On fully dynamic constant-factor approximation algorithms for clustering problems

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Abstract

Clustering is an important task with applications in many fields of computer science. We study the fully dynamic setting in which we want to maintain good clusters efficiently when input points (from a metric space) can be inserted and deleted. Many clustering problems are APX-hard but admit polynomial time $O(1)$-approximation algorithms. Thus, it is a natural question whether we can maintain $O(1)$-approximate solutions for them in subpolynomial update time, against adaptive and oblivious adversaries. Only a few results are known that give partial answers to this question. There are dynamic algorithms for $k$-center, $k$-means, and $k$-median that maintain constant factor approximations in expected $\tilde{O}(k^2)$ update time against an oblivious adversary. However, for these problems there are no algorithms known with an update time that is subpolynomial in $k$, and against an adaptive adversary there are even no (non-trivial) dynamic algorithms known at all. Also, for the $k$-sum-of-radii and the $k$-sum-of-diameters problems it is open whether there is any dynamic algorithm, against either type of adversary.

In this paper, we complete the picture of the question above for all these clustering problems.

• We show that there is no fully dynamic $O(1)$-approximation algorithm for any of the classic clustering problems above with an update time in $n^{o(1)}h(k)$ against an adaptive adversary, for an arbitrary function $h$. So in particular, there are no such deterministic algorithms.
• We give a lower bound of $\Omega(k)$ on the update time for each of the above problems, even against an oblivious adversary. This rules out update times with subpolynomial dependence on $k$.
• We give the first $O(1)$-approximate fully dynamic algorithms for $k$-sum-of-radii and for $k$-sum-of-diameters with expected update time of $\tilde{O}(k^{O(1)})$ against an oblivious adversary.
• Finally, for $k$-center we present a fully dynamic $(6 + \epsilon)$-approximation algorithm with an expected update time of $\tilde{O}(k)$ against an oblivious adversary. This is the first dynamic $O(1)$-approximation algorithm for any of the clustering problems above whose update time is $\tilde{O}(k)$ and in particular the first whose update time is asymptotically optimal in $k$.

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1 Introduction

Clustering is a central task in data science and in many fields of computer science such as machine learning, information retrieval, computer vision, data compression, and resource allocation. The goal is to partition a set of points into non-overlapping subsets so that “close” points belong to the same subset. To compare the quality of different clusterings, each partition is assigned a cost. Static and also online clustering (where data points arrive in an online manner and earlier decisions cannot be revoked) are well-studied, e.g., [24, 15, 14, 11, 25, 1, 6].

However, in various settings dynamic clustering is needed, e.g., for dynamically evolving data set or interactive data analysis. Research on dynamic clustering was already initiated in 1987 [7], but there are still many fundamental open questions. In fully dynamic clustering data points can be inserted as well as deleted and the goal is to maintain a set of clusters whose cost is within a (small) multiplicative factor of the optimal clustering for the current set of points. Dynamic clustering algorithms process a sequence of update operations and they can answer two types of queries: in a value-query an approximation $\alpha$- of the optimal clustering cost is returned, in a solution-query a solution is returned. Such an algorithm is an $\alpha$-approximation algorithm if it always holds that $OPT \leq apx \leq \alpha OPT$ and each returned solution has cost at most $\alpha OPT$. It is evaluated based on (i) its approximation ratio $\alpha$ and (ii) its running time per operation.

We study the following dynamic setting: at any point in time, there is a set $P$ of active points. We use $n$ to denote the current size of $P$. In each update, the adversary adds or deletes a point. The algorithm can query distances between pairs of points from the adversary. The reported distances stem from a metric on the set of points consisting of $P$ and the already deleted points, and an upper bound $\Delta$ on the aspect ratio of this metric is given to the algorithm at the beginning. For a given $k$, the algorithm needs to maintain a set $C$ of $k$ centers. For each point $p$ denote by $d(p, C)$ the distance of $p$ to its closest center in $C$. All points that are assigned to the same center $c \in C$ (i.e., for which $c$ is the closest center) form a cluster.

There are different clustering problems with different cost functions that one seeks to minimize. In the $k$-center problem, we want to minimize the maximum distance of a point $p$ to its assigned cluster center, i.e., we minimize the $L_\infty$-norm of the vector $(d(p, C))_p \in P$. In the $k$-median and $k$-means problems we instead minimize $\sum_{p \in S} d(p, C)$ and $\sum_{p \in S} d(p, C)^2$, respectively, which is equivalent to minimizing the $L_1$- and the $L_2$-norms of $(d(p, C))_p \in P$. More generally, in the $(k, p)$-clustering problem we minimize $\sum_{p \in S} d(p, C)^p$ (corresponding to the $L_p$-norm). Another way to interpret the $k$-center problem is that each cluster center $c_i$ has a radius $r_i$ assigned to it, $c_i$ “covers” all points within radius $r_i$, every point needs to be covered by a center, and we seek to minimize the maximum of these radii, i.e., $\max_i r_i$. A natural variation of this is the $k$-sum-of-radii problem (also known as the $k$-cover problem) in which we assign again each center $c_i$ a radius $r_i$ but seek instead to minimize their sum $\sum_i r_i$. Related to this is the $k$-sum-of-diameters problem in which we seek to minimize the sum of the diameters of the clusters. For $k$-center, $k$-median, and $k$-means we assume that the solution returned by an algorithm is the set of centers, for $k$-sum-of-radii the centers together with the radii, and for $k$-sum-of-diameter an actual partitioning of the points into clusters.

All above problems are NP-hard, and $k$-center, $k$-median, $k$-means, and $k$-sum-of-diameters are also known to be APX-hard [11, 15, 14, 27, 26, 13]. Hence, we aim for $O(1)$-approximation algorithms for them. In the offline setting, such algorithms with polynomial running time are known for all problems above [15, 22, 23, 2]. In this paper, we study clustering problems in the dynamic setting. Of course, after each update we could recompute the solution from scratch using a known offline algorithm, which leads to an update time that is polynomial in $n$ and $k$. But can we do better, i.e., achieve subpolynomial update time? For solution-queries the size of the output can be linear in $k$ (since all centers are returned), but for value-queries no such obvious lower bound exists. In this paper, we investigate the following basic question.

Can we maintain $O(1)$-approximate solutions for clustering problems dynamically, with update times that are subpolynomial in $n$ and/or $k$?
Table 1: An overview of our running time bounds (in red) in comparison to previous work. Our lower bounds against an adaptive adversary hold for an arbitrary function \( h \), also if \( k = 1 \), and if the algorithm can open \( O(k) \) centers instead of \( k \). Apart from \( k \)-center, they hold even if the algorithm needs to report only its current solution but not its (estimated) cost \( \text{apx} \). All our lower bounds hold additionally in the case where the algorithm and the optimal solution are allowed to select already deleted points as centers.

For this we need to distinguish between an adaptive adversary that can choose each operation based on all the algorithm’s query answers so far (e.g., which points are centers) and an oblivious adversary that knows the algorithm, but does not see the actual answers of the algorithm (which might depend on random bits). So far there are only dynamic \( O(1) \)-approximation algorithms known for \( k \)-center, \( k \)-median, and \( k \)-means with expected update times of \( \tilde{O}(k^2/\epsilon^{O(1)}) \) against an oblivious adversary \cite{8,19} (where \( \tilde{O} \) suppresses factors that are polylogarithmic in \( n \), \( k \), and \( \Delta \)). In particular, for none of the problems listed above there is a dynamic algorithm known that works against an adaptive adversary. Considering the recent progress on designing fast dynamic graph algorithms against an adaptive adversary \cite{30,5,21,35,17,12}, it is an obvious questions whether such fast algorithms are also possible for dynamic clustering. Furthermore, it is not known whether their quadratic dependence on \( k \) in the running time against an oblivious adversary is necessary, or whether we could improve it to, e.g., \( k \log k \). Moreover, for \( k \)-sum-of-radii and \( k \)-sum-of-diameters there is even no non-trivial dynamic algorithm known at all (for either type of adversary). This leaves many open questions.

1.1 Our results

In this paper we answer the above questions for each of the classic clustering problems defined earlier. To this end, we present novel upper and lower bounds for the remaining open settings, see Table 1 for an overview of all our results.

**Lower bounds.** Adaptive adversary. We present the first lower bounds for dynamic clustering problems against an adaptive adversary. Thus, they imply the same bounds for deterministic algorithms against an arbitrary adversary. Our bounds are very strong: they imply that for none of the above problems we can maintain \( O(1) \)-approximations with an update time that is subpolynomial in \( n \), even for the case that \( k = 1 \). So in particular this holds independently of how the running time depends on \( k \). Also, our bounds hold even if the algorithm can open \( O(k) \) centers instead of only \( k \). Formally, we prove the following tradeoff:
any algorithm with an update time of \( f(k, n) \) that opens up to \( O(k) \) centers must have an approximation ratio (compared to the optimal solution with only \( k \) centers) of \( \Omega((\log n / (\log f(1, 2n))) \) for \( k \)-center, \( k \)-sum-of-radii, \( k \)-diameter, of \( \Omega((\log n / (10 + 2 \log f(1, 2n)))) \) and \( k \)-median, of \( \Omega((\log n / (12 + 2 \log f(1, 2n))))^2 \) for \( k \)-means, and of \( \Omega((\log n / (2p + 8 + 2 \log f(1, 2n)))^p) \) for \((k, p)\)-clustering, see Theorem\(^5\). Thus, to achieve an \( O(1) \)-approximation, the update time needs to be \( n^{\Omega(1)} h(k) \) for any function \( h \).

For \( k \)-center, if \( k = 1 \), then one obtains a 2-approximation if one outputs simply an arbitrary point; our lower bound above hinges on the fact that the algorithm also needs to output an approximation \( \text{apx} \) for its cost. However, even if we require the algorithm to output only its current solution, we prove a lower bound of \( \Omega((\min\{k, \log n / (k \log f(k, 2n))\}) \) on the approximation ratio for any \( k \). Therefore, for sufficiently large \( k \) we again need an update time of \( n^{\Omega(1)} h(k) \) in order to maintain a \( O(1) \)-approximation. We show that our lower bound is (asymptotically) tight by giving a deterministic algorithm with \( O(k \log n \log \Delta / \epsilon) \) update time that achieves a \( \tilde{O}(\min\{k, \log(n/k)\}) \) approximation for arbitrary \( k \). Our lower bounds for all other problems hold directly also if the algorithm needs to output only its current solution. Even more, all above bounds hold already under the weaker assumption that only the amortized number of distance queries of the algorithm per operation is bounded by \( f(k, n) \), rather than its running time.

**Oblivious adversary.** Also, we provide a lower bound of \( \Omega(k) \) for the update time of a \((\Delta - \epsilon)\)-approximation algorithm for any of the above problems against an oblivious adversary (and hence also against an adaptive adversary). For \( k \)-means and \((k, p)\)-clustering this lower bound holds even for maintaining a \((\Delta^2 - \epsilon)\)-approximation or a \((\Delta^p - \epsilon)\)-approximation, respectively.

All our lower bounds hold in the setting where the algorithm can use as centers only points in \( P \), but also when additionally it can also use any point that it has seen so far, including points that are already deleted.

**Upper bounds.** \( k \)-sum-of-radii and \( k \)-sum-of-diameters. As discussed above, the \( k \)-sum-of-radii and \( k \)-sum-of-diameters problems are classical clustering problems for which no non-trivial dynamic approximation algorithm is known. We fill this gap by giving a fully dynamic \((13.008 + \epsilon)\)-approximation algorithm for \( k \)-sum-of-radii with an expected update time of \( k^{O(1/\epsilon)} \log \Delta \), which implies a dynamic \((26.016 + \epsilon)\)-approximation algorithm for \( k \)-sum-of-diameters with the same update time. This completes the picture that all above clustering problems admit fully dynamic \( O(1) \)-approximation algorithms with update time \( \tilde{O}(k^{O(1)}) \).

Our algorithms for \( k \)-sum-of-radii and \( k \)-sum-of-diameters use a new idea in the setting of dynamic clustering algorithms: we maintain a bi-criteria approximation, i.e., an \( O(1) \)-approximate solution that might open more than \( k \) centers, up to \( O(k/\epsilon) \) many. After each update, we transform this larger set of centers to a solution with only \( k \) centers, losing only a constant factor in this step. For the last step, our machinery allows to use any static offline \( \alpha \)-approximation algorithm for \( k \)-sum-of-radii in a black-box manner, yielding a dynamic \((6 + 2\alpha + \epsilon)\)-approximation algorithm for the problem (we use the algorithm due to Charikar and Panigrahy \([9]\) for which \( \alpha = 3.504 + \epsilon \)). Inserting all points of a fixed set \( P \) into our data structure even leads to the fastest known static \( O(1) \)-approximation algorithm for \( k \)-sum-of-radii.

**\( k \)-center.** We then demonstrate that our idea of using bi-criteria approximations has more applications: we apply it to \( k \)-center for which we give a fully dynamic \((6 + \epsilon)\)-approximation algorithm with an expected amortized update time of \( O(k \log^2 n \log \Delta / \epsilon) = \tilde{O}(k/\epsilon) \). This is the first dynamic \( O(1) \)-approximation algorithm for any of the classic clustering problems above with an update time whose dependence on \( k \) is asymptotically optimal (while being polylogarithmic in \( n \) and \( \Delta \)). In fact, before not even a dynamic \( O(1) \)-approximation algorithm with an update time in \( \tilde{O}(k^2) \) was known for the simpler deletion-only case of \( k \)-center or any other of the clustering problems above.

Note that improving the dependency on \( k \) to linear (even with an additional dependency on \( \log n \)) is beneficial in all settings where \( k \) has a poly-logarithmic or larger dependency on \( n \). This is the case in a variety of applications. For example, setting the number \( k \) of cluster centers to be super-logarithmic or even near-linear in the number of input points can improve the quality of spam and abuse detection \([32, 34]\) and
near-duplicate detection [18].

1.2 Technical overview

Lower bounds against an adaptive adversary. To sketch the idea behind the lower bounds we discuss here a simplified setting in which the algorithm (i) can query only the distances between points that are currently in \( P \) (i.e., that have been introduced already but not yet removed) and (ii) has a worst-case update time of \( f(k, n) \) (rather than amortized update time) for some function \( f \). (These assumptions are not needed for the lower bound proof presented in the later part of the paper.) Our goal is to show that the dynamic algorithm cannot maintain \( O(1) \)-approximate solutions in subpolynomial update time.

The adversary starts adding points into \( P \) and maintains an auxiliary graph \( G = (V, E) \) with one vertex \( v_p \) for each point \( p \). Whenever the algorithm queries the distance between two points \( p, p' \in P \), the adversary reports that they have a distance of 1 and adds an edge \( \{v_p, v_{p'}\} \) of length 1 to \( G \). Intuitively, the adversary uses \( G \) to keep track of the previously reported distances. Whenever there is a vertex \( v_p \) with a degree of at least \( 100f(k, n) \), in the next operation the adversary deletes the corresponding point \( p \). There could be several such vertices, and then the adversary deletes them one after the other. Thanks to assumption (i), once a point \( p \) is deleted, the degree of \( v_p \) cannot increase further. Hence, the degrees of the vertices in \( G \) cannot grow arbitrarily. More precisely, one can show that the degree of each vertex can grow to at most \( O(\log n \cdot f(k, n)) \). Thus, for each vertex \( v_p \) and each \( \ell \in \mathbb{N} \), there are at most \( O(\log^\ell n \cdot f(k, n)^\ell) \) vertices at distance \( \ell \) to \( v_p \). In particular, at least half of the vertices in \( G \) are at distance at least \( \Omega(\log O(\log n \cdot f(k, n)) n) = \Omega(\log n / [\log \log n \cdot \log(f(k, n))]) \) to \( v_p \).

Now observe that the algorithm knows only the point distances that it queried, i.e., those that correspond to edges in \( G \). For all other distances the triangle inequality imposes only an upper bound for any pairs of points whose corresponding vertices are connected in \( G \). Thus, the algorithm cannot distinguish the setting where the underlying metric is the shortest path metric in \( G \), from the setting where all points are at distance 1 to each other. If \( k = 1 \) for any of the problems under consideration, then for the selected center \( c \) the algorithm cannot distinguish whether all points are at distance 1 to \( c \) or if half of the points in \( P \) are at distance \( \Omega(\log n / [\log \log n \cdot \log(f(1, n))]) \) to \( c \). Therefore, the reported cost \( \text{apx} \) can be larger than \( \text{OPT} \) by a factor of up to \( \Omega(\log n / [\log \log n \cdot \log(f(1, n))]) \).

We improve the above construction so that it works also for algorithms with amortized update times, which are allowed to query distances to points that are already deleted and which may output \( O(k) \) instead of \( k \) centers, and also already if only the amortized number of distance queries per operation is bounded by \( f(k, n) \). To this end, we adjust the construction so that for points with degree of at least \( 100f(k, n) \) we report distances that might be larger than 1 and that still allow us to use the same argumentation as above. At the same time, we remove the factor of \( \log \log n \) in the denominator.

Lower bounds against an oblivious adversary. We sketch our lower bound of \( \Omega(k) \) for the needed update time in order to maintain solutions with an approximation ratio of \( \Delta - \epsilon \) (and of \( \Delta^2 - \epsilon \) and \( \Delta^p - \epsilon \) for \( k \)-means and \( (k, p) \)-clustering, respectively). First, the adversary introduces \( k \) points \( P_0 \) that are at pairwise distance \( \Delta \) to each other. Then, the adversary introduces a point \( p_1 \) and queries the cost of the current solution. The intuition is that the algorithm needs \( \Omega(k) \) queries in order to be able to distinguish whether \( p_1 \) is close to some point in \( P_0 \), e.g., at distance 1 (in which case the cost of \( \text{OPT} \) is 1) or whether \( p_1 \) is at distance \( \Delta \) from each point in \( P_0 \) (in which case the cost of \( \text{OPT} \) is \( \Delta, \Delta^2, \) or \( \Delta^p \), depending on the problem). Note that the algorithm needs to be able to distinguish these two cases in order to achieve its approximation guarantees.

Then the adversary removes \( p_1 \), introduces another point \( p_2 \), and repeats this process. Using this idea, we can generate a distribution over inputs and then apply Yao’s minmax principle to it. In this way we show that \( \Omega(k) \) queries per update are needed, even for a randomized algorithm against an oblivious adversary, in order to be able to distinguish the two cases above for every point \( p_i \) (and hence in order to obtain non-trivial...
approximation ratios of $\Delta - \epsilon$, $\Delta^2 - \epsilon$, and $\Delta^p - \epsilon$, respectively).

**Upper bounds via bi-criteria approximations.** Our algorithms against an oblivious adversary for $k$-sum-of-radii, $k$-sum-of-diameters, and $k$-center use the following paradigm: we maintain bi-criteria approximations, i.e., solutions with a small approximation ratio that might use more than $k$ centers (at most $O(k/\epsilon)$ or $O(k \log^2 n)$ centers, depending on the problem). In a second step, we use the centers of this solution as the input to an auxiliary dynamic instance for which we maintain a solution with only $k$ centers. These centers then form our solution to the actual problem, by increasing their radii appropriately. Since the input to our auxiliary instance is much smaller than the input to the original instance, we can afford to use algorithms for its update times have a much higher dependence on $n$. Depending on the desired overall update time, we can afford update times of $O(n)$ here, or even recompute the whole solution from scratch.

In all our algorithms we first “guess” a $(1 + \epsilon)$-approximate estimate $\text{OPT}'$ on the value $\text{OPT}$ of the optimal solution. For the simplicity of presentation, we assume that $d(p, q) \geq 1$ for any $p, q$ in the metric space. If this is not the case, we simply scale all distances linearly. More formally, for $O((\log \Delta)/\epsilon)$ values $\text{OPT}'$ that are powers of $1 + \epsilon$, we build a dynamic clustering data structure that uses $\text{OPT}'$ as an estimate of $\text{OPT}$. It has to either maintain a solution whose cost is within a constant factor of $\text{OPT}'$ or certify that $\text{OPT} > \text{OPT}'$. We describe our approaches for $k$-sum-of-radii and $k$-center below in more detail.

