Abstract

Traditional deep learning models are trained at a centralized server using labeled data samples collected from end devices or users. Such data samples often include private information, which the users may not be willing to share. Federated learning (FL) is an emerging approach to train such learning models without requiring the users to share their possibly private labeled data. In FL, each user trains its copy of the learning model locally. The server then collects the individual updates and aggregates them into a global model. A major challenge that arises in this method is the need of each user to efficiently transmit its learned model over the throughput limited uplink channel. In this work, we tackle this challenge using tools from quantization theory. In particular, we identify the unique characteristics associated with conveying trained models over rate-constrained channels, and propose a suitable quantization scheme for such settings, referred to as universal vector quantization for FL (UVeQFed). We show that combining universal vector quantization methods with FL yields a decentralized training system in which the compression of the trained models induces only a minimum distortion. We then theoretically analyze the distortion, showing that it vanishes as the number of users grows. We also characterize the convergence of models trained with the traditional federated averaging method combined with UVeQFed to the model which minimizes the loss function. Our numerical results demonstrate the gains of UVeQFed over previously proposed methods in terms of both distortion induced in quantization and accuracy of the resulting aggregated model.

1 Introduction

Machine learning methods have demonstrated unprecedented performance in a broad range of applications [1]. This is achieved by training a deep network model based on a large number of labeled training samples. Often, these samples are gathered on end devices or users, such as smartphones, while the deep model is maintained by a computationally powerful centralized server [2]. Traditionally, the users send their labeled data to the server, who in turn uses the massive amount of samples to train the model. However, data often contains private information, which the users may prefer not to share, and having each user transmit large volumes of training data to the server may induce a substantial load on the communication link. This gives rise to the need to adapt the network on the end-devices., i.e., train a centralized model in a distributed fashion [3]. Federated learning (FL)
proposed in [4], is a method to update such decentralized models. Instead of requiring the users to share their possibly private labeled data, each user trains the network locally, and conveys its trained model updates to the server. The server then iteratively aggregates these updates into a global network [5][6], commonly using some weighted average, also known as federated averaging [4].

One of the major challenges of FL is the transfer of a large number of updated model parameters over the uplink communication channel from the users to the server, whose throughput is typically constrained [4][6][8]. This challenge can be tackled by reducing the number of participating users, via, e.g., scheduling policies [9]. An alternative strategy is to reduce the volume of data each user conveys, via sparsification or scalar quantization [10][19]. The work [10] proposed various methods for compressing the updates sent from the users to the server. These methods include random masks, subsampling, and probabilistic quantization. Sparsifying masks for compressing the gradients were proposed in [11][13]. Additional forms of probabilistic scalar quantization for FL were considered in [14][18]. However, these approaches are suboptimal from a quantization theory perspective, as, e.g., discarding a random subset of the gradients can result in dominant distortion, while scalar quantization is inferior to vector quantization [20 Ch. 23]. This motivates the design and analysis of quantization methods for facilitating updated model transfer in FL, which minimize the error induced by quantization in the aggregated global model.

Here, we design quantizers for distributed training by tackling the uplink compression for FL problem from a quantization theory perspective. We first discuss the requirements which arise in FL setups under quantization constraints. We specifically identify the lack of a unified statistical model and the availability of a source of local randomness as characteristics of such setups. Based on these properties, we propose a mapping scheme following concepts from universal quantization [21], referred to as universial vector quantization for federated learning (UVeQFed). UVeQFed implements subtractive dithered lattice quantization, which is based on solid information theoretic arguments. In particular, such schemes are known to approach the most accurate achievable finite-bit representation, dictated by rate-distortion theory, to within a controllable gap [22], while meeting the aforementioned requirements. Consequently, UVeQFed allows FL to operate reliably under strict bit rate constraints, due to its ability to reduce the distortion induced by the need to quantize the model updates.

We theoretically analyze the ability of the server to accurately recover the updated model when UVeQFed is employed, showing that the error induced by UVeQFed is mitigated by conventional federated averaging, as well as analyzing the convergence of the global model to the one which minimize the loss function. Specifically, our analysis shows that this error can be bounded by a term which vanishes as the number of users grows, regardless of the statistical model from which the data of each user is generated, rigorously proving that the quantization distortion can be made arbitrarily small when a sufficient number of users contribute to the overall model. Then, we study the convergence of stochastic gradient descent (SGD)-based federated averaging with UVeQFed in a statistically heterogeneous setup, where the training available at each user obeys a different distribution, as is commonly the case in FL [6][8]. We prove that for strongly convex and smooth objectives, the expected distance between the resulting FL performance and the optimal one asymptotically decays as one over the number of iterations, which is the same order of convergence reported for FL without communication constraints in heterogeneous setups [23]. Finally, we show that these theoretical gains translate into FL performance gains in a numerical study. We demonstrate that FL with UVeQFed yields more accurate global models and faster convergence compared to previously proposed quantization approaches for such setups when operating under tight bit constraints of two and four bits per sample, considering synthetic data as well as the MNIST and CIFAR-10 data sets.

The rest of this paper is organized as follows: Section 2 presents the system model and identifies the requirements of FL quantization. Section 3 details the proposed quantization system, and Section 4 theoretically analyzes its performance. Experimental results are presented in Section 5. Section 6 concludes the paper.

Notations Throughout the paper, we use boldface lower-case letters for vectors, e.g., \( \mathbf{x} \); Matrices are denoted with boldface upper-case letters, e.g., \( \mathbf{M} \); calligraphic letters, such as \( \mathcal{X} \), are used for sets. Finally, \( \mathcal{R} \) and \( \mathcal{Z} \) are the sets of real numbers and integers, respectively.

Related Work UVeQFed is based on methods studied and derived in quantization theory, a field of research dealing with the ability to represent continuous-amplitude values using discrete quantities [24]. The fundamental limits of quantization, namely, the most accurate representation achievable using a given number of bits, is dictated by rate-distortion theory [25]. While achieving the minimal
Federated averaging, also referred to as local SGD \cite{mcmahan2017communication}, aims at recovering the model weights. We consider the FL system detailed in \cite{mcmahan2017communication}. A centralized server is training a model consisting of $\ell$ layers, with the same number of bits \cite{li2018federated, wang2019fed-model}, as we also numerically demonstrate in our experimental study. This can be achieved using one bit quantization \cite{zhang2018communication}, ternary quantization \cite{zhang2019ternary}, randomized uniform quantization \cite{zhang2020uniform}, and randomized non-uniform quantization \cite{zhang2021non-uniform}. The communication load can be further reduced by limiting the number of users participating in each training iteration \cite{zhang2021limit}. Most closely related to UVeQFed is the QSGD scheme proposed in \cite{zhang2019quantization}, which essentially implements non-subtractive dithered scalar quantization \cite{zhang2020quantization}. Our proposed UVeQFed combines dimensionality reduction methods with quantization schemes by jointly mapping sets of samples into discrete representations. Furthermore, UVeQFed exploits the availability of a source of common randomness, i.e., that the server can share a random seed with the users, to implement subtractive dithering, which is known to reduce the quantization error compared to non-subtractive dithered quantization operating with the same number of bits \cite{zhang2020quantization, zhang2021quantization}, as we also numerically demonstrate in our experimental study.

2 System Model

In this section we detail the FL with quantized model updates setup. To that aim, we first review the conventional FL setup in Section 2.1. Then, in Section 2.2, we formulate the problem and identify the unique requirements of quantizers utilized in FL systems.

