The Cauchy problem for the Pavlov equation with large data

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Abstract

The Pavlov equation is one of the simplest integrable systems of vector fields arising from various problems of mathematical physics and differential geometry which are intensively studied in recent literature. In this report, solving a nonlinear Riemann-Hilbert problem via a Newtonian iteration scheme, we complete the inverse scattering theory and prove a short time unique solvability of the Cauchy problem of the Pavlov equation with large initial data.

1 Introduction

Integrable dPDEs (dispersionless partial differential equations), including the dispersionless Kadomtsev-Petviashvili equation [11, 25, 28], the first and second heavenly equations of Plebanski [21], the dispersionless 2D Toda (or Boyer-Finley) equation [3, 8], and the Pavlov equation [6, 7, 23], are defined by a commutation $[L, M] = 0$ of pairs of one-parameter families of vector fields. They arise in various problems of mathematical physics and are intensively studied recently.

Due to lack of dispersion, integrable dPDEs may or may not exhibit a gradient catastrophe at finite time. Since the Lax operators are vector fields, the Fourier transform theory used in soliton theory for proving the existence of eigenfunctions fails and the inverse problem is intrinsically nonlinear for integrable dPDEs, unlike the $\bar{\partial}$-problem formulated for general soliton equations [1, 2]. At last, no explicit regular localized solutions, like solitons or lumps, exist for integrable dPDEs. Therefore, it is important to solve the inverse scattering problem for integrable dPDEs. A formal inverse scattering theory has been recently constructed, including i) to solve their Cauchy problem, ii) obtain the longtime behavior of solutions, iii) construct distinguished classes of exact implicit solutions, iv) establish if, due to the lack of dispersion, the nonlinearity of the PDE is “strong enough” to cause the gradient catastrophe of localized multidimensional disturbances, and v) to study analytically the breaking mechanism [12-22].
The Pavlov equation,

\[ v_{xt} + v_{yy} + v_{x}v_{xy} - v_{y}v_{xx} = 0, \]
\[ v = v(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}, \]

(1.1)
arising in the study of integrable hydrodynamic chains [23], and in differential geometry as a particular example of Einstein-Weyl metric [6], is the simplest integrable dPDE available in the literature [23], [7], [6]. It was first derived in [5] as a conformal symmetry of the second heavenly equation. In our previous work [10], we solve the forward problem via a Beltrami-type equation, a first order PDE, and a shifted Riemann-Hilbert problem and the inverse problem by a nonlinear integral equation under a small data constraint. More precisely, we justify the complex eigenfunction \( \Phi(x, z, \lambda) \), defined by

\[ \partial_y \Phi(x, y, \lambda) + (\lambda + v_x) \partial_x \Phi(x, y, \lambda) = 0, \quad x, y \in \mathbb{R}, \quad \lambda \in \mathbb{C}^\pm \]
\[ \Phi(x, y, \lambda) - (x - \lambda y) \to 0, \quad |x|, |y| \to \infty, \]

(1.2)
is link to the real eigenfunction \( \varphi(x, y, \lambda) \), defined by

\[ \partial_y \varphi + (\lambda + v_x) \partial_x \varphi = 0, \quad x, y \in \mathbb{R}, \quad \lambda \in \mathbb{R} \]
\[ \varphi - (x - \lambda y) \to 0, \quad y \to -\infty \]

(1.3)
by the equation

\[ \Phi^- (x, y, \lambda) = \varphi(x, y, \lambda) + \chi^- (\varphi(x, y, \lambda), \lambda) \]

(1.4)
where \( \chi^- (\xi, \lambda) \) satisfies a shifted Riemann-Hilbert problem

\[ \sigma(\xi, \lambda) + \chi^+ (\xi + \sigma(\xi, \lambda), \lambda) - \chi^- (\xi, \lambda) = 0, \quad \xi \in \mathbb{R}, \]
\[ \sigma(\xi, \lambda) = \lim_{y \to \infty} (\varphi(\xi, y, \lambda) - \xi). \]

(1.5)
Therefore, the inverse scattering theory reduces to deriving uniform \( \xi-, \lambda- \) asymptotic estimates of \( \chi(\xi, \lambda) \) in the direct problem, and solving the associated Riemann-Hilbert problem via the nonlinear integral equation

\[ \psi(x, y, t, \lambda) + \chi^-_R (\psi(x, y, t, \lambda), \lambda) = x - \lambda y - \lambda^2 t - \frac{1}{\pi} \int_R \frac{\chi_T (\psi(x, y, t, \zeta), \zeta)}{\zeta - \lambda} d\zeta \]

(1.6)
in the inverse problem. However, for technical reasons, we have to impose two strong assumptions, the compact support condition and small data constraint on the initial data \( v(x, y, 0) \), to make the above resolution scheme possible [10].
In this report, we aim to solving a large data problem. The difficulties are to find an effective scheme to solve the nonlinear inverse problem with large data and to establish a Fredholm alternative for the linearization. To do it, we transform the conjunction formula (1.7) to

$$\Phi^+(x, y, \lambda) = \Phi^-(x, y, \lambda) + R(\Phi^-(x, y, \lambda), \lambda).$$

and the inverse problem (1.6) to the nonlinear Riemann-Hilbert problem

$$\begin{align*}
\Psi^+(x, y, t, \lambda) &= \Psi^-(x, y, t, \lambda) + R(\Psi^-, \lambda), & \lambda \in \mathbb{R}, \\
\partial_t \Psi(x, y, t, \lambda) &= 0, & \lambda \in \mathbb{C}^+, \\
\Psi(x, y, t, \lambda) - (x - \lambda y - \lambda^2 t) &\to 0, & |\lambda| \to \infty,
\end{align*}$$

(1.8)

Furthermore, we adopt a Newtonian iteration approach to study (1.8). The linearization of the Newtonian process turns out to be a nonhomogeneous Riemann-Hilbert problem of which a Fredholm theory has been well investigated [9]. Consequently, an index zero condition on the linearization $1 + \partial_t R$, vital for a Fredholm theory, and a deformation property of $R(\Phi^-(x, y, \lambda), \lambda)$, needed in the Newtonian iteration scheme, should be justified to make the Newtonian iteration approach feasible.

The dispersion term $x - \lambda y - \lambda^2 t$ causes the estimates, necessary for the above Newtonian iteration approach, growing inevitably unbounded when $|y| \to \infty$ or $t$ gets larger. Precisely, the obstruction to the global solvability is the non existence of $R(\omega^- + x - \lambda y - \lambda^2 t, \lambda)$ when $t$ gets larger. Hence only a local solvability of the nonlinear Riemann-Hilbert problem is achieved in general. Moreover, without compact support constraints, the quadratic dispersion term destroys the $L^1(\mathbb{R}, d\lambda)$ property of $\partial R/\partial t$ as $t > 0$. So only a local solvability of the Lax pair of the Pavlov equation holds. To derive a local solvability of the Cauchy problem of the Pavlov equation, we still need to impose the compact support condition [10].

The contents of the paper are as follows. In Section 2 and 3, removing the compact support condition on the initial data $v(x, y)$ and using the $L^p(\mathbb{R}, d\xi)$ - Hilbert transform theory, we derive $\lambda$-asymptotics of the real eigenfunction $\varphi(x, y, \lambda)$ and the solution to the shifted Riemann-Hilbert problem $\chi(\xi, \lambda)$ via uniform $L^p(\mathbb{R}, d\xi)$ estimates on them. In Section 4, we derive the conjunction formula (1.7) between complex eigenfunctions $\Phi^\pm(x, y, \lambda)$ and define $R(\zeta, \lambda)$ as the spectral data. An index zero condition on $1 + \partial_t R$ and a deformation property of $R(\Phi^-(x, y, \lambda), \lambda)$ are justified as well. In Section 5 via a Newtonian iteration method and using the $L^p(\mathbb{R}, d\lambda)$ - Hilbert transform theory, we solve the short time unique existence of the nonlinear
Riemann-Hilbert problem (1.8) with prescribed initial data. The short time unique existence of the Lax pair and of the Cauchy problem of the Pavlov equation (1.1) are established in Section 6.

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2 The forward problem I: the real eigenfunctions

The forward problem of the Pavlov equation with compactly supported initial data has been solved in [10]. Removing the compact support restriction, we prove the existence of real eigenfunction \( \varphi(x, y, \lambda) \) and derive uniform estimates of scattering data \( \sigma(\xi, \lambda) \) in this section.

Throughout this report,

\[
\mathcal{S} = \{ f : \mathbb{R}^2 \to \mathbb{R} \mid f(x, y) \text{ is Schwartz in } x, y \},
\]

\[
L^p(\mathbb{R}, d\lambda) = \{ f : \mathbb{R} \to \mathbb{C} \mid |f|_{L^p(\mathbb{R}, d\lambda)} = \left( \int_{\mathbb{R}} |f(\lambda)|^p d\lambda \right)^{\frac{1}{p}} < \infty \},
\]

\[
H^p(\mathbb{R}, d\lambda) = \{ f : \mathbb{R} \to \mathbb{C} \mid f, \partial_\lambda f \in L^p(\mathbb{R}, d\lambda) \}.
\]

Consider, for each fixed \( \lambda \in \mathbb{R} \),

\[
\partial_y \varphi_\pm + (\lambda + v_x(x, y)) \partial_x \varphi_\pm = 0, \quad \text{for } x, y \in \mathbb{R},
\]

\[
\varphi_\pm(x, y, \lambda) - \xi \to 0, \quad \text{as } y \to \pm \infty,
\]

where \( \xi = x - \lambda y \). The solvability and uniqueness of the boundary value problem of the first order partial differential equation (2.1), (2.2) is shown by solving the ordinary differential equation (2.1), (2.2) is shown by solving the ordinary differential equation [10]

\[
\frac{dx}{dy} = \lambda + v_x(x, y), \quad x = x(y; x_0, y_0, \lambda), \quad x(y_0; x_0, y_0, \lambda) = x_0,
\]

or, equivalently,

\[
\frac{dh}{dy} = v_x(h + \lambda y, y),
\]

\[
h = h(y; \xi_0, y_0, \lambda) = x(y; x_0, y_0, \lambda) - \lambda y,
\]

\[
h(y_0; \xi_0, y_0, \lambda) = x_0 - \lambda y_0 = \xi_0.
\]
Hence

\[ h(y'; x - \lambda y, y, \lambda) = \xi + \int_y^{y'} v_x(h(y''; x - \lambda y, y, \lambda) + \lambda y'', y'') dy'', \quad (2.5) \]

\[ \varphi_\pm(x, y, \lambda) = h(\pm \infty; x - \lambda y, y, \lambda) \]

\[ = \xi + \int_{\pm \infty}^y v_x(h(y''; x - \lambda y, y, \lambda) + \lambda y'', y'') dy'', \quad (2.6) \]

and

\[ |\partial^2_y \partial^\alpha \partial_\xi^\beta (h(y'; \xi, y, \lambda) - \xi)| \leq C_{M, \alpha, \beta}. \quad (2.7) \]

