Heavy Quark Hadronic Lagrangian
for S-Wave Quarkonium

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Abstract

We use Heavy Quark Effective Theory (HQET) techniques to parametrize certain non-perturbative effects related to quantum fluctuations that put both heavy quark and antiquark in quarkonium almost on shell. The large off-shell momentum contributions are calculated using Coulomb type states. The almost on-shell momentum contributions are evaluated using an effective 'chiral' lagrangian which incorporates the relevant symmetries of the HQET for quarks and antiquarks. The cut-off dependence of both contributions matches perfectly. The decay constants and the matrix elements of bilinear currents at zero recoil are calculated. The new non-perturbative contributions from the on-shell region are parametrized by a single constant. They turn out to be $O(\alpha^2/\Lambda_{QCD}a_n)$, $a_n$ being the Bohr radius and $\alpha$ the strong coupling constant, times the non-perturbative contribution coming from the multipole expansion (gluon condensate). We discuss the physical applications to $\Upsilon$, $J/\Psi$ and $B_c$ systems.
1. Introduction

The so-called Heavy Quark Effective Theory (HQET) [1-5] has become a standard tool to study the properties of hadrons containing a single heavy quark (see [6] for reviews). The hadron momentum is essentially the momentum of the heavy quark which may then be considered almost on-shell. The dynamics becomes independent of the spin and the mass of the heavy quark giving rise to the so-called Isgur-Wise symmetries [1,2]. The relevant modes are momentum fluctuations of the order of $\Lambda_{QCD}$ which are described by the HQET [3-5]. One cannot actually carry out reliable perturbative calculations at that scale, but one can certainly use the Isgur-Wise symmetries to obtain relations between physical observables.

For hadrons containing two heavy quarks or more the HQET is not believed to be a suitable approximation. The reason being that a system of two heavy quarks is mainly governed by the perturbative Coulomb-type interaction. The relevant modes are momentum fluctuations of the order of the invers Bohr radius, which is flavor dependent, and not of the order of $\Lambda_{QCD}$. Still, if one is interested in subleading non-perturbative contributions related to the ”on-shellness” of the heavy quarks, the HQET may provide some useful information. Irrespectively of the above, the HQET has already been used in phenomenological approaches to two heavy quark systems [7].

We shall argue that the leading non-perturbative contributions in the on-shell region to the quarkonium decay constants and to the matrix elements of heavy-heavy currents between quarkonia states can be described by a suitably modified HQET. The well-known non-perturbative contributions in the off-shell region arising from the multipole expansion [8,9] are $O(\Lambda_{QCD} a_n/\alpha^2)$, $a_n$ being the Bohr radius and $\alpha$ the strong coupling constant, times the contributions we find. The key observation is that when the heavy quarks are almost on-shell the non-perturbative effects must be important. In that regime the multipole expansion breaks down, but it is precisely there where HQET techniques become
In ref. [10] it was pointed out that when fields describing both heavy quarks and heavy antiquarks with the same velocity are included in the HQET lagrangian, the latter has extra symmetries beyond the well known flavor and spin symmetries [1,2]. In ref. [11] the extra symmetries were thoroughly analysed (see [12] for related elaborations). It was shown that they are spontaneously broken down to the spin and flavor symmetries, even if the gluons are switched off. The Goldstone modes turn out to be two particle states with the quantum numbers of s-wave quarkonia. Translating these findings into phenomenologically useful statements was the original motivation of this work.

The main hypothesis in what follows is that whenever we have a heavy quark field we may split it in two momentum regimes. The momentum regime where the heavy quark is almost on shell (small relative three momentum), and the momentum regime where the heavy quark is off shell (large relative three momentum). The main observation is that the HQET should always be a good approximation for a heavy quark in the almost on-shell momentum regime of QCD [10,12], no matter whether the heavy quark is accompanied by another heavy quark in the hadron or not. What makes a hadron containing a single heavy quark qualitatively different from a hadron containing, say, two heavy quarks are the large off-shell momentum effects. In the former the large off-shell momentum effects are small and can be evaluated order by order in QCD perturbation theory [1,5,13,14]. In the latter the large off-shell momentum effects are dominant giving rise to Coulomb-type bound states. However, once this is taken into account there is no a priori reason not to use HQET in the almost on-shell momentum regime for systems with two heavy quarks. Then the extra symmetries found in [10,11], which naturally involve quarkonium systems, should be relevant.

Suppose we have two quarks $Q$ and $Q'$ which are sufficiently heavy so that the formalism below can be readily applicable. Let us denote by $\psi_Q, \eta_Q, Q_Q^*,$ and $Q_{Q'}$ the vector
\( \bar{Q}Q \), pseudoscalar \( \bar{Q}Q \), vector \( \bar{Q}Q' \) and pseudoscalar \( \bar{Q}Q' \) states. Our main results follow.

(i) The masses do not receive new non-perturbative contribution from the on-shell momentum region. Consequently, the leading non-perturbative correction comes from the multipole expansion [8,9]. This allows to extract \( m_Q \) in a model independent way from \( m_{\psi_Q} \), and hence fix the parameter \( \bar{\Lambda} \) relating \( m_Q \) with the mass of the \( Qq \) systems [6].

(ii) The new non-perturbative effects from the on-shell momentum region in the decay constants \( f_{\psi_Q}, f_{\eta_Q}, f_{Q_Q'}, \) and \( f_{Q_Q'} \) are given in terms of a single non-perturbative parameter \( f_H \).

(iii) The new non-perturbative effects from the on-shell momentum region in the matrix elements of bilinear heavy quark currents at zero recoil are given in terms of the same non-perturbative parameter \( f_H \). In particular, this implies that the semileptonic decays \( (m_Q > m_Q') \)

\[
\psi_Q, \eta_Q \rightarrow Q'_Q, Q_Q' \\
Q'_Q, Q_Q' \rightarrow \psi_Q', \eta_Q'
\]

at zero recoil are known in terms of \( f_{\psi_Q}, f_{\eta_Q}, f_{Q_Q'}, \) and \( f_{Q_Q'} \).

We distribute the paper as follows. In sect. 2 we perform some short distances calculations in the kinematical region we are interested in. In sect. 3 we summarize the main results of ref. [11] and match the results from sect. 2 with the HQET. In sect. 4 we construct a hadronic effective lagrangian for on-shell modes in quarkonium. In sect. 5 we calculate the decay constant. In sect. 6 we calculate the matrix elements of any bilinear heavy quark current between quarkonia states. This is relevant for the study of semileptonic decays at zero recoil. In sect. 7 we briefly discuss the possible use of our formalism for \( \Upsilon, B_c, B'_c, J/\Psi \) and \( \eta_c \) physics. Section 8 is devoted to the conclusions. In Appendix A we show how to include \( 1/m \) corrections in the hadronic effective lagrangian for the on-shell modes. A few technical details are relegated to Appendix B.
2. Short distance contributions in the on-shell momentum regime

As mentioned in the introduction, what makes a \( \bar{Q}Q \) system qualitatively different from a \( \bar{Q}q \) system are the short distance contributions. In a \( \bar{Q}q \) system these are well understood. They amount to Wilson coefficients in the currents and in the operators of the HQET lagrangian, with anomalous dimensions which are computable in the loop expansion of QCD. For a \( \bar{Q}Q \) system the short distance contributions cannot be accounted for by just anomalous dimensions in Wilson coefficients. Indeed, the anomalous dimension of a current containing a heavy quark field and a heavy antiquark field with the same velocity becomes imaginary and infinite [15]. For large \( m_Q \), the two quarks in a \( \bar{Q}Q \) system appear to be very close. Due to asymptotic freedom the system can be understood in a first approximation as a Coulomb-type bound state. In perturbation theory this is equivalent to sum up an infinite set of diagrams (ladder approximation) whose kernel is the tree level one gluon exchange (see [16] for a review).

We shall assume that the dominant short distance contribution to heavy quarkonia is the existence of Coulomb-type bound states. Typically we shall be interested in Green functions of the kind

\[
G_\Gamma(p_1, p_2) := \int d^4x_1d^4x_2 e^{ip_1x_1+ip_2x_2} \langle 0| \{ \bar{Q}^a \Gamma Q^b(0) \bar{Q}^{b_1}_{\alpha_1}(x_1) Q^{a_2}_{\alpha_2}(x_2) \} |0 \rangle ,
\]

(2.1)

for the range of momentum

\[
p_1 = -m_b v - k_1 , \quad p_2 = -m_a v - k_2 ,
\]

(2.2)
k_1 and \( k_2 \) being small.

Since the quarks are very massive, for the range of momentum (2.2) the leading contribution to (2.1) is only given by the following ordering

\[
G_\Gamma(p_1, p_2) = \int d^4x_1d^4x_2 e^{ip_1x_1+ip_2x_2} \theta (-max(x_1^0, x_2^0))
\]

\[
\times \langle 0| \bar{Q}^a \Gamma Q^b(0) \bar{Q}^{b_1}_{\alpha_1}(x_1) Q^{a_2}_{\alpha_2}(x_2) \} |0 \rangle .
\]

(2.3)
We insert the identity between the current and the fields and we approximate it by the vacuum plus the Coulomb-type states (the states above threshold shall not give contribution when we sit in the relevant pole). We treat then the fields as being free.

\[
1 \simeq |0\rangle\langle 0| + \sum_{n,s} \int \frac{d^3 \vec{F}_n}{(2\pi)^3 2P_n} |s, \vec{F}_n = m_{ab,n}\vec{v}| \langle s, \vec{P}_n = m_{ab,n}\vec{v}| (2.4)
\]

The Coulomb state in the center of mass frame (CM) reads

\[
|s, \vec{P}_n = m_{ab,n}\vec{v}\rangle = \frac{1}{\sqrt{N_c m_{ab,n}}} \int \frac{d^3 \vec{k}}{(2\pi)^3} \vec{\Psi}_{ab,n}(\vec{k}) \frac{1}{\sqrt{2p_1^0 p_2^0}} \times \sum_{\alpha,\beta} \bar{u}\alpha(p_1) \Gamma_s v^\beta(p_2) a_\alpha^\dagger(p_1) b_\beta^\dagger(p_2)|0\rangle ,
\]

(2.5)

where

\[
\vec{p}_1 = m_a \vec{v} + \vec{k} + \frac{\vec{k}.\vec{v}}{1 + v^0} \vec{v} , \quad \vec{p}_2 = m_b \vec{v} - \vec{k} - \frac{\vec{k}.\vec{v}}{1 + v^0} \vec{v} ,
\]

\[
p_1^0 = m_a v^0 + \vec{k}.\vec{v} , \quad p_2^0 = m_b v^0 - \vec{k}.\vec{v} ,
\]

\[
m_{ab} := m_a + m_b , m_{ab,n} := m_{ab} - E_{ab,n} , \quad \Gamma_s = i\gamma_5 p_- , i\vec{\gamma} p_- ,
\]

\[
v^2 = 1 , \quad p_{\pm} := \frac{1 \pm p^0}{2} , \quad e^j.v = 0 .
\]

(2.6)

\[E_{ab,n}, \Psi_{ab,n}(\vec{x}) \text{ and } \vec{\Psi}_{ab,n}(\vec{k})\text{ are the energy, the coordinate space wave function and the momentum space wave function of a Coulomb-type state with principal quantum number } n. \text{ } v \text{ is the bound state 4-vector velocity. } a_\alpha^\dagger(p_1) \text{ and } b_\beta^\dagger(p_2) \text{ are creation operators of particles and anti-particles respectively. } u^\alpha(p_1) \text{ and } v^\beta(p_2) \text{ are spinors normalized in such a way that in the large } m \text{ limit the following holds}
\]

\[
\sum_\alpha u^\alpha(p_1) \bar{u}^\alpha(p_1) = p_+ , \quad \sum_\alpha v^\alpha(p_1) \bar{v}^\alpha(p_1) = -p_- .
\]

