Randomized double and triple Kaczmarz for solving extended normal equations

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Abstract

The randomized Kaczmarz algorithm has received considerable attention recently because of its simplicity, speed, and the ability to approximately solve large-scale linear systems of equations. In this paper we propose randomized double and triple Kaczmarz algorithms to solve extended normal equations of the form $A^\top Ax = A^\top b - c$. The proposed algorithms avoid forming $A^\top A$ explicitly and work for arbitrary $A \in \mathbb{R}^{m \times n}$ (full rank or rank deficient, $m \geq n$ or $m < n$). Tight upper bounds showing exponential convergence in the mean square sense of the proposed algorithms are presented and numerical experiments are given to illustrate the theoretical results.

Keywords. Extended normal equations, Randomized Kaczmarz, Exponential convergence, Tight upper bounds

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1 Introduction

We consider the following extended normal equations

$$A^\top Ax = A^\top b - c$$

with arbitrary $A \in \mathbb{R}^{m \times n}$ (full rank or rank deficient, $m \geq n$ or $m < n$), $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. The linear system (1) arises in some applications, such as multilevel Levenberg-Marquardt methods for training artificial neural networks [4] or Fletcher’s exact penalty function approach [8]. Because of the existence of $c$, standard methods for least squares problems can not be used directly. Conjugate gradient-type methods based on full matrix-vector multiplications for solving (1) with full column rank $A$ have been proposed recently in [5]. However, these methods are not feasible when full matrix-vector multiplications are not available or “expensive” to obtain (e.g., the data matrix $A$ is dynamically growing or so large that it does not fit in computer memory).

In recent years, randomized iterative algorithms for solving large-scale linear systems or linear least squares problems have been greatly developed due to low memory footprints (these methods do not need to load the entire coefficient matrix into memory, and each iteration only requires a sample of rows and/or columns) and good numerical performance, such as the randomized Kaczmarz (RK) algorithm [16], the randomized coordinate descent algorithm [9], and their extensions, e.g., [20, 10, 14, 11, 2, 3, 11, 13, 19, 7, 12, 15, 17, 18]. In this paper, we propose two variants of the randomized Kaczmarz algorithm [16] to solve the extended normal equations (1). More specifically, we propose a randomized double Kaczmarz (RDK) algorithm for a solution of the linear system (1) if it is consistent ($c \in \text{range}(A^\top)$) and a randomized triple Kaczmarz (RTK) algorithm for a least squares solution of the linear system (1) if it is
inconsistent \((c \notin \text{range}(A^\top))\). We make no assumptions about the dimensions or rank of \(A\). We present tight upper bounds for the exponential convergence in the mean square sense of the proposed algorithms.

The organization of this paper is as follows. In the rest of this section, we give notation and preliminary. In Section 2, we review the RK algorithm. In Section 3 we describe the RDK algorithm and the RTK algorithm, and we also establish their convergence theory. In Section 4 we report the numerical results to illustrate the theoretical results. Finally, we present brief concluding remarks in Section 5.

**Notation and preliminary.** For any random variable \(\xi\), we use \(E[\xi]\) to denote the expectation of \(\xi\). For an integer \(m \geq 1\), let \([m] := \{1, 2, 3, \ldots, m\}\). For any vector \(b \in \mathbb{R}^m\), we use \(b_i, b^\top\) and \(\|b\|_2\) to denote the \(i\)th entry, the transpose and the Euclidean norm of \(b\), respectively. We use \(I\) to denote the identity matrix whose order is clear from the context. For any matrix \(A \in \mathbb{R}^{m \times n}\), we use \(A_{i,:}, A_{:,j} A^\top, A^\dagger, \|A\|_2, \|A\|_F, \text{range}(A), \text{rank}(A), \sigma_{\max}(A)\) and \(\sigma_{\min}(A)\) to denote the \(i\)th row, the \(j\)th column, the transpose, the Moore-Penrose pseudoinverse, the 2-norm, the Frobenius norm, the column space, the rank, the maximum and the minimum nonzero singular values of \(A\), respectively. All the convergence results depend on the positive number \(\rho\) defined as

\[
\rho := 1 - \frac{\sigma_{\min}^2(A)}{\|A\|_F^2}.
\]

