COARSE RICCI CURVATURE OF HYPERGRAPHS AND ITS GENERALIZATION

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Abstract. In the present paper, we introduce a concept of Ricci curvature on hypergraphs for a nonlinear Laplacian. We prove that our definition of the Ricci curvature is a generalization of Lin-Lu-Yau’s coarse Ricci curvature for graphs to hypergraphs. We also show a lower bound of nonzero eigenvalues of Laplacian, gradient estimate of heat flow, and diameter bound of Bonnet-Myers type for our curvature notion. This research leads to understanding how nonlinearity of Laplacian causes complexity of curvatures.

1. Introduction

The Ricci curvature of Riemannian manifolds plays an important role to analyze the geometric and analytic properties of the manifolds. It was a problem how to define a concept of Ricci curvature on generic metric spaces. Recently, on geodesic metric measure spaces, the synthetic notion of lower bound of Ricci curvature is defined, called the curvature-dimension condition [26, 40, 41]. On the other hand, there are some different notions of lower Ricci curvature bound on discrete spaces. Even for graphs, we have many notions, Ollivier’s coarse Ricci curvature, Lin-Lu-Yau’s coarse Ricci curvature, the (Bakry-Émery type) curvature-dimension condition, the exponentially curvature-dimension inequality, etc. The former two are defined by the contraction of $L^1$-Wassertein distance. The latter two are defined by holding Bakry-Émery type functional inequalities. Both coincides in the setting of Riemannian manifolds. Such notions of Ricci curvature on graphs are widely used to network analysis such as community detection [39].

In this paper, we introduce a definition of a coarse Ricci curvature on hypergraph of Lin-Lu-Yau type. Here, hypergraph is a generalization of graph to be able to represent relations among not only two, but also three or more entities. On hypergraphs, there is no crucial canonical definition of random walks as on graphs. Hence, we cannot naturally define the curvature notion on hypergraphs in Olliver’s manner. Asoodeh et al. [5] introduced a notion of coarse Ricci curvature on hypergraph by the random walk obtained by reducing hypergraph to a graph with clique

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expansion. Also there are several other notions of Ricci curvature on hypergraphs. See for instance [1, 14, 15, 24].

Our coarse Ricci curvature on hypergraphs is defined by using a nonlinear multivalued Laplacian called “submodular hypergraph Laplacian”, which introduced by [12, 27, 45]. In [42], they empirically showed that the hypergraph Laplacian defined in [20] outperforms the clique expansion for the quality of solutions of a clustering problem. This indicates that the hypergraph Laplacian in [20] is more capable to capture the geometric structure of hypergraph than the clique expansion. Although the our hypergraph Laplacian is multivalued and nonlinear, in [20], it was proven that this operator is maximal monotone. This means that our Laplacian is in a tractable class of nonlinear operators.

Our motivation of this research is to understand whether the nonlinearity of Laplacian causes some difficulty of curvatures or not. Our nonlinear Laplacian is an appropriate candidate for this attempt. We explain more details about this in the following.

For Riemannian manifolds, though the Ricci tensor needs $C^2$ smooth structure on them, lower bound condition of Ricci curvature can be described by the metric and measure, that we do not need the smoothness. More precisely, von Renesse and Sturm [44, Section 1] proved that for the smooth, complete, connected Riemannian manifold $(M, g)$, a volume measure $\text{vol}_g$ on it, the Ricci curvature $\text{Ric}_g(x, v, v)$ for $x \in M$ and $v \in T_x M$, and any $K \in \mathbb{R}$, the following (1)-(5) are equivalent:

1. (lower bound of Ricci curvature) $\text{Ric}_g \geq Kg$, which means that $\text{Ric}_g(x, v, v) \geq K|v|^2$ for any $x \in M$ and $v \in T_x M$.
2. (convexity of relative entropy) The relative entropy $\text{Ent}_{\text{vol}_g}$ defined in [44, P.924] is the displacement $K$-convex on the $L^2$-Wasserstein space $(\mathcal{P}_2(M), W_2)$ defined in [44, P.923–924] (cf.[13]).
3. (transportation inequality) For the measure restricted to the ball of radius $r$ whose center is $x \in M$

$$m_{r,x}(A) := \frac{\text{vol}_g(B_r(x) \cap A)}{\text{vol}_g(B_r(x))},$$

the following holds:

$$W_1(m_{r,x}, m_{r,y}) \leq \left(1 - \frac{K}{2(n+2)}r^2 + o(r^2)\right)d(x, y).$$

4. (contraction property of the gradient flow of entropy) For the gradient flow $\Phi : \mathbb{R}_+ \times \mathcal{P}_2(M) \rightarrow \mathcal{P}_2(M)$ of $\text{Ent}_m$,

$$W_2(\Phi(t, \mu), \Phi(t, \nu)) \leq e^{-Kt}W_2(\mu, \nu)$$

holds for $t > 0$ and any $\mu, \nu \in \mathcal{P}_2(M)$.

5. (gradient estimate of heat flow) For any $f \in C^{\infty}_c(M), x \in M$, and $t > 0$,

$$|\nabla h_t f|^2(x) \leq e^{-2Kt}h_t|\nabla f|^2(x)$$
holds. Here, $h_t : L^2(M) \to L^2(M)$ is the heat flow on $M$.

We note that (5) is deeply related to the property of Dirichlet form rather than the smoothness of the structure of manifolds. Moreover, the following Bochner inequality (or Bakry-Émery curvature dimension condition) is also equivalent to them (cf. [3, 4]):

(6) (Bochner inequality, curvature-dimension condition of Bakry-Émery type)

For any $f \in C_c^\infty(M)$, the following holds:

\[
\frac{1}{2} \Delta |\nabla f|^2 \geq \langle \nabla \Delta f, \nabla f \rangle + K|\nabla f|^2.
\]

Based on these facts, CD space, which is the metric measure space which is not necessarily manifold and looks like satisfying a lower Ricci curvature condition (with upper bound condition of dimension), is introduced by using the convexity of entropy on the $L^2$-Wasserstein space (Sturm [40, 41], Lott-Villani [26] for finite dimension, using not relative entropy but Rényi entropy). An important point is that the definition is only described in terms of measures and metrics. For the CD space whose dimension is bounded above, many important geometric and functional inequalities such as Bishop-Gromov inequality, Poincaré inequality and Brunn-Minkowski inequality were proved ([26, 34, 41]). However, Ohta-Sturm showed that the gradient estimate of the heat flow does not hold for generic CD spaces ([33]).

After that, an RCD space [2, 3] is defined, which is a CD space equipped with the infinitesimal Hilbertianity condition (defined by Gigli [19]) that any Sobolev space becomes a Hilbert space. On any RCD space, several theorems such as the $W_2$-contraction of the gradient flow of the relative entropy, the Bochner inequality (Bakry-Émery’s curvature dimension condition) and the gradient estimate of the heat flow have been proved and these are known as equivalent conditions on manifolds.

Many geometric results known on Riemannian manifolds are also proven such as Cheeger-Gromoll’s splitting theorem [18], Cheng’s maximum diameter theorem [23], isoperimetric inequalities [10] and so on. Furthermore, some results on RCD spaces include theorems which had not been proved even for Ricci limit spaces. It is known that both of CD and RCD spaces become geodesic metric spaces. RCD spaces established a firm position as geodesic spaces whose Ricci curvature is bounded from below.

On the other hand, although several definitions of discrete spaces whose Ricci curvature is bounded below were introduced, there has not been canonical definition. It can been seen that for graphs, coarse Ricci curvatures of Ollivier [34] and Lin-Lu-Yau [25] are related to the above (3) or (4), the curvature dimension condition of Bakry-Émery type [37] is related to (6), exponentially curvature-dimension condition is related to the Li-Yau inequality [7, 30, 31], and the definition by Maas
and by Bonciocat-Sturm [8] is related to (2). Although all of these definitions stem from the definitions or known facts for geodesic spaces, the relations among them are still not clear in discrete spaces. Laplacian is a key ingredient to define the curvature-dimension conditions of Bakry-Émery type. In spite of the Laplacian on usual graph is defined as a linear self-adjoint operator, this is not diffusion, thus the property as a differential operator is different from that on RCD spaces. (However, the exponentially curvature-dimension condition tries to overcome the difficulty. We also remark that though the Laplacian on RCD space is linear, that of general CD space might be nonlinear. In this sense, graphs can be considered to be in intermediate position between them).

As mentioned above, in this paper, we introduce a notion of coarse Ricci curvature on hypergraphs using the resolvent of the hypergraph Laplacian of [20]. Although our Laplacian is non-singular and multivalued, under the assumption of the lower bound condition of curvature of Lin-Lu-Yau type, we can deduce a diameter bound, a lower bound of nonzero eigenvalues, and a gradient estimate of the heat flow (of $L^\infty$ type). These properties do not hold for general CD spaces. This suggests that the nonlinearity of Laplacian does not necessarily make the heat flow intractable. The authors believe that this observation will drive us to investigate deeper properties of curvatures with our submodular Laplacian as a candidate.

Our arguments are applicable to more general settings for submodular transformations, which are vector valued set functions consisting of submodular functions. In Appendix A we give a sufficient condition for a submodular transformation to be able to straightforwardly generalize the curvature notions and our theorems.

This paper is organized as follows: In Section 2, we review fundamental properties of hypergraphs (Section 2.1), submodular hypergraph Laplacian and its resolvent (Section 2.2), $L^1$-Wasserstein distance (Section 2.3). In Section 3.1 we review the definition of the coarse Ricci curvature by Lin-Lu-Yau and rephrase this definition by resolvent of graph Laplacian. Based on this observation, in Section 3.2 and Section 3.3 we introduce a definition of $\lambda$-nonlinear Kantorovich difference, and (lower and upper) coarse Ricci curvature on hypergraphs. In Section 3.4 we show that for graphs, our coarse Ricci curvature is equal to that of Lin-Lu-Yau. In Section 3.5 we show applications of our coarse Ricci curvature to a bound of eigenvalues (Section 3.5.1), a gradient estimate for heat flow (Section 3.5.2), and a Bonnet-Myers type diameter bound (Section 3.5.3). We give some calculation of our curvatures in Section 6. In Appendix A we give a complete proof of the coincidence between upper and lower coarse Ricci curvature for hypergraphs. In Appendix B we review submodular functions (Section B.1), submodular transformations and the submodular Laplacian (Section B.2) and argue about generalization our curvature notion and theorems to submodular transformations. We also show examples of
submodular transformations such as directed graph, mutual information, directed hypergraphs etc. in Section B.3.

2. Preliminaries

2.1. Hypergraphs. A (weighted undirected) hypergraph \( H = (V, E, w) \) is a triple of a set \( V \), a set \( E \) of nonempty subsets of \( V \), and a function \( w: E \rightarrow \mathbb{R}_{>0} \). We call \( V \) a set of vertices, \( E \) a set of hyperedges, and \( w \) a weight function. We remark that if \(|e| = 2\) for any \( e \in E \), \( H \) is an undirected weighted graph. We say that \( H \) is finite if the set \( V \) is finite. For \( x, y \in V \), we write \( x \sim y \) if there exists a hyperedge \( e \in E \) including \( x \) and \( y \). We say that hypergraph \( H \) is connected if for any \( x, y \in V \), there is a sequence of vertices \( \{z_i\}_{i=1}^k \) such that 

\[
z_1 = x, \ z_k = y, \ \text{and} \ z_i \sim z_{i+1} \ (i = 1, \ldots, k - 1).
\]

Throughout this paper, we assume that any hypergraph is finite and connected. Let \( H = (V, E, w) \) be a weighted, undirected, connected, finite hypergraph. We define the degree of \( x \) by 

\[
d_x := \sum_{e \ni x} w_e
\]

and the degree matrix by 

\[
D := \text{diag}(d_x).
\]

We remark that if the hypergraph \( H \) is connected, then \( d_x > 0 \) for any \( x \in V \), thus \( D \) is non-singular. For a subset \( S \subseteq V \), the volume \( \text{vol}(S) \) of \( S \) is defined by 

\[
\text{vol}(S) = \sum_{x \in S} d_x.
\]

For two vertices \( x, y \in V \), we consider a distance on \( V \) as 

\[
d(x, y) := \min\{n : \exists \{z_i\}_{i=0}^n, z_0 = x, z_n = y, z_i \sim z_{i+1}\}.
\]

Then, \((V, d)\) becomes a metric space. We define the diameter \( \text{diam}(H) \) of the hypergraph \( H \) as that of the metric space \((V, d)\), i.e.,

\[
\text{diam}(H) := \max_{x, y \in V} d(x, y).
\]

We identify the \( \mathbb{R} \)-valued map on \( V \) with the set \( \mathbb{R}^V \) of vectors indexed by \( V \). We set \( \delta_x \in \mathbb{R}^V \) as the characteristic function at \( x \in V \):

\[
\delta_x(z) := \begin{cases} 1 & \text{if } z = x, \\ 0 & \text{otherwise.} \end{cases}
\]

We define the stationary distribution \( \pi \in \mathbb{R}^V \) by \( \pi(z) = d_z/\text{vol}(V) \) for any \( z \in V \).

