A RISK-SENSITIVE PORTFOLIO OPTIMIZATION PROBLEM WITH FIXED INCOMES SECURITIES

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Abstract. We discuss a class of risk-sensitive portfolio optimization problems. We consider the portfolio optimization model investigated by Nagai in 2003. The model by its nature can include fixed income securities as well in the portfolio. Under fairly general conditions, we prove the existence of optimal portfolio in both finite and infinite horizon problems.

Key words. Risk-sensitive control, fixed income securities, non-stationary optimal strategies.

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1. Introduction

In this paper, we study a class of portfolio optimization problems in continuous trading framework where the returns of the individual assets are explicitly being affected by underlying economic factors. The continuous time portfolio management has its origin in the pioneering work of Merton, see [15, 16]. Since then there were several contributions to the stochastic control applications to portfolio management, see [12, 13] for details. But most of these works deal with equities. Literature on portfolio optimization with fixed income assets is limited. A stochastic control model suitable for fixed income assets was first formulated by Merton [15]. Bielecki and Pliska in [3] and later in [4], investigated the following linear version of Merton’s model [15] with risk-sensitive criterion,

\[
\begin{align*}
\frac{dS_i(t)}{S_i(t)} &= (a + AX(t))_i dt + \sum_{k=1}^{m+n} \sigma_{ik} dW_k(t), \quad S_i(0) = s_i, \quad i = 1, 2, \ldots, m, \\
\frac{dX(t)}{X(t)} &= (b + BX(t)) dt + \Lambda dW(t), \quad X(0) = x,
\end{align*}
\]

where \( S_i(t) \) denote the price of \( i \)th security and \( X_j(t) \) the level of the \( j \)th factor at time \( t \) and \( W(\cdot) \) is an \( \mathbb{R}^{m+n} \) valued standard Brownian motion with components \( W_k(\cdot) \). In [4], authors improved their earlier work [3] by relaxing the assumption \( \Sigma \Lambda^\perp = 0 \). Hence, the portfolio model become capable of incorporating fixed income securities such as rolling horizon bonds (it is a portfolio of bonds).

Also Nagai in [17], considered the following general diffusion model and addressed the portfolio optimization problem with risk-sensitive criterion.
They assumed that the set of securities includes one bond, whose price is defined by the ODE:

\[
dS_0(t) = r(X(t))S_0(t)dt, \quad S_0(0) = s_0,
\]

where \( r(\cdot) \) is a nonnegative bounded function. The other security prices \( S_i(\cdot), \; i = 1, 2, \ldots, m \) and factors \( X(\cdot) \) are assumed to satisfy the SDEs

\[
\begin{cases}
    dS_i(t) = S_i(t)[g_i(X(t))dt + \sum_{k=1}^{m+n} \sigma_{ik}(X(t))dW_k(t)], \\
    S_i(0) = s_i, \quad i = 1, 2, \ldots, m, \\
    dX(t) = b(X(t))dt + \lambda(X(t))dW(t), \\
    X(0) = x \in \mathbb{R}^n.
\end{cases}
\]

Nagai proved the existence of optimal portfolios under the following assumptions:

(i) The functions \( g, \sigma, b, \lambda \) are Lipschitz continuous and \( \sigma \sigma \perp, \lambda \lambda \perp \) are uniformly elliptic.

(ii) There exists \( r_0 \) and \( \kappa \) positive such that

\[
\frac{1}{2} \text{tr}(\lambda \lambda \perp(x)) + x \perp [b(x) - \lambda \sigma \perp(\sigma \sigma \perp)^{-1}(g - r \bar{1}))(x)] + \frac{\kappa}{2} \frac{x \perp \lambda \lambda \perp(x)x}{\sqrt{1 + \|x\|^2}} \leq 0
\]

for all \( \|x\| \geq r_0 \), \( \bar{1} = (1, \cdots, 1) \perp \)

(iii) Let \( \hat{u} \) is the solution to (4.2), then

\[
\frac{4}{g^2}(g - r \bar{1}) \perp(\sigma \sigma \perp)^{-1}(g - r \bar{1}) - (\nabla \hat{u}) \perp \lambda \sigma \perp(\sigma \sigma \perp)^{-1} \sigma \lambda \perp \nabla \hat{u} \to \infty \text{ as } \|x\| \to \infty.
\]

Ergodic risk sensitive control problem for the linear case is well studied, see [3, 4, 10, 7] for example. But for the nonlinear case, most of the related works deals with the small noise case, see for example [10] [8] [5]. The nonlinear case, suited for the continuous portfolio optimization is studied in [9] and later in [17]. The work [9] also assumes the a condition which is similar to the condition (ii) in [17] given above. In this paper we consider the model given in [17]. Our main contribution is that we prove the existence of ergodic optimal investment strategy without the assumption (ii) and the assumption (iii) replaced with the assumption (A3) which is standard in the literature of stochastic control.

Rest of our paper is organized as follows: In Section 2, we give a formal description of the problem. In Section 3, we investigate the finite horizon problem. We prove the existence of optimal investment strategy in Theorem 3.1 and give an explicit form for the optimal investment strategy in Theorem 3.2. In Section 4, we prove the existence of optimal non stationary investment strategy under (A1)-(A3). Note that the main challenge is in establishing the uniqueness of the pde (4.2). This is achieved without the condition (ii) of [17] in Theorem 4.1.
2. Problem Formulation

We consider an economy with \( m \geq 2 \) securities and \( n \geq 1 \) factors, which are continuously traded on a frictionless market. All traders are assumed to be price takers. The set of securities may include stock, bonds and savings account and the set of factors may include dividend yields, price-earning ratios, short term interest rates, the rate of inflation.

