Deformation Expression for Elements of Algebras (VII)
–Vacuum/Pseudo-vacuum Representations–

Hideki Omori*  Yoshiaki Maeda†  Akira Yoshioka ‡
Tokyo University of Science  Keio University  Tokyo University of Science

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*Department of Mathematics, Faculty of Sciences and Technology, Tokyo University of Science, 2641, Noda, Chiba, 278-8510, Japan, email: omori@ma.noda.tus.ac.jp
†Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1, Hiyoshi, Yokohama, 223-8522, Japan, email: maeda@math.keio.ac.jp
‡Department of Mathematics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Tokyo, 102-8601, Japan, email: yoshioka@rs.kagu.tus.ac.jp
6 Vacuum representations of \(\ast\)-exponential functions of quadratic forms

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Thinking back the long history of physics, we see that the calculation used by physicists was nothing but the ordinary calculus. Another word, physicists have never wrote theories beyond the basic axioms of the calculus. This is not to declare of the victory of calculus or algebraic topology. On the contrary, we are thinking that every theory of mathematical physics must suggest new frontier of ordinary calculus, which are never viewed by classical geometers.

Weyl algebras or Heisenberg algebras are naturally involved in slightly extended systems of the algebra of ordinary calculus, and are supported by the classical notion of phase spaces on which the general mechanics are based. The theory of deformation quantizations gives a notion of quantization of “phase space”. To explain its essence in brief we proposed in the previous note the notion of \(\mu\)-regulated algebra.

Now, as it is a quantization of phase space, it does not give Schrödinger quantizations which are written by partial differential operators on “configuration spaces”.

For instance, the classical motion described by the Hamiltonian \(H(q, p)\) is given by the equation on the phase space

\[
\frac{d}{dt} A_t(q, p) = \{A_t, H\}
\]

by using a Poisson bracket. Its deformation quantization is simply to write this in the form

\[
\frac{d}{dt} A_t(u, v) = [A_t, H_*]
\]

by using the commutator bracket in a certain \(\mu\)-regulated algebra. The solution is then given by using the \(\ast\)-exponential function \(\text{Ad}(e_{\ast}^{tH_*})A_0\). But \(e_{\ast}^{tH_*}\) and \(\text{Ad}(e_{\ast}^{tH_*})A_0\) are only elements of transcendentally extended \(\mu\)-regulated algebra. This does not give the solution of the Schrödinger equation. We have to consider the (time independent) “state vector” \(\psi\) and the expectation value \(\langle \psi, A_t \psi \rangle\) to obtain the Schrödinger picture. Here we have to note that the group \(e_{\ast}^{tH_*}\) is often a covering group of \(\text{Ad}(e_{\ast}^{tH_*})\).

Note that there is no mathematical criterion for the state vectors, or the configuration space in the quantized phase space. We are vaguely thinking that state vectors are elements of a certain representation space and the configuration space is a place where one can draw differential geometrical pictures to objects.

In this series, we have introduced elements, called “vacuums” to consider the state vectors and the configuration spaces within the world of extended algebra of calculus with various expressions. We have found several strange elements, called polar elements, and an extended notions of vacuums, which were called pseudo-vacuums in \[13\]. These are not established notions in mathematical physics, but we are thinking that these must propose new frontier for mathematical physics. We are thinking that vacuums and pseudo-vacuums are not unique, but the function algebra of the configuration spaces must be an algebra similar to the Frobenius algebra defined by vacuums.

The point in this note is that to obtain classical pictures one has often to restrict the expression parameters, and there are two essentially different expression parameters.
1 Preliminaries

First we recall the summary of the notions through the papers [10]-[15] and former results together which will be used in this note. Throughout this note, we use notations as follows:

\[ u = (u_1, u_2, \ldots, u_{2m}) = (\tilde{u}, \tilde{v}), \quad \bar{u} = (\tilde{u}_1, \ldots, \tilde{u}_m), \quad \bar{v} = (\tilde{v}_1, \ldots, \tilde{v}_m) \]
\[ x = (x_1, x_2, \ldots, x_{2m}) = (\tilde{x}, \tilde{y}), \quad \xi = (\xi_1, \xi_2, \ldots, \xi_{2m}) = (\tilde{\xi}, \tilde{\eta}). \]

We also recall the Weyl algebra \( W_n[\hbar] \), the associative algebra generated by \( \tilde{u}, \tilde{v} \) with the fundamental relations

\[ [\tilde{u}_k, \tilde{v}_l] = -i\hbar \delta_{kl}, \quad [\tilde{u}_k, \bar{u}_l] = 0 = [\bar{v}_k, \tilde{v}_l], \quad \hbar > 0. \]

It is wellknown that this algebra is expressed by giving a product formula on the space of polynomials.

1.1 General product formulas and intertwiners

Let \( \mathcal{S}_\mathbb{C}(n), \; n=2m, \) be the space of complex symmetric matrices. For a fixed \( K \in \mathcal{S}_\mathbb{C}(n) \) and the standard skew-symmetric matrix \( J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \) we set \( \Lambda = K + J \) and define a product \( *_K \) on the space of polynomials \( \mathbb{C}[u] \) by the formula

\[ f *_K g = f e^{\frac{i\hbar}{2} \sum \Lambda_{ij} \partial_i \partial_j} g = \sum_k (\frac{i\hbar}{k!})^k \Lambda_{ij_1} \cdots \Lambda_{ij_k} \partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f \partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g. \]

(\( \mathbb{C}[u], *_K \)) is known to be an associative algebra isomorphic to the Weyl algebra \( W_n[\hbar] \).

For another symmetric matrix \( K' \), we have the following formula:

\[ e^{\frac{i\hbar}{2} \sum K'_{ij} \partial_i \partial_j} \left( (e^{-\frac{i\hbar}{2} \sum K_{ij} \partial_i \partial_j} f) *_K (e^{-\frac{i\hbar}{2} \sum K_{ij} \partial_i \partial_j} g) \right) = f *_{K+K'} g. \]

The next one is proved immediately by the formula [3]:

**Corollary 1.1** Let \( I_0^K(f) = e^{\frac{i\hbar}{2} \sum K_{ij} \partial_i \partial_j} f \) and \( I_K^0(f) = e^{-\frac{i\hbar}{2} \sum K_{ij} \partial_i \partial_j} f \). Then \( I_0^K \) is an isomorphism of \( (\mathbb{C}[u]; *) \) onto \( (\mathbb{C}[u]; *_K) \).

The isomorphism class is denoted by \( (W_{2m}[\hbar], *) \) and called the Weyl algebra. The operator

\[ I_K^{K'}(f) = \left[ \exp \left( \frac{i\hbar}{4} \sum_{i,j} (K_{ij} - K'_{ij}) \partial_i \partial_j \right) \right](f) = I_0^K (I_0^K)^{-1}(f) \]

will be called the **intertwiner**. Intertwiners do not change the algebraic structure *, but do change the expression of elements by the ordinary commutative structure.

Let \( Hol(\mathbb{C}^n) \) be the space of all holomorphic functions on the complex \( n \)-plane \( \mathbb{C}^n \) with the uniform convergence topology on each compact domain. \( Hol(\mathbb{C}^n) \) is a Fréchet space defined by a countable family of seminorms. It is clear that the product \( f *_K g \) is defined if one of \( f, g \) is a polynomial and another is a smooth function.

**Proposition 1.1** For every polynomial \( p(u) \in \mathbb{C}[u] \), the left-multiplication \( f \to p(u) *_K f \) and the right-multiplication \( f \to f *_K p(u) \) are both continuous linear mappings of \( Hol(\mathbb{C}^n) \) into itself.

If two of \( f, g, h \) are polynomials, then associativity \((f *_K g) *_K h = f *_K (g *_K h) \) holds.
1.2 Generic expression parameters

Consider for instance an element $u_1^2 *_{K_0} u_2 *_{K} u_1 *_{K} u_2^3$, which may be expressed differently via commutation relations. Computing out this by the product formula \(2\) gives the way to express the element univalent way. In this sense, the \(*_{K}\)-product formula is the \(*_{K}\)-expression formula for elements of algebra. Note that according to the choice of $K = 0, K_0, -K_0, I$, where

\[
(0, K_0, -K_0, I) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},
\]

the product formulas \(2\) give the Weyl ordered expression and the normal ordered expression, the antinormal ordered expression respectively, but the unit ordered expression is not so familiar in physics.

For each ordered expression, the product formulas are given respectively by the following formula:

\[
f(u)*_{g} g(u) = f \exp \frac{hi}{2} \{ \partial_u \wedge \partial_u \} g, \quad \text{(Moyal product formula)}
\]

\[
f(u)*_{K_0} g(u) = f \exp h i \{ \partial_u \partial_u \} g, \quad \text{(\textup{\Psi}DO product formula)}
\]

\[
f(u)*_{-K_0} g(u) = f \exp -h i \{ \partial_u \partial_u \} g, \quad \text{(\textup{\Psi}DO product formula)}
\]

where \(\partial_u \wedge \partial_u = \sum_i (\partial_{u_i} \partial_{u_i} - \partial_{u_i} \partial_{u_i})\) and \(\partial_u \partial_u = \sum_i \partial_{u_i} \partial_{u_i}\).

1.2.1 \(V\)-class of expression parameters

The product formula for the unit ordered expression looks a bit complicated to write down, but there are many interesting phenomena which has never appeared in Weyl-, normal-, or anti-normal-ordered expressions. We also give another family of expression parameters which will be often used in this note and involving \(iI\) expression parameter as a special case.

Set $n = 2m$. Let $V$ be a real $n$-dimensional subspace of $\mathbb{C}^n$ spanned by an $O(n)$ frame such that $JV = V$ and $\mathbb{C}^n = V \oplus iV$.

We denote $\pi_{re}, \pi_{im}$ the projections onto $V, iV$ respectively. As $\mathbb{C}^n = \mathbb{C}_x^m \oplus \mathbb{C}_y^m$ and $V \oplus iV = \mathbb{C}^n$, we see

\[
V = V \cap \mathbb{C}_x^m \oplus V \cap \mathbb{C}_y^m, \quad J(V \cap \mathbb{C}_x^m) = V \cap \mathbb{C}_y^m.
\]

We define a bilinear form $\langle \hat{x}, \hat{x}' \rangle$ on $V$ by reducing the canonical bilinear form to the subspace $V$. Since $V$ is spanned by the $O(n)$ frame we see it is positive definite and $\langle \hat{x} J, \hat{x}' \rangle = -\langle \hat{x}, \hat{x}' J \rangle$.

For a complex symmetric matrix $K \in \mathbb{G}(n)$, $\pi_{re} K; V \rightarrow V$ and $\pi_{im} K; V \rightarrow iV$ are called the $V$-real part and the $V$-imaginary part of $K$ respectively.

A class of expression parameters will be called a $V$-class, denoted by $\mathcal{H}_+(V)$, is a special class of expression parameters such that the real part of $iK$ is on $V$ is positive definite, i.e.

\[
\mathcal{H}_+(V) = \{K; \frac{1}{\mathbb{R}} \langle \xi \pi_{re}(iK) \rangle, \xi \} \geq c_K |\xi|^2, \quad \forall \xi \in V\}.
\]
If $V=\mathbb{R}^n$, then $\pi_{re}(iK)$ is positive definite on $\mathbb{R}^n$, but if $V=(i\mathbb{R})^n$, then $\pi_{re}(iK)$ is negative definite on $\mathbb{R}^n$. If $V=(\sqrt{t}\mathbb{R})^n$, then $\pi_{re}K$ is positive definite on $\mathbb{R}^n$.

Let $dV$ be the standard volume element on $V$. We denote by $dV=(\frac{1}{\sqrt{2\pi i}})^n dV$.

**Lemma 1.1** If $K \in \mathbb{H}_+(V)$, then $\int_V e^{-\frac{1}{i\hbar} \langle (iK)\xi,\xi \rangle} dV = \frac{2^m}{\sqrt{\det(iK)}}$, where $\sqrt{\det(iK)}$ is determined without sign ambiguity in the form

$$\sqrt{|\det(iK)|} \prod_{i=1}^m \sqrt{(1+ia_i)}, \quad \sqrt{|\det(iK)|} > 0, \quad \text{Re} \sqrt{(1+ia_i)} > 0.$$

If $K$ is fixed, this does not depend on the choice of $V$ whenever $K \in \mathbb{H}_+(V)$.

**Proof** We first fix $V$. Set $iK=R^2+iS$, where $R: V \to V$ is symmetric and positive definite on $V$ and $S$ is any real linear mapping of $V \to V$. Let $\lambda_1, \ldots, \lambda_2m$ be the eigenvalues of $R$. By a suitable $T \in SO(V)$, $iK$ is changed into

$$T(iK)T^{-1} = \text{diag}\{\lambda_1^2, \ldots, \lambda_{2m}^2\} + iTST^{-1}.$$

Changing variables by setting $\eta_i=\lambda_i \xi_i$, the integral turns out

$$\frac{1}{\det R} \int_{\mathbb{R}^{2m}} e^{-\frac{1}{i\hbar} (\sum_{k=1}^{2m} \eta_k^2 + t \sum_{k,l=1}^{2m} (R^{-1}SR^{-1})_{kl} \eta_k \eta_l)} d\eta,$$

where $(R^{-1}SR^{-1})_{kl}$ is a real matrix. Hence by a suitable $T' \in SO(2m)$, the integral becomes

$$\frac{1}{\det R} \int_{\mathbb{R}^{2m}} e^{-\frac{1}{i\hbar} \sum_{k=1}^{2m} (1+i\mu_k) \eta_k^2} d\eta, \quad \mu_k \in \mathbb{R}.$$

Note that Cauchy’s integral theorem and rotation of the path of integration gives

$$\frac{1}{\sqrt{2\pi i}} \int_{\mathbb{R}} e^{-\frac{1}{2\pi i} (1+i\mu_k) \eta_k^2} d\eta = \frac{\sqrt{2\pi i}}{\sqrt{1+i\mu_k}}, \quad \text{Re} \sqrt{(1+i\mu_k)} > 0.$$

It follows

$$\int_V e^{-\frac{1}{i\hbar} \langle (iK)\xi,\xi \rangle} d\xi = \frac{1}{\det R} \frac{1}{\sqrt{\det(I+iR^{-1}(iS)R^{-1})}} = \frac{2^m}{\sqrt{\det(iK)}},$$

and $\sqrt{\det(iK)} = \sqrt{|\det(iK)|} \prod_{k=1}^m \sqrt{(1+i\mu_k)}$. The result depends only on $K$, hence the integral gives the same result for $K \in \mathbb{H}_+(V) \cap \mathbb{H}_+(V')$. \hfill $\square$

In what follows we set the constant by

$$C_0(K) = \frac{2^m}{\sqrt{\det iK}} = \int_V e^{\frac{1}{\sqrt{\det iK}} |\xi|^2} d\xi = \int_V e^{-\frac{1}{\sqrt{\det iK}} |\xi|^2} d\xi,$$

where $\sqrt{\det iK} = \frac{1}{\sqrt{2\pi i}} n/2$. (7)
1.2.2 Some remarks about Fourier transformations

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all rapidly decreasing functions. First, express this space as the projective limit space of a family of Hilbert spaces. Taking the topological completion of $\mathcal{S}(\mathbb{R}^n)$ by the norm topology defined by the weighted $C^k$-inner product $\langle \cdot, \cdot \rangle_k$. We denote it by $\mathcal{S}^k(\mathbb{R}^n)$. The Sobolev lemma gives that $\mathcal{S}(\mathbb{R}^n) = \bigcap_k \mathcal{S}^k(\mathbb{R}^n)$. Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{i\langle \xi, x \rangle}dx, \quad \text{where } dx = \left(\frac{1}{2\pi\hbar}\right)^{n/2}d\xi.$$  

$\hat{f}(\xi)$ is sometimes denoted by $\hat{f}(\xi)$. Fourier transform is defined for $L^1$-functions at first, and extends in various ways.

**Lemma 1.2** If $f$, $f'$, $f''$ are continuous and summable, then $\hat{f}(\xi)$ is summable, and it holds that $||f||_{L^2} = ||\hat{f}(\xi)||_{L^2}$.

Fundamental properties of Fourier transform are all proved by integration by parts:

$$\hat{\mathcal{F}}(\mathcal{F} f) = (\frac{i}{\hbar}\xi)^{n}\hat{f}(\xi), \quad \hat{\mathcal{F}}((\frac{i}{\hbar} x)^{n} f) = \partial^{n}\hat{f}(\xi).$$

It is wellknown that the Fourier transform $\hat{f}$ gives a topological isomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. Furthermore, it gives a topological isomorphism of $\mathcal{S}^k(\mathbb{R}^n)$ onto $\mathcal{S}^k(\mathbb{R}^n)$ for every $k$. Thus, $\hat{f}$ gives a topological isomorphism of the dual space $\mathcal{S}'(\mathbb{R}^n)$ onto $\mathcal{S}'(\mathbb{R}^n)$.

Let $\mathcal{S}^{-k}(\mathbb{R}^n)$ be the dual space of $\mathcal{S}^k(\mathbb{R}^n)$. $\mathcal{S}^{-k}(\mathbb{R}^n)$ is a Hilbert space. We see easily that

$$\mathcal{S}^{-\infty}(\mathbb{R}^n) = \bigcup_{k} \mathcal{S}^{-k}(\mathbb{R}^n)$$

with the inductive limit topology. Elements of $\mathcal{S}^{-\infty}(\mathbb{R}^n)$ are called **tempered distributions**. If a tempered distribution $f$ is a function, that is, the value $f(x)$ is defined for every $x$, then $f$ is called a **slowly increasing function**.

For the convenience of notations, we denote

$$\delta(x-a) = \hat{\mathcal{F}}^{-1}(1) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\langle \xi, x-a \rangle}d\xi, \quad \int_{\mathbb{R}^n} \delta(x-a)e^\frac{i}{\hbar}(\xi, x)dx = e^\frac{i}{\hbar}(\xi, a).$$

Now let $(\mathcal{B})$, called the space of all bounded derivatives class, be a little wider function space than $\mathcal{S}$ consisting of all smooth functions $f$ such that the differential $\partial^{\alpha}f$ is bounded on $\mathbb{R}^n$ for every $\alpha$. The dual space $(\mathcal{B})'$ is convenient to make the convolution product:

$$f* g(x) = \int_{\mathbb{R}^n} f(y) g(x-y)dy.$$  

**Remark** The convolution product is welldefined if one of them is a rapidly decreasing distribution, where $f(x)$ is a rapidly decreasing distribution, iff $f(x)(1+|\xi|^2)^k$ belongs to the dual space of all bounded smooth functions on $\mathbb{R}^n$ for every integer $k$ (i.e. $(\mathcal{B}')$). $(\mathcal{B})$ forms a commutative algebra. Hence $(\mathcal{B}')$ forms also a commutative algebra under the convolution product.
Twisted convolution product \(*_J\)

\[
\int_{\mathbb{R}^n} f(Jg(J,x))e^{i\frac{1}{\hbar}(J,J)\xi}d\xi. 
\]

As \(|e^{i\frac{1}{\hbar}(J,J)}| = 1\), the twisted convolution product is well-defined under the same condition as above. For instance setting \(\delta_a(x) = \delta(x-a)\), we have \(\mathcal{F}(\delta_a)(\xi) := \delta_{\xi}(\xi) := e^{-\frac{i}{\hbar}(\xi,a)}\) and

\[
\dot{\delta}_a *_J \dot{\delta}_b(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}(\xi,a)} e^{i\frac{1}{\hbar}(J,J)\xi} e^{-\frac{i}{\hbar}(\xi-c)b} d\xi = \delta(b-a-\frac{1}{2}J)e^{-\frac{i}{\hbar}(\xi,b)}. 
\]

2 Star-exponential functions of linear forms

Let \(\langle a, u \rangle\) be a linear function. Then, we have

\[
\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(a,u)}dt = e^{\frac{i}{\hbar}(a,K-K')a} e^{\frac{i}{\hbar}(a,u)}. 
\]

Hence, \(\{e^{\frac{i}{\hbar}(a,K')a} e^{\frac{i}{\hbar}(a,u)}; K \in \mathbb{C}(2m)\}\) is a family of mutually isomorphic one parameter groups. We shall denote this family symbolically by \(e^{\frac{i}{\hbar}(a,u)}\), but it is often better to view this as an element. Its \(K\)-ordered expression will be denoted by

\[
e^{\frac{i}{\hbar}(a,u)} = e^{\frac{i}{\hbar}(a,K,a)} e^{\frac{i}{\hbar}(a,u)} = e^{\frac{i}{\hbar}(a^K,a)+\frac{i}{\hbar}(a,u)} = e^{\frac{i}{\hbar}(a(K),a)+\frac{i}{\hbar}(a,u)}. 
\]

Now, we see why the \(V\)-class of expression parameters and Lemma [11] are crucial in this note. In a \(V\)-class expression, \(e^{\frac{i}{\hbar}(a,u)} :\mathbb{C}\) is rapidly decreasing w.r.t. \(\xi \in V\). Hence in a certain \(K\)-expression, integrals

\[
\int_{-\infty}^{0} e^{\frac{i}{\hbar}(a,u)}dt, \quad \int_{0}^{\infty} e^{\frac{i}{\hbar}(a,u)}dt
\]

give two different inverses \((\frac{1}{i\hbar}(a,u))^\pm_1\) such that \((\frac{1}{i\hbar}(a,u))^\pm_1 = \int_{-\infty}^{0} e^{\frac{i}{\hbar}(a,u)}dt\).