2.2. Laplacian on hypergraph. In this subsection, we review a submodular hypergraph Laplacian in the sense of Ikeda et al.\cite{IKS} and some properties of the resolvent of the Laplacian, which were derived via general theory of maximal monotone operators. For the details of this theory, see \cite{BG}, \cite{BC}, \cite{L}, and \cite{MT}.

We define an inner product \( \langle \ , \ \rangle \) on \( \mathbb{R}^V \) by 

\[
\langle f, g \rangle = f^\top D^{-1} g = \sum_{x \in V} f(x)g(x)d_x^{-1}.
\]
On the Hilbert space \((\mathbb{R}^V, \langle \cdot, \cdot \rangle)\), we define the (submodular) hypergraph Laplacian \(L: \mathbb{R}^V \to 2^{\mathbb{R}^V}\) for \(H = (V, E, w)\) by
\[
L(f) = L_f := \left\{ \sum_{e \in E} w_e b_e^\top f \ ; \ b_e \in \text{argmax}_{b_e \in B_e} b_e^\top f \right\},
\]
(2.1)
where \(B_e\) is the base polytope for \(e \in E\), i.e., the subset of \(\mathbb{R}^V\) defined by
\[
B_e = \text{Conv}(\{\delta_x - \delta_y; x, y \in e\}),
\]
(2.2)
where \(\text{Conv}(X)\) for \(X \subset \mathbb{R}^V\) is the convex hull of \(X\) in \(\mathbb{R}^V\). We note that this Laplacian \(L\) is a modification of the definition introduced in \([12, 27]\), and a realization of the submodular transformation introduced in \([45]\) when the submodular transformation is a hypergraph. As mentioned in \([45, \text{Section 2}]\) or \([11, \text{P.15:8}]\), this Laplacian \(L\) is the sub-differential of the convex function \(Q: \mathbb{R}^V \to \mathbb{R}\) defined by
\[
Q(f) = \frac{1}{2} \sum_{e \in E} w(e) \max_{x, y \in e} (f(x) - f(y))^2.
\]
When the hypergraph \(H\) is a graph, by the definition (2.1), our Laplacian \(L\) becomes single-valued \(L(f) = \{ (D - A) f \}\), where \(A\) is the adjacency matrix of the graph \(H\).

In this paper, we mainly treat the following normalized Laplacian \(\mathcal{L}: \mathbb{R}^V \to 2^{\mathbb{R}^V}\), which is related to random walks or heat diffusion:
\[
\mathcal{L}(f) = \mathcal{L} f := L(D^{-1} f).
\]
By \([20, \text{Lemma 14, Lemma 15}]\), \(\mathcal{L}\) is a maximal monotone operator (or \(-\mathcal{L}\) is an \(m\)-dissipative operator) on the Hilbert space \((\mathbb{R}^V, \langle \cdot, \cdot \rangle)\). The resolvent \(J_\lambda: \mathbb{R}^V \to 2^{\mathbb{R}^V}\) is defined as follows:
\[
J_\lambda(f) := (I + \lambda \mathcal{L})^{-1}(f).
\]
Here, \(A^{-1}(f)\) for a multivalued operator \(A\) is defined by
\[
A^{-1}(f) := \{ g \in \mathbb{R}^V; f \in A(g) \}.
\]
By \([20, \text{Corollary 2.10}]\), the resolvent \(J_\lambda\) is non-expansive, i.e., for any \(f, g \in \mathbb{R}^V\), and any \(f' \in J_\lambda(f)\), \(g' \in J_\lambda(g)\),
\[
\|f' - g'\| \leq \|f - g\|
\]
holds. This implies that the resolvent operator \(J_\lambda\) is single-valued and continuous w.r.t. variable \(f \in \mathbb{R}^V\). We state these properties as a lemma:

**Lemma 2.1.** *The resolvent operator \(J_\lambda\) of \(\mathcal{L}\) is non-expansive, single-valued, and continuous w.r.t. a variable \(f \in \mathbb{R}^V\). In particular \(J_\lambda\) is injective.*

Since \(L f\) is the sub-differential of \(Q\) at \(D^{-1} f\), it is known that the resolvent \(J_\lambda\) is characterized as
\[
J_\lambda f = \text{argmin} \left\{ \frac{1}{2\lambda} \|f - g\|^2 + Q(D^{-1} g) \ ; \ g \in \mathbb{R}^V \right\}.
\]
By [29 Lemma 2.11(iii)], the resolvent $J_{\lambda}$ satisfies the following equation: For any $\lambda, \mu > 0,$
\begin{equation}
(2.3) \quad J_{\lambda}f = J_{\mu} \left( \frac{\mu}{\lambda} f + \frac{\lambda - \mu}{\lambda} J_{\lambda}f \right).
\end{equation}

The following properties of $J_{\lambda}$ follows from the specified properties of our Laplacian $\mathcal{L}$:

**Lemma 2.2.** For any $\lambda > 0$, any $f \in \mathbb{R}^V$ and any $c \in \mathbb{R}$, the following hold:

1. $\mathcal{L}(cf) = c\mathcal{L}(f)$ and $J_{\lambda}(cf) = c J_{\lambda}(f)$,
2. $J_{\lambda}(\mathbb{R}^V) = \mathbb{R}^V$.

**Proof.** We shall first show (1). For $c = 0$, (1) is trivial. We assume $c > 0$. Then, we have
\[
\arg\max_{b \in B_c} \langle b, cf \rangle = \arg\max_{b \in B_c} \langle b, f \rangle.
\]
Hence, the image of Laplacian of $cf$ becomes
\[
\mathcal{L}(cf) = \left\{ \sum_{e \in E} w_e b_e (b_e^T D^{-1} f) ; b_e \in \arg\max_{b \in B_c} \langle b, cf \rangle \right\}
\]
\[
= c \left\{ \sum_{e \in E} w_e b_e (b_e^T D^{-1} f) ; b_e \in \arg\max_{b \in B_c} \langle b, f \rangle \right\} = c \mathcal{L}(f).
\]
We consider $-c$ for $c > 0$. We have
\[
\arg\max_{b \in \overline{B_c}} \langle b, (-c)f \rangle = -\arg\max_{b \in B_c} \langle b, cf \rangle = -\arg\max_{b \in B_c} \langle b, f \rangle
\]
where $B_c, \overline{B_c}$ are the base polytope with respect to $f$ and $-cf$ respectively. Hence, $\mathcal{L}(-cf) = (-c)\mathcal{L}f$ holds for $c > 0$ similarly.

For nonzero $c \in \mathbb{R}$, let $g = J_{\lambda}(cf)$. Then, $cf \in (I + \lambda \mathcal{L})(g)$ holds by definition of $J_{\lambda}$. Thus, we have $f \in (I + \lambda \mathcal{L})(c^{-1}g)$ by $c^{-1} \mathcal{L}(g) = \mathcal{L}(c^{-1}g)$. This implies $c^{-1}g = J_{\lambda}(f)$, hence $g = c J_{\lambda}(f)$.

Next, we shall prove (2). As in [29 Lemma 15], by modifying the argument in [17 Section 3.1], we can show that a solution $x$ of equation $x + \lambda \mathcal{L}(x) \geq b$ exists for any $\lambda > 0$ and $b \in \mathbb{R}^V$. This argument yields that the domain of $J_{\lambda} = (I + \lambda \mathcal{L})^{-1}$ is $\mathbb{R}^V$. Then, $J_{\lambda}(\mathbb{R}^V) \subset \mathbb{R}^V$. Thus, it suffices to show that for any $g \in \mathbb{R}^V$, there exists an $f \in \mathbb{R}^V$ such that $J_{\lambda}(f) = g$, i.e., $f \in (I + \lambda \mathcal{L})(g)$. This follows from the fact that the domain of $\mathcal{L}$ is $\mathbb{R}^V$, hence $(I + \lambda \mathcal{L})(g)$ is not empty. This concludes the proof. \qed

By the general theory of maximal monotone operators [29 Theorem 4.2 and Theorem 4.10], the heat semigroup $h_t := e^{-t\mathcal{L}}$ is defined for the normalized Laplacian $\mathcal{L}$ and the following holds:
\begin{equation}
(2.4) \quad h_t f = \lim_{\lambda \downarrow 0} J_{\lambda}^{\lfloor t/\lambda \rfloor} f.
\end{equation}
Here, $[a]$ for $a \in \mathbb{R}$ is the maximum integer which is less than or equal to $a$. 
We set
\[ \| \mathcal{L} f \| := \inf \{ \| f' \| \mid f' \in \mathcal{L} f \} . \]

Then, by [29, Lemma 2.11 (ii)], the following holds:
\[ \| J_\lambda f - f \| \leq \lambda \| \mathcal{L} f \| . \]

Since \( \mathcal{L} f \) is a closed convex set by [29, Lemma 2.15], there exists a unique \( f' \in \mathcal{L} f \) such that \( \| f' \| = \| \mathcal{L} f \| \). We set \( \mathcal{L}^0 f \) as this \( f' \). This defines a single-valued operator \( \mathcal{L}^0 : \mathbb{R} V \rightarrow \mathbb{R} V \), called the canonical restriction of \( \mathcal{L} \). Then, the identities
\[ -\mathcal{L}^0 f = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} (J_\lambda f - f) = \lim_{t \downarrow 0} \frac{1}{t} (h_t f - f) \]
hold by [29, Lemma 2.22 and Theorem 3.5].

We show some useful lemmas.

