Derived factorization categories of non-Thom–Sebastiani-type sums of potentials

Yuki Hirano¹  |  Genki Ouchi²

¹Department of Mathematics, Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto, Japan
²Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Japan

Correspondence
Yuki Hirano, Department of Mathematics, Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan.
Email: y.hirano@math.kyoto-u.ac.jp

Funding information
JSPS, Grant/Award Numbers: 17H06783, 19K14502, 19K14520; RIKEN

Abstract
We first prove semi-orthogonal decompositions of derived factorization categories arising from sums of potentials of gauged Landau–Ginzburg models, where the sums are not necessarily Thom–Sebastiani type. We then apply the result to the category $\text{HMF}_{\mathcal{L}_f}(f)$ of maximally graded matrix factorizations of an invertible polynomial $f$ of chain type, and explicitly construct a full strong exceptional collection $E_1, \ldots, E_\mu$ in $\text{HMF}_{\mathcal{L}_f}(f)$ whose length $\mu$ is the Milnor number of the Berglund–Hübsch transpose $\tilde{f}$ of $f$. This proves a conjecture, which postulates that for an invertible polynomial $f$ the category $\text{HMF}_{\mathcal{L}_f}(f)$ admits a tilting object, in the case when $f$ is a chain polynomial. Moreover, by careful analysis of morphisms between the exceptional objects $E_i$, we explicitly determine the quiver with relations $(Q,I)$ which represents the endomorphism ring of the associated tilting object $\bigoplus_{i=1}^\mu E_i$ in $\text{HMF}_{\mathcal{L}_f}(f)$, and in particular we obtain an equivalence $\text{HMF}_{\mathcal{L}_f}(f) \cong \text{Db}^{\text{mod}}(kQ/I)$.

MSC 2020
14F08 (primary), 13C14 (secondary)
1 | INTRODUCTION

1.1 | Backgrounds

Let \( f \in S^n := \mathbb{C}[x_1, \cdots, x_n] \) be a quasi-homogeneous polynomial. The Milnor number of \( f \), denoted by \( \mu(f) \), is the dimension of the Jacobian ring \( S^n / (\delta f / \partial x_1, \ldots, \delta f / \partial x_n) \) as a \( \mathbb{C} \)-vector space. We say that \( f \in S^n \) is an invertible polynomial if the following conditions hold.

(i) There is a \( n \times n \)-matrix \( E = (E_{i,j}) \) whose entries \( \{E_{i,j}\} \) are non-negative integers such that \( E \) is invertible over \( \mathbb{Q} \), and that \( f \) is of the form

\[
f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \left( \prod_{j=1}^{n} x_j^{E_{i,j}} \right).
\]

(ii) The Berglund–Hübsch transpose \( \tilde{f} \) of \( f \), which is defined by

\[
\tilde{f}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \left( \prod_{j=1}^{n} x_j^{E_{j,i}} \right),
\]

is also quasi-homogeneous.

(iii) We have \( 1 \leq \mu(f) < \infty \) and \( 1 \leq \mu(\tilde{f}) < \infty \).

Invertible polynomials are studied in [9, 10] to construct pairs of topological mirror Calabi–Yau manifolds as a generalization of the Green–Plesser construction [27]. By Kreuzer–Skarke [41], an invertible polynomial is the Thom–Sebastiani sums of several invertible polynomials of the following two types:

**Chain type**

\[
x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{n-2} x_{n-1}^{a_{n-1}} + x_{n-1} x_{n}^{a_n}
\]

(\( a_1 \geq 2 \) and \( a_n \geq 2 \))

**Loop type**

\[
x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1
\]

(\( n \geq 2 \)).

Here, for invertible polynomials \( f \in S^n \) and \( g \in S^m \), the Thom–Sebastiani sum \( f \boxplus g \) of \( f \) and \( g \) is defined by \( f \boxplus g := f \otimes 1 + 1 \otimes g \in S^n \otimes \mathbb{C} S^m \cong S^{n+m} \). We temporarily say that an invertible polynomial \( f \) is indecomposable if it cannot be decomposed into the Thom–Sebastiani sum of two invertible polynomials, or equivalently if it is of chain type or loop type. When \( n = 1 \), an indecomposable invertible polynomial is of the form \( x_1^{a_1} \) for some \( a_1 \geq 2 \), and this polynomial is said to be of Fermat type. For an invertible polynomial \( f \), we can associate two triangulated categories in the context of the study of singularities. One is the homotopy category \( \text{HMF}^{L_f}_{S^n}(f) \) of maximally graded matrix factorizations of \( f \). Here the maximal grading \( L_f \) of \( f \) is the abelian group defined by

\[
L_f := \left( \bigoplus_{i=1}^{n} \mathbb{Z} \tilde{x}_i \right) \oplus \mathbb{Z} \tilde{f} / \left( \tilde{f} - \sum_{j=1}^{n} E_{i,j} \tilde{x}_j \right)_{1 \leq i \leq n}.
\]

This category \( \text{HMF}^{L_f}_{S^n}(f) \) is equivalent to the maximally graded singularity category \( \text{D}^{L_f}_{\text{sg}}(S^n / f) \) of the hypersurface \( S^n / f \) and the stable category \( \text{CM}^{L_f}(S^n / f) \) of maximally graded Cohen–Macaulay
modules over $S^n/f$. Another one is the derived category $D^b \text{Fuk}^- (f)$ of a directed Fukaya category of $f$. The category $D^b \text{Fuk}^- (f)$ is the categorification of the Milnor lattice of the isolated hypersurface singularity $f$. Takahashi proposed the following conjecture as a version of homological mirror symmetry for isolated hypersurface singularities (see also [15, 16, 61]).

**Conjecture 1.1** [60, Conjecture 21]. *For an invertible polynomial $f$, there is a finite acyclic quiver $Q$ with admissible relations $I$ such that we have equivalences*

$$HMF^L_{S^n}(f) \simeq D^b(\text{mod } \mathbb{C}Q/I) \simeq D^b \text{Fuk}^- (f).$$

For ADE polynomials, the first equivalence in Conjecture 1.1 is proved by Kajiura–Saito–Takahashi [37] and the second equivalence is proved by Seidel [57]. Conjecture 1.1 for Brieskorn–Pham singularities and the Thom–Sebastiani sum of ADE polynomials of type A or D follow from Futaki–Ueda [23, 24]. Recently, Habermann–Smith [29] and Habermann [28] proved Conjecture 1.1 for the $n = 2$ cases by constructing tilting objects on the both sides. Conjecture 1.1 leads to the following conjecture.

**Conjecture 1.2.** *Let $f \in S^n$ be an invertible polynomial. The category $HMF^L_{S^n}(f)$ has a full strong exceptional collection of length $\mu(f)$.*

This conjecture is proved by Kajiura–Saito–Takahashi [37, 38] for special $n = 3$ cases, namely ADE singularities and 14 exceptional unimodular singularities, and recently Kravets [40] proved Conjecture 1.2 for the cases when $n \leq 3$. Conjecture 1.2 implies the following.

**Conjecture 1.3** [43, Conjecture 1.3]. *Let $f \in S^n$ be an invertible polynomial. The category $HMF^L_{S^n}(f)$ has a tilting object.*

Proving Conjecture 1.3 is important in Lekili–Ueda’s work [43, Theorem 1.6] and the study of Cohen–Macaulay representations of graded Gorenstein rings [36, Problem 3.4]. Conjecture 1.2 can be weakened to the following.

**Conjecture 1.4.** *Let $f \in S$ be an invertible polynomial. The category $HMF^L_{S}(f)$ has a full exceptional collection of length $\mu(f)$.*

In an unpublished work by Yoko Hirano and Takahashi, they proved that Conjecture 1.4 is true when $n \leq 3$, and Conjecture 1.2 is true if $n \leq 3$ and $f$ is an invertible polynomial of chain type. For an invertible polynomial $f$ of chain type, Aramaki and Takahashi [2] constructed the full exceptional collection of the category $HMF^L_{S^n}(f)$ as the algebraic counterpart of the Orlik–Randell conjecture, which is proved by Varolgunes [64] recently. After the previous version of our paper appeared, Conjecture 1.4 was proved by Favero–Kaplan–Kelly [21].

By recent progress of the derived Morita theory for factorizations by [6], Conjecture 1.2 reduces to the case of indecomposable invertible polynomials, that is, of chain type or loop type. In this paper, we prove Conjecture 1.2 for invertible polynomials of chain type.
1.2 Main result

For each positive integer \( i \in \mathbb{Z}_{\geq 1} \), choose a positive integer \( a_i \in \mathbb{Z}_{\geq 1} \). Then for \( n \in \mathbb{Z}_{\geq 1} \), we have the associated polynomial

\[
f_n := x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{n-1} x_n^{a_n},
\]

and we set \( f_0 := 0 \). This polynomial \( f_n \) is an invertible polynomial of chain type only when \( a_n > 1 \), but we also discuss the case when \( a_n = 1 \). Even when \( a_n = 1 \), we can define the maximal grading \( L_n := L_{f_n} \) of \( f_n \) similarly. More precisely, the maximal grading \( L_n \) is defined by

\[
L_n := \left( \bigoplus_{i=1}^{n} \mathbb{Z} \vec{x}_i \right) \oplus \mathbb{Z} \vec{f}_n \bigg/ \langle \vec{f}_n - a_1 \vec{x}_1, \vec{f}_n - \vec{x}_{i-1} - a_i \vec{x}_i \mid 2 \leq i \leq n \rangle, \tag{1.A}
\]

and we set \( L_0 := \mathbb{Z} \vec{f}_0 \cong \mathbb{Z} \). Note that \( L_1 \cong \mathbb{Z} \) and \( \text{rk}(L_n) = 1 \) for any \( n \geq 0 \). The \( n \)-dimensional polynomial ring \( S^n = \mathbb{C}[x_1, \ldots, x_n] \) has a natural \( L_n \)-grading such that \( \deg(x_i) := \vec{x}_i \) for any \( 1 \leq i \leq n \) and \( \deg(f) = \vec{f} \). For \( m \geq -1 \), we set

\[
C_m := \text{HMF}_{S_m}(f_m),
\]

where \( C_{-1} \) is the zero category. Then, we have a natural equivalence \( C_0 = \text{HMF}_{\mathbb{C}}(0) \cong \mathbb{D}(\text{coh Spec } \mathbb{C}) \). For each \( n \), we define the functors

\[
\psi_i : C_{n-1} \rightarrow C_n \quad (0 \leq i \leq a_n - 2)
\]

\[
\phi_j : C_{n-2} \rightarrow C_n \quad (0 \leq j \leq a_{n-1} - 1)
\]

explicitly (see Subsection 4.1 for the details). Using these functors, we inductively construct the full strong exceptional collection \( \mathcal{E}_n \) of the category \( C_n \) as follows:

For a sequence \( \mathcal{E} = (E_1, \ldots, E_r) \) of objects in \( C_{n-1} \) and for each \( 0 \leq i \leq a_n - 2 \), we define a sequence \( \psi_i \mathcal{E} \) of objects in \( C_n \) by

\[
\psi_i \mathcal{E} := (\psi_i E_1, \ldots, \psi_i E_r).
\]

Similarly, for a sequence \( \mathcal{F} = (F_1, \ldots, F_s) \) of objects in \( C_{n-2} \) and for \( 0 \leq j \leq a_{n-1} - 1 \), we define a sequence \( \phi_j \mathcal{F} \) of objects in \( C_n \) by

\[
\phi_j \mathcal{F} := (\phi_j F_1, \ldots, \phi_j F_s).
\]

Moreover, we set

\[
\mathcal{E}^{-1} := \emptyset \quad \text{and} \quad \mathcal{E}^0 := \{(0 \rightarrow \mathbb{C} \rightarrow 0)\}.
\]

Then, we inductively define the sequence \( \mathcal{E}^n \) by

\[
\mathcal{E}^n := \left( \psi_0 \mathcal{E}^{n-1}, \ldots, \psi_{(a_n-2)} \mathcal{E}^{n-1}, \phi_0 \mathcal{E}^{n-2}, \ldots, \phi_{(a_{n-1}-1)} \mathcal{E}^{n-2} \right).
\]

The following is our main result in this paper.
Theorem 1.5 (Theorem 4.1). For any \( n \geq 1 \), the sequence \( \mathcal{E}^n \) is a full strong exceptional collection in \( \mathcal{C}_n \), and if \( a_n \geq 2 \) the length of \( \mathcal{E}^n \) is equal to the Milnor number \( \mu(\tilde{f}_n) \) of \( \tilde{f}_n \). In particular, the category \( \mathcal{C}_n \) has a tilting object.

To prove Conjecture 1.1, it is important to determine a quiver with relation \((Q, I)\) such that \( \mathcal{C}_n \cong D^b(\mathbb{C}Q/I) \). By a careful analysis of morphisms between exceptional objects in \( \mathcal{E}^n \), we also determine a quiver with relations \((Q^n, I^n)\) such that the endomorphism ring \( \text{End}_{\mathcal{C}_n}(T_n) \) of the associated tilting object \( T_n := \bigoplus_{E \in \mathcal{E}^n} E \) is isomorphic to \( \mathbb{C}Q^n/I^n \), and in particular we obtain an equivalence

\[
\mathcal{C}_n \cong D^b(\mathbb{C}Q^n/I^n).
\]

We expect that this equivalence can be applied to the study of Conjecture 1.1 for chain polynomials.

1.3 Contribution to Cohen–Macaulay representations

For an abelian group \( L \) of rank one and an \( L \)-graded Gorenstein ring \( R \), the stable category \( \text{CM}^L(R) \) of \( L \)-graded Cohen–Macaulay modules over \( R \) is one of the important invariants in the study of Cohen–Macaulay representations of \( R \), and the existence of a tilting object in the category \( \text{CM}^L(R) \) is widely studied. For example, if \( R = \bigoplus_{i \geq 0} R_i \) has a positive grading of \( L = \mathbb{Z} \) with \( R_0 \) a field, \( \text{CM}^\mathbb{Z}(R) \) has a tilting object either \( \dim R = 0 \) by Yamaura [65] or \( \dim R = 1 \) and \( R \) is reduced by Buchweitz–Iyama–Yamaura [13]. When \( \dim R \geq 2 \), \( \text{CM}^\mathbb{Z}(R) \) does not have a tilting object in general, and there is no general existence result for a tilting object in \( \text{CM}^\mathbb{Z}(R) \). Therefore, it is important to find classes of \((R, L)\) containing higher dimensional \( R \) such that \( \text{CM}^L(R) \) has a tilting object, and it is proved by Herschend–Iyama–Minamoto–Oppermann that Geigle–Lenzing complete intersection, which are a generalization of weighted projective line [26], are one such class [31].

Our result gives a new class \((R, L)\) such that \( \text{CM}^L(R) \) has a tilting object. More precisely, with the same notation as in the previous subsection, Theorem 1.5 implies that the category

\[
\text{CM}^L(S^n/f_n)
\]

has a tilting object. Moreover, the algebras \( \mathbb{C}Q^n/I^n \) are new examples of finite-dimensional algebras whose derived categories are fractional Calabi–Yau, which might be of independent interest.

1.4 Outline of the proof of Theorem 1.5

First, we see that \( \mathcal{C}_n \) is equivalent to the derived factorization category, which is a geometric interpretation of \( \mathcal{C}_n \), as follows: To the invertible polynomial \( f_n \) of chain type, we associate the maximal symmetry group

\[
G_n := \left\{ (\lambda_1, \ldots, \lambda_n) \in (\mathbb{G}_m)^n \left| \lambda_1^{a_1} = \lambda_1 \lambda_2^{a_2} = \cdots = \lambda_{n-1} \lambda_n^{a_n} \right. \right\} \tag{1.B}
\]
of \( f_n \), which naturally acts on \( \mathbb{A}^n \). Then, the chain polynomial \( f_n \) is a semi-invariant regular function on \( \mathbb{A}^n \) with respect to the \( G_n \)-action and a character

\[
\chi_{f_n} : G_n \to \mathbb{G}_m; (\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1^{a_1}
\]

of \( G_n \), and these define a gauged Landau–Ginzburg model \((\mathbb{A}^n, \chi_{f_n}, f_n)^{G_n}\). Then, we have its derived factorization category

\[
D_{coh}^n := \text{Dcoh}_{G_n}(\mathbb{A}^n, \chi_{f_n}, f_n),
\]

and it is standard that the category \( D_{coh}^n \) is equivalent to \( C_n \).

Next, we prove a semi-orthogonal decomposition of \( C_n \), which describes relationships among \( C_{n-2}, C_{n-1} \) and \( C_n \). For this, we show that there are fully faithful functors

\[
\Psi_i : D_{coh}^{n-1} \to D_{coh}^n (-a_n + 2 \leq i \leq 0)
\]

\[
\Phi_j : D_{coh}^{n-2} \to D_{coh}^n (-a_{n-1} + 1 \leq j \leq 0)
\]

and a semi-orthogonal decomposition

\[
D_{coh}^n = \left\langle \text{Im}(\Psi_0), \ldots, \text{Im}(\Psi(-a_n + 2)), \bigoplus_{j=0}^{-a_{n-1}+1} \text{Im}(\Phi_j) \right\rangle. \tag{1.C}
\]

This semi-orthogonal decomposition is deduced from a more general semi-orthogonal decomposition associated to the sum of polynomials that are not necessarily Thom–Sebastiani sums. We prove this general semi-orthogonal decompositions by modifying arguments appearing in the proofs of \([5, \text{Corollary 3.4}; 32, \text{Theorem 5.3}]\), which are analogous to results of \([48, \text{Theorem 40}]\). By natural equivalences \( C_m \cong D_{coh}^m \) for \( n-2 \leq m \leq n \), we obtain corresponding fully faithful functors \( \Psi_i : C_{n-1} \to C_n \) and \( \Phi_j : C_{n-2} \to C_n \) and the corresponding semi-orthogonal decomposition

\[
C_n = \left\langle \text{Im}(\Psi_0), \ldots, \text{Im}(\Psi(-a_n + 2)), \bigoplus_{j=0}^{-a_{n-1}+1} \text{Im}(\Phi_j) \right\rangle.
\]

Finally, we describe the fully faithful functors \( \Psi_i \) and \( \Phi_j \) to obtain the following isomorphisms

\[
\Psi_i \cong \psi_{-i}[i] \quad \text{and} \quad \Phi_j \cong \phi_{-j}[-a_n + 1 + j].
\]

By these explicit descriptions and the above semi-orthogonal decomposition, we see that the sequence \( \mathcal{E}^n \) is a full exceptional collection, and we prove strongness by utilizing Serre duality and certain exact triangles (see Lemma \( 4.11 \)) in \( C_n \).

### 1.5 Organization of the paper

This paper is organized as follows: In Section 2, we provide the basic definitions and properties of equivariant coherent sheaves, graded modules, derived factorization categories, and homotopy
categories of graded matrix factorizations. In Section 3, we prove semi-orthogonal decompositions of derived factorization categories associated to the sums of certain functions, which are generalizations of (1.C). In Section 4, we prove Theorem 1.5 and compute the quiver with relations associated to the full strong exceptional collection. In Section 5, we discuss further applications of the general version of the semi-orthogonal decomposition proved in Section 3, and we prove a generalization of Kuznetsov–Perry’s semi-orthogonal decompositions of homotopy categories of graded matrix factorizations of the Thom–Sebastiani sum \( f \boxplus t^N \) of a quasi-homogeneous polynomial \( f \) and a monomial \( t^N \) in one variable \( t \). In the Appendix, we provide a brief summary of equivariant categories, which is necessary in Section 5.

1.6 | Notation and convention

- Unless stated otherwise, all categories and stacks are over an algebraically closed field \( k \) of characteristic zero.
- For a functor \( F : A \to B \), we denote by \( \text{Im}(F) \) the essential image of \( F \).
- For an integer \( l \in \mathbb{Z} \), \( \chi : \mathbb{G}_m \to \mathbb{G}_m \) denotes the character of \( \mathbb{G}_m \) defined by \( \chi(a) := a^l \) for \( a \in \mathbb{G}_m \).
- For a character \( \chi : G \to \mathbb{G}_m \) of an algebraic group \( G \), we denote by \( \mathcal{O}(\chi) \) the \( G \)-equivariant invertible sheaf on \( X \) associated to \( \chi \). For a \( G \)-equivariant quasi-coherent sheaf \( F \) on \( X \), we set \( F(\chi) := F \otimes \mathcal{O}(\chi) \).
- For a dg-category \( A \), we denote by \( [A] \) its homotopy category.

2 | PRELIMINARIES

2.1 | Equivariant coherent sheaves

We recall the basics of equivariant sheaves. Let \( G \) be an algebraic group acting on a scheme \( X \). A quasi-coherent \( G \)-equivariant sheaf on \( X \) is a pair \((F, \theta)\) of a quasi-coherent sheaf \( F \) and an isomorphism \( \theta : \pi^* F \to \sigma^* F \) such that

\[
t^\ast \theta = \text{id}_F \quad \text{and} \quad (\text{id}_G \times \sigma) \circ (s \times \text{id}_X))^\ast \theta \circ (\text{id}_G \times \pi)^\ast \theta = (m \times \text{id}_X)^\ast \theta,
\]

where \( m : G \times G \to G \) is the multiplication and \( s : G \times G \to G \times G \) is the switch of two factors. Note that if \((F, \theta)\) is a quasi-coherent \( G \)-equivariant sheaf, for any \( g \in G \) the equivariant structure \( \theta \) defines an isomorphism

\[
\theta_g : F \cong \sigma_g^* F,
\]

where \( \sigma_g : X \to X \) is the action by \( g \in G \). We call \((F, \theta)\) coherent (respectively, locally free, injective) if the sheaf \( F \) is coherent (respectively, locally free, injective).

We denote by \( \text{Qcoh}_G X \) (respectively, \( \text{coh}_G X \)) the category of quasi-coherent (respectively, coherent) \( G \)-equivariant sheaves on \( X \) whose morphisms are \( G \)-invariant morphisms. Here, a \( G \)-invariant morphism \( \varphi : (F_1, \theta_1) \to (F_2, \theta_2) \) of equivariant sheaves is a morphism of sheaves \( \varphi : F_1 \to F_2 \) that commutes with \( \theta_1 \), that is, \( \sigma^* \varphi \circ \theta_1 = \theta_2 \circ \pi^* \varphi \).
**Definition 2.1.** Let $G, H$ be algebraic groups and $f : H \to G$ a morphism of algebraic groups. Let $X$ be a $G$-variety, $Y$ an $H$-variety and $\varphi : Y \to X$ an $f$-equivariant morphism, that is, $\varphi(h \cdot y) = f(h) \cdot \varphi(y)$ for any $h \in H$ and $y \in Y$. Then, we define the pullback functor

$$
\varphi^*_f : \text{Qcoh}_G X \to \text{Qcoh}_H Y
$$

by $\varphi^*_f(F, \theta) := (\varphi^*F, (f \times \varphi)^*\theta)$. If $H = G$ and $f = \text{id}_G$, we denote the pullback by $\varphi^*$ and define the pushforward

$$
\varphi_* : \text{Qcoh}_G Y \to \text{Qcoh}_G X
$$

by $\varphi_*(F, \theta) := (\varphi_*F, (\text{id}_G \times \varphi)_*\theta)$. If $H$ is a closed subgroup of $G$ and $f$ is the inclusion, the pullback $\text{id}^*_f : \text{Qcoh}_G X \to \text{Qcoh}_H X$ is called the *restriction functor*, and is denoted by $\text{Res}^G_H$. We write

$$
\text{Res}^G : \text{Qcoh}_G X \to \text{Qcoh} X
$$

for the restriction functor $\text{Res}^G_{[1]}$.

**Definition 2.2.** Let $G$ be an algebraic group acting on a variety $X$. Assume that a restricted action $H \times X \to X$ of a closed normal subgroup $H \subseteq G$ is trivial.

(1) We define a functor

$$
(-)^H : \text{coh}_G X \to \text{coh}_{G/H} X
$$

as follows: For $(F, \theta) \in \text{coh}_G X$, we define a subsheaf $F^H \subseteq F$ by the following local sections on any open subspace $U \subseteq X$;

$$
F^H(U) := \{x \in F(U) \mid \theta_h(x) = x, \text{ for } \forall h \in H\}.
$$

Then, since $\theta : \pi^*F \sim \pi^*F$ maps the subsheaf $\pi^*(F^H) \subseteq \pi^*F$ to the subsheaf $\pi^*(F^H) \subseteq \pi^*F$, the pair $(F^H, \theta|_{\pi^*(F^H)})$ is a $G$-equivariant sheaf that naturally descends to a $G/H$-equivariant sheaf.

(2) For a character $\chi : H \to \mathbb{G}_m$ and $(F, \theta) \in \text{coh}_G X$, we define a subsheaf $F_\chi \subseteq F$ by the following local sections on any open subspace $U \subseteq X$;

$$
F_\chi(U) := \{x \in F(U) \mid \theta_h(x) = \chi(h)x, \text{ for } \forall h \in H\}.
$$

Then, since $\theta$ preserves $F_\chi$, we have a $G$-equivariant sheaf $(F_\chi, \theta|_{\pi^*F_\chi}) \in \text{coh}_G X$. We call $(F, \theta)$ is of weight $\chi$ if $F = F_\chi$, and we define a subcategory $(\text{coh}_G X)_{\chi} \subseteq \text{coh}_G X$ consisting of equivariant sheaves of weights $\chi$. Then, we have a functor

$$
(-)_\chi : \text{coh}_G X \to (\text{coh}_G X)_{\chi}.
$$
For later use, we provide a few fundamental lemmas. For lack of a suitable reference, we give brief proofs of the lemmas although it is well-known to experts.

**Lemma 2.3.** Let $G$ be an affine algebraic group acting on varieties $X$ and $Y$, and $H \subseteq G$ a finite normal subgroup. Let $\pi : X \to Y$ be a $G$-equivariant morphism. If $H$-action on $Y$ is trivial and $\pi$ is a principal $H$-bundle, we have an equivalence

$$\text{coh}_G X \cong \text{coh}_{G/H} Y.$$  

**Proof.** Since $\pi$ is a finite morphism, the direct image $\pi_* : \text{Qcoh}_G X \to \text{Qcoh}_G Y$ preserves coherent sheaves. We define a functor

$$(\pi_*)^H : \text{coh}_G X \to \text{coh}_{G/H} Y$$

as the composition of $\pi_* : \text{coh}_G X \to \text{coh}_G Y$ and $(-)^H : \text{coh}_G Y \to \text{coh}_{G/H} Y$. Then, $(\pi_*)^H$ is right adjoint to $\pi_p^* : \text{coh}_{G/H} Y \to \text{coh}_G X$ by [6, Corollary 2.24], where $p : G \to G/H$ is the natural projection. Hence, it is enough to show that the adjunction morphisms $\eta : \text{id} \to (\pi_*)^H \circ \pi_p^*$ and $\epsilon : \pi_p^* \circ (\pi_*)^H \to \text{id}$ are isomorphisms of functors. Consider the following commutative diagram:

![Diagram](image)

Then, $\eta$ and $\epsilon$ are isomorphisms if and only if so are $\text{Res}^{G/H}(\eta)$ and $\text{Res}^G_H(\epsilon)$. Hence, the equivalence $\text{coh}_G X \cong \text{coh}_{G/H} Y$ follows from the equivalence $\text{coh}_H X \cong \text{coh}_Y$ that is well-known to hold. \qed

**Lemma 2.4.** Let $G$ be an affine algebraic group acting on $X$ and $H \subseteq G$ an abelian closed normal subgroup of $G$ such that $H$ acts trivially on $X$. We denote by $p : G \to G/H$ the natural projection, and we write $H^\vee$ for the set of characters of $H$.

1. The functor $\text{id}^*_p : \text{coh}_{G/H} X \to \text{coh}_G X$ is fully faithful, and it induces an equivalence

$$\text{id}^*_p : \text{coh}_{G/H} X \cong (\text{coh}_G X)_{\chi_0},$$

where $\chi_0 : H \to \mathbb{G}_m$ is the trivial character.

2. The functor

$$\bigoplus_{\chi \in H^\vee} (-)_\chi : \text{coh}_G X \to \bigoplus_{\chi \in H^\vee} (\text{coh}_G X)_\chi$$

is an equivalence.
(3) For characters \( \eta : G \to \mathbb{G}_m \) and \( \chi : H \to \mathbb{G}_m \), the tensor product with \( \mathcal{O}(\eta) \) gives an equivalence:

\[
(-) \otimes \mathcal{O}(\eta) : (\text{coh}_G X)_\chi \sim (\text{coh}_G X)_{(\eta|_H)\chi}.
\]

**Proof.**

(1) The functor \((-)^H : \text{coh}_G X \to \text{coh}_{G/H} X \) is right adjoint to \( \text{id}_*^p : \text{coh}_{G/H} X \to \text{coh}_G X \) by [6, Lemma 2.22]. For any \( F \in \text{coh}_{G/H} X \), let \( \eta_F : F \to (\text{id}_*^p(F))^H \) be the adjunction morphism. Denote by \( \mathcal{O}_{[X/G]} \in \text{coh}_G X \) and \( \mathcal{O}_{[X/(G/H)]} \in \text{coh}_{G/H} X \) the structure sheaves with natural equivariant structures induced by group actions. Then, since there is a natural isomorphism \((\mathcal{O}_{[X/G]})^H \cong \mathcal{O}_{[X/(G/H)]}\), we have the following isomorphisms

\[
F \cong F \otimes \mathcal{O}_{[X/(G/H)]}^H \cong (\text{id}_*^p(F) \otimes \mathcal{O}_{[X/G]})^H \cong (\text{id}_*^p(F))^H,
\]

where the second isomorphism follows from [6, Lemma 2.23]. Since the composition of the above isomorphisms is equal to the adjunction morphism \( \eta_F, \text{id}_*^p \) is fully faithful. The latter claim is obvious by construction.

(2) It is enough to show that \( F \cong \bigoplus_{\chi \in H^*} F_\chi \) for any \( F \in \text{coh}_G X \). By definition we see that \( \bigoplus_{\chi \in H^*} F_\chi \subseteq F \), and hence it suffices to show that \( F/(\bigoplus_{\chi \in H^*} F_\chi) \cong 0 \). Since \( H \) is abelian and acts trivially on \( X \), we have a decomposition \( \text{coh}_H X \cong \bigoplus_{\chi \in H^*} (\text{coh}_H X)_\chi \). Hence, \( \overline{F} := \text{Res}_H^G(F) \in \text{coh}_H X \) is decomposed into a direct sum \( \overline{F} \cong \bigoplus_{\chi \in H^*} \overline{F}_\chi \). Since \( \text{Res}_H^G(F_\chi) = \overline{F}_\chi \), we have \( \text{Res}_H^G(F) \cong \bigoplus_{\chi \in H^*} (\text{Res}_H^G(F_\chi)) \). This implies \( \text{Res}_H^G(F/(\bigoplus_{\chi \in H^*} F_\chi)) \cong 0 \), since \( \text{Res}_H^G \) is a right exact functor. Hence, we have \( F/(\bigoplus_{\chi \in H^*} F_\chi) \cong 0 \).

(3) This is obvious. \( \square \)

**Lemma 2.5.** Let \( G \) be an affine algebraic group acting on a variety \( X \). For a character \( \chi : G \to \mathbb{G}_m \) of \( G \), define a \( G \times \mathbb{G}_m \)-action on \( X \times \mathbb{G}_m \) by \( (g, h_1) \cdot (x, h_2) = (g \cdot x, \chi(g)^{-1}h_1 h_2) \) and denote by \( \varphi \) the morphism \( \text{id}_G \times \chi : G \to G \times \mathbb{G}_m \). Then, the morphism \( e : X \to X \times \mathbb{G}_m \) defined by \( e(x) := (x, 1) \) is \( \varphi \)-equivariant, and the pullback

\[
e^*_\varphi : \text{coh}_{G \times \mathbb{G}_m} (X \times \mathbb{G}_m) \sim \text{coh}_G X
\]

is an equivalence.

