Interaction and propagation of waves in slotted waveguides

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Abstract. This paper addresses the application of the operator-valued function
and small-parameter methods to the analysis of the wave propagation and
interaction in a family of slotted waveguides. For narrow slots, the eigenwaves
and field distributions in the cross-sectional domains are calculated as segments of
asymptotic series in powers of the characteristic small parameter of the problems.
For larger slots, the method of infinite-matrix (summation) equations is applied.
The results show that the interaction of waves occurs in slotted structures and that
semi-analytical methods are an efficient tool to analyse this phenomenon.

Introduction

Determination of the electromagnetic wave fields in various guiding structures of complicated
cross section is one of the topical problems of modern mathematical physics [1] related, in
particular, to the development of advanced mathematical methods in electromagnetics [2].

Recently, it has become known (see, for example, [3] and [4]) that a small variation
of certain geometrical parameters leads to abrupt changes of the field behaviour in slotted
waveguide structures, development of hybrid field configurations, etc. According to recent
findings summarized in [3] and [5], the violation of stability is often caused by the intertype
interaction of waves (or oscillations). This phenomenon was called the intertype interaction.
Physically, the interactions of waves and oscillations are two different phenomena; however,
they have a common mathematical background: the presence of various critical points of the
operators specifying the dispersion equations.

In [3] and [5] a mathematical model is proposed to describe this phenomenon as a coupling
of modes occurring in waveguides perturbed by small inhomogeneities in the presence of a
double frequency degeneration and the existence of a double- (and higher-) multiplicity root
of the dispersion equation (DE). The problem is reduced [5] to the study of operator-valued
functions (OVFs), $I - A(\lambda)$, of the appropriate spectral (e.g. frequency) parameter $\lambda$, where $I$ is the identity operator and $A(\lambda)$ is a compact analytical OVF of $\lambda$. DE often has the form $F(\lambda) = \det(I - A(\lambda)) = 0$, where $F$ is the infinite determinant.

This paper focuses on the application of OVFs and the small-parameter method proposed in [6] to the analysis of the wave propagation and interaction in a family of slotted waveguides. For narrow slots, we calculate the eigenwaves and field distributions in the cross-sectional domains as segments of asymptotic series in powers of the characteristic small parameter of the problems (the relative slotwidth). For larger slots, we employ the method of infinite-matrix (summation) equations developed in [7]. The results demonstrate that the interaction of waves is inherent to slotted structures and that semi-analytical methods are an efficient tool to analyse this phenomenon.

Note that the majority of results known to us demonstrated the interaction in open structures. This paper seems to be one of the first to show (using a rigorous mathematical proof rather than numerical simulation) that the interaction may also occur in closed structures.

1. Statement of the problem

We consider a cylindrical waveguide $D$ with a homogeneous filling (the relative material constants are $\varepsilon = \mu = 1$ and $\sigma = 0$) formed by two regular cylindrical waveguides $D_1$ and $D_2$ connected through a slot (see figure 1). The cross section $\mathcal{R}$ of waveguide $D$ in the plane $z = 0$ is formed by two rectangular domains

\[
\mathcal{R}^1 = \{ r = (x, y) : 0 < x < a_1, 0 < y < b_1 \}, \\
\mathcal{R}^2 = \{ r = (x, y) : 0 < x < a_2, -b_2 < y < 0 \},
\]

with the common part of the boundary $\Gamma = \partial \mathcal{R}^1 \cap \partial \mathcal{R}^2 = \{ r : y = 0, 0 \leq x \leq \min(a_1, a_2) \}$, containing the interval $S = \{ r : y = 0, d - w < x < d + w \}$ (the slot) with the edges $\partial S = \{ d_1 = d - w; d_2 = d + w \}$.

We look for nontrivial solutions of Maxwell’s equations

\[
\text{rot } \mathbf{H} = -ik \mathbf{E}, \\
\text{rot } \mathbf{E} = ik \mathbf{H},
\]

Figure 1. Typical structure of the cross section of a rectangular slotted waveguide.
assuming that the waveguide walls are perfectly conducting, so that
\[ [nE]_\Sigma = 0, \]
(where \( k \) is the free-space wavenumber and \( n \) is normal vector to the surface \( \Sigma = \partial D \)) in the form of running (normal) waves (or eigenwaves)
\[ E, H = E, H(r)e^{i\gamma z}, \]
that correspond to certain values of parameter \( \gamma \) (the longitudinal wavenumber). These values of \( \gamma \) will be called eigenvalues of the problem or points of the spectrum \( \sigma_D \) of eigenwaves.

The determination of eigenwaves reduces [1] to the boundary eigenvalue problem for Laplace’s equation
\[ \Delta u + \lambda u = 0, \quad (x, y) \in \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2, \]
(1)
with the edge condition
\[ \frac{\partial u}{\partial n} |_{\Gamma} = 0, \]
(2)
and the conditions on the line \( S = \{ r : y = 0, d_1 < x < d_2 \} \) (the slot):
\[ \frac{\partial u}{\partial y} |_{S} \in L_{2p}(S) = \left\{ f(x) : \|f\|_{2p}^2 = \int_S |f(x)|^2 p(x) \, dx < \infty \right\}, \]
\[ p(x) = \frac{1}{\sqrt{(d_2 - x)(x - d_1)}}, \]
\[ u(x, y)|_S = u(x, 0) \in W^1_{2p}(S) = \left\{ f(x) : f \in L_{2p}(S), f' \in L_{2p-1}(S) \right\}. \]
(4)

We will refer to (1)–(4) as problem H.