**$k$-sum-of-radii.** Recall that in the static offline setting there is a $(3.504 + \epsilon)$-approximation algorithm with running time $n^{O(1/\epsilon)}$ for $k$-sum-of-radii [9]. We provide a black-box reduction that transforms any such offline algorithm with running time of $g(n)$ and an approximation ratio of $\alpha$ into a fully dynamic $(6 + 2\alpha + \epsilon)$-approximation algorithm with an update time of $O(((k/\epsilon)^5 + g(\text{poly}(k/\epsilon))) \log \Delta)$. To this end, we introduce a dynamic primal-dual algorithm that maintains a bi-criteria approximation with up to $O(k/\epsilon)$ centers. After each update, we use these $O(k/\epsilon)$ centers as input for the offline $\alpha$-approximation algorithm, run it from scratch, and increase the radii of the computed centers appropriately such that they cover all points of the original input instance.

For computing the needed bi-criteria approximation, there is a polynomial-time offline primal-dual $(1)$-approximation algorithm with a running time of $\Omega(n^2 k)$ [9] from which we borrow ideas. Note that a dynamic algorithm with an update time of $(k \log n)^{O(1)}$ yields an offline algorithm with a running time of $n(k \log n)^{O(1)}$ (by inserting all points one after another) and no offline algorithm is known for the problem that is that fast. Hence, we need new ideas. First, we formulate the problem as an LP $(P)$ which has a variable $x_p^{(r)}$ for each combination of a point $p$ and a radius $r$ from a set of (suitably discretized) radii $R$, and a constraint for each point $p$. Let $(D)$ denote its dual LP, see below, where $z := \epsilon \text{OPT}'/k$.

\[
\begin{align*}
\min & \sum_{p \in P} \sum_{r \in R} x_p^{(r)} (r + z) \\
\text{s.t.} & \sum_{p' \in P \mid d(p, p') \leq r} x_p^{(r)} \geq 1 \ \forall p \in P \\
& x_p^{(r)} \geq 0 \ \forall p \in P \ \forall r \in R
\end{align*}
\]

\[
\begin{align*}
\max & \sum_{p \in P} y_p \\
\text{s.t.} & \sum_{p' \in P \mid d(p, p') \leq r} y_{p'} \leq r + z \ \forall p \in P, r \in R \\
& y_p \geq 0 \ \forall p \in P
\end{align*}
\]

We select a point $c$ randomly and raise its dual variable $y_c$. Unlike [9], we raise $y_c$ only until the constraint for $c$ and some radius $r$ becomes half-tight, i.e., $\sum_{p' \mid d(c, p') \leq r} y_{p'} = r/2 + z$. We show that then we can guarantee that no other dual constraint is violated, which saves running time since we need to check only the (few, i.e. $|R|$ many) dual constraints for $c$, and not the constraints for all other input points when we raise $y_c$. We add $c$ to our solution and assign it a radius of $2r$. Since the constraint for $(c, r)$ is half-tight, our dual can still "pay" for including $c$ with radius $2r$ in our solution.

In primal-dual algorithms, one often raises all primal variables whose constraints have become tight, in particular in the corresponding routine in [9]. However, since we assign to $c$ a radius of $2r$, we argue
that we do not need to raise the up to $\Omega(n)$ many primal variables corresponding to dual constraints that have become (half-)tight. Again, this saves a considerable amount of running time. Then, we consider the points that are not covered by $c$ (with radius $2r$), select one of these points uniformly at random, and iterate.

After a suitable pruning routine (which is faster than the corresponding routine in [9]) this process opens $k' = O(k/\epsilon)$ centers at cost $(6 + \epsilon)OPT'$, or asserts that $OPT > OPT'$. We maintain the above primal-dual solution dynamically. The most interesting update operation occurs when a point $p$ is deleted whose dual variable $y_p$ was raised at some point. Then, we “warm-start” the algorithm, starting at the point where $y_p$ was raised, and select instead another point $p'$ randomly to increase its dual variable $y_{p'}$. Suppose that $U_p$ were the set of points from which $p$ was selected randomly. The work of the warm-start is at most $O(|U_p|g(k'))$. In expectation the adversary needs to delete $\Omega(|U_p|)$ points of $U_p$ before deleting $p$. Thus, we charge the cost of the warm-start to these points, resulting in a $g(k')$ time per operation.

As mentioned above, after each update we feed the centers of the bi-criteria approximation as input points to our (arbitrary) static offline $\alpha$-approximation algorithm for $k$-sum-of-radii and run it. Finally, we translate its solution to a $(6 + 2\alpha + \epsilon)$-approximate solution to the original instance. For the best known offline approximation algorithm [9] it holds that $\alpha = 3.504$, and thus we obtain a ratio of $13.008 + \epsilon$ overall. For $k$-sum-of-diameters the same solution yields a $(26.016 + 2\epsilon)$-approximation.

**$k$-center.** In our $(6 + \epsilon)$-approximation algorithm for $k$-center, we maintain a bi-criteria solution with cost at most $4OPT'$ that uses up to $O(k \log^2 n)$ centers. We feed its centers into an auxiliary data structure with an update time of $O(n + k)$ that maintains a solution with cost at most $2OPT'$; so unlike for $k$-sum-of-radii we do not simply recompute the solution from scratch. Note that our update time of $O(n + k)$ is still faster than recomputing a 2-approximate solution from scratch with the fastest known algorithm, which needs $O(nk)$ time [24]. We argue that each input point is at distance at most $6OPT'$ from a point computed by the auxiliary data structure, which yields a $(6 + \epsilon)$-approximation.

We describe first the auxiliary data structure with update time $O(n + k)$. We say that a center $c$ covers a point $p$ if $d(c, p) \leq 2OPT'$. We maintain a counter $a_p$ for each point $p$ of the number of current centers that cover $p$. If the counter $a_p$ becomes zero, then $p$ is defined as a new center and, hence, we increment the counters of each point $p'$ that $p$ covers. This yields $O(n)$ increments. If a center $c$ is deleted, we decrement the counters of all points that are covered by $c$, which yields $O(n)$ decrements and check for each of these points whether it has to become a center now. As $n$ is the current size of $P$, if a point that is inserted and deleted when $n$ is small, it cannot “pay” for the cost of $O(n)$ if it becomes a center when $n$ is large. Instead we use an amortized analysis that charges each update operation $\Theta(n + k)$ for this cost. Our centers are at distance at least $2OPT'$ to each other. Hence, if at least $k + 1$ centers exist, they assert that $OPT > OPT'$. If there are at most $k$ centers, they cover all points and, thus, represent a valid 2-approximation.

Next, we describe a deletion-only data structure maintaining a solution with cost at most $4OPT'$ which is allowed to use $O(k \log n)$ centers, instead of only $k$. We will use it as a subroutine to obtain a fully dynamic data structure that uses $O(k \log^2 n)$ centers. Already for the deletion-only case, no data structure was known that maintains a $O(1)$-approximation with update time $k(\log n)^{O(1)}$. Similarly as Chan et al. [8], we initialize our data structure by selecting the first center $c_1$ uniformly at random among all points in $P =: U_1$ and assign all points to $c_1$ that are covered by $c_1$. We say that they form a cluster with center $c_1$. We define $U_2$ to be the remaining points and continue similarly by selecting a point $c_2$ uniformly at random from $U_2$, etc. In contrast to [8], we assign the computed centers into buckets such that for all points $c_i$ in the same bucket, the size of the corresponding set $U_i$ differs by at most a factor of 2. We stop if we have covered all points or if a bucket $B^*$ has $2k$ centers; hence, $B^*$ must contain the last $2k$ chosen centers. This defines $O(k \log n)$ centers since there can be only $\log n$ buckets.

If a point $p$ is deleted and $p$ is not a center, we do not change our centers. If a center $c$ is deleted, we replace $c$ by an arbitrary point $p$ in the cluster of $c$. This update can be done in time $O(1)$, and then $p$
becomes the center corresponding to this cluster. Since the diameter of the cluster is at most $4\OPT'$, $p$
 covers all points in the cluster if we assign it a radius of $4\OPT'$. In particular, this is still the case if later we
replace $p$ by some other point in the same cluster in the same fashion.

As long as the bucket $B^*$ contains at least $k + 1$ of its initially chosen centers, this asserts that $\OPT > \OPT'$. Once $k$ of the original centers from $B^*$ are deleted, we “warm-start” the above computation at the
point where the first center $c_i \in B^*$ was chosen, discarding all centers from $c_i$ onwards and recomputing
them with the current set $P$. Since each center $c_i$ was chosen uniformly at random from its respective set
$U_i$, in expectation the adversary needs to delete $\Omega(|U_i|)$ points from $U_i$ before deleting $c_i$. Since we deleted
$k$ centers from $B^*$ and the sizes of their respective sets $U_i$ differ by at most a factor of 2, in expectation
the adversary must delete $\Omega(k|U_\ast|)$ points before the warm-start. The set $U_\ast$ contains all points that are
assigned to some center in $B^*$. Hence, we can charge the cost of the warm-start of $O(k|U_\ast|)$ to these
deletions.

Using a technique from Overmars [51] we turn our deletion-only data structure into a fully dynamic
one, at the expense of increasing the number of centers by a factor of $O(\log n)$. This yields a set $C$ with
$O(k \log^2 n)$ centers and a cost of at most $4\OPT'$. As mentioned above, we feed $C$ into the auxiliary
data structure with $O(n + k)$ update time. This yields a solution with cost at most $6\OPT' \leq (6 + \epsilon)\OPT-$
approximation overall. As one update in $P$ might lead to many changes in $C$ this requires a careful amortized
analysis to achieve an update time $O(k \log^2 n \log \Delta)$.

Related work and all missing proofs can be found in the appendix.

2 Bounds against adaptive adversaries

In this section we present our upper and lower bounds on the approximation guarantees of dynamic algo-
rithms for $k$-center, $k$-median, $k$-means, $k$-sum-of-radius, $k$-sum-of-diameters, and $(k, p)$-clustering against
an adaptive adversary. We start with the lower bounds. First, we define a generic strategy for an adversary
that in each operation creates or deletes a point and answers the distance queries of the algorithm. Then we
show how to derive lower bounds for all of our problems from this single strategy. Finally, we show that our
lower bounds for $k$-center are asymptotically tight by giving a matching upper bound.

Strategy of the adaptive adversary. Let $f(k, n)$ be a positive function that is, for every fixed $k$, non-
decreasing in $n$. Suppose that there is an algorithm (for any of the problems under consideration) that in an
amortized sense queries the distances between at most $f(k, n)$ pairs of points per operation of the adversary,
where $n$ denotes the number of points at the beginning of the respective operation. Note that any algorithm
with an amortized update time of at most $f(k, n)$ fulfills this condition. To determine the distance between
two points the algorithm asks a distance query to the adversary. We present now an adversary $\mathcal{A}$ whose
goal is to maximize the approximation ratio of the algorithm. To record past answers and to give consistent
answers, $\mathcal{A}$ maintains a graph $G = (V, E)$ which contains a vertex $v_p \in V$ for each point $p$ that has been
inserted previously (including points that have been deleted already). Intuitively, with the edges in $E$ the
adversary keeps track of previous answers to distance queries. Each vertex $v_p$ is labeled as active, passive,
or off. If a point $p$ has not been deleted yet, then its vertex $v_p$ is labeled as active or passive. Once point $p$
is deleted, then $v_p$ is labeled as off. Intuitively, if $p$ has not been deleted yet, then $v_p$ is active if it has small
degree and passive if it has large degree. In the latter case, $\mathcal{A}$ will delete $p$ soon. All edges in $G$ have length
1, and for two vertices $v, v' \in V$ we denote by $d_G(v, v')$ their distance in $G$.

When choosing the next update operation, $\mathcal{A}$ checks whether there is a passive vertex $v_p$. If yes, $\mathcal{A}$ picks
an arbitrary passive vertex $v_{p'}$, deletes the corresponding point $p$, and labels the vertex $v_{p'}$ as off. Otherwise,
it adds a new point $p$, adds a corresponding vertex $v_{p'}$ to $G$, and labels $v_{p'}$ as active.

Suppose now that the algorithm queries the distance $d(p, p')$ for two points $p, p'$ while processing an
operation. Note that $p$ and/or $p'$ might have been deleted already. If both $v_p$ and $v_{p'}$ are active then $\mathcal{A}$ reports
to the algorithm that \( d(p, p') = 1 \) and adds an edge \( \{v_p, v_{p'}\} \) to \( E \). Intuitively, due to the edge \( \{v_p, v_{p'}\} \) the adversary remembers that it reported the distance \( d(p, p') = 1 \) before and ensures that in the future it will report distances consistently. Otherwise, \( A \) considers an augmented graph \( G' \) which consists of \( G \) and has in addition an edge \( \{v_p, v_{p'}\} \) of length 1 between any pair of active vertices \( \bar{p}, \bar{p}' \). The adversary computes the shortest path \( P \) between \( p \) and \( p' \) in \( G' \) and reports that \( d(p, p') \) equals the length of \( P \). If \( P \) uses an edge between two active vertices \( \bar{p}, \bar{p}' \), then \( A \) adds the edge \( \{v_p, v_{p'}\} \) to \( G \). Note that \( P \) can contain at most one edge between two active vertices since it is a shortest path in a graph in which all pairs of active vertices have distance 1. Observe that if both \( v_p \) and \( v_{p'} \) are active then this procedure reports that \( d(p, p') = 1 \) and adds an edge \( \{v_p, v_{p'}\} \) to \( E \) which is consistent with our definition above for this case. If a vertex \( v_p \) has degree at least 100(f(k, t)) for the current operation \( t \), then \( v_p \) is labeled as passive. A passive vertex never becomes active again.

In the next lemma we prove some properties about this strategy of \( A \). For each operation \( t \), denote by \( G_t = (V_t, E_t) \) the graph \( G \) at the beginning of the operation \( t \). Recall that the value of \( n \) right before the operation \( t \) (which is the number of current points) equals the number of active and passive vertices in \( G_t \). We show that the the number of active vertices is \( \Theta(t) \), each vertex has bounded degree, and there exist arbitrarily large values \( t \) such that in \( G_t \) there are no passive vertices (i.e., only active and off vertices).

**Lemma 1.** For every operation \( t > 0 \) the strategy of the adversary ensures the following properties for \( G_t \)

1. The number of active vertices in \( G_t \) is at least 96t/100,
2. Each vertex in \( G_t \) has a degree of at most 100f(k, n),
3. There exists an operation \( t' \) with \( t < t' \leq 2t \) such that \( G_{t'} \) contains only active and off vertices, but no passive vertices.

We say that an operation \( t \in \mathbb{N} \) is a clean operation if in \( G_t \) there are no passive vertices. For any clean operation \( t \), denote by \( G_t = (\bar{V}_t, \bar{E}_t) \) the subgraph of \( G_t \) induced by the active vertices in \( V_t \).

**Consistent metrics.** The algorithm does not necessarily know the complete metric of the given points, it knows only the distances reported by the adversary. In particular, there might be many possible metrics that are consistent with the reported distances. For each \( t \in \mathbb{N} \) denote by \( Q_t \) the points that were inserted before operation \( t \), including all points that were deleted before operation \( t \), and let \( P_t \subseteq Q_t \) denote the points in \( Q_t \) that are not deleted. Given a metric \( M \) on any point set \( P' \), for all pairs of points \( p, p' \in P' \) we denote by \( d_M(p, p') \geq 0 \) the distance between \( p \) and \( p' \) according to \( M \). For any \( t \in \mathbb{N} \) we say that a metric \( M \) for the point set \( Q_t \) is consistent if for any pair of points \( p, p' \in Q_t \) for which the adversary reported the distance \( d(p, p') \) before operation \( t \), it holds that \( d(p, p') = d_M(p, p') \). In particular, any consistent metric might be the true underlying metric for the point set \( Q_t \).

The key insight is that for each clean operation \( t \), we can build a consistent metric with the following procedure. Take the graph \( G_t \) and insert an arbitrary set of edges of length 1 between pairs of active vertices (but no edges that are incident to off vertices), and let \( G'_t \) denote the resulting graph. Let \( M \) be the shortest path metric according to \( G'_t \). If a metric \( M \) for \( Q_t \) is constructed in this way, we say that \( M \) is an augmented graph metric for \( t \).

**Lemma 2.** Let \( t \in \mathbb{N} \) be a clean operation and let \( M \) be an augmented graph metric for \( t \). Then \( M \) is consistent.

In particular, there are no shortcuts via off vertices in \( G_t \) that could make the metric \( M \) inconsistent.

We fix a clean operation \( t \in \mathbb{N} \). We define some metrics that are consistent with \( Q_t \) that we will use later for the lower bounds for our specific problems. The first one is the “uniform” metric \( M_{uni} \) that we obtain by adding to \( G_t \) an edge between each pair of active vertices in \( G_t \). As a result, \( d_{M_{uni}}(p, p') = 1 \) for any \( p, p' \in P_t \).

**Lemma 3.** For each clean operation \( t \) the corresponding metric \( M_{uni} \) is consistent.

In contrast to \( M_{uni} \), our next metric ensures that there are distances of up to \( \Omega(\log n) \) between some
pairs of points. Let \( p^* \in P_t \) be a point such that \( v_{p^*} \) is active. For each \( i \in \mathbb{N} \) let \( V^{(i)} \subseteq V_t \) denote the active vertices \( v \in V_t \) with \( d_{G_t}(v_{p^*}, v) = i \), and let \( V^{(n)} \subseteq V_t \) denote the vertices in \( G_t \) that are in a different connected component than \( p^* \). Since the vertices in \( G_t \) have degree at most \( 100f(k, n) \), more than half of all vertices are in sets \( V^{(i)} \) with \( i \geq \Omega(\log n/\log f(k, n)) \).

We define now a metric \( M(p^*) \) as the shortest path metric in the graph defined as follows. We start with \( G_t \), for each \( i, i' \in \mathbb{N} \) we add to \( G_t \) an edge \( \{v_p, v_{p'}\} \) between any pair of vertices \( v_p \in V^{(i)}, v_{p'} \in V^{(i')} \) such that \( |i - i'| \leq 1 \). As a result, for any \( i, i' \in \mathbb{N} \) and any \( v_p \in V^{(i)}, v_{p'} \in V^{(i')} \) we have that \( d_{M(p^*)}(p, p') = \max\{|i - i'|, 1\} \), i.e., \( d_{M(p^*)}(p, p') = 1 \) if \( i = i' \) and \( d_{M(p^*)}(p, p') = |i - i'| \) otherwise.

**Lemma 4.** For each clean operation \( t \) and each point \( p^* \in V_t \) the metric \( M(p^*) \) is consistent.

For any two thresholds \( \ell_1, \ell_2 \in \mathbb{N}_0 \) with \( \ell_1 < \ell_2 \) we define a metric \( M_{\ell_1, \ell_2}(p^*) \) (which is a variation of \( M(p^*) \)) as the shortest path metric in the following graph. Intuitively, we group the vertices in \( \bigcup_{i=\ell_1}^{\ell_2} V^{(i)} \) to one large group and similarly the vertices in \( \bigcup_{i=\ell_1}^{\infty} V^{(i)} \). Formally, in addition to the edges defined for \( M(p^*) \), for each pair of vertices \( v_p \in V^{(i)}, v_{p'} \in V^{(i')} \) we add an edge \( \{v_p, v_{p'}\} \) if \( i \leq i' \leq \ell_1 \) or \( \ell_2 \leq i \leq i' \).

**Lemma 5.** For each clean operation \( t \), each point \( p^* \in V_t \), and each \( \ell_1, \ell_2 \in \mathbb{N}_0 \) the metric \( M_{\ell_1, \ell_2}(p^*) \) is consistent.