For any nonzero matrix \(A\) and any \(u \in \text{range}(A^\top)\), it holds

\[
u^\top \left( I - \frac{A^\top A}{\|A\|_F^2} \right) u \leq \rho \|u\|_2^2. \tag{2}\]

### 2 Randomized Kaczmarz

In each iteration, the RK algorithm orthogonally projects the current estimate vector onto the affine hyperplane defined by a randomly chosen row of \(Ax = b\). See Algorithm 1 for details. Theorem 1 shows that the sequence \(\{x^k\}_{k=0}^\infty\) in the RK algorithm with arbitrary initial vector \(x^0 \in \mathbb{R}^n\) for a consistent linear system \(Ax = b\) converges to \(x^*_0 = (I - A^\dagger A)x^0 + A^\dagger b\), which is the orthogonal projection of \(x^0\) onto the solution set \(\{x \in \mathbb{R}^n \mid Ax = b\}\). We emphasize that we make no assumptions about the dimensions or rank of \(A\). The proof of Theorem 1 can be found in, e.g., [6, 20, 13]. For completeness and clarity, we provide a proof.

**Algorithm 1: RK for \(Ax = b\)**

1. Initialize \(x^0 \in \mathbb{R}^n\)
2. For \(k = 1, 2, \ldots\)
   1. Pick \(i \in [m]\) with probability \(\|A_{i,:}\|_2^2 / \|A\|_F^2\)
   2. Set \(x^k = x^{k-1} - \frac{A_{i,:}x^{k-1} - b_i}{\|A_{i,:}\|_2^2} (A_{i,:})^\top\)

**Theorem 1.** Suppose that \(b \in \text{range}(A)\) (i.e., \(Ax = b\) is consistent). The sequence \(\{x^k\}_{k=0}^\infty\) in the RK algorithm with arbitrary \(x^0 \in \mathbb{R}^n\) satisfies

\[
E \left[ \|x^k - x^*_0\|_2^2 \right] \leq \rho^k \|x^0 - x^*_0\|_2^2, \tag{3}
\]

where \(x^*_0 = (I - A^\dagger A)x^0 + A^\dagger b\).
Proof. By $Ax^0_0 = b$, we have

$$x^k - x^0_0 = x^{k-1} - x^0_0 - \frac{A_i : x^{k-1} - b_i}{\|A_i\|_2^2} (A_i)^\top$$

$$= x^{k-1} - x^0_0 - \frac{A_i : x^{k-1} - A_i x^0_0}{\|A_i\|_2^2} (A_i)^\top$$

$$= \left( I - \frac{(A_i)^\top A_i}{\|A_i\|_2^2} \right) (x^{k-1} - x^0_0).$$

(4)

It follows that

$$\|x^k - x^0_0\|_2^2 = (x^{k-1} - x^0_0)^\top \left( I - \frac{(A_i)^\top A_i}{\|A_i\|_2^2} \right) (x^{k-1} - x^0_0).$$

Taking condition expectation gives

$$\mathbb{E} \left[ \|x^k - x^0_0\|_2^2 | x^{k-1} \right] = (x^{k-1} - x^0_0)^\top \left( I - \frac{A^\top A}{\|A\|_F^2} \right) (x^{k-1} - x^0_0).$$

Noting that $x^0 - x^0_0 = A^\top (Ax^0 - b) \in \text{range}(A^\top)$ and $(A_i)^\top A_i (x^{k-1} - x^0_0) \in \text{range}(A^\top)$, by (4), we can show that $x^k - x^0_0 \in \text{range}(A^\top)$ by induction. Then by (2), we have

$$\mathbb{E} \left[ \|x^k - x^0_0\|_2^2 | x^{k-1} \right] \leq \rho \|x^{k-1} - x^0_0\|_2^2.$$

By the law of total expectation we have

$$\mathbb{E} \left[ \|x^k - x^0_0\|_2^2 \right] \leq \rho \mathbb{E} \left[ \|x^{k-1} - x^0_0\|_2^2 \right] \leq \cdots \leq \rho^k \|x^0 - x^0_0\|_2^2.$$

This completes the proof.

