Harmonic and Trace Inequalities in Lipschitz Domains

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Abstract We prove boundary inequalities in arbitrary bounded Lipschitz domains on the trace space of Sobolev spaces. For that, we make use of the trace operator, its Moore–Penrose inverse, and of a special inner product. We show that our trace inequalities are particularly useful to prove harmonic inequalities, which serve as powerful tools to characterize the harmonic functions on Sobolev spaces of non-integer order.

Keywords: Moore–Penrose equality, trace inequalities, harmonic inequalities, Lipschitz domains, trace spaces, harmonic functions, Hilbert spaces.

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1 Introduction

In this article we establish some new and important operator inequalities connected with traces on Hilbert spaces. Trace inequalities find several interesting applications, e.g., to problems from quantum statistical mechanics and information theory [2, 5, 11]. Here we establish new trace inequalities in Lipschitz domains, that is, in a domain of the Euclidean space whose boundary is “sufficiently regular”, in the sense that it can be thought of as, locally, being the graph of a Lipschitz continuous function [14]. The study of Lipschitz domains is an important research area per se, since many of the Sobolev embedding theorems require them as the natural domain of study [19]. Consequently, many partial differential equations found in applications and variational problems are defined on Lipschitz domains [3, 6, 12]. In our case, we investigate the application of the obtained trace inequalities in Lipschitz domains to harmonic functions [8], which is a subject of strong current research [10, 15, 18, 22].

The paper is organized as follows. In Section 2, we fix notations and recall necessary definitions and results, needed in the sequel. Our contribution is then given in Section 3: we prove a Moore–Penrose inverse equality (Theorem 1), trace inequalities (Theorem 2), and harmonic inequalities (Theorems 3 and 4). As an application of Theorem 4, we obtain a functional characterization of the harmonic Hilbert spaces for the range of values $0 \leq s \leq 1$ (Corollary 1).

2 Preliminaries

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces with inner products $(\cdot, \cdot)_{\mathcal{H}_1}$ and $(\cdot, \cdot)_{\mathcal{H}_2}$ and associated norms $\| \cdot \|_{\mathcal{H}_1}$ and $\| \cdot \|_{\mathcal{H}_2}$, respectively. We denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ and $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_1)$ is briefly denoted by $\mathcal{L}(\mathcal{H}_1)$. For an operator $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, its domain, range, and null space, respectively. The set of all bounded operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ is denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, while $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$ is briefly denoted by $\mathcal{B}(\mathcal{H}_1)$. The set of all closed densely defined operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ is denoted by $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ and, analogously as before, $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_1)$ is denoted by $\mathcal{C}(\mathcal{H}_1)$. For $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$, its adjoint operator is denoted by $A^* \in \mathcal{C}(\mathcal{H}_2, \mathcal{H}_1)$.

The Moore–Penrose inverse of a closed densely defined operator $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$, denoted by $A^\dagger$, is defined as the unique linear operator in $\mathcal{C}(\mathcal{H}_2, \mathcal{H}_1)$ such that

\[
\mathcal{D}(A^\dagger) = \mathcal{R}(A) \oplus \mathcal{N}(A^*), \quad \mathcal{N}(A^\dagger) = \mathcal{N}(A^*),
\]

and

\[
\begin{align*}
\begin{cases}
AA^\dagger A = A, \\
A^\dagger AA^\dagger = A^\dagger,
\end{cases}
\quad \begin{cases}
AA^\dagger \subset \mathcal{P}_{\mathcal{R}(A)^\perp}, \\
A^\dagger A \subset \mathcal{P}_{\mathcal{R}(A)^\perp},
\end{cases}
\end{align*}
\]
where \( \overline{E} \) denotes the closure of \( E \), \( E \in \{ R(A), R(A^\dagger) \} \), and \( P_E \) the orthogonal projection on the closed subspace \( \overline{E} \). The following lemma is used in the proof of our Moore–Penrose inverse equality (Theorem 1).

**Lemma 1** (See Lemma 2.5 and Corollary 2.6 of [13]) Let \( A \in C(H_1, H_2) \) and \( B \in C(H_2, H_1) \) be such that \( B = A^\dagger \). Then,

1. \( A(I + A^\dagger A)^{-1} = B^\dagger(I + BB^\dagger)^{-1} \);
2. \( (I + A^\dagger A)^{-1} + (I + BB^\dagger)^{-1} = I + P_{N(B^\dagger)} \);
3. \( A^\dagger(I + AA^\dagger)^{-1} = B(I + B^\dagger B)^{-1} \);
4. \( (I + AA^\dagger)^{-1} + (I + B^\dagger B)^{-1} = I + P_{N(A^\dagger)} \);
5. \( (I + AA^\dagger)^{-1} + (I + B^\dagger B)^{-1} = I \) (if \( A^\dagger \) is injective);
6. \( N(A^\dagger(I + AA^\dagger)^{-1/2}) = N(A^\dagger) = N(B) \).

