ON THE DECOMPOSITION THEOREM FOR INTERSECTION DE RHAM COMPLEXES

MAO SHENG AND ZEBAO ZHANG

Abstract. We establish a positive characteristic analogue of intersection cohomology for polarized variations of Hodge structure. This includes: a) the decomposition theorem for the intersection de Rham complex; b) the $E_1$-degeneration theorem for the intersection de Rham complex of a periodic de Rham bundle; c) the Kodaira vanishing theorem for the intersection cohomology groups of a periodic Higgs bundle.

1. Introduction

In this paper, we aim to establish an algebraic theory on the $L^2$ and intersection cohomology theory of polarizable variations of Hodge structure [Zuc, CKS, KK]. It becomes possible after the sequence of works [DI], [OV] and [LSZ17]. In their foundational work in nonabelian Hodge theory in positive characteristic, Ogus-Vologodsky [OV], followed by Schepler in the logarithmic setting [Sche], established the following two fundamental results (in which $\text{MIC}^0_{p-1}(X_{\log}/k)$ (resp. $\text{HIG}^0_{p-1}(X'_{\log}/k)$) is a certain category of logarithmic flat (resp. Higgs) sheaves, see §3 for detail):

**Theorem 1.1** (Ogus-Vologodsky, Schepler). Let $k$ be a perfect field of positive characteristic $p$. Let $X$ be a smooth variety over $k$ and $D \subset X$ a normal crossing divisor. Suppose the pair $(X,D)$ is $W_2(k)$-liftable.

i) There is an equivalence of categories

\[ \text{MIC}^0_{p-1}(X_{\log}/k) \xrightarrow{C} \text{HIG}^0_{p-1}(X'_{\log}/k), \]

where $X' = X \times_{\sigma,k} k$ and $\sigma$ is the Frobenius automorphism of $k$.

ii) For $(E, \theta) \in \text{HIG}^0_{p-1}(X'_{\log}/k)$, set $(H, \nabla) = C^{-1}(E, \theta)$. Then, one has an isomorphism in the derived category $D(X')$:

\[ \tau_{<p-l}F_*\Omega^*(H, \nabla) \cong \tau_{<p-l}\Omega^*(E, \theta), \]

where $l$ is the level of $(H, \nabla)$, $F : X \to X'$ is the relative Frobenius, $\Omega^*(H, \nabla)$ is the logarithmic de Rham complex attached to $(H, \nabla)$ and $\Omega^*(E, \theta)$ is the logarithmic Higgs complex attached to $(E, \theta)$.

Part (i) is a positive characteristic analogue of the Simpson correspondence over the field of complex numbers [Sim]; Part (ii) for the neutral Higgs object $(\mathcal{O}_{X'}, 0)$.

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is the fundamental decomposition theorem of Deligne-Illusie [DI]. For the Gauß-Manin bundle associated to a semistable family over $k$, which is $W_2(k)$-liftable, the decomposition theorem of Ogus-Vologodsky was first established by Illusie [IL90]. Moreover, he proved also the $E_1$-degeneration property and a relative Kodaira type vanishing theorem (that has been recently generalized to general coefficients [Ax]). In our work, we shall establish the “intersection” version of all these results.

The technical heart of our theory is the “intersection” version of the decomposition theorem of Ogus-Vologodsky. Motivated by the theory over the field of complex numbers [CKS, KK], we are led to consider certain subcomplexes of the de Rham complex. We refer our reader to §2 for the precise definition of these subcomplexes.

**Theorem** (Theorem 3.1). Use notation in Theorem 1.1. One has an isomorphism in $D(X')$: $$\tau_{<p-l} F_\ast \Omega^*_{\text{int}}(H, \nabla) \cong \tau_{<p-l} \Omega^*_{\text{int}}(E, \theta).$$

As the intersection condition becomes empty when $D = \emptyset$, the above theorem in this case is nothing but the decomposition theorem of Ogus-Vologodsky. Our method of proof of Theorem 3.1 is to construct, in the spirit of Deligne-Illusie, an explicit quasi-isomorphism between $\tau_{<p-l} F_\ast \Omega^*_{\text{int}}(H, \nabla)$ and $\tau_{<p-l} \Omega^*_{\text{int}}(E, \theta)$ which induces a quasi-isomorphism on intersection complexes.

Our next main result concerns the $E_1$-degeneration property of de Rham complexes attached to periodic objects in $\text{MIC}^0_{p-1}(X_{\log}/k)$, as first introduced in [LSZ17]. We remark that a Gauß-Manin bundle, as considered in [IL90], is one-periodic. See §4.1 for details. By a de Rham sheaf, we mean a triple $(H, \nabla, \text{Fil})$ where the pair $(H, \nabla)$ is a flat sheaf, equipped with a Griffiths transverse and effective finite decreasing filtration $\text{Fil}$. Note that the de Rham complex $\Omega^*(H, \nabla)$ attached to a de Rham sheaf $(H, \nabla, \text{Fil})$ is naturally filtered.

**Theorem** (Corollary 4.13). Use notation in Theorem 1.1. Assume additionally that $X$ is proper over $k$. For a periodic de Rham bundle $(H, \nabla, \text{Fil})$ satisfying $\dim X + \text{rank}(H) \leq p$, the spectral sequence associated to the filtered complex $\Omega^*_{\text{int}}(H, \nabla)$ degenerates at $E_1$.

A more precise form of the above theorem appears in §4. As a corollary, we obtain the generalized Eichler-Shimura isomorphism in positive characteristic (see Corollary 7.13 and Proposition 5.2 [Zuc] for its $L^2$-analogue in one-dimensional case).

**Corollary** (Corollary 4.11). For a strict $p$-torsion logarithmic Fontaine module $(H, \nabla, \text{Fil}, \Phi)$, an object of the category $\text{MF}^\nabla_{(0, p-1)}(X_{\log}/k)$, one has equalities of cohomology groups:

$$\text{Gr}_{\text{Fil}} \mathbb{H}^a(X, \Omega^*_{\text{int}}(H, \nabla)) = \mathbb{H}^a(X, \text{Gr}_{\text{Fil}} \Omega^*_{\text{int}}(H, \nabla)) = \mathbb{H}^a(X, \Omega^*_{\text{int}} \text{Gr}_{\text{Fil}}(H, \nabla)).$$

Our last main result is a Kodaira type vanishing theorem in positive characteristic for intersection cohomology groups.
Theorem (Corollary 5.6). Use notation in Theorem 1.1. Assume additionally that $X$ is projective over $k$. Let $(E, \theta)$ be a periodic Higgs bundle. If $\dim X + \operatorname{rank}(E) \leq p$, then the following vanishing holds

$$\mathbb{H}^i(X, \Omega^*_\text{int}(E, \theta) \otimes L) = 0, \quad i > \dim X,$$

for any ample line bundle $L$ on $X$.

By spreading out, we obtain an algebraic proof of $E_1$-degeneration property of the Hodge to de Rham spectral sequence of the intersection de Rham complex associated to a semistable family in characteristic zero, first proved via the transcendental means by Cattani-Kaplan-Schmid [CKS] and Kashiwara-Kawai [KK].

2. Intersection de Rham complexes

In this section, we introduce the notion of intersection de Rham complex. It is convenient to do this in a bit more general setting, which includes also the notion of intersection Higgs complex. Let $X$ be a regular scheme which is smooth over a notherian scheme $S$ and is equipped with an $S$-relative normal crossing divisor $D \subset X$. Let $X_{\text{log}}$ be the log scheme posed by the divisor $D$ and $X_{\text{log}} \to S$ be the natural morphism of log schemes (where $S$ is equipped with the trivial log structure) [Ka]. Let $\lambda \in \Gamma(S, \mathcal{O}_S)$. A $\lambda$-connection on an $\mathcal{O}_X$-module $E$ is an additive map $\nabla : E \to \omega_{X/S} \otimes E$, satisfying the $\lambda$-Leibniz rule

$$\nabla(fs) = \lambda df \otimes s + f \nabla(s),$$

where $\omega_{X/S}$ is the sheaf of relative differentials of log schemes, $f \in \mathcal{O}_X$ and $s \in E$. One extends $\nabla : \omega_{X/S} \otimes E \to \omega_{X/S}^2 \otimes E$ by the usual rule:

$$\nabla(\omega \otimes s) = \lambda d\omega \otimes s - \omega \wedge \nabla(s).$$

We are interested in $\lambda$-connections which are flat, namely those satisfying the curvature vanishing condition $\nabla \circ \nabla = 0$. For a flat $\lambda$-connection $(E, \nabla)$, one forms as usual the complex $\Omega^*(E, \nabla)$ which is given by

$$E \xrightarrow{\nabla} \omega_{X/S} \otimes E \xrightarrow{\nabla} \omega_{X/S}^2 \otimes E \xrightarrow{\nabla} \cdots.$$  

The case $\lambda = 1$ (resp. $\lambda = 0$) is a log de Rham (resp. Higgs) complex. The above complex is naturally a graded module over the graded algebra $\omega_{X/S}^\bullet$. As there is a natural morphism of graded algebras

$$\Omega^\bullet_{X/S} \to \omega_{X/S}^\bullet,$$

we may regard $\Omega^*(E, \nabla)$ as a graded module over $\Omega^\bullet_{X/S}$. Let $x \in D$. It is not difficult to see that, under our assumption, there exists a set of prime elements $\{t_1, \cdots, t_n\} \subset \mathcal{O}_U$ ($n = \dim X/S$) for some étale open neighborhood $U$ of $x$ such that

(i) $D$ has $r = r(x)$ branches in $U$ which are defined by $\{t_i, 1 \leq i \leq r\}$;

(ii) the restriction of $\Omega_{X/S}$ to $U$ admits an $\mathcal{O}_U$-basis $\{dt_1, \cdots, dt_n\}$;
(iii) the restriction of $\omega_{X/S}$ to $U$ admits an $\mathcal{O}_U$-basis
\[
\{d \log t_1, \cdots, d \log t_r, dt_{r+1}, \cdots, dt_n\}.
\]
We call such a set $\{t_1, \cdots, t_n\} \subset \mathcal{O}_U$ a special system of log coordinates at $x$. It is the correct type of local coordinates we shall work with throughout the paper. Given a special system of local coordinates at $x$, we may write the operator into a sum of components
\[
\nabla = \nabla_1 \otimes d \log t_1 + \cdots + \nabla_r \otimes d \log t_r + \nabla_{r+1} \otimes dt_{r+1} + \cdots + \nabla_n \otimes dt_n,
\]
with $\nabla_i \in \text{End}(E|_U)$. For a multi-index $I = (i_1, \cdots, i_l)$, set
\[
\nabla_I = \nabla_{i_1} \circ \cdots \circ \nabla_{i_l}, \quad t_I = t_{i_1} \cdots t_{i_l}, \quad dt_I = dt_{i_1} \wedge \cdots \wedge dt_{i_l}, \quad d \log t_I = \frac{dt_I}{t_I}.
\]
Set also $\nabla_0 = Id$ and $dt_0 = 1 \in \mathcal{O}_U$.

**Lemma 2.1.** Notation as above. Then the $\Omega^\bullet_{X/S}|_U$-graded submodule of $\Omega^* (E, \nabla)|_U$, which is generated by
\[
\{\nabla_I(e) \otimes d \log t_I|I \subset \{1, \cdots, r(x)\}, \quad e \in E(U)\},
\]
induces the germ of a subcomplex in $\Omega^* (E, \nabla) \otimes \mathcal{O}_{X,x}$ which is independent of the choice of a special system of log coordinates at $x$.

**Proof.** Let $R_t \subset \Omega^* (E, \nabla)|_U$ be the submodule as defined in the lemma using the special system of log coordinates $t = \{t_1, \cdots, t_n\}$. Take a unit $u$ in $\mathcal{O}_U$ and set $t'_1 = u \cdot t_1$. Then
\[
d \log t'_1 = (1 + t_1 a_1) d \log t_1 + \cdots + (1 + t_r a_r) d \log t_r + a_{r+1} dt_{r+1} + \cdots + a_n dt_n,
\]
where $a_i \in \mathcal{O}_U, 1 \leq i \leq n$. With respect to the new special system of log coordinates $t' = \{t'_1, t'_2, \cdots, t'_n\}$, one writes
\[
\nabla = \nabla'_1 \otimes d \log t'_1 + \cdots + \nabla'_r \otimes d \log t_r + \nabla'_{r+1} \otimes dt_{r+1} + \cdots + \nabla'_n \otimes dt_n.
\]
It is easy to find that
\[
\nabla'_i = (1 + t_1 a_1)^{-1} \nabla_1, \quad \nabla'_i = -(1 + t_1 a_1)^{-1} a_i \nabla_1 + \nabla_i, \quad 2 \leq i \leq r.
\]
It follows that $\nabla'_1(e) \otimes d \log t'_1$ as well as
\[
\nabla'_i(e) \otimes d \log t_i = -(1 + t_1 a_1)^{-1} a_i \nabla_1(e) \otimes dt_i + \nabla_i(e) \otimes d \log t_i, \quad i \geq 2
\]
belong to $R_t$. Therefore $R_{t'} \subset R_t$. By symmetry, $R_{t'} = R_t$. But this actually shows that $R_t$ is independent of the choice of a special system of log coordinates on $U$. Now by a standard Galois descent argument, $R_t$ descends to a submodule of $\Omega^*(E, \nabla)$ over a Zariski open neighborhood of $x$ and hence a germ of subcomplexes in $\Omega^*(E, \nabla) \otimes \mathcal{O}_{X,x}$. By uniqueness in Galois descent, the induced germ is independent of the choices of an étale open neighborhood $U$ of $x$ and a special system of log coordinates on $U$, as claimed. \hfill $\square$

**Definition 2.2.** Notation as above. Let $E$ be a quasi-coherent $\mathcal{O}_X$-module on $X$ and $\nabla$ a flat $\lambda$-connection on $E$. The subcomplex of the de Rham complex $\Omega^*(E, \nabla)$ defined in Lemma 2.1 is called the intersection complex of $(E, \nabla)$ and denoted by $\Omega^*_{\text{int}}(E, \nabla)$. 
The $\lambda = 1$ case is called an intersection de Rham complex and the $\lambda = 0$ case is called an intersection Higgs complex. These two types of intersection complexes are our principal objects of study.

3. Decomposition theorem

In this section and the next, $k$ is a perfect field of characteristic $p > 0$, $X$ is a smooth variety over $k$ and $D \subset X$ a NCD. We assume the pair $(X, D)$ is $W_2(k)$-liftable (see Definition 8.11 [EV]), and we choose and then fix such a lifting $(\tilde{X}, \tilde{D})$ in our discussion (this is because the correspondence Theorem 1.1 depends on such a choice). As a convention, a subscript $'$ means the base change of an object via the Frobenius automorphism $\sigma$. The proof of Theorem 3.1 below is close in spirit to the one in [DI]. We refer our reader to [EV] §8-10 and [IL02] for excellent accounts of this theory.

3.1. Preliminary. Logarithmic nonabelian Hodge theory in characteristic $p$ studies the following categories: let $\text{MIC}^0_{p-1}(X_{\text{log}}/k)$ be the category of logarithmic flat sheaves $(H, \nabla)$ with poles along $D = \sum D_i$ with conditions

(i) the $p$-curvature of $\nabla$ is locally nilpotent of level $\leq p - 1$;

(ii) the residue of $\nabla$ along each component $D_i$ is locally nilpotent of level $\leq p - 1$;

(iii) $\text{Tor}_1^{O_{X,x}}(H_x, D_i, x) = 0$ for each $D_i$ and every closed point $x \in D_i$.

Correspondingly, let $\text{HIG}^0_{p-1}(X'_{\text{log}}/k)$ be the category of logarithmic Higgs sheaves $(E, \theta)$ with poles along $D'$ with conditions

(i) $\theta$ is locally nilpotent of level $\leq p - 1$;

(ii) $\text{Tor}_1^{O_{X,x}}(E_x, F_4 D_i, x) = 0$ for each component $D_i$ and every closed point $x \in D_i$.

We briefly recall the construction of the inverse Cartier transform $C^{-1}$ of Ogus-Volgodsky via the technique of exponential twisting [LSZ14] (see also Appendix [LSYZ] for the special log setting). Take an open affine covering $U = \{\tilde{U}_\alpha\}_{\alpha \in I}$ of $\tilde{X}$ and for each $\tilde{U}_\alpha$, take a log Frobenius lifting $\tilde{F}_\alpha : \tilde{U}_\alpha \to \tilde{U}_\alpha'$ (see Propositions 9.7 and 9.9 [EV]). For each $\alpha$, set $U_\alpha = \tilde{U}_\alpha \times k$ and $\omega_{U_\alpha}$ (resp. $\Omega_{U_\alpha}$) to be the restriction of $\omega_{X/k}$ (resp. $\Omega_{X/k}$) to $U_\alpha$. Then, the obstruction of lifting the relative Frobenius $F : X \to X'$ over $k$ to a morphism $\tilde{X} \to \tilde{X}'$ over $W_2(k)$ is a class in $H^1(X, F^* \Theta_{\tilde{X}/k})$ whose $\check{C}$ech representative is given by $\{h_{\alpha\beta} : F^* \omega_{U_{\alpha\beta}} \to \mathcal{O}_{U_{\alpha\beta}}\}$ with the relations

(i) Over $U_{\alpha\beta}$, $dh_{\alpha\beta} = \zeta_\beta - \zeta_\alpha$ with $\zeta_\alpha := \tilde{F}_\alpha[H_{[p]}] : F^* \omega_{U_\alpha} \to \omega_{U_\alpha}$.

(ii) Over $U_{\alpha\beta\gamma}$, $h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$.

