Local regularity criterion of the Beale-Kato-Majda type for the 3D Euler equations

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Abstract
We prove a localized non blow-up theorem of the Beale-Kato-Majda type for the solution of the 3D incompressible Euler equations.

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1 Introduction
We consider the homogeneous incompressible Euler equations\(^8\) describing the fluid flows in \(\mathbb{R}^3\),

\[
\partial_t v + (v \cdot \nabla)v = -\nabla p, \quad \nabla \cdot v = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, +\infty),
\]

where \(v = (v_1(x,t), v_2(x,t), v_3(x,t))\) is the velocity of the fluid, and \(p = p(x,t)\) represents the pressure. The local in time well-posedness in the Sobolev space \(W^{k,p}(\mathbb{R}^3)\), \(k > 3/p + 1, 1 < p < +\infty\), for the Cauchy problem of the system \((1.1)\) is well-known by the result of Kato-Ponce\(^{10}\). The question of finite time singularity for such local in time classical solution is still an outstanding open problem (see e.g.\(^{15} 5 13\) for surveys of the problem and the related results). We say a local in time smooth solution \(v \in C([0,T]; W^{k,p}(\mathbb{R}^3))\), \(k > 3/p + 1, 1 < p < +\infty\), does not blow up (or equivalently regular) at \(t = T\) if

\[
\limsup_{t \uparrow T} \|v(t)\|_{W^{k,p}(\mathbb{R}^3)} < +\infty.
\]

The celebrated Beale-Kato-Majda criterion\(^{2}\) shows the local in time well-posedness estimates that \((1.2)\) is guaranteed if

\[
\int_0^T \|\omega(t)\|_{L^\infty(\mathbb{R}^3)} dt < +\infty, \quad \omega = \nabla \times v.
\]
See also [6, 7] for geometric type criterion. In particular in [14] Kozono-Taniuchi improved (1.3), replacing \( \| \omega(t) \|_{L^\infty(\mathbb{R}^3)} \) in (1.3) by a weaker norm \( \| \omega(t) \|_{BMO(\mathbb{R}^3)} \).

The aim of the present paper is to deduce a sufficient condition of local regularity, which localizes both of the previous criteria in [2, 14]. For this purpose we use the local BMO space. For \( r > 0 \) and \( x \in \mathbb{R}^n \) we denote \( B(x, r) = \{ y \in \mathbb{R}^n \mid |x - y| < r \} \), and \( B(r) = B(0, r) \) below. By \( BMO(B(r)) \) we denote the space of all \( u \in L^1(B(r)) \) such that

\[
|u|_{BMO(B(r))} = \sup_{x \in B(r)} \int_{B(z, r) \cap B(r)} |u - u_{B(z, r) \cap B(r)}| dy < +\infty,
\]

where we used the following notation for the average of \( u \) over \( \Omega \subset \mathbb{R}^n \).

\[
u_\Omega = \int_\Omega u \, dx.
\]

The space \( BMO(B(r)) \) will be equipped with the norm

\[
\| u \|_{BMO(B(r))} = |u|_{BMO(B(r))} + r^{-n} \| u \|_{L^1(B(1))}.
\]

Note that \( BMO(B(r)) \) is continuously embedded into \( L^q(B(r)) \) for all \( 1 \leq q < +\infty \). Indeed, in view of (B.3) in the appendix below it holds

\[
\| u \|_{L^p(B(r))} \leq cr^n \| u \|_{BMO(B(r))}.
\]

For simplicity we assume the possible blow-up occurs at the space-time origin \((0, 0)\), and consider the system (1.1) in \( B(\rho) \times (-\rho, 0) \) throughout the paper. Our aim in this paper is the proof of the following form of local regularity criterion.

**Theorem 1.1.** Let \( v \in C([-\rho, 0); W^{2, q}(B(\rho))) \cap L^\infty(-\rho, 0; L^2(B(\rho))), 3 < q < +\infty \), be a local solution of (1.1) in \( B(\rho) \times (-\rho, 0) \). If \( v \) satisfies

\[
(1.4) \quad \int_{-\rho}^{0} |\omega(s)|_{BMO(B(\rho))} \, ds < +\infty,
\]

then it holds \( v \in C([-\rho, 0); W^{2, q}(B(\rho))) \) for all \( 0 < r < \rho \).

**Remark 1.2.** From the obvious inequality \( |\omega(s)|_{BMO(B(\rho))} \leq 2 \| \omega(s) \|_{L^\infty(B(\rho))} \) we see that one can replace (1.3) by \( \int_{-\rho}^{0} |\omega(s)|_{L^\infty(B(\rho))} \, ds < +\infty \) in the above theorem. Thus, it generalizes both the original Beale-Kato-Majda criterion [2] and its improved version by Kozono-Taniuchi [14]. Moreover, Theorem 1.1 also provides substantial advantage over the global criterions of [2, 14] in the computational test of the blow-up (see e.g. [11, 12] and references therein) at a specific point in a domain, since we only need to compute the vorticities at points in a small neighborhood of that point, not at whole points in the region.
The contents of the paper is the following. In Section 2 we prove a localized version of the logarithmic Sobolev inequality. This is done by introducing suitable extension operator of functions defined on a ball to the whole domain of $\mathbb{R}^n$. In Section 3 we establish several multiplicative inequalities to be used later. These amount to localized version of the Calderón-Zygmund type inequality in $\mathbb{R}^n$, which enables us to estimate the gradient of velocity in terms of the vorticity with lower order integral of velocity. In Section 4 we prove a local $L^\infty L^p_x$ estimate for the vorticity. In order to do this we first prove a localized version of the Kozono-Taniuchi inequality(see [13] for the original global version). The vorticity estimate deduced in this section, combined with our assumption of local energy bound, implies $v \in L^\infty L^\infty_x$ locally, which is an important step for our proof of the main theorem. In Section 5, using the results of previous sections, we complete the proof of Theorem 1.1. This last part of the proof is based on the two new ingredients. One is the transform of the Euler system into new equations, which is similar to the one in our previous paper[4]. The other one is use a new iteration scheme of the Gronwall type. The corresponding iteration lemma is proved in Appendix A.

2 Local version of logarithmic Sobolev’s inequality

Our aim in this section is to prove the following local version of the logarithmic Sobolev inequality.

Lemma 2.1. Let $B(r)$ be a ball in $\mathbb{R}^n$ with the radius $r > 0$. For every $u \in W^{1,q}(B(r))$, $n < q < +\infty$, the following inequality holds true

$$(2.1) \quad \|u\|_{L^\infty(B(r))} \leq c(1+\|u\|_{BMO(B(r))}) \log \left(e+c\|\nabla u\|_{L^q(B(r))} + cr^{-1+\frac{n}{q}-\frac{2}{p}}\|u\|_{L^2(B(r))}\right)$$

with a constant $c > 0$ depending on $n$ and $q$.

In order to prove the above lemma we construct an extension operator, which is bounded with respect to both the $BMO$ norm and the Sobolev norm.

Lemma 2.2. Let $u \in W^{1,q}(B(r)) \cap BMO(B(r))$ with $B(r) \subset \mathbb{R}^n$. Let $\phi \in C_c^\infty(B(3r))$ denote a cut function such that $0 \leq \phi \leq 1$ in $\mathbb{R}^n$, $\phi \equiv 1$ on $B(2r)$ and $|\nabla \phi|^2 + |\nabla^2 \phi| \leq cr^{-2}$. We define an extension

$$(2.2) \quad U(x) := \begin{cases} u(x) & \text{if } x \in B(r) \\ u(T(x))\phi(x) & \text{if } x \in B(r)^c, \end{cases}$$

where $T : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ stands for the reflexion map

$$T(y) = \frac{r^2}{|y|^2}y, \quad y \in \mathbb{R}^n \setminus \{0\}.$$ 

Then, $U \in W^{1,q}(\mathbb{R}^n) \cap BMO$. In addition, the following estimates hold true

$$(2.3) \quad \|\nabla U\|_{L^q} \leq c(\|\nabla u\|_{L^q(B(r))} + r^{-1+\frac{n}{q}-\frac{2}{p}}\|u\|_{L^2(B(r))}),$$

$$(2.4) \quad |U|_{BMO} \leq c\|u\|_{BMO(B(r))}.$$
We first prove Lemma 2.1 assuming Lemma 2.2 is true.

**Proof of Lemma 2.1** We denote by $U$ the extension introduced in Lemma 2.2. According to Lemma 2.2 we get $U \in W^{1,q}(\mathbb{R}^n)$. In view of the logarithmic Sobolev embedding we infer

\[(2.5) \quad \|u\|_{L^\infty(B(r))} \leq \|U\|_{L^\infty} \leq c(1 + \|U\|_{BMO}) \log(e + \|\nabla U\|_q).\]

Estimating the right-hand side of (2.5) by means of (2.3) and (2.4), the assertion follows.