2.1 Federated Learning

We consider the FL system detailed in \cite{mcmahan2017communication}. A centralized server is training a model consisting of $m$ parameters based on labeled samples available at a set of $K$ remote users, in order to minimize some loss function $\ell(\cdot; \cdot)$. Letting $\{x_i^{(k)}, y_i^{(k)}\}_{i=1}^{n_k}$ be the set of $n_k$ labeled training samples at the $k$th user, $k \in \{1, \ldots, K\} \triangleq \mathcal{K}$, FL aims at recovering the $m \times 1$ weights vector $w^o$ satisfying

$$
\hat{w}^o = \arg \min_{\mathbf{w}} \left\{ F(\mathbf{w}) \triangleq \sum_{k=1}^{K} \alpha_k F_k(\mathbf{w}) \right\}.
$$

Here, the weighting average coefficients $\{\alpha_k\}$ are non-negative satisfying $\sum \alpha_k = 1$, and the local objective functions are defined as the empirical average over the corresponding training set, i.e.,

$$
F_k(\mathbf{w}) \equiv F_k(\mathbf{w}; \{x_i^{(k)}, y_i^{(k)}\}_{i=1}^{n_k}) \triangleq \frac{1}{n_k} \sum_{i=1}^{n_k} \ell(\mathbf{w}; (x_i^{(k)}, y_i^{(k)})).
$$

**Federated averaging** \cite{mcmahan2017communication}, also referred to as local SGD \cite{mcmahan2017communication}, aims at recovering $\hat{w}^o$ using iterative subsequent updates. In each update of time instance $t$, the server shares its current model, represented by the vector $\mathbf{w}_t \in \mathbb{R}^m$, with the users. The $k$th user, $k \in \mathcal{K}$, uses its set of $n_k$ labeled training samples to retrain the model $\mathbf{w}_t$ over $\tau$ time instances into an updated model $\hat{w}_{t+\tau}^{(k)} \in \mathbb{R}^m$.

Having updated the model weights, the $k$th user should convey its model update, denoted as $h_{t+\tau}^{(k)} \triangleq \hat{w}_{t+\tau}^{(k)} - \mathbf{w}_t$, to the server. Since uploading throughput is typically more limited compared to its downloading counterpart \cite{zhang2021communication}, the $k$th user needs to communicate a finite-bit quantized representation.
of its model update. Quantization consists of encoding the model update into a set of bits, and decoding each bit combination into a recovered model update \( \hat{h}^{(k)}_{t+\tau} \). The \( k \)th model update \( h^{(k)}_{t+\tau} \) is therefore encoded into a digital codeword of \( R_k \) bits denoted as \( u^{(k)}_{t+\tau} \), using an encoding function whose input is \( h^{(k)}_{t+\tau} \), i.e.,

\[
e^{(k)}_{t+\tau} : \mathcal{R}^m \rightarrow \mathcal{U}_k.
\]

(3)

The server uses the received codewords \( \{u^{(k)}_{t+\tau}\}_{k=1}^K \) to reconstruct \( \hat{h}^{(k)}_{t+\tau} \in \mathcal{R}^m \), obtained via a joint decoding function

\[
d^{(k)}_{t+\tau} : \mathcal{U}_1 \times \ldots \times \mathcal{U}_K \rightarrow \mathcal{R}^m.
\]

(4)

The recovered \( \hat{h}^{(k)}_{t+\tau} \) is an estimate of the weighted average \( \sum_{k=1}^K \alpha_k h^{(k)}_{t+\tau} \). Finally, the global model \( w_{t+\tau} \) is updated via

\[
w_{t+\tau} = w_t + \hat{h}^{(k)}_{t+\tau}.
\]

(5)

An illustration of this FL procedure is depicted in Fig. 1. Clearly, if the number of allowed bits is sufficiently large, the distance \( \|\hat{h}^{(k)}_{t+\tau} - \sum_{k=1}^K \alpha_k h^{(k)}_{t+\tau}\|_2 \) can be made arbitrarily small, allowing the server to update the global model as the desired weighted average:

\[
w^{des}_{t+\tau} = \sum_{k=1}^K \alpha_k w^{(k)}_{t+\tau}.
\]

(6)

In the presence of a limited bit budget, i.e., small values of \( \{R_k\} \), distortion is induced which can severely degrade the ability of the server to update its model. To tackle this issue, various methods have been proposed for quantizing the model updates, commonly based on sparsification or probabilistic scalar quantization. These approaches are suboptimal from a quantization theory perspective \([20, \text{Ch. 23}]\), motivating the study of efficient and practical quantization methods for FL.

2.2 Problem Formulation

Our goal is to propose an encoding-decoding system which mitigates the effect of quantization errors on the ability of the server to accurately recover the updated model \([6]\). To faithfully represent the FL setup, we design our quantization strategy in light of the following requirements and assumptions:

- **A1** All users share the same encoding function, denoted as \( e^{(k)}_t(\cdot) = e^t(\cdot) \) for each \( k \in \mathcal{K} \). This requirement, which was also considered in \([10]\), significantly simplifies FL implementation.

- **A2** No a-priori knowledge or distribution of \( h^{(k)}_{t+\tau} \) is assumed.

- **A3** As in \([10]\), the users and the server share a source of common randomness. This is achieved by, e.g., letting the server share with each user a random seed. Once a different seed is conveyed to each user, it can be used to obtain a dedicated source for the entire FL procedure.

Requirement \( \text{[A2]} \) gives rise to the need for a **universal quantization** approach, namely, a scheme which operates reliably regardless of the distribution of the model updates and without its prior knowledge.
3 UVeQFed

We now propose UVeQFed, which conveys the model updates \( \{ h_{i+\tau}^{(k)} \} \) from the users to the server over the rate-constrained channel using a universal quantization method \([21]\). Specifically, the scheme encodes each model update using \textit{subtractive dithered lattice quantization}, which operates in the same manner for each user, satisfying \([A3]\). UVeQFed allows the server to recover the updates with small average error regardless of the distribution of \( \{ h_{i+\tau}^{(k)} \} \), as required in \([A2]\) by exploiting the source of common randomness assumed in \([A3]\). In addition to its compliance with the model requirements stated in Section 2.2, the proposed approach is particularly suitable for FL, as the distortion is mitigated by federated averaging. This significantly improves the overall FL capabilities, as numerically demonstrated in Section [5].

Quantization Scheme Here, we present the encoding and decoding functions, \( e_{i+\tau}(\cdot) \) and \( d_{i+\tau}(\cdot) \), beginning with some definitions in lattice quantization: We fix a positive integer \( L \), referred to henceforth as the lattice dimension, and a non-singular \( L \times L \) matrix \( G \), which denotes the lattice generator matrix. For simplicity, we assume that \( M \triangleq \frac{2^L}{L} \) is an integer, where \( m \) is the number of model parameters, although the scheme can also be applied when this does not hold by replacing \( M \) with \( \lfloor M \rfloor \). Next, we use \( L \) to denote the lattice, which is the set of points in \( \mathbb{R}^L \) that can be written as an integer linear combination of the columns of \( G \), i.e.,

\[
L \triangleq \{ x = GL : l \in \mathbb{Z}^L \}. 
\]

(7)

A lattice quantizer \( Q_L(\cdot) \) maps each \( x \in \mathbb{R}^L \) to its nearest lattice point, i.e., \( Q_L(x) = L_x \) if \( \| x - L_x \| \leq \| x - L_l \| \) for every \( l \in L \). Finally, let \( P_0 \) be the basic lattice cell \([23]\), i.e., the set of points in \( \mathbb{R}^L \) which are closer to 0 than to any other lattice point:

\[
P_0 \triangleq \{ x \in \mathbb{R}^L : \| x \| < \| x - p \|, \forall p \in L/\{0\} \}. 
\]

(8)

For example, when \( G = \Delta \cdot I_L \) for some \( \Delta > 0 \), then \( L \) is the square lattice, for which \( P_0 \) is the set of vectors \( x \in \mathbb{R}^L \) whose \( \ell_\infty \) norm is not larger than \( \frac{\Delta}{\sqrt{L}} \). For this setting, \( Q_L(\cdot) \) implements entry-wise scalar uniform quantization with spacing \( \Delta \) \([20, \text{Ch. 23}]\).