There is an obvious modification for the case of normal crossing divisor by Galois descent. One takes a finite Galois cover over an open neighborhood of $x \in D$ such that $D$ becomes simple normal crossing upstairs. By functoriality of the exponential twisting, one has the canonical descent datum in the construction and therefore it descends to the one over the original Zariski open neighborhood of $x$. 
For an object $(E, \theta) \in \text{HIG}_{p-1}(X'_{\log}/k)$, its inverse Cartier transform $(H, \nabla) := C^{-1}(E, \theta) \in \text{MIC}_{p-1}(X_{\log}/k)$ is obtained by gluing local flat sheaves

$$(H_\alpha, \nabla_\alpha) := (F^*E|_{U_\alpha}, \nabla_{\text{can}} + (\text{id} \otimes \zeta_\alpha)(F^*\theta))$$

via the gluing data $\{G_{\alpha\beta} := \exp(1 \otimes h_{\alpha\beta})F^*\theta = \sum_{i=0}^{p-1} \frac{(1 \otimes h_{\alpha\beta})F^*\theta}{i!}\}$. It was shown that $(H, \nabla)$ is independent of the choices of data $(\tilde{U}_\alpha, \tilde{F}_\alpha)$. Let $\{t_1, \ldots, t_n\}$ be a special system of log coordinates on $\tilde{U}_\alpha$ (whose existence can be easily shown). We call the morphism defined by $\tilde{F}_\alpha(t_i) = t_i^p, 1 \leq i \leq n$ the standard log Frobenius lifting. For a finite complex $K^\bullet$ of $O_X$-modules, let $C(U', K^\bullet)$ be the simple complex associated to the Čech bicomplex of the covering $U'$ with values in $K^\bullet$. Since $U'$ consists of open affines, the natural augmentation $\epsilon : K^\bullet \to C(U', K^\bullet)$ is a quasi-isomorphism. The next theorem is the main goal of this section.

**Theorem 3.1.** Notation as above. There is an explicit quasi-isomorphism

$$\varphi : \tau_{<p-1}\Omega^\bullet(E, \theta) \to C(U', F_\tau\tau_{<p-1}\Omega^\bullet(H, \nabla))$$

which induces the quasi-isomorphism on the intersection subcomplexes. Namely, $\varphi$ restricts to the following quasi-isomorphism:

$$\varphi_{\text{int}} : \tau_{<p-1}\Omega^\bullet_{\text{int}}(E, \theta) \to C(U', F_\tau\tau_{<p-1}\Omega^\bullet_{\text{int}}(H, \nabla)).$$

**3.2. Notations.** The following notations will be used only in this section:

- $I$ The totally ordered index set of the open affine covering $U$.
- $\alpha = \alpha_0, \ldots, \alpha_r$ with $\alpha_i \in I$ and $\alpha_0 < \cdots < \alpha_r$.
- $U_{\alpha_0} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}$.
- $j(\jmath^1, \ldots, \jmath^r) \in \mathbb{N}_0^r$.
- $P(r; s)$ The set of partitions of $s$ into $r + 1$ nonnegative integers.
- $a(\jmath, p) \in P(r, s)$.
- $s! \prod_{k=1}^{s} \pi_{k=1}^{a(\jmath, p)} = \sum_{j=1}^{r} j^{s_k} \sum_{j=1}^{r} s_k j + l - k + r + 1 \in \mathbb{F}_p$ for $|\jmath| + r + s < p$.
- $\zeta_{\alpha}^s = F^*\omega_{\alpha_{\alpha}}^s \to \omega_{\alpha_0}^s$, which is the $s$-wedge product of $\zeta_\alpha$ for $s \geq 0$.
- $\varphi_{\jmath, p}^s = \sigma_{\alpha}^{s_0} \sigma_{\beta}^{s_1} \cdots \sigma_{\gamma}^{s_r} \in \text{End}(F^*E|_{U_\beta})$ with $\sigma_{\alpha\beta} = (1 \otimes h_{\alpha\beta})F^*\theta$.
- $\zeta_{\alpha\beta}^s \otimes \cdots \otimes \zeta_{\alpha\gamma}^r \in \text{End}(F^*E|_{U_\beta})$ with the obvious restriction map and the wedge product.
- $h_{\alpha_{\alpha_0} \cdots \alpha_{\alpha_r}} = F^*\omega_{\alpha_{\alpha_0} \cdots \alpha_{\alpha_r}} \to \mathcal{O}_{U_\beta}$, which is the composite of $h_{\alpha_0 \alpha_1} \otimes \cdots \otimes h_{\alpha_{r-1} \alpha_r}$ with the obvious restriction map and the multiplication.
- $\delta_i : \omega_{X'/k}^i \to \otimes^{i} \omega_{X'/k}$ for $i < p$ which is the canonical section of the projection $\otimes^{i} \omega_{X'} \to \omega_{X'}^i$. Explicitly, it is given by

$$\delta_i(\omega_1 \wedge \cdots \wedge \omega_i) = \frac{1}{i!} \sum_{\sigma \in S_i} \text{sgn}(\sigma) \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)},$$

for $\omega_1, \ldots, \omega_i$ local sections of $\omega_{X'/k}$.
3.3. Expression of $\varphi$. We define morphisms of $\mathcal{O}_{X'}$-modules

$$\varphi(0, s) : E \otimes \omega_{X'/k}^s \to \mathcal{C}^0(U', F_*\Omega^s(H, \nabla))$$

whose component along $U_\alpha$, $\alpha \in I$ is given by

$$\varphi(0, s)|_{U_\alpha} = (id \otimes \zeta^s_\alpha)F^*,$$  \hspace{1cm} (3.1.1)

and

$$\varphi(r, s) : E \otimes \omega_{X'/k}^{r+s} \to \mathcal{C}^r(U', F_*\Omega^s(H, \nabla))$$

for $r \geq 1$ whose component along $U_\alpha$ is given by

$$\varphi(r, s)|_{U_\alpha} = \sum_{\sigma, \delta} a(\sigma, \delta, p, q)F^*(id \otimes \delta_{r+s}),$$  \hspace{1cm} (3.1.2)

where the summation runs over $\sigma, \delta \in \mathbb{N}_0$ satisfying $|\sigma| + r + s < p$ and $p \in P(r, s)$. An immediate explanation of Formula (3.1.2) is in order: in the identification of $F^*E|_{U_\alpha}$ with $H|_{U_\alpha}$, we use the first index in $\alpha$. Namely, we have the following commutative diagram:

Putting $\varphi(r, s)$s altogether, we obtain an $\mathcal{O}_{X'}$-morphism for each $0 \leq d \leq p - 1$,

$$\varphi_d = \bigoplus_{r, s \geq 0, r + s = d} \varphi(r, s) : E \otimes \omega_{X'/k}^d \to \bigoplus_{r, s \geq 0, r + s = d} \mathcal{C}^r(U', F_*\Omega^s(H, \nabla)) = \mathcal{C}(U', F_*\Omega^s(H, \nabla))^d.$$

We have two basic properties of $\varphi_d$.

**Proposition 3.2.** Notation as above. Then the sequence $\varphi = (\varphi_0, \cdots, \varphi_{p-1})$ restricts to a morphism of complexes:

$$\varphi : \tau_{<p-1}\Omega^*(E, \theta) \longrightarrow \tau_{<p-1}\mathcal{C}(U', F_*\Omega^s(H, \nabla)).$$

The proof of the above proposition is straightforward and elementary. For the sake of exposition, we put details of the proof in Appendix. However, we want to single out one important special case which may be verified easily. Along the way, a useful homotopy formula relating two different log Frobenius liftings shows up. It is essentially local but we put it in a bit more general setting for a later use.

Assume now there exists a global log Frobenius lifting over $\tilde{X}$. Then, the inverse Cartier transform of $(E, \theta)$ is equal to

$$(H, \nabla) = (F^*E, \nabla_{can} + (id \otimes \zeta_{\tilde{F}})F^*\theta)$$

up to isomorphism, where $\zeta_{\tilde{F}} := \tilde{F}_{[p]}$ for a global log Frobenius lifting $\tilde{F}$. Apparently, the connection part depends on the choice of $\tilde{F}$. However, there is a canonical way to identify them. Namely, if we index two liftings by $\tilde{F}_\alpha$ and $\tilde{F}_\beta$, then the gluing isomorphism $G_{\alpha\beta}$ in §3.1 provides the identification. On the other
Proof. The problem is local. Fix an index

\[
\alpha \in \mathcal{O}_X.
\]

That is, one has a morphism of complexes 

\[
\varphi : \mathcal{O}_X \to \mathcal{O}_X.
\]

Proposition 3.5. for 

\[
(3.4.1)
\]

liftings. \tilde{\varphi} homotopy to sheaves of respect to the chosen coordinate system. Note that

\[
\zeta
\]

Lemma 3.4. Let \( \tilde{F}_\alpha, \tilde{F}_\beta \) be two global log Frobenius liftings over \( \tilde{X} \). Then \( \tilde{\varphi}_{\tilde{F}_\alpha} \) is homotopy to \( \tilde{\varphi}_{\tilde{F}_\beta} \) over the truncated subcomplex \( \tau_{<p-l} \mathcal{O}_X^* \) via the formula:

\[
(3.4.1) \tilde{\varphi}_{\tilde{F}_\alpha,s} - \tilde{\varphi}_{\tilde{F}_\beta,s} = F_s \log \varphi(1, s - 1) \alpha + (-1)^s \varphi(1, s) \alpha \circ \theta
\]

for \( 0 \leq s \leq p - l - 1 \). Consequently, the induced morphism on the cohomology sheaves of \( \tau_{<p-l} \mathcal{O}_X^* \) by \( \tilde{\varphi} \) is independent of the choice of global log Frobenius liftings.

The second basic property of \( \varphi_d \) is the following

Proposition 3.5. \( \varphi_d \) preserves the intersection condition. Consequently, the morphism \( \varphi \) in Proposition 3.2 restricts to a morphism between the intersection complexes. That is, one has a morphism of complexes

\[
\varphi_{\text{int}} : \tau_{<p-l} \mathcal{O}_X^{\text{int}}(E, \theta) \to \tau_{<p-l} \mathcal{C}(U', F_* \mathcal{O}_X^{\text{int}}(H, \nabla)).
\]

Proof. The problem is local. Fix an index \( \alpha \) and take a special system of log coordinates \( \{t_1, \ldots, t_n\} \) over \( U_{\alpha} \) such that \( \tilde{D} \) is defined by \( \prod_{i=1}^n t_i = 0 \). It suffices to show each \( \varphi(r, s)_{\alpha} \) maps an element of form

\[
\omega = \theta_J(e) \otimes d \log t_J \wedge d \log t_K \in \Omega^i_{\text{int}}(E, \theta), \quad J \subset \{1, \ldots, r\}, K \subset \{r + 1, \ldots, n\}
\]

in \( \mathcal{C}(U', F_* \mathcal{O}_X^{\text{int}}(H, \nabla)) \). Let us check first the \( r = 0 \) case. Consider the even more special case that the log Frobenius lifting \( \tilde{F} \) over \( U_{\alpha} \) takes the standard form with respect to the chosen coordinate system. Note that \( \zeta \) is independent of the choice of global log Frobenius liftings. For an arbitrary log Frobenius lifting \( \tilde{F}_\beta \), we make use of the homotopy formula 3.4.1 by setting \( \tilde{F}_\alpha \) to be the
standard one. It suffices to verify that \( \varphi(1,s)_{\alpha\beta} \) in \[3.4.1\] preserves the intersection condition. A typical summand in \( \varphi(1,s)_{\alpha\beta}, \) up to an element of \( \mathcal{O}_{U_n} \) in the coefficient, is given by \( \sigma_{\alpha\beta} \otimes \zeta_a^\alpha \otimes \zeta_b^\beta. \) A simple but key observation is that \( \zeta_b \) differs from \( \zeta_a \) by an exact regular differential. More precisely,

\[
\zeta_b(F^*d \log t_i) = \zeta_a(F^*d \log t_i) + dh_{\alpha\beta}(F^*d \log t_i) = d \log t_i + da_i,
\]

where \( a_i \) is an element in \( \mathcal{O}_{U_n}. \) It follows that \( \sigma_{\alpha\beta} \otimes \zeta_a^\alpha \otimes \zeta_b^\beta(F^*\omega) \) is equal to \( F^*\theta_F(\sigma_{\alpha\beta}(e)) \otimes d \log t_J \wedge d \log t_K + F^*\theta_F(\sigma_{\alpha\beta}(e)) \otimes d \log t_J \wedge \eta = \nabla_J(e') \otimes d \log t_J \wedge \eta', \)

where \( e' = \sigma_{\alpha\beta}(e), \) \( J' \subset J \) and \( \eta, \eta' \in \Omega^*_U. \) This shows the \( r = 0 \) case. Now we consider a general \( r. \) In the expression \( \sigma_{\alpha}^j \otimes \zeta_{\alpha}^j \), one replaces \( \zeta_{\alpha} \) for each \( i \neq 0 \) with \( \zeta_{\alpha a} + dh_{\alpha a a_i}. \) The same argument as above shows that \( \sigma_{\alpha}^j \otimes \zeta_{\alpha}^j(F^*\omega) \) is equal to \( \varphi(0,s)_{\alpha a}(\omega') \wedge \eta \) for some \( \omega' \) in \( \Omega^*_\text{int}(E, \theta) \) and a regular form \( \eta \) over \( U_n'. \) Since we have shown that \( \varphi(0,s)_{\alpha a}(\omega') \) lies in the intersection subcomplex, it follows that \( \varphi(r,s)_{\alpha}^j(\omega) \) also lies in the intersection subcomplex, as claimed.

\[\square\]

A byproduct of the above proof is that Formula \[3.4.1\] is also a homotopy on intersection complexes.

3.4. Cartier isomorphism. A simple way to interpret the classical Cartier isomorphism (see e.g. Theorem 9.14 [EV], Theorem 3.5 [IL02]) is that it computes the cohomology sheaves of the Frobenius push-forward of the de Rham complex. A beautiful generalization was found by A. Ogus for an arbitrary (logarithmic) flat sheaf (Theorems 1.2.1, 3.1.1 [Ogus04]), which is one of threads leading to the nonabelian Hodge theory in positive characteristic [OV]. As the classical one, the generalized Cartier isomorphism of Ogus is of purely positive characteristic. Its relation with the decomposition theorem of Ogus-Vologodsky (with liftability assumption) is explained in Remark 2.30 [OV]. The Cartier isomorphism we are going to present, which is related to that remark, is essentially local (assuming liftings of schemes and Frobenius) but also with respect to the intersection condition.

**Theorem 3.6.** Notation as Theorem \[3.3\]. Assume additionally a global log Frobenius lifting exists. Then Formula \[3.3.1\] induces quasi-isomorphisms

\[
\tilde{\varphi} : \tau_{<p-1}^*\Omega^*(E, \theta) \cong \tau_{<p-1}^*F_\ast\Omega^*(H, \nabla)
\]

and

\[
\tilde{\varphi}_{\text{int}} : \tau_{<p-1}^*\Omega^*_{\text{int}}(E, \theta) \cong \tau_{<p-1}^*F_\ast\Omega^*_{\text{int}}(H, \nabla).
\]

In the remaining paragraphs, we shall prove that both \( \tilde{\varphi} \) and \( \tilde{\varphi}_{\text{int}} \) in Lemma \[3.3\] are quasi-isomorphisms. The problem is local and thus we may assume that \( \tilde{X} \) to be affine with a special system of log coordinates \( \{t_1, \ldots, t_n\}. \) By Lemma \[3.4\], we may also assume further that \( \tilde{F} \) takes the standard form \( t_i \mapsto t_i^p \) for all \( i. \) Therefore,

\[
(3.6.1) \zeta(F^*dt_i) = \frac{dt_i}{t_i}, \quad 1 \leq i \leq r, \quad \zeta(F^*dt_i) = t_i^{p-1}dt_i, \quad r + 1 \leq i \leq n.
\]
By fixing these gauges, the problem turns to be purely linear algebraic. Let us show the case $\tilde{\varphi}$ first. We set
\[
\omega_i = \frac{dt_i}{t_i}, \quad 1 \leq i \leq r, \quad \omega_i = dt_i, \quad r + 1 \leq i \leq n.
\]
and
\[
P_0 = \{1\}, \quad P_m = \{\omega_{j_1} \wedge \cdots \wedge \omega_{j_m} : 1 \leq j_1 < \cdots < j_m \leq n\}, \quad 1 \leq m \leq n.
\]
As $P_m$ is a basis for $\omega_X^m/k$, we may identify $\Omega^*(E, \theta)$ with the following complex $EP_*$:
\[
EP_0 \xrightarrow{\theta^\wedge} EP_1 \xrightarrow{\theta^\wedge} \cdots \xrightarrow{\theta^\wedge} EP_n,
\]
where $EP_i := \mathbb{F}_p P_i \otimes_{\mathbb{F}_p} E$. Likewise, one defines the set $B_m, 0 \leq m \leq n$ consisting of degree $m$ elements of form
\[
t_1^{i_1} \cdots t_n^{i_n} \cdot \omega_{\alpha_1} \wedge \cdots \wedge \omega_{\alpha_m},
\]
where $0 \leq i_j \leq p-1, 1 \leq j \leq n$ and $1 \leq \alpha_1 < \cdots < \alpha_m \leq n$. Clearly, $B_m$ is a basis for $F_*\omega_X^m/k$. Therefore, we may identify $F_*\Omega^*(H, \nabla)$ with the complex $EB_*$:
\[
EB_0 \xrightarrow{d_0^\wedge} EB_1 \xrightarrow{d_0^\wedge} \cdots \xrightarrow{d_0^\wedge} EB_n,
\]
where $EB_m := \mathbb{F}_p B_m \otimes_{\mathbb{F}_p} E = F_*(\omega_X^m \otimes_{\mathcal{O}_X} F^*E)$. By the expression of $\zeta$ (Formula 3.6.1), $\zeta^m$ induces an injective map $\zeta^m : P_m \to B_m$ for $0 \leq m \leq n$. Denote $Q_m$ for its image and $N_m$ the complement of $Q_m$ in $B_m$.