According to [8], there exist isolated (real) eigenvalues \( \lambda_n \) of problem H, and \( \lambda_n \to \infty \) as \( n \to \infty \). Below, we will prove this statement independently, at least for sufficiently narrow slots, by reducing problem H to an integral equation with a logarithmic singularity of the kernel.

2. Integral equation of problem H

In order to solve boundary eigenvalue problem H and determine eigenvalues (in the case of a narrow slot), we will reduce this problem to a boundary integral equation with the integration
over the slot. To this end, we introduce the Green function of the second boundary-value problem for Helmholtz equation $\Delta u + \lambda u = 0$ in rectangle $R_i, (i = 1, 2)$:

$$G_i(\lambda; x, y, x^0, y^0) = \frac{4}{a_i b_i} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_n \delta_m \frac{\psi_n(x, a_i) \psi_n(x^0, a_i) \psi_m(y, b_i) \psi_m(y^0, b_i)}{\lambda - \lambda_{nm}^i},$$

where $\delta_0 = 0.5, \delta_n = 1, n \geq 1$, $\psi(t, d) = \cos \pi nt/d$. The function $G(\lambda; x, y, x^0, y^0)$ is meromorphic with the set of poles

$$\lambda_{nm}^i = \pi^2 \left(\frac{n^2}{a_i^2} + \frac{m^2}{b_i^2}\right), \quad i = 1, 2; \ m, n = 0, 1, \ldots;$$

that are eigenvalues of the Laplacian in rectangle $R_i (i = 1, 2)$.

Using Green’s formulae one can represent the solutions in the form of potentials

$$u(x, y) = u_i(x, y) = (-1)^{i+1} \int_S G_i(\lambda; x, y, x^0, 0) \phi_i(x^0) \, dx^0,$$

$$(x, y) \in R_i, \ \phi_i(x^0) = \frac{\partial u(x, y)}{\partial y}, \ \lambda \notin \Lambda_i = \{\lambda_{nm}^i\}_{n,m=0}^{\infty}, (i = 1, 2)$$

and reduce problem H to the homogeneous integral equation with respect to the unknown function $\varphi(x_0)$:

$$K(\lambda) \varphi \equiv \int_S K(\lambda; x, x_0) \varphi(x_0) \, dx_0 = 0, \quad x \in S,$$

where

$$K(\lambda; x, x_0) = G_1(\lambda; x, 0, x_0, 0) + G_2(\lambda; x, 0, x_0, 0).$$

In [5] it is proved that $K(\lambda) : L_{2p}(S) \rightarrow W_{2p}^2(S)$ is a Fredholm operator and a finite-meromorphic OFV with respect to $\lambda$. This implies, in particular, that, according to definitions [2], we can consider (5) as a problem on characteristic numbers for $K(\lambda)$. Using the Fredholm properties of integral operator $K(\lambda)$, we can state the equivalence of problem H and the integral equation (5) in the following sense: the set of characteristic numbers of the integral operator $K(\lambda)$ coincides with the set of eigenvalues of problem H.

### 3. Properties of operator-valued function $K(\lambda)$

Let us summarize some important properties of OFV $K(\lambda)$.

The kernel of $K(\lambda)$ has a logarithmic singularity, so that $K(\lambda)$ can be represented in the form [2]

$$K(\lambda) \phi = \Upsilon \phi + N(\lambda) \phi,$$

where

$$\Upsilon \phi = \frac{1}{\pi} \int_S \ln \frac{1}{|x - x_0|} \phi(x_0) \, dx_0,$$

$$N(\lambda) \phi = \int_S N(\lambda; x, x_0) \phi(x_0) \, dx_0,$$

$$N(\lambda; x, x_0) = N_1(\lambda; x, x_0) + g(x, x_0),$$

$$g(x, x_0) = \frac{2}{\pi} \sum_{i=1}^{2} \left[ \ln \left| \frac{2 \sin \frac{\pi}{2} (x - x_0)}{x - x_0} \right| + \ln \left| \frac{2 \sin \frac{\pi}{2} (x + x_0)}{x + x_0} \right| \right],$$

$$N_1(\lambda; x, x_0) = \sum_{i=1}^{2} \left\{ q^{(i)}_0(\lambda) + \sum_{n=1}^{\infty} \left( g^{(i)}_n(\lambda) - \frac{2}{\pi n} \right) \psi^{(i)}_n(x) \psi^{(i)}_n(x_0) \right\}.$$
Here $N$ is a meromorphic OVF with the same set of poles as $K(\lambda)$. The series $N_1(\lambda; x, x_0)$ converges uniformly with respect to $x, x_0 \in S$ and admits partial differentiation with respect to these variables. In addition,

$$\left(q_n^{(i)}(\lambda) - \frac{2}{\pi n}\right) \sim \frac{1}{n} e^{-2n},$$

so we can state that

$$N(\lambda; x, x_0) \in C^1(S \times S);$$

$$\frac{\partial^2 N(\lambda; x, x_0)}{\partial x^2} \in L_2([S \times S], p(x), p(x_0)).$$

