Bayesian reconstruction of the velocity distribution of weakly interacting massive particles from direct dark matter detection data

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Abstract. In this paper, we extended our earlier work on the reconstruction of the (time-averaged) one-dimensional velocity distribution of Galactic Weakly Interacting Massive Particles (WIMPs) and introduce the Bayesian fitting procedure to the theoretically predicted velocity distribution functions. In this reconstruction process, the (rough) velocity distribution reconstructed by using raw data from direct Dark Matter detection experiments directly, i.e. measured recoil energies, with one or more different target materials, has been used as “reconstructed-input” information. By assuming a fitting velocity distribution function and scanning the parameter space based on the Bayesian analysis, the astronomical characteristic parameters, e.g. the Solar and Earth’s Galactic velocities, will be pinned down as the output results.

Keywords: dark matter simulations, dark matter experiments

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Currently, direct Dark Matter detection experiments searching for Weakly Interacting Massive Particles (WIMPs) are one of the promising methods for understanding the nature of Dark Matter (DM) and identifying them among new particles produced at colliders as well as studying the (sub)structure of our Galactic halo [1–4].

In our earlier work [5], we developed methods for reconstructing the (moments of the) time-averaged one-dimensional velocity distribution of halo WIMPs by using the measured
recoil energies directly. This analysis requires no prior knowledge about the WIMP density near the Earth nor about their scattering cross section on nucleus, the unique required information is the mass of incident WIMPs. We therefore turned to develop the method for determining the WIMP mass model-independently by combining two experimental data sets with two different target nuclei [6]. By combining these methods and using two or three experimental data sets with different detector materials, one could reconstruct the one-dimensional velocity distribution of Galactic WIMPs directly. However, as presented in ref. [5], with a few hundreds or even thousands recorded WIMP events, only estimates of the reconstructed velocity distribution with pretty large statistical uncertainties at a few (< 10) points could be obtained.

Therefore, in order to offer more detailed information about the Galactic WIMP velocity distribution, we introduce in this paper the Bayesian analysis into our model-independent reconstruction procedure developed in ref. [5] to be able to determine, e.g. the position of the peak of the one-dimensional velocity distribution function and the concrete values of the characteristic Solar and Earth’s Galactic velocities.

The remainder of this paper is organized as follows. In section 2, we first review the model-independent method for reconstructing the time-averaged one-dimensional velocity distribution of halo WIMPs by using data from direct DM detection experiments directly. Then we introduce the Bayesian analysis and give the basic formulae needed in the extended reconstruction process. In section 3, we present numerical results of the reconstructed WIMP velocity distribution functions based on Monte-Carlo simulations for different generating and fitting velocity distributions. Different input WIMP masses as well as impure (pseudo-)data sets mixed with (artificially added) unrejected background events will also be considered. We conclude in section 4.

2 Formalism

In this section, we develop the formulæ needed for our Bayesian reconstruction of the one-dimensional velocity distribution function of halo WIMPs \( f_1(v) \) by using direct Dark Matter detection data directly.

We first review the model-independent method for reconstructing the time-averaged WIMP velocity distribution by using experimental data, i.e. measured recoil energies, directly from direct detection experiments. These “reconstructed data” (with estimated statistical uncertainties) will be used as input information for the further Bayesian analysis. Then, in the second part of this section, we review the basic concept of the Bayesian analysis and give the formulæ needed in our extended reconstruction procedure.

2.1 Model-independent reconstruction of one-dimensional WIMP velocity distribution

In this subsection, we review briefly the method for reconstructing the one-dimensional WIMP velocity distribution from experimental data directly. Detailed derivations and discussions can be found in ref. [5].

2.1.1 From the recoil spectrum

The basic expression for the differential event rate for elastic WIMP-nucleus scattering is given by [1]:

\[
\frac{dR}{dQ} = A F^2(Q) \int_{v_{\text{min}}}^{v_{\text{max}}} \left[ \frac{f_1(v)}{v} \right] dv .
\]  

(2.1)
Here $R$ is the direct detection event rate, i.e. the number of events per unit time and unit mass of detector material, $Q$ is the energy deposited in the detector, $F(Q)$ is the elastic nuclear form factor, $f_1(v)$ is the one-dimensional velocity distribution function of the WIMPs impinging on the detector, $v$ is the absolute value of the WIMP velocity in the laboratory frame. The constant coefficient $A$ is defined as $A$ ≡ $\rho_0\sigma_0 / 2m_\chi^2 r_{r,N}$, where $\rho_0$ is the WIMP density near the Earth and $\sigma_0$ is the total cross section ignoring the form factor suppression. The reduced mass $m_{r,N}$ is defined by $m_{r,N} = m_\chi m_N / (m_\chi + m_N)$, where $m_\chi$ is the WIMP mass and $m_N$ that of the target nucleus. Finally, $v_{\text{min}}$ is the minimal incoming velocity of incident WIMPs that can deposit the energy $Q$ in the detector: $v_{\text{min}} = \alpha \sqrt{Q}$ with the transformation constant

$$\alpha = \sqrt{\frac{m_N}{2m_{r,N}^2}} , \quad (2.2)$$

and $v_{\text{max}}$ is the maximal WIMP velocity in the Earth’s reference frame, which is related to the escape velocity from our Galaxy at the position of the Solar system, $v_{\text{esc}} \gtrsim 600 \text{ km/s}$. In our earlier work [5], it was found that, by using a time-averaged recoil spectrum $dR/dQ$ and assuming that no directional information exists, the normalized one-dimensional velocity distribution function of incident WIMPs, $f_1(v)$, can be solved from eq. (2.1) directly as

$$f_1(v) = N \left\{ -2Q \cdot \frac{d}{dQ} \left[ \frac{1}{F^2(Q)} \left( \frac{dR}{dQ} \right) \right] \right\}_{Q=\alpha^2/v^2} , \quad (2.3)$$

where the normalization constant $N$ is given by

$$N = \frac{2}{\alpha} \left\{ \int_0^\infty \frac{1}{\sqrt{Q}} \left[ \frac{1}{F^2(Q)} \left( \frac{dR}{dQ} \right) \right] dQ \right\}^{-1} . \quad (2.4)$$

Note that, because $f_1(v)$ in eq. (2.3) is normalized and the constant $A$ can be cancelled out, the velocity distribution function of halo WIMPs reconstructed by eq. (2.3) is independent of the local WIMP density $\rho_0$ as well as of the WIMP-nucleus cross section $\sigma_0$. However, not only the overall normalization constant $N$ given in eq. (2.4), but also the shape of the velocity distribution, through the transformation $Q = \alpha^2/v^2$ in eq. (2.3), depends on the WIMP mass $m_\chi$ (involved in the coefficient $\alpha$ defined in eq. (2.2)).

### 2.1.2 From experimental data directly

In order to avoid some model dependence during giving a functional form for the recoil spectrum $dR/dQ$ needed in eqs. (2.3) and (2.4), expressions that allow to reconstruct $f_1(v)$ directly from data (i.e. measured recoil energies) have been developed [5]. We started by considering experimental data described by

$$Q_n - \frac{b_n}{2} \leq Q_{n,i} \leq Q_n + \frac{b_n}{2} , \quad i = 1, 2, \ldots, N_n, \quad n = 1, 2, \ldots, B. \quad (2.5)$$

Here the entire experimental possible energy range between the minimal and maximal cut-offs $Q_{\text{min}}$ and $Q_{\text{max}}$ has been divided into $B$ bins with central points $Q_n$ and widths $b_n$. In each bin, $N_n$ events will be recorded.

Since the recoil spectrum $dR/dQ$ is expected to be approximately exponential [5], in order to approximate the spectrum in a rather wider range, instead of the conventional standard linear approximation, the following exponential ansatz for the measured recoil spectrum

$$dR/dQ = N \left\{ -2Q \cdot \frac{d}{dQ} \left[ \frac{1}{F^2(Q)} \left( \frac{dR}{dQ} \right) \right] \right\}_{Q=\alpha^2/v^2} . \quad (2.6)$$

where the normalization constant $N$ is given by

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(before normalized by the exposure $\mathcal{E}$) in the $n$th bin has been introduced [5]:

$$\left(\frac{dR}{dQ}\right)_{\text{expt},n} \equiv \left(\frac{dR}{dQ}\right)_{\text{expt},Q>Q_n} \equiv r_n e^{k_n(Q-Q_{s,n})}. \quad (2.6)$$

Here $r_n$ is the standard estimator for $\left(\frac{dR}{dQ}\right)_{\text{expt}}$ at $Q = Q_n$: $r_n = N_n/b_n$, $k_n$ is the logarithmic slope of the recoil spectrum in the $n$th $Q$-bin, which can be computed numerically from the average value of the measured recoil energies in this bin:

$$Q - Q_n \big|_n = \left(\frac{b_n}{2}\right) \coth \left(\frac{k_n b_n}{2}\right) - \frac{1}{k_n}, \quad (2.7)$$

where

$$\frac{(Q - Q_n)^\lambda}{N_n} \equiv \frac{1}{N_n} \sum_{i=1}^{N_n} (Q_{n,i} - Q_n)^\lambda. \quad (2.8)$$

Then the shifted point $Q_{s,n}$ in the ansatz (2.6), at which the leading systematic error due to the ansatz is minimal [5], can be estimated by

$$Q_{s,n} = Q_n + \frac{1}{k_n} \ln \left[ \sinh \left(\frac{k_n b_n/2}{k_n b_n/2}\right) \right]. \quad (2.9)$$

Note that $Q_{s,n}$ differs from the central point of the $n$th bin, $Q_n$.

Now, substituting the ansatz (2.6) into eq. (2.3) and then letting $Q = Q_{s,n}$, we can obtain that

$$f_{1,\text{rec}}(v_{s,n}) = \mathcal{N} \left[ \frac{2Q_{s,n}r_n}{F^2(Q_{s,n})} \right] \left[ \frac{d\ln F^2(Q)}{dQ} \bigg|_{Q=Q_{s,n}} - k_n \right]. \quad (2.10)$$

Here $v_{s,n} = \alpha \sqrt{Q_{s,n}}$ and the normalization constant $\mathcal{N}$ given in eq. (2.4) can be estimated directly from the data by

$$\mathcal{N} = \frac{2}{\alpha} \left[ \sum_a \frac{1}{\sqrt{Q_a F^2(Q_a)}} \right]^{-1}, \quad (2.11)$$

where the sum runs over all events in the sample.

2.1.3 Windowing the data set

In order to reduce the statistical uncertainty on the velocity distribution reconstructed by eq. (2.10) and some uncontrolled systematic errors caused by neglecting terms of higher powers of $Q - Q_n$, as well as to offer a reasonable number of reconstructible velocity points of $f_1(v)$, i.e. $v_{s,n}$, it has been introduced in ref. [5] that one can first collect experimental data in relatively small bins with linearly increased widths and then combining varying numbers of bins into overlapping “windows”. One starts by binning the data, as in eq. (2.5), where the bin widths satisfy $b_n = b_1 + (n - 1)\delta$, i.e.

$$Q_n = Q_{\text{min}} + \left( n - \frac{1}{2} \right) b_1 + \left[ \frac{(n - 1)^2}{2} \right] \delta. \quad (2.12)$$

Here the increment $\delta$ satisfies $\delta = 2(Q_{\text{max}} - Q_{\text{min}} - Bb_1)/B(B - 1)$, $B$ being the total number of bins, and $Q_{(\text{min},\text{max})}$ are the experimental minimal and maximal cut-off energies.
Assume up to \( n_W \) bins are collected into a window, with smaller windows at the borders of the range of \( Q \).

In order to distinguish the numbers of bins and windows, hereafter Latin indices \( n, m, \cdots \) are used to label bins, and Greek indices \( \mu, \nu, \cdots \) to label windows. For \( 1 \leq \mu \leq n_W \), the \( \mu \)th window simply consists of the first \( \mu \) bins; for \( n_W \leq \mu \leq B \), the \( \mu \)th window consists of bins \( \mu - n_W + 1, \mu - n_W + 2, \cdots, \mu \); and for \( B \leq \mu \leq B + n_W - 1 \), the \( \mu \)th window consists of the last \( n_W - (\mu - B) \) bins. This can also be described by introducing the indices \( n_{\mu-} \) and \( n_{\mu+} \) which label the first and last bins contributing to the \( \mu \)th window, with

\[
n_{\mu-} = \begin{cases} 1, & \text{for } \mu \leq n_W, \\ \mu - n_W + 1, & \text{for } \mu \geq n_W, \end{cases}
\]

and

\[
n_{\mu+} = \begin{cases} \mu, & \text{for } \mu \leq B, \\ B, & \text{for } \mu \geq B. \end{cases}
\]

The total number of windows defined through eqs. (2.13) and (2.14) is evidently \( W = B + n_W - 1 \), i.e. \( 1 \leq \mu \leq B + n_W - 1 \).

For a “windowed” data set, one can easily calculate the number of events per window as

\[
N_\mu = \sum_{n=n_{\mu-}}^{n_{\mu+}} N_n,
\]

as well as the average value of the measured recoil energies

\[
\frac{Q - Q_\mu}{|Q_n|} = \frac{1}{N_\mu} \left( \sum_{n=n_{\mu-}}^{n_{\mu+}} N_n \frac{Q_n}{|n|} \right) - Q_\mu,
\]

where \( Q_\mu \) is the central point of the \( \mu \)th window. The exponential ansatz in eq. (2.6) is now assumed to hold over an entire window. We can then estimate the prefactor as \( r_\mu = N_\mu/w_\mu \), \( w_\mu \) being the width of the \( \mu \)th window. The logarithmic slope of the recoil spectrum in the \( \mu \)th window, \( k_\mu \), as well as the shifted point \( Q_s,\mu \) (from the central point of each “window”, \( Q_\mu \)) can be calculated as in eqs. (2.7) and (2.9) with “bin” quantities replaced by “window” quantities. Finally, the covariance matrix of the estimates of \( f_1(v) \) at adjacent values of \( v_s,\mu = \alpha \sqrt{Q_{s,\mu}} \) is given by

\[
\begin{align*}
\text{cov} (f_{1,\text{rec}}(v_{s,\mu}), f_{1,\text{rec}}(v_{s,\nu})) &= \left[ f_{1,\text{rec}}(v_{s,\mu}) f_{1,\text{rec}}(v_{s,\nu}) \right] 
\frac{1}{r_\mu r_\nu} \text{cov} (r_\mu, r_\nu) + (2N)^2 \left[ \frac{Q_{s,\mu} Q_{s,\nu} r_\mu r_\nu}{F^2(Q_{s,\mu}) F^2(Q_{s,\nu})} \right] \text{cov} (k_\mu, k_\nu) \\
& - \mathcal{N} \left\{ \left[ f_{1,\text{rec}}(v_{s,\mu}) \right] \left[ \frac{2Q_{s,\nu} r_\nu}{F^2(Q_{s,\nu})} \right] \text{cov} (r_\mu, k_\nu) + (\mu \leftrightarrow \nu) \right\} .
\end{align*}
\]

\(^1\text{Note that eq. (2.17) should in principle also include contributions involving the statistical error on the estimator for } \mathcal{N} \text{ in eq. (2.11). However, this error and its correlations with the errors on the } r_\mu \text{ and } k_\mu \text{ have been found to be negligible compared to the errors included in eq. (2.17) [5].}\)
2.2 Bayesian analysis

In this subsection, we review the basic concept of the Bayesian analysis [7] (for its applications in physics, see e.g. ref. [8], and for its recent applications in direct DM detection phenomenology, see e.g. refs. [9–20]) and extend this method to the use of the “reconstructed data” offered by our model-independent reconstruction process described in the previous subsection.