Remark 2. If $\sigma_{\max}(A) = \sigma_{\min}(A)$, then the inequality [2] becomes equality. This yields that all the inequalities in the proof of Theorem 1 become equalities. Therefore, the convergence bound in Theorem 1 is tight.

3 Algorithms and main results

3.1 The RDK algorithm for the case $c \in \text{range}(A^\top)$

The randomized extended Kaczmarz (REK) algorithm [20] solves $A^\top A x = A^\top b$ via intertwining an iterate of RK on $A^\top z = 0$ with an iterate of RK on $Ax = b - z$. More precisely, the $k$th iterate of the REK algorithm, $x^k$, is the iterate of RK on $Ax = b - z^k$ from $x^{k-1}$, where $z^k$ is the $k$th iterate of RK on $A^\top z = 0$ with $z^0 \in b + \text{range}(A)$. Inspired by the REK algorithm, we propose Algorithm 2 to solve the problem (1) for the case $c \in \text{range}(A^\top)$. We note that $z^k$ in Algorithm 2 is the $k$th iterate of RK on $A^\top z = c$ with $z^0 \in b + \text{range}(A)$, and $x^k$ is the iterate of RK on $Ax = b - z^k$ from $x^{k-1}$ with arbitrary $x^0 \in \mathbb{R}^n$. Since two RK iterates are used in each iteration of Algorithm 2, we call it a randomized double Kaczmarz (RDK) algorithm. By (3), we have

$$\mathbb{E} \left[ \|z^k - z^0_0\|_2^2 \right] \leq \rho^k \|z^0 - z^0_0\|_2^2,$$

(5)

where

$$z^0_0 = (I - AA^\dagger)z^0 + (A^\top)^\dagger c = (I - AA^\dagger)b + (A^\top)^\dagger c.$$
Algorithm 2: RDK for $A^\top Ax = A^\top b - c$ with $c \in \text{range}(A^\top)$

Initialize $z^0 \in b + \text{range}(A)$ and $x^0 \in \mathbb{R}^n$
for $k = 1, 2, \ldots$ do
  Pick $j \in [n]$ with probability $\|A_{:,j}\|^2/\|A\|^2_F$
  Set $z^k = z^{k-1} - (A_{:,j})^\top z^{k-1} - c_j A_{:,j}$
  Pick $i \in [m]$ with probability $\|A_{i,:}\|^2/\|A\|^2_F$
  Set $x^k = x^{k-1} - \frac{A_{i,:}x^{k-1} - b_i + z^k_i (A_{i,:})^\top}{\|A_{i,:}\|^2_F}$

Theorem 3. Suppose that $c \in \text{range}(A^\top)$ (i.e., the linear system $[1]$ is consistent). The sequence $\{x^k\}_{k=0}^\infty$ in the RDK algorithm with $z^0 \in b + \text{range}(A)$ and arbitrary $x^0 \in \mathbb{R}^n$ satisfies

$$\mathbb{E} \left[ \|x^k - x^*_o\|^2 \right] \leq \frac{k \rho^k}{\|A\|^2_F} \|z^0 - z^*_o\|^2 + k \rho^k \|x^0 - x^*_o\|^2,$$

where $z^0_o = (I - AA^\dagger)b + (A^\top)^\dagger c$, and $x^0_o = (I - A^\dagger A)x^0 + A^\dagger b - (A^\top)^\dagger c$ is a solution of $[1]$.

