Nucleon Spin Fluctuations and Neutrino-Nucleon Energy Transfer in Supernovae

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The formation of neutrino spectra in a supernova depends crucially on strength and inelasticity of weak interactions in hot nuclear matter. Neutrino interactions with nonrelativistic nucleons are mainly governed by the dynamical structure function for the nucleon spin density which describes its fluctuations. It has recently been shown that these fluctuations give rise to a new mode of energy transfer between neutrinos and nucleons which inside the neutrinosphere is of comparable or greater importance than ordinary recoil. We calculate numerically the spin density structure function in the limit of a dilute, non-degenerate medium from exact two-nucleon wave functions for some representative nuclear interaction potentials. We show that spectrum and magnitude of the energy transfer can deviate significantly from those based on the Born approximation. They are, however, rather insensitive to the particular nuclear potential as long as it reproduces experimental nucleon scattering phase shifts at energies up to a few tens of MeV. We also compare with calculations based on a one-pion exchange potential in Born approximation and briefly comment on their applicability near the center of a supernova core. Our study is relevant for numerical simulations of the neutrino spectra emerging from type-II supernovae.

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I. INTRODUCTION

The detection of roughly a dozen neutrinos from SN 1987 A is in good qualitative agreement with the neutrino signal expected from the early cooling phase of a hot neutron star born in the center of the collapsed core of a massive star [1]. It is therefore generally believed that type-II supernovae such as SN 1987 A are the optical counterparts of such catastrophic events.

The formation of the spectra of neutrinos emitted from a type-II supernova takes place in a region where weak neutral current scattering and pair processes involving electron, $\mu$, and $\tau$ neutrinos and charged current creation and absorption of electron neutrinos on nucleons, nuclei and electrons cease to be efficient in keeping the neutrinos in thermodynamical equilibrium with the medium. The interplay between (roughly) energy conserving scattering and energy changing reactions plays a crucial role in that respect [2]. In previous studies of neutrino transport, the lowest order neutrino opacities in vacuum have been used. Neutral current scattering processes on nucleons and nuclei have been approximated to be elastic [3]. As a result, whereas the energy fluxes predicted for the three neutrino flavors turn out to be very similar [3], the effective temperatures are significantly higher for $\mu$ and $\tau$ neutrinos compared to electron neutrinos which because of their more efficient energy exchange with the medium decouple from it further out.

However, weak interaction rates in a medium differ significantly from those taking place in vacuum. On the one hand, the spin-dependent strong force between nucleons will establish spatial correlations of the density and the spin-density at that frequency. For example, at finite density, the spin-dependent nucleon-nucleon interaction also causes the nucleon spins to fluctuate. This leads to a reduction of the average total axial-vector current neutrino scattering cross section compared to its vacuum value [8]. This effect is most important at the high temperatures pertaining in the first few seconds after formation of the hot neutron star. In addition, the nucleon spin fluctuations can, apart from recoil, imply an enhanced energy transfer between nucleons and neutrinos [10]. It is this effect which we are mostly concerned with in the present work because it could significantly change predictions of the neutrino spectra with a tendency to lower predicted effective temperatures of $\mu$ and $\tau$ neutrinos [3]. This is of some importance in view of new neutrino detectors such as Super-Kamiokande and
the Sudbury Neutrino Observatory, which have the capa-

Near the surface of last scattering the neutrino opac-

However, the Born approximation is only applicable if

\[ |V| \ll \frac{1}{m_N a^2} \]

\[ |V| \ll \frac{p}{m_N} \]  \hspace{1cm} (1)

where \( |V| \sim 100 \text{ MeV} \) is the typical magnitude of the nu-

clear interaction potential, \( a \sim 1 \text{ fm} \) is its range, \( p \) is the

The goal of this paper is therefore to compute the dy-

\[ H_{\text{int}} = \frac{G_F}{2\sqrt{2}} \sum_{i=n,p} \bar{\psi}_i \gamma_\mu [C_{V,i} - C_{A,i} \gamma_5] \psi_i \]

\[ \times \bar{\psi}_\nu \gamma^\mu (1 - \gamma_5) \psi_\nu , \]  \hspace{1cm} (4)

where \( G_F \) is the Fermi constant, \( \psi_i (i = n, p) \) and \( \psi_\nu \)

we define the nucleon spin-density structure function in

\[ \nu + n + p \leftrightarrow \nu + n + p \]  \hspace{1cm} (2)

as well as the free-bound and bound-free processes in-

The analogous processes involving neutrino pairs or ax-

The rest of the paper is organized as follows: In Sect. II

A. Definition and General Properties

The interaction of interest here, neutrino-nucleon neu-

tral-current scattering, is given by the Hamiltonian

Another possible type of weak process is the emission of

\[ H_{\text{int}} = \frac{1}{2f_a} \sum_{i=n,p} C_{a,i} \bar{\psi}_i \gamma_\mu \gamma_5 \psi_i \partial^\mu a , \]  \hspace{1cm} (5)

where \( a \) is the axion field, \( f_a \) the Peccei-Quinn scale,

In the limit of non-relativistic nucleons, only the axial-

In the limit of non-relativistic nucleons, only the axial-

We keep things numerically simple, we will restrict our-
define the dimensionless quantity $\tilde{\sigma}_{w}(x) \equiv \sum_{i=n,p} \frac{C_{i}}{C} \phi_{i}(x) \frac{T}{2} \phi_{i}(x)$. \hfill (6)

Here, $\phi_{i}(x)$ ($i = n, p$) is the non-relativistic field operator for protons and neutrons which is a Pauli two-spinor, $\tau$ is the vector of Pauli matrices, and $C^{2} = C_{n}^{2}Y_{n} + C_{p}^{2}Y_{p}$ is an average neutral-current axial weak coupling constant to the nucleons, weighted by the fractional neutron and proton abundances $Y_{n}$ and $Y_{p}$. Defining the Fourier transform in a normalization volume $V$ as $\sigma_{w}(t, k) = V^{-1/2} \int d^{3}r e^{-ik \cdot r} \sigma_{w}(t, r)$, one can then define the nucleon spin-density structure function [12,10,16]:

$$S_{\sigma}(\omega, k) = \frac{1}{3n_{b}} \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle \sigma_{w}(t, k) \cdot \sigma_{w}(0, -k) \rangle. \hfill (7)$$

Here, $(\omega, k)$ is the four-momentum transfer to the medium, $n_{b}$ is the baryon number density and the expectation value $\langle \cdot \cdot \cdot \rangle$ is taken over a thermal ensemble at the medium temperature $T$ of medium states normalized to unity.