Encoder: The proposed encoding function \( e_{i+\tau}(\cdot) \) includes the following steps:

\textbf{E1 Normalize and partition:} The \( k \)th user scales \( h_{i+\tau}^{(k)} \) by \( \zeta \| h_{i+\tau}^{(k)} \| \) for some \( \zeta > 0 \), and divides the result into \( M \) distinct \( L \times 1 \) vectors, denoted \( \{ \tilde{h}_i^{(k)} \}_{i=1}^M \). The scalar quantity \( \zeta \| h_{i+\tau}^{(k)} \| \) is quantized separately from \( \{ \tilde{h}_i^{(k)} \}_{i=1}^M \) using some fine-resolution quantizer.

\textbf{E2 Dithering:} The encoder utilizes the source of common randomness, e.g., a shared seed, to generate the set of \( L \times 1 \) dither vectors \( \{ z_i^{(k)} \}_{i=1}^M \), which are randomized in an i.i.d. fashion, independently of \( h_{i+\tau}^{(k)} \), from a uniform distribution over \( P_0 \).

\textbf{E3 Quantization:} The vectors \( \{ \tilde{h}_i^{(k)} \}_{i=1}^M \) are discretized by adding the dither vectors and applying lattice quantization, i.e., by computing \( \{ Q_L(\tilde{h}_i^{(k)} + z_i^{(k)}) \} \).

\textbf{E4 Entropy coding:} The discrete values \( \{ Q_L(\tilde{h}_i^{(k)} + z_i^{(k)}) \} \) are encoded into a digital codeword \( u_{i+\tau}^{(k)} \), in a lossless manner. This can be achieved using entropy coding schemes, e.g., Huffman coding, Lempel-Ziv methods, and arithmetic codes \([36, \text{Ch. 13}]\).

In order to utilize entropy coding in step \textbf{E4}, the discretized \( \{ Q_L(\tilde{h}_i^{(k)} + z_i^{(k)}) \} \) must take values on a \textit{finite set}. This is achieved by the normalization in Step \textbf{E1}, which guarantees that \( \{ h_i^{(k)} \}_{i=1}^M \) all reside inside the \( L \)-dimensional ball with radius \( \zeta^{-1} \), in which the number of lattice points is not larger than \( \zeta^{-1}L^{L/2} \text{det}(G) \) \([37, \text{Ch. 2}]\), where \( \Gamma(\cdot) \) is the Gamma function. The overhead in accurately quantize the single scalar quantity \( \zeta \| h_{i+\tau}^{(k)} \| \) is typically negligible compared to the number of bits required to convey the set of vectors \( \{ \tilde{h}_i^{(k)} \}_{i=1}^M \), hardly affecting the overall quantization rate.

Decoder: The decoding mapping \( d_{i+\tau}(\cdot) \) implements the following:
Wolf coding [36, Ch. 15.4]. In such cases, the server decodes the received codewords $Q_L(h_{i}^{(k)} + z_{i}^{(k)})$. Since the encoding is carried out using a lossless source code, the discrete values are recovered without any errors.

**D2 Dither subtraction:** Using the source of common randomness, the server generates the dither vectors $\{z_i^{(k)}\}$, which can be carried out rapidly and at low complexity using random number generators as the dither vectors obey a uniform distribution. The server then subtracts the corresponding vector from each lattice point, i.e., compute $\{Q_L(h_{i}^{(k)} + z_{i}^{(k)}) - z_{i}^{(k)}\}$.

**D3 Collecting and scaling:** The values $\{Q_L(h_{i}^{(k)} + z_{i}^{(k)}) - z_{i}^{(k)}\}$ are collected into an $m \times 1$ vector $\hat{h}_{i}^{(k)}$ using the inverse operation of the partitioning and normalization in Step $E3$. The remaining encoding-decoding procedure is carried out independently for each user.

**D4 Model recovery:** The recovered matrices are combined into an updated model based on (5). Namely,

$$w_{t+\tau} = w_t + \sum_{k=1}^{K} \alpha_k \hat{h}_{i}^{(k)}. \tag{9}$$

A block diagram of the proposed scheme along with an illustration of the subtractive dithered lattice quantization procedure in Steps $E3$ and $D2$ are depicted in Fig. 2. The joint decoding aspect of the proposed scheme is introduced in the final model recovery Step $D4$. UVeQFed has several clear advantages. First, while it is based on information theoretic arguments, the resulting architecture is rather simple to implement. In particular, both subtractive dithered quantization as well as entropy coding are concrete and established methods which can be realized with relatively low complexity and feasible hardware requirements. The source of common randomness needed for generating the dither vectors can be obtained by sharing a common seed between the server and users. The statistical characterization of the quantization error of such quantizers can be obtained in a universal manner, i.e., regardless of the distribution of the model updates. This analytical tractability allows us to rigorously show that its combination with federated averaging mitigates the quantization error in Section 4. As the updates are quantized for a specific task, i.e., to obtain the global model by averaging, FL with bit constraints can be treated as a task-based quantization scenario [38–41]. This task is accounted for in the selection of the quantization scheme, using one for which the distortion vanishes by averaging regardless of the values of $\{h_{i}^{(k)}\}$. The encoding Steps $E1$/$E2$ can be viewed as a generalization of probabilistic scalar quantizers, used in, e.g., QSGD [15]. When the lattice dimension is $L = 1$ and $\zeta = 1$, Steps $E1$/$E2$ implement the same encoding as QSGD. However, the decoder is not the same as in QSGD due to the dither subtraction in Step $D2$, which is known to reduce the distortion and yield an error term that does not depend on the model updates [23]. Furthermore, UVeQFed allows using vector quantizers, i.e., setting $L > 1$, which is known to further improve the quantization accuracy [22].

The encoding procedure is carried out independently for each user. The server first decodes each digital codeword $u_{i}^{(k)}$ into the discrete value $\{Q_L(h_{i}^{(k)} + z_{i}^{(k)})\}$. Since the encoding is carried out using a lossless source code, the discrete values are recovered without any errors.
We consider in our analysis the case in which the users compute a single stochastic gradient in each time instance. When scalar quantizers are used, i.e.,

\[ \text{Theorem 1.} \]

The quantization error vector \( \epsilon_t \) characterizes the distortion in quantizing the model updates using UVeQFed. Unlike the model updates are recovered in order to the updated global model \( \tilde{h}_t^{(k)} \). See Appendix A.