**Proof.** Let \( p : X \times \mathbb{G}_m \to X \) and \( \pi : G \times \mathbb{G}_m \to G \) be the natural projections. Then, \( p \) is \( \pi \)-equivariant, and the composition \( e^*_\varphi \circ p^*_\pi : \text{coh}_G X \to \text{coh}_G X \) is the identity functor. Hence, \( e^*_\varphi \) is essentially surjective. Since the set of morphisms in \( \text{coh}_{G \times \mathbb{G}_m} (X \times \mathbb{G}_m) \) and \( \text{coh}_G X \) are the \( G \)-invariant subspaces of the set of morphisms in \( \text{coh}_{G \times \mathbb{G}_m} (X \times \mathbb{G}_m) \) and \( \text{coh}_G X \), respectively, the fully faithfulness of \( e^*_\varphi \) reduces to the case that \( G \) is trivial, which follows from [62, Lemma 1.3] (see also [6, Lemma 2.13]). \( \square \)
2.2 | Graded modules

To fix notation, we give a quick review of the categories of graded modules over graded rings. Let $L$ be a finitely generated abelian group, and let $S = \bigoplus_{i \in L} S_i$ be a Noetherian commutative ring with $L$-grading. An element $m \in M$ of $L$-graded $S$-module $M = \bigoplus_{i \in L} M_i$ is called homogeneous if $m \in M_i$ for some $i \in L$. If $m \in M_i$, we say that the degree of $m$ is $i$, and we write $\deg(m) = i$. An $L$-graded $S$-module $M$ is called finitely generated if $M$ has finitely many homogeneous generators. Denote by $\text{mod}^L S$ the category of finitely generated $L$-graded $S$-modules whose morphisms are degree preserving morphisms. For $M \in \text{mod}^L S$ and an element $i \in L$, we define the $i$-shift $M(i)$ of $M$ to be the $L$-graded $S$-module

$$M(i) := \bigoplus_{i' \in L} M(i')$$

defined by $M(i)_i := M_{i+i'}$. The $i$-shift defines the exact autoequivalence

$$(\cdot)(i) : \text{mod}^L S \sim \text{mod}^L S$$

of the abelian category $\text{mod}^L S$.

Let $L'$ be another finitely generated abelian groups, and $S'$ an $L'$-graded Noetherian ring, and suppose that $\alpha : L \to L'$ is a group homomorphism. A ring homomorphism $\varphi : S \to S'$ is $\alpha$-equivariant if $\varphi(S_i) \subset S'_{\alpha(i)}$ for any $i \in L$. For $M \in \text{mod}^L S$, we define the $L'$-graded $S'$-module $\varphi_*^\alpha M$ to be the $S'$-module $M \otimes_S S'$ with the $L'$-grading structure given by

$$(M \otimes_S S')_i := \left\{ \sum m \otimes s' \bigg| m \in M_i, s' \in S'_{\alpha(i)} \text{ for some } i \in L, i'' \in L' \text{ with } \alpha(i) + i'' = i' \right\}.$$ 

This defines a right exact functor

$$\varphi_*^\alpha : \text{mod}^L S \to \text{mod}^{L'} S'.$$

Since $\varphi_*^\alpha S(l) \cong S'(\alpha(l))$ for any $l \in L$, this functor restricts to the functor $\text{proj}^L S \to \text{proj}^{L'} S'$. If $S = S'$, $\varphi = \text{id}$ and $\alpha : L \to L'$ is an inclusion, then we write simply

$$M^{L'} := \text{id}_\alpha^* M,$$

and note that we have

$$(M^{L'})_i = \begin{cases} M_i & i' \in L \\ 0 & i' \notin L. \end{cases}$$

If $L = L'$, $\alpha = \text{id}$ and $S'$ is a finitely generated $S$-module, we have a forgetful functor

$$\varphi_* : \text{mod}^L S' \to \text{mod}^L S$$

associated to the ring homomorphism $\varphi$, and we have $(\varphi_* M)_i := M_i$ for any $l \in L$. 
Notation 2.6. For a finitely generated abelian group $A$, we set $G(A) := \text{Spec} \ k[A]$, where $k[A]$ is the group ring associated to $A$. Then, $G(A)$ is a commutative affine algebraic group. Conversely, for a commutative affine algebraic group $G$, we denote by $G^\vee := \text{Hom}(G, \mathbb{G}_m)$ the character group of $G$, and then $G^\vee$ is a finitely generated abelian group. It is standard that we have natural isomorphisms $G(A)^\vee \cong A$ and $G(G^\vee) \cong G$.

The grading structure on $S$ (respectively, $S'$) induces the algebraic group action from $G(L)$ (respectively, $G(L')$) on $\text{Spec} \ S$ (respectively, $\text{Spec} \ S'$), and we have a natural exact equivalence

$$\Gamma : \text{coh}_{G(L)} \text{Spec} \ S \sim \text{mod}^L S$$

of abelian categories (see, for example, [7, section 2.1]) given by taking global sections. This equivalence restricts to the equivalence of $G(L)$-equivariant locally free sheaves and $L$-graded projective modules. If we write

$$\tilde{\varphi} : \text{Spec} S' \rightarrow \text{Spec} S$$

the morphism associated to $\varphi : S \rightarrow S'$ and we set $G(\alpha) : G(L') \rightarrow G(L)$ the morphism of algebraic groups induced by $\alpha : L \rightarrow L'$, then the following diagram is commutative:

Moreover, if $L = L'$, $\alpha = \text{id}$ and $S'$ is a finitely generated $S$-module, the following diagram commutes:

2.3 Derived factorization categories

In this subsection, we provide a brief summary of the derived factorization categories.

Definition 2.7. A gauged Landau–Ginzburg model, or simply gauged LG model, is data $(X, \chi, W)^G$ with $X$ a scheme, $G$ an algebraic group acting on $X$, $\chi : G \rightarrow \mathbb{G}_m$ a character of $G$ and $W : X \rightarrow \mathbb{A}^1$ a $\chi$-semi-invariant regular function, that is, $W(g \cdot x) = \chi(g)W(x)$ for any $g \in G$ and any $x \in X$. If $G$ is trivial, we denote the gauged LG model by $(X, W)$, and call it Landau–Ginzburg model or LG model.
For a gauged LG model, we consider its factorizations that can be considered as ‘twisted complexes’.

**Definition 2.8.** Let \((X, \chi, W)^G\) be a gauged LG model. A quasi-coherent factorization of \((X, \chi, W)^G\) is a sequence

\[
F = \left( \begin{array}{c}
F_1 \\ F_0 \\ F_1(\chi)
\end{array} \right),
\]

where, for each \(i = 0, 1\), \(F_i\) is a \(G\)-equivariant quasi-coherent sheaf on \(X\) and \(\varphi_i^F\) is a \(G\)-invariant homomorphism such that \(\varphi_0^F \circ \varphi_1^F = W \cdot \text{id}_{F_1}\) and \(\varphi_1^F(\chi) \circ \varphi_0^F = W \cdot \text{id}_{F_0}\). Equivariant quasi-coherent sheaves \(F_0\) and \(F_1\) in the above sequence are called the components of \(F\). If the components \(F_i\) of \(F\) are coherent (respectively, locally free coherent, injective) sheaves, then \(F\) is called a coherent factorization (respectively, a matrix factorization, an injective factorization). We will often call these sequences just factorizations of \((X, \chi, W)^G\).

**Definition 2.9.** For a gauged LG model \((X, \chi, W)^G\), we define the abelian category

\[
\text{Qcoh}_G(X, \chi, W)
\]

whose objects are quasi-coherent factorizations of \((X, \chi, W)^G\), and whose set of morphisms are defined as follows: For two objects \(E, F \in \text{Qcoh}_G(X, \chi, W)\), we define \(\text{Hom}(E, F)\) to be the set of pairs \((f_1, f_0)\) of \(f_i \in \text{Hom}_{\text{Qcoh}_G X}(E_i, F_i)\) such that the following diagram commutes

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\varphi_1^E} & E_0 \\
| & \downarrow{f_1} & | \\
F_1 & \xrightarrow{\varphi_1^F} & F_0 \\
| & f_0 & | \\
E_1(\chi) & \xrightarrow{\varphi_0^E} & E_0(\chi)
\end{array}
\quad \quad
\begin{array}{ccc}
E_1 & \xrightarrow{\varphi_1^E} & E_0 \\
| & \downarrow{f_1(\chi)} & | \\
F_1 & \xrightarrow{\varphi_1^F} & F_0 \\
| & f_0(\chi) & | \\
E_1(\chi) & \xrightarrow{\varphi_0^E} & E_0(\chi)
\end{array}
\]

We define full subcategories

\[
\text{MF}_G(X, \chi, W) \subset \text{coh}_G(X, \chi, W) \subset \text{Qcoh}_G(X, \chi, W)
\]

of \(\text{Qcoh}_G(X, \chi, W)\) whose objects are matrix factorizations and coherent factorizations, respectively. By construction, these subcategories are exact subcategories.

Since factorizations can be considered as ‘twisted complexes’, we can consider the homotopy category of factorizations.

**Definition 2.10.** Two morphisms \(f = (f_1, f_0) : E \to F\) and \(g = (g_1, g_0) : E \to F\) in \(\text{Qcoh}_G(X, \chi, W)\) are homotopy equivalent, denoted by \(f \sim g\), if there exist two homomorphisms in \(\text{Qcoh}_G X\)

\[
h_0 : E_0 \to F_1 \quad \text{and} \quad h_1 : E_1(\chi) \to F_0
\]

such that \(f_0 - g_0 = \varphi_1^F h_0 + h_1 \varphi_0^F\) and \(f_1(\chi) - g_1(\chi) = \varphi_0^F h_1 + (h_0(\chi))(\varphi_1^F(\chi))\).
The homotopy category of factorizations of $(X, \chi, W)^G$, denoted by

$$K\text{Qcoh}_G(X, \chi, W),$$

is defined by $\text{Obj}(K\text{Qcoh}_G(X, \chi, W)) := \text{Obj}(\text{Qcoh}_G(X, \chi, W))$ and the set of morphisms are defined as the set of homotopy equivalence classes;

$$\text{Hom}_{K\text{Qcoh}(X, \chi, W)}(E, F) := \text{Hom}_{\text{Qcoh}(X, \chi, W)}(E, F)/\sim.$$

Similarly, we can define the homotopy category $K\text{coh}_G(X, \chi, W)$ (respectively, $K\text{MF}_G(X, \chi, W)$) of coherent factorizations (respectively, matrix factorizations) of $(X, \chi, W)^G$.

Next we define the totalization of a bounded complex of factorizations, which is an analogy of the total complex of a double complex.

**Definition 2.11.** Let $F = (\cdots \rightarrow F^i \xrightarrow{\delta^i} F^{i+1} \rightarrow \cdots)$ be a bounded complex of $\text{Qcoh}_G(X, \chi, W)$. For $l = 0, 1$, set

$$T_l := \bigoplus_{i+j=-l} F^i_j(\chi^{[j/2]}),$$

and define

$$t_l : T_l \rightarrow T_{l+1}$$

to be the homomorphism given by

$$t_l|_{F^i_j(\chi^{[j/2]})} := \delta^i_j(\chi^{[j/2]}) + (-1)^i \varphi_j^{F^i}(\chi^{[j/2]}),$$

where $\bar{n}$ is $n$ modulo 2, and $[m]$ is the minimum integer which is greater than or equal to a real number $m$. We define the totalization $\text{Tot}(F) \in \text{Qcoh}_G(X, \chi, W)$ of $F$ by

$$\text{Tot}(F) := \left(T_1 \xrightarrow{t_1} T_0 \xrightarrow{t_0} T_1(\chi)\right).$$

In what follows, we will recall that the homotopy category $K\text{Qcoh}_G(X, \chi, W)$ has a structure of triangulated category, and $K\text{coh}_G(X, \chi, W)$ and $K\text{MF}_G(X, \chi, W)$ are full triangulated subcategories of $K\text{Qcoh}_G(X, \chi, W)$.

**Definition 2.12.** We define an automorphism $T$ on $K\text{Qcoh}_G(X, \chi, W)$, which is called the shift functor, as follows. For an object $F \in K\text{coh}_G(X, \chi, W)$, we define an object $T(F)$ as

$$T(F) := \left(F_0 \xrightarrow{-\varphi_0^F} F_1(\chi) \xrightarrow{-\varphi_1^F(\chi)} F_0(\chi)\right),$$

and for a morphism $f = (f_1, f_0) \in \text{Hom}(E, F)$ we set $T(f) := (f_0, f_1(\chi)) \in \text{Hom}(T(E), T(F))$. For any integer $n \in \mathbb{Z}$, denote by $(-)[n]$ the functor $T^n(-)$. 
**Definition 2.13.** Let \( f : E \to F \) be a morphism in \( \text{Qcoh}_G(X, \chi, W) \). We define its *mapping cone* \( \text{Cone}(f) \) to be the totalization of the complex

\[
(\cdots \to 0 \to E \xrightarrow{f} F \to 0 \to \cdots)
\]

with \( F \) in degree zero. Then, the mapping cone \( \text{Cone}(f) \) of \( f \) is of the following form

\[
\begin{pmatrix}
F_1 \oplus E_0 & 0 \\
F_0 & F_0(\chi)
\end{pmatrix}
\begin{pmatrix}
\varphi_1^F & f_0 \\
0 & -\varphi_0^F
\end{pmatrix}
\begin{pmatrix}
E_1(\chi) & 0 \\
0 & E_1(\chi)
\end{pmatrix}
\]

A *distinguished triangle* is a sequence in \( \text{KQcoh}_G(X, \chi, W) \) which is isomorphic to a sequence of the form

\[
E \xrightarrow{i} F \xrightarrow{p} \text{Cone}(f) \xrightarrow{p} E[1],
\]

where \( i \) and \( p \) are natural injection and projection, respectively.

The following is well-known.

**Proposition 2.14.** The homotopy category \( \text{KQcoh}_G(X, \chi, W) \) is a triangulated category with respect to the above shift functor and the above distinguished triangles. The full subcategories \( \text{Kcoh}_G(X, \chi, W) \) and \( \text{KMF}_G(X, \chi, W) \) are full triangulated subcategories.

Following Positselski [17, 52], we define derived factorization categories (see also [49, 50]).

**Definition 2.15.** Denote by \( \text{Acoh}_G(X, \chi, W) \) the smallest thick subcategory of \( \text{Kcoh}_G(X, \chi, W) \) containing all totalizations of short exact sequences in \( \text{coh}_G(X, \chi, W) \). We define the (absolute) derived factorization category of \( (X, \chi, W)^G \) as the Verdier quotient

\[
\text{Dcoh}_G(X, \chi, W) := \text{Kcoh}_G(X, \chi, W) / \text{Acoh}_G(X, \chi, W).
\]

Similarly, we consider the thick subcategory \( \text{AMF}_G(X, \chi, W) \) containing all totalizations of short exact sequences in the exact category \( \text{MF}_G(X, \chi, W) \), and define the (absolute) derived matrix factorization category by

\[
\text{DMF}_G(X, \chi, W) := \text{KMF}_G(X, \chi, W) / \text{AMF}_G(X, \chi, W).
\]

We define a larger category \( \text{Dco Qcoh}_G(X, \chi, W) \) as follows: Denote by \( \text{Aco Qcoh}_G(X, \chi, W) \) the smallest thick subcategory of \( \text{KQcoh}_G(X, \chi, W) \) that is closed under taking small direct sums and contain all totalizations of short exact sequences in \( \text{Qcoh}_G(X, \chi, W) \). Then, we define \( \text{Dco Qcoh}_G(X, \chi, W) \) by

\[
\text{Dco Qcoh}_G(X, \chi, W) := \text{KQcoh}_G(X, \chi, W) / \text{Aco Qcoh}_G(X, \chi, W).
\]
Factorizations in $\text{Acoh}_G(X, \chi, W)$ or $\text{AMF}_G(X, \chi, W)$ are said to be acyclic, and factorizations in $A^\text{co Qcoh}_G(X, \chi, W)$ are said to be coacyclic. Two quasi-coherent factorizations $E$ and $F$ are said to be quasi-isomorphic if $E$ and $F$ are isomorphic in $D^\text{co Qcoh}_G(X, \chi, W)$.

The following propositions are standard.

**Proposition 2.16** [32, Proposition 2.25(1)]. The natural functor

$$D\text{coh}_G(X, \chi, W) \to D^\text{co Qcoh}_G(X, \chi, W)$$

is fully faithful, and the thick closure $D\text{coh}_G(X, \chi, W)$ of the essential image of the functor is the subcategory of compact objects.

**Proposition 2.17** [6, Proposition 3.14]. If $X$ is a smooth quasi-projective variety, the natural functor

$$\text{DMF}_G(X, \chi, W) \to D\text{coh}_G(X, \chi, W)$$

is an equivalence.

**Proposition 2.18** (cf. [4, Lemma 2.24]). Assume that $X = \text{Spec } R$ is an affine scheme and $G$ is reductive. For $P \in K\text{MF}_G(X, \chi, W)$ and $A \in \text{Acoh}_G(X, \chi, W)$, we have

$$\text{Hom}_{K\text{coh}_G(X, \chi, W)}(P, A) = 0.$$ 

In particular, the Verdier localizing functor

$$K\text{MF}_G(X, \chi, W) \sim \text{DMF}_G(X, \chi, W)$$

is an equivalence.

**Proof.** Since $G$ is reductive, the restriction functor $\text{Res}^G : K\text{coh}_G(X, \chi, W) \to K\text{coh}(X, W)$ is faithful. Hence, the problem reduces to the case when $G$ is trivial, and it follows from [4, Lemma 2.24].

The following result follows from Lemma 2.4.

**Lemma 2.19.** Notation is same as in Lemma 2.4. Let $\overline{\chi} : G/H \to G_m$ be a character, and define a character $\chi : G \to G_m$ by $\overline{\chi} \circ p$. Let $W : X \to \mathbb{A}^1$ be a $\chi$-semi-invariant regular function. Then, $W$ is also $\overline{\chi}$-semi-invariant with respect to the induced $G/H$-action on $X$. For $\eta \in H^\vee$, we denote by $\text{coh}_{G/H}(X, \chi, W)_\eta$ the subcategory of $\text{coh}_G(X, \chi, W)$ consisting of factorizations whose components lie in $(\text{coh}_G(X))_\eta$.

1. The functor $\text{id}_p^\eta : \text{coh}_{G/H}(X, \overline{\chi}, W) \to \text{coh}_G(X, \chi, W)$ is fully faithful, and it induces an equivalence
id^*_p : \text{coh}_{G/H}(X, \chi, W) \cong \text{coh}_G(X, \chi, W)_{\eta_0},

where \( \eta_0 : H \to \mathbb{G}_m \) is the trivial character.

(2) There is a decomposition of \( \text{coh}_G(X, \chi, W) \) into a direct sum

\[
\text{coh}_G(X, \chi, W) \sim \bigoplus_{\eta \in H^\vee} \text{coh}_G(X, \chi, W)_{\eta}.
\]

(3) For a character \( \phi : G \to \mathbb{G}_m \) of \( G \), the tensor product with \( \mathcal{O}(\phi) \) gives an equivalence

\[
(-) \otimes \mathcal{O}(\phi) : \text{coh}_G(X, \chi, W)_{\eta} \sim \text{coh}_G(X, \chi, W)_{(\phi|_H)\eta}.
\]

The following says that the derived factorization categories are generalizations of derived categories of coherent sheaves.

**Corollary 2.20.** Assume that \( \mathbb{G}_m \) acts trivially on \( X \). Let \( n > 0 \) be a positive integer, and denote by \( \chi_n : \mathbb{G}_m \to \mathbb{G}_m \) the character defined by \( \chi_n(a) := a^n \). We write \( \mu_n := \langle \zeta \rangle \subset \mathbb{G}_m \) for the subgroup of \( \mathbb{G}_m \) generated by a primitive \( n \)th root of unity \( \zeta \). Then, we have an orthogonal decomposition

\[
\text{Dcoh}_{\mathbb{G}_m}(X, \chi_n, 0) \cong \bigoplus_{i=1}^n \text{D}^b(\text{coh} X).
\]

**Proof.** Since the kernel of the surjection \( \chi_n : \mathbb{G}_m \to \mathbb{G}_m \) is equal to the subgroup \( \mu_n \subset \mathbb{G}_m \), we have the following short exact sequence:

\[
1 \to \mu_n \to \mathbb{G}_m \xrightarrow{\chi_n} \mathbb{G}_m \to 1.
\]

Then, by Lemma 2.19, we obtain a decomposition \( \text{coh}_{\mathbb{G}_m}(X, \chi_n, 0) \cong \bigoplus_{i=1}^n \text{coh}_{\mathbb{G}_m/\mu_n}(X, \chi_1, 0) \), and this induces the following direct sum decomposition of the derived factorization category

\[
\text{Dcoh}_{\mathbb{G}_m}(X, \chi_n, 0) \cong \bigoplus_{i=1}^n \text{Dcoh}_{\mathbb{G}_m/\mu_n}(X, \chi_1, 0).
\]

Hence, the result follows from \([32, \text{Proposition 2.14}]\). \( \square \)

For a Noetherian scheme \( X \) with an action from an algebraic group \( G \), the \( G \)-equivariant singularity category \( \text{D}^\text{sg}_G(X) \) is defined by the Verdier quotient

\[
\text{D}^\text{sg}_G(X) := \text{D}^b(\text{coh}_G X)/\text{Perf}_G X.
\]

Similarly, for an abelian group \( L \) and a commutative \( L \)-graded Noetherian ring \( R \), we define the \( L \)-graded singularity category \( \text{D}^L_{\text{sg}}(R) \) by

\[
\text{D}^L_{\text{sg}}(R) := \text{D}^b(\text{mod}^L R)/\text{K}^b(\text{proj}^L R).
\]
The natural equivalence in (2.2) induces the following equivalence:

\[ \text{D}^{\text{sg}}_G(\text{Spec} R) \sim \text{D}^{\text{sg}}_{\text{sg}}(R). \]

**Theorem 2.21** [32, Theorem 3.6; 49, Theorem 2]. *Let \((X, \chi, W)^G\) be a gauged LG model. Assume that \(X\) is a smooth variety, \(G\) is a reductive affine algebraic group and \(W\) is flat. Denote by \(X_0\) the zero scheme of \(W\). Then, we have an equivalence*

\[ \text{D}^{\text{coh}}_G(X, \chi, W) \sim \text{D}^{\text{sg}}_{\text{sg}}(X_0). \]

### 2.4 Homotopy category of graded matrix factorizations

We use the same notation as in Subsection 2.2, and let \(f \in S\) be a homogeneous element with \(\deg(f) = l \in L\).

**Definition 2.22.** An \(L\)-graded factorization of \(f\) is a sequence

\[ F = \left( F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_1(l) \right), \tag{2.C} \]

where each \(F_i\) is an \(L\)-graded finitely generated \(S\)-module and \(\varphi_i\) are degree preserving homomorphisms such that \(\varphi_0 \circ \varphi_1 = f \cdot \text{id}_{F_1}\) and \(\varphi_1(l) \circ \varphi_0 = f \cdot \text{id}_{F_0}\). Each graded module \(F_i\) in (2.C) is called a *component* of \(F\), and we say that an \(L\)-graded factorization \(F\) is an *\(L\)-graded matrix factorization* if both components are projective \(S\)-modules.

For \(L\)-graded factorizations \(E\) and \(F\), a *morphism* from \(E\) to \(F\) is a pair \((\alpha_1, \alpha_0)\) of morphisms \(\alpha_i : E_i \to F_i\) in \(\text{mod}^L S\) such that it is compatible with \(\varphi_i^E\) and \(\varphi_i^F\) \((i = 0, 1)\).

We define the **homotopy category of \(L\)-graded factorizations of \(f\)**

\[ \text{Kmod}^L(S, f) \]

whose objects are \(L\)-graded factorizations of \(f\) and the morphisms are homotopy equivalence classes of morphisms, where the homotopy equivalence is defined similarly to Definition 2.10. We define the **homotopy category of \(L\)-graded matrix factorizations of \(f\)** to be the full subcategory

\[ \text{HMF}^L_S(f) \subset \text{Kmod}^L(S, f) \]

of \(L\)-graded matrix factorizations of \(f\). Similarly to Proposition 2.14, the categories \(\text{Kmod}^L(S, f)\) and \(\text{HMF}^L_S(f)\) have natural structures of triangulated categories. We define the category

\[ \text{Dmod}^L(S, f) := \text{Kmod}^L(S, f) / \text{Amod}^L(S, f) \]

to be the Verdier quotient of \(\text{Kmod}^L(S, f)\) by the subcategory \(\text{Amod}^L(S, f) \subset \text{Kmod}^L(S, f)\), where \(\text{Amod}^L(S, f)\) is defined similarly to \(\text{Acoh}_G(X, \chi, W)\) in Definition 2.15. By the identical argument
in Propositions 2.17 and 2.18, if $S$ is regular, we have a natural equivalence

$$\text{HMF}^L(f) \sim \text{Dmod}^L(S, f).$$

The element $f \in S$ induces a $\chi_L$-semi-invariant regular function $f : \text{Spec } S \to \mathbb{A}^1$, where $\chi_L \in G(L)^\vee$ the character of $G(L)$ corresponding to the isomorphism $G(L)^\vee \cong L$. The equivalence (2.19) implies the following:

**Proposition 2.23.** We have natural equivalences

$$\text{Dcoh}_{G(L)}(\text{Spec } S, \chi_L, f) \sim \text{Dmod}^L(S, f)$$

$$\text{KMF}_{G(L)}(\text{Spec } S, \chi_L, f) \sim \text{HMF}^L_S(f)$$

of triangulated categories, and the following diagram of natural equivalences commutes

$$\begin{array}{ccc}
\text{KMF}_{G(L)}(\text{Spec } S, \chi_L, f) & \rightarrow & \text{HMF}^L_S(f) \\
\downarrow & & \downarrow \\
\text{Dcoh}_{G(L)}(\text{Spec } S, \chi_L, f) & \rightarrow & \text{Dmod}^L(S, f).
\end{array}$$

If $S$ is regular, we write $\text{CM}^L(S/f)$ for the stable category of the Frobenius category of maximal Cohen–Macaulay $L$-graded modules over the hypersurface $S/f$. The following is well-known.

**Theorem 2.24** [12, 18, 48]. Let $L$ be an abelian group, and suppose that a polynomial ring $S^n := k[x_1, \ldots, x_n]$ has an $L$-grading. Let $f \in S^n$ be a non-zero homogeneous element. Then, there are exact equivalences

$$\text{HMF}^L_{S^n}(f) \sim \text{CM}^L(S^n/f) \sim \text{D}_{\text{sg}}^L(S^n/f).$$

By Auslander–Reiten duality (see [38, Theorem 3.8]) or more general result by Favero–Kelly [22, Theorem 1.2], we have the following.

**Theorem 2.25.** Set $\vec{x}^n := \sum_{i=1}^n \vec{x}_i \in L_f$. Then, the autoequivalence

$$S := (-)(-\vec{x})[n] : \text{HMF}^L_{S^n}(f) \rightarrow \text{HMF}^L_{S^n}(f)$$

is the Serre functor on $\text{HMF}^L_{S^n}(f)$.

### 2.5 Functors of derived factorization categories

In this subsection, we recall fundamental functors between factorization categories. Let $X$ and $Y$ be Noetherian schemes, and $G$ an algebraic group acting on $X$ and $Y$. Let $\chi : G \to G_m$ be a character of $G$. 
2.5.1 Direct image and inverse image

Let \( f : X \to Y \) be a \( G \)-equivariant morphism. If \( W : Y \to \mathbb{A}^1 \) is a \( \chi \)-semi-invariant regular function, there are gauged LG models \((X, \chi, f^*W)\) and \((Y, \chi, W)\). Then, the direct image \( f_* : \text{Qcoh}_G X \to \text{Qcoh}_G Y \) and the inverse image \( f^* : \text{Qcoh}_G Y \to \text{Qcoh}_G X \) of \( G \)-equivariant sheaves naturally define the direct image and the inverse image between factorizations

\[
 f_* : \text{Qcoh}_G(X, \chi, f^*W) \to \text{Qcoh}_G(Y, \chi, W) \\
 f^* : \text{Qcoh}_G(Y, \chi, W) \to \text{Qcoh}_G(X, \chi, f^*W).
\]

The inverse image \( f^* \) preserves coherent factorizations, and, if \( f \) is proper, the direct image \( f_* \) also preserves coherent factorizations. Since these functors preserves homotopy equivalences of morphisms, we have the induced functors of the homotopy categories of factorizations

\[
 f_* : \text{KQcoh}_G(X, \chi, f^*W) \to \text{KQcoh}_G(Y, \chi, W) \\
 f^* : \text{KQcoh}_G(Y, \chi, W) \to \text{KQcoh}_G(X, \chi, f^*W).
\]

Furthermore, if \( f \) is affine, \( f_* \) maps \( \text{Aco Qcoh}_G(X, \chi, f^*W) \) to \( \text{Aco Qcoh}_G(Y, \chi, W) \), and so \( f_* \) induces a functor

\[
 f_* : \text{Dco Qcoh}_G(Y, \chi, W) \to \text{Dco Qcoh}_G(X, \chi, f^*W).
\]

For general \( f \), by [4, Corollary 2.25] we can define the right derived functor of \( f_* \)

\[
 \mathcal{R}_f f_* : \text{Dco Qcoh}_G(Y, \chi, W) \to \text{Dco Qcoh}_G(X, \chi, f^*W).
\]

If \( f \) is proper, the functor \( \mathcal{R}_f f_* \) preserves coherent factorizations.

On the other hand, if \( f \) is flat, \( f^* \) preserves coacyclic factorizations, and hence \( f^* \) naturally induces the functor

\[
 f^* : \text{Dco Qcoh}_G(Y, \chi, W) \to \text{Dco Qcoh}_G(X, \chi, f^*W).
\]

If \( Y \) is a smooth variety, by [6, Proposition 3.14] we can define the left derived functor

\[
 \mathcal{L}_f f^* : \text{Dco Qcoh}_G(Y, \chi, W) \to \text{Dco Qcoh}_G(X, \chi, f^*W)
\]

of \( f^* \) without the assumption of flatness of \( f \), and this functor \( \mathcal{L}_f f^* \) preserves coherent factorizations. See, for example, [6, 33, 45] for more details.

2.5.2 Tensor product

Let \( W_1 : X \to \mathbb{A}^1 \) and \( W_2 : X \to \mathbb{A}^1 \) be \( \chi \)-semi-invariant functions. We define the tensor products

\[
 (-) \otimes (-) : \text{coh}_G(X, \chi, W_1) \times \text{coh}_G(X, \chi, W_2) \to \text{coh}_G(X, \chi, W_1 + W_2)
\]
of factorizations by

\[ E \otimes F := \left( \bigoplus_{i=0,1} (F_i \otimes E_{i+1}) \xrightarrow{\varphi_{i}^{E \otimes F}} \bigoplus_{i=0,1} (F_i \otimes E_i)(\chi^i) \xrightarrow{\varphi_{0}^{E \otimes F}} \bigoplus_{i=0,1} (F_i \otimes E_{i+1})(\chi) \right), \]

where \( \bar{n} \) is \( n \) modulo 2,

\[ \varphi_{0}^{E \otimes F} = \begin{pmatrix} \varphi_{0}^{E} \otimes 1 & 1 \otimes \varphi_{0}^{F} \\ -1 \otimes \varphi_{0}^{E} & \varphi_{0}^{E} \otimes 1 \end{pmatrix} \]

and

\[ \varphi_{1}^{E \otimes F} = \begin{pmatrix} \varphi_{1}^{E} \otimes 1 & -(1 \otimes \varphi_{1}^{F})(\chi) \\ 1 \otimes \varphi_{1}^{F} & (\varphi_{1}^{E} \otimes 1)(\chi) \end{pmatrix}. \]

Since this bi-functor preserves the homotopy equivalences of morphisms, we have the induced functor

\[ (-) \otimes (-) : \text{Kcoh}_{G}(X, \chi, W_1) \times \text{Kcoh}_{G}(X, \chi, W_2) \to \text{Kcoh}_{G}(X, \chi, W_1 + W_2). \]

If \( F \in \text{MF}_{G}(X, \chi, W_1) \), the functor \( F \otimes (-) : \text{Kcoh}_{G}(X, \chi, W_2) \to \text{Kcoh}_{G}(X, \chi, W_1 + W_2) \) preserves acyclic factorizations, and thus it induces the functor

\[ (-) \otimes (-) : \text{DMF}_{G}(X, \chi, W_1) \times \text{Dcoh}_{G}(X, \chi, W_2) \to \text{Dcoh}_{G}(X, \chi, W_1 + W_2). \]

If \( X \) is a smooth variety, by Proposition 2.17 we can define the \textit{left derived functor}

\[ (-) \otimes^{L} (-) : \text{Dcoh}_{G}(X, \chi, W_1) \times \text{Dcoh}_{G}(X, \chi, W_2) \to \text{Dcoh}_{G}(X, \chi, W_1 + W_2) \]

of tensor products.