The \textbf{real eigenfunction} of the Pavlov equation is defined by

\[ \varphi(x, y, \lambda) = \varphi_-(x, y, \lambda) \quad (2.8) \]

and the \textbf{scattering data} \( \sigma(\xi, \lambda) \) is defined as

\[ \varphi_+(x, y, \lambda) = \varphi_-(x, y, \lambda) + \sigma(\varphi_-(x, y, \lambda), \lambda), \quad (2.9) \]

where

\[ \sigma(\xi, \lambda) = h(\infty; \xi, -\infty, \lambda) - \xi = \int_{-\infty}^\infty v_x(h(y''; \xi, -\infty, \lambda) + \lambda y'', y'') dy''. \quad (2.10) \]

**Lemma 2.1.** Suppose \( v \in \mathcal{G} \). Then

\[ 0 < C_1 < 1 + \partial_\xi \sigma(\xi, \lambda) < C_2. \quad (2.11) \]

**Proof.** Comparing \( (2.5) \) and \( (2.10) \), to prove \( (2.11) \), it suffices to show that

\[ 0 < c_1 \leq \partial_x h(y'; x - \lambda y, y, \lambda) \leq c_2 \quad (2.12) \]

for two constants \( c_1 \) and \( c_2 \). Taking derivatives on \( (2.1) \), one obtains

\[ \frac{\partial}{\partial y'} \frac{\partial h(y'; x - \lambda y, y, \lambda)}{\partial x} = v_{xx}(h + \lambda y', y') \frac{\partial h(y'; x - \lambda y, y, \lambda)}{\partial x}, \quad (2.13) \]

\[ \frac{\partial h(y; x - \lambda y, y, \lambda)}{\partial x} = 1. \quad (2.14) \]

Therefore,

\[ -\int_{\mathbb{R}} |u_{xx}(h + \lambda y'', y'')| dy'' \leq \log \frac{\partial h(y'; x - \lambda y, y, \lambda)}{\partial x} \leq \int_{\mathbb{R}} |u_{xx}(h + \lambda y'', y'')| dy''. \]
and
\[ e^{-\int_R |u_{xx}(h+\lambda y'',y'')|dy''} \leq \frac{\partial h(y';x-\lambda y,y_\lambda)}{\partial x} \leq e^{\int_R |u_{xx}(h+\lambda y'',y'')|dy''} \] (2.15)

So
\[ 0 < C_1 \leq \partial_\xi h(y',\xi,y_\lambda) < C_2, \] (2.16)
by the Schwartz condition and (2.12) is proved.

**Proposition 1.** Suppose \( v \in \mathcal{S} \) and \( p \geq 1 \). Then the scattering data \( \sigma \) satisfies
\[ |\partial_\lambda \partial_\xi \sigma(\xi,\lambda)|_{L^\infty} \leq \frac{C}{1 + |\lambda|^{2+\mu+\nu}}, \] (2.17)
\[ |\partial_\lambda \partial_\xi \sigma(\xi,\lambda)|_{L^p(\xi)} \leq \frac{C}{1 + |\lambda|^{2+\mu+\nu} - \frac{1}{p}}. \] (2.18)

**Proof.** The method in the proof of Proposition 3.2 in [10] can be applied to the non-compact case as well. Indeed, let
\[ x = h(y';\xi,-\infty,\lambda) + \lambda y' = \xi + \lambda y' + \int_{-\infty}^{y'} v_x dy''. \] (2.19)
For \( \lambda \gg 1 \), from the implicit function theorem, there exist \( y' = H(\xi,x,\lambda) \) and \( H_1(\xi,x,\lambda) \) so that
\[ H(\xi,x,\lambda) = -\frac{\xi + x}{\lambda} + \frac{H_1(\xi,x,\lambda)}{\lambda^2}. \] (2.20)
So
\[ \sigma(\xi,\lambda) = -\frac{H_1(\xi,\infty,\lambda)}{\lambda}. \] (2.21)
From (2.19), one has
\[ \frac{\partial y'}{\partial x} = \frac{1}{\lambda + v_x(x,H(\xi,x,\lambda))}, \] (2.22)
or, equivalently
\[ \frac{\partial H_1(\xi,x,\lambda)}{\partial x} = -\frac{v_x(x,-\frac{\xi + x}{\lambda} + \frac{H_1}{\lambda^2})}{1 + v_x(x,-\frac{\xi + x}{\lambda} + \frac{H_1}{\lambda^2})/\lambda}, \] (2.23)
\[ H_1(\xi,-\infty,\lambda) = 0. \] (2.24)
Define
\[ \hat{\lambda} = \frac{1}{\lambda}, \quad \hat{\xi} = \frac{\xi}{\lambda}, \quad \hat{H}_1(\hat{\xi}, x, \hat{\lambda}) = H_1\left(\frac{\xi}{\lambda}, x, \frac{1}{\lambda}\right). \]

So (2.23) and (2.24) turn into
\[
\frac{\partial \hat{H}_1(\hat{\xi}, x, \hat{\lambda})}{\partial x} = -\frac{v_x(x, -\hat{\xi} + \hat{\lambda}x + \hat{\lambda}^2 \hat{H}_1(\hat{\xi}, x, \hat{\lambda}))}{1 + \lambda v_x(x, -\xi + \lambda x + \lambda^2 H_1(\xi, x, \lambda))},
\]
(2.25)
\[
\hat{H}_1(\hat{\xi}, -\infty, \hat{\lambda}) = 0.
\]
(2.26)

For \(|\hat{\lambda}| < \frac{1}{2 \max |v_x(x, y)|}\), the right hand side of (2.25) is smooth in \(\hat{\xi}, \hat{\lambda}\).

Expanding it at \(\lambda = 0\) and using the boundary condition (2.26), we obtain
\[
\hat{H}_1(\hat{\xi}, \infty, \hat{\lambda}) = -\int_{-\infty}^{\infty} v_x(x, -\hat{\xi}) dx + O(\hat{\lambda})
\]
(2.27)
\[
= O(\hat{\lambda})
\]
by the mean value theorem, fundamental theorem of calculus, and \(v \in \mathcal{G}\).

From the Hadamard’s lemma it follows
\[
\sigma(\hat{\xi}/\hat{\lambda}, 1/\hat{\lambda}) = -\frac{\hat{H}_1(\hat{\xi}, \infty, \hat{\lambda})}{\lambda^2}
\]
is regular in \(\hat{\xi}, \hat{\lambda}\) for \(|\hat{\lambda}| < \frac{1}{2 \max |v_x(x, y)|}\). Consequently, by
\[
\partial_\lambda = -\hat{\lambda}^2 \partial_\lambda - \hat{\lambda} \hat{\xi} \partial_\xi, \quad \partial_\xi = \hat{\partial}_\xi,
\]
(2.28)
we obtain
\[
|\partial^\mu_\xi \partial^\nu_\eta \sigma(\xi, \lambda)|_{L^\infty} \leq \frac{C}{1 + |\lambda|^{2+\mu+\nu}},
\]
(2.29)
if \(\lambda \gg 1\). By analogy, one can show (2.29) if \(\lambda \ll -1\). Thus (2.17) is proved. Moreover, (2.18) follows from (2.27), (2.28), and the Minkowski inequality. \(\Box\)
3 The forward problem II: the complex eigenfunctions

With the positivity property (2.11), one has the unique solvability of the shifted Riemann-Hilbert problem [9]

\[
\sigma(\xi, \lambda) + \chi^+(\xi + \sigma(\xi, \lambda), \lambda) - \chi^-(\xi, \lambda) = 0, \quad \xi \in \mathbb{R},
\]
\[
\partial_\xi \chi = 0, \quad \xi \in \mathbb{C}^\pm,
\]
\[
\chi \to 0, \quad |\xi| \to \infty, \quad \xi \in \mathbb{C}.
\] (3.1)

Applying (2.17), (2.18), and the boundedness of the Hilbert transform, we show that

**Proposition 2.** If \( v \in \mathcal{S} \) and \( p > 1 \), then the shifted Riemann-Hilbert problem (3.1) admits a unique bounded solution \( \chi \) satisfying

\[
|\partial_\nu^\mu \partial_\xi^\nu \chi^-(\xi, \lambda)|_{L^\infty} \leq \frac{C}{1 + |\lambda|^{2+\mu+\nu-p}}, \quad \forall \xi \in \mathbb{R}, \quad \forall \lambda \in \mathbb{R}, \quad (3.2)
\]
\[
|\partial_\nu^\mu \partial_\xi^\nu \chi(\xi, \lambda)|_{L^\infty} \leq \frac{C}{1 + |\lambda|^{2+\mu+\nu-p}}, \quad \forall \xi \in \mathbb{C}^\pm, \quad \forall \lambda \in \mathbb{R}. \quad (3.3)
\]

**Proof.** In [21], the unique solvability of the shifted Riemann-Hilbert problem (3.1) has been justified by converting it to the following linear equation

\[
\chi^-(\xi, \lambda) - \frac{1}{2\pi i} \int_\mathbb{R} f(\xi, \xi', \lambda) \chi^-(\xi', \lambda) d\xi' + g(\xi, \lambda) = 0, \quad (3.4)
\]

where

\[
f(\xi, \xi', \lambda) = \frac{\partial_\xi \sigma(\xi', \lambda)}{s(\xi', \lambda) - s(\xi, \lambda)} - \frac{1}{\xi' - \xi},
\]
\[
g(\xi, \lambda) = -\frac{1}{2} \sigma(\xi, \lambda) + \frac{1}{2\pi i} \int_\mathbb{R} \frac{\partial_\xi \sigma(\xi', \lambda)}{s(\xi', \lambda) - s(\xi, \lambda)} \sigma(\xi', \lambda) d\xi',
\]
\[
s(\xi, \lambda) = \xi + \sigma(\xi, \lambda).
\] (3.5)

Hence it yields to showing the uniform estimate (3.2) without the compactly supported condition.