2.2.1 Baye’s theorem

We start with the Baye’s theorem: the probability of $A$ given that $B$ is true multiplies the probability of that $B$ is true is equal to the probability of that $A$ and $B$ happen simultaneously, which is also equal to the probability of $B$ given that $A$ is true multiplies the probability of that $A$ is true. This can simply be expressed as

$$p(A|B) \cdot p(B) = p(A \cap B) = p(B|A) \cdot p(A),$$

(2.18)

where $p(A|B)$ is called the “conditional probability” of $A$ given that $B$ is true. As long as $p(B) \neq 0$, the above equation can be rewritten as

$$p(A|B) = \frac{p(B|A) \cdot p(A)}{p(B)} = \frac{p(B|A) \cdot p(A)}{p(B|A) \cdot p(A) + p(B|\overline{A}) \cdot p(\overline{A})},$$

(2.19)

where $p(\overline{A}) = 1 - p(A)$ is the probability of the complement of $A$, i.e. the probability of that $A$ is not happen.

2.2.2 Bayesian statistics

Applying the Baye’s theorem described above, one can directly have that

$$p(\text{theory}|\text{result}) = \frac{p(\text{result}|\text{theory}) \cdot p(\text{theory})}{p(\text{result})}.$$

(2.20)

This means that, given the observed result, the probability of that a specified theory is true is proportional to the probability of the observed result in the specified theory multiplies the probability of the specified theory.

This statement can be understood as follows. If the observed result is predicted by a specified theory to be highly unlikely or even impossible/forbidden, then this observation makes the “degree of belief” of this specified theory small or even disproves this theory. In contrast, the observation of a prediction by a specified theory with a high probability will strengthen one’s belief in this theory.

2.2.3 Bayesian analysis

By extending eq. (2.20), we can obtain that

$$p(\Theta|\text{data}) = \frac{p(\text{data}|\Theta) \cdot p(\Theta)}{p(\text{data})}.$$

(2.21)

Here $\Theta = \{a_1, a_2, \cdots, a_{N_{\text{Bayesian}}}$ \}$ denotes a specified (combination of the) value(s) of the fitting parameter(s); $p(\Theta)$, called “prior probability”, represents our degree of belief about $\Theta$.
being the true value(s) of fitting parameter(s), which is often given in form of the (multiplication of the) probability distribution(s) of the fitting parameter(s). p(data), called “evidence”, is the total probability of obtaining the particular set of data, which is in practice irrespective of the actual value(s) of the parameter(s) and can be treated as a normalization constant; it will not be of interest in our further discussion. p(data|Θ) denotes the probability of the observed result, once the specified (combination of the) value(s) of the fitting parameter(s) happens, which can usually be described by the likelihood function of Θ, L(Θ). Finally, p(Θ|data), called the “posterior probability density function” for Θ, represents the probability of that the specified (combination of the) value(s) of the fitting parameter(s) happens, given the observed result.

### 2.2.4 Bayesian reconstruction of \( f_1(v) \)

Now, we can describe the procedure of our Bayesian reconstruction of the one-dimensional WIMP velocity distribution function in detail.

First, by using eqs. (2.10) and (2.11), one can obtain \( W = B + n_W - 1 \) reconstructed data points: \( (v_{s,\mu}, f_{1,\text{rec}}(v_{s,\mu}) \pm \sigma_{f_1,s,\mu}) \), for \( \mu = 1, 2, \ldots, W \), where

\[
\sigma_{f_1,s,\mu} \equiv \sqrt{\text{cov} (f_{1,\text{rec}}(v_{s,\mu}), f_{1,\text{rec}}(v_{s,\mu}))}
\]  

(2.22)

denote the square roots of the diagonal entries of the covariance matrix given in eq. (2.17). Choosing a theoretical prediction of the one-dimensional velocity distribution of halo WIMPs: \( f_{1,\text{th}}(v; a_1, a_2, \ldots, a_{N_{\text{Bayesian}}}) \), where \( (a_1, a_2, \ldots, a_{N_{\text{Bayesian}}}) \) are the \( N_{\text{Bayesian}} \) fitting parameters, and assuming that the reconstructed data points is Gaussian-distributed around the theoretical predictions \( f_{1,\text{th}}(v_{s,\mu}; a_1, a_2, \ldots, a_{N_{\text{Bayesian}}}) \), the likelihood function for \( p(\text{data}|\Theta) \) can then be defined by

\[
\mathcal{L} (f_{1,\text{rec}}(v_{s,\mu}), \mu = 1, 2, \ldots, W; a_i, i = 1, 2, \ldots, N_{\text{Bayesian}}) \\
\equiv \prod_{\mu=1}^{W} \text{Gau} (v_{s,\mu}, f_{1,\text{rec}}(v_{s,\mu}), \sigma_{f_1,s,\mu}; a_1, a_2, \ldots, a_{N_{\text{Bayesian}}}) ,
\]  

(2.23)

where

\[
\text{Gau} (v_{s,\mu}, f_{1,\text{rec}}(v_{s,\mu}), \sigma_{f_1,s,\mu}; a_1, a_2, \ldots, a_{N_{\text{Bayesian}}}) \\
\equiv \frac{1}{\sqrt{2\pi} \sigma_{f_1,s,\mu}} e^{-\left\{ f_{1,\text{rec}}(v_{s,\mu}) - f_{1,\text{th}}(v_{s,\mu}; a_1, a_2, \ldots, a_{N_{\text{Bayesian}}}) \right\}^2 / 2\sigma_{f_1,s,\mu}^2}.
\]  

(2.24)

Or, equivalently, we can use the logarithmic likelihood function given by

\[
\ln \mathcal{L} (f_{1,\text{rec}}(v_{s,\mu}), \mu = 1, 2, \ldots, W; a_i, i = 1, 2, \ldots, N_{\text{Bayesian}}) \\
= -\frac{1}{2} \sum_{\mu=1}^{W} \left[ \frac{f_{1,\text{rec}}(v_{s,\mu}) - f_{1,\text{th}}(v_{s,\mu}; a_1, a_2, \ldots, a_{N_{\text{Bayesian}}})}{\sigma_{f_1,s,\mu}} \right]^2 - \sum_{\mu=1}^{W} \ln \left( \sqrt{2\pi} \sigma_{f_1,s,\mu} \right) ,
\]  

(2.25)

Note that in practical use the second term in eq. (2.25) can be neglected, since it is just a constant for all scanned (combinations of) values of the fitting parameter(s) \( (a_1, a_2, \ldots, a_{N_{\text{Bayesian}}}) \).
Table 1. The input setup for the simple Maxwellian velocity distribution \( f_{1,\text{Gau}}(v) \) used for generating WIMP events as well as the scanning range, the expectation value of and the 1\( \sigma \) uncertainty on the unique fitting parameter \( v_0 \).

Finally, choosing the probability distribution function for each fitting parameter \( a_i \), \( p_i(a_i) \), the posterior probability density on the left-hand side of eq. (2.21) can then be given by

\[
p \left( a_i, i = 1, 2, \ldots, N_{\text{Bayesian}} \bigg| f_{1,\text{rec}}(v_{s,\mu}), \mu = 1, 2, \ldots, W \right) \propto \mathcal{L} \left( f_{1,\text{rec}}(v_{s,\mu}), \mu = 1, 2, \ldots, W; \ a_i, i = 1, 2, \ldots, N_{\text{Bayesian}} \right) \prod_{i=1}^{N_{\text{Bayesian}}} p_i(a_i).
\]

(2.26)

3 Numerical results

In this section, we present numerical results of our Bayesian reconstruction of the one-dimensional velocity distribution function of halo WIMPs based on Monte-Carlo simulations.

In order to test whether we can reconstruct \( f_1(v) \) correctly with even an improper adopted model and/or incorrect expectation value(s) of the fitting parameter(s), different choices of the WIMP velocity distribution function for generating signal events as well as different assumptions of \( f_1(v) \) and/or (slightly) different expectation value(s) of the fitting parameter(s) from the commonly adopted values for the Bayesian fitting process will be considdered in our simulations. In tables 1, 3 and 8, we list the input setup for the chosen input velocity distribution functions used for generating WIMP signals as well as the scanning ranges, the expectation values of and the 1\( \sigma \) uncertainties on the fitting parameters used for different fitting velocity distributions. Additionally, a common maximal cut-off on the one-dimensional WIMP velocity distribution has been set as \( v_{\text{max}} = 700 \) km/s.

The WIMP mass \( m_\chi \) involved in the coefficient \( \alpha \) for estimating the reconstructed points \( v_{s,\mu} \), as well as the normalization constant \( N \) has been assumed to be known precisely with a negligible uncertainty from other (e.g. collider) experiments or determined from direct detection experiments with different data sets. As in ref. [5], a \(^{76}\text{Ge}\) nucleus has been chosen as our detector material for reconstructing \( f_1(v) \), whereas a \(^{28}\text{Si}\) target and a second \(^{76}\text{Ge}\) target have been used for determining \( m_\chi \) [6].

As in refs. [5, 21], the WIMP-nucleus cross section in the coefficient \( A \) has been assumed to be only spin-independent (SI), \( \sigma_{\chi p}^{\text{SI}} = 10^{-9} \) pb, and the commonly used analytic form for the elastic nuclear form factor:

\[
F_{\text{SI}}^2(Q) = \left[ \frac{3j_1(qR_1)}{qR_1} \right]^2 e^{-(qs)^2}
\]

(3.1)

has been adopted. Here \( Q \) is the recoil energy transferred from the incident WIMP to the target nucleus, \( j_1(x) \) is a spherical Bessel function, \( q = \sqrt{2m_NQ} \) is the transferred 3-momentum, for the effective nuclear radius we use \( R_1 = \sqrt{R_A^2 - 5s^2} \) with \( R_A \simeq 1.2 A^{1/3} \) fm and a nuclear skin thickness \( s \simeq 1 \) fm.
In our simulations, the experimental threshold energies have been assumed to be negligible \( (Q_{\text{min}} = 0) \)\(^2\) and the maximal cut-off energies are set as \( Q_{\text{max}} = 100 \text{ keV} \) for all target nuclei;\(^3\) the widths of the first energy bin have also been set commonly as \( b_1 = 10 \text{ keV} \). Additionally, we assumed that all experimental systematic uncertainties as well as the uncertainty on the measurement of the recoil energy could be ignored. The energy resolution of most existing and next-generation detectors should be good enough so that the very small measurement uncertainties can be neglected compared to the statistical uncertainties on our reconstructed results with only few events. Energy range between \( Q_{\text{min}} \) and \( Q_{\text{max}} \) have been divided into five bins and up to three bins have been combined to a window. \((3 \times 5) \times 5,000\) experiments with 500 total events on average in one experiment have been simulated. In sections 3.1 to 3.3, the input WIMP mass has been fixed as \( m_\chi = 100 \text{ GeV} \). In section 3.4, we consider the cases with a light WIMP mass of \( m_\chi = 25 \text{ GeV} \) and a heavy one of \( m_\chi = 250 \text{ GeV} \). In addition, in section 3.5, we consider also briefly the effects of unrejected background events for different input WIMP masses \([21, 22]\).

Note that for our numerical simulations presented in this section, the actual numbers of generated signal and background events in each simulated experiment are Poisson-distributed around their expectation values independently. This means that, for example, for simulations shown in figures 19 we generate 400 (100) events on average for WIMP-signals (backgrounds) and the total event number recorded in one experiment is then the sum of these two numbers.

Regarding our degree of belief about each fitting parameter \( a_i \), i.e. \( p_i(a_i) \) in eq. (2.26), two probability distribution functions have been considered. The simplest one is the flat-distribution:

\[
p_i(a_i) = 1, \quad \text{for } a_{i,\text{min}} \leq a_i \leq a_{i,\text{max}},
\]

where \( a_{i,\text{(min, max)}} \) denote the minimal and maximal bounds of the scanning interval of the fitting parameter \( a_i \). On the other hand, for the case that we have already prior knowledge about one fitting parameter, a Gaussian-distribution:

\[
p_i(a_i; \mu_{a,i}, \sigma_{a,i}) = \frac{1}{\sqrt{2\pi} \sigma_{a,i}} e^{-\frac{(a_i - \mu_{a,i})^2}{2\sigma_{a,i}^2}}
\]

with the expectation value \( \mu_{a,i} \) of and the 1σ uncertainty \( \sigma_{a,i} \) on the fitting parameter \( a_i \) is used.

Note that, in one simulated experiment, we scan the parameter space \((a_1, a_2, \cdots, a_N_{\text{Bayesian}})\) in the volume \( a_i \in [a_{i,\text{min}}, a_{i,\text{max}}], i = 1, 2, \cdots, N_{\text{Bayesian}}\) to find a particular point \((a_1^*, a_2^*, \cdots, a_N_{\text{Bayesian}}^*)\), which maximizes the (numerator of the) posterior probability density: \( p \left( a_i, i = 1, 2, \cdots, N_{\text{Bayesian}} \bigg| f_{1,\text{rec}}(v_\mu), \mu = 1, 2, \cdots, W \right) \). After that all simulations have been done, we determine the median value of the (1σ lower and upper bounds of the) velocity distribution reconstructed by eq. (2.10) from all experiments, denoted as \( f_{1,\text{median}}(\alpha_{\text{(median)}}, \sqrt{Q_{s,\mu,\text{Bayesian}}}) \), for \( \mu = 1, 2, \cdots, W \), and shown as solid black crosses in

\(^2\)Note that, once the experimental threshold energy of the data set used for reconstructing \( f_1(v) \) is non-negligible, the estimate (2.11) of the normalization constant \( \mathcal{N} \) would need to be modified properly, especially for light WIMPs.

\(^3\)Note that, due to the maximal cut-off on the one-dimensional WIMP velocity distribution, \( v_{\text{max}} \), a kinematic maximal cut-off energy

\[
Q_{\text{max, kin}} = \frac{v_{\text{max}}^2}{\alpha^2}
\]

has also been taken into account.
Figure 1. (a) The reconstructed simple Maxwellian velocity distribution function for an input WIMP mass of $m_\chi = 100$ GeV with a $^{76}$Ge target. The black crosses are the velocity distribution reconstructed by eq. (2.10): the vertical error bars show the square roots of the diagonal entries of the covariance matrix estimated by eq. (2.17) (i.e., $\sigma_{f_{s,\mu}}$ given in eq. (2.22)) and the horizontal bars indicate the sizes of the windows used for estimating $f_{1,\text{rec}}(v_{s,\mu})$, respectively. The solid red curve is the generating simple Maxwellian velocity distribution with an input value of $v_0 = 220$ km/s. While the dashed green curve indicates the reconstructed simple Maxwellian velocity distribution with the fitting parameter $v_0$ given by the median value of all simulated experiments, the dash-dotted blue curve indicates the reconstructed simple Maxwellian velocity distribution with $v_0$ which maximizes $p_{\text{median}}(a_i, i = 1, 2, \cdots, N_{\text{Bayesian}})$ defined in eq. (3.5). (b) The distribution of the Bayesian reconstructed fitting parameter $v_0$ in all simulated experiments. The red vertical line indicates the true (input) value of $v_0$, which has been labeled with the subscript “gen”. The green vertical line indicates the median value of the simulated results, whereas the blue one indicates the value which maximizes $p_{\text{median}}$. In addition, the horizontal thick (thin) green bars show the 1 (2) $\sigma$ ranges of the reconstructed results. Note that the bins at $v_0 = 160$ km/s and $v_0 = 300$ km/s are “overflow” bins, which contain also the experiments with the best-fit $v_0$ value of either $v_0 < 160$ km/s or $v_0 > 300$ km/s. See the text for further details.

the (top-)left frame(s) of, e.g. figures 1. And we define further

$$p_{\text{median}}(a_i, i = 1, 2, \cdots, N_{\text{Bayesian}})$$

$$\equiv p\left(a_i, i = 1, 2, \cdots, N_{\text{Bayesian}}\bigg|f_{1,\text{median}}\left(a_{\text{median}}\sqrt{Q_{s,\mu,\text{Bayesian}}}, \mu = 1, 2, \cdots, W\right)\right),$$

(3.5)

and check the points $(a_1^*, a_2^*, \cdots, a_{N_{\text{Bayesian}}}^*)$ obtained from all simulated experiments to find the special (“best-fit”) point $(a_{1,\text{max}}, a_{2,\text{max}}, \cdots, a_{N_{\text{Bayesian}},\text{max}})$, which maximizes $p_{\text{median}}(a_i, i = 1, 2, \cdots, N_{\text{Bayesian}})$.