Let \( S_i(t) \) denote the price of \( i \)th security and \( X_j(t) \), the level of the \( j \)th factor at time \( t \). Dynamics of the security prices and factors are assumed to follow SDE given by

\[
\begin{align*}
    dS_0(t) &= r(X(t)) \, dt, \quad S_0(0) = s_0 > 0, \\
    \frac{dS_i(t)}{S_i(t)} &= a_i(X(t)) \, dt + \sum_{k=1}^{m+n} \sigma_{ik}(X(t)) \, dW_k(t), \\
    S_i(0) &= s_i > 0, \quad i = 1, 2, \ldots, m, \\
    dX_i(t) &= \mu_i(X(t)) \, dt + \sum_{k=1}^{m+n} \lambda_{ik}(X(t)) \, dW_k(t), \\
    X_i(0) &= x_i, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

(2.1)

where \( a = (a_1, \cdots, a_m)^\perp \), \( \mu = (\mu_1, \cdots, \mu_n)^\perp \), \( \sigma = [\sigma_{ij}] \) and \( \Lambda = [\lambda_{ij}] \) with \( a : \mathbb{R}^n \to \mathbb{R}^m \), \( \mu : \mathbb{R}^n \to \mathbb{R}^n \), \( \sigma : \mathbb{R}^n \to \mathbb{R}^{m \times (m+n)} \), \( \Lambda : \mathbb{R}^n \to \mathbb{R}^{n \times (m+n)} \) and \( r : \mathbb{R}^n \to \mathbb{R} \).

We assume that

\begin{enumerate}
    \item[(A1)] The functions \( a_i, \mu_i, \sigma_{ij}, \lambda_{ij} \) are bounded Lipschitz continuous and \( r \) is positive bounded measurable.
    \item[(A2)] The functions \( \sigma \sigma^\perp, \Lambda \Lambda^\perp \) are uniformly elliptic with uniform ellipticity constant \( \delta_0 > 0 \).
\end{enumerate}

Under (A1) and (A2), the SDE (2.1) has unique strong solution.

If \( n_i(t) \) denote the amount held by the investor in the \( i \)th security at time \( t \), then the wealth \( V(t) \) of the investor at time \( t \) is given by

\[
V(t) = \sum_{i=0}^{m} n_i(t) S_i(t).
\]

Set \( h_i(t) = \frac{n_i(t) S_i(t)}{V(t)} \), i.e., \( h_i(t) \) is the fraction of the wealth in the \( i \)th security at time \( t \). Then for a self financing strategy wealth equation takes the form

\[
\begin{align*}
    dV(t) &= V(t) \left[ r(X(t)) + h(t)^\perp (a(X(t)) - r(X(t)) \, \bar{1} \right] \, dt \\
    &\quad + V(t) h(t)^\perp \sigma(X(t)) \, dW(t), \quad V(0) = v > 0,
\end{align*}
\]

(2.2)

where \( h(t) = (h_1(t), \cdots, h_m(t))^\perp \).

We use the following admissibility conditions for the investment process \( h(\cdot) \).

**Definition 2.1.** An investment process \( h(\cdot) \) is admissible if the following conditions are satisfied:

\[
\begin{align*}
    \frac{d}{dt} h_i(t) &= \frac{d}{dt} \left( \frac{n_i(t) S_i(t)}{V(t)} \right) \\
    &= \frac{V(t) h_i(t) \frac{dV(t)}{dt} - V(t) h_i(t) \frac{dV(t)}{dt} + n_i(t) S_i(t) \frac{dV(t)}{dt}}{V(t)^2} \\
    &= \frac{V(t) h_i(t) \left[ r(X(t)) + h(t)^\perp (a(X(t)) - r(X(t)) \, \bar{1} \right] \, dt + V(t) h(t)^\perp \sigma(X(t)) \, dW(t)}{V(t)} \\
    &= \frac{h_i(t) \left[ r(X(t)) + h(t)^\perp (a(X(t)) - r(X(t)) \, \bar{1} \right] \, dt + h(t)^\perp \sigma(X(t)) \, dW(t)}{1}
\end{align*}
\]
(i) $h(\cdot)$ takes values in $\mathbb{R}^m$.
(ii) The process $h(\cdot)$ is progressively measurable with respect to the filtration
$\mathcal{G}_t = \sigma(S_1(s), \cdots, S_m(s), X(s) | s \leq t)$.
(iii) $E\left(\int_0^T ||h(s)||^2 ds\right) < \infty$, $\forall T$.

The class of admissible investment strategies is denoted by $\mathcal{H}$.

For a prescribed admissible strategy $h(\cdot)$ (see [10] p.162 for the definition of prescribed strategy) there exists a unique strong and almost surely positive solution $V(\cdot)$ to the SDE (2.2) see, [18] p.192. Also for $h(\cdot) \in \mathcal{H}$, the SDE (2.2) admits a unique weak solution. For an admissible strategy $h(\cdot)$ and for the initial conditions $x \in \mathbb{R}^n$ and $v > 0$, the risk-sensitive criterion for the horizon $[0, T]$ is given by

$$J_\theta^T(v, x, h(\cdot)) = \left(\frac{-2}{\theta}\right) \ln E^{h(\cdot)}[e^{-(\frac{\theta^2}{2})\ln V(T)} | V(0) = v, X(0) = x].$$

For the infinite horizon problem, the criterion is

$$J_\theta(v, x, h(\cdot)) = \lim_{T \to \infty} \left(\frac{-2}{\theta}\right) T^{-1} \ln E^{h(\cdot)}[e^{-(\frac{\theta^2}{2})\ln V(T)} | V(0) = v, X(0) = x].$$

We assume that $\theta > 0$, i.e., the investor is risk averse. Now the investor’s optimization problem, is as follows:

For finite horizon

$$\max_{h(\cdot) \in \mathcal{H}} J_\theta^T(v, x, h(\cdot))$$

subject to (2.1) and (2.2),

for infinite horizon

$$\max_{h(\cdot) \in \mathcal{H}} J_\theta(v, x, h(\cdot))$$

subject to (2.1) and (2.2).

**Definition 2.2.** (i) An admissible strategy $h^*(\cdot)$ is said to be optimal for the finite horizon problem if

$$J_\theta^T(v, x, h(\cdot)) \leq J_\theta^T(v, x, h^*(\cdot)), \forall \text{ admissible } h(\cdot).$$

(ii) An admissible strategy $h^*(\cdot)$ is said to be optimal for the infinite horizon problem if

$$J_\theta(v, x, h(\cdot)) \leq J_\theta(v, x, h^*(\cdot)), \forall \text{ admissible } h(\cdot).$$

3. Finite Horizon Problem

In this section, we consider the finite horizon problem described in the previous section. Our objective is to prove the existence of optimal investment strategies for the payoff function

$$J_\theta^T(v, x, h(\cdot)) = \left(\frac{-2}{\theta}\right) \ln E^{h(\cdot)}[e^{-(\frac{\theta^2}{2})\ln V(T)} | V(0) = v, X(0) = x].$$
The above optimal control problem is equivalent to minimize over $h(\cdot) \in \mathcal{H}$, the objective function

$$E^{h(\cdot)}[V(T)v^{-\frac{\theta}{2}} | V(0) = v, X(0) = x],$$

where $(X(\cdot), V(\cdot))$ is governed by (2.1) and (2.2).