**Proof of Lemma 2.2** First let us provide some basic properties of the map $T$. We compute

\[\partial_j T_i(y) = \frac{r^2}{|y|^2} \delta_{ij} - 2r^2 \frac{y_i y_j}{|y|^4}, \quad y \in \mathbb{R}^n \setminus \{0\}, \quad i, j = 1, \ldots, n.\]

Furthermore we get for $k, l \in \{1, \ldots, n\}$

\[\partial_k T(y) \cdot \partial_l T(y) = \frac{r^4}{|y|^4} (\delta_{ik} - 2 \frac{y_i y_k}{|y|^2} ) (\delta_{il} - 2 \frac{y_i y_l}{|y|^2} ) = \frac{r^4}{|y|^4} (\delta_{kl} - 4 \frac{y_k y_l}{|y|^2} + 4 \frac{y_i y_j}{|y|^2} ) = \frac{r^4}{|y|^4} \delta_{kl}.\]

Accordingly,

\[(\det DT(y))^2 = \det(DT(y)^\top DT(y)) = \det \left( \frac{r^4}{|y|^4} I \right) = \left( \frac{r}{|y|} \right)^{4n}.\]

This show that

\[(2.6) \quad |\det DT(y)| = \left( \frac{r}{|y|} \right)^{2n}.\]

Next, we show that for every $x, y \in \mathbb{R}^n \setminus \{0\}$ it holds

\[(2.7) \quad |T(x) - T(y)| = r^2 \frac{|x - y|}{|x| |y|}.\]

Indeed, by an elementary calculus we find

\[|T(x) - T(y)|^2 = \left| \frac{r^2}{|x|^2} x - \frac{r^2}{|y|^2} y \right|^2 = \frac{r^4}{|x|^2 |y|^2} \left| \frac{y}{|y|} x - \frac{x}{|x|} y \right|^2 = \frac{r^4}{|x|^2 |y|^2} |x - y|^2.\]

Whence, (2.7).

Let $B(x_0, \rho) \subset \mathbb{R}^n$, $0 < \rho < r$ be any ball. We discuss the three cases (i) $B(x_0, \rho) \subset B(r)$, (ii) $B(x_0, \rho) \subset B(r)^c$, and (iii) $B(x_0, \rho) \cap \partial B(r) \neq \emptyset$ separately.
(i) The case $B(x_0, \rho) \subset B(r)$: As $U = u$ in $B(r)$ it follows that
\[
\int_{B(x_0, \rho)} |U(y) - U_{B(x_0, \rho)}|dy = \int_{B(x_0, \rho)} |u(y) - u_{B(x_0, \rho)}|dy \leq |u|_{BMO(B(r))}.
\]

(ii) The case $B(x_0, \rho) \subset B(r)^c$: By means of triangle inequality and Jensen’s inequality we estimate
\[
\int_{B(x_0, \rho)} |U - U_{B(x_0, \rho)}|dy \leq \int_{B(x_0, \rho)} \int_{B(x_0, \rho)} |\phi(y)u(T(y)) - \phi(y')u(T(y'))|dydy'.
\]

Using change of coordinates $z = T(y), z' = T(y') (y = T(z), y' = T(z'))$, and observing (2.6), we obtain with the help of the transformation formula of the Lebesgue integral
\[
\int_{B(x_0, \rho)} \int_{B(x_0, \rho)} |\phi(y)u(T(y)) - \phi(y')u(T(y'))|dydy' \leq \frac{1}{c_n^2 \rho^{2n}} \int_{T(B(x_0, \rho))} \int_{T(B(x_0, \rho))} |\phi(T(z))u(z) - \phi(T(z'))u(z')| \left( \frac{r}{|z|} \right)^{2n} \left( \frac{r'}{|z'|} \right)^{2n} dzdz'.
\]

In case $x_0 \in B(4r)^c$ it is readily seen that $B(x_0, \rho) \subset B(3r)^c$, and therefore $U \equiv 0$ on $B(x_0, \rho)$. Thus, we may assume $x_0 \in B(4r)$. By means of (2.7) we infer for every $y \in B(x_0, \rho)$
\[
|T(x_0) - T(y)| \leq r^2 \frac{|x_0 - y|}{|x_0||y|} \leq |x_0 - y| < \rho.
\]

This implies that
\[
T(B(x_0, \rho)) \subset B(T(x_0), \rho) \subset B(r).
\]

With the help of triangle inequality we obtain for $z = T(y)$ and $y \in B(x_0, \rho)$
\[
|z| = |T(y)| = \frac{r^2}{|y|} \geq \frac{r^2}{|y - x_0| + |x_0|} > 5r.
\]

Now, the right-hand side of (2.9) can be estimated by virtue of (2.10) and (2.11). This gives
\[
\int_{B(x_0, \rho)} \int_{B(x_0, \rho)} |\phi(y)u(T(y)) - \phi(y')u(T(y'))|dydy' \leq \frac{5^{4n}}{c_n^2 \rho^{2n}} \int_{T(B(x_0, \rho))} \int_{T(B(x_0, \rho))} |\phi(T(z))u(z) - \phi(T(z'))u(z')|dzdz' \leq \frac{5^{4n}}{c_n^2 \rho^{2n}} \int_{T(B(x_0, \rho))} \int_{T(B(x_0, \rho))} |u(z) - u(z')|dzdz' + \frac{5^{4n}}{c_n^2 \rho^{2n}} \int_{T(B(x_0, \rho))} \int_{T(B(x_0, \rho))} |(\phi(T(z)) - \phi(T(z')))u(z')|dzdz' = I + II.
\]
Clearly, \( I \leq 5^4 |u|_{BMO(B(r))} \). To estimate the second integral we first note that for all \( z \in T(B(x_0, \rho)) \)

\[
|\phi(T(z)) - \phi(T(z'))| \leq 2 \|\nabla \phi\|_{\infty, \rho} \cdot cr^{-1} \rho.
\]

This together with (2.10) shows that

\[
II \leq cr^{-1} \rho \int_{B(T(x_0, \rho))} |u(z')|dz' \leq c \|u\|_{BMO(B(r))} + cr^{-1} \rho |u_{B(T(x_0), \rho)}|.
\]

Furthermore, by using a standard iteration, we get

\[
|u_{B(T(x_0), \rho)}| \leq c |u|_{BMO(B(r))} + |u_{B(T(x_0), r) \cap B(r)}| \leq c |u|_{BMO(B(r))} + cr^{-n} \|u\|_{L^1(B(r))}.
\]

Accordingly, \( II \leq c \|u\|_{BMO(B(r))} \). Inserting the estimates of \( I \) and \( II \) into the right-hand side of (2.12), we obtain from (2.8)

\[
\int_{B(x_0, \rho)} |U - U_{B(x_0, \rho)}| dy \leq c \|u\|_{BMO(B(r))}.
\]

(iii) The case \( B(x_0, \rho) \cap \partial B(r) \neq \emptyset \): Elementary,

\[
\int_{B(x_0, \rho)} \int_{B(x_0, \rho)} |U(y) - U(y')|dy dy' = \frac{1}{c^2 \rho^{2n}} \int_{B(x_0, \rho) \cap B(r)} \int_{B(x_0, \rho) \cap B(r)} |u(y) - u(y')|dy dy' \\
+ \frac{2}{c^2 \rho^{2n}} \int_{B(x_0, \rho) \cap B(r)} \int_{B(x_0, \rho) \setminus B(r)} |u(T(y)) - u(y')|dy dy' \\
+ \frac{1}{c^2 \rho^{2n}} \int_{B(x_0, \rho) \setminus B(r)} \int_{B(x_0, \rho) \setminus B(r)} |u(T(y)) - u(T(y'))|dy dy' = I + II + III.
\]

We set \( \hat{x} := \frac{x_0}{r}, x_0 \in \partial B(r) \). Since \( B(x, \rho) \cap B(r) \neq \emptyset \) it must hold \( |x_0| < r + \rho \), and therefore \( \hat{x} \in B(x_0, \rho) \). In particular, \( |x_0 - \hat{x}| < \rho \). This shows that

\[
|x_0 - \hat{x}|^2 = |x_0|^2 + 2r|x_0| + r^2 = (|x_0| - r)^2 < \rho^2,
\]

which in turn implies

(2.13) \( |x_0| - r < \rho \).

By using triangle inequality, we infer for every \( y \in B(x, \rho) \)

\[
|y - \hat{x}| \leq |y - x_0| + |x_0 - \hat{x}| < 2\rho.
\]

For \( y \in B(x, \rho) \setminus B(r) \), noting that \( \hat{x} = T(\hat{x}) \), and using (2.7), we find

\[
|T(y) - \hat{x}| = |T(y) - T(\hat{x})| = r^2 |y - \hat{x}| = \frac{y - \hat{x}}{|y - \hat{x}|} < 2\rho.
\]
Consequently, (2.14)  
\[ B(x, \rho) \cap B(r) \subset B(\hat{x}, 2\rho) \cap B(r), \quad T(B(x, 2\rho) \setminus B(r)) \subset B(\hat{x}, 2\rho) \cap B(r). \]

Using (2.14), we obtain
\[ I \leq c \int_{B(\hat{x}, 2\rho) \cap B(r)} \int_{B(\hat{x}, 2\rho) \cap B(r)} |u(y) - u(y')| dy dy' \leq c \|u\|_{BMO(B(r))}. \]

To estimate \( II \) we proceed as in (ii). For \( y \in B(x_0, \rho) \) we set \( z = T(y) \). Since \( |x_0| < \frac{3}{2}r \) and \( |y - x_0| < \frac{r}{2} \), we estimate
\[ |z| = |T(y)| = \frac{r^2}{|y|} \geq \frac{r^2}{|y - x_0| + |x_0|} > 2r. \]

Arguing as in (ii), using the transformation formula of the Lebesgue integral together with (2.6), (2.15) and (2.14), we infer
\[ II = \frac{1}{c_0^2 \rho^{2n}} \int_{B(x, \rho) \cap B(r)} \int_{T(B(x, \rho) \setminus B(r))} |u(z) - u(y')| \left( \frac{r}{|z|} \right)^{2n} dz dy' \]
\[ \leq c \int_{B(\hat{x}, 2\rho) \cap B(r)} \int_{B(\hat{x}, 2\rho) \cap B(r)} |u(z) - u(y')| dz dy' \leq c \|u\|_{BMO(B(r))}. \]

Similarly, we also estimate
\[ III \leq c \|u\|_{BMO(B(r))}. \]

Therefore, we have
\[ \int_{B(x_0, \rho)} |U - U_{B(x_0, \rho)}| dy \leq c \|u\|_{BMO(B(r))}. \]

This completes the proof of the lemma.

