CHERN CLASS INEQUALITIES ON POLARIZED MANIFOLDS AND NEF VECTOR BUNDLES

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Abstract. This article is concerned with Chern class and Chern number inequalities on polarized manifolds and nef vector bundles. For a polarized pair \((M, L)\) with \(L\) very ample, our first main result is a family of sharp Chern class inequalities. Among them the first one is a variant of a classical result and the equality case of the second one is a characterization of hypersurfaces. The second main result is a Chern number inequality on it, which includes a reverse Miyaoka-Yau type inequality. The third main result is that the Chern numbers of a nef vector bundle over a compact Kähler manifold are bounded below by the Euler number. As an application, we classify compact Kähler manifolds with nonnegative bisectional curvature whose Chern numbers are all positive. A conjecture related to the Euler number of compact Kähler manifolds with nonpositive bisectional curvature is proposed, which can be regarded as a complex analogue to the Hopf conjecture.

1. Introduction

Unless otherwise stated, vector bundles, complex manifolds and their dimensions mentioned throughout this article are respectively holomorphic, compact and positive.

Positivity is a central issue in complex differential geometry and algebraic geometry. For line bundles the positivity in differential geometry and ampleness in algebraic geometry are equivalent, thanks to the Kodaira embedding theorem. Griffiths ([Gr69]) and Hartshorne ([Ha66]) respectively generalized these two notions to higher rank vector bundles by introducing Griffiths-positivity and ampleness. It turns out that Griffiths-positivity implies ampleness, and Griffiths conjectured in [Gr69] that these two notions are equivalent, which is true when the base manifold is a projective curve ([CF90]). In general constructing a Griffiths-positive Hermitian metric on an ample vector bundle seems to be quite difficult. Very recently Demailly proposed in [De20] a method to attack this problem.

Griffiths also raised in [Gr69] the question of characterizing the polynomials in the Chern classes and Chern forms for Griffiths-positive or ample vector bundles which are positive as cohomology classes and differential forms. At the cohomology class level this was completely answered by Fulton and Lazarsfeld ([FL83]), who showed that the set of such polynomials for ample vector bundles is exactly the cone generated by the Schur polynomials of Chern classes (an earlier special case was obtained by Bloch and Gieseker in [BG71]). This consequently implies that all the Chern numbers of ample vector bundles are positive. At the form level, Griffiths’s question is still largely unknown except in some special cases ([Gu06], [2010 Mathematics Subject Classification. 57R20,32Q55,53C55,57R22.

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Recently the first author examined this question in [Li20] for nonnegative Hermitian vector bundles in the sense of Bott and Chern ([BC65]) over (not necessarily Kähler) complex manifolds and showed that the Schur polynomials of Chern forms of such vector bundles are strongly nonnegative.

Griffiths-nonnegativity can also be defined and its counterpart in algebraic geometry is numerical effectiveness ("nefness" for short), where the former implies the latter. After some earlier works ([Zhe89], [CP91], [CP93]), Demailly, Peternell and Schneider ([DPS94]) investigated in detail the structure of Kähler manifolds with nef tangent bundles. Among other things, they showed that those inequalities of Fulton-Lazarsfeld type remain true for Chern classes of nef vector bundles on Kähler manifolds ([DPS94, §2]). As an application, they deduced that all the Chern numbers of a nef vector bundles on an n-dimensional Kähler manifolds are nonnegative and bounded from above by the Chern number $c_1^n$ ([DPS94, Coro. 2.6]).

This upper bound plays a crucial role in establishing the main structural theorem in [DPS94] as well as in some other related applications. For instance, Zhang ([Zha97, Thm 3]) applied it to show that the canonical line bundle of an immersed projective submanifold in an abelian variety is ample if and only if its signed arithmetic genus is positive. Yang ([Ya17, Thm 1.2]) applied it and some other results in [DPS94] to show that the holomorphic tangent bundle of a compact Kähler manifold with nonnegative holomorphic bisectional curvature is big if and only if it is Fano, and then classified such manifolds by using Mok’s uniformization theorem ([Mok88]).

Also motivated by this upper bound it was shown in [Li20, Thm 3.2] that for Bott-Chern nonnegative Hermitian vector bundles the Euler number and $c_1^n$ are respectively the lower and upper bounds at the form level, by making use of the special properties of Bott-Chern nonnegativity. Note that the condition of Bott-Chern nonnegativity is stronger than that of Griffiths nonnegativity and hence than that of nefness ([Li20, Example 4.4]). So the method in [Li20] can not be directly carried over to nef vector bundles, as remarked in [Li20, Remark 3.3].

The main purposes of this article are two-folded. The first main purpose is to apply some nonnegativity results in [Li20] by finding a good geometric model. To be more precise, given a polarized manifold $(M, L)$ with $L$ very ample, we associate it to a Bott-Chern nonnegative Hermitian vector bundle. This yields a family of sharp Chern class inequalities (Theorem 2.1), among which the first one is a special case of a classical result. We apply some arguments of algebro-geometric nature to characterize the second equality case (Theorem 2.2), which turns out to be a topological characterization of all hypersurfaces in complex projective spaces. A Chern number inequality involving $L$ and the first two Chern classes of $M$ is also deduced (Theorem 2.4), which includes a reverse Miyaoka-Yau type inequality (Corollary 2.5).

The second main purpose is, by fully utilizing the positivity of Schur polynomials, we show that the Euler number of a nef vector bundle over a Kähler manifold is indeed the lower bound (Theorem 2.8). As a major application, we classify compact Kähler manifolds with nonnegative holomorphic bisectional curvature whose Chern numbers are all positive (Theorem 3.1). In view of Theorem 3.1, a conjecture (Conjecture 4.1) related to the Euler number of compact Kähler manifolds with nonpositive holomorphic bisectional curvature is proposed and we provide some positive evidences to it.
The rest of the article is organized as follows. The aforementioned main results (Theorems 2.1, 2.2, 2.4, 2.8) as well as some consequences are stated in Section 2. In Section 3 some applications including Theorem 3.1 and examples are presented. We propose and discuss Conjecture 4.1 in Section 4, which can be regarded as a “dual version” to Theorem 3.1. Sections 5 and 6 are then devoted to the proofs of Theorems 2.1 and 2.4 and Theorems 2.8 and 3.1 respectively. Since the proof of Theorem 2.2 is a little more involved, we postpone it to the last section, Section 7.

2. Main results

Let $M^n$ be an $n$-dimensional projective manifold with $L$ an ample line bundle on it. A classical result (cf. [BS95, p. 159]) states that $K_M + (n + 1)L$ is always nef, and is ample unless $(M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Here $K_M$ is the canonical line bundle of $M$. Our first main result is a family of sharp Chern class inequalities including a variant of this classical result as a special case. To state it, let us introduce some more notations first.

The Segre classes of a vector bundle $E$ are defined to be formal inverse of the total Chern class of $E^*$, the dual of $E$, i.e.,

$$ s(E) = 1 + s_1(E) + s_2(E) + \cdots := c(E^*)^{-1} = (1 - c_1(E) + c_2(E) - \cdots)^{-1}. $$

That is,

$$ s_1(E) = c_1(E), \quad s_2(E) = c_1^2(E) - c_2(E), \quad s_3(E) = c_1^3(E) - 2c_1(E)c_2(E) + c_3(E), \quad \text{and so on.} $$

For simplicity we denote by $s_i(M) := s_i(TM)$ and use $L$ for its first Chern class $c_1(L)$. A real $(k, k)$-form $\varphi$ on $M^n$ ($1 \leq k \leq n$) is called nonnegative ([De12, Ch. 3, §1.A]) if it can be written as

$$ \varphi = (\sqrt{-1})^{k^2} \sum_i \psi_i \wedge \overline{\psi}_i, $$

where these $\psi_i$ are $(k, 0)$-forms. A cohomology class $\alpha \in H^{k, k}(M; \mathbb{R})$ is called nonnegative, denoted by $\alpha \geq 0$, if it contains a nonnegative $(k, k)$-form representative. Clearly if $\alpha \geq 0$ then $\int_Y \alpha \geq 0$ for any $k$-dimensional subvariety $Y \subset M$.