We investigate the optimization problem by studying the corresponding Hamilton Jacobi Bellman (HJB) equation given by

$$0 = \frac{\partial \phi}{\partial t} + \inf_{h \in \mathbb{R}^m} L^h(\cdot) \phi(t, x, v), \quad \phi(T, x, v) = v^{-\frac{\theta}{2}} \text{ for } t > 0, x \in \mathbb{R}^n, v > 0,$$

where

$$L^h \phi = [r(x) + h^+(a(x) - r(x) \bar{1})] v \frac{\partial \phi}{\partial v} + \sum_{i=1}^{n} \mu_i(x) \frac{\partial \phi}{\partial x_i}$$

$$+ \frac{1}{2} h^+ \sigma(x) \sigma(x) h v^2 \frac{\partial^2 \phi}{\partial v^2} + \frac{1}{2} \sum_{ij=1}^{n} m_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

$$+ \frac{v}{2} \sum_{i=1}^{n} \sum_{l=1}^{m} \sum_{k=1}^{m+n} \lambda_{ik}(x) \sigma_{lk}(x) h_i \frac{\partial^2 \phi}{\partial x_i \partial v},$$

$$m_{ij}(x) = \sum_{k=1}^{m+n} \lambda_{ik}(x) \lambda_{jk}(x).$$

We seek a solution to (3.1) in the form

$$\phi(t, x, v) = v^{-\frac{\theta}{2}} e^{-\frac{\theta}{2}u(t, x)},$$

for a suitable function $u$. Consider the following PDE

$$0 = \frac{\partial u}{\partial t} + \sum_{i=1}^{n} \mu_i(x) \frac{\partial u}{\partial x_i} + \frac{\theta}{2} \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} - K \theta(x, \nabla u), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

$$u(T, x) = 0, \quad x \in \mathbb{R}^n,$$
where,

\begin{equation}
K_{\theta}(x, \nabla u) = \inf_{h \in \mathbb{R}^m} \left\{ \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) h^+ \sigma(x) \sigma(x)^\top h - h^+ (a(x) - r(x) \bar{1}) - r(x) \left[ \frac{\theta}{4} \sum_{i=1}^{n} \sum_{l=1}^{m} \sum_{k=1}^{m+n} \lambda_{ik}(x) \sigma_{lk}(x) h_l \frac{\partial u}{\partial x_i} \right] \right\}
\end{equation}

Using straightforward calculations, one can show that, the function $u \in C^{1+\frac{\delta}{2},2+\delta}((0,T) \times \mathbb{R}^n), 0 < \delta < 1$ is a solution to (3.3) iff $\phi \in C^{1+\frac{\delta}{2},2+\delta}((0,T) \times \mathbb{R}^n)$ given by (3.2) is a solution to the HJB equation (3.1).

Set

\[ u(t,x) = -\frac{2}{\theta} \ln \psi(t,x), \quad (t,x) \in [0, \infty) \times \mathbb{R}^n. \]

Then we can show that $u \in C^{1+\frac{\delta}{2},2+\delta}((0,T) \times \mathbb{R}^n)$ is a solution of (3.3) iff $\psi \in C^{1+\frac{\delta}{2},2+\delta}((0,T) \times \mathbb{R}^n)$ is a positive solution of the PDE

\begin{equation}
0 = \frac{\partial \psi}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \mu_i(x) \frac{\partial \psi}{\partial x_i}
+ H(x, \psi, \nabla \psi),
\end{equation}

where

\begin{equation}
H(t,x,\psi,\nabla \psi) = \frac{\theta}{2} \inf_{h \in \mathbb{R}^m} \left\{ \left\{ \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) h^+ \sigma(x) \sigma(x)^\top h - h^+ (a(x) - r(x) \bar{1}) - r(x) \right\} \psi \right\}
- h^+ \sigma(x) \Lambda(x) \nabla \psi.
\end{equation}

**Lemma 3.1.** Assume (A1)-(A2). The PDE (3.3) has unique solution $\psi \in C^{1+\frac{\delta}{2},2+\delta}((0,T) \times \mathbb{R}^n)$.

See [2], pp. 94-97, [14], pp.419-423 for a proof.

**Theorem 3.1.** Assume (A1)-(A2). The HJB equation (3.1) has a unique solution $\phi$ in $C^{1,2}((0,T) \times \mathbb{R}^n)$. Moreover

(i) For $(s,x,v) \in [0, T) \times \mathbb{R}^n \times (0, \infty)$,

\[ \phi(s,x,v) \leq E^{h(\cdot)} \left[ V(T)^{-(\theta/2)} | V(s) = v, X(s) = x \right] \]

for any admissible strategy $h(\cdot)$.

(ii) If $h^*(\cdot)$ is an admissible strategy such that

\[ L^{h^*} \phi(t,x,v) = \inf_{h \in \mathbb{R}^m} L^h \phi(t,x,v), \quad \forall \; t > 0, \; x \in \mathbb{R}^n, \; v > 0 \]
then \( \phi(s, x, v) = E^{h^*(\cdot)} \left[ V^*(T)^{-\theta/2} | V^*(s) = v, X(s) = x \right] \),
for any solution \( V^*(\cdot) \) of (2.2) corresponding to \( h^*(\cdot) \) and initial condition \((v, x)\).