(2.7)

Choosing the momenta as in (2.6) is crucial in order to take into account that the CM of the bound state moves with a fix velocity \( v \) with respect to the laboratory frame [17]. (2.5) has the usual relativistic normalization

\[
\langle s, \vec{F}_n = m_{ab,n}\vec{v}|r, \vec{P}_m = m_{ab,m}\vec{v}'\rangle = 2m_{ab,n}v^0(2\pi)^3 \delta^{(3)}(m_{ab,n}(\vec{v} - \vec{v}')) \delta_{nm} \delta_{rs} .
\]

(2.8)
We have to consider the following kind of matrix elements

\[ \langle s, m_{ab,n} \bar{v}|Q_{\alpha_2}^a(x_2)\bar{Q}_{\alpha_1}^b(x_1)|0 \rangle = e^{im_{ab,n}v.X} \langle s, m_{ab,n} \bar{v}|Q_{\alpha_2}^a(x_2-X)\bar{Q}_{\alpha_1}^b(x_1-X)|0 \rangle \]

\[ = e^{im_{ab,n}v.X} \frac{m_{ab}}{m_{ab,n}}(\bar{\Gamma}_s)_{\alpha_2\alpha_1} \int \frac{d^3k}{(2\pi)^3} \tilde{\Psi}_{ab,n}^*(k)e^{i(k.\bar{x}^\alpha-\bar{x}(k+\frac{v}{1+v^2}\bar{v}))}, \]  

(2.9)

\[ X = \frac{m_ax_1+m_bx_2}{m_{ab}}, \quad x = x_1 - x_2. \]

where it is essential to extract the CM dependence in the fields before using the explicit expression (2.5) for the calculation of (2.9). As mention above the states \(|s, m_{ab,n} \bar{v}|\) have the explicit expression (2.5) only in the CM frame [16,17]. Factors of the kind \(m_{ab}/m_{ab,n}\) appearing in several expressions above have been approximated to 1 in the rest of the paper. Finally, performing the \(x_1, x_2\) integral and taking into account that

\[ \sum_s (\Gamma_s)_{\alpha_2\alpha_4}(\bar{\Gamma}_s)_{\alpha_1\alpha_3} = -2(p_+)_{\alpha_2\alpha_3}(p-)_{\alpha_1\alpha_4} \]  

(2.10)

we obtain

\[ G_{\Gamma}(p_1,p_2) = \sum_n \tilde{\Psi}_{ab,n}^*(0)\Psi_{ab,n}(0)(p_-\Gamma p_+)_{\alpha_2\alpha_3}\delta_{i_1i_2} \]

\[ \times \frac{1}{v.k_2 + \frac{m_a}{m_{ab}}E_{ab,n} + i\epsilon} \frac{1}{v.k_1 + \frac{m_b}{m_{ab}}E_{ab,n} + i\epsilon}, \]  

(2.11)

In the last expression we approximated \(\tilde{\Psi}_{ab,n}(e^{i.k}) \simeq \tilde{\Psi}_{ab,n}(0)\) (we neglect \(O((\frac{n|e^{i.k}|}{m_\alpha})^2))\). In (2.11) there is a sum over an infinite number of poles. Each term in the sum corresponds to a Coulomb-type bound state. At the hadronic level we want to describe only one of those states. This is achieved by tunning the external momenta to sit on the relevant pole. Suppose we are interested in \(\psi_Q(n)\) state. Then we take

\[ k_1 = k_1' - \frac{m_b}{m_{ab}}E_{ab,n}v, \quad k_2 = k_2' - \frac{m_a}{m_{ab}}E_{ab,n}v, \]

(2.12)

so that in the limit \(k_i' \to 0, (i = 1,2)\) we obtain

\[ G_{\Gamma}(p_1,p_2) = \tilde{\Psi}_{ab,n}^*(0)\Psi_{ab,n}(0)(p_-\Gamma p_+)_{\alpha_2\alpha_3}\delta_{i_1i_2} \]

\[ \times \frac{1}{v.k_2' + i\epsilon} \frac{1}{v.k_1' + i\epsilon}. \]

(2.13)
Notice from (2.2) and (2.12) that we must subtract from the momentum of the quark 
\[(m_a - \frac{m_a}{m_{ab}}E_{ab,n})v\] in order to get an expression suitable to be reproduced in the HQET. This may be interpreted as if integrating out off-shell short distance degrees of freedom produces an effective mass for the almost on-shell modes of a heavy quark inside quarkonium. This effective mass depends on the precise bound state the quark is in. We are almost on-shell when \[v.k_1', e^j.k_i' \sim \Lambda_{QCD} \quad (i=1,2).\]

This restricts the validity of our approximation to the case \[E_{ab,n} \sim \mu_{ab}\alpha^2/n^2 \gg \Lambda_{QCD}\] (\(\mu_{ab}\) is the reduced mass), otherwise momentum fluctuations of the order of \(\Lambda_{QCD}\) would take us from one pole to another. Notice also that for arbitrary large but fix \(\mu_{ab}\) there is always an \(n\) where this approximation fails. Therefore we shall always be dealing with a finite number of low laying energy levels.

Consider the four-point function.

\[
G(p_1, p_2, p_3, p_4) := \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{ip_1x_1 + ip_2x_2 + ip_3x_3 + ip_4x_4} \\
\times \langle 0 | T \left\{ Q_{b_1}^{\alpha_1}(x_1) Q_{a_2}^{\alpha_2}(x_2) \bar{Q}_{a_3}^{\alpha_3}(x_3) \bar{Q}_{b_4}^{\alpha_4}(x_4) \right\} | 0 \rangle .
\]

(2.14)

For the momenta
\[
p_1 = -(m_b - \frac{m_b}{m_{ab}}E_{ab,n})v - k_1',
\]
\[
p_2 = (m_a - \frac{m_a}{m_{ab}}E_{ab,n})v + k_2',
\]
\[
p_3 = -(m_a - \frac{m_a}{m_{ab}}E_{ab,n})v - k_3',
\]
\[
p_4 = (m_b - \frac{m_b}{m_{ab}}E_{ab,n})v + k_4',
\]

\((k_i' \to 0 \ , \ i = 1, ..., 4)\) working in the same way we obtain

\[
G(p_1, p_2, p_3, p_4) = (2\pi)^4 \delta^{(4)}(-k_1' + k_2' - k_3' + k_4') \frac{i}{2N_c} \sum_{\Gamma_n = i\gamma_{5p} \ldots \eta p_\ldots} (\Gamma_n)_{\alpha_2\alpha_4} (\Gamma_n)_{\alpha_1\alpha_3} \\
\times \delta_{i_1 i_3} \delta_{i_2 i_4} \Psi_{ab,n}^*(0) \Psi_{ab,n}(0) \frac{1}{v.k'_3 + i\epsilon} \frac{1}{v.k'_1 + i\epsilon} \left( \frac{1}{v.k'_2 + i\epsilon} + \frac{1}{v.k'_4 + i\epsilon} \right).
\]

(2.16)

We shall see in the next section that (2.13) and (2.16) can be reproduced (with suitable changes) by a HQET for quarks and antiquarks.
3. HQET for quarks and antiquarks

The lagrangian of the HQET for quarks and antiquarks moving at the same velocity $v_\mu (v_\mu v^\mu = 1)$ reads [4]

$$L_v = i\bar{h} v \mu D^\mu h_v = i\bar{h}_v^+ v \cdot D h_v^+ - i\bar{h}_v^- v \cdot D h_v^- ,$$

(3.1)

where $h_v = h_v^+ + h_v^-$ and $h_v^\pm = \frac{1 + v^\mu}{2} h_v$. $h_v^+$ contains annihilation operators of quarks with small momentum about $mv_\mu$ and $h_v^-$ contains creation operators of anti-quarks again with small momentum about $mv_\mu$. $D_\mu$ is the covariant derivative containing the gluon field.

The quark and antiquark sector of (3.1) are independently invariant under the well-known spin and flavour symmetry [1,2,4]

$$h_v^\pm \to e^{i\epsilon_i^\pm S_i^\pm} h_v^\pm \text{ and } \bar{h}_v^\pm \to \bar{h}_v^\pm e^{-i\epsilon_i^\pm S_i^\pm} ,$$

(3.2)

where $S_i^\pm = i\epsilon_{ijk} [\phi_j, \phi_k] (1 \pm \phi)/2$, with $e^\mu_j, j = 1, 2, 3$ being an orthonormal set of space like vectors orthogonal to $v_\mu$, and

$$h_v^\pm \to e^{i\theta_\pm} h_v^\pm \text{ and } \bar{h}_v^\pm \to \bar{h}_v^\pm e^{-i\theta_\pm} .$$

(3.3)

$\epsilon_i^\pm$ and $\theta_\pm$ are arbitrary real numbers corresponding to the parameters of the transformations.

The lagrangian (3.1) is also invariant under the following set of transformations

$$h_v \to e^{i\gamma_5 \epsilon} h_v ; \quad \bar{h}_v \to \bar{h}_v e^{i\gamma_5 \epsilon} ,$$

(3.4)

$$h_v \to e^{i\gamma_5 \phi_i} h_v ; \quad \bar{h}_v \to \bar{h}_v e^{i\gamma_5 \phi_i} ,$$

(3.5)

$$h_v \to e^{i\phi_i} h_v ; \quad \bar{h}_v \to \bar{h}_v e^{i\phi_i} ,$$

(3.6)

$$h_v \to e^{i\phi_i} h_v ; \quad \bar{h}_v \to \bar{h}_v e^{i\phi_i} .$$

(3.7)

The whole set of transformations (3.2)-(3.7) corresponds to a $U(4)$ symmetry for a single flavour. For $N_{hf}$ heavy flavours they correspond to a $U(4N_{hf})$ group. In the latter case
\( h_v \) must be considered a vector in flavour space and the parameters of the transformations (3.2)-(3.7) as hermitian matrices in that space.

When the gluons are switched off it is easy to prove that the \( U(4N_{hf}) \) symmetry breaks spontaneously down to \( U(2N_{hf}) \otimes U(2N_{hf}) \) (see [11]). The following currents correspond to the broken generators

\[
j_{5\pm}^{ab} := \bar{h}_v^a i \gamma_5 p_{\pm} h_v^b \quad \text{and} \quad j_{5\pm}^{ab i} := \bar{h}_v^a i \gamma_5 p_{\pm} h_v^b,
\]

(3.8)

\( a, b, c... = 1, ... N_{hf} \) are flavour indices. They transform according to two four dimensional irreducible representations of \( U(2N_{hf}) \otimes U(2N_{hf}) \). In what follows we are going to assume that the situation above is not modified when soft gluons are switched on. The currents (3.8) have the quantum numbers of pseudoscalar and vector quarkonium respectively. The heavy quark and antiquark fields interact with soft gluons according to the lagrangian (3.1).

For soft gluons, perturbation theory cannot be realiable applied. However, one can use effective lagrangian techniques, which fully exploite the symmetries above, to parametrize the non-perturbative contributions in this region. This shall be done in section 4.

For further purposes let us carry out some leading order perturbative calculations.