Taking derivatives of (3.4) and applying Proposition 1, we obtain

\[
\partial_\nu^\mu \partial_\xi^\nu \chi^-(\xi, \lambda) = \frac{1}{2\pi i} \int_\mathbb{R} \frac{\partial_\xi (\partial_\xi + \partial_\xi')^\mu [k(\xi, \xi', \lambda) \chi^-(\xi', \lambda)]}{\xi' - \xi} d\xi'. \quad (3.6)
\]

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\[-\frac{1}{2\pi i} \int_{\mathbb{R}} \partial_{\xi}^\nu (\partial_{\xi'} + \partial_{\xi'})^\mu \frac{[k(\xi, \xi', \lambda)\sigma(\xi', \lambda)]}{\xi' - \xi} d\xi' + \frac{1}{2\pi i} \int_{\mathbb{R}} \partial_{\lambda}^\nu \partial_{\xi}^\mu \sigma(\xi', \lambda) \frac{\xi' - \xi}{\xi - \xi} d\xi' + \frac{1}{2} \partial_{\lambda}^\nu \partial_{\xi}^\mu \sigma \]

where

\[k(\xi, \xi', \lambda) = \frac{s(\xi, \lambda) - s(\xi', \lambda) - \partial_{\xi'} s(\xi', \lambda)(\xi - \xi')}{s(\xi, \lambda) - s(\xi', \lambda)}. \tag{3.7}\]

Applying Proposition 1, Hadamard’s lemma, and an induction argument, one can derive

\[|\partial_{\lambda}^\nu \partial_{\xi}^\mu k(\xi, \xi', \lambda)|_{L^\infty} \leq \frac{C}{1 + |\lambda|^{2\mu+\nu - \frac{1}{p}}}, \tag{3.8}\]

and then

\[|\partial_{\lambda}^\nu (\partial_{\xi} + \partial_{\xi'})^\mu k\chi^- - k\partial_{\lambda}^\nu \partial_{\xi}^\mu \chi^-|_{L^p(\mathbb{R}, d\xi)} \leq \frac{C}{1 + |\lambda|^{2\mu+\nu - \frac{1}{p}}}, \tag{3.9}\]

Plugging (3.9) into (3.6), we obtain

\[|\partial_{\lambda}^\nu \partial_{\xi}^\mu \chi^- - \mathcal{K}(\partial_{\lambda}^\nu \partial_{\xi}^\mu \chi^-)|_{L^p(\mathbb{R}, d\xi)} \leq \frac{C}{1 + |\lambda|^{2\mu+\nu - \frac{1}{p}}}, \tag{3.10}\]

where

\[\mathcal{K}\psi(\xi, \lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} f(\xi, \xi', \lambda)\psi(\xi', \lambda) d\xi'.\]

Hence the boundedness of the Hilbert transform on \(L^p\) for \(p > 1\),

\[|\partial_{\lambda}^\nu \partial_{\xi}^\mu \chi^-(\xi, \lambda)|_{L^p(\mathbb{R}, d\xi)} \leq \frac{C}{1 + |\lambda|^{2\mu+\nu - \frac{1}{p}}}, \quad |\lambda| \gg 1. \tag{3.11}\]

So

\[|\partial_{\lambda}^\nu \partial_{\xi}^\mu \chi^-(\xi, \lambda)|_{L^\infty} \leq \frac{C}{1 + |\lambda|^{2\mu+\nu - \frac{1}{p}}}, \quad |\lambda| \gg 1, \tag{3.12}\]

from the Sobolev imbedding theorem. Finally (3.12) is achieved for \(\forall \lambda \in \mathbb{R}\) by continuity. \(\square\)
Theorem 1. If \( v \in \mathcal{S} \), then for fixed \( x, y \in \mathbb{R} \), there exists a unique continuous \( \Phi(x, y, \lambda) \) such that \( \Phi(x, y, \lambda) \) is holomorphic for \( \lambda \in \mathbb{C}^\pm \) and has continuous limits on both sides of \( \lambda \in \mathbb{R} \) satisfying

\[
\Phi^-(x, y, \lambda) = \varphi(x, y, \lambda) + \chi^-(\varphi(x, y, \lambda), \lambda), \quad (3.12)
\]

\[
\Phi^+(x, y, \lambda) = \overline{\Phi^-(x, y, \lambda)}. \quad (3.13)
\]

Moreover, \( \Phi(x, y, \lambda) \) is a complex eigenfunction, i.e. for \( \forall \lambda \in \mathbb{C}^\pm \) fixed,

\[
(\partial_y + (\lambda + v_x)) \partial_x \Phi = 0, \quad \text{for } \forall x, y \in \mathbb{R},
\]

and

\[
\Phi(x, y, \lambda) = \Phi(x, y, \lambda), \quad (3.15)
\]

\[
|\partial^\nu_x \partial^k_y (\Phi^\pm - x + \lambda y)|_{L^\infty} \leq \frac{C}{1 + |\lambda|^{2+\nu+k-\frac{1}{p}}}. \quad (3.16)
\]

Proof. We refer to Theorem 3.1 in [10] for a detailed proof of this theorem except for (3.12). Note that the whole proof there does not require any small data assumption nor a compact support condition.

Moreover, the conjunction formula (3.12) for complex and real eigenfunctions has been justified by changes of variables in the proof of Theorem 3.2 in [10] under a compact support restriction. However, the same proof can be refined and holds for \( v(x, y) \in \mathcal{G} \) as well. Indeed, denote \( \lambda = \lambda_R + i\lambda_I \), the first change of variables \( F_1 : (x, y) \rightarrow (x_1, y_1) \) is defined by

\[
\begin{align*}
x_1 &= \lim_{y' \rightarrow -\infty} h(y'; x - \lambda_R y, y, \lambda_R) = \varphi_-(x, y, \lambda_R), \quad y < 0 \\
x_1 &= \lim_{y' \rightarrow +\infty} h(y'; x - \lambda_R y, y, \lambda_R) = \varphi_+(x, y, \lambda_R), \quad y > 0 \\
y_1 &= y
\end{align*}
\]

(3.17)

In the new variables,

\[
L = \partial_y + (\lambda + v_x) \partial_x = \partial_{y_1} + i \lambda_I \kappa(x_1, y_1) \partial_{x_1},
\]

where

\[
\kappa(x_1, y_1) = \frac{\partial \varphi_\pm(x, y, \lambda_R)}{\partial x}|_{(x, y) = F_1^{-1}(x_1, y_1)} = \neq 0.
\]

(3.19)

By (2.15), there exists a pair of positive constants \( C_1, C_2 \) such that:

\[
0 < C_1 \leq \kappa(x_1, y_1) \leq C_2.
\]

(3.20)
Secondly, taking $\lambda_I < 0$ from now on without loss of generality, we introduce $F_2 : (x_1, y_1) \to (x_2, y_2)$

$$\begin{aligned}
x_2 &= x_1, \\
y_2 &= \lambda_I y_1, \\
z_2 &= x_2 - iy_2
\end{aligned} \quad (3.21)$$

In the new variables, (3.18) turns into

$$L = \lambda_I (\partial_{y_2} + \kappa \left( x_2, \frac{y_2}{\lambda_I} \right) \partial_{x_2})$$

$$= \lambda_I \left[ 1 + \kappa(x_2, \frac{y_2}{\lambda_I}) \right] \left( \partial_{x_2} + r(z_3, \bar{z}_3, \lambda) \partial_{z_2} \right) \quad (3.22)$$

where

$$r(z_2, \bar{z}_2, \lambda) = \frac{-1 + \kappa(x_2, \frac{y_2}{\lambda_I})}{1 + \kappa(x_2, \frac{y_2}{\lambda_I})} |_{(x_2, y_2) = F_3^{-1}(z_3, \bar{z}_3)} < 1 \quad (3.23)$$

by (2.15).

The last change of variables is $F_3 : (x_2, y_2) \to z_3$, defined by

$$z_3 = x_2 - iy_2 + \chi(x_2 - iy_2, \lambda_R), \quad (3.24)$$

where $\chi(\xi, \lambda_R)$ is the solution of the shifted Riemann-Hilbert problem (3.1).

Consequently, (3.22) or (3.14) takes the form

$$[\partial_{\bar{z}_3} + q(z_3, \bar{z}_3, \lambda) \partial_{z_3}] \Phi = 0 \quad (3.25)$$

where

$$|q(z_3, \bar{z}_3, \lambda)| = \left| \frac{1 + \partial \chi / \partial z_2}{1 + \partial \chi / \partial z_2} \left| \frac{-1 + \kappa(x_2, \frac{y_2}{\lambda_I})}{1 + \kappa(x_2, \frac{y_2}{\lambda_I})} \right|_{(x_2, y_2) = F_3^{-1}(z_3, \bar{z}_3)} \
\quad - \frac{-1 + \kappa(x_2, \frac{y_2}{\lambda_I})}{1 + \kappa(x_2, \frac{y_2}{\lambda_I})} \right|_{(x_2, y_2) = F_3^{-1}(z_3, \bar{z}_3)} \right| \leq 1 \quad (3.26)$$

by solvability of (3.1) and (3.23). It is then natural to consider Beltrami equation (3.25) in the space $L^{2+\epsilon}(dz_3 d\bar{z}_3) \cap L^{2-\epsilon}(dz_3 d\bar{z}_3)$ where $\epsilon$ is sufficiently small. We can write (26)

$$\Phi(z_3, \bar{z}_3, \lambda) = z_3 + \partial_{\bar{z}_3}^{-1} \alpha(z_3, \bar{z}_3, \lambda) \quad (3.27)$$
where
\[ [1 + q(z_3, \bar{z}_3, \lambda) \partial_{z_3} \partial_{\bar{z}_3}^{-1}] \alpha(z_3, \bar{z}_3, \lambda) + q(z_3, \bar{z}_3, \lambda) = 0. \] (3.28)

Taking into account (2.15), (3.19), (3.24), (3.26), \( v \in \mathfrak{G} \), Proposition 2, and the Fubini theorem, we see that
\[ |q(z_3, \bar{z}_3, \lambda)|_{L^p}^p < |1 + \kappa(x_2, \frac{y_2}{\lambda})|_{(x_2, y_2) = \mathcal{F}_3^{-1}(z_3, \bar{z}_3)}^p \]
\[ < \int \int d z_{3, R} \int d z_{3, R} \]
\[ < \int d z_{3, I} \int d z_{3, R} \int d y_2 \int R \frac{C dy''}{1 + |y''|^{p'}} \int R \frac{1 + |x - \lambda|^{p}}{1 + |y_2 - y''|^{p'}} (x_2, y_2) = \mathcal{F}_3^{-1}(z_3, \bar{z}_3) \]
\[ < C'' \int dy_2 \int d y_2 \int R \frac{dy''}{1 + |y''|^{p'}} \int R \frac{1 + |x - \lambda|^{p}}{1 + |y_2 - y''|^{p'}} \]
\[ = \mathcal{O}(|\lambda_I|), \] (3.29)
and hence
\[ |\alpha(z_3, \bar{z}_3, \lambda)|_{L^p} = \mathcal{O}(|\lambda_I|^{\frac{1}{p}}), \quad |p - 2| < \varepsilon. \] (3.30)