3.1 Simple Maxwellian velocity distribution

We consider first the simplest isothermal spherical Galactic halo model for generating WIMP events. The normalized one-dimensional simple Maxwellian velocity distribution function can
be expressed as [1, 5]:

\[ f_{1, \text{Gau}}(v) = \frac{4}{\sqrt{\pi}} \left( \frac{v^2}{v_0^3} \right) e^{-v^2/v_0^2}, \]  

(3.6)

where \( v_0 \approx 220 \text{ km/s} \) is the Solar orbital velocity in the Galactic frame.

In table 1, we list the input setup for the simple Maxwellian velocity distribution \( f_{1, \text{Gau}}(v) \) used for generating WIMP events as well as the scanning range, the expectation value of and the \( 1\sigma \) uncertainty on the unique fitting parameter \( v_0 \). Note that, for generating WIMP signals, \( v_0 = 220 \text{ km/s} \) has been used, whereas for Bayesian fitting of this unique parameter we used a slightly different value of \( v_0 = 230 \text{ km/s} \) and set a \( 1\sigma \) uncertainty of \( \sigma(v_0) = 20 \text{ km/s} \).

### 3.1.1 Simple Maxwellian velocity distribution

As the simplest Galactic halo model, we consider first the use of the simple Maxwellian velocity distribution \( f_{1, \text{Gau}}(v) \) given in eq. (3.6) with an unique fitting parameter \( v_0 \) to fit the reconstructed-input data given by eqs. (2.10), (2.11) and (2.17).

In figure 1a, we show the reconstructed simple Maxwellian velocity distribution function for an input WIMP mass of \( m_\chi = 100 \text{ GeV} \) with a \( ^{76}\text{Ge} \) target. Here the flat distribution given by eq. (3.3) for the fitting parameter \( v_0 \) has been used. The black crosses are the velocity distribution reconstructed by eq. (2.10): the vertical error bars show the square roots of the diagonal entries of the covariance matrix given in eq. (2.17) (i.e. \( \sigma_{f_1,s,\mu} \) given in eq. (2.22)) and the horizontal bars indicate the sizes of the windows used for estimating \( f_{1,\text{rec}}(v_{s,\mu}) \), respectively. The solid red curve is the generating simple Maxwellian velocity distribution with an input value of \( v_0 = 220 \text{ km/s} \). While the dashed green curve indicates the reconstructed simple Maxwellian velocity distribution with the fitting parameter \( v_0 \) given by the median value of all simulated experiments, the dash-dotted blue curve indicates the reconstructed simple Maxwellian velocity distribution with \( v_0 \) which maximizes \( p_{\text{median}}(a_i, i = 1, 2, \cdots, N_{\text{Bayesian}}) \) defined in eq. (3.5).

Meanwhile, the light-green (light-blue) area shown here indicate the \( 1 \) (2) \( \sigma \) statistical uncertainty bands of the Bayesian reconstructed velocity distribution function, which has been determined as follows. After scanning the reconstructed fitting parameter \( v_0 \) obtained from all simulated experiments and ordering according to their \( p_{\text{median}} \) values defined in eq. (3.5) descendingly, we can not only determine the point which maximizes \( p_{\text{median}} \) (labeled with the subscript “max” in our plots,\(^4\)) but also the smallest and largest values of the first 68.27\% (95.45\%) of all these reconstructed \( v_0 \)’s. We then use the smallest (largest) value of the first 68.27\% (95.45\%) reconstructed \( v_0 \)’s to give the 1 (2) \( \sigma \) lower (upper) boundaries of the Bayesian reconstructed velocity distribution function. This means that all of the velocity distributions with \( v_0 \)’s which give the largest 68.27\% (95.45\%) \( p_{\text{median}} \) values should be in the 1 (2) \( \sigma \) light-green (light-blue) areas.

On the other hand, figure 1b shows the distribution of the Bayesian reconstructed fitting parameter \( v_0 \) in all simulated experiments. The red vertical line indicates the true (input) value of \( v_0 \), which has been labeled with the subscript “gen”. The green vertical line indicates the median value of the simulated results, whereas the blue one indicates the value which maximizes \( p_{\text{median}} \). In addition, the horizontal thick (thin) green bars show the 1 (2) \( \sigma \)

\(^4\)Note that the subscript “input” in figures 1 and 2 indicate that the needed WIMP mass for reconstructing \( f_1(v) \) has been given as the input WIMP mass; whereas, the subscript “algo” in figures 3 and 4 indicate that the WIMP mass is reconstructed by the algorithmic procedure developed in ref. [6].
ranges of the reconstructed results.\footnote{Note that the 1\,(2)\,\(\sigma\) ranges given here mean that, according to the order of the reconstructed values of \(v_0\), the 68.27\% (95.45\%) of the reconstructed values in the median range.} Note that the bins at \(v_0 = 160\text{ km/s}\) and \(v_0 = 300\text{ km/s}\) are “overflow” bins, which contain also the experiments with the best-fit \(v_0\) value of either \(v_0 < 160\text{ km/s}\) or \(v_0 > 300\text{ km/s}\).

In figures 1, it can be seen clearly that, without a prior knowledge about the Solar Galactic velocity, one could in principle pin down the parameter \(v_0\) very precisely with 1\,(2)\,\(\sigma\) statistical uncertainties of only \(\pm 11.2\,(\pm 22.4)\) km/s (see table 2). Moreover, by using the Bayesian reconstruction of the one-dimensional velocity distribution function, the large (1\,\(\sigma\)) statistical uncertainty given by eq. (2.22) can be reduced significantly: the band of the 2\,\(\sigma\) statistical uncertainty would be approximately equal to or even smaller than the (solid black) vertical 1\,\(\sigma\) uncertainty bars!

Furthermore, in figures 2 we consider the case with a rough prior knowledge about the Solar Galactic velocity \(v_0\). It has been found that, firstly, by using a Gaussian probability distribution for \(v_0\) with a 1\,\(\sigma\) uncertainty of 20\,km/s, one could reduce the 1\,(2)\,\(\sigma\) statistical uncertainties on the Bayesian reconstructed parameter \(v_0\) to \(\pm 9.8\,(\pm 18.2)\) km/s (see table 2). Secondly and more importantly, although an expectation value of \(v_0 = 230\text{ km/s}\), which differs (slightly) from the true (input) one, is used, the value of the fitting parameter \(v_0\) could still be pinned down precisely with a tiny systematic deviation (< 2 km/s).

### Table 2

The reconstructed results of \(v_0\) for all four considered cases with the simple Maxwellian velocity distribution \(f_{1,\text{Gau}}(v)\) as well as the 1\,(2)\,\(\sigma\) uncertainty ranges of the median values.

| Parameter | Input | WIMP mass | Prob. dist. | Max. \(\Delta v_{\text{median}}\) | Median | 1\,\(\sigma\) range | 2\,\(\sigma\) range |
|-----------|-------|-----------|-------------|---------------------------------|--------|--------------------|--------------------|
| \(v_0\) [km/s] | Flat | 218.8 | 218.8\(\pm 12.6\,(\pm 33.6)\) | [207.6, 231.4] | [196.4, 252.4] |
| Gaussian | 221.6 | 221.6 \(\pm 9.8\,(\pm 23.8)\) | [211.8, 231.4] | [203.4, 254.4] |
| Flat | 220.2 | 218.8\(\pm 21.0\,(\pm 49.0)\) | [202.0, 239.8] | [185.2, 267.8] |
| Gaussian | 220.2 | 221.6\(\pm 15.4\,(\pm 36.6)\) | [209.0, 237.0] | [195.0, 255.2] |

### Table 3

The input setup for the shifted Maxwellian velocity distribution \(f_{1,\text{sh}}(v)\) used for generating WIMP signals as well as the theoretically estimated values, the scanning ranges, the expectation values of and the 1\,\(\sigma\) uncertainties on the fitting parameters used for different fitting velocity distribution functions.

| Fitting model | Parameter | Input/theoretical value | Scanning range | Expectation value | 1\,\(\sigma\) uncertainty |
|---------------|-----------|-------------------------|----------------|------------------|------------------------|
| Simple        | \(v_0\) [km/s] | \(\sim 295\) | [160, 400] | 280 | 40 |
| 1-para. shifted | \(v_0\) [km/s] | 220 | [160, 300] | 230 | 20 |
| Shifting      | \(v_0\) [km/s] | 220 | [160, 300] | 230 | 20 |
| \(v_e\) [km/s] | 231 | [160, 300] | 245 | 20 |
| Varied shifted | \(v_0\) [km/s] | 220 | [160, 300] | 230 | 20 |
| \(\Delta v\) [km/s] | 11 | [−50, 80] | 15 | 20 |
Figure 2. As in figures 1, except that the Gaussian probability distribution given in eq. (3.4) for $v_0$ with an expectation value of $v_0 = 230 \text{ km/s}$ and a 1$\sigma$ uncertainty of 20 km/s has been used.

Figure 3. As in figures 1, except that the needed WIMP mass $m_\chi$ for reconstructing $f_1(v)$ is reconstructed by the algorithmic procedure developed in ref. [6] with a $^{28}$Si target and a second $^{76}$Ge target. While the vertical bars show the 1$\sigma$ statistical uncertainties estimated by eq. (2.22) taking into account an extra contribution from the 1$\sigma$ statistical uncertainty on the reconstructed WIMP mass, the horizontal bars shown here indicate the 1$\sigma$ statistical uncertainties on the estimates of $v_{s,\mu}$ due to the uncertainty on the reconstructed WIMP mass; the statistical and systematic uncertainties due to estimating of $Q_{s,\mu}$ have been neglected here.

In figures 3 and 4, we consider the case that the needed WIMP mass $m_\chi$ for reconstructing $f_1(v)$ is reconstructed by the algorithmic procedure developed in ref. [6] with a $^{28}$Si target and a second $^{76}$Ge target. Note that, while the vertical bars show the 1$\sigma$ statistical uncertainties estimated by eq. (2.22) taking into account an extra contribution from the 1$\sigma$ statistical uncertainty on the reconstructed WIMP mass, the horizontal bars shown here indicate the 1$\sigma$ statistical uncertainties on the estimates of $v_{s,\mu}$ due to the uncertainty on the reconstructed WIMP mass; the statistical and systematic uncertainties due to estimating of $Q_{s,\mu}$ have been neglected here.
Figure 4. As in figures 3, except that the Gaussian probability distribution for $v_0$ with an expectation value of $v_0 = 230\,\text{km/s}$ and a $1\sigma$ uncertainty of $20\,\text{km/s}$ has been used.

It can be seen that, due to the extra statistical fluctuation on the reconstructed WIMP mass [6] and in turn the uncertainty on the estimated $v_{s,\mu}$ (the horizontal bars in the left frames), the statistical uncertainties on the Bayesian reconstructed $v_0$ with both of the flat and the Gaussian probability distributions become $\sim 30\%$ to $\sim 60\%$ larger. However, same as the case with an input WIMP mass, the use of the Gaussian probability distribution can not only reduce the statistical uncertainty on $v_0$ significantly and therefore improve the Bayesian reconstructed velocity distribution, but also alleviate the “imprecisely” expected value of $v_0$.

In table 2, we list the reconstructed results of $v_0$ for all four considered cases with the simple Maxwellian velocity distribution $f_1,\text{Gau}(v)$ as well as the 1 (2) $\sigma$ uncertainty ranges of the median values of $v_0$. It would be worth to emphasize that, the statistical uncertainties shown in figures 2 and 4 are (much) smaller than the input uncertainty on the expectation value of $v_0$ of 20 km/s.

3.2 Shifted Maxwellian velocity distribution

By taking into account the orbital motion of the Solar system around our Galaxy as well as that of the Earth around the Sun, a more realistic shifted Maxwellian velocity distribution of halo WIMPs has been given by [1, 5]:

$$f_{1,\text{sh}}(v) = \frac{1}{\sqrt{\pi}} \left( \frac{v}{v_0 v_e} \right) \left[ e^{-(v-v_e)^2/v_0^2} - e^{-(v+v_e)^2/v_0^2} \right]. \quad (3.7)$$

Here $v_e$ is the time-dependent Earth’s velocity in the Galactic frame [1, 23, 24]:

$$v_e(t) = v_0 \left[ 1.05 + 0.07 \cos \left( \frac{2\pi(t-t_p)}{1\,\text{yr}} \right) \right], \quad (3.8)$$

with $t_p \simeq$ June 2nd is the date on which the velocity of the Earth relative to the WIMP halo is maximal.\textsuperscript{6}

\textsuperscript{6}In our simulations, the time dependence of the Earth’s velocity in the Galactic frame, the second term of $v_e(t)$, will be ignored, i.e. $v_e = 1.05 v_0$ is used.
3.2.1 Simple Maxwellian velocity distribution

We consider first the simplest case of the simple Maxwellian velocity distribution function \( f_1, \text{Gau}(v) \) with the unique fitting parameter \( v_0 \) to fit the reconstructed-input data points.

As in section 3.1.1, in figures 5 we use first the flat probability distribution for the fitting parameter \( v_0 \) with either the precisely known (input) (upper) or the reconstructed (lower) WIMP mass, respectively. Figures 5a and 5c show clearly that, although an “improper” choice for the fitting velocity distribution function and a simple flat probability distribution for \( v_0 \) (i.e. without any prior knowledge about \( v_0 \)) have been used, the 1\( \sigma \) statistical uncertainty bands of the reconstructed WIMP velocity distribution function could in principle still cover the true (input) distribution. More precisely and quantitatively, the deviations of the peaks of the reconstructed velocity distributions from that of the true (input) one are only \( \sim 10 \) km/s. Note that the 1\( \sigma \) statistical uncertainty on the reconstructed fitting parameter \( v_0 \) is \( \sim 20 \) km/s or even \( \sim 40 \) km/s (see table 4).

However, our simulations show also that, with an “improper” assumption about the fitting velocity distribution function, one would obtain an “unexpected” result for the fitting parameter \( v_0 \): 2.5\( \sigma \) (with the reconstructed WIMP mass, figure 5d) to 4\( \sigma \) (with the input WIMP mass, figure 5b) deviations of the reconstructed Solar Galactic velocity from the theoretical estimate of \( v_0 \approx 220 \) km/s. Such observation would indicate clearly that our initial assumption about the fitting velocity distribution function would be incorrect or at least need to be modified.

Moreover, we considered also the case that a rough prior knowledge about the Solar Galactic velocity \( v_0 \) exists and use the Gaussian probability distribution for \( v_0 \) with an expectation value of \( v_0 = 280 \) km/s and a 1\( \sigma \) uncertainty of \( 40 \) km/s. As observed in section 3.1.1, with a prior knowledge about the fitting parameter \( v_0 \), one could reconstruct the velocity distribution function better: the 1 (2) \( \sigma \) statistical uncertainties on \( v_0 \) could be reduced to \( \sim 60\% \). Additionally, the reconstructed 1\( \sigma \) statistical uncertainties on \( v_0 \) with both of the input and the reconstructed WIMP masses are much smaller than the input 1\( \sigma \) value of 40 km/s.

