Optimal control of storage incorporating market impact and with energy applications

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Abstract

Large scale electricity storage is set to play an increasingly important role in the management of future energy networks. A major aspect of the economics of such projects is captured in arbitrage, i.e. buying electricity when it is cheap and selling it when it is expensive. We consider a mathematical model which may account for nonlinear—and possibly stochastically evolving—cost functions, market impact, input and output rate constraints and both time-dependent and time-independent inefficiencies or losses in the storage process. We develop an algorithm which is maximally efficient in the sense that it incorporates the result that, at each point in time, the optimal management decision depends only a finite, and typically short, time horizon. We give examples related to the management of a real-world system. Finally we consider a model in which the associated costs evolve stochastically in time. Our results are formulated in a perfectly general setting which permits their application to other commodity storage problems.

1 Introduction

How should one optimally control an energy store which is used to make money by buying electricity when it is cheap, and selling it when it is expensive? We are interested in the mathematics of this problem where the store has both finite capacity and rate constraints, and where we allow the possibility that the activities of the store are of a sufficient magnitude as to impact upon prices in the market in which it operates. The underlying mathematics required for this application has various novel features and needs to be carefully formulated so as to properly account for physical characteristics of different storage technologies and to deal with inherent nonlinearities which occur when prices are impacted by the store’s behaviour. Although we focus on this application area, the framework developed can be used to consider more general commodity storage problems. A closely related application is to the management of demand in such systems, where the ability to contract with consumers to postpone demand may be regarded as negative storage. For some recent discussion and work on these applications see, for example, [1, 12, 13, 16, 17, 22, 20] and the references therein; for work on the optimal placement of storage within a network, see [18, 19]. These works are concerned, as here, with the mathematics...
of storage for arbitrage, i.e. taking advantage of—and hence assisting in smoothing—price fluctuations over time. This mathematics is of course also quite generally applicable to the use of storage in other markets. (For the mathematics of other uses of storage in energy systems—notably for buffering against uncertainty—see, for example, [3, 4, 5, 10, 15, 14, 22].)

We think of the available storage as a single store. Its value is equal to the profit which can be made by a notional store “owner” buying and selling as above. Our particular interest is in the case where the activities of the store are sufficiently significant as to have a market impact (the store becomes a “price maker”). In this case the store owner sees nonlinear cost functions as, at any time, the marginal costs of buying or returns from selling vary with the amount being bought or sold. In the case where the system or societal value of the store is required, this may be similarly calculated by adjusting the notional buying and selling prices so that the store “owner” is required to bear also the external costs of the store’s activities. Thus our framework is in this respect completely general.

The nonlinearity of the cost functions means that the linear programming techniques which might otherwise be used in the solution of this problem are not generally available. (However, see Section 2 for some further discussion and references for the case where linear programming techniques may be used.) Neither are dynamic programming techniques—deterministic or stochastic (see, for example, [6, 7])—very practicable. The reason for the latter is that optimization is typically over extended periods of time, during which the costs involved may vary with time in an irregular manner. The computational complexity of a dynamic programming approach may therefore be unduly burdensome, may fail to provide insight, and may not even be feasible in the case where future costs are probabilistically uncertain.

In the present paper we develop an approach based on the use of strong Lagrangian techniques (convex optimization theory) which not only accounts for nonlinear cost functions, but also enables the development of a algorithm which is efficient for the solution of the problem in the sense we explain in Section 4. In particular this algorithm is efficient in the sense that the decisions to be made at each point in time typically depend only on a very short future horizon—which is identifiable, but not determined in advance. The length of this horizon is of the same order as that of the shortest period of time over which prices fluctuate significantly; this is important when we may wish to optimize over a very much longer period of time. Our approach also allows us to account for differences in buying and selling prices, for both time-dependent and time-independent inefficiencies in the storage process, and for input and output rate constraints.

Initially we work in a deterministic setting in which we assume that all relevant buying and selling prices are known in advance. However, in Section 7 we consider prices which evolve stochastically.

In Section 2 we formally define the relevant mathematical problem, while in Section 3 we use strong Lagrangian theory to characterise mathematically its optimal solution. We use this theory in Section 4 to develop the algorithm for the solution referred to above and in Section 5 to show how the value of the store changes with respect to variation in its characteristic parameters. Section 6 considers an example based on real data and a real pumped storage facility. Finally in Section 7 we consider a model in which the cost functions are stochastic, and which we believe is as realistic in this respect as it practicable for applications.
2 Problem formulation

We work in discrete time, which we take to be integer. We assume that the store has total capacity of $E$ (which, in the context of an energy system, would be total energy which could be stored) and input and output rate constraints of $P_i$ and $P_o$ respectively (which, for an energy system, would be in units of power). We consider two types of (in)efficiency associated with the store. The first of these (and usually much the more significant in practice) is a time-independent efficiency $\eta$ which may be defined as the fraction of energy bought which is available to sell. This may be captured in our model either by adjusting buy prices by a factor $1/\eta$ or by multiplying sell prices by $\eta$ (the values of $E$, $P_i$ and $P_o$ in the two cases differing by a factor of $\eta$). Hence, without loss of generality, we take $\eta = 1$ throughout. The second type of (in)efficiency may be regarded as leakage over time, and is modelled by assuming that at each successive time instant there is lost a fraction $1 - \rho$ of whatever is in the store at that time.

Let $X = \{x : -P_o \leq x \leq P_i\}$. Both buying and selling prices at time $t$ may conveniently be represented by a function $C_t$ with $C_t(0) = 0$, which is increasing and convex in $x$ and which, for positive $x$, is the price of buying $x$ units (for example of energy) and, for negative $x$, is the negative of the price for selling $-x$ units. Thus the cost of increasing the level of the store contents by $x$, positive or negative, is always $C_t(x)$. The convexity assumption corresponds, for each time $t$, to an increasing cost to the store of buying each additional unit, a decreasing revenue obtained for selling each additional unit, and every unit buying price being at least as great as every unit selling price. Note that this last condition remains natural when we model an inefficient store, i.e. one for which $\eta < 1$, by assuming instead $\eta = 1$ and correspondingly lowering selling prices.

As indicated above, if the problem is to determine the value of the store to the entire system in which it operates, or to society, then these prices are taken to be those appropriate to the system or to be societal costs. Thus, for example, for $x$ positive, $C_t(x)$ is the price paid by the store at time $t$ for $x$ units of, for example, energy plus the increased cost paid by other energy users at that time as a result of the store’s purchase increasing market prices.

Figure 1 thus illustrates a typical cost function $C_t$. While the function $C_t$ may be formally regarded as defined over the whole real line, the rate constraints means that for the purposes of the present problem its domain is effectively restricted to the set $X$ defined above. (We shall later wish to consider the effect of varying the rate constraints.)

A special case is that of a “small” store, whose operations do not influence the market (the store is a “price-taker” rather than a “price-maker”), and which at time $t$ buys and sells at given prices per unit of $c_t^{(b)}$ and $c_t^{(s)}$ respectively, where we assume that $c_t^{(b)} \geq c_t^{(s)}$. Here the function $C_t$ is given by

$$C_t(x) = \begin{cases} c_t^{(b)} x & \text{if } x \geq 0 \\ c_t^{(s)} x & \text{if } x < 0. \end{cases}$$

Finally, we assume for the moment that all prices are known in advance, so that the problem of controlling the store is deterministic. We consider a realistic stochastic model in Section 7.

