Perturbative Corrections to Kähler Moduli Spaces

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Abstract

We propose a general formula for perturbative-in-$\alpha'$ corrections to the Kähler potential on the quantum Kähler moduli space of Calabi–Yau $n$-folds, for any $n$, in their asymptotic large volume regime. The knowledge of such perturbative corrections provides an important ingredient needed to analyze the full structure of this Kähler potential, including nonperturbative corrections such as the Gromov–Witten invariants of the Calabi–Yau $n$-folds. We argue that the perturbative corrections take a universal form, and we find that this form is encapsulated in a specific additive characteristic class of the Calabi–Yau $n$-fold which we call the log Gamma class, and which arises naturally in a generalization of Mukai’s modified Chern character map. Our proposal is inspired heavily by the recent observation of an equality between the partition function of certain supersymmetric, two-dimensional gauge theories on a two-sphere, and the aforementioned Kähler potential. We further strengthen our proposal by comparing our findings on the quantum Kähler moduli space to the complex structure moduli space of the corresponding mirror Calabi–Yau geometry.

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1 Introduction

Two-dimensional $\mathcal{N} = (2, 2)$ nonlinear $\sigma$-models with Calabi–Yau target manifolds have Kähler moduli spaces, described semiclassically as the complexification of the Kähler cone of the Calabi–Yau manifold. The metrics on these moduli spaces admit perturbative as well as non-perturbative corrections from their semiclassical approximations. The perturbative corrections for Calabi–Yau threefolds were determined by Candelas, de la Ossa, Green, and Parkes [1] and are related to the four-loop $\sigma$-model contributions to the $\beta$-function calculated in [2].

Interestingly, there is no five-loop correction to the $\beta$-function [3]. However, a framework for discussing perturbative corrections to all orders was laid out in [4]. Nemeschansky and Sen [5] explained how to modify the Calabi–Yau metric with non-local field redefinitions to achieve vanishing $\beta$-function in spite of loop corrections (of apparently arbitrary order).

Recently, a connection has been found between the (ultraviolet) partition function on $S^2$ of certain two-dimensional supersymmetric field theories and the metric on the Kähler moduli spaces of the conformal theories to which they flow under renormalization. (The connection was discovered in [6] based on calculations from [7,8] and argued to be correct in [9].) Nonperturbative information about the conformal field theory, such as the values of Gromov–Witten invariants, can be extracted from this partition function, but only if one has first understood the perturbative contribution to the moduli space metric. That is our task here.

Nonperturbative information has been extracted in the case of Calabi–Yau threefolds in [6,10–14], and for a class of Calabi–Yau fourfolds in [15]. In this note, we will propose a general formula for the perturbative contribution which involves the so-called “Gamma class” that was first discovered in a prescient paper of Libgober [16] and later rediscovered by Iritani [17,18] and Katzarkov–Kontsevich–Pantev [19]. We will verify that our formula agrees with the known perturbative contribution for Calabi–Yau threefolds and is consistent with the formulas in [15]; we will also check it explicitly in some other cases. Note that the present formulation does not rely on special geometry (a spacetime property of Calabi–Yau threefold compactifications), and therefore applies to $\sigma$-models on Calabi–Yau $n$-folds for arbitrary $n$.

Libgober’s original observation concerned the behavior of period integrals on the mirror Calabi–Yau varieties. These period integrals had been computed explicitly for Calabi–Yau hypersurfaces in toric Fano varieties [20], and Libgober observed — generalizing an observation of Hosono, Klemm, Theisen and Yau [21] for Calabi–Yau threefolds — that if he used the period integrals to compute the $n$-point correlation functions and their derivatives (on a Calabi–Yau $n$-fold) then the leading behavior was captured by a combination of Chern classes derived from the power series expansion of the Gamma function. Our modification of Libgober’s idea is to work with the $S^2$-partition function of the original Calabi–Yau variety (which contains similar information to the period integrals of the mirror) and extract the leading behavior in
this case.

The question we are addressing is a question in closed string theory, but the
correction we are proposing has a close connection to some issues in open string theory.
In forthcoming work\(^1\) of Hori and Romo (announced in a lecture at the University
of Tokyo \(^2\), among other places), a computation of the partition function on a
hemisphere for these same supersymmetric field theories is directly related to the
Gamma class. Their work provides an alternative derivation of the results we give
here.

In section 2, we review the definition of the Gamma class and its appearance in
a natural modification of Mukai’s Chern character map; this is relevant to mapping
a particular representation of the Kähler potential on the complex structure moduli
space (as an integral of the holomorphic \(n\)-form wedged with its complex conjugate) to
an analogous representation of the Kähler potential on the Kähler moduli space of the
mirror manifold. In section 3, we discuss the general form of perturbative corrections
to \(\beta\)-functions of nonlinear \(\sigma\)-models as well as their connection with the perturbative
corrections to the Kähler potential on the Kähler moduli space of the target space \(X\).
Section 4 is the centerpoint of the paper: we formulate a general proposal for the
perturbative contributions to the Kähler potential based on the Gamma class; we
then compute the perturbative part of the two-sphere partition function of a class of
two-dimensional abelian gauge theories associated to hypersurface Calabi–Yau \(n\)-folds
in products of projective spaces (equivalently, this computes the perturbative part of
the Kähler potential on the Kähler moduli space); finally, we argue that the class of
examples considered here is sufficiently large that the inferred form of the Kähler
potential holds for any Calabi–Yau \(n\)-fold. In section 5, we discuss how to use mirror
symmetry to map the Kähler potential of the complex structure moduli space to the
Kähler potential of the Kähler moduli space of the mirror, further supporting our
proposal. We close the paper with a discussion of future directions.

## 2 The Mukai pairing and the Gamma class

The periods of Calabi–Yau hypersurfaces in toric Fano varieties are generalized hyper-
geometric functions of the type studied in \[^{26}\], and as such, they have a power series
expansion near any “large complex structure” limit point in the compactified moduli
space \[^{20}\]. As is typical for hypergeometric functions, such series expansions involve
Gamma functions. Building on work of Hosono, Klemm, Theisen and Yau \[^{21}\] in
the case of Calabi–Yau threefolds\(^3\), Libgober showed \[^{16}\] that the physically relevant
\(n\)-point correlation functions (on a Calabi–Yau \(n\)-fold – cf. \[^{27}\]) and their deriva-
tives have asymptotic expansions controlled by certain combinations of characteristic

\[^{1}\]This work has now appeared \[^{22}\], along with two closely related papers \[^{23},^{24}\].

\[^{2}\]The significance of subleading terms in the prepotential as encoding topological data such as the
second Chern class was first recognized in \[^{21}\].

\[^{3}\]Libgober showed that the physically relevant \(n\)-point correlation functions (on a Calabi–Yau \(n\)-fold – cf. \[^{27}\]) and their derivatives have asymptotic expansions controlled by certain combinations of characteristic
classes of the Calabi–Yau manifold that are closely related to the Gamma function.

2.1 Multiplicative characteristic classes

To explain this, we need to recall Hirzebruch’s notion \[28\] of a multiplicative characteristic class \(\hat{Q}_X\) defined for algebraic varieties \(X\), and satisfying \(\hat{Q}_{X \times Y} = \hat{Q}_X \hat{Q}_Y\). Such a characteristic class can be constructed out of any formal power series \(Q(z)\) in a variable \(z\) with constant term 1.\(^3\) Since the product \(Q(z_1)Q(z_2) \cdots Q(z_k)\) is symmetric in \(z_1, z_2, \ldots, z_k\), it can be written as a formal power series in the elementary symmetric functions \(\sigma_1, \sigma_2, \ldots, \sigma_k\):

\[
\hat{Q}(\sigma_1, \sigma_2, \ldots) = Q(z_1)Q(z_2) \cdots
\]

Evaluating \(\hat{Q}\) at the Chern classes \(c_1(X), c_2(X), \ldots\) of \(X\) gives the characteristic class \(\hat{Q}_X\). This takes values in the total cohomology of \(X\), and so only involves finitely many terms in the infinite sum \(\hat{Q}(\vec{\sigma})\) for any given \(X\).

