ON THE EQUIVALENCE OF QUATERNIONIC CONTACT STRUCTURES

IVAN MINCHEV AND JAN SLOVÁK

Abstract. Following the Cartans’s original method of equivalence supported by methods of parabolic geometry, we provide a complete solution for the equivalence problem of quaternionic contact structures. This includes an explicit construction of the corresponding Cartan geometry and detailed information on all curvature components.

 CONTENTS

1. Introduction 1
2. Preliminaries 4
  2.1. Conventions concerning the use of complex tensors and indices 4
  2.2. Quaternionic contact manifolds 5
3. Solution to the quaternionic contact equivalence problem 6
4. The Curvature and the Bianchi identities 18
5. The associated Cartan geometry 30
  5.1. A few algebraic constructions 30
  5.2. The normal Cartan connection 33
6. Appendix 34
  6.1. Cartan geometries 34
  6.2. Parabolic geometries 35
  6.3. Regular filtrations on manifolds 35
  6.4. The curvature 36
References 36

1. Introduction

There is the series of important geometries naturally appearing at the generic hypersurfaces in projective spaces. The Klein models $G \to G/P$ for all of them are spheres, i.e. the conformal Riemannian sphere $S^n \subset \mathbb{R}P^{n+1}$, the CR-sphere $S^{2n+1} \subset \mathbb{C}P^{n+1}$, and the quaternionic contact sphere $S^{4n+3} \subset \mathbb{H}P^{n+1}$, respectively, or other nice homogeneous spaces in the cases of other than positive definite signatures.

All these geometries appear as boundaries of domains, carrying a lot of information – let us mention the conformal horizons in mathematical physics, the boundaries of domains in complex analysis and function theory, and the boundaries of quaternionic-Kähler domains.

The corresponding Lie algebras enjoy very similar algebraic structures with gradings

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$
where \( \mathfrak{g}_0 \) further splits as \( \mathfrak{h} \oplus \mathfrak{g}'_0 \), as indicated symbolically in the matrix (the * entries mean those computed from the symmetries of the matrix)

\[
\begin{pmatrix}
\mathfrak{h} & \mathfrak{g}_1 & \mathfrak{g}_2 \\
\mathfrak{g}'_0 & \mathfrak{g}_0 & * \\
\mathfrak{g}_{-2} & * & *
\end{pmatrix}
\]

The corresponding Lie algebras \( \mathfrak{g} \) are \( \mathfrak{so}(p+1,q+1) \), \( \mathfrak{su}(p+1,q+1) \), and \( \mathfrak{sp}(p+1,q+1) \). Thus viewing them as matrix algebras over \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \), they always have columns and rows of width \( 1, n, 1 \), respectively, and \( \mathfrak{h} = \mathbb{K}, \mathfrak{g}_{-1} = \mathbb{K}^n, \mathfrak{g}_1 = \mathbb{K}^{n*}, \mathfrak{g}_2 \) is the imaginary part of \( \mathbb{K} \) (thus vanishing in the case \( \mathbb{K} = \mathbb{R} \) and \( \mathfrak{g}_0' \) is the algebra of the same type as \( \mathfrak{g} \) of signature \( (p,q) \).

All these geometries fit into the class of Cartan geometries with \( G \) semisimple and \( P \) parabolic and thus there is the rich general theory explaining the cohomological character of basic invariants. The constructions of the relevant normalized Cartan connections and detailed analysis of its curvature is well known for decades in the first two cases, but much less is known in the case of quaternionic contact geometries. This is perhaps due to the much higher complexity of the analysis to be expected.

Our aim is to fill this gap and provide a full analogy to the construction of the normal Cartan connection by Chern and Moser in their paper [5], including detailed information on all curvature components. We shall come back to further motivation for this endeavor below.

We are going to deliver our construction in terms of the most classical exterior calculus and it should be completely understandable without direct insight into the general cohomological structure of the curvature. But of course, it is this knowledge which allows us to know in advance that the individual steps will work.

The first step in understanding the difficulty is the definition of the geometry itself. While the conformal geometry has got the trivial filtration on the tangent bundle and the geometry itself is one of the most classical \( G \)-structures, the CR geometry is already defined by a contact distribution \( T^{-1}M \subset TM \) with a further reduction of the associated graded tangent bundle to a structure group respecting the additional complex structure on the distribution.

Let us now clarify the CR case carefully from a more abstract point of view. The book [4] can be consulted for both the general theory and details on CR structures.

The homogeneity zero component of the first Lie algebra cohomology \( H^1(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})_0 \) is nontrivial in this case, thus the extra reduction of the frame bundle. Moreover, the second cohomology \( H^2(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \) is nontrivial in homogeneities one and two (in all dimensions \( \dim M \geq 5 \)). In particular, there is no cohomology in homogeneity zero and thus the algebraic Lie bracket on \( \text{Gr } TM \) induced by the Lie bracket of vector fields has to coincide with the Lie bracket on \( \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \) from the Lie algebra in question. This means that algebraic bracket on \( \text{Gr } TM \) has to be the imaginary part of a hermitian form on \( T^{-1}M \). Further, the cochains generating the second cohomology in homogeneity one are of the type \( \Lambda^2 \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1} \) (obstructing the integrability of the complex structure \( J \) on \( T^{-1}M \)). This is the torsion of the canonical Cartan curvature, which automatically vanishes in the case of embedded hypersurfaces in \( \mathbb{C}^{n+1} \).

Another important cohomological information is the automatic vanishing of the cohomology with cochains of the type \( \mathfrak{g}_{-2} \otimes \mathfrak{g}'_{-1} \otimes \mathfrak{g}_{-2} \), also in homogeneity one. Indeed, any choice of a contact form \( \theta \) defining the CR-distribution will split the tangent bundle to \( TM = T^{-1}M \oplus T^{-2}M \), identify \( T^{-2}M \) with \( M \times \mathbb{R} \) (via the Reeb vector field of \( \theta \)), the algebraic Lie bracket will get a symplectic form, and thus together with the complex structure \( J \) we also get the Levi-Civita connection for all derivatives in the directions of \( T^{-1}M \). The latter cohomological information implies that all these objects are fully in compliance with the canonical Cartan connection for the structure.

Now, we come to the quaternionic contact geometries. Here, the first cohomology \( H^1(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \) appears only in negative homogeneities, thus the entire geometry is completely defined by the distribution \( T^{-1}M \subset TM \) of codimension three. This means that if there were a pre-quaternionic vector space structure on \( T^{-1}M \) for which an algebraic bracket \( [,]_{alg} : \Lambda^2 T^{-1}M \rightarrow TM/T^{-1}M \) would be an imaginary part of a hermitian form, then this structure is unique.
Again, let us skip the lowest dimension first, i.e. \( \dim M \geq 11 \). Then the second cohomology \( H^2(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \) has two components. One of them appears in homogeneity zero, with cochains of the type \( \Lambda \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-2} \). This is a tricky point, since this means that if this part of the torsion is non-zero, then the geometric structure is defined by the distribution and a choice of an algebraic bracket \([\cdot, \cdot]_{\text{alg}}\) such that the latter bracket allows for a pre-quaternionic structure such that it gets the imaginary part of a hermitian form. Of course, the difference of this bracket and the standard one (defined by the Lie bracket of vector fields) should be normalized (co-closed in the terms of the Lie algebra cohomology, cf. the appendix).

Dealing with generic hypersurfaces in \( \mathbb{H}^{n+1} \), we should thus expect three very much different possibilities. First, the distributions with the inherited pre-quaternionic structure and the Lie bracket will satisfy all the properties (i.e. the bracket will be the imaginary part of suitable hermitian form on \( T_{-1}M \)). This is extremely restrictive and, as shown in [7], can happen only if \( M \) is locally isomorphic to the homogeneous model—the 3-Sasakian sphere.

The second possibility is to require that the inherited distribution and the Lie bracket are allowing for a pre-quaternionic structure as above. This assumption is much less rigid, but we need an explicit construction and knowledge of the canonical Cartan connection in order to be able to deal with such examples properly. This has been the initial main motivation for this paper and we shall come back to such special class of hypersurfaces in \( \mathbb{H}^{n+1} \) in another future work. In the lowest dimension, seven, this is the most general case, but we there we have also a homogeneity one torsion component.

We shall not deal with the most difficult third option here at all. Let us just mention a preprint by Stuart Armstrong devoted to a general class of Cartan geometries with this kind of behavior, [1].

Thus, let us assume we are given an abstract quaternionic contact manifold \( M \), i.e. a distribution equipped with the right quaternionic contact structure (in any signature). Then, only the other second cohomology component can give rise to curvature with cochains of the type \( \Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0^* \), except for the lowest dimension \( \dim M = 7 \), where another torsion with cochains of the type \( \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2} \) may appear. In particular, exactly as in the CR-geometry case, if there is no curvature of the form \( \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2} \), then the general first homogeneity prolongation procedure will again produce the triples of contact forms and their corresponding Reeb vector fields corresponding to the reductions of the structure group to \( \mathfrak{g}_0^* \) (as exploited in the positive definite case in [2]).

The latter observation will be the starting point in our construction. Moreover, the general knowledge of the total curvature structure deduced in [3, Corollary 3.2] reveals that there is no curvature with values in \( \mathfrak{h} \). Thus, in full analogy with the construction by Chern and Moser, we may move straight to the appropriate frame bundle with structure Lie algebra \( \mathfrak{h} = \mathbb{R} \oplus \mathfrak{sp}(1) = \mathbb{H} \), and work out all our exterior calculus there.

The main results are spread through the text as follows: Theorem 3.3 provides the construction of the Cartan connection as the canonical coframe at the right principal fiber bundle. Then, right in the beginning of the next section, Proposition 4.1 displays the complete structure equations of the coframe, thus providing all the curvature components of the canonical coframe. Next the differential consequences of the Bianchi identities (cf. Proposition 4.2) are explicitly listed in Proposition 4.3.

The explanation how the coframe and its curvature are related to the Lie algebra structure is presented in section 5 on the associated Cartan geometry in terms of the principal fiber bundles and the algebraic normalization conditions. In particular, in 5.2 we verify that the curvature of the constructed canonical coframe is co-closed and thus coincides with the normal Cartan connection for the quaternionic contact manifolds. In the very end, the appendix collects brief information on the abstract theory of parabolic geometries and provides links of the general concepts to the the individual objects and formulae in the paper.

Acknowledgments. I.M. is supported by a SoMoPro II Fellowship which is co-funded by the European Commission\(^1\) from “People” specific program (Marie Curie Actions) within the EU Seventh Framework Program on the basis of the grant agreement REA No. 291782. It is further co-financed by the South-Moravian Region.

---

\(^1\)This article reflects only the author’s views and the EU is not liable for any use that may be made of the information contained therein.
2. Preliminaries

2.1. Conventions concerning the use of complex tensors and indices. Throughout this paper, we use without comment the convention of summation over repeating indices; the small Greek indices $\alpha, \beta, \gamma, \ldots$ will have the range $1, \ldots, 2n$, whereas the indices $s, t, k, l, m$ will be running from 1 to 3.

Consider the Euclidean vector space $\mathbb{R}^{4n}$ with its standard inner product $\langle, \rangle$ (with or without signature) and a quaternionic structure induced by the identification $\mathbb{R}^{4n} \cong \mathbb{H}^n$ with the quaternionic coordinate space $\mathbb{H}^n$. The latter means that we endow $\mathbb{R}^{4n}$ with a fixed triple $J_1, J_2, J_3$ of complex structures which are Hermitian with respect to $\langle, \rangle$ and satisfy $J_1 J_2 = -J_2 J_1 = J_3$. The complex vector space $\mathbb{C}^{4n}$, being the complexification of $\mathbb{R}^{4n}$, splits as a direct sum of $+i$ and $-i$ eigenspaces, $\mathbb{C}^{4n} = W \oplus \overline{W}$, with respect to the complex structure $J_1$. The complex 2-form

$$\pi(u, v) \overset{\text{def}}{=} \langle J_2 u, v \rangle + i \langle J_3 u, v \rangle, \quad u, v \in \mathbb{C}^{4n},$$

has type $(2,0)$ with respect to $J_1$, i.e., it satisfies $\pi(J_1 u, v) = \pi(u, J_1 v) = i\pi(u, v)$. Let us fix an $\langle, \rangle$-orthonormal basis (once and for all)

$$(2.1) \quad \{ e_\alpha \in W, e_\bar{\alpha} \in \overline{W} \}, \quad e_\alpha = \overline{e_\alpha},$$

with dual basis $\{ e^\alpha, e^{\bar{\alpha}} \}$ so that $\pi = e^1 \wedge e^{n+1} + e^2 \wedge e^{n+2} + \cdots + e^n \wedge e^{2n}$. Then, we have

$$(2.2) \quad \langle, \rangle = g_{\alpha\bar{\beta}} e^\alpha \otimes e^{\bar{\beta}} + g_{\beta\bar{\alpha}} e^\bar{\alpha} \otimes e^{\beta}, \quad \pi = \pi_{\alpha\bar{\beta}} e^\alpha \otimes e^{\bar{\beta}},$$

where, for the positive definite case, we take

$$(2.3) \quad g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha} = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta \end{cases}, \quad \pi_{\alpha\bar{\beta}} = -\pi_{\bar{\beta}\alpha} = \begin{cases} 1, & \text{if } \alpha + n = \beta \\ -1, & \text{if } \alpha = \beta + n \\ 0, & \text{otherwise} \end{cases},$$

and for the case of signature, we take $-1$ instead of 1 for the respective coefficients. In fact, the precise values of the constants $g_{\alpha\bar{\beta}}$ and $\pi_{\alpha\bar{\beta}}$ are completely irrelevant for the forthcoming developments; the only thing that matters is that $g_{\alpha\bar{\beta}}$ is non-degenerate and hermitian (i.e. $g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha}$), $\pi$ is non-degenerate and skew-symmetric (i.e. $\pi_{\alpha\bar{\beta}} = -\pi_{\bar{\beta}\alpha}$), and that

$$g^{\sigma\bar{\tau}} \pi_{\alpha\sigma} \pi_{\bar{\tau}\bar{\beta}} = -g_{\alpha\bar{\beta}}, \quad \pi_{\sigma\bar{\beta}} \overset{\text{def}}{=} \pi_{\bar{\tau}\bar{\beta}},$$

where $g^{\sigma\bar{\tau}} = g^{\bar{\beta}\alpha}$ denotes the inverse of $g_{\alpha\bar{\beta}}$, i.e. $g^{\sigma\bar{\tau}} g_{\bar{\tau}\beta} = \delta^\sigma_\beta$ ($\delta^\sigma_\beta$ is the Kronecker delta).

Any array of complex numbers indexed by lower and upper Greek letters (with and without bars) corresponds to a tensor, e.g., $\{ A_{\alpha\beta\gamma} \}$ corresponds to the tensor

$$A_{\alpha\beta\gamma} e^\alpha \otimes e_\beta \otimes e_\gamma.$$

Clearly, the vertical as well as the horizontal position of an index carries information about the tensor. For two-tensors, we take $B^\alpha_\beta$ to mean $B_\gamma^\alpha$, i.e., the lower index is assumed to be first. We use $g_{\alpha\bar{\beta}}$ and $g^{\alpha\bar{\beta}}$ to lower and raise indices in the usual way, e.g.,

$$A_{\alpha\beta\gamma} = g_{\sigma\gamma} A^{\beta\sigma}_{\alpha\cdot}, \quad A^{\beta\gamma}_{\cdot\cdot\cdot} = g^{\bar{\sigma}\gamma} A_{\cdot\cdot\cdot}^{\beta\bar{\sigma}}.$$

We use also the following convention: Whenever an array $\{ A_{\alpha\beta\gamma} \}$ appears, the array $\{ A_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \}$ will be assumed to be defined, by default, by the complex conjugation

$$A_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \overline{A_{\alpha\beta\gamma}}.$$

This means that we interpret $\{ A_{\alpha\beta\gamma} \}$ as a representation of a real tensor, defined on $\mathbb{R}^{4n}$, with respect to the fixed complex basis (2.1); the corresponding real tensor in this case is

$$A_{\alpha\beta\gamma} e^\alpha \otimes e_\beta \otimes e_\gamma + A_{\bar{\alpha}\bar{\beta}\bar{\gamma}} e^{\bar{\alpha}} \otimes e^{\bar{\beta}} \otimes e^{\bar{\gamma}}.$$
Notice that $\pi^\alpha_\beta \pi^\beta_\gamma = - \delta^\alpha_\gamma$. We introduce a complex antilinear endomorphism $j$ of the tensor algebra of $\mathbb{R}^{4n}$, which takes a tensor with components $T_{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l \ldots}$ to a tensor of the same type, with components $(iT)_{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l \ldots}$ by the formula

$$
(2.4) \quad (iT)_{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l \ldots} = \sum_{\sigma_1 \ldots \sigma_{k+1} \tau_1 \ldots \tau_{l+1}} \pi^\alpha_{\alpha_1} \ldots \pi^\alpha_{\alpha_k} \pi^\beta_{\beta_1} \ldots \pi^\gamma_{\beta_l} \ldots T_{\sigma_1 \ldots \sigma_{k+1} \tau_1 \ldots \tau_{l+1}} \ldots.
$$

By definition, the group $Sp(n)$ consists of all endomorphisms of $\mathbb{R}^{4n}$ that preserve the inner product $\langle \cdot, \cdot \rangle$ and commute with the complex structures $J_1, J_2$ and $J_3$. With the above notation, we can alternatively describe $Sp(n)$ as the set of all two-tensors $\{U^\alpha_\beta\}$ satisfying

$$
(2.5) \quad g_{\sigma \tau} U^\alpha_\sigma U^\beta_\tau = g_{\alpha \beta}, \quad \pi_{\sigma \tau} U^\alpha_\sigma U^\beta_\tau = \pi_{\alpha \beta}.
$$

For its Lie algebra, $sp(n)$, we have the following description:

**Lemma 2.1.** For a tensor $\{X_{\alpha \beta}\}$, the following conditions are equivalent:

1. $\{X_{\alpha \beta}\} \in sp(n)$.
2. $X_{\alpha \beta} = -X_{\beta \alpha}$, $(jX)_{\alpha \beta} = X_{\alpha \beta}$.
3. $X^\alpha_\beta = \pi^{\alpha \sigma} Y_{\sigma \beta}$ for some tensor $\{Y_{\alpha \beta}\}$ satisfying $Y_{\alpha \beta} = Y_{\beta \alpha}$ and $(jY)_{\alpha \beta} = Y_{\alpha \beta}$.

**Proof.** The equivalence between (1) and (2) follows by differentiating (2.5) at the identity. To obtain (3), we define the tensor $\{Y_{\alpha \beta}\}$ by $Y_{\alpha \beta} = -\pi_{\sigma \tau} X^\beta_\sigma = -\pi^{\alpha \sigma} X_{\beta \sigma}$.

\[ \square \]

### 2.2. Quaternionic contact manifolds.

Let $M$ be a $(4n+3)$-dimensional manifold and $H$ be a smooth distribution on $M$ of codimension three. The pair $(M, H)$ is said to be a quaternionic contact (abbr. qc) structure if around each point of $M$ there exist 1-forms $\eta_1, \eta_2, \eta_3$ with common kernel $H$, a non-degenerate inner product $\tilde{g}$ on $H$ (with or without signature), and endomorphisms $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3$ of $H$, satisfying

$$
(2.6) \quad (\tilde{I}_1)^2 = (\tilde{I}_2)^2 = (\tilde{I}_3)^2 = -\mathrm{id}_H, \quad \tilde{I}_1 \tilde{I}_2 = -\tilde{I}_2 \tilde{I}_1 = \tilde{I}_3,
$$

$$
\tilde{d}\eta_s(X,Y) = 2\tilde{g}(\tilde{I}_sX,Y) \quad \text{for all } X,Y \in H.
$$

As we mentioned in the introduction, the qc structures may be considered as a quaternion analog of the CR manifolds of hypersurface type known from the complex analysis; one should, however, be aware of a few differences. First, for each qc manifold $(M, H)$, the linear span of the pointwise quaternionic structure $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3$ is a 3-dimensional subbundle of $\text{End}(H)$ which is uniquely determined by the distribution $H$ and does not need to be prescribed in advance. Secondly, there is an essential part in the definition of a CR manifold, called integrability condition, that requires for the holomorphic CR distribution to satisfy the Frobenius condition. The qc counterpart to the latter is the existence of Reeb vector fields, namely a triple $\xi_1, \xi_2, \xi_3$ of vector fields on $M$ satisfying for all $X \in H$,

$$
(2.7) \quad \tilde{\eta}_s(\tilde{\xi}_t) = \delta^s_t, \quad \tilde{d}\eta_s(\tilde{\xi}_t, X) = -\tilde{d}\eta_s(\tilde{\xi}_s, X)
$$

($\delta^s_t$ being the Kronecker delta). As shown in [2], the Reeb vector fields always exist if $\dim(M) > 7$. In the seven dimensional case this is an additional integrability condition on the qc structure (cf. [6]) which we will assume to be satisfied.

Let us define the 2-forms $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3$ by

$$
(2.8) \quad \tilde{\xi}_s \tilde{\omega}_t = 0, \quad \tilde{\omega}_s(X,Y) = 2\tilde{g}(\tilde{I}_sX,Y), \quad X,Y \in H.
$$

Then, as it can be easily verified, the existence of Reeb vector fields allows us to express the exterior derivatives $\tilde{d}\eta_s$ in the form

$$
(2.9) \quad \tilde{d}\eta_s = -\tilde{\alpha}_{ts} \wedge \tilde{\eta}_t + 2\tilde{\omega}_s, \quad \tilde{\alpha}_{st} = -\tilde{\alpha}_{ts}.
$$
Explicitly, the one-forms $\hat{\alpha}_{st}$ are given by
\[
\begin{align*}
\hat{\alpha}_{12} &= \xi_1 \cdot d\hat{\eta}_2 + \frac{1}{4} \left( -d\hat{\eta}_1(\xi_2, \xi_3) + d\hat{\eta}_2(\xi_3, \xi_1) + d\hat{\eta}_3(\xi_1, \xi_2) \right) \hat{\eta}_3 + d\hat{\eta}_1(\xi_1, \xi_2) \hat{\eta}_1 \\
\hat{\alpha}_{23} &= \xi_2 \cdot d\hat{\eta}_3 + \frac{1}{4} \left( d\hat{\eta}_1(\xi_2, \xi_3) - d\hat{\eta}_2(\xi_3, \xi_1) + d\hat{\eta}_3(\xi_1, \xi_2) \right) \hat{\eta}_1 + d\hat{\eta}_2(\xi_2, \xi_3) \hat{\eta}_2 \\
\hat{\alpha}_{31} &= \xi_3 \cdot d\hat{\eta}_1 + \frac{1}{4} \left( d\hat{\eta}_1(\xi_2, \xi_3) + d\hat{\eta}_2(\xi_3, \xi_1) - d\hat{\eta}_3(\xi_1, \xi_2) \right) \hat{\eta}_2 + d\hat{\eta}_3(\xi_3, \xi_1) \hat{\eta}_3
\end{align*}
\]

3. Solution to the Quaternionic Contact Equivalence Problem

It is well known that to each qc manifold $(M, H)$ one can associate a unique, up to a diffeomorphism, regular, normal Cartan geometry, i.e., a certain principle bundle $P_1 \to M$ endowed with a Cartan connection that satisfies some natural normalization conditions (see the Appendix and the references therein for more details on the topic). Our goal here is to provide an explicit construction for both the bundle and the connection in terms of geometric data generated entirely by the qc structure of $M$. We are using essentially the original Cartan’s method of equivalence that had been applied later with a great success by Chern and Moser in [5] for solving the respective equivalence problem in the CR case. The method is based entirely on classical exterior calculus and does not require any preliminary knowledge concerning the theory of parabolic geometries or the related Lie algebra cohomology. The main result here is Theorem 3.3.

Let $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, I_1, I_2, I_3, \tilde{g}$ be as in (2.6). If $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3$ are any (other) 1-forms satisfying (2.6) for some symmetric and positive definite $\tilde{g} \in H^* \otimes H^*$ and endomorphisms $\tilde{I}_s \in \text{End}(H)$ in place of $\tilde{g}$ and $\tilde{I}_s$ respectively, then it is known (see for example the appendix of [7]) that there exists a positive real-valued function $\mu$ and an $SO(3)$-valued function $\Psi = (a_{st})_{3 \times 3}$ so that
\[
\tilde{\eta}_s = \mu a_{ts} \tilde{\eta}_t, \quad \tilde{g} = \mu \tilde{g}, \quad \tilde{I}_s = a_{ts} \tilde{I}_t.
\]

In intrinsic terms, this means that we have a principle bundle $P_o$ over $M$ with structure group $CSO(3) = \mathbb{R}^+ \times SO(3)$ whose local sections are exactly the triples of 1-forms $(\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3)$ satisfying (2.6). The functions $\mu \in \mathbb{R}^+$ and $(a_{st})_{3 \times 3} \in SO(3)$ may be considered as local fiber coordinates on $P_o$ with respect to a fixed local section $(\mu \hat{\eta}_1, \mu \hat{\eta}_2, \mu \hat{\eta}_3)$. On $P_o$, we have globally defined intrinsic one-forms $\eta_1, \eta_2, \eta_3$ which, in terms of the local fiber coordinates, have the expression
\[
\eta_s = \mu a_{ts} \pi^*_o(\hat{\eta}_t),
\]
with $\pi_o : P_o \to M$ being the principle bundle projection. We will call a differential forms on $P_o$ semibasic if its contraction with any vector field tangent to the fibers of $\pi_o$ vanishes.

