Initial-boundary value problems for the Maxwell equations in the quasi-stationary approximation

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Abstract. The initial-boundary value problem for the quasi-stationary magnetic approximation of the time-dependent Maxwell equations with magnetic boundary conditions is studied. The case of inhomogeneous media containing conductive and non-conductive inclusions is considered. The problem is reduced to the problems of determining vector magnetic and scalar electric potentials. The special gauges are used, that generalize the Coulomb and Lorenz gauges and allow to formulate the problems of the independent definitions of the vector magnetic potential. The correctness of the problems are established under general conditions on the coefficients. The relation between solutions of the problems with different gauges is studied. The equivalence of the problems for potentials to the original boundary value problem is proved.

1. Introduction

Many practically important modern technological processes can be studied in the framework of the quasi-stationary approximations to the Maxwell equations [1]. Under conditions of sufficiently high conductivity, for example, in the design of electromagnetic devices, in solving problems of magnetic levitation and in advanced medicine, the quasi-stationary magnetic approximation (or, so called, eddy current approximation) to the Maxwell equations is used [2], [3].

Applied problems for the quasi-stationary Maxwell equations can be formulated in terms of fields (\( \textbf{H} \) and \( \text{E} \) - formulations [4]-[7] and in terms of potentials (the \( \textbf{T} - \psi \) formulations [8]-[10], the \( \text{A} - \varphi \) formulations with different gauge relations [11]-[17]). The study of the correctness of these formulations is the essential stage in the development and justification of numerical methods for solving problems.

When investigating problems for Maxwell equations in the eddy-current approximation in the real physically heterogeneous media the situation is typical, when a given spatial region contains conductive and non-conductive inclusions [3]-[6]. In this case at using the time-dependent eddy current approximations formulated initial-boundary value problem belong to the elliptic-parabolic type [6]. Consideration of such problems requires the study of the properties of the functional spaces in which a solution is defined. In particular, the important role played by inequalities, combined norms of vector field, its curl and divergence in Lebesgue spaces and allowing at the proof of well-posedness of generalized formulations to establish the coercivity of the respective bilinear forms, which is one of the requirements of the classical Lions theorem [18].
In the present paper the formulations of problems for the time-dependent quasi-stationary Maxwell system in terms of fields and in terms of potentials (vector magnetic potential $A$ and scalar electric potential $\phi$) for inhomogeneous regions are considered. The modified Coulomb and Lorenz gauge relations are used, which lead to generalized problems of independent determination of the vector magnetic potential.

For time-periodic solutions of quasi-stationary problems, the modified Lorenz gauges were discussed, in particular, in [16], [19]. The papers [20], [21] investigated direct and inverse problems for the system of Maxwell's equations in a quasi-stationary magnetic approximation using modified Coulomb and Lorentz gauge relations in conducting regions. In this paper, the results obtained in [19]-[21] extend to the case of bounded domains containing conductive and non-conducting subdomains.

For the considered class of problems the well-posedness of the problems is justified and the relationship between solutions of the problems in terms of potentials at different gauges, as well as between the solutions of problems in terms of potentials and in terms of magnetic field is studied.

All the results obtained in the paper are valid under sufficiently general conditions on the coefficients of the system and are based on estimates of the scalar products of vector fields, that proved in [22], [23].

2. The initial-boundary value problem for magnetic field

The Maxwell equations in the quasi-stationary magnetic approximation can be presented in the form [1]

\[
\begin{align*}
\text{curl} \, H(x,t) &= J(x,t), \\
\text{div} \, B(x,t) &= 0, \\
\text{curl} \, E(x,t) &= -\frac{\partial}{\partial t} B(x,t), \\
\text{div} \, D(x,t) &= \rho(x,t),
\end{align*}
\]

where $x \in \Omega \subset \mathbb{R}^3$, $t \in (0,T)$, $Q = \Omega \times (0,T)$, $H$, $B$, $E$, $D$, $J : Q \to \mathbb{R}^3$ and $\rho : Q \to \mathbb{R}$ are unknown functions.

