A BAYESIAN APPROACH FOR THE SOLUTION OF INVERSE PROBLEMS TO ESTIMATE RADIATIVE PROPERTIES

ABORDAGEM BAYESIANA COMO SOLUÇÃO DE PROBLEMAS INVERSOS PARA ESTIMAÇÃO DE PROPRIEDADES RADIATIVAS

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Abstract: The main objective of the present work is related to the formulation and solution of inverse problems in radiative heat transfer phenomena. The analysis consists in estimating parameters and functions of a participating medium, such as optical thickness, single scattering albedo, diffusive reflectivities and phase function coefficients. It is performed with the numerical application of a Bayesian framework, which includes “Maximum a Posteriori” (MAP) and “Markov Chains Monte Carlo” (MCMC), within the Metropolis-Hastings procedure. These methodologies proved to be effective for solving such problems.

Keywords: Radiative transfer. Bayesian approach. Inverse problems. MAP. MCMC.

Resumo: Este trabalho é fundamentado na formulação e solução de problemas inversos em fenômenos de transferência de calor radiativa. Desta forma, objetivou-se a estimação de propriedades e funções de um meio participante, como espessura óptica, albedo de espalhamento simples, refletividades difusas e coeficientes da função de fase de espalhamento anisotrópico. A solução numérica foi obtida via abordagem Bayesiana, através da aplicação das metodologias “Maximum a Posteriori” (MAP) e “Cadeias de Markov via Método de Monte Carlo” (MCMC), com o algoritmo de Metropolis-Hastings. Os métodos aplicados mostraram-se satisfatórios para a solução de tais problemas.

Palavras-chave: Transferência radiativa. Abordagem bayesiana. Problemas inversos. MAP. MCMC.

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1 INTRODUCTION

The present work is related to the formulation and solution of both direct and inverse problems in Radiative Heat Transfer, in one-dimensional participating media. In what concerns the direct problem (DP), the emitted radiation intensity is computed throughout the surface of interest, with respect to a geometric position and polar angle, once boundary conditions are known. The inverse problem (IP), in turn, is based upon inferring radiative properties that must have had originated the measured radiation, with special attention to a property known as phase function.

For the IP solution, the Bayesian approach will be applied, throughout the Maximum a Posteriori (MAP) approach, with the objective function optimized by the Gauss-Newton technique, and Markov Chains Monte Carlo (MCMC), with the Metropolis-Hastings heuristics. In this case, uncertainties will be associated to the parameters involved in the model, so as to simulate experimental conditions from the information obtained by solving the DP. Hence, the parameters to be estimated are modelled as a probability distribution function, being also possible to easily include prior information (KNUPP; NETO, 2010).

2 DIRECT PROBLEM

For the direct problem in radiative transfer, the objective relies on solving the set of equations which models the phenomena involved, once both medium properties and boundary conditions are known, in order to calculate radiation intensity in specific directions.

By “participating medium”, one may consider an absorbing, scattering or emitting medium (NETO; BECCENERI; VELHO, 2016). In this context, besides radiation emission, its boundary surfaces may scatter the incoming radiation and/or absorb some fraction of it. For a one-dimensional case, the related medium is geometrically defined by its thickness L, and it is also considered external radiation incidence at the boundary surfaces ($\tau = 0$ and $\tau = \tau_0$), with intensities $f_1$ and $f_2$, respectively, which are diffusely reflecting. A schematic representation may be seen in Fig. 1, reproduced from Knupp and Neto (2010). The related dimensionless model contains also parameters which will be further discussed, such as optical thickness ($\tau_0$), diffusive reflectivities ($\rho_1$ and $\rho_2$) and $\mu$, which represents the cosine of the polar angle.
Figure 1: Schematic representation of a one-dimensional participating medium.

Source: (KNUPP; NETO, 2010).

Some other hypotheses are considered in the model used, such as steady-state, homogeneous medium in what concerns radiative transfer, anisotropic scattering and azimuthal symmetry. Thus, the mathematical formulation consists on solving the Boltmann’s equation in its linear and dimensionless form (Eq. 1), aiming at computing the radiation intensity $I(\tau, \mu)$ considering the domain as $0 \leq \tau \leq \tau_0$ and $-1 \leq \mu \leq 1$. The radiation intensity is then a function of an optical variable $\tau$ and $\mu$, the cosine of the polar angle $\theta$, which consists on the angle between the radiation beam and the horizontal axis (ÖZISIK, 1973):