Key examples of this construction are the Todd class \(\text{td}_X\) derived from

\[
\frac{z}{1 - e^{-z}} = 1 + \frac{1}{2} z^2 + \frac{1}{12} z^4 - \frac{1}{720} z^6 + \cdots,
\]

which plays an important role in the Hirzebruch–Riemann–Roch formula, and the \(\hat{A}\)-genus \(\hat{A}_X\) derived from

\[
\frac{z/2}{\sinh(z/2)} = 1 - \frac{1}{24} z^2 + \frac{7}{5760} z^4 - \frac{31}{967680} z^6 + \cdots,
\]

which plays an important role in anomaly inflow \[29\] and the the study of Ramond–Ramond charges \[30\]. Since only even powers of \(z\) occur in the \(\hat{A}\)-genus, this characteristic class can be rewritten in terms of Pontryagin classes and defined for more general manifolds, not just algebraic varieties. Note that

\[
\frac{z}{1 - e^{-z}} = e^{z/2} \frac{z}{e^{z/2} - e^{-z/2}} = e^{z/2} \frac{z/2}{\sinh(z/2)},
\]

which implies that

\[
\text{td}_X = e^{c_1(X)/2} \hat{A}_X.
\]

Libgober’s computations showed that the asymptotic behavior of the \(n\)-point functions and their derivatives are governed by a new multiplicative characteristic class based on the power series for \(1/\Gamma(1 + z)\). We will follow later authors \[17, 19\] instead,

\(^3\)One can analogously construct an additive characteristic class out of a formal power series whose constant term is zero.
and base the “Gamma class” \( \hat{\Gamma}_X \) on the power series expansion of \( \Gamma(1 + z) \) itself, which is

\[
\Gamma(1 + z) = 1 - \gamma z + \left( \zeta(2) + \gamma^2 \right) \frac{z^2}{2} - \left( 2\zeta(3) + 3\zeta(2)\gamma + \gamma^3 \right) \frac{z^3}{6} \\
+ \left( 6\zeta(4) + 8\zeta(3)\gamma + 6\zeta(2)\gamma^2 + \gamma^4 \right) \frac{z^4}{24} \\
- \left( 24\zeta(5) + 20\zeta(2)\zeta(3) + 27\zeta(2)^2\gamma + 20\zeta(3)\gamma^2 + 10\zeta(2)\gamma^3 + \gamma^5 \right) \frac{z^5}{120} + \ldots
\]

where \( \gamma \) is Euler’s constant. (This formula can be further simplified if one wishes by using the standard facts \( \zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \) and so on.)

### 2.2 The Mukai pairing

There is another motivation for the introduction of the Gamma class. As shown long ago by Atiyah and Hirzebruch \[31\], the Chern character defines a ring homomorphism from topological K-theory to cohomology

\[
\text{ch} : K(X) \to H^{\text{even}}(X, \mathbb{Q}) ,
\]

which is an isomorphism after tensoring the left side with \( \mathbb{Q} \). We will be primarily concerned with those elements of \( K(X) \) that can be realized as coherent sheaves on \( X \); the subgroup of such elements will be denoted \( K_{\text{hol}}(X) \), following \[32\].

In the case of K3 surfaces, Mukai \[33\] considered the natural bilinear pairing on \( K_{\text{hol}}(X) \) given by the holomorphic Euler characteristic

\[
\chi(\mathcal{E}, \mathcal{F}) := \sum_k (-1)^k \dim \text{Ext}^k_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) ,
\]

and explained how to construct a bilinear pairing on \( H^{\text{even}}(X, \mathbb{Z}) \) and a modified Chern character map

\[
\mu : K_{\text{hol}}(X) \to H^{\text{even}}(X, \mathbb{Q}) ,
\]

which is an isometry, i.e., preserves the bilinear pairings.\[4\]

Mukai’s construction is based on the Hirzebruch–Riemann–Roch formula \[28\], which in this context says

\[
\chi(\mathcal{E}, \mathcal{F}) = \int_X \text{ch} \mathcal{E}^\vee \wedge \text{ch} \mathcal{F} \wedge \text{td} X ,
\]

\[4\] The main point of \[32\] is the proposal that the correct integral structure in the A-model is provided by \( K_{\text{hol}}(X) \).

\[5\] In fact, in Mukai’s original construction \( \mu \) reversed the sign on the bilinear pairing, but we are following the more recent convention of preserving the sign.
where $td_X$ is the Todd class. The Todd class has a square root based on the square root power series with constant term 1:

$$
\sqrt{\frac{z}{1-e^{-z}}} = 1 + \frac{z}{4} + \frac{z^2}{96} - \frac{z^3}{384} - \frac{z^4}{10240} + \frac{19z^5}{368640} + \frac{79z^6}{61931520} - \frac{55z^7}{49545216} + \ldots
$$

Mukai’s modified Chern character map is defined by

$$
\mu(E) := \text{ch} \sqrt{td_X},
$$

where we have implicitly extended the definition to the entire K-theory group $K(X)$. (We omit the wedge product symbol since all differential forms here are of even degree and hence commute.) This construction works in principle for any algebraic variety $X$.\textsuperscript{6}

However, as Căldăraru has explained in section 3 of [35], defining the Mukai pairing for a general algebraic variety $X$ requires some care. The pairing takes the general form

$$
\langle v \mid w \rangle = \int_X v^\lor \wedge w
$$

for some “duality” involution $v \mapsto v^\lor$, and Căldăraru shows that this involution must be

$$
v^\lor = \frac{\tau(v)}{\sqrt{\text{ch} \omega_X}},
$$

where $\omega_X = \mathcal{O}_X(K_X)$ is the canonical divisor on $X$ whose Chern character is $\text{ch}(\omega_X) = \exp(-c_1(X))$, and where $v \mapsto \tau(v)$ is the linear operator that acts as multiplication by $(-1)^k$ on $H^k(X)$.\textsuperscript{7}

As was observed in [17–19], Mukai’s modified Chern character map is not the only possibility: we could instead define a map

$$
\mu_\Lambda(E) := \text{ch} E \sqrt{td_X} \exp(i\Lambda),
$$

where $\Lambda$ satisfies $\tau(\Lambda) = -\Lambda$. (This includes Mukai’s construction as the special case $\Lambda = 0$.) Let us verify that this map is an isometry, first checking how it behaves on

\textsuperscript{6}For Calabi–Yau threefolds the corresponding terms in periods and prepotentials were first computed in [21] with a generalization for Calabi–Yau $n$-folds proposed in [16]. For Calabi–Yau fourfolds certain integral periods are worked out in [34].

\textsuperscript{7}Note that to define the Mukai pairing on even cohomology, we only need $v \mapsto \tau(v)$ to act as multiplication by $(-1)^k$ on $H^{2k}(X)$; the formula in the text is Căldăraru’s natural extension of this to a pairing on the entire cohomology ring.
dual bundles. We have

\[
\begin{align*}
(\mu_\Lambda(\mathcal{E}))^\vee &= \frac{\tau(\mu_\Lambda(\mathcal{E}))}{\sqrt{\text{ch}\omega_X}} \\
&= \frac{\tau(\text{ch}\mathcal{E})\tau(\sqrt{\text{td}_X})\tau(\exp(i\Lambda))}{\sqrt{\text{ch}\omega_X}} \\
&= \frac{\text{ch}\mathcal{E}^\vee\sqrt{\text{td}_X}\sqrt{\text{ch}\omega_X}\exp(-i\Lambda)}{\sqrt{\text{ch}\omega_X}} \\
&= \text{ch}\mathcal{E}^\vee\sqrt{\text{td}_X}\exp(-i\Lambda),
\end{align*}
\]

where we used the fact that

\[
\tau(\text{td}_X) = \text{ch}(\omega_X)\text{td}_X,
\]
as explained in proposition I.5.2 of [36]. Therefore,

\[
\langle \mu_\Lambda(\mathcal{E}) \mid \mu_\Lambda(\mathcal{F}) \rangle = \int (\mu_\Lambda(\mathcal{E}))^\vee \wedge \mu_\Lambda(\mathcal{F}) \\
= \int \text{ch}\mathcal{E}^\vee\sqrt{\text{td}_X}\exp(-i\Lambda)\text{ch}\mathcal{F}\sqrt{\text{td}_X}\exp(i\Lambda) \\
= \int \text{ch}\mathcal{E}^\vee\text{ch}\mathcal{F}\text{td}_X.
\]

The specific modification proposed by Iritani [17,18] and Katzarkov–Kontsevich–Pantev [19], begins with a rewriting of the power series associated to the Todd class using a familiar identity from complex analysis:

\[
\frac{z}{1 - e^{-z}} = e^{z/2} \frac{z/2}{\sinh(z/2)} = e^{z/2} \Gamma(1 + \frac{z}{2\pi i})\Gamma(1 - \frac{z}{2\pi}) .
\]

We use that factorization to define an alternative to the square root of the Todd class: if we write

\[
\sqrt{\frac{z}{1 - e^{-z}}} \exp(i\Lambda(z)) = e^{z/4} \Gamma(1 + \frac{z}{2\pi i}),
\]

then since \(z\) is real, we can solve for \(\Lambda(z)\) as

\[
\Lambda(z) = \text{Im} \log \Gamma(1 + \frac{z}{2\pi i}) \\
= \text{Im} \left( -\frac{\gamma z}{2\pi i} + \sum_{n \geq 1} (-1)^n \frac{\zeta(n)}{n} \left( \frac{z}{2\pi i} \right)^n \right) \\
= \frac{\gamma z}{2\pi} + \sum_{k \geq 1} (-1)^k \frac{\zeta(2k + 1)}{2k + 1} \left( \frac{z}{2\pi} \right)^{2k+1}.
\]
This power series can be used to define an additive characteristic class $\Lambda_X$ which we call the “log Gamma class” of $X$. Note that since only odd powers of $z$ appear in the power series expansion, $\tau(\Lambda_X) = -\Lambda_X$.

