Quadratic invariants of elastic moduli

Andrew N. Norris

Mechanical and Aerospace Engineering, Rutgers University, Piscataway NJ 08854-8058, USA norris@rutgers.edu

Abstract

A quadratic invariant is defined as a quadratic form in the elements of a tensor that remains invariant under a group of coordinate transformations. It is proved that there are 7 quadratic invariants of the 21-element elastic modulus tensor under SO(3) and 35 under SO(2). This answers some open questions raised by Ting (1987) and Ahmad (2002) in this journal.

1 Introduction

The tensor of elastic moduli $c_{ijkl}$ is known to possess two linear invariants under arbitrary proper orthogonal coordinate transformations, or SO(3):

$$A_1 = c_{ijij} = c_{11} + c_{22} + c_{33} + 2(c_{44} + c_{55} + c_{66}),$$

$$A_2 = c_{iijj} = c_{11} + c_{22} + c_{33} + 2(c_{12} + c_{23} + c_{13}).$$

(1)

Ahmad (2002) presented four quadratic invariants under SO(3), the first two of which were reported by Ting (1987),

$$B_1 = c_{ijkl}^2, B_2 = c_{iikl}c_{jjkl}, B_3 = c_{iikl}c_{jklj}, B_4 = c_{kiil}c_{kjjl}.$$  

(2)

Ahmad demonstrated that the seven quadratic invariants \{A_1^2, A_2^2, A_1A_2, B_1, B_2, B_3, B_4\} are independent but did not show completeness. It is clear that the following is an eighth invariant,

$$B_5 = c_{ijkl}^2,$$

(3)

although it is not so obvious whether or not it is independent of the other seven. We will prove that there are at most seven independent quadratic invariants under SO(3), and that the seven identified by Ahmad form a complete basis. In particular, $B_5$ is a linear combination of this basis, specifically (see the Appendix)

$$B_5 = \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 - A_1A_2 + B_1 - 2B_2 + 4B_3 - 2B_4.$$  

(4)

Consequently, every fourth order elasticity tensor satisfies this identity:

$$2c_{ijkl}(c_{ijkl} - c_{ikjl}) + (c_{iijj} - c_{ijjj})^2 - 4(c_{ijkk} - c_{ikjk})(c_{ijll} - c_{iljl}) = 0.$$  

(5)

It is well known that there are 5 linear invariants under rotation about an axis, or SO(2). Taking the axis as $e_3$ these are

$$L_1 = c_{11} + c_{22} + 2c_{66}, \quad L_2 = c_{44} + c_{55}, \quad L_3 = c_{11} + c_{22} + 2c_{12}, \quad L_4 = c_{13} + c_{23}, \quad L_5 = c_{33}.$$  

(6)
Ahmad (2002) listed 17 quadratic invariants for SO(2):

$$E_1 = c_{34}^2 + c_{35}^2,$$
$$E_2 = (c_{15} + c_{25})^2 + (c_{14} + c_{24})^2,$$
$$E_3 = (c_{15} + c_{46})^2 + (c_{24} + c_{56})^2,$$
$$E_4 = (c_{14} + c_{24})c_{34} + (c_{15} + c_{25})c_{35},$$
$$E_5 = (c_{15} + c_{46})c_{35} + (c_{24} + c_{56})c_{34},$$
$$E_6 = (c_{15} + c_{25})(c_{15} + c_{46}) + (c_{14} + c_{24})(c_{24} + c_{56}),$$
$$E_7 = c_{13}^2 + c_{23}^2 + 2c_{36}^2,$$
$$E_8 = c_{44}^2 + c_{55}^2 + 2c_{45}^2,$$
$$E_9 = (c_{11} + c_{12})^2 + (c_{12} + c_{22})^2 + 2(c_{16} + c_{26})^2,$$
$$E_{10} = c_{13}c_{55} + c_{23}c_{44} + 2c_{36}c_{45},$$
$$E_{11} = (c_{11} + c_{12})c_{13} + (c_{12} + c_{22})c_{23} + 2(c_{16} + c_{26})c_{36},$$
$$E_{12} = (c_{11} + c_{12})c_{55} + (c_{12} + c_{22})c_{44} + 2(c_{16} + c_{26})c_{45},$$
$$E_{13} = c_{11}^2 + c_{22} + 2c_{12} + 4(c_{16} + c_{26} + c_{36}^2),$$
$$E_{14} = c_{14}^2 + c_{24} + c_{15} + c_{25} + 2(c_{56} + c_{26}^2),$$
$$E_{15} = (c_{44} - c_{55})c_{36} + (c_{13} - c_{23})c_{45},$$
$$E_{16} = (c_{13} - c_{23})(c_{16} + c_{26}) - (c_{11} - c_{22})c_{36},$$
$$E_{17} = (c_{44} - c_{55})(c_{16} + c_{26}) + (c_{11} - c_{22})c_{45}.\tag{7}$$

Ting (1987) had previously reported 15 invariants, and these can be shown (Ahmad, 2002) to be contained in Ahmad’s larger set. Ahmad also demonstrated that the 32 quadratic invariants formed from the 15 quadratic combinations of $L_1, \ldots, L_5$ plus the 17 invariants $E_1, \ldots, E_{17}$ are independent of one another. The question of completeness remained open.

The purpose of this paper is to finish the study initiated by Ting (1987) and subsequently expanded by Ahmad (2002). Two principal results are derived: first that the seven quadratic invariants under SO(3) identified by Ahmad (2002) are indeed complete, and hence any quadratic isotropic invariant must be a linear combination of these. Secondly, we prove that there are 35 quadratic invariants under SO(2). A complete basis for the 35-dimensional space of quadratic invariants under SO(2) is formed by the 32 of Ahmad (2002) augmented by the these three,

$$E_{18} = (c_{15} + c_{46})c_{14} - (c_{24} + c_{56})c_{25} - c_{15}c_{56} + c_{24}c_{46},$$
$$E_{19} = (c_{15} + c_{46})c_{34} - (c_{24} + c_{56})c_{35},$$
$$E_{20} = (c_{15} + c_{25})c_{34} - (c_{14} + c_{24})c_{35}.\tag{8}$$

We consider quadratic forms on $E_{la}$, the space of tensors of elastic moduli\footnote{To be precise, $E_{la}$ comprises tensors that are positive definite, but that distinction is not necessary here as we are concerned with quadratic forms on the moduli not defined by the moduli.} which possess the underlying symmetries

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{klij},$$

implying 21 independent elements, at most. The general quadratic form on $E_{la}$ is

$$\phi = F_{ijklpqrs}C_{ijkl}C_{pqrs},$$

where $F_{ijklpqrs}$ are the elements of an eighth order tensor. Based on the properties (9) of the moduli, the elements of $F$ satisfy

$$F_{ijklpqrs} = F_{jiklpqrs}, \quad F_{ijklpqrs} = F_{klijpqrs}, \quad F_{ijklpqrs} = F_{pqrsijkl}.\tag{11}$$
Identifying quadratic invariants is therefore equivalent to describing the properties of the eighth order tensor $F$, in particular the number of independent elements that survive under the group of transformations considered. The problem is linked to that of finding the integrity basis for fourth order tensors. As noted in a review on tensor functions: "Relatively little is known about representations of functions of tensors of order higher than two, for any of the transformation groups of interest in continuum mechanics" (Rychlewski and Zhang, 1991, p. 83). Considerable work has been done on this topic for second order tensors since this is critical to the form of elastic strain energy functions (Spencer, 1971) (see Xiao (1996) for recent developments on $n=2$). However, this literature is not directly applicable since we are concerned with tensors of higher order than normally considered.

The results derived here concern properties of 8th order symmetric tensors, and the number of constants they possess under transverse isotropy and isotropy, subject to the indicial symmetries (11).

We begin in Section 2 with a summary of the main results. Although the focus of the paper deals with transformations caused by rotation, it will prove useful to first consider invariance under reflection. Section 3 introduces the notion of invariance under reflection about a plane, and quadratic invariants of the elastic moduli are derived in Section 4 for transformations under one and two reflections. Transformations under rotation about an axis is then considered in Section 5 where the results are proved. Finally, issues of consistency and completeness are discussed in Section 6.

A note on notation: The vector triad $e_i$, $i = 1, 2, 3$ is an orthonormal basis in 3-dimensions. The summation convention on repeated suffices is assumed.

2 Summary of the principal results

The results of the paper are summarized in the form of two theorems, with the proofs given in the subsequent Sections.