The kernel $K(\lambda; x, x_0)$ of $K(\lambda)$ can be represented in the following form [2]

$$K(\lambda; x, x_0) = \sum_{i=1}^{2} \left(q_0^{(i)}(\lambda) + \sum_{n=1}^{\infty} q_n^{(i)}(\lambda) \psi_n^{(i)}(x) \psi_n^{(i)}(x_0)\right),$$

where

$$q_0^{(i)} = -\frac{1}{a_i} f(\lambda b_i^2), \quad q_n^{(i)}(\lambda) = \frac{2b_i}{a_i} \left(f(\lambda b_i^2) - \frac{n^2 \pi^2}{a_i^2} - \lambda\right), \quad i = 1, 2;$$

$$f(z) = \begin{cases} \tanh^{-1}\sqrt{z}, & z > 0, \\ \tan^{-1}\sqrt{-z}, & z < 0. \end{cases}$$

The kernel $K(\lambda; x, x_0)$ is a meromorphic function with the same set of poles $\Lambda = \Lambda^1 \cup \Lambda^2$ as Green functions $G_i, i = 1, 2$, have. If $\lambda \notin \Lambda$ the series (6) converges uniformly with respect to $x, x_0 \in S$, $|x - x_0| \geq \delta > 0$ and also converges in $L_{2p-1}(S)$.

Functions $q_n^{(i)}(\lambda)$ have poles at the points

$$Q_n = \{q_n^{(i)}\}_{m=0}^{\infty}, \quad q_n^{(i)} = \pi^2 \left(\frac{n^2}{a_i^2} + \frac{m^2}{b_i^2}\right), \quad i = 1, 2, m, n = 0, 1, \ldots .$$

These properties of the kernel enable one to prove [6] that for all $\lambda \notin \Lambda$ operator $K(\lambda) : L_{2p}(S) \rightarrow W^2_{2p}(S)$ is a Fredholm operator and a finite-meromorphic OVF on the complex plane $C_\Lambda$.

Let $Q_n = \{q_{nk}\}_{k=0}^{\infty}$ be a sequence of real nonnegative poles of functions $q_n(\lambda)$, which are ordered with respect to increasing values. In [6] it is also shown that OVF $K(\lambda)$ has at least one characteristic number on every interval $\Lambda_n = (q_n, q_{n+1}), \ n = 1, 2, \ldots$ between each two neighbouring poles.

### 4. Generalized DE

The main point of the mathematical model proposed in [3] and [5] is to consider equation (5) as an operator DE, called the generalized DE, and determine characteristic numbers of the OVF $K(\lambda) = K(\lambda, \bar{\eta})$ as implicit functions $\lambda(\bar{\eta})$ of the vector of nonspectral parameters $\bar{\eta} = (a_1, a_2, b_1, b_2, d, w)$; these implicit functions are called generalized dispersion curves (GDCs). It is often convenient to determine $\lambda = \lambda(\xi)$ with respect to one particular parameter $\xi \in \bar{\eta}$ (other parameters being fixed), and to analyse one particular GDC on the $(\lambda, \xi)$ plane.
for different values of another parameter \( \zeta \in \tilde{\gamma} \) or the behaviour of several GDCs on the same parameter plane in the vicinities of certain (critical) points. It turns out that sometimes GDCs exhibit very special types of behaviour that reveal the presence of different singular (e.g. saddle) points of OVF \( K(\lambda, \tilde{\gamma}) \) considered as a multi-parameter mapping. Consequently, the interaction can be simulated and explained as a phenomenon inherent to slotted structures.

Below we will consider the case when the parameter \( \xi \) is taken to be equal to the width \( a_1 \) or \( a_2 \) of one of the rectangular cross-sectional domains, and the slotwidth \( w \) is small, so that the characteristic (small) parameter of the slotted structure under study

\[
\beta = \left( \frac{1}{\pi} \ln \frac{1}{w} \right)^{-1}.
\]

In a certain domain of variation of nonspectral parameters, the sequences \( \lambda_{nm}^{(i)} = \lambda_{nm}^{(i)}(\zeta) \) (poles of OVF \( K(\lambda) \)), considered as functions of a parameter \( \zeta \), which may be chosen equal to any of the \( \tilde{\gamma} \) components except slotwidth \( w \), generate a set of curves which may be naturally called the pole curves. According to formulae (13) derived in the next section, the pole curves are the limiting set for certain GDCs \( \tilde{\lambda} = \tilde{\lambda}(\beta) \) when \( \beta \to 0 \) (i.e. the slotwidth \( w \to 0 \)). It is natural to denote these limiting GDCs by the same symbols, writing \( \tilde{\lambda}(\beta) = \tilde{\lambda}_{nm}^{(i)}(\beta) \). Using the conventional designations [1], one may also denote by \( H_{nm} = H_{nm}(\zeta) \) (for every fixed \( \zeta \)) the \( H_{nm} \) waves propagating in each rectangular waveguide with cross section \( \mathcal{R}_i \) (\( i = 1, 2 \)) that correspond to \( \lambda_{nm}^{(i)} \) (note that the latter are simply eigenvalues of the Laplacian in rectangle \( \mathcal{R}_i \)). The GDCs of eigenwaves propagating in the slotted waveguide with the cross section \( \mathcal{R} \) that correspond to eigenvalues \( \lambda_{nm}^{(i)}(\beta) \) ‘perturbed’ by the presence of a narrow slot may be then naturally denoted by the symbol \( H_{nm}^i = H_{nm}^i(\zeta) \) and called eigenwaves of type \( H_{nm}^i \) (\( m, n = 0, 1, 2, \ldots, i = 1, 2 \)).