In linear media the following constitutive relations are valid:

\[
B = \mu H, \quad D = \varepsilon E, \quad J = \sigma E + J^{\text{ext}},
\]

where $\mu$ is a magnetic permeability, $\varepsilon$ is a permittivity, $\sigma$ is an electrical conductivity, $J^{\text{ext}}$ is an exterior current density.

It is assumed in this paper, that $\Omega$ is an open bounded domain, homeomorphic to a ball, with a Lipschitz boundary $\Gamma$. Let $\nu(x)$ is the unit normal vector in $x \in \Gamma$. For function $u : \overline{\Omega} \to \mathbb{R}^3$ we denote by $u_n$, $u_t$ the normal and tangent components of $u$ on $\Gamma$.

The system (1)–(5) is considered with the boundary condition

\[
H_n(x,t) = 0, \quad x \in \Gamma, \quad t \in (0,T)
\]

and the initial condition

\[
H(x,0) = h(x), \quad x \in \Omega.
\]

The considered domain consisting of a conductor $\Omega_c$ and an isolator $\Omega_i = \Omega \setminus \overline{\Omega_c}$. Let $\Omega_c$ is an open bounded homeomorphic to a ball domain with Lipschitz boundary $\Gamma_c$, $\overline{\Omega_c} \subset \Omega$, so $\partial \Omega_i = \Gamma \cup \Gamma_c$. The unit normal vectors in $x \in \Gamma_c$ to $\Omega_c$ is denoted by $\nu_c(x)$. For functions
\[ u : \Omega \to \mathbb{R}^3 \text{ and } u : \Omega \to \mathbb{R}^4 \text{ their restrictions on } \Omega_c \text{ are denoted by } u_c, u_c, \text{ and their restrictions on } \Omega_j \text{ are denoted by } u_j, u_j. \]

The \( \varepsilon \) and \( \mu \) are self-ajoint linear operators from \( \{L_2(\Omega)\}^3 \) into \( \{L_2(\Omega)\}^3 \), satisfying the following conditions:

\[ \varepsilon_i \| u \|_{L_2(\Omega)}^2 \leq (\epsilon u, u)_{L_2(\Omega)} \leq \varepsilon_2 \| u \|_{L_2(\Omega)}^2, \quad \mu_i \| u \|_{L_2(\Omega)}^2 \leq (\mu u, u)_{L_2(\Omega)} \leq \mu_2 \| u \|_{L_2(\Omega)}^2, \]

\( \sigma = \sigma(x) \) is symmetric \( 3 \times 3 \) matrix of measurable functions on \( \Omega \), satisfying the conditions

\[ \sigma_{ij}(x) = 0, \quad i, j = 1, 2, 3, \text{ for almost all } x \in \Omega_j, \]

\[ \sigma_j \| \xi \|_{L_2(\Omega)}^2 \leq (\sigma(x) \xi, \xi) \leq \sigma_2 \| \xi \|_{L_2(\Omega)}^2 \text{ for almost all } x \in \Omega_c \text{ and for all } \xi \in \mathbb{R}^3, \]

where \( \varepsilon_i, \mu_i, \sigma_i, (i = 1, 2) \) are positive numbers, by \( \| u \|_{L_2(\Omega)} \) and \( \langle \cdot, \cdot \rangle_{L_2(\Omega)} \) the norm and the scalar product in \( \{L_2(\Omega)\}^3 \) are denoted.

\( J^* : \Omega \to \mathbb{R}^3 \) and \( h : \Omega \to \mathbb{R}^3 \) are square integrable functions, such that

\[ \text{div} \ J^*_i = 0, \quad \left( J^*_i \right)_t, \quad (x, t) = 0, \quad (x, t) \in \Gamma \times (0, T), \quad \text{curl} \ h_i = J^*_i(0). \quad (8) \]

