Generation of harmonics by a focused laser beam in the vacuum

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We consider generation of odd harmonics by a super strong focused laser beam in the vacuum. The process occurs due to the plural light-by-light scattering effect. In the leading order of perturbation theory, generation of \((2k + 1)\)th harmonic is described by a loop diagram with \((2k + 2)\) external incoming, and two outgoing legs. A frequency of the beam is assumed to be much smaller than the Compton frequency, so that the approximation of a constant uniform electromagnetic field is valid locally. Analytical expressions for angular distribution of generated photons, as well as for their total emission rate are obtained in the leading order of perturbation theory. Influence of higher-order diagrams is studied numerically using the formalism of Intense Field QED. It is shown that the process may become observable for the beam intensity of the order of \(10^{27}\) W/cm\(^2\).

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In a recent paper [1] T. Tajima and G. Mourou suggested a path to reach an extremely high intensity level \(10^{26}\)–\(10^{28}\) W/cm\(^2\) already in the coming decade, taking advantage of the megajoule facilities, see also Refs. [2, 3]. The field strength for such lasers will be very close to the characteristic QED value \(E_S = m^2c^3/\hbar = 1.32 \times 10^{16}\) V/cm, and thus nonlinear QED vacuum polarization effects will become measurable. Some of them have been already studied in literature. Among those, harmonic generation by an intense laser beam propagating in an external magnetic field [4], and by two colliding beams in the vacuum [5]. Recently the experimental feasibility of the light-by-light stimulated scattering via three laser beams has been studied in Ref. [6]. Pair creation by a focused laser pulse in the vacuum was considered in Ref. [7], and by two colliding pulses in Ref. [8]. Other references could be found in the recent review [9]. In this letter, we consider the effect of odd harmonics generation by a strong focused laser beam in the vacuum. Clearly, the probability of this process is smaller compared to an analogous light-by-light scattering effect in a combination of laser beams [10]. However it is of great importance, since the superhigh laser intensities can be achieved only in focused and very short laser pulses, and hence the harmonics generation effect will necessarily be present at any facility producing pulses with the peak field strength of the order of \(E_S\).

The process under consideration is described by the Feynman diagram depicted in Fig. 1. In the two leading orders of perturbation theory the effect is represented by diagrams in Fig. 1b,c. The diagrams without outgoing laser photons do not contribute since they would correspond to creation of an extraneous photon by a group of laser photons propagating in one direction. As is well known, such photons do not interact. For the fields weakly varying at distances of the order of the Compton length \(l_C\), and time intervals \(l_C/c\), nonlinear vacuum polarization effects can be described using the Heisenberg-Euler radiative correction to the electromagnetic field Lagrangian density \[10\] [11], which can be represented by the following asymptotic expansion

\[
\mathcal{L}' = \alpha \left\{ \frac{4\mathcal{F}^2 + 7\mathcal{G}^2}{360\pi^2 E_S^2} + \frac{\mathcal{F} (8\mathcal{F}^2 + 13\mathcal{G}^2)}{630\pi^2 E_S^4} + \ldots \right\},
\]

\(\alpha = e^2/\hbar c\) is the fine structure constant, \(\mathcal{F} = (\mathbf{E}^2 - \mathbf{H}^2)/2\) and \(\mathcal{G} = \mathbf{E} \cdot \mathbf{H}\) are the invariants of the electromagnetic field\(^1\). The relation \(\lambda \gg l_C\), where \(\lambda\) is the wave length of laser radiation, is valid for any of existing or projected lasers (with a possible exception of \(\gamma\), or hard X-ray lasers). Thus the Lagrangian density \[10\] can be evidently applied for description of focused laser fields.

The Lagrangian density \[10\] generates effective vacuum polarization \(\mathbf{P} = \partial \mathcal{L}'/\partial \mathbf{E}\) and magnetization \(\mathbf{M} = -\partial \mathcal{L}'/\partial \mathbf{H}\) and the effective vacuum charge and current densities

\[
\rho = -\nabla \cdot \mathbf{P}, \quad \mathbf{j} = \left( \frac{\partial \mathbf{P}}{\partial t} - \nabla \times \mathbf{M} \right).
\]

Then the modified vacuum Maxwell equations in the Lorentz gauge are reduced to the form

\[
\Box A_\mu = 4\pi \rho_\mu(\mathbf{E}, \mathbf{H}).
\]

