ATTRACTION, INVARIANCE AND ERGODICITY FOR SDES ON RIEMANNIAN MANIFOLDS

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Abstract. We give a sufficient condition on nonlinearities of an SDE on a compact connected Riemannian manifold $M$ which implies that laws of all solutions converge weakly to the normalized Riemannian volume measure on $M$. This result is further applied to characterize invariant and ergodic measures for various SDEs on manifolds.

1. Introduction

This work has its origins in an attempt (so far not concluded) by the authors to find a mathematically sound numerical approximation for the stochastic geometric wave equations. Such equations have been recently introduced by two of us in [6], and our aim is to produce a counterpart of the numerical scheme introduced by three of us in [1] for the stochastic Landau-Lifshitz-Gilbert equations (used in the theory of ferromagnetism). For the deterministic geometric wave equations such questions have recently been studied by two of us and Schätzle in [2]. During the initial stages we encountered problems related to large time behaviour of the solutions and we realised that we did not know an answer to such problems even for finite dimensional models. The reason being that the finite dimensional approximations to stochastic geometric wave equations, for instance those generated by finite elements methods, are highly degenerate with respect to noise. Moreover, there are not many papers on the stochastic Langevin equations on manifolds, but see [16] and [3].

On the other hand, the existence and uniqueness of invariant measures or convergence of solutions to an attracting measure was studied in case of uniformly elliptic diffusion operators e.g. in [15] and in case of hypoelliptic parabolic and elliptic systems in [13] and [14]. The variety of interesting problems is of course much wider than the cases described above as even quite simple SDE problems degenerate in such a way that none of the existing results covers them (see the examples at the end of this paper). A particular approach to such degenerated elliptic problems is always required and we present one of them - still quite general - which implies that laws of all solutions converge weakly to the normalized Riemannian volume measure on $M$. For instance, in Theorem 2.9 we formulate sufficient conditions for the existence and uniqueness of an attractive measure. We develop further this result to characterize invariant and ergodic measures for various degenerated SDEs on manifolds. To be more precise, in Theorems 3.4 and 3.6 we give a complete characterization of the set of invariant and invariant ergodic measures for certain classes of SDEs on the sphere $S^2$, and the special orthogonal group $SO_3$.

A further goal of this paper is to construct numerical schemes for solving certain classes of SDEs on manifolds, i.e., the finite dimensional stochastic Landau-Lifshitz-Gilbert (LLG) equation, and the geodesic equation on the sphere with stochastic forcing. The LLG equation has been widely studied in the physical literature, see for instance [7] and references therein. They have also received some attention from the mathematical point of view, see [17]. General stochastic LLG’s for non-uniform magnetisations have been studied by one of us and Goldys in [5], while Lelièvre et al in [19] have described certain numerical schemes for SDEs with constraints. A convergent discretisation in space and time which is based on finite elements is proposed in [4]; this scheme guarantees the sphere constraint to hold for approximate magnetisation processes, and thus inherits the Lyapunov structure of the problem. As a consequence, iterates may be shown to construct weak martingale solutions of the limiting equations. Main steps of this construction are detailed for the
finite dimensional LLG equation in Section 3.1 to then study long-time dynamics in Section 3.2. A corresponding numerical program for second order equations with stochastic forcing is detailed in Section 4.1 which requires different tools; in particular, a discrete Lagrange multiplier is used in Algorithm B for iterates to inherit the sphere constraint in a discrete setting. Overall convergence of iterates is asserted in Theorem 4.3 which holds for this particular SDE on the sphere, but which may also be considered as a first step to numerically approximate the stochastic geometric wave equation. Again, computational examples are provided in Section 4.2 to illustrate the results proved in this work, and motivate further analytical studies for computationally observed long-time behaviors which lack a sound analytical understanding at this stage.

2. THE PROBLEM AND PRELIMINARY RESULTS

Let $M$ be a compact connected $d$-dimensional Riemannian manifold whose normalized Riemannian volume measure is denoted by $\lambda$ (i.e. $\lambda(M) = 1$). Integration with respect to $\lambda$ will be denoted by $dx$. Let us assume that $F$ and $G_1, \ldots, G_m$ are smooth vector fields on $M$. Let $W^1, \ldots, W^m$ be independent $(\mathcal{F}_t)$-Wiener processes on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and consider the following Stratonovich SDE

$$du = F(u) \, dt + \sum_{k=1}^{m} G_k(u) \circ dW^k$$

on $M$.

The following (preliminary) results on SDE on manifolds in this section are generally known and can be found e.g. in [11], [15] or [18]. Denote by $P = (P_t)_{t \geq 0}$ the Feller semigroup associated with the SDE (2.1), i.e.

$$P_t h(x) = \mathbb{E}(h(u^x(t))), \quad h \in B_b(M), \quad (t, x) \in \mathbb{R}_+ \times M$$

by $B_b(M)$ we denote the Banach space of all real bounded and Borel measurable functions on $M$, and $u^x$ is a solution of (2.1) such that $u^x(0) = x$.

Let us also denote by $C(M)$ the separable Banach space of all real continuous functions on $M$, let us recall that the restriction of $P$ to $C(M)$ is a $C_0$-semigroup on $C(M)$, denote by $\mathcal{A}$ its infinitesimal generator and recall the following characterization:

$$C^2(M) \subseteq D(\mathcal{A})$$

$$\mathcal{A} f = F f + \frac{1}{2} \sum_{k=1}^{m} G_k(G_k f), \quad f \in C^2(M).$$

We will denote by $P^*$ the dual semigroup on the space $\mathcal{M}(M)$ of all $\mathbb{R}$-valued Borel measures on $M$, and by $\mathcal{A}^*$ the infinitesimal generator of the dual semigroup.

We convene that, throughout this paper, all measures on $M$ will be Borel, i.e. with the domain $\mathcal{B}(M)$.

Let us also introduce the transition kernel

$$p(t, x, A) = \mathbb{P}(\{u^x(t) \in A\}), \quad t \geq 0, \quad x \in M, \quad A \in \mathcal{B}(M).$$

Let $\nu$ be a probability measure on $M$. We say that $\nu$ is invariant provided

$$\int_M p(t, x, A) \, d\nu = \nu(A), \quad \text{for all } t \geq 0, \quad x \in M \text{ and } A \in \mathcal{B}(M).$$

An invariant probability measure is said to be ergodic if it is an extremal point of the convex set of all invariant probability measures on $M$.

A probability measure $\nu$ on $M$ is called attractive if

$$\lim_{t \to +\infty} P_t f(x) = \int_M f \, d\nu, \quad \text{for all } x \in M \text{ and } f \in C(M).$$

\footnote{We will also say that $\nu$ is invariant for the SDE (2.1) that generates the semigroup $P$.}
which, in other words, means that every solution \( u \) of (2.1) converges in law to \( \nu \) irrespectively of the initial condition (even if \( u(0) \) is random).

Let \( L_0 = \{F,G_1,\ldots,G_m\} \) and \( L_n = \{[X,Y]: X,Y \in \bigcup_{j=0}^{n-1} L_j\} \) for \( n \in \mathbb{N} \), and denote

\[
\mathcal{L} = \text{span} \bigcup_{n=0}^{\infty} L_n, \quad \mathcal{L}_F = \text{span} \left( (L_0 \setminus \{F\}) \cup \bigcup_{n=1}^{\infty} L_n \right)
\]

where \([X,Y] = XY - YX\). Then we say that the vector fields \( F,G_1,\ldots,G_m \) satisfy the hypothesis

- \( \mathbf{(H)} \) if \( \dim \{X_p: X \in \mathcal{L}\} = d \) for every \( p \in M \) (the Hörmander condition),
- \( \mathbf{(F)} \) if every \( f \in C^\infty(M) \) is constant on \( M \) provided \( LF = 0 \) on \( M \) for every \( L \in \mathcal{L}_F \),
- \( \mathbf{(D)} \) if \( \text{div} \, G_1 = \cdots = \text{div} \, G_m = \text{div} \, F = 0 \) on \( M \),
- \( \mathbf{(C)} \) if, for every \( l \in \mathbb{N} \), there exists a finite dimensional subspace \( C_l \) of \( C^2(M) \) such that \( A(C_l) \subseteq C_l \) and \( \text{span} \bigcup_{l \in \mathbb{N}} C_l \) is dense in \( C(M) \) with respect to the supremal norm.

**Remark 2.1.** Let us observe that the equation (2.1) and the hypotheses \( \mathbf{(H)} \), \( \mathbf{(F)} \) and \( \mathbf{(C)} \) do not depend on the Riemannian structure of the manifold \( M \). In particular, we can, if necessary (and possible), change the metric on \( M \) in order the vector fields \( F,G_1,\ldots,G_m \) satisfy the hypothesis \( \mathbf{(D)} \). Of course, a change of the metric on \( M \) changes accordingly the Riemannian volume measure on \( M \).

**Remark 2.2.** The hypothesis \( \mathbf{(F)} \) is clearly satisfied if either the union of closures of the components of \( \{p \in M : \dim \{X_p: X \in \mathcal{L}_F\} = d\} \) is a connected dense subset of \( M \), or if any two points of \( M \) can be connected by a piecewise smooth curve consisting of integral curves of the vector fields in \( \mathcal{L}_F \) (see the examples in Section 3 and 4).

