Irreducible highest-weight modules and
equivariant quantization

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1 Introduction

The notion of deformation quantization, motivated by ideas coming from both
physics and mathematics, was introduced in classical papers [2, 7, 8]. Roughly
speaking, a deformation quantization of a Poisson manifold \((P, \{,\})\) is a formal
associative product on \((\text{Fun} P)[[\hbar]]\) given by \(f_1 \star f_2 = f_1 f_2 + \hbar c(f_1, f_2) + O(\hbar^2)\) for
any \(f_1, f_2 \in \text{Fun} P\), where the skew-symmetric part of \(c\) is equal to \(\{,\}\), and the
coefficients of the series for \(f_1 \star f_2\) should be given by bi-differential operators.

The fact that any Poisson manifold can be quantized in this sense was proved
by Kontzevich in [15]. However, finding exact formulas for specific cases of Pois-
son brackets is an interesting separate problem. There are several well-known
examples of such explicit formulas. One of the first was the Moyal product quan-
tizing the standard symplectic structure on \(\mathbb{R}^{2n}\). Another one is the standard
quantization of the Kirilov-Kostant-Souriau bracket on the dual space \(\mathfrak{g}^*\) to a Lie
algebra \(\mathfrak{g}\) (see [10]). Relations between this quantization and the Yang-Baxter
equation was shown by Gekhtman and Stolin in [9].

Despite the formula for the standard quantization of the Kirilov-Kostant-
Souriau bracket is known already for a long time, the problem of finding explicit
formulas for equivariant quantization of its symplectic leaves, i.e., coadjoint orbits
on \(\mathfrak{g}^*\), was open. Recently this problem was solved in important cases in [1, 3,
4, 14] using the relationship with the dynamical Yang-Baxter equation and the
Shapovalov form on Verma modules.

This paper is a continuation of [14]. One of the main results obtained in [14] is
the connection between quantum dynamical twists and equivariant quantization.
More precisely, let \(\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+\) be a finite-dimensional complex semisimple Lie
algebra with a fixed triangular decomposition, \(F = \mathbb{C}[G]\) the algebra of all matrix
elements of all finite dimensional representations of \(\mathfrak{g}\), \(M(\lambda)\) the Verma module
with highest weight \(\lambda \in \mathfrak{h}^*\), \(J(\lambda)\) the universal fusion element that corresponds
to \(\lambda\). Assume that \(\lambda\) is generic, i.e., \(M(\lambda)\) is irreducible. We have a natural

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map $\text{Hom}_g(M(\lambda), M(\lambda) \otimes F) \to F[0]$ which is an isomorphism of vector spaces. The translation onto $F[0]$ of the natural product on $\text{Hom}_g(M(\lambda), M(\lambda) \otimes F)$ is given by the formula $f_1 \star \lambda f_2 = \mu \left( J(\lambda)(f_1 \otimes f_2) \right)$, where $\mu$ is the initial product on $F$ restricted onto $F[0]$. We may treat the obtained algebra as an equivariant quantization of the coadjoint orbit $O_{\lambda} \subset g^*$ equipped with the Kirillov-Kostant-Souriau bracket.

The main goal of this paper is to present some generalizations of the above mentioned results to the case of non-generic $\lambda$. In fact, we give explicit formulas for star-products on certain subspaces of $F[0]$, which are in general not closed under the original multiplication on $F$.

Consider the irreducible $g$-module $V(\lambda)$ with highest weight $\lambda \in h^*$. We have $V(\lambda) = M(\lambda)/K_\lambda 1$, where $K_\lambda \subset U_{n_-}$, and $1$ is the generator of $M(\lambda)$. Consider also the opposite Verma module $\hat{M}(\lambda)$ with the lowest weight $-\lambda \in h^*$ and the lowest weight vector $\hat{1}_{-\lambda}$. Note that the maximal $g$-submodule in $\hat{M}(\lambda)$ is of the form $\hat{K}_\lambda \cdot \hat{1}_{-\lambda}$, where $\hat{K}_\lambda \subset U_{n_+}$. We get a vector space isomorphism $\text{Hom}_g(V(\lambda), V(\lambda) \otimes F) \simeq F[0]^{K_\lambda + \hat{K}_\lambda}$, which allows one to consider the product on $F[0]^{K_\lambda + \hat{K}_\lambda}$ induced by the natural multiplication in $\text{End}_g(V(\lambda), V(\lambda) \otimes F)$. One can express this product on $F[0]^{K_\lambda + \hat{K}_\lambda}$ as $f_1 \star \lambda f_2 = \mu \left( J_{\text{red}}(\lambda)(f_1 \otimes f_2) \right)$, where the “reduced fusion element” $J_{\text{red}}(\lambda)$ can be computed in terms of the Shapovalov form on $V(\lambda)$.

For a special case when $\lambda \in h^*$ satisfies $\langle \lambda, \alpha^\vee \rangle = 0$ for any $\alpha$ in some simple root subset $\Delta$ and generic otherwise we show that $F[0]^{K_\lambda + \hat{K}_\lambda} \simeq F[0]^{K_\lambda}$. In this case $F[0]^{K_\lambda}$ is closed under the original multiplication on $F$. In fact, this is the algebra of regular functions on the coadjoint orbit $O_{\lambda}$. Hence the algebra $(F[0]^{K_\lambda}, \star)$ can be viewed as an equivariant quantization of $O_{\lambda}$.

Finally, we investigate limiting properties of the universal fusion element $J(\lambda)$. In particular we show that for some values of $\lambda_0 \in h^*$ we can guarantee that $f_1 \star \lambda f_2 \to f_1 \star \lambda_0 f_2$ as $\lambda \to \lambda_0$. We also show that for any $\lambda_0$ having a “good limiting property” of this type the action map $Ug \to (\text{End} V(\lambda_0))_{\text{fin}}$ is surjective (here $(\text{End} V(\lambda_0))_{\text{fin}}$ stands for the locally finite part of $\text{End} V(\lambda_0)$ with respect to the adjoint action of $Ug$). Note that this surjectivity question is known as the classical problem of Kostant (see [11, 12]). The complete answer to this question is still unknown. However, there are examples of $\lambda_0$ such that $Ug \to (\text{End} V(\lambda_0))_{\text{fin}}$ is not surjective (see [12]). There is also a known class of simple highest weight modules for which this map is surjective. We comment on the Kostant problem in other parts of the paper as well.

We also notice that most of the results of this paper have analogues for quantized universal enveloping algebras. We will discuss these questions in details elsewhere.

This paper is organized as follows. Section 2 contains some general Hopf-algebraic constructions that will be useful in the sequel. In Section 3 we provide
a construction of a star-product on $F[0]^{K+\tilde{K}}$ by means of the Shapovalov form on $V(\lambda)$. Subsection 3.1 is devoted to the general construction, and in Subsection 3.2 we discuss applications to symmetric spaces and coadjoint orbits. Finally, Section 4 is devoted to study of limiting properties of fusion elements and corresponding star-products.

Throughout this paper all Lie algebras are assumed to be finite-dimensional, and the ground field is $\mathbb{C}$.

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2 Hopf algebra preliminaries

Let $A$ be a Hopf algebra. As usual, we will denote by $\Delta$ (resp. $\varepsilon$, $S$) the comultiplication (resp. counit, antipode) in $A$. We will systematically use the Sweedler notation for comultiplication, i.e., $\Delta(x) = \sum (x^{(1)} \otimes x^{(2)})$, $(\Delta \otimes \text{id}) \Delta(x) = \sum (x^{(1)} \otimes x^{(2)} \otimes x^{(3)})$, etc.

Assume $M$ is a (left) $A$-module. An element $m \in M$ is called locally finite if $\dim Am < \infty$. Denote by $M_{\text{fin}}$ the subset of all locally finite elements in $M$. Clearly, $M_{\text{fin}}$ is a submodule in $M$. Similarly, we can consider locally finite elements in a right $A$-module $N$. For convenience, we will use the notation $N^r_{\text{fin}}$ for the submodule of all locally finite elements in this case.

Recall that the left (resp. right) adjoint action of $A$ on itself is defined by the formula $\text{ad}_x a = \sum (x^{(1)}aS(x^{(2)})$ (resp. $\text{ad}_x^r a = \sum S(x^{(1)})ax^{(2)}$). We denote by $A_{\text{fin}}$ (resp. $A^r_{\text{fin}}$) the corresponding submodules of locally finite elements. Since $\text{ad}_x(ab) = \sum (x^{(1)})a\text{ad}_x^r(b)$, we see that $A_{\text{fin}}$ is a (unital) subalgebra in $A$; the same holds for $A^r_{\text{fin}}$. If the antipode $S$ is invertible, then $S$ defines an isomorphism between $A_{\text{fin}}$ and $A^r_{\text{fin}}$. We will assume that $S$ is invertible.