\section{Multiplicative inequalities}

In our discussion below we shall make use of the following multiplicative inequalities involving cut-off functions.

\textbf{Lemma 3.1.} Let \( \psi \in C_c^\infty(B(r)), 0 < r < \infty, \) such that \( 0 \leq \psi \leq 1 \) in \( B(r) \). For all \( u \in W^{1,q}(B(r)), 2 < q < \infty \) with \( \nabla \cdot u = 0 \) a.e. in \( B(r) \) and for all \( m \geq \frac{5q - 6}{2q} \) it holds
\[ (3.1) \quad \| \nabla u \psi^m \|_q \leq c \| \nabla \times u \psi^m \|_q + c \| \nabla \psi \|_\infty^a \| u \psi^{m-a} \|_2, \]
\[ (3.2) \quad \| u \psi^{m-k} \|_q \leq c \| u \psi^{m-ka} \|_2 \| \nabla \times u \psi^m \|_q^{1-a} + c \| \nabla \psi \|_\infty^{a-1} \| u \psi^{m-ka} \|_2 \]

where
\[ a = \frac{5q - 6}{2q}, \quad 1 \leq k \leq \frac{m}{a}. \]
Proof: Let $\psi \in C^\infty_c(B(r))$ be a cut off function, such that $0 \leq \psi \leq 1$. Recalling that $\nabla \cdot u = 0$, we calculate

$$-\Delta(u\psi^m) = \nabla \times \nabla \times (u\psi^m) - \nabla (u \cdot \nabla \psi^m).$$

Applying $\nabla$ to both sides of the above identity, we find that

$$-\Delta(\nabla (u\psi^m)) = \nabla \nabla \times (\nabla \times (u\psi^m)) - \nabla \nabla (u \cdot \nabla \psi^m).$$

Using Calderón-Zygmund’s inequality, we see that

$$\|\nabla (u\psi^m)\|_q \leq c\|\nabla \times (\psi^m u)\|_q + c\|\nabla \psi\|_\infty \|u\psi^{m-1}\|_q$$

$$\leq c\|\nabla \times u\psi^m\|_q + c\|\nabla \psi\|_\infty \|u\psi^{m-1}\|_q.$$  (3.3)

Estimating the last term on the right-hand side of the above estimate by means of (B.1) with $n = 3$, and applying Young’s inequality, we get (3.1). The second inequality easily follows (B.1) together with (3.1).  

Arguing as in the proof of Lemma 3.1, we get the following

Lemma 3.2. Let $\psi \in C^\infty_c(B(r))$, $0 < r < +\infty$, with $0 \leq \psi \leq 1$. For every $u \in W^{1,1}(B(r))$ such that $\nabla \cdot u = 0$ and $\nabla \times u \in BMO(B(r))$ it holds

$$\|\nabla \psi^5\|_{BMO} \leq c\|\nabla \times u\psi^5\|_{BMO} + c\left\{r^{\frac{3}{4}}\|\nabla \psi\|_{\infty}^\frac{5}{2} + \|\nabla \psi\|_{\infty}^\frac{5}{4}\right\}\|u\psi\|_2.$$  (3.4)

Proof: As we have seen in the proof of Lemma 3.1 it holds in $\mathbb{R}^3$

$$-\Delta \nabla (u\psi^6) = \nabla \nabla \times (\nabla \times (u\psi^6)) - \nabla \nabla (u \cdot \nabla \psi^6).$$

Using Calderón-Zygmund’s inequality [16], we find that

$$\|\nabla \psi^6\|_{BMO} \leq c\|\nabla \times (\psi^6 u)\|_{BMO} + c\|u \cdot \nabla \psi^6\|_{BMO}$$

$$\leq c\|\nabla \times u\psi^6\|_{BMO} + c\|\nabla \psi\|_{\infty} \|u\psi^5\|_{\infty}.$$  (3.5)

On the other hand, in view of (B.2) with $n = 3$, $q = 6$ and $m = 5$ we get

$$\|u\psi^5\|_{\infty} \leq c\|u\psi\|_2^\frac{1}{2} \|\nabla u\psi^5\|_6^\frac{3}{2} + c\|\nabla \psi\|_{\infty}^\frac{3}{2} \|u\psi\|_2.$$  (3.6)

We estimate the first term on the right-hand side of (3.6) by (3.1) with $q = 6$ and $m = 5$. This together with (B.3) gives

$$\|u\psi^5\|_{\infty} \leq c\|u\psi\|_2^\frac{1}{2} \|\nabla u\psi^5\|_6^\frac{3}{2} + c\|\nabla \psi\|_{\infty}^\frac{3}{2} \|u\psi\|_2.$$  (3.7)

Finally, combining (3.5) and (3.7), and applying Young’s inequality, we obtain (3.4).  

Lemma 3.3. Let \( u \in W^{1,1}(B(r)) \) with \( \nabla \times u \in BMO(B(r)) \). Then for all \( \psi \in C_c^\infty(B(r)) \) with \( 0 \leq \psi \leq 1 \) we get

\[
\| (\nabla \times u)\psi^5 \|_{BMO} \leq c \left( 1 + r^5 \| \nabla \psi \|_\infty \right) \left( \| \nabla \times u \|_{BMO(B(r))} + cr^{-\frac{3}{2}} \| u \|_{L^2(B(r))} \right).
\]

Proof: Assume \( r = 1 \). Let \( \eta \in C_c^\infty(B(1)) \) such that \( |\nabla \eta| \leq c \) and \( \int_{B(1)} \eta dx \geq c \), where \( c > 0 \) stands for a constant depending only on \( n \). For \( f \in L^1(B(1)) \) we define the mean

\[
\tilde{f}_{B(1)} = \frac{1}{\int_{B(1)} \eta dx} \int_{B(1)} f \eta dx.
\]

First we see that

\[
\| \nabla \times u \|_{L^1(B(1))} = \| \nabla \times u - \nabla \times u_{B(1)} \|_{L^1(B(1))} + |\nabla \times u_{B(1)}| \leq c \| \nabla \times u \|_{BMO(B(1))} + c \| u \|_{L^1(B(1))}.
\]

Using (3.3), we estimate for \( \rho \geq \frac{1}{2} \) and \( x_0 \in \mathbb{R}^3 \)

\[
\int_{B(x_0,\rho)} |(\nabla \times u)\psi^5 - (\nabla \times u\psi^5)_{B(x_0,\rho)}| dx
\]

\[
\leq c \| \nabla \times u \|_{L^1(B(1))}
\]

\[
\leq c \| \nabla \times u \|_{BMO(B(1))} + c \| u \|_{L^1(B(1))}.
\]

In case \( \rho \leq \frac{1}{2} \) and \( B(x_0, \rho) \cap B(1) \neq \emptyset \) there exists \( y_0 \in B(1) \) such that \( B(x_0, \rho) \subset B(y_0, 2\rho) \) and

\[
\int_{B(x_0,\rho)} |\nabla \times u\psi^5 - (\nabla \times u\psi^5)_{B(x_0,\rho)}| dx
\]

\[
\leq c \int_{B(y_0, 2\rho)} \int_{B(y_0, 2\rho)} |\nabla \times u(x)\psi^5(x) - |\nabla \times u(y)\psi^5(y)| |dx dy
\]

\[
\leq c \| \nabla \times u \|_{BMO(B(1))} + c \int_{B(y_0, 2\rho)} \int_{B(y_0, 2\rho)} |\nabla \times u(x)||\psi^5(x) - \psi^5(y)||dx dy.
\]

By the fundamental theorem of differentiation and integration we calculate

\[
\psi^5(y) - \psi^5(x) \leq 5\psi^4(\xi_1)\nabla \psi(\xi_1) \cdot (y - x)
\]

\[
= 5\psi^4(x)\nabla \psi(\xi_1) \cdot (y - x) + 5(\psi^4(\xi_1) - \psi^4(x))\nabla \psi(\xi_1) \cdot (y - x)
\]

\[
= \sum_{k=1}^{2} \psi^{5-k}(x) \prod_{i=1}^{k} (6-i)\nabla \psi(\xi_i) \cdot (\xi_{i-1} - x)
\]

\[
+ \psi^2(\xi_3) \prod_{i=1}^{3} (6-i)\nabla \psi(\xi_i) \cdot (\xi_{i-1} - x).
\]

9
For some $\xi_i \in [x, y]$, $i = 1, 2, 3$, and $\xi_0 = y$. This along with (3.9) yields
\[
\int_{B(y_0, 2\rho)} \int_{B(y_0, 2\rho)} |\nabla \times u(x)| \psi^5(x) - \psi^5(y) \, dx \, dy \\
\leq c \sum_{k=1}^{2} \|\nabla \psi\|_\infty^k \rho^{k-3} \int_{B(y_0, 2\rho)} |\nabla \times u(x)| \psi^{5-k}(x) \, dx \\
+ c \|\nabla \psi\|_\infty^3 \|\nabla \times u\|_{L^1(B(1))} \\
\leq c \sum_{k=1}^{2} \|\nabla \psi\|_\infty^k \|\nabla \times u\|^{5-k} + c \|\nabla \psi\|_\infty^3 \|\nabla \times u\|_{L^1(B(1))}.
\]