**Lemma 2.3.** For any function \( h \in \mathbb{R} V \) and any \( a \in \mathbb{R} \), the following identity holds:
\[ \mathcal{L} h = \mathcal{L}(h + a\pi). \]

*Proof.* Let \( b \in B_e \). Because \( b \) is a convex combination of \( \delta_x - \delta_y \) for \( x, y \in e \) and \( D^{-1}\pi \) is a constant function, we have
\[ b^\top D^{-1}\pi = 0. \]

Hence, for any \( b \in B_e \), we have
\[ b^\top (D^{-1}(h + a\pi)) = b^\top (D^{-1}h) + ab^\top (D^{-1}\pi) = b^\top (D^{-1}h). \]

This implies that \( \mathcal{L} h = \mathcal{L}(h + a\pi) \) holds. \( \square \)

**Lemma 2.4.** For any function \( f \in \mathbb{R} V \) and any \( a \in \mathbb{R} \), the following identity holds:
\[ J_\lambda f = J_\lambda(f - a\pi) + a\pi. \]

*Proof.* We set \( g := J_\lambda f \) and \( h := J_\lambda(f - a\pi) \). Then there exist \( g' \in \mathcal{L} g \) and \( h' \in \mathcal{L} h \) such that the identities
\[ f = g + \lambda g', \quad f - a\pi = h + \lambda h' \]
hold. Thus, we have
\[ (I + \lambda\mathcal{L})(g) \ni g + \lambda g' = f = h + \lambda h' + a\pi \]
\[ \in (h + a\pi) + \lambda\mathcal{L}(h) = (h + a\pi) + \lambda\mathcal{L}(h + a\pi) = (I + \lambda\mathcal{L})(h + a\pi). \]

Here, the inclusion follows from Lemma 2.3. Therefore, acting \( J_\lambda \) to the both sides, we get \( g = h + a\pi \) because \( J_\lambda = (I + \lambda\mathcal{L})^{-1} \) is injective. \( \square \)
2.3. \(L^1\)-Wasserstein distance. Let \((X, d, m)\) be a metric measure space, that is, \((X, d)\) is a complete separable metric space and \(m\) is a locally finite Borel measure on \(X\). We set \(\mathcal{P}(X)\) as the set of all Borel probability measures. For \(\mu, \nu \in \mathcal{P}(X)\), a measure \(\xi \in \mathcal{P}(X \times X)\) is called a coupling between \(\mu\) and \(\nu\) if

\[
\begin{aligned}
\xi(A \times X) &= \mu(A), \\
\xi(X \times A) &= \nu(A)
\end{aligned}
\]

holds for any Borel set \(A \subset X\). We set \(\mathcal{Cpl}(\mu, \nu)\) as the set of all couplings between \(\mu\) and \(\nu\). Since \(\mu \otimes \nu\) is a coupling between \(\mu\) and \(\nu\), \(\mathcal{Cpl}(\mu, \nu)\) is nonempty. We also define \(\mathcal{P}_1(X)\) by

\[
\mathcal{P}_1(X) := \left\{ \mu \in \mathcal{P}(X) : \int_X d(x, o) \mu(dx) < \infty \text{ for a point } o \in X \right\}.
\]

For \(\mu, \nu \in \mathcal{P}_1(X)\), the \(L^1\)-Wasserstein distance between them, denoted by \(W_1(\mu, \nu)\), is defined as

\[
W_1(\mu, \nu) := \inf \left\{ \int_{X \times X} d(x, y) \xi(dx, dy) : \xi \in \mathcal{Cpl}(\mu, \nu) \right\}.
\]

It is known that \(W_1\) is a metric on \(\mathcal{P}_1(X)\). We have the following duality formula for \(W_1\).

**Proposition 2.5** (Kantorovich-Rubinstein duality). For \(\mu, \nu \in \mathcal{P}_1(X)\),

\[
W_1(\mu, \nu) = \sup \left\{ \int_X f d\mu - \int_X f d\nu : f \text{ is a } 1\text{-Lipschitz} \right\}
\]

(2.7) holds.

We call a Lipschitz function \(f\) that realizes the supremum of (2.7) (if it exists) a Kantorovich potential.

3. Definition of coarse Ricci curvatures on hypergraphs

3.1. Lin-Lu-Yau’s coarse Ricci curvature on graphs. We here review Lin-Lu-Yau’s curvature notion for graphs. Let \(G = (V, E)\) be a simple graph, that is, \(V\) is a set and \(E \subset (V \times V) \setminus \{(x, x) : x \in V\}\). Here we do not distinguish \(\{x, y\}\) and \(\{y, x\}\) ∈ \(E\). For \(x, y \in V\), \(x \sim y\) means \(\{x, y\} \in E\). Given \(x, y \in V\), a sequence of points \(\{z_i\}_{i=0}^n\) is called a path from \(x\) to \(y\) if \(z_0 = x\), \(z_n = y\), \(z_i \sim z_{i+1}\) for \(i = 0, \ldots, n-1\), and \(n\) is called the length of the path. The distance \(d(x, y)\) of \(x, y \in V\) as the least number of lengths of paths from \(x\) to \(y\). A path \(\{z_i\}_{i=0}^n\) is said to be geodesic if it realizes the distance between \(z_0\) and \(z_n\). A function \(w : V \times V \to \mathbb{R}_{\geq 0}\) is a weight function such that \(w(x, y) > 0\) if and only if \(x \sim y\). The degree of \(x \in V\) is defined by \(d_x = \sum_y w(x, y)\).
For $0 < \alpha < 1$ and $x \in V$, the function $m^\alpha_x$ is defined by
\[
m^\alpha_x(y) := \begin{cases} 
\alpha & \text{if } y = x, \\
\frac{1 - \alpha}{d(x, y)} w(x, y) & \text{if } y \sim x, \\
0 & \text{otherwise}.
\end{cases}
\]
We can see $m^\alpha_x$ as a probability measure on $V$. It is trivial that $m^\alpha_x \in P_1(V)$ holds.

We define the $\alpha$-lazy coarse Ricci curvature between $x$ and $y$ by
\[
\kappa^\alpha(x, y) := 1 - \frac{W_1(m^\alpha_x, m^\alpha_y)}{d(x, y)}.
\]
Lin-Lu-Yau [25] defined the coarse Ricci curvature $\kappa_{LLY}$ by
\[
\kappa_{LLY}(x, y) := \lim_{\alpha \uparrow 1} \kappa^\alpha(x, y) 1 - \alpha.
\]
They proved the limit (3.1) always exists in [25].

By the Kantorovich-Rubinstein duality (Proposition 2.5), $W_1(m^\alpha_x, m^\alpha_y)$ can be rephrased by
\[
W_1(m^\alpha_x, m^\alpha_y) = \sup \left\{ \int_X f \, dm^\alpha_x - \int_X f \, dm^\alpha_y \mid f \text{ is a 1-Lipschitz} \right\}
\]
We remark that $m^\alpha_x$ can be written as
\[
m^\alpha_x = (\alpha I + (1 - \alpha)AD^{-1})\delta_x = (I - (1 - \alpha)L)\delta_x,
\]
where $A = (w(x, y))_{x,y} \in \mathbb{R}^{V \times V}$ is the adjacency matrix of the graph $G$. By using this presentation, we have
\[
\int_X f \, dm^\alpha_x = \sum_{z \in V} f(z)m^\alpha_x(z) = f^T m^\alpha_x \\
= f^T (I - (1 - \alpha)L)\delta_x \\
= ((I - (1 - \alpha)L)Df)^T D^{-1} \delta_x \\
= \langle (I - (1 - \alpha)L)Df, \delta_x \rangle.
\]
Thus, $W_1(m^\alpha_x, m^\alpha_y)$ can be written as
\[
W_1(m^\alpha_x, m^\alpha_y) = \sup \left\{ \langle (I - (1 - \alpha)L)Df, \delta_x - \delta_y \rangle \mid f \text{ is a 1-Lipschitz} \right\}.
\]
We set $\lambda = 1 - \alpha$. Then, we remark that
\[
(I - \lambda L)(g) = J_\lambda(g) + O(\lambda^2)
\]
holds for any $g \in \mathbb{R}^V$. Indeed, in the case of the usual graph Laplacian $L$, for sufficiently small $\lambda$, the resolvent $J_\lambda = (I + \lambda L)^{-1}$ has a Neumann series expansion as
\[
J_\lambda g = g - \lambda Lg + \sum_{k=2}^{\infty} (-\lambda L)^k g = (I - \lambda L)g + O(\lambda^2).
\]
We define the $\lambda$-linear Kantorovich difference $KD_{\lambda}(x, y)$ using the resolvent $J_{\lambda}$ as
\begin{equation}
KD_{\lambda}(x, y) = \sup \{ \langle J_{\lambda}Df, \delta_x - \delta_y \rangle ; f \text{ is a } 1\text{-Lipschitz} \}.
\end{equation}
Then, by the estimate (3.3), we can show
\begin{equation}
W_1(m^\alpha_x, m^\alpha_y) = KD_{\lambda}(x, y) + o(\lambda).
\end{equation}
We prove this estimate rigorously in Section 4. The crucial point to extend the definition of curvature notion to hypergraphs is that although $W_1(m^\alpha_x, m^\alpha_y)$ cannot be extended to hypergraph due to the multivalued property of our Laplacian, $KD_{\lambda}(x, y)$ can be extended naturally due to the single-valued property of the resolvent $J_{\lambda}$ for our Laplacian. In the following section, we shall introduce a $\lambda$-nonlinear Kantorovich difference which is a natural generalization of (3.4).

3.2. Nonlinear Kantorovich difference. Let $K$ be a positive number. A function $f \in \mathbb{R}^V$ is said to be weighted $K$-Lipschitz if $D^{-1}f$ is a $K$-Lipschitz function with respect to the distance $d$, i.e., $f$ satisfies
\[
\left| \frac{f(x)}{d_x} - \frac{f(y)}{d_y} \right| \leq Kd(x, y)
\]
for any $x, y \in V$. It is equivalent to $\langle f, \delta_x - \delta_y \rangle \leq Kd(x, y)$ for any $x, y \in V$. We represent the set of weighted $K$-Lipschitz functions on $V$ as $\text{Lip}_w^K(V)$.

**Definition 3.1.** Let $\lambda > 0$. For two vertices $x, y \in V$, the $\lambda$-nonlinear Kantorovich difference $KD_{\lambda}(x, y)$ of $x$ and $y$ is defined by
\[
KD_{\lambda}(x, y) := \sup \{ \langle J_{\lambda}f, \delta_x \rangle - \langle J_{\lambda}f, \delta_y \rangle ; f \in \text{Lip}_w^1(V) \}.
\]

**Remark 3.2.** Formally, we write
\[
\int f \, d\mu^\lambda_x := \langle J_{\lambda}f, \delta_x \rangle.
\]
When the hypergraph $H$ is a usual graph, $\mu^\lambda_x$ becomes a measure. Definition 3.1 is inspired by the Kantorovich-Rubinstein duality. As shown later, for each $\lambda > 0$, the $\lambda$-nonlinear Kantorovich difference is a distance function on $V$.

We prove that we can restrict the class of functions in the definition of $KD_{\lambda}$ to a smaller subset.

**Proposition 3.3.** For a function $f$, we set
\[
\|f\|_\infty := \max_{z \in V} \left| \frac{f(z)}{d_z} \right|.
\]
Then, the identity
\[
KD_{\lambda}(x, y) = \sup \{ \langle J_{\lambda}f, \delta_x - \delta_y \rangle ; f \in \text{Lip}_w^1(V), \|f\|_\infty \leq \text{diam}(H) \}
\]
holds. Moreover, the subset of $\text{Lip}_w^1(V)$
\[
\tilde{\text{Lip}}_w^1(V) = \{ f \in \text{Lip}_w^1(V) ; \|f\|_\infty \leq \text{diam}(H) \}
\]
is a bounded and closed subset of the finite dimensional Euclidean space \((\mathbb{R}^V, \langle \cdot, \cdot \rangle)\), hence a compact subset.

**Proof.** Let \(f\) be a weighted 1-Lipschitz function. We fix \(y_0 \in V\) satisfying \(\langle f, \delta_{y_0} \rangle = \min_{x \in V} \langle f, \delta_x \rangle\). We set \(\alpha := \langle f, \delta_{y_0} \rangle\) and \(F := f - \alpha \cdot \text{vol}(V)\pi\). Then, as

\[
\langle F, \delta_x - \delta_y \rangle = \frac{f(x) - \alpha d_x}{d_x} - \frac{f(y) - \alpha d_y}{d_y} = \langle f, \delta_x - \delta_y \rangle,
\]

this \(F\) is also a weighted 1-Lipschitz function. By

\[
\langle F, \delta_{y_0} \rangle = \frac{f(y_0)}{d_{y_0}} - \alpha = 0,
\]

we have

\[
\langle F, \delta_x \rangle = \langle F, \delta_x \rangle - \langle F, \delta_{y_0} \rangle \leq d(x, y_0) \leq \text{diam}(H)
\]

for any \(x \in V\). This implies that \(\|F\|_{\infty} \leq \text{diam}(H)\). Moreover, by Lemma 2.4 the identity \(J_\lambda F = J_\lambda f - \alpha \cdot \text{vol}(V)\pi\) holds. Thus, by

\[
\frac{J_\lambda F(x)}{d_x} - \frac{J_\lambda F(y)}{d_y} = \frac{J_\lambda f(x) - \alpha d_x}{d_x} - \frac{J_\lambda f(y) - \alpha d_y}{d_y} = \frac{J_\lambda f(x)}{d_x} - \frac{J_\lambda f(y)}{d_y},
\]

we obtain the desired properties.