**Lower bounds.** Consider a clean operation \( t \). The algorithm cannot distinguish between \( M_{\text{uni}} \) and \( M(p^*) \) for any \( p^* \in P_t \). In particular, for the case that \( k = 1 \) (for any of our clustering problems) the algorithm selects a point \( p^* \) as the center, and then for each \( i \) it cannot determine whether the distance of the points in \( V^{(i)} \) to \( p^* \) equals 1 or \( i \). However, there are at least \( n/2 \) points in sets \( V^{(i)} \) with \( i \geq \Omega(\log n/\log f(k, n)) \) and hence they contribute a large amount to the objective function value. This yields the following lower bounds, already for the case that \( k = 1 \). With more effort, we can show them even for bi-criteria approximations, i.e., for algorithms that may output \( O(k) \) centers, but where the approximation ratio is still calculated with respect to the optimal cost on \( k \) centers.

**Theorem 6.** Against an adaptive adversary, any dynamic algorithm that queries amortized at most \( f(k, n) \) distances per operation and outputs at most \( O(k) \) centers

- for \( k \)-center, \( k \)-sum-of- radii or \( k \)-sum-of-diameters has an approximation ratio of \( \Omega(\frac{\log(n)}{\log f(1,2n)}) \).
- for \( k \)-median has an approximation ratio of \( \Omega(\frac{\log(n)}{10+2\log f(1,2n)}) \).
- for \( k \)-means has an approximation ratio of \( \Omega(\frac{\log(n)}{12+2\log f(1,2n)}) \).
- for \((k, p)\)-clustering has an approximation ratio of \( \Omega(\frac{\log(n)}{2p+8+2\log f(1,2n)}) \).

For \( k \)-center the situation changes if the algorithm does not need to be able to report an upper bound of the value of its computed solution (but only the solution itself), since if \( k = 1 \), then any point is a \( 2 \)-approximation. However, for arbitrary \( k \) we can argue that there must be \( 3k \) consecutive sets \( V^{(i)}, V^{(i+1)}, ..., V^{(i+3k-1)} \) such that the algorithm does not place any center on any point corresponding to the vertices in these sets and hence incurs a cost of at least \( 3k/2 \). On the other hand, for the metric \( M_{i,i+3k-1}(p^*) \) the optimal solution selects one center from each set \( V^{(i+1)}, V^{(i+4)}, V^{(i+7)}, \ldots \) which yields a cost of only 1.

**Theorem 7.** Against an adaptive adversary, any dynamic algorithm for \( k \)-center that queries amortized at most \( f(k, n) \) distances per operation and outputs at most \( O(k) \) centers has an approximation factor of at least \( \Omega\left(\min\left\{ k, \frac{\log(n)}{k\log f(k,2n)} \right\} \right) \), even if at query time it needs to output only a center set, but no estimate \( \text{apx on OPT} \).

We show that our lower bound for \( k \)-center is asymptotically tight by giving an algorithm that matches these bounds. Our algorithm is deterministic, and hence its guarantees hold against an adaptive adversary.

**Theorem 8.** (see Appendix B) There is a dynamic deterministic algorithm for \( k \)-center with update time of \( O(k \log n \log \Delta/\epsilon) \) and an approximation factor of \( (1 + \epsilon) \min\{4k, 4\log(n/k)\} \).
3 Algorithms for $k$-sum-of-radii and $k$-sum-of-diameters

In this section we present our (randomized) dynamic $(13.008 + \epsilon)$-approximation algorithm for $k$-sum-of-radii with an amortized update time of $k^O(1/\epsilon) \log \Delta$, against an oblivious adversary. Our strategy is to maintain a bi-criteria approximation with $O(k/\epsilon)$ clusters whose sum of radii is at most $(6 + \epsilon)OPT$ (and which covers all input points). We show how to use an arbitrary offline $\alpha$-approximation algorithm to turn this solution into a $(6 + 2\alpha + \epsilon)$-approximate solution with only $k$ clusters. Using the algorithm in [9] (for which $\alpha = 3.504 + \epsilon$), this yields a dynamic $(13.008 + \epsilon)$-approximation for $k$-sum-of-radii, and hence a $(26.015 + \epsilon)$-approximation for $k$-sum-of-diameters.

Assume that we are given an $\epsilon > 0$ such that w.l.o.g. it holds that $1/\epsilon \in \mathbb{N}$. We maintain one data structure for each value $OPT'$ that is a power of $1 + \epsilon$ in $[1, \Delta]$, i.e., $O(\log(1 + \epsilon) \Delta)$ many. The data structure for each such value $OPT'$ outputs a solution of cost at most $(13.008 + \epsilon)OPT'$ or asserts that $OPT > OPT'$. We output the solution with smallest cost which is hence a $(13.008 + O(\epsilon))$-approximation. We describe first how to maintain the mentioned bi-criteria approximation. We define $z := \epsilon OPT'/k$. Our strategy is to maintain a solution for an auxiliary problem based on a Lagrangian relaxation-type approach. More specifically, we are allowed to select an arbitrarily large number of clusters, however, for each cluster we need to pay a fixed cost of $z$ plus the radius of the cluster. For the radii we allow only integral multiples of $z$ that are bounded by $OPT'$, i.e., only radii in the set $R = \{z, 2z, ..., OPT' - z, OPT'\}$.

This problem can be modeled by an integer program whose LP-relaxation is given in Section 1.2; we will show below that it holds that $\bar{OPT}OPT$ for each such value $OPT'$.

Lemma 9. Each iteration $i$ needs a running time of $O(|U_i| \epsilon + ik/\epsilon)$.

We stop when in some iteration $i^*$ it holds that $U_{i^*} = \emptyset$ or if we completed the $(2k/\epsilon)^2$-th iteration and $U_{(2k/\epsilon)^2+1} \neq \emptyset$. Suppose that $x$ and $y$ are the primal and dual vectors after the last iteration. In case that $U_{(2k/\epsilon)^2} \neq \emptyset$ we can guarantee that our dual solution has a value of more than $OPT'$ from which we can conclude that $OPT > OPT'$; therefore, we stop the computation for the estimated value $OPT'$ in this case.

Lemma 10. If $U_{(2k/\epsilon)^2+(2k/\epsilon)} \neq \emptyset$ then $OPT > OPT'$.

If $U_{(2k/\epsilon)^2} = \emptyset$ we perform a pruning step in order to transform $x$ into a solution whose cost is at most by a factor 3 larger than the cost of $y$. We initialize $S := \emptyset$. Let $S = \{(p_1, r_1), (p_2, r_2), ...\}$ denote the set of pairs $(p, r)$ with $x_p(r) = 1$. We sort the pairs in $S$ non-increasingly by their respective radius $r$. Consider a pair $(p_j, r_j)$. We insert the cluster $(p_j, 3r_j)$ in our solution $\bar{S}$ and delete from $S$ all pairs $(p_j', r_j')$ such that $j' > j$ and $d(p_j, p_{j'}) < r_j + r_{j'}$. Note that $(p_j, 3r_j)$ covers all points that are covered by any deleted pair $(p_{j'}, r_{j'})$ due to our ordering of the pairs. Let $\bar{S}$ denote the solution obtained in this way and let $\bar{x}$ denote the corresponding solution to $\bar{(P)}$, i.e., $\bar{x}_p(r) = 1$ if and only if $(p, r) \in \bar{S}$. We will show below
that \( \bar{S} \) is a feasible solution to \((P)\) with at most \(O(k/\epsilon)\) clusters. Let \( \bar{C} \subseteq P \) denote their centers, i.e.,
\[
\bar{C} = \{p \mid \exists r \in R : (p, r) \in \bar{S} \}.
\]

**Lemma 11.** Given \( \bar{x} \) we can compute \( \bar{x} \) in time \(O((k/\epsilon)^3)\) and \( \bar{x} \) selects at most \(O(k/\epsilon)\) centers.

We transform now the bi-criteria approximate solution \( \bar{x} \) into a feasible solution \( \tilde{x} \) with only \( k \) clusters. To this end, we invoke the offline \((3.504 + \epsilon)\)-approximation algorithm from \([2]\) on the input points \( \bar{C} \). Let \( \hat{S} \) denote the set of pairs \((\hat{p}, \hat{r})\) that it outputs (and note that not necessarily \( \hat{r} \in R \) since we use the algorithm as a black-box). Note that \( \hat{S} \) covers only the points in \( \bar{C} \), and not necessarily all points in \( P \). On the other hand, the solution \( \hat{S} \) has a cost of at most \( 2OPT \) since we can always find a solution with this cost covering \( \bar{C} \), even if we are only allowed to select centers from \( \bar{C} \). Thus, based on \( \bar{S} \) and \( \hat{S} \) we compute a solution \( S \) with at most \( k \) clusters that covers \( P \). We initialize \( S := \emptyset \). For each pair \((\hat{p}, \hat{r}) \in \hat{S} \) we consider the points \( \bar{C}' \) that are covered by \((\hat{p}, \hat{r})\). Among all these points, let \( \bar{p} \) be the point with maximum radius \( \hat{r} \) such that \((\bar{p}, \hat{r}) \in \hat{S} \). We add to \( \hat{S} \) the pair \((\bar{p}, \hat{r} + \hat{r})\) and remove \( \bar{C}' \) from \( \bar{C} \). We do this operation with each pair \((\bar{p}, \hat{r}) \in \hat{S} \). Let \( \hat{S} \) denote the resulting set of pairs.

**Lemma 12.** Given \( S \) and \( \hat{S} \) we can compute \( \hat{S} \) in time \(O(k^3/\epsilon^2)\).

We show that \( \hat{S} \) is a feasible solution with small cost. We start by bounding the cost of \( \hat{x} \) via \( y \).

**Lemma 13.** We have that \( \bar{x} \) and \( y \) are feasible solutions to \((P)\) and \((D)\), respectively, for which we have that
\[
\sum_{p \in P} \sum_{r \in R} \bar{x}_p(r + z) \leq 6 \cdot \sum_{p \in P} y_p \leq 6OPT'.
\]

Next, we argue that \( \hat{S} \) is feasible and bound its cost by the cost of \( S' \) and the cost of \( \hat{x} \).

**Lemma 14.** We have that \( \hat{S} \) is a feasible solution with cost at most
\[
\sum_{(\bar{p}, \hat{r}) \in \hat{S}} \bar{r} + \sum_{p \in P} \sum_{r \in R} \bar{x}_p(r + z) \leq (13.008 + \epsilon)OPT'.
\]

### 3.1 Dynamic algorithm

We describe now how we maintain the solutions \( x, \bar{x}, y, S, \tilde{S}, \) and \( \hat{S} \) dynamically when points are inserted or deleted. Our strategy is similar to \([8]\).

Suppose that a point \( p \) is inserted. For each \( i \in \{1, ..., 2k/\epsilon + 1\} \) we insert \( p \) into the set \( U_i \) if \( U_i \neq \emptyset \) and \( p \) is not covered by a pair \((p_j, r_j)\) with \( j \in \{1, ..., i - 1\} \). If there is an index \( i \in \{1, ..., 2k/\epsilon + 1\} \) such that \( U_{i-1} \neq \emptyset \) (assume again that \( U_0 \neq \emptyset \)), \( U_i = \emptyset \), and \( p \) is not covered by any pair \((p_j, r_j)\) with \( j \in \{1, ..., i - 1\} \), we start the above algorithm in the iteration \( i \), being initialized with \( U_i = \{p\} \) and the solutions \( x, y \) as being computed previously.

Suppose now that a point \( p \) is deleted. We remove \( p \) from each set \( U_i \) that contains \( p \). If there is no \( r \in R \) such that \((p, r) \in S \) then we do not do anything else. Assume now that \((p, r) \in S \) for some \( r \in R \). The intuition is that this does not happen very often since in each iteration \( i \) we choose a point uniformly at random from \( U_i \). More precisely, in expectation the adversary needs to delete a constant fraction of the points in \( U_i \) before deleting \( p \). Consider the index \( i \) such that \((p, r) = (p_i, r_i) \). We restart the algorithm from iteration \( i \). More precisely, we initialize \( y \) to the values that they have after raising the dual variables \( y_{p_1}, ..., y_{p_{i-1}} \) in this order as described above until a constraint for the respective point \( p_j \) becomes half-tight. We initialize \( x \) to the corresponding primal variables, i.e., \( x_{p_j}^{(2r_j)} = 1 \) for each \( j \in \{1, ..., i - 1\} \) and \( x_{p_j}^{(r_j)} = 0 \) for all other values of \( p', r' \). Also, we initialize the set \( U_i \) to be the obtained set after removing \( p \). With this initialization, we start the algorithm above in iteration \( i \), and thus compute like above the (final) vectors \( y, x \) and based on them \( S, \tilde{S}, \) and \( \hat{S} \).

When we restart the algorithm in some iteration \( i \) then it takes time \( O(|U_i|k^2) \) to compute the new set \( S \). We can charge this to the points that were already in \( U_i \) when \( U_i \) was recomputed the last time and to the points that were inserted into \( U_i \) later. After a point \( p \) was inserted, it is charged at most \( O(k/\epsilon) \) times in the latter manner since it appears in at most \( O(k/\epsilon) \) sets \( U_i \). Finally, given \( S \), we can compute the sets
The algorithm from [8] takes time $n^{O(1/\epsilon)}$ if the input has size $n$. One can show that this yields an update time of $k^{O(1/\epsilon)} + (k/\epsilon)^{4}$ for each value OPT. Finally, the same set $\tilde{S}$ yields a solution for $k$-sum-of-diameters, increasing the approximation ratio by a factor of 2.

**Theorem 15.** There are dynamic algorithms for the $k$-sum-of-diameters and the $k$-sum-of-diameters problems with update time $k^{O(1/\epsilon)} \log \Delta$ and with approximation ratios of $13.008 + \epsilon$ and $26.016 + \epsilon$, respectively, against an oblivious adversary.

## 4 Algorithm for $k$-center

We present our algorithm for the $k$-center problem which maintains a $(6 + \epsilon)$-approximate solution with an amortized update time of $O(k \log^2 n \log \Delta)$ against an adaptive adversary. As subroutines, it uses a dynamic $(2 + \epsilon)$-approximation algorithm with an update time of $O(n+k)$, and, for the deletion-only case, a bi-criteria $(4 + \epsilon)$-approximation algorithm that is allowed to use $O(k \log n)$ centers. Again, we run $O(\log_{1+\epsilon} \Delta)$ data structures in parallel, one for each value OPT that is a power of $1 + \epsilon$ in $[1, \Delta]$, and the algorithm outputs the solution of the data structure corresponding to the minimum value OPT among all data structures that output a solution (rather than asserting that the respective OPT < OPT). In the appendix, we provide the details of the fully dynamic algorithm with linear update time that we sketched in the technical overview, corresponding to the following lemma.

**Lemma 16.** For any given value OPT there is a fully dynamic algorithm that, started on a (possibly empty) set $P$ with $|P| = n_0$ for some integer $n_0$, has preprocessing time $O(k n_0)$, and, for the deletion-only case, a bi-criteria $(4 + \epsilon)$-approximation algorithm that is allowed to use $O(k \log n)$ centers. Again, we run $O(\log_{1+\epsilon} \Delta)$ data structures in parallel, one for each value OPT that is a power of $1 + \epsilon$ in $[1, \Delta]$, and the algorithm outputs the solution of the data structure corresponding to the minimum value OPT among all data structures that output a solution (rather than asserting that the respective OPT < OPT). In the appendix, we provide the details of the fully dynamic algorithm with linear update time that we sketched in the technical overview, corresponding to the following lemma.

**4.1 Deletion-only algorithm with $O(k \log n)$ centers**

Suppose we are given a set of $n$ points $P$ and in each operation the adversary deletes a point in $P$, but the adversary cannot insert new points into $P$. We present a data structure for this case which we preprocess as follows. We define a set of centers $C$ and a partition $\mathcal{C}$ of $C$ into buckets $C_{\log n}, C_{\log n-1}, ..., C_1$ which we initialize by $C := \emptyset$ and $C_j := \emptyset$ for each $j \in \{1, ..., \log n\}$. At the beginning of each iteration $i$ we are given a set of points $U_i$ where for $i = 1$ we define $U_1 := P$. In each iteration $i$, we select a point $c_i \in U_i$ uniformly at random from $U_i$, add $c_i$ to $C$ (i.e., make it a center). Note that this induces a natural ordering on the centers, i.e., $c_i$ becomes the $i$-th center. We also assign $c_i$ to the bucket $C_j$ such that $2^{j-1} < |U_i| \leq 2^j$. Also, we assign to $c_i$ all points in $P(c_i) := \{p \in U_i \mid d(p, c_i) \leq 2\text{OPT}'\}$ which form the cluster of $c_i$. We define $U_{i+1} := \{p \in U_i \mid d(c, p) > 2\text{OPT}'\}$ and observe that $|U_i| > |U_{i+1}|$. Note that therefore every center $c_j$ with $1 \leq j \leq i - 1$ was assigned to a bucket in $C_{\log n}, C_{\log n-1}, ..., C_1$. We stop at the beginning of an iteration $i$ if $U_i = \emptyset$. Also, we stop after some iteration $i$ if the respective bucket $C_j$ of iteration $i$ contains $2k$ centers after adding $c_i$ to it. In particular, this ensures that $|C| = \sum_{j=1}^{\log n} |C_j| \leq 2k \log n$. We say that a bucket $C_j$ is small if $|C_j| \leq k$ and big otherwise. If $|C_j| = 2k$ the bucket is very big, and no bucket has ever more than $2k$ centers. Whenever a bucket $C_j$ has received $2k$ centers, the while loop stops and we call $C_j$ the (very big) bucket $C^\ast$. There can be at most one very big bucket $C^\ast$ and if so, $C^\ast$ must be the last bucket into which centers were added and out of all non-empty buckets the one with smallest index. Note also that as long as $C^\ast$ is big or very big, it provides a “witness” that no solution for the $k$-center problem with value OPT’ exists.

As data structure we keep (1) an array $\mathcal{C}$ of size $\log n$ with each entry pointing to the corresponding bucket if it is non-empty, and a nil-pointer otherwise, (2) for each bucket a list of its centers, (3) for each center $c$ the index of its bucket as well as a list to all points in $P(c)$, and for every point that is not a center,
a nil-pointer instead of $P(c)$, (4) a pointer to the index of $C^*$ in the array $C$, (5) the number of centers in $C^*$, (6) a binary flag $F$, described below, and if $F$ is set, a list $U^*$ which contains all points that are not within distance 2OPT' of any center as well as the sets $P(c)$ of some already deleted centers $c$ of $C^*$. A set flag $F$ indicates that no solution for the k-center problem with value OPT exists since the last bucket is big. If $F$ is not set, the set $C$ contains a set of at most $2k\log n$ centers that cover $P$ and $F$ will never set again.

**Update operation.** Suppose now that the adversary deletes a point $p \in P$. Intuitively, we do the following: if $p$ is not a center then we do not need to do anything. If $p$ is a center and $F$ is not set then we simply replace $p$ by some other point in its cluster. We do the same if $F$ is set but $p$ is not a center from the last bucket $C^*$. If $F$ is set and $p \in C^*$ then we might perform a partial rebuild. Formally, we update the clustering lazily as follows:

**Case 1:** If $p \notin C$ then we simply delete $p$ from the set $P(c)$ that it belongs to.

**Case 2:** If $p \in C$ and $p$ does not belong to $C^*$, we select another point $p' \in P(p)$ instead of $p$ and set $P(p')$ to $P(p)$; if $P(p) = \emptyset$, we do nothing and the size of $C$ decreases by one.