**Proof:**

Letting \( \bar{\sigma}_L \) be the normalized second order lattice moment, defined as \( \bar{\sigma}_L^2 = \int_{-\infty}^{\infty} |x|^2 \, dx / \int_{-\infty}^{\infty} \, dx \). The moments of quantization error satisfy the following:

**Theorem 1.** The quantization error vector \( \epsilon_t^{(k)} \) has zero-mean entries and satisfies

\[
\mathbb{E} \{ \| \epsilon_t^{(k)} \|^2 | \tilde{h}_t^{(k)} \} = \zeta^2 \| h_t^{(k)} \|^2 M \bar{\sigma}_L^2.
\]  

**Proof:** See Appendix A.

Theorem 1 characterizes the distortion in quantizing the model updates using UVeQFed. Unlike the corresponding characterization of previous quantizers used in FL, which obtained an upper bound on the quantization error, e.g., \([15\text{, Lem. 1}]\), the dependence of the expected error norm on the number of bits is not explicit in (11), but rather encapsulated in the lattice moment \( \bar{\sigma}_L^2 \). To observe that (11) indeed represents lower distortion compared to previous FL quantization schemes, we note that even when scalar quantizers are used, i.e., \( L = 1 \) for which \( \frac{1}{L} \bar{\sigma}_L^2 \) is known to be largest \([44]\), the resulting quantization is reduced by a factor of 2 compared to conventional probabilistic scalar quantizers, such as QSGD, due to the subtraction of the dither upon decoding in Step 2 [28 Thms. 1-2].

The model updates are recovered in order to the updated global model \( \tilde{h}_t^{(k)} = \sum \alpha_t \tilde{w}_t^{(k)} \). Next show that the statistical characterization of the distortion in Theorem 1 contributes to the accuracy in recovering the desired \( \hat{h}_t^{(k)} \) via \( \tilde{w}_t^{(k)} \). To that aim, we introduce the following assumption on the stochastic gradients, which is often employed in distributed learning studies, e.g., \([23\text{, 24, 35}\):
The expected squared $\ell_2$ norm of the random vector $\nabla F_k(w; (x_{t_{(i,k)}}, y_{t_{(i,k)}}))$, representing the stochastic gradient evaluated at $w$, is bounded by some $\xi_k^2 > 0$ for all $w \in \mathbb{R}^m$.

We can now bound the distance between the desired model $w_{des}^{t+\tau}$ and the recovered one $w_{t+\tau}$:

**Theorem 2.** When $\text{AS1}$ holds, the mean-squared distance between $w_{t+\tau}$ and $w_{des}^{t+\tau}$ satisfies

$$
E \left\{ \| w_{t+\tau} - w_{des}^{t+\tau} \|^2 \right\} \leq M \xi_k^2 \sigma_k^2 T^2 \left( \sum_{t'=t}^{t+\tau-1} \eta_{t'}^2 \right) \sum_{k=1}^K \alpha_k^2 \xi_k^2.
$$

(12)

**Proof:** See Appendix B

Theorem 2 implies that the recovered model can be made arbitrarily close to the desired one by increasing $K$, namely, the number of users. For example, when $\alpha_k = 1/K$, i.e., conventional averaging, it follows from Theorem 2 that the mean-squared error in the weights decreases as $1/K$.

**FL convergence:** We next study the convergence of FL with UVeQFed. Our analysis is carried out under the following assumptions, commonly used in FL convergence studies [23, 34]:

**AS2** The local objective functions $\{F_k(\cdot)\}$ are all $\rho_s$-smooth, namely, for all $v_1, v_2 \in \mathbb{R}^m$ it holds that $F_k(v_1) - F_k(v_2) \leq (v_1 - v_2)^T \nabla F_k(v_2) + \frac{1}{2} \rho_s \| v_1 - v_2 \|^2$.

**AS3** The local objective functions $\{F_k(\cdot)\}$ are all $\rho_c$-strongly convex, namely, for all $v_1, v_2 \in \mathbb{R}^m$ it holds that $F_k(v_1) - F_k(v_2) \geq (v_1 - v_2)^T \nabla F_k(v_2) + \frac{1}{2} \rho_c \| v_1 - v_2 \|^2$.

We consider a statistically heterogeneous scenario. Such heterogeneity is in line with assumption $\text{AS2}$ which does not impose any specific distribution structure on the underlying statistics of the training data. Following [23], we define the heterogeneity gap, $\psi = F(w^o) - \min_{w \in \mathbb{R}^m} F_k(w)$, quantifying the degree of heterogeneity. If the training data originates from the same distribution, then $\psi$ tends to zero as the training size grows. However, for heterogeneous data, its value is positive.

The convergence of UVeQFed with federated averaging is characterized in the following theorem:

**Theorem 3.** Set $\gamma = \tau \max(1, 4\rho_s/\rho_c)$ and consider a UVeQFed setup satisfying $\text{AS1, AS3}$. Under this setting, federated averaging with step size $\eta_t = \frac{\rho_s}{\gamma(t+\tau)}$ for each $t \in \mathcal{N}$ satisfies

$$
E\{F(w_t)\} - F(w^o) \leq \frac{\rho_s}{2(t+\gamma)} \max \left( \frac{\rho_s^2 + \tau^2 b}{\tau \rho_c}, \gamma \| w_0 - w^o \|^2 \right),
$$

(13)

where $b \triangleq (1 + 4M \xi_k^2 \sigma_k^2 T^2) \sum_{k=1}^K \alpha_k^2 \xi_k^2 + 6\rho_s \psi + 8(\tau - 1)^2 \sum_{k=1}^K \alpha_k \xi_k^2$.

**Proof:** See Appendix C

Theorem 3 implies that UVeQFed with federated averaging converges at a rate of $O(1/t)$, which is the same order of convergence as FL without quantization constraints for i.i.d. [34] as well as heterogeneous data [23, 46]. A similar order of convergence was also reported for previous probabilistic quantization schemes which typically considered i.i.d. data, e.g., [15, Thm. 3.2]. While it is difficult to identify the convergence gains of UVeQFed over previously proposed FL quantizers, such as QSGD, by comparing Theorem 3 to their corresponding convergence bounds, in Section 5 we empirically demonstrate that UVeQFed converges to more accurate global models compared to FL with probabilistic scalar quantizers, when trained using i.i.d. as well as heterogeneous data sets. Additionally, we note that the communication load on the uplink channel induced by UVeQFed can be further reduced by allowing only part of the nodes to participate in each set of iterations [9, 17]. We leave the analysis of UVeQFed with partial node participation for future work.
5 Numerical Evaluations

In this section we numerically evaluate UVeQFed. We first compare the quantization error induced by UVeQFed to competing methods utilized in FL. Then, we numerically demonstrate how the reduced distortion is translated in FL performance gains using both MNIST and CIFAR-10 data sets.

Quantization error: We begin by focusing only on the compression method, studying its accuracy using synthetic data. We evaluate the distortion induced in quantization of UVeQFed operating with a two-dimensional hexagonal lattice, i.e., $L = 2$ and $G = [2, 0; 1, 1/\sqrt{3}]$ [31], as well as with scalar quantizers, namely, $L = 1$ and $G = 1$. The normalization coefficient is set to $\zeta = 2^{R/3}$.