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} + I(\tau, \mu) = \frac{\omega_0}{2} \int_{-1}^{1} p(\mu, \mu') I(\tau, \mu') \, d\mu' \quad 0 < \tau < \tau_0, -1 \leq \mu \leq 1, \quad (1)$$

$$I(0, \mu) = f_1(\mu) + 2 \rho_1 \int_{0}^{1} I(0, -\mu') \, d\mu' \quad \mu > 0, \quad (2)$$

$$I(\tau_0, \mu) = f_2(\mu) + 2 \rho_2 \int_{0}^{1} I(\tau_0, -\mu') \, d\mu' \quad \mu < 0. \quad (3)$$

In the set of Eqs. (1–3), are also considered other radiative properties, for instance optical thickness ($\tau_0$), diffusive reflectivities $\rho_1$ and $\rho_2$ at the boundaries $\tau = 0$ and $\tau = \tau_0$, respectively, and constant single scattering albedo ($\omega_0$). A special attention is given to the phase function $p(\mu, \mu')$, since it is considered anisotropic scattering. This function, firstly, is modelled as an expansion on Legendre-Polynomials (NETO; ÖZISIK, 1995), as follows:

$$p(\mu, \mu') = \sum_{m=0}^{M} b_m \, P_m(\mu) \, P_m(\mu'). \quad (4)$$

The phase function coefficients $b_m$, being $b_0 = 1$ and $m = 0, 1, ..., M$, are available in the literature (CHU; CLARK; CHURCHILL, 1957), as reproduced in Tab. 1, for two different situations, with backward and forward scattering, here called FF1 ($M=5$) and FF2 ($M=5$), both
with \( \alpha = \pi D/\lambda = 1 \), for a particle diameter \( D \) and wavelength \( \lambda \). The schematic representation of those functions is presented in Figure 2. In 1, \( n \) represents the refraction index.

Table 1: Phase function coefficients.

| Coefficients | FF1 (\( M=5, m = 1, 4, \alpha = 1 \)) | FF2 (\( M=5, m=2, \alpha = 1 \)) |
|--------------|----------------------------------|---------------------------------|
| \( b_1 \)    | -0.56524                         | 0.82864                         |
| \( b_2 \)    | 0.29783                          | 0.58827                         |
| \( b_3 \)    | 0.08571                          | 0.09472                         |
| \( b_4 \)    | 0.01003                          | 0.00805                         |
| \( b_5 \)    | 0.000063                         | 0.00042                         |
| \( b_6 \)    | 0.000000                         | 0.00000                         |

Source: Chu et. al, 1957.

Figure 2: Phase functions obtained for backward (FF1) and forward (FF2) scattering.

Source: The authors, 2018.

In order to solve numerically the DP, the Chandrasekhar’s Discrete Ordinates Method is implemented (CHANDRASEKHAR, 1960), which requires the angular and spatial domain discretization. A computational routine was developed by (CANATO, 2016), using the software Wolfram Mathematica, in which the tool NDSolve is applied for the solution process. Some adjustments were made so as to decrease the required computational time, since, as it will be furthermore discussed, it is a desired condition for the IP solution.

3 INVERSE PROBLEM

The inverse analysis consists of, once the exit radiation intensity is known, estimating radiative properties that may have caused it. In this context, the IP is applied to infer unknown quantities by indirect measurements, in which uncertainties are considered (ORLANDE, 2012). It may express the experimentalist’s subjectivity about the prior information but also random errors, that could have been included in the measurements, as normally happens in experimental
real conditions. As real experimental data was not available, it was simulated by adding random noise to the calculated values of the radiation intensity, $I_i$:

$$Y_i = I_i + N(0, \sigma_{exp}),$$

(5)

considering $N(0, \sigma_{exp})$ the probability function density that describes those uncertainties, modelled with a normal distribution, centered at zero with standard deviation $\sigma_{exp}$. Also, $i = 1, 2, ..., N$, for the set of $N = 2N_d$ synthetic experimental data.

