A zero density estimate and fractional imaginary parts of zeros for GL₂ \( L \)-functions

BY OLIVIA BECKWITH
Mathematics Department, Tulane University,
New Orleans, 6823 St. Charles Ave., LA 70118, U.S.A.
e-mail: obbeckwith@tulane.edu

DI LIU, JESSE THORNER
Department of Mathematics, University of Illinois at Urbana–Champaign,
1409 West Green Street, Urbana, IL 61801, U.S.A.
e-mails: dil4@illinois.edu, jesse.thorner@gmail.com

AND ALEXANDRU ZAHARESCU
Department of Mathematics, University of Illinois at Urbana–Champaign,
1409 West Green Street, Urbana, IL 61801, U.S.A.
AND Simion Stoilow Institute of Mathematics of the Romanian Academy,
P.O. Box 1-764, RO-014700 Bucharest, Romania.
e-mail: zaharesc@illinois.edu

(Received 14 September 2021; revised 10 October 2022; accepted 27 July 2022)

Abstract

We prove an analogue of Selberg’s zero density estimate for \( \zeta(s) \) that holds for any GL₂ \( L \)-function. We use this estimate to study the distribution of the vector of fractional parts of \( \gamma \alpha \), where \( \alpha \in \mathbb{R}^n \) is fixed and \( \gamma \) varies over the imaginary parts of the nontrivial zeros of a GL₂ \( L \)-function.

2020 Mathematics Subject Classification: 11M41 (Primary)

1. Introduction and statement of results

1.1. Main results

Let \( \zeta(s) \) be the Riemann zeta function, and let \( N(\sigma, T) \) denote the number of zeros \( \beta + i\gamma \) of \( \zeta(s) \) with \( \beta \geq \sigma \geq 0 \) and \( |\gamma| \leq T \). The asymptotic

\[
N(T) := N(0, T) \sim \frac{1}{\pi} T \log T
\]

(1.1)

follows from the argument principle. The Riemann hypothesis (RH) asserts that \( \zeta(s) \neq 0 \) for \( \Re(s) > 1/2 \), so \( N(\sigma, T) = 0 \) for \( \sigma > 1/2 \). Selberg [Tit86, theorem 9-19C] proved a delicate zero density estimate that recovers the upper bound in (1.1) at \( \sigma = 1/2 \), namely

\[
N(\sigma, T) \ll T^{1 - \frac{1}{2}(\sigma - \frac{1}{2})} \log T, \quad \sigma \geq \frac{1}{2}.
\]

(1.2)
(See Baluyot [Bal17, theorem 1.2.1] for an improvement.) Selberg’s estimate implies that

\[
\frac{1}{N(T)} \# \left\{ \rho = \beta + i\gamma : \xi(\rho) = 0, \ |\gamma| \leq T, \ \beta \in \left[ \frac{1}{2} - \frac{4 \log \log T}{\log T}, \frac{1}{2} + \frac{4 \log \log T}{\log T} \right] \right\} \ll \frac{1}{\log T}.
\]

As an application of (1.2), Selberg proved a central limit theorem for \( \log |\zeta(1/2 + it)| \):

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ T \leq t \leq 2T : \ \log |\zeta(1/2 + it)| \geq V \sqrt{\frac{1}{2} \log \log T} \right\} \sim \frac{1}{\sqrt{2\pi}} \int_{V}^{\infty} e^{-u^2/2} du. \quad (1.3)
\]

Let \( \mathcal{A} \) be the set of cuspidal automorphic representations of \( GL_2 \) over \( \mathbb{Q} \) with unitary central character. For \( \pi \in \mathcal{A} \), let \( L(s, \pi) \) be its standard \( L \)-function. Define

\[
N_{\pi}(\sigma, T) := \# \{ \rho = \beta + i\gamma : \beta \geq \sigma, \ |\gamma| \leq T, \ L(\rho, \pi) = 0\}.
\]

As with \( \zeta(s) \), the argument principle can be used to prove that

\[
N_{\pi}(T) := N_{\pi}(0, T) \sim (2/\pi)T \log T. \quad (1.4)
\]

The generalised Riemann hypothesis (GRH) asserts that \( L(s, \pi) \neq 0 \) for \( \text{Re}(s) > 1/2 \). Selberg [Sel92] observed that analogues of (1.2) and (1.3) should also hold for Hecke–Maaß newforms. When \( \pi \in \mathcal{A} \) corresponds with a holomorphic cuspidal newform of even weight \( k \geq 2 \), Luo [Luo95, theorem 1.1] and Li [FZ15, section 7] proved that \( N_{\pi}(\sigma, T) \ll T^{1-c (\sigma - 1/2)} \log T \). We prove:

**Theorem 1.1.** Let \( \theta \in [0, 7/64] \) be an admissible exponent toward the generalised Ramanujan conjecture for Hecke–Maaß newforms (see (2.1)), and fix \( 0 < c < 1/4 - \theta/2 \). If \( \pi \in \mathcal{A} \), \( \sigma \geq 1/2 \), and \( T \geq 2 \), then \( N_{\pi}(\sigma, T) \ll T^{1-c (\sigma - 1/2)} \log T \). The implied constant depends at most on \( \pi \).

**Remark 1.** Under the generalised Ramanujan conjecture for all Hecke–Maaß newforms, we may take \( \theta = 0 \). In this case, our result is as strong as Selberg’s zero density estimate (1.2) for \( \zeta(s) \). Currently, the best unconditional bound is \( \theta \leq 7/64 \), so we may choose any \( c < 25/128 \). This noticeably improves the work of Luo and Li, and it holds for any \( \pi \in \mathcal{A} \). The constant \( 1/4 - \theta/2 \) is, as of now, the supremum over all \( \sigma \) for which we can unconditionally prove an asymptotic for the second mollified moment of \( L(s, \pi) \) on \( \text{Re}(s) = 1/2 + 1/\log T \) with a mollifier of length \( T^{\sigma} \).

**Corollary 1.2.** Let \( \pi \in \mathcal{A} \). If \( V \in \mathbb{R} \), then as \( T \to \infty \), we have:

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ T \leq t \leq 2T : \ \log |L(1/2 + it, \pi)| \geq V \sqrt{\frac{1}{2} \log \log T} \right\} = \frac{1}{\sqrt{2\pi}} \int_{V}^{\infty} e^{-u^2/2} du;
\]

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ T \leq t \leq 2T : \ \text{arg} L(1/2 + it, \pi) \geq V \sqrt{\frac{1}{2} \log \log T} \right\} = \frac{1}{\sqrt{2\pi}} \int_{V}^{\infty} e^{-u^2/2} du.
\]

**Proof.** Bombieri and Hejhal [BH95, theorem B] proved that this follows from a zero density estimate of the quality given by Theorem 1.1.
Remark 2. Radziwiłł and Soundararajan [RS17] recently found a second proof of (1·3) which avoids the use of zero density estimates. Their work was recently extended to holomorphic newforms by Das [Das20]. The proof relies on both the generalised Ramanujan conjecture and the Sato–Tate conjecture, neither of which is known for any Hecke–Maaß newform.

We study the distribution of imaginary parts of the nontrivial zeros of \( L(s, \pi) \) using Theorem 1·1. First, we give a partial history of such results for \( \zeta(s) \). Hlawka [Hla75] proved that if \( \alpha \in \mathbb{R} \) is fixed and \( h : \mathbb{T} \to \mathbb{C} \) is continuous, then

\[
\lim_{T \to \infty} \frac{1}{N(T)} \sum_{|\gamma| \leq T} h(\alpha \gamma) = \int_{\mathbb{T}} h(t) dt,
\]

where \( \gamma \) varies over the imaginary parts of the nontrivial zeros of \( \zeta(s) \) and \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \). Thus the sequence of fractional parts \( \{ \gamma \alpha \} \) is equidistributed modulo 1. However, given a rate of convergence, there exist continuous functions \( h \) such that the limit in (1·5) cannot be attained with said rate (see also [FZ05, theorem 7]). Therefore, (1·5) is the best that one can say for arbitrary \( h \).

Ford and Zaharescu [FZ05, corollary 2] established the existence of a second order term, proving that if \( h : \mathbb{T} \to \mathbb{C} \) is twice continuously differentiable, then

\[
\sum_{|\gamma| \leq T} h(\alpha \gamma) = N(T) \int_{\mathbb{T}} h(t) dt + T \int_{\mathbb{T}} h(t) g_\alpha(t) dt + o(T),
\]

where

\[
g_\alpha(t) = \begin{cases} 
\frac{\log p}{\pi} \Re \sum_{k=1}^{\infty} e^{-2\pi i q kt} p^{\alpha k/2} & \text{if there exists a prime } p \text{ and } a, q \in \mathbb{Z} \text{ such that } gcd(a, q) = 1 \text{ and } \alpha = \frac{a \log p}{q 2\pi}, \\
0 & \text{otherwise.}
\end{cases}
\]

Despite the limitations on the analytic properties of \( h \), Ford and Zaharescu still conjectured [FZ05, conjecture A] that for any interval \( I \subseteq \mathbb{T} \) of length \( |I| \), we have

\[
\sum_{|\gamma| \leq T \atop \{\alpha \gamma\} \in I} 1 = |I| N(T) + T \int_I g_\alpha(t) dt + o(T),
\]

which implies that

\[
D_\alpha(T) := \sup_{I \subseteq \mathbb{T}} \left| \frac{1}{N(T)} \sum_{|\gamma| \leq T \atop \{\alpha \gamma\} \in I} 1 - |I| \right| = \frac{T}{N(T)} \sup_{I \subseteq \mathbb{T}} \left| \int_I g_\alpha(t) dt \right| + o\left( \frac{1}{\log T} \right).
\]

Ford, Soundararajan, and Zaharescu [FSZ09] made some progress toward the conjectured asymptotics (1·8) and (1·9). Unconditionally, they proved that

\[
D_\alpha(T) \geq \frac{T}{N(T)} \sup_{I \subseteq \mathbb{T}} \left| \int_I g_\alpha(t) dt \right| + o\left( \frac{1}{\log T} \right).
\]

1 On RH, absolute continuity suffices.
Assuming RH, they proved that

$$ \left| \sum_{\|\gamma\| \leq T} 1 - \|\gamma\| N(T) - T \int_{1}^{T} g_\alpha(t) \, dt \right| \leq \left( \frac{\alpha}{2} + o(1) \right) T. \quad (1.11) $$

Along with making some appealing connections between the conjectured asymptotics (1.8) and (1.9) and other intriguing open problems like pair correlation of zeros of $\xi(s)$ and the distribution of primes in short intervals, they proved analogues for other $L$-functions of (1.10) (assuming a zero density estimate of the form (1.2)) and (1.11) (assuming GRH).