Relativistic neutrinos and possibly axions will have typical energies of order $3T$ but are in general not in chemical equilibrium with the medium. Weak interactions such as neutral current neutrino scattering and pair processes and axion emission thus probe the spin-density function typically at thermal energy-momentum transfers. Since the momenta involved in the nucleon-nucleon interactions are much larger than the thermal momenta of relativistic particles, we will often employ the long wavelength limit, $S_{\sigma}(\omega, k) \equiv \lim_{k \to 0} S_{\sigma}(\omega, k)$ for which we define the dimensionless quantity $S_{\sigma}(x) \equiv T S_{\sigma}(xT)$. In this limit, integration of Eq. (7) over $\omega$ yields the sum rule

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_{\sigma}(\omega) - 1 = N_{\sigma} \equiv \frac{4}{3n_{b} V} \left\langle \sum_{i \not= j} \sigma_{i,w} \cdot \sigma_{j,w} \right\rangle,$$ \hfill (8)

where we wrote the spatial integral $\int d^{3}r \sigma_{w}(t, r) = \sum \sigma_{i,w}$. Here, $\sigma_{i,w} \equiv \sigma_{i} \delta(\vec{C}_{p}, C_{n})/C$, where $\sigma_{j}$ are the spin operators of the individual nucleons, and the matrix $\delta(\vec{C}_{p}, C_{n})$ acts in isospin space. In Eq. (8), $N_{\sigma}$ describes correlations among different nucleon spins. In the absence of such correlations $\int_{-\infty}^{+\infty} (d\omega/2\pi) S_{\sigma}(\omega)$ reduces to 1 which motivated the introduction of the weighted spin operator Eq. (8).

We formally introduce the complete set of eigenfunctions $|n\rangle$ of the total Hamiltonian $H$ of the nuclear medium, $H |n\rangle = \omega_{n} |n\rangle$, where $\omega_{n}$ are the corresponding energy eigenvalues. By inserting the identity operator $I = |n\rangle \langle n|$, between the spin operators, Eq. (7) can be rewritten into

$$S_{\sigma}(\omega, k) = \frac{8\pi}{3n_{b} Z} \sum_{n,m} e^{-\omega_{n}/T} \delta(\omega + \omega_{n} - \omega_{m}) \times |\langle n | \sigma_{w}(0, k) | m \rangle|^{2},$$ \hfill (9)

where $Z = \sum_{n} e^{-\omega_{n}/T}$ is the partition function. This form will be useful later and it shows that the structure function satisfies detailed balance,

$$S_{\sigma}(\omega, k) = S_{\sigma}(-\omega, -k) e^{\omega/T}.$$ \hfill (10)

It is therefore sufficient to know the function, e.g. for positive energy transfer to the medium, $\omega \geq 0$.

Up to now no approximations have been made with regard to the nucleon-nucleon interactions which determine the nonperturbative though unknown structure function $S_{\sigma}(\omega, k)$. From now on we will make the assumption that only two-nucleon forces are present. The Hamiltonian then has the form

$$H = \sum_{i} \frac{p_{i}^{2}}{2m_{N}} + \frac{1}{2} \sum_{i \not= j} \frac{1}{2} V(r_{ij}, \sigma_{i}, \sigma_{j}),$$ \hfill (11)

where $r_{ij}$ is the radius vector between nucleon $i$ and $j$, $p_{i}$ is the nucleon momentum, $V(r_{ij}, \sigma_{i}, \sigma_{j})$ is the spin dependent two-nucleon interaction potential, and the sums run over all nucleons. The most general two-nucleon potential in the non-relativistic limit can be written as [17]

$$V(r, \sigma_{1}, \sigma_{2}) = U(r) + U_{\sigma}(r) \sigma_{1} \cdot \sigma_{2} + U_{T}(r) T_{12} + P_{T} [U^{r}(r) + U_{\sigma}^{r}(r) \sigma_{1} \cdot \sigma_{2} + U_{T}^{r}(r) T_{12}]$$ \hfill (12)

where $r = r_{12}, r = |r|, \hat{r} = r/r, P_{r}$ is the isospin exchange operator, and the tensor operator is given by

$$T_{12} = 3 \sigma_{1} \cdot \hat{r} \sigma_{2} \cdot \hat{r} - \sigma_{1} \cdot \sigma_{2}.$$ \hfill (13)

Useful information about structure functions is contained in their moments of which Eq. (8) is an example for the lowest one. The next higher moment is given by the so called f-sum rule which is often discussed in the literature in the context of the density structure function and for spin-conserving interactions [18]. In Ref. [19] we derived a generalized f-sum rule for the spin-density structure function for one species of nucleons interacting via spin-dependent forces of the form Eq. (12) in a non-degenerate medium:

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \omega S_{\sigma}(\omega) = -\frac{4}{n_{b} V} \left\langle H_{T} \right\rangle.$$ \hfill (14)