**Lemma 2** (See Theorem 3.5 of [21]) Let \( H_1 \) and \( H_2 \) be two Hilbert spaces, \( A \in B(H_1, H_2) \), and \( B \) be its Moore–Penrose inverse. Then, the operator \( B^\dagger(I + BB^\dagger)^{-1/2} \) is bounded with closed range and has a bounded Moore–Penrose inverse given by

\[
T_B = B(I + B^\dagger B)^{-1/2} + A^\dagger(I + B^\dagger B)^{-1/2}.
\]

Moreover, the adjoint operator of \( T_B \) is \( T_B^\dagger \), where

\[
T_B^\dagger = B^\dagger(I + BB^\dagger)^{-1/2} + A(I + BB^\dagger)^{-1/2}.
\]

**Lemma 3** (See Theorem 3.8 of [21]) Let \( H_1 \) and \( H_2 \) be two Hilbert spaces, \( A \in B(H_1, H_2) \), and \( B \) be its Moore–Penrose inverse. Then, the decomposition

\[
A = (I + B^\dagger B)^{-1/2}T_B^\dagger
\]

holds, where \( T_B^\dagger = B^\dagger(I + BB^\dagger)^{-1/2} + A(I + BB^\dagger)^{-1/2} \).

**Lemma 4** (See Corollary 3.7 of [21]) Let \( A \in B(H_1, H_2) \) and \( B \) be its Moore–Penrose inverse. Then, \( T_B \) is an isomorphism from \( R(B^\dagger) \) to \( N(B^\dagger)^\perp \), where \( N(B^\dagger)^\perp \) denotes the orthogonal complement of \( N(B^\dagger) \).

Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) with boundary \( \partial \Omega \) and closure \( \overline{\Omega} \). We say that \( \partial \Omega \) is Lipschitz continuous if for every \( x \in \partial \Omega \) there exists a coordinate system \( (\hat{y}, \gamma_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \), a neighborhood \( Q_{\delta, \delta'}(x) \) of \( x \), and a Lipschitz function \( \gamma_x : \hat{Q}_{\delta} \rightarrow \mathbb{R} \), with the following properties:

1. \( \Omega \cap Q_{\delta, \delta'}(x) = \{(\hat{y}, \gamma_d) \in \hat{Q}_{\delta, \delta'}(x) / \gamma_x(\hat{x}) < \gamma_d\} \);\n2. \( \partial \Omega \cap \hat{Q}_{\delta, \delta'}(x) = \{(\hat{y}, \gamma_d) \in \hat{Q}_{\delta, \delta'}(x) / \gamma_x(\hat{x}) = \gamma_d\} \);

where \( \hat{Q}_{\delta, \delta'}(x) = \{(\hat{y}, \gamma_d) \in \mathbb{R}^d / ||\hat{y} - \hat{x}||_{\mathbb{R}^{d-1}} < \delta \text{ and } |\gamma_d - \gamma_x| < \delta'\} \) and \( \hat{Q}_{\delta}(x) = \{\hat{y} \in \mathbb{R}^{d-1} / ||\hat{y} - \hat{x}||_{\mathbb{R}^{d-1}} < \delta\} \) for \( \delta, \delta' > 0 \). An open connected subset \( \Omega \subset \mathbb{R}^d \), whose boundary is Lipschitz continuous, is called a Lipschitz domain. In the rest of this paper, \( \Omega \) denotes a
bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. We denote by $C^k(\Omega)$, $k \in \mathbb{N}$ or $k = \infty$, the space of real $k$ times continuously differentiable functions on $\Omega$. The space $C^\infty$ of all real functions on $\Omega$ with a compact support in $\Omega$ is denoted by $C_c^\infty(\Omega)$. We say that a sequence $(\varphi_n)_{n \geq 1} \in C_c^\infty(\Omega)$ converges to $\varphi \in C_c^\infty(\Omega)$, if there exists a compact $Q \subset \Omega$ such that $\text{supp}(\varphi_n) \subset Q$ for all $n \geq 1$ and, for all multi-index $\alpha \in \mathbb{N}^d$, the sequence $(\partial^\alpha \varphi_n)_{n \geq 1}$ converges uniformly to $\partial^\alpha \varphi$, where $\partial^\alpha$ denotes the partial derivative of order $\alpha$. The space $C_c^\infty(\Omega)$, induced by this convergence, is denoted by $\mathcal{D}(\Omega)$, while $\mathcal{D}'(\Omega)$ is the space of distributions on $\Omega$. For $k \in \mathbb{N}$, $H^k(\Omega)$ is the space of all distributions $u$ defined on $\Omega$ such that all partial derivatives of order at most $k$ lie in $L^2(\Omega)$, i.e., $\partial^\alpha u \in L^2(\Omega)$ $\forall |\alpha| \leq k$. This is a Hilbert space with the scalar product

$$(u, v)_{k, \Omega} = \sum_{|\alpha| \leq k} \int_\Omega \partial^\alpha u \partial^\alpha v \, dx,$$

where $dx$ is the Lebesgue measure and $u, v \in H^k(\Omega)$. The corresponding norm, denoted by $\| \cdot \|_{k, \Omega}$, is given by

$$\|u\|_{k, \Omega} = \left( \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u|^2 \, dx \right)^{1/2}.$$