**Lemma 3.7.** The complex $EB_*$ decomposes into a direct sum:
\[
EB_* = EQ_* \oplus EN_*,
\]
where $EQ_m = \mathbb{F}_p Q_m \otimes_{\mathbb{F}_p} E$ and $EN_m = \mathbb{F}_p N_m \otimes_{\mathbb{F}_p} E$, and $\tilde{\varphi}$ induces an isomorphism
\[
\tilde{\varphi} : EP_* \cong EQ_*.
\]

**Proof.** By Lemma 3.3 it suffices to show $\tilde{\varphi}_m$ induces an isomorphism $EP_m \cong EQ_m$ for each $0 \leq m \leq n$. By the injectivity of $\zeta^m$, this is clear. \qed

The hard part of this linear algebraic problem is to show the following

**Lemma 3.8.** $EN_*$ is exact.

Note that the Cartier isomorphism in the logarithmic case, i.e. the first statement of Theorem 3.6, follows from Lemmas 3.7, 3.8. Before giving the proof of the foregoing lemma, we first observe that the collection of finite dimensional $\mathbb{F}_p$-vector spaces $\mathbb{F}_p B_*$, equipped with the exterior differential $d$, is naturally a subcomplex of $EB_*$ for $(E, \theta) = (\mathcal{O}_X^\prime, 0)$. Moreover, one has the following

**Sublemma 3.9.** The complex $\mathbb{F}_p B_*$ decomposes into the direct sum of $\mathbb{F}_p Q_*$ whose differential is zero and $\mathbb{F}_p N_*$ which is exact.
Proof. Note that Theorem 3.6 holds for the case $(\mathcal{O}_{X'}, 0)$ since it is nothing but the classical Cartier isomorphism. Under the morphism $\hat{\phi}$, $\mathbb{F}_pP_s \subset EP_s$, which has zero differential, is mapped isomorphically onto $\mathbb{F}_pQ_s$. Thus $\mathbb{F}_pQ_s$ with zero differential is subcomplex of $\mathbb{F}_pB_s$. Because $\mathbb{F}_pB_s = \mathbb{F}_pQ_s \oplus \mathbb{F}_pN_s$ and $H^*(\mathbb{F}_pB_s) = H^*(\mathbb{F}_pP_s) = H^*(\mathbb{F}_pQ_s)$ (which results from the Cartier isomorphism), it follows that $H^*(\mathbb{F}_pN_s) = 0$ and hence $\mathbb{F}_pN_s$ is exact. \[\square\]

The next sublemma gives an explicit expression of the $\mathcal{O}_{X'}$-linear map $d_\theta : EB_0 \to EB_1$, whose proof is a straightforward calculation.

Sublemma 3.10. Write $\theta = \sum_{i=1}^n \theta_i \otimes \omega_i$ where the coefficients $\{\theta_i\}_{1 \leq i \leq n} \subset \text{End}_{\mathcal{O}_{X'}}(E)$ are mutually commutative and nilpotent. Then it holds that

$$d_\theta(e \otimes t_i^j) = \begin{cases}
(j \cdot id_E + \theta_i)(e) \otimes t_i^{j-1}dt_i, & 1 \leq i \leq r, \ 0 \leq j \leq p - 1; \\
(j \cdot id_E + t_i\theta_i)(e) \otimes t_i^{j-1}dt_i, & r + 1 \leq i \leq n, \ 1 \leq j \leq p - 1; \\
\theta_i(e) \otimes t_i^{p-1}dt_i, & r + 1 \leq i \leq n, \ j = 0.
\end{cases}$$

We proceed now to the proof of Lemma 3.8

Proof. Write the differential in the complex $EN_s$ at the $m$-th place by $(d_\theta)_m$. For $m = -1, n + 1$, we put $EN_m = 0$. Our aim is to show the equality

$$\ker((d_\theta)_m) = \text{im}((d_\theta)_{m-1}).$$

We shall prove the special case $\theta = 0$ first, in a way which can be adapted for the general case. To start with, we claim that there exists a basis $f_i$ for $\mathbb{F}_pN_i$ for $i = m - 1, m, m + 1$ such that the differential $d$ at $m - 1$ resp. $m$-th places, denoted by $d_{m-1}$ resp. $d_m$, take the following format:

$$d_{m-1}(f^I_{m-1}, f^H_{m-1}) = (f^I_m, f^H_m) \begin{pmatrix} A_{m-1} & B_{m-1} \\ C_{m-1} & D_{m-1} \end{pmatrix};$$

$$d_m(f^I_m, f^H_m) = (f^I_{m+1}, f^H_{m+1}) \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix},$$

where the minors $C_{m-1}$ and $A_m$ are of maximal rank. It follows from Sublemma 3.9 that

$$\text{rank}(d_{m-1}) + \text{rank}(d_m) = \dim \mathbb{F}_pN_m.$$

Hence, if either $\text{rank}(C_{m-1}) = 0$ or $\text{rank}(A_m) = 0$, the existence of such bases is straightforward. So we shall assume $\text{rank}(A_m) > 0$. We first choose bases $f_m$ and $f_{m+1}$ such that $d_m$ takes the claimed format, and an arbitrary basis $f_{m-1}$. Let $f_m \begin{pmatrix} u_m \\ v_m \end{pmatrix}$ be an element of $\mathbb{F}_pN_m$ lying in $\ker(d_{m+1})$. The column vectors $u_m$ and $v_m$ satisfy the relation

$$u_m = -A_m^{-1}B_m v_m.$$

It follows from the maximality of $\text{rank}(A_m)$ that $v_m$ can be arbitrary (by dimension counting). By the exactness of $\mathbb{F}_pN_s$ at the $m$-th place, the equation

$$\begin{pmatrix} A_{m-1} & B_{m-1} \\ C_{m-1} & D_{m-1} \end{pmatrix} \begin{pmatrix} u_{m-1} \\ v_{m-1} \end{pmatrix} = \begin{pmatrix} u_m \\ v_m \end{pmatrix}$$

is satisfied.
has always solutions. This implies that the submatrix \( (C_{m-1} \ D_{m-1}) \) has the maximal rank. Therefore, after a possible rearrangement of basis elements in \( f_{m-1} \), we may indeed assume that the square submatrix \( C_{m-1} \) is of maximal rank. The claim is proved. Using such bases, we may always take
\[
(3.10.1) \quad u_{m-1} = C_{m-1}^{-1}v_m; \quad v_{m-1} = 0
\]
as a solution of the above equation. An inspection of the expression of \( d_\theta \) shows that \( d_0 = id_E \otimes d \). Start with an element in \( \ker(d_0)_m \subset EN_m \) is represented by \( (f^I_m, f^{II}_m) \otimes \begin{pmatrix} u_m \\ v_m \end{pmatrix} \), where \( u_m \) and \( v_m \) are \( E \)-valued vectors. Therefore, by Formula 3.10.1, the element \( (f^I_{m-1}, f^{II}_{m-1}) \otimes \begin{pmatrix} C_{m-1}^{-1}v_m \\ 0 \end{pmatrix} \) \( \in EN_{m-1} \) maps under \( (d_0)_{m-1} \) onto the starting element. This proves the trivial Higgs field case. For the general case, we exploit on the fact \( \theta \) is a nilpotent operator. Let \( S = \Gamma(X', \mathcal{O}_{X'}) \) and \( R \subset \text{End}_{\mathcal{O}_{X'}}(E) \) be the sub \( S \)-algebra defined as the image of the \( S \)-morphism
\[
S[x_1, \ldots, x_n] \rightarrow \text{End}_{\mathcal{O}_{X'}}(E), \ x_i \mapsto \theta_i.
\]
Let \( R^+ \) be the ideal of \( R \) which is the image of the big ideal of \( S[x_1, \ldots, x_n] \). As each \( \theta_i \) is nilpotent of bounded level, any element in \( R^+ \) is nilpotent. The expression of \( d_\theta \) reveals the following fact:

\[
(d_\theta)_{m-1}((f^I_{m-1}, f^{II}_{m-1}) \otimes \begin{pmatrix} u_{m-1} \\ v_{m-1} \end{pmatrix}) = (f^I_m, f^{II}_m) \otimes \begin{pmatrix} (A_{m-1})_{\theta} & (B_{m-1})_{\theta} \\ (C_{m-1})_{\theta} & (D_{m-1})_{\theta} \end{pmatrix} \begin{pmatrix} u_m \\ v_m \end{pmatrix},
\]

\[
(d_\theta)_m((f^I_m, f^{II}_m) \otimes \begin{pmatrix} u_m \\ v_m \end{pmatrix}) = (f^I_{m+1}, f^{II}_{m+1}) \otimes \begin{pmatrix} (A_m)_{\theta} & (B_m)_{\theta} \\ (C_m)_{\theta} & (D_m)_{\theta} \end{pmatrix} \begin{pmatrix} u_m \\ v_m \end{pmatrix},
\]

with the important property that the minors
\[
(C_{m-1})_{\theta} = C_{m-1} \mod R^+, \quad (A_m)_{\theta} = A_m \mod R^+.
\]
Hence both \( (C_{m-1})_{\theta} \) and \( (A_m)_{\theta} \) are invertible. The rest argument is exactly the same as the trivial Higgs field case. As a consequence of Formula 3.10.1, the solution vector for \( d_\theta \) takes the following form which we shall use in the proof below for the intersection complex.
\[
(3.10.2) \quad u_{m-1} = (C_{m-1})_{\theta}v_m, \quad v_{m-1} = 0.
\]
This completes the proof. \( \square \)

We shall modify the above proofs to show the quasi-isomorphism for \( \hat{\varphi}_{\text{int}} \). In order to get a good description of intersection complexes, we need to introduce the notion of multi-weight.

**Definition 3.11.** Let \( \beta \in P_m \) or \( B_m, 0 \leq m \leq n \). We define the multi-weight of \( \beta \) by
\[
w(\beta) = (\epsilon_1, \ldots, \epsilon_r) \in \{0, 1\}^r \subset \mathbb{N}_0^r,
\]
where \( \epsilon_i = 1 \) if \( \frac{d_i}{t_i} \) is a factor of \( \beta \), otherwise \( \epsilon_i = 0 \).
Definition 3.12. For \( w = (w_1, ..., w_r) \in \mathbb{N}_0^r \), we define \( E_w \subset E \) by
\[
E_w = \sum_{\alpha + \beta = w; \alpha, \beta \in \mathbb{N}_0^r} t^\alpha \theta^\beta E,
\]
where \( t^\alpha = t_1^{\alpha_1} \cdots t_r^{\alpha_r} \) and \( \theta^\beta = \theta_1^{\beta_1} \cdots \theta_r^{\beta_r} \). If \( e \in E_w \), then we say \( e \) has multi-weight \( \geq w \) and write it as \( w(e) \geq w \). Analogously, we define \( R_w \subset R \) by
\[
R_w = \sum_{\alpha + \beta = w; \alpha, \beta \in \mathbb{N}_0^r} t^\alpha \theta^\beta R.
\]
An element \( \vartheta \in R_w \) is of weight \( \geq w \) and denoted by \( \vartheta(\vartheta) \geq w \).

The notion of multi-weight has the following basic properties: for two elements \( w, w' \in \mathbb{N}_0^r \), we say \( w \leq w' \) if \( w_i \leq w'_i \) for each \( 1 \leq i \leq r \). Clearly, if \( w \leq w' \), then \( E_w \subset E_{w'} \). Moreover, \( R_w R_{w'} \subset R_{w+w'} \) and \( R_w E_{w'} \subset E_{w+w'} \).

The first merit of the notion is an easy identification of an intersection complex. We may identify \( \Omega^*_{\text{int}}(E, \theta) \subset \Omega^*(E, \theta) \) with the following complex \( EP_{\text{int}*} \subset EP_* \):
\[
\bigoplus_{\beta \in P_0} E_{w(\beta)} \beta \xrightarrow{\theta_1} \bigoplus_{\beta \in P_1} E_{w(\beta)} \beta \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_{n-1}} \bigoplus_{\beta \in P_n} E_{w(\beta)} \beta,
\]
where \( E_{w(\beta)} = \mathbb{P}_{\mathbb{R}} \beta \otimes_{\mathbb{P}} E_{w(\beta)} \). Similarly, \( F_* (\Omega^*_{\text{int}}(H, \nabla)) \subset F_* (\Omega^*(H, \nabla)) \) can be identified with the following complex \( EB_{\text{int}*} \subset EB_* \):
\[
\bigoplus_{\beta \in B_0} E_{w(\beta)} \beta \xrightarrow{d_{\theta_1}} \bigoplus_{\beta \in B_1} E_{w(\beta)} \beta \xrightarrow{d_{\theta_2}} \cdots \xrightarrow{d_{\theta_{n-1}}} \bigoplus_{\beta \in B_n} E_{w(\beta)} \beta.
\]

Sublemma 3.13. Let \( \beta \in B_m \) and write
\[
(d_\theta)_m (e \otimes \beta) = \sum_i \vartheta_i (e) \otimes \beta_i
\]
with \( \beta_i \in B_{m+1} \). Then either \( \vartheta_i \in R \) is a unit and \( w(\beta) = w(\beta_i) \) or \( \vartheta_i \in R^+ \) and \( w(\beta_i) = w(\beta) + (0, \ldots, 0, 1, 0, \ldots, 0) \).

Proof. For a disjoint union \( I \coprod J \coprod K = \{1, ..., n\} \) with \( |I| + |J| = m \) and multi-indexes
\[
\alpha_J \in \{0, ..., p - 2\}^{J_1} \times \{0, ..., p - 1\}^{J_2}, \quad \beta_K \in \{0, ..., p - 1\}^K,
\]
with \( J_1 = J \cap \{1, ..., r\}, J_2 = J - J_1 \), we have
\[
d_\theta (e \otimes (t_k^{\beta_k} dt_{I_k} \wedge t_j^{\alpha_j} dt_{J_j})) = \sum_{k \in K} d_\theta (e \otimes t_k^{\beta_k}) \wedge (t_k^{\beta_{K-k}} dt_{I_k} \wedge t_j^{\alpha_j} dt_{J_j}),
\]
where \( \beta_K = \{\beta_i\}_{i \in K}, \beta_K - \{k\} = \{\beta_i\}_{i \in K - \{k\}} \). The sublemma then follows from the explicit expression in Sublemma 3.10.

The second half of Theorem 3.6 follows from the next lemma.

Lemma 3.14. One has the decomposition \( EB_{\text{int}*} = EQ_{\text{int}*} \oplus EN_{\text{int}*} \), where \( \varphi_{\text{int}} \) induces isomorphism \( EP_{\text{int}*} \cong EQ_{\text{int}*} \) and \( EN_{\text{int}*} \) is exact.
Proof. Only the last part requires a proof. Let us continue with the proof of Lemma 3.8. Using the notation therein, we claim that, given an \( E \)-valued vector \( v_m \) such that

\[
(f_{m}^{I}, f_{m}^{II}) \otimes \left(-\left(A_{m}\otimes (B_{m})_{\theta}v_m\right)\right) \in EN_{int,m},
\]

the corresponding solution vector

\[
(f_{m-1}^{I}, f_{m-1}^{II}) \otimes \begin{pmatrix} u_{m-1} & 0 \\
0 & 0 \end{pmatrix} = (f_{m-1}^{I}, f_{m-1}^{II}) \otimes \begin{pmatrix} (C_{m})_{\theta}^{-1}v_m \\\n0 \end{pmatrix}
\]

actually belongs to \( EN_{int,m-1} \). Clearly, the claim implies the exactness of \( EN_{int,s} \) at the \( m \)-th place.

Let \( s \) be the rank of \( d_{m-1} \). Say \( f_{m-1}^{I} = \{ f_{m-1}^{1}, \ldots, f_{m-1}^{s} \} \) and \( f_{m}^{II} = \{ f_{m}^{1}, \ldots, f_{m}^{s} \} \).