Fix a Hopf subalgebra $F$ of the Hopf algebra $A^*$ dual to $A$. In the sequel we will use the left and right regular actions of $A$ on $F$ defined respectively by the formulas $(\tilde{f})(x) = f(xa)$ and $(f^\mu)(x) = f(ax)$.

Now let $M$ be a (left) $A$-module. Equip $F$ with the left regular $A$-action and consider the space $\text{Hom}_A(M, M \otimes F)$. For any $\varphi, \psi \in \text{Hom}_A(M, M \otimes F)$ define

$$\varphi \ast \psi = (\text{id} \otimes \mu) \circ (\varphi \otimes \text{id}) \circ \psi,$$

where $\mu$ is the multiplication in $F$. It is straightforward to verify that $\varphi \ast \psi \in \text{Hom}_A(M, M \otimes F)$, and this definition equips $\text{Hom}_A(M, M \otimes F)$ with a unital associative algebra structure.
Consider the map $\Phi : \text{Hom}_A(M, M \otimes F) \to \text{End} M$, $\varphi \mapsto u_\varphi$, defined by $u_\varphi(m) = (\text{id} \otimes \varepsilon)(\varphi(m))$; here $\varepsilon(f) = f(1)$ is the counit in $F$. In other words, if $\varphi(m) = \sum_i m_i \otimes f_i$, then $u_\varphi(m) = \sum_i f_i(1)m_i$. Using the fact that $\varepsilon$ is an algebra homomorphism it is easy to show that $\Phi$ is an algebra homomorphism as well.

**Lemma 1.** The map $\Phi$ embeds $\text{Hom}_A(M, M \otimes F)$ into $\text{End} M$.

**Proof.** If $\varphi \in \text{Hom}_A(M, M \otimes F)$, $\varphi(m) = \sum_i m_i \otimes f_i$, then

$$\varphi(am) = a\varphi(m) = \sum_i \sum_{(a)} a_{(1)}m_i \otimes \overrightarrow{a_{(2)}}f_i,$$

and

$$u_\varphi(am) = \sum_i \sum_{(a)} (\overrightarrow{a_{(2)}}f_i)(1)a_{(1)}m_i = \sum_{(a)} a_{(1)} \left( \sum_i f(a_{(2)})m_i \right).$$

Assume now that $u_\varphi = 0$, i.e., $\sum_{(a)} a_{(1)} \left( \sum_i f(a_{(2)})m_i \right) = 0$ for any $a \in A$ and $m \in M$. Then, in particular,

$$0 = \sum_{(a)} S(a_{(1)})a_{(2)} \left( \sum_i f(a_{(3)})m_i \right) =$$

$$\sum_{(a)} \varepsilon(a_{(1)}) \left( \sum_i f(a_{(2)})m_i \right) = \sum_i f_i(a)m_i$$

for any $a \in A$ and $m \in M$. Obviously, this means that $\varphi = 0$. \qed

From now on we assume that $F$ contains all matrix elements of the (left) adjoint action of $A$ on $A_{\text{fin}}$. Since $F$ is closed under the antipode $(Sf)(x) = f(S(x))$, we see that this assumption is equivalent to the fact that $F$ contains all matrix elements of the right adjoint action of $A$ on $A_{\text{fin}}$.

Let $a \in A_{\text{fin}}$, i.e., for any $x \in A$ we have $\text{ad}^r_x a = \sum_i f_i(x)a_i$, where $f_i \in A^*$, $a_i \in A$. In fact, we see that $f_i \in F$ by the assumption above. Define a linear map $\varphi_a : M \to M \otimes F$ by the formula $\varphi_a(m) = \sum_i a_i m \otimes f_i$. Clearly, $\varphi_a$ is well defined.

**Lemma 2.** For any $a \in A_{\text{fin}}$ we have $\varphi_a \in \text{Hom}_A(M, M \otimes F)$.

**Proof.** Let $b \in A$. Notice that

$$\sum_{(b)} b_{(1)} \text{ad}^r_{b_{(2)}} y = \sum_{(b)} b_{(1)} S(b_{(2)})yb_{(3)} = y \sum_{(b)} \varepsilon(b_{(1)})b_{(2)} = yb$$
for any $y \in A$. Therefore for any $x \in A$ we have
\[
\sum_i f_i(x) a_i b = (\text{ad}_x^r a) b = \sum_i b_{(1)} \text{ad}_{x_{(2)}}^r a = \sum_i b_{(1)} \text{ad}_{x_{(2)}}^r a = \sum_i f_i(x b_{(2)}) a_i = \sum_i (b_{(2)} f_i(x)) b_{(1)} a_i,
\]
and
\[
\varphi_a(bm) = \sum_i a_i bm \otimes f_i = \sum_i \sum_j (b_{(1)} a_i m \otimes \overrightarrow{b_{(2)} f_i}) = \varphi_a(bm).
\]

Denote by $\Psi : A_{\text{fin}}^r \to \text{Hom}_A(M, M \otimes F)$ the linear map constructed above (i.e., $\Psi : a \mapsto \varphi_a$).

**Lemma 3.** The map $\Psi$ is an algebra homomorphism.

**Proof.** Let $a, b \in A_{\text{fin}}^r$, $x \in A$, $\text{ad}_x^r a = \sum_i f_i(x) a_i$, $\text{ad}_x^r b = \sum_j g_j(x) b_j$. Then
\[
\text{ad}_x^r(ab) = \sum_{(x)} \text{ad}_{x_{(1)}}(a) \text{ad}_{x_{(2)}}(b) = \sum_{i,j} f_i(x_{(1)}) g_j(x_{(2)}) a_i a_j = \sum_{i,j} (f_i g_j)(x) a_i b_j.
\]
Thus
\[
\varphi_{ab}(m) = \sum_{i,j} a_i b_j m \otimes f_i g_j = (\varphi_a * \varphi_b)(m)
\]
for any $m \in M$. \qed

**Remark 1.** It follows directly from the definitions that the composition $\Phi \Psi$ equals the restriction to $A_{\text{fin}}^r$ of the canonical homomorphism $A \to \text{End} M$, $a \mapsto a_M$.

Now consider $A_{\text{fin}}^r$, $\text{Hom}_A(M, M \otimes F)$ and $\text{End} M$ as right $A$-modules: $A_{\text{fin}}^r$ via right adjoint action, $\text{Hom}_A(M, M \otimes F)$ via right regular action on $F$ (i.e., $(\varphi \cdot a)(m) = (\text{id} \otimes \overrightarrow{a})(\varphi(m)))$, and $\text{End} M$ in a standard way (i.e., $u \cdot a = \sum_{(a)} S(a_{(1)})_M u a_{(2)}_M$). Note that $A_{\text{fin}}^r$, $\text{Hom}_A(M, M \otimes F)$ and $\text{End} M$ equipped with these structures are indeed right $A$-module algebras, i.e., the multiplication map is a module morphism, and the unit is invariant.

**Lemma 4.** The maps $\Phi$ and $\Psi$ are morphisms of right $A$-modules.
**Proof.** Straightforward.

**Corollary 5.** We have the following morphisms of right $A$-module algebras:

$$A^r_{\text{fin}} \xrightarrow{\Psi} \text{Hom}_A(M, M \otimes F)^r_{\text{fin}} \xrightarrow{\Phi} (\text{End} M)^r_{\text{fin}},$$

and $\Phi \Psi$ is the restriction of the canonical morphism $A \to \text{End} M$.

Now let us assume that $F$ contains all matrix elements of the canonical right $A$-action on $(\text{End} M)^r_{\text{fin}}$ (in particular, it is enough to require that $F$ contains all matrix elements of all finite dimensional representations of $A$).

**Proposition 6.** The map

$$\Phi : \text{Hom}_A(M, M \otimes F)^r_{\text{fin}} \longrightarrow (\text{End} M)^r_{\text{fin}}$$

is an isomorphism of right $A$-module algebras.

**Proof.** We already know that $\Phi$ is an embedding and homomorphism of right $A$-module algebras. Now let $u \in (\text{End} M)^r_{\text{fin}}$. Then $u \cdot x = \sum_{i=1}^{N} f_i(x) u_i$, where $f_i \in F$ and $u_i \in (\text{End} M)^r_{\text{fin}}$. We define $\Xi : (\text{End} M)^r_{\text{fin}} \rightarrow \text{Hom}_A(M, M \otimes F)$ by the formula $\Xi(u)(m) = \sum_{i=1}^{N} u_i(m) \otimes f_i$. It is straightforward to verify that $\Xi$ is a morphism of right $A$-module algebras. Therefore the image of $\Xi$ lies in $\text{Hom}_A(M, M \otimes F)^r_{\text{fin}}$. Since $u = \sum_{i=1}^{N} f_i(1) u_i$, we conclude that $\Phi \Xi = \text{id}$. Thus $\Phi$ is surjective and it follows that $\Phi$ is an isomorphism.