In the Calabi–Yau case, when $c_1 = 0$, we have

$$\Lambda_X = -\frac{\zeta(3)}{(2\pi)^3} c_3 + \frac{\zeta(5)}{(2\pi)^5} (c_5 - c_2c_3) - \frac{\zeta(7)}{(2\pi)^7} (c_7 - c_3c_4 - c_2c_5 + c_2^2c_3) + \ldots .$$

(2.1)

Notice that if $X$ is a K3 surface then $\Lambda_X = 0$ so the original version of Mukai’s proposal is unchanged. Notice also that for $X$ a Calabi–Yau threefold, the modification is proportional to $\zeta(3) \chi / \pi^3$, where $\chi$ is the topological Euler characteristic of $X$.

The “replacement” for the square root of the Todd class is thus a multiplicative characteristic class which we call the “complex Gamma class”:

$$\hat{\Gamma}_X^c = \sqrt{\text{td}_X} \exp(i\Lambda_X).$$

(2.2)

To compare this to the Gamma class, let $\nu$ be the operator on the cohomology of $X$ which acts on $H^k(X)$ as multiplication by $(2\pi i)^{k/2}$ (or the operator on power series which multiplies $z$ by $2\pi i$). Then since

$$\nu \left( \sqrt{\frac{z}{1 - e^{-z}}} \exp(i\Lambda(z)) e^{-z/4} \right) = \nu \left( \Gamma(1 + \frac{z}{2\pi i}) \right) = \Gamma(1 + z),$$

we see that

$$\nu \left( \hat{\Gamma}_X^c \sqrt{\text{ch}(\omega_X)} \right) = \hat{\Gamma}_X$$

is the same Gamma class defined in the previous subsection.

3 Perturbative nonlinear $\sigma$-model analysis

In this section, we discuss perturbative effects in the $\mathcal{N} = (2, 2)$ supersymmetric two-dimensional nonlinear $\sigma$-model on a Calabi–Yau manifold $X$ of arbitrary dimension. Among the marginal operators in such a theory, we single out the ones corresponding to the variation of the Kähler class of $X$, which are parameterized by the space $H^{1,1}(X)$. A metric on this space can be calculated in terms of two-point correlation functions of the corresponding operators.

Since this theory has $\mathcal{N} = (2, 2)$ supersymmetry, the moduli space metric will be Kähler and can be described in terms of a Kähler potential $K$. If we pick (complexified) Kähler coordinates $t_1, \ldots, t_s$ of $H^{1,1}(X)$ with respect to a basis of divisors $H_1, \ldots, H_s$, we will argue that the exponentiated sign-reversed Kähler potential takes a particularly nice form

$$e^{-K(t)} = \int_X \exp \left( 4\pi \sum_{\ell=1}^s \text{Im} \, t_\ell \, H_\ell \right) \cup \sum_{k=0}^n \chi_k + O(e^{2\pi it}) ,$$

(3.1)
for some cohomology classes $\chi_k \in H^{2k}(X, \mathbb{R})$, which specify the perturbative corrections, and where $O(e^{2\pi t})$ represents instanton corrections. We normalize things so that $\chi_0 = 1 \in H^0(X)$ corresponds to the leading order term.

3.1 Nonlinear $\sigma$-model action and the effective action

Under the renormalization group the $\mathcal{N} = (2, 2)$ supersymmetric, two-dimensional, nonlinear $\sigma$-model with Kähler target space $X$ (of complex dimension $n$), flows in the infrared to a conformal fixed point characterized by vanishing $\beta$-functions. In this work, the $\beta$-function of the target space Kähler form is of particular interest, which vanishes at tree level but is nonzero at one-loop:

$$\frac{1}{\alpha'} \beta_{ij} = R_{ij} + \Delta \omega_{ij}(\alpha') = R_{ij} + \alpha'^3 \frac{\zeta(3)}{48} T_{ij} + O(\alpha'^5) .$$

(3.2)

Here $\alpha'$ is the coupling constant in the nonlinear $\sigma$-model. At leading one-loop order, the Ricci tensor $R_{ij}$ appears; $\Delta \omega_{ij}$ then comprises all higher loop corrections, which are exact in cohomology, i.e., $\Delta \omega = d\rho$ with some global one form $\rho$ on $X$.

The tensor $T_{ij}$ is the first non-vanishing subleading correction at four loops [11], which has been explicitly calculated in ref. [42]. Thus, at leading order the vanishing $\beta$-function $\beta_{ij} = 0$ requires a Ricci-flat Kähler metric and hence a Calabi–Yau target space. However, this Ricci-flat Calabi–Yau target space metric gets further corrected at higher loops.

For our purposes it is useful to adopt an effective action point of view for the target space geometry. Namely, we interpret the condition for the vanishing $\beta$-function as the Euler–Lagrange equation for the metric $g_{ij}$ emerging from an action functional [41, 43]. The relevant effective action $S_{\text{eff}}[g]$ takes the form

$$S_{\text{eff}}[g] = \int \sqrt{g} [R(g) + \Delta S(\alpha', g)] ,$$

(3.3)

with the corrections $\Delta S(\alpha', R)$. The leading correction arises at fourth loop order $\alpha'^3$ and enjoys the expansion

$$\Delta S(\alpha', g) = \alpha'^3 S^{(4)}(g) + \alpha'^5 S^{(6)}(g) + \ldots .$$

Here the $n$-th loop correction $S^{(n)}(g)$ is a scalar functional of the metric tensor and the Riemann tensor. A proposal for the structure of these terms has been put forward in ref. [4].

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8The finiteness of $\mathcal{N} = (4, 4)$ supersymmetric non-linear $\sigma$-models with hyper-Kähler target spaces implies that $\Delta \omega_{ij}(\alpha') = 0$ [38-40].

9The five-loop correction at order $O(\alpha'^4)$ is absent [3]. Hence, the next non-vanishing correction is expected at order $O(\alpha'^5)$.
3.2 Weil–Petersson metric of the Kähler moduli space

The critical locus of the effective action functional $S_{\text{eff}}[g]$ encodes the moduli space of Kähler metrics in the presence of perturbative loop corrections. We are particularly interested in the Kähler moduli space $\mathcal{M}_K$ of metric deformations, i.e., we consider the effective action (3.3) for the class of hermitian metrics for a fixed complex structure of the target space $X$. At leading order, the moduli space $\mathcal{M}_K$ yields the Weil–Petersson metric

$$\sum_{k,\ell=1}^{s} G_{k\ell}(t) \, dt_k \overline{dt}_\ell = \partial \overline{\partial} K(t) ,$$  

(3.4)

in terms of the Kähler potential \[^{[44]}\]

$$e^{-K(t)} = \frac{1}{n!} \int_X \left( 4\pi \sum_{\ell=1}^{s} \text{Im} \, t_\ell \, H_\ell \right)^n + O(\alpha'^3) .$$  

(3.5)

(Notice that the choice of (positive) constant multiplying this expression can be altered by changing the Kähler potential without changing the Kähler metric; our choice is a convenient normalization used in this paper.) Clearly, since the effective action $S_{\text{eff}}[g]$ gets corrected beyond the leading contribution, the Weil–Petersson metric receives further corrections from higher loop orders. By means of mirror symmetry, for Calabi–Yau threefolds the four-loop correction has been determined to be \[^{[1]}\]

$$e^{-K(t)} = \frac{1}{n!} \int_X \left( 4\pi \sum_{\ell=1}^{s} \text{Im} \, t_\ell \, H_\ell \right)^n + \frac{\alpha'^3}{(n-3)!} \int_X \left( 4\pi \sum_{\ell=1}^{s} \text{Im} \, t_\ell \, H_\ell \right)^{n-3} \, \chi_3 + O(e^{2\pi i t}) ,$$  

(3.6)

for $n = 3$, with the characteristic class

$$\chi_3 = -2 \zeta(3) c_3(X) ,$$  

(3.7)

in terms of the third Chern class $c_3(X)$ and the Riemann $\zeta$-function. The appearance of the $\zeta$-value $\zeta(3)$ (of transcendental weight three) indicates its origin as a four-loop counterterm of the $\mathcal{N} = (2, 2)$ supersymmetric nonlinear $\sigma$-model \[^{[12]}\].