The distortion of UVeQFed is compared to QSGD [15], as well as to uniform quantizers with random unitary rotation [10], and to subsampling by random masks followed by uniform three-bit quantizers [10], all operating with the same quantization rate, i.e., the same overall number of bits. Let $H$ be a $128 \times 128$ matrix with Gaussian i.i.d. entries, and let $\Sigma$ be a $128 \times 128$ matrix whose entries are given by $(\Sigma)_{i,j} = e^{-0.2|i-j|}$, representing an exponentially decaying correlation. In Figs. 3a-3b we depict the per-entry squared-error in quantizing $H$ and $\Sigma H \Sigma^T$, representing independent and correlated data, respectively, versus the quantization rate $R$, defined as the ratio of the number of bits to the number of entries of $H$. The distortion is averaged over 100 independent realizations of $H$. We observe in Figs. 3a-3b that UVeQFed achieves a more accurate digital representation compared to previously proposed methods. It is also observed that UVeQFed with vector quantization, outperforms its scalar counterpart, and that the gain is more notable when the quantized entries are correlated, demonstrating the ability of vector quantization in exploiting statistical correlation.

FL convergence: Next, we demonstrate that the reduced distortion of UVeQFed also translates into FL performance gains. To that aim, we evaluate its application for training neural networks using the MNIST and CIFAR-10 data sets, and compare its performance to that achievable using QSGD. For MNIST, we use a fully-connected network with a single hidden layer of 50 neurons and an intermediate sigmoid activation. Each of the $K = 15$ users has 1000 training samples, which are distributed sequentially among the users, i.e., the first user has the first 1000 samples in the data set, and so on. The users update their weights using gradient descent, where federated averaging is carried out on each iteration. The resulting accuracy versus the number of iterations is depicted in Figs. 3c-3d for quantization rates $R = 2$ and $R = 4$, respectively. For CIFAR-10, we train the deep convolutional neural network architecture used in [47], whose trainable parameters constitute three convolution layers and two fully-connected layers. Here, we consider two methods for distributing the 50000 training images of CIFAR-10 among the $K = 10$ users: an i.i.d. division, where each user has the same number samples from each of the 10 labels, and a heterogenous division, in which at least 25% of the samples of each user correspond to a single distinct label. Each user completes a single epoch
of SGD with mini-batch size 60 before the models are aggregated. The resulting accuracy versus the number of epochs is depicted in Figs. 3e-3f for quantization rates $R = 2$ and $R = 4$, respectively.

We observe in Figs. 3c-3f that UVeQFed with vector quantizer, i.e., $L = 2$, results in convergence to the most accurate model for all the considered scenarios. The gains are more dominant for $R = 2$, implying that the usage of UVeQFed with multi-dimensional lattices can notably improve the performance over low rate channels. Particularly, we observe in Figs. 3e-3f that similar gains of UVeQFed are noted for both i.i.d. as well as heterogeneous setups, while the heterogeneous division of the data degrades the accuracy of all considered schemes compared to the i.i.d division. It is also observed that UVeQFed with scalar quantizers, i.e., $L = 1$, achieves improved convergence compared to QSGD for most considered setups, which stems from its reduced distortion. These results demonstrate that the theoretical benefits of UVeQFed, which rigorously hold under AS1-AS3, translate into improved convergence when operating under rate constraints with non-synthetic data.

6 Conclusions

In this work we proposed UVeQFed, which utilizes universal vector quantization methods to mitigate the effect of limited communication in FL. We first identified the specific requirements from quantization schemes used in FL setups. Then, we proposed an encoding-decoding strategy based on dithered lattice quantization. We analyzed UVeQFed, proving that its error term is mitigated by federated averaging, and characterizing its convergence profile. Our numerical study demonstrates that UVeQFed allows achieving more accurate recovery of model updates in each FL iteration compared to previously proposed schemes for the same number of bits, and that its reduced distortion is translated into improved convergence with non-synthetic data.

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A Proof of Theorem 1

To prove the theorem, we note that by decoding step \( D3 \) the error vector \( \epsilon_{i,t}^{(k)} \) scaled by \( \zeta ||h_{t+1}^{(k)}|| \), consists of \( M \) vectors \( \{\tilde{\epsilon}_i^{(k)} \} \). Each \( \epsilon_{i,t}^{(k)} \) is an \( L \times 1 \) vector representing the \( i \)th subtractive dithered quantization error, defined as \( \epsilon_{i,t}^{(k)} \triangleq Q \zeta(h_{i,t}^{(k)} + z_{i,t}^{(k)}) - z_{i,t}^{(k)} - h_{i,t}^{(k)} \). The fact that we have used subtractive dithered quantization via encoding steps \( E2 \) and decoding step \( D2 \) implies that, regardless of the statistical model of \( \{h_{i,t}^{(k)}\} \), the quantization error vectors \( \{\epsilon_{i,t}^{(k)}\} \) are zero-mean, i.i.d (over both \( i \) and \( k \)), and uniformly distributed over \( \mathcal{P}_0 [\tilde{w}] \). Consequently,

\[
E \left\{ \left\| \epsilon_{i,t}^{(k)} \right\|^2 | h_{t+1}^{(k)} \right\} = \zeta^2 ||h_{t+1}^{(k)}||^2 \sum_{i=1}^{M} E \left\{ \left\| \epsilon_{i}^{(k)} \right\|^2 \right\} = \zeta^2 ||h_{t+1}^{(k)}||^2 M \bar{\sigma}_z^2, 
\]

thus proving the theorem.

B Proof of Theorem 2

To prove the theorem, we first express the updated global model using as a sum of the desired global model and the quantization noise, and then we show that, due to the statistical properties of subtractive dithered quantization error \( \mathcal{P}_0 [\tilde{w}] \), the distance between \( w_{t+1} \) and \( w_{t+1}^{\text{des}} \) can be bounded via (12). To formulate this distance between \( w_{t+1} \) and the desired \( w_{t+1}^{\text{des}} \), we use \( \{w_{t+1,i}^{(k)}\}_{i=1}^{M} \), \( \{w_{t+1,\tau,i}^{(k)}\}_{\tau,i=1}^{M} \), and \( \{w_{t+1,\tau,i}^{\text{des}}\}_{i=1}^{M} \) to denote the partitions of \( w_{t+1} \), \( w_{t+1,\tau} \), and \( w_{t+1,\tau}^{\text{des}} \) into \( M \) distinct \( L \times 1 \) vectors, as done in Step \( E1 \). To formulate this distance, we use \( \{w_{t+1,i}^{\text{des}}\}_{i=1}^{M} \) to denote the partition of \( w_{t+1} \) into \( M \) distinct \( L \times 1 \) vectors via step \( E1 \), similarly to the definitions of \( \{w_{t+1,\tau,i}^{(k)}\} \) and \( \{w_{t+1,\tau,i}^{\text{des}}\} \).

