Large Scale Inhomogeneity Versus Source Evolution
— Can We Distinguish Them Observationally?

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ABSTRACT

We reconsider the issue of proving large scale spatial homogeneity of the universe, given isotropic observations about us and the possibility of source evolution both in numbers and luminosities. Two theorems make precise the freedom available in constructing cosmological models that will fit the observations. They make quite clear that homogeneity cannot be proven without either a fully determinate theory of source evolution, or availability of distance measures that are independent of source evolution. We contrast this goal with the standard approach that assumes spatial homogeneity \textit{a priori}, and determines source evolution functions on the basis of this assumption.

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1 INTRODUCTION

Ever since the earliest cosmological models, the Einstein and de Sitter models, we have been trying to fit observations to the Friedmann-Lemaître-Robertson-Walker (FLRW) spatially homogeneous and isotropic family of models. The successes of reproducing a Hubble redshift-distance law, calculating the correct cosmic helium & deuterium abundances, and the prediction of a cosmic microwave background radiation (CMBR), have convinced us of its validity as a bulk description of the universe. However, \textit{proving} that the geometry of the universe is FLRW on the largest scales is not easy. In fact, the history of observational cosmology shows that each improved instruments permit deeper surveys, the new data soon reveals inhomogeneities on the new scale.

The best evidence for homogeneity comes from limits on anisotropy of both galaxy counts and the CMBR, obtained in each case by comparison of observations in different directions. However this is strictly speaking only evidence for isotropy about the earth; homogeneity follows only if we introduce a Copernican principle, either for galaxies or for the Cosmic Background Radiation \textsuperscript{3}. Without this assumption, the models indicated are isotropic about us, but allow a spatial variation of the geometry and matter content that is spherically symmetric about our position.\textsuperscript{3} The Copernican principle is not really in dispute on a sub-horizon scale\textsuperscript{3} but could be incorrect on a super-horizon scale if we accept theories such as chaotic inflation \textsuperscript{4}; see also \textsuperscript{4}). Therefore one would like to actually prove homogeneity for the observable region of the universe, rather than assuming it on principle, which is essentially what happens in the usual approach. Similar issues have been discussed by Goodman \textsuperscript{5}.

There are several problems with demonstrating homogeneity from observed data. The deeper observations are not only fainter and redshifted, they are also affected by proper motion, reddening and absorption due to interstellar matter. These contribute to selection effects which are tricky to compensate for. But the main problem at large distances is the evolution of sources, since deeper observations are received from earlier cosmic epochs. Evolution can take place in source colours, luminosities, and sizes; at high redshift it can affect the type of source as well as their numbers. However in this paper, for simplicity, we shall only consider bolometric observations of one type of source.

How does the number and brightness of the observable sources relate to the local density at different times? Recent evidence for a sharp fall off of the space density of quasars above a redshift of \( z = 5 \) \textsuperscript{6} could be taken as ev-
idence of inhomogeneity, though most attribute it to source evolution. The large population of faint blue objects found by sensitive optical surveys is thought to be young starforming galaxies at high redshifts, and therefore constitute evidence for evolution [11, 12, 13] — if we assume the universe has a FLRW geometry. Without evolution, these observations are inconsistent with that geometry. One difficulty is that redshifts of faint objects are scarce and difficult to obtain. A redshift of \( z \approx 2 \) was deduced [14] by combining number counts versus magnitudes in 3 colours with galaxy evolution models and cosmological models, and comparing with the few measured redshifts available. Again this required the assumption of homogeneity [1]. Indeed, deducing the effects of source evolution by comparing observations with predictions in a FLRW model is a standard technique. Similarly studies of a luminosity-size relation also assume a FLRW model — recent examples are [15, 16]. However this cannot lead to certainty [1]. The new discovery of a radio galaxy which is apparently an immature giant elliptical galaxy at \( z = 4.41 \) [5], provides a more striking example of probable source evolution. The problem is to show that this is not rather evidence of spatial inhomogeneity, manifested in a change of the evolutionary history or the nature of objects observed at larger spatial distances from us. Also, claims of a periodicity on top of the Hubble law in the redshift-distance relation [17] indicate significant deviations from standard FLRW observational relations, which could be due to spatial inhomogeneities (e.g. [24]) or to a temporal variation in the cosmic expansion rate.

If suitably smoothed observations are isotropic, the principal observations of discrete sources one can hope to make are the number counts \( n(z) \) of sources as a function of “distance”, conveniently taken as given by cosmological redshift \( z \), and the magnitudes and angular diameters of sources, also as a function of redshift. If the assumed linear size and absolute luminosities of the sources are correct, the latter two give the luminosity and area distances \( R(z) \), which should be equal. This is rather fortunate since in practice it is often difficult to separate the two measures — one has to define an edge of a galaxy image in order to measure its luminosity, and conversely, one often defines the edge relative to the central brightness; and both measures are significant in determining selection effects [2]. In any case, at the largest distances angular diameters cannot be measured, and it is the luminosity distance that is used.

We consider two types of source evolution; absolute luminosity \( L(z) \), and mass per source \( m(z) \), i.e. total density over source number density, which represents evolution in source number via sources turning on, galaxy mergers etc. Since source evolution is likely to be determined as a function of age \( \tau \), these functions could usefully be expressed as \( L(\tau(z)) \) and \( m(\tau(z)) \); however it is analytically easier to solve the observational equations if they are considered as functions of the observable \( z \). This also helps to emphasise that if large-scale inhomogeneity were in fact to occur, the age of the universe would vary with spatial position and so becomes difficult to handle.

Earlier work (section 15.3 of [22], section 7 of [23]) showed that if the observational relations are isotropic and of the FLRW form, then the universe is indeed homogeneous, provided we can assume the matter stress tensor is that of pressure-free matter [4]. However that analysis did not fully consider the effects of source evolution.

In this paper we show that any given isotropic set of observations \( n(z) \& R(z) \), together with any given evolution functions \( L(z) \) and \( m(z) \), can be fitted by a spherically symmetric dust matter source in synchronous comoving coordinates and of what source evolution may occur [25]. Conversely we show that, given any spherically symmetric geometry and any set of observations, we can find evolution functions that will make the model compatible with the observations.

The purpose is to demonstrate explicitly — developing the ideas in [1] — that the relationship between the large scale isotropy of observations and large scale cosmic homogeneity is weaker than is commonly assumed. Indeed, apart from any other problems, we can’t have a good demonstration of homogeneity without observational tests of our source evolution theories that are independent of cosmological model, or distance measures that are not influenced by source evolution. This emphasises the conclusion that if the demonstration of homogeneity depends on knowing the source evolution, and validation of source evolution theories depends on knowing the cosmological model is homogeneous, then neither is proved. Indeed if we do not make the FLRW assumption, our results can be used to determine the degree of inhomogeneity from the observations and any given source evolution functions. If we do make the FLRW assumption, they can be used to determine the source evolution functions required to make the model compatible with that model. The latter is the way theory is usually run. The point of our paper is to emphasise that there are other options, and so such source evolution results should be viewed with caution.