Proof. Let

$$\hat{x}^k = x^{k-1} - \frac{A_{i,:}(x^{k-1} - A^\dagger b + (A^\top)^\dagger c) (A_{i,:})^\top}{\|A_{i,:}\|^2_F}. \quad (6)$$

By $A(A^\top A)^\dagger = (A^\top)^\dagger$, we have

$$x^k - \hat{x}^k = \frac{b_i - A_{i,:} A^\dagger b + A_{i,:} (A^\top)^\dagger c - z^k_i (A_{i,:})^\top}{\|A_{i,:}\|^2_F}$$

$$= \frac{I_{i,:}((I - AA^\dagger)b + (A^\top)^\dagger c - z^k) (A_{i,:})^\top}{\|A_{i,:}\|^2_F}$$

$$= \frac{I_{i,:}(z^k_o - z^k) (A_{i,:})^\top}{\|A_{i,:}\|^2_F}. \quad (7)$$

and by $A(I - A^\dagger A)x^0 = 0$, we have

$$\hat{x}^k - x^*_o = x^{k-1} - x^*_o - \frac{A_{i,:}(x^{k-1} - (I - A^\dagger A)x^0 - A^\dagger b + (A^\top)^\dagger c) (A_{i,:})^\top}{\|A_{i,:}\|^2_F}$$

$$= x^{k-1} - x^*_o - \frac{A_{i,:}(x^{k-1} - x^*_o) (A_{i,:})^\top}{\|A_{i,:}\|^2_F}$$

$$= \left( I - \frac{(A_{i,:})^\top A_{i,:}}{\|A_{i,:}\|^2_F} \right) (x^{k-1} - x^*_o). \quad (8)$$

By the orthogonality $(\hat{x}^k - x^*_o)^\top (x^k - \hat{x}^k) = 0$ (which is obvious from $[7]$ and $[8]$), we have

$$\|x^k - x^*_o\|^2 = \|x^k - \hat{x}^k\|^2 + \|\hat{x}^k - x^*_o\|^2. \quad (9)$$

Let $E_{k-1}[]$ denote the conditional expectation given the first $k - 1$ iterations of RDK. Let $E_{k-1}^{j}[\cdot]$ denote the expectation with respect to the $k$th row chosen and $E_{k-1}^{j}[\cdot]$ denote the expectation with respect to the $k$th column chosen. Then by the law of total expectation we have $E_{k-1}[\cdot] = E_{k-1}^{j}[E_{k-1}^{j}[\cdot]]$. It follows from

$$E_{k-1}[\|x^k - \hat{x}^k\|^2_F] = E_{k-1}\left[ \left( \frac{I_{i,:}(z^k_o - z^k)^2}{\|A_{i,:}\|^2_F} \right) \right] = E_{k-1}^{j}\left[ E_{k-1}^{j}[\|x^k - x^*_o\|^2_F] \right]$$

$$= \frac{1}{\|A\|^2_F} E_{k-1} [\|z^k - z^*_o\|^2_F]$$
that
\[ \mathbb{E} \left[ \|x^k - \hat{x}^k\|_2^2 \right] = \frac{1}{\|A\|_F^2} \mathbb{E} \left[ \|z^k - z_0^k\|_2^2 \right] \leq \rho^k \frac{\rho}{\|A\|_F^2} \left\| z^0 - z_0^0\right\|_2^2. \] (by (5))

By \( x^0 - x_0^0 \in \text{range}(A^\top) \), it is easy to show that \( x^{k-1} - x_0^0 \in \text{range}(A^\top) \) by induction. It follows from
\[
\mathbb{E}_{k-1} \left[ \|\hat{x}^k - x_0^0\|_2^2 \right] = \mathbb{E}_{k-1} \left[ (\hat{x}^k - x_0^0)^\top (\hat{x}^k - x_0^0) \right]
= \mathbb{E}_{k-1} \left[ (x^{k-1} - x_0^0)^\top \left( I - \frac{(A_{i,:})\top A_{i,:}}{\|A_{i,:}\|_2^2} \right) (x^{k-1} - x_0^0) \right]
= \mathbb{E}_{k-1} \left[ (x^{k-1} - x_0^0)^\top \left( I - \frac{A_{i,:}\top A_{i,:}}{\|A_{i,:}\|_2^2} \right) (x^{k-1} - x_0^0) \right]
\leq \rho \|x^{k-1} - x_0^0\|_2^2 \quad \text{(by (2))}
\]
that
\[ \mathbb{E} \left[ \|\hat{x}^k - x_0^0\|_2^2 \right] \leq \rho \mathbb{E} \left[ \|x^{k-1} - x_0^0\|_2^2 \right]. \] (11)

