Light-cone Hamiltonian flow for positronium

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Abstract

The technique of Hamiltonian flow equations is applied to the canonical Hamiltonian of quantum electrodynamics in the front form and 3+1 dimensions.

The aim is to generate a bound state equation in a quantum field theory, particularly to derive an effective Hamiltonian which is practically solvable in Fock-spaces with reduced particle number. The effective Hamiltonian, obtained as a solution of flow equations to the second order, is solved numerically for positronium spectrum. The impact of different similarity functions is explicitly studied.

The approach discussed can ultimately be used to address the same problem for quantum chromodynamics.

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1 Introduction

Over the past twenty years two fundamentally different pictures of hadrons have developed. One, the constituent quark model is closely related to experimental observation and phenomenology. The other, quantum chromodynamics (QCD) is based on a covariant non-abelian quantum field theory. Disregarding lattice gauge calculations, one has several reasons\cite{1} why the front form of Hamiltonian dynamics\cite{2}, as reviewed recently in\cite{3}, is one of the very few candidates for reconciling the two approaches. Particularly the simple vacuum and the simple boost properties confront with the complicated vacuum and the complicated boosts in the conventional Hamiltonian theory. Wilson and collaborators\cite{4,5} have proposed a scheme in which one presumes a potential for the bound-states and handles the relativistic effects by structures imposed by the needs of renormalization. The available numerical examples\cite{6,7} however violate admittedly some symmetries of the Lagrangian and it is not clear how to restore them systematically.

There are two major problems when one addresses to solve a Hamiltonian bound state equation

\[ H\ket{\psi} = E\ket{\psi} \quad (1) \]

in a covariant relativistic field theory. First, the canonical field-theoretical Hamiltonian \( H \) contains states (fields) with arbitrarily large energies. Second, the number of particles in a field theory is unlimited and \( H \) contains the impact of arbitrarily many particles. An eigenfunction \( \ket{\psi} \), for example a meson wave function, has contributions from arbitrarily many Fock-space sectors \( \ket{\psi} = \varphi_{qq}\ket{qq} + \varphi_{qqg}\ket{qqg} + \ldots \). Therefore, in general, the Hamiltonian operator \( H \) can be understood as a matrix with infinite dimensions both with respect to ‘energy’ and with respect to ‘particle number’. The method displayed below, the ‘Hamiltonian flow equations’ of Wegner\cite{8}, cope with either of them.

It is a subject of its own and not completely trivial to write down a suitable Hamiltonian (operator) \( H \)\cite{3}. In the sequel we shall use the canonical (light-cone) Hamiltonian of gauge theory in the light-cone gauge (\( A^+ = 0 \)). In the light-front Hamiltonian approach one faces then two classes of problems that are related with each other: the problems associated with the light-front formulation and the problems in formulating an effective Hamiltonian theory and the difficulties in the renormalization program. In order not to ponder all these problems with the problems of confinement and chirality, one disregards in this work QCD and restricts to QED as a model case. Unlike in other work\cite{1,2,10} on QED and other models we apply in this work the method of flow equations\cite{8,12,13,14} with the objective to derive from the Hamiltonian in the front form a well-founded effective low-energy Hamiltonian which acts in the space of a few particles and which can be solved explicitly for bound states.

Since we aim at a pedagogical presentation we sketch shortly the ingredients of front-form QED in Sec.\cite{3}. In Sec.\cite{4} the general aspects of the Hamiltonian flow equations are collected in such a form that the application of the flow equations to QED in Sec.\cite{5} becomes more transparent. It is here where the bulk of the present work is displayed in a formal way. The implications are discussed in Sec.\cite{5}, and the numerical calculations are given in Sec.\cite{5}. The work continues to be actively pursued, and this is only the first in a series of papers.
2 Flow equations and bound state problem

In this work, we shall focus on the flow equations for Hamiltonians formulated first by Wegner \[8\] aiming at the construction of an effective bound state Hamiltonian for field theories. The general aspects of the method do not depend on the nature of the Hamiltonian and before plunging into the paraphernalia of the field theoretical details it is useful to outline the general ideas in a manner slightly different from the original formulation \[8\].

It is always possible to divide the complete Fock space (with its different particle number sectors) into two arbitrary subspaces, called the $P$- and the $Q$-space. The Hamiltonian matrix of Eq.(1) has then the form

$$H = \begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix},$$  

(2)

with $P$ and $Q = 1 - P$ being projection operators. Suppose now we are unable to solve the eigenvalue equation for the whole matrix, because of, say, computer limitations. The method of flow equations allows then to unitarily transform the Hamiltonian matrix into a block-diagonal form

$$H_{\text{eff}} = \begin{pmatrix} PH_{\text{eff}}P & 0 \\ 0 & QH_{\text{eff}}Q \end{pmatrix}$$  

(3)

by analytical procedures. The two blocks are then decoupled, and the eigenvalue problem with an effective Hamiltonian $H_{\text{eff}}$ can be solved and diagonalized separately for either of the two spaces, which technically might (or might not) be easier than the solution of the full problem.

Since one can choose the number of particles in the $P$-space one reduces in this way in general the many-body problem to a few particle bound state problem at the expense of finding a more complicated effective Hamiltonian operating in a limited particle number space. This general idea is similar to the procedure of Tamm and Dancoff \[15, 16\] where an effective interaction in a few particle sector was obtained by eliminating the ‘virtual scatterings’ to the higher Fock-space sectors in the $Q$-space, see also \[9\].

The method of flow equations \[8, 12, 13, 14\] works with a unitary transform which is governed by a continuous parameter $l$. The unitarily transformed Hamiltonian is then a function of this parameter, \textit{i.e.}

$$\frac{dH}{dl} = [\eta(l), H(l)].$$  

(4)

The generator of the transformation is subject to some choice but taken here as \[8\]

$$\eta(l) = [H_d(l), H(l)].$$  

(5)

The Hamiltonian is separated conveniently into a block-diagonal part

$$H_d(l) = \begin{pmatrix} PH(l)P & 0 \\ 0 & QH(l)Q \end{pmatrix},$$  

(6)
and into the rest

\[ H(l) - H_d(l) = \begin{pmatrix} 0 & PH(l)Q \\ QH(l)P & 0 \end{pmatrix}, \]

which is purely off-diagonal. It is precisely this rest which due to \( PHQ \) ‘changes the particle number’. In many cases of practical interest one can interpret this rest as a ‘residual interaction’. If the rest vanishes, or if it is exponentially small, one has solved the most important part of the problem. In the sequel we convene that the flow parameter changes from \( l = 0 \) to \( l \rightarrow \infty \), corresponding to a change from the initial canonical Hamiltonian to block-diagonal effective Hamiltonian. According to Eq.(5) the generator is always off-diagonal.

\[ \eta(l) = \begin{pmatrix} 0 & P\eta(l)Q \\ Q\eta(l)P & 0 \end{pmatrix}. \]

The flow equations Eq.(4) for the diagonal and the rest sectors can then be disentangled into

\[ \frac{d}{dl}PHP = P\eta QHP - PHQ\eta P, \]
\[ \frac{d}{dl}PHQ = P\eta QHQ - PHP\eta Q, \]

and the trivial identity

\[ P\eta Q = PHPHQ - PHQHQ. \]

For \( QHQ, QHP, \) and \( Q\eta P \) one proceeds correspondingly. Since \( \text{Tr} PHQHP \) is restricted from above, Wegner’s choice for the generator Eq.(5) results in a monotonously decreasing measure for the off-diagonal particle-number changing interaction

\[ \frac{d}{dl} \text{Tr} PHQHP = \text{Tr} \left( P\eta Q(QHQHP - QHPHP) \right) + \text{Tr} \left( (PHPHQ - PHQHQ)\eta P \right) = 2\text{Tr}(P\eta Q\eta P) \leq 0. \]

For the generator this implies that \( \eta(l) \rightarrow 0 \) in the limit \( l \rightarrow \infty \) and that the block-diagonal part of the Hamiltonian commutes with the Hamiltonian itself. Thus, the effective Hamiltonian \( H_{\text{eff}} \) becomes ‘more and more block-diagonal’ with increasing flow parameter.

In general the solution of these equations will become quite involved. One reason is, that the equations are nonlinear. Another is, that starting with a two-particle interaction one generates due to the commutators three-particle, four-particle etc. interactions. It is however possible to solve the equations in certain limits or approximations. A limit in which the equations can be solved exactly to a large extend is the \( n \rightarrow \infty \) limit of an \( n \)-orbital model \[8\]. In this limit the equations for the two-particle interaction are closed, that is generated three-particle interactions do not couple back to the flow equations for the leading two-particle interaction. For realistic systems, which normally do not obey such a limit one can truncate the equations, which turned out to give very good
results for the Anderson impurity model \[18\] and the spin-Boson model \[19\]. Another approach is to perform a perturbation expansion in some coupling. This has been applied to the elimination of the electron-phonon coupling \[12\]. Similarly to \[13\] we will use this approach for the positronium on the light-cone here.

Before we enter this calculation it seems appropriate to explain, how this procedure works and to compare it to the similarity renormalization by Glazek and Wilson \[4\]. Suppose we would know approximately the eigenstates of the sector Hamiltonians \( PH(l)P \) and \( QH(l)Q \) and their eigenvalues \( E_p(l) \) and \( E_q(l) \). The indices \( p \) and \( q \) run over all states in the \( P \)- and \( Q \)-space, respectively. Suppose further, that this basis is \( l \)-independent. This means in other words, that we assume, the off-diagonal matrix elements \( h_{pq} \) and \( h_{q'q'} \) of \( PHP \) and \( QHQ \) are supposed to be small. Then in evaluating the commutators in Eqs. \( (14) \) and \( (11) \) we neglect the small off-diagonal matrix elements \( h_{pq} \) and \( h_{q'q'} \) and take into account only the diagonal matrix elements \( E_p \) and \( E_q \). In Eq. \( (4) \) however we keep all matrix elements on the right-hand side. Then Eqs. \( (9)-(11) \) yield

\[
\frac{dh_{pq}}{dl} = -\sum_q \left( \eta_{pq} h_{qq'} - h_{pq} \eta_{q'q'} \right),
\]

(13)

\[
\frac{dh_{pp}}{dl} = -(E_p - E_q) \eta_{pq},
\]

(14)

\[
\eta_{pq} = (E_p - E_q) h_{pq}.
\]

(15)

Substituting \( \eta_{pq} \) yields

\[
\frac{dh_{pq}(l)}{dl} = -\sum_q \left( \frac{dh_{pq}(l)}{dl} \frac{1}{E_p(l) - E_q(l)} h_{qq'}(l) + h_{pq}(l) \frac{1}{E_p(l) - E_q(l)} \frac{dh_{qq'}(l)}{dl} \right),
\]

(16)

where

\[
\frac{dh_{pq}(l)}{dl} = -(E_p(l) - E_q(l))^2 h_{pq}(l).
\]

(17)

The analogous equation for \( h_{q'q'} \) is obtained by interchanging \( p \rightarrow q \), \( p' \rightarrow q' \).

For the off-diagonal rest part one gets

\[
\eta_{pq}(l) = h_{pq}(0) \exp \left( -\int_0^l dl'(E_p(l') - E_q(l'))^2 \right).
\]

(18)

The \( l \)-dependence in this equation can become important. If in the limit \( l \rightarrow \infty \) the difference \( E_p - E_q \) vanishes, then the \( l \)-dependence can be quite crucial. If was first observed in the spin-Boson model \[19\] and later also for the electron-phonon coupling \[12\], that a self-consistent solution yields a decay \( E_p(l) - E_q(l) \propto 1/\sqrt{l} \) in the case of asymptotic degeneracy, so that the corresponding off-diagonal matrix element \( h_{pq} \) decays algebraically to zero. This procedure, however, goes beyond perturbation theory.

One has thus reached the goal: As the flow parameter tends to infinity the rest sector \( PHQ \) tends to zero and is eliminated. Simultaneously, this elimination gives rise to an effective Hamiltonian which has the block-diagonal structure \( PH(\infty)P \) and \( QH(\infty)Q \). The block \( PH(\infty)P \) is defined by

\[
h_{pp}(\infty) = h_{pp}(0) - \int_0^\infty dl \sum_q \left( \frac{dh_{pq}(l)}{dl} \frac{h_{qq'}(l)}{E_p(l) - E_q(l)} + h_{pq}(l) \frac{dh_{qq'}(l)}{dl} \right) + \frac{dh_{pp}(l)}{dl} \frac{h_{pp}(l)}{E_p(l) - E_q(l)}. \]

(19)
The first term represents the initial interaction in $P$-space and the second term originates from the elimination of the off-diagonal rest sectors.

Here we have given a rough idea on how the flow equations work. We have to consider how they work in perturbation theory and we have to decide on the blocks defined by the projectors $P$ and $Q$. Obviously the $P$-space should contain the states with one electron, one positron, and zero photons. The rest may be covered by the $Q$-space. Since the explicit calculations are done in terms of creation and annihilation operators, it is easier, to introduce not two but infinitely many blocks. Each block contains all states with a fixed number of electrons, a fixed number of positrons, and a fixed number of photons. The above equations can be easily generalized to this case. It is not necessary to write down the blocks explicitly, since the expressions in terms of creation and annihilation operators show explicitly, whether the number of particles is conserved or not and thus, which terms contribute to the diagonal and which to the off-diagonal part of the Hamiltonian. In quantum electrodynamics the small coupling is the charge $g$. The leading contribution is the kinetic energy of order $g^0$. (The interactions are explicitly given in the next section). It yields the leading diagonal matrix elements $E_p, E_q$ and no off-diagonal contributions. The vertex interaction, which describes emission and absorption of photons is of order $g$ and purely off-diagonal, since it changes the number of photons. The instantaneous interaction is of order $g^2$ and contains both particle-number conserving and particle-number violating contributions. In zeroth order in $g$ we have only the kinetic energy. This zeroth order contribution does not change. To first order in $g$ we have Eqs. (17,18), where the $l$-dependence of $E$ has to be neglected. Then one enters Eqs. (16,19), neglects the $l$-dependence of $E$ on the right hand-side and obtains $h_{pp'}$ to second order in $g^2$. The result is the effective interaction between electrons and positrons. It also includes a change of the one-particle energies, which become $l$-dependent in this order.

Equation (18) may be written

$$h_{pq}(l) = h_{pq}(0) f(z_{pq}).$$

with

$$f(z) = \exp(-z), \quad z_{pq}(l) = \int_0^l dl'(E_p(l') - E_q(l'))^2.$$  

We discuss now the choice of a more general ‘similarity function’ $f$. Such a more general function was first by Glazek and Wilson [4].

Correspondingly, the generator of the transformation is written as

$$\eta_{pq}(l) = -\frac{h_{pq}(l)}{E_p(l) - E_q(l)} \frac{d}{dl} \left( \ln f(z_{pq}) \right).$$

An example of a different choice for $f$ is given by

$$\eta_{pq}(l) = \text{sign}(E_p - E_q) h_{pq},$$

which yields an exponential decay of $f$

$$f = \exp\left(-\int_0^l dl'|E_p(l') - E_q(l')|\right).$$
This similarity function is good, if the sign of the difference \( E_p - E_q \) does not depend on the momenta of the interacting particles. This is the case for the absorption and emission of the photons in the light-cone frame and can thus be used here. Other possibilities are a sharp cut-off, if the energy difference is larger than a given energy of \( l \). Glazek and Wilson used a continuous elimination if the energy difference lies between two energies which decrease with \( l \).

