Dilaton Dynamics in (A)dS × S^5

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We investigate a stabilization of extra dimensions in a ten dimensional Kaluza-Klein theory and IIB supergravity. We assume (A)dS_5 × S^5 compactification, and calculate quantum effects to find an effective potential for the radius of internal space. The effective potential has a minimum, and if the universe is created on the top of the potential hill, the universe evolves from dS_5 to AdS_5 after exponential expansion. The internal space S^5 stays to be small and its radius becomes constant.

Our model in IIB supergravity contains the 4-form gauge field with classically vacuum expectation value, which is role of ten dimensional cosmological constant. If the universe evolves into AdS, the five dimensional Randall & Sundrum setup with stabilized dilaton is obtained from the type IIB supergravity model.

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I. INTRODUCTION

Randall & Sundrum (RS) brane world model[1] has been investigated by many authors in the cosmological and gravitational points of view. This model shows several interesting properties. The hierarchy problem may be solved by the warp factor. Or the gravity can be confined in non-compact spacetime. However, a higher dimensional realistic model including RS setup is not so far known. The most plausible candidate is the AdS_5 × S^5 compactification of the ten dimensional type IIB supergravity theory[2]. The scale of S^5, which is expected to be much smaller than that of AdS_5, is determined by the dilaton arising from a spontaneous compactification and is generally dynamical variable in the cosmological point of view. The dilaton in type IIB supergravity model has not ever been discussed as a dynamical variable but assumed to be a constant parameter. In order to discuss whether such a background is realized in the universe, we have to derive an effective potential for the dilaton, and analyze its stability. Unfortunately, in a pure gravitational system without quantum effects, there is no solution to stabilize a dilaton.

When we discuss dynamical evolution of the universe just after compactification, an inflationary expansion is required in our external spacetime. In RS setup, in which we assume the effective five-dimensional spacetime with a negative cosmological constant, the bulk spacetime may be explained naturally if we have inflation in five-dimensional external spacetime. However, it have well known that de Sitter supergravity can not arise from simple compactification of supergravity, string or M-theory[2]. It arises from nonstandard way. For example, the R symmetry group corresponding to conserved charge is not well defined and typically non-compact in de Sitter spacetime. Moreover the signature of kinetic term for RR scalar field in de Sitter spacetime is negative[2]. It is far from a realistic model in a supergravity theory. In cosmology, however, we would not need to assume a spacetime supersymmetry. Then some quantum effects will become important in the dynamics of the universe.

The purpose of this paper is to investigate a stabilization mechanism of the internal space via quantum effects in the ten dimensional Kaluza-Klein theory and IIB supergravity model. We assume that the vacuum ten dimensional spacetime is compactified into the direct product of five dimensional (anti-)de Sitter spacetime and compact five dimensional sphere, i.e (A)dS_5 × S^5. We consider the quantum fluctuations associated with several matter fields in order to stabilize the scale of internal space S^5. Many works suggest that quantum correction of higher dimensional matter field might provide a physical mechanism which is capable of accounting for extreme smallness of the extra dimensions.

The low energy effective action in our model is obtained by the integration of internal space, which is often called dimensional reduction. After the dimensional reduction, the dilaton couples the matter fields in external spacetime. The quantum fluctuations around a classical solution are computed in the form of quantum effective
potential as a function of dilaton. Then the quantum correction of the matter field naturally contributes to the dilaton potential. Since the quantum correction is dominant effect at the small scale of internal space, this correction at nearly Planck scale is expected to be very important.

This paper is organized as follows. In II we will calculate the 1-loop quantum correction and investigate its property for several fields in (A)dS$_5 \times S^5$. We will apply our approach to more realistic model, i.e. type IIB supergravity in dS$_5 \times S^5$ in [III]. Conclusion follows in [IV]. In Appendix, we also present the zeta functions for the case of AdS$_5 \times S^5$ compactification.

II. DYNAMICS OF DILATON IN (A)dS$_5 \times S^5$

First we discuss the dynamics of extra dimension in pure gravity system. We consider the ten dimensional Einstein-Hilbert action with a cosmological constant:

$$I_{EH} = \frac{1}{2\kappa^2} \int d^{10}X \sqrt{-g}(\bar{R} - 2\Lambda), \quad (1)$$

where $\kappa$ is a positive constant, $\bar{R}$ is the ten dimensional Ricci scalar, and $\Lambda$ is the cosmological constant. The vacuum state is assumed to be a five dimensional de Sitter space (dS$_5$) with a small extra sphere ($S^5$). Our ansatz for the metric is the following:

$$\bar{g}_{MN}dx^Mdx^N = \left(\frac{b}{b_0}\right)^{-10/3} g_{\mu\nu}dx^\mu dx^\nu + b^2 \Omega_i^0 dy^i dy^j, \quad (2)$$

where $g_{\mu\nu}$ is the metric of a five dimensional de Sitter spacetime, $b$ is the scale of a five dimensional sphere (i.e. a radius of $S^5$), $b_0$ is an initial value of $b$, and $\Omega_i^0 dy^i dy^j$ is the line element of a unit five dimensional sphere. $g_{\mu\nu}$ and $b$ depend only on the 5-dimensional coordinate $\{x^i\} (i = 0, 1, 2, \cdots)$. According to the ansatz (2), we truncate our model to a five dimensional effective spacetime, which is a real function depending only on the five dimensional coordinate $\{x^i\}$ and $\sigma$. According to the ansatz (2), we truncate our model to a five dimensional effective spacetime, which is a real function depending only on the five dimensional coordinate $\{x^i\}$. In order to evaluate the quantum correction in curved spacetime, here we adopt the path integral to compute the dilaton effective potential. Any divergence appeared in calculation must be removed by regularization technique. This paper uses the zeta function regularization, which was developed for performing the path integral in curved spacetime[4]. To calculate the quantum correction, we consider the 1-loop quantum correction for several matter fields. In the following, we review how it leads to Gaussian functional integrals, which can be expressed as functional determinants. In order to evaluate the functional integrals, we introduce the generalized zeta function which is the sum of the operator eigenvalues. We adopt this method to determine the 1-loop effective potential in dS$_5$ spacetime.

(A) Massless scalar field

First, we consider the ten dimensional massless scalar field:

$$I_S = -\frac{1}{2} \int d^{10}X \sqrt{-g} \bar{g}^{MN} \partial_M \tilde{\phi} \partial_N \tilde{\phi}. \quad (6)$$

The ten dimensional line element is assumed to be given by Eq. (4). To derive a five dimensional effective action, it is convenient to expand the field in terms of harmonics on the five dimensional sphere:

$$\tilde{\phi} = b^{-5/2}_0 \sum_{l,m} \phi_{l,m}(x^\mu) Y^{(5)}_{lm}(y^i), \quad (7)$$

where $Y^{(5)}_{lm}$ are real harmonics on the 5-sphere, which satisfy

$$\frac{1}{\sqrt{\Omega^{(5)}}} \partial_i \left( \sqrt{\Omega^{(5)}} \Omega_i^{(5)ij} \partial_j Y^{(5)}_{lm} \right) + (l + 4) Y^{(5)}_{lm} = 0. \quad (8)$$

$$\int d^5y \sqrt{\Omega^{(5)}} Y^{(5)}_{lm} Y^{(5)}_{lm'} = \delta_{ll'} \delta_{mm'}. \quad (9)$$

$l = 1, 2, \cdots$; and $m$ denote a set of four numbers, which is required in order for a set of all $Y^{(5)}_{lm}$ to be a complete set of $L^2$ functions on the 5-sphere, and $\phi_{l,m}(x^\mu)$ is a real function depending only on the five dimensional coordinate $\{x^\mu\}$. 

Next we consider the quantum matter fluctuations as an origin of an energy momentum tensor in dS$_5 \times S^5$ spacetime. The quantum correction arising from matter field is very important to stabilize the scale of extra dimension. If this correction does not exist in our model, it is impossible of dilaton to stabilize. Then the extra dimension finally collapses to singularity or expands forever. We have so far a lot of work on calculation of quantum correction in curved spacetime. Here we adopt the path integral to compute the dilaton effective potential. Any divergence appeared in calculation must be removed by regularization technique. This paper uses the zeta function regularization, which was developed for performing the path integral in curved spacetime[4]. To calculate the quantum correction, we consider the 1-loop quantum correction for several matter fields. In the following, we review how it leads to Gaussian functional integrals, which can be expressed as functional determinants. In order to evaluate the functional integrals, we introduce the generalized zeta function which is the sum of the operator eigenvalues. We adopt this method to determine the 1-loop effective potential in dS$_5$ spacetime.
By the substituting the expansion (7) into the action (4), the five dimensional effective action is given by

$$I_S = -\frac{1}{2} \sum_{l,m} \int d^5x \sqrt{-g} \left[ g^{\mu\nu} \partial_{\mu} \phi_{lm} \partial_{\nu} \phi_{lm} + M_\phi^2 \phi_{lm}^2 \right],$$

(10)

where the mass $M_\phi^2$ of scalar field is given by

$$M_\phi^2 = \frac{l(l+4)}{b_0^2} e^{-\frac{16 \kappa \sigma}{15}}.$$

(11)

Now we compute the quantum correction of the scalar field in 1-loop level. The calculation of the effective potential is carried out using path integral method. The fields are split into a classical part $\phi_{lm,c}$ and quantum part $\delta \phi_{lm}$. The action is then expanded in the quantum fields around arbitrary classical background field. We expand the fields to second order to calculate all 1-loop diagrams with any number of lines of external fields.

In the path integral approach to quantum field theory, the amplitude is given by an expression

$$Z = \int \mathcal{D}[\phi_{lm}] \exp \left( i I_I[\sigma, \phi_{lm}] \right),$$

(12)

where $\mathcal{D}[\phi_{lm}]$ is a measure on the functional space of scalar fields, and $I_I[\sigma, \phi_{lm}]$ is the total action.

