DEFINING RELATIONS FOR THE EXCEPTIONAL LIE SUPERALGEBRAS OF VECTOR FIELDS PERTAINING TO THE
STANDARD MODEL

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Abstract. We list defining relations for the four of the five exceptional simple Lie superalgebras some of which, as David Broadhurst conjectured and Kac demonstrated, may pertain to The Standard Model of elementary particles. For the fifth superalgebra the result is not final: there might be infinitely many relations. Contrarywise, for the same Lie superalgebra with Laurent polynomials as coefficients there are only finitely many relations.

The ground field is \( \mathbb{C} \).

David Broadhurst conjectured \([\text{Ka}2]\) that some of the exceptional infinite dimensional simple vectorial Lie superalgebras might be related to a Standard Model, and Kac laboriously elucidated and popularized this conjecture \([\text{Ka}3]\). Here we describe the exceptional algebras in terms of generators and relations. To make the paper of interest to a wider audience, we give an elucidation of the list of simple vectorial superalgebras and append with some not very well known definitions.

Our choice of contribution to Kirillov’s Fest was prompted in 1977 by A. A. Kirillov who suggested to one of us a joint project: calculation of certain Lie superalgebra cohomology. For various reasons the project was put aside. Still, we remember even passing wishes of our teacher especially if we cannot answer immediately; another reason was to thank one of the editors by reminding of the problems from the happy childhood, see \([\text{K}]\).

Observe that even the conventional simple finite dimensional Lie algebras admit several distinct presentations. For example, there are (1) the Serre relations between Chevalley generators (convenient and simple looking but numerous: for the non-exceptional algebras \((\mathfrak{g} = \mathfrak{sl}(n), \mathfrak{sp}(2n), \mathfrak{o}(n))\) the number of generators and relations grows with \(\text{rk} \mathfrak{g} \sim n\)); (2) relations between just a pair of Jacobson generators, see \([\text{GL}3]\) (the number of the relations between the Jacobson generators does not depend on \(n\) but, though indispensable in some questions, they are less convenient than Serre relations). There are other natural choices of generators even for \(\mathfrak{sl}(n)\).

Among various presentations, Chevalley generators (satisfying Serre relations) are the first choice because in terms of them we split the algebra into the sum of its “positive” and “negative” parts \(\mathfrak{g}_\pm \simeq \mathfrak{n}\) which are nilpotent. For any nilpotent Lie algebra the very notions “generators” and “relations” are most transparently defined. Moreover, they admit a homological interpretation. Suppose, as will be the case in our examples, there is a set of outer derivations acting on \(\mathfrak{n}\) so that \(\mathfrak{n}\) splits into the direct sum of 1-dimensional eigenspaces. Then the choice of a basis is unique up to scalar factors; moreover, usually, \(\mathfrak{n} \simeq \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \oplus [\mathfrak{n}, \mathfrak{n}]\) so any basis of the space \(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] = H_1(\mathfrak{n})\) is a set of generators of \(\mathfrak{n}\).

\hspace{1cm} \text{1991 Mathematics Subject Classification.} 17A70 (Primary) 17B35 (Secondary).

\textit{Key words and phrases.} Lie superalgebra, Cartan prolongation, nonholonomic structures, defining relations.

Financial support of NFR and RFBR grant 99-01-00245 is thankfully acknowledged by D. L. and I. Shch., respectively. We are also thankful to ESI for hospitality in 2000 and 2001, in particular, to A. Kirillov and V. Kac, organizers of a miniprogram in 2000.
To describe relations between the generators, consider the standard homology complex for $n$ with trivial coefficients ($[Fu]$):

$$0 \leftarrow n \xleftarrow{d_1} n \wedge n \xleftarrow{d_2} n \wedge n \wedge n \xleftarrow{d_3} \ldots .$$

By definition

$$d_1(x \wedge y) = [x, y], \quad d_2(x \wedge y \wedge z) = [x, y] \wedge z + [y, z] \wedge x + [z, x] \wedge y.$$ 

The condition $d_1d_2 = 0$ expresses the Jacobi identity. Obviously, the elements of $\text{Ker}d_1$ are relations. The consequences of the Jacobi identity are considered as trivial relations, they constitute $\text{Im}d_2$. Thus, for a nilpotent Lie algebra $n$, we identify a convenient basis of $H_1(n)$ with the generators and that of $H_2(n)$ with defining relations. Observe that the same arguments apply as well to locally nilpotent Lie (super)algebras (as considered below).

There is, actually, one more, “zeroth” piece, $g_0$, consisting of Cartan subalgebra $h$, but the relations expressing the elements of $h$ via positive and negative generators, as well as the action of $h$ on these generators, though vital and introduce Cartan matrix (or its analog), are obvious or, at least, so easy to compute, that we skip them. The most difficult to describe are relations between generators “of the same sign” (positive or negative). Such presentations for Lie algebras became very fashionable in connection with $q$-quantizations.

For Lie superalgebras, even with Cartan matrix, there are some subtleties: it turned out that there are non-Serre relations even between analogs of Chevalley generators, see [FLV], [Y], [GL2].

For simple finite dimensional Lie superalgebras, even with Cartan matrix, there are several inequivalent sets of Chevalley generators and, respectively, analogs of Serre relations, see [GL2]. For the Lie superalgebras without Cartan matrix the analogs of such sets are not described very explicitly at the moment, besides, there are infinitely many (non-equivalent) of them for infinite dimensional Lie (super)algebras. So we only confine ourselves to “a most simple” (or most convenient) one; for the series it corresponds to the so-called standard grading, for the exceptional vectorial Lie superalgebras the grading consistent with parity (if exists) is the most convenient one.

In §1, we describe simple vectorial Lie superalgebras, [Li, Sh1, Sh2, Sh14; CK2]. Five of them, if considered as abstract algebras, are exceptional; these 5 algebras have exactly 15 “incarnations” as particularly graded or filtered Lie superalgebras of vector fields on some supermanifolds. There are, of course, lots of gradings, but only finitely many (just one for simple Lie algebras) are determined by a maximal subalgebra of finite codimension. Such particular (Weisfeiler) grading is described below; it is of special interest. Among these 15 realizations some are distinguished by the fact that the corresponding $Z$-grading is consistent (with parity), these realizations are easier to deal with in our problem — description of defining relations.

Actually, our shining goal is the description of defining relations for any realization; unlike the finite dimensional case this might be impossible: there does not seem to be a way to pass from relations in terms of one set of simple roots to that in terms of another (even for $\mathfrak{sl}(m|n)$).

In §2 we state our main result. To formulate it, we represent the Lie superalgebras we consider as $g = \bigoplus_{i \geq -d} g_i$; we further set $g_- = \bigoplus_{i < 0} g_i$ and $g_+ = \bigoplus_{i > 0} g_i$.

Observe that $g_0$ has its own $Z$-grading, so it can be represented as $g_0 = \bigoplus_i g_{0,i}$, where $g_{0,0}$ is the Cartan subalgebra in $g_0$ and in $g$. Set $g_{0-} = \bigoplus_{i < 0} g_{0,i}$ and $g_{0+} = \bigoplus_{i > 0} g_{0,i}$.

Finally, set $g_+ = g_{0+} \oplus g_+$ and $g_- = g_{0-} \oplus g_-$.
Our main result is a description of defining relations for the locally nilpotent Lie super-algebras $G_+$ and $G_-$. Observe a remarkable difference in presentations for $kas$ generated by polynomials (in this paper) and $kas^L$ generated by Laurent polynomials, see [GLS1]. Our results (description of defining relations for $G_+$ and $G_-$) are final in all the cases, except $kas_+$; we even suspect that $kas_+$ might be not finitely presented. This never happens with Lie algebras of class SZGPG (simple $Z$-graded of polynomial growth) we are studying, but does happen with certain most innocent-looking loop superalgebras, cf. [GL2].

The results are verified by means of Grozman’s Mathematica-based package SuperLie, see [GL1]. We could not apply the general arguments of [FF1] to the unfinished case of $kas_+$; observe that these arguments should be applied very carefully: the original idea [FF2] that all relations for $g_+$ for any simple vectorial Lie algebras are of degree 2 is wrong, as pointed by Kochetkov, for the Hamiltonian series which always has relations of degree 3, in particular, for Lie superalgebras of Hamiltonian fields, cf. [GLP].

Related results. For description of defining relations in other cases see [LP] (simple and close to simple Lie algebras of vector fields with polynomial coefficients), [GLP] (nonexceptional Lie superalgebras; for series $q$ see [LSe]).

§1. Description of simple vectorial Lie superalgebras

0.0. On setting of the problem. Selection of Lie algebras with reasonably nice properties is a matter of taste and is under the influence of the problem considered. One of the usual choices is the class of simple algebras: they are easier to study and illuminate important symmetries.

Of simple Lie algebras, finite dimensional algebras are the first to study. They can be neatly encoded by very simple graphs — Dynkin diagrams — or Cartan matrices.

Next on the agenda are $Z$-graded Lie algebras of polynomial growth (let us call them SZGLAPGs for short). They resemble finite dimensional simple Lie algebras very much and their theory is very similar, see [Ka1]; they proved very useful in various branches of mathematics and theoretical physics. Some of them have no Cartan matrix, but are no less useful; e.g., such are vectorial Lie superalgebras.

0.0.1. Types of the classical Lie algebras. Let us qualitatively describe the simple Lie algebras of polynomial growth to better visualize them. They break into the disjoint union of the following types:

1) finite dimensional (growth 0).
2) loop algebras, perhaps, twisted (growth 1); more important in applications are their “relatives” called affine Kac–Moody algebras, cf. [Ka1].
3) vectorial algebras, i.e., Lie algebras of vector fields with polynomial coefficients (growth is equal to the number of indeterminates) or their completions with formal power series as coefficients.

These algebras are sometimes known under the recently introduced clumsy name “algebras of Cartan type”: just imagine a “Cartan subalgebra in a Lie algebra of Cartan type”.
4) the stringy algebra\(\text{vect}^L(1) = \mathcal{der} \mathbb{C}[t^{-1}, t]\) (the superscript stands for Laurent). This algebra is often called Witt algebra \(\text{witt}\) and by physicists the centerless Virasoro algebra because its nontrivial central extension, \(\text{vir}\), is called the Virasoro algebra.

Strictly speaking, stringy algebras are vectorial, but we retain the generic term vectorial for algebras with polynomial or formal coefficients.

The algebras of types 1)–4) are \(\mathbb{Z}\)-graded. Several of the algebras of types 2) and 3) (in supercase all four types) have deformations and some of the deformed algebras are not \(\mathbb{Z}\)-graded of polynomial growth. These deformations (studied insufficiently so far) naturally indicate one more type: filtered Lie (super)algebras of polynomial growth:

5) the Lie algebra of matrices of complex size and its generalizations, cf. \([GL3, LS]\). These algebras are simple filtered Lie algebras of polynomial growth whose associated graded are not simple.

0.0.2. Superization. After Wess and Zumino made importance of supersymmetries manifest, cf. \([D, WZ]\), it was natural to list simple “classical” Lie superalgebras.

— Finite dimensional superalgebras. All except vectorial ones were classified by Scheunert, Nahm, Rittenberg \([SNR]\) and I. Kaplansky. The unpublished preprint-1975 by Kaplansky appeared with some extensions as \([FK, K]\); it helped Kac to repair gaps in his independent proof, cf. \([KIC]\). For details of the proof (further elucidated in \([Sch]\)) in all cases see \([Ka4]\) where Kac also considered representation theory, infinite dimensional case and real forms.

— (Twisted) loop superalgebras. For an intrinsic characterization of loop superalgebras without appeal to Cartan matrix (rewritten from O. Mathieu) together with same for stringy superalgebras see \([GLS1]\): both types of algebras \(g = \bigoplus_{i=-d}^\infty g_i\) are of infinite depth \(d\) but

\[
\text{for the loop algebras every root vector corresponding to the real root acts locally nilpotently in the adjoint representation,}
\]

\[
\text{for the stringy algebras this is not so.}
\]

Leites conjectured that, as for Lie algebras, simple twisted loop superalgebras correspond to outer automorphisms of \(g\). Serganova listed these automorphisms and amended the conjecture, see \([SI]\). J. van de Leur classified twisted loop superalgebras with symmetrizable Cartan \([vdL]\); his list supports Leites-Serganova’s conjecture \([SI]\).

— Stringy superalgebras. For their conjectural list and partial proof of the completeness of this list see \([GLS1, KvdL]\). Observe that some of the stringy superalgebras possess Cartan matrix, though nonsymmetrizable ones \([GLS1]\).

In this paper we consider the remaining type of simple graded Lie superalgebras of polynomial growth:

— Vectorial Lie superalgebras. Main examples to keep before mind’s eye are the filtered Lie algebra \(L = \mathcal{der} \mathbb{C}[x]\) of formal vector fields and \(L = \mathcal{der} \mathbb{C}[x]\) of polynomial vector fields with grading and filtration given by setting \(\text{deg } x_i = 1\) for all \(i\).

More exactly, we present the exceptional simple vectorial superalgebras in terms of generators and defining relations.

\(^2\)The term induced by the lingo of imaginative physicists who now play with (either cherish, as we do, \([GSW]\) or take a calmer attitude \([WL]\)) the idea that an elementary particle is not a point but rather a slinky springy string; the term “stringy algebra” means “pertaining to string theory” but also mirrors their structure as a collection of several strings — the modules over the Witt algebra. In our setting this means “vectorial (super)algebra on a supermanifold whose base is a circle or a “relative” of such an algebra (its central extension, or an algebra of differentiatioins, etc.)."
### 1.0. Linear algebra in superspaces. Generalities.

A superspace is a \( \mathbb{Z}/2 \)-graded space; for any superspace \( V = V_0 \oplus V_1 \) denote by \( \Pi(V) \) another copy of the same superspace: with the shifted parity, i.e., \( (\Pi(V))_i = V_{i+1} \). The superdimension of \( V \) is \( \dim V = p + q \varepsilon \), where \( \varepsilon^2 = 1 \) and \( p = \dim V_0, \ q = \dim V_1 \). (Usually, \( \dim V \) is expressed as a pair \( (p, q) \) or \( p|q \); this obscures the fact that \( \dim V \otimes W = \dim V \cdot \dim W \); this fact is clear with the use of \( \varepsilon \).

A superspace structure in \( V \) induces the superspace structure in the space \( \text{End}(V) \). A basis of a superspace is always a basis consisting of homogeneous vectors; let \( \text{Par} = (p_1, \ldots, p_{\dim V}) \) be an ordered collection of their parities. We call \( \text{Par} \) the format of (the basis of) \( V \). A square supermatrix of format (size) \( \text{Par} \) is a \( \dim V \times \dim V \) matrix whose \( i \)th row and \( i \)th column are of the same parity \( p_i \).

One usually considers one of the simplest formats \( \text{Par} \), e.g., \( \text{Par} \) of the form \( (\bar{0}, \ldots, \bar{0}; \bar{1}, \ldots, \bar{1}) \) is called standard. In this paper we can do without nonstandard formats. But they are vital in the classification of systems of simple roots that the reader might be interested in connection with applications to \( q \)-quantization or integrable systems. Besides, systems of simple roots corresponding to distinct nonstandard regradings. (For an approach to superroots see [S2].)

The matrix unit \( E_{ij} \) is supposed to be of parity \( p_i + p_j \) and the bracket of supermatrices (of the same format) is defined via Sign Rule:

**if something of parity \( p \) moves past something of parity \( q \) the sign \( (-1)^{pq} \) accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.**

Examples of how to apply the Sign Rule: setting \( [X, Y] = XY - (-1)^{p(X)p(Y)} YX \) we get the notion of the supercommutator and the ensuing notions of the supercommutative superalgebra and the Lie superalgebra (which in addition to superskew-commutativity satisfies the super Jacobi identity, i.e., the Jacobi identity amended with the Sign Rule). The derivation (better say, superderivation) of a superalgebra \( A \) is a linear map \( D : A \rightarrow A \) that satisfies the Leibniz rule (and Sign rule)

\[
D(ab) = D(a)b + (-1)^{p(D)p(a)}aD(b).
\]

In particular, let \( A = \mathbb{C}[x] \) be the free supercommutative polynomial superalgebra in \( x = (x_1, \ldots, x_n) \), where the superstructure is determined by the parities of the indeterminates: \( p(x_i) = p_i \). Partial derivatives are defined (with the help of super Leibniz Rule) by the formulas \( \frac{\partial x_i}{\partial x_j} = \delta_{i,j} \). Clearly, the collection \( \partial \text{err} A \) of all superdifferentiations of \( A \) is a Lie superalgebra whose elements are of the form \( \sum f_i(x) \frac{\partial}{\partial x_i} \).

Given the supercommutative superalgebra \( \mathcal{F} \) of “functions” in indeterminates \( x \), define the supercommutative superalgebra \( \Omega \) of differential forms as polynomial algebra over \( \mathcal{F} \) in \( dx \), where \( p(d) = \bar{1} \). Since \( dx \) is even for an odd \( x \), we can consider not only polynomials in \( dx \). Smooth or analytic functions in the differentials of the \( x \) are called pseudodifferential forms on the supermanifold with coordinates \( x \), see [BL1]. We will need them to interpret \( \mathfrak{h}_\lambda(2)[2] \). The exterior differential is defined on (pseudo) differential forms by the formulas (mind Leibniz and Sign Rules):

\[
d(x_i) = dx_i \text{ and } d^2 = 0.
\]

The Lie derivative is defined (minding same Rules) by the formula

\[
L_D(df) = (-1)^{p(D)}d(D(f)).
\]

In particular,

\[
L_D((df)^\lambda) = \lambda(-1)^{p(D)(D(f))}(df)^{\lambda-1} \text{ for any } \lambda \in \mathbb{C}.
\]

The general linear Lie superalgebra of all supermatrices of size \( \text{Par} \) is denoted by \( \mathfrak{gl}(\text{Par}) \); usually, \( \mathfrak{gl}(\bar{0}, \ldots, \bar{0}, \bar{1}, \ldots, \bar{1}) \) is abbreviated to \( \mathfrak{gl}(\dim V_0 \dim V_1) \). Any matrix from \( \mathfrak{gl}(\text{Par}) \)
can be expressed as the sum of its even and odd parts; in the standard format this is the following block expression:
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.
\]
\[
p\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = 0, \quad p\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = 1.
\]

The supertrace is the map \(\mathfrak{gl}(\text{Par}) \to \mathbb{C}\), \((A_{ij}) \mapsto \sum (-1)^{p_i} A_{ij}\). Since \(\text{str}[x, y] = 0\), the subsuperspace of supertraceless matrices constitutes the special linear Lie subsuperalgebra \(\mathfrak{sl}(\text{Par})\).

There are, however, two super versions of \(\mathfrak{gl}(n)\), not one. The other version is called the queer Lie superalgebra and is defined as the one that preserves the complex structure given by an odd operator \(J\), i.e., is the centralizer \(\text{C}(J)\) of \(J\):
\[
q(n) = \text{C}(J) = \{X \in \mathfrak{gl}(n|n) \mid [X, J] = 0\}, \ \text{where} \ J^2 = -\text{id}.
\]

It is clear that by a change of basis we can reduce \(J\) to the form \(J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1 & 0 \end{pmatrix}\). In the standard format we have
\[
q(n) = \left\{\begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}.
\]

On \(q(n)\), the queertrace is defined: \(\text{qtr} : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \text{tr}B\). Denote by \(\mathfrak{qs}(n)\) the Lie superalgebra of queertraceless matrices.

Observe that the identity representations of \(q\) and \(qs\) in \(V\), though irreducible in supersetting, are not irreducible in the nongraded sense: take homogeneous (with respect to parity) and linearly independent vectors \(v_1, \ldots, v_n\) from \(V\); then \(\text{Span}(v_1 + J(v_1), \ldots, v_n + J(v_n))\) is an invariant subspace of \(V\) which is not a subsuperspace.

A representation is irreducible of general type or just of \(G\)-type if there is no invariant subspace, otherwise it is called irreducible of \(Q\)-type (\(Q\) is after the general queer Lie superalgebra — a specifically “superish” analog of \(\mathfrak{gl}\)); an irreducible representation of \(Q\)-type has no invariant subsuperspace but has an invariant subspace.

**Lie superalgebras that preserve bilinear forms: two types.** To the linear map \(F : V \to W\) of superspaces there corresponds the dual map \(F^* : W^* \to V^*\) between the dual superspaces. In a basis consisting of the vectors \(v_i\) of format \(\text{Par}\), the formula \(F(v_j) = \sum v_i A_{ij}\) assigns to \(F\) the supermatrix \(A\). In the dual bases, to \(F^*\) the supertransposed matrix \(A^t\) corresponds:
\[
(A^t)_{ij} = (-1)^{(p_i + p_j)(p_i + p(A))} A_{ji}.
\]

The supermatrices \(X \in \mathfrak{gl}(\text{Par})\) such that
\[
X^t B + (-1)^{p(X)p(B)} BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(\text{Par})
\]
constitute the Lie superalgebra \(\text{aut}(B)\) that preserves the bilinear form \(B^f\) on \(V\) whose matrix \(B\) is given by the formula \(B_{ij} = (-1)^{p(B^f)p(v_i)} B^f(v_i, v_j)\) for the basis vectors \(v_i\).

Recall that the supersymmetry of the homogeneous form \(B^f\) means that its matrix \(B\) satisfies the condition \(B^u = B\), where \(B^u = \begin{pmatrix} R^t & (-1)^{p(B)} T^t \\ (-1)^{p(B)} S^t & -U^t \end{pmatrix}\) for the matrix \(B = \begin{pmatrix} R & S \\ T & U \end{pmatrix}\). Similarly, skew-supersymmetry of \(B\) means that \(B^u = -B\). Thus, we see that the upsetting of bilinear forms \(u : \text{Bil}(V, W) \to \text{Bil}(W, V)\), which for the spaces and when \(V = W\) is expressed on matrices in terms of the transposition, is a new operation.
Most popular canonical forms of the nondegenerate supersymmetric form are the ones whose supermatrices in the standard format are the following canonical ones, $B_{ev}$ or $B'_{ev}$:

\[ B_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where} \quad J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \]

or

\[ B_{ev}(m|2n) = \begin{pmatrix} \text{antidiag}(1, \ldots, 1) & 0 \\ 0 & J_{2n} \end{pmatrix}. \]

The usual notation for $\text{aut}(B_{ev}(m|2n))$ is $\mathfrak{osp}(m|2n)$ or, more precisely, $\mathfrak{osp}^{sy}(m|2n)$. Observe that the passage from $V$ to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones, preserved by the “symplectico-orthogonal” Lie superalgebra, $\mathfrak{sp}'(2n|m)$ or, better say, $\mathfrak{osp}^{sk}(m|2n)$, which is isomorphic to $\mathfrak{osp}^{sy}(m|2n)$ but has a different matrix realization. We never use notation $\mathfrak{sp}'(2n|m)$ in order not to confuse with the special Poisson superalgebra.