2 THE LTB MODEL AND ITS NULL CONE

We here outline the metric and our notation and null cone solution; for more details in this notation see [26].

The general spherically symmetric metric for an irrotational dust matter source in synchronous comoving coordinates

\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - \frac{2m(r)}{a(t)^2 r^2}} + r^2 d\Omega^2 \right) \]

is often difficult to handle.

\[ \rho + \frac{3}{2} \frac{\dot{a}^2}{a^2} = \text{constant} \]

\[ \rho = \frac{4\pi G}{3} \frac{m(r)}{a(t)^3 r^2} \]

\[ \tau = \int \frac{dt}{a(t)} \]

\[ \frac{\dot{a}}{a} = \frac{1}{3} \dot{\tau} \]

\[ \dot{m} = \frac{4\pi G}{3} \frac{\dot{a}}{a^3} \]

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coordinates is the Lemaitre-Tolman-Bondi (LTB) \[26\] \[27\] \[28\] metric
\[
ds^2 = -dt^2 + \frac{(R'(t,r))^2}{1 + 2E(r)}dr^2 + R^2(t,r)d\Omega^2, \tag{1}
\]
where \( R'(t,r) = \partial R(t,r)/\partial r \), and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). The function \( R = R(t,r) \) is the areal radius, since the proper area of a sphere of coordinate radius \( r \) on a time slice of constant \( t \) is \( 4\pi R^2 \). Solving the Einstein field equations gives a generalised ‘Friedmann’ equation for \( R(t,r) \),
\[
\dot{R}(t,r) = \pm \sqrt{\frac{2M(r)}{R(t,r)}} + 2E(r), \tag{2}
\]
and an expression for the density
\[
4\pi \rho(t,r) = \frac{M'(r)}{R^2(t,r)R'(t,r)}, \tag{3}
\]
Eq. (3) can be solved in terms of a parameter \( \eta = \eta(t,r) \):
\[
R(t,r) = \frac{M(r)}{\mathcal{E}(r)} \phi_0(t,r), \quad \xi(t,r) = \left( \frac{\mathcal{E}(r)}{R(t,r)} \right)^{3/2} (t - t_B(r)) \tag{4}
\]
where
\[
\mathcal{E}(r) = \begin{cases} 2E(r), & \text{if } \cos \eta > 1, \\ 1, & \text{if } \eta = \frac{1}{2}\eta^2, \\ -2E(r), & \text{if } \cos \eta = 1, \end{cases} \\
\phi_0 = \begin{cases} \cosh \eta - 1, & \text{if } 0 > E, \\ \sinh \eta, & \text{if } 0 = E, \\ 1 - \cosh \eta, & \text{if } 0 < E, \end{cases} \\
\xi = \begin{cases} \sinh \eta - \eta, & \text{if } E > 0, \\ \frac{1}{2}\eta^3, & \text{if } E = 0, \\ \eta - \sin \eta, & \text{if } E < 0 \end{cases}
\]
for hyperbolic, parabolic and elliptic solutions respectively.

The LTB model is characterised by 3 arbitrary functions of coordinate radius \( r \). \( E = E(r) \geq -1/2 \) has a geometric role, determining the local ‘embedding angle’ of spatial slices, and also a dynamic role, determining the local energy per unit mass of the dust particles, and hence the type of evolution of \( R \). \( M = M(r) \) is the effective gravitational mass with comoving radius \( r \), \( t_B = t_B(r) \) is the local time at which \( R = 0 \), i.e. the local time of the big bang — we have a non-simultaneous bang surface. Specification of these three arbitrary functions — \( M(r), E(r) \) and \( t_B(r) \) — fully determines the model, and whilst all have some type of physical or geometric interpretation, they admit a freedom to choose the radial coordinate, leaving two physically meaningful choices, e.g. \( r = r(M), E = E(M), t_B = t_B(M) \).

2.1 The Observer’s Null Cone

We now take \( R \) and \( \rho \) as given on the observer’s past null cone, and we express the 3 arbitrary LTB functions in terms of them, so characterising the LTB model that fits the observations.

We generalise the gauge choice used in \(28\) to the case where the spatial sections are in general non-flat, i.e. all values of \( E \). Human observations of the sky are essentially a single event on cosmological scales, so we only need to be able to locate a single null cone; we don’t need a general solution. On radial null geodesics, \( ds^2 = 0 = d\theta^2 = d\phi^2 \); so from (1) if the past null cone of the observation event \( (t = t_0, r = 0) \) is given by \( t = \hat{t}(r) \), then \( \hat{t} \) satisfies
\[
\dot{\hat{t}} = -\frac{R'(\hat{t}(r),r)}{\sqrt{1 + 2E}} \tag{6}
\]
We will denote a quantity evaluated on the observer’s null cone, \( t = \hat{t}(r) \), by \( \hat{\cdot} \); for example \( R(\hat{t}(r),r) \equiv \hat{R} \). Now if we choose \( r \) so that, on the past light cone of \((l_0,r)\),
\[
\frac{R'(\hat{t}(r),r)}{\sqrt{1 + 2E}} = \frac{\hat{R}}{\sqrt{1 + 2E}} = 1, \tag{7}
\]
then the incoming radial null geodesics are given by
\[
\hat{t}(r) = t_0 - r. \tag{8}
\]
With our coordinate choice (6), the density (8) and the Friedmann equation (3) become
\[
4\pi \rho \hat{R}^2 = \frac{M'}{\sqrt{1 + 2E}} \tag{9}
\]
\[
\hat{R} = \pm \sqrt{\frac{2M}{R} + 2E}. \tag{10}
\]
The gauge equation is found from the total derivative of \( R \) on the null cone
\[
\frac{d\hat{R}}{dr} = \hat{R} + \hat{R} \frac{\dot{\hat{t}}}{dr} \tag{11}
\]
and with (8) and (10) substituted, it follows that
\[
\frac{d\hat{R}}{dr} - \sqrt{1 + 2E} = -\hat{R} = -\pm \sqrt{\frac{2M}{R} + 2E}. \tag{12}
\]
When we solve this for \( 2E(r) \) by squaring both sides and rearranging, we get
\[
1 + 2E = \left( \frac{1}{2} \left( \frac{d\hat{R}}{dr} \right)^2 + 1 \right) - \frac{M}{R} \left( \frac{d\hat{R}}{dr} \right)^2. \tag{13}
\]
This expression will tell us under what circumstances (or for which regions) the spatial sections are hyperbolic \( 1 + 2E > 1 \), parabolic \( 1 + 2E = 1 \) or elliptic \( 1 + 2E < 1 \), based on data obtained from the null cone. We now use the expression for the density on the null cone to find a linear, first order differential equation for \( M(r) \). Eliminating \( 1 + 2E \) between (3) and (9), we get
\[
\frac{dM}{dr} + \left( \frac{4\pi \rho \hat{R}}{dr} \right) \frac{dM}{dr} = \left( \frac{2\pi \rho \hat{R}^2}{dr} \right) \left( \frac{d\hat{R}}{dr} \right)^2 + 1. \tag{14}
\]
By evaluating (9) and (10) on the null cone we find
\[
\hat{R} = \frac{M}{\mathcal{E}} \hat{\phi}_0, \quad \hat{\xi} = \frac{\mathcal{E}^{3/2}}{M}, \tag{15}
\]
where
\[
\mathcal{E}(r) = \begin{cases} 2E(r), & \text{if } \cos \hat{\eta} > 1, \\ 1, & \text{if } \eta = \frac{1}{2}\eta^2, \\ -2E(r), & \text{if } \cos \hat{\eta} = 1, \end{cases} \\
\hat{\phi}_0 = \begin{cases} \cosh \hat{\eta} - 1, & \text{if } 0 > E, \\ \sinh \hat{\eta}, & \text{if } 0 = E, \\ 1 - \cosh \hat{\eta}, & \text{if } 0 < E, \end{cases} \\
\hat{\xi} = \begin{cases} \sinh \hat{\eta} - \hat{\eta}, & \text{if } E > 0, \\ \frac{1}{2}\hat{\eta}^3, & \text{if } E = 0, \\ \hat{\eta} - \sin \hat{\eta}, & \text{if } E < 0 \end{cases}
\]
†† Strictly speaking, the hyperbolic, parabolic and elliptic solutions obtain when \( RE/M > 0, = 0, \& < 0 \) respectively, since \( E = 0 \) at a spherical origin in both hyperbolic and elliptic models.
and