Consider first

\[
G_{\Gamma\Gamma'}(k) = \int d^4 x e^{-ik.x} \langle 0 | T \{ \bar{h}_v^a - \Gamma h_v^b (0) \bar{h}_v^b + \Gamma' h_v^a - (x) \} | 0 \rangle
\]

\[
= -iN_c \frac{\mu^3}{6\pi^2} tr (p_+ \Gamma' p_- \Gamma) \frac{1}{v.k + i\epsilon},
\]

(3.9)

where \( \mu \) is an ultraviolet symmetric cut-off in three-momentum (see [11] for more details).

Consider also

\[
G_{\Gamma\Gamma'\Gamma''}(k_1', k_2')
\]

\[
= \int d^4 x_1 d^4 x_2 e^{ik_1.x_1 - ik_2.x_2} \langle 0 | T \{ \bar{h}_v^a - \Gamma'' h_v^b (x_1) \bar{h}_v^b + \Gamma' h_v^a - (x_2) \} | 0 \rangle
\]

\[
= N_c \frac{\mu^3}{6\pi^2} tr (p_- \Gamma'' p_+ \Gamma' \Gamma) \frac{1}{v.k_1' + i\epsilon} \frac{1}{v.k_2' + i\epsilon}.
\]

(3.10)
The flavor indices \((a, b, c)\) are not summed up unless otherwise indicated. Colour indices are not explicitly displayed in the colour singlet currents. Otherwise they will be denoted by \(i_1, i_2, \ldots = 1, \ldots, N_c\), \(N_c\) being the number of colours. We shall drop the subscript \(v\) from \(h_v\) and change the superscripts \(\pm\) into subscripts in the following.

The strong cut-off dependence of (3.9)-(3.10) is puzzling. We shall see later on that it cancels against suitable short distance contributions.

As claimed before, it is easy to see that (2.13) is reproduced by the following HQET Green function at tree level

\[
G^\Gamma (k'_1, k'_2) = \int d^4x_1 d^4x_2 e^{-ik'_1 x_1 - ik'_2 x_2} \langle 0 | T \left\{ C^\Gamma \bar{h}^a (0) h^b (0) \bar{h}^b_{\alpha_1} (x_1) h^a_{\alpha_2} (x_2) \right\} | 0 \rangle \quad (3.11)
\]

with \(C^\Gamma\) being a Wilson coefficient.

\[
C^\Gamma = \bar{\Psi}^*_ab,n (0) \Psi_{ab,n} (0) . \quad (3.12)
\]

Analogously, (2.16) is reproduced in the HQET by *\footnote{One may be tempted to include (3.13) as a perturbation in the HQET lagrangian. This is not quite correct. The Green function}

\[
G(k'_1, k'_2, k'_3, k'_4) = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-ik'_1 x_1 + ik'_2 x_2 - ik'_3 x_3 + ik'_4 x_4}
\]

\[
\times \langle 0 | T \left\{ h^b_{\alpha_1} (x_1) h^a_{\alpha_2} (x_2) \bar{h}^b_{\alpha_3} (x_3) \bar{h}^a_{\alpha_4} (x_4) \right\} | 0 \rangle
\]

gives a non-zero contribution in the HQET which does not correspond to (2.14)-(2.16). It is (3.13) which gives the leading contribution to (2.14) in the HQET and hence the last term in (3.13) must not be included in the lagrangian. This means that unlike in the case of heavy-light systems, the short distance effects here cannot always be accounted for by only modifications of the currents and the lagrangian, as we may have na"ively expected.

We have to content ourselves by identifying for a given Green function, the Green function in the HQET which gives the same result.
\[ G(k_1', k_2', k_3', k_4') = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-ik_1'x_1 + ik_2'x_2 - ik_3'x_3 + ik_4'x_4} \times \langle 0\left| T\left\{ h_{-\alpha_1}^{i_1}(x_1) h_{+\alpha_2}^{i_2}(x_2) h_{+\alpha_3}^{i_3}(x_3) h_{-\alpha_4}^{i_4}(x_4) \right\} \sum_{\Gamma_n} \bar{h}_n^a \Gamma_n h_n^b(y) iv.\partial(\bar{h}_n^b \Gamma_n h_n^a(y)) \right| 0 \rangle. \]

4. Effective hadronic lagrangian for the on-shell contributions of s-wave quarkonia

We have seen that for the on-shell kinematical regime certain correlators can be reproduced in the HQET. We shall see in the sect. 5 and 6 that the contributions from this region to the decay constants and matrix elements reduce to the evaluation of heavy quark-antiquark currents in the HQET. For the range of momentum we are interested in these Green functions cannot reliable be evaluated in perturbation theory. We shall use in this section effective lagrangian techniques, very similar to those used in Chiral perturbation theory, to parametrize the nonperturbative contribution.

There are well-known rules [18] (see also [19]) to construct phenomenological lagrangians for Goldstone bosons associated to the symmetry breaking of a group \( G \) down to a subgroup \( H \) for relativistic theories. These rules need two slight modifications to become applicable to our case:

(i) The HQET is formally relativistic only after assigning transformation properties to the fix velocity \( v^\mu \). We must take into account that the velocity \( v^\mu \) as well as the \( e^i_\mu \) can also be used to build up relativistic invariant terms.

(ii) The HQET is not only globally \( U(4N_{hf}) \) invariant, but locally \( U(4N_{hf}) \) gauge invariant under transformations which only depend on the components \( x^i = x^\mu e^i_\mu \). We shall also require the phenomenological lagrangian to be local gauge invariant under the corresponding transformations.
With the above modifications (i) and (ii) we shall apply the rules [18] to the case $G = U(4N_{hf})$, $H = U(2N_{hf}) \otimes U(2N_{hf})$. Let us first associate to the currents (3.8) fields in the phenomenological lagrangian which have the same transformation properties under $H$

\[ H^{ab} \rightarrow \bar{h}^{a} i \gamma_{5} p^{b}, \quad H^{ab}_{i} \rightarrow \bar{h}^{a} i \psi^{i} p^{b}, \quad (4.1) \]

\[ H^{ba*} \rightarrow \bar{h}^{b} i \gamma_{5} p^{a}, \quad H^{ba i*} \rightarrow - \bar{h}^{b} i \psi^{i} p^{a}. \]

We build up the following object

\[ H = i \gamma_{5} p - i \psi^{i} p H^{i} + i \gamma_{5} p^{+} H^{+} + i \psi^{i} p^{+} H^{i+}, \quad (4.2) \]

\[ \bar{H} := \gamma^{0} H^{+} \gamma^{0} = H, \]

where we use matrix notation for $H^{ab}$ and $H^{ab i}$. $H$ transforms under the unbroken subgroup as follows

\[ H \rightarrow h H h^{-1}, \quad h \in U(2N_{hf}) \otimes U(2N_{hf}). \quad (4.3) \]

We assign non-linear transformations under the full group $U(4N_{hf})$ in the standard manner [18]

\[ g(\theta) e^{H} =: e^{H'} h(H, \theta), \]

\[ g \in U(4N_{hf}), \quad h \in U(2N_{hf}) \otimes U(2N_{hf}), \quad e^{H} \in U(4N_{hf})/U(2N_{hf}) \otimes U(2N_{hf}), \quad (4.4) \]

where $H'$ is the transformed field. Then

\[ e^{H} \rightarrow e^{H'} = ge^{H} h^{-1} = he^{H} \bar{g}, \quad (4.5) \]

where $\bar{g} = \gamma^{0} g^{+} \gamma^{0}$. The following property holds

\[ \bar{\phi} e^{H} = e^{-H} \bar{\phi}, \quad (4.6) \]

which implies that

\[ S := e^{2H} \bar{\phi} = \bar{\phi} e^{-2H}, \quad S^{2} = 1, \quad e^{2H} \bar{\phi} \rightarrow ge^{2H} \bar{\phi} g^{-1}. \quad (4.7) \]
Because of the local gauge symmetry we can only build the following connection and covariant tensor

\[
V := \frac{1}{2} \left( e^{-H} v.\partial e^H + e^H v.\partial e^{-H} \right) , \quad V \rightarrow hVh^{-1} + hv.\partial h^{-1} , \quad \phi V = V\phi ,
\]

\[
A := \frac{1}{2} \left( e^{-H} v.\partial e^H - e^H v.\partial e^{-H} \right) , \quad A \rightarrow hAh^{-1} , \quad \phi A = -A\phi , \quad (4.8)
\]

\[
v.\partial S = e^H Ae^H\phi .
\]

Notice that any derivative with respect to \( x^i := e^i_\mu x^\mu \) acting on functions of \( x^i \) which are not scalars will not be covariant under the local transformations.

The \( u(4N_{hf}) \) algebra and the HQET lagrangian are invariant under the following discrete symmetry

\[
e^i_\mu \rightarrow -e^i_\mu , \quad v_\mu \rightarrow -v_\mu , \quad (4.9)
\]

which is reminiscent of charge conjugation. They are also invariant under the \( SO(3) \) transformations \( e^i_\mu \rightarrow R^i_j e^j_\mu \) and, of course, under Lorentz transformations if we assign \( v_\mu \rightarrow \Lambda_\mu^\nu v_\nu , e^i_\mu \rightarrow \Lambda_\mu^\nu e^i_\nu \). All these symmetries should also be implemented in the effective lagrangian.

We can start at this point the construction of the effective lagrangian, order by order in derivatives, using the objects defined above. At first order it turns out that there is no invariant term. Still there is a term which is invariant up to a total derivative. It reads

\[
Tr(\phi V) \simeq -4tr(H^\dagger v.\partial H + H^i_\dagger v.\partial H^i) + \ldots ,
\]

\[
Tr(\phi V) \rightarrow Tr(\phi V) + Tr(\phi hv.\partial h^{-1}). \quad (4.10)
\]

\( Tr \) means trace over flavour and Dirac indices whereas \( tr \) means trace over flavour indices only. We keep \( tr \) for trace over Dirac indices only. It is not difficult to prove that \( Tr(\phi hv.\partial h^{-1}) \) is indeed a total derivative. This is analogous to the case of the Heisenberg ferromagnet where the leading order term in the effective lagrangian for the Goldstone mode is also invariant up to a total derivative [20]. Then at leading order the long distance
properties of heavy quarkonia are governed by a single constant. At next to leading order we have the term

\[ \text{Tr}(AA) \simeq -4\text{tr}(v.\partial H^\dagger v.\partial H + v.\partial H^\dagger v.\partial H^i) + ... . \] (4.11)

Terms containing \( x^i \) derivatives start appearing at sixth order. Notice that there is no vertex involving an odd number of fields. This holds at any order in derivatives and it is a consequence of the separate conservation of the number of heavy quarks and antiquarks.

For convenience we normalize the effective lagrangian as follows

\[ -i\frac{f_H^2}{4}\text{Tr}(\bar{\psi}V) = i\text{tr}(\Pi^\dagger v.\partial \Pi + \Pi^\dagger v.\partial \Pi^i) + ... , \]

\[ H = \frac{\Pi}{f_H} , \quad H^i = \frac{\Pi^i}{f_H} . \] (4.12)

\( f_H^2 \) is a dimension 3 parameter of the order of \( \Lambda_{QCD}^3 \). The effective lagrangian built above makes sense by itself as a toy model. If we ignore the matching with high energies we can withdraw some consequences out of the lowest order lagrangian. These and the 1/\( m \) corrections to this toy model are worked out in the appendix B.