Combining (9), (10), and (11) yields
\[
\mathbb{E} \left[ \|x^k - x_0^0\|_2^2 \right] = \mathbb{E} \left[ \|x^k - \hat{x}^k\|_2^2 \right] + \mathbb{E} \left[ \|\hat{x}^k - x_0^0\|_2^2 \right]
\leq \frac{\rho^k}{\|A\|_F^2} \left\| z^0 - z_0^0\right\|_2^2 + \rho \mathbb{E} \left[ \|x^{k-1} - x_0^0\|_2^2 \right]
\leq \frac{2\rho^k}{\|A\|_F^2} \left\| z^0 - z_0^0\right\|_2^2 + \rho^2 \mathbb{E} \left[ \|x^{k-2} - x_0^0\|_2^2 \right]
\leq \cdots
\leq \frac{k\rho^k}{\|A\|_F^2} \left\| z^0 - z_0^0\right\|_2^2 + \rho^k \left\| x^0 - x_0^0\right\|_2^2.
\]

It is trivial to verify that \( x_0^0 \) is a solution of (1). This completes the proof. \( \Box \)

**Remark 4.** If \( \sigma_{\text{max}}(A) = \sigma_{\text{min}}(A) \), then the inequalities (2) and (3) become equalities. This yields that all the inequalities in the proof of Theorem 3 become equalities. Therefore, the convergence bound in Theorem 3 is tight.

### 3.2 The RTK algorithm for the case \( c \notin \text{range}(A^\top) \)

Given \( U \in \mathbb{R}^{m \times k} \), \( V \in \mathbb{R}^{k \times n} \) and \( y \in \mathbb{R}^m \), the REK-RK algorithm \( \square \) Algorithm 2 \( \square \) solves the factorized linear system \( UVx = y \) for the case \( y \notin \text{range}(U) \) via intertwining an iterate of REK for solving \( U^\top Uz = U^\top y \) with an iterate of RK on \( Vx = z \). Inspired by the REK-RK algorithm, we propose Algorithm 3 for the linear system (1) with \( c \notin \text{range}(A^\top) \). We note that \( y^k \) in Algorithm 3 is the \( k \)th iterate of RK on \( Ay = 0 \) with \( y^0 \in c + \text{range}(A^\top) \), \( z^k \) is the iterate of RK on \( A^\top z = c - y^k \) from \( z^{k-1} \) with \( z^0 \in b + \text{range}(A) \), and \( x^k \) is the iterate of RK on \( Ax = b - z^k \) from \( x^{k-1} \) with arbitrary \( x^0 \in \mathbb{R}^n \). Since three RK iterates are used in each iteration of Algorithm 3, we call it a randomized triple Kaczmarz (RTK) algorithm. Actually, \( y^k \) and \( z^k \) of RTK are exactly the iterates of RDK applied for the system \( AA^\top z = Ac \) (or \( Ay = 0 \) and \( A^\top z = c - y \)). By Theorem 3 we have
\[ \mathbb{E} \left[ \|z^k - z_0^k\|_2^2 \right] \leq \frac{k\rho^k}{\|A\|_F^2} \left\| y^0 - y_0^0\right\|_2^2 + \rho^k \left\| z^0 - z_0^0\right\|_2^2, \] (12)
where \( y^*_0 = (I - A^\dagger A)c \) and \( z^0 = (I - AA^\dagger)b + (A^\top)^\dagger c \). We show that the sequence \( \{x^k\}_{k=0}^\infty \) in the RTK algorithm converges to a least squares solution of (1) in Theorem 5. We emphasize that we make no assumptions about the dimensions or rank of \( A \).