Since \(\text{Lip}_1^w(V)\) is a subset of the finite dimensional Euclidean space \((\mathbb{R}^V, \langle \cdot, \cdot \rangle)\), it is bounded with respect to \(\|\cdot\|_{\infty}\). Also since \(f(v)/d_v = \langle f, \delta_v \rangle\), \(L^2\)-convergence, \(L^2\)-weak convergence, and point-wise convergence are all equivalent. Hence \(\text{Lip}_1^w(V)\) is bounded closed, that is, compact.

We prove the finiteness of \(\text{KD}_\lambda(x, y)\).

**Lemma 3.4.** Let \(\lambda > 0\). For any \(x, y \in V\), \(0 \leq \text{KD}_\lambda(x, y) < \infty\).

**Proof.** Non-negativity is obvious since \(0 \in \mathbb{R}^V\) is weighted 1-Lipschitz. The finiteness of \(\text{KD}_\lambda(x, y)\) follows from the continuity of \(J_\lambda\) and the compactness of \(\text{Lip}_1^w(V)\) by Proposition 3.3. However, as we show the estimate (3.7) which we use later, we shall demonstrate a down-to-earth proof here.

To prove the finiteness of \(\text{KD}_\lambda(x, y)\), we first show that for any \(f \in \text{Lip}_1^w(V)\), \(\|\mathcal{L} f\| \leq \text{vol}(V)^{1/2}\) holds. Since \(f\) is weighted 1-Lipschitz,

\[
\mathbf{b}_e^\top (D^{-1} f) = \max_{u, v \in e} \left| \frac{f(u)}{d_u} - \frac{f(v)}{d_v} \right| \leq 1
\]

holds for any \(\mathbf{b}_e \in \argmax_{x \in B_e} \mathbf{b}_e^\top (D^{-1} f)\). We set \(f' = \sum_e w_e \mathbf{b}_e (\mathbf{b}_e^\top (D^{-1} f)) \in \mathcal{L} f\). We note that \(|\mathbf{b}_e(x)| \leq 1\) for any \(x \in V\) since \(\mathbf{b}_e \in B_e\). Then

\[
|f'(x)| = \left| \sum_{e \ni x} w_e \mathbf{b}_e^\top (D^{-1} f) \mathbf{b}_e(x) \right| \leq \sum_{e \ni x} w_e = d_x.
\]

Consequently, we obtain

\[
\|\mathcal{L} f\|^2 \leq \langle f', f' \rangle = \sum_{x \in V} f'(x)^2 d_x^{-1} \leq \sum_{x \in V} d_x = \text{vol}(V).
\]
Now, we go back to the proof. Define \( M := \max_{x \in V} d_x^{-1/2} \). For any weighted 1-Lipschitz \( f \),
\[
\int f \, d\mu_x^\lambda - \int f \, d\mu_y^\lambda = \langle J_\lambda f - f, \delta_x \rangle + \langle f, \delta_y \rangle - \langle J_\lambda f - f, \delta_y \rangle
\]
\[
\leq \| J_\lambda f - f \| (\| \delta_x \| + \| \delta_y \|) + d(x, y)
\]
\[
\leq \lambda \| Lf \| (d_x^{-1/2} + d_y^{-1/2}) + d(x, y)
\]
\[
\leq 2\lambda \text{vol}(V)^{1/2} M + d(x, y).
\]
Because the last quantity is independent of \( f \), we take the suprimum with respect to \( f \) to get
\[
(3.7) \quad \text{KD}_\lambda(x, y) \leq 2\lambda \text{vol}(V)^{1/2} M + d(x, y) < \infty.
\]

The following proposition shows that \( \text{KD}_\lambda \) is a distance function on \( V \):

**Proposition 3.5.** Let \( \lambda > 0 \). The following hold:

1. \( \text{KD}_\lambda(x, y) = 0 \iff x = y \).
2. \( \text{KD}_\lambda(x, y) = \text{KD}_\lambda(y, x) \).
3. \( \text{KD}_\lambda(x, z) \leq \text{KD}_\lambda(x, y) + \text{KD}_\lambda(y, z) \).

**Proof.** (1) \((\Leftarrow)\) and (2) follow from the definition. First, we prove (1) \((\Rightarrow)\). We assume that \( \text{KD}_\lambda(x, y) = 0 \). Then, for any \( f \in \text{Lip}_1^\lambda(V) \),
\[
\langle J_\lambda f, \delta_x - \delta_y \rangle = 0.
\]
On the other hand, the equality of sets
\[
\{ c f; f \in \text{Lip}_1^\lambda(V), c \in \mathbb{R} \} = \mathbb{R}^V
\]
holds. Thus, by combining \( \text{R} \) and the properties \( J_\lambda(cf) = cJ_\lambda(f) \) for any \( c \in \mathbb{R} \) and \( J_\lambda(\mathbb{R}^V) = \mathbb{R}^V \) as in Lemma 2.2, we have
\[
\langle g, \delta_x - \delta_y \rangle = 0
\]
for any \( g \in \mathbb{R}^V \). By the non-degeneracy of the inner product, we have \( \delta_x = \delta_y \), hence \( x = y \). Next, we prove the triangle inequality (3). For any \( \epsilon > 0 \), there exists a weighted 1-Lipschitz function \( f \) such that
\[
\text{KD}_\lambda(x, z) \leq \int f \, d\mu_x^\lambda - \int f \, d\mu_z^\lambda + \epsilon.
\]
Thus, we have
\[
\text{KD}_\lambda(x, z) \leq \int f \, d\mu_x^\lambda - \int f \, d\mu_z^\lambda + \epsilon
\]
\[
= \int f \, d\mu_x^\lambda - \int f \, d\mu_y^\lambda + \int f \, d\mu_y^\lambda - \int f \, d\mu_z^\lambda + \epsilon
\]
\[
\leq \text{KD}_\lambda(x, y) + \text{KD}_\lambda(y, z) + \epsilon.
\]
Since \( \epsilon > 0 \) is any positive number, the triangle inequality holds. \( \square \)
Remark 3.6. Although the $\lambda$-nonlinear Kantorovich difference $KD_\lambda$ is a distance function on $V$ by Proposition 3.3, it is still unclear how to change this distance with respect to $\lambda$. We only know the following inequality;

$$|KD_\lambda(x, y) - KD_\mu(x, y)| \leq 2M\vol(V)^{1/2}|\lambda - \mu|.$$ 

In fact, by the equation (2.20), we have

$$\langle J_\lambda f - J_\mu f, \delta_x - \delta_y \rangle = \langle J_\mu \left( \frac{\mu}{\lambda} f + \frac{\lambda - \mu}{\lambda} J_\lambda f \right) - J_\mu f, \delta_x - \delta_y \rangle \leq \left\| J_\mu \left( \frac{\mu}{\lambda} f + \frac{\lambda - \mu}{\lambda} J_\lambda f - J_\mu f \right) \right\| \cdot \|\delta_x - \delta_y\|

\leq 2M(\lambda - \mu)||L^0 f|| \leq 2\vol(V)^{1/2}M|\lambda - \mu|.$$

For any $\epsilon > 0$, there exists a function $f \in \lip_1^\lambda(V)$ such that $\langle J_\lambda f, \delta_x - \delta_y \rangle + \epsilon \geq KD_\lambda(x, y)$. Thus

$$KD_\lambda(x, y) - \epsilon \leq \langle J_\lambda f, \delta_x - \delta_y \rangle = (J_\lambda f - J_\mu f, \delta_x - \delta_y) + (J_\mu f, \delta_x - \delta_y) \leq 2\vol(V)^{1/2}M|\lambda - \mu| + KD_\mu(x, y).$$

Since $\epsilon > 0$ is arbitrary, we obtain $KD_\lambda(x, y) - KD_\mu(x, y) \leq 2\vol(V)^{1/2}M|\lambda - \mu|$. Changing the role of $\mu$ and $\lambda$, we have the conclusion.

Next, we prove existence of a weighted 1-Lipschitz function which attains the supremum in the definition of $KD_\lambda$.

**Proposition 3.7.** For any $x, y \in V$ and $\lambda > 0$, there exists a weighted 1-Lipschitz function $f$ such that the following identity holds:

$$\langle J_\lambda f, \delta_x \rangle - \langle J_\lambda f, \delta_y \rangle = KD_\lambda(x, y). \quad (3.9)$$

**Proof.** Let $\{f_n\} \subset \lip_1^\lambda(V)$ be a maximizing sequence of $KD_\lambda(x, y)$. As mentioned in Proposition 3.3, without loss of generality, we may assume $\|f_n\|_{\infty} \leq \diam(H)$ for all $n$. Now $\lip_1^\lambda(V)$ is a compact subset of the finite dimensional Euclidean space $(\mathbb{R}^V, \langle \cdot, \cdot \rangle)$ by Proposition 3.3 again, thus a sequentially compact subset. Hence, $\{f_n\} \subset \lip_1^\lambda(V) \subset \mathbb{R}^V$ has a subsequence $\{f_{n_j}\}$ which converges to an element $f'$ in $\lip_1^\lambda(V)$. Because $J_\lambda$ is continuous by Lemma 2.3, we have $J_\lambda(f_{n_j}) \to J_\lambda(f')$ as $n \to \infty$. Therefore,

$$KD_\lambda(x, y) = \lim_{j \to \infty} \langle J_\lambda(f_{n_j}), \delta_x - \delta_y \rangle = \langle J_\lambda(f'), \delta_x - \delta_y \rangle$$

holds. This implies the equation (3.9). □

We call a function satisfying the equality (3.9) a $\lambda$-nonlinear Kantorovich potential.
3.3. **Coarse Ricci curvature on hypergraphs.** In this subsection, we define the \( \lambda \)-coarse Ricci curvature \( \kappa_\lambda(x, y) \) on hypergraph and show some properties of it.

**Definition 3.8.** Let \( \lambda > 0 \). We fix \( x, y \in V \). The \( \lambda \)-coarse Ricci curvature \( \kappa_\lambda(x, y) \) along with \( x, y \) on the hypergraph \( H \) is defined by

\[
\kappa_\lambda(x, y) := 1 - \frac{\text{KD}_\lambda(x, y)}{d(x, y)}.
\]

The lower coarse Ricci curvature \( \underline{\kappa}(x, y) \) is defined by

\[
\underline{\kappa}(x, y) := \liminf_{\lambda \downarrow 0} \frac{\kappa_\lambda(x, y)}{\lambda}.
\]

In the similar manner, the upper coarse Ricci curvature \( \overline{\kappa}(x, y) \) is defined by

\[
\overline{\kappa}(x, y) := \limsup_{\lambda \downarrow 0} \frac{\kappa_\lambda(x, y)}{\lambda}.
\]

If \( \underline{\kappa}(x, y) = \overline{\kappa}(x, y) \) holds, we call this the coarse Ricci curvature for \( x, y \), denoted by \( \kappa(x, y) \).

**Remark 3.9.** By the inequality (3.7), the following holds:

\[
\kappa_\lambda(x, y) = 1 - \frac{\text{KD}_\lambda(x, y)}{d(x, y)} \geq 1 - 2\lambda\text{vol}(V)^{1/2}M/d(x, y) \geq -\lambda\text{vol}(V)^{1/2}M/d(x, y).
\]

This estimate implies that \( \underline{\kappa}(x, y) \) satisfies \( \underline{\kappa}(x, y) \geq -2\text{vol}(V)^{1/2}M/d(x, y) \).