Let us next discuss how to represent quark currents in the effective lagrangian. Consider

\[ j_{\Gamma}^{ab} = \bar{h}^a \Gamma h^b . \] (4.13)

Let us introduce a source \( a_{\Gamma}^{ab} \) for each of these currents and write all possible currents up in the lagrangian

\[ L_v = i\bar{h}\gamma_v D^\mu h + \bar{h}\gamma_v h , \]

\[ a := \sum_{\Gamma} a_{\Gamma}^{ab} \Gamma . \] (4.14)

\( L \) is now locally invariant under \( U(4N_{hf}) \) if we assign to \( a \) the transformation property

\[ a \rightarrow g a g^{-1} + g iv.\partial g^{-1} . \] (4.15)
At the hadronic level we may also require local gauge invariance upon the introduction of \( a \). This is easily achieved by changing \( v.\partial \) into \( v.\partial - ia \) in the definition of \( V \) in (4.8). We obtain

\[
L = -i f_H^2 \left[ Tr(p V) - i Tr(a S) \right].
\]

Then we may identify

\[
\bar{h}^a \Gamma h^b \rightarrow - f_H^2 Tr \left( \Gamma T^{ab} e^{2H} \right),
\]

where \( T^{ab} \) is the zero matrix in flavor space except for a 1 in row \( a \) column \( b \). It is interesting to observe that the \( U(4 N_{hf}) \) symmetry is so large that any bilinear current of the kind (4.13) can be written in terms of a generator of the \( U(4 N_{hf}) \) symmetry. This is the actual reason why the identification (4.17) does not involve any extra unknown parameter. It is analogous to the case of the vector and axial-vector currents in the Chiral Lagrangian [21].

Let us next calculate for further convenience the correlators (3.9) and (3.10) in the hadronic effective lagrangian. For (3.9) we have

\[
G_{\Gamma \Gamma'}(k) = \int d^4 x e^{-i k \cdot x} \langle 0 | T \left\{ \bar{h}_-^a \Gamma h_+^b (0) \bar{h}_+^b \Gamma' \bar{h}_-^a (x) \right\} | 0 \rangle
\]

\[
= \int d^4 x e^{-i k \cdot x} \langle 0 | T \left\{ \left[ - \frac{f_H^2}{4} Tr(p_- \Gamma p_+ T^{ab} e^{2H(0)}) \right] \left[ - \frac{f_H^2}{4} Tr(p_+ \Gamma' p_- T^{ba} e^{2H(x)}) \right] \right\} | 0 \rangle
\]

\[
\simeq \int d^4 x e^{-i k \cdot x} \langle 0 | T \left\{ \left[ - \frac{f_H^2}{4} Tr(p_- \Gamma p_+ T^{ab} e^{2H(0)}) \right] \left[ - \frac{f_H^2}{4} Tr(p_+ \Gamma' p_- T^{ba} e^{2H(x)}) \right] \right\} | 0 \rangle
\]

\[
= - i \frac{f_H^2}{2} \text{tr} (p_+ \Gamma' p_- \Gamma) \frac{1}{v \cdot k + i \epsilon}.
\]

\[(4.18)\]
For (3.10) we have
\[
G_{\Gamma'\Gamma''}(k_1', k_2') = \int d^4x_1 d^4x_2 e^{ik_1'x_1 - ik_2'x_2} \langle 0 | T \{ \bar{h}_-^{a'}\Gamma'' h^b_+ (x_1) \bar{h}_+^b \Gamma c (0) \bar{h}_+^{c'} \Gamma' h^a_-(x_2) \} | 0 \rangle
\]
\[
= \int d^4x_1 d^4x_2 e^{ik_1'x_1 - ik_2'x_2} \langle 0 | T \left\{ \left[ -\frac{f_H^2}{4} Tr(p_-\Gamma'' p_+ T^{ab} e^{2\mathcal{H}(x_1)}) \right] \times \left[ -\frac{f_H^2}{4} Tr(p_+\Gamma' p_+ T^{ca} e^{2\mathcal{H}(x_2)}) \right] \right\} | 0 \rangle
\]
\[
\approx \int d^4x_1 d^4x_2 e^{ik_1'x_1 - ik_2'x_2} \langle 0 | T \left\{ \left[ -\frac{f_H^2}{4} Tr(p_-\Gamma'' p_+ T^{ab} 2\mathcal{H}(x_1)) \right] \times \left[ -\frac{f_H^2}{4} Tr(p_+\Gamma' p_+ T^{ca} 2\mathcal{H}(x_2)) \right] \right\} | 0 \rangle
\]
\[
\approx \frac{f_H^2}{2} tr (p_-\Gamma'' p_+ \Gamma') \frac{1}{v.k_1' + i\epsilon} \frac{1}{v.k_2' + i\epsilon}.
\]
(4.19)

Notice at this point that we may obtain (3.9) and (3.10) from (4.18) and (4.19) by taking \(f_H^2/2 \to N_c \mu^2/6\pi^2\). Hence \(f_H^2\) at the hadronic level plays the role of the cut-off \(\mu\) at quark level. Observe also that the dependence on the \(\Gamma\)-matrices in (4.18)-(4.19) is explicit. All decay constants and matrix elements of bilinear currents are given in terms of the only non-perturbative parameter \(f_H\). This is a direct consequence of the \(U(4N_{hf})\) symmetry being spontaneously broken down to \(U(2N_{hf}) \otimes U(2N_{hf})\).

5. Example: the decay constant, \(f_\Upsilon\)

5.1. Separating and evaluating off-shell and on-shell contributions

Consider the current-current correlator
\[
G_\Gamma(p) := \int d^4x e^{ipx} \langle 0 | T \{ \bar{Q}^a \Gamma Q^b (0) \bar{Q}^b \Gamma Q^a (x) \} | 0 \rangle,
\]
\[
p = -m_{ab,n}v - k, \quad k \to 0.
\]
We separate
\[
\bar{Q}^a \Gamma Q^b = (\bar{Q}^a \Gamma Q^b)_{on} + (\bar{Q}^a \Gamma Q^b)_{off}.
\]
(5.2)

Where \((\bar{Q}^a \Gamma Q^b)_{on}\) and \((\bar{Q}^a \Gamma Q^b)_{off}\) means that both heavy quark fields in the current have momenta almost on-shell and off-shell respectively. Our goal is to obtain a representation
in terms of the HQET of any Green function containing an \(( \bar{Q}^a \Gamma Q^b )_{on} \). In order to enforce "on-shellness" it is convenient to make the substitution

\[
\int d^4 x (\bar{Q}^b \Gamma Q^a (x))_{on} e^{ipx} \rightarrow \int d^4 x_1 \bar{Q}^{b i_1}(x_1) e^{ip_1 x_1} \int d^4 x_2 Q^{a i_2}(x_2) e^{ip_2 x_2} (\Gamma)_{\alpha_1 \alpha_2} \delta_{i_1 i_2},
\]

(5.3)

\[
p_1 = -(m_a - \frac{m_a}{m_{ab}} E_{ab,n}) v - k'_1,
\]

\[
p_2 = -(m_b - \frac{m_b}{m_{ab}} E_{ab,n}) v - k'_2,
\]

(5.4)

\[
k = k'_1 + k'_2,
\]

\[
k'_1, k'_2 \rightarrow 0
\]

and see whether the new Green function admits a representation in terms of the HQET. This is nothing but the calculations carried out above. Then we undo (5.3) by putting the fields depending on \(x_1\) and \(x_2\) in the HQET at the same point \(x\). We have (from (2.1), (2.13) and (3.11))

\[
\int d^4 x e^{ipx} \langle 0 | T \{ (\bar{Q}^a \Gamma Q^b (0))_{off} (\bar{Q}^b \Gamma Q^a (x))_{on} \} | 0 \rangle
= \int d^4 x e^{-ikx} \langle 0 | T \{ C\bar{h}^a_+ \Gamma h^b_+ (0) \bar{h}^b_+ \Gamma h^a_+ (x) \} | 0 \rangle.
\]

(5.5)

Analogously, using (2.14), (2.16) and (3.13) we have

\[
\int d^4 x e^{ipx} \langle 0 | T \{ (\bar{Q}^a \Gamma Q^b (0))_{on} (\bar{Q}^b \Gamma Q^a (x))_{on} \} | 0 \rangle
= \int d^4 x e^{-ikx} \langle 0 | T \{ \bar{h}^a_+ \Gamma h^b_+ (0) \bar{h}^b_+ \Gamma h^a_+ (x) \}
\times i \int d^4 y \left( -\frac{1}{2N_c} \tilde{\Psi}_{ab,n}^*(0) \tilde{\Psi}_{ab,n} (0) \right) \sum_{\Gamma_n = i\gamma_{\alpha p_-}, i\gamma_{\beta p_-}} \bar{h}^b_\Gamma \Gamma_n h^a_\Gamma (y) i v.\partial(h^a_\Gamma \Gamma_n h^b_\Gamma (y)) \} | 0 \rangle.
\]

(5.6)

The contribution involving only off shell quarks has the familiar form

\[
\int d^4 x e^{ipx} \langle 0 | T \{ (Q^a \Gamma Q^b (0))_{off} (Q^b \Gamma Q^a (x))_{off} \} | 0 \rangle
= -i N_c tr(\Gamma p_+ \Gamma p_-) |\Psi_{ab,n}(0)|^2 \frac{1}{v.k + i\epsilon},
\]

(5.7)
The expressions (5.5) and (5.6) correspond to corrections $O(\Lambda_{QCD}^3 a_{ab,n}^3)$ and $O(\Lambda_{QCD}^6 a_{ab,n}^6)$ respectively to the leading result (5.7), $a_{ab,n} \sim n/(\alpha \mu_{ab})$ is the Bohr radius. Since we are only interested in the leading non-perturbative corrections we shall neglect (5.6) in the following. Let us only remark that the hadronization of the four quark operator in (5.6) introduces new parameters. This is because it is not a generator of the $U(4N_{hf})$ symmetry as the currents of the kind (4.17) are.

The r.h.s. of (5.5) can be hadronized and calculated using the effective lagrangian discussed in section 4. From (4.18) we obtain

$$\int d^4xe^{ipx}\langle 0|T\{(\bar{Q}^a \Gamma Q^b(0))_{off}(\bar{Q}^b \bar{\Gamma}Q^a(x))_{on}\}|0\rangle = -\frac{i}{2} tr(p_- \Gamma p_+ \bar{\Gamma}) \bar{\Psi}_{ab,n}^*(0) \Psi_{ab,n}(0) f_H^2 \frac{1}{v.k + i\epsilon}.$$ (5.8)

Notice that the result is spin independent and the flavor dependence resides only in the wave function, which is known. We finally obtain

$$|f_{\psi_Q(n)}|^2 = 4m_{ab,n} \left(N_c |\Psi_{ab,n}(0)|^2 + \frac{1}{2} \left( \bar{\Psi}_{ab,n}^*(0) \Psi_{ab,n}(0) + \bar{\Psi}_{ab,n}(0) \Psi_{ab,n}(0) \right) f_H^2 \right),$$

$$|f_{\eta_Q(n)}| = \frac{|f_{\psi_Q(n)}|}{m_{ab,n}}.$$ (5.9)

Notice that the non-perturbative correction we find to the decay constant is $O(\Lambda_{QCD}^3 a_{ab,n}^3)$ and hence presumably more important that the correction arising from the multipole expansion which is $O((\Lambda_{QCD} a_{ab,n})^4/\alpha^2)$ [8,9] (we count the quark condensate as $O(\Lambda_{QCD}^4)$).