Define the 21-vector of elastic moduli

$$c = \left(c_{11} c_{22} c_{33} c_{12} c_{13} c_{44} c_{45} c_{55} c_{66} c_{24} c_{25} c_{34} c_{35} c_{14} c_{15} c_{16} c_{26} c_{17} c_{27} c_{37} c_{47} c_{57} c_{67} c_{45} c_{46} c_{47} c_{56} c_{57} c_{67} \right)^t,$$

then we have the following:

**Theorem 1** All quadratic invariants of $c$ under $SO(3)$ are of the form $c^t F c$ where the $21 \times 21$ symmetric matrix $F$ is of the form

$$F = \begin{pmatrix}
A_{9 \times 9} & 0_{9 \times 12} \\
0_{12 \times 9} & B_{12 \times 12}
\end{pmatrix},$$

with symmetric matrices $A$ and $B$ defined by 12 and 7 independent elements, respectively. These matrices have the form

$$A = \begin{pmatrix}
 f_{11} & f_{12} & f_{12} & f_{14} & f_{15} & f_{15} & f_{17} & f_{18} & f_{18} \\
 f_{11} & f_{12} & f_{12} & f_{14} & f_{15} & f_{15} & f_{17} & f_{18} & f_{18} \\
 f_{11} & f_{15} & f_{15} & f_{14} & f_{18} & f_{18} & f_{17} & f_{18} & f_{17} \\
 f_{44} & f_{45} & f_{45} & f_{47} & f_{48} & f_{48} & f_{47} & f_{48} & f_{48} \\
 f_{44} & f_{45} & f_{48} & f_{47} & f_{48} & f_{48} & f_{47} & f_{48} & f_{48} \\
 f_{77} & f_{78} & f_{78} & f_{77} & f_{77} & f_{77}
\end{pmatrix},$$

$$B = \begin{pmatrix}
 Y & M \\
 f_{77} & f_{78} & f_{78} & f_{77} & f_{77} & f_{77}
\end{pmatrix}. $$
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Quadratic invariants

are arbitrary as long as they satisfy the 12 linearly independent conditions

\[ f_{11}, f_{12}, f_{14}, f_{15}, f_{17}, f_{18}, f_{44}, f_{45}, f_{47}, f_{48}, f_{77}, f_{78}, b_{11}, b_{14}, b_{11,10}, b_{44}, b_{47}, b_{4,10}, b_{10,10}; \]

are arbitrary as long as they satisfy the 12 linearly independent conditions

\[
2f_{11} + 2f_{12} - 4f_{15} + f_{44} - f_{77} = 0, \quad (16a)
\]
\[
2f_{11} + 2f_{12} - 2f_{18} - f_{44} - f_{47} = 0, \quad (16b)
\]
\[
2f_{12} - f_{14} - f_{17} = 0, \quad (16c)
\]
\[
f_{14} + f_{15} - f_{45} - f_{48} = 0, \quad (16d)
\]
\[
f_{17} + f_{18} - f_{48} - f_{78} = 0, \quad (16e)
\]
\[
2f_{44} - 2f_{15} - b_{11} = 0, \quad (16f)
\]
\[
2f_{14} - 2f_{15} + b_{14} = 0, \quad (16g)
\]
\[
2f_{47} - 2f_{48} - b_{110} = 0, \quad (16h)
\]
\[
4f_{11} - 2f_{15} - 2f_{18} - b_{14} = 0, \quad (16i)
\]
\[
4f_{12} - 2f_{15} - 2f_{18} + b_{47} = 0, \quad (16j)
\]
\[
2f_{17} - 2f_{18} + b_{410} = 0, \quad (16k)
\]
\[
2f_{77} - 2f_{78} - b_{10,10} = 0. \quad (16l)
\]

The number of linearly independent quadratic invariants under SO(3) is seven.

An immediate corollary of Theorem 1 is that the seven invariants derived by Ahmad (2002), based partly on the work of Ting (1987), are linearly independent and complete. The second result is

**Theorem 2** All quadratic invariants of \( c \) under SO(2) are of the form \( c^t F c \) where the 21×21 symmetric matrix \( F \) is

\[
F = F^{(1)} + F^{(2)},
\]

\( F^{(1)} \) and \( F^{(2)} \) define subspaces of dimension 29 and 6, respectively, and

\[
F^{(1)} = \begin{pmatrix}
A_{9\times 9} & 0_{9\times 12} \\
0_{12\times 9} & B^{(1)}_{12\times 12}
\end{pmatrix}, \quad F^{(2)} = \begin{pmatrix}
0_{9\times 9} & D_{9\times 12} \\
D^{(1)}_{12\times 9} & B^{(2)}_{12\times 12}
\end{pmatrix}.
\]

The symmetric matrices \( A, B^{(1)} \) and \( B^{(2)} \) and the matrix \( D \) are defined by 17, 27, 3 and 3 independent elements, respectively.

\[
B = \begin{pmatrix}
b_{11} & 0 & 0 & b_{14} & 0 & 0 & b_{14} & 0 & 0 & b_{110} & 0 & 0 \\
b_{11} & 0 & 0 & b_{14} & 0 & 0 & b_{14} & 0 & 0 & b_{110} & 0 \\
b_{11} & 0 & 0 & b_{14} & 0 & 0 & b_{14} & 0 & 0 & b_{110} & 0 \\
b_{44} & 0 & 0 & b_{47} & 0 & 0 & b_{47} & 0 & 0 & b_{110} & 0 \\
b_{44} & 0 & 0 & b_{47} & 0 & 0 & b_{47} & 0 & 0 & b_{110} & 0 \\
b_{44} & 0 & 0 & b_{47} & 0 & 0 & b_{47} & 0 & 0 & b_{110} & 0 \\
b_{44} & 0 & 0 & b_{47} & 0 & 0 & b_{47} & 0 & 0 & b_{110} & 0 \\
b_{44} & 0 & 0 & b_{47} & 0 & 0 & b_{47} & 0 & 0 & b_{110} & 0 \\
b_{10,10} & 0 & 0 & b_{10,10} & 0 & 0 & b_{10,10} & 0 & 0 & b_{10,10}
\end{pmatrix}, \quad (15)
\]
With the axis of rotation \( \mathbf{e}_3 \),

\[
A = \begin{pmatrix}
  f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{17} & f_{18} & f_{19} \\
  f_{11} & f_{13} & f_{14} & f_{15} & f_{16} & f_{17} & f_{18} & f_{19} & f_{33} & f_{34} & f_{35} & f_{36} & f_{37} & f_{38} & f_{39} \\
  f_{44} & f_{45} & f_{46} & f_{47} & f_{48} & f_{49} & f_{54} & f_{55} & f_{56} & f_{57} & f_{58} & f_{59} & f_{66} & f_{67} & f_{68} & f_{69} \\
  f_{77} & f_{78} & f_{79} & f_{88} & f_{99} \\
  S & Y & M
\end{pmatrix},
\]

(19)

\[
B^{(1)} = \begin{pmatrix}
  b_{11} & 0 & 0 & b_{14} & 0 & 0 & b_{17} & 0 & 0 & b_{110} & 0 & 0 \\
  b_{11} & 0 & 0 & b_{17} & 0 & 0 & b_{14} & 0 & 0 & b_{110} & 0 \\
  b_{33} & 0 & 0 & b_{36} & 0 & 0 & b_{36} & 0 & 0 & b_{312} \\
  b_{44} & 0 & 0 & b_{47} & 0 & 0 & b_{410} & 0 & 0 \\
  b_{55} & 0 & 0 & b_{57} & 0 & 0 & b_{511} & 0 \\
  b_{66} & 0 & 0 & b_{69} & 0 & 0 & b_{612} \\
  b_{55} & 0 & 0 & b_{511} & 0 & 0 \\
  b_{44} & 0 & 0 & b_{410} & 0 \\
  b_{66} & 0 & 0 & b_{612} \\
  b_{1010} & 0 & 0 \\
  b_{1010} & 0 \\
  b_{1212}
\end{pmatrix}.
\]

(20)