It is nontrivial that \( \overline{\kappa}(x, y) \) is finite. We prove this in the following.

**Lemma 3.10.** For any \( x, y \in V \), \( \overline{\kappa}(x, y) < \infty \) holds.

**Proof.** We fix \( x, y \in V \). Let \( f \) be a weighted 1-Lipschitz function satisfying \( f(x)/d_x = f(y)/d_y = d(x, y) \). Then

\[
\text{KD}_\lambda(x, y) \geq \langle J_\lambda f, \delta_x \rangle - \langle J_\lambda f, \delta_y \rangle = \langle J_\lambda f - f, \delta_x \rangle + \langle f, \delta_x \rangle - \langle f, \delta_y \rangle - \langle J_\lambda f - f, \delta_y \rangle = \langle J_\lambda f - f, \delta_x \rangle + d(x, y) - \langle J_\lambda f - f, \delta_y \rangle
\]

hold. Thus since

\[
\frac{1}{\lambda} \left( 1 - \frac{\text{KD}_\lambda(x, y)}{d(x, y)} \right) \leq \frac{\lambda^{-1}(J_\lambda f - f, \delta_y) - \lambda^{-1}(J_\lambda f - f, \delta_x)}{d(x, y)},
\]

we have

\[
\overline{\kappa}(x, y) \leq \frac{\langle \ell^0 f, \delta_x \rangle - \langle \ell^0 f, \delta_y \rangle}{d(x, y)} < \infty
\]

by the equation (2.6). \( \square \)

We prove that for any hypergraph \( H \), the identity \( \overline{\kappa}(x, y) = \underline{\kappa}(x, y) \) holds for any \( x, y \in V \). We leave the complete proof in Appendix [A] since it is a bit longer.

**Theorem 3.11.** For any hypergraph \( H = (V, E, w) \), the identity \( \overline{\kappa}(x, y) = \underline{\kappa}(x, y) \) holds for any \( x, y \in V \).
We prove that Theorem 3.11 for graphs in Section 4 by a more straightforward way than the case for hypergraphs.

In the hypergraph case, we already know that both curvatures coincide. However, as shown in Appendix B, we can extend the concept of upper and lower coarse Ricci curvature for submodular Laplacians. From now on, we obtain several results in which we assume a lower bound of either \( \kappa \) or \( \overline{\kappa} \) taking the case for submodular Laplacian into account.

Next, we show a relation between the inferiors of \( \kappa(x, y) \) for any pairs of vertices and for adjacent vertices.

**Lemma 3.12.** Let \( a \) be a real number. Then, the following holds:

\[
\inf_{x \sim y} \kappa(x, y) \geq a \implies \inf_{x, y} \kappa(x, y) \geq a.
\]

In particular, \( \inf_{x \sim y} \kappa(x, y) = \inf_{x, y} \kappa(x, y) \) holds.

**Proof.** Let \( n = d(x, y) \) and \( \{x_i\} \) be a shortest path connecting \( x \) and \( y \). Then, the following inequality holds for any \( \lambda > 0 \):

\[
\kappa(x, y) = 1 - \frac{\text{KD}_\lambda(x, y)}{d(x, y)} \geq 1 - \frac{\sum_{i=0}^{n-1} \text{KD}_\lambda(x_i, x_{i+1})}{n} \geq \frac{1}{n} \sum_{i=0}^{n-1} \left( 1 - \frac{\text{KD}_\lambda(x_i, x_{i+1})}{d(x_i, x_{i+1})} \right).
\]

Here, the inequality follows from Proposition 3.5 (3). Hence, we have

\[
\kappa(x, y) \geq \frac{1}{n} \sum_{i=0}^{n-1} \kappa(x_i, x_{i+1}) \geq a.
\]

We show another property of the inferior of the lower coarse Ricci curvatures.

**Lemma 3.13.** We set \( \kappa_\lambda := \inf_{x, y} \kappa_\lambda(x, y) \) and \( \overline{\kappa}_0 := \inf_{x, y} \overline{\kappa}(x, y) \). Then, the identity \( \lim_{\lambda \to 0^+} \kappa_\lambda / \lambda = \overline{\kappa}_0 \) holds.

**Proof.** Because \( V \) is a finite set, so is \( V \times V \). Thus, \( \kappa_\lambda = \min_{x, y} \kappa_\lambda(x, y) \) and \( \overline{\kappa}_0 = \min_{x, y} \overline{\kappa}(x, y) \) hold. We set \( \{z_\lambda\} := \{(x_\lambda, y_\lambda)\}_\lambda \) and assume that this satisfies \( \kappa_\lambda(x_\lambda, y_\lambda) = \kappa_\lambda \). By a simple argument, we can show that there is a pair \( (x_\infty, y_\infty) \in V \times V \) such that

\[
\liminf_{\lambda \to 0} \frac{\kappa_\lambda(x_\infty, y_\infty)}{\lambda} = \liminf_{\lambda \to 0} \frac{\kappa_\lambda}{\lambda}
\]

holds due to the finiteness of \( V \times V \). Let \( \kappa_0 = \kappa(x_0, y_0) \). Then, because

\[
\frac{\kappa_\lambda(x_\infty, y_\infty)}{\lambda} \leq \frac{\kappa_\lambda(x_0, y_0)}{\lambda},
\]

by taking the limit \( \lambda \to 0 \), we have

\[
\overline{\kappa}_0 \leq \overline{\kappa}(x_\infty, y_\infty) = \liminf_{\lambda \to 0} \frac{\kappa_\lambda(x_\infty, y_\infty)}{\lambda} = \liminf_{\lambda \to 0} \frac{\kappa_\lambda}{\lambda} = \frac{\kappa_0}{\lambda},
\]

\[
\leq \liminf_{\lambda \to 0} \frac{\kappa_\lambda(x_0, y_0)}{\lambda} = \overline{\kappa}(x_0, y_0) = \overline{\kappa}_0.
\]

Here, the first inequality is by the definitions of \( \overline{\kappa}_0 \). This concludes the proof. \( \square \)
4. Comparison with Lin-Lu-Yau’s coarse Ricci curvature on graphs

In this section, we show that for graphs, \(\tau(x, y) = \kappa(x, y)\) holds for any \(x, y \in V\) and that this coarse Ricci curvature \(\kappa(x, y) = \tau(x, y) = \kappa_L(x, y)\) is same as the one of Lin-Lu-Yau \([25]\) denoted by \(\kappa_{LLY}\). In this sense, our coarse Ricci curvature can be regarded as a generalization of Lin-Lu-Yau’s to hypergraphs.

**Proposition 4.1.** Let \(G = (V, E, \omega)\) be a weighted undirected graph. Then, \(\tau(x, y) = \kappa_L(x, y)\) holds for any \(x, y \in V\). Moreover, \(\kappa(x, y) = \kappa_{LLY}(x, y)\) holds for any \(x, y \in V\), where \(\kappa_{LLY}(x, y)\) is the coarse Ricci curvature introduced by Lin-Lu-Yau \([25]\).

**Proof.** The definition of Lin-Lu-Yau’s coarse Ricci curvature \(\kappa_{LLY}(x, y)\) is

\[\kappa_{LLY}(x, y) := \lim_{\alpha \uparrow 1} \frac{\kappa_\alpha(x, y)}{1 - \alpha}, \quad \kappa_\alpha(x, y) = 1 - \frac{W_1(m^\alpha_x, m^\alpha_y)}{d(x,y)}.\]

On the other hand, the definition of our curvature \(\kappa(x, y)\) (if it exists) is

\[\kappa(x, y) := \lim_{\lambda \downarrow 0} \frac{\kappa_\lambda(x, y)}{\lambda}, \quad \kappa_\lambda(x, y) = 1 - \frac{\text{KD}_\lambda(x, y)}{d(x,y)}.\]

We evaluate the difference of them with \(\lambda = 1 - \alpha\). In what follows, we assume that \(\lambda > 0\) is sufficiently small. Because the equation

\[(4.1) \quad \frac{1}{\lambda} |\kappa_\alpha(x, y) - \kappa_\lambda(x, y)| = \frac{1}{\lambda} |W_1(m^\alpha_x, m^\alpha_y) - \text{KD}_\lambda(x, y)| d(x,y)\]

holds, it is sufficient to evaluate \(|W_1(m^\alpha_x, m^\alpha_y) - \text{KD}_\lambda(x, y)|/\lambda\). There exist some potentials in the both case of \(W_1\) (cf. \([43]\)) and \(\text{KD}_\lambda\) by Proposition 3.7. We use these potentials for the following arguments. Let \(f^\alpha\) be a Kantorovich potential for \((m^\alpha_x, m^\alpha_y)\). Then, we obtain

\[W_1(m^\alpha_x, m^\alpha_y) - \text{KD}_\lambda(x, y) \leq \left( \int f^\alpha \ dm^\alpha_x - \int f^\alpha \ dm^\alpha_y \right) - (J_\lambda f^\alpha(x) - J_\lambda f^\alpha(y)) = \{(I - \lambda L) f^\alpha - J_\lambda f^\alpha\}(x) - \{(I - \lambda L) f^\alpha - J_\lambda f^\alpha\}(y).\]

Here, because \(J_\lambda = (I + \lambda L)^{-1}\) holds and \(\lambda\) is sufficiently small, we can take the Neumann series expansion as

\[J_\lambda f = f - \lambda Lf + \sum_{i=2}^{\infty} (-\lambda L)^i f.\]

Hence, we have

\[\frac{1}{\lambda}(W_1(m^\alpha_x, m^\alpha_y) - \text{KD}_\lambda(x, y)) \leq \sum_{i=2}^{\infty} (-\lambda)^{i-1} L^i f^\alpha(x) - \sum_{i=2}^{\infty} (-\lambda)^{i-1} L^i f^\alpha(y) \rightarrow 0\]

when \(\lambda \rightarrow 0\). Hence, we have

\[\lim_{\lambda \downarrow 0} \frac{1}{\lambda}(W_1(m^\alpha_x, m^\alpha_y) - \text{KD}_\lambda(x, y)) \leq 0.\]
By exchanging the rolls of $\text{KD}_\lambda$ and $W_1$, we obtain the similar result
\[
\lim_{\lambda \downarrow 0} \frac{1}{\lambda} (\text{KD}_\lambda(x, y) - W_1(m_x^\alpha, m_y^\alpha)) \leq 0.
\]
Consequently, we have
\[
\lim_{\lambda \downarrow 0} \left| W_1(m_x^\alpha, m_y^\alpha) - \text{KD}_\lambda(x, y) \right| = 0.
\]
In particular, because the limit $\lim_{\alpha \uparrow 1} \kappa^\alpha(x, y)/(1 - \alpha)$ exists by [25, P.609], the limit $\lim_{\lambda \downarrow 0} \kappa_\lambda(x, y)/\lambda$ also exists. This shows that Theorem 3.11 holds for graphs. Moreover, by combining the estimate (4.2) and the equation (4.1), we have $\kappa_{LLY}(x, y) = \kappa(x, y)$ for any $x, y \in V$. □

Remark 4.2. Any arguments so far are applicable for some other situations. In particular, by similar arguments, we can show that the Ricci curvature on directed graphs defined by Sakurai-Ozawa-Yamada [35] is same as a modification of our Ricci curvature to directed graphs. Indeed, because the Laplacian $\Delta$ in their definition is self-conjugate and nonpositive definite operator ([35, Prop. 2.4]) and the measure appears in their definition can be calculated as
\[
\int_V f \, dv_x^\epsilon = (I + \epsilon \Delta) f(x)
\]
([35, Lemma 3.1]), we can accomplish the similar proof as in Proposition 4.1.