Denote the successive levels of the store by a vector $S = (S_0, \ldots, S_T)$ where $S_t$ is the level of the store at each successive time $t$. Define also the vector $x(S) = (x_1(S), \ldots, x_T(S))$ by
Figure 1: Illustrative cost function $C_t$. The domain of the function is effectively restricted to the set $X = \{x : -P_0 \leq x \leq P_i\}$.

$x_t(S) = S_t - \rho S_{t-1}$ for each $t \geq 1$. Here $\rho$ is the leakage measure defined above, so that $x_t(S)$ represents the addition to the store at time $t$. It is convenient to assume that both the initial level $S_0$ and the final level $S_T$ of the store are fixed in advance at $S_0 = S_0^*$ and $S_T = S_T^*$. (If the final level $S_T$ is not fixed and the cost function $C_T$ is strictly increasing, then, for an optimal control, we may take $S_T$ to be minimised; however, we might, for example, require $S_T = S_0$—as a contribution to a toroidal solution.)

The problem thus becomes:

**P:** choose $S$ so as to minimise

$$G(S) := \sum_{t=1}^{T} C_t(x_t(S))$$

subject to the capacity constraints

$$S_0 = S_0^*, \quad S_T = S_T^*, \quad 0 \leq S_t \leq E, \quad 1 \leq t \leq T - 1. \quad (3)$$

and the rate constraints

$$x_t(S) \in X, \quad 1 \leq t \leq T. \quad (4)$$

We shall say that a vector $S$ is **feasible** for the problem **P** if it satisfies both the capacity constraints (3) and the rate constraints (4). We shall assume that $S_0^*$ and $S_T^*$ are sufficiently close that it is possible to change the level of the store from $S_0^*$ to $S_T^*$ between times 0 and $T$, i.e. that the set of feasible vectors $S$ is nonempty. Note that this set is then closed and convex and that the function $G$ defined by (2) is convex, and strictly so when the functions $C_t$ are strictly convex. Hence a solution to the problem **P** always exists, and is unique when the functions $C_t$ are strictly convex.

In the case where the cost functions $C_t$ are linear, or piecewise linear, as in the “small store” case given by (1), the problem **P** may be reformulated as a linear programming problem, and solved by, for example, the use of the minimum cost circulation algorithm (see, for example, [8, 2]). Our aim in the present paper is to deal with the general case, to show that the optimal choice of $S_t$ at each time $t$ depends only on a typically very short time horizon, and to develop an algorithm which is efficient both in the general and in the linear case.
3 Lagrangian formulation and characterisation of solution

We develop the strong Lagrangian theory [8, 21] associated with the problem \( P \) defined above. Theorem 1 gives sufficient conditions for a value \( S^* \) of \( S \) to solve the problem, while Theorem 2 guarantees the existence of such a value of \( S^* \), together with the associated vector \( \mu^* \) defined there.

**Theorem 1.** Suppose that there exists a vector \( \mu^* = (\mu^*_1, \ldots, \mu^*_T) \) and a value \( S^* = (S^*_0, \ldots, S^*_T) \) of \( S \) such that

(i) \( S^* \) is feasible for the stated problem,
(ii) for each \( t \) with \( 1 \leq t \leq T \), \( x_t(S^*) \) minimises \( C_t(x) - \mu^*_t x \) in \( x \in X \),
(iii) the pair \((S^*, \mu^*)\) satisfies the complementary slackness conditions, for \( 1 \leq t \leq T - 1 \),

\[
\begin{align*}
\rho \mu_{t+1}^* &= \mu_t^* & \text{if } 0 < S_t^* < E, \\
\rho \mu_{t+1}^* &\leq \mu_t^* & \text{if } S_t^* = 0, \\
\rho \mu_{t+1}^* &\geq \mu_t^* & \text{if } S_t^* = E.
\end{align*}
\]

Then \( S^* \) solves the stated problem \( P \).

**Proof.** Let \( S \) be any vector which is feasible for the problem (with \( S_0 = S_0^* \) and \( S_T = S_T^* \)). Then, from the condition (ii),

\[
\sum_{t=1}^{T} [C_t(x_t(S^*)) - \mu_t^* x_t(S^*)] \leq \sum_{t=1}^{T} [C_t(x_t(S)) - \mu_t^* x_t(S)].
\]

Rearranging and recalling that \( S \) and \( S^* \) agree at 0 and at \( T \), we have

\[
\sum_{t=1}^{T} C_t(x_t(S^*)) - \sum_{t=1}^{T} C_t(x_t(S)) \leq \sum_{t=1}^{T} \mu_t^* (S_t^* - \rho S_{t-1}^* - S_t + \rho S_{t-1})
\]

\[
= \sum_{t=1}^{T-1} (S_t^* - S_t)(\mu_t^* - \rho \mu_{t+1}^*)
\]

\[
\leq 0,
\]

by the condition (iii), so that the result follows.

Note that the vector \( \mu^* \) of Theorem 1 may always be taken to be nonnegative, i.e. to have nonnegative components: if \( \mu^* \) does not satisfy this condition then its negative components may all be increased to 0 and the pair \((S^*, \mu^*)\) will continue to satisfy the conditions of the theorem. (In the case of condition (ii) this follows since each of the functions \( C_t \) is increasing.) The vector \( \mu^* \) is a cumulative form of the vector of Lagrange multipliers associated with the capacity constraints (3) (see the proof of Theorem 2 below). It has the interpretation that, for each \( t \), the quantity \( \mu_t^* \) may be regarded as a notional reference value per unit volume in storage at that time. Thus, in the condition (ii) of the theorem, \( C_t(x) \) is the cost at time \( t \) of increasing the level of the store by \( x \) (again positive or negative) and \( \mu_t^* x \) may be regarded as a current offsetting measure of value added to the store; the quantity \( C_t(x) - \mu_t^* x \) is thus to be minimised in \( x \in X \). The relations (5) of condition (iii) of the theorem are then such that, were they to be violated, \( x_t \) and \( x_{t+1} \) could in general be adjusted so as to leave unchanged the level of the store at the end of
time $t+1$ while reducing the overall cost of operating the store throughout the period consisting of the times $t$ and $t+1$.

Note also that, in the condition (ii) of Theorem 1, the minimisation takes place without reference to the capacity constraints (as is appropriate given the above Lagrangian interpretation of $\mu^*$). However, the minimisation of that condition is required to respect the rate constraints $x \in X$—for which no Lagrange multiplier is introduced at this stage (but see Section 5). The reason for the apparent asymmetry of treatment of the two constraint types is that it is only the capacity constraints which introduce complexity into the optimisation problem, by introducing interactions between the amounts which may be bought and sold at different times. The rate constraints could, if we wished, be dropped from the formal statement of the problem by suitably modifying the cost functions so that the violation of these constraints was simply prohibitively expensive.

Before considering Theorem 2, which guarantees the existence of the pair $(S^*, \mu^*)$, we give a couple of simple examples, in each of which the reference vector $\mu^*$ is identified. Theorem 1 is not, however, needed for the solution of the first, very simple, example. It is needed in the second example only in the case where the store is sufficiently large as to have market impact (i.e. be a price maker).