The action is expanded in a neighborhood of these classical background fields as follows:

$$I_I[\sigma, \phi_{lm}] = I[\sigma, \phi_{lm,c}] + I[\sigma, \delta \phi_{lm}] + O \left( (\delta \phi)^2 \right),$$

(13)

where $\phi_{lm} = \phi_{lm,c} + \delta \phi_{lm}$. The action $I_I[\sigma, \phi_{lm}]$ is quadratic in $\delta \phi_{lm}$. The linear terms of $\delta \phi_{lm}$ has disappeared due to the classical equations of motion. We neglect all higher order terms than quadratic one in the 1-loop approximation. Then, the expression becomes

$$\ln Z = i I_I[\sigma, \phi_{lm,c}] + \ln \left\{ \int \mathcal{D}[\delta \phi_{lm}] \exp \left( i I_I[\sigma, \delta \phi_{lm}] \right) \right\}.$$  

(14)

We note that the integral is ill-defined because the operators in Eq. (14) are unbounded from below in the $dS_5$ spacetime with Lorentz signature. We have to perform a Wick rotation in order to redefine it and rewrite it in the Euclidean form. We then obtain the expression

$$\ln Z = -I_E[\sigma, \phi_{lm,c}] + \ln \left\{ \int \mathcal{D}[\delta \phi_{lm}] \exp \left( -I_E[\sigma, \delta \phi_{lm}] \right) \right\},$$

(15)

where $I_E$ is the Euclidean action expressed by

$$I_E[\sigma, \phi_{lm}] = \frac{1}{2} \sum_{l,m} \int d^5x \sqrt{-g} \left( \partial_{\mu} \phi_{lm} \partial^{\mu} \phi_{lm} + M_\phi^2 \phi_{lm}^2 \right).$$

(16)

Using the assumption $\phi_{lm} = \phi_{lm,c} + \delta \phi_{lm}$, we can integrate the kinetic term in the action by parts, resulting in

$$I_E[\sigma, \delta \phi_{lm}] = \frac{1}{2} \int d^5x \sqrt{-g} \delta \phi_{lm} \left\{ -\nabla^2_{(5)} + M_\phi^2 \right\} \delta \phi_{lm},$$

(17)

where $\nabla^2_{(5)}$ denotes the Laplacian in the five dimensional de Sitter spacetime.

The effective potential due to quantum correction ($V_{qc}$) is defined by the relation

$$\exp \left( -\int d^5x V_{qc} \right) = \int \mathcal{D}[\phi_{lm}] \exp \left( -I_E[\sigma, \delta \phi_{lm}] \right) = \left\{ \text{det} \mu^{-2} \left( \nabla^2_{(5)} - M_\phi^2 \right) \right\}^{-\frac{1}{2^2}},$$

(18)

where $\mu$ is a normalization constant with mass dimension. To compute the effective potential to 1-loop, we define it by

$$Z = \exp \left( -\int d^5x V_{eff} \right) = \exp \left( -\Omega_{vol} V_{eff} \right),$$

(19)

where $\Omega_{vol}$ is the volume of five dimensional de Sitter spacetime. Using the Eqs. (15), (18), (19), we find that 1-loop effective potential is

$$V_{eff}(\sigma) = U_0(\sigma) + V_{qc}(\sigma) = U_0(\sigma) + \frac{1}{2\Omega_{vol}} \ln \text{det} \left\{ \mu^{-2} \left( \nabla^2_{(5)} - M_\phi^2 \right) \right\}.$$

(20)

We shall evaluate the functional determinant on a background manifold in $dS_5$. We apply the standard technique of zeta function regularization. The functional determinants in terms of the generalized zeta function is the sum of operator eigenvalue

$$\zeta_{\phi}(s) = \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \frac{2(l+2)!}{4^l l!} d_{\phi}(l') \left\{ \lambda_{\phi}(l') + \frac{l(l+1)}{2b_0^2} e^{-\frac{16 \kappa \sigma}{15}} \right\}^{-s},$$

(21)

where $a$ is the scale of $dS_5$, and $\lambda_{\phi}(l')$ is the eigenvalue of scalar field $\phi$ on $dS_5$, and $d_{\phi}(l')$ is its degeneracy, respectively. This expansion is well defined and converge for $Re(s) > 5$. Using this function, the effective potential (20) is written by

$$V_{eff}(\sigma) = U_0(\sigma) - \frac{1}{2\Omega_{vol}} \left\{ \zeta_{\phi}'(0) + \zeta_{\phi}(0) \ln(a^2) \right\},$$

(22)

where in order to get the second term, we have used the relation

$$\text{det} (\mu M) = \mu^{C(0)} \text{det} M.$$  

(23)

Our task is now to calculate $\zeta_{\phi}(s)$ and to analytically continue to $s = 0$ to evaluate $\zeta_{\phi}(0)$ and $\zeta_{\phi}'(0)$. Giving eigenvalue of operator in de Sitter spacetime, the zeta function $\zeta_{\phi}$ is evaluated. The $dS_5$ spacetime is a five dimensional hyperboloid with a constant curvature and has a unique Euclidean section $S^5$ with a radius $a$. The degeneracy $d_{\phi}(l)$ and the eigenvalue $\lambda_{\phi}(l)$ of massless scalar
field in $dS_5$ spacetime are well known because this spacetime equal to $S^5$ in the Euclidean section. These are given by

$$d\phi(l') = l'(l' + 1), \quad \lambda_{\phi}(l') = \frac{2(l'(l' + 1))!}{4l!(l')!}. \quad (24)$$

If the condition of $a \gg b$ is satisfied, we easily calculate the value of $\zeta_{\phi}(0)$ and $\zeta_{\phi}'(0)$. Then the effective potential is given by

$$V_{\text{eff}} = \frac{1}{\kappa^2} e^{-\frac{4\phi}{\kappa_\sigma}} - \frac{10}{\kappa^2 b_0^2} e^{-\frac{4\phi_\sigma}{\kappa_\sigma}} + \frac{1}{b_0} e^{-\frac{4\phi_{\sigma}}{\kappa_\sigma}} \times \left[-\ln(\mu^2 a^2) + C_{\phi} \ln \left(\frac{a}{b_0}\right)^2 e^{-\frac{4\phi_{\sigma}}{\kappa_\sigma}}\right], \quad (25)$$

where $C_{\phi}$ is given by

$$C_{\phi} = \frac{15}{16\pi^2} \zeta_{\phi}'(0) e^{\frac{4\phi_{\sigma}}{\kappa_\sigma}} = \frac{25}{8192\pi}. \quad (26)$$

(B) U(1) gauge field

Next we compute the quantum correction of $U(1)$ gauge field $A_M$ described the action;

$$S_{U(1)} = \int d^5x \sqrt{-g} F_{MN} F^{MN}, \quad (27)$$

where $F_{MN} = \nabla_M A_N - \nabla_N A_M$. In order to perform the dimension reduction for the $U(1)$ field action in $dS_5 \otimes S^5$ spacetime, it is convenient to expand it by the vector harmonics on the $S^5$ as:

$$\tilde{A}_M dX^M = b_0^{-\frac{2}{7}} \sum_{l, m} \left[A^{(5)}_{\mu lm} Y^{(5)}_{lm} dx^\mu + \left(A^{(5)}_{(T) lm} Y^{(5)}_{(T) lm}\right)_i \right] dy^i, \quad (28)$$

where $A^{(5)}_{\mu lm}$, $A^{(5)}_{(T) lm}$ and $A^{(5)}_{(L) lm}$ depend only on the five dimensional coordinate $x^\mu$, $Y^{(5)}_{lm}$ and $V^{(5)}_{(T) lm}$, $V^{(5)}_{(L) lm}$ are the scalar harmonics, transverse vector harmonics, and longitudinal vector harmonics respectively. As $A^{(5)}_{(L) lm}$ represents gauge degrees of freedom, we eliminate them after the gauge fixing (See the Appendix in Ref. for definition and properties of these harmonics). By substituting the expansion into the action, we find the five dimensional effective action.

As this effective action still has dilaton coupling for vector and scalar modes, in order to evaluate the eigenvalues in the path integral, we integrate it by part and then rewrite the integrand to the proper form. We divide the field $A_{\mu}$ to the transverse and longitudinal parts as,

$$A^{(5)}_{\mu} dx^\mu = \left(A^{(5)}_{(T) \mu} + A^{(5)}_{(L) \mu}\right) dx^\mu. \quad (29)$$

For the quantization of $U(1)$ gauge field $A^{(5)}_{\mu}$, we choose a Lorentz gauge. The action for $U(1)$ gauge field $A^{(5)}_{\mu}$ is finally given by

$$S_{U(1)} = S_{(V)} + S_{(T)} + \delta S,$$

$$S_{(V)} = -\frac{1}{2} \int d^5x \sqrt{-g} A^{(5) \mu}_{(T) lm} \left\{ \frac{1}{2} \left(g_{\mu\nu} \nabla^{(5)}_{(T) lm} - \nabla_{(T) lm} \gamma_{\nu}\right) e^{-\frac{4\phi_{\sigma}}{\kappa_\sigma}} - e^{-\frac{4\phi_{\sigma}}{\kappa_\sigma}} \Delta_{(T) lm} + g_{\mu\nu} M^2_{(V)} \right\} A^{(5) \nu}_{(T) lm},$$

$$S_{(T)} = -\frac{1}{2} \int d^5x \sqrt{-g} A^{(5)}_{(T) lm} \left\{ \frac{1}{2} \nabla^2_{(5)} e^{-2\kappa_{\sigma}} - e^{-2\kappa_{\sigma}} \nabla^2 + M^2_{(T)} \right\} A^{(5)}_{(T) lm},$$

$$\delta S = -\frac{1}{2} \int d^5x \sqrt{-g} A^{(5) \mu}_{(L) lm} \nabla_{(L) lm} A^{(5) \nu}_{(L) lm}, \quad (29)$$

where $S_{(V)}$ is the gauge fixing action and $\alpha$ is positive constant and $\Delta_{(T) lm} = g_{\mu\nu} \nabla^2_{(5) lm} + R_{\mu\nu}$. $M^2_{(V)}$ and $M^2_{(T)}$ are mass of the five dimensional vector field $A^{(5) \mu}_{(T) lm}$ and that of the five dimensional scalar field $A^{(5)}_{(T) lm}$, respectively, which are given by

$$M^2_{(V)} = e^{-2\kappa_{\sigma}} \left\{ \frac{l(l + 4)}{b_0^2} \right\}, \quad (30)$$

$$M^2_{(T)} = e^{-2\kappa_{\sigma}} \left\{ \frac{l(l + 4) + 3}{b_0^2} \right\}. \quad (32)$$