**Lemma 3.1.** In a neighborhood of each point of $P_o$, we can find real one-forms $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ and semibasic complex one-forms $\theta^\alpha$ so that
\[
\begin{align*}
&d\eta_1 = -\varphi_0 \wedge \eta_1 - \varphi_2 \wedge \eta_3 + \varphi_3 \wedge \eta_2 + 2i g_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \\
&d\eta_2 = -\varphi_0 \wedge \eta_2 - \varphi_3 \wedge \eta_1 + \varphi_1 \wedge \eta_3 + \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\bar{\beta} + \pi_{\bar{\alpha}\bar{\beta}} \theta^\bar{\alpha} \wedge \theta^\bar{\beta} \\
&d\eta_3 = -\varphi_0 \wedge \eta_3 - \varphi_1 \wedge \eta_2 + \varphi_2 \wedge \eta_1 - i \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + i \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\bar{\beta} + i \pi_{\bar{\alpha}\bar{\beta}} \theta^\bar{\alpha} \wedge \theta^\bar{\beta},
\end{align*}
\]
where $g_{\alpha\beta} = g_{\beta\alpha}$ and $\pi_{\alpha\beta} = -\pi_{\beta\alpha}$ are the same (fixed) constants as in Section 2.1.

**Proof.** Let us consider the distribution $H \subset TM$ as a vector bundle over $M$ and take $\mathcal{H} \to P_o$ to be the corresponding pull-back bundle via $\pi_o$. To each point $p \in P_o$, we can associate a natural triple of endomorphisms $I_1, I_2, I_3$ and a symmetric 2-tensor $g$ of the fibers $\mathcal{H}_p$ of $\mathcal{H}$ given by
\[
I_s = a_{ts} \hat{I}_t, \quad g = \mu \tilde{g}
\]
(with the obvious identification $\mathcal{H}_p \cong H_{\pi_o(p)}$). Then,
\[
(I_1)^2 = (I_2)^2 = (I_3)^2 = -\text{id}_{\mathcal{H}}, \quad I_1 I_2 = -I_2 I_1 = I_3.
\]
The complexification of $\mathcal{H}_p$ (which we will denote again by $\mathcal{H}_p$) splits as $\mathcal{H}_p = W_p \oplus \overline{W}_p$ with $W_p$ and $\overline{W}_p$ being the eigenspaces of $+i$ and $-i$ with respect to the endomorphism $I_1$. We denote by $\pi$ the skew-symmetric 2-tensor on $\mathcal{H}$ given by
\[
\pi(u, v) = g(I_2 u, v) + i g(I_3 u, v), \quad u, v \in \mathcal{H},
\]
which is easily seen to be of type $(2,0)$ with respect to $I_1$, i.e., we have $\pi(I_1 u, v) = \pi(u, I_1 v) = i\pi(u, v)$.

Let us pick a local coframing
\begin{equation}
\{\theta^\alpha \in W^*, \theta^\beta \in W^*\}, \quad \theta^\beta = \overline{\theta^\alpha}
\end{equation}
for the (complexified) vector bundle $\mathcal{H}$ so that
\begin{equation}
g = g_{\alpha\beta} \theta^\alpha \otimes \theta^\beta + g_{\bar{\alpha}\bar{\beta}} \overline{\theta^\beta} \otimes \theta^\alpha \quad \text{and} \quad \pi = \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta,
\end{equation}
where $g_{\alpha\beta} = g_{\bar{\beta}\bar{\alpha}}$ and $\pi_{\alpha\beta} = -\pi_{\beta\alpha}$ are given by (2.3) (this is always possible by a standard linear algebra argumentation).

The two-forms $\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3$ given by (2.8) may be regarded as two-tensors on the fibers of $\mathcal{H}$. We will need the following identities:
\begin{equation}
\begin{aligned}
2\mu a_{s1} \hat{\omega}_s(u, v) &= 2g(a_{s1} \hat{I}_s u, v) = 2g(I_1 u, v) = 2i\mu a_{s1} \theta^\alpha(u) \theta^\beta(v) - 2i\mu a_{s1} \theta^\beta(u) \theta^\alpha(v) \\
2\mu a_{s2} \hat{\omega}_s(u, v) &= 2g(a_{s2} \hat{I}_s u, v) = 2g(\hat{I} u, v) = \pi(u, v) + \overline{\pi(u, v)}.
\end{aligned}
\end{equation}

The third identity we obtain similarly by
\begin{equation}
2\mu a_{s3} \hat{\omega}_s(u, v) = 2g(a_{s3} \hat{I}_s u, v) = 2g(I_3 u, v) = -i\pi(u, v) - i\overline{\pi(u, v)}.
\end{equation}

Let $\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$ be the Reeb vector fields corresponding to $\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$ (cf. (2.7)). For each $p \in P_\omega$, we have the map
\begin{equation}
T_pP_o \xrightarrow{\pi_p} T_{\pi_p(p)}M \rightarrow H_{\pi_p(p)} \xrightarrow{\cong} \mathcal{H}_p,
\end{equation}
where the second arrow denotes the projection on the first factor in $T_{\pi_p(p)}M = H_{\pi_p(p)} \oplus \text{span}\{\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3\}$. By (3.7), we can consider $\theta^\alpha, \overline{\theta^\beta}$ and $\hat{\omega}_s$ as 1-forms on $P_o$. Then, clearly, the identities (3.6) remain valid. Notice also that, since $(a_{st})_{3 \times 3} \in SO(3)$, we have $a_{st} \delta_{tt} = \delta_{tt}$ and thus the expression $a_{st} \delta_{tt}$ is skew-symmetric in $s, t$. By differentiating (3.1), we get
\begin{equation}
d\hat{\eta}_s = d\mu \wedge a_{ts} \hat{\eta}_t + \mu d a_{ts} \wedge \hat{\eta}_t + \mu a_{ts} \left( -\alpha_{s1} \hat{\eta}_t + 2\hat{\omega}_t \right)
\end{equation}
by (2.9)
\begin{equation}
= \mu a_{ts} \left( \mu^{-1} \delta_{ts} \mu + a_{kl} a_{ks} - a_{kl} a_{ms} \hat{\alpha}_{km} \right) \wedge \hat{\eta}_t + 2\mu a_{ts} \hat{\omega}_t
= \mu^{-1} d\mu \wedge \eta_s + a_{kl}(a_{ks} - a_{ms} \hat{\alpha}_{km}) \wedge \eta_l + 2\mu a_{ts} \hat{\omega}_t.
\end{equation}

Since $a_{kl}(a_{ks} - a_{ms} \hat{\alpha}_{km})$ is skew-symmetric in $l, s$, it can be represented by a triple of one-forms $\varphi_1, \varphi_2, \varphi_3$. Explicitly, we define
\begin{align*}
\varphi_0 &= -\mu^{-1} d\mu \\
\varphi_1 &= -a_{k2}(a_{k3} - a_{m3} \hat{\alpha}_{km}) \\
\varphi_2 &= -a_{k3}(a_{k1} - a_{m1} \hat{\alpha}_{km}) \\
\varphi_3 &= -a_{k1}(a_{k2} - a_{m2} \hat{\alpha}_{km}).
\end{align*}

Then, the Lemma follows by (3.8) and (3.6).

\begin{lemma}
If $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \theta^\alpha$ are any one-forms that satisfy the assertion of Lemma 3.1, then $\eta_1, \eta_2, \eta_3, \theta^\alpha, \overline{\theta^\beta}, \varphi_0, \varphi_1, \varphi_2, \varphi_3$ are pointwise linearly independent.
\end{lemma}
Any other one-forms $\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\theta}^\alpha$ will satisfy the assertion of Lemma 3.1 if and only if they are given by the formulas

$$
\begin{aligned}
\tilde{\theta}^\alpha &= U^\alpha_\beta \theta^\beta + i r^\alpha \eta_1 + \pi^\alpha_\beta r^\beta (\eta_2 + i \eta_3) \\
\tilde{\varphi}_0 &= \varphi_0 + 2U^\beta_\alpha r^\beta \theta^\alpha + 2U^\beta_\beta r^\beta \theta^\beta + \lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3 \\
\tilde{\varphi}_1 &= \varphi_1 - 2iU^\beta_\beta r^\beta \theta^\alpha + 2iU^\beta_\alpha r^\alpha \theta^\beta + 2r_\sigma r^\sigma \eta_1 - \lambda_3 \eta_2 + \lambda_2 \eta_3, \\
\tilde{\varphi}_2 &= \varphi_2 - 2\pi_\alpha U^\beta_\beta r^\beta \theta^\alpha - 2\pi_\beta U^\beta_\beta r^\beta \theta^\beta + \lambda_3 \eta_1 + 2r_\sigma r^\sigma \eta_2 - \lambda_1 \eta_3, \\
\tilde{\varphi}_3 &= \varphi_3 + 2\pi_\alpha U^\beta_\beta r^\beta \theta^\alpha - 2\pi_\beta U^\beta_\beta r^\beta \theta^\beta - \lambda_2 \eta_1 + \lambda_1 \eta_2 + 2r_\sigma r^\sigma \eta_3,
\end{aligned}
$$

where $U^\alpha_\beta, r^\alpha, \lambda_\alpha$ are some appropriate functions; $\lambda_1, \lambda_2, \lambda_3$ are real, and $\{U^\alpha_\beta\}$ satisfy (2.5), i.e., $\{U^\alpha_\beta\} \in Sp(n) \subset \text{End}(\mathbb{R}^{1n})$.

**Proof.** Let us begin by taking $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \theta^\alpha$ to be the one-forms constructed in the proof of Lemma 3.1. For these one-forms, it is obvious that $\{\eta_1, \eta_2, \eta_3, \theta^\alpha, \theta^\beta, \varphi_0, \varphi_1, \varphi_2, \varphi_3\}$ is a coframing for $TP_\sigma$. Let $\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\theta}^\alpha$ be any other one-forms satisfying the assertion of Lemma 3.1. Then, by subtracting the corresponding equations, we obtain

$$
\begin{aligned}
0 &= - (\tilde{\varphi}_0 - \varphi_0) \wedge \eta_1 - (\tilde{\varphi}_2 - \varphi_2) \wedge \eta_3 + (\tilde{\varphi}_3 - \varphi_3) \wedge \eta_2 + 2i\pi_{\alpha\beta} (\tilde{\theta}^\alpha \wedge \tilde{\theta}^\beta - \theta^\alpha \wedge \theta^\beta) \\
0 &= - (\tilde{\varphi}_0 - \varphi_0) \wedge \eta_2 - (\tilde{\varphi}_1 - \varphi_1) \wedge \eta_3 + (\tilde{\varphi}_1 - \varphi_1) \wedge \eta_1 + \pi_{\alpha\beta} (\tilde{\theta}^\alpha \wedge \tilde{\theta}^\beta - \theta^\alpha \wedge \theta^\beta) \\
0 &= - (\tilde{\varphi}_0 - \varphi_0) \wedge \eta_3 - (\tilde{\varphi}_1 - \varphi_1) \wedge \eta_2 + (\tilde{\varphi}_2 - \varphi_2) \wedge \eta_1 - i\pi_{\alpha\beta} (\tilde{\theta}^\alpha \wedge \tilde{\theta}^\beta - \theta^\alpha \wedge \theta^\beta) \\
&+ i\pi_{\alpha\beta} (\tilde{\theta}^\alpha \wedge \tilde{\theta}^\beta - \theta^\alpha \wedge \theta^\beta),
\end{aligned}
$$

Since, by assumption, $\tilde{\theta}^\alpha$ are semibasic, i.e., their contractions with vector fields tangent to the fibers of $\pi_o$ vanish, we have that

$$
\tilde{\theta}^\alpha = U^\alpha_\beta \theta^\beta + U^\alpha_\beta \theta^\beta + A^\alpha_\beta \eta_s
$$

for some appropriate coefficients $U^\alpha_\beta, U^\alpha_\beta, A^\alpha_\beta$.

Wedgeing the first identity of (3.10) with $\eta_2 \wedge \eta_3 \wedge \theta^1 \wedge \cdots \wedge \theta^{2n} \wedge \bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^{2n}$ yields

$$
(\tilde{\varphi}_0 - \varphi_0) \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \theta^1 \wedge \cdots \wedge \theta^{2n} \wedge \bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^{2n} = 0
$$

and hence $\tilde{\varphi}_0 - \varphi_0 \in \text{span}\{\eta_1, \eta_2, \eta_3, \theta^\alpha, \theta^\beta\}$. Proceeding similarly for $\tilde{\varphi}_1 - \varphi_1, \tilde{\varphi}_2 - \varphi_2$ and $\tilde{\varphi}_3 - \varphi_3$, we arrive at the equations

$$
\begin{aligned}
\tilde{\varphi}_0 &= \varphi_0 + b_\alpha \theta^\alpha + b_\alpha \theta^\alpha + b_s \eta_s \\
\tilde{\varphi}_1 &= \varphi_1 + c_\alpha \theta^\alpha + c_\alpha \theta^\alpha + c_s \eta_s \\
\tilde{\varphi}_2 &= \varphi_2 + d_\alpha \theta^\alpha + d_\alpha \theta^\alpha + d_s \eta_s \\
\tilde{\varphi}_3 &= \varphi_3 + e_\alpha \theta^\alpha + e_\alpha \theta^\alpha + e_s \eta_s
\end{aligned}
$$
with \( b^\alpha, c^\alpha, d^\alpha, e^\alpha, \alpha, \beta, \gamma, \delta \) being some appropriate functions. Substituting (3.11) and (3.12) back into (3.10) gives

\[
0 = \left( b_2 + e_1 + 2i g_{\alpha \beta} (A_2^\alpha A_2^\beta - A_1^\alpha A_1^\beta) \right) \eta_1 \eta_2 - \left( d_2 + e_3 - 2i g_{\alpha \beta} (A_2^\alpha A_2^\beta - A_1^\alpha A_1^\beta) \right) \eta_2 \eta_3
- \left( b_3 - d_1 + 2i g_{\alpha \beta} (A_2^\alpha A_2^\beta - A_1^\alpha A_1^\beta) \right) \eta_1 \eta_3
+ \left( - b_\alpha - 2i U_{\alpha \varphi} A_1^\varphi + 2i U_{\alpha \varphi} A_1^\varphi \right) \theta^\alpha \eta_1
+ \left( - b_\alpha + 2i U_{\alpha \varphi} A_1^\varphi - 2i U_{\alpha \varphi} A_1^\varphi \right) \theta^\alpha \eta_1
+ \left( c_\alpha - 2i U_{\alpha \varphi} A_2^\varphi + 2i U_{\alpha \varphi} A_2^\varphi \right) \theta^\alpha \eta_2
+ \left( c_\alpha + 2i U_{\alpha \varphi} A_2^\varphi - 2i U_{\alpha \varphi} A_2^\varphi \right) \theta^\alpha \eta_3
- 2i \left( g_{\alpha \beta} - g_{\alpha \tau} U_2^\alpha U_2^\beta + g_{\alpha \tau} U_2^\alpha U_2^\beta \right) \theta^\alpha \eta_3
+ i g_{\alpha \tau} \left( U_2^\alpha U_2^\tau - U_2^\alpha U_2^\tau \right) \theta^\alpha \theta^\beta + i g_{\alpha \tau} \left( U_2^\alpha U_2^\tau - U_2^\alpha U_2^\tau \right) \theta^\alpha \theta^\beta;
\]

\[
0 = - \left( b_1 - e_2 - 2i \pi_{\alpha \beta} A_1^\alpha A_1^\beta - 2i \pi_{\alpha \beta} A_1^\alpha A_1^\beta \right) \eta_1 \eta_2
+ \left( b_3 + e_2 + 2i \pi_{\alpha \beta} A_2^\alpha A_2^\beta + 2i \pi_{\alpha \beta} A_2^\alpha A_2^\beta \right) \eta_1 \eta_3
- \left( c_1 + e_3 - 2i \pi_{\alpha \beta} A_3^\alpha A_3^\beta + 2i \pi_{\alpha \beta} A_3^\alpha A_3^\beta \right) \eta_2 \eta_3
+ \left( - c_\alpha + 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau + 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau \right) \theta^\alpha \eta_1
+ \left( - c_\alpha + 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau - 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau \right) \theta^\alpha \eta_1
+ \left( c_\alpha + 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau + 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau \right) \theta^\alpha \eta_2
+ \left( c_\alpha + 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau - 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau \right) \theta^\alpha \eta_3
+ 2 \left( \pi_{\alpha \tau} U_1^\alpha U_1^\alpha + \pi_{\alpha \tau} U_1^\alpha U_1^\alpha \right) \theta^\alpha \eta_3
- \left( \pi_{\alpha \beta} - \pi_{\alpha \tau} U_1^\alpha U_1^\tau - \pi_{\alpha \tau} U_1^\alpha U_1^\tau \right) \theta^\alpha \theta^\beta
- \left( \pi_{\alpha \beta} - \pi_{\alpha \tau} U_1^\alpha U_1^\tau - \pi_{\alpha \tau} U_1^\alpha U_1^\tau \right) \theta^\alpha \theta^\beta
\]

\[
0 = - \left( c_1 + d_2 + 2i \pi_{\alpha \beta} A_2^\alpha A_2^\beta - 2i \pi_{\alpha \beta} A_2^\alpha A_2^\beta \right) \eta_1 \eta_2
+ \left( - b_2 + c_3 - 2i \pi_{\alpha \beta} A_3^\alpha A_3^\beta + 2i \pi_{\alpha \beta} A_3^\alpha A_3^\beta \right) \eta_1 \eta_3
+ \left( b_1 + d_3 - 2i \pi_{\alpha \beta} A_1^\alpha A_1^\beta + 2i \pi_{\alpha \beta} A_1^\alpha A_1^\beta \right) \eta_2 \eta_3
+ \left( d_\alpha - 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau + 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau \right) \theta^\alpha \eta_1
+ \left( d_\alpha + 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau - 2i \pi_{\alpha \tau} U_1^\alpha A_1^\tau \right) \theta^\alpha \eta_1
- \left( c_\alpha + 2i \pi_{\alpha \tau} U_2^\alpha A_2^\tau - 2i \pi_{\alpha \tau} U_2^\alpha A_2^\tau \right) \theta^\alpha \eta_2
- \left( c_\alpha - 2i \pi_{\alpha \tau} U_2^\alpha A_2^\tau + 2i \pi_{\alpha \tau} U_2^\alpha A_2^\tau \right) \theta^\alpha \eta_2
- \left( b_\alpha + 2i \pi_{\alpha \tau} U_3^\alpha A_3^\tau - 2i \pi_{\alpha \tau} U_3^\alpha A_3^\tau \right) \theta^\alpha \eta_3
- \left( b_\alpha - 2i \pi_{\alpha \tau} U_3^\alpha A_3^\tau + 2i \pi_{\alpha \tau} U_3^\alpha A_3^\tau \right) \theta^\alpha \eta_3
+ 2i \left( - \pi_{\alpha \tau} U_1^\alpha U_1^\alpha + \pi_{\alpha \tau} U_1^\alpha U_1^\alpha \right) \theta^\alpha \theta^\beta
+ i \left( \pi_{\alpha \beta} - \pi_{\alpha \tau} U_1^\alpha U_1^\tau + \pi_{\alpha \tau} U_1^\alpha U_1^\tau \right) \theta^\alpha \theta^\beta.<br>

Since \( \eta_1, \eta_2, \eta_3, \theta^\alpha, \theta^\beta \) are pointwise linearly independent, all the coefficients in (3.13),(3.14) and (3.15) must vanish. The vanishing of the coefficients of \( \theta^\alpha \theta^\beta \) in (3.14) and (3.15) gives

\[
\pi_{\alpha \beta} = \pi_{\alpha \tau} U_1^\alpha U_1^\beta
\]

This implies that the array \( \{ U_1^\alpha \} \) corresponds to an invertible endomorphism of \( \mathbb{R}^{4n} \). Furthermore, the vanishing of the coefficients of \( \theta^\alpha \theta^\beta \) in (3.14) and (3.15) yields \( \pi_{\alpha \tau} U_2^\alpha U_2^\beta = 0 \) and hence \( U_2^\alpha = 0 \), since both \( \pi_{\alpha \beta} \) and \( U_2^\alpha \) are invertible. Furthermore, the vanishing of the coefficients of \( \theta^\alpha \theta^\beta \) in (3.13) implies

\[
g_{\alpha \beta} = g_{\alpha \tau} U_1^\alpha U_1^\beta
\]
and thus \( \{U^\alpha_\beta\} \in Sp(n) \).

The vanishing of the coefficients of \( \theta^\alpha \wedge \eta_s \) in (3.13), (3.14) and (3.15) gives

\[
\begin{align*}
  b_\alpha &= 2i\, U^\alpha_\sigma A^\sigma_1 = 2\pi^\sigma r^\sigma U^\sigma_\alpha A^\sigma_2 \\
  c_\alpha &= -2\pi^\sigma r^\sigma U^\sigma_\alpha A^\sigma_3 = -2i\pi^\sigma r^\sigma U^\sigma_\alpha A^\sigma_2 \\
  d_\alpha &= 2i\, U^\alpha_\sigma A^\sigma_3 = 2\pi^\sigma r^\sigma U^\sigma_\alpha A^\sigma_1 \\
  e_\alpha &= -2i\, U^\alpha_\sigma A^\sigma_2 = 2\pi^\sigma r^\sigma U^\sigma_\alpha A^\sigma_1
\end{align*}
\]

(3.18)

from which we deduce that \( A^\alpha_3 = iA^\alpha_2 = -\pi^\sigma r^\sigma A^\sigma_1 \). Thus, by setting \( r^\alpha \overset{def}{=} -iA^\alpha_1 \), we obtain

\[
\begin{align*}
  A^\alpha_1 &= i r^\alpha, & A^\alpha_2 &= \pi^\sigma r^\sigma, & A^\alpha_3 &= i\pi^\sigma r^\sigma, \\
  b_\alpha &= 2U^\alpha_\sigma r^\sigma, & c_\alpha &= -2iU^\alpha_\sigma r^\sigma, & d_\alpha &= -2\pi^\sigma r^\sigma r^\tau, & e_\alpha &= 2i\pi^\sigma r^\sigma r^\tau.
\end{align*}
\]

(3.19)

We substitute (3.19) back into (3.13), (3.14), (3.15) and consider the coefficients of \( \eta_1 \wedge \eta_2, \eta_2 \wedge \eta_3 \) and \( \eta_3 \wedge \eta_1 \) to obtain

\[
0 = b_2 + e_1 = b_1 - d_1 = b_1 - e_2 = b_3 + c_2 = b_2 + c_3 = b_1 + d_3,
\]

(3.20)

\[
4r^\alpha r^\alpha = c_1 + d_2 = c_1 + e_3 = d_2 + e_3.
\]

Let us define

\[
\lambda_1 = b_1, \quad \lambda_2 = b_2, \quad \lambda_3 = b_3.
\]

Then, the equations (3.20) imply that

\[
\begin{align*}
  c_1 &= 2r^\alpha r^\alpha, & c_2 &= -\lambda_3, & c_3 &= \lambda_2, \\
  d_1 &= \lambda_3, & d_2 &= 2r^\alpha r^\alpha, & d_3 &= -\lambda_1, \\
  e_1 &= -\lambda_2, & e_2 &= \lambda_1, & e_3 &= 2r^\alpha r^\alpha.
\end{align*}
\]

(3.22)

Now, the equations (3.9) follow by substituting (3.19), (3.21) and (3.22) into (3.11) and (3.12). It remains only to show that the one-forms \( \eta_1, \eta_2, \eta_3, \hat{\theta}^\alpha, \hat{\theta}^\alpha, \hat{\varphi}_0, \hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3 \) are pointwise linearly independent. Indeed, we have the relation

\[
\begin{pmatrix}
  \eta_1 \\
  \eta_2 \\
  \eta_3 \\
  \hat{\theta}^\alpha \\
  \hat{\theta}^\alpha \\
  \hat{\varphi}_0 \\
  \hat{\varphi}_1 \\
  \hat{\varphi}_2 \\
  \hat{\varphi}_3
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  i r^\alpha & \pi^\alpha r^\sigma & i\pi^\alpha r^\tau & U^\alpha_\beta & 0 & 0 & 0 & 0 & 0 \\
  -i r^\alpha & \pi^\alpha r^\sigma & -i\pi^\alpha r^\tau & 0 & U^\alpha_\beta & 0 & 0 & 0 & 0 \\
  \lambda_1 & \lambda_2 & \lambda_3 & 2U^\beta_\sigma r^\tau & 2U^\beta_\sigma r^\alpha & 1 & 0 & 0 & 0 \\
  2r^\alpha r^\tau & -\lambda_3 & \lambda_2 & -2iU^\beta_\sigma r^\tau & 2iU^\beta_\sigma r^\alpha & 0 & 1 & 0 & 0 \\
  -\lambda_2 & \lambda_1 & 2r^\alpha r^\tau & 2i\pi^\sigma r^\alpha r^\tau & -2i\pi^\sigma r^\alpha r^\tau & 0 & 0 & 1 & 0
\end{pmatrix},
\]

(3.23)

which is clearly a non-singular transformation, since \( \{U^\alpha_\beta\} \in Sp(n) \) is non-singular.

\[
\square
\]

Let us denote by \( G_1 \) the set of all matrices (transformations) \( A(U^\alpha_\beta, r^\alpha, \lambda_s) \) given by (3.23) for some real numbers \( \lambda_s \), complex numbers \( r^\alpha \), and \( \{U^\alpha_\beta\} \in Sp(n) \). Then, it is easy to see that \( G_1 \) is a group; for any two matrices \( A(U^\alpha_\beta, r^\alpha, \lambda_s), (U^\alpha_\beta, \bar{r}^\alpha, \bar{\lambda}_s) \in G_1 \),

\[
A(U^\alpha_\beta, r^\alpha, \lambda_s) \cdot A(U^\alpha_\beta, \bar{r}^\alpha, \bar{\lambda}_s) = A(U^\alpha_\beta, \bar{r}^\alpha, \bar{\lambda}_s) \in G_1,
\]
\[
\begin{align*}
\tilde{U}^\alpha = U^\alpha U^\beta \\
\tilde{r}^\alpha = U^\alpha \tilde{r}^\beta + r^\alpha
\end{align*}
\]
\[\begin{align*}
\tilde{\lambda}_1 &= \lambda_1 + \tilde{\lambda}_1 + 2iU_{\alpha\beta} \tilde{r}^\alpha - 2iU_{\bar{\alpha}\bar{\beta}} \tilde{r}^\beta \\
\tilde{\lambda}_2 &= \lambda_2 + \tilde{\lambda}_2 + 2\pi^\alpha U_{\bar{\alpha}\beta} \tilde{r}^\beta + 2\pi_{\bar{\alpha}} U_{\alpha\beta} \tilde{r}^\beta \\
\tilde{\lambda}_3 &= \lambda_3 + \tilde{\lambda}_3 - 2i\pi^\alpha U_{\bar{\alpha}\beta} \tilde{r}^\beta + 2i\pi_{\bar{\alpha}} U_{\alpha\beta} \tilde{r}^\beta.
\end{align*}
\]

We have also the following formula for the inverse matrix
\[
\left( A(U^\alpha, r^\alpha, \lambda_s) \right)^{-1} = A\left( (U^\alpha)_{\beta}, (\lambda_s^\alpha), (-\lambda_s) \right).
\]

(Notice that, according to our conventions, \(U^\alpha_{\beta} \) is the inverse of \(U^\beta_{\alpha} \); this is because \(U^\beta_{\alpha} \in Sp(n) \) is an orthogonal transformation of \(\mathbb{R}^{4n} \).