5. Applications

5.1. Eigenvalues of Laplacian. We call an element $\mu \in \mathbb{R}_{>0}$ an eigenvalue of $\mathcal{L}$ if there is a function $f \in \mathbb{R}^V$ satisfying $\mathcal{L}^0 f = \mu f$. Then, the inferior of coarse Ricci curvatures bounds the eigenvalues.

**Theorem 5.1.** If $\inf_{x, y} \kappa(x, y) = \kappa_0$, then $\kappa_0 \leq \mu$ holds.

**Proof.** We assume that $\mathcal{L}^0 f = \mu f$ holds. By multiplying some constant if we need, we may assume that $f$ is weighted 1-Lipschitz. Moreover, without loss of generality, we may assume that $f(x)/d_x - f(y)/d_y = d(x, y)$ holds for some $x, y \in V$. Then, we have
\[
\text{KD}_\lambda(x, y) \geq (J_\lambda f, \delta_x) - (J_\lambda f, \delta_y)
\]
\[
= (J_\lambda f - f, \delta_x) + \frac{f(x)}{d_x} - \frac{f(y)}{d_y} - (J_\lambda f - f, \delta_y)
\]
\[
= (J_\lambda f - f, \delta_x) + d(x, y) - (J_\lambda f - f, \delta_y).
\]
Therefore, we have
\[
\overline{\kappa} \leq \underline{\kappa}(x, y) = \limsup_{\lambda \downarrow 0} \frac{\kappa_{\lambda}(x, y)}{\lambda} \\
\leq \limsup_{\lambda \downarrow 0} \lambda^{-1} \left(1 - \frac{\langle J_{\lambda}f - f, \delta_x \rangle + d(x, y) - \langle J_{\lambda}f - f, \delta_y \rangle}{d(x, y)}\right) \\
\leq \limsup_{\lambda \downarrow 0} \frac{1}{d(x, y)} \left(\langle \lambda^{-1}(J_{\lambda}f - f), \delta_y \rangle - \langle \lambda^{-1}(J_{\lambda}f - f), \delta_x \rangle\right) \\
= \frac{1}{d(x, y)} \left(\langle L_0^0 f, \delta_x \rangle - \langle L_0^0 f, \delta_y \rangle\right) \\
= \frac{\mu}{d(x, y)} \left(\frac{f(x)}{d_x} - \frac{f(y)}{d_y}\right) = \mu.
\]

\[\square\]

5.2. Gradient estimate. By using our coarse Ricci curvature, we can show a gradient estimate.

**Theorem 5.2.** Let \(\overline{\kappa}_0 = \inf_{x, y} \overline{\kappa}(x, y)\). Then, for any \(x, y \in V\) and \(t > 0\), the following inequality holds:
\[
\frac{h_t f(x)}{d_x} - \frac{h_t f(y)}{d_y} \leq e^{-\overline{\kappa}_0 t} d(x, y).
\]

**Proof.** By multiplying some constant if we need, we may assume that \(f\) is weighted 1-Lipschitz. We set \(\kappa_{\lambda} := \inf_{x, y} \kappa_{\lambda}(x, y)\). By the definition of \(K\Delta_{\lambda}\) and \(\kappa_{\lambda}\), we have
\[
\frac{J_{\lambda} f}{d_x} - \frac{J_{\lambda} f}{d_y} = \langle J_{\lambda} f, \delta_x \rangle - \langle J_{\lambda} f, \delta_y \rangle \leq K\Delta_{\lambda}(x, y) \leq (1 - \kappa_{\lambda}) d(x, y).
\]

Hence, \(J_{\lambda} f / (1 - \kappa_{\lambda})\) is also weighted 1-Lipschitz. By a similar calculation, we have
\[
\frac{J_{\lambda} f}{d_x} - \frac{J_{\lambda} f}{d_y} = \langle J_{\lambda} f, \delta_x \rangle - \langle J_{\lambda} f, \delta_y \rangle \\
= (1 - \kappa_{\lambda}) \left(\left\langle J_{\lambda} f, \left(\frac{1}{1 - \kappa_{\lambda}}\right), \delta_x \right\rangle - \left\langle J_{\lambda} f, \left(\frac{1}{1 - \kappa_{\lambda}}\right), \delta_y \right\rangle\right) \\
\leq (1 - \kappa_{\lambda})^2 d(x, y).
\]

Iterating similar calculations implies that \(\langle J_{\lambda}^n f, \delta_x \rangle - \langle J_{\lambda}^n f, \delta_y \rangle \leq (1 - \kappa_{\lambda})^n d(x, y)\) holds for any non-negative integer \(n\). Therefore, by the equation (2.4), we have
\[
\frac{h_t f(x)}{d_x} - \frac{h_t f(y)}{d_y} = \lim_{\lambda \downarrow 0} \left(\langle J_{\lambda}^{\lfloor t/\lambda \rfloor} f, \delta_x \rangle - \langle J_{\lambda}^{\lfloor t/\lambda \rfloor} f, \delta_y \rangle\right) \\
\leq \liminf_{\lambda \downarrow 0} (1 - \kappa_{\lambda})^{\lfloor t/\lambda \rfloor} d(x, y) \\
\leq \liminf_{\lambda \downarrow 0} e^{\frac{-\mu}{\lambda} t} d(x, y) \\
= e^{-\overline{\kappa}_0 t} d(x, y).
\]

Here, the second inequality follows from the well-known inequality \(e^{xt} \geq (1 + x)^t\) for given \(|x| < 1\) and any \(t > 0\).\[\square\]
5.3. Diameter bound. We also have a geometric consequence under the upper Ricci curvature being bound from below.

**Theorem 5.3** (Bonnet-Myers type diameter bound). We assume \( \inf_{x \neq y} \pi(x, y) \geq \kappa_0 > 0 \) holds. Then, the following holds:

\[
\text{diam}(H) \leq \frac{2}{\kappa_0}.
\]

**Proof.** Let \( p, q \in V \) be vertices which satisfy \( d(p, q) = \text{diam}(H) \). Then, for \( f \in \text{Lip}_w^1(V) \) with \( \langle f, \delta_p - \delta_q \rangle = d(p, q) \), by the inequality (3.10), we have

\[
\kappa_0 \leq \frac{\langle \mathcal{L}^0 f, \delta_p - \delta_q \rangle}{d(p, q)} \leq \frac{2}{d(p, q)}
\]

since such \( f \) satisfies \( |\mathcal{L}^0 f(x)| \leq d_x \) for any \( x \in V \) as the estimate (3.6). \( \square \)

6. Examples

In this section, we calculate some examples. A key formula for calculation is

\[
J_\lambda f = \arg\min \left\{ \frac{\|f - g\|^2}{2\lambda} + Q(D^{-1}g) : g \in \mathbb{R}^V \right\}.
\]

**Example 6.1.** We consider the hypergraph \( H = (V, E, w) \), where \( V := \{x, y, z\} \), \( E := \{xy, yz, zx, xyz\} \), and \( w(e) = 1 \) for any \( e \in E \). We here calculate the coarse Ricci curvature for this hypergraph \( H \).

More precisely, we calculate \( KD_\lambda(x, y) \) for \( x, y \in V \) and a sufficiently small \( \lambda > 0 \).

Let \( f \) be a weighted 1-Lipschitz function. We set the value \( f(x) = 3\alpha, f(y) = 3\beta, \) and \( f(z) = 3\gamma \). By Proposition 3.3, we may assume \( \beta = 0 \). We divide our argument into some cases as

1. \( \alpha > \gamma > 0 \),
2. \( \gamma > \alpha > 0 \),
3. \( \gamma = 0 \),
4. \( \gamma = \alpha \),
5. \( \alpha > 0 > \gamma \).

Moreover, we divide the cases for the values of \( J_\lambda f \). We remark that \( 0 \leq \alpha, \gamma \leq 1 \) holds, because \( f \) is weighted 1-Lipschitz. We set \( g = J_\lambda f \). Since \( J_\lambda f \to f \) as \( \lambda \to 0 \), we write \( g(x) = 3\alpha + 3a, g(y) = 3b, g(z) = 3\gamma + 3c \). We define

\[
F(g) := \frac{1}{2\lambda}\|f - g\|^2 + Q(D^{-1}g).
\]

(1) \( \alpha > \gamma > 0 \). Because \( \lambda > 0 \) is sufficiently small, \( J_\lambda f \) is closed to \( f \). Hence, we may assume \( \alpha + \alpha > \gamma + c > b \). Then, the Laplacian \( \mathcal{L} \) is determined uniquely and

\[
\begin{align*}
   b_{xy}^\top(D^{-1}g) &= \alpha + a - b, \\
   b_{xz}^\top(D^{-1}g) &= \alpha + a - \gamma - c, \\
   b_{yz}^\top(D^{-1}g) &= \gamma + c - b, \\
   b_{xyz}^\top(D^{-1}g) &= \alpha + a - b
\end{align*}
\]
hold. Hence, we have
\[ F(g) = \frac{1}{2\lambda} (3a^2 + 3b^2 + 3c^2) + \frac{1}{2} \left( 2(a + a - b)^2 + (\gamma + c - b)^2 + (\alpha + a - \gamma - c)^2 \right). \]

We set \( r := \lambda^{-1} \). Since \( J_\lambda f \) is a critical point for \( F \), \( F_a = F_b = F_c = 0 \). Hence we have
\[
\begin{pmatrix}
3(1 + r) & -2 & -1 \\
-2 & 3(1 + r) & -1 \\
-1 & -1 & 2 + 3r
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
= \begin{pmatrix}
-3\alpha + \gamma \\
2\alpha + \gamma \\
\alpha - 2\gamma r
\end{pmatrix}.
\]
This can be solved for \( \lambda > 0 \), and we have that \((a, b, c)^T\) is equal to
\[
\frac{1}{9r(1 + r)(3r + 5)}
\begin{pmatrix}
3(1 + r)(2 + 3r) - 1 & 2(2 + 3r) + 1 & 2 + 3(1 + r) \\
2(2 + 3r) + 1 & 3(1 + r)(2 + 3r) - 1 & 3(1 + r) + 2 \\
2 + 3(1 + r) & 3(1 + r) + 2 & 9(1 + r)^2 - 4
\end{pmatrix}
\begin{pmatrix}
-3\alpha + \gamma \\
\alpha - 2\gamma r
\end{pmatrix}.
\]

Because the inner product \( \langle J_\lambda f, \delta_x - \delta_y \rangle \) which we claim can be represented as \( \alpha + a - b \), we have
\[
\langle J_\lambda f, \delta_x - \delta_y \rangle = \langle g, \delta_x - \delta_y \rangle = \alpha + a - b
= \alpha + \frac{1}{9r(1 + r)(3r + 5)} \{( -3\alpha + \gamma)(3(1 + r)(2 + 3r) - 1 - (2(2 + 3r) + 1))
+ (2\alpha + \gamma)(2(2 + 3r) + 1 - (3(1 + r)(2 + 3r) - 1))
+ (\alpha - 2\gamma)(2 + 3(1 + r) - (2 + 3(1 + r)))\}
= \alpha - \frac{5\alpha}{9r(1 + r)(3r + 5)} (9r^2 + 9r)
= \frac{3\alpha}{3r + 5} \leq \frac{3r}{3r + 5}.
\]
Here, the last inequality is obtained by evaluating for \( \alpha = 1 \).

(2) \( \gamma > \alpha > 0 \). By a similar argument as above, we may assume \( \gamma + c > \alpha + a > b \).

