Functional inequalities for the heat flow on time-dependent metric measure spaces

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Abstract

We prove that synthetic lower Ricci bounds for metric measure spaces – both in the sense of Bakry-Émery and in the sense of Lott-Sturm-Villani – can be characterized by various functional inequalities including local Poincaré inequalities, local logarithmic Sobolev inequalities, dimension independent Harnack inequality, and logarithmic Harnack inequality.

More generally, these equivalences will be proven in the setting of time-dependent metric measure spaces and will provide a characterization of super-Ricci flows of metric measure spaces.

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1 Introduction

1.1 Setting

Huge research interest and extensive literature is devoted to the study of functional inequalities for the heat equation, both on Riemannian manifolds and on more abstract spaces. Of particular importance are functional inequalities which are equivalent to a uniform lower bound on the Ricci curvature, say $\text{Ric}_g \geq K \cdot g$. In F.-Y. Wang’s monograph \cite{Wang1992}, Thm 2.3.3., an impressive collection of 15 equivalent properties is listed.

In principle, all these properties and equivalences should hold – and indeed most of them do hold – in much more general settings. Many of them have been re-formulated and proven in the setting of Markov diffusion semigroups and $\Gamma$-calculus, initiated by the seminal work of Bakry \& Émery \cite{BakryEmery1985} and culminating now in the monograph \cite{BakryGentilLedoux2006} of Bakry, Gentil and Ledoux, see Theorems 4.7.2, 5.5.2, 5.5.5, 5.6.1 and Remark 5.6.2 in \cite{BakryGentilLedoux2006}.

Another, more recent, important setting for the study of heat equations and functional inequalities are metric measure spaces, in particular, such mm-spaces which are infinitesimally Hilbertian and which satisfy a synthetic lower Ricci bound as introduced in the foundational works of Sturm \cite{Sturm1996} and Lott \& Villani \cite{LottVillani2009}. In a series of groundbreaking papers, Ambrosio, Gigli \& Savaré \cite{AmbrosioGigliSavare2008a,AmbrosioGigliSavare2008b,AmbrosioGigliSavare2008c} introduced and analyzed the heat flow on such spaces and derived various functional inequalities. In particular, they proved that both the Bochner inequality (without dimensional term) and the $L^2$-gradient estimate are equivalent to the synthetic Ricci bound $\text{CD}(K, \infty)$; and they deduced the local Poincaré inequality and the logarithmic Harnack inequality. Savaré \cite{Savare2011} extended the powerful self-improvement property of Bochner’s inequality to mm-spaces and utilized it to deduce the $L^1$-gradient estimate; based on the latter, H. Li \cite{Li2011} proved the dimension-independent Harnack inequality which in turn implies the logarithmic Harnack inequality.

Only recently, some of these properties and equivalences have been extended to the heat flow on time-dependent Riemannian manifolds, e.g. by Cheng \& Thalmaier \cite{ChengThalmaier2012}, Haslhofer \& Naber \cite{HaslhoferNaber2015}, McCann \& Topping \cite{McCannTopping2014}, and Cheng \cite{Cheng2015}. The authors of the current paper had been the first to study the heat flow on time-dependent metric measure spaces \cite{ChengDegerinFournier2016}, to introduce the time-dependent counterpart of synthetic lower Ricci bounds, and to derive various functional inequalities equivalent to it.

Here and throughout this paper, the setting will be as follows. $(X, d_t, m_t)_{t \in I}$ is a time-dependent metric measure space where $I = (0,T)$ and $X$ is a topological space. The Borel measures $m_t = e^{-f_t} m$ and the geodesic distances $d_t$ are assumed to be logarithmic Lipschitz continuous in time. Moreover, the maps $x \mapsto f_t(x)$ are assumed to be bounded and Lipschitz continuous. That is, there exists a constant $L > 0$ such that for all $x, y \in X$ and $s, t \in I$

$$|f_s(x) - f_t(y)| \leq L|t - s| + Ld(x, y), \quad \left| \log \frac{d_t(x, y)}{d_s(x, y)} \right| \leq L|t - s|. \tag{A1.a}$$

Furthermore, for some $K \in \mathbb{R}$ and each $t \in I$ the static mm-space

$$(X, d_t, m_t) \text{ satisfies the condition } \text{RCD}(K, \infty). \tag{A1.b}$$

Given $t \in I$, let $W_t$ denote the $L^2$-Kantorovich-Wasserstein metric with respect to $d_t$ and let $S_t$ denote the relative Boltzmann entropy with respect to $m_t$.

The static mm-space $(X, d_t, m_t)$ defines a Dirichlet form $\mathcal{E}_t$, a Laplacian $\Delta_t$, and a square field operators $\Gamma_t$ related to each other via

$$-\int_X u \Delta_t v \, dm_t = \mathcal{E}_t(u, v) = \int_X \Gamma_t(u, v) \, dm_t \quad \forall u \in \mathcal{D}(\mathcal{E}_t), v \in \mathcal{D}(\Delta_t).$$

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whereas $s \mapsto P_{t,s}^* v$ provides solutions to the adjoint heat equation $\partial_t v_s = -\Delta_s v_s + v_s f_s$ on $(0,t) \times X$ with $v_t = v$.

The main result of our previous paper is the characterization of super-Ricci flows of mm-spaces in terms of the heat flow on them.

**Theorem 1.1 ([14]).** The following assertions are equivalent:

(i) For a.e. $t \in (0,T)$ and every $W_t$-geodesic $(\mu^a)_{a \in [0,1]}$ in $\mathcal{P}(X)$ with $\mu^0, \mu^1 \in \mathcal{D}(S)$

$$\partial_a S_t(\mu^a)|_{a=1} - \partial_a S_t(\mu^a)|_{a=0} \geq -\frac{1}{2} \partial_t \mathcal{W}^2_t(\mu^0, \mu^1).$$  \hspace{1cm} (E1)

(ii) For all $0 < s < t < T$ and $\mu, \nu \in \mathcal{P}(X)$

$$W_s(\hat{P}_{t,s} \mu, \hat{P}_{t,s} \nu) \leq W_t(\mu, \nu)$$  \hspace{1cm} (E2)

(iii) For all $u \in \mathcal{D}(\mathcal{E})$ and all $0 < s < t < T$

$$\Gamma_t(P_{t,s} u) \leq P_{t,s}(\Gamma_s(u))$$  \hspace{1cm} (E3)

(iv) For all $0 < s < t < T$ and for all $u_s, g_t \in \mathcal{D}(\mathcal{E})$ with $g_t \geq 0$, $g_t \in L^\infty$, $u_s \in \text{Lip}(X)$ and for a.e. $r \in (s,t)$

$$\Gamma_{2,r}(u_r)(g_r) \geq \frac{1}{2} \int \Gamma_r(u_r) g_r dm_r$$  \hspace{1cm} (E4)

where $u_r = P_{r,s} u_s$ and $g_r = P_{r,s}^* g_t$.

Here

$$\Gamma_{2,r}(u_r)(g_r) := \int \left[ \frac{1}{2} \Gamma_r(u_r) \Delta_r g_r + (\Delta_r u_r)^2 g_r + \Gamma_r(u_r, g_r) \Delta_r u_r \right] dm_r$$

denotes the distribution valued $\Gamma_2$-operator (at time $r$) applied to $u_r$ and tested against $g_r$ and

$$\Gamma_r(u_r) := \text{w-}\lim_{\delta \to 0} \frac{1}{\delta} \left( \Gamma_{r+\delta}(u_r) - \Gamma_r(u_r) \right)$$

denotes any subsequential weak limit of $\frac{1}{\delta} (\Gamma_{r+\delta} - \Gamma_{r+\delta})(u_r)$ in $L^2((s,t) \times X)$.

We say that a one-parameter family of mm-spaces $(X, d_t, m_t)_{t \in I}$ is a super-Ricci flow – or that it evolves as a super-Ricci flow – if it satisfies one/each assertion of the previous Theorem. This is a canonical extension of the notion of super-Ricci flows of Riemannian manifolds $(M, g_t)$ defined through the tensor inequality

$$\text{Ric}_t \geq -\frac{1}{2} \partial_t g_t.$$  

Property (i) above is called dynamic convexity of the Boltzmann entropy. This concept has been introduced by the second author in [20]; it provides a canonical generalization of the synthetic
Ricci bound \( CD(0, \infty) \) defined in terms of the semiconvexity of the Boltzmann entropy in the static setting.

Property (iv) is the appropriate generalization of Bochner’s inequality or, in other words, of the Bakry–Émery condition to the time-dependent setting. It will be called \textit{dynamic Bochner inequality (integrated in time)}. In contrast to that, we say that the \textit{dynamic Bochner inequality holds true pointwise in time} if

\[
\forall t \in I, \forall u, g \in D(\Delta) \cap L^\infty(X) \text{ with } \Gamma_t(u) \in L^\infty(X) \text{ and } g \geq 0
\]

\[
\int \left[ \Gamma_t(u) \Delta_t g + 2(\Delta_t u)^2 g + 2\Gamma_t(u, g) \Delta_t u - \partial_t \Gamma_t(u) g \right] \, dm_t \geq 0. \quad (E5)
\]

In the static case, Bochner’s inequality has the remarkable and powerful ‘self-improvement property’ which allows to deduce improved versions of the assertions in the previous Theorem, in particular, to derive the \( L^1 \)-gradient estimate. This self-improvement strategy in the time-dependent case requires additional time regularity of the involved quantities. It was carried out by the first author in [13] and can be reformulated with the notation from the current paper as follows.

\textbf{Theorem 1.2 (13).} Assume \((A2.a+c)\), see Section 2. Then the \( L^2 \)-gradient estimate \((E3)\) is equivalent to the \( L^1 \)-gradient estimate: for all \( u \in D(E) \) and all \( 0 < s < t < T \)

\[
\left( \Gamma_t(P_{t,s} u) \right)^{1/2} \leq P_{t,s} \left( \Gamma_s(u) \right)^{1/2} \quad (E6)
\]

Moreover, the dynamic Bochner inequality (integrated in time) implies the dynamic Bochner inequality pointwise in time which in turn implies the \( L^1 \)-gradient estimate as formulated above.

Additional assumptions on time regularity (e.g. continuity of \( t \mapsto \Delta_t P_{t,s} u \) in appropriate spaces) will be also requested for various results of the current paper; we will formulate these assumptions tailor-made in the subsequent sections.

\textbf{1.2 Summary of the main results}

Let us summarize the main results of the current paper. To simplify and unify the presentation here in the introduction, we will restrict ourselves to the case \( m_t(X) < \infty \) and in addition to our standing assumptions \((A1.a+b)\) we will request now all the assumptions which ever will be made in the sequel. Besides our standing assumptions \((A1.a+b)\), these are assumptions \((A2.a+b)\) formulated in Section 2, \((A3.a+b)\) formulated in Section 3, and assumptions \((A5.a+b)\) formulated in Section 5. We emphasize that all these extra assumptions are always fulfilled in the static case and they are also satisfied in the case of Riemannian manifolds with metric tensors which smoothly depend on time.

\textbf{Theorem 1.3.} Under the previously mentioned assumptions, the following assertions are equivalent:

(i) \( (X, d_t, m_t)_{t \in I} \) is a super-Ricci flow.

(ii) One/each of the local Poincaré inequalities holds

\[
P_t,s(u^2)(x) - (P_t,s u)^2(x) \leq 2(t - s) \Gamma_t(u) (x) \quad (E7)
\]

\[
P_t,s(u^2)(x) - (P_t,s u)^2(x) \geq 2(t - s) \Gamma_t(P_{t,s} u)(x). \quad (E8)
\]
(iii) One/each of the local logarithmic Sobolev inequalities holds

\[ P_{t,s}(u \log u) - P_{t,s} u \log P_{t,s} u \leq (t-s) P_{t,s} \left( \frac{\Gamma_t(u)}{u} \right), \quad (E9) \]

\[ P_{t,s}(u \log u) - P_{t,s} u \log P_{t,s} u \geq (t-s) \frac{\Gamma_t(P_{t,s} u)}{P_{t,s} u}. \quad (E10) \]

(iv) The dimension independent Harnack inequality holds for one/each \( \alpha \in (1, \infty) \)

\[ (P_{t,s} u)^\alpha(y) \leq P_{t,s}(u^\alpha)(x) \exp \left\{ \frac{\alpha d_i^2(x, y)}{4(\alpha - 1)(t-s)} \right\}. \quad (E11) \]

(iv) The logarithmic Harnack inequality holds

\[ P_{t,s}(\log u)(x) \leq \log(P_{t,s} u)(y) + \frac{d_i^2(x, y)}{4(t-s)}. \quad (E12) \]

The formulation “one/each” in particular means that one of the respective properties implies each of the respective properties.