The unknown parameters to be estimated, are represented by $\vec{Z}$, which should be computed by solving the IP. In this work it is used $f_1 = 1.0$ and $f_2 = 0$ for all simulations, for the incident radiation intensity at the boundaries. The optical thickness ($\tau_0$), scattering albedo ($\omega_0$), diffusive reflectivities ($\rho_1$ and $\rho_2$) and phase function coefficients ($b_i$) are the unknown to be estimated with the solution of the IP, i.e.,

$$\vec{Z} = \{\tau_0, \omega_0, \rho_1, \rho_2, b_1, b_2, b_3, b_4, b_5\}^T.$$  

(6)

The Bayesian approach will be applied as the solution method, since. The unknowns are handled as random quantities, to which will be related probability distributions associated to the values that they may assume. It means that it will be evaluated which of them is more or less likely to be assumed. Altogether, it should be stressed (KAPIO; SOMERSALO, 2005): (i) the model variables are treated as random; (ii) randomness describes the uncertainty level related to the parameters; (iii) prior information is modelled as a probability distribution function (PDF); and (iv) the IP solution, known as posteriori information, is also modelled as a PDF.

The objective function expresses mathematically the aspect to be optimized. Using Bayes’ Theorem in the formulation of the inverse problem (KAPIO; SOMERSALO, 2005):

$$\pi_{post}(\vec{Z}) = \pi(\vec{Y} | \vec{Z}) \pi(\vec{Z}) / \pi(\vec{Y}),$$

(7)

where $\pi_{post}$ is the posterior probability distribution function, $\pi_{pr}$ is the prior information and $\pi(\vec{Y})$, the marginal density, which acts as a normalizing constant. Besides, $\pi(\vec{Y} | \vec{Z})$ is called the Likelihood Function. Considering additive and non-related measurement errors, a normal distribution centered at $\mu_0 = 0$ and constant standard deviation, it may be represented as follows, considering the vector of parameters to be estimated $\vec{Z}$ and the set of experimental data $\vec{Y}$, for $N$ observations (BECK; ARNOLD, 1977; ORLANDE, 2015):
in which $W^{-1}$ is the inverse of the covariance matrix, calculated in function of each experimental observation standard deviation. The residuals vector $\vec{F}$ is given by the differences between the experimental data and the synthetic one:

$$\vec{F} = \vec{Y} - \vec{I}_{\text{calc}}.$$

The Maximum a Posteriori (MAP) objective function will be applied, which shall be minimized by the Gauss-Newton’s algorithm for single point estimates. The Markov Chain Monte Carlo (MCMC) will also be applied, by implementing the Metropolis-Hastings algorithm, from which will emerge the posteriori distribution.

### 3.1 Maximum a Posteriori (MAP)

Here, the use of MAP objective function turns the IP into an optimization problem, whereas minimization algorithms, such as Gauss-Newton, are able to provide single point estimates within the Bayesian context. This is possible since the amount of variables to be estimated $N_u$ is inferior to the number of empirical observations ($N = 2.N_d$). In this framework, the prior information may be modelled as (KNUPP; NETO, 2010):

$$\pi_{pr}(\vec{Z}) = (2\pi)^{-N/2} |V|^{-1/2} \exp\left(-\frac{1}{2} (\vec{Z} - \vec{\mu}_m)^T V^{-1} (\vec{Z} - \vec{\mu}_m)\right).$$

Which leads, finally, to the MAP function (KNUPP; NETO, 2010):

$$S_{\text{MAP}}(\vec{Z}) = F^T W^{-1} F + (\vec{\mu}_m - \vec{Z})^T V^{-1} (\vec{\mu}_m - \vec{Z}).$$

In Eq. (11), $\vec{\mu}_m$ contains the average values for the parameters to be estimated and $V$ is the covariance matrix of the parameter standard deviation. Those are part of the available prior information.

The Gauss-Newton’s method may be applied as the optimization solution approach, in order to maximize the posterior distribution. In this work, the routine, is based on the iterative process (ORLANDE, 2012):

$$\vec{Z}^{k+1} = \vec{Z}^k + [J^T W^{-1} J + V^{-1}]^{-1} [J^T W^{-1} \vec{F}^k + V^{-1} (\vec{\mu}_m - \vec{Z}^k)],$$
in which $J$ is the Jacobian matrix, which may be computed considering $i = 1, 2, ..., N$ and $j = 1, 2, ..., N_u$, for $N_u$ parameters to be estimated (ORLANDE, 2012):

$$J_{ij} = \frac{\partial I_{calc_i}}{\partial Z_j}. \quad (13)$$

The computational routine should be run until a stopping criterion ($\varepsilon$) is achieved:

$$\max (|\tilde{Z}^{k+1} - \tilde{Z}^k|) < \varepsilon. \quad (14)$$

### 3.2 Markov Chain Monte Carlo (MCMC)

In the MCMC approach, instead of evaluating the PDF for each parameter at certain points, as in MAP, the density itself generates random candidates that may satisfy the problem conditions. The advantage is that, by the end of the estimation process, a distribution function known as the posterior distribution is obtained, instead of single point estimates. It causes this stochastic algorithm to be viable, simple and efficient to be applied, especially in situations where there is few or no prior information available, since each sample is independent from the previous one (KAIPIO; SOMERSALO, 2005; ORLANDE, 2015).

