M-convex Function Minimization
Under L1-Distance Constraint

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Abstract

In this paper we consider a new problem of minimizing an M-convex function under L1-distance constraint (MML1); the constraint is given by an upper bound for L1-distance between a feasible solution and a given “center.” This is motivated by a nonlinear integer programming problem for re-allocation of dock capacity in a bike sharing system discussed by Freund et al. (2017). The main aim of this paper is to better understand the combinatorial structure of the dock re-allocation problem through the connection with M-convexity, and show its polynomial-time solvability using this connection. For this, we first show that the dock re-allocation problem can be reformulated in the form of (MML1). We then present a pseudo-polynomial-time algorithm for (MML1) based on steepest descent approach. We also propose two polynomial-time algorithms for (MML1) by replacing the L1-distance constraint with a simple linear constraint. Finally, we apply the results for (MML1) to the dock re-allocation problem to obtain a pseudo-polynomial-time steepest descent algorithm and also polynomial-time algorithms for this problem. The proposed algorithm is based on a proximity-scaling algorithm for a relaxation of the dock re-allocation problem, which is of interest in its own right.

1 Introduction

The concepts of M-convexity and M♮-convexity for functions in integer variables play a primary role in the theory of discrete convex analysis [11]. M-convex function, introduced by Murota [9] [10], is defined by a certain exchange axiom (see Section 2 for a precise definition), and enjoys various nice properties as “discrete convexity” such as a local characterization for global minimality, extensibility to ordinary convex functions, conjugacy, duality, etc. M♮-convex function is introduced by Murota and Shioura [14] as a variant of M-convex function. While the class of M♮-convex functions properly contains that of M-convex functions, the concept of M♮-convexity is essentially equivalent to M-convexity in some sense (see, e.g., [11]). Minimization of an M-convex function is the most fundamental optimization problem concerning M-convex functions, and a common generalization of the separable convex resource allocation problem under a submodular constraint and some classes of nonseparable convex function minimization on integer lattice points. M-convex function minimization can be solved by a steepest descent algorithm (or greedy algorithm) that runs in pseudo-polynomial time [11] [12], and various polynomial-time algorithms have been proposed [8] [15] [16] [17].
In this paper, we consider a new problem of minimizing an M-convex function under the L1-distance constraint, which is formulated as follows:

\[
\begin{align*}
\text{(MML1)} & \quad \text{Minimize} & & f(x) \\
& \quad \text{subject to} & & \sum_{i=1}^{n} x(i) = \theta, \\
& & & ||x - x_c||_1 \leq 2\gamma, \\
& & & x \in \text{dom } f,
\end{align*}
\]

where $\theta, \gamma \in \mathbb{Z}$, $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is an M-convex function such that $\sum_{i=1}^{n} x(i) = \theta$ holds for every $x \in \mathbb{Z}^n$ with $f(x) < +\infty$, and $x_c$ is a vector (called the “center”) with $f(x_c) < +\infty$ and $\sum_{i=1}^{n} x_c(i) = \theta$. This problem is motivated by a nonlinear integer programming problem for re-allocation of dock-capacity in a bike sharing system \[1\].

In a bike sharing system, many bike stations are located around a city so that users can rent and return bikes there. Each bike station has several docks and bikes; some docks are equipped with bikes, and the other docks are kept open so that users can return bikes at the station. The numbers of docks with bike and of open docks change as time passes, and it is possible that some users cannot rent or return a bike at a station due to the shortage of bikes or open docks, and in such situation users feel dissatisfied. To reduce users’ dissatisfaction, operators of a bike sharing system need to re-allocate docks (and bikes) among bike stations appropriately. Change to a new allocation, however, requires the movement of docks and bikes, which yields some amount of cost. Therefore, it is desirable that a new allocation is not so different from the current allocation. Hence, the task of operators in a bike sharing system is to minimize users’ dissatisfaction by changing the allocation of docks, while bounding the number of docks to be moved in the re-allocation.

This problem, which we refer to as the dock re-allocation problem, is discussed by Freund, Henderson, and Shmoys \[1\] and formulated as follows:

\[
\begin{align*}
\text{(DR)} & \quad \text{Minimize} & & \sum_{i=1}^{n} c_i(d(i), b(i)) \\
& \quad \text{subject to} & & \sum_{i=1}^{n} d(i) + b(i) = D + B, \\
& & & \sum_{i=1}^{n} b(i) \leq B, \\
& & & \sum_{i=1}^{n} [(d(i) + b(i)) - (\bar{d}(i) + \bar{b}(i))] \leq 2\gamma, \\
& & & \ell(i) \leq d(i) + b(i) \leq u(i), \quad d(i), b(i) \in \mathbb{Z}_+^n \quad (i \in N).
\end{align*}
\]

Here, $n \in \mathbb{Z}$ denotes the number of bike stations and $N = \{1, 2, \ldots, n\}$. For a station $i \in N$, we denote by $b(i), d(i) \in \mathbb{Z}_+$, respectively, the decision variables representing the numbers of docks with bike and of open docks allocated at the station. The expected number of dissatisfied users at the station $i$ is represented by a function $c_i : \mathbb{Z}_+^n \rightarrow \mathbb{R}$ in variables $d(i)$ and $b(i)$, and shown to have the property of multimodularity (see Section 2 for the definition).

The first constraint in (DR) means that the total number of docks (i.e., docks with bike and open docks) is equal to a fixed constant $D + B$. The second constraint gives an upper bound for the total number of docks with bike. The third constraint, given in the form of L1-distance constraint, means that the difference between the current and the new allocations of docks should be small, where $\bar{d}(i)$ and $\bar{b}(i)$ denote, respectively, the numbers of docks with bike and of open docks at the station $i$ in the current allocation. In addition, the number of docks $d(i) + b(i)$ at each station $i$ should be between lower and upper bounds $[\ell(i), u(i)]$, as represented by the fourth constraint.

\[\text{footnote}{\text{While the first constraint is given as an inequality } \sum_{i=1}^{n} (d_i + b_i) \leq D + B \text{ in } [1], \text{ it is implicitly assumed in } [1] \text{ that the inequality holds with equality. Indeed, the algorithm in } [1] \text{ applies only to the problem with the equality constraint.}}\]
For the problem (DR), Freund et al. [1] propose a steepest descent (or greedy) algorithm that repeatedly update a constant number of variables by ±1, and prove by using the multimodularity of the objective function that the algorithm finds an optimal solution of (DR) in at most $\gamma$ iterations. Hence, the problem (DR) can be solved in pseudo-polynomial time, while it is not known so far whether (DR) can be solved in polynomial time.

**Our Contribution** The main aim of this paper is to better understand the combinatorial structure of the problem (DR) through the connection with M-convexity, and to provide polynomial-time algorithms for (DR) by using the connection.

We first show that the dock reallocation problem (DR) can be reformulated in the form of the minimization of an M-convex function under the L1-distance constraint (MML1), where we regard $d(i) + b(i)$ as a single variable (see Section 3 for details).

We then consider the problem (MML1) and present a steepest descent algorithm that runs in pseudo-polynomial time. While it is known that unconstrained M-convex function minimization (without the L1-distance constraint) can be solved by a certain steepest descent algorithm (see [11, 12]; see also Section 4 for details), a naive application of the algorithm does not work for the problem (MML1), due to the L1-distance constraint. Nevertheless, we prove in Section 4 that if the center $x_c$ is used as an initial solution of the algorithm, then the steepest descent algorithm finds an optimal solution in $\gamma$ iterations. Moreover, we prove a stronger statement that for each $k = 0, 1, 2, \ldots$, the vector generated in the $k$-th iteration of the steepest descent algorithm is an optimal solution of the M-convex function minimization under the constraint $\|x - x_c\|_1 = 2k$.

As a byproduct of this result, we obtain new properties of the steepest descent algorithm for unconstrained M-convex function minimization. In particular, we provide a nontrivial tight bound on the number of iterations required by the algorithm, and show that the trajectory of the solutions generated by the algorithm is a geodesic (i.e., a "shortest" path) to the nearest optimal solution from the initial solution.

While the problem (MML1) can be solved by a steepest descent algorithm, its running time is pseudo-polynomial time. To obtain faster algorithms, we present in Section 5 two approaches to solve (MML1) in polynomial time. For this, we show that by using a minimizer of the M-convex objective function, the L1-distance constraint in (MML1) can be replaced with a simple linear constraint; the two approaches proposed in this section solve the M-convex function minimization under the simple linear constraint instead of the original problem. The first approach is to reduce the problem to the minimization of the sum of two M-convex functions, for which polynomial-time algorithms are available. The second approach is based on the reduction to the minimization of another M-convex function with smaller number of variables, and the resulting algorithm is faster than the first approach.

Finally, in Section 6 we apply the algorithms for (MML1) presented in Sections 4 and 5 to the dock reallocation problem (DR), which can be regarded as a special case of (MML1). We aim at obtaining fast algorithms by making use of the special structure of (DR).

In Section 6.1, we discuss an application of the steepest descent algorithm in Section 4 to (DR). A naive application of the algorithm takes $O(n^3 \log(B/n))$ time in each iteration since it requires $O(n \log(B/n))$ time for the evaluation of the M-convex function $f$ used in the reformulation of (DR). To reduce the time complexity, we present a useful property of the M-convex function $f$ that the update of function value $f(x)$ can be done quickly in $O(\log n)$ time if the vector $x$ is updated to a vector in a neighborhood. Furthermore, we make full use of this property to implement the steepest descent algorithm so that the algorithm works for the original formulation and each iteration requires $O(\log n)$ time only. We also discuss the connection with the steepest descent algorithm in [1].

Section 6.2 is devoted to polynomial-time algorithms for (DR). While the polynomial-time
The solvability of (DR) follows from the results in Section 5. Naive application of an algorithm in Section 5 leads to a polynomial-time but rather slow algorithm for (DR); a faster implementation is difficult this time since the algorithms in Section 4 are more involved. Instead, we use an idea in Section 4 and the structure of (DR) to obtain a faster polynomial-time algorithm. For this, we replace the L1-distance constraint in (DR) with a simple linear constraint, as in Section 5. This new formulation, together with the use of a new problem parameter, makes it possible to decompose the problem (DR) into two independent subproblems, both of which can be reduced to M-convex function minimization and therefore can be solved efficiently. We show that an algorithm based on this approach runs in \( O(n \log n \log((D + B)/n) \log B) \) time. To obtain this time bound, we prove a proximity theorem for a relaxation of the problem (DR) and devise a proximity-scaling algorithm for the relaxation; the proximity theorem and the algorithm are of interest in their own right.

Most of proofs are provided in Appendix.

2 Preliminaries on M-convexity

Throughout the paper, let \( n \) be a positive integer with \( n \geq 2 \) and put \( N = \{1, 2, \ldots, n\} \). We denote by \( \mathbb{R} \) the sets of real numbers, and by \( \mathbb{Z} \) (resp., by \( \mathbb{Z}_+ \)) the sets of integers (resp., nonnegative integers); \( \mathbb{Z}_{++} \) denotes the set of positive integers.

Let \( x = (x(1), x(2), \ldots, x(n)) \in \mathbb{R}^n \) be a vector. We define \( \text{supp}^+(x) = \{i \in N \mid x(i) > 0\} \) and \( \text{supp}^-(x) = \{i \in N \mid x(i) < 0\} \). For a subset \( Y \subseteq N \), we define \( x(Y) = \sum_{i \in Y} x(i) \). We define \( \|x\|_1 = \sum_{i \in N} |x(i)| \) and \( \|x\|_\infty = \max_{i \in N} |x(i)| \).

We define \( 0 = (0, 0, \ldots, 0) \in \mathbb{Z}^n \). For \( Y \subseteq N \), we denote by \( \chi_Y \in \{0, 1\}^n \) the characteristic vector of \( Y \), i.e., \( \chi_Y(i) = 1 \) if \( i \in Y \) and \( \chi_Y(i) = 0 \) otherwise. In particular, we denote \( \chi_i = \chi_{\{i\}} \) for every \( i \in N \). We also denote \( \chi_0 = 0 \). Inequality \( x \leq y \) for vectors \( x, y \in \mathbb{R}^n \) means component-wise inequality \( x(i) \leq y(i) \) for all \( i \in N \).

2.1 M-convex and Multimodular Functions

Let \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) be a function. The effective domain of \( f \) is defined by \( \text{dom} f = \{x \in \mathbb{Z}^n \mid f(x) < +\infty\} \), and the set of minimizers of \( f \) is denoted by \( \text{arg min} f \). Function \( f \) is said to be \( M^i \)-convex if it satisfies the following exchange property:

\[
(M^{i,\text{EXC}}) \forall x, y \in \text{dom} f, \forall i \in \text{supp}^+(x - y), \exists j \in \text{supp}^-(x - y) \cup \{0\} : f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).
\]

For an \( M^i \)-convex function \( f \), if \( \text{dom} f \) is contained in a hyperplane \( \{x \in \mathbb{Z}^n \mid x(N) = \theta\} \) for some \( \theta \in \mathbb{Z} \), then \( f \) is called an \( M^i \)-convex function, in particular. It is known that a function \( f \) is \( M \)-convex if and only if it satisfies the following exchange property:

\[
(M^{\text{EXC}}) \forall x, y \in \text{dom} f, \forall i \in \text{supp}^+(x - y), \exists j \in \text{supp}^-(x - y) : f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).
\]

\( M^i/M^2 \)-convex functions can be characterized by seemingly weaker exchange properties.

**Theorem 2.1** ([11] Theorem 6.4, [14]). Let \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) be a function.

(i) \( f \) is \( M \)-convex if and only if it satisfies the following condition:

\[
\forall x, y \in \text{dom} f \text{ with } x \neq y, \exists i \in \text{supp}^+(x - y), \exists j \in \text{supp}^-(x - y) : f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).
\]
(ii) $f$ is $M^\#$-convex if and only if it satisfies the following condition:
\[
\forall x, y \in \text{dom } f \text{ with } x \neq y \text{ and } x(N) \geq y(N), \exists i \in \text{supp}^+(x-y), \exists j \in \text{supp}^-(x-y) \cup \{0\} : \\
f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).
\]

$M^\#$-convexity of a function implies the following exchange properties.