Remark 1.4. a) Upper and lower local Poincaré inequalities together obviously imply the \( L^2 \)-gradient estimate \([E3]\). Upper and lower local logarithmic Sobolev inequality together imply

\[ \frac{\Gamma_t(P_{t,s} u)}{P_{t,s} u} \leq P_{t,s} \left( \frac{\Gamma_t(u)}{u} \right), \]

which is a priori weaker than the \( L^1 \)-gradient estimate \([E6]\). Indeed the \( L^1 \)-gradient estimate together with Jensen’s inequality applied to the function \( \beta(z, w) = z^2/w \) imply

\[ \frac{\Gamma_t(P_{t,s} u)}{P_{t,s} u} \leq \frac{(P_{t,s} \sqrt{\Gamma_t(u)})^2}{P_{t,s} u} \leq P_{t,s} \left( \frac{\Gamma_t(u)}{u} \right). \]

b) The dimension independent Harnack inequality for \( \alpha_1 \) and for \( \alpha_2 \) implies the dimension independent Harnack inequality for \( \alpha_1 \cdot \alpha_2 \), \([21]\), Thm. 1.4.2. The dimension independent Harnack inequality for a sequence \( \alpha_n \to \infty \) implies the log-Harnack inequality. In particular, the dimension independent Harnack inequality for some \( \alpha \in (1, \infty) \) implies the dimension independent Harnack inequality for all \( \alpha_k, k \in \mathbb{N} \), and thus the log-Harnack inequality, \([21]\), Cor. 1.4.3.

The proof of the above theorem will be presented in the subsequent sections, decomposed into a variety of theorems devoted to individual implications. In these theorems, we also specify in detail the spaces of functions \( u \) for which the respective inequalities are supposed to hold. In Section 2 we prove the implications \([E3] \Rightarrow [E7] \Rightarrow [E4] \) and \([E3] \Rightarrow [E8] \Rightarrow [E4] \) as well as the implication \([E4] \Rightarrow [E5] \). Section 3 is devoted to the proof of the implications \([E6] \Rightarrow [E9] \Rightarrow [E5] \) and \([E6] \Rightarrow [E10] \Rightarrow [E5] \). In Section 4 we prove the implications \([E6] \Rightarrow [E11] \Rightarrow [E10] \) and in Section 5 the implication \([E12] \Rightarrow [E5] \). This completes the proof of our theorem since \([E11] \Rightarrow [E12] \) according to the previous remark, \([E5] \Rightarrow [E6] \) according to Theorem 1.2 and trivially \([E6] \Rightarrow [E3] \).

The previous characterizations of super-Ricci flows easily extend to characterizations of \( K \)-super-Ricci flows for any \( K \neq 0 \) by considering reparametrized mm-spaces \( (X, \tilde{d}_t, \tilde{m}_t)_{t \in I} \) with \( \tilde{d}_t = e^{-K \tau(t)} d_{\tau(t)} \), \( \tilde{m}_t = m_{\tau(t)} \), and \( \tilde{I} = \{ t : \tau(t) \in I, 2Kt < C \} \) where \( C \in \mathbb{R} \) and \( \tau(t) = -\frac{1}{2K} \log(C - 2Kt) \), see Theorem 1.11 in \([14]\). Let us restrict ourselves to formulate this in the most simple case of static mm-spaces.
Corollary 1.5. For every infinitesimally Hilbertian \(mm\)-space \((X,d,m)\) the following assertions are equivalent:

(i) \((X,d,m)\) satisfies \(\text{CD}(K,\infty)\).

(ii) One/each of the local Poincaré inequalities holds

\[
P_t(u^2)(x) - (P_t u)^2(x) \leq \frac{1 - e^{-2Kt}}{K} P_t(\Gamma u)(x) \quad (1a)
\]

\[
P_t(u^2)(x) - (P_t u)^2(x) \geq \frac{e^{2Kt} - 1}{K} \Gamma(P_t u)(x).
\]

(iii) One/each of the local logarithmic Sobolev inequalities holds

\[
P_t(u \log u) - P_t u \log P_t u \leq \frac{1 - e^{-2Kt}}{2K} P_t \left( \frac{\Gamma(u)}{u} \right), \quad (2a)
\]

\[
P_t(u \log u) - P_t u \log P_t u \geq \frac{e^{2Kt} - 1}{2K} \frac{\Gamma(P_t u)}{P_t u}, \quad (2b)
\]

(iv) The dimension independent Harnack inequality holds for one/each \(\alpha \in (1,\infty)\)

\[
(P_t u)^\alpha(y) \leq P_t(u^\alpha)(x) \exp \left\{ \frac{\alpha K d^2(x,y)}{2(\alpha - 1)(1 - e^{-2Kt})} \right\}. \quad (3)
\]

(v) The logarithmic Harnack inequality holds

\[
P_t(\log u)(x) \leq \log(P_t u)(y) + \frac{K d^2(x,y)}{2(1 - e^{-2Kt})}. \quad (4)
\]

Remark 1.6. So far, in the setting of \(mm\)-spaces only the implications (i) \(\Rightarrow\) (iib), (i) \(\Rightarrow\) (v), and (i) \(\Rightarrow\) (iv) were known ([7] Thm. 6.8, [7] Lemma 4.6, and [13] Thm. 3.1). The implications (i) \(\Rightarrow\) (iia) and (i) \(\Rightarrow\) (iii) are new also in the static case. In particular, none of the reverse implications (iia) \(\Rightarrow\) (i), (iib) \(\Rightarrow\) (i), (ii), (iii), (iv), or (v) \(\Rightarrow\) (i) was proven before for \(mm\)-spaces.

Also so far, for the implication (v) \(\Rightarrow\) (i) no proof exists in the setting of \(\Gamma\)-calculus for diffusion semigroups.

1.3 Preliminaries

Let us recall some basic properties of the heat propagators \(P_{t,s}\) and their adjoints \(P_{t,s}^*\). We call \(u\) a solution to the heat equation on \((s,\tau) \times X\) if \(u \in \mathcal{F}_{(s,\tau)}\) and

\[
- \int \mathcal{E}_r(u_r, w_r) dr = \int \langle \partial_t u_r, e^{-fr} w_r \rangle dr
\]

for all \(w \in \mathcal{F}_{(s,\tau)}\). Here, \(\mathcal{F}_{(s,\tau)} = L^2((s,\tau) \to \mathcal{D}(\mathcal{E})) \cap H^1((s,\tau) \to \mathcal{D}(\mathcal{E})^*)\) and note that \(\mathcal{F}_{(s,\tau)} \subset C([s,\tau] \to L^2(X))\) so that the values at \(t = s\) and \(t = \tau\) exist. Indeed, however, \(u\) lies in a much smaller class and (as a consequence) \(w\) can be chosen from a much larger class of ‘test functions’. More precisely, if \(u_s \in \mathcal{D}(\mathcal{E})\) then \(u \in L^2((s,\tau) \to \mathcal{D}(\Delta)) \cap H^1((s,\tau) \to L^2(X))\) and \(\mathcal{D}(\mathcal{E})\), appropriately reformulated as

\[
\int \int \Delta_r u_r w_r dm_r dr = \int \int \partial_r u_r w_r dm_r dr,
\]

holds for all \(w \in L^2((s,\tau) \to L^2(X))\). An analogous reformulation holds true for solutions to the adjoint heat equation.

We collect the following properties from [14].
Lemma 1.7 ([14], Prop. 2.14). For all \( u \in L^2(X) \) and all \( s < t \)

1. \( u \geq 0 \iff P_{t,s}u \geq 0, \quad u \leq M \iff P_{t,s}u \leq M. \)

2. \( v \geq 0 \iff P_{t,s}^*v \geq 0, \quad v \leq M \iff P_{t,s}^*v \leq Me^{L(t-s)}. \)

3. \( \|P_{t,s}u\|_{L^p(m_t)} \leq e^{L(t-s)/p} \cdot \|u\|_{L^p(m_t)}, \quad \|P_{t,s}^*v\|_{L^p(m_s)} \leq e^{L(t-s)(1-1/p)} \cdot \|v\|_{L^p(m_s)}. \)

The latter estimates allow to extend the propagators \( P_{t,s} \) and their adjoints \( P_{t,s}^* \) in the canonical way from operators on \( L^2(X,m) \) to operators on \( L^p(X,m) \) for any \( p \in [1,\infty] \).

Proposition 1.8 ([14], Theorem 2.12). The following properties hold.

1. Let \( u_t = P_{t,s}u \). Then \( u_t \in D(\Delta_t) \) for a.e. \( t > s \) and

\[
\int_s^t \int |\Delta_t u_t|^2 \, dm_t \, dt \leq C(\mathcal{E}_s(u_s) - \mathcal{E}_r(u_r)),
\]

where \( s < r < T \) and \( C > 0 \) only depends on the Lipschitz constants of \( t \mapsto f_t \) and \( t \mapsto \log d_t \). Moreover

\[
\lim_{h \to 0} \frac{1}{h} (u_{t+h} - u_t) = \Delta_t u_t
\]

in \( L^2(X) \) for a.e. \( t > s \).

2. Let \( v_s = P_{t,s}^*v \). Then \( v_s \in D(\Delta_s) \) for a.e. \( s < t \) and

\[
\int_s^t \int |\Delta_s v_s|^2 \, dm_s \, ds \leq C(\mathcal{E}_t(v_t) - \mathcal{E}_r(v_r)) + C \int_s^t \int |v_s|^2 \, dm_s \, ds,
\]

where \( 0 < \sigma < t \) and \( C > 0 \) only depends on the Lipschitz constants of \( t \mapsto f_t \) and \( t \mapsto \log d_t \). Moreover

\[
\lim_{h \to 0} \frac{1}{h} (v_{s+h} - v_s) = -\Delta v_s + v_s f_s
\]

in \( L^2(X) \) for a.e. \( s < t \).

For later purposes it will be convenient to introduce the notion of semigroup mollification introduced in [4, Sec. 2.1].

Definition 1.9. Let \( t \in (0,T) \) and \( \kappa \in C_c^\infty(0,\infty) \) with \( \kappa \geq 0 \) and \( \int_0^\infty \kappa(r) \, dr = 1 \). Let \( (H^l_{r})_{r \geq 0} \) denote the heat semigroup in the static mm-space \((X,d_t,m_t)\). For \( \varepsilon > 0 \) and \( \psi \in D(\mathcal{E}) \cap L^\infty(X) \) we define

\[
\psi_\varepsilon = \frac{1}{\varepsilon} \int_0^\infty H^l_{r} \psi \kappa(r/\varepsilon) \, dr.
\]

It is immediate to verify that \( \psi_\varepsilon, \Delta_\varepsilon \psi_\varepsilon \in D(\Delta_t) \cap \text{Lip}_b(X) \) and \( \psi_\varepsilon \to \psi \) in \( D(\mathcal{E}) \) as \( \varepsilon \to 0 \), see e.g. [4 Sec 2.1].
2 The local and the reverse local Poincaré inequalities

2.1 From $L^2$-gradient estimate to local and reverse local Poincaré inequalities

**Theorem 2.1.** Suppose that the $L^2$-gradient estimate

\[ \Gamma_t(P_{t,s}u) \leq P_{t,s} \left( \Gamma_s u \right) \quad \text{a.e. on } X \]  

holds for all $u \in \mathcal{D}(\mathcal{E})$ and all $s < t$. Then we have

\[ P_{t,s}(u^2) - (P_{t,s}u)^2 \leq 2(t-s)P_{t,s}(\Gamma_s u) \quad \text{a.e. on } X \]  

for all $u \in \mathcal{D}(\mathcal{E})$ and

\[ P_{t,s}(u^2) - (P_{t,s}u)^2 \geq 2(t-s) \Gamma_t(P_{t,s}u) \quad \text{a.e. on } X \]  

for all $u \in L^2(X)$. In particular, for $u \in L^2(X) \cap L^\infty(X)$

\[ \Gamma_t(P_{t,s}u) \leq \frac{||u||^2_\infty}{2(t-s)}. \]  

**Proof.** Let $u = u_s, g = g_t \in \mathcal{F} \cap L^\infty$ be given and consider on $(s,t) \times X$ the solutions to the heat equation and adjoint heat equation

\[ u_r := P_{r,s}u_s, \quad g_r = P_{t,r}^* g_t. \]

Then by the defining properties of the heat equation (6) and the Leibniz rule for the weak time derivative

\[ -2 \int_s^t \int g_r(u_r)dm_r dr = \int_s^t \int -2 \Gamma_r(g_r u_r, u_r) + \Gamma_r(u_r^2, g_r) dm_r dr \]

\[ = \int_s^t \int \{2g_r \partial_r u_r + u_r^2 \partial_r g_r - u_r^2 g_r \partial_r f_r \} dm_r dr \]

\[ = \int_s^t \frac{d}{dr} \left( \int u_r^2 g_r dm_r \right) dr = \int u_r^2 g_r dm_t - \int u^2_s g_s dm_s. \]

This proves

\[ \int g((P_{t,s}u)^2 - P_{t,s}(u^2)) dm_t = -2 \int_s^t \int P_{t,r}^* g(\Gamma_r(P_{r,s}u)) dm_r dr. \]  

Applying (7) to $\Gamma_r(P_{r,s}u)$ on the right hand side gives

\[ \int g((P_{t,s}u)^2 - P_{t,s}(u^2)) dm_t \geq -2(t-s) \int gP_{t,s}(\Gamma_s(u)) dm_t, \]

and applying (7) to $P_{t,r} \Gamma_r$ gives

\[ \int g((P_{t,s}u)^2 - P_{t,s}(u^2)) dm_t \leq -2(t-s) \int g\Gamma_t(P_{t,s}(u)) dm_t. \]

Since $g$ is arbitrary, this proves the first two claims of the theorem in the case of bounded $u \in \mathcal{D}(\mathcal{E})$. The claim (9) for bounded $u \in L^2(X)$ follows by applying the latter estimate with $s + \delta$ in the place of $s$ to the function $P_{s+\delta,s}u$ as $\delta \to 0$, which lies in $\mathcal{D}(\mathcal{E})$ and from $\int gP_{t,s+\delta}((P_{s+\delta,s}u)^2)dm_t \to$
\[ \int gP_{t,s}(u^2)dm_t \] which in turn is a consequence of the continuity of \( \delta \mapsto P^*_{t,s+\delta}g \) and of \( \delta \mapsto P_{s+\delta,s}u \) in \( L^2 \) and the uniform boundedness of the latter in \( L^\infty \).