Suppose that the canonical map $A^r_{\text{fin}} \rightarrow (\text{End} M)^r_{\text{fin}}$ is an epimorphism.

**Proposition 7.** Let $N$ be a submodule of $M$. Then $u(N) \subset N$ for any $u \in (\text{End} M)^r_{\text{fin}}$ and $\varphi(N) \subset N \otimes F$ for any $\varphi \in \text{Hom}_A(M, M \otimes F)^r_{\text{fin}}$.

**Proof.** In this case there exists $a \in A^r_{\text{fin}}$ such that $u(m) = am$ for any $m \in M$. Hence $u(n) = an \in N$ for any $n \in N$. The second statement follows now from Proposition 6.

3 Irreducible highest weight modules and equivariant quantization for non-generic $\lambda$

3.1 General construction

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra. Fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-.$$
Let $\mathbf{R}$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, $\Pi$ the set of simple roots that corresponds to $\mathfrak{h}^+$, and $\mathbf{R}_+$ the corresponding set of positive roots. We denote by $\rho$ the sum of fundamental weights. For any $\alpha \in \mathbf{R}_+$ fix non-zero elements $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$.

For any $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ be the Verma module with the highest weight $\lambda$ and the highest weight vector $1_\lambda$.

Let $U\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$ equipped with the standard Hopf algebra structure. Clearly, $(U\mathfrak{g})_{\text{fin}} = U\mathfrak{g}$ and it is well known that the canonical map $U\mathfrak{g} \to (\text{End } M(\lambda))_{\text{fin}}$ is epimorphic for any $\lambda \in \mathfrak{h}^*$.

Let $F = \mathbb{C}[G]$, that is, $F$ consists of all matrix elements of all finite dimensional representations of $U\mathfrak{g}$. Then $\text{Hom}_\mathfrak{g}(M(\lambda), (M(\lambda) \otimes F)_{\text{fin}} = \text{Hom}_\mathfrak{g}(M(\lambda), (M(\lambda) \otimes F))$.

Let $K(\lambda)$ be the maximal $\mathfrak{g}$-submodule of $M(\lambda)$ and $V(\lambda) = M(\lambda)/K(\lambda)$ be the irreducible $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$. Applying Proposition we get the canonical maps $(\text{End } M(\lambda))_{\text{fin}} \to (\text{End } V(\lambda))_{\text{fin}}$ and $\text{Hom}_\mathfrak{g}(M(\lambda), (M(\lambda) \otimes F)) \to \text{Hom}_\mathfrak{g}(V(\lambda), (V(\lambda) \otimes F))$.

We have the following

**Proposition 8.** Let $\Phi_M$ be the map from Proposition 7. Then the diagram

$$
\begin{array}{ccc}
\text{Hom}_\mathfrak{g}(M(\lambda), (M(\lambda) \otimes F)) & \longrightarrow & \text{Hom}_\mathfrak{g}(V(\lambda), (V(\lambda) \otimes F)) \\
\downarrow \Phi_{M(\lambda)} & & \downarrow \Phi_{V(\lambda)} \\
(\text{End } M(\lambda))_{\text{fin}} & \longrightarrow & (\text{End } V(\lambda))_{\text{fin}}
\end{array}
$$

is commutative. 

Denote by $\mathbf{T}_\lambda$ the image of $1_\lambda$ in $V(\lambda)$. For any $\varphi \in \text{Hom}_\mathfrak{g}(V(\lambda), (V(\lambda) \otimes F))$ the formula $\varphi(\mathbf{T}_\lambda) = \mathbf{T}_\lambda \otimes f_\varphi + \sum_{\mu < \lambda} v_\mu \otimes f_\mu$ defines a map $\Theta : \text{Hom}_\mathfrak{g}(V(\lambda), (V(\lambda) \otimes F)) \to F[0]$, $\varphi \mapsto f_\varphi$.

**Theorem 9.** $\Theta$ is an embedding.

We want also to describe the image of $\Theta$. We will need some extra notation.

Denote by $x \mapsto (x)_0$ the projection $U\mathfrak{g} \to U\mathfrak{h}$ along $\mathfrak{n}_- \cdot U\mathfrak{g} + U\mathfrak{g} \cdot \mathfrak{n}_+$. For any $\lambda \in \mathfrak{h}^*$ consider a pairing $\pi_\lambda : \mathfrak{n}_- \otimes \mathfrak{n}_- \to \mathbb{C}$ defined by $\pi_\lambda(x \otimes y) = (\overline{xy})_0(\lambda)$ (here $S : x \mapsto \overline{x}$ is the antipode in $U\mathfrak{g}$). Denote by $\omega$ the Chevalley involution in $U\mathfrak{g}$. Then the map $\theta : x \mapsto \omega(\overline{x})$ is an isomorphism $\mathfrak{n}_- \to \mathfrak{n}_+$, and $S_\lambda(x \otimes y) = \pi_\lambda(\theta(x) \otimes y) = (\omega(xy))_0(\lambda) = \pi_\lambda(\theta(y) \otimes x)$ is the Shapovalov form on $\mathfrak{n}_-$.

Set

$$
K_\lambda = \{y \in \mathfrak{n}_- \mid \pi_\lambda(x \otimes y) = 0 \text{ for all } x \in \mathfrak{n}_+\},
$$

$$
\tilde{K}_\lambda = \{x \in \mathfrak{n}_+ \mid \pi_\lambda(x \otimes y) = 0 \text{ for all } y \in \mathfrak{n}_-\}.
$$

Clearly, $\tilde{K}_\lambda$ is the kernel of $S_\lambda$, $\tilde{K}_\lambda = \omega(\overline{K_\lambda})$. Notice also that $K(\lambda) = K_\lambda \cdot 1_\lambda$. 

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Lemma 11. \( \text{Hom} \tilde{\iota} \) the lowest weight vector \( \tilde{\iota} \) in \( M \) we will denote by \( \rho \) (here \( \varepsilon \) stands for the standard counit in \( U \)). In particular,

\[ F[0]^P = \{ f \in F[0] \mid \bar{a}f = \varepsilon(a)f \text{ for all } a \in P \} \]

Theorem 10. The image of \( \Theta \) is \( F[0]^{|K_\lambda + \bar{K}_\lambda|} \).

In order to prove Theorems 9 and 10 we need some preparations.

In the sequel \( L \) stands for a \( g \)-module which is a direct sum of finite dimensional \( g \)-modules.

For a \( g \)-module \( M \) which is a direct sum of finite-dimensional \( h \)-weight spaces we will denote by \( \tilde{\iota} M \) its restricted dual.

Let \( \tilde{M}(\iota) \) be the “opposite Verma module” with the lowest weight \( \iota \in h^* \) and the lowest weight vector \( \tilde{\iota} \iota \). It is clear that \( \tilde{K}_\iota \cdot \tilde{\iota} \lambda \) is the maximal \( g \)-submodule in \( \tilde{M}(\iota) \).

Lemma 11. \( \text{Hom}_{n_-}(M(\iota), L) = (M(\iota)^* \otimes L)^{n_-} \), \( \text{Hom}_{n_+}(\tilde{M}(\iota), L) = (\tilde{M}(\iota)^* \otimes L)^{n_+} \).

Proof. For any \( \varphi \in \text{Hom}_{n_-}(M(\iota), L) \) the image of \( \varphi \) is equal to the finite-dimensional \( n_- \)-submodule \( U_{n_-} \cdot \varphi(1) \). Therefore for any \( x \in U_{n_-} \) such that \( x1_{\iota} \) is a weight vector whose weight is large enough we have \( \varphi(x1_{\iota}) = x\varphi(1) = 0 \). Thus \( \varphi \) corresponds to an element in \( (\tilde{M}(\iota)^* \otimes L)^{n_-} \).

The second part of the lemma can be proved similarly. \( \square \)

Choose vectors \( 1^1_{\iota} \in M(\iota)^*[-\iota] \) and \( \tilde{1}^1_{\iota} \in \tilde{M}(-\iota)^*[\iota] \) such that \( (1^1_{\iota}, 1_{\iota}) = (\tilde{1}^1_{\iota}, \tilde{1}_{\iota}) = 1 \). Define maps \( \zeta : \text{Hom}_g(M(\iota), \tilde{M}(\iota)^* \otimes L) \to L[0] \) and \( \tilde{\zeta} : \text{Hom}_g(\tilde{M}(-\iota), M(\iota)^* \otimes L) \to L[0] \) by the formulas \( \varphi(1) = \tilde{1}^1_{\iota} \otimes \zeta_{\varphi} + \text{lower order terms} \), \( \varphi(\tilde{1}_{\iota}) = 1^1_{\iota} \otimes \zeta_{\varphi} + \text{higher order terms} \).