Let $L$ be a very ample line bundle on $M$, $\dim \mathbb{C}H^0(M, L) = N + 1$ and $\{s_0, s_1, \ldots, s_N\}$ a basis. Very ampleness means that $M$ can be holomorphically embedded into a complex projective space as a nonsingular projective variety via the following Kodaira map:

$$ M \hookrightarrow \mathbb{P}(H^0(M, L)^*) \cong \mathbb{P}^N $$

$$ x \mapsto [s_0(x) : s_1(x) : \cdots : s_N(x)]; \quad i^*(\mathcal{O}_{\mathbb{P}^N}(1)) = L. $$

We remark that in this case the image $i(M)$ is nondegenerate in the sense that it is not contained in a hyperplane. Otherwise some linear combination of $s_i$ would vanish on $M$, a contradiction.

With the notation above understood, it comes our first result.

**Theorem 2.1.** Let $L$ be very ample on $M^n$. We have for every $k \geq 1$ the following sharp Chern class inequalities

$$ \sum_{i=0}^{k} (-1)^i \binom{n+k}{k-i} \cdot s_i(M) \cdot L^{k-i} \geq 0, $$

where the equality case of (2.3) occurs if $(M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or $k > \min\{n, N - n\}$. 
Note that for $k = 1$, the left hand side of (2.3) is $(n + 1)L + K_M$. In this case, the inequality $(n + 1)L + K_M \geq 0$ is a special case of the aforementioned classical result, where $L$ is only needed to be ample, and the equality $K_M + (n + 1)L = 0$ occurs if and only if $(M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ due to a classical result of Kobayashi and Ochiai ([KO73]).

For $k = 2$, the left hand side of (2.3) becomes

$$\frac{1}{2}(n + 2)(n + 1)L^2 - (n + 2)Lc_1 + c_2^2 - c_2.$$

It turns out that the equality case of this gives exactly a characterization of hypersurfaces. In other words we have the following

**Theorem 2.2.** Let $L$ be very ample on $M^n$. Then by the $k = 2$ case of Theorem 2.1 we have

(2.4) $$\frac{1}{2}(n + 2)(n + 1)L^2 - (n + 2)Lc_1 + c_2^2 - c_2 \geq 0. $$

The equality occurs if and only if either $(M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or the Kodaira map (2.2) embeds $M^n$ as a hypersurface in $\mathbb{P}^{n+1}$ (i.e., $N - n = 1$) of degree $L^n$.

When $c_1 = 0$, i.e., for Calabi-Yau manifolds, the expressions in (2.3) can be simplified a bit to lead to

**Corollary 2.3.** Let $M^n$ be a Calabi-Yau manifold with $n \geq 2$. Then for any very ample line bundle $L$ on $M^n$, it holds

(2.5) $$\frac{1}{2}(n + 2)(n + 1)L^2 - c_2 \geq 0; $$

(2.6) $$\frac{1}{6}(n + 3)(n + 2)(n + 1)L^3 - (n + 3)Lc_2 - c_3 \geq 0. $$

The first equality case holds if and only if the Kodaira map (2.2) embeds $M^n$ in $\mathbb{P}^{n+1}$, and in this case the degree of the hypersurface is necessarily $L^n = n + 2$.

Our next result is the following Chern number inequality for $(M, L)$.

**Theorem 2.4.** Suppose $L$ is very ample on $M^n$. Then the following Chern number inequality holds:

(2.7) $$\left[ \frac{n(n + 1)}{2}L^2 - nc_1L + c_2 \right] \left[ - c_1 + (n + 1)L \right]^{n-2} \leq \left[ - c_1 + (n + 1)L \right]^n. $$

Fujita’s very ampleness conjecture ([Fuj85]) asserts that $K_M + (n + 2)L$ is very ample whenever $L$ is ample. This conjecture holds true if $L$ is further assumed to be globally generated ([Ke08, Thm 1.1]). So replacing $L$ in (2.7) with $-c_1 + (n + 2)L$ and with some calculations (see Example 3.10 for details), we have

**Corollary 2.5.** Suppose $L$ is ample and globally generated on $M^n$. Then

(2.8) $$\left[ - nc_1^2 + 2(n + 1)c_2 \right] \left[ - c_1 + (n + 1)L \right]^{n-2} \leq (n + 2)^3 \left[ - c_1 + (n + 1)L \right]^n. $$

In particular, if $K_M$ is ample and globally generated, we have

(2.9) $$c_2(-c_1)^{n-2} \leq \frac{(n + 2)^5 + n}{2(n + 1)}(-c_1)^n. $$

**Remark 2.6.**
(1) Yau’s celebrated Chern number inequality ([Yau77]) says that if $K_M$ is ample, then
\[
\frac{n}{2(n+1)}(-c_1)^n \leq c_2(-c_1)^{n-2},
\]
to which (2.9) can be viewed as a complement when $K_M$ is further assumed to be globally generated. Note that the inequality (2.9) is not optimal due to the method we employ (see Corollary 2.7 below). Even so, we are unaware of any this kind of reverse Miyaoka-Yau type inequality in the literature, to the best of our knowledge.

(2) In general, if $L$ is ample and $aL$ is very ample for some $a \in \mathbb{Q}^+$, we may from (2.7) have an inequality involving an extra constant $a$. It is also known by the work of Demailly ([De93]) that $2K_M + 12n^2L$ is always very ample for any ample $L$. So we may always take $a = 2 + 12n^2$ when $K_M$ is ample, which of course also leads to a reverse Miyaoka-Yau inequality but with a very large constant.

As a direct corollary of Theorem 2.2, we have the following

Corollary 2.7. Let $M^n$ be projective manifold with $c_1 < 0$ (or $c_1 > 0$) and $a \in \mathbb{Q}^+$ (or $a \in \mathbb{Q}^-$) such that $L = aK_M$ is a very ample line bundle. Replacing $L$ in (2.4) with $-ac_1$ (or $ac_1$) and multiplying by $(-c_1)^{n-2}$ (or $c_1^{n-2}$), it holds
\[
\frac{1}{2}(n+2)(n+1)a^2 + (n+2)a + 1]} \geq c_2(\varepsilon c_1)^{n-2},
\]
where $\varepsilon = -1$ (or 1). If the equality holds, then (see Example 3.9) $\frac{1}{\alpha} = \varepsilon(n + 2 - L^n)$, and the Kodaira map (2.2) of $L = aK_M$ embeds $M^n$ as a hypersurface in $\mathbb{P}^{n+1}$ with degree $L^n$. In particular, for those with very ample $K_M$, it holds that
\[
\frac{1}{2}(n^2 + 5n + 8)(-c_1)^n \geq c_2(-c_1)^{n-2},
\]
and the equality occurs if and only if $M^n \subset \mathbb{P}^{n+1}$ is a hypersurface of degree $n + 3$.