In the standard format the matrix realizations of these algebras are:

\[
\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^{sk}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},
\]

where $\left( \begin{array}{cc} A & B \\ C & -A^t \end{array} \right) \in \mathfrak{sp}(2n)$, $E \in \mathfrak{o}(m)$.

A nondegenerate supersymmetric odd bilinear form $B_{odd}(n|n)$ can be reduced to a canonical form whose matrix in the standard format is $J_{2n}$. A canonical form of the superskew odd nondegenerate form in the standard format is $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. The usual notation for $\text{aut}(B_{odd}(\text{Par}))$ is $\mathfrak{pe}(\text{Par})$. The passage from $V$ to $\Pi(V)$ establishes an isomorphism $\mathfrak{pe}^{sy}(\text{Par}) \cong \mathfrak{pe}^{sk}(\text{Par})$. This Lie superalgebra is called, as A. Weil suggested, periplectic. The matrix realizations in the standard format of these superalgebras is shorthanded to:

\[
\mathfrak{pe}^{sy}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\}, \quad \text{where} \quad B = -B^t, \quad C = C^t \\
\mathfrak{pe}^{sk}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\}, \quad \text{where} \quad B = B^t, \quad C = -C^t.
\]

Observe that though the Lie superalgebras $\mathfrak{osp}^{sy}(m|2n)$ and $\mathfrak{pe}^{sk}(2n|m)$, as well as $\mathfrak{pe}^{sy}(n)$ and $\mathfrak{pe}^{sk}(n)$, are isomorphic, the difference between them is sometimes crucial, see [Sh3].

The special periplectic superalgebra is $\mathfrak{spe}(n) = \{ X \in \mathfrak{pe}(n) \mid \text{str}X = 0 \}$. Of particular interest to us will be also $\mathfrak{spe}(n)_{a,b} = \mathfrak{spe}(n) \in C(az + bd)$, where $z = 1_{2n}$, $d = \text{diag}(1_n, -1_n)$. Hereafter $a \in b$ or $b \in a$ denote the semidirect product in which $a$ is the ideal.

**1.0.1. What a Lie superalgebra is.** Dealing with superalgebras it sometimes becomes useful to know their definition. Lie superalgebras were distinguished in topology in 1930’s or earlier. So when somebody offers a “better than usual” definition of a notion which seemed to have been established about 70 year ago this might look strange, to say the least. Nevertheless, the answer to the question “what is a Lie superalgebra?” is still not a common knowledge. Indeed, the naive definition (“apply the Sign Rule to the definition of the Lie algebra”) is manifestly inadequate for considering the (singular) supervarieties of deformations and applying representation theory to mathematical physics, for example, in the study of the coadjoint representation of the Lie supergroup which can act on a supermanifold but never on a superspace (an object from another category). So, to deform Lie superalgebras and apply group-theoretical methods in “super” setting, we must be able to recover a supermanifold from a superspace, and vice versa.
A proper definition of Lie superalgebras is as follows, cf. [L3, L2]. The Lie superalgebra in the category of supermanifolds corresponding to the “naive” Lie superalgebra $L = L_0 \oplus L_1$ is a linear supermanifold $L = (L_0, \mathcal{O})$, where the sheaf of functions $\mathcal{O}$ consists of functions on $L_0$ with values in the Grassmann superalgebra on $L_1$; this supermanifold should be such that for “any” (say, finitely generated, or from some other appropriate category) supercommutative superalgebra $C$, the space $\mathcal{L}(C) = \text{Hom}(\text{Spec} C, \mathcal{L})$, called the space of $C$-points of $\mathcal{L}$, is a Lie algebra and the correspondence $C \mapsto \mathcal{L}(C)$ is a functor in $C$. (A. Weil introduced this approach in algebraic geometry in 1954; in super setting it is called the language of points or families, see [1, 2].) This definition might look terribly complicated, but fortunately one can show that the correspondence $\mathcal{L} \mapsto L$ is one-to-one and the Lie algebra $\mathcal{L}(C)$, also denoted $L(C)$, admits a very simple description: $L(C) = (L \otimes C)_0$.

A Lie superalgebra homomorphism $\rho : L_1 \rightarrow L_2$ in these terms is a functor morphism, i.e., a collection of Lie algebra homomorphisms $\rho_C : L_1(C) \rightarrow L_2(C)$ compatible with morphisms of supercommutative superalgebras $C \rightarrow C'$. In particular, a representation of a Lie superalgebra $L$ in a superspace $V$ is a homomorphism $\rho : L \rightarrow \mathfrak{gl}(V)$, i.e., a collection of Lie algebra homomorphisms $\rho_C : L(C) \rightarrow (\mathfrak{gl}(V) \otimes C)_0$.

Example. Consider a representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. The tangent space of the moduli superspace of deformations of $\rho$ is isomorphic to $H^1(\mathfrak{g}; V \otimes V^*)$. For example, if $\mathfrak{g}$ is the 0$|n$-dimensional (i.e., purely odd) Lie superalgebra (with the only bracket possible: identically equal to zero), its only irreducible representations are the trivial one, 1, and $\Pi(1)$. Clearly, $1 \otimes 1^* \simeq \Pi(1) \otimes \Pi(1)^* \simeq 1$, and because the superalgebra is commutative, the differential in the cochain complex is trivial. Therefore, $H^1(\mathfrak{g}; 1) = E^1(\mathfrak{g}^*) \simeq \mathfrak{g}^*$, so there are dim $\mathfrak{g}$ odd parameters of deformations of the trivial representation. If we consider $\mathfrak{g}$ “naively” all of the odd parameters will be lost.

Which of these infinitesimal deformations can be extended to a global one is a separate much tougher question, usually solved ad hoc.

Examples that lucidly illustrate why one should always remember that a Lie superalgebra is not a mere linear superspace but a linear supermanifold are, e.g., deformations $\mathfrak{svect}(0|2n + 1)$ and $\mathfrak{sb}_n(2n - 1|2n)$ with odd parameters considered below and viewed as Lie algebras. In the category of supermanifolds these are simple Lie superalgebras.

1.0.2. Projectivization. If $\mathfrak{s}$ is a Lie algebra of scalar matrices, and $\mathfrak{g} \subset \mathfrak{gl}(n|n)$ is a Lie subsuperalgebra containing $\mathfrak{s}$, then the projective Lie superalgebra of type $\mathfrak{g}$ is $\mathfrak{pg} = \mathfrak{g} / \mathfrak{s}$. Lie superalgebras $\mathfrak{g}_1 \bigoplus \mathfrak{g}_2$ described in sect. 3.1 are also projective.

Projectivization sometimes leads to new Lie superalgebras, for example: $\mathfrak{pgl}(n|n), \mathfrak{psl}(n|n), \mathfrak{pq}(n), \mathfrak{psq}(n)$; whereas $\mathfrak{pgl}(p|q) \cong \mathfrak{sl}(p|q)$ if $p \neq q$.

1.0.3. What is a semisimple Lie superalgebra. These algebras are needed in description of primitive Lie superalgebras of vector fields — a geometrical natural problem though wild for Lie superalgebras, see [ALSh]. Recall that the Lie superalgebra $\mathfrak{g}$ without proper ideals and of dimension $> 1$ is called simple. Examples: $\mathfrak{sl}(m|n)$ for $m > n \geq 1$, $\mathfrak{psl}(n|n)$ for $n > 1$, $\mathfrak{psq}(n)$ for $n > 2$, $\mathfrak{osp}(m|2n)$ for $mn \neq 0$ and $\mathfrak{sp}(n)$ for $n > 2$.

We will not need the remaining simple finite dimensional Lie superalgebras of non-vectorial type. These superalgebras, discovered by I. Kaplansky (a 1975-preprint, see [K]) are $\mathfrak{osp}_n(4|2)$, the deforms of $\mathfrak{osp}(4|2)$, and the two exceptions that we denote by $\mathfrak{ag}_2$ and $\mathfrak{ab}_3$. For their description we refer to [K, FR, Sud], see also [GL3] for the description of the system of simple roots see [Ka4] completed in [vdL, S1, S2].

We say that $\mathfrak{h}$ is almost simple if it can be sandwiched (non-strictly) between a simple Lie superalgebra $\mathfrak{s}$ and the Lie superalgebra $\mathfrak{der} \mathfrak{s}$ of derivations of $\mathfrak{s}$; $\mathfrak{s} \subset \mathfrak{h} \subset \mathfrak{der} \mathfrak{s}$.
By definition, $\mathfrak{g}$ is semisimple if its radical is zero. Literally following the description of semisimple Lie algebras over the fields of prime characteristic, V. Kac [Ka4] gave the following description of semisimple Lie superalgebras. Let $\mathfrak{s}_1, \ldots, \mathfrak{s}_k$ be simple Lie superalgebras, let $n_1, \ldots, n_k$ be pairs of non-negative integers $n_j = (n_j^0, n_j^1)$, let $\mathcal{F}(n_j)$ be the supercommutative superalgebra of polynomials in $n_j^0$ even and $n_j^1$ odd indeterminates, and $\mathfrak{s} = \bigoplus_j (\mathfrak{s}_j \otimes \mathcal{F}(n_j))$. Then
\begin{equation}
\text{der } \mathfrak{s} = \bigoplus_j \left( (\text{der } \mathfrak{s}_j) \otimes \mathcal{F}(n_j) \right) \oplus \text{id}_{\mathfrak{s}_j} \otimes \text{vect}(n_j)). \tag{0.1.3}
\end{equation}

Let $\mathfrak{g}$ be a subalgebra of $\text{der } \mathfrak{s}$ containing $\mathfrak{s}$. If the projection of $\mathfrak{g}$ on $1 \otimes \text{vect}(n_j)_{-1}$ is onto for each $j$, then $\mathfrak{g}$ is semisimple and all semisimple Lie superalgebras arise in the manner indicated, i.e., as sums of almost simple superalgebras corresponding to the summands of (0.1.3).

1.0.4. A. Sergeev’s central extension. In 70’s A. Sergeev proved that there is just one nontrivial central extension of $\text{spe}(n)$. It exists only for $n = 4$ and we denote it by $\mathfrak{as}$. Let us represent an arbitrary element $A \in \mathfrak{as}$ as a pair $A = x + d \cdot z$, where $x \in \text{spe}(4)$, $d \in \mathbb{C}$ and $z$ is the central element. The bracket in $\mathfrak{as}$ is
\begin{equation}
\left[ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + d' \cdot z \right] = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + \text{tr } cc' \cdot z, \tag{0.1.4.1}
\end{equation}
where $-$ is extended via linearity from matrices $c_{ij} = E_{ij} - E_{ji}$ on which $\bar{c}_{ij} = c_{kl}$ for any even permutation $(1234) \mapsto (ijkl)$.

The Lie superalgebra $\mathfrak{as}$ can also be described by means of the spinor representation. For this we need several vectorial superalgebras defined in sect. 0.3. Consider $\mathfrak{po}(0|6)$, the Lie superalgebra whose superspace is the Grassmann superalgebra $\Lambda(\xi, \eta)$ generated by $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ and the bracket is the Poisson bracket (0.3.6). Recall that $\mathfrak{h}(0|6) = \text{Span}(\mathcal{H}_f \mid f \in \Lambda(\xi, \eta))$.

Now, observe that $\text{spe}(4)$ can be embedded into $\mathfrak{h}(0|6)$. Indeed, setting $\text{deg } \xi_i = \text{deg } \eta_i = 1$ for all $i$ we introduce a $\mathbb{Z}$-grading on $\Lambda(\xi, \eta)$ which, in turn, induces a $\mathbb{Z}$-grading on $\mathfrak{h}(0|6)$ of the form $\mathfrak{h}(0|6) = \bigoplus_{i \geq -1} \mathfrak{h}(0|6)_i$. Since $\mathfrak{s}(4) \cong \mathfrak{o}(6)$, we can identify $\text{spe}(4)_1$ with $\mathfrak{h}(0|6)_0$.

It is not difficult to see that the elements of degree $-1$ in the standard gradings of $\text{spe}(4)$ and $\mathfrak{h}(0|6)$ constitute isomorphic $\mathfrak{s}(4) \cong \mathfrak{o}(6)$-modules. It is subject to a direct verification that it is possible to embed $\text{spe}(4)_1$ into $\mathfrak{h}(0|6)_1$.

Sergeev’s extension $\mathfrak{as}$ is the result of the restriction to $\text{spe}(4) \subset \mathfrak{h}(0|6)$ of the cocycle that turns $\mathfrak{h}(0|6)$ into $\mathfrak{po}(0|6)$. The quantization deforms $\mathfrak{po}(0|6)$ into $\mathfrak{gl}(\Lambda(\xi))$: the through maps $T_\lambda : \mathfrak{as} \rightarrow \mathfrak{po}(0|6) \rightarrow \mathfrak{gl}(\Lambda(\xi))$ are representations of $\mathfrak{as}$ in the 4|4-dimensional modules spin$_\lambda$ isomorphic to each other for all $\lambda \neq 0$. The explicit form of $T_\lambda$ is as follows:
\begin{equation}
T_\lambda : \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z \mapsto \begin{pmatrix} a & b - \lambda \bar{c} \\ c & -a^t \end{pmatrix} + \lambda d \cdot 1_{4|4}, \tag{0.1.4.2}
\end{equation}
where $1_{4|4}$ is the unit matrix and $\bar{c}$ is defined in formula (0.1.4.1). Clearly, $T_\lambda$ is an irreducible representation for any $\lambda$.

1.0.5. Vectorial Lie superalgebras. The standard realization. The elements of the Lie algebra $\mathcal{L} = \text{der } \mathbb{C}[u]$ are considered as vector fields. The Lie algebra $\mathcal{L}$ has only one maximal subalgebra $\mathcal{L}_0$ of finite codimension (consisting of the fields that vanish at the origin). The subalgebra $\mathcal{L}_0$ determines a filtration of $\mathcal{L}$: set
\begin{equation}
\mathcal{L}_{-1} = \mathcal{L} \text{ and } \mathcal{L}_i = \{ D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}] \subset \mathcal{L}_{i-1} \} \text{ for } i \geq 1. \tag{0.2.1}
\end{equation}
The associated graded Lie superalgebra $L = \bigoplus_{i \geq -1} L_i$, where $L_i = \mathcal{L}_i / \mathcal{L}_{i+1}$, consists of the vector fields with polynomial coefficients.

**Superization.** For a simple Lie superalgebra $\mathcal{L}$ (for example, take $\mathcal{L} = \text{der} \mathbb{C}[u, \xi]$), suppose $\mathcal{L}_0 \subset \mathcal{L}$ is a maximal subalgebra of finite codimension. Let $\mathcal{L}_{-1}$ be a minimal subspace of $\mathcal{L}$ containing $\mathcal{L}_0$, different from $\mathcal{L}_0$ and $\mathcal{L}_0$-invariant. A Weisfeiler filtration of $\mathcal{L}$ is determined by setting for $i \geq 1$:

$$\mathcal{L}_{i-1} = [\mathcal{L}_{-1}, \mathcal{L}_{-1}] + \mathcal{L}_{-1} \quad \text{and} \quad \mathcal{L}_i = \{ D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1} \}. \quad (0.2.2)$$

Since the codimension of $\mathcal{L}_0$ is finite, the filtration takes the form

$$\mathcal{L} = \mathcal{L}_{-d} \supset \cdots \supset \mathcal{L}_0 \supset \cdots \quad (0.2.3)$$

for some $d$. This $d$ is the depth of $\mathcal{L}$ and of the associated graded Lie superalgebra $L$.

Considering the subspaces (0.2.1) as the basis of a topology, we can complete the graded or filtered Lie superalgebras $L$ or $\mathcal{L}$; the elements of the completion are the vector fields with formal power series as coefficients. Though the structure of the graded algebras is easier to describe, in applications the completed Lie superalgebras are usually needed.

Observe that not all filtered or graded Lie superalgebras of finite depth are vectorial, i.e., realizable with vector fields on a supermanifold of the same dimension as that of $\mathcal{L} / \mathcal{L}_0$; only those with faithful $L_0$-action on $L_-$ are.

Unlike Lie algebras, simple vectorial superalgebras possess several nonisomorphic maximal subalgebras of finite codimension, see 1.2.2.

1) **General algebras.** Let $x = (u_1, \ldots, u_n, \theta_1, \ldots, \theta_m)$, where the $u_i$ are even indeterminates and the $\theta_j$ are odd ones. Set $\text{vect}(n|m) = \text{der} \mathbb{C}[x]$; it is called the general vectorial Lie superalgebra.

2) **Special algebras.** The divergence of the field $D = \sum_i f_i \frac{\partial}{\partial u_i} + \sum_j g_j \frac{\partial}{\partial \theta_j}$ is the function (in our case: a polynomial, or a series)

$$\text{div}D = \sum_i \frac{\partial f_i}{\partial u_i} + \sum_j (-1)^{p(g_j)} \frac{\partial g_j}{\partial \theta_j}. \quad (0.2.2)$$

- The Lie superalgebra $\text{svect}(n|m) = \{ D \in \text{vect}(n|m) \mid \text{div}D = 0 \}$ is called the special (or divergence-free) vectorial superalgebra.

It is clear that it is also possible to describe $\text{svect}(n|m)$ as $\{ D \in \text{vect}(n|m) \mid L_D \text{vol}_x = 0 \}$, where $\text{vol}_x$ is the volume form with constant coefficients in coordinates $x$ (see 0.6) and $L_D$ the Lie derivative with respect to $D$.

- The Lie superalgebra $\text{svect}_\lambda(0|m) = \{ D \in \text{vect}(0|m) \mid \text{div}(1 + \lambda \theta_1 \cdots \theta_m)D = 0 \}$, where $p(\lambda) \equiv m \pmod{2}$, — the deform of $\text{svect}(0|m)$ — is called the deformed special (or divergence-free) vectorial superalgebra. Clearly, $\text{svect}_\lambda(0|m) \cong \text{svect}_\mu(0|m)$ for $\lambda \mu \neq 0$. So we briefly denote these deformations by $\text{svect}(0|m)$.

Observe that for odd $m$ the parameter of deformation, $\lambda$, is odd.

**Remark.** As is customary in differential geometry, where meaningful notations prevail, we sometimes write $\text{vect}(x)$ or $\text{vect}(V)$ if $V = \text{Span}(x)$ and use similar notations for the subalgebras of $\text{vect}$ introduced below. Some algebraists sometimes abbreviate $\text{vect}(n)$ and $\text{svect}(n)$ to $W_n$ (in honor of Witt) and $S_n$, respectively.

3) **The algebras that preserve Pfaff equations and differential 2-forms.**

- Set $u = (t, p_1, \ldots, p_n, q_1, \ldots, q_n)$; let

$$\tilde{\alpha}_1 = dt + \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq m} \theta_j d\theta_j \quad \text{and} \quad \tilde{\omega}_0 = d\tilde{\alpha}_1.$$
The form $\tilde{\alpha}_1$ is called contact, the form $\tilde{\omega}_0$ is called symplectic. Sometimes it is more convenient to redenote the $\theta$'s and set

$$\xi_j = \frac{1}{\sqrt{2}}(\theta_j - i\theta_{r+j}); \quad \eta_j = \frac{1}{\sqrt{2}}(\theta_j + i\theta_{r+j})$$

for $j \leq r = \lfloor m/2 \rfloor$ (here $i^2 = -1$), $\theta = \theta_{2r+1}$

and in place of $\tilde{\omega}_0$ or $\tilde{\alpha}_1$ take $\alpha_1$ and $\omega_0 = d\alpha_1$, respectively, where

$$\alpha_1 = dt + \sum_{1 \leq i \leq n}(p_idq_i - q_idp_i) + \sum_{1 \leq j \leq r}(\xi_jdn_j + n_jd\xi_j) \begin{cases} +\theta d\theta & \text{if } m = 2r \\
\theta d\theta & \text{if } m = 2r + 1. \end{cases}$$

The Lie superalgebra that preserves the Pfaff equation $\alpha_1 = 0$, i.e., the superalgebra

$$\mathfrak{e}(2n+1|m) = \{D \in \text{vect}(2n+1|m) \mid L_D\alpha_1 = f_D\alpha_1 \text{ for some } f_D \in \mathbb{C}[t, p, q, \theta]\},$$

is called the contact superalgebra. The Lie superalgebra

$$\mathfrak{po}(2n|m) = \{D \in \mathfrak{e}(2n+1|m) \mid L_D\alpha_1 = 0\}$$

is called the Poisson superalgebra. (A geometric interpretation of the Poisson superalgebra: it is the Lie superalgebra that preserves the connection with form $\alpha$ in the line bundle over a symplectic supermanifold with the symplectic form $d\alpha$.)

- Similarly, set $u = q = (q_1, \ldots, q_n)$, let $\theta = (\xi_1, \ldots, \xi_n; \tau)$ be odd. Set

$$\alpha_0 = d\tau + \sum_i(\xi_idq_i + q_id\xi_i), \quad \omega_1 = d\alpha_0$$

and call these forms the odd-contact and periplectic, respectively.

The Lie superalgebra that preserves the Pfaff equation $\alpha_0 = 0$, i.e., the superalgebra

$$\mathfrak{m}(n) = \{D \in \text{vect}(n|n+1) \mid L_D\alpha_0 = f_D \cdot \alpha_0 \text{ for some } f_D \in \mathbb{C}[q, \xi, \tau]\}$$

is called the odd-contact superalgebra.

The Lie superalgebra

$$\mathfrak{b}(n) = \{D \in \mathfrak{m}(n) \mid L_D\alpha_0 = 0\}$$

is called the Buttin superalgebra \cite{But}. (A geometric interpretation of the Buttin superalgebra: it is the Lie superalgebra that preserves the connection with form $\alpha_1$ in the line bundle of rank $\varepsilon = (0|1)$ over a periplectic supermanifold, i.e., over a supermanifold with the periplectic form $d\alpha_0$.)