### Table 4

| Parameter | WIMP mass | Prob. dist. | Max. \( p_{\text{median}} \) | Median | 1\( \sigma \) range | 2\( \sigma \) range |
|-----------|-----------|-------------|--------------------------|--------|-------------------|-------------------|
| \( v_0 \) [km/s] | Input | Flat | 296.8 | 296.8 ±21.6 \[+45.6,-36.0\] | [277.6, 318.4] | [260.8, 342.4] |
| | Gaussian | 294.4 | 292.0 ±16.8 \[+31.8,-26.4\] | [280.0, 308.8] | [265.6, 323.8] |
| Reconst. | Flat | 299.2 | 296.8 ±40.8 \[+86.4,-60.0\] | [265.6, 337.6] | [236.8, 383.2] |
| | Gaussian | 294.4 | 294.4 ±26.4 \[±52.8\] | [268.0, 320.8] | [241.6, 347.2] |

In table 3, we list the input setup for the shifted Maxwellian velocity distribution \( f_{1,\text{sh}}(v) \) used for generating WIMP signals as well as the theoretically estimated values, the scanning ranges, the expectation values of and the 1\( \sigma \) uncertainties on the fitting parameters used for different fitting velocity distribution functions.
In Table 4, we list the reconstructed values of $v_0$ for all four considered cases with the simple Maxwellian velocity distribution $f_{1,Gau}(v)$ as well as the 1 (2)σ uncertainty ranges of the median values.

3.2.2 One-parameter shifted Maxwellian velocity distribution

In the previous section 3.2.1, we have found that, by assuming (improperly) the simple Maxwellian velocity distribution $f_{1,Gau}(v)$, in both cases with and without an expectation value of the fitting parameter $v_0$, one would obtain a much higher reconstruction result: $v_{0,rec} \approx 295 \text{ km/s}$, which is 2σ to 4σ apart from the theoretical estimate of $v_0 \approx 220 \text{ km/s}$. This observation implies the need of a more suitable fitting velocity distribution function. Hence, as the second trial, we consider now the use of the shifted Maxwellian velocity distribution $f_{1,sh}(v)$ given in eq. (3.7) with only one fitting parameter, i.e. the Solar Galactic velocity $v_0$. In this case, we fix simply that

$$v_e = 1.05 v_0,$$

and neglect the time-dependence of $v_e(t)$.

In Figures 6, we consider the flat probability distribution for the fitting parameter $v_0$ with either the precisely known (input) (upper) or the reconstructed (lower) WIMP mass, respectively. It can be seen clearly that, with a more suitable assumption about the fitting function, one could indeed reconstruct the WIMP velocity distribution much closer to the true (input) one. Although no prior knowledge about $v_0$ is used, this most important characteristic parameter could in principle be pinned down very precisely: the difference between the median values of the reconstructed $v_0$ and the true (input) one would be $\lesssim 5 \text{ km/s}$ (see also Table 5).

In addition, with the input WIMP mass, the 1 (2)σ statistical uncertainties on the reconstructed $v_0$ are only $^{+14.0}_{-11.2}$ ($^{+29.4}_{-23.5}$) km/s. Even with the reconstructed WIMP mass, the 1 (2)σ statistical uncertainties could still be limited as small as only $^{+25.2}_{-22.4}$ ($^{+56.0}_{-44.8}$) km/s. It would be worth to emphasize that, compared to the uncertainty on the astronomical measurement of $v_0$ of $\sim 20 \text{ km/s}$ (or probably larger), the result offered by our Bayesian reconstruction method would be a pretty precise estimate and could help us to confirm the astronomical measurement of $v_0$.

Table 5. The reconstructed results of $v_0$ for all four considered cases with the one-parameter shifted Maxwellian velocity distribution $f_{1,sh,v_0}(v)$ as well as the 1 (2)σ uncertainty ranges of the median values.

| Parameter | WIMP mass | Prob. dist. | Max. $v_{\text{median}}$ | Median | 1σ range | 2σ range |
|-----------|-----------|-------------|--------------------------|--------|----------|----------|
| $v_0 \text{ [km/s]}$ | Input | Flat | 217.4 | $217.4^{+14.0}_{-11.2}$ ($^{+29.4}_{-23.5}$) | [206.2, 231.4] | [193.6, 246.8] |
| | Gaussian | 221.6 | $221.6^{+9.8}_{-8.4}$ (±18.2) | [213.2, 231.4] | [203.4, 239.8] |
| Reconst. | Flat | 218.8 | $218.8^{+25.2}_{-22.4}$ (±56.0) | [196.4, 244.0] | [174.0, 274.8] |
| | Gaussian | 221.6 | $221.6^{+16.8}_{-15.4}$ (±33.6) | [206.2, 238.4] | [188.0, 255.2] |

Note that hereafter we use $f_{1,sh,v_0}(v)$ to denote the “one-parameter” shifted Maxwellian velocity distribution function, in order to distinguish this from the “original” one given in eq. (3.7) with $v_0$ and $v_e$ as two independent fitting parameters.
Moreover, with the Gaussian probability distribution for \( v_0 \) with an expectation value of \( v_0 = 230 \text{ km/s} \) and a 1\( \sigma \) uncertainty of 20\( \text{ km/s} \), as summarized in table 5, the 1 (2)\( \sigma \) statistical uncertainties could be reduced significantly to be \( \lesssim 70\% \). Remind here that the expectation value of the Gaussian probability distribution of \( v_0 \) has been set as \( v_0 = 230 \text{ km/s} \), a bit different from the input value. Nevertheless, our simulations show that this “artificial” (systematic) error could be corrected in our reconstruction process for \( v_0 \).

In table 5, we list the reconstructed results of \( v_0 \) for all four considered cases with the one-parameter shifted Maxwellian velocity distribution \( f_{1,\text{sh},v_0}(v) \) as well as the 1 (2)\( \sigma \) uncertainty ranges of the median values of \( v_0 \).

3.2.3 Shifted Maxwellian velocity distribution

Now we release the fixed relation between \( v_0 \) and \( v_e \) given in eq. (3.9) and consider the reconstruction of these two parameters simultaneously and independently. In addition, we assume here that, from the (naive) trials with the simple and one-parameter shifted Maxwellian ve-
Figure 6. As in figures 5, except that the one-parameter shifted Maxwellian velocity distribution function \( f_{1,\text{sh},v_0}(v) \) with the unique fitting parameter \( v_0 \) has been used as the fitting velocity distribution.

Velocity distributions done previously, one could already obtain a rough idea about the shape of the velocity distribution of halo WIMPs. This information would in turn give us the prior knowledge about the expectation values of the Solar and Earth’s Galactic velocities \( v_0 \) and \( v_e \). Hence, we consider here only the Gaussian probability distribution for both fitting parameters with expectation values of \( v_0 = 230 \text{ km/s} \) and \( v_e = 245 \text{ km/s} \) and a common 1\( \sigma \) uncertainty of 20 km/s (see table 3).\(^8\)

In figure 7a, we show the reconstructed shifted Maxwellian velocity distribution function as well as the 1 (2) \( \sigma \) statistical uncertainty bands. Here the reconstructed WIMP mass has been used. Comparing to results given in table 5, it can be seen clearly that, with a prior knowledge about the Solar and Earth’s Galactic velocities \( v_0 \) and \( v_e \), one could in principle reconstruct the velocity distribution function with two fitting parameters more precisely: the

\(^8\)Remind that both of the expectation values of the fitting parameters \( v_0 \) and \( v_e \) differ slightly from the true (input) values. Note also that the time-dependence of the Earth’s Galactic velocity is ignored here and \( v_e \) is thus treated as a time-independent fitting parameter.
Note however that, for using velocity distributions with two or more fitting parameters without constraints on these parameters (i.e. the use of the “flat” probability distribution), the distribution of the reconstructed results by our Bayesian analysis would be pretty wide, a part of them would even be on the boundary of the scanning ranges of these parameters.

1 (2) \sigma statistical uncertainty bands are much thinner and the deviations of \( v_0 \) and \( v_c \) are only a few km/s (see also table 6).\(^9\)

Figure 7b shows the distribution of the Bayesian reconstructed fitting parameter \( v_0 \) and \( v_c \) in all simulated experiments on the \( v_0 - v_c \) plane. The light-green (light-blue, gold) points indicate the 1 (2) \( > 2 \) \sigma areas of the reconstructed combination of \( v_0 \) and \( v_c \). The red upward-triangle indicates the input values of \( v_0 \) and \( v_c \), which has been labeled with the subscript “gen”. The green disk shows the median values of the simulated results, whereas the blue downward-triangle the point which maximizes \( p_{median} \). The meaning of the horizontal thick (thin) green bars are the same as in figures 6b and 6d.

Figure 7. (a) As in figure 6c (with the reconstructed WIMP mass), except that the shifted Maxwellian velocity distribution function given in eq. (3.7) with two fitting parameters \( v_0 \) and \( v_c \) is used. The Gaussian probability distribution for both fitting parameters with expectation values of \( v_0 = 230 \) km/s and \( v_c = 245 \) km/s and a common 1\( \sigma \) uncertainty of 20 km/s has been used. (b) The distribution of the Bayesian reconstructed fitting parameters \( v_0 \) and \( v_c \) in all simulated experiments on the \( v_0 - v_c \) plane. The light-green (light-blue, gold) points indicate the 1 (2) \( > 2 \) \sigma areas of the reconstructed combination of \( v_0 \) and \( v_c \). The red upward-triangle indicates the input values of \( v_0 \) and \( v_c \), which has been labeled with the subscript “gen”. The green disk shows the median values of the simulated results, whereas the blue downward-triangle the point which maximizes \( p_{median} \). The meaning of the horizontal thick (thin) green bars are the same as in figures 6b and 6d. (c) As in figure 6d. (d) Similar to (c): the distribution of the Bayesian reconstructed second fitting parameter \( v_c \). See the text for further details.
points indicate the 1 (2) (> 2) σ areas of the reconstructed combination of \(v_0\) and \(v_e\). Note here that these 1 (2) (> 2) σ areas are determined according to the descending order of the \(p_{\text{median}}\) values of the reconstructed combination of \(v_0\) and \(v_e\). This means that the light-green (light-blue, gold) areas are the reconstructed combinations of \(v_0\) and \(v_e\) which give the largest 68.27% (95.45%) \(p_{\text{median}}\) values in all of the simulated experiments.

Moreover, the red upward-triangle indicates the input values of \(v_0\) and \(v_e\), which has been labeled with the subscript “gen”. The green disk shows the median values of the simulated results, whereas and blue downward-triangle the point which maximizes \(p_{\text{median}}\). In addition, the thick (thin) green crosses show the 1 (2) σ (68.27% (95.45%)) ranges of the reconstructed results according to the order of the reconstructed values of \(v_0\) or \(v_e\) along (centered at their median values \(v_{0,\text{median}}\) or \(v_{e,\text{median}}\)).

Additionally, in figures 7c and 7d, we give the distributions of the Bayesian reconstructed fitting parameters \(v_0\) and \(v_e\) as well as the 1 (2) σ statistical uncertainty ranges on the \(v_0\)- and \(v_e\)-axes separately. Note here that we project all reconstructed combinations of \((v_0, v_e)\) on the \(v_0\)- or \(v_e\)-axis. Comparing results given in table 6 to those given in table 5, it can be found that, while the deviations of \(v_0\) are a little bit larger then the results with the “one-parameter” fitting velocity distribution, the 1 (2) σ statistical uncertainties are reduced to \(\sim 70\%\).

In table 6, we list the reconstructed results of \(v_0\) and \(v_e\) with the shifted Maxwellian velocity distribution \(f_{1,\text{sh}}(v)\) as well as the 1 (2) σ uncertainty ranges of the median values.

### 3.2.4 Varied shifted Maxwellian velocity distribution

In the previous section 3.2.3, it has been found that, by using the shifted Maxwellian velocity distribution given in eq. (3.7) with two fitting parameters: \(v_0\) and \(v_e\), one could reconstruct the (1 (2) σ statistical uncertainty bands of the) velocity distribution function as well as pin down the Solar Galactic velocity \(v_0\) pretty precisely. However, as shown in figures 7d and table 6, the deviations of the reconstructed Earth’s Galactic velocity \(v_e\) from the true (input) value seem to be (much) larger than the deviations of the reconstructed \(v_0\). This might be caused by the strong (anti-)correlation between \(v_0\) and \(v_e\). Hence, we consider now a variation of the shifted Maxwellian distribution function, in the hope that this pretty large systematic deviation of \(v_e\) could be reduced.

We rewrite the shifted Maxwellian velocity distribution given in eq. (3.7) to the following “varied” form:

\[
f_{1,\text{sh},\Delta v}(v) = \frac{1}{\sqrt{\pi}} \left[ \frac{v}{v_0(v_0 + \Delta v)} \right] \left\{ e^{-[v-(v_0+\Delta v)]^2/v_0^2} - e^{-[v+(v_0+\Delta v)]^2/v_0^2} \right\}, \tag{3.10}\]
Input: shifted Maxwellian velocity distribution \( f_{1,sh}(v) \)

| Parameter | WIMP mass | Prob. dist. | Max. \( \mu_{\text{median}} \) | Median | 1\( \sigma \) range | 2\( \sigma \) range |
|-----------|-----------|-------------|----------------|--------|----------------|----------------|
| \( v_0 \) [km/s] | Input | Gaussian | 221.6 | 221.6 \( \pm \) 8.4 (\( +15.4 \)) | [213.2, 230.0] | [206.2, 237.0] |
| Reconst. | Gaussian | 221.6 | 221.6 \( \pm \) 14.0 (\( +29.4 \)) | [207.6, 235.6] | [192.2, 251.0] |
| \( \Delta v \) [km/s] | Input | Gaussian | 11.1 | 11.1 \( \pm \) 5.2 (\( +10.4 \), \( -11.1 \)) | [5.9, 16.3] | [−0.6, 21.5] |
| Reconst. | Gaussian | 11.1 | 11.1 \( \pm \) 7.8 (\( +15.6 \), \( -14.3 \)) | [3.3, 18.9] | [−3.2, 26.7] |

Table 7. The reconstructed results of \( v_0 \) and \( \Delta v \) with the variated shifted Maxwellian velocity distribution \( f_{1,sh,\Delta v}(v) \) as well as the 1\( \sigma \) uncertainty ranges of the median values.

where

\[
\Delta v \equiv v_e - v_0 \tag{3.11}
\]

is the difference between \( v_0 \) and \( v_e(t) \).

As in the previous section 3.2.3, we assume here that, we have already a rough idea about the shape of the velocity distribution of halo WIMPs, and thus the prior knowledge about the expectation values of the (difference between the) Solar and Earth’s Galactic velocities \( v_0 \) and \( \Delta v \). Hence, we consider here only the Gaussian probability distribution for both of the fitting parameters with expectation values of \( v_0 = 230 \text{ km/s} \) and \( \Delta v = 15 \text{ km/s} \) and a common 1\( \sigma \) uncertainty of 20 km/s (see table 3).

By comparing figures 8 to figures 7, it can be seen obviously that the reconstructed velocity distribution function (with the reconstructed WIMP mass) could indeed match the true (input) one much precisely, with however (slightly) wider 1\( \sigma \) statistical uncertainty bands. The systematic deviations of both fitting parameters from the true (input) values would also be much smaller than those given in the \( v_0 - v_e \) case. This implies importantly that, the use of our variated shifted velocity distribution function given in eq. (3.10) could indeed offer an estimate of the Earth’s Galactic velocity \( v_e = v_0 + \Delta v \) with a much higher precision. Such a trick would be helpful to improve the estimation(s) of the Earth’s Galactic velocity \( v_e \) (and/or other fitting parameter(s)).