From the decoding and model recovery steps \( D3 \), it follows that

\[
\hat{w}_{t+1,i} = w_{t+1,i} + \sum_{k=1}^{K} \alpha_k \zeta ||h_{t+1}^{(k)}|| \left( Q \zeta(h_i^{(k)} + z_{i}^{(k)}) - z_{i}^{(k)} - h_i^{(k)} \right)
\]

where \( \tilde{\epsilon}_i^{(k)} \) is the subtractive dithered quantization error, defined in Appendix A. Now, since \( h_i^{(k)} = \hat{w}_{t+1,i} - w_{t+1,i} \) combined with \( \mathcal{P}_0 [\tilde{w}] \) and the fact that \( \sum_{k=1}^{K} \alpha_k = 1 \), it holds that \( \hat{w}_{t+1,i} + \sum_{k=1}^{K} \alpha_k \zeta ||h_{t+1}^{(k)}|| \hat{h}_i^{(k)} = \hat{w}_{t+1,i}^{\text{des}} \). Substituting this into (B.1) yields

\[
\hat{w}_{t+1,i} - \hat{w}_{t+1,i}^{\text{des}} = \sum_{k=1}^{K} \alpha_k \zeta ||h_{t+1}^{(k)}|| \tilde{\epsilon}_i^{(k)}. 
\]

As discussed in Appendix A, \( \{\tilde{\epsilon}_i^{(k)}\} \) are zero-mean, i.i.d (over both \( i \) and \( k \)), and independent of \( h_i^{(k)} \). Consequently, by the law of total expectation

\[
E \left\{ \left\| w_{t+1} - w_{t+1}^{\text{des}} \right\|^2 \right\} = E \left\{ \left\| \sum_{i=1}^{M} \sum_{k=1}^{K} \alpha_k \zeta ||h_{t+1}^{(k)}|| \tilde{\epsilon}_i^{(k)} \right\|^2 \right\}
\]

\[
= E \left\{ \left\| \sum_{i=1}^{M} \sum_{k=1}^{K} \alpha_k \zeta ||h_{t+1}^{(k)}|| \tilde{\epsilon}_i^{(k)} \right\|^2 \right\}
\]

\[
= E \left\{ \sum_{k=1}^{K} \alpha_k^2 \zeta^2 ||h_{t+1}^{(k)}||^2 \bar{\sigma}_z^2 \right\}. 
\]

Next, we note that by (10), the model update \( h_{t+1}^{(k)} = \hat{w}_{t+1,i} - \hat{w}_{t+1,i}^{(k)} \) can be written as the sum of the stochastic gradients \( h_{t+1}^{(k)} = \sum_{i'=t}^{t+\tau-1} \eta \nabla F_k(w_{i'}^{(k)}; (x_{i'}^{(k)}; y_{i'}^{(k)})) \). Since the indices \( \{i'_k\} \) are i.i.d.
over $t$ and $k$, applying the law of total expectation to (B.3) yields

$$E \left\{ \| w_{t+\tau} - \hat{w}_{t+\tau} \|_2^2 \right\} = E \left\{ E \left\{ M\zeta^2 \sigma_L^2 \sum_{k=1}^{K} \alpha_k^2 \| h_t^{(k)} \|_2^2 \big| \{ \tilde{w}_t^{(k)} \} \right\} \right\}$$

$$= E \left\{ M\zeta^2 \sigma_L^2 \sum_{k=1}^{K} \alpha_k^2 E \left\{ \| h_t^{(k)} \|_2^2 \big| \{ \tilde{w}_t^{(k)} \} \right\} \right\}$$

$$= E \left\{ M\zeta^2 \sigma_L^2 \sum_{k=1}^{K} \alpha_k^2 \tau \sum_{t'=t}^{t+\tau-1} \eta_t^2 E \left\{ \| \nabla F_k(\tilde{w}_{t'}^{(k)}; (x_{t'}^{(k)}, y_{t'}^{(k)})) \|_2^2 \big| \{ \tilde{w}_t^{(k)} \} \right\} \right\} \leq E \left\{ \| \nabla F_k(\tilde{w}_t^{(k)}; (x_t^{(k)}, y_t^{(k)})) \|_2^2 \big| \{ \tilde{w}_t^{(k)} \} \right\} \leq \xi_k^2 \text{ by [AS] Equation (B.4)}$$

where in (a) we used the inequality $\| \sum_{t'=t}^{t+\tau-1} \eta_t r_t \|_2^2 \leq \tau \sum_{t'=t}^{t+\tau-1} \| r_t \|_2^2$, which holds for any multivariate sequence $\{r_t\}$; and (b) holds since the uniform distribution of the random index $i_k$ implies that the expected value of the stochastic gradient is the full gradient, i.e., $E\{\nabla F_k(w; (x_{i_k}^{(k)}, y_{i_k}^{(k)}))\} = \nabla F_k(w)$, and consequently, $E\{\| \nabla F_k(\tilde{w}_t^{(k)}; (x_{i_k}^{(k)}, y_{i_k}^{(k)})) - \nabla F_k(\tilde{w}_t^{(k)}) \|_2^2 \} \leq E\{\| \nabla F_k(\tilde{w}_t^{(k)}; (x_t^{(k)}, y_t^{(k)})) \|_2^2 \} \leq \xi_k^2$ by [AS]. Equation (B.4) proves the theorem.

C Proof of Theorem 3

Our proof follows a similar outline to that used in [23,34], with the introduction of additional arguments for handling the quantization constraints imposed on the uplink channels. The unique characteristics of the quantization error which arise from the dithered strategy presented in Section 3 allow us to rigorously incorporate its contribution into the overall flow of the proof.

C.1 Recursive Bound on Weights Error

From [22] it follows that the effect of subtractive dithered quantization can be modeled as additive noise, independent of the quantized value, whose distribution depends only on the properties of the lattice. In particular, it holds that the distortion induced in quantizing the model update $h_t^{(k)}$, denoted $\epsilon_t^{(k)}$, is an $m \times 1$ zero-mean additive noise vector independent of $h_t^{(k)}$, and thus also of $\tilde{w}_t^{(k)}$ and $i_t^{(k)}$. Consequently, by defining the sequence $\tilde{e}_t^{(k)}$ such that $\tilde{e}_t^{(k)} = \epsilon_t^{(k)}$ if $t$ is an integer multiple of $\tau$ and $\tilde{e}_t^{(k)} = 0$ otherwise, it follows that (10) can be written as

$$\tilde{w}_{t+1}^{(k)} = \begin{cases} \tilde{w}_t^{(k)} - \eta_t \nabla F_k(\tilde{w}_t^{(k)}; (x_t^{(k)}, y_t^{(k)})) + \tilde{e}_t^{(k)} & t+1 \notin \mathcal{T}_\tau, \\ \sum_{k'=1}^{K} \alpha_{k'} \tilde{w}_t^{(k')} - \eta_t \nabla F_k(\tilde{w}_t^{(k)}; (x_t^{(k)}, y_t^{(k)})) + \tilde{e}_t^{(k)} & t+1 \in \mathcal{T}_\tau. \end{cases} \quad (C.1)$$

The equivalent model update representation (C.1) allows us to model the effect of subtractive dithered quantization on the overall FL procedure as additional noise corrupting the computation of the stochastic gradients. Building upon this representation, we now follow the strategy proposed in [34] and adapted to heterogeneous data in [23]. This is achieved by defining a virtual sequence $\{v_t\}$ from $\{\tilde{w}_t^{(k)}\}$ which can be shown to behave almost like mini-batch SGD with batch size $\tau$, while being within a bounded distance of the FL model weights $\{\hat{w}_t^{(k)}\}$, by properly setting the step size $\eta_t$. In
particular, we define the virtual sequence \( \{ v_t \} \) via
\[
v_t \triangleq \sum_{k=1}^{K} \alpha_k \tilde{w}_t^{(k)}, \tag{C.2}
\]
which coincides with \( \tilde{w}_t^{(k)} \) when \( t \) is an integer multiple of \( \tau \). Further define the averaged noisy stochastic gradients and the averaged full gradients as
\[
\hat{g}_t \triangleq \sum_{k=1}^{K} \alpha_k \left( \nabla F_k (\tilde{w}_t^{(k)}; (x_t^{(k)}; y_t^{(k)})) - \frac{1}{\eta} \tilde{E}_{t+1} \right), \quad g_t \triangleq \sum_{k=1}^{K} \alpha_k \nabla F_k (\tilde{w}_t^{(k)}),
\]
respectively. Note that since the quantization error is zero-mean and the sample indexes \( \{ i_t^{(k)} \} \) are independent and uniformly distributed, it holds that \( E \{ \hat{g}_t \} = g_t \). Additionally, the virtual sequence \( \{ \tilde{v}_t \} \) satisfies \( v_{t+1} = v_t - \eta_t \hat{g}_t \). The resulting model is thus equivalent to that used in [23, App. A], and as a result, by assumptions AS2 and AS3, it follows from [23, Lemma 1] that if \( \eta_t \leq 4 \rho \), then
\[
E \left\{ \| v_{t+1} - w^o \|^2 \right\} \leq (1 - \eta_t \rho_c) E \left\{ \| v_t - w^o \|^2 \right\} + \eta_t^2 E \left\{ \| \hat{g}_t - g_t \|^2 \right\} + 6 \rho \eta_t^2 \psi + 2E \left\{ \sum_{k=1}^{K} \alpha_k \| v_t - \tilde{w}_t^{(k)} \|^2 \right\}. \tag{C.3}
\]
Expression \( \text{(C.3)} \) bounds the expected distance between the virtual sequence \( \{ v_t \} \) and the optimal weights \( w^o \) in a recursive manner. We further bound the summations in \( \text{(C.3)} \), using the following lemmas:

Lemma C.1. If the step size \( \eta_t \) is non-increasing and satisfies \( \eta_t \leq 2 \eta_{t+1} \) for each \( t \geq 0 \), then, when Assumption AS1 is satisfied, it holds that
\[
\eta_t^2 E \left\{ \| \hat{g}_t - g_t \|^2 \right\} \leq (1 + 4M \zeta^2 \sigma^2 \tau^2) \eta_t^2 \sum_{k=1}^{K} \alpha_k^2 \xi_k^2. \tag{C.4}
\]

Lemma C.2. If the step size \( \eta_t \) is non-increasing and satisfies \( \eta_t \leq 2 \eta_{t+1} \) for each \( t \geq 0 \), then, by AS1 it holds that
\[
E \left\{ \sum_{k=1}^{K} \alpha_k \| v_t - \tilde{w}_t^{(k)} \|^2 \right\} \leq 4(\tau - 1)^2 \eta_t^2 \sum_{k=1}^{K} \alpha_k^2 \xi_k^2. \tag{C.5}
\]

Next, we define \( \delta_t \triangleq E \left\{ \| v_t - w^o \|^2 \right\} \). When \( t \in T_{\tau} \), the term \( \delta_t \) represents the \( \ell_2 \) norm of the error in the weights of the global model. Using Lemmas C.1, C.2, By substituting \( \text{(C.5)} \) and \( \text{(C.4)} \) into \( \text{(C.3)} \), we obtain the following recursive relationship on the weights error
\[
\delta_{t+1} \leq (1 - \eta_t \rho_c) \delta_t + \eta_t^2 b, \tag{C.6}
\]
where
\[
b = (1 + 4M \zeta^2 \sigma^2 \tau^2) \sum_{k=1}^{K} \alpha_k^2 \xi_k^2 + 6 \rho \psi + 8(\tau - 1)^2 \sum_{k=1}^{K} \alpha_k^2 \xi_k^2.
\]
The relationship in \( \text{(C.6)} \) is used in the sequel to prove the FL convergence bound stated in Theorem 5.

### C.2 FL Convergence Bound

Here, we prove Theorem 5 based on the recursive relationship in \( \text{(C.6)} \). This is achieved by properly setting the step-size and the FL systems parameters in \( \text{(C.6)} \) to bound \( \delta_t \triangleq E \left\{ \| v_t - w^o \|^2 \right\} \), and combining the resulting bound with the strong convexity of the objective AS3 to prove \( \text{(13)} \).

In particular, we set the step size \( \eta_t \) to take the form \( \eta_t = \frac{\beta}{t + \gamma} \), for some \( \beta > 0 \) and \( \gamma \geq \max \left( 4 \rho \beta, \tau \right) \), for which \( \eta_t \leq \frac{1}{4 \rho} \) and \( \eta_t \leq 2 \eta_{t+1} \), implying that \( \text{(C.3)} \) and \( \text{(C.5)} \) hold.
Under such settings, we show that there exists a finite $\nu$ such that $\delta_l \leq \frac{\nu}{t+\gamma}$ for all integer $l \geq 0$. We prove this by induction, noting that setting $\nu \geq \gamma \delta_0$ guarantees that it holds for $t = 0$. Consequently, we next show that if $\delta_l \leq \frac{\nu}{t+\gamma}$, then $\delta_{l+1} \leq \frac{\nu}{t+1+\gamma}$. It follows from (C.6) that

$$
\delta_{l+1} \leq \left( 1 - \frac{\beta}{t+\gamma} \rho_c \right) \frac{\nu}{t+\gamma} + \left( \frac{\beta}{t+\gamma} \right)^2 b
$$

$$
= \frac{1}{t+\gamma} \left( \left( 1 - \frac{\beta}{t+\gamma} \rho_c \right) \nu + \frac{\beta^2}{t+\gamma} b \right).
$$

Consequently, $\delta_{l+1} \leq \frac{\nu}{t+1+\gamma}$ holds when

$$
\frac{1}{t+\tau} \left( \left( 1 - \frac{\beta}{t+\gamma} \rho_c \right) \nu + \frac{\beta^2}{t+\gamma} b \right) \leq \nu.
$$

or, equivalently,

$$
\left( 1 - \frac{\beta}{t+\gamma} \rho_c \right) \nu + \frac{\beta^2}{t+\gamma} b \leq \frac{t+\gamma}{t+1+\gamma} \nu.
$$

By setting $\nu \geq \frac{1+\beta^2b}{\beta \rho_c}$, the left hand side of (C.8) satisfies

$$
\left( 1 - \frac{\beta}{t+\gamma} \rho_c \right) \nu + \frac{\beta^2}{t+\gamma} b = \frac{t-1+\gamma}{t+\gamma} \nu + \left( \frac{1-\beta \rho_c}{t+\gamma} \nu + \frac{\beta^2}{t+\gamma} b \right)
$$

$$
= \frac{t-1+\gamma}{t+\gamma} \nu + \frac{1}{t+\gamma} ((1-\beta \rho_c) \nu + \beta^2 b)
$$

$$
\leq \frac{t-1+\gamma}{t+\gamma} \nu,
$$

where (a) holds since $\nu \geq \frac{1+\beta^2}{\beta \rho_c}$. As the right hand side of (C.9) is not larger than that of (C.8), it follows that (C.8) holds for the current setting, which in turn proves that $\delta_{l+1} \leq \frac{\nu}{t+1+\gamma}$. Finally, the smoothness of the objective AS2 implies that

$$
\mathbb{E}\{F(\mathbf{w}_l)\} - F(\mathbf{w}^*) \leq \frac{\rho_s \delta_l}{2} \leq \frac{\rho_s \nu}{2(t+\gamma)},
$$

which, in light of the above setting, holds for $\nu \geq \max \left( \frac{(1+\beta^2)}{\beta \rho_c}, \gamma \delta_0 \right)$, $\gamma \geq \max (\tau, 4 \beta \rho_s)$, and $\beta > 0$. In particular, setting $\beta = \frac{\rho}{\rho_c}$ results in $\gamma \geq \tau \max (1, 4 \rho_s / \rho_c)$ and $\nu \geq \max (\frac{1+\beta^2}{\beta \rho_c}, \gamma \delta_0)$, which, when substituted into (C.10), proves (13).

