Some results on the stability of quasi-static paths of elastic-plastic systems with hardening

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Abstract. In this paper, a concept of stability of quasi-static paths is discussed. This takes into account the existence of fast (dynamic) and slow (quasi-static) times scales in the mechanical systems that have an elastic-plastic behavior with linear hardening. The proposed concept is essentially a continuity property relative to the size of the initial perturbations (as in Lyapunov stability) and relative to the smallness of the rate of application of the forces (which here plays the role of the small parameter in singular perturbation problems). Existence and uniqueness results for the dynamic and quasi-static problems are recalled and the stability of quasi-static paths for elastic-plastic systems with hardening is obtained.

1. Introduction

The computation of quasi-static evolutions of mechanical systems (frequently involving physical and geometric nonlinearities, plasticity, damage, etc.) is done with the presumption that (some portions of) the computed quasi-static paths are a reliable approximation of the actual paths that, in our physical world, are subjected to the universal rule of Newton’s law. The common use of static equilibrium equations instead of Newton’s law in the computation of the slow evolution of some mechanical systems requires a justification and an assessment of the limits of its validity. That is the purpose of the research work initiated in [6], and continued in [10, 9, 7]. To the best of our knowledge, Martins and coworkers [6, 10] were the first to recognize the distinct time scales involved in dynamic and quasi-static problems, as well as the singular perturbation nature of the problem of using static equilibrium equations to approximate governing dynamic equations, when the time rate of change of the load is decreased to zero. They performed a change of variables in the governing system of dynamic equations that consists of replacing the (fast) physical time \( t \) by a (slow) loading parameter \( \lambda \) whose rate of change with respect to time, \( \varepsilon = d\lambda/dt \), is eventually decreased to zero. This leads to a system of singularly perturbed equations or inclusions, in which some of the highest order derivatives with respect to the loading parameter appear multiplied by the small time rate of change \( \varepsilon \) of the load.

In the present paper, the definition of stability of quasi-static paths given in [10] is adapted to a continuum (1D) case that involves an elastic-plastic material with kinematic hardening,
and we shall prove that in this case the dynamic evolutions remain close to a quasi-static path when they start sufficiently close to it and the load is applied sufficiently slowly.

Johnson [3, 4] developed the variational formulation of plasticity problems with hardening. For the corresponding quasi-static problem he proved existence of a strong solution and, under some additional assumptions, a regularity result for the velocity was obtained. A similar formulation had already been obtained for perfect elastic-plastic and elastic-visco-plastic systems by Duvaut and Lions [2]. In what concerns the dynamic problems in elasto-plasticity with hardening, we address the reader to the works of Krejčí [5], Showalter and Shi [11, 12], Visintin [13], and the references therein.

The structure of the article is the following. In Section 2, the mathematical formulations for dynamic and quasi-static elastic-plastic systems with hardening are presented. In Section 3, existence and uniqueness results, which use the theory of m-accretive operators, are recalled [1, 2, 11, 12, 14]; some results are also shown for the elastic-visco-plastic (Yosida) regularization of the original dynamic problem followed by its finite dimensional (Galerkin) approximation. In Section 4, the definition of stability of a quasi-static path is adapted from [7, 9, 10] to the present elastic-plastic problems, and stability in that sense for these problems is proved.

2. Governing equations
We consider an elastic-plastic bar with linear kinematic hardening that has length $L$ along the $x$ axis. Geometrical linearity is assumed. The governing dynamic equation can be non-dimensionalized by using the non-dimensional time ($\tau$) and load parameter ($\lambda$, $\lambda = \lambda_1 + \varepsilon \tau$), yielding

$$\varepsilon^2 u'' - \sigma_x = f(x, \lambda),$$

where $u, r, f$ are the non-dimensional axial displacement, the stress in the plastic element and the applied force per unit length along the bar, respectively; $\sigma$ is the stress in the elastic-plastic bar, and the subscript $x$ denotes a derivative with respect to $x$. For simplicity, the mass density of the bar has been taken as unitary.