**Case 3:** If $p$ belongs to $C^*$ and $F$ is not set, we perform the same update as in Case 2.

**Case 4:** If $p$ belongs to $C^*$ and $F$ is set, we simply delete $p$ as center $C^*$, add its set $P(p)$ to $U^*$, and decrease the center counter of $C^*$. If $C^*$ becomes small due to this deletion, let $c_i \in C^*$ denote the first center that was inserted into $C^*$, i.e., it is the center with the smallest index in $C^*$. We define $U_i := \{p \in P \mid \forall i' < i : d(p, c_{i'}) > 2OPT'\}$. Note that this set can be created by taking the union of the sets $P(c)$ stored in the list of pointers of $C^*$ with $U^*$. Let $C := \bigcup_{j=1}^{i-1} C_j$ which can be created by removing all centers $c_i, c_{i+1}, \ldots$ (stored at $C^*$) from $C$. Then we perform a partial rebuild by calling a partial rebuild routine $PR(U_i, C)$.

We describe the partial rebuild routine $PR(U_i, C)$ now. We start the routine by initializing it with $U_i$ and $C$. Then we select a point $c_i$ uniformly at random from $U_i$, add $c_i$ to $C$, and assign $c_i$ to the bucket $C_j$ such that $2^{i-1} < |U_i| \leq 2^i$. As we are in the deletion-only case, $C_j$ is necessarily the non-empty bucket (after $c_i$ was added) with smallest index. Also, by processing all points in $U_i$ we create the set $P(c_i) := \{p \in U_i \mid d(p, c_i) \leq 2OPT'\}$, assign it to $c_i$, and remove them from $U_i$. Let us call the resulting set $U_{i+1}$, i.e., $U_{i+1} := U_i \setminus P(c_i)$, and iterate. Let $C_j$ be the bucket into which $c_i$ was placed. Like before, we stop if $U_i = \emptyset$ at the beginning of some iteration $i$ or if after some iteration $i$ the bucket $C_j$ contains $2k$ centers. In the former case we unset the flag $F$, and then no further partial rebuild will happen and also no very big bucket will ever exist again. In the latter case we keep the flag $F$ set and denote $C_j$ by $C^*$.

**Update time.** We want to show that the algorithm has an amortized update time of $O(k)$ per operation. To argue this, we partition the operations into phases $Q_1, Q_2, \ldots$ such that whenever an operation $t$ causes us to partially rebuild, the current phase ends. Note that each update in a phase, except the last one, only performs $O(1)$ work. Thus, if no partial rebuild happens at the end of a phase $Q$ (because the flag $F$ is not set or because after the last operation in $Q$ the adversary does not delete any more points), then we only perform $O(1)$ work per update. Thus, consider a phase $Q$ at whose end we perform a partial rebuild. Then, at the beginning of $Q$ there is exactly one very big bucket, namely $C^*$, which contains exactly $2k$ centers at the beginning of $Q$ and exactly $k$ at the end. Thus, exactly $k$ centers were deleted from $C^*$ during $Q$. Let $t$ be the last operation of that phase. For each operation $t \in Q$ we define a budget $a_t$ as follows. Let $P(t)$ denote the set of existing points before operation $t$ and for each center $c_{i'} \in C^*$ we define $U_{i'}^{(t)} := \{p \in P(t) \mid \forall i'' < i' : d(p, c_{i''}) > 2OPT'\}$. As we are in the deletion-only case, $U_{i'}^{(t)} \subseteq U_{i'}^{(t)}$ for $t' \leq t$. If operation $t$ does not delete a center in $C^*$, then we define $a_t := k$. Otherwise, suppose that operation $t$ deletes a center $c_i \in C^*$. We set $a_t := |U_i^{(t)}| + k$. For each of the at most $2k$ centers $c_{i'} \in C^*$ we know that the adversary deletes $c_{i'}$ with probability at most $1/|U_{i'}^{(t)}|$. This holds since $c_{i'}$ was chosen uniformly at random from the respective set $U_{i'}$ when $c_{i'}$ was selected the last time in a partial rebuild or during preprocessing. In case that $c_{i'}$ is deleted it holds that $a_t = |U_{i'}^{(t)}| + k$. Since $|C^*| \leq 2k$ one can show that $E[a_t] \leq 1 \cdot k + \sum_{t \in [2k]} 1 \leq O(k)$.
Lemma 17. For every \( t \geq 0 \) we have that \( \mathbb{E}[a_t] \leq O(k) \).

Let \( c_i \) be the first center that was added to \( C^* \) in phase \( Q \) and let \( U_i \) be the corresponding set during the last partial rebuild. The overall work during phase \( Q \) is bounded by \( O(|Q| + k|U_i|) \), where \( |Q| \) denotes the number of iterations during \( Q \): for the at most \( 2k \log n \) new centers chosen during the partial rebuild at the very end, it takes time \( \Theta(|U_i| + |U_{i+1}| + \ldots) \) to compare, for all \( j \geq i \), every point in \( U_j \) to determine \( P(c_j) \). Since \( \sum_{j=0}^{k \log n} |U_{i+j}| \leq \sum_{j=0}^{k \log n} k|U_{i+j}k| \leq \sum_{j=0}^{k \log n} k|U_i|/2^j \leq 2k|U_i| \), the partial rebuild takes time \( \Theta(k|U_i|) \). All other operations in \( Q \) cause \( O(1) \) work each. Now the key insight is that \( |Q| + k|U_i| \leq O(\sum_{t \in Q} a_t) \). If \( |Q| \geq |U_i|/4 \), this is immediate since each \( a_t \geq k \). Otherwise, we observe that if a center \( c_{t'} \) is deleted by operation \( t \), then \( a_t = |U_{t'}^{[t]}| \geq |U_i|/4 \) as at most \( |Q| < |U_i|/4 \) many points where deleted during phase \( Q \) and at the beginning of the phase \( |U_{t'}| > |U_i|/2 \). Since \( k \) centers are deleted before a partial rebuild, we obtain the following lemma.

Lemma 18. The running time during phase \( Q \) is bounded by \( O(\sum_{t \in Q} a_t) \).

Now Lemmas 17 and 18 together imply that the expected running time during phase \( Q \) is bounded by \( O(|Q|k) \) which yields an amortized expected update time of \( O(k) \) per operation.

Lemma 19. The algorithm has an amortized expected update time of \( O(k) \) per operation.

For the fully dynamic algorithm in the next subsection we also need to bound the amortized number of changes in the set \( C \).

Lemma 20. There are at least \( k \) deletions between two partial rebuilds. If there are more than 2 changes to \( C \) after an update operation, then there was a partial rebuild at this operation.

Proof. Each phase consists of all deletions between two consecutive rebuilds. At the beginning of a phase \( C^* \) contains \( 2k \) centers, at the end of the phase it only contains \( k \) centers. Thus there are at least \( k \) deletions during a phase, i.e., between two partial rebuilds.

Each update operation that is not the last operation of a phase deletes at most one and creates at most one new center. Thus, after a deletion there are only two changes to \( C \), except if a partial rebuild was performed. \( \square \)

Query operation. The full details appear in the appendix (see the technical overview for a short version).

Lemma 21. There is an algorithm for the deletion-only case with an expected update time of \( O(k) \) that either outputs a solution of value \( 4OPT' \) with \( O(k \log n) \) centers, or asserts that there is no solution with value \( OPT' \) that uses at most \( k \) centers.

4.2 General case

We use the algorithms from Lemma 16 and Section 4.1 and techniques from Overmars [31] in order to obtain a fully dynamic data structure with \( \tilde{O}(k) \) time per operation. We partition the points \( P \) into \( O(\log n) \) groups \( P_1, \ldots, P_{O(\log n)} \). This partition is dynamically created by the algorithm. The data structure consists of three parts: (1) For \( P_1 \) we use the fully dynamic data structure with linear update time from Lemma 16 to which we refer to as \( D_1 \). (2) For each \( P_j \) with \( j \geq 2 \) we maintain the data structure from Section 4.1 which we will refer to as \( D_{2,j} \). As this data structure can only handle deletions, we describe below how we handle insertions. (3) For each \( j \in \{1, \ldots, \log n\} \), let \( C_j \subseteq P_j \) denote the centers selected by the data structure for \( P_j \). Let \( C := \bigcup_j C_j \). Note that \( |C| = O(k \log^2 n) \) at all times. We use another instance of the data structure from Lemma 16 to which we refer to as \( D_3 \), with \( C \) as input points. Let \( C' \subseteq C \) denote the centers selected by this data structure. If \( |C'| \leq k \), the algorithm outputs \( C' \), otherwise \( |C'| = k + 1 \) and it outputs that \( OPT' < OPT \).
Theorem 23. There is a randomized dynamic $(6 + \epsilon)$-approximation algorithm for the $k$-center problem.
with amortized update time of $O(k \log \Delta \log^2 n)$ against an oblivious adversary.

We prove that our update time is tight up to logarithmic factors because an algorithm needs to make at least amortized $\Omega(k)$ queries in expectation to obtain any non-trivial approximation ratio. This holds for $k$-center and also for all other clustering problems studied in this paper.

**Theorem 24.** Let $\epsilon > 0$. Any (randomized) dynamic algorithm for finite metric spaces that can provide against an oblivious adversary an $(\Delta - \epsilon)$-approximation to the cost of $k$-center, $k$-sum-of-radii, $k$-sum-of-diameters or an $(\Delta^p - \epsilon)$-approximation to the cost of $(1, p)$-clustering (where $p = 1$ yields $k$-median and $p = 2$ yields $k$-means) queries amortized $\Omega(k)$ distances in expectation after $O(k^2)$ operations, also if the algorithm needs to provide only its center set and not an approximation $\text{apx}$ to the cost.

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A Related work

We summarize prior work on offline and dynamic results for the clustering problems we consider in this paper. Most prior work on fully dynamic clustering algorithms is for special metric spaces that are a generalization of the Euclidean metric space, namely in metric spaces with bounded doubling dimension.

\textbf{k-center clustering.} There is a simple greedy 2-approximation algorithm for k-center \cite{24} which is best possible \cite{15}. This is the first clustering problem for which dynamic algorithms were found. Indeed, already in 2004 Charikar et al. \cite{3} gave two insertions-only algorithms: a deterministic 8-approximation and a randomized 5.43-approximation, both with $O(k \log k)$ update time per operation, which was improved to an (almost optimal) insertion-only $(2+\epsilon)$-approximation algorithm with update time $O((k \log k)/\epsilon^4)$ \cite{28} for any small $\epsilon > 0$. The best known fully dynamic algorithm was given by Chan et al. \cite{8}. It is randomized and achieves an approximation ratio of $2 + \epsilon$ with an expected update time of $O((k^2 \log \Delta)/\epsilon)$ against an oblivious adversary. In metric spaces with bounded doubling dimension Goranci et al. recently gave a $(2+\epsilon)$-approximation algorithm with update time independent of $k$, namely $O((\log \Delta \log \log \Delta)/\epsilon)$ time per operation \cite{16}. More generally, if the doubling dimension is $\kappa$, the time per operation is $O((2/\epsilon)^{O(\kappa)} \log \Delta \log \log \Delta \log \epsilon^{-1})$. Note that $\kappa$ can be $\Theta(\log n)$, in which case the above algorithm takes time polynomial in $n$. Independently, a 16-approximation algorithm with $O(D^2 (\log \Delta)^2 \log n)$ time per operation was presented by Schmidt and Sohler in \cite{33} if the metric space is a $D$-dimensional Euclidean space. This algorithm also works for hierarchical k-center clustering.

\textbf{k-sum-of-radii and k-sum-of-diameters.} The best known polynomial-time result for k-sum-of-radii is a $(3.504 + \epsilon)$-approximation algorithm \cite{9} and there is a QPTAS \cite{14}. Any solution for the k-sum-of-radii problem can be turned into a solution for the k-sum-of-diameters problem with an increase in the cost of at most a factor of 2 (and vice versa without any increase). The k-sum-of-diameters problem is NP-hard to approximate with a factor better than 2 \cite{11}. Henzinger et al \cite{20} recently developed a fully dynamic algorithm for a variant of the sum-of-radii problem which does not limit the number of centers. Instead there is a set $F$ of facilities and a set $C$ of clients and the algorithm must assign a radius $r_i$ to each facility $i$ such that every client is within distance at most $r_i$ for some facility $i$. The cost of the solution is the sum of the radii $\sum_i r_i$. Additionally each facility can be assigned an opening cost $f_i$ and the algorithm then tries to minimize $\sum_i r_i + f_i$. In metric spaces with bounded doubling dimension the algorithm achieves a constant approximation in $O(\log \Delta)$ time per operation. More generally, if the doubling dimension is $\kappa$, the algorithm maintains a $O(2^{2\kappa})$-approximation in time $O(2^{6\kappa} \log \Delta)$ per operation. Note that $\kappa$ can be $\Theta(\log n)$, in which case the above algorithm takes time polynomial in $n$.

Note that the k-sum-of-radii problem is related to the SET COVER problem where there is one set for each combination of a center $c$ and a radius $r$ that contains all points $p$ with $d(p,c) \leq r$. However, the number of sets in a solution of SET COVER is not limited and, thus, it cannot be used to solve the k-sum-of-radii problem.

\textbf{k-means and k-median.} In practice, the k-median and k-means problems are very popular. The best static polynomial-time approximation algorithm for k-median achieves an approximation ratio of 2.675 $+$ $\epsilon$ \cite{6} and the best such algorithm for k-means achieves a ratio of $9 + \epsilon$ \cite{1}. Furthermore, there is a lower bound of
\(\Omega(nk)\) on the running time of any (static) constant-factor approximation algorithm [29], implying that there is no insertions-only algorithm with \(o(k)\) time per operation. For both problems Henzinger and Kale [19] gave randomized \(O(1)\)-approximation algorithms with \(\bar{O}(k^3)\) worst-case update time against an adaptive adversary. Cohen-Addad et al. [10] gave fully dynamic \(O(1)\)-approximation algorithms for \(k\)-means and \(k\)-median with expected amortized update time of \(\bar{O}(n+k^3)\). Note, however, that their algorithm is consistent, i.e., it additionally tries to minimize the number of center changes.

**B Lower bounds for algorithms against an adaptive adversary**

We introduce some formal notation and definitions we use to revisit the adversarial strategy that generates an input stream and answers distance queries on the set of currently known points. Then, we derive lower bound constructions for the aforementioned problems that are based on the metric space defined by the stream and the answers to the algorithm.

We describe a strategy for an adversary \(\mathcal{A}\) that generates a stream of update operations \(\sigma\) and answers distance queries \(q\) on pairs of points by any dynamic algorithm with a guarantee on its amortized complexity. In the following presentation, the adversary constructs the underlying metric space ad hoc. More precisely, the adversary constructs two metric spaces simultaneously that cannot be distinguished by the algorithm and its queries. All subsequent lower bounds stem from the fact that the problem at hand has different optimal costs on the input for the two metric spaces. When the algorithm outputs a solution, the adversary can fix a metric space that induces high cost for the centers chosen by the algorithm.

During the execution of the algorithm, the adversary maintains a graph \(G\). Each point that was inserted by the adversary is represented by a node in \(G\). All query answers given to the algorithm by the adversary can be derived from \(G\) using the shortest path metric \(d_G(\cdot, \cdot)\) on \(G\). We denote the algorithm’s \(i\)th query after update operation \(t\) by \(q_{t,i}\), and the adversary’s answer by \(\text{ans}(q_{t,i})\). For \(t > 0\), we denote the number of queries asked by the algorithm between the \(t\)th and the \((t+1)\)th update operation by \(c(t)\). If \(t\) is clear from context, we simplify notation and write \(c := c(t)\). Note that, in this section, we use a slightly extended notation when indexing graphs when compared to other sections. Details follow.

We number the update operations consecutively starting with 1 using index \(t\) and after each update operation, we index the distance queries that the algorithm issues while processing the update operation and the immediately following value- or solution-queries using index \(i\). Let \(c > 0\) and let \(G_{0,c(0)}\) be the empty graph. For every \(t > 0, i \in [c(t)]\), consider \(i\)-th query issued by the algorithm processing the \(t\)-th update operation. The graph \(G_{t,i}\) has the following structure. For every point \(x\) that is inserted in the first \(t\) operations, \(V(G_{t,i})\) contains a node \(x\). All edges have length 1, and it holds that \(V(G_{t,i}) \supseteq V(G_{t,i-1})\) and \(E(G_{t,i}) \supseteq E(G_{t,i-1})\).

Edges are inserted by the adversary as detailed below. Let \(\leq\) denote the predicate that corresponds to the lexicographic order. In particular, the adversary maintains the following invariant, which is parameterized by the update operation \(t\) and the corresponding query \(i\): for all \((t', j) \leq (t, i)\) and \((u, v) := q_{t',j}, \text{ans}(q_{t',j}) = d_{G_{t,i}}(u, v)\). In other words, any query given by the adversary remains consistent with the shortest path metric on all versions of \(G\) after the query was answered. The adversary distinguishes the following types of nodes in \(G_{t,i}\) to answer a query. Recall that \(f := f(k, n)\) is an upper bound on the amortized complexity per update operation of the algorithm, which is non-decreasing in \(n\) for fixed \(k\).

**Definition 25** (type of nodes). Let \(t \geq 0, i \in [c]\) and let \(u \in V(G_{t,i})\). If \(u\) has degree less than \(100f(k, i)\) for all \(i \in [t]\), it is active after update \(t\), otherwise it is passive. In addition, the adversary can mark passive nodes as off. We denote the set of active, passive and off nodes in \(G_{t,i}\) by \(A_{t,i}, P_{t,i}\) and \(D_{t,i}\), respectively.

For operation \(t\), the adversary answers the \(i\)th query \(q_{t,i}\) according to the shortest path metric on \(G_{t,i-1}\) with the additional edge set \(A_{t,i-1} \times A_{t,i-1}\). In other words, the adversary (virtually) adds edges between all active nodes in \(G_{t,i-1}\) and reports the length of a shortest path between the query points in the resulting
After the adversary answered query $q_{t,i}$, the resulting graph $G_{t,i}$ is $G_{t,i-1}$ plus all edges of the shortest path that was used to answer the query. A key element of our analysis is that all answers up to operation $t$ and query $i$ are equal to the length of the shortest paths between the corresponding query points in $G_{t,i}$. The generation of the input stream and the answers to all queries are formally given by Algorithm 1 and 2, respectively.

Function $\text{GenerateStream}(t)$

\begin{verbatim}
if there exists a passive node $x \in V(G_{t-1,c})$ then
  mark $x$ as off in $G_{t,0}$
  return $\langle \text{delete } x \rangle$
else
  let $x$ be a new point, i.e., that was not returned by the adversary before
  return $\langle \text{insert } x \rangle$
\end{verbatim}

Algorithm 1: Construction of element $\sigma_t$ of $\sigma$

Function $\text{AnswerQuery}(q_{t,i} = (x, y))$

\begin{verbatim}
let $H_{t,i} = (V(G_{t,i-1}), E(G_{t,i-1}) \cup (A_{t,i-1} \times A_{t,i-1}))$
let $p = (e_1, \ldots, e_k)$ be a shortest path between $x$ and $y$ in $H_{t,i}$
set $G_{t,i} := G_{t,i-1}$
foreach $e_i$ do
  insert $e_i$ into $G_{t,i}$
return length of $p$
\end{verbatim}

Algorithm 2: Answer of the adversary to query $q_{t,i}$

B.1 Adversarial strategy

Let $n_t$ be the number of active and passive nodes, i.e., the number of current points for the algorithm, after operation $t$. In the whole section we use the notations of $G_{t',i}, A_{t,i}$ etc. from Definition 25.

B.1.1 Proof of Lemma 1

The next three lemmas prove the three claims in Lemma 1.

Lemma 26 (Lemma 1 (1)). For every $t > 0$, the number of active nodes in $G_{t,0}$ is at least $96t/100$.

