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Constantin Schrade and Liang Fu

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Parity-controlled $2\pi$ Josephson effect mediated by Majorana Kramers pairs

Constantin Schrade and Liang Fu

Department of Physics, Massachusetts Institute of Technology, 77 Massachusetts Ave., Cambridge, MA 02139

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We study a time-reversal-invariant topological superconductor island hosting spatially separated Majorana Kramers pairs, with weak tunnel couplings to two $s$-wave superconducting leads. When the topological superconductor island is in the Coulomb blockade regime, we predict that a Josephson current flows between the two leads due to a non-local transfer of Cooper pairs mediated by the Majorana Kramers pairs. Interestingly, we find that the sign of the Josephson current is controlled by the joint parity of all four Majorana bound states on the island. Consequently, this parity-controlled Josephson effect can be used for qubit read-out in Majorana-based quantum computing.

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The past years have shown rapid progress towards the realization of topological superconductors (TSCs) hosting spatially separated Majorana bound states (MBSs) [1–3], which may be useful in building a robust quantum computer. Promising platforms for TSCs to date include hybrid superconductor (SC) - semiconductor nanowire devices under magnetic fields [4–9], chains of magnetic atoms on top of a SC substrate [10–16] as well as vortices in SC-topological insulator heterostructures [17–19]. While all of these setups are designed to search for unpaired MBSs, it was predicted that topological superconductivity also exists in time-reversal-invariant (TRI) systems and gives rise to Kramers doublets of MBSs or Majorana Kramers pairs (MKPs) [20]. In particular, a one-dimensional TRI TSC wire hosts spatially separated MKPs at its two ends. Despite consisting of two MBSs, an isolated MKP is a robust zero-energy degree of freedom protected by time reversal symmetry.

Candidate systems for realizing such TRI topological superconductors comprise nanowires contacted to unconventional SCs [21–24], Josephson $\pi$-junctions in proximitized nanowires and topological insulators [25–28] as well as setups of two nanowires or two topological insulator systems coupled via a conventional $s$-wave SC [29–32]. Additionally, it was pointed out recently that TSCs could appear in systems with an emergent time-reversal symmetry [33–36]. While various schemes were put forward to detect the MKPs in such systems [37–42], novel properties of MKPs and TRI TSCs remain to be explored.

Here, we study the Josephson effect in a mesoscopic TRI TSC island tunnel coupled to two $s$-wave superconducting leads via two spatially separated MKPs, see Fig. 1(a). When the island is in the Coulomb blockade regime, we show that a finite Josephson current flows due to higher order co-tunnelling processes in which Cooper pairs in the SC leads tunnel in and out of the spatially separated MKPs localized at opposite ends of the island. We find that the sign of the resulting Josephson current is controlled by the joint parity of the two MKPs. For the case of odd joint parity, the two SC leads form a Josephson $\pi$-junction, whereas for even joint parity the two SC leads form a Josephson 0-junction. Besides being a robust and easily accessible property of MKPs, we hope that the sign reversal of the Josephson current will prove useful for qubit read-out in Majorana-based quantum computing [43–51].

Model. We consider a mesoscopic TRI TSC island connected to the ground by a capacitor and weakly coupled to two $s$-wave SC leads, see Fig. 1(a). The two SC leads $\ell = L,R$ are described by the BCS (Bardeen-Cooper-
Schrieffer) Hamiltonian,
\[ H_0 = \sum_{\ell = L, R} \sum_k \Psi_{\ell, k}^\dagger \left( \xi_k \eta_z + \Delta_\ell \eta_c e^{i\phi_\ell} \eta_z \right) \Psi_{\ell, k}. \]  
(1)

Here, \( \Psi_{\ell, k} = (c_{\ell, k}, c_{\ell, k}^\dagger)^T \) is a Nambu spinor with \( c_{\ell, ks} \) the electron annihilation operator, where \( k \) denotes single-particle states with normal state dispersion \( \xi_k \) and Kramers index \( s = \uparrow, \downarrow \). By definition, s-wave pairing occurs between Kramers pairs \((k, s)\) and \((-k, -s)\), resulting in the superconducting gap \( \Delta_\ell \). The SC leads through the TRI TSC island. For states with \( n \) electrons on the island, the transfer of a \( 2\lambda \) Cooper pair across the junction entails four steps of sub-tunneling between the TRI TSC island and the two SC leads, \( \tau_0 \) and \( \tau\). This allows us to truncate the Hilbert space of the island retaining only the states with \( n_0 \) and \( n_0 + 1 \) units of charge, see Fig. 1(b). All remaining charge states are separated from this low-energy subspace by a large charging energy, \( U \gg \Delta \gg \pi \nu \lambda |\lambda|_c^2 \), and hence have negligible contribution to the Josephson current.