**Example 1.** As a simple (toy) example, suppose that $T = 2$ and that the cost functions $C_t$, $t = 1, 2$, in addition to being increasing and convex, are differentiable (with necessarily continuous first derivatives); however, as an exception and in order to allow for a distinction between buying and selling prices we allow a difference between the left and right derivatives of the functions $C_t$ at 0, denoting these one-sided derivatives by $C_t'(0-)$ and $C_t'(0+)$ respectively (with, necessarily, $C_t'(0-) \leq C_t'(0+)$ for $t = 1, 2$). We suppose additionally, and again for simplicity, that the input and output rate constraints are equal, setting $P_1 = P_2 = P$, and that there is no leakage (i.e. $\rho = 1$). Finally we suppose $S_0^* = S_2^* = 0$ so that the store starts empty and is required to finish empty. Thus the only possible control of the store lies in the choice of the amount $x \geq 0$ which is bought at time 1 and sold again at time 2.

For this example, the optimal policy is of course easily determined. Our concern is merely to identify, in this very simple case, the vector $\mu^*$ of Theorem 1. This vector plays a crucial role in more complex optimization over longer time periods. We consider the three possible cases.

(i) If $C_1'(0+) \geq C_2'(0-) \geq C_2'(0-)$ then clearly the optimal policy is buy and sell nothing and we take $x = 0$. For the vector $\mu^*$ of Theorem 1 we may take $\mu_1^* = C_1'(0+)$ and $\mu_2^* = C_2'(0-)$. 

(ii) If $C_1'(0+) < C_2'(0-)$ and there exists $x$ such that both $0 \leq x \leq \min(E, P)$ and $C_1'(x) = C_2'(-x)$, then this choice of $x$ is again clearly optimal. The vector $\mu^*$ is given (uniquely) by $\mu_1^* = \mu_2^* = C_1'(x)$.

(iii) Finally, if $C_1'(x) < C_2'(-x)$ for all $x$ such that $0 \leq x \leq \min(E, P)$, then the optimal choice of $x$ is given by $x = \min(E, P)$. In the case where $P \leq E$ we require $C_1'(P) \leq \mu_1^* = \mu_2^* \leq C_2'(-P)$, while in the case where $E < P$ we require $C_1'(E) = \mu_1^* \leq \mu_2^* = C_2'(-E)$.

Note that the actual solution to this very simple problem depends on $E$ and $P$ only through $\min(E, P)$. However, as previously observed, $\mu^*$ plays an asymmetric rôle with respect to capacity and rate constraints and thus formally differs in the case (iii) according to which of $E$ or $P$ is the greater.
Example 2. Periodic costs. As a second simple example, we suppose that the cost functions vary over time in a manner which is completely periodic. To begin with, we consider the “small store”, or price-taker, case in which the cost functions $C_t$ are given by (1) (with $c_t^{(b)} \geq c_t^{(s)}$ for all $t$). We suppose that the periodic behaviour is such that, at some time $t_1$ in a cycle, both $c_t^{(b)}$ and $c_t^{(s)}$ are simultaneously at a minimum; the unit costs $c_t^{(b)}$ and $c_t^{(s)}$ then increase monotonically up to a time $t_2 > t_1$ where they are simultaneously at a maximum, before decreasing monotonically again to the same minimum value as previously at further time $t_3 > t_2$; this pattern is then repeated indefinitely with period $t_3 - t_1$. We suppose also that the minimum value of the unit buy costs $c_t^{(b)}$ is less than the maximum value of the unit sell costs $c_t^{(s)}$ (otherwise the store remains unused). We again assume, for simplicity, that there is no leakage (i.e. $\rho = 1$), that $P_t = P_o = P$ and that time is sufficiently finely discretised that $E/P$ (the minimum time in which the store may completely empty or fill) may be taken to be integer. The optimal control policy depends (up to a multiplicative constant) on $E$ and $P$ only through the ratio $E/P$; hence, without loss of generality, we assume $P = 1$.

The simplicity of this example is such that the optimal control of the store is again immediately clear: for all $E$ there exist reference costs $\mu_t^{(b)} \leq \mu_t^{(s)}$ such that the store buys the maximum value of one unit at those times such that $c_t^{(b)} < \mu_t^{(b)}$ and sells the maximum value of one unit at those times such that $c_t^{(s)} > \mu_t^{(s)}$; for $E$ sufficiently small we may take $\mu_t^{(b)} < \mu_t^{(s)}$ and the store completely empties and fills on each cycle; however, as $E$ increases it reaches a value at which the reference costs $\mu_t^{(b)}$ and $\mu_t^{(s)}$ equalise, and for this and larger values of $E$ the capacity constraint is no longer binding.

As in the case of the previous example, this “small store” problem is too simple for its solution to require the use of the reference vector $\mu^*$ of Theorem 1 (but see below for where it is needed). We note, however, that this vector may be given by $\mu_t^* = \mu_t^{(b)}$ at those times $t$ at which the store is buying, and by $\mu_t^* = \mu_t^{(s)}$ at those times $t$ at which it is selling; at other times (at each of which the store will either be completely full or completely empty) $\mu_t^*$ is merely required to satisfy the condition (iii) of Theorem 1 together with the condition $c_t^{(s)} \leq \mu_t^* \leq c_t^{(b)}$ (so that the condition (ii) of Theorem 1 is satisfied).

We also comment briefly on the effect of varying the frequency of the cost variation. If, in what should strictly be a continuous-time setting, this frequency is increased by a factor $\alpha$ with the rate constraint $P$ being similarly increased by the same factor, then this corresponds to a simple time speed-up, with the store’s revenue per unit time also being increased by the factor $\alpha$. However, suppose instead that while the frequency of the cost variation is increased by the factor $\alpha$, the rate constraint $P$ is held constant at its original value and that the capacity constraint $E$ is replaced by $E/\alpha$. It then follows, from the earlier observation that the optimal control depends on $E$ and $P$ only through their ratio, that the optimal control is here a rescaled version of the original and that the store’s revenue per unit time remains unchanged from the original. Thus we have the well-known result that more frequent cost variation enables the same revenue to be obtained with a smaller store capacity.

When we consider the general case in which the store is a price-maker, and in which the cost functions $C_t$ have the same general periodicity over time, but no longer have the simple structure given by (1), then the store may fill and empty over periods of time which are longer than the minimum necessary, so as to avoid the higher costs or penalties of buying or selling too much at once. The reference vector $\mu^*$ of Theorem 1 then becomes essential in deciding the correct volume of each transaction.
Theorem 1 does not require the convexity of the cost functions $C_t$ of the problem $P$ defined in Section 2. This condition is, however, required to ensure the existence of the vector $\mu^*$ of that theorem, as is given by Theorem 2 below. The latter theorem identifies $\mu^*$ as essentially a cumulative Lagrange multiplier for capacity constraint variation. It is a further application of arguments to be found in strong Lagrangian theory (again see [21]).