Sobolev spaces $H^s(\Omega)$, for non-integers $s$, are defined by the real interpolation method \[16\] [20]. The trace spaces $H^s(\partial \Omega)$ can be defined by using charts on $\partial \Omega$ and partitions of unity subordinated to the covering of $\partial \Omega$. If $\Omega$ is a Lipschitz hypograph, then there exists a Lipschitz function $\gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ such that $\Omega = \{ x \in \mathbb{R}^{d-1} / x_d < \gamma(\tilde{x}) \text{ for all } \tilde{x} \in \mathbb{R}^{d-1} \}$. This allows to construct Sobolev spaces on the boundary $\partial \Omega$, in terms of Sobolev spaces on $\mathbb{R}^{d-1}$ \[16\]. This is done as follows. For $g \in L^2(\partial \Omega)$, we define $g_\gamma(\tilde{x}) = g(\tilde{x}, \gamma(\tilde{x}))$ for $\tilde{x} \in \mathbb{R}^{d-1}$, we let

$$H^s(\partial \Omega) = \left\{ g \in L^2(\partial \Omega) \mid g_\gamma \in H^s(\mathbb{R}^{d-1}) \text{ for } 0 \leq s \leq 1 \right\},$$

and equip this space with the inner product $(g, h)_{H^s(\partial \Omega)} = (g_\gamma, h_\gamma)_{\mathbb{R}^{d-1}}$, where

$$(u, v)_{s, \mathbb{R}^{d-1}} = \int_{\mathbb{R}^{d-1}} (1 + |\xi|^2)^s \hat{u}(\xi) \hat{v}(\xi) \, d\xi,$$

and $\hat{u}$ denotes the Fourier transform of $u$. Recalling that any Lipschitz function is almost everywhere differentiable, we know that any Lipschitz hypograph $\Omega$ has a surface measure $\sigma$ and an outward unit normal $\nu$ that exists $\sigma$-almost everywhere on $\partial \Omega$. If $\Omega$ is a Lipschitz hypograph, then $d\sigma(x) = \sqrt{1 + (\nabla \gamma(\tilde{x}))^2} \, d\tilde{x}$ and

$$\nu(x) = \frac{(\nabla \gamma(\tilde{x}), 1)}{\sqrt{1 + (\nabla \gamma(\tilde{x}))^2}}$$

for $\tilde{x} \in \mathbb{R}^{d-1}$. This allows to construct Sobolev spaces on $\mathbb{R}^{d-1}$ via interpolation with $L^2(\partial \Omega)$. The resulting spaces are denoted by $H^s(\mathbb{R}^{d-1})$.

\[1\] [16] [20]
for almost every $x \in \partial \Omega$. Suppose now that $\Omega$ is a Lipschitz domain. Because $\partial \Omega \subset \bigcup_{x \in \partial \Omega} Q_{\delta, \delta'}(x)$ and $\partial \Omega$ is compact, there exist $x^1, x^2, \ldots, x^n \in \partial \Omega$ such that

$$\partial \Omega \subset \bigcup_{j=1}^n Q_{\delta, \delta'}(x^j).$$

It follows that the family $(W_j) = (Q_{\delta, \delta'}(x^j))$ is a finite open cover of $\partial \Omega$, i.e., each $W_j$ is an open subset of $\mathbb{R}^d$ and $\partial \Omega \subseteq \bigcup_j W_j$. Let $(\varphi_j)$ be a partition of unity subordinate to the open cover $(W_j)$ of $\partial \Omega$, i.e., $\varphi_j \in C(\overline{W_j})$ and $\sum_j \varphi_j(x) = 1$ for all $x \in \partial \Omega$.

The inner product in $H^s(\partial \Omega)$ is then defined by

$$(u, v)_{\partial \Omega} = \sum_j (\varphi_j u, \varphi_j v)_{H^s(\partial \Omega)},$$

where $\Omega_j$ can be transformed to a Lipschitz hypograph by a rigid motion, i.e., by a rotation plus a translation, and satisfies $W_j \cap \Omega = W_j \cap \Omega_j$ for each $j$. The associated norm will be denoted by $\| \cdot \|_{s, \partial \Omega}$. It is interesting to mention that a different choice of $(W_j), (\Omega_j)$ and $(\varphi_j)$ would yield the same space $H^s(\partial \Omega)$ with an equivalent norm, for $0 \leq s \leq 1$. For more on the subject we refer the interested reader to [1, 2, 16].