Here, $H_{T}$ is the part of the total Hamiltonian involving the tensor operator $T_{ij}$. In the present paper we will consider both neutrons and protons but assume a central two-nucleon potential, i.e. absence of tensor contributions. The f-sum rule is then modified to
\[
\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\omega S_\sigma(\omega) = -\frac{4}{3n_b\sqrt{V}} \left( \frac{C_p - C_n}{C} \right)^2 \langle H^{np}_\sigma \rangle, \tag{15}
\]
where \(H^{np}_\sigma\) is the spin-dependent central part of the total Hamiltonian which contributes to neutron-proton interactions. Note from Eqs. (10) and (14) that for only one nucleon species a tensor interaction is required to give a non-trivial spin-density structure function. This is because the central part of the interaction conserves the total nucleon spin and thus does not contribute to its fluctuations. In contrast, for two nucleon species, a central spin-dependent proton-neutron interaction is sufficient for a non-trivial structure function as long as the neutral-current axial weak coupling constants for protons and neutrons are different [see Eq. (15)]. We stress, however, that the actual (positive) value of the \(f\) sum depends on all interaction terms via the states entering the thermal average.

### B. Relevance for Weak Interactions

The differential axial-vector-current neutrino-nucleon cross section is determined by the dynamical nucleon spin-density structure function \(S_\sigma(\omega,k)\), taken at the difference of initial and final neutrino four-momentum \((\varepsilon_1,k_1)\) and \((\varepsilon_2,k_2)\) via [13,16]:

\[
d\sigma_A = G_F^2 C_A^2 \frac{3}{4} S_\sigma(\varepsilon_1 - \varepsilon_2, k_1 - k_2) d^3k_2 \left( \frac{2\pi}{\tilde{\sigma}} \right)^3, \tag{16}
\]
where \(\theta\) is the angle between \(k_1\) and \(k_2\). In our convention, the neutral current axial-vector contribution to neutrino scattering rates on the ensemble of all nucleons is \(n_b d\sigma_A\).

The axion emission rate per volume, \(Q_a\), is governed by a structure function \(S_{\sigma,a}\) which is obtained from Eqs. (8), (9) by substituting \(C_i \rightarrow C_{\sigma,i}\) \((i = n, p)\),

\[
Q_a = \frac{C_{\sigma,n}^2 n_b}{(4\pi)^2 f_a^2} \int_0^{\infty} d\omega \omega^4 S_{\sigma,n}(\omega, \omega) \tag{17}
\]
where \(C_{\sigma,n}^2 = C_{\sigma,n}^2 Y_n + C_{\sigma,p}^2 Y_p\). We have assumed an isotropic medium such that \(S_{\sigma}(\omega, k)\) only depends on \(k = |k|\).

Various quantities relevant for neutrino diffusion are determined by the spin-density structure function. For the remainder of this section, we assume a Maxwell-Boltzmann distribution at temperature \(T_\nu\) for the neutrinos. Furthermore, we make use of the normalization given by Eq. (8). Contributions form spin-spin correlations represented by \(N_\sigma\) are mainly induced by the presence of nucleon bound states and by the Pauli exclusion principle which becomes important in a degenerate medium [19]. Both effects are small in the post-collapse phase of a supernova in which we are interested. We therefore assume \(N_\sigma \ll 1\) in Eq. (8). The average energy transfer per collision in a dilute medium can then be written as [10,11]

\[
\langle \Delta \varepsilon \rangle = \int_0^{\infty} \frac{dx}{2\pi} S_\sigma(x) \left( x + \beta x^2 + \frac{\beta^2 x^3}{12} \right) (e^{-x} - e^{-\beta x}), \tag{18}
\]
with \(\beta \equiv T/T_\nu\). This should be compared to the average energy transfer by nucleon recoils [20],

\[
\langle \Delta \varepsilon \rangle_{\text{recoil}} = \frac{30(\beta - 1)}{\beta^2} \frac{T^2}{m_N}. \tag{19}
\]

Another interesting quantity is the reduction of the average total axial-vector scattering cross section \(\langle \sigma_A \rangle\) [see Eq. (16)] in the nuclear medium [8,9]. First we note that a term of the form \(A\delta(\omega)\) in \(S_\sigma(\omega)\) corresponds to a total elastic scattering cross section

\[
\sigma_{el}(\varepsilon_1) = \frac{3A}{8\pi^2} G_F^2 C_A^2 \varepsilon_1^2. \tag{20}
\]
For \(n_b \rightarrow 0\) there are no spin fluctuations and correlations, and Eq. (8) implies \(S_\sigma(\omega) = 2\pi\delta(\omega)\) and thus \(\sigma_0 = \left(9/\pi\right) C_A^2 G_F^2 T^2\) for the thermally averaged cross section. In Ref. [8] we obtained the expression

\[
\frac{\delta(\sigma_A)}{\sigma_0} = \frac{\langle \sigma_A \rangle - \sigma_0}{\sigma_0} = N_\sigma - \int_0^{\infty} \frac{dx}{2\pi} S_\sigma(x) \left[ 1 - \left(1 + x + \frac{x^2}{6} \right) e^{-x} \right],
\]
which again holds in the dilute medium. The physical quantities discussed here will be calculated for the supernova environment in Sect. V.