Consider also the natural maps

\( r : \text{Hom}_g(M(\iota) \otimes \tilde{M}(-\iota), L) \to \text{Hom}_g(M(\iota), \tilde{M}(-\iota)^* \otimes L), \)

\( \tilde{r} : \text{Hom}_g(\tilde{M}(-\iota) \otimes M(\iota), L) \to \text{Hom}_g(\tilde{M}(-\iota), M(\iota)^* \otimes L). \)

Proposition 12. Maps \( \zeta \), \( \tilde{\zeta} \), \( r \), and \( \tilde{r} \) are vector space isomorphisms, and the diagram

\[
\begin{array}{ccc}
\text{Hom}_g(M(\iota) \otimes \tilde{M}(-\iota), L) & \xrightarrow{r} & \text{Hom}_g(M(\iota), \tilde{M}(-\iota)^* \otimes L) \\
\downarrow{\tilde{r}} & & \downarrow{\zeta} \\
\text{Hom}_g(\tilde{M}(-\iota), M(\iota)^* \otimes L) & \xrightarrow{\tilde{\zeta}} & L[0]
\end{array}
\]

is commutative.
Proof. First of all notice that we have the natural identifications
\[ \text{Hom}_g(M(\lambda), \tilde{M}(\lambda) \otimes L) = (\tilde{M}(\lambda)^* \otimes L)^{n_+}[\lambda], \]
\[ \text{Hom}_g(\tilde{M}(\lambda), (M(\lambda) \otimes L) = (M(\lambda)^* \otimes L)^{n_-}[\lambda]. \]

Further on, we have
\[ \text{Hom}_g(M(\lambda) \otimes \tilde{M}(\lambda), L) = \text{Hom}_g(M(\lambda), \text{Hom}_g(\tilde{M}(\lambda), L)) = \]
\[ \text{Hom}_{n_+}(\tilde{M}(\lambda), L)[\lambda] = L[0]. \]

On the other side, \( \text{Hom}_{n_+}(\tilde{M}(\lambda), L)[\lambda] = (\tilde{M}(\lambda)^* \otimes L)^{n_+}[\lambda] \) by Lemma \( \square \)

Now it is easy to see that the map \( r \) (resp. \( \zeta \)) corresponds to the identification \( \text{Hom}_g(M(\lambda) \otimes \tilde{M}(\lambda), L) = (\tilde{M}(\lambda)^* \otimes L)^{n_+}[\lambda] \) (resp. \( (\tilde{M}(\lambda)^* \otimes L)^{n_-}[\lambda] = L[0] \)).

The second part of the proposition concerning \( \tilde{r} \) and \( \tilde{\zeta} \) can be verified similarly. \( \square \)

Now note that the pairing \( \pi_\lambda : U_n \otimes U_{\lambda} \to \mathbb{C} \) naturally defines the pairing \( \tilde{M}(\lambda) \otimes M(\lambda) \to \mathbb{C} \). Denote by \( \chi_\lambda : M(\lambda) \to \tilde{M}(\lambda)^* \) the corresponding morphism of \( g \)-modules. The kernel of \( \chi_\lambda \) is equal to \( K(\lambda) = K_\lambda \cdot 1_\lambda \), and the image of \( \chi_\lambda \) is \( (K_\lambda \cdot 1_{\lambda})^\perp \). Therefore, \( (K_\lambda \cdot 1_{\lambda})^\perp \simeq V(\lambda) \), and \( \chi_\lambda \) can be naturally represented as \( \chi''_\lambda \circ \chi'_\lambda \), where
\[ M(\lambda) \xrightarrow{\chi'_\lambda} V(\lambda) \xrightarrow{\chi''_\lambda} \tilde{M}(\lambda)^*. \]

The morphisms \( \chi'_\lambda \) and \( \chi''_\lambda \) induce the commutative diagram of inclusions
\[
\begin{array}{ccc}
\text{Hom}_g(V(\lambda), V(\lambda) \otimes L) & \longrightarrow & \text{Hom}_g(M(\lambda), V(\lambda) \otimes L) \\
\downarrow & & \downarrow \\
\text{Hom}_g(V(\lambda), \tilde{M}(\lambda)^* \otimes L) & \longrightarrow & \text{Hom}_g(M(\lambda), \tilde{M}(\lambda)^* \otimes L).
\end{array}
\]

It is clear that the following lemma holds:

**Lemma 13.** The image of \( \text{Hom}_g(V(\lambda), V(\lambda) \otimes L) \) in \( \text{Hom}_g(M(\lambda), \tilde{M}(\lambda)^* \otimes L) \) under the inclusion above consists of the morphisms \( \phi : M(\lambda) \to \tilde{M}(\lambda)^* \otimes L \) such that \( \phi(K_\lambda 1_\lambda) = 0 \) and \( \phi(M(\lambda)) \subset (K_\lambda \cdot 1_{\lambda})^\perp \otimes L. \) \( \square \)

**Proposition 14.** Let \( \phi \in \text{Hom}_g(M(\lambda), \tilde{M}(\lambda)^* \otimes L) \). Then \( \phi(M(\lambda)) \subset (K_\lambda \cdot 1_{\lambda})^\perp \otimes L \) iff \( \phi(K_\lambda) = 0. \)

**Proof.** First notice that \( \phi(M(\lambda)) \subset (K_\lambda \cdot 1_{\lambda})^\perp \otimes L \) iff \( \phi(1_\lambda) \in (K_\lambda \cdot 1_{\lambda})^\perp \otimes L \).

Indeed, for any \( x \in U_\lambda \) we have \( \phi(x1_\lambda) = \sum(x)(x_{(1)} \otimes x_{(2)})) \phi(1_\lambda) \) and \( U_\lambda \cdot \tilde{K}_\lambda \tilde{1}_{\lambda} \cdot \tilde{K}_\lambda \tilde{1}_{\lambda} \).

9
Denote by $\psi$ the element in $\text{Hom}_{k}(\tilde{M}(-\lambda), L)$ that corresponds to $\varphi(1_{\lambda}) \in (\tilde{M}(-\lambda)^* \otimes L)^{n^+}$ (see Lemma 11). Under this notation $\varphi(1_{\lambda}) \in (\tilde{K}_{\lambda}\tilde{1}_{-\lambda})^{\perp} \otimes L$ iff $\psi(\tilde{K}_{\lambda}\tilde{1}_{-\lambda}) = 0$. On the other hand, $\zeta_{\varphi} = \psi(\tilde{1}_{-\lambda})$ and $\psi(\tilde{K}_{\lambda}\tilde{1}_{-\lambda}) = \tilde{K}_{\lambda}\psi(\tilde{1}_{-\lambda}) = \tilde{K}_{\lambda}\zeta_{\varphi}$. This completes the proof. \qed

**Proposition 15.** Let $\varphi \in \text{Hom}_{g}(M(\lambda), \tilde{M}(-\lambda)^* \otimes L)$. Then $\varphi(K_{\lambda}1_{\lambda}) = 0$ iff $K_{\lambda}\zeta_{\varphi} = 0$.

**Proof.** Consider $\tilde{\varphi} = r^{-1}(\varphi) : \text{Hom}_{g}(M(\lambda) \otimes \tilde{M}(-\lambda), L)$ and $\tilde{\tilde{\varphi}} = \tilde{r}(\tilde{\varphi}) = \tilde{r}(r^{-1}(\varphi)) : \text{Hom}_{g}(\tilde{M}(-\lambda), \tilde{M}(\lambda)^* \otimes L)$ (see Proposition 14). Clearly, $\varphi(K_{\lambda}1_{\lambda}) = 0$ iff $\tilde{\tilde{\varphi}}(K_{\lambda}1_{\lambda} \otimes \tilde{M}(-\lambda)) = 0$ iff $\tilde{\tilde{\varphi}}(\tilde{M}(-\lambda)) \subseteq (K_{\lambda}1_{\lambda})^{\perp} \otimes L$.

Arguing as in the proof of Proposition 14 we see that $\tilde{\tilde{\varphi}}(\tilde{M}(-\lambda)) \subseteq (K_{\lambda}1_{\lambda})^{\perp} \otimes L$ iff $K_{\lambda}\tilde{\tilde{\varphi}} = 0$. Now it is enough to notice that $\tilde{\zeta}_{\varphi} = \tilde{\tilde{\varphi}}(1_{\lambda} \otimes \tilde{1}_{-\lambda}) = \zeta_{\varphi}$. \qed

Define a map $u : \text{Hom}_{g}(V(\lambda), V(\lambda) \otimes L) \to L[0]$ via $\varphi \mapsto u_{\varphi}$, where $\varphi(\tilde{1}_{\lambda}) = \tilde{1}_{\lambda} \otimes u_{\varphi}$ + lower order terms.