\(^1\) From now on, the natural units \(\hbar = c = 1\) are used.
The vacuum current \( \mathbf{J}_v \) contains the fine structure constant \( \alpha \). Therefore we will look for a solution to Eq. \( 3 \) (as well as the fields \( \mathbf{E}, \mathbf{H} \)) in the form of a superposition of functions \( \mathbf{a} \) (or fields \( \mathbf{E}, \mathbf{H} \)) in the form of a superposition of \( e^{i\mathbf{k}_0 \cdot \mathbf{r} + \phi} \), where the fields with zero index obey the classical vacuum Maxwell equations. For the field in the zeroth-order approximation we will use the model of a focused laser beam developed in Ref. \( 13 \). This model is based on an exact solution of Maxwell equations and was successfully employed to explain experimental results on scattering of relativistic electrons by a focused laser beam \( 14 \). A similar model was used in Ref. \( 15 \).

Following Ref. \( 13 \) we describe a focused laser beam as a superposition of monochromatic plane waves of the same frequency \( \omega \) and with wave vectors lying inside a cone with aperture angle \( 2\Delta \). The vector potential of the beam propagating along the \( z \) axis can be written then as

\[
\mathbf{A}^{(0)}(\mathbf{r}, z, t) = \int_{k_z < \omega} d^2k \mathbf{a}(k_\perp)e^{i(k_\perp \cdot \mathbf{r} + k_z z - \omega t)},
\]

where

\[
\mathbf{k} = k_\perp + k_z \mathbf{z}, \quad k_z = \sqrt{\omega^2 - k_\perp^2}, \quad \mathbf{k} \cdot \mathbf{a}(k_\perp) = 0.
\]

The configuration of the beam is determined by the function \( \mathbf{a}(k_\perp) \) and may vary strongly. However, independently of the form of \( \mathbf{a}(k_\perp) \), any polarized beam may be expressed as a superposition of \( e- \) and \( h- \) polarized waves, i.e., the waves respectively with the vectors \( \mathbf{E}^{(0)} = i\omega \mathbf{a}^{(0)} \) and \( \mathbf{H}^{(0)} = \nabla \times \mathbf{A}^{(0)} \) perpendicular to the propagation direction of the beam. We will choose the following form for functions \( \mathbf{a}^{e,h}(k_\perp) \) for \( e- \) and \( h- \) circularly polarized beams

\[
\mathbf{a}^{(h)}(k_\perp) = \frac{\mathbf{E}^{(0)}(k_\perp)}{4\sqrt{2}\pi\omega^5\Delta^4} \mathbf{k}_\perp \left( \frac{\mathbf{e} \cdot \mathbf{l}(0)}{\mathbf{k} \cdot \mathbf{l}(0)} \right) e^{-k_\perp^2/4\omega^2\Delta^2},
\]

where \( \mathbf{k}_\perp = \mathbf{k} \times \hat{\mathbf{z}}/k_z, \mathbf{l}(k) = (\mathbf{k} \times \hat{\mathbf{z}})/\omega \) and \( \mathbf{e} = (\cos \phi \mathbf{z} + \sin \phi \mathbf{y})/\sqrt{2} \). The resulting expressions for the fields can be found in Refs. \( 15 \). They describe a focused beam with the focal spot of the characteristic radius \( R = 1/\omega \Delta \) and the diffraction length \( L = 1/\omega \Delta^2 \). It will be important for us later on in this Letter that the fields acquire the following structure

\[
\mathbf{E}^{(0)}, \mathbf{H}^{(0)} = E^{(0)}_0 e^{-i\varphi} \mathbf{F}(\xi, \chi),
\]

where \( \varphi = \omega(t - z), \xi = r_\perp/R, \chi = z/L \). The quantity \( E^{(0)}_0 \) is the amplitude of electric (magnetic) field at the focal axis of the beam. It is worth noting that the type of polarization of the beam is determined by the field configuration near the focal axis (see Refs. \( 13 \) for details).

For \( \Delta \ll 1 \) the integral in \( 11 \) can be evaluated analytically. The formulas for the field invariants \( F, G \) can be found in Ref. \( 12 \). Here we give explicit expressions only for the invariants \( F, G \) in the plane focal plane \( z = 0 \) (\( \phi \) is the polar angle in that plane),

\[
F^c = E^2_0 \Delta^2 e^{-2\xi^2} \left\{ -8\xi^6 + 32\xi^4 + 6\xi^2 \cos[2(\omega t + \phi)] - 28\xi^2 + 4 \right\},
\]

\[
G^c = 2E^2_0 \Delta^2 \xi^2 e^{-2\xi^2} \sin[2(\omega t + \phi)],
\]

the upper (lower) sign corresponds to the clockwise- (counterclockwise-) polarized waves. The fields in \( e- \) and \( h- \) polarized waves are related as

\[
\mathbf{E}^{(0)} = \mathbf{E}^{(0)h} = -\mathbf{H}^{(0)} \quad \mathbf{H}^{(0)h} = -\mathbf{E}^{(0)}.
\]

It is well known, see, e.g., Ref. \( 16 \), that Fourier components of magnetic strength of the radiation emitted by the vacuum polarization currents.