By noting however that, as shown in tables 6 and 7, by using the variated shifted Maxwellian velocity distribution, the 1\( \sigma \) statistical uncertainties on the reconstructed \( v_0 \) would be \( \sim 10\% \) larger. Meanwhile, from eq. (3.11) the statistical uncertainty on \( v_e \) can be estimated by

\[
\sigma (v_e) = \sqrt{\sigma^2 (v_0) + \sigma^2 (\Delta v) + 2 \text{ cov} (v_0, \Delta v) \sigma (v_0) \sigma (\Delta v)} \leq \sqrt{\sigma^2 (v_0) + \sigma^2 (\Delta v)}. \tag{3.12}
\]

Then, according to the results given in table 7, the 1\( \sigma \) statistical uncertainties on \( v_e \) would be maximal \( \leq 15\% \) enlarged, whereas the systematic deviation of \( v_e \) is much smaller.

In table 7, we list the reconstructed results of \( v_0 \) and \( \Delta v \) with the variated shifted Maxwellian velocity distribution \( f_{1,sh,\Delta v}(v) \) as well as the 1\( \sigma \) uncertainty ranges of the median values.

---

\(^{10}\)As in the previous section 3.2.3, the time-dependence of the Earth’s Galactic velocity is ignored here and \( \Delta v \) is thus treated as a *time-independent* fitting parameter.

\(^{11}\)Since, according to eq. (3.11), for a fixed value of \( v_e \) two fitting parameters \( v_0 \) and \( \Delta v \) should be “anti-correlated”.
Here \( \Delta v \) is the maximal cut-off velocity of \( f_{1,Gau,k}(v) \) and \( N_{f,k} \) is the normalization constant depending on the value of the power index \( k \).

In figure 9, we give the normalized modified Maxwellian velocity distribution function \( f_{1,Gau,k}(v) \) with a common value of the Solar Galactic velocity \( v_0 = 220 \text{ km/s} \) and different power indices \( k \): \( k = 1 \) (dashed light-green), \( k = 2 \) (solid red), \( k = 3 \) (dash-dotted blue).
Figure 9. The normalized modified Maxwellian velocity distribution function $f_{1, \text{Gau}, k}(v)$ given in eq. (3.13) with a common value of the Solar Galactic velocity $v_0 = 220 \text{ km/s}$ and different power indices $k$: $k = 1$ (dashed light-green), $k = 2$ (solid red), $k = 3$ (dash-dotted blue) and $k = 4$ (dotted magenta). As a comparison, the simple Maxwellian velocity distribution $f_{1, \text{Gau}}(v)$ with $v_0 = 220 \text{ km/s}$ is also given as the short-dashed black curve.

and $k = 4$ (dotted magenta). As a comparison, the simple Maxwellian velocity distribution $f_{1, \text{Gau}}(v)$ with $v_0 = 220 \text{ km/s}$ is also given as the short-dashed black curve.

In table 8, we list the input setup for the modified Maxwellian velocity distribution $f_{1, \text{Gau}, k}(v)$ used for generating WIMP signals as well as the theoretically estimated values, the scanning ranges, the expectation values of and the $1\sigma$ uncertainties on the fitting parameters used for different fitting velocity distribution functions.

### 3.3.1 Simple Maxwellian velocity distribution

As in section 3.2, we start to fit to the reconstructed-input data points with the simple Maxwellian velocity distribution function $f_{1, \text{Gau}}(v)$.

In figures 10, we use the flat probability distribution for the fitting parameter $v_0$ with either the precisely known (input) (upper) or the reconstructed (lower) WIMP mass, respectively. As shown in figures 1 and 3, although prior knowledge about the Solar Galactic velocity is absent, one could in principle pin down the fitting parameter $v_0$ very precisely with systematic deviations of only a few km/s and $1\sigma$ statistical uncertainties of only $+11.2^{-12.6}_{-12.4} \text{ km/s}$ and $+21.0^{-32.2}_{-32.2} \text{ km/s}$.

On the other hand, by using the Gaussian probability distribution for $v_0$ with an expectation value of $v_0 = 230 \text{ km/s}$ and a $1\sigma$ uncertainty of $20 \text{ km/s}$, as summarized in table 9, the statistical uncertainties on the reconstructed $v_0$ could now in principle be reduced by $\sim 10\%$ to $\sim 30\%$. In addition, even with the reconstructed WIMP mass and thus a larger statistical fluctuation, the $1\sigma$ statistical uncertainties are (much) smaller than the given uncertainty on the expectation value of $v_0$ (20 km/s). Moreover, although an expectation value of $v_0 = 230 \text{ km/s}$, which differs (slightly) from the true (input) one, is used, the value of the fitting parameter $v_0$ could still be pinned down precisely with a negligible systematic deviation.
As in figures 5, except that the modified Maxwellian velocity distribution function given in eq. (3.13) has been used for generating WIMP signals.

Furthermore, remind here that, either, as shown in figures 10, two reconstructed (dashed green and dash-dotted blue) velocity distribution functions match the true (input) (solid red) one very well, but the reconstructed Solar Galactic velocity $v_0$ would shift slightly away from the true (input) value, or $v_0$ could be determined very precisely, but the reconstructed velocity distribution functions would differ slightly from the true (input) one. This is because of the fact that we have artificially used an input value of $k = 2$ and, as shown in figure 9, the modified Maxwellian velocity distribution function with $k = 2$ is slightly different from the simple Maxwellian distribution.

In table 9, we list the reconstructed results of $v_0$ for all four considered cases with the simple Maxwellian velocity distribution $f_1(v)$ as well as the 1 (2) $\sigma$ uncertainty ranges of the median values of $v_0$.

### 3.3.2 One-parameter shifted Maxwellian velocity distribution

Now, as a comparison of section 3.3.1, we consider as the next trail the reconstruction with the one-parameter shifted Maxwellian velocity distribution function to fit the modified simple Maxwellian velocity distribution.
Table 8. The input setup for the modified Maxwellian velocity distribution \( f_{1, \text{Gau}, k}(v) \) used for generating WIMP signals as well as the theoretically estimated values, the scanning ranges, the expectation values of and the 1\( \sigma \) uncertainties on the fitting parameters used for different fitting velocity distribution functions.

| Fitting model          | Parameter | Input/theoretical value | Scanning range | Expectation value | 1\( \sigma \) uncertainty |
|------------------------|-----------|-------------------------|----------------|------------------|---------------------------|
| Simple                 | \( v_0 \) [km/s] | \( \sim 220 \) | \([160, 300]\) | 230              | 20                        |
| 1-para. shifted        | \( v_0 \) [km/s] | \( \sim 175 \) | \([100, 300]\) | 200              | 40                        |
| Shifted                | \( v_0 \) [km/s] | \( \sim 175 \) | \([100, 300]\) | 200              | 40                        |
|                        | \( v_v \) [km/s] | \( \sim 185 \) | \([50, 300]\)  | 200              | 40                        |
| Shifted                | \( v_0 \) [km/s] | \( \sim 175 \) | \([100, 300]\) | 200              | 40                        |
|                        | \( \Delta v \) [km/s] | \( \sim 10 \) | \([-120, 80]\) | -20              | 40                        |
| Variated shifted       | \( v_0 \) [km/s] | \( \sim 175 \) | \([100, 300]\) | 200              | 40                        |
|                        | \( \Delta v \) [km/s] | \( \sim 10 \) | \([-120, 80]\) | -20              | 40                        |
| Modified               | \( v_0 \) [km/s] | 220          | \([160, 300]\) | 230              | 20                        |
|                        | \( k \)          | 2            | \([0.5, 3.5]\) | 1                | 0.5                       |

Table 9. The reconstructed results of \( v_0 \) for all four considered cases with the simple Maxwellian velocity distribution \( f_{1, \text{Gau}}(v) \) as well as the 1 (2)\( \sigma \) uncertainty ranges of the median values.

As usual, in figures 11 we use the flat probability distribution for the fitting parameter \( v_0 \) with either the precisely known (input) (upper) or the reconstructed (lower) WIMP mass, respectively. It can be seen that, firstly, although the 1 (2)\( \sigma \) statistical uncertainty bands could still cover the true (input) velocity distribution (somehow well), the systematic deviations of the peaks of the reconstructed velocity distributions from that of the true (input) one would be \( \sim 15 \) km/s. The comparisons between reconstructed results shown in figures 11a and 11c to figures 10a and 10c could indicate a high probability of the improper assumption of the fitting (one-parameter) shifted Maxwellian velocity distribution.

Moreover, figures 11b and 11d (see also table 10) show clearly 4\( \sigma \) to even 6\( \sigma \) deviations of the reconstructed \( v_0 \) from the true (input) value of \( v_0 = 220 \) km/s. As discussed in section 3.2.1, this observation implies also an improper use of the (one-parameter) shifted Maxwellian velocity distribution as our fitting function.

On the other hand, by using the Gaussian probability distribution for \( v_0 \) with a slightly smaller expectation value of \( v_0 = 200 \) km/s and a 1\( \sigma \) uncertainty of 40 km/s, we found that, in contrast to our observations presented before, for this case the use of the Gaussian probability distribution would not reduce the 1 (2)\( \sigma \) statistical uncertainties for both cases with the true (input) and the reconstructed WIMP masses (see table 10). This would be caused by the large difference between the given expectation value and the (theoretically estimated) most
Figure 11. As in figures 10, except that the one-parameter shifted Maxwellian velocity distribution function \( f_{1,sh,v_0}(v) \) with the unique fitting parameter \( v_0 \) has been used as the fitting velocity distribution.

| Parameter \( v_0 \) [km/s] | Input  | Flat    | 162.0   | 160.0 ± 10.0 | 152.0, 170.0 | 146.0, 188.0 |
|-------------------------|--------|---------|---------|--------------|--------------|--------------|
|                         | Gaussian| 162.0   | 162.0 ± 10.0 | 154.0, 172.0 | 148.0, 188.0 |
|                         | Reconst.| Flat    | 162.0 ± 14.0 | 148.0, 176.0 | 138.0, 198.0 |
|                         |        | Gaussian| 164.0   | 162.0 ± 16.0 | 150.0, 178.0 | 138.0, 200.0 |

Table 10. The reconstructed results of \( v_0 \) for all four considered cases with the one-parameter shifted Maxwellian velocity distribution \( f_{1,sh,v_0}(v) \) as well as the 1 (2) \( \sigma \) uncertainty ranges of the median values.

suitable one of the parameter \( v_0 \) (175 km/s, see table 8). This would indicate that, in practical use of our Bayesian reconstruction method, some trial-and-error tests for determining a more
suitable expectation value of $v_0$ (and the other fitting parameters) would be necessary and could then improve the fitting results.

In table 10, we list the reconstructed results of $v_0$ for all four considered cases with the one-parameter shifted Maxwellian velocity distribution $f_{1,sh,v_0}(v)$ as well as the 1 (2) $\sigma$ uncertainty ranges of the median values of $v_0$.

### 3.3.3 Shifted Maxwellian velocity distribution

As in section 3.2.3, we release now the fixed relation between $v_0$ and $v_e$ given in eq. (3.9) and consider the reconstruction of these two parameters simultaneously and independently. In addition, we also assume here that, from the (naive) trials with the simple and one-parameter shifted Maxwellian velocity distributions done previously, one could already obtain a rough idea about the shape of the velocity distribution of halo WIMPs as well as expectation values of the Solar and Earth’s Galactic velocities $v_0$ and $v_e$. Hence, we consider here only the Gaussian probability distribution for both fitting parameters with a common expectation value of $v_0 = v_e = 200$ km/s and a common 1 $\sigma$ uncertainty of 40 km/s (see table 8). Note also that, after some trial-and-error tests, we set the scanning ranges of $v_0$ and $v_e$ as $v_0 \in [100, 300]$ km/s and $v_e \in [50, 300]$ km/s.

In figures 12, we show the case with the reconstructed WIMP mass. As observed in section 3.3.2, although the (low-velocity parts of the) 1 (2) $\sigma$ statistical uncertainty bands could cover the true (input) velocity distribution, the systematic deviations of the peaks of the reconstructed velocity distributions from that of the true (input) one would be $\sim 10$ km/s (a little bit better then results shown in figures 11a and 11c). Moreover, figures 12c and 12d show $\gtrsim 3\sigma$ deviations of the reconstructed values of $v_0$ and $v_e$ from the true (input, estimated) values of $v_0 = 220$ km/s and $v_e = 231$ km/s. Additionally, the best-fit value of $v_e$ is now even ($\sim 10$ km/s) smaller than the best-fit value of $v_0$. Hence, as discussed in section 3.3.2, this observation (combined with the results given in section 3.3.2) would indicate a high probability of the improper assumption of the fitting shifted Maxwellian velocity distribution.

Figures 12 as well as table 11 show that one could fit an improper model to experimental data (somehow) well with combinations of special, probably unusual values of the fitting parameters. On one hand, the reconstructed results could offer us (rough) information about the (shape of the) velocity distribution of Galactic WIMPs. On the other hand, however, the observation that the reconstructed values of $v_0$ and $v_e$ are 2$\sigma$ to even $\gtrsim 4\sigma$ different from our (theoretically) expected values would indicate evidently the improper assumption about the fitting velocity distribution function.

In table 11, we list the reconstructed results of $v_0$ and $v_e$ with the shifted Maxwellian velocity distribution $f_{1,sh}(v)$ as well as the 1 (2) $\sigma$ uncertainty ranges of the median values.

### 3.3.4 Variated shifted Maxwellian velocity distribution

As in section 3.2.4, in order to correct the systematic deviations of the results given with the shifted Maxwellian velocity distribution function in eq. (3.7), we consider here the use of its variation given in eq. (3.10). As previously, we consider here only the Gaussian probability distribution for both fitting parameters $v_0$ and $\Delta v$, with a common 1$\sigma$ uncertainty of 40 km/s. Moreover, according to the fitting results given in section 3.3.3 and some trial-and-error tests, we set $v_0 = 200$ km/s and $\Delta v = -20$ km/s as the expectation values as well as $v_0 \in [100, 300]$ km/s and $\Delta v \in [-120, 80]$ km/s as the scanning ranges.

Comparing figures 13 to figures 12, it can be seen clearly that the use of the variated shifted Maxwellian velocity distribution could indeed offer a more reasonable and preciser
Figure 12. As in figures 7, except that the modified Maxwellian velocity distribution function given in eq. (3.13) has been used for generating WIMP signals. Note that the solid red vertical line shown in (b) indicates the input value of the parameter $v_0$ (since no input value for $v_e$). The Gaussian probability distribution for both fitting parameters with a common expectation value of $0 = e$ as well as the $1 \sigma$ uncertainty ranges of the median values.

### Table 11

| Parameter | WIMP mass | Prob. dist | Max. $\text{median}$ | Median | $1 \sigma$ range | $2 \sigma$ range |
|-----------|-----------|------------|----------------------|--------|------------------|------------------|
| $v_0$ [km/s] | Input | Gaussian | 172.0 | $170.0 \pm 12.0 \pm 38.0$ | [158.0, 182.0] | [146.0, 208.0] |
| Reconst. | Gaussian | 172.0 | $170.0 \pm 12.0 \pm 44.0$ | [156.0, 188.0] | [140.0, 214.0] |
| $v_e$ [km/s] | Input | Gaussian | 162.5 | $162.5 \pm 17.5 \pm 42.5$ | [147.5, 180.0] | [135.0, 205.0] |
| Reconst. | Gaussian | 162.5 | $162.5 \pm 20.0 \pm 49.1$ | [147.5, 182.5] | [132.5, 210.6] |

Table 11. The reconstructed results of $v_0$ and $v_e$ with the shifted Maxwellian velocity distribution $f_1,_{\text{sh}}(v)$ as well as the $1 \sigma$ uncertainty ranges of the median values.
reconstruction: not only that the (1 (2) σ statistical uncertainty bands of the) reconstructed velocity distribution functions can match the true (input) one more closely, the 1 (2) σ statistical uncertainties on \( v_0 \) is also only \( \lesssim 70\% \) of the uncertainties shown in figure 12c. Additionally, by using eq. (3.12), the upper bound of the 1 (2) σ statistical uncertainties on \( v_0 \) can be (approximately) given as: \(+16.1^{+23.4}_{-12.8}\) km/s, which is also maximal (approximately) equal to or even smaller than the uncertainties shown in figure 12c. Hence, we would like to conclude that our introduction of the variated shifted Maxwellian velocity distribution (and/or probably some other variations) could indeed be a useful trick for the practical use of our Bayesian reconstruction procedure.

