Stochastic backgrounds of gravitational waves and spherical detectors

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Abstract
The analysis of how a stochastic background of gravitational radiation interacts with a spherical detector is given in detail, which leads to explicit expressions for the system response functions, as well as for the cross-correlation matrix of different readout channels. It is shown that distinctive features of GW-induced random detector excitations, relative to locally generated noise, are, in practice, insufficient for separating the signal from the noise by means of a single sphere.

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1. Introduction

Stochastic backgrounds of gravitational waves (GWs) are amongst the most interesting sources to be detected by the upcoming new generation of antennae: this kind of gravitational radiation will convey to us, for example, information on the structure of the universe at its earliest evolutionary stages. This is quite unique to gravitational waves, due to their extremely weak coupling to other forms of matter and radiation, and will therefore greatly enhance our understanding of cosmology as well as fundamental physics. Literature on the possible forms, origins and spectral shapes of various gravitational wave backgrounds is rich—see, for example, [1] for a detailed review and further references on the subject.

The stochastic nature of gravitational background signals, together with the same stochastic origin of the local noise generated in the detector itself, makes it very difficult to tell local noise from an actual signal. Therefore, detection strategies for extracting that kind of gravitational signal from detector data are based on finding differential properties capable of distinguishing between both processes.

The standard procedure for identifying a stochastic signal is to use two (or more) antennae, and cross-correlate their outputs. The concept is that, while local noises in separate detectors are uncorrelated, the stochastic signal is common to both. The signal-to-noise ratio therefore
builds up on the basis of long-term average compensation of local noise fluctuations, and can be seen to develop as $T^{1/4}$, where $T$ is the integration time [2]. Specific strategies for GW background detection have been considered for two bar detectors [3], two interferometers [4], a bar and an interferometer [5] and two spherical detectors [6].

A spherical GW detector is a multimode device which, in a number of senses, behaves like an array of bars [7–9]. This suggests that one might try and take advantage of such multimode capacity to perform a search for a GW background by suitably cross-correlating the different detector readouts. A procedure like this would therefore enable the determination of the GW background spectral density with a single detector—a significant advantage for ridding the usual method (sketched above) of inherent uncertainties bound to the fact that no two real detectors are exactly identical.

In this direction, one might expect a priori that specific signal correlation patterns happen between pairs of sensor readouts which are not shared by the local sources of noise, thereby making it possible to filter the latter out. This paper is concerned with the analysis of possibilities offered by the sphere as a multimode device for the detection of a stochastic background of GWs, defined by a spectral density function.

The outline of the article is as follows. In section 2 we set up definitions and notation for the stochastic background of GWs, and in section 3 we derive the sphere’s response to this signal. In section 4 we present the correlation functions between the sphere’s responses induced by the gravitational signal at two points on its surface, where motion sensors will be attached. Section 5 is dedicated to calculating the same correlation function presented in section 4 but in this case induced by local noise, and discusses, in light of these results, various possible strategies for retrieving information about the incoming gravitational signal spectrum with a single spherical detector. We close with a summary of conclusions in section 6, and also include a brief mathematical appendix.

2. The stochastic signal

In this section we give a few definitions and notation for a stochastic background of gravitational radiation, and its characterization through the spectral density $S_h(\omega)$.

A background of gravitational waves can be expressed as a linear superposition of plane waves coming from all directions and with all possible polarizations,

$$h_{ij}(t, \mathbf{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d^2\Omega e^{-i\omega(t - \Omega \cdot \mathbf{x})} \sum_A \tilde{h}_A(\omega; \Omega, \psi) G_{ij}(\Omega, \psi)$$