### 5.2 Cut-off independence

Let us next discuss the important issue of the cut-off independence. Even though we have not written it down explicitly, the introduction of a cut-off to separate almost on-shell momenta from off-shell momenta is necessary. Of course, the final results must not depend on the particular value of the cut-off. At the short distance end of the calculation, the cut-off must exclude momenta which are almost on-shell. This is easily achieved by
cutting off small momenta from the wave function

\[ \Psi_{ab,n}(0) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \tilde{\Psi}_{ab,n}(\vec{k}) \rightarrow \int \frac{d^3 \vec{k}}{(2\pi)^3} \tilde{\Psi}_{ab,n}(\vec{k}) =: \Psi_{ab,n}^{(\mu)}(0), \quad (5.10) \]

where \( \mu \) is a symmetric IR cut-off in three momentum. The wave functions in (5.9) must be understood as the cut-off wave functions (5.10). On the HQET side the cut-off must be ultraviolet. It has already been displayed in the leading order perturbative calculation at quark level in section 3. In particular, from (3.9) we obtain

\[
\int d^4x e^{i p \cdot x}\langle 0| T \{(\bar{Q}^a \Gamma Q^b(0))_{off}(\bar{Q}^b \bar{\Gamma} Q^a(x))_{on}\}|0\rangle = -\frac{i}{2} N_c tr(p_- \Gamma p_+ \bar{\Gamma}) \tilde{\Psi}_{ab,n}^{*}(0) \Psi_{ab,n}(0) \frac{1}{v.k + i\epsilon}. \quad (5.11)
\]

This strong cut-off dependence, however, is totally compensated by (5.10). Indeed, once (5.10) is used we have

\[
\frac{d}{d\mu} |\Psi_{ab,n}^{(\mu)}(0)|^2 = -\frac{\mu^2}{2\pi^2} \left( \Psi_{ab,n}^{*}(\mu) \Psi_{ab,n}^{(\mu)}(0) + \Psi_{ab,n}^{(\mu)}(0) \tilde{\Psi}_{ab,n}(\mu) \right) = -\frac{\mu^2}{2\pi^2} \left( \tilde{\Psi}_{ab,n}^{*}(0) \Psi_{ab,n}(0) + \Psi_{ab,n}^{*}(0) \tilde{\Psi}_{ab,n}(0) \right.
\]

\[+ O((\mu a_{ab,n})^2) \right), \quad (5.12)\]

\[
\frac{d}{d\mu} \left( \tilde{\Psi}_{ab,n}^{*}(0) \Psi_{ab,n}^{(\mu)}(0) \frac{\mu^3}{6\pi^2} \right) = \tilde{\Psi}_{ab,n}^{*}(0) \Psi_{ab,n}(0) \frac{\mu^2}{2\pi^2} \left[ 1 + O((\mu a_{ab,n})^2) \right],
\]

\[
\frac{d}{d\mu} \left( \Psi_{ab,n}^{(\mu)}^{*}(0) \tilde{\Psi}_{ab,n}(0) \frac{\mu^3}{6\pi^2} \right) = \Psi_{ab,n}^{*}(0) \tilde{\Psi}_{ab,n}(0) \frac{\mu^2}{2\pi^2} \left[ 1 + O((\mu a_{ab,n})^2) \right].
\]

Notice that the way in which the cut-off dependence cancels is remarkable. The strong cut-off dependence of (5.11) was first found in [11]. It was not clear at all which short distance contribution it should cancel against. (5.10) gives the solution to that puzzle. It is apparent from (5.8) and (5.11) that \( f_H \) in the hadronic theory plays the role of the UV cut-off in the HQET at quark level. From (5.12) it is clear that the cut-off \( \mu \) must be
much smaller than the inverse Bohr radius. Therefore our formalism becomes exact in the following situation

\[ m_a, m_b \gg 1/a_{ab,n} \gg \mu \gg \Lambda_{QCD} \gg k', \]

\[ \frac{\mu_{ab}a^2(1/a_{ab,n})}{n^2} \gg k'. \]  

(5.13)

Furthermore, we have to assume that \( \mu \) can be taken large enough so that we may enter the asymptotic freedom regime from the HQET side. Otherwise the matching we have carried out at tree level would not make much sense.

From the discussion above it should also be clear that (5.9) can be written in a cut-off independent way at \( O((\mu a_{ab,n})^3) \) by just replacing

\[ f_H^2 \rightarrow \tilde{f}_H^2 := f_H^2 - \frac{N_c \mu^3}{3\pi^2}, \]

(5.14)

where \( \tilde{f}_H^2 \) need not be positive.

### 5.3. Physical state normalization

There is still a subtle point which makes eq. (5.9) with the replacement (5.14) not quite correct. It has to do with the normalization of physical states. It will be clear later on (see eq. (6.14) below) that the states we obtain by this procedure do not have the standard relativistic normalization as they are supposed to. When we evaluate the Green function (5.1) we insert resolutions of the identity which are approximated by Coulomb-type states. This is all right. However the low momentum tale of these states is cut-off and substituted by a quantity evaluated using the effective hadronic theory. After doing so there is no guarantee that the resolution of the identity we introduced is still properly normalized. This can be fixed up by changing

\[ \sum_n \int \frac{d^3 \vec{P}_n}{(2\pi)^3 2P_n^0} |n\rangle\langle n| \rightarrow \sum_n \int \frac{d^3 \vec{P}_n}{(2\pi)^3 2P_n^0} |n\rangle\langle n|^{(\mu)} N_n(\mu, f_H) \]

(5.15)

where \( |n\rangle\langle n|^{(\mu)} \) symbolises the cut-off Coulomb states whose low energy tale is evaluated in the hadronic effective theory. We present a heuristic calculation of \( N_n(\mu, f_H) \).
We start from the Coulomb-type bound state (2.5) and separate high and low relative momentum according to

\[
|\Gamma_n, \vec{P}_n = m_{ab,n} \vec{v} \rangle = |\Gamma_n, \vec{P}_n = m_{ab,n} \vec{v} \rangle^{k>\mu} + |\Gamma_n, \vec{P}_n = m_{ab,n} \vec{v} \rangle^{k<\mu}
\]

(5.16)

The high momentum part of the physical state can be well approximated by the Coulomb-type contribution so we may leave it as it stands. However, the low momentum part receives non-perturbative corrections, which we evaluate using the effective hadronic lagrangian.

We proceed as follows. Since \( a_{ab,n} \mu \ll 1 \), we can approximate the low momentum region by

\[
|\Gamma_n, \vec{P}_n = m_{ab,n} \vec{v} \rangle^{k<\mu} \simeq \frac{v^0}{\sqrt{N_c}} \frac{\tilde{\Psi}_{ab,n}(\vec{0})}{\sqrt{2m_av^02m_bv^0}} \frac{m_{ab}^3}{m_{ab,n}} \int \frac{d^3k}{(2\pi)^3} \sum_{\alpha,\beta} \bar{u}^\alpha(p_1) \Gamma_n v^\beta(p_2) \bar{a}^\dagger_\alpha(p_1) \bar{b}^\dagger_\beta(p_2) |0\rangle,
\]

(5.17)

Observe now that (5.17) is nothing but the integral of a local HQET current.

\[
|\Gamma_n, \vec{P}_n = m_{ab,n} \vec{v} \rangle^{k<\mu} \simeq \frac{v^0}{\sqrt{N_c}} \tilde{\Psi}_{ab,n}(\vec{0}) \frac{m_{ab}^3}{m_{ab,n}} \int d^3xe^{-ikx} \tilde{h}_a^{\dagger} \Gamma_n \tilde{h}_b(x) |0\rangle,
\]

(5.18)

where \( k \to 0 \) and only low momenta are allowed.

At this point, we can hadronize the current (see (4.17)) and calculate the low momentum contribution to \( N_n(\mu, f_H) \)

\[
k<\mu|\Gamma_n, \vec{P}_n = m_{ab,n} \vec{v}|\Gamma_n, \vec{P}_n' = m_{ab,n} \vec{v}'\rangle^{k<\mu}
\]

\[
= 2m_{ab,n} v^0(2\pi)^3 \delta(3)(m_{ab,n}(\vec{v} - \vec{v}')) \frac{f_H^2}{2N_c} |\tilde{\Psi}_{ab,n}(\vec{0})|^2.
\]

(5.19)

Then, putting together high and low momentum contributions, we have

\[
\langle \Gamma_n, \vec{P}_n = m_{ab,n} \vec{v} | \Gamma_n, \vec{P}_n' = m_{ab,n} \vec{v}' \rangle
\]

\[
= 2m_{ab,n} v^0(2\pi)^3 \delta(3)(m_{ab,n}(\vec{v} - \vec{v}')) \left[ \int_{k>\mu} \frac{d^3k}{(2\pi)^3} |\tilde{\Psi}_{ab,n}(\vec{k})|^2 + \frac{f_H^2}{2N_c} |\tilde{\Psi}_{ab,n}(\vec{0})|^2 \right] + \left[ 1 + \frac{f_H^2}{2N_c} |\tilde{\Psi}_{ab,n}(\vec{0})|^2 \right].
\]

(5.20)
Where $f_H^2$ is defined in (5.14) and notice that the result is cut-off independent.

Finally, the normalization factor reads

$$N_n(\mu, f_H) = \frac{1}{1 + \frac{f_H^2}{2N_c}|\tilde{\Psi}_{ab,n}(0)|^2}. \quad (5.21)$$

$N_n(\mu, f_H)$ can also be obtained from requiring

$$\langle \Gamma_n, \vec{P}_n = m_{ab,n}v | \int d^3\bar{x}\bar{Q}^b\gamma^0Q^b(\bar{x})|\Gamma_n, \vec{P}_n' = m_{ab,n}v' \rangle = 2m_{ab,n}v^0(2\pi)^3\delta^{(3)}(m_{ab,n}(\bar{v} - \bar{v}')) \quad (5.22)$$

as we shall see later on. Once we have taken into account the correct normalization (5.9) reads

$$|f_{\psi(n)}|^2 = 4m_{ab,n} \left[N_c|\Psi_{ab,n}(0)|^2 \right. \left. + \frac{1}{2} \left( \tilde{\Psi}^*_{ab,n}(0)\Psi_{ab,n}(0) + \Psi^*_{ab,n}(0)\tilde{\Psi}_{ab,n}(0) \right)f_H^2 \right. \left. - |\Psi_{ab,n}(0)|^2|\tilde{\Psi}_{ab,n}(0)|^2\frac{f_H^2}{2} \right] \quad (5.23)$$

We shall relegate to section 7 the discussion on the applicability of the limit (5.13) and formula (5.23) to physical situations.