\[ \tau(r) \equiv \tilde{t}(r) - t_B(r) = t_0 - r - t_B(r) \]  \hspace{1cm} (17)

can be interpreted as proper time from the bang surface to the past null cone along the particle world lines. Thus, with \( M \) given by (14) and then \( E \) by (13), we can solve for \( \bar{\eta} \) from

\[ \phi_0 = \frac{\partial R}{\partial \bar{\eta}} \]  \hspace{1cm} (18)

and (13), \( \tau(r) \) from

\[ \tau = \frac{M}{\Sigma^{3/2}} \xi \]  \hspace{1cm} (19)

with (16) again, and hence \( t_B(r) \) from (17).

### 2.2 Origin Conditions

At the origin of spherical coordinates, \( r = 0 \), where \( R(t, 0) = 0 \) and \( \dot{R}(t, 0) = 0 \) for all \( t \), we assume that the density is non-zero, that the type of time evolution (hyperbolic, parabolic or elliptic) is not different from its immediate neighbourhood, and that all functions are smooth — i.e. functions of \( r \) have zero first derivative there. Thus eq (1) tells us that \( RE/M \) and \( E^{3/2}/M \) must be finite at \( r = 0 \), (3) shows us that \( E \rightarrow 0 \) and hence \( M \rightarrow 0 \) and \( E \sim M^{2/3} \) at \( r = 0 \). Eqs (1) and (2) become

\[ \frac{\partial \dot{R}}{\partial r} \bigg|_{r=0} = \dot{R} \big|_{r=0} = \frac{d\dot{R}}{dr} \bigg|_{r=0} = \sqrt{1 + 2E} = 1 \]  \hspace{1cm} (20)

and thus \( \dot{R} \sim r \) to lowest order near \( r = 0 \). From (1) we find

\[ M' \approx 4\pi \rho_0 r^2 \]  \hspace{1cm} (21)

and so

\[ E \sim \left( \frac{4}{3} \pi \rho_0 \right)^{2/3} r^2 \]  \hspace{1cm} (22)

We verify these origin conditions satisfy (4) to order \( r^2 \) and (3) trivially to order \( r^0 \).

### 2.3 Redshift-distance formula

We use the fact that in the geometric optics limit, for two light rays emitted on the worldline at \( r_{em} \) with time interval \( \delta t_{em} = t^+(r_{em}) - t^-(r_{em}) \) and observed on the central worldline with time interval \( \delta t_{ob} = t^+(0) - t^-(0) \)

\[ 1 + z = \frac{\delta t_{ob}}{\delta t_{em}} \]  \hspace{1cm} (23)

The incoming radial null geodesics are given by

\[ dt = -R'(t, r) / \sqrt{1 + 2E} \]  \hspace{1cm} dr,

so for two successive light rays, \(- & +, \) passing through two nearby comoving worldlines \( r_A \) \& \( r_B = r_A + dr \) at times \( t_A, t_B \), \( t'_A \) \& \( t'_{B} \)

\[ d(\delta t) = \delta t_B - \delta t_A = dt^+ - dt^- = \frac{[R'(t^+, r) + R'(t^-, r)]}{\sqrt{1 + 2E}} \]  \hspace{1cm} dr

Consequently

\[ d\ln t = -\left( \frac{\partial}{\partial t} [R(t, r)] \right) / \sqrt{1 + 2E} \]  \hspace{1cm} dr

which means that, integrating along the light ray and applying this to the log of (24), the redshift is given by

\[ \ln(1 + z) = \int_0^{r_{em}} \frac{\dot{R}'(t, r)}{\sqrt{1 + 2E}} \]  \hspace{1cm} dr

for the central observer at \( r = 0 \), receiving signals from an emitter at \( r = r_{em} \).

We need to find the redshift \( z \) explicitly in terms of observables. We differentiate (3) with respect to \( r \):

\[ \frac{\dot{R}'\dot{R}}{\sqrt{1 + 2E}} = - \frac{M'}{R\sqrt{1 + 2E}} - \frac{MR'}{R^2\sqrt{1 + 2E}} + \frac{E'}{\sqrt{1 + 2E}} \]  \hspace{1cm} (25)

so when evaluated on the observer’s past null cone, we get

\[ \frac{\dot{R}'\dot{R}}{\sqrt{1 + 2E}} = 1 - \left( \frac{M'}{R \sqrt{1 + 2E}} - \frac{M R'}{R^2} + (\sqrt{1 + 2E})' \right) \]  \hspace{1cm} (26)

Now, from (13), the derivative of \( \sqrt{1 + 2E} \) is given by

\[ (\sqrt{1 + 2E})' = \frac{d^2 \dot{R}}{dr^2} - \frac{M'}{R} \frac{dR}{dr} + \frac{M}{R^2} - \frac{\dot{E}'}{\sqrt{1 + 2E}} \frac{d^2 \dot{R}}{dr^2} \]  \hspace{1cm} (27)

so, after eliminating \( M' \) by substituting from equation (1), it follows that

\[ \frac{d^2 \dot{R}}{dr^2} + \frac{4 \pi \rho_0 \dot{R} - 4 \pi \rho_0 \dot{R} \sqrt{1 + 2E}}{\sqrt{1 + 2E}} = 1 \]  \hspace{1cm} (28)

where we have used equation (2) to provide the second equality. From (2) it now follows that

\[ \frac{d}{dr} \ln(1 + z) = \left[ - \frac{d^2 \dot{R}}{dr^2} + 4 \pi \rho_0 \right] / \left( \frac{d \dot{R}}{dr} \right) \]  \hspace{1cm} (29)

which theoretically gives the redshift in terms of coordinate \( r \), directly from \( \dot{R}(r) \) and \( \dot{\rho}(r) \), viz

\[ \ln(1 + z) = - \int_0^r \left[ \frac{d^2 \dot{R}}{dr^2} + 4 \pi \rho_0 \right] / \left( \frac{d \dot{R}}{dr} \right) \]  \hspace{1cm} dr

However, we will be given observations in terms of \( z \), rather than the unobservable coordinate \( r \). This will be addressed in the next section.