Using the estimates from \[ 26 \] we see, that
\[ \| \Phi(z_3, \bar{z}_3, \lambda) - z_3 \|_{L^\infty(d z_3 d \bar{z}_3)} = \mathcal{O}(|\lambda_I|^{\frac{1}{p}}), \]
and \( \Phi(z_3, \bar{z}_3, \lambda) \) uniformly converges to \( z_3 \). So (3.12) is obtained by composing \( \mathcal{F}_j, j = 1, 2, 3 \) and taking \( \lambda_I \to 0. \)

4 The forward problem III: the nonlinear Riemann Hilbert problem

Theorem 1 yields
\[ \Phi^+(x, y, \lambda) - \Phi^-(x, y, \lambda) = -2i\chi \phi(x, y, \lambda), \quad \lambda \in \mathbb{R}, \] (4.1)
where \( \chi = \chi_R + \chi_I \). So the Plemelji formula and (3.14) imply
\[ \Phi(x, y, \lambda) = x - \lambda y - \frac{1}{\pi} \int_R \frac{\chi \phi(x, y, \zeta)}{\zeta - \lambda} d\zeta, \] (4.2)
\[ v(x, y) = -\frac{1}{\pi} \int_{\mathbb{R}} \chi_f^- (\varphi(x, y, \lambda), \lambda) d\lambda. \] (4.3)

From (3.13) and (4.2), we then derive the non linear integral equation

\[ \varphi(x, y, \lambda) + \chi_R^- (\varphi(x, y, \lambda), \lambda) = x - \lambda y - \frac{1}{\pi} \int_{\mathbb{R}} \frac{\chi_f^- (\varphi(x, y, \zeta), \zeta)}{\zeta - \lambda} d\zeta. \] (4.4)

In [10], under a small data assumption, (4.4) plays a key role in solving the inverse problem of the Pavlov equation. However, without small data constraints, it is difficult to study the inverse problem via (4.4) since a Fredholm alternative theorem for the linearization is hard to justify.

On the other hand, it is more direct to adopt the nonlinear Riemann-Hilbert approach [15]-[19] because the associated linearized operator is a standard non homogeneous Riemann-Hilbert problem (Section 5) with which the Fredholm alternative theory has been well understood [9]. Therefore, we will transform the non linear integral equation (4.4) to a nonlinear Riemann-Hilbert problem (4.19) in this section.

**Lemma 4.1.** If \( v \in \mathcal{G} \) is compactly supported in \( y \), then there exists a constant \( C_{\lambda, k} \), determined by \( \lambda \), \( k \), such that

\[ \left| \frac{\partial \sigma}{\partial \xi} (\xi, \lambda) \right| \leq \frac{C_{\lambda, k}}{1 + |\xi|^k}. \] (4.5)

**Proof.** From (2.5) and (2.16), there exist uniform constants \( C_i \)

\[ 0 < C_1 \left( |\xi| + 1 \right) \leq |h(y'; \xi, -\infty, \lambda)| \leq C_2 \left( |\xi| + 1 \right), \]
\[ 0 < C_1 \leq |\frac{\partial h}{\partial \xi} (y'; \xi, -\infty, \lambda)| \leq C_2 \] (4.6)

For each fixed \( \lambda \in \mathbb{R} \), plugging (4.6) into (2.10), and using (2.7), \( v \in \mathcal{G}, v \) is compactly supported in \( y \), we obtain

\[ \left| \frac{\partial \sigma}{\partial \xi} (\xi, \lambda) \right| = C \int_{-\infty}^{\infty} |v_{xx} (h(y''; \xi, -\infty, \lambda) + \lambda y'', y'')| dy'' \]
\[ \leq \frac{C_{\lambda, k}}{1 + |\xi|^k}. \]

\[ \square \]

**Lemma 4.2.** If \( v \in \mathcal{G} \) is compactly supported in \( y \), then for each fixed \( \lambda \in \mathbb{R} \), the map

\[ \Xi : \mathbb{R} \to \partial \mathcal{O} \subset \mathbb{C}, \]
\[ \Xi (\xi) = \xi + \chi^- (\xi, \lambda), \] (4.7)
diffeomorphically maps $\mathbb{R}$ to its image, denoted as $\partial \mathfrak{D} = \partial \mathfrak{D}(\lambda)$. Namely, $\Xi$ has an inverse, denoted as $\mathcal{M}(\xi, \lambda)$, satisfying
\begin{equation}
\mathcal{M}(\xi + \chi^-(\xi, \lambda), \lambda) = \xi, \quad \forall \xi, \forall \lambda \in \mathbb{R}.
\tag{4.8}
\end{equation}
Moreover, denote $\overline{\mathfrak{D}} = \partial \mathfrak{D}(\lambda) \cup \{\xi + \sigma(\xi, \lambda)\mid \xi \in \mathbb{C}^-\} = \{\xi + \chi(\xi, \lambda)\mid \xi \in \mathbb{C}^-\}$. Then there exist a uniform constant $\hbar > 0$, $\tilde{\mathfrak{D}} = \tilde{\mathfrak{D}}(\lambda)$, $\tilde{\mathcal{U}}(\lambda)$, $\tilde{\chi}(\xi, \lambda)$, and $\tilde{\mathcal{M}}(\zeta, \lambda)$, such that
\begin{enumerate}
\item $\xi + \tilde{\chi}(\xi, \lambda) : \tilde{\mathcal{U}} \times \mathbb{R} \to \tilde{\mathfrak{D}}$, $\overline{\mathfrak{D}} \subset \tilde{\mathcal{U}} \subset \mathfrak{D}$, $\mathfrak{D} \subset \tilde{\mathfrak{D}} \subset \mathfrak{C}$
\item distance of $\partial \mathfrak{D}(\lambda)$ and $\partial \tilde{\mathfrak{D}}(\lambda)$ is larger than $\hbar$ for $\forall \lambda \in \mathbb{R}$,
\item $\tilde{\chi}(\xi, \lambda)$ is holomorphic in $\xi \in \tilde{\mathcal{U}}$,
\item the inverse $(I + \chi)^{-1}(\zeta, \lambda)$ on $\mathfrak{D}$ has a holomorphic extension $\tilde{\mathcal{M}}(\zeta, \lambda) = (I + \tilde{\chi})^{-1}(\zeta, \lambda)$ on $\tilde{\mathfrak{D}}$.
\end{enumerate}
\begin{equation}
\tag{4.9}
\end{equation}
and
\begin{align*}
|\partial^\mu \chi^\nu \tilde{\chi}(\xi, \lambda)|_{L^\infty} & \leq \frac{C}{1 + |\lambda|^{2 + \mu + \nu - \frac{1}{p}}}, \forall \xi \in \tilde{\mathcal{U}}, \forall \lambda \in \mathbb{R}, \mu + \nu \geq 0; \\
|\partial^\mu \chi^\nu \tilde{\mathcal{M}}(\zeta, \lambda)|_{L^\infty} & \leq \frac{C}{1 + |\lambda|^{\max(0, \mu + \nu - 1) - \frac{1}{p}}}, \forall \zeta \in \tilde{\mathfrak{D}}, \forall \lambda \in \mathbb{R}, \mu + \nu > 0.
\tag{4.10}
\end{align*}
Proof. Note $\chi$ is the solution of the shifted Riemann-Hilbert problem \cite{9},
\begin{equation}
\xi + \sigma(\xi, \lambda) + \chi^+(\xi + \sigma(\xi, \lambda), \lambda) = \xi + \chi^-(\xi, \lambda).
\tag{4.11}
\end{equation}
Differentiating both sides of (4.11) with respect to $\xi$, one obtains a Riemann-Hilbert problem
\begin{equation}
\left(1 + \frac{\partial \chi}{\partial \xi}\right)^+ \left(1 + \frac{\partial \sigma}{\partial \xi}\right) = \left(1 + \frac{\partial \chi}{\partial \xi}\right)^-.
\tag{4.12}
\end{equation}
From Lemma \cite{2} $\log(1 + \frac{\partial \sigma}{\partial \xi})$ is well-defined. Combining with Lemma \cite{4} and Proposition \cite{2} one then derives \cite{9}
\begin{equation}
1 + \frac{\partial \chi}{\partial \xi}(\xi, \lambda) = \exp\left(-\frac{1}{2\pi i} \int_\mathbb{R} \frac{\log(1 + \frac{\partial \sigma}{\partial \xi})}{t - \xi} dt\right).
\tag{4.13}
\end{equation}
So for $\xi \in \mathbb{R}$,

$$1 + \frac{\partial \chi^-}{\partial \xi} (\xi, \lambda) = \exp \left( -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log(1 + \frac{\partial \sigma}{\partial \xi})}{t - \xi} dt + \frac{1}{2} \log(1 + \frac{\partial \sigma}{\partial \xi}) \right). \quad (4.14)$$

Therefore,

$$\left( 1 + \frac{\partial \chi^-}{\partial \xi} (\xi, \lambda) \right)^2 + \left( \frac{\partial \chi^-}{\partial \xi} (\xi, \lambda) \right)^2 \neq 0, \quad \text{for all } \xi, \lambda \in \mathbb{R}. \quad (4.15)$$

Moreover, applying Proposition 2, (4.14), and (4.15), we have

$$0 < C_1 \leq \left( 1 + \frac{\partial \chi^-}{\partial \xi} (\xi, \lambda) \right)^2 + \left( \frac{\partial \chi^-}{\partial \xi} (\xi, \lambda) \right)^2 \leq C_2, \quad \text{for all } \xi, \lambda \in \mathbb{R}. \quad (4.16)$$

So $\Xi$ maps $\mathbb{R}$ to its image $\partial \mathcal{D}$ diffeomorphically. Combining with Proposition 2 we conclude that, for $\forall \lambda \in \mathbb{R}$, $\xi + \chi(\xi, \lambda)$ is a degree one holomorphic map from $\xi \in \mathbb{C}^-$ onto its image $\mathcal{D} = \mathcal{D}(\lambda)$. So the existence of $h > 0$, $\hat{\mathcal{D}} = \hat{\mathcal{D}}(\lambda)$, $\hat{\mu} = \hat{\mu}(\lambda)$, $\hat{\chi}(\xi, \lambda)$, and $\mathcal{M}(\zeta, \lambda)$, satisfying (4.19) and (4.10), follows from the inverse function theorem.