To calculate the 1-loop quantum correction by the $U(1)$ gauge field, we follow the procedure discussed in Ref. Finally, we obtain the effective potential due to $A^{(5) \mu}_{(T) lm}$ and $A^{(5) \nu}_{(T) lm}$ as follows:

$$V_{U(1)} \equiv \frac{\Lambda}{\kappa^2} e^{-\frac{4\phi_{\sigma}}{\kappa_\sigma}} - \frac{10}{\kappa^2 b_0^2} e^{-\frac{4\phi_{\sigma}}{\kappa_\sigma}} + \frac{1}{b_0} e^{-\frac{4\phi_{\sigma}}{\kappa_\sigma}} \times \left[-\ln(\mu^2 a^2) + C_{\phi} \ln \left(\frac{a}{b_0}\right)^2 e^{-\frac{4\phi_{\sigma}}{\kappa_\sigma}}\right], \quad (33)$$

where $C_{\phi}$ is given by

$$C_{\phi} = \frac{15}{16\pi^2} \zeta_{\phi}'(0) e^{\frac{4\phi_{\sigma}}{\kappa_\sigma}} = \frac{1999}{49152\pi}. \quad (34)$$

where

$$\zeta_{\phi}(s) = \sum_{l=0}^{\infty} \sum_{l'=1}^{\infty} D_{V}(l) d_{V}(l') \left[ e^{-\frac{4\phi_{\sigma}}{\kappa_\sigma} \lambda_{V}(l') a^2} + e^{-2\kappa_{\sigma} \Lambda_{V}^2 l'} \right], \quad (35)$$

where degeneracies $D_{V}(l)$, $d_{V}(l')$ and eigenvalues $\lambda_{V}(l')$, $\Lambda_{V}^2(l')$ are given by

$$D_{V}(l) = \frac{(l + 1)(l + 2)(l + 3)}{12},$$

$$d_{V}(l') = \frac{1}{3} \left((l + 2)^2 + 4\right),$$

$$\Lambda_{V}^2(l) = \frac{L(l + 4)}{b_0^2},$$

$$\lambda_{V}(l') = \{l(l + 4) - 3\}. \quad (36)$$
(C) Dirac spinor field

We then calculate the quantum correction associated with massless Dirac spinor field on $dS_5 \times S^5$. The action is given by

$$S_\psi = i \int d^{10}X \sqrt{-g} (\gamma^M \nabla_M \psi). \tag{37}$$

The spinor representation of $O(1, 5 + 5)$ is a direct product of the spinor representation of $O(1, 4)$ and $O(5)$. The ten dimensional gamma matrix $\tilde{\gamma}$ is given by

$$\tilde{\gamma}^\mu = \gamma^\mu \otimes 1, \quad \tilde{\gamma}^i = \gamma^5 \otimes \gamma^i, \quad (\gamma^5)^2 = 1,$$

$$\{\tilde{\gamma}^M, \tilde{\gamma}^N\} = 2g^{MN}, \tag{38}$$

where the $\gamma^\mu$ ($\mu = 0, 1, 2, 3, 4$) are Dirac matrices in $dS_5$ while the $\gamma^i$ ($i = 5, 6, \cdots, 9$) are those in $S^5$. The Dirac spinor field $\psi$ is expanded as spinor harmonics analogous to scalar field;

$$\tilde{\psi}(x^\mu, y^i) = b_0^{5/2} \sum_{l,m} \psi_{lm}(x^\mu)Y_{\psi}^{(5)}(y), \tag{39}$$

where $\psi_{lm}(x^\mu)$ is the Dirac spinor field in the five dimensional spacetime. $Y_{\psi}^{(5)}$ are real spinor harmonics on the $S^5$ satisfying

$$i \tilde{\gamma}^i \nabla_i Y_{\psi}^{(5)} = \lambda_\psi Y_{\psi}^{(5)}, \tag{40}$$

$$\int d^5y \sqrt{\Omega^{(5)}} Y_{\psi}^{(5)} \nabla_i Y_{\psi}^{(5)} \nabla_i Y_{\psi}^{(5)} = \delta_{lm} \delta_{mm'}, \tag{41}$$

and $\psi_{lm}(x^\mu)$ is a real function depending only on the five dimensional coordinates $x^\mu$. $l = 1, 2, \cdots$, and $m$ denote a set of six numbers which is required in order for $Y_{\psi}^{(5)}$ to be a complete set of $L^2$ functions on the $S^5$. Here $\tilde{\gamma}^i \nabla_i$ is the Dirac operator on the unit seven sphere $S^7$ and $\Lambda_\psi(l)$ denotes an eigenvalue for the Dirac spinor field $\psi$. Using the relation for the ten dimensional Dirac operator $\tilde{\gamma}^M \nabla_M = \tilde{\gamma}^5 \nabla_\mu \otimes 1 + \gamma^5 \otimes \tilde{\gamma}^i \nabla_i$, we obtain the five dimensional effective action

$$S(\psi_{lm}, \bar{\psi}_{lm}) = \sum_{l,m} \int d^5x \sqrt{-g} \times \bar{\psi}_{lm} \left(i \gamma^\mu \nabla_\mu + \Lambda_\psi \gamma^5\right) \psi_{lm}. \tag{42}$$

The partition function $Z$ for massless Dirac spinor field on $dS_5$ is

$$Z = \int D[\psi_{lm}] D[\bar{\psi}_{lm}] \exp \left\{-iS(\psi_{lm}, \bar{\psi}_{lm})\right\}, \tag{43}$$

where $D[\psi_{lm}]$ and $D[\bar{\psi}_{lm}]$ are the functional measure of the spinor field $\psi_{lm}$ and its Dirac adjoint field $\bar{\psi}_{lm}$, respectively. Using the definition of a Gaussian functional for anti-commuting fields, we obtain the partition function as

$$\ln Z = \ln \det \{\mu^{-1} \left(i \gamma^\mu \nabla_\mu + \Lambda_\psi(l)\gamma^5\right)\}
= \frac{1}{2} \ln \det \left\{\mu^{-2} \left(- (\gamma^\mu \nabla_\mu)^2 + (\Lambda_\psi(l))^2\right)\right\}, \tag{44}$$

where five dimensional Dirac operator $(\gamma^\mu \nabla_\mu)^2$ is given by

$$(\gamma^\mu \nabla_\mu)^2 = \nabla_5^2 + \frac{1}{4} R_{(5)}. \tag{45}$$

The effective potential is then rewritten as

$$V_{eff}(\sigma) = U_0(\sigma) - \frac{1}{4 \Omega_{efl}} \{\zeta'_f(0) + \zeta_f(0) \ln(\mu^2 a^2)\}, \tag{46}$$

where $\zeta_f(0)$ is the generalized zeta function for the Dirac spinor field:

$$\zeta_f(s) = \sum_{l=2}^{\infty} \sum_{l'=0}^{\infty} D_\psi(l)d_\psi(l') \left[\frac{\lambda_\psi(l')}{a^2} + \frac{\Lambda_\psi^2}{b_0^5}\right]. \tag{47}$$

Here degeneracies $D_\psi(l)$, $d_\psi(l')$ and eigenvalues $\lambda_\psi(l')$, $\Lambda_\psi(l)$ are given by

$$D_\psi(l) = \frac{(l + 4)(l + 3)(l + 2)(l + 1)}{4l!},$$

$$d_\psi(l') = \frac{(l' + 4)(l' + 3)(l' + 2)(l' + 1)}{4l'!},$$

$$\Lambda_\psi^2(l) = e^{-\frac{24}{\kappa^2}\sigma} \left(l + \frac{5}{2}\right)^2,$$

$$\Lambda_\psi^2(l') = \left(l' + \frac{5}{2}\right)^2 - 5. \tag{48}$$

Following the method given in [1], we find the effective potential to 1-loop order is

$$V_{eff} = \frac{\Lambda}{\kappa^2} e^{-\frac{24}{\kappa^2}\sigma} - \frac{10}{\kappa^2 b_0^5} e^{-\frac{24}{\kappa^2}\sigma} + \frac{C_f}{b_0^5} e^{-\frac{24}{\kappa^2}\sigma}, \tag{49}$$

where $C_f$ is given by

$$C_f = \frac{15}{16\pi^2} \zeta'_f(0) e^{\frac{24}{\kappa^2}\sigma} \approx -3.335245 \times 10^{-6}. \tag{50}$$

Note that the logarithmic term does not appear because $\zeta_f(0)$ vanishes. The same problem arises in Minkowski spacetime[2].

(D) gravitational field (scalar mode)

Finally, we investigate the quantum correction by gravitational field. In our model, it is assumed that the distance scale of the extra dimension is by a few order magnitude larger than Planck scale. We then apply the method of the conventional loop expansion approach to the quantization of the gravitational field theory[23].