### III. BEYOND THE BORN APPROXIMATION

#### A. Classical versus General Quantum Result

In the limit \(|\omega| \ll T\) the nucleon spin can be treated as a classical spin \(\mathbf{s}\) being changed abruptly by some random amount \(\Delta \mathbf{s}\) in a typical nucleon-nucleon collision event which takes place on a time scale \(\sim 1/T\) and thus appears to be “hard”. In this case we expect [21]

\[
S_\sigma(\omega) \simeq \frac{\Gamma_{\sigma}}{\omega^2 + \Gamma_{\sigma}^2/4}, \tag{22}
\]
where the spin fluctuation rate \(\Gamma_{\sigma}\) is related to the collision rate \(\Gamma_{\text{coll}}\) by

\[
\Gamma_{\sigma} = \frac{\langle |\Delta \mathbf{s}|^2 \rangle}{\langle \mathbf{s}^2 \rangle} \Gamma_{\text{coll}}. \tag{23}
\]
Note that the spin fluctuation rate suppresses the \(\omega^{-2}\) bremsstrahlung spectrum which otherwise would violate the existence of the normalization Eq. (8). This is known as the Landau-Pomeranchuk-Migdal (LPM) effect [22,23]. In previous work [24,25] it has been
discussed how the Lorentzian shape Eq. (22) might influence weak interaction rates at high densities where $\Gamma_{\sigma} \geq T$. The high density behavior of the spin-density structure function can also influence limits on the axion mass [23]. For $|\omega| \gg \Gamma_{\sigma}$ multiple scattering effects can be ignored and the spin-density structure function can be computed by a quantum mechanical treatment of two-nucleon scattering. From the generic $\omega^{-2}$ divergence of all bremsstrahlung processes for $\omega \to 0$ one expects the general form [9]

$$S_\sigma(\omega) = \frac{\Gamma_{\sigma}}{\omega^2} s(\omega/T) \times \left\{ \begin{array}{ll} e^{\omega/T} & \text{for } \omega < 0, \\ 1 & \text{for } \omega > 0, \end{array} \right. \quad (24)$$

where $s(x)$ is a nonsingular even function with $s(0) = 1$. The specific shape of $s(x)$ for $x \gtrsim 1$ depends on the nucleon-nucleon interaction potential Eq. (12), and its calculation for realistic interaction potentials is the main goal of this paper. Comparing Eqs. (22) and (24) in their common range of validity, $\Gamma_{\sigma} \ll |\omega| \ll T$, shows that the coefficient $\Gamma_{\sigma}$ of the bremsstrahlung divergence in Eq. (24) can be interpreted as a nucleon spin fluctuation rate and that the classical limit of hard collisions corresponds to $s(x) = 1$. The existence of the $f$ sum Eqs. (13), (14) shows that $s(x)$ has to decrease for large $x$ due to quantum corrections.

**B. Exact Treatment in the Limit of High Energy Transfers**

For $\omega \gtrsim \Gamma_{\sigma}$ where scattering involving more than two nucleons is negligible, we can numerically compute twonucleon wave functions from a given nucleon-nucleon interaction potential and use it in the general expression Eq. (4). For a central potential the eigenfunctions for the relative motion in the proton-neutron center of mass system

$$|P\rangle \equiv |p, l, m, S\rangle = R_{p\ell S}(r)Y_m(\Omega)|S\rangle \quad (25)$$

are characterized by the quantum numbers for the radial momentum, $p$, the orbital angular momentum, $l$ and $m$, and the total spin, $S$, where $R_{p\ell S}(r)$ is the radial wave function and $Y_m(\Omega)$ are the spherical harmonics. The corresponding energy eigenvalues $\omega_p$ have a $(2l + 1)(2S + 1)$ fold degeneracy. The Pauli exclusion principle then also determines the isospin to $I = \frac{1}{2} + (-1)^l \left( \frac{1}{2} - S \right)$. Assuming an isotropic medium and using $\sigma_\omega(0,k) = V^{-1/2} \sum_n \sigma_n e^{-ik\cdot r_\ell} \delta(\omega_p + \omega_\ell - k^2)/(4\pi m_N)^2/(2\pi)^3$ with a normalization volume $V = 1/n_b$, after some algebraic manipulations we obtain

$$S_\sigma(\omega, k) = \frac{16\pi^{1/2}}{3C^2} \frac{1}{k} \left( \frac{m_N}{T} \right)^{1/2} \frac{1}{Y_p Y_n} \frac{1}{Z_{CM}} \times \sum_{P,Q} e^{-\omega_p/T - m_N(\omega + \omega_\ell - \omega_Q - k^2)/(4m_N)^2/(2\pi)^3}$$

$$\times \left| \left\langle P \mid C_\ell \sigma_\ell e^{-i\mathbf{k} \cdot \mathbf{r}_\ell} + C_n \sigma_n e^{i\mathbf{k} \cdot \mathbf{r}_n} \mid Q \right\rangle \right|^2, \quad (26)$$

$$Z_{CM} = \sum_P e^{-\omega_p/T}. \quad (27)$$

where $Z_{CM}$ is the sum over all nucleon energy levels $\omega_p$. For $k \to 0$ this expression transforms into

$$S_\sigma(\omega) = 4\pi Y_p Y_n \left( \frac{C_\ell - C_n}{C} \right)^2 \frac{1}{Z_{CM}} \times \sum_{P,Q} \sum_{(2l + 1)} \frac{1}{\omega_\ell} e^{-\omega_p/T} \delta(\omega + \omega_\ell - \omega_Q)$$

$$\times \left| \int_{r_{\text{max}}} dr' R_{p\ell S}(r)R_{q\ell(1-S)}(r') \right|^2, \quad (27)$$

where we have made use of the orthogonality of the system of eigenfunctions which are supposed to be normalized to unity. In practice one constructs bound and scattering states of the stationary radial Schrödinger equation within a finite spherical volume of radius $r_{\text{max}}$ and computes the matrix elements appearing in Eq. (27). For a nucleon interaction potential that is not radially symmetric, the eigenstates $|P\rangle \equiv |p, jP\rangle$ are characterized by the total angular momentum $J$ and parity $P$ and are superpositions of orbital angular momentum eigenstates. With this modification, Eq. (22) still holds but we will not pursue this more complicated case here which would lead to coupled radial equations for the corresponding radial functions $R_p jP$.