Next, for two given positive integers $k$ and $r$, let us denote by $\Gamma(k, r)$ the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of weight $k$ by nonnegative integers $\lambda_j \leq r$:
\[
r \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0, \quad \sum_{j=1}^{k} \lambda_j = k.
\]

Let $E$ be a rank $r$ vector bundle. For each partition $\lambda \in \Gamma(k, r)$, the Chern monomial $c_\lambda(E)$ is defined by
\[
c_\lambda(E) := \prod_{j=1}^{k} c_{\lambda_j}(E) \in H^{k,k}(M; \mathbb{Z}).
\]

With this notation understood, we come to our third main result.

Theorem 2.8. Let $E$ be a rank $r$ nef vector bundle on an $n$-dimensional compact Kähler manifold $(M, \omega)$, and $c_i(E)$ the $i$-th Chern classes of $E$ ($0 \leq i \leq r$). Then for each $1 \leq k \leq n$ and $\lambda \in \Gamma(k, r)$ we have
\[
\int_M c_\lambda(E) \wedge [\omega]^{n-k} \geq \int_M c_k(E) \wedge [\omega]^{n-k} \geq 0.
\]
In particular, all the Chern numbers of $E$ are bounded below by the (nonnegative) Euler number $\int_M c_n(E)$. 

3. Applications and examples

3.1. Some applications to Theorem 2.8. Mori ([Mor79]) and Siu-Yau’s ([SY80]) independent solution to the Frankel conjecture asserts that an $n$-dimensional compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{P}^n$. Building on a splitting result of Howard-Smyth-Wu ([HSW81], [Wu81]) and combining analytic and algebraic tools, Mok solved the generalized Frankel conjecture by showing the following uniformization theorem. If a compact Kähler manifold has nonnegative holomorphic bisectional curvature, then its universal cover is holomorphically isometric to

$$(3.1) \quad (\mathbb{C}^q, g_0) \times (\mathbb{P}^{N_1}, \theta_1) \times \cdots \times (\mathbb{P}^{N_k}, \theta_k) \times (M_1, g_1) \times \cdots \times (M_p, g_p),$$

where $g_0$ is flat, $\theta_i$ are Kähler metrics on $\mathbb{P}^{N_i}$ carrying nonnegative holomorphic bisectional curvature, and $(M_i, g_i)$ are irreducible compact Hermitian symmetric spaces of rank $\geq 2$ equipped with the canonical metrics.

As a major application of Theorem 2.8, we can, with the help of Mok’s uniformization theorem, classify compact Kähler manifolds with nonnegative holomorphic bisectional curvature whose Chern numbers are all positive.

**Theorem 3.1.** Let $M$ be an $n$-dimensional compact Kähler manifold with nonnegative holomorphic bisectional curvature. Then

1. either all the Chern numbers of $M$ are positive, in which case $M$ is holomorphically isometric to

   $$(3.2) \quad (\mathbb{P}^{N_1}, \theta_1) \times \cdots \times (\mathbb{P}^{N_k}, \theta_k) \times (M_1, g_1) \times \cdots \times (M_p, g_p)$$

   with $\theta_i$ and $(M_i, g_i)$ are the same as those in (3.1);

2. or all the Chern numbers of $M$ vanish, in which case $\pi_1(M)$ is infinite and its universal cover splits off a nontrivial complex Euclidean factor $(\mathbb{C}^q, g_0)$ in (3.1).

Combining the results in [DPS94] and [Ya17] with Theorem 3.1, we have now the following equivalent conditions to characterize simply-connected compact Kähler manifolds with nonnegative holomorphic bisectional curvature.

**Theorem 3.2.** Let $M$ be an $n$-dimensional compact Kähler manifold with nonnegative holomorphic bisectional curvature. Then the following four statements are equivalent.

1. $M$ is Fano, i.e., $c_1(M) > 0$;
2. the holomorphic tangent bundle $T_M$ is big;
3. the Chern number $c_1^n > 0$;
4. all the Chern numbers of $M$ are positive.

**Remark 3.3.** “(1) $\iff$ (3)” is due to Demailly-Peternell-Schneider ([DPS94, §4]), where they indeed showed this for compact Kähler manifolds with nef tangent bundles. “(2) $\iff$ (3)” is due to Yang ([Ya17, Thm 1.2]). “(4) $\iff$ (3)” follows from our Theorem 3.1.

Recall that, for a (possibly non-Kähler) Hermitian metric $g$ on a complex manifold $M$, the holomorphic bisectional curvature of $g$ can still be defined in terms of the Chern connection. Denote by $TM$ the holomorphic tangent bundle of $M$. Then the Hermitian vector
bundle \((TM, g)\) (resp. \((T^*M, g)\)) is Griffiths-nonnegative if and only if the holomorphic bisectional curvature of \(g\) is nonnegative (resp. nonpositive) ([Zhe00, p. 177]). Since Griffiths-nonnegativity implies nefness, combining the lower bound in Theorem 2.8 with the upper bound in [DPS94] we have the following

**Corollary 3.4.** Let \(M\) be an \(n\)-dimensional compact Kähler manifold with a (possibly different) Hermitian metric \(g\) whose holomorphic bisectional curvature is nonnegative (resp. nonpositive). Then the Chern numbers \(c_\lambda[M]\) of \(M\) satisfy

\[
0 \leq c_n[M] \leq c_\lambda[M] \leq c_1^n[M]
\]

(resp. \(0 \leq (-1)^n c_n[M] \leq (-1)^n c_\lambda[M] \leq (-1)^n c_1^n[M]\)).

The famous Hopf conjecture asserts that the Euler number \(\chi(M)\) of a closed 2\(n\)-dimensional Riemannian manifold \(M\) with sectional curvature \(K < 0\) (resp. \(K \leq 0\)) satisfies \((-1)^n\chi(M) > 0\) (resp. \((-1)^n\chi(M) \geq 0\)), which is true when \(n \leq 2\) ([Ch55]) but is still open in its full generality for \(n \geq 3\). Gromov ([Gr91]) introduced the notion of “Kähler hyperbolicity”, which includes Kähler metrics with \(K < 0\) as special cases, and showed that the Euler number of Kähler hyperbolic manifolds have the expected sign. This notion was extended independently by Cao-Xavier ([CX01]) and Jost-Zuo ([JZ00]) to the nonpositive case. These consequently settled the above Hopf Conjecture for Kähler manifolds. Indeed what they achieved is a solution of a stronger conjecture, the Singer conjecture in the Kählerian case ([Liu02, §11]).

Note that the sign of holomorphic bisectional curvature of a Kähler metric is dominated by that of (Riemannian) sectional curvature ([Zhe00, p. 178]). So our following corollary provides more information on Chern numbers of compact Kähler manifolds with nonpositive sectional curvature.

**Corollary 3.5.** Let \((M, \omega)\) be an \(n\)-dimensional compact Kähler manifold with nonpositive (Riemannian) sectional curvature. Then its Chern numbers \(c_\lambda[M]\) satisfy

\[
0 \leq (-1)^n \chi(M) = (-1)^n c_n[M] \leq (-1)^n c_\lambda[M] \leq (-1)^n c_1^n[M].
\]

**Remark 3.6.** By refining Gromov’s idea, the first author recently deduced that ([Li19, Thm 2.1]) \(n\)-dimensional Kähler hyperbolic manifolds indeed satisfy a family of optimal Chern number inequalities and the first one is exactly \((-1)^n c_n \geq n + 1\), which is an improved inequality expected by the Hopf conjecture.

3.2. **Examples.** We give in this subsection some examples to illustrate some main results in Section 2.

The following tensor product formulas for Segre classes and total Chern class are well-known (cf. [Ful98, p. 49-50, p. 56]) and shall be used in the sequel.