**Theorem 2.** Suppose $v \in \mathcal{S}$ is compactly supported in $y$. There exists a function

$$R(\zeta, \lambda) : \tilde{\mathcal{D}}(\lambda) \times \mathbb{R} \to \mathbb{C},$$

$$R(\zeta, \lambda) \text{ is holomorphic in } \zeta \in \tilde{\mathcal{D}},$$

with $\tilde{\mathcal{D}}$ defined by (4.9), satisfying the algebraic constraint

$$\mathcal{R}(\zeta, \lambda) = \mathcal{R}(\mathcal{R}(\zeta, \lambda), \lambda),$$

$$\mathcal{R}(\zeta, \lambda) = \zeta + R(\zeta, \lambda), \quad (4.17)$$

and the analytical constraint

$$|\partial_\zeta^{\mu} \partial_\lambda^\nu R(\zeta, \lambda)|_{L^\infty} \leq \frac{C}{1 + |\lambda|^{2+\mu+\nu-\frac{1}{p}}}, \quad (4.18)$$

such that the nonlinear Riemann-Hilbert problem

$$\Phi^+(x, y, \lambda) = \Phi^-(x, y, \lambda) + R(\Phi^-(x, y, \lambda), \lambda), \quad \lambda \in \mathbb{R},$$

$$\partial_\lambda \Phi(x, y, \lambda) = 0, \quad \lambda \in \mathbb{C}^\pm, \quad (4.19)$$

$$\Phi(x, y, \lambda) - (x - \lambda y) \to 0, \quad |\lambda| \to \infty$$
holds. Moreover, the representation formula for the complex eigenfunction and the potential are

\[
\Phi(x, y, \lambda) = x - \lambda y + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{R(\Phi^-(x, y, \lambda'), \lambda')}{\lambda' - \lambda} d\lambda',
\]

\[
v(x, y) = \frac{1}{2\pi i} \int_{\mathbb{R}} R(\Phi^-(x, y, \lambda), \lambda) d\lambda.
\]

**Proof.** Via (3.12), (4.8), one has

\[
\phi(x, y, \lambda) = M(\Phi^-(x, y, \lambda), \lambda).
\]

Moreover, if we define

\[
R(\zeta, \lambda) = -2i\tilde{\chi}_{I}(\tilde{M}(\zeta, \lambda), \lambda) \quad \zeta \in \tilde{D}, \lambda \in \mathbb{R}
\]

with \(\tilde{D}, \tilde{\chi} = \chi_R + i\chi_I,\) and \(\tilde{M}\) defined by (4.9), then Theorem 1, (4.1) imply (4.19). Therefore, the analytical constraint (4.18) can be justified by (4.10) and (4.22). Besides, from (4.19) and the reality condition (3.15), we have

\[
\Phi^+(x, y, \lambda)
= R(\Phi^-(x, y, \lambda), \lambda)
= R(R(\Phi^-(x, y, \lambda), \lambda), \lambda)
= R(R(R(\Phi^-(x, y, \lambda), \lambda), \lambda), \lambda), \quad \forall \lambda \in \mathbb{R}
\]

Thus the algebraic constraint (4.17) can be derived from the holomorphy of \(R(\zeta, \lambda)\) (from that of \(\tilde{\chi}(\cdot, \lambda), \tilde{M}(\cdot, \lambda)\) and (4.23).

Equations (4.20) is immediate from the Plemelj formula and the jump formula (4.19). Equation (4.21) can be derived from

\[
0 = [\partial_y + (\lambda + v_x(x, y))\partial_x] \Phi(x, y, \lambda)
= v_x(x, y) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(\partial_y + (\lambda' + v_x(x, y))\partial_x)R(\Phi^-(x, y, \lambda'), \lambda')}{\lambda' - \lambda} d\lambda'
- \frac{1}{2\pi i} \left( \partial_x \int_{\mathbb{R}} R(\Phi^-(x, y, \lambda'), \lambda') d\lambda' \right)
= v_x(x, y) - \frac{1}{2\pi i} \left( \partial_x \int_{\mathbb{R}} R(\Phi^-(x, y, \lambda'), \lambda') d\lambda' \right).
\]
Definition 1. If \( v \in S \) is compactly supported in \( y \), then the spectral data of \( v(x,y) \) is defined by

\[
R(\zeta, \lambda) = -2i\tilde{\chi}_I(\tilde{M}(\zeta, \lambda), \lambda), \quad \zeta \in \tilde{D}, \lambda \in \mathbb{R},
\]

where \( \tilde{\chi} = \tilde{\chi}_R + i\tilde{\chi}_I \), \( \tilde{M} \), and \( \tilde{D} \) are defined by (4.9).

Remark 1. From (4.19) and (4.24), the spectral data for the Pavlov equation is purely continuous and nonlinear, unlike the \( \bar{\partial} \)-problem formulated for general soliton equations [1], [2], [27].

Remark 2. The holomorphy condition on the spectral data \( R(\cdot, \lambda) \) is crucial for the reality condition (4.17) and is indispensable for the nonlinear Riemann-Hilbert approach in Section 6.

Proposition 3. Suppose \( 0 \neq v \in S \) is compactly supported in \( y \). There exist uniform constants \( \epsilon_0, \delta_0 \),

\[
0 < \epsilon_0, \delta_0 < \infty,
\]

such that if

\[
|\omega^{-}(x,y,t,\lambda) - (\Phi^{-}(x,y,\lambda) - x + \lambda y)|_{H^p(\mathbb{R}, d\lambda)} \leq \delta_0,
\]

\[
0 \leq t \leq \epsilon_0,
\]

then

\[
\omega^{-}(x,y,t,\lambda) + x - \lambda y - \lambda^2 t \in \tilde{D}(\lambda),
\]

\[
\forall \lambda \in \mathbb{R}, \quad 0 \leq t \leq \epsilon_0.
\]

Proof. Let \( h \) be defined by (4.9). Proposition 2 implies there exists \( N \) such that

\[
\sup_{\zeta \in \partial D} |\zeta_I| \leq \frac{h}{4}, \quad \text{for} \forall |\lambda| > N.
\]

Denote \( \Im(\zeta) = \zeta_I \). Hence (4.26) and (4.27) imply, for \( \lambda \in \mathbb{R}, |\lambda| > N,

\[
|\Im \left( \omega^{-}(x,y,t,\lambda) + x - \lambda y - \lambda^2 t \right) |
\]

\[
= |\Im \left( \omega^{-}(x,y,t,\lambda) + x - \lambda y \right) |
\]

\[
\leq |\Im \left[ \omega^{-}(x,y,t,\lambda) - (\Phi^{-}(x,y,\lambda) - (x - \lambda y)) \right] |
\]

\[
+ |\Im (\Phi^{-}(x,y,\lambda))| |
\]

\[
= |\Im \left[ \omega^{-}(x,y,t,\lambda) - (\Phi^{-}(x,y,\lambda) - (x - \lambda y)) \right] |
\]

\[
+ |\Im (\Phi^{-}(x,y,\lambda))| |
\]

\[
\leq |\Im \left[ \omega^{-}(x,y,t,\lambda) - (\Phi^{-}(x,y,\lambda) - (x - \lambda y)) \right] |
\]

\[
+ \sup_{\zeta \in \partial D} |\zeta_I|
\]

\[
\leq C \delta_0 + \frac{h}{4}
\]

(4.28)
Here $C$ is a uniform constant determined by Sobolev’s theorem. So if

$$\delta_0 = \frac{\hbar}{2C},$$

then (4.29), (4.27), and (4.28) yield

$$\omega^-(x, y, t, \lambda) + x - \lambda y - \lambda^2 t \in \tilde{\mathcal{D}},$$

for $t \leq \epsilon_0$, $\forall \lambda \in \mathbb{R}$, $|\lambda| > N$. (4.29)

On the other hand, for $|\lambda| \leq N$. Since

$$\Phi^-(x, y, \lambda) = \varphi(x, y, \lambda) + \chi^-(\varphi(x, y, \lambda), \lambda) \subset \partial \mathcal{D}.$$ 

We have, if

$$|\omega^-(x, y, t, \lambda) + x - \lambda y - \lambda^2 t - \Phi^-(x, y, \lambda)|_{L^\infty} \leq \hbar,$$ (4.30)

then

$$\omega^-(x, y, t, \lambda) + x - \lambda y - \lambda^2 t \in \tilde{\mathcal{D}}$$ (4.31)

by (4.9). Condition (4.30) can be assured by (4.26), $0 \leq t \leq \epsilon_0$, $|\lambda| \leq N$, and

$$\epsilon_0 = \frac{\hbar}{2N^2}.$$ 

Finally, $0 < \epsilon_0$, $\delta_0 < \infty$ follows from $0 < \hbar < \infty$ which is implied by Proposition 2 (4.16), $v \neq 0$, and Liouville’s theorem. \hfill $\square$

The following lemma will be used to compute the index of the linearized Riemann-Hilbert problem [9] for the inverse problem.

**Lemma 4.3.** Suppose $v \in \mathcal{S}$ and is compactly supported in $y$. Then

$$\text{Ind} \left( 1 + \frac{\partial R}{\partial \zeta}(\Phi^-(x, y, \lambda), \lambda) \right)$$

$$\equiv \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial \lambda \partial \zeta R(\Phi^-(x, y, \lambda), \lambda)}{1 + \frac{\partial R}{\partial \zeta}(\Phi^-(x, y, \lambda), \lambda)} d\lambda$$ (4.32)

$$= 0.$$
Proof. A direct computation yields

\[
|1 + \frac{\partial R}{\partial \zeta}(\Phi^-(x, y, \lambda), \lambda)|
\]

\[
= |1 - 2i\partial_\zeta \left[\chi^- \left(\mathbf{I} + \chi^-\right)^{-1}\right](\Phi^-(x, y, \lambda), \lambda)|
\]

\[
= |1 - 2i\partial_\xi \frac{\partial_\xi}{\partial \zeta}(\Phi^-(x, y, \lambda), \lambda)|
\]

\[
= |1 - \frac{2i\partial_\xi}{1 + \partial_\xi}(\varphi(x, y, \lambda), \lambda)|
\]

\[
= \frac{1 + \partial_\xi}{1 + \partial_\xi}(\varphi(x, y, \lambda), \lambda)|
\]

\[= 1. \tag{4.33}\]

Therefore \(\text{Ind} \left(1 + \frac{\partial R}{\partial \zeta}(\Phi^-(x, y, \lambda), \lambda)\right)\) is well defined for \(x, y, \lambda \in \mathbb{R}\).

Furthermore, noting that the index function \([9]\) is continuous and integer valued and using (4.33), it suffices to show

\[
|\partial_\lambda \partial_\zeta R(\Phi^-(x, y, \lambda), \lambda)|_{L^1(\mathbb{R}; d\lambda)} \ll 1, \text{ for } y = 0, \text{ and } x \to \infty. \tag{4.34}\]

To verify (4.34), we first note

\[
\partial_\lambda \partial_\zeta R(\Phi^-(x, y, \lambda), \lambda) \text{ are uniformly bounded by an } L^1(\mathbb{R}, d\lambda) \text{ function.} \tag{4.35}\]

from (4.18). Moreover, for arbitrary positive constant \(M, |\lambda| < M\), as \(x \to \infty\),

\[
|\partial_\lambda \partial_\zeta R(\Phi^-(x, 0, \lambda), \lambda)|
\]

\[
\leq C|\partial_\lambda \partial_\zeta \chi^- \varphi(x, 0, \lambda), \lambda)|
\]

\[
= C|\partial_\lambda \partial_\zeta \chi^- (x + \varphi_0(x, 0, \lambda), \lambda)|
\]

\[\to 0 \tag{4.36}\]

by (2.5), (2.6), Lemma 4.2 and Proposition 2. Therefore (4.34) is justified by (4.35), (4.36), the Lebesque’s Dominated Convergence Theorem, Proposition 2 and taking \(M\) sufficiently large.