It is convenient to expand the fields \( \mathbf{E}, \mathbf{H} \) generated by vacuum currents in a Fourier series, e.g., \( \mathbf{H} = \sum_r \mathbf{H}^\prime(r)e^{-i\omega t} \). It is well known, see, e.g., Ref. \( 17 \), that Fourier components of magnetic strength of the radiation field in the wave zone are given by

\[
\mathbf{H}^\prime(r) = \frac{i\hbar \alpha e^{i\omega t}}{\gamma T} \int_0^T dt \int d^3r e^{i\omega(t' - nr)} \mathbf{n} \times \mathbf{j}(r', t'),
\]

\( \mathbf{n} = r/r \). Using Eqs. \( 2 \) we reduce Eq. \( 8 \) after integration by parts to the form

\[
\mathbf{h}^\prime = \frac{\omega^2}{r} e^{i\omega t} \left( h^{(1)}_l(n) + h^{(1)}_l(n) \right),
\]

\[
h^{(i)}_l(t', r') = \left( \frac{\partial \mathbf{e}'}{\partial \mathbf{F}} \right)_0 l_1(n) \cdot \left( \mathbf{E}^{(0)} + \mathbf{n} \times \mathbf{H}^{(0)} \right)
\]

\[
- \left( \frac{\partial \mathbf{e}'}{\partial \mathbf{F}} \right)_0 l_1(n) \cdot \left( \mathbf{H}^{(0)} - \mathbf{n} \times \mathbf{E}^{(0)} \right).
\]

The angular distribution of the radiation power at frequency \( \omega l \) is given by \( dI = |\mathbf{H}^{(1)}|^2 d\Omega_{\mathbf{k}}/2\pi \), see Ref. \( 10 \). To obtain the rate \( P_1 \) of photon emission we should divide this quantity by the photon frequency \( \omega l \) and then integrate it over the spatial angle,

\[
P_1 = \frac{\omega^3}{2\pi} \int d\Omega \langle |h^{(1)}_l|^2 + |h^{(1)}_l|^2 \rangle.
\]
We arrive at the same result computing the Feynman diagram Fig. 1, where the bold line corresponds to the exact electron propagator in a locally constant electromagnetic field [12].

Expanding the vacuum current in a Fourier series, we represent the probability of photon emission as a sum of partial probabilities, each with its own conservation law corresponding to conversion of some number \( n \) of laser photons into \( n' \) laser photons \((n' < n)\) and an extraneous one with the frequency \( \omega' = \omega \). It immediately follows from the Furry theorem that the frequency multiplier \( k \) can be an odd number only, \( l = 2k + 1 \). This can be seen also from Eq. (11), since the derivatives of the effective Lagrangian (1) with respect to the invariants are even functions of electromagnetic field strengths.

Let us estimate now the rate of photon generation by under assumption \( \Delta \ll 1 \). First we note, that due to Eq. (8), \( \tilde{h}_l^{(i)}(t', \mathbf{r}') \) is a function of variables \( \varphi', \xi', \phi', \chi' \). Passing on to these variables, we write down Eq. (10) in the integral (4) are of the order \( k \) under assumption \( \Delta \). We have

\[
\tilde{h}_l^{(i)}(t', \mathbf{r}') = \frac{V_f}{2\pi} \int d\varphi' \int \xi' d\xi' d\phi' d\chi' \tilde{h}_l^{(i)}(\varphi', \xi', \phi', \chi') \times \exp \left\{ ilc\varphi' - il\frac{\sin \theta}{\Delta} \xi' \sin \phi' + il \frac{1 - \cos \theta}{\Delta^2} \chi' \right\},
\]

where \( \theta \) is the azimuthal angle of the emitted photon, \( \cos \theta = (\mathbf{n} \cdot \mathbf{z}) \). It can be easily be seen from Eq. (5), the effective values of the variable of integration \( k_\perp \) in the integral (4) are of the order \( k_\perp \sim \omega \Delta \) when \( \Delta \ll 1 \). Then, it can be shown using Eqs. (5) and the definitions of the fields \( \mathbf{E}^{(0)}, \mathbf{H}^{(0)} \), that the combinations