**Example 3.7.** Let \(E\) be a vector bundle of rank \(n + 1\) (resp. \(n\)) and \(L\) a line bundle. Then we have

\[
s_k(E \otimes L) = \sum_{i=0}^k \binom{n+k}{k-i} \cdot s_i(E) \cdot L^{k-i}, \quad \forall k.
\]

(resp. \(c(E \otimes L) = \sum_{i=0}^n c_i(E) \cdot (1 + L)^{n-i}\).)

Note that our Segre classes \(s_k\) defined in (2.1) are different from those in [Ful98] by a sign \((-1)^k\) (cf. [Ful98, p. 50]).
Example 3.8. Let $L := \mathcal{O}_{\mathbb{P}^n}(1)$ be the hyperplane line bundle on $\mathbb{P}^n$. We show that the equality cases of (2.3) are satisfied by $(\mathbb{P}^n, L)$. Indeed we have the relation $T\mathbb{P}^n \oplus C = (n+1)L$, where $C$ denotes the trivial line bundle, and thus

$$\tag{3.3} C^{n+1} = (T^*\mathbb{P}^n \oplus C) \otimes L.$$ 

Taking the $k$-th Segre class $s_k(\cdot)$ on both sides of (3.3) leads to

$$\tag{3.4} 0 = s_k((T^*\mathbb{P}^n \oplus C) \otimes L) = \sum_{i=0}^{k} \left( \begin{array}{c} n+k \\ k-i \end{array} \right) \cdot s_i(T^*\mathbb{P}^n \oplus C) \cdot L^{k-i} \quad \text{(by Example 3.7)}$$

Example 3.9. (1) If $c_1 = 0$, (2.5) says that $\frac{1}{2}(n+2)(n+1)L^2 \geq c_2$ for any very ample line bundle $L$, and equality occurs when and only when the Kodaira map (2.2) for $L$ embeds $M^n$ as a degree $n+2$ hypersurface in $\mathbb{P}^{n+1}$.

(2) For a projective manifold $M^n$ with $c_1 < 0$, one may take a very ample line bundle $L = -ac_1$ with $a \in \mathbb{Q}^+$, and get from (2.4) that

$$\left[ \frac{1}{2}(n+2)(n+1)a^2 + (n+2)a + 1 \right] c_1^2 \geq c_2,$$

with equality if and only if the Kodaira map for $L$ embeds $M^n$ as a hyperplane in $\mathbb{P}^{n+1}$ of degree $L^n = a^n(-c_1)^n \geq n+3$. In this case we have $c_1 = (n+2 - L^n)L$ and so $a = 1/\left[ L^n - (n+2) \right]$. 

(3) Similarly, for a Fano manifold $M^n$, one may take a very ample line bundle $L = ac_1$ with $a \in \mathbb{Q}^+$ and get from (2.4) the inequality

$$\left[ \frac{1}{2}(n+2)(n+1)a^2 - (n+2)a + 1 \right] c_1^2 \geq c_2,$$

where the equality case occurs if and only if either $L^n = 1$ and $(M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, or the Kodaira map embeds $M^n$ as a hyperplane in $\mathbb{P}^{n+1}$ with degree $L^n \leq n+1$. In the latter case we have $a = 1/[(n+2) - L^n]$.

Example 3.10. In this example we indicate how to derive (2.8) from (2.7). Indeed, direct calculations imply that

$$\tag{3.5} \frac{n(n+1)}{2} \left[ -c_1 + (n+2)L \right]^2 - nc_1 \left[ -c_1 + (n+2)L \right] + c_2$$

Replacing $L$ in (2.7) with $-c_1 + (n+2)L$ and using (3.5) it yields

$$\tag{3.6} \left\{ \frac{n(n+2)}{2(n+1)} \left[ -c_1 + (n+1)L \right]^2 - \frac{n}{2(n+1)} c_1^2 + c_2 \right\} \left[ -c_1 + (n+1)L \right]^{n-2} \leq \left( n+2 \right)^2 \left[ -c_1 + (n+1)L \right]^n.$$ 

Multiplying by $2(n+1)$ on both sides of (3.6) and cancelling the terms involving $[-c_1 + (n+1)L]^n$ yields (2.8). Further replacing $L$ with $-c_1$ in (2.8) leads to (2.9).
In view of Theorem 3.1, it is natural to wonder, for compact Kähler manifolds with nonpositive holomorphic bisectional curvature, whether their signed Chern numbers have the similar phenomenon of simultaneous positivity like Theorem 3.1. In contrast to Mok’s uniformization theorem in the nonnegative situation, our current knowledge on the structure of nonpositive holomorphic bisectional curvature compact Kähler manifolds is still much less satisfactory. So no appropriate structure theorem is available to deduce a similar conclusion, to our best knowledge. Nevertheless, we believe the validity of the following conjecture, which can be regarded as the complex analogue to the Hopf conjecture.

**Conjecture 4.1.** Let $(M,\omega)$ be an $n$-dimensional compact Kähler manifold with nonpositive holomorphic bisectional curvature whose Ricci curvature is quasi-negative. Then the signed Euler number $(-1)^{n}c_n[M] > 0$.

**Remark 4.2.** If a Kähler metric $\omega$ has nonpositive bisectional curvature, then its Ricci curvature is nonpositive. So the quasi-negativity of Ricci curvature is equivalent to $(-1)^{n}c_1^2[M] > 0$. In view of Corollary 3.4, Conjecture 4.1 is equivalent to the simultaneous positivity and vanishing of Chern numbers for such manifolds. Unfortunately, so far we are unable to solve it.

A positive evidence to Conjecture 4.1 indeed has been provided in [Li20]. It turns out that the holomorphic cotangent bundles of (immersed) complex submanifolds in complex tori admit Bott-Chern nonnegative Hermitian metrics ([Li20, Ex 4.3]). As an application of the main results, it is shown in ([Li20, Thm 7.3]), among other things, that their signed Chern numbers have the phenomena of simultaneous positivity and vanishing. Note that complex submanifolds in complex tori can be equipped with Kähler metrics with nonpositive holomorphic bisectional curvature (the induced metrics from the flat complex tori). So this result indeed partially confirms Conjecture 4.1.

It is worth mentioning that the following splitting result for nonpositive bisectional curvature compact Kähler manifolds, which is dual to the famous splitting result of Howard-Smyth-Wu in the nonnegative situation ([HSW81], [Wu81]) and originally conjectured by S.-T. Yau, has been recently confirmed by Liu ([Liu14]) building on earlier works of Wu and the second author ([Zhe02], [WZ02]).

**Theorem 4.3 (Liu, Wu-Zheng).** If $(M,\omega)$ is an $n$-dimensional compact Kähler manifold with nonpositive holomorphic bisectional curvature whose maximal rank of the Ricci form is $r$ (0 ≤ r ≤ n), then there exists a finite cover $M'$ of $M$ such that $M'$ is holomorphically isometric to a flat torus bundle $T^{n-r}$ over a compact Kähler manifold $N^r$ with nonpositive bisectional curvature and $c_1(N) < 0$.

More precise statement and various corollaries can be found in [Liu14]. Although it seems to us that the information provided by this splitting theorem is not enough to reach Conjecture 4.1 in its full generality, we can still have the following result.