**Proposition 2.3.** A probability measure \( \nu \) on \( M \) is invariant if and only if

\[
\int_M Ah \, d\nu = 0, \quad h \in C^2(M).
\]

**Proof.** See the identity (4.58) on p. 292 in [15]. \( \square \)

**Proposition 2.4** (Krylov-Bogolyubov). There exists at least one invariant measure.

**Proof.** See Corollary 3.1.2 in [9]. \( \square \)

**Proposition 2.5.** Let \( R \in C^\infty(M) \) be a non-negative function. Then the measure \( d\mu = Rd\lambda \) is an invariant measure if and only if the following equality holds on \( M \)

\[
\frac{1}{2} \sum_{k=1}^{d} \text{div} \{\text{div}(RG_k)G_k\} = \text{div}(RF).
\]

**Proof.** This result can be proved directly as for instance the last but one claim of Theorem 6.3.2 from [10] or it can be deduced from that result. \( \square \)

**Corollary 2.6.** If \( \mathbf{(D)} \) holds then

\[
A^* f = -Ff + \frac{1}{2} \sum_{k=1}^{m} G_k^2(f), \quad f \in C^2(M).
\]

Moreover, the condition (2.7) is equivalent to \( A^* R = 0 \).

**Proof.** Since for a vector field \( X \) on \( M \) and any \( f \in C^2(M) \), we have

\[
X^* f = -\text{div}(fX) = -f \text{div} X - X(f),
\]

\[
(X^2)^* f = f(\text{div} X)^2 + 2X(f)\text{div} X + fX(\text{div} X) + X^2(f),
\]

by Proposition 2.5 we get the result. \( \square \)

**Proposition 2.7.** If \( \mathbf{(H)} \) holds then every invariant measure has a density \( R \in C^\infty(M) \) with respect to the Riemannian volume measure \( \lambda \) on \( M \).

**Proof.** This follows from [12]. \( \square \)
The following is a consequence of Theorem 6.3.2 from \cite{10} (or of our Proposition \ref{2.5}).

**Theorem 2.8.** The normalized Riemannian volume measure $\lambda$ on $M$ is invariant iff

\begin{equation}
\text{div}(F) + \sum_{k=1}^{m} [G_k(\text{div} G_k) - (\text{div} G_k)^2] = 0 \quad \text{on } M.
\end{equation}

In particular, if the condition (D) is satisfied, then $\lambda$ is invariant. Moreover, if both conditions (D) and (H) are satisfied, then $\lambda$ is the unique invariant probability measure.

**Proof.** It is sufficient to prove the last claim. So let us assume that $\mu$ is an invariant probability measure. Then, in view of Proposition \ref{2.7}, $\mu$ has a density $R$ with respect to $\lambda$ and $R \in C^\infty(M)$. Moreover, by Corollary \ref{2.6} $A^* R = 0$. On the other hand, in view of the first formula in the proof of Corollary \ref{2.6} from the assumption (D) we infer that

\begin{equation}
\int_M R \cdot A^* R \, dx = \frac{1}{2} \sum_{k=1}^{m} \int_M |G_k R|^2 \, dx.
\end{equation}

Hence $G_1 R = \cdots = G_m R = 0$ and consequently $FR = -A^* R = 0$. Thus

\[ LR = 0 \quad \text{for every } L \in \mathcal{A}(F, G_1, \ldots, G_m). \]

The condition (H) yields that $R$ is a constant function. \hfill $\Box$

**Theorem 2.9.** Assume that the hypothesis (F), (D) and (C) hold. Assume that a process $u$ is a solution to SDE \textup{(2.1)}. Then there exists a probability measure $\theta$ on $M$ such that

\begin{equation}
\lim_{t \to \infty} E f(u(t)) = \int_M f \, d\theta, \quad f \in C(M).
\end{equation}

If, in addition, (H) holds then $\theta = \lambda$.

**Proof.** Let us fix a natural number $l \in \mathbb{N}$. Let us choose a basis $\{f_1, \ldots, f_n\}$ of $C_l$ and let $(a_{ij})_{i,j=1}^{n}$ be a $(n \times n)$-matrix of the restriction of linear operator $A$ to $C_l$ with respect to this basis.

In particular,

\[ A f_i(p) = \sum_{j=1}^{n} a_{ij} f_j(p), \quad p \in M, \quad i \in \{1, \ldots, n\}. \]

Let us denote by $A$ the linear operator in $\mathbb{R}^n$ whose matrix in the canonical basis $\{e_1, \ldots, e_n\}$ is equal to $(a_{ij})_{i,j=1}^{n}$.

Let a process $u$ be a solution to SDE \textup{(2.1)}. Then for every $i \in \{1, \ldots, n\}$

\[ E f_i(u(t)) = E f_i(u(0)) + E \int_0^t (A f_i)(u(s)) \, ds \]

\[ = E f_i(u(0)) + \sum_{j=1}^{n} a_{ij} \int_0^t E f_j(u(s)) \, ds, \quad t \geq 0. \]

Hence

\begin{equation}
\sum_{i=1}^{n} E f_i(u(t)) e_i = e^{At} \left[ \sum_{j=1}^{n} E f_j(u(0)) e_j \right], \quad t \geq 0.
\end{equation}

Since by Lemma \ref{2.10} below the operator valued function $e^{At}$ is convergent as $t \to \infty$, in view of \textup{(2.12)} we infer that so are the functions $E f_i(u(t))$. By the density part of assumption (C), we conclude that for every $f \in C(M)$, $E f(u(t))$ is convergent as $t \to \infty$.

Now we will prove the last claim and so we assume that the condition (H) holds. Then by the Krylov-Bogolyubov Theorem in conjunction with Theorem \ref{2.8}

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \nu_s(B) \, ds = \lambda(B), \quad B \in \mathcal{B}(M), \]

where $\nu_t$ is the law of $u(t)$. This concludes the proof. \hfill \Box
Lemma 2.10. The operator valued function $e^{tA}$ from the proof of Theorem 2.9 is convergent as $t \to \infty$.

Proof of Lemma 2.10. Let us begin by introducing the following notation:

$$y(p) = (f_j(p))_{j=1}^n = \sum_{j=1}^n f_j(p) e_j \in \mathbb{R}^n, \ p \in M.$$ \hspace{1cm} (2.13)

If follows from equality (2.12) that for any $p \in M$, $\sup_{t \geq 0} |e^{At}y(p)| < \infty$. On the other hand, since the functions $f_1, \ldots, f_n$ are linearly independent in $C^2(M)$, the vectors $\{y(p) : p \in M\}$ span $\mathbb{R}^n$. Hence, we infer that

$$\sup_{t \geq 0} \|e^{At}\| < \infty.$$ \hspace{1cm} (2.14)

In the above we considered $A$ and $e^{At}$ as linear operators on $\mathbb{R}^n$. We may naturally extend them to the complex space $\mathbb{C}^n$. Since the euclidean norms of $e^{At}$ in $\mathbb{R}^n$ and $\mathbb{C}^n$ coincide, we infer that

$$\sup_{t \geq 0} \|e^{At}\|_{L(\mathbb{C}^n)} < \infty.$$ \hspace{1cm} (2.15)

It follows from (2.13) that the spectrum $\sigma(A)$ is contained in $\mathbb{C}_-$, the closed real-negative halfplane of $\mathbb{C}$.

We will show that $(i\mathbb{R} \setminus \{0\}) \cap \sigma(A) = \emptyset$. Arguing by contradiction let us consider $\lambda \in (i\mathbb{R} \setminus \{0\}) \cap \sigma(A)$. Then $\lambda = i\alpha$ for some $\alpha \in \mathbb{R} \setminus \{0\}$ belongs to the spectrum of the adjoint operator $A^*$. Hence and there is $z = u + iv \in \mathbb{C}^n \setminus \{0\}$ such that $A^*z = i\alpha z$, i.e. $A^*u = -\alpha u$ and $A^*v = \alpha v$. If we denote $g = \sum_{j=1}^n u_j f_j$ and $h = \sum_{j=1}^n v_j f_j$, then $A^*g = -\alpha g$ and $A^*h = \alpha h$. Hence, by identity (2.10), we get

$$0 = \int_M g (A^*g + h A^*h) \ dx = -\frac{1}{2} \sum_{j=1}^m \int_M [(G_j g)^2 + |G_j h|^2] \ dx.$$ \hspace{1cm} (2.16)

Therefore $G_j g = G_j h = 0$ on $M$ for every $j \in \{1, \ldots, m\}$. Consequently, $Fg = -\alpha g$ and $Fh = \alpha g$ and so

$$[G_j, F]g = G_j Fg = -\alpha G_j h = 0,$$

$$[G_j, F]h = G_j Fh = \alpha G_j g = 0.$$ \hspace{1cm} (2.17)

By an induction, we infer that $Xg = Xh = 0$ for every $X \in \mathcal{L}_F$. Hence, by assumption (F), the functions $g$ and $h$ are constant on $M$. Thus $h = -\alpha^{-1} Fg = 0$ and $g = \alpha^{-1} Fh = 0$ what implies that $z = 0$. This contradiction concludes the proof of our claim.