Due to the superconducting gap \( \Delta \) in the SC leads, single charge transfer across the TRI TSC island is suppressed at low energy. Cooper pair transport occurring separately between each SC lead and the island is also forbidden, as these processes alter the charge of the island by \( 2e \) and thereby leak out of the low-energy Hilbert space. Hence, up to fourth order in the tunneling amplitudes \( \lambda \), only two types of co-tunneling processes give rise to coherent Josephson coupling between the two SC leads. These processes transfer charge \( e \) between the two SC leads through the TRI TSC island. For states with \( n_0 \) electrons on the island, the transfer of a \( 2e \)-charged Cooper pair across the junction entails four steps of subsequently adding and removing electrons on the island, see Fig. 2(a). This is also the case for states with \( n_0 + 1 \) electrons on the island, but the intermediate steps of adding and removing charges are reversed, see Fig. 2(b).

The amplitude of these processes at and near resonances are derived in the limit of weak tunnel coupling, \( \Gamma_\ell \equiv \pi \nu_\ell |\lambda|_c^2 \ll \Delta \) with \( \nu_\ell \) the normal-state density of states per spin of the \( \ell\)-SC at the Fermi energy [52]. The resulting effective Hamiltonian acting on the reduced Hilbert space consisting of the BCS ground states of the SC leads and the charge states \( n_0 \) and \( n_0 + 1 \) of the mesoscopic TRI TSC island reads,
\[ H_{\text{eff}} = \frac{\delta}{2} \tau \pm (\gamma_R^\uparrow \gamma_L^\uparrow \gamma_R^\downarrow \gamma_L^\downarrow) \left( J_0 + J_1 \delta \tau \right) \cos(\varphi_L - \varphi_R) \]  
(5)

Here, the tunneling amplitudes at the junction between the island and the lead \( \lambda_{\ell ss'} \) are allowed to take the most general form, i.e., complex and spin-dependent. Time reversal symmetry implies \( \lambda_{\ell ss'}^* = (s_y)_s \lambda_{\ell ss'} (s_y)_{s'} \) with \( s_{x,y,z} \) denoting the Pauli matrices in spin-space. We note that one can always choose a proper spin basis transformation so that the tunneling amplitude becomes real and spin-independent, i.e., \( \lambda_{\ell ss'} = \lambda_0 \delta_{ss'} \). Without loss of generality, this choice will be adopted below [52].

The operator \( e^{i\theta_0/2} \) in Eq. (4) increases/decreases the total charge of the TRI TSC island by one charge unit, \( [n, e^{i\theta_0/2}] = \pm e^{i\theta_0/2} \) while the MBS operators change the electron number parity in the TRI TSC island [53]. We remark that the MBSs at one end of the island do not couple to the SC lead at the opposite end, as the MBS localization length is assumed to be much shorter than the island length. To summarize, the full Hamiltonian of our setup is \( H = H_0 + U_C(n) + H_R \).

**Josephson current near a resonance.** We now show that a Josephson current occurs due to Cooper pair tunneling between the TRI TSC island and the two SC leads enabled by the two MKPs. We first focus on the near-resonant case, \( |\delta| \ll \Delta \) with \( \delta \equiv U_C(n_0) - U_C(n_0 + 1) \). This means that the MBSs at one end of the island do not couple to the SC lead at the opposite end, as the MBS localization length is assumed to be much shorter than the island length. To summarize, the full Hamiltonian of our setup is \( H = H_0 + U_C(n) + H_R \).

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Due to the superconducting gap \( \Delta \) in the SC leads, single charge transfer across the TRI TSC island is suppressed at low energy. Cooper pair transport occurring separately between each SC lead and the island is also forbidden, as these processes alter the charge of the island by \( 2e \) and thereby leak out of the low-energy Hilbert space. Hence, up to fourth order in the tunneling amplitudes \( \lambda \), only two types of co-tunneling processes give rise to coherent Josephson coupling between the two SC leads. These processes transfer charge \( 2e \) between the two SC leads through the TRI TSC island. For states with \( n_0 \) electrons on the island, the transfer of a \( 2e \)-charged Cooper pair across the junction entails four steps of subsequently adding and removing electrons on the island, see Fig. 2(a). This is also the case for states with \( n_0 + 1 \) electrons on the island, but the intermediate steps of adding and removing charges are reversed, see Fig. 2(b).