**Proof:** Existence of the solution of (3.1) follows from Lemma 3.1. Let \( \phi \in C^{1,2}((0, T) \times \mathbb{R}^n) \) be a solution to (3.1). For each admissible \( h(\cdot) \) we have

\[
0 \leq \frac{\partial \phi}{\partial t} + L^{h(\cdot)} \phi(t, X(t), V(t)), \quad t \geq 0,
\]

where \((X(\cdot), V(\cdot))\) is given by (2.1)–(2.2) with initial conditions \( X(s) = x, V(s) = v \). For every integer \( n \geq 1 \) define the stopping time

\[
\tau_n = T \wedge \inf \{ t \geq s | \| (X(t), V(t)) \| \geq n \},
\]

where \( \| \cdot \| \) is the usual norm in \( \mathbb{R}^{n+1} \). Clearly, \( \tau_n \uparrow T \). Now using Ito’s formula, we have

\[
\phi(\tau_n, X(\tau_n), V(\tau_n)) - \phi(s, x, v) = \int_s^{\tau_n} \left[ \frac{\partial \phi}{\partial t} + L^{h(\cdot)} \phi \right] dt + \int_s^{\tau_n} \left[ \sum_{i=1}^n \lambda_i(X(r)) \frac{\partial \phi}{\partial x_i} + h(r)^{1/2} \sigma(X(r)) V(r) \frac{\partial \phi}{\partial v} \right] dW(r)
\]

where \( I_{[s, \tau_n]} \) denote the indicator function on \([s, \tau_n]\) and \( \lambda_i \) is the \( i \)th row of matrix \( \Lambda \).

Using \( 0 \leq \frac{\partial \phi}{\partial t} + L^{h(\cdot)} \phi(t, x, v), \forall t > 0, v > 0, x \in \mathbb{R}^n \) and taking the expectation on the both side, we have

\[
E^{h(\cdot)}[\phi(\tau_n, X(\tau_n), V(\tau_n)) - \phi(s, x, v)] | V(s) = v, X(s) = x \geq 0.
\]

Now let \( n \to \infty \) we get,

\[
0 \leq E^{h(\cdot)}[\phi(T, X(T), V(T)) | V(s) = v, X(s) = x] - E^{h(\cdot)}[\phi(s, x, v) | V(s) = v, X(s) = x].
\]

\[
\phi(s, x, v) \leq E^{h(\cdot)}[\phi(T, X(T), V(T)) | V(s) = v, X(s) = x].
\]

For the proof of (ii), note that from the definition of \( h^*(\cdot) \), we have

\[
L^{h^*(\cdot)} \phi(t, x, v) = 0
\]

Now using Ito’s formula as above, it follows that

\[
\phi(s, x, v) = E^{h^*(\cdot)} \left[ V^*(T)^{-(\theta/2)} | V^*(s) = v, X(s) = x \right],
\]

where \( V^*(\cdot) \) is a solution to (2.2) corresponding to \( h^*(\cdot) \). Hence

\[
\phi(s, x, v) = \inf_{h \in \mathbb{R}^m} E^{h(\cdot)} \left[ V(T)^{-(\theta/2)} | V(s) = v, X(s) = x \right].
\]
Theorem 3.2. Assume (A1)-(A2). Let $H_\theta(t, x)$ denote a minimizing selector in (3.4), that is,

$$H_\theta(t, x) = \left(\frac{2}{\theta + 2}\right)\left[a(x) - r(x)\bar{1} + \frac{\theta}{2} \sigma \Lambda \nabla u \right](\sigma \Lambda)\nabla u^{-1}(x).$$

Then the investment process

$$(3.7) \quad h_\theta(t) := H_\theta(t, X(t)),$$

is optimal, i.e.

$$(3.8) \quad J^T_\theta(v, x, h(\cdot)) \leq J^T_\theta(v, x, h_\theta(\cdot)),$$

for all admissible $h(\cdot), v > 0, x \in \mathbb{R}^n$.

Proof: Let $\phi$ be as in (3.2). Then it follows from Theorem 3.1 that $\phi$ is the unique solution to the HJB equation (3.1). Since $H_\theta$ is a minimizing selector in equation (3.4), we have

$$L^{H_\theta(\cdot)}(t, x, v) = \inf_{h \in \mathbb{R}^m} L^{h(\cdot)}(t, x, v), \forall t > 0, v > 0, x \in \mathbb{R}^n.$$

Now (i) and (ii) of Theorem 3.1 implies that

$$E^{h_\theta(\cdot)}\left[V^*(T)^{-(\theta/2)}|V^*(s) = v, X(s) = x]\right] \leq E^{h(\cdot)}\left[V(T)^{-(\theta/2)}|V(s) = v, X(s) = x] \right],$$

for all admissible $h(\cdot) \text{ and } V^*(\cdot)$ is the unique solution to (2.2) for the prescribed admissible strategy $h_\theta(\cdot)$. Hence,

$$J^T_\theta(v, x, h(\cdot)) \leq J^T_\theta(v, x, h_\theta(\cdot)),$$

for all admissible strategy $(h(\cdot), v > 0, x \in \mathbb{R}^n$.

4. Infinite Horizon Problem

In this section, we consider the infinite horizon problem. The method is to treat the problem as the asymptotic limit of the finite horizon problem. Thus we investigate the asymptotic behavior of the HJB equation of the finite horizon problem. Hence we require the following Lyapunov type stability condition.

(A3) There exists a function $v : \mathbb{R}^m \rightarrow \mathbb{R}$ such that

(i) $v \in C^2(\mathbb{R}^m), \quad v \geq 0$

(ii) The function $\|
abla v\|$ has polynomial growth.
(iii) \( L^{h,\omega}u(x) \to -\infty \) as \( \|x\| \to \infty \) for all \( h \) and \( \omega \), where

\[
L^{h,\omega} \phi = \sum_{i=1}^{n} \left[ \mu_i(x) + \sum_{k=1}^{m+n} \lambda_{ik}(x) \omega_k + \frac{\theta}{2} \sum_{l=1}^{n} h_l \left( \sum_{k=1}^{m+n} \lambda_{ik}(x) \sigma_{lk}(x) \right) \right] \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} m_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}.
\]

Consider the following auxiliary PDE

\[
\left\{ \begin{array}{l}
\frac{\partial \tilde{u}}{\partial t} = \sum_{i=1}^{n} \mu_i(x) \frac{\partial \tilde{u}}{\partial x_i} + \frac{1}{2} \left[ \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j} - K(x, \nabla \tilde{u}), \ t > 0, x \in \mathbb{R}^n, \\
\tilde{u}(0, x) = 0, \ \forall x \in \mathbb{R}^n,
\end{array} \right.
\]

We can show that \( \tilde{u} \in C^{1+\frac{\delta}{2},2+\delta}((0,T) \times \mathbb{R}^n) \) is a solution to (4.1) iff \( u \in C^{1+\frac{\delta}{2},2+\delta}((0,T) \times \mathbb{R}^n) \) is unique solution to (3.5). Hence (4.1) has unique solution \( \tilde{u} \in C^{1+\frac{\delta}{2},2+\delta}((0,T) \times \mathbb{R}^n) \). Using Feynman-Kac representation of (4.1), see [11], p.366 and (A3), we can show that \( \tilde{u} \geq 0, \ \frac{\partial \tilde{u}}{\partial t} \geq 0 \). Now we state the following estimate which is crucial to study the asymptotic behavior of (4.1).