(1)

where $\Omega$ is a unit vector pointing to a generic location, $\psi$ is a rotation angle around this vector, $\tilde{h}_A(\omega; \Omega, \psi)$ are the wave amplitudes and $G_{ij}(\Omega, \psi)$ the polarization matrices. General relativity predicts only two transverse polarization modes. Nevertheless, another four degrees of freedom must generally be allowed for. We thus have six independent $3 \times 3$ symmetric polarization matrices altogether. If we choose a right-handed triad of orthonormal vectors $m$, $n$ and $\Omega$, where $\Omega$ is aligned with the propagation direction, a suitable parametrization for the $G$-matrices is the following:

$$
G^{+}_{ij}(\Omega, \psi) = \sqrt{\frac{2}{3}} \delta_{ij}, \quad G^{-}_{ij}(\Omega, \psi) = \sqrt{\frac{1}{3}} (3\Omega_i \Omega_j - \delta_{ij})
$$

$$
G^{m}_{ij}(\Omega, \psi) = m_i \Omega_j + m_j \Omega_i, \quad G^{n}_{ij}(\Omega, \psi) = n_i \Omega_j + n_j \Omega_i \quad (2)
$$

$$
G^{T}_{ij}(\Omega, \psi) = m_i n_j + n_i m_j, \quad G^{T^*}_{ij}(\Omega, \psi) = m_i m_j - n_i n_j
$$
where we give to the index $A$ in $G^A_{ij}(\Omega, \psi)$ the following meanings: $A = M$ is the monopole component, $A = L_0$, $L_m$, $L_n$ are longitudinal quadrupole components, and $A = T_+, T_\times$ are the two transverse quadrupole components. These matrices satisfy normalization conditions

$$\sum_{ij} G^A_{ij} G^B_{ji} = 2\delta^{AB}. \quad (3)$$

If $(\theta, \phi, \psi)$ are Euler angles relating the laboratory frame to the wave frame just described then

$$m = (\cos \psi \cos \theta \cos \phi - \sin \psi \sin \phi, \cos \psi \cos \theta \sin \phi + \sin \psi \cos \phi, -\cos \psi \sin \phi)$$
$$n = (-\sin \psi \cos \theta \cos \phi - \cos \psi \sin \phi, -\sin \psi \cos \theta \sin \phi + \cos \psi \cos \phi, \sin \psi \cos \phi)$$
$$\Omega = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (4)$$

where we can see that a choice for the angles $\theta, \phi$ gives a direction in space and a choice for $\psi$ gives the definition of the two transverse polarization states for a particular direction.

The ‘electric’ components $R_{00ij}(t)$ of the Riemann tensor at the centre of mass of the sphere are

$$R_{00ij}(t) = \frac{1}{2} h_{ij,00}(0, t) = - \int_{-\infty}^{\infty} d\omega \omega^2 e^{-i\omega t} \int d^2 \Omega \int d\psi \sum_{A=0}^{5} \tilde{h}_A(\omega; \Omega, \psi) G^A_{ij}(\Omega, \psi), \quad (5)$$

and these are the only components carrying information on an incoming GW [11].

In this paper we will consider an unpolarized stochastic bath of GWs which is also isotropic and stationary. Its statistical properties are therefore encoded in its power spectrum function, defined by the following equation:

$$\langle \tilde{h}_A^* (\omega, \Omega, \psi) \tilde{h}_A (\omega', \Omega', \psi') \rangle = \delta(\omega - \omega') \left\{ \frac{1}{4\pi} \delta^2(\Omega, \Omega') \frac{1}{2\pi} \delta(\psi - \psi') \right\} \left\{ \delta_{AA'} \frac{1}{2} S_B(\omega) \right\} \quad (6)$$

where $\langle - \rangle$ stands for ensemble average. There are alternative ways to characterize the frequency spectrum, for example, through an energy density per unit logarithmic interval of frequency, or in terms of a characteristic amplitude of the stochastic background. The definitions for such quantities, as well as their mutual relationships can be found in [1].

3. The sphere’s response

In this section we calculate the sphere’s response to the background of gravitational waves given by equation (1). Henceforth we shall use the general formalism and notation of [10].