In general, further corrections in $\alpha'$ appear for Calabi–Yau target spaces of higher dimension $n > 3$. They take the following form\[^{[10]}\]

$$e^{-K(t)} = \int_X \exp \left( 4\pi \sum_{\ell=1}^{s} \text{Im} \, t_\ell \, H_\ell \right) \cup \sum_{k=0}^{n} \alpha'^k \chi_k + O(e^{2\pi i t}) .$$  

(3.8)

In fact, the characteristic classes $\chi_k$ arise from the perturbative loop corrections at loop order $k+1$. Due to the appearance of higher curvature tensors in the corrections\[^{[10]}\]

\[^{[10]}\]Compared to eq. (3.1), we have included here the coupling constant $\alpha'$ of the nonlinear $\sigma$-model, which — for ease of notation — we set to $\alpha' = 1$ in the other sections of this note.
\( \Delta \omega_{ij} \) of the \( \beta \)-function \((3.2)\), integrating such curvature tensors can be expressed in terms of the Chern classes of the tangent bundle of the target space \( X \). Furthermore, the loop corrections appearing in \( \Delta \omega_{ij} \) at a given loop order \( k + 1 \), i.e., at order \( \alpha'^k \), give rise to corrections with transcendentality degree \( k \), which is a general property of loop corrections of supersymmetric two-dimensional \( \sigma \)-models \([45]\). As a result, the cohomology classes \( \chi_k \) are homogeneous elements of transcendental degree \( k \) in the graded polynomial ring over all products of multiple \( \zeta \)-values up to transcendental weight \( k \).

\[
\chi_k \in H^{2k}(X, \mathbb{Q})[\zeta(m)_{m=2, \ldots, k}, \ldots, \zeta(m_1, m_2)_{2 \leq m_1 + m_2 \leq k}, \ldots, \zeta(1, \ldots, 1)]_k. \tag{3.9}
\]

The transcendental weight of a multiple \( \zeta \)-value \( \zeta(m_1, \ldots, m_a) \) is given by the sum \( m_1 + \ldots + m_a \), and the multiple zeta functions \( \zeta(m_1, \ldots, m_a) \) generalize the Riemann zeta function according to \([46]\)

\[
\zeta(m_1, \ldots, m_a) = \sum_{n_1 > n_2 > \ldots > n_a} \frac{1}{n_1^{m_1} \cdots n_a^{m_a}}.
\]

Note that there are many non-trivial relations over \( \mathbb{Q} \) among such multiple \( \zeta \)-values, see for instance \([47]\).

The four-loop correction as specified by the characteristic class \( \chi_3 \) takes the universal form \((3.7)\). By studying a sufficiently large class of higher-dimensional Calabi–Yau target spaces \( X \), we will determine by universality the explicit characteristic classes \( \chi_k \) for general \( k > 3 \) in the following as well.

### 4 Corrections from the partition function

In the previous section, we argued that the perturbative contributions to the metric on the Kähler moduli space take a universal form represented by certain polynomials in the Chern classes at each fixed degree. We now wish to determine those Chern class polynomials, and we will do so using the equivalence between the two-sphere partition function and the exponentiated sign-reversed Kähler potential for a certain large class of gauged linear sigma models \([6]\).

First, let us state the result. Given an integral basis \( H_1, \ldots, H_s \) of \( H^{1,1}(X) \) with corresponding coordinates \( t_1, \ldots, t_s \) on the complexified Kähler moduli space, we will verify below that the Kähler potential \( K \) satisfies

\[
e^{-K} = (2\pi i)^n \int_X e^{-\sum (t_\ell - \bar{t}_\ell) H_\ell} \left( \frac{\hat{\Gamma}_X}{\Gamma_X} \right) \chi_k \hat{\Gamma}_X + O(e^{2\pi i t}). \tag{4.1}
\]

That is, the ratio of the complex Gamma class to its complex conjugate (computed formally in the cohomology ring) completely captures the perturbative contributions to the metric.
There is an alternative formulation using the map $\nu$ introduced in (2.3) and the log Gamma class $\Lambda_X$:

$$e^{-K} = \int_X \exp \left( 4\pi \sum_{\ell} \text{Im} \ t_{\ell} H_{\ell} \right) \nu(e^{2i\Lambda_X}) + O(e^{2\pi i t}),$$

(4.2)

from which we see that the log Gamma class contains all of the information we need to determine the classes $\chi_k$.

For a Calabi–Yau manifold $X$ of complex dimension $n$ with generically non-vanishing Chern classes $c_2(X)$ through $c_n(X)$, there are $p(k) - p(k - 1)$ distinct degree $k$ monomials in the Chern classes, where $p(m)$ is the number of partitions of the integer $m$. In the following, we consider smooth $n$-dimensional Calabi–Yau hypersurfaces in projective simplicial toric Fano varieties of dimension $n + 1$. A subset of such examples of Calabi–Yau manifolds is given by smooth Calabi–Yau hypersurfaces embedded in $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_s}$ (with $n_1 \leq n_2 \leq \ldots \leq n_s$) of degree $(n_1 + 1, \ldots, n_s + 1)$ and complex dimension $n = n_1 + \ldots + n_s - 1$. These latter examples already give rise to $p(n + 1)$ Calabi–Yau geometries of complex dimension $n$. In appendix A we demonstrate that, indeed, up to Calabi–Yau ninefolds this class of Calabi–Yau hypersurfaces yields a sufficiently generic set of examples. Therefore, it seems reasonable that the smooth Calabi–Yau hypersurfaces in projective simplicial toric Fano varieties will in general furnish sufficiently many examples to confirm our universal answer (4.1).

### 4.1 Gauged linear $\sigma$-model for toric hypersurfaces

Let us consider a projective simplicial toric Fano variety $\mathbb{P}_\Sigma$ of dimension $n + 1$ defined in terms of a complete fan $\Sigma$ in $\mathbb{N}_\mathbb{R}$ of dimension $n + 1$. The one-dimensional cones $\rho \in \Sigma(1)$ in the fan $\Sigma$ are associated to toric divisors $D_\rho$. If the toric variety $\mathbb{P}_\Sigma$ has only zero-dimensional singularities, a generic section of the anti-canonical bundle $O_{\mathbb{P}_\Sigma}(\sum_{\rho \in \Sigma(1)} D_\rho)$ describes a smooth Calabi–Yau hypersurface $X_\Sigma$ of dimension $n$.

The gauged linear $\sigma$-model realizes the toric ambient space $\mathbb{P}_\Sigma$ via symplectic reduction with respect to the moment map

$$\mu_\Sigma : \mathbb{C}^{n+s+1} \rightarrow A_n(\mathbb{P}_\Sigma) \otimes \mathbb{R} \simeq H^{1,1}(\mathbb{P}_\Sigma) \simeq \mathbb{R}^s,$$

$$(\Phi_1, \ldots, \Phi_{n+s+1}) \mapsto \frac{1}{2} \left( \sum_{k=1}^{n+s+1} q^1_k |\Phi_k|^2, \ldots, \sum_{k=1}^{n+s+1} q^s_k |\Phi_k|^2 \right).$$

(4.3)

Here $A_n(\mathbb{P}_\Sigma)$ denotes the $n$-th Chow group of $\mathbb{P}_\Sigma$. The integral charges $q^\ell_n$ are constrained by $0 = \sum_{\rho \in \Sigma(1)} q^\ell_n v_\rho$ for $\ell = 1, \ldots, s$ in terms of the one-dimensional integral generators $v_\rho$ of the cones $\Sigma(1)$. For a Kähler form $\omega \in H^{1,1}(\mathbb{P}_\Sigma)$ the toric variety $\mathbb{P}_\Sigma$ is now described by the symplectic quotient

$$\mathbb{P}_\Sigma \simeq \mu_\Sigma^{-1}(\omega)/U(1)^s,$$

(4.4)
with respect to the $U(1)^s$ action associated to the moment map $\mu_\Sigma$.\footnote{Further details on the method of symplectic reduction are reviewed for instance in ref. \cite{48}.}

The abelian $\mathcal{N} = (2, 2)$ gauged linear $\sigma$-model description of the Calabi–Yau hypersurface $X_\Sigma$ arises from the charged $\mathcal{N} = (2, 2)$ chiral spectrum

| Fields | $P$ | $\Phi_1$ | $\Phi_2$ | $\cdots$ | $\Phi_{n+s+1}$ |
|--------|-----|---------|---------|---------|----------------|
| $U(1)_1$ | $-q_0^1$ | $q_1^1$ | $q_2^1$ | $\cdots$ | $q_{n+s+1}^1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $U(1)_s$ | $-q_0^s$ | $q_1^s$ | $q_2^s$ | $\cdots$ | $q_{n+s+1}^s$ |

(4.5)

The gauge charges of the $P$-field are given by

$q_0^\ell = \sum_{k=1}^{n+s+1} q_k^\ell, \quad \ell = 1, \ldots, n + s + 1 ,

in terms of the integral charges with respect to the remaining chiral fields. The gauge invariant superpotential reads

$$W(P, \Phi) = P G(\Phi) .$$

(4.6)

Here $G(\Phi)$ is a generic polynomial in the chiral fields $\Phi_1, \ldots, \Phi_{n+s+1}$ such that $W(P, \Phi)$ forms a gauge invariant superpotential. The polynomial $G(\Phi)$ is readily identified with a section of the anti-canonical bundle $O_{\mathbb{P}_\Sigma}(\sum_{\rho \in \Sigma(1)} D_\rho)$.