The Lie superalgebras

$$\mathfrak{sm}(n) = \{D \in \mathfrak{m}(n) \mid \text{div } D = 0\}, \quad \mathfrak{sb}(n) = \{D \in \mathfrak{b}(n) \mid \text{div } D = 0\}$$

are called the divergence-free (or special) odd-contact and special Buttin superalgebras, respectively.

Remark. A relation with finite dimensional geometry is as follows. Clearly, $\ker \alpha_1 = \ker \tilde{\alpha}_1$. The restriction of $\tilde{\omega}_0$ to $\ker \alpha_1$ is the orthosymplectic form $B_{ev}(m|2n)$; the restriction of $\omega_0$ to $\ker \tilde{\alpha}_1$ is $B'_{ev}(m|2n)$. Similarly, the restriction of $\omega_1$ to $\ker \alpha_0$ is $B_{odd}(n|n)$.

1.0.5.1. Generating functions. A laconic way to describe $\mathfrak{e}$, $\mathfrak{m}$ and their subalgebras is via generating functions.

- Odd form $\alpha_1$. For $f \in \mathbb{C}[t, p, q, \theta]$ set:

$$K_f = (2 - E)(f) \frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t} E,$$

$$H_f = f$$

(0.3.1)
where $E = \sum \frac{\partial}{\partial y_i}$ (here the $y_i$ are all the coordinates except $t$) is the Euler operator (which counts the degree with respect to the $y_i$), and $H_f$ is the Hamiltonian field with Hamiltonian $f$ that preserves $d\tilde{\alpha}_1$:

$$H_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial}{\partial \theta_j}. \quad (0.3.2)$$

The choice of the form $\alpha_1$ instead of $\tilde{\alpha}_1$ only affects the shape of $H_f$ that we give for $m = 2k + 1$:

$$H_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (-1)^{p(f)} \left( \sum_{j \leq k} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial q_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial}{\partial q_j} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right) \right).$$

- Even form $\alpha_0$. For $f \in \mathbb{C}[q, \xi, \tau]$ set:

$$M_f = (2 - E)(f) \frac{\partial}{\partial \tau} - Lf - (-1)^{p(f)} \frac{\partial f}{\partial \tau} E, \quad (0.3.3)$$

where $E = \sum y_i \frac{\partial}{\partial y_i}$ (here the $y_i$ are all the coordinates except $\tau$), and

$$L_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \xi_i} \right). \quad (0.3.4)$$

Since

$$L_{K_f}(\alpha_1) = 2 \frac{\partial f}{\partial \tau} \alpha_1 = K_1(f) \alpha_1, \quad L_{M_f}(\alpha_0) = -(-1)^{p(f)} 2 \frac{\partial f}{\partial \tau} \alpha_0 = -(-1)^{p(f)} M_1(f) \alpha_0, \quad (0.3.5)$$

it follows that $K_f \in \mathfrak{g}(2n + 1|m)$ and $M_f \in \mathfrak{m}(n)$. Observe that

$$p(Lf) = p(Mf) = p(f) + 1.$$

- To the (super)commutators $[K_f, K_g]$ or $[M_f, M_g]$ there correspond contact brackets of the generating functions:

$$[K_f, K_g] = [K_{\{f, g\}}]_{\text{k.b.}}; \quad [M_f, M_g] = [M_{\{f, g\}}]_{\text{m.b.}}.$$

The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on $t$ (resp. $\tau$).

The Poisson bracket $\{\cdot, \cdot\}_{\text{p.b.}}$ (in the realization with the form $\omega_0$) is given by the formula

$$\{f, g\}_{\text{p.b.}} = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} \right) - (-1)^{p(f)} \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_j}. \quad (0.3.6)$$

and in the realization with the form $\omega_0$ for $m = 2k + 1$ it is given by the formula

$$\{f, g\}_{\text{p.b.}} = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} \right) - (-1)^{p(f)} \left( \sum_{j \leq m} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \xi_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j} + \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right) \right).$$

The Buttin bracket $\{\cdot, \cdot\}_{\text{b.b.}}$ is given by the formula

$$\{f, g\}_{\text{b.b.}} = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i} \right). \quad (0.3.7)$$
Remark. The what we call here “Buttin bracket” was discovered in pre-super era by Schouten; Buttin was the first to prove that this bracket establishes a Lie superalgebra structure. The interpretations of the Buttin superalgebra similar to that of the Poisson algebra and of the elements of \( \mathfrak{le} \) as analogs of Hamiltonian vector fields was given in \( \text{[L1]} \). The Buttin bracket and “odd mechanics” introduced in \( \text{[L1]} \) was rediscovered by Batalin with Vilkovisky; it gained a great deal of currency under the name antibracket, cf. \( \text{GPS} \).

The Schouten bracket was originally defined on the superspace of polyvector fields on a manifold, i.e., on the superspace of sections of the exterior algebra (over the algebra \( \text{F} \) of functions) of the tangent bundle, \( \Gamma(\Lambda^r(T(M))) \cong \Lambda^r_{\mathfrak{p}}(\text{Vect}(M)) \). The explicit formula of the Schouten bracket (in which the hatted slot should be ignored, as usual) is

\[
[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_l. 
\]

With the help of Sign Rule we easily superize formula (\( * \)) for the case when \( M \) is replaced with a supermanifold \( \mathcal{M} \). Let \( x \) and \( \xi \) be the even and odd coordinates on \( \mathcal{M} \). Setting \( \theta_i = \Pi(\frac{\partial}{\partial x^i}) = \hat{x}_i, q_j = \Pi(\frac{\partial}{\partial \xi_j}) = \xi_j \) we get an identification of the Schouten bracket of polyvector fields on \( \mathcal{M} \) with the Buttin bracket of functions on the supermanifold \( \mathcal{N} \) whose coordinates are \( x, \xi \) and \( \hat{x}, \hat{\xi} \).

In terms of the Poisson and Buttin brackets, respectively, the contact brackets are

\[
\{f, g\}_{k, b.} = (2 - E)(f) \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} (2 - E)(g) - \{f, g\}_{P, b.} 
\]

and

\[
\{f, g\}_{m, b.} = (2 - E)(f) \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} (2 - E)(g) - \{f, g\}_{B, b.}.
\]

The Lie superalgebras of Hamiltonian fields (or Hamiltonian superalgebra) and its special subalgebra (defined only if \( m = 0 \)) are

\[
\mathfrak{h}(2n|m) = \{D \in \text{vect}(2n|m) \mid L_D \omega_0 = 0\} \quad \text{and} \quad \mathfrak{sh}(m) = \{H_f \in \mathfrak{h}(0|m) \mid \int f \text{vol}_\theta = 0\}.
\]

The “odd” analogues of the Lie superalgebra of Hamiltonian fields are the Lie superalgebra of vector fields \( \text{Le}_f \) introduced in \( \text{[L1]} \) and its special subalgebra:

\[
\mathfrak{le}(n) = \{D \in \text{vect}(n|n) \mid L_D \omega_1 = 0\} \quad \text{and} \quad \mathfrak{set}(n) = \{D \in \mathfrak{le}(n) \mid \text{div} D = 0\}.
\]

It is not difficult to prove the following isomorphisms (as superspaces):

\[
\mathfrak{t}(2n + 1|m) \cong \text{Span}(K_f \mid f \in \mathbb{C}[t, p, q, \xi]); \quad \mathfrak{le}(n) \cong \text{Span}(\text{Le}_f \mid f \in \mathbb{C}[q, \xi]);
\]

\[
\mathfrak{m}(n) \cong \text{Span}(M_f \mid f \in \mathbb{C}[\tau, q, \xi]); \quad \mathfrak{h}(2n|m) \cong \text{Span}(H_f \mid f \in \mathbb{C}[p, q, \xi]).
\]

Set \( \mathfrak{sp}(m) = \{K_f \in \mathfrak{po}(0|m) \mid \int f \text{vol}_\xi = 0\} \) and \( \mathfrak{sh}(m) = \mathfrak{sp}(m)/\mathbb{C} \cdot K_1 \).

1.0.5.2. Divergence-free subalgebras. Since, as is easy to calculate,

\[
\text{div} K_f = (2n + 2 - m) K_1(f),
\]

it follows that the divergence-free subalgebra of the contact Lie superalgebra either coincides with it (for \( m = 2n + 2 \)) or is the Poisson superalgebra. For the “odd” contact series the situation is more interesting: the divergence free subalgebra is simple and new (as compared with the above list).

Since

\[
\text{div} M_f = (-1)^{p(f)} 2 \left( 1 - E \right) \frac{\partial f}{\partial \tau} - \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i},
\]

(0.4.1)
it follows that
\[ \text{sm}(n) = \text{Span} \left( M_f \in \text{m}(n) \mid (1 - E) \frac{\partial f}{\partial \tau} = \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right). \]

In particular,
\[ \text{div}Le_f = (-1)^p(f)2 \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i}. \]  
(0.4.2)

The odd analog of the Laplacian, namely, the operator
\[ \Delta = \sum_{i \leq n} \frac{\partial^2}{\partial q_i \partial \xi_i} \]  
(0.4.3)
on a periplectic supermanifold appeared in physics under the name of BRST operator, cf. [GPS]. The divergence-free vector fields from \( \text{sl}(n) \) are generated by harmonic functions, i.e., such that \( \Delta(f) = 0 \).

Lie superalgebras \( \text{sl}(n) \), \( \text{sb}(n) \) and \( \text{svect}(1|n) \) have ideals \( \text{sl}(n) \), \( \text{sb}(n) \) and \( \text{svect}(n) \) of codimension 1 defined from the exact sequences
\[ 0 \longrightarrow \text{sl}(n) \longrightarrow \text{sl}(n) \longrightarrow C \cdot L\xi_1...\xi_n \longrightarrow 0, \]
\[ 0 \longrightarrow \text{sb}(n) \longrightarrow \text{sb}(n) \longrightarrow C \cdot M\xi_1...\xi_n \longrightarrow 0, \]
\[ 0 \longrightarrow \text{svect}(n) \longrightarrow \text{svect}(1|n) \longrightarrow C \cdot \xi_1...\xi_n \frac{\partial}{\partial t} \longrightarrow 0. \]

1.0.5.3. The Cartan prolongs. We will repeatedly use the Cartan prolong. So let us recall the definition and generalize it somewhat. Let \( g \) be a Lie algebra, \( V \) a \( g \)-module, \( S^i \) the operator of the \( i \)-th symmetric power. Set \( g_{-1} = V; g_0 = g \) and for \( i > 0 \) define the \( i \)-th Cartan prolong (the result of Cartan’s prolongation) of the pair \( (g_{-1}, g_0) \) as
\[ g_k = \{ X \in \text{Hom}(g_{-1}, g_{k-1}) \mid X(v_0)(v_1, v_2, \ldots v_k) = X(v_1)(v_0, v_2, \ldots, v_k) \text{ for any } v_i \in g_{-1} \}. \]

Equivalently, let
\[ i : S^{k+1}(g_{-1})^* \otimes g_{-1} \longrightarrow S^k(g_{-1})^* \otimes g_{-1} \]  
(0.5.1)
be the natural embedding and
\[ j : S^k(g_{-1})^* \otimes g_0 \longrightarrow S^k(g_{-1})^* \otimes g_{-1} \]  
(0.5.2)
the natural map. Then \( g_k = i(S^{k+1}(g_{-1})^* \otimes g_{-1}) \cap j(S^k(g_{-1})^* \otimes g_0). \)

The Cartan prolong of the pair \( (V, g) \) is \( (g_{-1}, g_0)_* = \bigoplus_{k \geq -1} g_k \).

In what follows \( \cdot \) in superscript denotes, as is now customary, the collection of all degrees, while \( \ast \) is reserved for dualization; in the subscripts we retain the old-fashioned \( \ast \) instead of \( \cdot \) to avoid too close a contact with the punctuation marks.)

Suppose that the \( g_0 \)-module \( g_{-1} \) is faithful. Then, clearly,
\[ (g_{-1}, g_0)_* \subset \text{vect}(n) = \text{der} C[x_1, \ldots, x_n], \text{ where } n = \text{dim } g_{-1} \text{ and } \]
\[ g_i = \{ D \in \text{vect}(n) \mid \deg D = i; [D, X] \in g_{i-1} \text{ for any } X \in g_{-1} \}. \]

It is subject to an easy verification that the Lie algebra structure on \( \text{vect}(n) \) induces same on \( (g_{-1}, g_0)_* \).

Of the four simple vectorial Lie algebras, three are Cartan prolongs: \( \text{vect}(n) = (\text{id}, gl(n))_* \), \( \text{svect}(n) = (\text{id}, sl(n))_* \) and \( h(2n) = (\text{id}, sp(n))_* \). The fourth one — \( \mathfrak{f}(2n + 1) \) — is the result of a trifle more general construction described as follows.
1.0.5.4. A generalization of the Cartan prolong. Let \( g_- = \bigoplus_{d \leq i \leq 1} g_i \) be a nilpotent \( \mathbb{Z} \)-graded Lie algebra and \( g_0 \subset \text{der}_0 g \) a Lie subalgebra of the \( \mathbb{Z} \)-grading-preserving derivations. Let

\[
i : S^{k+1}(g_-)^* \otimes g_- \longrightarrow S^k(g_-)^* \otimes g_-^* \otimes g_-
\]

and

\[
j : S^k(g_-)^* \otimes g_0 \longrightarrow S^k(g_-)^* \otimes g_-^* \otimes g_-
\]

be the natural embeddings similar to (0.5.1) and (0.5.2), respectively. For \( k > 0 \) define the \( k \)-th prolong of the pair \((g_-, g_0)\) to be:

\[
g_k = (j(S^*(g_-)^* \otimes g_0) \cap i(S^*(g_-)^* \otimes g_-))_k,
\]

where the subscript \( k \) in the right hand side singles out the component of degree \( k \).

Set \((g_-, g_0)_* = \bigoplus_{i \geq -d} g_i^*\); then, as is easy to verify, \((g_-, g_0)_*\) is a Lie algebra.

What is the Lie algebra of contact vector fields in these terms? Denote by \( \text{hei}(2n) \) the Heisenberg Lie algebra: its space is \( W \oplus i \cdot z \), where \( W \) is a 2\( n \)-dimensional space endowed with a nondegenerate skew-symmetric bilinear form \( B \) and the bracket in \( \text{hei}(2n) \) is given by the following relations:

\[
z \text{ is in the center and } [v, w] = B(v, w) \cdot z \text{ for any } v, w \in W.
\]

Clearly, \( \mathfrak{g}(2n + 1) \cong (\text{hei}(2n), \mathfrak{cosp}(2n))_* \).

1.0.5.5. Lie superalgebras of vector fields as Cartan’s prolongs. The superization of the constructions from sec. 0.5 are straightforward: via Sign Rule. We thus get infinite dimensional Lie superalgebras

\[
\text{vect}(m|n) = (\text{id}, \mathfrak{gl}(m|n))_*; \quad \text{s vect}(m|n) = (\text{id}, \mathfrak{sl}(m|n))_*;
\]

\[
\mathfrak{h}(2m|n) = (\text{id}, \mathfrak{osp}^{sk}(m|2n))_*;
\]

\[
\mathfrak{le}(n) = (\text{id}, \mathfrak{pe}^{sk}(n))_*; \quad \mathfrak{sl}(n) = (\text{id}, \mathfrak{sp}^{sk}(n))_*.
\]

Remark. Observe that the Cartan prolongs \((\text{id}, \mathfrak{osp}^{su}(m|2n))_*\) and \((\text{id}, \mathfrak{pe}^{su}(n))_*\) are finite dimensional.

The generalization of Cartan’s prolongations described in sec. 0.5.1 was first defined in [ALSI] (and repeatedly used later, e.g., in [CK1]). Observe that after superization it has \textbf{two} analogs associated with the contact series \( \mathfrak{g} \) and \( \mathfrak{m} \), respectively.

- Define the Lie superalgebra \( \text{hei}(2n|m) \) on the direct sum of a \((2n, m)\)-dimensional superspace \( W \) endowed with a nondegenerate skew-symmetric bilinear form and a \((1, 0)\)-dimensional space spanned by \( z \).

Clearly, we have \( \mathfrak{g}(2n + 1|m) = (\text{hei}(2n|m), \mathfrak{cosp}^{sk}(m|2n))_* \) and, given \( \text{hei}(2n|m) \) and a subalgebra \( g \) of \( \mathfrak{cosp}^{sk}(m|2n) \), we call \((\text{hei}(2n|m), g)_*\) the \( k \)-prolong of \((W, g)\), where \( W \) is the identity \( \mathfrak{osp}^{sk}(m|2n) \)-module.

- The “odd” analog of \( \mathfrak{g} \) is associated with the following “odd” analog of \( \text{hei}(2n|m) \). Denote by \( \mathfrak{ab}(n) \) the antibracket Lie superalgebra: its space is \( W \oplus i \cdot z \), where \( W \) is an \( n|n \)-dimensional superspace endowed with a nondegenerate skew-symmetric odd bilinear form \( B \); the bracket in \( \mathfrak{ab}(n) \) is given by the following relations:

\[
z \text{ is odd and lies in the center; } [v, w] = B(v, w) \cdot z \text{ for } v, w \in W.
\]

Set \( \mathfrak{m}(n) = (\mathfrak{ab}(n), \mathfrak{pe}^{sk}(n))_* \) and, given \( \mathfrak{ab}(n) \) and a subalgebra \( g \) of \( \mathfrak{pe}^{sk}(n) \), we call \((\mathfrak{ab}(n), g)_*\) the \( m \)-prolong of \((W, g)\), where \( W \) is the identity \( \mathfrak{pe}^{sk}(n) \)-module.

Generally, given a nondegenerate form \( B \) on a superspace \( W \) and a superalgebra \( g \) that preserves \( B \), we refer to the above generalized prolongations as to \textit{mk-prolongation} of the pair \((W, g)\).
1.0.5.6. A partial Cartan prolong of \(( \oplus \mathfrak{g}_i, \mathfrak{h}_1)\). Consider the generalized Cartan prolong \((\mathfrak{g}_-, \mathfrak{g}_0)_s\). Take a \(\mathfrak{g}_0\)-submodule \(\mathfrak{h}_1\) in \(\mathfrak{g}_1\) such that \([\mathfrak{g}_-, \mathfrak{h}_1] = \mathfrak{g}_0\). If such \(\mathfrak{h}_1\) exists (usually, the inclusion \([\mathfrak{g}_-, \mathfrak{h}_1] \subset \mathfrak{g}_0\) is strict), define the \(i\)th prolongation of \((\oplus \mathfrak{g}_i, \mathfrak{h}_1)\) for \(i \geq 2\) to be \(\mathfrak{h}_i = \{D \in \mathfrak{g}_i \mid [D, \mathfrak{g}_-] \subset \mathfrak{h}_{i-1}\}\). Set \(\mathfrak{h}_i = \mathfrak{g}_i\) for \(i < 0\) and \(\mathfrak{h}_s = \sum \mathfrak{h}_i\).

Examples: \(\text{vect}(1|n)\) is a subalgebra of \(\mathfrak{t}(1|2n)\). The former is obtained as Cartan’s prolong of the same nonpositive part as \(\mathfrak{t}(1|2n)\) and a submodule of \(\mathfrak{t}(1|2n)\), cf. Table 1.2.1. The simple exceptional superalgebra \(\mathfrak{f}\) as introduced in 1.2.3 is another example.

1.0.6. The modules of tensor fields. To advance further, we have to recall the definition of the modules of tensor fields over \(\text{vect}(m|n)\) and its subalgebras, see [BL1], [L2]. For any other \(\mathbb{Z}\)-graded vectorial Lie superalgebra the construction is identical.

Let \(\mathfrak{g} = \text{vect}(m|n)\) and \(\mathfrak{g}_s = \oplus \mathfrak{g}_i\). Clearly, \(\text{vect}_0(m|n) \cong \mathfrak{gl}(m|n)\). Let \(V\) be the \(\mathfrak{gl}(m|n)\)-module with the lowest weight \(\lambda = \text{lwt}(V)\). Make \(V\) into a \(\mathfrak{g}_s\)-module setting \(\mathfrak{g}_s \cdot V = 0\) for \(\mathfrak{g}_+ = \oplus \mathfrak{g}_i\). Let us realize \(\mathfrak{g}\) by vector fields on the \(m|n\)-dimensional linear supermanifold \(C^{m|n}\) with coordinates \(x = (u, \xi)\). The superspace \(T(V) = \text{Hom}_{U(\mathfrak{g}_s)}(U(\mathfrak{g}), V)\) is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to \(\mathbb{C}[[x]] \otimes V\). Its elements have a natural interpretation as formal tensor fields of type \(V\). When \(\lambda = (a, \ldots, a)\) we will simply write \(T(\bar{a})\) instead of \(T(\lambda)\). We will usually consider \(\mathfrak{g}\)-modules induced from irreducible \(\mathfrak{g}_0\)-modules.

Examples: \(\text{vect}(m|n)\) as \(\text{vect}(m|n)\)- and \(\text{svect}(m|n)\)-modules is \(T(\text{id})\). More examples:

\(T(\bar{0})\) is the superspace of functions; \(\text{Vol}(m|n) = T(1, \ldots, 1; -1, \ldots, -1)\) (the semicolon separates the first \(m\) (“even”) coordinates of the weight with respect to the matrix units \(E_{ii}\) of \(\mathfrak{gl}(m|n)\)) is the superspace of densities or volume forms. We denote the generator of \(\text{Vol}(m|n)\) corresponding to the ordered set of coordinates \(x\) by \(\text{vol}(x)\). The space of \(\lambda\)-densities is \(\text{Vol}^\lambda(m|n) = T(\lambda, \ldots, \lambda; -\lambda, \ldots, -\lambda)\). In particular, \(\text{Vol}^\lambda(m|0) = T(\bar{\lambda})\) but \(\text{Vol}^\lambda(0|n) = T(-\bar{\lambda})\). We set: \(\text{Vol}_0(0|m) = \{v \in \text{Vol} \mid f \cdot v = 0\}\) and \(T_0(\bar{0}) = \Lambda(m)/\mathbb{C} \cdot 1\).