The elements \( f_{33}, f_{34}, f_{37} \) of \( A \) and \( b_{44} \) of \( B^{(1)} \) are arbitrary, while the remaining 40 distinct elements in \( A \) and \( B^{(1)} \) are arbitrary as long as they satisfy the 15 linearly independent conditions

\begin{align*}
2f_{11} + 2f_{12} - 4f_{16} + f_{66} - f_{99} &= 0, \quad (21a) \\
2f_{11} + 2f_{12} - 2f_{19} - f_{66} - f_{69} &= 0, \quad (21b) \\
2f_{13} - f_{36} - f_{39} &= 0, \quad (21c) \\
f_{14} + f_{15} - f_{46} - f_{49} &= 0, \quad (21d) \\
f_{17} + f_{18} - f_{67} - f_{79} &= 0, \quad (21e) \\
2f_{44} - 2f_{45} - b_{33} &= 0, \quad (21f) \\
2f_{41} - 2f_{15} - b_{36} &= 0, \quad (21g) \\
2f_{47} - 2f_{48} - b_{312} &= 0, \quad (21h) \\
4f_{11} - 2f_{16} - 2f_{19} - b_{66} &= 0, \quad (21i) \\
4f_{12} - 2f_{16} - 2f_{19} + b_{69} &= 0, \quad (21j) \\
2f_{17} - 2f_{18} + b_{612} &= 0, \quad (21k) \\
2f_{77} - 2f_{78} - b_{1212} &= 0, \quad (21l) \\
b_{14} - b_{47} + b_{410} &= 0, \quad (21m) \\
b_{11} - 2b_{17} + b_{55} - b_{1010} &= 0, \quad (21n) \\
b_{11} + b_{110} - b_{55} + b_{511} &= 0. \quad (21o)
\end{align*}
Also,
\[
B^{(2)} = \begin{pmatrix}
0 & 0 & 0 & 0 & b_{15} & 0 & 0 & b_{18} & 0 & 0 & b_{15} & 0 \\
0 & 0 & -b_{18} & 0 & 0 & -b_{15} & 0 & 0 & -b_{15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_{45} & 0 & 0 & 0 & 0 & 0 & 0 & b_{1,8} + b_{4,5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -b_{15} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -b_{45} & 0 & 0 & 0 & 0 & b_{15} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -b_{1,8} - b_{4,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S & Y & M
\end{pmatrix},
\] (22)

and
\[
D = \begin{pmatrix}
0 & 0 & d_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{1,12} \\
0 & 0 & -d_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d_{1,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_{13} & 0 & 0 & 0 & 0 & d_{1,12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -d_{13} & 0 & 0 & -d_{13} & 0 & 0 & -d_{1,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -d_{4,12} & 0 & 0 & d_{1,12} & 0 & 0 & d_{1,12} & 0 & 0 & 0 & 0 \\
0 & 0 & d_{4,12} & 0 & 0 & -d_{1,12} & 0 & 0 & -d_{1,12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\] (23)

where the 6 elements \(b_{15}, b_{18}, b_{45}, d_{1,3}, d_{1,12}, d_{4,12}\) are arbitrary.

The number of linearly independent quadratic invariants under SO(2) is thirty five.

## 3 Invariants under reflection about a plane

### 3.1 Linear and quadratic invariants under reflection

In order to fix ideas consider a tensor of order one, a vector \(v = v_i e_i\). Invariants are defined in relation to a transformation, usually associated with some material symmetry. The simplest transformation is that of reflection about a plane orthogonal to a direction \(e\), which we denote \(R(e)\). This transformation is associated with monoclinic symmetry (Cowin and Mehrabadi, 1995). The action of \(R(e)\) on \(v\) is defined by
\[
R(e)v = v - 2(v \cdot e)e.
\] (24)

Thus all \(v\) orthogonal to \(e\) are unchanged, or invariant under \(R(e)\). The linear invariant of an arbitrary vector may be defined as its projection onto the invariant subspace, i.e. \(v - (v \cdot e)e\). The number of linear invariants is the dimension of the invariant subspace, in this case 2.

Quadratic invariants are defined by quadratic forms, which for a vector require a symmetric second order tensor, say \(F = F^t\). Let
\[
\phi(v) = v \cdot F \cdot v,
\] (25)

then we seek \(F\) which leave \(\phi\) unchanged under the action of \(R(e)\) on \(v\):
\[
\phi = R(e)\phi \quad \text{where } R(e)\phi \equiv \phi(R(e)v).
\] (26)
Since
\[ R(e)\phi = [v - 2(v \cdot e)] \cdot F \cdot [v - 2(v \cdot e)] = \phi - 4(v \cdot e)[v - (v \cdot e)e] \cdot F \cdot e, \]  
we see that \( R(e)\phi = \phi \) under two circumstances: (i) for all \( F \) such that \( F \cdot e = 0 \), which defines a three dimensional subspace of second order symmetric tensors; (ii) for all \( F \) of the form \( F = \mu e \otimes e \), a one dimensional subspace. It may be checked that all quadratic forms that leave \( \phi \) fixed must be combinations of these, and hence the number of quadratic invariants for vectors is \( 3+1=4 \).

As we deal with tensors of higher order it is simpler to work with the components relative to the basis \( e_i, i = 1, 2, 3 \). To be specific, we consider \( e = e_3 \), then the 2 linear invariants are \( v_1 \) and \( v_2 \). Similarly, the 4 quadratic invariants are defined by \( \phi \) of the form \( (25) \) where \( f_{13} = f_{23} = 0 \) but \( F \) is otherwise arbitrary. In summary,
\[ v = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}, \quad F = \begin{pmatrix} f_{11} & f_{12} & 0 \\ f_{12} & f_{22} & 0 \\ 0 & 0 & f_{33} \end{pmatrix}, \quad n = 1 \text{ under } R(e_3), \]  
where \( n = 1 \) indicates that these are the linear and quadratic invariants for tensors of order \( n \).

Note that we do not present the quadratic invariants as, for instance, the set \( \{v_1^2, v_2^2, v_1 v_2, v_3^2\} \), but use notions from linear algebra which are natural for quadratic forms.

### 3.1.1 \( n=2 \)

A second order symmetric tensor defines a unique quadratic form, through eq. \( (25) \). Based on the analysis for \( n = 1 \) we can immediately identify the linear invariants under \( R(e_3) \) any second order symmetric tensor \( V = v_{ij}e_i \otimes e_j \) as \( v_{11}, v_{22}, v_{12}, v_{33} \).

Quadratic forms on \( V \) are
\[ \phi = v_{ij}F_{ijkl}v_{kl}, \]  
where the elements of \( F \) satisfy
\[ F_{ijkl} = F_{jikl}, \quad F_{ijkl} = F_{klij}. \]  
These are the same relations that define \( E_{la} \), see eq. \( (9) \). Thus, the symmetries of \( F_{ijkl} \) under \( R(e_3) \) follow by standard arguments: all elements with index 3 occurring an odd number of times must vanish \( (\text{Cowin and Mehrabadi, 1995}) \). In summary,
\[ V = \begin{pmatrix} v_{11} & v_{12} & 0 \\ v_{12} & v_{22} & 0 \\ 0 & 0 & v_{33} \end{pmatrix}, \quad F = \begin{pmatrix} f_{11} & f_{12} & f_{13} & 0 & 0 & f_{16} \\ f_{12} & f_{22} & f_{23} & 0 & 0 & f_{16} \\ f_{13} & f_{23} & f_{33} & 0 & 0 & f_{36} \\ 0 & 0 & 0 & f_{44} & 0 & f_{46} \\ 0 & 0 & 0 & 0 & f_{55} & 0 \\ f_{16} & f_{26} & f_{36} & 0 & 0 & f_{66} \end{pmatrix}, \quad n = 2 \text{ under } R(e_3). \]  

The thirteen elements of \( F \) define the subspace of \( E_{la} \) invariant under the single reflection \( R(e_3) \). That is, there are 13 linear invariants for fourth order symmetric elasticity tensors. Since these tensors are defined by the quadratic form (energy) acting on second order symmetric tensors (strain) the 13 linear invariants of \( n = 4 \) correspond to the quadratic invariants of \( n = 2 \), in the same way that the linear invariants for \( n = 2 \) correspond to the quadratic invariants of \( n = 1 \), eqs. \( (28) \) and \( (31) \). It should be clear that the same equivalence holds for arbitrary tensor order \( n \), and the result may be stated as follows:

**Lemma 1** The quadratic invariants of a tensor of order \( n \) are defined by the linear invariants of the corresponding symmetric tensor of order \( 2n \).
3.2 Remarks on reflections and rotations

We are concerned with invariants under the group of proper orthogonal transformations, SO(3), and the group of rotations about an axis, SO(2). The orthogonality requires that right-handed triads of coordinate axes remain right-handed under transformation, or equivalently that the determinants of the 3×3 transformation matrices are +1. But three dimensional inversion is also acceptable by virtue of the fact that the tensor we are concerned with is of even order, viz. 8. Therefore, SO(3) should be considered extended, to include the group $I_3$ of transformations $1, 2, 3 \to -1, 2, 3$, etc. Thus $O(3) = SO(3) \cup I_3$. Similarly, $O(2) = SO(2) \cup I_2$ where $I_2$ is the group of transformations formed from $1, 2 \to -1, 2; 1, 2 \to 1, -2$. Another way to understand why we can replace $SO(2) \to O(2)$ is to consider reflection about a plane, say $R(e_3)$, followed by inversion. This is equal to rotation about the normal to the plane by $\pi$, and hence $O(2)$ is equivalent to $SO(2) \cup R(e_3)$. We will find this particularly useful as a starting point later, and it motivates consideration of the invariants under the action of $R(e_3)$ first.

We are now ready to determine the quadratic invariants of $E\!\!\!l\!\!\!a$, starting with invariants under reflection.