On the other hand, we can find that, although the width of the 1 (2) σ statistical uncertainty bands of the reconstructed velocity distribution functions as well as the 1 (2) σ statistical uncertainties on \( v_0 \) would be clearly larger, the 1 (2) σ statistical uncertainties on \( \Delta v \) are approximately equal to results with the true (input) WIMP mass. Meanwhile, even for the case with the reconstructed WIMP mass, the 1 (2) σ statistical uncertainties on \( v_0 \) and \( \Delta v \) are (much) smaller than the given 1σ uncertainty of their Gaussian probability distributions (40 km/s).

Remind that, our simulations of the use of the variated shifted Maxwellian velocity distribution function for fitting data generated by (modified) simple Maxwellian one shown here indicates again that, by using an “improper” assumption about the fitting velocity distribution function with prior knowledge about the fitting parameters (\( v_0, v_e \) or \( \Delta v \)), one would still reconstruct an approximate shape of the WIMP velocity distribution; the deviations of the peaks of the reconstructed velocity distributions from that of the true (input) one could even be \( \lesssim 10 \) km/s, much smaller than the commonly used 1σ uncertainty on the Solar Galactic velocity of \( \sim 20 \) km/s.

However, our simulations show also that, with an “improper” assumption about the fitting velocity distribution, one would obtain an “unexpected” result about each single fitting parameter. E.g. here we get 3σ to 5σ deviations of the reconstructed Solar Galactic velocity from the theoretical estimate (see table 12) and large “negative” values for the difference between the Solar and Earth’s Galactic velocities. This observation indicates clearly that our initial assumption about the fitting velocity distribution function would be incorrect or at least need to be modified.

In table 12, we list the reconstructed results of \( v_0 \) and \( \Delta v \) with the variated shifted Maxwellian velocity distribution \( f_{1,sh,\Delta v}(v) \) as well as the 1 (2) σ uncertainty ranges of the median values.

### 3.3.5 Modified Maxwellian velocity distribution

As the last tested fitting velocity distribution function with data generated by the modified Maxwellian velocity distribution given by eq. (3.13), we consider now the reconstruction of the modified Maxwellian velocity distribution itself with two fitting parameters: the Solar Galactic velocity \( v_0 \) and the power index \( k \).

In figures 14, we use the Gaussian probability distribution for the fitting parameter \( v_0 \) with an expectation value of \( v_0 = 230 \) km/s and a 1σ uncertainty of 20 km/s but the flat distribution for the parameter \( k \) with the reconstructed WIMP mass. Remind that, in figure 14d the bins at \( k = 0.5 \) and \( k = 3.5 \) are “overflow” bins. This means that they contain also the experiments whose best-fit values of \( k \) would be either \( k < 0.5 \) or \( k > 3.5 \).

First, as shown in section 3.3.1, the Solar Galactic velocity \( v_0 \) could be pinned down very precisely: a small systematic deviation of < 10 km/s and 1σ statistical uncertainty of \( \lesssim 10 \) km/s could be achieved. However, figures 14b and 14d show that, due to the small
Figure 13. As in figures 12, except that the variated shifted Maxwellian velocity distribution function given in eq. (3.10) with two fitting parameters \( v_0 \) and \( \Delta v \) is used. The Gaussian probability distribution for both fitting parameters with expectation values of \( v_0 = 200 \) km/s and \( \Delta v = -20 \) km/s and a common 1\( \sigma \) uncertainty of 40 km/s has been used.

| Parameter | WIMP mass | Prob. dist. | Max. \( p_{\text{median}} \) | Median | 1\( \sigma \) range | 2\( \sigma \) range |
|-----------|-----------|-------------|-----------------|--------|----------------|----------------|
| \( v_0 \) [km/s] | Input | Gaussian | 180.0 | 178.0 ± 8.0 | (160.0, 196.0) | (160.0, 204.0) |
| | Reconst. | Gaussian | 180.0 | 178.0 ± 14.0 | (162.0, 194.0) | (156.0, 212.5) |
| \( \Delta v \) [km/s] | Input | Gaussian | -32.0 | -32.0 ± 14.0 | (-42.0, -20.0) | (-52.0, -1.5) |
| | Reconst. | Gaussian | -32.0 | -32.0 ± 30.0 | (-42.0, -20.0) | (-52.0, -1.5) |

Table 12. The reconstructed results of \( v_0 \) and \( \Delta v \) with the variated shifted Maxwellian velocity distribution \( f_{1,\text{sh,}\Delta v}(v) \) as well as the 1 (2) \( \sigma \) uncertainty ranges of the median values.
difference between the modified Maxwellian velocity distribution with different power indices $k$ (see figure 9) and the large statistical fluctuation and 1σ reconstruction uncertainty with only 500 WIMP events (on average), our Bayesian reconstruction of the velocity distribution would be totally non-sensitive on the second fitting parameter (power index) $k$.

Nevertheless, the wide spread of the reconstructed power index $k$ (in particular, the high cumulative numbers in the (overflow) bin $k = 3.5$) and, in contrast, the narrow widths of the 1 (2)σ statistical uncertainty bands of the reconstructed velocity distribution functions imply that, for reconstructing the rough information about the one-dimensional WIMP velocity distribution, a precise value of the power index $k$ would not be crucial, and the simple Maxwellian velocity distribution $f_{1,\text{Gau}}(v)$ given in eq. (3.6) would already be a good approximation.\footnote{As described in ref. [25], the modification of the simple Maxwellian velocity distribution function $f_{1,\text{Gau},k}(v)$ given in eq. (3.13) has significant difference from the simple Maxwellian one given in eq. (3.6) in the}

Figure 14. As in figures 12, except that the modified Maxwellian velocity distribution function given in eq. (3.13) with two fitting parameters $v_0$ and $k$ is used. The Gaussian probability distribution for $v_0$ with an expectation value of $v_0 = 230 \text{ km/s}$ and a 1σ uncertainty of 20 km/s but the flat distribution for $k$ have been used. Remind that the bins at $k = 0.5$ and $k = 3.5$ are “overflow” bins, which contain also the experiments with the best-fit $k$ value of either $k < 0.5$ or $k > 3.5$.\footnote{As described in ref. [25], the modification of the simple Maxwellian velocity distribution function $f_{1,\text{Gau},k}(v)$ given in eq. (3.13) has significant difference from the simple Maxwellian one given in eq. (3.6) in the}
In table 13, we give the reconstructed results of $v_0$ and $k$ with the modified Maxwellian velocity distribution $f_{1,Gau,k}(v)$ as well as the $1$ ($2$) $\sigma$ uncertainty ranges of the median values. Note that, since the $1\sigma$ lower and upper bounds of the median values of $k$ are already beyond our scanning range, the $2\sigma$ bounds are meaningless to give here.

### 3.4 For different WIMP masses

In the previous sections 3.1 to 3.3, we fixed the input WIMP mass as $m_\chi = 100\text{ GeV}$. As a further test of our Bayesian reconstruction method for the one-dimensional WIMP velocity distribution, in this subsection, we consider the cases for either a light WIMP mass of $m_\chi = 25\text{ GeV}$ or a heavy WIMP mass of $m_\chi = 250\text{ GeV}$.

Here we consider only the *shifted* Maxwellian velocity distribution given in eq. (3.7) for generating WIMP events and show results with the reconstructed WIMP mass and two fitting functions: simple and shifted Maxwellian velocity distributions. All input setup and fitting parameters are the same as in section 3.2 (see table 3).

#### 3.4.1 For a light WIMP mass

We consider first a rather light WIMP mass of $m_\chi = 25\text{ GeV}$. Note that, firstly, since for our tested targets $^{28}\text{Si}$ and $^{76}\text{Ge}$, the kinematic maximal cut-off energies given in eq. (3.2) are only $Q_{\text{max,kin, Si}} = 68.12\text{ keV}$ and $Q_{\text{max,kin, Ge}} = 52.65\text{ keV}$, respectively, the maximal experimental cut-off energy for both targets in our simulations demonstrated here has been reduced to only $Q_{\text{max}} = 50\text{ keV}$. Secondly, since the lighter the WIMP mass, the sharper the expected recoil energy spectrum, the width of the first energy bin has been set as $b_1 = 5\text{ keV}$ and the total number of bins has been reduced to only four $\text{bins}$ between $Q_{\text{min}}$ and $Q_{\text{max}}$ ($B = 4$); up to two bins have been combined to a window and thus four windows ($W = 4$) will be reconstructed.

In figures 15 (cf. figures 5c and 5d), we use first the (improper) simple Maxwellian velocity distribution function with one parameter $v_0$ to fit the reconstructed-input data. As the first trial of the reconstruction of the one-dimensional WIMP velocity distribution *without* prior knowledge about the Solar Galactic velocity, the flat probability distribution has been used here (results with the Gaussian probability distribution are given in table 14).

Although we use the improper assumption about the fitting velocity distribution and have only four available data points (solid black vertical bars), the $1$ ($2$) $\sigma$ statistical uncertainty bands could still cover the true (input) velocity distribution with a systematic deviation of the peak of the reconstructed velocity distribution of $\lesssim 15\text{ km/s}$ from that of the true (input) one. However, the best-fit values of the Solar Galactic velocity are now $\simeq 294\text{ km/s}$ and $\sim 3\sigma$ apart from the theoretically expected value. Moreover, our further

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13 The last window is neglected automatically in the AMIDAS code, due to a very few expected event number in the last bin (window).

14 It has been found that, by reducing the total number of the energy bins (and in turn that of the windows) and thus collecting more events in one bin (window), the Bayesian reconstructed velocity distribution could be improved (significantly).
Figure 15. As in figures 5c and 5d: shifted and simple Maxwellian velocity distributions have been used for generating WIMP events and as the fitting function, respectively; the flat probability distribution for \( v_0 \) and the reconstructed WIMP mass has been used, except that the input WIMP mass has been set as \( m_\chi = 25 \text{ GeV} \). See the text for further details about the simulation setup.

### Table 13

| Parameter | Input: modified Maxwellian velocity distribution \( f_{1,Gau,k=2}(v) \) | Reconstruction: modified Maxwellian velocity distribution \( f_{1,Gau,k}(v) \) |
|-----------|---------------------------------------------------------------|---------------------------------------------------------------|
|           | Input mass | Prob. dist. | Max. \( \chi_{\text{median}} \) | Median | 1σ range | 2σ range |
| \( v_0 \) [km/s] | Input | Gaussian | 228.6 | \( 227.2^{+7.0}_{-8.4} \) (\( 218.8, 234.2 \)) | \( [219.0, 246.8] \) |
|           | Reconst. | Gaussian | 227.2 | \( 227.2^{+9.8}_{-12.6} \) (\( 214.6, 237.0 \)) | \( [199.2, 256.6] \) |
| \( k \) | Input | Flat | 3.47 | 3.38 | [0.5, 3.5] | × |
|           | Reconst. | Flat | 2.72 | 3.29 | [0.5, 3.5] | × |

The reconstructed results of \( v_0 \) and \( k \) with the modified Maxwellian velocity distribution \( f_{1,Gau,k}(v) \) as well as the 1 (2)σ uncertainty ranges of the median values. Note that, firstly, we use here the Gaussian probability distribution for \( v_0 \) but the flat one for \( k \). Secondly, since the 1σ lower and upper bounds of the median values of \( k \) are already beyond our scanning range, the 2σ bounds are meaningless to give here.

Simulations with the Gaussian probability distribution for the fitting parameter \( v_0 \) with an expectation value of \( v_0 = 280 \text{ km/s} \) and a 1σ uncertainty of 40 km/s show that, the 1 (2)σ statistical uncertainties for such a light WIMP mass could be reduced to \( \sim 60\% \) to \( \sim 80\% \) (compare table 14 to table 4).

In figures 16 (cf. figures 7), the shifted Maxwellian velocity distribution with two fitting parameters \( v_0 \) and \( v_e \) has been tested to fit to the reconstructed-input data. Only the Gaussian probability distribution for both fitting parameters with expectation values of \( v_0 = 230 \text{ km/s} \) and \( v_e = 245 \text{ km/s} \) and a common 1σ uncertainty of 20 km/s is considered here. Astonishingly, with only four available data points the reconstructed velocity distribution functions could match the true (input) one very precisely: the systematic deviation of \( v_0 \) is negligible and that of \( v_e \) is only a few km/s, the 1 (2)σ statistical uncertainties on two fitting parameters are also only \( +12.6 \) (\( +28.0 \)) and \( \pm 12.6(\pm 25.2) \), respectively.
Table 14. The reconstructed results with four fitting velocity distribution functions for the input WIMP mass of $m_\chi = 25$ GeV.

Moreover, we considered also the use of the varied shifted Maxwellian velocity distribution with two parameters $v_0$ and $\Delta v$ to fit to the reconstructed-input data. Only the Gaussian probability distribution for both fitting parameters with expectation values of $v_0 = 230$ km/s and $\Delta v = 15$ km/s and a common 1σ uncertainty of 20 km/s is considered here. It has been found that, although the 1(2)σ statistical uncertainty bands are a bit wider than those given with the shifted Maxwellian distribution, with only four available data points the reconstructed velocity distribution function could also match the true (input) one very well. Moreover, as shown in sections 3.2.4 and 3.3.4, the second fitting parameter $\Delta v$ could also be pinned down very precisely with a negligible systematic deviation (see table 14).

In table 14, we give the reconstructed results with all four fitting velocity distribution functions for the input WIMP mass of $m_\chi = 25$ GeV. Both cases with the true (input) and the reconstructed WIMP masses have been simulated and summarized.

### 3.4.2 For a heavy WIMP mass

We consider now a rather heavy WIMP mass of $m_\chi = 250$ GeV. The maximal experimental cut-off energy for both targets in our simulations demonstrated here has been set again as $Q_{\text{max}} = 100$ keV. And, as usual, the width of the first energy bin has been set as $b_1 = 10$ keV,
five bins between $Q_{\text{min}}$ and $Q_{\text{max}}$ are used ($B = 5$) and up to three bins have been combined to a window ($W = 6$).

For a heavy WIMP mass, due to the (large) statistical fluctuation discussed in ref. [6], our reconstructed velocity distribution functions given in figure 17a (cf. figures 5c and 15a) have clearly a (much) wider $1 (2) \sigma$ statistical uncertainty bands; the $1 (2) \sigma$ statistical uncertainties on the reconstructed Solar Galactic velocity are also (much) larger as $\pm 33.6 \ (\pm 64.8)$ km/s. And, as shown in figure 17b, a considerable fraction of the reconstructed $v_0$ would exceed our scanning upper bound of 400 km/s. Remind that the bin at $v_0 = 400$ km/s is an “overflow” bin, which contains also the experiments with the best-fit $v_0$ value of $> 400$ km/s. Nevertheless, comparing to the much larger $1 \sigma$ statistical uncertainty on the reconstructed-input data (solid black vertical bars), our Bayesian reconstruction with an in fact improper fitting velocity distribution could still offer an approximation with only $\lesssim 15$ km/s systematic deviation of the peak of the reconstructed velocity distribution from that of the true (input) one.
Figure 17. As in figures 15: the flat probability distribution for \( v_0 \) and the reconstructed WIMP mass have been used, except that the input WIMP mass has been set as \( m_\chi = 250 \) GeV. Remind that the bin at \( v_0 = 400 \) km/s is an “overflow” bin, which contains also the experiments with the best-fit \( v_0 \) value of > 400 km/s.