**Proposition 4.4.** Let $(M,\omega)$ be as in Conjecture 4.1. Then the signed Chern number $(-1)^{n}c_2c_1^{n-2}[M] > 0$. In particular, Conjecture 4.1 is true for $n = 2$. 
Proof. Let \( M' \) be the finite cover of \( M \) as in Theorem 4.3. The quasi-negativity of the Ricci curvature on \( M \) implies that \( c_1(M') < 0 \). Then Yau’s Chern number inequality tells us that
\[
(-1)^n c_2 c_1^{n-2} [M'] \geq \frac{n}{2(n+1)} (-1)^n c_1^n [M].
\]
Since Chern numbers are multiplicative under a finite cover, (4.1) leads to
\[
(-1)^n c_2 c_1^{n-2} [M] \geq \frac{n}{2(n+1)} (-1)^n c_1^n [M],
\]
from which the conclusion follows. \( \square \)

5. PROOFS OF THEOREMS 2.1 AND 2.4

Let \((M^n, L)\) be a polarized manifold with \( L \) a very ample line bundle. This \( L \) embeds \( M \) into some complex projective space \( \mathbb{P}^N \) as a nondegenerate smooth projective variety via the Kodaira map (2.2).

The embedding \( i \) induces a Gauss map \( \gamma \) which sends \( p \in M \) to \( T_p M \), the \( n \)-dimensional projective tangent space of \( M \) at \( p \) in \( \mathbb{P}^N \):
\[
M \xrightarrow{\gamma} G_n(\mathbb{P}^N) \cong G_{n+1}(\mathbb{C}^{N+1})
\]
\[
p \mapsto T_p M,
\]
where \( G_n(\mathbb{P}^N) \) is the Grassmannian variety of \( n \)-dimensional projective subspaces in \( \mathbb{P}^N \), which is isomorphic to \( G_{n+1}(\mathbb{C}^{N+1}) \), the usual complex Grassmannian of \((n+1)\)-dimensional linear spaces in \( \mathbb{C}^{N+1} \).

Let \( S \) be the rank \( n+1 \) tautological subbundle over \( G_{n+1}(\mathbb{C}^{N+1}) \). The bundles \( \gamma^*(S) \), \( L \) and the tangent bundle \( TM \) are famously related to each other via the following exact sequence (cf. [At57, p. 198])
\[
0 \longrightarrow \mathcal{L} \longrightarrow \gamma^*(S) \otimes L \longrightarrow TM \longrightarrow 0,
\]
where as before \( \mathcal{L} \) denotes the trivial line bundle on \( M \).

Remark 5.1. The geometric model described above has been used in several papers to deduce Chern number inequalities in their context. For instance, Tai used in [Ta89] the exact sequence (5.2) and those symmetric polynomials invariant by translation to deduce Chern number inequalities for complete intersections in \( \mathbb{P}^N \). This idea was further push forwarded by Manivel in [Ma94]. Du and Sun applied it in [DS17] to treat the boundedness of the region given by the Chern ratios.

5.1. Proof of Theorem 2.1. Note that the rank \( N - n \) universal quotient bundle \( Q \) over \( G_{n+1}(\mathbb{C}^{N+1}) \) is globally generated and so is \( \gamma^*(Q) \) over \( M^n \), the pull back under the Gauss map \( \gamma \) in (5.1). Bott and Chern introduced in [BC65] a nonnegativity notion for holomorphic vector bundles, which is called Bott-Chern nonnegativity in [Li20]. Its precise definition is not important in this article, and we only need the fact that any globally generated vector bundle can be equipped with a Bott-Chern nonnegative Hermitian metric ([Li20, (4.2)]). So \( \gamma^*(Q) \) admits such a Hermitian metric, say \( h \). In this case the Chern forms \( c_k(\gamma^*(Q), h) \) with respect to the canonical Chern connection are nonnegative as real \((k, k)\)-forms ([Li20, Prop. 3.1]). We remark that in [Li20] they are called “strongly nonnegative” to distinguish from another weaker nonnegativity. So the Chern classes \( c_k((\gamma^*Q)) \geq 0 \) as cohomology classes.
We claim that
\begin{equation}
(5.3) \quad c_k(\gamma^*(Q)) = \sum_{i=0}^{k} (-1)^i \binom{n + k}{k - i} \cdot s_i(M) \cdot L^{k-i},
\end{equation}
from which the inequalities (2.3) follows. Since \( \gamma^*(Q) \) is a rank \( N - n \) vector bundle over an \( n \)-dimensional manifold. So \( c_k(\gamma^*(Q)) = 0 \) for \( k > \min\{n, N - n\} \).

We now prove (5.3). Note that the total Chern classes of \( Q \) and the universal subbundle \( S \) are related by \( c(Q)c(S) = 1 \) and therefore \( c(\gamma^*(Q))c(\gamma^*(S)) = 1 \). This implies that
\begin{equation}
(5.4) \quad c(\gamma^*(Q)) = \frac{1}{c(\gamma^*(S))} = s(\gamma^*(S^*))
\end{equation}
by the definition of Segre classes in (2.1). On the other hand, the exact sequence (5.2) tells us that
\begin{equation}
(5.5) \quad s(\gamma^*(S^*)) = s(\mathbb{C} \oplus T^*M) \otimes L).
\end{equation}
Combining (5.4) with (5.5) yields
\begin{equation}
(5.6) \quad c_k(\gamma^*(Q)) = s_k(\gamma^*(S^*)) = s_k(\mathbb{C} \oplus T^*M) \otimes L]
\end{equation}
\begin{equation}
= \sum_{i=0}^{k} \binom{n + k}{k - i} \cdot s_i(\mathbb{C} \oplus T^*M) \cdot L^{k-i} \quad \text{(by Example 3.7)}
\end{equation}
This completes the proof of (5.3) and hence of the inequalities in (2.3).

In Example 3.8 we have showed by directly calculation that the equality cases in (2.3) are satisfied by the pair \((M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\). This fact is now obvious from our proof as in this case the Gauss map \( \gamma \) is a constant map and so \( \gamma^*(Q) \) is trivial.

5.2. **Proof of Theorem 2.4.** For simplicity we denote by \( c_i \) the \( i \)-th Chern class of \( M \). The exact sequence (5.2) says that the total Chern class of \( \gamma^*(S^*) \) is given by
\begin{equation}
(5.7) \quad c(\gamma^*(S^*)) = (1 + L) \cdot c(T^*M \otimes L)
\end{equation}
\begin{equation}
= (1 + L) \left[ \sum_{i=0}^{n} (-1)^i c_i \cdot (1 + L)^{n-i} \right]. \quad \text{(by Example 3.7)}
\end{equation}
In particular
\begin{equation}
(5.8) \quad \begin{cases}
  c_1(\gamma^*(S^*)) = -c_1 + (n + 1)L \\
  c_2(\gamma^*(S^*)) = \frac{n(n+1)}{2}L^2 - nc_1L + c_2.
\end{cases}
\end{equation}
Note that \( S^* \) is also globally generated and so is \( \gamma^*(S^*) \). As mentioned above \( \gamma^*(S^*) \) can be endowed with a Bott-Chern nonnegative Hermitian metric. Then we have the following Chern number inequality, which is a special case of [Li20, Thm 3.2]
\begin{equation}
(5.9) \quad c_2(\gamma^*(S^*)) \left[ c_1(\gamma^*(S^*)) \right]^{n-2} \leq \left[ c_1(\gamma^*(S^*)) \right]^n.
\end{equation}
Substituting (5.7) into (5.8) yields the desired inequality (2.7). Note that (5.8) can also be deduced from [DPS94, Coro. 2.6] as \( \gamma^*(S^*) \) is also nef.
6. Proofs of Theorems 2.8 and 3.1

6.1. Proof of Theorem 2.8. Under the notations introduced in Section 2, we start with the following

Definition 6.1. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Gamma(k, r)$, the Schur polynomial $S_{\lambda}(c_1, \ldots, c_r) \in \mathbb{Z}[c_1, \ldots, c_r]$ is defined as follows.

$$S_{\lambda}(c_1, \ldots, c_r) := \det(c_{\lambda_i-j+j})_{1 \leq i, j \leq k} \quad (i : \text{row}, j : \text{column})$$

where we adopt the convention that $c_0 := 1$ and $c_i := 0$ if $i \notin [0, r]$.