To describe the corresponding representation of the Lie algebra \(g_1 \), we differentiate the representation (3.23) of \(G_1 \) at the identity matrix \(Id = A(\delta^\alpha_{\beta}, 0, 0) \) and introduce the following parameterization for \(g_1 \) (being the tangent space of \(G_1 \) at \(Id \)):
\[
\Gamma_{\alpha\beta} \overset{def}{=} \pi_{\alpha\sigma}(dU^\sigma_{\beta})_{|Id}, \quad \phi^\alpha \overset{def}{=} -(d\phi^\alpha)_{|Id}, \quad \psi_s \overset{def}{=} -(d\psi)_{|Id}.
\]

Then, the representation of \(g_1 \) on \(TP_0 \) is given by the transformations of the form
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\eta_1 & \eta_2 & \eta_3 & \theta^3 \\
-\phi^\alpha & -\pi^\alpha_{\sigma} \phi^\beta & -i\pi^\alpha_{\sigma} \phi^\beta & -\pi^\alpha_{\sigma} \Gamma_{\sigma\beta} & 0 & 0 & 0 & 0 & \theta^3 \\
i\phi^\bar{\alpha} & -\pi^\bar{\alpha}_{\sigma} \phi^\bar{\beta} & i\pi^\bar{\alpha}_{\sigma} \phi^\bar{\beta} & -\pi^\bar{\alpha}_{\sigma} \Gamma_{\bar{\sigma}\bar{\beta}} & 0 & 0 & 0 & 0 & \theta^3 \\
-\psi_1 & -\psi_2 & -\psi_3 & -2\phi^\beta & -2\phi^\bar{\beta} & 0 & 0 & 0 & \varphi_0 \\
0 & -\psi_1 & -\psi_2 & 2i\phi^\beta & -2i\phi^\bar{\beta} & 0 & 0 & 0 & \varphi_1 \\
-\psi_3 & 0 & \psi_1 & 2\pi_{\sigma\beta} \phi^\sigma & -2\pi_{\bar{\sigma}\bar{\beta}} \phi^\bar{\sigma} & 0 & 0 & 0 & \varphi_2 \\
\psi_2 & -\psi_1 & 0 & 2i\pi_{\sigma\beta} \phi^\sigma & -2i\pi_{\bar{\sigma}\bar{\beta}} \phi^\bar{\sigma} & 0 & 0 & 0 & \varphi_3
\end{pmatrix}
\]

where \(\Gamma_{\alpha\beta}, \phi^\alpha \) are complex, \(\psi_s \) are real and the following equations are satisfied:
\[
\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}, \quad (j\Gamma)_{\alpha\beta} = \Gamma_{\alpha\beta},
\]

i.e., \(\{\pi^\alpha_{\sigma} \Gamma_{\sigma\beta}\} \in sp(n) \) (cf. Lemma 2.1).

By Lemma 3.1 and Lemma 3.2, it follows that the manifold \(P_0 \) has an induced \(G_1\)-structure. Denote by \(P_1 \) its principle \(G_1\)-bundle and let \(\pi_1 : P_1 \to P_0 \) be the corresponding principle bundle projection. The local sections of \(P_1 \) are precisely the local coframings \(\{\eta_1, \eta_2, \eta_2, \theta^\alpha, \theta^\bar{\alpha}, \varphi_0, \varphi_1, \varphi_2, \varphi_3\} \) for \(TP_0 \) for which the assertion of Lemma 3.1 is satisfied. In \(P_1 \) there are intrinsically (and hence globally) defined one-forms (for which we keep the same notation) \(\eta_1, \eta_2, \eta_2, \theta^\alpha, \theta^\bar{\alpha}, \varphi_0, \varphi_1, \varphi_2, \varphi_3 \) which are everywhere linearly independent and satisfy the structure equations
\[
\begin{align*}
d\eta_1 &= -\varphi_0 \wedge \eta_1 - \varphi_2 \wedge \eta_3 + \varphi_3 \wedge \eta_2 + 2i\eta_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \\
d\eta_2 &= -\varphi_0 \wedge \eta_2 - \varphi_2 \wedge \eta_1 + \varphi_3 \wedge \eta_3 + \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + \pi_{\bar{\alpha}} \theta^\bar{\alpha} \wedge \theta^\bar{\beta} \\
d\eta_3 &= -\varphi_0 \wedge \eta_3 - \varphi_1 \wedge \eta_2 + \varphi_2 \wedge \eta_1 - i\pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + i\pi_{\bar{\alpha}} \theta^\bar{\alpha} \wedge \theta^\bar{\beta}.
\end{align*}
\]

**Theorem 3.3.** On \(P_1 \), there exists a unique set of complex one-forms \(\Gamma_{\alpha\beta}, \phi^\alpha \) and real one-forms \(\psi_1, \psi_2, \psi_3 \) such that
\[
\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}, \quad (j\Gamma)_{\alpha\beta} = \Gamma_{\alpha\beta}.
\]
and the equations

\[
\begin{align*}
    d\theta^\alpha &= -i\phi^\alpha \wedge \eta_1 - \pi^\alpha_\beta \phi^\beta \wedge (\eta_2 + i\eta_3) - \pi^\alpha_\beta \Gamma_{\alpha\beta} \wedge \theta^\beta - \frac{1}{2}(\varphi_0 + i\varphi_1) \wedge \theta^\alpha - \frac{1}{2} \pi^\alpha_\beta (\varphi_2 + i\varphi_3) \wedge \theta^\beta, \\
    d\varphi_0 &= -\psi_1 \wedge \eta_1 - \psi_2 \wedge \eta_2 - \psi_3 \wedge \eta_3 - 2\phi_\beta \wedge \theta^\beta - 2\phi_\beta \wedge \theta^\beta, \\
    d\varphi_1 &= -\varphi_2 \wedge \varphi_3 - \psi_2 \wedge \eta_3 + \psi_3 \wedge \eta_2 + 2i\phi_\beta \wedge \theta^\beta - 2i\phi_\beta \wedge \theta^\beta, \\
    d\varphi_2 &= -\varphi_3 \wedge \varphi_1 - \psi_3 \wedge \eta_1 + \psi_1 \wedge \eta_3 - 2\pi_{\alpha\beta} \phi^\beta \wedge \theta^\beta - 2\pi_{\alpha\beta} \phi^\beta \wedge \theta^\beta, \\
    d\varphi_3 &= -\varphi_1 \wedge \varphi_2 - \psi_1 \wedge \eta_2 + \psi_2 \wedge \eta_1 + 2i\pi_{\alpha\beta} \phi^\beta \wedge \theta^\beta - 2i\pi_{\alpha\beta} \phi^\beta \wedge \theta^\beta,
\end{align*}
\]

(3.31)

are satisfied.

Furthermore, the set

\[
(\eta_1, \eta_2, \eta_3, \theta^\alpha, \varphi_0, \varphi_1, \varphi_2, \varphi_3) \cup \{\Gamma_{\alpha\beta} : \alpha \leq \beta\} \cup \{\phi^\alpha, \phi^\beta, \psi_1, \psi_2, \psi_3\}
\]

is a global coframing for the (complexified) tangent bundle $TP_1$.

Proof. The exterior differentiation of the structure equations (3.29) gives:

\[
\begin{align*}
    0 &= -d\varphi_0 \wedge \eta_1 + (d\varphi_3 + \varphi_1 \wedge \varphi_2) \wedge \eta_2 - (d\varphi_2 + \varphi_3 \wedge \varphi_1) \wedge \eta_3 - i\gamma_{0\alpha\beta} \left(0 + \pi^\alpha_\beta \phi^\beta \right) \wedge \theta^\alpha - i\gamma_{0\alpha\beta} \left(0 + \pi^\alpha_\beta \phi^\beta \right) \wedge \theta^\beta; \\
    0 &= -d\varphi_3 + \varphi_1 \wedge \varphi_2 \wedge \eta_1 - d\varphi_3 \wedge \eta_2 + d\varphi_1 + \varphi_2 \wedge \varphi_3 \wedge \eta_3 + \pi_{\alpha\beta} \left(0 + \pi^\alpha_\beta \phi^\beta \right) \wedge \theta^\alpha + \pi_{\alpha\beta} \left(0 + \pi^\alpha_\beta \phi^\beta \right) \wedge \theta^\beta; \\
    0 &= (d\varphi_2 + \varphi_3 \wedge \varphi_1) \wedge \eta_1 - (d\varphi_1 + \varphi_2 \wedge \varphi_3) \wedge \eta_2 - d\varphi_0 + \eta_3 + i\gamma_{0\alpha\beta} \left(0 + \pi^\alpha_\beta \phi^\beta \right) \wedge \theta^\alpha + \pi_{\alpha\beta} \left(0 + \pi^\alpha_\beta \phi^\beta \right) \wedge \theta^\beta.
\end{align*}
\]

(3.33)

From this, it follows that $d\theta^\alpha \equiv 0$ modulo $\{\eta_1, \theta^\beta, \varphi_3\}$. Letting

\[
d\theta^\alpha \equiv X^\alpha_\beta \wedge \theta^\beta + X^\alpha_\beta \wedge \theta^\beta \mod \eta_1,
\]

for some one forms $X^\alpha_\beta, X^\alpha_\beta$, and substituting back into (3.33), we compute modulo $\eta_1$ (i.e., by ignoring the terms involving $\eta_1$):

\[
\begin{align*}
    0 &= -\left(X_{\alpha\beta} - X_{\beta\alpha} + \pi_{\alpha\beta} (\varphi_2 - i\varphi_3) \right) \wedge \theta^\alpha \wedge \theta^\beta + \left(X_{\alpha\beta} - X_{\beta\alpha} + \pi_{\alpha\beta} (\varphi_2 - i\varphi_3) \right) \wedge \theta^\alpha \wedge \theta^\beta; \\
    0 &= \left(\pi^\alpha_\beta X_{\alpha\sigma} - \pi^\alpha_\beta X_{\alpha\sigma} + \pi_{\alpha\beta} (\varphi_0 + i\varphi_1) \right) \wedge \theta^\alpha \wedge \theta^\beta; \\
    0 &= -i\left(\pi^\alpha_\beta X_{\alpha\sigma} - \pi^\alpha_\beta X_{\alpha\sigma} + \pi_{\alpha\beta} (\varphi_0 + i\varphi_1) \right) \wedge \theta^\alpha \wedge \theta^\beta;
\end{align*}
\]

(3.35)

Adding the third equation of (3.35), multiplied by $i$, to the second gives

\[
2 \left(\pi^\alpha_\beta X_{\beta\alpha} - \pi^\alpha_\beta X_{\alpha\beta} + \pi_{\alpha\beta} (\varphi_0 + i\varphi_1) \right) \wedge \theta^\alpha \wedge \theta^\beta \equiv 0 \mod \eta_1,
\]

from which we deduce that both the expressions in the large parentheses vanish modulo $\{\theta^\alpha, \theta^\beta, \eta_1\}$. Let

\[
2X_{\beta\alpha} + \pi_{\beta\alpha} (\varphi_2 + i\varphi_3) \equiv Y_{\beta\alpha\gamma} \theta^\gamma + Y_{\beta\alpha\gamma} \theta^\gamma \mod \eta_1
\]

(3.36)
for some functions $Y_{\tilde{\gamma} \tilde{\sigma} \tilde{\rho}}$, $Y_{\tilde{\gamma} \tilde{\sigma} \tilde{\rho}}$. Then, by substituting back into (3.36) and considering only the coefficient of $\theta^\alpha \wedge \theta^\beta \wedge \tilde{\theta}^\gamma$, we obtain the symmetry $Y_{\tilde{\gamma} \tilde{\sigma} \tilde{\rho}} = Y_{\tilde{\gamma} \tilde{\rho} \tilde{\sigma}}$. Therefore, we have that

$$
(3.38) \quad d\theta^\alpha \equiv X^\alpha_\beta \wedge \theta^\beta + g^{\alpha\beta} X^\beta_\sigma \wedge \theta^\sigma + \frac{1}{2} g^{\alpha\beta} \left( X^\beta_\sigma - \frac{1}{2} Y_{\tilde{\beta} \tilde{\sigma} \tilde{\rho}} \tilde{\theta}^\gamma \right) \wedge \theta^\beta \equiv X^\alpha_\beta \wedge \theta^\beta + g^{\alpha\beta} \left( -\frac{1}{2} \pi_{\tilde{\sigma} \tilde{\beta}} (\varphi_2 + i\varphi_3) + \frac{1}{2} Y_{\tilde{\beta} \tilde{\sigma} \tilde{\rho}} \tilde{\theta}^\gamma \right) \wedge \theta^\beta
$$

This means that among all the one-forms $X^\alpha_\beta$, $X^\alpha_\beta$, for which (3.34) is satisfied, we can find such that

$$
(3.39) \quad X^\alpha_\beta = -\frac{1}{2} \pi_{\tilde{\beta} \tilde{\alpha} \tilde{\gamma}} \tilde{\theta}^\gamma.
$$

Assuming (3.39), we have

$$
(3.40) \quad d\theta^\alpha \equiv X^\alpha_\beta \wedge \theta^\beta - \frac{1}{2} \pi_{\tilde{\beta} \tilde{\alpha} \tilde{\gamma}} \tilde{\theta}^\gamma \wedge \theta^\beta \mod \eta_s
$$

and the equations (3.35) (modulo $\eta_s$) become:

$$
0 \equiv \left( X_{\alpha \beta} + X_{\tilde{\beta} \alpha} + g_{\alpha \beta} \varphi_0 \right) \wedge \theta^\alpha \wedge \theta^\beta;
$$

$$
0 \equiv \left( \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \frac{1}{2} \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \pi_{\alpha \beta} (\varphi_0 + i\varphi_1) \right) \wedge \theta^\alpha \wedge \theta^\beta
$$

$$
0 \equiv \left( \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \frac{1}{2} \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \pi_{\alpha \beta} (\varphi_0 + i\varphi_1) \right) \wedge \theta^\alpha \wedge \theta^\beta
$$

$$
0 \equiv \left( \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \frac{1}{2} \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \pi_{\alpha \beta} (\varphi_0 + i\varphi_1) \right) \wedge \theta^\alpha \wedge \theta^\beta
$$

$$
0 \equiv \left( \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \frac{1}{2} \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \pi_{\alpha \beta} (\varphi_0 + i\varphi_1) \right) \wedge \theta^\alpha \wedge \theta^\beta
$$

$$
- \left( \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \frac{1}{2} \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \pi_{\alpha \beta} (\varphi_0 + i\varphi_1) \right) \wedge \theta^\alpha \wedge \theta^\beta.
$$

Let us define

$$
(3.42) \quad \tilde{\Gamma}_{\alpha \beta} \overset{df}{=} \pi^\beta_{\tilde{\alpha}} \left( X_{\tilde{\beta} \sigma} + \frac{1}{2} \pi_{\tilde{\beta} \sigma} (\varphi_0 + i\varphi_1) \right).
$$

Then, (3.40) and (3.41) yield

$$
\left\{ \begin{array}{ll}
0 & \equiv \left( \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \frac{1}{2} \pi^\beta_{\tilde{\alpha}} X_{\tilde{\beta} \sigma} + \pi_{\alpha \beta} (\varphi_0 + i\varphi_1) \right) \wedge \theta^\alpha \wedge \theta^\beta
\end{array} \right. \mod \eta_s
$$

The third equation of (3.43) implies the the existence of (unique) functions $C_{\alpha \beta \gamma}$ satisfying

$$
(3.44) \quad \begin{array}{ll}
\tilde{\Gamma}_{\alpha \beta} - \tilde{\Gamma}_{\beta \alpha} & \equiv C_{\alpha \beta \gamma} \theta^\gamma \mod \eta_s
\end{array}
$$

$$
C_{\alpha \beta \gamma} = -C_{\beta \alpha \gamma}
$$

$$
C_{\alpha \beta \gamma} + C_{\beta \gamma \alpha} + C_{\gamma \alpha \beta} = 0.
$$

Let $D_{\alpha \beta \gamma} \overset{df}{=} \frac{1}{2} (C_{\alpha \beta \gamma} + C_{\alpha \gamma \beta})$ and

$$
\tilde{\Gamma}_{\alpha \beta} \overset{df}{=} \tilde{\Gamma}_{\alpha \beta} - D_{\alpha \beta \gamma} \theta^\gamma.
$$
Then, modulo $\eta_s$,
\[
\hat{\Gamma}_{\alpha\beta} - \hat{\Gamma}_{\beta\alpha} \equiv \bar{\Gamma}_{\alpha\beta} - \bar{\Gamma}_{\beta\alpha} - (D_{\alpha\beta\gamma} - D_{\beta\alpha\gamma})\theta^\gamma \equiv C_{\alpha\beta\gamma}\theta^\gamma - (D_{\alpha\beta\gamma} - D_{\beta\alpha\gamma})\theta^\gamma \\
\equiv C_{\alpha\beta\gamma}\theta^\gamma - \frac{1}{3}(C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\alpha\gamma})\theta^\gamma \\
\equiv C_{\alpha\beta\gamma}\theta^\gamma - \frac{1}{3}(3C_{\alpha\beta\gamma} - C_{\alpha\beta\gamma} - C_{\gamma\alpha\beta} - C_{\beta\gamma\alpha})\theta^\gamma \equiv 0.
\]
And since $D_{\alpha\beta\gamma} = D_{\alpha\beta\gamma}$, the equations (3.43) become
\[
\begin{align*}
\begin{cases}
\hat{\Gamma}_{\alpha\beta} = \hat{\Gamma}_{\beta\alpha} + \hat{\Gamma}_{\alpha\beta} \\
(\alpha \beta) \quad \text{mod} \quad \eta_s
\end{cases}
\end{align*}
\]
By the second identity of (3.45), $\pi^\alpha_{\alpha\beta} \hat{\Gamma}_{\alpha\beta} + \pi^\beta_{\beta\beta} \hat{\Gamma}_{\beta\beta} \equiv 0 \mod \{\theta^\alpha, \theta^\beta, \eta_s\}$. Hence there exists functions $A_{\alpha\beta\gamma}, B_{\alpha\beta\gamma}$ so that
\[
(3.46) \quad \pi^\alpha_{\alpha\beta} \hat{\Gamma}_{\alpha\beta} + \pi^\beta_{\beta\beta} \hat{\Gamma}_{\beta\beta} \equiv \pi^\alpha_{\alpha\beta} A_{\alpha\beta\gamma} + \pi^\beta_{\beta\beta} B_{\beta\beta\gamma} \equiv 0 \mod \eta_s
\]
and $A_{\alpha\beta\gamma} = A_{\gamma\beta\alpha}, B_{\alpha\beta\gamma} = B_{\alpha\beta\gamma}$. We multiply (3.46) by $\pi^\beta_{\alpha\beta}$ and sum over $\beta$ to obtain
\[
(3.47) \quad (j\hat{\Gamma})_{\alpha\tau} - \hat{\Gamma}_{\tau\alpha} \equiv \hat{A}_{\alpha\tau\gamma} + \hat{B}_{\alpha\tau\gamma} \equiv \hat{A}_{\alpha\tau\gamma} + \hat{B}_{\alpha\tau\gamma} \equiv 0 \mod \eta_s.
\]
Since, by the third identity of (3.45), the LHS of the (3.47) is symmetric (modulo $\eta_s$) in the indices $\alpha, \tau, \beta, \gamma$, so is the RHS. This implies that both $A_{\alpha\beta\gamma}$ and $B_{\alpha\beta\gamma}$ are totally symmetric in $\alpha, \beta, \gamma$.

Applying the map $j$ to both sides of (3.47) gives
\[
\hat{\Gamma}_{\alpha\tau} - (j\hat{\Gamma})_{\alpha\tau} \equiv \pi^\beta_{\alpha\beta} \pi^\tau_{\beta\beta} \left(- A_{\alpha\beta\gamma} + \pi^\gamma_{\alpha\beta} \pi^\beta_{\beta\beta} B_{\beta\beta\gamma}\theta^\gamma\right) \equiv \hat{B}_{\alpha\tau\gamma} - \pi^\gamma_{\alpha\beta} \pi^\beta_{\beta\beta} A_{\alpha\beta\gamma} \equiv 0 \mod \eta_s.
\]
which, by comparison with the initial equation (3.47), yields $A_{\alpha\beta\gamma} = B_{\alpha\beta\gamma}$.

If we set
\[
\hat{\Gamma}_{\alpha\beta} \overset{\text{def}}{=} \hat{\Gamma}_{\alpha\beta} - A_{\alpha\beta\gamma} \theta^\gamma,
\]
then, by (3.45), (3.47), and the just established properties of $A_{\alpha\beta\gamma} = B_{\alpha\beta\gamma}$, we obtain that
\[
(3.48) \quad \begin{cases}
(\hat{\Gamma})_{\alpha\beta} \equiv \hat{\Gamma}_{\alpha\beta} \\
\hat{\Gamma}_{\alpha\beta} \equiv \hat{\Gamma}_{\beta\alpha} \quad \text{mod} \quad \eta_s
\end{cases}
\]
and
\[
(3.49) \quad d\theta^\alpha = -\pi^\alpha_{\alpha\beta} \hat{\Gamma}_{\alpha\beta} \wedge \theta^\beta - \frac{1}{2}(\varphi_0 + i\varphi_1) \wedge \theta^\beta - \frac{1}{2}\pi^\gamma_{\alpha\beta} (\varphi_2 + i\varphi_3) \wedge \theta^\beta + (X_s)^\alpha \wedge \eta_s,
\]
for some one-forms $(X_s)^\alpha$.

Substituting (3.49) back into (3.33) yields:
\[
0 = -\left(d\varphi_0 - 2ig_{\alpha\beta}(X_1)^\alpha \wedge \theta^\beta + 2ig_{\alpha\beta}(X_1)^\alpha \wedge \theta^\beta\right) \wedge \eta_1 + \left(d\varphi_3 + \varphi_1 \wedge \varphi_2 + 2ig_{\alpha\beta}(X_2)^\alpha \wedge \theta^\beta\right) \wedge \eta_3
\]
\[
- 2ig_{\alpha\beta}(X_2)^\alpha \wedge \theta^\beta \wedge \eta_3 - \left(d\varphi_3 + \varphi_1 \wedge \varphi_2 + 2ig_{\alpha\beta}(X_3)^\alpha \wedge \theta^\beta\right) \wedge \eta_3
\]
\[
\equiv \left(d\varphi_0 + 2ig_{\alpha\beta}(X_1)^\alpha \wedge \theta^\beta\right) \wedge \eta_1 - \left(d\varphi_0 + 2ig_{\alpha\beta}(X_2)^\alpha \wedge \theta^\beta\right) \wedge \eta_3
\]
\[
+ 2\pi_{\alpha\beta}(X_2)^\alpha \wedge \theta^\beta \wedge \eta_3 + \left(d\varphi_1 + \varphi_2 \wedge \varphi_3 - 2\pi_{\alpha\beta}(X_3)^\alpha \wedge \theta^\beta\right) \wedge \eta_1
\]
\[
+ 2\pi_{\alpha\beta}(X_2)^\alpha \wedge \theta^\beta \wedge \eta_3 + \left(d\varphi_0 + 2ig_{\alpha\beta}(X_2)^\alpha \wedge \theta^\beta\right) \wedge \eta_3
\]
\[
+ 2\pi_{\alpha\beta}(X_3)^\alpha \wedge \theta^\beta \wedge \eta_3 + 2\pi_{\alpha\beta}(X_2)^\alpha \wedge \theta^\beta \wedge \eta_3;
\]
the equations (3.48), where the asterisk represents coefficients which are irrelevant at the moment. It follows that

\[ H \Rightarrow Z \]

we obtain that

\[ (3.52) \]

\[ \alpha \beta \]

Therefore, if we set

\[ (3.54) \]

we have

\[
\begin{align*}
0 &= \left( -2i\bar{g}_{\alpha\beta}((X_2)^{\alpha} - i\pi_2^{\alpha} (X_2)^{\gamma}) \right. \\
& \left. - 2i\pi_{a\beta}(X_2)^{\alpha} \wedge \theta^\beta \right) \wedge \eta_1 + (d\varphi_0 + 2i\pi_{a\beta}(X_2)^{\alpha} \wedge \theta^\beta) \wedge \eta_2 - (d\varphi_2 + \varphi_2 \wedge \varphi_3)
\end{align*}
\]

\[ (3.53) \]

\[ \theta \]

Therefore, if we set

\[ (3.56) \]

the equations (3.54) become

\[
\begin{align*}
(X_1)^\alpha &= -i\hat{\phi}^\alpha + i\pi_2^{\alpha} Y_2^\beta \theta^\beta \\
(X_2)^\alpha &= -\pi_2^{\alpha} \hat{\theta}^\beta + Y_2^\alpha \theta^\beta + \ldots \\
(X_3)^\alpha &= -i\pi_2^{\alpha} \hat{\sigma} + Z_2^\alpha \theta^\beta + \ldots
\end{align*}
\]

(3.55)

(3.56)
Substituting (3.56) and (3.55) into (3.49), we obtain

\[ d\theta^\alpha = -\pi^\alpha_\sigma \Gamma_{\sigma\beta} \wedge \theta^\beta - \frac{1}{2} (\varphi_0 + i \varphi_1) \wedge \theta^\alpha - \frac{1}{2} \pi^\beta_\delta (\varphi_2 + i \varphi_3) \wedge \theta^\delta - i \delta^\alpha \wedge \eta_1 \\
- \pi^\alpha_\sigma \theta^\sigma \wedge (\eta_2 + i \eta_3) + (C_{st})^\alpha \eta_s \wedge \eta_t, \]

where \((C_{st})^\alpha = -(C_{st})^\alpha\) are some appropriate coefficients.