In figures 18 (cf. figures 7 and 16), the shifted Maxwellian velocity distribution with two fitting parameters \( v_0 \) and \( v_e \) has been tested to fit to the reconstructed-input data. Only the Gaussian probability distribution for both fitting parameters with expectation values of \( v_0 = 230 \) km/s and \( v_e = 245 \) km/s and a common 1σ uncertainty of 20 km/s is considered here.

With a proper fitting velocity distribution and one more fitting parameter, the reconstructed velocity distribution could now match the true (input) one very precisely with much narrower 1 (2) σ statistical uncertainty bands. Additionally, the 1 (2) σ statistical uncertainties on the fitting parameters \( v_0 \) and \( v_e \) can be significantly reduced to only \( \pm 9.8 \) (\( \pm 22.4 \)) and \( \pm 11.2 \) (\( \pm 26.6 \)), respectively. Note that, as shown in table 15, the use of the one-parameter shifted Maxwellian velocity distribution with only one fitting parameter \( v_0 \) and the fixed relation between \( v_0 \) and \( v_e \) would give a (much) wider 1 (2) σ statistical uncertainty bands of the reconstructed velocity distribution as well as a (much) larger 1 (2) σ statistical uncertainties on the reconstructed Solar Galactic velocity: \( \sim 50\% \) to a factor of \( \sim 2 \) larger.

Moreover, we tested also the possibility of improving the reconstruction precision by the use of the variated shifted Maxwellian velocity distribution with two fitting parameters \( v_0 \) and \( \Delta v \). Only the Gaussian probability distribution for both fitting parameters with expectation values of \( v_0 = 230 \) km/s and \( \Delta v = 15 \) km/s and a common 1σ uncertainty of 20 km/s is considered here.

While the reconstructed velocity distribution function could still match the true (input) one very precisely with however wider 1 (2) σ statistical uncertainty bands, the “best-fit” values of both parameters \( v_0 \) and \( \Delta v \) (and in turn \( v_e \)) could again be very precisely determined with negligible systematic deviations (see table 15).

Note that, as shown in figures 17 and 18, once the WIMP mass is heavy, for an experimental maximal cut-off energy \( Q_{\text{max}} \approx 100 \) GeV, the reconstructible velocity range of our model-independent data analysis method would be much smaller than our maximal cut-off velocity \( v_{\text{max}} \) (e.g. \( \sim 285 \) km/s for \( m_\chi = 250 \) GeV and the \(^{76}\text{Ge} \) target). Therefore, our simulations shown in this subsection demonstrate meaningfully that, our Bayesian reconstruction
of the one-dimensional WIMP velocity distribution would be an important improvement for offering more and precise information about the Galactic halo, e.g. the position of the peak of the WIMP velocity distribution, for the WIMP mass between $\mathcal{O}(20)$ GeV and even $\mathcal{O}(500)$ GeV.

In table 15, we give the reconstructed results with all four fitting velocity distribution functions for the input WIMP mass of $m_\chi = 250$ GeV. Both cases with the true (input) and the reconstructed WIMP masses have been simulated and summarized.

### 3.5 Background effects

In this last part of our presentation of the numerical simulations of the Bayesian reconstruction of the WIMP velocity distribution function, we consider the effects of unrejected background events. Similar to our earlier works in refs. [21, 22], we take into account a small fraction of artificially generated background events in the fake experimental data sets and want to study how well the WIMP velocity distribution as well as the fitting parameters could be reconstructed.

| Parameter | WIMP mass | Prob. dist. | Max. $p_\text{median}$ | Median | $1\sigma$ range | $2\sigma$ range |
|-----------|-----------|-------------|------------------------|--------|-----------------|-----------------|
| $v_0$ [km/s] | Input | Flat | 287.2 | $287.2^{+3.28}_{-2.16}$ ($^{+4.08}_{-4.08}$) | [265.6, 316.0] | [246.4, 356.8] |
| | | Gaussian | 284.8 | $284.8^{+19.2}_{-16.8}$ ($^{+36.0}_{-31.2}$) | [268.0, 304.0] | [253.6, 320.8] |
| Reconst. | Flat | 289.6 | $289.6^{+5.5}_{-4.3}$ ($^{+10.4}_{-7.4}$) | [246.4, 344.8] | [215.2, 400.0] |
| | Gaussian | 289.6 | $287.2^{+33.6}_{-31.2}$ ($^{+44.8}_{-40.0}$) | [256.0, 320.8] | [227.2, 352.0] |

**Table 15.** The reconstructed results with four fitting velocity distribution functions for the input WIMP mass of $m_\chi = 250$ GeV.
Figure 18. As in figures 16: the Gaussian probability distribution for both fitting parameters $v_0$ and $v_e$ as well as the reconstructed WIMP mass have been used, except that the input WIMP mass has been set as $m_\chi = 250$ GeV.

In all simulations demonstrated in this subsection, a combination of the target-dependent exponential form of the residue background spectrum introduced in ref. [22] with a small constant component has been used:

$$\left( \frac{dR}{dQ} \right)_{bg} = \left( \frac{dR}{dQ} \right)_{bg,ex} + \alpha \left( \frac{dR}{dQ} \right)_{bg,\text{const}},$$

where

$$\left( \frac{dR}{dQ} \right)_{bg,ex} = \exp \left( -\frac{Q}{\mathrm{keV}} A^{0.6} \right),$$

and

$$\left( \frac{dR}{dQ} \right)_{bg,\text{const}} = 1.$$

Here $Q$ is the recoil energy, $A$ is the atomic mass number of the target nucleus. The power index of $A$, 0.6, is an empirical constant, which has been chosen so that the exponential back-
ground spectrum is somehow similar to, but still different from the expected recoil spectrum of the target nuclei; otherwise, there is in practice no difference between the WIMP scattering and background spectra.\footnote{Note that, among different possible choices, we use in our simulations the atomic mass number $A$ as the simplest, unique characteristic parameter in the general analytic form (3.15) for defining the artificial residue background spectra for different target nuclei. However, it does not mean that the (superposition of the real) background spectra would depend simply/primarily on $A$ or on the mass of the target nucleus, $m_N$. In other words, it is practically equivalent to use the expression (3.15) or $(dR/dQ)_{bg,ex} = e^{-Q/13.5 \text{ keV}}$ directly for a $^{76}$Ge target (cf. [31]).} Additionally, $r_{\text{const}}$ is the ratio between the exponential and constant components in the total \textit{background} spectrum, which has been fixed as $r_{\text{const}} = 0.05$ for all simulations.

Note that, firstly, as argued in ref. [22], the exponential form of background spectrum is rather naive; but, since we consider here only \textit{a few tens residue} background events induced by \textit{several different} sources, pass all discrimination criteria, and then mix with other WIMP-induced events in our data sets of a few hundreds total events, an exact form of background spectrum for each target nucleus would not be crucial and the exponential + constant form of background spectrum in eq. (3.14) should practically not be unrealistic. Secondly, as demonstrated in refs. [5, 6] and in the previous subsections, our Bayesian reconstruction of the one-dimensional WIMP velocity distribution requires only measured recoil energies and occasionally prior knowledge about the Solar and Earth’s Galactic velocities. Hence, for applying this method to future real experimental data, prior knowledge about (different) background source(s) is \textit{not required at all}.

In figures 19, we show the measured recoil energy spectrum (solid red histogram) for a $^{76}$Ge (a) and a $^{28}$Si (b) targets with an input WIMP mass of $m_\chi = 100$ GeV. The dotted blue curve is the elastic WIMP-nucleus scattering spectrum for generating signal events, whereas the dashed green curve shows the \textit{artificial} background spectrum: the exponential background spectrum given in eq. (3.15) accompanied with an extra constant component, normalized to fit to the background ratio of 20%.

3.5.1 For a moderate WIMP mass

We consider first a moderate input WIMP mass of $m_\chi = 100$ GeV. The \textit{shifted} Maxwellian velocity distribution given in eq. (3.7) is used for generating WIMP signals. All input setup and fitting parameters are the same as in section 3.2 (see table 3) and a fraction of 20\% background events has been taken into account. Additionally, as in section 3.4, we consider only the use of the Gaussian probability distribution for the fitting parameters $v_0$ and $v_e$ as well as the use of the reconstructed WIMP mass.

We consider at first the one-parameter shifted Maxwellian velocity distribution. It has been found that, due to the extra background events in both of the low and high energy ranges (see figures 19), the reconstructed-input data (solid black vertical bars) would be shifted (strongly) to the low-velocity range: the peak of the solid black crosses is now at $\sim 220 \text{ km/s}$, i.e. $\sim 90 \text{ km/s}$ smaller then the position of the true (input) velocity distribution. However, our simulation indicates clearly and importantly that, by assuming the shifted Maxwellian WIMP velocity distribution and the \textit{time-averaged} relation between the Solar and Earth’s Galactic velocities, the reconstructed WIMP velocity distributions could alleviate this systematic shift: the deviations of the peaks of the (1 (2) $\sigma$ statistical uncertainty bands of the) reconstructed velocity distributions would only be $\sim 30^{+30}_{-20} \text{ (}^{+60}_{-40}\text{)} \text{ km/s}$.\footnote{Note that, as shown in figure 20a, the reconstructed WIMP mass is now \textit{overestimated}: $m_\chi, \text{rec} \approx 137 \text{ GeV.}
In fact, it has also been found that, once an (approximately) precisely determined (true) WIMP mass could be used, the reconstructed WIMP velocity distribution could match the true (input) one very precisely: the deviation of the reconstructed $v_0$ would be $\lesssim 10$ km/s (flat) or even negligible (Gaussian) (see table 16).

Note that, although a fraction of 20% unrejected background events has been mixed (artificially) into the analyzed (pseudo-)data sets, the $1 \sigma$ statistical uncertainty on the median value of the reconstructed $v_0$’s (202.0$^{+18.2}_{-16.8}$ km/s) would still cover the true (input) Solar Galactic velocity of $v_0 = 220$ km/s. Moreover, once we take into account the statistical fluctuation of the reconstructed-input data, the effect of 20% residue background events on reconstructing information about the (shape of the) WIMP velocity distribution function would not be very significant.

Then, we release the fixed relation between $v_0$ and $v_e$ and use the shifted Maxwellian velocity distribution to determine these two parameters simultaneously and independently (shown in figures 20). It has been found that, firstly, the $1\, (2)\, \sigma$ statistical uncertainty bands are narrower than those obtained with the one-parameter shifted Maxwellian velocity distribution; the deviations of the peaks of the reconstructed velocity distributions from that of the true (input) one would be reduced to only $\lesssim 15$ km/s. In addition, the systematic deviations and the $1\sigma$ statistical uncertainties on the median values of the reconstructed fitting parameters $v_0$ and $v_e$ shown in figures 20c and 20d are also (much) smaller (see table 16). Note here that, as given in table 16, once an (approximately) precisely determined (true) WIMP mass could be used, one could reconstruct the WIMP velocity distribution function very precisely with very small or even negligible systematic deviations of both two fitting parameters. This means that, our Bayesian reconstruction method for the WIMP velocity distribution function would not be affected (significantly) by a fraction of $\sim 20\%$ unrejected background events mixed in the analyzed data sets (for a WIMP mass of $\mathcal{O}(100)$ GeV).

In table 16, we give the reconstructed results with all four fitting velocity distribution functions for data sets mixed with 20% background events and the input WIMP mass of
Figure 20. As in figures 7: the shifted Maxwellian velocity distribution function and the Gaussian probability distribution for both fitting parameters $v_0$ and $v_e$ as well as the reconstructed WIMP mass have been used, except that a fraction of 20% background events generated by the spectrum given in eq. (3.14) has been taken into account.

$m_\chi = 100\text{ GeV}$. Both cases with the true (input) and the reconstructed WIMP masses have been simulated and summarized.

3.5.2 For a light WIMP mass

Now, we consider the case with a light input WIMP mass of $m_\chi = 25\text{ GeV}$. Simulation setup is the same as in section 3.4.1 and a fraction of 20% background events has been taken into account.

For the case with the one-parameter shifted Maxwellian velocity distribution, since the reconstructed WIMP mass would be $\sim 30\%$ overestimated ($m_{\chi,\text{rec}} \approx 31.5\text{ GeV}$) due to the extra background events and only four reconstructed-input data points are available, the “best-fit” one-parameter shifted Maxwellian velocity distribution functions would match not the true (input) velocity distribution, but the analyzed data points well. Nevertheless, at least, the 2$\sigma$ statistical uncertainty band of the reconstructed velocity distributions could cover the true (input) one; the systematic deviations of the peaks of the reconstructed velocity
## Input: shifted Maxwellian velocity distribution $f_{1,sh}(v)$

| Parameter | WIMP mass | Prob. dist. | Max. $\Delta v_{\text{median}}$ | Median | $1\sigma$ range | $2\sigma$ range |
|-----------|-----------|-------------|-------------------------------|--------|----------------|----------------|
| $v_0$ [km/s] | Input | Flat | 284.8 | 284.8 $^{+2.4}_{-0.2}$ | [265.6, 308.8] | [244.0, 337.6] |
| | | Gaussian | 284.8 | 284.8 $^{+16.8}_{-12.6}$ | [268.0, 301.6] | [251.2, 320.8] |
| | Reconst. | Flat | 253.6 | 253.6 $^{+30.0}_{-31.2}$ | [222.4, 289.6] | [193.6, 332.8] |
| | | Gaussian | 258.4 | 258.4 $^{+28.8}_{-26.4}$ | [232.0, 287.2] | [203.2, 316.0] |

### Reconstruction: one-parameter shifted Maxwellian velocity distribution $f_{1,sh,v_0}(v)$

| Parameter | WIMP mass | Prob. dist. | Max. $\Delta v_{\text{median}}$ | Median | $1\sigma$ range | $2\sigma$ range |
|-----------|-----------|-------------|-------------------------------|--------|----------------|----------------|
| $v_0$ [km/s] | Input | Flat | 211.8 | 211.8 $^{+16.8}_{-15.4}$ | [196.4, 228.6] | [181.0, 248.2] |
| | | Gaussian | 217.4 | 218.8 $^{+9.8}_{-11.2}$ | [207.6, 228.6] | [195.0, 239.8] |
| | Reconst. | Flat | 188.0 | 188.0 $^{+26.6}_{-22.4}$ | [165.6, 214.6] | [160.0, 244.0] |
| | | Gaussian | 202.0 | 202.0 $^{+18.2}_{-16.8}$ | [185.2, 220.2] | [167.0, 238.4] |

### Reconstruction: shifted Maxwellian velocity distribution $f_{1,sh}(v)$

| Parameter | WIMP mass | Prob. dist. | Max. $\Delta v_{\text{median}}$ | Median | $1\sigma$ range | $2\sigma$ range |
|-----------|-----------|-------------|-------------------------------|--------|----------------|----------------|
| $v_0$ [km/s] | Input | Gaussian | 225.8 | 225.8 $^{+7.0}_{-8.4}$ | [217.4, 232.8] | [189.4, 239.8] |
| | Reconst. | Gaussian | 214.6 | 214.6 $^{+12.6}_{-12.6}$ | [202.0, 226.8] | [189.4, 239.8] |
| $v_e$ [km/s] | Input | Gaussian | 231.4 | 231.4 $^{+8.4}_{-8.4}$ | [223.0, 239.8] | [213.2, 248.2] |
| | Reconst. | Gaussian | 220.2 | 220.2 $^{+12.6}_{-11.2}$ | [209.0, 232.8] | [196.4, 245.4] |

### Reconstruction: variated shifted Maxwellian velocity distribution $f_{1,sh,\Delta v}(v)$

| Parameter | WIMP mass | Prob. dist. | Max. $\Delta v_{\text{median}}$ | Median | $1\sigma$ range | $2\sigma$ range |
|-----------|-----------|-------------|-------------------------------|--------|----------------|----------------|
| $v_0$ [km/s] | Input | Gaussian | 220.2 | 220.2 $^{+8.4}_{-9.0}$ | [210.4, 228.6] | [200.6, 239.8] |
| | Reconst. | Gaussian | 204.8 | 204.8 $^{+16.8}_{-15.4}$ | [189.4, 221.6] | [174.0, 237.0] |
| $\Delta v$ [km/s] | Input | Gaussian | 5.9 | 5.9 $^{+5.2}_{-6.5}$ | [−0.6, 11.1] | [−7.1, 16.3] |
| | Reconst. | Gaussian | −0.6 | −0.6 $^{+7.8}_{-15.6}$ | [−8.4, 7.2] | [−16.2, 15.0] |

### Table 16. The reconstructed results with four fitting velocity distribution functions for data sets mixed with 20% background events and the input WIMP mass of $m_x = 100$ GeV.