**Proposition 16.** The map $u$ defines the isomorphism $\text{Hom}_{g}(V(\lambda), V(\lambda) \otimes L) \simeq L[0]^{K_{\lambda}+\tilde{K}_{\lambda}}$.

**Proof.** Observe that $u$ can be decomposed as

$$
\text{Hom}_{g}(V(\lambda), V(\lambda) \otimes L) \longrightarrow \text{Hom}_{g}(M(\lambda), \tilde{M}(-\lambda)^* \otimes L) \longrightarrow L[0],
$$

where the first arrow is the natural inclusion considered in Lemma 13. Now it is enough to apply the above mentioned lemma and Propositions 14 and 15. \qed

Applying the last proposition to the case $L = F$ we get Theorems 9 and 10.

Now we describe $\Theta^{-1} : F[0]^{K_{\lambda}+\tilde{K}_{\lambda}} \to \text{Hom}_{g}(V(\lambda), V(\lambda) \otimes F)$ explicitly. We are going to obtain a formula for $\Theta^{-1}$ in terms of the Shapovalov form. By means of the standard identification $M(\lambda) \simeq \text{Un}_{-}$ we can regard $S_{\lambda}$ as a bilinear form on $M(\lambda)$. Denote by $\overline{S}_{\lambda}$ the corresponding bilinear form on $V(\lambda)$. Set

$$
Q_{+} = \left( \sum_{\alpha \in \Pi} \mathbb{Z}_{+}^{\alpha} \right) \setminus \{0\}.
$$

For any $\beta \in Q_{+}$ denote by $\overline{S}_{\lambda}^{\beta}$ the restriction of $\overline{S}_{\lambda}$ to $V(\lambda)[\lambda - \beta]$. Let $x^{i}_{\beta} \cdot \tilde{1}_{\lambda}$ be an arbitrary basis in $V(\lambda)[\lambda - \beta]$, where $x^{i}_{\beta} \in \text{Un}_{-}[-\beta]$.

Take $f \in F[0]^{K_{\lambda}+\tilde{K}_{\lambda}}$ and set $\varphi = \Theta^{-1}(f)$, i.e.,

$$
\varphi(\tilde{1}_{\lambda}) = \tilde{1}_{\lambda} \otimes f + \sum_{\beta \in Q_{+}} \sum_{i} x^{i}_{\beta} \cdot \tilde{1}_{\lambda} \otimes f^{\beta,i}.
$$
Proposition 17. \( f_{\beta,i} = \sum_j \left( \overleftarrow{S}^\beta_{\lambda} \right)^{-1}_{ij} \omega \left( \overrightarrow{x_i^\beta} \right) f. \)

Proof. Set \( \xi = \varphi(\overleftarrow{1}_\lambda) \). Clearly, \( \xi \) is a singular element in \( V(\lambda) \otimes F \). In particular, \((e \otimes 1 + 1 \otimes e) \xi = 0\), i.e., \((e \otimes 1) \xi = (1 \otimes \overrightarrow{1}) \xi\) for any \( e \in \overrightarrow{n_+} \). By induction we get \((x \otimes 1) \xi = (1 \otimes \overrightarrow{x}) \xi\) for any \( x \in U \overrightarrow{n_+} \). Therefore

\[
(\overrightarrow{S}_\lambda \otimes \text{id}) (\overrightarrow{1}_\lambda \otimes (\omega \left( \overrightarrow{x_i^\beta} \right) \otimes 1)) = (\overrightarrow{S}_\lambda \otimes \text{id}) \left( \overrightarrow{1}_\lambda \otimes (1 \otimes \omega \left( \overrightarrow{x_i^\beta} \right)) \xi \right).
\]

Calculating both sides of this equation we get

\[
\sum_i S_\lambda(x_i^\beta \otimes x_i^\beta) f_{\beta,i} = \omega \left( \overrightarrow{x_i^\beta} \right) f,
\]

and the proposition follows.

Let us define an associative product \( \ast_\lambda \) on \( F[0]^{K_\lambda + \overrightarrow{K}_\lambda} \) by means of \( \Theta \). We are going to obtain an explicit formula for \( \ast_\lambda \) in terms of the Shapovalov form.

Theorem 18. For any \( f_1, f_2 \in F[0]^{K_\lambda + \overrightarrow{K}_\lambda} \) we have

\[
f_1 \ast_\lambda f_2 = \mu \left( \overrightarrow{J_{\text{red}}(\lambda)}(f_1 \otimes f_2) \right),
\]

where

\[
J_{\text{red}}(\lambda) = 1 \otimes 1 + \sum_{\beta \in Q_+} \sum_{i,j} \left( \overrightarrow{S}^\beta_{\lambda} \right)^{-1}_{ij} x_i^\beta \otimes \omega \left( \overrightarrow{x_j^\beta} \right).
\]

Proof. We have \( f_1 \ast_\lambda f_2 = \Theta(\varphi_1 \ast \varphi_2) \), where \( \varphi_1 = \Theta^{-1}(f_1) \), \( \varphi_2 = \Theta^{-1}(f_2) \), and \( \ast \) is the product on \( \text{Hom}_g(V(\lambda), V(\lambda) \otimes F) \) given by \((\mathbb{I})\). Now observe that

\[
(\varphi_1 \ast \varphi_2)(\overleftarrow{1}_\lambda) = (\text{id} \otimes \mu)(\varphi_1 \otimes \text{id})(\varphi_2(\overleftarrow{1}_\lambda)) = (\text{id} \otimes \mu)(\varphi_1 \otimes \text{id}) \left( \overleftarrow{1}_\lambda \otimes f_2 + \sum_{\beta \in Q_+} \sum_i x_i^\beta \cdot \overleftarrow{1}_\lambda \otimes f_{2,\beta,i} \right) = (\text{id} \otimes \mu) \left( \varphi_1(\overleftarrow{1}_\lambda) \otimes f_2 + \sum_{\beta \in Q_+} \sum_i (\Delta(x_i^\beta) \varphi_1(\overleftarrow{1}_\lambda)) \otimes f_{2,\beta,i} \right) = \overleftarrow{1}_\lambda \otimes \left( f_1 f_2 + \sum_{\beta \in Q_+} \sum_i (\overrightarrow{x_i^\beta} f_1) f_{2,\beta,i} \right) + \text{lower order terms}.
\]

Therefore

\[
f_1 \ast_\lambda f_2 = f_1 f_2 + \sum_{\beta \in Q_+} \sum_i (\overrightarrow{x_i^\beta} f_1) f_{2,\beta,i}.
\]

To finish the proof it is enough now to apply Proposition 17 to \( f_2 \). \( \square \)
3.2 Application to symmetric spaces

Now let us apply the construction above to some specific values of \( \lambda \in \mathfrak{h}^* \).

Let \( \Delta \subset \Pi \). Assume that \( \lambda \in \mathfrak{h}^* \) is such that \( \langle \lambda, \alpha^\vee \rangle = n_\alpha \in \mathbb{Z}_+ \) for any \( \alpha \in \Delta \), and \( \langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N} \) for \( \beta \in \mathbb{R}_+ \setminus \text{span} \Delta \).

**Proposition 19.** Let \( L \) be a \( \mathfrak{g} \)-module which is a direct sum of finite-dimensional \( \mathfrak{g} \)-modules, \( l \in L[0] \). Then \( l \in L[0]^\mathcal{K}_\lambda \) iff \( l \in L[0]^\tilde{\mathcal{K}}_\lambda \).

**Proof.** Recall that in our case \( K_\lambda \) (resp. \( \tilde{K}_\lambda \)) is generated by \( Y_{\alpha}^{n_\alpha+1} \) (resp. \( X_{\alpha}^{n_\alpha+1} \)) for all \( \alpha \in \Delta \).

Now take any \( \alpha \in \Delta \) and regard \( L \) as an \( sl(2)_\alpha \)-module, where \( sl(2)_\alpha \subset \mathfrak{g} \) is a subalgebra generated by \( X_\alpha \) and \( Y_\alpha \). By standard structure theory of finite-dimensional \( sl(2) \)-modules we see that \( Y_{\alpha}^{n_\alpha+1}l = 0 \) iff \( X_{\alpha}^{n_\alpha+1}l = 0 \), which completes the proof. \( \square \)

**Corollary 20.** \( F[0]^{K_\lambda+\tilde{K}_\lambda} = F[0]^{\mathcal{K}_\lambda} \).

Therefore in this case we get the associative algebra \( (F[0]^{\mathcal{K}_\lambda}, \star_\lambda) \).