The following lemma is from [10] (cf. Proposition 3.5 and its proof therein) and will be used in Theorem 5.
Lemma 4.4. Suppose \( v \in \mathfrak{S} \) and is compactly supported. Then for fixed \( t > 0 \) and \( \omega(x, y, t, \lambda) \in L^\infty(\mathbb{R}, d\lambda) \), as \( \lambda \to \infty \),
\[
|\partial_\xi^\mu \chi^- (\omega(x, y, t, \lambda) + x - \lambda y - \lambda^2 t, \lambda)| \leq \mathcal{O} \left( \frac{1}{1 + |\lambda|^{2\mu + 3 - \frac{1}{p}}} \right). \quad (4.37)
\]

5 The inverse problem I: the nonlinear Riemann-Hilbert problem

We turn to the inverse problem of the Pavlov equation. Recall that for \( t = 0 \), we have
\[
\Phi^+(x, y, \lambda) = \Phi^-(x, y, \lambda) + R(\Phi^-(x, y, \lambda), \lambda)
\]
\[
\equiv \mathcal{R}(\Phi^-(x, y, \lambda), \lambda). \quad (5.1)
\]

Our goal is to solve the nonlinear Riemann-Hilbert problem
\[
\Psi^+(x, y, t, \lambda) = \mathcal{R}(\Psi^-(x, y, t, \lambda), \lambda), \quad \lambda \in \mathbb{R}, \ t > 0,
\]
\[
\partial_\lambda \Psi(x, y, t, \lambda) = 0, \quad \lambda \in \mathbb{C}^+, \ t > 0,
\]
\[
\Psi(x, y, 0, \lambda) = \Phi(x, y, \lambda). \quad (5.2)
\]

To tackle the nonlinear Riemann-Hilbert problem, we will adopt a nonlinear Newtonian iteration approach [4]. More precisely, first normalize
\[
\Psi(x, y, t, \lambda) = \omega(x, y, z, t) + x - \lambda y - \lambda^2 t,
\]
and let
\[
\mathcal{R}(\omega^- + x - \lambda y - \lambda^2 t, \lambda) - (\omega^+ + x - \lambda y - \lambda^2 t)
\]
\[
= R(\omega^- + x - \lambda y - \lambda^2 t, \lambda) + \omega^- - \omega^+
\]
\[
= \mathcal{G}(\omega^+, \omega^-). \quad (5.4)
\]

Define \( \omega_n^\pm(x, y, t, \lambda) \) recursively by
\[
\mathbf{G}_n \delta \Omega_{n+1} = \mathbf{G}_n \delta \Omega_n - \mathcal{G}(\Omega_n), \quad \text{for } n \geq 0, \quad (5.5)
\]
with
\[
\Omega_n = (\omega_n^+, \omega_n^-), \quad (5.6)
\]
\[
\omega_0^\pm(x, y, t, \lambda) \equiv \Phi^\pm(x, y, \lambda) - (x - \lambda y), \quad (5.7)
\]
\[
\delta \Omega_n = \Omega_n - \Omega_0 = (\delta \omega_n^+, \delta \omega_n^-). \quad (5.8)
\]
and $G_n$ is the linearization of $G$ at $\Omega_n$, i.e.,

$$G_n \delta \Omega = - \delta \omega^+ + \frac{\partial R}{\partial \zeta} \mid_{\omega_n^++xy-\lambda t} \delta \omega^-$$

(5.9)

Thus (5.5) can be written as the non homogeneous Riemann-Hilbert problem

$$\delta \omega^+_{n+1} = J_n \delta \omega^-_{n+1} - (G_n \delta \Omega_n - G(\Omega_n)),$$

(5.10)

Owing to Proposition 3, we will justify

$$\delta \omega_n^- \leq \delta_0 \quad \text{for} \quad 0 \leq t \leq \epsilon_1 \leq \epsilon_0$$

(5.11)

to make the above algorithm eligible. Hence from (5.11) to (5.9), it is easy to see that if $\{\Omega_n\}$ converge to $\Omega$ in $H^p(\mathbb{R}, d\lambda)$, then $G(\Omega)$ is well-defined and $G(\Omega) = 0$. Combining with the Cauchy integral formula, consequently, we will obtain a solution for the nonlinear Riemann-Hilbert problem (5.2).

**Lemma 5.1. (Estimate of $G_{n}^{-1}$)** Suppose $p > 1$ and $v \in \mathcal{S}$ is compactly supported in $y$. If (5.11) is valid, $g(\lambda) \in H^p(\mathbb{R}, d\lambda)$, then the non homogeneous Riemann-Hilbert problem

$$f^+ = J_n f^- + g, \quad \lambda \in \mathbb{R},$$

$$\partial_\lambda f = 0, \quad \lambda \in \mathbb{C}^\pm.$$  

(5.12)

admits a unique solution $f$ such that

$$|f^\pm|_{H^p(\mathbb{R}, d\lambda)} \leq C_{\omega_0} |g|_{H^p(\mathbb{R}, d\lambda)}.$$  

(5.13)

Here $C_{\omega_0}$ is a constant determined by $|\omega_0^\pm|_{H^p(\mathbb{R}, d\lambda)}$.

**Proof.** Using (4.33) and (5.11) (shrink $h$ if necessary) and adapting the proof of Lemma 4.3, one can prove $\text{Ind} J_n$ is well defined and $\text{Ind} J_n = 0$. Hence one has the unique solvability of the Riemann-Hilbert problem

$$X^+(\lambda) = J_n X^-(\lambda), \quad \lambda \in \mathbb{R},$$

$$\partial_\lambda X(\lambda) = 0, \quad \lambda \in \mathbb{C}^\pm,$$

$$X^\pm \to 1, \quad |\lambda| \to \infty,$$

(5.14)

$$\partial_\lambda X^\pm, \partial_\lambda X^{\pm-1} \in H^p(\mathbb{R}, d\lambda),$$

$$X^\pm, X^{\pm-1} \in L^\infty,$$

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and of the non homogeneous Riemann-Hilbert problem \((5.12)\) and \((5.19)\)

\[ f = X \left( \frac{1}{2\pi i} \int_R \frac{x}{\zeta - \lambda} \, d\zeta \right). \]  \hspace{1cm} (5.15)

So \((5.13)\) follows.

\textbf{Lemma 5.2. (Small time estimate and regularities)} If \(p > 2\) and \(v \in \mathcal{S}\) is compactly supported in \(y\), then

\[ |G(\omega_0^+)_{\mathcal{H}^p} < C_{\omega_0} (1 + |y|) t, \quad \text{for } 0 \leq t \leq \epsilon_0. \]  \hspace{1cm} (5.16)

Moreover, if \((5.11)\) is valid, then

\[ |\partial^\mu_\xi \mathcal{R}(\omega^- + x - \lambda y - \lambda^2 t, \lambda)|_{\mathcal{H}^p(\mathbb{R}, d\lambda)} < C_{\omega_0}(1 + |y|), \quad 0 \leq t \leq \epsilon_1. \]  \hspace{1cm} (5.17)

Here \(C_{\omega_0}\) is determined by \(|\omega_0^+(x, y, \lambda)|_{\mathcal{H}^p(\mathbb{R}, d\lambda)}\).

\textbf{Proof.} Formula \((5.1)\) and \((5.4)\) imply

\[ \mathcal{R}(\omega_0^- + x - \lambda y, \lambda) - (\omega_0^+ + x - \lambda y) = 0. \]  \hspace{1cm} (5.18)

Hence

\[
\begin{align*}
|G(\omega_0^+)_{\mathcal{H}^p} = |\mathcal{R}(\omega_0^- + x - \lambda y - \lambda^2 t, \lambda) - (\omega_0^+ + x - \lambda y - \lambda^2 t)|_{\mathcal{H}^p} \\
\leq |\mathcal{R}(\omega_0^- + x - \lambda y, \lambda) - (\omega_0^+ + x - \lambda y)|_{\mathcal{H}^p} + |\mathcal{R}(\omega_0^- + x - \lambda y - \lambda^2 t, \lambda) - \mathcal{R}(\omega_0^- + x - \lambda y, \lambda) + \lambda^2 t|_{\mathcal{H}^p} \\
= |\mathcal{R}(\omega_0^- + x - \lambda y - \lambda^2 t, \lambda) - \mathcal{R}(\omega_0^- + x - \lambda y, \lambda)|_{\mathcal{H}^p}.
\end{align*}
\]

Let \(\zeta_t = \omega_0^- + x - \lambda y - \lambda^2 t\). Therefore, to prove \((5.16)\), it yields to showing

\[
|\mathcal{R}(\xi_0, \lambda) - \mathcal{R}(\zeta_t, \lambda)|_{L^p} \leq C(1 + |y|) t, \hspace{1cm} (5.19)
\]

\[
|\partial_\lambda \mathcal{R}(\xi_0, \lambda) - \partial_\lambda \mathcal{R}(\zeta_t, \lambda)|_{L^p} \leq C(1 + |y|) t. \hspace{1cm} (5.20)
\]

Theorem \(2\) and the mean value theorem (or inequality) imply

\[
\begin{align*}
&|\partial_\lambda \mathcal{R}(\xi_0, \lambda) - \partial_\lambda \mathcal{R}(\zeta_t, \lambda)|_{L^p} \\
\leq & |\partial_\xi \mathcal{R}(\zeta, \lambda)|_{\xi = \xi_0} \frac{\partial \xi}{\partial \lambda} - |\partial_\xi \mathcal{R}(\zeta, \lambda)|_{\xi = \zeta_t} \frac{\partial \xi}{\partial \lambda} |_{L^p} \\
&+ |\partial_\lambda \mathcal{R}(\zeta, \lambda)|_{\xi = \xi_0} - |\partial_\lambda \mathcal{R}(\zeta, \lambda)|_{\xi = \zeta_t} |_{L^p} \\
\leq & |\partial_\xi \mathcal{R}(\zeta, \lambda)|_{\xi = \xi_0} \frac{\partial \xi}{\partial \lambda} - |\partial_\xi \mathcal{R}(\zeta, \lambda)|_{\xi = \xi_0} \frac{\partial \xi}{\partial \lambda} |_{L^p}
\end{align*}
\]
\[\begin{align*}
+|\partial_\xi R(\zeta, \lambda)|_{\zeta=\xi_0} \frac{\partial_\xi}{\partial \lambda} &- \partial_\lambda R(\zeta, \lambda)|_{\zeta=\xi_0} \frac{\partial_\lambda}{\partial \lambda}|_{L^p} \\
+|\partial_\lambda R(\zeta, \lambda)|_{\zeta=\xi_0} - \partial_\lambda R(\zeta, \lambda)|_{\zeta=\xi_0}|_{L^p} \\
\leq |\partial_\xi R(\zeta, \lambda)|_{\zeta=\xi_0} 2\lambda t |_{L^p} + C_{\omega_0} \sup_{0 \leq t' \leq 1} |\partial_\xi^2 R(\zeta', \lambda) t \lambda^2 (1 + |y| + 2|\lambda| t)|_{L^p} \\
+|\partial_\lambda^2 R(\zeta', \lambda) \lambda^2 t|_{L^p} \\
\leq C_{\omega_0} t \left| \frac{1 + |y|}{1 + |\lambda|^{1-\frac{1}{p}}} \right|_{L^p} \\
\leq C_{\omega_0} (1 + |y|) t,
\end{align*}\]

where the constant \(C_{\omega_0}\) is determined by \(|\omega_0(x, y, \cdot)|_{H^p}\). Note we have used (5.11) valid for \(\omega_0\) and \(p > 2\) in the above derivation. Thus (5.20) is proved.