The extension $e$ is the derivative in space of the displacement $u$, and it can be decomposed into elastic, $e^e$, and plastic, $e^p$, parts:

$$e = u_x = e^e + e^p.$$  \hfill (2)

The stress $\sigma$ in the bar is related to the stress in the plastic element and the plastic strain by the linear kinematic hardening rule

$$\sigma = r + He^p,$$  \hfill (3)

and is related to the elastic part of the extension by means of Hooke’s law

$$\sigma = Ee^e = E(e - e^p),$$  \hfill (4)

where the elasticity and hardening moduli $E$ and $H$ are positive constants. Therefore (2), (3), and (4) lead to

$$\sigma = \sigma(e, r) = D(e + H^{-1}r)$$

where $D = (E^{-1} + H^{-1})^{-1}$. Carrying (5) and (2) into (1), we obtain

$$\varepsilon^2 u'' - Du_{xx} - DH^{-1}r_x = f.$$  \hfill (6)
The behavior of the plastic element is characterized by the non-dimensional inequality and flow
rule:
\[
|r| \leq 1, \quad (ep)' \begin{cases} 
\geq 0 & \text{if } r = +1, \\
= 0 & \text{if } -1 < r < +1, \\
\leq 0 & \text{if } r = -1.
\end{cases}
\] (7)

The governing dynamic equations (6), together with the conditions (7) can be put in the form
of a singular perturbation system of first order differential equations and inclusion. For that
purpose, let \( \mathcal{C} \) denote the following closed convex set in \( L^2(0,L) \)
\[
\mathcal{C} = \{ r \in L^2(0,L) : |r| \leq 1 \},
\] (8)
and let \( \text{sign}^{-1}(r) \) be the normal cone to \( \mathcal{C} \) at \( r \in L^2(0,L) \). Then we observe that (7) can be
written in the differential inclusion form:
\[
(ep)' \in \text{sign}^{-1}(r).
\] (9)

Relations (4) lead to
\[
(ep)' = \tilde{D}^{-1}(Eu_x' - r') = DH^{-1}u_x' - \tilde{D}^{-1}r', \quad \text{where } \tilde{D} = E + H.
\] (10)
Substituting (10) in (9), we get
\[
\tilde{D}^{-1}r' - DH^{-1}u_x' + \text{sign}^{-1}(r) \ni 0.
\] (11)

We introduce now the following spaces
\[
\mathcal{H} = L^2(0,L), \quad \mathcal{V} = H^1(0,L), \quad \mathcal{V}_0 = H^1_0(0,L).
\] (12)

We denote the norm in \( \mathcal{H} \) (resp. \( \mathcal{V} \)) by \( |\cdot| \) (resp. \( || \cdot || \)) and the scalar product in \( \mathcal{H} \) by \( (\cdot, \cdot) \).

From (6) and (11) we finally obtain the governing dynamic system
\[
\begin{cases}
\epsilon v' - Du_{xx} - DH^{-1}r_x = f, \\
\epsilon u' - v = 0, \\
\tilde{D}^{-1}r' - DH^{-1}u_x' + \text{sign}^{-1}(r) \ni 0,
\end{cases}
\] (13)

with the Dirichlet boundary conditions
\[
v = u = 0 \text{ on } (0,L) \times (\lambda_1, \lambda_2),
\] (14)
and the initial conditions (recall (5) and (12))
\[
(v(\lambda_1), u(\lambda_1), r(\lambda_1)) = (v_1, u_1, r_1) \in \mathcal{V}_0 \times \mathcal{V}_0 \times \mathcal{C}, \quad \text{with } \sigma(u_x, r) \in \mathcal{V}.
\] (15)

Note that the differential inclusion system (13) can be written in the variational inequality form:
\[
\begin{cases}
(u, r) \in \mathcal{V}_0 \times \mathcal{C} \text{ and } \forall (u^*, r^*) \in \mathcal{V}_0 \times \mathcal{C} : \\
(\epsilon^2 u'', u^*) + a(u, u^*) + b(r, u^*) = (f, u^*), \\
(r^* - r, \tilde{D}^{-1}r') - b(r^* - r, u') \geq 0,
\end{cases}
\] (16)
where the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are given by
\[
a(u, u^*) = (Du_x, u_x^*), \\
b(r, u^*) = (DH^{-1}r, u_x^*),
\]
for all $u, u^* \in V$ and all $r \in H$. Clearly there exist positive constants $\alpha_m, \alpha_M$ and $\beta_M$ such that, for all $u, u^* \in V$ and all $r \in H$:

\[
\alpha_m \|u\|^2 \leq a(u, u), \quad a(u, u^*) \leq \alpha_M \|u\| \|u^*\|,
\]

\[
b(r, u^*) \leq \beta_M |r| \|u^*\|.
\]

These inequalities will be used throughout the rest of the paper in order to obtain all the estimates that are based on variational statements.