The trace operator maps each continuous function $u$ on $\Omega$ to its restriction onto $\partial \Omega$ and may be extended to be a bounded surjective operator, denoted by $\Gamma$, from $H^s(\Omega)$ to $H^{s-\frac{1}{2}}(\partial \Omega)$ for $1/2 < s < 3/2$ [4, 10]. The range and null space of $\Gamma$ are given by $\mathcal{R}(\Gamma) = H^{s-\frac{1}{2}}(\partial \Omega)$ and $\mathcal{N}(\Gamma) = H^s_0(\Omega)$, respectively, where $H^s_0(\Omega)$ is defined to be the closure in $H^s(\Omega)$ of infinitely differentiable functions compactly supported in $\Omega$. For $s = 3/2$, this is no longer valid. For $s > 3/2$, the trace operator from $H^s(\Omega)$ to $H^1(\partial \Omega)$ is bounded [4].

Let us set $\Gamma = T_1 \Gamma_1$, where $\Gamma_1$ is the trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial \Omega)$ and $T_1$ is the embedding operator from $H^{1/2}(\partial \Omega)$ into $L^2(\partial \Omega)$. According to a classical result of Gagliardo [9], we know that $\mathcal{R}(\Gamma) = H^{1/2}(\partial \Omega)$. Since $\Gamma_1$ is bounded and $T_1$ is compact [17], the trace operator $\Gamma$ from $H^1(\Omega)$ to $L^2(\partial \Omega)$ is also compact.

Now, let us induce $H^1(\Omega)$ with the following inner product:

$$(u, v)_{\partial \Omega} = \int_\Omega \nabla u \nabla v dx + \int_{\partial \Omega} \Gamma u \Gamma v d\sigma \quad \forall u, v \in H^1(\Omega).$$

The associated norm $\| \cdot \|_{\partial \Omega}$ is given by

$$\|u\|_{\partial \Omega} = \left( \|\nabla u\|_{\partial \Omega}^2 + \|\Gamma u\|_{\partial \Omega}^2 \right)^{1/2}$$

and $H^1(\Omega)$, induced with the inner product $(\cdot, \cdot)_{\partial \Omega}$, is denoted by $H^1_{\partial \Omega}(\Omega)$. A well-known result of Nečas [17], asserts that under the condition that $\Omega$ is a bounded Lipschitz domain, the norms $\| \cdot \|_{\partial \Omega}$ and $\| \cdot \|_1$ are equivalent.

The following characterization is useful to prove our trace inequalities in Section 3.

**Lemma 5 (See Corollary 6.9 of [21])** Let $\Gamma$ be the trace operator from $H^s_0(\Omega)$ to $L^2(\partial \Omega)$, $\Lambda$ its Moore–Penrose inverse, and $\Lambda^*$ be its adjoint operator. Then, for
0 \leq s \leq 1$, we have that $H^s(\partial \Omega) = \mathcal{H}^s(\partial \Omega)$ with equivalence of norms, where $\mathcal{H}^s(\partial \Omega) = \{(I + \Lambda^* \Lambda)^{-s}g \mid g \in L^2(\partial \Omega)\}.$

3 Main Results

We begin by proving an important equality that, together with the trace inequalities of Theorem 2, will be useful to prove our harmonic inequality of Theorem 3.

Theorem 1 (The Moore–Penrose inverse equality) Let $\Gamma \in \mathcal{B}(H^1_0(\Omega), L^2(\partial \Omega))$ be the trace operator and $\Lambda \in C(L^2(\partial \Omega), H^1_0(\Omega))$ its Moore–Penrose inverse. Then, for a real $s$, the following equality holds:

$$T_{\Lambda^*}(I + \Lambda \Lambda^*)^{-s} = (I + \Lambda^* \Lambda)^{-s}T_{\Lambda^*},$$

where $T_{\Lambda^*} = \Lambda^*(I + \Lambda \Lambda^*)^{-1/2} + \Gamma(I + \Lambda \Lambda^*)^{-1/2}$.

Proof From Lemma 1

$$N(T_{\Lambda^*}(I + \Lambda \Lambda^*)^{-s}) = N(T_{\Lambda^*})$$

$$= N((I + \Lambda^* \Lambda)^{-1/2})$$

$$= N(\Lambda^*)$$

$$= H^1_0(\Omega)$$

$$= N((I + \Lambda^* \Lambda)^{-s}T_{\Lambda^*}),$$

and we have

$$T_{\Lambda^*}(I + \Lambda \Lambda^*)^{-s}v = 0 = (I + \Lambda^* \Lambda)^{-s}T_{\Lambda^*}v$$

for all $v \in N(\Lambda^*)$. Now let us consider the operator $\Gamma^*\Gamma : H^1_0(\Omega) \longrightarrow H^1_0(\Omega)$, where $\Gamma^*$ is the adjoint of the trace operator $\Gamma$. Given the compactness of $\Gamma$, the operator $\Gamma^*\Gamma$ is compact and self-adjoint. Then there exists a sequence of pairs $(s_k, v_k)_{k \geq 1}$ associated to $\Gamma^*\Gamma$ such that

$$\Gamma^*\Gamma v_k = s_k^2 v_k.$$ 

To prove the equality on $N(\Lambda^*)^2$, we show that

$$T_{\Lambda^*}(I + \Lambda \Lambda^*)^{-s}v_k = (I + \Lambda^* \Lambda)^{-s}T_{\Lambda^*}v_k.$$ 