The generalized solutions of the problems will be considered, that is, all equalities have to be satisfied in the sense of distributions and boundary conditions have to be satisfied in the sense of the trace theory. The following Hilbert spaces with the respective scalar products are defined [24]:

\[ H(\text{div}; \Omega) = \left\{ u \in \{L_2(\Omega)\}^3 : \text{div} u \in L_2(\Omega) \right\}, \quad K(\text{div}; \Omega) = \left\{ u \in \{L_2(\Omega)\}^3 : \text{div} u = 0 \right\}, \]

\[ (u, v)_{\text{div}; \Omega} = \int_{\Omega} (u \cdot v) \, dx + \int_{\Omega} \text{div} u \, \text{div} v \, dx, \]

\[ H(\text{curl}; \Omega) = \left\{ u \in \{L_2(\Omega)\}^3 : \text{curl} u \in L_2(\Omega) \right\}, \quad K(\text{curl}; \Omega) = \left\{ u \in \{L_2(\Omega)\}^3 : \text{curl} u = 0 \right\}, \]

\[ (u, v)_{\text{curl}; \Omega} = \int_{\Omega} (u \cdot v) \, dx + \int_{\Omega} (\text{curl} u \cdot \text{curl} v) \, dx. \]

\( H_0(\text{div}; \Omega), \quad H_0(\text{curl}; \Omega) \) denote the closures of the set of test vector-functions in \( H(\text{div}; \Omega) \) and \( H(\text{curl}; \Omega) \) respectively, \( K_0(\text{div}; \Omega) = K(\text{div}; \Omega) \cap H_0(\text{div}; \Omega). \)

The solution of the problem (1)–(7) is the set of functions

\[ H \in L_2(0, T, H(\text{div}; \Omega)), \quad B \in L_2(0, T, K(\text{div}; \Omega)), \quad J \in L_2(0, T, K_0(\text{div}; \Omega)), \quad E \in L_2(0, T, H(\text{curl}; \Omega)), \quad D \in \{L_2(\Omega)\}^3, \quad \rho \in L_2(0, T, H^{-1}(\Omega)), \]

satisfying (1), (3)–(5), (7).

The functional space

\[ W(\mu, \Omega) = \left\{ u \in H_0(\text{curl}; \Omega) : \mu u \in H(\text{div}; \Omega) \right\} \]

is introduced. Then (1)–(7) is reduced to the problem of determining the function \( H \in L_2(0, T, W(\mu, \Omega)) \) satisfying (7), that for all \( v \in W(\mu, \Omega) \), such that \( \text{curl} v = 0 \),

\[ \frac{d}{dt} \int_{\Omega_c} (\mu H \cdot v) \, dx + \int_{\Omega_c} \left( \sigma^{-1} \text{curl} H \cdot \text{curl} v \right) \, dx = \int_{\Omega_c} \left( \sigma^{-1} J^* \cdot \text{curl} v \right) \, dx \quad (9) \]

and

\[ \text{curl} H_i = J^*_i. \quad (10) \]
If $H$ is a solution of problem (7), (9), (10), then from (1) and (5) functions $B = \mu H$, $J = \text{curl} H$ and $E_c = \sigma^{-1}(J - J^{ext})$ are uniquely determined. To uniquely identify function $E$ (and, consequently, functions $D = \varepsilon E$ and $\rho = \text{div} D$) in the whole domain $\Omega$, additional conditions are required. For example, let
\[ \rho I = \rho_0, \quad (\text{curl} E)_I(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T), \tag{11} \]
where $\rho_0 \in L^2(\Omega_I)$ is a given function.

3. The initial-boundary value problems for vector potential

The vector magnetic potential $A$ and the scalar electric potential $\varphi$ are introduced as new unknown variables by the formulas
\[ B = \text{curl} A, \quad E = -\text{grad} \varphi - \frac{\partial}{\partial t} A. \tag{12} \]

In this case, the system (1)–(5) is reduced to one equation
\[ \frac{\partial}{\partial t} \sigma A + \text{curl}^{-1} \text{curl} A = -\text{grad} \varphi + J^{ext}. \tag{13} \]

Equation (13) is provided by the boundary condition corresponding to (6)
\[ \left( \mu^{-1} \text{curl} A \right)_I(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T). \tag{14} \]

and the initial condition
\[ A(x, 0) = a(x), \quad x \in \Omega_c, \quad a \in \{L^2(\Omega_c)\}^3. \tag{15} \]

The solution of the problem (13)-(15) is functions $A \in L^2(0, T, H(\text{curl}; \Omega))$, $\varphi \in L^2(0, T, H^1(\Omega))$, satisfying (13), (15) in the sense of the distribution on $Q$ and (14) in the sense of the trace theory, that is $\mu^{-1} \text{curl} A \in L^2(0, T, H_0(\text{curl}; \Omega))$.