The corresponding quasi-static system is then (let $\varepsilon = 0$ in (13))

\[
\begin{align*}
-D\ddot{u}_{xx} - DH^{-1}\ddot{r}_x &= f, \\
\tilde{D}^{-1}\dot{r} - DH^{-1}\ddot{u}_x + \text{sign}^{-1}(\tilde{r}) &\ni 0,
\end{align*}
\]

with the Dirichlet boundary conditions

\[
\bar{u} = 0 \text{ on } \{0, L\} \times (\lambda_1, \lambda_2),
\]

and the initial conditions (recall again (5) and (12))

\[
(u(\lambda_1), \bar{r}(\lambda_1)) = (\bar{u}_1, \bar{r}_1) \in V_0 \times C, \text{ with } \sigma(\bar{u}_x, \bar{r}_1) \in V.
\]

Note that, consistent with the above, the quasi-static displacement rate with respect to the physical time vanishes ($\dot{v} \equiv 0$). On the other hand, the differential inclusion system (17) can be written in the variational form:

\[
\begin{align*}
(u, \bar{r}) &\in V_0 \times C \text{ and } \forall (u^*, r^*) \in V_0 \times C : \\
a(u, u^*) + b(r, u^*) &= (f, u^*), \\
(r^* - \bar{r}, \tilde{D}^{-1}\dot{r}') - b(r^* - \bar{r}, \ddot{u}') &\geq 0.
\end{align*}
\]

3. Existence, uniqueness and approximation results

In this section we recall existence and uniqueness results for the dynamic and quasi-static problems that were formulated earlier. The theory of $m$-accretive operators is used with this purpose. We also recall the elastic-visco-plastic regularization of the original dynamic elastic-plastic problem, as well as the finite dimensional approximation of the regularized problem. This has the purpose of recalling or proving some properties of those approximations that will be used subsequently.

3.1. Existence and uniqueness of solution for the dynamic and quasi-static problems

The dynamic and the quasi-static systems introduced in Section 2 can be rewritten in a form that may be studied with the theory of $m$-accretive operators. Consider the differential inclusion problem that involves a multivalued operator $\mathcal{A}$ in the Hilbert space $Y$, with domain $\mathcal{D} = \{x \in Y : \mathcal{A}x \neq \emptyset\}$:

\[
\begin{align*}
x(\lambda) &\in \mathcal{D}(\lambda), \quad \forall \lambda \in [\lambda_1, \lambda_2], \\
x' + \mathcal{A}x &\ni g \text{ a.e. on } (\lambda_1, \lambda_2), \\
x(\lambda_1) &= x_1.
\end{align*}
\]

Recall that existence and uniqueness of solution to this problem can be obtained from the following (cf. [4] or [3]):
Proposition 3.1 Assume that $A$ is an $m$-accretive operator in the Hilbert space $Y$, $g$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H})$ and $x_1 \in D(A)$. Then there exists a unique solution $x$ of (21) belonging to $W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{Y})$.

In what concerns the dynamic problem, we differentiate the second equation in the system (13) with respect to $x$, we use $e = u_x$ as an alternative unknown function, and we denote $x = (v, D^{1/2}e, D^{-1/2}e)$. Then the governing dynamic system (13) yields the inclusion (21b) with

$$A = \frac{1}{\varepsilon} \begin{pmatrix} 0 & -D^{1/2}\partial / \partial x & -E\tilde{D}^{-1/2}\partial / \partial x \\ -D^{1/2}\partial / \partial x & 0 & 0 \\ -E\tilde{D}^{-1/2}\partial / \partial x & 0 & \varepsilon\tilde{D}^{1/2}\text{sign}^{-1}(\tilde{D}^{1/2}()) \end{pmatrix} \quad \text{and} \quad g = \frac{1}{\varepsilon} \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}. \quad (22)$$

Letting $Y = \mathcal{H}$, it can be proved (see [11] and [12]) that $A$ is an $m$-accretive operator in $Y$. Then Proposition 3.1, yields the following:

**Corollary 3.2** Assume that $f$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H})$ and that (15) holds. Then there exists a unique $x = (v, D^{1/2}e, D^{-1/2}e) \in W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H})$ that solves (21) with $A$ and $g$ given by (22), and with $r(\lambda) \in C$ for all $\lambda \in [\lambda_1, \lambda_2]$, $v \in L^{\infty}(\lambda_1, \lambda_2; \mathcal{V}_0)$, and $\sigma(e, r) \in L^{\infty}(\lambda_1, \lambda_2; \mathcal{V})$.