6. Matrix elements at zero recoil

We are interested in Green functions of the kind

$$G_{\Gamma\Gamma'\Gamma''}(p_1, p_2) = \int d^4x_1d^4x_2e^{ip_1x_1+ip_2x_2}\langle 0|T \{ Q^a\Gamma''Q^b(\bar{x}_1)\bar{Q}^b\Gamma'Q^c(0)\bar{Q}^cQ^a(\bar{x}_2) \}|0 \rangle. \quad (6.1)$$

For the momentum range

$$p_1 = m_{ab,n}v + k'_1, \quad p_2 = -m_{ac,m}v - k'_2.$$  

$$k'_1, k'_2 \rightarrow 0 \quad (6.2)$$
We separate each current in almost on-shell momenta and off-shell momenta according to (5.2). The leading contribution is given by the term
\[
G_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2) = \int d^4x_1 d^4x_2 e^{ip_1 x_1 + ip_2 x_2} \langle 0 | T \left\{ (\tilde{Q}^a \Gamma'' Q^b(x_1))_{\text{off}} (\bar{Q}^b \Gamma Q^c(0))_{\text{off}} (\bar{Q}^c \Gamma' Q^a(x_2))_{\text{off}} \right\} | 0 \rangle,
\]
and the next to leading contribution by the term
\[
G^\text{on}_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2) := G^\text{on,1}_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2) + G^\text{on,2}_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2) + G^\text{on,3}_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2),
\]
\[
G^\text{on,1}_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2) = \int d^4x_1 d^4x_2 e^{ip_1 x_1 + ip_2 x_2} \langle 0 | T \left\{ (\tilde{Q}^a \Gamma'' Q^b(x_1))_{\text{on}} (\bar{Q}^b \Gamma Q^c(0))_{\text{off}} (\bar{Q}^c \Gamma' Q^a(x_2))_{\text{off}} \right\} | 0 \rangle,
\]
\[
G^\text{on,2}_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2) = \int d^4x_1 d^4x_2 e^{ip_1 x_1 + ip_2 x_2} \langle 0 | T \left\{ (\tilde{Q}^a \Gamma'' Q^b(x_1))_{\text{off}} (\bar{Q}^b \Gamma Q^c(0))_{\text{on}} (\bar{Q}^c \Gamma' Q^a(x_2))_{\text{off}} \right\} | 0 \rangle,
\]
\[
G^\text{on,3}_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2) = \int d^4x_1 d^4x_2 e^{ip_1 x_1 + ip_2 x_2} \langle 0 | T \left\{ (\tilde{Q}^a \Gamma'' Q^b(x_1))_{\text{off}} (\bar{Q}^b \Gamma Q^c(0))_{\text{off}} (\bar{Q}^c \Gamma' Q^a(x_2))_{\text{on}} \right\} | 0 \rangle.
\]
The calculation of (6.5) and (6.7) is analogous to the ones carried out in section 2. We obtain
\[
G^\text{on,1}_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2) = \int d^4x_1 d^4x_2 e^{ik'_1 x_1 - ik'_2 x_2} i C_1 \langle 0 | T \left\{ \bar{h}^a \Gamma'' h_+(x_1) \bar{h}^b \Gamma p_+ \Gamma' h^a_-(0) \right\} | 0 \rangle \times \int d^4q \frac{e^{iq x_2}}{v . q + i \epsilon},
\]
\[
C_1 = \Psi^*_{ac,m}(0) \bar{\Psi}_{ab,n}(0) \int \frac{d^3\vec{k}}{(2\pi)^3} \bar{\Psi}^*_{ab,n}(\vec{k}) \bar{\Psi}_{ac,m}(\vec{k}),
\]
\[
G^\text{on,3}_{\Gamma\Gamma^\prime\Gamma^\prime'}(p_1, p_2) = \int d^4x_1 d^4x_2 e^{ik'_1 x_1 - ik'_2 x_2} i C_3 \langle 0 | T \left\{ \bar{h}^a \Gamma'' p_+ \Gamma h^c_+(0) \bar{h}^c_+ \Gamma' h^a_-(x_2) \right\} | 0 \rangle \times \int d^4q \frac{e^{-iq x_1}}{v . q + i \epsilon},
\]
\[ C_3 = \bar{\Psi}_{ac,m}(0)\Psi_{ab,n}(0) \int \frac{d^3\vec{k}}{(2\pi)^3} \bar{\Psi}_{ab,n}(\vec{k}) \bar{\Psi}_{ac,m}(\vec{k}). \]

Notice that (6.5) and (6.7) cannot be written in terms of local Green functions in the HQET. One propagator must be kept explicit.

The calculation of (6.6) is more subtle. We describe it in some detail in the Appendix B. We obtain

\[ G_{\Gamma \Gamma' \Gamma''}^{on,1}(p_1, p_2) = \int d^4x_1 d^4x_2 e^{ik_1^i x_1 - ik_2^i x_2} C_2 \langle 0| T \{ \bar{h}_a \Gamma'' h^b (x_1) \bar{h}_+^b \Gamma h^c_+ (0) \bar{h}^c_+ \Gamma' h^a_-(x_2) \} |0 \rangle, \]

(6.10)

\[ C_2 = \bar{\Psi}_{ac,m}(0)\Psi_{ab,n}(0) \bar{\Psi}_{ac,m}(0)\bar{\Psi}_{ab,n}(0). \]

This term is the only one in (6.4) which remains in the matrix elements (see (6.14) below).

We calculate (6.8)-(6.10) using the hadronic effective lagrangian (see formulas (4.18) and (4.19)). We obtain

\[ G_{\Gamma \Gamma' \Gamma''}^{on,2}(p_1, p_2) = \int d^4x_1 d^4x_2 e^{ik_1^i x_1 - ik_2^i x_2} C_2 \langle 0| T \{ \bar{h}_a \Gamma'' h^b (x_1) \bar{h}_+^b \Gamma h^c_+ (0) \bar{h}^c_+ \Gamma' h^a_-(x_2) \} |0 \rangle, \]

(6.11)

\[ G_{\Gamma \Gamma' \Gamma''}^{on,3}(p_1, p_2) = \int d^4x_1 d^4x_2 e^{ik_1^i x_1 - ik_2^i x_2} C_2 \langle 0| T \{ \bar{h}_a \Gamma'' h^b (x_1) \bar{h}_+^b \Gamma h^c_+ (0) \bar{h}^c_+ \Gamma' h^a_-(x_2) \} |0 \rangle, \]

(6.12)

The matrix element at zero recoil then reads

\[ \langle \Gamma_n, \vec{p}_n = m_{ab,n} \vec{v}|Q^b T Q^c(0)|\Gamma_m, \vec{p}_m = m_{ac,n} \vec{v} \rangle \]

\[ = -\sqrt{m_{ab,n}m_{ac,m}} tr(\Gamma_n \Gamma_m) \left( \int \frac{d^3\vec{k}}{(2\pi)^3} \bar{\Psi}_{ab,n}(\vec{k}) \bar{\Psi}_{ac,m}(\vec{k}) + \frac{f_H^2}{2N_c} \bar{\Psi}_{ac,m}(0) \bar{\Psi}_{ab,n}(0) \right), \]

(6.14)

\[ \Gamma_n = i\gamma_5 p_-, i\vec{\gamma} \cdot \vec{p}_- \text{ for the pseudoscalar and vector particle respectively. The integral in (6.14) must be understood with an infrared cut-off } \mu. \text{ From (6.14) it is apparent that our physical states are not properly normalized. Indeed, for } b = c \text{ and } \Gamma = \gamma^0 \text{ one should obtain (5.22) but one does not. The reason for this has been discussed at the end of sect.} \]
5. The solution consist of introducing the normalization factor $N_n(\mu, f_H)$ defined in (5.21). The properly normalized result reads

$$
\langle \Gamma_n, \vec{P}_n = m_{ab,n} \vec{v} | \bar{Q}^b \Gamma Q^c(0) | \Gamma_m, \vec{P}_m = m_{ac,m} \vec{v} \rangle = -\sqrt{m_{ab,n} m_{ac,m}} \text{tr} (\bar{\Gamma}_n \Gamma_m) \\
\times \left[ \int \frac{d^3 \vec{k}}{(2\pi)^3} \bar{\Psi}_{ab,n}^* (\vec{k}) \Psi_{ac,m} (\vec{k}) \left( 1 - \frac{f_H^2}{4N_c} |\bar{\Psi}_{ab,n} (0)|^2 - \frac{f_H^2}{4N_c} |\bar{\Psi}_{ac,m} (0)|^2 \right) \\
+ \frac{f_H^2}{2N_c} \bar{\Psi}_{ac,m} (0) \Psi_{ab,n}^* (0) \right],
$$

(6.15)

Notice that the non-perturbative correction depends only on a single parameter $f_H^2$ which may be extracted from the decay constants calculated in section 5. This is a non-trivial prediction which turns out to be a direct consequence of the $U(4N_h f) \otimes U(2N_h f)$ symmetry being spontaneously broken down to $U(2N_h f) \otimes U(2N_h f)$.

7. Applications

If the charm and bottom mass were large enough we could apply the results above to the physics of $\Upsilon$, $\eta_b$, $B_c$, $B_c^*$, $J/\Psi$ and $\eta_c$. (The top is believed to be too heavy to form hadronic bound states and will be ignored). We analyse in this section whether this is so or not. In the systems where the formalism actually applies, we are mainly interested in estimating the importance of the new non-perturbative contribution rather than in obtaining accurate results. The latter is a much harder task which is definitely beyond the scope of the present work.

Let us first focus on bottom. The fact that the almost 'on-shell' momentum excitations in heavy quarkonium are Goldstone modes [11] implies that the $\Upsilon$ and $\eta_b$ spectrum does not received additional non-perturbative contributions. We may then extract the bottom mass from the $\Upsilon$ mass by means of the formulas given in [8,9] which take into account the leading order in the multipole expansion. Since we have established a link between quarkonium and the HQET we can next use $m_b$ to extract $\Lambda$, the non-perturbative parameter relating the
mass of the $B$-meson to $m_b$. Moreover, taking into account that $\bar{\Lambda}$ is flavour independent, we may next extract the charm mass $m_c$. We summarize the results in the Table I.

In Table I the values we obtain for $m_b$ are about a 3% lower than those obtained in QCD sum rules [22] but compatible with a recent QCD-based evaluation [23] and with the lattice calculation [24]. The values we obtain for $\bar{\Lambda}$ are a bit lower but otherwise compatible with those extracted from QCD sum rules [6]. Our values for $m_c$ are again about a 6% lower than the typical values in QCD sum rules [22]. We should emphasize that our numbers in Table I are model independent.

We can next extract the non-perturbative parameter $\bar{f}_H^2$ from $f_T$ (this is done in Table II). We use the following formula

$$f_T = 2\sqrt{3m_T\tilde{\Psi}_{bb,0}(0)}$$

$$\times \left[ 1 + \frac{\tilde{\Psi}_{bb,0}(0)\bar{f}_H^2}{6\tilde{\Psi}_{bb,0}(0)} - \frac{|\tilde{\Psi}_{bb,0}(0)|^2\bar{f}_H^2}{12} - \frac{8\alpha(m_b)}{3\pi} + 8.77m_b^2 < B^2 > (\frac{a_{bb,0}}{2})^6 \right], \quad (7.1)$$

where the 1-loop QCD corrections and the leading correction from the multipole expansion [9] * are taken into account.

The numbers in Table II are very sensitive to the scale at which $\alpha$ is taken. Notice that we choose $\alpha = \alpha(1/a_{bb,0})$ in the Bohr radius and binding energy but $\alpha = \alpha(m_b)$ in the 1-loop perturbative correction included in (7.1). From Table II we see that for the actual values of $m_b$ and $\Lambda_{QCD} = 100, 150 MeV$ the 'on-shell' contribution $(\bar{f}_H)$ does not dominate over the condensate, but it is certainly sizeable. For $\Lambda_{QCD} = 200 MeV$ all corrections are about the same order and for any value of $m$ the 'on-shell' contribution dominates over the condensate.

Observe that the conditions (5.13), in particular $a_{bb,0}^{-1} \gg \mu \gg \Lambda_{QCD}$, may be considered as reasonable well fulfilled if we take the cut-off $\mu \sim 700$ MeV (see table III below).