By using Hölder’s inequality, we find that
\[
\|\nabla \times u\|^{5-k}_\infty \leq \|\nabla \times u\|_{L^1(B(1))}^{\frac{2k-1}{2k}} \|\nabla \times u\|_{L^5(B(1))}^{\frac{k}{2-k}} \|\nabla \times u\|_{BMO}^{\frac{6-2k}{6}}.
\]

Applying the embedding $L^6(B(1)) \hookrightarrow BMO(B(1))$ (cf. Lemma B.3), we get
\[
\|\nabla \times u\|^{5-k}_\infty \leq c \|\nabla \times u\|_{L^5(B(1))}^{\frac{2k-1}{2k}} \|\nabla \times u\|_{BMO}^{\frac{6-2k}{6}}.
\]

Combining the above inequalities, and applying Young’s inequality together with (3.9), we arrive at
\[
\|\nabla \times u\|^{5}_\infty \leq c \|\nabla \times u\|_{BMO(B(1))} + c \|\nabla \psi\|_\infty \|\nabla \times u\|_{L^1(B(1))} \\
\leq c \left(1 + \|\nabla \psi\|_{\infty} \right) \left(\|\nabla \times u\|_{BMO(B(1))} + c \|\nabla \times u\|_{L^1(B(1))} \right).
\]

Whence, (3.8) follows immediately from (3.10) by using a standard scaling argument.

\[\text{(3.10)}\]

Combining Lemma 3.2 and Lemma 3.3, we get

**Corollary 3.4.** Let $\psi \in C_c^\infty(B(r))$, $0 < r < +\infty$, with $0 \leq \psi \leq 1$. For every $u \in W^{1,1}(B(r))$ such that $\nabla \cdot u = 0$ and $\nabla \times u \in BMO(B(r))$ it holds
\[
\|\nabla u\psi^5\|_{BMO(B(r))} \leq c \left\{1 + r^5 \|\nabla \psi\|_\infty^5 \right\} \|\nabla \times u\|_{BMO(B(r))} \\
+ c \left\{r^{-2} + r^{2} \|\nabla \psi\|_\infty^5 \right\} \|u\|_{L^2(B(r))}.
\]

(3.11)
Proof: Combining (3.11) and (3.8) along with Young’s inequality, we infer
\[
\|\nabla u\psi^m\|_{BMO(B(r))}^q \leq c\left\{1 + r^5\|
abla\psi\|_\infty^5\right\} (|
abla \times u|_{BMO(\bar{B}(r))} + cr^{-\frac{5}{2}}\|u\|_{L^2(B(r))})
\]
\[+ c\left\{r^{-\frac{5}{2}}\|
abla\psi\|_\infty^5 + \|
abla\psi\|_\infty^5\right\} \|u\|_{L^2(B(r))}
\]
\[\leq c\left\{1 + r^5\|
abla\psi\|_\infty^5\right\} \|\nabla \times u|_{BMO(B(r))}
\]
\[+ c\left\{r^{-\frac{5}{2}} + r^{-\frac{5}{2}}\|
abla\psi\|_\infty^5\right\} \|u\|_{L^2(B(r))}.
\]
(3.12)
Whence, (3.11).

Remark 3.5. Note that thanks to (3.11) for every $v \in L^\infty(-\rho, 0; L^2(B(\rho)))$, which satisfies the local Beale-Kato-Majda condition (1.4) it holds for all $0 < r < \rho$

\[
\int_{-1}^{0} \|\omega(t)\|_{BMO(\bar{B}(r))} dt < +\infty.
\]

In particular, (3.7) with $n = 3$ together with (3.13) implies that for all $0 < r < \rho$

(3.14) $v \in L^\infty(-\rho, 0; L^\infty(B(r)))$.

Lemma 3.6. Let $u \in W^{2,q}(B(r))$, $2 \leq q < +\infty$. Let $m, k \in \mathbb{R}$ such that $2 \leq m < +\infty$ and $0 < k < 2m$. Then for every $\psi \in C_c^\infty(B(r))$ it holds

(3.15) $\|\nabla u\psi^m\|_q \leq c\|u\psi^{2m-k}\|_q^{1/2}\|\nabla^2 u\psi^k\|_q^{1/2} + c\|\nabla\psi\|_\infty\|u\psi^{m-1}\|_q$.

If in addition, if $\nabla \cdot u = 0$ almost everywhere in $B(r)$, then for $2 \leq m < +\infty$ and $m+1 \leq k \leq 2m$ it holds

(3.16) $\|\nabla^2 u\psi^k\|_q \leq c\|\nabla^2 u\times u\psi^k\|_q + c\|\nabla\psi\|_\infty^2\|u\psi^{k-2}\|_q$.

(3.17) $\|\nabla u\psi^m\|_q \leq c\|u\psi^{2m-k}\|_q^{1/2}\|\nabla^2 u\times u\psi^k\|_q^{1/2} + c\|\nabla\psi\|_\infty\|u\psi^{m-1}\|_q$.

Proof: Applying integration by parts, and using Hölder’s inequality, we get
\[
\|\nabla u\psi^m\|_q^q = -\int_{B(\rho)} u\nabla u \cdot \nabla |\nabla u|^{q-2} \psi^q dx - \int_{B(\rho)} u\nabla u |\nabla u|^{q-2} \cdot \nabla\psi^m dx
\]
\[\leq c\|u\psi^{2m-k}\|_q^{1/2}\|\nabla^2 u\psi^k\|_q^{1/2}\|\nabla u\psi^m\|_q^{q-2}
\]
\[+ c\|\nabla\psi\|_\infty\|u\psi^{m-1}\|_{L^2(B(\rho))}\|\nabla u\psi^m\|_q^{q-1},
\]
and Young’s inequality gives (3.15).

In case $\nabla \cdot u = 0$ almost everywhere in $B(r)$ we may apply (3.3) with $\nabla u$ in place of $u$ in order to estimate in (3.15) the norm involving the second gradient of $u$. This together with (3.15) with $m = k-1$ gives
\[
\|\nabla^2 u\psi^k\|_q \leq c\|\nabla\psi\|_\infty\|\nabla u\psi^{k-1}\|_q
\]
\[\leq c\|\nabla\psi\|_\infty\|\nabla^2 u\psi^{k-2}\|_q + c\|\nabla\psi\|_\infty\|\nabla u\psi^{k-2}\|_q^{1/2}\|\nabla^2 u\psi^k\|_q^{1/2}
\]
\[+ \|\nabla\psi\|_\infty\|u\psi^{k-2}\|_q.
\]
Then we apply Young’s inequality to obtain (3.16). The estimate (3.17) is now an immediate consequence of (3.15) and (3.16).

Combining Lemma 3.1 and Lemma 3.6, we get the following Corollary 3.7.

**Corollary 3.7.** For all $u \in W^{2,q}(B(r))$, for all $\psi \in C^\infty_c(B(r))$ with $0 \leq \psi \leq 1$ and for all $k > 5$ we get

$$\|\nabla^2 u \psi^k\|_q \leq c \|\nabla \nabla \times u\|_1 \|\nabla^2 \psi^k\|_{q^{-2}}^2 + c \|\nabla\psi\|_{q^{-1}} \|u\psi^k\|_2.$$  

**Proof:** Let $k > q$. The estimate (3.17) with $m = k$ and $k = 2$ reads

$$\|u \psi^k - 2\|_q \leq c \|u \psi^k - 2\|_1 \|\nabla \times u\|_{q^{-2}}^2 + c \|\nabla \psi\|_{q^{-1}} \|u\psi^k - 2\|_2.$$  

Combining this inequality with (3.16), and applying Young’s inequality, we obtain (3.18).

**4 Local estimates of the vorticity**

**Theorem 4.1.** Under the assumption of Theorem 1.1 it holds for all $0 < r < \rho$ and for all $1 \leq q < +\infty$

$$\omega \in L^\infty(-\rho, 0; L^q(B(r))).$$  

**Proof:** Applying curl to both sides of (1.1), we get the vorticity equation

$$\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v \quad \text{in} \quad B(\rho) \times (-\rho, 0).$$

Fix, $2 \leq q < +\infty$. Let $0 < r < \sigma < \rho$ be arbitrarily chosen. Let $0 < r < r_1 < r_2 < \sigma$, and set $\tilde{r} := \frac{r_1 + r_2}{2}$. Let $\phi \in C^\infty_c(B(\tilde{r}))$ denote a cut off function such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B(r)$, and $|\nabla \phi| \leq c(r_2 - r_1)^{-1}$. Next, let $-\rho < t_0 < t_1 < 0$ be fixed, where $t_0$ will be taken sufficiently small, specified below. We multiply (4.2) by $\omega|\omega|^{q-2}\phi^q$, integrate the result over $B(r_2) \times (t_0, t)$, $t_0 < t < t_1$, and apply integration by parts. This yields

$$\|\omega(t)\phi\|_q^q = \|\omega(t_0)\phi\|_q^q + \int_{t_0}^t \int_{B(r_2)} v(s) \cdot \nabla \|\omega(s)\|_q^q \phi^q ds + q \int_{t_0}^t \int_{B(r_2)} \omega(s) \cdot \nabla v(s) \cdot \omega(s)\|\omega(s)\|_q^q \phi^q ds.$$  

$$= \|\omega(t_0)\phi\|_q^q + I + II.$$  

12
First, by using Hölder’s inequality and Lemma $\text{B.3}$, we estimate

$$I \leq c(r_2 - r_1)^{-1} \int_{t_0}^{t} \left\| v(s) \right\|_{L^\infty(B(\sigma))} \left\| \omega(s) \phi^{1 - \frac{1}{q}} \right\| g ds$$

$$\leq c(r_2 - r_1)^{-1} \int_{t_0}^{t} \left\| v(s) \right\|_{L^\infty(B(\sigma))} \left\| \omega(s) \phi^{1 - \frac{1}{q}} \right\| \frac{1}{4} ds \text{ ess sup}_{s \in (t_0, t_1)} \left\| \omega(s) \right\|_{L^q(B(r_2))}^\frac{q-\frac{1}{4}}{q}$$

$$\leq c(r_2 - r_1)^{-1} \left\| v \right\|_{L^\frac{4}{3}(-\rho,0;L^\infty(B(\sigma)))} \left( \int_{-\rho}^{0} \left\| \omega(s) \right\|_{BMO(B(\sigma))} ds \right)^\frac{1}{q} \times$$

$$\times \text{ ess sup}_{s \in (t_0, t_1)} \left\| \omega(s) \right\|_{L^q(B(r_2))}^\frac{q-\frac{1}{4}}{q}.$$ 