4 Invariants of elastic moduli under reflection

We examine $E\!\!\!l\!\!\!a$ under reflection first about a single plane, and then about two. Based on the Voigt notation for indexing elastic moduli with 21 elements, the indices for $F$ of (10) and (11) as a square symmetric matrix $f_{ij}$ run from 1, 2, …, 21, according to the following

$$i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$$

$$IJ = \{11, 22, 33, 23, 12, 44, 55, 66, 14, 25, 36, 34, 15, 26, 24, 35, 16, 56, 46, 45\}$$

where $IJ$ are the Voigt indices, which are capital suffices taking the values 1, 2, …, 21, and the 3×3 coordinate axes remain right-handed under transformation, or equivalently that the determinants of the 3×3 transformation matrices are +1. But some of these coincide with members of $O(2)$: the elements indicated by the indices in boldface, plus the elements defined by the remaining indices. The elements that are distinct from (33) are

$$f_{j_3k_3} = 0, \quad j_3 \in \{1, \ldots, 9, 12, 15, 18, 21\}, \quad k_3 \in \{10, 11, 13, 14, 16, 17, 19, 20\}. \quad (33)$$

That leaves $231 - 104 = 127$ non-zero elements that remain under reflection about a symmetry plane, or monoclinic symmetry.

Next consider reflection about two planes with perpendicular normals, say $R(e_3)$ and $R(e_1)$. As before, elements that have indices $i j k l p q r s$ where 1 occurs an odd number of times must vanish. The 104 elements with 1 occurring an odd number of times are

$$f_{j_1k_1} = 0, \quad j_1 \in \{1, \ldots, 9, 10, 13, 16, 19\}, \quad k_1 \in \{11, 12, 14, 15, 17, 18, 20, 21\}. \quad (34)$$

But some of these coincide with members of $O(2)$: the elements indicated by the indices in boldface, plus the elements defined by the remaining indices. The elements that are distinct from (33) are

$$f_{j_1k_1} = 0, \quad \tilde{j}_1 \in \{1, \ldots, 9\}, \quad \tilde{k}_1 \in \{12, 15, 18, 21\}, \quad (35)$$

and the $4 \times 4 = 16$:

$$f_{j_1k_1} = 0, \quad \hat{j}_1 \in \{10, 13, 16, 19\}, \quad \hat{k}_1 \in \{11, 14, 17, 20\}, \quad (36)$$

In summary, $F$ is of the form

$$F = \begin{pmatrix} A_{9\times 9} & 0_{9\times 12} \\ 0_{12\times 9} & B_{12\times 12} \end{pmatrix}, \quad (37)$$
where $A$ and $B$ are symmetric. The matrix $A$ is full, indicating $9 \times 10/2 = 45$ independent elements, and letting the indices of $B$ run from 1 to 12, we have
\[ b_{ij} = 0, \quad (ij) \in \{(3, 6, 9, 12) \times (1, 2, 4, 5, 7, 8, 10, 11)\} \oplus \{(1, 4, 7, 10) \times (2, 5, 8, 11)\}. \] (38)

Therefore, $B$ has $12 \times 13/2 - 4 \times 12 = 30$ for a total of 75 independent elements in $F$. In fact, $B$ has banded structure
\[
B = \begin{pmatrix}
    b_{11} & 0 & 0 & b_{14} & 0 & 0 & b_{17} & 0 & 0 & b_{10} & 0 & 0 \\
    0 & 0 & b_{22} & 0 & 0 & b_{25} & 0 & 0 & b_{28} & 0 & 0 & b_{211} & 0 \\
    0 & 0 & b_{33} & 0 & 0 & b_{36} & 0 & 0 & b_{39} & 0 & 0 & b_{312} & 0 \\
    b_{44} & 0 & 0 & b_{47} & 0 & 0 & b_{410} & 0 & 0 & b_{511} & 0 \\
    0 & 0 & b_{55} & 0 & 0 & b_{58} & 0 & 0 & b_{511} & 0 \\
    b_{66} & 0 & 0 & b_{69} & 0 & 0 & b_{612} & 0 \\
    b_{77} & 0 & 0 & b_{710} & 0 & 0 \\
    0 & 0 & b_{88} & 0 & 0 & b_{811} & 0 \\
    b_{99} & 0 & 0 & b_{912} & 0 \\
    b_{1010} & 0 & 0 \\
    0 & 0 & b_{1111} & 0 \\
    0 & 0 & b_{1212} & 0
\end{pmatrix}. \] (39)

Note the indices for $B$ as a square symmetric matrix $b_{ij}$ run from 1, 2, . . . , 12, according to the following

\[
\begin{array}{c}
\text{B} \\
\text{Voigt} \\
_\text{F(21} \times 21) \\
\end{array}
\begin{array}{c}
i = 1 \\
I J = 14 \\
\times 12 \\
\end{array}
\begin{array}{c}
2 \\
36 \\
34 \\
15 \\
26 \\
24 \\
35 \\
16 \\
56 \\
46 \\
45 \\
\end{array}
\]

The 8th order tensor $F$ of eqs. (37) and (39) is unchanged under reflection about a third plane orthogonal to the others, i.e. $R(\mathbf{e}_2)$. In order to see this, note that $R(\mathbf{e}_2)$ implies that all elements with index 2 occurring an odd number of times must vanish. However, since the tensor is of even order, and we have eliminated elements with 1 and 3 occurring odd numbers of times, the tensor is automatically invariant under $R(\mathbf{e}_2)$. This effect is well known in elasticity: that a material with two orthogonal planes of symmetry automatically has a third (Cowin and Mehrabadi, 1995).

In summary,

Lemma 2 All quadratic invariants of $c$ under reflection about a plane are of the form $c^t F c$ where the $21 \times 21$ symmetric matrix $F$ is
\[
F = \begin{pmatrix}
    A_{9 \times 9} & D_{9 \times 12} \\
    D_{12 \times 9} & B_{12 \times 12}
\end{pmatrix},
\] (40)

$A = A^t$ is full and $B = B^t$ and $D$ are of banded form,
\[
B = \begin{pmatrix}
    b_{11} & b_{12} & 0 & b_{14} & b_{15} & 0 & b_{17} & b_{18} & 0 & b_{10} & 0 & b_{11} & 0 \\
    0 & 0 & b_{22} & b_{24} & b_{25} & 0 & b_{27} & b_{28} & 0 & b_{210} & 0 & b_{211} & 0 \\
    0 & 0 & b_{33} & 0 & b_{36} & 0 & b_{39} & 0 & 0 & b_{312} & 0 \\
    b_{44} & 0 & b_{45} & 0 & b_{47} & b_{48} & 0 & b_{410} & 0 & b_{411} & 0 \\
    0 & b_{55} & 0 & b_{57} & b_{58} & 0 & b_{510} & b_{511} & 0 \\
    0 & 0 & b_{66} & 0 & b_{69} & 0 & 0 & b_{612} & 0 \\
    b_{77} & 0 & b_{78} & 0 & b_{710} & b_{711} & 0 \\
    0 & b_{88} & 0 & b_{810} & b_{811} & 0 \\
    b_{99} & 0 & 0 & b_{912} & 0 \\
    b_{1010} & 0 & 0 & b_{1011} & 0 \\
    b_{1111} & 0 & 0 & b_{1212} & 0
\end{pmatrix}. \] (41)
and

\[ D = \begin{pmatrix}
0 & d_{13} & 0 & d_{16} & 0 & 0 & d_{19} & 0 & 0 & d_{112} \\
0 & d_{23} & 0 & d_{26} & 0 & 0 & d_{29} & 0 & 0 & d_{212} \\
0 & d_{33} & 0 & d_{36} & 0 & 0 & d_{39} & 0 & 0 & d_{312} \\
0 & d_{43} & 0 & d_{46} & 0 & 0 & d_{49} & 0 & 0 & d_{412} \\
0 & d_{53} & 0 & d_{56} & 0 & 0 & d_{59} & 0 & 0 & d_{512} \\
0 & d_{63} & 0 & d_{66} & 0 & 0 & d_{69} & 0 & 0 & d_{612} \\
0 & d_{73} & 0 & d_{76} & 0 & 0 & d_{79} & 0 & 0 & d_{712} \\
0 & d_{83} & 0 & d_{86} & 0 & 0 & d_{89} & 0 & 0 & d_{812} \\
0 & d_{93} & 0 & d_{96} & 0 & 0 & d_{99} & 0 & 0 & d_{912}
\end{pmatrix}, \quad (42)

where the second index of \( D \) is the same as that for \( B \), i.e. runs from 1, 2, \ldots, 12. The number of linearly independent quadratic invariants is 127.