The solution of the problem (13)-(15) is obviously not unique. The application of the classical Coulomb gauge $\text{div} A = 0$ or Lorentz gauge $\text{div} A + \mu \text{grad} \varphi = 0$ [12]-[15] leads in heterogeneous media to the study of the coupled system of equations for the potentials. However, of practical interest are problem statements, which allow to determine a magnetic potential independently of the electrical potential. The following two types of gauges are discussed:
\[ \text{div} \sigma A_c = 0, \quad (\sigma A)_{c I}(x, t) = 0, \quad (x, t) \in \Gamma_c \times (0, T), \quad \text{div} A_I = 0, \quad A_v(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T) \tag{16} \]

and
\[ \varphi_c = -\kappa \text{div} \sigma A_c, \quad (\sigma A)_{c I}(x, t) = 0, \quad (x, t) \in \Gamma_c \times (0, T), \quad \text{div} A_I = 0, \quad A_v(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T), \tag{17} \]

where $\kappa > 0$ is an arbitrary constant.

For the variational formulation of the problems the following Gilbert spaces are defined:
\[ W(\sigma; \Omega_c) = \{ u \in H(\text{curl}; \Omega_c) : \text{cu} \in H^1_0(\text{div}; \Omega_c) \}, \quad V(\sigma; \Omega_c) = \{ u \in W(\sigma; \Omega_c) : \text{div} \text{cu} = 0 \}, \]
\[ W'(\sigma; \Omega_c) = \int_{\Omega_c} (u \cdot v) dx + \int_{\Gamma_c} (\text{curl} u \cdot \text{curl} v) dx + \int_{\Gamma_c} (\text{div} \text{cu} \cdot \text{div} \sigma v) dx, \]

\[ W(\sigma; \Omega_c, \Omega) = \{ u \in H(\text{curl}; \Omega) : \text{cu} \in H^1_0(\text{div}; \Omega_c), \text{div} u_I = 0, (u_I)_v(x) = 0, x \in \partial \Omega \}, \]
\[ V(\sigma; \Omega_c, \Omega) = \{ u \in W(\sigma; \Omega_c, \Omega) : \text{div} \text{cu} = 0 \}. \]
\[ (u, v)_w = \int_\Omega (u \cdot v) dx + \int_\Omega (\text{curl } u \cdot \text{curl } v) dx + \int_{\Omega_c} (\text{div } \sigma u \cdot \text{div } \sigma v) dx. \]

The problem (13)-(16) is reduced to the following problem: to find a function \( A \in L_2(0, T, V(\sigma; \Omega_c, \Omega)) \) satisfying (15) such that for all \( v \in V(\sigma; \Omega_c, \Omega) \)

\[
\frac{d}{dt} \int_{\Omega_c} (\sigma A \cdot v) dx + \int_{\Omega_c} (\mu^{-1} \text{curl } A \cdot \text{curl } v) dx + \int_{\Omega_c} (\text{div } \sigma A \cdot \text{div } \sigma v) dx = \int_{\Omega_c} (J^{ext} \cdot v) dx,
\]

the problem (13)-(15), (17) is reduced to the problem of determining \( A \in L_2(0, T, W(\sigma; \Omega_c, \Omega)) \)
satisfying (15) such that for all \( v \in W(\sigma; \Omega_c, \Omega) \)

\[
\frac{d}{dt} \int_{\Omega_c} (\sigma A \cdot v) dx + \int_{\Omega_c} (\mu^{-1} \text{curl } A \cdot \text{curl } v) dx + \int_{\Omega_c} (\text{div } \sigma A \cdot \text{div } \sigma v) dx = \int_{\Omega} (J^{ext} \cdot v) dx.
\]

**Theorem 1.** For any \( a \in \{L_2(\Omega)\}^3 \), \( \text{div } \sigma a = 0 \), and \( J^{ext} \in \{L_2(Q)\}^3 \) satisfying (8) the problem (18), (15) has unique solution \( A \in L_2(0, T, \{L_2(\Omega)\}^3) \). If \( a \in V(\sigma; \Omega_c), \) \( \frac{\partial}{\partial t} J^{ext} \in L_2(0, T, \{L_2(\Omega_t)\}^3) \), then \( \frac{\partial}{\partial t} A \in L_2(0, T, \{L_2(\Omega_c)\}^3) \).

