{2} - \epsilon)K} \right)^2
\]

and

\[
\|S_{\theta_i}(X)\|_2 \leq \frac{1}{(\frac{1}{2} - \epsilon)\sqrt{\log 2}}.
\]

By the rotation invariance of the \( \Psi_2 \) norm \( [15] \), this implies

\[
\| \langle u, S_{\theta_i}(X) \rangle \|_2 \leq \frac{C}{(\frac{1}{2} - \epsilon)\sqrt{\log 2}}
\]

for some absolute constant \( C \). Thus by Theorem \( [2] \)

\[
Tr(I_M(\theta)) \leq \left( \frac{8}{\epsilon^2} + 4 \right) \left( \frac{C}{(\frac{1}{2} - \epsilon)\sqrt{\log 2}} \right)^2 k.
\]

For the other bound,

\[
Tr(I_X(\theta)) \leq d \left( \frac{1}{2} - \epsilon \right)^2.
\]

Now consider the sparse case \( \Theta = [(\frac{1}{2} - \epsilon)\frac{1}{2}, (\frac{1}{2} + \epsilon)\frac{1}{2}]^d \). We have

\[
\text{var} \left( S_{\theta_i}(X) \right) = \frac{1}{\theta_i^2} - \frac{1}{(1 - \theta_i)^2} (1 - \theta_i)
\]

\[
= \frac{1}{\theta_i^2} - \frac{1}{1 - \theta_i} \leq \frac{2d}{\frac{1}{2} - \epsilon},
\]

and therefore by independence,

\[
\text{var} \left( \langle u, S_{\theta_i}(X) \rangle \right) \leq \frac{2d}{\frac{1}{2} - \epsilon}.
\]

Thus by Theorem \( [1] \)

\[
Tr(I_M(\theta)) \leq \frac{2d}{\frac{1}{2} - \epsilon} \min\{d, 2\kappa \}.
\]

C. Proof of Theorem \( [2] \)

In order to bound the minimax risk under the blackboard model, we will proceed by writing down the Fisher information from the transcript \( Y \) that is described in Section III. Let \( b_{v,y}(x_{I_v}) = b_{v,y}(x_{I_v}) \) if the path \( \tau(y) \) takes the “1” branch after node \( v \), and \( b_{v,y}(x_{I_v}) = 1 - b_{v,y}(x_{I_v}) \) otherwise. The probability distribution of \( Y \) can be written as

\[
P(Y = y) = \mathbb{E} \left[ \prod_{v \in \tau(y)} b_{v,y}(X_{I_v}) \right]
\]

so that by the independence of the \( X_i \),

\[
P(Y = y) = \prod_{i=1}^{n} E \left[ \prod_{v \in \tau(y): i = v} b_{v,y}(X_i) \right]
\]

\[
= \prod_{i=1}^{n} E [p_{v,y}(X_i)]
\]

where \( p_{v,y}(x_i) = \prod_{v \in \tau(y): i = v} b_{v,y}(x_i) \). The score for component \( \theta_i \) is therefore

\[
\frac{\partial}{\partial \theta_i} \log \mathbb{P}(Y = y) = \sum_{j=1}^{n} \mathbb{E} [S_{\theta_j}(X_j)p_{j,y}(X_j)] \frac{\mathbb{E}[S_{\theta_j}(X_j)p_{j,y}(X_j)]}{\mathbb{E}[p_{j,y}(X_j)]}.
\]

The Fisher information from \( Y \) for estimating the component \( \theta_i \) is then

\[
\mathbb{E} \left[ \left( \frac{\partial}{\partial \theta_i} \log \mathbb{P}(Y = y) \right)^2 \right] = \sum_{j,k,y} \mathbb{P}(Y = y) \mathbb{E} [S_{\theta_j}(X_j)p_{j,y}(X_j)] \mathbb{E} [S_{\theta_k}(X_k)p_{k,y}(X_k)] \mathbb{E}[p_{j,y}(X_j)] \mathbb{E}[p_{k,y}(X_k)].
\]

Note that when \( j \neq k \) the terms within this summation are zero:

\[
\sum_{y} \mathbb{P}(Y = y) \mathbb{E} [S_{\theta_j}(X_j)p_{j,y}(X_j)] \mathbb{E} [S_{\theta_k}(X_k)p_{k,y}(X_k)] \mathbb{E}[p_{j,y}(X_j)] \mathbb{E}[p_{k,y}(X_k)]
\]

\[
= \mathbb{E} [S_{\theta_j}(X_j)S_{\theta_k}(X_k) \sum_{y} \prod_{l=1}^{n} p_{l,y}(X_l)]
\]

\[
= \mathbb{E} [S_{\theta_j}(X_j)S_{\theta_k}(X_k) 0].
\]

The step in \( (22) \) follows since \( \prod_{l=1}^{n} p_{l,y}(x_l) \) describes the probability that \( Y = y \) for fixed samples \( x_1, \ldots, x_n \), and thus \( \sum_{y} \prod_{l=1}^{n} p_{l,y}(x_l) = 1 \).