Since we neglect interactions among more than two nucleons, our formalism does only account for neutrons, protons and deuterons. Higher nuclei such as helium are not included. In this sense, strictly, $n_b$, $Y_p$, and $Y_n$ have to be interpreted within the ensemble of neutrons, protons and deuterons only. Nuclear statistical equilibrium shows that in practice this does not make a big difference in our situation where $Y_p \ll Y_n$. Keeping this in mind we can now calculate $Z_{CM}$ analytically. Around the neutronosphere the nucleons are at best mildly degenerate. We therefore assume a Maxwell-Boltzmann distribution $f(p) = e^{-p^2/(2\mu T)}$ for the unbound proton-neutron states. Here, $p = s^{1/2}/2$ is the nucleon momentum in the center of mass system, expressed in terms of the squared center of mass energy $s$ (excluding the nucleon rest mass), and $\mu = m_N/2$ is the reduced nucleon mass. Taking into account the spin degrees of freedom we then have

$$Z_{CM} = 3e^{-\varepsilon_d/T} + \frac{4}{n_b} \left( \frac{\mu T}{2\pi} \right)^{3/2} e^{-\varepsilon_d/T}. \quad (28)$$

where $\varepsilon_d \approx 2.2$ MeV is the deuteron binding energy (the deuteron has $S = 1$). The degree of dissociation, i.e. the fractional abundance of unbound states is then

$$f_u = \left[ 1 + \frac{3}{4} n_b \left( \frac{2\pi}{\mu T} \right)^{3/2} e^{-\varepsilon_d/T} \right]^{-1}. \quad (29)$$

As a consequence, $S_\sigma(\omega)$ from Eqs. (26) and (27) does not exhibit a simple linear scaling with the nucleon density $n_b$, except for the dilute limit, $n_b \to 0, f_b \to 1$. It
can be seen that the numerator in Eqs. (23) and (27) is independent of $n_b$ and the density dependence of $S_p$ thus exclusively stems from $Z_{CM}$.

In the limit of zero temperature, $f_\nu \to 0$ and only the deuteron bound state will be populated in thermal equilibrium. In this limit, Eq. (14) describes the cross section for the weak neutral current deuteron break up process, Eq. (3). Integration over the phase space for the outgoing neutrino yields the total cross section

$$\sigma_{\nu d}^{NC}(\varepsilon_1) = \frac{3G_F^2}{16\pi^2} \bigg(\frac{C_p - C_n}{Y}\bigg)^2 \times \int_{\varepsilon_1}^{\omega_1} d\omega (\varepsilon_1 - \omega)^2 \lim_{T \to 0} S_\sigma(\omega)$$

for incident neutrino energy $\varepsilon_1$.

Another instructive limiting case is the absence of spin flip interactions. Scattering on protons and neutrons then has to be added incoherently with the states $|P\rangle$ now being plane waves. Eq. (26) then reduces to the ordinary recoil expression

$$S_\sigma(\omega, k) = \frac{1}{k} \bigg(\frac{2\pi m_N}{T}\bigg)^{1/2} e^{-m_N[\omega-k^2/(2m_N)]^2/(2Tk^2)}.$$  

(31)

Let us now get back to the general case. In agreement with Eq. (13) and Ref. 14, only proton-neutron scattering contributes to Eq. (27), and only if the neutral-current axial weak coupling constants for protons and neutrons are different. Since the total spin is conserved by a central potential, the spin-density structure function is governed by the fluctuations of the difference of the proton and neutron spins, $\sigma_p - \sigma_n$. To compare with the general results Eqs. (23) and (24), we study the corresponding spin-flip cross section which is defined as

$$\sigma_{sf}(s) = \frac{\langle \Delta (\sigma_p - \sigma_n) \rangle^2}{\langle (\sigma_p - \sigma_n)^2 \rangle} \sigma_{np}(s).$$  

(32)

Here, $\sigma_{np}(s) = \sum_l \sigma_{np}(l, s)$, where the average total proton-neutron scattering cross section in angular momentum state $l$ is given by

$$\sigma_{np}(l, s) = \frac{4\pi}{s} (2l + 1) \bigg[ 3 \sin^2 \delta_{l1}(s) + \sin^2 \delta_{l0}(s) \bigg]$$  

(33)
in terms of the phase shifts $\delta_{lS}(s)$. The latter are defined by the asymptotic behavior

$$R_{plS}(r) \propto \sin \bigg[ (2l + 1)r/2 \pm \delta_{lS}(8\mu_\nu P) \bigg],$$  

(34)
of the scattering states $\omega_p > 0$ for $(2l + 1)r/2 \gg l$. The nucleon spin flip rate is now just defined as

$$\Gamma_{sf} = Y_pY_n n_b \int_0^{\infty} dp dp f(p)(p/\mu)\sigma_{sf}(4p^2) \int_0^{\infty} dp dp f(p),$$  

(35)

where $p/\mu$ is the relative velocity of proton and neutron.

As can be seen from phase shift analysis, the spin flip cross section Eq. (23) is

$$\sigma_{sf}(s) = \frac{16\pi}{3} \sum_l (2l + 1) [\sin \delta_{l1}(s) - \sin \delta_{l0}(s)]^2.$$  

(36)

Note that this vanishes if the phase shifts for $S = 0$ and $S = 1$ are equal, as expected. Given $\sigma_{sf}$, one can compute $\Gamma_{sf}$ from Eq. (23) and compare it with the Born approximation to be discussed below and with Eq. (23). This will be done in the following two sections.