If the generator \(\text{vol}\) of \(\text{Vol}\) is fixed, then \(\text{Vol} \cong T(\bar{0})\), as \(\text{svect}(m|n)\)-modules. We set \(T_0(\bar{0})\) to denote the \(\text{svect}(m|n)\)-module \(\text{Vol}_0(0|m)/\mathbb{C} \cdot \text{vol}(\xi)\).

Remark. To view the volume element as \(d^m u d^n \xi\) is totally wrong; the superdeterminant can never appear as a factor under the changes of variables. We can try to use the usual notations of differentials provided all the differentials anticommute. Then the linear transformations that do not intermix the even \(u\)’s with the odd \(\xi\)’s multiply the volume element \(\text{vol}(x)\), viewed as the fraction \(\frac{d u_1 \cdots d u_m}{d \xi_1 \cdots d \xi_n}\), by the Berezinian of the transformation. But how could we justify this? Let \(X = (x, \xi)\). If we consider the usual, exterior, differential forms, then the \(dX_i\)’s super anti-commute, hence, the \(d \xi_i\) commute; whereas if we consider the symmetric product of the differentials, as in the metrics, then the \(dX_i\)’s supercommute, hence, the \(d \xi_i\) commute. However, the \(\frac{\partial}{\partial \xi_i}\) anticommute and, from transformations’ point of view, \(\frac{\partial}{\partial \xi_i} = -\frac{\partial}{\partial \xi_i}\). The notation, \(du_1 \cdots du_m \cdot \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_n}\), is, nevertheless, still wrong: almost any transformation \(A : (u, \xi) \mapsto (v, \eta)\) sends \(du_1 \cdots du_m \cdot \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_n}\) to the correct element, ber\((A) du^m \cdot \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_n}\), plus extra terms. Indeed, the fraction \(du_1 \cdots du_m \cdot \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_n}\) is the highest weight vector of an indecomposable \(\mathfrak{gl}(m|n)\)-module and \(\text{vol}(x)\) is the notation of the image of this vector in the 1-dimensional quotient module modulo the invariant submodule that consists precisely of all the extra terms.

1.0.7. Deformations of the Buttin superalgebra. (After [ALSI].) As is clear from the definition of the Buttin bracket, there is a regrading (namely, \(\mathfrak{b}(n; g)\) given by deg \(\xi_i = 0\), deg \(q_i = 1\) for all \(i\)) under which \(\mathfrak{b}(n)\), initially of depth 2, takes the form \(\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i\) with \(\mathfrak{g}_0 = \text{vect}(0|n)\) and \(\mathfrak{g}_{-1} \cong \Pi(\mathbb{C}[\xi])\). Replace now the \(\text{vect}(0|n)\)-module \(\mathfrak{g}_{-1}\) of functions (with
inverted parity) with the module of $\lambda$-densities, i.e., set $\fg_{-1} \cong \Pi(\text{Vol}(0|n)\lambda)$, where

$$L_D(\text{vol}_\xi)^\lambda = \lambda \text{div} D \cdot \text{vol}_\xi$$

and $p(\text{vol}_\xi)^\lambda = \overline{I}$.

0.7.1. Define $\fb_\lambda(n; n)$ as the Cartan’s prolong $(\fg_{-1}, \fg_0)_* = (\Pi(\text{Vol}(0|n)\lambda), \text{vect}(0|n))_*$. Clearly, it is a deform of $\fb(n; n)$. The collection of these $\fb_\lambda(n; n)$ for all $\lambda$’s is called the main deformation. (Though main, this deformation is not the quantization of the Butt in bracket, cf. [Ko1], [L4], [Sh3].)

The deformation $\fb_\lambda(n)$ of $\fb(n)$ is a regrading of $\fb_\lambda(n; n)$ described as follows. Set

$$\fb_\lambda(n) = \{M_f \in \fm(n) \mid a \text{ div} M_f = (-1)^{p(f)} 2(a - bn) \frac{\partial f}{\partial \tau}\},$$

(0.7)

Taking into account the explicit form (0.4.1) of the divergence of $M_f$ we get

$$\fb_\lambda(n) = \{M_f \in \fm(n) \mid (bn - aE) \frac{\partial f}{\partial \tau} = a \Delta f\} = \{D \in \text{vect}(n|n + 1) \mid L_D(\text{vol}_\eta^n) a_{\alpha - bn} = 0\}.$$

It is subject to a direct check that $\fb_{\alpha, b}(n)$ is isomorphic to $\fb_\lambda(n)$ for $\lambda = \frac{2\alpha}{m(a - b)}$. This isomorphism shows that $\lambda$ actually runs over $\mathbb{CP}^1$, not $\mathbb{C}$.

Observe that $\fb_{\alpha, b}(n) \cong \text{sm}(n)$. Observe also that $\fb_{\alpha, -a}(2; 2) \cong \fz_1(2; 2)$, see (OI).

As follows from the description of $\text{vect}(m|n)$-modules ([BL1]) and the criteria for simplicity of $Z$-graded Lie superalgebras ([Kad]), the Lie superalgebras $\fb_\lambda(n)$ are simple for $n > 1$ and $\lambda \neq 0, 1, \infty$. It is also clear that the $\fb_\lambda(n)$ are nonisomorphic for distinct $\lambda$’s for $n > 2$.

The Lie superalgebra $\fb(n) = \fb_0(n)$ is not simple: it has an $\varepsilon$-dimensional, i.e., $(0|1)$-dimensional, center. At $\lambda = 1$ and $\infty$ the Lie superalgebras $\fb_\lambda(n)$ are not simple either: they have an ideal of codimension $\varepsilon^n$ and $\varepsilon^{n+1}$, respectively. The corresponding exact sequences are

$$0 \longrightarrow \mathbb{C}M_1 \longrightarrow \fb(n) \longrightarrow \mathbb{C}\cdot e(n) \longrightarrow 0,$$

$$0 \longrightarrow \fb_{\lambda_0}(n) \longrightarrow \fb_1(n) \longrightarrow \mathbb{C} \cdot M_{\xi_1, \ldots, \xi_n} \longrightarrow 0,$$

$$0 \longrightarrow \fb_{\omega}(n) \longrightarrow \fb_\infty(n) \longrightarrow \mathbb{C} \cdot M_{\tau \xi_1, \ldots, \xi_n} \longrightarrow 0.$$

1.0.7.2. A correction. G. Shmelev [Sm] interpreted $\fz_2(2|2)$ of the Lie superalgebra $\fz(2|2)$ of Hamiltonian vector fields as preserving either the pseudodifferential form

$$d\eta^\frac{1}{2 - \lambda} \left(\lambda dq dp + (1 - \lambda)d\xi d\eta\right)$$

or, equivalently, as preserving the pseudodifferential form

$$d\eta^\frac{1}{2 - 2} (dq dp + d\xi d\eta).$$

Careful calculations reveal that these interpretations (that we rewrote, e.g. in [ALSh]), are incorrect.

1.0.8. The exceptional simple vectorial Lie superalgebras as Cartan’s prolongs. For a more detailed description of the “standard” realizations of the exceptions see [Sh1], [ShP].

1.0.9. The structures preserved. It is always desirable to find the structure preserved by the Lie superalgebra under the study. To see what do the vectorial superalgebras in nonstandard realizations preserve, we have to say, first of all, what is the structure that $\fg_0 = \text{vect}(0|n)$ preserves on $\fg_{-1} = \Lambda(n)$.

Let $\fg = \text{vect}(0|n)$, set further

$$W = \Lambda(n), \ V = \Lambda(n)/\mathbb{C} \cdot 1, \ V_0 = \{\varphi \in \Lambda(n) \mid \varphi(0) = 0\}.$$
The projection \( p : W \rightarrow V \) establishes a natural isomorphism between \( V \) and \( V_0 \). Let \( i : V_0 \rightarrow W \) be the “inverse” embedding.

Denote by \( \text{mult} : W \otimes W \rightarrow W \) the tensor of valency \((2, 1)\) on \( W \) which determines the multiplication on \( W \). Since \( V_0 \) is an ideal in the associative supercommutative superalgebra \( W \), the image \( \text{mult}|_{V_0 \otimes V_0} \) is contained in \( V_0 \). Denote by \( \text{mult}^\circ \) the tensor which coincides with \( \text{mult} \) on \( V_0 \otimes V_0 \) and vanishes on \( \mathbb{C} \cdot 1 \otimes W \oplus W \otimes \mathbb{C} \cdot 1 \). By means of the projection \( p \) and the embedding \( i \) we can \( \mathfrak{g} \)-invariantly transport \( \text{mult}^\circ \) to \( V \). The tensor obtained will be also denoted by \( \text{mult}^\circ \).

For any monomial \( \varphi \in W \) denote by \( \varphi^* \) the dual functional (in the monomial basis \( B(W) \)). Then

\[
\text{mult} = \text{mult}^\circ + \sum_{\varphi \in B(W)} 1^* \otimes \varphi^* \otimes \varphi = \text{mult}^\circ + 1^* \otimes \sum_{\varphi \in B(W)} \varphi^* \otimes \varphi.
\]

By definition of \( \mathfrak{g} \), it preserves \( \text{mult} \), i.e., \( L_D(\text{mult}) = 0 \) for any \( D \in \mathfrak{g} \). Hence,

\[
L_D(\text{mult}^\circ) = -L_D(1^*) \otimes \sum_{\varphi \in B(W)} \varphi^* \otimes \varphi - (1^*) \otimes L_D \left( \sum_{\varphi \in B(W)} \varphi^* \otimes \varphi \right).
\]

Under the restriction onto \( V_0 \otimes V_0 \) the second summand vanishes. Observe that \( \sum_{\varphi \in B(W)} \varphi^* \otimes \varphi \) is the identity operator on \( W \). Thus,

\[
L_D(\text{mult}^\circ|_{V_0 \otimes V_0}) = -L_D(1^*) \otimes \text{id}|_{V_0}.
\]

The lift of this identity operator to \( W \) reads as follows:

\[
L_D(\text{mult}) = \alpha(D) \otimes \text{id}|_W \quad \text{for a 1-form } \alpha \text{ on } W.
\]

Thus, all the structures preserved by \( \mathfrak{g}_0 \) on \( \mathfrak{g}_{-1} \) are clear, except for those preserved by several of the exceptional algebras. Namely, these structures are: (1) the tensor products \( B \otimes \text{mult} \) of a bilinear or a volume form \( B \) preserved (perhaps, conformally, up to multiplication by a scalar) in the fiber of a vector bundle over a \( 0|\tau \)-dimensional supermanifold on which the structure governed by \( \text{mult} \) is preserved, (2) \( \text{mult}^\circ \), or \( \text{mult} \) twisted by divergence with factor \( \lambda \). Observe that the volume element \( B \) may be not just \( \text{vol}(\xi) \) but \( (b + \alpha\xi_1 \ldots \xi_n)\text{vol}(\xi) \) as well.

The structures of another type, namely certain pseudodifferential forms, preserved by \( \mathfrak{h}_\lambda(2|2) \), are already described.

1.1. Description of algebras. Consider infinite dimensional complex filtered Lie superalgebras \( \tilde{\mathcal{L}} \) with decreasing filtration of the form

\[
\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{-\tilde{d}} \supset \tilde{\mathcal{L}}_{-\tilde{d}+1} \supset \cdots \supset \tilde{\mathcal{L}}_0 \supset \tilde{\mathcal{L}}_1 \supset \cdots
\]

(PF)

where depth \( \tilde{d} \) is finite and where

1) \( \tilde{\mathcal{L}}_0 \) is a maximal subalgebra of finite codimension;

2) for any non-zero \( x \in \tilde{\mathcal{L}}_k \) for \( k \geq 0 \), where \( \tilde{\mathcal{L}}_k = \tilde{\mathcal{L}}_k / \tilde{\mathcal{L}}_{k+1} \), there exists \( y \in \tilde{\mathcal{L}}_{-1} \) such that \([x, y] \neq 0\);

3) \( \tilde{\mathcal{L}}_0 \) does not contain ideals.

The pair \( (\tilde{\mathcal{L}}, \tilde{\mathcal{L}}_0) \) is called a primitive Lie algebra.

It turns out that there are virtually as many primitive Lie algebras as there are simple ones of polynomial growth and finite depth. Contrarywise, the classification problem of primitive Lie superalgebras is wild, see [ALSh].

We assume that these Lie superalgebras \( \tilde{\mathcal{L}} \) are complete with respect to a natural topology whose basis of neighborhoods of zero is formed by the spaces of finite codimension, e.g.,
the $\tilde{L}_i$. (In the absence of odd indeterminates this topology is the most natural one: we consider two vector fields $k$-close if their coefficients coincide up to terms of degree $> k$.) This topology is naturally (see. \S 1) called projective limit topology but a more clumsy term “linearly compact topology” is also used.

Observe that the very term “filtered algebra” implies that $[\tilde{L}_i, \tilde{L}_j] \subset \tilde{L}_{i+j}$ whereas conditions 1) and 2) manifestly imply that $\dim \tilde{L}_i < \infty$ for all $k$ and the $\mathbb{Z}$-graded Lie superalgebra $\tilde{L} = \bigoplus_{k \geq -d} \tilde{L}_k$ associated with $\tilde{L}$ grows polynomially, i.e., $\dim \bigoplus_{k \leq n} \tilde{L}_k$ grows as a polynomial in $n$.

Weisfeiler endowed every such filtered Lie algebra $\tilde{L}$ with another, refined, filtration $\tilde{L} = \mathcal{L}$:

$$\mathcal{L} = \mathcal{L}_{-d} \supset \mathcal{L}_{-d+1} \supset \cdots \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots$$

(WF1)

where $\mathcal{L}_{-1}$ is a minimal $\mathcal{L}_0$-invariant subspace and the other terms are defined by the formula (for $i \geq 1$):

$$\mathcal{L}_{-i-1} = [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} \quad \text{and} \quad \mathcal{L}_i = \{ D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1} \}.$$  

(WF2)

This $d$ is the depth of $\mathcal{L}$ and of the associated graded Lie superalgebra $L$.

An advantage of the Weisfeiler filtrations is that for the corresponding regraded Lie superalgebra $L$ the $L_0$-action on $L_{-1}$ is irreducible (see sec. 1.0.2). These refined filtrations are called Weisfeiler filtrations even for Lie superalgebras, where Weisfeiler’s construction is literally applied; we will shortly write $W$-filtrations and call the gradings associated with $W$-filtrations $W$-gradings.

When the $L_0$-module $L_{-1}$ is faithful, as is always the case for simple Lie superalgebras $L$, such filtered Lie superalgebras $\mathcal{L}$ (and the associated with them graded ones, $L$) can be realized by vector fields on $(\mathcal{L}/\mathcal{L}_0)^*$ with formal (resp. polynomial) coefficients. So, being primarily interested in simple Lie superalgebras, we will refer to $W$-filtered and $W$-graded Lie superalgebras as vectorial ones for short.

In \cite{LSp} we have announced and in \cite{LSp} we have classified simple $W$-graded vectorial Lie superalgebras. Observe that the classification results of Cheng and Kac concerning simple $W$-filtered complete vectorial Lie superalgebras (\cite{Ka}, \cite{CK}) should be completed: they have only considered filtered deformations corresponding to one filtration; other filtrations might produce new filtered deformations....

**On notations.** To simplify grasping the general picture from the displayed formulas of the following theorem, let us immediately inform that the prime example, $\mathfrak{vect}(m|n; r)$, is the Lie algebra of vector fields whose coefficients are formal power series (or polynomials, depending on the problem) in $m$ commuting and $n$ anticommuting indeterminates with the filtration (and grading) determined by equating the degrees of $r$ ($0 \leq r \leq m$) odd indeterminates to $0$, the degrees of the remaining indeterminates being equal to 1. The regradings of other series are determined similarly, see 1.2.2. We omit indicating parameter $r$ if $r = 0$.

In the sequel for any $\mathfrak{g}$ we write $\mathfrak{c}g = \mathfrak{g} \oplus \mathbb{C} \cdot z$ or $\mathfrak{c}(\mathfrak{g})$ to denote the trivial central extension with the 1-dimensional even center generated by $z$. Any nontrivial central extension is shorthanded as $\mathfrak{c}(\mathfrak{g})$.

If $d$ is the operator that determines the $\mathbb{Z}$ grading of $\mathfrak{g}$ and does not belong to $\mathfrak{g}$, then the Lie algebra $\mathfrak{g} \oplus \mathbb{C} \cdot d$ is denoted by $\mathfrak{d}(\mathfrak{g})$. If the operators $d$ and $z$ described above may be considered as having the same degree with respect to some grading, we write $\mathfrak{g}_{a,b}$ to shorthand $\mathfrak{g} \oplus \mathbb{C}(az + bd)$.

In proofs we use notations of roots in the simple finite dimensional Lie algebras from \cite{OV}, Table 5. The standard (identity) representation of a matrix Lie superalgebra $\mathfrak{g}$, i.e., a subalgebra of $\mathfrak{gl}(V)$, in $V$ is denoted by $\text{id}$ or, for clarity, $\text{id}_\mathfrak{g}$. 
Further elucidations. 1) In formulas (S) and (E) below: parentheses contain the superdimension of the superspace of indeterminates. The semicolon separates the superdimension from a shorthand description of the regrading.

Passing from one regrading to another one we take a “minimal” realization (i.e., with a minimal $\dim L/L_0$) as the point of reference. For the exceptional Lie superalgebras another point of reference is often more convenient, the consistent regrading $K$.

The regradings of the series are governed by a parameter $r$ described in sec. 1.2. All regradings are given in suggestive notations, e.g., $\mathfrak{kas}^c(3\eta)$ means that having taken $\mathfrak{kas}^c$ as the point of reference we set $\deg \eta = 0$ for each of the three $\eta$’s (certainly, this imposes some conditions on the degrees of the other indeterminates). The exceptional grading $E_H$ of $\mathfrak{h}_\lambda(2|2)$ is described in passing in the list of occasional isomorphisms (OI); it is described in detail for another incarnation of this algebra (1.2.1.1). By $K$ we initially denoted consistent gradings spelled with Russian accent, later we decided to retain it in order to acknowledge Kac’s skill in using it, $CK$ is an exceptional grading found by Cheng and Kac.

2) Several algebras are “drop-outs” from the series. For example: $\text{svect}^c(1|n;r)$ are “drop-outs” from the series $\text{svect}(m|n;r)$ since the latter are not simple for $m = 1$ but contain the simple ideal $\text{svect}^c(1|n;r)$. Similarly, $\mathfrak{le}(n|n;r)$, $\mathfrak{b}_1^c(n|n+1;r)$ and $\mathfrak{b}_\infty^c(n|n+1;r)$ are “drop-outs” from the series $\mathfrak{b}_\lambda(n|n+1;r)$ corresponding to $\lambda = 0$, 1 and $\infty$, respectively. Though $\text{sm}$ is not a dropout due to the above reason, it is singled out by its divergence free property, hence, deserves a separate line.

The finite dimensional Lie superalgebra $\mathfrak{h}(0|n)$ of hamiltonian vector fields is not simple either and contains a simple ideal $\mathfrak{sh}(0|n)$.

Theorem. The simple $W$-graded vectorial Lie superalgebras $L$ constitute the following series (S) and five exceptional families of fifteen individual algebras (E). They are pairwise non-isomorphic, as graded and filtered superalgebras, bar for occasional isomorphisms (OI). Parameters at which $\dim L$ becomes finite are marked by “FD”.

All these algebras are either Cartan prolongs or results of the generalized Cartan prolongations (described in sec 0.5) and, therefore, are determined by the terms $L_i$ with $i \leq 0$ (or $i \leq 1$ in some cases). These terms are listed in sec. 1.2 (and their form might drastically vary with $n$ and $r$).

\begin{align*}
\textbf{vect}(m|n;r) & \text{ for } (m|n) \neq (0|1) \text{ and } 0 \leq r \leq n; \text{ FD for } m = 0, n > 1 \\
\textbf{svect}(m|n;r) & \text{ for } m > 1, 0 \leq r \leq n; \text{ FD for } m = 0 \text{ and } n > 2 \\
\textbf{svect}^c(1|n;r) & \text{ for } n > 1, 0 \leq r \leq n \\
\textbf{svect}(0|n) & \text{ for } n > 2 \text{ (FD)}
\end{align*}

\begin{align*}
\mathfrak{e}(2m+1|n;r) & \text{ for } 0 \leq r \leq \left[\frac{n}{2}\right] \text{ unless } (m|n) = (0|2k) \\
\mathfrak{e}(1|2k;r) & \text{ for } 0 \leq r \leq k \text{ except } r = k - 1 \\
\mathfrak{h}(2m|n;r) & \text{ for } m > 0 \text{ and } 0 \leq r \leq \left[\frac{n}{2}\right] \\
\mathfrak{h}_\lambda(2|2;r) & \text{ for } \lambda \neq 0, 1, \infty, \text{ and } r = 0, 1 \text{ and } E_H \text{ (see (OI) and sec. 1.2.1.1)} \\
\mathfrak{sh}(0|n) & \text{ for } n > 3 \text{ (FD)}
\end{align*}
Our original notations of exceptional simple vectorial superalgebras, though reflect the way they are constructed and the geometry preserved, are rather long. But to write just $e(dim)$ is to create confusion: the superdimensions of the superspaces $(L/L)^*$ on which the algebra $L$ is realized coincide sometimes for different regradings. So we simplify notations only for $\mathfrak{e}^\ell \mathfrak{e}$ and, in accordance with description of nonstandard gradings in sec. 1.2, we set:

| Lie superalgebra     | its regradings (shorthand)                                      |
|----------------------|----------------------------------------------------------------|
| $\mathfrak{vle}(4|3; r)$, $r = 0, 1, K$  | $\mathfrak{vle}(4|3)$, $\mathfrak{vle}(5|4)$, and $\mathfrak{vle}(3|6)$ |
| $\mathfrak{vas}(4|4)$                     | $\mathfrak{vas}(4|4)$                                       |
| $\mathfrak{vas}(1|6; r)$, $r = 0, 1, 2\xi, 3\eta$ | $\mathfrak{vas}(1|6)$, $\mathfrak{vas}(5|5)$, $\mathfrak{vas}(4|4)$, and $\mathfrak{vas}(4|3)$ |
| $\mathfrak{mb}(4|5; r)$, $r = 0, 1, K$    | $\mathfrak{mb}(4|5)$, $\mathfrak{mb}(5|6)$, and $\mathfrak{mb}(3|8)$ |
| $\mathfrak{tse}(9|6; r)$, $r = 0, 2, K, CK$ | $\mathfrak{tse}(9|6)$, $\mathfrak{tse}(11|9)$, $\mathfrak{tse}(5|10)$, and $\mathfrak{c}(9|11)$ |

Hereafter we inconsistently abbreviate $m(n|n + 1; r)$, $b_\lambda(n|n + 1; r)$, $l(e(n|n; r)$, etc., i.e., algebras from $(S_m)$, to $m(n; r)$, $b_\lambda(n; r)$, etc., respectively.