Thanks to the monotonicity (w.r.t. \( C \mapsto u \land C \) or \( C \mapsto u \lor -C \)) of all the involved quantities, the claims for unbounded \( u \) will follow by a simple truncation argument. Indeed, \( u \land C \lor -C \rightarrow u \) in \( L^2 \) and thus, since \( g \) is bounded, \( \int g(P_{t,s}u \land C \lor -C)^2dm_t \rightarrow \int g(P_{t,s}u)^2dm_t \) as well as \( \int (u \land C \lor -C)^2P^*_{t,s}g dm_s \rightarrow \int u^2P^*_{t,s}g dm_s \). Moreover, under the heat flow the initial \( L^2 \)-convergence will be improved to a \( D(\mathcal{E}) \)-convergence. Thus

\[
\int g\Gamma_t(P_{t,s}(u \land C \lor -C))\,dm_t \rightarrow \int g\Gamma_t(P_{t,s}(u))\,dm_t.
\]

Finally, for the remaining term it suffices to observe that

\[
\int gP_{t,s}(\Gamma_t(u \land C \lor -C))\,dm_t \leq \int gP_{t,s}(\Gamma_t(u))\,dm_t.
\]

\[ \square \]

### 2.2 From reverse local Poincaré inequality to dynamic Bochner inequality

**Theorem 2.2.** Suppose that the reverse local Poincaré inequality holds:

for all \( s < t \) and for all \( u \in D(\mathcal{E}) \cap L^\infty(X) \)

\[
P_{t,s}(u^2) - (P_{t,s}u)^2 \geq 2(t-s)\Gamma_t(P_{t,s}u) \quad \text{a.e. on } X.
\]

Then the dynamic Bochner inequality \((\text{EB4})\) holds true (‘integrated in time’):

\[
\forall S,T \in I, \forall u,g \in \mathcal{F} \text{ with } g \in L^\infty, u \in \text{Lip}(X) \text{ and for a.e. } q \in (S,T)
\]

\[
\int \left[ (\Delta_q g_q)\Gamma_q(u_q) + 2(\Delta_q u_q)^2 g_q + 2\Gamma_q(u_q,g_q)\Delta_q u_q - \frac{\partial}{\partial t} \Gamma_q(u_q)g_q \right] \,dm_q \geq 0
\]

where \( u_q := P_{q,S}u, g_q = P^*_T g \).

**Proof.** Given \( u \in D(\mathcal{E}) \cap L^\infty(X) \) and nonnegative \( g \in L^1(X) \cap L^\infty(X) \) we have shown in (II) that for all \( s < t \)

\[
\int g(P_{t,s}(u^2) - (P_{t,s}u)^2)\,dm_t = \int_s^t \int P^*_{t,r}g\Gamma_r(P_{r,s}u)\,dm_r dr.
\]

Approximation by truncated \( u \)’s easily allows to extend the assertion to all \( u \in D(\mathcal{E}) \). The local Poincaré inequality, therefore, implies

\[
0 \leq \frac{1}{(t-s)^2} \int P_{t,s}u^2 - (P_{t,s}u)^2 - 2(t-s)\Gamma_t(P_{t,s}u) \,g \,dm_t
\]

\[
= \frac{2}{(t-s)^2} \int_s^t \int g \left[ P_{t,r}\Gamma_r(P_{r,s}u) - \Gamma_t(P_{t,s}u) \right] \,dm_t \,dr.
\]

Now let us fix \( S,T \in I \) and choose \( g_T, u_S \in \mathcal{F} \) with \( g_T \in L^\infty \) and \( u_S \in \text{Lip}(X) \). Given \( s,t \) with \( S < s < t < T \), we put

\[
g_t = P^*_T g_T, \quad u_s = u_{s,S} u_S
\]

and apply the previous estimate with \( g_t, u_s \) in the place of \( g, u \). Then

\[
0 \leq \frac{2}{(t-s)^2} \int_s^t \int g_t \left[ P_{t,r}\Gamma_r(u_r) - \Gamma_t(u_t) \right] \,dm_t \,dr = \frac{1}{(t-s)^2} \int_s^t [\Psi(r) - \Psi(t)] \,dr
\]

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where we now define

$$\Psi(q) := \int g_q \Gamma_q(u_q) \, dm_q = - \int g_q u_q \Delta_q u_q \, dm_t + \frac{1}{2} \int u_q^2 \Delta_q g_q \, dm_q$$

Following the proof of Theorem 5.7 in [14] we have

$$\Psi(r) - \Psi(t) \leq \int_t^r \int \left[ (\Delta_q g_q) \Gamma_q(u_q) + 2(\Delta_q u_q)^2 g_q + 2\Gamma_q(u_q, g_q) \Delta_q u_q) - \dot{\Gamma}_q(u_q) g_q \right] \, dm_q dq$$

and hence

$$0 \leq \frac{1}{(t-s)^2} \int_s^t \int \left[ (\Delta_q g_q) \Gamma_q(u_q) + 2(\Delta_q u_q)^2 g_q + 2\Gamma_q(u_q, g_q) \Delta_q u_q) - \dot{\Gamma}_q(u_q) g_q \right] \, dm_q dq$$

Since this holds for all \((s, t) \subset (S, T)\), it implies (by Lebesgue’s density theorem) that

$$0 \leq \int \left[ (\Delta_q g_q) \Gamma_q(u_q) + 2(\Delta_q u_q)^2 g_q + 2\Gamma_q(u_q, g_q) \Delta_q u_q) - \dot{\Gamma}_q(u_q) g_q \right] \, dm_q$$

for a.e. \(q \in (S, T)\). This is the claim, namely the dynamic Bochner inequality (E4).

**2.3 From local Poincaré inequality to dynamic Bochner inequality**

For the proof of the following implication, we will make the additional a priori assumption that

$$\sup_t \|\Gamma_t(P_t u)\|_\infty < \infty \quad \text{(A2.a)}$$

for each \(u \in \text{Lip}(X)\). Note that this assumption is always fulfilled in the time-independent case thanks to the RCD\((K, \infty)\)-condition as one of our standing assumptions.

**Theorem 2.3.** Suppose [(A2.a)] and that the local Poincaré inequality holds: for all \(s < t\) and for all \(u \in \mathcal{D}(E) \cap L^\infty(X)\)

$$P_t \Gamma_s u) - (P_{t-s} u)^2 \leq 2(t-s) P_t \Gamma_s u \quad \text{a.e. on } X.$$

Then the dynamic Bochner inequality (E4) holds true (‘integrated in time’).

**Proof.** The proof is very similar to that of the previous theorem. Now the a priori assumption is required to guarantee appropriate integrability of the involved quantities (which in the previous case was a simple consequence of the assumption, cf. estimate (10)). The local Poincaré inequality then implies

$$0 \geq \frac{1}{(t-s)^2} \int g \left[ (\Delta_q \Gamma_s u)^2 - (P_{t-s} \Gamma_s u)^2 - 2P_{t-s} \Gamma_s u \right] \, dm_t$$

$$= \frac{2}{(t-s)^2} \int_s^t \frac{d}{dq} \int g_q \Gamma_q(u_q) \, dm_q dq dr$$

$$= -\frac{2}{(t-s)^2} \int_s^t \int \left[ (\Delta_q g_q) \Gamma_q(u_q) + 2(\Delta_q u_q)^2 g_q + 2\Gamma_q(u_q, g_q) \Delta_q u_q) - \dot{\Gamma}_q(u_q) g_q \right] \, dm_q dq dr$$

$$= -\frac{2}{(t-s)^2} \int_s^t (t-q) \left[ (\Delta_q g_q) \Gamma_q(u_q) + 2(\Delta_q u_q)^2 g_q + 2\Gamma_q(u_q, g_q) \Delta_q u_q) - \dot{\Gamma}_q(u_q) g_q \right] \, dm_q dq.$$
Again by Lebesgue’s density theorem this implies that
\[ 0 \leq \int \left[ (\Delta_q g_q) \Gamma_q(u_q) + 2(\Delta_q u_q)^2 g_q + 2\Gamma_q(u_q, g_q)\Delta_q u_q - \Gamma_q(u_q)g_q \right] dm_q \]
for a.e. \( q \in (S, T) \).

2.4 From dynamic Bochner inequality (‘integrated in time’) to dynamic Bochner inequality pointwise in time

In addition to our standing assumptions, let us now assume that

- the domains \( \mathcal{D}(\Delta_t) \) are independent of \( t \in (0, T) \) and for \( u, g \in \mathcal{D}(\Delta) \) with \( \Delta_t u, \Delta_t g \in L^\infty(X) \) the functions
  \[
  s \mapsto \Delta_s u, \quad q \mapsto \Delta_q P_{q,s} u, \quad q \mapsto \Delta_q P_{t,q}^* g,
  \]
  are continuous in \( L^2(X) \) and bounded in \( L^\infty(X) \);

- for \( u \in \mathcal{D}(\mathcal{E}) \) the function \( \partial_s \Gamma_s(u) \) exists in \( L^1(X) \) and
  \[
  s \mapsto \partial_s \Gamma_s(u), \quad q \mapsto \partial_q \Gamma_q(P_{q,s} u)
  \]
  are continuous in \( L^1(X) \).

Note that all these assumptions are trivially satisfied in the static case.

**Lemma 2.4.** The assumption (A2.b) implies that for \( u, g \in \mathcal{D}(\Delta) \) with \( \Delta_t u, \Delta_t g \in L^\infty(X) \) the functions
\[
q \mapsto P_{q,s} u, \quad q \mapsto P_{t,q}^* g
\]
are continuous in \( \mathcal{D}(\mathcal{E}) \).

**Proof.** This follows from integration by parts. \( \square \)

**Theorem 2.5.** Under the previous assumptions, the dynamic Bochner inequality (E3) implies the following ‘dynamic Bochner inequality pointwise in time’
\[
\forall t \in I, \forall u, g \in \mathcal{D}(\Delta) \cap L^\infty(X) \text{ with } \Gamma_t(u) \in L^\infty(X) \text{ and } g \geq 0
\]
\[ \int \left[ (\Delta_t g) \Gamma_t(u) + 2(\Delta_t u)^2 g + 2\Gamma_t(u, g)\Delta_t u - \partial_t \Gamma_t(u)g \right] dm_t \geq 0. \]  

(12)

**Proof.** Given \( t \in I, u, g \in \mathcal{D}(\Delta) \cap L^\infty(X) \) with \( \Gamma_t(u), \Delta_t u, \Delta_t g \in L^\infty(X) \) and \( g \geq 0 \), choose \( s < t \) and define \( u_{q,s} := P_{q,s} u, g_q = P_{t,q}^* g \) for \( q \in [s, t] \). Then the dynamic Bochner inequality in its integrated version and (A2.c) imply that the function
\[
q \mapsto \int \left[ (\Delta_q g_q) \Gamma_q(u_{q,s}) + 2(\Delta_q u_{q,s})^2 g_q + 2\Gamma_q(u_{q,s}, g_q)\Delta_q u_{q,s} - \partial_q \Gamma_q(u_{q,s})g_q \right] dm_q
\]
is nonnegative for a.e. \( q \). Moreover, according to (A2.b), Lemma 2.4 and (A2.c), this function is continuous. Thus, in particular, it is nonnegative for \( q = s \), i.e.
\[
\int \left[ (\Delta_s P_{t,s}^* g) \Gamma_s(u) + 2(\Delta_s u)^2 P_{t,s}^* g + 2\Gamma_s(u, P_{t,s}^* g)\Delta_s u - \partial_s \Gamma_s(u)P_{t,s}^* g \right] dm_s \geq 0.
\]

Now finally we consider the limit \( s \to t \) which implies \( P_{t,s}^* g \to g \) in \( L^2(X) \) as well as \( \Delta_s P_{t,s}^* u \to \Delta_t u \) by (A2.b). According to Lemma 2.4, \( P_{t,s}^* g \to g \) in \( \mathcal{D}(\mathcal{E}) \). Therefore,
\[
\int \left[ (\Delta_t g) \Gamma_t(u) + 2(\Delta_t u)^2 g + 2\Gamma_t(u, g)\Delta_t u - \partial_t \Gamma_t(u)g \right] dm_t \geq 0.
\]

To obtain the estimate for general \( u, g \), we approximate them using the static \((X, d_t, m_t)\)-heat semigroup mollifier from Definition 1.9 \( \square \)
3 The local logarithmic Sobolev inequalities

3.1 From $L^1$-gradient estimate to local logarithmic Sobolev inequality

**Theorem 3.1.** Suppose that the $L^1$-gradient estimate

$$\sqrt{\Gamma_t(P_{t,s}u)} \leq P_{t,s}\sqrt{\Gamma_s(u)}$$

holds for every $s < t$ and $u \in \mathcal{D}(\mathcal{E})$. Then for every $s < t$ and $u \geq 0$ such that $u \in \mathcal{D}(S)$ and $\sqrt{u} \in \mathcal{D}(\mathcal{E})$

$$P_{t,s}(u \log u) - P_{t,s}u \log P_{t,s}u \leq (t-s) P_{t,s} \left( \frac{\Gamma_s(u)}{u} \right)$$

(14)

$$P_{t,s}(u \log u) - P_{t,s}u \log P_{t,s}u \geq (t-s) \frac{\Gamma_t(P_{t,s}u)}{P_{t,s}u}$$

(15)

$m$-a.e.. Estimate (15) holds more generally for all nonnegative $u \in \mathcal{D}(S) \cap L^1(X)$.