Below, we will use these designations and omit symbol ‘ when considering \( H_{nm}^i \) eigenwaves in the slotted waveguide \( \mathcal{R} \) with a narrow slot.

5. Narrow slots

In the case of a narrow slot (\( \beta \ll 1 \)), it is natural to apply the small-parameter method proposed in [6] and to determine GDCs \( \lambda = \lambda(\beta) \) (and thus eigenvalues of problem H) explicitly by evaluating characteristic numbers of OVF \( K(\lambda) \) in the form of segments of asymptotical series in powers of \( \beta \).

Replacing variables by the formulae \( x = wt + d \) and \( x_0 = ws + d \) we change to the canonical interval of integration (−1, 1). Using the representation of the kernel of OVF (5)

\[
K(\lambda; t, s) = \frac{2}{\pi} \ln \frac{1}{|t - s|} + \frac{2}{\pi} \ln \left( \frac{1}{w} + \sum_{i=1}^{2} \frac{2}{a_i b_i} \sum_{m=0}^{\infty} \frac{\delta_m}{\lambda_{nm} - \lambda} \right) \cos \frac{\pi n(wt + d)}{a_i} \cos \frac{\pi n(ws + d)}{a_i} \\
+ \frac{4}{a_i b_i} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \delta_m \left( \frac{1}{\lambda_{nm} - \lambda} - \frac{1}{\lambda_{nm}} \right) \cos \frac{\pi n(wt + d)}{a_i} \cos \frac{\pi n(ws + d)}{a_i} \\
+ \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-\pi n b_i / a_i} \cos \frac{\pi n(wt + d)}{a_i} \cos \frac{\pi n(ws + d)}{a_i} \\
- \frac{1}{\pi} \ln \left( \frac{2 \sin \frac{\pi x}{2a_i}(x - x_0)}{\pi w} \right) - \frac{1}{\pi} \ln \left( \frac{2 \sin \frac{\pi x}{2a_i}(w(t + s) + 2d)}{\pi w} \right),
\]

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we rewrite operator $K(\lambda)$ in the form

$$K(\lambda, w)\varphi = \alpha L\varphi + \frac{\alpha}{\beta}(\varphi, 1)1 + M_0(\lambda)(\varphi, 1)1 + wR(\lambda, w)\varphi,$$

where the integral operator $L\varphi = \frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{|s-t|} \varphi(s) \, ds$, $(\varphi, 1) = \int_{-1}^{1} \varphi(s) \, ds$ denotes the inner product, $\alpha = 2$, $M_0(\lambda) = N(\lambda; 0, 0)$, and $R(\lambda, w)$ is OVF holomorphic with respect to $\lambda$ and uniformly bounded as $w \to 0$. In order to perform the semi-inversion of $K(\lambda)$ in the vicinity of a pole associated with eigenvalues of the Laplacian in rectangle $\mathcal{R}_1$, we use here the small parameter

$$\beta = \left(1 + \frac{1}{\pi} \ln \frac{a_1}{w}\right)^{-1}$$

involving the slot diameter normalized to the width of the upper domain and modify accordingly the kernel representation of the OVF (5) by adding and subtracting a constant. Let us introduce an OVF

$$L_1(\lambda, \beta)\varphi = \alpha L\varphi + \left(\frac{\alpha}{\beta} + M_0(\lambda)\right)(\varphi, 1)1.$$

The inverse $L_1^{-1}(\lambda, \beta)$ is given by the formula [6, p 50],

$$L_1^{-1}(\lambda, \beta)f = \frac{1}{\alpha} L^{-1}f - \frac{1}{\alpha} \left(\frac{\beta}{\alpha} + \frac{M_0(\lambda)}{\alpha}\right)(L^{-1}f, 1) L^{-1}1,$$

where operator $L^{-1}$ is defined in [9]

$$L^{-1}f \equiv -\frac{1}{\pi} \int_{a}^{b} \frac{\sqrt{(b-s)(s-a)f'(s)}}{\sqrt{(b-t)(t-a)(s-t)}} + \frac{C}{\sqrt{(b-t)(t-a)}}.$$

If $\beta$ is sufficiently small, then $L_1^{-1}$ can be expanded in powers of $\beta$

$$L_1^{-1}(\lambda, \beta)f = \frac{1}{\alpha} L^{-1}f - \ln 2 \frac{1 - \beta}{\pi} f + \beta^2 \ln 2 \left(\frac{M_0(\lambda)}{\alpha} + \frac{\ln 2}{\pi}\right)(L^{-1}f, 1)L^{-1}1 + O(\beta^3)$$

as $\beta \to 0$. The following statements are proved in [6] (see formula (1.103), p 33 and lemma 5, p 47).

For any fixed $\lambda$, there exists a value $\beta_0$ and a constant $C > 0$ such that, for any $\beta < \beta_0$, the estimate $\|L_1^{-1}(\lambda, \beta)\| < C$ holds.

At any regular point $\lambda$ of OVF $K(\lambda, w)$, the inverse $K^{-1}(\lambda, w)$ admits the representation in the form of the Neumann series:

$$K^{-1}(\lambda, w) = \sum_{n=0}^{\infty} (-1)^n w^n (L_1^{-1} N)^n L_1^{-1},$$

which converges in the operator norm uniformly with respect to $\lambda$ and $w$ in the vicinity of $\lambda$ and in a certain interval $(0, w_0)$ respectively.