In this paper, we extend the work in [FZ05, FSZ09, FMZ17, LZ21] to $L(s, \pi)$ for any $\pi \in \mathcal{A}$ using Theorem 1.1. Let $n \geq 1$. Consider the $\alpha \in \mathbb{R}^n$ for which there exists a constant $C_\alpha > 0$ such that

$$ |m \cdot \alpha| \geq C_\alpha e^{-\|m\|_2^2} \quad \text{for all } m \in \mathbb{Z}^n \setminus \{0\}, $$

where $\|m\|_p$ is the $\ell^p$ norm on $\mathbb{R}^n$ for $1 \leq p \leq \infty$. This is a technical artifact of our extension to $\mathbb{R}^n$; when $n = 1$, the condition reduces to $\alpha \neq 0$. Our density function $g_{\pi, \alpha}(t)$, which extends (1.7) for $n \geq 2$, is identically zero unless there exists a matrix $M = (b_{jk}) \in \mathcal{M}_{n \times n}(\mathbb{Z})$ with linearly independent row vectors $b_j$ and $\gcd(b_{j1}, \ldots, b_{jn}) = 1$ for all $1 \leq j \leq r$; fully reduced rationals $a_1/q_1, \ldots, a_r/q_r$; and distinct primes $p_1, \ldots, p_r$ such that

$$ M \alpha^\top = \begin{pmatrix} a_1 \log p_1/2\pi & \cdots & a_r \log p_r/2\pi \end{pmatrix}^{\top}. \quad (1.13) $$

Among such possible matrices $M$, choose one with maximal $r$, which uniquely determines the row vectors $b_j = (b_{j1}, \ldots, b_{jn})$. If such an $M$ exists, then define

$$ g_{\pi, \alpha}(t) := \frac{2}{\pi} \Re \sum_{j=1}^{r} \sum_{l=1}^{\infty} \frac{\Lambda_{\pi}(p_j^{a_{jl}})}{p_j^{a_{jl}/2}} e^{-2\pi i q_j(l-1) t}, \quad (1.14) $$

where

$$ \frac{L'(s, \pi)}{L(s, \pi)} = \sum_{n=1}^{\infty} \frac{\Lambda_{\pi}(n)}{n^s}, \quad \Re(s) > 1. $$

Let $\pi \in \mathcal{A}$, $n \geq 1$, and $B \subseteq \mathbb{T}^n$ be a product of $n$ subintervals of $\mathbb{T}$. The conjecture in (1.8) can be extended to $\pi \in \mathcal{A}$ as follows:

$$ \sum_{\|\gamma\| \leq T} 1 = \text{vol}(B) N_\pi(T) + T \int_{B} g_{\pi, \alpha}(t) \, dt + o(T), \quad (1.15) $$

where $\gamma$ ranges over the imaginary parts of the nontrivial zeros of $L(s, \pi)$. As progress toward (1.15), we prove an unconditional $n$-dimensional version of (1.6) for $\text{GL}_2$

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2The set of vectors $\alpha$ for which there exists $C_\alpha > 0$ such that (1.12) holds have full Lebesgue measure in $\mathbb{R}^n$ by work of Kemble [Kem05] and Khintchine [Khi24].
$L$-functions. In what follows, let $C^n(\mathbb{T}^n)$ be the set of $u$-times continuously differentiable functions $h: \mathbb{T}^n \to \mathbb{R}$. Let $\nu$ vary over the imaginary parts of the nontrivial zeros of $L(s, \pi)$.

**Theorem 1.3.** Let $\pi \in \mathcal{A}$. Let $\alpha \in \mathbb{R}^n$ satisfy (1.12). If $h \in C^{n+2}(\mathbb{T}^n)$, then

$$
\sum_{|\nu| \leq T} h(\nu \alpha) = N_\pi(T) \int_{\mathbb{T}^n} h(t) dt + T \int_{\mathbb{T}^n} h(t) g_{\pi, \alpha}(t) dt + o(T),
$$

where $dt$ is Lebesgue measure on $\mathbb{T}^n$. The implied constant depends on $\pi$, $h$, and $\alpha$.

**Remark 3.** With extra work, we may allow $h \in C^{n+1}(\mathbb{T}^n)$. Also, under some extra conditions on $\alpha$ which exclude a density zero subset of $\mathbb{R}^n$, we expect to further quantify the $o(T)$ error term in (1.16).

Define the discrepancy

$$
D_{\pi, \alpha}(T) := \sup_{B \subseteq \mathbb{T}^n} \left| \frac{1}{N_\pi(T)} \sum_{|\nu| \leq T, |\nu \alpha| \in B} 1 - \text{vol}(B) \right|.
$$

Our next result follows quickly from Theorem 1.3.

**Corollary 1.4.** Let $\pi \in \mathcal{A}$. If $\alpha \in \mathbb{R}^n$ satisfies (1.12), then

$$
D_{\pi, \alpha}(T) \geq \frac{T}{N_\pi(T)} \int_{\mathbb{T}^n} g_{\pi, \alpha}(t) dt + o\left(\frac{1}{\log T}\right).
$$

When $n = 1$, we recover an unconditional analogue of (1.10) for all $\pi \in \mathcal{A}$. This special case of Corollary 1.4 was proved in [FSZ09] under the hypothesis of a zero density estimate of the form proved in Theorem 1.1.

**Remark 4.** Let $d \geq 3$ be an integer, and let $L(s, \pi)$ be the standard $L$-function associated to a cuspidal automorphic representation $\pi$ of $GL_d$ over $\mathbb{Q}$. Let $N_\pi(\sigma, T)$ be the number of nontrivial zeros $\beta + i\nu$ of $L(s, \pi)$ with $\beta \geq \sigma$ and $|\nu| \leq T$. If there exist constants $c_\pi > 0$ and $d_\pi > 0$ such that $N_\pi(\sigma, T) \ll T^{1-c_\pi(\sigma-1)}(\log T)^{d_\pi}$, then one can prove an analogue of Theorem 1.3 and Corollary 1.4 for $L(s, \pi)$. Such an estimate for $N_\pi(\sigma, T)$ is not yet known for any $d \geq 3$, and it appears to be quite difficult to prove.

1.2. Application to “zero races”

In a letter to Fuss, Chebyshev observed that the generalised Riemann hypothesis implies that primes $p \equiv 3 \pmod{4}$ tend to be more numerous than primes $p \equiv 1 \pmod{4}$. Using the generalised Riemann hypothesis and other hypotheses, Rubinstein and Sarnak [RS94] began a systematic study of “prime number races” in which they determine how often $\pi(x; 4, 3) > \pi(x; 4, 1)$, where $\pi(x; q, a)$ equals $\#\{p \leq x: p = a \pmod{q}\}$. They proved that

$$
\lim_{X \to \infty} \frac{1}{\log X} \int_{\pi(r; 4, 3) > \pi(r; 4, 1)} \frac{dt}{t} = 0.9959 \ldots
$$

Thus the “bias” toward primes of the form $4n + 3$ is quite strong. The literature on prime number races which study such inequities is quite vast. See, for instance, the work of Fiorilli, Ford, Harper, Konyagin, Lamzouri and Martin [FM13, FK02, FLK13, FHL19].
We use Theorem 1.3 to study inequities not between primes in different residue classes, but between zeros of different \( L \)-functions. Define

\[
\mathcal{C}^T(\mathbb{T}^n) := \{ h \in C^T(\mathbb{T}^n) : h \text{ is nonnegative and not identically zero}\}.
\]

Let \( h \in \mathcal{C}^{n+2}(\mathbb{T}^n) \), and let \( \alpha \) satisfy (1.12). We consider two holomorphic cusp forms \( f_1 \) and \( f_2 \) with trivial nebentypus, where

\[
f_j(z) = \sum_{n=1}^{\infty} \lambda_{f_j}(n) n^{\frac{k_j-1}{2}} e^{2\pi i n z} \in S^\text{new}_{k_j}(\Gamma_0(q_j)), \quad j \in \{1, 2\}
\]

have trivial nebentypus, even integral weights \( k_j \geq 2 \) and squarefree levels \( q_j \geq 1 \). We assume that \( f_j \) is normalised so that \( \lambda_{f_j}(1) = 1 \) and that \( f_j \) is an eigenfunction of all of the Hecke operators. We call such cusp forms \( \text{newforms} \) (see [Ono04, section 2.5]). It is classical that there exists \( \pi_j \in \mathcal{A} \) such that \( L(s, f_j) = L(s, \pi_j) \), so \( \lambda_{f_j}(n) = \lambda_{\pi_j}(n) \), \( g_{f_j, \alpha}(t) = g_{\pi_j, \alpha}(t) \), etc.

We say \( f_1 \) wins the \((\alpha, h)\)-race (against \( f_2 \)) if for all large \( T \), we have

\[
\sum_{|\gamma_1| \leq T} h(\alpha, \gamma_1) > \sum_{|\gamma_2| \leq T} h(\alpha, \gamma_2),
\]

where \( \gamma_j \) runs over the imaginary parts of the nontrivial zeros of \( L(s, f_j) \). If neither \( f_1 \) nor \( f_2 \) wins the \((\alpha, h)\)-race, we say the race is undecided. The results of [LZ21] suggest that the winner of a decided \((\alpha, h)\)-race is determined by the levels \( q_1 \) and \( q_2 \) and the behavior of \( g_{f_1, \alpha} \) and \( g_{f_2, \alpha} \).

First, we show that if \( q_1 = q_2 \) and \( h \in \mathcal{C}^3(\mathbb{T}) \), then the proportion of primes \( p \) such that \( f_1 \) wins the \((\log p)/(2\pi), h)\)-race is \( 1/2 \).

**Corollary 1.5.** For \( j = 1, 2 \), let \( q_j \) be squarefree and \( f_j \in S^\text{new}_{k_j}(\Gamma_0(q_j)) \) be non-CM newforms. Suppose that \( f_1 \neq f_2 \otimes \chi \) for all primitive Dirichlet characters \( \chi \). If \( h \in \mathcal{C}^3(\mathbb{T}) \), then

\[
\frac{\#\{p \leq X : f_1 \text{ wins the } (\frac{\log p}{2\pi}, h)\text{-race}\}}{\#\{p \leq X\}} = \frac{1}{2} + O(\frac{\sqrt{\log \log \log X}}{(\log \log X)^{1/4}}).
\]

Next we look at the distribution of

\[
H(f_1, f_2, h, \alpha) := \lim_{T \to \infty} \frac{1}{T} \sum_{|\gamma_1| < T} h(\alpha, \gamma_1) - \sum_{|\gamma_2| < T} h(\alpha, \gamma_2) - \frac{1}{\pi} \log \frac{q_1}{q_2} \int_{\mathbb{T}} h(t) dt
\]

as \( \alpha \) varies over values of \( \log p/2\pi \) for prime values of \( p \). Given \( I \subseteq [-2, 2] \), let \( \mu_{ST}(I) \) be the Sato–Tate measure \( 1/(2\pi) \int_I \sqrt{4 - t^2} dt \), and let \( \mu_{ST,2} \) be the product measure defined on boxes \( I_1 \times I_2 \subseteq [-2, 2] \times [-2, 2] \) by \( \mu_{ST,2}(I_1 \times I_2) = \mu_{ST}(I_1) \mu_{ST}(I_2) \). For \( \mathcal{I} \subseteq [-4, 4] \), we define

\[
v(\mathcal{I}) := \mu_{ST,2}((x, y) \in [-2, 2]^2 : x - y \in \mathcal{I}).
\]

For \( h \in \mathcal{C}^3(\mathbb{T}) \), we set

\[
k_h := \int_0^1 h(t) \cos (2\pi t) dt.
\]
In the following statement and throughout the paper, for any interval $I$ and $\beta \in \mathbb{R}$, we let $\beta I = \{\beta x : x \in I \}$. 

**Corollary 1.6.** For $j = 1, 2$, let $q_j$ be squarefree and $f_j \in S_{kj}^{\text{new}}(\Gamma_0(q_j))$ be normalised holomorphic non-CM newforms with trivial nebentypus. Suppose that $f_1 \neq f_2 \otimes \chi$ for all primitive Dirichlet characters $\chi$. Let 

$$\varepsilon_X := \frac{(\log \log \log X)^{1/4}}{(\log \log X)^{1/8}}.$$ 

Assume $h \in \mathcal{C}^3(\mathbb{T})$ is such that $k_h \neq 0$. Then for any interval $I \subseteq [-4, 4]$, we have 

$$\frac{\#\{p \in [(1 - \varepsilon_X)X, X] : \frac{\log X}{\sqrt{X}} H(f_1, f_2, h, \frac{\log p}{2\pi}) \in I\}}{\#\{p \in [(1 - \varepsilon_X)X, X]\}} = v\left(\frac{\pi}{2k_h} I\right) + O(\varepsilon_X)$$

with an implied constant independent of $I$.