Write \( f_{m}^{II} \otimes v_m \) explicitly as \( \sum_{1 \leq i \leq s} f_{m}^{i} \otimes v_m^i \) and \( f_{m-1}^{I} \otimes u_{m-1} \) as \( \sum_{1 \leq i \leq s} f_{m-1}^{i} \otimes w_{m-1}^i \). By the above assumption, \( w(v_m^i) \geq w(f_{m-1}^{i}) \) for each \( 1 \leq i \leq s \). By the basic property of multi-weight, the claim pins down to the following inequality

\[ w(u_{m-1}^i) \geq w(f_{m-1}^{i}), \quad 1 \leq i \leq s. \]

Without loss of generality, we prove the case \( i = 1 \) only. This becomes mostly obvious if we expand out \( u_{m-1} = (C_{m-1})_{\theta}v_m \) (see Formula 3.10.2). Write \( (C_{m-1})_{\theta} \) by \( P = (p_{ij}) \) temporarily, and let \( P_{\sigma} = (p_{\sigma ij}) \) be the adjacent matrix of \( P \). Then

\[
u_{m-1}^{1} = \det(P)^{-1} \cdot \sum_{1 \leq i \leq s} p_{1i}^{\sigma} v_{m}^{i}.
\]

It suffices to show for each \( 1 \leq i \leq s \), \( w(p_{1i}^{\sigma} v_{m}^{i}) \geq w(f_{m-1}^{i}) \). Again, without loss of generality, we prove the case \( i = 1 \) only. That is, we claim that

\[ w(p_{11}^{\sigma}(v_{m}^{1})) \geq w(f_{m-1}^{1}). \]

As

\[ p_{11}^{\sigma} = \sum_{\sigma} sgn(\sigma)p_{2\sigma(2)} \cdots p_{s\sigma(s)}, \]

it suffices to show

\[ w(p_{2\sigma(2)} \cdots p_{s\sigma(s)})(v_{m}^{1})) \geq w(f_{m-1}^{1}), \]

for any permutation \( \sigma \) of the set \( \{2, \ldots, s\} \). By Sublemma 3.13, we have for any \( i \),

\[ w(p_{\sigma(i)}^{\sigma}) \geq w(f_{m}^{\sigma(i)}) - w(f_{m-1}^{i}). \]

Moreover, as the matrix \( P \) is invertible, there must exist some permutation \( \tau \) such that each component in the product \( p_{2\tau(2)} \cdots p_{s\tau(s)} \) is a unit. By Sublemma 3.13 again,

\[
\sum_{i=1}^{s} w(f_{m}^{i}) = \sum_{i=1}^{s} w(f_{m-1}^{i}).
\]
Therefore, we compute that
\[
\begin{align*}
  w(p_{2r(2)} & \cdots p_{s\sigma(s)}(v_{m_1}^1)) \\
  \geq & \sum_{i=2}^{s} \left[w(f_{m_i}^1) - w(f_{m_i-1})\right] + w(f_{m_1}^1) \\
  = & \sum_{i=1}^{s} w(f_{m_i}^1) - \sum_{i=2}^{s} w(f_{m_i-1}) \\
  = & \sum_{i=1}^{s} w(f_{m_i-1}) - \sum_{i=2}^{s} w(f_{m_i-1}) \\
  = & w(f_{m-1}).
\end{align*}
\]

The claim is proved. \(\square\)

3.5. **Quasi-isomorphism.** We are in the position to prove that the morphisms of complexes \(\varphi, \varphi_{\text{int}}\), constructed in Subsection 3.3, are quasi-isomorphisms. Note that the problem is local in nature and therefore we may assume the existence of log Frobenius lifting. Our task is therefore to relate it with the Cartier isomorphism (Theorem 3.6). This is fulfilled by the next lemma, which results directly from Propositions 3.2 and 3.5 (that is, the precise content of \(\varphi\) is irrelevant).

**Lemma 3.15.** Notation and assumption as in Theorem 3.6. Consider the following diagram:

\[
\begin{array}{ccc}
\tau_{<p-l} F_s \Omega^*(H, \nabla) & \to & \tau_{<p-l} \mathcal{C}(U', F_s \Omega^*(H, \nabla)) \\
\uparrow\varphi & & \uparrow\varphi \\
\tau_{<p-l} \Omega^*(E, \theta) & & \tau_{<p-l} \Omega^*(E, \theta),
\end{array}
\]

where \(\epsilon\) is the augmentation morphism. The composite morphism \(\epsilon \circ \tilde{\varphi}\) and the morphism \(\varphi\) induce the same morphism on the cohomology sheaves. The same statement holds for \(\varphi_{\text{int}}\) of the intersection complexes.

**Proof.** The proof for the intersection complexes is completely analogous. So we give the proof for the whole complex only. By definition, the composite \(\epsilon \circ \tilde{\varphi}\) is nothing but the (0, \(i\))-component of \(\varphi\) acting on an element of degree \(i\) for \(0 \leq i \leq p-l-1\). For \(i = 0\), there is nothing to prove. So we may assume \(i \geq 1\) in the following. Let \(\omega \in E \otimes \omega^{i',j}/\kappa\) be such an element representing the class \([\omega]\) in the \(i\)-th cohomology sheaf of the complex \(\tau_{<p-l} \Omega^*(E, \theta)\). For simplicity we denote \(C^{r,s}\) for \(\mathcal{C}(U', \tau_{<p-l} F_s \Omega^*(H, \nabla))\). We claim that there exist elements \(\varpi^{r,s} \in C^{r,s}\) with \(r+s = i-1\) such that

\[
\varphi(\omega) = \epsilon \circ \tilde{\varphi}(\omega) + \nabla(\varpi^{0,i-1}) + \sum_{r+s=i-1} D(\varpi^{r,s}),
\]

where \(\nabla(\varpi^{0,i-1})\) satisfies the relation \(\delta(\nabla(\varpi^{0,i-1})) = 0\). The lemma follows from the claim: since \(\delta(\nabla(\varpi^{0,i-1})) = 0\), there is a (unique) section \(\varpi'\) of \(F_s \Omega^*(H, \nabla)\) such that \(\epsilon(\varpi') = \nabla(\varpi^{0,i-1})\). The section \(\varpi'\) has the property that \(\nabla(\varpi') = 0\) and hence defines a class in the \(i\)-th cohomology sheaf of the complex \(\tau_{<p-l} F_s \Omega^*(H, \nabla)\). By its very expression, the class is zero since it is locally \(\nabla\)-exact. Set \(\varpi = \sum_{r+s=i-1} (\varpi^{r,s})\) and rewrite the above formula by

\[
\varphi(\omega) = \epsilon(\tilde{\varphi}(\omega) + \varpi') + D(\varpi).
\]
Taking classes in corresponding cohomology sheaves, the formula yields the middle one of the following equalities:

\[ \varphi(\omega) = [\varphi(\omega)] = \epsilon([\tilde{\varphi}(\omega)]) = \epsilon \circ \tilde{\varphi}(\omega). \]

To show the claim, it suffices to observe that, as \( \omega \) is \( \theta \)-closed, Proposition 3.2 implies that \( \varphi(\omega) \) is \( D \)-closed. For each fixed \( i \), the augmentation \( \epsilon \) is a resolution. It implies that

\[ D(\varphi(i,0)(\omega)) = \nabla(\epsilon \circ \tilde{\varphi}(\omega)) \pm \delta(\epsilon \circ \tilde{\varphi}(\omega)) = 0. \]

Therefore \( \sum_{r+s=i, r>0} \varphi(r,s)(\omega) \) is indeed \( D \)-closed. Spelling this out, one finds a set of relations which allow us to conclude the claim. These relations read:

(i) \( \delta(\varphi(i,0)(\omega)) = 0; \)
(ii) \( \nabla(\varphi(r,i-r)(\omega)) = \pm \delta(\varphi(r-1,i-r+1)(\omega)), \quad 2 \leq r \leq i; \)
(iii) \( \nabla(\varphi(1,i-1)(\omega)) = 0. \)

By Relation (i), there exists an \( \varpi^{i-1,0} \in C^{i-1,0} \) such that the following relation holds

(iv) \( \delta(\varpi^{i-1,0}) = \varphi(i,0)(\omega). \)

By Relation (ii) (assuming \( i \geq 2 \), one obtains consecutively elements \( \{\varpi^{i-2,1}, \ldots, \varpi^{0,i-1}\} \) with \( \varpi^{r-1,i-r} \in C^{r-1,i-r} \) satisfying

(v) \( \delta(\varpi^{r-2,i-r+1}) = \varphi(r-1,i-r+1)(\omega) + \nabla(\varpi^{r-1,i-r}). \)

Summing up Relations (iv) and (v), one obtains the claimed equality. Finally, Relation (iii) implies the relation \( \delta(\nabla(\varpi^{0,i-1})) = 0 \) as required. The claim is proved. \( \square \)

The truth of Theorem 3.1 becomes immediate.

Proof. Proposition 3.2 constructs the claimed morphism \( \varphi \), which restricts to a morphism between the intersection subcomplexes by Proposition 3.5. From Lemma 3.15 and Theorem 3.6 the theorem follows. \( \square \)

4. \( E_1 \)-Degeneration Theorem

Whether the Hodge to de Rham spectral sequence associated to a certain de Rham complex degenerates at \( E_1 \) is one of central problems in Hodge theory. The decomposition theorem, as introduced by Deligne-Illusie [DI], is the best device for this problem in positive characteristic. Therefore, we shall address on it in this section, as a natural continuation of our study on decomposition theorem.

4.1. Periodic de Rham sheaves. Let us first remark that the works [LSZ14], [LSZ17] extend in an obvious way to the special logarithmic setting\(^2\). In the next definition, which was introduced in [LSZ17] when \( D \) is absent, the notation \( C^{-1} \) (by abuse of notation) is actually the composite of the inverse Cartier transform of Ogus-Vologodsky with the base change functor \( \pi^* \).

\(^2\)As we have demonstrated in Appendix [LSYZ], which extends the major part of [LSZ14] in the current setting, the extension can be realized by replacing Frobenius liftings by log Frobenius liftings and the Taylor formula by its log analogue (Formula 6.7.1 [Ka]).
Definition 4.1. A de Rham sheaf $(H, \nabla, \text{Fil})$ is said to be periodic if $H$ is a coherent $\mathcal{O}_X$-module and there exists a natural number $f$ and a sequence of Griffiths transverse filtrations $\{\text{Fil}_i, 0 \leq i \leq f - 1\}$ of level $\leq p - 1$ inductively defined on

$$(H_i, \nabla_i) = C^{-1} \circ \text{Gr}_{\text{Fil}_{i-1}}(H_{i-1}, \nabla_{i-1})$$
with $(H_0, \nabla_0, \text{Fil}_0) = (H, \nabla, \text{Fil})$ such that the flat sheaf $(H_f, \nabla_f)$ is isomorphic to the initial flat sheaf $(H, \nabla)$. A graded Higgs sheaf $(E, \theta) \in \text{HIG}^p_{p-1}(X_{\log}/k)$ is said to be periodic if $E$ is coherent and there exists a natural number $f$ and a sequence of Griffiths transverse filtrations $\{\text{Fil}_i, 0 \leq i \leq f - 1\}$ of level $\leq p - 1$ inductively defined on

$$(H_i, \nabla_i) = C^{-1}(E_i, \theta_i)$$
with $(E_0, \theta_0) = (E, \theta)$ and $(E_{i+1}, \theta_{i+1}) = \text{Gr}_{\text{Fil}_i}(H_i, \nabla_i)$ such that $(E_f, \theta_f)$ is isomorphic to the initial Higgs module $(E, \theta)$ as graded Higgs sheaves.

By the equivalence of categories Theorem 1.1, the graded Higgs sheaf $\text{Gr}_{\text{Fil}}(H, \nabla)$ for a periodic de Rham sheaf $(H, \nabla, \text{Fil})$ is periodic and conversely, the de Rham sheaf $(C^{-1}(E, \theta), \text{Fil}_0)$ appearing in the definition of a periodic Higgs sheaf is periodic. By Proposition 3.12 [LSZ17], a periodic Higgs sheaf is locally free, and a periodic de Rham sheaf is also locally free (and the filtration is a filtration of locally free subsheaves and locally split). Therefore, we shall speak of periodic de Rham/Higgs bundles instead of sheaves from now on.

Proposition 4.2. Let $MF^{\nabla}_{[0,p-1]}(X_{\log}/k)$ be the category of strict $p$-torsion Fontaine modules over $X_{\log}$, and let $(H, \nabla, \text{Fil}, \Phi)$ be an object in it. Then the triple $(H, \nabla, \text{Fil})$ forms a one-periodic de Rham bundle.

Proof. One adapts the proof of Proposition 4.1 [LSZ14] to the current setting. By the calculation therein, the relative Frobenius $\Phi$ induces a natural isomorphism

$$\tilde{\Phi} : C^{-1} \circ \text{Gr}_{\text{Fil}}(H, \nabla) \cong (H, \nabla),$$
which implies the result. □

4.2. Intersection-adaptedness theorem. Let $(H, \nabla) \in \text{MIC}^0_{p-1}(X_{\log}/k)$. The decomposition theorem we obtained in §3 may be restated as the following equality in the derived category $D(X')$:

$$\Omega^*_\text{int}C(H, \nabla) = F_\ast \Omega^*_\text{int}(H, \nabla).$$
Now for a de Rham sheaf $(H, \nabla, \text{Fil})$, it is natural to ask whether the following analogous equality holds or not:

$$\Omega^*_\text{int} \text{Gr}_{\text{Fil}}(H, \nabla) = \text{Gr}_{\text{Fil}} \Omega^*_\text{int}(H, \nabla).$$
Note that, without the intersection condition, the above equality is tautologically true. In particular, the above equality holds away from the divisor $D$ at infinity. However, it is not difficult to produce a counterexample to the above equality at infinity. In this subsection, we are going to establish an intersection-adaptedness theorem, which is the intersection version in positive characteristic of the $L^2$-adaptedness theorem in the complex analytic setting which was shown in the
one-variable case by Zucker ([5 Zuc]). His tool is the $SL_2$-orbit theorem of Schmid [Schm].

**Theorem 4.3.** Let $(H, \nabla, \Fil)$ be a one-periodic de Rham bundle. Then the following equality of complexes holds:

$$Gr_{Fil}^* \Omega_{int}^*(H, \nabla) = \Omega_{int}^* Gr_{Fil}(H, \nabla).$$

To approach the problem, it is to good to look at it in a more general setting. Let $S$ be a noetherian scheme and $(X, D)$ be a log pair over $S$ (by which we mean $X$ is a smooth scheme over $S$ and $D \subset X$ a NCD relative to $S$). Let $(H, \nabla, \Fil)$ be a de Rham sheaf over $X_{\log}/S$. Set $(E = \oplus_i E^i, \theta)$ to be graded Higgs sheaf.

**Lemma 4.4.** Notation as above. Then, one has the natural inclusion of complexes

$$\Omega_{int}^*(E, \theta) \subset Gr_{Fil}^* \Omega_{int}^*(H, \nabla) \subset \Omega^*(E, \theta).$$

**Proof.** This is a local problem around the divisor $D$. So we may assume $X$ to be affine, equipped with a special system of log coordinates $\{t_1, \cdots, t_n\}$ at a closed point $x \in X$. Assume there are $r$-branches of $D$ passing through $x$ which are defined by $t_1 \cdots t_r = 0$. For $I \subset \{1, \cdots, r\}$ and $J \subset \{r + 1, \cdots, n\}$, set $\beta_{I,J} = \frac{dt_I}{t_I} \land dt_J$. Using the notation in §3.4, we may write

$$\Omega_{int}^*(H, \nabla) = \bigoplus_{I \subset \{1, \cdots, r\}, J \subset \{r + 1, \cdots, n\}} H_{w(\beta_{I,J})} \beta_{I,J}$$

and

$$\Omega_{int}^*(E, \theta) = \bigoplus_{I \subset \{1, \cdots, r\}, J \subset \{r + 1, \cdots, n\}} E_{w(\beta_{I,J})} \beta_{I,J},$$

with $H_{w(\beta_{I,J})} = \sum_{K \subset I} t_{I-K} \nabla_K H$ and $E_{w(\beta_{I,J})} = \sum_{K \subset I} t_{I-K} \theta_K E$. As the filtration $Fil^*$ on $\Omega_{int}^*(H, \nabla)$ is compatible with the direct sum, it is equivalent to show $E_{w(\beta_{I,J})} \subset Gr_{Fil} H_{w(\beta_{I,J})}$ for each $I, J$. By the Griffiths transversality, we know that

$$\sum_{K \subset I} t_{I-K} \nabla_K Fil^{l+|K|} H \subset Fil^l \sum_{K \subset I} t_{I-K} \nabla_K H = Fil^l H_{w(\beta_{I,J})}.$$ 

By the commutativity of the following diagram (with natural projections as vertical maps)

$$\xymatrix{Fil^{l+|K|} H \ar[r]^{\nabla_K} \ar[d] & Fil^l H \ar[d] \\
E^{l+|K|} \ar[r]^{\theta_K} & E^l},$$

it follows that the image in $E^l$ of the summand $t_{I-K} \nabla_K Fil^{l+|K|} H$ in the left hand side of the inclusion \ref{4.4.1} is equal to $t_{I-K} \nabla_K E^{l+|K|}$. Summing up over various $K$s, the inclusion \ref{4.4.1} leads to the inclusion $E_{w(\beta_{I,J})} \subset Gr_{Fil} H_{w(\beta_{I,J})}$. As $l$ is arbitrary, the requested inclusion holds. □
In the proof of Lemma 4.4, one sees clearly that if the inclusion becomes an equality, that is if the following condition holds for any \( l \), where \( I^c \) is the complement of \( I \) in the set \( \{1, \ldots, r\} \), the equality \( \Omega^*_\text{int}(E, \theta) = Gr_{\text{Fil}}\Omega^*_\text{int}(H, \nabla) \) holds around the point \( x \). Our method to prove the intersection-adaptedness theorem is to establish the above equality in positive characteristic.

Use the notation in §3.1. As in Lemma 4.4, we assume \( X = \text{Spec}(A) \) to be affine with a special system of log coordinates \( \{t_1, \ldots, t_n\} \) at \( x \in D \). Moreover, shrinking \( X \) if necessary, we may assume that \( (H, \text{Fil}) \) is filtered free: it means that there exists a basis \( \{e_0, \ldots, e_l\} \) of \( H \) such that

\[
\text{Fil}^lH = \bigoplus_{j \geq l} A\{e_j\}, \quad 0 \leq i \leq l,
\]

where \( l \leq p - 1 \) is the level of \( \text{Fil} \) and \( A\{e_j\} \) is the free module generated by elements in \( \{e_j\} \). The residue of \( \nabla \) along the component \( D_i \) is denoted by

\[
\text{res}_{D_i} \nabla = \nabla|_{D_i} \in \text{End}_{\mathcal{O}_{D_i}}(H|_{D_i}).
\]

For any nonempty set \( I \subset \{1, \ldots, r\} \), we denote \( \text{res}_{D_i} \nabla \) for the composite \( \prod_{i \in I} \text{res}_{D_i} \nabla|_{D_i} \), which is said to be the residue of \( \nabla \) along the strata \( D_I \). Two special properties about \( \{\text{res}_{D_i} \nabla\} \) are crucial to the proof.