Hence, we have
\[
b_{xy}^T(D^{-1}g) = \alpha + a - b, \quad b_{xz}^T(D^{-1}g) = \gamma + c - \alpha - a, \quad b_{yz}^T(D^{-1}g) = \gamma + c - b.
\]

Thus we have
\[
F(g) = \frac{1}{2\lambda} (3a^2 + 3b^2 + 3c^2) + \frac{1}{2} \left( (\alpha + a - b)^2 + 2(\gamma + c - b)^2 + (\gamma + c - \alpha - a)^2 \right). \]

From \( F_a = F_b = F_c = 0 \), we obtain
\[
\begin{pmatrix}
3r + 2 & -1 & -1 \\
-1 & 3(1 + r) & -2 \\
-1 & -2 & 3(1 + r)
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
= \begin{pmatrix}
-2\alpha + \gamma \\
\alpha + 2\gamma \\
\alpha - 3\gamma
\end{pmatrix}.\]
This simultaneous equation can also be solved as follows: \((a, b, c)^\top\) is equal to
\[
\frac{1}{9r(1+r)(3r+5)} \begin{pmatrix}
9(1+r)^2 - 4 & 3(1+r) + 2 & 3(1+r) + 2 \\
3(1+r) + 2 & 3(1+r)(2+3r) - 1 & 2(2+3r) + 1 \\
3(1+r) + 2 & 2(2+3r) + 1 & 3(1+r)(2+3r) - 1
\end{pmatrix}
\begin{pmatrix}
-2\alpha + \gamma \\
\alpha + 2\gamma \\
\alpha - 3\gamma
\end{pmatrix}.
\]

Then, we have
\[
\langle J_\lambda f, \delta x - \delta y \rangle = \alpha + a - b
\]
\[
= \alpha - \frac{1}{(1+r)(3r+5)} \{\alpha(3r+5) + \gamma r\}
\]
\[
= \frac{r(\alpha(3r+5) - \gamma r)}{(1+r)(3r+5)}.
\]

(3) \(\gamma = 0\). In this case, \(b = c\) by the symmetry of \(H\) and \(f\). Then we have
\[
b_{xy}^\top(D^{-1}g) = \alpha + a - b, \quad b_{xz}^\top(D^{-1}g) = \alpha + a - b, \quad b_{yz}^\top(D^{-1}g) = 0,
\]
\[
b_{xyz}^\top(D^{-1}g) = \alpha + a - b.
\]

Thus we have
\[
F(g) = \frac{1}{2\lambda} \left(3a^2 + 6b^2\right) + \frac{3}{2}(\alpha + a - b)^2.
\]

\(F_a = F_b = 0\) leads
\[
\begin{pmatrix}
1 + r & -1 \\
-1 & 1 + 2r
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \begin{pmatrix}
-\alpha \\
\alpha
\end{pmatrix}.
\]

Therefore, we have
\[
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \frac{1}{r(2r+3)} \begin{pmatrix}
1 + 2r & 1 \\
1 & 1 + r
\end{pmatrix}
\begin{pmatrix}
-\alpha \\
\alpha
\end{pmatrix}
= \frac{\alpha}{2r+3} \begin{pmatrix}
-2 \\
1
\end{pmatrix}.
\]

Hence, we have
\[
\langle J_\lambda f, \delta x - \delta y \rangle = \alpha + a - b = \frac{2\alpha r}{2r+3} \leq \frac{2r}{2r+3}.
\]

The last inequality follows from evaluating the value for \(\alpha \leq 1\).

(4) \(\gamma = \alpha\). In this case, we also have \(a = c\) similarly as the above. Then, we have
\[
b_{xy}^\top(D^{-1}g) = \alpha + a - b, \quad b_{xz}^\top(D^{-1}g) = 0, \quad b_{yz}^\top(D^{-1}g) = \alpha + a - b,
\]
\[
b_{xyz}^\top(D^{-1}g) = \alpha + a - b,
\]

thus
\[
F(g) = \frac{1}{2\lambda} \left(6a^2 + 3b^2\right) + \frac{3}{2}(\alpha + a - b)^2.
\]
The equations \( F_a = F_b = 0 \) can be written as
\[
\begin{pmatrix}
1 + 2r & -1 \\
-1 & 1 + r
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \begin{pmatrix}
-\alpha \\
\alpha
\end{pmatrix},
\]
hence we have
\[
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \frac{\alpha}{2r + 3} \begin{pmatrix}
-1 \\
2
\end{pmatrix}.
\]
Consequently, we have
\[
\langle J_\lambda f, \delta_x - \delta_y \rangle = \alpha + a - b = \frac{2\alpha r}{2r + 3} \leq \frac{2r}{2r + 3}.
\]
The last inequality follows from \( \alpha \leq 1 \).

(5) \( \alpha > 0 > \gamma \). Note that since \( f \in \text{Lip}^1(V) \), \( \alpha - \gamma \leq 1 \) and \( \alpha < 1 \). In this case, we have
\[
\begin{align*}
b^\top_{xy}(D^{-1}g) &= \alpha + a - b, \quad b^\top_{xz}(D^{-1}g) = \alpha + a - \gamma - c, \\
b^\top_{yz}(D^{-1}g) &= b - \gamma - c,
\end{align*}
\]
Thus, we have
\[
F(g) = \frac{1}{2\lambda} \left( 3a^2 + 3b^2 + 3c^2 \right) + \frac{1}{2} \left( (\alpha + a - b)^2 + 2(\alpha + a - \gamma - c)^2 + (b - \gamma - c)^2 \right).
\]
In the same manner as before, we obtain
\[
\begin{pmatrix}
3(1 + r) & -1 & -2 \\
-1 & 3r + 2 & -1 \\
-2 & -1 & 3(1 + r)
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
= \begin{pmatrix}
-3\alpha + 2\gamma \\
\alpha + \gamma \\
2\alpha - 3\gamma
\end{pmatrix}.
\]
Hence, \((a \, b \, c)^\top\) is equal to
\[
\frac{1}{9r(1 + r)(3r + 5)} \begin{pmatrix}
3(1 + r)(3r + 2) - 1 & 3(1 + r) + 2 & 2(2 + 3r) + 1 \\
3(1 + r) + 2 & 9(1 + r)^2 - 4 & 3(1 + r) + 2 \\
2(2 + 3r) + 1 & 3(1 + r) + 2 & 3(1 + r)(2 + 3r) - 1
\end{pmatrix}.
\]
Finally we have
\[
\langle g, \delta_x - \delta_y \rangle = \alpha + a - b = \frac{r}{(1 + r)(3r + 5)} \{ \alpha(3r + 4) + \gamma \}.
\]

By comparing the above cases, we can show that the values \( \langle J_\lambda f, \delta_x - \delta_y \rangle \) for the cases (1) is less than or equal to \( 2r/(2r + 3) \) which is attained for the case (3) and
(4). Thus, it is sufficient to compare the cases (2), (3), and (5). We can calculate the differences as

\[
(2) - (3) = \frac{2r}{2r + 3} \geq \frac{r(2(1 - \alpha)r + 2 - 3\alpha)}{(2r + 3)(1 + r)}, \quad \text{and}
\]

\[
(2) - (5) = \frac{2r}{2r + 3} - \left( \frac{r}{(1 + r)(3r + 5)} \{\alpha(3r + 4) + \gamma\} \right) \geq \frac{2r}{2r + 3} - \frac{\alpha r}{1 + r} \geq 0.
\]

Then, the most right hand sides are non-negative, because \( r = \lambda^{-1} \) is sufficiently large and \( 1 \geq \gamma > \alpha \) in the case (3) and \( \alpha < 1, \gamma < 0 \) in the case (5). Thus, we have

\[
\text{KD}_{\lambda}(x, y) = \frac{2\lambda^{-1}}{2\lambda^{-1} + 3}.
\]

Consequently, the coarse Ricci curvature \( \kappa(x, y) \) exists for \( x, y \in V \) and becomes

\[
\kappa(x, y) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \left( 1 - \frac{\text{KD}_{\lambda}(x, y)}{d(x, y)} \right) = \frac{3}{2}.
\]

**Remark 6.2.** We predict that if we consider the hypergraph \( H = (V, E, w) \) such that \( \#V = n, E = 2^V \setminus \{\emptyset, V\} \) and \( w(e) = 1 \) for any \( e \in E \), the \( \lambda \)-nonlinear Kantorovich potential \( f \) satisfies that \( f(x) = d_x \) and \( f(z) = 0 \) (\( z \neq x \)).

**Remark 6.3.** We predict that

\[
\xi(x, y) = \pi(x, y) = \kappa(x, y)
\]

\[
= C(x, y) = \inf \left\{ \frac{\mathcal{L}^0 f, \delta_x - \delta_y}{d(x, y)} \mid f \in \text{Lip}^1_\lambda(V), \langle f, \delta_x - \delta_y \rangle = d(x, y) \right\}
\]

holds. A similar formula holds for graph cases \(^{32}\). If this is true, it could be somewhat easier to calculate our curvatures. By assuming this equation once here, we try to show some predicted curvatures in some cases.

**Example 6.4.** Let \( H = (V, E, w) \) be the hypergraph such that \( V = \{x, y, z\}, E = \{e = \{x, y, z\}\} \), and \( w_e = 1 \). We consider \( f : V \to \mathbb{R} \) such that \( f(x) = 1, f(y) = 0, f(z) = 0 \). Then, we have \( \mathcal{L}^0 f(x) = 1, \mathcal{L}^0 f(y) = -1/2, \mathcal{L}^0 f(z) = -1/2 \), and

\[
\kappa(x, y) = C(x, y) \leq \langle \mathcal{L}^0 f, \delta_x - \delta_y \rangle = 1 - (-1/2) = 3/2.
\]

Actually, we are able to prove that the curvature of this hypergraph \( H \) is 3/2.

**Example 6.5** (complete hypergraph). We consider the hypergraph \( H = (V, E, w) \) as \( V = \{v_1, v_2, \ldots, v_n\}, E = 2^V \setminus \{\{v_1\}, \ldots, \{v_n\}, \emptyset\} \), and \( w_e = 1 \). Then, we have \( \#E = 2^n - n - 1 \) and \( d_x = d = 2^{n-1} - (n - 1) - 1 = 2^{n-1} \) for any \( x \in V \). We count the number of hyperedges \( e \in E \) including \( v_1 \) and \( v_2 \). The number of such \( e \) satisfying \( \#e = k \) is \( \binom{n-2}{k-2} \).
Theorem A.1. The coarse Ricci curvature

\[ \kappa(x_0, y_0) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left( 1 - \frac{\text{KD}_\lambda(x_0, y_0)}{d(x_0, y_0)} \right) \]

along with \( x_0, y_0 \in V \) on a hypergraph \( H \) exists, where \( \text{KD}_\lambda(x_0, y_0) \) is the \( \lambda \)-nonlinear Kantorovich difference, given in Definition 3.1.

Let \( f : V \to \mathbb{R} \) be the function satisfying \( f(v_1) = d \), and \( f(v_i) = 0 \) (\( i = 2, \ldots, n \)). Then, we have \( L_0^0 f(v_1) = d \). Moreover, for \( e \) including \( v_1 \) and \( v_2 \) such that \#\( e = k \), we may choose \( \delta_{v_1} \) \( (k - 1)^{-1} \sum_{i \geq 2, v_i \in e} \delta_{v_i} \), \( v_1, v_2 \in e \) as \( b_e \). Thus, we have

\[
L_0^0 f(v_2) = -\sum_{k=2}^{n} \frac{1}{k-1} (n-2)! \left( \begin{array}{c} n-2 \\ k-2 \end{array} \right) = -\sum_{k=2}^{n} \frac{(n-2)!}{(k-1)!} = -\frac{1}{n-1} \sum_{k=2}^{n} \frac{(n-2)!}{(n-k)!} = -\frac{1}{n-1} \left\{ (1 + 1)^{n-1} - 1 \right\} = -\frac{2^{n-1} - 1}{n-1}.
\]

Therefore, we have the following inequality:

\[
C(v_1, v_2) \leq \frac{1}{d} (L_0^0 f(v_1) - L_0^0 f(v_2)) = \frac{1}{d} \left( d + \frac{2^{n-1} - 1}{n-1} \right) = 1 + \frac{d}{d(n-1)} = \frac{n}{n-1}.
\]

By this observation, we predict that the curvature of the complete hypergraph \( H \) with \( n \) vertices is \( n/(n-1) \). This prediction agrees with calculation in Example 6.1.