**Theorem 2.2** ([14]). Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an $M^\#$-convex function and $x, y \in \text{dom } f$.

(i) If $x(N) \leq y(N)$, then for every $i \in \text{supp}^+(x-y)$ there exists some $j \in \text{supp}^-(x-y)$ such that
\[
f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).
\]

(ii) If $x(N) < y(N)$, then there exists some $j \in \text{supp}^-(x-y)$ such that
\[
f(x) + f(y) \geq f(x + \chi_j) + f(y - \chi_j).
\]

We then explain the concept of multimodularity and its connection with $M^\#$-convexity. A function $\varphi : \mathbb{Z}^2_+ \to \mathbb{R}$ in two variables is called multimodal if it satisfies the following conditions:
\[
\varphi(\eta + 1, \zeta + 1) - \varphi(\eta, \zeta + 1) \geq \varphi(\eta, \zeta) - \varphi(\eta, \zeta) \quad (\forall \eta, \zeta \in \mathbb{Z}_+),
\]
\[
\varphi(\eta - 1, \zeta + 1) - \varphi(\eta - 1, \zeta) \geq \varphi(\eta, \zeta) - \varphi(\eta, \zeta - 1) \quad (\forall \eta, \zeta \in \mathbb{Z}_+),
\]
\[
\varphi(\eta + 1, \zeta - 1) - \varphi(\eta, \zeta - 1) \geq \varphi(\eta, \zeta) - \varphi(\eta - 1, \zeta) \quad (\forall \eta, \zeta \in \mathbb{Z}_+).
\]

For functions in two variables, multimodularity and $M^\#$-convexity are essentially equivalent.

**Proposition 2.3** (cf. [7]). A function $\varphi : \mathbb{Z}^2_+ \to \mathbb{R}$ in two variables is multimodal if and only if the function $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{+\infty\}$ given by
\[
\text{dom } f = \mathbb{Z}^2_+, \quad f(\alpha, \beta) = \varphi(\alpha, \beta) \quad (\alpha, \beta) \in \text{dom } f
\]
is $M^\#$-convex.

This relationship and Theorem 2.2 immediately imply the following property of multimodal functions.

**Proposition 2.4.** Let $\varphi : \mathbb{Z}^2_+ \to \mathbb{R}$ be a multimodal function, and $\eta, \zeta, \eta', \zeta' \in \mathbb{Z}_+$.

(i) If $\eta > \eta'$ and $\zeta < \zeta'$, then it holds that
\[
\varphi(\eta, \zeta) + \varphi(\eta', \zeta') \geq \varphi(\eta - 1, \zeta + 1) + \varphi(\eta' + 1, \zeta' - 1).
\]

(ii) If $\eta > \eta'$ and $\eta + \zeta > \eta' + \zeta'$, then it holds that
\[
\varphi(\eta, \zeta) + \varphi(\eta', \zeta') \geq \varphi(\eta - 1, \zeta) + \varphi(\eta' + 1, \zeta').
\]

**Proof.** By Proposition 2.3, $\varphi$ can be seen as an $M^\#$-convex function. We first prove the claim (i). Theorem 2.2 (i) implies that if $\eta + \zeta \leq \eta' + \zeta'$ then $\varphi(\eta, \zeta) + \varphi(\eta', \zeta') \geq \varphi(\eta - 1, \zeta + 1) + \varphi(\eta' + 1, \zeta' - 1)$, and if $\eta' + \zeta' \leq \eta + \zeta$ then $\varphi(\eta, \zeta') + \varphi(\eta', \zeta) \geq \varphi(\eta' + 1, \zeta' - 1) + \varphi(\eta - 1, \zeta + 1)$. In either case, the inequality holds.

We then prove the claim (ii). If $\eta \geq \zeta'$, then the inequality follows immediately from Theorem 2.2 (ii). If $\eta + \zeta > \eta' + \zeta'$ and $\zeta < \zeta'$, then the inequality follows immediately from (M$^\#$-EXC).
2.2 Minimization of an M-convex Function

We consider the minimization of an M-convex function. A minimizer of an M-convex function can be characterized by a local optimality condition.

Theorem 2.5 (cf. [11, Theorem 6.26]). For an M-convex function \( f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \), a vector \( x^* \in \text{dom} f \) is a minimizer of \( f \) if and only if \( f(x^* - \chi_i + \chi_j) \geq f(x^*) \) (\( \forall i, j \in \mathbb{N} \)).

This theorem immediately implies that the minimization of an M-convex function can be solved by the following steepest descent algorithm (see, e.g., [11, Section 10.1.1]):

Algorithm SteepestDescent
Step 0: Let \( x_0 \in \text{dom} f \) be an appropriately chosen initial vector. Set \( k := 1 \).
Step 1: If \( f(x_{k-1} + \chi_i - \chi_j) \geq f(x_{k-1}) \) for every \( i, j \in \mathbb{N} \), then output \( x_{k-1} \) and stop.
Step 2: Find \( i_k, j_k \in \mathbb{N} \) that minimizes \( f(x_{k-1} + \chi_{i_k} - \chi_{j_k}) \).
Step 3: Set \( x_k := x_{k-1} + \chi_{i_k} - \chi_{j_k} \), \( k := k + 1 \), and go to Step 1.

Theorem 2.6 (cf. [11, Section 10.1.1]). Let \( f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) be an M-convex function \( f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) with bounded \( \text{dom} f \). Then, the algorithm SteepestDescent outputs a minimizer of \( f \) after a finite number of iterations.

Polynomial-time algorithms based on proximity scaling algorithms are proposed for M-convex function minimization [8, 15, 16, 17], and the current best time complexity bounds are given as follows. For a set \( S \subseteq \mathbb{Z}^n \), we define the \( L_\infty \)-diameter of \( S \) by

\[
L = \max\{\|x - y\|_\infty \mid x, y \in S\}.
\]

Theorem 2.7 ([16, 17]). Minimization of an M-convex function \( f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) can be done in \( O(n^3 \log(L/n)F) \) time, where \( L \) is the \( L_\infty \)-diameter of \( \text{dom} f \) and \( F \) denotes the time to evaluate the function value of \( f \).

We also consider the minimization of an \( \Gamma \)-convex function \( f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) under the constraint that \( x(\bar{N}) = \theta \) for a given \( \theta \in \mathbb{Z} \). While this problem is essentially equivalent to M-convex function minimization, it can be solved faster if \( \text{dom} f \) is given by an interval.

Theorem 2.8 (cf. [10]). Let \( f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) be an \( \Gamma \)-convex function such that \( \text{dom} f \) is given by an interval, and \( \theta \in \mathbb{Z}_{++} \). Then, the minimization of \( f \) under the constraint \( x(\bar{N}) = \theta \) can be solved in \( O(n^2 \log(L/n)F) \) time, where \( L \) is the \( L_\infty \)-diameter of the set \( \{x \in \text{dom} f \mid x(\bar{N}) = \theta \} \) and \( F \) denotes the time to evaluate the function value of \( f \).

3 Reformulation of Dock Re-allocation Problem as (MML1)

We consider the dock re-allocation problem (DR) explained in Introduction. Using vector notation, the problem (DR) can be simply rewritten as follows:

\[
\text{(DR)} \quad \begin{array}{ll}
\text{Minimize} & c(d, b) \\
\text{subject to} & d(N) + b(N) = D + B, \\
& b(N) \leq B, \\
& ||(d + b) - (\bar{d} + \bar{b})||_1 \leq 2\gamma, \\
& \ell \leq d + b \leq u, \quad d, b \in \mathbb{Z}_u^n,
\end{array}
\]

6
where \( c : \mathbb{Z}_+^n \times \mathbb{Z}_+^n \to \mathbb{R} \) is a function given by \( c(d, b) = \sum_{i=1}^n c_i(d(i), b(i)) \). In this section, we show that (DR) can be reformulated as the problem (MML1).

We define a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) by

\[
\begin{align*}
\text{dom } f &= \{ x \in \mathbb{Z}^n \mid x(N) = D + B, \ell \leq x \leq u, d + b - \gamma 1 \leq x \leq d + b + \gamma 1 \}, \\
f(x) &= \min \{ c(d, b) \mid d, b \in \mathbb{Z}_+^n, d + b = x, b(N) \leq B \} \quad (x \in \text{dom } f).
\end{align*}
\]

As shown below, \( f \) is an \( M^2 \)-convex function. With this function \( f \), the problem (DR) can be reformulated as

\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{subject to} & \quad x(N) = D + B, \\
& \quad \|x - (d + b)\|_1 \leq 2\gamma, \\
& \quad x \in \text{dom } f.
\end{align*}
\]

Hence, (DR) is reformulated as (MML1). We note that in the reformulation of (DR) above, the constraint \( \|x - (d + b)\|_1 \leq 2\gamma \) implies the inequality \( d + b - \gamma 1 \leq x \leq d + b + \gamma 1 \) that appears in the definition of \( \text{dom } f \) in (3.1). Hence, addition of this constraint in the definition of \( \text{dom } f \) is not necessary in the reformulation above, but it is added to obtain a better time complexity in the following section.

**Theorem 3.1.** Function \( f \) in (3.1) is \( M \)-convex.

We also consider a function \( \hat{f} : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) by

\[
\begin{align*}
\text{dom } \hat{f} &= \{ x \in \mathbb{Z}^n \mid \ell \leq x \leq u, d + b - \gamma 1 \leq x \leq d + b + \gamma 1 \}, \\
\hat{f}(x) &= \min \{ c(d, b) \mid d, b \in \mathbb{Z}_+^n, d + b = x, b(N) \leq B \} \quad (x \in \text{dom } \hat{f}).
\end{align*}
\]

The difference from the function \( f \) in (3.1) is that the equation \( x(N) = D + B \) is missing in the definition of \( \hat{f} \). It is easy to see that for every \( x \in \mathbb{Z}^n \) with \( x(N) = D + B \), we have \( \hat{f}(x) = f(x) \). In a similar way as \( f \), we can show that \( \hat{f} \) is an \( M^2 \)-convex function. Hence, instead of \( f \), we may use \( \hat{f} \) as an objective function of the reformulation of (DR). This objective function is useful in obtaining a faster algorithm. We note that the effective domain of \( \hat{f} \) is given by an interval. This fact is used in Section 6.

## 4 Steepest Descent Algorithm for (MML1)

In this section, we show that an optimal solution of the problem (MML1) can be obtained by using a variant of the steepest descent algorithm STEEPEST DESCENT in Section 2.2 for unconstrained \( M \)-convex function minimization. While we are mainly interested in the case where the center \( x_c \) is a feasible solution to (MML1), we also consider the case with infeasible \( x_c \).

We assume that the effective domain \( \text{dom } f \) of the function \( f \) is bounded; this assumption implies that \( \text{arg min } f \neq \emptyset \), in particular.

Let \( \sigma \in \mathbb{Z}_+ \) be the half of L1-distance between \( x_c \) and a nearest vector in \( \text{dom } f \), and \( \tau \in \mathbb{Z}_+ \) the half of L1-distance between \( x_c \) and a nearest minimizer of \( f \), i.e.,

\[
\sigma = (1/2) \min \{ \|x - x_c\|_1 \mid x \in \text{dom } f\}, \quad \tau = (1/2) \min \{ \|x - x_c\|_1 \mid x \in \text{arg min } f\}. \tag{4.1}
\]

We have \( \sigma = 0 \) if \( x_c \) is a a feasible solution. Also, note that a minimizer \( x^* \) of \( f \) with \( \|x^* - x_c\|_1 = 2\tau \) is given by a minimizer of a function \( f(x) + \varepsilon \|x - x_c\|_1 \) with a sufficiently small positive \( \varepsilon \). Since the sum of an \( M \)-convex function and a separable-convex function is \( M \)-convex

\[7\]
Theorem 6.13, a minimizer of \( f(x) + \varepsilon \|x - x_c\|_1 \) can be obtained by any algorithm for unconstrained M-convex function minimization. If \( \tau \leq \gamma \), then the vector \( x^* \) is optimal for (MML1). Hence, we assume \( \tau > \gamma \) in the following.

In the following, we denote by (MML1(\( k \))) the problem (MML1) with the constant \( \gamma \) in the L1-distance constraint is replaced with a parameter \( k \in \mathbb{Z}_+ \). We first present a property of optimal solutions of the problem (MML1(\( k \))). For every \( k \), we denote by \( M_k \subseteq \mathbb{Z}^n \) and by \( \mu_k \in \mathbb{R} \), respectively, the set of optimal solutions and the optimal value of the problem (MML1(\( k \))). We have \( M_0 = \{ x_c \} \) and \( \mu_0 = f(x_c) \) if \( x_c \) is feasible; we also have \( M_k = \{ x \in \arg \min f \mid \|x - x_c\|_1 \leq 2k \} \) and \( \mu_k = \min f \) for every \( k \geq \tau \).

**Theorem 4.1.**

(i) It holds that \( \mu_\sigma > \mu_{\sigma+1} > \cdots > \mu_\tau \) and \( M_k \subseteq \{ x \in \mathbb{Z}^n \mid \|x - x_c\|_1 = 2k \} \) for \( k \in [\sigma, \tau] \).

(ii) For every integer \( k \in [\sigma, \tau - 1] \) and \( y \in M_k \), there exists some \( y \in M_{k+1} \) such that \( y = y + \chi_i - \chi_j \) for some \( i \in N \setminus \text{supp}^+ (y - x_c) \) and \( j \in N \setminus \text{supp}^+ (y - x_c) \).