In what concerns the quasi-static problem, we differentiate the first identity in (17) with respect to the load parameter $\lambda$ and we get

$$-D\dddot{u}_{xx} = DH^{-1}\dddot{r}_x + f',$$  \hspace{1cm} (23)

together with the Dirichlet boundary conditions (18). Since this is an elliptic problem for $\dddot{u}$ we conclude that there exists a unique solution. For such solution $\dddot{u}_x + H^{-1}\dddot{r}$ depends linearly and continuously on $f'$, i.e.

$$\dddot{u}_x + H^{-1}\dddot{r} = Bf',$$ \hspace{1cm} (24)

where $B$ is a continuous linear operator between the appropriate spaces. Inserting this in the inclusion in (17) we finally get the differential inclusion

$$\dddot{r} + H\text{sign}^{-1}(\dddot{r}) \ni DBf'.$$ \hspace{1cm} (25)

The sub-differential $\partial \varphi(\dddot{r}) = \text{sign}^{-1}(\dddot{r})$ is an $m$-accretive operator since $\varphi(\dddot{r})$ is a proper convex function and lower semi-continuous function. For $x = \dddot{r}$, $A = H\text{sign}^{-1}$, $g = DBf'$ and $Y = \mathcal{H}$, we apply Proposition 3.1 and we obtain the following Corollary:

**Corollary 3.3** Assume that $f$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H})$ and (19) holds. Then there exists a unique solution $(\dddot{u}, \dddot{\varphi})$ of (17)-(19) such that $(\dddot{u}, \dddot{\varphi}) \in W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{V}_0 \times \mathcal{H})$, with $\dddot{\varphi} \in C$ for all $\lambda \in [\lambda_1, \lambda_2]$, and $\sigma(\dddot{u}_x, \dddot{\varphi}) \in L^{\infty}(\lambda_1, \lambda_2; \mathcal{V})$.

### 3.2. Elastic-visco-plastic and Galerkin approximations

We consider first the elastic-visco-plastic problem:

$$\begin{cases} \varepsilon^2 u''_x - Du'''_{xxx} - DH^{-1}r_{xx} = f, \\ \tilde{D}^{-1}r''_x - DH^{-1}u'''_{xx} + \tilde{J}_x(r_x) = 0, \end{cases} \text{ where } \tilde{J}_x(r_x) = \frac{1}{\mu} (r_x - \text{proj}_x r_x), \quad (26)$$

with the Dirichlet boundary conditions

$$u_\mu = 0 \text{ on } (0, L) \times (\lambda_1, \lambda_2), \quad (27)$$
and the initial conditions

\[ (\varepsilon u'_\mu(\lambda_1), u_\mu(\lambda_1), r_\mu(\lambda_1)) = (v_1, u_1, r_1) \in V_0 \times V_0 \times C, \] with \( \sigma(u_{x1}, r_1) \in \mathcal{V} \).

Here \( \mu > 0 \) is the viscosity parameter and \( \text{proj}_C \) denotes the projection on the convex \( C \).

The variational formulation of the regularized problem (26)–(28) reads

\[
\begin{cases}
(u_\mu, r_\mu) \in V_0 \times H \text{ such that } \forall (u^*, r^*) \in V_0 \times H, \\
(\varepsilon^2 u''_\mu, u^*) + a(u_\mu, u^*) + b(r_\mu, u^*) = (f, u^*), \\
(r^*, \tilde{D}^{-1}r'_\mu) - b(r^*, u'_\mu) + (r^*, \mathcal{J}_\mu(r_\mu)) = 0,
\end{cases}
\]

also with the initial conditions (28). Note that this elastic-visco-plastic problem is an Yosida regularization of the original elastic-plastic problem. For a similar approximation in the theory of operators, see \[8\]. Whenever convenient we shall use the notation \( v_\mu = \varepsilon u'_\mu \) in the following text.