In general, for any holomorphic function \(H_t\) of \(u\), the \(*\)-exponential function \(e^{tH_t}\) may be defined as the collection \(e^{tH_t};\mathbb{C}\) of power series \(\sum \frac{t^n}{n!}H^n;\mathbb{C}\) formally rearranged as power series of \(t\). In the case \(h\) is not formal, the \(*\)-exponential function \(e^{tH_t}\) is not defined by a power series. Instead, these are considered as the real analytic solution of an evolution equation

\[
\frac{d}{dt}H_t^* = [H_t^*, K]^*;H_t^*;K, \quad H_0^* = 1. 
\]

This is a differential equation, if \(H_t\) is a polynomial, but the solution may not exist in general. The solution (if exists uniquely) with initial data \(H_0^*\) will be denoted by \(e^{tH_t}\). In particular, it is easy to have the exponential law with an ordinary exponential functions

\[
e^{tH_t} e^{tK} = e^{tH_t + tK}. 
\]

Fundamental formulas for calculations are easily obtained by the product formula:
Proposition 2.1 For every expression parameter $K$
\[
\hat{e}^{\frac{1}{i K}}(\mathbf{u}) \cdot \hat{e}^{\frac{1}{i K}}(\mathbf{b}) = e^{\frac{1}{i K} \langle \mathbf{a}, \mathbf{b} \rangle} \cdot \hat{e}^{\frac{1}{i K}}(\mathbf{a}) \cdot \hat{e}^{\frac{1}{i K}}(\mathbf{u}).
\]
Hence this identity may be written without suffix $: _K$. On the other hand, the $*_K$-product of ordinary exponential function and a holomorphic function $f(\mathbf{u})$ is
\[
e^{\frac{1}{i K} \langle \mathbf{a}, \mathbf{u} \rangle} f(\mathbf{u}) = e^{\frac{1}{i K} \langle \mathbf{a}, \mathbf{u} \rangle} f(\mathbf{u} + \frac{1}{2} \mathbf{a}(K + J)).
\]
Since the notations $\mathbf{u} = (\tilde{u}, \tilde{v}) = (\tilde{u}, 0) + (0, \tilde{v})$, $\xi = (\tilde{\xi}, \tilde{\eta}) = (\tilde{\xi}, 0) + (0, \tilde{\eta})$, $\mathbf{x} = (\tilde{x}, \tilde{y}) = (\tilde{x}, 0) + (0, \tilde{y})$, we see
\[
e^{\frac{1}{i K} \langle \mathbf{u}, \mathbf{x} \rangle} = e^{-\frac{1}{2 i K} \langle \xi, \eta \rangle} e^{\frac{1}{i K} \langle \tilde{\xi}, \tilde{\eta} \rangle},
\]
where we make attention to tricky notations $\langle \xi J, \tilde{\eta} \rangle = -\sum_{i=1}^{m} \tilde{\xi}_i \tilde{\eta}_i$ and $\langle \tilde{\eta} J, \tilde{\xi} \rangle = \sum_{i=1}^{m} \tilde{\eta}_i \tilde{\xi}_i$.

According to the notation $\xi = (\tilde{\xi}, \tilde{\eta})$ we denote the expression parameter $K$ by
\[
K = \begin{bmatrix} K_{**} & K_{*} \\ K_{*} & K_{\infty} \end{bmatrix}.
\]
If $K \in \mathcal{B}_+(V)$, then $(iK_{**})_{re}$ and $(iK_{\infty})_{re}$ are positive definite on $V \cap \mathbb{C}^m$, $V \cap \mathbb{C}^m_{\tilde{\xi}}$, respectively. Hence
\[
\hat{e}^{\frac{1}{i K}}(\tilde{\xi}, \mathbf{u}) = e^{\frac{1}{i K} \langle \tilde{\xi} K, \mathbf{u} \rangle} \hat{e}^{\frac{1}{i K}}(\mathbf{u}), \quad \hat{e}^{\frac{1}{i K}}(\tilde{\eta}, \mathbf{v}) = e^{\frac{1}{i K} \langle \mathbf{v} K_{\infty}, \tilde{\eta} \rangle} \hat{e}^{\frac{1}{i K}}(\mathbf{v})
\]
are rapidly decreasing on $\xi \in V \cap \mathbb{C}^m$, $\tilde{\eta} \in V \cap \mathbb{C}^m_{\tilde{\xi}}$, respectively.

The next theorem is fundamental and very useful:

Theorem 2.1 In a $V$-class expression $K(\in \mathcal{B}_+(V))$,
\[
\int_V e^{\frac{1}{i K} \langle \mathbf{u}, \mathbf{x} \rangle} dV_{:K} = \frac{2^m}{\sqrt{\det i K}} e^{\frac{1}{i K} \langle \mathbf{u}, \mathbf{x} \rangle} + \langle \mathbf{u}, \mathbf{x} \rangle}
\]
holds without sign ambiguity and rapidly decreasing w.r.t. $\mathbf{x} \in V$ with the order of $e^{-\frac{1}{i K} \langle \mathbf{u}, \mathbf{x} \rangle^2}$, and it is an entire function w.r.t. $\mathbf{u}$. Similarly,
\[
\int_{V \cap \mathbb{C}^m_{\tilde{\xi}}} e^{\frac{1}{i K} \langle \mathbf{u}, \mathbf{x} \rangle} dV_{:K_{**}} = \frac{\sqrt{2^m}}{\sqrt{\det i K_{\infty}}} e^{\frac{1}{i K_{\infty}} \langle \mathbf{u}, \mathbf{x} \rangle} + \langle \mathbf{u}, \mathbf{x} \rangle}
\]
is defined without sign ambiguity and rapidly decreasing on $V \cap \mathbb{C}^m_{\tilde{\xi}}$ with the order of $e^{-\frac{1}{i K_{\infty}} \langle \mathbf{u}, \mathbf{x} \rangle^2}$, and it is an entire function of $\mathbf{u}$.

Proof We show first the first statement. Recall that $\hat{e}^{\frac{1}{i K} \langle \mathbf{u}, \mathbf{x} \rangle} = e^{\frac{1}{i K} \langle \xi K, \mathbf{u} \rangle} \hat{e}^{\frac{1}{i K} \langle \mathbf{u}, \mathbf{x} \rangle}$, we see
\[
\frac{1}{4 i h} (\langle \xi K, \mathbf{u} \rangle + \frac{i}{h} \langle \mathbf{u}, \mathbf{x} \rangle) = -\frac{1}{4 i h} (\langle i K K, 4 i \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{x} \rangle)
\]
\[
= -\frac{1}{4 i h} (\langle i K K + 2 i \langle \mathbf{u}, \mathbf{x} \rangle \rangle \langle \mathbf{u}, \mathbf{x} \rangle) - \frac{1}{4 i h} (\langle \mathbf{u}, \mathbf{x} \rangle) + \langle \mathbf{u}, \mathbf{x} \rangle).
\]
By Lemma 1.1, we get

\[ \int_V e^{-\frac{1}{\hbar} \left( (\xi+\alpha)\sqrt{\eta} R \right)} dV_{\xi} \]

converges, and is independent of \( u - x \), for replacing \( 2\imath (u - x) \frac{1}{\sqrt{\eta} R} \) by \( \alpha \sqrt{\eta} R \)

\[
\frac{\partial}{\partial \alpha_i} \int_V e^{-\frac{1}{\hbar} \left( (\xi+\alpha)\sqrt{\eta} R \right)} dV_{\xi} = \int_V \frac{\partial}{\partial \xi_i} e^{-\frac{1}{\hbar} \left( (\xi+\alpha)\sqrt{\eta} R \right)} dV_{\xi} = 0.
\]

By Lemma [11], we get

\[
\int_V : e^{\frac{1}{\hbar} (\xi, u - x) :}_K dV_{\xi} - \int_V : e^{\frac{1}{\hbar} (\xi, u - x) :}_K dV_{\xi} e^{-\frac{1}{\hbar} (u - x) \frac{1}{\hbar} (u - x)} = \frac{2m}{\hbar^4} e^{-\frac{1}{\hbar} (u - x) \frac{1}{\hbar} (u - x)}.
\]

Clearly, this is rapidly decreasing w.r.t. \( x \in V \). The second one is similar.

\[ \square \]

### 2.1 Star-delta functions in V-class expressions

Keeping the Fourier transform of 1 in mind, we define *delta functions of full-variables* \( \delta^V_s(u - x) \), and *delta functions of half-variables* \( \delta^V_s(u - x), \delta^V_s(v - y) \) in a V-class expression as follows:

\[
\begin{align*}
\delta^V_s(u - x) : K & = \int_V : e^{\frac{1}{\hbar} (\xi, u - x) :}_K dV_{\xi} = \int_V : e^{\frac{1}{\hbar} (\xi, u - x) :} dV_{\xi} dV'_{\eta}, \\
\delta^V_s(u - x) : K & = \int_{V \cap C^m_\xi} : e^{\frac{1}{\hbar} (\xi, u - x) :}_K dV_{\xi} = \int_{V \cap C^m_\xi} : e^{\frac{1}{\hbar} (\xi, u - x) :}_K dV_{\xi},
\end{align*}
\]

where \( dV_{\xi}, dV'_{\eta} \) are standard volume element of \( V \cap C^m_\xi, V \cap C^m_\eta \) fractionalized by \((2\pi \hbar)^{m/2}\). By Theorem 2.1, these are rapidly decreasing w.r.t. \( \tilde{x}, \tilde{y} \) and entire functions w.r.t. \( \tilde{u}, \tilde{v} \) respectively.

As \( \frac{1}{i\hbar} \tilde{u}_i \ast e^{\frac{1}{\hbar} (\xi, \tilde{u})} = \partial_{\xi_i} e^{\frac{1}{\hbar} (\xi, \tilde{u})} \) we have the property similar to a usual delta function:

\[
\frac{1}{i\hbar} (\tilde{u}_i - \tilde{x}_i) \ast \delta^V_s(\tilde{u} - \tilde{x}) = 0 = \delta^V_s(\tilde{u} - \tilde{x}) \ast \frac{1}{i\hbar} (\tilde{u}_i - \tilde{x}_i).
\]

As changing the variables shows that \( dV'_{\tilde{\xi}} = e^{i\theta} d\tilde{x}', \tilde{x}' \in \mathbb{R}^n \), and there is a linear transformation \( T \)

\[
\int_{V \cap C^m_\xi} e^{\frac{1}{\hbar} (\xi, \tilde{x}) :} dV_{\tilde{\xi}} = e^{i\theta} \int_{\mathbb{R}^n} e^{\frac{1}{\hbar} (\xi^T \tilde{x}) :} d\tilde{x}' = e^{i\theta} \delta(\xi T).
\]

For the convenience of computations we define

\[ \delta^V_s(\xi) = e^{i\theta} \delta(\xi T). \]

Hence

\[ \int_{V \cap C^m_\xi} : \delta^V_s(u - x) :_K dV_{\tilde{u}} = 1, \quad \int_{V \cap C^m_\eta} : \delta^V_s(v - y) :_K dV_{\tilde{v}} = 1. \]
The $*$-product of $*$-delta functions is given as follows:

\[(16) \quad :\delta_s^V(\bar{u} - \bar{x}) \ast \delta_s^V(\bar{u} - \bar{x}') :_K = \int_V \delta^V(\bar{x} - \bar{x}') : e^{\frac{i}{\hbar} \langle \bar{\xi}, \bar{u} - \bar{x} \rangle} :_K dV_{\bar{\xi}} = \delta^V(\bar{x} - \bar{x}') : \delta_s^V(\bar{u} - \bar{x}') :_K.\]

We have also

\[(17) \quad :\delta_s^V(\bar{u} - \bar{y}) \ast \delta_s^V(\bar{u} - \bar{y}') :_K = \delta^V(\bar{y} - \bar{y}') : \delta_s^V(\bar{u} - \bar{y}') :_K.\]

For a tempered distribution $h(\bar{x})$ on $V$, we define the inverse Fourier transform by

\[\hat{h}(\bar{\xi}) = \int_V h(\bar{x}) e^{-\frac{i}{\hbar} \langle \bar{\xi}, \bar{x} \rangle} dV_{\bar{x}}, \quad \bar{\xi} \in V \cap \mathbb{C}_m.\]

Putting the reciprocity $h(\bar{x}' - \bar{x}) = \int_V \hat{h}(\bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \bar{x}' - \bar{x} \rangle} dV_{\bar{\xi}}$ in mind, we define for $K \in \mathcal{D}(V)$, a $*$-function

\[(18) \quad :h_s^V(\bar{u} - \bar{x}) :_K = \int_V \hat{h}(\bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \bar{u} - \bar{x} \rangle} :_K dV_{\bar{\xi}} = \int_V \int_V h(\bar{x}') e^{\frac{i}{\hbar} \langle \bar{\xi}, \bar{u} - \bar{x}' \rangle} :_K dV_{\bar{x}} dV_{\bar{\xi}}.\]

As $e^{\frac{i}{\hbar} \langle \bar{\xi}, \bar{u} \rangle} :_K$ is rapidly decreasing under $V$-class expressions, one may write this by

\[ :h_s^V(\bar{u} - \bar{x}) :_K = \int_V h(\bar{x}) : \delta_s^V(\bar{u} - \bar{x}') :_K dV_{\bar{x}}.\]

By Theorem 2.1, $h_s^V(\bar{u} - \bar{x}) :_K$ is rapidly decreasing w.r.t. $\bar{x} \in V$ in a $V$-class expression, and an entire function w.r.t. $\bar{u}$.

Let $f(\bar{\xi})$, $\hat{g}(\bar{\xi})$ be tempered distributions on $V$. Suppose the convolution product is well defined as a tempered distribution. Then,

\[ :f_s^V(\bar{u}) \ast g_s^V(\bar{u}) :_K = \int_V \left( \int_V f(\bar{\xi}) \hat{g}(\bar{\xi} - \bar{\xi}) : \hat{\xi} e^{\frac{i}{\hbar} \langle \bar{\xi}, \bar{u} - \bar{\xi} \rangle} :_K dV_{\bar{\xi}} dV_{\hat{\xi}} \right) \]

by noting that $e^{\frac{i}{\hbar} \langle \bar{\xi}, \bar{u} \rangle} :_K$ is rapidly decreasing in $\bar{\xi}$ under any $V$-class expression parameters.

The basic properties of these $*$-functions are

\[ f_s^V(\bar{u}) \ast g_s^V(\bar{u}) = \int_V (f \hat{g})(\bar{x}) \delta_s^V(\bar{u} - \bar{x}) dV_{\bar{x}}, \quad [\hat{g}_s^V(\bar{u})] = \int_V i h \partial_{\bar{x}} h(\bar{x}) \delta_s^V(\bar{u} - \bar{x}) dV_{\bar{x}}.\]

By setting $\bar{x} = (\bar{x}, \bar{y})$, the above arguments may be applied to the case of “full-variables.”

**Proposition 2.2** If $f(\bar{y})$ is a tempered distribution on $V \cap \mathbb{C}_m$, then $\int_V f(\bar{y}) : \delta_s^V(\bar{u} - \bar{x}) :_K dV_\bar{x}$ is a half-variable $*$-function

\[ \int_V f(\bar{y}) : \delta_s^V(\bar{u} - \bar{x}) :_K dV_\bar{x} = \int_{V \cap \mathbb{C}_m} f(\bar{y}) : \delta_s(\hat{\bar{u}} - \hat{\bar{y}}) :_K dV_\hat{\bar{y}} = :f_s^V(\bar{v}) :_K.\]
Proof Note that \( \mathbf{x} = (\tilde{x}, \tilde{y}), \mathbf{u} = (\tilde{u}, \tilde{v}), \mathbf{dV}_x = dV_x dV_y, \mathbf{dV}_\xi = dV_\xi dV_\eta \) and (12) gives
\[
\int \int : \delta^V(u - x) :_K dV_x dV_y = \int \int e^{-\frac{i}{\hbar} \langle \xi / \eta, \cdot \rangle} \cdot e^{\frac{i}{\hbar} \langle \eta, \eta - \tilde{y} \rangle} \cdot e^{\frac{i}{\hbar} \langle \eta, \tilde{v} - \tilde{y} \rangle} :_K dV_x dV_y = : e^{-\frac{i}{\hbar} \langle \eta, \tilde{v} - \tilde{y} \rangle} :_K,
\]
for integrating by \( \tilde{x} \) first gives \( \int e^{-\frac{i}{\hbar} \langle \xi / \eta, \eta \rangle} \cdot e^{\frac{i}{\hbar} \langle \eta, \tilde{v} - \tilde{y} \rangle} \cdot :_K = \delta^V(\xi) \). Plugging this we obtain the result. \( \square \)

Now, let \( \rho(\tilde{y}) \) be a tempered distribution on \( V \cap \mathbb{C}_y^m \) or a continuous function on \( V \cap \mathbb{C}_y^m \) such that \( e^{i \rho(\tilde{y})} \) is the growth order \( e^{\epsilon \tilde{y}^n} \), \( \alpha < 2 \). Then, we have a remarkable result as follows:

**Proposition 2.3** In \( V \)-class expressions, \( :e^{i \rho(\tilde{y})} :_K = \int_V e^{i \rho(\tilde{y})} : \delta^V(u - x) :_K dV_x \) is same to
\[ :e^{i \rho(\tilde{y})} :_K = \int_{V \cap \mathbb{C}_y^m} e^{i \rho(\tilde{y})} : \delta^V(\tilde{v} - \tilde{y}) :_K dV_\tilde{y} \]
and it is welldefined one parameter group w.r.t. \( t \) and an entire function w.r.t. \( \tilde{v} \).

**Proof** It is enough to prove the group property. For this, we show that this satisfies the equation
\[
\frac{d}{dt} : e^{i t \rho(\tilde{y})} :_K = i \rho(\tilde{v}) : e^{i t \rho(\tilde{y})} :_K + \int_{\mathbb{R}^m} e^{i t \rho(\tilde{y})} : \delta^V(\tilde{v} - \tilde{y}) :_K dV_\tilde{y}.
\]
Note that \( \frac{d}{dt} : e^{i t \rho(\tilde{y})} :_K = \int_{V \cap \mathbb{C}_y^m} i \rho(\tilde{y}) : e^{i t \rho(\tilde{y})} :_K dV_\tilde{y} \), and \( i \rho(\tilde{v}) :_K = \int_{V \cap \mathbb{C}_y^m} i \rho(\tilde{y}) : \delta^V(\tilde{v} - \tilde{y}) :_K dV_\tilde{y} \). Hence applying (17), we have
\[
\int \int i \rho(\tilde{y}) e^{i t \rho(\tilde{y})} : \delta^V(\tilde{v} - \tilde{y}) :_K dV_\tilde{y} dV_x = \int i \rho(\tilde{y}) e^{i t \rho(\tilde{y})} : \delta^V(\tilde{v} - \tilde{y}) :_K dV_\tilde{y}.
\]
\( \square \)

Note also that \( e^{i t \rho(\tilde{y})} \) may not be real analytic in \( t \). But the above Theorem shows that for any polynomial \( p(\tilde{y}) \), the \( * \)-exponential function \( e^{i t \rho(\tilde{y})} \) is welldefined in any \( V \)-class expression as a real one parameter group. However, we have seen in [10] that such one parameter group must have singularities in the complex domain. Indeed, if \( \deg p(\tilde{y}) \geq 3 \) then the radius of convergence of the series \( \sum_k \frac{1}{k!} p_k(\tilde{v}) \) is 0 (cf. [10], Proposition 1.2).

Proposition 2.3 can be applied to the hybrid case. Let \( \rho(\tilde{x}, \tilde{y}) \) be a tempered distribution on \( V \cap \mathbb{C}_y^m \) w.r.t. \( \tilde{y} \), or a continuous function on \( V \cap \mathbb{C}_y^m \) such that \( e^{i \rho(\tilde{x}, \tilde{y})} \) is growth order \( e^{\epsilon \tilde{y}^n} \), \( \alpha < 2 \) w.r.t. \( \tilde{y} \) at each fixed \( \tilde{x} \).

**Proposition 2.4** Under a \( V \)-class expression, the \( * \)-exponential function
\[
e^{i \rho(\tilde{x}, \tilde{y})} = \int_{V \cap \mathbb{C}_y^m} e^{i \rho(\tilde{x}, \tilde{v})} : \delta^V(\tilde{v} - \tilde{y}) :_K dV_x
\]
is welldefined one parameter group w.r.t. \( t \) and an entire function w.r.t. \( \tilde{v} \).
For $K \in \mathfrak{H}_+(V)$, we have defined $\delta^{(V)}_{*}(\hat{v})$, $\delta^{(V)}_{*}(\hat{u})$ separately. By setting $u=(\hat{u}, \hat{v})$, $\xi=(\hat{\xi}, \hat{\eta})$, $x=(\hat{x}, \hat{y})$, the $K$-ordered expression of the $*$-product $\delta^{(V)}_{*}(\hat{v}-\hat{y}) \ast \delta^{(V)}_{*}(\hat{u}-\hat{x})$ is computed as follows:

$$\delta^{(V)}_{*}(\hat{v}-\hat{y}) \ast \delta^{(V)}_{*}(\hat{u}-\hat{x}):_K = \int \int e^{\frac{i}{\hbar} (\hat{\eta}J, \hat{\xi})} e^{\frac{1}{2\pi} (\xi, u-x)} \eta \, dV_\eta \, dV_{\xi} = \int_V e^{-\frac{i}{\hbar} (\xi, C) \cdot \xi} e^{\frac{i}{\hbar} (\xi, u-x)} dV_{\xi},$$

where $(\hat{\eta}J, \hat{\xi}) = \sum_i \hat{\eta}_i \hat{\xi}_i$, $C = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. So, suppose $K+ \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is non-singular. Then

$$\delta^{(V)}_{*}(\hat{v}-\hat{y}) \ast \delta^{(V)}_{*}(\hat{u}-\hat{x}):_K = \frac{2^m}{\sqrt{\det(i(K+C))}} e^{-\frac{i}{\hbar} (u-x) \cdot (K+C) \cdot (u-x)}, \quad u=(\hat{u}, \hat{v}), \quad x=(\hat{x}, \hat{y}).$$

Note that the r.h.s. makes sense whenever $K+C$ is non-singular, but this has $\pm$ sign ambiguity in $\sqrt{\det(i(K+C))}$. In § II it will be shown that $\delta^{(V)}_{*}(\hat{v}-\hat{y}) \ast \delta^{(V)}_{*}(\hat{u}-\hat{x}):_K$ is an idempotent element, and this relates what we called the vacuum.

### 2.2 Several properties of $*$-functions of full-variables

On the other hand, we have defined the $*$-delta function (of full variables) as follows:

$$\delta^{(V)}_{*}(u-x):_K = \int_V e^{\frac{1}{2\pi} (\xi, u-x)} \eta \, dV_{\xi}, \quad x \in V.$$ 

We note here that the exponential law gives

$$e^{\frac{i}{2\pi} (\xi, u-x)} = e^{\frac{i}{2\pi} (\xi, u)} e^{-\frac{i}{\hbar} (\xi, x)} \quad \text{and} \quad \int_V e^{\frac{i}{2\pi} (\xi, u-x)} dV_{\xi} = \delta^{(V)}(\xi) e^{\frac{i}{2\pi} (\xi, u)}.$$ 

It follows

$$\int_V \delta^{(V)}_{*}(u-x) \, d\xi = 1.$$ 

Let $f(x)$ be a tempered distribution on $V$ and let $\check{f}^{(V)}(\xi) = \int_V f(x) e^{-\frac{i}{\hbar}(\xi, x)} dV_x$ be the inverse Fourier transform. Noting the wellknown reciprocity formula $f(x) = \int_V \check{f}^{(V)}(\xi) e^{\frac{i}{2\pi}(\xi, x)} d\xi$, we define $*$-function corresponding to $f(x)$ as

$$\check{f}^{(V)}(u-x):_K = \int_V \check{f}^{(V)}(\xi) e^{\frac{i}{2\pi}(\xi, u-x)} \eta \, dV_{\xi} = \int_V \int_V f(x') e^{-\frac{i}{\hbar}(\xi, x')} e^{\frac{i}{2\pi}(\xi, u-x')} \eta \, dV_{\xi} \, dV_{\eta}.$$ 

The Weyl ordered ($K=0$) expression of $\delta^{(V)}_{*}(u-x)$ is given by $\delta^{(V)}_{*}(u-x):_0 = \delta^{(V)}(u-x)$. Thus we see

$$\check{f}^{(V)}(u):_0 = \int f(x) \delta^{(V)}(u-x) \, d\xi = f(u).$$

**Proposition 2.5** The inverse of the correspondence $f(x) \rightarrow \check{f}^{(V)}(u)$ is given by its Weyl ordered expression and replacement of $u$ by $x$. 