We consider a gravitational perturbation $h_{MN}$ around a background metric $\bar{g}_{MN}^{(0)}$, which we shall specify later;

$$\bar{g}_{MN} = \bar{g}_{MN}^{(0)} + h_{MN}. \tag{51}$$
Substituting Eq. (11) into Eq. (11), we obtain the perturbed Einstein-Hilbert action as follows.

\[ I_{EH} = \int d^4x \sqrt{-g^{(0)}} \left[ \mathcal{L}_0 + \mathcal{L}_2 \right], \quad (52) \]

where

\[ \mathcal{L}_0 = \frac{1}{2\kappa^2} \left[ \tilde{\mathcal{R}}^{(0)} - 2\Lambda \right] \]

\[ \mathcal{L}_2 = \frac{1}{2\kappa^2} \left\{ \begin{array}{c}
\frac{1}{2} \left( h^{MN} \nabla_{(M} h_{N)} - h^{MN} \right) \tilde{\mathcal{R}}^{(0)} \\
+ \frac{1}{4} \left( \nabla^{MN} h^{(M} h_{N)} - h^{MN} \right) \tilde{\mathcal{R}}^{(0)} \\
+ \frac{1}{4} \left( \nabla^{MN} h^{(M} h_{N)} - h^{MN} \right) \tilde{\mathcal{R}}^{(0)} \\
+ \nabla^M (h^{MN} h_{MN}) \\
- \Lambda \left( \frac{1}{4} h^2 - \frac{1}{2} h^{MN} h_{MN} \right) + O(h^3) \end{array} \right\}, \quad (53) \]

where \( \tilde{\mathcal{R}}^{(0)} \) denotes the covariant derivative with respect to \( g^{(0)}_{MN} \), and \( \tilde{\mathcal{R}}^{(0)} \) and \( \tilde{\mathcal{R}}^{(0)} \) are the Ricci tensor and scalar of the background metric \( g^{(0)}_{MN} \). As for the background geometry, we compactify it on a five dimensional sphere \( S^5 \).

After gauge-fixing and redefining \( g_{\mu\nu} \) and \( b \), the perturbation \( h_{MN} \) is expanded as \( \tilde{h}_{(T)} \):

\[ h_{MN} = \sum_{l,m} \left[ \tilde{h}_{(T)}^{lm} Y_{lm} \right]_i d\sigma^i + 2h_{(T)}^{lm} \Omega_i Y_{lm} \right]_i d\sigma^i \]

\[ + \left\{ \begin{array}{c}
\tilde{h}_{(T)}^{lm} \nabla_i Y_{lm} \\
\tilde{h}_{(T)}^{lm} \nabla_i Y_{lm} \\
\tilde{h}_{(T)}^{lm} \nabla_i Y_{lm} \end{array} \right\}, \quad (55) \]

where \( Y_{lm}, \Omega_i Y_{lm}, \) and \( T^{lm} \) are the scalar harmonics, the vector harmonics, and tensor harmonics, respectively. The coefficients \( \tilde{h}_{(T)}^{lm} \), \( \tilde{h}_{(T)}^{lm} \), \( \tilde{h}_{(T)}^{lm} \), \( \tilde{h}_{(T)}^{lm} \), \( \tilde{h}_{(T)}^{lm} \), \( \tilde{h}_{(T)}^{lm} \), and \( \tilde{h}_{(T)}^{lm} \) depend only on the five dimensional coordinates \( y^i \), while the harmonics depend only on the coordinates \( \sigma^i \) on \( S^5 \). The summations are taken over \( l \geq 1 \) for the scalar and vector harmonics, and over \( l \geq 2 \) for the tensor harmonics.

Herewith we only calculate the scalar mode \( \tilde{h}_{(T)}^{lm} \), just for a technical reason. Substituting the above harmonic expansion into the Einstein-Hilbert action, we then obtain the following action.

\[ I = I_g + I_\chi, \quad (56) \]

where \( I_g \) is five dimensional Einstein-Hilbert action \( \tilde{I}_g \) and

\[ I_\chi = -\frac{1}{2} \int d^4x \sqrt{-g^{(0)}} \left[ e^{-4\kappa_\sigma} g^\mu_\nu \partial_\mu \chi \partial_\nu \chi + M^2_\chi \chi \right], \quad (57) \]

The scalar mode field \( \chi^{lm} \) and its mass \( M^2_\chi \) are defined as follows.

\[ M^2_\chi = \frac{1}{8\sqrt{2\pi}} \sqrt{\frac{h_{(T)}^{lm}}{b_0}} \quad (l \geq 2). \quad (59) \]

In order to analyze quantum correction, we rewrite the above expression because of the existence of dilaton coupling to kinetic term of \( \chi \). After some calculation, we obtain

\[ I_\chi = -\frac{1}{2} \int d^4x \sqrt{-g^{(0)}} \chi \left[ \frac{\Lambda_\chi}{2} e^{-2\kappa_\sigma} - e^{-4\kappa_\sigma} \nabla^{(5)} \chi \right], \quad (59) \]

Using the results of Appendix \( \Box \) in \( \Box \), we can evaluate \( \zeta(s) \) and \( \zeta'(s) \) for \( \chi \) at \( s = 0 \). Under the assumption of \( a \gg b \), we get the following expression;

\[ V_{eff}(b) = \frac{\Lambda_\chi}{\kappa^2} e^{-2\kappa_\sigma} - \frac{21}{\kappa^2 b_0^2} e^{-2\kappa_\sigma} + \frac{1}{b_0^2} e^{-4\kappa_\sigma} \]

\[ \left[ -\ln \left( \frac{a^2}{b_0^2} \right) + C_\chi \ln \left( \frac{a^2}{b_0^2} \right) \right], \quad (63) \]

where \( C_\chi \) is given by

\[ C_\chi = \frac{15}{16\pi^2} \zeta'(0) e^{-4\kappa_\sigma} = \frac{111}{16\pi} \quad (64) \]

This result indicates that one-loop graviton contributions are roughly three orders of magnitude bigger than scalar contribution. These are well-known results in Minkowski spacetime \( \Box \).
B. Quantum correction in AdS$_5 \times S^5$

As we will discuss in the next subsection, the minimum of the effective potential turns out to be negative in most plausible cases. Therefore, in order to show a consistency of our results, we have to calculate the quantum correction in AdS$_5 \times S^5$ back ground and derive the effective potential.

Since the procedure to get the effective potential in AdS$_5 \times S^5$ is almost the same as that in dS$_5 \times S^5$ except for the zeta functions, we do not repeat it here. Instead, showing how to derive the zeta functions in AdS$_5 \times S^5$ in Appendix, we just summarize our results as follows:

The effective potentials for bosonic fields (scalar field $\phi$, $U(1)$ gauge field $A_M$, and scalar mode of gravitational field $h^{(3)}_{\mu\nu}$) is given by

$$V_{eff}(\sigma) = \frac{\Lambda}{\kappa^2} e^{-\frac{4\pi^2}{L} \kappa \sigma} - \frac{10}{\kappa^2 b_0^2} e^{-\frac{4\pi^2}{L} \kappa \sigma} + \frac{1}{b_0} e^{-\frac{4\pi^2}{L} \kappa \sigma}$$

$$\times \left[-\ln (\mu^2 a^2) + D_c \ln \left( \left( \frac{a}{b_0} \right)^2 e^{-\frac{4\pi^2}{L} \kappa \sigma} \right) \right], \quad (65)$$

where $D_c$ is given by

$$D_c = \frac{\pi}{4320} \quad \text{for a scalar field}$$

$$= \frac{47\pi}{55296} \quad \text{for a gauge field}$$

$$= \frac{11\pi}{10} \quad \text{for a gravitational field}. \quad (66)$$

For a Dirac spinor field, the effective potential is given by

$$V_{eff}(\sigma) = \frac{\Lambda}{\kappa^2} e^{-\frac{4\pi^2}{L} \kappa \sigma} - \frac{10}{\kappa^2 b_0^2} e^{-\frac{4\pi^2}{L} \kappa \sigma} + \frac{D_f}{b_0} e^{-\frac{4\pi^2}{L} \kappa \sigma}, \quad (67)$$

where $D_f = -7.29353 \times 10^{-6}$.

C. Dynamics of dilaton and stabilization of $S^5$

Once we know the effective potential for the dilaton, it is easy to analyze a stability of extra dimensions [17]. To stabilize the extra dimensions ($S^5$), the dilaton potential has to have a minimum or at least a local minimum. From (65) the effective potential in dS$_5 \times S^5$ is given by

$$V_{eff}(\sigma) = \frac{\Lambda}{\kappa^2} e^{-\frac{4\pi^2}{L} \kappa \sigma} - \frac{10}{\kappa^2 b_0^2} e^{-\frac{4\pi^2}{L} \kappa \sigma}$$

$$+ \frac{C}{b_0^2} e^{-\frac{4\pi^2}{L} \kappa \sigma} + \frac{C_F}{b_0} e^{-\frac{4\pi^2}{L} \kappa \sigma}, \quad (68)$$

where $C$ and $C_F$ are given by

$$C_B = \left[ -N \ln (\mu^2 a^2) + C \ln \left( \left( \frac{a}{b_0} e^{-\frac{4\pi^2}{L} \kappa \sigma} \right)^2 \right) \right]. \quad (69)$$

with $N = N_0 + N_V + N_f + 1$, $C = N_0 \kappa c_0 + N_V \kappa c_V + \kappa c_\chi$ and $C_F = N_f \kappa c_f$. $N_0$, $N_V$, and $N_f$ are numbers of scalar, vector and fermion field.