To this end, let $\Gamma v_k = s_k z_k$. It follows that $\Gamma^* z_k = s_k v_k$ and, from Lemma 2

$$\Gamma v_k = (I + \Lambda \Lambda^*)^{-1/2}T_{\Lambda^*}v_k.$$ 

On the other hand,

$$T_{\Lambda^*}v_k = \Lambda^*(I + \Lambda \Lambda^*)^{-1/2}v_k + \Gamma(I + \Lambda \Lambda^*)^{-1/2}v_k.$$ 

By putting $(I + \Lambda \Lambda^*)^{-1}v_k = w_k$, we have $v_k = w_k + \Lambda \Lambda^* w_k$ and

$$\Gamma^*\Gamma v_k = s_k^2 v_k = \Gamma^* v_k + w_k.$$ 

which implies that
\[
(I + \Gamma^* \Gamma)^{-1} \Gamma^* \Gamma v_k = s_k^2 (I + \Gamma^* \Gamma)^{-1} v_k \\
= w_k \\
= \Gamma^* \Gamma (I + \Gamma^* \Gamma)^{-1} v_k.
\] (1)

Using Lemma 1, it follows from (1) that
\[
(I + \Gamma^* \Gamma)^{-1} \Gamma^* \Gamma v_k = \Gamma^* \Lambda^* (I + \Lambda \Lambda^*)^{-1} v_k.
\]

This leads, again from Lemma 1, to
\[
(I + \Lambda \Lambda^*)^{-1} v_k = s_k^2 (I + \Gamma^* \Gamma)^{-1} v_k \\
= s_k^2 (v_k - (I + \Lambda \Lambda^*)^{-1} v_k),
\]
so that
\[
(1 + s_k^2) (I + \Lambda \Lambda^*)^{-1} v_k = s_k^2 v_k.
\]

Thus,
\[
(I + \Lambda \Lambda^*)^{-1} v_k = \frac{s_k^2 v_k}{1 + s_k^2} \\
= (1 + s_k^2)^{-1} s_k^2 v_k,
\]
which implies that
\[
(I + \Lambda \Lambda^*)^{-s} v_k = \left( s_k^2 \left( 1 + s_k^2 \right)^{-1} \right)^s v_k.
\]

In particular,
\[
(I + \Lambda \Lambda^*)^{-1/2} v_k = s_k (1 + s_k^2)^{-1/2} v_k.
\]

Consequently,
\[
T_{\Lambda^*} v_k = \Lambda^* (I + \Lambda \Lambda^*)^{-1/2} v_k + \Gamma (I + \Lambda \Lambda^*)^{-1/2} v_k \\
= \Lambda^* \left( \frac{s_k}{\sqrt{1 + s_k^2}} v_k \right) + \Gamma \left( \frac{s_k}{\sqrt{1 + s_k^2}} v_k \right) \\
= \frac{1}{\sqrt{1 + s_k^2}} z_k + \frac{s_k^2}{\sqrt{1 + s_k^2}} z_k \\
= \sqrt{1 + s_k^2} z_k.
\]

Therefore,
\[ T_{\Lambda^*} (I + \Lambda \Lambda^*)^{-s} v_k = \left( \frac{s_k^2}{1 + s_k^2} \right)^s \left( \frac{1 + s_k^2}{\sqrt{1 + s_k^2}} \right) z_k = \left( \frac{s_k^2}{1 + s_k^2} \right)^s \sqrt{1 + s_k^2} z_k. \]

On the other hand, \( \Gamma^* z_k = \Gamma z_k = s_k^2 z_k \). By putting \( (I + \Lambda^* \Lambda)^{-1} z_k = e_k \), we have

\[ z_k = e_k + \Lambda^* \Lambda e_k, \]

which implies that

\[ \Gamma^* z_k = s_k^2 z_k = \Gamma^* e_k + e_k \]

and

\[ (I + \Gamma^*)^{-1} \Gamma^* z_k = s_k^2 (I + \Gamma^*)^{-1} z_k = e_k = \Gamma^* (I + \Gamma^*)^{-1} z_k. \]

Using Lemma 1, it follows that

\[ (I + \Gamma^*)^{-1} \Gamma^* z_k = \Gamma \Lambda (I + \Lambda^* \Lambda)^{-1} z_k. \]