We assume that the sphere’s response $\mathbf{u}(\mathbf{x}, t)$ to an incoming signal is the solution to the partial differential equation

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \left\{ \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) \right\} = \mathbf{f}(\mathbf{x}, t) \quad (7)$$

with suitable initial and boundary conditions. Here, $\lambda$ and $\mu$ are the material’s elastic Lamé coefficients [12], and $\rho$ is the sphere’s density. In the rhs of equation (7), $\mathbf{f}(\mathbf{x}, t) = \rho c^2 R_{00ij}(t) x_j$ is the density of gravitational wave tidal forces, ensuing from the geodesic
deviation equation—see, e.g., [13]. The latter term can be split up into its monopole and quadrupole components according to [10]

\[ f(x, t) = f^{(00)}(x)g^{(00)}(t) + \sum_{m=-2}^{2} f^{(2m)}(x)g^{(2m)}(t) \]  

with

\[ f^{(00)}(x) = \rho E^{(00)} x_j, \quad g^{(00)}(t) = \frac{4\pi}{3} \epsilon^{(00)} R_{00}(t) c^2 \]  

\[ f^{(2m)}(x) = \rho E^{(2m)} x_j, \quad g^{(2m)}(t) = \frac{8\pi}{15} \epsilon^{(2m)} R_{00}(t) c^2 \quad (m = -2, \ldots, 2). \]  

The definition for the matrices \( E^{(00)} \) and \( E^{(2m)} \) can be found in appendix A. The generic response function \( u(x, t) \) can be expressed as an orthogonal series expansion, which only involves quadrupole and monopole terms:

\[ u(x, t) = \sum_{n=0}^{\infty} \sum_{m=-2}^{2} a_{n0} \omega_{n0} u_{n0}^{(00)}(x) g_{n0}^{(00)}(t) + \sum_{n=1}^{\infty} \sum_{m=-2}^{2} a_{n2} \omega_{n2} \sum_{m=-2}^{2} u_{n2m}^{(2m)}(x) g_{n2m}^{(2m)}(t) \]  

where \( u_{nlm} \) are spheroidal wavefunctions corresponding to the \((2l + 1)\)-degenerate eigenfrequency \( \omega_{nl} \), i.e.,

\[ \mathcal{L} u_{nlm}(x) = -\omega_{nl}^2 u_{nlm}(x) \]  

with \( \mathcal{L} \) the differential operator defined in equation (7); \( a_{n0} \) and \( a_{n2} \) are projection coefficients—see [14] for details on definitions and values of these quantities.

Finally, \( g_{nlm}^{(lm)}(t) \) are convolution products between the driving terms \( g_{nlm}(t) \) and the corresponding eigenmode oscillation function—in this case represented by an ideally non-dissipating, purely sinusoidal vibration:

\[ g_{nlm}^{(lm)}(t) = \int_{t_0}^{t} \int_{-\infty}^{\infty} d\omega \omega^2 T(t; \omega, \omega_{nl}) \int d^2\Omega d\psi \sum_{A} \tilde{h}_{A}(\omega; \Omega, \psi) G_{ij}^{A}(\Omega, \psi) E_{ij}^{nlm} \]  

where

\[ T(t; \omega, \omega_{nl}) = \int_{0}^{t} d\tau e^{-i\omega \tau} \sin[\omega_{nl}(t - \tau)] \]  

is a function strongly peaked at \( \omega = \pm \omega_{nl} \) for large values of \( t \). The sphere’s response \( u(x, t) \) is then given by equation (10) with \( g_{nlm}^{(lm)}(t) \) given by equation (13).

In actual practice, the sphere’s motions are sensed at a number of different positions on its surface, where motion sensors are attached. We consider devices which are only sensitive to radial displacements, and attached to the sphere’s surface at positions

\[ x_a = R n_a, \quad a = 1, \ldots, J \]
where $R$ is the radius of the sphere, and $\mathbf{n}_a$ a unit vector pointing outward at the $a$th position. There will consequently be $J$ readout channels with output displacements

$$u_a(t) \equiv \mathbf{n}_a \cdot \mathbf{u}(\mathbf{x}_a, t), \quad a = 1, \ldots, J. \quad (16)$$

If equation (10) is now used, these are seen to be

$$u^{GW}_a(t) = \sum_{n=1}^{\infty} \frac{a_{0n}}{a_{00}} A_{a0}(R) Y_{00}(n_a) g_{a0}^{(00)}(t)$$

$$+ \sum_{n=1}^{\infty} \frac{a_{0n}^2}{a_{02}} A_{a2}(R) \left[ \sum_{m=-2}^{2} Y_{2m}(n_a) g_{a2}^{(2m)}(t) \right], \quad a = 1, \ldots, J \quad (17)$$

where $Y_{lm}$ are spherical harmonics and $A_{nl}(R)$ is a radial function coefficient in the spheroidal wavefunction $u_{nlm}(\mathbf{x})$; this one depends on whether we are dealing with a solid or a hollow sphere—see [10] or [15] for each case, respectively.