**Definition 4.5.** Let \( d \) be a natural number, \( (d_0, \ldots, d_l) \) a partition of \( d \) into a sum of nonnegative integers and \( s \) an integer with \( 0 \leq s \leq l \). A matrix \( M \in M_{d \times d}(k) \) is lower-triangular of type \( (d_0, \ldots, d_l; s) \) if \( M = (a_{i,j})_{0 \leq i,j \leq l} \) with \( a_{i,j} \in M_{d_i \times d_j}(k) \) and \( a_{i,i+t} = 0 \) for any \( t > s \). If the partition is clear in the context, we shall briefly say that \( M \) is lower-triangular of type \( s \). If furthermore the equality

\[
\text{rank}(M) = \sum_{i=0}^{l-s} \text{rank}(a_{i,i+s})
\]

holds, \( M \) is said to be special lower-triangular of type \( s \). An endomorphism \( \mathbb{M} \) on a \( d \)-dimensional vector space \( V \) over \( k \) is said to be (special) lower-triangulizable of type \( s \) if the representation matrix of \( \mathbb{M} \) with respect to some basis of \( V \) is (special) lower-triangular of type \( s \).

In our setting, set \( d_i \) to be the rank of \( E^i = \text{Fil}^iH/\text{Fil}^{i+1}H \) for \( 0 \leq i \leq l \). By Griffiths transversality, the connection matrix of one-forms under the basis \( \{e_0, \ldots, e_l\} \) takes the following form:

\[
\nabla(e_0, \ldots, e_l) = (e_0, \ldots, e_l) \begin{pmatrix}
a_{00} & a_{01} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
a_{l-10} & \cdots & \cdots & a_{l-1l-1} & a_{l-1l} \\
a_{00} & \cdots & \cdots & a_{ll-1} & a_{ll}
\end{pmatrix},
\]

where \( a_{ij} = \sum_k a_{ij,k} \omega_k \) with \( a_{ij,k} \in M_{d_i \times d_j}(A) \) is a matrix of one-forms. It is easy to see that the associated representation matrix of the residue \( \text{res}_{D_i}(\nabla) \) at
$x \in D_I$ is lower-triangular of type $\leq \min\{|I|, l\}$. A stronger property holds for the residues of one-periodic de Rham bundles.

**Lemma 4.6.** The morphism $\text{res}_{D_I} \nabla$ at $x \in D_I$ is special lower-triangularizable of type $\leq \min\{|I|, l\}$.

*Proof.* As $(H, \nabla, \text{Fil})$ is one-periodic, we may choose and then fix an isomorphism of flat bundles:

$$\psi : C^{-1} \circ \text{Gr}_{\text{Fil}}(H, \nabla) = C^{-1}(E, \theta) \cong (H, \nabla).$$

Since we are considering a local problem, we may assume that the log Frobenius lifting with respect to the coordinate system $\{t_1, \cdots, t_n\}$ takes the standard form. Let $\{\bar{e}_i\}$ be the basis of $E^\circ$ induced by $\{e_i\}$. Then, the representation matrix of the component $\theta_i$ along $D_i$ of the Higgs field $\theta$ with respect to $\{\bar{e}_0, \cdots, \bar{e}_l\}$ takes the following form:

$$\theta_i(\bar{e}_0, \cdots, \bar{e}_l) = (\bar{e}_0, \cdots, \bar{e}_l) \begin{pmatrix}
0 & a_{01,i} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & a_{l-1l,i} \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix},$$

which is clearly special lower-triangular of type $\leq 1$. Now the bundle part of $C^{-1}(E, \theta)$ is just $F_X^*E$ and therefore admits a free basis $\{F_X^*\bar{e}_i\}_{0 \leq i \leq l}$. Via $\psi$, we obtain another basis $\{f_0, \cdots, f_l\}$ of $H$ which is defined by $f_i = \psi(F_X^*\bar{e}_i)$. Now using the connection part in the isomorphism $\psi$, we know that, with respect to the new basis $\{f_i\}$, the connection component $\nabla_i$ along $D_i$, $1 \leq i \leq r$ takes the following form

$$\nabla_i(f_0, \cdots, f_l) = (f_0, \cdots, f_l) \begin{pmatrix}
0 & a_{01,i}^p & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & a_{l-1l,i}^p \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}.$$  

From the above expression, one deduces immediately that $\text{res}_{D_I} \nabla$ at any $x \in D_I$ is special lower-triangular of type $\leq \min\{|I|, l\}$. The lemma is proved. \hfill \square

The second property asserts that the ranks of $\text{res}_{D_I} \nabla(x)$ keep constant along $x \in D_I$.

**Lemma 4.7.** The kernel and the cokernel of the morphism $\text{res}_{D_I} \nabla : H|_{D_I} \to H|_{D_I}$ are locally free.

*Proof.* As $H|_{D_I}$ is locally free of finite rank, it suffices to show that the cokernel of $\text{res}_{D_I} \nabla$ is locally free. This is again a local problem. So we use notations in the proof of Lemma 4.6. Set $A_I$ to be the coordinate ring of $D_I$. Let $L_I$ be the cokernel of the morphism $\text{res}_{D_I} \nabla$ between free $A_I$-modules:

$$H|_{D_I} \xrightarrow{\text{res}_{D_I} \nabla} H|_{D_I}.$$
Let \( \text{Fitt}_j(L_I) \) be the \( j \)-th Fitting ideal of \( L_I \). We are going to show that these Fitting ideals are either zero or the whole ring \( A_I \), from which one deduces that \( L_I \) is projective (see e.g Proposition 20.8 [E1]). Therefore, the localization of \( L_I \) at any prime ideal of \( A_I \) is free, which implies that the cokernel (as sheaf) of \( \text{res}_{D_I} \nabla \) is locally free as claimed. Consider first the case \( I = \{ i \} \) for some \( 1 \leq i \leq r \). Then one notices that the representation matrix of \( \nabla_i \) under the basis \( \{ e_0, \cdots, e_i \} \) takes the form

\[
M_i = \begin{pmatrix}
a_{00,i} & a_{01,i} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
a_{l-10,i} & \cdots & \cdots & a_{l-1l-1,i} & a_{l-1l,i} \\
a_{0l,i} & \cdots & \cdots & a_{ll-1,i} & a_{ll,i}
\end{pmatrix},
\]

while under the other basis \( \{ f_0, \cdots, f_l \} \) is the \( p \)-th power of

\[
N_i = \begin{pmatrix}
0 & a_{01,i} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & a_{ll-1,i} \\
0 & \cdots & \cdots & 0 & 0
\end{pmatrix}.
\]

Under the natural projection \( A \to A_I \), \( M_i \) and \( N_i^p \) give rise to two matrices \( M_i, N_i^p \) with entries in \( A_I \) which are two representation matrices of \( \text{res}_{D_i} \nabla \). For a matrix \( Z \) with entries in \( A_I \), denote \( I_j(Z) \) for the ideal of \( A_I \) generated by minors of size \( j \times j \) in \( Z \). Then by the explicit forms of \( M_i \) and \( N_i \) given as above, we get the following relations:

\[
I_j(\bar{N}_i) \subseteq I_j(M_i) = \text{Fitt}_j(L_i) = I_j(\bar{N}_i^p) \subseteq I_j(N_i).
\]

Therefore, we get the equalities:

\[
\text{Fitt}_j(L_i) = I_j(\bar{N}_i) = I_j(\bar{N}_i^p).
\]

As \( A_I \) is a noetherian domain, the latter equality implies that \( I_j(\bar{N}_i) \) is either (0) or (1) by Krull’s theorem. Therefore, the assertion for \( |I| = 1 \) case follows. For a general \( I \subset \{ 1, \cdots, r \} \), the above proof works verbatim: one forms the matrices \( M_I = \prod_{i \in I} M_i \) and \( N_I = \prod_{i \in I} N_i \) whose \(|I|\)-th diagonals coincide. Replacing \( M_i \) and \( \bar{N}_i \) with the images of \( M_I \) and \( N_I \) in \( M_{d \times d}(A_I) \) in the above argument, we obtain the assertion for this case too. \( \square \)

Now we can proceed to the proof of Theorem 4.3.

**Proof.** As remarked above, we may assume \( X \) to be affine local with a special system of log coordinates at \( x \), and it suffices to show the equality in 14.2. If the number of branches \( r \) at \( x \) is zero, there is nothing to prove. So we assume \( r \) to be positive. We prove by induction and shall prove the following induction step: let \( s \) be a local section of \( Fill_{i}^{l} \sum_{I \subset \{ 1, \cdots, r \}} t_{I} \nabla_{I} H \). If \( s \) vanishes along \( D_I \) for any \( I \subset \{ 1, \cdots, r \} \) with \(|I| = m \geq 1\), then we find a new section \( s' \) such that \( s - s' \in \sum_{I \subset \{ 1, \cdots, r \}} t_{I} \nabla_{I} Fill_{I}^{l+|I|} H \) and \( s' \) vanishes along any \( D_{I'} \) with \(|I'| = m - 1\).
(as convention, \( D_0 = X \)). Before doing that, let us show first that the initial condition in the induction step can be realized. Denote \([r] = \{1, \cdots, r\}\). Applying Sublemma 4.8 below to \(\text{res}_{DJ} \nabla\) in Lemma 4.6 we conclude that for any closed points \(x' \in D_{[r]}\), the value \(s(x')\) of \(s\) at \(x'\) lies in

\[
\text{Fil}^l \text{res}_{D_{[r]}} \nabla H(x') = \text{res}_{D_{[r]}} \nabla \text{Fil}^{l+r} H(x')
\]

By Lemma 4.7, the cokernel of the following morphism

\[
\text{Fil}^{l+r} H|_{D_{[r]}} \xrightarrow{\text{res}_{D_{[r]}} \nabla} \text{Fil}^l H|_{D_{[r]}}
\]

is locally free. As the image of \(s\) in each closed fiber of the cokernel is zero, it is identically zero in the cokernel. In other words, there is some section \(s_r \in \text{Fil}^{l+r} H\) such that \(\nabla_{[r]}(s_r)\) has its restriction along \(D_{[r]}\) equal to \(s|_{D_{[r]}}\). Therefore, the section \(s' = s - \nabla_{[r]}(s_r)\) will satisfy the requested initial condition. Now we assume the induction hypothesis is satisfied. Let us consider the case \(m = 1\) first. In this case, \(s\) vanishes along all components of \(D\). Thus it belongs to the subsheaf \(\mathcal{F}_D\text{Fil}^l H \subset \text{Fil}^l H\) and therefore may be written as \(t_1 \cdots t_r \cdot s_0\) for some \(s_0 \in \text{Fil}^l H\). The section \(s' = s - t_1 \cdots t_r \cdot s_0\) satisfies the requested condition for \(m = 1\). Assume \(m \geq 2\). Take an \(I \subset [r]\) with \(|I| = m\). Fix an \(i \in I\) and set \(J = I \setminus \{i\}\). We claim that we may take an \(s_J \in \text{Fil}^{l+|J|} H\) such that the restriction of \(t_i \nabla_{I} s_J\) to \(D_J\) is equal to \(s_{D_J}\). By Lemma 4.7, the kernel and the cokernel of the morphism \(\text{res}_{D_J} \nabla : \text{Fil}^{l+|J|} H_{D_J} \to \text{Fil}^l H_{D_J}\) are locally free. Then, the following short exact sequence of locally free sheaves over \(D_J\) is split

\[
0 \to \ker(\text{res}_{D_J} \nabla) \to \text{Fil}^{l+|J|} H|_{D_J} \xrightarrow{\text{res}_{D_J} \nabla} \text{im}(\text{res}_{D_J} \nabla) \to 0.
\]

Replacing \([r]\) with \(J\), the argument in the initial step shows that the restriction \(s|_{D_J}\), which is a priori a section of \(\text{Fil}^l H|_{D_J}\), actually lies in the subsheaf \(\text{Fil}^{l+|J|} H|_{D_J}\). To get a good lifting of \(s|_{D_J}\), we take a local basis \(\{e_I, e_{II}\}\) of \(\text{Fil}^{l+|J|} H|_{D_J}\) such that \(e_I\) belongs to \(\ker(\text{res}_{D_J} \nabla)\) and \(e_{II} = \{e_{II,J}\}\) projects to a local basis \(e_{II} = \{e_{II,J}\}\) of \(\ker(\text{res}_{D_J} \nabla)\). Write \(s|_{D_J} = \sum_j f_j \cdot e_{II,J}\) with \(f_j \in \mathcal{O}_{D_J}\). As \(D_I \subset D_J\) is defined by \(t_i = 0\) and \(s|_{D_I} = 0\) by hypothesis, it follows that \(t_i\) divides \(f_j\) for any \(j\). Write \(f_j = t_i g_j\) with \(g_j \in \mathcal{O}_{D_J}\). So we take the lifting to be \(t_i \sum_j g_j e_{II,J}\). Taking any lifting \(g_j\) of \(\tilde{g}_j\) in \(\mathcal{O}_X\) and any lifting \(\tilde{e}_{II,J}\) of \(e_{II,J}\) in \(\text{Fil}^{l+|J|} H\), we obtain the section \(s_J = \sum_j g_j \tilde{e}_{II,J}\) of \(\text{Fil}^{l+|J|} H\) which has the property that

\[
t_i [\nabla_{J} s_J]|_{D_J} = t_i \text{res}_{D_J} \nabla [\sum_j (g_j \tilde{e}_{II,J})]|_{D_J} = \text{res}_{D_J} \nabla (t_i \sum_j g_j e_{II,J}) = s|_{D_J},
\]

as claimed. By construction, \(t_i \nabla_{J} s_J\) belongs to \(\sum_{I \in \{1, \cdots, r\}} \sum_{i \in I} t_i \nabla_i \text{Fil}^{l+|I|} H\). Finally, we set \(s' = s - \sum_{i \in I} t_i \nabla_{I \setminus \{i\}} s_{I \setminus \{i\}}\), which satisfies the requested property for \(m\). The theorem is proved. \(\Box\)

Let \(M\) be a lower-triangular matrix of type \((d_0, \cdots, d_i \leq s)\). Let \(V = k^d\), equipped with the standard basis \(\{e_0, \cdots, e_l\}\) where the elements in \(\{e_i\}\) span the \(k\)-linear subspace \(V_i\) of dimension \(d_i\). Let \(M\) be the endomorphism on \(V\) determined by \(M\) under the above basis. We equip \(V\) with the finite decreasing
filtration $Fil$ defined by $Fil^i V = \oplus_{j \geq i} V_j$ for $0 \leq i \leq l$. Under this definition, it is obvious that

$$\mathbb{M}(Fil^{i+s} V) \subseteq \text{im}(\mathbb{M}) \cap Fil^i V.$$  

**Sublemma 4.8.** Notation as above. If $M$ is special, then the equality

$$\mathbb{M}(Fil^{i+s} V) = \text{im}(\mathbb{M}) \cap Fil^i V$$

holds.

**Proof.** The problem is purely linear algebraic. So we take a vector

$$v = \sum_{j=0}^{l} e_j \lambda_i \in V, \quad \lambda_i = 0, \ 0 \leq j \leq i - 1$$

with $v = \mathbb{M}(v')$ for some $v' \in V$. We aim to show that $v \in \mathbb{M}(Fil^{i+s} V)$. We first show a special case of the sublemma, which serves as the starting step of an induction argument. Namely, we claim that if $i > l - s$, then $v = 0$. In order to prove it, we put the matrix $M = (a_{ij})$ into the following form: without loss of generality, we may assume that $r_i = \text{rank}(a_{ii+s}) > 0$ for each $0 \leq i \leq l - s$. For each such $i$, we may moreover assume that the first $r_i$ columns of $a_{ii+s}$ are linearly independent. For any $0 \leq j \leq l$, set $a_{ji+s} \in M_{d_j \times r_j}(k)$ to be the submatrix consisting of the first $r_i$ columns. Now we set $M_i \in M_{d_i \times r_i}(k)$ to be submatrix of $M$ consisting of the $r_i$ columns which contain $a'_{ii+s}$ as submatrix. By the condition on the matrix $M$, we know that $M_i$ is of form

$$
(0 \ 0 \ a'_{ii+s} \cdots \ a'_{ii+s})^t,
$$

and there exists a unique set of matrices $\{B_0, \cdots, B_{l-s}\}$ with

$$B_i = (b_{i0}, \cdots, b_{i1}) \in M_{r_i \times d}(k)$$

such that

$$M = \sum_{i=0}^{l-s} M_i \cdot B_i.$$  

Then, with respect to the basis $\{e_i\}$, the equation $v = \mathbb{M}(v')$ and the expression (4.8.2) yield the following equality for $0 \leq i \leq l$:

$$\lambda_i = a'_{is}(B_0 \lambda') + \cdots + a'_{ij+s}(B_j \lambda') + \cdots + a'_{il}(B_{l-s} \lambda').$$

As $a'_{0j+s} = 0$ for $j > 0$, the equality (4.8.3) for $i = 0$ gives rise to

$$\lambda_0 = a'_{0s}(B_0 \lambda').$$

As $a'_{0s}$ is of full rank and $\lambda_0 = 0$ by assumption, we obtain $B_0 \lambda' = 0$. Substituting it into the equality (4.8.3) for $i = 1$, the same reasoning yields $B_1 \lambda' = 0$. Continuing this argument up to the equality (4.8.3) for $i = l - s$, we may conclude that

$$B_0 \lambda' = 0, \cdots, B_{l-s} \lambda' = 0.$$  

Inserting them back to the equation (4.8.3) for each $l - s + 1 \leq i \leq l$, we conclude that $\lambda_i = 0$ for all $i$s. This proves the claim. The following induction step shall complete the proof of the sublemma:
Claim 4.9. Assume \( i \leq l - s + 1 \). Then

\[
\text{im}(M) \cap Fil^i V \subset \text{im}(M) \cap Fil^{i+1} V + M(Fil^{i+s} V).
\]