Sometimes instead of $b_\lambda(n; r)$, where $\lambda = \frac{2a}{n(a-b)} \in \mathbb{C} \cup \{\infty\}$, we write $b_{a,b}(n; r)$ for clarity. Parameters $a$, $b$, as natural as $\lambda$, are introduced in sec. 0.7. Observe that the exceptional regrading $E$ of $b_\lambda(2)$ and isomorphisms $h_\lambda(2|2) \cong b_\lambda(2; 2)$ determine the exceptional grading $E_H$ of $h_\lambda(2|2)$.

(OI): Occasional isomorphisms:

\[
\begin{align*}
\operatorname{vect}(1|1) & \cong \operatorname{vect}(1|1); \\
\operatorname{svect}(2|1) & \cong \mathfrak{l}(2; 2); \quad \operatorname{svect}(2|1; 1) \cong \mathfrak{l}(2); \\
\mathfrak{sm}(n) & \cong b_{(n-1)/2}(n); \text{ in particular, } \mathfrak{sm}(2) \cong b_2(2), \text{ and} \\
\mathfrak{sm}(3) & \cong b_1(3), \text{ hence, } \mathfrak{sm}(3) \text{ is not simple}; \\
b_{1/2}(2; 2) & \cong h_{1/2}(2; 2) = h(2; 2); \quad b_\lambda(2|2) \cong b_\lambda(2; 2); \quad h_\lambda(2|2) \cong b_\lambda(2); \\
b_{a,b}(2; E) & \cong b_{-b-a}(2) \cong b_{a}(2); \text{ for } a \neq b \text{ and } b \neq 0; \\
b_{a,0}(2; E) & \cong \mathfrak{l}(2) \text{ and } b_{a,a}(2; E) \cong h_\infty(2); \\
h_\lambda(2|2) & \cong h_{-\lambda}(2|2); \\
\mathfrak{s}b_\mu(2^{n-1} - 1|2^{n-1}) & \cong \mathfrak{s}b_\nu(2^{n-1} - 1|2^{n-1}) \text{ for } \mu \nu \neq 0.
\end{align*}
\]
Though \( b_\lambda(2) \) and \( h_\lambda(2|2) \) are isomorphic, we consider them separately because they preserve very distinct structures.

**Warning.** Isomorphic abstract Lie superalgebras might be quite distinct as filtered: e.g., regradings provide us with isomorphisms of algebras

\[ \mathfrak{g}(1|2) \cong \mathfrak{ve}(1|1) \cong \mathfrak{m}(1) \] as abstract algebras.

Observe that of the above three nonisomorphic filtered algebras only one is W-filtered.

Even if the regraded algebra can be realized by vector fields on the superspace of the same dimension, the structures preserved are totally different: e.g., \( \mathfrak{g}(1|2) \cong \mathfrak{m}(1) \) and \( \mathfrak{ksle}(9|6; 2) = \mathfrak{ksle}(11|9) \cong \mathfrak{ct}(9|11) \).

**Remark.** 1) The excluded regradings \( \mathfrak{g}(1|2k; k-1) \) as well as \( m(n|n + 1; n - 1) \) and the ones they induce on \( b_\lambda, \mathfrak{le} \) and \( \mathfrak{slc} \) are often considered in applications (at least for small values of \( k \) and \( n \)) though these gradings/filtrations are not Weisfeiler ones: for them the \( g_0 \)-module \( g_{-1} \) is reducible.

2) \( \mathfrak{svect}(0|n) \), as well as \( \mathfrak{sh}_{2}(2n-1|1)[2n-1] \), depend on an odd parameter if \( n \) is odd.

3) The Lie superalgebras \( \mathfrak{svect} \) and \( \mathfrak{sh} \), as well as \( \mathfrak{ve}, \mathfrak{ve} \) for \( m = 0 \), are finite dimensional.

The above Lie superalgebras sometimes admit deformations that do not possess Weisfeiler filtrations. These deformations are considered in detail in [L4], [Ko1], [Ko2], [LSh3], [LSh4].

The five types of exceptional Lie superalgebras are given below in their minimal realizations as Cartan’s prolongs \( (g_{-1}, g_0) \), or generalized (see sect. 0.5) Cartan’s prolongs \( (g_{-1}, g_0)_{m} \) for \( g_-. = \bigoplus_{-d \leq i \leq -1} g_{i} \), expressed for \( d = 2 \) as \( (g_{-2}, g_{-1}, g_0)_{m} \) together with one of the Lie superalgebras (from \( S \)) as an ambient which contains the exceptional one as a maximal Lie subalgebra.

| \( \mathfrak{vle}(4|3; r) = (\Pi(\Lambda(3))/C \cdot 1, \mathfrak{ve}(0|3)_{*}) \subset \mathfrak{ve}(4|3), \ r = 0, 1, K \) |
| \( \mathfrak{vas}(4|4) = (\mathfrak{spin}, \mathfrak{as})_{*} \subset \mathfrak{ve}(4|4) \) |
| \( \mathfrak{vas}(5; r) \subset \mathfrak{g}(1|6; r), \ r = 0, 1, 3 \xi \) |
| \( \mathfrak{vas}(5; 3\eta) = (\mathfrak{vol}(0|3), c(\mathfrak{ve}(0|3)))_{*} \subset \mathfrak{ve}(4|3) \) |
| \( \mathfrak{mb}(4|5; r) = (\mathfrak{ab}(4), \mathfrak{ve}(0|3))_{*} \subset \mathfrak{m}(4|5), \ r = 0, 1, K \) |
| \( \mathfrak{ksle}(9|6; r) = (\mathfrak{hei}(8|6), \mathfrak{ve}(0|4)_{3,3})_{\xi}^{k} \subset \mathfrak{f}(9|6), \ r = 0, 2 \) |
| \( \mathfrak{ksle}(9|6; K) = (\mathfrak{id}(5), \Lambda^{2}(\mathfrak{id}(5))^{k}_{*} \mathfrak{sl}(5))_{k}^{k} \subset \mathfrak{ve}(5|10; 2, 3, 2, 2, 2|1, \ldots, 1) \) |

1.2.1. **Nonstandard realizations.** The following are all the nonstandard gradings of the Lie superalgebras indicated. In particular, the gradings in the series \( \mathfrak{ve} \) induce the gradings in the series \( \mathfrak{ve} \) and the exceptional algebras \( \mathfrak{vle}(4|3) \) and \( \mathfrak{vas}(4|4) \); the gradings in \( \mathfrak{m} \) induce the gradings in \( b_\lambda, \mathfrak{le}, \mathfrak{slc}, \mathfrak{slc}^{o}, \mathfrak{b}, \mathfrak{sh}, \mathfrak{sh}^{o} \) and the exceptional algebra \( \mathfrak{mb} \); the gradings in \( \mathfrak{f} \) induce the gradings in \( \mathfrak{po}, \mathfrak{h} \) and the exceptional algebras \( \mathfrak{fas} \) and \( \mathfrak{fslc} \).

In what follows we consider \( \mathfrak{g}(2n + 1|m) \) as preserving the Pfaff equation \( \tilde{\alpha} = 0 \), where

\[ \tilde{\alpha} = dt + \sum_{i \leq n} (p_i dq_i - q_i dp_i) + \sum_{j \leq r} (\xi_j d\eta_j + \eta_j d\xi_j) + \sum_{k \geq m - 2r} \theta_k d\theta_k. \]

The standard realizations correspond to \( r = 0 \), they are marked by (\( * \)). Observe that the codimension of \( L_0 \) attains its minimum in the standard realization.
| Lie superalgebra                      | its \(\mathbb{Z}\)-grading                                                                 |
|--------------------------------------|-------------------------------------------------------------------------------------------|
| \(\text{vect}(n|m;r),\)             | \begin{align*}                                                                                       
| 0\leq r\leq m & \ deg u_i &= \deg \xi_j = 1 \text{ for any } i,j \\
| & \ deg \xi_j &= 0 \text{ for } 1\leq j \leq r; \ 
| & \ deg u_i &= \deg \xi_{r+s} = 1 \text{ for any } i,s \end{align*} |
| \(\text{m}(n;r),\)                  | \begin{align*}                                                                                       
| 0\leq r\leq n & \ deg \tau = 2, \ deg q_i = \deg \xi_i = 1 \text{ for any } i \\
| r\neq n-1 & \ deg \tau = \deg q_i = 1, \ deg \xi_i = 0 \text{ for any } i \\
| & \ deg \tau = \deg q_i = 2, \ deg \xi_i = 0 \text{ for } 1\leq i \leq r < n; \ 
| & \ deg u_{r+j} = \deg \xi_{r+j} = 1 \text{ for any } j \end{align*} |
| \(\mathfrak{f}(2n+1|m;r),\)         | \begin{align*}                                                                                       
| 0\leq r\leq \left\lfloor \frac{m}{2} \right\rfloor & \ deg p_i = \deg q_i = \deg \xi_j = \deg \eta_j = \deg \theta_k = 1 \text{ for any } i,j,k \\
| r\neq k-1 \text{ for } m=2k \text{ and } n=0 & \ deg t = \deg \xi_i = 2, \ deg \eta_i = 0 \text{ for } 1\leq i \leq r \leq \left\lfloor \frac{m}{2} \right\rfloor; \\
| & \ deg p_i = \deg q_i = \deg \theta_j = 1 \text{ for } j \geq 1 \text{ and all } i \end{align*} |
| \(\mathfrak{f}(1|2m;m)\)            | \begin{align*}                                                                                       
| & \ deg t = \deg \xi_i = 1, \ deg \eta_i = 0 \text{ for } 1\leq i \leq m \end{align*} |

1.2.1.1. The exceptional nonstandard regrading \(E\). This is a regrading of \(\mathfrak{b}_{a,b}(2;E)\) given by the formulas:

\[
\deg \tau = 0; \quad \deg \xi_1 = \deg \xi_2 = -1; \quad \deg q_1 = \deg q_2 = 1. \quad (r = E)
\]

Then for the generic \(a\) and \(b\) we have:

\[
\mathfrak{g}_{-2} = \Pi(\mathbb{C}\xi_2) \text{ and } \mathfrak{g}_{-1} = \text{Span}\{\text{Le}_{\xi_1}, \text{Le}_{\xi_2}, \text{Le}_{Q_1}, \text{Le}_{Q_2}\},
\]

where \(Q_1 = A\xi_1\xi_2q_1 + Br\xi_2, \ Q_2 = A\xi_1\xi_2q_2 - B\tau\xi_1\) and where \(A\) and \(B\) are some coefficients determined by \(a\) and \(b\). The bracket on \(\mathfrak{g}_{-1}\) is determined by the odd form \(\omega = c\sum dQ_id\xi_i\), so \(\mathfrak{g}_0\) must be contained in \(\mathfrak{m}(2)_0\). The direct calculations show that \(\dim \mathfrak{g}_0 = 4|4\) and

\[
\mathfrak{g}_0 = \mathfrak{sp}(2) \oplus \mathbb{C}X, \text{ where } X = \text{Le}_{a\tau+b}\sum_{q_i\xi_i}.
\]

Indeed, \(\mathfrak{sp}(2)_0 \cong \mathfrak{so}(2) = \text{Span}\{\text{Le}_{q_1\xi_1}, \text{Le}_{q_2\xi_1}, \text{Le}_{q_1\xi_2}, \text{Le}_{q_2\xi_2}\}, \ \mathfrak{sp}(2)_{-1} = \mathbb{C} \cdot \text{Le}_1\) and \(\mathfrak{sp}(2)_1 = \mathbb{C}\text{Le}_{a\xi_2+P(q)+\beta r\Delta(\xi_2P(q))}\), where \(P(q)\) is a monomial of degree 2 and \(\alpha, \beta\) are some coefficients. The eigenvalues of \(X\) on \(\mathfrak{g}_{-1}\) are \(-a + b\) on the even part and \(a + b\) on the odd part. So \(\mathfrak{b}_{a,b}(2;E) \cong \mathfrak{b}_{-b-a}(2) \cong \mathfrak{b}_{a,b}(2)\).

For \(b = 0\) and \(a = b\) formula (*) should be modified because then \(\mathfrak{g}_{-2} = 0\). In these cases \(\mathfrak{b}_{a,0}(2;E) \cong \mathfrak{f}(2)\) and \(\mathfrak{b}_{a,a}(2;E) \cong \mathfrak{f}_{\infty}(2)\), respectively.

For \(n > 2\), as well as for \(\mathfrak{m}(n)\) for \(n > 1\), similar regradings are not Weisfeiler ones.

The exceptional grading \(E\) of \(\mathfrak{b}_1(2)\) induces the exceptional grading \(E_H\) of the isomorphic algebra \(\mathfrak{b}_1(2)\), see (OI).

1.2.1.2. The W-regradings of exceptional algebras.

**Theorem.** ([Sh14], [CK2]) The W-regradings of the exceptional simple vectorial Lie superalgebras are given by the following regradings of their “standard” ambients listed in sec. 1.1:

1) \(\mathfrak{ve}(4|3;r) = (\Pi(\Lambda(3)/\mathbb{C} \cdot 1), \mathfrak{vect}(0|3), \mathfrak{vect}(4|3)), \ r = 0, 1, K; \)

\[
\begin{align*}
\ r = 0: & \ deg y = \deg u_i = \deg \xi_i = 1 \\
\ r = 1: & \ deg y = \deg \xi_1 = 0, \ deg u_2 = \deg u_3 = \deg \xi_2 = \deg \xi_3 = 1, \ deg u_1 = 2 \\
\ r = K: & \ deg y = 0, \ deg u_i = 2; \ deg \xi_i = 1
\end{align*}
\]

2) \(\mathfrak{vas}(4|4) = (\text{spin, as}, \mathfrak{vect}(4|4)); \)
3) \( \mathfrak{kas} \subset \mathfrak{k}(1|6; r), \ r = 0, 1, 3 \xi; \ \mathfrak{kas}(3\eta) \subset \mathfrak{svect}(4|3); \)

\[ r = 0: \deg t = 2, \ \deg \eta_1 = 1; \ \deg \xi_1 = 1; \]

\[ r = 1: \deg \xi_1 = 0, \ \deg \eta_1 = \deg t = 2, \ \deg \xi_2 = \deg \xi_3 = \deg \eta_2 = \deg \eta_3 = 1; \]

\[ r = 3\xi: \deg \xi_1 = 0, \ \deg \eta_i = \deg t = 1; \]

\[ r = 3\eta: \deg \eta_i = 0, \deg \xi_i = \deg t = 1 \]

4) \( \mathfrak{m}(4|5; r) = (\mathfrak{ab}(4), \mathfrak{svect}(0|3))^m \subset \mathfrak{m}(4), \ r = 0, 1, K; \)

\[ r = 0: \deg \tau = 2, \ \deg u_i = \deg \xi_i = 1 \text{ for } i = 0, 1, 2, 3; \]

\[ r = 1: \deg \tau = \deg \xi_0 = \deg u_1 = 2, \ \deg u_2 = \deg u_3 = \deg \xi_2 = \deg \xi_3 = 1; \ \deg \xi_1 = \deg u_0 = 0 \]

\[ r = K: \deg \tau = \deg \xi_0 = 3, \ \deg u_0 = 0, \ \deg u_1 = 2; \ \deg \xi_i = 1 \text{ for } i > 0 \]

5) \( \mathfrak{fe}(9|6; r) = (\mathfrak{hei}(8|6), \mathfrak{svect}(4)) \subset \mathfrak{k}(9|6), \ r = 0, 2, K, CK \)

\[ r = 0: \deg t = 2, \ \deg p_i = \deg q_i = \deg \xi_i = \deg \eta_i = 1; \]

\[ r = 2: \deg t = \deg q_3 = \deg q_4 = \deg \eta_1 = 2, \ \deg q_1 = \deg q_2 = \deg p_1 = \deg p_2 = \deg \eta_2 = \deg \eta_3 = \deg \xi_2 = \deg \xi_3 = 1; \ \deg p_3 = \deg p_4 = \deg \xi_1 = 0; \]

\[ r = K: \deg t = \deg q_1 = 2, \ \deg p_1 = 0; \ \deg \xi_i = \deg \eta_i = 1; \]

\[ r = CK: \deg t = \deg q_1 = 3, \ \deg p_1 = 0; \ \deg q_2 = \deg q_3 = \deg q_4 = \deg \xi_1 = \deg \xi_2 = \deg \xi_3 = 2; \ \deg p_2 = \deg p_3 = \deg p_4 = \deg \eta_i = \deg \eta_2 = \deg \eta_3 = 1 \]

Thus, from the point of view of classification of the W-filtered Lie superalgebras, there are five families of exceptional algebras consisting of 15 individual algebras.

### 1.2.2. Several first terms that determine the Cartan prolongs

To facilitate the comparison of various vectorial superalgebras, we offer the following Table. The most interesting phenomena occur for extremal values of parameter \( r \) and small values of superdimension \( m|n \).

The central element \( z \in \mathfrak{g}_0 \) is supposed to be chosen so that it acts on \( \mathfrak{g}_k \) as \( k \cdot \text{id} \) and \( \Lambda(r) = \mathbb{C}[\xi_1, \ldots, \xi_r] \) is the Grassmann superalgebra generated by the \( \xi_i \) of degree 0 for all \( i \).

We set \( \Lambda(0) = \mathbb{C} \), the \( \mathfrak{vect}(0|n) \)-module \( \Lambda(m)/\mathbb{C} \cdot 1 \) is denoted by \( T_0(\mathbb{0}) \); set \( \text{Vol}_0(0|m) = \{ v \in \text{Vol}(0|m) \mid \int v = 0 \} \) whereas the \( \mathfrak{svect} \)-module \( T_0^\Lambda(\mathbb{0}) \) is defined as \( \text{Vol}_0(0|m)/\mathbb{C} \cdot 1 \) (for more elucidations see sec. 0.6).

Recall the range of the regrading parameter \( r: 0 \leq r \leq m \), where \( m \) is the number of odd indeterminates, except for the series \( \mathfrak{f} \) and \( \mathfrak{h} \) when \( 0 \leq r \leq \lceil \frac{m}{2} \rceil \), and \( \mathfrak{m}(n), \mathfrak{b}_\lambda(n), \text{ etc. when } 0 \leq r \leq n \) with value \( r = n - 1 \) excluded. We exclude certain values of \( r \), namely, \( r = k - 1 \) for \( \mathfrak{f}(1|2k) \) and \( r = n - 1 \) for \( \mathfrak{m}(n) \) and their subalgebras because for these values of \( r \) the corresponding grading is not a W-grading: the \( \mathfrak{g}_0 \) module \( \mathfrak{g}_{-1} \) is reducible.

| \( \mathfrak{g} \) | \( \mathfrak{g}_{-2} \) | \( \mathfrak{g}_{-1} \) | \( \mathfrak{g}_0 \) |
|---|---|---|---|
| \( \mathfrak{vect}(n|m; r) \) | \( \text{id} \otimes \Lambda(r) \) | \( \mathfrak{gl}(n|m - r) \otimes \Lambda(r) \notin \mathfrak{vect}(0|r) \) | \( \mathfrak{gl}(n|m - r) \otimes \Lambda(r) \notin \mathfrak{vect}(0|m) \) |
| \( \mathfrak{vect}(1|m; m) \) | \( \Lambda(m) \) | \( \Lambda(m) \notin \mathfrak{vect}(0|m) \) | \( \Lambda(m) \notin \mathfrak{vect}(0|m) \) |
| \( \mathfrak{svect}(n|m; r) \) | \( \text{id} \otimes \Lambda(r) \) | \( \mathfrak{sl}(n|m - r) \otimes \Lambda(r) \notin \mathfrak{vect}(0|r) \) | \( \mathfrak{sl}(n|m - r) \otimes \Lambda(r) \notin \mathfrak{vect}(0|m) \) |
| \( \mathfrak{svect}^c(1|m; m) \) | \( \text{Vol}_0(0|m) \) | \( \Lambda(m) \notin \mathfrak{vect}(0|m) \) | \( \Lambda(m) \notin \mathfrak{vect}(0|m) \) |
| \( \mathfrak{svect}^c(1|2) \) | \( \Pi(T_0^\Lambda(\mathbb{0})) \) | \( \mathfrak{vect}(0|2) \cong \mathfrak{sl}(1|2) \) | \( \mathfrak{vect}(0|2) \cong \mathfrak{sl}(2|1) \) |
We set \( p(\mu) \equiv n \pmod{2} \), so \( \mu \) can be odd indeterminate. The Lie superalgebras \( \tilde{\mathfrak{vect}}_\mu(0|n) \) are isomorphic for nonzero \( \mu \)'s; hence, so are the algebras \( \mathfrak{sb}_\mu(2^{n-1} - 1|2^{n-1}) \). So for \( n \) even we can set \( \mu = 1 \), while if \( \mu \) is odd we should consider it as an additional indeterminate on which the coefficients depend.