Let us now assume that $0 \in \sigma(A)$. Then there exists a natural number $N \in \mathbb{N}$, matrices $A_k$, $k \in \{1, \ldots, N\}$, a positive number $\varepsilon > 0$ and a holomorphic $\mathcal{L}(\mathbb{R}^n)$-valued function $V$ defined on an open disc in $\mathbb{C}$ of radius $\varepsilon$ such that

$$(\lambda - A)^{-1} = \sum_{k=1}^N \lambda^{-k} A_k + V(\lambda), \quad 0 < |\lambda| < \varepsilon.$$ \hspace{1cm} (2.18)

By the residuum theorem,

$$e^{tA} = \sum_{k=1}^N \frac{t^{k-1}}{(k-1)!} A_k + o(t)$$

where $o(t) \to 0$ as $t \to \infty$. We however already know that $e^{tA}$ is bounded and hence, necessarily, $N = 1$. Thus we infer that $e^{tA} \to A_1$ as $t \to \infty$ what concludes the proof.

□
3. Example I

Assume that $A, B_1, \ldots, B_m$ are antisymmetric $(n \times n)$-matrices and $W_1, \ldots, W_m$ are independent ($\mathcal{F}_t$)-Wiener processes. Consider the following stochastic differential equation in the Stratonovich form,

$$(3.1) \quad dz = Az \, dt + \sum_{k=1}^{m} B_k z \circ dW_k$$

on the sphere $S^{n-1} \subseteq \mathbb{R}^n$. Denoting by $\mathcal{L}(\mathbb{R}^n)$ the space of real $(n \times n)$-matrices, every solution of (3.1) can be written as $Z(t)z_0$, where $Z$ is an $\mathcal{L}(\mathbb{R}^n)$-valued solution of the following Stratonovich stochastic differential equation

$$(3.2) \quad Z(t) = AZ \, dt + \sum_{k=1}^{m} B_k Z \circ dW^k, \quad t \geq 0$$

satisfying the initial condition $Z(0) = I$. In fact, the solution $Z$ takes values in the set $SO_n$, called the special orthogonal group and consisting of all unitary operators in $\mathbb{R}^n$ with determinant 1. Let us recall that

- the set $SO_n$ is a compact connected Lie group and a submanifold of $\mathcal{L}(\mathbb{R}^n)$,
- for $Z \in SO_n$,

$$T_Z SO_n = \{ V \in \mathcal{L}(\mathbb{R}^n) : V + Z V^* Z = 0 \} = \{ V \in \mathcal{L}(\mathbb{R}^n) : V Z^* + Z V^* = 0 \},$$

- there exists a bi-invariant Riemannian metric on $SO_n$ and the corresponding normalized Riemannian volume measure $\lambda$ on $SO_n$ is bi-invariant (with respect to multiplication by elements of $SO_n$).

For an antisymmetric $M \in \mathcal{L}(\mathbb{R}^n)$, we denote by $M$ the following vector field on $SO_n$,

$$M : SO_n \ni Z \mapsto MZ \in TSO_n.$$  

**Lemma 3.1.** In the above framework, the conditions (D) and (C) are satisfied for the vector fields $A$ and $B_1, \ldots, B_k$.

**Proof.** Let us fix an antisymmetric $M \in \mathcal{L}(\mathbb{R}^n)$. We have $\text{div } M = 0$. Indeed, if $(E_i)$ are orthonormal vector fields on $SO_n$, then, see [35],

$$\text{div } M = \sum_{i=1}^{n} (\nabla E_i M, E_i)_{\mathbb{R}^{n \times n}}.$$ 

On the other hand, since $M$ is antisymmetric,

$$(\nabla E_i M, E_i)_{\mathbb{R}^{n \times n}} = \lim_{t \to 0} t^{-1} \langle M(z + tE_i z) - M z, E_i z \rangle_{\mathbb{R}^{n \times n}} = \langle M E_i z, E_i z \rangle_{\mathbb{R}^{n \times n}} = 0.$$

Combining the above two identities we infer that the condition (D) is satisfied.

Let us prove the the hypothesis (C) is also satisfied. Towards this end we identify the space $\mathcal{L}(\mathbb{R}^n)$ with $\mathbb{R}^{n \times n}$. Let us denote by $C_i$ the vector space spanned by the restriction to $O_n$ of polynomials on $\mathbb{R}^{n \times n}$ of order smaller or equal to $l$, i.e. $C_l = \text{span}\{ f_\alpha : |\alpha| \leq l \}$, where

$$f_\alpha(X) = X^\alpha = \prod_{j=1}^{n} \prod_{k=1}^{n} x_{j,k}^{\alpha_{j,k}}, \quad X = [x_{ij}] \in \mathbb{R}^{n \times n}, \quad \alpha \in \mathbb{N}^n_0 \times n.$$

Observe that

$$(M f_\alpha)(Z) = \sum_{l,j,k=1}^{n} m_{lj,k} \alpha_{j,k} z^{\alpha-e^k+l} = \sum_{l,j,k=1}^{n} m_{lj,k} \alpha_{j,k} f_{\alpha-e^k+l}(Z), \quad Z = [z_{ij}] \in \mathbb{R}^{n \times n},$$

where $e^k$ is the zero $(n \times n)$-matrix except for the position $(j,k)$ where $e^{jk} = 1$. In particular, since the degree of $f_{\alpha-e^k+l}$ is $\leq l$, we infer that $C_l$ is invariant for the operator $A$ defined in [22]. Finally, by the Stone-Weierstrass Theorem, $\bigcup_{l=0}^{\infty} C_l$ is dense in $C(SO_n)$. □
Corollary 3.2. The normalized Riemannian volume measure $\lambda$ on $SO_n$ is invariant with respect to the Feller semigroup generated by the SDE (3.2).

Theorem 3.3. If the Lie algebra $L_A(A, B_1, \ldots, B_m)$ contains all antisymmetric $(n \times n)$-matrices, then $\lambda$ is an attractive and unique invariant probability measure for the Feller semigroup generated by the SDE (3.2).

As a special case let us assume that $A$ and $B$ are two antisymmetric $(3 \times 3)$-matrices such that $B \neq 0$. Consider the following Stratonovich SDE on $SO_3$:

$$dZ(t) = AZ(t) dt + BZ(t) dW.$$  

Since both $A$ and $B$ are antisymmetric, we infer that

(i) $[A, B] = 0$ if $A$ and $B$ are linearly dependent,
(ii) $\{A, B, [A, B]\}$ spans the 3-dimensional space of antisymmetric matrices iff $A$ and $B$ are linearly independent

Hence, two possibilities may arise. We begin with the first one.

Theorem 3.4. If $[A, B] \neq 0$ then the measure $\lambda$ on $SO_3$ is an attractive and unique invariant probability measure for for the Feller semigroup generated by the SDE (3.3).

If $[A, B] = 0$ then $A$ and $B$ are commuting and so $\exp \{tA + WtB\}$ is the unique solution of (3.5). Assume that $B \neq 0$. Let us put $\rho = (\frac{1}{2} \|B\|_{Q_3, 3})^\frac{1}{2}$. For $K \in \{S^2, SO_3\}$, define

$$S : S^1 \times K \ni (p, Z) \mapsto s(p)Z \in K,$$

$$s(x, y) = \frac{1 - x}{\rho^2} B^2 + \frac{y}{\rho} B + I, \quad (x, y) \in S^1,$$

Note that for antisymmetric $(3 \times 3)$-matrix $B$, $B^3 = -\rho^2 B$. Thus the definition (3.4-3.5) of the map $S$ is correct. Moreover, since $s(p)s(q) = s(pq)$ for all $p, q \in S^1$, the map

$$S^3 \ni p \mapsto S(p, \cdot) \in K$$

is a group homomorphism, i.e. $S(p, S(q, Z)) = S(pq, Z)$ for $p, q \in S^1$ and $Z \in K$. For $Z \in K$ we will denote by $S^Z$ the orbit of the group $(S_p)_{p \in S^1}$ through $Z$, i.e. the function

$$S^Z : S^1 \ni p \mapsto S(p, Z) \in K$$

Let also denote by $H$ be the space of all such orbits, i.e.

$$H = \{S^Z[S^1] : Z \in K\}.$$

The space $H$ is equipped with the quotient topology for the surjective projection

$$\pi : K \ni Z \mapsto S^Z[S^1] \in H$$

with respect to which $H$ is compact. The quotient topology of $H$ is metrizable by the classical Hausdorff metric

$$\rho(X, Y) = \max\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\}.$$  

Let us denote by $dx$ the normalized Haar measure on $S^1$. If $X \in H$ and $U, V \in K$ are two elements belonging to the orbit of $X$, then $S^U(dx) = S^V(dx)$ on $\mathcal{B}(K)$. Therefore we can define a measure $\mu_X := S^U(dx)$ on $K$ with the support on $X$. Moreover for every $J \in \mathcal{B}(K)$, the function

$$H \ni X \mapsto \mu_X(J) \in [0, 1]$$

is Borel measurable.