The amplitude of these processes at and near resonances are derived in the limit of weak tunnel coupling, \( \Gamma_\ell \equiv \pi \nu_\ell |\lambda|_c^2 \ll \Delta \) with \( \nu_\ell \) the normal-state density of states per spin of the \( \ell\)-SC at the Fermi energy [52]. The resulting effective Hamiltonian acting on the reduced Hilbert space consisting of the BCS ground states of the SC leads and the charge states \( n_0 \) and \( n_0 + 1 \) of the mesoscopic TRI TSC island reads,
\[ H_{\text{eff}} = \frac{\delta}{2} \tau \pm (\gamma_R^\uparrow \gamma_L^\uparrow \gamma_R^\downarrow \gamma_L^\downarrow) \left( J_0 + J_1 \delta \tau \right) \cos(\varphi_L - \varphi_R) \]  
(5)

where \( \tau \) denotes the charge state \( n_0 \) and \( n_0 + 1 \) in the island, respectively. Here, the first term describes the energy splitting \( \delta \) of the two charge states due to detuning the gate charge \( Q_0 \) away from the resonant point \( Q_0/e = n_0 + 1/2 \). Moreover, \( J_0 \) is the Josephson coupling.
Josephson current between the SC leads is given by
\( I = \frac{2e}{\hbar} \left( J_0 + J_1 \frac{\delta}{\Delta} \right) \sin(\varphi_L - \varphi_R). \)  

where \( \sigma \equiv (-1)^{n_0} \tau_z \pm 1 \) denotes the even- (odd-) parity member of the two nearly-degenerate charge states \( n_0 \) and \( n_0 + 1 \) in the TRI TSC island, and \( \delta \equiv (-1)^{n_0} \delta \) is the energy difference of the \( \sigma \equiv +1 \) and \( \sigma \equiv -1 \) state. Four aspects are noteworthy:

1. The Josephson current between the SC leads is mediated solely by the MKPs localized at opposite boundaries of the TRI TSC island with its magnitude being determined by the coupling strengths at the two junctions only. This is remarkable because the MKPs have essentially zero wavefunction overlap and so no direct coupling exists between the MKPs in Eq. (4). We remark that the proposed Josephson effect may require longer measurement times as it emerges only at fourth order in the tunnelling amplitudes.

2. For a given phase difference \( \varphi_L - \varphi_R \), the sign of the Josephson current in Eq. (7) depends on the joint fermion parity of the four MBs. For a given fermion parity state, the Josephson current is \( 2\pi \) periodic with respect to the phase. We hence refer to Eq. (7) as the parity-controlled \( 2\pi \) Josephson effect. When the joint parity of the four MBs is even, \( \gamma_{n_0+1} \gamma_{n_0} \gamma_{n_0+1} \gamma_{n_0} = +1 \), the critical current is positive, \( I_c > 0 \), and the superconducting weak link forms a Josephson 0-junction. When the joint parity of the four MBs is odd, \( \gamma_{n_0+1} \gamma_{n_0} \gamma_{n_0+1} \gamma_{n_0} = -1 \), the sign of the critical current is negative, \( I_c < 0 \), and the weak link forms a Josephson \( \pi \)-junction, see also Fig. 2(c).

The parity-controlled Josephson effect has two immediate applications: First, for sufficiently long parity lifetimes, the sign of the critical current permits a direct measurement of the joint parity of four MBs in the island, an essential element for Majorana-based quantum computing [43-51]. In contrast, for time-reversal-breaking TSCs with unpaired MBs, the sign of the Josephson current in the \( 4\pi \)-periodic Josephson effect only permits measuring the parity of two MBs in the weak limit [55, 56]. Second, the switching times between positive and negative critical currents through the island...
provide a way of measuring the quasiparticle poisoning rate.

(3) Eq. (9) shows that on resonance ($\delta = 0$), $J_1 = 0$, i.e., the magnitude of critical current in even and odd parity branches is identical. Away from resonance ($\delta \neq 0$), $J_1 \neq 0$. Hence this symmetry is lifted and the critical current mediated by the TSC island in even or odd configurations differs in magnitude. When the even parity state is higher (lower) in energy $\delta > 0$ ($< 0$), the corresponding critical current is larger (smaller) in magnitude, see Eq. (9) and Fig. 2(c).

(4) The Josephson coupling for the limit of large charging energy ($U \gg \pi \nu \ell |\lambda|^2$) is qualitatively different from the zero charging energy case. In the latter case the dominant contributions are of second-order in the tunnel amplitudes leading to a Josephson coupling $\propto \sin \varphi_\ell$ [37]. In the intermediate charging energy regime ($U \sim \pi \nu \ell |\lambda|^2$) both sinusoidal and cosinusoidal contributions are present yielding an interaction-dependent anomalous phase shift in the current-phase relation that interpolates between the zero and large charging energy limit.

Josephson current near a Coulomb valley. In this section, we show that the proposed parity-controlled Josephson effect is more general and also arises near a Coulomb valley when $Q_0/e$ is close to an integer value, $2N + 1$ or $2N$, so that the ground states of the island consist of either an odd number of electrons, $n_0 = 2N + 1$, or an even number of electrons, $n_0 = 2N$.