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Applying the exponential law in the r.h.s. gives

\[(24) \quad f^{(V)}_s(u-x)_K = \int_V \int_V f(x') e^{\frac{1}{\imath \hbar} \langle x' - x, x - x' \rangle} \delta^{(V)}(u-x'-x)_K \, dV_x \, dV_{\xi} = \int_V f(x') \delta_s^{(V)}(u-x'-x)_K \, dV_{x'}.
\]

The next theorem is the main tool to extend the class of $*$-functions via Fourier transform.

**Theorem 2.2** For every tempered distribution $f(x)$ on $V$, the $V$-class expression \[24\] of the integral $\int_V f(x') \delta^{(V)}_s(u-x'-x) \, dV_x$ is rapidly decreasing w.r.t. $x$ and an entire function of $u$. In particular we see

$$
\delta^{(V)}_s(u-a)_K = \int_V \delta^{(V)}(x-a) \delta^{(V)}_s(u-x)_K \, dV_x.
$$

Moreover, even if $f(x) = e^{\lambda \alpha x}$-growth on $V$ with $0 < \alpha < 2$, the integral $\int_V f(x) \delta^{(V)}(u-x)_K \, dV_x$ is well-defined to give an entire function w.r.t. $u$.

**Proof** The $K$-expression is $\int_V \int_V f(x) e^{-\frac{1}{\imath \hbar} \langle x', x \rangle} e^{\frac{1}{\imath \hbar} \langle x', x \rangle} \, dV_x \, dV_{\xi}$. By restricting $u$ to an arbitrary compact subset of $C^{2m}$, Theorem 2.1 of \[24\] gives that $\int_V e^{-\frac{1}{\imath \hbar} \langle x', x \rangle} e^{\frac{1}{\imath \hbar} \langle x', x \rangle}$ is rapidly decreasing w.r.t. $x \in V$ with the growth order $e^{-\epsilon_K |x|^2}$. Hence, the integral $\int_V f(x) \delta_{\alpha}(u-x) \, dV_x$ exists for every tempered distribution $f(x)$ on $V$. Since the complex differentiation $\partial_u$ does not suffer the convergence, we see that this is holomorphic w.r.t. $u$. \qed

**Warning** $\partial_u e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K = \partial_u e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K = \partial_u e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K$. But this is not $\frac{\imath}{\hbar} u_i \ast e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K$. It is very easy to make a confusion. We have in fact

$$
\frac{\imath}{\hbar} u_i \ast e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K = e^{\frac{1}{\imath \hbar} \langle x, K \rangle} \frac{1}{\hbar} u_i \ast e^{\frac{1}{\imath \hbar} \langle x, u \rangle} = e^{\frac{1}{\imath \hbar} \langle x, K \rangle} \left( \frac{1}{\hbar} u_i \ast e^{\frac{1}{\imath \hbar} \langle x, K \rangle} \sum_j \frac{1}{2} (K+J)_{ij} + (K-J)_{ij} \right) e^{\frac{1}{\imath \hbar} \langle x, u \rangle},
$$

$$
\frac{1}{\hbar} u_i \ast e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K = e^{\frac{1}{\imath \hbar} \langle x, K \rangle} \frac{1}{\hbar} u_i \ast e^{\frac{1}{\imath \hbar} \langle x, u \rangle} = e^{\frac{1}{\imath \hbar} \langle x, K \rangle} \left( \frac{1}{\hbar} u_i \ast e^{\frac{1}{\imath \hbar} \langle x, K \rangle} \sum_j \frac{1}{2} (K+J)_{ij} + (K-J)_{ij} \right) e^{\frac{1}{\imath \hbar} \langle x, u \rangle}.
$$

Summing up to eliminate the terms involving $J$, we see

\[(25) \quad \frac{\imath}{\hbar} u_i \ast e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K + e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K \frac{\imath}{\hbar} u_i \ast e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K = \partial_u e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K.
\]

Thus, we see that

**Proposition 2.6** In a $V$-class expression, $\delta^{(V)}_s(u)$ anti-commutes with every generator i.e.

$$
\frac{\imath}{\hbar} u_i \ast \delta^{(V)}_s(u) + \delta^{(V)}_s(u) \ast \frac{\imath}{\hbar} u_i \ast = \int_V \partial_u e^{\frac{1}{\imath \hbar} \langle x, u \rangle}_K \, d\xi = 0, \quad i = 1 \sim 2m.
$$

In particular $\delta^{(V)}_s(u) \ast f^{(V)}_s(u) = f^{(V)}_s(-u) \ast \delta^{(V)}_s(u)$.  

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Proof The proof is given also by using (12) and the integration by parts. For the second identity
\( \delta^\nu(u) \cdot f^\nu(u) = f^\nu(-u) \cdot \delta^\nu(u) \), note first that \( e^\nu(\xi) \cdot \delta^\nu(u) = e^\nu(u) = e^\nu(\xi) \cdot e^\nu(u) \), and hence
\[ e^\nu(\xi - u) \cdot \delta^\nu(u) = \delta^\nu(u) \cdot e^\nu(\xi + u). \]

Since \( f^\nu(u) = \int f(x) \delta^\nu(u-x) dV_x \), the changing variables in the integration gives the result. \( \square \)

Adjoint actions. On the other hand, it is easy to see that in every \(*_\nu\)-product
\[ \text{ad} \big( \frac{1}{i\hbar}(a, u) \big)(u_1, u_2, \ldots, u_{2m}) = (a_1, a_2, \ldots, a_{2m}).J. \]

It follows \( \text{ad}(\frac{1}{i\hbar}(a, u))(u_1, u_2, \ldots, u_{2m}) = (u_1, u_2, \ldots, u_{2m}) + (a_1, a_2, \ldots, a_{2m}).J. \), and
\[ \text{Ad}(e^\nu(\xi)) = (u_1, u_2, \ldots, u_{2m}) + (a_1, a_2, \ldots, a_{2m}).J. \]

Since \( \text{Ad}(e^\nu(\xi)) \) gives a \(*\)-isomorphism, we have
\[ (26) :e^\nu(\xi) * p_\nu(u) * e^\nu(\xi) :_K = :p_\nu(u+a.J) :_K, \quad :e^\nu(\xi) * e^\nu(\xi) * e^\nu(\xi) :_K = :e^\nu(\xi + a.J) :_K. \]

The identity \( (26) \) extends easily to
\[ (27) :e^\nu(\xi) * \delta^\nu(u-x) * e^\nu(\xi) :_K = :\delta^\nu(u-x+a.J) :_K, \quad :e^\nu(\xi) * \delta^\nu(v-y) * e^\nu(\xi) :_K = :\delta^\nu(v-y-a) :_K, \quad :e^\nu(\xi) * \delta^\nu(u-x) * e^\nu(\xi) :_K = :\delta^\nu(u-x+b) :_K, \]

where \( a=(\tilde{a}, \tilde{b}) \) and \( a.J=(\tilde{b}, -\tilde{a}) \).

A \(*\)-delta function \( \delta^\nu(u) \) of full variables has a peculiar property. The first equality of \( (26) \) gives that \( f_t(u) = \text{Ad}(e^\nu(\xi)) \delta^\nu(u) \) satisfies the Heisenberg equation
\[ \frac{d}{dt} f_t(u) = \frac{1}{i\hbar} \langle [a, u], f_t(u) \rangle, \quad f_0(u) = \delta^\nu(u). \]

On the other hand, since \( \delta^\nu(u) \) anti-commutes with generators, \( f_t(u) \) satisfies the evolution equation
\[ \frac{d}{dt} f_t(u) = \frac{2}{i\hbar} \langle a, u \rangle * f_t(u), \quad f_0(u) = \delta^\nu(u). \]

Although \( f^\nu(u-x) \) is defined similarly to \( f^\nu(u-x) \), the nature of those are very different. As it will be seen below, \( \delta^\nu(v-x) * \delta^\nu(u-x) :_K = 2^{2m} \), while \( \delta^\nu(v-x) * \delta^\nu(u-x) :_K = \delta^\nu(\tilde{x} - \tilde{x}) :_K \).

For tempered distribution \( f(x), g(x) \) on \( V \), let \( \hat{f}(\xi), \hat{g}(\eta) \) be their Fourier transforms, and suppose the convolution product is welldefined as a tempered distribution \( \hat{f} \cdot \hat{g}(\xi) = \int V \hat{f} \hat{g}(\xi - \eta) dV_\eta = \hat{f} \hat{g}(\xi) \). Then as \( V \) is assumed \( JV=V \), the twisted convolution product \( f \cdot g(\xi) \) (cf.,(8)) is also welldefined. We denote
\[ (28) (f \cdot g)(x) = \int V \hat{f} \hat{g}(\xi) e^{i\xi x} dV_\xi, \quad x \in V. \]
Hence we have

\begin{equation}
\delta^{(29)}(\xi) = \int_{V} f^{(29)}(\xi) \xi e^{\frac{\pi}{12}(\xi,\xi)} :_{:K} d\xi d\zeta = \int_{V} f^{(30)}(\xi) \xi e^{\frac{\pi}{12}(\xi,\xi)} :_{:K} d\xi.
\end{equation}

If \( K \in \mathcal{F}_{+}(V) \), then

\begin{equation}
\delta^{(31)}(u) \xi e^{\frac{\pi}{12}(\xi,\xi)} :_{:K} \xi = \int_{V} f^{(u)}(\xi) \xi e^{\frac{\pi}{12}(\xi,\xi)} :_{:K} d\xi d\zeta.
\end{equation}

For instance setting \( \delta^{(32)}(x) = \delta^{(33)}(x-a) \), we have \( \delta^{(34)}(\xi) = e^{-\frac{i}{4}(\xi,\xi)} \) and

\begin{equation}
\delta^{(35)}(\xi) = \int_{V} \left( \int_{V} e^{-\frac{1}{2}(\xi,\xi)} :_{:K} \xi \right) e^{\frac{1}{2}(\xi,\xi)} :_{:K} d\xi d\zeta.
\end{equation}

by setting \( \zeta = \xi' \) in the delta function. It follows

\begin{equation}
(\delta^{(36)}(x) \xi e^{\frac{\pi}{12}(\xi,\xi)} :_{:K} \xi = \int_{V} f^{(37)}(\xi) \xi e^{\frac{\pi}{12}(\xi,\xi)} :_{:K} d\xi d\zeta.
\end{equation}

By \( \text{[30]} \), we have a remarkable formula

\begin{equation}
\delta^{(38)}(\xi) = \int_{V} \left( \int_{V} e^{-\frac{1}{2}(\xi,\xi)} :_{:K} \xi \right) e^{\frac{1}{2}(\xi,\xi)} :_{:K} d\xi d\zeta.
\end{equation}

In particular, we have

\begin{equation}
\delta^{(39)}(\xi) = \int_{V} \left( \int_{V} e^{-\frac{1}{2}(\xi,\xi)} :_{:K} \xi \right) e^{\frac{1}{2}(\xi,\xi)} :_{:K} d\xi d\zeta.
\end{equation}

This is obtained also by another direct calculation as follows:

\begin{equation}
\delta^{(40)}(\xi) = \int_{V} \left( \int_{V} e^{-\frac{1}{2}(\xi,\xi)} :_{:K} \xi \right) e^{\frac{1}{2}(\xi,\xi)} :_{:K} d\xi d\zeta.
\end{equation}

By the exponential law and \( \text{[12]} \), we have

\begin{equation}
\delta^{(41)}(\xi) = \int_{V} \left( \int_{V} e^{-\frac{1}{2}(\xi,\xi)} :_{:K} \xi \right) e^{\frac{1}{2}(\xi,\xi)} :_{:K} d\xi d\zeta.
\end{equation}
Noting that $\langle \zeta, J, \xi \rangle = -\langle \zeta, \xi, J \rangle$, we have
\[
\frac{1}{i\hbar} \langle \zeta, J, \xi \rangle + \frac{2}{i\hbar} \langle \xi, J, \zeta \rangle + \frac{4}{i\hbar} \langle \zeta, u-b \rangle \\
= \frac{1}{i\hbar} \left( \zeta \sqrt{iK} - \langle \zeta, J-2(u-b) \rangle \frac{i}{\sqrt{iK}} \right) \left( \xi \sqrt{iK} - \langle \xi, J-2(u-b) \rangle \frac{i}{\sqrt{iK}} \right) \\
- \frac{1}{i\hbar} \left( \langle \zeta, J-2(u-b) \rangle \frac{i}{\sqrt{iK}} \right) \left( \xi \sqrt{iK} - \langle \xi, J-2(u-b) \rangle \frac{i}{\sqrt{iK}} \right).
\]
By the same reasoning as in Theorem 2.1 and by setting $C_0(K) = \frac{2m}{\sqrt{\det(iK)}}$, we have
\[
\langle \delta_s^{(V)}(u-a) * \delta_s^{(V)}(u-b) \rangle_K = C_0(K) \int_V e^{-\frac{i}{4\hbar} \left( \langle \xi, J-2(u-b) \rangle \frac{i}{\sqrt{iK}} \right) \left( \langle \xi, J-2(u-b) \rangle \frac{i}{\sqrt{iK}} \right)} \cdot \frac{1}{iK} \langle a-b \rangle \, dV_x.
\]
Noting that
\[
- \frac{1}{4\hbar} \left( \langle \xi, J-2(u-a) \rangle \frac{i}{\sqrt{iK}} \right) \left( \langle \xi, J-2(u-a) \rangle \frac{i}{\sqrt{iK}} \right) = -4 i \langle \xi, J \rangle \langle a-b \rangle \right)
\]
and $J \frac{1}{K} J$ is positive definite as $JV=V$, we have by [31] that
\[
\langle \delta_s^{(V)}(u-a) * \delta_s^{(V)}(u-b) \rangle_K = C_0(K) C_0(J) e^{-H(a-b)(iK) \langle a-b \rangle} = e^{H(a-b)} e^{-H(a-b)(iK) \langle a-b \rangle}.
\]
For a fixed $V$, we define $V_k$ by $V_k = V \cap \{ s \bar{u}_k + t \bar{v}_k : (s, t) \in \mathbb{C}^2 \}$ and by $dV_k$ we denote the volume form on $V_k$ divided by $2\pi \hbar$. Define as follows and note that this is not a hybrid *-delta function but a partial *-delta function:
\[
\langle \delta_s^{(V)}(\bar{u}_k, \bar{v}_k) \rangle_K = \int_{V_k} e^{H(s \bar{u}_k + t \bar{v}_k)} \, dV_k, \quad K \in \mathcal{S}_+(V).
\]
By Theorem 2.1 the $K$-ordered expression is
\[
\langle \delta_s^{(V)}(\bar{u}_k, \bar{v}_k) \rangle_K = \frac{2}{\sqrt{\det(iK)}} e^{-\frac{i}{K}(\bar{u}_k, \bar{v}_k) iK(\bar{u}_k, \bar{v}_k)}, \quad \text{where } K(k) = \begin{bmatrix} K_{k,k} & K_{k,m+k} \\ K_{m+k,k} & K_{m+k,m+k} \end{bmatrix}.
\]
As $[\bar{u}_i, \bar{v}_j] = 0$ for $i \neq j$, we see $\langle \delta_s^{(V)}(\bar{u}_i, \bar{v}_j) \rangle_K = \delta_s^{(V)}(\bar{u}_i, \bar{v}_j)$ and $\delta_s^{(V)}(\bar{u}) \delta_s^{(V)}(\bar{v}) = \delta_s^{(V)}(\bar{u}_1, \bar{v}_1) \delta_s^{(V)}(\bar{u}_2, \bar{v}_2) \cdots \delta_s^{(V)}(\bar{u}_m, \bar{v}_m)$. For simplicity, we denote $\delta_s(k) = \delta_s^{(V)}(\bar{u}_k, \bar{v}_k)$ in what follows. Then we see
\[
\delta_s(k)^2 = 4, \quad \delta_s(k) \delta_s(l) = \delta_s(l) \delta_s(k)
\]
and
\[
\bar{u}_k \delta_s(k) = -\delta_s(k) \bar{u}_k, \quad \bar{v}_k \delta_s(k) = -\delta_s(k) \bar{v}_k, \quad \bar{u}_l \delta_s(k) = \delta_s(k) \bar{u}_l \quad (k \neq l).
\]
In the next section, we see that every $\delta_s(\bar{u}_i, \bar{v}_i)$ is on a compact one parameter subgroup of a *-exponential function of a quadratic form. Moreover, Theorem 3.1 in §3.1.1 below shows that $\frac{1}{2} \delta_s(k)$ is one of $\pm \varepsilon_{00}(k)$, $\pm i \varepsilon_{00}(k)$. 

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3 Star-exponential functions of quadratic forms

In the previous section we have treated elements obtained by the integrations of $*$-exponential functions of linear forms. In this section, we give some interesting relations between these elements and the $*$-exponential functions of quadratic forms.

Let $H = \sum_{k=1}^{m} \frac{1}{i\hbar} \tilde{x}_k \tilde{y}_k$. Then, by [23], we have

\[ \int_{\mathbb{R}^{2m}} \left( \sum_{k=1}^{m} \frac{1}{i\hbar} \tilde{x}_k \tilde{y}_k \right) \delta_\epsilon(u-x) dx = \sum_{k=1}^{m} \frac{1}{i\hbar} \tilde{u}_k \tilde{v}_k, \]

where $a \ast b = \frac{1}{2}(a \ast b + b \ast a)$. We denote $H_\epsilon = \sum_{k=1}^{m} \frac{1}{i\hbar} \tilde{u}_k \tilde{v}_k$. The $*$-exponential functions of "half-variables" can be managed by the method mentioned in the previous sections, but $\int e^{itH} \delta_\epsilon(u-x) dx$ does not form a group. The $*$-exponential functions of quadratic forms of full variables are defined by the real analytic solutions of evolution equations [11]. Indeed, this was the main tool in the previous notes [12], [13].

3.1 Summary of blurred Lie groups

The space of quadratic forms is isomorphic to the Lie algebra of $Sp(m, \mathbb{C})$, i.e.

\[ \{ \langle uA, u \rangle : A \in \mathfrak{G}(2m) \} \cong \mathfrak{sp}(m, \mathbb{C}) = \{ \alpha : \alpha J + J\alpha = 0 \} \]

as Lie algebras under the commutator bracket product. Let $E_{2m} = \{ \{ \xi, u \} : \xi \in \mathbb{C}^{2m} \}$. For every quadratic form $\frac{1}{2\hbar} \langle uA, u \rangle$, $\text{ad}(\frac{1}{2\hbar} \langle uA, u \rangle)$ is welldefined as a linear mapping independent of expression parameters:

\[ \text{ad}(\frac{1}{2\hbar} \langle uA, u \rangle) : E_{2m} \to E_{2m}, \quad \text{ad}(\frac{1}{2\hbar} \langle uA, u \rangle) : \text{Hol}(\mathbb{C}^{2m}) \to \text{Hol}(\mathbb{C}^{2m}) \]

It is easy to see that $\text{ad}(\langle u(\frac{1}{2\hbar} J), u \rangle) = -\alpha \in \mathfrak{sp}(m, \mathbb{C})$.

For every $\alpha \in \mathfrak{sp}(m, \mathbb{C})$, the $K$-ordered expression of the $*$-exponential function is given as follows:

\[ \iota_{\mathfrak{e}}^{\frac{1}{2\hbar} \langle u(\alpha J), u \rangle} : \mathfrak{e}_\epsilon^{(\frac{1}{2\hbar} \langle u(\alpha J), u \rangle)} = \frac{2^m}{\sqrt{\det(I-K+e^{-2\alpha}(I+\kappa))}} e^{\frac{i}{\hbar} \langle u(\alpha J), u \rangle} \iota_\epsilon^{\frac{2}{\hbar} \langle u(\alpha J), u \rangle} \]

where $\kappa = JK$. As [23] has double branched singular points in generic $K$, we have to use two sheets by setting slits in the complex plane to treat $\iota_{\mathfrak{e}}^{\frac{1}{2\hbar} \langle u(\alpha J), u \rangle}$ univalent way. Because of these singularities, $*$-products of these $*$-exponential functions of quadratic forms are defined only for some open subsets depending on expression parameters. Thus, expression parameters may be viewed as “local coordinate system” for that object, and intertwiners are “changing coordinate”. However in general intertwiners applied to $*$-exponential functions do not satisfy the cocycle conditions. Hence $\iota_{\mathfrak{e}}^{\frac{1}{2\hbar} \langle u(\alpha J), u \rangle} : K$ do not form a group, but by considering generic $K$ all together, these generate an object $Sp_C^{(\frac{1}{2})}(m)$, called the blurred Lie group which looks like a double covering group of $Sp(m, \mathbb{C})$ which is known to be simply connected.

In spite of this, $Sp_C^{(\frac{1}{2})}(m)$ can contain several genuine groups, when intertwiners on the object $G$ satisfies the cocycle conditions. Some of them are given in [12]. Here we give another comment which may be used later.
Proposition 3.1 If the object $G$ is expressed by the collection of $:U_1;K_1:, :U_2;K_2$ with the intertwiner $I_{K_1}^{K_2}$, then the joint object $(:U_1;K_1:, :U_2;K_2)$ gives a point set picture to the object $G$.

Note that $J \in \mathfrak{sp}(m, \mathbb{C})$ and also $J \in \mathfrak{Sp}(m, \mathbb{C}) = \{ g \in \text{GL}(2m, \mathbb{C}); gJg^{-1} = J \}$. For every $g \in \text{Sp}(m, \mathbb{C})$, $\tilde{J} = gJg^{-1}$ is both an element of Lie algebra and a group element satisfying

$$\tilde{J}^2 = -I, \quad e^{t\tilde{J}} = \cos tI + (\sin t)\tilde{J}.$$ 

Setting $\alpha = \tilde{J}$ in (35) and noting $\alpha J = gJg^{-1}J = -g^{-1}g$, we have in [12] in generic $K$-ordered expression that

$$e^{tJ} = \frac{1}{\sqrt{\det(\cos tI - (\sin t)gKg)}} e^{\frac{t}{2\pi}(ug, ug)}.$$ 

Now, one may assume in generic ordered expressions, $tKg$ has disjoint $2m$ simple eigenvalues. Considering the diagonalization of $tKg$ in (35), we easily see that

Lemma 3.1 In a generic (open dense) ordered expression, the singular points distributed $\pi$-periodically along $2m$ lines parallel to the real axis, and the singular points are all simple double branched singular points. Moreover, $e^{\frac{t}{2\pi}(ug, ug)}$ is rapidly decreasing along lines parallel to the pure imaginary axis of the growth order $e^{-|\nu|^m}$, where $2m = n$.