In AdS$_5 \times S^5$ background, the effective potential is given by Eq. (65) with replacing the constants $C_0$, $C_V$, $C_\chi$ and $C_F$ with $D_\phi$, $D_V$, $D_\chi$ and $D_f$, respectively. The stable point $\sigma = \sigma_s$ must satisfy the following conditions,

$$\frac{\partial V_{eff}(\sigma_s)}{\partial \sigma} = 0, \quad \frac{\partial^2 V_{eff}(\sigma_s)}{\partial \sigma^2} > 0. \quad (70)$$

These conditions are satisfied for wide range of parameters in our model. To show it, we first analyze the potential under some approximation. In naive analysis, we ignore the fermion contribution and assume that $C_\beta$ is constant because the dependence of variables $a$ and $\sigma$ is logarithmic. In this case, the condition (70) is given as

$$m_A^2 - 16m_{b_0}^2 e^{-2\pi^2 \kappa \sigma_s} + 4Cm_{b_0}^2 e^{-10\pi^2 \kappa \sigma_s} = 0$$

$$5m_A^2 - 128m_{b_0}^2 e^{-2\pi^2 \kappa \sigma_s} + 80Cm_{b_0}^2 e^{-10\pi^2 \kappa \sigma_s} > 0, \quad (71)$$

where we introduce the dimensionless mass scales $m_A = M_A / M_\chi = \sqrt{\Lambda} \kappa^{1/3}$ and $m_{b_0} = M_{b_0} / M_\chi = \kappa^{1/3} / b_0$, which denote the ratios of the mass scale of a cosmological constant and that of internal space to the 5-dimensional Planck mass, respectively. Introducing Eq. (71) into Eq. (64), we find the condition

$$C_B m_{b_0}^3 e^{-8\pi^2 \kappa \sigma_s} > 4. \quad (72)$$

The value of the potential at minimal point $\sigma = \sigma_s$ is then found to be

$$V_{eff}(\sigma_s) = M_\chi^5 e^{-\frac{4\pi^2}{L} \kappa \sigma_s} \left[ m_A^2 - 10m_{b_0}^2 e^{-2\pi^2 \kappa \sigma_s} + C_B m_{b_0}^2 e^{-10\pi^2 \kappa \sigma_s} \right]$$

$$= 3M_\chi^5 m_{b_0}^3 e^{-\frac{4\pi^2}{L} \kappa \sigma_s} \left[ 2 - C_B m_{b_0}^2 e^{-8\pi^2 \kappa \sigma_s} \right]$$

$$< -6M_\chi^5 m_{b_0}^3 e^{-\frac{4\pi^2}{L} \kappa \sigma_s}. \quad (73)$$

In the last inequality, we have used (72). This result means that the stable minimum of the potential, if it exists, is always negative.

The above naive analysis is confirmed numerically. Including the dependence of $\sigma$ in $C_B$, we survey the parameter space and find that the potential minimum is always negative when we have a stable minimum point. For example, choosing $N_0 = 1$, $N_V = 5$, $N_f = 2$, and setting $\Lambda = 1.59992 \times 10^{-3} M_\chi^5$, $b_0 = 10^2 L_\chi$, $\mu = b_0^{-1}$, and $a = 49.1490 \times 10^5 L_\chi$, we find the minimum value $V_{eff}(\sigma_s) = -0.33577 M_\chi^5$ at $\sigma_s = -5.11012 L_\chi$ for dS$_5 \times S^5$ background, while we obtain the minimum value $V_{eff}(\sigma_s) = -0.30375 M_\chi^5$ at $\sigma_s = -5.03668 L_\chi$ for AdS$_5 \times S^5$ background. The potential form is given in Fig. 14.

The universe may be created one the top of the potential hill. Then inflation will take place. However, such state is not stable in the dS$_5 \times S^5$ background. The universe rolls down to the potential minimum, which value is
negative. Thus we have to switch the effective potential calculated in $AdS_5 \times S^5$ background. In this case, we also have a negative potential minimum. Then the universe evolves into the static $AdS_5 \times S^5$, which is a stable spacetime. The parameter $b_0$ is the scale of renormalization in the quantum correction. The ratio of the parameters $(b_0/a)^4$ must be much smaller than unity in order to be renormalizable.

From the consistency condition, we find that $V_{eff}(\sigma_s) = -0.30375 M_5^4$ at $\sigma_s = -5.03658 L_5$ for $a = 4.91490 \times 10^6 L_5$. We have chosen the same values as the above ones for $N_\phi = 1, N_V = 5, N_f = 2, b_0 = 10^2 L_5, \mu = b_0^{-1}$ and $a = 4.91490 \times 10^6 L_5$. The minimum of the potential is located at $\sigma_s = -5.11012 L_5$ ($b_0 = b_0 \exp(\kappa \sigma_s) = 29.4772 L_5$) and its value is $V_{eff}(\sigma_s) = -0.33577 M_5^4$. $L_5$ and $M_5$ denote the five dimensional Planck length and mass, respectively.

III. DILATON DYNAMICS IN TYPE IIB SUPERGRAVITY MODEL

Now we consider ten dimensional type IIB supergravity model. This model has been discussed by many authors from the motivation of AdS/CFT correspondence\[4]. Stelle et al.\[4] studied type IIB supergravity in $AdS_5 \times S^5$ as the candidate for the RS brane world model. However, the stability of $S^5$ and its dynamics in their suggestion was not shown. In order to construct a realistic RS model, we should treat the scale of $S^5$ as a dynamical variable and show that it stays at small value compared with that of $AdS_5$.

In the cosmological point of view, de Sitter spacetime is more interesting than anti-de Sitter spacetime because the de Sitter spacetime corresponds to a geometry of inflationary era in the cosmology. In fact, the brane world cosmology in $dS_5$ spacetime is discussed by several authors with the issues of inflation. In the following, we investigate the stabilization of $S^5$ in the $(A)dS_5 \times S^5$ spacetime.

Let us consider ten dimensional type IIB supergravity action\[13]:

$$S_{IIB} = S_{NS} + S_R + S_{CS},$$

$$S_{NS} = \frac{1}{2\kappa^2} \int d^{10} X \sqrt{-g} e^{-2\Phi} \left[ \bar{R} + 4\tilde{g}^{MN} \partial_M \Phi \partial_N \Phi - \frac{1}{2} |H_3|^2 \right],$$

$$S_R = \frac{1}{4\kappa^2} \int d^{10} X \sqrt{-g} \left( |F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |F_5|^2 \right),$$

$$S_{CS} = -\frac{1}{4\kappa^2} \int d^{10} X \sqrt{-g} C_4 \wedge H_3 \wedge F_3,$$

where

$$\tilde{F}_3 = F_3 - C_3 \wedge H_3,$$

$$\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3,$$

and $\bar{R}$ is a Ricci scalar with respect to the metric $\bar{g}_{MN}$. Note that the subscript numbers in $H_3, F_1, F_3, F_5$ etc denote the rank of tensor fields. If we assume the $F_1 = F_3 = H_3 = 0$, the action is given by

$$S_{IIB} = \frac{1}{2\kappa^2} \int d^{10} X \sqrt{-g} e^{-2\Phi} \left[ \bar{R} + 4\tilde{g}^{MN} \partial_M \Phi \partial_N \Phi - \frac{1}{4} |F_5|^2 \right] + \text{terms involving } F_1, F_3, H_3.$$

Since we consider the case when the Neveu-Schwarz (NS) scalar field $\Phi$ is time dependent, the factor $e^{2\Phi}$ before the Ricci scalar $\bar{R}$ and the kinetic term $\tilde{g}^{MN} \partial_M \Phi \partial_N \Phi$ of the $\Phi$ is not constant. Hence the action of NS fields looks quite different from the usual massless scalar field. Then we perform the following conformal transformation

$$\tilde{g}_{MN} = e^{\Phi/2} \bar{g}_{MN}.$$  

Note that we calculate the quantum effect at the local minimum of potential $V(\Phi)$ (i.e. $\Phi = 0$) which arises after the compactification of $S^5$. Then the conformal factor becomes unity after $\Phi$ settles to $\Phi = 0$. Consequently the change of the frame does not affect any result, provided that our results are interpreted after $\Phi$ settles to $\Phi = 0$. Using the conformal transformation\[17] and rescaling $\Phi$ as $\Phi \rightarrow \Phi/(2\sqrt{17})$, the action is given by

$$S = \frac{1}{2\kappa^2} \int d^{10} X \sqrt{-g} \left[ \bar{R} - \tilde{g}^{MN} \partial_M \Phi \partial_N \Phi - \frac{1}{4} |F_5|^2 \right],$$

where $\bar{R}$ is a Ricci scalar of the ten dimensional metric tensor $\tilde{g}_{\mu\nu}$. The background metric is expressed as Eq.\[23, 24]. We consider the Freund-Rubin type solution

$$V_{eff}/(M \kappa)^5$$

FIG. 1: The dilaton effective potential for $dS_5 \times S^5$ background given in Eq.\[25] is depicted. We set $\bar{\Lambda} = 1.59992 \times 10^{-3} M_5^2, N_\phi = 1, N_V = 5, N_f = 2, b_0 = 10^2 L_5, \mu = b_0^{-1}$ and $a = 4.91490 \times 10^6 L_5$. The minimum of the potential is located at $\sigma_s = -5.11012 L_5$ ($b_0 = b_0 \exp(\kappa \sigma_s) = 29.4772 L_5$) and its value is $V_{eff}(\sigma_s) = -0.33577 M_5^4$. $L_5$ and $M_5$ denote the five dimensional Planck length and mass, respectively.
as 5-form field strength:

\[ F_{M_1 \cdots M_5} = \begin{cases} \left( f / \sqrt{\Omega(5)} \right) \epsilon_{M_1 \cdots M_5}, & M_1 = i, \cdots, M_5 = m, \\ 0, & \text{otherwise}, \end{cases} \]

where \( f \) is a constant and \( \sqrt{\Omega(5)} \) is the determinant of five dimensional sphere \( S^5 \). This field strength is wrapped around the \( S^5 \). We then set \( |F_5|^2 = 8\Lambda \) and redefine the NS scalar \( \varphi = \Phi / \kappa \). The ten dimensional action \(^{(79)}\) is now rewritten by

\[ S = \int d^{10}X \sqrt{-g} \left[ \frac{1}{2\kappa^2} (\bar{R} - 2\Lambda) - \frac{1}{2} g^{MN} \partial_M \varphi \partial_N \varphi \right]. \]

To write down the five dimensional effective action, the NS scalar \( \varphi \) is expanded as

\[ \varphi = b_0^{-5/2} \sum_{l,m} \varphi_{lm} (x^\mu) \chi_{lm}^{(5)} (y^j), \]

where \( b_0 \) is the initial value of \( b \). Substituting the metric \(^{24}\) and the NS scalar \(^{31}\) into the action \(^{31}\), we get the five dimensional effective action

\[ S = \int d^5x \sqrt{-g} \left[ \left( \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - U(\sigma) \right) \right. \]

\[ \left. - \sum_{l,m} \frac{1}{2} \left( g^{\mu\nu} \partial_\mu \varphi_{lm} \partial_\nu \varphi_{lm} + M_\varphi^2 \varphi_{lm}^2 \right) \right], \]

where \( R \) is Ricci scalar of five dimensional metric tensor \( g_{\mu\nu} \), \( \kappa \) is a positive constant defined by \( \kappa^2 = \bar{\kappa}^2 / (25 b_0^2 \pi) \), \( \sigma \) is given by \(^{24}\) the mass \( M_\varphi^2 \) of five dimensional NS scalar field \( \varphi \) is given by

\[ M_\varphi^2 = \frac{l(l + 4)}{b_0^2} e^{-\frac{36}{\kappa_\sigma} \sigma}, \]

and the dilaton potential \( U(\sigma) \) is given by

\[ U(\sigma) = \frac{\bar{\Lambda}}{\kappa^2} e^{-\frac{36}{\kappa_\sigma} \sigma} - \frac{10}{\kappa^2 b_0^2} e^{\frac{36}{\kappa_\sigma} \sigma}. \]

Note that the classical NS scalar fields \( \varphi_{lm} \) in the action \(^{22}\) has a potential, which vanishes at \( \varphi_{lm} = 0 \), unless \( l = m = 0 \) which does not couple to the dilaton. Furthermore, the kinetic term of \( \varphi \) is also approximately zero because the NS scalar is stable at \( \varphi_{lm} = 0 \) (\( l \neq 0 \)). Then we expect that the massive scalar field \( \varphi_{lm} = 0 \) (\( l \neq 0 \)) has a zero vacuum expectation value at least classical level, resulting in that this scalar field does not produce the energy momentum except for the massless mode. This background geometry has 3-brane which act as source for the 4-form field.