The MCMC will be applied through the Metropolis-Hastings technique, which consists in an acceptance algorithm to achieve the posterior distribution. Thus, the process consists on accepting or not the candidates in the chain, comparing the probability for it to be accepted with the other ones, which should stimulate convergence as it proceeds (ORLANDE, 2015). This iterative routine may be summarized by the following steps:

1. Generate a random candidate sample $\tilde{Z}^*$, with an auxiliary distribution $q(\tilde{Z}^*, \tilde{Z}^{(t-1)})$;

2. Compute the acceptance factor ($\beta$):

$$\beta = \min \left[1, \frac{\pi(\tilde{Z}^* | \tilde{Y}) \ p(\tilde{Z}^{(t-1)}, \tilde{Z}^*)}{\pi(\tilde{Z}^{(t-1)} | \tilde{Y}) \ p(\tilde{Z}^*, \tilde{Z}^{(t-1)})} \right]; \quad (15)$$

3. Generate a random value $U$, evenly distributed within $(0,1)$;

4. If $U \leq \beta$, the candidate is accepted into the chain: $\tilde{Z}^{(t)} = \tilde{Z}^*$. Otherwise, it is rejected: $\tilde{Z}^{(t)} = \tilde{Z}^{(t-1)}$;

5. Return to the initial step, so as to achieve a chain $\tilde{Z}^0, \tilde{Z}^1, \ldots, \tilde{Z}^t$. 
The values obtained before an equilibrium condition is achieved are discarded, which is called burn-in period (ORLANDE, 2012). In order to achieve a single equilibrium distribution, i.e., posterior distribution, it is needed that the chain be (ORLANDE, 2015): (i) Homogeneous, with invariant transition probabilities from one state to another; (ii) irreductive, as each state may be obtained from any other, since the states are independent; and (iii) non-periodic, as there is not absorbing states.

4 RESULTS AND DISCUSSION

The objective is to solve the IP, considering a homogeneous participating medium with anisotropic scattering. The results obtained when applying MAP are shown in section 4.1, whereas in 4.2, those associated with MCMC.

The chosen methodology is based on the computational routine developed by Canato (2016), in the platform Wolfram Mathematica, applying tools such as the function NDSolve. It showed good results when numerically solving differential equations. Some adjustments were proposed to this process, so as to reduce the computational time required. When solving the direct problems, a 50% decrease was achieved, which was mandatory to apply such robust techniques for the solution of the inverse problem. The Gaussian Quadrature order for the MAP problems was $N_d = 10$, while for MCMC, $N_d = 5$ and the tolerance for the iterative processes, $\epsilon = 10^{-3}$. The simulations were executed in a laptop with an Intel Core i3 processor, with 2.8 GHz.

4.1 Inverse Problem Solved with MAP

In this work, since experimental data was not available, this role was played by the results obtained by solving the PD, when some uncertainties were added, as shown in Eq. (5). In this case, the MAP function was firstly used to verify how the results were impacted by the prior information and noise level in the experimental data. In this case, the exact values for the parameters were chosen to model the prior, as the point in which the normal distribution is centered. Besides, regarding the standard deviation, it was implemented 1% of those exact values for each parameter, i.e., $\sigma_i = 0.01 \tilde{Z}$, here simply represented by $\sigma_i = 0.01$.

Different values of experimental noise were simulated to verify the convergence of the model. Here, it is represented as a standard deviation ($\sigma_{exp}$) when modelling the synthetic data as seen in 5. It was assumed FF1 for the values of the phase function coefficients and, for radiative properties, $\tau_0 = 1.0$, $\omega_0 = 0.5$, $\rho_1 = \rho_2 = 0.2$, as shown in Tab. 2.