Finally, if we set

\[ \phi^\alpha \overset{\text{def}}{=} \hat{\phi}^\alpha + \left( \pi^\alpha_\sigma (C_{12})^\sigma - (C_{23})^\alpha - i \pi^\alpha_\sigma (C_{31})^\sigma \right) \eta_1 + i \left( -(C_{12})^\alpha + \pi^\alpha_\sigma (C_{23})^\sigma + i (C_{31})^\alpha \right) \eta_2 \\
+ \left( -(C_{12})^\alpha - \pi^\alpha_\sigma (C_{23})^\sigma + i (C_{31})^\alpha \right) \eta_3, \]

then, by (3.57), we obtain that

\[ d\theta^\alpha = -\pi^\alpha_\sigma \Gamma_{\sigma\beta} \wedge \theta^\beta - \frac{1}{2} (\varphi_0 + i \varphi_1) \wedge \theta^\alpha - \frac{1}{2} \pi^\beta_\delta (\varphi_2 + i \varphi_3) \wedge \theta^\delta - i \phi^\alpha \wedge \eta_1 - \pi^\alpha_\sigma \theta^\sigma \wedge (\eta_2 + i \eta_3). \]

So far, we have shown, by (3.48) and (3.56), that the equations \(\Gamma_{\alpha\beta} \equiv \Gamma_{\beta\alpha}\) and \((i\Gamma)_{\alpha\beta} \equiv \Gamma_{\alpha\beta}\) are satisfied only modulo \(\eta_\kappa\). Let us assume

\[ \begin{align*}
\pi^\alpha_\beta \left( (i\Gamma)_{\alpha\sigma} - \Gamma_{\sigma\alpha} \right) &= (A_s)_{\alpha\beta} \eta_s \\
\Gamma_{\alpha\beta} - \Gamma_{\beta\alpha} &= (B_s)_{\alpha\beta} \eta_s
\end{align*} \]

for some functions \((A_s)_{\alpha\beta}\) and \((B_s)_{\alpha\beta} = -(B_s)_{\beta\alpha}\).

Substituting (3.58) and (3.59) back into (3.33) gives:

\[ \begin{align*}
0 &= - \left( d\varphi_0 + 2 \phi^\beta \wedge \theta^\beta + 2 \phi^\beta \wedge \theta^\beta - 2 i (A_1)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \right) \wedge \eta_1 \\
&+ \left( d\varphi_3 + \varphi_1 \wedge \varphi_2 - 2 i \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + 2 i \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + 2 i (A_2)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \right) \wedge \eta_2 \\
&- \left( d\varphi_2 + \varphi_3 \wedge \varphi_1 + 2 \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + 2 \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta - 2 i (A_3)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \right) \wedge \eta_3;
\end{align*} \]

\[ \begin{align*}
0 &= - \left( d\varphi_3 + \varphi_1 \wedge \varphi_2 - 2 i \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + 2 i \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta - (B_1)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta - (B_1)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \right) \wedge \eta_1 \\
&- \left( d\varphi_0 + 2 \phi^\beta \wedge \theta^\beta + 2 \phi^\beta \wedge \theta^\beta - (B_2)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta - (B_2)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \right) \wedge \eta_2 \\
&+ \left( d\varphi_1 + \varphi_2 \wedge \varphi_3 - 2 i \phi^\beta \wedge \theta^\beta + 2 i \phi^\beta \wedge \theta^\beta + (B_3)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + (B_3)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \right) \wedge \eta_3;
\end{align*} \]

\[ \begin{align*}
0 &= \left( d\varphi_2 + \varphi_3 \wedge \varphi_1 + 2 \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + 2 \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta - i (B_1)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + i (B_1)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \right) \wedge \eta_1 \\
&- \left( d\varphi_1 + \varphi_2 \wedge \varphi_3 - 2 i \phi^\beta \wedge \theta^\beta + 2 i \phi^\beta \wedge \theta^\beta + i (B_2)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta - i (B_2)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \right) \wedge \eta_2 \\
&- \left( d\varphi_0 + 2 \phi^\beta \wedge \theta^\beta + 2 \phi^\beta \wedge \theta^\beta + i (B_3)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta - i (B_3)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \right) \wedge \eta_3.
\end{align*} \]

By wedging the equation (3.60) with \(\eta_2 \wedge \eta_3\) and subtracting from the result the equation (3.60) wedged with \(\eta_3 \wedge \eta_1\), we see that

\[ 2 i (A_1)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 = - (B_2)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 - 2 (B_2)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \wedge \eta_1 \wedge \eta_2 \wedge \eta_3, \]

and hence \((A_1)_{\alpha\beta} = 0, (B_2)_{\alpha\beta} = 0\). Proceeding similarly, we obtain \((A_3)_{\alpha\beta} = 0, (B_3)_{\alpha\beta} = 0\) and thus, by (3.59), the properties (3.30). A substitution back into (3.60), (3.61) and (3.62) yields:
It follows that

(3.64)

Substituting \((\alpha\beta\gamma, (\alpha\beta)\gamma, (\alpha\beta)\gamma, \gamma)\) into the first equation of

(3.65)

These imply the existence of real one-forms \(A_s, B_s\) so that

\[
\begin{align*}
0 &= -\left( d\varphi_0 + 2\phi_\beta \wedge \theta^\beta + 2\phi_\beta \wedge \theta^\bar{\beta} \right) \wedge \eta_1 + \left( d\varphi_3 + \varphi_1 \wedge \varphi_2 - 2i\pi_{\alpha\beta}\phi^\alpha \wedge \theta^\beta \\
&\quad + 2i\pi_{\alpha\bar{\beta}}\phi^{\bar{\alpha}} \wedge \theta^\beta \right) \wedge \eta_2 - \left( d\varphi_2 + \varphi_3 \wedge \varphi_1 + 2\pi_{\alpha\beta}\phi^\alpha \wedge \theta^\beta + 2\pi_{\bar{\alpha}\bar{\beta}}\phi^{\bar{\alpha}} \wedge \theta^\beta \right) \wedge \eta_3;
\end{align*}
\]

\[
0 = -\left( d\varphi_3 + \varphi_1 \wedge \varphi_2 - 2i\pi_{\alpha\beta}\phi^\alpha \wedge \theta^\beta + 2i\pi_{\alpha\bar{\beta}}\phi^{\bar{\alpha}} \wedge \theta^\beta \right) \wedge \eta_1 - \left( d\varphi_0 + 2\phi_\beta \wedge \theta^\beta \\
&\quad + 2\phi_\beta \wedge \theta^\bar{\beta} \right) \wedge \eta_2 + \left( d\varphi_2 + \varphi_3 \wedge \varphi_1 - 2i\phi_\beta \wedge \theta^\beta + 2i\phi_\bar{\beta} \wedge \theta^\bar{\beta} \right) \wedge \eta_3;
\]

\[
0 = \left( d\varphi_2 + \varphi_3 \wedge \varphi_1 + 2\pi_{\alpha\beta}\phi^\alpha \wedge \theta^\beta + 2\pi_{\alpha\bar{\beta}}\phi^{\bar{\alpha}} \wedge \theta^\bar{\beta} \right) \wedge \eta_1 - \left( d\varphi_1 + \varphi_2 \wedge \varphi_3 - 2i\phi_\beta \wedge \theta^\beta \\
&\quad + 2i\phi_\beta \wedge \theta^\bar{\beta} \right) \wedge \eta_2 - \left( d\varphi_0 + 2\phi_\beta \wedge \theta^\beta + 2\phi_\bar{\beta} \wedge \theta^\bar{\beta} \right) \wedge \eta_3.
\]

Substituting (3.65) into (3.64), we obtain the relations

\[
B_{11} = B_{22} = B_{33} = 0, \quad B_{23} = -B_{32} = A_1, \quad B_{31} = -B_{13} = A_2, \quad B_{12} = -B_{21} = A_3.
\]

Hence, if setting

\[
\psi_s \overset{def}{=} -A_s,
\]

the one-forms \(\Gamma_{\alpha\beta}, \phi^\alpha, \psi_s\) satisfy (3.31) and are as required in the theorem.

To prove the uniqueness, assume that \(\tilde{\Gamma}_{\alpha\beta}, \tilde{\phi}^\alpha, \tilde{\psi}_s\) are any other one-forms satisfying the requirements of the theorem and let

(3.66)

Then, \(L_{\alpha\beta} = L_{\beta\alpha}\), \((\mathcal{L})_{\alpha\beta} = L_{\alpha\beta}\) and by subtraction, we obtain the identities

(3.67)

It follows that

(3.68)

where \(L_{\alpha\beta\gamma}, L_{\alpha\beta\gamma}, (\mathcal{L})_{\alpha\beta\gamma}, (\mathcal{L})_{\alpha\beta\gamma}, (\mathcal{L})_{\alpha\beta\gamma}, (\mathcal{L})_{\alpha\beta\gamma}, (\mathcal{L})_{\alpha\beta\gamma}, (\mathcal{L})_{\alpha\beta\gamma}\) are some appropriate coefficients, and also

(3.69)

We substitute (3.68) into the first equation of (3.67). The vanishing of the coefficient of \(\theta^\beta \wedge \theta^\bar{\beta}\) gives \(L_{\alpha\beta\gamma} = 0\) which, by the second identity of (3.69), implies that also \(L_{\alpha\beta\gamma} = 0\). Proceeding similarly, we easily obtain that the rest of the coefficients in (3.68) vanish as well and hence the uniqueness of the one-forms \(\Gamma_{\alpha\beta}, \phi^\alpha, \psi_s\).

Finally, the fact that the one-forms

(3.70)

\[\{\eta_1, \eta_2, \eta_3, \theta^\alpha, \theta^\bar{\alpha}, \theta^\beta, \theta^\bar{\beta}, \varphi_0, \varphi_1, \varphi_2, \varphi_3\} \cup \{\Gamma_{\alpha\beta} : \alpha \leq \beta\} \cup \{\phi^\alpha, \phi^\bar{\alpha}, \psi_1, \psi_2, \psi_3\}\]
are pointwise linearly independent is easily derived form the observation that, by construction, 
η₁, η₂, η₃, θ₀, θ₁, φ₀, φ₁, φ₂, φ₃ are pointwise linearly independent and semibasic (w.r.t. the projection 
π₁ : P₁ → P₀), whereas

\[ \{ \Gamma_{αβ} : α ≤ β \} ∪ \{ φ'^{α}, φ'^{β}, ψ_1, ψ_2, ψ_3 \} \]

are independent modulo \{ η₁, η₂, η₃, θ₀, θ₁, φ₀, φ₁, φ₂, φ₃ \}. The latter is a consequence of the structure equations (3.31).

4. The Curvature and the Bianchi identities

In order to understand the curvature components and their properties, we shall have to compute the full 
structure equations, the corresponding Bianchi identities and some of their differential consequences, all in 
parallel. Thus we shall first provide three collections of the resulting formulae (in the next three propositions) 
and then go through all the computations in one package. Perhaps reading the computations will enlighten 
the three propositions best. Further links are available in the appendix.

Proposition 4.1 (Curvature components). On P₁, there exist unique, globally defined, complex-valued functions

\[ S_{αβγδ}, V_{αβγ}, L_{αβ}, M_{αβ}, C_α, K_α, P, Q, R \]

so that:

(I) Each of the arrays \{ S_{αβγδ} \}, \{ V_{αβγ} \}, \{ L_{αβ} \}, \{ M_{αβ} \} is totally symmetric in its indices.

(II) We have

\[ \begin{align*}
(ιS)_{αβγδ} &= S_{αβγδ} \\
(ιL)_{αβ} &= L_{αβ} \\
ιR &= R.
\end{align*} \]

(III) The exterior derivatives \( dΓ_{αβ} \), \( dφ_α \) and \( dv_α \) are given by

\[ \begin{align*}
dΓ_{αβ} &= -π^{ασ}Γ_{ασ} ∧ Γ_{ββ} + 2π^{α}_α(φ_β ∧ θ_α - φ_α ∧ θ_β) + 2π^{α}_α(φ_α ∧ θ_α - φ_σ ∧ θ_α) \\
&+ π^{σ}_α S_{σβγ} θ_σ ∧ θ_γ + \left( V_{αβγ} θ_γ + π^{σ}_α π^{τ}_β V_{στγ} θ_γ \right) ∧ η_1 \\
&- iπ^{γ}_γ V_{αβσ} θ_σ ∧ (η_2 + iη_3) + i(ιV)_{αβγ} θ_γ ∧ (η_2 - iη_3) \\
&- iL_{αβ} (η_2 + iη_3) ∧ (η_2 - iη_3) + M_{αβ} η_1 ∧ (η_2 + iη_3) + (ιM)_{αβ} η_1 ∧ (η_2 - iη_3),
\end{align*} \]

\[ \begin{align*}
dφ_α &= \frac{1}{2}(φ_0 + iφ_1) ∧ φ_α + \frac{1}{2}π_α(φ_2 - iφ_3) ∧ φ_γ - π^{σ}_α Γ_{σγ} ∧ φ_γ ∧ iψ_1 ∧ θ_α \\
&- iπ^{σ}_γ (ψ_2 - iψ_3) ∧ θ_γ - iπ^{σ}_γ V_{αγσ} θ_σ ∧ θ_γ + M_{αγ} θ_γ ∧ η_1 + π^{σ}_α L_{σγ} θ_γ ∧ η_1 \\
&+ iL_{αγ} θ_γ ∧ (η_2 - iη_3) - iπ^{σ}_γ M_{ασ} θ_γ ∧ η_2 + iη_3 - C_α (η_2 + iη_3) ∧ (η_2 - iη_3) \\
&+ K_α η_1 ∧ (η_2 + iη_3) + iπ_α C_σ η_1 ∧ (η_2 - iη_3),
\end{align*} \]

\[ \begin{align*}
dψ_1 &= φ_0 ∧ ψ_2 ∧ φ_3 ∧ ψ_3 - 4iφ_2 ∧ φ_γ ∧ θ_γ + 4π^{σ}_σ L_{σγ} θ_σ ∧ θ_γ + 4C_γ θ_γ ∧ η_1 \\
&+ 4C_γ θ_γ ∧ η_1 - 4iπ^{σ}_σ C_σ θ_γ ∧ (η_2 + iη_3) + 4iπ^{σ}_σ C_σ θ_γ ∧ (η_2 - iη_3) \\
&+ Π η_1 ∧ (η_2 + iη_3) + Π η_1 ∧ (η_2 - iη_3) + Π (η_2 + iη_3) ∧ (η_2 - iη_3),
\end{align*} \]

\[ \begin{align*}
dψ_2 + dψ_3 &= (φ_0 - iφ_1) ∧ (ψ_2 + iψ_3) + i(ψ_2 + iψ_3) ∧ ψ_1 + 4π^{σ}_σ φ_γ ∧ φ_γ + 4iπ^{σ}_σ M_{σβ} θ_γ ∧ θ_γ \\
&+ 4iπ^{σ}_σ C_σ θ_γ ∧ η_1 - 4θ_γ θ_γ ∧ η_1 - 4iC_γ θ_γ ∧ (η_2 + iη_3) - 4iπ^{σ}_σ H_σ θ_γ ∧ (η_2 - iη_3) \\
&- iΠ η_1 ∧ (η_2 + iη_3) + Π η_1 ∧ (η_2 - iη_3) - Π (η_2 + iη_3) ∧ (η_2 - iη_3).
\end{align*} \]
In order to describe further relations and differential consequences between the curvature components defined by Proposition 4.1, we introduce the following list of one-forms:

\begin{equation}
S^*_{\alpha \beta \gamma \delta} \stackrel{def}{=} dS_{\alpha \beta \gamma \delta} - \tilde{S}^*_{\alpha \beta \gamma \delta}
\end{equation}

\begin{equation}
V^*_{\alpha \beta \gamma} \stackrel{def}{=} dV_{\alpha \beta \gamma} - \tilde{V}^*_{\alpha \beta \gamma}
\end{equation}

\begin{equation}
L^*_{\alpha \beta} \stackrel{def}{=} dL_{\alpha \beta} - \tilde{L}^*_{\alpha \beta}
\end{equation}

\begin{equation}
M^*_{\alpha \beta} \stackrel{def}{=} dM_{\alpha \beta} - \tilde{M}^*_{\alpha \beta}
\end{equation}

\begin{equation}
\mathcal{E}^*_{\alpha} \stackrel{def}{=} d\mathcal{E}_{\alpha} - \tilde{\mathcal{E}}^*_{\alpha}
\end{equation}

\begin{equation}
\mathcal{H}^*_{\alpha} \stackrel{def}{=} d\mathcal{H}_{\alpha} - \tilde{\mathcal{H}}^*_{\alpha}
\end{equation}

\begin{equation}
\mathcal{R}^* \stackrel{def}{=} d\mathcal{R} - \tilde{\mathcal{R}}^*\quad \tilde{\mathcal{R}}^* = 3\mathcal{P}_{\varphi_0} - (\varphi_2 + i\varphi_3)\mathcal{P} - (\varphi_2 - i\varphi_3)\mathcal{Q} + 8\phi^*\mathcal{E}_\tau + 8\phi^*\mathcal{E}_\tau
\end{equation}

\begin{equation}
\mathcal{P}^* \stackrel{def}{=} d\mathcal{P} - \tilde{\mathcal{P}}^*\quad \tilde{\mathcal{P}}^* = (3\mathcal{P}_{\varphi_0} + i\varphi_1)\mathcal{P} - \frac{i}{2}(\varphi_2 - i\varphi_3)\mathcal{Q} + \frac{3}{2}(\varphi_2 - i\varphi_3)\mathcal{P} + 4i\phi^*\mathcal{H}_\tau - 12\pi_{\tau \phi}^*\mathcal{E}_\phi
\end{equation}

\begin{equation}
\mathcal{Q}^* \stackrel{def}{=} d\mathcal{Q} - \tilde{\mathcal{Q}}^*\quad \tilde{\mathcal{Q}}^* = (3\mathcal{P}_{\varphi_0} + 2i\varphi_1)\mathcal{Q} - 2i(\varphi_2 - i\varphi_3)\mathcal{P} + 16\pi_{\tau \phi}^*\mathcal{E}_\phi
\end{equation}

Using (4.2) we easily obtain that one-forms $S^*_{\alpha \beta \gamma \delta}$, $V^*_{\alpha \beta \gamma}$, $L^*_{\alpha \beta}$, $M^*_{\alpha \beta}$ are totally symmetric in their indices and

\begin{equation}
\begin{cases}
(iS^*_{\alpha \beta \gamma \delta} = S^*_{\alpha \beta \gamma \delta} \\
(iL^*_{\alpha \beta} = L^*_{\alpha \beta} \\
(R^* = R^*.
\end{cases}
\end{equation}
Proposition 4.2 (Bianchi identities). The following identities are satisfied:

\[ d^2 \Gamma_{\alpha \beta} = \pi^\sigma \delta^i_{\alpha \beta \gamma \sigma} \wedge \theta^\gamma \wedge \theta^\delta + V_i^\alpha_{\beta \gamma} \wedge \eta_1 + \pi^\sigma \delta^i_{\alpha \beta \gamma \sigma} \wedge \theta^\gamma \wedge \eta_1 \]

\[ - i \pi^\sigma \delta^i_{\alpha \beta \gamma} \wedge \theta^\gamma \wedge (\eta_2 + i \eta_3) + i \pi^\sigma \delta^i_{\alpha \beta \gamma} \wedge \theta^\gamma \wedge (\eta_2 - i \eta_3) \]

\[- i \mathcal{L}_{\alpha \beta} \wedge (\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3) + M^*_{\alpha \beta} \wedge \eta_1 \wedge (\eta_2 + i \eta_3) \]

[\quad + \pi^\sigma \delta^i_{\alpha \beta} M^*_{\mu \nu} \wedge \eta_1 \wedge (\eta_2 - i \eta_3) - 0;\]

\[ d^2 \phi_\alpha = \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\delta \wedge \theta^\gamma + \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\delta \wedge \eta_1 + M^*_{\alpha \beta} \wedge \theta^\gamma \wedge \eta_1 \]

\[- i \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\delta \wedge (\eta_2 + i \eta_3) + i \mathcal{L}_{\alpha \beta} \wedge \theta^\gamma \wedge (\eta_2 - i \eta_3) - C_{\alpha i} \wedge (\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3) \]

[\quad + \pi^\gamma \delta^i_{\alpha \beta} \wedge \eta_1 \wedge (\eta_2 - i \eta_3) + M^*_{\alpha \beta} \wedge \eta_1 \wedge (\eta_2 + i \eta_3) - 0;\]

\[ d^2 \psi_1 = 4 \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\delta \wedge \theta^\gamma + 4 \mathcal{E}_{\gamma i} \wedge \theta^\gamma \wedge \eta_1 + 4 \mathcal{E}_{\gamma i} \wedge \theta^\gamma \wedge \eta_1 + 4 i \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\gamma \wedge (\eta_2 - i \eta_3) \]

\[- 4 i \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\gamma \wedge (\eta_2 + i \eta_3) + \mathcal{E}_{\gamma i} \wedge \eta_1 \wedge (\eta_2 + i \eta_3) + \pi^\gamma \delta^i_{\alpha \beta} \wedge \eta_1 \wedge (\eta_2 - i \eta_3) \]

[\quad + i R^* \wedge (\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3) - 0;\]

\[ d^2 (\psi_2 + i \psi_3) = 4 \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\delta \wedge \theta^\gamma + 4 \mathcal{E}_{\gamma i} \wedge \theta^\gamma \wedge \eta_1 - 4 \mathcal{E}_{\gamma i} \wedge \theta^\gamma \wedge \eta_1 - 4 \mathcal{E}_{\gamma i} \wedge \theta^\gamma \wedge (\eta_2 + i \eta_3) \]

\[- 4 \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\gamma \wedge (\eta_2 - i \eta_3) - i \mathcal{R}^* \wedge \eta_1 \wedge (\eta_2 + i \eta_3) + \pi^\gamma \delta^i_{\alpha \beta} \wedge \eta_1 \wedge (\eta_2 - i \eta_3) \]

[\quad - \mathcal{R}^* \wedge (\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3) = 0.\]

Proposition 4.3 (The secondary derivatives). On \( P_1 \), there exist unique, globally defined, complex valued functions

\[ A_{\alpha \beta \gamma \delta}, \ B_{\alpha \beta \gamma \delta}, \ C_{\alpha \beta \gamma \delta}, \ D_{\alpha \gamma}, \ E_{\alpha \gamma}, \ F_{\alpha \gamma}, \ G_{\alpha \beta}, \ H_{\alpha \beta}, \ Y_{\alpha \beta}, \ Z_{\alpha \beta}, \]

\[ (N_1)_\alpha, \ (N_2)_\alpha, \ (N_3)_\alpha, \ (N_4)_\alpha, \ (N_5)_\alpha, \ U_\alpha, \ W_\alpha \]

so that:

(I) Each of the arrays \( \{ A_{\alpha \beta \gamma \delta} \}, \{ B_{\alpha \beta \gamma \delta} \}, \{ C_{\alpha \beta \gamma \delta} \}, \{ D_{\alpha \gamma} \}, \{ E_{\alpha \gamma} \}, \{ F_{\alpha \gamma} \}, \{ G_{\alpha \beta} \}, \{ H_{\alpha \beta} \}, \{ Y_{\alpha \beta} \}, \{ Z_{\alpha \beta} \} \)

is totally symmetric in its indices.