**Proof.** Define for $s < r < t$, $g \in L^1(X) \cap L^\infty(X)$ such that $g \geq 0$ and $u \in \mathcal{D}(S) \cap L^\infty(X)$ such that $u \geq 0$ and $\sqrt{u} \in \mathcal{D}(\mathcal{E})$

$$\Psi_\varepsilon(r) = \int P^*_{t,r} g \psi_\varepsilon(P_{r,s}u) \, dm_r,$$

where $\psi_\varepsilon(z) : [0, \infty) \to \mathbb{R}$ by setting $\psi_\varepsilon(z) = \log(z + \varepsilon) + 1$ and $\psi_\varepsilon(0) = 0$. Since $r \mapsto P_{r,s}u$ and $r \mapsto P^*_{t,r} g$ are continuous in $L^2(X)$, the map $r \mapsto \Psi(r)$ is continuous. Then similarly as in the proof of Theorem 2.1

$$\frac{d}{dr} \Psi_\varepsilon(r) = \int \Gamma_r(P^*_{t,r}g, P_{r,s}u) \psi_\varepsilon'(P_{r,s}u) - \Gamma_r(P^*_{t,r}g \psi_\varepsilon(P_{r,s}u), P_{r,s}u) \, dm_r$$

$$= - \int P^*_{t,r} g \psi_\varepsilon'''(P_{r,s}u) \Gamma_r(P_{r,s}u) \, dm_r$$

$$= - \int P^*_{t,r} g \frac{\Gamma_r(P_{r,s}u)}{P_{r,s}u + \varepsilon} \, dm_r.$$

Using the Cauchy-Schwartz inequality and (13) we find for the integrand

$$P_{t,r} \left( \frac{\Gamma_r(P_{r,s}u)}{P_{r,s}u + \varepsilon} \right) \leq P_{t,r} \left( \frac{|\nabla s u|}{u + \varepsilon} \right)^2 \leq P_{t,r} \left( \frac{|\nabla s u|}{u + \varepsilon} \right)^2 \leq P_{t,r} \left( \frac{P_{r,s}(\nabla s u)^2}{P_{r,s}u + \varepsilon} \right) \leq P_{t,r} \left( \frac{P_{r,s}(\nabla s u)^2}{P_{r,s}u + \varepsilon} \right).$$

Integration over $(s, t)$ yields

$$\int g \psi_\varepsilon(P_{t,s}u) \, dm_t - \int g P_{t,s}(\psi_\varepsilon(u)) \, dm_t \geq -(t-s) \int g P_{t,s} \left( \frac{\Gamma_s(u)}{u + \varepsilon} \right) \, dm_t.$$

(16)

Since $u \in \mathcal{D}(S)$ we have by Proposition 2.8 in [14] that $P_{t,s}u \in \mathcal{D}(S)$ and we find by dominated convergence that the left hand side converges as $\varepsilon \to 0$ to

$$\int g P_{t,s}u \log(P_{t,s}u) \, dm_t - \int g P_{t,s}(u \log u) \, dm_t.$$
while by monotone convergence the right hand side converges to

\[- (t - s) \int g P_{t, s} \left( \frac{\Gamma_s(u)}{u} \right) \, dm_t,\]

and hence

\[\int g P_{t, s} u \log(P_{t, s} u) \, dm_t - \int g P_{t, s} u \log u \, dm_t \geq - (t - s) \int g P_{t, s} \left( \frac{\Gamma_s(u)}{u} \right) \, dm_t. \tag{17}\]

By taking $u^n := u \land n$ and letting $n \to \infty$ we obtain (17) for general $u \in \mathcal{D}(S)$ with $\sqrt{u} \in \mathcal{D}(\mathcal{E})$, since $u^n \to u$ and $P_{t, s} u^n \to P_{t, s} u$ in $L^1(X)$, and $\Gamma(u^n) = \Gamma(u) 1_{\{u < n\}}$ a.e.

Since $g$ is arbitrary we find for a.e. $x \in X$

\[P_{t, s}(u \log u) - P_{t, s} u \log P_{t, s} u \leq (t - s) P_{t, s} \left( \frac{\Gamma_s(u)}{u} \right).\]

To obtain the reverse bound (15) we apply Jensen’s inequality to the functions $\eta(z) = z^2$ and $\beta(z, w) = z^2 / w$, which amounts to

\[P_{t, r} \left( \frac{\Gamma_r(P_{r, s} u)}{P_{r, s} u} \right) \geq \frac{P_{t, r} \Gamma_r(P_{r, s} u)}{P_{t, s} u} \geq \frac{(P_{t, r} \nabla_r(P_{r, s} u))^2}{P_{t, s} u} \geq \frac{|\nabla_t(P_{t, s} u)|^2}{P_{t, s} u}.\]

A similar argumentation as above yields the desired estimate. \hfill \Box

### 3.2 From local logarithmic Sobolev inequalities to dynamic Bochner inequality

For this subsection we will additionally assume that (A2.a-c) hold. Moreover, we assume that $m_t(X) < \infty$ for some (hence all) $t \in (0, T)$ and that

- for all fixed $s \in (0, T)$ and all $u \in \mathcal{D}(\Delta) \cap L^\infty(X)$ such that $\Delta_s u \in L^\infty(X)$
  \[q \mapsto P_{q, s} u \text{ is continuous in } L^\infty(X); \tag{A3.a}\]

- for $u \in \mathcal{D}(\mathcal{E}) \cap \mathcal{D}(\Delta)$ such that $u \geq c > 0$ the function
  \[q \mapsto \partial_q \Gamma_q(\log u_{q, s}) \text{ is continuous in } L^1(X). \tag{A3.b}\]

Note that (A3.a+b) are always satisfied for the usual heat flow $(P_t)_{t \geq 0}$ on RCD($K, \infty$)-spaces, for (A3.a) see also Lemma 5.3.

We show the following,

**Theorem 3.2.** Assume that one of the local log-Sobolev inequalities, (14) or (15), holds. Then the pointwise dynamic Bochner holds for $t$, i.e. for all $v \in \mathcal{D}(\Delta_t) \cap L^\infty(X)$ such that $\Gamma_t(v) \in L^\infty(X)$ and all $g \in \mathcal{D}(\Delta_t) \cap L^\infty(X)$ with $g \geq 0$ holds

\[
\frac{1}{2} \int \Gamma_t(v) \Delta_t g \, dm_t + \int (\Delta_t v)^2 g + \Gamma_t(v, g) \Delta_t v \, dm_t \geq \frac{1}{2} \int (\partial_t \Gamma_t)(v) g \, dm_t. \tag{18}\]

**Proof.** Let $v, \Delta_t v \in \mathcal{D}(\Delta_t) \cap \text{Lip}_b(X)$. Define $u = e^v$. Then $u \in \mathcal{D}(S) \cap \text{Lip}_b(X) \cap \mathcal{D}(\Delta_t)$ with $\Delta_t u \in L^\infty(X) \cap \mathcal{D}(\mathcal{E})$ and $u \geq \varepsilon > 0$. Let $g \in \mathcal{D}(\Delta_t) \cap L^\infty(X)$ with $g \geq 0$. Then we claim

\[
\int g(P_{t, s}(u \log u) - P_{t, s} u \log P_{t, s} u) \, dm_t = - \int_s^t \frac{d}{dr} \int P_{t, r}^* g(P_{r, s} u \log P_{r, s} u) \, dm_r \, dr
= \int_s^t \int P_{t, r}^* g(P_{r, s} u \Gamma_r(\log P_{r, s} u)) \, dm_r \, dr.
\]
For this we need to show
\[ r \mapsto \int P^*_t g P_{r,s} u \log P_{r,s} u \, dm_r \]
is absolutely continuous. Call \( g_r = P^*_t g \) and \( u_r = P_{r,s} u \) and let \( r_1 < r_2 \). Then using the splitting
\[
\left| \int \Delta u_r \log u_r \, dm_r \right| \leq \left| \int g_r \Delta u_r \log u_r \, dm_r \right|
\]
we see the above mentioned map is absolutely continuous by virtue of Proposition 1.8. To compute its derivative we consider the difference quotient
\[
\frac{1}{h} \left( \int g_{r+h} u_{r+h} \log u_{r+h} \, dm_{r+h} - \int g_r u_r \log u_r \, dm_r \right)
\]
and let \( h \to 0 \) which is
\[
\lim_{h \to 0} \frac{1}{h} \left( \int g_{r+h} u_{r+h} \log u_{r+h} \, dm_{r+h} - \int g_r u_r \log u_r \, dm_r \right)
\]
for a.e. \( r \).

Hence we found that (14) implies
\[
0 \geq \int g(P_{t,s}(u \log u)) - P_{t,s}(u) \log P_{t,s}(u) - (t-s)P_{t,s}(\frac{G_s(u)}{u}) \, dm_t
\]
\[
= \int_s^t \int g_r u_r \Gamma_r(u \log u) \, dm_r - \int g P_{t,s}(\frac{G_s(u)}{u}) \, dm_t \, dr.
\]

We now claim that the map
\[
q \mapsto \int g_q u_q \Gamma_q(u \log u) \, dm_q
\]
is absolutely continuous. To this end we let \( s < q_1 < q_2 < t \) outside of an exceptional set of measure zero and compute

\[
\left| \int g_{q_2} u_{q_2} \Gamma_{q_2} (\log u_{q_2}) \, dm_{q_2} - \int g_{q_1} u_{q_1} \Gamma_{q_1} (\log u_{q_1}) \, dm_{q_1} \right|
\leq \left| \int_{q_1}^{q_2} \Delta_q g_{q_2} u_{q_2} \Gamma_{q_2} (\log u_{q_2}) \, dq \right|
+ \left| \int_{q_1}^{q_2} g_{q_1} \Delta_q u_{q_2} \Gamma_{q_2} (\log u_{q_2}) \, dq \right|
+ C(q_2 - q_1) \left| \int g_{q_1} u_{q_1} \Gamma_{q_1} (\log u_{q_2}) \, dm_{q_1} \right|
+ \left| \int g_{q_1} u_{q_1} (\Gamma_{q_1} (\log u_{q_2}) - \Gamma_{q_1} (\log u_{q_1})) \, dm_{q_1} \right|.
\]

The first three terms are finite by virtue of (A2.a) and Proposition 1.8. For the last one we further compute

\[
\left| \int g_{q_1} u_{q_1} (\Gamma_{q_1} (\log u_{q_2}) - \Gamma_{q_1} (\log u_{q_1})) \, dm_{q_1} \right|
= \left| \int g_{q_1} u_{q_1} \Gamma_{q_1} (\log u_{q_2} - \log u_{q_1}, \log u_{q_2} + \log u_{q_1}) \, dm_{q_1} \right|
\leq \left| \int g_{q_1} u_{q_1} (\log u_{q_2} - \log u_{q_1}) \Delta_q (\log u_{q_2} + \log u_{q_1}) \, dm_{q_1} \right|
+ \left| \int \Gamma_{q_1} (\log u_{q_2} + \log u_{q_1}, g_{q_1} u_{q_1}) (\log u_{q_2} - \log u_{q_1}) \, dm_{q_1} \right|
\leq \frac{1}{\varepsilon} \left( \int_{q_1}^{q_2} \left| \Delta_q u_{q_1} \right|^2 \, dq \right)^{1/2}
+ (q_2 - q_1) \left| \int \Gamma_{q_1} (\log u_{q_2} + \log u_{q_1}) \Gamma_{q_1} (\log u_{q_1}) \, dm_{q_1} \right|
\]

where we used that \( u_q \geq \varepsilon \) and \( \Delta \log u_q \in L^\infty(X) \) for a.e. \( q \).