Here we consider the quantum effect of NS scalar. The NS scalar denotes the scale of eleventh dimension in eleven dimensional supergravity, which is compactified a la Kaluza-Klein. This length scale is not too far above Planck length. Even though a satisfactory quantum theory of gravity is not known so far, we expect that the quantum effect is presumably very important. As the action \(^{22}\) is same form as \(^{31}\) and \(^{24}\), the 1-loop effective potential for NS scalar is given by

\[ V_{\text{eff}}(\sigma) = \frac{\bar{\Lambda}}{\kappa^2} e^{-\frac{36}{\kappa_\sigma} \sigma} - \frac{10}{\kappa^2 b_0^2} e^{\frac{36}{\kappa_\sigma} \sigma} + \frac{1}{b_0^2} e^{\frac{36}{\kappa_\sigma} \sigma} \]

\[ \times \left[ -\ln(\mu^2 a^2) + C \ln \left( \frac{a}{b_0} \right)^2 e^{\frac{36}{\kappa_\sigma} \sigma} \right], \]

where \( C = 25/(8192\pi) \) (See Eq.(26)). This is exactly same as that calculated in \(^{31}\) We have other fields which give the contribution to the effective potential. Such a contribution has also been calculated in \(^{31}\). In the background is \( AdS_5 \times S^5 \), we again find Eqs. (80) and \(^{34}\).

In the present model, we have \( N_\phi = 1, N_\nu = 5, \) and \( N_f = 2 \). Setting \( \bar{\Lambda} = 1.59992 \times 10^{-3} M_\kappa^2 b_0 = 10^3 L_\kappa \) and \( \mu = b_0^{-1} \), we find the stable minimum point \( \sigma_a = -5.03658 L_\kappa \) where \( V_{\text{eff}}(\sigma_a) = -0.30375 M_\kappa^2 \) for \( a = 4.91490 \times 10^6 L_\kappa \). We may conclude that the stable RS model is realized from Type IIB supergravity theory.

We may also have to include further contributions from other modes of gravitons and gravitinos, which are ignored in the present analysis because of a technical reason. It might be justified from the following reason. If supersymmetry is not broken, quantum correction should vanish at the \( AdS_5 \) minimum and then a stable compactification is not obtained. Hence, here, we assume that supersymmetry is broken by an unknown mechanism in order to find a stable minimum.

**IV. CONCLUSION**

We have calculated quantum effects in the \((A)dS_5 \times S^5\) compactified background of the ten dimensional Kaluza-Klein theory and type IIB supergravity and discussed its stability using the effective potential. In their pure gravity systems, a curvature term of the internal space gives a dominant contribution to the dilaton potential at small \( b \), while a cosmological constant term becomes dominant for large \( b \). Hence the dilaton potential is unbounded from below as \( b \to 0 \) and drops exponentially as \( b \to \infty \). Then the extra dimension either shrinks to zero volume or is decompactified. However, if we include quantum effects, we find a stable minimum for the dilaton potential. In the ten dimensional Kaluza-Klein model, the cosmological constant and quantum correction of various matter fields force to expand the extra dimension while the curvature of the internal spacetime forces to contract it. These combination produces a local minimum of the effective potential. In ten dimensional type IIB supergravity model, the scale of extra dimension is stabilized
by balancing the five form gauge field strength wrapped around the $S^5$, the curvature term of $S^5$ and quantum correction term induced by the NS scalar. The NS scalar is originally characterized by the direction of eleventh dimension in the eleven dimensional supergravity. Thus the quantum effect of NS scalar becomes important if the eleventh dimension is compactified near the Planck scale.

When the universe is created, the dilaton may be located near the top of the potential hill, and then the background geometry is almost $dS_5$ spacetime. We then have exponential expansion of the 5-dimensional spacetime. As the dilaton rolls down the potential hill, the five dimensional spacetime geometry deviates from the $dS_5$. The dilaton potential eventually turns out to be negative, and the dilaton finds a stable minimum point. Then, the five dimensional spacetime becomes $AdS_5$. Hence, a five dimensional de Sitter ($dS_5$) spacetime evolves into a five dimensional anti-de Sitter($AdS_5$) when the dilaton settles down to the potential minimum. Associated with dilaton stabilization, the change of spacetime geometry takes place.

Our solution contains a 3-brane because the background geometry has the five form gauge field strength. Hence, the ten dimensional $(A) dS_5 \times S^5$ compactification may provide a realistic model of the RS brane world\[1\]. However, if we have branes in our $AdS_5$, we may find additional contributions of quantum effect because we have new boundaries of branes. It may change some part of the present results, although we believe that $AdS_5$ is still stable. We leave this calculation as a future work.

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APPENDIX A: ZETA FUNCTION REGULARIZATION IN $AdS_n \times S^d$

1. scalar field

In this Appendix, we provide the method to calculate the zeta function regularization for scalar field in the product spacetime $AdS_n \times S^d$. The Euclidean section for $AdS_n$ spacetime is the $n$-dimensional hyperbolic space $H^n$. The calculation of zeta function for $AdS_n$ is discussed in \[19\]. We extend their calculation technique to the zeta function for $AdS_n \times S^d$. On the compact Euclidean section, the zeta function is given by\[4\]

$$
\zeta_\phi(s) = \sum_{l=0}^{\infty} D_l \Lambda_l^{-s}, \quad (A1)
$$

where $\Lambda_l$ is the discrete eigenvalue of the Laplace-Beltrami operator and $D_l$ is the degeneracy of the eigenvalue. The calculation of zeta function on the $S^d$ is performed using well-known spectrum of the Laplace-Beltrami operator on $S^d$. On the other hand, the zeta function for the noncompact manifold is not same as that in a compact case. On the homogeneous $n$-dimensional hyperbolic space $H^n$, the zeta function takes the form\[10\]

$$
\zeta_\phi(s) = \int_0^\infty d\lambda \mu(\lambda) \Lambda(\lambda)^{-s}, \quad (A2)
$$

where $\Lambda(\lambda)$ is the eigenvalue of the Laplace-Beltrami operator on the $H^n$ and $\lambda$ is the real parameter which labels the continuous spectrum and $\mu(\lambda)$ is the spectrum function (or Plancherel measure) on $H^n$ corresponding to the discrete degeneracy on the $S^d$. The spectrum function for the scalar and spinor fields on the $H^n$ was already calculated in \[20, 21\]. The spectrum function for the transverse-traceless tensor field on $H^n$ was also given in \[22\]. Define the generalized zeta function in $H^n \times S^d$ for the scalar field as

$$
\zeta_\phi(s) = \sum_{l=0}^{\infty} \frac{(l + d - 2)! (2l + d - 1)!}{(d - 1)! l!} \int_0^\infty d\lambda \mu(\lambda)
\times \left\{ \frac{\lambda^2 + \{n - 1\}/2}{a^2} \right\}^{l} \left[ \frac{l + d - 1}{b^2} e^{-[2d/(n-2)+2]\sigma \cdot \sigma} \right]^{l} \cdot \cdot \cdot \right\}^{-s}, \quad (A3)
$$

For the odd dimension, Plancherel measure $\mu(\lambda)$ is given by \[10\]

$$
\mu(\lambda) = \frac{\pi}{2^{2(n-2)} \Gamma(n/2)^2} \prod_{j=0}^{(n-3)/2} (\lambda^2 + j^2). \quad (A4)
$$

Using $D = (d - 1)/2$, $N = (n - 1)/2$ instead of $d$, $n$ and running variables $L = l + D$, we rewrite \[A3\] as

$$
\zeta_\phi(s) = \sum_{L=0}^{\infty} D_\phi(L) \int_0^\infty d\lambda \mu(\lambda) \left( \frac{\lambda^2 + N^2}{a^2} + M^2 \right)^{-s}, \quad (A5)
$$

where the $D_\phi(L)$, $\Lambda_\phi(L)$ are given by

$$
D_\phi(L) = \frac{2l^2}{(2D)!} \left\{ L^2 - (D - 1)^2 \right\} \cdot \left\{ L^2 - 1 \right\}, \quad (A6)
$$

$$
\Lambda_\phi(L) = L^2 - D^2, \quad (A7)
$$

$$
M_\phi^2 = \frac{\Lambda_\phi(L)}{b^2} e^{-[2d/(n-2)+2]\sigma \cdot \sigma}. \quad (A8)
$$
Integrating Eq. (A5) with respect to \( \lambda \), we find the expression

\[
\zeta_{\phi}(s) = \frac{\pi^{3/2}a^{2s}}{2^{2n-1} \{ \Gamma(2N + 1) \}^2 \Gamma(s)} \sum_{L=D}^{\infty} D_{\phi}(L) \\
\times \left[ 3 \left( N^2 + a^2 M^2 \right)^{-s+5/2} \Gamma \left( s - \frac{5}{2} \right) \\
+ 2 \left( N^2 + a^2 M^2 \right)^{-s+3/2} \Gamma \left( s - \frac{3}{2} \right) \right]. \tag{A7}
\]