Using Young’s inequality, we obtain

$$I \leq c(r_2 - r_1)^{-4q} \left\| v \right\|_{L^\frac{4}{3}(-\rho,0;L^\infty(B(\sigma)))}^{4q} \left( \int_{-\rho}^{0} \left\| \omega(s) \right\|_{BMO(B(\sigma))} ds \right)^q$$

$$+ \varepsilon \text{ ess sup}_{s \in (t_0, t_1)} \left\| \omega(s) \right\|_{L^q(B(r_2))}^q.$$

Secondly, we get by the aid of Hölder’s inequality

$$II \leq c \int_{t_0}^{t} \left\| \omega(s) \cdot \nabla v(s) \right\|_{L^q(B(\bar{r}))} ds \text{ ess sup}_{s \in (t_0, t_1)} \left\| \omega(s) \right\|_{L^q(B(r_2))}^{q-1}.$$

To proceed further we prove the following localization of Kozono-Taniuchi’s inequality$\text{[13]}$.

**Lemma 4.2.** Let $f, g \in BMO(B(r)) \cap L^q(B(r))$, $1 < q < +\infty$, then $f \cdot g \in L^q(B(r))$ and it holds

$$\left\| f \cdot g \right\|_{L^q(B(r))} \leq c \left( |f|_{BMO(B(r))} \left\| g \right\|_{L^q(B(r))} + |g|_{BMO(B(r))} \left\| f \right\|_{L^q(B(r))} \right)$$

$$+ cr^{-\frac{q}{2}} \left\| f \right\|_{L^q(B(r))} \left\| g \right\|_{L^q(B(r))},$$

where the constant $c > 0$ depends on $q$ only.

**Proof:** We define the extension

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in B(r) \\ f\left(\frac{x}{|x|}\right) \phi(x) & \text{if } x \in B(r)^c. \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in B(r) \\ g\left(\frac{x}{|x|}\right) \phi(x) & \text{if } x \in B(r)^c. \end{cases}$$

13
where \( \phi \in C_c^\infty(B(4r)) \) denotes a suitable cut off function such that \( \phi \equiv 1 \) on \( B(2r) \). According to Lemma 2.2 it holds \( f, g \in BMO \cap L^q(\mathbb{R}^3) \) together with the estimates

\[
\begin{align*}
(4.6) \quad & \| f \|_q \leq c \| f \|_{L^q(B(r))}, \quad \| g \|_q \leq c \| g \|_{L^q(B(r))} \\
(4.7) \quad & \| \tilde{f} \|_{BMO} \leq c |f|_{BMO(B(r))} + cr^{-3} \| f \|_1, \\
(4.8) \quad & \| \tilde{g} \|_{BMO} \leq c |g|_{BMO(B(r))} + cr^{-3} \| g \|_1.
\end{align*}
\]

Thanks to Kozono-Taniuchi’s inequality [13, Lemma 1(i)], combined with (4.6), (4.7), and (4.8) we find that

\[
\| f \cdot g \|_{L^q(B(r))} \leq \| \tilde{f} \cdot \tilde{g} \|_{L^q(B(r))}
\]

\[
\leq c \left( \| f \|_{BMO(B(r))} \| \tilde{g} \|_{L^q(B(r))} + \| \tilde{g} \|_{BMO(B(r))} \| \tilde{f} \|_{L^q(B(r))} \right)
\]

\[
\leq c \left( |f|_{BMO(B(r))} \| g \|_{L^q(B(r))} + |g|_{BMO(B(r))} \| f \|_{L^q(B(r))} \right)
\]

\[
+ cr^{-3} \left( \| f \|_{L^1(B(r))} \| g \|_{L^q(B(r))} + \| g \|_{L^1(B(r))} \| f \|_{L^q(B(r))} \right).
\]

Whence, by using Jensen’s inequality we get (4.5).

Thanks to (4.5) we find

\[
\| \omega(s) \cdot \nabla v(s) \|_{L^q(B(\bar{r}))}
\]

\[
\leq c \left( |\omega(s)|_{BMO(B(\bar{r}))} \| \nabla v(s) \|_{L^q(B(\bar{r}))} + |\nabla v(s)|_{BMO(B(\bar{r}))} \| \omega(s) \|_{L^q(B(\bar{r}))} \right)
\]

\[
+ cr^{-\frac{4}{q}} \| \omega(s) \|_{L^q(B(\bar{r}))} \| \nabla v(s) \|_{L^q(B(\bar{r}))}.
\]

By the aid of Corollary 3.4 we get

\[
(4.10) \quad \| \nabla v(s) \|_{BMO(B(\bar{r}))} \leq c \left( |\omega(s)|_{BMO(B(\rho))} + (r_2 - r_1)^{-\frac{2}{q}} \| v(s) \|_{L^2(\rho)} \right).
\]

Furthermore, by virtue of Lemma 3.1 we see that

\[
(4.11) \quad \| \nabla v(s) \|_{L^q(B(\bar{r}))} \leq c \left( |\omega(s)|_{L^q(B(r_2))} + (r_2 - r_1)^{-\beta} \| v(s) \|_{L^2(\rho)} \right),
\]

where

\[
\beta = \frac{5q - 6}{2q}.
\]

We now estimate the right-hand side of (4.9) by (4.10) and (4.11). This yields

\[
\| \omega(s) \cdot \nabla v(s) \|_{L^q(B(\bar{r}))}
\]

\[
\leq c |\omega(s)|_{BMO(B(\rho))} \| \omega(s) \|_{L^q(B(r_2))}
\]

\[
+ c (r_2 - r_1)^{-\frac{2}{q}} \| v(s) \|_{L^2(B(\rho))} \| \omega(s) \|_{L^q(B(r_2))}
\]

\[
+ c (r_2 - r_1)^{-\beta} |\omega(s)|_{BMO(B(\rho))} \| v(s) \|_{L^2(B(\rho))} + c (r_2 - r_1)^{-\beta - \frac{2}{q}} \| v(s) \|_{L^2(B(\rho))}^2.
\]
By using a standard iteration argument, we deduce from (4.13) that

\[ \| \omega(s) \|_{L^q(B(r_2))} \leq c(r_2 - r_1)^{\frac{3}{4}} \left( \| \omega(s) \|_{BMO(B(\rho))} + (r_2 - r_1)^{-\frac{\beta}{2}} \| v(s) \|_{L^2(B(\rho))} \right) \]

due to (B.3), we are led to

\[ \| \omega(s) \cdot \nabla v(s) \|_{L^q(B(\bar{r}))} \]

(4.12)

\[ \leq c a(s) \| \omega(s) \|_{L^q(B(r_2))} + c(r_2 - r_1)^{-\beta} a(s) \| v(s) \|_{L^2(B(\rho))}, \]

where we have set

\[ a(s) = \| \omega(s) \|_{BMO(B(\rho))} + (r_2 - r_1)^{-\frac{\beta}{2}} \| v(s) \|_{L^2(B(\rho))}. \]

Integrating (4.12) over \((t_0, t)\), then combining the result with (4.4), and applying Young’s inequality, we obtain

\[ \begin{align*}
II & \leq c \int_{t_0}^{0} a(s) ds \sup_{s \in (t_0, t_1)} \| \omega(s) \|^q_{L^q(B(r_2))} + \varepsilon \sup_{s \in (t_0, t_1)} \| \omega(s) \|^q_{L^q(B(r_2))} \\
& \quad + c(r_2 - r_1)^{-q\beta} \| v \|^q_{L^\infty(-\rho, 0; L^2(B(\rho)))} \left( \int_{t_0}^{0} a(s) ds \right)^q.
\end{align*} \]

We take \( \varepsilon = \frac{1}{6} \), and choose \( t_0 \) such that

\[ c \int_{t_0}^{0} a(s) ds \leq \frac{1}{6}. \]

Inserting the estimates of \( I \) and \( II \) into the right-hand side of (4.3), and taking into account that \( \beta \leq 4 \), we get

\[ \begin{align*}
\sup_{s \in (t_0, t_1)} \| \omega(s) \|^q_{L^q(B(r_1))} \\
\leq \frac{1}{2} \sup_{s \in (t_0, t_1)} \| \omega(s) \|^q_{L^q(B(r_2))} \\
+ \| \omega(t_0) \|_{L^q(B(\sigma))} + c(r_2 - r_1)^{-4q} \left( \| v \|^q_{L^\infty(-\rho, 0; L^2(B(\rho)))} + \| v \|_{L^4(-\rho, 0; L^\infty(B(\rho)))} \right)^q.
\end{align*} \]

(4.13)

By using a standard iteration argument, we deduce from (4.13) that

\[ \begin{align*}
\sup_{s \in (t_0, t_1)} \| \omega(s) \|^q_{L^q(B(r_1))} \\
\leq c(\sigma - r)^{-4q} \left( \| \omega(t_0) \|_{L^q(B(\sigma))} + \| v \|^q_{L^\infty(-\rho, 0; L^2(B(\rho)))} + \| v \|_{L^4(-\rho, 0; L^\infty(B(\rho)))} \right)^q.
\end{align*} \]

Since \( c \) is independent of \( t_1 \), using the fact \( \omega \in L^\infty(-\rho, t_0; L^q(B(\bar{r}))) \) and (3.14), combined with the hypothesis \( v \in L^\infty(-\rho, 0; L^2(B(\rho))) \), we get (4.1).