(II) We have

\[ dS_{\alpha \beta \gamma} = \tilde{S}_{\alpha \beta \gamma} \wedge \theta^\delta + \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge (B_{\alpha \beta \gamma} + i B_{\alpha \beta \gamma}) \eta_1 \wedge (i C_{\alpha \beta \gamma}) (\eta_2 + i \eta_3) \]

\[- i (i E_{\alpha \beta \gamma}) (\eta_2 - i \eta_3);\]

\[ dV_{\alpha \beta} = V_{\alpha \beta} + \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\gamma + \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\gamma \wedge \eta_1 + \mathcal{E}_{\gamma i} \wedge (\eta_2 + i \eta_3) + \mathcal{F}_{\alpha \beta} \wedge (\eta_2 - i \eta_3);\]

\[ d\mathcal{L}_{\alpha \beta} = \mathcal{L}_{\alpha \beta} - (i \mathcal{J})_{\alpha \beta} \wedge \theta^\gamma + \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge (i \mathcal{I})_{\alpha \beta} \wedge (\eta_2 - i \eta_3) \]

\[- i \mathcal{G}_{\alpha \beta} \wedge (\eta_2 + i \eta_3) - i \mathcal{G}_{\alpha \beta} \wedge (\eta_2 - i \eta_3);\]

\[ dM_{\alpha \beta} = \tilde{M}_{\alpha \beta} \wedge \theta^\gamma + \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\gamma \wedge (N_1)_\alpha \eta_1 \wedge (N_2)_\alpha \eta_1 \wedge (N_3)_\alpha \eta_1 \wedge (N_4)_\alpha \eta_1 \wedge (N_5)_\alpha \eta_1 \wedge (\eta_2 + i \eta_3);\]

\[ d\mathcal{E}_{\alpha} = \tilde{\mathcal{E}}_{\alpha} + \mathcal{G}_{\alpha} \wedge \theta^\gamma \wedge (i \mathcal{I})_{\alpha \beta} \wedge (\eta_2 + i \eta_3) + \mathcal{F}_{\alpha \beta} \wedge (\eta_2 - i \eta_3);\]

\[ d\mathcal{K}_\alpha = \tilde{\mathcal{K}}_\alpha - \mathcal{K}_\alpha \wedge \theta^\gamma + i \pi^\gamma \delta^i_{\alpha \beta \gamma \delta} \wedge \theta^\gamma \wedge (N_1)_\alpha \eta_1 \wedge (N_2)_\alpha \eta_1 \wedge (N_3)_\alpha \eta_1 \wedge (N_4)_\alpha \eta_1 \wedge (N_5)_\alpha \eta_1 \wedge (\eta_2 + i \eta_3);\]

\[ d\mathcal{R} = \tilde{\mathcal{R}} + 4 \pi^\gamma \delta^i_{(N_3)_\alpha} \wedge \theta^\gamma + 4 \pi^\gamma \delta^i_{(N_3)_\alpha} \wedge \theta^\gamma \wedge i (\mathcal{L}_{\alpha} - \mathcal{M}_{\alpha}) \eta_1 \wedge (\eta_2 + i \eta_3) \]

\[- i (\mathcal{L}_{\alpha} + \mathcal{M}_{\alpha}) (\eta_2 - i \eta_3);\]

\[ dP = \tilde{\mathcal{P}} - 4 \mathcal{N}_2 \wedge (N_1) \wedge \theta^\gamma + 4 \mathcal{N}_2 \wedge \theta^\gamma \wedge (N_1)_\alpha \eta_1 \wedge (\eta_2 + i \eta_3) \]

\[ + (N_1) \wedge (N_3) \eta_1 \wedge (\eta_2 - i \eta_3);\]

\[ dQ = \tilde{\mathcal{Q}} + 4 \mathcal{N}_2 \wedge (N_1) \wedge \theta^\gamma + 4 \mathcal{N}_2 \wedge \theta^\gamma \wedge (N_1)_\alpha \eta_1 \wedge (\eta_2 + i \eta_3) \]

\[ + (N_1) \wedge (N_5) \eta_1 \wedge (\eta_2 - i \eta_3).\]
Proof of Propositions 4.1, 4.2 and 4.3. A differentiation of the first equation of (3.31), after some calculations using (3.31), yields

\[ 0 = d^2 \theta^\alpha = \pi^\alpha \sigma \left( -d \Gamma_\sigma \beta - \pi^\nu \Gamma_\sigma \nu \wedge \Gamma_\tau \beta + 2 \pi^\tau_\sigma (\phi_\beta \wedge \theta_\tau - \phi_\tau \wedge \theta_\beta) + 2 \pi^\tau_\beta (\phi_\sigma \wedge \theta_\tau - \phi_\tau \wedge \theta_\sigma) \right) \wedge \theta^\beta \]

\[ + i \left( -d \phi^\alpha - \pi^\nu \Gamma_\sigma \gamma \wedge \phi^\gamma + \frac{1}{2} (\phi_0 - i \phi_1) \wedge \phi^\alpha - \frac{1}{2} \pi^\alpha (\phi_2 + i \phi_3) \wedge \phi^\gamma \right) \wedge \eta_1 \]

\[ + \pi^\alpha_\beta \left( -d \phi^\beta - \pi^\gamma \Gamma_\sigma \gamma \wedge \phi^\gamma + \frac{1}{2} (\phi_0 + i \phi_1) \wedge \phi^\gamma - \frac{1}{2} \pi^\gamma (\phi_2 - i \phi_3) \wedge \phi^\gamma \right) \wedge (\eta_2 + i \eta_3) \]

If we set

\[ X_{\alpha \beta} = d \Gamma_\alpha \beta + \pi^\nu \Gamma_\alpha \nu \wedge \Gamma_\tau \beta - 2 \pi^\tau_\alpha (\phi_\beta \wedge \theta_\tau + \phi_\tau \wedge \theta_\beta) - 2 \pi^\tau_\beta (\phi_\alpha \wedge \theta_\tau - \phi_\tau \wedge \theta_\alpha) \]

\[ Y^\alpha = d \phi^\alpha + \pi^\nu \phi_\nu \wedge \phi^\gamma + \frac{1}{2} (\phi_0 - i \phi_1) \wedge \phi^\alpha + \frac{1}{2} \pi^\gamma (\phi_2 + i \phi_3) \wedge \phi^\gamma \]

\[ - \frac{i}{2} \psi_1 \wedge \theta^\alpha - \frac{1}{2} \pi^\gamma (\psi_2 + i \psi_3) \wedge \theta^\gamma, \]

the above equation reads as

\[ \pi^\alpha X_{\sigma \beta} \wedge \theta^\beta + i Y^\alpha \wedge \eta_1 + \pi^\beta_\gamma Y^\beta \wedge (\eta_2 + i \eta_3) = 0. \]

Consequently, there exist one-forms \( A_{\alpha \beta \gamma} \) so that

\[ X_{\alpha \beta} \equiv A_{\alpha \beta \gamma} \wedge \theta^\gamma \mod \{ \eta_1, \eta_2 + i \eta_3 \}. \]

A small calculation using (4.23) shows that

\[ X_{\alpha \beta} = X_{\beta \alpha}, \quad (iX)_{\alpha \beta} = X_{\alpha \beta}, \]

and therefore,

\[ A_{\alpha \beta \gamma} \wedge \theta^\gamma = \pi^\alpha \pi^\beta A_{\alpha \beta \gamma} \wedge \theta^\gamma \mod \{ \eta_1, \eta_2 + i \eta_3, \eta_2 - i \eta_3 \}. \]

It follows that \( A_{\alpha \beta \gamma} \equiv 0 \) modulo \( \{ \theta^\gamma, \theta^\alpha, \eta_1, \eta_2 + i \eta_3, \eta_2 - i \eta_3 \} \) and thus, there exist functions \( A_{\alpha \beta \gamma \delta}, A_{\alpha \beta \gamma \delta} \) so that

\[ X_{\alpha \beta} \equiv A_{\alpha \beta \gamma \delta} \theta^\gamma \wedge \theta^\delta + A_{\alpha \beta \gamma \delta} \theta^\gamma \wedge \theta^\delta \mod \{ \eta_1, \eta_2 + i \eta_3, \eta_2 - i \eta_3 \} \]

and

\[ \begin{cases} 
A_{\alpha \beta \gamma \delta} = A_{\beta \alpha \gamma \delta} \\
A_{\alpha \beta \gamma \delta} = -A_{\alpha \beta \delta \gamma} \\
A_{\alpha \beta \gamma \delta} = A_{\beta \alpha \gamma \delta}.
\end{cases} \]

Substituting back into (4.24) gives \( A_{\alpha \beta \gamma \delta} \theta^\gamma \wedge \theta^\delta = 0 \) and therefore, the array \( \{ A_{\alpha \beta \gamma \delta} \} \) is totally symmetric in the indices \( \alpha, \beta, \gamma \). We have also

\[ 0 = (iX)_{\alpha \beta} - X_{\alpha \beta} = \pi^\alpha \pi^\beta A_{\beta \gamma \delta} \theta^\gamma \wedge \theta^\delta + \pi^\alpha \pi^\beta A_{\alpha \beta \gamma \delta} \theta^\gamma \wedge \theta^\delta \]

\[ - A_{\alpha \beta \gamma \delta} \theta^\gamma \wedge \theta^\delta - A_{\alpha \beta \gamma \delta} \theta^\gamma \wedge \theta^\delta + \ldots \]

where the omitted terms are vanishing modulo \( \{ \eta_1, \eta_2 + i \eta_3, \eta_2 - i \eta_3 \} \). Therefore, \( A_{\alpha \beta \gamma \delta} = 0 \) (in view of the second line of (4.26)) and

\[ A_{\alpha \beta \gamma \delta} = -\pi^\alpha \pi^\beta A_{\beta \gamma \delta}. \]

Let us define

\[ S_{\alpha \beta \gamma \delta} \overset{\text{def}}{=} -\pi^\alpha \pi^\beta A_{\alpha \beta \gamma \delta}. \]
Then, since the array \( \{ A_{\alpha\beta\gamma}\} \) is totally symmetric in the indices \( \alpha, \beta, \gamma \), equation (4.27) implies that the array \( \{ S_{\alpha\beta\gamma}\} \) is totally symmetric in all of its indices and \( (iS)_{\alpha\beta\gamma} = S_{\alpha\beta\gamma} \). Furthermore, we have

\[
(4.29) \quad X_{\alpha\beta} \equiv \pi^\sigma \delta_{\alpha\beta\gamma} \theta^\gamma \land \theta^\delta + (B_1)_{\alpha\beta} \land \eta_1 + (B_2)_{\alpha\beta} \land (\eta_2 + i\eta_3) + (B_3)_{\alpha\beta} \land (\eta_2 - i\eta_3)
\]

for some appropriate one-forms \( (B_\alpha)_{\alpha\beta} \).

On the other hand, (4.24) implies that the two-forms \( Y^\alpha \) and \( \pi^\sigma \beta Y^\beta \) are vanishing modulo \( \{ \theta^\gamma, \eta_1, \eta_2 + i\eta_3 \} \) and thus, the same holds true for \( Y^\alpha \). Therefore, there exist functions \( C_{\alpha\beta\gamma} \) for which

\[
(4.30) \quad Y^\alpha = C_{\alpha\beta\gamma} \theta^\beta \land \theta^\gamma \text{ mod } \eta_3.
\]

We substitute (4.29) and (4.30) into (4.24) to obtain, modulo \( \{ \theta^\gamma, \eta_2, \eta_3, \eta_1 \} \),

\[
(4.31) \quad \pi^\alpha \sigma (B_1)_{\sigma\beta} \land \eta_1 \land \theta^\beta + \pi^\alpha \sigma (B_2)_{\sigma\beta} \land (\eta_2 + i\eta_3) \land \theta^\beta + \pi^\alpha \sigma (B_3)_{\sigma\beta} \land (\eta_2 - i\eta_3) \land \theta^\beta + iC_{\alpha\beta\gamma} \theta^\beta \land \theta^\gamma \land \eta_1 + \pi^\sigma \rho C_{\alpha\beta\gamma} \theta^\rho \land \theta^\gamma \land (\eta_2 + i\eta_3) \equiv 0.
\]

Consequently,

\[
\begin{align*}
(B_1)_{\alpha\beta} &\equiv 0 \text{ mod } \{ \theta^\gamma, \theta^\delta, \eta_3 \} \\
(B_3)_{\alpha\beta} &\equiv 0 \text{ mod } \{ \theta^\gamma, \eta_3 \}
\end{align*}
\]

By (4.25), we have that \( (B_2)_{\alpha\beta} = \pi^\alpha \beta \pi^\delta \gamma (B_3)_{\delta\gamma} \) and therefore, there exist functions \( (B_1)_{\alpha\beta\gamma}, (B_1)_{\alpha\beta\gamma} \) and \( (B_3)_{\alpha\beta\gamma} \) so that

\[
\begin{align*}
(B_1)_{\alpha\beta} &\equiv (B_1)_{\alpha\beta\gamma} \land \theta^\gamma + (B_1)_{\alpha\beta\gamma} \land \theta^\gamma \land \eta_1 \\
(B_2)_{\alpha\beta} &\equiv \pi^\alpha \beta \pi^\delta \gamma (B_3)_{\delta\gamma} \land \theta^\gamma \land \eta_1 \\
(B_3)_{\alpha\beta} &\equiv (B_3)_{\alpha\beta\gamma} \land \theta^\gamma.
\end{align*}
\]

On the account of (4.25) and (4.29), we obtain that the arrays \( \{ (B_1)_{\alpha\beta\gamma} \} \), \( \{ (B_1)_{\alpha\beta\gamma} \} \), \( \{ (B_3)_{\alpha\beta\gamma} \} \) are symmetric in \( \alpha, \beta \) and satisfy

\[
(4.32) \quad (B_1)_{\alpha\beta\gamma} = \pi^\alpha \beta \pi^\delta \gamma (B_3)_{\delta\gamma\gamma}.
\]

Substituting back into (4.31) yields

\[
\begin{align*}
\pi^\alpha \sigma (B_1)_{\sigma\beta\gamma} \land \theta^\beta \land \theta^\gamma \land \eta_1 + g^\alpha \sigma (\pi^\beta \gamma (B_1)_{\sigma\gamma\gamma} + iC_{\sigma\beta\gamma}) \land \theta^\beta \land \theta^\gamma \land \eta_1 \\
+ \pi^\alpha \sigma (\pi^\delta \gamma (B_3)_{\delta\gamma\gamma} + C_{\sigma\gamma\beta}) \land \theta^\beta \land \theta^\gamma \land (\eta_2 + i\eta_3) + \pi^\alpha \sigma (B_3)_{\sigma\gamma\beta} \land \theta^\beta \land \theta^\gamma \land (\eta_2 - i\eta_3) \equiv 0.
\end{align*}
\]

It follows that the arrays \( \{ (B_1)_{\alpha\beta\gamma} \} \) and \( \{ (B_3)_{\alpha\beta\gamma} \} \) are totally symmetric in their indices, and we have the identities

\[
-\pi^\delta \beta (B_1)_{\delta\gamma\gamma} + iC_{\sigma\beta\gamma} = 0, \quad \pi^\delta \gamma (B_3)_{\delta\gamma\gamma} + C_{\sigma\gamma\beta} = 0.
\]

Hence,

\[
C_{\alpha\beta\gamma} = -i\pi^\delta \beta (B_1)_{\delta\gamma\gamma}
\]

and also

\[
(B_3)_{\alpha\beta\gamma} = -\pi^\delta \beta \pi^\gamma \gamma C_{\sigma\gamma\gamma} = i\pi^\delta \beta \pi^\gamma \gamma (B_1)_{\delta\gamma\gamma}.
\]

Setting

\[
V_{\alpha\beta\gamma} \overset{def}{=} (B_1)_{\alpha\beta\gamma},
\]

we obtain

\[
\begin{align*}
(B_1)_{\alpha\beta} &\equiv V_{\alpha\beta\gamma} \land \theta^\gamma + \pi^\delta \beta \pi^\gamma \gamma V_{\delta\gamma\gamma} \land \theta^\gamma \\
(B_2)_{\alpha\beta} &\equiv -i\pi^\delta \beta V_{\alpha\beta\gamma} \land \theta^\gamma \\
(B_3)_{\alpha\beta} &\equiv i(\bar{V})_{\alpha\beta\gamma} \land \theta^\gamma \text{ mod } \eta_3 \\
Y^\alpha &\equiv -i\pi^\delta \beta \bar{V}_{\alpha\beta\gamma} \land \theta^\beta \land \theta^\gamma.
\end{align*}
\]
Thus, for some appropriate functions \((A_s)_{\alpha\beta}\), we have
\[
X_{\alpha\beta} = \pi^\alpha_{\beta} S_{\alpha\beta\gamma\delta} \theta^\gamma \wedge \theta^\delta + \left( V_{\alpha\beta\gamma\delta} \theta^\gamma + \pi^\alpha_{\beta} \pi^\gamma_{\delta} V_{\alpha\beta\gamma\delta} \theta^\gamma \right) \wedge \eta_1 \\
- i \pi^\alpha_{\beta} V_{\alpha\beta\sigma} \theta^\sigma \wedge (\eta_2 + i \eta_3) + i (i V)_{\alpha\beta\sigma} \theta^\gamma \wedge (\eta_2 - i \eta_3) \\
+ (A_1)_{\alpha\beta} \eta_1 \wedge (\eta_2 + i \eta_3) + (A_2)_{\alpha\beta} \eta_1 \wedge (\eta_2 - i \eta_3) + (A_3)_{\alpha\beta}(\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3).
\]
(4.33)

Clearly, by (4.25), \(\{(A_s)_{\alpha\beta}\}\) are symmetric in the \(\alpha, \beta\) and satisfy
\[
(A_2)_{\alpha\beta} = \pi^\alpha_{\beta} \pi^\gamma_{\delta} (A_1)_{\gamma\delta}, \quad (A_3)_{\alpha\beta} = - \pi^\alpha_{\beta} \pi^\gamma_{\delta} (A_3)_{\gamma\delta}.
\]
(4.34)

Using one more time the argument that both \(Y^\alpha\) and \(\bar{Y}^\alpha\) are vanishing modulo \(\{\theta^\gamma, \eta_1, \eta_2 + i \eta_3\}\), we deduce that there exist functions \(C_\alpha, D_{\alpha\beta}, F_{\alpha\beta}\), and one-forms \(E_\alpha\) so that
\[
Y^\alpha = - i \pi^\alpha_{\beta} V_{\alpha\beta\gamma} \theta^\gamma \wedge \theta^\gamma + D^\alpha_{\beta} \theta^\gamma \wedge (\eta_2 + i \eta_3) + F^\alpha_{\beta} \theta^\gamma \wedge (\eta_2 - i \eta_3) \\
+ C^\alpha \eta_2 \wedge (\eta_2 + i \eta_3) + (\eta_2 - i \eta_3) + C^\alpha \wedge \eta_1.
\]
(4.35)

Substituting (4.33) and (4.35) back into (4.24) gives
\[
\pi^\alpha_{\sigma} (A_1)_{\sigma\beta} \theta^\beta \wedge \eta_1 \wedge (\eta_2 + i \eta_3) + \pi^\alpha_{\sigma} (A_2)_{\sigma\beta} \theta^\beta \wedge \eta_1 \wedge (\eta_2 - i \eta_3) \\
+ \pi^\alpha_{\sigma} (A_3)_{\sigma\beta} \theta^\beta \wedge \eta_1 \wedge (\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3) - i F^\alpha_{\beta} \theta^\beta \wedge \eta_1 \wedge (\eta_2 - i \eta_3) \\
- i D^\alpha_{\beta} \theta^\beta \wedge \eta_1 \wedge (\eta_2 + i \eta_3) - \pi^\alpha_{\beta} D^\alpha_{\beta} \theta^\beta \wedge \eta_1 \wedge (\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3) \\
+ i C^\alpha \eta_1 \wedge (\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3) + \pi^\alpha_{\beta} E^3_{\beta} \wedge \eta_1 \wedge (\eta_2 + i \eta_3) = 0.
\]
(4.36)

By considering the coefficients of \(\theta^\beta \wedge \eta_1 \wedge (\eta_2 - i \eta_3)\) and \(\theta^\beta \wedge (\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3)\), we obtain
\[
\begin{cases}
F^\alpha_{\beta} = - i \pi^\alpha_{\sigma} (A_2)_{\sigma\beta} = \pi^\alpha_{\beta} (A_1)_{\alpha\beta} \\
D_{\alpha\beta} = -(A_3)_{\alpha\beta}
\end{cases}
\]
(4.37)

and thus, (4.36) simplifies to
\[
\pi^\alpha_{\sigma} (A_1)_{\sigma\beta} \theta^\beta \wedge \eta_1 \wedge (\eta_2 + i \eta_3) - i D^\alpha_{\beta} \theta^\beta \wedge \eta_1 \wedge (\eta_2 + i \eta_3) \\
+ i C^\alpha \eta_1 \wedge (\eta_2 + i \eta_3) \wedge (\eta_2 - i \eta_3) + \pi^\alpha_{\beta} E^3_{\beta} \wedge \eta_1 \wedge (\eta_2 + i \eta_3) = 0.
\]
(4.38)

By (4.38), we obtain
\[
E^\alpha = E^3_{\beta} \theta^\beta + E^3_{\bar{\beta}} \bar{\theta}^\beta + (E_1)_{\alpha} \eta_1 + (E_2)_{\alpha}(\eta_2 + i \eta_3) + (E_3)_{\alpha}(\eta_2 - i \eta_3),
\]
where \(E_1^3\), \(E_2^3\) and \((E_s)^{\alpha}\) are some appropriate coefficients. Furthermore, by substituting (4.39) into (4.38), we get
\[
\begin{cases}
E_{\alpha\beta} = (A_1)_{\alpha\beta} \\
E^3_{\beta} = - i \pi^\alpha_{\sigma} D_{\sigma\beta} = \pi^\alpha_{\beta} (A_3)_{\alpha\beta} \\
(E_2)^{\alpha} = - i \pi^\alpha_{\beta} C^\alpha.
\end{cases}
\]
(4.40)

Setting
\[
\begin{cases}
\mathcal{L}_{\alpha\beta} \overset{\text{def}}{=} i (A_3)_{\alpha\beta} \\
\mathcal{M}_{\alpha\beta} \overset{\text{def}}{=} (A_1)_{\alpha\beta} \\
\mathcal{C}^\alpha \overset{\text{def}}{=} C^\alpha \\
\mathcal{H}^\alpha \overset{\text{def}}{=} - (E_3)^{\alpha},
\end{cases}
\]
(4.41)

we obtain, by (4.23), (4.33), (4.35), (4.37), (4.37), (4.40) and (4.41), the relations (4.3) and (4.4).
We proceed by differentiating (4.3) and (4.4) one more time. After some rather long but straightforward calculations, we obtain

\[(4.42)\quad 0 = d^2 \Gamma_{\alpha\beta} = \pi^2 \left[ dS_{\alpha\beta\gamma} - \pi^{\rho\tau} \Gamma_{\rho\alpha} S_{\tau\beta\gamma} - \pi^{\tau\rho} \Gamma_{\tau\beta} S_{\rho\alpha\gamma} - \pi^{\rho\tau} \Gamma_{\rho\gamma} S_{\alpha\beta\tau} - \pi^{\tau\rho} \Gamma_{\tau\alpha} S_{\rho\beta\gamma} - \varphi_0 S_{\alpha\beta\gamma} - 2i(\pi_{\alpha\tau} V_{\sigma\beta\gamma} + \pi_{\beta\tau} V_{\alpha\sigma\gamma} + \pi_{\gamma\tau} V_{\alpha\beta\sigma} + \pi_{\sigma\tau} V_{\alpha\gamma\beta}) \theta^\tau - 2i(\pi_{\alpha\tau} V_{\alpha\beta\gamma} + \pi_{\beta\tau} V_{\alpha\gamma\beta} + \pi_{\gamma\tau} V_{\alpha\beta\sigma} + \pi_{\sigma\tau} V_{\alpha\gamma\beta}) \theta^\tau \right] \wedge \theta^\gamma \wedge \theta^\delta \\
\]

\[
+ \left[ dV_{\alpha\beta\gamma} - \pi^{\rho\tau} \Gamma_{\rho\alpha} V_{\tau\beta\gamma} + \pi^{\tau\rho} \Gamma_{\tau\beta} V_{\rho\alpha\gamma} - \pi^{\rho\tau} \Gamma_{\rho\gamma} V_{\alpha\beta\tau} + \pi^{\tau\rho} \Gamma_{\tau\alpha} V_{\rho\beta\gamma} + \frac{1}{2} \varphi_2 (iV)_{\alpha\beta\gamma} + 2(\pi_{\alpha\tau} M_{\beta\sigma} + \pi_{\beta\tau} M_{\alpha\sigma} + \pi_{\gamma\tau} M_{\alpha\beta}) \theta^\tau \\
+ 2(i\pi^2) (iV)_{\alpha\beta\gamma} + 2(i\pi^2) (iV)_{\alpha\beta\gamma} - (3\varphi_0 + i\varphi_1) V_{\alpha\beta\gamma} \right] \wedge \theta^\gamma \wedge \eta_1 \\
\]

\[
+ \pi^2 \frac{\eta_2}{\eta_3} \left[ dV_{\mu\nu\xi} - \pi^{\tau\varphi} \Gamma_{\tau\mu} V_{\xi\nu\varphi} - \pi^{\varphi\tau} \Gamma_{\varphi\nu} V_{\tau\xi\varphi} - \pi^{\varphi\tau} \Gamma_{\varphi\xi} V_{\tau\nu\varphi} - \pi^{\tau\varphi} \Gamma_{\tau\nu} V_{\varphi\xi\varphi} + \frac{1}{2} \varphi_2 (iV)_{\mu\nu\xi} + 2(\pi_{\mu\tau} M_{\nu\xi} + \pi_{\nu\tau} M_{\mu\xi} + \pi_{\xi\tau} M_{\mu\nu}) \theta^\tau \\
+ 2(i\pi^2) (iV)_{\mu\nu\xi} + 2(i\pi^2) (iV)_{\mu\nu\xi} - (3\varphi_0 + i\varphi_1) V_{\mu\nu\xi} \right] \wedge \theta^\gamma \wedge (\eta_2 + i\eta_3) \\
\]

\[
+ i \left[ dM_{\alpha\beta} - \pi^{\rho\tau} \Gamma_{\rho\alpha} M_{\tau\beta} - \pi^{\tau\rho} \Gamma_{\tau\beta} M_{\rho\alpha} - \varphi_0 M_{\alpha\beta} - \frac{1}{2} (\varphi_2 + i\varphi_3) M_{\alpha\beta} - \frac{1}{2} (\varphi_2 + i\varphi_3) (iM)_{\alpha\beta} \\
- \phi^\varphi V_{\alpha\beta\sigma} - \pi^{\rho\tau} \pi^\varphi_{\rho\sigma} \phi^\varphi V_{\mu\nu\varphi} - 2i(\pi_{\alpha\tau} \mathcal{C}_{\beta\rho} - \pi_{\beta\tau} \mathcal{C}_{\alpha\rho}) \theta^\tau \\
- 2i(\pi_{\alpha\tau} \mathcal{C}_{\beta\rho} + \pi_{\beta\tau} \mathcal{C}_{\alpha\rho}) \theta^\tau \right] \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3) \\
\]

\[
+ \left[ dM_{\alpha\beta} - \pi^{\rho\tau} \Gamma_{\rho\alpha} M_{\tau\beta} - \pi^{\tau\rho} \Gamma_{\tau\beta} M_{\rho\alpha} - (2\varphi_0 + i\varphi_1) M_{\alpha\beta} + (\varphi_2 - i\varphi_3) L_{\alpha\beta} + 2\pi^\varphi \phi^\varphi V_{\alpha\beta\sigma} \\
+ 2(\pi_{\alpha\tau} K_{\beta\rho} + \pi_{\beta\tau} K_{\alpha\rho}) \theta^\tau - 2i(\pi_{\alpha\tau} \mathcal{C}_{\beta\rho} + \pi_{\beta\tau} \mathcal{C}_{\alpha\rho}) \theta^\tau \right] \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3) \\
\]
\[ + \pi^\mu_\alpha \pi^\nu_\beta \left[ dM_{\mu\beta} - \pi^{\tau\sigma} \Gamma_{\sigma\beta} M_{\tau\nu} - \pi^{\tau\sigma} \Gamma_{\sigma\nu} M_{\tau\beta} - (2\varphi_0 - i\varphi_1) M_{\mu\beta} + (\varphi_2 + i\varphi_3) L_{\mu\beta} + 2\pi^\sigma_\nu \phi^\tau V_{\mu\nu\sigma} \right] + 2(\pi_{\mu\tau} H_{\nu} + \pi_{\nu\tau} H_{\mu}) \theta^\tau + 2i(\varphi_{\tau\mu} c_\nu + g_{\tau\nu} c_\mu) \theta^\tau \right] \land \eta_1 \land (\eta_2 - i\eta_3) \]