We have already observed that, under strict convexity of the cost functions $C_t$, the solution $S^*$ to the problem $P$ is unique. However, we further remark that even this condition is insufficient to guarantee uniqueness of $\mu^*$ as above. We address this issue in Section 5, where we assume sufficient differentiability conditions on the cost functions $C_t$ as to ensure uniqueness of $\mu^*$ and to derive sensitivity results for variation of the minimised cost function of $P$ with respect to both its capacity and rate constraints.

Prior to Theorem 2 it is convenient to introduce the more general problem $P(a, b)$ in which $S_0$ is kept fixed at the value $S_0^*$ of interest above, but in which $S_1, \ldots, S_T$ are allowed to vary between quite general upper and lower bounds:

$$P(a, b): \text{ minimise } \sum_{t=1}^{T} C_t(x_t(S)) \text{ over all } S = (S_0, \ldots, S_T) \text{ with } S_0 = S_0^* \text{ and subject to the further constraints}$$

$$a_t \leq S_t \leq b_t, \quad 1 \leq t \leq T,$$  \hspace{1cm} (6)

and $x_t(S) \in X$ for $1 \leq t \leq T$, where $a = (a_1, \ldots, a_T)$ and $b = (b_1, \ldots, b_T)$ are such that $a_t \leq b_t$ for all $t$.

Note that the convexity of the functions $C_t$ guarantees their continuity, and, since for each $a, b$ as above the space of allowed values of $S$ is compact, a solution $S^*(a, b)$ to the problem $P(a, b)$ always exists. Let $V(a, b)$ be the corresponding minimised value of the objective function, i.e. $V(a, b) = \sum_{t=1}^{T} C_t(x_t(S^*(a, b)))$. Note also that it follows easily from the convexity of the functions $C_t$, the linearity in $a$ and $b$ of the constraints (6), and the observation that the pointwise minimum of a set of convex functions is again convex, that $V(a, b)$ is itself convex in $a$ and $b$. Define also $a^*$ and $b^*$ to be the values of $a$ and $b$ corresponding to our particular problem $P$ of interest, i.e. $a^*_t = 0$ and $b^*_t = E$ for $1 \leq t \leq T - 1$, and $a^*_T = b^*_T = S^*_T$. Further, let $S^* = (S_0^*, \ldots, S_T^*) = S^*(a^*, b^*)$ denote the solution to this problem.

**Theorem 2.** Under the given convexity condition on the cost functions $C_t$, there always exists a pair $(S^*, \mu^*)$ which solves the problem $P$ as in Theorem 1.

**Proof.** Consider the more general problem $P(a, b)$ defined above. Introduce slack (or surplus) variables $z = (z_1, \ldots, z_T)$ and $w = (w_1, \ldots, w_T)$ and rewrite this problem as:

$$P(a, b): \text{ minimise } \sum_{t=1}^{T} C_t(x_t(S)) \text{ over all } S = (S_0, \ldots, S_T) \text{ with } S_0 = S_0^*, \text{ all } z \geq 0, \text{ all } w \geq 0, \text{ and subject to the further constraints}$$

$$S_t - z_t = a_t, \quad 1 \leq t \leq T,$$  \hspace{1cm} (7)

$$S_t + w_t = b_t, \quad 1 \leq t \leq T,$$  \hspace{1cm} (8)

and, again, $x_t(S) \in X$ for $1 \leq t \leq T$.

Since, as already observed, the function $V(a, b)$ is itself convex in $a$ and $b$, it follows by the supporting hyperplane theorem (see [8] or [21]), that there exist vectors (Lagrange multipliers) $\alpha^* = (\alpha_1^*, \ldots, \alpha_T^*)$ and $\beta^* = (\beta_1^*, \ldots, \beta_T^*)$ such that

$$V(a, b) \geq V(a^*, b^*) + \sum_{t=1}^{T} \alpha^*_t (a_t - a^*_t) + \sum_{t=1}^{T} \beta^*_t (b_t - b^*_t) \quad \text{for all } a, b. \hspace{1cm} (9)$$
Thus also, for all $S$ with $S_0 = S_0^*$ and such that $x_t(S) \in X$ for $1 \leq t \leq T$, for all $z \geq 0$, and for all $w \geq 0,$

$$
\sum_{t=1}^{T} [C_t(x_t(S)) - \alpha_t^*(S_t - z_t) - \beta_t^*(S_t + w_t)] \\
\geq \sum_{t=1}^{T} [C_t(x_t(S^*)) - \alpha_t^*(S_t^* - z_t^*) - \beta_t^*(S_t^* + w_t^*)] \quad (10)
$$

Since the components of $z$ and $w$ may take arbitrary positive values, we deduce immediately the following usual complementary slackness conditions for the vectors of Lagrange multipliers $\alpha^*$ and $\beta^*$:

$$
\alpha_t^* \geq 0, \quad \alpha_t^* = 0 \text{ whenever } z_t^* > 0, \quad 1 \leq t \leq T; \quad (11)
$$

$$
\beta_t^* \leq 0, \quad \beta_t^* = 0 \text{ whenever } w_t^* > 0, \quad 1 \leq t \leq T. \quad (12)
$$

Thus, from (10)–(12) and by taking $z_t = w_t = 0$ for all $t$ on the left side of (10), it follows that, for all $S$ with $S_0 = S_0^*$ and $x_t(S) \in X$ for $1 \leq t \leq T,$

$$
\sum_{t=1}^{T} [C_t(x_t(S)) - (\alpha_t^* + \beta_t^*)S_t] \geq \sum_{t=1}^{T} [C_t(x_t(S^*)) - (\alpha_t^* + \beta_t^*)S_t^*]. \quad (13)
$$

Thus also, for all $x = (x_1, \ldots, x_t)$ such that $x_t \in X$ for $1 \leq t \leq T$, by defining $S$ by $S_0 = S_0^*$ and $S_t = \rho S_{t-1} + x_t$ for $1 \leq t \leq T,$ it follows that

$$
\sum_{t=1}^{T} [C_t(x_t) - \mu_t^* x_t] \geq \sum_{t=1}^{T} [C_t(x_t(S^*)) - \mu_t^* x_t(S^*)]. \quad (14)
$$

where, for each $1 \leq t \leq T,$ we define

$$
\mu_t^* = \sum_{u=t}^{T} \rho^{u-t} (\alpha_u^* + \beta_u^*). \quad (15)
$$

It now follows that the pair $(S^*, \mu^*)$ satisfies the conditions (i) and (ii) of Theorem 1. Further, on recalling from (7) and (8) respectively that, for $1 \leq t \leq T - 1,$ we have $z_t^* = 0$ if and only if $S_t^* = 0$ and $w_t^* = 0$ if and only if $S_t^* = E,$ it follows also from (11), (12) and the definition (15) of the vector $\mu^*$, that the pair $(S^*, \mu^*)$ satisfies the complementary slackness conditions (iii) of Theorem 1. \qed

Recall the earlier interpretation of each successive $\mu_t^*$ as providing a unit reference value determining the quantity $x_t$ (positive or negative) which should be added to the level of the store at that time. In Section 4 we give an efficient algorithm for the determination of the successive values of $\mu_t^*$.