Such similarity functions can be used in two cases:

(i) In the first case one does not introduce blocks, but aims to diagonalize single states. Then starting from plane waves the two-particle interaction becomes negligible, which may prevent the procedure from diagonalization. This happened in a first attempt in [8], and Jones, Perry, Glazek [7] could perform the elimination of the off-diagonal interaction only down to energy differences of order Rydberg. This problem which shows up in the continuum, might be overcome, if one can introduce a discretization.

(ii) For block-diagonalization they can be used to the order of perturbation theory as discussed here. If one goes beyond this order, then it is not obvious how to use a general similarity function. Despite the fact, that there is a lot of freedom to choose \( \eta \), one has to make sure, that the off-diagonal matrix elements really decay, as shown in Eq. (12) for the choice (5).

The most important properties of the similarity function \( f(z) \) are

\[
\begin{align*}
  f(0) &= 1, \\
  f(z \to \infty) &= 0.
\end{align*}
\]

(25)

Its functional dependence on \( z \) is less important. As to be shown below in Sec. 5 and in Sec. 6 by way of example, different choices for the similarity function have almost no impact on physical observables like the spectrum or the wave functions of bound states.

This leaves us finally with

\[
\begin{align*}
  h_{\text{eff},pp'} &= h_{pp'}(\infty) = h_{pp'}(0) - \sum_q h_{pq}(0)h_{qp'}(0) \\
  &\times \left( \frac{1}{E_p - E_q} \int_0^\infty dl \frac{df(z_{pq}(l))}{dl} f(z_{qp'}(l)) + \frac{1}{E_{p'} - E_q} \int_0^\infty dl f(z_{pq}(l)) \frac{df(z_{qp'}(l))}{dl} \right). \quad (26)
\end{align*}
\]

in order \( g^2 \).

For \( p = p' \) this equation may contain ultra-violet divergences as \( l \) goes to 0. Remember that the elimination of the rest sector in Eq.(18) by means of flow equations is performed not in one step, as in the Tamm-Dancoff approach [15, 16], but rather sequentially for different energy differences, i.e.

\[
\lambda = \frac{1}{\sqrt{l}} \leq |E_p - E_q| \leq \frac{1}{\sqrt{l_0}} \to 0 = \Lambda \to \infty. \quad (27)
\]

As it turns out the parameter \( \lambda = 1/\sqrt{l} \) plays the role of an ultra-violet cut-off \( \Lambda \) [4]. The elimination of the matrix element \( h_{pq} \) in Eq.(18) reminds us to the standard concept of renormalization by Wilson, where the high energy modes are integrated out in the path integral representation resulting in the effective action for the low energy scales. Indeed,
performing the $l$ ($\lambda$)-integration to the leading order one gets for the diagonal elements $h_{pp'}$, (for $p = p'$ in Eq.(16)),

$$E_p(\lambda) = E_p(\Lambda) + \left. \frac{\sum_q h_{pq}(\Lambda')h_{qp}(\Lambda')}{E_p(\Lambda) - E_q(\Lambda)} \right|_{\lambda}$$

that defines the connection of energies at different energy scales and coincides with the second order of conventional perturbation theory. In the case of QED$_{3+1}$, as the bare cut-off $\Lambda$ tends to infinity, the sum in Eq.(28) diverges, and one has to introduce the corresponding counter term. Note that the sum at the upper limit $\lambda$ is regulated by the similarity factor in $h_{pq}(\lambda)$, since only the energy differences $|E_p - E_q| \leq \lambda$ are present. The ultra-violet renormalization can be attacked with the technique of flow equations order by order in a systematic way, and further work is in preparation.

In the remainder of this paper, the above schematic equations of Hamiltonian flow are worked out explicitly for QED$_{3+1}$ on the light-cone.

3 Canonical QED Hamiltonian on the light-front

Canonical QED$_{3+1}$ in the front form has been reviewed recently [3]. Therefore, only the most salient features are recollected in this section, mostly for the purpose to shape notation. The Lagrangian density for QED

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma_\mu + eA_\mu - m)\psi$$

is considered here in the light-cone gauge $A^+ = A^0 + A^3 = 0$. Zero modes will be disregarded. The constrained degrees of freedom $A^-$ and $\psi_-$ ($\Lambda = \frac{1}{2}\gamma^0\gamma^\pm$ are projection operators, thus $\psi_\pm = \Lambda_\pm \psi$, and $\psi = \psi_+ + \psi_-$) are removed explicitly and produce the canonical QED Hamiltonian. It is defined through the independent physical fields $A_{\perp}$ and $\psi_+$ [22]. To solve the constrained equations for $A^- \text{ and } \psi_-$ the auxiliary fields

$$\tilde{A}_+ = A_+ - \frac{9}{(i\partial^+)^2} J^+,$$

$$\tilde{\Psi} = \Psi_+ + \left( m\beta - i\alpha^i\partial_{\perp i} \right) \frac{1}{2i\partial_-} \Psi_+,$$

are introduced. The fermion current is $\tilde{J}^\mu(x) = \overline{\Psi}\gamma^\mu\tilde{\Psi}$. The resulting canonical Hamiltonian $H = P_+$ is given as a sum of the free Hamiltonian and the interaction

$$H = P_+ = H_0 + V + W.$$  

The free Hamiltonian $H_0$, the ‘kinetic energy’, is

$$H_0 = \frac{1}{2} \int dx_+ d^2x_{\perp} \left( \overline{\Psi} \gamma^+ \frac{m^2 + (i\nabla_{\perp})^2}{i\partial^+} \Psi + \tilde{A}^\mu(i\nabla_{\perp})^2 \tilde{A}_\mu \right).$$
In the ‘interaction energy’ \( V + W \), the vertex interaction \( V \) is the light-cone analogue of the minimal coupling interaction in covariant QED and \( W = W_1 + W_2 \) is the sum of the instantaneous-gluon \( W_1 \) and the instantaneous-fermion interactions \( W_2 \). The latter arise from the constraint equations. More explicitly, the interaction is given by

\[
V = g \int dx_+ d^2 x_\perp \bar{\Psi} \gamma^\mu \Psi \mu \tilde{A}_\mu,
\]

\[
W_1 = \frac{g^2}{2} \int dx_+ d^2 x_\perp \bar{J}^+ \frac{1}{(i\partial^+)^2} J^+,
\]

\[
W_2 = \frac{g^2}{2} \int dx_+ d^2 x_\perp \bar{\Psi} \gamma^\mu \tilde{A}_\mu \frac{\gamma^\nu}{i\partial^\nu} (\gamma^\nu \tilde{A}_\nu \Psi).
\]

By definition, the fields \( \bar{\Psi} = \bar{\Psi}_+ + \bar{\Psi}_- \) and \( \tilde{A}_\mu = (0, \tilde{A}_1, \tilde{A}_2) \) are the free solutions which in momentum space are parametrized as

\[
\bar{\Psi}_\alpha(x) = \sum_\lambda \int \frac{dp^+ d^2 p_{\perp}}{\sqrt{2p^+ (2\pi)^3}} \left( b(p) u_\alpha(p, \lambda) e^{-ipx} + \bar{d}(p) \bar{v}_\alpha(p, \lambda) e^{ipx} \right),
\]

\[
\tilde{A}_\mu(x) = \sum_\lambda \int \frac{dp^+ d^2 p_{\perp}}{\sqrt{2p^+ (2\pi)^3}} \left( a(p) \epsilon_\mu(p, \lambda) e^{-ipx} + a^\dagger(p) \epsilon_\mu^*(p, \lambda) e^{ipx} \right).
\]

The Dirac spinors and the polarization vectors are given explicitly in [3]. The single particle operators obey the commutation relations

\[
[a(p), a^\dagger(p')] = \{b(p), \bar{b}(p')\} = \{d(p), \bar{d}(p')\} = \delta(p^+ - p'^+) \delta^{(2)}(\vec{p}_\perp - \vec{p'}_\perp) \delta_\lambda^\nu.
\]

Inserting the free fields into the Hamiltonian yields for the vertex interaction

\[
V = \frac{g}{\sqrt{(2\pi)^3}} \int \frac{dp^1 d^2 p_{1\perp}}{\sqrt{2p^1}} \int \frac{dp^2 d^2 p_{2\perp}}{\sqrt{2p^2}} \int \frac{dp^3 d^2 p_{3\perp}}{\sqrt{2p^3}} \times \int \frac{dx_+ d^2 x_\perp}{(2\pi)^3} \left[ \left( b^>(p_1) \bar{v}_\alpha(p_1, p_1, p_1) e^{-ip_1x} + \bar{d}(p_1) \bar{v}_\alpha(p_1, p_1, p_1) e^{ip_1x} \right) \right.
\]

\[
\times \gamma_{\alpha\beta} \left( d^>(p_2) v_\beta(p_2, p_2, p_2) e^{ip_2x} + b(p_2) v_\beta(p_2, p_2, p_2) e^{-ip_2x} \right)
\]

\[
\times \left. \left( a^\dagger(p_3) \epsilon_\mu^*(p_3, p_3, p_3) e^{-ip_3x} + a(p_3) \epsilon_\mu^*(p_3, p_3, p_3) e^{ip_3x} \right) \right] .
\]

The integration over configuration space yields the vertex interaction as a Fock-space operator. The integration produces Dirac-delta functions in the single-particle spatial momenta,

\[
\int \frac{dx_+}{2\pi} e^{i x_+ (\sum_j p_j^+) } = \delta \left( \sum_j p_j^+ \right),
\]

\[
\int \frac{d^2 x_\perp}{(2\pi)^2} e^{-i \vec{x}_\perp \cdot (\sum_j \vec{p}_\perp)} = \delta^{(2)} \left( \sum_j \vec{p}_\perp \right). \]

(37)
and often in the sequel these will be used as a 3-dimensional $\delta$-function

$$\delta^{(3)} \left( \sum_j p_j \right) = \delta \left( \sum_j p_j^\perp \right) \delta^{(2)} \left( \sum_j \vec{p}_{\perp j} \right). \tag{38}$$

The sums over $j$ run over all respective single particles. The Dirac-delta’s reflect three-momentum conservation, as always in a Hamiltonian approach. $W_1$ and $W_2$ as Fock-space operators are obtained correspondingly and found explicitly in [3].

### 4 Flow equations applied to QED

In the sequel we consider the canonical Hamiltonian for QED as given in Eq.(31) and work out the details when straightforwardly applying the flow equations as given in Eqs.(9)-(11). Since we restrict ourselves to solve them up to second order in the coupling constant, we can content ourselves to include explicitly only two Fock-space sectors: The sector with one electron and one positron ($e\bar{e}$) can be identified with the $P$-space discussed above, and the sector with one electron, one positron, and one photon ($e\bar{e}\gamma$) with the $Q$-space. As a result one aims at the effective Hamiltonian in the $e\bar{e}$ space. It is helpful to know that the effective interaction must turn as the Coulomb potential, to lowest order of approximation.

As a technical trick and for gaining more transparency we omit first the instantaneous interactions $W$. They will be re-installed at the end of the calculation. The physical argument is that the instantaneous interaction is already of order $g^2$; the changes due to the flow are of higher order in the coupling constant and have to be omitted here by consistency. The light-cone Hamiltonian Eq.(31) is then $H = H_0 + V$ and has a very simple structure. Since the vertex interaction can not have diagonal matrix elements, both diagonal sector Hamiltonians $PHP$ and $QHQ$ are diagonal operators from the outset. The case studied here is thus a realization of the paradigmatic case discussed in the second part of Sec. 2. To the leading order in the coupling constant the particle number changing part $PHQ$ is given by the vertex interaction, i.e. $H_r(l) = \tilde{V}(l)$. By reasons to become clear soon, we shall put hats on the operators in this section.

At finite flow parameter $l$ the Hamiltonian is $H(l) = H_d(l) + H_r(l)$. The unitary transformation of the flow equations diminishes $\tilde{V}(l)$ and generates a new interaction, $\tilde{U}(l)$. This interaction is (up to second order) diagonal in particle number and contributes to $H_d(l) = H_0(l) + U(l)$. The flow equations Eqs.(9)-(11) become then consecutively

$$\frac{d\tilde{U}(l)}{dl} = [\tilde{\eta}(l), \tilde{V}(l)], \tag{39}$$

$$\frac{d\tilde{V}(l)}{dl} = [\tilde{\eta}(l), \tilde{H}_0(l)], \tag{40}$$

$$\tilde{\eta}(l) = [\tilde{H}_0(l), \tilde{V}(l)]. \tag{41}$$

In this section all of these operators will be evaluated explicitly. The free part is

$$\tilde{H}_0(l) = \sum_\lambda \int dp^+ d^2 p_\perp E(p; l) \left( b^\dagger(p, \lambda) b(p, \lambda) + d^\dagger(p, \lambda) d(p, \lambda) + a^\dagger(p, \lambda) a(p, \lambda) \right). \tag{42}$$
The single particle energies \((E = p^-)\) depend on the 3-momentum \(p = (p^+, \vec{p}_\perp)\)
\[
E(p; l) = \frac{m^2(p; l) + \vec{p}_\perp^2}{p^+},
\tag{43}
\]
and potentially on the flow parameter through the mass \(m^2(p; l)\) of the particle in question. The interaction term is obtained from Eq.(36)
\[
\hat{V}(l) = \frac{1}{\sqrt{(2\pi)^3}} \int [d^3p_1] \int [d^3p_2] \int [d^3p_3] \delta^{(3)}(p_1 - p_2 - p_3) g(p_1, p_2, p_3; l)
\times \left[ b_1^\dagger b_2 a_3 (\overline{\varphi}_1 \varphi_3 u_2) - d_1^\dagger d_2 a_3 (\overline{\varphi}_2 \varphi^*_3 v_1) + a_1^\dagger d_2 b_3 (\overline{\varphi}_2 \varphi^*_3 u_3) \right] + h.c. \tag{44}
\]
The integration symbols denote
\[
[d^3p] = dp^+ d^2p_\perp \sum_\lambda,
\tag{45}
\]
for example, and the abbreviations \(u_1 \equiv u(p_1, \lambda_1)\) and \(\varphi_3^\dagger \equiv \gamma^\mu \epsilon_\mu^\ast(p_3, \lambda_3)\) are introduced for the sake of a compact notation. The effective coupling ‘constant’ has the initial value
\[
g(p_1, p_2, p_3; l = 0) = g = e, \tag{46}
\]
with the fine structure constant \(\alpha = e^2/4\pi \sim 1/137\). Correspondingly, the generator of the unitary transformation \(\hat{\eta} = [\hat{H}_0, \hat{V}]\) becomes
\[
\hat{\eta}(l) = \frac{1}{\sqrt{(2\pi)^3}} \int [d^3p_1] \int [d^3p_2] \int [d^3p_3] \delta^{(3)}(p_1 - p_2 - p_3) \eta(p_1, p_2, p_3; l)
\times \left[ b_1^\dagger b_2 a_3 (\overline{\varphi}_1 \varphi_3 u_2) - d_1^\dagger d_2 a_3 (\overline{\varphi}_2 \varphi^*_3 v_1) + a_1^\dagger d_2 b_3 (\overline{\varphi}_2 \varphi^*_3 u_3) \right] - h.c. \tag{47}
\]
The structure of \(\hat{\eta}\) is very similar to \(\hat{V}\) because \(\hat{H}_0\) is diagonal, thus
\[
\eta(p_1, p_2, p_3; l) = g(p_1, p_2, p_3; l) D(p_1, p_2, p_3; l). \tag{48}
\]
It is convenient to introduce the difference of single particle energies
\[
D(p_1, p_2, p_3; l) = E(p_1; l) - E(p_2; l) - E(p_3; l), \tag{49}
\]
with the \(E\)’s being defined in Eq.(33). The derivative \(d\hat{V}(l)/dl = [\hat{\eta}(l), \hat{H}_0(l)]\) becomes
\[
\frac{d\hat{V}(l)}{dl} = - \int [d^3p_1] \int [d^3p_2] \int [d^3p_3] \delta^{(3)}(p_1 - p_2 - p_3) \frac{\eta(p_1, p_2, p_3; l)}{\sqrt{(2\pi)^3}}
\times D(p_1, p_2, p_3; l) \left[ b_1^\dagger b_2 a_3 (\overline{\varphi}_1 \varphi_3 u_2) - d_1^\dagger d_2 a_3 (\overline{\varphi}_2 \varphi^*_3 v_1) + a_1^\dagger d_2 b_3 (\overline{\varphi}_2 \varphi^*_3 u_3) \right] + h.c. \tag{50}
\]
Finally, we calculate the new interactions \(\hat{U}(l)\) which are defined through the derivative \(d\hat{U}(l)/dl = [\hat{\eta}(l), \hat{V}(l)]\). Their calculation is somewhat cumbersome but straightforward. Inserting the six terms from Eq.(34) and the six terms from Eq.(47) gives 36 terms for \(\hat{U}(l)\) which by symmetries reduce to six interactions in the 2-particle sectors. In the present work one restricts to calculate \(\hat{U}_{ee}\), the effective interaction between an electron and a positron, as illustrated in Fig. 1. It has an exchange part and an annihilation part.
4.1 The exchange part