In the spectrum (4.5), we have chosen the $U(1)$-gauge group factors of the rows such that $\mathbb{R}_{>0} \subset \mathbb{R}^s \simeq H^{1,1}(\mathbb{P}_\Sigma) = \text{Image}(\mu_\Sigma)$ corresponds to the Kähler cone of the simplicial toric variety $\mathbb{P}_\Sigma$. (This amounts to identifying the rows of the charges (4.5) with the Mori cone vectors of the toric variety $\mathbb{P}_\Sigma$.) Then for any $\omega \in \mathbb{R}_{>0}$ the Calabi–Yau hypersurface

$$X_\Sigma \simeq \left\{ \mu_\Sigma^{-1}(\omega)/U(1)^s \subset \mathbb{C}^{n+s+1} \mid G(\Phi) = 0 \right\} ,$$

(4.7)

arises as the semiclassical vacuum manifold of the geometric phase of the described gauged linear $\sigma$-model with the spectrum (4.5). The choice of $\omega \in \mathbb{R}_{>0}$ — which specifies the Fayet–Iliopoulos terms of the gauged linear $\sigma$-model — determines the Kähler class of the hypersurface $X_\Sigma$.
4.2 Partition function for toric Calabi–Yau hypersurfaces

The two-sphere partition function of the abelian gauged linear $\sigma$-model with the spectrum (4.5) is given by [6–8]

$$Z_{S^2}(r, \theta) = \sum_{m_1, \ldots, m_s} \int \frac{d\sigma_1}{2\pi i} \cdots \frac{d\sigma_s}{2\pi i} (-1)^s e^{-\sum_{\ell=1}^s 4\pi i \sigma_\ell m_\ell}$$

in terms of the complexified Kähler class $r$ of integration in the partition function (4.8). The partition function integral can be explicitly evaluated by multi-dimensional residue calculus along the lines of ref. [49].

The perturbative quantum corrections to the partition function (4.8) arise from the multi-dimensional residue at $(\sigma_1, \ldots, \sigma_s) = 0$, the remaining poles give rise to non-perturbative quantum corrections of order $O(e^{-r})$. Thus the perturbative contributions to the partition function — which are the focus of this note — are given by

$$Z_{S^2}^{\text{pert}} = \int_{\gamma + i \mathbb{R}^s} d\sigma_1 \cdots d\sigma_s (-1)^n \frac{\sum_{\ell=1}^s q_\ell \sigma_\ell}{\prod_{k=1}^{n+s+1} (\sum_{\ell=1}^s q_\ell \sigma_\ell)}$$

$$\times e^{-\sum_{\ell=1}^s 4\pi i \sigma_\ell} \Gamma(1 + \sum_{\ell=1}^s q_\ell \sigma_\ell) \prod_{k=1}^{n+s+1} \Gamma(1 + \sum_{\ell=1}^s q_\ell \sigma_\ell),$$

where the integral is taken over the $s$-dimensional plane $\gamma + i \mathbb{R}^s$ with $\gamma$ as defined in (4.9). Carrying out the residues yields

$$Z_{S^2}^{\text{pert}} = (2\pi i)^n \int_{X_S} e^{-\sum_{\ell=1}^s (\xi_\ell - \xi_\ell^0) H_\ell} \frac{\Gamma(1 - \frac{1}{2\pi i} \sum_{\rho} D_\rho)}{\Gamma(1 + \frac{1}{2\pi i} \sum_{\rho} D_\rho)} \frac{\prod_{\rho} \Gamma(1 + \frac{1}{2\pi i} D_\rho)}{\Gamma(1 - \frac{1}{2\pi i} \sum_{\rho} D_\rho)},$$

in terms of the toric divisors $D_\rho$, $\rho = 1, \ldots, n + s + 1$, and the generators $H_\ell$, $\ell = 1, \ldots, s$, of the Kähler cone, which are linearly-equivalent to the toric divisors according to

$$D_\rho \sim \sum_{\ell=1}^s q_\ell \cdot H_\ell.$$  

\[\text{We have dropped an irrelevant prefactor in } Z_{S^2}(r, \theta) \text{ coming from the } R\text{-charges of the chiral fields (4.5). For further details see ref. [6].}\]
The complexified algebraic Kähler parameters

\[ \xi_\ell = -\theta_\ell + i r_\ell , \quad t_\ell = \xi_\ell + O(e^{-r}) . \] (4.12)
correspond to the Kähler coordinates \( t_\ell \) up to non-perturbative corrections.

The technical details of the derivation of the general expression (4.11) are deferred to ref. [50]. Here, we briefly present the derivation for a hypersurface Calabi–Yau \( n \)-fold \( X \) in a product of projective spaces

\[ \mathbb{P}^{n+1}_\otimes := \mathbb{P}^{n_1}_\otimes \times \mathbb{P}^{n_2}_\otimes \times \ldots \times \mathbb{P}^{n_s}_\otimes , \quad \dim \mathbb{P}^{n+1}_\otimes = n_1 + \ldots + n_s = n + 1 \]

(as argued before, this already furnishes a sufficient class of examples for our universality argument). The toric divisors \( D_\rho_\ell, \rho_\ell = 1, \ldots, n_\ell + 1 \), associated to the factors \( \mathbb{P}^{n_\ell}_\otimes \) in \( \mathbb{P}^{n+1}_\otimes \), are linearly equivalent to the hyperplane classes \( H_\ell \) of \( \mathbb{P}^{n_\ell}_\otimes \), while the Calabi–Yau hypersurface is a section of the anti-canonical bundle \( \mathcal{O}_{\mathbb{P}^{n+1}_\otimes}((n_1 + 1)H_1 + \ldots + (n_s + 1)H_s) \). As a consequence, all non-vanishing intersection numbers of such a Calabi–Yau hypersurface \( X \) read

\[ n_\ell + 1 = \int_X H_1^{n_1} \cup H_2^{n_2} \cup \ldots \cup H_{\ell-1}^{n_{\ell-1}} \cup \ldots \cup H_{s-1}^{n_{s-1}} \cup H_s^{n_s}, \quad \ell = 1, \ldots, s , \]

which allows us to rewrite any integral over the Calabi–Yau \( n \)-fold \( X \) of a (formal) power series \( p(H_\ell) \) in the hyperplane classes \( H_\ell \) as

\[
\begin{aligned}
\int_X p(H_\ell) &= \left( \sum_{\ell=1}^s \frac{n_\ell + 1}{(n_1)! \ldots (n_{\ell-1})! \ldots (n_s)!} \frac{d^n}{dH_1^{n_1} \ldots dH_{\ell-1}^{n_{\ell-1}} \ldots dH_s^{n_s}} \right) \bigg|_{H_\ell=0} p(H_\ell) \\
&= \oint_0 \frac{d\sigma_1}{2\pi i} \cdots \oint_0 \frac{d\sigma_s}{2\pi i} \frac{(n_1 + 1)\sigma_1 + \ldots + (n_s + 1)\sigma_s}{\sigma_1^{n_1+1} \cdots \sigma_s^{n_s+1}} p(\sigma_1) .
\end{aligned}
\]

Furthermore, for the Calabi–Yau hypersurface \( X \) in \( \mathbb{P}^{n+1}_\otimes \), the perturbative partition function (4.10) simplifies to

\[
Z_{S^2}^{\text{pert}} = \oint_0 \frac{d\sigma_1}{2\pi i} \cdots \oint_0 \frac{d\sigma_s}{2\pi i} \frac{(-1)^n}{\sigma_1^{n_1+1} \cdots \sigma_s^{n_s+1}} \\
\times e^{-\sum_{\ell=1}^s 4\pi r_\ell \sigma_\ell} \frac{\Gamma(1 + \sum_{\ell=1}^s (n_\ell + 1)\sigma_\ell)}{\Gamma(1 - \sum_{\ell=1}^s (n_\ell + 1)\sigma_\ell)} \prod_{\ell=1}^s \frac{\Gamma(1 - \sigma_\ell)^{n_\ell+1}}{\Gamma(1 + \sigma_\ell)^{n_\ell+1}} ,
\]

and we observe that the multi-dimensional residue calculus realizes the identity (4.13). Therefore, we find for such a Calabi–Yau hypersurface \( X \) in a product of projective spaces \( \mathbb{P}^{n+1}_\otimes \) agreement with the general expression (4.11).
Let us now make contact with the characteristic classes discussed in section 2. Since the complex Gamma class \( \hat{\Gamma}^C \) is multiplicative, the \( \hat{\Gamma}^C \)-class of the Calabi–Yau hypersurface \( X_\Sigma \) embedded in \( \mathbb{P}_\Sigma \) becomes

\[
\hat{\Gamma}^C_{X_\Sigma} = \frac{\hat{\Gamma}^C_{\mathbb{P}_\Sigma}}{\hat{\Gamma}^C_{N_{X_\Sigma}}} = \frac{\prod_\rho \Gamma(1 + \frac{1}{2\pi i} D_\rho)}{\Gamma(1 + \frac{1}{2\pi i} \sum_\rho D_\rho)},
\]

where the last equality follows from the adjunction formula \( N_{X_\Sigma} \simeq \mathcal{O}_{\mathbb{P}_\Sigma}(\sum_\rho D_\rho) \) for the normal bundle \( N_{X_\Sigma} \) of \( X_\Sigma \) and from the identity \( c(\mathbb{P}_\Sigma) = c(\oplus_\rho D_\rho) \) of the total Chern classes \([51]\). As a consequence, inserting (4.14) into (4.11), we obtain the perturbative part of the \( S^2 \) partition function expressed as

\[
Z_{S^2}^{\text{pert}} = (2\pi i)^n \int_X e^{-\sum_\ell(t_\ell - \bar{t}_\ell)H_\ell} \left( \hat{\Gamma}^C_{X_\Sigma} \hat{\Gamma}^C_X \right),
\]

which, thanks to \([6]\), verifies our claim about the form of the metric for a large class of Calabi–Yau \( n \)-folds.