### 4 Algorithm

We now give an explicit construction of a pair $(S^*, \mu^*)$ (with $\mu^*$ nonnegative) as in Theorem 1. This construction provides an algorithm for the solution of the problem $P$ in the general case.
We assume for the moment that there is no leakage from the store over time, i.e. that \( \rho = 1 \). With this assumption, the algorithm below may briefly be described as that of attempting to choose \((S^*, \mu^*)\) so as to satisfy the conditions of Theorem 1, by choosing the components of these vectors successively in time and by keeping \( \mu^*_i \) as constant as possible over \( t \), changes only being allowed at those times when the store is either empty or full. Once the algorithm is understood, the modifications required to deal with the case \( \rho < 1 \) are easily seen and are indicated in brief at the end of this section.

For further simplicity, we suppose first that the cost functions \( C_t \) are all strictly convex. Then, as already noted, the vector \( S^* \) of Theorem 1 is unique—though the corresponding vector \( \mu^* \) need not be. We give a construction of \((S^*, \mu^*)\) which is sequential in time. For any \( t \) such that \( 1 \leq t \leq T \) and any (scalar) \( \mu \geq 0 \), define \( x_t^*(\mu) \) to be the unique value of \( x \) which minimises \( C_t(x) - \mu x \) in \( x \in X \). Note that \( x_t^*(\mu) \) is then continuous and increasing (though not necessarily strictly so) in \( \mu \). Define a sequence of times \( 0 = T_0 < T_1 < \cdots < T_k = T \) and the pair \((S^*, \mu^*)\) inductively as follows. Suppose that \( i \geq 0 \) is such that \( T_0, \ldots, T_i \) together with \( S_{T_i}^*, \ldots, S_T^* \) and \( \mu_{T_i}^*, \ldots, \mu_T^* \), are all defined. For each (scalar) \( \mu \geq 0 \), define a vector \( S(\mu) = (S_1(\mu), \ldots, S_T(\mu)) \) by

\[
S_t(\mu) = \begin{cases} 
S_t^*, & 1 \leq t \leq T_i \\
S_{t-1}(\mu) + x_t^*(\mu), & T_i + 1 \leq t \leq T.
\end{cases}
\]  

(16)

Define the sets

\[
M_i = \{ \mu \geq 0 : \exists T'(\mu) \text{ with } T_i + 1 \leq T'(\mu) \leq T \text{ such that } 0 \leq S_t(\mu) \leq E \text{ for } T_i + 1 \leq t < T'(\mu) \}
\]

and

\[
M'_i = \{ \mu \geq 0 : \exists T'(\mu) \text{ with } T_i + 1 \leq T'(\mu) \leq T \text{ such that } 0 \leq S_t(\mu) \leq E \text{ for } T_i + 1 \leq t < T'(\mu) \}
\]

and

\[
either S_{T'(\mu)}(\mu) < 0 \text{ or } T'(\mu) = T, S_T(\mu) < S_T^* \}.
\]

Thus \( M_i \) and \( M'_i \) are the sets of \( \mu \) for which \( S(\mu) \) violates one of the capacity constraints and first does so respectively below or above—in either case at a time which we denote by \( T'(\mu) \). Since each \( x_t^*(\mu) \) is increasing in \( \mu \), it follows that if \( \mu \in M_i \) then \( \mu' \in M_i \) for all \( \mu' < \mu \) and that if \( \mu \in M'_i \) then \( \mu' \in M'_i \) for all \( \mu' > \mu \); further the sets \( M_i \) and \( M'_i \) are disjoint, and (since the pair \((S^*, \mu^*)\) exists) neither \( M_i \) nor \( M'_i \) can be the entire set \([0, \infty) \). Let \( \bar{\mu}_i = \sup M_i \) (where we take \( \bar{\mu}_i = 0 \) in the case where \( M_i \) is empty). We now consider the behaviour of \( S(\bar{\mu}_i) \), for which there are three possibilities:

(a) the vector \( S(\bar{\mu}_i) \) is feasible; in this case we take \( T_{i+1} = T \) and \( S_t^* = S_t(\bar{\mu}_i) \) with \( \mu_t^* = \bar{\mu}_i \) for \( T_i + 1 \leq t \leq T \) (thus also \( S_t^* = S_t(\bar{\mu}_i) \) for all \( t \));

(b) the scalar \( \bar{\mu}_i \) belongs to the set \( M_i \); here there necessarily exists at least one \( t < T'(\bar{\mu}_i) \) such that \( S_t(\bar{\mu}_i) = E \) (for otherwise, by the continuity of each \( S_t(\mu) \) in \( \mu, \mu \) could be increased above \( \bar{\mu}_i \) while still belonging to the set \( M_i \)); define \( T_{i+1} \) to be any such \( t \), say the largest, and (again) take \( S_t^* = S_t(\bar{\mu}_i) \) and \( \mu_t^* = \bar{\mu}_i \) for all \( t \) such that \( T_i + 1 \leq t \leq T_{i+1} \); note also that we then have \( \bar{\mu}_i \in M_{i+1} \) so that we shall necessarily have \( \bar{\mu}_{i+1} \geq \bar{\mu}_i \);

(c) the scalar \( \bar{\mu}_i \) belongs to the set \( M'_i \); here, similarly to the case (b), there necessarily exists at least one \( t < T'(\bar{\mu}_i) \) such that \( S_t(\bar{\mu}_i) = 0 \); define \( T_{i+1} \) to be any such \( t \), again say the largest, and again take \( S_t^* = S_t(\bar{\mu}_i) \) and \( \mu_t^* = \bar{\mu}_i \) for all \( t \) such that \( T_i + 1 \leq t \leq T_{i+1} \); further, in this case we have \( \bar{\mu}_i \not\in M_{i+1} \) so that we shall necessarily have \( \bar{\mu}_{i+1} \leq \bar{\mu}_i \).
In the case where the cost functions $C_t$ are all strictly convex, it now follows immediately from the above construction of the pair $(S^*, \mu^*)$ that this pair satisfies the conditions (i)–(iii) of Theorem 1.

In the case where, for at least some $t$, the cost function $C_t$ is convex, but not necessarily strictly convex, a little extra care is required. Here, for such $t$, the function $\mu \mapsto x_t^*(\mu)$ is not in general uniquely defined, and, for any given choice, this function is not in general continuous. However, in essence, the above construction of $(S^*, \mu^*)$ continues to hold—it is simply a matter, where necessary, of choosing the right value of $x_t^*(\mu)$. (One way to see this more formally is as follows. For each $\epsilon > 0$, consider the problem $P(\epsilon)$ in which each of the functions $C_t$ is replaced by $C_t(\epsilon)$ given by $C_t(\epsilon)(x) = C_t(x) + \epsilon x^2$ and so becomes strictly convex. It is easily seen that, as $\epsilon \to 0$, the solutions to the problem $P(\epsilon)$ converge pointwise to a solution $(S^*, \mu^*)$ to the original problem, satisfying the conditions of (i)–(iii) of Theorem 1. In the implementation of the present algorithm, implicit consideration of this limit enables the right choice, where necessary, of each $x_t^*(\mu)$ as discussed above.)

We summarise our results in Theorem 3 below.

**Theorem 3.** Assume $\rho = 1$. Then the pair $(S^*, \mu^*)$ given by the above recursive construction satisfies the conditions (i)–(iii) of Theorem 1.