Let us next turn our attention to charm. The charm mass is known not to be heavy enough as for the multipole expansion to work [8]. This means that the non-perturbative

* We use the formula given in ref. [9] which differs from the ones in ref. [8].
contribution overwhelms the perturbative one. Therefore any approximation whose leading order is a perturbative contribution, like our approach, will not be able to say much about charmonium. In particular, for the 'on-shell' contributions the difficulty lies on the second last condition in (5.13) being fulfilled. There is little room to accommodate the cut-off $\mu$ between the inverse Bohr radius and $\Lambda_{QCD}$ as should be clear from Table III. We refrain ourselves from giving any numbers for charmonium.

Unfortunately, the situation is not much better for the $B_c$, which has received considerable attention lately [25-27]. Nonetheless, once we have $\bar{f}_H^2$, we shall give some numbers in this case in Table IV.

From Table IV we see that for $\Lambda_{QCD} = 100, 150 \, MeV$ the contribution of the condensate is too large for the approach to be reliable. For $\Lambda_{QCD} = 200 \, MeV$ we are at the boundary of its validity since the 'on-shell' correction is large. We may thus give a rough estimate for $f_{B_c}$ only for $\Lambda_{QCD} \sim 200 \, MeV$, which turns out to be compatible with the estimate obtained by QCD sum rules [26], but about a 30\% lower than potential model estimates [27].

From the Table V it follows that the new non-perturbative contribution is not very important in the matrix elements between $\Upsilon - B_c$ states.

The decay constants and matrix elements above receive contributions from corrections of several types:

(i) QCD perturbative corrections to the Coulomb potential $O(\alpha(1/a_n))$. These have been evaluated at one loop level in [28] (see also [23]).

(ii) Relativistic corrections to the Coulomb potential $O(\alpha(1/a_n))$ (see also [28,23]).

(iii) QCD perturbative corrections to the Green functions $O(\alpha(m))$. These corrections have been taken into account in (7.1). They correspond to the only QCD corrections in heavy-light systems. In our case they are important for the calculation of matrix elements at non-zero recoil.
(iv) Non-perturbative corrections arising from the multipole expansion in the off-shell momentum region $O(\Lambda_{QCD}^4 a_n^4/\alpha^2(1/a_n))$ [8,9]. These corrections have also been taken into account in (7.1).

(v) Finite mass corrections $O(\Lambda_{QCD}^2/m)$ in the hadronic HQET lagrangian.

8. Conclusions

We have demonstrated that, contrary to the common belief, HQET techniques are also useful for the study of systems composed of two heavy quarks. In particular, we have identified new nonperturbative contributions to the decay constants and to certain matrix elements which are described by a hadronic lagrangian based on the HQET. All these new contributions are parametrized at leading order by a single constant $f_H$. This is non-trivial and can be traced back to the fact that a $U(4N_{hf})$ symmetry is spontaneously broken down to $U(2N_{hf}) \otimes U(2N_{hf})$.

It is remarkable that strong cut-off dependences coming from a totally different origin match perfectly. Indeed, at the off-shell end the cut-off arises from an integral over a Coulomb type wave function, whereas at the on-shell end it arises from a Feynman integral.

We should also stress that we have been able to put in the same context (i.e. the HQET) both heavy-heavy and heavy-light systems. This allows for a model independent determination of heavy quark masses from quarkonium, which may then be used to extract the parameter $\bar{\Lambda}$ relating the mass of the heavy-light systems to the mass of the heavy quark.

As far as practical applications is concerned, our formalism is suitable for the ground state of the $\Upsilon$ and $\eta_b$ family. Unfortunately the charm mass is too small for the formalism to become applicable in general to $J/\Psi$ and $B_c$ systems. Nevertheless one may stretch it in some cases to obtain information on the mass and decay constant of the latter.
We have presented a technique which allows to disentangle the on-shell contributions from the rest and match them to the HQET. The matching has been carried out at tree level. We have already shown in [29] that the matching also goes through at one loop level. Nevertheless, a word of caution is needed. It would be desirable to have a more direct and systematic derivation of these results from QCD. Progress in this direction is being made [30].

Let us finally mention that the hadronic HQET lagrangian can easily incorporate heavy-light mesons. The formalism can then be extended to the calculation of matrix elements between quarkonium and heavy-light systems. The leading non-perturbative contributions to those are also given by $f_H$ and another non-perturbative parameter which is related to heavy-light decay constants. Non-recoil contributions can also be evaluated within the formalism.

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**Appendix A: a toy model**

Because of the similarity, both in physics and techniques, to the Chiral perturbation theory it is interesting to consider a toy model which contains the on-shell contributions only. At quark level the model is described by the HQET with quarks and antiquarks with the same velocity of sect. 3. At hadronic level is described by the effective hadronic lagrangian of sect. 4.
Within this model, the interactions between \((\eta Q, \eta Q'), (\eta Q, \psi Q'), (\psi Q, \psi Q')\), when the two particles move roughly at the same velocity, are described by a single unknown constant. This is analogous to the fact that at lowest order in 3-flavour chiral perturbation theory the elastic scattering of \((\pi, \pi), (K, K)\) and \((\pi, K)\) is also described by a single constant. When heavy-light mesons are included in the effective lagrangian the same constant describes the elastic scattering of heavy-light mesons with quarkonium. This is also analogous to the fact that the local vertex \(\pi-\pi-N-N\) at leading order in the Chiral Lagrangian is described by the same constant as the \((\pi, \pi)\) elastic scattering. Let us mention at this point that when one actually calculates the scattering amplitudes, one obtains zero. This has to do with the universality of the leading order effective lagrangians for Goldstone modes [18,19,20]. Any theory undergoing a \(U(4N_{hf})\) spontaneous symmetry breaking down to \(U(2N_{hf}) \otimes U(2N_{hf})\) has the same low energy effective lagrangian (4.12) provided the rest of the symmetries in the theory are also the same. It was shown in [11], that even when the gluons are switched off the spontaneous symmetry breaking occurs in the HQET. In that case there is no interaction in the fundamental theory and hence it is not surprising that the scattering amplitudes in the effective lagrangian vanish. Universality implies that there will be vanishing scattering amplitudes when the gluons are switched on as well.

Within this model one can also treat \(1/m\) corrections in a similar way that small quarks masses are dealt with in Chiral perturbation theory. At the quark level the leading \(1/m\) corrections to the HQET are given by a kinetic term

\[
- \sum_{a=1}^{N_{hf}} \frac{1}{2m_a} D_i \bar{h}^a D_i h^a \tag{A.1}
\]

and a spin breaking term

\[
\sum_{a=1}^{N_{hf}} \frac{1}{4m_a} \bar{h}^a S^l G^l h^a
\]

\[
G^l = -\frac{1}{2} \epsilon^{jkl} \epsilon_j^\mu \epsilon_k^\nu G_{\mu\nu}. \tag{A.2}
\]
The kinetic term (A.1) does not break the global $U(2N_{hf}) \otimes U(2N_{hf})$ symmetry but it breaks its local version. In order to construct at the hadronic level terms which break the $U(4N_{hf})$ symmetry in the same fashion as (A.1) does, we introduce the $u(4N_{hf})$-valued sources $\phi$ and $a_i$ transforming as

$$\phi \rightarrow g\phi g^{-1},$$  
$$a_i \rightarrow g a_i g^{-1} + g \partial_i g^{-1}. \quad (A.3)$$

Then the term

$$-d_i\bar{h}\phi d_ih,$$
$$d_ih := (D_i + a_i)h \quad , \quad d_i\bar{h}\phi := D_i\bar{h}\phi - \bar{h}\phi a_i \quad (A.4)$$

is on one hand invariant under $U(4N_{hf})$ and on the other reduces to (A.1) upon setting $a_i = 0$, $\phi = \begin{pmatrix} \frac{1}{2m_u} & \frac{1}{2m_b} & \cdots \end{pmatrix} \psi$. \quad (A.5)

At the hadronic level, we must then construct invariant terms linear in $\phi$, which may also contain $a_i$. Up to two space derivatives we have

$$tr(S\phi),$$
$$tr(S\phi d_i S d_i S), \quad (A.6)$$
$$tr(S\phi) tr(d_i S d_i S), \quad (A.7)$$
$$d_i S := \partial_i S + a_i S - S a_i. \quad (A.8)$$

We have not written down terms which coincide or vanish upon using (A.5).

For the spin breaking term (A.2) we may introduce a $u(4N_{hf})$-valued source $R^l$ transforming as

$$R^l \rightarrow g R^l g^{-1} \quad (A.9)$$

so that (A.2) is substituted by

$$\bar{h}\phi R^l G^l h. \quad (A.10)$$
We recover (A.2) upon setting
\[ R^l = \left( \frac{1}{4m_a} \quad \frac{1}{4m_b} \quad \ldots \right) \neq S^l. \]  

(A.11)

There are no terms at the hadronic level with the same symmetry transformation properties at lower orders in derivatives. The first possible term appears at third order.

Therefore the leading \( 1/m \) corrections introduce 3 new parameters. (A.6) provides a mass term \( O(\Lambda^2_{QCD}/m) \) and (A.7)-(A.8) give rise to the usual non-relativistic kinetic term. The procedure above can easily be extended to any finite order in \( 1/m \).

**Appendix B**

We present in this appendix some technical details on the evaluation of the off-shell short distance effects carried out in sect. 6.

Consider the following matrix element
\[ \langle \Gamma_n, m_{ab,n\vec{v}}|\bar{Q}^{b}\Gamma Q^{c}(x)|\Gamma_m, m_{ac,n\vec{v}} \rangle. \]  
(B.1)

Since two different bound states are involved, it is not clear \textit{a priori} which CM dependence one should subtract before using (2.5). Nevertheless, translation invariance implies that the result of the calculation must fulfill
\[ \langle \Gamma_n, m_{ab,n\vec{v}}|\bar{Q}^{b}\Gamma Q^{c}(x+a)|\Gamma_m, m_{ac,n\vec{v}} \rangle = e^{im_{ab,n\vec{v},a}} - e^{im_{ac,m\vec{v},a}} \times \langle \Gamma_n, m_{ab,n\vec{v}}|\bar{Q}^{b}\Gamma Q^{c}(x)|\Gamma_m, m_{ac,n\vec{v}} \rangle. \]  
(B.2)

We also have
\[ \langle \Gamma_n, m_{ab,n\vec{v}}|\bar{Q}^{b}\Gamma Q^{c}(x)|\Gamma_m, m_{ac,n\vec{v}} \rangle = e^{im_{ab,n\vec{v},\xi}} - e^{im_{ac,m\vec{v},\xi}} \times \langle \Gamma_n, m_{ab,n\vec{v}}|\bar{Q}^{b}\Gamma Q^{c}(x-\xi)|\Gamma_m, m_{ac,n\vec{v}} \rangle. \]  
(B.3)

If we assign \( \xi \to \xi + a \) under translations (B.3) fulfills (B.2). If we also require \( \xi \) to be a linear function of \( x \), then necessarily \( \xi = x \) and the result is well defined.

\[ \langle \Gamma_n, m_{ab,n\vec{v}}|\bar{Q}^{b}\Gamma Q^{c}(x)|\Gamma_m, m_{ac,n\vec{v}} \rangle = e^{im_{ab,n\vec{v},x}} - e^{im_{ac,m\vec{v},x}} \langle \Gamma_n, m_{ab,n\vec{v}}|\bar{Q}^{b}\Gamma Q^{c}(0)|\Gamma_m, m_{ac,n\vec{v}} \rangle \]
\[ = e^{im_{ab,n\vec{v},x}} - e^{im_{ac,m\vec{v},x}} \left( -\text{tr}(\bar{\Gamma}_n \Gamma_m) \int \frac{d^3k}{(2\pi)^3} \bar{\Psi}_{ab,n}(\vec{k}) \tilde{\Psi}_{ac,m} (\vec{k}) \right). \]
\[ (B.4) \]
Consider next
\[
\langle \Gamma_n, m_{ab,n} \vec{v} \mid \bar{Q}_{\alpha_3}^{b i_3}(x_3)Q_{\alpha_4}^{c i_4}(x_4) \mid \Gamma_m, m_{ac,m} \vec{v} \rangle .
\] (B.5)