Now assume additionally that \( n_\alpha = 0 \) for all \( \alpha \in \Delta \). In this case \( F[0]^{\mathcal{K}_\lambda} \) is closed under the original product in \( F \). Moreover, \( F[0]^{\mathcal{K}_\lambda} \simeq \text{Fun}(G/U) \), where \( U \subset G \) is the reductive Levi subgroup that corresponds to \( \Delta \). Notice that in this case formula \( 13 \) defines an equivariant quantization of the Kirillov-Kostant-Souriau bracket on the coadjoint orbit through \( \lambda \). A formula of this type appears also in \( 11 \).

**Remark 2.** Assume again that \( \lambda \in \mathfrak{h}^* \) is as described in the beginning of this subsection \( (n_\alpha \text{ should not necessary be } 0) \). It is known that in this case \( M(\lambda) \) is projective (see \( 11 \)). In particular, the natural map \( \text{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes F) \to \text{Hom}_\mathfrak{g}(M(\lambda), V(\lambda) \otimes F) \) is surjective. This map factors as \( \text{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes F) \to \text{Hom}_\mathfrak{g}(V(\lambda), V(\lambda) \otimes F) \to \text{Hom}_\mathfrak{g}(M(\lambda), V(\lambda) \otimes F) \), where the first map defined by Proposition \( 7 \). Hence the map \( \text{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes F) \to \text{Hom}_\mathfrak{g}(V(\lambda), V(\lambda) \otimes F) \) is also surjective. By Proposition \( 8 \) the canonical map \( (\text{End } M(\lambda))_{\text{fin}}^r \to (\text{End } V(\lambda))_{\text{fin}}^r \) is also surjective. Since the natural map \( U_{\mathfrak{g}} \to (\text{End } M(\lambda))_{\text{fin}}^r \) is surjective for any \( \lambda \), we recover the fact that the map \( U_{\mathfrak{g}} \to \text{End } V(\lambda))_{\text{fin}}^r \is also surjective in this case (see \( 12 \)).

4 Limiting properties of the fusion element

In this section we will freely use the notation of the previous one. For any generic \( \lambda \in \mathfrak{h}^* \) (i.e., \( \langle \lambda_0 + \rho, \beta^\vee \rangle \notin \mathbb{N} \) for all \( \beta \in \mathbb{R}_+ \)) we denote by \( J(\lambda) \) the fusion element related to the Verma module \( M(\lambda) \) (see, e.g., \( 3 \)). Notice that in this case \( V(\lambda) = M(\lambda) \) and \( J_{\text{red}}(\lambda) = J(\lambda) \).
4.1 One distinguished root case

Fix $\alpha \in \mathfrak{R}_+$. Take $\lambda_0 \in \mathfrak{h}^*$ such that $\langle \lambda_0 + \rho, \alpha^\vee \rangle = n \in \mathbb{N}$, $\langle \lambda_0 + \rho, \beta^\vee \rangle \notin \mathbb{N}$ for all $\beta \in \mathfrak{R}_+ \setminus \{ \alpha \}$.

**Theorem 21.** Let $N$ be an arbitrary $\mathfrak{n}_+$-module. Consider the family of operators $J(\lambda)_N : F[0]^{\mathcal{K}_{\lambda_0}} \otimes N \to F \otimes N$ naturally defined by $J(\lambda)$. Then this family is regular at $\lambda = \lambda_0$.

**Proof.** Fix an arbitrary line $l \subset \mathfrak{h}^*$ through $\lambda_0$, $l = \{ \lambda_0 + t \nu \mid t \in \mathbb{C} \}$, transversal to the hyperplane $\langle \lambda + \rho, \alpha^\vee \rangle = n$.

Identify $M(\lambda)$ with $U \mathfrak{n}_-$ in the standard way. Recall that we have a basis $x_\beta^i \in U \mathfrak{n}_-[-\beta]$ for $\beta \in Q_+$. Let $L \left( S^\beta_\lambda \right) \in \text{End} U \mathfrak{n}_-[-\beta]$ be given by the matrix $\left( S^\beta_\lambda \right)_{ij}$ in the basis $x_\beta^i$. Notice that $\text{Ker} L \left( S^\beta_{\lambda_0} \right) = \text{Ker} S^\beta_{\lambda_0} = \mathcal{K}_{\lambda_0}[-\beta] := \mathcal{K}_{\lambda_0} \cap U \mathfrak{n}_-[-\beta]$.

For any $\lambda \in l$ sufficiently close to $\lambda_0$, $\lambda \neq \lambda_0$ we have $M(\lambda)$ is irreducible, and $L \left( S^\beta_\lambda \right)$ is invertible for any $\beta \in Q_+$. In this notation we have

$$J(\lambda) = 1 \otimes 1 + \sum_{\beta \in Q_+} \sum_j L \left( S^\beta_\lambda \right)^{-1} x_\beta^j \otimes \omega \left( x_\beta^j \right).$$

Take $\lambda = \lambda_0 + t \nu \in l$. Fix any $\beta \in Q_+$ and set $V = U \mathfrak{n}_-[-\beta]$, $A_t = L \left( S^\beta_\lambda \right)$, $V_0 = \text{Ker} A_0 = \mathcal{K}_{\lambda_0}[-\beta] \subset V$. Write $A_t = A_0 + t B_t$, where $B_t$ is regular at $t = 0$. It is known (see, e.g., [3]) that we have $A_t^{-1} = \frac{1}{t} C + D_t$, where $D_t$ is regular at $t = 0$.

**Lemma 22.** $\text{Im} C \subset V_0$.

**Proof.** We have $A_t A_t^{-1} = \text{id}$ for any $t \neq 0$, i.e., $\frac{1}{t} A_0 C + A_0 D_t + B_t C + t B_t D_t = \text{id}$. Since the left hand side should be regular at $t = 0$, we have $A_0 C = 0$, which proves the lemma.

For $t \neq 0$ set $J_t = \sum_j A_t^{-1} x_j \otimes \omega(x_j)$ (from now on we are omitting the index $\beta$ for the sake of brevity). By Lemma 22 we have $C x_j \in V_0 = \mathcal{K}_{\lambda_0}[-\beta]$. Hence for $f \in F[0]^{\mathcal{K}_{\lambda_0}}$ we have $\overrightarrow{C x_j f} = 0$. Therefore $A_t^{-1} x_j f = \frac{1}{t} \overrightarrow{C x_j f} + \overrightarrow{D_t x_j f} = \overrightarrow{D_t x_j f}$. This proves the regularity of $(J_t)_N (f \otimes \cdot)$ at $t = 0$, i.e., the regularity of $J(\lambda)_N (f \otimes \cdot)$ at $\lambda = \lambda_0$.

Similarly to Theorem 21 one can prove the following

**Theorem 23.** Let $M$ be an arbitrary $\mathfrak{n}_-$-module. Consider the family of operators $J(\lambda)^M : M \otimes F[0]^{\mathcal{K}_{\lambda_0}} \to M \otimes F$ naturally defined by $J(\lambda)$. Then this family is regular at $\lambda = \lambda_0$. 

\[ \square \]
Theorem 24. Let $f \in F[0]^{K_{\lambda_0}}$, $g \in F[0]^{\tilde{K}_{\lambda_0}}$. Then $\overrightarrow{J(\lambda)}(f \otimes g) \to \overrightarrow{J_{\text{red}}(\lambda_0)}(f \otimes g)$ as $\lambda \to \lambda_0$.

Proof. We will use the notation defined in the proof of Theorem 21.

Lemma 25. $D_0 A_0 = \text{id}$ on $V/V_0$.

Proof. Arguing in the same manner as in the proof of Lemma 22 but starting from $A_t^{-1} A_t = \text{id}$ and setting $t = 0$, we get $D_0 A_0 + C B_0 = \text{id}$. Now notice that for any $v \in V$ we have $C B_0 v \in V_0$, which proves the lemma.

Since the Shapovalov form is symmetric, we may choose $V_1 \subset V$ such that $V = V_0 \oplus V_1$, $A_0(V_1) = V_1$, and $A_0$ is non-degenerate on $V_1$. Assume that the basis $x_j$ is compatible with this decomposition. We see that $A_t^{-1} x_j f \to \overrightarrow{D_0 x_j f}$ as $t \to 0$.

For any $x_j \in V_0$ we have $\omega(x_j) \in \tilde{K}_{\lambda_0} \cap U n_{+}[\beta]$. This implies that $\overrightarrow{D_0 x_j f} \otimes \omega(x_j) g = 0$ by our assumptions on $g$.