**Algorithm 3:** RTK for \( A^\top Ax = A^\top b - c \) with \( c \not\in \text{range}(A^\top) \)

Initialize \( y^0 \in c + \text{range}(A^\top) \), \( z^0 \in b + \text{range}(A) \), and \( x^0 \in \mathbb{R}^n \)

for \( k = 1, 2, \ldots \) do

Pick \( i \in [m] \) with probability \( \|A_{i,:}\|_2^2/\|A\|_F^2 \)

Set \( y^k = y^{k-1} - A_{i,:}y^{k-1}/\|A_{i,:}\|_2^2 (A_{i,:})^\top \)

Pick \( j \in [n] \) with probability \( \|A_{.,j}\|_2^2/\|A\|_F^2 \)

Set \( z^k = z^{k-1} - (A_{.,j})^\top z^{k-1} - c_j + y^k \)

Pick \( i \in [m] \) with probability \( \|A_{i,:}\|_2^2/\|A\|_F^2 \)

Set \( x^k = x^{k-1} - A_{i,:}x^{k-1} - b_i + z^k \)

end do

**Theorem 5.** Suppose that \( c \not\in \text{range}(A^\top) \) (i.e., the linear system (1) is inconsistent). The sequence \( \{x^k\}_{k=0}^\infty \) in the RTK algorithm with \( y^0 \in c + \text{range}(A^\top) \), \( z^0 \in b + \text{range}(A) \), and arbitrary \( x^0 \in \mathbb{R}^n \) satisfies

\[
\mathbb{E} \left[ \|x^k - x^*_0\|_2^2 \right] \leq \frac{k(k + 1)\rho^k}{2\|A\|_F^2} \|y^0 - y^*_0\|_2^2 + \frac{k\rho^k}{\|A\|_F^2} \|z^0 - z^*_0\|_2^2 + \rho^k \|x^0 - x^*_0\|_2^2,
\]

where \( y^*_0 = (I - A^\dagger A)c \), \( z^*_0 = (I - AA^\dagger)b + (A^\top)^\dagger c \), and \( x^0 \) is a least squares solution of (1).

**Proof.** Let \( \hat{x}^k \) be the vector given in (6). We note that the equalities (7)–(9), and the inequality (11) in the proof of Theorem 3 still hold. By (12), the estimate (10) becomes

\[
\mathbb{E} \left[ \|x^k - \hat{x}^k\|_2^2 \right] = \frac{1}{\|A\|_F^2} \mathbb{E} \left[ \|z^k - z^*_0\|_2^2 \right] \leq \frac{k\rho^k}{\|A\|_F^2} \|y^0 - y^*_0\|_2^2 + \frac{\rho^k}{\|A\|_F^2} \|z^0 - z^*_0\|_2^2. \tag{13}
\]

Combining (9), (11), and (13) yields

\[
\mathbb{E} \left[ \|x^k - x^*_0\|_2^2 \right] = \mathbb{E} \left[ \|x^k - \hat{x}^k\|_2^2 \right] + \mathbb{E} \left[ \|\hat{x}^k - x^*_0\|_2^2 \right] \\
\leq \frac{k\rho^k}{\|A\|_F^2} \|y^0 - y^*_0\|_2^2 + \frac{\rho^k}{\|A\|_F^2} \|z^0 - z^*_0\|_2^2 + \rho \mathbb{E} \left[ \|x^{k-1} - x^*_0\|_2^2 \right] \\
\leq \frac{k\rho^k}{\|A\|_F^2} \|y^0 - y^*_0\|_2^2 + \frac{(k-1)\rho^k}{\|A\|_F^2} \|y^0 - y^*_0\|_2^2 \\
+ \frac{2\rho^k}{\|A\|_F^2} \|z^0 - z^*_0\|_2^2 + \rho^2 \mathbb{E} \left[ \|x^{k-2} - x^*_0\|_2^2 \right] \\
\leq \cdots \\
\leq \frac{k(k + 1)\rho^k}{2\|A\|_F^2} \|y^0 - y^*_0\|_2^2 + \frac{k\rho^k}{\|A\|_F^2} \|z^0 - z^*_0\|_2^2 + \rho^k \|x^0 - x^*_0\|_2^2.
\]