IV. THE BORN APPROXIMATION

By expanding the unbond states $|P\rangle$ into plane waves within first order perturbation theory and inserting the result into Eq. (4), one obtains the spin-density structure function in Born approximation. For $\omega > 0$, the result in the long wavelength limit is

$$S_\sigma^{Born}(\omega) = \frac{1}{\omega^2} \frac{\mu}{2\pi} \int_0^{\infty} dp f(p) \bigg[ U_{\sigma p}(k) \bigg]^2.$$  

(37)

where $k_{\text{max, min}} = (p^2 + 2\mu^2)^{1/2} \pm p$, and $U_{\sigma p}(k)$ is the Fourier transform of the coefficient of $\sigma_p \cdot \sigma_n$ in the proton-neutron interaction ($\sigma_p$ and $\sigma_n$ being the proton and neutron spins). Only the relative motion between proton and neutron influences Eq. (37) because neither energy nor momentum can be transferred to the center of mass motion in the long wavelength limit. In contrast to Eq. (27), $S_\sigma^{Born}(\omega)$ scales linearly with $n_b$.

In Ref. 14 we considered one species of nucleons coupling to a classical, external scattering center via an interaction of the form Eq. (4), where one of the spins was replaced by a classical spin $s$ associated with the external scatterer. The result for the spin-density structure function in the long wavelength limit in Born approximation is very similar to Eq. (37), for the case of a central two-nucleon potential and a medium of protons and neutrons with different neutral-current axial weak coupling constants.

In Born approximation, the spin flip cross section Eq. (23) evaluates to

$$\Gamma_{sf}^{Born} = \frac{20}{27\pi} \frac{\mu^2}{s} \int_0^{1/2} dk \bigg[ U_{\sigma p}(k) \bigg]^2,$$  

(38)

where $\langle (\sigma_p - \sigma_n)^2 \rangle = 3/2$ was used. Comparing Eqs. (23), (37), and (38) yields

$$\Gamma_{sf}^{Born} = \frac{27}{10} \bigg(\frac{C_p - C_n}{C}\bigg)^2 \Gamma_{sf}^{Born}.$$  

(39)
i.e. the spin fluctuation rate in $S_\sigma(\omega)$ and the average spin flip rate indeed agree within a factor of order unity, apart from the factor $[(C_p - C_n)/C]^2$ involving the weak coupling constants which results from our specific definition of $S_\sigma$.

As an example, we consider the usually adopted OPE potential which is a good approximation to the nucleon-nucleon interaction for distances greater than the inverse pion mass $m_\pi$. With $f \approx 1$ being the pion-nucleon coupling constant, its Fourier transform is

$$V_{\text{OPE}}(k, \sigma_1, \sigma_2) = -\left(\frac{2f}{m_\pi}\right)^2 \frac{(\sigma_1 \cdot k)(\sigma_2 \cdot k)}{k^2 + m_\pi^2} (2P_x - 1)$$

and it clearly has a tensor contribution. The spin-density structure function corresponding to this potential has been calculated in Born approximation [13]. Translated into our notation, the contribution from proton-neutron scattering takes the form of Eq. (24) with $s(x) \equiv \tilde{s}(x)/\tilde{s}(0)$ given by the function

$$\tilde{s}(x) = \int_0^\infty dv [v(v+x)]^{1/2} e^{-v}$$

$$\times \left[ 5C_4^2 + 3C_2^2 \right] s_1(v,x) + 2 \left( C_2^2 + C_2^2 \right) s_2(v,x)$$

$$- \left( 6C_2^2 + 2C_2^2 \right) s_3(v,x) \right],$$

where $C_\pm = (C_p \pm C_n)/2$ and

$$s_i(v,x) = \int_{-1}^{+1} dz \left[ \begin{array}{c} \frac{(2v+x-2z[v(v+x)])^{1/2}}{(2v+x-2z[v(v+x)])^{1/2} + y} \\ \frac{(2v+x)^2 - 4z(v+x)^2}{(2v+x+y)^2 - 4z(v+x)^2} \\ (2v+x+y)^2 - 4z(v+x)^2 \end{array} \right]$$

$$i = 1, 2, 3,$$

with $y = m_\pi^2/(m_N T)$. Furthermore, the contribution to the nucleon spin fluctuation rate $\Gamma_\sigma$ is

$$\Gamma_{\text{Born}}^{\text{OPE}} = \frac{2}{3} Y_p Y_n \frac{\tilde{s}(0)}{C^2} \Gamma_A,$$

with

$$\Gamma_A = 4\sqrt{\pi} \alpha_\pi \frac{n_0 T^{1/2}}{m_N^{5/2}} = 8.6 \text{ MeV} \rho_{13} T^{1/2}.$$  

More generally, as can be seen from Eq. (37), one has $S_\sigma^{\text{Born}}(\omega) \propto \omega^{-3/2 - r}$ for $\omega \to \infty$ if $U_\sigma(k) \propto k^{-r}$ for $k \to \infty$, corresponding to existence and square integrability of the $(r-2)$th derivative of the interaction potential. This should hold independently of the Born approximation which is viable in the limit of high energies as we will see now.

V. A NUMERICAL MODEL FOR THE SUPERNova ENVIRONMENT

We first note that $S_\sigma(\omega)$ from Eqs. (37) and (26) is proportional to the dimensionless factor

$$Y = Y_p Y_n \left( \frac{C_p - C_n}{C} \right)^2 = \left( C_p - C_n \right)^2 \frac{Y_p Y_n}{C_p Y_p + C_n Y_n},$$

which describes its compositional and coupling constant dependence for fixed $n_b$ and $T$. For processes involving only protons or neutrons this factor would be replaced by $Y_p^2$ and $Y_n^2$, respectively. Since interaction rates are proportional to $n_b S_\sigma$, by definition, $Y$ is a rough measure of the contribution of proton-neutron scattering to weak neutral current inelastic interaction rates. For the neutrino-nucleon coupling in a nuclear medium we will adopt $C_{A,p} \approx 1.09$, and $C_{A,n} \approx -0.91$ [13], so that $Y \approx 0.5$ for $Y_p \approx 0.1$.