3.1.1 Polar elements

For any fixed $g \in \text{Sp}(m, \mathbb{C})$, the behavior of $*$-exponential function $e^{\frac{t}{2\pi}(ug, ug)_*}$ is very strange. At $t=0$, the initial direction $\frac{1}{2\pi}(ug, ug)_*$ depends on $g \in \text{Sp}(m, \mathbb{C})$. At $t=\pi$, we have $e^{\frac{t}{2\pi}(ug, ug)_*}:K = \sqrt{1} = \pm 1$. As $\text{Sp}(m, \mathbb{C})$ is connected, this looks determined without $\pm$ ambiguity. If $K=0$, then (35) is

$$e^{\frac{t}{2\pi}(ug, ug)_*}:0 = \frac{1}{(\cos t)^m} e^{\frac{t}{2\pi}(ug, ug)_*}:0 = (-1)^m$$

without sign ambiguity by requesting 1 at the initial point $t = 0$. However choosing a suitable $K$, there is a case where $e^{\frac{\pi}{2\pi}(ug, ug)_*}:K = 1$.

On the other hand, $e^{\frac{\pi}{2\pi}(ug, ug)_*}:K$ also looks to be a focal point.

Lemma 3.2 Denoting by $[0 \sim a]$ a path starting from the origin 0 ending at the point $a$ avoiding singular points, but evaluated at $a$, we have

$$e^{[0 \sim \pi]\frac{1}{2\pi}(ug, ug)_*}:K = \frac{1}{\sqrt{\det K}} e^{-\frac{1}{2\pi}(u, \frac{1}{K} u)}.$$ 

The sign of $\sqrt{\det K}$ is determined by the sheet on which the end point of the path $[0 \sim \pi]$ is sitting.

At a first glance, as $\text{Sp}(m, \mathbb{C})$ is connected, this looks to be determined without $\pm$ ambiguity. Note however that for every $g \in \text{Sp}(m, \mathbb{C})$ there is an opposite $k \in \text{Sp}(m, \mathbb{C})$ such that $-\langle ug, ug \rangle_* = \langle uk, uk \rangle_*$. This is shown for instance

$$g \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} g = -g^t g.$$ 

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Thus, we have
\[ :e^\frac{i}{\pi} \pi_\epsilon(u,g,u_g) :_K = \frac{1}{\sqrt{\det K}} e^{-\frac{i}{\pi} \pi_\epsilon(u,g,u_g)} = :e^\frac{i}{\pi} \pi_\epsilon(u,k,u_k) :_K. \]

Considering a path \( g(t) \) from \( g \) to \( k \) in \( Sp(m, \mathbb{C}) \) together with line segment \( st, 0 \leq s \leq 1 \), for each \( t \).
We have also
\[ :e^\frac{i}{\pi} \pi_\epsilon(u,g,u_g) :_K = \frac{1}{\sqrt{\det K}} e^{-\frac{i}{\pi} \pi_\epsilon(u,g,u_g)} = :e^\frac{i}{\pi} \pi_\epsilon(u,k,u_k) :_K. \]

Thus, if \( :e^\frac{2i}{\pi} \pi_\epsilon(u,g,u_g) :_K = -1 \), it looks to cause a contradiction. The reason of this strange phenomenon is that a singular point appears on the path \( g(st) \) from \( g \) to \( k \). One cannot set the path \([0 \sim \pi]\) continuously in \( g \in Sp(m, \mathbb{C}) \), the sign changes discontinuously at some \( g \). In fact, the sign depends on the path \( e^\frac{t}{\pi} \pi_\epsilon(u,g,u_g) \) of the \( \ast \)-exponential function. Two paths are equivalent only when these can continuously change each other by keeping the parity of crossing slits invariant. To distinguish the sign, we use the notation
\[(39) \quad \varepsilon_0[g] :_K = :e^\frac{[0 \sim \pi]}{\pi} \pi_\epsilon(u_g,u_g) :_K = \frac{1}{\sqrt{\det(\cos((0 \sim \pi)I - (\sin((0 \sim \pi)yKg) e^{-\frac{i}{\pi} \pi_\epsilon(u,g,u_g)})},
\]

where \([0 \sim a]\) is the path along the straight line segment. Note that \( \varepsilon_0[g] :_K \) may not be defined at some \( g \), when a singular point appears in the interval \([0, \pi]\). \( \varepsilon_0[g] \) is not continuous w.r.t. \( g \). The sign changes discontinuously at some \( g \). For a generic \( K \), there is \( g \in Sp(m, \mathbb{C}) \) such that \( ygKg \) is a real diagonal matrix. Hence \( :e^\frac{t}{\pi} \pi_\epsilon(u,g,u_g) :_K \) has a singular point. One can make a detour of a singular point by a slight change of path, but a double branched singular point forces to change the sheets.

For every \( k, \varepsilon_0(k) = e^\frac{\pi i}{\pi} \pi_\epsilon_k \) is called a polar element. It is interesting that polar element \( \varepsilon_0(k) \) behaves just like a scalar, but it behaves various ways. Sometimes, it behaves as if it were \(-1\), and sometimes it looks as if \( i \) depending on \( K \). We call such elements \( g \)-scalars. But, to treat this as a univalent element, we have to distinguish more strictly, or it is better to treat \( \varepsilon_0(k) \) as two-valued elements. Although the additive structure is difficult to control, the multiplicative structure is treated within such a multi-valued structure by calculations such as \( \sqrt{a} \sqrt{b} = \sqrt{ab} \).

Polar elements are obtained also by the formula (12) in the case \( m = 1 \) at \( t = \pm \frac{\pi}{2} \). Set \( t = \pi i, \frac{1}{2} \pi i \) in (12). Then
\[ :e^\frac{i}{\pi} \pi_\epsilon \frac{2u^2}{2i} :_K = \frac{2}{\sqrt{4}}, \quad :e^\frac{i}{\pi} \pi_\epsilon \frac{2i2u^2}{2i} :_K = \frac{2}{\sqrt{4(\delta^2 - c^2)}} e^{\frac{1}{4}(\delta^2 - c^2) (2i)(\delta^2 + \delta^2) + 2cc^2} \]

By the bumping identity we have already seen in the previous note remarkable properties of polar elements:
\[ \varepsilon_0(k) * \tilde{u}_i = (-1)^{\delta k i} \tilde{u}_i * \varepsilon_0(k), \quad \varepsilon_0(k) * \tilde{v}_i = (-1)^{\delta k i} \tilde{v}_i * \varepsilon_0(k), \quad (k = 1 \sim m). \]

On the other hand, Proposition 2.6 shows that in a \( V \)-class expression \( \delta_\epsilon(u) \) anti-commutes with every generator. Now, comparing with the \( K \)-ordered expression in Theorem 2.1 of \( \delta_\epsilon(u) \), we see

**Theorem 3.1** A polar element \( :e^\frac{[0 \sim \pi]}{\pi} \pi_\epsilon(u_g,u_g) :_K \) is one of \( \pm 2^{-m} \delta_\epsilon(u), \pm i2^{-m} \delta_\epsilon(u) \). The constant is determined by \( K \) and the path \([0 \sim \pi]\) and the parity of times of crossing slits.
Note that \( \varepsilon_{\pi}^{[0,\pi]}(su(2),u_2) \) is defined in generic ordered expression, while \( \delta_s(u) \) is defined in V-class expressions without sign ambiguity. However, \( 2^{-m}\delta_s(u) \) belongs to various *-one parameter subgroups given in the form \( e^{i\omega}e^{\frac{\pi}{m}(u_2,u_2)} \) such that \( e^{\pi i\omega}e^{\frac{\pi}{m}(u_2,u_2)} = 1 \). We define the total polar element by

\[
\varepsilon_{00}(L) = \varepsilon_{00}(1)*\varepsilon_{00}(2)*\ldots*\varepsilon_{00}(m).
\]

In the case \( m = 1 \), setting \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{C}) \), the quadratic form \( \langle u_2, u_2 \rangle_s \) is

\[
(\tilde{u}, \tilde{v}) \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = (a^2+b^2)\tilde{u}^2 + (c^2+d^2)\tilde{v}^2 + 2(ac+bd)\tilde{u}\tilde{v}.
\]

In particular this covers the quadratic forms given by the Lie algebra \( su(2) \) of \( SU(2) \). Let \( su_1(2) \) be the space of all traceless skew-hermitian matrices with determinant 1. Thus by identifying \( su_1(2)J \) with a space of bilinear forms, we set

\[
(40) \quad \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} x+iy & i\rho \\ i\rho & x-iy \end{bmatrix}, \quad x, y, \rho \in \mathbb{R}, \quad x+iy = \sqrt{1-\rho^2} e^{i\theta}, \quad |\rho| < 1.
\]

Then we easily see

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4(1-\rho^2) e^{i\theta/2} \cosh \xi & 4\sqrt{1-\rho^2} e^{i\theta/2} \sinh \xi \\ 4\sqrt{1-\rho^2} e^{-i\theta/2} \sinh \eta & 4(1-\rho^2) e^{-i\theta/2} \cosh \eta \end{bmatrix}, \quad \cosh(\xi+\eta) = \frac{1}{\sqrt{1-\rho^2}} \cosh(\xi+\eta) = \frac{\rho}{\sqrt{1-\rho^2}},
\]

where \( \xi, \eta \) are real variables. If \( \rho = \pm 1 \), we set

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.
\]

Denote by \( S' \) the subset

\[
(41) \quad S' = \{ g \in SL(2, \mathbb{C}); \frac{1}{ih} \langle u_2, u_2 \rangle_s \in su_1(2)J \}.
\]

To control the sign ambiguity of (36), we have to prepare two sheets and have to fix slits. However, note that the way of setting slits is not unique. Let us consider the case \( m = 1 \) and a quadratic form \( 2\tilde{u}\tilde{v} \) by setting \( g = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \). We take a general expression parameter \( K = \begin{bmatrix} \delta & c \\ c & \delta' \end{bmatrix} \). By setting \( \Delta = e^t + e^{-t} - c(e^t - e^{-t}) \), the \( K \)-ordered expression of \( e^{\frac{1}{2}i\Delta\tilde{u}\tilde{v}} \) is given in [12] by

\[
(42) \quad :e^{\frac{1}{2}i\Delta\tilde{u}\tilde{v}} :_{K} = \frac{2}{\sqrt{\Delta^2 - (e^t - e^{-t})^2\delta\delta'}} e^{\frac{i}{\Delta} \frac{(e^t - e^{-t})^2\delta\delta'}{2} (K'\tilde{u}\tilde{v})^2}.
\]

### 3.2 Some special classes of expression parameters

As \( \varepsilon_{00} = \pm 1 \), polar elements have double valued nature, but \( \varepsilon_{00} \) is contained in various one parameter subgroups. Moreover as the periodicity depends also on the expression parameter, polar elements must be considered together with that one parameter subgroup. If \( \varepsilon_{00} \) is given as \( e^{\pi iH} \), then there are \( K, K' \) such that \( :e^{2\pi iH} :_{K} = 1 \), \( :e^{2\pi iH} :_{K'} = -1 \), and if \( :e^{2\pi iH} :_{K} = -1 \), then the argument in the previous subsection shows there must be a quadratic form \( H'_s \) such that \( :e^{2\pi iH'_s} :_{K'} = 1 \).
### 3.2.1 A special class $\mathfrak{K}_{re}$

We found in [12] there is a special class $\mathfrak{K}_{re}$ of expression parameters:

**Proposition 3.2** If $K_{re} = \begin{bmatrix} \rho & i^{c'} \\ ic' & \rho \end{bmatrix}$ with $c' = c + i\theta$, $c, \rho \in \mathbb{R}$, $|\theta|$ is small, satisfies $|\frac{\rho + ic'}{\rho - ic'} - 1| \neq 1$, then $K_{re}$ ordered expressions of $\ast$-exponential functions

$$e^{\frac{\pi}{\hbar} (ug, ug)_*}, \quad \forall g \in S' \quad (cf. [11]),$$

in particular

$$e^{\frac{\pi}{\hbar} 2\bar{u} \bar{v}}, \quad e^{\frac{\pi}{\hbar} (\bar{u}^2 + \bar{v}^2)}, \quad e^{\frac{\pi}{\hbar} (\bar{u}^2 - \bar{v}^2)},$$

have no singular point on the real axis and $\pi$-periodic, but each of them has singular points sitting $\pi$-periodically along two lines parallel to the real axis on both upper and lower half plane.

Hence, the polar element $\varepsilon_{00}$ may be written in the $K_{re}$-expression by

$$\varepsilon_{00} : e^{\frac{\pi}{\hbar} (u, u)}, \quad \forall g \in S'.$$

We have $\varepsilon_{00}^2 = 1$, but $\varepsilon_{00}$ is not a scalar, as this anti-commutes with generators. Moreover $\varepsilon_{00}$ has three square roots

$$e_1 = e^{\frac{\pi}{\hbar} 2\bar{u} \bar{v}}, \quad e_2 = e^{\frac{\pi}{\hbar} (\bar{u}^2 + \bar{v}^2)}, \quad e_3 = e^{\frac{\pi}{\hbar} (\bar{u}^2 - \bar{v}^2)}$$

such that $e_1^2 = \varepsilon_{00}$. Furthermore, we see $e_1e_2e_3 = e_3$ in the $K_{re}$-ordered expression.

Generally, adjoint relations of quadratic forms give the following master relations (cf. [13]) for elements of square roots of the polar element.

**Lemma 3.3** Let $H_*$ be a quadratic form with the discriminant $1$. Then, $e^{\pi \text{ad}(H_*)}_* e_j = e_j^{-1}$. This implies that $e_i e_j e_i^{-1} = e_j^{-1}$. These relations hold without sign ambiguity.

By this master relation, we have in general

$$e_i e_j = e_j^{-1} e_i = \varepsilon_{00} e_j e_i.$$

By the identity $e_3 = e_1 e_2$, we have

$$e_2 e_3 = e_2 e_1 e_2 = e_2 e_2^{-1} e_1 = e_1.$$

Similarly,

$$e_3 e_1 = e_3 e_2 e_3 = e_3 e_3^{-1} e_2 = e_2.$$

The bumping identity gives the interesting commutation relations:

$$\tilde{u} e_1 = -i e_1 \tilde{u}, \quad \tilde{u} e_2 = -e_2 \tilde{u}, \quad \tilde{u} e_3 = i e_3 \tilde{u},$$

$$\tilde{v} e_1 = i e_1 \tilde{v}, \quad \tilde{v} e_2 = e_2 \tilde{u}, \quad \tilde{v} e_3 = i e_3 \tilde{u}.$$

The polar element anti-commutes with generators, and we have
On the other hand, the subalgebra $e_1$. Obviously, the open intervals $(0, \pi)$ and $\pi, \infty$ exist such that

$$1 = \frac{1}{2}(1 + \varepsilon_{00}) + \frac{1}{2}(1 - \varepsilon_{00}), \quad \frac{1}{2}(1 + \varepsilon_{00})\frac{1}{2}(1 - \varepsilon_{00}) = 0.$$ 

The subalgebra $\frac{1}{2}(1 - \varepsilon_{00})$ is naturally isomorphic to the complexification $\mathbb{C}\mathbb{H}$ of the quaternion field $\mathbb{H}$ such that by denoting $\hat{1} = \frac{1}{2}(1 - \varepsilon_{00})$, $\hat{e}_i = \frac{1}{2}(1 - \varepsilon_{00})e_i$:

$$\hat{e}_{00} = \frac{1}{2}(1 - \varepsilon_{00}) = -\hat{1}, \quad \hat{e}_i = -\hat{1}^i \hat{e}_j = -\hat{1} \hat{e}_j \hat{e}_i, \quad 1 \leq i, j \leq 3.$$ 

On the other hand, the subalgebra $\frac{1}{2}(1 + \varepsilon_{00})$ is the group ring over $\mathbb{C}$ of the Klein’s four group. Obviously, $\frac{1}{2}(1 - \varepsilon_{00})A$ and $\frac{1}{2}(1 + \varepsilon_{00})A$ are not isomorphic.

Viewing $\varepsilon_{00}$ as the representative of $\varepsilon_{00}(k)$, we define for each $k$ square roots $e_1(k), e_2(k), e_3(k)$ of $\varepsilon_{00}(k)$, and denote by $A(k)$ the algebra generated by $e_{00}(k), e_1(k), e_2(k), e_3(k)$. Then

$$e_{i_1}^1(1) e_{i_2}^2(2) \cdots e_{i_m}^m(m); \quad \varepsilon_i = 0, 1, 2, 3$$

form an $m$-tensor algebra $A(1) \otimes A(2) \otimes \cdots \otimes A(m)$.

### 3.2.2 Another special class $\mathfrak{A}_s$

On the other hand, we have shown in [11] the following:

**Proposition 3.3** There is a small class $\mathfrak{A}_s$ (called another special class) of expression parameters such that polar elements $\varepsilon_{00}(1), \varepsilon_{00}(2), \ldots, \varepsilon_{00}(m)$ given by $\varepsilon_{00}(k) = \varepsilon_{00}(u_k = \varepsilon_{00})$ form a Clifford algebra $\mathcal{C}(m)$ of $m$ generators such that $\varepsilon_{00}(k) = -1$ for every $k$.

The shape of matrices in $\mathfrak{A}_s$ is given mainly as follows:

$$K_s = \begin{bmatrix} i\rho I_m & (c) \\ (c) & i\rho I_m \end{bmatrix}, \quad \rho, \ c \in \mathbb{R}, \quad \rho > 0, \quad (c) = m \times m \text{ matrix, all entries are } c$$

This is an expression parameter such that $\text{Re}(iK_s)$ is negative definite.

The proof of Proposition 3.3 is based on the fact that $\varepsilon_s^{\hat{u}_{k}(u_k = \varepsilon_{00})} A_{\mathfrak{A}_s}$ has singular points on the open intervals $(0, \pi)$ and $(\pi, 2\pi)$, but no other singular point in

$$e_{\hat{u}_{s}(u_k = \varepsilon_{00})} \mathfrak{A}_s, \quad (s, t) \in [0, \pi] \times [0, \pi].$$

For simplicity of notations we denote in what follows $\varepsilon_{00}(k)$ in the special class expression $K_s$ by $\varepsilon_k$, i.e. $\varepsilon_{K_s} = \varepsilon_{00}(k)$. The next formula is easy to see

$$e_{\hat{u}_{s}(u_k = \varepsilon_{00})} = \cos t + (\varepsilon_k \cos \theta + \varepsilon_{00} \sin \theta) \sin t.$$

That is, $\varepsilon_k, \varepsilon_{00}$ are “infinitesimal operators”, although these are $*$-exponentiated elements.
Proposition 3.4 A system of polar elements \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\} forms a Clifford algebra in a \(\mathfrak{R}_n\)-ordered expression such that \(\varepsilon_i \ast \varepsilon_j + \varepsilon_j \ast \varepsilon_i = -2 \delta_{ij}\). If \(m=2d\) then
\[
\xi_k = \frac{1}{2}(\varepsilon_{2k-1} + i\varepsilon_{2k}), \quad \eta_k = \frac{1}{2}(\varepsilon_{2k-1} - i\varepsilon_{2k}), \quad k = 1 \sim d,
\]
satisfy in + bracket notations
\[
\{\xi_k, \xi_l\} = 0 = \{\eta_k, \eta_l\}, \quad \{\xi_k, \eta_l\} = -\delta_{k,l}.
\]
Hence, \(\xi_1, \ldots, \xi_d, \eta_1, \ldots, \eta_d\) respectively generate the Grassmann algebra \(\Lambda_d(\xi), \Lambda_d(\eta)\).

Remark 1 In the previous subsection we have shown the relation that the polar element \(\varepsilon_k\) defined above is one of \(\pm 2^{-1} \delta(\tilde{u}_k, \tilde{v}_k), \pm 2^{-1} i \delta(\tilde{u}_k, \tilde{v}_k)\). However, note this is a very anomalous phenomenon. In spite that the commutativity \(\delta(\tilde{u}_l, \tilde{v}_l) \ast \delta(\tilde{u}_l, \tilde{v}_l) = \delta(\tilde{u}_l, \tilde{v}_l) \ast \delta(\tilde{u}_l, \tilde{v}_l)\) is easily checked in a \(\mathfrak{H}_n(\mathbb{R}^n)\)-class expression, Proposition 3.3 insists that \(\varepsilon_k \ast \varepsilon_l = -\varepsilon_l \ast \varepsilon_k\). Indeed, a polar element \(\varepsilon_k\) should be defined together with path in a \(\ast\)-exponential function from the origin. The equality above means only the endpoint of the path is one of \(\pm 2^{-1} \delta(\tilde{u}_k, \tilde{v}_k), \pm 2^{-1} i \delta(\tilde{u}_k, \tilde{v}_k)\). In this sense, polar elements are not elements of \(\text{Hol}(\mathbb{C}^{2m})\), but elements of a certain space which is one step up the degree of fine identifications. This does not seem to be a simple homotopical phenomena.

3.2.3 A class \(\mathfrak{R}_{im}\)

Note now that all \(\varepsilon_k\) in this subsection is defined by \(e_*^{\pm} u_k e_{\mp} v_k\), but a polar element is a member of various one parameter subgroups of \(\ast\)-exponential functions of quadratic forms. Since \(\varepsilon_k^2 = -1\), the argument in §3.3.1 shows that there must exist a quadratic form \(H_*\) such that \(e_*^{\pm} H_* : K_* = e_*^{\pm} K_*\), but \(e_*^{\pm} H_* : K_* = 1\). Hence, it seems to be better to think \(e_*^{\pm} H_* : K_* \neq e_*^{\pm} K_*\), but how the properties of square roots \(e_*^{\pm} H_* : e_*^{\pm} H_* = e_*^{\pm} K_* = e_*^{\pm} K_*\) are explained.

In the previous note [12], we have treated the case \(m=1\) and \(K\) is given by
\[
\begin{pmatrix}
i\rho & c' \\
c & \rho
\end{pmatrix}, \quad \rho > 0, \quad c' = c + i\theta, \quad c \in \mathbb{R}, \quad \theta \neq 0 \text{ but small.}
\]
In this expression, the polar element \(\varepsilon_{00}\) splits into three cases:
\[
(e_*^{\pm} u_{00})^2 : K_{im} = -1, \quad (e_*^{\pm} (u^2 - \tilde{v}^2))^2 : K_{im} = -1, \quad (e_*^{\pm} (u^2 + \tilde{v}^2))^2 : K_{im} = 1.
\]
Note The replacement \((u, v) \rightarrow (e_*^{\mp} u, e_*^{\pm} v)\) of generators gives the exchange of the second and third elements.