This proves absolute continuity and differentiating yields

\[
0 \geq \int_{s}^{t} \frac{d}{dq} \left\{ \int u_q g_q \partial_q \Gamma_q (\log u_q) \, dq \right\} \, dq \, dr
= \int_{s}^{t} \left\{ \int u_q g_q \partial_q \Gamma_q (\log u_q) - \Delta_q (u_q g_q) \Gamma_q (\log u_q) \, dq \right\} \, dq \, dr
- 2 \int (\Delta_q \log u_q)^2 u_q g_q + \Gamma_q (\log u_q, u_q g_q) \Delta_q \log u_q \, dm_q \, dr
= \int_{s}^{t} \left\{ \int u_q g_q \partial_q \Gamma_q (\log u_q) - \Delta_q (u_q g_q) \Gamma_q (\log u_q) \, dq \right\} \, dq \, dr
- 2 \int (\Delta_q \log u_q)^2 u_q g_q + \Gamma_q (\log u_q, u_q g_q) \Delta_q \log u_q \, dm_q \, dq.
\]

Define

\[
\Phi(q) = \int u_q g_q \partial_q \Gamma_q (\log u_q) - \Delta_q (u_q g_q) \Gamma_q (\log u_q) \, dm_q
- 2 \int (\Delta_q \log u_q)^2 u_q g_q + \Gamma_q (\log u_q, u_q g_q) \Delta_q \log u_q \, dm_q,
\]

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where \( u_q = u_{q,s} \).

We want to show that \( \Phi : [s,t] \to \mathbb{R} \) defines a continuous function. In order to do so, we consider each term separately.

The first term \( q \mapsto \int u_q g_q \partial_t \Gamma_q (\log u_q) \, dm_q \) is continuous since \( q \mapsto g_q \) is continuous in \( L^\infty(X) \) by (A3.a) and since \( u \geq \varepsilon \), \( q \mapsto \partial_t \Gamma_q (\log u_q) \) is continuous in \( L^1(X) \) by (A3.b) and \( q \mapsto e^{-f_s} \) is continuous in \( L^\infty(X) \).

The second term \( q \mapsto \int \Delta_q (u_q g_q) \Gamma_q (\log u_q) \, dm_q \) is continuous since \( q \mapsto \Delta_q u_q \), \( q \mapsto \Delta_q g_q \), \( q \mapsto g_q \) and \( q \mapsto u_q^{-1} \) are continuous in \( L^2(X) \) by (A2.b) and (A3.a), \( q \mapsto \Gamma_q (u_q) \) is weak\(^*\) continuous in \( L^\infty(X) \) by Lemma 2.4 and \( A2.a \), \( q \mapsto u_q^{-2} \) is continuous in \( L^\infty(X) \) by (3) and (A3.a), and \( q \mapsto \Gamma_q (u_q) \) is continuous in \( L^1(X) \) by Lemma 2.4.

The third term \( q \mapsto \int (\Delta_q \log u_q)^2 u_q g_q \, dm_q \) is continuous since \( q \mapsto \Delta_q u_q \) is continuous in \( L^2(X) \) by (A2.b), and \( q \mapsto \Gamma_q (u_q) \) is weak\(^*\) continuous by (A2.a) and Lemma 2.4 and \( q \mapsto u_q \), \( q \mapsto g_q \) are continuous in \( L^\infty(X) \cap L^2(X) \) by (3) and (A3.a).

Finally the last term \( q \mapsto \int \Gamma_q (\log u_q, u_q g_q) \Delta_q \log u_q \, dm_q \) is continuous since \( q \mapsto \Delta_q u_q \) is continuous in \( L^2(X) \) and weak\(^*\)-continuous in \( L^\infty(X) \) by (A2.b), and \( q \mapsto \Gamma_q (u_q, u_q g_q) \) is continuous in \( L^1(X) \) by Lemma 2.4, \( q \mapsto \Gamma_q (u_q) \) is weak\(^*\) continuous by (A2.a) and Lemma 2.4 and \( q \mapsto u_q^{-2} \), \( q \mapsto u_q^{-3} \) is continuous in \( L^\infty(X) \) by (3), (A3.a) and \( u \geq \varepsilon \).

Then it holds by Lebesgue differentiation

\[
0 \geq \int u_q \partial_t \Gamma_q (\log u) - \Delta_q (u_q s) \Gamma_q (\log u_q) \, ds - 2 \int (\Delta_q \log u)^2 u_q s + \Gamma_q (\log u, u_q s) \Delta_q \log u \, ds.
\]

Similarly as before we let \( s \to t \) and obtain after choosing \( \tilde{g} = e^{-v} g \in \mathcal{D}(\Delta_t) \cap L^\infty(X) \) and obtain recalling \( \bar{u} = e^v \)

\[
0 \geq \int \tilde{g} \partial_t \Gamma_t (v) - \Delta_t (\bar{u}) \Gamma_t (v) \, dt - 2 \int (\Delta_t v)^2 \tilde{g} + \Gamma_t (v, \tilde{g}) \Delta_t v \, dt
\]

for all \( v, \Delta_t v \in \mathcal{D}(\Delta_t) \cap \text{Lip}_c(X) \) and \( \tilde{g} \in \mathcal{D}(\Delta_t) \cap L^\infty(X) \) with \( \tilde{g} \geq 0 \). The result for general \( v \in \mathcal{D}(\Delta_t) \cap L^\infty(X) \) such that \( \Gamma_t (v) \in L^\infty(X) \) and all \( \tilde{g} \in \mathcal{D}(\Delta_t) \cap L^\infty(X) \) with \( \tilde{g} \geq 0 \) follows by approximation with the semigroup mollifier from Definition 1.9.

Similarly one deduces Bochner from the reverse local logarithmic Sobolev bound. Indeed by (15) it holds by the same argument as above

\[
0 \leq \int_s^t \int g_r u_r \Gamma_t (\log u_r) \, dr - \int g_{\Gamma_t (u_t)} \, dm_t \, dr
\]

and since \( q \mapsto \int g_q u_q \Gamma_q (\log u_q) \, dm_q \)

\[
0 \geq \int_s^t \frac{d}{dq} \int_r^t g_q u_q \Gamma_q (\log u_q) \, dm_q \, dq \, dr,
\]

which is the same as in line (19).

\[ \square \]

4 The dimension independent Harnack inequality

4.1 From \( L^1 \)-gradient estimate to dimension independent Harnack inequality

This section will be devoted to derive the following result.

**Theorem 4.1.** Fix \( \alpha > 1 \). Suppose that the \( L^1 \)-gradient estimate \( \text{[13]} \) holds. Then for all \( u \in L^2(X) \) such that \( u \geq 0 \), m.a.e. \( x, y \in X \) and \( t > s \) we have

\[
(P_{t,s} u)^\alpha (y) \leq (P_{t,s} u^\alpha)(x) \exp \left\{ \frac{\alpha d_t^2(x, y)}{4(\alpha - 1)(t - s)} \right\}.
\]
Before starting with the proof of this results, let us recall the notion of regular curves as introduced in [4] and refined in [3], as well as the notion of velocity densities taken from [5]. A curve \((\mu_r)_{r \in [0,1]}\) with \(\mu_r = \rho_r m\) is called regular if the following are satisfied:

- \(\mu \in \text{Lip}([0,1]; (P_2(X), W))\)
- There exists a constant \(R > 0\) such that \(\rho_r \leq R\) a.e. for every \(s \in [0,1]\)
- \(\sqrt{\rho_r} \in \mathcal{D}(\mathcal{E})\) such that \(\mathcal{E}(\sqrt{\rho_r}) \leq E\) for every \(s \in [0,1]\).

We recall the following result (Lemma 12.2 in [5]).

**Lemma 4.2.** For every geodesic \((\mu_r)_{r \in [0,1]}\) there exist regular curves \(\mu^n\) such that \(\mu^n_r \to \mu_r\) in \(L^2\)-Kantorovich sense for all \(r \in [0,1]\) and

\[
\limsup_n \int_0^1 |\dot{\mu}^n_r|^2 \, dr \leq W^2(\mu_0, \mu_1).
\]

A 2-absolutely continuous curve \(\mu\) admits a velocity density \(v \in L^2(X \times [0,1], \int \mu_t \, dt)\) in the sense that for every \(\varphi \in \text{Lip}_b(X)\)

\[
|\int \varphi \, d\mu_t - \int \varphi \, d\mu_s| \leq \int_s^t |\nabla \varphi| v \, d\mu_r \, dr
\]

and there exists a unique velocity density with minimal \(L^2(X \times [0,1], \int \mu_t \, dt)\)-norm satisfying

\[
|\dot{\mu}_t|^2 = \int v^2_t \, d\mu_t \quad \text{for a.e. } t \in [0,1],
\]

cf. Theorem 6.7 in [5].

**Proof of the Theorem.** Let \(u \in L^2(X) \cap L^\infty(X)\), with \(u \leq M\) a.e.. Fix \(s < t\) and define for \(s < r < t\)

\[
\psi^\varepsilon_r(u) := P_{r,s} \eta \epsilon(P_{r,s}u),
\]

\[
\Psi^\varepsilon(r) := \int \omega \epsilon(\psi^\varepsilon_r(u)) \, d\mu_r,
\]

where \(\mu_r = \rho_r m_r\) is a regular curve in \(P_2(X)\), and define functions on \(\mathbb{R}\)

\[
\eta \epsilon(z) = (z + \varepsilon)^\alpha - \varepsilon^\alpha, \quad \omega \epsilon(z) = \log(z + \varepsilon), \quad 0 < \varepsilon < 1.
\]

Note that \(\eta \epsilon' \in \text{Lip}_b([0, M]), \omega \epsilon \in \text{Lip}_b([0, M])\) and

\[
\eta \epsilon(z) + \varepsilon \geq (z + \varepsilon)^\alpha, \quad \eta \epsilon(z) \leq z^\alpha.
\]

Then, \(r \mapsto \Psi^\epsilon(r)\) is locally absolutely continuous due to the splitting

\[
|\Psi^\epsilon(r + h) - \Psi^\epsilon(r)| \leq \int_0^h \omega \epsilon'(\psi^\epsilon_r(u))(\psi^\epsilon_{r+h}(u) - \psi^\epsilon_r(u)) \, d\mu_r + \int_r^{r+h} \int |\omega \epsilon'(\psi^\epsilon_{r+h}(u))| \text{lip}_a(\psi^\epsilon_{r+h}(u))v_s \, d\mu_s \, ds,
\]

(21)
where $\zeta \in (r, r + h)$ and $v$ is the unique velocity density of $\mu$. Indeed the first term is absolutely continuous since it can be rewritten as

$$\int \omega_{\varepsilon}'(\psi^\varepsilon_r(u))(\psi^\varepsilon_{r+h}(u) - \psi^\varepsilon_r(u)) \, d\mu_r$$

$$= \int P^*_{t,r} \left( \frac{\rho_r}{\psi^\varepsilon_r(u) + \varepsilon} \eta_{\varepsilon}(P_{r,s} u) \right) \, d\mu_r - \int P^*_{t,r} \left( \frac{\rho_r}{\psi^\varepsilon_r(u) + \varepsilon} \eta_{\varepsilon}(P_{r,s} u) \right) \, d\mu_r$$

and because of the 2-absolute continuity of $r \mapsto P^*_{t,r} g$, $r \mapsto P_{r,s} u$ by Proposition 1.8 the Lipschitz continuity of $\eta_{\varepsilon}$, and the Lipschitz continuity of $r \mapsto f_r$.

For the second term in (21) note that for all $s < r < t$, $\psi^\varepsilon_r(u)$ is in Lip$^b(X)$ by virtue of the $L^1$-gradient estimate (13), which can be seen by

$$|\nabla_t(P_{t,r}\eta_{\varepsilon}(P_{r,s} u))| \leq P_{t,r} \|\nabla_r(\eta_{\varepsilon}(P_{r,s} u))\| \leq P_{t,r}(\eta_{\varepsilon}'(P_{r,s} u)|\nabla_r(P_{r,s} u)|) \leq \frac{P_{t,r}(\eta_{\varepsilon}'(P_{r,s} u)||P_{r,s} u||_\infty)}{\sqrt{2(r-s)}}$$

(22)

where we used Theorem 2.1 in the last step.