The values shown in Tab. 2 suggest that, since the uncertainties about each parameter are...
Table 2: Radiative properties estimated with MAP, homogeneous participating medium and anisotropic scattering, $\tau_0 = 1.0$, $\omega_0 = 0.5$, $\rho_1 = \rho_2 = 0.2$, $\sigma_i = 0.01$ and FF2.

| $\sigma_{exp}$ | $\tau_0$ | $\omega_0$ | $\rho_1$ | $\rho_2$ | b1    | b2    | b3    | b4    | b5    |
|---------------|----------|------------|----------|----------|-------|-------|-------|-------|-------|
| 0.001         | 1.000    | 0.499      | 0.200    | 0.200    | -0.56573 | 0.29790 | 0.08572 | 0.01003 | 0.00006 |
| 0.01          | 1.001    | 0.498      | 0.200    | 0.199    | -0.56511 | 0.29783 | 0.08571 | 0.01003 | 0.00006 |
| 0.1           | 1.001    | 0.499      | 0.200    | 0.200    | -0.56527 | 0.29783 | 0.08571 | 0.01003 | 0.00006 |
| 0.25          | 0.999    | 0.500      | 0.200    | 0.199    | -0.56523 | 0.29783 | 0.08571 | 0.01003 | 0.00006 |
| 0.5           | 0.999    | 0.500      | 0.200    | 0.199    | -0.56524 | 0.29783 | 0.08571 | 0.01003 | 0.00006 |
| 0.75          | 0.999    | 0.500      | 0.200    | 0.199    | -0.56524 | 0.29783 | 0.08571 | 0.01003 | 0.00006 |
| 1             | 1.000    | 0.499      | 0.200    | 0.200    | -0.56524 | 0.29783 | 0.08571 | 0.01003 | 0.00006 |
| Exact         | 1.00     | 0.50       | 0.20     | 0.20     | -0.56524 | 0.29783 | 0.08571 | 0.01003 | 0.00006 |

Source: The authors, 2018.

not large ($\sigma_i = 0.01$, i.e., 1% over the exact values), the experimental noise is not enough to avoid convergence, since even for larger experimental standard deviation, the results were close to the exact ones. In this context, the convergence is mostly associated with the quality of the prior information, since, for each parameter, the exact values were used as the center of the normal distribution, with a fraction of 1% of them as standard deviation. All simulations were executed in approximately 10 minutes, which is a strong quality of the MAP technique.

In order to verify the implications of the values assumed for the parameters when modelling the prior information, it was also considered a case in which from a phase function of a forward scattering medium (FF2), wrongly assumed, the IP should recover properties related to a backward scattering one. It is useful when the experimentalist is not sure about the values of the parameters to be estimated. It should also be stressed that assuming that one of the parameters is null will lead the iterative routine to indetermination problems, since there will be a division by zero. The expansion coefficients for FF2 are presented in1.

The objective is now to consider different levels of experimental noise ($\sigma_{exp}$). Regarding the prior information, the values chosen as the center of the normal distribution were $\tau_0 = 1.0$, $\omega_0 = 0.5$, $\rho_1 = \rho_2 = 0.2$ and the FF2 coefficients, shown in Tab. 1, for the phase function. Uncertainties represented in the form of standard deviation ($\sigma_i$) were again assumed as a fraction of the main values for each parameter, now at 10%.

The parameters calculated with the IP may be found in Tab. 3 for $\tau_0$, $\omega_0$, $\rho_1$, $\rho_2$. In Fig. 3 there is a comparison between the obtained phase function and the prior one.
Table 3: Estimation for phase function coefficients by a forward scattering medium, with MAP, $\tau_0 = 1.0, \omega_0 = 0.5, \rho_1 = \rho_2 = 0.2$.

| $\sigma_{exp}$ | $\tau_0$ | $\omega_0$ | $\rho_1$ | $\rho_2$ |
|----------------|----------|------------|----------|----------|
| 0.01           | 0.971    | 0.501      | 0.200    | 0.211    |
| 0.1            | 1.058    | 0.513      | 0.198    | 0.203    |
| 0.25           | 1.026    | 0.480      | 0.200    | 0.201    |
| 0.50           | 0.997    | 0.5045     | 0.199    | 0.200    |
| 0.75           | 0.989    | 0.505      | 0.199    | 0.199    |
| 1              | 0.999    | 0.502      | 0.199    | 0.199    |
| **Exact**      | 1.00     | 0.50       | 0.20     | 0.20     |

**Source:** The authors, 2018.

Figure 3: Comparison between estimated, prior and exact phase function for different levels of experimental noise.

![Comparison between estimated, prior and exact phase function for different levels of experimental noise.](image)

**Source:** The authors, 2018.

It is noticeable that there was no impact on the properties shown in Tab. 3, as the results converge to the exact ones. Nevertheless, in what concerns the phase function coefficients, it is possible to observe that the larger the experimental noise, the more distant the estimated function was from the exact one. It is suggested that for $\sigma_{exp} \geq 0.25$, the function that was estimated was close to the prior itself. The simulations were executed in approximately 12 minutes.