This leads, again from Lemma 1, to

\[ (I + \Lambda^* \Lambda)^{-1} z_k = s_k^2 (I + \Gamma^*)^{-1} z_k = s_k^2 \left( z_k - (I + \Lambda^* \Lambda)^{-1} z_k \right), \]

so that

\[ (1 + s_k^2) (I + \Lambda^* \Lambda)^{-1} z_k = s_k^2 z_k \]

and

\[ (I + \Lambda^* \Lambda)^{-1} z_k = s_k^2 (1 + s_k^{-2})^{-1} z_k. \]

Consequently,

\[ (I + \Lambda^* \Lambda)^{-s} z_k = \left( \frac{s_k^2}{1 + s_k^2} \right)^{-s} z_k, \]

which implies that

\[ (I + \Lambda^* \Lambda)^{-s} T_{\Lambda^*} v_k = (I + \Lambda^* \Lambda)^{-s} \sqrt{1 + s_k^2} z_k = \sqrt{1 + s_k^2} \left( \frac{s_k^2}{1 + s_k^2} \right)^s z_k. \]

Hence, one has \( (I + \Lambda^* \Lambda)^{-s} T_{\Lambda^*} v_k = T_{\Lambda^*} (I + \Lambda \Lambda^*)^{-s} v_k \) for all \( k \geq 1 \) and the proof is complete. \( \square \)
Moreover, from Lemma 3, we have that there exist a positive constant $c_{\text{rem}}$ such that

\[ \|u\|_{\mathcal{H}^s} = \|\Gamma_s u\|_{s-1/2, \partial \Omega}. \]

**Theorem 2 (The trace inequalities)** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain with boundary $\partial \Omega$. Consider, for $1 < s < 3/2$, the trace operators $\Gamma_s$ and $\Gamma$ from $\mathcal{H}^s(\Omega)$ to $\mathcal{H}^{s-1/2}(\partial \Omega)$ and from $\mathcal{H}_d^s(\Omega)$ to $L^2(\partial \Omega)$, respectively, and let $\Lambda$ be the Moore–Penrose inverse of $\Gamma$. Then there exist two positive constants $c_1$ and $c_2$ such that the inequalities

\[ c_1 \|\Gamma_s v\|_{s-1/2, \partial \Omega} \leq \|(I + \Lambda^* \Lambda)^{s-1/2} \Gamma \tilde{v}\|_{0, \partial \Omega} \leq c_2 \|\Gamma_s v\|_{s-1/2, \partial \Omega} \]  \hspace{1cm} (2)

hold for all $v \in \mathcal{H}^s(\Omega)$, where $\tilde{v}$ is the embedding of $v$ in $\mathcal{H}^1(\Omega)$.

**Proof** Assume $1 < s < 3/2$ and let $v \in \mathcal{H}^s(\Omega)$ and $\tilde{v}$ be its embedding in $\mathcal{H}^1(\Omega)$. Clearly, $\Gamma_s v \in \mathcal{H}^{s-1/2}(\partial \Omega)$ and $\Gamma \tilde{v} \in L^2(\partial \Omega)$. From Lemma 5, it follows that

\[ \mathcal{H}^{s-1/2}(\partial \Omega) = \mathcal{H}^{s-1/2}(\partial \Omega) \]

with equivalence between the norm $\| \cdot \|_{s-1/2, \partial \Omega}$ and the graph norm defined for $g \in L^2(\partial \Omega)$ by

\[ g \mapsto \|(I + \Lambda^* \Lambda)^{s-1/2} g\|_{0, \partial \Omega}. \]

Equivalently, there exist two positive constants $c_1$ and $c_2$ such that (2) holds for all $v \in \mathcal{H}^s(\Omega)$. □

As an application of Theorems 1 and 2 we prove the harmonic inequality (3).

**Theorem 3 (The harmonic inequality for $1 < s < 3/2$)** Assume $1 < s < 3/2$. Then, for all $v \in \mathcal{H}^s(\Omega)$, the following inequality holds:

\[ \|v\|_{\mathcal{H}^s(\Omega)} \leq \|T_{\Lambda^*}||v||_{s-1/2, \partial \Omega}, \]  \hspace{1cm} (3)

where

\[ T_{\Lambda^*} = \Lambda^*(I + \Lambda \Lambda^*)^{-1/2} + \Gamma(I + \Lambda \Lambda^*)^{-1/2} \]

and $\tilde{v}$ is the embedding of $v$ in $\mathcal{H}^1(\Omega)$.