Because the time-dependent terms $g_{a2}^{(2m)}(t)$ in (17) are random in nature, the readouts $u_a(t)$ are random too. In addition, the linear character of the relationship between both ensures that certain properties of the driving terms, such as, e.g., Gaussianity, directly carry over to the outputs. We now propose to investigate the relevant statistical properties of the latter.

4. Cross-correlation functions

As already stated in the introduction, the essential ingredients in our analysis are the cross-correlations between different output channels. In order to make them useful, however, we first need to establish their theoretical structure, i.e., to assess their behaviour under ideal conditions of infinite integration times, and relate them to the unique characteristic function $S_0(\omega)$—the signal spectral density.

Correlations between the outputs at the readout channels are naturally defined by

$$R_{ab}(t, \tau) \equiv \langle u^*_a(t) u_b(t + \tau) \rangle \quad (18)$$

where $\langle \cdot \rangle$ stands for ensemble average again. We must now introduce the function $u^{GW}_a(t)$, given by equation (17), into equation (18) to calculate the correlation functions we are looking for. When averages are taken, equation (6) for the ensemble average of the wave amplitudes comes into play. Dirac $\delta$-functions appearing in that equation allow us to easily perform the integrals in $\Omega$, $\psi$ and $\omega$. The remaining integrals in $\Omega$, $\psi$ can then be done after evaluating

$$G_{ijkl} \equiv \int \frac{d\psi}{2\pi} \int \frac{d^2\Omega}{4\pi} \sum_A e^A_{ij}(\Omega, \psi) e^A_{kl}(\Omega, \psi) \quad (19)$$

which yields the following result:

$$G_{ijkl} = \frac{1}{4} \delta_{ij} \delta_{kl} + 2N \left[ -\frac{1}{4} \delta_{ij} \delta_{kl} + \frac{1}{10} (\delta_{ij} \delta_{jl} + \delta_{il} \delta_{jl}) \right]. \quad (20)$$

The index $A$ in equation (19) runs over the polarization modes of the incoming gravitational signal. And the result given in equation (20) takes into account the monopole mode and $N$ quadrupole modes (note that $N \leq 5$, for example $N = 2$ in general relativity, and that all of them have the same contribution to $G_{ijkl}$).

It is now readily seen that there is no cross-correlation between monopole and quadrupole modes, and that the correlation functions between the outputs induced by the gravitational
signal split up as
\[ R^{GW}_{ab}(t, \tau) = R^{(0)GW}_{ab}(t, \tau) + R^{(2)GW}_{ab}(t, \tau) \quad (21) \]
where \( R^{(0)GW}_{ab} \) and \( R^{(2)GW}_{ab} \) are monopole and quadrupole terms, respectively, and have the form
\[ R^{(0)GW}_{ab}(t, \tau) = 2 P_0(\Omega_a \cdot \Omega_b) \sum_{n,n'=1}^{\infty} \frac{a_{n0}a_{n'0}A_{00}(R)A_{00}(R)}{\omega_{n0}\omega_{n'0}} f^{(0)}_{nn}(t, \tau) \quad (22) \]
\[ R^{(2)GW}_{ab}(t, \tau) = \frac{4N}{15} P_2(\Omega_a \cdot \Omega_b) \sum_{n,n'=1}^{\infty} \frac{a_{n2}a_{n'2}A_{22}(R)A_{22}(R)}{\omega_{n2}\omega_{n'2}} f^{(2)}_{nn}(t, \tau) \]
with \( P_0 \) the zeroth Legendre polynomial, \( P_2 \) is the second Legendre polynomial, and
\[ f^{(l)}_{nn}(t, \tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\tilde{R}^{(l)}(\omega)}{\omega^4} \text{Re}[T^*(t, \omega, \omega_{nl})T(t+\tau, \omega, \omega_{n'l})] S_b(\omega), \quad l = 0, 2. \quad (23) \]
Here we see how the spectral density \( S_b(\omega) \) characterizing the incoming signal appears, and its relation to the correlation functions.