In what follows \( \lambda = \frac{2n}{n(a,b)} \neq 0, 1, \infty \); the three exceptional cases (corresponding to the “dropouts” \( \mathfrak{le}(n) \), \( \mathfrak{b}_0(n) \) and \( \mathfrak{b}_\infty(n) \), respectively) are considered separately. The irreducibility condition of the \( g_0 \)-module \( g_{-1} \) for \( g = \mathfrak{b}_\infty \) excludes \( r = n - 1 \). The case \( r = n - 2 \) is extra exceptional, so in the following tables

\[
0 < r < n - 2; \quad \text{additionally, unless specified, } a \neq b \text{ and } (a, b) \neq k(n, n - 2).
\]

To further clarify the following tables, denote the superspace of the identity \( k|k \)-dimensional representation of \( \mathfrak{spe}(k) \) by \( V \); let \( d = \text{diag}(1_k, -1_k) \in \mathfrak{pe}(k) \). Let \( W = V \otimes \Lambda(r) \) and \( D \in \mathfrak{vect}(0|r) \). Let \( \Xi = \xi_1 \cdots \xi_n \in \Lambda(\xi_1, \ldots, \xi_n) \). Denote by \( T^r \) the representation of \( \mathfrak{vect}(0|r) \) in \( \mathfrak{spe}(n - r) \otimes \Lambda(r) \) given by the formula

\[
T^r(D) = 1 \otimes D + d \otimes \frac{1}{n - r} \text{div}D. \quad (T^r)
\]

In \( \mathfrak{sl}_\circ(n; r)_0 \) for \( r \neq n - 2 \):

\[
\mathfrak{vect}(0|r) \text{ acts on the ideal } \mathfrak{spe}(n - r) \otimes \Lambda(r) \text{ via } T^r; \\
\text{any } X \otimes f \in \mathfrak{spe}(n - r) \otimes \Lambda(r) \text{ acts in } g_{-2} \text{ as } \text{id} \otimes f \text{ and in } g_{-2} \text{ as } 0; \\
\text{any } D \in \mathfrak{vect}(0|r) \text{ acts in } g_{-1} \text{ via } T^r \text{ and in } g_{-2} \text{ as } D.
\]

In \( \mathfrak{sl}_\circ(n; n - 2)_0 \), observe that \( \mathfrak{spe}(2) \cong \mathbb{C}(\mathfrak{Le}_{\xi_1} \cdot \mathfrak{Le}_{\xi_2}) \subset \mathbb{C} \mathfrak{Le}_{\xi_1} \mathfrak{Le}_{\xi_2} \), whereas \( g_{-2} \) and \( g_{-1} \) are as above, for \( r < n - 2 \). Set \( \mathfrak{h} = \mathbb{C}(\mathfrak{Le}_{\xi_1} \cdot \mathfrak{Le}_{\xi_2}) \). In this case

\[
\mathfrak{g}_0 \cong (\mathfrak{h} \otimes \Lambda(n - 2) \subset \mathbb{C} \mathfrak{Le}_{\xi_1} \mathfrak{Le}_{\xi_2} \otimes (\Lambda(n - 2) \setminus \mathbb{C} \xi_3 \cdots \xi_n)) \subset T^1(\mathfrak{vect}(0|n - 2)). \quad (1.2.1.2)
\]

The action of \( \mathfrak{vect}(0|n - 2) \), the quotient of \( \mathfrak{g}_0 \) modulo the underlined ideal is via \((1.1.2.1)\). In the subspace \( \xi_1 \mathfrak{Le}_{\xi_2} \otimes \Lambda(n - 2) \subset \mathfrak{g}_0 \) this action is as in the space of volume forms. So we can throw away \( \Xi \).
In the following table the terms "\(g_{-1}\)" denote the superspace isomorphic to the listed one but with the action given by formulas (1.2.1.1) and (1.2.1.2), as indicated.

| \(g\)       | \(g_{-2}\) | \(g_{-1}\) | \(g_0\)          |
|------------|------------|------------|------------------|
| \(le(n)\) |            | id         | \(pe(n)\)       |
| \(le(n; r)\) | \(\Pi(T_0(\bar{0}))\) | \(id \otimes \Lambda(r)\) | \(pe(n - r) \otimes \Lambda(r) \in \text{vect}(0|r)\) |
| \(le(n; n)\) |            | \(\Pi(T_0(\bar{0}))\) | \(\text{vect}(0|n)\) |
| \(\text{sle}^\circ(n)\) |            | id         | \(\text{spe}(n)\) |
| \(\text{sle}^\circ(n; r)\) | "\(\Pi(T_0(\bar{0}))\)" | "\(id \otimes \Lambda(r)\)" | \(\text{spe}(n - r) \otimes \Lambda(r) \in \text{vect}(0|r)\) |
| \(\text{sle}^\circ(n; n - 2)\) | "\(\Pi(T_0(\bar{0}))\)" | "\(id \otimes \Lambda(r)\)" | see (1.2.1.2) |
| \(\text{sle}^\circ(n; n)\) |            | \(\Pi(T_0(\bar{0}))\) | \(\text{svect}(0|n)\) |

We consider \(b_{a,b}(n; r)\) for \(0 < r < n - 2\) and \(ar - bn \neq 0\); in particular, this excludes \(b^\circ_\infty(n; n) = b^\circ_{a,a}(n; n)\) and \(b^\circ_1(n; n - 2) = b^\circ_{a,n-2}(n; n - 2)\). Set

\[
c = \frac{a}{ar - bn}.
\]

The case \(ar = bn\), i.e., \(\lambda = \frac{2}{n - r}\) is an exceptional one, the \(\text{vect}(0|r)\)-action on the ideal \(\text{cspe}(n - r) \otimes \Lambda(r) \in \text{vect}(0|r)\) of \(g_0\) and on \(g_{-1}\) is the same as for \(\text{sle}^\circ\), see (1.2.1.1).

If \(z\) is the central element of \(\text{cspe}(n - r)\) that acts on \(g_{-1}\) as \(id\), then

\[
z \otimes \psi\text{ acts on }g_{-1}\text{ as }id \otimes \psi, \text{ and on }g_{-2}\text{ as }2id \otimes \psi.
\]
1.2.3. The exceptional vectorial Lie subsuperalgebras. Here are the terms \( g_i \) for \( i \leq 0 \) of 14 of the 15 exceptional algebras, the last column gives \( \dim g_- \):

| \( g \)          | \( g_- \) | \( g_-^1 \) | \( g_0 \) | \( \dim g_- \) |
|------------------|----------|------------|---------|----------------|
| \text{vle}(4;3)  | \vdash   | \Pi(\Lambda(3)/C1) | \text{c}(\text{vect}(0;3)) | 4/3 |
| \text{vle}(4;3;1) | \C - 1   | \id \otimes \Lambda(2) | \text{c}(\text{sl}(2) \otimes \Lambda(2) + T^{1/2}(\text{vect}(0;2))) | 5/4 |
| \text{vle}(4;3; K) | \id_{\text{sl}(3)} | \id_{\text{sl}(3)}^* \otimes \text{id}_{\text{sl}(2)} \otimes 1 | \text{sl}(3) \oplus \text{sl}(2) \oplus \C z | 3/6 |
| \text{vas}(4;4)  | \vdash   | \C - 1 | \text{spin} | \text{as} | 4/4 |
| \( \text{tas} \) | \C - 1   | \Pi(\id) | \text{co}(6) | 1/6 |
| \( \text{tas}(-1) \) | \Lambda(1) | \id_{\text{sl}(2)} \otimes (\id_{\text{sl}(2)} \otimes \Lambda(1)) | \text{c}(\text{sl}(2) \otimes \Lambda(2) \otimes (\text{vect}(0;1))) | 5/5 |
| \( \text{tas}(-3; \xi) \) | \vdash   | \Lambda(3) | \Lambda(3) \oplus \text{id}(13) | 4/4 |
| \( \text{tas}(-3; \eta) \) | \vdash   | \text{Vol}(0;3) | \text{c}(\text{vect}(0;3)) | 4/3 |
| \text{mb}(4;5)   | \Pi(\C - 1) | \C - 1 | \text{c}(\text{vect}(0;3)) | 4/5 |
| \text{mb}(4;5;1) | \Lambda(1) | \text{sl}(2) \otimes \Lambda(2) | \text{c}(\text{sl}(2) \otimes \Lambda(1) \otimes (\text{vect}(0;2))) | 5/6 |
| \text{mb}(4;5; K) | \id_{\text{sl}(3)} | \Pi(\text{id}_{\text{sl}(2)} \otimes \text{sl}(2) \otimes \C) | \text{sl}(3) \oplus \text{sl}(2) \oplus \C z | 3/8 |
| \text{tsle}(9;6) | \C - 1   | \Pi(T^0_{\text{sl}(5)}) | \text{svect}(0;4) \otimes \text{sl}(5) | 9/6 |
| \text{tsle}(9;6; 2) | \id_{\text{sl}(3;1)} | \id_{\text{sl}(2)} \otimes \Lambda(3) | \text{c}(\text{sl}(2) \otimes \Lambda(3) \otimes \text{sl}(13); 11/9 |
| \text{tsle}(9;6; K) | \id | \Pi(\text{L}^2(\text{id}^*)) | \text{sl}(5) | 5/10 |

Observe that none of the simple W-graded vectorial Lie superalgebras is of depth > 3 and only two algebras are of depth 3: one of the above, \( \text{mb}(4;5; K) \), for which we have \( \text{mb}(4;5; K)_{-3} \cong \Pi(\text{id}_{\text{sl}(2)}) \), and another one, \( \text{tsle}(9;6; CK) = \text{ct}(9;11) \).

This \( \text{ct}(9;11) \) is the 15-th exceptional simple vectorial Lie superalgebra; its non-positive terms are as follows (we assume that the \( \text{sl}(2) \)- and \( \text{sl}(3) \)-modules are purely even):

\[
\begin{align*}
\text{ct}(9;11)_0 & \cong (\text{sl}(2) \oplus \text{sl}(3) \otimes \Lambda(1)) \otimes \text{vect}(0;1); \\
\text{ct}(9;11)_{-1} & \cong \text{id}_{\text{sl}(2)} \otimes (\text{id}_{\text{sl}(3)} \otimes \Lambda(1)); \\
\text{ct}(9;11)_{-2} & \cong \text{id}_{\text{sl}(3)}^* \otimes \Lambda(1); \\
\text{ct}(9;11)_{-3} & \cong \Pi(\text{id}_{\text{sl}(2)} \otimes \C).
\end{align*}
\]

1.2.3.1. The exceptional Lie subsuperalgebra \( \text{tas} \) of \( \text{t}(1;6) \). This Lie superalgebra is not determined by its nonpositive part and requires a closer study. The Lie superalgebra \( g = \text{t}(1;2n) \) is generated by the functions from \( \C[t, \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n] \). The standard Z-grading of \( g \) is induced by the Z-grading of \( \C[t, \xi, \eta] \) given by \( \deg t = 2 \), \( \deg \xi_i = \deg \eta_i = 1 \); namely, \( \deg K_f = \deg f - 2 \). Clearly, in this grading \( g \) is of depth 2. Let us consider the functions that generate several first homogeneous components of \( g = \bigoplus_{i \geq -2} g_i \):

| component | \( g_- \) | \( g_-^1 \) | \( g_0 \) | \( g_1 \) |
|-----------|----------|------------|---------|-------|
| its generators | \( 1 \) | \( \Lambda^1(\xi, \eta) \) | \( \Lambda^2(\xi, \eta) \oplus \C \cdot t \) | \( \Lambda^3(\xi, \eta) \oplus t \Lambda^1(\xi, \eta) \) |

As one can prove directly, the component \( g_1 \) generates the whole subalgebra \( g_+ \) of elements of positive degree. The component \( g_1 \) splits into two \( g_0 \)-modules \( g_1 = \Lambda^3 \) and \( g_1 = t \Lambda^1 \). It is obvious that \( g_{12} \) is always irreducible and the component \( g_{11} \) is trivial for \( n = 1 \).

The partial Cartan prolongs of \( g_{11} \) and \( g_{12} \) are well-known:

\[
\begin{align*}
(\text{g}_- \oplus \text{g}_0, \text{g}_{11})_{\text{smk}} & \cong \text{po}(0|2n) \oplus \C \cdot K_z \cong \text{p}(\text{po}(0|2n)); \\
(\text{g}_- \oplus \text{g}_0, \text{g}_{12})_{\text{smk}} & = \text{g}_- \oplus \text{g}_- \oplus \text{g}_0 \oplus \text{g}_{12} \oplus \C \cdot K_{z2} \cong \text{osp}(2n|2).
\end{align*}
\]
Observe a remarkable property of \( \mathfrak{t}(1|6) \): only for it does the component \( g_{11} \) split into 2 irreducible modules that we will denote \( g_{11}^\xi \) and \( g_{11}^\eta \): one is generated by \( \xi_1 \xi_2 \xi_3 \), the other one by \( \eta_1 \eta_2 \eta_3 \).

Observe further, that \( g_0 = \mathfrak{co}(6) \cong \mathfrak{gl}(4) \). As \( \mathfrak{gl}(4) \)-modules, \( g_{11}^\xi \) and \( g_{11}^\eta \) are the symmetric squares \( S^2(\text{id}) \) and \( S^2(\text{id}^*) \) of the identity 4-dimensional representation and its dual, respectively.

**Theorem.** ([Sh13], [Sh14]) The Cartan prolong \( \mathfrak{kas}^\xi = (g_- \oplus g_0, g_{11}^\xi \oplus g_{12})_{nk} \) is infinite dimensional and simple. It is isomorphic to \( \mathfrak{kas}^\eta = (g_- \oplus g_0, g_{11}^\eta \oplus g_{12})_{nk} \).

When it does not matter which of isomorphic algebras \( \mathfrak{kas}^\xi \cong \mathfrak{kas}^\eta \) to take we will write \( \mathfrak{kas} \).

For their explicit presentation by generating functions and for related structures on the “stringy” Lie superalgebras see [GLS1].

**1.2.5. Grozman’s theorem and a description of \( g \) as \( g_0 \) and \( g_1 \).** In [CK2] the exceptional algebras are described as \( g = g_0 \oplus g_1 \). For most of the series such description is of little value because each homogeneous component \( g_0 \) and \( g_1 \) has a complicated structure. For the exceptions (and for twisted polyvector fields) the situation is totally different!

Observe that apart from being beautiful, such description is useful for the construction of Volichenko algebras, cf. [L2].

Recall in this relation a theorem of Grozman [G]. He completely described bilinear differential operators acting in tensor fields and invariant under all changes of coordinates. It turned out that almost all of the first order operators determine a Lie superalgebra on its domain. Some of these superalgebras are simple or close to simple. In the constructions below we use some of these invariant operators.

\[ g = \mathfrak{e}(5|10); \quad g_0 = \mathfrak{svect}(5|0) \cong d\Omega^3, \quad g_1 = \Pi(d\Omega^1) \] with the natural \( g_0 \)-action on \( g_1 \) and the bracketing of odd elements being their product; we identify for any permutation \( (ijklm) \) of (12345):

\[ dx_i \wedge dx_j \wedge dx_k \wedge dx_l \otimes \text{vol}^{-1} = \text{sign}(ijklm) \frac{\partial}{\partial x_m}. \]

\[ g = \mathfrak{vas}(4|4); \quad g_0 = \mathfrak{vect}(4|0), \text{ and } g_1 = \Omega^1 \otimes \text{Vol}^{-1/2} \] with the natural \( g_0 \)-action on \( g_1 \) and the bracketing of odd elements being

\[ [\omega_1 \otimes \text{vol}^{-1/2}, \omega_2 \otimes \text{vol}^{-1/2}] = (d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2) \otimes \text{vol}^{-1}, \]

where we identify

\[ dx_i \wedge dx_j \wedge dx_k \otimes \text{vol}^{-1} = \text{sign}(ijkl) \frac{\partial}{\partial x_l} \text{ for any permutation } (ijkl) \text{ of (1234)}. \]

\[ g = \mathfrak{vlc}(3|6); \quad g_0 = \mathfrak{vect}(3|0) \oplus \mathfrak{sl}(2)^{(1)}_{\geq 0}, \text{ where } g^{(1)}_{\geq 0} = g \otimes \mathbb{C}[t], \text{ and } g_1 = \left( \Omega^1 \otimes \text{Vol}^{-1/2} \right) \otimes \text{id}_{\mathfrak{sl}(2)^{(1)}_{\geq 0}} \] with the natural \( g_0 \)-action on \( g_1 \).

Recall that \( \text{id}_{\mathfrak{sl}(2)} \) is the irreducible \( \mathfrak{sl}(2) \)-module \( L^1 \) with highest weight 1; its tensor square splits into \( L^2 \cong \mathfrak{sl}(2) \) and the trivial module \( L^0 \); accordingly, denote by \( v_1 \wedge v_2 \) and \( v_1 \bullet v_2 \) the projections of \( v_1 \otimes v_2 \in L^1 \otimes L^1 \) onto the skew-symmetric and symmetric components, respectively. For \( f_1, f_2 \in \Omega^0, \omega_1, \omega_2 \in \Omega^1 \) and \( v_1, v_2 \in L^1 \) we set

\[ [(\omega_1 \wedge v_1)\text{vol}^{-1/2}, (\omega_2 \otimes v_2)\text{vol}^{-1/2}] = (\omega_1 \wedge \omega_2) \otimes (v_1 \wedge v_2) + d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2) \otimes (v_1 \bullet v_2) \text{vol}^{-1}, \]
where we identify $\Omega^0$ with $\Omega^2 \otimes_{\Omega^0} \text{Vol}^{-1}$ and $\Omega^2 \otimes_{\Omega^0} \text{Vol}^{-1}$ with $\text{vect}(3|0)$ by setting
\[
dx_i \wedge dx_j \otimes \text{vol}^{-1} = \text{sign}(ijk) \frac{\partial}{\partial x_k}
\]
for any permutation $(ijk)$ of (123).

\[g = \text{mb}(3|8): \quad g_0 = \text{vect}(3|0) \oplus \mathfrak{sl}(2)_{\geq 0}^{(1)}, \quad \text{and} \quad g_1 = g_{-1} \oplus g_1, \quad \text{where}
\]
\[
g_{-1} = (\Pi \text{Vol}^{-1/2}) \otimes \text{id}_{\mathfrak{sl}(2)_{\geq 0}^{(1)}} \quad \text{and} \quad g_1 = \left(\Omega^1 \otimes \text{Vol}^{-1/2}\right) \otimes \text{id}_{\mathfrak{sl}(2)_{\geq 0}^{(1)}};
\]
clearly, one can interchange $g_{\pm 1}$.

Multiplication is similar to that of $g = \mathfrak{fe}(3|6)$. For $f_1, f_2 \in \Omega^0$, $\omega_1, \omega_2 \in \Omega^1$ and $v_1, v_2 \in L^1$ we set
\[
[(\omega_1 \otimes v_1) \text{vol}^{-1/2}, (\omega_2 \otimes v_2) \text{vol}^{-1/2}] = 0,
\]
\[
[(f_1 \otimes v_1) \text{vol}^{-1/2}, (f_2 \otimes v_2) \text{vol}^{-1/2}] = (df_1 \wedge df_2) \otimes (v_1 \wedge v_2) \text{vol}^{-1},
\]
\[
[(f_1 \otimes v_1) \text{vol}^{-1/2}, (\omega_1 \otimes v_2) \text{vol}^{-1/2}] =
\]
\[
(f_1 \omega_1 \otimes (v_1 \wedge v_2) + (df_1 \omega_1 + f_1 d\omega_1) \otimes (v_1 \cdot v_2)) \text{vol}^{-1}.
\]

\[g = \mathfrak{fas}: \quad g_0 = \text{vect}(1|0) \oplus \mathfrak{sl}(4)_{\geq 0}^{(1)}, \quad \text{and} \quad g_1 = g_{-1} \oplus g_1, \quad \text{where} \quad g_{-1} = \Pi \left(\Lambda^2(\text{id}_{\mathfrak{sl}(2)_{\geq 0}^{(1)}})\right) \quad \text{and}
\]
\[g_1 = \Pi \left(S^2(\text{id}_{\mathfrak{sl}(2)_{\geq 0}^{(1)}})\right); \quad \text{clearly, one can interchange} \quad g_{\pm 1}.
\]

\[g = \mathfrak{b}_{\lambda}(n; \bar{n}): \quad \text{here} \quad \bar{n} \quad \text{denotes the grading given by the formulas} \quad \deg q_i = 0, \quad \deg \xi_i = 1 \quad \text{for} \quad i = 1, \ldots, n. \quad \text{Then for} \quad i = -1, 0, \ldots, n - 1 \quad \text{we have} \quad g_i = (\Pi^i(\Lambda^{i-1}(\mathfrak{fe}(n|0)))) \otimes \text{Vol}^{-(i-1)\lambda}.
\]

Consider $n = 2$ more attentively. Clearly, one can interchange $g_{\pm 1}$; this possibility explains a mysterious isomorphism mentioned in (OI); if $\lambda = -1 - \lambda$ we have additional automorphisms, whereas for $\lambda = \frac{1}{2}$ (and $\lambda = -\frac{3}{2}$) there is a nontrivial central extension missed in [Ko1], [Ko2] and studied in [LSH3].