Here one may set \(e_*^{\mp} \tilde{v} \varepsilon_0 = e_*^{\pm} (u^2 - \tilde{v}^2)\), but \(e_*^{\mp} (u^2 + \tilde{v}^2)\) does not relate others. Although we have three square roots as above
\[
e_1 = e_*^{\pm} \tilde{v} \varepsilon_0, \quad e_2 = e_*^{\pm} (u^2 - \tilde{v}^2), \quad e_3 = e_*^{\pm} (u^2 + \tilde{v}^2),
\]
23
As it is mentioned in the preface, there is no mathematical definition of “vacuum”, but this is a kind of target to which one makes actions to create the “space-time” or the “configuration space”.

**Note** So far, $K$ was called an expression parameter and it was treated as a supplemental parameter to express the true nature of elements. However, the observation for the polar elements in this section shows that expression parameters represent certain essential nature of elements. Here we note that $R_s \cap R_{re} = \emptyset$, $R_{im} \cap R_{re} = \emptyset$. In the next section, we give another notion of elements which depend essentially on the expression parameters.

## 4 Vacuums, pseudo-vacuums

As it is mentioned in the preface, there is no mathematical definition of “vacuum”, but this is a kind of target to which one makes actions to create the “space-time” or the “configuration space”.

Recall first several properties of the $*$-exponential function $e^{zH_z}$, $H_z = \sum_{k=1}^{m} z_k \tilde{u}_k \tilde{v}_k$.

**(a)** In generic ordered expression, one may suppose that there is no singular point on the real line. $e^{zH_z}$ is $4\pi$ periodic w.r.t. $t$, i.e. $e^{(z+i\pi)H_z} = e^{zH_z}$. More precisely, the periodicity depends on the real part of $z$. In the case $m=1$, there is an interval $[a,b]$, called the exchanging interval in [13] such that $e^{(s+t)H_z}$ is $2\pi$ periodic if $a < s < b$, and alternating $2\pi$ periodic if $s \notin [a,b]$.

**(b)** $e^{zH_z}$ is rapidly decreasing on $\mathbb{R}$ of $e^{-|t|}$ order. Hence the integral $\int_{\mathbb{R}} e^{zH_z} dt$ is well-defined, and it follows: $H_z \int_{\mathbb{R}} e^{zH_z} dt = 0$.

**(c)** Double branched singular points are distributed $\pi i$ periodically along $2m$ lines parallel to the imaginary axis whose positions depend on expression parameters. In the case $m=1$, the real part of these lines are $a$ and $b$.

**(d)** $\lim_{t \to -\infty} e^{-t} e^{zH_z}$ is a nontrivial element denoted by $\tilde{c}L$. We have called this the (total) vacuum in [13]. The precise statement for the case $m = 1$ is as follows:

**Proposition 4.1** In a generic ordered expression $e^{\frac{1}{2m} \tilde{u} \tilde{v}}$ is rapidly decreasing with the growth order $e^{-|t|}$ along lines parallel to the real axis. Noting $\tilde{v} \tilde{u} = \tilde{u} \tilde{v} + \frac{1}{2} i \hbar$, we see the following: In generic ordered expressions such that there is no singular point on the real axis, but the limit exist

$$
\lim_{t \to -\infty} e^{\frac{1}{2m} \tilde{u} \tilde{v}} = \infty_0, \quad \lim_{t \to -\infty} e^{\frac{1}{2m} \tilde{u} \tilde{v}} = \infty_0, \quad \lim_{t \to \infty} e^{\frac{1}{2m} \tilde{u} \tilde{v}} = 0, \quad \lim_{t \to \infty} e^{\frac{1}{2m} \tilde{u} \tilde{v}} = 0.
$$

More precisely, in a fixed generic expression parameter $K = [\begin{smallmatrix} s & 0 \\ c & s \end{smallmatrix}]$, $e^{\frac{1}{2m} \tilde{u} \tilde{v}}$ is smooth rapidly decreasing in $\pm$ directions and [42] gives

$$
\begin{align*}
\infty_0 : K & = \lim_{t \to -\infty} e^{\frac{1}{2m} \tilde{u} \tilde{v}} : K = \frac{2}{(1+c)^2 - \delta \delta} e^{\frac{1}{2m} \tilde{u} \tilde{v} - (1+c) \tilde{u} \tilde{v} + \delta \delta}, \\
\infty_0 : K & = \lim_{t \to \infty} e^{\frac{1}{2m} \tilde{u} \tilde{v}} : K = \frac{2}{(1-c)^2 - \delta \delta} e^{\frac{1}{2m} \tilde{u} \tilde{v} + (1-c) \tilde{u} \tilde{v} + \delta \delta}.
\end{align*}
$$

$$
\begin{align*}
\lim_{t \to -\infty} e^{\frac{1}{2m} \tilde{u} \tilde{v}} : K & = 0, \quad \lim_{t \to -\infty} e^{\frac{1}{2m} \tilde{u} \tilde{v}} : K = 0.
\end{align*}
$$
without sign ambiguity by requesting that this is in the same sheet as the starting point \( t = 0 \). The idempotent properties of \( \varpi_0 \) and \( \varpi_0 \) follows immediately, but the product \( \varpi_0 \* \varpi_0 \) is not defined. We call \( \varpi_0 \) and \( \varpi_0 \) vacuum and bar-vacuum respectively.

Now compare \([14]\) with the formula \([19]\) in the case \( m = 1 \). If \( K \in \mathcal{D}_+(V) \). Then \( \delta_x(V) \* \delta_x(V) \) is welldefined, but the \( K \)-ordered expression in \([19]\) has a sign ambiguity.

On the other hand, \( \varpi_{00} :_K \) is defined without sign ambiguity. We see that the \( \pm \) sign of \( \delta_x(V) \* \delta_x(V) \) must be chosen so that these equalize in \( V \)-class expressions.

For every \( k \), \( \varpi_{00}(k) = \lim_{s \to -\infty} e^{-\frac{i}{k} \sum \varepsilon_k^* \varepsilon_k} \) and \( \varpi_{00}(k) = \lim_{t \to \infty} e^{\frac{i}{k} \sum t \varepsilon_k \varepsilon_k} \) are called the partial vacuum and the partial bar-vacuum respectively. As they are given by \( \lim \), the exponential law gives the idempotent property \( \varpi_{00}(k) \* \varpi_{00}(k) = \varpi_{00}(k) \), \( \varpi_{00}(k) \* \varpi_{00}(k) = \varpi_{00}(k) \). In these definitions, the \( * \)-product \( \varpi_{00}(k) \* \varpi_{00}(k) \) is not fixed depending on \( s + t \). We call

\[
\varpi(L) = \varpi_{00}(1) \* \varpi_{00}(2) \* \cdots \varpi_{00}(m)
\]
a standard vacuum, where \( \varpi_{00}(k) = \lim_{t \to -\infty} e^{-\frac{i}{k} \sum \varepsilon_k^* \varepsilon_k} \), but to fix this without sign ambiguity, we have the mention about the path to \( t \to -\infty \) so that the path does not change sheets, which is always possible, but we can select the another sheet to obtain \( - \varpi(L) \). Such a selection rule does not suffer the definition of vacuums. As

\[
\delta_x(V) \* \delta_x(V) = \prod_k \delta_x(V) \* \delta_x(V) \text{ and } \varpi(L) = \prod_k \lim_{t \to -\infty} e^{-\frac{i}{k} \sum \varepsilon_k^* \varepsilon_k},
\]

the next one follows:

**Proposition 4.2** \( \delta_x(V) \* \delta_x(V) \) is what we called the (total) vacuum \( \varpi(L) \) in previous notes. Precisely, in every \( V \)-class expression we have \( \delta_x(V) \* \delta_x(V) :_K = : \varpi(L) :_K \), but the r.h.s. is defined for generic ordered expressions. Similarly, we have \( \delta_x(V) \* \delta_x(V) :_K = : \varpi(L) :_K \).

As \( \tilde{v}_k^* \varpi(L) = 0 \) we see in generic ordered expression

\[(45) \quad \varepsilon_{00}(L) \* \varpi(L) = i^m \varpi(L) .
\]

In the case \( \varpi(L) \), the regular representation space w.r.t. \( \varpi(L) \) is \( \mathbb{C}[\tilde{u}] \* \varpi(L) \). This is obviously ordinary commutative space of functions \( f(\tilde{u}) \). The detail will be mentioned in \( \S 5 \) Here we give only a several comments.

Extending \( \mathbb{C} \)-linearly the natural mapping \( \varepsilon(\tilde{u}) = \tilde{v} \), we obtain a nondegenerate bilinear form

\[
\langle f(\tilde{u}), g(\tilde{u}) \rangle = \varpi(L) \* t(f(\tilde{u})) \* g(\tilde{u}) \* \varpi(L)
\]
such that

\[
\langle f(\tilde{u}), g(\tilde{u}) \rangle = \langle g(\tilde{u}), f(\tilde{u}) \rangle, \quad \langle f(\tilde{u}), g(\tilde{u}) \* h(\tilde{u}) \rangle = \langle f(\tilde{u}) \* g(\tilde{u}), h(\tilde{u}) \rangle.
\]

We call this the Frobenius algebra w.r.t. \( \varpi(L) \). (See Wikipedia for a quick view of Frobenius algebra.)

Although \( K \)-ordered expressions may have sign ambiguity, \( \ast \)-products of vacuums are defined without ambiguity. Namely under \( V \)-class expressions, the family of elements \( \delta_x(V) \ast \delta_x(V) \) for \( (\tilde{x}, \tilde{y}) \in V \) generates a noncommutative associative algebra.
Proposition 4.3. The family of elements \( \{ \delta_{\nu}^{(V)}(\tilde{y}-\tilde{y}) \} \) associatively closed under the \( \ast \)-product and every element is an idempotent element. By Theorem 2.7 together with changing variables gives the product formula:

\[
(\delta_{\nu}^{(V)}(\tilde{v}-\tilde{y}) \ast \delta_{\nu}^{(V)}(\tilde{u}-\tilde{x})) \ast (\delta_{\nu}^{(V)}(\tilde{v}-\tilde{y}) \ast \delta_{\nu}^{(V)}(\tilde{u}-\tilde{x})) = e^{-\frac{1}{\hbar} \langle \tilde{y} J, \tilde{z} \rangle} \delta_{\nu}^{(V)}(\tilde{v}-\tilde{y}) \ast \delta_{\nu}^{(V)}(\tilde{u}-\tilde{x})
\]

where \( \langle (\tilde{y}-\tilde{y}) J, \tilde{z} \rangle = \sum_{j=1}^{m}(\tilde{y}_j-\tilde{y}'_j)(\tilde{y}_j-\tilde{y}'_j) \). Denote by \( \{ \tilde{y} \} \ast \delta_{\nu}^{(V)}(\tilde{v}-\tilde{y}) \ast \delta_{\nu}^{(V)}(\tilde{u}-\tilde{x}) \) for simplicity and call it the field of vacuums. Hence (46) is the product formula of fields of vacuums.

**Proof.** The associativity follows by the direct calculation. It is enough to show the product formula. By definition, we have

\[
\delta_{\nu}^{(V)}(\tilde{v}-\tilde{y}) \ast \delta_{\nu}^{(V)}(\tilde{u}-\tilde{x}) = \int dV_{\tilde{x}} dV_{\tilde{y}} dV_{\tilde{y}'} \int dV_{\tilde{u}} dV_{\tilde{v}} dV_{\tilde{v}'}
\]

Proposition 2.7 gives \( e_{\nu}^{(\tilde{v}, \tilde{u} \ast)} e_{\nu}^{(\tilde{y}, \tilde{y}')} = e_{\nu}^{(\tilde{v} + \tilde{y}, \tilde{u} + \tilde{y}')} \where \( \langle \tilde{v} J, \tilde{y} \rangle = - \sum_{j=1}^{m} \tilde{v}_j \tilde{y}_j \). Plugging this with the exponential law, we have

\[
\int dV_{\tilde{v}} dV_{\tilde{u}} dV_{\tilde{v}'} \int dV_{\tilde{y}} dV_{\tilde{y}'} dV_{\tilde{z}}
\]

This becomes

\[
\int dV_{\tilde{v}} dV_{\tilde{u}} dV_{\tilde{v}'} \int dV_{\tilde{y}} dV_{\tilde{y}'} dV_{\tilde{z}}
\]

where \( \{ \tilde{y} \} = \delta_{\nu}^{(V)}(\tilde{v}-\tilde{y}) \ast \delta_{\nu}^{(V)}(\tilde{u}-\tilde{x}) \). Note that \( \int dV e_{\nu}^{(\tilde{v}, \tilde{y}'+ \tilde{z} J)} dV_{\tilde{y}} = \delta_{\nu}^{(V)}(\tilde{v}-\tilde{y}'+ \tilde{z} J) \). This is supported only at \( \tilde{z} = (\tilde{y} J, \tilde{y}') J \). Hence

\[
\int dV_{\tilde{v}} dV_{\tilde{u}} dV_{\tilde{v}'} \int dV_{\tilde{y}} dV_{\tilde{y}'} dV_{\tilde{z}}
\]

Plugging this into the integral, we obtain (46).}

The product formulas (46) are rewritten as

\[
\{ \tilde{y} \} \ast \{ \tilde{y}' \} = C_{\tilde{y}, \tilde{y}'} \{ \tilde{y} \} \ast \{ \tilde{y}' \} = e^{-\frac{1}{\hbar} \langle \tilde{y} J, \tilde{z} \rangle \ast \delta_{\nu}^{(V)}(\tilde{u}-\tilde{x})}
\]

It is easy to see that

\[
\{ \tilde{y} \} \ast \{ \tilde{y}' \} \ast \{ \tilde{y}'' \} = C_{\tilde{y}, \tilde{y}', \tilde{y}''} \{ \tilde{y} \} \ast \{ \tilde{y}' \} \ast \{ \tilde{y}'' \} = 1
\]

Setting \( \tilde{y} = \phi(\tilde{x}) \) for a smooth mapping, \( \{ \phi(\tilde{x}) \} \ast \{ \tilde{x} \} \) will be called a field of vacuums on \( V \sqcap \mathbb{C}^m \).
Setting \( C[\tilde{u}]*\{\tilde{\phi}(\tilde{x})\} \) as the regular representation space, \( \tilde{u}_i \) is represented as the multiplication operator, and \( \frac{1}{i\hbar} \tilde{u}_i \) does as the differential operator

\[
\frac{1}{i\hbar} \tilde{u}_i * p_s(\tilde{u})*\{\tilde{\phi}(\tilde{x})\}_s = (\partial_{\tilde{u}} p_s(\tilde{u}) + \frac{1}{i\hbar} \tilde{\phi}_i(\tilde{x}) p_s(\tilde{u})) * \{\tilde{\phi}(\tilde{x})\}_s.
\]

This is often denoted by the notation of covariant differentiation

\[
\frac{1}{i\hbar} \tilde{u}_i * p_s(\tilde{u})*\{\tilde{\phi}(\tilde{x})\}_s = (\nabla_{\tilde{u}} p_s(\tilde{u}))*\{\tilde{\phi}(\tilde{x})\}_s.
\]

Recall the definition of \(*\)-delta functions \([20]\) of full variables. The next formula gives a relation between \( \delta_s^{(V)}(u-x) \) and “half-variable” \(*\)-delta functions.

\[
:\delta_s^{(V)}(u-x)_:K = \int_V e^{\frac{i}{\pi} (\xi \cdot (u-x))} :\delta_s^{(V)}(\xi):K d\xi = \int_V e^{-\frac{i}{\pi}(\xi,J,\eta)} :\delta_s^{(V)}(\eta,\bar{\eta}-\bar{y})_:K d\xi d\eta = \int_V :\delta_s^{(V)}(u-x') + \frac{1}{\eta} :\delta_s^{(V)}(\eta,\bar{\eta}-\bar{y})_:K d\xi d\eta.
\]

Replacing \( \frac{1}{\eta}J \) by \( \xi' \), we see

\[
\delta_s^{(V)}(u-x') = \int_V 2^m \delta_s^{(V)}(\tilde{u} - \tilde{x}' + \xi') :\delta_s^{(V)}(\xi') d\xi'.
\]

As \( \langle \xi', J, \bar{\eta} - \bar{y} \rangle s \delta_s(\tilde{v} - \bar{y}) = 0 \) gives \( e^{-\frac{i}{\pi}(\xi, J, \bar{\eta} - \bar{y})} \delta_s(\tilde{v} - \bar{y}) = e^{-\frac{i}{\pi}(\xi', J, \bar{\eta} - \bar{y})} \delta_s(\tilde{v} - \bar{y}) \), we have

\[
\delta_s^{(V)}(u-x') * \{y\bar{x}\} = 2^m \int_{V \cap C^m} e^{-\frac{i}{\pi}(\xi', J, \bar{\eta} - \bar{y})} :\delta_s^{(V)}(u-x' + \xi') d\xi' * \{y\bar{x}\} = 2^m e^{\frac{1}{\eta}(\tilde{u} - \tilde{x}', \bar{J}, \bar{\eta} - \bar{y})} \{y\bar{x}\}_s.
\]

In particular, \( \delta_s^{(V)}(u)*\{0\}_s = 2^m \{00\}_s \). Several interesting properties will be given in the next section.

On the other hand, if \( |\text{Re} s| \) is sufficiently large, then \( e^{(s+i\sigma)\frac{i}{\pi}\tilde{u} \tilde{x}} \) and \( e^{(s+i\sigma)\frac{i}{\pi}\tilde{v} \tilde{y}} \) are both \( 2\pi \)-periodic w.r.t. \( \sigma \). More precisely, these are \( 2\pi \)-periodic w.r.t. \( \sigma \), if \( s \) is outside of the exchanging interval \([a, b]\). Thus, it is better to define vacuums as the limits of period integral:

\[
2\pi \varpi_{00} = \lim_{t \to -\infty} \int_{-\pi}^{\pi} e^{(t+i\sigma)\frac{i}{\pi}\tilde{u} \tilde{x}} d\sigma, \quad 2\pi \varpi_{00} = \lim_{s \to \infty} \int_{-\pi}^{\pi} e^{(s+i\sigma)\frac{i}{\pi}\tilde{v} \tilde{y}} d\sigma.
\]

In fact, we have no need to take the limit. Cauchy’s integral theorem gives

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(s+i\sigma)\frac{i}{\pi}\tilde{u} \tilde{x}} d\sigma = \begin{cases} \varpi_{00}(\tilde{x}_s) & s < a \\ 0 & s > b \end{cases}, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(s+i\sigma)\frac{i}{\pi}\tilde{v} \tilde{y}} d\sigma = \begin{cases} 0 & s < a \\ \varpi_{00}(\tilde{y}_s) & s > b \end{cases}.
\]

The product \( \varpi_{00} * \varpi_{00} \) can not be defined directly by the definition, but the product

\[
\int_{-\pi}^{\pi} e^{(s+i\sigma)\frac{i}{\pi}\tilde{u} \tilde{x}} d\sigma * \int_{-\pi}^{\pi} e^{(s'+i\sigma')\frac{i}{\pi}\tilde{v} \tilde{y}} d\sigma' = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{(s+i\sigma)\frac{i}{\pi}\tilde{u} \tilde{x} + (s'+i\sigma')\frac{i}{\pi}\tilde{v} \tilde{y}} d\sigma d\sigma'
\]

can be defined always to give 0, for by using \( \frac{1}{\pi} \tilde{u} \tilde{v} = \frac{1}{\pi} \tilde{u} \tilde{v} - \frac{1}{\pi} \), and \( \frac{1}{\pi} \tilde{u} \tilde{v} = \frac{1}{\pi} \tilde{u} \tilde{v} + \frac{1}{\pi} \), the change of variables gives

\[
\int_{-\pi}^{\pi} e^{i\tau} d\tau \int_{-\pi}^{\pi} e^{i\tau' (s-s') \frac{1}{\pi}\tilde{u} \tilde{v}} d\tau' = 0.
\]

Thus, we have

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Proposition 4.4 For every polynomial $p(u, v)$, $\omega_{00} * p(\tilde{u}, \tilde{v}) * \omega_{00} = 0 = \omega_{00} * p(\tilde{u}, \tilde{v}) * \omega_{00}$ in generic ordered expression.

Note The next identities are easy to see

$$\omega_{00} * \frac{1}{i\hbar} \tilde{u} \tilde{v} = \frac{1}{2} \omega_{00}, \quad \frac{1}{i\hbar} \tilde{u} \tilde{v} * \omega_{00} = -\frac{1}{2} \omega_{00}.$$ 

Hence in order to keep the associativity $(\omega_{00} * \frac{1}{i\hbar} \tilde{u} \tilde{v}) * \omega_{00} = \omega_{00} * (\frac{1}{i\hbar} \tilde{u} \tilde{v} * \omega_{00})$, we have to define $\frac{1}{2} \omega_{00} * \omega_{00} = -\frac{1}{2} \omega_{00} * \omega_{00} = 0$. To avoid such a strange phenomenon, we have to indicate how an element has been defined.

4.1 Clifford vacuum

Recall Propositions 3.3 and 3.4. Note that as $\varepsilon^2 = -1$ in $\mathfrak{su}$-ordered expressions we have

$$ \frac{1}{2} (1 + i \varepsilon) + \frac{1}{2} (1 - i \varepsilon) = 1, \quad \frac{1}{2} (1 + i \varepsilon) * \frac{1}{2} (1 - i \varepsilon) = 0, \quad \frac{2}{2} (1 + i \varepsilon)^2 = \frac{1}{2} (1 + i \varepsilon), \quad \frac{2}{2} (1 - i \varepsilon)^2 = \frac{1}{2} (1 - i \varepsilon) .$$

$$(1 + i \varepsilon) * \overline{\omega}_* (L) = 0 = \overline{\omega}_* (L) * (1 + i \varepsilon), \quad \frac{1}{2} (1 - i \varepsilon) * \overline{\omega}_* (L) = \overline{\omega}_* (L), \quad \frac{1}{2} (1 + i \varepsilon) * \overline{\omega}_* = \overline{\omega}_* * \frac{1}{2} (1 - i \varepsilon).$$

Note that

Proposition 4.5 In generic ordered expression, \( \frac{1}{i\hbar} \int_{-\infty}^{t} \varepsilon^{\frac{1}{2}} \tilde{u}_k \tilde{\varepsilon} dt \) is welldefined to give the *-inverse \((\tilde{u}_k * \tilde{\varepsilon})^{-1}\). Using these, \( \tilde{u}_k = (\tilde{u}_k * \tilde{\varepsilon})^{-1} * \tilde{\varepsilon} \) satisfies \( \tilde{u}_k * \tilde{u}_k = 1 \), \( \tilde{\varepsilon} * \tilde{u}_k = 1 - \omega_{00}(k) \). \( \tilde{u}_k \) is called a (left) half-inverse of \( \tilde{\varepsilon} \). As \( (\tilde{u}_k * \tilde{\varepsilon})^{-1} \) commutes with \( \tilde{\varepsilon}_k \) and \( \tilde{\varepsilon} * \tilde{u}_k = -\tilde{u}_k * \tilde{\varepsilon}_k \), \( \tilde{u}_k \) anti-commutes with \( \tilde{\varepsilon}_k \). Moreover as \( \tilde{u}_k * \omega_{00}(k) = 0 \), we see \( \tilde{u}_k * \overline{\omega}_* = 0 \) for every \( k \).