Under reflection about two orthogonal planes the number of linearly independent quadratic invariants reduces to 75, and \( F \) then has the form \( \text{(40)} \) with \( D \equiv 0 \) and the 16 elements in bold in \( \text{(41)} \) are zero, i.e. \( B \) reduces to \( B^{(1)} \) of eq. \( \text{(20)} \).

Thus, invariance under reflection about one and then two orthogonal planes reduces the number of independent elements in \( F \) from 231 to 127 to 75. The case of a single plane is monoclinic symmetry, and two (or three) corresponds to orthorhombic symmetry.

Reflection symmetry is an important tool in developing the structure of anisotropic elastic tensors (Cowin and Mehrabadi, 1995). In fact, all eight fundamental symmetries can be cast in terms of repeated application of the reflection operator (Chadwick et al., 2001). The definition of transverse isotropy implies that every plane containing the axis of symmetry is a symmetry plane, in addition to the plane orthogonal to the axis. However, this equivalence does not hold in the present problem, dealing with the form of the 8\textsuperscript{th} order tensor \( F \) invariant under SO(2). Consider rotation about \( e_3 \) by \( \pi \), which transforms \( \{e_1, e_2\} \rightarrow \{e_2, -e_1\} \). Since elements with odd numbered occurrences of 3 have already been eliminated by the requirement of invariance under \( R(e_3) \), it therefore eliminates terms with 1 (or 2) occurring an odd number of times. Thus, \( c_{3312} = 0 \) for TI elasticity, but it does not eliminate elements such as \( f_{33122233} \) from \( F \). Hence SO(2) invariance for \( F \) is not the same as invariance under \( R(e) \) for all \( e \perp e_3 \).

In other words, SO(2) invariance of \( F \) is not a subspace of orthorhombic symmetry, but it is a subspace of monoclinic symmetry. This allows us to start the search for SO(2) invariance with only 127 of the 231 elements. Furthermore, SO(3) invariance of \( F \) is a proper subspace of orthorhombic symmetry, meaning it can be sought starting from the 75 element form of \( F \) in \( \text{(63)} \).

First we need to introduce the rotation matrix for \( \text{Ela} \).

5 Invariants under rotation about an axis

We commence by casting the problem in general terms, with some simpler examples before considering the problem for the 8\textsuperscript{th} order tensor.

5.1 General theory

Let \( c \) be a vector, not restricted to the elastic modulus vector, and consider the quadratic form

\[ \phi = c^t F c, \quad (43) \]

where \( F \) is symmetric\(^2\)

\[ F = F^t. \quad (44) \]

\(^2\)If \( F = M + S \) where \( M = M^t \) and \( S = -S^t \) then \( \phi \) defined in \( \text{(43)} \) is \( \phi = c^t F c \) independent of \( S \). The skew-symmetric part of \( F \) is irrelevant and can be set to zero.
Under rotation through angle $\theta$ about axis $p$, a 3-vector,
\begin{equation}
    c \rightarrow c' = Qc, \tag{45}
\end{equation}
where $Q = Q(\theta, p)$ is the rotation. Accordingly,
\begin{equation}
    \phi \rightarrow \phi'(\theta) = c'Q^tFQc. \tag{46}
\end{equation}
We want $\phi'(\theta)$ to be independent of $\theta$, that is, its derivative should vanish for all $\theta$. The derivative is
\begin{equation}
    \dot{\phi'} = c'Q^tF\dot{Q}c + c'\dot{Q}^tFQc, \tag{47}
\end{equation}
where the dot indicates differentiation with respect to $\theta$. By an application of standard Lie group and Lie algebra theory (Fegan, 1991), $Q(\theta)$ is a one parameter subgroup of SO($N$) where $N$ is the vector length. $Q$ is generated by $P = -P^t$, an element from its Lie algebra of skew-symmetric matrices. Thus, $Q(\theta) = \exp \theta P$, from which the derivative is
\begin{equation}
    \dot{Q} = PQ. \tag{48}
\end{equation}
The particular form of $P$ for elasticity tensors ($N = 21$) was derived by Norris (2007).

Equations (47) and (48) imply
\begin{equation}
    \dot{\phi'} = c'(Q^tFPQ - Q^tPFQ)c. \tag{49}
\end{equation}
Note that the matrix $(Q^tFPQ - Q^tPFQ)$ is symmetric, hence $\phi'(\theta) = 0$ for all possible $c$ implies the SO(2) invariance condition:
\begin{equation}
    Q^t(FP - PF)Q = 0 \quad \text{for all } \theta. \tag{50}
\end{equation}
Hence we have

**Lemma 3** The quadratic form $\phi = c^tFc$ is an SO(2) invariant if and only if $F$ commutes with the generator $P$:
\begin{equation}
    FP - PF = 0. \tag{51}
\end{equation}
Equation (51) is a necessary condition since (50) must hold at $\theta = 0$ where $Q(0) = I$. The sufficiency of (51) is clear, hence eq. (51) is the condition that $F$ must satisfy.

The question of quadratic invariants under SO(2) therefore reduces to finding all symmetric $F$ which commute with the skew rotation generator $P$, or equivalently, sym($FP$) vanishes. We next demonstrate this approach by application to increasingly higher order tensors, associated with quadratic forms on tensors of order $n = 1, 2$ and ultimately the desired $n = 4$.

### 5.2 Quadratic forms on vectors: $n=1$

In this case the generator is the standard and well known skew symmetric second order tensor defined by the axis of rotation $p$,
\begin{equation}
    P_{ij} = -\epsilon_{ijk}p_k. \tag{52}
\end{equation}
Let $p = e_3$ so that
\begin{equation}
    P = \begin{pmatrix}
        0 & 1 & 0 \\
        -1 & 0 & 0 \\
        0 & 0 & 0
    \end{pmatrix}, \tag{53}
\end{equation}
and let
\begin{equation}
    F = \begin{pmatrix}
        f_{11} & f_{12} & f_{13} \\
        f_{12} & f_{22} & f_{23} \\
        f_{13} & f_{23} & f_{33}
    \end{pmatrix}. \tag{54}
\end{equation}
Then
\[ FP - PF = \begin{pmatrix} -2f_{12} & f_{11} - f_{22} & -f_{23} \\ f_{11} - f_{22} & 2f_{12} & f_{13} \\ -f_{23} & f_{13} & 0 \end{pmatrix}. \quad (55) \]

Setting each of the elements to zero gives four conditions: \( f_{11} - f_{22} = f_{12} = f_{13} = f_{23} = 0 \), so that the most general \( F \) is of the form
\[ F = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad (56) \]
for \( a, b \neq 0 \).

The associated invariants are
\[ \phi_a = c_1^2 + c_2^2, \quad \phi_b = c_3^2. \quad (57) \]
For arbitrary direction \( p \) these are
\[ \phi_a = c^t c - (p^t c)^2, \quad \phi_b = (p^t c)^2, \quad (58) \]
corresponding to
\[ F_a = -P^2, \quad F_b = I + P^2. \quad (59) \]
Any \( F \) comprised of even powers of \( P \) will obviously commute with \( P \). For \( n = 1 \) there are only two independent \( F \) of this form, i.e. \( F = I, P^2 \), since \( P \) satisfies the characteristic equation
\[ P^3 + P = 0, \quad (60) \]
and hence \( P^{2+2m} = (-1)^m P^2 \).

### 5.3 Quadratic forms on tensors: \( n=2 \)

The relation between the second order symmetric tensor \( C \) and the 6-vector \( c \) is
\[ C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} \Rightarrow c \equiv (C_{11} \ C_{22} \ C_{33} \ \sqrt{2}C_{23} \ \sqrt{2}C_{13} \ \sqrt{2}C_{12})^t. \quad (61) \]

\( P \) is now a 6×6 skew symmetric matrix,
\[ P = \begin{pmatrix} 0 & \sqrt{2}(Z-Y^t) \\ \sqrt{2}(Y^t-Z) & X-X^t \end{pmatrix}. \quad (62) \]
where
\[ X = \begin{pmatrix} 0 & p_3 & 0 \\ 0 & 0 & p_1 \\ p_2 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & p_2 & 0 \\ 0 & 0 & p_3 \\ p_1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & p_2 \\ p_3 & 0 & 0 \end{pmatrix}. \quad (63) \]