Proof. Recall that $f(k, n)$ is a positive function that is non-decreasing in $n$. We prove the claim by induction. By the properties of $f$, the case $t = 1$ follows trivially. Let $t \geq 2$. For any $i \in [t]$, the algorithm’s query budget increases by $f(k, n_i)$ queries after the $i^{th}$ update. Since $f(k, i)$ is non-decreasing, nodes inserted after update operation $i - 1$ can only become passive if their degrees increase to at least $100f(k, i)$. Therefore, it holds that $|P_{t,0}| \leq \sum_{i \in [t]} 2f(k, n_i)/(100f(k, i)) \leq \sum_{i \in [t]} 2f(k, i)/(100f(k, i)) \leq 2t/100$. It follows that the adversary will delete at most $2t/100$ points in the first $t$ operations and insert points in the other at least $(1 - 2/100)t$ operations. The number of active nodes after update $t$ is $|A_{t,0}| \geq t - |P_{t,0}| - 2t/100 \geq 96t/100$. \qed

Lemma 27 (Lemma 1 (2)). For every $t > 0, i \in [c]$, all nodes have degree at most $100f(k, t)$ in $G_{t,i}$.

Proof. By definition, the claim is true for active nodes. Edges are only inserted into $G$ if the algorithm queries for the distance between two nodes $x, y$ and the adversary determines a shortest path between $x$ and $y$ that contains edges that are not present in $G_{t,i-1}$ (see Algorithm 2). Since all such edges are edges between active nodes, only degrees of active nodes in $G_{t,i}$ increase. The adversary finds a shortest path on
Proof. Let this be a contradiction to the assumption. We prove the claim by induction over the operations $t$ with the properties that $G_{t,0}$ contains no passive node, but $G_{t+1,0}$ contains at least one passive node. The claim is true for the initial (empty) graph $G_{0,0}$. Let $t > 0$. We prove that in at least one operation $t' \in \{t + 1, \ldots, 2t\}$, the number of passive nodes is 0. For the sake of contradiction, assume that for all $t'$, $t < t' \leq 2t$, the number of passive nodes is non-zero, i.e., $|P_{t',0}| > 0$. We call an active node semi-active if it has degree greater than 50$f(k, i)$ after some update $i \in [2t]$. Otherwise, we call it fully-active. Recall that, similarly, a vertex becomes passive if it has degree at least 100$f(k, i)$ after some update $i \in [2t]$ (and never becomes active again).

For any $i \in [2t]$, the algorithm’s query budget increases by $f(k, n_i)$ queries after the $i$th update. Since $f(k, i)$ is non-decreasing, nodes inserted after update operation $i - 1$ can only become semi-active if their degrees increase to at least 50$f(k, i)$ (resp. 100$f(k, i)$) by the definition of semi-active (resp. passive). Also due to the monotonicity of $f$, the number of semi-active or passive nodes is maximized if the algorithm invests its query budget as soon as possible. It follows that the number of semi-active or passive nodes up to operation $2t$ is at most $\sum_{j=0}^{2t} f(k, j) / (50f(k, j)) \leq 2t/50$. Without loss of generality, we may assume that all semi-active nodes are passive (so the algorithm does not need to invest budget to make them passive).

After update operation $2t$, the algorithm’s total query budget from all update operations is at most $\sum_{i=0}^{2t} f(k, n_i) \leq \sum_{i=0}^{2t} f(k, i) \leq 2tf(k, 2t)$. The algorithm may use its budget to increase the degree of at most $2tf(k, 2t) / (50f(k, 2t)) \leq 2t/50$ fully-active nodes to at least 100$f(k, 2t)$, i.e., to make them passive. Recall our assumption that $|P_{t,0}| > 0$ for all $i \in \{t + 1, \ldots, 2t\}$. The adversary deletes one point corresponding to a passive node in each update operation from $\{t + 1, \ldots, 2t\}$. Therefore, the number of passive nodes after update operation $2t$ is

$$|P_{2t,0}| \leq \frac{2t}{50} + \frac{2t}{50} - t \leq \frac{4t}{50} - t < 0.$$

This is a contradiction to the assumption. 

**B.1.2 Proofs of Lemmas 2 to 5**

The following observation follows immediately from the properties of shortest path metrics.

**Observation 29.** Let $G = (V, E)$ and $G' = (V, E')$ be two graphs so that $E \subseteq E'$. If a sequence of queries is consistent with the shortest path metric on $G$ as well as on $G'$, then, for any $E''$, $E \subseteq E'' \subseteq E'$, it is also consistent with the shortest path metric on $(V, E'')$.

The following lemma together with Observation 29 implies Lemmas 2 to 5 and 31 by setting $G = G_{t, i-1}$ and $G' = (V(G_{t, i-1}), E(G_{t, i-1}) \cup (A_{t, i-1} \times A_{t, i-1}))$ in Observation 29.

**Lemma 30.** For any $t, t' > 0$, $i, i' \in [c]$ so that $(t', i') < (t, i)$, the answer given to query $q_{t', i'}$ is consistent with the shortest path metric on $G'$. 

**Proof.** Let $t' > 0$, $i' \in [c]$ so that $(t', i') < (t, i)$ and denote $(x, y) := q := q_{t', i'}$. We prove that $\text{ans}(q) = d_{G_{t, i}}(x, y)$. As neither vertices nor edges are deleted after they have been inserted, for every $(t_1, i_1) < (t_2, i_2)$, $G_{t_2, i_2}$ is a supergraph of $G_{t_1, i_1}$. Thus, if there exists a path $P$ between $x$ and $y$ in $G_{t', i'}$, a shortest path in $G_{t, i}$ between $x$ and $y$ cannot be longer than $P$.

It remains to prove $\text{ans}(q) \leq d_{G_{t, i}}(x, y)$. For the sake of contradiction, assume that there exist $t'', i''$
so that \( \text{ans}(q) = d_{G_{t^*}, t^* - 1}(x, y) \), but \( \text{ans}(q) > d_{G_{t^*}, t^*}(x, y) \). By Definition 24, passive nodes never become active. For any passive node \( v \in P_{t^*, t^* - 1} \), it follows that \( d_{G_{t^*}, t^*}(v, A_{t,i}) \geq d_{G_{t^*}, t^* - 1}(v, A_{t,i - 1}) \) as Algorithm 2 only inserts edges between vertices in \( A_{t,i} \) into \( G_{t,i} \). Therefore, any shortest path between \( x \) and \( y \) in the graph \( (V(G_{t^*}, t^* - 1), E(G_{t^*}, t^* - 1) \cup (A_{t^*}, t^* - 1 \times A_{t^*}, t^* - 1)) \) has length at least \( d_{G_{t^*}, t^* - 1}(x, y) = \text{ans}(q) \).

### B.2 Lower bounds for clustering

Our lower bounds apply for the case that the algorithm is allowed to choose as centers any points that have ever been inserted as well as to the case where centers must belong to the set of current points. In the whole section we use the notations of \( G_{t^*}, A_{t,i} \) etc. from the introduction of Appendix B.

#### B.2.1 Proof of Theorem 6

For any \( \ell \in \mathbb{N}_0 \), we define a metric \( M_{t}(P^*) \) on a subset of points \( P^* \) as the shortest path metric on the following graph. For each pair of active vertices \( u, v \) we add an edge if \( d(u, P^*) \geq \ell \) and \( d(v, P^*) \geq \ell \).

**Lemma 31.** For each clean update operation \( t \), each subset of points \( P^* \in V_t \), and each \( \ell \in \mathbb{N}_0 \) the metric \( M_{t}(P^*) \) is consistent.

**Proof.** The metric \( M_{t}(P^*) \) is an augmented graph metric for \( t \) and, thus, for a clean update operation \( t \), it is consistent by Lemma 26.

**Lemma 32.** Consider any dynamic algorithm for \((1,p)\)-clustering that queries amortized \( f(1,n) \) distances per operation, where \( n \) is the number of current points, and outputs at most \( 1 \leq g \leq n \) centers. For any \( t \geq 1 \) such that \( t \) is a clean operation, the approximation factor of the algorithm’s solution (with respect to the optimal \((1,p)\)-clustering cost) against an adaptive adversary right after operation \( t \) is at least \( \left[ \frac{\log(t/4g)}{p + \log(100f(1,t))} \right]^p/4 \) and \( 96t/100 \leq n \leq t \).

**Proof.** Denote \( G := (V, E) := G_{t,0} \) and \( A := A_{t,0} \). By Lemma 26, the number of active nodes in \( G \) is at least \( 96t/100 \geq t \), which implies that \( |A| \geq 96t/100 \). Thus, after operation \( t, n \geq 96t/100 \).

Let \( C \) be the centers that are picked by the algorithm after operation \( t \). By the assumption of the lemma, \( |C| \leq g \). Consider \( M_t(C) \), where \( \ell := \log(t/4g)/(p + \log(100f(1,t))) \). By Lemma 27, for any \( s \in C \), the size of \(|\{x \mid x \in A \land d(x, s) < \ell\}| \) is at most \( \sum_{i=1}^{\ell} (100f(1,t))^{i} < (100f(1,t))^{\ell} \).

Let \( V_+ := \{x \mid x \in A \land d(x, C) \geq \ell \} \). Since \( |C| \leq g \), it holds that \( |V_+| > |A| - g(100f(1,t))^{\ell} \geq 96t/100 - g(t/4g)^{\log((100f(1,t))/(p + \log(100f(1,t))))} \geq 96t/100 - t/4 \geq t/2 \). Since \( d(s, V_+) \geq \ell \) for any \( s \in C \), the \((1,p)\)-clustering cost of \( S \), and thus the cost of the algorithm, is at least \( |V_+| \cdot \ell p \geq t/2 \cdot \ell p \).

We will show that if instead a single point corresponding to a vertex of \( V_+ \) is picked as center, then the cost is at most \( 2t \), which provides an upper bound on the cost of the optimum solution. It follows that the approximation factor achieved by the algorithm is at least \( \ell p / 4 \).

To complete the proof consider a point \( x \) whose corresponding point \( v_x \) belongs to \( V_+ \). One can easily show that for each \( a \in \mathbb{N} \) it holds that \( a^p \leq (100f(1,t) \cdot 2p)^{2a/3} \) since for each \( a \in \mathbb{N} \) it holds that \( a^p = 2^p \log a \leq 2^p \frac{a^p}{2p} \). Thus, the \((1,p)\)-clustering cost if a single point \( x \) is chosen as center is at most \( \ell p / 4 \).
As the algorithm cannot tell whether \( i \) or 1 is the correct answer, and it always has to output a value that
\[ \ell \geq \log (100f(1,t)+2) \cdot t/4g \]
This yields an approximation ratio of at least \( t/2^p = \Omega((p)) \).

**Lemma 33.** Consider any dynamic algorithm for computing the diameter of a dynamic point set that queries
amortized \( f(1,n) \) distances per operation, where \( n \) is the number of current points, and outputs at most
\( g \geq 1 \) centers. For any \( t \geq 2 \) such that \( t \) is a clean operation, the approximation factor of the algorithm's
solution (with respect to the correct diameter) against an adaptive adversary right after operation \( t \) is at
least \( \log(96t/100)/(\log 102f(1,t)) - 1 \) and \( 96t/100 \leq n \leq t \).

**Proof:** Denote \( G := (V,E) := G_{t,0} \) and \( A := A_{t,0} \). By Lemma \( 26 \) the number of active nodes in \( G \) is at
least \( 96t/100 \geq t \), which implies that \( |A| \geq 96t/100 \). Thus, after operation \( t \), \( n \geq 96t/100 \).

Without loss of generality, we assume that \( G[A] \) has exactly one connected component: If this is not the
case, let \( C_1, \ldots, C_s \) be the connected components of \( G[A] \) and observe that we may insert a path connecting
the connected components by inserting into \( G[A] \) edges \((v_1,v_2), \ldots, (v_{s-1},v_s)\) of length 1, where \( v_i \in C_i \)
are arbitrary vertices. This increases the maximum degree of nodes in \( G \) by at most \( 2 \) and the shortest-path metric
\( M \) on the resulting graph that is constructed in this way is an augmented graph metric for \( t \). As \( t \) is clean, Lemma \( 2 \) shows that \( M \) is consistent.

Let \( x \in V \) be any node. By Lemma \( 27 \) the number \( n(i) \) of nodes that have distance \( i \) to \( x \) is at most
\[ \sum_{j \in [i]} (100f(1,t)+2)^j \leq (100f(1,t)+2)^{i+1} \]
Consider the largest \( i \) such that \( (100f(1,t)+2)^{i+1} \leq n \).
It follows that there exists a node at distance \( i+1 \) to \( x \). Furthermore \( (100f(1,t)+2)^{i+2} \geq n(i+1) \geq n \),
which implies that \( i + 2 \geq \log_{100f(1,t)+2} n \). As \( f(1,t) \geq 1 \) for all values of \( t \), there exists a shortest path
\( P \) starting at \( x \) of length at least \( i + 1 \geq \log_{100f(1,t)+2} n - 1 \geq (\log n/ \log (102f(1,t))) - 1 =: \ell \).
It follows that the diameter is at least \( \ell \). On the other hand, \( M \) can be extended by adding an edge between
any pair of active nodes, resulting in the consistent metric \( M_{\text{uni}} \).
For this metric the diameter of \( G \) is 1.
As the algorithm cannot tell whether \( \ell \) or 1 is the correct answer, and it always has to output a value that
is as least as large as the correct answer, it will output at least \( \ell \). Thus, the approximation ratio is at least
\( \ell \geq \log(96t/100)/\log 102f(1,t) - 1 \).

Note that this implies a lower bound for the approximation ratio for 1-center, 1-sum-of-radii, and 1-sum-of-diameter.

**Proof (of Theorem 3).** Let \( t \in \mathbb{N} \). By Lemma \( 1 \) there is a value \( t' \) with \( t < t' \leq 2t' \) such that \( t' \) is a clean
operation. Let \( n \) be the number of active points at iteration \( t' \). By Lemma \( 26 \) we know that \( t' \geq n \geq
96t'/100. Note that hence t' \leq 2n.

Recall that we assumed that the function f(k, n) is non-decreasing in n (for any fixed k). Suppose that after operation t' we query the solution value of an algorithm for 1-center, 1-sum-of-radii, or 1-sum-of-diameter. By Lemma 33 its approximation ratio is at least

\[
\frac{\log(96t'/100)}{\log(102f(k,t'))} - 1 \geq \frac{\log(96n/100)}{\log(102f(k,2n))} - 1 = \Omega \left( \frac{\log(n)}{\log(f(k,2n))} \right).
\]

Suppose that instead we query the solution from the algorithm for (1, p)-clustering. By Lemma 32 its approximation ratio is at least

\[
\left[ \frac{\log(t'/4g)}{p + \log(100f(1,t'))} \right]^{p/4} \geq \left[ \frac{\log(n/4g)}{p + \log(100f(1,2n))} \right]^{p/4}
\geq \left[ \frac{\log(n) - \log(4g)}{p + \log(100f(1,2n))} \right]^{p/4}
\geq \left[ \frac{\log(n)}{1.1p + 1.1\log(100f(1,2n))} \right]^{p/4}
\geq \left[ \frac{\log(n)}{1.1p + 1.1(7 + \log(f(1,2n)))} \right]^{p/4}
\geq \left[ \frac{\log(n)}{1.1p + 8 + 1.1\log(f(1,2n))} \right]^{p/4}
\geq \left[ \frac{\log(n)}{2p + 8 + 2\log(f(1,2n))} \right]^{p/4}
\geq \Omega \left( \frac{\log n}{2p + 8 + 2\log f(1,2n)} \right)^p
\]

using that g = O(1). Hence, for k-median and k-means if we take p = 1 and p = 2, respectively, this yields bounds of \( \Omega \left( \frac{\log n}{10 + 2\log f(1,2n)} \right) \) and \( \Omega \left( \frac{\log n}{12 + 2\log f(1,2n)} \right)^2 \), respectively.

\[ \square \]

### B.2.2 Proof of Theorem 7

**Lemma 34** (Theorem 7). Let \( k \geq 2 \). Consider any dynamic algorithm for maintaining an approximate k-center solution of a dynamic point set that (1) queries amortized \( f(k, n) \) distances per operation, where \( n \) is the number of current points, and (2) outputs at most \( g(k) \in O(k) \) centers. For any \( t \geq 2 \) such that \( t \) is a clean operation, the approximation factor of the algorithm’s solution (with respect to an optimal k-center solution) against an adaptive adversary right after operation \( t \) is at least \( \Omega \left( \min \left\{ k, \frac{\log n}{k \log f(k,2n)} \right\} \right) \).

**Proof.** Denote \( G := (V, E) := G_{t,0} \) and \( A := A_{t,0} \). By Lemma 26 the number of active nodes in \( G \) is at least \( 96t/100 \geq t \), which implies that \( |A| \geq 96t/100 \). Thus, after operation \( t \), the number \( n \) of current points is at least \( 96t/100 \).

Without loss of generality, we assume that \( G[A] \) has exactly one connected component: If this is not the case, let \( C_1, \ldots, C_s \) be the connected components of \( G[A] \) and observe that we may insert a path connecting the connected components by inserting into \( G[A] \) edges \((v_1, v_2), \ldots, (v_{s-1}, v_s)\) of length 1, where \( v_i \in C_i \) are arbitrary vertices. This increases the maximum degree of nodes in \( G \) by at most 2 and the shortest-path metric \( M \) on the resulting graph that is constructed in this way is an augmented graph metric for \( t \). As \( t \) is clean, Lemma 2 shows that \( M \) is consistent.

Let \( x \in V \) be any node. By Lemma 27 the number \( n(i) \) of nodes that have distance \( i \) to \( x \) is at most \( \sum_{j \in [t]} (100f(1,t) + 2)^j \leq (100f(k,t) + 2)^{t+1} \). Consider the largest \( \ell \) such that \( (100f(k,t) + 2)^\ell + 1 < n.\)
It follows that there exists a node $z$ at distance $\ell + 1$ to $x$. Furthermore $(100f(k,t) + 2)^{\ell+2} \geq n^{(\ell+1)} \geq n$, which implies that $\ell + 2 \geq \log_{100f(k,t)+2} n$.

Let $S = \{s_1, \ldots, s_{g(k)}\}$ be the solution of the algorithm. For $i \in [\ell + 1]$, let us define $V^{(i)}$ to be the set of vertices $v$ in $G[A]$ with $d_{G[A]}(x,v) = i$. By pigeon hole principle, there must exist a consecutive sequence $(i_1, \ldots, i_m)$ so that $m \geq \ell/(g(k) + 1)$ and, for all $i \in \{i_1, \ldots, i_m\}$, $S \cap V^{(i)} = \emptyset$. Let $k' = \min\{3k - 1, m/2\}$. Consider the metric $M_{i_1,i_{k'}}(x)$ and let $y_{i_1}, \ldots, y_{k'}$ be elements from the respective sets $V^{(i_1)}, \ldots, V^{(i_{k'})}$.

The algorithm’s solution $S$ has cost at least $k'$ because $d_G(S, V_{i_{k'}}) \geq k'$. The solution $\{y_{3j-2} \mid j \in \mathbb{N} \land 3j - 2 \in [k']\}$ is optimal and has cost 1. It follows that the approximation factor of $S$ is greater than or equal to $k' = \min\{3k - 1, m/2\}$.

Let $n$ be the number of points at iteration $t$. We calculate that

$$\min \left\{ \frac{3k}{2}, \frac{m}{2} \right\} \geq \Omega \left( \min \left\{ k, \frac{\ell}{g(k) + 1} \right\} \right)$$

$$\geq \Omega \left( \min \left\{ k, \frac{1}{g(k)} \left( \frac{\log n}{\log (102f(k,t))} - 1 \right) \right\} \right)$$

$$\geq \Omega \left( \min \left\{ k, \frac{\log n}{k \log (102f(k,2n))} \right\} \right)$$

$$\geq \Omega \left( \min \left\{ k, \frac{\log n}{k \log (\log f(k,2n))} \right\} \right)$$

$$\geq \Omega \left( \min \left\{ k, \frac{\log n}{k \log 102 + k \log f(k,2n)} \right\} \right)$$

using that $g(k) = O(k)$. \hfill \footnote{C}

### C Algorithm for $k$-Sum-of-Radii and $k$-Sum-of-Diameter

In this section, we prove the correctness of Algorithm 3 for the $k$-sum-of-radii and the $k$-sum-of-diameter problem as described in Section 3. To compute a solution in the static setting, the algorithm is invoked as PrimalDual($P$, $\emptyset$, $R$, $z$, $k$, $\epsilon$, $0$, $0$, $0$), where $P$ is the set of input points, $R$ is the set of radii, $z = \epsilon \text{OPT}'/k$ is the facility cost and $k$ is the number of clusters. Algorithm 3 is a pseudo-code version of the algorithm described in Section 3.