We consider now a finite dimensional approximation of the above elastic-visco-plastic problem, which is obtained in the following classical manner. Let \( \{w_j\}_{j=1}^\infty \) be a complete orthonormal sequence in \( H \) whose elements belong to \( H^2(0, L) \). Let \( u_{\mu n} = \sum_{i=1}^n g_{in}(\lambda)w_i(x) \) and \( r_{\mu n} = \sum_{i=1}^n h_{in}(\lambda)w_i(x) \) satisfy the following variational formulation

\[
\begin{cases}
\forall u^* = \sum_{i=1}^n g^*_i(\lambda)w_i(x) \text{ and } r^* = \sum_{i=1}^n h^*_i(\lambda)w_i(x), \\
(\varepsilon^2 u''_{\mu n}, u^*) + a(u_{\mu n}, u^*) + b(r_{\mu n}, u^*) = (f, u^*), \\
(r^*, \tilde{D}^{-1}r'_{\mu n}) - b(r^*, u'_{\mu n}) + (r^*, \mathcal{J}_\mu(r_{\mu n})) = 0,
\end{cases}
\]

with

\[
\varepsilon \lim_{n \to \infty} \sum_{i=1}^n g'_{in}(\lambda_1)w_i(x) = v_1, \quad \lim_{n \to \infty} \sum_{i=1}^n g_{in}(\lambda_1)w_i(x) = u_1, \quad \lim_{n \to \infty} \sum_{i=1}^n h_{in}(\lambda_1)w_i(x) = r_1.
\]

The following results can be proved for the above approximations, when the dimension parameter \( n \) tends to \( \infty \), and the viscosity parameter \( \mu \) tends to \( 0 \).

**Proposition 3.4** Assume that \( f \) belongs to \( W^{1,\infty}(\lambda_1, \lambda_2; H) \) and that (28) holds. Then there exists a unique solution \((v_\mu, u_\mu, r_\mu)\) of (26)–(28) such that \((v_\mu, u_\mu, r_\mu)\) and \((v'_\mu, u'_\mu, r'_\mu)\) belong respectively to \( L^\infty(\lambda_1, \lambda_2; V_0 \times V_0 \times H) \) and \( L^\infty(\lambda_1, \lambda_2; H \times H \times H) \) and \( \sigma(u_{\mu x}, r_\mu) \in L^\infty(\lambda_1, \lambda_2; \mathcal{V}) \). Moreover, as \( \mu \) tends to zero, \( u_\mu \) and \( \sigma(u_{\mu x}, r_\mu) \) converge strongly to their limits.

The Galerkin approximation described above together with a priori estimates based on the variational formulations (29), (30) can be used to prove these results. The reader can find detailed proofs in the Appendix of \[8\] or in \[2\]. This Proposition can also be proved using the theory of \( m \)-accretive operators.

**Lemma 3.5** Assume that (28) holds and that \( f \) belongs to \( W^{1,\infty}(\lambda_1, \lambda_2; H) \). Then independently of \( \mu > 0 \), for all \( \lambda \) belonging to \((\lambda_1, \lambda_2)\), \( v_\mu(\lambda), u_{\mu x}(\lambda) \) and \( r_\mu(\lambda) \) are bounded in \( H \). Moreover, as \( \mu \) tends to \( 0 \), it holds that, for all \( \lambda \) belonging to \((\lambda_1, \lambda_2)\),

\[
\begin{align*}
v_\mu(\lambda) & \to v(\lambda) \text{ strongly in } H, \\
u_{\mu x}(\lambda) & \to u_x(\lambda) \text{ strongly in } H, \\
r_\mu(\lambda) & \to r(\lambda) \text{ strongly in } H,
\end{align*}
\]

and also

\[
\varepsilon v'_\mu \rightharpoonup \varepsilon v' \text{ weakly }* \text{ in } L^\infty(\lambda_1, \lambda_2; V_0^*)
\]

(32)
Energy estimation based on the variational formulation (29) leads to the above boundedness properties. The strong convergence properties are obtained by the energy estimation of the difference between the elastic-visco-plastic system (29) and the elastic-plastic system with hardening (16). The same difference and the strong convergence of \( \sigma(u_{\mu x}, r) = D(u_{\mu x} + H^{-1}r_\mu) \) lead to the desired result.

**Lemma 3.6** Assume that (28) holds and that \( f \) belongs to \( W^{2,\infty}(\lambda_1, \lambda_2; \mathcal{H}) \). Then there exists a subsequence, still denoted by \( v'_{\mu n} \), such that

\[
v'_{\mu n} \rightharpoonup v'_{\mu} \quad \text{weakly}^* \quad \text{in} \quad L^\infty(\lambda_1, \lambda_2; \mathcal{H}).
\]