The matrix \( P \) was first derived by Mehrabadi et al. (1995), and the present form is due to Norris (2007). We will find this useful when we consider the analogous \( P \) for \( n = 4 \) later. We note that the characteristic equation of \( P \) is now of fifth \((2n+1)\) order (Mehrabadi et al., 1995; Norris, 2007),
\[ P(P^2 + I)(P^2 + 4I) = 0. \quad (64) \]
Let \( p = e_3 \) then

\[
P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\
0 & 0 & 0 & 0 & 0 & \sqrt{2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{65}
\]

and the SO(2) commutator is

\[
FP - PF = \begin{pmatrix}
-2f_{16} & f_{16} - f_{26} & -f_{36} & \frac{1}{\sqrt{2}}f_{15} - f_{46} & -\frac{1}{\sqrt{2}}f_{14} - f_{56} & f_{11} - f_{12} - f_{66} \\
2f_{26} & f_{36} & \frac{1}{\sqrt{2}}f_{25} + f_{46} & -\frac{1}{\sqrt{2}}f_{24} + f_{56} & f_{12} - f_{22} + f_{66} \\
0 & \frac{1}{\sqrt{2}}f_{35} & -\frac{1}{\sqrt{2}}f_{34} & f_{13} - f_{23} \\
\sqrt{2}f_{45} & \frac{1}{\sqrt{2}}(f_{55} - f_{44}) & f_{14} - f_{24} + \frac{1}{\sqrt{2}}f_{56} \\
S & Y & M & -\sqrt{2}f_{45} & f_{15} - f_{25} - \frac{1}{\sqrt{2}}f_{46} \\
2f_{16} - 2f_{26}
\end{pmatrix}. \tag{66}
\]

Setting all elements to zero and solving the resulting linear equations, we find that \( F \) has five independent elements (and the expected form of a TI tensor of elastic moduli),

\[
F = \begin{pmatrix}
f_{11} & f_{12} & f_{13} & 0 & 0 & 0 \\
f_{12} & f_{11} & f_{13} & 0 & 0 & 0 \\
f_{13} & f_{13} & f_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & f_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & f_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & f_{11} - f_{12}
\end{pmatrix}. \tag{67}
\]

Thus, there are five quadratic invariants,

\[
\phi_{11} = C_{11}^2 + C_{22}^2 + 2C_{12}^2, \quad \phi_{12} = C_{11}C_{22} - C_{12}^2, \quad \phi_{33} = C_{33}^2, \\
\phi_{13} = (C_{11} + C_{22})C_{33}, \quad \phi_{44} = C_{13}^2 + C_{23}^2,
\]

\[
\tag{68}
\]

(corresponding to \( f_{11}, f_{12}, f_{33}, f_{13} \), and \( f_{44} \), respectively.

The linear invariants of \( C \) under SO(2) are

\[
\lambda_1 = C_{11} + C_{22}, \quad \lambda_2 = C_{33}. \tag{69}
\]

The five quadratic invariants can be recast as three defined by the linear invariants and two new ones, e.g.,

\[
\phi_1 = \lambda_1^2, \quad \phi_2 = \lambda_2^2, \quad \phi_3 = \lambda_1\lambda_2, \quad \phi_4 = C_{11}C_{22} - C_{12}^2, \quad \phi_5 = C_{13}^2 + C_{23}^2.
\]

\[
\tag{70}
\]

In keeping with the statement of Lemma \[\text{[1]}\] we note that the five independent elements of \( F \) correspond to the linear invariants of \( \mathcal{E}_{\text{la}} \) under SO(2). We next consider the quadratic invariants of \( \mathcal{E}_{\text{la}} \) under SO(2).
5.4 Quadratic forms on the elastic moduli: n=4

The fundamental quadratic form is now expressed in alternative ways,

\[ \phi = c^t F c = \tilde{c}^t \tilde{F} \tilde{c}, \]  

(71)

where \( \tilde{c} \) is a 21-vector related to the 21-vector of moduli \( c \) in (12) by

\[ \tilde{c} = T c, \]  

(72)

and

\[ T = \text{diag} (1 1 1 \sqrt{2} \sqrt{2} \sqrt{2} 2 2 2 2 2 2 2 2 2 2 \sqrt{8} \sqrt{8} \sqrt{8}). \]  

(73)

We introduce \( \tilde{F} \) in (71) to remove the \( \sqrt{2} \) factors from the final expressions. The two are simply related by

\[ F = T \tilde{F} T. \]  

(74)

Results below are given for the elements of \( F \) which do not have the \( \sqrt{2} \) factors.

5.4.1 The rotation matrix

The 21×21 skew symmetric generator is \( \text{(Norris, 2007)} \)

\[ P = R - R^t, \]  

(75)

where

\[ R = \begin{pmatrix} 0 & 0 & 0 & 0 & 2Y & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2}Y & 0 & \sqrt{2}N & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N & -\sqrt{2}Y \\ 0 & -\sqrt{2}Z & 0 & 0 & X & 0 & -\sqrt{2}X \\ 0 & \sqrt{2}N & 2N & 0 & 0 & X & 0 \\ 2Z & 0 & 0 & -\sqrt{2}X & 0 & 0 & \sqrt{2}X \\ 0 & 0 & -\sqrt{2}Z & -\sqrt{2}X & \sqrt{2}X & 0 & -X \end{pmatrix}, \]  

(76)

with \( X, Y \) and \( Z \) as before in eq. (63), and

\[ N = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}. \]  

(77)

Note that the characteristic equations for \( P \) is

\[ P(P^2 + I)(P^2 + 4I)(P^2 + 9I)(P^2 + 16I) = 0. \]  

(78)

The conditions (51) are that the commutator of \( \tilde{F} \) and \( P \) vanish,

\[ \tilde{F} P - P \tilde{F} = 0. \]  

(79)

This may be rewritten in a form involving the elements of \( F \) of (73), with the intent of removing the factors of \( \sqrt{2} \). The resulting equation is

\[ \hat{P} F + F \hat{P}^t = 0, \quad \text{where } \hat{P} = TPT^{-1}. \]  

(80)

The modified generator \( \hat{P} \) is no longer skew symmetric, but it still satisfies the characteristic equation (78), and, like \( P \), can be used to compute the rotation \( Q = \exp(\theta P) \), or its modified form, \( \hat{Q} = \exp(\theta \hat{P}) = TQT^{-1} \), as a polynomial of degree 8 in \( \hat{P} \) \( \text{(Norris, 2007)} \). With \( p = e_3 \), \( R \) has 18 non-zero elements, hence \( \hat{P} \) has 36 non-zero elements.
5.4.2 Quarter turn conditions on the elements of F

Before applying the SO(2) condition \[ (80) \] to \( F \), we perform a preliminary simplification by invoking invariance under a quarter turn about the axis of rotation. The transformed elements \( F_{ijklpqrs} \) should be unaltered under the interchanges of indices 1, 2 \( \rightarrow \) 2, \(-1\). This implies a total of 52 relations among the 127 elements of \( F \), which we split into two categories depending on whether the index 1 occurs an even or an odd number of times. In the former category are the following 31 identities,

\[
\begin{align*}
& f_{11} = f_{22}, \quad f_{13} = f_{23}, \quad f_{14} = f_{25}, \quad f_{15} = f_{24}, \\
& f_{16} = f_{26}, \quad f_{17} = f_{28}, \quad f_{18} = f_{27}, \quad f_{19} = f_{29}, \\
& f_{34} = f_{35}, \quad f_{37} = f_{38}, \quad f_{44} = f_{55}, \quad f_{46} = f_{56}, \\
& f_{47} = f_{58}, \quad f_{48} = f_{57}, \quad f_{49} = f_{59}, \quad f_{67} = f_{68}, \\
& f_{77} = f_{88}, \quad f_{79} = f_{89}, \quad f_{710} = f_{1111}.
\end{align*}
\]

(81)

and in the second group we have the following 21 connections,

\[
\begin{align*}
& b_{15} + b_{27} = 0, \quad b_{18} + b_{24} = 0, \quad b_{111} + b_{210} = 0, \\
& b_{45} + b_{78} = 0, \quad b_{411} + b_{810} = 0, \quad b_{510} + b_{711} = 0, \\
& d_{13} + d_{23} = 0, \quad d_{43} + d_{53} = 0, \quad d_{73} + d_{83} = 0, \\
& d_{16} + d_{29} = 0, \quad d_{19} + d_{26} = 0, \quad d_{36} + d_{39} = 0, \\
& d_{46} + d_{59} = 0, \quad d_{49} + d_{56} = 0, \quad d_{66} + d_{69} = 0, \\
& d_{76} + d_{89} = 0, \quad d_{79} + d_{86} = 0, \quad d_{96} + d_{99} = 0, \\
& d_{112} + d_{212} = 0, \quad d_{412} + d_{512} = 0, \quad d_{712} + d_{812} = 0.
\end{align*}
\]

(82)