Estimate (5.19) can be proved similarly.

Similarly, (5.17) can be derived by

\[|\partial_\mu^\mu R(\omega^- + x - \lambda y - \lambda^2 t, \lambda)|_{H^p}\]

\[\leq C_{\omega_0} \sum_{k=0}^1 \left| \partial_\xi^{\mu+k} R(\omega^- + x - \lambda y - \lambda^2 t, \lambda)(1 + |y| + |\lambda| t)^k \right|_{L^p} \\
\leq C_{\omega_0} \left| \frac{1 + |y| + t}{1 + |\lambda|^{2+\mu-\frac{1}{p}}} \right|_{L^p} \\
\leq C_{\omega_0} (1 + |y| + t).
\]

\(\square\)

**Lemma 5.3. (Uniform boundedness)** Suppose \(p > 2\) and \(v \in \mathcal{G}\) is compactly supported in \(y\). There exist uniform constants \(\epsilon_1 = \epsilon_1(y, p, \omega_0^\pm) \leq \epsilon_0\) and \(\delta_1 = \delta_1(y, p, \omega_0^\pm) \leq \delta_0\) such that

\[|\delta\Omega_n|_{H^p} = |\delta\omega_n^+|_{H^p} + |\delta\omega_n^-|_{H^p} \leq \delta_1, \quad \forall n, \quad \text{for } 0 \leq t \leq \epsilon_1, \quad (5.22)\]

**Proof.** If \(t \leq \epsilon_1 \leq \epsilon_0, \quad |\delta\Omega_n|_{H^p} \leq \delta_1 \leq \delta_0, \quad (5.23)\)

then Lemma \textbf{5.1, 5.2} Theorem \textbf{2} and the mean value inequality imply

\[|\delta\Omega_{n+1}|_{H^p}\]

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\[
\begin{align*}
\leq & \ C_{\omega_0} | - G(\Omega^\pm_n) + G_n \delta\Omega_n |_{H^p} \\
= & \ C_{\omega_0} | - G(\Omega_0) + G(\Omega_0) - G(\Omega_n) + G_n \delta\Omega_n |_{H^p} \\
= & \ C_{\omega_0} | - G(\Omega_0) + G_n \delta\Omega_n - \left[ \int_0^1 \partial_\zeta G(\theta\Omega_0 + (1 - \theta)\Omega_n) d\theta \right] \delta\Omega_n |_{H^p} \\
\leq & \ C_{\omega_0} (|G(\Omega_0)|_{H^p} + \sup_{\zeta \in \tilde{D}} |\partial^2_\zeta R(\zeta, \lambda)| |(\delta\omega^\pm_n)^2|_{H^p}) \\
= & \ C_*(t + \delta_1^2) \quad (5.24)
\end{align*}
\]

where \(C_{\omega_0}\) is a constant determined by \(|\omega^\pm_0(x, y, \cdot)|_{H^p}\) and \(C_* = C_{\omega_0}(1 + |y|)\).

Applying an induction argument, condition (5.23) and the lemma can be proved by successively choosing

\[
\delta_1 = \min\left\{ \frac{1}{2C_*}, \delta_0 \right\}, \quad \epsilon_1 = \min\left\{ \frac{\delta_1}{2C_*}, \epsilon_0 \right\}. \quad (5.25)
\]

Hence the recursive formula in (5.4)-(5.9) are eligible by Proposition 3 and Lemma 5.3. Moreover,

**Lemma 5.4. (Convergence)** Suppose \(p > 2\) and \(v \in \mathcal{S}\) is compactly supported in \(y\). Let \(\epsilon_1\) and \(\delta_1\) be defined by (5.25). Then, for \(0 \leq t \leq \epsilon_1\), \(\{\omega^\pm_n\}\) converges to some \(\omega^\pm(x, y, t, \lambda)\), with \(|\omega^\pm - (\Phi^\pm - (x - \lambda y))|_{H^p} \leq \delta_1\).

**Proof.** Let

\[
\Delta\Omega_{n+1} = \Omega_{n+1} - \Omega_n = (\omega^+_n - \omega^-_n, \omega^-_n - \omega^-_{n+1}).
\]

From (5.4)-(5.9), one has

\[
G_n \Delta\Omega_{n+1} = G_{n-1} \Delta\Omega_n - (G(\Omega_n) - G(\Omega_{n-1})).
\quad (5.26)
\]

Similarly, we obtain

\[
| - G_{n-1} \Delta\Omega_n + G(\Omega_n) - G(\Omega_{n-1})|_{H^p} \leq C_{\omega_0} (1 + |y|)|\Delta\Omega_n|_{H^p}. \quad (5.27)
\]

Here \(C_{\omega_0}\) is a constant determined by \(|\omega^\pm_0(x, y, \cdot)|_{H^p}\). Applying Lemma 5.1 and 5.2 we derive

\[
|\Delta\Omega_{n+1}|_{H^p} \leq C_1|\Delta\Omega_n|_{H^p} \quad (5.28)
\]

from (5.26), (5.27). Here \(C_1 = C_{\omega_0}'(1 + |y|)\) is a constant independent of \(n\). Thus by Lemma 5.3,

\[
|\Delta\Omega_{n+1}|_{H^p} \leq C_1 \delta_1|\Delta\Omega_n|_{H^p}.
\]

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For small $\delta_1$ satisfying $2C_1\delta_1 < 1$ (redo the proof of Lemma 5.3 if necessary), we obtain
\[ |\Delta \Omega_{n+1}|_{H^p} \leq \frac{1}{2} |\Delta \Omega_n|_{H^p}, \]
which implies the convergence of $\{\Omega_n\}$.

**Theorem 3. (Local solvability for the NRH problem)** For $p > 2$, $\forall x$, $\forall y \in \mathbb{R}$, and $v \in \mathfrak{S}$ is compactly supported in $y$. Let $\epsilon_1 = \epsilon_1(y, p, \omega_0^\pm)$, $\delta_0$ be defined by (5.25) and Proposition 3. Let
\[ \epsilon_2 = \min\{\epsilon_1(y, p, \omega_0^\pm), \epsilon_1(y, 2p, \omega_0^\pm)\}. \]
(5.29)

Then the nonlinear Riemann-Hilbert problem
\[ \Psi^+(x, y, t, \lambda) = \mathcal{R}(\Psi^-(x, y, t, \lambda), \lambda), \quad \lambda \in \mathbb{R}, \quad 0 \leq t \leq \epsilon_2, \]
\[ \partial_{\lambda} \Psi^+(x, y, t, \lambda) = 0, \quad \lambda \in \mathbb{C}^\pm, \quad 0 \leq t \leq \epsilon_2, \]
\[ \Psi(x, y, 0, \lambda) = \Phi(x, y, \lambda) \]
(5.30)

has a unique solution satisfying the reality condition
\[ \Psi^+(x, y, t, \lambda) = \overline{\Psi^-(x, y, t, \lambda)} \]
(5.31)
and
\[ |\Psi^\pm(x, y, t, \lambda) - (\Phi^\pm(x, y, \lambda) - \lambda^2 t)|_{H^p} < \delta_0, \]
\[ \partial_t^h \partial_y^k \partial_x^\mu [\Psi^\pm - (x - \lambda y - \lambda^2 t)] \in L^p, \]
\[ 0 \leq h + k + \mu \leq 2, \quad h < 2. \]
(5.32)

**Proof.** Therefore, for each $x, y \in \mathbb{R}$, from (5.4)-(5.9), Lemma 5.4, Plemelji formula, and

\[ \Psi(x, y, t, \lambda) = x - \lambda y - \lambda^2 t + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{R(\omega^- + x - \lambda' y - \lambda'^2 t, \lambda')}{\lambda' - \lambda} d\lambda', \]

we obtain a local solution for the nonlinear Riemann-Hilbert problem (5.30).

The reality condition in (5.31) follows from the algebraic constraint (4.17) (adapting (4.23) for the extension $\mathcal{R}$). Suppose
\[ \Psi_i^+(x, y, t, \lambda) = \mathcal{R}(\Psi_i^-(x, y, t, \lambda), \lambda), \]
\[ \Psi_i^+(x, y, 0, \lambda) = \Phi_i^+(x, y, \lambda), \]
\[ |\Psi_i^+(x, y, t, \lambda) - (\Phi_i^+(x, y, \lambda) - \lambda^2 t)|_{H^p} < \delta_0 \]
(5.33)

for $i = 1, 2$. Denote
\[ \Pi^\pm(x, y, t, \lambda) = \Psi_1^\pm(x, y, t, \lambda) - \Psi_2^\pm(x, y, t, \lambda), \]
\[ \kappa(x, y, t, \lambda) = \int_0^1 \frac{\partial}{\partial \theta} (\theta \Psi_1^+(x, y, t, \lambda) + (1 - \theta) \Psi_2^+(x, y, t, \lambda), \lambda) d\theta. \]
Then
\[ \Pi^+(x, y, t, \lambda) = \kappa(x, y, t, \lambda) \Pi^-(x, y, t, \lambda), \]
\[ \Pi^\pm(x, y, t, \lambda) \in H^p. \]
Therefore, justifying the index zero condition by (5.33), \(|\kappa(x, y, 0, \lambda)| = |\partial R/\partial \zeta(\Phi^-(x, y, \lambda), \lambda)| \neq 0, \) and Lemma 4.3 one can derive \( \Pi^\pm(x, y, t, \lambda) \equiv 0 \) and verify the uniqueness.