**Lemma 4.1.** Let \( \tilde{u} \) be the unique solution to (4.1). Then for each \( c > 0 \)

\[
|\nabla x \tilde{u}(t, x)|^2 - \frac{4(1+c)(\theta + 2)}{\theta \delta_0} \left| \frac{\partial \tilde{u}(t, x)}{\partial t} \right| 
\leq K \left( |\nabla Q|^2_{2r} + |\nabla (\lambda \lambda^\perp)|^2_{2r} + |\nabla B|_{2r} + |B|_{2r}^2 + |U|_{2r} + |\nabla U|_{2r}^2 + 1 \right),
\]

\( t > 0, \ x \in B(0, r), \)

where \( \delta_0 \) is the uniform ellipticity constant of \( \Lambda \Lambda^\perp, \)

\[
\begin{align*}
Q(x) &= \lambda^\frac{\theta}{4} [I - \frac{\theta}{\theta + 2} \sigma^\perp (\sigma \sigma^\perp)^{-1} \sigma^\perp], \\
B(x) &= \mu(x) - \frac{\theta}{\theta + 2} \lambda^\perp (\sigma \sigma^\perp)^{-1} [a(x) - r(x) \mathbb{1}], \\
U(x) &= \frac{1}{\theta + 2} (a - r \mathbb{1})^\perp (\sigma \sigma^\perp)^{-1} (a - r \mathbb{1}) + r(x).
\end{align*}
\]

\( | \cdot |_{2r} = \| \cdot \|_{L^\infty(B(0,r))} \) and \( K > 0 \) is a constant depending on \( c, \delta_0, n \).

The proof of Lemma 4.1 follows from the proof of [17, Theorem 2.1 (i), Remark (i)]. Now using the above estimate we prove the following lemma, see appendix for the proof.
Lemma 4.2. Let \( \hat{u} \) be the solution to (4.1) and \( x_0 \in \mathbb{R}^n \), then there exists a subsequence \( \{T_i\} \subset \mathbb{R}^n \) such that \( \hat{u}(T_i, x) - \hat{u}(T_i, x_0) \) converges to a function \( \hat{u} \in C^2(\mathbb{R}^n) \) uniformly on compact sets and strongly in \( W_{loc}^{1,2} \) and \( \frac{\partial \hat{u}(T_i, \cdot)}{\partial t} \) to \( \rho \in \mathbb{R} \) uniformly on each compact set. Moreover, \((\hat{u}(\cdot), \rho)\) satisfies

\[
\rho = \frac{1}{2} \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial^2 \hat{u}}{\partial x_i \partial x_j} - \frac{\theta}{4} \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial \hat{u}}{\partial x_i} \frac{\partial \hat{u}}{\partial x_j} + \sum_{i=1}^{n} \mu_i(x) \frac{\partial \hat{u}}{\partial x_i} - K_\theta(x, \nabla \hat{u}),
\]

\[
\lim_{\|x\| \to \infty} \hat{u}(x) = \infty, \quad x \in \mathbb{R}^n.
\]

To show the uniqueness of the above PDE (4.2) we rewrite (4.2) as

\[
\rho = \frac{1}{2} \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial^2 \hat{u}}{\partial x_i \partial x_j} - \inf_{\omega \subset \mathbb{R}^{m+n}} \left[ \frac{1}{\theta} \|\omega\|^2 - \omega \Lambda(x) \nabla \hat{u} \right] + \mu(x) \nabla \hat{u}
\]

\[
- \sup_{h \in \mathbb{R}^{m}} \left[ h^\perp (a(x) - r(x) \mathbf{I}) + r(x) - \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) h^\perp \sigma(x) \sigma(x)^{\perp} h - \frac{\theta}{2} h^\perp \sigma(x) \Lambda(x)^{\perp} \nabla \hat{u} \right],
\]

\[
\lim_{\|x\| \to \infty} \hat{u}(x) = \infty, \quad x \in \mathbb{R}^n.
\]

Hence the PDE (4.2) takes the form (4.3)

\[
\rho = \frac{1}{2} \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial^2 \hat{u}}{\partial x_i \partial x_j}
\]

\[
+ \sup_{\omega \subset \mathbb{R}^{m+n}} \inf_{h \in \mathbb{R}^{m}} \left[ \left( \mu(x)^{\perp} + \omega^{\perp} \Lambda(x)^{\perp} + \frac{\theta}{2} h^\perp \sigma(x) \Lambda(x)^{\perp} \right) \nabla \hat{u} - \frac{1}{\theta} \|\omega\|^2 \right]
\]

\[
+ \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) h^\perp \sigma(x) \sigma(x)^{\perp} h - h^\perp (a(x) - r(x) \mathbf{I}) - r(x)
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial^2 \hat{u}}{\partial x_i \partial x_j}
\]

\[
+ \inf_{h \in \mathbb{R}^{m}} \sup_{\omega \subset \mathbb{R}^{m+n}} \left[ \left( \mu(x)^{\perp} + \omega^{\perp} \Lambda(x)^{\perp} + \frac{\theta}{2} h^\perp \sigma(x) \Lambda(x)^{\perp} \right) \nabla \hat{u} - \frac{1}{\theta} \|\omega\|^2 \right]
\]

\[
+ \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) h^\perp \sigma(x) \sigma(x)^{\perp} h - h^\perp (a(x) - r(x) \mathbf{I}) - r(x)
\]

\[
\lim_{\|x\| \to \infty} \hat{u}(x) = \infty.
\]
Consider the SDE (4.4)
\[
dx_i(t) = \left[ \mu_i(X(t)) + \sum_{k=1}^{m+n} \lambda_{ik}(X(t)) \omega_k(X(t)) + \theta \sum_{l=1}^{m} \sum_{k=1}^{m+n} \lambda_{lk}(X(t)) \sigma_{lk}(X(t)) h_l(t) \right] dt
\]
\[+ \sum_{k=1}^{m+n} \lambda_{ik}(X(t)) dW_k(t), \quad i = 1, \ldots, n.\]