(iii) For every integer \( k \in [\sigma, \tau - 1] \) and \( y \in M_{k+1} \), there exists some \( y' \in M_k \) such that \( y' = y + \chi_i + \chi_j \) for some \( i \in \text{supp}^+ (y - x_c) \) and \( j \in \text{supp}^- (y - x_c) \).

This is the key property to prove the validity of the algorithms presented in this section. In particular, we see from the claim (i) in the theorem that the set of optimal solutions of (MML1) is given by \( M_\gamma \).

Theorem 4.1 implies that a variant of the steepest descent algorithm for unconstrained M-convex function minimization finds an optimal solution of (MML1).

**Algorithm SteepestDescentMML1**

**Step 0:** Compute \( \sigma \) in (4.1) and \( x^0 \in M_\sigma \). Set \( x_\sigma := x^0 \) and \( k := \sigma + 1 \).

**Step 1:** If \( k - 1 = \gamma \), then output \( x_{k-1} \) and stop.

**Step 2:** Find \( i_k, j_k \in N \) that minimizes \( f(x_{k-1} + \chi_{i_k} - \chi_{j_k}) \).

**Step 3:** Set \( x_k := x_{k-1} + \chi_{i_k} - \chi_{j_k} \), \( k := k + 1 \), and go to Step 1.

**Theorem 4.2.** The algorithm SteepestDescentMML1 applied to an M-convex function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) outputs an optimal solution of (MML1) in \( \gamma - \sigma \) iterations. Moreover, the vector \( x_k \) generated in each iteration of the algorithm satisfies \( x_k \in M_k \).

**Proof.** We prove by induction that \( x_k \in M_k \) for each \( k \). Assume that \( x_{k-1} \in M_{k-1} \) holds for some \( k < \gamma \). By the behavior of the algorithm and Theorem 4.1, \( x_k \) is given as \( x_k = x_{k-1} + \chi_{i_k} - \chi_{j_k} \) with \( i_k \neq j_k \) and satisfies \( x_k \in M_k \). \( \square \)

Note that the running time of the algorithm SteepestDescentMML1, except for Step 0, is \( O(n^2 (\gamma - \sigma)) \), provided that the evaluation of function value can be done in constant time. Computation of \( \sigma \) and \( x^0 \) in Step 0 can be done by finding a minimizer \( x^0 \) of a function \( f(x) + \Upsilon \|x - x_c\|_1 \) with a sufficiently large positive \( \Upsilon > \max \{ f(x) \mid x \in \text{dom} f \} \) and then setting \( \sigma = \|x^0 - x_c\|_1 \). Function \( f(x) + \Upsilon \|x - x_c\|_1 \) is also M-convex, and therefore its minimization can be done by any algorithm for M-convex function minimization, even if the value \( \Upsilon \) is not given specifically.

Using Theorem 4.1 (iii), we can also consider another variant of steepest descent algorithm that starts from a nearest minimizer \( x^* \) of \( f \) and greedily approaches \( x_c \); see Appendix.
5 Polynomial-Time Algorithms for (MML1)

In this section we show that the problem (MML1) can be solved in polynomial time. As in Section 4 we assume that the value \( \tau \) in (4.1) satisfies \( \tau > \gamma \), and let \( x^* \in \text{dom } f \) be a minimizer of \( f \) with \( \|x^* - x_c\|_1 = 2\tau \), which is fixed throughout this section.

We note that every vector \( x \) satisfying the constraint \( \|x - x_c\|_1 \leq 2\gamma \) is contained in the interval \( [x_c - \gamma \mathbf{1}, x_c + \gamma \mathbf{1}] \). Hence, we assume in this section that the effective domain \( \text{dom } f \) of \( f \) is also contained in the interval \( [x_c - \gamma \mathbf{1}, x_c + \gamma \mathbf{1}] \); if the given \( f \) does not satisfy this condition, then it suffices to consider the restriction of \( f \) on this interval. This assumption implies that the \( L_\infty \)-diameter of \( f \) is bounded by \( 2\gamma \); we use this fact in the analysis of algorithms.

5.1 Reduction to Problem with Linear Constraints

We first show that the \( L_1 \)-distance constraint \( \|x - x_c\|_1 \leq 2\gamma \) in (MML1) can be replaced with a system of linear constraints. Let us consider the following problem:

\[
\text{(MM-L)} \quad \begin{array}{l}
\text{Minimize } f(x) \\
\text{subject to } \begin{align*}
& x(N) = \theta, \\
& x(P) = x_c(P) + \gamma, \\
& \ell \leq x \leq u, \\
& x \in \text{dom } f,
\end{align*}
\end{array}
\]

where \( P = \text{supp}^+(x^* - x_c) \), and \( \ell, u \in \mathbb{Z}^n \) are vectors given by

\[
\ell(i) = \begin{cases} 
  x_c(i) & (i \in P), \\
  \max \{x^*(i), x_c(i) - \gamma\} & (i \in N \setminus P),
\end{cases}
\quad
\quad
u(i) = \begin{cases} 
  \min \{x^*(i), x_c(i) + \gamma\} & (i \in P), \\
  x_c(i) & (i \in N \setminus P).
\end{cases}
\]

Lemma 5.1. Every optimal solution of (MM-L) is also optimal for (MML1).

While the problem (MM-L) does not fit into the framework of M-convex function minimization problem, due to the constraint \( x(P) = x_c(P) + \gamma \), it can be formulated as the minimization of the sum of two M-convex functions. Indeed, (MM-L) is equivalent to the minimization of the sum of functions \( f_1, f_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\} \) given by

\[
f_1(x) = \begin{cases} 
  f(x) & \text{(if } x(N) = \theta), \\
  +\infty & \text{(otherwise)},
\end{cases}
\quad
f_2(x) = \begin{cases} 
  0 & \text{(if } x(N) = \theta, x(P) = x_c(P) + \gamma, \ell \leq x \leq u), \\
  +\infty & \text{(otherwise)}.
\end{cases}
\]

It is not difficult to see that \( f_1 \) and \( f_2 \) satisfy (M-EXC), i.e., the two functions are M-convex.

It is known that minimization of the sum of two M\(^2\)-convex functions \( f_1, f_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\} \) can be solved in polynomial time (see, e.g., [11]), and the fastest algorithm runs in \( O(n^6(\log L)^2 \log(nK)) \) time [5], where \( L \) is the maximum of the \( L_\infty \)-diameter of \( \text{dom } f_1 \) and of \( \text{dom } f_2 \) (see (2.3) for the definition of \( L_\infty \)-diameter) and \( K \) is given by \( K = \max_{h=1,2} \max \{|f_h(x) - f_h(y)| : x, y \in \text{dom } f_h\} \). For the functions \( f_1 \) and \( f_2 \) defined above, the \( L_\infty \)-diameter of \( f_1 \) and \( f_2 \) is bounded by \( \max_{i \in N} \{\hat{u}(i) - \ell(i)\} \leq \gamma \). Hence, we obtain the following result.

Theorem 5.2. The problem (MML1) can be solved in \( O(n^6(\log \gamma)^2 \log(nK_f)) \) time, where \( K_f = \max\{|f(x) - f(y)| : x, y \in \text{dom } f\} \).
5.2 Reduction to M-convex Function Minimization

We now explain an alternative approach to solve the problem (MM-L) by the reduction to the minimization of an M-convex function.

For a vector \( y \in \mathbb{Z}^N \setminus P \) we define a set \( T(y) \subseteq \mathbb{Z}^n \) by
\[
T(y) = \{ x \in \text{dom} f \mid x(i) = y(i) \ (i \in N \setminus P), \ \hat{\ell}(i) \leq x(i) \leq \hat{u}(i) \ (i \in P) \}.
\]
Then, the function \( g : \mathbb{Z}^N \setminus P \rightarrow \mathbb{R} \cup \{ +\infty \} \) is defined as follows:
\[
g(y) = \begin{cases} 
\min \{ f(x) \mid x \in T(y) \} & \text{if } y(N \setminus P) = \theta - (x_c(P) + \gamma) \\
+\infty & \text{and } \hat{\ell}(i) \leq y(i) \leq \hat{u}(i) \ (\forall i \in N \setminus P) 
\end{cases}, \quad (5.1)
\]
By definition, \( x \in \mathbb{Z}^n \) is a feasible solution of (MM-L) if and only if the vector \( y \in \mathbb{Z}^N \setminus P \) given by \( y(i) = x(i) \ (i \in N \setminus P) \) satisfies \( y \in \text{dom} g \) and \( x \in T(y) \). Therefore, the problem (MM-L) can be reduced to the minimization of function \( g \): for a minimizer \( y^* \in \mathbb{Z}^N \setminus P \) of \( g \), the vector \( x^* \in T(y^*) \) with \( g(y^*) = f(x^*) \) is an optimal solution of (MM-L).

**Proposition 5.3.** Function \( g \) is M-convex.

We analyze the running time of the algorithm. By Theorem 2.7, the minimization of \( g \) can be done in \( O(n^3 \log(\gamma/n) F_g) \) time, where \( F_g \) denotes the time to evaluate the function value of \( g \). The evaluation of the value of function \( g \) can be seen as the minimization of an M-convex function. Since the \( L_\infty \)-diameter of \( f \) is bounded by \( \gamma \), the evaluation of \( g \) can be done in \( O(n^3 \log(\gamma/n)) \) time by Theorem 2.7 provided that the function evaluation of \( f \) can be done in constant time. Hence, we obtain the following time complexity result:

**Theorem 5.4.** The problem (MML1) can be solved in \( O(n^6(\log(\gamma/n))^2) \) time.

6 Application to Dock Re-allocation Problem

As observed in Section 3, the dock re-allocation problem (DR) can be seen as a special case of the problem (MML1). In this section, we apply the results obtained in Sections 4 and 5 for (MML1) to obtain algorithms for (DR). In particular, we show that the problem (DR) can be solved in polynomial time.

6.1 Steepest Descent Algorithm

We first present a steepest descent algorithm for (DR) by applying the algorithm in Section 4 for (MML1). We also show that a fast implementation of the steepest descent algorithm coincides with the greedy algorithm proposed by Freund et al. [11].

Recall that (DR) can be reformulated in the form of (MML1) as
\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{subject to} & \quad x(N) = D + B, \ |x - (\bar{d} + \bar{b})|_1 \leq 2\gamma, \ x \in \text{dom} f,
\end{align*}
\]
where the M-convex function \( f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{ +\infty \} \) is given by (5.4). By definition, the function value \( f(x) \) for a given \( x \in \text{dom} f \) can be computed by solving the following problem:
\[
\text{(SRA}(x)) \quad \begin{align*}
\text{Minimize} & \quad c(x - b, b) \equiv \sum_{i=1}^n c_i (x(i) - b(i), b(i)) \\
\text{subject to} & \quad b(N) \leq B, \ 0 \leq b \leq x, \ b \in \mathbb{Z}^n.
\end{align*}
\]
It is observed that for each $i \in N$, $c_i(x(i) - b(i), b(i))$ is a convex function in variable $b(i)$ since $c_i$ is a multimodular (or $M^2$-convex) function. Hence, the problem (SRA($x$)) can be seen as a simple resource allocation problem and therefore the evaluation of the function value of $f$ can be done in $O(n \log(B/n))$ time (see, e.g., [2]).

The algorithm SteepestDescentMML1 is rewritten in term of the problem (DR) as follows. Recall that $(\bar{d}, \bar{b})$ is a feasible solution of the problem (DR), and therefore the vector $\bar{x} = \bar{d} + \bar{b}$ is used as the initial solution of the steepest descent algorithm.

**Algorithm SteepestDescentDR**

**Step 0:** Set $x_0 := \bar{d} + \bar{b}$ and $k := 1$.

**Step 1:** If $k = 1$, then output the solution $(x_{k-1} - b_{k-1}, b_{k-1})$ and stop.

**Step 2:** For every distinct $i, j \in N$, compute the value $f(x_{k-1} + \chi_i - \chi_j)$ by solving (SRA($x_{k-1} + \chi_i - \chi_j$)), and find $i_k, j_k \in N$ minimizing $f(x_{k-1} + \chi_{i_k} - \chi_{j_k})$.

**Step 3:** Let $\hat{b}$ be an optimal solution of the problem (SRA($x_{k-1} + \chi_{i_k} - \chi_{j_k}$)), set $x_k := x_{k-1} + \chi_{i_k} - \chi_{j_k}$, $k := k + 1$, and go to Step 1.

Since the evaluation of the function value $f(x)$ requires $O(n \log(B/n))$ time, each iteration requires $O(n^3 \log(B/n))$ time, and the total running time of the algorithm is $O(\gamma n^3 \log(B/n))$.

The next lemma shows that the evaluation of the value $f(x)$ can be done faster by maintaining an optimal solution of the problem (SRA($x$)) for each $k$. This lemma is essentially equivalent to Lemma 6 in [1], while the statement of the lemma is described differently in our notation.

**Lemma 6.1** ([1] Lemma 6). Let $x \in \text{dom } f$, and $b \in \mathbb{Z}^n$ be an optimal solution of the problem (SRA($x$)). Also, let $i, j \in N$ be distinct elements such that $x + \chi_i - \chi_j \in \text{dom } f$. Then, there exists an optimal solution $\hat{b} \in \mathbb{Z}^n$ of the problem (SRA($x + \chi_i - \chi_j$)) such that

$$\hat{b} \in \{b, b + \chi_i, b - \chi_j, b + \chi_i - \chi_j\} \cup \{b + \chi_i - \chi_t \mid t \in N \setminus \{i, j\}\} \cup \{b + \chi_s - \chi_j \mid s \in N \setminus \{i, j\}\}. \quad (6.1)$$

It follows from Lemma 6.1 that for each $i, j \in N$, an optimal solution of the problem (SRA($x + \chi_i - \chi_j$)) can be found in $O(n)$ time, provided that an optimal solution of the problem (SRA($x$)) is available. Therefore, the running time of the algorithm SteepestDescentDR can be reduced to $O(\gamma n^3)$.