Moreover denoting by \( c \) a positive constant that is independent of \( \mu \) and \( n \), and letting \( v'_{\mu n}(\lambda_1) \) be the initial acceleration of the \( \mu_n \) solution, the following inequality holds for all \( \lambda \) belonging to \((\lambda_1, \lambda_2)\)

\[
|\varepsilon v'_{\mu n}(\lambda)|^2 \leq c(||v_1||^2 + \varepsilon^2|v'_{\mu n}(\lambda_1)|^2 + \varepsilon^2|f'(\lambda_1)|^2 + \varepsilon^2||f'\|vert_{L^\infty(\lambda_1, \lambda_2; \mathcal{H})}^2 + \varepsilon^2||f''\rvert_{L^2(\lambda_1, \lambda_2; \mathcal{H})}^2).
\]

Unlike the previous results, the proof of this Lemma requires the *differentiation of the governing system* (30) with respect to \( \lambda \). Taking \( u^* = \varepsilon^2u''_{\mu n} \) and \( r^* = \varepsilon^2r'_{\mu n} \) in the resulting system, adding the two resulting identities, and using the monotonicity of \( r \mapsto J_\mu(r) \), yields an appropriate energy inequality. Its integration in \( \lambda \), taking into account the solution values at the initial \( \lambda_1 \), and the application of Gronwall’s Lemma, leads to the final result. The detailed proof can be found in [8].

### 4. Stability of quasi-static paths of elastic-plastic systems with linear hardening

In Section 4.1, we adapt the definition of stability of a quasi-static path ([7], [9], [10]) to the present elastic-plastic problem with hardening. In Section 4.2, *a priori* estimates on the elastic-visco-plastic systems are obtained, which prove that the dynamic and the quasi-static solutions remain close to each other if the dynamic solution of (13) is initially close to the quasi-static solution of (17) and the loading rate \( \varepsilon \) is sufficiently small.

#### 4.1. Definition of stability of a quasi-static path

The mathematical definition of stability of a quasi-static path at an equilibrium point is presented in the context of the governing dynamic system (13)–(15) and the quasi-static system (17)–(19).

**Definition 4.1** The quasi-static path \((\bar{u}(\lambda), \bar{r}(\lambda))\) is said to be stable at \( \lambda_1 \) if there exists \( 0 < \Delta \lambda \leq \lambda_2 - \lambda_1 \), such that, for all \( \delta > 0 \) there exists \( \bar{\rho}(\delta) > 0 \) and \( \bar{\varepsilon}(\delta) > 0 \) such that for all initial conditions \( u_1, v_1, r_1 \) and \( \bar{u}_1, \bar{r}_1 \) and all \( \varepsilon > 0 \) such that

\[
|v_1|^2 + \|u_1 - \bar{u}_1\|^2 + |r_1 - \bar{r}_1|^2 \leq \bar{\rho}(\delta) \quad \text{and} \quad \varepsilon \leq \bar{\varepsilon}(\delta),
\]

the solution \((u(\lambda), v(\lambda), r(\lambda))\) of the dynamic system (13)–(15) satisfies

\[
|v(\lambda)|^2 + \|u(\lambda) - \bar{u}(\lambda)\|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \leq \delta,
\]

for all \( \lambda \in [\lambda_1, \lambda_1 + \Delta \lambda] \).

For more details, the reader is referred to [10].
4.2. Stability of a quasi-static path

In order to prove the stability of the quasi-static path, we obtain an energy estimate of the difference between the dynamic and the quasi-static solution. In order to do this we will consider an auxiliary “special dynamic solution” \( (\tilde{v}, \tilde{u}, \tilde{r}) \) that solves (26) with the Dirichlet boundary conditions (14) and with the initial conditions

\[
\left( \tilde{v}(\lambda_1), \tilde{u}(\lambda_1), \tilde{r}(\lambda_1) \right) = (\varepsilon \bar{u}', \bar{u}_1, \bar{r}_1) \in \mathcal{V}_0 \times \mathcal{V}_0 \times \mathcal{C}, \quad \text{with } \sigma(\bar{u}_1, \bar{r}_1) \in \mathcal{V}.
\]  

(35)

This means that the initial conditions for this special dynamic solution coincide with or result from the quasi-static solution. Besides these “quasi-static initial conditions” (35) the special dynamic solution thus satisfies the variational statements:

\[
\begin{cases}
(\tilde{u}, \tilde{r}) \in \mathcal{V}_0 \times \mathcal{C} \text{ and } \forall(u^*, r^*) \in \mathcal{V}_0 \times \mathcal{C}, \\
(\varepsilon^2 \tilde{u}'', u^*) + a(\tilde{u}, u^*) + b(\tilde{r}, u^*) = (f, u^*), \\
(r^* - \tilde{r}, \tilde{D}^{-1} \tilde{r}' - b(r^* - \tilde{r}, \tilde{u}') \geq 0.
\end{cases}
\]