In a stationary random process, the second-order correlation functions are time-shift invariant (by definition), i.e., if we use definition (18) then the function \( R \) should only depend on its second argument, \( \tau \). Time-shift invariance has been assumed for the exciting GW bath, as we have characterized it by the spectral density \( S_b(\omega) \). Nevertheless, the sphere’s response is calculated with the series expansion (10) in which a choice for the state of the sphere at \( t = 0 \) (initial conditions) is assumed, which, therefore, breaks that invariance.

The correlation functions we are looking for can actually be obtained from (21) and (22) by taking their limit when \( t \) approaches infinity, because in this limit the property of time-shift invariance is recovered, as the system looses memory of any particular initial condition in the remote past.

Memory is, however, not lost in the kind of ideal, non-dissipative elastic body we have described in the preceding sections. Meaningful results can only be obtained if some kind of dissipation, no matter how large or small, is present in the system. This is therefore the appropriate place to introduce dissipation and we shall do it in the standard way of assuming an exponentially decaying amplitude for the oscillation eigenmodes (see [16] for a thorough discussion on dissipation in GW detectors in general), i.e., we make the replacement
\[ \sin(\omega_{nl}\tau) \rightarrow e^{-\gamma_{nl}\tau} \sin(\omega_{nl}\tau) \quad (24) \]
where \( \gamma_{nl} \) is proportional to the inverse of the decay time of the associated mode, which is in all cases of interest much larger than the period of oscillation of that mode, or
\[ \gamma_{nl} \ll \omega_{nl}. \quad (25) \]
If we express the limits of \( R^{(0)GW}_{ab}(t, \tau) \) and \( R^{(2)GW}_{ab}(t, \tau) \) by dropping the first argument then the following results are readily obtained:
\[ R^{(0)GW}_{ab}(\tau) = 2 P_0(\Omega_a \cdot \Omega_b) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{R}^{(0)}(\omega) \frac{1}{2} S_b(\omega) e^{i\omega \tau} \quad (26) \]
\[ R^{(2)GW}_{ab}(\tau) = \frac{4N}{15} P_2(\Omega_a \cdot \Omega_b) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{R}^{(2)}(\omega) \frac{1}{2} S_b(\omega) e^{i\omega \tau} \quad (27) \]
where
\[ \tilde{R}^{(l)}(\omega) = \sum_n |a_{nl}A_{nl}(R)|^2 \omega^4 |L_{nl}(\omega)|^2 \quad (28) \]
and \( L_{nl}(\omega) \) is a Lorentzian curve:
\[
L_{nl}(\omega) \equiv \frac{1}{\omega^2 - \omega_{nl}^2 + 2i\gamma_{nl}\omega},
\]
(29)

Equations (26) and (27) have a foreseeable functional structure: the correlation functions are Fourier transforms of the spectral density of the driving stochastic signal times the sphere’s transfer function \( \tilde{R}^{(1)}(\omega) \), and this is a characteristic sum of Lorentzian curves centred on the resonant frequencies, and with appropriate weights. Note also that different harmonics \((n, n')\) appear to be uncorrelated.

4.1. Mode channels

The algebraic structure of the correlation function matrix requires some further consideration. \( P_2(\Omega_a \cdot \Omega_b) \) is a symmetric matrix whose rank\(^1\) is at most 5, which means one can form five linear combinations
\[
y^{(m)}(t) = \sum_{a=1}^{J} v^{(m)}_a u_a(t), \quad m = -2, \ldots, 2
\]
(30)

where the coefficient \( v^{(m)}_a \) is the \( a \)th component of the (normalized) eigenvector associated with the \( m \)th non-null eigenvalue of \( P_2(\Omega_a \cdot \Omega_b) \), \( \xi^2_m \), say, which is always positive.