\[\square\]

C.3 Deferred Proofs

C.3.1 Proof of Lemma C.1

To prove the lemma, we note that

$$
\eta^2_k \mathbb{E} \left\{ \| \mathbf{g}_l - \mathbf{g}_i \|^2 \right\} = \eta^2_k \mathbb{E} \left\{ \left\| \sum_{k=1}^{K} \alpha_k \left( \nabla F_k \left( \mathbf{w}_l \left( k \right) \right); \left( \mathbf{x}_i \left( k \right), \mathbf{y}_i \left( k \right) \right) \right) - \nabla F_k \left( \mathbf{w}_i \left( k \right) \right) - \frac{1}{\eta_k} e_\left( k \right) \right\|^2 \right\}
$$

$$
\leq \eta^2_k \sum_{k=1}^{K} \alpha^2_k \mathbb{E} \left\{ \left\| \nabla F_k \left( \mathbf{w}_l \left( k \right) \right); \left( \mathbf{x}_i \left( k \right), \mathbf{y}_i \left( k \right) \right) \right) - \nabla F_k \left( \mathbf{w}_i \left( k \right) \right) \right\|^2 \right\}
$$

$$
+ \sum_{k=1}^{K} \alpha^2_k \mathbb{E} \left\{ \left\| e_\left( k \right) \right\|^2 \right\}
$$

$$
\leq \eta^2_k \sum_{k=1}^{K} \alpha^2_k \left\| \mathbf{e}_l \left( k \right) \right\|^2 + \sum_{k=1}^{K} \alpha^2_k \mathbb{E} \left\{ \left\| e_\left( k \right) \right\|^2 \right\},
$$

where (a) follows since the quantization noise and the stochastic gradients are mutually independent; and (b) holds since the uniform distribution of the random index $i_k$ implies that the expected value of
the stochastic gradient is the full gradient, i.e., $E\{\nabla F_k(w; (x^{(k)}_{i_t}, y^{(k)}_{i_t}))\} = \nabla F_k(w)$, and consequently, $E\{\|\nabla F_k(\bar{w}_t^{(k)}; (x^{(k)}_{i_t}, y^{(k)}_{i_t})) - \nabla F_k(\bar{w}_t^{(k)}; (x^{(k)}_{i_t}, y^{(k)}_{i_t}))\|^2\} \leq E\{\|\nabla F_k(\bar{w}_t^{(k)}; (x^{(k)}_{i_t}, y^{(k)}_{i_t}))\|^2\} \leq \zeta_k^2$ by [AST]. Furthermore, the definition of $e_{t+1}^{(k)}$ implies that $E\{\|e_{t+1}^{(k)}\|^2\} = 0$ for $t+1 \notin T$, while for $t + 1 \in T$ it holds that $E\{\|e_{t+1}^{(k)}\|^2\} = E\{\|e_{t+1}^{(k)}\|^2\} = M\sigma_k^2$. Now, similarly to the derivation in [B.4], the quantization error induced by UVeQFed satisfies

$$E\{\|e_{t+1}^{(k)}\|^2\} \leq M\zeta_k^2\sigma_k^2 E\left\{\sum_{t'=t+1-\tau}^{t+1} \eta_{t'} \|\nabla F_k(\bar{w}_{t'}^{(k)}; (x^{(k)}_{i_{t'}}, y^{(k)}_{i_{t'}}))\|^2\right\} \leq M\zeta_k^2\sigma_k^2 \tau \sum_{t'=t+1-\tau}^{t+1} \eta_{t'}^2 E\left\{\|\nabla F_k(\bar{w}_{t'}^{(k)}; (x^{(k)}_{i_{t'}}, y^{(k)}_{i_{t'}}))\|^2\right\} \leq M\zeta_k^2\sigma_k^2 \tau^2 \eta_{t+1-\tau}^2 \zeta_k^2$$

(C.12)

where in (a) we used the inequality $\|\sum_{t'=t+1-\tau}^{t+1} \eta_{t'} r_{t'}\|^2 \leq \tau \sum_{t'=t+1-\tau}^{t+1} \eta_{t'}^2 \|r_{t'}\|^2$, which holds for any multivariate sequence $\{r_{t'}\}$; (b) is obtained from assumption [AST] and (c) follows since $\eta_{t+1-\tau} \leq 2\eta_{t+1} \leq 2\eta_t$. Substituting (C.12) into (C.11) proves the lemma. □

C.3.2 Proof of Lemma [C.2]

Note that for $t_0 = \lfloor t/\tau \rfloor \tau$, which is an integer multiple of $\tau$, it holds that $v_{t_0} = \bar{w}_{t_0}$. Since (C.5) trivially holds for $t = t_0$, we henceforth focus on the case where $t > t_0$. We now write

$$E\left\{\sum_{k=1}^K \alpha_k \|\bar{w}_t^{(k)} - v_t\|^2\right\} = E\left\{\sum_{k=1}^K \alpha_k \|\bar{w}_t^{(k)} - \bar{w}_{t_0}^{(k)} - (v_t - v_{t_0})\|^2\right\} \leq E\left\{\sum_{k=1}^K \alpha_k \|\bar{w}_t^{(k)} - \bar{w}_{t_0}^{(k)}\|^2\right\} \leq \sum_{k=1}^K \alpha_k E\left\{\|\bar{w}_t^{(k)} - \bar{w}_{t_0}^{(k)}\|^2\right\},$$

(C.13)

where in (a) we used the fact that for every set $\{r^{(k)}\}$, one can define a random vector $r$ such that $Pr(r = r^{(k)}) = \alpha_k$, and thus

$$\sum_{k=1}^K \alpha_k \|r^{(k)} - \sum_{l=1}^K \alpha_l r^{(l)}\|^2 = E\{||r - E(r)||^2\} = E\{||r||^2\} = \sum_{k=1}^K \alpha_k ||r^{(k)}||^2.$$  

Next, we recall that $e_{t'} = 0$ for each $t' = t_0 + 1, \ldots, t$. Consequently, similarly to the derivation in (C.12),

$$E\left\{\|\bar{w}_t^{(k)} - \bar{w}_{t_0}^{(k)}\|^2\right\} = E\left\{\sum_{t'=t_0}^{t-1} \eta_{t'} \nabla F_k(\bar{w}_{t'}^{(k)}; (x^{(k)}_{i_{t'}}, y^{(k)}_{i_{t'}}))\right\} \leq (\tau - 1) \sum_{t'=t_0}^{t-1} \eta_{t'}^2 E\left\{\|\nabla F_k(\bar{w}_{t'}^{(k)}; (x^{(k)}_{i_{t'}}, y^{(k)}_{i_{t'}}))\|^2\right\} \leq (\tau - 1)^2 \eta_{t_0}^2 \zeta_k^2 \leq 4(\tau - 1)^2 \eta_{t_0}^2 \zeta_k^2,$$

(C.14)
where in (a) we used the inequality $\| \sum_{t=t_0}^{t-1} r_t \|^2 \leq (t - 1 - t_0) \sum_{t=t_0}^{t-1} \| r_t \|^2 \leq (\tau - 1) \sum_{t=t_0}^{t-1} \| r_t \|^2$, which holds for any multivariate sequence $\{r_t\}$; (b) is obtained from assumption (\textcolor{red}{\textit{AS1}}) and (c) follows since $\eta_{t_0} \leq \eta_{t-\tau} \leq 2\eta_t$. Substituting (C.14) into (C.13) proves the lemma.