### 3 OBSERVABLES AND SOURCE EVOLUTION

For simplicity we shall confine ourselves to one type of cosmic source and only consider bolometric luminosities. We shall assume that the luminosity of each source can evolve with time, and that the number density of sources can also evolve. The former we write as an absolute bolometric luminosity \( L \), and the latter we shall represent as an evolving mass per source, \( m \), which gives the total local density when multiplied by the source number density. As mentioned, we assume isotropy about the earth (once our proper motion has been accounted for), and also that the post decoupling universe is well described by zero pressure matter — “dust”.

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The particles of this dust are galaxies (or perhaps clusters of galaxies). This means we can use the simplest inhomogeneous cosmology — the LTB metric, which is spherically symmetric and inhomogeneous in the radial direction only, and is written in comoving coordinates.

The two source evolution functions are most naturally expressed as functions of local proper time since the big bang, \(\tilde{\tau}(\tau)\) and \(m(\tau)\). However, in a LTB model the time of the bang may vary from point to point, so that the age of objects at redshift \(z\) is uncertain both because the bang time is uncertain and because the location of the null cone is uncertain. The proper time from bang to null cone will be a function of redshift, \(\tau(z)\), and the projections of the evolution functions on the null cone we will write as \(\tilde{L}\) and \(\tilde{m}\). Of course, \(\tau(z)\) is unknown until we have solved for the LTB model that fits the data. However, for the sake of simplicity, we will take \(\tilde{L}\) and \(\tilde{m}\) to be given as functions of \(z\), to illustrate how the 3 quantities, cosmic evolution, cosmic spatial variation, and source evolution are mixed together in the luminosity and number count observations, \(\ell\) and \(n\). A treatment dealing with evolution functions based on \(\tau\) would involve solving a much more complicated set of differential equations in parallel.

### 3.1 Relating Observables to the LTB Model

The area distance or equivalently the diameter distance is the true linear extent of the source over the measured angular size. This is by definition the same as the areal radius in the LTB model \(R\), which multiplies the angular displacements to give proper distances tangentially. The projection onto the observer’s null cone gives the observable quantity \(\tilde{R}\).

The luminosity distance is theoretically the same as the diameter distance \((\tilde{d}_L)\), and is measurable provided we know the true absolute luminosity of the source at the time of emission \(\tilde{L}\). If the observed apparent luminosity is \(\ell(z)\) then

\[
\tilde{R}(z) = \sqrt{\tilde{L}(\tau) / \tilde{\tau}(\tau)}.
\] (31)

Let the observed number density of sources in redshift space be \(n(z)\) per steradian per unit redshift interval, so that the number observed in a given redshift interval and solid angle is

\[
nd\Omega dz.
\] (32)

and over the whole sky this is

\[
4\pi ndz.
\] (33)

Thus the total rest mass between \(z\) and \(z + dz\) is

\[
4\pi \tilde{m}ndz
\] (34)

where \(\tilde{m}(z) = m(\tau(z))\) is the mass per source — i.e. the true density over the source number density. This primarily represents the evolution in the number density of sources. Given a local proper density \(\rho = \rho(\tilde{r}, \tau)\), and its value on the null cone \(\tilde{\rho}\), the total rest mass between \(r\) and \(r + dr\) is

\[
\tilde{\rho}dV = \rho R^2 dr / \sqrt{1 + 2\frac{\rho}{\tilde{\rho}}}
\] (35)

where \(dV\) is the proper volume on a constant time slice, evaluated on the null cone. Hence by (32), (33) and (35)

\[
\tilde{R}^2 \tilde{\rho} = \tilde{m}n dz / dr.
\] (36)

Thus we may substitute for \(\tilde{R}\) and \(\tilde{\rho}\) from (33) and (36).

We transform \(\tilde{R}\) to be in terms of redshift \(z\) instead of coordinate \(r\) by writing it as

\[
d\tilde{R} dz / dr + d\tilde{R}^2 dz^2 (1 + z) + 4\pi \tilde{\rho} \tilde{R}(1 + z) = 0
\] and applying

\[
d\tilde{R} dz / dr = d\tilde{R} / dr, \quad d\tilde{R}^2 dz^2 = d\tilde{R} dz / dr + d\tilde{R}^2 dz^2 / dz (dz / dr)^2
\] to get

\[
d\tilde{R} dz / dr + d\tilde{R}^2 dz^2 (1 + z) + 4\pi \tilde{\rho} \tilde{R}(1 + z) \left(\frac{dz}{dr}\right)^2 = -4\pi \tilde{\rho} \tilde{R}
\] (37)

Integrating with respect to \(r\) and using (36) gives

\[
\int_0^z \frac{d\tilde{R}}{dz} \frac{d\tilde{R}}{dz} (1 + z) dz = -\int_0^z 4\pi \tilde{\rho}(\tilde{z})(1 + \tilde{z}) \left(\frac{dz}{dr}\right)^2 d\tilde{z}
\] (38)

and we used the origin conditions \([dz/dr)(d\tilde{R}/dz)]_0 = [dR/dr)]_0 = 1\), and \(z(0) = 0\). It follows that

\[
\frac{dz}{dr} = \left[\frac{d\tilde{R}}{dz} (1 + z) \right]^{-1} \left\{ 1 - 4\pi \int_0^z \frac{\tilde{m}(\tilde{z})n(\tilde{z})}{\tilde{R}(\tilde{z})} (1 + \tilde{z}) d\tilde{z} \right\}
\] (40)

Note that this equation differs from the analogous one in Stoeger et al (24) --- their equation (32) --- by a factor of \((1 + z)\), and perhaps aptly illustrates the difference in the coordinate systems. To get the full model we have to solve the null Raychaudhuri equation (57) to get \(r(z)\) (and thus \(\tau(z)\)). Equation (40) is a first integral of (37). This has to be integrated one more time to obtain \(r(z)\). We must also specify boundary conditions at the origin \(r = 0\), which we have already used in getting to (40):

\[
\frac{dz}{dr}(0) = \frac{dz}{d\tilde{R}}(0) \frac{d\tilde{R}}{dr}(0) = 1 / \frac{d\tilde{R}}{dz}(0)
\] and also

\[
z(0) = 0 \quad \Leftrightarrow \quad r(z = 0) = 0.
\]

so that, integrating \(dr/dz\) gives

\[
r(z) = \int_0^z \left[\frac{d\tilde{R}}{dz} (1 + z) \right]^{-1} \left\{ 1 - 4\pi \int_0^z \frac{\tilde{m}(\tilde{z})n(\tilde{z})}{\tilde{R}(\tilde{z})} (1 + \tilde{z}) d\tilde{z} \right\} dz .
\] (41)

### 4 THEOREMS

†† There they use \(M_0\) which equals \(8\pi \tilde{m}n / \tilde{R}^2\) in the current notation.
4.1 Theorem (A):

Subject to the conditions of appendix B, for any given isotropic observations \( \ell(z) \) & \( n(z) \) with any given source evolution functions \( \dot{L}(z) \) & \( \dot{m}(z) \), a set of LTB functions can be found to make the LTB observational relations fit the observations.