Returning to the Fisher information from \( Y \) we have that

\[
\sum_{i=1}^{d} \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta_i} \log \mathbb{P}(Y = y) \right)^2 \right] = \sum_{y} \mathbb{P}(Y = y) \sum_{j,d} \left( \mathbb{E}[S_{\theta_j}(X_j)p_{j,y}(X_j)] \mathbb{E}[p_{j,y}(X_j)] \mathbb{E}[p_{j,y}(X_j)] \right)^2.
\]

Let \( E_{j,y} \) denote taking expectation with respect to the new density

\[
p_{j,y}(x_j)f(x_j) \frac{\mathbb{E}[S_{\theta_j}(X_j)p_{j,y}(X_j)]}{\mathbb{E}[p_{j,y}(X_j)]}.
\]
Now we can simplify (23) as
\[
\sum_{i=1}^{d} \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta_i} \log \mathbb{P}(Y = y) \right)^2 \right] = \sum_{y} \mathbb{P}(Y = y) \sum_{j} \| E_{j,y}[S_\theta(X_j)] \|^2 . \tag{24}
\]

(i) Suppose that \( \| (u, S_\theta(X)) \|_2 \leq N \) for any unit vector \( u \in \mathbb{R}^d \). Then letting
\[
u = \frac{E_{j,y}[S_\theta(X)]}{\| E_{j,y}[S_\theta(X)] \|} \]
we have
\[
2 \geq \mathbb{E}[\exp((\| (u, S_\theta(X)) \|/N)^2)] \\
\geq \mathbb{E}[E_{j,y}(X) \exp((\| (u, S_\theta(X)) \|/N)^2]) \\
\geq \mathbb{E}[E_{j,y}(X) \| E_{j,y}[\exp((\| (u, S_\theta(X)) \|/N)^2)] \\
\geq \mathbb{E}[E_{j,y}(X) \exp(\left( \frac{1}{N} \mathbb{E}_{j,y}[\| (u, S_\theta(X)) \|^2] \right)) \\
\geq \mathbb{E}[E_{j,y}(X) \exp(\left( \frac{1}{N} \mathbb{E}_{j,y}[\| (u, S_\theta(X)) \|^2 \right))
\]
and
\[
\| E_{j,y}[S_\theta(X_j)] \| \leq N \left( \log \frac{2}{\mathbb{E}[E_{j,y}(X) \|]} \right)^{\frac{1}{2}} .
\]
Continuing from (24),
\[
\sum_{i=1}^{d} \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta_i} \log \mathbb{P}(Y = y) \right)^2 \right] \leq N^2 \sum_{y} \mathbb{P}(Y = y) \sum_{j} \log \frac{2}{\mathbb{E}[E_{j,y}(X) \|]} \\
= N^2 \sum_{y} \mathbb{P}(Y = y) \left( \log \frac{1}{\mathbb{P}(Y = y)} + n \log 2 \right) .
\tag{25}
\]
Finally, by the concavity of \( x \mapsto x \log \frac{1}{x} \),
\[
\sum_{i=1}^{d} \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta_i} \log \mathbb{P}(Y = y) \right)^2 \right] \leq N^2 (nk + n \log 2) \\
\leq 2N^2 nk . \tag{26}
\]
(ii) Now suppose we instead just have the finite variance condition that
\[
\text{var}(\langle u, S_\theta(X) \rangle) \leq I_0
\]
for any unit vector \( u \in \mathbb{R}^d \). Picking again
\[
u = \frac{E_{j,y}[S_\theta(X)]}{\| E_{j,y}[S_\theta(X)] \|} ,
\]
the Cauchy-Schwarz inequality implies
\[
\mathbb{E}[E_{j,y}(X) \| E_{j,y}[S_\theta(X)] \|^2 \| E_{j,y}[S_\theta(X)] \|^2] \\
= \frac{1}{\mathbb{E}[E_{j,y}(X) \|]} \mathbb{E}[E_{j,y}(X) \| E_{j,y}[S_\theta(X)] \|^2] \\
\leq \frac{1}{\mathbb{E}[E_{j,y}(X) \|]} \mathbb{E}[E_{j,y}(X) \| E_{j,y}(X) \|^2] \\
\leq \frac{1}{\mathbb{E}[E_{j,y}(X) \|]} \mathbb{E}[E_{j,y}(X) \| E_{j,y}(X) \|^2] \\
= \mathbb{E}[E_{j,y}(X) \| E_{j,y}(X)] .
\]
Together with (24), this gives
\[
\sum_{i=1}^{d} \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta_i} \log \mathbb{P}(Y = y) \right)^2 \right] \\
\leq \sum_{y} \mathbb{P}(Y = y) \sum_{j} \| E_{j,y}[S_\theta(X_j)] \|^2 \\
\leq \sum_{j} I_0 \prod_{i \neq j} \mathbb{E}[E_{i,y}(X)] \\
= I_0 n I_0 .
\]
The last equality follows from the nontrivial fact that
\[
\sum_{y} \pi_{i \neq j} \mathbb{E}[E_{i,y}(X)] = 2^k \text{ for each } j \text{ (see (3)).}
\]
In both the sub-Gaussian (i) and finite variance (ii) cases, we apply the van Trees inequality just as in Theorem 3 to arrive at the final result.

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