The matrix element $\tilde{U}_{ex}$ of the exchange part

$$\tilde{U}_{ex}(l) = \int [d^3 p_1] \int [d^3 p_2] \int [d^3 p'_1] \int [d^3 p'_2] \delta^{(3)}(p_1 + p_2 - p'_1 - p'_2) \tilde{U}_{ex}(p_1, p_2; p'_1, p'_2; l) b^\dagger(p_1, \lambda_1)b(p'_1, \lambda'_1) d^\dagger(p_2, \lambda_2)d(p'_2, \lambda'_2).$$  \hspace{1cm} (51)

is calculated next to some detail. By direct substitution of Eqs. (44) and (17) one gets

$$\frac{d\tilde{U}_{ex}^{gen}(l)}{dl} = -\frac{1}{(2\pi)^3} \int [d^3 p_1] \int [d^3 p_2] \int [d^3 p_3] \int [d^3 p'_1] \int [d^3 p'_2] \int [d^3 p'_3] \delta^{(3)}(p_1 - p'_1 - p'_3) \delta^{(3)}(p'_2 - p_2 - p'_3) \left[ b^\dagger_1 b^\dagger_3 a^3_3 d^3_3 d^3_2 \right] \left( \tilde{\Pi}_{12}^3 \tilde{u}_{12} \right) \left( \tilde{\Pi}_{23}^3 \tilde{v}_{23} \right)$$

$$+ \delta^{(3)}(p'_1 - p_1 - p_3) \delta^{(3)}(p_2 - p'_2 - p'_3) \left[ d^3_2 d^3_2 a^3_3 a^3_1 b^\dagger_1 b^\dagger_2 \right] \left( \tilde{\Pi}_{12}^3 \tilde{u}_{12} \right) \left( \tilde{\Pi}_{23}^3 \tilde{v}_{23} \right)$$

$$\left( \eta(p_1, p'_1, p_3; l) g(p'_2, p_2, p'_3; l) + \eta(p'_2, p_2, p'_3; l) g(p_1, p'_1, p_3; l) \right) \right).$$  \hspace{1cm} (52)

The commutation relations Eq. (33) for the photon induce a three-momentum delta-function $\delta^{(3)}(p_3 - p_3')$. Since

$$\eta(p_1, p'_1, p_1 - p'_1; l) = -\eta(p'_1, p_1, p'_1 - p_1; l),$$

$$g(p'_1, p_1, p'_1 - p_1; l) = g(p_1, p'_1, p_1 - p'_1; l),$$  \hspace{1cm} (53)

all of the integrations in Eq. (52) can be performed trivially. The sum over the photon helicity is carried out by introducing the polarization tensor $\tilde{\Pi}_{\mu\nu}$.

$$d_{\mu\nu}(q) \equiv \sum_\lambda \epsilon^\mu_\lambda(q, \lambda)\epsilon^\nu_\lambda(q, \lambda) = -g_{\mu\nu} + \frac{\eta_\mu q_\nu + \eta_\nu q_\mu}{q^+}. \hspace{1cm} (54)$$

The null vector $\eta^\mu$ has the components $(\eta^+, \eta^\perp, \eta^-) = (0, \bar{0}, 2)$ and should not be confused with the generator $\eta$. Dropping hence forward the argument $l$ in $g$ or $\eta$ for the reason of notational compactness the $l$-derivative of $\tilde{U}_{ex}^{gen}$ becomes

$$\frac{d\tilde{U}_{ex}^{gen}}{dl} = -\frac{1}{2(2\pi)^3} \left( \theta(p_1^+ - p'_1^+) + \theta(p'_1^+ - p_1^+) \right).$$

Figure 1: The effective interaction between an electron ($e$) and a positron ($\bar{e}$).

Figure 2: The graph of the instantaneous exchange interaction. Taken from [14].
To the order considered here it is independent of \( l \), i.e. \( d\tilde{U}^{\text{inst}}_{ex}/dl = 0 \).

### 4.2 The annihilation part

Having been so explicit for the exchange part one can proceed rather quickly for the annihilation channel where the calculation proceeds quite correspondingly. One defines first the matrix element \( \tilde{U}_{an} \) by

\[
\tilde{U}_{ee:an}(l) = \int [d^3 p_1] [d^3 p_2] \int [d^3 p'] \int [d^3 p''] \delta^{(3)}(p_1 + p_2 - p'_1 - p''_2) \tilde{U}_{an}(p_1, p_2; p'_1, p''_2; l) b^†(p_1, \lambda_1) b(p'_1, \lambda'_1) d^†(p'_2, \lambda'_2) d(p'_2, \lambda'_2).
\]

Its \( l \)-derivative is defined by the flow equations, i.e.

\[
\frac{d\tilde{U}^{\text{gen}}_{ee:an}(l)}{dl} = -\frac{1}{2(2\pi)^3} \int [d^3 p_1] [d^3 p_2] [d^3 p_3] [d^3 p_1'] [d^3 p_2'] \delta^{(3)}(p_1 + p_2 - p_3 - p') \left[ b^† d^x_{a_3} d^x_2 b_1 \right] \left( \tau_1 \psi_3 \psi_2 \right) \left( \tau_2 \psi_3 \psi_1 \right)
\]

\[
\left( (\eta(p_3, p_2, p_1) g(p'_2, p'_3, p'_1) + 0) g(p_3, p_2, p_1) \right).
\]

All integrations can be performed explicitly and one arrives at

\[
\frac{d\tilde{U}^{\text{gen}}_{an}(l)}{dl} = -\frac{1}{2(2\pi)^3} \frac{\bar{\psi}(p_1, \lambda_1) \gamma^\mu v(p_2, \lambda_2)}{\sqrt{2p_1^+ \sqrt{2p_2^+}}} \frac{\gamma^\nu u(p'_1, \lambda'_1)}{\sqrt{2p_1'^+ \sqrt{2p_2'^+}}} \times \left( (\eta(p_1 + p_2, p_2, p_1) g(p'_1 + p'_2, p'_1) + 0) g(p_1 + p_2, p_2, p_1) \eta(p'_1 + p'_2, p'_1) \right)
\]

The 4-momentum of the photon is denoted here by \( p \). The instantaneous interaction in the exchange channel

\[
\tilde{U}^{\text{inst}}_{an}(l) = \frac{e^2}{2(2\pi)^3} \frac{\bar{\psi}(p_1, \lambda_1) \gamma^\mu v(p_2, \lambda_2)}{\sqrt{2p_1^+ \sqrt{2p_2^+}}} \frac{\gamma^\nu u(p'_1, \lambda'_1)}{\sqrt{2p_1'^+ \sqrt{2p_2'^+}}} \eta \eta
\]

is again independent of \( l \) in the lowest non-trivial order of the coupling constant.
4.3 Integrating the flow equations for the exchange

The first order flow equations have been given in Eqs. (40) and (41) in operator form. After evaluating all matrix elements they reduce simply to two coupled equations:

\[
\frac{dg(p_1, p_2, p_3; l)}{dl} = -D(p_1; p_2, p_3; l) \eta(p_1, p_2, p_3; l),
\]

\[
\eta(p_1, p_2, p_3; l) = D(p_1, p_2, p_3; l) g(p_1, p_2, p_3; l).
\]

Replacing \( g \) by the suitably normalized similarity function \( f \) according to

\[
g(p_1, p_2, p_3; l) = g(0) f(p_1, p_2, p_3; l) = e f(p_1, p_2, p_3; l),
\]

see also Eq. (46), one gets

\[
f(p_1, p_2, p_3; l) = \exp \left( -l D^2(p_1, p_2, p_3) \right)
\]

as the explicit solution. It describes the decay rate of the off-diagonal vertex interaction \( V \). However, because of the considerations in Sec. 2 particularly Eq. (22) and the considerations below we want to keep \( f = f(D; l) \) as a general function. Correspondingly, we rewrite \( \eta \) as

\[
\eta(l) = -\frac{1}{D} \left( \frac{d \ln f(D; l)}{dl} \right) g(l).
\]

The formal integration of the flow Eqs. (61) and (62) can be treated more compactly in a reasonably short notation. We therefore define the always negative quantities

\[
D_e = p_1^- - p_1^- - (p_1' - p_1)^-,
\]

\[
D_e^- = p_2^- - p_2'^- - (p_2 - p_2')^-.
\]

They represent the energy differences along the electron and the positron line, respectively, and are in simple relationship to both the 4-momentum of the exchanged photon

\[
q_\mu = p_1'\mu - p_1\mu - \eta_\mu \frac{D_e}{2} = p_2\mu - p_2'\mu - \eta_\mu \frac{D_e^-}{2},
\]

and to the (Feynman-) 4-momentum transfers along the two lines

\[
Q^2_e = -(p_1' - p_1)^2 = -q^+ D_e,
\]

\[
Q^2_e^- = -(p_2 - p_2')^2 = -q^+ D_e^-,
\]

which need not be equal in a Hamiltonian approach. Since the always positive Feynman-momentum transfer \( Q \) is a more physical quantity than the energy difference, the \( D \)'s will be substituted in the sequel by the \( Q \)'s as long as no misunderstanding can arise. In fact, we shall use often the mean-square momentum transfer and the mean-square difference

\[
Q^2 = \frac{1}{2}(Q^2_e + Q^2_e^-) = -\frac{q^+}{2}(D_e + D_e^-),
\]

\[
\delta Q^2 = \frac{1}{2}(Q^2_e - Q^2_e^-) = -\frac{q^+}{2}(D_e - D_e^-),
\]
respectively. The above definitions are also useful to simplify the polarization tensor appearing in Eq. (55). Since \( d_{\mu\nu}(q) \) appears always in combinations with the spinors one can make use of the Dirac equation \((p_1 - p'_1)_\mu \overline{\psi}(p_1) \gamma^\mu \psi(p'_1) = 0 \) and write
\[
g_{\mu} \overline{\psi}(p_1, \lambda_1) \gamma^\mu \psi(p'_1, \lambda'_1) = -\frac{D_e}{2} \eta_{\mu} \overline{\psi}(p_1, \lambda_1) \gamma^\mu \psi(p'_1, \lambda'_1). \tag{72}
\]
One can replace thus in Eq. (55) \( d_{\mu\nu}(q) \rightarrow -g_{\mu\nu} + \eta_{\mu} \eta_{\nu}(Q/q^+)^2 \). With these definitions we return now to the problem of integrating up Eq. (55). Substituting Eqs. (61) and (63) one has for its \( l \)-dependent part
\[
\eta(p'_1, p_1, p'_1 - p_1; l) g(p_2, p'_2, p_2 - p'_2; l) + \eta(p_2, p'_2, q; l) g(p'_1, p_1, q; l)
= -e^2 \left( \frac{1}{D_e} \frac{df(D_e; l)}{dl} f(D_e; l) + \frac{1}{D_e} \frac{df(D_e; l)}{dl'} f(D_e; l') \right).	ag{73}
\]
For the formal integration it turns out useful to introduce the abbreviation
\[
\Theta(D_e, D_e) = -\int_0^\infty dl' \frac{df(D_e; l')}{dl'} f(D_e; l) \equiv \Theta_{ee},
\]
which is not symmetric in the arguments but which satisfies by means of Eq. (72)
\[
\Theta(D_e, D_e) + \Theta(D_e, D_e) = \Theta_{ee} + \Theta_{ee} = 1. \tag{74}
\]
When \( l \)-integrating Eq. (72), the \( l \)-dependence of \( D_e(l) \) in the denominator can be neglected to the order considered here, therefore
\[
\int_0^\infty dl' \left( \eta(p'_1, p_1, p'_1 - p_1; l') g(p_2, p'_2, p_2 - p'_2; l') + \eta(p_2, p'_2, q; l') g(p'_1, p_1, q; l') \right)
= e^2 \left( \frac{\Theta_{ee}}{D_e} + \frac{\Theta_{ee}}{D_e} \right) = -e^2 q^+ \left( \frac{\Theta_{ee}}{Q_e^2} + \frac{\Theta_{ee}}{Q_e^2} \right). \tag{76}
\]
The latter combination appears repeatedly, see for example Eq. (138). Putting things together the generated interaction Eq. (53) becomes
\[
\tilde{U}^{gen}_{ex} = + \frac{e^2}{2(2\pi)^3} \frac{\overline{\psi}(p_1, \lambda_1) \gamma^\mu \psi(p'_1, \lambda'_1)}{\sqrt{2p_1^+} \sqrt{2p'_1^+}} \frac{(\overline{\psi}(p_2, \lambda_2) \gamma^\nu \psi(p_2, \lambda_2))}{\sqrt{2p_2^+} \sqrt{2p_2^+}}
\times d_{\mu\nu}(q) \left( \frac{\Theta_{ee}}{Q_e^2} + \frac{\Theta_{ee}}{Q_e^2} \right), \tag{77}
\]
and after substituting \( d_{\mu\nu}(q) \)
\[
\tilde{U}^{gen}_{ex} = - \frac{e^2}{2(2\pi)^3} \frac{\overline{\psi}(p_1, \lambda_1) \gamma^\mu \psi(p'_1, \lambda'_1)}{\sqrt{2p_1^+} \sqrt{2p'_1^+}} \frac{(\overline{\psi}(p_2, \lambda_2) \gamma^\nu \psi(p_2, \lambda_2))}{\sqrt{2p_2^+} \sqrt{2p_2^+}}
\times \left( g_{\mu\nu} - \eta_{\mu} \eta_{\nu} \frac{Q_e^2}{q^+} \right) \left( \frac{\Theta_{ee}}{Q_e^2} + \frac{\Theta_{ee}}{Q_e^2} \right). \tag{78}
\]
Adding the instantaneous exchange interaction Eq.(56), using Eq.( 75), one gets
\[\tilde{U}_{ex} = -\frac{e^2}{2(2\pi)^3} \frac{(\bar{u}(p_1, \lambda_1)\gamma^\mu u(p'_1, \lambda'_1)) (\bar{v}(p'_2, \lambda'_2)\gamma^\nu v(p_2, \lambda_2))}{\sqrt{2p^+_1}\sqrt{2p^{'+}_1} \sqrt{2p^+_2}\sqrt{2p^{'+}_2}} \times \left[ g_{\mu\nu} \left( \frac{\Theta_{\bar{e}e}^a}{Q^2_e} + \frac{\Theta_{\bar{e}e}^b}{Q^2_e} \right) + \eta_{\mu} \eta_{\nu} \left( \frac{\Theta_{\bar{e}e}^a}{Q^2_e} - \frac{\Theta_{\bar{e}e}^b}{Q^2_e} \right) \frac{\delta Q^2}{q^*} \right]\]  
for the matrix element of the total exchange interaction.