**Proof** Consider $v \in \mathcal{H}^s(\Omega)$ and $\tilde{v}$ its embedding in $\mathcal{H}^1(\Omega)$. It follows from Theorem 2 that there exist a positive constant $c_3$ such that

\[ \|v\|_{\mathcal{H}^s(\Omega)} = \|\Gamma_s v\|_{s-1/2, \partial \Omega} \leq c_3 \|(I + \Lambda^* \Lambda)^{s-1/2} \Gamma \tilde{v}\|_{0, \partial \Omega}. \]

Moreover, from Lemma 3 we have
\[ \Gamma = (I + \Lambda^*\Lambda)^{-1/2}T_{\Lambda^*}, \]

where

\[ T_{\Lambda^*} = \Lambda^*(I + \Lambda\Lambda^*)^{-1} + \Gamma(I + \Lambda\Lambda^*)^{-1/2}. \]

It follows from Theorem 1 that

\[ \|v\|_{\mathcal{H}(\Omega)} = \|(I + \Lambda^*\Lambda)^{-1}T_{\Lambda^*}v\|_{0,\partial\Omega} = \|T_{\Lambda^*}(I + \Lambda\Lambda^*)^{-1}v\|_{0,\partial\Omega}. \]

Lemma 2 asserts that \( T_{\Lambda^*} \) is bounded and the intended inequality (3) follows. \( \square \)

Now consider the embedding operator \( E \) from \( H^1(\Omega) \) into \( L^2(\Omega) \) and its adjoint \( E^* \), which is the solution operator of the following Robin problem for the Poisson equation:

\[ \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_{\nu}u + \Gamma u = 0 & \text{on } \partial\Omega, \end{cases} \]

where \( f \in L^2(\Omega) \) and \( \partial_{\nu} \) denotes the normal derivative operator with exterior normal \( v \). Let \( E_0^* \) be the solution operator of the Dirichlet problem for the following Poisson equation:

\[ \begin{cases} -\Delta u^0 = f & \text{in } \Omega, \\ \Gamma u^0 = 0 & \text{on } \partial\Omega. \end{cases} \]

By setting \( E_1^* = E^* - E_0^* \) and \( u^1 = E_1^* f \), it follows that \( u^1 \) is the solution of the Dirichlet problem for the following Laplace equation:

\[ \begin{cases} -\Delta u^1 = 0 & \text{in } \Omega, \\ \Gamma u^1 = \Gamma u & \text{on } \partial\Omega. \end{cases} \]

Let \( 0 \leq s \leq 1 \), \( F_1 \) be the Moore–Penrose inverse of \( E_1 \), \( F_1^* \) be its adjoint operator, and denote

\[ X^s(\Omega) = \left\{ (I + F_1^*F_1)^{-s/2}v \mid v \in \mathcal{H}(\Omega) \right\}, \]

where \( \mathcal{H}(\Omega) \) is the Bergman space. Next we prove the harmonic inequalities for the case \( s = 1 \).

**Theorem 4 (The harmonic inequalities for \( s = 1 \))** Let \( \Omega \subset \mathbb{R}^d \), \( d \geq 2 \), be a bounded Lipschitz domain. Then, for all \( v \in \mathcal{H}^1(\Omega) \), there exist two positive constants \( c'_1 \) and \( c'_2 \), not depending on \( v \), such that

\[ c'_1\|v\|_{\partial\Omega} \leq \|(I + F_1^*F_1)^{1/2}E_1v\|_{0,\partial\Omega} \leq c'_2\|v\|_{\partial\Omega}. \] (4)

**Proof** From Lemma 3, the decomposition

\[ E_1 = (I + F_1^*F_1)^{-1/2}T_{F_1} \]

holds, where
\[ T_F^* = F^*_1 (I + F_1 F^*_1)^{-1/2} + E_1 (I + F_1 F^*_1)^{-1/2}. \]

Moreover, since \( R(E_1) = \mathcal{H}^1(\Omega) \), it follows that \( \mathcal{H}^1(\Omega) = \mathcal{X}^1(\Omega) \). Now consider the graph norm

\[ ||u||_{\mathcal{X}^1(\Omega)} = ||(I + F_1^* F_1)^{1/2} u||_{0,\Omega} \]

for \( u \in \mathcal{X}^1(\Omega) \). It follows that \( E_1 v \in \mathcal{X}^1(\Omega) \) for \( v \in \mathcal{H}^1(\Omega) \) and

\[ ||(I + F_1^* F_1)^{1/2} E_1 v||_{0,\Omega} = ||T_{F^*} v||_{0,\Omega}. \]

In agreement with Lemma 4, we can view \( T_{F^*} \) as an isomorphism from \( \mathcal{H}^1(\Omega) \) into \( \mathcal{H}(\Omega) \), and there exist two positive constants \( c'_1 \) and \( c'_2 \), not depending on \( v \), such that the conclusion holds.

The harmonic inequalities of Theorem 4 are a useful tool to provide a functional characterization of the harmonic Hilbert spaces for the range of values \( 0 \leq s \leq 1 \).