To complete the proof, we need to consider cases of higher derivatives. By (5.30), we obtain the Riemann-Hilbert problems
\begin{align*}
\Psi_+^x &= \partial R/\partial \zeta(\Psi_-, \lambda)\Psi_-, \\
\Psi_+^y &= \partial R/\partial \zeta(\Psi_-, \lambda)\Psi_-, \\
\Psi_+^t &= \partial R/\partial \zeta(\Psi_-, \lambda)\Psi_-. 
\end{align*}
(5.34)

(5.35)

(5.36)
Using the renormalization \( \Psi(x, y, t, \lambda) = \omega(x, y, t, \lambda) + x - \lambda y - \lambda^2 t \), the Riemann-Hilbert problems (5.34)-(5.36) turn into
\begin{align*}
\omega_+^x &= \partial R/\partial \zeta(\omega_- + x - \lambda y - \lambda^2 t, \lambda), \\
\omega_+^y &= \partial R/\partial \zeta(\omega_- - \lambda y, \lambda) - \partial R/\partial \zeta, \\
\omega_+^t &= \partial R/\partial \zeta(\omega_- - \lambda^2 t, \lambda). 
\end{align*}
(5.37)
(5.38)
(5.39)

where \( \partial R/\partial \zeta = \partial R/\partial \zeta(\omega^- + x - \lambda y - \lambda^2 t, \lambda) \). They are linear Riemann-Hilbert problem (5.31) with non homogeneous terms

\[ \partial R/\partial \zeta, \partial R/\partial \zeta \lambda, \partial R/\partial \zeta \lambda^2 \in L^p, \quad p > 2. \]
(5.40)

Therefore, by Lemma 4.3 (cf. (5.14) and (5.15)), one can derive the unique solvability in \( L^p \) of (5.37)-(5.39).

Furthermore, for higher derivatives, we first use the same method to prove
\[ \partial^h \partial^k \partial^\mu \Psi^\pm - (x - \lambda y - \lambda^2 t) \in L^p \cap L^{2p}, \]
(5.41)
for \( 0 \leq t \leq \epsilon_2 \), where \( \epsilon_2 \) is defined by (5.29). On the other hand, taking
derivatives of (5.37)-(5.39), one obtains

\[
\begin{align*}
\omega_{xx}^+ &= \frac{\partial R}{\partial \zeta} \omega_x^- + \frac{\partial^2 R}{\partial \zeta^2} \left( (\omega_x^-)^2 + 2\omega_x^- + 1 \right), \\
\omega_{yt}^+ &= \frac{\partial R}{\partial \zeta} \omega_y^- + \frac{\partial^2 R}{\partial \zeta^2} \left( \omega_y^- \omega_t^- - \lambda^2 \omega_y^- - \lambda \omega_t^- - \lambda^3 \right), \\
\omega_{xy}^+ &= \frac{\partial R}{\partial \zeta} \omega_{xy}^- + \frac{\partial^2 R}{\partial \zeta^2} \left( \omega_x^- \omega_y^- - \lambda \omega_x^- + \omega_y^- - \lambda \right), \\
\omega_{xt}^+ &= \frac{\partial R}{\partial \zeta} \omega_{xt}^- + \frac{\partial^2 R}{\partial \zeta^2} \left( \omega_x^- \omega_t^- - \lambda^2 \omega_x^- + \omega_t^- - \lambda^2 \right), \\
\omega_{yy}^+ &= \frac{\partial R}{\partial \zeta} \omega_{yy}^- + \frac{\partial^2 R}{\partial \zeta^2} \left( (\omega_y^-)^2 - 2\lambda \omega_y^- + \lambda^2 \right). 
\end{align*}
\]
(5.42)

Theorem 2 and (5.41) imply all non homogeneous terms in the right hand sides of (5.42) are \(L^p\) functions. Therefore, the estimates (5.32) can be derived as above.

Remark 3. If the condition (5.32) is dropped, the uniqueness is no longer true. A counterexample is given by \(\Phi^\pm(x - \lambda^2 t, y, \lambda)\) which satisfies (5.30) but does not satisfy (5.32).

Remark 4. The Newtonian iteration in this section can be elucidated to prove the set

\[
S = \{T \in \mathbb{R} \mid \text{the nonlinear Riemann-Hilbert problem} \}
\]

\[
\Psi^+(x, y, t, \lambda) = \mathcal{R}(\Psi^-(x, y, t, \lambda), \lambda), \quad \lambda \in \mathbb{R},
\]

\[
\partial_{\lambda} \Psi(x, y, t, \lambda) = 0, \quad \lambda \in \mathbb{C}^\pm,
\]

\[
\Psi(x, y, 0, \lambda) = \Phi(x, y, \lambda)
\]

is solved for \(0 \leq t \leq T\}.

is open. On the other hand, the dispersion relation \(x - \lambda y - \lambda^2 t\) causes the estimates, in particular,

\[
|R(\omega^\pm(x, y, t, \lambda) + x - \lambda y - \lambda^2 t, \lambda)|_{H^p(\mathbb{R}, d\lambda)}
\]

growing inevitably unbounded as \(t \gg 1\). That is, when \(t \gg 1\), one can not exclude the possibilities

- the index zero condition on \(1 + \partial_{\lambda} R\) breaks;
the deformation property of $R(\Phi^-(x, y, \lambda), \lambda)$ fails, since $\hbar < \infty$. Therefore, $S$ is not closed in general. So only a local solvability for the nonlinear Riemann-Hilbert problem (5.30) is achieved and it is not practical to recover the global small data solution obtained in [10] via the Newtonian iteration scheme.

6 The inverse problem II: the Lax equation and the Cauchy problem

Theorem 4. (Local solvability of the Lax pair) Suppose $v_0 \in \mathcal{S}$ is compactly supported in $y$. Let $\Psi^\pm(x, y, t, \lambda)$ and $\epsilon_2 = \epsilon_2(y)$ be the solution of the nonlinear Riemann-Hilbert problem (5.30) obtained in Theorem 3 (replacing $v$ by $v_0$). Define

$$v(x, y, t) = \frac{1}{2\pi i} \int_{\mathbb{R}} R(\Psi^-(x, y, t, \lambda'), \lambda') d\lambda', \quad (6.1)$$

$$\Psi(x, y, t, \lambda) = x - \lambda y - \lambda^2 t + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{R(\Psi^-(x, y, t, \lambda'), \lambda')}{\lambda' - \lambda} d\lambda', \quad (6.2)$$

$$\forall x, y \in \mathbb{R}, \quad \lambda \in \mathbb{C}^\pm, \quad 0 \leq t \leq \epsilon_2.$$

Then for $0 \leq \mu + k \leq 1$,

$$v(x, y, t) = v(x, y, t), \quad (6.3)$$

$$v(x, y, 0) = v_0(x, y), \quad (6.4)$$

$$\partial_y^k \partial_x^\mu v(x, y, t) \in C(\mathbb{R} \times \mathbb{R} \times [0, \epsilon_2]) \cap L^\infty(\mathbb{R} \times \mathbb{R} \times [0, \epsilon_2]); \quad (6.5)$$

and for $0 \leq \mu + k + h \leq 2$, $h < 2$, and $\lambda \in \mathbb{C}^\pm$,

$$\partial_t^h \partial_y^k \partial_x^\mu [\Psi - (x - \lambda y - \lambda^2 t)] \in L^\infty(\mathbb{R} \times \mathbb{R} \times [0, \epsilon_2]). \quad (6.6)$$

Moreover, for $\lambda \in \mathbb{C}^\pm$, the Lax pair

$$L \Psi = \partial_y \Psi + (\lambda + v_x) \partial_x \Psi = 0, \quad (6.7)$$

$$M \Psi = \partial_t \Psi + (\lambda^2 + \lambda v_x - \epsilon_2) \partial_x \Psi = 0, \quad (6.8)$$

for $\forall x, y \in \mathbb{R}, \quad 0 \leq t \leq \epsilon_2$

exist uniquely.
Proof. Theorem 4, (5.32), (6.1), (6.2), Sobolev’s theorem, and Hölder inequality imply (6.5) and (6.6).

Applying $L$ and $M$ to both sides of (5.30), and using (5.32), (6.5), $p > 2$, we obtain

\[ L \Psi^+ = \frac{\partial R}{\partial \zeta} L \Psi^-, \quad L \Psi^\pm \in L^p, \]
\[ M \Psi^+ = \frac{\partial R}{\partial \zeta} M \Psi^-, \quad M \Psi^\pm \in L^p. \]

(6.9)

Applying Lemma 4.3 (cf. (5.14) and (5.15)), we conclude $L \Psi^+ = L \Psi = 0$ and $M \Psi^+ = M \Psi = 0$. Hence (6.7) and (6.8) are justified.

From (5.31), one can prove $\Psi(x, y, t, \lambda) = \Psi(x, y, t, \bar{\lambda})$. Together with (6.7), we obtain the reality condition (6.3).

\[ \square \]

Theorem 5. (Local solvability of the Cauchy problem) Suppose $v_0 \in \mathcal{G}$ with a compact support. Then there exists $\epsilon_2(y) > 0$ such that the Cauchy problem of the Pavlov equation

\[ v_{xt} + v_{yy} = yv_{xx} - x v_{xy}, \quad \forall x, y \in \mathbb{R}, \quad 0 < t \leq \epsilon_2(y), \]
\[ v(x, y, 0) = v_0(x, y) \]

(6.10)

admits a unique real solution. Moreover,

\[ \partial^\mu_y \partial^\nu_x v \in C(\mathbb{R} \times \mathbb{R} \times [0, \epsilon_2]) \cap L^{\infty}(\mathbb{R} \times \mathbb{R} \times [0, \epsilon_2]), \]
\[ 0 \leq \mu + k \leq 1. \]

Proof. Applying Theorem 4 and Lemma 4.3, one can derive higher order (non uniform) decay of $R(\Psi(x, y, t, \lambda))$ in $\lambda$ if $0 < t$. Hence $v_{xt}$, $v_{yy}$, $v_{xy}$, and $v_{xx}$ exist. So we can compute the compatibility of the Lax pair (6.7) and (6.8) and obtain

\[ (v_{xt} + v_{yy} - v_y v_{xx} + v_x v_{xy}) \partial_x \Psi \equiv 0, \quad 0 < t \leq \epsilon_2(y). \]

\[ \square \]

Remark 5. Unlike Theorem 4 where the Lax pair holds up to $t = 0$, we cannot prove the Pavlov equation is valid at $t = 0$.

Remark 6. The property $\hbar < \infty$ in (4.9) is the obstruction to global solvability of the Lax pair and Cauchy problem with large initial data.
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