To obtain asymptotic representations for the operators and characteristic numbers, we will use the first two terms of expansion (7):

$$K^{-1}(\lambda, w)f = L_1^{-1}f - w L_1^{-1}N L_1^{-1}f + O(w^2) \quad \text{as } w \to 0,$$
and the following simplification:

\[ K^{-1}(\lambda, w)f = L^{-1}(\lambda, w) + O(w) \]

\[ = \frac{1}{\alpha} L^{-1}f - \frac{1}{\alpha} \frac{1}{\beta} + \frac{M_0(\lambda)}{\alpha} (L^{-1}f, 1) L^{-1}1 + O(w), \quad \text{as } w \to 0. \]

For sufficiently small \( w \) the following expansion in powers of \( \beta \) can be used:

\[ K^{-1}(\lambda, \beta)f = \frac{1}{\alpha} L^{-1}f - \frac{\ln 2}{\alpha \pi} \left( 1 - \frac{\beta \ln 2}{\pi} \right) + \beta^2 \frac{\ln 2}{\pi} \left( \frac{M_0(\lambda)}{\alpha} + \ln 2 \right) (L^{-1}f, 1) 1 + O(\beta), \quad \beta \to 0. \] (8)

Referring to the operator (5) and using (6) we can write an exact representation of \( K(\lambda, w) \) in the vicinity of the chosen pole \( \lambda^{(i)}_{NM} \)

\[ K(\lambda, w)\varphi = K^i_{NM}(\lambda, w)\varphi + \frac{4}{a_i b_i} \frac{\delta_M}{\lambda_{NM} - \lambda} (\varphi, \varphi_N) \varphi_N. \] (9)

where

\[ K^i_{NM}(\lambda, w)\varphi = \alpha \Upsilon \varphi + \frac{\alpha}{\pi} \ln \frac{1}{w}(\varphi, 1) 1 + C^i_{NM}(\varphi, 1) 1 + wR(\lambda, a) \]

is holomorphic in the vicinity of \( \lambda_{\nu} \), and

\[ C^i_{NM}(\lambda) = M_0(\lambda) - \frac{4}{a_i b_i} \frac{\varphi_N(0)}{\lambda^{(i)}_{NM} - \lambda} \]

\[ = \sum_{i=1}^2 \left\{ \frac{2}{a_i b_i} \sum_{m=0} \delta_m \frac{\lambda_{m0} - \lambda}{\lambda_{nm} - \lambda} + \frac{4}{a_i b_i} \sum_{n=1}^\infty \sum_{m=0}^\infty \delta_m \left( \frac{1 - \delta_{nm}}{\lambda_{nm} - \lambda} - \frac{1}{\lambda_{nm}} \right) \cos^2 \frac{\pi nd}{a_i} \right\} \]

\[ + \frac{2 \varepsilon_i}{\pi} \sum_{n=1}^\infty \frac{e^{-\pi nb/a_i}}{\sinh(\pi nb/a_i)} \cos^2 \frac{\pi nd}{a_i} - \varepsilon_i \ln \frac{2 \sin \frac{\pi d}{a}}{a} \]

and \( \delta_{nm} \) is the Kronecker delta. With the help of (8) we can evaluate the expression for the operator \( (K^i_{NM})^{-1}(\lambda, w) \)

\[ (K^i_{NM})^{-1}(\lambda, w)f = L^{-1}f + O(w) \]

\[ = \frac{L^{-1}f}{\alpha} - \frac{\ln 2}{\alpha} \left( 1 - \beta \frac{\ln 2}{\pi} \right) (C^i_{NM}(\lambda) + \ln 2) (L^{-1}f, 1) L^{-1}1 + O(\beta^3). \] (10)

Operator (10) can be applied to the calculation of characteristic numbers of OVF (5).

According to [2] the number \( \lambda \) is the characteristic number of the OVF \( K(\lambda, w) \) if there exists the function \( \varphi \neq 0 \), satisfying the homogeneous equation

\[ K^i_{NM}(\lambda, w)\varphi + \frac{4}{a_i b_i} \frac{\delta_M}{\lambda_{NM} - \lambda} (\varphi, \varphi_N) \varphi_N = 0. \]

For sufficiently small \( \beta \) operator \( K^i_{NM}(\lambda, w) \) is invertible in the vicinity of \( \lambda_{NM}^i \), so the number \( \lambda \) is the characteristic number of \( K(\lambda, w) \) if it satisfies the equation