**Discussion.** The above algorithm requires the determination, at each of the successive times $T_i$, $0 \leq i \leq k - 1$, of the succeeding time $T_{i+1}$ and of the common value $\bar{\mu}_i$ of $\mu^*_t$ for $T_i + 1 \leq t \leq T_{i+1}$. This is done by looking ahead for the minimum time horizon necessary for the above determination; the process then restarts at the time $T_{i+1}$. A lengthening of the total time $T$ over which the optimization is to be performed does not in general change the values of the times $T_i$, but rather simply creates more of them. In this sense both the solution to the problem $P$ and the above algorithm are local in time, so that the solution to $P$ involves computation which grows essentially linearly in $T$. The typical length of the intervals between the successive times $T_i$ depends on the shape of the cost functions $C_t$ (notably the difference between buying and selling prices), together with the rate at which these functions fluctuate in time. This is to be expected as the store operates by selling at prices above those at which it bought, and what is important is the frequency with which such events can occur. For example, such fluctuations may occur an a 24-hour cycle, and, depending on the shape of the cost functions, the typical length of the intervals between the successive times $T_i$ may then be of the order of around 12 hours. These points are illustrated further in the example of the following section.

We observe also that, in any numerical implementation of the above algorithm, for each time $T_i$ as above, the determination of the succeeding time $T_{i+1}$ and of $\bar{\mu}_i$ involves some form of search over an interval of the real line and as such may typically only be carried out to a specified degree of precision. This is inevitable given general convex cost functions.

We now consider briefly the case of general $\rho \leq 1$, i.e. where we also model possible leakage from the store. Only small and readily understood modifications are required to the above algorithm. Here, as before, the essence of the argument is to attempt to choose $(S^*, \mu^*)$ so as to satisfy the conditions of Theorem 1, again by choosing the components of these vectors successively in time, but now maintaining the relationship $\rho \mu^*_t = \mu^*_t$, except at those times $t$ such that the store is either empty or full. Thus we proceed as previously,
except that the relation (16) now becomes

\[ S_t(\mu) = \begin{cases} 
S_t^*, & 1 \leq t \leq T_i \\
\rho S_{t-1}(\mu) + x_t^*(\rho T_{i+1} - t \mu), & T_i + 1 \leq t \leq T,
\end{cases} \]

and corresponding and obvious small modifications are required in the three cases (a)–(c) considered previously.

### 5 Sensitivity of store value with respect to constraint variation

Under suitable differentiability assumptions, the Lagrangian theory of the preceding sections enables an immediate determination of the effect on the cost of operating the store (the negative of its value) of marginal variations in either the capacity or the rate constraints. The capacity variation result is almost immediate, while the rate constraint result requires a modest extension of the earlier theory. Throughout we again consider the more general problem \( P(a, b) \) introduced in Section 3, together with its minimised objective function \( V(a, b) \)—corresponding to the minimum cost of operating the store. We again let \( a^* \) and \( b^* \) to be the values of \( a \) and \( b \) corresponding to our particular problem \( P \) of interest—as previously defined. We assume throughout this section that the minimised objective function \( V(a, b) \) is differentiable with respect to (each of the components of) the vectors \( a \) and \( b \) at \( (a^*, b^*) \)—as will be the case when, for example, the cost functions \( C_t \) are differentiable at the solution to the problem \( P \).

Under this differentiability condition the vector \( \mu^* \) of Theorem 1 is uniquely defined. This follows from consideration of the algorithm of Section 4, which sequentially constructs a pair \( (S^*, \mu^*) \) satisfying the conditions of Theorem 1. Here the differentiability condition above implies easily that any attempt to vary \( \mu^* \) as constructed by that algorithm leads to a violation of the complementary slackness conditions (iii) of Theorem 1. (Alternatively, the uniqueness may here be argued directly from the conditions (ii) and (iii) of Theorem 1, again by considering infinitesimal variation of \( \mu^*_t \) at those times \( t \) such that the capacity constraints are binding.) This vector \( \mu^* \) is thus as identified by Theorem 2—and has the interpretation in terms of Lagrange multipliers given there—and is as constructed by the algorithm of Section 4.

It is convenient to write \( V^* \) for the value \( V(a^*, b^*) \) of the minimised objective function for our particular problem of interest \( P = P(a^*, b^*) \). For the sensitivity of the cost of operating the store with respect to variation in the capacity constraint, we have the following result.

**Theorem 4.** The derivative of the cost of operating the store with respect to variation of the capacity \( E \) is given by

\[
\frac{\partial V^*}{\partial E} = \sum_{t \in \tau} (\mu^*_t - \rho \mu^*_{t+1}),
\]

where \( \tau \) is the set of times \( t \) such that \( 1 \leq t \leq T - 1 \) and \( S^*_t = E \), and where \( \mu^* \) is as identified above.

**Proof.** Let \( \alpha^* \) and \( \beta^* \) be the vector Lagrange multipliers introduced in the proof of Theorem 2. Recall also the definition of \( b^* \) above. From the standard interpretation of Lagrange
multipliers in the presence of differentiability of an objective function,
\[
\frac{\partial V^*}{\partial E} = \sum_{1 \leq i \leq T-1} \beta^*_i
\]
\[
= \sum_{t \in \tau} (\alpha^*_t + \beta^*_t),
\]  \hspace{1cm} (18)
where (18) above follows from the conditions (11) and (12) (which imply that for \(1 \leq t \leq T - 1\), we have \(\beta^*_t = 0\) for \(t \notin \tau\) and \(\alpha^*_t = 0\) for \(t \in \tau\)). The required result now follows on using (15).

We now consider the sensitivity of the cost of operating the store with respect to variation in the rate constraints. We here have the following result.

**Theorem 5.** Assume additionally that the cost functions \(C_i\) are differentiable at the points \(P_i\) and \(-P_o\) corresponding to the input and output rate constraints. Then the derivatives of the cost \(V(a^*, b^*)\) of operating the store with respect to variation of the input and output rate constraints \(P_i\) and \(P_o\) are given respectively by

\[
\frac{\partial V^*}{\partial P_i} = \sum_{t \in \tau_i} (C'_i(P_i) - \mu^*_i)
\]
\[
\frac{\partial V^*}{\partial P_o} = \sum_{t \in \tau_o} (\mu^*_i - C'_i(-P_o)),
\]
where \(\tau_i\) is the set of times \(1 \leq t \leq T\) such that \(x_t(S^*) = P_i\) and \(\tau_o\) is the set of times \(1 \leq t \leq T\) such that \(x_t(S^*) = -P_o\) (i.e. \(\tau_i\) and \(\tau_o\) are respectively the sets of times such that the input and output rate constraints are binding at the solution \(S^*\) to the problem \(P\)), and where again \(\mu^*\) is as identified above.