and

\[
(4.43) \quad 0 = d^2 \phi^\alpha = \frac{i}{2} \left[ d\psi_1 - \varphi_0 \land \psi_1 + \varphi_2 \land \psi_3 - \varphi_3 \land \psi_2 + 4i\phi_\beta \land \phi^\beta - 4\pi^\sigma_\nu L_{\beta\alpha} \phi^\beta \land \phi^\gamma \\
- 4(\pi^\sigma_\beta \phi^\beta \land \phi^\beta) \land \eta_1 + 4i\pi^\sigma_\beta \pi^\gamma_\sigma \phi^\beta \land (\eta_2 + i\eta_3) - 4i\pi^\sigma_\beta \phi^\beta \land (\eta_2 - i\eta_3) \right] \land \theta^\alpha
\]

\[
+ \frac{1}{2} \pi^\nu_\beta \left[ d(\varphi_2 + i\varphi_3) (V)_{\mu\nu\gamma} + 2(\pi_{\mu\tau} M_{\rho\gamma} + \pi_{\nu\tau} M_{\rho\gamma} + \pi_{\gamma\tau} M_{\rho\mu}) \theta^\tau \right.
\]

\[
- 2(\varphi_{\tau\mu} L_{\nu\gamma} + \varphi_{\tau\nu} L_{\mu\gamma} + \varphi_{\tau\gamma} L_{\mu\nu}) \theta^\tau \right] \land \theta^\beta \land \theta^\gamma
\]

\[
+ \pi^\alpha_\mu \left[ dL_{\mu\beta} - \pi^{\tau\sigma} \Gamma_{\sigma\mu} L_{\tau\beta} - \pi^{\tau\sigma} \Gamma_{\sigma\beta} L_{\mu\tau} - 2\varphi_0 L_{\mu\beta} - \frac{1}{2} M_{\mu\beta}(\varphi_2 + i\varphi_3) - \frac{1}{2}(iM)_{\mu\beta}(\varphi_2 - i\varphi_3) \right.
\]

\[
- \phi^\sigma V_{\mu\beta\gamma} - \pi^\rho_\mu \pi^\rho_\nu \phi^\sigma V_{\tau\nu\sigma} - 2i(\pi_{\mu\tau} c_\beta + \pi_{\nu\tau} c_\mu) \theta^\tau - 2i(\varphi_{\mu\tau} c_\beta + \pi_{\nu\tau} c_\mu) \theta^\tau \right]
\]

\[
- 2(\varphi_{\tau\mu} \pi^\sigma_\beta c_\sigma + \varphi_{\tau\nu} \pi^\sigma_\mu c_\sigma) \theta^\tau \right] \land \theta^\beta \land \eta_1
\]

\[
+ \left[ dM^\alpha_\beta + \pi^{\alpha\sigma} \Gamma_{\sigma\tau} M^\beta_\tau - \pi^{\tau\sigma} \Gamma_{\sigma\beta} M^\alpha_\tau - (2\varphi_0 - i\varphi_1) M^\alpha_\beta + (\varphi_2 + i\varphi_3) L^\alpha_\beta + 2\pi^\sigma_\nu \phi^\tau V_{\alpha\nu\sigma} \right.
\]

\[
+ 2(\pi^\rho_\nu H_{\beta} + \pi^\rho_\beta H_{\nu}) \theta^\rho + 2i(\delta^\rho_\sigma c_\beta + \varphi_{\rho\beta} c_\alpha) \theta^\rho \right] \land \theta^\beta \land \eta_1
\]

\[
+ i\pi^\beta_\alpha \left[ dM^\alpha_\beta + \pi^{\alpha\sigma} \Gamma_{\sigma\tau} M^\beta_\tau - \pi^{\tau\sigma} \Gamma_{\sigma\beta} M^\alpha_\tau - (2\varphi_0 - i\varphi_1) M^\alpha_\beta + (\varphi_2 + i\varphi_3) L^\alpha_\beta + 2\pi^\sigma_\nu \phi^\tau V_{\alpha\nu\sigma} \right.
\]

\[
+ 2(\pi^\rho_\nu H_{\beta} + \pi^\rho_\beta H_{\nu}) \theta^\rho + 2i(\delta^\rho_\sigma c_\beta + \varphi_{\rho\beta} c_\alpha) \theta^\rho \right] \land \theta^\beta \land \eta_1
\]

\[
- i \left[ dL^\alpha_\beta + \pi^{\alpha\sigma} \Gamma_{\sigma\tau} L^\beta_\tau - \pi^{\tau\sigma} \Gamma_{\sigma\beta} L^\alpha_\tau - 2\varphi_0 L^\alpha_\beta - \frac{1}{2} (\varphi_2 - i\varphi_3) M^\alpha_\beta - \frac{1}{2}(iM)_{\alpha\beta}(\varphi_2 + i\varphi_3) \right.
\]

\[
- \phi^\sigma V_{\alpha\beta\gamma} - \pi^{\rho\mu} \pi^\rho_\nu \phi^\sigma V_{\mu\nu\sigma} + 2i(\varphi_{\rho\beta} c_\sigma + \pi_{\rho\beta} c_\alpha) \theta^\rho \right]
\]

\[
+ 2i(\delta^\rho_\sigma c_\beta + \varphi_{\rho\beta} c_\alpha) \theta^\rho \right] \land \theta^\beta \land \eta_2 + i\eta_3
\]
\[ + \left[ d\Theta^\alpha + \pi^\alpha \Gamma_{\sigma r} \Theta^r - \frac{1}{2} \left( 5\varphi_0 - i\varphi_1 \right) \Theta^\alpha - \pi^\alpha (\varphi_2 + i\varphi_3) \Theta^\sigma + 2i\pi^\alpha \phi^\sigma \mathcal{L}_{\sigma r} + i\phi^\sigma M^\alpha_{\sigma r} \right. \\
\left. + \frac{i}{2} (\varphi_2 - i\varphi_3) \mathcal{K}^\alpha \right] \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3) \]
\[ + i\pi^\alpha \left[ d\Theta^\alpha + \pi^\mu (\mu \neq r, \sigma) \Theta^r - \frac{1}{2} \left( 5\varphi_0 + i\varphi_1 \right) \Theta^\alpha - \pi^\mu (\varphi_2 - i\varphi_3) \Theta^\sigma - 2i\pi^\mu \phi^\sigma \mathcal{L}_{\sigma r} - i\phi^\sigma M^\alpha_{\sigma r} \right. \\
\left. - \frac{i}{2} (\varphi_2 + i\varphi_3) \mathcal{K}^\alpha \right] \wedge (\eta_1 \wedge (\eta_2 + i\eta_3)) \]
\[ + \left[ d\Theta^\alpha + \pi^\alpha \Gamma_{\sigma r} \Theta^r - \frac{1}{2} \left( 5\varphi_0 - 3i\varphi_1 \right) \Theta^\alpha + \frac{3i}{2} (\varphi_2 + i\varphi_3) \Theta^\alpha - 3\pi^\alpha \phi^\sigma (i\mathcal{M})_{\sigma r} \right) \wedge (\eta_2 - i\eta_3) \]

Using (4.7) - (4.12), by (4.42) we obtain the first of the Bianchi identities (4.17), whereas (4.43) reads as

\[ 0 = d^2 \phi_\alpha = -\frac{i}{2} g_{\alpha\beta} \Psi \wedge \theta^\beta = -\frac{i}{2} \pi_{\alpha\beta} \bar{\Psi} \wedge \theta^\beta - i\pi^\alpha_\beta \mathcal{V}_{\alpha\beta\gamma} \wedge \theta^\gamma + \pi^\alpha_\beta \bar{\mathcal{L}}^\alpha_{\beta r} \wedge \theta^r \wedge (\eta_2 + i\eta_3) \]
\[ + \mathcal{M}_{\alpha\beta} \wedge \theta^\beta \wedge \eta_1 - i\pi^\alpha_\beta \mathcal{M}_{\alpha\beta} \wedge \theta^\beta \wedge (\eta_2 + i\eta_3) + i\mathcal{L}^*(\alpha\beta) \wedge \theta^\beta \wedge (\eta_2 - i\eta_3) \]
\[ - \Theta^\alpha_c \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3) + i\pi^\alpha_c \Theta^\alpha_c \wedge \eta_1 \wedge (\eta_2 - i\eta_3) + \Theta^\alpha_c \wedge \eta_1 \wedge (\eta_2 + i\eta_3) = 0, \]

with \( \Psi, \Phi \) being two-forms defined by

\[ \Psi \equiv \left[ d\psi_1 - \varphi_0 \wedge \psi_1 + \varphi_2 \wedge \psi_3 - \varphi_3 \wedge \psi_2 + 4i\phi_2 \wedge \phi^\beta - 4\pi^\beta_\gamma \mathcal{L}_{\beta \gamma} \theta^\beta \wedge \theta^\gamma - 4(\mathcal{E}_{\beta} \theta^\beta + \mathcal{E}_{\beta} \theta^\gamma) \right] \wedge \eta_1 \]
\[ + 4\pi^\beta_\gamma \Theta_{\beta \gamma} \wedge (\eta_2 + i\eta_3) - 4i\pi^\beta_\gamma \Theta_{\beta \gamma} \wedge (\eta_2 - i\eta_3), \]

\[ \Phi \equiv \left[ d(\psi_2 + i\psi_3) - (\varphi_0 - i\varphi_1) \wedge (\psi_2 + i\psi_3) - i(\varphi_2 + i\varphi_3) \wedge \psi_1 - 4\pi^\beta_\gamma \phi^\beta \wedge \phi^\gamma \right. \\
\left. - 4i\pi^\beta_\gamma \mathcal{M}_{\beta \gamma} \theta^\beta \wedge \theta^\gamma + 4(\pi^\beta_\gamma \mathcal{E}_{\beta \gamma} \theta^\beta) \wedge (\eta_2 + i\eta_3) \right. \\
\left. + 4\pi^\beta_\gamma \mathcal{H}_{\beta \gamma} \theta^\beta \wedge (\eta_2 - i\eta_3), \right. \]

and \( \Theta^\alpha_c, \mathcal{K}_{\alpha} \) being one-forms given by

\[ \Theta^\alpha_c \equiv \left[ d\Theta^\alpha - \pi^\alpha \Gamma_{\alpha \sigma} \Theta^\sigma - \frac{1}{2} \left( 5\varphi_0 + i\varphi_1 \right) \Theta^\alpha + \pi^\alpha_\sigma (\varphi_2 + i\varphi_3) \Theta^\sigma + 2i\pi^\alpha \phi^\sigma \mathcal{L}_{\sigma \alpha} - i\phi^\sigma M^\alpha_{\sigma} \right. \\
\left. - \frac{i}{2} (\varphi_2 + i\varphi_3) \mathcal{K}_{\alpha} \right] \]

An immediate consequence of (4.17) and (4.44) is that

\[ \begin{align*}
S^\alpha_{\beta \gamma \delta} &= S_{\alpha \beta \gamma \delta} \theta^\delta + S_{\alpha \beta \gamma \delta} \theta^\delta + (S_{\alpha \beta \gamma \delta} \eta_1 + (S_{\beta \gamma \delta} \eta_2 + i\eta_3) + (S_{\gamma \delta} \eta_2 - i\eta_3) \\
V^\alpha_{\beta \gamma \delta} &= V_{\alpha \beta \gamma \delta} \theta^\delta + V_{\alpha \beta \gamma \delta} \theta^\delta + (V_{\beta \gamma \delta} \eta_1 + (V_{\gamma \delta} \eta_2 + i\eta_3) + (V_{\delta} \eta_2 - i\eta_3) \\
\mathcal{L}^\alpha_{\beta \gamma \delta} &= \mathcal{L}_{\beta \gamma \delta} \eta_1 + (\mathcal{L}_{\gamma \delta} \eta_2 + i\eta_3) + (\mathcal{L}_{\delta} \eta_2 - i\eta_3) \\
\mathcal{M}^\alpha_{\beta \gamma \delta} &= \mathcal{M}_{\alpha \beta \gamma \delta} \theta^\delta + \mathcal{M}_{\alpha \beta \gamma \delta} \theta^\delta + (\mathcal{M}_{\beta \gamma \delta} \eta_1 + (\mathcal{M}_{\gamma \delta} \eta_2 + i\eta_3) + (\mathcal{M}_{\delta} \eta_2 - i\eta_3) \\
\Theta^\alpha_c &= \Theta^\alpha_c \eta_1 + \Theta^\alpha_c \eta_1 + (\Theta^\alpha_c \eta_1 + \Theta^\alpha_c \eta_1 + (\mathcal{H}_{\alpha} \eta_2 + i\eta_3) + (\mathcal{H}_{\alpha} \eta_2 - i\eta_3) \\
\mathcal{K}_{\alpha} &= \mathcal{K}_{\alpha} \theta^\delta + \mathcal{K}_{\alpha} \theta^\delta + (\mathcal{K}_{\alpha} \eta_1 + (\mathcal{K}_{\alpha} \eta_2 + i\eta_3) + (\mathcal{K}_{\alpha} \eta_2 - i\eta_3).
\end{align*} \]

for some appropriate coefficients \( S_{\alpha \beta \gamma \delta}, S_{\alpha \beta \gamma \delta}, (S_{\alpha}), (S_{\beta}), (S_{\gamma}), (S_{\delta}), \mathcal{L}_{\alpha \beta \gamma \delta}, \mathcal{L}_{\alpha \beta \gamma \delta}, (\mathcal{L}_{\alpha}), \mathcal{M}_{\alpha \beta \gamma \delta}, \mathcal{M}_{\alpha \beta \gamma \delta}, (\mathcal{M}_{\alpha}), (\mathcal{M}_{\beta}), (\mathcal{M}_{\gamma}), (\mathcal{M}_{\delta}), \mathcal{H}_{\alpha \beta \gamma \delta}, \mathcal{H}_{\alpha \beta \gamma \delta}, (\mathcal{H}_{\alpha}), (\mathcal{H}_{\beta}), (\mathcal{H}_{\gamma}), (\mathcal{H}_{\delta}). \)
Substituting (4.49) back into (4.17), we consider only the terms involving \( \theta^\gamma \wedge \theta^\delta \wedge \eta_1 \) and \( \theta^\gamma \wedge \theta^\delta \wedge (\eta_2 + i\eta_3) \). Then,
\[
\pi_\delta^\gamma (S_{\alpha \beta \gamma \sigma, \epsilon} \theta^\epsilon + S_{\alpha \beta \gamma \sigma, \mu} \theta^\mu) \wedge \theta^\gamma \wedge \theta^\delta = 0
\]
and it follows that the array \( \{ S_{\alpha \beta \gamma \sigma, \epsilon} \} \) must be totally symmetric. By the first line of (4.2) (which we have already proved),
\[
S^*_\alpha \beta \gamma \delta = \pi^\mu_\alpha \pi^\sigma_\beta \pi^\tau_\gamma \pi^\pi_\delta S_{\mu \nu \sigma \tau}
\]
and therefore,
\[
S_{\alpha \beta \gamma \delta, \epsilon} = \pi^\mu_\alpha \pi^\sigma_\beta \pi^\tau_\gamma \pi^\pi_\delta S_{\mu \nu \sigma \tau, \epsilon}.
\]
Hence, defining
\[
(4.50) \quad A_{\alpha \beta \gamma \delta, \epsilon} \overset{\text{def}}{=} S_{\alpha \beta \gamma \delta, \epsilon},
\]
we obtain
\[
(4.51) \quad S_{\alpha \beta \gamma \delta, \epsilon} = -\pi^\gamma_\epsilon (iA)_{\alpha \beta \gamma \delta \epsilon}.
\]
The vanishing of the coefficients of \( \theta^\gamma \wedge \theta^\delta \wedge \eta_1 \), \( \theta^\gamma \wedge \theta^\delta \wedge (\eta_2 + i\eta_3) \) and \( \theta^\gamma \wedge \theta^\delta \wedge (\eta_2 - i\eta_3) \) (after substituting (4.49) into (4.17)) yields
\[
\begin{align*}
\pi^\gamma_\delta (S_1)_{\alpha \beta \gamma \sigma} &- V_{\alpha \beta \gamma, \delta} + \pi^\mu_\alpha \pi^\nu_\beta V_{\mu \nu \delta, \gamma} = 0 \quad (S_1)_{\alpha \beta \gamma \delta} = B_{\alpha \beta \gamma \delta} + (iB)_{\alpha \beta \gamma \delta}, \\
\pi^\gamma_\delta (S_2)_{\alpha \beta \gamma \sigma} &- i\pi^\mu_\delta V_{\alpha \beta \sigma, \gamma} = 0 \quad (S_2)_{\alpha \beta \gamma \delta} = iC_{\alpha \beta \gamma \delta} \quad (S_3)_{\alpha \beta \gamma \delta} = -i(iC)_{\alpha \beta \gamma \delta}, \\
\pi^\gamma_\delta (S_3)_{\alpha \beta \gamma \sigma} &- i\pi^\mu_\alpha \pi^\nu_\beta \pi^\tau_\gamma V_{\mu \nu \tau \delta} = 0.
\end{align*}
\]
Therefore, if we define
\[
\begin{align*}
B_{\alpha \beta \gamma \delta} &\overset{\text{def}}{=} -\pi^\gamma_\delta V_{\alpha \beta \gamma, \sigma} \\
C_{\alpha \beta \gamma \delta} &\overset{\text{def}}{=} -i(S_2)_{\alpha \beta \gamma \delta},
\end{align*}
\]
we obtain that the arrays \( \{ B_{\alpha \beta \gamma \delta} \} \) and \( \{ C_{\alpha \beta \gamma \delta} \} \) are totally symmetric and
\[
(4.52) \quad (S_1)_{\alpha \beta \gamma \delta} = B_{\alpha \beta \gamma \delta} + (iB)_{\alpha \beta \gamma \delta}, \quad (S_2)_{\alpha \beta \gamma \delta} = iC_{\alpha \beta \gamma \delta}, \quad (S_3)_{\alpha \beta \gamma \delta} = -i(iC)_{\alpha \beta \gamma \delta}.
\]
Similarly, the coefficients of \( \theta^\gamma \wedge \eta_1 \wedge (\eta_2 + i\eta_3) \), \( \theta^\gamma \wedge \eta_1 \wedge (\eta_2 + i\eta_3) \) and \( \theta^\gamma \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3) \) give the equations
\[
\begin{align*}
(V_2)_{\alpha \beta \gamma, \sigma} + M_{\alpha \beta, \gamma} &\overset{\text{def}}{=} 0 \\
i\pi^\mu_\gamma (V_1)_{\alpha \beta \sigma, \gamma} + M_{\alpha \beta, \gamma} + \pi^\mu_\alpha \pi^\nu_\beta (V_3)_{\mu \nu \delta, \gamma} &\overset{\text{def}}{=} 0 \\
i\pi^\mu_\alpha \pi^\nu_\beta \pi^\tau_\gamma (V_3)_{\mu \nu \tau \delta} + i\mathcal{L}_{\alpha \beta, \gamma} &\overset{\text{def}}{=} 0.
\end{align*}
\]
Defining
\[
\begin{align*}
D_{\alpha \beta \gamma} &\overset{\text{def}}{=} (V_1)_{\alpha \beta \gamma} \\
E_{\alpha \beta \gamma} &\overset{\text{def}}{=} (V_2)_{\alpha \beta \gamma} \\
F_{\alpha \beta \gamma} &\overset{\text{def}}{=} (V_3)_{\alpha \beta \gamma},
\end{align*}
\]
we deduce that
\[
(4.53) \quad \mathcal{L}_{\alpha \beta, \gamma} = -(i\mathcal{F})_{\alpha \beta, \gamma}, \quad \mathcal{L}_{\alpha \beta, \gamma} = -\pi^\gamma_\gamma \mathcal{F}_{\alpha \beta, \sigma}, \\
M_{\alpha \beta, \gamma} = -E_{\alpha \beta, \gamma}, \quad M_{\alpha \beta, \gamma} = -i\pi^\gamma_\gamma D_{\alpha \beta, \sigma} - \pi^\mu_\alpha \pi^\nu_\beta \mathcal{F}_{\mu \nu \gamma}.
\]
Another consequence of (4.44) is that the two-forms $\Psi$ and $\Phi$ must be contained in $\Lambda^2\{\theta^\alpha, \theta^\beta, \eta_s\}$ and therefore, there should exist functions $(X_s)_{\alpha\beta} = -(X_s)_{\beta\alpha}$ and $(Y_s)_{\alpha\beta} = -(Y_s)_{\beta\alpha}$ so that
\[
d\psi_1 - \varphi_0 \wedge \psi_1 + \varphi_2 \wedge \psi_3 - \varphi_3 \wedge \psi_2 + 4i\phi_\beta \wedge \phi^\beta + 4\pi_\alpha^\beta E_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \equiv
\]
\[
\equiv (X_1)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + (X_2)_{\alpha\beta} \theta^5 \wedge \theta^\beta + (Y_1)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta
\]
\[
\]
\[
d\psi_2 - \varphi_0 \wedge \psi_2 + \varphi_3 \wedge \psi_1 - \varphi_1 \wedge \psi_3 + 2\pi_\alpha^\beta \phi^\alpha \wedge \phi^\beta
\]
\[
\equiv (X_2)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + (X_2)_{\alpha\beta} \theta^5 \wedge \theta^\beta + (Y_2)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta
\]
\[
\]
\[
d\psi_3 - \varphi_0 \wedge \psi_3 + \varphi_1 \wedge \psi_2 - \varphi_2 \wedge \psi_1 - 2\pi_\alpha^\beta \phi^\alpha \wedge \phi^\beta
\]
\[
\equiv (X_3)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + (X_3)_{\alpha\beta} \theta^5 \wedge \theta^\beta + (Y_3)_{\alpha\beta} \theta^\alpha \wedge \theta^\beta
\]
\[
\]
Substituting (4.54) into (4.44), we obtain that
\[
(4.55) \quad 0 = \frac{i}{2} \left( (X_1)_{\beta\gamma} \theta^\beta \wedge \theta^\gamma + (X_1)_{\gamma\delta} \theta^\gamma \wedge \theta^\delta + (Y_1)_{\beta\gamma} \theta^\beta \wedge \theta^\gamma \right) \wedge \theta^\alpha
\]
\[
+ \frac{1}{2} \pi_3^\beta \left[ \left( (X_2)_{\beta\gamma} + i(X_3)_{\beta\gamma} \right) \theta^\beta \wedge \theta^\gamma + \left( (X_2)_{\gamma\delta} + i(X_3)_{\gamma\delta} \right) \theta^\gamma \wedge \theta^\delta \right] \wedge \theta^\alpha
\]
\[
\]
which yields the system of equations
\[
(X_1)_{\alpha\beta} = 0,
\]
\[
(X_2)_{\alpha\beta} - i(X_3)_{\alpha\beta} = 0,
\]
\[
(4.56) \quad i \left( g_{\alpha\delta}(Y_1)_{\gamma\delta} - g_{\gamma\delta}(Y_1)_{\beta\delta} \right) = 2\pi_{\delta\delta} \left( (X_2)_{\beta\gamma} + i(X_3)_{\beta\gamma} \right),
\]
\[
\]
\[
\pi_{\alpha\delta} \left( Y_2)_{\gamma\delta} - \pi_{\alpha\gamma} (Y_2)_{\beta\delta} + i\pi_{\alpha\delta} (Y_3)_{\beta\gamma} - i\pi_{\alpha\gamma} (Y_3)_{\beta\delta} \right) = 2i g_{\alpha\delta}(X_1)_{\gamma\delta}.
\]
\[
\]
If we multiply the third line of (4.56) by $g_{\delta\delta}$ and take the sum in $\alpha$ and $\beta$ we obtain that
\[
(Y_1)_{\gamma\delta} = -\frac{2i}{4n-1} \pi_{\delta\delta} \left( (X_2)_{\gamma\sigma} + i(X_3)_{\gamma\sigma} \right).
\]
\[
\]
Substituting back into (4.56) gives
\[
\frac{2}{4n-1} \left[ g_{\alpha\delta} \pi_{\delta\delta} \left( (X_2)_{\gamma\sigma} + i(X_3)_{\gamma\sigma} \right) - g_{\gamma\delta} \pi_{\delta\delta} \left( (X_2)_{\beta\sigma} + i(X_3)_{\beta\sigma} \right) \right] = 2\pi_{\alpha\delta} \left( (X_2)_{\beta\gamma} + i(X_3)_{\beta\gamma} \right)
\]
\[
\]
Now, we multiply the latter by $\pi_{\alpha\delta}$ and take the sum in $\alpha$ and $\delta$ to arrive at
\[
-\frac{4}{4n-1} \left( (X_2)_{\beta\gamma} + i(X_3)_{\beta\gamma} \right) = 8n \left( (X_2)_{\beta\gamma} + i(X_3)_{\beta\gamma} \right).
\]
\[
\]
This together with the second line of (4.56) implies that $(X_2)_{\alpha\beta} = (X_3)_{\alpha\beta} = 0$. Proceeding similarly with the forth line of (4.56), we conclude that $(X_s)_{\alpha\beta} = 0$ and $(Y_s)_{\alpha\beta} = 0$.