We are in a similar situation as above. However now translation invariance does not fix completely the result. Under the same assumptions we obtain
\[
\langle \Gamma_n, m_{ab,n} \vec{v} \mid \bar{Q}_{\alpha_3}^{b i_3}(x_3)Q_{\alpha_4}^{c i_4}(x_4) \mid \Gamma_m, m_{ac,m} \vec{v} \rangle 
= e^{i(\alpha x_3 + (1 - \alpha) x_4)}(E_{ac,m} - E_{ab,n}) + im_b v.x_3 - im_c v.x_4 \left( -\frac{1}{N_c} (\Gamma_m \bar{\Gamma}_n)_{\alpha_4 \alpha_3} \delta_{i_3 i_4} \right)
\times \int \frac{d^3k}{(2\pi)^3} \bar{\Psi}_{ab,n}(\vec{k}) \bar{\Psi}_{ac,m}(\vec{k}) e^{i(\vec{k} \cdot \vec{v} x_3 - \vec{x}_3) - (\vec{x}_3 - \vec{x}_4)(\vec{k} + \frac{\vec{v}}{1 + \vec{v} \cdot \vec{v}})},
\] (B.6)

where \( \alpha \) is arbitrary and parametrizes the ambiguity. Usually one never runs into calculations of the kind (B.5) but rather of matrix elements of currents as in (B.1), which are not ambiguous. We find expressions like (B.5) in our calculation because we insist in enforcing "on-shellness" in certain currents. In our concrete case we have a current with a momentum insertion
\[
\bar{Q}^b T Q^c(x) e^{ip.x}, \quad p = (-m_b + m_c + E_{ab,n} - E_{ac,m})v .
\] (B.7)

In order to enforce on-shellness we substitute it by
\[
\bar{Q}_{\alpha_3}^{b i_3}(x_3)Q_{\alpha_4}^{c i_4}(x_4)e^{ip_3x_3 + ip_4x_4}(\Gamma)_{\alpha_3 \alpha_4} \delta_{i_3 i_4} ,
\] (B.8)

as mentioned in (5.3). However in doing so there is a momentum mismatch \((m_a/m_{ab})E_{ab,n} - m_a/m_{ac}E_{ac,m})v\) which should be fixed somehow in order to get back (B.7) in the \( x_3 = x_4 = x \) limit. The most general way of distributing this momentum mismatch between \( x_3 \) and \( x_4 \) is by inserting in (B.8)
\[
e^{i(\beta x_3 + (1 - \beta) x_4)(m_a/m_{ab})E_{ab,n} - m_a/m_{ac}E_{ac,m})v}.
\] (B.9)

Any \( \beta \) is equally good since we are eventually interested in the limit \( x_3 = x_4 = x \). Notice that the ambiguity in \( \alpha \) in (B.6) is proportional to the ambiguity in \( \beta \) in (B.9). Since we
can choose $\beta$ at will, we do it in such a way that the dependence in both $\alpha$ and $\beta$ cancels. This is how we are able to obtain a representation of (6.6) in terms of the HQET (6.10).

References

[1] M.B. Voloshin and M.A. Shifman, Yad. Fiz. 45 (1987) 463 [Sov. J. Nucl. Phys. 45 (1987) 292].
   H.D. Politzer and M.B. Wise, Phys. Lett. B206 (1988) 681; Phys. Lett. B208 (1988) 504.
[2] N. Isgur and M.B. Wise, Phys. Lett. B232 (1989) 113; Phys. Lett. B237 (1990) 527.
[3] E. Eichen and B. Hill, Phys. Lett. B234 (1990) 511.
[4] H. Georgi, Phys. Lett. B240 (1990) 447.
[5] B. Grinstein, Nucl. Phys. B339 (1990) 253.
[6] B. Grinstein, in Proceedings of the Workshop on High Energy Phenomenology, Mexico City Mexico, Jul. 1-12, 1991, 161-216 and in Proceedings Intersections between particle and nuclear physics Tucson 1991, 112-126.
   T. Mannel, Chinese Journal of Physics 31 (1993) 1.
   M. Neubert, ’Phys. Rep.’ 245 (1994) 259.
[7] R. Casalbuoni, A. Deandrea, N. Di Bartolomeo, R. Gatto, F. Ferruglio and G. Nar- dulli, Phys. Lett. B309 (1993) 163; Phys. Lett. B302 (1993) 95.
   M.A. Sanchis, Phys. Lett. B312 (1993) 333; Z. Phys. C 62 (1994) 271.
[8] H. Leutwyler, Phys. Lett. B98 (1981) 447.
[9] M. B. Voloshin, Nucl. Phys. B154 (1979) 365; Sov. J. Nucl. Phys., 36, 143 (1982).
[10] J. Soto and R. Tzani, Phys. Lett. B297 (1992) 358.
[11] J. Soto and R. Tzani, Int. J. Mod. Phys. A9 (1994) 4949.
[12] A. Das and V.S. Mathur, Phys. Rev. D49 (1994) 2508.
   A. Das and M. Hott, ’Chiral invariance of massive fermions’, UR-1352 preprint.
[13] T. Mannel, W. Roberts and Z. Ryzak, Nucl. Phys. B368 (1992) 204.
[14] A. F. Falk, H. Georgi, B. Grinstein and M.B. Wise, *Nucl. Phys.* **B343** (1990) 1.

[15] B. Grinstein, W. Kilian, T. Mannel and M. Wise, *Nucl. Phys.* **B363** (1991) 19.

W. Kilian, T. Mannel and T. Ohl, *Phys. Lett.* **B304** (1993) 311.

[16] W. Lucha, F.F. Schorbel and D. Gromes, *Phys. Rep.* **200** (1991) 127.

[17] F.E. Close and Z. Li, *Phys. Lett.* **B289** (1992) 143.

[18] S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **D177** (1969) 2239.

C. Callan, S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **D177** (1969) 2247.

[19] H. Leutwyler, *Ann. Phys.* **235** (1994) 165.

[20] H. Leutwyler, *Phys. Rev.* **D49** (1994) 3033.

[21] J. Gasser and H. Leutwyler, *Ann. Phys.* (N.Y.) **158** (1984) 142; *Nucl. Phys.* **B250** (1985) 465.

[22] S. Narison, *Lecture Notes in Physics*, Vol. 26, 'QCD Spectral Sum Rules' (World Scientific, Singapore, 1989).

[23] S. Titard and F.J. Ynduráin, *Phys. Rev.* **D49** (1994) 6007.

[24] C.T.H. Davis, K. Hornbostel, A. Langau, G.P. Lepage, A. Liedsey, C.J. Morningstar, J. Shigenitsu and J. Sloan, *Phys. Rev. Lett.* **73** (1994) 2654.

[25] E. Jenkins, M. Luke, A. V. Manohar and M. Savage, *Nucl. Phys.* **B390** (1993) 463.

[26] E. Bagan, H.G. Dosch, P. Gosdzinsky, S. Narison and J.-M. Richard, *Z. Phys.* **C64** (1994) 57.

[27] E.J. Eichten and C. Quigg, *Phys. Rev.* **D49** (1994) 5845.

V.V. Kiselev, A.K. Likhoded and A.V. Tkabladze, 'B_{c} spectroscopy', IHEP 94-51 preprint.

[28] J. Pantaleone, S.-H. Henry Tye and Y. J. Ng, *Phys. Rev.* **D33** (1986) 777.

[29] A. Pineda and J. Soto, *Phys. Lett.* **B361** (1995) 95.

[30] A. Pineda and J. Soto, in progress.

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Table I. We use $\Lambda_{QCD}$ as an input and take the one loop running coupling constant $\alpha$ at the scale of the invers Bohr radius, i.e. $\alpha = \alpha(1/a_{bb,0})$. For the gluon condensate we take the fix value $< B^2 > = (585\, MeV)^4$. The error in $m_b$ has been taken from estimations of the hiperfine splittings $O(\alpha^2)$, which are also the main source of error in $\bar{\Lambda}$. For $m_c$ the error comes both from $\bar{\Lambda}$ and the $1/m_c$ corrections. The last column gives our model independent determination of $m_{B_c}$.

Table II. We display the relative weight, with its sign, of the 1-loop ($\alpha(m_b)$), the condensate ($< B^2 >$) and the ’on-shell’ ($\bar{f}_H$) contribution with respect to the Coulomb type contribution (normalized to 1). The last columns display the mass $m_{cr}$ from which the ’on-shell’ contribution dominates over the condensate and the value of $(\bar{f}_H^2)^{1/3}$.

Table III. We give the $\bar{c}c, \bar{b}c, \bar{b}b$ invers Bohr radius as a function of $\Lambda_{QCD}$.

Table IV. We display the analogous to Table II for $B_c$. We have also given our predictions for $f_{B_c}$ in the last column.

Table V. We give the relative weight, with its sign, of the ’on-shell’ contribution with respect to the Coulomb-type contribution (normalized to 1) in the matrix elements (6.15) between $\Upsilon - B_c$ states.
Table I.

| $\Lambda_{QCD}$ (MeV) | $m_b$ (MeV) | $\Lambda$ (MeV) | $m_c$ (MeV) | $m_{B_c}$ (MeV) |
|------------------------|-------------|-----------------|-------------|-----------------|
| 200                    | 4877±35     | 436±35          | 1539±70     | 6212±110        |
| 150                    | 4843±35     | 470±35          | 1505±70     | 6242±110        |
| 100                    | 4802±35     | 511±35          | 1464±70     | 6312±110        |

Table II.

| $\Lambda_{QCD}$ (MeV) | $\alpha(m_b)$ | $<B^2>$ | $f_H$ | $m_{cr}$ (GeV) | $(f_H^2)^{1/3}$ (MeV) |
|------------------------|---------------|---------|-------|----------------|-----------------------|
| 200                    | -0.19         | 0.10    | -0.11 | —              | 260                   |
| 150                    | -0.17         | 0.19    | -0.08 | 90             | 210                   |
| 100                    | -0.15         | 0.41    | -0.12 | 160            | 210                   |

Table III.

| $\Lambda_{QCD}$ (MeV) | $1/a_{cc,0}$ (MeV) | $1/a_{bc,0}$ (MeV) | $1/a_{bb,0}$ (MeV) |
|------------------------|--------------------|--------------------|--------------------|
| 200                    | 630                | 790                | 1240               |
| 150                    | 540                | 700                | 1120               |
| 100                    | 450                | 590                | 980                |

Table IV.

| $\Lambda_{QCD}$ (MeV) | $\alpha(2\mu_{bc})$ | $<B^2>$ | $f_H$ | $f_{B_c}$ (MeV) |
|------------------------|----------------------|---------|-------|-----------------|
| 200                    | -0.24                | 0.35    | -0.44 | 370             |
| 150                    | -0.22                | 0.74    | -0.34 | 540             |
| 100                    | -0.19                | 1.93    | -0.54 | 780             |

Table V.

| $\Lambda_{QCD}$ (MeV) | $B_c - \Upsilon$ |
|------------------------|------------------|
| 200                    | -0.10            |
| 150                    | -0.08            |
| 100                    | -0.14            |