By means of Sobolev’s embedding theorem we immediately deduce from (4.1) the following
Corollary 4.3. Let $v \in L^{\infty}(-\rho,0; L^{2}(B(\rho)))$ with $|\omega(\cdot)|_{BMO(B(\rho))} \in L^{1}(-\rho,0)$ be a solution to the Euler equations (1.1). Then for all $0 < r < \rho$,

\[ v \in L^{\infty}(-\rho,0; C^{0,\gamma}(B(r))) \quad \forall 0 < \gamma < 1. \]

In particular, $v \in L^{\infty}(B(r) \times (-\rho,0))$ for all $0 < r < \rho$.

5 Proof of Theorem 1.1

We are now ready to prove our main theorem.

Proof of Theorem 1.1 We take $-\rho < t_* < 0$ such that

\[ \frac{1}{2} \rho^2 (-t_*)^{-\frac{1}{2}} - 4C_0 > 0. \]

Let $t_* < t_0 < 0$. We consider the following transformation of $(v,p) \mapsto (V,P)$ defined by

\[ V(y,t) = v((1 + (-t)^{\frac{1}{2}})y,t), \]
\[ P(y,t) = \frac{1}{1 + (-t)^{\frac{1}{2}}} p((1 + (-t)^{\frac{1}{2}})y,t), \]

which was first introduced by the authors of this paper in [4]. Thanks to Corollary 4.3 we have

\[ \|V(t) \cdot y\|_{L^{\infty}(B(\rho_0))} \leq C_0 \quad \forall t \in [t_0,0), \]

where $\rho_0 := \frac{\rho}{(1 + (-t_0)^{\frac{1}{2}})} > \frac{\rho}{2}$. We also define

\[ W(y,t) = \frac{\frac{1}{2}(-t)^{-1/2} y + V(y,t)}{1 + (-t)^{1/2}}, \quad (y,t) \in B(\rho_0) \times (t_0,0). \]

We claim that

\[ W(y,t) \cdot y > 0 \quad \text{for all} \quad (y,t) \in \left(\overline{B(\rho_0)} \setminus B(\rho/2)\right) \times (t_0,0). \]

Indeed, according to (5.2) together with (5.1) we estimate

\[ W(y,t) \cdot y \geq \frac{1}{4}\left(\frac{1}{2} \rho^2 (-t_*)^{-\frac{1}{2}} - 4C_0 \right) \geq \frac{1}{4}\left(\frac{1}{2} \rho^2 (-t_*)^{-\frac{1}{2}} - 4C_0 \right) > 0. \]

Using the chain rule, we see that (1.1) turns into the following equations, which hold in $B(\rho_0) \times (t_0,0)$.

\[ \partial_t V + W \cdot \nabla V = -\nabla P, \quad \nabla \cdot V = 0. \]
We set
\[ \Omega = \nabla \times V \quad \text{in} \quad B(\rho_0) \times (t_0, 0). \]

Then applying \( \nabla \times \) to (5.4), we obtain the following equations
\[
(5.5) \quad \partial_t \Omega + \frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1 + (-t)^{\frac{1}{2}}} \Omega + W \cdot \nabla \Omega = \nabla \cdot \nabla V,
\]

Applying the operator \( \partial_i (i \in \{1, 2, 3\}) \) to both sides of (5.5), we get the equations
\[
(5.6) \quad \partial_t \partial_i \Omega + \frac{(-t)^{-\frac{1}{2}}}{1 + (-t)^{\frac{1}{2}}} \partial_i \Omega + W \cdot \nabla \partial_i \Omega = (\partial_i \Omega) \cdot \nabla V + \Omega \cdot \partial_i \nabla V.
\]

Let \( 3 < q < +\infty \) and \( \frac{q}{2} < r < \rho_0 \) be arbitrarily chosen. Set \( \rho_* = \frac{r + \rho_0}{2} \), and define
\[
\rho_m := \rho_* - (\rho_* - r)^{m+1} \rho_*^{-m}, \quad m \in \mathbb{N} \cup \{0\}.
\]

Clearly,
\[
r_{m+1} - r_m = r \left( 1 - \frac{r}{\rho_*} \right)^{m+1}, \quad \text{and} \quad r_m \nearrow \rho_*,
\]

Let \( \eta_m \in C^\infty(\mathbb{R}) \) denote a cut off function such that \( 0 \leq \eta_m \leq 1 \) in \( \mathbb{R} \), \( \eta_m \equiv 1 \) on \( (-\infty, \rho_m] \), \( \eta_m \equiv 0 \) in \( (\rho_m, +\infty) \), and \( 0 \leq -\eta'_m \leq \frac{2}{r_{m+1} - r_m} = 2r^{-1} \left( \frac{\rho_0 + r}{\rho_0 - r} \right)^{m+1} \). Next, we multiply both sides of (5.6) by \( \partial_i \Omega |\nabla \Omega|^{q-2} \phi_{m}^{6q} \), where \( \phi_m(y) = \eta_m(||y||) \), integrate the result over \( B(r_{m+1}) \times (t_0, t), \ t_0 < t < 0 \), sum over \( i = 1, 2, 3 \). Then, applying the integration by parts, we have
\[
|\nabla \Omega(\tau)\phi_{m}^6|_{q}^{q} - 6q \int_{t_0}^{t} \int_{B(r_{m+1}) \setminus B(r_m)} \frac{W \cdot y}{|y|} |\nabla \Omega(s)|^{q-2} \phi_{m}^{6q-1} \eta'_m(||y||) dy ds
\]
\[
\leq |\nabla \Omega(t_0)\phi_{m}^6|_{q}^{q} + 2 \int_{t_0}^{t} \int_{B(r_{m+1})} \frac{(-s)^{-\frac{1}{2}}}{1 + (-s)^{\frac{1}{2}}} |\nabla \Omega(s)|^{q} \phi_{m}^{6q} dy ds
\]
\[
- 2q \int_{t_0}^{t} \int_{B(r_{m+1})} \nabla V(s) : \partial_i \Omega(s) \otimes \partial_i \Omega(s) |\nabla \Omega(s)|^{q-2} \phi_{m}^{6q} dy ds
\]
\[
- 2q \int_{t_0}^{t} \int_{B(r_{m+1})} \partial_i \nabla V(s) : \partial_i \Omega(s) \otimes \Omega(s) |\nabla \Omega(s)|^{q-2} \phi_{m}^{6q} dy ds.
\]

Since \( B(r_{m+1}) \setminus B(r_m) \subset B(\rho_0) \setminus B(\rho/2) \), the fact (5.3) ensures that \( W \cdot y > 0 \) in \( B(r_{m+1}) \setminus B(r_m) \times (t_0, 0) \). Furthermore, recalling that \( \eta'_m \leq 0 \), we see that the sign of
the integral on the left-hand side of the above inequality is non-negative. Consequently, taking into account \( q > 3 \), it follows that

\[
\| \nabla \Omega(t) \phi_m^6 \|_q^q 
\leq \| \nabla \Omega(t_0) \phi_m^6 \|_q^q - 2q \int_{t_0}^{t} \int_{B(r_{m+1})} \nabla V(s) : \partial_t \Omega(s) \otimes \partial_t \Omega(s) |\nabla \Omega(s)|^{q-2} \phi_m^6 dy ds 
- 2q \int_{t_0}^{t} \int_{B(r_{m+1})} \partial_i \nabla V(s) : \partial_i \Omega(s) \otimes \Omega(s) |\nabla \Omega(s)|^{q-2} \phi_m^6 dy ds 
= \| \nabla \Omega(t_0) \phi_m^6 \|_q^q + I + II.
\]

For notational simplicity we define

\[
Z_m(s) = \| \nabla^2 V(s) \phi_m^6 \|_q^q, \quad t_0 \leq s < 0.
\]

We immediately see that

\[
I + II \leq q^{2q+2} \int_{t_0}^{t} \| \nabla V(s) \|_{L^\infty(B(r_{m+1}))} Z_m(s) ds.
\]

Inserting the estimates of \( I + II \) into (5.7), we deduce

\[
\| \nabla \Omega(t) \phi_m^6 \|_q^q \leq 2Z_m(t_0) + q^{2q+2} \int_{t_0}^{t} \| \nabla V(s) \|_{L^\infty(B(r_{m+1}))} Z_m(s) ds.
\]

Furthermore, in view of (3.18), we see that

\[
Z_m(t) \leq c\| \nabla \Omega(t) \phi_m^6 \|_q^q + c\| \nabla \phi_m \|_{L^\infty(B(r_{m+1}))}^{\frac{7q-6}{2q}} \| V(t) \phi_m \|_2
\leq c\| \nabla \Omega(t) \phi_m^6 \|_q^q + c^{q-\frac{7q-6}{2q}} \left( \frac{\rho_0 + r}{\rho_0 - r} \right)^{\frac{(7q-6)m}{2q}}.
\]