## 2.6 Supports of factorizations

We recall the supports of factorizations, and we provide a lemma that we need for a slight generalization of global Knörrer periodicity in the next subsection. In this subsection, we only consider the case that \( G \) is trivial. Let \( X \) be a smooth variety, and \( W : X \to \mathbb{A}^1 \) a regular function on \( X \).

For a point \( p \in X \) in the scheme \( X \), we set \( X_p := \text{Spec} \; O_{X,p} \). Since the functor \((-)_p : \text{coh} \; X \to \text{coh} \; X_p \) defined by taking the stalk at \( p \) is exact, it induces a functor between derived factorization categories

\[ (-)_p : \text{Dcoh}(X, W) \to \text{Dcoh}(X_p, W_p), \]

where \( W_p : X_p \to \mathbb{A}^1 \) is the stalk of \( W \) at \( p \). Obviously this functor preserves matrix factorizations

\[ (-)_p : \text{DMF}(X, W) \to \text{DMF}(X_p, W_p). \]
Definition 2.26. For $F \in \text{Dcoh}(X, W)$, we define the subset $\text{Supp}(F) \subset X$ by

$$\text{Supp}(F) := \{ p \in X \mid F_p \neq 0 \text{ in } \text{Dcoh}(X_p, W_p) \}.$$ 

By Proposition 2.17, we have an equivalence $\Phi : \text{DMF}(X, W) \cong \text{Dcoh}(X, W)$, and $\Phi$ commutes with the functor $(-)_p$. Hence, we have

$$\text{Supp}(F) = \text{Supp}(\Phi^{-1}(F)),$$

and thus $\text{Supp}(F)$ is a closed subset of $X$ by [34, Proposition 2.20(2)]. We say that a thick subcategory $\mathcal{T} \subset \text{Dcoh}(X, W)$ is closed under tensor action from $\text{DMF}(X, 0)$ if for any $E \in \text{DMF}(X, 0)$ and any $F \in \mathcal{T}$ we have $E \otimes F \in \mathcal{T}$. By [34, Lemma 2.25], for any subset $S \subseteq X$ of $X$, the full subcategory

$$\{ F \in \text{Dcoh}(X, W) \mid \text{Supp}(F) \subseteq S \}$$

is a triangulated subcategory that is closed under direct summands and tensor action from $\text{Dcoh}(X, 0)$. The following lemma will be necessary in the proof of Theorem 2.28(3).

Lemma 2.27 [34, Theorem 1.1 and Proposition 5.3(2)]. Let $F \in \text{Dcoh}(X, W)$. Assume that $W$ is a non-zero-divisor, and denote by $X_0 \subset X$ the zero scheme of $W$.

1. The support $\text{Supp}(F)$ of $F$ is contained in the singular locus $\text{Sing}(X_0)$ of $X_0$.
2. There is an object $E \in \text{Dcoh}(X, W)$ such that $\text{Supp}(E) = \text{Sing}(X_0)$.
3. If $\mathcal{T} \subseteq \text{Dcoh}(X, W)$ is a thick subcategory closed under tensor action from $\text{DMF}(X, 0)$, then there exists a unique specialization-closed subset $Z \subseteq X$ such that

$$\mathcal{T} = \{ F \in \text{Dcoh}(X, W) \mid \text{Supp}(F) \subseteq Z \}.$$  

In particular, if $\mathcal{T}$ contains an object $F$ with $\text{Supp}(F) = \text{Sing}(X_0)$, then $\mathcal{T} = \text{Dcoh}(X, W)$.

2.7 | Knörrer periodicity

In this subsection, we prove a slight generalization of the Knörrer periodicity by [32, 35, 47, 58], that will be necessary in the proof of Theorem 3.1.

Let $X$ be a smooth quasi-projective variety over $k$, and let $G$ be a reductive affine algebraic group acting on $X$. Let $\mathcal{E}$ be a $G$-equivariant locally free sheaf on $X$ of finite rank, and choose a $G$-invariant regular section $s \in \Gamma(X, \mathcal{E}^\vee)^G$ of the dual $\mathcal{E}^\vee := \text{Hom}_X(\mathcal{E}, \mathcal{O}_X)$ of $\mathcal{E}$. Denote by $Z_s \subset X$ the zero scheme of $s$. Let $\chi : G \to \mathbb{G}_m$ be a character of $G$. Then, $\mathcal{E}(\chi)$ induces a vector bundle $V(\mathcal{E}(\chi)) := \text{Spec}(\text{Sym}(\mathcal{E}(\chi)^\vee))$ over $X$ with a $G$-action induced by the equivariant structure of $\mathcal{E}(\chi)$. Let $q : V(\mathcal{E}(\chi)) \to X$ and $p : V(\mathcal{E}(\chi))|_Z \to Z_s$ be natural projections, and let $i : V(\mathcal{E}(\chi))|_{Z_s} \hookrightarrow V(\mathcal{E}(\chi))$ and $j : Z_s \hookrightarrow X$ be the closed immersions. Now we have the following...
The regular section $s$ induces a $\chi$-semi-invariant regular function $Q_s : V(\mathcal{E}(\chi)) \to \mathbb{A}^1$. Let $W : X \to \mathbb{A}^1$ be a $\chi$-semi-invariant regular function.

**Theorem 2.28** (cf. [32, 35, 47, 58]). Assume that one of the following conditions holds.

1. The restricted function $W|_{Z_s} : Z_s \to \mathbb{A}^1$ is flat.
2. There is a reductive algebraic group $H$ such that $G = H \times G_m$ and $1 \times G_m \subseteq G$ acts trivially on $X$. Moreover, $W = 0$ and $\chi : H \times G_m \to G_m$ is the projection.
3. $W|_{Z_s} = 0$, $Z_s$ is smooth, and $\text{Dcoh}_G(Z_s, \chi, W|_{Z_s})$ is idempotent complete.

Then, we have an equivalence

$$i_* p^* : \text{Dcoh}_G(Z_s, \chi, W|_{Z_s}) \cong \text{Dcoh}_G(V(\mathcal{E}(\chi)), \chi, q^* W + Q_s).$$

**Proof.**

1. This is [32, Theorem 1.2].
2. This is [35, 58] for trivial $H$, and the result for general $H$ is [32, Proposition 4.8].
3. To ease notation, we denote $Z := Z_s$, $V := V(\mathcal{E}(\chi))$ and $Q := Q_s : V \to \mathbb{A}^1$. We prove the result by the following three steps.

**Step 1:** In the first step, we consider the case when $G = 0$, and we prove the functor

$$i_* p^* : \text{DcoQcoh}(Z, W|_Z) \to \text{DcoQcoh}(V, q^* W + Q)$$

between larger categories is an equivalence. By [32, Lemma 4.5], this functor is fully faithful, and has a right adjoint

$$p_* i! : \text{DcoQcoh}(V, q^* W + Q) \to \text{DcoQcoh}(Z, W|_Z).$$

by [17, Theorem 3.8]. Hence, it suffices to show that $p_* i!$ is also fully faithful. For this, we show that, for any $F \in \text{DcoQcoh}(V, q^* W + Q_s)$, the adjunction morphism

$$\sigma_F : i_* p^* p_* i!(F) \to F$$

is an isomorphism, or equivalently, the cone $C(\sigma_F) := \text{Cone}(\sigma_F)$ is the zero object.

By [17, Proposition 1.10], the set of objects in the category $\text{DcoQcoh}(V, q^* W + Q)$ is a set of compact generators in $\text{DcoQcoh}(V, q^* W + Q)$. Therefore, by [55, Lemma 2.1.1], the smallest triangulated subcategory of $\text{DcoQcoh}(V, q^* W + Q)$ that contains $\text{DcoQcoh}(V, q^* W + Q)$ and is closed under (infinite) coproducts is equal to the whole category $\text{DcoQcoh}(V, q^* W + Q)$. Moreover, since $i_* p^*$ and
admit right adjoint functors, these functors commute with coproducts, and so does the direct image \( p_* \) by \([46, \text{Lemma 1.4}]\). Thus, the composition \( i_* p^* \circ p_* i^! \) also commutes with coproducts. Therefore, to show that the adjunction \( \sigma_F \) is an isomorphism, we may assume that \( F \) lies in \( \text{Dcoh}(V, q^* W + Q) \). Denote by \( \mathcal{T} := \text{Im}(i_* p^*) \) the thick closure of the triangulated subcategory \( \text{Im}(i_* p^*) \subset \text{Dcoh}(V, q^* W + Q) \). We claim that \( \sigma_F \) is an isomorphism if \( F \in \mathcal{T} \).

First, we claim that \( \mathcal{T} \) is closed under tensor action from \( \text{DMF}(V, 0) \). Since \( X \) is quasi-projective, \( X \) has an ample line bundle \( \mathcal{L} \). If we denote \( \mathcal{L} := q^* \mathcal{L}_X \), \( \mathcal{L} \) is an ample line bundle on \( V \) since \( q \) is quasi-affine. Then, the category \( \text{DMF}(V, 0) \) is generated by objects of the form \( \tilde{\mathcal{L}}_n := (0 \to \mathcal{L}_n \to 0) \). Hence, it suffices to show that \( \mathcal{T} \) is closed under tensor product with \( \tilde{\mathcal{L}}_n \) for any \( n \in \mathbb{Z} \).

To prove the claim, we only need to show that \( Y = \text{Sing}(V_0) \) by \([2.27, \text{Lemma 3}]\). Since \( Z \) is smooth, \( \text{Sing}(V_0) = Z \), where \( Z \) is considered as a closed subscheme of \( V \) via the zero section \( Z \hookrightarrow V|_Z \). For a point \( x \in Z \), we denote by the same notation \( x \) the points in \( V|_Z \) and \( V \) that correspond to \( x \) via the zero section. Then, we can show that the localized functor \( (i_x)_* \circ (p_x)^* : \text{Dcoh}(Z, 0) \to \text{Dcoh}(V|_Z, (q^* W + Q)|_Z) \) is also fully faithful by using \([32, \text{Lemma 4.4}]\) and an argument that is identical to \([32, \text{Lemma 4.5}]\), where \( p_x : (V|_Z)_x \to Z_x \) and \( i_x : (V|_Z)_x \to V_x \) are the localized morphisms. Consider a matrix factorization \( \tilde{\mathcal{O}}_Z \) defined by \( \tilde{\mathcal{O}}_Z := (0 \to \mathcal{O}_Z \to 0) \in \text{Dcoh}(Z, 0) \). Then, it is easy to see that \( \text{Supp}(\tilde{\mathcal{O}}_Z) = Z \) by \([45, \text{Proposition 2.30}]\). Since we have \( i_* p^*((\tilde{\mathcal{O}}_Z)_x) \cong (i_x)_* \circ (p_x)^* \) by flat base changes and the localized functor \( (i_x)_* \circ (p_x)^* \) is fully faithful, we have \( \text{Supp}(i_* p^*((\tilde{\mathcal{O}}_Z)_x)) = \text{Supp}(\tilde{\mathcal{O}}_Z) = Z \). Hence, \( Y = Z \), since \( i_* p^*((\tilde{\mathcal{O}}_Z)_x) \in \mathcal{T} \).

**Step 2:** In the second step, we show that the functor

\[ i_* p^* : \text{D}^{\text{coQcoh}}_G(Z, \chi, W|_Z) \to \text{D}^{\text{coQcoh}}_G(V, \chi, q^* W + Q) \]

is an equivalence. The restriction functors \( \text{Res}^G \) between equivariant quasi-coherent sheaves induces the following functors

\[ \text{Res}^G : \text{D}^{\text{coQcoh}}_G(Z, \chi, W|_Z) \to \text{D}^{\text{coQcoh}}_G(Z, W|_Z) \]

\[ \text{Res}^G : \text{D}^{\text{coQcoh}}_G(V, \chi, q^* W + Q) \to \text{D}^{\text{coQcoh}}_G(V, q^* W + Q), \]
and these functors have right adjoint functors, denoted by \( \text{Ind}_G \) (see [6, Definition 2.14] for the definition of \( \text{Ind}_G \)). Then, by Example A.2 we have the induced comonads, denoted by \( \mathbb{T} \), on \( D^\text{coQcoh}(Z, W|_Z) \) and \( D^\text{coQcoh}(V, q^*W + Q) \). Since \( G \) is reductive, by [33, Lemma 4.56], the comparison functors

\[
\Gamma : D^\text{coQcoh}_G(Z, \chi, W|_Z) \to D^\text{coQcoh}(Z, W|_Z)\mathbb{T}
\]

\[
\Gamma : D^\text{coQcoh}_G(V, \chi, q^*W + Q) \to D^\text{coQcoh}(V, q^*W + Q)\mathbb{T}
\]

are equivalences (note that in our setting, by [33, Remark 4.4], \( D\text{Qcoh}_G(\phi) \) in [33] is equivalent to \( D^\text{coQcoh}_G(\phi) \) since \( V \) and \( Z \) are smooth). By [33, Lemma 2.11], we see that the functor \( i_* p^* : D^\text{coQcoh}(Z, W|_Z) \to D^\text{coQcoh}(V, q^*W + Q) \) is a linearizable functor, and we have an induced functor

\[
(i_* p^*)_\mathbb{T} : D^\text{coQcoh}(Z, W|_Z)\mathbb{T} \to D^\text{coQcoh}(V, q^*W + Q)\mathbb{T}
\]

that commutes with \( i_* p^* : D^\text{coQcoh}_G(Z, \chi, W|_Z) \to D^\text{coQcoh}_G(V, \chi, q^*W + Q) \) and the comparison functors. Proposition A.9 implies that the induced functor \( (i_* p^*)_\mathbb{T} \) is an equivalence by Step 1, and hence the functor \( i_* p^* : D^\text{coQcoh}_G(Z, \chi, W|_Z) \to D^\text{coQcoh}_G(V, \chi, q^*W + Q) \) is also an equivalence.

**Step 3:** In the final step, we finish the proof. By Step 2, we have already shown that the functor \( i_* p^* : \text{Dcoh}_G(Z, \chi, W|_Z) \to \text{Dcoh}_G(V, \chi, q^*W + Q) \) is fully faithful. By Step 2 and Proposition 2.16, the equivalence \( i_* p^* : D^\text{coQcoh}_G(Z, \chi, W|_Z) \to D^\text{coQcoh}_G(V, \chi, q^*W + Q) \) gives the equivalence

\[
i_* p^* : \text{Dcoh}_G(Z, \chi, W|_Z) \sim \text{Dcoh}_G(V, \chi, q^*W + Q)
\]

of compact objects, where \( \text{Dcoh}_G(Z, \chi, W|_Z) \sim \text{Dcoh}_G(V, \chi, q^*W + Q) \) denotes the thick closure of \( \text{Dcoh}_G(Z, \chi, W|_Z) \). But since \( \text{Dcoh}_G(Z, \chi, W|_Z) \) is idempotent complete, we have \( \text{Dcoh}_G(Z, \chi, W|_Z) = \text{Dcoh}_G(Z, \chi, W|_Z) \). Hence, the functor \( i_* p^* : \text{Dcoh}_G(Z, \chi, W|_Z) \to \text{Dcoh}_G(V, \chi, q^*W + Q) \) is essentially surjective. This finishes the proof. \( \square \)

### 2.8 DG-enhancements and derived Morita theory

We recall the definition of the Thom–Sebastiani sum of gauged LG models:

**Definition 2.29.** The *Thom–Sebastiani sum* of two gauged LG models \((X_1, \chi_1, W_1)^G_1\) and \((X_2, \chi_2, W_2)^G_2\) is defined to be the gauged LG model

\[
(X_1 \times X_2, \chi_1 \times \text{Gm} \chi_2, W_1 \boxplus W_2)^{G_1 \times \text{Gm} G_2},
\]

where \( G_1 \times \text{Gm} G_2 := \{(g_1, g_2) \in G_1 \times G_2 \mid \chi_1(g_1) = \chi_2(g_2)\} \), \((X_1 \times \text{Gm} \chi_2)(g_1, g_2) := \chi_1(g_1) = \chi_2(g_2)\), and \( W_1 \boxplus W_2 := p_1^* W_1 + p_2^* W_2 \) (here \( p_i \) are natural projections). We also call the potential \( W_1 \boxplus W_2 \) the *Thom–Sebastiani sum* of \( W_1 \) and \( W_2 \).

We recall the derived Morita theory for factorizations by [6] which is a general version of [14, 51]. For this, we recall dg-enhancements of derived factorization categories.
Definition 2.30. Let $(X, \chi, W)^G$ be a gauged LG model. We define the dg-category

$$\text{Qcoh}_G(X, \chi, W)$$

to be the category of quasi-coherent factorizations of $(X, \chi, W)^G$ whose Hom-complexes are defined as follows: For any $E, F \in \text{Qcoh}_G(X, \chi, W)$, we define the complex $\text{Hom}(E, F)$ of morphisms from $E$ to $F$ as the following graded vector space

$$\text{Hom}(E, F) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(E, F)^n$$

with a differential $d^i : \text{Hom}(E, F)^i \to \text{Hom}(E, F)^{i+1}$ given by

$$d^i(f) := \varphi^F \circ f - (-1)^i f \circ \varphi^E,$$

where

$$\text{Hom}(E, F)^{2m} := \text{Hom}(E_1, F_1(\chi^m)) \oplus \text{Hom}(E_0, F_0(\chi^m))$$

$$\text{Hom}(E, F)^{2m+1} := \text{Hom}(E_1, F_0(\chi^m)) \oplus \text{Hom}(E_0, F_1(\chi^{m+1})).$$

We denote by $\text{Inj}_G(X, \chi, W)$ (respectively, $\text{MF}_G(X, \chi, W)$) the full dg-subcategory of $\text{Qcoh}_G(X, \chi, W)$ consisting of injective factorizations (respectively, matrix factorizations). We also consider the full dg-subcategory $\text{Inj}_G(X, \chi, W) \subset \text{Inj}_G(X, \chi, W)$ consisting of injective factorizations that are quasi-isomorphic to coherent factorizations. Note that the homotopy category $[\text{MF}_G(X, \chi, W)]$ of the dg-category $\text{MF}_G(X, \chi, W)$ is equal to $\text{KMF}_G(X, \chi, W)$. Hence, if $X$ is an affine scheme and $G$ is reductive, then we have the following equivalence by Lemma 2.18

$$[\text{MF}_G(X, \chi, W)] \sim \text{DMF}_G(X, \chi, W).$$

The following standard result provides dg-enhancements of general gauged LG models.

Proposition 2.31 [4, Corollary 2.25; 32, Lemma 2.12]. Let $(X, \chi, W)^G$ be a gauged LG model, and assume that $X$ is Noetherian. Then, the natural functors

$$[\text{Inj}_G(X, \chi, W)] \to \text{D}^{co} \text{Qcoh}_G(X, \chi, W)$$

$$[\text{Inj}_G(X, \chi, W)] \to \text{Dcoh}_G(X, \chi, W)$$

are equivalences.

In what follows, we recall the derived Morita theory for derived factorization categories. We freely use the terminology and notation from [63]. Recall that the natural inclusion functor $\text{Ho}(\text{dg-cat}^{tr}) \to \text{Ho}(\text{dg-cat})$ has a left adjoint functor $(\sim)_{\text{pe}} : \text{Ho}(\text{dg-cat}) \to \text{Ho}(\text{dg-cat}^{tr})$ that sends a dg-category $\mathcal{T}$ to its triangulated hull $\tilde{\mathcal{T}}_{\text{pe}}$. For two objects $\mathcal{T}, S \in \text{Ho}(\text{dg-cat})$, the Morita product $\mathcal{T} \otimes S \in \text{Ho}(\text{dg-cat}^{tr})$ of $\mathcal{T}$ and $S$ is defined by
The following is a special case of the derived Morita theory in [6].

**Theorem 2.32** [6, Corollary 5.18]. Let $X_1$ and $X_2$ be smooth varieties, and $G_1$ and $G_2$ affine algebraic groups. For each $i = 1, 2$, assume that $G_i$ acts on $X_i$, and let $W_i : X_i \to \mathbb{A}^1$ be a $\varphi_i$-semi-invariant regular functions, where $\varphi_i : G_i \to \mathbb{G}_m$ is a character. Assume the following conditions.

(i) The singular locus $\text{Sing}(Z_{W_1} \times Z_{W_2})$ of the product of the zero schemes $Z_{W_i}$ of $W_i$ is contained in $\text{Sing}(Z_{W_1} \boxplus W_2)$.
(ii) The category $\text{Dcoh}_{G_1 \times \mathbb{G}_m G_2}(X_1 \times X_2, \varphi_1 \times \mathbb{G}_m \varphi_2, W_1 \boxplus W_2)$ is idempotent complete.

Then, we have the following quasi-equivalence

$$\text{inj}_{G_1}(X_1, \varphi_1, W_1) \otimes \text{inj}_{G_2}(X_2, \varphi_2, W_2) \sim \text{inj}_{G_1 \times \mathbb{G}_m G_2}(X_1 \times X_2, \varphi_1 \times \mathbb{G}_m \varphi_2, W_1 \boxplus W_2).$$

Here the affine algebraic group $G_1 \times \mathbb{G}_m G_2$ and its character $\varphi_1 \times \mathbb{G}_m \varphi_2$ are defined as in Definition 2.29.

### 3 | SEMI-ORTHOGONAL DECOMPOSITIONS FROM SUMS OF POTENTIALS

In this section, we first prove semi-orthogonal decompositions of derived factorization categories of sums of potentials. The key ingredients of this semi-orthogonal decompositions are Theorem 2.28 and semi-orthogonal decompositions arising from variations of GIT quotients [8, 30, 56]. We then discuss its application to invertible polynomials of chain type.

#### 3.1 | Semi-orthogonal decompositions of the sums of potentials: General version

Denote by $\zeta \in k$ a primitive $N$th root of unity, and define a cyclic group $\mu_N = \langle \zeta \rangle$ to be the subgroup of $\mathbb{G}_m$ generated by $\zeta$. Let $G$ be a reductive affine algebraic group together with an injection $\iota : \mu_N \hookrightarrow G$ of algebraic groups such that its image $\iota(\mu_N) \subset G$ is a normal subgroup. Denote by $\pi : G \to G/\mu_N$ the natural projection. Let $X$ be a smooth quasi-projective variety with an action from $G/\mu_N$ such that $X$ has a $G/\mu_N$-invariant affine open covering. Let $\psi : G \to \mathbb{G}_m$ be a character of $G$ such that the composition $\psi \circ \iota : \mu_N \to \mathbb{G}_m$ is the natural inclusion. Since $\psi^N \circ \iota : \mu_N \to \mathbb{G}_m$ is trivial, there exists a character $\psi^N : G/\mu_N \to \mathbb{G}_m$ such that $\psi^N \circ \iota = \psi^N$. For a character $\phi : G/\mu_N \to \mathbb{G}_m$, we define characters $\overline{\phi} : G/\mu_N \to \mathbb{G}_m$ and $\chi : G \to \mathbb{G}_m$ by $\overline{\phi} := \phi \psi^N$ and $\chi := \overline{\phi} \circ \pi$, respectively. We fix a positive integer $m > 0$ and a vector $d := (d_1, \ldots, d_m) \in \mathbb{Z}_0^m$, and we define a set

$$I(d, N) := \{(i_1, \ldots, i_m) \in \mathbb{Z}_0^m \mid d_1 i_1 + \cdots + d_m i_m = N\}.$$
Let $\mathbb{A}_t^m$ be the $m$-dimensional affine space with coordinate $t = (t_1, \ldots, t_m)$, and define a $G$-action on $X \times \mathbb{A}_t^m$ by
\[
g \cdot (x, t_1, \ldots, t_m) := \left( \pi(g) \cdot x, \psi^{d_1}(g)t_1, \ldots, \psi^{d_m}(g)t_m \right).
\]

Let $W : X \to \mathbb{A}^1$ be a $\chi$-semi-invariant regular function and $F : X \times \mathbb{A}_t^m \to \mathbb{A}^1$ a non-constant regular function of the form
\[
F = f_{I_1}t_1 + \ldots + f_{I_r}t_r \in \Gamma(X, \mathcal{O}_X)[t_1, \ldots, t_m],
\]
where $I_k = (i_{k,1}, \ldots, i_{k,m}) \in I(d, N)$, $t^k := t_1^{i_{k,1}} \cdots t_m^{i_{k,m}}$, and each $f_{I_k} \in \Gamma(X, \mathcal{O}(\phi)^{G/\mu_N})$ is a $\phi$-semi-invariant regular function on $X$. Then, $F$ is a $\chi$-semi-invariant regular function on $X \times \mathbb{A}_t^m$ with respect to the above $G$-action. Denote by $\mathbb{P}(d) := [\mathbb{A}_t^m \setminus \{ 0 \}/G_m]$ the quotient stack of the $G_m$-action on $\mathbb{A}_t^m \setminus \{ 0 \}$ with the weight of $t_i$ being $d_i$, and denote by $Z_F \subset X \times \mathbb{P}(d)$ the hypersurface defined by $F$. Then, $Z_F$ is isomorphic to the quotient stack $[Y_F/G_m]$, where $Y_F \subset Y := X \times (\mathbb{A}_t^m \setminus \{ 0 \})$ is the zero scheme in $Y$ defined by $F$. Denote by $\mathcal{O}(1) \in \text{coh}(\mathbb{P}(d))$ the line bundle corresponding to $\mathcal{O}(\text{id}_{G_m})$ via the natural equivalence $\text{coh}(\mathbb{P}(d)) \cong \text{coh}(\mathbb{A}_t^m \setminus \{ 0 \})$, and let $\mathcal{L} \in \text{coh}(\mathbb{G}/\mu_N)$ be the pullback of $\mathcal{O}(1) \in \text{coh}(\mathbb{G}/\mu_N)$ by the morphism $Z_F \to \mathbb{P}(d)$ defined as the composition of the inclusion $Z_F \hookrightarrow X \times \mathbb{P}(d)$ and the projection $X \times \mathbb{P}(d) \to \mathbb{P}(d)$. By abuse of notation, we write $W$ for the pullbacks of $W : X \to \mathbb{A}^1$ by the natural projections $X \times \mathbb{A}_t^m \to X$ and $X \times \mathbb{P}(d) \to X$.

**Theorem 3.1.** Fix $\ell \in \mathbb{Z}$, and set $\mu := \sum_{i=1}^m d_i$. Assume either of the following conditions.

(i) The restricted function $W|_{Y_F} : Y_F \to \mathbb{A}^1$ is flat.

(ii) The restricted function $W|_{Y_F} : Y_F \to \mathbb{A}^1$ is the zero map, $Y_F$ is smooth and the category $\text{Dcoh}_{\mathbb{G}/\mu_N}(Z_F, \mathcal{X}, W|_{Z_F})$ is idempotent complete.

Then, we have the following.

(1) If $N < \mu$, there are fully faithful functors
\[
\Phi_\ell : \text{Dcoh}_{\mathbb{G}}(X \times \mathbb{A}_t^m, \mathcal{X}, W + F) \hookrightarrow \text{Dcoh}_{\mathbb{G}/\mu_N}(Z_F, \mathcal{X}, W|_{Z_F}),
\]
\[
\Psi_\ell : \text{Dcoh}_{\mathbb{G}/\mu_N}(X, \mathcal{X}, W) \hookrightarrow \text{Dcoh}_{\mathbb{G}/\mu_N}(Z_F, \mathcal{X}, W|_{Z_F})
\]
and there is a semi-orthogonal decomposition
\[
\text{Dcoh}_{\mathbb{G}/\mu_N}(Z_F, \mathcal{X}, W|_{Z_F}) = \langle \text{Im}(\Psi_\ell(N - \mu + 1)), \ldots, \text{Im}(\Psi_\ell), \text{Im}(\Phi_\ell) \rangle,
\]
where $\Psi_\ell(k) := ((- ) \otimes \mathcal{L}^k) \circ \Psi_\ell$.

(2) If $N = \mu$, we have an equivalence
\[
\Phi_\ell : \text{Dcoh}_{\mathbb{G}}(X \times \mathbb{A}_t^m, \mathcal{X}, W + F) \sim \text{Dcoh}_{\mathbb{G}/\mu_N}(Z_F, \mathcal{X}, W|_{Z_F}).
\]

(3) If $N > \mu$, there are fully faithful functors
\[
\Phi_\ell : \text{Dcoh}_{\mathbb{G}/\mu_N}(Z_F, \mathcal{X}, W|_{Z_F}) \hookrightarrow \text{Dcoh}_{\mathbb{G}}(X \times \mathbb{A}_t^m, \mathcal{X}, W + F),
\]
\[ \Psi_\epsilon : \text{Dcoh}_G/\mu_N(X, \overline{X}, W) \hookrightarrow \text{Dcoh}_G(X \times \mathbb{A}_t^m, \mathcal{X}, W + F) \]

and there is a semi-orthogonal decomposition

\[ \text{Dcoh}_G(X \times \mathbb{A}_t^m, \mathcal{X}, W + F) = \langle \text{Im}(\Psi_\epsilon), \ldots, \text{Im}(\Psi_\epsilon(\mu - N + 1)), \text{Im}(\Phi_\epsilon) \rangle, \]

where \( \Psi_\epsilon(k) := ((-)^k \otimes \mathcal{O}(\psi^k)) \circ \Psi_\epsilon \).