In the next step we calculate the actual derivative of $\Psi^\varepsilon(r)$. For this we split the difference quotient into two terms as in (21). The first one looks like and can be estimated using the concavity of $\omega_{\varepsilon}$ in the following way

$$\frac{1}{h} \int \omega_{\varepsilon}(\psi^\varepsilon_{r+h}(u)) - \omega_{\varepsilon}(\psi^\varepsilon_r(u)) \, d\mu_r \leq \frac{1}{h} \int \frac{1}{\psi^\varepsilon_r(u) + \varepsilon} (\psi^\varepsilon_{r+h}(u) - \psi^\varepsilon_r(u)) \, d\mu_r$$

$$\leq \frac{1}{h} \int \left( P^*_{t,r} - P^*_{t,r+h} \right) \left( \frac{\rho_r}{\psi^\varepsilon_r(u) + \varepsilon} \right) \eta_{\varepsilon}(P_{r,s} u) \, d\mu_r$$

$$+ \frac{1}{h} \int P^*_{t,r+h} \left( \frac{\rho_r}{\psi^\varepsilon_r(u) + \varepsilon} \right) (\eta_{\varepsilon}(P_{r+h,s} u) - \eta_{\varepsilon}(P_{r,s} u)) \, d\mu_r$$

$$+ \frac{1}{h} \int P^*_{t,h} \left( \frac{\rho_r}{\psi^\varepsilon_r(u) + \varepsilon} \right) \eta_{\varepsilon}(P_{h+r} u) d(m_{r+h} - m_r).$$

Taking the limit $h \to 0$, by Proposition 1.8 the first and the last term together converge to $\int \Gamma_r \left( P^*_{t,r} \left( \frac{\rho_r}{\psi^\varepsilon_r(u)} \right), \eta_{\varepsilon}(P_{r,s} u) \right) \, d\mu_r$ whereas the second can be bounded from above by

$$\limsup_{h \to 0} \frac{1}{h} \int P^*_{t,r+h} \left( \frac{\rho_r}{\psi^\varepsilon_r(u) + \varepsilon} \right) (\eta_{\varepsilon}(P_{r+h,s} u) - \eta_{\varepsilon}(P_{r,s} u)) \, d\mu_r$$

$$\leq \int P^*_{t,r} \left( \frac{\rho_r}{\psi^\varepsilon_r(u) + \varepsilon} \right) \eta_{\varepsilon}'(P_{r,s} u) \Delta_r P_{r,s} u \, d\mu_r,$$

for a.e. $r$. The last inequality holds by the convexity of $\eta_{\varepsilon}$, since $\eta_{\varepsilon}'(P_{r+h,s} u) \mapsto P_{r,s} u$ in $L^2(X)$, $\frac{1}{h}(P_{r+h,s} u - P_{r,s} u) \mapsto \Delta_r P_{r,s} u$ in $L^2(X)$ for a.e. $r$ and $P^*_{t,r+h} \left( \frac{\rho_r}{\psi^\varepsilon_r(u) + \varepsilon} \right) \mapsto P^*_{t,r} \left( \frac{\rho_r}{\psi^\varepsilon_r(u) + \varepsilon} \right)$ weak-* in $L^\infty(X)$ due to the uniform boundedness.

For the second term of the difference quotient of $\Psi^\varepsilon(r)$ note that $\omega_{\varepsilon}(P_{r,t} \eta_{\varepsilon}(P_{r,s} u))$ belongs to Lip$^b(X)$ by virtue of the local Poincaré inequality (Theorem 2.1). Hence for the unique minimal velocity density $v$ for $\mu$ we find

$$\int \omega_{\varepsilon}(P_{t,r} \eta_{\varepsilon}(P_{r,s} u)) \rho_r \, dt \leq \int \lip_{t} \omega_{\varepsilon}(P_{t,r} \eta_{\varepsilon}(P_{r,s} u)) v_r \, d\mu_t$$

$$\leq \int |\omega_{\varepsilon}'(\psi^\varepsilon_r(u))| \lip_{t} \psi^\varepsilon_r(u) v_r \, d\mu_t.$$
Summarizing we find by applying the chain- and Leibniz rule

\[ \frac{d^+}{dr} \Psi^\varepsilon(r) \leq \int \Gamma_r(\eta^\varepsilon_r(P_{r,s}u), P_{t,r}^* \frac{\rho_r}{\psi_r^\varepsilon + \varepsilon} + \eta^\varepsilon_r(P_{r,s}u) \Delta_r P_{r,s}u P_{t,r}^* \frac{\rho_r}{\psi_r^\varepsilon + \varepsilon}) \, dm_r + \int |\omega_r^\varepsilon(\psi_r^\varepsilon)| \left| \text{lip}_t(\psi_r^\varepsilon) \right| v_r \, dm_r. \]

From \( \eta^\varepsilon_r(x) \) we know that for each \( s < r < t \), \( \psi_r^\varepsilon(u) \) belongs to Lip(\( X \)) and thus we know that \( \text{lip}_t(\psi_r^\varepsilon(u)) \leq P_{t,r} \nabla_r(\eta^\varepsilon_r(P_{r,s}u)) \). (33)

From this we deduce

\[ \frac{d^+}{dr} \Psi^\varepsilon(r) \leq \int (-\eta^\varepsilon_r(P_{r,s}u) \Gamma_r(P_{r,s}u)/(\psi_r^\varepsilon + \varepsilon) \frac{\rho_r}{\psi_r^\varepsilon + \varepsilon}) \, dm_r + \int \frac{\rho_r}{\psi_r^\varepsilon(u) + \varepsilon} P_{t,r} \eta^\varepsilon_r(P_{r,s}u) v_r \, dm_t \]

\[ = \int \frac{\alpha \rho_r}{\psi_r^\varepsilon(u) + \varepsilon} \left( -\alpha \frac{1}{(\psi_r^\varepsilon + \varepsilon) \frac{\rho_r}{\psi_r^\varepsilon + \varepsilon}} \left( \frac{P_{r,s}u}{P_{t,r} \eta^\varepsilon_r(P_{r,s}u)} \right)^2 \right) + v_r P_{t,r} \left( (P_{r,s}u + \varepsilon) \alpha \frac{\nabla_r P_{r,s}u}{P_{r,s}u + \varepsilon} \right) \, dm_t \]

\[ \leq \int \frac{\alpha \rho_r}{\psi_r^\varepsilon(u) + \varepsilon} \sup_{\kappa} \left( -\alpha \frac{1}{(\psi_r^\varepsilon + \varepsilon) \frac{\rho_r}{\psi_r^\varepsilon + \varepsilon}} \left( \frac{P_{r,s}u}{P_{t,r} \eta^\varepsilon_r(P_{r,s}u)} \right)^2 \right) + v_r P_{t,r} \left( (P_{r,s}u + \varepsilon) \alpha \frac{\nabla_r P_{r,s}u}{P_{r,s}u + \varepsilon} \right) \, dm_t \]

Calculating the supremum and using (31) further yields

\[ \frac{d^+}{dr} \Psi^\varepsilon(r) \leq \int \frac{\alpha \rho_r}{\psi_r^\varepsilon(u) + \varepsilon} \left( \frac{v_r^2}{4(\alpha - 1)} \right) \, dm_t \leq \frac{\alpha}{4(\alpha - 1)} \int v_r^2 \, d\mu_r = \frac{\alpha}{4(\alpha - 1)} |\mu_r|^2, \]

where we used that \( v \) is the minimal velocity density for \( \mu \). Integrating from \( s \) to \( t \) yields

\[ \Psi^\varepsilon(t) - \Psi^\varepsilon(s) \leq \frac{\alpha}{4(\alpha - 1)} \int_s^t |\mu_r|^2 \, dr. \]

Hence, by approximating \( W^2 \)-geodesics with regular curves and taking the scaling into account we end up with

\[ \Psi^\varepsilon(t) - \Psi^\varepsilon(s) \leq \frac{\alpha}{4(\alpha - 1)(t - s)} W^2(\mu_s, \mu_t). \]

We get for a.e. \( x, y \in X \), after letting \( \mu_s \to \delta_x \) and \( \mu_t \to \delta_y \) with respect to \( L^2 \)-Kantorovich distance,

\[ \log \frac{\eta^\varepsilon_r(P_{t,s}u)(y)}{\eta^\varepsilon_r(P_{t,s}u)(x)} \leq \frac{\alpha d^2_l(x, y)}{4(\alpha - 1)(t - s)}. \]

Now we let \( \varepsilon \to 0 \). Since \( \eta^\varepsilon_r(P_{t,s}u) \to (P_{t,s}u)^\alpha \), and \( P_{t,s} \eta^\varepsilon_r(u) \to P_{t,s}(u^\alpha) \) a.e. by monotone convergence we find

\[ \frac{(P_{t,s}u)^\alpha(y)}{P_{t,s}(u^\alpha)(x)} \leq \exp \left\{ \frac{\alpha d^2_l(x, y)}{4(\alpha - 1)(t - s)} \right\}. \]

which is the result for \( u \in L^2(X) \cap L^\infty(X) \). The result for general \( u \) follows by a truncation argument. \( \square \)
4.2 From dimension independent Harnack inequality to local logarithmic Sobolev inequality

We assume in this section that $m_t(X) < \infty$ for some and thus for all $t \in (0, T)$.

**Theorem 4.3.** Assume that the Harnack inequality (3) holds. Then the local logarithmic Sobolev inequality holds

$$P_{t,s}(u \log u) - P_{t,s}u \log P_{t,s}u \geq (t-s)\frac{\Gamma_t(P_{t,s}u)}{P_{t,s}u},$$

for all $u \in D(S) \cap L^1(X)$ such that $u \geq 0$ m.-a.e..

**Proof.** Let $u \in L^1(X) \cap L^\infty(X)$ with $u \geq c > 0$. From the Harnack inequality it follows that

$$\int \alpha \log(P_{t,s}u) \, d\mu - \int \log(P_{t,s}(u^\alpha)) \, d\nu \leq \frac{\alpha W^2_2(\mu,\nu)}{4(\alpha - 1)(t-s)}$$

holds for each probability measures $\mu,\nu$ which are absolutely continuous with respect to $m_t$. This follows from integrating (3) with respect to an optimal transport plan.

Now choose $\mu = \varphi m_t$ with $g \geq 0$ and $g \in D(E) \cap L^\infty(X)$. Consider the associated Dirichlet form $E^g(u) := \int \Gamma_t(u)g \, dm_t$ with heat semigroup $(H^g_t)_{t \geq 0}$ and generator $\Delta^g$. We introduce for fixed $\varepsilon > 0$ the function

$$\psi = \frac{1}{\varepsilon} \int_0^\infty H^g_\varepsilon(\psi_0) \kappa(r/\varepsilon) \, dr,$$

where $\kappa \in C^\infty_c(0,\infty)$ with $\kappa \geq 0$ and $\int_0^\infty \kappa(r) \, dr = 1$ and $\psi_0 \in D(E^g) \cap L^\infty(gm_t)$. Note that $||\Delta^g\psi||_\infty \leq M$ for some $M \geq 0$ and hence $\mu_\tau := g(1 - \tau \Delta^g\psi)m_t$ is a probability measure for all $\tau < 1/2M$. First we will show that

$$\limsup_{\tau \to 0} \frac{1}{2\tau^2} W^2_2(\mu, \mu_\tau) \leq \frac{1}{2} \int \Gamma_t(\psi) g \, dm$$

(25)

using the Hopf-Lax semigroup $(Q_\tau)_{\tau \geq 0}$ with respect to $d_t$. For $\varphi \in C_b(X)$ we find for $r \leq \tau$

$$\frac{d}{dr} \int Q_\tau(\varphi) \, d\mu \leq \int \left(-\frac{1}{2} \nabla_t Q_\tau(\varphi)^2 (1 - \tau \Delta^g\psi) - Q_\tau(\varphi) \Delta^g(\psi)\right) g \, dm_t$$

$$\leq \int \left(-\frac{1}{2} \nabla_t Q_\tau(\varphi)^2 (1 - \tau M) + \Gamma_t(\varphi, \varphi) g \right) \, dm_t$$

$$\leq \frac{1}{2(1 - \tau M)} \int \Gamma_t(\psi) g \, dm_t.$$

Integrating on $[0, \tau]$, taking the supremum over all $\varphi$, dividing by $\tau$ and letting $\tau \to 0$ yields (25). For $\alpha = 1 + \tau$, $\tau > 0$ (24) reads as

$$(1 + \tau) \int \log(P_{t,s}u) \, d\mu - \int \log(P_{t,s}(u^{1+\tau})) \, d\mu_\tau \leq \left\{ \frac{(1 + \tau) W^2_2(\mu, \mu_\tau)}{4\tau(t-s)} \right\}.$$

We divide by $\tau > 0$ and let $\tau \to 0$. By (25) the right hand side can be estimated from above by

$$\frac{1}{4(t-s)} \int \Gamma_t(\psi) g \, dm_t.$$
We claim that together with the left hand side this amounts to
\[
\int \log(P_{t,s}u) \, d\mu - \int \frac{P_{t,s}(u \log u)}{P_{t,s}u} \, d\mu - \int \Gamma_t(\log(P_{t,s}u), \psi) \, d\mu \leq \frac{1}{4(t-s)} \int \Gamma_t(\psi) \, d\mu. \tag{27}
\]
Indeed, it is straightforward to check that \( r \mapsto \int \log(P_{t,s}u^{1+r}) \, d\mu \) is absolutely continuous with derivative
\[
\Psi(r) := \int \frac{P_{t,s}(u^{1+r} \log u)}{P_{t,s}u^{1+r}} \, d\mu - \int \log(P_{t,s}u^{1+r}(\Delta^g_t \psi)g \, dm_t.
\]
Since \( u \geq c > 0 \) we see that \( r \mapsto \Psi(r) \) is continuous. Hence
\[
\frac{1}{\tau} \left( \int \log(P_{t,s}u) \, d\mu - \int \log(P_{t,s}(u^{1+\tau})) \, d\mu \right) = -\frac{1}{\tau} \int_0^\tau \Psi(r) \, dr
\]
\[
\stackrel{\tau \to 0}{\rightarrow} - \int \frac{P_{t,s}(u \log u)}{P_{t,s}u} \, d\mu - \int \Gamma_t(\log(P_{t,s}u), \psi) \, d\mu.
\]
Together with (26) this yields (27).