Combining (5.10) with (5.9), we find that

\[
Z_m(t) \leq cZ_m(t_0) + c^{q-\frac{7q-6}{2q}} \left( \frac{\rho_0 + r}{\rho_0 - r} \right)^{\frac{(7q-6)m}{2q}} + c \int_{t_0}^{t} \| \nabla V(s) \|_{L^\infty(B(r_{m+1}))} Z_m(s) ds.
\]

By means of the local version of the logarithmic Sobolev embedding inequality (cf. Lemma 2.1) we find for every \( s \in (t_0, 0) \)

\[
\| \nabla V(s) \|_{L^\infty(B(r_{m+1}))} 
\leq c(1 + \| \nabla V(s) \|_{BMO(B(\rho_0))}) \log(e + \| \nabla^2 V(s) \|_{L^2(B(r_{m+1}))}) + \| V(s) \|_{L^2(B(r_{m+1}))}) 
\leq c(1 + \| \nabla V(s) \|_{BMO(B(\rho_0))}) \log(e + \| v(s) \|_{L^2(B(\rho_0))}) + Z_m+1
\leq c(1 + \| \nabla V(s) \|_{BMO(B(\rho_0))}) \log(e + Z_m+1),
\]

(5.12)
where the constant $c > 0$ depends only on $\rho$.

We continue our discussion by estimating the term on the right-hand side involving the $BMO$ norm of $\nabla V$. For this purpose let $\eta \in C^\infty(\mathbb{R})$ be a cut off function such that $0 \leq \eta \leq 1$ in $\mathbb{R}$, $\eta \equiv 1$ on $B(\rho_*)$, $\eta \equiv 0$ in $(\rho_0, +\infty)$, and $0 \leq -\eta' \leq 2(\rho_0 - \rho_*)^{-1} = 4(\rho_0 - r)^{-1}$. We set $\psi(y) = \eta(|y|)$. By means of Jensen’s inequality we get for every ball $B(x_0, \sigma)$ with $x_0 \in B(\rho_0/2)$ and $0 < \sigma < \rho_0$

\[
\int_{B(\rho_0) \cap B(x_0, \sigma)} |\nabla V(s) - (\nabla V(s))_{B(\rho_0/2) \cap B(x_0, \sigma)}|dy \\
\leq c \int_{B(\rho_0) \cap B(x_0, \sigma)} \int_{B(\rho_0/2) \cap B(x_0, \sigma)} |\nabla V(y, s) - \nabla V(y', s)| dy dy' \\
\leq c \int_{B(x_0, \sigma)} |(\nabla V(s))\psi^6 - (\nabla V(s)\psi^6)_{B(x_0, \sigma)}|dy.
\]

This together with (3.11) with $r = \rho_0$ yields

\[
\|\nabla V(s)\|_{BMO(B(\rho_0))} \\
\leq \|\nabla V(s)\psi^6\|_{BMO} \\
\leq c \left\{1 + \rho_0^\frac{5}{2} \|\nabla \psi\|_{BMO} \right\} \|\Omega(s)\|_{BMO(B(\rho_0))} + c \left\{\rho_0^{-\frac{\rho_0}{2}} + \rho_0^{-\frac{\rho_0}{2}} \|\nabla \psi\|_{BMO} \right\} \|V(s)\|_{L^2(B(\rho_0))} \\
\leq c \left(\frac{\rho_0}{\rho_0 - \rho_*}\right)^5 \|\Omega(s)\|_{BMO(B(\rho_0))} + c \rho_0^{-\frac{\rho_0}{2}} \left(\frac{\rho_0}{\rho_0 - \rho_*}\right)^5 \|V(s)\|_{L^2(B(\rho_0))} \\
\leq c \left(\frac{\rho_0}{\rho_0 - \rho_*}\right)^5 \left\{\|\omega(s)\|_{BMO(B(\rho))} + c \rho_0^{-\frac{\rho_0}{2}} \|\omega\|_{L^\infty(B(\rho))}\right\} \\
(5.13) \quad \leq c (\|\omega(s)\|_{BMO(B(\rho))} + 1),
\]

where we used Corollary 4.3 in the last step. Combining (5.12) and (5.13), we get

\[
\|\nabla V(s)\|_{L^\infty(B(r_{m+1}))} \leq c \left\{1 + \|\omega(s)\|_{BMO(B(\rho))}\right\} \log(e + Z_{m+1}).
\]

Combining (5.11) and (5.14), we arrive at

\[
Z_m(t) \leq d^m + \int_{t_0}^t a(s)Z_m(s) \log(e + Z_{m+1})ds.
\]

where

\[
d = c \left(\frac{\rho_0 + r}{\rho_0 - r}\right)^{\frac{\rho_0 - a}{2\rho}}, \quad a(s) = c (1 + \|\omega(s)\|_{BMO(B(\rho))}),
\]

while $c$ denotes a positive constant independently on $m \in \mathbb{N}$. Setting

\[
Y_m(s) = Z_m(s) + e,
\]

and eventually replacing $c$ by a larger constant independent on $m$, the estimate (5.13) turns into

\[
e + Y_m(t) \leq d^m + \int_{t_0}^t a(s)Y_m(s) \log(e + Y_{m+1})ds.
\]

19
We define

\[
X_m(t) = d^m + \int_{t_0}^{t} a(s)Y_m(s) \log(e + Y_{m+1}(s)) ds, \quad t \in [t_0, 0).
\]

Then, (5.17) implies \(e + Y_m \leq X_m\), and thus the differential inequality

\[
X'_m = aY_m \left( \log(Y_{m+1}) + 1 \right) \leq aX_m(t) \log(X_m+1) \quad \text{in} \quad [t_0, 0).
\]

Dividing both sides by \(X_m\), we are led to

\[
\beta'_m \leq a(t) \beta_{m+1}, \quad \text{where} \quad \beta_m(t) := \log(X_{m+1}(t)).
\]

Integrating the both sides over \((t_0, t)\) with \(t_0 \leq t < 0\), we find

\[
\beta_m(t) \leq m \log d + \int_{t_0}^{t} a(s) \beta_{m+1}(s) ds.
\]

We now verify that the sequence \(\{\beta_m(t)\}\) satisfy the condition \((A.4)\) of Lemma \(A.2\) below. From the definitions (5.18), (5.16) and (5.8) one has

\[
\beta_m(t) = \log(X_{m+1}(t))
\leq m \log d + \log \left( \int_{t_0}^{t_1} |a(s)| ds \sup_{t_0 < s < t} \log \left\{ |Y_m(s)| \log(e + Y_{m+1}(s)) \right\} \right) + 1
\leq m \log d
\]

\[
+ 3 \log \left( \int_{t_0}^{t_1} |a(s)| ds \log \left\{ \sup_{t_0 < s < t} \|\nabla^2 V(s)\|_{L^q(B(\rho_0))} + e \right\} \right) + 1
\leq M(t)^m
\]

for all \(m \in \mathbb{N}\), where we set

\[
M(t) = \log d + 3 \log \left( \int_{t_0}^{t_1} |a(s)| ds \log \left\{ \sup_{t_0 < s < t} \|\nabla^2 V(s)\|_{L^q(B(\rho_0))} + e \right\} \right) + 2 < +\infty
\]

for each \(t \in [t_0, t_1)\). Therefore the condition \((A.4)\) is satisfied. Applying Lemma \(A.2\) it follows that

\[
\log(e + Y_0(t)) \leq \beta_0(t) \leq \log d \int_{t_0}^{t} a(s) ds e^{\int_{t_0}^{s} a(s) ds}.
\]

According to the hypothesis \((1.4)\) we see that \(\sup_{t \in (-\rho, 0)} \log(e + Y_0(t)) < +\infty\). This yields

\[
\sup_{t \in (-\rho, 0)} \|\nabla^2 v(t)\|_{L^q(B(\rho))} \leq \sup_{t \in (-\rho, 0)} \|\nabla^2 V(t)\|_{L^q(B(\rho_0))} < +\infty.
\]
This completes the proof of Theorem 1.1.