In fact, Lemma 6.1 implies that the running time $O(n^3)$ in each iteration can be further reduced by computing elements $i_k, j_k \in N$ minimizing the value $f(x_{k-1} + \chi_{i_k} - \chi_{j_k})$ and an optimal solution of the problem (SRA($x + \chi_{i_k} - \chi_{j_k}$)) simultaneously. We denote

$$R = \{(d, b) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid d(N) + b(N) = D + B, b(N) \leq B, \ell \leq d + b \leq u, d \geq 0, b \geq 0\},$$

i.e., $R$ is the set of vectors $(d, b) \in \mathbb{Z}^n \times \mathbb{Z}^n$ satisfying the constraints of the problem (DR), except for the L1-distance constraint $\|\langle d \rangle - \langle b \rangle\|_1 \leq 2\gamma$. We also denote $N(d, b) = N_1(d, b) \cup N_2(d, b) \cup \cdots \cup N_6(d, b)$, where

- $N_1(d, b) = \{(d + \chi_i - \chi_j, b) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid i, j \in N, i \neq j\}$,
- $N_2(d, b) = \{(d - \chi_j, b + \chi_i) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid i, j \in N, i \neq j\}$,
- $N_3(d, b) = \{(d + \chi_i, b - \chi_j) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid i, j \in N, i \neq j\}$,
- $N_4(d, b) = \{(d, b + \chi_i - \chi_j) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid i, j \in N, i \neq j\}$,
- $N_5(d, b) = \{(d - \chi_j + \chi_i, b + \chi_i - \chi_j) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid i, j \in N, i \neq j, t \in N \setminus \{i, j\}\}$,
- $N_6(d, b) = \{(d - \chi_s + \chi_i, b + \chi_s - \chi_j) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid i, j \in N, i \neq j, s \in N \setminus \{i, j\}\}$.

The following property follows immediately from Lemma 6.1.
Lemma 6.2. For $x \in \text{dom } f$, and an optimal solution $b \in \mathbb{Z}^n$ of $(\text{SRA}(x))$, we have

$$\min \{f(x + \chi_i - \chi_j) \mid i, j \in N, \ell \leq x + \chi_i - \chi_j \leq u\} = \min \{c(d', b') \mid (d', b') \in N(d, b) \cap R\}.$$ 

By Lemma 6.2, the algorithm SteepestDescentDR can be rewritten as follows in terms of original variables $(d, b)$ as follows, which is nothing but the greedy algorithm by Freund et al. [1].

Algorithm SteepestDescentDR'

Step 0: Set $d_0 := d$, $b_0 := b$, and $k := 1$.

Step 1: If $k - 1 = \gamma$, then output the solution $(d_{k-1}, b_{k-1})$ and stop.

Step 2: Find $(d', b') \in N(d_{k-1}, b_{k-1}) \cap R$ that minimizes $c(d', b')$.

Step 3: Set $(d_k, b_k) := (d', b')$ and go to Step 1.

For $h = 1, 2, \ldots, 6$, the value $\min \{c(d', b') \mid (d', b') \in N_h(d_{k-1}, b_{k-1}) \cap R\}$ can be computed in $O(\log n)$ time by using six binary heaps that maintain the following six sets of numbers, as in [1, Section 3.1]:

$$\{c_i(d_{k-1}(i) + 1, b(i)) - c_i(d_{k-1}(i), b(i)) \mid i \in N\},$$
$$\{c_i(d_{k-1}(i) - 1, b(i)) - c_i(d_{k-1}(i), b(i)) \mid i \in N\},$$
$$\{c_i(d(i), b(i) + 1) - c_i(d(i), b(i)) \mid i \in N\},$$
$$\{c_i(d(i), b(i) - 1) - c_i(d(i), b(i)) \mid i \in N\},$$
$$\{c_i(d(i) + 1, b(i) - 1) - c_i(d(i), b(i)) \mid i \in N\},$$
$$\{c_i(d(i) - 1, b(i) + 1) - c_i(d(i), b(i)) \mid i \in N\}.$$

Hence, each iteration of the algorithm can be done in $O(\log n)$ time. Since the initialization of the heaps requires $O(n)$ time, we obtain the following result:

Theorem 6.3 ([1]). The algorithm SteepestDescentDR (and also SteepestDescentDR') can be implemented so that it runs in $O(n + \gamma \log n)$ time.

6.2 Polynomial-Time Solvability of (DR)

The running time of the algorithm SteepestDescentDR is proportional to the problem parameter $\gamma$ and therefore pseudo-polynomial time. We show that (DR) can be solved in polynomial time by using the approach in Section 5.

To apply the approach in Section 5 consider the minimization problem of the $\mathbb{M}^\mathbb{Z}$-convex function $\hat{f}$ in (3.2) under the constraint $x(N) = D + B$, which is equivalent to the following:

\[
\begin{align*}
(\text{DA}) \quad \text{Minimize} & \quad c(d, b) \\
\text{subject to} & \quad d(N) + b(N) = D + B, \\
& \quad b(N) \leq B, \\
& \quad \ell \leq d + b \leq u, \quad d, b \in \mathbb{Z}^n_+, \\
& \quad d + b - \gamma 1 \leq d + b \leq d + b + \gamma 1.
\end{align*}
\]

We analyze the time complexity required to solve the problem (DA). Since the effective domain of the function $\hat{f}$ is an interval, we can apply Theorem 2.8 to obtain the following time bound.

Proposition 6.4. The problem (DA) can be solved in $O(n^3 \log(\gamma/n) \log(B/n))$ time.
By using a special structure of (DA), we can prove the following proximity theorem, which leads to a faster algorithm for (DA).

**Theorem 6.5.** Let \((d, b) \in \mathbb{Z}^n \times \mathbb{Z}^n\) be a feasible solution of (DA) that minimizes the value \(c(d, b)\) under the condition that all components of \(d\) and \(b\) are even integers. Then, there exists some optimal solution \((d^*, b^*) \in \mathbb{Z}^n \times \mathbb{Z}\) of (DA) such that \(\|d^* + b^* - (d + b)\|_1 \leq 16n\).

**Theorem 6.6.** A proximity-scaling algorithm finds an optimal solution of the problem (DA) in \(O(n \log n \log((D + B)/n))\) time.

Details of the proximity theorem and the proximity-scaling algorithm are given in Appendix.

We then analyze the time complexity for solving the dock reallocation problem (DR), provided that an optimal solution (DA) is available. An application of Theorem 5.4 to (DR) immediately implies the following time bound.

**Proposition 6.7.** The problem (DR) can be solved in \(O(n^2(\log(\gamma/n))^2 \log(B/n))\) time.

To obtain a better time bound for (DR), we consider a different approach. The discussion in Section 5 shows that if we have an optimal solution \((d^*, b^*)\) of (DA), then the problem (DR) can be reformulated as a problem without L1-distance constraint:

\[
(DR-L) \quad \text{Minimize} \quad c(d, b)
\]

subject to

\[
\begin{align*}
&d(N) + b(N) = D + B, \\
&b(N) \leq B, \\
&d(P) + b(P) = \bar{d}(P) + \bar{b}(P) + \gamma, \\
&d(N \setminus P) + b(N \setminus P) = \bar{d}(N \setminus P) + \bar{b}(N \setminus P) - \gamma, \\
&\ell \leq d + b \leq u, \ d, b \in \mathbb{Z}_+^n, \\
&\bar{d} + \bar{b} - \gamma 1 \leq d + b \leq \bar{d} + \bar{b} + \gamma 1,
\end{align*}
\]

where \(P \subseteq N\) is a set given as \(P = \text{supp}^+((d^* + b^*) - (\bar{d} + \bar{b}))\). To solve the problem (DR-L) efficiently, we consider the two problems (DR-L-A(\(\alpha\))) and (DR-L-B(\(\alpha\))) with parameter \(\alpha\):

\[
(DR-L-A(\alpha)) \quad \text{Minimize} \quad \sum_{i \in P} c_i(d(i), b(i))
\]

subject to

\[
\begin{align*}
&b(P) \leq \alpha, \\
&d(P) + b(P) = \bar{d}(P) + \bar{b}(P) + \gamma, \\
&\ell(i) \leq d(i) + b(i) \leq u(i), \ d(i), b(i) \in \mathbb{Z}_+ \ (i \in P), \\
&\bar{d}(i) + \bar{b}(i) - \gamma 1 \leq d(i) + b(i) \leq \bar{d}(i) + \bar{b}(i) + \gamma 1 \ (i \in P);
\end{align*}
\]

(DR-L-B(\(\alpha\))) is defined similarly to (DR-L-A(\(\alpha\))), where \(P\) is replaced with \(N \setminus P\) and the first constraint \(b(P) \leq \alpha\) is replaced with \(b(N \setminus P) \leq B - \alpha\). The two problems above have (almost) the same structure as the problem (DA), and therefore can be solved in \(O(n \log n \log((D+B)/n))\) time by Theorem 6.6.

We denote by \(\psi_A(\alpha)\) (resp., \(\psi_B(\alpha)\)) the optimal value of the problem (DR-L-A(\(\alpha\))) (resp., (DR-L-B(\(\alpha\)))). Then, it is not difficult to see that the optimal value of the problem (DR-L) is given by \(\min_{0 \leq \alpha \leq B} [\psi_A(\alpha) + \psi_B(\alpha)]\). The next property shows that the minimum value of \(\psi_A(\alpha) + \psi_B(\alpha)\) can be computed by binary search with respect to \(\alpha\).

**Proposition 6.8.** The values \(\psi_A(\alpha)\) and \(\psi_B(\alpha)\) are convex functions in \(\alpha \in [0, B]\).

Since the binary search terminates in \(O(\log B)\) iterations and each iteration requires \(O(n \log n \log((D + B)/n))\) time by Theorems 6.6, we obtain the following time bound.

**Theorem 6.9.** The problem (DR) can be solved in \(O(n \log n \log((D + B)/n) \log B)\) time.
References

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A Appendix: Proofs

A.1 Proof of Theorem 3.1

By Theorem 2.1(i), it suffices to show that the following condition holds for every \(x', x'' \in \text{dom } f\) with \(x' \neq x''\):

\[
\exists i \in \text{supp}^+(x' - x''), \ \exists j \in \text{supp}^-(x' - x'') : 
\]
\[
f(x') + f(x'') \geq f(x' - \chi_i + \chi_j) + f(x'' + \chi_i - \chi_j). 
\]  

(A.1)

For \(x \in \text{dom } f\), we denote

\[
S(x) = \{(d, b) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n \mid y + z = x, \ b(N) \leq B\}. 
\]

Let \(x', x'' \in \text{dom } f\) be distinct vectors, and let \((d', b') \in S(x')\) (resp., \((d'', b'') \in S(x'')\)) be a pair of vectors such that \(f(x') = c(d', b')\) (resp., \(f(x'') = c(d'', b'')\)). We denote

\[
N^+ = \text{supp}^+(x' - x''), \quad N^- = \text{supp}^-(x' - x''), \quad N^0 = N \setminus (N^+ \cup N^-). 
\]

In the following, we consider only the case with \(b(N) = b''(N) = B\) since the remaining case can be proved similarly and more easily. Note that this assumption and the equation \(x'(N) = x''(N)\) implies \(d'(N) = d''(N)\).

We first show by using Proposition 2.4 that the condition (A.1) holds if at least one of the following four conditions holds:

(C1) \(N^+ \cap \text{supp}^+(d' - d'') \neq \emptyset, \quad N^- \cap \text{supp}^-((d' - d'') \neq \emptyset, \)

(C2) \(N^+ \cap \text{supp}^+(b' - b'') \neq \emptyset, \quad N^- \cap \text{supp}^-((b' - b'') \neq \emptyset, \)

(C3) \(N^+ \cap \text{supp}^+(d' - d'') \neq \emptyset, \quad N^- \cap \text{supp}^-((b' - b'') \neq \emptyset, \quad N^0 \cap \text{supp}^-((d' - d'') \neq \emptyset, \)

(C4) \(N^+ \cap \text{supp}^+(b' - b'') \neq \emptyset, \quad N^- \cap \text{supp}^-((d' - d'') \neq \emptyset, \quad N^0 \cap \text{supp}^-((b' - b'') \neq \emptyset. \)

In the following, we give a proof for only the case with (C3); the proof for other cases are similar and omitted. Let \(i, j, s \in N\) be distinct elements such that

\[
i \in N^+ \cap \text{supp}^+(d' - d''), \quad j \in N^- \cap \text{supp}^-((b' - b'') \quad s \in N^0 \cap \text{supp}^-((d' - d''). \)
\]

Note that the choice of \(s\) implies \(s \in \text{supp}^+(b' - b'')\). We define vectors \(d', d'', b', b''\), \(\tilde{x}', \tilde{x}'' \in \mathbb{Z}^n\) by

\[
d' = d' - \chi_i + \chi_s, \quad d'' = d'' + \chi_i - \chi_s, \quad b' = b' + \chi_j - \chi_s, \quad b'' = b'' - \chi_j + \chi_s, 
\]

\[
\tilde{x}' = \tilde{x}' + \tilde{b}' = (x' - \chi_i + \chi_j), \quad \tilde{x}'' = \tilde{x}'' + \tilde{b}'' = (x'' + \chi_i - \chi_j). 
\]

It is not difficult to see that \(\tilde{x}', \tilde{x}'' \in \text{dom } f\), \((\tilde{d}', \tilde{b}') \in S(\tilde{x}')\), and \((\tilde{d}'', \tilde{b}'') \in S(\tilde{x}'')\) hold. Hence, we have

\[
f(\tilde{x}') \leq c(\tilde{d}', \tilde{b}'), \quad f(\tilde{x}'') \leq c(\tilde{d}'', \tilde{b}''). 
\]  

(A.2)

By the choice of \(i, j, s \in N\), the following inequalities follow from Proposition 2.4

\[
c_i(d'(i), b'(i)) + c_i(d''(i), b''(i)) 
\]

\[
\geq c_i(d'(i) - 1, b'(i)) + c_i(d''(i) + 1, b''(i)) = c_i(\tilde{d}'(i), \tilde{b}'(i)) + c_i(\tilde{d}''(i), \tilde{b}''(i)), 
\]

\[c_j(d'(j), b'(j)) + c_j(d''(j), b''(j)) \]

\[
\geq c_j(d'(j), b'(j) + 1) + c_j(d''(j), b''(j) - 1) = c_j(\tilde{d}'(j), \tilde{b}'(j)) + c_j(\tilde{d}''(j), \tilde{b}''(j)), 
\]

\[c_s(d'(s), b'(s)) + c_s(d''(s), b''(s)) \]

\[
\geq c_s(d'(s) + 1, b'(s) - 1) + c_s(d''(s) - 1, b''(s) + 1) = c_s(\tilde{d}'(s), \tilde{b}'(s)) + c_s(\tilde{d}''(s), \tilde{b}''(s)). 
\]  

(A.5)
From these inequalities and $\text{(A.2)}$ follows that

$$f(x') + f(x'') = c(d', b') + c(d'', b'') \geq c(d', b') + c(d'', b'') \geq f(x') + f(x'') = f(x' - \chi_i + \chi_j) + f(x'' + \chi_i - \chi_j).$$

This shows that the inequality $\text{(A.1)}$ holds.