By considering the coefficients of $\theta^\delta \wedge \theta^\gamma \wedge \eta_1$, $\theta^\beta \wedge \theta^\gamma \wedge \eta_1$, $\theta^\beta \wedge \theta^\gamma \wedge (\eta_2 + i\eta_3)$, $\theta^\beta \wedge \theta^\gamma \wedge (\eta_2 - i\eta_3)$, $\theta^\beta \wedge \theta^\gamma \wedge (\eta_2 + i\eta_3)$, $\theta^\beta \wedge \theta^\gamma \wedge \eta_1$, $\theta^\beta \wedge \theta^\gamma \wedge (\eta_2 + i\eta_3)$, $\theta^\beta \wedge \theta^\gamma \wedge (\eta_2 - i\eta_3)$, $\theta^\beta \wedge \theta^\gamma \wedge (\eta_2 + i\eta_3)$, $\theta^\beta \wedge \theta^\gamma \wedge (\eta_2 - i\eta_3)$ and $\theta^\beta \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3)$ separately, we obtain the equations
\[
(4.57) \quad \Psi = U \eta_1 \wedge (\eta_2 + i\eta_3) + \frac{1}{2n} \eta_1 \wedge (\eta_2 - i\eta_3) + iW \eta_2 + i\eta_3 \wedge (\eta_2 - i\eta_3),
\]
\[
\Phi = V_1 \eta_1 \wedge (\eta_2 + i\eta_3) + V_2 \eta_1 \wedge (\eta_2 - i\eta_3) + V_3 \eta_2 + i\eta_3 \wedge (\eta_2 - i\eta_3)
\]
for some appropriate coefficients $U$, $V_s$ and $W = \overline{W}$. Substituting back into (4.44) and considering the coefficients of $\theta^\beta \wedge \eta_1 \wedge (\eta_2 + i\eta_3)$, $\theta^\beta \wedge \eta_1 \wedge (\eta_2 + i\eta_3)$, $\theta^\beta \wedge \eta_1 \wedge (\eta_2 - i\eta_3)$, $\theta^\beta \wedge \eta_1 \wedge (\eta_2 - i\eta_3)$, $\theta^\beta \wedge \eta_1 \wedge (\eta_2 - i\eta_3)$, $\theta^\beta \wedge \eta_1 \wedge (\eta_2 - i\eta_3)$ and $\theta^\beta \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3)$ separately, we obtain the equations
\begin{align*}
\mathcal{K}_{\alpha,\beta} + (M_2)_{\alpha\beta} - \frac{1}{2} \pi_{\alpha\beta} V_2 &= 0 \\
\mathcal{K}_{\alpha,\beta} - \frac{i}{2} g_{\alpha\beta} U + \pi^\sigma_\alpha (L_3)_{\bar{\mu}\bar{\beta}} + i\pi^\mu_\beta (M_1)_{\alpha\nu} &= 0 \\
\mathcal{E}_{\alpha,\beta} - i\pi^\sigma_\alpha (M_3)_{\bar{\sigma}\bar{\beta}} + \pi^\sigma_\beta (L_1)_{\bar{\sigma}\bar{\beta}} - \frac{i}{2} g_{\alpha\beta} V_1 &= 0 \\
\mathcal{E}_{\alpha,\beta} + \frac{1}{2} \pi_{\alpha\beta} U + i(L_2)_{\alpha\beta} &= 0 \\
\mathcal{E}_{\alpha,\beta} - \frac{1}{2} \pi_{\alpha\beta} V_3 + i(L_2)_{\alpha\beta} &= 0 \\
\mathcal{E}_{\alpha,\beta} + i\pi^3_\beta (M_3)_{\alpha\sigma} - \frac{1}{2} g_{\alpha\beta} W &= 0.
\end{align*}
(4.58)

By the third and the sixth lines of (4.58), we have
\[ \frac{1}{2} \pi_{\alpha\beta} (W + i\overline{V}_1) = (L_1)_{\alpha\beta} + i(M_3)_{\alpha\beta} - i\pi^\mu_\alpha \pi^\nu_\beta (M_3)_{\bar{\mu}\bar{\nu}} \]
and hence
\[ V_1 = -iW, \quad (L_1)_{\alpha\beta} = -i(M_3)_{\alpha\beta} + i\pi^\mu_\alpha \pi^\nu_\beta (M_3)_{\bar{\mu}\bar{\nu}}. \]
Whereas, the forth and the fifth lines of (4.58) yield
\[ V_3 = -\overline{V}. \]
Therefore, if we define
\[
\begin{align*}
\mathcal{P} &\overset{\text{def}}{=} U \\
\mathcal{Q} &\overset{\text{def}}{=} \overline{V}_2 \\
\mathcal{R} &\overset{\text{def}}{=} W,
\end{align*}
\]
we get the structure equations (4.5) and (4.6), which completes the proof of Proposition 4.1.

Furthermore, defining
\[
\begin{align*}
\mathcal{G}_{\alpha\beta} &\overset{\text{def}}{=} -i(L_2)_{\alpha\beta} \\
\mathcal{X}_{\alpha\beta} &\overset{\text{def}}{=} (M_1)_{\alpha\beta} \\
\mathcal{Y}_{\alpha\beta} &\overset{\text{def}}{=} (M_2)_{\alpha\beta} \\
\mathcal{Z}_{\alpha\beta} &\overset{\text{def}}{=} (M_3)_{\alpha\beta},
\end{align*}
\]
we obtain that
\[
\begin{align*}
(L_1)_{\alpha\beta} &= i \left( (i\mathcal{Z}_{\alpha\beta}) - \mathcal{X}_{\alpha\beta} \right), \\
(L_2)_{\alpha\beta} &= i\mathcal{G}_{\alpha\beta}, \\
(L_3)_{\alpha\beta} &= -i(\mathcal{G})_{\alpha\beta},
\end{align*}
(4.59)
\[
\begin{align*}
\mathcal{E}_{\alpha,\beta} &= \mathcal{G}_{\alpha\beta} - \frac{1}{2} \pi_{\alpha\beta} \mathcal{P}, \\
\mathcal{E}_{\alpha,\beta} &= -i\pi^\sigma_\beta \mathcal{Z}_{\alpha\sigma} + \frac{1}{2} g_{\alpha\beta} \mathcal{R}, \\
\mathcal{K}_{\alpha,\beta} &= -\mathcal{Y}_{\alpha\beta} + \frac{1}{2} \pi_{\alpha\beta} \mathcal{Q}, \\
\mathcal{K}_{\alpha,\beta} &= i\pi^\sigma_\beta \left( \mathcal{G}_{\alpha\sigma} - \mathcal{X}_{\alpha\sigma} \right) + \frac{i}{2} g_{\alpha\beta} \mathcal{P}.
\end{align*}
\]

Now, substituting (4.57) into (4.44) and using the above relations, we get the second of the Bianchi identities (4.18).

We proceed by differentiating both sides of the equations (4.5) and (4.6). After some straightforward calculations we arrive at the third (4.19) and the forth (4.20) of the Bianchi identities, which completes the proof of Proposition 4.2.

One immediate consequence of (4.19) and (4.20) is that the one forms \( \mathcal{P}^*, \mathcal{Q}^* \) and \( \mathcal{R}^* \) must be vanishing modulo \( \{ \theta^{\alpha}, \theta^{\bar{\alpha}}, \eta_{\alpha} \} \). Let
\[
\begin{align*}
\mathcal{P}^* &= X_\epsilon \theta^\epsilon + Y_\epsilon \theta^\epsilon + P_1 \eta_1 + P_2 (\eta_2 + i\eta_3) + P_3 (\eta_2 - i\eta_3) \\
\mathcal{Q}^* &= Z_\epsilon \theta^\epsilon + W_\epsilon \theta^\epsilon + Q_1 \eta_1 + Q_4 (\eta_2 + i\eta_3) + Q_3 (\eta_2 - i\eta_3) \\
\mathcal{R}^* &= U_\epsilon \theta^\epsilon + U_\epsilon \theta^\epsilon + R_1 \eta_1 + R_2 (\eta_2 + i\eta_3) + R_2 (\eta_2 - i\eta_3),
\end{align*}
(4.60)
where \( X, Y, Z, W, U, V, \mathcal{P}, Q, \mathcal{R} \) are some appropriate functions. Substituting (4.60) and (4.49) into (4.19), (4.20) and considering the coefficients of \( \theta^\alpha \wedge \eta_1 \wedge (\eta_2 + i\eta_3), \theta^\alpha \wedge \eta_1 \wedge (\eta_2 + i\eta_3), \theta^\alpha \wedge \eta_1 \wedge (\eta_2 - i\eta_3), \theta^\alpha \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3), \theta^\alpha \wedge \eta_1 \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3) \) separately, we obtain the equations
\[
\begin{align*}
X_\alpha + 4(i\alpha_2)_\alpha &= 0 \\
U_\alpha - 4i\pi^\sigma_\alpha (\mathcal{E}_3)_\sigma &= 0 \\
Y_\alpha + 4(i\alpha_3)_\alpha - 4i\pi^\sigma_\alpha (\mathcal{E}_1)_\sigma &= 0 \\
U_\alpha + 4i(\mathcal{H}_3)_\alpha - 4i(\mathcal{E}_1)_\alpha &= 0 \\
W_\alpha + 4i\pi^\bar{\sigma}_\alpha \left((\mathcal{E}_2)_\bar{\sigma} + (\mathcal{H}_1)_\bar{\sigma}\right) &= 0 \\
Z_\alpha - 4(\mathcal{H}_2)_\alpha &= 0 \\
Y_\alpha - 4i\pi^\sigma_\alpha (\mathcal{H}_3)_\sigma &= 0 \\
\mathcal{R}_1 - i(\mathcal{P}_3 - \mathcal{P}_3) &= 0 \\
\mathcal{R}_2 + i(\mathcal{P}_1 + \mathcal{Q}_3) &= 0.
\end{align*}
\]

(4.61)

From these we easily deduce that if we define
\[
\begin{align*}
\mathcal{U}_1 \overset{\text{def}}{=} & \mathcal{P}_1, \\
\mathcal{U}_2 \overset{\text{def}}{=} & \mathcal{P}_2, \\
\mathcal{U}_3 \overset{\text{def}}{=} & \mathcal{P}_3, \\
\mathcal{W}_1 \overset{\text{def}}{=} & \mathcal{Q}_1, \\
\mathcal{W}_2 \overset{\text{def}}{=} & \mathcal{Q}_2, \\
\mathcal{W}_3 \overset{\text{def}}{=} & \mathcal{Q}_3, \\
(N_1)_\alpha \overset{\text{def}}{=} & (\mathcal{E}_1)_\alpha, \\
(N_2)_\alpha \overset{\text{def}}{=} & (\mathcal{E}_2)_\alpha, \\
(N_3)_\alpha \overset{\text{def}}{=} & (\mathcal{E}_3)_\alpha, \\
(N_4)_\alpha \overset{\text{def}}{=} & (\mathcal{H}_1)_\alpha, \\
(N_5)_\alpha \overset{\text{def}}{=} & (\mathcal{H}_2)_\alpha,
\end{align*}
\]

then we have that
\[
\begin{align*}
X_\alpha &= -4(N_2)_\alpha, \\
Y_\alpha &= 4i\pi^\sigma_\alpha (N_1)_\alpha - (N_3)_\alpha, \\
Z_\alpha &= 4(N_5)_\alpha, \\
W_\alpha &= -4i\pi^\bar{\sigma}_\alpha (N_2)_\bar{\sigma} - (N_4)_\bar{\sigma}, \\
U_\alpha &= 4\pi^\sigma_\alpha (N_3)_\sigma, \\
\mathcal{R}_1 &= i(\mathcal{U}_3 - \mathcal{U}_3), \\
\mathcal{R}_2 &= -i(\mathcal{U}_1 + \mathcal{W}_3), \\
(\mathcal{H}_3)_\alpha &= (N_1)_\alpha + i\pi^\sigma_\alpha (N_3)_\sigma.
\end{align*}
\]

This completes the proof of Proposition 4.3. \(\square\)

5. The associated Cartan geometry

Our next goal is to check that the construction of the canonical coframe from Theorem 3.3 coincides with the general normalization used for all parabolic geometries (and explained briefly in the appendix).

First we compare the structure equations from the Proposition 4.1 with those of the homogeneous model \( G \to G/P \). This verifies that our coframe lives on the principal fibre bundle with the right structure group.

Next, we express the Kostant’s codifferential on the cochains explicitly, and we obtain that indeed, the curvature components from Proposition 4.1 are normalized in the canonical way.

5.1. A few algebraic constructions. Consider the standard action of the group \( Sp(n + 1, 1) \) on \( \mathbb{R}^{4n+8} \) defined by some (fixed) identification \( \mathbb{R}^{4n+8} \cong \mathbb{H}^{n+2} \). Let \( J_1, J_2, J_3 \) be the induced invariant quaternionic structure on \( \mathbb{R}^{4n+8} \) and let \( \langle , \rangle \) be the corresponding inner product of signature \( (+ (4n + 4), -4) \). The complexification \( \mathbb{C}^{4n+8} \) of \( \mathbb{R}^{4n+8} \) splits as a direct sum of \( i \) and \( -i \) eigenspaces with respect to the complex structure \( J_1 \),
\[
\mathbb{C}^{4n+8} = W \oplus \overline{W}.
\]

Let us fix a basis \( \{v_1, v_2, e_\alpha, w_1, w_2\} \) of \( W \) for which
\[
\begin{align*}
J_2(v_1) &= \overline{v}_2, \\
J_2(e_\alpha) &= \pi^\beta_\alpha e^\beta, \\
J_2(w_1) &= \overline{w}_2.
\end{align*}
\]

Let us fix a basis \( \{v_1, v_2, e_\alpha, w_1, w_2\} \) of \( W \) for which
\[
\begin{align*}
J_2(v_1) &= \overline{v}_2, \\
J_2(e_\alpha) &= \pi^\beta_\alpha e^\beta, \\
J_2(w_1) &= \overline{w}_2.
\end{align*}
\]
and
\begin{align*}
\langle v_1, \bar{v}_1 \rangle &= 0, & \langle v_1, v_2 \rangle &= 0, & \langle v_1, e_\beta \rangle &= 0, & \langle v_1, \bar{v}_1 \rangle &= 1, & \langle v_1, \bar{v}_2 \rangle &= 0, \\
\langle v_2, \bar{v}_1 \rangle &= 0, & \langle v_2, \bar{v}_2 \rangle &= 0, & \langle v_2, e_\alpha \rangle &= 0, & \langle v_2, \bar{v}_1 \rangle &= 0, & \langle v_2, \bar{v}_2 \rangle &= 1. \\
\langle \epsilon_\alpha, \bar{\epsilon}_1 \rangle &= 0, & \langle \epsilon_\alpha, \bar{v}_2 \rangle &= 0, & \langle \epsilon_\alpha, e_\beta \rangle &= g_{\alpha\beta}, & \langle \epsilon_\alpha, \bar{\epsilon}_1 \rangle &= 0, & \langle \epsilon_\alpha, \bar{\epsilon}_2 \rangle &= 0. \\
\langle w_1, \bar{v}_1 \rangle &= 1, & \langle w_1, \bar{v}_2 \rangle &= 0, & \langle w_1, e_\alpha \rangle &= 0, & \langle w_1, \bar{v}_1 \rangle &= 0, & \langle w_1, \bar{v}_2 \rangle &= 0, \\
\langle w_2, \bar{v}_1 \rangle &= 0, & \langle w_2, \bar{v}_2 \rangle &= 1, & \langle w_2, e_\alpha \rangle &= 0, & \langle w_2, \bar{v}_1 \rangle &= 0, & \langle w_2, \bar{v}_2 \rangle &= 0.
\end{align*}

The group $Sp(n+1,1)$ consists of all endomorphisms of $W$ that take $\{v_1, v_2, \epsilon_\alpha, w_1, w_2\}$ into a bases with the same properties (5.1), (5.2). By differentiating these at the identity, we obtain the Lie algebra $\mathfrak{g} = sp(n+1,1)$ as the set of all matrices of the form
\begin{equation}
\begin{pmatrix}
- \frac{1}{2} (\varphi_0 + i\varphi_1) & - \frac{1}{2} (\varphi_2 - i\varphi_3) & 2i\eta_{\beta\bar{\alpha}} \phi^\beta & i\psi_1 & (\psi_2 - i\psi_3) \\
\frac{1}{2} (\varphi_2 + i\varphi_3) & - \frac{1}{2} (\varphi_0 - i\varphi_1) & 2i\eta_{\alpha\bar{\beta}} \phi^\alpha & -i\psi_2 + i\psi_3 & -i\psi_1 \\
\frac{i}{2} \eta_1 & - \frac{i}{2} (\eta_2 + i\eta_3) & ig_{\beta\bar{\alpha}} \theta^\beta & \frac{1}{2} (\varphi_0 - i\varphi_1) & - \frac{1}{2} (\varphi_2 - i\varphi_3) \\
- \frac{i}{2} (\eta_2 + i\eta_3) & \frac{i}{2} \eta_1 & i\pi_{\alpha\beta} \phi^\alpha & \frac{1}{2} (\varphi_2 + i\varphi_3) & \frac{1}{2} (\varphi_0 + i\varphi_1)
\end{pmatrix},
\end{equation}

where $\eta_\alpha, \varphi_\alpha, \psi_\alpha$ are real, and $\theta^\alpha, \phi^\alpha, \Gamma_{\alpha\beta}$ are complex so that
$$\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}, \quad (i\Gamma)_{\alpha\beta} = \Gamma_{\alpha\beta}.$$

We may interpret $\eta_\alpha, \theta^\alpha, \varphi_\alpha, \psi_\alpha, \Gamma_{\alpha\beta, \phi^\alpha}$ as left-invariant one-forms on the Lie group $Sp(n+1,1)$. We immediately derive (by using just matrix multiplication) the structure equations of the group:
\begin{align}
d\eta_1 &= -\varphi_0 \wedge \eta_1 - \varphi_2 \wedge \eta_3 + \varphi_3 \wedge \eta_2 + 2i\eta_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^\beta \\
d\eta_2 &= -\varphi_0 \wedge \eta_2 - \varphi_3 \wedge \eta_1 + \varphi_1 \wedge \eta_3 + \pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + \pi_{\alpha\bar{\beta}} \phi^\alpha \wedge \phi^\beta \\
d\eta_3 &= -\varphi_0 \wedge \eta_3 - \varphi_1 \wedge \eta_2 + \varphi_2 \wedge \eta_1 - i\pi_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + i\pi_{\alpha\bar{\beta}} \phi^\alpha \wedge \phi^\beta \\
d\theta^\alpha &= -i\phi^\alpha \wedge \eta_1 - \pi_{\alpha\beta} \phi^\beta \wedge (\eta_2 + i\eta_3) - \pi_{\alpha\bar{\beta}} \Gamma_{\alpha\beta} \wedge \theta^\beta - \frac{1}{2} (\varphi_0 + i\varphi_1) \wedge \theta^\alpha - \frac{1}{2} \pi_{\beta\bar{\alpha}} \Gamma_{\alpha\beta} \wedge (\varphi_0 + i\varphi_1) \\
d\varphi_0 &= -\psi_1 \wedge \eta_1 - \psi_2 \wedge \eta_2 - \psi_3 \wedge \eta_3 - 2\phi_\beta \wedge \theta^\beta - 2\phi_\beta \wedge \theta^\beta \\
d\varphi_1 &= -\psi_2 \wedge \varphi_3 - \psi_2 \wedge \eta_3 + \psi_3 \wedge \eta_2 + 2\phi_\beta \wedge \theta^\beta - 2\phi_\beta \wedge \theta^\beta \\
d\varphi_2 &= -\varphi_3 \wedge \psi_1 \wedge \eta_1 + \psi_1 \wedge \eta_3 - 2\pi_{\alpha\beta} \phi^\alpha \wedge \theta^\beta - 2\pi_{\alpha\bar{\beta}} \phi^\alpha \wedge \phi^\beta \\
d\varphi_3 &= -\psi_1 \wedge \varphi_2 - \psi_1 \wedge \eta_2 + \psi_2 \wedge \eta_1 + 2i\pi_{\alpha\beta} \phi^\alpha \wedge \theta^\beta - 2i\pi_{\alpha\bar{\beta}} \phi^\alpha \wedge \phi^\beta \\
d\theta_{\alpha\beta} &= -\pi_{\alpha\bar{\gamma}} \Gamma_{\alpha\beta} \wedge \Gamma_{\gamma\bar{\beta}} + \pi_{\alpha\beta} \phi_{\alpha\beta} \wedge \theta^\beta + \pi_{\alpha\beta} \phi_{\alpha\beta} \wedge \phi^\beta \\
d\phi^\alpha &= \frac{1}{2} (\varphi_0 - i\varphi_1) \wedge \phi^\alpha - \frac{1}{2} \pi_{\bar{\alpha}\gamma} \phi_{\bar{\alpha}\gamma} \wedge \phi^\gamma - \pi_{\bar{\alpha}\beta} \Gamma_{\bar{\alpha}\beta} \wedge \phi^\gamma + \frac{i}{2} \psi_1 \wedge \theta^\alpha + \frac{1}{2} \pi_{\alpha\beta} \alpha \wedge \psi_2 + 4i\phi_\gamma \wedge \phi^\gamma \\
d\psi_2 + i d\psi_3 &= (\varphi_0 - i\varphi_1) \wedge (\psi_2 + i\psi_3) + i(\varphi_2 + i\varphi_3) \wedge \psi_1 + 4\pi_{\alpha\beta} \phi^\gamma \wedge \phi^\delta.
\end{align}

Notice that the equations (5.4) are formally identical with the corresponding structure equations (3.29), (3.31), (4.3), (4.4), (4.5), (4.6) for the global coframing on $P_1$ constructed in Theorem 3.3, if assuming that all curvature components vanish,
$$S_{\alpha\beta\gamma} = V_{\alpha\beta\gamma} = L_{\alpha\beta} = M_{\alpha\beta} = C_{\alpha} = H_{\alpha} = \mathcal{P} = Q = \mathcal{R} = 0.$$

The Killing form of the Lie algebra $\mathfrak{g} = sp(n+1,1)$, which we will denote by $\mathbb{B}$, is defined as $\mathbb{B}(A, B) = \text{trace}(C \mapsto [A, [B, C]])$, $A, B \in \mathfrak{g}$. Using the above notation, we compute
\begin{align}
\mathbb{B}(A, B) &= -4(4n+6)\left(\eta_{\alpha}(A)\psi_\beta(B) + \psi_\beta(A)\eta_{\alpha}(B)\right) + (2n+6)\varphi_0(A)\varphi_0(B) - (2n+4)\varphi_0(A)\varphi_0(B) \\
&- 4(2n+7)\left(\theta_{\alpha}(A)\phi^\alpha(B) + \phi^\alpha(A)\theta_{\alpha}(B) + \theta_{\alpha}(A)\phi^\alpha(B) + \phi^\alpha(A)\theta_{\alpha}(B)\right) - 7\Gamma_{\alpha\beta}(A)\Gamma_{\alpha\beta}(B)
\end{align}
Notice that the sum $\Gamma_{\alpha\beta}(A)\Gamma^{\alpha\beta}(B)$ produces always a real number, since we have

$$
\Gamma_{\alpha\beta}(A)\Gamma^{\alpha\beta}(B) = (i\Gamma)_{\alpha\beta}(A)(i\Gamma)^{\alpha\beta}(B) = \left(\sigma^\alpha\sigma^\beta\Gamma_\sigma(A)\right)\left(\sigma^\alpha\sigma^\beta\Gamma_\sigma(B)\right) = \Gamma_{\alpha\beta}(A)\Gamma^{\alpha\beta}(B).
$$

Furthermore, the Lie algebra $\mathfrak{g}$ has a splitting (which is also a $|2|$-grading)

$$
\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,
$$

that dualizes the splitting

$$
\{\eta_\alpha\}, \{\theta^\alpha\}, \{\varphi_\alpha\}, \{\varphi_s\}, \{\Gamma_\alpha\}, \{\phi^\alpha\}, \{\psi_s\}
$$

of the left-invariant one-forms. Let

$$(5.6)\quad \{E_s \in \mathfrak{g}_{-2}\}, \{Z_\alpha, Z_\beta \in \mathfrak{g}_{-1}\}$$

be a frame (of the complexification) of $\mathfrak{g}_- \cong \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ dual to the coframe $\{\eta_\alpha\}, \{\theta^\alpha, \theta^\beta\}$, i.e. such that

$$\eta_\alpha(E_t) = \delta_{st}, \quad \theta^\alpha(Z_\beta) = \delta^\alpha_\beta, \quad \theta^\beta(Z_\beta) = 0, \quad Z_\beta = \overline{Z_\beta},$$

and let $\{\hat{E}_s \in \mathfrak{g}_2\}, \{\hat{Z}^\alpha, \hat{Z}^\beta \in \mathfrak{g}_1\}$ be the corresponding frame of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ dual to $(5.6)$ with respect to the Killing form $\mathfrak{B}$, i.e. such that

$$\mathfrak{B}(E_s, \hat{E}_t) = \delta_{st}, \quad \mathfrak{B}(Z_\alpha, \hat{Z}^\beta) = \delta^\alpha_\beta, \quad \mathfrak{B}(Z_\alpha, \hat{Z}^\beta) = 0, \quad \hat{Z}^\alpha = \overline{Z_\alpha}.$$

Then, the map $\partial^*: \Lambda^2(\mathfrak{g}_-)\otimes \mathfrak{g} \rightarrow (\mathfrak{g}_-)\otimes \mathfrak{g}$, given, for any $A \in \mathfrak{g}_-$, by

$$(5.7)\quad (\partial^*K_\alpha)(A) = 2[\hat{E}_s, K(A, E_s)] + 2[\hat{Z}^\alpha, K(A, Z_\alpha)] + 2[\hat{Z}^\alpha, K(A, Z_\alpha)]
$$

$$- K([\hat{E}_s, A]_-, E_s) - K([\hat{Z}^\alpha, A]_-, Z_\alpha) - K([\hat{Z}^\alpha, A]_-, Z_\alpha),$$

where $X_-$ denotes the projection of $X \in \mathfrak{g}$ onto $\mathfrak{g}_-$, is known as the Kostant codifferential (cf. [4], p. 261 or [8], p. 468).