Therefore, \( A, B \) and \( D \) have 45-18=27, 46-19=27 and 36-15=21 independent elements, respectively, for a total of 127- 52= 75 independent elements in \( F \). \( A \) has the form given in \[ (19) \], while \( B \)
and $D$ are of the form

$$B = \begin{pmatrix}
  b_{11} & b_{12} & 0 & b_{14} & b_{15} & 0 & b_{17} & b_{18} & 0 & b_{110} & b_{111} & 0 \\
  b_{11} & 0 & -b_{18} & b_{17} & 0 & -b_{18} & b_{14} & 0 & -b_{111} & b_{110} & 0 \\
  b_{33} & 0 & 0 & b_{36} & 0 & 0 & b_{36} & 0 & 0 & b_{312} & 0 \\
  b_{44} & b_{45} & 0 & b_{47} & b_{18} & 0 & b_{410} & b_{411} & 0 & 0 & 0 \\
  b_{55} & 0 & b_{57} & b_{17} & 0 & b_{510} & b_{511} & 0 & 0 & 0 & 0 \\
  b_{66} & 0 & 0 & b_{69} & 0 & 0 & b_{612} & 0 & 0 & b_{612} & 0 \\
  b_{55} & -b_{45} & 0 & b_{511} & -b_{510} & 0 & 0 & 0 & 0 & 0 & 0 \\
  b_{44} & 0 & -b_{111} & b_{14} & 0 & 0 & b_{111} & 0 & 0 & b_{111} & 0 \\
  b_{66} & 0 & 0 & b_{69} & 0 & 0 & b_{612} & 0 & 0 & b_{612} & 0 \\
  b_{1010} & 0 & 0 & b_{1011} & 0 & 0 & b_{1011} & 0 & 0 & b_{1011} & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  b_{1212} & \end{pmatrix}, \quad (83)$$

$$D = \begin{pmatrix}
  0 & 0 & d_{13} & 0 & 0 & d_{16} & 0 & 0 & d_{19} & 0 & 0 & d_{112} \\
  0 & 0 & -d_{13} & 0 & 0 & -d_{19} & 0 & 0 & -d_{16} & 0 & 0 & -d_{112} \\
  0 & 0 & d_{33} & 0 & 0 & d_{36} & 0 & 0 & -d_{36} & 0 & 0 & d_{312} \\
  0 & 0 & d_{43} & 0 & 0 & d_{46} & 0 & 0 & d_{49} & 0 & 0 & d_{412} \\
  0 & 0 & -d_{43} & 0 & 0 & -d_{49} & 0 & 0 & -d_{46} & 0 & 0 & -d_{412} \\
  0 & 0 & d_{63} & 0 & 0 & d_{66} & 0 & 0 & -d_{66} & 0 & 0 & d_{612} \\
  0 & 0 & d_{73} & 0 & 0 & d_{76} & 0 & 0 & d_{79} & 0 & 0 & d_{712} \\
  0 & 0 & -d_{73} & 0 & 0 & -d_{79} & 0 & 0 & -d_{76} & 0 & 0 & -d_{712} \\
  0 & 0 & d_{93} & 0 & 0 & d_{96} & 0 & 0 & -d_{96} & 0 & 0 & d_{912} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \end{pmatrix}, \quad (84)$$

5.4.3 SO(2) conditions on the elements of $F$  

We are now ready to apply the SO(2) invariance condition \([80]\) to the simplified form of $F$ with 75 independent elements, where the matrix $\tilde{P}$ defined by \([76]\) with $p = e_3$ has 36 nonzero elements. The system of equations generated is large but relatively straightforward to solve, particularly with the aid of an electronic computer. We simply state the results, which the reader may check.

We find that the matrix in \([80]\) vanishes if and only if 40 additional relations among the 75 independent elements are met. These may be split into two sets, of 15 and 25 respectively. The first set of 15 relations \([21]\) involve only those elements of $F$ in which the index 1 (and hence 2) occurs an even number of times, and therefore they act only on the elements of $A_{9 \times 9}$ and the non-zero elements of $B_{12 \times 12}^{(1)}$ of \([20]\). The second set of 25 relations involve only elements of $F$ with the index 1 (and 2) occurring an odd number of times:

\begin{align*}
  b_{1,2} &= 0, & b_{4,8} &= 0, & b_{5,7} &= 0, & b_{10,11} &= 0, \\
  b_{1,5} - b_{1,11} &= 0, & b_{1,5} + b_{5,10} &= 0, & b_{1,8} + b_{4,5} - b_{4,11} &= 0, \\
  d_{1,6} &= 0, & d_{3,9} &= 0, & d_{3,3} &= 0, & d_{3,6} &= 0, \\
  d_{3,12} &= 0, & d_{4,3} &= 0, & d_{6,3} &= 0, & d_{6,6} &= 0, \\
  d_{6,12} &= 0, & d_{7,12} &= 0, & d_{9,3} &= 0, & d_{9,6} &= 0, \\
  d_{9,12} &= 0, & d_{1,3} - d_{4,6} &= 0, & d_{1,3} - d_{4,9} &= 0, & d_{1,12} - d_{7,6} &= 0, \\
  d_{1,12} - d_{7,9} &= 0, & d_{4,12} + d_{7,3} &= 0. \\
\end{align*}
Thus,

$$B = \begin{pmatrix} 0 & 0 & \mathbf{b}_{14} & \mathbf{b}_{15} & 0 & \mathbf{b}_{17} & \mathbf{b}_{18} & 0 & \mathbf{b}_{110} & \mathbf{b}_{15} & 0 \\ 0 & 0 & -\mathbf{b}_{18} & \mathbf{b}_{17} & 0 & \mathbf{b}_{14} & 0 & -\mathbf{b}_{15} & -\mathbf{b}_{110} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{b}_{36} & 0 & 0 & \mathbf{b}_{36} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{b}_{45} & 0 & 0 & \mathbf{b}_{47} & 0 & 0 & \mathbf{b}_{410} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{b}_{47} & 0 & 0 & \mathbf{b}_{47} & 0 & 0 & \mathbf{b}_{410} & 0 \\ 0 & 0 & \mathbf{b}_{55} & 0 & 0 & \mathbf{b}_{66} & 0 & 0 & \mathbf{b}_{66} & 0 & 0 \\ -\mathbf{b}_{45} & 0 & 0 & \mathbf{b}_{55} & -\mathbf{b}_{45} & 0 & \mathbf{b}_{55} & -\mathbf{b}_{45} & 0 & 0 & 0 \\ \mathbf{b}_{14} & 0 & -\mathbf{b}_{18} - \mathbf{b}_{4,5} & \mathbf{b}_{44} & 0 & -\mathbf{b}_{18} - \mathbf{b}_{4,5} & \mathbf{b}_{44} & 0 & -\mathbf{b}_{18} - \mathbf{b}_{4,5} & \mathbf{b}_{44} & 0 \\ \mathbf{b}_{66} & 0 & \mathbf{b}_{10,10} & -\mathbf{b}_{18} - \mathbf{b}_{4,5} & \mathbf{b}_{66} & 0 & \mathbf{b}_{10,10} & -\mathbf{b}_{18} - \mathbf{b}_{4,5} & \mathbf{b}_{66} & 0 & \mathbf{b}_{10,10} \\ \mathbf{b}_{10,10} & 0 & 0 & \mathbf{b}_{10,10} & 0 & 0 & \mathbf{b}_{10,10} & 0 & 0 & \mathbf{b}_{10,10} & 0 & \mathbf{b}_{10,10} \\ \mathbf{b}_{12,12} & 0 & 0 & \mathbf{b}_{12,12} & 0 & 0 & \mathbf{b}_{12,12} & 0 & 0 & \mathbf{b}_{12,12} & 0 & \mathbf{b}_{12,12} \end{pmatrix}$$

and $D$ is given by (23). The terms in bold in (86) are essentially decoupled from the others, and we therefore split $F$ according to eq. (17) in order to emphasize the disjoint nature of the subspaces generated by the elements with index 1 occurring an even and an odd number of times.

The partition (17) also allows us to easily determine the dimensionality of $F$. Thus, $F^{(2)}$ clearly has 6 independent elements. There are 44 distinct elements in $F^{(1)}$, and all but 4 of these, $f_{33}$, $f_{44}$, $f_{37}$ and $b_{14}$, occur in the relations (21). These 15 conditions are of rank 15, indicating that the conditions are linearly independent. This can be verified by noting that each of the 15 equations involves at least one element (the final one in the left member) not contained in the other 14 equations. Hence, there are a total of 15 constraints on the 50 elements of $F$, which therefore has $50 - 15 = 35$ independent elements. This is the dimension of $F$, and the number of independent quadratic forms on $F$.

### 5.5 SO(3) conditions on the elements of $F$

There are various ways to deduce the SO(3) form of $F$ using the results for SO(2) (actually O(2)). As discussed before, the required group of transformations is simply $O(3)$. We also note the following as a consequence of Theorem 2 and Lemma 2.

**Lemma 4** All quadratic invariants of $c$ under $O(2) \cup R(e_1)$ where $e_1$ is perpendicular to the axis of rotation are of the form $c^T F^{(1)} c$ where the 21×21 symmetric matrix $F^{(1)}$ is given by eq. (15).