If $q_1 > q_2$, then in contrast to Corollary 1.5, $f_1$ wins the $((\log p)/(2\pi), h)$-race for all except finitely many primes $p$.

**Corollary 1.7.** For $j = 1, 2$, let $f_j \in S_{kj}^{\text{new}}(\Gamma_0(q_j))$ be normalised holomorphic newforms with trivial nebentypus. If $q_1 > q_2$, then for any $h \in \mathcal{C}^3(\mathbb{T})$, there are at most finitely many primes $p$ such that $f_2$ wins the $((\log p)/(2\pi)$-race.

Corollary 1.7 shows that it is rare for $f_1$ to win an $(\alpha, h)$ race against $f_2$ if $q_2 > q_1$, but we can show this occurs infinitely often. Our result can be stated neatly in terms of *local races* rather than in terms of the $(\alpha, h)$-races described above. For $t_0 \in \mathbb{T}$, we say that $f_1$ wins the local $(\alpha, t_0)$-race against $f_2$ if there exists a neighbourhood $U$ of $t_0 \in \mathbb{T}$ such that the $(\alpha, h)$-race is won by $f_1$ for all $h \in \mathcal{C}^3(\mathbb{T})$ which are supported on $U$.

**Corollary 1.8.** For $j = 1, 2$, let $q_j$ be squarefree and $f_j \in S_{kj}^{\text{new}}(\Gamma_0(q_j))$ be normalised holomorphic non-CM newforms with trivial nebentypus. Suppose that $f_1 \neq f_2 \otimes \chi$ for all primitive Dirichlet characters $\chi$. Fix $t_0 \in [0, 1)$. Let $q_1$ be sufficiently large, and let $q_2 \in (q_1, q_1 + q_1^{1/2}]$, and let $k_1$ and $k_2$ be fixed. There exists $\alpha \in \mathbb{R}$ such that $f_1$ wins the local $(\alpha, t_0)$-race against $f_2$.

In addition to Theorem 1.3, the proofs of these corollaries rely on a quantifiable understanding of the joint distribution of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ as $p$ varies over the primes. Such an understanding follows from the effective version of the Sato–Tate conjecture which counts the number of primes $p \leq X$ such that $(\lambda_{f_1}(p), \lambda_{f_2}(p)) \in I_1 \times I_2$ proved by the third author in [Tho21] (see Theorem 2.1 below). One can prove analogues of Corollaries 1.5, 1.6 and 1.8 for Dirichlet $L$-functions by replacing the effective Sato–Tate estimates with results on primes in arithmetic progressions.

**2. Preliminaries**

**2.1. GL$_2$ $L$-functions over $\mathbb{Q}$**

Let $\pi \in \mathcal{A}$. Here, we state the essential properties of $L$-functions of $L(s, \pi)$ that we use throughout our proofs. See [IK04, chapter 5] for a convenient summary. Given $\pi \in \mathcal{A}$ with
level \( q_\pi \), there exist suitable complex numbers \( \alpha_{1,\pi}(p) \) and \( \alpha_{2,\pi}(p) \) such that
\[
L(s, \pi) = \prod_{p \text{ prime}} \prod_{j=1}^{2} (1 - \alpha_{j,\pi}(p)p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}.
\]
The sum and product both converge absolutely for \( \text{Re}(s) > 1 \). There also exist spectral parameters \( \kappa_\pi(1) \) and \( \kappa_\pi(2) \) such that if we define
\[
L(s, \pi) = \prod_{\text{prime } p} \frac{1}{\frac{1}{2} \prod_{j=1}^{2} (1 - \alpha_{j,\pi}(p)p^{-s})^{-1}} = \sum_{n=1}^{\infty} \lambda_\pi(n)n^s.
\]
the completed \( L \)-function \( \Lambda(s, \pi) := \pi^{s/2}L(s, \pi)L(s, \pi) \) is entire of order 1.

Let \( \tilde{\pi} \in \mathcal{A} \) be the contragredient representation. We have \( \alpha_{j,\tilde{\pi}}(p) = \overline{\alpha_{j,\pi}(p)} \) and \( \kappa_{\tilde{\pi}}(j) = \overline{\kappa_{\pi}(j)} \) for \( j = 1, 2 \). Moreover, there exists a complex number \( W(\pi) \) of modulus 1 such that for all \( s \in \mathbb{C}, \)
\[
\Lambda(s, \pi) = W(\pi)\Lambda(1-s, \tilde{\pi}).
\]
Building on work of Kim and Sarnak [Kim03, appendix], Blomer and Brumley [BB11] proved that there exists \( \theta \in [0, 7/64] \) such that we have the uniform bounds
\[
\log_p |\alpha_{j,\pi}(p)|, -\text{Re}(\kappa_{\pi}(j)) \leq \theta.
\]
The generalised Ramanujan conjecture and the Selberg eigenvalue conjecture assert that \( \theta = 0 \).

The Rankin–Selberg \( L \)-function\(^3\)
\[
L(s, \pi \otimes \tilde{\pi}) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi \times \tilde{\pi}}(n)}{n^s} = \prod_{p|q_\pi} \prod_{j=1}^{2} \prod_{j'=1}^{2} (1 - \alpha_{j,\pi}(p)\overline{\alpha_{j',\pi}(p)p^{-s}})^{-1}, \quad \text{Re}(s) > 1
\]
factors as \( \zeta(s)L(s, \text{Ad}^2\pi) \), where the adjoint square lift \( \text{Ad}^2\pi \) is an automorphic representation of \( \text{GL}_3(\mathbb{A}_Q) \). Thus, \( L(s, \text{Ad}^2\pi) \) is an entire automorphic \( L \)-function. This fact and the bound \(|\lambda_{\pi}(n)|^2 \leq \lambda_{\pi \times \tilde{\pi}}(n) \) [JLW21, lemma 3.1] enable us to prove via contour integration that
\[
\sum_{n \leq X} |\lambda_{\pi}(n)|^2 \leq \sum_{n \leq X} \lambda_{\pi \times \tilde{\pi}}(n) \ll_{\pi} X.
\]
It follows from [IK04, theorem 5.42] applied to \( \zeta(s) \) and \( L(s, \text{Ad}^2\pi) \) that there exists an effectively computable constant \( c_{\pi} > 0 \) such that \( L(s, \pi \times \tilde{\pi}) \neq 0 \) in the region
\[
\text{Re}(s) \geq 1 - \frac{c_{\pi}}{\log(\|\text{Im}(s)\| + 3)}.
\]
\(^3\)The \( \approx \) suppresses the more complicated Euler factors at primes \( p|q_\pi \). Their explicit description does not arise in our proofs.
2.2. Holomorphic newforms

Many of our corollaries pertain specifically to $\pi \in \mathcal{A}$ corresponding to holomorphic newforms. As above, let

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{k-1}e^{2\pi i nz} \in S_k^{\text{new}}(\Gamma_0(q))$$

be a holomorphic cuspidal newform (normalised so that $\lambda_f(1) = 1$) of even integral weight $k \geq 2$, level $q \geq 1$, and trivial nebentypus. If $\pi_f \in \mathcal{A}$ corresponds with $f$, then $L(s, f) = L(s, \pi_f)$, $\lambda_f(n) = \lambda_{\pi_f}(n)$, etc.

For these newforms, it follows from Deligne’s proof of the Weil conjectures that the generalised Ramanujan conjecture holds. Thus, for $f$ a holomorphic cuspidal newform as above, we may take $\theta = 0$ in (2.1). Since we assume that $f$ has trivial central character, Deligne’s bound implies that there exists $\theta_p \in [0, \pi]$ such that

$$\lambda_f(p) = 2 \cos \theta_p.$$ 

The Sato–Tate conjecture, now a theorem due to Barnet-Lamb, Geraghty, Harris, and Taylor [BLGHT11], states that the sequence $(\theta_p)$ is equidistributed in the interval $[-2, 2]$ with respect to the measure $2/\pi (\sin t)^2 dt$. In other words, if $I \subseteq [0, \pi]$ is a subinterval, then

$$\lim_{X \to \infty} \frac{\#\{p \leq X : \theta_p \in I\}}{\#\{p \leq X\}} = \frac{1}{2} \int_I (\sin t)^2 dt; \quad \pi(X) := \#\{p \text{ prime} : p \leq X\}.$$ 

After a change of variables, this implies that for any interval $I \subseteq [-2, 2]$, we have

$$\lim_{X \to \infty} \frac{\#\{\lambda_f(p) \in I : p \leq X\}}{\#\{p \leq X\}} = \frac{1}{2\pi} \int_I \sqrt{4-t^2} dt =: \mu_{ST}(I). \quad (2.4)$$

A recent paper by Thorner [Tho21] provides both an unconditional and a GRH-conditional rate of convergence in (2.4).

Our corollaries of Theorem 1.3 require a natural refinement of the Sato–Tate conjecture. For $j = 1, 2$, let $f_j \in S_k^{\text{new}}(\Gamma_0(q_j))$ be a holomorphic cuspidal newform as above. Suppose that $f_1 \neq f_2 \otimes \chi$ for all primitive nontrivial Dirichlet characters $\chi$. Building on work of Harris [Har09], Wong [Won19] proved that the sequences $(\lambda_{f_1}(p))$ and $(\lambda_{f_2}(p))$ exhibit a joint distribution: If $I_1, I_2 \subseteq [-2, 2]$, then

$$\lim_{X \to \infty} \frac{\#\{p \leq X : \lambda_{f_1}(p) \in I_1, \lambda_{f_2}(p) \in I_2\}}{\#\{p \leq X\}} = \mu_{ST}(I_1)\mu_{ST}(I_2). \quad (2.5)$$

Our corollaries of Theorem 1.3 require a nontrivial unconditional bound on the rate of convergence in (2.5). Such a bound was recently proved in [Tho21, theorem 1.2].

**Theorem 2.1.** For $j = 1, 2$, let $q_j$ be squarefree and $f_j \in S_k^{\text{new}}(\Gamma_0(q_j))$ be a normalised holomorphic cuspidal newform with trivial nebentypus. Suppose that $f_1 \neq f_2 \otimes \chi$ for all primitive nontrivial Dirichlet character $\chi$. Let $I_1, I_2 \subseteq [-2, 2]$ be subintervals. There exists an absolute and effectively computable constant $c > 0$ such that
The lemma now follows from (2.1).

COROLLARY 2.2. Let $f_1$ and $f_2$ be as in Theorem 2.1. Recall the definition of $v$ in (1.17). If $\mathcal{I} \subseteq [-4, 4]$, then

$$\frac{\# \{ p \leq X : \lambda_{f_1}(p) - \lambda_{f_2}(p) \in \mathcal{I} \}}{\# \{ p \leq X \}} = v(\mathcal{I}) + O\left(\frac{\sqrt{\log \log \log X}}{(\log \log X)^{1/4}}\right),$$

with an implied constant independent of $\mathcal{I}$.

Proof. Let $g(X) := \frac{(\log \log X)^{1/4}}{\sqrt{\log \log X}}$, $S := \{(x, y) \in [-2, 2]^2 : x - y \in \mathcal{I}\}$.