To achieve this, the key is to explore the more detailed information on the matrices \( B_i \)'s appearing in the equation \( 4.8.2 \). In fact, we are going to show that

\[
\begin{align*}
(4.9.1) & \quad b_{ij} = 0, \quad j > i + s \\
(4.9.2) & \quad b_{ii+s} = (Id, b'_{ii+s}).
\end{align*}
\]

We prove it by induction on \( i \). Comparing the two sides of the equality \( 4.8.2 \), we get

\[ a_{0j} = a'_{0s} b_{0j}, \quad 0 \leq j \leq l. \]

As \( a_{0j} = 0 \) for \( j > s \) and \( a'_{0s} \) is of full rank, \( b_{0j} = 0 \) for \( j > s \). From the equation

\[ a_{0s} = (a'_{0s}, a''_{0s}) = a'_{0s} b_{0s}, \]

we see that \( b_{0s} \) is of form \( (Id, b'_{0s}) \). Thus, the \( i = 0 \) case is proved. Assume the truth for the cases \( \leq i \). Then we have the equality

\[ a_{i+1} = a'_{i+1} b_{0j} + a'_{i+1+s} b_{1j} + \cdots + a'_{i+s} b_{s-j}. \]

Note that for \( j > i + 1 + s \), each summand except the term \( a'_{i+1+s} b_{i+1+s} \) in the left hand side of the above expression is zero. It follows that \( b_{i+1+s} = 0 \) for \( j > i + 1 + s \). Look at \( j = i + 1 + s \). The above equality is reduced to the following equality

\[ a_{i+1+s} = a'_{i+1+s} b_{i+1+s}, \]

from which it follows that \( b_{i+1+s} = (Id, b'_{i+1+s}) \). Now we come to the proof of Claim \( 4.9 \) It suffices to construct a vector

\[ v'' = \sum_j e_j \lambda''_j \in Fil^{i+s} V \]

such that \( M(v' - v'') \in Fil^{i+1} V \). Such a vector \( v'' \) can be given as follows: set \( \lambda''_j = 0 \) for \( j \neq i + s \) and

\[ \lambda''_{i+s} = \begin{pmatrix} B_i \lambda' \\ 0 \end{pmatrix}. \]

(Note that \( B_i \lambda' \in M_{r_i \times 1}(k) \) and \( r_i = \text{rank}(a_{ii+s}) \leq d_{i+s} \), so that we need possibly an additional zero vector of size \( (d_{i+s} - r_i) \times 1 \) to make the above defined \( \lambda''_{i+s} \in M_{d_{i+s} \times 1}(k) \)). Clearly, \( v'' \in Fil^{i+s} V \) by construction. Applying the first property in \( 4.9.1 \) to the expression \( 4.8.3 \) we find that for \( 0 \leq i \leq l \), \( (M \lambda'')_i = 0 \), and

\[ (M \lambda'')_{i+1} = a'_{i+s} B_i \lambda''. \]

By the second property in \( 4.9.1 \) we find further that

\[ B_i \lambda'' = (Id, b'_{i+s}) \begin{pmatrix} B_i \lambda' \\ 0 \end{pmatrix} = B_i \lambda'. \]

Therefore, \( (M \lambda'')_{i+1} = a'_{i+s} B_i \lambda' = (M \lambda')_{i+1} \). It implies that \( M(v' - v'') \in Fil^{i+1} V \) as claimed.

\( \square \)
4.3. \( E_1 \)-degeneration. In this subsection, the scheme \( X \) is assumed additionally to be proper over \( k \), if not otherwise specified. It is useful to record the data of a one-periodic de Rham bundle \((H, \nabla, Fil)\) in the following triangle:

\[
\begin{array}{c}
(H, \nabla) \\
\downarrow \text{Gr}_{Fil} \\
\text{Gr}_{Fil}(H, \nabla) = (E, \theta).
\end{array}
\]

The following \( E_1 \)-degeneration theorem has a one-sentence proof.

**Theorem 4.10.** Let \((H, \nabla, Fil)\) be as above. If \( \dim X + l \leq p \), then the Hodge to de Rham spectral sequence associated to the intersection de Rham complex \( \Omega^*_\text{int}(H, \nabla) \) degenerates at \( E_1 \).

**Proof.** In the above triangle, applying the intersection-adaptedness theorem (Theorem 4.3) to the left down side and then the decomposition theorem (Theorem 3.1) to the right down side, one obtains the following (in)equality on the dimensions of finite dimensional \( k \)-vector spaces:

\[
\dim \text{Gr}_{Fil} \mathbb{H}^i(X, \Omega^*_\text{int}(H, \nabla)) \leq \dim \mathbb{H}^i(X, \text{Gr}_{Fil} \Omega^*_\text{int}(H, \nabla)) = \dim \mathbb{H}^i(X, \Omega^*_\text{int} \text{Gr}_{Fil}(H, \nabla)) = \dim \mathbb{H}^i(X, \Omega^*_\text{int} C^{-1}(E, \theta)) = \dim \mathbb{H}^i(X, \Omega^*_\text{int} C^{-1}(E, \theta)) = \dim \mathbb{H}^i(X, \Omega^*_{\text{int}}(H, \nabla)) = \dim \text{Gr}_{Fil} \mathbb{H}^i(X, \Omega^*_\text{int}(H, \nabla)),
\]

which implies the first inequality is actually an equality and hence the theorem. \( \square \)

**Corollary 4.11.** Let \((H, \nabla, Fil, \Phi)\) be an object in \( MF_{[0,p-1]}^\nabla(X_{\text{log}}/k) \). Then the Hodge to de Rham spectral sequence associated to the intersection de Rham complex \( \Omega^*_{\text{int}}(H, \nabla) \) degenerates at \( E_1 \). Consequently, one has the following equalities

\[
\text{Gr}_{Fil} \mathbb{H}^*(X, \Omega^*_\text{int}(H, \nabla)) = \mathbb{H}^*(X, \text{Gr}_{Fil} \Omega^*_\text{int}(H, \nabla)) = \mathbb{H}^*(X, \Omega^*_\text{int} \text{Gr}_{Fil}(H, \nabla)).
\]

**Proof.** It follows from Proposition 4.2 and Theorem 4.10. \( \square \)

It is useful to extend the intersection-adaptedness theorem and hence the \( E_1 \)-degeneration theorem to a general periodic de Rham bundle.

**Corollary 4.12.** Notation and assumption as Theorem 4.3. Let \((H, \nabla, Fil)\) be a periodic de Rham bundle. Then it holds that

\[
\text{Gr}_{Fil} \Omega^*_\text{int}(H, \nabla) = \Omega^*_\text{int} \text{Gr}_{Fil}(H, \nabla).
\]
Proof. Assume the period of \((H, \nabla, Fil)\) to be \(f\). One observes that the direct sum

\[
\bigoplus_{i=0}^{f-1}(H_i, \nabla_i, Fil_i)
\]

is a one-periodic de Rham bundle. It is clear that the grading functor \(Gr_{Fil}\) and the intersection functor \(\Omega^*_{int}\) commute with direct sum. Hence, by Theorem 4.3 we get that

\[
\bigoplus_{i=0}^{f-1} Gr_{Fil_i}(H_i, \nabla_i) = \bigoplus_{i=0}^{f-1} \Omega^*_{int}(H_i, \nabla_i).
\]

By Lemma 4.4, we have the natural inclusion for each \(i\):

\[
\Omega^*_{int}Gr_{Fil_i}(H_i, \nabla_i) \subset Gr_{Fil_i}\Omega^*_{int}(H_i, \nabla_i).
\]

Therefore, we have the equality for each \(i\),

\[
\Omega^*_{int}Gr_{Fil_i}(H_i, \nabla_i) = Gr_{Fil_i}\Omega^*_{int}(H_i, \nabla_i).
\]

The \(i = 0\) case of the above equality is the claimed result. \(\square\)

Corollary 4.13. Notation and assumption as Theorem 4.10. Let \((H, \nabla, Fil)\) be a periodic de Rham bundle. If \(\text{dim} X + \text{rank} H \leq p\), then the Hodge to de Rham spectral sequence of the intersection de Rham complex \(\Omega^*_{int}(H, \nabla)\) degenerates at \(E_1\).

Proof. By Corollary 4.12 the proof of Theorem 4.10 yields the following sequence of inequalities:

\[
\dim \mathbb{H}^i(X, \Omega^*_{int}(H_0, \nabla_0)) \leq \dim \mathbb{H}^i(X, \Omega^*_{int}(H_1, \nabla_1)) \leq \cdots \leq \dim \mathbb{H}^i(X, \Omega^*_{int}(H_f, \nabla_f)).
\]

Since the last term is equal to the first term by the periodicity, the above inequalities are all equalities. The first one is the claim. \(\square\)

Remark 4.14. In the above corollary, the Hodge to de Rham spectral sequence associated to the de Rham complex \(\Omega^* (H, \nabla)\) also degenerates at \(E_1\). The proof is the same but much easier. This generalizes the \(E_1\)-degeneration theorem of Illusie attached to semistable families \([IL90]\) (see also \([Ogus94]\)).

5. Applications

In this section, \(K\) is a field of characteristic zero. Given a log pair \((X, D)\) defined over \(K\), where \(X\) is smooth projective of dimension \(d\) and \(D\) a NCD in \(X\), we consider rational polarized variation of Hodge structure \((\mathbb{Q}\text{-PVHS})\) over \(X^0 = X - D\) arising from geometry: let \(f : (Y, E) \to (X, D)\) be a semistable family of relative dimension \(n\) over \(K\) by which we mean that \(f\) is projective with smooth locus \(f^0 : Y^0 = Y - E \to X^0\), \(Y\) is smooth projective and \(E = f^{-1}D \subset Y\) is a NCD. Choose and then fix an embedding \(K \to \mathbb{C}\). The local system \(V = R^n f^!_\mathbb{Q}Y_{an}\) on \(X^0_{an}\) admits the structure of \(\mathbb{Q}\text{-PVHS}\). Let \(V^\pi\) be the intermediate extension of \(V\) (which is a perverse sheaf) over \(X\). The following fundamental fact in Hodge theory was first established by Zucker \([Zuc]\) in the curve case, i.e. \(\text{dim} X = 1\) and conjectured by Deligne.
Theorem 5.1 (Cattani-Kaplan-Schmid [CKS], Kashiwara-Kawai [KK]). The intersection cohomology group $IH^m(X_{an}, V) := \mathbb{H}^*(X_{an}, V^m), 0 \leq m \leq d$ has a canonical pure Hodge structure of weight $n+m$.

Among other things, the theory of $L^2$-integrable harmonic forms plays a pivotal role in the proofs. With a bit more works, one may deduce from the above result the following $E_1$-degeneration theorem which is purely algebraic.

**Theorem 5.2.** Notation as above. Let $(H, \nabla, Fil)$ be the Gauss-Manin system associated to the semistable family $f$. Then the spectral sequence of the filtered intersection de Rham complex $\Omega^*_{int}(H, \nabla)$ degenerates at $E_1$.

Note that one has explicitly $(H, \nabla) = (R^nf_*\Omega^*_Y /X (\log E/D), \nabla^{GM})$, and $Fil$ is the Hodge filtration. Our theory in positive characteristic, developed in the previous sections, yields an algebraic proof of the above theorem. Also, it gives a clear description of Hodge $(P, Q)$-components of the pure Hodge structure attached to $IH^m(X_{an}, V)$ in Theorem 5.1 (which seems to be unknown). Set $$(E = \bigoplus_{i+j=n} E^{i,j} = R^jf_*\Omega^*_Y /X (\log E/D), \theta = \bigoplus_{i+j=n} \theta^{i,j})$$ to be the graded Higgs bundle $Gr_Fil(H, \nabla)$. One notices that the intersection complex of $(E, \theta)$ is a direct sum: $$\Omega^*_{int}(E, \theta) = \bigoplus_{P=0}^{n+d} \Omega^*_{int,P}(E, \theta),$$ where $\Omega^*_{int,P}(E, \theta)$ is a subcomplex of $\Omega^*_P(E, \theta)$ whose $l$-term is given by $$\Omega^l_P(E, \theta) := E^{P-l, n-P+l} \otimes \Omega^l_X (\log D).$$

**Corollary 5.3.** Notation as above. Then one has a natural isomorphism $$IH^m(X_{an}, V_C) \cong \mathbb{H}^m(X, \Omega^*_{int}(E, \theta)) \otimes_K \mathbb{C}.$$ Moreover, let $Fil$ be the Hodge filtration on the intersection cohomology group $IH^m(X, V_C), 0 \leq m \leq d$. Then one has a natural isomorphism $$Gr^{Fil}_PH^m(X_{an}, V_C) \cong \mathbb{H}^m(X, \Omega^*_{int,P}(E, \theta)) \otimes_K \mathbb{C}.$$ The proof of Corollary 5.3 comes after that of Theorem 5.2 to which we now proceed. First a simple lemma.

**Lemma 5.4.** Let $S$ be a noetherian scheme of finite type over $\mathbb{Z}$. Let $f : X \to S$ be a projective morphism of noetherian schemes. Let $A^*$ and $B^*$ be finite complexes of coherent $\mathcal{O}_X$-modules together with an injective morphism $\iota : A^* \to B^*$ of complexes. Then, there is a nonempty Zariski open $U \subset S$ such that for any closed point $x$ in $U$ (with the residue field $k(x)$), the morphism $$\iota \otimes k(x) : A^* \otimes k(x) \to B^* \otimes k(x)$$ is injective.
Thus, using the \( E \) properties are satisfied:
\[
\text{namely the case of an injective morphism } \iota : A \to B \text{ of coherent } \mathcal{O}_X\text{-modules.}
\]
Completing it into a short exact sequence of \( \mathcal{O}_X\)-modules:
\[
0 \to A \xrightarrow{\iota} B \to C \to 0.
\]
By the flattening stratification (see e.g. Theorem 2.15 [HL]), there is a nonempty Zariski open subset \( U \subset X \) such that \( B \) and \( C \) are flat over \( U \). By Lemma 2.14 [HL], it follows that for any closed point \( x \in U \), the morphism \( \iota \otimes k(x) \) is injective as claimed.

**Proof.** We have the a priori inequality
\[
dim_K \mathbb{H}^n(X, \Omega^*_{int}(H, \nabla)) \leq \dim_K \mathbb{H}^n(X, Gr_{Fil} \Omega^*_{int}(H, \nabla)),
\]
and it is an equality if and only if the spectral sequence degenerates at \( E_1 \). Our proof follows the mod \( p \) reduction argument of Deligne-Illusie [DI]. We divide it into several small steps for clarity:

Step 1: Write \( K \) as the inductive limit of its sub \( \mathbb{Z} \)-algebras of finite type. There exists a sub \( \mathbb{Z} \)-algebra \( A \subset K \) of finite type and a semistable family \( \hat{f} : (\mathfrak{Y}, \mathcal{E}) \to (\mathfrak{X}, \mathcal{D}) \) defined over \( S = \text{Spec}(A) \) such that \( f \) is induced by the base change \( \text{Spec}(K) \to S \). That is, we ‘spread-out’ the family \( f \) to a good integral model \( \hat{f} \). Let \( \alpha : \mathfrak{X} \to S \) be the structure morphism.

Step 2: Let \( (\hat{\mathfrak{H}}, \nabla, \hat{\mathfrak{g}}) \) be the Gauß-Manin system associated to the semistable family \( \hat{f} \) defined over \( S \) and \( (\mathcal{E}, \theta) \) the graded Higgs module. There exists a principal open subset \( T = \text{Spec}(A_t) \subset S \) for some \( t \in A \) over which the following properties are satisfied:

(i) \( \hat{\mathfrak{H}}, \mathcal{E} \) are locally free;
(ii) \( (\hat{\mathfrak{H}}, \nabla, \hat{\mathfrak{g}}) \) as well as \( (\mathcal{E}, \theta) \) satisfy the base change property for arbitrary morphism \( S' \to T \) of noetherian schemes;
(iii) The \( A \) modules \( R^n \alpha_* (\Omega^*_{int}(\hat{\mathfrak{H}}, \nabla)) \) and \( R^n \alpha_* (\Omega^*_{int}(\mathcal{E}, \theta)) \) are free of finite rank.

Replace \( S \) by \( T \) in the following but keep the notation.

Step 3: By Lemma 5.4, we may assume that for any closed point \( s \in S \) by shrinking \( S \) if necessary,
\[
\Omega^*_{int}(\hat{\mathfrak{H}}, \nabla) \otimes k(s) \cong \Omega^*_{int}(H_s, \nabla_s)
\]
and
\[
\Omega^*_{int}(\mathcal{E}, \theta) \otimes k(s) \cong \Omega^*_{int}(E_s, \theta_s),
\]
where \( (H_s, \nabla_s, Fil_s) \) is the Gauß-Manin system of the closed fiber \( f_s \) of \( f \) over \( s \) and \( (E_s, \theta_s) \) is the associated graded Higgs bundle. Moreover, we may also assume that the characteristic of \( k(s) \) is larger than \( d + n \).