If we now define a new correlation function matrix
\[
\mathcal{R}_{GW}^{mm'}(\tau) \equiv \langle y^{(m)}(t)y^{(m')}(t+\tau) \rangle, \quad m, m' = -2, \ldots, 2
\]
(31)

we readily see that
\[
\mathcal{R}_{GW}^{mm'}(\tau) = \frac{4N}{15} \xi^2_m \delta_{mm'} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{R}^{(1)}(\omega) \frac{1}{2} \tilde{S}_b(\omega) e^{i\omega\tau}.
\]
(32)

Thus the five quadrupole channels \( y^{(m)}(t) \) are uncorrelated with one another. They are therefore particularly well suited for improved estimation of the spectral density \( \tilde{S}_b(\omega) \), since the latter is common to all of them. One may recall here that \( y^{(m)}(t) \) are actually mode channels for the well-known TIGA [17] and PHC [18] sensor distributions, in which cases
\[
v^{(m)}_a \propto Y_{2m}(\Omega_a) \quad \text{for TIGA and PHC.}
\]
(33)

Note that the fact that the \( y^{(m)}(t) \) are uncorrelated is independent of whether the motion sensors are resonant or not, so it applies to a system like the recently proposed dual sphere [19] too. Note also that sensor geometries such as TIGA and PHC, for which the \( y^{(m)}(t) \) are mode channels (i.e., quantities whose frequency spectrum \( \tilde{y}^{(m)}(\omega) \) is directly proportional to that of the GW quadrupole amplitudes \( \tilde{g}^{(2m)}(\omega) \)), allow for a very clear and direct physical interpretation of the correlation coefficients (32). They are therefore also preferred from the specific point of view of the signals considered in this paper.

Summing up, the main result so far is that cross-correlations between different output channels of a spherical GW detector, whether direct sensor readouts or quadrupole (mode) channels, possess a specific pattern, which is given by equations (27) or (32), respectively. The problem is of course that a random signal must be distinguished from local noise that is also random; we thus next review the relevant properties of the latter.

\(^1\) Actually, rank \( \{P(\Omega_a \cdot \Omega_b)\} = 2J + 1 \), provided no two of the \( J \) vectors \( \Omega_a \) are parallel, and provided of course that \( J \geq 2l + 1 \). See [14] for full details.
5. Local noise and detection strategies

In this section we study the multipole characteristics of the correlation functions between the outputs at two points of the sphere induced by the local noise.

The actual readouts of the spherical antenna are in fact superpositions of those induced by the GW and by the local noise, i.e.,
\[ u_a(t) = u_a^{GW}(t) + u_a^{LN}(t). \]

We shall generically consider that
\[ u_a^{LN}(t) = \sum_{n,l,m} q_{nlm}(t) A_{nl}(R) Y_{lm}(\Omega_a) \]
where \( q_{nlm}(t) \) are narrow band stochastic time series, with correlation times of the order of \( 1/\gamma_{nl} \). Because these \( q_{nlm}(t) \) are the amplitudes of different normal modes of the solid, they are statistically uncorrelated, or
\[ \langle q^*_{nlm}(t) q_{n'l'm'}(t + \tau) \rangle = \delta_{nn'} \delta_{ll'} \delta_{mm'} \mathcal{F}_{nl}(\tau) \]
with
\[ \mathcal{F}_{nl}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |L_{nl}(\omega)|^2 \frac{1}{2} S_{LN}(\omega) e^{i\omega \tau} \]
where \( L_{nl}(\omega) \) is given by (29) and \( S_{LN}(\omega) \) is the input spectral density of local detector noise. We need not go into the details of the specific form of this function here, which is a superposition of thermal and transducer back-action noise—see [19].