If $\nu$ is a probability measure on $K$, then by $\nu_* = \pi(\nu)$ we denote the image probability measure on $H$ by the map $\pi$. Finally, we define a measure $\tilde{\nu}$ on $K$ by

$$\tilde{\nu}(J) = S(dx \otimes \nu)(J) = \int_H \mu_X(J) d\nu_*(X), \quad J \in \mathcal{B}(K).$$

In the following three theorems, the equation (3.3) will be considered on $K \in \{S^2, SO_3\}$. The joint proof is given after Theorem 3.7.
Theorem 3.5. Let us assume that $|A, B| = 0$ and $B \neq 0$. Let $Z$ be a solution of the equation (3.3) on $K \in \{S^2, SO_3\}$ and let us denote by $\nu$ the law of $Z(0)$. Then the law of $Z(t)$ converges weakly as $t \to \infty$ to the measure $\nu_\infty$ on $\mathcal{B}(K)$.

The following notation will be useful in formulation of the next result. If $\theta$ is a probability measure on $H$, then $\nu_\theta$ is a probability measure on $K$ defined via the following averaging formula.

\begin{equation}
\nu_\theta(J) = \int_H \mu_X(J) d\theta(X), \quad J \in \mathcal{B}(K).
\end{equation}

Theorem 3.6. Let us assume that $|A, B| = 0$ and $B \neq 0$. Then the mapping $\theta \mapsto \nu_\theta$ is a bijection between the set of probability measures on $H$ and the set of invariant probability measures for the equation (3.3) on $K \in \{S^2, SO_3\}$.

Theorem 3.7. Let $|A, B| = 0$ and $B \neq 0$. Then a probability measure $\nu$ on $K \in \{S^2, SO_3\}$ is ergodic for the equation (3.3) if and only if there exists $X \in H$ such that $\nu = \mu_X$.

Proof. Let us observe that since $B^2 = -\rho B$,

\begin{equation}
e^tB = \frac{1 - \cos(ps)}{\rho^2}B^2 + \frac{\sin(ps)}{\rho}B + I, \quad s \in \mathbb{R}.
\end{equation}

Since $A$ and $B$ are linearly dependent, there exists $\kappa$ such that $A = \kappa B$ and, as already mentioned, every solution $Z$ of the equation (3.3) has the form

\begin{equation}
Z(t) = e^{tA+W_1B}Z(0) = e^{(\kappa t+W_1)B}Z(0) = S(\gamma(t), Z(0))
\end{equation}

where $\gamma(t) = (\cos(\kappa t + W_1), \sin(\kappa t + W_1))$. Using the Fourier series, we may easily prove that $\gamma(t)$ converges in law to the normalized uniform measure $dx$ on $\mathcal{B}(S^1)$, hence $Z(t)$ converges in law to $\nu = S(dx \otimes \nu)$ where $\nu$ is the law of $Z(0)$ on $\mathcal{B}(K)$. Consequently, Theorem 3.5 is proved on one hand, and on the other hand, we proved that if $\nu$ is invariant then $\nu = \nu_\infty$.

However, $S(\mu \otimes \nu) = \nu_\infty$ for every probability measure $\mu$ on $\mathcal{B}(S^1)$ (in particular, for the law of $\gamma(t)$) hence $\nu$ is invariant if and only if $\nu = \nu_\infty$. Moreover, $S(\mu \otimes \nu_\theta) = \nu_\theta$ for every probability measure $\mu$ on $\mathcal{B}(S^1)$ and every probability measure $\theta$ on $\mathcal{B}(H)$. Hence $\{\nu_\theta : \theta$ probability measure on $\mathcal{B}(H)\}$ coincides with the set of invariant probability measures for the equation (3.3). The injectivity of $\theta \mapsto \nu_\theta$ follows from the fact that the image measure $\pi(\nu_\theta)$ coincides with $\theta$ on $\mathcal{B}(H)$ and Theorem 3.6 is thus proved.

To prove Theorem 3.7 realize the following: If a probability measure $\theta$ on $\mathcal{B}(H)$ is not a Dirac measure then there exists an open set $H_0 \subseteq H$ such that $\theta(H_0) \in (0, 1)$. If we define $H_i = H \setminus H_0$ and $\theta_i(J) := \theta(J \cap H_i)/\theta(H_i)$ for $i \in \{0, 1\}$ and $J \in \mathcal{B}(H)$ then $\nu_\theta = \theta(H_0)\nu_{\theta_0} + \theta(H_1)\nu_{\theta_1}$ and thus $\nu_\theta$ is not extremal. On the other hand, if $\mu_X = \nu_\delta_X = \lambda_0\nu_{\theta_0} + \lambda_1\nu_{\theta_1} = \lambda_0\theta_0 + \lambda_1\theta_1$ holds for some $X \in H$, $\lambda_i > 0$, $\lambda_0 + \lambda_1 = 1$ and for some probability measures $\theta_0, \theta_1$ on $\mathcal{B}(H)$ then $\delta_X = \pi(\nu_\delta_X) = \pi(\lambda_0\theta_0 + \lambda_1\theta_1) = \lambda_0\theta_0 + \lambda_1\theta_1$, hence $\theta_0 = \theta_1$ and $\mu_X$ is thus extremal.

Remark 3.8. If $|A, B| = 0$, $B \neq 0$ and $\nu$ is a probability measure on $K$ then the following three conditions are equivalent.

(i) $\nu$ is an invariant measure for the equation (3.3),

(ii) $S_p(\nu) = \nu$ for every $p \in S^1$, 

(iii) $\nu = S(dx \otimes \nu)$.

Remark 3.9. In a finer look, Theorem 3.5 tells us what happens, as $t \to \infty$, on fibres of $\pi$. If $|A, B| = 0$ then the diffusion (3.3) uniformizes, as $t \to \infty$, the initial distribution $\nu$ along the orbits $S^Z = \{S(p, Z) : p \in S^1\}$ indexed by $Z \in K$. Indeed, let $\nu'$ be the law of the solution (with the initial law $\nu^t$) at time $t \geq 0$ and consider the $\nu$-unique system of conditional probabilities $(\nu' [\cdot|X = x]_H = (\nu'_X)_x \in H$ on $\mathcal{B}(K)$, $\nu'(X) = 1$, to which the measure $\nu'$ desintegrates with respect to $\nu_\infty$. Then it can be verified by the definition of a desintegrated measure that $\nu'_X = S(l_i \otimes \nu'_X)$ where $l_i$ is the law of $\gamma(t)$, hence $\nu'_X$ converges weakly to $\mu_X$ by Theorem 3.5.
3.1. Numerical approximation. The stochastic Landau-Lifshitz-Gilbert equation describes (uniform) atomistic ferromagnetic spin dynamics at finite temperatures; it is of the form \( (3.3) \), with \( A\mathbf{z} = -\mathbf{z} \times h \) and \( B\mathbf{z} = -\mathbf{z} \times (h + h_\perp) \), such that
\[
dz = -\mathbf{z} \times h \, dt - \mathbf{z} \times (h + h_\perp) \, d\mathbf{W}, \quad t \geq 0,
\]
with \( z(0) = z_0 \). Here, \( z_0, h \in \mathbb{R}^3 \) are some unit vectors, and \( h_\perp \in \mathbb{R}^3 \) be perpendicular to \( h \). The Hamiltonian \( \mathcal{E}(\varphi) = -\langle h, \varphi \rangle_{\mathbb{R}^3} \) is conserved by the flow for absent stochastic forcing, or \( h_\perp = 0 \).

We propose a non-dissipative, symmetric, and convergent discretization of \( (3.10) \). Let \( I_k \) be an equi-distant mesh of size \( k > 0 \) covering \([0, T]\). We denote \( \varphi^{n+1/2} := \frac{1}{2}(\varphi^{n+1} + \varphi^n) \).

**Algorithm A.** Let \( Z^n := z_0 \). For \( n \in \mathbb{N} \), find \( Z^{n+1} \in \mathbb{R}^3 \) such that
\[
Z^{n+1} - Z^n = -kZ^{n+1/2} \times h - Z^{n+1/2} \times (h + h_\perp) \Delta W_{n+1}.
\]

where \( \Delta W_{n+1} := W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, k) \).

Let \( k < 1 \). Similar to \([1]\), the \( \mathbb{R}^3 \)-valued random variables \( \{Z^{n+1}\}_{n \geq 0} \) exist, are unique, satisfy \( |Z^{n+1}| = 1 \) for all \( n \geq 0 \), and converge to the solution of \( (3.10) \) for \( k \to 0 \). Moreover, pathwise conservation of energy \( \mathcal{E}(Z^{n+1}) = \mathcal{E}(z_0) \) holds for \( h_\perp = 0 \).