4.2 Proof: — Algorithm (A):

To obtain the LTB mass, energy and bangtime functions \((M, E \) and \( t_B \) respectively) from observational data and source evolution we would proceed as follows.

- Take the discrete observed data for \( \ell(z, \theta, \phi) \) and \( n(z, \theta, \phi) \), average it over all angles to obtain \( \ell(z) \) and \( n(z) \), and fit it to some smooth analytic functions, such as polynomials. We may wish to first correct the data for known distortions and selection effects due to proper motions, absorption, shot noise, image distortion, etc;
- Choose evolution functions \( \dot{L}(z) \) and \( \dot{m}(z) \) based on whatever theoretical arguments may be mustered;
- Determine \( \dot{R}(z) \) from \( \dot{L}(z) \) and \( \ell(z) \) using \((\ref{LTB.null})\);
- Solve \((\ref{LTB.1})\) for \( r(z) \) and hence \( z(r) \), then convert functions of \( z \) to functions of \( r \) — see appendix B for existence conditions;
- Solve \((\ref{LTB.4})\) and \((\ref{LTB.16})\) for \( M(r) \) — existence conditions are given in appendix B;
- Determine \( E(r) \) from \((\ref{LTB.13})\);
- Solve for \( \ddot{r} \) from \((\ref{LTB.16}) \) and \((\ref{LTB.18}) \);
- Solve for \( \tau(r) \) from \((\ref{LTB.18}) \) and \((\ref{LTB.14}) \) — \( L(\tau) \) and \( m(\tau) \) could now be found;
- Determine \( t_0(\tau) \) from \((\ref{LTB.16}) \).

In practice, these equations would be solved numerically, and in parallel rather than sequentially, nevertheless the above would determine the numerical procedure within each integration step. □

By determining the 3 arbitrary functions, we have specified the LTB model that fits the given observations and evolution functions. This result simply asserts we can construct a (generally inhomogeneous) spherically symmetric exact solution of the field equations that will fit any given source observations combined with any chosen source evolution functions.

We assert, without proof, that if the given observations and source evolution functions are reasonable, then the LTB arbitrary functions will generate a reasonable LTB model. Our definition of ‘reasonable’ is intentionally rather vague. By reasonable observations we obviously include the actual data, suitably processed to account for selection effects. We also include ‘realistic’ hypothetical alternatives, but not functions that are wildly different from reality. Reasonable evolution functions are hard to define since the actual ones are not well known, especially at larger \( z \) values. By a reasonable LTB model, we mostly mean that the density and expansion rate will be within realistic ranges. A less crucial criterion is that there will be no shell crossings too close to the past null cone. Evolving the model a long time away from the null cone, either forwards or backwards, may introduce shell crossings because the data is imprecise. In general we don’t expect shell crossings on the large scale — i.e. two or more different large scale flows of galaxies in the same region — nevertheless it is conceivable and in that case the LTB description is inapplicable.

4.3 Corollary (B):

A LTB model can be found to fit the observations with zero evolution — \( \dot{m} = \text{constant}, \dot{L} = \text{constant}. \)

4.4 Proof:

This is an obvious consequence of (A). □

Given realistic data, these models will be inhomogeneous. Indeed this is the reason that non-zero evolution functions have been introduced (otherwise, observations are incompatible with a FLRW universe).

4.5 Theorem (C):

Subject to the conditions of appendix B, for any given isotropic observations \( \ell(z) \) & \( n(z) \), and any given LTB model, source evolution functions \( \dot{L}(z) \) & \( \dot{m}(z) \) can be found that make the LTB observational relations fit these observations.

4.6 Proof: — Algorithm (C):

We adapt the above algorithmic procedure to prove this.

- As before, average the data over all angles, and fit it to smooth functions \( \ell(z) \) and \( n(z) \);
- Specify two of the three functions \( M(r) \), \( E(r) \) and \( t_B(r) \), the third being determined by the coordinate condition \((\ref{LTB.12})\);
- It seems expedient to choose \( M(r) \) and \( E(r) \);
- Determine \( \dot{R}(r) \) from the first order differential equation in \( \dot{R} \) and its \( r \) derivative — equation \((\ref{LTB.19})\). The functions should be chosen to satisfy the origin conditions of section 2.2 — see existence conditions in appendix B;
- Calculate \( \dot{\rho}(r) \) from \((\ref{LTB.2})\);
- Solve for \( \dot{L}(r) \) as well as \( \tau(r) \) from \((\ref{LTB.18}) \), \((\ref{LTB.14}) \) and \((\ref{LTB.17}) \) with \((\ref{LTB.20}) \) defining \( \dot{\rho}(r) \);
- Integrate \((\ref{LTB.19}) \) to get \( z(r) \) — appendix B gives the existence conditions;
- Use the given \( \ell(z) \) and \( n(z) \), to find \( \dot{L}(z) \) from \((\ref{LTB.21}) \) and \( \dot{m}(z) \) from \((\ref{LTB.24}) \). From these and \( \tau(z) \) solve for \( \dot{L}(\tau) \) and \( \dot{m}(\tau) \), if needed. □

Again we assert that if the given observations and LTB model are ‘reasonable’, then the derived evolution functions will be ‘reasonable’. The idea is that we can vary the LTB model to which we fit the observations to some extent, but still keep the required source evolution functions within a ‘realistic’ range.

\footnote{Data can be extended through a shell crossing \((\ref{LTB.21})\), but not within the LTB formalism.}

\footnote{Though we don’t strictly know the sign of \( \dot{R} = \sqrt{2M/R + 2E} \), it is fairly safe to assume it is positive on our past null cone on the large scales we are considering.}

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4.7 Corollary (D):

Source evolution functions can be found that make the dust FLRW observational relations fit any observations.

4.8 Proof:

An obvious consequence of (C). □

Loosely put these theorems say
(i) You can always fit isotropic observations with an LTB model, whatever the source evolution;
(ii) If you fiddle the source evolution hard enough, you can fit the observations to any LTB or dust FLRW model.