It is trivial to verify that \( x^0 \) is a least squares solution of (1). Then we complete the proof. \( \square \)

**Remark 6.** If \( \sigma_{\max}(A) = \sigma_{\min}(A) \), then the inequalities (2) and (12) become equalities. This yields that all the inequalities in the proof of Theorem 5 become equalities. Therefore, the convergence bound in Theorem 5 is tight.
4 Numerical results

In this section, we report the numerical results of the RDK algorithm and the RTK algorithm for solving (1). The purpose is to illustrate our theoretical results (Theorems 3 and 5) via simple examples. All experiments are performed using MATLAB on a laptop with 2.7-GHz Intel Core i7 processor, 16-GB memory, and Mac operating system.

The matrix $A$ and the vectors $b$ and $c$ in (1) are generated by using the MATLAB functions \texttt{diag}, \texttt{null}, \texttt{ones}, \texttt{qr}, \texttt{rand}, and \texttt{randn} as follows. Given $m$, $n$, $r = \text{rank}(A)$, and $\kappa \geq 1$, we construct the matrix $A$ by $A = UV^\top$, where $U \in \mathbb{R}^{m \times r}$, $D \in \mathbb{R}^{r \times r}$ and $V \in \mathbb{R}^{n \times r}$ are given by $[U, \sim] = \text{qr} (\text{randn}(m, r), 0)$, $D = \text{diag} (\text{ones}(r, 1) + (\kappa - 1)*\text{rand}(r, 1))$ and $[V, \sim] = \text{qr} (\text{randn}(n, r), 0)$. So the condition number of $A$, which is defined as $\sigma_{\text{max}}(A)/\sigma_{\text{min}}(A)$, is upper bounded by $\kappa$. The vector $b$ is taken to be $b = \text{randn}(m, 1)$. For the case $c \in \text{range}(A^\top)$, the vector $c$ is constructed by $c = A^\top \text{randn}(m, 1)$. For the case $c \not\in \text{range}(A^\top)$, the vector $c$ is constructed by $c = \text{randn}(n, 1) + \text{null}(A) \text{randn}(n-r, 1)$.

In all experiments we use $y^0 = c$, $z^0 = b$, and $x^0 = 0$. In Figures 1 and 2 we plot the error $\|x^k - A^\dagger b + (A^\top A)^\dagger c\|^2_2$ (average of 50 independent trials) of RDK and RTK. For all cases, RDK and REK converge. In particular, for $\kappa = 1$, which means all nonzero singular values of $A$ are the same, the convergence bounds in Theorems 3 and 5 are attained (see Figure 1). All these experimental results support the theoretical findings presented in Theorems 3 and 5.

![Figure 1: The error $\|x^k - A^\dagger b + (A^\top A)^\dagger c\|^2_2$ (average of 50 independent trials) for $m = 500$, $n = 250$, $r = 150$, and $\kappa = 1$. Left: RDK for the case $c \in \text{range}(A^\top)$. Right: RTK for the case $c \not\in \text{range}(A^\top)$.](image)

5 Concluding remarks

In this work, we propose randomized iterative algorithms that solve the extended normal equations. We prove that the RDK algorithm exponentially converges to a solution of the extended normal equations for the consistent case and prove that the RTK algorithm exponentially converges to a least squares solution of the extended normal equations for the inconsistent case. Our convergence analysis applies to arbitrary matrix $A$ and the convergence upper bounds are attained for the case that all nonzero singular values of $A$ are the same. Numerical experiments confirm the theoretical results.

We remark that for the scenarios where $A$ is so large that it does not fit in computer memory, iterative methods based on full matrix-vector multiplications (e.g., Krylov subspace methods) are inefficient because the entire matrix $A$ must be accessed in each step (which leads huge
communication costs). If memory is a concern, the proposed RDK and RTK algorithms are appropriate alternatives because at each step only a sample of rows and columns are required.

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