We set now

$$\xi_k = \frac{1}{2} (1 + i \varepsilon) * \tilde{u}_k, \quad \eta_k = \frac{1}{2} (1 - i \varepsilon) * \tilde{u}_k = \frac{1}{2} (1 - i \varepsilon) * (\tilde{u}_k * \tilde{\varepsilon})^{-1} * \tilde{\varepsilon}.$$ 

The next formulas are easy to see

$$\xi_k^2 = 0 = \eta_k^2, \quad \xi_k * \eta_k + \eta_k * \xi_k = 1 - \frac{1}{2} (1 + i \varepsilon) * \omega_{00}(k) = 1, \quad \eta_k * \overline{\omega}_* (L) = 0 = \overline{\omega}_* (L) * \xi_k.$$ 

Hence, \( \{1, \xi_k, \eta_k\} \) generates 2×2 matrix algebra \( M_2(\mathbb{C}) \). As \( \varepsilon_k * \varepsilon_\ell = -\varepsilon_\ell * \varepsilon_k \) for \( k \neq \ell \), we see in general

$$\xi_k * \xi_\ell + \xi_\ell * \xi_k = 0 = \eta_k * \eta_\ell + \eta_\ell * \eta_k, \quad \xi_k * \eta_k + \eta_k * \xi_k = \delta_{k\ell}.$$ 

Hence \( \xi_1, \cdots, \xi_m \) (resp. \( \eta_1, \cdots, \eta_m \)) form a Grassmann algebra \( \bigwedge_m (\xi) \) (resp. \( \bigwedge_m (\eta) \)). Grassmann algebra is called often a “super commutative” algebra. Now setting \( \overline{\omega}(\wedge) = \overline{\omega}(L) \), we call \( \overline{\omega}(\wedge) \) the Clifford vacuum.

The regular representation space is spanned by the space of all differential forms \( \sum f_\alpha (\tilde{u}) \xi^\alpha * \overline{\omega}(\wedge) \), where \( \xi^\alpha = \xi_i \wedge \xi_j \wedge \cdots \wedge \xi_p \) using notations of Grassmann algebra. By this representation, the computations on Clifford algebra is translated into the calculus on the Grassmann algebra. Note that

$$\tilde{\varepsilon}_k * \xi^\alpha * \overline{\omega}(\wedge) = 0, \quad k = 1 \sim m.$$
It follows
\[ \tilde{v}_k \ast f_\alpha (\tilde{u}) \xi^a \ast \tilde{\varpi} (\wedge) = i h (\partial_k f_\alpha (\tilde{u})) \xi^a \ast \tilde{\varpi} (\wedge). \]

It is easy to see
\[ (51) \]
\[ \tilde{\varpi} (\wedge) \ast \mathbb{C}[\tilde{u}, \tilde{v}] \ast (\bigwedge_m (\xi) + \bigwedge_m (\eta)) \tilde{\varpi} (\wedge) = \mathbb{C} \tilde{\varpi} (\wedge). \]

Hence, the regular representation space has a natural nondegenerate bilinear form over \( K \). It is difficult to fix the formula of the exterior differentiation. Hence, it is difficult to fix the formula of the exterior differentiation.

Note however that
\[ \xi_k \ast (\tilde{u} \ast \tilde{v}) = (\tilde{u} \ast \tilde{v}) \ast \xi_k, \quad (\tilde{u} \ast \tilde{v}) \ast \xi \ast \tilde{\varpi} (\wedge) = 0, \quad \frac{1}{2} \tilde{u} \ast \tilde{v} = - i h - \tilde{u} \ast \tilde{\varpi}_{00} (k) \ast \tilde{v}. \]

Since \( \tilde{u} \ast \tilde{\varpi}_{00} (k) \ast \tilde{v} = 0 \), these two are canonical conjugate in a sense. Then, one may use
\[ \frac{1}{2} \tilde{u}^{2k}, \frac{1}{2} \tilde{u}^{2}, \cdots, \frac{1}{2} \tilde{u}^{2m}, \tilde{u} \ast \tilde{v}, \tilde{u} \ast \tilde{v}, \cdots, \tilde{u} \ast \tilde{v} \]

instead of original generators \( \tilde{u}, \tilde{v}, \cdots, \tilde{u}, \tilde{v} \). We denote
\[ (x_1, \cdots, x_m, y_1, \cdots, y_m) = (\frac{1}{2} \tilde{u}^{2k}, \frac{1}{2} \tilde{u}^{2}, \cdots, \frac{1}{2} \tilde{u}^{2m}, \tilde{u} \ast \tilde{v}, \tilde{u} \ast \tilde{v}, \cdots, \tilde{u} \ast \tilde{v}). \]

The regular representation space is spanned by elements such as
\[ f_\alpha (x) \ast \xi^a \ast \tilde{\varpi} (\wedge). \]

Thus, the exterior differential is defined by
\[ d(\xi^a \ast \tilde{\varpi} (\wedge)) = \sum_k \xi_k \ast y_k \ast f_\alpha (x) \xi^a \ast \tilde{\varpi} (\wedge), \quad (i.e. \quad d = \sum_k \xi_k \ast y_k \ast ). \]

On this space, the computational rule is the same to those of differential forms. As \( x_k = \frac{1}{2} \tilde{u}^{2k} \) and this looks “positive” in a sense, the algebra generated by \( (x_1, \cdots, x_m, y_1, \cdots, y_m) \) is not isomorphic to the original Weyl algebra, but it looks to be the algebra of \( m \) copies of upper half planes.

### 4.2 Pseudo-vacuums

Let \( I_s (K) = [a, b] \) be the exchanging interval (cf. [19]) of \( e_*(s+i \tilde{u} \ast \tilde{v}) \). If \( a < b, \quad e_*(s+i \tilde{u} \ast \tilde{v}) \) is \( 2 \pi i \)-periodic in \( t \), and if \( s < a \) or \( b < s \) \( e_* (s+i \tilde{u} \ast \tilde{v}) \) is alternating \( 2 \pi i \)-periodic. For \( a < s < b \), we set
\[ (52) \]
\[ : \varpi_*(s) :_K = \frac{1}{2 \pi} \int_0^{2 \pi} : e_*(s+i \tilde{u} \ast \tilde{v}) :_K dt, \]

This is independent of \( s \) whenever \( a < s < b \) and \( (\frac{1}{2 \pi} \tilde{u} \ast \tilde{v}) \ast \int_0^{2 \pi} e_*(s+i \tilde{u} \ast \tilde{v}) = 0. \) We denote by \( K_0 \) the totality of expression parameter \( K \) such that \( I_s (K) \) contains the origin 0.
Proposition 4.6 Suppose \( K \in \mathcal{R}_0 \). Then \( e^{\frac{i\pi}{m}\tilde{u}_k\tilde{v}_k} \) is 2\( \pi \)-periodic and \( \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{i\pi}{m}\tilde{u}_k\tilde{v}_k} dt \) has the idempotent property. This is called the pseudo-vacuum. We denote this by \( \tilde{\omega}_s(0) \). This is a nontrivial element, but note that the pseudo-vacuum is expressed only by expression parameter \( K \in \mathcal{R}_0 \).

Proof It is enough to show that \( \tilde{\omega}_s(0) \neq 0 \). To see this, note that \( \lim_{s \to -\infty} e^{(s+i\pi)\frac{i\pi}{m}\tilde{u}_k\tilde{v}_k} = \tilde{\omega}_0 \neq 0 \) for every fixed \( t \). Hence \( \lim_{s \to -\infty} e^{(s+i\pi)\frac{i\pi}{m}\tilde{u}_k\tilde{v}_k} = 2\pi \tilde{\omega}_0 \neq 0 \).

By Proposition 4.2 and the definition of exchanging interval, we see

Corollary 4.1 Obviously \( \mathcal{R}_{re} \subset \mathcal{R}_0 \). Hence one may treat the pseudo-vacuum under \( K_{re} \)-expressions.

Important remark \( \varepsilon_{00} \) satisfies \( \varepsilon_{00} \varepsilon_{00} = 1 \) in \( K_{re} \)-expression, while \( \varepsilon_{00} \tilde{\omega}(L) = i\tilde{\omega}(L) \). This may sound contradiction, but recall the definition of the product \( e^{\frac{i\pi}{m}\tilde{u}_k\tilde{v}_k} * F \). This is defined as the solution of evolution equation

\[
\frac{d}{dt} f_t = i\frac{\tilde{u}_k \tilde{v}_k}{\pi} \alpha^*_K f_t, \quad f_0 = :F^*_K.
\]

Note that the property \( \varepsilon_{00} \tilde{\omega}(L) = 1 \) is the property of the solution with the initial data 1. \( e^{\frac{i\pi}{m}\tilde{u}_k\tilde{v}_k} * \tilde{\omega}(L) \) is alternating 2\( \pi \)-periodic in generic ordered expressions, for the vacuum is very far from origin 1.

It should be noted that the identity \( e^{\frac{i\pi}{m}\tilde{u}_k\tilde{v}_k} * \tilde{\omega}(L) = 0 \) holds only for \( z \) is pure imaginary or for a small real part.

In the case \( m > 1 \) we define

\[
\tilde{\omega}_s(0) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{it_1 \frac{i\pi}{m} \tilde{u}_1 \tilde{v}_1} \cdots * e^{it_m \frac{i\pi}{m} \tilde{u}_m \tilde{v}_m} d^m t \subset K_{re}.
\]

We see then

\[
\frac{1}{i\hbar} \tilde{u}_k \tilde{v}_k = 0, \quad e^{it \frac{i\pi}{m} \tilde{u}_k \tilde{v}_k} \tilde{\omega}_s(0) = \tilde{\omega}_s(0), \quad t \in \mathbb{R}, \quad k = 1 \sim m.
\]

Hence the regular representation space w.r.t. \( \tilde{\omega}_s(0) \) is \((\mathbb{C}[u] + \mathbb{C}[v]) \tilde{\omega}_s(0) \), on which \( u_k \partial_{u_k} - v_k \partial_{v_k}, (u_k \partial_{u_k})^2 + (v_k \partial_{v_k})^2 \) act as operators.

Moreover, repeated use of the bumping identity gives the following:

Proposition 4.7 \( \tilde{\omega}_s(0) * \mathbb{C}[u,v] \tilde{\omega}_s(0) = \tilde{\omega}_s(0) \).

Proof It is enough to show that \( \tilde{\omega}_s(0) \tilde{u} \tilde{\omega}_s(0) = 0 = \tilde{\omega}_s(0) \tilde{v} \tilde{\omega}_s(0) \) in the case \( m = 1 \). Note that the bumping identity and the exponential law give

\[
\int_0^{2\pi} e^{is \frac{i\pi}{m} \tilde{u}_k \tilde{v}_k} d s \tilde{u} \tilde{v} = \tilde{u} \int_0^{2\pi} e^{is \frac{i\pi}{m} \tilde{u}_k \tilde{v}_k} d s \int_0^{2\pi} e^{is \frac{i\pi}{m} \tilde{u}_k \tilde{v}_k} d t = \tilde{u} \int_0^{2\pi} e^{is \frac{i\pi}{m} \tilde{u}_k \tilde{v}_k} d s d t = \tilde{u} \int_0^{2\pi} e^{is \frac{i\pi}{m} \tilde{u}_k \tilde{v}_k} d s d t = 0.
\]

The similar computation gives the second one.

In particular, we have

\[
\tilde{\omega}_s(0) \tilde{u}_k \tilde{v}_k \tilde{u}_k \tilde{v}_k \tilde{\omega}_s(0) = (i\hbar)^p (1/2)_p \tilde{\omega}_s(0), \quad \tilde{\omega}_s(0) \tilde{u}_k \tilde{v}_k \tilde{u}_k \tilde{v}_k \tilde{\omega}_s(0) = (-i\hbar)^p (1/2)_p \tilde{\omega}_s(0),
\]

(53)
where \((a)_p = a(a+1) \cdots (a+p-1)\), all others are 0. Hence by defining an involution

\[ \tau : \mathbb{C}[\bar{u}] + \mathbb{C}[\bar{v}] \to \mathbb{C}[\bar{v}] + \mathbb{C}[\bar{u}], \quad \tau(\bar{a}_k) = \bar{v}_k, \quad \tau(\bar{v}_k) = -\bar{a}_k, \quad \tau(1) = 1, \]
the regular representation space \((\mathbb{C}[\bar{u}] + \mathbb{C}[\bar{v}]) \ast \tilde{\omega}_s(0)\) has a natural nondegenerate bilinear product \(\tilde{\omega}_s(0) \ast f \ast g' \ast \tilde{\omega}_s(0)\).

On the other hand by the property (b) mentioned in §4 integrals \(\int_{-\infty}^{0} e_s^{s \frac{1}{i\hbar} \bar{u}_s \bar{v}} ds, - \int_{0}^{\infty} e_s^{s \frac{1}{i\hbar} \bar{u}_s \bar{v}} ds\) converge to give a \(*\)-inverse of \(\frac{1}{i\hbar} \bar{u} \ast \bar{v}\). Similarly, integrals \(\int_{-\infty}^{0} e_s^{s \frac{1}{i\hbar} \bar{v}_s \bar{u}} ds, - \int_{0}^{\infty} e_s^{s \frac{1}{i\hbar} \bar{v}_s \bar{u}} ds\) converge to give \(*\)-inverses of \(\frac{1}{i\hbar} \bar{v} \ast \bar{u}, \frac{1}{i\hbar} \bar{u} \ast \bar{v}\) respectively. By setting \(\bar{u}^* = \bar{v} \ast (\frac{1}{i\hbar} \bar{u} \ast \bar{v})^{-1}\), we have

\[ \bar{u}^* \bar{u}^* = 1, \quad \bar{u} \ast \bar{u}^* = 1 - \tilde{\omega}_{00}. \]

Note also that

\[ \tilde{\omega}_{00} \ast \tilde{\omega}_{00} = \tilde{\omega}_{00}, \quad \bar{u} \ast \tilde{\omega}_{00} = 0 = \tilde{\omega}_{00} \ast \bar{v}, \quad (\bar{u}^*)^k \ast \tilde{\omega}_{00} \ast \bar{v}^\ell \text{ is the } (k, \ell)-\text{matrix element}. \]

Since \((\bar{u}^* \bar{v}) \ast \tilde{\omega}_s(0) = - \frac{\hbar}{2} \tilde{\omega}_s(0)\), if the associativity is expected, then one may set

\[(54) \quad \bar{u}^* \ast \tilde{\omega}_s(0) = \bar{v} \ast (\frac{1}{i\hbar} \bar{u} \ast \bar{v})^{-1} \ast \tilde{\omega}_s(0) = - 2 \bar{v} \ast \tilde{\omega}_s(0). \]

However note that the double integral

\[ \int_{0}^{\infty} \int_{0}^{2\pi} e_s^{s \frac{1}{i\hbar} \bar{u}_s \bar{v}} \ast e_s^{s \frac{1}{i\hbar} \bar{v}_s \bar{u}} ds dt \]

does not converge suffered by a singular point in the domain. However, if we take the integral by \(dt\) first, then this gives an interesting result. Note that \(\tilde{\omega}_s(0)\) is given also by \(\tilde{\omega}_s(0) = \frac{1}{\pi} \int_{0}^{2\pi} e_s^{s \frac{1}{i\hbar} \bar{u}_s \bar{v}} dt\) and \(\int_{0}^{2\pi} e_s^{s \frac{1}{i\hbar} \bar{u}_s \bar{v}} dt = 0\) for \(s < a\) or \(s > b\), where \([a, b]\) is the exchanging interval of \(:e_s^{s \frac{1}{i\hbar} \bar{u}_s \bar{v}} :_K\).

**Lemma 4.1** Under the expression \(K \in \mathfrak{R}_0\), it holds \(e_s^{(s+i)t \frac{1}{i\hbar} \bar{u} \bar{v}} \ast \tilde{\omega}_s(0) = \tilde{\omega}_s(0)\) for \(a < s < b\) and vanishes outside. Hence

\[ \int_{0}^{\infty} e_s^{s \frac{1}{i\hbar} \bar{u}_s \bar{v}} \ast \tilde{\omega}_s(0) ds = 2(1 - e^{-\frac{b}{\hbar}}) \tilde{\omega}_s(0), \quad \bar{u}^* \ast \tilde{\omega}_s(0) = - 2(1 - e^{-\frac{b}{\hbar}}) \bar{v} \ast \tilde{\omega}_s(0) \]

which is (54) only in the case that the exchanging interval is \([a, \infty)\).

We think this might be the true nature of the formula of the product \(\bar{u}^* \ast \tilde{\omega}_s(0)\) depending on the expression parameter.

Anyhow, by setting \(\bar{u}^* \ast \tilde{\omega}_s(0) = a_0 \bar{v} \ast \tilde{\omega}_s(0)\), we compute

\[ (\bar{u}^*)^2 \ast \tilde{\omega}_s(0) = a_0 \bar{v} \ast \left(\frac{1}{i\hbar} \bar{u} \ast \bar{v}\right)^{-1} \ast \tilde{\omega}_s(0) = - a_0 \bar{v} \ast \int_{0}^{\infty} e_s^{s \frac{1}{i\hbar} \bar{u}_s \bar{v}} \ast \tilde{\omega}_s(0) dt. \]

The bumping identity and the argument about the exchanging interval give

\[ (\bar{u}^*)^2 \ast \tilde{\omega}_s(0) = - a_0 \bar{v} \ast \int_{0}^{b} e_s^{s \frac{1}{i\hbar} \bar{u}_s \bar{v} - 1} \ast \tilde{\omega}_s(0) dt = - a_0 \bar{v} \ast \int_{0}^{b} e^{-\frac{b}{\hbar}} dt. \]

Repeating these we have the next result:
Lemma 4.2 Depending on the expression parameter $K \in \mathfrak{K}_0$, there are constants $\alpha_n \neq 0$ such that $(\tilde{u}^*)^n \ast \varpi_0(0) = \alpha_n v^n \ast \varpi_0(0)$.

Thus the regular representation space is the linear space $\mathbb{C}[\tilde{u}^*, \tilde{u}]$, which is linearly isomorphic to the Laurent polynomial space $\mathbb{C}[z^{-1}, z]$. However, the space $\mathbb{C}[\tilde{u}^*, \tilde{u}]$ is not closed under the $\ast$-product. This generates an algebra $\mathcal{A}$ which is the direct sum of the Laurent polynomial space and the space of matrices:

$$\mathcal{A} = \mathbb{C}[\tilde{u}^*, \tilde{u}] \oplus \mathcal{M}, \quad \text{where } \mathcal{M} = \{(\tilde{u}^*)^k \ast \varpi_00 \ast \tilde{u}^\ell; k, \ell \in \mathbb{N}\}.$$

$\mathcal{A}$ is a noncommutative associative algebra containing the two-sided ideal $\mathcal{M}$ of matrices of finite rank such that the quotient algebra is the Laurent polynomial ring $\mathbb{C}[z^{-1}, z]$.

$$0 \to \mathcal{M} \to \mathcal{A} \to \mathbb{C}[z^{-1}, z] \to 0.$$

Hence $\mathcal{A}$ may be regarded as a nontrivial extension of $\mathbb{C}[z^{-1}, z]$ by $\mathcal{M}$ such that

$$z \ast z^{-1} = 1, \quad z^{-1} \ast z = 1 - \varpi_00, \quad z^{-2} \ast z = z^{-1} - \tilde{u}^* \ast \varpi_00, \quad z^{-2} \ast z^2 = 1 - \varpi_00 - \tilde{u}^* \ast \varpi_00 \ast \tilde{u}, \ e.t.c.$$

4.3 $SU(2)$-vacuum

Note that the pseudo-vacuum $\tilde{\varpi}_0(0)$ is given by the period integral $\int_{S^3} dt$ of a $\ast$-exponential function. By recalling the argument in §3.2.1 it is natural to think such a period integral may extend to higher dimensions, e.g. $\int_{S^3} dg$. In this section, we restrict our attention to the case $m = 1$.

By using $x, y, \rho$ in (40), $\ast$-exponential functions $\epsilon_x \frac{it}{\pi} H_x$ for $H_x = (\tilde{u}g, \tilde{u}g) \in \mathfrak{su}_1(2)J$ is written in the form

$$\gamma_x(t, g) : _K = : \epsilon_x \frac{it}{\pi} (\tilde{u}^2 + \varpi^2) + iy(\tilde{u}^2 - \varpi^2) + 2\rho \tilde{u} \varpi \tilde{v}^\prime \cdot \epsilon_{J_K}.$$

Recalling Proposition 3.1 we apply the formula (36) for the cases

$$K = \begin{bmatrix} 1 & ic \\ ic & 1 \end{bmatrix} \in \mathfrak{K}_{re}, \quad K' = \begin{bmatrix} 1 & ic' \\ ic' & 1 \end{bmatrix}, \quad c' = c + i\theta, \quad \theta \neq 0 \quad \text{but small for simplicity}.$$

The $K'$-ordered expression is given as

$$(55) \quad \gamma_x(t, g) : _{K'} = \frac{1}{\sqrt{\text{det}(\cos t - (\sin t)gK'g)}} e^{\frac{1}{2}(ug \cos t - \sin t y \sqrt{g} \varpi_00 \ast y \varpi_00 \ast g)}.$$

Note that $\gamma_x(t, g) : _{K'}$ is holomorphic in $(t, g) \in V \times S'$ where $V$ is a neighborhood of $[0, 2\pi]$ in $\mathbb{C}$.