**Corollary 1** Assume \( 0 \leq s \leq 1 \). Then \( \mathcal{H}^s(\Omega) \) form an interpolatory family. Moreover,

\[ \mathcal{H}^s(\Omega) = \mathcal{X}^s(\Omega) \]

with equivalence of norms.

**Proof** For \( s = 0 \), one has the equality \( \mathcal{X}(\Omega) = \mathcal{H}(\Omega) \) by definition. For \( s = 1 \), Theorem 4 asserts that \( \mathcal{X}^1(\Omega) = \mathcal{H}^1(\Omega) \) with the equivalence of the norm \( || \cdot ||_{\mathcal{X}^1(\Omega)} \) with the norm on \( \mathcal{H}^1(\Omega) \), which is the same as the one on \( \mathcal{H}^1_\partial(\Omega) \). The intended equality \( \mathcal{H}^s(\Omega) = \mathcal{X}^s(\Omega) \) with equivalence of norms, \( 0 < s < 1 \), follows from classical results on the theory of positive self-adjoint operators, which assert that both \( \mathcal{X}^s(\Omega) \) and \( \mathcal{H}^s(\Omega) \) form an interpolating family for \( 0 < s < 1 \).

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**References**

1. R. A. Adams and J. F. Fournier, *Sobolev spaces*, Elsevier/Academic Press, Amsterdam, 2003.
2. E. Carlen, Trace inequalities and quantum entropy: an introductory course, in *Entropy and the quantum*, 73–140, Contemp. Math., 529, Amer. Math. Soc., Providence, RI, 2010.
3. C. D. Collins and J. L. Taylor, Eigenvalue convergence on perturbed Lipschitz domains for elliptic systems with mixed general decompositions of the boundary, J. Differential Equations 265 (2018), no. 12, 6187–6209.
4. M. Costabel, Boundary integral operators on Lipschitz domains: elementary results, SIAM J. Math. Anal. 19 (1988), no. 3, 613–626.
5. X. Chen, R. Jiang and D. Yang, Hardy and Hardy-Sobolev spaces on strongly Lipschitz domains and some applications, Anal. Geom. Metr. Spaces 4 (2016), 336–362.
6. B. Dacorogna, Introduction to the calculus of variations, third edition, Imperial College Press, London, 2015.
7. R. Dautray and J.-L. Lions, Mathematical analysis and numerical methods for science and technology. Vol. 2. Functional and variational methods. Springer-Verlag, Berlin, 1988.
8. L. C. Evans, Partial differential equations. Graduate Studies in Mathematics, 19, American Mathematical Society, Providence, RI, 1998.
9. E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, Rend. Sem. Mat. Univ. Padova 27 (1957), 284–305.
10. V. K. Gupta and P. Sharma, Hypergeometric inequalities for certain unified classes of multivalent harmonic functions, Appl. Anal. Math. 13 (2018), no. 1, 315–332.
11. M. Hayajneh, S. Hayajneh and F. Kittaneh, On some classical trace inequalities and a new Hilbert-Schmidt norm inequality, Math. Inequal. Appl. 21 (2018), no. 4, 1175–1183.
12. M. Kohr and W. L. Wendland, Variational approach for the Stokes and Navier–Stokes systems with nonsmooth coefficients in Lipschitz domains on compact Riemannian manifolds, Calc. Var. Partial Differential Equations 57 (2018), no. 6, 57:165.
13. J.-Ph. Labrousse, Inverses généralisés d’opérateurs non bornés, Proc. Amer. Math. Soc. 115 (1992), no. 1, 125–129.
14. C. R. Loga, An extension theorem for matrix weighted Sobolev spaces on Lipschitz domains, Houston J. Math. 43 (2017), no. 4, 1209–1233.
15. G. Maze and U. Wagner, A note on the weighted harmonic-geometric-arithmetic means inequalities, Math. Inequal. Appl. 15 (2012), no. 1, 15–26.
16. W. McLean, Strongly elliptic systems and boundary integral equations, Cambridge University Press, Cambridge, 2000.
17. J. Nečas, Les méthodes directes en théorie des équations elliptiques, Masson et Cie, Éditeurs, Paris, 1967.
18. S. M. Nikol’skii and P. I. Lizorkin, Inequalities for harmonic, spherical and algebraic polynomials, Dokl. Akad. Nauk SSSR 289 (1986), no. 3, 541–545.
19. M. Prats, Sobolev regularity of the Beurling transform on planar domains, Publ. Mat. 61 (2017), no. 2, 291–336.
20. L. Tartar, An introduction to Sobolev spaces and interpolation spaces, Lecture Notes of the Unione Matematica Italiana, 3, Springer, Berlin, 2007.
21. S. Touhami, A. Chaira and D. F. M. Torres, Functional characterizations of trace spaces in Lipschitz domains, Banach J. Math. Anal., submitted.
22. B. Wei and W. Wang, Some inequalities for general Lp-harmonic Blaschke bodies, J. Math. Inequal. 10 (2016), no. 1, 63–73.