Let \( \mathcal{M}_1 \) denote the set of all Markov strategies in \( \mathcal{H} \) and
\[
\mathcal{M}_2 = \{ \omega : \mathbb{R} \to \mathbb{R}^{n+m} \mid \text{measurable and } E \int_0^T \|\omega(X(t))\|^2 dt < \infty \text{ for all } T > 0 \}.
\]

For \( h \in \mathbb{R}^m, w \in \mathbb{R}^{n+m}, \phi : \mathbb{R}^n \to \mathbb{R}, \) set
\[
L_{h,w} \phi = \sum_{i=1}^{n} \left[ \mu_i(x) + \sum_{k=1}^{m+n} \lambda_{ik}(x) \omega_k + \frac{\theta}{2} \sum_{l=1}^{m} \sum_{k=1}^{m+n} h_l \left( \sum_{k=1}^{m+n} \lambda_{ik}(x) \sigma_{lk}(x) \right) \right] \frac{\partial \phi}{\partial x_i}
\]
\[+ \frac{1}{2} \sum_{i=1}^{n} m_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}.\]

and
\[
r(x, h, \omega) = \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) h^+ \sigma(x) \sigma(x)^+ h - \frac{1}{\theta} \|\omega\|^2 - h^+ (a(x) - r(x)) - r(x).
\]

Let \( \tilde{\omega}(\cdot), \tilde{h}(\cdot) \) be such that
\[
\sup_{\omega \in \mathbb{R}^{m+n}} \inf_{h \in \mathbb{R}^m} \left[ L_{h,\omega} \hat{u} + r(h, \omega) \right] = \inf_{h \in \mathbb{R}^m} \left[ L_{\tilde{h},\tilde{\omega}(\cdot)} \hat{u} + r(h, \tilde{\omega}(\cdot)) \right]
\]
\[= \sup_{\omega \in \mathbb{R}^{m+n}} \left[ L_{\tilde{h},\tilde{\omega}(\cdot)} \hat{u} + r(h, \tilde{\omega}(\cdot)) \right] = \inf_{h \in \mathbb{R}^m} \sup_{\omega \in \mathbb{R}^{m+n}} \left[ L_{\tilde{h},\tilde{\omega}(\cdot)} \hat{u} + r(h, \omega) \right]
\]
\[= L_{\tilde{h}(\cdot),\tilde{\omega}(\cdot)} \hat{u} + r(\tilde{h}(\cdot), \tilde{\omega}(\cdot)).\]

Fix \( \tilde{h}(\cdot) \in \mathcal{M}_1, \) let \( X_1(\cdot) \) denote the process (4.4) with initial condition \( x \in \mathbb{R}^n \) corresponding to \( (\tilde{h}(\cdot), \tilde{\omega}(\cdot)) \), then using Ito’s formula, we have
\[ \hat{u}(X_1(T)) - \hat{u}(x) \]
\[ = \int_0^T L^{h(\cdot),\omega(\cdot)} \hat{u}(X_1(t)) dt + \text{Martingale (Zero-mean)} \]
\[ = \int_0^T \left[ L^{h(\cdot),\omega(\cdot)} \hat{u}(X_1(t)) + r(X_1(t), h(X_1(t)), \omega(X_1(t))) \right] dt \]
\[ - \int_0^T r(X_1(t), h(X_1(t)), \omega(X_1(t))) dt + \text{Martingale (Zero-mean)} \]
\[ \geq \inf_{h \in \mathcal{M}_1} \sup_{\omega \in \mathcal{M}_2, h \in \mathcal{M}_1} \int_0^T \left[ L^{h(\cdot),\omega(\cdot)} \hat{u}(X_1(t)) + r(X_1(t), h(X_1(t)), \omega(X_1(t))) \right] dt \]
\[ - \int_0^T r(X_1(t), h(X_1(t)), \omega(X_1(t))) dt + \text{Martingale (Zero-mean)} \]
\[ = T\rho - \int_0^T r(X_1(t), h(X_1(t)), \omega(X_1(t))) dt + \text{Martingale (Zero-mean)}. \]

Taking expectation, we have
\[ (4.5) \quad E[\hat{u}(X_1(T))] - \hat{u}(x) \geq \rho T - E \left[ \int_0^T r(X_1(t), h(X_1(t)), \omega(X_1(t))) dt \right]. \]

Now mimicking the arguments in [1] (see appendix for a proof), using (A3) we can show that \( \hat{u} \in o(v(\cdot)) \) and
\[ (4.6) \quad \lim_{T \to \infty} \frac{1}{T} E[\hat{u}(X_1(T))] = 0, \]

Now divide (4.5) by and let \( T \to \infty \) we have
\[ \rho \leq \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T r(X_1(t), \tilde{h}(X_1(t)), \tilde{\omega}(X_1(t))) dt \right] \quad \forall \ h(\cdot) \in \mathcal{M}_1. \]

Therefore
\[ \rho \leq \sup_{h(\cdot) \in \mathcal{M}_1} \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T r(X_1(t), h(X_1(t)), \omega(X_1(t))) dt \right]. \]

Hence
\[ (4.7) \quad \rho \leq \inf_{\omega(\cdot) \in \mathcal{M}_2} \sup_{h(\cdot) \in \mathcal{M}_1} \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T r(X(t), h(X(t)), \omega(X(t))) dt \right], \]

where \( X(\cdot) \) is the process (4.4) corresponding to \( (h(\cdot), \omega(\cdot)) \). Now a similar argument shows that
\[ (4.8) \quad \rho \geq \sup_{h(\cdot) \in \mathcal{M}_1} \inf_{\omega(\cdot) \in \mathcal{M}_2} \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T r(X(t), h(X(t)), \omega(X(t))) dt \right]. \]
Combining (4.7) and (4.8), we get
\[
\rho = \sup_{h(\cdot) \in \mathcal{M}_1, \omega(\cdot) \in \mathcal{M}_2} \inf_{w(\cdot) \in \mathcal{M}} \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T r(X(t), h(X(t)), \omega(X(t))) dt \right]
\]
\[
= \inf_{\omega(\cdot) \in \mathcal{M} \cap \mathcal{M}_2} \sup_{h(\cdot) \in \mathcal{M}_1} \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T r(X(t), h(X(t)), \omega(X(t))) dt \right]
\]