In the following we are interested in the environment given inside but not far from the neutrinosphere in a supernova. For the rest of this section we choose the representative numbers $\rho = 10^{13} \text{ g cm}^{-3}$, $T = 8 \text{ MeV}$, and $T_\nu = 10 \text{ MeV}$. For these parameters, $f_\alpha \approx 0.36$, corresponding to a fractional deuterium abundance $Y_d \approx (1-f_\alpha) Y_p$. Nuclear statistical equilibrium involving higher nuclei gives values that are within $20 - 30\%$ of this if $Y_p \lesssim 0.2$. We stress again that due to the presence of bound states the spin-density structure function calculated by phase shift analysis and weak interaction rates computed from it do not exhibit a simple scaling behavior with density and/or temperature, as discussed below Eq. (28). Our choice represents a typical case of interest for the neutrino-nucleon energy transfer.

For the proton-neutron interaction potential $V_{S_{np}}^p(r)$ for total spin $S$ we chose the following Gaussian potentials [such that $U_\sigma = V_{S_{np}}^p - V_{S_{np}}$ and $U_{np} = (V_{S_{np}}^p + V_{S_{np}}^p)/2$ in the notation of Eq. (28)]:

$$V_{0_{np}}^p(r) = -33.6 e^{-(r/1.77\text{ fm})^2} \text{ MeV}.$$  

Its strengths and ranges were fit to reproduce the experimental values for the scattering lengths $a_{np}$ and effective ranges $r_{eff,S}$ which determine the low-energy expansion of the phase shifts $\delta_{0,S}$ [17]:

$$V_{1_{np}}^p(r) = -84.7 e^{-(r/1.36\text{ fm})^2} \text{ MeV}.$$  

(46)
As a result, the \(s\)-wave proton-neutron scattering cross section predicted by Eq. (46) agrees with the experimental one to within less than 5% in the laboratory energy range between 0 and \(\simeq 20\) MeV (see Fig. 1). In addition, the energy of the bound state resulting for \(S = 1\) coincides with the deuteron binding energy within 5%. A central potential describes the deuteron rather well since the contribution of the D state to the bound state wave function is only about 6%. Finally, we have compared the numbers for the weak neutral current deuteron break up cross section resulting from Eq. (38) with calculations in the literature [27]. In the energy range between \(\simeq 5\) MeV and 40 MeV we found agreement to within about 10%. This serves as a further check for the correct normalization of our calculation.

Also shown in Fig. 1 is the spin flip cross section as calculated from the potential Eq. (46) both in Born approximation [Eq. (38)] and numerically from the phase shifts [Eq. (35)]. It is clearly seen that the Born approximation is far from being good. The integrated spin flip rate Eq. (35), \(\Gamma_{sf} \simeq 0.46 Y_p Y_n \) MeV, differs by about a factor 2.5 from the Born approximation \(\Gamma_{sf}^{\text{Born}} \simeq 0.21 Y_p Y_n \) MeV. This is also reflected by the nucleon spin-density structure function calculated from Eqs. (44) and (27) for \(k \to 0\), as shown in Fig. 2. The phase shift analysis was performed by computing the radial eigenfunctions up to some maximal orbital angular momentum \(l_{\text{max}} = 3\) above which they are close enough to the free eigenfunctions to make a negligible contribution to Eq. (27). To achieve a sufficient resolution in the energy range of interest, about 500 eigenfunctions had to be computed. We verified that the resulting \(S_\sigma(\omega)/Y\) satisfies the \(f\) sum rule Eq. (15) to within 10%. Note, furthermore, that the Born approximation and the phase shift calculation of the quantities shown in Figs. 1 and 2 converge at high energies where the second condition in Eq. (1) is asymptotically satisfied. We also verified that for a weak interaction potential satisfying the first condition in Eq. (1), the Born approximation agrees well with the phase shift analysis over the whole energy range as expected.

For comparison, Fig. 1 also shows Eq. (24) with the perturbative expressions Eqs. (41, 44) for the proton-neutron scattering contribution to \(S_\sigma\) based on the OPE potential (thin dashed curve). After all, this curve reproduces the general normalization of the spin-density structure function quite well, but it cannot reproduce the quite prominent deuteron resonance. The non-vanishing
pion mass is taken into account in this curve and suppresses it by roughly a factor 2 compared to calculations neglecting the pion mass. Note the steepening at high $\omega$ of the curves for the potential Eq. (14) in contrast to $\sigma_{\sigma,\text{OPE}}(\omega)$ which guarantees or violates $\sum$ integrability, respectively.

The quantities of interest for neutrino diffusion as discussed in Sect. II are shown in Table I. In calculating the cross section reduction Eq. (12) we neglected the term $N_\sigma$ involving spin-spin correlations. It is easy to see that the $S = 1$ deuteron bound state yields the contribution