To regularize the mode sum in Eq. (A7), we replace the infinite sum for \( L \) by complex integration. The generalized zeta function is

\[
\zeta_{\phi}(s) = \frac{\pi^{3/2}a^{2s}}{2^{2n-1} \{ \Gamma(2N + 1) \}^2 \Gamma(s)} \\
\times \left[ \frac{3i}{2} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-\left[ 2d/(n-2) + 2 \right] \kappa_\sigma} \right\}^{-s+5/2} \right. \\
\times \int_{C_1} dz \ D_{\phi}(z) \cot(\pi z) \left( z^2 - A_L^2 \right)^{-s+5/2} \Gamma \left( s - \frac{5}{2} \right) \\
+ \left. i \left\{ \left( \frac{a}{b_0} \right)^2 e^{-\left[ 2d/(n-2) + 2 \right] \kappa_\sigma} \right\}^{-s+3/2} \right. \\
\times \int_{C_1} dz \ D_{\phi}(z) \cot(\pi z) \left( z^2 - A_L^2 \right)^{-s+3/2} \Gamma \left( s - \frac{3}{2} \right) \right]. \tag{A8}
\]

and the contour \( C_1 \) in the complex plane is showed in Fig. 2. Note that there are two branch points \( z = \pm A_L \) in the integration. We introduce the function \( \tilde{\zeta}_k(s) \) given by

\[
\tilde{\zeta}_k(s) = \frac{\pi^{3/2}a^{2s}}{2^{2n-1} \{ \Gamma(2N + 1) \}^2 \Gamma(s)} \times \right. \\
\left. \left\{ \left( \frac{a}{b_0} \right)^2 e^{-\left[ 2d/(n-2) + 2 \right] \kappa_\sigma} \right\}^{-s+k/2} \right. \\
\times \int_{C_1} dz \ D_{\phi}(z) \cot(\pi z) \left( z^2 - A_L^2 \right)^{-s+k/2} \Gamma \left( s - \frac{k}{2} \right) \right]. \tag{A10}
\]

Using the above definition, \( \zeta_{\phi}(s) \) is then expressed by

\[
\zeta_{\phi}(s) = 3 \tilde{\zeta}_5(s) + 2 \tilde{\zeta}_3(s). \tag{A11}
\]

Now we move the contour \( C_1 \) to the parallel line along the imaginary axis in order to account the poles for \( \cot(\pi z) \) in Eq. (A10) (See Fig. 1). The contour \( C_2 \) is replaced with lines passing just above the cuts associated with \( z = \pm A_L \). \( \zeta(s) \) is then given by

\[
A_L^2 = D^2 - N^2 \left( \frac{b_0}{a} \right)^2 e^{\left[ 2d/(n-2) + 2 \right] \kappa_\sigma}. \tag{A9}
\]
\[\tilde{\zeta}_k(s) = \frac{\pi^{3/2} a^{2s}}{2^{2n-1} \{\Gamma(2N + 1)\}^2 \Gamma(s)} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-[2d/(n-2)+2]k_s \sigma} \right\}^{-s+k/2} \sin \left\{ \pi \left( s - \frac{k}{2} \right) \right\} \]
\[\times \left\{ \int_0^\infty dx \ D_\phi(ix) \left( x^2 + A_L^2 \right)^{-s+k/2} \coth(\pi x) - \int_0^{A_L} dx \ \cot(\pi x) \ D_\phi(x) \left( A_L^2 - x^2 \right)^{-s+k/2} \right\}. \quad \text{(A12)}\]

where \(D_\phi(ix)\) is the polynomial with coefficients \(r_{Nk}\);

\[D_\phi(ix) = (-1)^D \frac{2a^2}{(2D)!} \left\{ x^2 + (D - 1)^2 \right\} \ldots \{ x^2 + 1 \}\]
\[\equiv (-1)^D \sum_{p=0}^{D-1} r_{Np} x^{2p+2}. \quad \text{(A13)}\]

The first term in Eq. (A10) comes from the integral along the imaginary axis and the second term in Eq. (A10) is contribution from the contours along the cuts of on the real axis, respectively. Substituting the relation

\[\coth(\pi x) = 1 + \frac{2}{e^{2\pi x} - 1}, \quad \text{(A14)}\]

into the first term of (A12), the function \(\tilde{\zeta}_k(s)\) is finally given by

\[\tilde{\zeta}_k(s) = \frac{\pi^{3/2} a^{2s}}{2^{2n-1} \{\Gamma(2N + 1)\}^2 \Gamma(s)} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-[2d/(n-2)+2]k_s \sigma} \right\}^{-s+k/2} \sin \left\{ \pi \left( s - \frac{k}{2} \right) \right\} \]
\[\times \left\{ \int_0^\infty dx \ D_\phi(ix) \left( x^2 + A_L^2 \right)^{-s+k/2} \frac{2}{e^{2\pi x} - 1} + \int_0^{A_L} dx \ D_\phi(x) \left( A_L^2 - x^2 \right)^{-s+k/2} \cot(\pi x) \right\}. \quad \text{(A15)}\]

2. spinor field

Here we discuss the zeta function regularization for the spin \(1/2\) field in \(H^n \times S^d\) when both \(n\) and \(d\) are odd. We define the generalized zeta function in \(H^n \times S^d\) for the spinor field as follows

\[\zeta_f(s) = \sum_{l=0}^\infty \frac{\Gamma \left( \frac{l+d}{2} \right)}{\Gamma \left( \frac{d}{2} \right) l!} \int_0^\infty d\lambda \ \mu(\lambda) \]
\[\times \left\{ \left( \frac{\lambda^2 - n(n-1)}{4a^2} + \left( \frac{l+d}{2} \right) \right)^{s} \right\}, \quad \text{(A16)}\]

where the spectral function \(\mu(\lambda)\) in odd dimension is given by \[21\]

\[\mu(\lambda) = \frac{\pi}{2^{2(n-2)} \{\Gamma(n/2)\}^2} \prod_{j=1/2}^{(n-2)/2} (\lambda^2 + j^2). \quad \text{(A17)}\]

Using \(D = d/2\), \(N = n/2\) instead of \(d\), \(n\) and running variables \(L = l + D\), we rewrite (A10) as

\[\zeta_f(s) = \sum_{L=D}^\infty D_f(L) \int_0^\infty d\lambda \ \mu(\lambda) \]
\[\times \left\{ \left( \frac{\lambda^2 - N(N-\frac{d}{2})}{a^2} + M^2 \right)^{s} \right\}, \quad \text{(A18)}\]

where the \(D_f(L)\), \(\Lambda_f(L)\) are given by

\[D_f(L) = \frac{1}{(2D)!} \left\{ L^2 - (D - 1)^2 \right\} \ldots \left\{ L^2 - \frac{1}{4} \right\}, \quad \text{A19}\]

\[\Lambda_f(L) = L^2 - D^2, \quad M_f^2 = \frac{\Lambda_f(L)}{b^2}. \quad \text{(A19)}\]

Integrating Eq. (A18) with respect to \(\lambda\), we find the expression
\( \zeta_f(s) = \frac{\pi^{3/2} a^{2s}}{2^{2n-1} \Gamma(2N)} \sum_{L=0}^{\infty} \frac{1}{\Gamma(s)} D_f(L) \left[ 12 \left\{ N \left( N - \frac{1}{2} \right) + a^2 M^2 \right\}^{s+5/2} - \frac{1}{2} \right] \Gamma \left( s - \frac{5}{2} \right) \\
+ 20 \left\{ N \left( N - \frac{1}{2} \right) + a^2 M^2 \right\}^{-s+3/2} \Gamma \left( s - \frac{3}{2} \right) + 9 \left\{ N \left( N - \frac{1}{2} \right) + a^2 M^2 \right\}^{-s+1/2} \Gamma \left( s - \frac{1}{2} \right) \right] \). \tag{A20}

Replacing the infinite sum for \( L \) by complex integration, the generalized zeta function is found to be

\[
\zeta_f(s) = \frac{\pi^{3/2} a^{2s}}{2^{2n-1} \Gamma(2N)^2} \Gamma(s) \times \left[ 6i \left\{ \left( \frac{a}{b_0} \right)^2 e^{-[2d/(n-2)+2] \kappa_\sigma} \right\}^{s+5/2} \int_{F_1} dz D_f(z) \tan(\pi z) \left( z^2 - A_L^2 \right)^{-s+5/2} \Gamma \left( s - \frac{5}{2} \right) \\
+ 10i \left\{ \left( \frac{a}{b_0} \right)^2 e^{-[2d/(n-2)+2] \kappa_\sigma} \right\}^{s+3/2} \int_{F_1} dz D_f(z) \tan(\pi z) \left( z^2 - A_L^2 \right)^{-s+3/2} \Gamma \left( s - \frac{3}{2} \right) \\
+ \frac{9i}{2} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-[2d/(n-2)+2] \kappa_\sigma} \right\}^{s+1/2} \int_{F_1} dz D_f(z) \tan(\pi z) \left( z^2 - A_L^2 \right)^{-s+1/2} \Gamma \left( s - \frac{1}{2} \right) \right], \tag{A21}
\]

where the contour \( F_1 \) in the complex plane is showed in Fig.3 and \( A_L^2 \) is given by

\( A_L^2 = -N \left( N - \frac{1}{2} \right) \left( \frac{b_0}{a} \right)^2 e^{[2d/(n-2)+2] \kappa_\sigma}. \tag{A22} \)

Note that there are two branch points \( z = \pm A_L \). \( \zeta_f(s) \) is then given by

\[
\tilde{\zeta}_k(s) = \frac{\pi^{3/2} a^{2s}}{2^{2n-1} \Gamma(2N)^2} \Gamma(s) \sin \left\{ \pi \left( s - \frac{k}{2} \right) \right\} \times \left[ \left\{ \left( \frac{a}{b_0} \right)^2 e^{-[2d/(n-2)+2] \kappa_\sigma} \right\}^{s+k/2} \int_0^\infty dx D_f(ix) \left( x^2 + A_L^2 \right)^{s+k/2} \tanh(\pi x) \\
- \int_0^{A_L} dx \tan(\pi x) D_f(x) \left( A_L^2 - x^2 \right)^{-s+k/2} \right], \tag{A25}
\]

where \( D_f(ix) \) is the polynomial with coefficients \( f_{N_k} \);

\[
D_f(ix) = (-1)^{(2D-1)/2} \frac{x^2 + (D - 1)^2 \cdots \{x^2 + 1\}}{(2D)!} \\
= (-1)^{(2D-1)/2} \sum_{p=0}^{(2D-1)/2} f_{N_p} x^{2p}. \tag{A26}
\]

The first term in Eq. \( \text{A25} \) comes from the integral along the imaginary axis and the second term in Eq. \( \text{A25} \) is
FIG. 3: The contour $F_1$ in Eq. (A21) is replaced by the contour $F_2$. Note that the contour $F_2$ avoid the branch point at $z = \pm A_L$.