Step 4: By the result in §4.1, the de Rham bundle \( (H_s, \nabla_s, Fil_s) \) is one-periodic. Thus, using the \( E_1 \)-degeneration theorem in positive characteristic (Theorem 4.13), one obtains the equality
\[
dim_{k(s)} \mathbb{H}^n(X_s, \Omega^*_{int}(H_s, \nabla_s)) = \dim_{k(s)} \mathbb{H}^n(X_s, Gr_{Fil_s} \Omega^*_{int}(H_s, \nabla_s)).
\]
Step 5: By Steps 2 and 3, we have the following equalities
\[
\dim_K \mathbb{H}^n(X, \Omega^\ast_{\text{int}}(H, \nabla)) = \dim_K R^n\alpha_s(\Omega^\ast_{\text{int}}(\mathcal{F}, \nabla)) \otimes A K \\
= \dim_{k(s)} R^n\alpha_s(\Omega^\ast_{\text{int}}(\mathcal{F}, \nabla)) \otimes k(s) \\
= \dim_{k(s)} \mathbb{H}^n(X_s, \Omega^\ast_{\text{int}}(H_s, \nabla_s)).
\]
Similarly, \( \dim_K \mathbb{H}^n(X, Gr_{\text{Fil}}\Omega^\ast_{\text{int}}(H, \nabla)) = \dim_{k(s)} \mathbb{H}^n(X_s, Gr_{\text{Fil}}\Omega^\ast_{\text{int}}(H_s, \nabla_s)). \) Therefore, the theorem follows from Step 4. \( \square \)

We prove the following intersection-adaptedness theorem in characteristic zero.

Theorem 5.5. Let \((H, \nabla, Fil)\) be the Gauß-Manin system associated to the semistable family \(f\) defined over \(K\). Then the following equality holds:
\[
\Omega^\ast_{\text{int}}Gr_{\text{Fil}}(H, \nabla) = Gr_{\text{Fil}}\Omega^\ast_{\text{int}}(H, \nabla).
\]

Proof. We use the notations in the proof of Theorem 5.2. By replacing \(S\) with the Zariski closure of the point \(\text{Spec}(K)\), in the proof of Theorem 5.2 we may assume additionally that \(\text{Spec}(K)\) in \(S\) is the generic point. By Lemma 4.4, we have an inclusion of complexes \(\Omega^\ast_{\text{int}}(\mathcal{E}, \theta) \subseteq Gr_{\text{Fil}}\Omega^\ast_{\text{int}}(\mathcal{F}, \nabla)\). Let \(Q^\ast\) be the cokernel of the inclusion. Since it is a finite complex of coherent \(\mathcal{O}_X\)-modules, the union \(Z\) of supports of all terms in \(Q^\ast\) is a closed subset of \(\mathfrak{X}\). By Step 3 in the proof of Theorem 5.2, for any closed point \(s \in S\), the base change \(Q^\ast \otimes \mathcal{O}_{X_s}\), where \(X_s := \mathfrak{X} \times S \text{Spec}(k(s))\) is the closed fiber of \(\mathfrak{X}\) over \(s\), is equal to the cokernel of the inclusion \(\Omega^\ast_{\text{int}}(\mathcal{E}_s, \theta_s) \subseteq Gr_{\text{Fil}}\Omega^\ast_{\text{int}}(H_s, \nabla_s)\). By Theorem 4.3, \(Q^\ast \otimes \mathcal{O}_{X_s}\) is zero. Since the structure morphism \(\mathfrak{X} \to S\) is proper, any closed point \(x \in \mathfrak{X}\) is contained in some closed fiber \(X_s\). Therefore the complex \(Q^\ast\) is zero at all closed points of \(\mathfrak{X}\). So \(Z\) is empty and \(Q^\ast\) is identically zero. Therefore, the equality \(\Omega^\ast_{\text{int}}(\mathcal{E}, \theta) = Gr_{\text{Fil}}\Omega^\ast_{\text{int}}(\mathcal{F}, \nabla)\) holds. By the flat base change to the generic point, we obtain the equality for the starting family \(f\) over \(K\). The theorem is proved. \( \square \)

Now we come to the proof of Corollary 5.3.

Proof. By Kashiwara-Kawai [KK], the middle perverse sheaf \(V^\ast \otimes \mathbb{C}\) is quasi-isomorphic to \(\Omega^\ast_{\text{int}}(H_{an}, \nabla_{an})\). Using the algebraic de Rham theorem of Grothendieck, for each degree \(m\), there is a natural isomorphism of filtered vector spaces
\[
\mathbb{H}^m(X, \Omega^\ast_{\text{int}}(H, \nabla)) \cong \mathbb{H}^m(X_{an}, \Omega^\ast_{\text{int}}(H_{an}, \nabla_{an})).
\]
The rest follows from Theorem 5.2 and Theorem 5.5. \( \square \)

We conclude our paper with a Kodaira type vanishing result.

Corollary 5.6. Let \(X\) be an \(n\)-dimensional smooth projective variety over \(k\) with \(D \subset X\) a simple normal crossing divisor. Assume \((X, D)\) is \(W_2\)-liftable. Let \((E, \theta)\) be a periodic logarithmic Higgs bundle of rank \(r\) with pole along \(D\). If the inequality \(n + r \leq p\) holds, one has the following vanishing
\[
\mathbb{H}^i(X, \Omega^\ast_{\text{int}}(E, \theta) \otimes L) = 0, \quad i > n
\]
for any ample line bundle \(L\) on \(X\).
Proof. Assume the period of \((E, \theta)\) to be \(f\). Let \((E_i, \theta_i), i \geq 0\) be the Higgs terms of an \(f\)-periodic Higgs-de Rham flow with the initial term \((E_0, \theta_0) = (E, \theta)\). The decomposition theorem [3.1] and the \(E_1\)-degeneration theorem [1.13] together yield a sequence of equalities

\[
\dim \mathbb{H}^i(\Omega^*_{\text{int}}(E_0, \theta_0) \otimes L) = \dim \mathbb{H}^i(\Omega^*_{\text{int}}(E_1, \theta_1) \otimes L^p) = \cdots = \dim \mathbb{H}^i(\Omega^*_{\text{int}}(E_f, \theta_f) \otimes L^{p^r}).
\]

As \((E_f, \theta_f) \cong (E_0, \theta_0)\) by assumption, we obtain the following equality

\[
\dim \mathbb{H}^i(\Omega^*_{\text{int}}(E, \theta) \otimes L) = \dim \mathbb{H}^i(\Omega^*_{\text{int}}(E, \theta) \otimes L^{p^r}).
\]

It follows that for any \(m \in \mathbb{N}\),

\[
\dim \mathbb{H}^i(\Omega^*_{\text{int}}(E, \theta) \otimes L) = \dim \mathbb{H}^i(\Omega^*_{\text{int}}(E, \theta) \otimes L^{p^mj}).
\]

On the other hand, the Serre vanishing theorem implies that for \(i > n\),

\[
\dim \mathbb{H}^i(\Omega^*_{\text{int}}(E, \theta) \otimes L^{p^mj}) = 0, \quad m \gg 1.
\]

Therefore \(\dim \mathbb{H}^i(\Omega^*_{\text{int}}(E, \theta) \otimes L) = 0\) for \(i > n\) as claimed. \(\square\)

**Theorem 5.7.** Notation as above. Then, for any ample line bundle \(L\) over \(X\), it holds that

\[
\mathbb{H}^i(X, \Omega^*_{\text{int}}(E, \theta) \otimes L) = 0, \quad i > d.
\]

**Proof.** This follows directly from the proof of Theorem 5.2 and Corollary 5.6. \(\square\)

### 6. Appendix

In the appendix, we verify the truth of Proposition [3.2]. It comes down to compare the following two expressions:

\[
(6.0.1) \quad (F_s \nabla_{\alpha_0} \varphi(r, s) \omega)_\alpha + ((-1)^{s+1} \delta \varphi(r - 1, s + 1) \omega)_\alpha,
\]

\[
(6.0.2) \quad (\varphi(r, s + 1) \theta \omega)_\alpha,
\]

where \(r > 0, s \geq -1, r + s \leq \min\{n, p - 1\}, \omega \in E \otimes \omega^{r+s}\), \(\alpha = (\alpha_0, \ldots, \alpha_r)\).

As the problem is local, we choose a system of special log system \(\{t_i\}\) on \(\tilde{U}\) such that the divisor \(\tilde{D}\) is defined by the product of the first \(r\) coordinate functions. Shrinking \(\tilde{U}\) if necessary, we may assume that \(\{d \log t_i\}\) form a local basis for \(\omega_{X/k}\). We write

\[
\theta = \sum_i \theta_i \otimes d \log t_i, \quad \theta_i \in \text{End}_{\mathcal{O}_{\tilde{U}}}(E|_{\tilde{U}}).
\]

#### 6.1. Notations.

- \(W \{1, \ldots, n\}\).
- \(W_0^s \{ \tilde{i} \in W^s : i_1 < \cdots < i_s \}\).
- \(W^s \{ \tilde{i} \in W^s : \#\{i_1, \ldots, i_s\} = s \}\).
- \(W^s_0 = W_0^{s_0} \times \cdots \times W_0^{s_r}\) (if \(s_k = 0\) for some \(k\), then we delete \(W_0^{s_k}\) in this product).
- \(\hat{a}(i)\) Deleting the \(i\)-th entry \(a_i\) of \(\alpha\), \(0 \leq i \leq r\).
- \(\hat{p}(i)\) Deleting the \(i\)-th entry \(s_i\) of \(p = (s_0, \ldots, s_r)\), \(0 \leq i \leq r\).


\[ c(i, \ell) = k - 1, \text{ where } i \in \mathbb{I} = (i_1, \ldots, i_s) \in W^s \text{ and } k \text{ is the smallest number such that } i = i_k. \]

\[ \chi(i, \bar{t}, k) (i_1, \ldots, i_{k-1}, i, i_k, \ldots, i_r), \text{ where } \bar{t} = (i_1, \ldots, i_r) \in W^r, i \in W, 1 \leq i \leq r + 1. \]

\[ \rho(i) \text{ Replacing the } i^{\text{th}} \text{ entry } s_i \text{ of } \rho \text{ by } s_i - 1. \]

\[ \theta(t)(\bar{t}, \bar{\bar{t}}) \prod_{q=1}^{n} \theta_{q}^{j_1 + j_2 + \cdots + \delta j_{m+1} - \delta j_{m+1}}, \text{ where we denote } \theta_{i}^k = id \text{ if } k \leq 0. \]

\[ \zeta_{\omega, \rho}(F^* \log t_i) \zeta_{\omega, \rho}(F^* \log t_i \otimes \cdots \otimes F^* \log t_i). \]

\[ h_{\omega}(F^* \log t_i) \zeta_{\omega, \rho}(F^* \log t_i \otimes \cdots \otimes F^* \log t_i). \]

\[ \frac{1}{j!} \prod_{q=1}^{n} \prod_{i=1}^{r} h_{\omega_{\rho}}(F^* \log t_i \rho_{i}^{j}). \]

\[ \sum_{j(k, l) \in \mathbb{N}^n, \text{ where } j(k, l) = j_i^k, (k', l') \neq (k, l); j(k, l)_{k} = j_i^k + 1, 1 \leq k \leq r, 1 \leq i \leq n. \]

\[ \frac{1}{j!} (\sum_{j}^{j_0}, \sum_{j}^{j_r}), \text{ where } \sum_{j}^{j_i} = j_1^0 + \cdots + j_i^0 \text{ for } i = 0, \ldots, r. \]

\[ \frac{1}{j!} (\sum_{j}^{j_1}, \sum_{j}^{j_2}, \sum_{j}^{j_r}) \in \mathbb{F}_p \text{ if } j^r + s_r > 0, \sum_{k=1}^{r} j^k + \sum_{k=1}^{r} s_k < p \text{ otherwise.} \]

### 6.2. Local description of \( \varphi \).

Let us assume \( \omega \in E|_{U'_j \otimes \omega_{U}'_j/k} \) and let us further write

\[ \omega = \frac{1}{(r + s)!} \sum_{i_1, \ldots, i_{r+s}} e_{i_1, \ldots, i_{r+s}} \log t_{i_1} \wedge \cdots \wedge \log t_{i_{r+s}} \]

for some \( e_{i_1, \ldots, i_{r+s}} \in E|_{U'_j} \) (we also write it by \( \omega_{i_1, \ldots, i_{r+s}} \)) satisfying

\[ e_{i_1, \ldots, i_{r+s}} = \text{sgn}(\sigma)e_{i_{\sigma(1)}, \ldots, i_{\sigma(r+s)}} \]

where \( \sigma \) is a permutation of \( \{1, \ldots, r + s\} \). Let us introduce a family \( \hat{\mathfrak{A}}_{r,s,k} \) by

\[ \left\{ l_{\omega}^{j_{\bar{t}, \bar{\bar{t}}}} : F_{*} \frac{\hat{h}_{\omega}(F^* \log t_i)}{j!} \zeta_{\omega}(F^* \log t_i) \right\} E_{P(r, s), \mathbb{F}_p} \mathbb{F}_p W_{r,s}^1. \]

Using the skew-symmetric form of \( \omega \), the natural identification \( F_{*}(H_{\omega_0} \otimes \omega_{U'_{\omega_0}}/k) = E_{\omega_{U_0}} \otimes F_{*}(\omega_{U'_{\omega_0}}^r/k) \) together with a direct computation, we have the following local description of \( (\varphi(r, s)\omega)_{\mathfrak{A}} \). The proof of the following lemma is straightforward.

**Lemma 6.1.** There exists a natural family

\[ \left\{ e_{1}^{j_{\bar{t}, \bar{\bar{t}}}} : a'(\sum_{j}^{j_0}, \sum_{j}^{j_r}) \theta(t)(\bar{t}, \bar{\bar{t}}) E_{\mathbb{F}_p} \right\} E_{P(r, s), \mathbb{F}_p} W_{r,s}^1 \]

such that \( (\varphi(r, s)\omega)_{\mathfrak{A}} = \sum e_{1}^{j_{\bar{t}, \bar{\bar{t}}}} \otimes j_{1}^{j_{\bar{t}, \bar{\bar{t}}}}. \)

### 6.3. Verification.

Let us write

\[ \theta_{\omega} = \frac{1}{(r + s + 1)!} \sum_{i_1, \ldots, i_{r+s+1}} (\theta_{\omega})_{i_1, \ldots, i_{r+s+1}} \log t_{i_1} \wedge \cdots \wedge \log t_{i_{r+s+1}}, \]

where \( (\theta_{\omega})_{i_1, \ldots, i_{r+s+1}} = \sum_{k=1}^{r+s+1} (-1)^{k-1} \theta_{i_1, \ldots, i_k, \ldots, i_{r+s+1}} \) and \( r + s + 1 < p. \)
Lemma 6.2. Notations as above. There exist two natural families
\[
\{ e^{j,i}_{\underline{\lambda},p} \in E|_{U'_\underline{\alpha}} \}, \quad \underline{\lambda} \in \mathbb{N}^n, \underline{p} \in P(r,s+1), \underline{\lambda} \in W_0^r \cap W_1^{s+1}, \quad \{ e^{n,j}_{\underline{\lambda},p} \in E|_{U'_\underline{\alpha}} \}, \quad \underline{\lambda} \in \mathbb{N}^n, \underline{p} \in P(r,s+1), \underline{\lambda} \in W_0^r \cap W_1^{s+1},
\]

such that (6.0.1) \( \sum e^{j,i}_{\underline{\lambda},p} \in F_{\underline{\alpha}} \) and (6.0.2) \( \sum e^{n,j}_{\underline{\lambda},p} \in F_{\underline{\alpha}} \). For any \( j \in \mathbb{N}^n, \underline{p} \in P(r,s+1), \underline{\lambda} \in W_0^r \cap W_1^{s+1}, e^{j,i}_{\underline{\lambda},p} = e^{n,j}_{\underline{\lambda},p} \) if the corresponding condition in following tabular is satisfied:

| \( j+s+1 \leq p \) | \( j+s+1 \leq p \) | \( j+s+1 \leq p \) | \( j+s+1 \leq p \) |
|-----------------|-----------------|-----------------|-----------------|
| \( r+s \leq p+1 \) | \( r+s \leq p+1 \) | \( r+s \leq p+1 \) | \( r+s \leq p+1 \) |
| \( j+s+1 < p \) | \( j+s+1 < p \) | \( j+s+1 < p \) | \( j+s+1 < p \) |
| \( j+s+1 \leq p \) | \( j+s+1 \leq p \) | \( j+s+1 \leq p \) | \( j+s+1 \leq p \) |

where \( l \) is the nilpotent level of \((E,\theta)\) and

\[
\underline{j} = (j^k_q), \quad 1 \leq k \leq r, \quad 1 \leq q \leq n, \quad j = \sum_{k,q} j^k_q, \quad j^r = \sum_{q} j^r_q, \quad \underline{p} = (s_0, ..., s_r).
\]

Moreover, for any one of last three cases in the above tabular, if the corresponding condition is satisfied, then \( e^{j,i}_{\underline{\lambda},p} = 0 = e^{n,j}_{\underline{\lambda},p} \). For the first case in the above tabular, we have

\[
e^{j,i}_{\underline{\lambda},p} = e^{n,j}_{\underline{\lambda},p} = \sum_{\tau \in W^r} a^*(\Sigma j^\tau p) j^\tau \theta(\tau j, \tau p)(\theta \omega)|_{U'_\underline{\alpha}}.
\]

By the tabular in the Lemma, we know that the truncation

\[
\tau_{<p-1} \phi : \tau_{<p-1}(E \otimes \omega_{X_{\log/k}}^*) \rightarrow \tau_{<p-1}C^*(U'_\underline{\alpha}, F_*(H \otimes \omega_{X_{\log/k}}^*))
\]

is an \( O_X \)-morphism of complexes of \( O_X \)-modules. Therefore, Proposition 3.2 follows.