We shall use the following two special Schur polynomials.

Example 6.2. We have

$$S_{(i,0,\ldots,0)}(c_1, \ldots, c_r) = c_i$$

and

$$S_{(k-i,i,0,\ldots,0)}(c_1, \ldots, c_r) = \begin{vmatrix} c_{k-i} & c_{k-i+1} & \cdots & \ast & \cdots & \ast \\ c_{i-1} & c_i & \cdots & \ast & \cdots & \ast \\ 0 & 0 & 1 & \cdots & \ast & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & \ast & \ast \end{vmatrix} (0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor)$$

$$= c_{k-i}c_i - c_{k-i+1}c_{i-1}.$$ 

Schur polynomials have appeared and played important roles in algebraic combinatorics, representation theory, algebraic geometry and so on. We refer to [Ful97] and [Mac95] for various facts on them. What we need in the proof is the following special case of the remarkable Littlewood-Richardson rule ([Mac95, p. 142]).

Lemma 6.3. Denote by

$$P(k, r) := \left\{ \sum_{\lambda \in \Gamma(k, r)} a_{\lambda} S_{\lambda}(c_1, \ldots, c_r) \mid a_{\lambda} \geq 0 \right\}.$$ 

Then

$$P(k_1, r) \cdot P(k_2, r) \subset P(k_1 + k_2, r).$$

Now we are ready to prove Theorem 2.8. For convenience, we denote by

$$C_i := c_i(E), \quad S_{\lambda} := S_{\lambda}(c_1(E), \ldots, c_r(E)).$$

The following Fulton-Lazarsfeld type inequalities for nef vector bundles over compact Kähler manifolds are due to Demailly, Peternell and Schneider ([DPS94, §2]).

$$\int_M S_{\lambda} \wedge [\omega]^{n-k} \geq 0, \quad (1 \leq k \leq n, \lambda \in \Gamma(k, r)).$$

(6.3)
In view of (6.3) and the definition of $P(k, r)$, in order to prove Theorem 2.8, it suffices to show
\begin{equation}
C_{\lambda_1} C_{\lambda_2} \cdots C_{\lambda_k} - C_{\lambda_1 + \cdots \lambda_k} \in P(\sum_{i=1}^{k} \lambda_i, r),
\end{equation}
which follows from the following two lemmas.

**Lemma 6.4.** We have
\begin{equation}
C_{\lambda_1} + \cdots + \lambda_{t-1} - C_{\lambda_t} - C_{\lambda_1} + \cdots + \lambda_{t} \in P(\lambda_1, r) \quad \forall \ 2 \leq t \leq k.
\end{equation}

**Proof.**
\begin{align*}
C_{\lambda_1} + \cdots + \lambda_{t-1} - C_{\lambda_t} - C_{\lambda_1} + \cdots + \lambda_{t} &= \sum_{i=0}^{\lambda_t-1} (C_{\lambda_1} + \cdots + \lambda_{i-1} + C_{\lambda_{i-1}+1} - C_{\lambda_1} + \cdots + \lambda_{i-1} + C_{\lambda_t-1} + C_{\lambda_t}) \\
&= \sum_{i=0}^{\lambda_t-1} S(\lambda_1, \lambda_{i-1}, 0, \ldots, 0) \quad \text{(by (6.2))} \\
&\in P(\sum_{i=1}^{t} \lambda_i, r).
\end{align*}

**Lemma 6.5.** The inequality (6.4) holds true.

**Proof.**
\begin{align*}
C_{\lambda_1} C_{\lambda_2} \cdots C_{\lambda_k} - C_{\lambda_1 + \cdots \lambda_k} &= \sum_{i=1}^{k-1} (C_{\lambda_1} + \cdots + \lambda_i \cdot C_{\lambda_{i+1}} \cdots C_{\lambda_k} - C_{\lambda_1} + \cdots + \lambda_{i+1} \cdot C_{\lambda_{i+2}} \cdots C_{\lambda_k}) \\
&= \sum_{i=1}^{k-1} (C_{\lambda_1} + \cdots + \lambda_i \cdot C_{\lambda_{i+1}} - C_{\lambda_1} + \cdots + \lambda_{i+1}) \cdot C_{\lambda_{i+2}} \cdots C_{\lambda_k} \\
&= \sum_{i=1}^{k-1} (C_{\lambda_1} + \cdots + \lambda_i \cdot C_{\lambda_{i+1}} - C_{\lambda_1} + \cdots + \lambda_{i+1}) \cdot S(\lambda_{i+2}, 0, \ldots, 0) \cdots S(\lambda_k, 0, \ldots, 0) \quad \text{(by (6.1))} \\
&\in \sum_{i=1}^{k} P(\lambda_1 + \cdots + \lambda_{i+1}, r) \cdot P(\lambda_{i+2}, r) \cdots P(\lambda_k, r) \quad \text{(by (6.5))} \\
&\subset P(\sum_{i=1}^{k} \lambda_i, r). \quad \text{(by Lemma 6.3)}
\end{align*}

This completes the proof of (6.4) and hence Theorem 2.8. \qed

### 6.2. Proof of Theorem 3.1

Its proof is an application of the following Howard-Smyth-Wu’s splitting theorem ([HSW81], [Wu81]) and Mok’s uniformization theorem ([Mok88]).

**Theorem 6.6.** Let $(M, \omega)$ be an $n$-dimensional compact Kähler manifold with nonnegative holomorphic bisectional curvature. Let $q \in \mathbb{Z}_{\geq 0}$ be the irregularity of $M$, which is one half of the first Betti number of $M$. 
(1) (Mok) If $M$ is simply-connected, then it is holomorphically isometric to
\begin{equation}
(6.6) \quad (\mathbb{P}^{N_1}, \theta_1) \times \cdots \times (\mathbb{P}^{N_k}, \theta_k) \times (M_1, g_1) \times \cdots \times (M_p, g_p),
\end{equation}
where $\theta_i$ are Kähler metrics on $\mathbb{P}^{N_i}$ carrying nonnegative holomorphic bisectional curvature, and $(M_i, g_i)$ are irreducible compact Hermitian symmetric spaces of rank $\geq 2$ equipped with the canonical metrics.

(2) (Howard-Smyth-Wu) If $\pi_1(M)$ is nontrivial, then it is infinite. Thus $q > 0$ and the Albanese map $M \rightarrow T^q_C$ is a locally isometrically trivial holomorphic fiber bundle, where $T^q_C$ is equipped with flat metric and the fiber is holomorphically isometric to \((6.6)\).

We can now proceed to prove Theorem 3.1.

**Proof.** First note that the Euler number $c_n(\cdot)$ of the manifolds of the form \((6.6)\) is strictly positive. Indeed, Since odd-dimensional homologies of irreducible compact Hermitian symmetric spaces are zero and so all $c_n(M_i) > 0$. Therefore
\begin{equation}
(6.7) \quad c_n(\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_k} \times M_1 \times \cdots \times M_p) = c_n(\mathbb{P}^{N_1}) \cdots c_n(\mathbb{P}^{N_k}) c_n(M_1) \cdots c_n(M_p) > 0.
\end{equation}

If $M$ is simply-connected, $M$ is of the form \((6.6)\), whose Euler number is strictly positive due to \((6.7)\). Then Theorem 2.8 implies that all the Chern numbers of $M$ are positive.