Note also that $\text{det}(\cos tI - (\sin t)y (gK')g) = \text{det}(\cos tI - (\sin t)g'ygK')$ and set

$$g'y = \begin{bmatrix} x + iy & ip \\ ip & x - iy \end{bmatrix}, \quad g \in SL(2, \mathbb{C}).$$

Then, $x^2 + y^2 + \rho^2 = 1$, and the amplitude part is

$$\left( \text{det} \begin{bmatrix} \cos t - (x - \rho c' + iy) \sin t & -i(\rho + c'(x + iy)) \sin t \\ -i(\rho + c'(x - iy)) \sin t & \cos t - (x - \rho c' - iy) \sin t \end{bmatrix} \right)^{-1/2} \left( (\cos t - (x - \rho c') \sin t)^2 + y^2 \sin^2 t + ((\rho + c')^2 + (c')^2 y^2) \sin^2 t \right)^{-1/2}.$$
This is singular at a point \( y=0, \rho+c'x=0, \cot t=x(1+(c')^2) \). Hence under the \( K' \)-ordered expression, there is only one singular point \((\cot t, x, y, \rho)=(0, 0, 0, 0)\). As we use only \( K' \)-ordered expression, we need not care about cocycle conditions, and this expressions determine a point set \( D^4 \). We easily see that the following:

**Lemma 4.3** There is only one singular point at \((\cos t, x, y, \rho)=(0, 0, 0, 0)\) in the unit 4-disk

\[
D^4=\{(\cos t, x, y, \rho); \cos^2 t+\sin^2 t(x^2+y^2+\rho^2) \leq 1\}
\]

Note that \( \gamma=(\cos t, x, y, \rho) \) such that \( \cos^2 t+\sin^2 t(x^2+y^2+\rho^2)=1 \) corresponds to the element

\[
\gamma_h=e^{i\frac{\pi}{4}(x(\bar{a}^2+\bar{b}^2)+iy(\bar{a}^2-\bar{b}^2)+2i\rho\bar{a}\bar{b})} \in SU(2)
\]

and \( :SU(2)_K: \), forms a group under the \( *_{K'} \)-product. Thus, one may identify the boundary \( \partial D^4 \) with the group \( SU(2) \). Letting \( dm_\gamma \) be the standard invariant measure on \( SU(2)=S^3 \), we set

\[
:\Omega_{*}:_{K'}=\int_{\partial D^4} \gamma_{*}:_{K'} dm_\gamma
\]

and call \( \Omega_{*} \) the \( SU(2) \)-vaccum, although this is a collection of expressed elements. Obviously,

\[
:SU(2)_K:*_{K'}:SU(2)_K: = :\Omega_{*}:_{K'} = :\Omega_{*}:_{K'}:*_{K'}:SU(2)_K: , \quad \text{and hence }:SU(2)_K:*_{K'}:SU(2)_K: = 0.
\]

**Proposition 4.8** The \( SU(2) \)-vacuum does not vanish, i.e. \( \Omega_{*} \neq 0 \).

**Proof** Recall that \( [55] \) is holomorphic in particular on \((t, g) \in V \times V_{SU(2)} \) where \( V \) is a neighborhood of \([0, \pi] \) in \( \mathbb{C} \) and \( V_{SU(2)} \) is a neighborhood of \( SU(2) \) in \( SL(2, \mathbb{C}) \). We use the projection \( \iota : \mathcal{A} \to \frac{1}{2}(1+\varepsilon_{00})\mathcal{A} \) defined in Theorem 3.2 where \( \iota(\varepsilon_{00}) \) may be treated as 1. It is enough to prove \( \iota(\Omega_{*}) \neq 0 \).

For every fixed \( g, h \in S' \), consider a mapping \( C : \partial D \to V \times V_{SU(2)} \) as in the l.h.s.figure. Note that \( C \) extends to a holomorphic mapping \( \tilde{C} : D \to V \times V_{SU(2)} \). By Cauchy’s integral theorem we see \( \iota(\tilde{\gamma}(g))=\int_{\partial D^4} \iota(\tilde{\gamma}(g);_K) \) does not depend on \( g \). Setting \( g=1 \), we have

\[
\iota(\tilde{\gamma}(g);_K) = \int_{\partial D^4} \iota(\tilde{\gamma}(g);_K) dt.
\]

This is non vanishing by the same reasoning as in Proposition 14.6. Now, note that \( SU(2)/S^1=S^2 \), then integrating by the standard volume form \( dm(S^2) \) on \( S^2 \) gives

\[
\int_{\partial D^4} \iota(\tilde{\gamma}(g);_K) dm_\gamma = \int_{S^2} \iota(\tilde{\gamma}(g);_K) dm(S^2) = 4\pi \int_{\partial D^4} \iota(\tilde{\gamma}(g);_K) \iota(\tilde{\gamma}(g);_K) dm_\gamma
\]

In a sense, \( \Omega_{*} \) is the group \( SU(2) \) itself, just as the invariant 3-form on \( SU(2) \).

Note that \( e^{i\Omega_{*}(\bar{u},\bar{v})} = \Omega_{*} \) holds only for \( t \in \mathbb{R} \). Note also that the space of quadratic forms is given as \( \mathfrak{sl}(2, \mathbb{C}) J \) and \( \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{h}(2) \) where \( \mathfrak{h}(2) \) is the space of all traceless hermitian matrices. The space \( \mathfrak{h}(2) J \) forms a Lie algebra under a new bracket product \([X, Y] = i[X, iY] \). But this is isomorphic to \( \mathfrak{su}(2) J \) and hence \( \mathfrak{h}(2) J \) may be regarded as the Lie algebra of \( SU(2) \).

Now, it is a problem how the regular representation space w.r.t. \( \Omega_{*} \) should be defined. One may think that this must contain the enveloping algebra of \( \mathfrak{h}(2) J \) under the \( * \)-product, that is,
(\mathfrak{h}(2)J)^\ast \ast \Omega_\ast$ are contained. But this refuses $i$ in the constant term, for $i(\mathfrak{h}(2)J)=\mathfrak{su}(2)J$. Hence it is difficult to make an infinite dimensional algebra under the computation of modulo $\mathfrak{su}(2)J$.

On the other hand $\{1, i, \tilde{u}, i\tilde{u}, \tilde{v}, i\tilde{v}, \mathfrak{h}(2)J\}$ is a Lie algebra over $\mathbb{R}$ under the new bracket product $[[X, Y]]$, and there are Lie subalgebras

$$E_0=\{1, \tilde{u}, \tilde{v}, \frac{1}{2\hbar}(\tilde{u}^2+\tilde{v}^2)\}, \quad E_+=\{1, e^{\frac{\pi i}{2}\tilde{u}}, e^{-\frac{\pi i}{2}\tilde{v}}, \frac{i}{2\hbar}(\tilde{u}^2-\tilde{v}^2)\}$$

$$E_-=\{1, e^{\frac{\pi i}{2}}(\tilde{u}+\tilde{v}), e^{-\frac{\pi i}{2}}(\tilde{u}-\tilde{v}), \frac{i}{\hbar}\tilde{u}\tilde{v}\}.$$  

These are mutually isomorphic under suitable linear change of generators, and their enveloping algebras are infinite dimensional. In general, every closed one-parameter subgroup $T_s$ of $SU(2)$ causes a rotation on a 2-dimensional $\mathbb{R}$ linear subspace $V_2$ of $\mathbb{C}\tilde{u}\oplus\mathbb{C}\tilde{v}$ such that $[V_2, V_2] \subset \mathbb{R}$.

The $\mathbb{R}$-linear subspaces where the rotations occurs by $\frac{1}{2\hbar}(\tilde{u}^2+\tilde{v}^2)$, $\frac{1}{2\hbar}(\tilde{u}^2-\tilde{v}^2)$ and $\frac{i}{\hbar}\tilde{u}\tilde{v}$ span a 3-dimensional subspace $E^3$ over $\mathbb{R}$:

$$E^3=\mathbb{R}(\tilde{u}, \tilde{v}) \oplus \mathbb{R}(e^{\frac{\pi i}{2}\tilde{u}}, e^{-\frac{\pi i}{2}\tilde{v}}) \oplus \mathbb{R}(\frac{\tilde{u}^2+\tilde{v}^2}{2}, \frac{\tilde{u}^2-\tilde{v}^2}{2}, \frac{i}{\hbar} \tilde{u}\tilde{v})$$

with complementary subspace $E_0=\mathbb{R}((1-i)\tilde{u}-(1+i)\tilde{v})$.

Setting

$$(s, \xi_1, \xi_2, t)=s+\xi_1 \tilde{u}+\xi_2 \tilde{v}+t\frac{1}{\hbar}(\tilde{u}^2+\tilde{v}^2),$$

the Lie bracket product on $E_0$ is

$$[[s, \xi_1, \xi_2, t], (s', \eta_1, \eta_2, t')]=((\xi_1\eta_2-\xi_2\eta_1), t\eta_2-t'\xi_2, -t\eta_1+t'\xi_1, 0).$$

The next one is proved by direct calculations:

**Proposition 4.9** $E_0$ forms a Lie subalgebra under the bracket product $[[X, Y]]$. Furthermore, $E_0$ has a Lorentz bilinear form

$$<(s, \xi_1, \xi_2, t), (s', \eta_1, \eta_2, t')>=st'+ts' - \xi_1\eta_1 - \xi_2\eta_2,$$

which is adjoint invariant:

$$<(s, \xi_1, \xi_2, t), (s', \eta_1, \eta_2, t')>, (s'', \eta_1', \eta_2', t'')]+<(s', \eta_1, \eta_2, t'), [(s, \xi_1, \xi_2, t), (s'', \eta_1', \eta_2', t'')]> = 0.$$

In particular, $E_0$ is a Minkowski space.

It is not hard to see that the Lie group $G$ with the Lie algebra $(E_0, [\cdot, \cdot])$ is obtained by giving a group structure on the space $E_0$. Note that $E_0$ is not invariant under the Lorentz group $SO(1, 3)$.

As it is naturally expected, $E_\pm$ are also Minkowski spaces, and these three $\{E_0, E_+, E_-\}$ jointly define the standard Minkowski metric on $\mathbb{R}^{1+3}$ such that the positive cone $V_+$ in $\mathbb{R}^{1+3}$ is the subset

$$V_+=\{AA^*; A\in SL(2, \mathbb{C})\}$$

of all hermitian matrices $\mathbb{R}^2\mathfrak{h}(2)$. The Lorentz group $SO(1, 3)$ acts only on such a joint object. This gives a big change of the role of the Lorentz group $SO(1, 3)$ and gives an entrance key to the relativistic quantum theory in physics. These will be discussed in the next note.


5 Pseudo-differential operators as vacuum representations

Before entering into the field theory, we have to make it clear what the infinite dimensional regular representation space by a vacuum makes a configuration space. In this subsection we set \( V = \mathbb{R}^{2m} \) for simplicity, and we show that ordinary pseudo-differential operators are obtained by the vacuum representation of a certain \( \mu \)-regulated algebra.

Let \( C^\infty(\mathbb{R}^m) \), \( C^\infty_0(\mathbb{R}^m) \) be the space of smooth functions on \( \mathbb{R}^m \) and the space of all compactly supported smooth functions on \( \mathbb{R}^m \) respectively. First for every \( f(\tilde{x}) \in C^\infty_0(\mathbb{R}^m) \) consider the correspondence

\[
 f(\tilde{x}) \to f_*(\tilde{x} + \tilde{u}) = \int_{\mathbb{R}^m} f(\tilde{x} + \tilde{x}') \delta_*(\tilde{u} - \tilde{x}') \, d\tilde{x}'.
\]

This may be regarded as the \( * \)-function field defined by \( f \in C^\infty_0(\mathbb{R}^m) \). The inverse correspondence is obtained by taking the Weyl ordered expression of the r.h.s. and replacing \( \tilde{u} \) by 0.

Let \( h(\tilde{x}, \tilde{y}') \) be a smooth function w.r.t. \( \tilde{x} \) and a tempered distribution w.r.t. \( \tilde{y}' \) on \( \mathbb{R}^{2m} \), we define hybrid \( * \)-function, \( h_*(\tilde{x}, \tilde{v}) \) by

\[
 h_*(\tilde{x}, \tilde{v}) = \int \int h(\tilde{x}, \tilde{y}') \delta_*(\tilde{v} - \tilde{y}') \, d\tilde{y}'.
\]

Consider the operator

\[
 f_*(\tilde{u} + \tilde{x}) \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \} \to h_*(\tilde{x}, \tilde{v}) \ast f_*(\tilde{u} + \tilde{x}) \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \}.
\]

Note that

\[
 h_*(\tilde{x}, \tilde{v}) \ast f_*(\tilde{u} + \tilde{x}) \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \} = \int \int h(\tilde{x}, \tilde{y}') f(\tilde{x} + \tilde{x}') \{ \tilde{y}' \tilde{x}' \} \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \}, \, d\tilde{y}' \, d\tilde{x}'.
\]

Following Proposition 2.5, we would like to write the above in the form

\[
 h_*(\tilde{x}, \tilde{v}) \ast f_*(\tilde{u} + \tilde{x}) \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \} = \int_{\mathbb{R}^m} P(h, f)(\tilde{x} + \tilde{x}') \delta_*(\tilde{u} - \tilde{x}') d\tilde{x}' \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \},
\]

and to take out \( P(h, f)(\tilde{x} + \tilde{x}') \)-part. For that purpose, rewrite \( \{ \tilde{y}' \tilde{x}' \} \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \} \) as follows

\[
 \delta_*(\tilde{v} - \tilde{y}') \ast \delta_*(\tilde{u} - \tilde{x}') \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \} = \int \int e^{\frac{i}{\hbar} (\eta, \tilde{\xi} + \tilde{x}')} e^{\frac{i}{\hbar} (\eta, \tilde{\phi}(\tilde{x}) - \tilde{y}')} e^{\frac{i}{\hbar} (\tilde{\xi}, \tilde{\phi}(\tilde{x}) - \eta)} \{ \tilde{\phi}(\tilde{x}) \tilde{x} \}, d\tilde{y}' \, d\tilde{\xi}
\]

by using \( (\eta, \tilde{\xi}) = (\tilde{\xi}, \eta) \). Integrating by \( \tilde{\xi} \), we have

\[
 h_*(\tilde{x}, \tilde{v}) \ast f_*(\tilde{u} + \tilde{x}) \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \} = \int \int h(\tilde{x}, \tilde{y}') f(\tilde{x} + \tilde{x}') e^{\frac{i}{\hbar} (\tilde{\xi}, \tilde{\phi}(\tilde{x}) - \tilde{y}')} \delta_*(\tilde{u} - \tilde{x}' + \tilde{\eta}) d\tilde{y}' d\tilde{\xi} \ast \{ \tilde{\phi}(\tilde{x}) \tilde{x} \}.
\]
Now, note that the $*$-variable $\hat{v}$ does not contained in the integral. By Proposition 2.3, we take its Weyl ordered expression. Since $\delta_\nu((\hat{u}-\hat{x}+\hat{\eta})\gamma_0)=\delta((\hat{u}-\hat{x}+\hat{\eta})J)$ and this is supported only at $\hat{\eta}=(\hat{u}-\hat{x})J$, we replace $\hat{\eta}$ in the integral by $(\hat{u}-\hat{x})J$. Then putting $\hat{u}=0$, we have

$$h_\nu(\hat{x}, \hat{v}) * f_\nu(\hat{u} + \hat{x})*\{\hat{\phi}(\hat{x})\hat{x}\}_\nu = \left( \int \int h(\hat{x}, \hat{y}) f(\hat{x} + \hat{\nu}e^{-\frac{1}{\hbar}JF(x, \hat{y})}) d\hat{x} d\hat{y} \right) \{\hat{\phi}(\hat{x})\hat{x}\}_\nu.$$ 

If $\hat{\phi}(\hat{x})=0$, the operator in the big parentheses may be regarded as the pseudo-differential operator of Hörmander. The product formula is given by PDO-product formula. In general, $\hat{\phi}(\hat{x})$ is regarded as the electromagnetic potential.

**Note** $\int \int h(\hat{x}', \hat{y}') f(\hat{x}'') e^{\frac{1}{\hbar}JF(\hat{x}', \hat{y}'-\hat{x}'')} d\hat{x}' d\hat{y}'$ does not converge in general, even if $f(\hat{x}'') \in C_0^\infty(V)$, but if $h(\hat{x}', \hat{y}')=(\hat{y}'-\hat{y})^\alpha$, then the integration by parts gives that

$$\int (\hat{y}' - \hat{y})^\alpha f(\hat{x}'') * e^{\frac{1}{\hbar}JF(\hat{x}', \hat{y}'-\hat{x}'')} d\hat{x}' d\hat{y}' = (ih\partial_x)^\alpha f(\hat{x}').$$

By this reason, what is essential for the convergence is the remainder term of the expansion

$$h(\hat{x}', \hat{y}') = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \partial^\alpha_\nu h(\hat{x}', \hat{y})(\hat{y}' - \hat{y})^\alpha + h(k)(\hat{x}', \hat{y}).$$

The integral converges if $h(k)(\hat{x}', \hat{y})$ belongs to $L_1(V \cap C^m)$ for some $k$. To make the required condition clear, we have to define the notion of “symbol class” of functions. Such a method of calculation of integrals is often called oscillatory integrals and it is denoted by

$$os-\int \int h(\hat{x}', \hat{y}') f(\hat{x}'') e^{\frac{1}{\hbar}JF(\hat{x}', \hat{y}'-\hat{x}'')} d\hat{x}' d\hat{y}'$$

Note also that the variables of configuration space play only parameters. In contrast, these play essential role in pseudo-differential operators of Weyl type mentioned below.

Setting $u=(\hat{u}, \hat{v})$, $x=(\hat{x}, \hat{y})$, and using the correspondence

$$f_\nu(\hat{u}) = \int f(\hat{x}) \delta_\nu(\hat{u} - \hat{x}) d\hat{x}, \quad h_\nu(u) = \int \int h(x) \delta_\nu(u - x) d\hat{x},$$

we consider the operator

$$h_\nu(u) * f_\nu(\hat{u}) * \{0\}_\nu = \int \int h(x) f(x') \delta_\nu(u - x) * \delta_\nu(\hat{u} - \hat{x}) d\hat{x} d\hat{y} d\hat{x} * \{0\}_\nu.$$ 

But we want to rewrite this in the form $\int P(h, f)(\hat{x}') \delta_\nu(\hat{u} - \hat{x}') * \{0\}_\nu$. Note that by Proposition 2.1 we have

$$\delta_\nu(\hat{u} - \hat{x}) * \delta_\nu(\hat{u} - \hat{x}') = \int \int e^{-\frac{1}{\hbar}JF(\xi, \eta)} e^{\frac{i}{\hbar}JF(\hat{u} - \hat{x})} * e^{\frac{i}{\hbar}JF(\eta, \hat{y})} d\xi d\eta d\xi' = \int \int e^{-\frac{1}{\hbar}JF(\xi, \eta)} e^{\frac{i}{\hbar}JF(\eta, \hat{y})} d\xi d\eta d\xi'.$$
Since $\tilde{u}_*\{00\}_s=0$, we see $e^{\frac{1}{i\hbar}(\tilde{u}-\tilde{x})}\ast\{0\}_s=e^{\frac{1}{i\pi}(\tilde{y},\tilde{y})}\ast\{0\}_s$ and

$$
\delta_*(u-x)\ast\delta_*(\tilde{u}-\tilde{x})\ast\{00\}_s
$$

$$
=\int\int e^{-\frac{1}{i\pi}(\tilde{\xi},\tilde{\eta})}e^{\frac{1}{2i\hbar}(\tilde{\xi},\tilde{\eta})}e^{\frac{1}{i\hbar}(\tilde{\xi},\tilde{u}+\tilde{x})}d\tilde{\xi}d\tilde{\eta}d\tilde{\xi}'\{00\}_s
$$

$$
=\int\int e^{-\frac{1}{i\pi}(\tilde{\xi},\tilde{\eta})}e^{\frac{1}{2i\hbar}(\tilde{\xi},\tilde{x}'-\tilde{x})}e^{\frac{1}{i\hbar}(\tilde{\xi},\tilde{u}-\tilde{x}+\tilde{\eta})}d\tilde{\xi}d\tilde{\eta}d\tilde{\xi}'\{00\}_s
$$

$$
=\int\int e^{-\frac{1}{i\pi}(\tilde{\xi},\tilde{\eta})}e^{\frac{1}{2i\hbar}(\tilde{\xi},\tilde{x}'-\tilde{x}-\tilde{\eta})}e^{\frac{1}{i\hbar}(\tilde{\xi},\tilde{u}-\tilde{x}+\tilde{\eta})}d\tilde{\xi}d\tilde{\eta}d\tilde{\xi}'\{00\}_s.
$$

Integrating by $\tilde{\xi}$ and $\tilde{\xi}'$, we have

$$
\int\int h(\tilde{x},\tilde{y})f(\tilde{x}')\delta_*(u-x)\ast\delta_*(\tilde{u}-\tilde{x}')d\tilde{x}d\tilde{y}d\tilde{x}'\ast\{00\}_s
$$

$$
=\int\int h(\tilde{x},\tilde{y})f(\tilde{x}')e^{-\frac{1}{i\pi}(\tilde{\eta},\tilde{y})}\delta(\tilde{x}'-\tilde{x})\delta_*(\tilde{u}-\tilde{x}+\tilde{\eta})d\tilde{\eta}d\tilde{x}d\tilde{y}d\tilde{x}'\{00\}_s.
$$

Now, recall that the Weyl ordered expression of $\delta_*(\tilde{u}-\tilde{x}'+\tilde{\eta}J)$ is $\delta(\tilde{u}-\tilde{x}'+\tilde{\eta}J)$. Hence setting $\tilde{u}=\tilde{x}'$, there must be the relation

$$
2(\tilde{x}'-\tilde{x})=\tilde{\eta}J=\tilde{x}'-\tilde{x}'.
$$

It follows

$$
\tilde{x}=(\tilde{x}'+\tilde{x}')/2.
$$

Changing variables gives

$$
(56)\ h_*(u)\ast f_*(\tilde{u})\ast\{\tilde{y}x\}_s=\int\left(\int h\left(\frac{1}{2}(\tilde{x}'+\tilde{x}''),\tilde{y}\right)f(\tilde{x}')e^{\frac{i}{\pi}(\tilde{y}',(\tilde{x}'-\tilde{x}''))d\tilde{x}'d\tilde{y}'}\delta_*(\tilde{u}-\tilde{x}'')d\tilde{x}'\{00\}_s.
$$

The operator in the big parentheses is known as the pseudo-differential operator of Weyl-type. The product formula is given by the Moyal product formula.