Letting \( \varepsilon \to 0 \) we conclude
\[
\int \log(P_{t,s}u) \, d\mu - \int \frac{P_{t,s}(u \log u)}{P_{t,s}u} \, d\mu - \int \Gamma_t(\log(P_{t,s}u), \psi_0) \, d\mu \leq \frac{1}{4(t-s)} \int \Gamma_t(\psi_0) \, d\mu.
\]
Now we may choose \( \psi_0 = -2(t-s) \log(P_{t,s}u) \) and obtain
\[
\int \log(P_{t,s}u) \, d\mu - \int \frac{P_{t,s}(u \log u)}{P_{t,s}u} \, d\mu + (t-s) \int \Gamma_t(\log(P_{t,s}u)) \, d\mu \leq 0.
\]
Since this holds for all \( \mu = gm_t \), we recover the local logarithmic Sobolev inequality
\[
P_{t,s}(u \log u) - P_{t,s}u \log P_{t,s}u \geq (t-s) \frac{\Gamma_t(P_{t,s}u)}{P_{t,s}u},
\]
for all \( u \in L^1(X) \cap L^\infty(X) \) with \( u \geq c > 0 \). We obtain the estimate for all nonnegative \( u \in D(S) \cap L^1(X) \) by a truncation argument. \( \square \)

5 The logarithmic Harnack inequality

We already noted in Remark 1.5, the dimension-independent Harnack inequality (for some exponent \( \alpha \)) implies the logarithmic Harnack inequality.

This section is devoted to prove that the logarithmic Harnack inequality implies the dynamic Bochner inequality. To do so, in addition to our standing assumptions, in particular, the validity of a \( \text{RCD}(K, \infty) \)-condition for each \((X, d_t, m_t)\) and a log-Lipschitz dependence on \( t \) for \( d_t \) and \( m_t \), we have to impose various continuity assumptions (all of which are satisfied in the static case).

We assume that \( m_t(X) < \infty \) for \( t \in (0, T) \), \((\text{A2.a-c})\) and \((\text{A3.b})\) hold. Moreover, writing \( u_{q,s} = P_{q,s}u \), we assume that

- for \( u \in D(\mathcal{E}) \cap D(\Delta) \) the functions
  \[
  q \mapsto u_{q,s}, \quad s \mapsto \Delta_s u, \quad q \mapsto \Delta_q u_{q,s}
  \tag{A5.a}
  \]
  are continuous in \( D(\mathcal{E}) \cap L^1(X) \);
for $w, w_q \in D(\Delta)$ as $q \to t$, and $\Delta_t w_q \to \Delta_t w$ in $L^1(X)$

$$\Delta_q P^s_{t,q} w_q \to \Delta_t w \quad \text{in} \ L^1(X). \quad (A5.b)$$

Let us emphasize that (A5.a+b) are always satisfied in the static case.

**Theorem 5.1.** If the logarithmic Harnack inequality

$$P_{t,s}(\log u)(x) \leq \log(P_{t,s} u)(y) + \frac{d_t^2(x,y)}{4(t-s)} \quad (28)$$

holds for all nonnegative $u \in L^1(X) \cap L^\infty(X)$, then the pointwise dynamic Bochner inequality holds at time $t$, i.e.

$$\frac{1}{2} \int \Gamma_t(f) \Delta_t g \ dm_t + \int (\Delta_t f)^2 g + \Gamma_t(f, g) \Delta_t f \ dm_t \geq \frac{1}{2} \int (\partial_t \Gamma_t)(f) g \ dm_t \quad (29)$$

for all $f \in D(\Delta_t) \cap L^\infty(X)$ such that $\Gamma_t(f) \in L^\infty(X)$ and all nonnegative $g \in D(\Delta_t) \cap L^\infty(X)$.

**Proof.** Let $f, \Delta_t f \in D(\Delta_t) \cap \text{Lip}^b(X)$. Then by Lemma 5.3 $u = e^f \in D(\Delta_t) \cap \text{Lip}^b(X)$ with $\Delta_t e^f, \Delta_t^2 e^f \in L^\infty(X) \cap D(\mathcal{E})$ and $u \geq e^{-\|f\|_{L^\infty}} =: \varepsilon > 0$.

Let us introduce some function $g$ satisfying $C \geq g \geq c > 0$. Moreover we will assume that $g \in D(\Delta_t) \cap \text{Lip}(X)$ such that $\Delta_t g \in D(\mathcal{E})$. We define the Cheeger energy $\frac{1}{2} \mathcal{E}_t^g$ associated with $d_t$ and finite measure $g_m_t$. The relaxed gradient $|\nabla f|_s$ is invariant under this perturbations, hence $|\nabla f|_s^2 = |\nabla f|_s$ and $D(\mathcal{E}_t^g) = D(\mathcal{E})$. We refer to [2, Section 4] for these facts. This leads to the following integral representation of $\mathcal{E}_t^g$

$$\mathcal{E}_t^g(f) = \int \Gamma_t(f) g \ dm_t,$$

which makes it a symmetric bilinear form. We denote the associated (Markovian) semigroup by $P^g_t$ and its generator by $\Delta_t^g$, which satisfies the following integration by parts formula

$$\int \Delta_t^g fh \ dm_t = - \int \Gamma_t(f, h) g \ dm_t$$

for all $f \in D(\Delta_t^g)$ and $h \in D(\mathcal{E}_t^g)$. Since $\log g \in D(\mathcal{E})$ this can be rewritten into

$$\Delta_t^g = \Delta_t + \Gamma_t(\log g, \cdot)$$

and thus $D(\Delta_t) \subset D(\Delta_t^g)$.

For $s \leq t$ we set

$$v_s = P^g_{t-s} e^{-2f} \quad \text{and} \quad \mu_s = v_s g m_t.$$

Note that $v_s \in D(\Delta_t^g) \cap L^\infty(X)$ for all $s \leq t$ by Lemma 5.3. Without restriction, we may assume that $\mu_t$, and hence $\mu_s$ for every $s < t$, is a probability measure. Otherwise, simply replace $f$ by $f + C$ for a suitable constant $C$.

Assume that the logarithmic Harnack inequality holds for the function $u = e^f$. We integrate the inequality w.r.t. the $W_t$-optimal coupling of $\mu_t$ and $\mu_s$ to obtain for any $s < t$

$$\int P_{t,s} \log u \ dm_{\mu_s} - \int \log P_{t,s} u \ dm_{\mu_t} \leq \frac{1}{4(t-s)} W_t^2(\mu_t, \mu_s). \quad (30)$$
Consider the map \( r \mapsto \int P_{t,r} \log P_{t,s} u \, d\mu_r \). This map is absolutely continuous since for a.e. \( s < r_1 < r_2 < t \)

\[
| \int_{r_2}^{r_1} P_{t,r} \log P_{t,r} u \, d\mu_r - \int P_{t,r_1} \log P_{t,r_1} u \, d\mu_r | \\
\leq | \int_{r_1}^{r_2} \Gamma_r \log u_r, P^*_r (v_r g) \, d\mu_r | dr | \\
+ \frac{1}{2} \int_{r_1}^{r_2} | \Delta_r u_r |^2 \, d\mu_r | dr |
\]

Hence for the left hand side of (30) we find

\[
\int P_{t,s} \log u \, d\mu_s - \int P_{t,s} u \, d\mu_t = - \int_s^t \frac{d}{dr} \int P_{t,r} \log P_{t,s} u \, d\mu_r | dr |
\]

\[
= \int_s^t - P_{t,r} \Delta_r \log P_{t,s} u - P_{t,r} \frac{\Delta_r P_{t,s} u}{P_{t,s} u} - \Gamma_t (P_{t,r} \log P_{t,s} u, \log v_r) \, d\mu_r | dr |
\]

\[
= - \int_s^t P_{t,r} \Gamma_r (\log P_{t,s} u) + \Gamma_t (P_{t,r} \log P_{t,s} u, \log v_r) \, d\mu_r | dr |
\]

and for the right hand side Kuwada’s Lemma ([2] Lemma 6.1) yields

\[
\frac{1}{4} (t-s) \text{W}^2_t (\mu_s, \mu_t) \leq \frac{1}{4} \int_s^t \Gamma_t (\log v_r) \, d\mu_r | dr |
\]

Hence (30) can be rewritten as follows

\[
\int_s^t \int - P_{t,r} \Gamma_r (\log P_{t,s} u) - \Gamma_t (P_{t,r} \log P_{t,s} u, \log v_r) - \frac{1}{4} \Gamma_t (\log v_r) \, d\mu_r | dr \leq 0. (31)
\]

Now let us consider the map

\[
r \mapsto \int - P_{t,r} \Gamma_r (\log P_{t,s} u) - \Gamma_t (P_{t,r} \log P_{t,s} u, \log v_r) - \frac{1}{4} \Gamma_t (\log v_r) \, d\mu_r | dr
\]

\[
= : I(r) + II(r) + III(r).
\]

From the second part of Lemma 5.2 we know that the map \( r \mapsto III(r) \) is absolutely continuous with derivative

\[
\frac{d}{dr} III(r) = \int \left( \frac{1}{2} \Gamma_t (\log v_r, \Delta^q v_r) - \frac{1}{4} \Gamma_t (\log v_r) \Delta^q v_r \right) \, d\mu_t
\]

\[
= \frac{1}{2} \int \Gamma_t (\log v_r, \Delta^q v_r) - \frac{1}{4} \Gamma_t \left( \Delta^q v_r, \log v_r \right) \, d\mu_r.
\]

For I we calculate for a.e. \( r_1 < r_2 \)

\[
| I(r_1) - I(r_2) | \leq | \int_{r_2}^{r_1} \int \frac{\Gamma_{r_2} (u_{r_2})}{u_{r_2}^2} \Delta_r P^*_r (v_r g) \, d\mu_r | dr |
\]

\[
+ | \int \left( \frac{\Gamma_{r_1} (u_{r_1})}{u_{r_1}^2} - \frac{\Gamma_{r_1} (u_{r_1})}{u_{r_1}^2} \right) P^*_r (v_r g) \, d\mu_r | dr |
\]

\[
+ | \int_{r_1}^{r_2} P_{r,r_1} \left( \frac{\Gamma_{r_1} (u_{r_1})}{u_{r_1}^2} \right) \Delta^q v_r g \, d\mu_r | dr |
\]

\[
= 23
\]
The second term of this subdivision can be estimated as follows

\[ \left| \int \left( \frac{\Gamma_r(u_{r_2})}{u_{r_2}^2} - \frac{\Gamma_r(u_{r_1})}{u_{r_1}^2} \right) P^*_{t,r_1}(v_{r_2} g) \, dm_{r_1} \right| \leq C(r_2 - r_1) \int \frac{1}{\varepsilon} \int_{r_1}^{r_2} T \Delta \theta \Gamma_\lambda \psi \left( \frac{\Delta^q v_{r_2}}{v_{r_2}} \right) \, dm_{r_1} \, dr \]

Finally for \( II \) we argue similarly as for \( I \)

\[ |II(r_1) - II(r_2)| \leq | \int_{r_1}^{r_2} \log u_{r_2} \Delta r P^*_{t,r}(\Delta^q v_{r_2} g) \, dm_{r_1} \, dr \]

Thus \( r \to I(r) + II(r) + III(r) \) is absolutely continuous and subtracting \( 0 = I(t) + II(t) + III(t) \) from both sides in (31) we can rewrite

\[ \int_s^t \frac{d}{dq} \int_r^t \left[ P_{t,q} \left( \log P_{t,q} u \right) + \Gamma_q \left( \log P_{t,q} u \right) \right] \, dq \, dr \leq (t - s) \int \frac{1}{\varepsilon} \int_{r_1}^{r_2} \frac{T}{\varepsilon} \Delta \theta \Gamma_\lambda \psi \left( \frac{\Delta^q v_{r_2}}{v_{r_2}} \right) \, dm_{r_1} \, dr \]

Recall that \( \mu_q = v_q g \, m_t \) and put \( u_{q,s} = P_{q,s} u \). Then the term on the LHS of (32) takes the form

\[ \int_s^t \int_r^t \left[ P_{t,q} \left( \log u_{q,s} + \log v_q \right) \right] \, dq \, dr = \int_s^t \int_r^t \left[ P_{t,q} \left( - \Delta \theta \Gamma_\lambda \psi \left( \frac{\Delta^q v_{q,s}}{v_q} \right) + \frac{1}{\varepsilon} \Gamma_q \left( \log v_q \right) + \frac{1}{\varepsilon} \Gamma_q \left( \log v_q \right) \right) \, dq \, dr \right] \]

We decompose \( \Psi \) into five terms and verify the continuity of each of them. For the first one,

\[ \Psi_1(q) := - \int \Delta \theta \Gamma_\lambda \psi \left( \frac{\Delta^q v_{q,s}}{v_q} \right) \, dq \, dr = - \int \Gamma_q \left( \log u_{q,s} \right) \Delta \theta \Gamma_\lambda \psi \left( \frac{\Delta^q v_{q,s}}{v_q} \right) \, dq \, dr \]
continuity follows from the fact that \( q \mapsto \Gamma_q(\log u_{q,s}) \) is weak*-continuous in \( L^\infty(X) \) by (A5.a) and (A2.a), and \( q \mapsto \Delta_q P_{t,q}^*(v_q g) \) is continuous in \( L^1(X) \) by assumption (A5.b) together with the fact that \( q \mapsto \Delta_t(v_q g) \) is continuous in \( L^1(X) \).