3.2 Proof of Theorem 3.1

We will prove Theorem 3.1 by modifying the arguments in [5, section 3; 32, section 5] rather than by generalizing them. The key new technique is to consider an unramified cyclic cover in (3.E) below. Set

\[ Q := X \times \mathbb{A}_t^m \times \mathbb{A}_u^1. \]

We define a \( \mathbb{G}_m \times (G/\mu_N) \)-action on \( Q \) by

\[ (a, g) \cdot (x, t_1, \ldots, t_m, u) := (g \cdot x, a^{d_1}t_1, \ldots, a^{d_m}t_m, a^{-N}\overline{\psi}_N(g)u). \tag{3.A} \]

Then, the regular function

\[ W_Q := W + Fu : Q \to \mathbb{A}_1^1 \]

on \( Q \) is \((1 \times \mathcal{X})\)-semi-invariant. Denote by \( \lambda : \mathbb{G}_m \to \mathbb{G}_m \times (G/\mu_N) \) the one-parameter subgroup defined by \( \lambda(a) := (a, 1) \). We denote by \( Z_\lambda \subset Q \) the fixed locus with respect to the \( \lambda \)-action. Then, we have \( Z_\lambda = \{(x, t_1, \ldots, t_m, u) \in Q \mid t_1 = \cdots = t_m = u = 0\} \cong X \). Set \( S_+ := \{q \in Q \mid \lim_{a \to 0} \lambda(a)q \in Z_\lambda \} \) and \( S_- := \{q \in Q \mid \lim_{a \to 0} \lambda(a)^{-1}q \in Z_\lambda \} \). If we define \( Q_\pm \) to be the complement of \( S_\pm \) in \( Q \), then we have

\[ Q_+ = X \times \mathbb{A}_t^m \times (\mathbb{A}_u^1 \setminus \{0\}) \quad \text{and} \quad Q_- = X \times (\mathbb{A}_t^m \setminus \{0\}) \times \mathbb{A}_u^1. \]

Then, the stratifications

\[ (\mathcal{R}^+ : Q = Q_+ \sqcup S_+) \quad \text{and} \quad (\mathcal{R}^- : Q = Q_- \sqcup S_-) \]

are elementary HKKN stratifications, and the pair of these stratifications defines an elementary wall crossing in the sense of [8]. Denote by \( t(\mathcal{R}^\pm) \) the \( \lambda^\pm \)-weight of the \( \mathbb{G}_m \)-action on the fiber \( \text{Spec}(\text{Sym}(\omega_{S_\pm/Q})) \) of the geometric vector bundle associated to the \( \lambda^\pm \)-equivariant relative canonical bundle \( \omega_{S_\pm/Q} \) at a point \( x \) in the fixed locus \( Z_\lambda = Z_{\lambda^{-1}} \). Since \( Z_\lambda \) is connected, the numbers \( t(\mathcal{R}^\pm) \) do not depend on the choice of \( x \) (see [8, Lemma 2.1.18]). Then, we have

\[ t(\mathcal{R}^+) = -N \quad \text{and} \quad t(\mathcal{R}^-) = -\mu. \]
Let \( \lambda \subset G_m \times (G/\mu_N) \) be the image of \( \lambda : G_m \to G_m \times (G/\mu_N) \), and denote by \( C(\lambda) \) the centralizer of \( \lambda \subset G_m \times (G/\mu_N) \). Then, we have \( C(\lambda) = G_m \times (G/\mu_N) \). Furthermore, we define \( G_\lambda := C(\lambda)/\langle \lambda \rangle \), and let

\[
\theta : G_m \times (G/\mu_N) \to G_m
\]

be the character of \( G_m \times (G/\mu_N) = C(\lambda) \) defined by \( \theta(a, g) := a \). Then, we have a natural equivalence

\[
\text{Dcoh}_{G_\lambda}(Z_\lambda, 1 \times \overline{X}, W_q|_{Z_\lambda}) \cong \text{Dcoh}_{G/\mu_N}(X, \overline{X}, W). \quad (3.B)
\]

Let

\[
i_\pm : Q_\pm \hookrightarrow Q \quad \text{and} \quad j_\pm : S_\pm \hookrightarrow Q
\]

be the open immersions and the closed immersions, respectively, and denote by

\[
\pi_\pm : S_\pm \to X
\]

the natural projections. Then, \( \pi_+ \) is a trivial vector bundle of rank \( m \), and \( \pi_- \) is a trivial vector bundle of rank 1. Identifying \( X \) with the fixed locus \( Z_\lambda = X \times \{0\} \times \{0\} \subset Q \), we have an induced \( G_m \times (G/\mu_N) \)-action on \( X \) given by \( (a, g) \cdot x = g \cdot x \). Then, by Lemma 2.4 the projection \( p : G_m \times (G/\mu_N) \to G/\mu_N \) induces a fully faithful functor

\[
id^*_p : \text{coh}_{G/\mu_N}X \leftrightarrow \text{coh}_{G_m \times (G/\mu_N)}X
\]

since \( G_m \)-action on \( X \) is trivial, and we have the following decomposition

\[
\text{coh}_{G_m \times (G/\mu_N)}X \cong \bigoplus_{k \in \mathbb{Z}} \text{coh}_{G_m \times (G/\mu_N)}(X \times \overline{X}, W)(\theta|_{G_m})^k. \quad (3.C)
\]

We define \( \text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times G, W)_k \subset \text{Dcoh}_{G_m \times G}(X, 1 \times G, W) \) to be the subcategory consisting of factorizations \( F \) whose components \( F_i \) lie in \( \text{coh}_{G_m \times G}(X \times \overline{X}, W)(\theta|_{G_m})^k \). Note that the subcategory \( \text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times G, W)_k \) is the subcategory of factorizations with \( \lambda \)-weights \( k \) (or equivalently, with \( -\lambda \)-weights \( -k \)) in the sense of [8, Definition 3.4.3]. Then, the above decomposition \((3.C)\) gives rise to a decomposition of the derived factorization category

\[
\text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times \overline{X}, W) \cong \bigoplus_{k \in \mathbb{Z}} \text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times \overline{X}, W)_k.
\]

Denote by

\[
\tau_k : \text{Dcoh}_{G/\mu_N}(X, \overline{X}, W) \sim \text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times \overline{X}, W)_k
\]

the equivalence defined by \( \tau_k(F) := \text{id}^*_p(F) \otimes \mathcal{O}(\theta^k) \). We also denote by

\[
t_k : \text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times \overline{X}, W)_k \hookrightarrow \text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times \overline{X}, W)
\]

the natural inclusion.
***Definition 3.2.*** We define functors

\[ \Upsilon^\pm_k : \text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times \overline{X}, W)_k \to \text{Dcoh}_{G_m \times (G/\mu_N)}(Q, 1 \times \overline{X}, W_Q) \]

as follows: First consider the (underived) inverse image

\[ \pi^\pm_* : \text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times \overline{X}, W) \to \text{Dcoh}_{G_m \times (G/\mu_N)}(S^\pm, 1 \times \overline{X}, \pi^\pm_* W) \]

of the projection map \( \pi_\pm : S^\pm \to X \). Next, since the pullback \((j^\pm)_* W_Q : S^\pm \to \mathbb{A}^1\) equals to the pullback \(\pi^\pm_* W\), we have the (underived) direct image

\[ j^\pm_* : \text{Dcoh}_{G_m \times (G/\mu_N)}(S^\pm, 1 \times \overline{X}, \pi^\pm_* W) \to \text{Dcoh}_{G_m \times (G/\mu_N)}(Q, 1 \times \overline{X}, W_Q). \]

Then, we define \( \Upsilon^\pm_k \) to be the composition \( j^\pm_* \circ \pi^\pm_* \circ \iota_k \).

***Lemma 3.3*** [8, Lemma 3.4.5]. *The functor*

\[ \Upsilon^\pm_k : \text{Dcoh}_{G_m \times (G/\mu_N)}(X, 1 \times \overline{X}, W)_k \to \text{Dcoh}_{G_m \times (G/\mu_N)}(Q, 1 \times \overline{X}, W_Q) \]

*is fully faithful.*

Next, we consider windows of \( \text{Dcoh}_{G_m \times (G/\mu_N)}(Q, 1 \times \overline{X}, W_Q) \). For an interval \( I \subset \mathbb{Z} \), let

\[ \mathcal{W}^\pm_{\lambda^\pm, I} \subset \text{Dcoh}_{G_m \times (G/\mu_N)}(Q, 1 \times \overline{X}, W_Q) \]

be the \( I \)-grade-windows with respect to \( \lambda^\pm \) in the sense of [8, Definition 3.1.2]. For each integer \( \ell \in \mathbb{Z} \), we set \( I^\pm_\ell := [\ell + t(\mathfrak{K}^\pm) + 1, \ell] \subset \mathbb{Z} \), and we define subcategories

\[ \mathcal{W}^\pm_\ell := \mathcal{W}^\pm_{\lambda^\pm, I^\pm_\ell} \subset \text{Dcoh}_{G_m \times (G/\mu_N)}(Q, 1 \times \overline{X}, W_Q). \]

Then, by [8, Corollary 3.2.2, Proposition 3.3.2], the pullbacks

\[ i^\pm_* (\ell) := i^\pm_* |_{\mathcal{W}^\pm_\ell} : \mathcal{W}^\pm_\ell \to \text{Dcoh}_{G_m \times (G/\mu_N)}(Q, 1 \times \overline{X}, W_Q) \]

of the open immersions \( i^\pm_* \) are equivalences. Since \( \mathcal{W}^\pm_\ell = \mathcal{W}^\pm_{\lambda^\mp, [-\ell, -\ell - t(\mathfrak{K}^\pm) - 1]} \), we have the following.

- **If** \( t(\mathfrak{K}^+) \leq t(\mathfrak{K}^-) \), we have \( \mathcal{W}^\pm_{-\ell} \subset \mathcal{W}^\pm_{-t(\mathfrak{K}^+) + \ell - 1} \). In this case, we define a fully faithful functor
  \[ \Phi^+_\ell : \text{Dcoh}_{G_m \times (G/\mu_N)}(Q_-, 1 \times \overline{X}, W_Q) \hookrightarrow \text{Dcoh}_{G_m \times (G/\mu_N)}(Q_+, 1 \times \overline{X}, W_Q) \]
  to be the composition \( i^+_\ell (t(\mathfrak{K}^+) + \ell - 1) \circ (i^+_\ell (\ell))^{-1} \).

- **If** \( t(\mathfrak{K}^+) \geq t(\mathfrak{K}^-) \), we have \( \mathcal{W}^\pm_{-\ell} \subset \mathcal{W}^\pm_{-t(\mathfrak{K}^-) + \ell - 1} \). In this case, we define a fully faithful functor
  \[ \Phi^-_\ell : \text{Dcoh}_{G_m \times (G/\mu_N)}(Q_+, 1 \times \overline{X}, W_Q) \hookrightarrow \text{Dcoh}_{G_m \times (G/\mu_N)}(Q_-, 1 \times \overline{X}, W_Q) \]
  to be the composition \( i^-\ell (t(\mathfrak{K}^-) + \ell - 1) \circ (i^-\ell (\ell))^{-1} \).
Furthermore, by [8, Lemma 3.4.6], we have the following.

- If $t(\mathfrak{K}^+) < t(\mathfrak{K}^-)$ and $-t(\mathfrak{K}^-) + \ell \leq j \leq -t(\mathfrak{K}^+) + \ell - 1$, the essential image of $Y^+_j$ lies in $\mathcal{W}_{-t(\mathfrak{K}^+) + \ell - 1}$. In this case, we define a fully faithful functor
  \[ \Psi_j : D_{coh\mathcal{G}\times(G/\mathcal{G}_N)}(Q_+, 1 \times \overline{X}, W_Q) \]
  to be the composition $i^+_*(t(\mathfrak{K}^-) + \ell - 1) \circ Y^+_j \circ \tau_j$. Since the essential image of $Y^+_{j+1}$ is equal to that of the composition $((-) \otimes \mathcal{O}(\mathfrak{g})) \circ Y^+_j$, we have
  \[ \text{Im}(\Psi_{j+1}) = \text{Im}\left( ((-) \otimes \mathcal{O}(\mathfrak{g})) \circ \Psi_j \right). \]

- If $t(\mathfrak{K}^+) > t(\mathfrak{K}^-)$ and $-t(\mathfrak{K}^+) + \ell \leq j \leq -t(\mathfrak{K}^-) + \ell - 1$, the essential image of $Y^-_j$ lies in $\mathcal{W}_{-t(\mathfrak{K}^-) + \ell - 1}$. In this case, we define a fully faithful functor
  \[ \Psi_j : D_{coh\mathcal{G}\times(G/\mathcal{G}_N)}(Q_-, 1 \times \overline{X}, W_Q) \]
  to be the composition $i^-_*(t(\mathfrak{K}^-) + \ell - 1) \circ Y^-_j \circ \tau_j$. Since the essential image of $Y^-_{j+1}$ is equal to that of the composition $((-) \otimes \mathcal{O}(\mathfrak{g}^{-1})) \circ Y^-_j$, we have
  \[ \text{Im}(\Psi_{j+1}) = \text{Im}\left( ((-) \otimes \mathcal{O}(\mathfrak{g}^{-1})) \circ \Psi_j \right). \]

Then, by iterated applications of [8, Proposition 3.4.7], the above fully faithful functors give rise to the following semi-orthogonal decompositions.

**Proposition 3.4.** Fix $\ell \in \mathbb{Z}$.

1. **If** $N > \mu$, there is a semi-orthogonal decomposition
   \[ D_{coh\mathcal{G}\times(G/\mathcal{G}_N)}(Q_+, 1 \times \overline{X}, W_Q) = \langle \text{Im}(\Psi_{N+\ell-1}), ..., \text{Im}(\Psi_{\mu+\ell}), \text{Im}(\Phi_\ell) \rangle. \]

2. **If** $N = \mu$, we have an equivalence
   \[ \Phi_\ell : D_{coh\mathcal{G}\times(G/\mathcal{G}_N)}(Q_+, 1 \times \overline{X}, W_Q) \sim D_{coh\mathcal{G}\times(G/\mathcal{G}_N)}(Q_-, 1 \times \overline{X}, W_Q). \]

3. **If** $N < \mu$, there is a semi-orthogonal decomposition
   \[ D_{coh\mathcal{G}\times(G/\mathcal{G}_N)}(Q_-, 1 \times \overline{X}, W_Q) = \langle \text{Im}(\Psi_{\mu+\ell-1}), ..., \text{Im}(\Psi_{N+\ell}), \text{Im}(\Phi_\ell) \rangle. \]

Now it is enough to show the following two lemmas:

**Lemma 3.5.** With the same notation as above, we have the following.
We have an equivalence
\[ \Phi_+ : \text{Dcoh}_{\mathbb{G}_m \times G}(Q_+, 1 \times \overline{X}, W_Q) \cong \text{Dcoh}_G(X \times \mathbb{A}^m, \chi, W + F). \]

Under the assumption of Theorem 3.1, we have an equivalence
\[ \Phi_- : \text{Dcoh}_{\mathbb{G}_m \times G}(Q-, 1 \times \overline{X}, W_Q) \cong \text{Dcoh}_{G/\mathbb{N}}(Z_F, \chi, W|_{Z_F}). \]

Proof.

1. We consider \( \mathbb{G}_m \times G \)-action on \( Q_+ = X \times \mathbb{A}^m \times (\mathbb{A}^1_u \setminus \{0\}) \) defined by
\[ \left( a, g \right) \cdot \left( x, t_1, \ldots, t_m, u \right) \mapsto \left( \pi(g) \cdot x, \psi(g)^{d_1}t_1, \ldots, \psi(g)^{d_m}t_m, \psi(g)^{-1}au \right). \]

Denote by \( \tilde{Q}_+ \) this new \( \mathbb{G}_m \times G \)-variety. Furthermore, we consider an unramified cyclic cover \( p_+ : \tilde{Q}_+ \to Q_+ \) defined by
\[ p_+(x, t_1, \ldots, t_m, u) : = (x, t_1 u^{d_1} \ldots, t_m u^{d_m}, u^{-N}). \]

Note that \( \mathbb{G}_m \times (G/\mathbb{N}) \)-action on \( Q_+ \) lifts to an action from \( \mathbb{G}_m \times G \) on \( Q_+ \), and the subgroup \( 1 \times \mathbb{N} \subset \mathbb{G}_m \times G \) acts trivially on \( Q_+ \). Then, \( Q_+ \) is a \( \mathbb{G}_m \times G \)-variety, and \( p_+ \) is a \( \mathbb{G}_m \times G \)-equivariant morphism that is a principal \( \mathbb{N} \)-bundle, where \( \mathbb{N} \)-action on \( \tilde{Q}_+ \) is given by
\[ \zeta \cdot (x, t_1, \ldots, t_m, u) : = (x, \zeta^{d_1}t_1, \ldots, \zeta^{d_m}t_m, \zeta^{-1}u) \]

since \( \psi \circ \iota : \mathbb{N} \to \mathbb{G}_m \) is the natural inclusion. Therefore, we have the following equivalences
\[ \text{Dcoh}_{\mathbb{G}_m \times G}(Q_+, 1 \times \overline{X}, W_Q) \cong \text{Dcoh}_{\mathbb{G}_m \times G}(\tilde{Q}_+, 1 \times \overline{X}, \pi^* W_Q) \cong \text{Dcoh}_G(X \times \mathbb{A}^m, \chi, W + F), \]

where the first equivalence follows from Lemma 2.3 and the second one follows from Lemma 2.5.

2. We define \( Y := X \times (\mathbb{A}^m \setminus \{0\}) \). Then, \( Q_- = Y \times \mathbb{A}^1 \), and so \( Y \) has the induced \( \mathbb{G}_m \times (G/\mathbb{N}) \)-action. Recall that \( \chi_{-N} : \mathbb{G}_m \to \mathbb{G}_m \) denotes the character defined by \( \chi_{-N}(a) := a^{-N} \), and set \( E := \mathcal{O}(\chi_{-N} \times (\phi^{-1})) \). Then, \( F \in \Gamma(Y, E^\vee \otimes_{\mathbb{G}_m \times G}(G/\mathbb{N}), \text{and } Q_- \) is the \( \mathbb{G}_m \times (G/\mathbb{N}) \)-vector bundle on \( Y \) associated to \( E(1 \times \overline{X}) \cong \mathcal{O}(\chi_{-N} \otimes \psi_N) \). More precisely, we have an isomorphism
\[ Q_- \cong \text{Spec}(\text{Sym}_Y(E(1 \times \overline{X})^\vee)). \]

Then, the quotient stack \( Z_F = [Y_F/\mathbb{G}_m] \subset [Y/\mathbb{G}_m] \cong X \times \mathbb{P}(d) \) is the closed substack of \( X \times \mathbb{P}(d) \). Therefore, we have the following equivalences
\[ \text{Dcoh}_{\mathbb{G}_m \times G}(Q_-, 1 \times \overline{X}, W_Q) \cong \text{Dcoh}_{\mathbb{G}_m \times G}(Y_F, 1 \times \overline{X}, W|_{Y_F}) \cong \text{Dcoh}_{G/\mathbb{N}}(Z_F, \overline{X}, W|_{Z_F}), \]

where the first equivalence follows from Theorem 2.28. \( \square \)
Lemma 3.6. Let $\Phi_{\pm}$ be the equivalences in the above lemma. We have the following functor isomorphisms:

$$
\Phi_{+}((-) \otimes \mathcal{O}(\theta)) \cong \Phi_{+}(-) \otimes \mathcal{O}(\psi)
$$

$$
\Phi_{-}((-) \otimes \mathcal{O}(\theta)) \cong \Phi_{-}(-) \otimes \mathcal{L}.
$$

Proof. Denote by $\gamma : \mathbb{G}_m \times G \to \mathbb{G}_m \times (\mathbb{G}/\mu_N)$ and $\eta : G \to \mathbb{G}_m \times G$ the morphisms defined by $\gamma(a, g) := (a, \pi(g))$ and $\eta(g) := (\psi(g), g)$, respectively. Let $e : X \times A_1^m \to \tilde{Q}^+ = X \times A_1^m \times (\mathbb{A}^1_\mathbb{C} \setminus \{0\})$ be the inclusion defined by $e(x, t) := (x, t, 1)$. Then, $\Phi_{+} = e^*_\eta \circ (p_+)\gamma_\eta$ by definition, and hence the former functor isomorphism follows from the following sequence of isomorphisms

$$
\Phi_{+}((-) \otimes \mathcal{O}(\theta)) 
\cong e^*_\eta((p_+)\gamma_\eta((-) \otimes \mathcal{O}(\theta)))
\cong e^*_\eta((p_+)\gamma_\eta(-) \otimes \mathcal{O}(\theta \circ \gamma \circ \eta))
\cong e^*_\eta((p_+)\gamma_\eta(-) \otimes \mathcal{O}(\theta \circ \gamma \circ \eta))
= \Phi_{+}(-) \otimes \mathcal{O}(\psi).
$$

The latter functor isomorphism is also checked by an easy direct calculation.

3.3 One-dimensional reduction of factorizations

In this subsection, we refine the semi-orthogonal decompositions in Theorem 3.1 in the case when $m = d_1 = \mu = 1$. We keep the notation above, and let $s \in \Gamma(X, \mathcal{O}(\phi))^{G/\mu_N}$ be a non-zero $\phi$-semi-invariant regular function on $X$. Denote by $Z_s \subset X$ the zero scheme of $s$. Then, if $s$ is non-constant and $m = d_1 = 1$, $X \times \mathbb{P}(d)$ is isomorphic to $X$ and the algebraic stack $Z_F$ is isomorphic to $Z_s$. Hence, by Proposition 3.4 and Lemmas 3.5(2) and 3.8, we obtain a semi-orthogonal decomposition that compares derived factorization categories associated to the potentials $W : X \to \mathbb{A}^1$ and $W + st^N : X \times \mathbb{A}^1_\mathbb{C} \to \mathbb{A}^1_\mathbb{C}$.

Corollary 3.7. There is a fully faithful functor

$$
\Phi : D\text{coh}_{G/\mu_N}(X, X_s, W) \hookrightarrow D\text{coh}_G(X \times \mathbb{A}^1_\mathbb{C}, X, W + st^N).
$$

Furthermore, we have the following semi-orthogonal decomposition:

(1) If $s = 1$ and $N > 1$, we have a semi-orthogonal decomposition

$$
D\text{coh}_G(X \times \mathbb{A}^1_\mathbb{C}, X, W + t^N) = \langle \text{Im}(\Phi_0), \ldots, \text{Im}(\Phi_{-N+2}) \rangle.
$$

(2) If $s$ is non-constant and $W|_{Z_s} : Z_s \to \mathbb{A}^1$ is flat, there is a fully faithful functor

$$
\Psi : D\text{coh}_{G/\mu_N}(Z_s, \mathbb{X}, W|_{Z_s}) \hookrightarrow D\text{coh}_G(X \times \mathbb{A}^1_\mathbb{C}, X, W + st^N),
$$
and we have a semi-orthogonal decomposition
\[
\text{Dcoh}_G(X \times \mathbb{A}^1, X, W + st^N) = \langle \text{Im}(\Phi_0), \ldots, \text{Im}(\Phi_{-N+2}), \text{Im}(\Psi) \rangle.
\]
Here \(\Phi_i\) denotes the composition \((-\otimes O(\psi^i)) \circ \Phi\).

**Lemma 3.8.** If \(s = 1\), the category \(\text{Dcoh}_{\mathbb{G} \times (G/\mu_N)}(Q_-, 1 \times \overline{X}, W_Q)\) in the proof of Theorem 3.1 is the zero category.

**Proof.** Since \(\mathbb{G} \times (G/\mu_N)\) is reductive, by [32, Lemma 2.33] the restriction functor
\[
\text{Dcoh}_{\mathbb{G} \times (G/\mu_N)}(Q_-, 1 \times \overline{X}, W_Q) \rightarrow \text{Dcoh}(Q_-, W_Q)
\]
is faithful. Hence, it is enough to show that \(\text{Dcoh}(Q_-, W_Q) = 0\). Since \(W_Q\) is flat, \(\text{Dcoh}(Q_-, W_Q)\) is equivalent to the singularity category \(\text{D}^\text{sg}(Q_0)\) of the zero scheme \(Q_0\) of \(W_Q : Q_- \rightarrow \mathbb{A}^1\) by Theorem 2.21. Since \(Q_0\) is smooth, we have \(\text{Dcoh}(Q_-, W_Q) \cong \text{D}^\text{sg}(Q_0) = 0\). \(\square\)

**Remark 3.9.** By [59] or [48], there are quasi-fully faithful dg-functors \(F_i : \text{inj}_{\mathbb{G} \times (G/\mu_N)}(\text{Spec } k, X_1, 0) \rightarrow \text{inj}_{\mathbb{G} \times (G/\mu_N)}(\mathbb{A}^1, X_N, t^N)\) \((-N + 2 \leq i \leq 0\)) and the semi-orthogonal decomposition
\[
\text{inj}_{\mathbb{G} \times (G/\mu_N)}(\mathbb{A}^1, X_N, t^N) = \langle \text{Im}(F_0), \ldots, \text{Im}(F_{-N+2}) \rangle.
\]
Moreover, [6, Corollary 5.18] implies the following semi-orthogonal decomposition
\[
\text{inj}_{\mathbb{G} \times (G/\mu_N)}(X, \overline{X}, W) \cong \text{inj}_{\mathbb{G} \times (G/\mu_N)}(X, \overline{X}, W) \otimes \text{inj}_{\mathbb{G} \times (G/\mu_N)}(\text{Spec } k, X_1, 0),
\]
where \(\text{inj}\) denotes the dg-subcategory of \(\text{Inj}\) consisting of compact objects. Hence, by [6, Corollary 5.18; 7, Lemma 2.49] there are quasi-fully faithful functors
\[
\overline{\Phi}_i : \text{inj}_{\mathbb{G} \times (G/\mu_N)}(X, \overline{X}, W) \rightarrow \text{inj}_{\mathbb{G} \times (G/\mu_N)}(X \times \mathbb{A}^1, X, W + t^N)
\]
for \(-N + 2 \leq i \leq 0\) and the semi-orthogonal decomposition
\[
\text{inj}_{\mathbb{G} \times (G/\mu_N)}(X \times \mathbb{A}^1, X, W + t^N) = \langle \text{Im}(\overline{\Phi}_0), \ldots, \text{Im}(\overline{\Phi}_{-N-2}) \rangle.
\]
Hence, Corollary 3.7(1) follows from the derived Morita theory if \(\text{Dcoh}_G(X \times \mathbb{A}^1, X, W + t^N)\) and \(\text{Dcoh}_{\mathbb{G} \times (G/\mu_N)}(X, \overline{X}, W)\) are idempotent complete.

### 3.4 Semi-orthogonal decomposition for chain polynomials

For each positive integer \(i \in \mathbb{Z}_{\geq 1}\), choose a positive integer \(a_i \in \mathbb{Z}_{\geq 1}\) with \(a_i \geq 2\). Then, for each \(n \in \mathbb{Z}_{\geq 0}\), we have the associated polynomial
\[
f_n := x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{n-1} x_n^{a_n},
\]
where we set $f_0 := 0$. We denote by $L_n$ the maximal grading of $f_n$, which is defined in (1.A). Then, we have $L_0 = \mathbb{Z} \cong \mathbb{Z}$, $L_1 \cong \mathbb{Z}$ and $\operatorname{rk}(L_n) = 1$ for any $n \geq 0$. Then, the $n$-dimensional polynomial ring $S^n = k[x_1, \ldots, x_n]$ has a natural $L_n$-grading such that $\deg(x_i) := \vec{e}_i$ and $\deg(f) = \vec{f}$. For $n \geq -1$, we set

$$C_n := \operatorname{HMF}_{S^n}(f_n)$$

$$D_n := \operatorname{Dmod}_{S^n}(S^n, f_n),$$

where $C_{-1}$ and $D_{-1}$ are defined to be the zero categories. It is well-known that the category $C_n$ is idempotent complete for any $n \geq 0$ [2, Proposition 2.7].

Any element $l \in L_n$ can be represented by $\sum_{i=1}^n l_i \vec{e}_i$ for some $l_i \in \mathbb{Z}$, and it induces a character $\chi_l : G_n \to \mathbb{G}_m$ defined by

$$\chi_l(\lambda_1, \ldots, \lambda_n) := \prod_{i=1}^n \lambda_l^i.$$ 

Note that this character does not depend on the choice of $l_i$, and this defines an isomorphism

$$L_n \sim (G_n)^{\vee} = \operatorname{Hom}(G_n, \mathbb{G}_m), \quad (3.F)$$

where $G_n$ is the algebraic group defined in (1.B). Thus, if we write

$$D_n^{\text{coh}} := \operatorname{Dcoh}_{G_n}(\mathbb{A}_x^n, \chi_{f_n}, f_n)$$

as in the introduction, by Proposition 2.23, we have natural equivalences

$$C_n \sim D_n \sim D_n^{\text{coh}}. \quad (3.G)$$

Now we apply Theorem 3.1 in the case when $\ell = a_n - 1$ for chain polynomials as follows: We use the same notation as in Subsection 3.2, and we consider the case when $m = \mu = d_1 = 1, X = \mathbb{A}_x^{n-1}, N = a_n$ and $G = G_n$. The natural projection $\pi_n : (G_m)^n \to (G_m)^{n-1}$ induces a surjective morphism

$$\pi := \pi_n : G_n \to G_{n-1}, (\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1, \ldots, \lambda_{n-1}) \quad (3.H)$$

of algebraic groups, and we have $\operatorname{Ker}(\pi) = \{(1, \ldots, 1, \lambda_n) \in G_n \mid \lambda_d \in \mu_{a_n} \} \cong \mu_{a_n}$. Hence, there is an exact sequence

$$1 \to \mu_{a_n} \xrightarrow{\iota} G_n \xrightarrow{\pi} G_{n-1} \to 1,$$

where $\iota : \mu_{a_n} \to G_n$ is defined by $\iota(\lambda) := (1, \ldots, 1, \lambda)$, and so $G_{n-1} \cong G/\mu_{a_n}$.

Let $\psi : G = G_n \to G_m, \psi_{a_n} : G/\mu_{a_n} \cong G_{n-1} \to G_m$ and $\phi : G/\mu_{a_n} \cong G_{n-1} \to G_m$ be the characters in Subsection 3.2 defined by

$$\psi(\lambda_1, \ldots, \lambda_n) := \lambda_n$$

$$\psi_{a_n}(\lambda_1, \ldots, \lambda_{n-1}) := \lambda_1^{a_1} \lambda_{n-1}^{-1}$$

$$\phi(\lambda_1, \ldots, \lambda_n) := \phi(\lambda_1, \ldots, \lambda_{n-1}) := \psi(\lambda_1, \ldots, \lambda_{n-1})$$

As $G/\mu_{a_n} \cong G_{n-1}$, we have the natural isomorphism $\phi \cong \psi_{a_n}$.
\[ \phi(\lambda_1, \ldots, \lambda_{n-1}) := \lambda_{n-1}, \]

respectively. Then, \( \psi \circ t : \mu_{a_n} \to G_m \) is the natural inclusion, and \( \overline{\psi}^{a_n} \circ \pi = \psi^a.n \). Recall that \( \overline{\chi} : G\,_{n-1} \to G_m \) is the character defined by \( \overline{\chi} := \phi \psi^{a_n} \) and \( \chi = \overline{\chi} \circ \pi : G_n \to G_m \). Then, \( \overline{\chi} = \chi_{f_{n-1}} \) and \( \chi = \chi_{f_n} \). We write \( x_n \) instead of \( i \), and so

\[ Q = \mathbb{A}^{n-1}_x \times \mathbb{A}^1_{x_n} \times \mathbb{A}^1_u \cong \text{Spec} \, S^{n-1}[x_n, u], \]

and \( Q \) has an action from \( G_m \times G/\mu_{a_n} \cong G_m \times G_{n-1} \) defined by \( (3.A) \). We set

\( R := S^{n-1}[x_n, u] \)

\( R_+ := S^{n-1}[x_n, u^{\pm 1}] \)

\( R_- := S^{n-1}[x_{\pm 1}^1, u] \).

Then, \( Q \cong \text{Spec} \, R \), and we have

\[ Q_+ = \mathbb{A}^{n-1}_x \times \mathbb{A}^1_{x_n} \times (\mathbb{A}^1_u \setminus \{0\}) \cong \text{Spec} \, R_+ \]

\[ Q_- = \mathbb{A}^{n-1}_x \times (\mathbb{A}^1_{x_n} \setminus \{0\}) \times \mathbb{A}^1_u \cong \text{Spec} \, R_- \).

Consider the case when \( W = f_{n-1} \) and \( F = x_{n-1}x_n^a, \) and thus \( W_Q = f_{n-1} + x_{n-1}x_n^a u \). By Proposition 3.4 in the case when \( \ell' := -a_n + 1, \) the fully faithful functors

\[ \Psi_i : D_{n-1}^{\text{coh}} \hookrightarrow \text{Dcoh}_{G_m \times G_{n-1}}(Q_+, 1 \times \chi_{f_{n-1}}, W_Q) \]

for \( -a_n + 2 \leq i \leq 0 \) and

\[ \Phi_{(-a_n+1)} : \text{Dcoh}_{G_m \times G_{n-1}}(Q_-, 1 \times \chi_{f_{n-1}}, W_Q) \hookrightarrow \text{Dcoh}_{G_m \times G_{n-1}}(Q_+, 1 \times \chi_{f_{n-1}}, W_Q), \]

gives rise to a semi-orthogonal decomposition

\[ \text{Dcoh}_{G_m \times G_{n-1}}(Q_+, 1 \times \chi_{f_{n-1}}, W_Q) = \langle \text{Im} (\Psi_0), \ldots, \text{Im} (\Psi_{-a_n+2}), \text{Im} (\Phi_{-a_n+1}) \rangle. \]

Moreover, by Lemma 3.5 we have equivalences

\[ \Phi_+ : \text{Dcoh}_{G_m \times G_{n-1}}(Q_+, 1 \times \chi_{f_{n-1}}, W_Q) \sim D_{n}^{\text{coh}} \]

\[ \Phi_- : \text{Dcoh}_{G_m \times G_{n-1}}(Q_-, 1 \times \chi_{f_{n-1}}, W_Q) \sim \text{Dcoh}_{G_{n-1}}(\mathbb{A}^{n-2}_x, \chi_{f_{n-1}}, f_{n-2}). \]

Since the subgroup \( \mu_{a_n} \hookrightarrow G_{n-1}; \lambda \mapsto (1, \ldots, 1, \lambda) \) of \( G_{n-1} \) trivially acts on \( \mathbb{A}^{n-2}_x \), by Lemma 2.4 we have orthogonal decomposition

\[ \text{coh}_{G_{n-1}} \mathbb{A}^{n-2}_x \cong \bigoplus_{j=0}^{a_{n-1}-1} \text{Im} \left( \left( (-) \otimes \mathcal{O}(\phi^j) \right) \circ \text{id}^* \right), \]
where $\text{id}_{\pi_{n-1}}^*: \text{coh}_{\mathbb{A}^n_{x}} \rightarrow \text{coh}_{\mathbb{A}^{n-2}_{x}}$ is the fully faithful functor associated to the surjection $\pi_{n-1}: G_{n-1} \rightarrow G_{n-2}$. Since $\phi^j+a_{n-1}|_{\mu_{n-1}} = \phi^j|_{\mu_{n-1}}$ for any $j \in \mathbb{Z}$, we have an equality $\text{Im}\bigg(((-) \otimes \mathcal{O}(\phi^j+a_{n-1})) \circ \text{id}_{\pi_{n-1}}^*\bigg) = \text{Im}\bigg(((-) \otimes \mathcal{O}(\phi^j)) \circ \text{id}_{\pi_{n-1}}^*\bigg)$ by Lemma 2.4. Thus, we have

$$\text{coh}_{G_{n-1}}^* \mathbb{A}^n_{x} \cong \bigoplus_{j=0}^{-a_{n-1}+1} \text{Im}\bigg(((-) \otimes \mathcal{O}(\phi^j)) \circ \text{id}_{\pi_{n-1}}^*\bigg).$$

We define the functors

$$\Psi^\text{coh} : D_{n-1}^\text{coh} \rightarrow D_n^\text{coh} \quad (3.0)$$

$$\Phi^\text{coh} : D_{n-2}^\text{coh} \rightarrow D_n^\text{coh} \quad (3.1)$$

to be the following compositions of fully faithful functors

$$\Psi^\text{coh} := \Phi_+ \circ \Psi_0 \quad (3.2)$$

$$\Phi^\text{coh} := \Phi_+ \circ \Phi_{(-a+1)} \circ (\Phi_-)^{-1} \circ \text{id}_{\pi_{n-1}}^* \quad (3.3)$$

Denote by

$$\Psi : C_{n-1} \hookrightarrow C_n, \quad \Phi : C_{n-2} \hookrightarrow C_n \quad (3.4)$$

$$\Psi^\text{mod} : D_{n-1} \hookrightarrow D_n, \quad \Phi^\text{mod} : D_{n-2} \hookrightarrow D_n \quad (3.5)$$

the corresponding functors via the natural equivalences (3.6).