$$N_\sigma = \frac{C_{A,p}C_{A,n}Y_d}{C_A^2} 3 \tag{48}$$

which implies a further reduction of $\langle \sigma_A \rangle$ because of the opposite sign of $C_{A,p}$ and $C_{A,n}$. This corresponds to the fact that the cross section for elastic scattering on deuterons is significantly smaller than that on free nucleons. Also note that Eqs. (18) and (19) imply that $\langle \Delta\varepsilon \rangle \propto T_0 - T$ for $T_0 - T \ll T$. Finally, we observe that $\Gamma_\sigma/\Gamma_{sf} \approx \Gamma_{\sigma,\text{Born}}/\Gamma_{sf,\text{Born}}$, i.e., the relation Eq. (22) between the spin fluctuation rate $\Gamma_\sigma$ appearing in $S_\sigma(\omega)$ and the spin flip rate $\Gamma_{sf}$ also holds beyond the Born approximation.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$\Gamma_\sigma/Y$ [MeV] & $\langle \Delta\varepsilon \rangle/(T_0 - T)$ & $\delta \langle \sigma_A \rangle/\langle \sigma_0 Y \rangle$ \\
\hline
recoil & - & 0.32 & - \\
OPE, Born & 3.6 & 0.18Y & -0.14 \\
Eq. (10), Born & 0.66 & 0.029Y & -0.018 \\
Eq. (10), exact & 1.5 & 0.15Y & -0.076 \\
\hline
\end{tabular}
\caption{Comparison of the quantities shown in the top row for the cases specified in the first column for proton-neutron interactions. The first entry in the third column is the proton-neutron potential used, and the second entry indicates whether the Born approximation ["Born", according to Eq. (22)] or the exact two-nucleon wave functions ["exact", according to Eq. (22)] have been used. For the OPE case, $C_p = 1.99$ and $C_n = -0.91$ have been assumed. Spin-spin correlations have been neglected in $\delta \langle \sigma_A \rangle/\langle \sigma_0 \rangle$ (see text).}
\end{table}

Clearly, the Born approximation for the potential Eq. (10) does not give a good estimate to any of these quantities. However, the numbers based on the OPE potential in Born approximation give a reasonable estimate to most of the integrated quantities in Table I. In particular, these results confirm that the average inelastic neutrino-nucleon energy transfer $\langle \Delta\varepsilon \rangle$ is indeed comparable to the recoil energy $\langle \Delta\varepsilon \rangle_{\text{recoil}}$, as suggested by calculations employing the Born approximation for the OPE potential Eq. (10). This energy transfer is, however, differently distributed with a much longer tail to high energy transfers, as can be seen in Fig. 3.

We have furthermore checked that the results for $S_\sigma$ calculated from the phase shift analysis Eq. (27) are insensitive to the detailed shape of $U^{\sigma,p}(r)$ as long as it reproduces the experimental phase shifts in the corresponding energy range. In particular, properties of the potential at short distances $r$ influence $S_\sigma(\omega)$ only for $\omega = p^2/m_N \gtrsim 1/(m_Nr^2)$. For the conditions near the neutrinosphere it is therefore sufficient that the potential reproduces nucleon-nucleon scattering up to a few tens of MeV.

Towards the center of the hot neutron star, at densities around nuclear density and $T \approx 30 - 50$ MeV, predictions for the quantities shown in Table I by the OPE potential in Born approximation are about 10 times higher than corresponding predictions based on the potential Eq. (10) for which Born approximation and phase shift analysis become rather similar. This shows that in this environment weak interaction rates become quite sensitive to the short distance behavior of the two-nucleon interaction potential which is different for these two potentials. This can have important ramifications for neutrino opacities and axion emissivities in the supernova core that are usually based on these OPE calculations [11,12]. Whereas calculating assuming an OPE potential should be a reasonable approximation in the context of a "cold" neutron star this is not necessarily the case for the much
higher thermal energies involved in a hot protoneutron star. Neutrino opacities govern the cooling time scale of the protoneutron star \cite{4} while axion emissivities determine axion mass bounds based on supernovae \cite{5}. Apart from taking into account many-body effects such as multiple scattering \cite{5}, \cite{6}, \cite{7}, \cite{8}, \cite{9} a more reliable calculation of these quantities thus requires to use nuclear potentials that fit nucleon-nucleon scattering data also at energies above a few tens of MeV to ensure the correct small distance behavior. We leave that to a separate study.

VI. SUMMARY AND CONCLUSIONS

We have discussed weak axial-vector current interactions involving nucleons in hot non-degenerate nuclear matter at temperatures around 10 MeV and densities of a few percent of nuclear saturation density, i.e. for conditions given in the vicinity of the neutrinosphere in a type-II supernova. To describe such interactions in the limit of non-relativistic nucleons we adopted the structure function formalism for the nucleon spin-density. Our special emphasis was on the energy transfer between the weak “probe” and the nuclear medium which is induced by the nucleon spin fluctuations caused by the spin-dependent nucleon-nucleon interactions. To lowest order in the nucleon-nucleon interactions, i.e. in Born approximation, this is represented by nucleon bremsstrahlung. We have shown, however, that the Born approximation is in general not a reliable estimate for these effects. As an alternative, we have performed computations using exact two-nucleon wave functions for a spherically symmetric two-nucleon interaction potential that was fit to experimental data. In this case, only proton-neutron interactions contribute to inelastic weak neutral-current interactions with nucleons. We compared our calculations with results for the corresponding contribution based on the usually adopted OPE potential in Born approximation. One gets rather good agreement for most integrated cross sections and the average energy transfer in a scattering event. In particular, we confirm that the latter is comparable to the recoil energy, as suggested by the results for the OPE potential in Born approximation. In contrast, differential quantities such as the distribution of energy transfers deviate significantly from predictions based on the Born approximation. Our calculations specifically predict that the energy transfer peaks at the deuteron binding energy. Resonances from higher nuclei which have not been taken into account here such as helium are probably less important for \( T \lesssim 10 \text{ MeV} \) because they would appear at higher energies and contribute less to the energy transfer distribution (see Fig. 3).

The formalism presented here can be extended to two-nucleon potentials that are not spherically symmetric and to finite momentum transfer [see Eq. \( (26) \)]. Our results might have a significant impact on the formation of neutrino spectra from type-II supernovae. A quantitative understanding will, however, require detailed numerical simulations. Finally, we demonstrated that weak interaction rates in the hot supernova core are sensitive to the small distance behavior of the nucleon-nucleon interaction potential which is not well described by the usually adopted OPE potential. This should be taken into account in future investigations.

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