**Remark 3.10.** Let \( \mathcal{Z}^{n+1} := \mathbb{E}Z^{n+1} \). Then for \( \mathcal{H} := \frac{4n}{n+1} \),
\[
\left| \mathcal{Z}^{n+1} - \mathcal{Z}^n + k\mathcal{Z}^{n+1/2} \times h + \frac{k}{2}|h + h_\perp|^2 \left( \mathcal{Z}^{n+1/2} - \langle \mathcal{Z}^{n+1/2}, \mathcal{H} \rangle_{\mathcal{H}} \right) \right| \leq Ck^2.
\]

Hence, the stochastic forcing exerts a damping in direction perpendicular to \( h + h_\perp \). Estimate \( (3.12) \) follows as in the (more detailed) Remark \([4.3]\) below for a second order equation with stochastic forcing. The main ideas are to repeatedly use \( (3.11) \), and approximation arguments that base on \( |Z^{n+1}| = 1 \). As a consequence, the limiting equation reads
\[
\mathcal{Z}_t + \mathcal{Z} \times h + \frac{1}{2}|h + h_\perp|^2 (\mathcal{Z} - \langle \mathcal{Z}, \mathcal{H}\rangle_{\mathcal{H}}) = 0.
\]

3.2. Numerical experiments. We perform computational studies of the stochastic Landau-Lifshitz-Gilbert equation \( (3.10) \) using Algorithm A. The nonlinear system in Algorithm A is solved up to machine accuracy by a fixed-point algorithm, cf. \([1]\). By \( N \) we denote the number of discrete sample paths of the Wiener process used in the computations, and \( Z^n_l \) denotes the numerical solution at time \( t^n \) computed for the \( l \)-th sample path.

The unit sphere is divided into segments \( \omega_{ij} \subset \mathbb{S}^2 \) associated with points
\[
\mathbf{x}_{ij} = (\sin(i\pi/16) \cos(j\pi/16), \sin(i\pi/16) \sin(j\pi/16), \cos(i\pi/16)),
\]
i = 0, ..., 16, \( j = 0, ..., 31 \) such that \( \omega_{ij} = \{ x \in \mathbb{S}^2 | \mathbf{x}_{ij} = \arg \min_{\mathbf{x}_{lm}} |x - \mathbf{x}_{lm}| \} \). For the above partition of the sphere, at a fixed time level \( t^n \), we construct a piecewise constant empirical probability density function \( \hat{f}^n(x) : \mathbb{S}^2 \to \mathbb{R} \) as
\[
\hat{f}^n(x) |_{\omega_{ij}} = \frac{\# \{ l | Z^n_l \in \omega_{ij} \} }{|\omega_{ij}| N},
\]
for \( i = 0, ..., 16, j = 0, ..., 31 \), where \( \# \Omega \) denotes the cardinality of the set \( \Omega \).

The presented results below were computed with \( k = 0.01 \) for \( N = 20000 \) sample paths.

For the first numerical experiment we consider \( z_0 = (0, 1/\sqrt{2}, 1/\sqrt{2}) \), \( h = (0, 0, 1) \), \( h_\perp = (0, 0, 0) \), which corresponds to \([A, B] = 0 \). The resulting probability density for any \( z_0 \) is uniform on a circle with the center \((z_0, h)\), see Figure ???. Further, \( \mathbb{E}(z(t)) \to (z_0, h) \) for \( t \to \infty \) and \( (z(t), h) = (z_0, h) \) for any \( z_0 \) (i.e. the pathwise angle between \( z \) and \( h \) is constant in time) for any \( t \). The computations agree with the statements of Theorems \([3.5, 3.7]\).

In the second experiment we set \( z_0 = (0, 1/\sqrt{2}, 1/\sqrt{2}) \), \( h = (0, 0, 1) \), \( h_\perp = (0, 1, 0) \). In this case we have \([A, B] \neq 0 \) and the initial probability density which is a Dirac measure concentrated at \( z_0 \) is expected to approach the uniform probability density function of the unit sphere \( f^S = (4\pi)^{-1} \approx 0.0796 \). We also expect that \( \mathbb{E}(z(t)) \to 0 \) for \( t \to \infty \). Because of the finite approximation of the problem, the uniform probability density \( f^S \) is only attained approximately. Once the
computed probability density becomes diffuse, it fluctuates randomly around the uniform state \( f^3 \), see Figure ?? . Analogically, the trajectory \( \mathbb{E}(u(t)) \) approaches the center of the sphere and for long times fluctuates randomly around the center, see Figure ?? . The random fluctuations in the probability density function \( \bar{f}^n \) can be significantly reduced by taking the average over a sufficiently large number of time levels. Here, we compute the time averaged probability density function \( \overline{f} \) over the last 100 time levels, i.e., we take \( \overline{f}(x) = \frac{1}{100} \sum_{k=100}^{T} \bar{f}^n(x) \), see Figure ?? . The results in the second numerical experiment agree with the assertion of Theorem 3.4.

4. Example II

The geodesic equation on the sphere has the form \( d\dot{u} = -|\dot{u}|^2 u \ dt \). Consider this second order equation with a stochastic perturbation

\[
(4.1) \quad \dot{u} = -|\dot{u}|^2 u \ dt + (u \times \dot{u}) \circ dW
\]

which is a Stratonovich SDE \( dz = F(z) \ dt + G(z) \circ dW \) on the tangent bundle \( TS^2 \subseteq \mathbb{R}^6 \) driven by a standard one-dimensional Wiener process \( W \) where \( u \times \dot{u} \in \mathbb{R}^3 \) is the outer product in \( \mathbb{R}^3 \) and

\[
F(p,\xi) = \begin{pmatrix} \xi \\ -|\xi|^2 p \end{pmatrix}, \quad G(p,\xi) = \begin{pmatrix} 0 \\ p \times \xi \end{pmatrix}, \quad [G,F](p,\xi) = \begin{pmatrix} p \times \xi \\ 0 \end{pmatrix}, \quad (p,\xi) \in TS^2
\]

are tangent vector fields on \( TS^2 \). Observe that \( F \), \( G \) and \( [G,F] \) are also orthogonal tangent vector fields on the 3-dimensional submanifold

\[
(4.2) \quad M_r = \{(p,\xi) \in TS^2 : |\xi| = r\}, \quad r > 0
\]

hence the hypothesis (H) is verified. If we put \( E_1 = F/(r^2 + r^4)^{\frac{1}{2}}, \quad E_2 = G/r, \quad E_3 = [G,F]/r \) then \( \{E_1, E_2, E_3\} \) is an othonormal frame on \( M_r \) and

\[
\text{div} \ Y = \sum_{j=1}^3 (dE_j Y, E_j)_{\mathbb{R}^6} = 0, \quad Y \in \{F,G,[F,G]\}
\]

where \( d_X Y(p) = \lim_{t \to 0} t^{-1}[Y(p + tX) - Y(p)] \) and so the hypothesis (D) is holds. Next, if \( [G,F]f = Gf = 0 \) on \( M_r \) for some \( f \in C^\infty(M_r) \) then \( f \) is constant along any integral curve \( \dot{\gamma} = [G,F] \gamma \) or \( \dot{\delta} = G \delta \). Since

\[
\text{Rng} \gamma = \{(p,\xi) \in M_r : p = \gamma_1(0)\} \quad \text{and} \quad \text{Rng} \delta = \{(p,\xi) \in M_r : \xi = \delta_2(0)\},
\]

any two points in \( M_r \) can be connected by a piecewise-smooth curve consisting of at most three integral curves of \( [G,F] \) and \( G \) hence the hypothesis (F) is satisfied.

Finally, let \( C_l \) denote the space spanned by polynomials on \( \mathbb{R}^6 \) of order smaller or equal to \( l \). Then \( F[C_l] \cup G[C_l] \subseteq C_l \) hence \( C_l \) is invariant for the operator \( A \) defined in (2.2) and \( \bigcup_{l=0}^\infty C_l \) is dense in \( C(M_r) \) by the Stone-Weierstrass theorem and the hypothesis (C) holds.

Let \( \mu_r \) be the probability measure on \( TS^2 \) supported on \( M_r \) for \( r > 0 \) such that the restriction of \( \mu_r \) on \( R(M_r) \) coincides with the normalized Riemannian volume measure on \( M_r \). If \( \nu \) is a probability measure on \( TS^2 \), we define the measures

\[
\nu_*(U) = \nu \{(p,\xi) \in TS^2 : |\xi| \in U\}, \quad U \in \mathcal{B}(0,\infty)
\]

\[
\bar{\nu}(A) = \nu (A \cap M_0) + \int_{(0,\infty)} \mu_r(A) \ d\nu_*, \quad A \in \mathcal{B}(TS^2).
\]

**Theorem 4.1.** Let \( (u,\dot{u}) \) be a solution of (4.1) on \( TS^2 \) with an initial distribution \( \nu \) on \( \mathcal{B}(TS^2) \). Then the laws of \( (u,\dot{u}) \) converge weakly to \( \bar{\nu} \) as \( t \to \infty \). Moreover, \( \nu \) is invariant for (4.1) iff \( \nu = \bar{\nu} \) and \( \nu \) is ergodic for (4.1) iff \( \nu = \delta_{(p,0)} \) for some \( p \in S^2 \) or \( \nu = \mu_r \) for some \( r > 0 \).