We will define rectangles whose unions approximate $S$. Let $x_j = -2 + j[4/g(X)]$. If $\mathcal{I} = (c, d)$, set $R_j = [x_j, x_j+1] \times [x_j - d, x_j+1 - c]$, and similarly $T_j := [x_j, x_j+1] \times [x_j+1 - d, x_j - c]$. By construction, we have $\bigcup T_j \subseteq S \subseteq \bigcup R_j$, which implies that

$$\frac{\# \{ p \leq X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in \bigcup T_j \}}{\# \{ p \leq X \}} \leq \frac{\# \{ p \leq X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in S \}}{\# \{ p \leq X \}} \leq \frac{\# \{ p \leq X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in \bigcup R_j \}}{\# \{ p \leq X \}}. \quad (2.6)$$

We apply Theorem 2.1 to count the primes $p$ with $(\lambda_{f_1}(p), \lambda_{f_2}(p))$ in $\bigcup R_j$.

$$\frac{\# \{ p \leq X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in \bigcup R_j \}}{\# \{ p \leq X \}} = \sum_{j=1}^{g(X)} \left( \mu_{ST,2}(R_j) + O\left(\frac{\log \log \log X}{\sqrt{\log \log X}}\right) \right) = \mu_{ST,2}(\bigcup R_j) + O\left(\frac{g(X) \log \log \log X}{\sqrt{\log \log X}}\right).$$

Since the area of $\bigcup R_j \setminus S$ is at most $g(X)[4/g(X)]^2$, we have

$$\mu_{ST,2}(\bigcup R_j) - \mu_{ST,2}(S) = \mu_{ST,2}(\bigcup R_j \setminus S) = O(g(X)^{-1}),$$

and we conclude that

$$\frac{\# \{ p \leq X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in \bigcup R_j \}}{\# \{ p \leq X \}} = v(\mathcal{I}) + O\left(g(X)^{-1} + g(X) \frac{\log \log \log X}{\sqrt{\log \log X}}\right) = v(\mathcal{I}) + O\left(\frac{\sqrt{\log \log \log X}}{(\log \log X)^{1/4}}\right). \quad (2.7)$$

The same argument shows that (2.7) holds with $R_j$ replaced with $T_j$ on the left-hand side. The lemma now follows from (2.6).

Finally, we require a refinement of (1.4), namely

$$N_f(T) = T \log \left( q_f \left( \frac{T}{2\pi e} \right)^2 \right) + O(\log (k_f q_f T)). \quad (2.8)$$

This is [IK04, theorem 5.8] applied to $L(s, f)$.
A zero density estimate for GL₂ L-functions

3. Proof of Theorem 1.1

Let \( \pi \in \mathcal{A} \). We detect the zeros of \( L(s, \pi) \) by estimating a mollified second moment of \( L(s, \pi) \) near the line \( \text{Re}(s) = 1/2 \). To describe our mollifier, we define \( \mu_\pi(n) \) by the convolution identity

\[
\sum_{d|n} \mu_\pi(d) \lambda_\pi(n/d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1 \end{cases}
\]

so that

\[
\frac{1}{L(s, \pi)} = \sum_{n=1}^\infty \frac{\mu_\pi(n)}{n^s}, \quad \text{Re}(s) > 1.
\]

Let \( T > 0 \) be a large parameter, and let \( 0 < \sigma < 1/4 \). Define \( P(t) \) by

\[
P(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq T^{\sigma}/2, \\ 2(1 - \frac{\log t}{\log T^{\sigma}}) & \text{if } T^{\sigma}/2 < t \leq T^{\sigma}, \\ 0 & \text{if } t > T^{\sigma}. \end{cases}
\]

Our mollifier is

\[
M_\pi(s, T^{\sigma}) = \sum_{n \leq T^{\sigma}} \frac{\mu_\pi(n)}{n^s} P(n).
\]

As a proxy for detecting zeros of \( L(s, \pi) \) near \( \text{Re}(s) = 1/2 \), where \( L(s, \pi) \) oscillates wildly, we detect zeros of the mollified L-function \( L(s, \pi)M_\pi(s, T^{\sigma}) \) near \( \text{Re}(s) = 1/2 \). To this end, we let \( w : \mathbb{R} \to \mathbb{R} \) be an infinitely differentiable function whose support is a compact subset of \( [T/4, 2T] \) and whose \( j \)-th derivative satisfies \( |w^{(j)}(t)| \ll_{w,j} ((\log T)/T)^j \) for all \( j \geq 0 \). Also, let \( w(t) = 1 \) for \( t \in [T/2, T] \). We will estimate

\[
I_f(\alpha, \beta) = \int_{-\infty}^\infty w(t)L\left(\frac{1}{2} + \alpha + it, \pi\right)L\left(\frac{1}{2} + \beta - it, \tilde{\pi}\right)|M_\pi\left(\frac{1}{2} + \frac{1}{\log T} + it, T^{\sigma}\right)|^2 dt,
\]

eventually choosing \( \alpha \) and \( \beta \) to equal \( 1/\log T \). Define

\[
G(s) = e^{\frac{(\alpha + \beta)^2 - (2s)^2}{(\alpha + \beta)^2}},
\]

\[
g_{\alpha, \beta}(s, t) = \frac{L\left(\frac{1}{2} + \alpha + s + it, \pi_{\infty}\right)L\left(\frac{1}{2} + \beta + s - it, \tilde{\pi}_{\infty}\right)}{L\left(\frac{1}{2} + \alpha + it, \pi_{\infty}\right)L\left(\frac{1}{2} + \beta - it, \tilde{\pi}_{\infty}\right)},
\]

\[
V_{\alpha, \beta}(s, t) = \frac{1}{2\pi i} \int_{\text{Re}(s)=1} \frac{G(s)}{s} g_{\alpha, \beta}(s, t) x^{-s} ds,
\]

\[
X_{\alpha, \beta}(t) = \frac{L\left(\frac{1}{2} - \alpha - it, \pi_{\infty}\right)L\left(\frac{1}{2} - \beta + s + it, \tilde{\pi}_{\infty}\right)}{L\left(\frac{1}{2} + \alpha + it, \pi_{\infty}\right)L\left(\frac{1}{2} + \beta - it, \tilde{\pi}_{\infty}\right)}.
\]

As in [Ber15, AT21], it follows from the approximate functional equation (see [IK04, section 5.2] also) that

\[
I_f(\alpha, \beta) = \sum_{a,b \leq T^{\sigma}} \frac{\mu_\pi(a)\mu_\pi(b)}{\sqrt{ab}} \frac{P(a)P(b)}{(ab)^{\frac{1}{2\sigma}}} \left(D_{a,b}(\alpha, \beta) + D_{a,b}^-(\alpha, \beta) + N_{a,b}^+(\alpha, \beta) + N_{a,b}^-(\alpha, \beta)\right),
\]
where we have split the sum into diagonal terms

\[ D_{a,b}^+(\alpha, \beta) = \sum_{am=bn} \frac{\lambda_\pi(m)\lambda_\pi(n)}{m^{\frac{1}{2}+\alpha}n^{\frac{1}{2}+\beta}} \int_{-\infty}^{\infty} w(t) V_{\alpha,\beta}(mn, t) \, dt, \]

\[ D_{a,b}^-(\alpha, \beta) = \sum_{am=bn} \frac{\lambda_\pi(m)\lambda_\pi(n)}{m^{\frac{1}{2}-\beta}n^{\frac{1}{2}-\alpha}} \int_{-\infty}^{\infty} w(t) X_{\alpha,\beta}(t) V_{-\beta,-\alpha}(mn, t) \, dt \]

and off-diagonal terms

\[ N_{a,b}^+(\alpha, \beta) = \sum_{am\neq bn} \frac{\lambda_\pi(m)\lambda_\pi(n)}{m^{\frac{1}{2}+\alpha}n^{\frac{1}{2}+\beta}} \int_{-\infty}^{\infty} w(t) \left( \frac{bn}{am} \right)^{it} V_{\alpha,\beta}(mn, t) \, dt, \]

\[ N_{a,b}^-(\alpha, \beta) = \sum_{am\neq bn} \frac{\lambda_\pi(m)\lambda_\pi(n)}{m^{\frac{1}{2}-\beta}n^{\frac{1}{2}-\alpha}} \int_{-\infty}^{\infty} w(t) \left( \frac{bn}{am} \right)^{it} X_{\alpha,\beta}(t) V_{-\beta,-\alpha}(mn, t) \, dt. \]

**Lemma 3.1.** Let \( \varepsilon > 0 \). If \( \alpha, \beta \in \mathbb{C} \) satisfy \( |\alpha|, |\beta| \ll 1/\log T \) and \( |\alpha + \beta| \gg 1/\log T \), then for any integers \( a, b \geq 1 \), we have that \( N_{a,b}^\pm(\alpha, \beta) \ll \varepsilon (ab)^{\frac{1}{2}}T^{1/2+\theta}(abT)^\varepsilon \).

**Proof.** This is [AT21, proposition 3.4].

**Corollary 3.2.** Fix \( 0 < \sigma < 1/4 - \theta/2 \). There exists a constant \( \delta > 0 \) such that

\[ \left| \sum_{a,b \leq T^\sigma} \frac{\mu_\pi(a)\mu_\pi(b)}{\sqrt{ab}} \frac{P(a)P(b)}{(ab)^{1/2}} (N_{a,b}^+(\alpha, \beta) + N_{a,b}^-(\alpha, \beta)) \right| \ll T^{1-\delta}. \]

**Proof.** Let \( \varepsilon > 0 \). First, observe that \( |\mu_\pi(a)| \ll 1 + |\lambda_\pi(a)| \ll 1 + |\lambda_\pi(a)|^2 \). We then apply Lemma 3.1 and bound everything else trivially to obtain

\[ \sum_{a,b \leq T^\sigma} \frac{\mu_\pi(a)\mu_\pi(b)}{\sqrt{ab}} \frac{P(a)P(b)}{(ab)^{1/2}} \sum_{\pm} N_{a,b}^\pm(\alpha, \beta) \ll T^{1/2+\theta+\varepsilon} \left( \sum_{a \leq T^\sigma} \sum_{a \leq T^\sigma} |\lambda_\pi(a)|^2 \right)^2. \]

By (2.2), this is \( \ll f,T^\frac{1}{2}+\theta+2\sigma+\varepsilon \). If \( \sigma \leq 1/4 - \theta/2 - \varepsilon \), then the above display is \( \ll T^{1-\varepsilon} \).

**Proposition 3.3.** If \( T \) is sufficiently large and \( \sigma \in (0, 1/4 - \theta/2) \) is fixed, then

\[ \int_{T/2}^T \left| L\left( \frac{1}{2} + \frac{1}{\log T} + it, \pi \right) M_\pi\left( \frac{1}{2} + \frac{1}{\log T} + it, T^{\sigma} \right)^2 \right| \, dt \ll T. \]

**Proof.** In light of Corollary 3.2, it remains to bound the diagonal contribution. We first note by a calculation identical to [Ber15, lemma 11] that

\[ \sum_{a,b \leq T^\sigma} \frac{\mu_\pi(a)\mu_\pi(b)}{\sqrt{ab}} \frac{P(a)P(b)}{(ab)^{1/2}} (D_{a,b}^+(\alpha, \beta) + D_{a,b}^-(\alpha, \beta)) \]

\[ = \sum_{a,b \leq T^\sigma} \frac{\mu_\pi(a)\mu_\pi(b)}{\sqrt{ab}} \frac{P(a)P(b)}{(ab)^{1/2}} (D_{a,b}^+(\alpha, \beta) + T^{-2(\alpha+\beta)}D_{a,b}^+(\alpha, \beta) - \alpha, -\beta)) + O\left( \frac{T}{\log T} \right). \]
By a computation identical to \( \sum_{a,b} \frac{\mu(a)\mu(b)}{\alpha} P(a)P(b) \frac{1}{ab} D_a^+(\alpha, \beta) \) with \( \alpha = \beta = 1/\log T \).