\[ 1 + \frac{4}{a_i b_i} \frac{\delta_M}{\lambda_{NM} - \lambda} ((K^i_{NM})^{-1} \varphi_N, \varphi_N) = 0. \] (11)
For functions $\phi_N$ we have
$$
\varphi_N^i(wt) = c^i_N + O(w), \quad \text{as } w \to 0,
$$
where $c^i_N = \cos(\pi N d/a_i)$. Then the following approximate representation is valid:
$$
(K_{NM}^{-1}(\lambda, w)\varphi_N, \varphi_N^i) = |c^i_N|^2(L_1^{-1}(\lambda, \beta)1, 1) + O(w) \quad \text{as } w \to 0.
$$
The values $L^{-1}$ and $(L^{-1}, 1)$ can be determined explicitly:
$$
L^{-1}(\lambda, \beta)1 = \frac{1}{\ln 2} \frac{1}{\sqrt{1-t^2}}, \quad (L^{-1}(\lambda, \beta)1, 1) = \frac{\pi}{\ln 2}.
$$
Then,
$$
(L^{-1}(\lambda, \beta)1, 1) = \frac{\beta}{\alpha} + O(\beta^2) \quad \text{as } \beta \to 0,
$$
and, therefore,
$$
(K_{NM}^{-1}(\lambda, w)\varphi_N, \varphi_N^i) = \beta \frac{|c^i_N|^2}{\alpha} + O(\beta^2) = O(\beta) \quad \text{as } \beta \to 0. \quad (12)
$$
Substituting (12) into equation (11) we evaluate analytical formulae for the determination of characteristic numbers
$$
\lambda_{NM}^{i\theta} = \lambda_{NM}^{(i)} + \beta \frac{4c^i_N}{a_ib_i} \frac{\delta_M}{\alpha} - \beta^2 \frac{4c^i_N}{a_ib_i} \frac{\delta_M}{\alpha} \left( \frac{C_{NM}^i(\lambda_{NM})}{\alpha} + \frac{\ln 2}{\pi} \right) + O(\beta^3). \quad (13)
$$
Below we will use these segments of asymptotic series of order $n_a = 3$ as initial data to calculate propagation constants of normal waves in the case of exponentially narrow slots with $\beta < \beta_0$, $\beta_0 \approx 0.1$, which gives approximately the range of applicability of (13). Note, however, that using the semi-inversion (10) and (11) and taking more terms in asymptotic series (10), one can theoretically obtain an arbitrary number $n_a$ of terms in (13), thus extending the range of applicability of the asymptotic series.

6. Summation equations

In order to calculate approximations to characteristic numbers of the Fredholm integral OVF $K(\lambda)$ for larger slots we use the method of summation equations [7].

Consider on the interval $[-1, 1]$ an auxiliary class $\Phi$ of functions $W_{2p}^1(-1, 1)$ (this weighted Sobolev space is defined in (4) with $S = (-1, 1)$ and $p(x) = 1/\sqrt{1-x^2}$) admitting representation in the form of the Fourier–Chebyshev series
$$
\varphi(x) = \frac{\xi_0}{\sqrt{2}} T_0(x) + \sum_{n=1}^{\infty} \xi_n T_n(x), \quad -1 \leq x \leq 1,
$$
where $T_n(s) = \cos(n \cos^{-1} s)$ are Chebyshev polynomials of the first kind and $\xi = (\xi_0, \xi_1, \ldots, \xi_n, \ldots) \in h_2 = \{\xi : \sum_{n=0}^{\infty} |\xi_n|^2 n^2 < \infty\}$.

In order to correctly reduce an integral equation to an infinite system, we will use the following statement proved in [2]: spaces $W_{2p}^1(-1, 1)$ and $\Phi$ are isomorphic.

Separating explicitly the weight function $p(x) = 1/\sqrt{1-x^2}$, we can rewrite the equation (5) for the new unknown function $\psi \in W_{2p}^1(-1, 1)$
$$
K(\lambda, \eta)\psi \equiv c_0 L\psi + N(\lambda, \eta)\psi = 0, \quad (14)
$$
where

\[ L\psi = \frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{|t-s|} \frac{\psi(s)}{\sqrt{1-s^2}} \, ds, \]

\[ \Phi_N(t) = N(\lambda, \tilde{\eta})\psi = \frac{1}{\pi} \int_{-1}^{1} N(\lambda, \tilde{\eta}; t, s) \frac{\psi(s)}{\sqrt{1-s^2}} \, ds. \]

Here \( \tilde{\eta} = (a_1, a_2, b_1, b_2, d, \omega) \) is the vector of nonspectral parameters.

**Summation operators (SOs)** act in the Hilbert space of infinite number sequences

\[ h_p = \left\{ \{a_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |a_n|^2 n^p < \infty \right\} \]

and are defined by their infinite matrices introduced below.

The SO of the problem \( H \) can be evaluated by expanding the unknown function and the functions entering the kernel of (5) in the Fourier–Chebyshev series. As a result, we reduce (5) to an infinite equation system

\[ K(\lambda)\xi = L\xi + A(\lambda)\xi = 0, \quad \xi = (\xi_0, \xi_1, \ldots, \xi_n, \ldots) \in h_2, \]

where

\[ L = \begin{pmatrix}
\ln 2 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 1 & 0 & \cdots & 0 & \cdots \\
0 & 0 & 1/2 & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1/n & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \]

\[ A(\lambda) = \|a_{nj}(\lambda)\|_{n,j=1}^{\infty}; \quad a_{nj} = \frac{\varepsilon_{nj}}{\pi^2} \int_{\Pi} N(\lambda, s, t) \frac{T_n(t)}{\sqrt{1-t^2}} \frac{T_j(s)}{\sqrt{1-s^2}} \, ds \, dt, \]

\[ \varepsilon_{nj} = \begin{cases} 
1, & n = j = 0; \\
2, & n \geq 1; \quad j \geq 1; \quad \Pi = [-1, 1] \times [-1, 1]; \\
\sqrt{2}, & n j = 0, \quad n \neq j.
\end{cases} \]

Here \( K(\lambda) : h_2 \to h_4 \) is a holomorphic Fredholm operator.

Let us denote by \( \rho(K) = \{ \lambda : \exists K^{-1}(\lambda) : h_4 \to h_2 \} \) the resolvent set and by \( \sigma(K) = C(\lambda) \setminus \rho(K) \) the spectrum (set of characteristic numbers) of the summation OVF \( K(\lambda) \).