To conclude the proof, we show that at least one of the four conditions (C1)–(C4) holds. Assume, to the contrary, that neither of the four conditions holds. Since $x'(N) = x''(N)$ and $x' \neq x''$, we have $N^+ \neq \emptyset$, which implies at least one of $N^+ \cap \text{supp}^+(d' - d'') \neq \emptyset$ and $N^+ \cap \text{supp}^+(b' - b'') \neq \emptyset$ holds; we may assume that the former holds. Since (C1) does not hold, we have $N^- \cap \text{supp}^-(d' - d'') = \emptyset$, which implies that $N^- \subseteq \text{supp}^-(b' - b'')$. Since (C2) does not hold, we have $N^+ \cap \text{supp}^+(b' - b'') = \emptyset$, which implies that $N^+ \subseteq \text{supp}^+(d' - d'')$. Since (C3) does not hold, we have $N^0 \cap \text{supp}^-(d' - d'') = \emptyset$. Hence, we have

$$d'(N^+) > d''(N^+), \quad d'(N^-) \geq d''(N^-), \quad d'(N^0) \geq d''(N^0),$$

implying that $d'(N) > d''(N)$, a contradiction to the equation $d'(N) = d''(N)$. This concludes the proof.

### A.2 Proof of Theorem 4.1

We prove Theorem 4.1 in this section. For this, we show some technical lemmas.

**Lemma A.1.** Let $y, \tilde{y} \in \mathbb{Z}^n$ be distinct vectors satisfying $y(N) = \tilde{y}(N)$. If $\|y - x_c\|_1 \leq \|\tilde{y} - x_c\|_1$, then we have $\tilde{y}(i) > x_c(i)$ for some $i \in \text{supp}^+(\tilde{y} - y)$ or $\tilde{y}(j) < x_c(j)$ for some $j \in \text{supp}^-(\tilde{y} - y)$ (or both).

**Proof.** We prove the statement by contradiction. Assume, to the contrary, that $\tilde{y}(i) \leq x_c(i)$ for all $i \in \text{supp}^+(\tilde{y} - y)$ and $\tilde{y}(j) \geq x_c(j)$ for all $j \in \text{supp}^-(\tilde{y} - y)$. Then, it holds that

$$\|\tilde{y} - x_c\|_1 - \|y - x_c\|_1 = \sum_{i \in \text{supp}^+(\tilde{y} - y)} (|\tilde{y}(i) - x_c(i)| - |y(i) - x_c(i)|) + \sum_{j \in \text{supp}^-(\tilde{y} - y)} (|\tilde{y}(j) - x_c(j)| - |y(j) - x_c(j)|) \leq \sum_{i \in \text{supp}^+(\tilde{y} - y)} [(x_c(i) - \tilde{y}(i)) - (x_c(i) - y(i))] + \sum_{j \in \text{supp}^-(\tilde{y} - y)} [(\tilde{y}(j) - x_c(j)) - (y(j) - x_c(j))] = \sum_{i \in \text{supp}^+(\tilde{y} - y)} (-\tilde{y}(i) + y(i)) + \sum_{j \in \text{supp}^-(\tilde{y} - y)} (\tilde{y}(j) - y(j)) < 0,$$

a contradiction to the inequality $\|y - x_c\|_1 \leq \|\tilde{y} - x_c\|_1$. \qed

**Lemma A.2.** Let $x, y, z \in \mathbb{Z}^n$, $i \in \text{supp}^+(x - y)$, and $j \in \text{supp}^-(x - y)$. Then, we have

$$\|x - z\|_1 + \|y - z\|_1 \geq \|(x - \chi_i + \chi_j) - z\|_1 + \|(y + \chi_i - \chi_j) - z\|_1.$$
Hence, we have $\|x - z\|_1 = \sum_{i=1}^{h} |x_i - z_i|$ and each term $|x_i - z_i|$ is a univariate convex function in $x_i$, the claim follows. \hfill \blacksquare

We say that a sequence $y_0, y_1, \ldots, y_h \in \text{dom} \ f$ of vectors is monotone if $\|y_k - y_0\|_1 = 2k$ holds for $k = 0, 1, \ldots, h$. This condition can be rewritten as follows:

for $k = 0, 1, \ldots, h - 1$, it holds that $y_{k+1} = y_k - \chi_i + \chi_j$ for some $i \in \text{supp}^+ (y_k - y_h)$ and $j \in \text{supp}^- (y_k - y_h)$.

Recall that by the definition of $\tau$, every optimal solution of the problem (MML1(\tau)) is a minimizer of $f$.

**Lemma A.3.** Let $y \in \text{dom} \ f$ be a vector with $\|y - x_c\|_1 < 2\tau$, and $x^* \in M_\tau$ be a vector minimizing the value $\|x^* - y\|_1$. Then, there exists a monotone sequence $y_0, y_1, \ldots, y_h \in \text{dom} \ f$ with $h = (1/2)\|y - x^*\|_1$ such that $y_0 = y$, $y_h = x^*$, and $f(y_0) > f(y_1) > \cdots > f(y_h)$.

**Proof.** We prove the claim by induction on $h$. It suffices to show that there exists some $i \in \text{supp}^+ (y - x^*)$ and $j \in \text{supp}^- (y - x^*)$ such that $f(y - \chi_i + \chi_j) < f(y)$ since $(1/2)\|y - \chi_i + \chi_j\|_1 = h - 1$.

Since $\|x^* - x_c\|_1 = 2\tau > \|y - x_c\|_1$, it follows from Lemma [\text{A.4}] that $x^*(i) > x_c(i)$ for some $i \in \text{supp}^+ (x^* - y)$ or $x^*(j) < x_c(j)$ for some $j \in \text{supp}^- (x^* - y)$ (or both); we assume, without loss of generality, that the former holds. Then, the exchange property (M-EXC) of $M$-convex function $f$ applied to $x^*, y$, and $i$ implies that there exists some $j \in \text{supp}^- (x^* - y)$ such that

$$f(x^*) + f(y) \geq f(x^* - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

Hence, if we have $f(x^*) < f(x^* - \chi_i + \chi_j)$, then \text{[A.6]} implies the desired inequality $f(y - \chi_i + \chi_j) < f(y)$. In the following, we prove $f(x^*) < f(x^* - \chi_i + \chi_j)$.

By the choice of $i$, we have $\|x^* - \chi_i + \chi_j\|_1 - x_c|_1 - \|x^* - x_c\|_1 \in \{0, 2\}$. If $\|x^* - \chi_i + \chi_j\|_1 - x_c|_1 - \|x^* - x_c\|_1 = 0$ then we have $f(x^*) < f(x^* - \chi_i + \chi_j)$ by the choice of $x^*$ since $\|x^* - \chi_i + \chi_j\|_1 = \|x^* - \chi_i + \chi_j\|_1 < \|x^* - y\|_1$. If $\|x^* - \chi_i + \chi_j\|_1 - x_c|_1 - \|x^* - x_c\|_1 = -2$ then we have $\|x^* - \chi_i + \chi_j\|_1 - x_c|_1 < 2\tau$ and therefore $f(x^*) < f(x^* - \chi_i + \chi_j)$ holds by the definition of $\tau$. Hence, we have $f(x^*) < f(x^* - \chi_i + \chi_j)$ in either case. \hfill \blacksquare

We now prove the claims (i), (ii), and (iii) of Theorem 4.1 in turn.

**Proof of Theorem 4.1 (i).** We first show that $\mu_k > \mu_{k+1}$ for each integer $k \in [\sigma, \tau - 1]$. Let $y \in M_\sigma$, and $x^* \in M_\tau$ be a vector that minimizes the value $\|x^* - y\|_1$. Note that $x^* \in \arg \min f$, and by the induction hypothesis we have $\|y - x_c\|_1 = 2k$. By Lemma [\text{A.3}], there exists a monotone sequence $y_0, y_1, \ldots, y_h \in \text{dom} \ f$ with $h = (1/2)\|y - x^*\|_1$ such that $y_0 = y$, $y_h = x^*$, and $\mu_h = f(y_h) > f(y_1) > \cdots > f(y_0)$. Since $\|y_{t+1} - x_c\|_1 - \|y_t - x_c\|_1 \in \{-2, 0, 2\}$ for every integer $t \in [0, h - 1]$ and $\|y_h - x_c\|_1 = 2\tau > 2k$, $\|y_0 - x_c\| < 1$, there exists some integer $s \in [1, h]$ such that $\|y_s - x_c\| = 2(k + 1)$; such $s$ satisfies $\mu_{k+1} \leq f(y_s) < f(y_0) = \mu_k$.

The inclusion $M_k \subseteq \{x \in \mathbb{Z}^n \mid \|x - x_c\|_1 = 2k\}$ follows from the inequality $\mu_k < \mu_{k-1}$ since $f(x) \geq \mu_{k-1} > \mu_k$ holds for every $x \in \text{dom} \ f$ with $\|x - x_c\|_1 < 2k$. \hfill \blacksquare

**Proof of Theorem 4.1 (ii).** We fix $y \in M_k$, and let $\tilde{y}$ be a vector in $M_{k+1}$ that minimizes $\|\tilde{y} - y\|_1$. By Lemma [\text{A.1}] it suffices to consider the following two cases:
In the following we give a proof for Case 1 only since Case 2 can be proven in a similar way. Suppose that there exists some $i \in \text{supp}^+(\tilde{y} - y) \cap \text{supp}^+(\tilde{y} - x_c)$. By (M-EXC) applied to $\tilde{y}$ and $y$, there exists some $j \in \text{supp}^-(\tilde{y} - y)$ such that

$$f(\tilde{y}) + f(y) \geq f(\tilde{y} - x_i + \chi_j) + f(y + \chi_i - \chi_j). \quad (A.7)$$

Put $\tilde{z} = \tilde{y} - \chi_i + \chi_j$, $z = y + \chi_i - \chi_j$, and

$$\alpha = \|z - x_c\|_1 - \|\tilde{y} - x_c\|_1, \quad \beta = \|z - x_c\|_1 - \|y - x_c\|_1.$$

Then, we have $\beta \in \{-2,0,+2\}$ and $\alpha \in \{-2,0\}$ since $\tilde{y}(i) > x_c(i)$.

Assume first that $\alpha = 0$ holds. By Lemma A.2, we have

$$\alpha + \beta = \|z - x_c\|_1 + \|z - x_c\|_1 - \|\tilde{y} - x_c\|_1 - \|y - x_c\|_1 \leq 0.$$

This, together with $\alpha = 0$, implies $\beta \leq 0$. Hence, it holds that $\|z - x_c\|_1 \leq \|y - x_c\|_1 = 2k$, implying $f(z) \geq \mu_{\|z - x_c\|_1/2} \geq \mu_k$ by Theorem 4.1 (ii). Since $\|z - x_c\|_1 = \|\tilde{y} - x_c\|_1 = 2(k + 1)$, we have $f(z) \geq \mu_{k+1}$. Combining these inequalities with (A.7), we have

$$\mu_{k+1} + \mu_k = f(\tilde{y}) + f(y) \geq f(\tilde{z}) + f(z) \geq \mu_{k+1} + \mu_k,$$

from which follows that $f(\tilde{z}) = \mu_{k+1}$, a contradiction to the choice of $\tilde{y}$ since $\|\tilde{z} - y\|_1 = \|\tilde{y} - y\|_1 - 2$. This shows that $\alpha = 0$ cannot occur. Hence, we have $\alpha = -2$.

Since $\alpha = -2$, we have $\|z - x_c\|_1 = \|\tilde{y} - x_c\|_1 - 2 = 2k$, from which follows that $f(z) \geq \mu_k$. We also have $\|z - x_c\|_1 \leq \|y - x_c\|_1 + 2 = 2(k + 1)$, and therefore $f(z) \geq \mu_{\|z - x_c\|_1/2} \geq \mu_{k+1}$, where the last inequality is by Theorem 4.1 (i). Combining these inequalities with (A.7), we have

$$\mu_{k+1} + \mu_k = f(\tilde{y}) + f(y) \geq f(\tilde{z}) + f(z) \geq \mu_k + \mu_{k+1},$$

from which follows that $f(z) = \mu_{k+1}$. This implies $\|z - x_c\|_1 = 2(k + 1)$ since $\mu_{k-1} > \mu_k > \mu_{k+1}$ by Theorem 4.1 (i). Hence, we have $z = y + \chi_i - \chi_j \in M_{k+1}, \ i \in N \setminus \text{supp}^-(y - x_c)$, and $j \in N \setminus \text{supp}^+(y - x_c)$. \hfill \Box

**Proof of Theorem 4.1 (iii).** The proof below is quite similar to that for Theorem 4.1 (ii).

We fix $y' \in M_{k+1}$, and let $y$ be a vector in $M_k$ that minimizes $\|y - y'\|_1$. By Lemma A.1 it suffices to consider the following two cases:

Case 1: $\text{supp}^+(y' - y) \cap \text{supp}^+(y' - x_c) \neq \emptyset$.