#### C.1 Proof of Lemma 9

We bound the running time of a single primal-dual step. This implies Lemma 9

**Lemma 35** (Lemma 9). *The running time of the $i$th iteration of the while-loop in PrimalDual is $O(ik/\epsilon + |U_i|)$.*

**Proof:** In each iteration, at most one entry of $y$, i.e., $y_{b_i}$, increases. Therefore, at most $i - 1$ entries of $y$ are non-zero at the beginning of iteration $i$. By keeping a list of non-zero entries, each sum corresponding to a constraint of the dual program can be computed in time $O(i)$. Since $|R| \leq k/\epsilon$, it follows that $\delta_i$ and $r_i$ can be computed in time $O(ik/\epsilon)$. To construct $U_{i+1}$, it is sufficient to iterate over $U_i$ once. \hfill \footnote{D}

#### C.2 Proof of Lemma 10

We show that our choice of $z = \epsilon \text{OPT}'/k$ will result in a solution $\bar{S}$ if OPT $\leq \text{OPT}'$.
**Data:** point set $P$, unassigned point sets $U = \{U_0, \ldots\}$, radii set $R$, facility cost $z$, number of clusters $k$, precision $\epsilon$, primal vector $x = \{x_p^{(i)} \mid r \in R \land p \in P\}$, dual vector $y = \{y_p \mid p \in P\}$

**Function** $\text{PrimalDual}(P, U, R, z, k, \epsilon, x, y, i)$

```plaintext
while $U_i \neq \emptyset$ do
    $p_i \leftarrow$ uniformly random point from $U_i$
    $\delta_i \leftarrow \max\{0\} \cup \{\delta' \mid \delta' \in \mathbb{R} \land \forall r' \in R : \sum_{p' \in P, d(p, p') \leq r'} y_{p'} + \delta' \leq r'/2 + z\}$
    if $\delta_i > 0$ then
        $r_i \leftarrow \max\{r' \mid r' \in R \land \sum_{p' \in P, d(p, p') \leq r'} y_{p'} + \delta_i = r'/2 + z\}$
    else
        $r_i \leftarrow \max\{r' \mid r' \in R \land \sum_{p' \in P, d(p, p') \leq r'} y_{p'} \geq r'/2 + z\}$
        $y_{p_i} \leftarrow \delta_i; x_{p_i} = 1$
        $U_{i+1} \leftarrow U_i \setminus \{p' \mid p' \in U_i \land d(p_i, p') \leq 2r_i\}$
        $i \leftarrow i + 1$
    if $i > (2k/\epsilon)^2$ then
        return “OPT’ < OPT”
    $S \leftarrow \emptyset; S \leftarrow \text{sort}((p_j, r_j)_{j \in[i−1]} \text{ non-increasingly according to } r_j$
    forall $(p, r) \in S$ do
        if $\notin(p_j, r_j) \in S : d(p, p_j) \leq r + r_j$ then
            $S \leftarrow S \cup \{(p, 3r)\}$
    return $S, U, x, y, r, i$
```

Algorithm 3: Pseudo code of the primal-dual algorithm for $k$-sum-of-radii as described in Section 3.

**Lemma 10.** If $U_{(2k/\epsilon)^2+(2k/\epsilon)} \neq \emptyset$ then OPT > OPT’. 

**Proof.** Since, by weak duality, $\sum_{p \in P} y_p$ is a lower bound to the optimum value of $(P)$, we bound the number of iterations that are sufficient to guarantee $\sum_{p \in P} y_p > OPT'$. Call an iteration of the while-loop in $\text{PrimalDual}$ successful if $\delta_i > 0$, and unsuccessful otherwise. First, observe that for each successful iteration $i$, the algorithms increases $y_{p_i}$ by at least $z/2$. This is due to the fact that all values in $R$ are multiples of $z$, and therefore $r'/2 + z$ is a multiple of $z/2$ for any $r \in R$. Since the algorithm increases $y_{p_i}$ as much as possible, all $y_{p_i}$ are multiples of $z/2$. Therefore, after OPT’/(z/2) + 1 = 2k/\epsilon + 1 successful iterations, it holds that OPT $\geq \sum_{p \in P} y_p > OPT’$.

Now, we prove that for each successful iteration, there are at most $|R|$ unsuccessful iterations. Then, it follows that after $((2k/\epsilon) + 1) \cdot |R| \leq (2k/\epsilon)^2 + (2k/\epsilon)$ iterations, OPT > OPT’. Let $i$ be an unsuccessful iteration. The crucial observation for the following argument is that for the maximum $r_i \in R$ so that $(p_i, r_i)$ is at least half-tight and for any $j < i$ so that $d(p_i, p_j) \leq r_i$ and $y_{p_i} > 0$, we have $r_j < r_i$: otherwise, $p_i$ would have been removed from $U_j$. On the other hand, such $p_j$ must exist because the dual constraint $(p_i, r_i)$ is at least half-tight but $y_{p_i} = 0$. Now, we charge the radius $r_i$ to the point $p_j$ and observe that we will never charge $r_i$ to $p_j$ again. This is due to the fact that all points $p \in U_{i−1}$ with distance $d(p_j, p) \leq r_j$ are removed from $U_i$ because $d(p_i, p) \leq d(p_i, p_j) + d(p_j, p) \leq 2r_i$. Therefore, for any successful iteration $j$ and any $r \in R$, there is at most one $i > j$ so that $r = r_i$ is charged to $p_j$, each accounting for an unsuccessful iteration.

**C.3 Proof of Lemma 11**

We argue that the pruning step has running time poly$(k/\epsilon)$ to prove Lemma 11.

**Lemma 36 (Lemma 11).** The for-loop in $\text{PrimalDual}$ has running time $O((k/\epsilon)^4)$. 

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Proof: Since \( i \leq 2(2k/\varepsilon)^2 \), it holds that \(|S| \leq |\{ p \in P \mid \exists r \in R : x_p^{(r)} > 0\}| \leq 2(2k/\varepsilon)^2\). Therefore, sorting \( S \) requires at most \( O((2k/\varepsilon)^2 \log(2k/\varepsilon)) \) time. In each iteration of the loop, it suffices to iterate over \( \tilde{S} \). Since \(|\tilde{S}| \leq |S|\), the claim follows. \( \square \)

C.4 Proof of Lemma 12

Lemma 12. Given \( \tilde{S} \) and \( \hat{S} \) we can compute \( \hat{S} \) in time \( O(k^3/\varepsilon^2) \).

Proof. Recall that we sort points in \( S \) non-increasingly. Therefore, for every point \( p \), there exists only one \((p, r) \in S\) that is inserted into \( \tilde{S}\). It follows that \(|\tilde{S}| \leq |S| \leq (2k/\varepsilon)^2 \) and since \(|\tilde{S}| \leq k\), it suffices to iterate over \( \tilde{S} \) for every \((\tilde{p}, \tilde{r}) \in \hat{S}\). \( \square \)

C.5 Proof of Lemma 13

We turn to the feasibility and approximation guarantee of the solution that is computed by Algorithm 3. First, we observe that the dual solution is always feasible.

Lemma 37. During the entire execution of Algorithm 3 no dual constraint \((p, r)\) is violated.

Proof. For the sake of contradiction, let \( i \) be the first iteration of the while-loop in PrimalDual after which there exists \((p, r)\) such that \( \sum_{p' \in P : d(p, p') \leq r} y_{p'} > r + z \). From this definition, it follows that \( d(p, p_i) \leq r \) as only \( y_{p_i} \) has increased in iteration \( i \). By the triangle inequality, for all \( p' \in P \) so that \( d(p, p') \leq r \), we have that \( d(p_i, p') \leq 2r \). Therefore, the dual constraint \((p_i, 2r)\) is more than half-tight:

\[
\sum_{p' \in P : d(p_i, p') \leq 2r} y_{p'} \geq \sum_{p' \in P : d(p, p') \leq r} y_{p'} > r + z = \frac{2r}{2} + z.
\]

This is a contradiction to the choice of \((p_i, r_i)\). \( \square \)

The primal solution is also feasible to the LP, and its cost is bounded by \( 6OPT' \).

Lemma 13. We have that \( \tilde{x} \) and \( y \) are feasible solutions to \((P)\) and \((D)\), respectively, for which we have that \( \sum_{p \in P} \sum_{r \in R} \tilde{x}_p^{(r)} (r + z) \leq 6 \cdot \sum_{p \in P} y_p \leq 6OPT' \).

Proof. Since \( U_i = \emptyset \), \( x \) is feasible at the end of Algorithm 3. By the construction of \( \tilde{S} \), it follows that \( \tilde{x} \) is feasible. By Lemma 37, \( y \) is always feasible for \((D)\). It remains to bound the cost of \( \tilde{S} \).

By the construction of \( \tilde{S} \), for any \((p, r) \in \tilde{S}\), \( \tilde{x}_p^{(r/3)} = 1 \) and \((p, r/3)\) is at least half-tight in \( y \). In other words, \( r + z \leq 6 \cdot (r/(2 \cdot 3) + z) = 6 \sum_{d(p, p') \leq r/3} y_{p'} \). Let \( \tilde{S}(p, r) = \{ p' \in P : d(p, p') \leq r \} \). For all \((p_1, r_1), (p_2, r_2) \in \tilde{S}, p_1 \neq p_2\), we have that \( \tilde{S}(p_1, r_1/3) \) and \( \tilde{S}(p_2, r_2/3) \) are disjoint due to the construction of \( \tilde{S} \). Therefore, it holds that

\[
\sum_{p \in P} \sum_{r \in R} \tilde{x}_p^{(r)} (r + z) = \sum_{(p, r) \in \tilde{S}} (r + z) \leq 6 \cdot \sum_{p \in P} y_p.
\]

By weak duality, \( 6 \cdot \sum_{p \in P} y_p \leq 6OPT' \). \( \square \)

C.6 Proof of Lemma 14

Finally, we prove that the pruned solution is a feasible \( k \)-sum-of-radii solution with cost bounded by \( (13.008 + \varepsilon)OPT' \).

Lemma 14. We have that \( \tilde{S} \) is a feasible solution with cost at most \( \sum_{(\tilde{p}, \tilde{r}) \in \tilde{S}} \tilde{r} + \sum_{p \in P} \sum_{r \in R} \tilde{x}_p^{(r)} (r + z) \leq (13.008 + \varepsilon)OPT' \).
Proof. First observe that the cost of $\hat{S}$ is bounded by $2 \cdot 3.504 \cdot \text{OPT}$ since we can construct a solution with cost at most $2\text{OPT}$ that covers $\bar{C}$ using only centers from $\bar{C}$: for each point $p \in \bar{C}$, take the point $p' \in \text{OPT}$ covering $p$ with some radius $r'$, and select $p$ with radius $2r'$.

Let $p \in P$, let $(\hat{p}, \hat{r}) \in S$ so that $d(p, \hat{p}) \leq \hat{r}$ and let $(\hat{p}, \hat{r}) \in \hat{S}$ that was chosen to cover $\bar{p}$. By the triangle inequality, $d(p, \hat{p}) \leq \hat{r} + \bar{r}$. Let $(\hat{p}, \hat{r})$ be the corresponding tuple in $\hat{S}$. By the construction of $\hat{S}$, we have that $\hat{r} + \bar{r} \leq \bar{r}$. Therefore, $\hat{S}$ is feasible. It follows that the cost of $\hat{S}$ is at most

$$\sum_{(\hat{p}, \hat{r}) \in \hat{S}} \hat{r} + \sum_{p \in P} \sum_{r \in R} \hat{d}_p(r + z) \leq (2 \cdot 3.504 + 6 + \epsilon)\text{OPT}'. \quad \square$$

C.7 Proof of Theorem 15

Theorem 15. There are dynamic algorithms for the $k$-sum-of-radii and the $k$-sum-of-diameters problems with update time $k^{O(1/\epsilon)} \log \Delta$ and with approximation ratios of $13.008 + \epsilon$ and $26.016 + \epsilon$, respectively, against an oblivious adversary.

Proof. Consider a fixed choice of $\text{OPT}'$. This assumption will be removed at the end of the proof. We invoke Algorithm 5 on the initial point set as for the static setting. Consider an operation $t$, and let $\hat{S}, \hat{U}, \hat{x}, \hat{y}, \hat{r}, \hat{i}$ be the state before this operation.

If a point $p$ is inserted, the algorithm checks, for every $j \in \{1, \ldots, i - 1\}$, if $d(p_j, p) \leq 2r_i$. If this is not the case for any $j$, $p$ is added to $U_j$ and the algorithm proceeds. Otherwise, the algorithm stops. If the last check fails and $i > (2k/\epsilon)^2$, the algorithm stops, too. Otherwise, it runs $\text{PrimalDual}(P, \hat{U}, R, z, k, \epsilon, x, y, i)$ with the updated $\hat{U}$. In any case, the point $p$ deposits a budget of $2k/\epsilon$ tokens for each possible $U_i, i \in [2k/\epsilon]$, i.e., $(2k/\epsilon)^2$ tokens in total. By Lemmas 9 and 36, the total running time is $O((2k/\epsilon)^2k/\epsilon + 0 + (k/\epsilon)^4 + (2k/\epsilon)^2) = O((k/\epsilon)^4)$.

If a point $p$ is deleted, the algorithm deletes $p$ from all $\hat{U}_i$ it is contained in. If for all radii $r \in R$ we have that $\hat{x}_p = 0$ then the algorithm stops. Otherwise, let $j$ be so that $p \in \hat{U}_j \setminus \hat{U}_{j+1}$, i.e., $p$ is the center of the $j$th cluster. The algorithm sets, for all $r \in R$ and all $j' \geq j$, $\hat{x}_p^{(r)} = 0$, $\bar{y}_{p, r} = 0$ and, for all $j' > j$, $\hat{U}_{j'} = \emptyset$. Then, it calls $\text{PrimalDual}(P, \hat{U}, R, z, k, \epsilon, \hat{x}, \bar{y}, j)$. By Lemmas 9 and 36, the total running time is $O((2k/\epsilon)^2k/\epsilon + (2k/\epsilon)|U_j| + (k/\epsilon)^4)$.

The correctness of the algorithm follows from the fact that if $\text{OPT}' \geq \text{OPT}$, the algorithm produces a feasible solution irrespective of the choice of the $p_i, i \in [2k/\epsilon]$, by Lemma 14 and observing that the procedure described above simulates a valid run of the algorithm for $P = P_{i-1} \cup \{p\}$ and $P = P_i \setminus \{p\}$, respectively. Finally, we prove that the expected time to process all deletions up to operation $t$ is bounded by $O(t \cdot 2k/\epsilon)$. The argument runs closely along the running time analysis in 8.

Let $t' \leq t, j \in [2k/\epsilon]$ and let $\hat{U}_j(t')$ be the set $U_j(t')$ that was returned by $\text{PrimalDual}$ after the last call that took place before operation $t'$ so that the argument $i$ is such that $i \leq j$. Note that this is the last call to $\text{PrimalDual}$ before operation $t'$ when $U_j$ is reclusted. We decompose $U_j(t')$ into $A_j(t') = U_j(t') \setminus \hat{U}_j(t')$ and $B_j(t') = \hat{U}_j(t') \cap U_j(t')$ and define the random variable $T_i(t')$, where $T_i(t') = |B(t')|$ if operation $t'$ deletes center $p_i$ and $T_i(t') = 0$ otherwise. Next, we bound $E[\sum_{t' < t} \sum_{i \in [2k/\epsilon]} T_i(t')]$. For $t' < t$ and $i \in [2k/\epsilon]$, consider $E[T_i(t')]$. Since $p_i$ was picked uniformly at random from $B(t')$, the probability that operation $t'$ deletes $p_i$ is $1/|B(t')|$. Therefore, $E[T_i(t')] = 1$. By linearity of expectation, $E[\sum_{t' < t} \sum_{i \in [2k/\epsilon]} T_i(t')] \leq 2k/\epsilon$. If operation $t'$ deletes $p_j$, $U_j$ is reclusted at operation $t'$ and any point in $A$ is not in $A(t')$ for any $t'' \geq t'$. Therefore, each point $p \in A$ can pay $(2k/\epsilon)$ tokens from its insertion budget if $p_j$ is deleted. The expected amortized cost for all operations up to operation $t$ is therefore at most $O(2k/\epsilon^2k/\epsilon + (2k/\epsilon)^2 + (k/\epsilon)^4) = O((k/\epsilon)^4)$.

Now, we remove the assumption that $\text{OPT}'$ is known. Recall that $d(x, y) \geq 1$ for every $x, y \in X$, and
\[ \Delta = \max_{x,y \in X} d(x, y) \] For every \( \Gamma \in \{(1 + \varepsilon)^i \mid i \in \lceil \log_{1+\varepsilon}(k\Delta) \rceil \}, \) the algorithm maintains an instance of the LP with \( \text{OPT}' = (1 + \varepsilon)^\Gamma. \) After every update, the algorithm determines the smallest \( \Gamma \) for which a solution is returned. Recall that the algorithm from [9] takes time \( O(n^{O(1/\varepsilon)}) \). The total expected amortized cost is \( O(k^{O(1/\varepsilon)} \log \Delta). \) \hfill \square

\section{Algorithm for \( k \)-center}

\subsection{Proof of Lemma 16}

Our strategy is to maintain a set of at most \( k + 1 \) points \( C \subseteq P \) so that any two points \( c, c' \in C \) satisfy that \( d(p, p') > 2\text{OPT}' \). If \( |C| = k + 1 \) then this asserts that there can be no solution with value \( \text{OPT}' \). If \( |C| \leq k \) then we will ensure that \( C \) forms a feasible solution of value \( 2\text{OPT}' \).

The main idea is as follows: We say that two points \( p, p' \in P, p \neq p' \), are neighbors if \( d(p, p') \leq 2\text{OPT}' \). We dynamically maintain a counter \( a_p \in \{0, \ldots, k+1\} \) for each point \( p \in P \) such that \( a_p \) denotes the number of neighbors of \( p \) in \( C \). As shown in the next lemma, this is sufficient to maintain a suitable set \( C \) and leads to an algorithm with amortized update time \( O(n + k) \).

\textbf{Lemma 16.} For any given value \( \text{OPT}' \) there is a fully dynamic algorithm that, started on a (possibly empty) set \( P \) with \( |P| = n_0 \) for some integer \( n_0 \) has preprocessing time \( O(kn_0) \) and amortized update time \( O(n + k) \) such that after each operation it either returns a solution \( C \) of value at most \( 2\text{OPT}' \) or asserts that \( \text{OPT} > \text{OPT}' \), where \( n \) is the current size of \( P \). Any point in \( P \) becomes a center at most once, and then stays a center until it is deleted.

\textbf{Proof:} The algorithm maintains a set of at most \( k + 1 \) points \( C \subseteq P \) so that any two points \( c, c' \in C \) satisfy that \( d(p, p') > 2\text{OPT}' \). If \( |C| = k + 1 \) then this asserts that there can be no solution with value \( \text{OPT}' \). If \( |C| \leq k \) then we will ensure that \( C \) forms a feasible solution of value \( 2\text{OPT}' \). Our algorithm maintains as data structure for each point \( p \in P \) a counter \( a_p \in \{0, \ldots, k+1\} \) such that \( a_p \) denotes the number of neighbors of \( p \) in \( C \). In particular, if \( p \in C \) then it will always hold that \( a_p = 0 \).