### 4.3 Further checks

We have argued, based on the universal nature of the answer, and a verification in a sufficient number of examples, that the metric on the Kähler moduli space satisfies

\[
e^{-K} = \int_X e^{4\pi \sum \im H_\ell} \nu(e^{2i\Lambda_X}) + O(e^{2\pi i t}) .
\]

Upon expanding the exponentiated \( \Lambda_X \)-class, this formula makes direct contact with the perturbative cohomology classes \( \chi_k \) in (3.8). In particular, inserting the expansion (2.1) into (4.16) predicts the first few perturbative corrections

\[
\begin{align*}
\chi_3 &= -2\zeta(3)c_3 , \quad \chi_4 = 0 , \quad \chi_5 = 2\zeta(5) (c_2c_3 - c_5) , \quad \chi_6 = 2\zeta(3)^2 c_3^2 , \\
\chi_7 &= -2\zeta(7) (c_2^2c_3 - c_3c_4 - c_2c_5 + c_7) , \quad \chi_8 = 4\zeta(3)\zeta(5) (c_3c_5 - 2c_2^2c_3) , \\
\chi_9 &= -\frac{4}{3}\zeta(3)^3 c_3^3 + 2\zeta(9) \left( c_2^3c_3 - \frac{1}{3} c_3^3 - 2c_2c_3c_4 + c_2^2c_5 + c_2c_5 + c_3c_6 + c_2c_7 - c_9 \right) ,
\end{align*}
\]

in terms of the Chern classes \( c_k \) of the Calabi–Yau \( n \)-fold \( X \). The contributions \( \chi_3 \) and \( \chi_4 \) are exactly what appear in \([15]\). We will verify these contributions up through \( \chi_9 \) explicitly in appendix \([A]\).

At four-loops, the result for \( \chi_3 \) agrees with the four-loop correction (3.7) predicted by means of mirror symmetry in ref. \([1]\). Furthermore, \( \chi_4 = 0 \) confirms the absence of the five-loop correction as has been established in ref. \([3]\). In general the perturbative corrections \( \chi_k \) are in accord with the predicted ring structure (3.9). We further observe that — due to the structure of the \( \Lambda_X \) class — only ordinary \( \zeta \)-values occur. The
absence of multiple $\zeta$-values is a bit surprising from the two-dimensional $\mathcal{N} = (2, 2)$ non-linear $\sigma$-model point of view. From a mirror symmetry perspective this feature has previously been observed and explained in ref. [52], where it is demonstrated that multiple $\zeta$-values always appear in such combinations that they can be traded by ordinary $\zeta$-values due to non-trivial transcendental relations among them.

5 Motivation from mirror symmetry

Now we outline a derivation of eq. (4.1) from mirror symmetry to further support our proposal. Let us begin with the mirror $Y$ of our desired Calabi–Yau manifold $X$, and study a large complex structure limit point in the complex structure moduli space of $Y$. There are various assumptions we could make about the local structure near such a point (analyzed thoroughly in [53]), but let us take the simplest form: we assume that our (complex structure) moduli space $\mathcal{M}_0$ is compactified to $\mathcal{M}$ so that the boundary $\partial \mathcal{M} = \mathcal{M} - \mathcal{M}_0$ is a divisor with simple normal crossings, and so that our large complex structure limit point is a point of maximal depth in the boundary. That is, we assume there are local coordinates $z_1, \ldots, z_s$ on the compactified moduli space $\mathcal{M}$ such that our point is the origin, and such that the boundary divisor has components $\{z_\ell = 0\}$. Our task is to study the asymptotic behavior of the so-called Weil–Petersson metric on the moduli space.

The basic structure of the metric was found long ago by Tian [54] and Todorov [55], who showed that if $\Omega_z$ is a family of holomorphic $n$-forms depending on the parameter $z$, then the Kähler potential $K$ of the Weil–Petersson metric (with a suitable normalization) satisfies

$$e^{-K} = (-1)^{n(n-1)/2}(2\pi i)^n \int_Y \Omega_z \wedge \overline{\Omega_z},$$

where, as elsewhere in this paper, $n$ is the dimension of the Calabi–Yau manifold $Y$. This is the quantity we wish to compare to our results about the metric on the Kähler moduli space of the mirror.

Let $Y_z$ be the Calabi–Yau manifold associated to $z$, for $z$ not in the boundary. Then the cohomology groups $H^n(Y_z, \mathbb{Z})$ do not form a single-valued $\mathbb{Z}$-bundle, but rather undergo a monodromy transformation $T_\ell$ as we loop around $\{z_\ell = 0\}$ — such a structure is known as a local system. Let $N_\ell = \log T_\ell$. We can find some single-valued objects defined throughout our coordinate patch of the form

$$\exp\left( - \sum_\ell \frac{1}{2\pi i} N_\ell \log z_\ell \right) v$$

for any $v \in H^n(Y_z, \mathbb{Z})$. The point is that as the argument of $z_\ell$ increases through $2\pi$, following $v$ continuously takes it to $T_\ell v$, but this is then canceled by the factor of $\exp(-N_\ell) = T_\ell^{-1}$ we inserted into the expression.
These single-valued objects capture the possible asymptotic behaviors of Hodge-theoretic quantities near the large complex structure limit point. In particular, the asymptotic behavior of a family $\Omega_z$ of holomorphic $n$-forms, which depend holomorphically on $z$, must take the form

$$
\exp\left(-\sum_\ell \frac{1}{2\pi i} N_\ell \log z_\ell\right) v_{\text{holo}} + O(z)
$$

(5.1)

for some appropriate vector $v_{\text{holo}} \in H^n(Y_z, \mathbb{C})$. We use complex coefficients for cohomology here because the holomorphic form can be modified by an arbitrary constant.

When written in this form, we can calculate the asymptotic behavior of the Tian–Todorov exponentiated sign-reversed Kähler potential $e^{-K} = (-1)^{\frac{n(n-1)}{2}} (2\pi)^n \int \Omega_z \wedge \overline{\Omega_z}$. It is

$$
(-1)^{\frac{n(n-1)}{2}} (2\pi)^n \int_Y \exp\left(-\sum_\ell \frac{1}{2\pi i} N_\ell \log z_\ell\right) v_{\text{holo}} \wedge \exp\left(\sum_\ell \frac{1}{2\pi i} N_\ell \log \overline{z_\ell}\right) \overline{v_{\text{holo}}}
$$

$$
\sim (-1)^{\frac{n(n-1)}{2}} (2\pi)^n \int_Y \exp\left(-\sum_\ell \frac{1}{2\pi i} N_\ell (\log z_\ell + \log \overline{z_\ell})\right) v_{\text{holo}} \wedge \overline{v_{\text{holo}}},
$$

where $\sim$ means that the two sides differ by $O(z)$. If we rewrite this in terms of $t_\ell = \frac{1}{2\pi i} \log z_\ell$, the asymptotic expression becomes

$$
e^{-K} \sim (-1)^{\frac{n(n-1)}{2}} (2\pi)^n \int_Y \exp\left(-\sum_\ell N_\ell (t_\ell - \overline{t_\ell})\right) v_{\text{holo}} \wedge \overline{v_{\text{holo}}}
$$

$$
\sim \sum_k \frac{1}{k!} \int_Y \left(4\pi \sum_\ell N_\ell \text{Im} t_\ell\right)^k v_{\text{holo}} \wedge (-1)^{\frac{n(n-1)}{2}} (2\pi)^n v_{\text{holo}},
$$

(5.2)

which we can recognize as being the same general form as (3.1), once a specific choice for $v_{\text{holo}}$ has been made.