However a certain care about the convergence as in the case of Hörmander type is requested. In precise, we have to use the oscillatory integrals:

$$
(57)\ \int_0\left(\int h\left(\frac{1}{2}(\tilde{x}'+\tilde{x}''),\tilde{y}\right)f(\tilde{x}')e^{\frac{i}{\pi}(\tilde{y}',(\tilde{x}'-\tilde{x}''))d\tilde{x}'d\tilde{y}'}\delta_*(\tilde{u}-\tilde{x}'')d\tilde{x}'\{00\}_s.
$$

### 5.1 Fourier integral operators as vacuum representations

For the vacuum representation of a $*$-exponential function $e^{\frac{1}{i\pi}(a,u)}$, we first apply the decomposition [12] or Proposition 2.1 and note that $e^{\frac{1}{i\pi}(\tilde{b},\tilde{x})}\ast\{\tilde{\phi}(\tilde{z})\tilde{x}\}_s=e^{\frac{1}{i\pi}(\tilde{b},\tilde{x})}\{\tilde{\phi}(\tilde{z})\tilde{x}\}_s$. We have then

$$
e^{\frac{1}{i\pi}(\tilde{b},\tilde{v})}\ast f(\tilde{u})\ast\{\tilde{\phi}(\tilde{x})\tilde{x}\}_s=e^{\frac{1}{i\pi}(\tilde{b},\tilde{v})}\ast f(\tilde{u})\ast e^{\frac{1}{i\pi}(\tilde{b},\tilde{v})}\ast\{\tilde{\phi}(\tilde{x})\tilde{x}\}_s=f(\tilde{u}+\tilde{b})\ast e^{\frac{1}{i\pi}(\tilde{b},\tilde{x})}\ast\{\tilde{\phi}(\tilde{z})\tilde{x}\}_s.
$$

We can apply this formula to obtain a vacuum representation of certain diffeomorphisms of the configuration space. Let $\tilde{x} \rightarrow \tilde{\psi}(\tilde{x})$ be a diffeomorphism on $\mathbb{R}^m$ which is the identity except on a compact subset. Then, we see the following vacuum representation is the desired one:

$$
e^{\frac{1}{i\pi}(\tilde{\psi}(\tilde{x}),\tilde{v})}\ast f(\tilde{u})\ast\{\tilde{\phi}(\tilde{x})\tilde{x}\}_s=f(\tilde{u}+\tilde{\psi}(\tilde{x}))\ast e^{\frac{1}{i\pi}(\tilde{\psi}(\tilde{x}),\tilde{b})}\ast\{\tilde{\phi}(\tilde{z})\tilde{x}\}_s.
$$
One may extend this to make the vacuum representations of symplectic transformations of positively homogeneous of degree 1 which is near the identity. This is called a Fourier integral operator. Here we only mention what is symplectic transformations of positively homogeneous of degree 1. Regard $\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$ as the cotangent bundle $T^* \mathbb{R}^m \setminus \{0\}$ with removed 0-section. A symplectic diffeomorphism $\varphi=(\varphi_1, \varphi_2) : T^* \mathbb{R}^m \setminus \{0\} \rightarrow T^* \mathbb{R}^m \setminus \{0\}$ is of positively homogeneous of degree 1, if it satisfies

$$\varphi_1(\tilde{x}, r \tilde{y})=\varphi_1(\tilde{x}, \tilde{y}), \quad \varphi_2(\tilde{x}, r \tilde{y})=r \varphi_2(\tilde{x}, \tilde{y}), \quad r > 0.$$ 

Therefore, this method cannot be applied to the vacuum representations of *-exponential functions of quadratic forms. As it will be seen in the next section, such an equation appears in Schrödinger equation of harmonic oscillators.

However, as it is wellknown, Schrödinger equations are not invariant under the Lorentz group. To obtain Lorentz invariance, we have to use the “square root” of the Hamiltonian by loosing the locality of the operator, but such a non-local operator of degree 1 can be treated by Fourier integral operators.

### 6 Vacuum representations of *-exponential functions of quadratic forms

The vacuum representations of *-exponential functions of quadratic forms are little strange. This is because the vacuum representations give a kind of double cover of the adjoint representations.

Setting $e^{itH} \ast f(\tilde{u}) \ast \overline{\omega}(L)=f_i(\tilde{u}) \ast \overline{\omega}(L)$, we have to solve the initial value problem of

$$\frac{d}{dt} f_i(\tilde{u}) \ast \overline{\omega}(L)=iH_\ast f_i(\tilde{u}) \ast \overline{\omega}(L), \quad f_i(\tilde{u}) \ast \overline{\omega}(L)|_{t=0} = f(\tilde{u}) \ast \overline{\omega}(L).$$

Since this is a very difficult problem in general, we restrict our attention to the case $H_\ast$ is a quadratic form $\frac{1}{\sqrt{2}}(\tilde{u}g, \tilde{u}g)$, $g \in Sp(m, \mathbb{C})$ so that $e^{itH_\ast}$ has a periodical property. In this case, we consider the eigenvalue problem first

$$iH_\ast f_n(\tilde{u}) \ast \overline{\omega}(L)=\lambda_n f_n(\tilde{u}) \ast \overline{\omega}(L).$$

If $H_\ast$ is fixed, then this belongs to the ordinary representation theory of a compact group $S^1$, the eigenvectors $\{f_n(\tilde{u}), n=0, 1, 2, \cdots\}$ form an orthonormal system to make a Hilbert space. The initial value problem is solved by making the initial function by a linear combination of eigenvectors $\{f_n(\tilde{u})\}$.

Here we have to care about the two possible cases where $e^{itH_\ast}$ is alternating $\pi$-periodic and $\pi$-periodic.

Set $e^{itH_\ast} \ast \overline{\omega}(L)=\phi_i(\tilde{u}) \ast \overline{\omega}(L)$, and according to the periodicity, we define elements

$$\Phi_i(\tilde{u}) \ast \overline{\omega}(L)=\frac{1}{2\pi} \int_0^{2\pi} e^{it} \phi_i(\tilde{u}) \ast \overline{\omega}(L)dt, \quad \Phi_0(\tilde{u}) \ast \overline{\omega}(L)=\frac{1}{2\pi} \int_0^{2\pi} \phi_i(\tilde{u}) \ast \overline{\omega}(L)dt.$$ 

We easily see that

$$H_\ast \Phi_i(\tilde{u}) \ast \overline{\omega}(L)=i\Phi_i(\tilde{u}) \ast \overline{\omega}(L), \quad H_\ast \Phi_0(\tilde{u}) \ast \overline{\omega}(L)=\Phi_0(\tilde{u}) \ast \overline{\omega}(L).$$

For the initial value problem we set the initial function as $f(\tilde{u})=\phi(\tilde{u}) \ast \Phi_i(\tilde{u})$ or $\phi(\tilde{u}) \ast \Phi_0(\tilde{u})$. Then

$$e^{itH_\ast} \ast \phi(\tilde{u}) \ast \Phi_i(\tilde{u}) \ast \overline{\omega}(L)=(\text{Ad}(e^{itH_\ast}) \phi(\tilde{u})) \ast e^{it} \Phi_i(\tilde{u}) \ast \overline{\omega}(L),$$

38
Hence, $\Phi_+(\hat{u})$ is not a strict state vector of Heisenberg, but the combination with the ground state vibration, which makes the periodicity change from the periodicity of adjoint representation.

**Note** If $\operatorname{Ad}(e^{iHt}_t)\phi(\hat{u})$ is $\pi$-periodic (resp. alternating $\pi$-periodic), then $e^{it}\operatorname{Ad}(e^{iHt}_t)\phi(\hat{u})$ is alternating $\pi$-periodic (resp. $\pi$-periodic).

Now, we want to see the above observation more concretely for the case $m=1$. Setting for each $k$

$$e^{it}\frac{1}{a}(a\hat{u}^2_k + b\hat{\nu}_k^2 +2c\hat{u}_k\hat{\nu}_k) + f(\hat{u})*\hat{\omega}(L) = f_t(\hat{u})*\hat{\omega}(L),$$

$f_t(\hat{u})$ must satisfy the equation involving Schrödinger equation in a special case

$$i\frac{d}{dt}f_t(\hat{u}_k)*\hat{\omega}(L) = \frac{1}{\hbar} \left( a\hat{u}^2_k + b\hat{\nu}_k^2 +2c\hat{u}_k\hat{\nu}_k \right)*f_t(\hat{u}_k)*\hat{\omega}(L)$$

$$= \left( \frac{1}{\hbar} a\hat{u}^2_k*f_t(\hat{u}_k) - \hbar b\partial\hat{\nu}_k\partial\hat{u}_k f_t(\hat{u}_k) +2ci\hat{\nu}_k\partial\hat{u}_k + \frac{1}{2} f_t(\hat{u}_k) \right)*\hat{\omega}(L).$$

As we are interested in the periodical solutions, we assume $ab-c^2=1$ in what follows. The feature of this equation is that the term $2ci\frac{1}{a}f_t(\hat{u}_k)$ of (trivial) central extension term (Schwinger term) appears. If $c=0$, a big difference appears between the cases $a, b$ are reals and pure imaginaries. (See the last part of this section.)

If $b=0$, then setting $c = \pm i$, the initial value problem is

$$i\left( \partial_t + 2i\hat{\nu}_k\partial u_k \right)f_t = \left( \frac{1}{\hbar} a\hat{u}^2_k + 1 \right)f_t, \quad f_0 = 1.$$  

Changing variables $(t, \hat{u}_k)=(t', e^\mp 2it'x)$ gives

$$i\partial_{t'}f_{t'}(e^\mp 2it'x) = \left( \frac{1}{\hbar} a_e^\mp 4it'x^2 + 1 \right)*f_{t'}(e^\mp 2it'x), \quad f_{0}(x) = 1.$$  

Hence, $f_{t'}(e^\mp 2it'x) = C(x)e^{\mp it}e^{\frac{a}{\hbar}e^\mp 4it'x^2}$ and adjusting the initial condition gives

$$e^{it}e^{\mp \frac{a}{\hbar}(a\hat{u}^2_k + 2i\hat{u}_k\hat{\nu}_k)}*\hat{\omega}(L) = f_t(\hat{u}_k)*\hat{\omega}(L) = e^{\mp it}e^{\frac{a}{\hbar}(1-e^{4it})\hat{u}^2_k}*\hat{\omega}(L).$$

This is alternating $\pi$-periodic and

$$\epsilon_{00}(k)*\hat{\omega}(L) = e^{\frac{a}{\hbar}(a\hat{u}^2_k + 2i\hat{u}_k\hat{\nu}_k)}*\hat{\omega}(L) = \mp i\hat{\omega}(L), \quad \forall a.$$  

On the other hand, the Fourier expansion solves the eigenvalue problem:

$$e^{-it}e^{\mp \frac{a}{\hbar}(1-e^{-4it})\hat{u}^2_k} = e^{-it}\sum_{n=-\infty}^{\infty} a_n(\hat{u}_k)e^{int}$$

However, the eigenvalue problem is solved by itself. $(\frac{1}{\hbar}(a\hat{u}^2_k + 2i\hat{u}_k\hat{\nu}_k + 1)f_{\lambda}(\hat{u}_k) = i\lambda f_{\lambda}(\hat{u}_k))$. Setting $f_{\lambda}(\hat{u}_k) = g_e^{\pm \frac{a}{\hbar}\hat{u}^2_k}$. We easily see that $f_{\lambda}(\hat{u}_k) = \hat{u}_k^{\frac{a}{\hbar}+\frac{1}{2}}e^{\mp \frac{a}{\hbar}\hat{u}^2_k}$. This is smooth only if $i\lambda = \mp (2n+1)$. It follows

$$e^{it\frac{1}{\hbar}(a\hat{u}^2_k + 2i\hat{u}_k\hat{\nu}_k)}*\hat{\omega}(L) = e^{it}\sum_{n=-\infty}^{\infty} a_n(\hat{u}_k)e^{int}\hat{\omega}(L).$$
for every integer $n$ and these are alternating $\pi$-periodic and

$$
\varepsilon_{00}(k)*u_k^n*e^{\pm \hbar \tilde{u}_k^2*v}(L)=e^{i(2n+1)*u_k^n*e^{\pm \hbar \tilde{u}_k^2*v}(L)}=i(-u_k^n)*e^{\pm \hbar \tilde{u}_k^2*v}(L)
$$

for every $a$. In particular $\varepsilon_{00}(k)*u_k^n*e^{\pm \hbar \tilde{u}_k^2*v}(L)=-u_k^n*e^{\pm \hbar \tilde{u}_k^2*v}(L)$ for every $a$ and $n$.

To make a Hilbert space by using $\{e^{\pm \hbar \tilde{u}_k^2}, \tilde{u}_k e^{\pm \hbar \tilde{u}_k^2}, \tilde{u}_k^2 e^{\pm \hbar \tilde{u}_k^2}, \cdots \}$, we have to restrict the variable $\tilde{u}_k$ so that $\text{Re}(\frac{\tilde{u}_k^2}{\hbar})<0$.

**Note** If there is no constant term, then the eigenvalues are $2n$, $n=1, 2, 3, \cdots$. Hence the solutions of the initial value problems are $\pi$-periodic.

Suppose next that $b\neq 0$. We set first $f_\ell(\tilde{u}_k)=h(t, \tilde{u}_k)e_{\alpha_\ell}^{\tilde{u}_k^2}$, $h(0, \tilde{u}_k)=e^{-\alpha_\ell}^{\tilde{u}_k^2}$ and

$$
i c f_\ell(\tilde{u}_k)=ic h(t, \tilde{u}_k)e_{\alpha_\ell}^{\tilde{u}_k^2},
\frac{1}{\hbar} a \tilde{u}_k^2 f_\ell(\tilde{u}_k)=\frac{1}{\hbar} a \tilde{u}_k^2 h(t, \tilde{u}_k)e_{\alpha_\ell}^{\tilde{u}_k^2},
2ic \tilde{u}_k \partial_{\tilde{u}_k} f_\ell(\tilde{u}_k)=2ic \tilde{u}_k \partial_{\tilde{u}_k} h(t, \tilde{u}_k)e_{\alpha_\ell}^{\tilde{u}_k^2}+4i\alpha e_{\alpha_\ell}^{\tilde{u}_k^2} h(t, \tilde{u}_k)e_{\alpha_\ell}^{\tilde{u}_k^2},
-\hbar b \partial_{\tilde{u}_k}^2 f_\ell(\tilde{u}_k)=-\hbar b \left( \partial_{\tilde{u}_k}^2 h(t, \tilde{u}_k)+4\alpha \tilde{u}_k \partial_{\tilde{u}_k} h(t, \tilde{u}_k)+2\alpha h(t, \tilde{u}_k)+4\alpha_\ell \tilde{u}_k h(t, \tilde{u}_k) \right)e_{\alpha_\ell}^{\tilde{u}_k^2}.
$$

Plugging these into (58) we have

$$i\partial_t h(t, \tilde{u}_k)=-\hbar b \partial_{\tilde{u}_k}^2 h(t, \tilde{u}_k)+2(ic-2\hbar \alpha)\tilde{u}_k \partial_{\tilde{u}_k} h(t, \tilde{u}_k)+\frac{1}{\hbar}(4\alpha \tilde{u}_k \partial_{\tilde{u}_k} h(t, \tilde{u}_k)+4\alpha_\ell \tilde{u}_k h(t, \tilde{u}_k))e_{\alpha_\ell}^{\tilde{u}_k^2}.
$$

(60)

Setting $\alpha=\frac{ic}{2\hbar}$ to eliminate the second and third terms and recalling $ab-c^2=1$, we have

$$i\partial_t h(t, \tilde{u}_k)=-\hbar b \partial_{\tilde{u}_k}^2 h(t, \tilde{u}_k)+\frac{1}{\hbar b} \tilde{u}_k^2 h(t, \tilde{u}_k).
$$

Putting $\tilde{u}_k=\sqrt{\hbar b} x$ changes the above equation to

$$
 x^2 h_t(x)-\partial_x^2 h_t(x)=i\partial_t h_t(x), \quad h_0(x)=e^{-\frac{\hbar x^2}{2}}=e^{-\frac{\hbar |u|^2}{2}}.
$$

This is the Schrödinger equation of standard harmonic oscillator. The eigenvalue problem is well-known. The eigenvectors are given by using Hermite polynomials $H_n(x)$

$$(x^2-\partial_x^2)H_n(x)e^{-\frac{\hbar x^2}{2}}e^{int}=(2n+1)H_n(x)e^{-\frac{\hbar x^2}{2}}e^{int}, \quad n=1, 2, \cdots.
$$

By the original variables these are

$$H_n(x)e^{-\frac{\hbar x^2}{2}}e^{-\frac{\hbar |u|^2}{2}}=H_n(\sqrt{\hbar b} \tilde{u}_k) e^{-\frac{\hbar|\tilde{u}_k|^2}{2}}=e^{i(2n+1)\tilde{u}_k^2}H_n(\sqrt{\hbar b} \tilde{u}_k) e^{-\frac{\hbar|\tilde{u}_k|^2}{2}}\tilde{u}_k^2*\tilde{u}(L).
$$

It follows

$$e^{i\frac{1}{\hbar b}((\alpha\tilde{u}_k^2+6\hbar c\tilde{u}_k+4\hbar b\tilde{u}_k^2)u_k)}*H_n(\sqrt{\hbar b} \tilde{u}_k) e^{i\frac{1}{\hbar b}((\alpha\tilde{u}_k^2+6\hbar c\tilde{u}_k+4\hbar b\tilde{u}_k^2)u_k)}u_k*\tilde{u}(L)=e^{i(2n+1)\tilde{u}_k^2}H_n(\sqrt{\hbar b} \tilde{u}_k) e^{i\frac{1}{\hbar b}((\alpha\tilde{u}_k^2+6\hbar c\tilde{u}_k+4\hbar b\tilde{u}_k^2)u_k)}.$$
and
\[
\varepsilon_{00}(k) * H_n(\sqrt{\hbar} \tilde{u}_k) e^{-\left(\frac{1}{2\hbar^2} + \frac{i}{2\hbar} \right) \tilde{u}_k^2 + \frac{i}{2} \tilde{u}_k} (L) \\
= e^{i \frac{1}{2\hbar} \left( a\tilde{u}_k^2 + \tilde{u}_k + 2\tilde{u}_k \tilde{v}_k \right)} * H_n(\sqrt{\hbar} \tilde{u}_k) e^{-\left(\frac{1}{2\hbar^2} + \frac{i}{2\hbar} \right) \tilde{u}_k^2 + \frac{i}{2} \tilde{u}_k} (L) \\
= i^{2n+1} H_n(\sqrt{\hbar} \tilde{u}_k) e^{-\left(\frac{1}{2\hbar^2} + \frac{i}{2\hbar} \right) \tilde{u}_k^2 + \frac{i}{2} \tilde{u}_k} (L).
\]

To make a Hilbert space we have to restrict the variable to an \( \mathbb{R} \) linear subspace so that
\[
\Re \frac{1+ci}{2\hbar^2} \tilde{u}_k^2 < 0.
\]
But here the \( \sqrt{\hbar} \) causes a delicate sign change, as we have to use \( \sqrt{-\hbar} \) in the opposite quadratic form.

For simplicity, we investigate the case \( ab=1 \), but \( a, b \) are reals and the case \( a, b \) are pure imaginary. If \( a, b \) are reals, then (60) is nothing but the equation of standard harmonic oscillators, but if \( a, b \) are pure imaginary, then by setting \( \alpha = \pm i \frac{1}{2\hbar} \) the equation (60) turns out
\[
\partial_t h_{it}(\tilde{u}_k) = \left( ih\partial_{\tilde{u}_k}^2 \pm \tilde{u}_k \partial_{\tilde{u}_k} \pm 1 \right) h_{it}(\tilde{u}_k), \quad h_{0}(\tilde{u}_k) = e^{i \frac{1}{2\hbar} \tilde{u}_k^2}.
\]
Hence setting \( h_{it}(\tilde{u}_k) = e^{it} g_{it}(\tilde{u}_k) \), we have to solve
\[
\partial_t g_{it}(\tilde{u}_k) = \left( ih\partial_{\tilde{u}_k}^2 \pm \tilde{u}_k \partial_{\tilde{u}_k} \right) g_{it}(\tilde{u}_k), \quad g_{0}(\tilde{u}_k) = e^{i \frac{1}{2\hbar} \tilde{u}_k^2}.
\]
These looks similar to the equation (57), but the point here is that we have to use the Fourier transform. This procedure makes the periodicity change.

By setting \( g_{t}(\tilde{u}_k) = \int \hat{g}_t(\tilde{u}_k) e^{i \pi \xi \tilde{u}_k} d\xi \) we have \( \hat{g}_t(\xi) = e^{i \pi \xi^2} \hat{g}_0(\xi^2) \), but adjusting the initial condition gives
\[
\hat{g}_{it}(\xi) = \int \frac{e^{i \pi \xi^2(e^{2it}-1)}}{2\hbar^2} e^{i \frac{1}{2\hbar} \xi^2 u^2} e^{i \pi \xi^2 u} d\xi.
\]
Hence change variables \( u \) to \( e^{it} u \), we see that \( \hat{g}_{it}(\xi) \) is alternating \( \pi \)-periodic. As the result, \( h_{it}(\tilde{u}_k) \) is \( \pi \)-periodic.

**Note**
As it was seen, solutions of Schrödinger equations of harmonic oscillators have no singular point on the real line, as we choose a generic ordered expressions where the \( * \)-exponential functions has no singular point on the real line. However in computations after (58) are done without specifying the expression parameter. As these are computations only for \( \tilde{u}_k \) variable, these are in fact same to the computations under the normal ordered expressions.

On the other hand, there is another set up of Schrödinger equations of harmonic oscillators from classical harmonic oscillators. This is set up so that the Weyl ordered expression coincide to the classical one. Those two approaches to quantum harmonic oscillators give slightly different results. The later contains singular points on the real line. The reason is that the Weyl ordered expression
\[
: e^{i \frac{1}{\hbar} \left( (a+ib)^2 + 2cuv \right) } :_0, \quad ab-c^2=1,
\]
has singular points on real line corresponding to polar elements (cf. (58)). Here, Maslov’s theory can be applied to control the discontinuous jump. However, we need not to use Maslov’s theory
by selecting a suitable expression parameters. Thus, we are thinking that Maslov’s theory may be applied to understand what does the next Theorem 6.1 implies.

Recall the argument in §3.1.1, (38) and take the opposite quadratic form. The next Theorem shows that it is difficult to make a group of Fourier integral operators on $\mathbb{R}^4$ which extends the group $SU(2)$.

**Theorem 6.1** In any fixed generic ordered expression, there is no common Hilbert space on which

$$e^{t\frac{1}{i\hbar}(a\hat{u}_k^2+b\hat{v}_k^2+2c\hat{u}_k\cdot\hat{v}_k)}, \quad ab-c^2=1,$$

is regularly represented w.r.t. the vacuum $\hat{\omega}(L)$.

**Proof** Suppose there is a Hilbert space on which $e^{t\frac{1}{i\hbar}(a\hat{u}_k^2+b\hat{v}_k^2+2c\hat{u}_k\cdot\hat{v}_k)}$ is represented for all $(a, b, c)$ such that $ab-c^2=1$. Then, the argument above shows that $\epsilon_{00}(k)^2$ is represented by $i^\ell$ by some integer $\ell$. As the space $ab-c^2=1$ is connected this must be fixed throughout the space $ab-c^2=1$. Since there is a case that $\epsilon_{00}(k)^2=-1$, $i^\ell$ must be $-1$. However, the existence of opposite quadratic forms gives a contradiction by the same argument as in §3.1.1.

**Note** As it is mentioned in [12], $e^{t\frac{1}{i\hbar}(a\hat{u}_k^2+b\hat{v}_k^2+2c\hat{u}_k\cdot\hat{v}_k)}$, $ab-c^2=1$, generate a group-like object which looks a “double cover” of $SL(2, \mathbb{C})$ and this contains the “double cover” of $SU(2)$. Therefore, if an expression parameter $K$ is fixed, these objects cannot form genuine groups and they must contain some singular points. Theorem 6.1 shows such singular points are not removable by the vacuum representation.

To overcome the difficulty, we have to use the $SU(2)$-vacuum, but we have to restrict the expression parameters.

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