### 4.4 Integrating the flow equations for the annihilation

For the annihilation term we define the (now always positive) energy differences as
\[D_a = p'_1^- + p'_2^- - (p'_1 + p'_2)^-\]
\[D_b = p_1^- + p_2^- - (p_1 + p_2)^-\].  

They are related to the 4-momentum of the photon \(p^\mu\) in the \(t\)-channel by
\[p_\mu = p'_{1\mu} + p'_{2\mu} - \eta_\mu \frac{D_a}{2} = p_{1\mu} + p_{2\mu} - \eta_\mu \frac{D_b}{2}.\]

Rather than the momentum transfer \(Q\) the free invariant mass-squares of the initial and final states are introduced by
\[M^2_a = (p'_1 + p'_2)^2 = p^+ D_a,\]
\[M^2_b = (p_1 + p_2)^2 = p^+ D_b,\]

as well as their mean and difference
\[M^2 = \frac{1}{2}(M^2_a + M^2_b) = \frac{p^+}{2}(D_a + D_b),\]
\[\delta M^2 = \frac{1}{2}(M^2_a - M^2_b) = \frac{p^+}{2}(D_a - D_b),\]

respectively. The generated annihilation interaction becomes
\[\tilde{U}_{an}^{gen} = +\frac{e^2}{2(2\pi)^3} \frac{(\bar{u}(p_1, \lambda_1)\gamma^\mu v(p_2, \lambda_2)) (\bar{v}(p'_2, \lambda'_2)\gamma^\nu u(p'_1, \lambda'_1))}{\sqrt{2p^+_1}\sqrt{2p^{'+}_1} \sqrt{2p^+_2}\sqrt{2p^{'+}_2}} \times \frac{\delta Q^2}{q^*} d_{\mu\nu}(p) \left( \frac{\Theta_{ab}}{M^2_a} + \frac{\Theta_{ba}}{M^2_b} \right).\]

The Dirac equation \((p_1 + p_2)_\mu \bar{\sigma}(p_1)\gamma^\mu v(p_2) = 0\) allows to write
\[p_\mu \bar{\sigma}(p_1, \lambda_1)\gamma^\mu v(p_2, \lambda_2) = -\frac{D_b}{2} \eta_\mu \bar{\sigma}(p_1, \lambda_1)\gamma^\mu v(p_2, \lambda_2).\]
After substituting $d_{\mu\nu}(p) \rightarrow -g_{\mu\nu} - \eta_{\mu}\eta_{\nu}M^2/p^+ + 2\eta_{\mu}\eta_{\nu}M^2/p^2 + 2\eta_{\mu}\eta_{\nu}M^2/p^2 + 2\eta_{\mu}\eta_{\nu}M^2/p^2$ one gets for $\tilde{U}^{\text{gen}}_{\text{an}}$

\[
\tilde{U}^{\text{gen}}_{\text{an}} = - \frac{e^2}{2(2\pi)^3} \frac{(\pi(p_1, \lambda_1)\gamma^{\mu\nu}(p_2, \lambda_2))(\pi(p'_2, \lambda'_2)\gamma'^{\mu\nu}(p'_1, \lambda'_1))}{\sqrt{2p_1^+ 2p_2^+} \sqrt{2p'_2^+ 2p'_1^+}} \times \left( g_{\mu\nu} + \eta_{\mu}\eta_{\nu} \frac{M^2}{p^2} \right) \left( \frac{\Theta_{ab}}{M_a^2} + \frac{\Theta_{ba}}{M_b^2} \right). \tag{88}
\]

Adding the instantaneous term yields the effective interaction in the annihilation channel

\[
\tilde{U}_{an} = - \frac{e^2}{2(2\pi)^3} \frac{(\pi(p_1, \lambda_1)\gamma^{\mu\nu}(p_2, \lambda_2))(\pi(p'_2, \lambda'_2)\gamma'^{\mu\nu}(p'_1, \lambda'_1))}{\sqrt{2p_1^+ 2p_2^+} \sqrt{2p'_2^+ 2p'_1^+}} \times \left[ g_{\mu\nu} \left( \frac{\Theta_{ab}}{M_a^2} + \frac{\Theta_{ba}}{M_b^2} \right) - \eta_{\mu}\eta_{\nu} \left( \frac{\Theta_{ab}}{M_a^2} - \frac{\Theta_{ba}}{M_b^2} \right) \frac{\delta M^2}{p^+} \right], \tag{89}
\]

all in perfect analogy to the exchange term.

### 4.5 The effective $e\bar{e}$-interaction

Thus far, we have been studying the structure of the Hamiltonian proper $H = P_+ = P^-/2$.

In dealing with its spectra it is advantageous to study the spectrum of the ‘light-cone Hamiltonian’ $H_{LC} = P_\mu P^\mu = P_+ P^- - P_{\perp}^2$. It is a Lorentz scalar \[3\] with the eigenvalues having the dimension of an invariant mass-squared. Combining the effective interaction of the exchange and annihilation channels one introduces therefore

\[
U_{\text{eff}} = P^+ (\tilde{U}_{\text{ex}} + \tilde{U}_{\text{an}}), \tag{90}
\]

see Eq.(79) and Eq.(89). One gets

\[
U_{\text{eff}} = - \frac{\alpha}{4\pi^2} \gamma^{\mu\nu} <_{\text{ex}} \left[ g_{\mu\nu} \left( \frac{\Theta_{ee}}{Q_e^2} + \frac{\Theta_{\bar{e}e}}{Q_{\bar{e}}^2} \right) + \eta_{\mu}\eta_{\nu} \frac{\delta Q^2}{q^2} \left( \frac{\Theta_{ee}}{Q_e^2} - \frac{\Theta_{\bar{e}e}}{Q_{\bar{e}}^2} \right) \right]
- \frac{\alpha}{4\pi^2} \gamma^{\mu\nu} <_{\text{an}} \left[ g_{\mu\nu} \left( \frac{\Theta_{ab}}{M_a^2} + \frac{\Theta_{ba}}{M_b^2} \right) - \eta_{\mu}\eta_{\nu} \frac{\delta M^2}{p^2} \left( \frac{\Theta_{ab}}{M_a^2} - \frac{\Theta_{ba}}{M_b^2} \right) \right]. \tag{91}
\]

The symbols $\gamma^{\mu\nu}$ are introduced conveniently as

\[
\gamma^{\mu\nu} = \frac{(\pi(p_1, \lambda_1)\gamma^{\mu\nu}(p'_1, \lambda'_1))(\pi(p'_2, \lambda'_2)\gamma'^{\mu\nu}(p_2, \lambda_2))}{\sqrt{x'x'(1-x')(1-x')}} = \frac{(j^{\mu}j^{\nu})_{\text{ex}}}{\sqrt{x'x'(1-x')(1-x')}} \tag{92}
\]

where for example $x = p_1^+/P^+$. The longitudinal momentum fraction \[3\].

The effective interaction $U_{\text{eff}}$ is the kernel of the integral equation

\[
M^2 \langle x, \vec{k}_\perp; \lambda_1, \lambda_2 | \psi \rangle = \frac{m^2 + \vec{k}_\perp^2}{x(1-x)} \langle x, \vec{k}_\perp; \lambda_1, \lambda_2 | \psi \rangle + \sum \int d'x'd^2k_\perp \langle x, \vec{k}_\perp; \lambda_1, \lambda_2 | U_{\text{eff}} | x', \vec{k}_\perp'; \lambda_1, \lambda_2' \rangle \langle x', \vec{k}_\perp'; \lambda_1, \lambda_2' | \psi \rangle. \tag{93}
\]
In the equation appear only intrinsic transversal momenta $\vec{k}_\perp$ and longitudinal momentum fractions $x = p_1^+/P^+$, defined by \( p_1^\mu = (xP^+, xP_\perp + \vec{k}_\perp, p^-) \). Its spectrum is thus manifestly independent of the kinematical state of the bounded system, particularly of $P^+$ and $P_\perp$, which reflects the boost invariances peculiar to the front form [3]. The integral equation replaces in some way Eq.(1). The first term on the r.h.s is the free part of the Hamiltonian in analogy to a 'kinetic energy', and the second term is an 'interaction energy' which is the relativistically correct interaction, correct up to the second order in the coupling constant.

The integration domain $D$ is restricted by the covariant cut-off condition of Brodsky and Lepage [22],

$$ \frac{m^2 + k_\perp^2}{x(1-x)} \leq \Lambda^2 + 4m^2, \tag{94} $$

which allows for states having a kinetic energy below the bare cut-off $\Lambda$.

4.6 Dependence on the cut-off function

Before discussing the dependence of the effective potential in Eq.(91) on the cut-off function $f$ one has to determine the dependence of $\Theta$ on its two energy arguments $D$, as given in Eq.(74). It is natural to assume that the similarity function $f(D; l)$ is a homogeneous function of its arguments

$$ f(D; l) = f(D^\kappa l) \tag{95} $$

with some exponent $\kappa$. As examples we consider three types of similarity functions and the corresponding $\Theta$ functions:

the exponential cut-off

$$ f(D; l) = \exp(-|D|l), \quad \kappa = 1, \quad \Theta(D_e, D_\bar{e}) = \frac{D_e}{D_e + D_\bar{e}}, \tag{96} $$

the Gaussian cut-off

$$ f(D; l) = \exp(-D^2 l), \quad \kappa = 2, \quad \Theta(D_e, D_\bar{e}) = \frac{D_e^2}{D_e^2 + D_\bar{e}^2}, \tag{97} $$

and the sharp cut-off

$$ f(D; l) = \theta(1/l - |D|^\kappa) = \theta(1 - |D|^\kappa l), \quad \Theta(D_e, D_\bar{e}) = \theta(|D_e| - |D_\bar{e}|) = \theta(1 - \frac{D_\bar{e}}{D_e}). \tag{98} $$

In the last case $\kappa$ is an arbitrary positive number. In the first and in the last case we have used that $D_e$ and $D_\bar{e}$ have the same sign. The second case corresponds to the solutions of the original flow equations, see Eqs.(18) and (64).

Assuming this homogeneity of $f$, $\Theta$ is quite generally a function of the ratio of its two arguments since

$$ \Theta(D_e; D_\bar{e}) = -\int_0^\infty dl' \frac{df(D_e^\kappa l')}{dl'} f(D_e^\kappa l') = -\int_0^\infty dz \frac{df(z)}{dz} f \left( \left( \frac{D_e}{D_\bar{e}} \right)^\kappa z \right). \tag{99} $$
Instead we may express it as a function of
\[ \xi = \frac{D_e - D_\bar{e}}{D_e + D_\bar{e}} = \frac{\delta Q^2}{Q^2}, \quad (100) \]
\[ \Theta(D_e; D_\bar{e}) = \frac{1}{2} (1 + \vartheta(\xi)). \quad (101) \]

Due to Eq.(75) \( \vartheta(\xi) \) is an odd function, \( \vartheta(-\xi) = -\vartheta(\xi) \).

Then the first expression in square brackets in (91) which contains the singular part
of the effective interaction reads
\[
B_\text{ex}^\mu\nu = \left[ g_{\mu\nu} \left( \frac{\Theta_{ee}}{Q_e^2} + \frac{\Theta_{\bar{e}\bar{e}}}{Q_{\bar{e}}^2} \right) + \eta_\mu \eta_\nu \frac{\delta Q^2}{q^4} \left( \frac{\Theta_{ee}}{Q_e^2} - \frac{\Theta_{\bar{e}\bar{e}}}{Q_{\bar{e}}^2} \right) \right] + \frac{\xi^2 - \xi \vartheta(\xi)}{1 - \xi^2} \left( \frac{g_{\mu\nu}}{Q^2} - \frac{\eta_\mu \eta_\nu}{q^4} \right). \quad (102) \]

For the three similarity functions mentioned above one obtains

**exponential** \( \vartheta(\xi) = \xi \), \( \frac{\xi^2 - \xi \vartheta(\xi)}{1 - \xi^2} = 0 \), \( (103) \)

**Gaussian** \( \vartheta(\xi) = \frac{2z}{1+z^2} \), \( \frac{\xi^2 - \xi \vartheta(\xi)}{1 - \xi^2} = \frac{-\xi^2}{1 + \xi^2} \), \( (104) \)

**sharp** \( \vartheta(\xi) = \text{sign}(\xi) \), \( \frac{\xi^2 - \xi \vartheta(\xi)}{1 - \xi^2} = \frac{-|\xi|}{1 + |\xi|} \). \( (105) \)

We observe that the effective interaction depends explicitly on the similarity function. The
requirement of block diagonalization of the Hamiltonian determines the generator only
up to a unitary transformation of the blocks. This explains why the effective interaction
may depend on the similarity function.

We will discuss the dependence on this function further in the next section, but mention
that for the elimination of the electron-phonon interaction in solid-state physics which
yields the effective attractive interaction between electron pairs responsible for super-
conductivity, one may also choose different similarity functions, see [12] and [20, 21].
For realistic spectra, Mielke [21] has found that the critical temperature calculated from
the Gaussian similarity function and that suggested by Glazek and Wilson [4] differ by only
2%, the difference to those calculated by the conventional Eliashberg theory was only 5%.

Since the kernel of the integral equation is manifestly frame-independent, one can
evaluate it in the particular frame \( P^+ = 2m \) and \( \vec{P}_\perp = 0 \). For the further discussions we
choose to express the momenta as
\[
p_1^+ = m + p_\parallel, \quad \vec{p}_1^\perp = \vec{p}_\perp,
\]
\[
p_2^+ = m - p_\parallel, \quad \vec{p}_2^\perp = -\vec{p}_\perp,
\]
and similarly for \( p_1' \) and \( p_2' \). With \( q^+ = q_\parallel = p_1' - p_\parallel, \ p_2^2 = \vec{p}_1^2 + p_\parallel^2 \) and \( q^2 = q_\perp^2 + q_\parallel^2 \) the
energy differences become
\[
D_e = \frac{m^2 + \vec{p}_\perp^2}{m + p_\parallel} - \frac{m^2 + p_\parallel^2}{m - p_\parallel} - \frac{q^2}{q^+} = -\frac{q^2}{m - p_\parallel} + \frac{p_\parallel^2}{m - p_\parallel}, \quad (107) \]
\[ D_e = \frac{m^2 + \vec{p}_\perp^2}{m - p_\parallel} - \frac{m^2 + \vec{p}'_\perp^2}{m - p'_\parallel} - \frac{\vec{q}^2}{q^2} = -\frac{q^2}{m - p_\parallel} - \frac{p^2}{m - p'_\parallel} + \frac{p'^2}{m - p'_\parallel}. \] (108)

The other quantities become correspondingly
\[ D_e - D_{\bar{e}} = 2m \left( \frac{p'^2}{m^2 - p'^2} - \frac{p^2}{m^2 - p^2} \right), \] (109)
\[ \xi^2 = \left( \frac{\delta Q^2}{Q^2} \right)^2 = \frac{q^2}{Q^2} \frac{(D_e - D_{\bar{e}})^2}{Q^2}, \] (110)
\[ Q^2 = q^2 + q_\parallel \left( \frac{p^2}{m^2 - p^2} - \frac{p'^2}{m^2 - p'^2} \right). \] (111)

Note that \(-m \leq p_\parallel \leq m\) and that \(p_\parallel\) may not be interpreted as the z-component of a single-particle momentum.