The system readout correlation functions of local noise, defined by
\[ R_{ab}^{LN}(\tau) \equiv \langle u_a^{LN}(t) u_b^{LN}(t + \tau) \rangle \]
are therefore given by
\[ R_{ab}^{LN}(\tau) = \sum_l R_{ab}^{(l)LN}(\tau) \]
where
\[ R_{ab}^{(l)LN}(\tau) = \frac{2l + 1}{4\pi} P_l(\Omega_a \cdot \Omega_b) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_n |A_{nl}(R)|^2 |L_{nl}(\omega)|^2 \frac{1}{2} S_{LN}(\omega) e^{i\omega \tau}. \]

These expressions show that local noise correlations split up in a way similar to (21), except of course that we now have contributions from all multipoles rather than just monopole and quadrupole modes, characteristic of GW signals. In particular, we can easily assess how quadrupole channels—as defined in (30)—correlate with one another in a local noise dominated detector:
\[
\langle y_{LM}^{(m)a}(t) y_{LM}^{(m)b}(t + \tau) \rangle = \frac{5}{4\pi} \sum_{mnmm'} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_n |A_{n2}(R)|^2 |L_{n2}(\omega)|^2 \frac{1}{2} S_{LN}(\omega) e^{i\omega \tau} \]
\[
+ \sum_{l \neq 2} \frac{2l + 1}{4\pi} T_{mn}^{(l)mm'} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_n |A_{nl}(R)|^2 |L_{nl}(\omega)|^2 \frac{1}{2} S_{LN}(\omega) e^{i\omega \tau}
\]
\[ T_{mn}^{(l)mm'} = \sum_{a,b} v_a^{(m)a} v_b^{(m')b} P_l(\Omega_a \cdot \Omega_b), \quad m = -2, \ldots, 2 \]
is generally a non-diagonal matrix.²

² Exceptionally, it is diagonal for certain values of \( l \) in certain sensor configurations—see the discussion in [20].
The important conclusion to be drawn from equations (40) and (41) is that monopole and quadrupole correlations follow precisely the same algebraic pattern, whether they are dominated by local noise or by a bath of random GWs. Extra terms, such as those in the last line of (41), do not really provide a distinctive feature of local noise, for they are only significant at frequencies different from those of the quadrupole modes, as determined by the presence of the frequency dependent coefficients $L_{nl}(\omega)$ in each case.

This is a very unfortunate circumstance indeed for, in principle, it renders useless the cross-correlation algorithm between different detector channels as an efficient means of filtering out local noise. Are there expedient alternatives?

5.1. Filtering strategies

There are two interesting possibilities one can try for filtering out local noise effects with a single multimode detector.

(i) Since different normal modes are statistically uncorrelated, one is tempted to consider, e.g., the first and second quadrupole modes of a sphere [9, 15, 19] as a pair of independent antennae, then use cross-correlation between them to filter local noise out.

However, this cannot possibly work because, even if local noise induced fluctuations are independent in either mode, so also are GW-induced fluctuations. And, as we have just seen, there is no distinctive feature as to how noise gets added to the signal at different quadrupole modes. Actually, as seen in equations (31) and (41), there is no way to tell which part in $y^{\Omega}(t)$ is due to GWs and which is due to local noise.

(ii) The fundamental difference between a bath of random GWs and local noise is that, while the former can only affect quadrupole modes (or, at most, also monopole modes), local noise excites all modes instead—not just quadrupole/monopole. Given that, in a realistic GW antenna, mode linewidths are extremely narrow, i.e., energy transfer between different modes is extremely slow, an attractive possibility for using a single sphere as a detector of a continuous flow of stochastic GWs would be to identify the effect of the latter as a temperature excess in the quadrupole modes, relative to other modes—dipole, octupole, etc.

While this may look like a reasonable approach, it is in fact an impractical one, as we can appreciate by the following argument. The temperature of a mode is of course a measure of the variance of its random oscillations; since these are a narrow band stochastic process, their correlation time, $\tau_{\text{corr}}$, is quite long, hence a reliable estimate of the variance requires taking data during a suitable number of correlation times for averaging. If $t_{\text{obs}}$ is such a time then the relative error in the temperature estimation is of the order of

$$\frac{\delta T}{T} \sim \frac{\tau_{\text{corr}}}{t_{\text{obs}}}. \quad (43)$$