(36)

This special dynamic solution is used in the stability proof in the following manner: the difference between the dynamic and the quasi-static solution is estimated by evaluating separately the difference between the dynamic solution and the special dynamic solution, and the difference between the latter and the quasi-static solution:

\[
\begin{align*}
|v(\lambda)|^2 &+ \|u(\lambda) - \bar{u}(\lambda)\|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \\
&\leq 2(|v(\lambda) - \tilde{v}(\lambda)|^2 + \|u(\lambda) - \bar{u}(\lambda)\|^2 + |r(\lambda) - \tilde{r}(\lambda)|^2 \\
&+ |\tilde{v}(\lambda)|^2 + \|\tilde{u}(\lambda) - \bar{u}(\lambda)\|^2 + |\tilde{r}(\lambda) - \bar{r}(\lambda)|^2),
\end{align*}
\]

(37)

In what concerns the difference between the dynamic solution and the special dynamic solution, the desired estimate is nothing but a continuity property of the solutions of the dynamic problem relatively to the initial conditions. The corresponding energy estimate is obtained in Lemma 4.2. A similar “well-posedness” result was recently obtained by Visintin [13].

In Lemma 4.3 an energy estimate of the difference between the special dynamic solution and the quasi-static one is obtained which makes use of the results of the Yosida and the Galerkin approximations of the previous section.

**Lemma 4.2** Assume that (15) and (35) hold and that \( f \) belongs to \( W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H}) \). Then there exists a positive constant \( c \) such that:

\[
|v(\lambda) - \tilde{v}(\lambda)|^2 + \|u(\lambda) - \bar{u}(\lambda)\|^2 + |r(\lambda) - \tilde{r}(\lambda)|^2 \\
\leq c(|v_1 - \varepsilon \bar{u}'(\lambda_1)|^2 + \|u_1 - \bar{u}_1\|^2 + |r_1 - \bar{r}_1|^2).
\]

(38)

**Proof.** Subtracting the equalities in (16) and (36), and choosing \( u^* = u' - \bar{u}' \) in both, we have

\[
(\varepsilon^2 (u'' - \bar{u}''), u' - \bar{u}') + a(u - \bar{u}, u' - \bar{u}') + b(r - \bar{r}, u' - \bar{u}') = 0.
\]

(39)

On the other hand, taking \( r^* = \tilde{r} \) in the inequality of (16) and \( r^* = r \) in the inequality of (36), we get, after adding the resulting inequalities:

\[
(r - \tilde{r}, \tilde{D}^{-1}(r' - \tilde{r}')) \leq b(r - \bar{r}, u' - \bar{u}').
\]

(40)

Carrying (40) into (39) and since \( v = \varepsilon u' \) and \( \tilde{v} = \varepsilon \bar{u}' \), we obtain

\[
\frac{d}{d\lambda} (|v - \tilde{v}|^2 + a(u - \bar{u}, u - \bar{u}) + |\tilde{D}^{-1/2}(r - \bar{r})|^2) \leq 0.
\]

(41)

Integrating (41) over \( (\lambda_1, \lambda), \lambda \in [\lambda_1, \lambda_2] \), and using the initial conditions (15) and (35) leads to the desired result.
Lemma 4.3 Assume that \( f \) belongs to \( W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H}) \) and that (19) and (35) hold. Then there exists a positive constant \( c \), such that

\[
|\tilde{v}(\lambda)|^2 + \|\tilde{u}(\lambda) - \tilde{u}(\lambda)\|^2 + |r(\lambda) - \tilde{r}(\lambda)|^2 \leq c[\varepsilon^2|\tilde{u}'(\lambda)|^2 + \int_{\lambda_1}^{\lambda} (\varepsilon\tilde{v}', \tilde{u}') \, d\xi]
\]

Proof. Proceeding as in the previous Lemma, but dealing now with the difference between the “special dynamic solution” and the quasi-static one, we obtain successively

\[
\varepsilon^2(\tilde{u}'', \tilde{u}') + a(\tilde{u} - \tilde{u}, \tilde{u}' - \tilde{u}') + b(\tilde{r} - \tilde{r}, \tilde{u}' - \tilde{u}') = \varepsilon^2(\tilde{u}'', \tilde{u}'),
\]