**Theorem 2.** For any \( a \in \{L_2(\Omega)\}^3 \) and \( J^{ext} \in \{L_2(Q)\}^3 \) satisfying (8) the problem (19), (15) has unique solution \( A \in L_2(0, T, W(\sigma; \Omega_c, \Omega)) \). Herewith \( A \in C(0, T, \{L_2(\Omega_c)\}^3) \). If \( a \in W(\sigma; \Omega_c), \) \( \frac{\partial}{\partial t} J^{ext} \in L_2(0, T, \{L_2(\Omega_t)\}^3) \), then \( \frac{\partial}{\partial t} A \in L_2(0, T, \{L_2(\Omega_c)\}^3) \).

To prove theorems 1 and 2 we introduce the closed subspaces

\[ W_0 = \{ u \in W(\sigma; \Omega_c); \text{curl } \mu^{-1} \text{curl } u = 0, \mu^{-1} \text{curl } u, (x) = 0, x \in \Gamma \}, \]

\[ W_1 = \{ u \in W(\sigma; \Omega_c); u_c = 0 \}, \]

so \( W(\sigma; \Omega_c, \Omega) = W_0 + W_1 \), and define the linear continuation operator \( F : W(\sigma; \Omega_c) \to W_0 \). Then \( A = A_1 + FA_0 \), where \( A_1 \in L_2(0, T, W_1) \) and for all \( v \in W_1 \) and almost all \( t \in (0, T) \) it satisfies the equality

\[
\int_{\Omega_t} (\mu^{-1} \text{curl } A(t) \cdot \text{curl } v) dx = \int_{\Omega_t} (J^{ext}(t) \cdot v) dx. \quad (20)
\]

In problem (19), (15) \( A_{oc} \in L_2(0, T, W(\sigma; \Omega_c)) \) and for all \( v \in W(\sigma; \Omega_c) \)

\[
\frac{d}{dt} \int_{\Omega_c} (\sigma A_{oc} \cdot v) dx + \int_{\Omega_c} (\mu^{-1} \text{curl } FA_{oc} \cdot \text{curl } Fv) dx + \int_{\Omega_c} (\text{div } \sigma A_{oc} \cdot \text{div } \sigma Fv) dx = \int_{\Omega} (J^{ext} \cdot Fv) dx \quad (21)
\]

Similarly, in problem (18), (15) \( A_{oc} \in L_2(0, T, V(\sigma; \Omega_c)) \) and (21) is true for all \( v \in V(\sigma; \Omega_c) \).

The existence of a solution of problem (20) is proved by applying the Lax-Milgram lemma, the existence of a solution of problem (21) follows from the Lions theorem [18]. In this case, the coercivity of the corresponding bilinear forms can be justified by the estimates, which follow from the obtained [23] inequalities (1) for scalar products of vector fields.

**Lemma 1.** There exists a constant \( C > 0 \), which depends only on \( \Omega \) and \( \Omega_c \), such that for any \( u \in W \) the inequality

\[
\|u\|_{L_2(\Omega)}^2 \leq C \left( \|\text{curl } u\|_{L_2(\Omega)}^2 + \|\text{div } \sigma u\|_{L_2(\Omega_c)}^2 \right)
\]

holds.
Theorem 3. Let $A \in L_2(0,T, V(\sigma; \Omega_c, \Omega))$ is the solution of the problem (18), (15), where $a \in V(\sigma; \Omega_c)$, and $\frac{\partial}{\partial t} \vec{J}_{\varepsilon}^a \in L_2(0,T, [L_2(\Omega_t)]^3)$. Then there is a function $\phi \in L_2(0,T, H^1(\Omega))$, such that $A, \phi$ is a solution of the problem (13)-(15).