For any $x_j \in V_1$ we see, by Lemma 25, that $\overrightarrow{D_0 x_j f} = \overrightarrow{A_t^{-1} x_j f}$. Thus

$$\overrightarrow{J_t}(f \otimes g) \to \sum_{j: x_j \in V_1} \overrightarrow{A_t^{-1} x_j f} \otimes \overrightarrow{\omega(x_j) g}.$$  

Clearly, this means, by definition of $J_{\text{red}}(\lambda_0)$, that $\overrightarrow{J(\lambda)}(f \otimes g) \to \overrightarrow{J_{\text{red}}(\lambda_0)}(f \otimes g)$ as $\lambda \to \lambda_0$.

Corollary 26. Let $f_1, f_2 \in F[0]^{K_{\lambda_0} + \tilde{K}_{\lambda_0}}$. Then $f_1 \star_\lambda f_2 \to f_1 \star_{\lambda_0} f_2$ as $\lambda \to \lambda_0$.

Example 1. Let $g = \mathfrak{sl}(2)$. In [14] we considered the star-product on polynomial functions on coadjoint orbits $O_\lambda$ of $g$ defined by the natural action of the fusion element $J(\lambda)$. In particular, we obtained the formula

$$f_a \star_\lambda f_b = \left(1 - \frac{1}{\lambda}\right) f_a f_b + \frac{1}{2} f_{[a,b]} + \frac{\lambda}{2} \langle a, b \rangle,$$

where $f_x$ is the restriction onto $O_\lambda$ of the linear function on $g^*$ defined by $x \in g$, and $\langle a, b \rangle = \text{Tr}(ab)$. Despite $J(\lambda)$ has a singularity at $\lambda = 1$ we see that $f_a \star_1 f_b$ is well defined, and the set $\{f_x \mid x \in g\}$ generates an algebra under $\star_1$ isomorphic to $\text{End} V(1) \simeq \text{Mat}(2, \mathbb{C})$.

Similarly, one can also show that for any $\lambda \in \mathbb{Z}_+$ the set $\{f_x \mid x \in g\}$ generates an algebra under $\star_\lambda$ isomorphic to $\text{End} V(\lambda) \simeq \text{Mat}(\lambda + 1, \mathbb{C})$. Corollary 26 explains these phenomena.
4.2 Regularity properties

Let \( \lambda_0 \in \mathfrak{h}^* \). We will say that \( \lambda_0 \) has the good regularity property if for any \( n_- \)-module \( M \) the family of operators \( J(\lambda)^T : M \otimes F[0]K_{\lambda_0} \rightarrow M \otimes F \) naturally defined by \( J(\lambda) \) is regular at \( \lambda = \lambda_0 \). Clearly, if \( \lambda_0 \) is generic (i.e., \( V(\lambda_0) = M(\lambda_0) \) is irreducible), then \( \lambda_0 \) has the good regularity property. We have seen that \( \lambda_0 \) as in Subsection 4.1 also has the good regularity property.

**Theorem 27.** Assume that \( \lambda_0 \in \mathfrak{h}^* \) has the good regularity property. Then for any \( f \in F[0]K_{\lambda_0}, g \in F[0]K_{\lambda_0} \) we have \( J(\lambda)(f \otimes g) \rightarrow J_{\text{red}}(\lambda_0)(f \otimes g) \) as \( \lambda \rightarrow \lambda_0 \).

**Proof.** For any \( \lambda \in \mathfrak{h}^* \) we may naturally identify \( M(\lambda) \) with \( \mathfrak{u}n_- \) as \( n_- \)-modules. Therefore we know by definition of a good regular property that \( J(\lambda)^{M(\lambda)}(1_\lambda \otimes g) \) is regular at \( \lambda = \lambda_0 \). Thus \( J(\lambda)^{M(\lambda)}(1_\lambda \otimes g) \rightarrow Z \in M(\lambda_0) \otimes F \) as \( \lambda \rightarrow \lambda_0 \). In an arbitrary basis \( x_\beta \in \mathfrak{u}n_-[-\beta] \) we have

\[
J(\lambda)^{M(\lambda)}(1_\lambda \otimes g) = 1_\lambda \otimes g + \sum_{\beta \in Q_+} \sum_{i,j} (S^\beta_{ij})^{-1} x_\beta^i \lambda_\lambda \otimes \omega \left( \frac{x_\beta^j}{g} \right) g,
\]

and

\[
Z = 1_{\lambda_0} \otimes g + \sum_{\beta \in Q_+} \sum_{i,j} a_{ij}^\beta \left( x_\beta^i 1_{\lambda_0} \right) \otimes \omega \left( \frac{x_\beta^j}{g} \right) g
\]

for some coefficients \( a_{ij}^\beta \in \mathbb{C} \).

Now choose a basis \( x_\beta \in \mathfrak{u}n_-[-\beta] \) in the following way: first take a basis in \( K_{\lambda_0}[-\beta] = K_{\lambda_0} \cap \mathfrak{u}n_-[-\beta] \) and then extend it arbitrarily to a basis in the whole \( \mathfrak{u}n_-[-\beta] \). In this basis the projection \( \overline{Z} \in V(\lambda_0) \otimes F \) of the element \( Z \) is given by

\[
\overline{Z} = 1_{\lambda_0} \otimes g + \sum_{\beta \in Q_+} \sum_{x_\beta^i, x_\beta^j \in K_{\lambda_0}[-\beta]} a_{ij}^\beta \left( x_\beta^i 1_{\lambda_0} \right) \otimes \omega \left( \frac{x_\beta^j}{g} \right) g.
\]  

(5)

Now notice that \( Z \), being the limit of singular vectors of weight \( \lambda \) in \( M(\lambda) \otimes F \), defines the intertwining operator \( \varphi_Z \in \text{Hom}_{\mathfrak{g}}(M(\lambda_0), M(\lambda_0) \otimes F) \), \( \varphi_Z(1_{\lambda_0}) = Z \). Under the natural map \( \text{Hom}_{\mathfrak{g}}(M(\lambda_0), M(\lambda_0) \otimes F) \rightarrow \text{Hom}_{\mathfrak{g}}(V(\lambda_0), V(\lambda_0) \otimes F) \) we have \( \varphi_Z \mapsto \varphi_{\overline{Z}} \), where \( \varphi_{\overline{Z}}(1_{\lambda_0}) = \overline{Z} \). Therefore \( \overline{Z} = J_{\text{red}}(\lambda_0)^{M(\lambda_0)}(1_{\lambda_0} \otimes g) \) by Proposition 17 and the definition of \( J_{\text{red}}(\lambda_0) \). Comparing this with (5) we conclude that for all \( i, j \) such that \( x_\beta^i, x_\beta^j \in K_{\lambda_0}[-\beta] \) we have \( a_{ij}^\beta = (S^\beta_{ij})^{-1} \).
Finally,

\[ \overline{J}(\lambda)(f \otimes g) \to fg + \sum_{\beta \in Q_+} \sum_{i,j} a_{ij}^\beta x_j^\beta f \otimes \omega(x_j^\beta)g = \]

\[ fg + \sum_{\beta \in Q_+} \sum_{x_j^\beta \in K_{\lambda_0}[\beta]} a_{ij}^\beta x_j^\beta f \otimes \omega(x_j^\beta)g = \]

\[ fg + \sum_{\beta \in Q_+} \sum_{x_j^\beta \in K_{\lambda_0}[\beta]} \left( \mathcal{S}_\lambda \right)_{ij}^{-1} x_j^\beta f \otimes \omega(x_j^\beta)g = \]

\[ \overline{J}_{\text{red}}(\lambda_0)(f \otimes g) \]

as \( \lambda \to \lambda_0 \). \( \square \)

**Remark 3.** Theorem \[27\] provides another proof of Theorem \[24\].

**Corollary 28.** Assume that \( \lambda_0 \in \mathfrak{h}^* \) has the good regularity property. Let \( f_1, f_2 \in F[0]^{K_{\lambda_0} + \tilde{K}_{\lambda_0}} \). Then \( f_1 \ast \lambda f_2 \to f_1 \ast \lambda_0 f_2 \) as \( \lambda \to \lambda_0 \). \( \square \)

**Proposition 29.** Assume that \( \lambda_0 \in \mathfrak{h}^* \) has the good regularity property. Then \( F[0]^{K_{\lambda_0}} = F[0]^{\tilde{K}_{\lambda_0}} = F[0]^{K_{\lambda_0} + \tilde{K}_{\lambda_0}} \).