(43)

\[
(\tilde{r} - \tilde{r}, \tilde{D}^{-1}(\tilde{r}' - \tilde{r}')) \leq b(\tilde{r} - \tilde{r}, \tilde{u}' - \tilde{u}'),
\]

(44)

\[
\frac{d}{d\lambda} (|\tilde{v}|^2 + a(\tilde{u} - \tilde{u}, \tilde{u} - \tilde{u}) + |\tilde{D}^{-1/2}(\tilde{r} - \tilde{r})|^2) \leq 2\varepsilon^2(\tilde{u}'', \tilde{u}').
\]

(45)

The desired result follows from integrating (45) over \((\lambda_1, \lambda)\) with \( \lambda \in [\lambda_1, \lambda_2] \), since the initial conditions (35) for the special dynamic solution are such that \( \tilde{v}(\lambda_1) = \varepsilon\tilde{u}', \tilde{u}(\lambda_1) = \tilde{u}_1, \) and \( \tilde{r}(\lambda_1) = \tilde{r}_1 \).

In order to obtain the final stability result, the estimates (38) and (42) of the previous lemmas are inserted in the inequality (37). We have

\[
|v(\lambda)|^2 + \|u(\lambda) - \tilde{u}(\lambda)\|^2 + |r(\lambda) - \tilde{r}(\lambda)|^2 \leq c[|v_1| + \varepsilon\tilde{u}'(\lambda_1)|^2 + \|u_1 - \tilde{u}_1\|^2 + |r_1 - \tilde{r}_1|^2
\]

(46)

\[
+ \varepsilon^2|\tilde{u}'(\lambda_1)|^2 + \int_{\lambda_1}^{\lambda} (\varepsilon\tilde{v}', \tilde{u}') \, d\xi],
\]

where, again, \( c \) is some positive constant that is independent of \( \varepsilon \). In order to estimate the last integral, we consider the finite dimensional (Galerkin) approximation of the elastic-visco-plastic (Yosida) regularization of the inertia term \( \varepsilon\tilde{v}' \):

\[
\int_{\lambda_1}^{\lambda} (\varepsilon\tilde{v}', \tilde{u}') \, d\xi = \int_{\lambda_1}^{\lambda} (\varepsilon(\tilde{v}' - \tilde{v}'_{\mu}), \tilde{u}') \, d\xi + \int_{\lambda_1}^{\lambda} (\varepsilon(\tilde{v}'_{\mu} - \tilde{v}'_{\mu n}), \tilde{u}') \, d\xi + \int_{\lambda_1}^{\lambda} (\varepsilon\tilde{v}'_{\mu n}, \tilde{u}') \, d\xi.
\]

(47)

The conclusion follows then from Lemma 3.6: the estimate (34) of \( \varepsilon\tilde{v}'_{\mu n} \) and the weak * convergence (32) and (33) of the terms that involve differences of inertia terms. Note, in particular, that the initial terms involved in the estimate (34) of \( \varepsilon\tilde{v}'_{\mu n} \) have the following simplifications: \( \|\tilde{v}_1\|^2 = \varepsilon^2\|\tilde{u}'(\lambda_1)\|^2 \) and \( \tilde{v}'_{\mu n}(\lambda_1) = 0 \). It is because of these simplifications that we use the special dynamic solution in our proof and, in Lemma 4.3, we estimate the distance between the quasi-static solution and the special dynamic solution (rather than directly estimate the distance between the quasi-static solution and an arbitrary dynamic solution).

Before stating the desired stability result, we remark that the proof given here differs from the one presented in [8], namely, on the use made here of the weak * convergence (32) of \( \varepsilon\tilde{v}'_{\mu} \) to \( \varepsilon\tilde{v}' \), and on some aspects of the notation and of the order of the presentation, which hopefully will improve its clarity.

Proposition 4.4 (Stability). Assume that \( f \) belongs to \( W^{2,\infty}(\lambda_1, \lambda_2; \mathcal{H}) \) and that (15) and (19) hold. Then there exists \( \gamma > 0 \) such that for \( 0 < \varepsilon < 1 \),

\[
|v(\lambda)|^2 + \|u(\lambda) - \tilde{u}(\lambda)\|^2 + |r(\lambda) - \tilde{r}(\lambda)|^2 \leq \gamma(|v_1|^2 + |u_1 - \tilde{u}_1|^2 + |r_1 - \tilde{r}_1|^2 + \varepsilon).
\]
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