**Proof.** We proceed as in the proof of Theorem 2. However, we rewrite the problem \(P(a, b)\) by relaxing the rate constraints \(x_t(S) \in X\) to \(x_t(S) \in \mathbb{R}\) and introducing instead the additional functional constraints

\[
x_t(S) + u_t = P_i, \quad 1 \leq t \leq T,
\]
\[
x_t(S) - v_t = -P_o, \quad 1 \leq t \leq T,
\]  \hspace{1cm} (21)
(22)
for slack (or surplus) variables \(u = (u_1, \ldots, u_T)\) and \(v = (v_1, \ldots, v_T)\) constrained to be positive. We thus introduce additional vectors \(\gamma^* = (\gamma^*_1, \ldots, \gamma^*_T)\) and \(\delta^* = (\delta^*_1, \ldots, \delta^*_T)\) of Lagrange multipliers to deal respectively with the additional functional constraints (21) and (22). Arguing as before we have the further complementary slackness conditions (in addition to (11) and (12))

\[
\gamma^*_t \leq 0, \quad \gamma^*_t = 0 \text{ whenever } u^*_t > 0, \quad 1 \leq t \leq T,
\]
\[
\delta^*_t \geq 0, \quad \delta^*_t = 0 \text{ whenever } v^*_t > 0, \quad 1 \leq t \leq T.
\]  \hspace{1cm} (23)
(24)
where \(u^*\) and \(v^*\) are the values of \(u\) and \(v\) at the solution \(S^*\) to the original problem \(P\). Again arguing as in the proof of Theorem 2, we now have that, for each \(1 \leq t \leq T\),

\[
x_t(S^*) \text{ minimises } C_t(x) - (\mu^*_t + \gamma^*_t + \delta^*_t)x \text{ in } x \in \mathbb{R},
\]  \hspace{1cm} (25)
where the vector \(\mu^* = (\mu^*_1, \ldots, \mu^*_T)\) remains as identified in Theorem 2—since the interpretations as derivatives of the Lagrange multipliers \(\alpha^*\) and \(\beta^*\) of that theorem remain
unchanged and $\mu^*$ remains as identified by (15). (We observe in passing that the relation (25) stands formally in contrast to the result in the proof of Theorem 2 where, from (14), $x_t(S^*)$ minimised $C_t(x) - \mu^*_t x$ in $x \in X$).

We now note that, once again from the differentiability assumptions of the present theorem, and standard Lagrangian theory,

$$\frac{\partial V^*}{\partial P_i} = \sum_{t \in \tau_i} \gamma^*_t.$$

Further, for $t \in \tau_i$, we have $v^*_t = P_i + P_o > 0$ and so $\delta^*_t = 0$ (from (24)) and also $C'_t(P_i) = \mu^*_t + \gamma^*_t$ (from (25)). The result (19) now follows. The result (20) follows similarly.

Remark 1. Note that the results (19) and (20) of Theorem 5 are also intuitively clear from the interpretation of $\mu^*_t$ given in Section 3 as a notional unit reference value for additions to the store at each time $t$. Thus for (19), note that, for each $t \in \tau_i$, increasing the maximum input rate $P_i$ by $dP_i$ permits the addition of increased value $\mu^*_t dP_i$—corresponding to the addition to the level of the store—at a cost of $C'_t(P_i)dP_i$.

6 Examples: pumped storage

In this section we illustrate some of our results with an example storage facility using parameters motivated by the Dinorwig pumped-storage power station in Snowdonia, north Wales—see [23] for a good description of this power station and its uses. (Note, however, that Dinorwig is not currently primarily used for price arbitrage, but rather for the provision of fast response services to the GB energy network.) We use half-hourly time units and the “small store” cost structure (1), with $c_s(t) = 0$ for all $t$, reflecting the approximate efficiency $\eta = 0.75$ of the Dinorwig plant. We assume also a common input and output rate constraint $P_i = P_o = P$, say. The cost series are proportional to the real half-hourly spot market wholesale electricity prices during the period corresponding to the example. As might be expected these prices show a strong daily cyclical behaviour. It follows from, for example, the condition (ii) of Theorem 1, that, for the “small store” essentially linear cost structure (1), the optimal control is (or may be taken to be) bang-bang. Thus in each time period $t$ the store either buys the maximum permitted by its rate and capacity constraints, sells the maximum, or does nothing—according to whether respectively the current unit reference value $\mu^*_t$ of Theorem 1 (which exists by Theorem 2) is greater than $c^{(b)}_t$, less that $c^{(s)}_t$, or lies between these two values. (This “threshold” behaviour in the case where the cost structure is linear has been noted by other authors—see, for example, [12]).

It follows from simple scaling arguments that the optimal strategy in the “small store” case depends on $E$ and $P$ through the ratio $E/P$. In the first of our examples we take the ratio $E/P$ to correspond to 10 half-hourly periods—the total length of time which the Dinorwig facility takes to either fill or empty. Figure 2 shows the two price series $c^{(b)}_t$ and $c^{(s)}_t$ for the 7-day period Sunday 9 January 2011 to Saturday 15 January 2011 inclusive. The decisions to buy, sell or keep the level of the store unchanged are indicated by the blue and red dots for the buy and sell decisions. In the lower panel we show the series of storage values, $S_t$, over this one-week period. Each day storage is emptied when prices are sufficiently high and filled when prices are low. In our second example we double the capacity of the
storage plant so that the ratio $E/P$ corresponds to 20 half-hourly periods, keeping the
two price series as before. Figure 3 shows the corresponding plots. In particular we note
that doubling the ratio $E/P$ has the effect of ensuring that the storage facility is only
partially filled during some of the daily cycles, the precise behaviour depending on the
price fluctuations over the days concerned. Were the ratio $E/P$ to be further increased,
to the point where the capacity constraint were significantly less critical, we should expect
to see continuous nonempty storage over much longer periods of time, taking advantage
of price difference between, for example, different seasons of the year.
For some further numerical results in the context of this particular example, see [9].

7 A stochastic model

We now suppose that the cost functions $C_t$ evolve randomly in time. The general approach
to the determination of an optimal storage strategy is then via stochastic dynamic pro-
gramming. However, while general problems may be solved numerically, little analytical
progress is possible without further assumptions. Reasonable such assumptions might be
that uncertainties in future costs evolve multiplicatively as we proceed backwards in time.
This seems reasonable if buying and selling prices are determined by markets which evolve
similarly, as is usually the case. Further this probably represents as accurate a stochas-
tic model as may be formulated in practical applications. (Recall that, as we proceed
forwards in time, the stochastic structure of the future may be subject to revision as a
result of unmodelled external events, and that, at each successive time, there is then the
opportunity to re-optimise control strategies.) More precisely we assume that the cost
functions $C_t$ are given by

$$C_t = \xi_t \tilde{C}_t, \quad 1 \leq t \leq T,$$

where $(\tilde{C}_1, \ldots, \tilde{C}_T)$ is a sequence of deterministic cost functions and where $(\xi_1, \ldots, \xi_T)$ is
a sequence of strictly positive real-valued random variables forming a martingale, i.e. such that

$$E(\xi_t | F_{t-1}) = \xi_{t-1}, \quad 1 \leq t \leq T;$$

(26)

here $E$ denotes expectation and each $F_t$ is the $\sigma$-algebra generated by $\xi_1, \ldots, \xi_t$ (with $F_0$
the trivial $\sigma$-algebra). Note that, since the functions $\tilde{C}_t$ may if necessary be rescaled, there
is no loss of generality in omitting a multiplicative constant from (26). The deterministic
functions $\tilde{C}_t$ are assumed to satisfy the same conditions as the cost functions $C_t$ of the
deterministic problem given in Section 2, and hence the random cost functions $C_t$ also
satisfy these conditions.