**Proof.** The spaces \( \{(p,0)\}_{p \in S^2} \) and \( M_r \) for \( r > 0 \) are invariant for (4.1). Hence, if \( f \in C(TS^2) \) then \( \int_{TS^2} f \ dp_{\nu(t,u)} \) converges to \( f(p,0) \) if \( x = (p,0) \) or it converges to \( \int_{TS^2} f \ d\mu_r \) if \( x = (p,\xi) \) and \( |\xi| = r > 0 \) by Theorem 2.3 since \( \mu_r \) is the unique invariant and also attractive probability measure for (4.1) on \( M_r \). The ergodicity follows from Remark 4.2 as ergodic probability measures
Theorem 4.3. Let \( n \) unique \( R \) product formula auxiliary problem are in fact solutions of (4.3).

Remark 4.2. Invariant measures for (4.1) can be uniquely described as measures

\[
\nu_{a,b}(A) = a(A \cap M_0) + \int_{(0,\infty)} \mu_r(A) \, db, \quad A \in \mathcal{B}(TS^2)
\]

where \( a \) and \( b \) are finite measures on \( \mathcal{B}(M_0) \) and on \( \mathcal{B}(0, \infty) \) so that \( a(M_0) + b(0, \infty) = 1 \).

4.1. Numerical approximation. We propose a non-dissipative, symmetric discretization of (4.1) to construct strong solutions and numerically study long-time asymptotics. Let \( \{(U^n, V^n)\}_n \) be approximate iterates of \( \{(u(t_n), \dot{u}(t_n))\}_n \) on an equi-distant mesh \( I_k \) of size \( k > 0 \), covering \([0, T]\). We denote \( d_t \varphi^{n+1} := \lambda_k (\varphi^{n+1} - \varphi^n) \).

Algorithm B. Let \((U^0, V^0) := (u_0, \dot{u}(0))\), and \( U^{-1} := U^0 - kV^0 \). For \( n \geq 0 \), find \((U^{n+1}, V^{n+1}, \lambda^{n+1}) \in \mathbb{R}^{3+3+1} \), such that for \( \Delta W_{n+1} := W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0,k) \) holds

\[
V^{n+1} - V^n = \frac{k}{2} (U^{n+1} + U^n) + \frac{1}{4} (U^{n+1} + U^n - 1) \times (V^{n+1} + V^n) \Delta W_{n+1} \tag{4.3}
\]

\[
d_t U^{n+1} = V^{n+1}
\]

\[
\lambda^{n+1} = \begin{cases} 
0 & \text{for } \frac{1}{2}(U^{n+1} + U^n) = 0, \\
 \frac{\langle V^n, V^{n-1} \rangle}{(U^{n+1} + U^n - 1)^2} & \text{for } \frac{1}{2}(U^{n+1} + U^n) \neq 0 \text{ and } n \geq 1, \\
 \frac{\langle V^n, V^{n-1} \rangle - \frac{1}{2} |V^n|^2}{(U^{n+1} + U^n - 1)^2} & \text{for } \frac{1}{2}(U^{n+1} + U^n) \neq 0 \text{ and } n = 0.
\end{cases}
\]

The choice of the Lagrange multiplier \( \lambda^{n+1} \) ensures that \( |U^{n+1}| = 1 \) for \( n \geq 0 \); the case \( n = 0 \) has to compensate for the fact that the defined \( U^{-1} \) is not necessarily of unit length.

To see the formula (4.3) for \( n \geq 1 \), we multiply (4.3) with \( \frac{1}{4}(U^{n+1} + U^n) \) and use the discrete product formula

\[
(d_t \varphi^n, \psi^n) = -(-\varphi^{n-1}, d_t \psi^n) + d_t (\varphi^n, \psi^n)
\]

to find

\[
\frac{1}{2}(d_t V^{n+1}, U^{n+1} + U^n) = -\frac{1}{2}(V^n, V^{n+1} + V^n) + \frac{1}{2} d_t (V^{n+1}, U^{n+1} + U^n - U^n + U^{n-1}),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^3 \), and \( |\cdot| = (\cdot, \cdot)^{1/2} \). Since \( (V^{n+1}, U^{n+1} + U^n) = 0 \), we further obtain

\[
= -\frac{1}{2} \left( (V^n, V^{n+1} + V^{n-1}) + kd_t (V^{n+1}, -V^n) \right)
\]

\[
= -\frac{1}{2} \left( (V^n, V^{n+1} + V^{n-1} - (V^{n+1}, V^n) + (V^n, V^{n-1}) \right) = -(V^n, V^{n-1}).
\]

Hence \( -(V^n, V^{n-1}) = \lambda^{n+1} \frac{1}{2}(U^{n+1} + U^{n-1})^2 \), which yields the formula for \( \lambda^{n+1} \) in (4.3).

For \( n = 0 \), we conclude similarly, using \( \langle U^0, V^0 \rangle = 0 \), and the definition of \( U^{-1} \).

Theorem 4.3. Let \( T > 0 \), and \( k \leq k_0(U^0, V^0) \) be sufficiently small. For every \( n \geq 0 \), there exist unique \( \mathbb{R}^{3+3} \)-valued random variables \((U^{n+1}, V^{n+1})\) of Algorithm A such that \( |U^{n+1}| = 1 \) for all \( n \geq 0 \), and

\[
E(V^{n+1}) = E(V^0) \quad \text{where } E(\varphi) = \frac{1}{2} |\varphi|^2.
\]

Moreover, \( \{(U^{n+1}, V^{n+1})\}_{n \geq 0} \) construct strong solutions of (4.3) for \( k \rightarrow 0 \), in the way specified in the proof below.

Solvability of (4.3) will be shown by an inductive argument that is based on Brouwer’s fixed point theorem: an auxiliary problem is introduced which excludes the case where \( \frac{1}{2}(U^{n+1} + U^{n-1}) = 0 \) when computing \( \lambda^{n+1} \); for sufficiently small \( k > 0 \), constructed solutions of the auxiliary problem are in fact solutions of (4.3).
Proof. 1. Auxiliary problem. Fix \( n \geq 1 \). For every \( 0 < \epsilon \leq \frac{1}{2} \), define the continuous function \( \mathcal{F}_\epsilon : \mathbb{R}^3 \to \mathbb{R}^3 \) where

\[
\mathcal{F}_\epsilon(W) := \frac{2}{k} (W - U^n) + k \frac{(V^n, V^{n-1})}{\max\{|W|^2, \epsilon\}} W - W \times (V^n - \frac{2}{k} U^n) \Delta W_{n+1}.
\]

We compute respectively,

\[
\frac{2}{k} (W - U^n, W) \geq \frac{2}{k} (|W| - |U^n|) |W|,
\]

\[
k \frac{(V^n, V^{n-1})}{\max\{|W|^2, \epsilon\}} |W|^2 \geq -k |V^n| |V^{n-1}|.
\]

Since the stochastic term in (4.4) vanishes after multiplication with \( W \), there exists some function \( R_n > 0 \) such that

\[
(\mathcal{F}_\epsilon(W), W) \geq 0 \quad \forall W \in \{ \varphi \in \mathbb{R}^3 : |\varphi| \geq R_n(U^n, V^n, V^{n-1}) \}.
\]

Then, Brouwer’s fixed point theorem implies the existence of \( W^* \equiv \frac{1}{2}(U^{n+1} + U^{n-1}) \), such that \( \mathcal{F}_\epsilon \left( \frac{1}{2}(U^{n+1} + U^{n-1}) \right) = 0 \) for all \( \omega \in \Omega \).

The argument easily adapts to the case \( n = 0 \).

2. Solvability and energy identity. We show that \( \frac{1}{2}(U^{n+1} + U^{n-1}) \) solves \( \mathcal{F}_0 \left( \frac{1}{2}(U^{n+1} + U^{n-1}) \right) = 0 \).

By induction, it suffices to verify that

\[
\left| \frac{1}{2}(U^{n+1} + U^{n-1}) \right| = \left| \frac{k}{2} (V^n + U^n) + \frac{1}{2} (U^n + U^{n-1}) \right| \geq \left| \frac{1}{2} (U^n + U^{n-1}) \right| - \frac{k}{2} |V^{n+1}|
\]

\[
\geq |U^{n-1}| - \frac{k}{2} (|V^n| + |V^{n+1}|)
\]

\[
\geq 1 - \frac{1}{4} + \frac{k}{2} |V^{n+1}| > \frac{1}{2},
\]

for \( k \leq k_0(U^0, V^0) < 1 \) sufficiently small.