Case 2: $\text{supp}^-(y' - y) \cap \text{supp}^-(y' - x_c) \neq \emptyset$.

In the following we give a proof for Case 1 only since Case 2 can be proven in a similar way. Suppose that there exists some $i \in \text{supp}^+(y' - y) \cap \text{supp}^+(y' - x_c) \neq \emptyset$. By (M-EXC) applied to $y'$ and $y$, there exists some $j \in \text{supp}^-(y' - y)$ such that

$$f(y') + f(y) \geq f(y' - x_i + \chi_j) + f(y + \chi_i - \chi_j). \quad (A.8)$$

Put $z' = y' - \chi_i + \chi_j$, $z = y + \chi_i - \chi_j$, and

$$\alpha = \|z' - x_c\|_1 - \|y' - x_c\|_1, \quad \beta = \|z - x_c\|_1 - \|y - x_c\|_1.$$
Then, we have $\beta \in \{-2, 0, +2\}$ and $\alpha \in \{-2, 0\}$ since $y'(i) > x_c(i)$.

Assume first that $\alpha = 0$ holds. By Lemma A.2, we have

$$\alpha + \beta = \|z' - x_c\|_1 + \|z - x_c\|_1 - \|y' - x_c\|_1 - \|y - x_c\|_1 \leq 0.$$ 

This, together with $\alpha = 0$, implies $\beta \leq 0$. Hence, it holds that $\|z - x_c\|_1 \leq \|y - x_c\|_1 = 2k$, and therefore $f(z) \geq \mu \|z - x_c\|_1 / 2 \geq \mu_k$ by Theorem 4.1 (i). Since $\|z' - x_c\|_1 = \|y' - x_c\|_1 = 2(k + 1)$, we have $f(z') \geq \mu_{k+1}$. Combining these inequalities with (A.8), we have

$$\mu_k + \mu_k = f(y') + f(y) \geq f(z') + f(z) \geq \mu_k + \mu_{k+1},$$

from which follows that $f(z') = \mu_{k+1}$, a contradiction to the choice of $y'$ since $\|z' - y\|_1 = \|y' - y\|_1 - 2$. This shows that $\alpha = 0$ cannot occur. Hence, we have $\alpha = -2$.

Since $\alpha = -2$, we have $\|z' - x_c\|_1 = \|y' - x_c\|_1 = 2k$, implying that $f(z') \geq \mu_k$. We also have $\|z - x_c\|_1 \leq \|y - x_c\|_1 = 2(k + 1)$, and therefore $f(z) \geq \mu \|z - x_c\|_1 / 2 \geq \mu_{k+1}$, where the last inequality is by Theorem 4.1 (i). Combining these inequalities with (A.8), we have

$$\mu_k + \mu_k = f(y') + f(y) \geq f(z') + f(z) \geq \mu_k + \mu_{k+1},$$

from which follows that $f(z') = \mu_k$. Hence, we have $z' = y' - \chi_i + \chi_j \in M_k$, $i \in \text{supp}^+(y - x_c)$ and $j \in \text{supp}^{-}(y - x_c)$.

**A.3 Reverse Steepest Descent Algorithm for (MML1)**

We can also consider another variant of steepest descent algorithm that starts from a nearest minimizer $x^\bullet$ of $f$ and greedily approaches $x_c$. This algorithm finds an optimal solution of (MML1) faster than STEEPESTDESCENTMML1 if $\tau - \gamma$ is smaller than $\gamma - \sigma$.

**Algorithm** REVERSESTEEPESTDESCENTMML1

**Step 0:** Compute the value $\tau$ in (4.1) and a minimizer $x^\bullet$ of $f$ with $\|x^\bullet - x_c\|_1 = 2\tau$. Set $x_\tau := x^\bullet$, and $k := \tau - 1$.

**Step 1:** If $k + 1 = \gamma$, then output $x_{k+1}$ and stop.

**Step 2:** Find $i_k, j_k \in N$ that minimizes $f(x_{k+1} - \chi_{i_k} + \chi_{j_k})$.

**Step 2:** Set $x_k := x_{k+1} - \chi_{i_k} + \chi_{j_k}$, $k := k - 1$, and go to Step 1.

**Theorem A.4.** The algorithm REVERSESTEEPESTDESCENTMML1 applied to an M-convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ outputs an optimal solution of (MML1) in $\tau - \gamma$ iterations.

**Proof.** In a similar way as in the proof of Theorem 4.2 we can show that $x_k \in M_k$ holds for $k = \tau, \tau - 1, \ldots, \gamma$. Hence, the output $x_\gamma$ of the algorithm is an optimal solution of (MML1).

**A.4 Remarks in Section 4**

As an immediate corollary of Theorem 4.2 we can obtain the following property of the algorithm STEEPESTDESCENT in the case with $x_c \in \text{dom } f$. Note that the behavior of the algorithm STEEPESTDESCENTMML1 coincides with that of STEEPESTDESCENT if $x_c \in \text{dom } f$ and $\gamma \geq \tau$.

**Corollary A.5.** Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an M-convex function with $x_c \in \text{dom } f$. Suppose that the algorithm STEEPESTDESCENT is applied to $f$ with $x_c$ as an initial vector. Then, the algorithm terminates in exactly $\tau$ iterations and outputs an optimal solution of (MML1).
Remark A.6. In Corollary A.3 we obtained the exact bound on the number of iterations required by the algorithm STEEPDEST_DESCENT. While this bound is shown for some special case of M-convex functions and for some variants of the algorithm, it is not proven so far for the "naive" steepest descent algorithm (i.e., STEEPDEST_DESCENT).

The same bound for STEEPDEST_DESCENT is obtained by [12] for the special case where an M-convex function has a unique minimizer. Based on this fact, the same bound for general M-convex functions is obtained in [12], where a variant of STEEPDEST_DESCENT with certain tie-breaking rules in the choice of $i_k$ and $j_k$ in Step 1 is used. The same bound can be also obtained by using another variant of STEEPDEST_DESCENT in [16], where a region containing an optimal solution is explicitly maintained by lower and upper bound vectors. Corollary A.5 shows that no modification of the algorithm STEEPDEST_DESCENT is necessary to obtain the same exact bound.

\[ \square \]

Remark A.7. It can be shown that the sequence of optimal values $\mu_k$ for (MML1($k$)) is a convex sequence.

Theorem A.8. For every integer $k \in [1, \tau - 1]$, it holds that $\mu_{k-1} + \mu_{k+1} \geq 2\mu_k$.

\[ \square \]

Proof of Theorem A.8. For $k = 1, 2, \ldots, \tau - 1$, let $x_{k-1} \in M_{k-1}$ and $x_{k+1} \in M_{k+1}$ be vectors such that

$$x_{k+1} = x_{k-1} - \chi_i - \chi_j + \chi' \quad \text{for some } i, i', j, j' \in N \text{ with } \{i, i'\} \cap \{j, j'\} = \emptyset, i, i' \in \text{supp}^-(x_{k+1} - x_c), \text{ and } j, j' \in \text{supp}^+(x_{k+1} - x_c);$$

the existence of such $x_{k-1}$ and $x_{k+1}$ follows from the claim (ii) (or (iii)) of Theorem 4.1. By (M-EXC) applied to $x_{k-1}$ and $x_{k+1}$, we have $f(x_{k-1}) + f(x_{k+1}) \geq f(y) + f(z)$ with $(y, z) = (x_{k-1} - \chi_i + \chi_j, x_{k+1} - \chi_i' + \chi_j')$ or $(y, z) = (x_{k-1} - \chi_i + \chi_j', x_{k+1} - \chi_i' + \chi_j)$. In either case we have $\|y - x_c\|_1 = \|z - x_c\|_1 = 2k$, and therefore follows that

$$\mu_{k-1} + \mu_{k+1} = f(x_{k-1}) + f(x_{k+1}) \geq f(y) + f(z) \geq 2\mu_k.$$

\[ \square \]

A.5 Proof of Lemma 5.1

We first show that every feasible solution $x$ of (MM-L) satisfies the L1-distance constraint $\|x - x_c\|_1 \leq 2\gamma$. Under the condition $\ell \leq x \leq \hat{u}$ we have

$$\|x - x_c\|_1 = (x(P) - x_c(P)) + (x_c(N \setminus P) - x(N \setminus P)),$$

and the equation $x(N) = \theta = x_c(N)$ implies that $x(P) - x_c(P) = x_c(N \setminus P) - x(N \setminus P)$. Since $x(P) = x_c(P) + \gamma$, the L1-distance $\|x - x_c\|_1$ is bounded by $2\gamma$.

To conclude the proof, it suffices to show that there exists an optimal solution $x^*$ of (MML1) such that $x^*(P) = x_c(P) + \gamma$ and $\ell \leq x^* \leq \hat{u}$. Repeated use of Theorem 4.1(iii) implies that there exists an optimal solution $x^* \in \text{dom} f$ of (MML1) such that $\|x^* - x_c\|_1 = 2\gamma$, $x^*(P) = x_c(P) + \gamma$, $x^*(N \setminus P) = x_c(N \setminus P) - \gamma$, and

$$x_c(i) \leq x^*(i) \leq x^*_c(i) \quad (i \in P), \quad x^*(i) \leq x^*(i) \leq x_c(i) \quad (i \in N \setminus P). \quad (A.9)$$

By the equation $x^*(P) = x_c(P) + \gamma$, for $i \in P$ the upper bound of $x^*(i)$ in (A.9) can be replaced with $\min\{x^*_c(i), x_c(i) + \gamma\}$. Similarly, for $i \in N \setminus P$ the lower bound of $x^*(i)$ in (A.9) can be replaced with $\max\{x^*_c(i), x_c(i) - \gamma\}$. This concludes the proof.
\[ f'(x) = \begin{cases} f(x) & \text{if } x(N) = \theta, \; \hat{\ell}(i) \leq x(i) \leq \hat{u}(i) \; (i \in P), \\ +\infty & \text{otherwise}. \end{cases} \]

Then, \( f' \) is an M-convex function \([11, \text{Theorem 6.13 (5)}]\). The function \( g' : \mathbb{Z}^N \rightarrow \mathbb{R} \cup \{+\infty\} \) given by

\[
g'(y) = \min \{f'(x) \mid x(i) = y(i) \; (i \in N \setminus P)\} \quad (y \in \mathbb{Z}^N) \]

is an \( M^2 \)-convex function since \( g' \) is a projection of the M-convex function \( f' \) \([11, \text{Theorem 6.15 (2)}]\). Finally, function \( g \) is given as

\[
g(y) = \begin{cases} g'(y) & \text{if } y(N \setminus P) = \theta - (x_c(P) + \gamma), \; \hat{\ell}(i) \leq x(i) \leq \hat{u}(i) \; (i \in N \setminus P), \\ +\infty & \text{otherwise}, \end{cases} \]

and therefore \( g \) is M-convex (cf. \([11, \text{Theorem 6.13}])\).

### A.6 Proof of Proposition 5.3

We show that \( g \) is M-convex. Define a function \( f' : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) by

\[
f'(x) = \begin{cases} f(x) & \text{if } x(N) = \theta, \; \hat{\ell}(i) \leq x(i) \leq \hat{u}(i) \; (i \in P), \\ +\infty & \text{otherwise}. \end{cases} \]

Then, \( f' \) is an M-convex function \([11, \text{Theorem 6.13 (5)}]\). The function \( g' : \mathbb{Z}^N \rightarrow \mathbb{R} \cup \{+\infty\} \) given by

\[
g'(y) = \min \{f'(x) \mid x(i) = y(i) \; (i \in N \setminus P)\} \quad (y \in \mathbb{Z}^N) \]

is an \( M^2 \)-convex function since \( g' \) is a projection of the M-convex function \( f' \) \([11, \text{Theorem 6.15 (2)}]\). Finally, function \( g \) is given as

\[
g(y) = \begin{cases} g'(y) & \text{if } y(N \setminus P) = \theta - (x_c(P) + \gamma), \; \hat{\ell}(i) \leq x(i) \leq \hat{u}(i) \; (i \in N \setminus P), \\ +\infty & \text{otherwise}, \end{cases} \]

and therefore \( g \) is M-convex (cf. \([11, \text{Theorem 6.13}])\).

### A.7 Proof of Lemma 6.1

We denote \( \hat{x} = x + \chi_i - \chi_j \), and let \( \hat{b} \in \mathbb{Z}^n \) be an optimal solution of the problem \( \text{(SRA(\hat{x}))} \) that minimizes the value \( ||\hat{b} - b||_1 \). We show that the vector \( \hat{b} \) satisfies the condition \([6.11]\).