4.2 Inverse Problem Solved with MCMC

With respect to the MCMC framework, the main advantage of its application relies on the fact that the methodology leads to results as distribution functions, with statistical significance as results may be read within a confidence interval, instead of single point estimates.
Under such circumstances, the chains were built with 80,000 states, from which at least 30,000 were related to the burn-in steps. In order to accelerate the iterative problem, for the Gaussian Quadrature order was set at $N_d = 5$. The simulation was executed in two steps: for 70000 states was associated a variable density transition for the generation of new candidates, obtained by the product of the values in the actual state ($\vec{Z}'$) and a pre-defined uncertainty for the MCMC process, ($\sigma_{MCMC} = 0.001 \times \vec{Z}'$). For the 10,000 states remaining, this value was assumed to be constant ($\sigma_{MCMC} = 0.001$).

In the MCMC framework, it is possible to compute results in situations where there is few or no prior information available (ORLANDE, 2015). This work considers two cases: one in which the prior was included and another without this information. In this circumstance, when not absent, it was modelled as a normal distribution, centered at the exact values for each parameter and with a standard deviation of 1% of the main values, here represented by $\sigma_i = 0.01$. Regarding experimental noise, $\sigma_{exp} = 0.001$. It was chosen a smaller experimental noise than in the MAP approach to avoid greater execution time. The main values for the posterior distributions may be seen in Tab. 4.

### Table 4: Mean values for posterior distribution, solved MCMC, with and without prior information.

| Posterior  | $\tau_0$ | $\omega_0$ | $\rho_1$ | $\rho_2$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ |
|------------|---------|-----------|---------|---------|-------|-------|-------|-------|-------|
| With priori| 0.99    | 0.50      | 0.19    | 0.20    | -0.56616 | 0.28024 | 0.09755 | 0.00970 | 0.00006 |
| No prior   | 1.04    | 0.53      | 0.16    | 0.23    | 0.01011 | 0.79808 | -0.05272 | 0.39041 | 0.01677 |
| Exact      | 1.00    | 0.50      | 0.20    | 0.20    | -0.56524 | 0.29783 | 0.08571 | 0.01003 | 0.00006 |

Source: The authors, 2018.

The histograms for the estimated radiative properties, regarding the case with prior information are shown in Fig. 4. They represent the posterior information, pointing the values that are more likely to be assumed. In this case, it is possible to consider a set of information, which conducts to a probability statement about the object under analysis. Thus, the estimation may be evaluated as its mean value, within a confidence interval, instead of single point estimates.
Figure 4: Histograms for estimated phase function coefficients, MCMC with prior information, experimental noise as $\sigma_{exp} = 0.001$.

Source: The authors, 2018.

It is shown in Fig. 5 the estimated phase function in which the prior information was considered. It suggests that the methodology leads to a good estimation also for those parameters, unlike the results when the prior information was not included, as seen in Tab. 4.

Figure 5: Phase function estimated by MCMC, considering experimental noise as $\sigma_{exp} = 0.001$.

Source: The authors, 2018.
In what is referred to the estimation of radiative parameters, both methodologies proved to be efficient and led to good estimate, except for the phase function coefficients, in the event that there was no a priori information. Evidently, the existence of a set of prior information acted helping convergence, which led to better estimations, and presented a total time of 13.6 hours and 6.4% acceptance rate. For the case where there was no prior information, the process was executed during 11 hours with a higher acceptance rate, 8.3%.

5 CONCLUSIONS

The Bayesian Framework applied in the solution for the inverse radiative transfer problem, with MAP function and MCMC, in radiative transfer showed to be efficient. For the MAP case, the results information have a strict relationship with the quality of the prior information, and hence, the precision of the estimation process. For the simulations in which the exact values of the parameters are known, even with a great level of experimental noise it was possible to obtain accurate the parameter values. Nevertheless, when wrong assumptions about them are made, larger experimental noises may lead to wrong estimates. All simulations were executed in a laptop with an Intel Core i3 processor (2.8 GHz), in less than 15 minutes. In what concerns to the MCMC approach, the existence of prior information improved the performance of the estimation problem, converging in 13 hours. Without the prior, it was possible to acquire good estimations only for the scattering albedo, optical thickness and diffusive reflectivities, while diverging for the phase functions coefficients, in a 11-hour process. In this case, simulating more states and/or applying regulating methods may produce better estimations.

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