## 5 Discussion and interpretation

The integral equation (93) seems to have two kinds of singularities: The ‘Coulomb singularities’ \(1/Q_e^2\) or \(1/Q_{\bar{e}}^2\) and the ‘collinear singularity’ \(1/q^2\). Either denominator can become zero in the integral equation. The Coulomb singularity is square-integrable and welcome since it provides the binding. The collinear singularity is disastrous. If the coefficient of \(1/q^2\) is finite at \(q \to 0\) the integral equation is not solvable. The singularity structure of the final result must therefore be discussed carefully.

First we observe, that in the case of the exponential similarity function the effective interaction between an electron and a positron becomes
\[ U_{\text{eff}} = -\frac{\alpha}{4\pi^2} \frac{1}{\sqrt{x(1-x)x'(1-x')}} \left( \frac{<j^\mu J_\mu>_{\text{ex}}}{Q^2} + \frac{<c^\mu C_\mu>_{\text{an}}}{M^2} \right). \] (112)

The collinear singularity is wiped out since the coefficient related to the similarity function vanishes, see Eq.(103). This astoundingly compact formula exactly agrees with the Tamm-Dancoff approach [10]. The explicit \(x\)-dependence in the denominator of Eq.(92) looks like the only remnant of the light-cone formulation; all other quantities are Lorentz-contracted scalars.

With the method of Hamiltonian flow one can calculate also the scattering amplitude for \(e\bar{e}\)-scattering
\[ T_{\text{scattering}} = -\frac{\alpha}{4\pi^2} \frac{1}{\sqrt{x(1-x)x'(1-x')}} \left( \frac{<j^\mu J_\mu>_{\text{ex}}}{Q^2} + \frac{<c^\mu C_\mu>_{\text{an}}}{M^2} \right), \] (113)

see App. A. The expression agrees identically with the Feynman amplitude. In the scattering process the free 4-momentum \(P_\mu\) is conserved, and thus the momentum transfer along the electron and the positron line are the same, i.e. \(Q_e^2 = Q_{\bar{e}}^2 = Q^2\). Therefore
\( \delta Q^2 = 0 \) and thus \( \xi = 0 \). According to Eq. (102), the coefficient of the collinear singularity vanishes identically.

The scattering amplitude can be investigated also in light-cone perturbation theory [22]. Its exchange part is presented in all details in Sec. 3.4 of [3]. As can be pursued there, the collinear singularity appears in both the instantaneous interaction and the second order amplitude \( VGV \) of the vertex interaction, however such that the two contributions cancel each other exactly. Only the integrable Coulomb singularity remains. One concludes that Hamiltonian flow equations are governed de facto by the same mechanism of cancelation but that the disappearance of the collinear singularity is somewhat more subtle. To understand that better two approximations are discussed in the sequel.

First, let us expand the kernel up to terms linear in \( p/m \). To this order holds

\[
\bar{u}(p', \lambda') \gamma^\mu u(p, \lambda) = \begin{cases} 
2m\delta^\lambda_{\lambda'}, & \mu = +, -, \\
0, & \mu = 1, 2.
\end{cases}
\]

The bracket symbols of Eq. (102) become then

\[
\langle \gamma^\mu \gamma^\nu \rangle < g_{\mu\nu} = \langle \gamma^\mu \gamma^\nu \rangle \eta_{\mu\nu} = 16m^2 \delta^\lambda_{\lambda'} \delta^\lambda_{\lambda'}. \]

According to Eqs. (107) and (108) one has to this order

\[
D_e = D_{\bar{e}} = -q^2/q_{\parallel}.
\]

Consequently, \( \delta Q^2 = 0 \) and that alone is sufficient to wipe out all dependence on the similarity function. The exchange part becomes

\[
U_{\text{eff}} = -\frac{\alpha}{\pi^2} \frac{1}{q^2} (2m)^2 \delta^\lambda_{\lambda'} \delta^\lambda_{\lambda'}. \]

The factor \((2m)^2\) is a light-cone peculiarity, see Sec. 4.9 of [3], and the factor in front of it is precisely the Fourier transform of the familiar Coulomb potential in three space-dimensions. Hence the effective electron-positron interaction Eq. (91) is bound to produce the Bohr spectrum.

Next, expand the kernel up to second order in \( p^2/m^2 \). This gives the familiar Breit-Fermi spin-spin and tensor interactions [4], that insures the correct spin-splittings for the positronium ground state and restores the rotational invariance [5]. Brisudova et al. [6] state also that the ‘\( \eta_{\mu\nu} \)’ term may influence the spin-orbit coupling in second-order bound-state perturbation theory. The correct singlet-triplet splitting is observed also in the numerical solutions of the integral equation [10, 17] with the effective interaction (112) as well as with flow equations (see further).

Let us now consider the case of a general similarity function. In order to do this we have to discuss the expressions of the previous subsection. We realize, that as the momentum transfer \( q = p' - p \) tends to zero \( D_e - D_{\bar{e}} \) tends linearly to zero and \( Q^2 \) vanishes quadratically in \( q \). This approach to zero is anisotropic. Only at \( p = 0 \) itself \( Q^2 \) approaches 0 isotropically and the linear contribution in \( D_e - D_{\bar{e}} \) disappears. One can also show, that \( Q^2 \) can only vanish, if \( q = 0 \). It is not sufficient, that \( q^+ \) vanishes. Consequently \( \xi \) is always finite (which also follows from the observation, that \( Q_e^2 \) and \( Q_{\bar{e}}^2 \)
are positive and thus \(-1 \leq \xi \leq +1\). However as \(q\) vanishes \(\xi\) will in general depend on the direction from which \(q\) approaches zero. Moreover we realize, that for sufficiently smooth similarity functions \(f\) the function \(\vartheta(\xi)\) will be analytic as in the cases of the exponential and Gaussian cut-off. Then the pre-factor \((\xi^2 - \xi \vartheta(\xi))/(1 - \xi^2)\) in (102) contains a factor \(\xi^2\), which itself contains a factor \(q^2\), which cancels against the denominator \(q^2\) of the \(\eta_\mu\eta_\nu\) term. Thus the interaction becomes only singular if \(q\) approaches 0 where it diverges like \(1/q^2\). This is however not true for the sharp cut-off, where only one factor \(q^2\) can be cancelled and one \(q^+\) remains in the denominator of the \(\eta_\mu\eta_\nu\) term. Thus for a smooth cut-off one gets rid of the collinear singularity.

We realize further that \(\xi\) is of order \(p/m\). Since however the Bohr momentum is of order \(m\alpha\) the contribution due to the second term in (102) is smaller by a factor \(\alpha^2\) in comparison to the leading term \(g_\mu\nu/Q^2\).

Thus we realize that in this order we have in addition to the leading term a contribution

\[-\frac{\alpha}{4\pi^2} <\gamma^\mu\gamma^\nu>_{\text{ex}} (1 - \vartheta'(0))\xi^2 \left(\frac{g_\mu\nu}{Q^2} - \frac{\eta_\mu\eta_\nu}{q^{+2}}\right)\]

(118)

to the interaction. It obviously depends on the similarity function via the derivative \(\vartheta'(0)\).

We will shortly discuss the leading contribution and use Eq. (115). Collecting terms one obtains

\((1 - \vartheta'(0))\frac{16\alpha}{\pi^2} q^2 \frac{(p'^2 - p^2)^2}{q^{+6}}\)

(119)
as the contribution to the effective interaction. This interaction is spin-independent but anisotropic. It depends on the similarity function and is of order \(\alpha^2\) in comparison to that of the leading Coulomb interaction. We emphasize as already pointed out in [13], that in this order in \(\alpha^2\) also one- and two-loop terms contribute. Since they will also in general depend on the cut-off chosen, there should be a cancelation between the term found here and loop terms. It was argued there, that in order \(\alpha^2\) the one- and two-loop terms only contribute spin-independent interactions and interactions of one spin (spin-orbit coupling), but not interactions between both spins (spin-spin and tensor interaction), which is in agreement with our finding that the interaction (119) does not depend on the spin.

### 6 Numerical solution for positronium spectrum

We solve the integral equation Eq.(133), with interaction kernel given in Eq.(111), for positronium mass spectrum numerically. Effective interaction with different choice of cut-offs is summarized in Appendix (B).

#### 6.1 Formulation of the problem

In polar coordinates the light-front variables are \((\vec{k}_\perp; x) = (k_\perp, \varphi; x)\); therefore the matrix elements of the effective interaction Eq.(111) depend on the angles \(\varphi\) and \(\varphi'\), i.e. \(\langle x, k_\perp, \varphi; \lambda_1, \lambda_2 | V_{\text{eff}} | x', k'_\perp, \varphi'; \lambda'_1, \lambda'_2 \rangle\). In order to introduce the spectroscopic notation for
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positronium mass spectrum we integrate out the angular degree of freedom, $\varphi$, introducing a discrete quantum number $J_z = n, n \in \mathbb{Z}$ (actually for the annihilation channel only $|J_z| \leq 1$ is possible),

$$\langle x, k_\perp; J_z, \lambda_1, \lambda_2 | \tilde{V}_{\text{eff}} | x', k'_\perp; J'_z, \lambda'_1, \lambda'_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{-iL_z\varphi} \int_0^{2\pi} d\varphi' e^{iL'_z\varphi'} \langle x, k_\perp, \varphi; \lambda_1, \lambda_2 | V_{\text{eff}} | x', k'_\perp, \varphi'; \lambda'_1, \lambda'_2 \rangle$$

(120)

where $L_z = J_z - S_z$; $S_z = \frac{1}{2} + \frac{1}{2}$ and the states can be classified (strictly speaking only for rotationally invariant systems) according to their quantum numbers of total angular momentum $J$, orbit angular momentum $L$, and total spin $S$. Definition of angular momentum operators in light-front dynamics is problematic because they include interactions.

The matrix elements of the effective interaction before integrating over the angles, $\langle x, k_\perp, \varphi; \lambda_1, \lambda_2 | V_{\text{eff}} | x', k'_\perp, \varphi'; \lambda'_1, \lambda'_2 \rangle$, and after the integration introducing the total momentum, $J_z$, $\langle x, k_\perp; J_z, \lambda_1, \lambda_2 | \tilde{V}_{\text{eff}} | x', k'_\perp; J'_z, \lambda'_1, \lambda'_2 \rangle$ for different cut-off functions are given in the exchange and annihilation channels in Appendices (C) and (D), respectively.

Now we proceed to solve for the positronium spectrum in all sectors of $J_z$. For this purpose we formulate the light-front integral equation Eq. (93) in the form where the integral kernel is given by the effective interaction for the total momentum $J_z$, Eq. (120).

We introduce instead of Jacobi momentum $(x, k_\perp)$ the three momentum in the center of mass frame $\vec{p} = (p_z, k_\perp) = (\mu \cos \theta, \mu \sin \theta \cos \varphi, \mu \sin \theta \sin \varphi)$ as follows

$$x = \frac{1}{2} \left(1 + \frac{p_z}{\sqrt{\vec{p}^2 + m^2}}\right),$$

(121)

where the Jacobian of this transformation $dx/dp_z$ is

$$J = \frac{1}{2} \frac{m^2 + \mu^2 \sin^2 \theta}{(m^2 + \mu^2)^{3/2}}.$$  

(122)

One obtains then the integral equation

$$\langle M_n^2 - 4(m^2 + \mu^2) \rangle \tilde{\psi}_n (\mu, \cos \theta; J_z, \lambda_1, \lambda_2)$$

$$+ \sum_{J'_z, \lambda'_1, \lambda'_2} \int_D d\mu' \int_0^{\pi} d\cos \theta' \frac{\mu'^4 m^2 + \mu'^2 (1 - \cos^2 \theta')}{(m^2 + \mu'^2)^{3/2}}$$

$$\times \langle \mu, \cos \theta; J_z, \lambda_1, \lambda_2 | \tilde{V}_{\text{eff}} | \mu', \cos \theta'; J'_z, \lambda'_1, \lambda'_2 \rangle \tilde{\psi}_n (\mu', \cos \theta'; J'_z, \lambda'_1, \lambda'_2) = 0.$$  

(123)

The integration domain $D$, defined in Eq. (94), is given now by $\mu \in [0; \frac{1}{2}]$. Neither $L_z$ nor $S_z$ are good quantum numbers; therefore we set $L_z = J_z - S_z$.

The integral equation Eq. (123) is used to calculate positronium mass spectrum numerically. Note, that if one succeeds to integrate out the angular degrees of freedom for the effective interaction Eq. (120) analytically, one has 2-dimensional integration in Eq. (123) instead of 3-dimensional one in the original integral equation (93) to perform numerically.
Table 1: Binding coefficients, $B_n = 4(2 - M_n)/\alpha^2$ ($\alpha = 0.3$), for the lowest modes of the positronium spectrum at $J_z = 0$ for the equal time perturbation theory up to order $\alpha^4$ ($B_{ETPP}$ [10]) compared to our calculations with exponential ($B_E$), Gaussian ($B_G$) and sharp ($B_S$) cut-offs. $B_G$ is obtained using only $g_{\mu\nu}$ part of interaction; for $B_G^\eta\eta'$ term is included. Exchange channel is considered.

| $n$ | Term | $B_{ETPP}$ | $B_E$ | $B_G^\eta$ | $B_G$ | $B_S$ |
|-----|------|------------|------|-------------|------|------|
| 1   | $1S_0$ | 1.118125   | 1.049550 | 1.101027    | 1.026170 | 0.920921 |
| 2   | $1S_1$ | 0.998125   | 1.001010 | 1.049700    | 0.981969 | 0.885347 |
| 3   | $2S_0$ | 0.268633   | 0.260237 | 0.266490    | 0.260642 | 0.242607 |
| 4   | $2S_1$ | 0.253633   | 0.253804 | 0.259506    | 0.254765 | 0.234312 |
| 5   | $2P_1$ | 0.253633   | 0.257969 | 0.263056    | 0.257664 | 0.237611 |
| 6   | $2P_0$ | 0.261133   | 0.267070 | 0.273826    | 0.266563 | 0.243075 |
| 7   | $2P_1$ | 0.255508   | 0.259667 | 0.265412    | 0.260127 | 0.238135 |
| 8   | $2P_2$ | 0.251008   | 0.255258 | 0.260345    | 0.255498 | 0.236383 |

We use the numerical code [11], worked out by Uwe Trittmann for the similar problem [10]. This code includes for the numerical integration the Gauss-Legendre algorithm (Gaussian quadratures). To improve the numerical convergence the technique of Coulomb counterterms is included. The problem has been solved for all components of the total angular momentum, $J_z$.

Positronium spectrum is mainly defined by the Coulomb singularity

$$\vec{q} \to 0,$$

(124)

which is an integrable one analytically and also, by use of technique of Coulomb counterterms, numerically. In this region the effective interaction Eq.(111) has leading Coulomb behavior Eq.(112), independent on the cut-off function. We use therefore in numerical procedure standard Coulomb counterterms, introduced for the Coulomb problem Eq.(112) [10, 11], for all cut-offs.

Also we expect therefore the same pattern of levels for different cut-offs, that is proved numerically further.