Let \( \sigma_0 = 1/2 + 1/\log T \). We observe via the Mellin inversion that \( I^D(\alpha, \beta) \) equals

\[
\frac{4}{(2\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{1} \left( \frac{T^{\sigma(u+v)}}{2 - (T^{\sigma(u+v)})} - 1 \right) G(s) g_{\alpha, \beta}(s, t) \times \sum_{a,b,m,n \geq 1 \atop \alpha, m = \infty} \frac{\mu(a)\mu(b)\lambda(m)\lambda(n)}{a^{\sigma_0+v}b^{\sigma_0+u}m^{1/2+\alpha+s}n^{1/2+\beta+s}} du dv dt.
\]

By a computation identical to [Ber15, lemma 6], there exists a product of half-planes containing an open neighborhood of the point \( u = \nu = s = 0 \) and an Euler product \( A_{\alpha, \beta}(u, \nu, s) \), absolutely convergent for \( (u, \nu, s) \) in said product of half-planes, such that

\[
\sum_{a,b,m,n \geq 1 \atop \alpha, m = \infty} \frac{\mu(a)\mu(b)\lambda(m)\lambda(n)}{a^{\sigma_0+v}b^{\sigma_0+u}m^{1/2+\alpha+s}n^{1/2+\beta+s}} = A_{\alpha, \beta}(u + \frac{1}{\log T}, \nu + \frac{1}{\log T}, s)
\]

\[
L(1 + \alpha + \beta + 2s, \pi \otimes \tilde{\pi}) L(1 + \frac{2}{\log T} + u + \nu, \pi \otimes \tilde{\pi})
\]

\[
L(1 + \frac{1}{\log T} + \alpha + u + s, \pi \otimes \tilde{\pi}) L(1 + \frac{1}{\log T} + \beta + v + s, \pi \otimes \tilde{\pi}).
\]

By Möbius inversion and a continuity argument, one can prove that \( A_{0,0}(0,0,0) = 1 \) [Ber15, lemma 7].

Upon choosing \( \delta \) sufficiently small and shifting the contours to \( \text{Re}(u) = \delta, \text{Re}(\nu) = \delta \), and \( \text{Re}(s) = -\delta/2 \), we find that

\[
I^D(\alpha, \beta) = \frac{4L(1 + \alpha + \beta, \pi \otimes \tilde{\pi})}{(2\pi i)^2} \int_{\mathbb{R}} T^{(u+v)} dt \int_{(\delta)} \frac{T^{(u+v)}}{2 - (T^{(u+v)})} - 1 \right) G(s) g_{\alpha, \beta}(s, t) \times \sum_{a,b,m,n \geq 1 \atop \alpha, m = \infty} \frac{\mu(a)\mu(b)\lambda(m)\lambda(n)}{a^{\sigma_0+v}b^{\sigma_0+u}m^{1/2+\alpha+s}n^{1/2+\beta+s}} du dv dt.
\]

The contribution from the pole at \( u = \nu = 0 \) determines the magnitude of the double integral over \( u \) and \( v \), and this magnitude is \( \asymp (\log T)^2 \) for \( \alpha \) and \( \beta \) in our prescribed range. (This follows once we push the \( u- \) and \( v- \)contours to the left using (2.3).) Since \( \int_{\mathbb{R}} T^{u} dt \asymp T \) by hypothesis, we combine all preceding contributions and choose \( \alpha = \beta = 1/\log T \) to conclude the desired bound.

**Corollary 3.4.** If \( T \) is sufficiently large and \( \sigma \in (0, 1/4 - \theta/2) \) is fixed, then

\[
\int_{T/2}^T |L(\frac{1}{2} + \frac{1}{\log T} + it, \pi)M_{\pi}(\frac{1}{2} + \frac{1}{\log T} + it, T^{\sigma}) - 1|^2 dt \ll T.
\]

**Proof.** This follows from Proposition 3.3 and the Cauchy–Schwarz inequality.

We prove a corresponding estimate on a vertical line to the right of \( \text{Re}(s) = 1 \).
Lemma 3.5. If $\sigma \in (0, 1/4 - \theta/2)$ and $A > \theta$ are fixed and $T$ is sufficiently large, then

$$\int_{T/2}^{T} |L(1 + A + it, \pi)M_{\pi}(1 + A + it, T^{\sigma}) - 1|^2 dt \ll T^{1 + (\theta - A - \frac{1}{2})\sigma} (\log T)^{15}.$$  

Proof. Let $A > \theta$. It follows immediately from [MV74, corollary 3] that if $T > 1$ and $(b_{n})$ is any sequence of complex numbers satisfying $\sum_{n} n|b_{n}|^2 < \infty$, then

$$\int_{T/2}^{T} \left| \sum_{n=1}^{\infty} b_{n}n^{-it} \right|^2 dt = \sum_{n=1}^{\infty} |b_{n}|^2 (T/2 + O(n)).$$  \hspace{1cm} (3.4)

We define $a_{n}$ by the identity

$$L(1 + A + it, \pi)M_{\pi}(1 + A + it, T^{\sigma}) - 1 = \sum_{n=1}^{\infty} a_{n}n^{-1 - A - it}.$$  

By (3.1), (3.2), and (3.3), we have that $a_{n} = 0$ for all $n \leq T^{\sigma}/2$ and $|a_{n}|^2 \ll (n^\theta d_{4}(n))^2 \ll n^{2\theta} d_{16}(n)$, where $d_{k}(n)$ is the $n$-th Dirichlet coefficient of $\zeta(s)^k$. The desired result follows once we apply (3.4) with $b_{n} = a_{n}n^{-1 - A}$.

Proof of Theorem 1.1. We use Gabriel’s convexity principle [Tit86, section 7.8] to interpolate the bounds in Corollary 3.4 and Lemma 3.5. In particular, if $0 < \sigma < 1/4 - \theta/2$ and $A > \theta$ are fixed and $c = \sigma(1 + 2A - 2\theta)/(1 + 2A)$, then in the range $1/2 + 1/\log T \leq \sigma \leq 1 + A$, we have

$$\int_{T/2}^{T} |L(\sigma + it, \pi)M_{\pi}(\sigma + it, T^{\sigma}) - 1|^2 dt \ll T^{1+\frac{A-\sigma}{(1 + A - \frac{1}{2})\log T}} (1 + \log T)^{15} T^{-c \sigma - \frac{1}{2}}.$$  \hspace{1cm} (3.5)

Define $\Phi(s) := 1 - (1 - L(s, \pi)M_{\pi}(s, T^{\sigma}))^2$. By construction, if $\alpha \in \mathbb{C}$, then

$$\operatorname{ord}_{s=\alpha} \Phi(s) \geq \operatorname{ord}_{s=\alpha} L(s, \pi)M_{\pi}(s, T^{\sigma}).$$

For any $M \geq 1$, let $C_{M}$ be the rectangular contour with corners $\sigma + iT/2, \sigma + iT, M + iT/2,$ and $M + iT$. Applying Littlewood’s lemma ([Tit58, pp. 132–133]) and letting $M \to \infty$, we have

$$\int_{\sigma}^{1} (N_{\pi}(\sigma’, T) - N_{\pi}(\sigma’, T/2))d\sigma’ \leq \lim_{M \to \infty} \frac{1}{2\pi} \int_{C_{M}} \log \Phi(s) ds$$

$$= \frac{1}{2\pi} \int_{T/2}^{T} \log |\Phi(\sigma + it)| dt + \frac{1}{2\pi} \int_{\sigma}^{\infty} \arg (\Phi(x + iT)) dx - \frac{1}{2\pi} \int_{\sigma}^{\infty} \arg (\Phi(x + \frac{it}{2})) dx.$$  

In view of the bound $\log |1 + z| \leq |z|$, it follows from (3.5) that

$$\frac{1}{2\pi} \int_{T/2}^{T} \log |\Phi(\sigma + it)| dt \leq \int_{T/2}^{T} |L(\sigma + it, \pi)M_{\pi}(\sigma + it, T^{\sigma}) - 1|^2 dt \ll T^{1 - c(\sigma - \frac{1}{2})}.$$
For the second and third integrals, we consider the integrals over $x \leq 1$ and over $x > 1$ separately. For $x \in (1, \infty)$ we can trivially bound the Dirichlet series as

$$|1 - L(x + iT, \pi)M_{\pi}(x + iT, T^s)| \leq \sum_{n=2}^{\infty} |a_n|n^{-1} \ll 2^{-x},$$

where $(a_n)$ is a certain sequence of complex numbers such that $|a_n| \ll \epsilon n^{\sigma+\epsilon}$ for any fixed $\epsilon > 0$ and all $n \geq 2$. Thus, we have

$$|\arg (1 - (1 - L(x + iT, \pi)M_{\pi}(x + iT, T^s)^2))| \ll 2^{-x},$$

so

$$\left|\int_1^\infty \arg (\Phi(x + iT))dx\right|, \left|\int_1^\infty \arg (\Phi(x + 2iT))dx\right| \ll 1.$$

To handle the integrals for $1/2 + 1/\log T \leq x \leq 1$, we use the trivial bound

$$\left|\int_{\sigma}^1 \arg (\Phi(x + 2iT))dx\right| \leq (1 - \sigma) \max_{\sigma \leq x \leq 1} |\arg (\Phi(x + 2iT))| \ll \log T.$$

A proof of the final bound is contained within the proof of [IK04, theorem 5.8]. The corresponding integral for $\Phi(x + iT)$ has the same bound.

By the preceding work, it follows by dyadic decomposition that

$$\int_{\sigma}^1 N_{\pi}(\sigma', T)d\sigma' \ll T^{1-c(\sigma - \frac{1}{2})}, \quad \frac{1}{2} + \frac{1}{\log T} \leq \sigma \leq 1.$$

This estimate, the mean value theorem for integrals, and the fact that $N_{\pi}(\sigma, T)$ is monotonically decreasing as $\sigma$ increases together imply that

$$N_{\pi}(\sigma, T) \leq \frac{1}{\sigma - (\sigma - \frac{1}{\log T})} \int_{\sigma - \frac{1}{\log T}}^{\sigma} N_{\pi}(\sigma', T)d\sigma' \ll \frac{1}{\sigma - (\sigma - \frac{1}{\log T})} \int_{\sigma - \frac{1}{\log T}}^{1} N_{\pi}(\sigma', T)d\sigma' \ll T^{1-c(\sigma - \frac{1}{2})} \log T.$$

If $1/2 \leq \sigma \leq 1/2 + 1/\log T$, then (1.4) implies that $N_{\pi}(\sigma, T) \ll T \log T \times T^{1-c(\sigma - 1/2)} \log T$.

To finish the proof, note that if $\theta = 0$, then for all $A > 0$, we have $c = \omega$. If $\theta > 0$, then fix $0 < \epsilon < \theta \sigma / (\theta + 1/2)$ and choose $A = \theta \sigma / \epsilon - 1/2$. With these choices, we find that $c > \omega - \epsilon$. Theorem 1.1 now follows.

4. Proof of Theorem 1.3

We begin with a few preliminary lemmas. Throughout the section, $\theta$ is an admissible exponent toward the generalised Ramanujan conjecture as in Theorem 1.1. Our first result is an $n$-dimensional version of the Riemann–Lebesgue lemma.