§2. Main result

Recall that we usually skip the wedge sign in the product of supercommuting differential forms; $\Pi$ is the change of parity sign. The relations will be divided as in introduction: into Serre relations (S); lowest (for the positive part) or highest (for the negative part) weight relations, (LW) and (HW), respectively; other, new type relations (N). Observe that (LW) and (HW) are of the same form as (S).

\[\mathfrak{fe}(5|10) \quad \text{Set (positive generators)}:
\]
\[X_i = x_i \frac{\partial}{\partial x_{i+1}} \quad \text{for} \quad i = 1, 2, 3, 4, \quad \text{and} \quad Z = \Pi x_5 dx_4 dx_5.
\]

Relations in $\mathfrak{fe}_+$ are:
\[
S \begin{array}{c}
[X_1, X_3] = 0, \quad [X_1, X_4] = 0, \quad \text{ad}^2_{X_4} X_3 = 0, \quad [X_2, X_4] = 0, \\
\text{ad}^2_{X_1} X_2 = 0, \quad \text{ad}^2_{X_2} X_1 = 0,
\end{array}
\]
\[
S \begin{array}{c}
\text{ad}^2_{X_2} X_3 = 0, \quad \text{ad}^2_{X_3} X_2 = 0, \quad \text{ad}^2_{X_4} X_3 = 0,
\end{array}
\]
\[
\text{LW} \begin{array}{c}
\text{ad}^2_{X_1} Z = 0, \quad [X_1, Z] = 0, \quad [X_2, Z] = 0, \quad \text{ad}^2_{X_4} Z = 0,
\end{array}
\]
\[
\text{N}_2 \begin{array}{c}
[X, Z] = 0, \quad [[X_3, Z], [X_4, Z]] = 0, \quad [[X_4, [X_3, Z]], [X_3, X_4], Z] = 0,
\end{array}
\]
\[
\text{N}_4 \begin{array}{c}
[[[X_1, X_2], [X_3, Z]], [Z, [X_2, X_3]], [Z, [X_3, X_4]]] = 0.
\end{array}
\]

Set (negative generators):
\[Y_i = x_{i+1} \frac{\partial}{\partial x_i} \quad \text{for} \quad i = 1, 2, 3, 4, \quad \text{and} \quad Z = \Pi dx_1 dx_2.
\]
The relations in $\mathfrak{tslc}_-$ are:

\begin{align*}
S & \quad [Y_1, Y_2] = 0, \quad [Y_1, Y_3] = 0, \quad [Y_2, Y_3] = 0, \\
S & \quad \text{ad}^2_{Y_1} Y_2 = 0, \quad \text{ad}^2_{Y_2} Y_1 = 0, \quad \text{ad}^2_{Y_2} Y_3 = 0, \\
S & \quad \text{ad}^2_{Y_3} Y_2 = 0, \quad \text{ad}^2_{Y_3} Y_4 = 0, \quad \text{ad}^2_{Y_4} Y_3 = 0, \\
HW & \quad [Y_1, Z] = 0, \quad [Y_3, Z] = 0, \quad [Y_4, Z] = 0, \quad \text{ad}^2_{Y_2} Z = 0, \\
N_2 & \quad [Z, Z] = 0; \quad [[[Y_2, Z], [Y_3, Y_4]], [[Z, [Y_1, Y_2]], [Z, [Y_2, Y_3]]]] = 0.
\end{align*}

\textbf{mb(4|5)} Set (negative generators):

\begin{align*}
Y_1 = \xi_0, \quad Y_2 = q_2 \xi_1, \quad Y_3 = q_3 \xi_2, \quad Z = -q_0 q_1 + \xi_2 \xi_3.
\end{align*}

The relations in $\text{mb}(4|5)_-$ are:

\begin{align*}
S & \quad [Y_1, Y_2] = 0, \quad [Y_1, Y_3] = 0, \quad \text{ad}^2_{Y_3} Y_2 = 0, \quad \text{ad}^2_{Y_2} Y_3 = 0, \\
HW & \quad \text{ad}^2_{Y_1} Z = 0, \quad \text{ad}^2_{Y_3} Z = 0, \quad [Y_3, Z] = 0, \\
N_2 & \quad [Z, Z] = 0; \quad [[[Y_1, Z], [Y_2, Z]], [[Y_1, Z], [Z, [Y_2, Y_3]]]] = 0.
\end{align*}

Set (positive generators):

\begin{align*}
X_1 &= q_0^2 \xi_0 - q_0 q_1 \xi_1 - q_0 q_2 \xi_2 - q_0 q_3 \xi_3 + \tau q_0 + 2 \xi_1 \xi_2 \xi_3, \\
X_2 &= q_1 \xi_2, \quad X_3 = q_2 \xi_3, \quad Z_1 = q_3^2, \quad Z_2 = \xi_0 \xi_1.
\end{align*}

The relations in $\text{mb}(4|5)_+$ are:

\begin{align*}
S & \quad [X_1, X_2] = 0, \quad [X_1, X_3] = 0, \quad \text{ad}^2_{X_3} X_2 = 0, \quad \text{ad}^2_{X_3} X_2 = 0, \\
HW & \quad [X_2, Z_1] = 0, \quad [X_3, Z_2] = 0, \quad \text{ad}^2_{X_1} Z_1 = 0, \\
LW & \quad \text{ad}^2_{X_3} Z_1 = 0, \quad \text{ad}^2_{X_1} Z_2 = 0, \quad \text{ad}^2_{X_2} Z_2 = 0, \\
N_2 & \quad [Z_1, Z_1] = 0, \quad [Z_1, Z_2] = 0, \quad [Z_2, Z_2] = 0, \\
N_2 & \quad [Z_1, [X_3, [Z_1, X_3]]] = 0, \\
N_2 & \quad [[[X_1, Z_1], [X_2, [X_3, Z_2]]] = 4 [Z_2, [X_1, Z_1]], \quad [[[X_1, Z_2], [X_2, Z_2]] = 0; \\
N_3 & \quad 2 [[Z_2, [X_1, Z_1]], [X_2, X_3], [X_3, Z_1]] = \\
& \quad - ([Z_1, [X_2, X_3]], [X_1, Z_2], [X_3, Z_1]]; \\
N_4 & \quad [[[X_3, Z_1], [X_2, [X_1, Z_2]]], [[X_2, [X_1, Z_2]], [X_3, [X_3, Z_1]]] = \\
& \quad - 4 [[[X_1, Z_1], [X_2, X_3]], [[Z_2, [X_1, Z_1]], [Z_2, [X_2, X_3]]] \\
& \quad - [[[X_2, Z_2], [X_3, [X_1, Z_1]], [[X_2, [X_1, Z_2]], [X_3, [X_3, Z_1]]]].
\end{align*}

\textbf{vfc(4|3; K)} Set (negative generators):

\begin{align*}
Y_1 &= -x_4^2 \frac{\partial}{\partial x_4} - x_1 x_3 \frac{\partial}{\partial x_5} - x_4 x_6 \frac{\partial}{\partial x_5} - x_4 x_7 \frac{\partial}{\partial x_7} + x_3 x_6 \frac{\partial}{\partial x_4} - x_5 x_7 \frac{\partial}{\partial x_2} + x_6 x_7 \frac{\partial}{\partial x_5}, \\
Y_2 &= -x_2 \frac{\partial}{\partial x_1} + x_5 \frac{\partial}{\partial x_4}, \quad Y_3 = -x_3 \frac{\partial}{\partial x_2} + x_6 \frac{\partial}{\partial x_5}, \quad Z = \frac{\partial}{\partial x_5}.
\end{align*}

The relations in $\text{vfc}(4|3; K)_-$ are:

\begin{align*}
S & \quad [Y_1, Y_2] = 0, \quad [Y_1, Y_3] = 0, \quad \text{ad}^2_{Y_2} Y_3 = 0, \quad \text{ad}^2_{Y_3} Y_2 = 0, \\
HW & \quad \text{ad}^2_{Y_1} Z = 0, \quad \text{ad}^2_{Y_3} Z = 0, \quad [Y_3, Z] = 0, \\
N_2 & \quad [Z, Z] = 0; \quad [[[Y_2, Z], [Y_3, Y_4]], [[Z, [Y_1, Y_2]], [Z, [Y_2, Y_3]]]] = 0.
\end{align*}
Set (positive generators):

\[
X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = -x_1 \frac{\partial}{\partial x_2} + x_6 \frac{\partial}{\partial x_5}, \quad X_3 = -x_2 \frac{\partial}{\partial x_3} + x_7 \frac{\partial}{\partial x_4},
\]

\[
Z_1 = x_3 x_4 \frac{\partial}{\partial x_7} - x_3 x_5 \frac{\partial}{\partial x_2} + x_3 x_6 \frac{\partial}{\partial x_1} + x_5 x_6 \frac{\partial}{\partial x_7},
\]

\[
Z_2 = x_2 x_4 \frac{\partial}{\partial x_7} - x_2 x_5 \frac{\partial}{\partial x_2} + x_2 x_6 \frac{\partial}{\partial x_1} + x_4 x_5 \frac{\partial}{\partial x_4} + x_5 x_6 \frac{\partial}{\partial x_6} - x_3 x_4 \frac{\partial}{\partial x_6} - x_3 x_5 \frac{\partial}{\partial x_3} + x_3 x_7 \frac{\partial}{\partial x_1} + x_4 x_5 \frac{\partial}{\partial x_4} + x_5 x_7 \frac{\partial}{\partial x_7}.
\]

The relations in \(\mathfrak{vl}(4|3; K)_+\) are:

\[
S \quad [X_1, X_2] = 0, \quad [X_1, X_3] = 0, \quad \text{ad}^2_{X_2} X_3 = 0, \quad \text{ad}^2_{X_3} X_2 = 0,
\]

\[
LW \quad [X_2, Z_1] = 0, \quad [X_3, Z_2] = 0, \quad \text{ad}^2_{X_2} Z_1 = 0, \quad \text{ad}^2_{X_1} Z_2 = 0,
\]

\[
LW \quad \text{ad}^3_{X_3} Z_1 = 0, \quad \text{ad}^2_{X_2} Z_2 = 0,
\]

\[
N_2 \quad [Z_1, Z_2] = 0, \quad [Z_1, Z_2] = 0, \quad [Z_2, Z_2] = 0,
\]

\[
N_2 \quad [Z_1, [X_3, [X_3, Z_1]]] = 0,
\]

\[
N_2 \quad [[X_1, Z_1], [X_3, Z_1]] = [Z_2, [X_1, Z_1]],
\]

\[
N_2 \quad [[X_3, Z_1], [X_2, [X_1, Z_2]]] = 2 [[X_1, Z_2], [X_2, Z_2]] + 3 [[X_1, Z_2], [Z_1, [X_2, X_3]]] - 2 [[X_2, X_3], [Z_2, [X_1, Z_1]]] - 2 [[X_2, Z_2], [X_3, [X_1, Z_1]]];
\]

\[
N_3 \quad [[Z_1, [X_2, X_3]], [Z_2, [X_1, Z_1]]] = -3 [[X_2, Z_2], [X_2, [X_1, Z_1]]];
\]

\[
N_3 \quad [[Z_2, [X_2, X_3]], [[X_1, Z_2], [X_2, Z_2]]] = 0,
\]

\[
N_4 \quad [[Z_2, [X_1, Z_1]], [X_1, Z_2], [X_2, Z_2]] = 0;
\]

\[
N_4 \quad [[[[X_3, Z_1], [Z_2, [X_2, X_3]]], [X_2, [X_1, Z_2]]], [X_3, [X_1, Z_1]]] = 6 [[[X_1, Z_2], [Z_1, [X_2, X_3]]], [[X_1, Z_2], [X_2, [X_2, X_3]]]] + 4 [[[X_1, Z_2], [Z_2, [X_2, X_3]]], [[X_2, X_3], [Z_2, [X_1, Z_1]]]] + 2 [[[X_1, Z_2], [Z_2, [X_2, X_3]]], [[X_2, Z_2], [X_3, [X_1, Z_1]]]];
\]

\[
N_5 \quad 6 [[[Z_2, [X_1, Z_1]], [X_2, Z_2]], [[X_2, Z_2]], [[X_2, [X_2, X_3]], [X_1, Z_1]], [X_2, Z_2]]] = [[[X_1, Z_2], [X_2, Z_2]], [[[X_1, Z_2], [X_2, Z_2]], [[X_2, Z_2], [X_2, [X_1, Z_1]]]]];
\]

\[
N_6 \quad [[[Z_2, [X_1, Z_1]], [X_1, Z_2], [X_2, Z_2]], [[[X_1, Z_1], [X_2, Z_2]], [X_2, Z_2], [X_2, [X_1, Z_1]]]] = 0.
\]

**Generators and relations for \(\mathfrak{vas}(4|4)\)** A preliminary step to description of presentation of \(\mathfrak{vas}(4|4)\) is the description of \(\mathfrak{vas}(4|4)\) by \(x\) and the negative by \(y\); each even (odd) element has even (odd) index. Set

\[
x_1 = \eta_3, \quad x_2 = \zeta_2 \eta_1, \quad x_4 = \zeta_3 \eta_2, \quad x_3 = \zeta_1 \eta_1 \eta_3 - \zeta_2 \eta_2 \eta_3
\]

\[
y_1 = \zeta_3, \quad y_2 = \zeta_2 \eta_3, \quad y_4 = \zeta_1 \eta_2 \eta_3, \quad y_3 = -\zeta_1 \zeta_3 \eta_1 - \zeta_2 \zeta_3 \eta_2;
\]

and let Cartan subalgebra be spanned by

\[
h_0 = 1, \quad h_i = \zeta_i \eta_i.
\]

The relations for positive elements are:

\[
S \quad \text{ad}^2_{x_2} x_4 = 0, \quad \text{ad}^2_{x_4} x_2 = 0, \quad \text{ad}^2_{x_4} x_1 = 0,
\]

\[
LW \quad [x_1, x_2] = 0, \quad \text{ad}^2_{x_4} x_3 = 0, \quad \text{ad}^2_{x_2} x_3 = 0,
\]

\[
N_2 \quad [x_1, x_1] = 0, \quad [x_1, x_3] = 0, \quad [x_3, x_3] = 0,
\]

\[
N_3 \quad 2 [x_3, [x_2, x_4]] = [x_4, [x_2, x_3]].
\]
The relations for negative elements are

\[
S \quad [y_1, y_1] = 0, \quad [y_1, y_3] = 0, \quad [y_2, y_4] = 0, \\
S \quad [y_4, y_4] = 0, \quad [y_4, y_3] = 0, \quad [y_3, y_3] = 0, \\
S \quad \text{ad}_{y_2}y_1 = 0, \quad \text{ad}_{y_2}y_3 = 0, \\
N \quad [[y_1, y_4], [y_4, [y_1, y_2]]] = 0; \quad [[y_3, [y_1, y_2]], [[y_1, y_2], [y_1, y_4]]] = 0.
\]

The weights with respect to the \( h_i \) are:

| \( x_1 \) | (0 0 0 1) | \( y_1 \) | (0 0 0 -1) |
| \( x_2 \) | (0 1 -1 0) | \( y_2 \) | (0 0 0 -1) |
| \( x_4 \) | (0 0 1 -1) | \( y_4 \) | (0 -1 1 1) |
| \( x_3 \) | (0 0 0 1) | \( y_3 \) | (0 0 0 -1) |

Other relations between positive and negative elements:

\[
[x_1, y_1] = -h_0, \quad [x_1, y_2] = 0, \quad [x_1, y_4] = 0, \\
[x_1, y_3] = -h_1 - h_2, [x_2, y_1] = 0, \quad [x_2, y_2] = 0, \quad [x_2, y_4] = x_3, \\
[x_2, y_3] = 0, \quad [x_4, y_1] = 0, \quad [x_4, y_2] = h_2 - h_3, \\
[x_4, y_4] = 0, \quad [x_3, y_1] = -h_1 + h_2, \\
[x_3, y_2] = 0, \quad [x_3, y_4] = 0, \quad [x_3, y_3] = 0.
\]

**Generators and relations for \( \mathfrak{vas}(4|4) \).** We denote \( \partial_i = \frac{\partial}{\partial x_i} \) and \( \delta_i = \frac{\partial}{\partial y_i} \). Set (positive generators):

\[
X_1 = x_1 \delta_4 - x_4 \delta_1 + \theta_3 \partial_2 - \theta_2 \partial_3, \quad X_2 = x_2 \partial_1 - \theta_1 \delta_2, \\
X_3 = x_3 \delta_3, \quad X_4 = x_1 \delta_4 + x_4 \delta_1, \\
Z = -x_1^2 \partial_4 + 2 x_1 \theta_3 \delta_1.
\]

Relations in \( \mathfrak{vas}_+ \) (since only \( X_2 \) is even and the representations of \( \mathfrak{g}_0 \) in \( \mathfrak{g}_{\pm 1} \) are not, strictly speaking neither highest nor lowest weight one, all these relations are of \( (N) \) type; the same applies for \( \mathfrak{vas}_- \) since only \( Y_2 \) and \( Y_3 \) are even):

\[
[X_1, X_1] = 0, \quad [X_1, X_4] = 0, \quad [X_2, X_3] = 0, \quad [X_3, X_3] = 0, \\
[X_3, X_4] = 0, \quad [X_3, Z] = 0, \quad [X_4, X_4] = 0, \\
[X_4, Z] = 3 [X_1, Z], \quad \text{ad}_{X_2}^2 X_1 = 0, \\
\text{ad}_{X_2}^2 X_4 = 0, \quad \text{ad}_{X_2}^2 Z = 0, \\
2 [Z, [X_2, X_4]] = - [X_4, [X_2, Z]]; \\
\text{ad}_{X_2}^2 X_2 = 0, \quad \text{ad}_{X_2}^3 Z = 0, \\
[[X_1, X_3], [X_3, [X_1, X_2]]] = 0, \\
[[X_4, [X_1, X_2]], [X_1, X_2], [X_1, X_3]] = 0, \\
[[X_1, X_2], [X_2, Z]], [[X_1, X_3], [X_2, Z]] = 0.
\]

**Generators in \( \mathfrak{vas}(4|4)_- \).** Set:

\[
Y_1 = x_2 \delta_3 - x_3 \delta_2 + \theta_4 \partial_1 - \theta_1 \partial_4, \quad Y_2 = x_2 \partial_3 - \theta_3 \delta_2, \\
Y_3 = x_1 \partial_2 - \theta_2 \delta_1, \quad Y_4 = x_2 \delta_3 + x_3 \delta_2, \quad Z = \delta_4.
\]
Relations in $\mathfrak{vas}(4|4)$:

\begin{align*}
S & | \quad \text{ad}_{Y_3}^2 Y_3 = 0, \quad \text{ad}_{Y_3}^2 Y_2 = 0, \\
N & | \quad \text{ad}_{Y_3}^2 Y_1 = 0, \quad \text{ad}_{Y_4}^2 Y_4 = 0, \quad \text{ad}_{Y_3}^2 Y_4 = 0, \\
N & | \quad [Y_1, Y_1] = 0, \quad [Y_1, Y_2] = 0, \quad [Y_1, Y_4] = 0, \quad [Y_2, Z] = 0, \\
N & | \quad [Y_3, Z] = 0, \quad [Y_4, Y_4] = 0, \quad [Y_4, Z] = 0, \quad [Z, Z] = 0, \\
N & | \quad 2 [Y_1, [Y_2, Y_3]] = [Y_3, [Y_2, Y_4]], \quad [[Y_1, Z], [Y_3, Y_4]] = [[Y_1, Y_3], [Y_1, Z]].
\end{align*}

$\mathfrak{vas}(4|3; K)$ Set (negative generators):

\[ Y_1 = \xi_2 \eta_1, \quad Y_2 = \xi_3 \eta_2, \quad Y_3 = \eta_2 \eta_3, \quad Z = \xi_1. \]

The relations in $\mathfrak{vas}(4|3; K)$ are:

\begin{align*}
S & | \quad [Y_2, Y_3] = 0, \quad \text{ad}_{Y_2}^2 Y_1 = 0, \quad \text{ad}_{Y_3}^2 Y_1 = 0, \\
S & | \quad \text{ad}_{Y_1}^2 Y_2 = 0, \quad \text{ad}_{Y_3}^2 Y_3 = 0, \\
HW & | \quad [Y_2, Z] = 0, \quad [Y_3, Z] = 0, \quad \text{ad}_{Y_1}^2 Z = 0, \\
N_2 & | \quad [Z, Z] = 0.
\end{align*}

Set (positive generators):

\[ X_1 = \xi_1 \eta_2, \quad X_2 = \xi_2 \eta_3, \quad X_3 = \xi_2 \xi_3, \quad Z_1 = t \eta_1, \quad Z_2 = \xi_3 \eta_1 \eta_2. \]

The relations in $\mathfrak{vas}(4|3; K)$ (computed up to deg $\leq 40$) consist of Serre-type relations:

\begin{align*}
[X_1, Z_2] & = 0, \quad [X_2, X_3] = 0, \quad [X_2, Z_1] = 0, \quad [X_3, Z_1] = 0, \\
[X_3, Z_2] & = 0, \quad [Z_1, Z_1] = 0, \quad [Z_1, Z_2] = 0, \quad [Z_2, Z_2] = 0, \\
\text{ad}_{X_1}^2 X_2 & = 0, \quad \text{ad}_{X_1}^2 X_3 = 0, \quad \text{ad}_{X_1}^2 Z_1 = 0, \\
\text{ad}_{X_2}^2 X_1 & = 0, \quad \text{ad}_{X_3}^2 X_1 = 0, \quad \text{ad}_{X_2}^2 Z_2 = 0.
\end{align*}
and new (awful) type of relations:

\[ [X_2, [X_2, [X_2, Z_2]]] = 0, \]

\[ \{[[X_1, X_3], [X_2, Z_2]], [[X_1, X_3], [X_2, Z_2]]\} = -[[Z_1, [X_1, X_3]], [X_2, Z_2]], \]

\[ 4 \{[[Z_1, [X_1, X_3]], [Z_1, [X_3, X_2]]], [[Z_1, [X_1, X_3]], [[X_2, Z_2], [Z_1, [X_1, X_3]]] = \]

\[ \{[[Z_1, [X_1, X_3]], [Z_1, [X_1, X_2]]], [[Z_1, [X_1, X_3]], [[X_2, Z_2], [Z_1, [X_1, X_2]]] = + \]

\[ 12 \{[[Z_1, [X_1, X_3]], [Z_1, [X_1, X_3]], [[X_2, Z_2], [Z_1, [X_1, X_3]]] = \]

\[ 24 \{[[Z_1, [X_1, X_3]], [Z_1, [X_1, X_3]], [[X_2, Z_2], [Z_1, [X_1, X_2]]] = \]

\[ 24 \{[[Z_1, [X_1, X_3]], [Z_1, [X_1, X_3]], [[X_2, Z_2], [Z_1, [X_1, X_3]]] = \]

\[ 120 \{[[Z_1, [X_1, X_3]], [Z_1, [X_1, X_3]], [[X_2, Z_2], [Z_1, [X_1, X_3]]] = \]

\[ 120 \{[[Z_1, [X_1, X_3]], [Z_1, [X_1, X_3]], [[X_2, Z_2], [Z_1, [X_1, X_3]]] = \]

\[ 120 \{[[Z_1, [X_1, X_3]], [Z_1, [X_1, X_3]], [[X_2, Z_2], [Z_1, [X_1, X_3]]] = \]

Comment. Lie superalgebra was has only one W-grading, so we consider it. For the remaining exceptional simple Lie superalgebras \( g \) we take the consistent grading. So \( g_0 \) is a Lie algebra, moreover, a reductive one; \( g_1 \) generates the positive part, and \( g_{-1} \) generates the negative part, moreover, \( g_{-1} \) is an irreducible \( g_0 \)-module. Therefore, \( g \) is generated by the Chevalley generators \( X_i^\pm \) and \( H_i = [X_i^+, X_i^-] \) of \( g_0 \), the lowest weight vectors \( v_{\lambda_k} \) of \( g_1 \) and the highest weight vector \( w_\Lambda \) of \( g_{-1} \). A number of relations between these generators are already known: the Serre relations for \( X_i^\pm \) and relations involving \( H_i \); and also

\[ (ad X_i^+)^{\lambda_k}(X_i^+)^{+1}(v_{\lambda_k}) = 0 \quad (LW) \]

\[ (ad X_i^-)^{\lambda_k}(X_i^-)^{1}(w_\Lambda) = 0 \quad (HW) \]

and the weight relations \( H_i(v_{\lambda_k}) = \lambda_k^{(i)} \cdot v_{\lambda_k} \), etc. Since the finite diminsional representations of the reductive algebras are completely reducible, we have to compare \( g_1 \wedge g_1 \wedge g_2 \) (the lowest weight vectors of \( g_1 \wedge g_1 / g_2 \) are relations) and similarly for the negative part. If this does not suffice \((g_1 \wedge g_1 \wedge g_1 \mod \text{relations is greater than } g_3)\) we add new relations, etc.
For positive parts of simple vectorial Lie algebras we can use Fuchs’ theorem with Kochetkov’s correction (and direct calculations described above for small dimensions). For exceptional algebras we consider we have no general theory, so having obtained the whole \( \mathfrak{g} \) and no new relations for several \( i > 0 \) we conjecture that we got all. For \( \mathfrak{f} \) as we keep getting new relations in every \( i \).