Another important limiting case to study effective interaction Eq.(111), namely its exchange part Eq.(102), is the collinear limit

$$q^+ \to 0,$$

(125)

that is special for light-front calculations. Exchange part of the effective interaction is given by Eq.(118), which is finite in this limit. This is true for the regular cut-off functions, as in the case of exponential and gaussian cut-offs, where the derivative $d\vartheta(0)/d\xi$ is well defined. For sharp cut-off this condition is not fulfilled, and the effective interaction contains the $1/q^+$ type of singularity (see Appendix (B)). We do not associate any physics with this singularity, and consider it as a consequence of artificial choice of cut-off, which corresponds to singular generator of unitary transformation Eq.(15). In numerical calculations we omit $'\eta_\mu\eta'_\nu$ term in exchange channel for sharp cut-off.
### Table 2: Binding coefficients, $B_n = 4(2 - M_n)/\alpha^2$ ($\alpha = 0.3$), for the lowest modes of the positronium spectrum at $J_z = 0$ for our calculations with exponential ($B_E$), Gaussian ($B_G$) and sharp ($B_S$) cut-offs. $B'_G$ includes $\eta_\mu \eta_\nu$ term in exchange channel; $B_G$ does not. Exchange and annihilation channels are considered.

| $n$ | Term  | $B_E$  | $B'_G$ | $B_G$  | $B_S$  |
|-----|-------|--------|--------|--------|--------|
| 1   | $1^3S_0$ | 1.049550 | 1.101270 | 1.026170 | 0.920921 |
| 2   | $1^3S_1$ | 0.936800  | 0.978018  | 0.921847  | 0.834004 |
| 3   | $2^3S_0$ | 0.260237  | 0.266490  | 0.260642  | 0.242624 |
| 4   | $2^3S_1$ | 0.255292  | 0.260383  | 0.255615  | 0.234338 |
| 5   | $2^3P_1$ | 0.257969  | 0.263056  | 0.257664  | 0.236383 |
| 6   | $2^3P_0$ | 0.267090  | 0.273847  | 0.266626  | 0.243075 |
| 7   | $2^3P_1$ | 0.259667  | 0.265412  | 0.260127  | 0.237611 |
| 8   | $2^3P_2$ | 0.245615  | 0.250821  | 0.247091  | 0.230901 |

### Table 3: Difference in the corresponding energy levels between $J_z = 0$ and $J_z = 1$ states for exponential ($\delta B_E$), Gaussian ($\delta B_G$) and sharp ($\delta B_S$) cut-offs. Exchange channel is considered.

| $n$ | Term  | $\delta B_E$   | $\delta B_G$   | $\delta B_S$ |
|-----|-------|----------------|----------------|--------------|
| 2   | $1^3S_1$ | 6.30 $10^{-4}$ | 1.76 $10^{-3}$ | 1.18 $10^{-3}$ |
| 4   | $2^3S_1$ | 8.40 $10^{-5}$ | 1.77 $10^{-4}$ | 9.0 $10^{-5}$ |
| 5   | $2^3P_1$ | -1.30 $10^{-5}$ | -7.47 $10^{-4}$ | -9.1 $10^{-5}$ |
| 7   | $2^3P_1$ | -4.08 $10^{-4}$ | -4.08 $10^{-4}$ | 1.4 $10^{-4}$ |
| 8   | $2^3P_2$ | 5 $10^{-6}$     | -7.7 $10^{-5}$ | 4.15 $10^{-4}$ |

### Table 4: Difference in the corresponding energy levels between $J_z = 0$ and $J_z = 1$ states for exponential ($\delta B_E$), Gaussian ($\delta B_G$) and sharp ($\delta B_S$) cut-offs. Exchange and annihilation channels are considered.

| $n$ | Term  | $\delta B_E$   | $\delta B_G$   | $\delta B_S$ |
|-----|-------|----------------|----------------|--------------|
| 2   | $1^3S_1$ | -1.411 $10^{-3}$ | -7.86 $10^{-4}$ | -1.65 $10^{-3}$ |
| 4   | $2^3S_1$ | -4.1 $10^{-5}$   | -4.0 $10^{-5}$   | -1.15 $10^{-4}$ |
| 5   | $2^1P_1$ | -6.4 $10^{-5}$   | -6.52 $10^{-4}$  | -4.60 $10^{-4}$ |
| 7   | $2^3P_1$ | -4.69 $10^{-4}$  | -4.74 $10^{-4}$  | -1.40 $10^{-4}$ |
| 8   | $2^3P_2$ | -1.96 $10^{-4}$  | -1.36 $10^{-4}$  | -2.44 $10^{-4}$ |
6.2 Discussion of numerical results

We place the results of calculations for three different cut-offs, performed in exchange and including both exchange and annihilation channels, in Tables (1) and (2), respectively. The corresponding set of figures is presented in Fig.(3) and Fig.(4). We get the ionization threshold at $M^2 \sim 4m^2$, the Bohr spectrum, and the fine structure. Including annihilation part increases the splittings twice as large for the lowest multiplets.

We argued that the region of Coulomb singularity, and hence $'g'_{\mu\nu}$ part of effective interaction, determines mainly the positronium spectrum. However, including $'\eta\eta'_{\mu\nu}$ part for gaussian cut-off shifts spectrum as a whole down to about 5 – 7%, since this part is diagonal in spin space (Appendix (C)), and improves the data to be near the result obtained in covariant equal time calculations (Table(1)). For the sharp cut-off the lowest multiplet is placed higher than the one in case of exponential and gaussian cut-offs. The reason is in disregarding the infrared divergent $'\eta\eta'_{\mu\nu}$ part. Presumably, it is necessary to take into account $'\eta\eta'_{\mu\nu}$ term in exchange channel also for sharp cut-off after the proper regularization of infrared longitudinal divergences is done.

As one can see from presents figures, certain mass eigenvalues at $J_z = 0$ are degenerate with certain eigenvalues at other $J_z$ to a very high degree of numerical precision. As an example, consider the second lowest eigenvalue for $J_z = 0$. It is degenerate with the lowest eigenvalue for $J_z = \pm 1$, and can thus be classified as a member of the triplet with $J = 1$. Correspondingly, the lowest eigenvalue for $J_z = 0$ having no companion can be classified as the singlet state with $J = 0$. Quite in general one can interpret degenerate multiplets as members of a state with total angular momentum $J = 2J_{z,max} + 1$. One can get the quantum number of total angular momentum $J$ from the number of degenerate states for a fixed eigenvalue $M^2_n$. One can make contact with the conventional classification scheme $2S+1L_J^J_z$, as indicated in Tables (1)-(2).

Such pattern of spectrum is driven by rotational invariance. To trace rotational symmetry we calculate the difference of energy levels between $J_z = 0$ and $J_z = 1$ states for the lowest multiplets. The data are given for exchange and including annihilation channel in Tables (3) and (4), respectively. Including annihilation channel improves the extent of degeneracy (see Table(3) and Figure(4)).

Concerning the spin-splittings the best agreement with covariant calculations is obtained for gaussian cut-off, the worst results are for sharp cut-off. Rotational invariance is traced on the level of spectrum by studying the degree of degeneracy of corresponding states with the same total momentum but different projection $J_z$ in the multiplet. Again, better results are obtained for exponential and gaussian cut-off functions than for sharp cut-off. This suggests, that smooth cut-off functions are preferable to perform calculations.

Generally, the impact of the different choice of cut-off functions on the spectrum is small.

In this work we solve the bound state integral equation for the one fixed integration interval. Integration domain introduces the ultraviolet cut-off dependence of invariant mass squared $M^2(\Lambda)$, that reflects renormalization group properties of the effective coupling constant. We leave this question for the future study.
Figure 3: The invariant mass-squared spectrum $M^2_i$ for positronium versus the projection of the total spin, $J_z$, excluding annihilation with exponential, Gaussian and sharp cut-offs. The number of integration points is $N_1 = N_2 = 21$.

Figure 4: The invariant mass-squared spectrum $M^2_i$ for positronium versus the projection of the total spin, $J_z$, including annihilation with exponential, Gaussian and sharp cut-offs. The number of integration points is $N_1 = N_2 = 21$.

7 Summary and conclusions

We have applied the method of Hamiltonian flow to the canonical Hamiltonian of quantum electrodynamics in the front form and derived an effective interaction for an electron and a positron, which acts only in the $ee$-sector of Fock space. To lowest order of approximation is the familiar Coulomb interaction. In this first of a series of papers we have restricted ourselves to include terms up to the second order in the coupling constant $e$. By reasons of simplicity (almost) all aspects of renormalization theory within a Hamiltonian approach have been disregarded in this first assault. Depending on the particular choice of the
similarity function one gets perfect agreement with other approaches particularly with the method of iterated resolvents. Special emphasis is put on the impact of the collinear singularity. Depending on the similarity function, the collinear singularity is either absent from the outset, or it is shown explicitly that it has no impact on the solubility of the final integral equation.

The numerical solution of positronium bound state problem, with the effective electron positron interaction obtained by the flow equations, is presented. No approximations along numerical procedure are done.

One concludes that the method of Hamiltonian flow equations looks like an excellent tool to progress further, particularly to attack the severe problems of renormalization theory within a non-perturbative Hamiltonian approach to field theory and thus to apply it eventually to non-abelian gauge field theory and quantum chromodynamics.

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A Flow equations and Feynman amplitudes

There must be a connection between the Feynman diagrammatic technique which is based on the action and the flow equations which are based on the Hamiltonian. It will be studied here up to second order in the coupling constant.

The difference resides in the $l$-ordering of the generator $\eta(l)$ in the unitary operator

$$ U(l) = T_l \exp \left( \int_0^l \eta(l') dl' \right). \tag{126} $$

The transformed Hamiltonian reads

$$ H(l) = U(l) H(0) U^+(l). \tag{127} $$

With the full unitary transformation

$$ U(\infty) = e^{-S} \tag{128} $$

one has thus up to the second order in the coupling constant

$$ H(\infty) = e^{-S} H(0) e^S = H(0) + [H(0), S] + [[H(0), S], S] + ..., \tag{129} $$

and

$$ S = S^{(1)} + S^{(2)} + ... = - \int_0^\infty dl \eta(l) - \frac{1}{2} \int_0^\infty dl \int_0^l dl' [\eta(l), \eta(l')] + .... \tag{130} $$

The series for the effective Hamiltonian reads

$$ H(\infty) = H_0(0) + V(0) + [H_0(0), S^{(1)}] + [V(0), S^{(1)}] + \frac{1}{2} [[H_0(0), S^{(1)}], S^{(1)}] + [H_0(0), S^{(2)}] + .... \tag{131} $$

The first order term vanishes

$$ V(0) + [H_0(0), S^{(1)}] = 0, \tag{132} $$

which gives rise to

$$ H(\infty) = H_0(0) + \frac{1}{2} [V(0), S^{(1)}] + [H_0(0), S^{(2)}]. \tag{133} $$

Denoting by $S^{(i)}$ the order with respect to $e$ and omitting the instantaneous terms gives

$$ S^{(1)} = \frac{e}{\sqrt{(2\pi)^3}} \int \frac{d^3 p_1}{\sqrt{2p_1^+}} \int \frac{d^3 p_2}{\sqrt{2p_2^+}} \int \frac{d^3 p_3}{\sqrt{2p_3^+}} \delta^{(3)}(p_1 - p_2 - p_3) \frac{1}{D(p_1, p_2, p_3)} \times \left[ b_1^\dagger b_2 a_3 (\bar{u}_1 \gamma_3 u_2) - d_1^\dagger d_2 a_3 (\bar{v}_2 \gamma_3 v_1) + a_1^\dagger d_2 b_3 (\bar{v}_2 \gamma_1 u_3) \right] - h.c. \tag{134} $$
and

\[
S^{(2)} = \frac{e^2}{(2\pi)^3} \sum_{\lambda_1, \lambda_2, \lambda'_1, \lambda'_2} \int \frac{[d^3p_1]}{\sqrt{2p_1^+}} \int \frac{[d^3p_2]}{\sqrt{2p_2^+}} \int \frac{[d^3p'_1]}{\sqrt{2p'_1^+}} \int \frac{[d^3p'_2]}{\sqrt{2p'_2^+}}
\]

\[
\times \delta^{(3)}(p_1 + p_2 - p'_1 - p'_2) \frac{d_{\mu\nu}(q)}{q^+} \left( -\frac{1}{2D_1D_2} \right) (\Theta(D_1, D_2) - \Theta(D_2, D_1))
\]

\[
\times \frac{\Theta(p_1, \lambda_1) \gamma^\mu u(p'_1, \lambda'_1) (\Theta(p_2, \lambda_2) \gamma^\nu v(p_2, \lambda_2))}{\sqrt{2p_1^+} \sqrt{2p_1^{'+}} \sqrt{2p_2^+} \sqrt{2p_2^{'+}}}
\]

\[
\times : b^i(p_1, \lambda_1) b(p'_1, \lambda'_1) d^i(p_2, \lambda_2) d(p'_2, \lambda'_2) : .
\tag{135}
\]

The $D$ are of course the energy differences defined in Eq.(138), particularly $D_1 = p'_1^- - p_1^- - (p'_1 - p_1)^-$ and $D_2 = p'_2^- - p_2^- - (p'_2 - p_2)^-$, and $q$ denotes the 4-momentum of the exchanged gluon. The matrix elements in the $e\bar{e}$-sector become correspondingly

\[
\langle e\bar{e}|[V(0), S^{(1)}]|e\bar{e}\rangle = -\frac{e^2}{2(2\pi)^3} \frac{\Theta(p_1, \lambda_1) \gamma^\mu u(p'_1, \lambda'_1) (\Theta(p_2, \lambda_2) \gamma^\nu v(p_2, \lambda_2))}{\sqrt{2p_1^+} \sqrt{2p_1^{'+}} \sqrt{2p_2^+} \sqrt{2p_2^{'+}}}
\]

\[
\times \frac{d_{\mu\nu}(q)}{q^+} \left( \frac{1}{D_1} + \frac{1}{D_2} \right)
\tag{136}
\]

and

\[
\langle e\bar{e}|[H_0(0), S^{(2)}]|e\bar{e}\rangle = -\frac{e^2}{2(2\pi)^3} \frac{\Theta(p_1, \lambda_1) \gamma^\mu u(p'_1, \lambda'_1) (\Theta(p_2, \lambda_2) \gamma^\nu v(p_2, \lambda_2))}{\sqrt{2p_1^+} \sqrt{2p_1^{'+}} \sqrt{2p_2^+} \sqrt{2p_2^{'+}}}
\]

\[
\times \frac{d_{\mu\nu}(q)}{q^+} \left( -\frac{1}{2} \frac{D_1 - D_2}{D_1 D_2} \right) (\Theta(D_1, D_2) - \Theta(D_2, D_1)) .
\tag{137}
\]

Inserting the latter two equation into Eq.(133) gives the generated interaction in Eq.(137). Note that the $\Theta$-factor in Eq.(138) satisfies

\[
\frac{\Theta_{12}}{D_1} + \frac{\Theta_{21}}{D_2} = \frac{1}{2} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) + \frac{1}{2} \left( \frac{1}{D_1} - \frac{1}{D_2} \right) (\Theta_{12} - \Theta_{21}) .
\tag{138}
\]

The first term in Eq.(137) corresponds therefore to the result of perturbation theory and comes from Eq.(136). The second term originates from the $l$-ordering (137) and vanishes on the mass shell.
B  Defining different cut-offs

In this appendix we summarize the results for the effective electron-positron interaction, generated by the flow equations with different similarity functions. In the practical work, three different similarity function will be studied explicitly:

1. the exponential cut-off, 2. the gaussian cut-off, and 3. the sharp cut-off.