Let \( n \geq 1 \). For all \( 0 \leq \ell \leq n \), there holds \( |U^\ell| = 1 \), and

\[
E(V^\ell) = E(V^0).
\]

Then \( W^* = \frac{1}{2}(U^{n+1} + U^{n-1}) \) from Step 1. solves

\[
kd_{n} V^{n+1} = \frac{\lambda_{\epsilon}^{n+1} k}{2} (U^{n+1} + U^{n-1}) + \frac{1}{4} (U^n + U^{n-1}) \times (V^n + V^{n+1}) \Delta W_{n+1},
\]

where

\[
\lambda_{\epsilon}^{n+1} = - \frac{\max\{|\epsilon|, \frac{1}{2}(U^{n+1} + U^{n-1})|2\}}{\max\{\epsilon, (\frac{1}{2}U^{n+1} + U^{n-1})|2\}}.
\]

Testing (4.7) with \( \frac{1}{2}(U^{n+1} - U^{n-1}) = \frac{k}{2}(V^n + V^{n+1}) \), and using binomial formula \( \frac{1}{2}(U^{n+1} + U^{n-1}) = \frac{1}{2}(|U^{n+1}|^2 - 1) \), as well as \( |U^{n+1}|^2 \leq k^2 |V^{n+1}|^2 + |U^n|^2 \), and the inductive assumption \( |U^n|^2 = 1 \),

\[
d_{\epsilon} |V^{n+1}|^2 \leq \frac{\lambda_{\epsilon}^{n+1} k^2 |V^{n+1}|^2}{4}
\]

\[
\leq \frac{|V^n| |V^{n-1}| |k^2 |V^{n+1}|^2}{4 \max\{|\epsilon|, (1 - \frac{1}{2} - k|V^{n+1}|)|2\}} \leq \frac{k^2}{4\epsilon} |V^n| |V^{n-1}| |V^{n+1}|^2,
\]

for \( \epsilon \leq \frac{1}{2} \). By a (repeated) use of the discrete version of Gronwall’s inequality, there exists a constant \( C > 0 \) independent on time \( T > 0 \), such that

\[
|V^{n+1}|^2 \leq C |V^0|^2.
\]

As a consequence, (4.3) is valid, and hence \( \mathcal{F}_\epsilon \left( \frac{1}{2}(U^{n+1} + U^{n-1}) \right) = 0 \) for \( \epsilon = 0 \); therefore, \( U^{n+1} \) solves (4.3), satisfies the sphere constraint, and conserves the Hamiltonian, i.e., (4.6) holds for \( 0 \leq \ell \leq n + 1 \).
For $n = 0$, we argue correspondingly, starting in (4.3) with
\[
d_{t}|V|^{2} \leq C\lambda_{1}^{2}|U^{1} + U^{-1}|^{2} \leq \frac{k|V^{0}|(|V^{0}| + |V^{1}|)^{2}}{\max\{\epsilon, \frac{1}{2}(U^{1} + U^{-1})\}},
\]
from which we again infer (4.10), and (4.15).

Uniqueness of \((U^{n}, V^{n})_{n}\) follows by an energy argument, using (4.5), (4.6), $k \leq k_{0}$, and the discrete version of Gronwall’s inequality.

3. Convergence. We rewrite (4.3) in the form
\[
\begin{align*}
dv &= -|v|^{2}u \, dt + u \times v \, dW, \\
du &= v \, dt;
\end{align*}
\]
(4.10)
\[
(\psi_{k}^{t}, \varphi_{k}^{t}) \to (u, v) \quad \text{in } C([0, T]; \mathbb{R}) \quad \text{as } (k' \to 0),
\]
where \((u, v)\) is strong solution of (4.3).

It is because of the discrete sphere constraint and the (discrete) energy identity that sequences
\[
\{(\psi_{k}^{t}, \varphi_{k}^{t})\}_{k} \subset C([0, T]; \mathbb{R})
\]
are uniformly bounded. Moreover, there holds for all $t \geq 0$
\[
\begin{align*}
\psi(t) &= \psi(0) + \int_{0}^{t} \frac{\lambda}{2}[\psi^{+} + \psi^{-} - k\psi^{-}] \, ds \\
&\quad + \frac{1}{4} \int_{0}^{t} (\psi^{+} + \psi^{-} - k\psi^{-}) \times (\psi^{+} + \psi^{-}) \, dW(s),
\end{align*}
\]
(4.12)
\[
\begin{align*}
\varphi(t) &= \varphi(0) + \int_{0}^{t} \psi(s) \, ds.
\end{align*}
\]
Then, by (4.12), (4.3), and Hölder continuity property of $W$, sequences \{(\psi_{k}, \varphi_{k})\}_{k} are equi-continuous. Hence, by Arzela-Ascoli theorem, there exist sub-sequences \{(\psi_{k'}^{t}, \varphi_{k'}^{t})\}_{k'} and continuous processes \((\psi, \varphi)\) on $[0, T]$ such that
\[
\begin{align*}
\|\psi_{k'} - u\|_{C([0, T]; \mathbb{R}^{3})} + \|\varphi_{k'} - v\|_{C([0, T]; \mathbb{R}^{3})} \to 0 \quad \text{as } (k' \to 0) \quad \mathbb{P} \text{- a.s.}
\end{align*}
\]
We identify limits in (4.12). The only crucial term is the stochastic (Ito) integral term which may be stated in the form
\[
\begin{align*}
\frac{1}{2} \int_{0}^{t} (\psi^{+} + \psi^{-} - k\psi^{-}) \times (\psi^{-} + k\psi^{+}) \, dW(s).
\end{align*}
\]
(4.14)
We easily find for every $t \in [0, T]$,
\[
\frac{1}{2} \int_{0}^{t} (\psi^{+} + \psi^{-} - k\psi^{-}) \times \psi^{-} \, dW(s) \to \int_{0}^{t} u \times v \, dW(s) \quad \text{as } (k \to 0) \quad \mathbb{P} \text{- a.s.}
\]
(4.15)
The remaining term in (4.14) involves $\frac{1}{2} \psi^{+}$, which will be substituted by identity (4.3.1),
\[
\begin{align*}
\frac{1}{2} \int_{0}^{t} (\psi^{+} + \psi^{-} - k\psi^{-}) \times (\psi^{-} + \frac{k}{4}(\psi^{+} + \psi^{-} - k\psi^{-}) \\
&\quad + \frac{1}{4}(\psi^{+} + \psi^{-} - k\psi^{-}) \times (\psi^{-} + \frac{k}{2} \psi^{+}) \Delta W_{n+1}) \, dW(s).
\end{align*}
\]
If compared to (4.14), the critical factor \( \frac{1}{2} \dot{Y} \) is now scaled by an additional \( \Delta W_{n+1} \); using again (4.3), Ito’s isometry, and the estimate \( \mathbb{E}[\Delta W_{n+1}]^2 \leq Ck^{2p-1} \) then lead to

\[
\frac{1}{8} \int_0^t (\mathbb{W}^+ + \mathbb{W}^-) \times \left( (\mathbb{W}^+ + \mathbb{W}^- - kY^-) \times (Y^- + \frac{1}{2} \dot{Y}) \right) (W^+ - W^-) dW(s)
\]

(4.16)

\[
\rightarrow \frac{1}{2} \int_0^t u \times (u \times v) ds \quad (k \to 0) \quad \mathbb{P} \text{-a.s.,}
\]

for all \( t \in [0,T] \). As a consequence, there holds

\[
v(t) = v(0) + \int_0^t |v|^2 u \, ds + \int_0^t u \times v \, dW(s) + \frac{1}{2} \int_0^t u \times (u \times v) ds,
\]

where the last term is the Stratonovich correction. The proof is complete.

\[\square\]

**Remark 4.4.** 1. Let \( |V^0| \) be constant, and \( (\mathbb{Y}^{n+1}, \mathbb{W}^{n+1}) := (\mathbb{E}V^{n+1}, \mathbb{E}U^{n+1}) \). Then

\[
|\mathbb{Y}^{n+1} - \mathbb{Y}^n - k[\mathbb{E}[V^{n+1}]^2 \mathbb{W}^{n+1} - \frac{1}{2} \mathbb{Y}^{n+1}]| \leq Ck^2,
\]

(4.17)

i.e., the stochastic forcing term exerts a damping in direction \( \mathbb{Y}^{n+1} \). To show this result, we start with

\[
\mathbb{Y}^{n+1} - \mathbb{Y}^n = \frac{k}{2} \mathbb{E}[\Lambda^{n+1}(U^{n+1} + U^{n-1})] + \frac{1}{2} \mathbb{E}[(U^{n+1} + U^{n-1}) \times V^{n+1/2} \Delta W_{n+1}]
\]

(4.18)

\( =: I + II \).

We use Theorem 4.3 and an approximation argument to conclude that

\[
I = -\frac{k}{2} \mathbb{E}[(V^n, V^{n-1})(1 - \frac{1}{(U^{n+1} + U^{n-1})^2})(U^{n+1} + U^{n-1})]
\]

\[
= -\frac{k}{2} \mathbb{E}[(V^n, V^{n-1})(\mathbb{W}^{n+1} + \mathbb{W}^{n-1})] + \mathcal{O}(k^3)
\]

\[
= -k \mathbb{E}[V^n \mathbb{W}^{n+1}] + \mathcal{O}(k^2),
\]

thanks to the power property of expectations, and \( |\frac{1}{2}(U^{n+1} + U^{n-1})^2 - 1| \leq Ck^2 \).