**Lemma 4.1.** Let $J \geq 1$. Suppose that $h \in C^{n+2}(\mathbb{T}^n)$ has the Fourier expansion

$$h(t) = \sum_{m \in \mathbb{Z}^n} c_m e^{2\pi i m \cdot t}.$$
We have $|c_m| \ll h \|m\|_2^{-n-2}$, and consequently, we have

$$h(t) = \sum_{m \in \mathbb{Z}^n \atop \|m\|_2 \leq J} c_m e^{2\pi i (m \cdot t)} + O(h(J^{-2})).$$

Proof. We have

$$c_m = \int_{\mathbb{T}^n} h(t) e^{-2\pi i (m \cdot t)} dt. \tag{4.1}$$

Let $m = (m_1, m_2, \ldots, m_n)$. Choose $j \in \{1, \ldots, n\}$ such that $|m_j| = \|m\|_{\infty}$. Integrate (4.1) by parts for $n + 2$ times with respect to the coordinate $t_j$ of $t$ so that

$$c_m = \frac{(-1)^{n+2}}{(2\pi m_j)^{n+2}} \int_{\mathbb{T}^n} \left( \frac{\partial^{n+2}}{\partial t_j^{n+2}} h(t) \right) e^{-2\pi i (m \cdot t)} dt. \tag{4.2}$$

Since $\sqrt{n}\|m\|_{\infty} \geq \|m\|_2$, the desired result follows from the triangle inequality.

Lemma 4.2. Let $x > 1$ and $T \geq 2$, and let $\langle x \rangle$ being the closest integer to $x$. We have

$$\sum_{|\gamma| \leq T} x^{\gamma} = -\frac{\Lambda_\pi(\langle x \rangle)}{\pi} \cdot e^{T \log(x/\langle x \rangle)} - \frac{1}{i \log(x/\langle x \rangle)} + O(x^{1+\theta}(\log(2x) + \log T) + \log T/\log x).$$

Proof. This is [FSZ09, lemma 2] with $\epsilon = \theta$.

Using Theorem 1.1 and Lemma 4.2, we prove an analogue of [FZ05, (3.8)].

Lemma 4.3. Let $c$ be as in Theorem 1.1. If $1 < x < \exp((c/3)(\log T/\log \log T))$, then

$$\sum_{|\gamma| \leq T} x^{i\gamma} = \sum_{|\gamma| \leq T} x^{\gamma - \frac{1}{2}} + O\left(\frac{T(\log x)^2}{\log T} + \frac{T}{(\log T)^2}\right).$$

Proof. Let $\delta = ((3/c)(\log \log T/\log T))$, so $0 < \delta \log x < 1$. By Theorem 1.1, we have that

$$\left| \sum_{|\gamma| \leq T \atop |\beta - \frac{1}{2}| \geq \delta} (x^{i\gamma} - x^{\gamma - \frac{1}{2}}) \right| \ll \sum_{|\gamma| \leq T \atop \beta \geq \frac{1}{2} + \delta} x^{\beta - \frac{1}{2}} \ll x^\delta N_{\pi}(\frac{1}{2} + \delta, T)$$

$$+ \log x \int_{\frac{1}{2} + \delta}^1 x^{\sigma - \frac{1}{2}} N_{\pi}(\sigma, T) d\sigma \ll \frac{T}{(\log T)^2}.$$

By the functional equation for $L(s, \pi)$, $\beta + i\gamma$ is a nontrivial zero if and only if $1 - \beta + i\gamma$ is a nontrivial zero. Therefore, we have

$$\left| \sum_{|\gamma| \leq T \atop |\beta - \frac{1}{2}| < \delta} (x^{i\gamma} - x^{\gamma - \frac{1}{2}}) \right|$$

$$= \left| \sum_{|\gamma| \leq T \atop \frac{1}{2} < \beta < \frac{1}{2} + \delta} (x^{i\gamma} (1 - x^{\beta - \frac{1}{2}}) + x^{i\gamma} (1 - x^{\frac{1}{2} - \beta}) \right|$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press
The desired result follows.

By Lemma 4

\[
|x^{i\gamma} (1 - x^{-\beta - \frac{1}{2}}) + x^{i\gamma} (1 - x^{-(\beta - \frac{1}{2})})| = \sum_{|\gamma| \leq T \atop 0 < \beta - \frac{1}{2} < \delta} (x^{\beta - \frac{1}{2}} + x^{-(\beta - \frac{1}{2})} - 2). \tag{4.3}
\]

Note that \(x^{\beta - \frac{1}{2}} + x^{-(\beta - \frac{1}{2})} - 2 = (2 \sinh (1/2(\beta - 1) \log x))^2\). Note that if \(0 < \beta - 1/2 < \delta\), then \(0 < (1/2)(\beta - 1/2) \log x < 1/2\). Since \(y < \sinh (y) < 2y\) for \(0 < y < 1/2\), (4.3) is

\[
\ll (\log x)^2 \sum_{|\gamma| \leq T \atop 0 < \beta - \frac{1}{2} < \delta} (\beta - \frac{1}{2})^2 \ll (\log x)^2 \int_0^\delta \sigma N\pi(\frac{1}{2} + \sigma, T)d\sigma \ll \frac{T(\log x)^2}{\log T}.
\]

The desired result follows.

**Proof of Theorem 1.3.** Let \(h \in C^{n+2} (\mathbb{T}^n)\), and let \(\alpha\) satisfy \((1.12)\). Let \(J \in [1, 100 \log (eT)]\). We begin with the expansion

\[
\sum_{|\gamma| \leq T} h(\gamma \alpha) = \sum_{|\gamma| \leq T} \sum_{m \in \mathbb{Z}^n} c_m e^{2\pi i \gamma (m \alpha)} = N\pi(T) \int_{\mathbb{T}^n} h(t)dt + \sum_{|\gamma| \leq T} \left( \sum_{1 \leq |m|_2 \leq J \atop |m|_2 > J} c_m e^{2\pi i \gamma (m \alpha)} \right).
\]

By Lemma 4.1, we have

\[
\sum_{|\gamma| \leq T} h(\gamma \alpha) - N\pi(T) \int_{\mathbb{T}^n} h(t)dt = \sum_{|\gamma| \leq T} \sum_{1 \leq |m|_2 \leq J} c_m e^{2\pi i \gamma (m \alpha)} + O\left(\frac{N\pi(T)}{J^2}\right). \tag{4.4}
\]

Write \(x_m = e^{2\pi (m \alpha)}\). Since \(x_{-m}^{i\gamma} = x_m^{i\gamma}\) and \(c_{-m} = -c_m\), we find that (4.4) equals

\[
2\mathfrak{N} \sum_{1 \leq |m|_2 \leq J \atop m \alpha > 0} c_m \sum_{0 < \gamma \leq T} x_m^{i\gamma} + O\left(\frac{N\pi(T)}{J^2}\right).
\]

Choose \(J\) so that \(|m|_2 \leq J\) implies \(\log x_m < ((c/3) (\log T/\log \log T))\) (with \(c\) as in Theorem 1.1). By Lemma 4.3 and the above display, (4.4) equals

\[
2\mathfrak{N} \sum_{1 \leq |m|_2 \leq J \atop m \alpha > 0} c_m \sum_{|\gamma| \leq T} x_m^{\frac{1}{2}} + O\left(\frac{N\pi(T)}{J^2} + \frac{T}{\log T} \sum_{1 \leq |m|_2 \leq J} \sum_{|m|_2 \leq J} \frac{1}{|m|_2^{n+2}} \left( (\log x_m)^2 + \frac{1}{\log T} \right) \right).
\]

Since \(\log x_m \leq 2\pi |m|_2 |\alpha|_2\), it follows from our preliminary bound for \(J\) that (4.4) equals

\[
2\mathfrak{N} \sum_{1 \leq |m|_2 \leq J \atop m \alpha > 0} c_m \sum_{|\gamma| \leq T} x_m^{\frac{1}{2}} + O\left(\frac{N\pi(T)}{J^2} + \frac{T}{\log T} \log T \right). \tag{4.5}
\]
We apply Lemma 4·2 to conclude that (4·5) equals

\[-\frac{2T}{\pi} \Re \sum_{1 \leq \|m\|_2 \leq J, m \cdot \alpha > 0} \frac{c_m \Lambda_\pi(\langle x_m \rangle)}{\sqrt{x_m}} \cdot e^{iT \log \frac{x_m}{\langle x_m \rangle}} - 1 \cdot iT \log \frac{x_m}{\langle x_m \rangle} + \mathcal{E}, \quad (4·6)\]

where \( \mathcal{E} \) satisfies (note that \( \log x_m = 2\pi(m \cdot \alpha) \))

\[|\mathcal{E}| \ll \sum_{1 \leq \|m\|_2 \leq J, m \cdot \alpha > 0} x_m^{1+\theta}(\log (2x_m) + \log T) + \sum_{1 \leq \|m\|_2 \leq J, m \cdot \alpha > 0} \log \frac{T}{\|m\|_2} + \left( \frac{N_\pi(T)}{J^2} + \frac{T \log \log T}{\log T} \right).\]

Our choice of \( J \) ensures that \( \log (2x_m) \ll \log T \), so it follows from (1·12) that

\[|\mathcal{E}| \ll e^{\theta \log (2x_m)} \log T + \frac{N_\pi(T)}{J^2} + \frac{T \log \log T}{\log T}. \quad (4·7)\]

We choose \( J = (\log T)^{2/3} \). Since \( N_\pi(T) \ll \log T \), (4·6) equals

\[-\frac{2T}{\pi} \Re \sum_{1 \leq \|m\|_2 \leq J, m \cdot \alpha > 0} \frac{c_m \Lambda_\pi(\langle x_m \rangle)}{\sqrt{x_m}} \cdot e^{iT \log \frac{x_m}{\langle x_m \rangle}} - 1 \cdot iT \log \frac{x_m}{\langle x_m \rangle} + O\left( \frac{T}{(\log T)^{1/3}} \right). \quad (4·8)\]

Observe that if \( x_m \neq \langle x_m \rangle \), then \( |e^{iT \log \frac{x_m}{\langle x_m \rangle}} - 1| \leq |iT \log \langle x_m / \langle x_m \rangle \rangle| \). Also, since \( 0 \leq \theta < 1/2 \), it follows that \( |\Lambda_\pi(\langle x_m \rangle)| \ll \sqrt{x_m} \). The proof of Lemma 4·1 ensures that \( |c_m| \ll \|m\|_2^{-n-2} \), so the sum over \( m \) converges absolutely. By the decay of \( |c_m| \), our choice of \( J \), and (4·7), (4·8) equals

\[-\frac{2T}{\pi} \Re \sum_{m \cdot \alpha > 0, x_m = \langle x_m \rangle} \frac{c_m \Lambda_\pi(\langle x_m \rangle)}{\sqrt{x_m}} + \sum_{m \cdot \alpha > 0, x_m \neq \langle x_m \rangle} \frac{c_m \Lambda_\pi(\langle x_m \rangle)}{\sqrt{x_m}} \cdot e^{iT \log \frac{x_m}{\langle x_m \rangle}} - 1 \cdot iT \log \frac{x_m}{\langle x_m \rangle} + O\left( \frac{T}{(\log T)^{1/3}} \right). \quad (4·9)\]

In particular, each sum over \( m \) converges absolutely.

To handle the sum over \( m \) such that \( x_m \neq \langle x_m \rangle \), we note (by absolute convergence) that for all \( \varepsilon > 0 \), there exists \( M_\varepsilon = M_\varepsilon(\alpha, h) > 0 \) such that

\[\left\| \sum_{\|m\|_2 > M_\varepsilon, m \cdot \alpha > 0, x_m \neq \langle x_m \rangle} \frac{c_m \Lambda_\pi(\langle x_m \rangle)}{\sqrt{x_m}} \cdot e^{iT \log \frac{x_m}{\langle x_m \rangle}} - 1 \right\| < \varepsilon.\]

Consequently, we have
\begin{align*}
\left| - \frac{2T}{\pi} \Re \sum_{\|m\| \geq 1, m \not= 0, x_m \neq \langle x_m \rangle} \frac{c_m \Lambda_\pi (\langle x_m \rangle)}{\sqrt{x_m}} \cdot e^{iT \log \frac{x_m}{\langle x_m \rangle} - 1} \right|
\leq \left| \Re \sum_{\|m\| \leq M, m \not= 0, x_m \neq \langle x_m \rangle} \frac{c_m \Lambda_\pi (\langle x_m \rangle)}{\sqrt{x_m}} \cdot e^{iT \log \frac{x_m}{\langle x_m \rangle} - 1} \right| + \varepsilon T.
\end{align*}

As we let \( \varepsilon \to 0 \) sufficiently slowly, we conclude that (4.6) equals \( o(T) \), as desired.