1) Exponential cut-off

\[ f(D; l) = \exp (-|D|l) \]

\[ \Theta(D_e, D_\ell) = \frac{D_e}{D_e + D_\ell} ; \ \vartheta(\xi) = \xi \]

\[ V_{\text{eff}} = \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} \frac{g_{\mu\nu}}{q^+ D} - \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \frac{g_{\mu\nu}}{p^+ D} \]

\[ = -\frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} \frac{1}{Q^2} - \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \frac{1}{M^2}, \]  \hspace{1cm} (139)  

where \( D = 1/2(D_e + D_\ell) \) and \( \tilde{D} = 1/2(D_a + D_b) \). The first choice of similarity function gives exactly the result of perturbation theory.

2) Gaussian cut-off

\[ f(D; l) = \exp (-D^2 l) \]

\[ \Theta(D_e, D_\ell) = \frac{D_e^2}{D_e^2 + D_\ell^2} ; \ \vartheta(\xi) = \frac{2\xi}{1 + \xi^2} \]

\[ V_{\text{eff}} = \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} \left[ \frac{g_{\mu\nu} D_e + D_\ell}{q^+ D_e^2 + D_\ell^2} - \frac{\eta_\mu \eta_\nu (D_e - D_\ell)^2}{2q^{+2} D_e^2 + D_\ell^2} \right] \]

\[ - \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \left[ \frac{g_{\mu\nu} D_a + D_b}{p^+ D_a^2 + D_b^2} - \frac{\eta_\mu \eta_\nu (D_a - D_b)^2}{2p^{+2} D_a^2 + D_b^2} \right] \]

\[ = -\frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} \left[ \frac{g_{\mu\nu}}{Q^2} + \frac{\eta_\mu \eta_\nu \delta Q^4}{q^{+2} Q^4} \right] \frac{Q^4}{Q^4 + \delta Q^4} \]

\[ - \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \left[ \frac{g_{\mu\nu}}{M^2} - \frac{\eta_\mu \eta_\nu \delta M^4}{p^{+2} M^4} \right] \frac{M^4}{M^4 + \delta M^4}, \]  \hspace{1cm} (140)  

where we understand under \( Q^4 = (Q^2)^2 \) and \( \delta Q^4 = (\delta Q^2)^2 \) with \( Q^2 \) and \( \delta Q^2 \) defined in Eq. (71).

3) Sharp cut-off

\[ f(D; l) = \theta (1 - |D|l) \]

\[ \Theta(D_e, D_\ell) = \theta (|D_e| - |D_\ell|) ; \ \vartheta(\xi) = \text{sign}(\xi) \]

\[ V_{\text{eff}} = \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} \left[ \frac{g_{\mu\nu}}{q^+} \left( \frac{\theta(|D_e| - |D_\ell|)}{D_e} + \frac{\theta(|D_\ell| - |D_e|)}{D_\ell} \right) \right] \]

\[ - \frac{\eta_\mu \eta_\nu}{2q^{+2}} (D_e - D_\ell) \left( \frac{\theta(|D_e| - |D_\ell|)}{D_e} - \frac{\theta(|D_\ell| - |D_e|)}{D_\ell} \right) \]

\[ - \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \left[ \frac{g_{\mu\nu}}{p^+} \left( \frac{\theta(|D_a| - |D_b|)}{D_a} + \frac{\theta(|D_b| - |D_a|)}{D_b} \right) \right] \]
\[-\frac{\eta_\mu \eta_\nu}{2p^{+2}}(D_a - D_b) \left( \frac{\theta(|D_a| - |D_b|)}{D_a} - \frac{\theta(|D_b| - |D_a|)}{D_b} \right) \]

\[= - \frac{\alpha}{4\pi^2} \left( \langle \gamma^\mu \gamma^\nu \rangle \right)_{ex} \left[ \frac{g_{\mu\nu}}{Q^2} + \frac{\eta_\mu \eta_\nu \delta Q^2}{Q^2} \right] \frac{Q^2}{Q^2 + |\delta Q^2|} \]

\[- \frac{\alpha}{4\pi^2} \left( \langle \gamma^\mu \gamma^\nu \rangle \right)_{an} \left[ \frac{g_{\mu\nu}}{M^2} - \frac{\eta_\mu \eta_\nu \delta M^2}{M^2} \right] \frac{M^2}{M^2 + |\delta M^2|}, \] (141)

The motivation to choose these cut-off functions is the following. Using exponential cut-off in flow equations one generates the same interaction as obtained also in Tamm-Dancoff approach, where numerical calculations of positronium spectrum are performed \[10\], and we use this numerical code here. Note also, that for this cut-off the effective interaction looks very much as in covariant calculations: it contains only \(g_{\mu\nu}\) part, and \(\eta_\mu \eta_\nu\) part is identically zero, so that there is no collinear problem. Gaussian cut-off corresponds to the original choice of generator Eqs. (5,41) by Wegner as commutator of diagonal, particle number conserving, and off-diagonal, particle number changing, parts of Hamiltonian. Sharp cut-off is used often in the alternative similarity scheme to perform calculations \[4\].
C  The matrix elements in the exchange channel

In this Appendix we follow the scheme of the work [10] to calculate the matrix elements of the effective interaction in the exchange channel.\(^1\) Here, we list the general, angle-dependent matrix elements defining the effective interaction in the exchange channel and the corresponding matrix elements of the effective interaction for arbitrary \(J_z\), after integrating out the angles. Exchange part of the effective interaction for three different cut-offs Eqs. (139–141) can be written

\[
V_{\text{eff}} = -\frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle B_{\mu\nu},
\]

where explicitly one has

(1) **Exponential cut-off**

\[
B_{\mu\nu} = \frac{g_{\mu\nu}}{Q^2},
\]

(2) **Gaussian cut-off**

\[
B_{\mu\nu} = g_{\mu\nu} \text{Re} \left( \frac{1}{Q^2 + i\delta Q^2} \right) - \frac{\eta_{\mu}\eta_{\nu}}{q^4} \delta Q^2 \text{Im} \left( \frac{1}{Q^2 + i\delta Q^2} \right),
\]

where Re and Im are real and imaginary parts, respectively, and \(i^2 = -1\).

(3) **Sharp cut-off**

\[
B_{\mu\nu} = g_{\mu\nu} \left( \frac{\theta(-\delta Q^2)}{Q^2 - \delta Q^2} + \frac{\theta(\delta Q^2)}{Q^2 + \delta Q^2} \right) - \frac{\eta_{\mu}\eta_{\nu}}{q^4} \delta Q^2 \left( \frac{\theta(-\delta Q^2)}{Q^2 - \delta Q^2} - \frac{\theta(\delta Q^2)}{Q^2 + \delta Q^2} \right),
\]

where \(q = p_1' - p_1\) is the momentum transfer; and \(\langle \gamma^\mu \gamma^\nu \rangle\) for the exchange channel is given in Eq. (12). We omit index ‘ex’ everywhere.

It is convenient to extract the angular dependence in the functions

\[
Q^2_{\text{e}} = a_1 - b \cos t
\]

\[
Q^2_{\text{e}} = a_2 - b \cos t
\]

\[
t = \varphi - \varphi',
\]

where we define

\[
\tilde{k}_\perp = k_\perp (\cos \varphi, \sin \varphi)
\]

in polar system; here the terms are given

\[
a_1 = \frac{x'k_\perp^2 + xk_\perp'^2 + m^2(x - x')^2}{xx'}
\]

\(^1\) Some of these calculations can be found in [14].
\[ a_2 = \frac{1 - x'}{1 - x} k_\perp^2 + \frac{1 - x}{1 - x'} k'_\perp^2 + m^2 \frac{(x - x')^2}{(1 - x)(1 - x')} \]
\[ b = 2k_\perp k'_\perp \]  

(148)

Then the functions in Eqs. (143–145) are given

\[ Q^2 = a - b \cos t \]
\[ \delta Q^2 = \delta a, \]  

(149)

where

\[ a = \frac{1}{2} (a_1 + a_2) \]
\[ \delta a = \frac{1}{2} (a_1 - a_2), \]  

(150)

It is useful to display the matrix elements of the effective interaction in the form of tables. The matrix elements depend on the one hand on the momenta of the electron and positron, respectively, and on the other hand on their helicities before and after the interaction. The dependence on the helicities occur during the calculation of these functions \( E(x, \vec{k}_\perp; \lambda_1, \lambda_2|x', \vec{k}'_\perp; \lambda'_1, \lambda'_2) \) in part I and \( G(x, k_\perp; \lambda_1, \lambda_2|x', k'_\perp; \lambda'_1, \lambda'_2) \) in part II as different Kronecker deltas \([22]\). These functions are displayed in the form of helicity tables. We use the following notation for the elements of the tables

\[ F_i(1, 2) \rightarrow E_i(x, \vec{k}_\perp; x', \vec{k}'_\perp); \quad G_i(x, k_\perp; x', k'_\perp) \]  

(151)

Also we have used in both cases for the permutation of particle and anti-particle

\[ F_3^*(x, \vec{k}_\perp; x', \vec{k}'_\perp) = F_3(1 - x, -\vec{k}_\perp; 1 - x', -\vec{k}'_\perp) \]  

(152)

one has the corresponding for the elements of arbitrary \( J_z \); in the case when the function additionally depends on the component of the total angular momentum \( J_z = n \) we have introduced

\[ \tilde{F}_i(n) = F_i(-n) \]  

(153)

C.1 The helicity table

To calculate the matrix elements of the effective interaction in the exchange channel we use the matrix elements of the Dirac spinors listed in Table (3) \([22]\). Also the following holds \( \bar{v}_\lambda(p) \gamma^\alpha v_\lambda(q) = \bar{u}_{\lambda}(q) \gamma^\alpha u_{\lambda}(p) \).

We introduce for the matrix elements entering in the effective interaction Eqs. (143–145)

\[ 2E^{(1)}(x, \vec{k}_\perp; \lambda_1, \lambda_2|x', \vec{k}'_\perp; \lambda'_1, \lambda'_2) = \langle \gamma^\mu \gamma^\nu \rangle g_{\mu\nu}, \]  

(154)
Table 5: Matrix elements of the Dirac spinors.

\[
\gamma_\mu \gamma_\nu g_{\mu\nu} = \frac{1}{2} \gamma^{+} \gamma^{−} + \frac{1}{2} \gamma^{−} \gamma^{+} - \gamma_{2} \gamma_{2} - \gamma_{2} \gamma_{2} \quad \text{and} \quad 2E^{(2)}(x, k_{\perp}; \lambda_1, \lambda_2 | x', \bar{k}_{\perp}; \lambda_1', \lambda_2') = \gamma^{\mu} \gamma^{\nu} \eta_{\mu} \eta_{\nu} \frac{1}{q_{x'}},
\]

with \( \gamma^{\mu} \gamma^{\nu} \eta_{\mu} \eta_{\nu} = \gamma^{+} \gamma^{+} \); where

\[
\gamma^{\mu} \gamma^{\nu} = \frac{\langle \bar{u}(x, k_{\perp}; \lambda_1) \gamma^{\mu} u(x', \bar{k}_{\perp}; \lambda_1') \rangle (\bar{v}(1 - x', -\bar{k}_{\perp}; \lambda_2') \gamma^{\nu} v(1 - x, -\bar{k}_{\perp}; \lambda_2))}{\sqrt{x x'} \sqrt{(1 - x)(1 - x')}}
\]

These functions are displayed in Table (3).

| final : initial | \( \lambda_1', \lambda_2' = \uparrow \uparrow \) | \( \lambda_1', \lambda_2' = \uparrow \downarrow \) | \( \lambda_1', \lambda_2' = \downarrow \uparrow \) | \( \lambda_1', \lambda_2' = \downarrow \downarrow \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \lambda_1, \lambda_2 = \uparrow \uparrow \) | \( E_1(1, 2) \) | \( E_3(1, 2) \) | \( 0 \) | \( 0 \) |
| \( \lambda_1, \lambda_2 = \uparrow \downarrow \) | \( E_3(1, 2) \) | \( E_2(1, 2) \) | \( -E_3(2, 1) \) | \( -E_3(2, 1) \) |
| \( \lambda_1, \lambda_2 = \downarrow \uparrow \) | \( E_3(2, 1) \) | \( E_2(1, 2) \) | \( -E_3(2, 1) \) | \( E_1(1, 2) \) |
| \( \lambda_1, \lambda_2 = \downarrow \downarrow \) | \( 0 \) | \( -E_3(1, 2) \) | \( -E_3(1, 2) \) | \( E_1(1, 2) \) |

Table 6: General helicity table defining the effective interaction in the exchange channel.

The matrix elements \( E_i^{(n)}(1, 2) = E_i^{(n)}(x, k_{\perp}; x', \bar{k}_{\perp}) \) with \( n = 1 \) and \( n = 2 \) for \( \gamma_{\mu} \gamma_{\nu} \) terms, respectively, are the following

\[
E_1^{(1)}(x, k_{\perp}; x', \bar{k}_{\perp}) = m^2 \left( \frac{1}{x x'} + \frac{1}{(1 - x)(1 - x')} \right) + \frac{k_{\perp} \bar{k}_{\perp}'}{x x'(1 - x)(1 - x')} e^{-i(\varphi - \varphi')}
\]
\[ E_2^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = m^2 \left( \frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right) + k'_\perp \frac{1}{x(1-x)} + k''_\perp \frac{1}{x'(1-x')} + k'_\perp \frac{e^{i(\phi-\phi')}}{xx'} + \frac{e^{-i(\phi-\phi')}}{(1-x)(1-x')} \]

\[ E_3^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -\frac{m}{xx'} \left( k'_\perp e^{i\phi'} - k'_\perp \frac{1-x'}{1-x} e^{i\phi} \right) \]

\[ E_4^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -m^2 \frac{(x-x')^2}{xx'(1-x)(1-x')} \]

and

\[ E_1^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = E_2^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = \frac{2}{(x-x')^2} \]

\[ E_3^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = E_4^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = 0. \]

### C.2 The helicity table for arbitrary \( J_z \).

Following the description given in the main text Eq. (120) we integrate out the angles in the effective interaction in the exchange channel. For the matrix elements of the effective interaction for an arbitrary \( J_z = \pm n \) with \( n \in \mathbb{Z} \) we introduce the functions \( G(x, k_\perp; \lambda_1, \lambda_2 | x', k'_\perp; \lambda'_1, \lambda'_2) = \langle x, k_\perp; J_z; \lambda_1, \lambda_2 | \tilde{V}_{\text{eff}} | x', k'_\perp; J'_z; \lambda'_1, \lambda'_2 \rangle \) in the exchange channel and obtain the helicity Table (7).

| final : initial | \((\lambda'_1, \lambda'_2) = \uparrow\uparrow\) | \((\lambda'_1, \lambda'_2) = \uparrow\downarrow\) | \((\lambda'_1, \lambda'_2) = \downarrow\uparrow\) | \((\lambda'_1, \lambda'_2) = \downarrow\downarrow\) |
|----------------|-----------------|-----------------|-----------------|-----------------|
| \((\lambda_1, \lambda_2) = \uparrow\uparrow\) | \(G_1(1,2)\) | \(G_2(1,2)\) | \(G_3(1,2)\) | \(0\) |
| \((\lambda_1, \lambda_2) = \uparrow\downarrow\) | \(G_3^*(2,1)\) | \(G_4(1,2)\) | \(-\hat{G}_4(1,2)\) | \(\hat{G}_3(2,1)\) |
| \((\lambda_1, \lambda_2) = \downarrow\uparrow\) | \(G_3(2,1)\) | \(G_2(1,2)\) | \(-\hat{G}_2(1,2)\) | \(\hat{G}_3^*(2,1)\) |
| \((\lambda_1, \lambda_2) = \downarrow\downarrow\) | \(0\) | \(-\hat{G}_3(1,2)\) | \(-\hat{G}_3^*(1,2)\) | \(\hat{G}_1(1,2)\) |

Table 7: Helicity table of the effective interaction for \( J_z = \pm n, x > x' \).