Although theorem (C) is an extreme case, and is likely to generate highly unphysical evolution functions if the LTB model is chosen arbitrarily, it is just a generalisation of (D) which is regularly used in an attempt to determine evolution functions from cosmological observations. Theorem (C) highlights the dangers of this approach.

A complication arises if the redshift is not monotonically increasing with distance. We have seen from the well behaved numerical example in (29) for a parabolic case, that \( \hat{R}(z) \) and \( \hat{\rho}(z) \) may not be single valued, and that the \( \hat{R} - z \) and \( \hat{\rho} - z \) plots can loop. However, in compiling the real observational data, we merely add all the galaxies we see at a particular redshift, to get a number count. Similarly, we merely take an average over the luminosities observed at a particular redshift, ascribing the variation to natural scatter in intrinsic properties and observational error, rather than to a multiply valued function. Thus we make \( \hat{R}(z) \) and \( \hat{\rho}(z) \) single valued by construction. So the data functions we are trying to fit may not lead to such a good model. In other words, assuming we succeed in constructing a well behaved LTB model from the data, it may not be the LTB model that best represents the real universe. It seems unlikely — though not entirely impossible — that there will be a reliable way of de-convolving the superposed parts of these observational data curves, or even of discerning whether loops are present. It is hard to predict how likely this scenario is.

5 CONCLUSIONS

We have shown that a LTB model (a Lemaître-Tolman-Bondi spherically symmetric dust cosmology) can be found to fit any given set of observations of source counts \( n(z) \) and luminosity/area distance \( \hat{R}(z) \), averaged over all angles, and any evolution functions for source luminosity \( \hat{L}(z) \) and mass per source \( \hat{n}(z) \). In other words, even if we accept isotropy, then demonstrating homogeneity — rather than assuming it must hold because of the Copernican principle — requires more than these observations. Conversely, our result can be used to determine the degree of inhomogeneity from the observations and given source evolution functions.

If the demonstration of homogeneity depends on knowing the source evolution, and validation of source evolution theories depends on knowing the cosmological model is homogeneous, then neither is proved. Thus we need methods of validating source evolution models that don’t depend on assumptions of homogeneity to establish the age at any given \( z \). Similarly deep cosmological distance measures that don’t depend on luminosity and are not influenced by source evolution would help pin down the cosmological model better.

There are various promising developments, in particular:
(a) distance measurement by Supernovae;
(b) determinations of cosmological parameters via gravitational lensing measurements;
(c) accurate measurements of the Sunyaev-Zel’dovich effect;
(d) observation of CMBR Doppler peaks by the MAP and COBRAS/SAMBA satellites. This will only determine parameters in the neighbourhood of \( z = 1000 \), but is independent of source evolution all the same.
(e) the increasing number of source evolution studies that look for tell-tale signs of early stages of galaxy evolution, such as intense star formation, etc.

Once again, the FLRW assumption is usually if not always made in analyses of these effects. A re-analysis that permits inhomogeneity would be very worthwhile, as these techniques may well provide information complementary to the principal cosmological measures, that would help separate out the effects of cosmic evolution, spatial inhomogeneity, and source evolution. Some of these issues are discussed in (31).

In fact, it is already difficult to constrain the values of \( H_0, q_0 \) and \( \Lambda \) within a homogeneous dust model because of the uncertainty in source evolution, as pointed out in (31). In this case the value of \( \Lambda \) affects the time evolution of the scale factor, and so the deviation of the angular diameter-redshift relation from expectation for a \( \Lambda = 0 \) FLRW model could be due to non-zero \( \Lambda \) or to source evolution. Similarly the possible presence of non-baryonic dark matter — or for that matter, the possibility that gravity obeys field equations other than Einstein’s — could significantly affect the cosmic time evolution, and introduce further uncertainty.

The introduction of multi-colour observations does not resolve the problem in any simple way. If we have observations in various colour bands — say U B & V — then we must replace the source luminosity evolution function by a set of evolution functions for the luminosity in each colour. Thus, if we find deviations of the observations from FLRW expectations, we still have a freedom to attribute this either to inhomogeneity or to source evolution. It’s true that young galaxies with lots of star formation are very blue. But, having introduced colour observations, and permitted evolution in colour, we must also admit the possibility of spatial inhomogeneities in the intrinsic colours of sources. We come back to the same problem — are the differences between observations in different colours due to source evolution or spatial inhomogeneity? The only difference here is that cosmic evolution is fairly easily factored out, as the redshift is measured.

We are not here asserting that the observable universe is inhomogeneous, nor are we suggesting that source evolution studies that assume homogeneity are not worthwhile.

The purpose of this paper is to emphasise that we don’t

Indeed a very interesting option is to use the bounds on the inhomogeneity obtained from the Sunyaev-Zel’dovich effect to constrain the LTB model chosen, bearing in mind that this method still suffers from excessive error due to absorption effects.

*** i.e. Planck Survey
have unquestionable evidence for spatial homogeneity; and that we can’t have a good demonstration of homogeneity — or even homogeneity on average — without a reliable theory of source evolution, supported by measurements that are independent of cosmological model, and/or cosmic distance measures that don’t depend on knowing the luminosity evolution of sources. Our best basis for assuming spatial homogeneity is the Stoeger-Maartens-Ellis theorem or “almost EGS theorem” ([2]), which says that, if the universe is expanding and the CMBR (cosmic microwave background radiation) is almost isotropic for all observers since decoupling, then the universe is almost homogeneous, and more specifically, the scale of CMBR anisotropy puts a limit on the degree of cosmic inhomogeneity. But this result depends on a weak form of the Copernican principle; and however convincing that principle is in general terms, we shouldn’t overstate it. This line of thought says that the earth is just another planet around the sun, but it doesn’t say all planets are the same size or composition. It says that our galaxy and our supercluster are one among many, but allows several types of galaxy and considerable variety in galaxy clustering. Thus the principle does not insist on uniformity on any scale, or even that the observable portion of the universe has a density particularly close to the “global average” — assuming we can define such a thing. And above all, while it may be true in the real universe, it is also possible that this is not so.

We are entitled to deduce homogeneity on the basis of untested philosophical principles, such as a Copernican principle; but we must be quite clear what we are doing when we make such a deduction, and how it relates to possible observational tests. This paper helps throw light on the latter issue.