\section*{Appendix}

\subsection*{1.0. Linear algebra in superspaces. Generalities.} A \textit{superspace} is a \( \mathbb{Z}/2 \)-graded space; for any superspace \( V = V_0 \oplus V_1 \) denote by \( \Pi(V) \) another copy of the same superspace: with the shifted parity, i.e., \( (\Pi(V))_i = V_{i+1} \). The \textit{superdimension} of \( V \) is \( \dim V = p + q\varepsilon \), where \( \varepsilon^2 = 1 \) and \( p = \dim V_0, \; q = \dim V_1 \). (Usually, \( \dim V \) is expressed as a pair \( (p, q) \) or \( p|q \); this obscures the fact that \( \dim V \otimes W = \dim V \cdot \dim W \); this fact is clear with the use of \( \varepsilon \).

A superspace structure in \( V \) induces the superspace structure in the space \( \operatorname{End}(V) \). A \textit{basis of a superspace} is always a basis consisting of homogeneous vectors; let \( \operatorname{Par} = (p_1, \ldots, p_{\dim V}) \) be an ordered collection of their parities. We call \( \operatorname{Par} \) the \textit{format} (of the basis of) \( V \). A square \textit{supermatrix} of format (size) \( \operatorname{Par} \) is a \( \dim V \times \dim V \) matrix whose \( i \)th row and \( i \)th column are of the same parity \( p_i \).

One usually considers one of the simplest formats \( \operatorname{Par} \), e.g., \( \operatorname{Par} \) of the form \((\overline{0}, \ldots, \overline{0}; \overline{1}, \ldots, \overline{1})\) is denoted \((\dim V_0, \dim V_1)\) and called \textit{standard}. In this paper we can do without nonstandard formats. But they are vital in the classification of systems of simple roots that the reader might be interested in in connection with applications to \( q \)-quantization or integrable systems. Besides, systems of simple roots corresponding to distinct nonstandard formats are related by odd reflections — analogs of our nonstandard regradings. (For an approach to superroots see \cite{[S2]}.)

The matrix unit \( E_{ij} \) is supposed to be of parity \( p_i + p_j \) and the bracket of supermatrices (of the same format) is defined via \textbf{Sign Rule}:

\textit{if something of parity \( p \) moves past something of parity \( q \) the sign \((-1)^{pq} \) accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.}

Examples: by setting \([X, Y] = XY - (-1)^{p(X)p(Y)} YX\) we get the notion of the supercommutator and the ensuing notions of the supercommutative superalgebra and the Lie superalgebra (which in addition to superskew-commutativity satisfies the super Jacobi identity, i.e., the Jacobi identity amended with the Sign Rule). The derivation (better say, superderivation) of a superalgebra \( A \) is a linear map \( D : A \to A \) that satisfies the Leibniz rule (and Sign rule)

\[ D(ab) = D(a)b + (-1)^{p(a)p(b)} aD(b). \]

In particular, let \( A = \mathbb{C}[x] \) be the free supercommutative polynomial superalgebra in \( x = (x_1, \ldots, x_n) \), where the superstructure is determined by the parities of the indeterminates: \( p(x_i) = p_i \). Partial derivatives are defined (with the help of super Leibniz Rule) by the formulas \( \frac{\partial}{\partial x_i} = \delta_{i,j} \). Clearly, the collection \( \partial_{\text{er}} A \) of all superdifferentiations of \( A \) is a Lie superalgebra whose elements are of the form \( \sum f_i(x) \frac{\partial}{\partial x_i} \).

Given the supercommutative superalgebra \( \mathcal{F} \) of “functions” in indeterminates \( x \), define the supercommutative superalgebra \( \Omega \) of differential forms as polynomial algebra over \( \mathcal{F} \) in \( dx \), where \( p(d) = 1 \). Since \( dx \) is even for an odd \( x \), we can consider not only polynomials in \( dx \). Smooth or analytic functions in the differentials of the \( x \) are called \textit{pseudodifferential forms} on the supermanifold with coordinates \( x \), see \cite{[BLT]}.

We will need them to interpret \( \mathfrak{h}(\mathfrak{g}) \). The exterior differential is defined on (pseudo) differential forms by the formulas (mind Leibniz and Sign Rules):

\[ d(x_i) = dx_i \quad \text{and} \quad d^2 = 0. \]
The Lie derivative is defined (minding same Rules) by the formula

\[ L_D(df) = (-1)^{p(D)} d(D(f)). \]

In particular,

\[ L_D((df)^\lambda) = \lambda(-1)^{p(D)} d(D(f))((df)^\lambda)^{-1} \text{ for any } \lambda \in \mathbb{C}. \]

The general linear Lie superalgebra of all supermatrices of size Par is denoted by \( gl(Par) \) (usually, \( gl(\dim V_0|\dim V_1) \)). Any matrix from \( gl(Par) \) can be expressed as the sum of its even and odd parts; in the standard format this is the following block expression:

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
= 
\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix}
+ 
\begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix},
\quad
p\left(\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix}\right) = \bar{0},
p\left(\begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix}\right) = 1.
\]

The supertrace is the map \( gl(Par) \rightarrow \mathbb{C} \), \( (A_{ij}) \mapsto \sum (-1)^{p_i} A_{ij} \). Since \( \text{str}[x,y] = 0 \), the subsuperspace of supertraceless matrices constitutes the special linear Lie subsuperalgebra \( \mathfrak{sl}(Par) \).

There are, however, two super versions of \( \mathfrak{gl}(n) \), not one. The other version is called the queer Lie superalgebra and is defined as the one that preserves the complex structure given by an odd operator \( J \), i.e., is the centralizer \( C(J) \) of \( J \):

\[ q(n) = C(J) = \{ X \in \mathfrak{gl}(n|n) \mid [X,J] = 0 \}, \text{ where } J^2 = -\text{id}. \]

It is clear that by a change of basis we can reduce \( J \) to the form \( J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1 & 0 \end{pmatrix} \). In the standard format we have

\[ q(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}. \]

On \( q(n) \), the queertrace is defined: \( qtr : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \text{tr}B \). Denote by \( \mathfrak{sq}(n) \) the Lie superalgebra of queertraceless matrices.

Observe that the identity representations of \( q \) and \( \mathfrak{sq} \) in \( V \), though reducible in supersetting, are not reducible in the nongraded sense: take homogeneous (with respect to parity) and linearly independent vectors \( v_1, \ldots, v_n \) from \( V \); then \( \text{Span}(v_1 + J(v_1), \ldots, v_n + J(v_n)) \) is an invariant subspace of \( V \) which is not a subsuperspace.

A representation is irreducible of general type or just of \( G \)-type if there is no invariant subspace, otherwise it is called irreducible of \( Q \)-type (\( Q \) is after the general queer Lie superalgebra — a specifically “superish” analog of \( \mathfrak{gl} \)); an irreducible representation of \( Q \)-type has no invariant subsuperspace but has an invariant subspace.

**Lie superalgebras that preserve bilinear forms: two types.** To the linear map \( F : V \rightarrow W \) of superspaces there corresponds the dual map \( F^* : W^* \rightarrow V^* \) between the dual superspaces. In a basis consisting of the vectors \( v_i \) of format \( \text{Par} \), the formula \( F(v_j) = \sum v_i A_{ij} \) assigns to \( F \) the supermatrix \( A \). In the dual bases, to \( F^* \) the supertransposed matrix \( A^{st} \) corresponds:

\[ (A^{st})_{ij} = (-1)^{(p_i + p_j)(p_i + p(A))} A_{ji}. \]

The supermatrices \( X \in \mathfrak{gl}(Par) \) such that

\[ X^{st} B + (-1)^{p(X)p(B)} BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(Par) \]

constitute the Lie superalgebra \( \mathfrak{aut}(B) \) that preserves the bilinear form \( B^f \) on \( V \) whose matrix \( B \) is given by the formula \( B_{ij} = (-1)^{p(B_j)p(v_i)} B^f(v_i, v_j) \) for the basis vectors \( v_i \).
Recall that the supersymmetry of the homogeneous form $B^t$ means that its matrix $B$ satisfies the condition $B^u = B$, where for the matrix $B = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$ we set $B^u = \begin{pmatrix} R^t & (-1)^{\mu(B)} T^t \\ (-1)^{\nu(B)} S^t & -U^t \end{pmatrix}$.

Similarly, skew-supersymmetry of $B$ means that $B^u = -B$. Thus, we see that the upsetting of bilinear forms $u : \text{Bil}(V,W) \to \text{Bil}(W,V)$, which for the spaces and when $V = W$ is expressed on matrices in terms of the transposition, is a new operation.

Most popular canonical forms of the nondegenerate supersymmetric form are the ones whose supermatrices in the standard format are the following canonical ones, $B_{ev}$ or $B'_{ev}$:

$$B'_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where} \quad J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

or

$$B_{ev}(m|2n) = \begin{pmatrix} \text{antidiag}(1, \ldots, 1) & 0 \\ 0 & J_{2n} \end{pmatrix}.$$

The usual notation for $\text{aut}(B_{ev}(m|2n))$ is $\text{osp}(m|2n)$ or, more precisely, $\text{osp}^{yu}(m|2n)$. Observe that the passage from $V$ to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones, preserved by the “symplectico-orthogonal” Lie superalgebra, $\text{sp}'\text{o}(2n|m)$ or, better say, $\text{osp}^{sk}(m|2n)$, which is isomorphic to $\text{osp}^{yu}(m|2n)$ but has a different matrix realization. We never use notation $\text{sp}'\text{o}(2n|m)$ in order not to confuse with the special Poisson superalgebra.

In the standard format the matrix realizations of these algebras are:

$$\text{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\}; \quad \text{osp}^{sk}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},$$

where $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n)$, $E \in \mathfrak{o}(m)$.

A nondegenerate supersymmetric odd bilinear form $B_{odd}(n|n)$ can be reduced to a canonical form whose matrix in the standard format is $J_{2n}$. A canonical form of the superskew odd nondegenerate form in the standard format is $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. The usual notation for $\text{aut}(B_{odd}(\text{Par}))$ is $\text{pe}(\text{Par})$. The passage from $V$ to $\Pi(V)$ establishes an isomorphism $\text{pe}^{yu}(\text{Par}) \cong \text{pe}^{sk}(\text{Par})$. This Lie superalgebra is called, as A. Weil suggested, periplectic. The matrix realizations in the standard format of these superalgebras is shorthanded to:

$$\text{pe}^{yu}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\}, \quad \text{where} \quad B = -B^t, \ C = C^t;$$

$$\text{pe}^{sk}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\}, \quad \text{where} \quad B = B^t, \ C = -C^t.$$
algebra”) is manifestly inadequate for considering the (singular) supervarieties of deformations and applying representation theory to mathematical physics, for example, in the study of the coadjoint representation of the Lie supergroup which can act on a supermanifold but never on a superspace (an object from another category). So, to deform Lie superalgebras and apply group-theoretical methods in “super” setting, we must be able to recover a supermanifold from a superspace, and vice versa.

A proper definition of Lie superalgebras is as follows, cf. [L3]–[L5]. The Lie superalgebra in the category of supermanifolds corresponding to the “naive” Lie superalgebra $L=L_0 \oplus L_1$ is a linear supermanifold $L=(L_0,\mathcal{O})$, where the sheaf of functions $\mathcal{O}$ consists of functions on $L_0$ with values in the Grassmann superalgebra on $L_1^*$; this supermanifold should be such that for “any” (say, finitely generated, or from some other appropriate category) supercommutative superalgebra $C$, the space $\mathcal{L}(C)=\text{Hom}(\text{Spec}C,\mathcal{L})$, called the space of C-points of $\mathcal{L}$, is a Lie algebra and the correspondence $C \mapsto \mathcal{L}(C)$ is a functor in $C$. (A. Weil introduced this approach in algebraic geometry in 1954; in super setting it is called the language of points or families, see [D], [L4].) This definition might look terribly complicated, but fortunately one can show that the correspondence $\mathcal{L} \hookrightarrow L$ is one-to-one and the Lie algebra $\mathcal{L}(C)$, also denoted $L(C)$, admits a very simple description: $L(C)=(L \otimes C)_0$.

A Lie superalgebra homomorphism $\rho : L_1 \longrightarrow L_2$ in these terms is a functor morphism, i.e., a collection of Lie algebra homomorphisms $\rho_C : L_1(C) \longrightarrow L_2(C)$ compatible with morphisms of supercommutative superalgebras $C \longrightarrow C'$. In particular, a representation of a Lie superalgebra $L$ in a superspace $V$ is a homomorphism $\rho : L \longrightarrow \mathfrak{gl}(V)$, i.e., a collection of Lie algebra homomorphisms $\rho_C : L(C) \longrightarrow (\mathfrak{gl}(V) \otimes C)_0$.

**Example.** Consider a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. The tangent space of the moduli superspace of deformations of $\rho$ is isomorphic to $H^1(\mathfrak{g}; V \otimes V^*)$. For example, if $\mathfrak{g}$ is the 0/|n-dimensional (i.e., purely odd) Lie superalgebra (with the only bracket possible: identically equal to zero), its only irreducible representations are the trivial one, 1, and $\Pi(1)$. Clearly, $1 \otimes 1^* \simeq \Pi(1) \otimes \Pi(1)^* \simeq 1$, and because the superalgebra is commutative, the differential in the cochain complex is trivial. Therefore, $H^1(\mathfrak{g}; 1) = E^1(\mathfrak{g}^*) \simeq \mathfrak{g}^*$, so there are $\dim \mathfrak{g}$ odd parameters of deformations of the trivial representation. If we consider $\mathfrak{g}$ “naively” all of the odd parameters will be lost.

Which of these infinitesimal deformations can be extended to a global one is a separate much tougher question, usually solved *ad hoc*.

Examples that lucidly illustrate why one should always remember that a Lie superalgebra is not a mere linear superspace but a linear supermanifold are, e.g., deforms $\text{Vect}(0|2n+1)$ and $\text{B}_{\mu}(2^{2n} - 1|2^{2n})$ with odd parameters considered below and viewed as Lie algebras. In the category of supermanifolds these are simple Lie superalgebras.

### 1.0.2. Projectivization.

If $\mathfrak{s}$ is a Lie algebra of scalar matrices, and $\mathfrak{g} \subset \mathfrak{gl}(n|n)$ is a Lie subsuperalgebra containing $\mathfrak{s}$, then the projective Lie superalgebra of type $\mathfrak{g}$ is $\mathfrak{pg} = \mathfrak{g}/\mathfrak{s}$. Lie superalgebras $\mathfrak{g}_1 \circ \mathfrak{g}_2$ described in sect. 3.1 are also projective.

Projectivization sometimes leads to new Lie superalgebras, for example: $\mathfrak{pgl}(n|n)$, $\mathfrak{psl}(n|n)$, $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$; whereas $\mathfrak{pgl}(p|q) \cong \mathfrak{sl}(p|q)$ if $p \neq q$.

### 1.0.3. What is a semisimple Lie superalgebra.

These algebras are needed in description of primitive Lie superalgebras of vector fields — a geometrically natural problem though wild for Lie superalgebras, see [ALS]. Recall that the Lie superalgebra $\mathfrak{g}$ without proper ideals and of dimension $>1$ is called *simple*. Examples: $\mathfrak{sl}(m|n)$ for $m > n \geq 1$, $\mathfrak{psl}(n|n)$ for $n > 1$, $\mathfrak{psq}(n)$ for $n > 2$, $\mathfrak{osp}(m|2n)$ for $mn \neq 0$ and $\mathfrak{spe}(n)$ for $n > 2$. 
We will not need the remaining simple finite dimensional Lie superalgebras of non-vectorial type. These superalgebras, discovered by I. Kaplansky (a 1975-preprint, see [K]) are \(osp_n(4|2)\), the deforms of \(osp(4|2)\), and the two exceptions that we denote by \(\mathfrak{a}q_2\) and \(\mathfrak{a}b_3\). For their description we refer to [K], [FR], [Sud], see also [GL2] for the description of the system of simple roots see [Ka4] completed in [vdL, S1, S2].

We say that \(\mathfrak{h}\) is *almost simple* if it can be sandwiched (non-strictly) between a simple Lie superalgebra \(\mathfrak{s}\) and the Lie superalgebra \(\mathfrak{der}\mathfrak{s}\) of derivations of \(\mathfrak{s}\): \(\mathfrak{s} \subset \mathfrak{h} \subset \mathfrak{der}\mathfrak{s}\).

By definition, \(\mathfrak{g}\) is *semisimple* if its radical is zero. Literally following the description of semisimple Lie algebras over the fields of prime characteristic, V. Kac [Ka4] gave the following description of semisimple Lie superalgebras. Let \(\mathfrak{s}_1, \ldots, \mathfrak{s}_k\) be simple Lie superalgebras, let \(n_1, \ldots, n_k\) be pairs of non-negative integers \(n_j = (n^0_j, n^1_j)\), let \(\mathcal{F}(n_j)\) be the supercommutative superalgebra of polynomials in \(n^0_j\) even and \(n^1_j\) odd indeterminates, and \(\mathfrak{s} = \bigoplus_j (\mathfrak{s}_j \otimes \mathcal{F}(n_j))\).

Then
\[
\mathfrak{der}\mathfrak{s} = \bigoplus \left( (\mathfrak{der}\mathfrak{s}_j) \otimes \mathcal{F}(n_j) \ni \text{id}_{\mathfrak{s}_j} \otimes \text{vect}(n_j) \right).
\]

Let \(\mathfrak{g}\) be a subalgebra of \(\mathfrak{der}\mathfrak{s}\) containing \(\mathfrak{s}\). If the projection of \(\mathfrak{g}\) on \(1 \otimes \text{vect}(n_j)\) is onto for each \(j\), then \(\mathfrak{g}\) is semisimple and all semisimple Lie superalgebras arise in the manner indicated, i.e., as sums of *almost simple superalgebras* corresponding to the summands of (0.1.3).

**1.0.4. A. Sergeev’s central extension.** In 70’s A. Sergeev proved that there is just one nontrivial central extension of \(\mathfrak{spe}(n)\). It exists only for \(n = 4\) and we denote it by \(\mathfrak{as}\). Let us represent an arbitrary element \(A \in \mathfrak{as}\) as a pair \(A = x + d \cdot z\), where \(x \in \mathfrak{spe}(4), d \in \mathbb{C}\) and \(z\) is the central element. The bracket in \(\mathfrak{as}\) is

\[
\left[ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + d' \cdot z \right] = \left[ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} \right] + \text{tr} cc' \cdot z,
\]

where \(\text{tr}\) is extended via linearity from matrices \(c_{ij} = E_{ij} - E_{ji}\) on which \(\bar{c}_{ij} = c_{kl}\) for any even permutation \((1234) \mapsto (ijkl)\).

The Lie superalgebra \(\mathfrak{as}\) can also be described with the help of the spinor representation. For this we need several vectorial superalgebras defined in sect. 0.3. Consider \(\mathfrak{po}(0|6)\), the Lie superalgebra whose superspace is the Grassmann superalgebra \(\Lambda(\xi, \eta)\) generated by \(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3\) and the bracket is the Poisson bracket (0.3.6). Recall that \(\mathfrak{h}(0|6) = \text{Span}(H_f | f \in \Lambda(\xi, \eta))\).

Now, observe that \(\mathfrak{spe}(4)\) can be embedded into \(\mathfrak{h}(0|6)\). Indeed, setting \(\deg \xi_i = \deg \eta_i = 1\) for all \(i\) we introduce a \(\mathbb{Z}\)-grading on \(\Lambda(\xi, \eta)\) which, in turn, induces a \(\mathbb{Z}\)-grading on \(\mathfrak{h}(0|6)\) of the form \(\mathfrak{h}(0|6) = \bigoplus_{i \geq -1} \mathfrak{h}(0|6)_i\). Since \(\mathfrak{s}(4) \cong \mathfrak{o}(6)\), we can identify \(\mathfrak{spe}(4)_0\) with \(\mathfrak{h}(0|6)_0\).

It is not difficult to see that the elements of degree \(-1\) in the standard gradings of \(\mathfrak{spe}(4)\) and \(\mathfrak{h}(0|6)\) constitute isomorphic \(\mathfrak{s}(4) \cong \mathfrak{o}(6)\)-modules. It is subject to a direct verification that it is possible to embed \(\mathfrak{spe}(4)_1\) into \(\mathfrak{h}(0|6)_1\).

Sergeev’s extension \(\mathfrak{as}\) is the result of the restriction to \(\mathfrak{spe}(4) \subset \mathfrak{h}(0|6)\) of the cocycle that turns \(\mathfrak{h}(0|6)\) into \(\mathfrak{po}(0|6)\). The quantization deforms \(\mathfrak{po}(0|6)\) into \(\mathfrak{gl}(\Lambda(\xi))\); the through maps \(T_\lambda : \mathfrak{as} \longrightarrow \mathfrak{po}(0|6) \longrightarrow \mathfrak{gl}(\Lambda(\xi))\) are representations of \(\mathfrak{as}\) in the 4|4-dimensional modules \(\text{spin}_\lambda\) isomorphic to each other for all \(\lambda \neq 0\). The explicit form of \(T_\lambda\) is as follows:

\[
T_\lambda : \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z \mapsto \begin{pmatrix} a & b - \lambda \bar{c} \\ c & -a^t \end{pmatrix} + \lambda d \cdot 1_{4|4},
\]

where \(1_{4|4}\) is the unit matrix and \(\bar{c}\) is defined in formula (0.1.4.1). Clearly, \(T_\lambda\) is an irreducible representation for any \(\lambda\).
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