Claim 1: If \( s \in N \) satisfies either \( s \in \text{supp}^+ (\hat{b} - b) \setminus \{i\} \) or \( s = i \) and \( \hat{b}(i) - b(i) \geq 2 \), then

\[
c_s(\hat{x}(s) - \hat{b}(s), \hat{b}(s)) + c_s(x(s) - b(s), b(s)) \\
\geq c_s(\hat{x}(s) - \hat{b}(s) + 1, \hat{b}(s) - 1) + c_s(x(s) - b(s) - 1, b(s) + 1). \tag{A.10} \]

If \( t \in N \) satisfies either \( t \in \text{supp}^- (\hat{b} - b) \setminus \{j\} \) or \( t = j \) and \( \hat{b}(j) - b(j) \leq -2 \), then

\[
c_t(\hat{x}(t) - \hat{b}(t), \hat{b}(t)) + c_t(x(t) - b(t), b(t)) \\
\geq c_t(\hat{x}(t) - \hat{b}(t) - 1, \hat{b}(t) + 1) + c_t(x(t) - b(t) + 1, b(t) - 1). \tag{A.11} \]

[Proof of Claim] We prove the inequality \((A.10)\) only since \((A.11)\) can be shown similarly. If \( s \in \text{supp}^+ (\hat{b} - b) \setminus \{i\} \) then we have \( \hat{b}(s) > b(s) \) and \( \hat{x}(s) - \hat{b}(s) < x(s) - b(s) \). If \( s = i \) and \( \hat{b}(i) - b(i) \geq 2 \), then we have \( \hat{x}(i) - \hat{b}(i) \leq x(i) + 1 - (b(i) + 2) < x(i) - b(i) \). In either case, \((A.10)\) follows by Proposition 2.4 (i). [End of Claim]

To prove the lemma, we consider the following two conditions:

(a) \( \text{supp}^+ (\hat{b} - b) \subseteq \{i\} \) and \( \hat{b}(i) - b(i) \leq 1 \),

(b) \( \text{supp}^- (\hat{b} - b) \subseteq \{j\} \) and \( \hat{b}(j) - b(j) \geq -1 \).

Claim 2: At least one of the conditions (a) and (b) holds. Moreover, the condition (a) holds if \( b(N) < B \), and the condition (b) holds if \( \hat{b} < B \).

[Proof of Claim] To prove the former statement, assume, to the contrary, that there exist some \( s, t \in N \) satisfying the following two conditions:

\[
s \in \text{supp}^+ (\hat{b} - b) \setminus \{i\} \text{ or } s = i \text{ and } \hat{b}(i) - b(i) \geq 2, \\
t \in \text{supp}^- (\hat{b} - b) \setminus \{j\} \text{ or } t = j \text{ and } \hat{b}(j) - b(j) \leq -2. \]
Then, it holds that
\[
c(\hat{x} - \hat{b}, \hat{b}) + c(x - b, b)
- c(\hat{x} - (\hat{b} - \chi_s + \chi_t), \hat{b} - \chi_s + \chi_t) - c(x - (b + \chi_s - \chi_t), b + \chi_s - \chi_t)
\]
\[
= \left[ c_s(\hat{x}(s) - \hat{b}(s), \hat{b}(s)) + c_s(x(s) - b(s), b(s))
- c_s(\hat{x}(s) - \hat{b}(s) + 1, \hat{b}(s) - 1) - c_s(x(s) - b(s) - 1, b(s) + 1) \right]
+ \left[ c_t(\hat{x}(t) - \hat{b}(t), \hat{b}(t)) + c_t(x(t) - b(t), b(t))
- c_t(\hat{x}(t) - \hat{b}(t) - 1, \hat{b}(t) + 1) - c_t(x(t) - b(t) + 1, b(t) - 1) \right]
\geq 0,
\]
(A.12)
where the inequality is by (A.10) and (A.11) in Claim 1. Note that \(b + \chi_s - \chi_t\) is a feasible solution of (SRA(x)) since \((b + \chi_s - \chi_t)(N) = b(N) \leq B\), \(b(s) < \hat{b}(s) \leq x(s)\), and \(\hat{b}(t) > b(t) \geq 0\). Therefore, we have
\[
c(x - b, b) \leq c(x - (b + \chi_s - \chi_t), b + \chi_s - \chi_t),
\]
which, together with (A.12), implies
\[
c(\hat{x} - \hat{b}, \hat{b}) \geq c(\hat{x} - (\hat{b} - \chi_s + \chi_t), \hat{b} - \chi_s + \chi_t).
\]
Since \(\hat{b}\) is an optimal solution of (SRA(x + \chi_i - \chi_j)) and \(\hat{b} - \chi_s + \chi_t\) is a feasible solution of (SRA(x + \chi_i - \chi_j)), the vector \(\hat{b} - \chi_s + \chi_t\) is also an optimal solution of (SRA(x + \chi_i - \chi_j)), a contradiction to the choice of \(\hat{b}\) since \(||(\hat{b} - \chi_s + \chi_t) - b||_1 = ||\hat{b} - b||_1 - 2\). This concludes the proof of the former statement.

To prove the latter statement, we assume \(b(N) < B\); the case \(\hat{b}(N) < B\) can be proven in a similar way. Assume, to the contrary, that there exist some \(s \in N\) satisfying \(s \in \text{supp}^+(\hat{b} - b) \setminus \{i\}\), or \(s = i\) and \(\hat{b}(i) - b(i) \geq 2\). Then, we have
\[
c(\hat{x} - \hat{b}, \hat{b}) + c(x - b, b) - c(\hat{x} - (\hat{b} - \chi_s), \hat{b} - \chi_s) - c(x - (b + \chi_s), b + \chi_s)
\]
\[
= c_s(\hat{x}(s) - \hat{b}(s), \hat{b}(s)) + c_s(x(s) - b(s), b(s))
- c_s(\hat{x}(s) - \hat{b}(s) + 1, \hat{b}(s) - 1) - c_s(x(s) - b(s) - 1, b(s) + 1)
\geq 0,
\]
(A.13)
where the inequality is by (A.10) in Claim 1. The vector \(b + \chi_s\) is a feasible solution of (SRA(x)) since \(b(N) < B\). Hence, we have
\[
c(x - b, b) \leq c(x - (b + \chi_s), b + \chi_s),
\]
which, together with (A.13), implies
\[
c(\hat{x} - \hat{b}, \hat{b}) \geq c(\hat{x} - (\hat{b} - \chi_s), \hat{b} - \chi_s).
\]
(A.14)
Since \(\hat{b} - \chi_s\) is a feasible solution of (SRA(x + \chi_i - \chi_j)), optimality of \(\hat{b}\) and the inequality (A.14) imply that \(\hat{b} - \chi_s\) is also an optimal solution of (SRA(x + \chi_i - \chi_j)), a contradiction to the choice of \(\hat{b}\) since \(||(\hat{b} - \chi_s) - b||_1 = ||\hat{b} - b||_1 - 1\). Hence, the condition (a) holds. [End of Claim]

We now prove the lemma. It is easy to see from Claim 2 that the following properties hold:
• if \( b(N) = \hat{b}(N) < B \), then \( \hat{b} \in \{ b, b + \chi_i - \chi_j \} \),
• if \( b(N) < \hat{b}(N) \leq B \), then \( \hat{b} = b + \chi_i \),
• if \( B \geq b(N) > \hat{b}(N) \), then \( \hat{b} = b - \chi_j \).

We next consider the case with \( b(N) = \hat{b}(N) = B \). Then, one of (a) and (b) holds by Claim 2. Suppose that (a) holds. If \( \text{supp}^+(\hat{b} - b) = \emptyset \), then \( \hat{b} = b \) follows since \( b(N) = \hat{b}(N) \). If \( \text{supp}^+(\hat{b} - b) \neq \emptyset \), then we have \( \text{supp}^+(\hat{b} - b) = \{ i \} \) and \( \hat{b}(i) = b(i) + 1 \). Since \( b(N) = \hat{b}(N) \), there exists a unique element \( t \) in \( \text{supp}^-(\hat{b} - b) \) and it satisfies \( i \neq i \) and \( \hat{b}(t) = b(t) - 1 \). Hence, we have \( \hat{b} = b \) or \( \hat{b} = b + \chi_i - \chi_j \) for some \( t \in N \{ i \} \). If the condition (b) holds, then we can show in a similar way that \( \hat{b} = b \) or \( \hat{b} = b + \chi_s - \chi_j \) for some \( s \in N \setminus \{ j \} \).

### A.8 Proof of Proposition 6.8

We consider a variant of the problem (DA), where the constant \( B \) is replaced with a parameter \( \alpha \):

\[
\text{(DA[\alpha])}
\begin{align*}
\text{Minimize} & \quad c(d, b) \\
\text{subject to} & \quad d(N) + b(N) = D + B, \\
& \quad b(N) \leq \alpha, \\
& \quad \ell \leq d + b \leq u, \\
& \quad d + b - \gamma 1 \leq d + b \leq \tilde{d} + \tilde{b} + \gamma 1, \\
& \quad d, b \in \mathbb{Z}_n^+.
\end{align*}
\]

We denote by \( \psi(\alpha) \) the optimal value of (DA[\alpha]). To prove Proposition 6.8, it suffices to show the following property of \( \psi(\alpha) \).

**Lemma A.9.** The value \( \psi(\alpha) \) is a convex function in \( \alpha \in [0, B] \).

**Proof.** For \( \alpha \in [0, B - 2] \), we show that \( \psi(\alpha) + \psi(\alpha + 2) \geq 2\psi(\alpha + 1) \) holds.

Let \((d, b) \in \mathbb{Z}^n \times \mathbb{Z}^n \) be an optimal solution of (DA[\alpha]). Also, let \((\hat{d}, \hat{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n \) be an optimal solution of (DA[\alpha + 2]) that has the minimum value of \( \| \tilde{d} - d \|_1 + \| \tilde{b} - b \|_1 \). Note that we have \( c(d, b) = \psi(\alpha) \) and \( c(\hat{d}, \hat{b}) = \psi(\alpha + 2) \). By the definition of \( \psi \), we have \( \psi(\alpha) \geq \psi(\alpha + 1) \geq \psi(\alpha + 2) \). Hence, if \((d, b) \) is a feasible solution of (DA[\alpha + 2]), then we have \( \psi(\alpha + 1) \leq \psi(\alpha + 2) \) and therefore the inequality \( \psi(\alpha) + \psi(\alpha + 2) \geq 2\psi(\alpha + 1) \) follows. Therefore, we may assume that \( \hat{b}(N) = \alpha + 2 \) in the following.

Since \( b(N) = \alpha + 2 > \alpha = b(N) \) and \( \hat{d}(N) + \hat{b} = d(N) + b(N) \), it holds that \( \text{supp}^+(\hat{b} - b) \neq \emptyset \) and \( \text{supp}^-(\hat{d} - d) \neq \emptyset \). We first consider the case where there exists some \( i \in \text{supp}^+(\hat{b} - b) \) with \( i \in \text{supp}^-(\hat{d} - d) \). Then, Proposition 2.4(i) implies that

\[
c_i(\hat{d}(i), \hat{b}(i)) + c_i(d(i), b(i)) \geq c_i(\hat{d}(i) + 1, \hat{b}(i) - 1) + c_i(d(i) - 1, b(i) + 1),
\]

from which follows that the vectors \( \hat{d}' = \hat{d} + \chi_i, \hat{b}' = \hat{b} - \chi_i, d' = d - \chi_i, \) and \( b' = b + \chi_i \) satisfy the inequality

\[
\psi(\alpha + 2) + \psi(\alpha) = c(\hat{d}, \hat{b}) + c(d, b) \geq c(\hat{d}', \hat{b}') + c(d', b') \geq 2\psi(\alpha + 1),
\]

where the last inequality is by the fact that \((\hat{d}', \hat{b}')\) and \((d', b')\) are feasible solutions of (DA[\alpha + 1]).

We next consider the case where there exists no \( i \in \text{supp}^+(\hat{b} - b) \) with \( i \in \text{supp}^-(\hat{d} - d) \), i.e., we have \( \text{supp}^+(\hat{b} - b) \subseteq N \setminus \text{supp}^-(\hat{d} - d) \) and therefore \( \hat{d}(i) + \hat{b}(i) > d(i) + b(i) \) holds for every \( i \in \text{supp}^+(\hat{b} - b) \). Then, we have \( \text{supp}^-(\hat{d} - d) \subseteq N \setminus \text{supp}^+(\hat{b} - b) \) and therefore \( d(j) + b(j) <
Theorem A.10. We first propose a steepest descent algorithm for (DA). By using the fact that the problem (DA) can be reformulated as the minimization of the M-convex function $f$ given by (3.1), we can show that (DA) can be solved by a steepest descent algorithm similar to SteepestDescentDA. Difference from SteepestDescentDA is in the choice of the initial vector and in the termination condition. In the algorithm below, the initial vector can be any feasible solution that is bike-optimal, and the termination condition is given by a local optimality. Here, we say that a feasible solution $(d, b)$ of (DA) is bike-optimal if $b$ is an optimal solution of the problem (SRA$(d + b)$).

**Algorithm SteepestDescentDA**

**Step 0:** Set $(d_0, b_0)$ be an arbitrarily chosen bike-optimal feasible solution, and $k := 1$.

**Step 1:** If $c(d', b') \geq c(d_{k-1}, b_{k-1})$ for every $(d', b') \in N(d_{k-1}, b_{k-1}) \cap R$, then output the solution $(d_{k-1}, b_{k-1})$ and stop.

**Step 2:** Find $(d', b') \in N(d_{k-1}, b_{k-1}) \cap R$ that minimizes $c(d', b')$.

**Step 3:** Set $(d_k, b_k) := (d', b')$ and go to Step 1.

By applying Corollary A.8 to $f$ in (3.1) and also using the same analysis in Section 6.1, we obtain the following time complexity bound.

**Theorem A.10.** The algorithm SteepestDescentDA outputs an optimal solution in $O(n + \nu \log n)$ time with

$$\nu = \min\{\|(d, b) - (d_0, b_0)\|_1 \mid (d, b) \text{ is an optimal solution of (DA)}\}.$$
where \((\bar{d}, \bar{b})\) is some (fixed) feasible solution of \((DA)\). Note that for \(i \in N\), the function
\[
c_i^x(\eta, \zeta) = c_i(\lambda \eta + \bar{d}(i), \lambda \zeta + \bar{b}(i))
\]
is also a multimodular function in \((\eta, \zeta)\), the problem \((DA(\lambda))\) has the same combinatorial structure as \((DA)\), and therefore any algorithm for \((DA)\) can be applied to \((DA(\lambda))\). Our proximity-scaling algorithm is based on this fact and the following proximity theorem for \((DA)\):

**Theorem A.11.** Let \(\lambda\) be a positive integer with \(\lambda \geq 2\), and \((d, b) \in \mathbb{Z}^n \times \mathbb{Z}^n\) be a \(\lambda\)-optimal solution of \((DA)\). Then, there exists some optimal solution \((d^*, b^*) \in \mathbb{Z}^n \times \mathbb{Z}^n\) of \((DA)\) such that
\[
\| (d^* + b^*) - (d + b) \|_1 \leq 8\lambda n.
\]

Proof is given later in this subsection.