Under this condition, Cooper pair transport occurs via virtually excited states of order $U$ on the island. Up to fourth order in the couplings $\lambda_\ell$, three types of cotunnelling processes contribute to the Josephson coupling: The first type of process involves subsequently adding and removing a unit of charge on the island. For the second type of process, the first two intermediate steps involve adding/removing a charge on the island, while in the final two intermediate steps this order of adding/removing a charge is reversed. In the third type of process, a Cooper pair from one lead is added/removed on the island in the first two intermediate steps, which alters the island charge by $2e$. Subsequently, the Cooper pair is again removed/added from/to the other lead in the final two intermediate steps so that the island returns to its ground state. Importantly, the processes of the second and third type involve intermediate charge states $n_0 - 1$, $n_0 + 2$, which are energetically unfavourable in the close-to-resonance case, but in the Coulomb valley case, should be included.

The amplitudes of the processes can be calculated in the limit of weak tunnel couplings, $\Gamma_\ell \ll \Delta, U$, using fourth-order perturbation theory. The resulting effective Hamiltonian acting on the BCS ground states of the SC leads and the charge ground states on the island reads,

$$H_{\text{eff}} = -(\gamma_{R, \uparrow} \gamma_L \uparrow \gamma_{R, \downarrow} \gamma_L \downarrow)J' \cos(\varphi_L - \varphi_R).$$

Here, we have introduced the coupling constant $J' = J'_0 + J'_1 + J'_2$ with,

$$J'_0 = \frac{32 \Gamma_L \Gamma_R}{\pi^2 \Delta} \int_1^\infty \frac{dx \, dy}{f(x)f(y)[f(x) + f(y)]g(x)g(y)}$$

$$J'_1 = \frac{32 \Gamma_L \Gamma_R}{\pi^2 \Delta} \int_1^\infty \frac{dx \, dy}{f(x)f(y)[f(x) + f(y)]g(x)^2}$$

$$J'_2 = \frac{8 \Gamma_L \Gamma_R}{\pi^2 U} \left[ \int_1^\infty \frac{dx}{f(x)g(x)} \right]^2,$$

and $g(x) = \sqrt{1 + x^2 + U^2/\Delta}$. The effective Hamiltonian given in Eq. (10) is the second main result of our work.

Crucially, we observe that the direct coupling of the effective Hamiltonian to the joint parity $\gamma_{R, \uparrow} \gamma_L \uparrow \gamma_{R, \downarrow} \gamma_L \downarrow$ is preserved near a Coulomb valley. For the simplest case when the joint parity is fixed by the total island charge mod 2,

$$\gamma_L \uparrow \gamma_R \uparrow \gamma_L \downarrow \gamma_R \downarrow = (-1)^{n_0}.$$  \hspace{1cm} (12)

the resulting Josephson current is given by,

$$I' = (-1)^{n_0}(2e/h)J' \sin(\varphi_L - \varphi_R).$$  \hspace{1cm} (13)

We want to emphasize three features of this result:

(1) Unlike in the close-to-resonance case, the Josephson current consists of only a single branch for either an even parity ground state, $n_0 = 2N$, or an odd parity ground state, $n_0 = 2N + 1$. However, the sign of the critical current $I'_c = (-1)^{n_0}(2e/h)J'$ remains to be a direct measure of the joint parity $\gamma_{R, \uparrow} \gamma_L \uparrow \gamma_{R, \downarrow} \gamma_L \downarrow$ through the gauge constraint given in Eq. (12).

(2) In comparison to the close-to-resonance case, the sign of the supercurrent is expected to be more stable against quasiparticle poisoning events due to the large charging energy.

(3) At the Coulomb valleys, the magnitude of the critical current is identical for both even and odd configurations. This behavior is in contrast with weak links of two SC leads coupled via a quantum dot, where odd and even charge states of the quantum dot create Josephson 0- and $\pi$-junctions, respectively, but with critical current generally of different magnitude [57].

Before closing, we point out that under rather general conditions no Josephson current is observed when the TRI TSC island is replaced by a time-reversal-breaking TSC island in symmetry class D [58]. This is because after a proper spin basis transformation, a non-degenerate MBS in the TSC island couple only to a single spin species [59] and not to both spin species as MKPs do in the case for a TRI TSC island. Finally, a parity-controlled Josephson effect can also appear for a trivially superconducting wire with a low-lying Andreev bound state described by two hybridized MBSs at each end. In this case, the parity-controlled Josephson effect enables the read-out of non-topological Andreev qubits.

Conclusions. We have shown that in a weak link of two s-wave SCs coupled via a TRI TSC island, a Josephson current can flow due to Cooper pairs tunneling in
and out of spatially separated MKPs. We have demonstrated that the sign of the resulting Josephson current is fixed by the joint parity of the four MBSs on the island. This parity-controlled Josephson effect can hence be used as a read-out mechanism for the joint parity, a key requirement in Majorana-based quantum computing [43–51].

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