**Lemma 5.1.** If $K \in \Lambda^2(\mathfrak{g}_-)\otimes (\mathfrak{sp}(n) \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2) \subset \Lambda^2(\mathfrak{g}_-)^* \otimes \mathfrak{g}$, then for any $A \in \mathfrak{g}_-$, we have

$$(\partial^*K_\alpha)(A) = \frac{1}{4(2n + 7)} \left( i \left( K(Z^\alpha, Z_\alpha) - K(\hat{Z}^\alpha, \hat{Z}^\alpha) \right) \eta_\alpha(A) - \pi^{\alpha\beta}(K(Z_\alpha, Z_\beta) - \eta_\alpha(A)) \right)
$$

$$- \pi^{\alpha\beta} K(Z_\alpha, Z_\beta) \left( \eta_\alpha(A) - i\eta_3(A) \right) - 2\pi^{\beta\gamma} \Gamma^\beta_\sigma(K(A, Z_\alpha)) \hat{Z}_\beta - 2\pi^{\beta\gamma} \Gamma^\alpha_\sigma(K(A, Z_\alpha)) \hat{Z}_\beta
$$

$$+ \frac{i4(2n + 3)}{2n + 7} \left( \phi^\alpha(K(A, Z_\alpha)) - \phi^\beta(K(A, Z_\alpha)) \right) \hat{E}_1 + \frac{4(2n + 3)}{2n + 7} \pi^{\alpha\beta}_\sigma \phi^\beta(K(A, Z_\alpha)) \hat{E}_1 + \frac{4(2n + 3)}{2n + 7} \pi^{\alpha\beta}_\sigma \phi^\beta(K(A, Z_\alpha)) \hat{E}_1
$$

$$+ \frac{i4(2n + 3)}{2n + 7} \pi^{\alpha\beta}_\sigma \phi^\beta(K(A, Z_\alpha)) \hat{E}_1 + \frac{4(2n + 3)}{2n + 7} \pi^{\alpha\beta}_\sigma \phi^\beta(K(A, Z_\alpha)) \hat{E}_1
$$

$$- g^{\alpha\beta} \pi^{\alpha\beta}_\sigma \Gamma_\sigma(B) \hat{Z}_\beta + \frac{i4(2n + 6)}{2n + 7} \phi^\beta(B) \hat{E}_1 + \frac{4(2n + 6)}{2n + 7} \phi^\beta(B) \hat{E}_1 + \frac{i4(2n + 6)}{2n + 7} \phi^\beta(B) \hat{E}_1
$$

**Proof.** By $(5.5)$, it follows that

$$\psi_\alpha(\hat{E}_t) = - \frac{\delta_{st}}{4n + 6}, \quad \phi_\alpha(\hat{Z}^\beta) = - \frac{\delta^\alpha_\beta}{4(2n + 7)}, \quad \phi_\alpha(\hat{Z}^\beta) = 0.$$

Let $A \in \mathfrak{g}_-$ and $B \in \mathfrak{sp}(n) \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be any two matrices. Then, using the structure equations $(5.4)$ of $\mathfrak{g}$, we compute that

$$[\hat{E}_s, B] = - (4n + 6)\psi_\alpha([\hat{E}_s, B]) \hat{E}_t = (4n + 6)d\psi_\alpha(\hat{E}_s, B) \hat{E}_t = 0,$$

$$[\hat{Z}^\alpha, B] = - 4(2n + 7) \phi^\beta([\hat{Z}^\alpha, B]) \hat{Z}_\beta - 4(2n + 7) \phi^\beta([\hat{Z}^\alpha, B]) \hat{Z}_\beta - (4n + 6) \psi_\alpha([\hat{Z}^\alpha, B]) \hat{E}_s
$$

$$= 4(2n + 7) d\phi^\beta(\hat{Z}^\alpha, B) \hat{Z}_\beta + 4(2n + 7) d\phi^\beta(\hat{Z}^\alpha, B) \hat{Z}_\beta + (4n + 6) d\psi_\alpha(\hat{Z}^\alpha, B) \hat{E}_s
$$

$$= - g^{\alpha\beta} \pi^{\alpha\beta}_\sigma \Gamma_\sigma(B) \hat{Z}_\beta + \frac{i4(2n + 6)}{2n + 7} \phi^\beta(B) \hat{E}_1 + \frac{4(2n + 6)}{2n + 7} \phi^\beta(B) \hat{E}_1 + \frac{i4(2n + 6)}{2n + 7} \phi^\beta(B) \hat{E}_1
$$

$$[\hat{Z}^\alpha, B] = - g^{\alpha\beta} \pi^{\alpha\beta}_\sigma \Gamma_\sigma(B) \hat{Z}_\beta - \frac{i4(2n + 6)}{2n + 7} \phi^\beta(B) \hat{E}_1 + \frac{4(2n + 6)}{2n + 7} \phi^\beta(B) \hat{E}_1 + \frac{i4(2n + 6)}{2n + 7} \phi^\beta(B) \hat{E}_1,$$
and similarly,

\[
\hat{E}_s, A \_ = 0,
\]

\[
\hat{Z}, A \_ = \theta^\beta(\hat{Z}, A) + \theta^\beta(\hat{Z}, A) Z_\beta + \eta_s(\hat{Z}, B) E_s
\]

\[
= -d\theta^\beta(\hat{Z}, A) Z_\beta - d\theta^\beta(\hat{Z}, A) Z_\beta - d\eta_s(\hat{Z}, A) E_s
\]

\[
(5.9)
\]

Substituting (5.8) and (5.9) into (5.7) gives the lemma.

\[
\Box
\]

The subalgebra \( p = g_0 \oplus g_1 \oplus g_2 \subset g \) determines a parabolic subgroup \( \mathfrak{P} \subset Sp(n+1,1) \), which, alternatively, can be described as the stabilizer of the complex 2-plane span \( \{\mathbf{v}_1, \mathbf{v}_2\} \subset W \) in \( Sp(n+1,1) \). Explicitly, \( \mathfrak{P} \) consists of all matrices of the form

\[
\begin{pmatrix}
A & -\left(A \begin{pmatrix}
U_{\beta \gamma} & \pi^\gamma_r & r^\gamma
\end{pmatrix}
\right)
\end{pmatrix}
\]

\[
(5.10)
\]

where

\[
A \in CSp(1) = \left\{ \left( \begin{array}{cc}
a_1 & -a_2 \\
a_2 & a_1
\end{array} \right) : a_1, a_2 \in \mathbb{C}, \quad a_1^2 + a_2^2 \neq 0 \right\},
\]

\[
U = (U_{\beta \gamma}) \in Sp(n), \quad r = (r^\alpha) \in \mathbb{C}^{2n} \quad \text{and} \quad \lambda_1, \lambda_2, \lambda_3 \text{ are real numbers.}
\]

5.2. The normal Cartan connection. Let \((M,H)\) be a quaternionic contact manifold and take \( pr : P_1 \to M \) to be the composition of the two principal bundle projections

\[
P_1 \overset{\pi_1}{\to} P_0 \overset{\pi_0}{\to} M,
\]

as constructed in Section 3. We can consider the global coframing

\[
(5.11)
\]

\[
\{\eta_1, \eta_2, \eta_2, \theta^\alpha, \theta^\beta, \varphi_0, \varphi_1, \varphi_2, \varphi_3\} \cup \{\Gamma_\alpha : \alpha \leq \beta\} \cup \{\phi^\alpha, \phi^\beta, \psi_1, \psi_2, \psi_3\}
\]

of the (complexified) tangent bundle \( TP_1 \) (cf. Theorem 3.3) as a map \( \omega : TP_1 \to g \ (g = sp(n+1,1)) \), by declaring \( \omega = \Omega(\eta_\alpha, \theta^\alpha, \theta^\beta, \varphi_0, \varphi_1, \varphi_2, \varphi_3) \) to be the matrix given by formula (5.3).

Recall that the local sections of \( P_1 \) are precisely the local coframings \( \eta_\alpha, \theta^\alpha, \theta^\beta, \varphi_0, \varphi_1, \varphi_2, \varphi_3 \) on \( TP_0 \) for which the assertion of Lemma 3.1 is satisfied, and observe that the equations in the lemma coincide with the first three of the structure equations (5.4) for the corresponding left-invariant one-forms on \( Sp(n+1,1) \). Since the adjoint action of \( \mathfrak{P} \) (cf. (5.10)) on \( g \) preserves (5.4), we can use it to define a natural action of \( \mathfrak{P} \) on the manifold \( P_1 \) that will preserve the fibers of the projection \( pr \). In fact, one can show that \( pr : P_1 \to M \) is a principal bundle with structure group \( \mathfrak{P}/Z_2 \). Moreover, the uniqueness part of Theorem 3.3 ensures that the \( g \)-valued form \( \omega \) on \( P_1 \) will be \( \mathfrak{P} \)-equivariant and therefore it gives a Cartan connection on \( pr : P_1 \to M \).

The curvature of the Cartan connection \( \omega \) is a function \( K \in C^\infty \left( P_1, \Lambda^2(g_-)^* \otimes (sp(n) \oplus g_1 \oplus g_2) \right) \) which, by (4.3), (4.4), (4.5), (4.6) and (5.4), is given by
\[
\Gamma_{\alpha\beta}(K) = \pi^\alpha_{\beta\gamma\sigma} \theta^\gamma \wedge \theta^\sigma + \left( \mathcal{V}_{\alpha\beta\gamma} \theta^\gamma + \pi^\alpha_{\alpha\beta} \mathcal{V}_{\alpha\beta\gamma} \theta^\gamma \right) \wedge \eta_1 \\
- i \pi^\alpha_{\alpha\beta} \mathcal{V}_{\alpha\beta\gamma} \theta^\gamma \wedge (\eta_2 + i\eta_3) + i(\mathcal{V})_{\alpha\beta\gamma} \theta^\gamma \wedge (\eta_2 - i\eta_3) \\
- i \mathcal{L}_{\alpha\beta} (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3) + \mathcal{H}_{\alpha\beta} \eta_1 \wedge (\eta_2 + i\eta_3) \\
+ (iM)_{\alpha\beta} \eta_1 \wedge (\eta_2 - i\eta_3),
\]
\[
\phi_\alpha(K) = -i \pi^\alpha_{\beta\gamma} \mathcal{V}_{\alpha\beta\gamma} \theta^\gamma \wedge \theta^\beta + \mathcal{M}_{\alpha\beta\gamma} \theta^\gamma \wedge \eta_1 + \pi^\alpha_{\alpha\beta} \mathcal{L}_{\alpha\beta\gamma} \theta^\gamma \wedge \eta_1 \\
+ i \mathcal{L}_{\alpha\beta\gamma} \theta^\gamma \wedge (\eta_2 - i\eta_3) - i \pi^\alpha_{\alpha\beta} \mathcal{M}_{\alpha\beta\gamma} \theta^\gamma \wedge (\eta_2 + i\eta_3) - \mathcal{C}_{\alpha\beta} (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3) \\
+ \mathcal{H}_{\alpha\beta} \eta_1 \wedge (\eta_2 + i\eta_3) + i \pi^\alpha_{\alpha\beta} \mathcal{C}_{\alpha\beta} \mathcal{C}_{\alpha\beta} \eta_1 \wedge (\eta_2 - i\eta_3),
\]
\[
\psi_1(K) = 4 \pi^\alpha_{\beta\gamma} \mathcal{L}_{\alpha\beta\gamma} \theta^\gamma \wedge \theta^\beta + 4 \mathcal{C}_{\alpha\beta} \mathcal{L}_{\alpha\beta\gamma} \theta^\gamma \wedge \eta_1 + 4 \mathcal{C}_{\alpha\beta} \mathcal{L}_{\alpha\beta} \theta^\beta \wedge \eta_1 - 4 i \pi^\alpha_{\alpha\beta} \mathcal{C}_{\alpha\beta} \mathcal{C}_{\alpha\beta} \theta^\beta \wedge (\eta_2 + i\eta_3) \\
+ 4 i \pi^\alpha_{\alpha\beta} \mathcal{C}_{\alpha\beta} \mathcal{L}_{\alpha\beta\gamma} \theta^\gamma \wedge (\eta_2 - i\eta_3) + \mathcal{P} \eta_1 \wedge (\eta_2 + i\eta_3) + \mathcal{P} \eta_1 \wedge (\eta_2 - i\eta_3) \\
+ i \mathcal{R} \eta_1 \wedge (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3),
\]
\[
\psi_2(K) + i \psi_3(K) = 4 i \pi^\alpha_{\alpha\beta} \mathcal{M}_{\alpha\beta\gamma} \theta^\gamma \wedge \theta^\beta + 4 i \pi^\alpha_{\alpha\beta} \mathcal{M}_{\alpha\beta} \mathcal{L}_{\alpha\beta\gamma} \theta^\gamma \wedge \eta_1 - 4 i \mathcal{C}_{\alpha\beta} \mathcal{L}_{\alpha\beta} \theta^\beta \wedge \eta_1 - 4 i \mathcal{C}_{\alpha\beta} \mathcal{L}_{\alpha\beta\gamma} \theta^\gamma \wedge (\eta_2 + i\eta_3) \\
- 4 i \pi^\alpha_{\alpha\beta} \mathcal{H}_{\alpha\beta} \theta^\beta \wedge (\eta_2 - i\eta_3) - i \mathcal{R} \eta_1 \wedge (\eta_2 + i\eta_3) + \mathcal{P} \eta_1 \wedge (\eta_2 - i\eta_3) \\
- \mathcal{P} (\eta_2 + i\eta_3) \wedge (\eta_2 - i\eta_3),
\]

The properties of the curvature components (cf. Proposition 4.1)

\[
S_{\alpha\beta\gamma\delta}, \mathcal{V}_{\alpha\beta\gamma}, \mathcal{L}_{\alpha\beta}, \mathcal{M}_{\alpha\beta}, \mathcal{C}_{\alpha}, \mathcal{H}_{\alpha}, \mathcal{P}, \mathcal{Q}, \mathcal{R}
\]

imply that

\[
g^{\alpha\beta} K(Z_\alpha, Z_\beta) = 0, \quad \pi^{\alpha\beta} K(Z_\alpha, Z_\beta) = 0, \quad \Gamma_{\alpha\beta}(K(Z^\beta, \cdot)) = 0,
\]
\[
\phi_\alpha(K(Z^\alpha, \cdot)) = 0, \quad \pi^{\alpha\beta} \phi_\alpha(K(Z^\beta, \cdot)) = 0,
\]
and therefore, by Lemma (5.1), we have \( \partial^* K = 0 \). Thus \( (P_1, \omega) \) coincides with the regular, normal Cartan geometry associated with the quaternionic contact manifold \( (M, H) \).

6. Appendix

This section serves as a brief collection of basic facts on Cartan geometries. All the details and much more information can be found in the book [4]. At the same time, we provide links to the general structure theory to our computations.

6.1. Cartan geometries. Elie Cartan’s generalized spaces (espace generalisé) are curved analogs of the homogeneous space \( G/P \) for Lie groups \( P \subset G \). They are defined as right invariant absolute parallelism \( \omega \) on a principal \( P \)-bundle \( \mathcal{G} \) reproducing the fundamental vector fields. Let us write \( g \) and \( p \) for the Lie algebras of \( G \) and \( P \), respectively.

A Cartan geometry \((\mathcal{G}, \omega)\) of type \( G/P \) is a principal fiber bundle \( \mathcal{G} \) with structure group \( P \), equipped with a smooth one-form \( \omega \in \Omega^1(\mathcal{G}, g) \) satisfying

1. \( \omega(\zeta_Z)(u) = Z \) for all \( u \in \mathcal{G} \) and fundamental fields \( \zeta_Z, Z \in p \)
2. \( (r^p)^* \omega = \text{Ad}(p^{-1}) \omega \) for all \( u \in P \)
3. \( \omega|_{T_u \mathcal{G}} : T_u \mathcal{G} \rightarrow g \) is a linear isomorphism for all \( u \in \mathcal{G} \).

In particular, each \( X \in g \) defines the constant vector field \( \omega^{-1}(X) \) defined by \( \omega(\omega^{-1}(X)(u)) = X, u \in \mathcal{G} \).

The one forms with latter three properties are called Cartan connections.

The homogeneous model \( G \rightarrow G/P \) together with the Maurer Cartan form \( \omega \) is an example of such geometry.

The morphisms between parabolic geometries \((\mathcal{G}, \omega)\) and \((\mathcal{G}', \omega')\) are principal fiber bundle morphisms \( \phi \) which preserve the Cartan connections, i.e. \( \phi : \mathcal{G} \rightarrow \mathcal{G}' \) and \( \phi^* \omega' = \omega \).
The structure equations \( d\omega + \frac{1}{2}[\omega, \omega] = K \) define the horizontal smooth form \( K \in \Omega^2(\mathfrak{s}, \mathfrak{g}) \) called the curvature of the Cartan connection \( \omega \). The curvature function \( \kappa : \mathfrak{s} \rightarrow \wedge^2 \mathfrak{g}^* \otimes \mathfrak{g} \) is then defined by means of the parallelism

\[
\kappa(u)(X, Y) = K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) = [X, Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)]).
\]

In particular, the curvature function is valued in the cochains for the second cohomology \( H^2(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \).

The curvature vanishes if and only if the geometry is locally equivalent to its homogeneous model.

### 6.2. Parabolic geometries

If we consider a semisimple Lie group \( G \) and its parabolic subgroup \( P \), we call the Cartan geometries parabolic.

It well known that the choice of the parabolic subalgebra \( \mathfrak{p} \subset \mathfrak{g} \) is equivalent to its grading

\[
\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k
\]

where \( \mathfrak{g}_i = \mathfrak{g}_{-i}^+ \) with respect to the Killing form.

Then there are two ways how to split the curvature function \( \kappa \) now. We may consider the target components \( \kappa_\ell \), \( \ell \geq 0 \), according to the values in \( \mathfrak{g}_\ell \). The whole \( \mathfrak{g}_- \)-component \( \kappa_- \) is called the torsion of the Cartan connection \( \omega \). The other possibility is to consider the homogeneity of the bilinear maps \( \kappa(u) \), i.e.

\[
\kappa = \sum_{\ell=-k+2}^{3k} \kappa^{(\ell)}: \kappa^{(\ell)}|_{\mathfrak{g}_i \times \mathfrak{g}_j} : \mathfrak{g}_i \times \mathfrak{g}_j \rightarrow \mathfrak{g}_{i+j+\ell}.
\]

Since we deal with semisimple algebras only, there is the codifferential \( \partial^* \) which is adjoint to the Lie algebra cohomology differential \( \partial \). Consequently, there is the Hodge theory on the cochains which enables us to deal effectively with the curvatures. In particular, we may use several restrictions on the values of the curvature which turn out to be quite useful. In particular, we may use several restrictions on the values of the curvature which turn out to be quite useful.

If all curvature \( \kappa^{(j)} \) vanish for \( j < 0 \), then the filtration obtained from the grading of the Lie algebra and the absolute parallelism \( \omega \) is compatible with brackets of Lie vector fields, and if \( k^{(0)} \) vanishes as well, then these brackets even coincide with the algebraic bracket inherited by the absolute parallelism from \( \mathfrak{g}_{\leq 0} \).

### 6.3. Regular filtrations on manifolds

Let us fix the graded \( \mathfrak{g} \) and \( \mathfrak{p} \) as above. Starting with a filtration \( TM = T^{-k}M \supset \cdots \supset T^{-1}M \), assume that its associated graded vector bundle \( \text{Gr} TM \), with its algebraic bracket induced by the Lie bracket of vector fields, is pointwise isomorphic to the negative part of the graded Lie algebra \( \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \). If \( \mathfrak{g}_0 \) is smaller than the entire \( \mathfrak{g}(\mathfrak{g}_{-1}) \), then assume the frame bundle of \( \text{Gr} TM \) has been reduced to the structure group \( G_0 \). We call such filtrations regular infinitesimal flag structures of type \( \mathfrak{g}/\mathfrak{p} \).

**Theorem 6.1.** There is the bijective correspondence between the isomorphism classes of regular normal parabolic geometries of type \( G/P \) and the regular infinitesimal flag structures of type \( \mathfrak{g}/\mathfrak{p} \) on \( M \), except for one series of one–graded, and one series of two–graded Lie algebras \( \mathfrak{g} \) for which \( H^1(\mathfrak{g}_-, \mathfrak{g}) \) is nonzero in homogeneous degree one.

Although this general theorem is proved in a constructive way, cf. [4] or [8], the explicit and effective construction is far from trivial in the individual cases as soon as the grading is of length \( k \geq 2 \). Thus the arguments leading to the theorem are rather serving as guidelines for the explicit constructions.

The entire section 5 provides the links of the Cartan connection from the above theorem to our construction in the paper. In particular, the matrix of one-forms (5.3) is just the explicit expression of \( \omega \), the comparison of the structure equations for the Lie group \( Sp(n + 1, 1) \) with the structure equations for this \( \omega \) shows the right equivariance and we have computed there that the normality condition is satisfied too.

The uniqueness part from the latter theorem then implies that we have constructed the canonical regular and normal Cartan connection \( \omega \).
6.4. The curvature. We are now in position to say more about the structure of the curvature. Each representation $\rho$ of the entire group $G$ on a vector space $\mathcal{V}$ defines the natural bundle $\mathcal{F} \times_\rho \mathcal{V}$ over the manifolds $M$ with the regular infinitesimal flag structures. Moreover, the unique extension $\hat{\omega} : T\mathcal{F} \rightarrow \mathfrak{g}$ of the canonical Cartan connection $\omega$ to the extended bundle $\hat{\mathcal{F}} = \mathcal{F} \times_\rho G$ provides the canonical covariant derivative on all such natural bundles (as a principal connection on $\hat{\mathcal{F}}$). The adjoint representation of $G$ on $\mathfrak{g}$ is the best example leading to the so called adjoint tractor bundle $A$ and the curvature $\kappa$ can be interpreted as a two-form on the manifold $M$ with values in $A$.

The splitting of $\mathfrak{g}$ into irreducible $G_0$ components corresponds to the splitting of the adjoint tractor bundle into components seen whenever we reduce the structure group to $G_0$. In our case, we have $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(n)$ and there is the only harmonic component $S_{\alpha\beta\gamma\delta}$ which corresponds to cochains $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{sp}(n)$. This is the only component of homogeneity two and all the potentially nonzero components of $\kappa$ as deduced in the Proposition 4.1, are listed in the table.

| homogeneity | the cochains | object in Proposition 4.1 |
|-------------|--------------|---------------------------|
| 2           | $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \rightarrow \mathfrak{sp}(n)$ | $S_{\alpha\beta\gamma\delta}$ |
| 3           | $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1} \rightarrow \mathfrak{sp}(n)$ | $V_{\alpha\beta\gamma}$ |
| 4           | $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1$ | $\mathcal{V}_{\alpha\beta\gamma}$ |
| 4           | $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1} \rightarrow \mathfrak{sp}(n)$ | $\mathcal{L}_{\alpha\beta}, \mathcal{M}_{\alpha\beta}$ |
| 4           | $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_2$ | $\mathcal{L}_{\alpha\beta}, \mathcal{M}_{\alpha\beta}$ |
| 5           | $\mathfrak{g}_{-2} \wedge \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_1$ | $\mathcal{C}_\alpha, \mathcal{H}_\alpha$ |
| 5           | $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_2$ | $\mathcal{C}_\alpha, \mathcal{H}_\alpha$ |
| 6           | $\mathfrak{g}_{-2} \wedge \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_2$ | $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ |

Notice that it is the $\partial^* \kappa = 0$ normalization which enforces several potentially different components to coincide.

Since the second Lie algebra cohomology $H^2(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ is completely reducible with trivial action of $\mathfrak{g}_{>0}$, the harmonic curvature components living in the kernel of both $\partial^*$ and $\partial$ are well defined tensors on $M$. A great general result of the so called BGG calculus reads that the entire curvature of a regular and normal Cartan connection is obtained as the image of the harmonic part under a suitable natural linear differential operator.

Although we have not delivered this linear differential operator explicitly, we came quite close in Proposition 4.3. Indeed, note that the expression for the differential $dS_{\alpha\beta\gamma\delta}$ contains known combination of the curvature components and the new quantities identified in the proposition. A straightforward check of the involved symmetries and relations reveals that we can actually compute the quantities $V_{\alpha\beta\gamma}$ in terms of the differential of $S_{\alpha\beta\gamma\delta}$. Similarly, the next line allows to express the component composed of $M_{\alpha\beta}$ and $\mathcal{L}_{\alpha\beta}$ depending on second order derivatives of $S_{\alpha\beta\gamma\delta}$ and first derivatives of $V_{\alpha\beta\gamma}$. This goes on, until we finally express $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ from the lines involving the differentials of $\mathcal{C}_\alpha$ and $\mathcal{H}_\alpha$. The latter expression will involve fourth derivatives of $S_{\alpha\beta\gamma\delta}$, as expected.

References

[1] Armstrong, S., Non-regular [2]–graded geometries I: general theory, arxiv.org (2009), 0902.1133, 23 pp.
[2] Biquard, O., Métriques d’Einstein asymptotiquement symétriques, Astérisque 265 (2000).
[3] Čap, A., Correspondence spaces and twistor spaces for parabolic geometries, J. Reine Angew. Math. 582 (2005), 143172.
[4] Čap, A., Slovák, J. Parabolic geometries. I. Background and general theory, Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI. 2, 32, 34, 35.
[5] Chern, S.-S., Moser, J.K., Real hypersurfaces in complex manifolds, Acta Math., 133 (1974), 219-271; Erratum Acta Math, 150 (1983) 297. 2, 6.
[6] Duchemin, D., Quaternionic contact structures in dimension 7, Ann. Inst. Fourier, Grenoble 56, 4 (2006) 851–885.
[7] Ivanov, S., Minchev, I., Vassilev, D., Quaternionic contact hypersurfaces in hyper-Kähler manifolds, to appear in Ann. Mat. Pura Appl., arXiv:1406.4256.
[8] Yamaguchi, K., Differential systems associated with simple graded Lie algebras, Advanced Stud. in Pure Math 22 (1993) 413-494.
(Ivan Minchev) University of Sofia, Faculty of Mathematics and Informatics, blvd. James Bourchier 5, 1164 Sofia, Bulgaria; Department of Mathematics and Statistics, Masaryk University, Kotlarska 2, 61137 Brno, Czech Republic
E-mail address: minchev@fmi.uni-sofia.bg

(Jan Slovák) Department of Mathematics and Statistics, Masaryk University, Kotlarska 2, 61137 Brno, Czech Republic
E-mail address: slovak@math.muni.cz