If $\pi_1(M)$ is nontrivial, then by Howard-Smyth-Wu’s splitting result we have nontrivial Albanese variety $T^q_C$. Since the Ricci curvature is quasi-positive along the fiber and vanishes along $T^q_C$, the maximal rank of Ricci curvature of $\omega$ is less than $n$. This implies that the Chern number $c^*_1[M] = 0$ and consequently all the Chern numbers vanish. This completes the desired proof. \hfill \Box

**Remark 6.7.** We can also apply a result in [DPS94] to give a slightly different proof. If $c^*_1[M] > 0$, then $M$ is Fano due to [DPS94, Prop. 3.10]. Since a Fano manifold is simply-connected ([Zhe00, p. 225]), this reduces to the first case above. Otherwise $c^*_1[M] = 0$ and this reduces to the second case.

7. Proof of Theorem 2.2

Assume that the equality case of \((2.4)\) in Theorem 2.2 holds. This, as we have seen in the proof of Theorem 2.1, is equivalent to the second Segre class $s_2(\gamma^* S^*) = 0$. We want to deduce from it that either $(M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, or $N - n = 1$, i.e., the Kodaira map \((2.2)\) embeds $M^n$ as a hypersurface in $\mathbb{P}^{n+1}$. To this end, we first introduce a quantity $d$, which is the maximum of the dimensions of the osculating spaces of order 2 on $M$, and apply some arguments of algebro-geometric nature to show that the desired conclusion holds true if $d \leq n + 1$. Then we shall show that the inequality $d \leq n + 1$ follows from $s_2(\gamma^* S^*) = 0$.

7.1. The osculating space of order 2. Let $z = (z^1, \ldots, z^n)$ be a local holomorphic coordinate system centered at $p \in U \subset M$, and $\phi : U \rightarrow \mathbb{C}^{N+1} \setminus \{0\}$ a local lifting of the embedding $i$ in \((2.2)\) around $p$. The osculating space of order $k$ at $p$ is defined to be the projective subspace $T^k_p(M)$ in $\mathbb{P}^N$ passing through $p$, spanned by $[\frac{\partial \phi}{\partial z^I}(0)]$ for all multi-indices $I = (i_1, \ldots, i_n)$ with length $|I| = i_1 + \cdots + i_n \leq k$. It turns out that $T^k_p(M)$ is independent of the local coordinate
and lifting chosen. By definition $\mathbb{T}_p(M) \subset \mathbb{T}_p^k(M)$ and $\mathbb{T}_p^1(M) = \mathbb{T}_p(M) \cong \mathbb{P}^n$ is precisely the projective tangent space at $p$.

Let $\text{Tan}(M)$ and $\text{Sec}(M)$ be the tangent variety and secant variety of $M$, which are defined to be

$$\text{Tan}(M) := \bigcup_{p \in M} \mathbb{T}_p(M), \quad \text{Sec}(M) := \{\text{lines } uv \mid u, v \in M, \ u \neq v\},$$

whose expected (maximal) dimensions are $2n$ and $2n + 1$ respectively.

Below we focus on $\mathbb{T}_p^2(M)$, the osculating spaces of order 2. Unlike the case of $k = 1$, the dimensions of $\mathbb{T}_p^k(M)$ ($k \geq 2$) may vary and thus let $d := \max_{p \in M} \dim \mathbb{T}_p^2(M)$, the maximum of the dimensions of $\mathbb{T}_p^2(M)$ on $M$. The following lemma shows that the condition of \textit{\"d \leq n+1\"} shall yield the desired conclusion.

**Lemma 7.1.** If $d \leq n + 1$, then either $(M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, or $N - n = 1$.

**Proof.** It is easy to see that (cf. [BF04, Lemma 1]), for a generic point $q \in \text{Tan}(M)$, where $q \in \mathbb{T}_p(M)$ for $p$ generic in $M$, $\mathbb{T}_q(\text{Tan}(M)) \subset \mathbb{T}_p^2(M)$. So $\dim \text{Tan}(M) \leq n + 1$ by the assumption. On the other hand, $\dim \text{Tan}(M) \geq n$. We distinguish two different cases:

1. $\dim \text{Tan}(M) = n$, then all the $\mathbb{T}_p(M)$ coincide and thus $M = \mathbb{P}^n$.
2. $\dim \text{Tan}(M) = n + 1$ and $\text{Tan}(M) = \mathbb{P}^{n+1}$, then $M$ is contained in this $\mathbb{P}^{n+1}$ and thus $n + 1 = N$, due to the nondegeneracy of $M$ in $\mathbb{P}^N$ (see the remarks after (2.2)).

Now it suffices to rule out the possibility of $\dim \text{Tan}(M) = n + 1$ and $\text{Tan}(M) \neq \mathbb{P}^{n+1}$. Indeed, since $n \geq 2$, in this case $\dim \text{Tan}(M) < 2n$, the expected dimension. A beautiful result of Fulton and Hansen ([FH79, Coro. 4]) says that when $M$ is smooth and $\dim \text{Tan}(M)$ is strictly less than the expected dimension, one always has $\text{Tan}(M) = \text{Sec}(M)$. So $\dim \text{Sec}(M) = n + 1$. On the other hand, by Zak’s result on linear normality ([Za93, Ch.II, Coro. 2.11]), one has

$$\dim \text{Sec}(M) \geq \frac{3}{2} n + 1 = n + \frac{n}{2} + 1 \geq n + 2,$$

which of course contradicts $\dim \text{Sec}(M) = n + 1$. \hfill $\square$

### 7.2. Completion of the proof

It remains to show that, under the condition $s_2(\gamma^* S^*) = 0$, the inequality $d \leq n + 1$ indeed holds. Note that $\gamma^* S^*$ is a quotient of the trivial bundle $\mathbb{C}^{N+1}$, and recall from [Li20] that the induced metric on $\gamma^* S^*$ from the trivial one on $\mathbb{C}^{N+1}$ is Bott-Chern nonnegative and hence its second Segre form is nonnegative as a $(2, 2)$-form. The condition $s_2(\gamma^* S^*) = 0$ then guarantees that this form is identically zero, which implies that $d \leq n + 1$. For technical reason we do it on the dual bundle $\gamma^* S$, which is a subbundle of $\mathbb{C}^{N+1}$. We shall carry out the details in the sequel.