The number of linearly independent quadratic invariants under $SO(2) \cup R(e_1)$ is 29.

We can therefore start with $F^{(1)}$ of Theorem 2, which corresponds to $O(2) \cup R(e_1)$. The next step is to consider quarter turns about $e_1$ and $e_2$, which reduces $A$ and $B$ to the forms (14) and (15), respectively. The 12 distinct elements in $A$ and 7 in $B$ are related by the 15 conditions of Theorem 2 in eqs. (21). We find that only 12 of these are linearly independent, or in other words, the system of equations is rank(12). A linearly independent system can be obtained by, for instance, ignoring the final three conditions in (21), to give a system of 12 equations on the 19 elements, i.e. eqs. (16).

In summary, there are 12 relations between the 19 elements of $F$. Hence there are 7 quadratic invariants under SO(3).

It remains to discuss these results with respect to the invariants proposed by Ting (1987) and Ahmad (2002).

### 6 Consistency and completeness

A given quadratic form can be checked to see if it is consistent with one of the invariant forms defined in Theorems 1 and 2. We will describe how to do this and discuss the consistency of the
quadratic invariants proposed by Ting and Ahmad. We note in passing that $A_1, A_2$ and $B_1, \ldots, B_4$ of eqs. (1) and (2) are obviously consistent with SO(3) invariance, but some of the SO(2) invariants proposed by Ting and Ahmad are not immediately obvious. We also discuss the completeness of Ting and Ahmad’s quadratic invariants.

### 6.1 SO(3)

#### 6.1.1 Consistency

Given a quadratic form $\phi$ in $c$, for instance $A_1^2$ of eq. (1), define a 21×21 symmetric matrix $F$ according to

$$f_{ij} = \frac{1}{2} \frac{\partial^2 \phi}{\partial c_i \partial c_j}.$$  \hfill (87)

where $c_i$ is the $i$–th component of the modulus vector $c$ of (12). The quadratic form $\phi$ is an invariant under SO(3) iff (i) the elements of $F$ have the form as defined by eqs. (13)-(15), and (ii) they satisfy the 12 equations (16).

It may be confirmed that Ahmad’s seven quadratic forms \{A_1^2, A_2^2, A_1 A_2, B_1, B_2, B_3, B_4\} are SO(3) invariants.

#### 6.1.2 Completeness

Assuming a quadratic form $\phi$ is consistent with SO(3) invariance, define the vector of the 19 distinct elements in $F$ of (14) and (15), specifically

$$u = u_{19 \times 1} = (f_{11} f_{12} f_{13} f_{14} f_{15} f_{17} f_{18} f_{44} f_{45} f_{47} f_{48} f_{77} f_{78} b_{11} b_{14} b_{11} b_{14} b_{44} b_{47} b_{410} b_{1010})^t.$$  \hfill (88)

Now suppose we have a set of different quadratic forms, $\phi_1, \ldots, \phi_N$, where $N \leq 7$ (we do not need to consider $N > 7$ since there can be no more than 7 linearly independent forms). Let $u_i$ be the 19-vector for $\phi_i$, $i = 1, \ldots, N$, and define the matrix

$$M = M_{19 \times N} = (u_1 u_2 \ldots u_N).$$  \hfill (89)

Then the quadratic forms are linearly independent iff $\text{rank}(M) = N$. By definition, a set of 7 quadratic forms is complete if they are linearly independent.

It may be checked that Ahmad’s seven quadratic forms define a matrix $M$ of rank 7, and are therefore complete.

### 6.2 SO(2)

Checking a given quadratic form for consistency is analogous to the procedure described for SO(3). Thus, first compute $F$ according to (87), then check that (i) the elements of $F$ have the form as defined by eqs. (13)-(15), and (ii) they satisfy the 15 equations (21).

It may be confirmed that Ahmad’s 17 quadratic forms $E_1, \ldots, E_{17}$ in eq. (7) and the new quadratic forms $E_{18}, E_{19}, E_{20}$ are all SO(2) invariants, as are the 15 quadratic forms defined by the five linear invariants $L_1, \ldots, L_5$ of eq. (6).

These 35 quadratic forms are also complete. In order to see this define $v = v_{50 \times 1}$ as the vector of the 50 distinct elements in $F$ of (19) - (23), specifically

$$v = (f_{11} f_{12} f_{13} f_{14} f_{15} f_{16} f_{17} f_{18} f_{19} f_{33} f_{34} f_{36} f_{37} f_{39} f_{44} f_{45} f_{46} f_{47} f_{48} f_{66} f_{67} f_{77} f_{78} f_{79} f_{99} b_{11} b_{14} b_{15} b_{17} b_{18} b_{11} b_{33} b_{36} b_{312} b_{44} b_{45} b_{47} b_{410} b_{55} b_{511} b_{66} b_{69} b_{612} b_{1010} b_{1212} d_{13} d_{112} d_{412})^t.$$  \hfill (90)
Let \( M \) be the matrix formed from the \( v \)-vectors of the 35 quadratic forms. Then it may be checked that \( \text{rank}(M) = 35 \), indicating that they are a complete set.

Finally, we note that 

\[
\frac{1}{2} c^t F^{(2)} c = -d_{412} E_{15} - d_{13} E_{16} + d_{112} E_{17} + b_{15} E_{18} + (b_{18} + b_{45}) E_{19} - b_{18} E_{20}. \tag{91}
\]

This illustrates how the partition of \( F \) in eq. (17) splits the set of invariants into distinct subsets, one associated with elements of \( F \) that have index 1 (and 2) an even number of times, which is a 29 dimensional subspace. The other is the six dimensional subspace associated with the fact that SO(2) is not a subspace of orthorhombic symmetry for this type of 8th order tensor.

## Appendix

### A Proof of equation (4)

Define the totally symmetric and the asymmetric parts of the elasticity tensor as 

\[
C_{ijkl}^{(s)} = \frac{1}{3} (C_{ijkl} + C_{ikjl} + C_{iljk}), \quad C_{ijkl}^{(a)} = \frac{1}{3} \left( 2 C_{ijkl} - C_{ikjl} - C_{iljk} \right), \tag{A.1}
\]

respectively. These partition the elastic moduli, 

\[
C_{ijkl} = C_{ijkl}^{(s)} + C_{ijkl}^{(a)}, \tag{A.2}
\]

and are orthogonal in the sense that 

\[
C_{ijkl}^{(s)} C_{ijkl}^{(s)} = C_{ijkl}^{(s)} C_{ijkl}^{(a)} C_{ijkl}^{(a)} C_{ijkl}^{(a)} C_{ijkl}^{(a)} C_{ijkl}^{(a)}. \tag{A.3}
\]

Hence, 

\[
2 C_{ijkl}^{(s)} C_{ikjl} = C_{ijkl}^{(s)} (C_{ikjl} + C_{iljk}) \\
= C_{ijkl}^{(s)} (3 C_{ijkl}^{(s)} - C_{ijkl}^{(a)}) \\
= C_{ijkl}^{(s)} (2 C_{ijkl}^{(s)} - C_{ijkl}^{(a)}) \\
= 2 C_{ijkl}^{(s)} C_{ijkl}^{(s)} - C_{ijkl}^{(a)} C_{ijkl}^{(a)} C_{ijkl}^{(a)} \\
= 2 C_{ijkl}^{(s)} C_{ijkl}^{(s)} - 3 C_{ijkl}^{(a)} C_{ijkl}^{(a)}, \tag{A.4}
\]

and therefore 

\[
B_5 = B_1 - \frac{3}{2} C_{ijkl}^{(a)} C_{ijkl}^{(a)}. \tag{A.5}
\]

While it is easy to verify that 

\[
C_{ijkl}^{(a)} = \frac{2}{3} D_{ij}, \tag{A.6}
\]

where 

\[
D_{ij} = C_{ijkl} - C_{ikjl}, \tag{A.7}
\]

it is not as obvious that \( C^{(a)} \) is completely defined by \( D \). It may be shown (Backus, 1970; Norris, 2008) that 

\[
C_{ijkl}^{(a)} = \frac{1}{3} \left[ 2 D_{ij} \delta_{kl} + 2 D_{kl} \delta_{ij} - D_{ik} \delta_{jl} - D_{il} \delta_{jk} - D_{jk} \delta_{il} - D_{jl} \delta_{ik} \right].
\]
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\[ + \frac{1}{2} D_{mn}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}) \], \quad (A.8)

and hence,

\[ C^{(a)}_{ijkl} C^{(a)}_{ijkl} = \frac{4}{3} D_{ij} D_{ij} - \frac{1}{3} D_{ii} D_{jj} \]. \quad (A.9)

Finally,

\[ B_5 = B_1 + \frac{1}{2} D_{ij} D_{jj} - 2 D_{ij} D_{ij} \], \quad (A.10)

from which eq. (I) follows.

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