The distributions from that of the true (input) one would also only be $\sim 30$ km/s. Meanwhile, the $2\sigma$ statistical uncertainty on the median value of the reconstructed $v_0$’s ($206.6^{+29.4}_{-26.6}$ km/s) would still cover the true (input) Solar Galactic velocity of $v_0 = 220$ km/s. This could be further improved by using an (approximately) precisely determined (true) WIMP mass to be $227.2^{+9.4}_{-8.4}$ km/s (flat) and $228.6 \pm 7.0$ km/s (Gaussian) (see table 17).

As the case of the 100 GeV WIMP mass shown in figure 20a, the $1(2)\sigma$ statistical uncertainty bands reconstructed with the shifted Maxwellian velocity distribution with two independent fitting parameters $v_0$ and $v_e$ shown in figure 21a would clearly be much narrower than those reconstructed with only one parameter $v_0$. Meanwhile, in contrast to other (presented) cases, our simulations with the true (input) WIMP mass show that the reconstructed-input data as well as the $1(2)\sigma$ statistical uncertainty bands of the) reconstructed velocity distribution function would slightly shift to the high-velocity range.

Furthermore, comparing results given in table 17 to those in table 16, it has been found interesting and probably importantly that, for an (input) WIMP mass of $O(20)$ GeV, the use of our variated shifted Maxwellian velocity distribution given in eq. (3.10) could offer preciser reconstruction results with (relatively) smaller statistical uncertainties, although fewer (four in our simulations) data points are available.

\[ – 42 – \]
would be only of the reconstructed WIMP velocity distribution functions from that of the true (input) one though the input WIMP mass is pretty heavy, the systematic deviations of the peak positions of the background spectrum used in our simulations would cause a strongly mixed with 20% background events and the input WIMP mass of $m_\chi = 250$ GeV. Simulation setup is the same as in section 3.4.2. Note however that, since the constant component of the background spectrum used in our simulations would cause a strongly overestimated WIMP mass, in particular, once WIMPs are heavy (e.g. the 250 GeV input WIMP mass would now be reconstructed as $\approx 338$ GeV) [22], the ratio of the background events in the analyzed data sets has been set as only 10% [21].

As usual, we consider first the one-parameter shifted Maxwellian velocity distribution function to fit the reconstructed-input data points. Unexpectedly, we have found that, although the input WIMP mass is pretty heavy, the systematic deviations of the peak positions of the reconstructed WIMP velocity distribution functions from that of the true (input) one would be only $\sim 10$ km/s (for 10% background ratio!). This might be due to that, as shown

| Parameter | WIMP mass | Prob. dist. | Max. $p_{\text{median}}$ | Median | $1\sigma$ range | $2\sigma$ range |
|-----------|-----------|-------------|----------------------|--------|----------------|----------------|
| $v_0$ [km/s] | Input | Flat | 308.8 | 306.4 $^{+10.8}_{-12.0}$ ($^{+25.6}_{-23.0}$) | [294.4, 323.2] | [282.4, 342.4] |
| | Gaussian | 304.0 | 304.0 ± 12.0 ($\pm 24.0$) | [292.0, 316.0] | [280.0, 328.0] |
| | Reconst. Flat | 260.8 | 260.8 $^{+24.0}_{-21.6}$ ($^{+51.0}_{-40.8}$) | [239.2, 284.8] | [220.0, 311.8] |
| | Gaussian 265.6 | 263.2 $^{+21.6}_{-19.2}$ ($^{+44.2}_{-38.4}$) | [244.0, 284.8] | [224.8, 306.4] |

Table 17. The reconstructed results with four fitting velocity distribution functions for data sets mixed with 20% background events and the input WIMP mass of $m_\chi = 25$ GeV.

In table 17, we give the reconstructed results with all four fitting velocity distribution functions for data sets mixed with 20% background events and the input WIMP mass of $m_\chi = 25$ GeV. Both cases with the true (input) and the reconstructed WIMP masses have been simulated and summarized.

3.5.3 For a heavy WIMP mass

As the last case, we consider here a heavy input WIMP mass of $m_\chi = 250$ GeV. Simulation setup is the same as in section 3.4.2. Note however that, since the constant component of the background spectrum used in our simulations would cause a strongly overestimated WIMP mass, in particular, once WIMPs are heavy (e.g. the 250 GeV input WIMP mass would now be reconstructed as $\approx 338$ GeV) [22], the ratio of the background events in the analyzed data sets has been set as only 10% [21].

As usual, we consider first the one-parameter shifted Maxwellian velocity distribution function to fit the reconstructed-input data points. Unexpectedly, we have found that, although the input WIMP mass is pretty heavy, the systematic deviations of the peak positions of the reconstructed WIMP velocity distribution functions from that of the true (input) one would be only $\sim 10$ km/s (for 10% background ratio!). This might be due to that, as shown
in section 3.4.2, for our used experimental maximal cut-off energy \( Q_{\text{max}} = 100 \) GeV and the \(^{76}\text{Ge}\) target, the reconstructible velocity range would only be \( \sim 270 \) km/s (shifted slightly to the low-velocity range due to the overestimated WIMP mass) and thus this maximal reconstructible velocity is theoretically smaller than the position of the peak of the velocity distribution function (see e.g. figure 22a). This means that the approximately monotonically increased shape of the reconstructed-input data points (solid black vertical bars) with pretty large 1\( \sigma \) statistical uncertainties would alleviate the effects of the overestimations of the analyzed (reconstructed-input) data points and the reconstructed WIMP mass caused by the extra background events.

Then we use the shifted Maxwellian velocity distribution function with two independent fitting parameters \( v_0 \) and \( v_e \) to fit the reconstructed-input data points. Astonishingly and unexpectedly (probably accidentally), figures 22 show that both of the “best-fit” results of the parameters \( v_0 \) and \( v_e \) are almost exact as the true (input) values and the 1\( \sigma \) statistical uncertainties on \( v_0 \) and \( v_e \) are only \( \sim 10 \) km/s.

Meanwhile, in contrast to our simulation results with the variated shifted Maxwellian velocity distribution function shown previously, for the case of the 250 GeV WIMP mass with
10% background ratio, the Bayesian reconstructed parameter $\Delta v$ could have a (much) larger deviations from the true (estimated) value (see table 18)!

In table 18, we give the reconstructed results with all four fitting velocity distribution functions for data sets mixed with 10% background events and the input WIMP mass of $m_\chi = 250$ GeV. Both cases with the true (input) and the reconstructed WIMP masses have been simulated and summarized.

### 4 Summary and conclusions

In this paper, we extended our earlier work on the development of the model-independent data analysis method for the reconstruction of the (time-averaged) one-dimensional velocity distribution of Galactic WIMPs and introduced the Bayesian fitting procedure of the theoretical velocity distribution functions.

In this fitting procedure, the (rough) velocity distribution reconstructed by using raw experimental data, i.e. measured recoil energies, with one or more different target nuclei has been used as reconstructed-input data (points). By assuming a fitting WIMP velocity...
Table 18. The reconstructed results with four fitting velocity distribution functions for data sets mixed with 10% background events and the input WIMP mass of $m_\chi = 250$ GeV.

distribution function and scanning the parameter space based on the Bayesian analysis, the (fitting) astronomical characteristic parameters, e.g. the Solar and Earth’s Galactic velocities $v_0$ and $v_e$, would be pinned down as the output results and thus the functional form of the one-dimensional velocity distribution can be reconstructed (instead of only a few discrete points).

As the first test of our Bayesian reconstruction method for the one-dimensional WIMP velocity distribution function, we used the simplest isothermal spherical Galactic halo model for both generating WIMP-signal events and as the assumed velocity distribution with the unique fitting parameter: the Solar Galactic velocity $v_0$. Our simulations show that, with (only) 500 recorded events (on average) and without prior knowledge about the Solar Galactic velocity, $v_0$ could in principle be pinned down with a negligible deviation and a 1σ statistical uncertainty of only $\sim 12$ km/s (with a precisely known WIMP mass) or $\sim 20$ km/s (with a reconstructed WIMP mass), respectively. Moreover, once (rough) information about the Solar Galactic velocity can be given, the statistical uncertainties on the reconstructed $v_0$ could even be reduced to $\sim 70\%$.

For more realistic consideration, we then took into account the orbital motion of the Solar system around our Galaxy as well as that of the Earth around the Sun and turned...
to use the shifted Maxwellian velocity distribution function for generating WIMP signals. As comparisons, four different fitting functions have been considered: the simple and the (one-parameter and variated) shifted Maxwellian velocity distributions. It has been found that, firstly, with an improper assumed fitting function (e.g. the simple Maxwellian velocity distribution here), the WIMP velocity distribution could still be reconstructed and offer some important information about Galactic WIMPs, e.g. the rough position of the peak of the one-dimensional velocity distribution function. The deviations of the peaks of the reconstructed velocity distributions from that of the true (input) one would be only $\sim 10 \text{ km/s}$. However, the best-fit value(s) of the fitting parameter(s) would be unexpected/unreasonable. For instance, the reconstructed Solar Galactic velocity $v_0$ would be $2\sigma$ (with the reconstructed WIMP mass) or even $4\sigma$ (with the input WIMP mass) apart from its theoretical estimate. Such an observation could in turn be an important criterion on the assumption of fitting velocity distribution function.

Moreover, our simulations with the (one-parameter and variated) shifted Maxwellian velocity distributions show that, although in all of these three cases the reconstructed velocity distributions could match the true (input) one pretty precisely, with two fitting parameters the $1(2)\sigma$ statistical uncertainty bands of the reconstructed velocity distributions would be narrower than those with only one fitting parameter. In addition, the use of the variation of the shifted Maxwellian velocity distribution could (strongly) reduce the systematic deviations of the determinations of the characteristic Solar and Earth’s Galactic velocities $v_0$ and $v_e$, with however a bit larger statistical uncertainties.

Furthermore, we considered also a modification of the simple Maxwellian velocity distribution with an extra power index as the generating WIMP velocity distribution. First, we used the simple Maxwellian velocity distribution without the power index as the test fitting function. Since the difference between the modification and the original simple Maxwellian velocity distributions are very tiny, the reconstructed velocity distribution function could match the true (input) one very precisely and the characteristic Solar Galactic velocity could also be reconstructed with negligible systematic deviation.

Meanwhile, our simulations with the (one-parameter and variated) shifted Maxwellian velocity distribution functions show that, although the positions of the peak of the reconstructed velocity distribution would be only $\lesssim 10 \text{ km/s}$ deviated from the true (input) one, a clear $2\sigma$ to even $6\sigma$ difference between the best-fit values of the Solar and/or the Earth’s Galactic velocities and the true (input) ones could be observed. Such results would in turn indicate evidently the improper assumption of the shifted Maxwellian velocity distribution function.

Moreover, from the simulations with the modified simple Maxwellian velocity distribution with the power index as the second fitting parameter, it has been found that our Bayesian reconstruction of the WIMP velocity distribution would be (totally) non-sensitive on the power index. This means that, unfortunately, with only a few hundreds of recorded WIMP events it would still be impossible to distinguish (evidently) different subtle variations of the (simple and shifted) WIMP velocity distribution functions.

As comparisons, we considered also a light and a heavy input WIMP masses. For the case of light WIMPs, due to the sharp shapes of the recoil energy spectra and the small kinetic maximal cut-off energies, the recorded WIMP events would need to be separated into fewer bins/windows. However, our simulations show that, with only four available reconstructed-input data points, the true (input) velocity distribution function could astonishingly be reconstructed very precisely. On the other hand, once WIMPs are heavy, the
statistical fluctuation on the reconstructed WIMP mass becomes pretty large and hence the Bayesian reconstructions of the velocity distribution as well as of the Solar and Earth’s Galactic velocities would have large statistical uncertainties. Nevertheless, the reconstructed velocity distribution function with the best-fit characteristic Solar and Earth’s Galactic velocities could still match the true (input) one very precisely.

Finally, the effects of residue (unrejected) background events mixed in data sets to analyze have also been considered. Three different WIMP masses with background ratios of 10% or 20% have been tested. Although, due to the choice of our artificial residue background spectrum, the reconstructed WIMP masses would be overestimated and the (rough shape of) the reconstructed-input data points would thus be shifted (significantly) to lower velocities, the functional forms of the chosen fitting velocity distributions could somehow alleviate these systematic shifts and the 1 (2)σ statistical uncertainty bands could still cover the true (input) velocity distribution. In particular, for heavy WIMPs, since the reconstructed-input data points should be in the velocity range smaller than the position of the peak of the velocity distribution function, its approximately monotonically increased shape with pretty large 1σ statistical uncertainties would alleviate the effects of the overestimations of the analyzed (reconstructed-input) data points and the reconstructed WIMP mass caused by the extra background events. The reconstructed velocity distribution function could then match the true (input) one pretty well.

It would be worth to emphasize that, first, comparing to the pretty large (1σ) statistical uncertainties on the reconstructed-input data points (offered by our model-independent method developed in ref. [5] with raw experimental measured recoil energies), our Bayesian reconstruction of the WIMP velocity distribution function introduced here with only a few km/s deviation and O(10) km/s 1σ statistical uncertainties on the reconstructed Solar and Earth’s Galactic velocities would be a remarkable improvement.

Second, all our simulations show importantly that, even initial values different slightly from the true (input) setup have been used as the expectation values for the Gaussian probability distribution of the fitting parameters, these fitting parameters could still be pinned down (pretty) precisely. As long as a proper assumed fitting velocity distribution function is used, the best-fit values of the reconstructed parameters could always be less than 1σ apart from the true (input/theoretical) values. This observation indicates that rough, slightly incorrect prior knowledge about our fitting parameters would not affect (significantly) the reconstructed results in our Bayesian reconstruction procedure.

Moreover, by rewriting the functional form of the (basic) fitting velocity distribution function, one could not only pin down the fitting parameters more precisely, but also occasionally reduce the statistical uncertainties on the reconstructed parameters.

In summary, we developed in this paper the Bayesian reconstruction procedure for fitting theoretically predicted models of the one-dimensional WIMP velocity distribution function to data (points), which can be reconstructed directly from experimental measured recoil energies. Hopefully, this extension of our earlier work could offer more useful information about the Dark Matter halo, which could further be used in e.g. indirect DM detection experiments.

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