**Algorithm** ProximityScalingDA

**Step 0:** Let \((d_0, b_0)\) be an arbitrarily chosen feasible solution of \((DA)\) and \(x_0 = d_0 + b_0\). Set \(\lambda = (D + B) / 4n\) and \(p := 1\).

**Step 1:** Let \(b'_{p-1} \in \mathbb{Z}^n\) be a vector such that \((x_{p-1} - b'_{p-1}, b'_{p-1})\) is a bike-optimal solution of \((DA(\lambda))\).

**Step 2:** Apply the algorithm SteepestDescentDA to \((DA(\lambda))\) with the initial solution \((x_{p-1} - b'_{p-1}, b'_{p-1})\) to find a \(\lambda\)-optimal solution \((d_p, b_p)\).

**Step 3:** If \(\lambda = 1\), then output \((d_p, b_p)\) and stop. Otherwise, set \(x_p = d_p + b_p\), \(\lambda := \lfloor (\lambda / 2) \rfloor\), \(p := p + 1\), and go to Step 1.

We analyze the time complexity of the algorithm ProximityScalingDA. The definition of the initial \(\lambda\) in Step 0 implies that there exists a \(\lambda\)-optimal solution \((d, b)\) with \(\| (d + b) - x_0 \|_1 \leq 8\lambda n\). Also, in the \(p\)-th iterations with \(p \geq 2\), Theorem A.11 implies that there exists a \(\lambda\)-optimal solution \((d, b)\) with \(\| (d + b) - x_{p-1} \|_1 \leq 8\lambda n\). Hence, it follows from Theorem A.10 that each iteration, except for Step 1, can be done in \(O(n \log n)\) time. We can also show in a similar way that in Step 1, the vector \(b'_{p-1}\) can be computed by using a variant of steepest descent algorithm with the initial vector \(b_{p-1}\), and prove that its running time is \(O(n \log n)\). Hence, each iteration of the algorithm runs in \(O(n \log n)\). Since the number of iterations is \(O(\log((D + B) / n))\), we obtain the following bound for the algorithm ProximityScalingDA.

**Theorem A.12.** The algorithm ProximityScalingDA finds an optimal solution of the problem \((DA)\) in \(O(n \log n \log((D + B) / n))\) time.

**A.9.2 Proof of Theorem A.11**

Let \((d^*, b^*)\) be an optimal solution of \((DA)\) that minimizes the value \(\| d^* - d \|_1 + \| b^* - b \|_1\). We prove that \((d^*, b^*)\) satisfies the inequality \(\| x^* - x \|_1 \leq 8\lambda n\) with \(x = d + b\) and \(x^* = d^* + b^*\).

In the proof we consider the following six sets.

\[
I_1 = \{ i \in N \mid d(i) - d^*(i) \geq \lambda, \ b(i) - b^*(i) \leq -\lambda \}, \quad (A.15)
\]
\[
I_2 = \{ i \in N \mid d(j) - d^*(j) \leq -\lambda, \ b(j) - b^*(j) \geq \lambda \}, \quad (A.16)
\]
\[
I_3 = \{ i \in N \mid x(i) - x^*(i) \geq \lambda, \ d(i) - d^*(i) \geq \lambda, \} \quad (A.17)
\]
\[
I_4 = \{ i \in N \mid x(j) - x^*(j) \leq -\lambda, \ d(j) - d^*(j) \leq -\lambda \}, \quad (A.18)
\]
\[
I_5 = \{ i \in N \mid x(i) - x^*(i) \geq \lambda, \ b(i) - b^*(i) \geq \lambda \}, \quad (A.19)
\]
\[
I_6 = \{ i \in N \mid x(j) - x^*(j) \leq -\lambda, \ b(j) - b^*(j) \leq -\lambda \}. \quad (A.20)
\]
Lemma A.13.

(i) At least one of $I_1$ and $I_2$ is an empty set.
(ii) If $b^*(N) < B$ then $I_2 = \emptyset$ holds; if $b(N) - B \leq -\lambda$ then $I_1 = \emptyset$ holds.

Proof. We first prove (i). Assume, to the contrary, that both of $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$ hold. Then, there exist distinct $i, j \in N$ such that

$$d(i) - d^*(i) \geq \lambda, \quad b(i) - b^*(i) \leq -\lambda, \quad d(j) - d^*(j) \leq -\lambda, \quad b(j) - b^*(j) \geq \lambda.$$ 

We consider the pair of vectors $(d', b') \equiv (d - \lambda \chi_i + \lambda \chi_j, b + \lambda \chi_i - \lambda \chi_j)$, which is a feasible solution of (DR-AP$(\lambda)$) since $d + b = d' + b'$ and $b(N) = b'(N)$. We show below that $c(d', b') < c(d, b)$ holds, a contradiction to the choice of $(d, b)$.

By Proposition 2.4 (i), we have

$$c(d(i), b(i)) + c(d^*(i), b^*(i)) \geq c(d(i) - 1, b(i) + 1) + c(d^*(i) + 1, b^*(i) - 1),$$

$$c(j(d(j), b^*(j))) + c(d^*(j), b(j)) \geq c(d(j) + 1, b(j) - 1) + c(d^*(j) - 1, b^*(j) + 1).$$

This implies

$$c(d, b) + c(d^*, b^*) \geq c(d - \chi_i + \chi_j, b + \chi_i - \chi_j) + c(d^* + \chi_j, b^* - \chi_i + \chi_j). \quad \text{(A.21)}$$

Note that $(d'', b'') = (d^* + \chi_i - \chi_j, b^* - \chi_i + \chi_j)$ is also a feasible solution of (DR-AP) since $d'' + b'' = d^* + b^*$ and $b''(N) = b'(N)$. Since $(d'' + \chi_i - \chi_j, b'' - \chi_i + \chi_j)$ satisfies

$$\|d'' + \chi_i - \chi_j\|_1 + \|b'' - \chi_i + \chi_j\|_1 < \|d^* - d\|_1 + \|b^* - b\|_1,$$

we have

$$c(d'', b'') < c(d^* + \chi_i - \chi_j, b^* - \chi_i + \chi_j),$$

which, together with (A.21), implies $c(d, b) > c(d - \chi_i + \chi_j, b + \chi_i - \chi_j)$.

In a similar way, we can also prove the inequalities

$$c(d - \chi_i + \chi_j, b + \chi_i - \chi_j) > c(d - 2\chi_i + 2\chi_j, b + 2\chi_i - 2\chi_j) > \cdots > c(d - \lambda \chi_i + \lambda \chi_j, b + \lambda \chi_i - \lambda \chi_j),$$

from which $c(d, b) > c(d', b')$ follows.

Proof of (ii) is similar to (i) and omitted.

Lemma A.14. At least one of $I_3 = \emptyset$ and $I_4 = \emptyset$ holds.

Proof. Proof is similar to that for Lemma A.13 and omitted.

Lemma A.15. At least one of $I_5 = \emptyset$ and $I_6 = \emptyset$ holds.

Proof. Proof is similar to that for Lemma A.13 and omitted.

Lemma A.16.

(i) At least one of $I_4$, $I_5$, and $I_6$ is an empty set.
(ii) If $b^*(N) < B$ then at least one of $I_4$ and $I_5$ is an empty set.
(iii) At least one of $I_3$, $I_6$, and $I_2$ is an empty set.
(iv) If $b(N) - B \leq -\lambda$ then at least one of $I_3$ and $I_6$ is an empty set.
Proof. We prove (i) only. Assume, to the contrary, that all of the sets \( I_4, I_5, \) and \( I_1 \) are nonempty, and let \( i \in I_4, j \in I_5, \) and \( s \in I_1. \) Then, elements \( i, j, s \) are distinct by the definitions of \( I_4, I_5, \) and \( I_1. \) We denote
\[
(d', b') = (d + \lambda \chi_i - \lambda \chi_s, b - \lambda \chi_j + \lambda \chi_s),
\]
\[
(d'', b'') = (d^* - \chi_i + \chi_s, b^* + \chi_j - \chi_s).
\]
Since \((d', b')\) and \((d'', b'')\) satisfy
\[
d'(P) + b'(P) = d(P) + b(P), \quad b'(N) = b(N), \quad d''(P) + b''(P) = d^*(P) + b^*(P), \quad b''(N) = b^*(N),
\]
\((d', b')\) (resp., \((d'', b'')\)) is a feasible solution of \((\text{DR-AP}(\lambda))\) (resp., \((\text{DR-AP})\)). Using this fact, we can derive a contradiction as in Lemma A.13.

\[\square\]

Lemma A.17. We have \( \|x - x^*\|_1 \leq 4\lambda n \) if at least one of the following two conditions holds:

(a) \( I_3 = I_5 = \emptyset, \)

(b) \( I_4 = I_6 = \emptyset. \)

Proof. Suppose that \( I_4 = I_6 = \emptyset \) holds. Then, we have \( x(i) - x^*(i) \geq -2\lambda \) for every \( i \in N. \) Let \( N_- = \text{supp}^-(x - x^*). \) Since \( x(N) - x^*(N) = 0, \) we have
\[
\|x - x^*\|_1 = [x(N \setminus N_-) - x^*(N \setminus N_-)] + [x^*(N_-) - x(N_-)]
\]
\[
= [x(N) - x^*(N)] + 2[x^*(N_-) - x(N_-)]
\]
\[
= 4\lambda |N_-| \leq 4\lambda n.
\]

Proof for the case with \( I_3 = I_5 = \emptyset \) is similar.

\[\square\]

Lemma A.18. We have \( \|x - x^*\|_1 \leq 8\lambda n \) if at least one of the following two conditions hold:

(a) \( I_2 = I_4 = I_5 = \emptyset \) and \( b(N) - b^*(N) > -\lambda, \)

(b) \( I_1 = I_3 = I_6 = \emptyset \) and \( b(N) - b^*(N) < \lambda. \)

Proof. We consider the case where (a) holds, and show that \( \|d - d^*\|_1 \leq 4\lambda n \) and \( \|b - b^*\|_1 \leq 4\lambda n \) hold, which implies
\[
\|x - x^*\|_1 \leq \|d - d^*\|_1 + \|b - b^*\|_1 \leq 8\lambda n.
\]

Since \( I_2 = I_4 = I_5 = \emptyset, \) it holds that
\[
d(i) - d^*(i) \geq -2\lambda, \quad b(i) - b^*(i) \leq 2\lambda \quad (i \in N).
\]
\[\text{(A.22)}\]

Since \( b(N) - b^*(N) > -\lambda \) and \( x(N) - x^*(N) = 0, \) we have \( d(N) - d^*(N) < \lambda. \)

To prove the inequality \( \|d - d^*\|_1 \leq 4\lambda n, \) let \( H = \text{supp}^-(d - d^*). \) If \( H = N, \) then we have \( d(i) - d^*(i) < 0 \) for every \( i \in N, \) implying that
\[
\|d - d^*\|_1 = \sum_{i \in N} |d(i) - d^*(i)| = \sum_{i \in N} [d^*(i) - d(i)] = d^*(N) - d(N) < \lambda \leq 4\lambda n.
\]

If \( H \neq N, \) then we have
\[
\|d - d^*\|_1 = \sum_{i \in N'} |d(i) - d^*(i)| = [d(N \setminus H) - d^*(N \setminus H)] + [d^*(H) - d(H)]
\]
\[
= [d(N) - d^*(N)] + 2[d^*(H) - d(H)]
\]
\[
< \lambda + 4\lambda |H| \leq 4\lambda n,
\]
where the first inequality is by \( d(N) - d^*(N) < \lambda \) and \( d(i) - d^*(i) \geq -2\lambda \) for \( i \in N, \) and the second inequality is by \( |H| < n. \) The inequality \( \|b - b^*\|_1 \leq 4\lambda n \) can be proved similarly by using the inequalities \( b(N) - b^*(N) < -\lambda \) and \( b(i) - b^*(i) \leq 2\lambda \) for \( i \in N. \)

\[\square\]

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Lemma A.19. We have $\|x - x^*\|_1 \leq 8\lambda n$.

Proof. By Lemmas A.14 and A.15, we have the following four possibilities:

(Case 1) $I_4 = I_6 = \emptyset$,   (Case 2) $I_3 = I_5 = \emptyset$,
(Case 3) $I_4 = I_5 = \emptyset$, $I_3 \neq \emptyset$, $I_6 \neq \emptyset$, (Case 4) $I_3 = I_6 = \emptyset$, $I_4 \neq \emptyset$, $I_5 \neq \emptyset$.

If Case 1 or 2 holds, then we have $\|x - x^*\|_1 \leq 8\lambda n$ by Lemma A.17. Below we give proofs for Cases 3 and 4.

[Proof for Case 3] By Lemma A.16 (iii) and (iv), we have $I_2 = \emptyset$ and $b(N) - B > -\lambda$; the second inequality implies $b(N) - b^*(N) > -\lambda$ since $b^*(N) \leq B$. Hence, we have $\|x - x^*\|_1 \leq 8\lambda n$ by Lemma A.18.

[Proof for Case 4] By Lemma A.16 (i) and (ii), we have $I_1 = \emptyset$ and $b^*(N) = B$; the second equation implies $b(N) - b^*(N) < \lambda$ since $b(N) \leq B$. Hence, we have $\|x - x^*\|_1 \leq 8\lambda n$ by Lemma A.18.

□