\textit{Insertions.} When a new point \( p \) is inserted into \( P \), we compute \( a_p \) by determining its distance to all centers. If \( a_p = 0 \) and \( |C| \leq k \) then we add \( p \) into \( C \). Then, we increment (by one) the counter \( a_p \) for each neighbor \( p' \in P \) of \( p \).

\textit{Deletions.} Suppose that a point \( p \in P \) is deleted. If \( p \notin C \) we do not change anything else. If \( p \in C \), we decrement (by one) the counter of each neighbor \( p' \in P \) of \( p \). Then, we iterate over \( P \setminus C \) in arbitrary order. If for a point \( p' \in P \setminus C \) we have that \( a_{p'} = 0 \) and \( |C| \leq k \), then we add \( p' \) into \( C \) and increment the counter \( a_{p'} \) for each neighbor \( p'' \in P \) of \( p' \). Then we consider the next point in \( P \setminus C \). Note that it could be that for a point \( p' \in P \setminus C \) its counter \( a_{p'} \) equals to \( 0 \) immediately after \( p \) was deleted, but by the time that \( p' \) is considered, i.e. \( a_{p'} \) is checked, it holds that \( a_{p'} > 0 \). In this case \( p' \) does, of course, not become a center.

\textit{Preprocessing.} When started on a non-empty set \( P \), the algorithm simply inserts every element of \( P \) into an initially empty data structure.

\textbf{Correctness.} Note that the algorithm maintains the following invariant.

\textit{(INV)} If \( |C| \leq k \), then for all points \( p \in P \setminus C \) it holds that \( a_p > 0 \).

This can be shown by an easy induction over the number of operations. It follows that whenever \( |C| \leq k \), then \( C \) forms a feasible solution with value at most \( 2\text{OPT}' \) as every point in \( P \) is within distance at most \( 2\text{OPT}' \) of a point in \( C \). If \( |C| = k + 1 \) then all points in \( C \) are at distance more than \( 2\text{OPT}' \) of each other, and hence we can assert that \( \text{OPT} > \text{OPT}' \).

\textit{Running time analysis.} We now show that our amortized update time is \( O(n + k) \). Let \( c_t \) be the number of centers after update operation \( t \) and \( n_t \) be the size of \( P \) after update operation \( t \). Operation 0 is the preprocessing and \( n_0 \) is the size of the initial set \( P \) and \( c_0 \) the number of centers after preprocessing.

\textit{Worst-case running time.} We first analyze the worst-case running time of update operation \( t \) for \( t \geq 0 \).
When update operation $t$ inserts a point $p$, we need $\Theta(k)$ time to initialize the counter of $p$ by counting its neighbors in $C$. Additionally, if $p$ becomes a center, it incurs a cost of $\Theta(n_t)$ to update the counters of its neighbors. Thus insertions have $\Theta(n_t)$ worst-case time. Furthermore, only insertions that create a new center take time $\Theta(n_t)$, all other insertions take time $\Theta(k)$. This implies that the worst-case preprocessing time is $\Theta(n_0k)$.

Consider next the case that update operation $t$ deletes a point $p$. When a point $p \in P \setminus C$ is deleted, the running time is constant. If a center $p$ is deleted, the algorithm incurs a cost of $O(n_t)$ to update the counters of its neighbors. Additionally, all the neighbors $q$ with $a_q = 0$ are placed on a queue and processed one after the other. If a point $q$ has $a_q = 0$ when it is pulled off the queue, it becomes a center and there is cost of $O(n_t)$ to update the counters of all neighbors of $q$. If $a_q > 0$, $q$ does not become a center and only $O(1)$ time is spent on $q$. Thus, the worst-case time per delete is $O((1 + \delta_t) \cdot n_t)$, where $\delta_t = c_t - c_{t-1}$.

**Token-charging scheme.** We next show how to pay for these operations using amortized analysis. We use a token-base approach such that each token can be used to pay for $O(1)$ amount of work. The preprocessing phase is charged $(2k+1)n_0$ tokens, $kn_0$ of its tokens are used to pay for the preprocessing time and the remaining $(k+1)n_0$ of the tokens are placed on the bank account. It follows that the amortized preprocessing time is $\Theta(kn_0)$, as is its worst-case running time.

Consider the $t$-th update operation. Each update operation is charged $n_t + k$ tokens, where $n$ is the current number of points. For insertions we use at most $n_t$ tokens to pay for the operation and place $k$ tokens on a bank account. It follows that the amortized insertion time is $O(n_t + k) = O(n + k)$.

Assume next that the $t$-th update operation is a deletion. We will show below that for any $t \geq 1$ the bank account contains at least $\delta_t n_{t-1}$ tokens right before operation $t$. We use the $n_t + k$ tokens charged to the deletion plus $\delta_t n_{t-1}$ tokens from the bank account to pay for the deletions and we put any leftover tokens on the bank account. As the worst-case running time of a deletion is $O((1 + \delta_t) \cdot n_t)$, it follows that the amortized time of the deletion is $O((1 + \delta_t) \cdot n_t - \delta_t n_t + (n_t + k)) = O(n_t + k) = O(n + k)$.

It remains to prove that the bank account contains at least $\delta_t n_{t-1}$ tokens right before the $t$-th operation. Note that $\delta_t = c_t - c_{t-1} \leq k + 1 - c_{t-1}$. Thus it suffices to show that the bank account contains at least $(k + 1 - c_{t-1}) \cdot n_{t-1}$ tokens before operation $t$ for $t \geq 1$, i.e. at least $(k + 1 - c_t) \cdot n_t$ tokens after operation $t$ for $t \geq 0$. To show this we perform an induction on $t$. For $t = 0$ note that preprocessing placed $(k + 1)n_0 \geq (k + 1 - c_0)n_0$ token on the bank account and thus, the claim holds. Assume the claim was true after $t - 1$ operations and we want to show that it holds after $t$ operation. We consider the following cases.

(a) If operation $t$ is an insertion, then the number of tokens on the bank account increases by $k$, while $c_t$ might be unchanged or increased by 1. Thus $c_t \geq c_{t-1}$ and $n_t = n_{t-1} + 1$. It follows that the number of tokens on the bank account is at least $(k + 1 - c_{t-1}) \cdot n_{t-1} + k \geq (k + 1 - c_t) \cdot (n_t - 1) + k \geq (k + 1 - c_t) \cdot n_t + k - (k + 1 - c_t) \geq (k + 1 - c_t) \cdot n_t$ as $c_t \geq 1$ after the insertion.

(b) If operation $t$ is a deletion, then $n_t = n_{t-1} - 1$, the operation is charged $n_t + k$ and the operation consumes $(1 + \delta_t)n_t$ tokens. Recall that $c_{t-1} = c_t - \delta_t$ and that $c_{t-1} \leq k + 1 \leq 2k + 1$. Thus the number of tokens on the bank account after operation $t$ is at least $(k + 1 - c_{t-1}) \cdot n_{t-1} + n_t + k - (1 + \delta_t)n_t = (k + 1 - c_t + \delta_t + 1 - 1 - \delta_t) \cdot n_t + 2k + 1 - c_{t-1} \geq (k + 1 - c_t) \cdot n_t$.

This completes the induction and, thus, the running time analysis.

It follows directly by construction of the algorithm that any point in $P$ becomes a center at most once, and then stays a center until it is deleted. Note that after a point $p$ has become a center, it might be that at some point $\text{OPT} > \text{OPT}'$ is reported and no center is output. However, if later on again a solution is output, then $p$ is again a center in this solution (unless $p$ has already been deleted). 

\[\square\]

**D.2 Proof of Lemma 21**

**Lemma 21.** There is an algorithm for the deletion-only case with an expected update time of $O(k)$ that either outputs a solution of value $4\text{OPT}'$ with $O(k \log n)$ centers, or asserts that there is no solution with
value $OPT'$ that uses at most $k$ centers.

Proof. Suppose that the current solution is queried. If each bucket is small (which happens if we stopped
the last partial rebuild because $U_i = \emptyset$ for some $i$), we output $C$. By construction, in this case each point
$p \in P$ is contained in some cluster $P(p')$ for some point $p'$, and we output one point from each such cluster
$P(p')$. Each cluster has a diameter of at most $4OPT'$, and therefore for each point $p \in P$ there is some
center $c \in C$ such that $d(p, c) \leq 4OPT'$. In addition, we have that $|C| \leq O(k \log n)$. On the other hand,
if there is a large bucket $C^*$ we report that $OPT > OPT'$. This is justified since $C^*$ contains at least $k + 1$
points such that for any two such distinct points $c, c' \in C^*$ it holds that $d(c, c') > 2OPT'$. □

E Lower bound for randomized algorithms against an oblivious adversary

In this section we prove Theorem 24 i.e., we show that if an algorithm returns a $(\Delta - \epsilon)$-approximation
(for some $\epsilon > 0$) of the cost of the optimal solution to $k$-center, then it needs to ask $\Omega(k)$ queries to the
adversary for each insertion after the first $k$ insertions. Then we argue that we get similar lower bounds also
for $k$-median, $k$-means, $k$-sum-of-radii, and $k$-sum-of-diameters.

To avoid confusion between the query operations that the adversary asks to the algorithm and the distance
queries that the algorithm asks the adversary, we formalize the latter kind of queries using the notion of
distance oracles. A distance oracle is a black-box that returns the distance between two points that are
given as input in constant time and its output is controlled by the adversary. This is the only way that the
algorithm can get information about the metric space, with each distance query “costing” constant time for
the algorithm. As the distance oracle needs to give consistent answers, the distance between two queried
points has been fixed, i.e. the adversary cannot change it later. The answers of the distance oracle are
determined by an adversary who also determines the sequence of operations given to the algorithm. Thus
there are two types of queries that should not be confused: (1) the $k$-center-cost query operations issued by
the adversary and (2) the distance-oracle queries issued by the algorithm. The goal of the adversary is to
maximize the running time of the algorithm. The general approach of the lower bound construction is to
give an adversary that gives a sequence of operations and reveals the metric in such a way that the algorithm
has to ask $\Omega(k)$ distance oracle queries per operation.

We first present the general idea: Assume the adversary uses a metric space $(X, d)$ with $\min_{x, y \in X} d(x, y) =
1$ and $\max_{x, y \in X} d(x, y) = \Delta$. Suppose by contradiction that there is a dynamic randomized algorithm for
$k$-center for which the update time is at most $k/4$, and in addition the algorithm needs at most $k/4$ time to
report the value $\text{apx}$.

Consider the following sequence of operations and queries. As usual, we assume that the adversary
needs to define the operations without seeing the random bits of the algorithm. First, the adversary intro-
duces $k$ points $P_0$ that are at pairwise distance $\Delta$ to each other. For simplicity, we assume that the algorithm
knows their pairwise distances, without having to query the distance between any pair of them. Thus, their
distances are fixed and cannot be changed by the adversary anymore. Then, the adversary introduces a point
$p_1$ and afterwards it issues as $k$-center-cost query.

If the algorithm issues at least $k$ distance-oracle queries for the insertion of $p_1$ with probability 1, then the
claim that the algorithm spends time $\Omega(k)$ per insertion holds already. Otherwise there is some probability
$q > 0$ such that if $d(p_1, p) = \Delta$ for each $p \in P_0$, then the algorithm queries the distance between $p_1$ and at
most $k - 1$ points $P_0^* \subseteq P_0$. Thus there is at least one point $p_1^*$ in $P_0^* \subseteq P_0$ such that the distance $d(p_1, p_1^*)$
has not yet been queried and, thus, not yet been fixed. Let $\text{apx}$ denote the value of the approximate solution
that the algorithm reports in this case. Consider the following two cases:

Case 1: If $\text{apx} \geq \Delta$, then the adversary takes the point $p_1^*$ and defines $d(p_1, p_1^*) = 1$ and $d(p_1, p) = \Delta$
for each $p \in P_0 \setminus \{p_1^*\}$. (Note that for this metric, with probability at least $q$ the algorithm makes the same
queries as for the metric in which $d(p_1, p) = \Delta$ for each $p \in P_0$, i.e. where $d(p_1, p_1^*) = \Delta$.) It follows that

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the optimum solution consists of all points in \( P_0 \) and has cost 1, i.e., \( OPT = 1 \leq \text{apx}/\Delta \) and hence the approximation ratio of the algorithm is at least \( \Delta \) (with probability at least \( q > 0 \)).

Case 2: If \( \text{apx} < \Delta \) then the adversary defines that \( d(p_1, p) = \Delta \) for each \( p \in P_0 \), also for \( p^*_i \) and, thus, the cost of the optimal solution is \( \Delta > \text{apx} \). Therefore, in this case the algorithm reports a wrong upper bound for its cost which is a contradiction.

Thus, the approximation ratio of the algorithm is again at least \( \Delta \). Finally, the adversary deletes \( p_1 \) again and repeats the above with a new point \( p_2 \) and so on. For each new point \( p_1, p_2, \ldots \), there are three operations of the adversary, namely an insertion, a \( k \)-center cost query, and a deletion. The running time of the algorithm is at most \( k/4 \) per operation, and thus, at most \( 3k/4 \) per point \( p_i \).

We formalize this bound and its proof in the following theorem.

**Theorem 38.** Let \( \epsilon > 0 \). Any (randomized) dynamic algorithm for finite metric spaces that can provide a \((\Delta - \epsilon)\)-approximation to the \( k \)-center cost of its input issues amortized \( \Omega(k) \) distance-oracle queries in expectation after \( O(k^2) \) operations generated by an oblivious adversary.

**Proof.** We use Yao’s principle to prove the lower bound, i.e., we give a distribution of the input and determine the worst-case complexity of any deterministic algorithm for such an input distribution. Let \( \text{apx} \) be any deterministic algorithm that (i) takes as input a sequence of \( t \) point insertions, deletions and cost queries, (ii) outputs, for each cost-query operation \( t' \), a \((\Delta - \epsilon)\)-approximation to the \( k \)-center cost of the set of input points after operation \( t' - 1 \) and (iii) issues at most \( tk/256 \) distance-oracle queries in expectation. Since any dynamic algorithm that beats the claimed lower bound in the statement of the lemma can be turned into \( \text{apx} \) by using it as a black-box, proving that \( \text{apx} \) does not exist proves the claim.

We define the uniform distribution over a set \( S \) of sequences, each consisting of \( t := k + 30k^2 \) operations. For the sake of analysis, we describe \( S \) by iteratively constructing an element from it according to the uniform distribution. Let \( P_0 \) be a set of \( k \) points that have pairwise distance \( \Delta \). The first \( k \) operations insert the points in \( P_0 \). For each \( i \in \{0, \ldots, 10k^2 - 1\} \), we choose a uniformly random point \( p^*_i \in P_0 \). Operation \( k + 3i \) inserts a point \( p_i \), where, for every \( p \in P_0 \setminus \{p^*_i\} \), we have \( d(p_i, p) = \Delta \). With probability \( 1/2 \), we set \( d(p_i, p^*_i) = 1 \), and \( d(p_i, p^*_i) = \Delta \) otherwise. Operation \( k + 3i + 1 \) is a \( k \)-center cost query, and operation \( k + 3i + 2 \) removes \( p_i \). This way, every sequence \( \sigma \in S \) is naturally partitioned into an initialization block of \( k \) operations and \( 10k^2 \) small blocks of 3 operation each. As a sequence is determined by \( 10k^2 \) many 2\( k \)-ary choices, it follows that \( |S| = (2k)^{10k^2} \).

For the sake of simplicity, we reveal the identity of \( P_0 \) and the pairwise distances of points in \( P_0 \) to the algorithm. This gives additional information to the algorithm without any cost for the algorithm, i.e., it only “helps” the algorithm. By an averaging argument, the algorithm must query at most \( tk/128 \) distances on at least half of the sequences from \( S \). Let \( T \subseteq S \) denote this subset of sequences. By another averaging argument, for at least half of the small blocks \( i \) in a sequence from \( T \), \( \text{apx} \) queries at most \((tk/128)/k^2 \leq 32k^3/(128k^2) = k/4 \) distances between \( p_i \) and points in \( P_0 \).

Therefore, there must exist a \( \sigma \in S \) and a corresponding small block \( j \) so that \( \text{apx} \) queries at most \( k/4 \) distances between \( p_j \) and points in \( P_0 \). Let \( \sigma' \in S \) be any sequence that equals \( \sigma \) on the first \( k + 3j \) operations (i.e., on its points and distances). When the \( j \)th cost query on \( \sigma \) or \( \sigma' \) is issued, the optimal cost can differ by a factor of \( \Delta \): If \( d(p_j, p^*_j) = 1 \), the optimal cost of the instance after operation \( k + 3i + 1 \) is 1; if \( d(p_j, p^*_j) = \Delta \), the optimal cost of the instance is \( \Delta \). To distinguish these two cases, the algorithm needs to determine whether there exists \( p^*_j \) so that \( d(p_j, p^*_j) = 1 \). Since \( p^*_j \) is chosen uniformly at random and the events \( d(p_j, p^*_j) = 1 \) and \( d(p_j, p^*_j) = \Delta \) both have non-zero probability, the probability that \( \text{apx} \) distinguishes these two events is at most \( k/4 \cdot 1/(|P_0| - k/4) \leq k/4 \cdot 1/(3k) < 1 \) by the union bound. This contradicts the assumption on \( \text{apx} \).

Finally, we observe that each of the \( k^2 \) points \( p_j \) is chosen from a set of \( k \) points, and therefore, a space of size \( |P_0| + k \cdot k^2 = O(k^3) \) suffices to construct \( S \). Choosing, for any \( i, j \), \( d(p_i, p_j) = 1 \) if \( p^*_i = p^*_j \) and \( d(p_i, p_j) = \Delta \) otherwise, the space is metric.
Also note that Yao’s minmax principle gives a lower bound against an oblivious adversary. □

Using a similar construction, we can obtain a lower bound on the approximation ratio of any center set that is computed by an algorithm (rather than an approximation on the cost of $\OPT$ computed by an algorithm) that queries amortized $o(k)$ distances $d(p, q)$ in expectation, where $p, q \in X$ (note that a $\Omega(k)$ bound on the running time follows already from the output complexity of a center set for a suitably chosen input). Here, we fix a set $P_0$ of $k - 1$ points at pairwise distance $\Delta$ and insert two points $q_i$ and $p_i$ in each iteration. The point $p_i$ has distance $\Delta$ to all other points, while $q_i$ has distance 1 to a uniformly random $p_i^{(s)} \in P_0$ and distance $\Delta$ to every other point. Intuitively, it is hard for the algorithm to decide whether it should place the $k^{th}$ center on $q_j$ or $p_j$. Choosing $P_0 \cup \{p_i\}$ or $P_0 \setminus \{p_i^*\} \cup \{q_j, p_j\}$ gives a solution with cost 1, while any other choice gives a solution with cost $\Delta$.

**Theorem 39.** Let $\epsilon > 0$. Any (randomized) dynamic algorithm for finite metric spaces that can provide a center set whose cost is a $(\Delta - \epsilon)$-approximation to the $k$-center cost of its input issues amortized $\Omega(k)$ distance-oracle queries in expectation after $O(k^2)$ operations generated by an oblivious adversary.