Analysing the rate of convergence of matrix elements we can prove [2] the existence of the infinite determinant. Thus the exact characteristic numbers can be determined as roots of the equation

\[ F(\lambda) = \det K(\lambda) = 0. \]

Using the theory of discrete convergence developed in [10] and estimates (4), one can prove that for every characteristic number \( \lambda_0 \) there exists a sequence of approximate characteristic numbers \( \lambda_0^N \) of finite-dimensional OVFs \( K_N(\lambda, \mu) \) (the truncated SO) such that \( \lambda_0^N \to \lambda_0, N \to \infty \). Hence we can determine the approximate characteristic numbers as roots of the equation

\[ F_N(\lambda) = \det K_N(\lambda) = 0. \]

We apply this method to calculate characteristic numbers of OVF \( K(\lambda) \) and then the propagation constants of normal waves, in the case of larger slots, using (13) as initial data.

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We calculated longitudinal wavenumbers $\gamma$ as implicit functions $\gamma = \gamma(\xi)$, $\xi \in \vec{\eta}$ (the vector $\vec{\eta}$ of nonspectral parameters is defined in section 4). The following parameters were taken: $a_1 = 8.0$, $b_1 = 7.0$, $a_2 = 2.5$, $b_2 = 2.7$, $d = 0.1$, and $w = 0.001$. We mainly considered the waves of types $H^1_{01}$, $H^1_{10}$, and $H^1_{20}$, denoting by $\gamma_{nm}^i$ the longitudinal wavenumber of the $H^i_{nm}$ wave.

Let us first analyse the pole curves $\gamma^* = \gamma_{nm}^*(\xi)$ which are actually the (unperturbed) dispersion curves (DCs) of the waves in the waveguide formed by the single upper domain $D_1$ without the slot. Figure 2 shows the dependences of $\gamma^*$ on the width $a_1$ of the upper rectangular domain $R_1$. Figure 3 displays the corresponding perturbed DCs for waveguide $D$ with a slot. One observes the interaction of the waves under study in the vicinity of certain points. In particular, points $a_1$ in figure 3 that correspond to the points of intersection of curves in figure 2 are the critical points (in a certain sense) of OVF $K(\lambda)$. A detailed consideration of such points and an explanation of the interaction phenomena associated with them is performed in the next section.

Figures 4 and 5 display the dependences on the slotwidth (parameter $w$). The DC behaviour agrees with analytical formulae (13) in the range of their applicability.
8. Interaction of waves

Set $\zeta = a_1$ (where $a_1$ is the width of the upper rectangular domain) and choose two arbitrary types of waves $H_{N_1 M_1}^{i_1}$ and $H_{N_2 M_2}^{i_2}$. Denote the triple of indices by $\nu = (i_1, N_1, M_1)$ and $\mu = (i_2, N_2, M_2)$ and the corresponding pole curves by $\lambda_\nu(\zeta)$ and $\lambda_\mu(\zeta)$. At a certain point $P^* = (\lambda^*, \zeta^*)$ on the $(\lambda, \zeta)$-plane two pole curves may intersect, i.e. $\lambda_\nu(\zeta^*) = \lambda_\mu(\zeta^*)$. In the vicinity of such a point, which will be called the degeneration point, the interaction of waves may occur. Note in this way that the set $Q_n$ introduced in section 3 is a multi-parameter family of points $\{q_{i m}^{(i)}\}_{m=0}^{\infty}$ ($i = 1, 2$) that can merge at different combinations of parameters. Here we analyse only one particular situation which leads to the mode interaction: two points of the set $Q_n$ of the same index $i = 1$ merge at a certain value $a_1 = \zeta^*$ when $a_1$ is varied and other parameters are fixed. The character of a degeneration point is governed by properties of the OVF $K(\lambda) = K(\lambda, \zeta)$ in the vicinity of this point. Depending on the parameter values, this
Figure 6. The saddle point of $F(\lambda, a)$.

point may be, in particular, a Morse critical point. In order to determine the character (type) of a degeneration point we will use below analytical representations of $K(\lambda, \zeta)$ (in this paper, we apply the form (9)) and then illustrate the results numerically. It should be noted that various local effects, such as intertype interaction and/or mode coupling, are connected with the presence of critical (in particular, degeneration) points of the operator of the problem, and Morse critical points constitute generally a subset of all critical points.

In order to study the interaction of waves in the vicinity of the degeneration point $P^*$, we analyse the GDC behaviour in the $(\lambda, \zeta)$ plane. To this end, take two poles $\lambda_\nu$ and $\lambda_\mu$, 

$$\lambda_\nu = \pi^2 \left( \frac{n^2_\nu a^2_\nu + m^2_\nu}{b^2_\nu} \right), \quad \lambda_\mu = \pi^2 \left( \frac{n^2_\mu a^2_\mu + m^2_\mu}{b^2_\mu} \right),$$

and use representation (9) to write OVF $K(\lambda)$ in the form with two separated poles

$$K(\lambda) \phi = K_{\nu\mu}(\lambda) \phi + \frac{m_\nu}{\lambda_\nu - \lambda}(\phi, \phi_\nu)\phi_\nu + \frac{m_\mu}{\lambda_\mu - \lambda}(\phi, \phi_\mu)\phi_\mu,$$

where $K_{\nu\mu}(\lambda)$ is holomorphic in the vicinity of $\lambda_\nu$ and $\lambda_\mu$. Our aim is to consider the behaviour of meromorphic OVF $K_{\nu\mu}(\lambda)$ when these two poles merge. To this end, we will perform semi-inversion of $K(\lambda)$. We rewrite the equation $K(\lambda) \phi = 0$ in the form

$$\phi + \frac{m_\nu}{\lambda_\nu - \lambda}(\phi, \phi_\nu)K_{\nu\mu}(\lambda)^{-1}\phi_\nu + \frac{m_\mu}{\lambda_\mu - \lambda}(\phi, \phi_\mu)K_{\nu\mu}(\lambda)^{-1}\phi_\mu = 0. \quad (15)$$