Let us consider a few likely numbers: an SQL detector—such as that described in [19]—should be capable of sensing energy innovations of magnitude $kT/Q$, where $k$ is Boltzmann’s constant, $T$ is the temperature of the mode (e.g. the first quadrupole), and $Q$ its mechanical quality factor. An optimized device should be able to reach $T/Q \sim 10^{-8}$ K$^{-1}$, for example with $Q = 10^7$ and $T = 100$ mK. A spherical antenna whose considered resonance happens at, say, 1 kHz thus has a correlation time of $2\pi Q/\omega_{12} = 10^4$ seconds. On the other hand, the precision required in the measurement of the mode temperature to match the SQL is $\delta T/T = 10^{-7}$ for the assumed temperature of 0.1 K.
If we now make use of (43) then the conclusion is that an integration time of about $3 \times 10^{10}$ years is required to detect such a small temperature excess as the optimized antenna permits . . . .

It can be argued that stochastic background signals will actually be hotter than the precision limit just described. However nucleosynthesis constraints, for instance, set bounds on the spectral density of GWs to values below $3 \times 10^{-24}$ Hz$^{-1/2}$ at frequencies in the 1 kHz range [1]. This translates into a GW bath temperature of fractions of a $\mu$K, for whose accurate estimation integration times of the order of several tens of thousands of years would be required. The use of fast variance estimation techniques [21] may improve this by between one and two orders of magnitude but, even so, one would still be in the range of 100 to 1000 years of integration time—an absurd figure.

There is also little hope that GW backgrounds originating in astrophysical sources, such as, e.g., supernovae [22], have an effective temperature above $10^{-7}$ K or so [1].

Altogether then, the multimode capabilities of a spherical GW antenna do not seem to offer the practical possibility of making reliable single detector measurements of the spectral density of a background of stochastic GWs. An exception to this, however, happens when some a priori information about the actual form of the spectral function is known ahead of time (template) [23]. If this is the case then optimum filter techniques can be used, of which a wealth is available in the literature [24].

6. Conclusion

In this paper we have investigated in detail how a spherical detector interacts with a background of gravitational waves, characterized by a frequency-dependent spectral energy density function, $S_0(\omega)$. In particular, we have found the generic response functions for a device with an arbitrary number of motion sensors, as well as their correlation matrix, both of the system readouts and for the five quadrupole channels—mode channels for certain specific sensor layout geometries.

We have discovered that there is no distinctive pattern in that correlation matrix, relative to the local noise induced pattern, except that the latter involves all the antenna’s oscillation eigenmodes rather than just the monopole/quadrupole harmonics characteristic of generic GWs. This difference, however, proves insufficient to efficiently filter the stochastic GWs from random local disturbances.

We are thus led to conclude that the multimode character of a spherical detector does not offer a useful alternative to the traditional cross-correlation between two (or more) detectors as a recipe for getting rid of local noise effects. However, we are exploring the possibilities a sphere may offer because of their other properties, like omnidirectionality or bigger cross sections.

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3 See [6] for a variant of the matched filter procedure. So called Capon filters are useful in certain circumstances.
Appendix

A suitable representation for the matrices $E_{ij}^{(0)}$ and $E_{ij}^{(2m)}$ used in equation (9) is the following:

\[
E_{ij}^{(0)} = \begin{pmatrix}
\frac{1}{4\pi} & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad E_{ij}^{(20)} = \begin{pmatrix}
\frac{5}{16\pi} & 1 & -1 & 0 \\
0 & -1 & 0 & 2
\end{pmatrix}
\]

\[
E_{ij}^{(2\pm 1)} = \begin{pmatrix}
\frac{15}{32\pi} & 1 & 0 & 0 \\
0 & 0 & \mp 1 & 0
\end{pmatrix}, \quad E_{ij}^{(2\pm 2)} = \begin{pmatrix}
\frac{15}{32\pi} & \pm i & 0 & 0 \\
0 & \mp i & 0 & 0
\end{pmatrix}
\]

having the properties

\[
E_{ij}^{(0)} \Omega_i \Omega_j = Y_{00}(\Omega), \quad E_{ij}^{(2m)} \Omega_i \Omega_j = Y_{2m}(\Omega)
\]

where $\Omega$ is a unit vector, and $Y_{lm}(\Omega)$ are spherical harmonics.

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A variety of other techniques have even been considered for bar GW detectors.

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