**Theorem 3.10.** Let $n \geq 1$ be a positive integer. The fully faithful functors

$$\Psi : C_{n-1} \hookrightarrow C_n \quad \text{and} \quad \Phi : C_{n-2} \hookrightarrow C_n$$

gives a semi-orthogonal decomposition

$$C_n = \left\langle \text{Im}(\Psi_0), ..., \text{Im}(\Psi_{(-a_{n-2})}), \bigoplus_{j=0}^{-a_{n-1}+1} \text{Im}(\Phi_j) \right\rangle,$$

where $\Psi_i$ is the composition $((-) \otimes (i\bar{x}_n)) \circ \Psi$ and $\Phi_j$ is the composition $((-) \otimes (j\bar{x}_{n-1})) \circ \Phi$.

**Proof.** By the above argument, we have a semi-orthogonal decomposition

$$D_n^\text{coh} = \left\langle \text{Im}(\Psi_0^\text{coh}), ..., \text{Im}(\Psi_{(-a_{n-2})}^\text{coh}), \bigoplus_{j=0}^{-a_{n-1}+1} \text{Im}(\Phi_j^\text{coh}) \right\rangle,$$

where $\Psi_i^\text{coh}$ is the composition $((-) \otimes \mathcal{O}(\psi^i)) \circ \Psi$ and $\Phi_j^\text{coh}$ is the composition $((-) \otimes \mathcal{O}(\phi^j)) \circ \Phi^\text{coh}$. Since the character $\phi : G_n \rightarrow \mathbb{G}_m$ (respectively, $\phi : G_{n-1} \rightarrow \mathbb{G}_m$) corresponds to the element $\bar{x}_n \in L_n$ (respectively, $\bar{x}_{n-1} \in L_{n-1}$) via the isomorphism (3.6), the result follows from the equivalence $D_n^\text{coh} \cong C_n$ in (3.6).
In this section, we prove Conjecture 1.2 for chain polynomials. We use the same notation as in Subsection 3.4.

### 4.1 Inductive construction of full strong exceptional collections

In this section, we construct a sequence $E^n$ of objects in $C_n$, which turn out to be full strongly exceptional. Set

$$E^0 := (0 \to k \to 0) \in C_0.$$  \hfill (4.A)

We define two objects $\psi E^0 \in C_1$ and $\phi E^0 \in C_2$ by

$$\psi E^0 := \left( S^1(-\vec{x}_1) \xrightarrow{x_1} S^1 \xrightarrow{x_1^a - 1} S^1(\vec{f} - \vec{x}_1) \right)$$

$$\phi E^0 := \left( S^2(-\vec{x}_1) \xrightarrow{x_1} S^2 \xrightarrow{x_1^a - 1 + x_2^a} S^1(\vec{f} - \vec{x}_1) \right),$$

and for each $0 \leq i \leq a_1 - 2$ and $0 \leq j \leq a_1 - 1$, we set

$$\psi_i E^0 := \psi E^0(-i\vec{x}_1)[i] \in C_1$$

$$\phi_j E^0 := \phi E^0(-j\vec{x}_1 + (-a_2 + 1)\vec{x}_2)[a_2 + j - 1] \in C_2.$$  

Similarly, for $n \geq 1$ and an object $F = (F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_1(\vec{f})) \in C_n$, we define objects

$$\psi F \in C_{n+1} \quad \text{and} \quad \phi F \in C_{n+2}$$

as follows: To ease notation, we set

$$x := x_n, \quad y := x_{n+1}, \quad z := x_{n+2}, \quad b := a_{n+1}, \quad \text{and} \quad c := a_{n+2},$$  \hfill (4.B)

and so $f_{n+1} = f_n + xy^b$ and $f_{n+2} = f_n + xy^b + yz^c$. The objects $\psi F$ and $\phi F$ are defined to be the matrix factorizations defined by

$$\psi F := \left( \begin{array}{c}
\varphi_1 \\
-y
\end{array} \right) \xrightarrow{x y^{b-1}} \left( \begin{array}{c}
\varphi_0 \\
-x y^{b-1}
\end{array} \right) \xrightarrow{y} \psi F \oplus \left( \begin{array}{c}
\varphi_1 \mid \\
\varphi_0 \mid
\end{array} \right) \xrightarrow{xy^{b-1}} \phi F \oplus \left( \begin{array}{c}
\varphi_1 \\
-x y^{b-1}
\end{array} \right) \xrightarrow{y} \phi F(\vec{f} - \vec{y}) \oplus \left( \begin{array}{c}
\varphi_0 \\
-x y^{b-1}
\end{array} \right) \xrightarrow{y} \phi F(\vec{f} - \vec{y}) \right)$$
\[ \phi F := \left\{ \begin{array}{c} \hat{F}_1 \oplus (\hat{F}_0(-\hat{\eta})) \\ \left(\hat{\phi}_1 - y\right) \\ -g \\ \hat{\phi}_0 \end{array} \right\}, \]

where \((\hat{\eta}) := (-)^{L_{n-1}} \otimes S^{n+1}, \quad (\hat{\phi}) := (-)^{L_{n+2}} \otimes S^{n+2}\) and \(g := x^by^1 + z^c\). Sometimes we simply write \(\phi_i\) for \(\hat{\phi}_i\) or \(\hat{\phi}_i\). For each \(0 \leq i \leq b - 2\) and each \(0 \leq j \leq b - 1\), we set

\[ \psi_i F := \psi F(-i\hat{\eta}) \quad [i] \in C_{n+1} \] (4.C)

\[ \phi_j F := \phi F(-j\hat{\eta} + (c + 1)\hat{\omega}) \quad [c + j - 1] \in C_{n+2}. \] (4.D)

Then, \(\psi_0 F = \psi F\) and \(\phi_0 F = \phi F((-c + 1)\hat{\omega})\).

For a sequence \(\mathcal{E} = (E_1, ..., E_r)\) of objects in \(C_n\) and for each \(0 \leq i \leq b - 2\), we define a sequence \(\psi_i \mathcal{E}\) of objects in \(C_{n+1}\) by

\[ \psi_i \mathcal{E} := (\psi_i E_1, ..., \psi_i E_r). \]

Similarly, for a sequence \(\mathcal{F} = (F_1, ..., F_s)\) of objects in \(C_n\) and for \(0 \leq j \leq b - 1\), we define a sequence \(\phi_j \mathcal{F}\) of objects in \(C_{n+2}\) by

\[ \phi_j \mathcal{F} := (\phi_j F_1, ..., \phi_j F_s). \]

We inductively define sequences \(\mathcal{E}^n\) of objects in \(C_n\) for any \(n \geq -1\) as follows. First set

\[ \mathcal{E}^{-1} := \emptyset \quad \text{and} \quad \mathcal{E}^0 := \{E^0\}. \]

Then, we define the sequence \(\mathcal{E}^n\) by

\[ \mathcal{E}^n := \left(\psi_0 \mathcal{E}^{n-1}, ..., \psi_{(a_n-2)} \mathcal{E}^{n-1}, \phi_0 \mathcal{E}^{n-2}, ..., \phi_{(a_n-1)} \mathcal{E}^{n-2}\right). \]

The following is our main result in this paper.

**Theorem 4.1.** For any \(n \geq 1\), the sequence \(\mathcal{E}^n\) is a full strong exceptional collection in \(C_n\), and, if \(a_n \geq 2\), the length of \(\mathcal{E}^n\) is equal to the Milnor number of \(\tilde{f}_n\). In particular, \(C_n\) has a tilting object.

In the study of Cohen–Macaulay representation, the existence of a tilting object in the stable category of graded Cohen–Macaulay modules over a graded Gorenstein ring is a fundamental problem [36, Problem 3.4]. By Theorems 2.24 and 4.1, we have the following.

**Corollary 4.2.** The category \(CM_{L_n}(S^n/f_n)\) has a tilting object.
4.2  Explicit descriptions of $\Psi$ and $\Phi$

In this section, we describe the functors $\Psi : C_{n-1} \to C_n$ and $\Phi : C_{n-2} \to C_n$ in Theorem 3.10 explicitly in terms of matrix factorizations. This descriptions enable us to apply certain distinguished triangles in $C_n$ (see Lemma 4.11) to our proof of Theorem 4.1. We use the notation in the above sections, for example, Subsections 3.2 and 3.4.

Lemma 4.3. The fully faithful functor $\Psi^{\text{coh}} : D_{C_{n-1}}^{\text{coh}} \to D_{C_n}^{\text{coh}}$ is isomorphic to the composition

$$D_{C_{n-1}}^{\text{coh}} \xrightarrow{id_{\pi_n}^*} \text{Dcoh}_{G_n}(\mathbb{A}_x^{n-1}, X_{f_{n-1}}) \xrightarrow{j_*} D_{n}^{\text{coh}},$$

where $\pi_n : G_n \to G_{n-1}$ is the surjection in (3.H) and $j : \mathbb{A}_x^{n-1} \hookrightarrow \mathbb{A}_x^n, (x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, 0)$ is the natural closed immersion.

Proof. By construction the functor $\Psi_0$ in (3.N) is given by

$$\Psi_0(F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F(\chi_{f_{n-1}})) = \left( \Theta(F_1) \xrightarrow{\Theta(\varphi_1)} \Theta(F_0) \xrightarrow{\Theta(\varphi_0)} \Theta(F_1)(1 \times \chi_{f_{n-1}}) \right),$$

where $\Theta : \text{coh}_{G_{n-1}} \mathbb{A}_x^{n-1} \rightarrow \text{coh}_{G_m \times G_{n-1}} \mathbb{A}_x^{n-1} \to \text{coh}_{G_m \times G_{n-1}} \mathbb{A}_x^{n-1}$ is the exact functor defined to be the compositions of the following exact functors:

$$\text{coh}_{G_{n-1}} \mathbb{A}_x^{n-1} \xrightarrow{id_{p_2}^*} \text{coh}_{G_m \times G_{n-1}} \mathbb{A}_x^{n-1} \xrightarrow{\pi_-^*} \text{coh}_{G_m \times G_{n-1}} \mathbb{S}_- \xrightarrow{j_-^*} \text{coh}_{G_m \times G_{n-1}} Q \xrightarrow{i_+^*} \text{coh}_{G_m \times G_{n-1}} Q_+.$$

By the flat base change, the functor $\Theta$ is isomorphic to the composition $q^* \circ j_* \circ id_{p_2}^*$, where

$$j_* : \text{coh}_{G_m \times G_{n-1}} \mathbb{A}_x^{n-1} \to \text{coh}_{G_m \times G_{n-1}} \mathbb{A}_x^{n-1} \times \mathbb{A}_{x_n}^1$$

is the direct image by the natural closed immersion $j : \mathbb{A}_x^{n-1} \hookrightarrow \mathbb{A}_x^{n-1} \times \mathbb{A}_{x_n}^1$ and

$$q^* : \text{coh}_{G_m \times G_{n-1}} \mathbb{A}_x^{n-1} \times \mathbb{A}_{x_n}^1 \to \text{coh}_{G_m \times G_{n-1}} Q_+$$

is the pullback by the natural projection $q : Q_+ = \mathbb{A}_x^{n-1} \times \mathbb{A}_{x_n}^1 \times (\mathbb{A}_{u}^1 \setminus \{0\}) \to \mathbb{A}_x^{n-1} \times \mathbb{A}_{x_n}^1$.

On the other hand, recall that the equivalence

$$\Phi_+ : \text{Dcoh}_{G_m \times G_{n-1}}(Q_+, 1 \times \chi_{f_{n-1}}, W_Q) \xrightarrow{\sim} \text{Dcoh}_{G_n}(\mathbb{A}_x^n, X_{f_n}, f_n)$$

in Lemma 3.5 is defined by

\[
\Phi_+ \left( F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_1(1 \times \chi_{f_{n-1}}) \right) = \left( \Sigma(F_1) \xrightarrow{\Sigma(\varphi_1)} \Sigma(F_0) \xrightarrow{\Sigma(\varphi_0)} \Sigma(F_1)(\chi_{f_n}) \right),
\]

where the functor \( \Sigma : \text{coh}_{G_m \times G_{n-1}} \to \text{coh}_{G_n} \mathbb{A}^n_x \) is the exact equivalence defined by the composition

\[
((p_+ \mathbb{a}_n)^{-1})^{-1} : \text{coh}_{G_m \times G_{n-1}} Q_+ \to \text{coh}_{G_m \times G_n} \bar{Q}_+ \to \text{coh}_{G_n} \mathbb{A}^n_x,
\]

where \( \bar{Q}_+ \) and \( p_+ : \bar{Q}_+ \to Q \) are as in the proof of Lemma 3.5, \( \bar{Q}_+ = \mathbb{A}^n_x \) is the embedding given by \( x \mapsto (x, 1) \) and \( \varphi : G_n \to G_m \times G_n \) is the character defined by \( \varphi(g) = (\psi(g), g) \). By [6, Corollary 2.24], the equivalence \( ((p_+ \mathbb{a}_n)^{-1})^{-1} \) is isomorphic to \( (p_+ \mathbb{a}_n)^{-1} \), and thus \( \Sigma \) is isomorphic to \( (p_+ \circ \bar{e})^{-1} \), where \( e : \mathbb{A}^n_x \to Q_+ = \mathbb{A}^n_x \times (\mathbb{A}^1_u \setminus \{0\}) \) is the map \( x \mapsto (x, 1) \) and \( \psi \times \pi : G_n \to G_m \times G_{n-1} \) is given by \( (\psi \times \pi)(\lambda_1, \ldots, \lambda_n) = (\lambda_n, (\lambda_1, \ldots, \lambda_{n-1})) \). Therefore, using [6, Lemma 2.19], we see that \( \Psi_{\text{coh}} \) is induced by the composition

\[
\text{coh}_{G_{n-1}} \mathbb{A}^{n-1}_x \xrightarrow{id_x} \text{coh}_{G_n} \mathbb{A}^{n-1}_x \xrightarrow{I_s} \text{coh}_{G_n} \mathbb{A}^n_x.
\]

This finishes the proof. \( \square \)

**Lemma 4.4.** For \( F = E^0 \in C_0 \) in (4.A) or any \( F \in C_{n-1} \) with \( n \geq 2 \), we have an isomorphism

\[
\Psi(F) \cong \psi F.
\]

**Proof.** We only prove the case when \( F \in C_{n-1} \) with \( n \geq 2 \). By Lemma 4.3, the functor \( \Psi_{\text{mod}} : D_{n-1} \to D_n \) is the composition

\[
D_{n-1} \xrightarrow{(-)_{\overline{f}_{n-1}}} \text{Dmod}_{\overline{f}_{n-1}}(S^{n-1}, f_{n-1}) \xrightarrow{q_s} D_n,
\]

where \( q : S^n \to S^n / (x_n) \) is the quotient map. Thus, it suffices to show that the object \( q_s(F_{\overline{f}_n}) \) is isomorphic to \( \psi F \) in \( D_n \). We define the functor \( \bar{e} : \text{proj}_{f_{n-1}} S^{n-1} \to \text{proj}_{f_n} S^n \) by \( \bar{F} := \text{proj}_{f_{n-1}} \otimes_{\mathbb{Q}_{n-1}} S^n \). Then, we have a short exact sequence

\[
0 \to \bar{F} \xrightarrow{x_n} \bar{F} \xrightarrow{q_p} q_s(F_{\overline{f}_n}) \to 0
\]

in \( \text{mod}_{\overline{f}_n} \), where \( q_p := q \otimes P_{\overline{f}_n} \). This induces a short exact sequence

\[
0 \to E^1 \xrightarrow{\alpha} E^0 \xrightarrow{\beta} q_s(F_{\overline{f}_n}) \to 0
\]

(4.E)
in the abelian category $\text{mod}^{L_n}(S^n, f_n)$ of graded factorizations, where $E^i$ and morphisms $\alpha, \beta$ are defined as follows: $E^0$ is defined by

$$E^0 : \begin{pmatrix}
E^0_1 & E^0_0
\end{pmatrix}
\xrightarrow{egin{pmatrix}
\bar{\varphi}_1 & x_n \\
-x_{n-1}x_n^{a_{n-1}} & \bar{\varphi}_0
\end{pmatrix}}
\xrightarrow{egin{pmatrix}
\bar{\varphi}_0 & -x_n \\
x_{n-1}x_n^{a_{n-1}} & \bar{\varphi}_1
\end{pmatrix}}
E^0_1(f_n^2),$$

where $E^0_1 := \bar{F}_1 \oplus (\bar{F}_0(-\bar{x}_n))$ and $E^0_0 := \bar{F}_0 \oplus (\bar{F}_1(\bar{f}_n - \bar{x}_n))$, and $E^1$ is defined by

$$E^1 : \begin{pmatrix}
E^1_1 & E^1_0
\end{pmatrix}
\xrightarrow{egin{pmatrix}
\bar{\varphi}_1 & 1 \\
-x_{n-1}x_n^{a_{n-1}} & \bar{\varphi}_0
\end{pmatrix}}
\xrightarrow{egin{pmatrix}
\bar{\varphi}_0 & -1 \\
x_{n-1}x_n^{a_{n-1}} & \bar{\varphi}_1
\end{pmatrix}}
E^1_0(f_n^2),$$

where $E^1_1 := \bar{F}_1(-\bar{x}_n) \oplus (\bar{F}_0(-\bar{x}_n))$ and $E^1_0 := \bar{F}_0(-\bar{x}_n) \oplus (\bar{F}_1(\bar{f}_n - \bar{x}_n))$. The morphism $\alpha = (\alpha_1, \alpha_0) : E^1 \to E^0$ is given by

$$\alpha_1 : E^1_1 \to E^0_1 \text{ and } \alpha_0 : E^1_0 \to E^0_0,$$

and the morphism $\beta = (\beta_1, \beta_0) : E^0 \to \Gamma(F)$ is given by

$$\beta_1 : E^0_1 \xrightarrow{(q_{F_1}, 0)} q_*(F_{L_n}^{L_n}) \text{ and } \beta_0 : E^0_0 \xrightarrow{(q_{F_0}, 0)} q_*(F_{L_n}^{L_n}).$$

By [45, Lemma 2.7(a)], the short exact sequence (4.6) induces a triangle

$$E^1 \xrightarrow{\alpha} E^0 \xrightarrow{\beta} q_*(F_{L_n}^{L_n}) \to E^1[1]$$

in $D_n$. If we set

$$h_0 : E^1_0 \xrightarrow{\begin{pmatrix}0 & 0 \\ 1 & 0 \end{pmatrix}} E^1_1 \text{ and } h_1 : \bar{E}^1_0 \xrightarrow{\begin{pmatrix}0 & 0 \\ -1 & 0 \end{pmatrix}} \bar{E}^1_1,$$

then we have $\text{id}_{E^1_0} = \varphi^E_1 h_0 + h_1 \varphi^E_0$ and $\text{id}_{E^1_1} = h_0 \varphi^E_1 + \varphi^E_0(-\bar{f}_n)h_1(-\bar{f}_n)$, which means that the identity map $\text{id} : E^1 \to E^1$ is homotopy equivalent to the zero map. Hence, $E^1$ is isomorphic to the zero object in $D_n$, and thus we have

$$\Psi(F) \xrightarrow{\sim} q_*(F_{L_n}^{L_n}) \xrightarrow{\sim} E^0 = \psi F.$$

This finishes the proof. □

By Proposition 2.23, we have equivalences...
\[ \text{Dcoh}_{G_m \times G_{n-1}}(Q, 1 \times \Gamma_{f_{n-1}}, W_Q) \cong \text{Dmod}^{\mathbb{Z} \oplus L_{n-1}}(R, W_Q), \]  

(4.F)

\[ \text{Dcoh}_{G_m \times G_{n-1}}(Q^+, 1 \times \Gamma_{f_{n-1}}, W_Q) \cong \text{Dmod}^{\mathbb{Z} \oplus L_{n-1}}(R^+, W_Q), \]  

(4.G)

\[ \text{Dcoh}_{G_m \times G_{n-1}}(Q^-, 1 \times \Gamma_{f_{n-1}}, W_Q) \cong \text{Dmod}^{\mathbb{Z} \oplus L_{n-1}}(R^-, W_Q). \]  

(4.H)

For an interval \( I \subset \mathbb{Z} \), denote by \( F_I \) the set of finite direct sums \( \oplus_i F_i \) of \( (\mathbb{Z} \oplus L_{n-1}) \)-graded rank one free \( R \)-modules \( F_i \) with \( F_i \cong R(a, l) \) for some \( a \in I \) and \( l \in L_{n-1} \). Then, the \( I \)-grade-window \( \mathcal{W}_\lambda, I \) with respect to \( \lambda \) in (3.D) corresponds to the following subcategory

\[ \{ F = ( F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_1(0, \vec{f}_{n-1})) | F_i \in F_I \} \subset \text{HMF}^{\mathbb{Z} \oplus L_{n-1}}(W_Q), \]  

(4.I)

which is denoted by the same notation \( \mathcal{W}_\lambda, I \), of \( \text{HMF}^{\mathbb{Z} \oplus L_{n-1}}(W_Q) \) via the natural equivalences.

\[ \mathcal{W}^+_0 = \mathcal{W}_\lambda, [-a_n+1, 0] = \{ F \in \text{HMF}^{\mathbb{Z} \oplus L_{n-1}}(W_Q) | F_i \in F_I[-a_n+1,0] \} \]  

(4.J)

\[ \mathcal{W}_{-a_n-1} = \mathcal{W}_\lambda, [-a_n+1] = \{ F \in \text{HMF}^{\mathbb{Z} \oplus L_{n-1}}(W_Q) | F_i \in F_I[-a_n+1] \}. \]  

(4.K)

**Lemma 4.5.** For \( F = E^0 \in C_0 \) in (4.A) or any \( F \in C_{n-2} \) with \( n \geq 3 \), we have an isomorphism

\[ \Phi(F) \cong \Phi(F)(-a_n + 1)\vec{x}_n. \]

**Proof.** The functor \( \Phi : C_{n-2} \to C_n \) is the composition

\[ C_{n-2} \xrightarrow{(-)^{f_{n-1}}} \text{HMF}_{L_{n-2}}^{L_n-1}(S_{n-2}, f_{n-2}) \]

\[ \xrightarrow{(\Phi^\text{mod})^{-1}} \text{Dmod}^{\mathbb{Z} \oplus L_{n-1}}(R_-, W_Q) \]

\[ \xrightarrow{\sim} \text{HMF}_{R_+}^{\mathbb{Z} \oplus L_{n-1}}(W_Q) \]

\[ \xrightarrow{\Phi^\text{proj}_{(-a_n+1)}} \text{HMF}_{R_+}^{\mathbb{Z} \oplus L_{n-1}}(W_Q) \]

\[ \xrightarrow{\Phi^\text{proj}_+} C_n, \]

where the first functor is induced by the functor \( (-)^{f_{n-1}} : \text{mod}^{L_{n-2}} S_{n-2} \to \text{mod}^{L_{n-2}} S_{n-2} \) (recall notation (2.A)) associated to the inclusion \( L_{n-2} \subset L_{n-1} \) corresponding to the surjection \( \pi_{n-1} : G_{n-1} \to G_{n-2} \), and \( \Phi^\text{mod}_-, \Phi^\text{proj}_{(-a_n+1)} \) and \( \Phi^\text{proj}_+ \) are the functors corresponding to \( \Phi_- \), \( \Phi_{(-a_n+1)} \) and \( \Phi_+ \) in Subsection 3.4, respectively, via natural equivalences.

First, we describe the composition

\[ (\Phi^\text{mod}_-)^{-1} \circ (-)^{f_{n-1}} : C_{n-2} \to \text{Dmod}^{\mathbb{Z} \oplus L_{n-1}}(R_-, W_Q). \]  

(4.L)
The quasi-inverse

\[(\Phi^{-1})^{-1} : \text{Dcoh}_{G_{n-1}}(\mathbb{A}_x^{n-2}, \chi_{f_{n-1}}, f_{n-2}) \sim \text{Dcoh}_{G_m \times G_{n-1}}(Q_{-}, 1 \times \chi_{f_{n-1}}, W_Q)\]

of \(\Phi^{-1}\) in Lemma 3.5 is given by

\[(\Phi^{-1})^{-1}(F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_1(\chi_{f_{n-1}})) = \left(\Delta(F_1) \xrightarrow{\Delta(\varphi_1)} \Delta(F_0) \xrightarrow{\Delta(\varphi_0)} \Delta(F_1)(1 \times \chi_{f_{n-1}})\right),\]

where \(\Delta : \text{coh}_{G_{n-1}}(\mathbb{A}_x^{n-2}) \to \text{coh}_{G_{m} \times G_{n-1}}Q_{-}\) is the following composition:

\[
\begin{align*}
\text{coh}_{G_{n-1}}(\mathbb{A}_x^{n-2}) & \xrightarrow{(p_1)^*} \text{coh}_{G_m \times G_{n-1}}(\mathbb{A}_x^{n-2} \times (\mathbb{A}_1_{x_n} \setminus \{0\}) \\
& \xrightarrow{p_2^*} \text{coh}_{G_m \times G_{n-1}}(\mathbb{A}_x^{n-2} \times (\mathbb{A}_1_{x_n} \setminus \{0\}) \times \mathbb{A}_u^1) \\
& \xrightarrow{k_*} \text{coh}_{G_{m} \times G_{n-1}}Q_{-},
\end{align*}
\]

where \(p_1 : \mathbb{A}_x^{n-2} \times (\mathbb{A}_1_{x_n} \setminus \{0\}) \to \mathbb{A}_x^{n-2}, p_2 : G_m \times G_{n-1} \to G_{n-1}\) and \(p_{1,2} : \mathbb{A}_x^{n-2} \times (\mathbb{A}_1_{x_n} \setminus \{0\}) \times \mathbb{A}_u^1 \to \mathbb{A}_x^{n-2} \times (\mathbb{A}_1_{x_n} \setminus \{0\}) \times \mathbb{A}_u^1 \subset Q_{-}\) is the closed immersion. Thus, the composition (4.L) sends an object \(F = (F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_1(\vec{f}_{n-2})) \in C_{n-2}\) to the object

\[
\Gamma(F) = \left(\Gamma(F_1) \xrightarrow{\Gamma(\varphi_1)} \Gamma(F_0) \xrightarrow{\Gamma(\varphi_0)} \Gamma(F_1)(\vec{W}_Q)\right),
\]

where \(\Gamma(\cdot) := q_* \circ e^* \circ (-)^{Z \oplus L_{n-1}}\) is the composition

\[
\text{proj}^{L_{n-2}}S^{n-2} \xrightarrow{(-)^{Z \oplus L_{n-1}}} \text{proj}^{Z \oplus L_{n-1}}S^{n-2} \\
\xrightarrow{e^*} \text{proj}^{Z \oplus L_{n-1}}S^{n-2}[x_n^{\pm 1}, u] \\
\xrightarrow{q_*} \text{mod}^{Z \oplus L_{n-1}}R_{-}
\]

and \(e : S^{n-2} \subset S^{n-2}[x_n^{\pm 1}, u]\) is the natural inclusion and \(q : R_{-} \to (S^{n-1}/(x_n-1))[x_n^{\pm 1}, u] \sim S^{n-2}[x_n^{\pm 1}, u]\) is the quotient morphism.