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A Gronwall type lemma

Lemma A.1. Let $-\infty < a < b < +\infty$. Let $f \in L^1(a, b)$ and $k \in \mathbb{N}$. Then it holds

\[(A.1) \int_a^b \int_a^{t_1} \cdots \int_a^{t_k-1} f(t_j) dt_k dt_{k-1} \cdots dt_1 = \frac{1}{k!} \left( \int_a^b f(t) dt \right)^k.\]

Proof: We prove the assertion by induction. For $k = 1$ (A.1) is obvious. Assume the assertion holds for $k - 1$. Using the hypothesis of induction, we find

\[(A.2) \int_a^b \int_a^{t_1} \cdots \int_a^{t_k-1} f(t_j) dt_k dt_{k-1} \cdots dt_1 = \int_{t_1}^{t_k} \left\{ \int_{t_1}^{t_k-1} \cdots \int_{t_1}^{t_2} f(t_j) dt_k dt_{k-1} \cdots dt_2 \right\} dt_1 = \frac{1}{(k-1)!} \int_a^b f(s) \left( \int_a^s f(t) dt \right)^{k-1} ds.\]

Applying integration by parts, we calculate

\[
\int_a^b \int_a^s f(s) \left( \int_a^s f(t) dt \right)^{k-1} ds = \int_a^b \frac{d}{ds} \int_a^s f(t) dt \left( \int_a^s f(t) dt \right)^{k-1} ds
\]

\[
= \left( \int_a^b f(t) dt \right)^k - (k-1) \int_a^b f(s) \left( \int_a^s f(t) dt \right)^{k-2} ds
\]

\[
= \left( \int_a^b f(t) dt \right)^k - (k-1) \int_a^b f(s) \left( \int_a^s f(t) dt \right)^{k-1} ds.
\]

This yields

\[(A.3) \int_a^b \int_a^s f(s) \left( \int_a^s f(t) dt \right)^{k-1} ds = \frac{1}{k} \left( \int_a^b f(t) dt \right)^k.\]

Replacing the integral on the right-hand side of (A.2) by the right-hand side of (A.3), we obtain (A.1) for $k$. Hence by induction (A.1) holds for all $k \in \mathbb{N}$.
Lemma A.2 (Iteration lemma). Let $\beta_m : [t_0, t_1] \to \mathbb{R}$, $m \in \mathbb{N} \cup \{0\}$ be a sequences of bounded functions. Suppose there exists $0 < K = K(t) < +\infty$ for each $t \in [t_0, t_1]$ such that

$$|\beta_m(t)| < K(t)^m \quad \forall t \in [t_0, t_1], \forall m \in \mathbb{N}. \quad (A.4)$$

Furthermore let $a \in L^1(t_0, t_1)$ with $a(t) \geq 0$ for almost every $t \in [t_0, t_1]$. We assume that the following recursive integral inequality holds true for a constant $C > 0$

$$\beta_m(t) \leq Cm + \int_{t_0}^{t} a(s)\beta_{m+1}(s)ds, \quad m \in \mathbb{N} \cup \{0\}. \quad (A.5)$$

Then the following inequality holds true for all $t \in [t_0, t_1]$

$$\beta_0(t) \leq C \int_{t_0}^{t} a(s)ds e^{\int_{t_0}^{t} a(s)ds}. \quad (A.6)$$

**Proof:** Iterating (A.5) $m$-times, and applying Lemma A.1 we see that for each $t \in [t_0, t_1]$ it follows

$$\beta_0(t) \leq C \int_{t_0}^{t} a(s_1)ds_1 + 2C \int_{t_0}^{t} a(s_1) a(s_2)ds_2ds_1$$

$$+ \ldots + Cm \int_{t_0}^{t} \int_{t_0}^{s_1} \ldots \int_{t_0}^{s_{m-1}} a(s_1)a(s_2) \ldots a(s_m)ds_m \ldots ds_2ds_1$$

$$+ \int_{t_0}^{t} \int_{t_0}^{s_1} \ldots \int_{t_0}^{s_m} a(s_1)a(s_2) \ldots a(s_{m+1})\beta_{m+1}(s_{m+1})ds_{m+1} \ldots ds_2ds_1$$

\[ (A.7) \quad = \sum_{k=1}^{m} \frac{C}{(k-1)!} \left( \int_{t_0}^{t} a(s)ds \right)^k + J_m(t), \]

where

$$|J_m(t)| \leq \sup_{0<s<t} |\beta_{m+1}(s)| \int_{t_0}^{t} \int_{t_0}^{s_1} \ldots \int_{t_0}^{s_m} a(s_1)a(s_2) \ldots a(s_{m+1})ds_{m+1} \ldots ds_2ds_1$$

$$\leq \frac{1}{(m+1)!} \left( \sup_{0<s<t} K(s) \int_{t_0}^{t} a(s)ds \right)^{m+1} \to 0 \quad \text{as} \quad m \to +\infty$$

for each $t \in [t_0, t_1)$. Therefore,

$$\beta_0(t) \leq \sum_{k=1}^{\infty} \frac{C}{(k-1)!} \left( \int_{t_0}^{t} a(s)ds \right)^k = C \int_{t_0}^{t} a(s)ds e^{\int_{t_0}^{t} a(s)ds}. \quad \blacksquare$$
B Gagliardo-Nirenberg’s inequality with cut-off

Lemma B.1. Let \( \psi \in C_c^\infty(\mathbb{R}^n) \) with \( 0 \leq \psi \leq 1 \), and \( k \geq 1 \). For all \( u \in W^{1,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), \( 2 < q < +\infty \), and for \( m \geq k\left(\frac{n(q-2)}{2q}\right) \) it holds

\[
\|u\psi^{m-k}\|_q \leq c\|u\psi^{m-k-n}\|_q^{\frac{2q}{n(q-2)+2q}} \|\nabla \psi\|_q^{\frac{n(q-2)}{n(q-2)+2q}} + c\|u\psi\|_q \|\nabla \psi\|_\infty \|u\psi^{m-k-n}\|_2.
\]

(B.1)

Proof: Let \( \psi \in C_c^\infty(\mathbb{R}^n) \) with \( 0 \leq \psi \leq 1 \), and \( k \geq 1 \). Let \( m \geq k\left(\frac{n(q-2)}{2q}\right) \). By means of Hölder’s inequality we estimate

\[
\|u\psi^{m-k}\|_q \leq \|u\psi^{m-k-n}\|_q^{\frac{2q}{n(q-2)+2q}} \|\nabla \psi\|_q^{\frac{n(q-2)}{n(q-2)+2q}} \|u\psi^{m-k-n}\|_2 \leq c\|u\psi^{m-k-n}\|_q^{\frac{2q}{n(q-2)+2q}} \|\nabla \psi\|_q^{\frac{n(q-2)}{n(q-2)+2q}} \|u\psi^{m-k-n}\|_2 ,
\]

where \( w = |u|^{q-1}\psi^{m-k-n} \). By Sobolev’s inequality along with Hölder’s inequality we infer

\[
\|w\|_{\frac{q}{n-1}} \leq \|
abla w\|_1 \leq c \int_{\mathbb{R}^n} |u|^{q-1} |\nabla u| \psi^{m-n}\|\nabla \psi\|_\infty \|u\psi^{m-k-n}\|_q \|\nabla \psi\|_\infty \|u\psi^{m-k-n}\|_q.
\]

Combining the last two estimates and applying Young’s inequality, we obtain the assertion (B.1).

Lemma B.2. Let \( u \in W^{1,q}(B(\rho)) \), \( n < q < +\infty \) such that \( \nabla \cdot u = 0 \) almost everywhere in \( B(\rho) \). Then for every \( \psi \in C_c^\infty(B(\rho)) \) with \( 0 \leq \psi \leq 1 \) and \( m \geq \frac{q}{2} \) it holds

\[
\|u\psi^m\|_\infty \leq c\|u\psi^{m-\frac{q}{2}}\|_2^{\frac{2(q-n)}{2q-2n+q}} \|\nabla u\psi^m\|_2^{\frac{nq}{2q-2n+q}} + c\|\nabla \psi\|_\infty \|u\psi^{m-\frac{q}{2}}\|_2.
\]

(B.2)

Proof: By virtue of Gagliardo-Nirenberg’s inequality we estimate

\[
\|u\psi^m\|_\infty \leq c\|u\psi^{m-1}\|_q^{\frac{n}{q}} \|\nabla \psi\|_q^{\frac{n}{q}} \|u\psi^{m-1}\|_q ,
\]

From \( L^q \)-interpolation we find

\[
\|u\psi^{m-1}\|_q \leq \|u\psi^{m-\frac{q}{2}}\|_2^{\frac{q}{2}} \|u\psi^m\|_\infty^{1-\frac{q}{2}}.
\]

Combining the two estimates above, and applying Young’s inequality, we obtain the assertion of the lemma.

Lemma B.3. Let \( u \in BMO(B(\rho)) \). Then \( u \in \cap_{1 \leq q < \infty} L^q(B(\rho)) \), and it holds

\[
\|u\|_{L^q(B(\rho))} \leq c r^{\frac{q}{q-1}} |u|_{BMO(B(\rho))} + c r^{\frac{n}{q} - n} \|u\|_{L^1(B(\rho))} = c r^{\frac{q}{q-1}} \|u\|_{BMO(B(\rho))}.
\]

(B.3)
Proof: Let $u \in BMO(B(1))$. By $U \in BMO$ we denote the extension defined in Section 2. By John-Nirenberg's inequality\[9\] it holds

$$m(\{ x \in Q(2) \mid |U - U_{Q(2)}| > \lambda \}) \leq c_1 e^{-\frac{c_2 \lambda}{|U|_{BMO(Q(2))}}} \quad \forall \lambda \in (0, +\infty)$$

with constants $c_1, c_2$, depending on $n$ and $q$ only, where $m(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^n$, and $Q(r)$ denotes the cube with the center at origin and the side length $2r$. Multiplying both sides by $\lambda^{q-1}$, integrating the result over $(0, +\infty)$, using a suitable change of coordinates, and employing \[2.4\], we arrive at

$$\|U - U_{Q(2)}\|_{L^q(Q(2))}^q = q \int_0^\infty \lambda^{q-1} m(\{ x \in Q(2) \mid |U - U_{Q(2)}| > \lambda \})$$

$$\leq q |U|_{BMO(Q(2))}^{q-1} \int_0^\infty \lambda^{q-1} c_1 e^{-c_2 \lambda}$$

$$\leq c \|u\|_{BMO(B(1))}^q.$$

Accordingly,

$$\|u\|_{L^q(B(1))} \leq \|U - U_{Q(2)}\|_{L^q(Q(2))} + c |U_{Q(2)}| \leq c \|u\|_{BMO(B(1))}.$$

Hence, \[B.3\] follows from the above estimate by using a standard scaling argument. ■

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