We use the identity \( U^{n+1} = kV^{n+1} + U^n \) for the leading term in \( I \), and properties of the vector product to conclude that

\[
II = \frac{k}{4} \mathbb{E}[(V^{n+1} - V^n) \times V^n \Delta W_{n+1}] + \frac{1}{4} \mathbb{E}[(U^n + U^{n-1}) \times (V^{n+1} - V^n) \Delta W_{n+1}] =: II_a + II_b.
\]

We easily verify \( |II_a| \leq Ck^2 \), thanks to (4.3), and properties of iterates given in Theorem 4.3. For \( II_b \), we use (4.3) as well, and the relevant term is then

\[
\frac{1}{16} \mathbb{E}[(U^n + U^{n-1}) \times ((U^{n+1} + U^{n-1}) \times (V^{n+1} + V^n)) \Delta_{n+1}^2]
\]

\[
= \frac{1}{2} \mathbb{E}[U^n \times (U^n \times V^n)] \Delta_{n+1}^2 + \mathcal{O}(k^2)
\]

\[
= -\frac{k}{2} \mathbb{Y}^n + \mathcal{O}(k^2) = -\frac{k}{2} \mathbb{Y}^{n+1} + \mathcal{O}(k^2),
\]

thanks to the power property of expectations, earlier boundedness results of iterates \( \{U^n, V^n\} \), the fact that \( |U^n| = 1 \) for all \( n \geq 0 \), the cross product formula \( a \times (b \times c) = b(a, c) - c(a, b) \), and another approximation argument. This observation then settles (4.17).

2. Strong solutions of (4.11) satisfy

\[
|u(t)| = 1, \quad E(v(t)) = E(v_0) \quad \forall t \in [0, T],
\]

and are unique, due to Lipschitz continuity of coefficients in (4.1); hence, the whole sequence \( \{ (\mathbb{W}_k, \mathbb{F}_k) \} \) converges to \( (u, v) \), for \( k \to 0 \).

3. Increments of a Wiener process in Algorithm A may be approximated by a sequence of general, not necessarily Gaussian random variables, which properly approximate higher moments.
of \( \Delta W_{n+1} \); martingale solutions of (4.1) may then be obtained by a more involved argumentation from Algorithm A as well, using theorems of Prohorov and Skorokhod; cf. [4].

4.2. Numerical experiments. In this section we present some numerical obtained by the Algorithm B that has been applied to a slightly more general problem than (4.1).

\[
d\dot{u} = -|\dot{u}|^2 u \, dt + \sqrt{D}(u \times \dot{u}) \circ dW,
\]

where \( D \) is a fixed constant that controls the intensity of the noise term. The Lagrange multiplier was computed as

\[
\lambda^{n+1} = \frac{-(V^n, U^{n+1} + U^{n-1}) + \frac{1}{2}(1 - \|U^n\|^2)}{\|1/2(U^{n+1} + U^{n-1})\|^2}.
\]

The above formula is equivalent to the corresponding expression in (4.3). However, the present formulation (4.19) is slightly more convenient for numerical computations, since it ensures that the round off errors in the constraint \( \|U^n\| = 1 \) do not accumulate over time. The solution of the nonlinear scheme (4.3) is obtained up to machine accuracy by a simple fixed-point algorithm, cf. [2].

The probability density function \( \hat{f}^n \) was constructed as in Example I with \( N = 20000 \) sample paths. For all computations in this section we take the time step size \( k = 0.001 \) and the initial conditions \( u(0) = (0, 1, 0), \dot{u}(0) = (1, 0, 0) \). The initial probability density function associated with the above initial conditions is a Dirac delta function concentrated around \( u(0) \).

In Figure 1 we display the computed probability density \( \hat{f}^n \) for \( D = 1, T = 60 \) at different time levels. Initially the probability density function is advected in the direction of the initial velocity and is simultaneously being diffused. For early times, the diffusion seems to act predominantly in the direction perpendicular to the initial velocity. In Figure 2 we display the time averaged probability density function \( \hat{f} \), the trajectory \( \mathbb{E}(u(t)) \) and a zoom at \( \mathbb{E}(u(t)) \) near the center of the sphere.

The evolution of the probability density for \( D = 10, T = 60 \) is shown in Figure 2. Similarly as in the previous experiment the probability density function diffuses and becomes uniform for large time. Some advection in the direction of the initial velocity can still be observed, however, the overall process has a predominantly diffusive character. We observe that the overall evolution damped due to the effects of the random forcing term, see Remark (4.3) (1) and Figure 1. In Figure 2 we display the time averaged probability density function, the trajectory \( \mathbb{E}(u(t)) \) and a zoom at \( \mathbb{E}(u(t)) \) near the center.

Figure 3 contains the computed trajectories of \( \mathbb{E}(u(t)) \) for \( D = 0.1 \) and \( D = 100 \). The respective probability densities asymptotically converge towards the uniform distribution for large time.

In Figure 4 we show the graphs of the time evolution of the approximate error \( \mathcal{E}_{\text{max}}^n : t^n \to \max_{\mathbf{x} \in S^2} |f^n(\mathbf{x}) - f^S| \) for \( D = 0.01, 0.1, 1, 10, 100 \). The quantity \( \mathcal{E}_{\text{max}}^n \) serves as an approximation of the distance from the uniform probability distribution in the \( L_\infty \) norm. Note that the oscillations in the error graphs are due to the approximation of the probability density. The numerical experiments provide evidence that the probability densities for all \( D \) converges towards the uniform probability density \( f^S \) for \( t \to \infty \). The probability density evolutions for decreasing values of \( D \) have an increasingly “advective” character and the evolutions for increasing values have an increasingly “diffusive” character. It is also interesting to note, that the convergence towards the uniform distribution becomes slower for both increasing and decreasing values of \( D \).

**Figure 1.** Convergence of the probability distribution of \( u \) towards a uniform distribution for different values of the coefficient \( D \).

In the last experiment we study the long time behavior of the pair \( (u, \dot{u}) \) for \( D = 1, N = 20000 \). Towards this end, we introduce a partition of the manifold \( M_1 \) defined in (4.2). First, we consider a partition of the unit sphere into segments \( \omega^i_N, i = 1, \ldots, 6 \) associated with the
Proposition A.2. Assume that $L$. BAŇAS, Z. BRZEŹNIAK, M. ONDREJÁT & A. PROHL

Proposition A.1. Let us write $\mathcal{L}(\mathcal{X})$ the smallest Lie algebra containing $\mathcal{X}$. If $A \subseteq \mathcal{L}$ and $p \in U$, then we define $A(p) = \{ A_p : A \in A \}$. One can verify by symmetries of this partition that the normalized surface volume of each $M_i$ is equal to $1/48$. For $n = 60000$ (i.e., at time $t^n = 60$) we have for $i = 1, \ldots, 6, j = 1, \ldots, 8$ #\{ |U^n_i | \in \omega_i^2 \} \approx N/6 = 3333 and #\{ |(U^n_i , V^n_i ) | \in M^n_i \} \in (386, 455) \approx N/6/8 = 417, see Figure ?? left and Figure ?? right, respectively. The numerical experiments indicate that the point-wise probability measure for $(u, \dot{u})$ converges to the invariant measure $\mu$. The (rescaled) approximate $L_\infty$ error $\mathcal{E}_{\text{max}}$ for $(u, \dot{u})$ has similar evolution as the approximate $L_\infty$ error for $u$. Moreover, it seems that the convergence of the error in time is exponential, see Figure ??.

APPENDIX A. LIE ALGEBRA

Let $U$ be an open set on a $C^\infty$-manifold.

- The set $\mathcal{L}$ of all smooth tangent vector fields on $U$ is a vector space with the Jacobi bracket. Any vector subspace of $\mathcal{L}$ closed under the Jacobi bracket is called a Lie algebra.
- If $\mathcal{X}$ is a set of smooth tangent vector fields on $U$, then we denote by $\mathcal{L}(\mathcal{X})$ the smallest Lie algebra containing $\mathcal{X}$.
- If $A \subseteq \mathcal{L}$ and $p \in U$, then we define $A(p) = \{ A_p : A \in A \}$.

Proposition A.1. Define $L_0 = \text{span} \{ \mathcal{X} \}$ and $L_n = \text{span} \{ L_{n-1} \cup \{ [A, B] : A, B \in L_{n-1} \} \}$. Then $\bigcup L_n = \mathcal{L}(\mathcal{X})$.

Proposition A.2. Assume that $\mathcal{X} \subset \mathcal{L}$. Let $X_1, \ldots, X_m, Y \in \mathcal{L}$ and let $f_i \in C^\infty(U)$. Then

$$\mathcal{L}(X_1, \ldots, X_m, Y)(p) = \mathcal{L}(X_1, \ldots, X_m, Y + \sum_{j=1}^m f_j X_j)(p), \quad p \in U.$$

Proof. Let us write $A_1 = \{ X_1, \ldots, X_m, Y \}$, $A_2 = \{ X_1, \ldots, X_m, Y + \sum_{j=1}^m f_j X_j \}$,

$$\mathcal{G}^i = \left\{ \sum_{k=1}^K h_k L_k : h_k \in C^\infty(U), L_k \in \mathcal{L}(A^j), K \in \mathbb{N} \right\}.$$

Apparently, $\mathcal{G}^i$ is a Lie algebra for $i \in \{ 1, 2 \}$, $\mathcal{A}^i \subseteq \mathcal{G}^j$ whenever $\{ i, j \} = \{ 1, 2 \}$ hence $\mathcal{L}(A^i) \subseteq \mathcal{G}^j$ whenever $\{ i, j \} = \{ 1, 2 \}$. But then

$$\mathcal{L}(A^i)(p) \subseteq \mathcal{G}^j(p) \subseteq \mathcal{L}(A^j)(p).$$

□

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