5.1 The Gauss–Manin connection and the Hodge filtration

The family of holomorphic $n$-forms $\Omega_z$ can be regarded as a section of a holomorphic bundle $\mathcal{H}$ over the moduli space whose fibers are the cohomology groups $H^n(Y_z, \mathbb{C})$. Sections of $\mathcal{H}$ can be differentiated with respect to parameters using the Gauss–Manin connection $\nabla : \mathcal{H} \rightarrow \mathcal{H} \otimes \Omega^1_{\mathcal{M}_0}$, which has the property that $\nabla(\varphi) = 0$ if $\varphi \in H^n(Y_z, \mathbb{Z})$ for all $z$. The holomorphic $n$-forms span a rank one subbundle $\mathcal{F}^n$ of $\mathcal{H}$ which is the first subbundle in the Hodge filtration:

$$
\mathcal{F}^p = \{ \phi \mid \phi_z \in H^{p,n-p}(Y_z, \mathbb{C}) \oplus H^{p+1,n-p-1}(Y_z, \mathbb{C}) \oplus \cdots \oplus H^{n,0}(Y_z, \mathbb{C}) \} \subset \mathcal{H}.
$$
We can think of elements of $\mathcal{F}^p$ as consisting of sums of forms in $H^n(Y, \mathbb{C})$, each of which has at least $p$ holomorphic differentials. The subbundles in this filtration are related by differentiation:

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_{\mathcal{M}_0}.$$ 

As mentioned above, the integer cohomology groups (contained in $\{ \varphi \mid \nabla(\varphi) = 0 \}$) form a local system with monodromy around the boundary divisors; any explicit description of this local system must involve logarithms of the coordinates. Since the derivative of a logarithm has a simple pole, it is natural to expect (and known to be true \[56\]) that the bundle $\mathcal{H}$ and the Gauss–Manin connection $\nabla$ extend over $\mathcal{M}$ to a bundle $\tilde{\mathcal{H}}$ and a connection $\tilde{\nabla}$ satisfying

$$\tilde{\nabla} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}} \otimes \Omega^1_{\mathcal{M}}(\log \partial \mathcal{M}),$$

where $\Omega^1_{\mathcal{M}}(\log \partial \mathcal{M})$ is the sheaf of differentials with logarithmic poles on the boundary (generated in our local coordinate system by $dz_1/z_1, \ldots, dz_s/z_s$). It is also known \[57\] that the Hodge filtration extends to a filtration $\tilde{\mathcal{F}}^p$ by holomorphic subbundles satisfying

$$\tilde{\nabla}(\tilde{\mathcal{F}}^p) \subset \tilde{\mathcal{F}}^{p-1} \otimes \Omega^1_{\mathcal{M}}(\log \partial \mathcal{M}).$$

### 5.2 The mirror

The mirror analogues of the local system, the extended Gauss–Manin connection, and the extended Hodge filtration were described in \[58\] (see also \[59\].) These provide the mirror analogue of the family of holomorphic $n$-forms, which can then be used to calculate the metric on the Kähler moduli space. However, one ingredient of this computation (the complex conjugation of the Hodge bundles) was left implicit in \[58\], and in fact needs to be modified as we shall describe.

The mirror of the logarithms $N_\ell$ of the monodromy transformations, which define the local system, are the transformations on $H^{even}(X, \mathbb{C})$ given by cup-product with the corresponding divisors $H_\ell$, which are the edges of the Kähler cone.\[13\] The vector that plays the rôle of $v_{\text{holo}}$ in \[51\] is the generator $1 \in H^{0,0}(X)$, so that a generator of the mirror of $\tilde{\mathcal{F}}^n$ is

$$\exp \left( - \sum_{\ell} \frac{1}{2\pi i} N_\ell \log z_\ell \right) 1 + O(z).$$

The local system $\bigcup H^n(Y_z, \mathbb{Z})$ has an integer structure, and the earliest guesses about that structure on the mirror were to use the integer structure on even cohomology $H^{even}(X_z, \mathbb{Z})$. However, as Hosono pointed out \[52\], the correct integer structure — the one which is compatible with mirror symmetry in known examples

\[13\] More precisely, we have chosen a simplicial cone within the Kähler cone, and the $H_\ell$ are the edges of it.
— is provided by $K_{\text{hol}}(X)$. Thus, whenever we need the integer structure of the local system $\bigcup H^{\text{even}}(X, \mathbb{C})$, we should apply the isomorphism between $K_{\text{hol}}(X) \otimes \mathbb{C}$ and $H^{\text{even}}(X, \mathbb{C})$ and use the integer structure coming from $K$-theory.

For the purpose of calculating the metric, we do not need the integer structure on the local system, but we do need the real structure so that we can perform complex conjugation. And as Iritani \cite{iritani2017,iritani2018} and Katzarkov–Kontsevich–Pantev \cite{katzarkov2019} have taught us, the most natural isomorphism to use is not the Chern character map, but rather

$$\mu_{\Lambda_X} : \mathcal{E} \mapsto \text{ch}(\mathcal{E}) \wedge \tilde{\Gamma}_X^c.$$  

(5.3)

Note that neither the Gauss–Manin connection, nor the Hodge filtration, nor even the local system with complex coefficients — the topics discussed in \cite{wein2020} — depend on this isomorphism. But the complex conjugation needed to define the metric does depend on it.

To compute the effect of complex conjugation, note that wedging with $\tilde{\Gamma}_X^c$ is an invertible operation, since the power series for $\Gamma(1 + \frac{z}{2\pi i})$ is invertible. Thus, the complex conjugate of $w$ is computed using (5.3) as

$$w \mapsto \text{ch}^{-1}(w \wedge \tilde{\Gamma}_X^c) \mapsto \text{ch}^{-1}(\overline{w} \wedge \tilde{\Gamma}_X^c) \mapsto \overline{w} \left( \tilde{\Gamma}_X^c / \tilde{\Gamma}_X^c \right).$$

Finally, the mirror of the wedge product pairing $\langle \varphi | \psi \rangle := (-1)^{(n-1)/2} \int_Y \varphi \wedge \psi$ is the Mukai pairing $\langle v | w \rangle := \int_X v^\vee \wedge w$. Thus, the asymptotic behavior of the metric, by a computation parallel to (5.2), is seen to be

$$(2\pi i)^n \int_X \exp \left( - \sum_{\ell} H_\ell(t_\ell - \overline{t_\ell}) \right) \wedge \left( \tilde{\Gamma}_X^c / \tilde{\Gamma}_X^c \right) \approx (2\pi i)^n \int_X \exp \left( - \sum_{\ell} H_\ell(t_\ell - \overline{t_\ell}) \right) \wedge \left( \tilde{\Gamma}_X^c / \tilde{\Gamma}_X^c \right),$$

which agrees with our previous proposal.

### 6 Discussion

Using the recently-proposed two-sphere partition function correspondence \cite{two-sphere}, we determined the perturbative $\alpha'$-corrections to the Kähler potential for the Weil–Petersson metric of the quantum Kähler moduli space of Calabi–Yau $n$-folds $X$. From the associated $\mathcal{N} = (2,2)$ nonlinear $\sigma$-model perspective we derived the perturbative corrections to the marginal Kähler deformations of their target spaces. As we

\footnote{In \cite{wein2020}, the complex conjugation was implicitly assumed to come from the integer structure on cohomology; we are correcting that assumption here.}
explained, these quantum corrections appear from the requirement of a vanishing $\beta$-function for the Kähler metric $g_{ij}$ of the target space manifold $X$. To leading order — i.e., to one-loop order — this condition restricts the target space metric to the Ricci-flat Calabi–Yau metric in a given Kähler class [60]. The subleading higher-loop corrections further modify this target space metric (within its original Kähler class) [5, 37]. Thus — although not spelled out in detail here — our calculation fixes coefficients of the counterterms proposed in [4], which come from higher-loop perturbative corrections to the $\beta$-function of the Kähler metric to the Calabi–Yau target space $X$. In particular, our proposal is in accord with the four-loop correction predicted by means of mirror symmetry in [1], and it confirms the absence of a five-loop correction established in [5]. Furthermore, our proposed perturbative corrections are in agreement with the quantum Kähler potential for Calabi–Yau fourfolds conjectured in [15].

We found that the structure of the perturbative $\alpha'$-corrections discussed above is captured by interesting characteristic classes of the Calabi–Yau $n$-fold $X$. The perturbative corrections were most conveniently extracted from an additive characteristic class which we introduced, the “log Gamma” class $\Lambda_X$, which in turns is closely related to the multiplicative Gamma class $\hat{\Gamma}_X$ introduced in [16–19]. We explained how the “log Gamma” class $\Lambda_X$ is part of a natural generalization to Mukai’s modified Chern character map. Previously, the characteristic class $\hat{\Gamma}_X$ had appeared in period integrals on mirror Calabi–Yau geometries [16–17] and their generalizations [18–19], and also in the context of deformation quantization of Poisson manifolds [19, 62]. It is gratifying to see that our two-sphere partition function calculation — which is mirror-symmetric to the period integral analysis of [16] — conforms with these other approaches.