Fix any $p \in M$. We can choose a basis $\sigma = \{s_0, s_1, \ldots, s_N\}$ of $H^0(M, L)$ such that

$$s_0(p) \neq 0, \quad s_1(p) = \cdots = s_N(p) = 0, \quad z^j := \frac{s_j}{s_0}, \quad dz^1 \wedge \cdots \wedge dz^n|_p \neq 0,$$

and

$$\frac{\partial z^\alpha}{\partial z^i}(p) = 0, \quad \forall \ 1 \leq i \leq n, \quad \forall \ n+1 \leq \alpha \leq N.$$

So $z = (z^1, \ldots, z^n)$ forms a local holomorphic coordinate system centered around $p$. As before the basis $\sigma$ gives us a holomorphic embedding $M \hookrightarrow \mathbb{P}^N$ and hence the Gauss map
\[ \gamma: M \to G_{n+1}(\mathbb{C}^{N+1}). \] Near \( p \), the submanifold \( M \subset \mathbb{P}^N \) is defined via equations
\[ z^\alpha = f^\alpha(z^1, \ldots, z^n), \quad n + 1 \leq \alpha \leq N. \]

At the origin \( p \), we have
\[ f^\alpha(0) = 0, \quad f^\alpha_i(0) := \frac{\partial f^\alpha}{\partial z^i}(0) = 0, \quad \forall \ n + 1 \leq \alpha \leq N, \quad \forall \ 1 \leq i \leq n. \]

Let \( \{e_0, \ldots, e_N\} \) be the standard frame of the \((N + 1)\)-dimensional trivial bundle \( \mathbb{C}^{N+1} \) on \( M \). A local frame of its subbundle \( \gamma^*(S) \) near \( p \) is given by \( \{X_0, X_1, \ldots, X_n\} \), where
\[
\begin{align*}
X_i &= e_i + \sum_{\alpha=n+1}^{N} f^\alpha_i e_\alpha, \quad 1 \leq i \leq n, \\
X_0 &= e_0 + \sum_{\alpha=n+1}^{N} h^\alpha e_\alpha, \\
h^\alpha &= f^\alpha - \sum_{j=1}^{n} z^j f^\alpha_j, \quad n + 1 \leq \alpha \leq N.
\end{align*}
\]

Now fix a flat metric \( \langle, \rangle \) on \( \mathbb{C}^{N+1} \) so that \( \{e_0, \ldots, e_N\} \) is unitary. Denote its restricted metric on \( \gamma^*S \) by \( g \). Then the matrix of \( g \) under the frame \( \{X_0, X_1, \ldots, X_n\} \) is
\[ g = (\langle X_i, X_j \rangle)_{0 \leq i, j \leq n} = I_{n+1} + FF^*, \]
where
\[
F = \begin{pmatrix}
 h^{(n+1)} & h^{(n+2)} & \cdots & h^N \\
 f^{(n+1)} & f^{(n+2)} & \cdots & f^1 \\
 \vdots & \vdots & \ddots & \vdots \\
 f^{(n+1)} & f^{(n+2)} & \cdots & f^N
\end{pmatrix}.
\]

Using the facts that \( g(0) = I_{n+1}, \ dg(0) = (0), \ F(0) = (0) \) and the entries of \( F \) are holomorphic with respect to \( z \), the curvature matrix \( \Theta = (\Theta_{ij})_{0 \leq i, j \leq n} \) of \( g \) at the origin \( p \) is given by
\[ (\Theta_{ij})(p) = \bar{\partial}[(\partial g) \cdot g^{-1}](0) \]
\[ = -\partial F \wedge (\partial F)^*(0) \]
\[ = -(\xi^\alpha_i) \wedge (\xi^\beta_j)^*(0) \]
\[ = -\sum_{\alpha=n+1}^{N} \xi^\alpha_i \wedge \overline{\xi^\alpha_j}(0), \]
where \( \xi^\alpha_0 := \partial h^\alpha \) and \( \xi^\alpha_i := \partial f^\alpha_i \) when \( 1 \leq i \leq n \).
We now compute $s_2(\gamma^*S, g)$, the second Segre form of $\gamma^*S$ with respect to $g$, at $p$:

\[
s_2(\gamma^*S, g) = [c_1(\gamma^*S, g)]^2 - c_2(\gamma^*S, g)\]

\[= \left[\text{tr}(\frac{\sqrt{-1}}{2\pi} \Theta)^2 - \frac{\text{tr}(\frac{\sqrt{-1}}{2\pi} \Theta)^2 - \text{tr}[(\frac{\sqrt{-1}}{2\pi} \Theta)^2]}{2}\right]\]

\[= -\frac{1}{8\pi^2} \left[\sum_{i=0}^{n} (\Theta_{ii})^2 + \sum_{0\leq i,j\leq n} \Theta_{ij} \Theta_{ji}\right]\]

\[= -\frac{1}{4\pi^2} \left[\sum_{i=0}^{n} (\Theta_{ii})^2 + \sum_{0\leq i,j\leq n} (\Theta_{ii} \Theta_{jj} + \Theta_{ij} \Theta_{ji})\right]\]

\[= \frac{1}{4\pi^2} \left[\sum_{i,\alpha,\beta} \xi_i^\alpha \wedge \xi_i^\beta \wedge \xi_i^\alpha \wedge \xi_i^\beta + \frac{1}{2} \sum_{i<j,\alpha,\beta} (\xi_i^\alpha \wedge \xi_j^\beta - \xi_i^\beta \wedge \xi_j^\alpha) \wedge (\xi_i^\alpha \wedge \xi_j^\beta - \xi_i^\beta \wedge \xi_j^\alpha)\right]\]

\[= \frac{1}{4\pi^2}(A + B),\]

which is a nonnegative $(2,2)$-form as so are $A$ and $B$. Now if $s_2(\gamma^*S^*) = 0$, then $s_2(\gamma^*S) = 0$ and so $A = B = 0$. At the origin we have

\[\xi_0^\alpha(0) = \partial h^\alpha(0) = 0, \quad \xi_i^\alpha(0) = \partial f_i^\alpha(0) =: \sum_{j=1}^{n} f_{ij}^\alpha(0) dz^j.\]

Write $H^\alpha = (f_{ij}^\alpha(0))$ for the Hessian matrices. Our goal is to show that the linear space $W = \mathbb{C}\{H^{n+1}, \ldots, H^N\}$ spanned by these Hessians at $p$ is at most one dimensional. For any $v \in V := \mathbb{C}^n$, let us write $H^\alpha_v = \xi_v^\alpha(0) = \sum_{i=1}^{n} v_i \xi_i^\alpha(0)$, which can be viewed as a vector in $V$ (under the coframe $dz^j$). Note that $A = B = 0$ means

(7.1) \[H^\alpha_v \wedge H^\beta_v = 0, \quad \forall n + 1 \leq \alpha, \beta \leq N, \forall v \in V.\]

Given any $u, v \in V$, if we replace $v$ in (7.1) by $u + tv$, where $t \in \mathbb{C}$, and look at the $t$-terms, we get

(7.2) \[H^\alpha_u \wedge H^\beta_v + H^\alpha_v \wedge H^\beta_u = 0.\]

To see that $W$ has dimension at most one, first let us assume that there is an $\alpha$ such that the rank of $H^\alpha$ is at least 2. Thus $H^\alpha_u \wedge H^\alpha_v \neq 0$ for generic $u, v \in V$. Let $U$ be the open dense subset of $V$ such that $H^\alpha_u \wedge H^\alpha_v \neq 0$ for any $u, v \in U$. For each $v \in U$, since $H^\alpha_v \neq 0$, by the equation (7.1), we know that there exists a unique constant $\lambda(v)$ such that $H^\alpha_v = \lambda(v)H^\alpha_v$. Also, for any $u, v$ in $U$, by (7.2) we get

(7.3) \[(\lambda(u) - \lambda(v))H^\alpha_u \wedge H^\alpha_v = 0,\]

hence $\lambda(u) = \lambda(v)$. This means that $\lambda$ is a constant function on $U$, hence we have $H^\beta = \lambda H^\alpha$. That is, any other Hessian is a constant multiple of this $H^\alpha$.

Now we are left with the case when each $H^\alpha$, or any of the linear combinations, has rank at most one. We know that for each $i$, the $i$-th rows of these matrices are all parallel, and all these matrices are symmetric, so each of them is a constant multiple of $v^t v$ for some fixed column vector $v$ in $V$. So all these Hessian matrices at $p$ form a linear space of dimension at most one. By definition, this means exactly that the second osculating space $T^2_p(M)$ at $p$ is at most $(n + 1)$-dimensional. This completes the proof of Theorem 2.2.
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