For the sum over \( m \) such that \( x_m = \langle x_m \rangle \), which means \( x_m \in \mathbb{Z} \), the terms which are not prime powers will vanish due to the presence of the von Mangoldt function. For the other terms which are prime powers, we have \( m \cdot \alpha = (k \log p)/(2\pi) \) for some \( k \in \mathbb{N} \) by the definition of \( x_m \). This will only happen when \( m \) is a multiple of \( q_j b_j \) for some \( j \in \{1, \ldots, r\} \) due to our choice of the vector \( \alpha \) in (1.13), so

\[- \frac{2T}{\pi} \Re \sum_{\|m\| \leq M, m \not= 0, x_m \neq \langle x_m \rangle} \frac{c_m \Lambda_\pi (\langle x_m \rangle)}{\sqrt{x_m}} \cdot e^{iT \log \frac{x_m}{\langle x_m \rangle} - 1} = - \frac{2T}{\pi} \Re \sum_{j=1}^{r} \sum_{l=1}^{\infty} \frac{\Lambda_\pi (p_j^{a_j})}{p_j^{a_j/2}} c_l b_j \int_{T} h(t) g_{f,\pi,\alpha} (t) dt.\]

The last equation holds because of (1.14) and (4.1).

5. Proof of Corollary 1.4

Let \( \mathbb{B} \subseteq \mathbb{T}^n \) be a product of \( n \) subintervals of \( \mathbb{T} \) for which \( \int_{\mathbb{B}} \left| g_{f,\pi,\alpha} (t) \right| dt \) attains its maximum. For \( \varepsilon > 0 \), let \( \varphi_\varepsilon : \mathbb{T}^n \to \mathbb{R} \) satisfy the following conditions:

(i) \( \varphi_\varepsilon \) is nonnegative and infinitely differentiable;

(ii) \( \varphi_\varepsilon \) is supported on a compact subset of \( U_\varepsilon := \{ t \in \mathbb{T}^n : \| t \|_2 < \varepsilon \} \); and

(iii) \( \int_{\mathbb{T}^n} \varphi_\varepsilon (t) dt = 1 \).

Let \( 1_\mathbb{B} \) be the indicator function of \( \mathbb{B} \), and define \( h_\varepsilon (t) = \int_{\mathbb{T}^n} \varphi_\varepsilon (t) 1_\mathbb{B} (x - t) dt \). Then \( h_\varepsilon \) is infinitely differentiable, and thus Theorem 1.3 holds with \( r \) arbitrarily large for \( h = h_\varepsilon \). Consequently, for any fixed \( r \geq n + 2 \), we have

\[
\int_{\mathbb{T}^n} h_\varepsilon (y) \left( \sum_{\| y \| \leq T, \{ y \alpha \} \in \mathbb{B} + y} 1 - \text{vol}(\mathbb{B}) N_\pi (T) \right) dy = T \int_{U_\varepsilon} h_\varepsilon (y) \int_{\mathbb{B} + y} g_{f,\pi,\alpha} (x) dx dy + o(T).
\]

It follows from our definition of \( g_{f,\pi,\alpha} (t) \) in (1.14) that \( g_{f,\pi,\alpha} (t) \ll 1 \), hence

\[
\left| \int_{\mathbb{B} + y} g_{f,\pi,\alpha} (x) dx - \int_{\mathbb{B}} g_{f,\pi,\alpha} (x) dx \right| \ll \varepsilon
\]

for all \( y \in U_\varepsilon \). Thus, we have

\[
\int_{\mathbb{T}^n} h_\varepsilon (y) \left( \sum_{\| y \| \leq T, \{ y \alpha \} \in \mathbb{B} + y} 1 - \text{vol}(\mathbb{B}) N_\pi (T) \right) dy = T \int_{\mathbb{B}} g_{f,\pi,\alpha} (t) dt + O(\varepsilon T) + o(T).
\]
By the mean value theorem, there exists $y \in U_\varepsilon$ such that

$$\left| \sum_{|\gamma| \leq T, \{\gamma \alpha\} \in \mathbb{B} + y} 1 - \text{vol}(\mathbb{B})N_\pi(T) \right| \geq T \left| \int_{\mathbb{B}} g_{f, \alpha}(t) dt \right| + O(\varepsilon T) + o(T).$$

The proof follows once we let $\varepsilon \to 0$ sufficiently slowly as a function of $T$.

### 6. Proofs of Corollaries 1.5-1.8

Throughout Sections 6.3-6.5, all levels are assumed to be squarefree.

#### 6.1. An estimate for the density function

We begin with a useful estimate for the density function $g_{f, \alpha}$ associated to a holomorphic cuspidal newform $f \in S_k(\Gamma_0(q))$ as in Section 2.2.

**Lemma 6.1.** Let $f \in S_k(\Gamma_0(q))$ be a newform and let $\alpha = (a \log p)/(2\pi q)$. Then we have

$$\left| g_{f, \alpha}(t) + \frac{2}{\pi} \frac{\Lambda_\pi(p^a)}{p^{a/2}} \cos(2\pi qt) \right| \leq \frac{4 \log p}{\pi p^a(1 - p^{-a/2})}.$$

**Proof.** In this case, Deligne’s bound implies that

$$\left| g_{f, \alpha}(t) + \frac{2}{\pi} \frac{\Lambda_\pi(p^a)}{p^{a/2}} \cos(2\pi qt) \right| = \left| \frac{2}{\pi} \sum_{\ell = 2}^{\infty} \frac{\Lambda_\pi(p^a\ell)}{p^{a\ell/2}} \cos(2\pi q\ell t) \right| \leq \sum_{\ell = 2}^{\infty} \frac{2 \log p}{p^{a\ell/2}} = \frac{2 \log p}{p^a(1 - p^{-a/2})}.$$

#### 6.2. Proof of Corollary 1.5

First, we prove a simple criterion for $f_1$ winning the $((\log p)/(2\pi), h)$-race. From Lemma 6.1, we obtain

$$\int_{\mathbb{T}} h(t)(g_{f_1, \alpha}(t) - g_{f_2, \alpha}(t)) dt > \frac{(\lambda_{f_2}(p) - \lambda_{f_1}(p))2 \log p}{\pi \sqrt{p}} \int_{\mathbb{T}} h(t) \cos(2\pi t) dt$$

$$- \int_{\mathbb{T}} h(t) dt \cdot 8 \log p \frac{1}{\pi p(1 - p^{-1/2})}.$$

Consequently, the inequality

$$(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \int_{\mathbb{T}} h(t) \cos(2\pi t) dt > \frac{4 \int_{\mathbb{T}} h(t) dt}{\sqrt{p}(1 - p^{-1/2})}, \quad (6.1)$$

implies that

$$\int_{\mathbb{T}} h(t)(g_{f_1, \alpha}(t) - g_{f_2, \alpha}(t)) dt > 0,$$

which by Theorem 1.3 tells us that $f_1$ wins the $((\log p)/(2\pi), h)$-race.

Throughout the proof, we let $k_h := \int_{\mathbb{T}} h(t) \cos(2\pi t) dt$. The number of $p$ for which $f_1$ wins the race is equal to $T_1 - T_2 + T_3$, where

$$T_1 := \#\{p \leq X : k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0\},$$
A zero density estimate for $GL_2$ $L$-functions

\[ T_2 := \# \{ p \leq X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0 \}, \]

\[ T_3 := \# \{ p \leq X : f_1 \text{ wins the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \leq 0 \}. \]

From the symmetry of $\mu_{ST,2}$, we see that $\nu((x, y) \in [-2, 2]^2 : x - y > 0) = 1/2$. By Corollary 2.2 with $\mathcal{I} = (0, 4)$, we have

\[ T_1 = \frac{1}{2} \pi(X) + O\left( \pi(X) \frac{\sqrt{\log \log \log X}}{(\log \log X)^{1/4}} \right). \]

We have to show that $T_2$ and $T_3$ are both $O(\pi(X)(\sqrt{\log \log \log X}/(\log \log X)^{1/4}))$. Note that

\[ T_2 = \# \{ \sqrt{X} \leq p \leq X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0 \} + O(\pi(\sqrt{X})). \]

If $p \geq \sqrt{X}$ and $f_1$ loses the $(\alpha, h)$-race and $k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0$, then by (6.1), we have

\[ k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in \left(0, \frac{4 \int_{1/3}^X h(t) dt}{\sqrt{p}(1 - p^{-1/2})} \right] \subseteq \left(0, \frac{4 \int_{1/3}^X h(t) dt}{X^{3/4}(1 - X^{-1/2})} \right]. \]

Denoting by $J_X$ the rightmost interval in the preceding display, we have

\[ \# \{ p \leq X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0 \} \]
\[ \leq \# \{ p \leq X : \lambda_{f_2}(p) - \lambda_{f_1}(p) \in J_X \} + O(\pi(\sqrt{X})). \]

By Corollary 2.2, this is at most

\[ \# \{ p \leq X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0 \} \]
\[ \leq \nu \left( |k_h|^{-1} J_X \right) \pi(X) + O\left( \pi(X) \frac{\sqrt{\log \log \log X}}{(\log \log X)^{1/4}} \right). \]

Since $\nu(|k_h|^{-1} J_X) = O(|J_X|) = O(X^{-1/4})$, it follows that

\[ \# \{ p \leq X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0 \} = O\left( \pi(X) \frac{\sqrt{\log \log \log X}}{(\log \log X)^{1/4}} \right). \]

If the conditions for $T_3$ are true, then

\[ k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in \left[ - \frac{4 \int_{1/3}^X h(t) dt}{\sqrt{p}(1 - p^{-1/2})}, 0 \right]. \]

Therefore, $T_3 \ll \pi(X)(\sqrt{\log \log \log X}/(\log \log X)^{1/4})$ by the argument used to bound $T_2$.

The result follows from the estimate shown for $T_1$ and the bounds for $T_2$ and $T_3$.