Let \( (\rho', \psi) \) be another solution in the class \( \mathbb{R} \times C^2(\mathbb{R}_+) \cap o(\hat{u}(\cdot)) \). Then using
the similar argument, one can easily check that
\[
\rho' = \sup_{h(\cdot) \in \mathcal{H}, \omega(\cdot) \in \mathcal{M}} \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T r(X(t), h(X(t)), \omega(X(t))) dt \right] = \rho.
\]
Let \( h_1 \in \mathcal{M}_1 \) be such that
\[
\rho = \inf_{w(\cdot) \in \mathbb{R}^{n+m}} \left[ L^{h_1(\cdot), w_1}(\hat{u} \cdot r(x, h_1(x), w_1)) \right],
\]
\( w_1 \in \mathcal{M}_2 \) be such that
\[
\rho = \sup_{h_1(\cdot) \in \mathbb{R}^{n+m}} \left[ L^{h_1(\cdot) \psi + r(x, h, w_1(x))} \right]
\]
and \( X(\cdot) \) be the solution to (4.4) corresponding to \( (h_1(\cdot), w_1(\cdot)) \). Then \( L^{h_1(\cdot), \omega(\cdot)}(\hat{u} \cdot \psi) \leq 0 \ \forall \ h(\cdot) \in \mathcal{M}_1, \ \omega(\cdot) \in \mathcal{M}_2 \).

Thus \( \hat{u}(X(t)) - \psi(X(t)), \ t \geq 0 \) is a submartingale satisfying
\[
\sup_t \mathbb{E}[|u(X(t)) - \psi(X(t))|] \leq k \lim_{t \to \infty} \frac{1}{t} \int_0^t |X(t)|^{2n} ds < \infty,
\]
for suitable \( k > 0, \ n \geq 1 \). We use here the fact that \( \psi \) and \( \hat{u} \) have polynomial growth. By the submartingale convergence theorem, it converges a.s. Since
\( \psi(x_0) = \hat{u}(x_0) = 0 \) and \( X(\cdot) \) visits any arbitrarily small neighborhood of zero infinitely often a.s., it can converge only to zero. The same argument proves that \( \psi - \hat{u} \) is identically zero: if not, \( \psi - \hat{u} > \delta > 0 \) for some \( \delta \) in some open ball which is visited infinitely often a.s. by \( X(\cdot) \), contradicting the convergence of \( \psi(X(\cdot)) - \hat{u}(X(\cdot)) \) to zero. Hence \( \psi - \hat{u} \) is identically zero. Thus we have the following theorem.

**Theorem 4.1.** Assume (A1)-(A3). The pde (4.1) has a unique solution \( (\rho, \hat{u}) \in \mathbb{R} \times C(\mathbb{R}_+) \) satisfying \( \hat{u}(x_0) = 0 \).

**Theorem 4.2.** Assume (A1)-(A3). Let \( h_{\theta}(\cdot) \) be as in Theorem 3.2. Then:
(i). For all \( v > 0 \) and \( x \in \mathbb{R}^n \) we have
\[
J_{\theta}(v, x, h_{\theta}(\cdot)) = \lim_{t \to \infty} \left( \frac{-2}{\theta} \right) t^{-1} \ln E^{h_{\theta}(\cdot)} \left[ e^{-(\theta/2) \ln V^*(t)} |V(0) = v, X(0) = x| \right]
\]
:= \rho(\theta)
where \( V^*(\cdot) \) is the unique solution of (2.1) corresponding to \( h_{\theta}(\cdot) \) and the initial condition \( (v, x) \).
(ii). The admissible strategy \( h_\theta(\cdot) \) is optimal.

**Proof:** From Theorem 3.2, we have

\[
\frac{1}{T} J^T_\theta(x, v, h_\theta(\cdot)) \geq \frac{1}{T} J^T_\theta(x, v, h(\cdot))
\]

for all \( h(\cdot) \) admissible. Now using Theorem 4.1, we have

\[
\frac{1}{T} J^T_\theta(x, v, h_\theta(\cdot)) = \frac{1}{T} \ln v - \frac{1}{T} u(T-t, x) \to \rho \text{ as } T \to \infty.
\]

Now from (4.9) and (4.10), we have

\[
\rho = \lim_{T \to \infty} \frac{1}{T} J^T_\theta(v, x, h_\theta(\cdot)) \geq \liminf_{T \to \infty} \frac{1}{T} J^T_\theta(v, x, h(\cdot)).
\]

Hence we have the theorem. \( \square \)

**Remark 4.1.** We have shown that the optimal strategies in both finite horizon and infinite horizon problems are functions of the economic factors only. This happens since the economic factors are what which drives the asset price movements. Another interesting observation is the same optimal strategy works for both finite and infinite horizon problems.

**Remark 4.2.** (i) If we assume that \( \sigma \Lambda \perp \equiv 0 \), then strategy given in Theorem 4.2 is stationary. But in this case portfolio cannot include bonds.

(ii) Instead of \( \sigma \Lambda \perp \equiv 0 \) if we assume the condition (ii) of [17], then a close mimicry of the proof of [17], Theorem 4.1] we can show that

\[
H_\theta(x) = \frac{\theta}{\theta + 2} (\sigma \sigma^\perp)^{-1} [a(x) - r(x) \bar{1} - \frac{\theta}{2} \sigma \lambda^\perp \nabla \tilde{u}(x)]
\]

is an optimal stationary strategy.

5. Conclusion

In this paper, we have investigated the risk-sensitive portfolio optimization problem where the assets are explicitly depending on the economic factors. Our portfolio model can also include fixed income securities such as rolling horizon bonds. We prove the existence of optimal investment strategies under very general conditions.