The “log Gamma” characteristic class $\Lambda_X$ also has interesting number theoretic properties due to the appearance of Riemann $\zeta$-values. In particular, the form-degrees of the terms in the characteristic class $\Lambda_X$ conform with the transcendenceality degrees of the $\zeta$-values which appear there. From a physics point of view this property indicated that the terms in $\Lambda_X$ originated from perturbative corrections of two-dimensional supersymmetric field theories [45], where both the form degree and the transcendenceality degree signal the loop order of the corresponding perturbative correction to the aforementioned $\beta$-function of the target space metric of $X$.

Finally, let us point out that the perturbative corrections to the Kähler potential of the quantum Kähler moduli space which we have determined, together with the generalization of Mukai’s modified Chern character map which we have studied, provide the necessary ingredients to define a variation of polarized Hodge structures on the closed-string topological $A$-periods in the asymptotic large volume limit [19, 58]. By deforming the ordinary cohomology ring to the $A$-model quantum cohomology ring this asymptotic variation of polarized Hodge structures canonically extends to the

\[\text{As well as quantum cohomology of Fano varieties [61].}\]
variation of polarized Hodge structures of topological A-periods beyond the asymptotic large volume limit. In this way, we can systematically derive the exponentiated sign-reversed Kähler potential for any Calabi–Yau $n$-fold — including both perturbative and non-perturbative corrections — from first principles. In particular, as proposed for Calabi–Yau threefolds in [6] and Calabi–Yau fourfolds in [15], this allows us to extract certain Gromov–Witten invariants of Calabi–Yau $n$-folds, for any $n$, from the sphere-partition function proposal [6]. We will discuss such implications of this work in detail in ref. [50].

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A Numerical evidence

In section 3 we argued that the perturbative corrections to the metric on the Kähler moduli space of two-dimensional $\mathcal{N} = (2, 2)$ nonlinear $\sigma$-models with $n$-dimensional Calabi–Yau target spaces $X$ are determined by universal polynomials $\chi_k$ in Chern classes up to degree $n$. We further argued in section 4 that these polynomials are calculated by the two-sphere partition function according to eq. (4.1).

In the derivation in section 4 we have assumed that the considered class of Calabi–Yau manifolds is sufficiently generic, i.e., that the Chern monomials are sufficiently distinct such that an unambiguous answer for the universal polynomials $\chi_k$ can be derived. By studying explicit examples we demonstrates here that up to Calabi–Yau ninefolds this assumption is indeed justified.

For this purpose it suffices to analyze the top-degree polynomial $\chi_n$ of Calabi–Yau $n$-folds, as the lower-degree polynomials $\chi_k$ (for $k < n$) have inductively already
been confirmed as the top-degree polynomial of Calabi–Yau $k$-folds. Thus, by studying explicit examples we confirm in this appendix that for Calabi–Yau $n$-folds the universal relation holds

$$
\int_X \chi_n = Z_{S^2}^{\text{pert}} |_{t=0},
$$

(A.1)

where $\chi_n$ is determined from the $\hat{\Gamma}_X^C$-class or from the $\Lambda_X$-class as discussed in section 4. We will do this by computing enough examples to ensure that $\chi_n$ is the only polynomial in Chern classes that could have given the same result as computed by the partition function $Z_{S^2}$. In this way, we explicitly verify the universal classes up through Calabi–Yau ninefolds as given in (4.17).

For a Calabi–Yau $n$-fold there are $p(n) - p(n-1)$ non-trivial Chern monomials of degree $n$, requiring at least as many examples. For $n = \{3, 4, 5, 6, 7, 8, 9\}$ the number of degree $n$ Chern monomials are $\{1, 2, 2, 4, 4, 7, 8\}$. (As an example, the Chern monomials for $n = 7$ are $c_7, c_5c_2, c_4c_3$ and $c_3c_2c_2$.) Thus, we must compute 28 examples in all.

Consider Calabi–Yau hypersurfaces in products of projective spaces $\mathbb{P}^{n+1}_\otimes = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ with $n+1 \equiv \sum n_s$. As argued in section 4, there are $p(n+1) > p(n) - p(n-1)$ such hypersurfaces, and therefore this class seems to be sufficient for our purposes. In table A.1 we present our results. For the considered Calabi-Yau manifolds at dimension $n$, the $(p(n) - p(n-1)) \times (p(n) - p(n-1))$ matrix, whose rows are the degree $n$ Chern monomials for the Calabi-Yau $n$-fold hypersurfaces in the ambient spaces $\mathbb{P}^{n+1}_\otimes$, has full rank. Therefore, if the match in equation (4.17) holds for each $n$-fold example, this full rank condition is sufficient to ensure that the universal polynomial argued for in section 3 must be $\chi_n$. We have computed both sides of eq. (A.1) and in all cases we find agreement. For the consider Calabi–Yau hypersurfaces the resulting perturbative corrections $\chi_n$ are list in table A.1.
| Toric Ambient Space $\mathbb{P}^{n+1}$ | Perturbative Correction $\chi_n$ |
|--------------------------------------|---------------------------------|
| $\mathbb{P}^1$                       | 400 $\zeta(3)$                  |
| $\mathbb{P}^5$                       | 0                               |
| $\mathbb{P}^4 \times \mathbb{P}^1$  | 0                               |
| $\mathbb{P}^6$                       | 47040 $\zeta(5)$                |
| $\mathbb{P}^5 \times \mathbb{P}^1$  | 37320 $\zeta(5)$                |
| $\mathbb{P}^7$                       | 451584 $\zeta(3)^2$             |
| $\mathbb{P}^6 \times \mathbb{P}^1$  | 357504 $\zeta(3)^2$             |
| $\mathbb{P}^5 \times \mathbb{P}^2$  | 321408 $\zeta(3)^2$             |
| $\mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 285696 $\zeta(3)^2$ |
| $\mathbb{P}^8$                       | 12299040 $\zeta(7)$             |
| $\mathbb{P}^7 \times \mathbb{P}^1$  | 9586976 $\zeta(7)$              |
| $\mathbb{P}^6 \times \mathbb{P}^2$  | 8470728 $\zeta(7)$              |
| $\mathbb{P}^6 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 7529536 $\zeta(7)$ |
| $\mathbb{P}^9$                       | 263973600 $\zeta(3)\zeta(5)$    |
| $\mathbb{P}^8 \times \mathbb{P}^1$  | 204909696 $\zeta(3)\zeta(5)$    |
| $\mathbb{P}^7 \times \mathbb{P}^2$  | 180006912 $\zeta(3)\zeta(5)$    |
| $\mathbb{P}^7 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 160006144 $\zeta(3)\zeta(5)$ |
| $\mathbb{P}^6 \times \mathbb{P}^3$  | 167713280 $\zeta(3)\zeta(5)$    |
| $\mathbb{P}^6 \times \mathbb{P}^2 \times \mathbb{P}^1$ | 141613920 $\zeta(3)\zeta(5)$ |
| $\mathbb{P}^6 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 125879040 $\zeta(3)\zeta(5)$ |
| $\mathbb{P}^{10}$                    | $\frac{3748096000}{3} \zeta(3)^4 + \frac{17291616320}{3} \zeta(9)$ |
| $\mathbb{P}^9 \times \mathbb{P}^1$  | $\frac{967032000}{3} \zeta(3)^3 + \frac{4444444440}{3} \zeta(9)$ |
| $\mathbb{P}^8 \times \mathbb{P}^2$  | $\frac{846106560}{3} \zeta(3)^3 + \frac{3874204890}{3} \zeta(9)$ |
| $\mathbb{P}^8 \times \mathbb{P}^1 \times \mathbb{P}^1$ | $\frac{752094720}{3} \zeta(3)^3 + \frac{3443737680}{3} \zeta(9)$ |
| $\mathbb{P}^7 \times \mathbb{P}^3$  | $\frac{2349916160}{3} \zeta(3)^3 + \frac{10737418240}{3} \zeta(9)$ |
| $\mathbb{P}^7 \times \mathbb{P}^2 \times \mathbb{P}^1$ | $\frac{661929984}{3} \zeta(3)^3 + \frac{3019898880}{3} \zeta(9)$ |
| $\mathbb{P}^7 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | $\frac{588382208}{3} \zeta(3)^3 + \frac{2684354560}{3} \zeta(9)$ |
| $\mathbb{P}^6 \times \mathbb{P}^4$  | $\frac{2255352400}{3} \zeta(3)^3 + \frac{10294287500}{3} \zeta(9)$ |

Table A.1: Perturbative corrections $\chi_n$ to the nonlinear $\sigma$-models with Calabi–Yau hypersurface target spaces embedded in the products of projective spaces $\mathbb{P}^{n+1}$.  

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