Theorem 4. Let $A \in L_2(0,T, W(\sigma; \Omega_c, \Omega))$ is the solution of the problem (19), (15), where $a \in W(\sigma; \Omega_c)$, and $\frac{\partial}{\partial t} \vec{J}_{\varepsilon}^a \in L_2(0,T, [L_2(\Omega_t)]^3)$. Then there is a function $\phi \in L_2(0,T, H^1(\Omega))$, such that $A, \phi$ is a solution of the problem (13)-(15) and $\phi_c = -\kappa \text{div} \vec{a}A_c$.

4. The relationship between solutions of the problems

The next theorem establishes the relation between the solutions of the problems under different gauges.

Theorem 5. Let $A$ is a solution of (18), (15) and $A_\kappa$ is a solution of (19) with the initial condition $A_\kappa(0) = a + q$, where $q \in K(\text{curl}; \Omega_c)$. Then $\text{curl}A = \text{curl}A_\kappa$ and $A_\kappa \rightarrow A$ in $L_2(0,T, W(\sigma; \Omega_c, \Omega))$ as $\kappa \rightarrow \infty$ and the following estimation is valid

$$
\left\| A_\kappa - A \right\|_{L_2(0,T,W)} \leq C \kappa^{-1/2} \left( \kappa^{-1} \left\| \vec{J}_{\varepsilon}^a \right\|_{L_2(\Omega)}^2 + \left\| \vec{q} \right\|_{L_2(\Omega)}^2 \right)^{1/2}.
$$

where the constant $C > 0$ depends only on $\Omega_c$ and $\Omega$. If $q = 0$ then $A_\kappa \rightarrow A$ in $C(0,T, [L_2(\Omega_c)]^3)$, for $t \in (0,T)$

$$
\left\| A_\kappa(t) - A(t) \right\|_{L_2(\Omega_c)} \leq C \kappa^{-1/2} \left\| \vec{J}_{\varepsilon}^a \right\|_{L_2(\Omega)}.
$$

We suppose now $\text{curl}a = \mu \vec{h}$. The using of potentials requires that $H, E$, which defined by (9), satisfy the Maxwell equations. Thus the correctness of application the gauges implies the equivalence the problems in terms of potentials and the origin boundary value problem.

Theorem 6. Let $A \in L_2(0,T, W(\sigma; \Omega_c, \Omega))$ is the solution of the problem (18), (15) or (19), (15), where $a \in W(\sigma; \Omega_c)$ and $\frac{\partial}{\partial t} \vec{J}_{\varepsilon}^a \in L_2(0,T, [L_2(\Omega_t)]^3)$. Then $H = \mu^{-1} \text{curl}A \in L_2(0,T, W(\mu; \Omega))$ is a solution of the problem (7), (9), (10).

The conditions (11) in terms of potentials means

$$
\text{div} \varepsilon \left( \frac{\partial}{\partial t} A + \text{grad} \phi \right) = -\rho_0, \varepsilon \left( \frac{\partial}{\partial t} A + \text{grad} \phi \right)(x,t) = 0, (x,t) \in \Gamma \times (0,T).
$$

Let in addition, the following conditions hold

$$
\vec{J}_{\varepsilon}^a \in H^1(0,T, [L_2(\Omega)]^3), \vec{h} \in W(\mu; \Omega).
$$

The following statements are valid.

Theorem 7. Let $A \in L_2(0,T, V(\sigma; \Omega_c, \Omega))$ is the solution of the problem (18), (15), where $a \in V(\sigma; \Omega_c)$. Then there is the function $\phi \in L_2(0,T, H^1(\Omega))$, unique up to an additive constant, such that $A, \phi$ is a solution of the problem (13)-(15), (22).

Theorem 8. Let $A \in L_2(0,T, W(\sigma; \Omega_c, \Omega))$ is the solution of the problem (19), (15), where $a \in W(\sigma; \Omega_c)$, $\text{div} \vec{a} \in H^1(\Omega_c)$. Then there is the unique function $\phi \in L_2(0,T, H^1(\Omega))$, such that $A, \phi$ is the solution of the problem (13)-(15), (22) and $\phi_c = -\kappa \text{div} \vec{a}A_c$.

From theorems 7, 8 the next theorem follows.
Theorem 9. The problem (1)–(7), (11) has unique solution. Moreover the relations (12) are valid, where $A$ is the solution of (18) (15) or (19), (15).

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