Here, the functions \( G_i(1,2) = G_i(x, k_\perp; x', k'_\perp) \) are given

\[ G_1(x, k_\perp; x', k'_\perp) = \left( \frac{m^2}{xx'} + \frac{m^2}{(1-x)(1-x')} \right) \text{Int}(|1-n|) \]

\[ + \frac{k_\perp k'_\perp}{xx'(1-x)(1-x')} \text{Int}(|n|) - \frac{2\delta a}{(x-x')^2} \text{Int}(|1-n|) \]

\[ G_2(x, k_\perp; x', k'_\perp) = \left( \frac{m^2}{xx'} + \frac{1}{(1-x)(1-x')} \right) + \frac{k^2}{x(1-x)} + \frac{k'_2}{x'(1-x')} \]

\[ \times \text{Int}(|n|) \]

\[ + \frac{k_\perp k'_\perp}{xx'} \text{Int}(|1-n|) + \frac{1}{(1-x)(1-x')} \text{Int}(|1+n|) \]
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\[- \frac{2\delta a}{(x - x')^2} \tilde{I}nt(|n|)\]

\[G_3(x, k_{\perp}; x', k'_{\perp}) = -\frac{m}{xx'} \left( k'_{\perp} \tilde{I}nt(|1 - n|) - k_{\perp} \frac{1 - x'}{1 - x} \tilde{I}nt(|n|) \right)\]

\[G_4(x, k_{\perp}; x', k'_{\perp}) = -m^2 \frac{(x - x')^2}{xx'(1 - x)(1 - x')} \tilde{I}nt(|n|)\]

we define

\[I(n; a, b) = -\frac{\alpha}{2\pi^2} \int_0^{2\pi} dt \frac{\cos nt}{a - b \cos t},\]

then in Eq. (159) the following functions are introduced

1) **Exponential cut-off**

\[\tilde{I}nt(n) = I(n; a, b)\]

\[\tilde{I}nt(n) = 0,\]

(161)

2) **Gaussian cut-off**

\[\tilde{I}nt(n) = \text{Re}I(n; a + i\delta a, b)\]

\[\tilde{I}nt(n) = \text{Im}I(n; a + i\delta a, b),\]

(162)

3) **Sharp cut-off**

\[\tilde{I}nt(n) = \theta(-\delta a)I(n; a - \delta a, b) + \theta(\delta a)I(n; a + \delta a, b)\]

\[\tilde{I}nt(n) = \theta(-\delta a)I(n; a - \delta a, b) - \theta(\delta a)I(n; a + \delta a, b),\]

(163)

also \(a + \delta a = a_1\) and \(a - \delta a = a_2\).

Explicitly is used

\[\int_0^{2\pi} dt \frac{\cos nt}{a - b \cos t} = 2\pi \frac{1}{\sqrt{a^2 - b^2}} \left( \frac{a - \sqrt{a^2 - b^2}}{b} \right)^n\]

\[\int_0^{2\pi} dt \frac{\sin nt}{a - b \cos t} = 0,\]

(164)

where \(a\) can contain imaginary part as in the case of gaussian cut-off.
The matrix elements in the annihilation channel

We repeat the same calculations for the matrix elements of the effective interaction in the annihilation channel. Annihilation part of the effective interaction can be written

\[ V_{\text{eff}} = -\frac{\alpha}{4\pi^2} (\gamma^\mu \gamma^\nu) C_{\mu\nu}, \]  

where one has

\[ C_{\mu\nu} = g_{\mu\nu} \left( \frac{\Theta_{ab}}{M_a^2} + \frac{\Theta_{ba}}{M_b^2} \right) - \frac{\eta_{\mu} \eta_{\nu}}{p^+}, \]  

in the frame \( p_\perp = 0. \) Explicitly the annihilation part of the effective interaction for different cut-offs Eq. (133)- Eq. (141) is given

1. **Exponential cut-off**

\[ C_{\mu\nu} = \frac{g_{\mu\nu}}{M^2}, \]  

2. **Gaussian cut-off**

\[ C_{\mu\nu} = \frac{g_{\mu\nu}}{M^4 + \delta M^4} \frac{M^2}{\eta_{\mu} \eta_{\nu}} p^+ M, \]  

3. **Sharp cut-off**

\[ C_{\mu\nu} = \frac{g_{\mu\nu}}{M^2} \left( \frac{\theta(M^2_a - M^2_b)}{M_a^2} + \frac{\theta(M^2_b - M^2_a)}{M_b^2} \right) - \frac{\eta_{\mu} \eta_{\nu}}{p^+ M^2}, \]  

where \( p^+ = p^+_1 + p^+_2 \) is the total momentum; and \( \langle \gamma^\mu \gamma^\nu \rangle \) for annihilation is defined in Eq. (92). The functions present in Eq. (167)- Eq. (169) are given in the light-front frame

\[ M_a^2 = \frac{k^*_a^2 + m^2}{x'(1 - x')}, \]  

\[ M_b^2 = \frac{k^*_b^2 + m^2}{x(1 - x)}, \]  

\[ \]  

Indeed \( \langle \gamma^\mu \gamma^\nu \rangle g_{\mu\nu} = \frac{1}{2} \langle \gamma^+ \gamma^- \rangle + \frac{1}{2} \langle \gamma^- \gamma^+ \rangle + \langle \gamma^\mu \gamma^\nu \rangle g_{\mu\nu} \); therefore it holds

\[ g_{\mu\nu} = g_{\mu\nu}^\perp + \frac{\eta_{\mu} (p_\nu - p_\nu^\perp) + \eta_{\nu} (p_\mu - p_\mu^\perp)}{p^+} - \frac{p^{\perp 2}}{p^+ \eta_{\mu} \eta_{\nu}}, \]  

The 4-momentum of the photon \( p_\mu \) in the \( t \)-channel can be written \( p_\mu = p^*_\mu \) with \( p^*_\mu = \frac{1}{2} p^+_1 \) with \( D_a, D_b \) defined in Eq. (80). The Dirac equation \( (p_1 + p_2) \bar{u}(p_1) \gamma^\mu v(p_2) = 0 \) allows then to write \( p_\mu \bar{u}(p_1, \lambda_1) \gamma^\mu v(p_2, \lambda_2) = -2 M^2 / (2 p^+ \eta_{\mu} \bar{u}(p_1, \lambda_1) \gamma^\mu v(p_2, \lambda_2)) \). Thus, when \( p_\perp = 0 \), one has

\[ g_{\mu\nu} \rightarrow g_{\mu\nu}^\perp - \frac{\eta_{\mu} \eta_{\nu}}{p^+ M^2}, \]  

where the arrow means that this tensor should be contracted with \( \langle \gamma^\mu \gamma^\nu \rangle \) in the annihilation channel.
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we remind also

\[ M^2 = \frac{1}{2} (M_a^2 + M_b^2) \]
\[ \delta M^2 = \frac{1}{2} (M_a^2 - M_b^2) \] (171)

Note that the energy denominators of the effective interaction in the annihilation channel do not depend on the angles \( \varphi, \varphi' \).

| \( \mathcal{M} \) | \( \frac{1}{\sqrt{k+k'}} \bar{v}(k', \lambda') \mathcal{M} u(k, \lambda) \) |
|----------------|--------------------------------------------------|
| \( \gamma^+ \) | \( 2\delta^\lambda_{-\lambda'} \) |
| \( \gamma^- \) | \( \frac{2}{k+k'} \left[ -\left( m^2 - k_\perp k'_\perp e^{+i\lambda(\varphi-\varphi')} \right) \delta^\lambda_{-\lambda'} - m\lambda \left( \frac{k'_\perp e^{+i\lambda}\varphi'}{k_\perp} + \frac{k_\perp e^{+i\lambda}\varphi}{k'_\perp} \right) \delta^\lambda_{\lambda'} \right] \) |
| \( \gamma^\perp_1 \) | \( \left( \frac{k'_\perp}{k'} e^{-i\lambda\varphi'} + \frac{k_\perp}{k} e^{+i\lambda\varphi'} \right) \delta^\lambda_{-\lambda'} - m\lambda \left( \frac{1}{k'+k} + \frac{1}{k} \right) \delta^\lambda_{\lambda'} \) |
| \( \gamma^\perp_2 \) | \( \frac{\imath}{\frac{k'}{k'} + \frac{k}{k}} e^{-i\lambda\varphi'} - \frac{k_\perp}{k} e^{+i\lambda\varphi} \) \( \delta^\lambda_{-\lambda'} - im \left( \frac{1}{k'+k} + \frac{1}{k} \right) \delta^\lambda_{\lambda'} \) |

Table 8: Matrix elements of the Dirac spinors.

D.1 The helicity table

For the calculation of matrix elements of effective interaction in the annihilation channel we use the matrix elements of the Dirac spinors listed in Table (8) [22]. Also the following holds (\( \bar{v}_\lambda(q)^\alpha u_\lambda(q)^+ = \bar{u}_\lambda(q)^\gamma u_\lambda(p) \)).

We introduce

\[ 2H^{(1)}(x, \bar{k}_\perp; \lambda_1, \lambda_2 | x', \bar{k}'_\perp; \lambda'_1, \lambda'_2) = \langle \gamma^\mu \gamma^\nu \rangle g^\perp_{\mu\nu} = -\langle \gamma^1 \rangle - \langle \gamma^2 \rangle \]
\[ 2H^{(2)}(x, \bar{k}_\perp; \lambda_1, \lambda_2 | x', \bar{k}'_\perp; \lambda'_1, \lambda'_2) = \langle \gamma^\mu \gamma^\nu \rangle \eta^\mu \eta^\nu \frac{1}{p^2} \] (172)

where

\[ \langle \gamma^\mu \gamma^\nu \rangle = \frac{\langle \bar{u}(1 - x', -\bar{k}_\perp; \lambda'_2) \gamma^\mu u(x', \bar{k}'_\perp; \lambda'_1) \rangle \langle \bar{u}(x, \bar{k}_\perp; \lambda_1) \gamma^\nu v(1 - x, -\bar{k}_\perp; \lambda_2) \rangle}{\sqrt{x'(1 - x')}} \] (173)
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| final:initial | $(\lambda_1', \lambda_2') = \uparrow \uparrow$ | $(\lambda_1', \lambda_2') = \uparrow \downarrow$ | $(\lambda_1', \lambda_2') = \downarrow \uparrow$ | $(\lambda_1', \lambda_2') = \downarrow \downarrow$ |
|--------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| $(\lambda_1, \lambda_2) = \uparrow \uparrow$ | $H_1(1, 2)$ | $H_3(2, 1)$ | $H_5^*(2, 1)$ | 0 |
| $(\lambda_1, \lambda_2) = \uparrow \downarrow$ | $H_3(2, 1)$ | $H_2^*(1, 2)$ | $H_4(2, 1)$ | 0 |
| $(\lambda_1, \lambda_2) = \downarrow \uparrow$ | $H_5^*(1, 2)$ | $H_4(1, 2)$ | $H_2(1, 2)$ | 0 |
| $(\lambda_1, \lambda_2) = \downarrow \downarrow$ | 0 | 0 | 0 | 0 |

Table 9: General helicity table defining the effective interaction in the annihilation channel.

These functions are displayed in the Table [3].

Here, the matrix elements $H_i^{(n)}(x, \vec{k}_\perp; x', \vec{k}'_\perp)$ are the following

\[
H_1^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -m^2 \left( \frac{1}{x} + \frac{1}{1 - x} \right) \left( \frac{1}{x'} + \frac{1}{1 - x'} \right)
\]

\[
H_2^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -k_\perp k'_\perp \left( \frac{e^{i(\varphi - \varphi')}}{xx'} \right)
\]

\[
H_3^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -m\lambda_1 \left( \frac{1}{x} + \frac{1}{1 - x} \right) \frac{k'_\perp}{1 - x'} e^{i\varphi}
\]

\[
H_4^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = k_\perp k'_\perp \left( \frac{e^{i(\varphi - \varphi')}}{x'(1 - x)} \right)
\]

(174)

and

\[
H_1^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = H_3^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = 0
\]

\[
H_2^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = H_4^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = 2
\]

(175)

D.2 The helicity table for $|J_z| \leq 1$

The matrix elements of the effective interaction for $J_z \geq 0$ $F(x, k_\perp; \lambda_1, \lambda_2|x', k'_\perp; \lambda'_1, \lambda'_2) = \langle x, k_\perp; J_z, \lambda_1, \lambda_2 | \hat{V}_{eff} | x', k'_\perp; J'_z, \lambda'_1, \lambda'_2 \rangle$ in the annihilation channel (the sum of the generated interaction for $J_z = +1$ and instantaneous graph for $J_z = 0$) are given in Table [3].

The function $F_i(1, 2) = F_i(x, k_\perp; x', k'_\perp)$ are the following

\[
F_1(x, k_\perp; x', k'_\perp) = \frac{\alpha}{\pi} \frac{1}{\Omega} \frac{m^2}{xx'(1 - x)(1 - x')} \delta_{|J_z|, 1}
\]

\[
F_2(x, k_\perp; x', k'_\perp) = \frac{\alpha}{\pi} \left( \frac{1}{\Omega} \frac{k_\perp k'_\perp}{xx'} \delta_{|J_z|, 1} + 2\delta_{J_z, 0} \right)
\]

\[
F_3(x, k_\perp; x', k'_\perp) = \frac{\alpha}{\pi} \frac{1}{\Omega} \frac{\lambda_1}{x'(1 - x')} \frac{m}{1 - x} k_\perp \delta_{|J_z|, 1}
\]
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
final:initial & \((\lambda_1', \lambda_2') = \uparrow \uparrow\) & \((\lambda_1', \lambda_2') = \uparrow \downarrow\) & \((\lambda_1', \lambda_2') = \downarrow \uparrow\) & \((\lambda_1', \lambda_2') = \downarrow \downarrow\) \\
\hline
\((\lambda_1, \lambda_2) = \uparrow \uparrow\) & \(F_1(1, 2)\) & \(F_3(2, 1)\) & \(F_3^*(2, 1)\) & 0 \\
\hline
\((\lambda_1, \lambda_2) = \uparrow \downarrow\) & \(F_3(1, 2)\) & \(F_2^*(1, 2)\) & \(F_4(2, 1)\) & 0 \\
\hline
\((\lambda_1, \lambda_2) = \downarrow \uparrow\) & \(F_3^*(1, 2)\) & \(F_4(1, 2)\) & \(F_2(1, 2)\) & 0 \\
\hline
\((\lambda_1, \lambda_2) = \downarrow \downarrow\) & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{Helicity table of the effective interaction in the annihilation channel for \(J_z \geq 0\).}
\end{table}

where we have introduced

(1) **Exponential cut-off**

\[
\frac{1}{\Omega} = \frac{1}{M^2},
\]

(2) **Gaussian cut-off**

\[
\frac{1}{\Omega} = \frac{M^2_a + M^2_b}{M^4_a + M^4_b},
\]

(3) **Sharp cut-off**

\[
\frac{1}{\Omega} = \frac{\theta(M^2_a - M^2_b)}{M^2_a} + \frac{\theta(M^2_b - M^2_a)}{M^2_b}.
\]

The table for \(J_z = -1\) is obtained by inverting all helicities, i.e.

\[
F(J_z = +1; \lambda_1, \lambda_2) = -\lambda_1 F(J_z = -1; -\lambda_1, -\lambda_2),
\]

The matrix elements of the effective interaction in the annihilation channel are nonzero only for \(|J_z| \leq 1\) due to the restriction on the angular momentum of the photon.

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