Continuity of the second one,

\[
\Psi_2(q) := \int \partial_t \Gamma_q(\log u_{q,s}) P_{t,q}^*(v_q g) \, dm_q,
\]

follows from \( L^1 \)-continuity of \( q \mapsto \partial_t \Gamma_q(\log u_{q,s}) \), as requested in assumption (A3.b), and the weak*-continuity of \( q \mapsto P_{t,q}^*(v_q g) \) in \( L^\infty(X) \), resulting from (A5.b) together with the uniform boundedness in \( L^\infty(X) \).

For the third one,

\[
\Psi_3(q) := 2 \int \Gamma_q \left( \log u_{q,s}, \frac{1}{u_{q,s}} \Delta q u_{q,s} \right) P_{t,q}^*(v_q g) \, dm_q,
\]

assumptions (A5.a) and (A3.b) yield continuity of \( q \mapsto \Gamma_q \left( \log u_{q,s}, \frac{1}{u_{q,s}} \Delta q u_{q,s} \right) \) in \( L^1(X) \) combined with (A2.a) and \( u \geq \varepsilon \). Together with the weak*-continuity of \( q \mapsto P_{t,q}^*(v_q g) \) in \( L^\infty(X) \), this yields the claim.

The fourth term,

\[
\Psi_4(q) := \int \Gamma_t(P_{t,q} \log u_{q,s}, \Delta^q v_q) \, g \, dm_t
\]

is continuous since \( q \mapsto P_{t,q} \log u_{q,s} \) is continuous in \( D(E) \) by (A5.a) and (A2.a), and \( q \mapsto \Delta^q v_q \) is continuous in \( D(E) \) by Lemma 5.3.

The final term

\[
\Psi_5(q) := \int \left[ \frac{1}{2} \Gamma_t(\log v_q, \frac{\Delta^q v_q}{v_q}) v_q + \frac{1}{4} \Gamma_t(\log v_q, \Delta^q v_q) \right] \, g \, dm_t
\]

is always continuous in \( q \) without any extra assumption.

Similarly one computes the right hand side of (32). Recalling that \( \log v_t = -2f \):

\[
\frac{1}{t-s} \int \left[ \Gamma_t(\log u_{t,s}) + \Gamma_t(\log u_{t,s}, \log v_t) + \frac{1}{4} \Gamma_t(\log v_t) \right] \, d\mu_t = \frac{1}{t-s} \int \Gamma_t \left( \log u_{t,s} - f \right) \, d\mu_t
\]

\[
= \frac{1}{t-s} \int_s^t \partial_q \int \Gamma_t \left( \log u_{q,s} - f \right) \, d\mu_t \, dq
\]

\[
= \frac{2}{t-s} \int_s^t \int \Gamma_t \left( \log u_{q,s} - f, \frac{\Delta q u_{q,s}}{u_{q,s}} \right) \, d\mu_t \, dq.
\]

Note that by the continuity of \( q \mapsto \log \) in \( D(E) \) and the continuity of \( q \mapsto \frac{\Delta q u_{q,s}}{u_{q,s}} \) in \( D(E) \) by virtue of (A5.a), (A2.a) and the fact that \( u \geq \varepsilon \), the map \( q \mapsto \int \Gamma_t \left( \log u_{q,s} - f, \frac{\Delta q u_{q,s}}{u_{q,s}} \right) \, d\mu_t \) is continuous. Then by the Lebesgue differentiation theorem and the continuity discussion above we deduce from (32) that (recalling that \( u = e^f \))

\[
\Psi_5(s) = \int \left( - \Delta s \Gamma_s(f) + \partial_s \Gamma_s(f) + 2 \Gamma_s(f, \frac{1}{e} \Delta s e^f) \right) P_{t,s}^*(v_s g) \, dm_s
\]

\[
+ \int \left[ \Gamma_t(P_{t,s} f, \Delta^q v_s) + \frac{1}{2} \Gamma_t(\log v_s, \frac{\Delta^q v_s}{v_s}) v_s + \frac{1}{4} \Gamma_t(\log v_s, \Delta^q v_s) \right] g \, dm_t \leq 0. \tag{34}
\]
Then, letting $s \to t$, by continuity we have (recalling that $v_t = e^{-2f}$)

$$\int \left[ \Gamma_t(f) \Delta_t(e^{-2f} g) - (\partial_t \Gamma_t(f) + 2 \Gamma_t(\Delta_t f, f)) e^{-2f} g \right] dm_t \geq 0.$$ 

Choose $g = (\tilde{g} + \varepsilon) e^{2f}$, where $\tilde{g} \in \text{Lip}_b(X) \cap D(\Delta)$ with $\Delta_t \tilde{g} \in D(\mathcal{E})$. Then $g \in D(\Delta_t) \cap \text{Lip}(X)$ such that $\Delta_t g \in D(\mathcal{E})$ by Lemma 5.3 and [18] Theorem 3.4, and there exists constants $c, C$ such that $0 < c \leq g \leq C$. With this choice we obtain

$$\int \left[ \Gamma_t(f) \Delta_t \tilde{g} - (\partial_t \Gamma_t(f) + 2 \Delta_t f)^2 \tilde{g} + 2 \Gamma_t(f, \tilde{g}) \Delta_t f \right] dm_t \geq 0$$

for all $f, \Delta_t f \in D(\Delta_t) \cap \text{Lip}_b(X)$ and nonnegative $\tilde{g} \in \text{Lip}_b(X) \cap D(\Delta_t)$ with $\Delta_t \tilde{g} \in D(\mathcal{E})$. The result for general $f \in D(\Delta_t) \cap \text{Lip}_b(X)$ and nonnegative $\tilde{g} \in L^\infty(X) \cap D(\Delta_t)$ follows by approximation with the standard $t$-semigroup mollifier from Definition 1.9.

**Lemma 5.2.** Let $g \in \text{Lip}_b(X)$ satisfying $C \geq g \geq c > 0$. Let $u \in \text{Lip}_b(X) \cap D(\Delta)$ such that $\Delta^g u \in L^\infty(X) \cap D(\mathcal{E})$. Moreover let $\psi \in C^1(\mathbb{R})$. Then for $v_t = P^u_t \psi \ r \mapsto \int \Gamma(u_r) \psi(u_r) g dm$ is absolutely continuous and

$$\frac{d}{dr} \int \Gamma(u_r) \psi(u_r) g dm = \int (2 \Gamma(u_r, \Delta^g u_r) \psi(u_r) + \Gamma(u_r) \psi'(u_r) \Delta^g u_r) g dm$$

for a.e. $r \geq 0$.

**Proof.** Let $0 < s < t$. Then it is well-known that, see e.g. [10] Theorem 4.8 or [11] Theorem 4.6,

$$| \int \Gamma(u_t) \psi(u_t) g dm - \int \Gamma(u_s) \psi(u_s) g dm|$$

\leq | \int (\Gamma(u_t) - \Gamma(u_s)) \psi(u_t) g dm | + | \int \Gamma(u_s) (\psi(u_t) - \psi(u_s)) g dm |$$

\begin{align*}
= | \int_s^t \int 2 \Gamma(u_r, \Delta^g u_r) \psi(u_t) g dm dr | + | \int_s^t \int \Gamma(u_s) \psi'(u_r) \Delta^g u_r g dm dr | \\
\leq ||\psi(u_t) g||_\infty (\int_s^t \mathcal{E}^g(u_r) + \mathcal{E}^g(P_r \Delta u) dr) + (t - s) \sup_r ||\psi'(u_r) g||_\infty \mathcal{E}^g(u_s) \sup_r ||P_r \Delta u||_\infty < \infty,
\end{align*}

which shows $r \mapsto \int \Gamma(u_r) \psi(u_r) g dm$ is absolutely continuous. From this we deduce that for a.e. $r \geq 0$

$$\frac{d}{dr} \int \Gamma(u_r) \psi(u_r) g dm = \int (2 \Gamma(u_r, \Delta^g u_r) \psi(u_r) + \Gamma^g(u_r) \psi'(u_r) \Delta^g u_r) g dm.$$ 

**Lemma 5.3.** Let $(X, d, m)$ be an $\text{RCD}(K, \infty)$-space. Let $f, \Delta f \in D(\Delta) \cap \text{Lip}_b(X)$. Then $e^f \in D(\Delta) \cap \text{Lip}_b(X)$ with $\Delta e^f, \Delta^g e^f \in L^\infty \cap D(\mathcal{E})$ and $e^f \geq c$ for some $c > 0$.

Moreover the functions $t \mapsto P_t e^f$ and $t \mapsto P^\phi_t e^f$ are continuous in $L^\infty(X)$.

**Proof.** Since $f$ is bounded, $e^f$ is bounded as well and $e^f \geq e^{-\|f\|_\infty} > 0$. By the chain rule we have $\Gamma(e^f) = e^{2f} \Gamma(f) \in L^\infty$ and

$$\Delta(e^f) = e^f (\Gamma(f) + \Delta f)$$

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which belongs to $L^2 \cap L^\infty$. Next we show that $\Delta e^f \in D(\mathcal{E})$. For this note that
\[
\mathcal{E}(e^f \Delta f) \leq 2 \int e^{2f} \Gamma(\Delta f) + (\Delta f)^2 e^{2f} \Gamma(f) \, dm
\]
is bounded and
\[
\mathcal{E}(e^f \Gamma(f)) \leq 2 \int e^{2f} \Gamma(\Gamma(f)) + \Gamma(f)^3 e^{2f} \, dm
\]
\[
\leq 2 \|e^f\|_{L^\infty} \int (\Gamma(f)^2 - \Gamma(f) \Gamma(f, \Delta f)) \, dm + 2 \int \Gamma(f)^3 e^f \, dm
\]
is bounded as well. In the last step we used [18, Lemma 3.2] to bound $\mathcal{E}(\Gamma(f))$. Summing $\mathcal{E}(e^f \Delta f)$ and $\mathcal{E}(e^f \Gamma(f))$ yields that $\Delta e^f \in D(\mathcal{E})$.

Similarly we show that $\Delta^g e^f \in D(\mathcal{E}^g)$. Recall first that $D(\Delta) \subset D(\Delta^g)$ and
\[
\Delta^g e^f = \Delta e^f + \Gamma(\log g, e^f)
\]
which is an $L^\infty$-function. Moreover note that
\[
\mathcal{E}(\Delta^g e^f) = \int \Gamma(\Delta e^f) + \Gamma(\Gamma(\log g, e^f)) \, dm.
\]
For the first summand we know already that it is bounded. For the second summand we use [18, Theorem 3.4] and obtain
\[
\int \Gamma(\Gamma(\log g, e^f)) \, dm \leq 2 \int \left( \gamma_2(\log g) - K \Gamma(\log g) \Gamma(e^f) + \gamma_2(e^f) - K \Gamma(e^f) \right) \Gamma(\log g) \, dm,
\]
where $\gamma_2(\log g), \gamma_2(e^f)$ are $L^1$-functions, since $\log g$ and $e^f$ belong to Lip$_b(X) \cap D(\Delta)$ with $\Delta \log g, \Delta e^f \in D(\mathcal{E})$.

For the last claim, note that
\[
P_t e^f - P_s e^f = \int_s^t \Delta P_r e^f \, dr,
\]
where the last integral has to be understood as a Bochner integral. Hence
\[
\|P_t e^f - P_s e^f\|_{L^\infty} = \|\int_s^t \Delta P_r e^f \, dr\|_{L^\infty} \leq \int_s^t \|\Delta e^f\|_{L^\infty} \, dr \leq (t-s) \|\Delta e^f\|_{L^\infty}.
\]
The other statement follows analogously.

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