Next we replace the object \(\Gamma(F)\) in (4.M) with a matrix factorization. For a graded projective module \(P \in \text{proj}^{L_{n-2}}S^{n-2}\), we write

\[(\ast) : \text{proj}^{L_{n-2}}S^{n-2} \to \text{proj}^{Z \oplus L_{n-1}}R_{-}\]

for the functor defined by

\[
\overline{P} := (P^{Z \oplus L_{n-2}}) \otimes S^{n-2}R_{-} \in \text{proj}^{Z \oplus L_{n-1}}R_{-}.
\]

Then, we have a short exact sequence

\[
0 \to \overline{P}(-d) \xrightarrow{x_{n-1}} \overline{P} \xrightarrow{q_P} \Gamma(P) \to 0
\]
in \( \text{mod}^{\mathbb{Z} \oplus L_{n-1}} R_- \), where \( d := \deg(x_{n-1}) = (0, \tilde{x}_{n-1}) \in \mathbb{Z} \oplus L_{n-1} \) and \( q_P := q \otimes P : \overline{F} \to \Gamma(P) \) is the surjection induced by the quotient map \( q : R_- \to S^{n-2}[x_{n-1}^\pm 1, u] \). This induces a short exact sequence

\[
0 \longrightarrow E^1 \xrightarrow{\alpha} E^0 \xrightarrow{\beta} \Gamma(F) \longrightarrow 0
\]

in the abelian category \( \text{mod}^{\mathbb{Z} \oplus L_{n-1}}(R_-, W_Q) \) of graded factorizations, where \( E^i \) and morphisms \( \alpha, \beta \) are defined as follows: \( E^0 \) is defined by

\[
E^0 := \begin{pmatrix}
\overline{\varphi}_1 & x_{n-1} \\
-g & \overline{\varphi}_0
\end{pmatrix}
\end{pmatrix}
\] \[E^0 \rightarrow E^0(0, \tilde{f}_{n-1})
\]

where \( g := x_{n-2}x_{n-1}^{a_{n-1}} + x_{n}a_{n}^\#u \), \( E^0_1 := \overline{F}_1 \oplus (\overline{F}_0(-d)) \) and \( E^0_0 := \overline{F}_0 \oplus (\overline{F}_1((0, \tilde{f}_{n-1}) - d)) \), and \( E^1 \) is defined by

\[
E^1 := \begin{pmatrix}
\overline{\varphi}_1 & 1 \\
-gx_{n-1} & \overline{\varphi}_0
\end{pmatrix}
\end{pmatrix}
\] \[E^1 \rightarrow E^1(0, \tilde{f}_{n-1})
\]

where \( E^1_1 := \overline{F}_1(-d) \oplus (\overline{F}_0(-d)) \) and \( E^1_0 := \overline{F}_0(-d) \oplus (\overline{F}_1((0, \tilde{f}_{n-1}) - d)) \). The morphism \( \alpha = (\alpha_1, \alpha_0) : E^1 \to E^0 \) is given by

\[
\alpha_1 : E^1_1 \xrightarrow{(x_{n-1}, 0)} E^0_1 \quad \text{and} \quad \alpha_0 : E^1_0 \xrightarrow{(0, 1)} E^0_0,
\]

and the morphism \( \beta = (\beta_1, \beta_0) : E^0 \to \Gamma(F) \) is given by

\[
\beta_1 : E^0 \xrightarrow{q_{F_1}} \Gamma(F_1) \quad \text{and} \quad \beta_0 : E^0 \xrightarrow{q_{F_0}} \Gamma(F_0).
\]

By the same argument as in the proof of Lemma 4.4, we see that there is an isomorphism

\[
\Gamma(F) \xrightarrow{\sim} E^0 \in \text{HMf}_{R_-}^{\mathbb{Z} \oplus L_{n-1}}(W_Q).
\]

Finally, we compute the image of \( E^0 \in \text{HMf}_{R_-}^{\mathbb{Z} \oplus L_{n-1}}(W_Q) \) by the composition

\[
\text{HMf}_{R_-}^{\mathbb{Z} \oplus L_{n-1}}(W_Q) \xrightarrow{\Phi^{proj}_{(-a_{n+1})}} \text{HMf}_{R_c}^{\mathbb{Z} \oplus L_{n-1}}(W_Q) \xrightarrow{\Phi^{proj}_c} C_n.
\]
The first functor $\Phi_{\text{proj}}^{(-a_n+1)}$ is the composition

$$
\text{HMF}^{\mathbb{Z} \oplus L_{n-1}}_R(W_Q) \xrightarrow{(i_+)^{-1}} \mathcal{W}_{a_n-1}^- \subset \mathcal{W}_0^{i_+} \xrightarrow{i_+^*} \text{HMF}^{\mathbb{Z} \oplus L_{n-1}}_R(W_Q),
$$

where $i_- : R \hookrightarrow R_-$ and $i_+ : R \hookrightarrow R_+$ are natural inclusions. Write

$$(\simeq) : \text{proj}_{S^{n-2}} S^{n-2} \to \text{proj} \mathbb{Z} \oplus L_{n-1} R$$

for the functor given by $\tilde{P} := P^{\mathbb{Z} \oplus L_{n-1}} \otimes_{S^{n-2}} R$, where $R$ is the $S^{n-2}$-algebra via the natural inclusion $S^{n-2} \hookrightarrow R$. We define an object $\gamma F \in \text{HMF}^{\mathbb{Z} \oplus L_{n-1}}_R(W_Q)$ by

$$
\gamma F := \begin{pmatrix}
\varphi_1 \\
x_n-1 \\
\varphi_0 \\
-\tilde{g} \\
\varphi_1 \\
\varphi_0 \\
\tilde{g} \\
\varphi_1 \\
\gamma F_1 \\
\gamma F_0 \\
\gamma F_1(0, \vec{f}_{n-1})
\end{pmatrix},
$$

where $g = x_n-2x_n^{a_{n-1}} + x_n^{a_n} u$, and $\gamma F_1 := \tilde{F}_1 \oplus (\tilde{F}_0(-d))$ and $\gamma F_0 := \tilde{F}_0 \oplus (\tilde{F}_1((0, \vec{f}_{n-1})-d))$. Then, $\gamma F \in \mathcal{W}_{\lambda,\{0\}}$, where $\{0\} := \{0,0\}$ and $\mathcal{W}_{\lambda,\{0\}}$ is the window subcategory defined in (4.1). Since $\deg(x_n) = (1,0)$, by the equality (4.4.1)

$$
\gamma F((-a_n+1)\vec{x}_n) \in \mathcal{W}^-_{a_n-1},
$$

and $i_-^*(\gamma F((-a_n+1)\vec{x}_n)) \cong E^0((-a_n+1)\vec{x}_n))$. Since $x_n \in R_-$ is a unit, we have an isomorphism $\chi_{a_n^2} : E^0((-a_n+1)\vec{x}_n) \cong E^0$. Thus, the image of $E^0$ by the equivalence $\text{HMF}^{\mathbb{Z} \oplus L_{n-1}}_R(W_Q) \xrightarrow{(i_+)^{-1}} \mathcal{W}_{a_n-1}^-$ is isomorphic to the object $\gamma F((-a_n+1)\vec{x}_n)$, and so we have

$$
\Phi(F) \cong (\Phi_{\text{proj}}^+ \circ i_+^*)(\gamma F((-a_n+1)\vec{x}_n)).
$$

As we saw in the proof of Lemma 4.3, the functor

$$
\Phi^+ : \text{Dcoh}_{G_n \times G_{n-1}}(Q_+, 1 \times \chi_{f_{n-1}}, W_Q) \xrightarrow{\sim} \text{Dcoh}_{G_n}(\mathbb{A}_x^n, \chi_{f_n}, \vec{f}_n)
$$

is given by $e^*_{\psi \times \pi} : \text{coh}_{G_m \times G_{n-1}} Q_+ \to \text{coh}_{G_n} \mathbb{A}_x^n$, where we use the same notation as in the proof of Lemma 4.3. Hence, the corresponding functor

$$
\Phi^\text{proj}_+ : \text{HMF}^{\mathbb{Z} \oplus L_{n-1}}_R(W_Q) \xrightarrow{\sim} C_n
$$

is given by $r^*_{\sigma} : \text{proj}^{\mathbb{Z} \oplus L_{n-1}}_R \to \text{proj}_L^n S^n$, where $r : R_+ \to S^n$ is the ring homomorphism given by substituting 1 to $u$, and $\sigma : \mathbb{Z} \oplus L_{n-1} \to L_n$ is the group homomorphism defined by $\sigma(a, \sum_{i=1}^{n-1} b_i \vec{x}_i) = \sum_{i=1}^{n-1} b_i \vec{x}_i + a \vec{x}_n$. It is obvious that we have

$$
(r^*_{\sigma} \circ i_+^*)(\gamma F((-a_n+1)\vec{x}_n)) \cong \Phi(F((-a_n+1)\vec{x}_n)),
$$

and therefore we have an isomorphism $\Phi(F) \cong \Phi(F((-a_n+1)\vec{x}_n))$. \qed

The following remark will be necessary in Subsection 4.4.
Remark 4.6. The assignment $E \mapsto \psi E$ for $E \in C_{n-1}$ defines an exact functor

$$\psi : C_{n-1} \to C_n,$$

where, for a morphism $\alpha = (\alpha_1, \alpha_0) : E \to F$ in $C_{n-1}$, the morphism

$$\psi \alpha = ((\psi \alpha)_1, (\psi \alpha)_0) : \psi E \to \psi F$$

is defined by

$$
\begin{align*}
(\psi \alpha)_1 : (\psi E)_1 &= E_1 \oplus E_0(-\vec{y}) \\
&\xrightarrow{\begin{pmatrix} \bar{\alpha}_1 & 0 \\ 0 & \bar{\alpha}_0 \end{pmatrix}}
F_1 \oplus F_0(-\vec{y}) = (\psi F)_1 \\
(\psi \alpha)_0 : (\psi E)_0 &= E_0 \oplus E_1(\vec{f} - \vec{y}) \\
&\xrightarrow{\begin{pmatrix} \bar{\alpha}_0 & 0 \\ 0 & \bar{\alpha}_1 \end{pmatrix}}
F_0 \oplus F_1(\vec{f} - \vec{y}) = (\psi F)_0.
\end{align*}
$$

Note that the following diagram commutes:

\[
\begin{array}{ccc}
\psi E & \xrightarrow{\sim} q_*(E^{L_n}) & \xrightarrow{\psi^\text{mod}(E)} \\
| & \downarrow \psi \alpha & \downarrow q_*(\alpha^{L_n}) \downarrow \psi^\text{mod}(\alpha) \\
\psi F & \xrightarrow{\sim} q_*(F^{L_n}) & \xrightarrow{\psi^\text{mod}(F)}
\end{array}
\]

where the horizontal isomorphisms are given by the morphism $\beta$ in (4.E). This means that the functor $\Psi : C_{n-1} \to C_n$ is isomorphic to the functor $\psi : C_{n-1} \to C_n$.

Similarly, the assignment $E \mapsto \phi E$ defines a functor

$$\phi : C_{n-2} \to C_n$$

such that the composition $\left((-)(-a_n + 1)\tilde{x}_n\right) \circ \phi$ is isomorphic to $\Phi : C_{n-2} \to C_n$. For a morphism $\alpha = (\alpha_1, \alpha_0) : E \to F$ in $C_{n-2}$, the morphism

$$\phi \alpha : \phi E \to \phi F$$

is defined by

$$
\begin{align*}
(\phi \alpha)_1 : (\phi E)_1 &= \tilde{E}_1 \oplus \tilde{E}_0(-\vec{y}) \\
&\xrightarrow{\begin{pmatrix} \bar{\alpha}_1 & 0 \\ 0 & \bar{\alpha}_0 \end{pmatrix}}
\tilde{F}_1 \oplus \tilde{F}_0(-\vec{y}) = (\phi F)_1 \\
(\phi \alpha)_0 : (\phi E)_0 &= \tilde{E}_0 \oplus \tilde{E}_1(\vec{f} - \vec{y}) \\
&\xrightarrow{\begin{pmatrix} \bar{\alpha}_0 & 0 \\ 0 & \bar{\alpha}_1 \end{pmatrix}}
\tilde{F}_0 \oplus \tilde{F}_1(\vec{f} - \vec{y}) = (\phi F)_0.
\end{align*}
$$
4.3 Proof of Theorem 4.1

In this subsection, we prove Theorem 4.1 by induction on $n$. If $a_n = 1$, by Theorem 3.10 the category $C^n$ is orthogonally decomposed into some copies of $C^{n-2}$, and so nothing to prove. Thus, we may assume $a_n \geq 2$. First, we compute the Milnor number $\mu_n := \mu(\tilde{f}_n)$ of $\tilde{f}_n$. Set

$$b_i := \prod_{j=0}^{n-i} a_{n-j} = a_n a_{n-1} \cdots a_i.$$ 

**Lemma 4.7.** Assume $a_n \geq 2$. We have

$$\mu_n = b_1 - b_2 + \cdots + (-1)^{n-1} b_n + (-1)^n.$$ 

**Proof.** Since $\tilde{f}_n$ is a quasi-homogeneous polynomial with an isolated singularity only at the origin, the Milnor number $\mu_n$ is equal to $\prod_{i=1}^n \left( \frac{1}{q_i} - 1 \right)$, where $q_i$ is the rational weight of $x_i$ such that the degree of $\tilde{f}_n$ with respect to $\{q_i\}$ is equal to 1. Set $c_{n+1} := 1$, and for each $1 \leq i \leq n$ set

$$c_i := b_i - b_{i+1} + \cdots + (-1)^{n-i} b_n + (-1)^{n-i+1}.$$ 

Since $q_i = c_{i+1}/b_i$, we have

$$\frac{1}{q_i} - 1 = \frac{c_i}{c_{i+1}}.$$ 

Thus,

$$\mu_n = \prod_{i=1}^n \left( \frac{1}{q_i} - 1 \right) = \prod_{i=1}^n \frac{c_i}{c_{i+1}} = c_1 = b_1 - b_2 + \cdots + (-1)^{n-1} b_n + (-1)^n. \quad \square$$

**Proposition 4.8.** The sequence $E^n$ is a full exceptional collection in $C_n$. If $a_n \geq 2$, the length $\nu_n$ of $E^n$ is equal to the Milnor number $\mu_n$ of $\tilde{f}_n$, where we set $\mu_0 := 1$.

**Proof.** Since the equivalence $C_0 = \text{HMFP}^c_k(0) \cong \text{D}^b(\text{mod } k)$ by Corollary 2.20 maps the object $E^0 \in C_0$ to the free module $k \in \text{D}^b(\text{mod } k)$, $E^0$ is a full strong exceptional collection of length 1. Since we have $C_1 = \langle \text{Im } \Psi_0, \ldots, \text{Im } \Psi_{a_1-2} \rangle$, $E_1$ is a full exceptional collection of length $\nu_1 = a_1 - 1$. By induction on $n$, the former statement follows from Theorem 3.10 and Lemmas 4.4 and 4.5.

By Theorem 3.10, we have $\nu_n = (a_n - 1)\nu_{n-1} + a_{n-1} \nu_{n-2}$. Since $\nu_0 = 1$ and $\nu_1 = a_1 - 1$, this recursion shows the equality

$$\nu_n = b_1 - b_2 + \cdots + (-1)^{n-1} b_n + (-1)^n.$$ 

Thus, the latter assertion follows from Lemma 4.7. \quad \square

Atsushi Takahashi pointed out that the semi-orthogonal decomposition in Theorem 3.10 may be related to the result of Gabrielov [25] via mirror symmetry.
Remark 4.9. From the semi-orthogonal decompositions in Theorem 3.10, we have a recursion
\[ \nu_n = (a_n - 1)\nu_{n-1} + a_{n-1}\nu_{n-2}. \]
The equivalent recursion \( \mu_n = (a_n - 1)\mu_{n-1} + a_{n-1}\mu_{n-2} \) can be deduced from the geometry of the singularity \((\tilde{f}_n, 0)\) \cite[Theorem 1]{25}.

The following lemma implies that \( \mathcal{E}^1 \) is a full strong exceptional collection.

**Lemma 4.10.** Let \( E \) and \( F \) be objects in \( C_{n-1} \). For \( 0 \leq i < j \leq a_n - 2 \), we have the following
\[
\text{Hom}(\psi_i E, \psi_j F[l]) \cong \begin{cases} 
\text{Hom}(E, F[l]) & j = i + 1 \\
0 & \text{otherwise.} 
\end{cases}
\]

In particular, if \( \mathcal{E}^{n-1} \) is strongly exceptional, the sequence
\[
(\psi_0 \mathcal{E}, \ldots, \psi_{b-2} \mathcal{E})
\]
is also strongly exceptional.

**Proof.** Applying the Serre functor and using the isomorphism \((\psi_j F)(\tilde{x}_n) \cong (\psi_{j-1} F)[1]\), we obtain isomorphisms of vector spaces
\[
\text{Hom}(\psi_i E, \psi_j F[l]) \cong \text{Hom}(\psi_j F[l], (\psi_i E)(-\tilde{x}^n)[n]) \\
\cong \text{Hom}(\psi_{j-1} F[l + 1], \psi_i(E(-\tilde{x}^{n-1}))[n]).
\]

First assume that \( j > i + 1 \). The object \( \psi_{j-1} F[l + 1] \) lies in \( \text{Im} \psi_{-j+1} \), and the object \( \psi_i(E(-\tilde{x}^{n-1}))[n] \) lies in \( \text{Im} \psi_{-i} \). Since \( -j + 1 < -i \), by the semi-orthogonal decomposition in Theorem 3.10, we have
\[
\text{Hom}(\psi_i E, \psi_j F[l]) \cong \text{Hom}(\psi_{j-1} F[l + 1], \psi_i(E(-\tilde{x}^{n-1}))[n]) = 0.
\]

Next assume that \( j = i + 1 \). Then, we have isomorphisms
\[
\text{Hom}(\psi_i E, \psi_j F[l]) \\
\cong \text{Hom}(\psi_j F[l + 1], \psi_i(E(-\tilde{x}^{n-1}))[n]) \\
\cong \text{Hom}(F[l + 1], E(-\tilde{x}^{n-1})[n]) \\
\cong \text{Hom}(E(-\tilde{x}^{n-1})[n], F(-\tilde{x}^{n-1})[l + n]) \\
\cong \text{Hom}(E, F[l]),
\]
where the second isomorphism follows from the fully faithfulness of the functor \( \psi_i \), and the third one follows from the Serre duality in \( C_{n-1} \). \( \square \)

Next we compute the set of morphisms \( \text{Hom}_{C_n}(\psi_i \mathcal{E}^{n-1}, \phi_j \mathcal{E}^{n-2}[l]) \) for any \( l \in \mathbb{Z} \). For this, we need the following lemmas.
Lemma 4.11. For each object \( F \in C_n \), we have the following triangle in \( C_{n+2} \):

\[
\phi F(-\bar{z}) \to \phi F \to \psi^2 F \to \phi F(-\bar{z})[1].
\]

Proof. We use the notation (4.B) and set \( g := xy^{b-1} + z^c \). By construction, the object

\[
\psi^2 F = \begin{pmatrix}
\varphi_{\psi^2 F} & y & z & 0 \\
-x y^{b-1} & \varphi_0 & 0 & z \\
- y z^{c-1} & 0 & \varphi_0 & -y \\
0 & - y z^{c-1} & x y^{b-1} & \varphi_1
\end{pmatrix}
\]

is given by

\[
\begin{align*}
(\psi^2 F)_1 & := \widehat{F}_1 \oplus (\widehat{F}_0(-\bar{y})) \oplus \left( \widehat{F}_0(-\bar{z}) \right) \oplus \left( \widehat{F}_1(\vec{f} - \bar{y} - \bar{z}) \right) \\
(\psi^2 F)_0 & := \widehat{F}_0 \oplus \left( \widehat{F}_1(\vec{f} - \bar{y}) \right) \oplus \left( \widehat{F}(\vec{f} - \bar{z}) \right) \oplus \left( \widehat{F}_0(\vec{f} - \bar{y} - \bar{z}) \right)
\end{align*}
\]

By applying elementary row and column operations, we see that \( \psi^2 F \) is isomorphic to the following matrix factorization

\[
\begin{pmatrix}
\varphi_1 & y & z & 0 \\
-x y^{b-1} & \varphi_0 & 0 & z \\
- y z^{c-1} & 0 & \varphi_0 & -y \\
0 & - y z^{c-1} & x y^{b-1} & \varphi_1
\end{pmatrix} \xrightarrow{\varphi_{\psi^2 F}} \begin{pmatrix}
\varphi_0 & -y & -z & 0 \\
x y^{b-1} & \varphi_1 & 0 & -z \\
y z^{c-1} & 0 & \varphi_1 & y \\
0 & y z^{c-1} & -x y^{b-1} & \varphi_0
\end{pmatrix}.
\]

which is nothing but the mapping cone of the multiplication \( z : \phi F(-\bar{z}) \to \phi F \) by \( z \). Indeed, the following matrices

\[
\begin{pmatrix}
\varphi_1 & y & z & 0 \\
-x y^{b-1} & \varphi_0 & 0 & z \\
- y z^{c-1} & 0 & \varphi_0 & -y \\
0 & - y z^{c-1} & x y^{b-1} & \varphi_1
\end{pmatrix} \xrightarrow{\alpha_1} \begin{pmatrix}
\varphi_0 & -y & -z & 0 \\
x y^{b-1} & \varphi_1 & 0 & -z \\
y z^{c-1} & 0 & \varphi_1 & y \\
0 & y z^{c-1} & -x y^{b-1} & \varphi_0
\end{pmatrix} \xrightarrow{\alpha_0} \begin{pmatrix}
\varphi_0 & -y & -z & 0 \\
x y^{b-1} & \varphi_1 & 0 & -z \\
y z^{c-1} & 0 & \varphi_1 & y \\
0 & y z^{c-1} & -x y^{b-1} & \varphi_0
\end{pmatrix}
\]

defines an isomorphism \( \alpha = (\alpha_1, \alpha_0) : \psi^2 F \to Cone(z : \phi F(-\bar{z}) \to \phi F) \).

\(\square\)

Lemma 4.12. For \( E \in C_{n-1} \) and \( 0 \leq j \leq a_{n-1} - 1 \), set

\[
\bar{E}_j := (\psi E)(j \bar{x}_{n-1} + (-a_n + 2)\bar{x}_n)[a_n - j - 2] \in C_n.
\]

(4.N)
For an object $F \in C_{n-2}$ and any $l \in \mathbb{Z}$, we have an isomorphism

$$
\text{Hom}(\tilde{E}_j, \phi_0 F[l]) \cong \text{Hom}(\tilde{E}_j, (\psi^2 F)((-a_n + 1)\vec{x}_n)[a_n - 1 + l]).
$$

**Proof.** By Lemma 4.11 and $\phi_0 F = (\phi F)((-a_n + 1)\vec{x}_n)[a_n - 1]$, we have following triangle

$$
(\phi_0 F)(-\vec{x}_n) \to \phi_0 F \to (\psi_2 F)((-a_n + 1)\vec{x}_n)[a_n - 1] \to (\phi_0 F)(-\vec{x}_n)[1].
$$

By the long exact sequence obtained by applying $\text{Hom}(\tilde{E}_j, -)$ to the above triangle, it suffices to show the vanishing

$$
\text{Hom}(\tilde{E}_j, \phi_0 F(-\vec{x}_n)[l]) = 0 \quad (4.0)
$$

for any $l \in \mathbb{Z}$. By the Serre duality, we have isomorphisms

$$
\text{Hom}(\tilde{E}_j, \phi_0 F(-\vec{x}_n)[l]) \cong \text{Hom}(\phi_0 F(-\vec{x}_n)[l], \tilde{E}_j(-\vec{x}_n)[n]) \cong \text{Hom}(\phi_0 F[l], \tilde{E}_j(-\vec{x}_n-1)[n]).
$$

Since $\tilde{E}_j(-\vec{x}_n-1) = \psi(E(-\vec{x}_n^{-2} + (j - 1)\vec{x}_{n-1}))(-a_n + j - 2)[a_n - j - 2] \in \text{Im} \Psi_{-a_n+2}$ and $\phi_0 F \in \text{Im} \Phi_0$, the semi-orthogonality in Theorem 3.10 implies the vanishing (4.0). \qed

Now we can compute the set $\text{Hom}(\psi_i E, \phi_j F[l])$.

**Lemma 4.13.** Let $E \in C_{n-1}$ and $F \in C_{n-2}$ be objects. For $0 \leq i \leq a_n - 2$ and $0 \leq j \leq a_n - 1$, we have the following

$$
\text{Hom}(\psi_i E, \phi_j F[l]) \cong \begin{cases} 
\text{Hom}(E, \psi_j F[l]) & i = a_n - 2 \\
0 & \text{otherwise}
\end{cases}
$$

**Proof.** First we show the vanishing $\text{Hom}(\psi_i E, \phi_j F[l]) = 0$ when $i < a_n - 2$. Applying the Serre functor and using isomorphism $\psi_i E(-\vec{x}_n)[1] \cong \psi_{i+1} E$, we have

$$
\text{Hom}(\psi_i E, \phi_j F[l]) \cong \text{Hom}(\phi_j F[l], (\psi_i E)(-\vec{x}_n)[n]) \cong \text{Hom}(\phi_j F[l], \psi_{i+1}(E(-\vec{x}_n-1))[n - 1]).
$$

By the semi-orthogonal decomposition in Theorem 3.10, the object $\phi_j F[l]$ is left orthogonal to the full subcategory $\text{Im} \Psi_{i+1}$, and thus $\text{Hom}(\phi_j F[l], \psi_{i+1}(E(-\vec{x}_n-1))[n - 1]) = 0$.

Now it suffices to show an isomorphism

$$
\text{Hom}(\psi_{a_n-2} E, \phi_j F[l]) \cong \text{Hom}(E, \psi_j F[l]).
$$

Using notation (4.N) and Lemma 4.12, we have isomorphisms

$$
\text{Hom}(\psi_{a_n-2} E, \phi_j F[l]) = \text{Hom}(\psi E((-a_n + 2)\vec{x}_n)[a_n - 2], (\phi_0 F)(-j\vec{x}_{n-1})[l + j]) \cong \text{Hom}(\tilde{E}_j, \phi_0 F[l])
$$
\[ \cong \text{Hom}(\tilde{E}_j, (\psi^2 F)((-a_n +1)\bar{x}_n)[a_n -1 + l]) \]
\[ \cong \text{Hom}((\psi E)(j\bar{x}_{n-1} + \bar{x}_n), (\psi^2 F)[l + j + 1]). \]  
(4.P)

Applying the Serre duality in \( C_n \) and the fully faithfulness of \( \Psi \), we have isomorphisms

\[ (4.P) \cong \text{Hom}((\psi^2 F)[l + j + 1], (\psi E)(j\bar{x}_{n-1} + \bar{x}_n - \bar{x}_n)[n]) \]
\[ \cong \text{Hom}((\psi^2 F)[l + j + 1], \psi(E(j\bar{x}_n - \bar{x}_n))[n]) \]
\[ \cong \text{Hom}(\psi F[l + j], E(j\bar{x}_{n-1} - \bar{x}_n)[n - 1]). \]  
(4.Q)

Again by the Serre duality in \( C_{n-1} \), we have the isomorphisms

\[ (4.Q) \cong \text{Hom}(E(j\bar{x}_{n-1} - \bar{x}_n)[n - 1], (\psi F)(-\bar{x}_n)[l + j + n -1]) \]
\[ \cong \text{Hom}(E, \psi F(-j\bar{x}_{n-1})[j + l]) \]
\[ \cong \text{Hom}(E, \psi_j F[l]). \]

This completes the proof. \( \square \)

Now for Theorem 4.1, it suffices to prove the following.

**Corollary 4.14.** If \( E^{n-1} \) and \( E^{n-2} \) are full strong exceptional collections, then \( E^n \) is also a full strong exceptional collection.

**Proof.** Write \( E^n = \{ E_1, \ldots, E_{\mu} \} (\mu := \mu(f_n)). \) By Proposition 4.8, it suffices to show that for any two objects \( E_s, E_{s'} \in E^n \) with \( s < s' \) and any non-zero integer \( l \neq 0 \), the vanishing

\[ \text{Hom}(E_s, E_{s'}[l]) = 0 \]

holds. Since the sequence \( E^{n-1} \) is strongly exceptional, Lemma 4.10 implies that the above vanishing holds if \( E_s \in \psi_i E^{n-1} \) and \( E_{s'} \in \psi_i' E^{n-1} \) for some \( 0 \leq i, i' \leq a_n - 2 \). Moreover, the above vanishing also holds when \( E_s \in \phi_j E^{n-2} \) and \( E_{s'} \in \phi_j' E^{n-2} \) for some \( 0 \leq j, j' \leq a_{n-1} - 1 \), since if \( j \neq j' \) then any elements \( \phi_j E^{n-2} \) is orthogonal to any elements in \( \phi_j' E^{n-2} \), and if \( j = j' \) the vanishing follows from the assumption that the sequence \( E^{n-2} \) is strongly exceptional. Therefore, it is enough to show the vanishing

\[ \text{Hom}(\psi_i E, \phi_j F[l]) = 0 \]

for any \( E \in E^{n-1}, F \in E^{n-1}, 0 \leq i \leq a_n - 2, 0 \leq j \leq a_{n-1} - 1 \) and \( l \neq 0 \). By Lemma 4.13, this vanishing holds for any \( l \in \mathbb{Z} \) and \( 0 \leq i < a_n - 2 \), and if \( i = a_n - 2 \), we have an isomorphism

\[ \text{Hom}(\psi_{a_n-2} E, \phi_j F[l]) \cong \text{Hom}(E, \psi_j F[l]). \]  
(4.R)

Since both of \( E \) and \( \psi_j F \) lie in \( E^{n-1} \) and the sequence \( E^{n-1} \) is strongly exceptional, the right-hand side of (4.R) vanishes for any \( l \neq 0 \). \( \square \)
The quiver with relations associated to the tilting object in $C_n$

Using the results in the previous subsection, we compute the quiver with relation $(Q^n, I^n)$ which represents the endomorphism ring $\text{End}_{C_n}(T_n)$ of the associated tilting object

$$T_n := \bigoplus_{E_i \in \mathcal{E}^n} E_i \in C_n. \quad (4.8)$$

4.4.1 Low-dimensional cases

Since $\mathcal{E}^0$ consists of a unique exceptional object

$$E^0 = (0 \to k \to 0) \in C_0,$$

the quiver $Q^0$ has a unique vertex, no arrow and no relations. Pictorially, $Q^0$ is

$$E^0 \bullet$$

Recall that $\mathcal{E}^1 = (\psi_0 E^0, \ldots, \psi_{(a_1 - 2)} E^0)$. By Lemma 4.10, there is a non-zero morphism from $\psi_i E$ to $\psi_j E$ if and only if $j = i + 1$, and in particular the non-zero morphisms are automatically irreducible. Since $\text{Hom}(\psi_i E, \psi_{i+1} E) = 1$, the corresponding quiver $Q^1$ is

$$\psi_0 E^0 \bullet \xrightarrow{\alpha_1} \psi_1 E^0 \bullet \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{a_1 - 2}} \psi_{a_1 - 2} E^0 \bullet$$

and the relations are

$$I^1 = \{ \alpha_{i+1} \circ \alpha_i = 0 \mid 1 \leq i \leq a_1 - 3 \}.$$

4.4.2 Lemmas for higher dimensional cases

Recall from Remark 4.6, the assignments $E \mapsto \psi E$ and $E \mapsto \phi E$ define functors $\psi : C_{n-1} \to C_n$ and $\phi : C_{n-2} \to C_n$, respectively. Following notation (4.C), for each $0 \leq i \leq a_n - 2$ and $0 \leq j \leq a_{n-1} - 1$, we define functors

$$\psi_i : C_{n-1} \to C_n$$

$$\phi_j : C_{n-2} \to C_n$$

by $\psi_i := (\bullet(-i\bar{x}_{a_n})[i]) \circ \psi$ and $\phi_j := ((-j\bar{x}_{a_n-1} + (-a_n + 1)\bar{x}_{a_n})[a_n + j - 1]) \circ \phi$.

To compute the quiver with relations $(Q^n, I^n)$ in the case when $n \geq 2$, we need the following lemmas.
Lemma 4.15. For each non-zero object \( E \in C_{n-1} \) and any \( 0 \leq i \leq a_n - 3 \), there is a non-zero map

\[
\lambda^E_i : \psi_i E \to \psi_{i+1} E
\]

such that the following conditions hold.

1. For any morphism \( \alpha = (\alpha_1, \alpha_0) : E \to E' \) between non-zero objects in \( C_{n-1} \), the following diagram commutes:

\[
\begin{array}{c}
\psi_i E \downarrow \quad \alpha \downarrow \quad \psi_i E' \\
\psi_{i+1} E \downarrow \quad \psi_{i+1} \alpha \downarrow \quad \psi_{i+1} E'
\end{array}
\]

In particular, the morphisms \( \{ \lambda^E_i \} \) define a functor morphism \( \lambda_i : \psi_i \to \psi_{i+1} \).

2. The composition \( \lambda^{E'}_i \circ \psi_i \alpha \) is the zero map if and only if \( \alpha \) is the zero map.

Proof. We only prove the result in the case when \( i = 0 \), since \( \lambda^E_i \) can be given by \( \lambda^E_0 = \lambda^E_{0}(-i\bar{x}_n)[i] \).

Write \( E = (E_1 \xrightarrow{\varphi_1} E_0 \xrightarrow{\varphi_0} E_1(\vec{f})) \in C_{n-1} \), and consider the following morphisms

\[
\begin{pmatrix}
0 & 1 \\
\chi y^{b-2} & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & -1 \\
-x\chi y^{b-2} & 0
\end{pmatrix}
\]

where we use the similar notation as in (4.B), namely \( x := x_{n-1} \), \( y := x_n \) and \( b := a_n \). These morphisms define a morphism

\[
\lambda^E_0 := (\lambda^E_1, \lambda^E_0) : \psi_1 E \to \psi_1 E.
\]

Since the object \( E \) is a non-zero object in \( C_{n-1} \), we may assume that the matrices \( \varphi_1 \) and \( \varphi_0 \) have no unit entry. On the other hand, the matrices \( (\lambda^E_1) \) and \( (\lambda^E_0) \) have unit entries. Thus, the morphism \( \lambda^E_0 \) cannot be homotopy equivalent to the zero map.