\[
\mathbf{l}_i(\mathbf{n}) \cdot \left( \mathbf{H}^{(0)} - \mathbf{n} \times \mathbf{E}^{(0)} \right), \quad \mathbf{l}_i(\mathbf{n}) \cdot \left( \mathbf{E}^{(0)} + \mathbf{n} \times \mathbf{H}^{(0)} \right),
\]

which appear in Eq. (11), are of the order \( E_0^{(0)} \Delta^2 \) for both \( e^- \) and \( h^- \) polarized waves. Both invariant \( \mathcal{F} \) and \( \mathcal{G} \) are \( \sim (\Delta E_0^{(0)})^2 \) in the focal region, see Eq. (7). Then, it follows from Eq. (11) that the derivatives \( \partial \mathcal{L}/\partial \mathcal{F}, \partial \mathcal{L}/\partial \mathcal{G} \) are of the form \( \alpha g(\vartheta; \varphi', \xi', \phi', \chi') \) near the focus, where \( g \) is a dimensionless function, and the dimensionless parameter \( \vartheta \) is defined as \( \vartheta = \Delta E_0^{(0)}/E_S \). Thus the coefficients \( k_1^{(i)} \) in Eq. (10) acquire at \( \Delta \ll 1 \) the form

\[
k_1^{(i)} = \alpha V_f E_0^{(0)} \Delta^2 g_{ii}(v, \vartheta), \quad \text{where} \quad v = \theta/\Delta.
\]

We have taken into account the fact that photons are emitted basically with \( \vartheta \leq \Delta \) when \( \Delta \ll 1 \), as it can be seen from Eq. (12). Substituting this estimate into Eq. (11) we obtain for the number of photons generated per one period, \( N_l = P/T \):

\[
N_l \sim \omega^2 \alpha^2 (V_f)^2 (E_0^{(0)} \Delta^2)^2 \Delta^2 \times \sum_i \int_0^\infty dv |g_{ii}(v, \vartheta)|^2 = \alpha \left( \frac{m}{\omega \Delta} \right)^4 \vartheta^2 f_i(\vartheta).
\]
$K'$, where the collision is head-on, the probability of our process per unit time and unit volume can be estimated using Eq. (4.38) of Ref. [5]. Since the probability is invariant, we have in the laboratory frame

$$\frac{dN_2^{(E-H)}}{dV dt} \sim \alpha^2 \omega' \left( \frac{E'_0}{E_S} \right)^8 dn_\psi. \quad (14)$$

Here $\omega'$ and $E'_0$ are the frequency and the peak field strength of the laser at the $K'$-frame. The velocity of the $K'$-frame is $v = \cos \psi \sim 1 - \psi^2/2$, and the corresponding Lorentz factor $\gamma \sim 1/\psi$. Hence, $\omega' = \omega \psi$ and $E'_0 = E_0 \psi$. The factor $dn_\psi$ arises due to the effect of stimulated photon emission, and has the meaning of the number of laser photons propagating at the interval of angles from $\psi$ to $\psi + d\psi$ relative to the direction of propagation of the laser beam. It is easy to see using (14) that $dn_\psi \sim |\mathbf{a}(k_\perp)|^2 dk_\perp^2$ with $k_\perp = \omega \psi$, and $\mathbf{a}(k_\perp)$ given by Eq. (9). After performing integration over $\psi$ in (14), and multiplying the result by the 4-volume of the focal region $V_T \sim 1/(\omega \Delta)^4$ we arrive, up to a numerical factor, to Eq. (13) with $l = 3$.

In the present letter we have studied the effect of harmonics generation by an intense focused laser pulse in vacuum. We have shown that for the laser pulse with $\lambda = 1 \mu m$ and $\Delta = 0.1$ the number of generated photons reaches the value of one photon per period at intensity $I \approx 5 \cdot 10^{27} W/cm^2 \approx 10^{-2} I_S$. The corresponding Lorentz factor is one order of magnitude less than the characteristic QED field $E_S$. This is explained by a very large value of the effective focal region as compared to the characteristic Compton 4-volume, $l_0^2/c$. It is very important that the rate of photon generation for $e$- and $h$-polarized waves are close to each other and reach an observable level at $\vartheta \approx 10^{-2}$. It follows from Ref. [5] that the effect of pair creation by a focused $e$-polarized laser pulse arises at the same values of $\vartheta$, while in an $h$-polarized wave production of pairs begins noticeably later. This means that in an $h$-polarized wave we will not have the background of secondary photons radiated by created particles, and hence, just such waves should be employed for observation of the effect of harmonics generation by a focused laser pulse in vacuum.

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