Proof. Let us figure out the natural constructions of the above two natural families. For any \( f \in \mathcal{F}_{\underline{\alpha},r,s,t}, e \in E|_{U'_\underline{\alpha}} \), we have \( F_*(\nabla_{can}(e \otimes f)) = e \otimes F_*(f) \), it is easy to see that \( F_*(f) \) is an \( \mathcal{F}_{\underline{\alpha}} \)-linear combination of the family \( \mathcal{F}_{\underline{\alpha},r,s+1,t} \) with coefficients \( \pm 1 \) and hence \( F_*(\nabla_{can}(\varphi(r,s)\omega)) \) is a combination of the family \( \mathcal{F}_{\underline{\alpha},r,s+1,t} \) with coefficients in \( E|_{U'_\underline{\alpha}} \). On the other hand, since

\[
F_*(1 \otimes \zeta_{\alpha_0}) F^* \theta(e \otimes f) = \sum_{1}^{n} \theta_{1} e \otimes (F_*(\zeta_{\alpha_0}) (F^* \theta_{1} f))
\]

we know that \( F_*(1 \otimes \zeta_{\alpha_0}) F^* \theta(\varphi(r,s)\omega) \) is a combination of elements in the family \( \mathcal{F}_{\underline{\alpha},r,s+1,t} \) with coefficients in \( E|_{U'_\underline{\alpha}} \). Consequently, we get that \( F_*(\nabla_{\alpha_0}(\varphi(r,s)\omega)) \) is a combination of elements in the family \( \mathcal{F}_{\underline{\alpha},r,s+1,t} \) with coefficients in \( E|_{U'_\underline{\alpha}} \).

Let us consider the second term in (6.0.1). Define families \( \mathcal{F}_{\underline{\alpha},t}(k,r-s+1,t), 0 \leq k \leq r \) in the similar way as \( \mathcal{F}_{\underline{\alpha},r,s+1,t} \). We claim that there is a natural way to expand

\[
((-1)^{s+1} \delta \varphi(r-1,s+1)\omega)|_{\underline{\alpha}}
\]

as combination of elements in the family \( \mathcal{F}_{\underline{\alpha},r,s+1,t} \) with coefficients in \( E|_{U'_\underline{\alpha}} \). But the definition of \( \delta \), we obtain

\[
((-1)^{s+1} \delta \varphi(r-1,s+1)\omega)|_{\underline{\alpha}} = \sum_{k=0}^{r} (-1)^{k+s+1} \varphi(r-1,s+1)|_{\underline{\alpha}(k)} \omega.
\]
If we can show that for any \(0 \leq k \leq r\), \(\varphi(r - 1, s + 1)\overline{A}(k)\omega\) is a combination of elements in the family \(\mathfrak{F}_{\Delta, r, s+1, t}\) with coefficients in \(E|_{U^{r'}}\), then our claim follows.

Here’s the natural ways: when \(k = 0\), using the transition morphism \(G_{a_{0}a_{1}}\); when \(0 < k < r\), replacing \(h_{a_{k}a_{k+1}}\) by \(h_{a_{k}a_{k}a_{k+1}}\); when \(k = r\), this is obvious.

The above discussion gives rise to the first natural family in this lemma. The second natural family comes from the skew-symmetric form of \(\theta\omega\) as introduced above for \(r + s < p - 1\) and we take it to be \(\{0\}\) if \(r + s + 1 = p\).

Let us give an outline of the verification of the above tabular. For the last three cases, it is easy to check that \(e_{\Delta, p}^{\prime} = 0\) and for the first case we have

\[
e_{\Delta, p}^{\prime} = \sum_{\bar{r} \in W^r} d'(\sum_{j, p} \bar{r}_j (\overline{r}_j, \overline{r}_j)(\theta\omega)_{\bar{r}_j}.
\]

Let us turn to the computation of \(e_{\Delta, p}^{\prime}\). The computation of \(e_{\Delta, p}^{\prime}\) in the first case of (6.2.1) is mostly technical. Hence we put it off till the end of this lemma. The computation in the second case is nothing but the one in the first case, combining with the fact that \(\theta_1 = \cdots = \theta_n = 0\) which ensures \(e_{\Delta, p}^{\prime} = 0\). The third case would follow immediately after the first case has been done. In the last case, \(e_{\Delta, p}^{\prime}\) is the sum of terms containing \(\theta_1, \ldots, \theta_n\) with a total power \((j+s+1)-(r+s) \geq p-(r+s)\). Thus if \(l < p - (r+s)\), then \(e_{\Delta, p}^{\prime} = 0\).

Given a term \(f_{\Delta, p}^{\prime} = 3_{\Delta, r, s+1, t}\) corresponding to the first case in tabular (6.2.1) with \(\bar{i} = (i_1^t), \bar{p} = (s_k)\). We can check that

\[
f_{\Delta, p}^{\prime} = f_{\Delta, p}^{j(1, p_0)}(\overline{i}_1^t), f_{\Delta, p}^{j(0)}(p_0) \text{ if } s_0 > 0 \text{ and } 1 \leq l \leq s_0
\]

\[
f_{\Delta, p}^{j(k, i_1^t)}(\overline{i}_1^t), f_{\Delta, p}^{j(k+1, i_1^t)}(\overline{i}_1^t) \text{ if } s_k > 0, 0 < k < r \text{ and } 1 \leq l \leq s_k
\]

\[
f_{\Delta, p}^{j(r, i_1^t)}(\overline{i}_1^t) \text{ if } s_r > 0 \text{ and } 1 \leq l \leq s_r
\]

are all the possible \(f \in 3_{\Delta, r, s+1, t}\) such that the natural expansion of \(F_\ast \nabla_{a_0}(e \otimes f)\) by the family \(3_{\Delta, r, s+1, t}\) with coefficients in \(E|_{U^{r'}}\) will provide a coefficient for \(f_{\Delta, p}^{\prime}\), where \(e \in E|_{U_{a_0}}\) is the natural coefficient of \(f\) in \((\varphi(r, s)\omega)_{\Delta}\). Hence the natural coefficient of \(f_{\Delta, p}^{\prime}\) in (6.0.1) provided by the first term of (6.0.1) is the sum of following terms:

\[
\sum_{0 \leq k < s_k > 0} B_k, \quad B_k = \sum_{l=1}^{s_k} (-1)^l e_{\Delta, p}^{j(l, i_1^t)}(\overline{i}_1^t)
\]

\[
\sum_{0 < k \leq s_k > 0} A_k, \quad A_k = \sum_{l=1}^{s_k} (-1)^l e_{\Delta, p}^{j(l, i_1^t)}(\overline{i}_1^t)
\]

and \(A_0 = \sum_{l=1}^{s_0} (-1)^{l-1} \theta_1 e_{\Delta, p}^{j(0)}(p_0)\) if \(s_0 > 0\). On the other hand, it is easy to see that the coefficient of \(f_{\Delta, p}^{\prime}\) in (6.0.1) provided by the second term of (6.0.1) is the sum of following terms:

\[
\sum_{0 < k < s_k = 0} (-1)^{k+s+1} e_{\Delta, p}^{j(k, k+1)(\overline{i}_1^t), (-1)^{s+1} e_{\Delta, p}^{j(1)}(\overline{i}_1^t)}\], \quad if \ s_0 = 0.
\]
Moreover, it is easy to check that for any $0 \leq k \leq r$, we know that

\begin{equation}
(6.2.2) \quad A_0 = \sum_{l=1}^{s_k} \sum_{\tau \in W^r} (-1)^{l-1} a'(\Sigma j, p(0)) j_\tau \theta(t) e_{\omega} (\tau, \tilde{\tau}) \tilde{e}_{\omega} (\tau, \tilde{\tau})
\end{equation}

for $0 < k \leq r$, $A_k$ is equal to

\[ \sum_{l=1}^{s_k} \sum_{\tau \in W^r} (-1)^{c(l, \tau)} a'(\Sigma j, p(k)) j_\tau \theta(t) e_{\omega} (\tau, \tilde{\tau}) \tilde{e}_{\omega} (\tau, \tilde{\tau}) \]

for $0 \leq k < r$, $B_k$ is equal to

\[ \sum_{l=1}^{s_k} \sum_{\tau \in W^r} (-1)^{c(l, \tau)} a'(\Sigma j, p(k)) j_\tau \theta(t) e_{\omega} (\tau, \tilde{\tau}) \tilde{e}_{\omega} (\tau, \tilde{\tau}) \]

Let us divide $B_k$ ($0 \leq k < r$) into two parts $B_{k1}, B_{k2}$ by replacing $j(k+1, i_k^r)$ by $j_{\tau}$, $j(k, i_k^r)$ by $j_{\tau}$ respectively in the expression of $B_k$. Similarly, let us divide $A_k$ ($0 < k \leq r$) into two parts $A_{k1}, A_{k2}$ by replacing $j(k, i_k^r)$ by $j_{\tau}$, $j(k, i_k^r)$ by $j_{\tau}$ respectively in the expression of $A_k$. For any $1 \leq l \leq s_k$, $0 < k < r$, the following equalities

\[ \theta_{\tau} \theta(t)(\tilde{\tau}, j_{\tau}) = \theta(t)(\tilde{\tau}, j(k, i_k^r)) = \theta(t)(\tilde{\tau}, j(k+1, i_k^r)), \]

\[ a'(\Sigma j, p(k)) - a'(\Sigma j, p(k)) = a'(\Sigma j', p) \]

hold, and we define

\[ \Delta_k := A_{k1} + B_{k1} = \sum_{l=1}^{s_k} \sum_{\tau \in W^r} (-1)^{c(l, \tau)} a'(\Sigma j, p) j_\tau \theta(t) e_{\omega} (\tau, \tilde{\tau}) \tilde{e}_{\omega} (\tau, \tilde{\tau}) \]

Similarly, we define

\[ \Delta_0 := A_0 + B_{01} = \sum_{l=1}^{s_k} \sum_{\tau \in W^r} (-1)^{c(l, \tau)} a'(\Sigma j, p) j_\tau \theta(t) e_{\omega} (\tau, \tilde{\tau}) \tilde{e}_{\omega} (\tau, \tilde{\tau}) \]

\[ \Delta_r := A_{r1} = \sum_{l=1}^{s_k} \sum_{\tau \in W^r} (-1)^{c(l, \tau)} a'(\Sigma j, p) j_\tau \theta(t) e_{\omega} (\tau, \tilde{\tau}) \tilde{e}_{\omega} (\tau, \tilde{\tau}) \]

In the definition of $\Delta_r$, one notices that $a'(\Sigma j, p(r)) = a'(\Sigma j, p)$. It is easy to see that $j(k+1, i_k^r) t = j_{\tau}$ ($0 < k \leq r$) (resp. $j(k, i_k^r) t = j_{\tau}$ ($0 < k < r$)) if $i_k^r$ is not the $k$-th (resp. $(k+1)$-th) entry of $\tilde{\tau}$ and

\[ j(k, i_k^r) t = (j_{\tau}^k + 1) j(k, i_k^r) \]

if $i_k^r$ is the $k$-th (resp. $(k+1)$-th) entry of $\tilde{\tau}$. Hence $A_{k2}$ (resp. $B_{k2}$) is equal to

\[ \sum_{\tau \in W^{r-1}} (-1)^{s+1+k} \sum_{l=1}^{s_k} (-1)^{s+1+k} \sum_{\tau \in W^r} (-1)^{c(l, \tau)} a'(\Sigma j, p) j_\tau \theta(t) e_{\omega} (\tau, \tilde{\tau}) \tilde{e}_{\omega} (\tau, \tilde{\tau}) \]

We should notice that although $A_{k2}$’s (resp. $B_{k2}$’s) are defined only when $s_k > 0$, the above expressions of $A_{k2}$’s (resp. $B_{k2}$’s) are defined when $s_k = 0$ as well. So let us redefine $A_{k2}$’s (resp. $B_{k2}$’s) by the right hand side of above equalities. Moreover, it is easy to check that for any $0 \leq k \leq r$, if $s_k = 0$, then the canonical $(E, \iota_0)$-coefficient of $f_{\Delta_k}^{i_k^r}$ in $[60, 0, 1]$ provided by $(-1)^{s+1}(\varphi(r, s+1, t) \iota_{\omega}) \tilde{\Delta}(k)$ is $B_{02}$ when $k = 0$, $A_{k2} + B_{k2}$ when $0 < k < r$ and $A_{r2} = 0$ when $k = r$. 

\[ \tilde{\Delta}(k) \]
Set

\[ M = B_{02} + \sum_{0<k<r} (A_{k+2} + B_{k+2}) + A_{r-2} = \sum_{k=0}^{r-1} (B_{k+2} + A_{(k+1)2}), \quad N = \sum_{0 \leq k \leq r, s_k \neq 0} \Delta_k. \]

It is easy to see that the coefficient of \( f_{\tau}^{i, j} \) in (6.0.1) is \( M + N \).

**Lemma 6.3.** \( M + N = \sum_{\tau \in W^r} a'(\Sigma j, p) j \theta(\tau, 1)(\theta \omega)_{\tau} \).

**Proof.** A direct computation shows that

\[
M = \sum_{k=0}^{r-1} \sum_{\tau \in W^{r-1}} (-1)^{s+k+1} j^{k+1} a'(\Sigma j, p) \hat{j} (k+1) \theta (t, \tau) e_{\tau},
\]

\[
N = \sum_{i \in \mathfrak{L}} \sum_{\tau \in W^r} \sum_{k} (-1)^{s+k+1} a'(\Sigma j, p) j \theta (t, \tau) e_{\tau},
\]

where the second summation takes over all \( 0 \leq k \leq r - 1, \tau \in W^{r-1} \) satisfying \( \chi (i, \tau , k + 1) = \tau \). On the other hand, we can verify that

\[
N = \sum_{i \in \mathfrak{L}} \sum_{\tau \in W^r} (-1)^{c(i, j)} a'(\Sigma j, p) j \theta (t, \tau) e_{\tau},
\]

from which it follows that

\[
M + N = \sum_{\tau \in W^r} a'(\Sigma j, p) j \theta (t, \tau) (\sum_{i \in \mathfrak{L}} (-1)^{c(i, j)} \theta e_{\tau}) + \sum_{i \in \mathfrak{L}} (-1)^{s+k+1} \theta e_{\tau}.
\]

By the definition of \( (\theta \omega)_{\tau} \), we know that

\[
(\theta \omega)_{\tau} = \sum_{i \in \mathfrak{L}} (-1)^{c(i, j)} \theta e_{\tau} + \sum_{i \in \mathfrak{L}} (-1)^{s+k+1} \theta e_{\tau}.
\]

Now the lemma follows. \( \square \)

**References**

[Ar] D. Arapura, Kodaira-Saito vanishing theorem via Higgs bundles in positive characteristic, Journal für angewandte und reine Mathematik, 2018.

[CKS] E. Cattani, A. Kaplan, W. Schmid, \( L^2 \) and intersection cohomologies for a polarizable variation of Hodge structure. Invent. Math. 87, 217-270, 1987.

[Di] P. Deligne, L. Illusie, Relèvements modulo \( p^2 \) et décomposition du complexe de de Rham, Invent. Math. 89 (1987), 247-270.

[Ei] D. Eisenbud, Commutative algebra with a view towards algebraic geometry, GTM 150, New York, Springer-Verlag, 1999.

[EV] H. Esnault, E. Viehweg, Lectures on vanishing theorems. DMV Seminar, 20. Birkhäuser Verlag, Basel, 1992.

[Fa] G. Faltings, Crystalline cohomology and \( p \)-adic Galois-representations, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 25-80, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[HL] D. Huybrechts, M. Lehn, The geometry of moduli spaces of sheaves. Second Edition, Cambridge University Press.
[IL90] L. Illusie, Rédaction semistable et décomposition de complexes de de Rham à coefficients, Duke Math. J., vol. 60, 1990, no. 1, 139-185.
[IL02] L. Illusie, Frobenius and Hodge degeneration, Introduction to Hodge theory, SMF/AMS Texts and Monographs, vol 8., 96-145, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2002.
 KK] M. Kashiwara, T. Kawai, Poincaré lemma for a variation of Hodge structure. Publ. RIMS, Kyoto University 23, 345-407, 1987.
 [Ka] K. Kato, Logarithmic structures of Fontaine-Illusie, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191-224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.
[LSYZ] G.-T. Lan, M. Sheng, Y.-H. Yang, K. Zuo, Uniformization of $p$-adic curves via Higgs-de Rham flows. Journal für angewandte und reine Mathematik, 2016.
[LSZ14] G.-T. Lan, M. Sheng, K. Zuo, Nonabelian Hodge theory in positive characteristic via exponential twisting, Mathematical Research Letter, 2014.
[LSZ17] G.-T. Lan, M. Sheng, K. Zuo, Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups, Journal of European Mathematical Society, 2017.
[Ogus94] A. Ogus, F-crystals, Griffiths transversality, and the Hodge decomposition, Astérisque 221, 1994.
[Ogus04] A. Ogus, Higgs cohomology, $p$-curvature, and the Cartier isomorphism. Compositio Math. 140 (2004), 145-164.
[OV] A. Ogus, V. Vologodsky, Nonabelian Hodge theory in characteristic $p$, Publ. Math. Inst. Hautes études Sci. 106 (2007), 1-138.
[Sche] D. Schepler, Logarithmic nonabelian Hodge theory in characteristic $p$. arXiv:0802.1977V1, 2008.
[Schm] W. Schmid, Variation of Hodge structure: the singularities of the period mapping, Inventiones Math. 22, 211-319, 1973.
[Sim] C. Simpson, Higgs bundles and local systems, Inst. Hautes Études Sci. Publ. Math. No. 75 (1992), 5-95.
[Zuc] S. Zucker, Hodge theory with degenerating coefficients: $L_2$ cohomology in the Poincaré metric, Ann. of Math. 109, 1979, 415-476.

E-mail address: msheng@ustc.edu.cn
E-mail address: zzb2@mail.ustc.edu.cn

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, China