Next, we provide the zeta function regularization for $\zeta L$. Note that the contour $F_2$ avoid the branch point at $z = \pm A_L$.

3. **vector field**

Next, we provide the zeta function regularization for the transverse vector field in the product geometry of $H^n \times S^d$. Now we define the generalized zeta function in $H^n \times S^d$ for the transverse vector field as

$$
\zeta V(s) = \sum_{l=0}^{\infty} \frac{(l + d - 2)!}{(d - 1)! \cdot l!} \int_0^\infty d\lambda \mu(\lambda) \left( \frac{\lambda^2 + \{a - \frac{(n - 1)}{2}\}^2 + 1}{a^2} + \frac{l(l + d - 1)}{b_0^2} e^{-2d/(n-2) + 2} \kappa \sigma \right)^{-s}, \quad (A29)
$$

where $\mu(\lambda)$ is the Plancherel measure. For the odd dimension, that is given by

$$
\mu(\lambda) = \frac{\pi \{\lambda^2 + (1 + \frac{n-3}{2})^2\}}{2^{(n-2)} \{\Gamma(n/2)\}^2} \prod_{j=0}^{(n-5)/2} \left( \lambda^2 + j^2 \right). \quad (A30)
$$

Using $D = (d - 1)/2$, $N = (n - 1)/2$ instead of $d$, $n$ and running variables $L = l + D$, we rewrite (A29) as

$$
\zeta V(s) = \sum_{L=D}^{\infty} D_V(L) \int_0^\infty d\lambda \mu(\lambda) \left( \frac{\lambda^2 + N^2}{a^2} + M^2 \right)^{-s}, \quad (A31)
$$

where the $D_V(L)$, $\Lambda_V(L)$ are given by

$$
D_V(L) = \frac{2L^2}{(2D)!} \{L^2 - (D - 1)^2\} \cdots \{L^2 - 1\},
$$

$$
\Lambda_V(L) = L^2 - D^2,
$$

$$
M_L^2 = \frac{\Lambda_V(L)}{b_0^2} e^{-2d/(n-2) + 2} \kappa \sigma. \quad (A32)
$$

Contribution from the contours along the cuts of on the real axis, respectively. Substituting the relation

$$
\tanh(\pi x) = 1 - \frac{2}{e^{2\pi x} + 1}, \quad (A27)
$$

into the first term of (A28), the function $\tilde{\zeta}_{k}(s)$ is finally given by

$$
\tilde{\zeta}_{k}(s) = \frac{\pi^{3/2} a^{2s}}{2^{2n-1} \{\Gamma(2N)\}^2} \sin \left\{ \pi \left( s - \frac{k}{2} \right) \right\} \left\{ \frac{a}{b_0} \right\}^2 e^{-2d/(n-2) + 2} \kappa \sigma \left[ 1 - \frac{2}{e^{2\pi x} + 1} \right]^{-s+k/2}
$$

$$
\times \left\{ \frac{1}{2} \sum_{p=0}^{(2D-1)/2} f_p(A_L^2)^{p+4-s} \Gamma(p + \frac{3}{2}) \Gamma(s - p - 4) \Gamma(s - \frac{k}{2}) \right\}^{-s+k/2}
$$

$$
+ \int_0^\infty dx D_f(ix) \left( x^2 + A_L^2 \right)^{-s+k/2} \frac{2}{e^{2\pi x} + 1} + \int_0^{A_L} dx D_f(x) \left( A_L^2 - x^2 \right)^{-s+k/2} \tan(\pi x) \right\}. \quad (A28)
$$
Integrating Eq. (A31) with respect to $\lambda$, we get the expression

$$\zeta_V(s) = \frac{\pi^{3/2}a^{2s}}{2^{2n-1} \Gamma(2N+1)^2} \frac{1}{\Gamma(s)} \sum_{L=D}^{\infty} D_V(L)$$

$$\times \left[ 3 \left( N^2 + a^2M^2 \right)^{-s+5/2} \Gamma \left( s - \frac{5}{2} \right) \right.$$

$$\left. + 8 \left( N^2 + a^2M^2 \right)^{-s+3/2} \Gamma \left( s - \frac{3}{2} \right) \right].$$

(A33)

Replacing the infinite sum for $L$ by complex integration, the generalized zeta function is given by

$$\zeta_V(s) = \frac{\pi^{3/2}a^{2s}}{2^{2n-1} \Gamma(2N+1)^2} \frac{1}{\Gamma(s)}$$

$$\times \left[ \frac{3i}{2} \left\{ \left( \frac{a}{b_0^2} \right)^2 e^{-[2d/(n-2)+2]s+\sigma} \right\}^{-s+5/2} \right.$$

$$\times \int_{C_1} dz D_V(z) \cot(\pi z) \left( z^2 - A_L^2 \right)^{-s+5/2} \Gamma \left( s - \frac{5}{2} \right)$$

$$\left. + i \left\{ \left( \frac{a}{b_0^2} \right)^2 e^{-[2d/(n-2)+2]s+\sigma} \right\}^{-s+3/2} \right.$$  

$$\times \int_{V_1} dz D_V(z) \cot(\pi z) \left( z^2 - A_L^2 \right)^{-s+3/2} \Gamma \left( s - \frac{3}{2} \right) \right].$$  

(A34)

Define the function $\tilde{\zeta}_k(s)$:

$$\tilde{\zeta}_k(s) = \frac{\pi^{3/2}a^{2s}}{2^{2n-1} \Gamma(2N+1)^2} \frac{1}{\Gamma(s)}$$

$$\times \left[ \frac{i}{2} \left\{ \left( \frac{a}{b_0^2} \right)^2 e^{-[2d/(n-2)+2]s+\sigma} \right\}^{-s+k/2} \right.$$  

$$\times \int_{C_1} dz D_V(z) \cot(\pi z) \left( z^2 - A_L^2 \right)^{-s+k/2} \right].$$  

(A36)

$\zeta_V(s)$ is then expressed by

$$\zeta_V(s) = 3 \tilde{\zeta}_s(s) + 8 \tilde{\zeta}_3(s).$$

(A37)

Now we move the contour $V_1$ to the parallel line along the imaginary axis in order to account the poles for $\cot(\pi z)$ in $A36$ (See Fig. 4). The contour $V_2$ is replaced with lines passing just above the cuts associated with $z = \pm A_L$. $\zeta(s)$ is then given by

$$\tilde{\zeta}_k(s) = \frac{\pi^{3/2}a^{2s}}{2^{2n-1} \Gamma(2N+1)^2} \frac{1}{\Gamma(s)}$$

$$\times \left\{ \left( \frac{a}{b_0^2} \right)^2 e^{-[2d/(n-2)+2]s+\sigma} \right\}^{-s+k/2}$$

$$\times \int_0^\infty dx D_V(ix) \left( x^2 - A_L^2 \right)^{-s} \coth(\pi x)$$

$$- \int_0^{A_L} dx \cot(\pi x) D_V(x) \left( A_L^2 - x^2 \right)^{-s+k/2} \right].$$

(A38)
where \( D_V(ix) \) is the polynomial with coefficients \( r_{nk} \):

\[
D_V(ix) = (-1)^D \frac{2\pi^2}{(2D)!} \left\{ x^2 + (D - 1)^2 \right\} \cdots \left\{ x^2 + 1 \right\}
\]

\[
\equiv (-1)^D \sum_{p=0}^{D-1} r_{np} x^{2p+2} . \tag{A39}
\]

The first term in Eq. (A36) comes from the integral along the imaginary axis and the second term in Eq. (A36) is contribution from the contours along the cuts of on the real axis, respectively. Substituting the relation

\[
\coth(\pi x) = 1 + \frac{2}{e^{\pi x} - 1} , \tag{A40}
\]

into the first term of (A38), the function \( \zeta_k(s) \) is finally given by

\[
\zeta_k(s) = \frac{\pi^{3/2} a^{2s}}{2^{2s-1} \left( \Gamma(2N + 1) \right)^2} \sin \left\{ \pi \left( s - \frac{k}{2} \right) \right\} \left\{ \frac{a}{b_0} \right\}^2 e^{-2d/(n-2)+2\kappa\sigma} \left[ \frac{\pi x}{2} \right]^{s+k/2}
\]

\[
\times \left\{ \frac{1}{2} \sum_{p=0}^{D-1} r_{np} \left( A_L^2 \right)^{p+4-s} \Gamma \left( s - \frac{4}{2} \right) \Gamma \left( s - p - 4 \right) \Gamma \left( s - \frac{k}{2} \right) \right\} - \int_0^\infty dx \left( x^2 + A_L^2 \right)^{-s+k/2} \left( 2 e^{\pi x} - 1 \right) + \int_0^{A_L} dx D_V(x) \left( A_L^2 - x^2 \right)^{-s+k/2} \cot(\pi x) \right\} . \tag{A41}
\]