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APPENDIX A: A CHARACTERISATION OF HOMOGENEITY FROM OBSERVATIONS IN AN ISOTROPIC UNIVERSE

We demonstrate the procedure in section 4 in the case where the input observations, after correcting for evolution, are in the RW form. It turns out that the LT flattish functions assume their RW form. This amounts to a proof that a radially inhomogeneous dust universe is RW iff the area distance and number count relations as a function of redshift take the RW form. It is a special case of Theorem (B) where we assume that there is no evolution. These RW relations are

\[
\hat{R}(z) = \frac{q_0 z + (1 - q_0)}{H_0 q_0^2 (1 + z)^2} \quad (A1)
\]

and

\[
\frac{4 \pi \hat{m}}{3} = \frac{[q_0 z + (1 - q_0) (1 - \sqrt{2 q_0 z + 1})^2]}{H_0 q_0^2 (1 + z)^3 \sqrt{2 q_0 z + 1}} \quad (A2)
\]

respectively. Then we can integrate the null Raychaudhuri equation (17) once obtaining

\[
d\tau = H_0 (1 + z)^2 \sqrt{2 q_0 z + 1} \quad (A3)
\]

This may be integrated once again (illustrating with the case \(q_0 < \frac{1}{2}\)) to obtain

\[
r = \frac{1}{H_0 (1 - 2 q_0)} \left[ 1 - \frac{\sqrt{2 q_0 z + 1}}{1 + z} + \frac{q_0}{\sqrt{1 - 2 q_0}} \ln \left( \frac{\sqrt{2 q_0 z + 1} + \sqrt{1 - 2 q_0}}{\sqrt{2 q_0 z + 1} - \sqrt{1 - 2 q_0}} \right) \right] \quad (A4)
\]

We continue by solving the first order differential equation for \(M(z)\) (the effective gravitational mass) (18). This equation may be written as

\[
\frac{d}{dz} \left[ \frac{M(z)}{(1 + z) d\hat{R} / dr} \right] = \frac{2 \pi \hat{m}}{(1 + z) + \frac{2 \pi \hat{m}}{(1 + z)(d\hat{R} / dr)^2}} \quad (A5)
\]

We substitute the RW area distance and number count relations into this and find that

\[
M(z) = H_0^2 q_0 \hat{R}^3 (1 + z)^3 \quad (A6)
\]

and from (13) it follows that

\[
2E(z) = (1 - 2 q_0) H_0^2 \hat{R}^2 (1 + z)^2 \quad (A7)
\]

These two relations show that \(M \propto (2E)^{3/2}\). We next show that this universe has a simultaneous bangtime. We need (A4) in addition to (11) and (12). Restricting ourselves to the case \(q_0 < \frac{1}{2}\), it follows that

\[
\tau = \frac{1}{H_0 (1 - 2 q_0)} \left[ \sqrt{2 q_0 z + 1} - \frac{q_0}{\sqrt{1 - 2 q_0}} \ln \left( \frac{\sqrt{2 q_0 z + 1} + \sqrt{1 - 2 q_0}}{\sqrt{2 q_0 z + 1} - \sqrt{1 - 2 q_0}} \right) \right] \quad (A8)
\]

and thus

\[
t_B(r) = t_0 - \frac{1}{H_0 (1 - 2 q_0)} \left[ 1 + \frac{q_0}{\sqrt{1 - 2 q_0}} \ln \left( \frac{1 - \sqrt{1 - 2 q_0} \sqrt{1 + 1 - 2 q_0}}{1 + 1 - 2 q_0} \right) \right] \quad (A9)
\]

which means that the bang surface is simultaneous. ** ** This, together with \(M \propto (2E)^{3/2}\) is all we need to show.

APPENDIX B: CONDITIONS FOR EXISTENCE OF SOLUTIONS

B1 Existence of solutions \(r(z)\) and \(z(r)\) to equation (11)

We know \(\hat{R}(z) \rightarrow 0\) as \(z \rightarrow 0\), but from (11) we expect \(\hat{R}(z) \sim R^2(z)\), assuming \(\hat{\rho}, \hat{\rho}(z)\) and \(d\hat{r} / dz\) all \(\sim\) constant as \(z \rightarrow 0\), and of course \(\hat{m}, n, R\) and \((1 + z)\) must all be \(\geq 0\). However the existence condition is less stringent.

Assume that near \(z = 0\), \(\hat{m}(1 + z) / \hat{R} = S(z) z^o\), where \(\sigma\) is a constant and \(S(0)\) is finite and non-zero. Then, to leading order near \(z = 0\),

\[
I_1(z) = \int_0^z \hat{m} \frac{(1 + \tau)}{R} d\tau = \int_0^z S(\tau) z^o d\tau
\]

\[
I_1 = \left[ \frac{S(0) \tau^{o+1}}{(\sigma + 1)} \right]_0^z
\]

which exists provided \(\sigma > -1\). Since all the terms in the integrand are positive, \(I_1\) is monotonically increasing.

We expect \(\hat{R}(z)\) to increase from 0 to a maximum, at \(z_m\) say, and then decrease asymptotically towards 0. We assume no looping, i.e. \(\hat{R}(z)\) is single valued and \(d\hat{r} / dz\) doesn’t diverge, and that there is only one maximum. Thus \(d\hat{R} / dz\) goes from positive to negative values, and approaches 0 asymptotically. It is evident from (13) that \(4 \pi I_1(z) = 1\) where \(d\hat{R} / dz = 0\) in LT models. Therefore we write \(d\hat{R} / dz(1 + z) = P(z) (z - z_m)^\alpha\) and \(\{1 - 4 \pi I_1\} = Q(z) (z - z_m)^\beta\), where \(\alpha\) and \(\beta\) are constants, and \(P_m = P(z_m)\) & \(Q_m = Q(z_m)\) are finite and non-zero. Then to leading order near \(z = z_m\), (13) is

\[
r - r_m = \int_{z_m}^z P(\tau) (\tau - z_m)^{\alpha - \beta} d\tau
\]

\[
r = r_m + \left[ \frac{P_m (z - z_m)^{\alpha - \beta + 1}}{Q_m (\alpha - \beta + 1)} \right]_{z_m}^z
\]

and so \(r(z)\) exists for \(\alpha - \beta + 1 > 0\).

Our conditions for the existence of \(r(z)\) are:

(i) \(\hat{m}, n, R\) and \((1 + z)\) are \(\geq 0\),
(ii) near \(z = 0\), \(\hat{m}/\hat{R} \sim z^o\) with \(\sigma > -1\),
(iii) \(d\hat{R} / dz\) is finite everywhere,
(iv) near \(z = z_m\), \(d\hat{R} / dz(1 + z) \sim (z - z_m)^\alpha\) and \(\{1 - 4 \pi I_1\} \sim (z - z_m)^\beta\), with \(\alpha - \beta + 1 > 0\).

The condition for the existence of \(z(r)\) is

(v) \(z(r)\) is monotonic.

** ** We could set \(\tau(0) = t_0\) in which case \(t_B(r) = 0\), but this is not necessary.
Conditions (i) & (ii) are manifestly reasonable. Conditions (iii) & (iv) are more problematic. As was shown previously \text{(23)}, large enough inhomogeneities can create maxima and minima in $z(r)$ and so make $r(z)$ multi-valued, especially near $d\hat{R}/dz = 0$, in which case neither (iii) nor (v) would be satisfied. However, a multi-valued $r(z)$ manifested itself in a $\hat{R}(z)$ graph that looped. In practice, we don’t expect to get a looping $\hat{R}(z)$ from the observational data. The values of $\ell$ and $n$ at each $z$ are averages over all measured values, and so are single valued by construction. Also $z(r)$ was always single valued in the numerical examples considered in \text{(23)}, so, if $r(z)$ exists, then inverting it should not be a problem. Unfortunately (iv) is unlikely to be satisfied \textit{exactly} for real data — the maximum in $\hat{R}$ will not be at exactly the same value as the locus of $4\pi I_1 = 1$, so one would have to tweak the fitted function to obtain a numerical solution. In other words, the function $r(z)$ is sensitive to observational error here. This however is not too serious, since there will be a measure of freedom in the smooth functions $\ell(z)$ and $n(z)$ that are fitted to the discrete data. In fact this problem exists even if the universe were genuinely homogeneous — even if we knew the source evolution functions exactly, the best-fit curves obtained from imprecise observational data would need adjustment to obtain a solution.