It suffices to prove that the non-zero morphism \( \lambda^E_0 \) satisfies the conditions (1) and (2). The condition (1) can be checked by direct computations, and so we omit the details. We check the condition (2). It is obvious that \( \alpha = 0 \) implies \( \lambda^{E'}_0 \circ \psi_1 \alpha = 0 \). Assume that \( \lambda^{E'}_0 \circ \psi_1 \alpha = 0 \). We set \( \beta := \lambda^{E'}_0 \circ \psi_1 \alpha \) and write \( E' = (E'_1 \xrightarrow{\varphi'_1} E'_0 \xrightarrow{\varphi'_0} E'_1(\vec{f})) \). Then, \( \beta \) is given by the following morphisms:

\[
\begin{pmatrix}
0 & \alpha_0 \\
\chi y^{b-2} & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & \alpha_0 \\
\chi y^{b-2} & 0
\end{pmatrix}
\]
\[
\beta_0 : (\psi E)_0 = \bar{E}_0 \oplus \bar{E}_1(\vec{f} - \vec{y}) \xrightarrow{\left( \begin{array}{cc} 0 & -\alpha_1 \\ -xy^{b-2} & 0 \end{array} \right)} \bar{E}_1'(\vec{f} - \vec{y}) \oplus \bar{E}_0'(\vec{f} - 2\vec{y}) = (\psi_1 E')_0,
\]

where we simply write \( \alpha_i : \bar{E}_i \to \bar{E}_i' \) for \( \bar{\alpha}_i \). Since \( \beta = 0 \), there are morphisms

\[
g : (\psi E)_0 = \bar{E}_0 \oplus \bar{E}_1(\vec{f} - \vec{y}) \xrightarrow{\left( \begin{array}{cc} g_1 & g_2 \\ g_3 & g_4 \end{array} \right)} \bar{E}_1'(-\vec{y}) \oplus \bar{E}_0'(\vec{f} - 2\vec{y}) = (\psi_1 E')_1
\]

\[
h : (\psi E)_1(\vec{f}) = \bar{E}_1(\vec{f}) \oplus \bar{E}_0(\vec{f} - \vec{y}) \xrightarrow{\left( \begin{array}{cc} h_1 & h_2 \\ h_3 & h_4 \end{array} \right)} \bar{E}_1'(\vec{f} - \vec{y}) \oplus \bar{E}_0'(\vec{f} - 2\vec{y}) = (\psi_1 E')_0
\]
such that

\[
\beta_1 = g \circ \varphi_{\psi E} + (\varphi_{1(E')}'(\vec{f})) \circ (h(-\vec{f}))
= \left( \begin{array}{cc} g_1 & g_2 \\ g_3 & g_4 \end{array} \right) \left( \begin{array}{cc} \varphi_1 & y \\ -xy^{b-1} & \varphi_0 \end{array} \right) + \left( \begin{array}{cc} -\varphi_1' & -y \\ xy^{b-1} & -\varphi_0' \end{array} \right) \left( \begin{array}{cc} h_1 & h_2 \\ h_3 & h_4 \end{array} \right)
\quad (4.T)
\]

and

\[
\beta_0 = h \circ \varphi_{\psi E} + \varphi_{1E'} \circ g
= \left( \begin{array}{cc} h_1 & h_2 \\ h_3 & h_4 \end{array} \right) \left( \begin{array}{cc} \varphi_0 & -y \\ xy^{b-1} & \varphi_1 \end{array} \right) + \left( \begin{array}{cc} -\varphi_0' & y \\ -xy^{b-1} & -\varphi_1' \end{array} \right) \left( \begin{array}{cc} g_1 & g_2 \\ g_3 & g_4 \end{array} \right)
\quad (4.U)
\]

By Equation (4.T), we have

\[
\alpha_0 = g_1 y + g_2 \varphi_0 - \varphi_1' h_2 + y h_4.
\]

Since \( (\alpha_i)|_{y=0} = \alpha_i \), \( (\varphi_i)|_{y=0} = \varphi_1 \) and \( (\varphi_i')|_{y=0} = \varphi_1' \), we have

\[
\alpha_0 = (g_2|_{y=0}) \varphi_0 + \varphi_1'(-h_2)|_{y=0}).
\]

Similarly, by (4.U), we have

\[
\alpha_1 = (-(h_2|_{y=0}) \varphi_1 + \varphi_0'(g_2|_{y=0})
\]

Hence, the morphism \( \alpha \) is homotopy equivalent to the zero map.

\[\square\]

**Lemma 4.16.** For each non-zero object \( F \in C_{n-2} \) and \( 0 \leq j \leq a_{n-1} - 2 \), there is a non-zero morphism

\[
\sigma_{F}^j : \psi_{a_{n-2}} F \to \phi_j F
\]
such that the following conditions hold.
For any morphism \( \alpha = (\alpha_1, \alpha_0) : F \to F' \) between non-zero objects in \( \mathcal{C}_{n-2} \), the following diagram commutes:

\[
\begin{array}{c}
\psi_{a_{n-2}} \psi_j F' \xrightarrow{\sigma'_j} \phi_j F'
\\
\psi_{a_{n-2}} \psi_j \alpha \downarrow \quad \downarrow \phi_j \alpha
\\
\psi_{a_{n-2}} \psi_j F \xrightarrow{\sigma_j} \phi_j F
\end{array}
\]

In particular, the morphisms \( \{\sigma_j\} \) define a functor morphism \( \sigma_j : \psi_{a_{n-2}} \psi_j \to \phi_j \).

(2) The composition \( \sigma_j' \circ \psi_{a_{n-2}} \psi_j \alpha \) is the zero map if and only if \( \alpha \) is the zero map.

**Proof.** We only give the non-zero morphism \( \sigma_j' \), since the other part can be checked by similar computations and arguments as in the proof of Lemma 4.15. We use the similar notation as in (4.B). Then, the morphism \( \sigma_j' : \psi_{a_{n-2}} \psi_j F \to \phi_j F \) is defined by

\[
\sigma_j' = \sigma((-(c - 2)\vec{z} - j\vec{y}))[c + j - 2],
\]

where

\[
\sigma = (\sigma_1, \sigma_0) : \psi^2 F \to \phi F(-\vec{z})[1]
\]

is given by

\[
\begin{align*}
\sigma_1 : (\psi^2 F)_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ z^{c-1} & 0 & 0 & 1 \end{pmatrix} \to (\phi F(-\vec{z})[1])_1 \\
\sigma_0 : (\psi^2 F)_0 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ z^{c-1} & 0 & 0 & 1 \end{pmatrix} \to (\phi F(-\vec{z})[1])_0,
\end{align*}
\]

where

\[
\begin{align*}
(\psi^2 F)_1 &= \hat{F}_1 \oplus \hat{F}_0(-\vec{y}) \oplus \hat{F}_0(-\vec{z}) \oplus \hat{F}_1(\vec{f} - \vec{y} - \vec{z}) \\
(\psi^2 F)_0 &= \hat{F}_0 \oplus \hat{F}_1(\vec{f} - \vec{y}) \oplus \hat{F}_1(\vec{f} - \vec{z}) \oplus \hat{F}_0(\vec{f} - \vec{y} - \vec{z}) \\
(\phi F(-\vec{z})[1])_1 &= \hat{F}_0(-\vec{z}) \oplus \hat{F}_1(\vec{f} - \vec{y} - \vec{z}) \\
(\phi F(-\vec{z})[1])_0 &= \hat{F}_1(\vec{f} - \vec{z}) \oplus \hat{F}_0(\vec{f} - \vec{y} - \vec{z}).
\end{align*}
\]

Since each matrix \( \sigma_i \) contains unit entries, \( \sigma_j' \) is not a zero-map. \qed

**Lemma 4.17.** For any non-zero object \( F \in \mathcal{C}_{n-2} \), there is a non-zero morphism

\[
\theta^F : \psi_{a_{n-2}} \psi_{a_{n-1}} F \to \phi_{a_{n-1}} F
\]

such that the following conditions hold.
For any morphism $\alpha = (\alpha_1, \alpha_0) : F \to F'$ between non-zero objects in $C_{n-2}$, the following diagram commutes:

\[
\begin{array}{c}
\psi_{a_{n-2}} \psi_{a_{n-1}} F \\
\downarrow \psi_{a_{n-2}} \psi_{a_{n-1}} \alpha \\
\psi_{a_{n-2}} \psi_{a_{n-1}} F' \\
\end{array}
\xrightarrow{\varphi} 
\begin{array}{c}
\phi_{a_{n-1}} F \\
\downarrow \phi_{a_{n-1}} \alpha \\
\phi_{a_{n-1}} F' \\
\end{array}
\]

In particular, the morphisms $\{\varphi^F\}$ define a functor morphism $\varphi : \psi_{a_{n-2}} \psi_{a_{n-1}} \to \phi_{a_{n-1}}$.

The composition $\varphi F' \circ \psi_{a_{n-2}} \psi_{a_{n-1}} \alpha$ is the zero map if and only if $\alpha$ is the zero map.

Proof. Again we only give the non-zero map $\varphi^F$, and we use the similar notation as in (4.B). The morphism $\varphi^F : \psi_{a_{n-2}} \psi_{a_{n-1}} F \to \phi_{a_{n-1}} F$ is defined by

\[
\varphi^F := \pi(-(b - 2)\vec{y} - (c - 2)\vec{z})[b + c - 4],
\]

where

\[
\varphi = (\varphi_1, \varphi_0) : \psi^2 \to \phi F(-\vec{y} - \vec{z})[2]
\]

is given by

\[
\varphi_1 : (\psi^2 F)_1 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\vec{z}c^{-1} & xy^{b-2} & 0 \end{pmatrix}} (\phi F(-\vec{y} - \vec{z})[2])_1
\]

\[
\varphi_0 : (\psi^2 F)_0 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\vec{z}c^{-1} & xy^{b-2} & 0 \end{pmatrix}} (\phi F(-\vec{y} - \vec{z})[2])_0,
\]

where

\[
(\psi^2 F)_1 = \hat{F}_1 \oplus \hat{F}_0(-\vec{y}) \oplus \hat{F}_0(-\vec{z}) \oplus \hat{F}_1(\vec{f} - \vec{y} - \vec{z})
\]

\[
(\psi^2 F)_0 = \hat{F}_0 \oplus \hat{F}_1(\vec{f} - \vec{y}) \oplus \hat{F}_1(\vec{f} - \vec{z}) \oplus \hat{F}_0(\vec{f} - \vec{y} - \vec{z})
\]

\[
(\phi F(-\vec{y} - \vec{z})[2])_1 = \hat{F}_1(\vec{f} - \vec{y} - \vec{z}) \oplus \hat{F}_0(\vec{f} - 2\vec{y} - \vec{z})
\]

\[
(\phi F(-\vec{y} - \vec{z})[2])_0 = \hat{F}_0(\vec{f} - \vec{y} - \vec{z}) \oplus \hat{F}_1(2\vec{f} - 2\vec{y} - \vec{z}).
\]

Since each matrix $\varphi_i$ contains a unit entry, $\varphi^F$ is not a zero-map.

4.4.3 Irreducible morphisms between the exceptional objects

In this section, using the above lemmas, we determine the irreducible morphisms between objects in $E^n$. For two objects $E, E' \in E^n$, we set

\[
\text{irr}(E, E') := \dim_k \text{Irr}(E, E'),
\]
where the vector space $\text{Irr}(E, E')$ is defined as the following quotient space:

$$\text{Irr}(E, E') := \text{Hom}(E, E') / \text{Im} \left( \bigoplus_{E'' \in \mathcal{E} \setminus \{E, E'\}} \text{Hom}(E'', E') \times \text{Hom}(E', E'') \right).$$

Then, the number of arrows from a vertex $E$ to another vertex $E'$ in $Q^n$ is equal to the number $\text{irr}(E, E')$. By Lemmas 4.10 and 4.13 and $\text{Hom}(E^0, E^0) \cong k$, for any $E, E' \in \mathcal{E}^n$ we have $\text{hom}(E, E') := \dim_k \text{Hom}(E, E') \in \{0, 1\}$, and in particular we have

$$\text{irr}(E, E') \in \{0, 1\}.$$  

The following lemmas determine the numbers $\text{irr}(E, F)$ for all $E, E' \in \mathcal{E}^n$. Write $\mathcal{E}^{n-1} = (E_1, \ldots, E_{\mu_{n-1}})$.

**Lemma 4.18.** For $0 \leq i \neq j \leq a_{n-1} - 2$ and $E_r, E_s \in \mathcal{E}^{n-1}$, we have

$$\text{irr}(\psi_i E_r, \psi_j E_s) = \begin{cases} 1 & r = s \text{ and } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** By Lemma 4.10, the vector space $\text{Hom}(\psi_i E_r, \psi_j E_s)$ vanishes when $j \neq i + 1$, so it is enough to show that

$$\text{irr}(\psi_i E_r, \psi_{i+1} E_s) = \begin{cases} 1 & r = s \\ 0 & \text{otherwise.} \end{cases}$$

First we prove $\text{irr}(\psi_i E_r, \psi_{i+1} E_r) = 1$. Since $\text{Hom}(\psi_i E_r, \psi_{i+1} E_r) \cong \text{Hom}(E_r, E_r) \cong k$, we have $\text{hom}(\psi_i E_r, \psi_{i+1} E_r) = 1$. It is enough to show that a non-zero map $\alpha : \psi_i E_r \to \psi_{i+1} E_r$ is an irreducible morphism. Assume that there are an exceptional object $E' \in \mathcal{E}^n$ and non-isomorphisms $\beta : \psi_i E_r \to E'$ and $\gamma : E' \to \psi_{i+1} E_r$ such that $\alpha = \gamma \circ \beta$. Since $\beta$ and $\gamma$ are non-zero morphisms, either of the following cases holds.

1. $E' \cong \psi_j E_t$ for some $E_t \in \mathcal{E}^{n-1}$ with $r < t$.
2. $E' \cong \psi_{i+1} E_u$ for some $E_u \in \mathcal{E}^{n-1}$ with $u < r$.

Assume the former case (1) holds. Since any non-zero endomorphism of an object in $\mathcal{E}^n$ has to be an isomorphism, we see that $r < t$. Then, there $\gamma$ has to be the zero-map since $\text{Hom}(\psi_j E_t, \psi_{i+1} E_r) \cong \text{Hom}(E_t, E_r) = 0$. Thus, the latter case (2) must hold. But since $u < r$, the morphism $\beta$ has to be the zero map, since $\text{Hom}(\psi_i E_r, \psi_{i+1} E_u) \cong \text{Hom}(E_r, E_u) = 0$. This is a contradiction, and thus $\alpha$ is irreducible.

We only need to prove the vanishing $\text{irr}(\psi_i E_r, \psi_{i+1} E_s) = 0$ if $r \neq s$. Since $\text{hom}(\psi_i E_r, \psi_{i+1} E_s) = \text{hom}(E_r, E_s)$, we may assume $\text{hom}(E_r, E_s) = 1$. Then, there is a non-zero morphism $\alpha : E_r \to E_s$. By Lemma 4.15, the composition

$$\psi_i E_r \xrightarrow{\psi_i E_r} \psi_{i+1} E_r \xrightarrow{\psi_{i+1} \alpha} \psi_{i+1} E_s$$
is a non-zero map. Since $\hom(\psi_iE_r, \psi_{i+1}E_s) = 1$, this composition spans $\hom(\psi_iE_r, \psi_{i+1}E_s)$, and thus we have $\text{irr}(\psi_iE_r, \psi_{i+1}E_s) = 0$. □

**Lemma 4.19.** For $0 \leq i \leq a_n - 2$, $0 \leq j \leq a_{n-1} - 1$ and $E \in \mathcal{E}^{n-1}$ and $F \in \mathcal{E}^{n-2}$, we have

$$\text{irr}(\psi_iE, \phi_jF) = \begin{cases} i = a_n - 2 \text{ and } E = \psi_jF \\ 1 \text{ or } i = a_n - 2, \ j = a_{n-1} - 1 \text{ and } E = \psi_{a_{n-1}-2}F \\ 0 \text{ otherwise} \end{cases}$$

**Proof.** This follows from a similar argument as in the proof of Lemma 4.18 by using Lemmas 4.13, 4.16 and 4.17. □

**4.4.4 Construction of $(Q^n, I^n)$**

Now we give an inductive construction of $(Q^n, I^n)$. We denote by $Q^n_0$ and $Q^n_1$ the set of vertices of $Q^n$ and the set of arrows in $Q^n$, respectively. Assume that the quiver with relations $(Q^n, I^n)$ is given for all $k < n$. For each $0 \leq i \leq a_n - 2$, we set

$$\psi_i Q^n_{0-1} := \{\psi_i v \mid v \in Q^n_{0-1}\}$$
$$\psi_i Q^n_{1-1} := \{\psi_i \alpha \mid \alpha \in Q^n_{1-1}\}.$$

Then, these defines a quiver

$$\psi_i Q^n := (\psi_i Q^n_{0-1}, \psi_i Q^n_{1-1})$$

that is isomorphic to the quiver $Q^{n-1}$. If we denote by $\psi_i I^n_{1-1}$ the relations of arrows in $\psi_i Q^n_{1-1}$ corresponding to $I^{n-1}$, we have the quiver with relations

$$\psi_i (Q^n_{1-1}, I^n_{1-1}) := (\psi_i Q^n_{1-1}, \psi_i I^n_{1-1})$$

that is isomorphic to $(Q^{n-1}, I^{n-1})$. Similarly, for each $0 \leq j \leq a_{n-1} - 1$, we consider the quiver with relations

$$\phi_j (Q^{n-2}, I^{n-2}) := (\phi_j Q^{n-2}_0, \phi_j I^{n-2})$$

that is isomorphic to $(Q^{n-2}, I^{n-2})$. More precisely, the quiver $\phi_j Q^{n-2} = (\phi_j Q^{n-2}_0, \phi_j Q^{n-2}_1)$ is given by

$$\phi_j Q^{n-2}_0 := \{\phi_j v \mid v \in Q^{n-2}_0\}$$
$$\phi_j Q^{n-2}_1 := \{\phi_j \alpha \mid \alpha \in Q^{n-2}_1\},$$

and its relations $\phi_j I^{n-2}$ corresponds to $I^{n-2}$. 
Then, the set of vertices $Q_0^n$ is defined by the following disjoint union

$$Q_0^n := \left( \bigcup_{i=0}^{a_n-2} \psi_i Q_{0}^{n-1} \right) \bigcup \left( \bigcup_{j=0}^{a_n-1} \phi_j Q_{0}^{n-2} \right),$$

and the set of arrows $Q_1^n$ is defined by

$$Q_1^n := \left( \bigcup_{i=0}^{a_n-2} \psi_i Q_{1}^{n-1} \right) \bigcup \left( \bigcup_{j=0}^{a_n-1} \phi_j Q_{1}^{n-2} \right) \bigcup \left( \bigcup_{i=0}^{a_n-3} \Lambda_i \right) \bigcup \left( \bigcup_{j=0}^{a_n-2} \Sigma_j \right) \bigcup \Theta,$$

where the sets $\Lambda_i$, $\Sigma_j$ and $\Theta$ are defined by

$$\Lambda_i := \{ \lambda_i^v : \psi_i v \to \psi_{i+1} v | v \in Q_0^{n-1} \},$$

$$\Sigma_j := \{ \sigma_j^v : \psi_{a_n-2} \psi_j v \to \phi_j v | v \in Q_0^{n-2} \},$$

$$\Theta := \{ \theta^v : \psi_{a_n-2} \psi_{a_n-1} v \to \phi_{a_n-1} v | v \in Q_0^{n-2} \}.$$

By setting $Q_{(0)}^{n-1} := \psi_i Q_{(0)}^{n-1}$ and $Q_{[j]}^{n-2} := \phi_j Q_{[j]}^{n-2}$, we can draw a rough picture of the quiver $Q^n$:

We can draw more precise picture as follows: To ease notation, we set $Q_{(i,i')}^{n-2} := \psi_i \psi_{i'} Q_{(i,i')}^{n-2}$, $Q_{(i,j)}^{n-3} := \psi_i \phi_j Q_{(i,j)}^{n-3}$, $b_n := a_n - 2$, $c_n := b_{n-1} = a_{n-1} - 2$ and $d_n := c_{n-1} + 1 = a_{n-2} - 1$. Then, the quiver $Q^n$ can be described by the following:
Next we construct the relations $I^n$. For this, consider the following set of relations

\[
J^\lambda_{\text{null}} := \bigcup_{i=0}^{a_{n-4}} \left\{ \lambda_{i+1}^v \circ \lambda_i^v = 0 \mid v \in Q_0^{n-1} \right\}
\]

\[
J^\lambda_{\text{comm}} := \bigcup_{i=0}^{a_{n-3}} \left\{ (\psi_{i+1} \alpha) \circ \lambda_i^v = \lambda_i^{v'} \circ (\psi_i \alpha) \mid \alpha : v \to v' \in Q_1^{n-1} \right\}
\]

\[
J^\sigma_{\text{null}} := \bigcup_{j=0}^{a_{n-1}-1} \left\{ \sigma_j^v \circ \lambda_{a_{n-3}}^v = 0 \mid v \in Q_0^{n-2} \right\}
\]

\[
J^\sigma_{\text{comm}} := \bigcup_{j=0}^{a_{n-1}-1} \left\{ (\phi_j \alpha) \circ \sigma_j^v = \sigma_j^{v'} \circ (\psi_{a_{n-2}} \psi_j \alpha) \mid \alpha : v \to v' \in Q_1^{n-2} \right\}
\]

\[
J^\theta_{\text{null}} := \left\{ \theta^v \circ \psi_{(a_{n-1}-2)} \lambda_{a_{n-3}}^v = 0 \mid v \in Q_0^{n-2} \right\}
\]

\[
J^\theta_{\text{comm}} := \left\{ (\phi_{a_{n-1}-1} \alpha) \circ \theta^v = \theta^{v'} \circ (\phi_{a_{n-2}} \psi_{a_{n-1}-2} \alpha) \mid \alpha : v \to v' \in Q_1^{n-2} \right\}
\]

Then, the set of relations $I^n$ is generated by the following relations

\[
\left( \bigcup_{i=0}^{a_{n-2}} \psi_i I^{n-1} \right) \bigcup \left( \bigcup_{j=0}^{a_{n-1}-1} \phi_j I^{n-2} \right) \bigcup \left( \bigcup_{* \in \{\lambda, \sigma, \theta\}} J^*_{\text{null}} \right) \bigcup \left( \bigcup_{* \in \{\lambda, \sigma, \theta\}} J^*_{\text{comm}} \right).
\]

**Example 4.20.** The quiver $Q^2$ is

![Quiver Diagram](image-url)
and the relations $I^2$ is given by

$$I^2 = \begin{cases} \alpha_k \alpha_{k+1} + \alpha_k \lambda_l = \lambda_l \alpha_k & 1 \leq k \leq a_1 - 3, 0 \leq l \leq a_2 - 3 \\ \lambda_{i+1} \lambda_i = \sigma_{j} \lambda_{a_2-3} = \theta \lambda_{a_2-3} = 0 & 0 \leq i \leq a_2 - 4, 0 \leq j \leq a_1 - 2 \end{cases}.$$ 

Consider the path algebra $kQ^n$ of $Q^n$, and denote by the same notation $I^n \subset kQ^n$ the two-sided ideal of $kQ^n$ associated to the relations $I^n$ in $Q^n$.

**Theorem 4.21.** Let $T_n \in C_n$ be the tilting object in (4.5). Then, the endomorphism ring $\text{End}_{C_n}(T_n)$ is isomorphic to the algebra $kQ^n/I^n$. In particular, we have an equivalence

$$C_n \sim D^b(\text{mod } kQ^n/I^n).$$

**Proof.** The former statement follows from the above results in Subsections 4.4.2 and 4.4.3. We prove the latter statement. Since $C_n$ is idempotent complete and algebraic, by [39] (see also [1, Proposition 2.3]) or [11], we have an exact equivalence

$$C_n \cong K^b(\text{proj } kQ^n/I^n).$$

Since the tilting object $T_n$ is a strong generator of $C_n$ in the sense of [54], $K^b(\text{proj } kQ^n/I^n) \cong D^b(\text{mod } kQ^n/I^n)$ by [54, Proposition 7.25].

**5 | FURTHER APPLICATIONS OF THEOREM 3.1**

**5.1 | Kuznetsoy–Perry’s semi-orthogonal decomposition revisited**

As a special case of Corollary 3.7, we obtain Kuznetsov–Perry type semi-orthogonal decompositions (cf. [42]) for ramified cyclic covers of weighted projective spaces. In this subsection, we prove the following theorem.

Let $n > 0$, $N > 1$ and $c > 0$ be positive integers. Let $W \in S : = k[x_1, ..., x_n]$ be a quasi-homogeneous polynomial of degree $Nc$ with respect to $\deg(x_i) = a_i \in \mathbb{Z}_{>0}$, and denote by $\mu_N \subset G_m$ the subgroup generated by a primitive $N$th root of unity $\zeta \in G_m$. Consider a $\mu_N$-action and a $G_m$-action on $\text{Spec} S[t] \cong \mathbb{A}_x^n \times \mathbb{A}_t^1$ defined by $\zeta \cdot (x, t) : = (x, \zeta t)$ and $g \cdot (x, t) : = (g^{a_1}x_1, ..., g^{a_n}x_n, gt)$, where $x = (x_1, ..., x_n)$ is a coordinate of $\mathbb{A}_x^n$. Then, the $\mu_N$-action on $\mathbb{A}_x^n \times \mathbb{A}_t^1$ induces a $\mu_N$-action on $\text{HMF}_{S[t]}^Z(W + t^N)$. Then, we have the following.

**Theorem 5.1.** There are fully faithful functors

$$\Phi_i : \text{HMF}_{S[t]}^Z(W) \hookrightarrow \text{HMF}_{S[t]}^Z(W + t^N)^{\mu_N}$$

for $-N + 2 \leq i \leq 0$ and a semi-orthogonal decomposition

$$\text{HMF}_{S[t]}^Z(W + t^N)^{\mu_N} = \langle \text{Im}(\Phi_0), ..., \text{Im}(\Phi_{-N+2}) \rangle,$$

where $\text{HMF}_{S[t]}^Z(W + t^N)^{\mu_N}$ denotes the $\mu_N$-equivariant category of $\text{HMF}_{S[t]}^Z(W + t^N)$. 

To prove Theorem 5.1, we need to prepare some results. Set $X := \mathbb{A}^n_x$ and $G := G_m \times \mu_N$. We define characters $\phi : G_m \to G_m$ and $\psi : G_m \times \mu_N \to G_m$ by $\phi(g) = 1$ and $\psi(g, \zeta) := g^{\zeta}$. Fix $(a_1, \ldots, a_n) \in \mathbb{Z}_{>0}^n$. Note that $W + t^N$ is a $X_{Ne}$-semi-invariant regular function on $X \times \mathbb{A}^1_t$. By Corollary 3.7(1), we have the following semi-orthogonal decomposition.

**Corollary 5.2.** Assume that $N > 1$. There is a fully faithful functor

$$\Phi : \text{Dcoh}_{G_m}(\mathbb{A}^n_x, X_{Ne}, W) \hookrightarrow \text{Dcoh}_{G_m \times \mu_N}(\mathbb{A}^n_x \times \mathbb{A}^1_t, X_{Ne} \times 1, W + t^N),$$

and we have the following semi-orthogonal decomposition:

$$\text{Dcoh}_{G_m \times \mu_N}(\mathbb{A}^n_x \times \mathbb{A}^1_t, X_{Ne} \times 1, W + t^N) = \langle \text{Im}(\Phi_0), \ldots, \text{Im}(\Phi_{-N+2}) \rangle,$$

Here $\text{Im}(\Phi_i)$ denotes the essential image of the composition $((-) \otimes \mathcal{O}(\psi_i)) \circ \Phi$.

We refine Corollary 5.2 in terms of equivariant categories. For equivariant categories, see [20] or the Appendix. We prove the following proposition.

**Proposition 5.3.** Let $X$ be a smooth variety, and suppose that a reductive affine algebraic $G$ and a finite group $H$ act on $X$. Let $W : X \to \mathbb{A}^1$ be a non-constant semi-invariant regular function with respect to some character $\chi : G \times H \to G_m$. Assume that $\text{Dcoh}_{G \times H}(X, \chi, W)$ is idempotent complete. Then, we have an equivalence

$$\text{Dcoh}_{G \times H}(X, \chi, W) \cong \text{Dcoh}_{G}(X, \chi|_{G \times \{1\}}, W)^H.$$

Before giving the proof of Proposition 5.3, we prove Lemma 5.4 and Proposition 5.5. In Lemma 5.4, we see the compatibility between equivariant categories and Verdier quotients.

**Lemma 5.4.** Let $\mathcal{T}$ be a triangulated category with a dg-enhancement, and let $S$ a thick subcategory of $\mathcal{T}$. Let $G$ be a finite group acting on $\mathcal{T}$. Assume that $S$ is stable under $G$-action, that is, for every $g \in G$ the autoequivalences $\sigma_g : \mathcal{T} \to \mathcal{T}$ defining the $G$-action preserves the subcategory $S$. Then, we have a fully faithful functor

$$\Sigma : \mathcal{T}^G / S^G \hookrightarrow (\mathcal{T} / S)^G.$$

Moreover, if $\mathcal{T}^G / S^G$ is idempotent complete, the above functor is an equivalence.

**Proof.** Since the subcategory $S$ is $G$-stable, the functor $p^* : \mathcal{T}^G \to \mathcal{T}$ maps the subcategory $S^G$ to the subcategory $S$, and the functor $p_* : \mathcal{T} \to \mathcal{T}^G$ maps $S$ to $S^G$, where the functors $p^*$ and $p_*$ are defined by Definition A.12. Hence, these functors induces the functors

$$\mathcal{T}^G / S^G \xrightarrow{p^*} \mathcal{T} / S \xrightarrow{p_*} \mathcal{T}^G / S^G,$$

and the adjunction $(p^* \dashv p_*)$ induces the adjunction $(\overline{p^*} \dashv \overline{p_*})$. By the argument in the proof of [20, Proposition 3.11(1)], the adjunction morphism $\eta : \text{id} \to p_*p^*$ is a split mono, that is, there
exists a functor morphism \( \zeta : p_* p^* \to \text{id} \) such that \( \zeta \circ \eta = \text{id} \). These functor morphisms naturally induces functor morphisms \( \tilde{\eta} : \text{id} \to p_* p^* \) and \( \tilde{\zeta} : p_* p^* \to \text{id} \) such that \( \tilde{\zeta} \circ \tilde{\eta} = \text{id} \). Since \( \tilde{\eta} \) is nothing but the adjunction morphism of the adjoint pair \((p^* \dashv p_*)\), by Proposition A.7, the comparison functor

\[ \Gamma : \mathcal{T}^G_S \to (\mathcal{T}/S)_{\mathbb{T}(p^*, p_*)} \]

is fully faithful. Moreover, if \( \mathcal{T}^G_S \) is idempotent complete, the functor \( \Gamma \) is an equivalence.

On the other hand, the \( G \)-stability of \( S \) also induces a \( G \)-action on the Verdier quotient \( \mathcal{T}/S \), and this \( G \)-action defines an adjoint pair \((q^* \dashv q_*)\) of functors

\[ (\mathcal{T}/S)^G \xrightarrow{q^*} \mathcal{T}/S \xrightarrow{q_*} (\mathcal{T}/S)^G. \]

Since two comonads \( \mathbb{T}(p^*, p_*) \) and \( \mathbb{T}(q^*, q_*) \) on \( \mathcal{T}/S \) are naturally isomorphic, we have a natural equivalence

\[ (\mathcal{T}/S)_{\mathbb{T}(p^*, p_*)} \cong (\mathcal{T}/S)_{\mathbb{T}(q^*, q_*)}, \]

and the latter category \( (\mathcal{T}/S)_{\mathbb{T}(q^*, q_*)} \) is equivalent to \( (\mathcal{T}/S)^G \) by Proposition A.13. This finishes the proof.

Proposition 5.5. Let \( Y \) be a quasi-projective scheme. Suppose that a reductive affine algebraic \( G \) and a finite group \( H \) act on \( Y \). Assume that the \( G \times H \)-equivariant singularity category \( D^\text{sg}_{G \times H}(Y) \) of \( Y \) is idempotent complete. Then, we have an equivalence

\[ D^\text{sg}_{G \times H}(Y) \cong D^\text{sg}_G(Y)^H. \]

Proof. First, note that there is a natural equivalence \( \text{coh}_{G \times H}(Y) \cong (\text{coh}_G Y)^H \), where \( (\text{coh}_G Y)^H \) denotes the \( H \)-equivariant category of the abelian category \( \text{coh}_G Y \). By [20, Theorem 7.1], we have a natural equivalence \( D^b(\text{coh}_{G \times H}(Y)) \cong D^b(\text{coh}_G Y)^H \), and it is easy to see that this equivalence restricts to an equivalence \( \text{Perf}_{G \times H} Y \cong \text{Perf}_G Y^H \). Then, we obtain the result by the following sequence of equivalences

\[ D^\text{sg}_{G \times H}(Y) = D^b(\text{coh}_{G \times H} Y)/\text{Perf}_{G \times H} Y \]

\[ \cong D^b(\text{coh}_G Y)^H/\text{Perf}_G Y^H \]

\[ \cong (D^b(\text{coh}_G Y)/\text{Perf}_G Y)^H \]

\[ = D^\text{sg}_G(Y)^H, \]

where the third line follows from Lemma 5.4.

Proof of Proposition 5.3. Denote by \( X_0 \) the zero scheme of \( W \). By Theorem 2.21, we have equivalences \( \text{Dcoh}_{G \times H}(X, \chi, W) \cong D^\text{sg}_{G \times H}(X_0) \) and \( \text{Dcoh}_{G}(X, \chi|_{G \times [1]}, W) \cong D^\text{sg}_G(X_0)_1 \). Hence, the result follows from Proposition 5.5.