**Proof.** Let \( u \in F[0]^{\tilde{K}_{\lambda_0}} \). If \( \lambda \in \mathfrak{h}^* \) is generic, then the element \( J(\lambda)M(\lambda)(1_\lambda \otimes u) \) is a singular vector of weight \( \lambda \) in \( M(\lambda) \otimes F \). Therefore \( Z := \lim_{\lambda \to \lambda_0} J(\lambda)M(\lambda)(1_\lambda \otimes u) \) is a singular vector of weight \( \lambda_0 \) in \( M(\lambda_0) \otimes F \), and hence we have \( \varphi_Z \in \text{Hom}_g(M(\lambda_0), M(\lambda_0) \otimes F), \varphi_Z(1_{\lambda_0}) = Z \).

Under the natural map \( \text{Hom}_g(M(\lambda_0), M(\lambda_0) \otimes F) \to \text{Hom}_g(V(\lambda_0), V(\lambda_0) \otimes F) \), we have \( \varphi_Z \to \varphi_Z \), where \( \varphi_Z(1_{\lambda_0}) = Z \) is the projection of \( Z \) onto \( V(\lambda_0) \otimes F \). Now notice that \( u = \Theta(\varphi_Z) \in F[0]^{K_{\lambda_0} + \tilde{K}_{\lambda_0}} \), which proves the proposition. \( \square \)

**Proposition 30.** Assume that \( \lambda_0 \in \mathfrak{h}^* \) has the good regularity property. Then the natural map \( \text{Hom}_g(M(\lambda_0), M(\lambda_0) \otimes F) \to \text{Hom}_g(V(\lambda_0), V(\lambda_0) \otimes F) \) is surjective.

**Proof.** Recall that we have the isomorphism

\[ \Theta : \text{Hom}_g(V(\lambda_0), V(\lambda_0) \otimes F) \to F[0]^{K_{\lambda_0} + \tilde{K}_{\lambda_0}} = F[0]^{K_{\lambda_0}}. \]

Now take \( u \in F[0]^{\tilde{K}_{\lambda_0}} \). Consider \( Z = \lim_{\lambda \to \lambda_0} J(\lambda)M(\lambda)(1_\lambda \otimes u) \in M(\lambda_0) \otimes F \). Since \( Z \) a singular vector of weight \( \lambda_0 \), we have \( \varphi_Z \in \text{Hom}_g(M(\lambda_0), M(\lambda_0) \otimes F), \varphi_Z(1_{\lambda_0}) = Z \). Clearly, under the mapping \( \text{Hom}_g(M(\lambda_0), M(\lambda_0) \otimes F) \to \text{Hom}_g(V(\lambda_0), V(\lambda_0) \otimes F) \) the image of \( \varphi_Z \) equals to \( \Theta^{-1}(u) \), which proves the proposition. \( \square \)

**Proposition 31.** Assume that \( \lambda_0 \in \mathfrak{h}^* \) has the good regularity property. Then the action map \( U \mathfrak{g} \to (\text{End} V(\lambda_0))_{\text{fin}}^{r} \) is surjective.
Proof. Recall that by Proposition 6 we have the isomorphisms
\[ \text{Hom}_g(M(\lambda_0), M(\lambda_0) \otimes F) \simeq (\text{End} M(\lambda_0))^r_{\text{fin}}, \]
\[ \text{Hom}_g(V(\lambda_0), V(\lambda_0) \otimes F) \simeq (\text{End} V(\lambda_0))^r_{\text{fin}}. \]

It is well known that the action map \( U\mathfrak{g} \to (\text{End} M(\lambda_0))^r_{\text{fin}} \) is surjective for any \( \lambda_0 \in \mathfrak{h}^* \) (see [12]). Since by Proposition 30 the map \( (\text{End} M(\lambda_0))^r_{\text{fin}} \to (\text{End} V(\lambda_0))^r_{\text{fin}} \) is surjective, the map \( U\mathfrak{g} \to (\text{End} V(\lambda_0))^r_{\text{fin}} \) is also surjective. \( \square \)

4.3 Symmetric space case

Let \( \Delta \subset \Pi \). Assume that \( \lambda_0 \in \mathfrak{h}^* \) is such that \( \langle \lambda_0, \alpha^\vee \rangle = 0 \) for any \( \alpha \in \Delta \), and \( \langle \lambda_0 + \rho, \beta^\vee \rangle \notin \mathbb{N} \) for \( \beta \in R_+ \setminus \text{span} \Delta \).

Theorem 32. Let \( N \) be an arbitrary \( n_+ \)-module. Consider the family of operators \( J(\lambda)_N : F[0]^{K\lambda_0} \otimes N \to F \otimes N \) naturally defined by \( J(\lambda) \). Then this family is regular at \( \lambda = \lambda_0 \).

Proof. It is known (see, e.g., [6]) that the only singularities of \( J(\lambda) \) near \( \lambda_0 \) are simple poles on the hyperplanes \( \langle \lambda, \alpha^\vee \rangle = 0 \) for \( \alpha \in R_+ \cap \text{span} \Delta \). Therefore it is enough to show that for any \( f \in F[0]^{K\lambda_0} \) the operator \( J(\lambda)_N(f \otimes \cdot) \) has no singularity at any such hyperplane.

Let \( \Delta = \{ \alpha_1, \ldots, \alpha_l \} \). For each \( i = 1, \ldots, l \) take an arbitrary \( \lambda_i \in \mathfrak{h}^* \) such that \( \langle \lambda_i, \alpha_i^\vee \rangle = 0 \), and \( \langle \lambda_i + \rho, \beta^\vee \rangle \notin \mathbb{N} \) for \( \beta \in R_+ \setminus \{ \alpha_i \} \). It is well known that \( K_{\lambda_0} = K_{\lambda_1} + \ldots + K_{\lambda_l} \). In particular, \( K_{\lambda_i} \subset K_{\lambda_0} \). Also, \( F[0]^{K_{\lambda_i}} \supset F[0]^{K_{\lambda_0}} \).

Therefore we may apply Theorem 21 and conclude that \( J(\lambda)_N(f \otimes \cdot) \) is regular at \( \lambda = \lambda_i \) for each \( i \).

Now consider a hyperplane \( \langle \lambda, \alpha^\vee \rangle = 0 \) for \( \alpha \in R_+ \cap \text{span} \Delta \) which may be composite. Take an arbitrary \( \lambda' \in \mathfrak{h}^* \) such that \( \langle \lambda', \alpha^\vee \rangle = 0 \), and \( \langle \lambda' + \rho, \beta^\vee \rangle \notin \mathbb{N} \) for \( \beta \in R_+ \setminus \{ \alpha \} \). It follows from the results of [16] that \( K_{\lambda'} \subset K_{\lambda_1} + \ldots + K_{\lambda_l} \), i.e., \( K_{\lambda'} \subset K_{\lambda_0} \). Arguing as above we see that \( J(\lambda)_N(f \otimes \cdot) \) is regular at \( \lambda = \lambda' \), which completes the proof. \( \square \)

By similar considerations one can prove the following

Theorem 33. Let \( M \) be an arbitrary \( n_- \)-module. Consider the family of operators \( J(\lambda)_M^M : M \otimes F[0]^{K\lambda_0} \to M \otimes F \) naturally defined by \( J(\lambda) \). Then this family is regular at \( \lambda = \lambda_0 \). \( \square \)

Hence we conclude that any \( \lambda_0 \) as described at the beginning of this subsection has the good regularity property. In particular, all results of Subsection 1.2 are applicable to this situation.
Remark 4. Recall that for $\lambda_0$ as described above the formula $f_1 \ast_{\lambda_0} f_2 = \mu \left( \overrightarrow{J}_{\text{red}}(\lambda_0)(f_1 \otimes f_2) \right)$ gives an equivariant quantization of the Kirillov-Kostant-Souriau bracket on the coadjoint orbit through $\lambda_0$. Applying results of Subsection 4.2 we conclude that

$$f_1 \ast_{\lambda_0} f_2 = \lim_{\lambda \to \lambda_0} f_1 \ast_{\lambda} f_2 = \lim_{\lambda \to \lambda_0} \mu \left( \overrightarrow{J}(\lambda)(f_1 \otimes f_2) \right).$$

4.4 Concluding remarks

Let $\Delta \subset \Pi$. It would be interesting to investigate whether our good regularity property still holds for any $\lambda_0 \in \mathfrak{h}^*$ such that $\langle \lambda_0, \alpha^\vee \rangle = n_\alpha \in \mathbb{Z}_+$ for any $\alpha \in \Delta$, and $\langle \lambda_0 + \rho, \beta^\vee \rangle \notin \mathbb{N}$ for $\beta \in \mathbb{R}_+ \setminus \text{span } \Delta$. This would imply that the action map $Ug \to (\text{End } V(\lambda_0))^\text{fin}$ is surjective. The latter fact is known for $\lambda_0$ of consideration (cf. also Remark 2). Proving the good regularity property will provide a new explanation of this result.

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