### B2 Existence of solutions $M(r)$ to equation \textbf{(14)}

Enc Eq \textbf{(14)} has the form of an inhomogeneous linear first order ODE,

$$\frac{dM}{dr} + a(r)M = b(r)$$

which has the formal solution

$$M = \mu^{-1} \left[ M_m \mu_m + \int_{r_m}^{r} b rdr \right], \quad \mu = e^{\int a dr}$$

where $M_m = M(r_m)$ and $\mu_m = \mu(r_m)$.

However we know that $d\hat{R}/dr$ goes through 0 at the maximum of $\hat{R}(r)$ — at $r_m$ say, so both $a(r)$ and $b(r)$ are divergent there. It is evident that $a$ & $b$ are finite everywhere else, so we just have to show $M(r)$ exists in the neighbourhood of this divergence. Suppose that, near $r = r_m$, $d\hat{R}/dr$ is of the form $d\hat{R}/dr \sim (r - r_m)^\nu$, where $\nu > 0$ and is constant, so $a(r) = F(r)(r - r_m)^{-\nu}$ and $b(r) = G(r)(r - r_m)^{-\nu}$, where $F(r)$ and $G(r)$ are finite, positive and non-zero. We expect $\nu = 1$. Then to leading order near $r_m$, for $\nu = 1$,

$$\mu = e^{F_m \ln(r - r_m)} = (r - r_m)^{F_m}$$

where $F_m = F(r_m)$, so

$$M = (r - r_m)^{-F_m} \left[ 0 + \int_{r_m}^{r} G(r)(r - r_m)^{F_m - 1} dr \right]$$

$$M = (r - r_m)^{-F_m} \left[ \frac{G_m(r - r_m)^{F_m}}{F_m} - 0 \right]$$

and thus

$$M = \frac{G_m}{F_m}$$

to leading order. Comparison with \textbf{(14)} shows $M_m = \hat{R}_m/2$, which is consistent with the fact that $d\hat{R}/dr = 0$ lies on the apparent horizon $R = 2M$. Thus $M(r)$ exists in the neighbourhood of $r_m$.

For completeness we consider $\nu \neq 1$, working to leading order.

$$\mu = e^{F_m(r - r_m)^{1/\nu - 1/\nu}}$$

$$M = e^{-F_m(r - r_m)^{1/\nu - 1/\nu}} \times \left[ \int_{r_m}^{r} G(r)(r - r_m)^{-\nu} e^{F_m(r - r_m)^{1/\nu - 1/\nu}} dr \right]$$

$$M = e^{-F_m(r - r_m)^{1/\nu - 1/\nu}} \left[ \frac{G_m e^{F_m(r - r_m)^{1/\nu - 1/\nu}}}{F_m} \right]_{r_m}$$

For $0 < \nu < 1$, we again get

$$M = \frac{G_m}{F_m}$$

to leading order, which is the expected value. But for $\nu > 1$ we get a divergence at $r = r_m$.

Thus our conditions for existence of $M(r)$ are that:

(i) the origin conditions of section \textbf{2.2} are satisfied.

(ii) $R(r) = \hat{R}(z(r))$ has a power-law maximum of the form $\hat{R} \sim (r - r_m)^\alpha$ with $1 < \alpha \leq 2$, with a quadratic maximum being the most reasonable.

### B3 Existence of solutions $\hat{R}(r)$ to equation \textbf{(12)}

The equation is

$$\frac{d\hat{R}}{dr} = \sqrt{1 + 2E} - \sqrt{2M/R + 2E}$$

assuming that we take the positive root on the right — i.e. that large scale recollapse has not begun anywhere on our past light cone. Near $r = 0$, $E$, $M$ & $\hat{R}$ all go to 0, but our origin conditions require $M \sim r^3$, $E \sim r^2$ & $\hat{R} \sim r$, so the solution exists here. Where $\hat{R} = 2M$ the r.h.s. is zero, so $\hat{R}$ has a maximum. We already have $2E \geq -1$ for a well behaved metric, and $2M/\hat{R} & 2E$ are separately positive for parabolic and hyperbolic models, while for elliptic models we see from \textbf{(13)} & \textbf{(14)} that $(-2E)\hat{R}M/(1 - \cos \eta) \leq 2$, so $2M/\hat{R} + 2E = \sqrt{M/\hat{R} + 2E/M}$ is always real.

Our only conditions for $\hat{R}(r)$ to exist are:

(i) the origin conditions of section \textbf{2.2} are satisfied.

### B4 Existence of solutions $z(r)$ to equation \textbf{(30)}

The origin conditions ensure that, near $r = 0$, $d^2\hat{R}/dr^2 \sim 0$, $d\hat{R}/dr \sim 1$, and $\hat{\rho} \sim$ constant, so that the integral exists in this neighbourhood.

Where the null cone crosses the apparent horizon, $\hat{R} = 2M$, we have $d\hat{R}/dr = 0$. However, we find from \textbf{(12)} & \textbf{(1)} that the integrand of \textbf{(21)} is

$$\left[ \frac{d^2 \hat{R}}{dr^2} + 4\pi \hat{\rho} \hat{R} / (d\hat{R} / dr) \right] \left[ \frac{d\hat{R}}{dr} \right]$$

$$= \left[ \frac{E'}{\sqrt{1 + 2E}} - \frac{M'}{R} - \frac{M}{R^2} \right] \left[ \sqrt{1 + 2E} - \frac{2M}{R} + 2E \right]$$

$$+ E'/\sqrt{2M/\hat{R} + 2E + M'/R + 2E}$$

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\[
\left( \sqrt{1 + 2E} - \sqrt{\frac{2M}{R} + 2E} \right) \\
= \left[ \left( \frac{M \sqrt{1 + 2E}}{R^2} - E' - \frac{M'}{R} \right) \left\{ \sqrt{1 + 2E} - \sqrt{\frac{2M}{R} + 2E} \right\} \right] / \left( \sqrt{1 + 2E} \sqrt{\frac{2M}{R} + 2E} \right)
\]

which is well behaved at \( \hat{R} = 2M \).

Our conditions for existence of \( \ln(1 + z) \) are merely
(i) the origin conditions, \( E \sim r^2, M \sim r^3 \) near \( r = 0 \).