By Corollary 5.2 and Propositions 2.23 and 5.3, we have Theorem 5.1.
5.2 Case of Thom–Sebastiani sum

In this subsection, we consider special cases of Theorem 3.1, where the sum $W + F$ is of Thom–Sebastiani type. We use the notation in Subsection 3.1.

Set $X := \mathbb{A}^n \times G$, $G := \mathbb{G}_m \times \mathbb{G}_m$, and define characters $\phi : \mathbb{G}_m \to \mathbb{G}_m$ and $\psi : G \to \mathbb{G}_m$ by $\phi(g) := 1$ and $\psi(g, \zeta) := g^\zeta$ for $g \in \mathbb{G}_m$. Then, $X = \psi^N = \chi_N$. Fix $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$. We define actions from $\mathbb{G}_m$ on $\mathbb{A}^n_x$ and on $\mathbb{A}^m_t$ such that $\deg(x_i) = a_i$ and $\deg(t_i) = d_i$, we also consider an action from $\mathbb{G}_m$ on $\mathbb{A}^n_x \times \mathbb{A}^m_t$ defined by

$$\zeta \cdot (x_1, \ldots, x_n, t_1, \ldots, t_m) := (x_1, \ldots, x_n, \zeta t_1, \ldots, \zeta t_m).$$

Assume that $W \in k[x] := k[x_1, \ldots, x_n]$ and $F \in k[t] := k[t_1, \ldots, t_m]$ are quasi-homogeneous polynomials of degree $N$ with respect to the weights $\deg(x_i) = a_i$ and $\deg(t_i) = d_i$. Then, $W + F$ is a $\chi$-semi-invariant regular function on $\mathbb{A}^n_x \times \mathbb{A}^m_t$ and $Z_F$ is the product $X \times V_F$, where $V_F \subset \mathbb{P}(d)$ is the hypersurface defined by $F$. For simplicity, we write $k[x, t] := k[x_1, \ldots, x_n, t_1, \ldots, t_m]$. By Theorem 3.1 and Propositions 2.23 and 5.3, we have the following theorem.

**Theorem 5.6.** Set $\mu := \sum_{i=1}^m d_i$.

1. If $N < \mu$, there are fully faithful functors

$$\Phi : \text{HMF}_{k[x, t]}^Z(W + F)^{\mu^N} \hookrightarrow \text{Dcoh}_{\mathbb{G}_m}^{\chi_N}(\mathbb{A}^n_x \times V_F, \chi_N, W),$$

$$\Psi_i : \text{HMF}_{k[x]}^Z(W) \hookrightarrow \text{Dcoh}_{\mathbb{G}_m}^{\chi_N}(\mathbb{A}^n_x \times V_F, \chi_N, W)$$

for $N - \mu + 1 \leq i \leq 0$ and there is a semi-orthogonal decomposition

$$\text{Dcoh}_{\mathbb{G}_m}^{\chi_N}(\mathbb{A}^n_x \times V_F, \chi_N, W) = \langle \text{Im}(\Psi_{N-\mu+1}), \ldots, \text{Im}(\Psi_0), \text{Im}(\Phi) \rangle.$$

2. If $N = \mu$, we have an equivalence

$$\Phi : \text{HMF}_{k[x, t]}^Z(W + F)^{\mu^N} \sim \text{Dcoh}_{\mathbb{G}_m}^{\chi_N}(\mathbb{A}^n_x \times V_F, \chi_N, W).$$

3. If $N > \mu$, there are fully faithful functors

$$\Phi : \text{Dcoh}_{\mathbb{G}_m}^{\chi_N}(\mathbb{A}^n_x \times V_F, \chi_N, W) \hookrightarrow \text{HMF}_{k[x, t]}^Z(W + F)^{\mu^N},$$

$$\Psi_i : \text{HMF}_{k[x]}^Z(W) \hookrightarrow \text{HMF}_{k[x, t]}^Z(W + F)^{\mu^N}$$

for $\mu - N + 1 \leq i \leq 0$ and there is a semi-orthogonal decomposition

$$\text{HMF}_{k[x, t]}^Z(W + F)^{\mu^N} = \langle \text{Im}(\Psi_0), \ldots, \text{Im}(\Psi_{\mu-N+1}), \text{Im}(\Phi) \rangle.$$

**Remark 5.7.** Note that we have $(\mathbb{A}^n_x \times V_F, \chi_N, W)^{\mathbb{G}_m} \cong (\mathbb{A}^n_x, \chi_N, W)^{\mathbb{G}_m} \boxtimes (V_F, \chi_1, 0)^{\mathbb{G}_m}$ via the algebraic group isomorphism $\mathbb{G}_m \sim \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m; a \mapsto (a, \chi_N(a))$. Then by [5, Lemma 4.8; 48], the categories $\text{HMF}_{k[x]}^Z(W)$ and $\text{HMF}_{k[x, t]}^Z(W + F)$ in Theorem 5.6 are idempotent complete. Hence, by
Lemma A.14, the category \( HMF^\mathbb{Z}_{k[x,t]}(W + F)^\mu_N \) in Theorem 5.6 is also idempotent complete. Consequently, by Theorem 5.6 and [5, Lemma 4.8] again, \( \text{Dcoh}_{\mathbb{Z}}(\mathbb{A}^n_x \times V_F, \chi_N, W) \) is also idempotent complete. Due to Proposition 2.31 and Theorem 5.6, we have

\[
\text{Dcoh}_{\text{Gm}}(\mathbb{A}^n_x \times V_F, \chi_N, W) \cong [\text{inj}_{\text{Gm}}(\mathbb{A}^n_x \times V_F, \chi_N, W)] \\
\cong [\text{inj}_{\text{Gm}}(\mathbb{A}^n_x, \chi_N, W) \otimes \text{inj}_{\text{Gm}}(V_F, \chi_1, 0)].
\]

The decomposition in Remark 5.7 gives an application to derived categories of products of Calabi–Yau hypersurfaces.

Remark 5.8. Thus, if \( N = n \), we have

\[
\text{Dcoh}_{\text{Gm}}(\mathbb{A}^n_x \times V_F, \chi_n, W) \cong [\text{inj}_{\text{Gm}}(\mathbb{A}^n_x, \chi_n, W) \otimes \text{inj}_{\text{Gm}}(V_F, \chi_1, 0)] \\
\cong [\text{inj}_{\text{Gm}}(V_W, \chi_1, 0) \otimes \text{inj}_{\text{Gm}}(V_F, \chi_1, 0)] \\
\cong [\text{inj}_{\text{Gm}}(V_W \times V_F, \chi_1, 0)] \\
\cong \text{D}^b(\text{coh} V_F \times V_W),
\]

where the first equivalence is by Remark 5.7 and the second equivalence follows from Orlov’s LG/CY correspondence (see [6, Theorem 6.13] for dg-version of it). In particular, if \( n = \mu = N \), that is, both of \( V_F \) and \( V_W \) are Calabi–Yau, we have an equivalence

\[
\text{D}^b(\text{coh} V_F \times V_W) \cong HMF^\mathbb{Z}_{k[x,t]}(W + F)^\mu_N.
\]

The situation in this subsection appeared in previous works [6, 44].

Remark 5.9. Assume that \( a_1 = \ldots = a_n = d_1 = \ldots = d_m = 1 \) and \( N \geq \max\{n, m\} \). Then, we have \( \mu = m \) and \( N \geq \mu \). In [44], Lim studied a semi-orthogonal decomposition of the derived category \( \text{D}^b([V_{W+F}/\mu_N]) \) of the quotient stack \( [V_{W+F}/\mu_N] \), where \( V_{W+F} \subset \mathbb{P}^{n+m} \) is the hypersurface defined by \( W + F \). By [6, Example 3.10], if \( N \geq \max\{n, m\} \), there is a fully faithful functor \( HMF^\mathbb{Z}_{k[x,t]}(W + F)^\mu_N \to \text{D}^b([V_{W+F}/\mu_N]).

APPENDIX: COMODULES OVER COMONADS AND EQUIVARIANT CATEGORY

In this appendix, following [19, 20], we recall definitions and basic properties of comodules over comonads and equivairant categories. The only new result in this appendix is Lemma A.14.

A.1 Comodules over comonads

Let \( C \) be a category. We begin by recalling the definitions of comonads on \( C \) and comodules over a comonad.
**Definition A.1.** A comonad \( \mathbb{T} = (T, \varepsilon, \delta) \) on \( C \) consists of a functor \( T : C \to C \) and functor morphisms \( \varepsilon : T \to \text{id}_C \) and \( \delta : T \to T^2 \) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
T & \xrightarrow{\delta} & T^2 \\
\downarrow{\varepsilon} & & \downarrow{T \varepsilon} \\
T^2 & \xrightarrow{\mathbb{T}} & T \\
\end{array}
\]

**Example A.2.** Let \( P = (P^* \dashv P_*) \) be adjoint functors \( P^* : C \to D \) and \( P_* : D \to C \), and let \( \eta_P : \text{id}_C \to P_*P^* \) and \( \varepsilon_P : P^*P_* \to \text{id}_D \) be the adjunction morphisms. Set \( T_P := P^*P_* \) and \( \delta_P := P^*\eta_PP_* \). Then, the triple \( \mathbb{T}(P) := (T_P, \varepsilon_P, \delta_P) \) is a comonad on \( D \).

**Definition A.3.** Let \( \mathbb{T} = (T, \varepsilon, \delta) \) be a comonad on \( C \). A comodule over \( \mathbb{T} \) is a pair \((C, \vartheta_C)\) of an object \( C \in C \) and a morphism \( \vartheta_C : C \to T(C) \) such that

(1) \( \varepsilon(C) \circ \vartheta_C = \text{id}_C \), and

(2) the following diagram is commutative:

\[
\begin{array}{ccc}
C & \xrightarrow{\vartheta_C} & T(C) \\
\downarrow{\delta_C} & & \downarrow{T(\delta_C)} \\
T(C) & \xrightarrow{\delta(C)} & T^2(C). \\
\end{array}
\]

Given a comonad \( \mathbb{T} \) on \( C \), we define the category \( C^\mathbb{T} \) of comodules over the comonad \( \mathbb{T} \):

**Definition A.4.** Let \( \mathbb{T} = (T, \varepsilon, \delta) \) be a comonad on \( C \). The category \( C^\mathbb{T} \) of comodules over \( \mathbb{T} \) is the category whose objects are comodules over \( \mathbb{T} \) and whose sets of morphisms are defined as follows:

\[
\text{Hom}_{C^\mathbb{T}}((C_1, \vartheta_{C_1}),(C_2, \vartheta_{C_2})) := \{ f \in \text{Hom}_C(C_1, C_2) \mid T(f) \circ \vartheta_{C_1} = \vartheta_{C_2} \circ f \}.
\]

For a full subcategory \( B \subseteq C \), we define the full subcategory \( C^\mathbb{T}_B \subseteq C^\mathbb{T} \) as

\[
\text{Ob}(C^\mathbb{T}_B) := \{(C, \vartheta_C) \in \text{Ob}(C^\mathbb{T}) \mid C \cong B \text{ for some } B \in B\}.
\]

**Remark A.5.** Let \( (C, \vartheta_C) \in C^\mathbb{T}_B \). By definition, there exist an object \( B \in B \) and an isomorphism \( \varphi : C \cong B \). If we set \( \vartheta_B := T(\varphi)\vartheta_C\varphi^{-1} \), then the pair \((B, \vartheta_B)\) is an object of \( C^\mathbb{T}_B \) and \( \varphi \) gives an isomorphism from \((C, \vartheta_C)\) to \((B, \vartheta_B)\) in \( C^\mathbb{T}_B \).

For a comonad which is given by an adjoint pair \( (P^* \dashv P_*) \), we have a canonical functor, called comparison functor, from the domain of \( P^* \) to the category of comodules over the comonad.
**Definition A.6.** Notation is same as in Example A.2. For an adjoint pair \( P = (P^* \dashv P^*) \), we define a functor
\[
\Gamma_P : C \to D_{\mathbb{T}(P)}
\]
as follows: For any \( C \in C \) we define \( \Gamma_P(C) := (P^*(C), P^*(\eta_P(C))) \), and for any morphism \( f \) in \( C \) we define \( \Gamma_P(f) := P^*(f) \). This functor is called the *comparison functor* of \( P \).

The following proposition gives sufficient conditions for a comparison functor to be fully faithful or an equivalence.

**Proposition A.7** [19, Theorem 3.9, Corollary 3.11]. Notation is same as above.

1. If for any \( C \in C \), the morphism \( \eta_P(C) : C \to P^*P^*(C) \) is a split mono, that is, there is a morphism \( \zeta_C : P^*P^*(C) \to C \) such that \( \zeta \circ \eta_P(C) = \text{id}_C \), then the comparison functor \( \Gamma_P : C \to D_{\mathbb{T}(P)} \) is fully faithful.
2. If \( C \) is idempotent complete and the functor morphism \( \eta_P : \text{id}_C \to P^*P^* \) is a split mono, that is, there exists a functor morphism \( \zeta : P^*P^* \to \text{id}_C \) such that \( \zeta \circ \eta = \text{id} \), then \( \Gamma_P : C \to D_{\mathbb{T}(P)} \) is an equivalence.

Next we recall linearizable functors which induce natural functors between categories of comodules following [33]. Let \( \mathbb{T}_A \) (respectively, \( \mathbb{T}_B \)) be a category and \( \mathbb{T}_A = (T_A, \epsilon_A, \delta_A) \) (respectively, \( \mathbb{T}_B = (T_B, \epsilon_B, \delta_B) \)) a command on \( A \) (respectively, \( B \)).

**Definition A.8.** A functor \( F : \mathbb{T}_A \to \mathbb{T}_B \) is said to be *linearizable* with respect to \( \mathbb{T}_A \) and \( \mathbb{T}_B \), or just *linearizable*, if there exists an isomorphism of functors
\[
\Omega : F \mathbb{T}_A \sim \mathbb{T}_B F
\]
such that the following two diagrams of functor morphisms are commutative:

\[
\begin{array}{ccc}
FT_A & \xrightarrow{\Omega} & T_B F \\
\downarrow F \epsilon_A & & \downarrow \epsilon_B F \\
F & \xrightarrow{\epsilon_B} & F
\end{array}
\quad \quad \quad
\begin{array}{ccc}
FT_A & \xrightarrow{\Omega} & T_B F \\
\downarrow F \delta_A & & \downarrow \delta_B F \\
T_B \Omega \ast \Omega T_A & \xrightarrow{T_B \Omega \ast \Omega T_A} & T_B^2 F
\end{array}
\]

We call the pair \((F, \Omega)\) a *linearized functor* with respect to \( \mathbb{T}_A \) and \( \mathbb{T}_B \), and the isomorphism \( \Omega : F \mathbb{T}_A \sim \mathbb{T}_B F \) of functors is called a *linearization* of \( F \) with respect to \( \mathbb{T}_A \) and \( \mathbb{T}_B \).

If \( F : \mathbb{T}_A \to \mathbb{T}_B \) is a linearizable functor with a linearization \( \Omega : F \mathbb{T}_A \sim \mathbb{T}_B F \), we have an induced functor
\[
F_\Omega : \mathbb{T}_A \to \mathbb{T}_B
\]
defined by
\[
F_\Omega(A, \delta_A) := (F(A), \Omega(A) \circ F(\delta_A)) \quad \text{and} \quad F_\Omega(f) := F(f).
\]

The following proposition is a special case of [33, Proposition 2.10].
**Proposition A.9** (cf. [33, Proposition 2.10]). Let $F : A \to B$ be a linearizable functor with a linearization $\Omega : FT_A \sim T_B F$. If $F : A \to B$ is fully faithful (respectively, an equivalence), then the induced functor $F_{\Omega} : A_{T_A} \to B_{T_B}$ is also fully faithful (respectively, an equivalence).

### A.2 Equivariant category

In this section, all categories are $k$-linear. Let $C$ be an additive category and $G$ a finite group.

**Definition A.10.** A (right) action of $G$ on $C$ is given by the following data (i) and (ii).

(i) For every $g \in G$, an autoequivalence $\sigma_g : C \sim C$.

(ii) For every $g, h \in G$, functor isomorphisms $c_{g,h} : \sigma_g \circ \sigma_h \sim \sigma_{hg}$ such that the diagram

\[
\begin{array}{ccc}
\sigma_f \sigma_g \sigma_h & \xrightarrow{c_{f,k}} & \sigma_f \sigma_{hg} \\
\downarrow{c_{f,k}} & & \downarrow{c_{f,k}} \\
\sigma_g \sigma_f \sigma_h & \xrightarrow{c_{g,f,k}} & \sigma_{hg} \sigma_f
\end{array}
\]

commutes for all $f, g, h \in G$.

For a $G$-action on $C$, the equivariant category $C^G$ of $C$ is defined as follows: An object of $C^G$ is a pair $(F, (\theta_g)_{g \in G})$ of an object $F \in C$ and $(\theta_g)_{g \in G}$ is a family of isomorphisms

\[\theta_g : F \sim \sigma_g(F)\]

such that the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\theta_f} & \sigma_g(F) \\
\downarrow{\sigma_{hg}} & & \downarrow{\sigma_f(\theta_h)} \\
\sigma_{hg}(F) & \xleftarrow{c_{g,f,k}} & \sigma_g(\sigma_h(F))
\end{array}
\]

commutes for all $g, h \in G$. A morphism $f : (F_1, (\theta_1^g)) \to (F_2, (\theta_2^g))$ in $C^G$ is defined as a morphism $f : F_1 \to F_2$ in $C$ such that $f$ is compatible with $\theta^g$.

**Remark A.11.** If a category $C$ has a (right) $G$-action, then for any two objects $(F_1, (\theta^1_g))$ and $(F_2, (\theta^2_g))$, the equivariant structures $(\theta^1_g)$ and $(\theta^2_g)$ defines a natural (right) $G$-action on the set $\text{Hom}_C(F_1, F_2)$ as follows: For any $\varphi \in \text{Hom}_C(F_1, F_2)$ and $g \in G$, the morphism $\varphi \cdot g \in \text{Hom}_C(F_1, F_2)$ is defined as the following composition of morphisms:

\[
F_1 \xrightarrow{\theta_1^g} \sigma_g(F_1) \xrightarrow{\sigma_g(\varphi)} \sigma_g(F_2) \xrightarrow{(\theta_2^g)^{-1}} F_2.
\]

By definition, the set of morphisms in the equivariant category $C^G$ is equal to the $G$-invariant subspace of the set of morphisms in $C$:

\[\text{Hom}_{C^G}((F_1, (\theta_1^g)), (F_2, (\theta_2^g))) = \text{Hom}_C(F_1, F_2)^G.\]
**Definition A.12.** Define a functor

\[ p^* : C^G \to C \]

to be the forgetful functor, and define a functor

\[ p_* : C \to C^G \]

by \( p_*(F) := \left( \bigoplus_{h \in G} \sigma_h(F), (\theta_g) \right) \).

Then, by [20, Lemma 3.8] we have the adjunctions

\[ p_\ast \dashv p^* \dashv p_* \]

Recall that \( \mathbb{T}(p^*, p_*) \) denotes the comonad induced by the adjunction \( p^* \dashv p_* \).

**Proposition A.13** [20, Proposition 3.11]. The comparison functor

\[ C^G \to \mathcal{C}_{\mathbb{T}(p^*, p_*)} \]

is an equivalence.

Note that any \( G \)-action on \( C \) naturally extends to the \( G \)-action on its idempotent completion \( \hat{C} \) (see [20, Proposition 3.13]).

**Lemma A.14.** Let \( C \) be an additive category with an action of a finite group \( G \). Then, there is an additive equivalence

\[ \mathcal{C}^G \cong (\hat{C})^G. \]

In particular, if \( C \) is idempotent complete, so is \( C^G \). Furthermore, if \( G \) is abelian, \( C^G \) is idempotent complete if and only if \( C \) is idempotent complete.

Before we prove this, we recall basic properties of idempotent completion. An additive functor \( F : A \to B \) between additive categories induces an additive functor \( \hat{F} : \hat{A} \to \hat{B} \) defined by \( \hat{F}(A, e) := (F(A), F(e)) \). If \( F_1, F_2 : A \to B \) are additive functors and \( \varphi : F_1 \to F_2 \) is a functor morphism, then \( \varphi \) induces a functor morphism \( \hat{\varphi} : \hat{F}_1 \to \hat{F}_2 \) defined by

\[ \hat{\varphi}(A, e) := F_2(e) \circ \varphi(A) \circ F_1(e). \]

Since \( \varphi(A) \circ F_1(e) = F_2(e) \circ \varphi(A) \), we have \( \hat{\varphi}(A, e) = F_2(e) \circ \varphi(A) = \varphi(A) \circ F_1(e) \). This implies that the equality \( \varphi_2 \circ \varphi_1 = \varphi_2 \circ \varphi_1 \) of functor morphisms, and for any additive functor \( F : A \to B \), we have \( \hat{id}_F = \hat{id}_F \) since the identity morphism of an object \( (A, e) \in \hat{A} \) is the idempotent \( e : A \to A \). Therefore, we see that if \( p^* : A \to B \) and \( p_* : B \to A \) are an adjoint pair, the induced functors \( \hat{p}^* : \hat{A} \to \hat{B} \) and \( \hat{p}_* : \hat{B} \to \hat{A} \) are also an adjoint pair.
Proof of Lemma A.14. The latter statement follows from the former one by [20, Theorem 4.2]. Let \( p^* : C^G \to C, p_* : C \to C^G \) and \( q^* : (\hat{C})^G \to \hat{C}, q_* : \hat{C} \to (\hat{C})^G \) be the adjoint pairs defined in Definition A.12. Then, the adjoint pair \((p^* \dashv p_*)\) induces the adjoint pair \((\hat{p}^* \dashv \hat{p}_*)\). Then, we obtain two comonads \( \mathbb{T}(q^*, q_*) \) and \( \mathbb{T}(\hat{p}^*, \hat{p}_*) \) on \( \hat{C} \), and these comonads are tautologically isomorphic to each other. Hence, by Proposition A.13 there is a sequence of equivalences

\[
(\hat{C})^G \cong \hat{C}_{\mathbb{T}(q^*, q_*)} \cong \hat{C}_{\mathbb{T}(\hat{p}^*, \hat{p}_*)}.
\]

Thus, we only need to show the comparison functor

\[
\hat{C}^G \to \hat{C}_{\mathbb{T}(\hat{p}^*, \hat{p}_*)}
\]

is an equivalence. For this, we use Proposition A.7(2). Since \( \hat{C}^G \) is idempotent complete, it suffices to verify that the functor morphism \( \hat{\eta} : \text{id} \to \hat{p}_*\hat{p}^* \) is a split mono. By the proof of [20, Proposition 3.11(1)], the functor morphism \( \eta : \text{id} \to p_*p^* \) splits, and so there exists a functor morphism \( \gamma : p_*p^* \to \text{id} \) such that \( \gamma \circ \eta = \text{id} \). It is easy to see that the induced functor morphism \( \hat{\gamma} : \hat{p}_*\hat{p}^* \to \text{id} \) is a retraction of \( \hat{\eta} \). \( \square \)

Let \( \mathcal{T} \) be a triangulated category with an action of a finite group \( G \), where the autoequivalences \( \sigma_g : \mathcal{T} \to \mathcal{T} \) of the action are supposed to be exact equivalences. Then, the equivariant category \( \mathcal{T}^G \) has natural shift functors and a class of distinguished triangles induced by the triangulated structure of \( \mathcal{T} \). The following result guarantees that, if \( \mathcal{T} \) has a dg-enhancement, \( \mathcal{T}^G \) is a triangulated category with respect to the natural shift functors and distinguished triangles.

**Proposition A.15** [20, Corollary 6.10]. Notation is same as above. If \( \mathcal{T} \) has a dg-enhancement, then \( \mathcal{T}^G \) is a triangulated category with respect to the natural shift functors and distinguished triangles.

**ACKNOWLEDGEMENTS**

The authors would like to thank Atsushi Takahashi for giving valuable comments on a draft version of this article, and for informing us about his unpublished work and deep insight into the homological LG mirror symmetry. They also thank the referee for carefully reading the paper, and giving valuable comments and suggestions. Yuki Hirano is supported by JSPS KAKENHI, Grant Numbers: 17H06783 and 19K14502. Genki Ouchi is supported by Interdisciplinary Theoretical and Mathematical Sciences Program (iTHEMS) in RIKEN and JSPS KAKENHI, Grant Number: 19K14520. A part of the paper was written during visits of both authors to the Max-Planck-Institute for Mathematics in Bonn. The authors would like to thank the Max-Planck-Institute for Mathematics for the hospitality and support.

**JOURNAL INFORMATION**

The *Proceedings of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.
REFERENCES

1. T. Aihara and O. Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 633–668.
2. D. Aramaki and A. Takahashi, *Maximally-graded matrix factorizations for an invertible polynomial of chain type*, Adv. math. **373** (2020), 107320, 23 p.
3. P. Balmer and M. Schlichting, *Idempotent completion of triangulated categories*, J. Algebra **236** (2001), no. 2, 819–834.
4. M. Ballard, D. Deliu, D. Favero, M. U. Isik, and L. Katzarkov, *Resolutions in factorization categories*, Adv. Math. **295** (2016), 195–249.
5. M. Ballard, D. Deliu, D. Favero, M. U. Isik, and L. Katzarkov, *On the derived categories of degree d hypersurface fibrations*, Math. Ann. **371** (2018), no. 12, 337–370.
6. M. Ballard, D. Favero, and L. Katzarkov, *A category of kernels for equivariant factorizations and its implications for Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 1–111.
7. M. Ballard, D. Favero, and L. Katzarkov, *A category of kernels for equivariant factorizations, II: further implications*, J. Math. Pure Appl. (9) **102** (2014), 702–757.
8. M. Ballard, D. Favero, and L. Katzarkov, *Variation of geometric invariant theory quotients and derived categories*, J. Reine Angew. Math. **746** (2018), no. 12, 337–370.
9. A. Bondal and M. Kapranov, *Enhanced triangulated categories*, Math. USSR Sb. **70** (1991), no. 1, 93–107.
10. R.-O. Buchweitz, *Maximal Cohen–Macaulay modules and Tate-cohomology over Gorenstein rings*, Preprint, 1986, 155 p.
11. A. Bondal and M. Kapranov, *Cohomological descent theory for a morphism of stacks and for equivariant derived categories*, Sb. Math. **202** (2011), no. 3–4, 495–526.
12. A. I. Efimov and L. Positselski, *Coherent analogues of matrix factorizations and relative singularity categories*, Algebra Number Theory **9** (2015), no. 5, 1159–1292.
13. D. Eisenbud, *Homological algebra on a complete intersection*, with an application to group representations, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64.
14. A. Gabrielov, *Dynkin diagrams of unimodal singularities*, Funct. Anal. Appl. **8** (1974), 192–196.
15. W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras. Singularities, representation of algebras, and vector bundles (Lambrecht, 1985)*, Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297.
16. B. R. Greene and M. R. Plesser, *Duality in Calabi–Yau moduli space*, Nuclear Phys. B. **338** (1990), no. 1, 15–37.
28. M. Habermann, *Homological mirror symmetry for invertible polynomials in two variables*, Quantum Topol. **13** (2022), no. 2, 207–253.

29. M. Habermann and J. Smith, *Homological Berglund–Hübsch mirror symmetry for curve singularities*, J. Symplectic Geom. **18** (2020), no. 6, 1515–1574.

30. D. Halpern-Leistner, *The derived category of a GIT quotient*, J. Amer. Math. Soc. **28** (2015), no. 3, 871–912.

31. M. Herschend, O. Iyama, H. Minamoto, and S. Oppermann, *Representation theory of Geigle-Lenzing complete intersections*, Mem. Amer. Math. Soc., to appear.

32. Y. Hirano, *Derived Knörrer periodicity and Orlov's theorem for gauged Landau–Ginzburg models*, Compos. Math. **153** (2017), no. 5, 973–1007.

33. Y. Hirano, *Equivalences of derived factorization categories of gauged Landau–Ginzburg models*, Adv. Math. **306** (2017), 200–278.

34. Y. Hirano, *Relative singular locus and Balmer spectrum of matrix factorizations*, Trans. Amer. Math. Soc. **371** (2019), no. 7, 4994–5021.

35. M. U. Isik, *Equivalence of the derived category of a variety with a singularity category*, Int. Math. Res. Not. IMRN (2013), no. 12, 2787–2808.

36. O. Iyama, *Tilting Cohen–Macaulay representations*, Proc. Internat. Congr. Math., vol. II, Rio de Janeiro, 2018. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 125–162.

37. H. Kajiura, K. Saito, and A. Takahashi, *Matrix factorization and representations of quivers. II. Type ADE case*, Adv. Math. **211** (2007), no. 1, 327–362.

38. H. Kajiura, K. Saito, and A. Takahashi, *Triangulated categories of matrix factorizations for regular systems of weights with $\varepsilon = -1$*, Adv. Math. **220** (2009), no. 5, 1602–1654.

39. B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.

40. O. Kravets, *Categories of singularities of invertible polynomials*, arXiv:1911.09859.

41. M. Kreuzer and H. Skarke, *On the classification of quasihomogeneous functions*, Comm. Math. Phys. **150** (1992), no. 1, 137–147.

42. A. Kuznetsov and A. Perry, *Derived categories of cyclic covers and their branch divisors*, Selecta Math. (N.S.) **23** (2017), no. 1, 389–423.

43. Y. Lekili and K. Ueda, *Homological mirror symmetry for milnor fibers via moduli of $A_\infty$-structures*, J. Topol. **15** (2022), no. 3, 1058–1106.

44. B. Lim, *Equivariant derived categories associated to a sum of two potentials*, J. Geom. Phys. **160** (2021), Paper No. 103944, 24 p.

45. V. A. Lunts and O. M. Schnürer, *Matrix factorizations and semi-orthogonal decompositions for blowing-ups*, J. Noncommut. Geom. **10** (2016), no. 3, 981–1042.

46. A. Neeman, *The Grothendieck duality theorem via Bousfield’s techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236.

47. D. O. Orlov, *Triangulated categories of singularities and equivalences between Landau–Ginzburg models*, Mat. Sb. **197** (2006), no. 12, 117–132; translation in Sb. Math. **197** (2006), no. 11–12, 1827–1840.

48. D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin., vol. II, Progr. Math., vol. 270, Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 503–531.

49. D. Orlov, *Matrix factorizations for nonaffine LG models*, Math. Ann. **353** (2012), no. 1, 95–108.

50. A. Polishchuk and A. Vaintrob, *Matrix factorizations and singularity categories for stacks*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 7, 2609–2642.

51. A. Polishchuk and A. Vaintrob, *Matrix factorizations and cohomological field theories*, J. Reine Angew. Math. **714** (2016), 1–122.

52. L. Positselski, *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence*, Mem. Amer. Math. Soc. **212** (2011), no. 966 , 133 pp.

53. A. Preygel, *Thom–Sebastiani & duality for matrix factorizations*, arXiv:1101.5834.

54. R. Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256.

55. S. Schwede and B. Shipley, *Stable model categories are categories of modules*, Topology **42** (2003), no. 1, 103–153.

56. E. Segal, *Equivalence between GIT quotients of Landau–Ginzburg B-models*, Comm. Math. Phys. **304** (2011), no. 2, 411–432.
57. P. Seidel, *More about vanishing cycles and mutation*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 429–465.
58. I. Shipman, *A geometric approach to Orlov’s theorem*, Compos. Math. 148 (2012), no. 5, 1365–1389.
59. A. Takahashi, *Matrix factorizations and representations of quivers I*, arXiv:0506347.
60. A. Takahashi, *Homological mirror symmetry for isolated hypersurface singularities*, talk at “Workshop on homological Mirror Symmetry and Related Topics”, University of Miami, https://math.berkeley.edu/~auroux/frg/miami09-notes/, 2009.
61. A. Takahashi, *Weighted projective lines associated to regular systems of weights of dual type*, Adv. Stud. Pure Math. 59 (2010), 371–388.
62. R. W. Thomason, *Equivariant resolution, linearization, and Hilbert’s fourteenth problem over arbitrary base schemes*, Adv. Math. 65 (1987), 16–34.
63. B. Toën, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. 167 (2007), no. 3, 615–667.
64. U. Varolgunes, *Seifert form of chain type invertible singularities*, arXiv:2002.10684.
65. K. Yamaura, *Realizing stable categories as derived categories*, Adv. Math. 248 (2013), 784–819.