Multiplying by $\phi_\nu$ and then by $\phi_\mu$ and integrating, we evaluate the inner product $(\phi, \phi_\nu)$ and finally obtain an equation equivalent to (15) (under the assumption that $(\phi, \phi_\nu) \neq 0$),

$$1 + \frac{m_\nu}{\lambda_\nu - \lambda}(K_{\nu\mu}(\lambda)^{-1}\phi_\nu, \phi_\nu) + \frac{m_\mu}{\lambda_\mu - \lambda}(\phi_\nu, \phi_\mu)K_{\nu\mu}(\lambda)^{-1}\phi_\mu, \phi_\mu) = 0.$$

Operator $K(\lambda)$ is noninvertible at the points $\lambda$ which are roots of the equation

$$(\lambda - \lambda_\nu)(\lambda_\mu - \lambda + m_\mu(K_{\nu\mu}(\lambda)^{-1}\phi_\mu, \phi_\mu))) = 0.$$

Using the estimate (12), one can represent the latter as

$$\lambda^2 - \lambda(\lambda_\nu + \lambda_\mu) + \lambda_\nu\lambda_\mu + O(\beta) = 0. \quad (16)$$

Considering the left-hand side in (16) as a function $F(\lambda, a)$ of two variables, we can represent $F(\lambda, a)$ as a canonical quadratic form

$$F(\lambda, a) = A_1\tilde{\lambda}^2 + A_2\tilde{a}^2 + O(\beta).$$
**Figure 7.** Field distribution before the interaction, $a_1 = 6.0$. (a) $H_{01}^1$, $\gamma = 1$, (b) $H_{10}^1$, $\gamma = 0.962$.

**Figure 8.** Field distribution in the left vicinity of the degeneration point, $\gamma_{10} \approx \gamma_{01} \approx 0.979$. (a) $H_{01}^1$, $a_1 = 7.0001$, (b) $H_{10}^1$, $a_1 = 6.999$.

In the case of $H_{10}^1$ and $H_{01}^1$ waves

$$\tilde{a} = a - a^*, \quad \tilde{\lambda} = \lambda - \lambda^* + \frac{A_2}{2A_1}(a - a^*), \quad A_1 = 1, \quad A_2 = -\frac{\pi^2}{(a^*)^6}.$$  

This canonical quadratic form exhibits a characteristic behaviour in the vicinity of the degeneration point which simulates a typical case of interaction of $H_{10}^1$ and $H_{01}^1$ waves.

The saddle-point behaviour of $F(\lambda, a)$ corresponding to the interaction of the $H_{10}^1$ and $H_{01}^1$ waves presented in figure 3 is shown in figure 6.
Figure 9. Field distribution in the right vicinity of the degeneration point, \( \gamma_{10} \approx \gamma_{01} \approx 0.98 \). (a) \( H_{01}, a_1 = 7.001 \), (b) \( H_{10}, a_1 = 6.9999 \).

Figure 10. Field distribution after the interaction, \( a_1 = 8.0 \). (a) \( H_{01}, \gamma = 0.988 \), (b) \( H_{01}, \gamma = 1 \).

Figures 7–10 illustrate the behaviour of the electromagnetic field in the upper rectangular domain \( \mathcal{R}^1 \) in the vicinity of the critical point \( a_1 = 7.0 \). The fixed parameters of the waveguide are \( a_2 = 2.5, b_1 = 7.0, b_2 = 2.7, d = 0.1, w = 0.001 \). Figure 7 shows the \( H_{01} \) and \( H_{10} \) field distributions ‘before’ the interaction at the point \( a_1 = 6.0 \) (i.e. when parameter \( a_1 \) is a little smaller than its critical value). Figure 10 shows the field distribution ‘after’ the interaction at the point \( a_1 = 8.0 \) (when \( a_1 \) is a little greater than its critical value). Figures 8 and 9 demonstrate the dynamic of interaction of the \( H_{10} \) and \( H_{01} \) waves. One can see that a slight increase in \( a_1 \) leads to essential changes in the field structure: close to the degeneration point one observes a setup of the hybrid \( H_{01 \rightarrow 10} \) and \( H_{10 \rightarrow 01} \) waves.
9. Conclusion

We have considered the propagation and intertype interaction of waves in a family of slotted waveguides of complicated cross section using a mathematical model proposed in [3] and [5]. We have simulated the interaction numerically and analytically, describing this phenomenon as coupling of modes in a waveguide perturbed by a narrow slot. We have shown that the interaction of waves is inherent to this slotted structure. In fact, the operator of the problem, considered as a meromorphic function of two (complex) variables, has specific critical points at which its poles coincide, and the intertype interaction of waves occurs in the vicinity of such points. There are other critical points and, consequently, other types of interaction that take place in slotted waveguides. These cases can also be studied using the approach set forth in the paper.

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