**Proof.** We use Yao’s principle to prove the lower bound. Let apx be any deterministic algorithm that (i) takes as input a sequence of $t$ point insertions, deletions and cost queries, (ii) outputs, for each center-query operation $t'$, a center set whose cost is a $(\Delta - \epsilon)$ to the $k$-center cost of the set of input points after operation $t' - 1$ and (iii) queries at most $tk/512$ distances in expectation over the input distribution. Since any dynamic algorithm that beats the claimed lower bound in the statement of the lemma can be turned into apx by using it as a black-box, proving that apx does not exist proves the claim. We define the uniform distribution over a set $S$ of sequences, each consisting of $t := k + 50k^2$ operations. For the sake of analysis, we describe $S$ by iteratively constructing an element from it according to the uniform distribution. Let $P_0$ be a set of $k - 1$ points that have pairwise distance $\Delta$. The first $k' := k - 1$ operations insert the points in $P_0$. For each $i \in \{0, \ldots, 10k^2 - 1\}$, we choose a uniformly random point $p_i^* \in P_0$. Operations $k' + 5i$ and $k' + 5i + 1$ insert points $q_i, p_i$ at distance $\Delta$ to each other. For every $p \in P_0 \setminus \{p_i^*\}$, we have $d(q_i, p) = \Delta$, and we set $d(q_i, p_i^*) = 1$. The distance between $p_i$ and any point $p \in P_0$ is $\Delta$. Operation $k' + 5i + 2$ queries the algorithm’s solution, and operations $k' + 5i + 3$ and $k' + 5i + 4$ remove $q_i, p_i$. This way, every sequence $\sigma \in S$ is naturally partitioned into an initialization block of $k'$ operations and $10k^2$ small blocks of 5 operations each. As a sequence is determined by $10k^2$ many $k$-ary choices, it follows that $|S| = k^{10k^2}$.

For the sake of simplicity, we reveal the identity of $P_0$ and the pairwise distances of points in $P_0$ to the algorithm. This gives additional information to the algorithm without any cost for the algorithm, i.e., it only “helps” the algorithm. By an averaging argument, the algorithm must query at most $tk/256$ distances on at least half of the sequences from $S$. Let $T \subseteq S$ denote this subset of sequences. By another averaging argument, for at least half of the small blocks $i$ in a sequence from $T$, apx queries at most $(tk/256)/k^2 \leq 64k^3/(256k^2) = k/4$ distances between $\{q_i, p_i\}$ and points in $P_0$.

Therefore, there must exist a $\sigma \in S$ and a corresponding small block $j$ so that apx queries at most $k/4$ distances between $\{q_j, p_j\}$ and points in $P_0$. Let $\sigma' \in S$ be any sequence that equals $\sigma$ on the first $k' + 5j$ operations (i.e., on its points and distances). When the $j$-th cost query on $\sigma$ or $\sigma'$ is issued, only the center sets $P_0 \cup \{p_j\}$ and $P_0 \setminus \{p_j^*\} \cup \{q_j, p_j\}$ have cost 1, all other center sets have cost $\Delta$. To distinguish these two cases, the algorithm needs to determine $q_j$ or $p_j^*$, i.e., issue a query that reveals $d(q_j, p_j^*)$. Since $p_j^*$ is chosen uniformly at random, the probability that apx distinguishes these two sequences is at most $k/4 \cdot 1/(2|P_0| - k/4) \leq k/4 \cdot 4/(7k) < 1$ by the union bound. This contradicts the assumption on apx.

Finally, we observe that each of the $2k^2$ points $q_j, p_j$ are chosen from a set of $2k$ points, and therefore, a space of size $|P_0| + 2k \cdot 2k^2 = O(k^3)$ suffices to construct $S$. Choosing, for any $i, j, d(p_i, p_j) = 1$ if $p_i^* = p_j^*$ and $d(p_i, p_j) = \Delta$ otherwise, the space is metric.

With exactly the same construction we obtain lower bounds of $\Delta$ for $k$-sum-of-radii, $k$-sum-of-diameters, $k$-median, a lower bound of $\Delta^2$ for $k$-means, and a lower bound of $\Delta^p$ for $(k, p)$-clustering for each $p > 0$. The reason is that in each of the possible metrics above, the cost of $\OPT$ is defined by $d(p_i, p)$ for one point
We show that Algorithm 5 maintains a clustering tree. Afterwards, the algorithm recurses on the parent of $u$. We explain in the proof of Theorem 48 how to get rid of this assumption. For the sake of simplicity, we identify a node $u$ with its associated blocking graph $N = (V, E)$ in the following. For each node $N = (V, E)$, at most $k$ points are marked as centers (isCenter in Algorithm 4), and the algorithm maintains the invariant that two centers $u, v \in V$ have distance at least $\OPT'$ by keeping record of blocking edges in the blocking graph between centers and points that have distance less than $\OPT'$ to one of these centers. We say that a center $u$ blocks a point $v$ (from being a center) if there is an edge $(u, v)$ in the blocking graph. In addition, the algorithm records whether $N$ contains more than $k$ points with pairwise distance greater than $\OPT'$ (lowerBoundWitness in Algorithms 4 and 5).}

**Insertions** (see InsertPoint). When a point $u$ is inserted into $T$, a node $N = (V, E)$ with less than $2k$ points is selected and it is checked whether $d(v, u) \leq \OPT'$ for any center $v \in V$. If this is the case, the algorithm inserts an edge $(u, v)$ for every such center $v$ into $E$ and terminates afterwards. Otherwise, the algorithm checks whether the number of centers is less than $k$. If this is the case, it marks $u$ as a center and inserts an edge $(u, w)$ for each point $w \in V$ with $d(u, w) \leq \OPT'$ and recurses on the parent of $N$. Otherwise, if there are more than $k - 1$ centers in $u$, the algorithm marks $N$ as witness and terminates.

**Deletions** (see DeletePoint). When a point $u$ is deleted from $T$, the point is first removed from the leaf $N$ (and the blocking graph) where it is stored. If $u$ was not a center, the algorithm terminates. Otherwise, the algorithm checks whether any points were unblocked (have no adjacent node in the blocking graph) and, if this is the case, proceeds by attempting to mark these points as centers and inserting them into the parent of $N$ one by one (after marking the first point as center, the remaining points may be blocked again). Afterwards, the algorithm recurses on the parent of $N$.

Given a rooted tree $T$ and a node $u$ of $T$, we denote the subtree of $T$ that is rooted at $u$ by $T(u)$. For a clustering tree $T$, we denote the set of all points stored at the leaves of $T(u)$ by $\mathcal{P}(u)$. Recall that the points directly stored at $u$ are denoted by $P_u$. Observe that for each node $u$ in a clustering tree, $T(u)$ is a clustering tree of $\mathcal{P}(u)$.

### F.1 Feasibility

We show that Algorithm 5 maintains a clustering tree.

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Data: isCenter is a boolean array on the elements of $V$, lowerBoundWitness is a boolean array on the nodes of $T$

Function $\text{InsertIntoNode}(N = (V, E), p, OPT')$

insert $p$ into $V$

foreach $v \in V \setminus \{p\}$ do
  if $\text{isCenter}[v] \wedge d(p, v) \leq OPT'$ then
    insert $(v, p)$ into $E$
  return $\text{TryMakeCenter}(G, p, OPT')$

Function $\text{DeleteFromNode}(N = (V, E), p, OPT')$

newCenters $\leftarrow \emptyset$
neighbors $\leftarrow \Gamma(p)$
delete $p$ from $N$
if $\text{isCenter}[p] = \text{true}$ then
  foreach $u \in$ neighbors do
    newCenters $\leftarrow$ newCenters $\cup$ $\text{TryMakeCenter}(N, u, OPT')$
  return newCenters

Function $\text{TryMakeCenter}(N = (V, E), p, OPT')$

if $\deg(p) = 0 \wedge |\{v \mid v \in V \wedge \text{isCenter}[v]\}| < k$ then
  $\text{isCenter}[p] \leftarrow \text{true}$
  foreach $v \in V \setminus \{p\}$ do
    if $d(p, v) \leq OPT'$ then
      insert $(p, v)$ into $E$
    lowerBoundWitness[N] $\leftarrow$ false
  return $\{p\}$
else if $\deg(p) = 0$ then
  lowerBoundWitness[N] $\leftarrow$ true
return $\emptyset$

Algorithm 4: Insertion and deletion of a point $P$ in a node of the clustering tree $T$ (represented by a blocking graph $G$).

**Lemma 41.** Let $T$ be a clustering tree on a point set $P$. After calling $\text{InsertPoint}(T, p)$ (see Algorithm 5) for some point $p \notin P$, $T$ is a clustering tree on $P \cup \{p\}$.

**Proof.** We prove the statement by induction over the recursive calls of $\text{InsertIntoNode}$ in $\text{InsertPoint}$.

Let $N = (V, E)$ be the leaf in $T$ where $p$ is inserted. Condition 1 in Definition 40 is guaranteed for $N$ by $\text{InsertPoint}$. The algorithm $\text{InsertIntoNode}$ ensures that $p$ is marked as a center only if it is not within distance $OPT'$ of any other center. Let $C$ be the center set of $N$ before inserting $p$. By condition 2, $C$ has cost $\leq OPT'$. Let $C'$ be the center set of any optimal solution on $V$. If $\text{cost}_k(C', V) \leq OPT' / 2$, each center of $C'$ covers at least one cluster of $C$. By pigeonhole principle, $p$ is not blocked by a center in $C$ if and only if $|C| < k$. Otherwise, if $\text{cost}_k(C', V) > OPT' / 2$, $p$ is chosen if no center in $C$ covers $p$ and $|C'| < k$, or $N$ is marked as witness. It follows that condition 2 is still satisfied for $N$ after $\text{InsertIntoNode}$ terminates.

Now, let $N = (V, E)$ be any inner node in a call to $\text{InsertIntoNode}(N, p)$. We note that such call is only made if $p$ was marked as a center in its child $N'$ on which $\text{InsertIntoNode}$ was called before by $\text{InsertPoint}$. Thus, $p$ is inserted into $V$ if and only if $p$ is a center in $N'$. Therefore, condition 2 is satisfied for $N$. By the above reasoning, condition 3 is also satisfied.

**Lemma 42.** Let $T$ be a clustering tree on a point set $P$. After calling $\text{DeletePoint}(T, p)$ (see Algorithm 6) for some point $p \notin P$, $T$ is a clustering tree on $P \setminus \{p\}$.
Data: lowerBoundWitness is a boolean array on the nodes of $T$

Function InsertPoint ($T,p,OPT'$)

\[ N \leftarrow \text{leaf in } T \text{ that contains less than } 2k \text{ elements} \]
\[ \text{do} \]
\[ \text{centers } \leftarrow \text{InsertIntoNode}(N,p) \]
\[ N \leftarrow N.\text{parent} \]
\[ \text{while centers } = \{ p \} \]

Function DeletePoint ($T,p,OPT'$)

\[ N \leftarrow \text{leaf in } T \text{ that contains } p \]
\[ \text{centers } \leftarrow \emptyset; \text{failed } \leftarrow \text{false} \]
\[ \text{do} \]
\[ \text{centers } \leftarrow \text{centers } \cup \text{DeleteFromNode}(N,p) \]
\[ \text{newCenters } \leftarrow \emptyset \]
\[ \text{foreach } v \in \text{centers} \]
\[ \text{newCenters } \leftarrow \text{newCenters } \cup \text{InsertIntoNode}(N.\text{parent},v) \]
\[ \text{centers } \leftarrow \text{newCenters} \]
\[ N \leftarrow N.\text{parent} \]
\[ \text{while } N \neq \text{null} \]

Algorithm 5: Insertion and deletion of a point $p$ in the clustering tree $T$.

**rithm 5** for some point $p \in P$, $T$ is a clustering tree on $P \setminus \{p\}$.

**Proof.** We prove the statement by induction over the loop’s iterations in DeletePoint. Let $N = (V,E)$ be the leaf in $T$ where $p$ was inserted. DeleteFromNode deletes $p$ from $N$ and iterates over all unblocked points to mark them as centers one by one. After deleting $p$, condition [1] holds for $N$. Let $U$ be the set of unblocked points after removing $p$, let $C$ be the center set after removing $p$ from $V$, and let $C'$ be the center set of any optimal solution on $V \setminus \{p\}$. If $\text{cost}_k(C',V) \leq \text{OPT}'/2$, each unblocked point from $U$ covers at least one cluster of $C'$ with cost $\text{OPT}'$. Otherwise, if $\text{cost}_k(C',V) > \text{OPT}'/2$, a set of at most $k - |C|$ points from $U$ is chosen, or $N$ is marked as witness. It follows that condition [2] is still satisfied for $N$ after DeleteFromNode terminates. Let $C''$ be the center set that is returned by DeleteFromNode. DeletePoint inserts all points from $C''$ into the parent node. This reinstates condition [3] on the parent node. Then, DeleteFromNode recurses on the parent. \qed

### F.2 Approximation guarantees

We use the following notion of super clusters and its properties to prove the $O(k)$ upper bound on the approximation ratio of the algorithm.

**Definition 43** (super cluster). Let $P$ be a set of points and let $C$ be a center set with $\text{cost}_k(C,P) \leq 2\text{OPT}'$. Consider the graph $G = (C,E)$, where $E = \{(u,v) \mid d(u,v) \leq 2\text{OPT}'\}$. For every connected component in $G$, we call the union of clusters corresponding to this component a super cluster.

**Lemma 44.** Let $T$ be a clustering tree constructed by Algorithm 5 with node-cost $\text{OPT}'$ and no node marked as witness. For any node $N$ in $T$, $N$ contains one point from each super cluster in $P(N)$ that is marked as center.

**Proof.** Let $S$ be a supercluster of $P$. Let $N$ be a node in $T$ that contains a point $p \in S$. For the sake of contradiction, assume that there exists no point $q \in S \cap N$ that is marked as center. By Definition 43, the $k$-center clustering cost of the centers in $N$ is greater than $\text{OPT}'$. By Definition 40, this implies that a point
must be marked as witness. This is a contradiction to the assumption that no node is marked as witness.

The following simple observation leads to the bound of $O(\log(n/k))$ on the approximation ratio.

**Observation 45.** Let $c > 0$, let $P, Q$ be sets of points and let $C, C'$ so that $\text{cost}_k(C, P) \leq c$ and $\text{cost}_k(C', Q) \leq c$. For every $k$-center set $C''$ with $\text{cost}_k(C'', C \cup C') \leq c'$ we have $\text{cost}_k(C'', P \cup Q) \leq c + c'$ by the triangle inequality.

We combine the previous results and obtain the following approximation ratio for our algorithm.

**Lemma 46.** Let $T$ be a clustering tree on a point set $P$ that is constructed by Algorithm 5 with node-cost $\text{OPT}'$. Let $C$ be the points in the root of $T$ that are marked as centers. If no node in $C$ is marked as witness, $\text{cost}_k(C, P) \leq \min\{k, \log(n/k)\} \cdot 4\text{opt}_k(P)$. Otherwise, $\text{cost}_k(C, P) \geq \text{OPT}'/2$.

**Proof.** Assume that $\text{opt}_k(P) \leq \text{OPT}'/2$, as otherwise, there exists a node that stores at least $k + 1$ points that have pairwise distance $\text{OPT}'$, which implies the claim. First, we prove $\text{cost}_k(C, P) \leq \log(n/k) \cdot \text{OPT}'$. It follows from Observation 45 that $C$ has $k$-center cost $2\text{OPT}'$ on the points stored in the root’s children. Since $T$ has depth at most $\log(n/k)$, it follows by recursively applying Observation 45 on the children that $C$ also has cost $\log(n/k) \cdot \text{OPT}'$ on $P$.

Now, we prove $\text{cost}_k(C, P) \leq k$. Let $p \in P$, and let $S$ be the super cluster of $p$ with corresponding optimal center set $C'$. We have $d(p, C') \leq \text{OPT}'/2$. By Lemma 44 there exists a center $q \in C$ so that $d(q, C') \leq \text{OPT}'/2$. By the definition of super clusters, for every $x, y \in C'$, $d(x, y) \leq (k - 1) \cdot 2\text{OPT}'$. It follows from the triangle inequality that $d(p, q) \leq 2k\text{OPT}'$. □

### F.3 Running time

**Lemma 47.** The amortized update time of Algorithm 5 is $O(k \log(n/k))$.

**Proof.** To maintain blocking graphs efficiently, the graphs are stored in adjacency list representation and the degrees of the vertices as well as the number of centers are stored in counters. When a point $p$ is inserted, $5k \log(n/k)$ tokens are paid into the account of $p$. Each token pays for a (universally) constant amount of work.

Each point is inserted at most once in each of the $\log(n/k)$ nodes from the leaf where it is inserted up to the root. The key observation is that it is marked as a center in each of these nodes at most once (when it is inserted, or later when a center is deleted): Marking a point as center is irrevocable until it is deleted. For each node $N$ and point $p \in N$, it can be checked in constant time whether $p$ can be marked as a center in $N$ by checking its degree in the blocking graph. Marking $p$ as a center takes time $O(2k)$ because it is sufficient to check the distance to all other at most $2k$ points in $N$ and insert the corresponding blocking edges. For each point $p$, we charge the time it takes to mark $p$ as a center to the account of $p$. Therefore, marking $p$ as center withdraws a most $2k \log(n/k)$ tokens from its account in total.

Consider the insertion of a point $p$ into a node. As mentioned, $p$ is inserted in at most $\log(n/k)$ nodes, and in each of these nodes, it is inserted at most once (when it is inserted into the tree, or when it is marked as a center in a child node). Inserting a point into a node $N$ requires the algorithm to check the distance to all at most $k$ centers in $N$ to insert blocking edges, which results in at most $k \log(n/k)$ work in total.

It remains to analyze the time that is required to update the tree when a point $p$ is deleted. For any node $N$, if $p$ is not a center in $N$, deleting $p$ takes constant time. Otherwise, if $p$ is a center, the algorithm needs to check its at most $2k$ neighbors in the blocking graph one by one whether they can be marked as centers. Checking a point $q$ takes only constant time, and marking $q$ as a center has already been charged to $q$ by the previous analysis. All centers that have been marked have to be inserted into the parent of $N$, but this has also been charged to the corresponding points. Therefore, deleting $p$ consumes at most $2k \log(n/k)$ tokens from the account of $p$. □
F.4 Main result

It only remains to combine all previous results to obtain Theorem 8.

**Theorem 48** (Theorem 8). Let $\epsilon, k > 0$. There exists a deterministic algorithm for the dynamic $k$-center problem that has amortized update time $O(k \log(n) \log(\Delta) / \log(1 + \epsilon))$ and approximation factor $(1 + \epsilon) \cdot \min\{4k, 4 \log(n/k)\}$.

**Proof.** Since $P = (P_1, \ldots, P_n)$ is a dynamic point set, its size $n_t := |P_t|$ can increase over time. Therefore, we need to remove the assumption that the clustering tree has depth $\log(\max_{t \in [n]} n_t)$. First, we note that we can insert and delete points so that the clustering tree $T$ is a complete binary tree (the inner nodes induce a full binary tree, and all leaves on the last level are aligned left): We always insert points into the left-most leaf on the last level of $T$ that is not full; when a point is deleted from a leaf $N$ that is not the right-most leaf $N'$ on the last level of $T$, we delete an arbitrary point $q$ from $N'$ and insert $q$ into $N$. Deleting and reinserting points this way can be seen as two update operations, and therefore, it can only increase time required to update the tree by a factor of 3. Furthermore, any leaf can be turned into an inner node by adding a copy of itself as its left child and adding an empty node as its right child. Vice versa, an empty leaf and its (left) sibling can be contracted into its parent. This way, the algorithm can guarantee that the depth of the tree is between $\lfloor \log(n_t/k) \rfloor$ and $\lceil \log(n_t/k) \rceil$ at all times $t \in [n]$.

Recall that $d(x, y) \geq 1$ for every $x, y \in X$, and $\Delta = \max_{x, y \in X} d(x, y)$. For every $\Gamma \in \{(1 + \epsilon)^i \mid i \in \lfloor \log_2(1+\epsilon(\Delta)) \rfloor\}$, the algorithm maintains an instance $T_\Gamma$ of a clustering tree with node-cost $\text{OPT}' = (1 + \epsilon)^\Gamma$ by invoking Algorithm 5. After every update, the algorithm determines the smallest $\Gamma$ so that no node in $T_\Gamma$ is marked as witness, and it reports the center set of the root of $T_\Gamma$. The bound on the cost follows immediately from Lemma 46. Since there are at most $\log(\Delta)/\log(1 + \epsilon)$ instances, the bound on the time follows from Lemma 47. \hfill \square