**6.3. Proof of Corollary 1.6**

**Proof.** From Theorem 1.3, Lemma 6.1, and (2.8) we have

\[ H\left(f_1, f_2, h, \frac{\log p}{2\pi}\right) = \frac{2k_h}{\pi} \cdot \frac{\log p \cdot (\lambda_{f_2}(p) - \lambda_{f_1}(p))}{p^{1/2}} + O_h\left( \frac{\log p}{p} \right). \]  

(6.2)

Consider the statements

\[ \frac{2k_h}{\pi} (\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in \mathcal{I} \]  

(6.3)
and
\[
\frac{\sqrt{X}}{\log X} H(f_1, f_2, h, \frac{\log p}{2\pi}) \in \mathcal{I}.
\]  
(6.4)

Defining
\[
T_1 := \#\{p \in [(1 - \varepsilon_X)X, X] : (6.3) \text{ holds}\},
\]
\[
T_2 := \#\{p \in [(1 - \varepsilon_X)X, X] : (6.3) \text{ holds and (6.4) fails}\},
\]
\[
T_3 := \#\{p \in [(1 - \varepsilon_X)X, X] : (6.3) \text{ fails and (6.4) holds}\},
\]
we have
\[
\#\{p \in [(1 - \varepsilon_X)X, X] : \frac{\sqrt{X}}{\log X} H(f_1, f_2, h, \frac{\log p}{2\pi}) \in \mathcal{I}\} = T_1 - T_2 + T_3.
\]

By Corollary 2.2 we have:
\[
T_1 = \#\{p \leq X : (6.3) \text{ holds}\} - \#\{p \leq (1 - \varepsilon_X)X : (6.3) \text{ holds}\}
\]
\[
= \nu\left(\frac{\pi}{2kh}\right)\pi(X) + O(\pi(X)\varepsilon_X^2)
\]
\[
- \nu\left(\frac{\pi}{2kh}\right)\pi((1 - \varepsilon_X)X) + O(\pi(X - \varepsilon_X)X\varepsilon_X^2)
\]
\[
= \nu\left(\frac{\pi}{2kh}\right)\cdot \left(\pi(X) - \pi((1 - \varepsilon_X)X)\right) + O\left(\pi(X)\varepsilon_X^3\right).
\]

We proceed to show that $T_2$ and $T_3$ are $O\left(\varepsilon_X^2\pi(X)\right)$ as $X \to \infty$. We first examine $T_2$. Set $\mathcal{I} = (\delta_1, \delta_2)$. If the condition in (6.4) is false, then
\[
\frac{\sqrt{X}}{\log X} H(f_1, f_2, h, \frac{\log p}{2\pi}) \notin [\delta_1, \delta_2].
\]
Applying (6.2), we deduce that
\[
\frac{\sqrt{X}}{\log X} \frac{2kh}{\pi} \cdot \frac{\log p \cdot (\lambda_{f_2}(p) - \lambda_{f_1}(p))}{p^{1/2}} \notin \left[\frac{\delta_1 + C \log p}{p}, \frac{\delta_2 - C \log p}{p}\right].
\]
where $C$ is an implied constant in (6.2). Then (6.3) gives us
\[
\frac{2kh}{\pi} (\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in \left(\delta_1, \delta_1 \left(\frac{p^{1/2} \log X}{X^{1/2} \log p}\right) + \frac{C}{\sqrt{p}}\right) \cup \left(\delta_2 \left(\frac{p^{1/2} \log X}{X^{1/2} \log p}\right) - \frac{C}{\sqrt{p}}, \delta_2\right).
\]  
(6.5)

Since $p \in [(1 - \varepsilon_X)X, X]$, it follows that $(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in I_X$, where $I_X$ is
\[
\frac{\pi}{2kh} \left(\delta_1 \frac{\delta_1 \log X}{\log X(1 - \varepsilon_X)} + \frac{C}{\sqrt{X(1 - \varepsilon_X)}}\right) \cup \frac{\pi}{2kh} \left(\delta_2 \sqrt{1 - \varepsilon_X} - \frac{C}{\sqrt{X(1 - \varepsilon_X)}}, \delta_2\right).
\]

By Corollary 2.2, we have
\[
\#\{(1 - \varepsilon_X)X \leq p \leq X : (\lambda_{f_1}(p) - \lambda_{f_2}(p)) \in I_X\}
\]
\[
= \nu(I_X)(\pi(X) - \pi((1 - \varepsilon_X)X)) + O\left(\pi(X)\varepsilon_X^3\right).
\]
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From the prime number theorem, we obtain

$$\pi(X) - \pi((1 - \varepsilon_X)X) \sim \varepsilon_X \pi(X).$$

Combining this with the fact that $\nu(I_X) = O(\varepsilon_X)$, we conclude the following:

$$\# \{(1 - \varepsilon_X)X \leq p \leq X : (\lambda_{f_1}(p) - \lambda_{f_2}(p)) \in I_X\} = O(\varepsilon_X^2 \pi(X)).$$

Therefore, $T_2 = O(\varepsilon_X^2 \pi(X))$. A very similar argument can be used to bound $T_3$. More specifically, if (6.4) holds, then we have

$$\frac{2k_h}{\pi} \frac{\log p \cdot (\lambda_{f_2}(p) - \lambda_{f_1}(p))}{p^{1/2}} \in (\delta_1 - \frac{C \log p}{p}, \delta_2 + \frac{C \log p}{p}),$$

If (6.3) fails and (6.4) holds, then, much like (6.5), we obtain

$$\frac{2k_h}{\pi} (\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in (\delta_1 \left( \frac{p^{1/2} \log X}{X^{1/2} \log p} - \frac{C}{\sqrt{p}}, \delta_1 \right) \cup \left( \delta_2, \delta_2 \left( \frac{p^{1/2} \log X}{X^{1/2} \log p} + \frac{C}{\sqrt{p}} \right) \right).$$

By Corollary 2.2, the number of such $p \in (1 - \varepsilon_X)X, X)$ is $O(\varepsilon_X^2 \pi(X))$.

6.4. Proof of Corollary 1.7

By Theorem 1.3, if $f_2$ wins the $(\alpha, h)$-race, where $\alpha = \log p/2\pi$, then we must have

$$0 < \frac{\log (q_1/q_2)}{\pi} \int_T h(t) dt < \int_T h(t)(g_{f_2,\alpha}(t) - g_{f_1,\alpha}(t)) dt.$$ 

By Lemma 6.1, we have

$$\left| \int_T (g_{f_2,\alpha}(t) - g_{f_1,\alpha}(t)) h(t) dt \right| \leq \int_T h(t) dt \cdot \frac{2 \log p}{\pi} \left( \frac{1}{p^{1/2}} + \frac{1}{(p^{1/2} - 1)p^{1/2}} \right).$$

It follows that $\log (q_1/q_2) \leq 2p^{-1/2}(1 + (p^{1/2} - 1)^{-1}) \log p$. The left hand side is independent of $\alpha$ and positive, while the right hand side tends to zero as $p$ grows. Thus, this inequality holds for only finitely many primes $p$.

6.5. Proof of Corollary 1.8

Fix $t_0 \in [0, 1)$, and $k_1, k_2 \in \mathbb{Z}$. By Theorem 1.3 and the same reasoning as in [LZ21, theorem 1.2], that $f_1$ wins the local $(\alpha, t_0)$-race against $f_2$ if

$$g_{f_1,\alpha}(t_0) + \frac{\log q_1}{2\pi} > g_{f_2,\alpha}(t_0) + \frac{\log q_2}{2\pi},$$

or, equivalently, if $(1/2\pi) \log (q_1/q_2) > g_{f_2,\alpha}(t_0) - g_{f_1,\alpha}(t_0)$.

We first assume $t_0 \neq 1/4, 3/4$. For $\alpha = \log p/2\pi$ and $q_2 \in (q_1, q_1 + \sqrt{q_1})$, by Lemma 6.1, the following is sufficient to guarantee that $f_1$ wins the $(\alpha, t_0)$ race:

$$\frac{\sqrt{p}}{2 \log p} \log (1 + q_1^{-1/2}) < \cos (2\pi t_0)(\lambda_{f_2}(p) - \lambda_{f_1}(p)) - \frac{4}{p^{1/2}(1 - p^{-1/2})}.$$ (6.6)

This inequality is automatically true if both

$$(\lambda_{f_1}(p) - \lambda_{f_2}(p)) \cos (2\pi t_0) \geq \frac{\left| \cos (2\pi t_0) \right|}{2}$$ (6.7)
and

\[
\frac{|\cos(2\pi t_0)|}{2} > \frac{p^{1/2}}{2 \log p} \log (1 + q_1^{-1/2}) + \frac{4}{p^{1/2}(1 - p^{-1/2})}
\] (6.8)

are satisfied.

Now set \(X = q_1^{1/4}\) and \(Y = q_1^{1/6}\). For \(p \in [Y, X]\), if \(q_1\) is sufficiently large, then (6.8) is satisfied. Choose \(I_1, I_2 \subseteq [-2, 2]\) such that \((\lambda_{f_1}(p), \lambda_{f_2}(p)) \in I_1 \times I_2\) implies (6.7). Suppose \(f_1 \in S_{k_1}^{\text{new}}(\Gamma_0(q_1)), f_2 \in S_{k_2}^{\text{new}}(\Gamma_0(q_2))\) are non-CM newforms, where \(q_2 \in [q_1, q_1 + \sqrt{q_1}]\), and assume, as in the statement of the Corollary, that \(f_2 \neq f_1 \otimes \chi\) for any primitive Dirichlet character \(\chi\). Let

\[
\pi_{f_1, f_2, I_1, I_2}(X) := \#\{p \leq X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in I_1 \times I_2\}.
\]

Then by Theorem 2.1 we have

\[
\pi_{f_1, f_2, I_1, I_2}(X) - \pi_{f_1, f_2, I_1, I_2}(Y) \geq \mu_{ST}(I_1)\mu_{ST}(I_2)(\pi(X) - \pi(Y))
\]

\[
- c\pi(X) \frac{\log \log \log (k_1 k_2 q_1^{1/4} + 1)}{(\log \log q_1^{1/4})^{1/2}} - c\pi(Y) \frac{\log \log \log (k_1 k_2 q_1^{1/4} + 1)}{(\log \log q_1^{1/6})^{1/2}}.
\]

So if \(q_1\) is sufficiently large, then \(\pi_{f_1, f_2, I_1, I_2}(X) > \pi_{f_1, f_2, I_1, I_2}(Y)\). So there exists \(p\) between \(Y\) and \(X\) such that (6.6) is satisfied and therefore \(f_1\) wins the local \((t_0, (\log p)/(2\pi))\) race.

Finally, if \(t_0 = 1/4, 3/4\), then instead of (6.6), we wish to find \(p\) such that

\[
\frac{p}{2 \log p} \log (1 + q_1^{-1/2}) < (\lambda_{f_1}(p))^2 - (\lambda_{f_2}(p))^2 - \frac{4}{p^{3/2}(1 - p^{-1/2})}.
\]

We obtain this the same way as the first case.

7. Example

We conclude with a numerical example to illustrate (1.15). For our example, we consider the \(L\)-function \(L(s, \Delta)\) associated to the discriminant modular form

\[
\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} = \sum_{n=1}^{\infty} \tau(n)e^{2\pi inz} \in S_{12}^{\text{new}}(\Gamma_0(1)),
\]

where \(\tau(n)\) denotes the Ramanujan tau function. We use Rubinstein’s lcalc package [Rub14] to calculate the \(2 \cdot 10^5\) nontrivial zeros \(L(s, \Delta)\) up to height \(T = 74920.77\).

Let \(M\) and \(\alpha\) satisfy the following relation for (1.13),

\[
M \alpha^T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{2\pi} \begin{pmatrix} \log 2 \\ \log 3 \end{pmatrix},
\]

so that (1.14) will define our density function \(g_{\Delta, \alpha}(x, y)\). We graph \(g_{\Delta, \alpha}(x, y)\) in Figure 1(a) below. Next, we partition the unit square \([0, 1) \times [0, 1)\) as

\[
[0, 1) \times [0, 1) = \bigcup_{a=0}^{29} \bigcup_{b=0}^{29} S_{a,b}, \quad S_{a,b} := \left[ \frac{a}{30}, \frac{a+1}{30} \right) \times \left[ \frac{b}{30}, \frac{b+1}{30} \right).
\]
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Fig. 1. Example.

Given $(x, y) \in [0, 1) \times [0, 1)$, there exists a unique pair of integers $a$ and $b$ with $0 \leq a, b \leq 29$ such that $(x, y) \in S_{a,b}$. Denoting this unique square as $S(x, y)$, we define

$$\tilde{g}_{\Delta, \alpha}(x, y) := \# \{ \rho = \beta + i \gamma : L(\rho, \Delta) = 0 \text{ and } (\{\alpha_1 \gamma\}, \{\alpha_2 \gamma\}) \in S(x, y) \}.$$

This gives us a discretised approximation to $g_{\Delta, \alpha}(x, y)$, which we plot in Figure 1(b) above.

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