The optimization problem $P$ of Section 2 now becomes

$P$: choose the random vector $S = (S_1, \ldots, S_T)$, with $S_t \in F_t$ for each $t$, so as to minimise

$$G(S) := E \left[ \sum_{t=1}^{T} C_t(x_t(S)) \right]$$

(27)

with $S_0 = S_0^*$ and $S_T = S_T^*$ (where $S_0$ and $S_T$ are fixed constants as previously),
and again subject to the capacity constraints

$$0 \leq S_t \leq E, \quad 1 \leq t \leq T - 1.$$
Figure 2: Example in which $E/P$ corresponds to 10 half-hourly periods: plots of price series with buy and sell times and of corresponding level of storage.
Figure 3: Example in which $E/P$ corresponds to 20 half-hourly periods: plots of price series with buy and sell times and of corresponding level of storage.
Note in particular that each \(S_t\) (or, equivalently, each \(x_t(S)\)) may be chosen based on the knowledge of the realised random variables \(\xi_1, \ldots, \xi_t\) to time \(t\). We now have the following result.

**Theorem 6.** The solution to the above problem remains deterministic, with the optimal sequence of store levels as given in the case where stochastic cost functions \(C_t\) are replaced by their deterministic counterparts \(\bar{C}_t\). Further the optimized value of the objective function (27) is the same as that for the deterministic variation of the problem.

**Remark 2.** This result is intuitively clear, since the stochastic aspect of the problem can be characterised as consisting of, at each successive time, a random but uniform scaling of all future costs, and each such scaling cannot change the optimal strategy. However, a formal proof is required.

**Proof of Theorem 6.** Consider first the case in which the stochastic cost functions \(C_t\) are replaced by their deterministic counterparts \(\bar{C}_t\). For each \(0 \leq t \leq T - 1\), and each fixed \(S_t\) such that \(0 \leq S_t \leq E\), with \(S_0 = S_0^*\), define

\[
\bar{V}_t(S_t) = \min_{S_{t+1}, \ldots, S_{T-1}} \sum_{u=t+1}^{T} \bar{C}_u(x_u(S)),
\]

where \(S = (S_t, \ldots, S_T)\) and, for each \(u > t\), we have \(0 \leq S_u \leq E\) with \(S_T = S_T^*\) and \(x_u(S) = S_u - \rho S_{u-1}\). Define also \(\bar{V}_T(S_T^*) = 0\). Thus \(\bar{V}_t(S_t)\) represents optimised future costs at time \(t\) given that the level of the store is then \(S_t\). Then, by the usual dynamic programming recursion, we have

\[
\bar{V}_t(S_t) = \min_{x_{t+1}} \left[ \bar{C}_{t+1}(x_{t+1}) + \bar{V}_{t+1}(\rho S_t + x_{t+1}) \right], \quad 0 \leq t \leq T - 1,
\]

where the above minimisation is taken over \(x_{t+1}\) such that \(0 \leq \rho S_t + x_{t+1} \leq E\) for \(0 \leq t \leq T - 2\) and \(\rho S_{T-1} + x_T = E\).

In the general stochastic case define similarly, for \(0 \leq t \leq T - 1\), and each fixed \(S_t\) such that \(0 \leq S_t \leq E\), again with \(S_0 = S_0^*\),

\[
V_t(S_t) = \mathbb{E} \left[ \min_{S_{t+1}, \ldots, S_{T-1}} \sum_{u=t+1}^{T} C_u(x_u(S)) \Big| \mathcal{F}_t \right],
\]

where the random vector \(S = (S_t, \ldots, S_T)\) and, for each \(u > t\), we have \(S_u \in \mathcal{F}_u\) and \(0 \leq S_u \leq E\) with \(S_T = S_T^*\) and where \(x_u(S) = S_u - \rho S_{u-1}\). Define also \(V_T(S_T^*) = 0\). Thus again \(V_t(S_t)\) represents optimised future costs at time \(t\) given that the level of the store is then \(S_t\).

We now assert that, for each \(t\) and \(S_t\) as above,

\[
V_t(S_t) = \xi_t \bar{V}_t(S_t).
\]

The proof of this assertion is by backwards induction in time \(t\). The result is trivially true for \(t = T\). Assume now that it is true for \(t = u + 1\), where \(0 \leq u \leq T - 1\). Then,
analogously to (28),

\[
V_u(S_u) = \mathbb{E} \left[ \min_{x_{u+1} \in F_{u+1}} \left( C_{u+1}(x_{u+1}) + V_{u+1}(\rho S_u + x_{u+1}) \right) \bigg| F_u \right] \\
= \mathbb{E} \left[ \min_{x_{u+1} \in F_{u+1}} \xi_{u+1} [\tilde{C}_{u+1}(x_{u+1}) + \tilde{V}_{u+1}(\rho S_u + x_{u+1})] \bigg| F_u \right] \\
= \mathbb{E} \left[ \xi_{u+1} \tilde{V}_u(S_u) \bigg| F_u \right] \\
= \xi_u \tilde{V}_u(S_u),
\]

where the above minimisation is taken over \( x_{u+1} \in F_{u+1} \) such that \( 0 \leq \rho S_u + x_{u+1} \leq E \), with \( \rho S_{T-1} + x_T = E \) in the case \( u = T - 1 \), and where (31) and (32) follow from (26) and (30) respectively. Hence the assertion (30) holds for all \( t \) and for all \( S_t \).

Note also that, from iteration of the argument leading to (32), for each \( t \) and \( S_t \), the optimising values of \( S_{t+1}, \ldots, S_{T-1} \) are as in the deterministic case. The theorem now follows from this observation and from (30) in the case \( t = 0 \).

8 Commentary and conclusions

In the preceding sections we have developed the optimization theory associated with the use of storage for arbitrage, and given an algorithm for determining the optimal control policy for, and hence the value of, storage when used for this purpose. In particular our algorithm captures the fact that the control policy is essentially local in time, in that, for a given system subject to given capacity and rate constraints, at each time optimal decisions are dependent only on the relevant cost functions for what is typically a very short time horizon.

Our model accounts for nonlinear cost functions, rate constraints, storage inefficiencies, and the effect of externalities caused by the activities of the store impacting the market. It further accounts for leakage over time from the store—something which may be expected to substantially further localise over time the character of optimal control policies. While our main model is deterministic in that it assumes that all the prices determining the cost functions are known in advance, we have also considered a realistic stochastic model, for which it is shown that the optimal one-step control decision at any point in time is given by replacing future cost functions by their expected values, and proceeding as in the deterministic case. In a more general stochastic model, we should expect the same result to be true to a good first approximation.

What we have not done in the present paper is to consider the use of storage for providing a reserve in case of unexpected system shocks, such as sudden surges in demand or shortfalls in supply. This problem is considered by other authors (see, for example, [5, 10, 11]) in the case where the probabilities of storage underflows or overflows are controlled to fixed levels. However, we believe that a further approach here would be to attach economic values to such underflows or overflows, translating to attaching an economic worth to the absolute level the store (as opposed to attaching a worth to a change in the level of the store as in the present paper). Since in practice storage is used both for arbitrage and for buffering or control as described above, this would provide a more integrated approach to the full economic valuation of such storage.
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