6. Appendix

**Proof of Lemma 4.2.**

Set \( \tilde{\phi}(T, x) = \tilde{u}(T, x) - \tilde{u}(T, x_0) \). Using Lemma 4.1, it can be shown that \( \{\tilde{\phi}(T, \cdot)\}_{T > 0} \) is uniformly bounded and equicontinuous on compact subsets of \( \mathbb{R}^n \). Therefore it has a subsequence \( \{\phi(T_i, \cdot)\} \) converging to a function \( \tilde{u}(\cdot) \in C^2(\mathbb{R}^n) \) uniformly on each compact set. Moreover, \( \frac{\partial \tilde{u}}{\partial T} \geq 0 \) and by Lemma [14] \( \{\tilde{\phi}(T, \cdot)\} \) forms a bounded subset of Hilbert space.
$W^{1,2}(B(0,R))$ for each $R > 0$ and we see that there exists a subsequence (w.o.l.g itself) \{\tilde{u}(T_i,\cdot)\} converging to $\tilde{u} \in W^{1,2}_{loc}([\mathbb{R}^n_t])$ weakly in $W^{1,2}_{loc}$ and strongly in $L^2_{loc}$. Taking a further subsequence (w.o.l.g itself), we can see that $\tilde{u}(T_i,\cdot) \rightarrow \tilde{u}(\cdot)$ a.s. and that $\tilde{u} \equiv \tilde{u}$.

Also we can show that $\nabla \phi(T_i,\cdot) \rightarrow \nabla \tilde{u}(\cdot)$ strongly in $L^2_{loc}(\mathbb{R}^n)$.

Put $\xi(\cdot) = \frac{\partial \tilde{u}}{\partial t}$. Then we obtain from (4.4)

$$\frac{\partial \tilde{\phi}(T_i, x)}{\partial t} = \sum_{i=1}^{n} \mu_i(x) \frac{\partial \tilde{\phi}(T_i, x)}{\partial x_i} + \frac{1}{2} \left[ -\frac{\theta}{2} \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial \tilde{\phi}(T_i, x)}{\partial x_i} \frac{\partial \tilde{\phi}(T_i, x)}{\partial x_j} - \theta \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial \tilde{\phi}(T_i, x)}{\partial x_i} \frac{\partial \tilde{\phi}(T_i, x)}{\partial x_j} \right]$$

$$\sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial^2 \tilde{\phi}(T_i, x)}{\partial x_i \partial x_j}$$

$$- K_{\theta}(x, \nabla \tilde{\phi}), \ (T_i, x) \in (0, \infty) \times \mathbb{R}^n,$$

$$\tilde{\phi}(0, x) = 0, \ \forall x \in \mathbb{R}^n,$$

we can see that $(\tilde{u}(\cdot), \rho(\cdot))$ satisfies (4.1). Now we show that $\rho(\cdot)$ is a constant.

Fix $x^1 \in B(0, R_0)$. For $x \in B(0, R)$, for $R \geq R_0$

$$\rho(x) = \lim_{n \rightarrow \infty} \frac{\partial \tilde{\phi}(T_n, x)}{\partial t} = \lim_{n \rightarrow \infty} \frac{\tilde{\phi}(T_n, x)}{T_n}$$

$$= \lim_{n \rightarrow \infty} \frac{\tilde{\phi}(T_n, x) - \tilde{u}(T_n, x^1)}{T_n} + \lim_{n \rightarrow \infty} \frac{\tilde{\phi}(T_n, x^1)}{T_n}$$

$$= \lim_{n \rightarrow \infty} \frac{\nabla \tilde{\phi}(T_n, x^1), (x - x^1)}{T_n} + \rho(x^1).$$

Now from Lemma 4.1 it follows that

$$\lim_{n \rightarrow \infty} \frac{\nabla \tilde{\phi}(T_n, x^1), (x - x^1)}{T_n} = 0 \ \text{whenever} \ x \in B(0, R).$$

Therefore $\rho(x) = \rho(x^1)$ whenever $x \in B(0, R)$, for any $R \geq R_0$.

Since $R_0$, $R$ can be chosen arbitrary, we have

$$\rho(x) = \rho(x^1) \ \forall \ y \in \mathbb{R}^n.$$
Hence $\rho$ is constant. \hfill $\Box$

**Proof of (4.6).** From (A3), there exists $r > 0$ such that

$$L^{\bar{h}, \bar{\omega}}v(x) \leq -1, \text{ whenever } \|x\| \geq r, \bar{h} \in \mathbb{R}^m, \bar{\omega} \in \mathbb{R}^{n+m}.$$ Let $X(\cdot)$ be the process corresponding to $(\bar{h}(\cdot), \bar{\omega}(\cdot))$ with $X(0) = x, \|x\| \geq r$. Note that

$$\bar{h}(x) = \frac{2}{\theta + 2}(\sigma \sigma^\perp)^{-1}[a(x) - r(x)\bar{1} - \frac{\theta}{4}\Lambda \sigma^\perp \nabla \hat{u}]$$

and

$$\bar{\omega}(x) = \frac{\theta}{2}\Lambda \nabla \hat{u}(x).$$

From Lemma 4.1 and (A1), it follows that

$$\|\nabla \hat{u}\|_{L^\infty(\mathbb{R}^n)} \leq c,$$ for some $c > 0$.

Hence there exists a constant $c_1 > 0$ such that

$$\|\bar{h}\|_{L^\infty(\mathbb{R}^n)} + \|\bar{\omega}\|_{L^\infty(\mathbb{R}^n)} \leq c_1.$$

Let $\tau_r$ be the first time the process $X(\cdot)$ hits the ball $B(0, r)$. Using Ito’s formula we have,

$$E \hat{u}(X(\tau_r)) - \hat{u}(x) = -E\left[\int_0^{\tau_r} r(X(s), \bar{h}(X(s)), \bar{\omega}(X(s))) \, ds\right]$$

Therefore from [1, Lemma 4.1, p. 166], there exists constants $c_2, c_3$ such that

$$\hat{u}(x) \leq c_2 + c_3 v(x), \|x\| \geq r.$$ i.e. $\hat{u} \in o(v(\cdot))$. Now mimicking the arguments from [1, pp.165-168], the equation (4.6) follows.

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