Appendix A. Existence of coarse Ricci curvatures

The purpose of this appendix is to show the existence of coarse Ricci curvature:

\[ \kappa(x_0, y_0) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left( 1 - \frac{\text{KD}_\lambda(x_0, y_0)}{d(x_0, y_0)} \right) \]

for any \( x_0, y_0 \in V \) on a hypergraph \( H \) exists, where \( \text{KD}_\lambda(x_0, y_0) \) is the \( \lambda \)-nonlinear Kantorovich difference, given in Definition 3.1.

Let us consider a generalized hypergraph \( H = (V, E, \omega, d) \) consisting of a finite set \( V \), a set \( E \) of nonempty subsets of \( V \), a function \( \omega : E \to \mathbb{R}_{>0} \), and a function \( d : V \to \mathbb{R}_{>0} \). For a while, the condition \( d_x = \sum_{e \ni x} \omega_e \) is not imposed. For simplicity, we put \#\( V = n \) and \( V = \{1, 2, \ldots, n\} \) and identify a real valued function \( f : V \to \mathbb{R} \) with \( f = (f_1, \ldots, f_n)^\top \in \mathbb{R}^n \). The vector space \( \mathbb{R}^n \) of real valued functions on \( V \) can be expressed as the disjoint union

\[ \mathbb{R}^n = \bigsqcup_{\rho \in R_n} U_m, \]

where two vectors \( f = (f_1, \ldots, f_n)^\top, g = (g_1, \ldots, g_n)^\top \in \mathbb{R}^n \) belong to the same component \( U_m \), for some \( \rho \in R_n \) if and only if the elements of \( f \) and \( g \) are in the same order, that is, \( \text{sgn}(f_x - f_y) = \text{sgn}(g_x - g_y) \) for any \( x, y \in V \) with \( \text{sgn} : \mathbb{R} \to \{-1, 0, 1\} \) given by

\[ \text{sgn}(r) = \begin{cases} 1 & (r > 0) \\ 0 & (r = 0) \\ -1 & (r < 0). \end{cases} \]
Let \( K := \mathbb{Q}(\{\omega_e\}_{e \in E}, \{d_x\}_{x \in V}) \subset \mathbb{R} \) be the subfield of \( \mathbb{R} \) generated by \( \{\omega_e\}_{e \in E} \) and \( \{d_x\}_{x \in V} \), and consider the field \( K(z) \) of rational functions in \( z \) with coefficients in \( K \). Moreover, let \( G = G_z = G_{H,z} : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) be the (multivalued) function defined by

\[
(A.2) \quad Gf = G_z f := (D + zL)(f) \quad (f \in \mathbb{R}^n),
\]

which defines \( G_\lambda : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) for any \( \lambda > 0 \), where \( D = \text{diag}(d_1, \ldots, d_n) \) and \( L : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) is the hypergraph Laplacian given in (2.1).

### A.1. Piecewise linear inverse

Let \( H \) be a generalized hypergraph. We will show the following proposition.

**Proposition A.2.** For any \( \rho \in R_n \), there exists a symmetric matrix \( N_{\rho,z} \in M_n(K(z)) \) such that \( N_{\rho,\lambda} \) has non-negative entries for any \( \lambda > 0 \) and \( (N_{\rho,\lambda} \circ G_\lambda)(f) = f \) holds for any \( f \in U_\rho \).

In order to prove the proposition, let us prepare the following notations.

- For each \( e \in E \), put \( f_e^+ := \max_{x \in e} f_x \) and \( f_e^- := \min_{x \in e} f_x \) with \( f = (f_1, \ldots, f_n)^\top \in \mathbb{R}^n \).
- For each \( \rho \in R_n \), let \( V = \sqcup_{I \in \mathcal{I}_\rho} I \) be a unique decomposition of \( V = \{1, \ldots, n\} \) such that each \( f = (f_1, \ldots, f_n)^\top \in U_\rho \) satisfies \( f_x = f_y \) if and only if \( x, y \in I \) for some \( I \in \mathcal{I}_\rho \). The decomposition is independent of the choice of \( f \in U_\rho \).
- For each \( e \in E \) and \( \rho \in R_n \), let \( I^\pm_{e,\rho} \subset V \) be the set of points \( x \in V \) with \( f_x = f_e^\pm \) for \( f \in U_\rho \), which is also independent of the choice of \( f \in U_\rho \). In particular, we have \( I^\pm_{e,\rho} \subset \mathcal{I}_\rho \). Moreover, either \( I^+_{e,\rho} \cap I^-_{e,\rho} = \emptyset \) or \( e \subset I^+_{e,\rho} = I^-_{e,\rho} \) holds.
- For each \( e \in E \) and \( \rho \in R_n \), put

\[
\mathcal{M}_{e,\rho} := \text{Conv}(\{S_{xy} : x \in e \cap I^+_{e,\rho}, y \in e \cap I^-_{e,\rho}\}),
\]

where \( S_{xy} \in M_n(\mathbb{Q}) \) is a symmetric matrix, given by

\[
S_{xy} = I_{xx} - I_{xy} - I_{yx} + I_{yy} = \begin{pmatrix} x & y \\ x & y \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]

with \( I_{xy} = (\delta_x(i) \cdot \delta_y(j))_{1 \leq i, j \leq n} \in M_n(\mathbb{Q}) \) (here \( S_{xy} = 0 \) if \( x = y \)). In particular, each element of \( \mathcal{M}_{e,\rho} \) is symmetric.
Now we notice that the hypergraph Laplacian $L: \mathbb{R}^n \to \mathbb{R}^n$ defined in (2.1) is expressed as

$$(A.3) \quad L(f) = \sum_{e \in E} \omega_e L_e(f) \quad \text{with} \quad L_e(f) = \{ b_e(b_e^T f) ; b_e \in \text{argmax}_{b \in B_e} b^T f \} ,$$

where $B_e$ is given in (2.2). Here the sum of subsets $A,B \subset \mathbb{R}^n$ stands for the Minkowski sum: $A + B := \{ a + b \in \mathbb{R}^n ; a \in A, b \in B \}$, and the multiplication of $A \subset \mathbb{R}^n$ by a scalar $c \in \mathbb{R}$ means $cA := \{ ca \in \mathbb{R}^n ; a \in A \}$. Hence the restriction of $L_e$ on each component $U_\rho$ is calculated as

$$(A.4) \quad L_e|_{U_\rho}(f) = (f_e - f_e^1) \text{Conv}(\{ \delta_x - \delta_y ; x \in e \cap I_{e,\rho}^+, y \in e \cap I_{e,\rho}^- \}) = M_{e,\rho} f ,$$

If $I_{e,\rho}^+ = I_{e,\rho}^-$, then one has $e \in I_{e,\rho}^+$ and $L_e(f) = M_{e,\rho} f = 0$ for any $f \in U_\rho$.

**Proof of Proposition A.3** We divide the proof into three steps.

**Step 1** Assume that $\rho \in R_n$ satisfies $\mathcal{I}_\rho = \{ \{1,2,\ldots,n\} \}$, which means that $f = (f_1,\ldots,f_n)^T \in U_\rho$ satisfies $f_x = f_y$ for any $x,y \in V$. In this case, one has $I_{e,\rho}^+ = I_{e,\rho}^-$ and thus $L_e|_{U_\rho} = M_{e,\rho} = 0$ for any $e \in E$. Therefore, $G_\rho|_{U_\rho} = D + \sum_{e \in E} \omega_e M_{e,\rho} = D$ is a single matrix and $N_{\rho,z} = D^{-1}$ is a symmetric matrix with $N_{\rho,\lambda} = D^{-1}$ having non-negative entries for any $\lambda > 0$.

**Step 2** Assume that $\rho \in R_n$ satisfies $\mathcal{I}_\rho = \{ \{1\},\{2\},\ldots,\{n\} \}$, which means that $f = (f_1,\ldots,f_n)^T \in U_\rho$ satisfies $f_x \neq f_y$ for any $x \neq y \in V$. In this case, one has $\#I_{e,\rho}^+ = \#I_{e,\rho}^- = 1$ and thus $L_e|_{U_\rho} = M_{e,\rho}$ is a single valued function for any $e \in E$. Moreover, $G_\rho|_{U_\rho} = D + \sum_{e \in E} \omega_e M_{e,\rho}$ is a symmetric matrix such that $G_\lambda|_{U_\rho}$ has positive diagonal entries and non-negative off-diagonal entries for any $\lambda > 0$, and it satisfies

$$(D + \sum_{e \in E} \omega_e M_{e,\rho}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} .$$

Hence it is known (see e.g. [6], Theorem 6.34) that there exists an inverse matrix $N_{\rho,z} = (D + \sum_{e \in E} \omega_e M_{e,\rho})^{-1} \in M_n(K(z))$, which is symmetric, and $N_{\rho,\lambda}$ has non-negative entries for any $\lambda > 0$.

**Step 3** We prove the proposition by induction on $n$. The proposition for the case $n = 1$ can be proved from Step 1 (or Step 2). Assume that the proposition holds for $n < m$, and consider the case $n = m$. If $\rho \in R_m$ satisfies $\mathcal{I}_\rho = \{ \{1,2,\ldots,m\} \}$ or $\mathcal{I}_\rho = \{ \{1\},\{2\},\ldots,\{m\} \}$, then the proposition holds from Step 1 or Step 2. Otherwise, by exchanging the indices if necessary, one may assume that there exists $1 < k < m$ such that $\{k,\ldots,m\} \in \mathcal{I}_\rho$, that is, $f = (f_1,\ldots,f_m)^T \in U_\rho$ satisfies $f_x \neq f_y$ for any $1 \leq x \leq k - 1$ and $k \leq y \leq m$ and $f_k = f_{k+1} = \cdots = f_m$. Now we
consider contractions of a function \( f = (f_1, \ldots, f_m)^\top \in \mathbb{R}^m \), given by

\[
\tilde{f} := \begin{pmatrix}
  f_1 \\
  \vdots \\
  f_{k-1} \\
  \sum_{x=k}^m f_x
\end{pmatrix} \in \mathbb{R}^k, \quad \tilde{f} := \begin{pmatrix}
  f_1 \\
  \vdots \\
  f_{k-1} \\
  f_k
\end{pmatrix} \in \mathbb{R}^k,
\]

and also consider a contraction of a matrix \( A = (a_{ij})_{1 \leq i,j \leq m} \in M_m(\mathbb{R}) \), given by

\[
\tilde{A} := \begin{pmatrix}
  a_{11} & \cdots & a_{1k-1} & \sum_{j=k}^m a_{1j} \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{k-11} & \cdots & a_{k-1k-1} & \sum_{j=k}^m a_{k-1j} \\
  \sum_{i=k}^m a_{i1} & \cdots & \sum_{i=k}^m a_{ik-1} & \sum_{i,j=k}^m a_{ij}
\end{pmatrix} \in M_k(\mathbb{R}).
\]

One may also consider contractions of a function in \( K(z)^m \) or in \( \mathbb{R}(z)^m \), and a matrix in \( M_m(K(z)) \) or in \( M_m(\mathbb{R}(z)) \) in the same manner. Note that if \( f \in U_\rho \), that is, \( f_k = \cdots = f_m \), then \( (A \tilde{f}) = \tilde{A} \tilde{f} \). Let \( \tilde{\rho} \in R_k \) be an index given by the relation \( U_{\tilde{\rho}} = \{ \tilde{f} \in \mathbb{R}^k : f \in U_\rho \} \subset \mathbb{R}^k \).

Then it is seen that there exists a contraction \( \tilde{H} = (\tilde{V}, \tilde{E}, \tilde{\omega}, \tilde{d}) \) of the hypergraph \( H \) with \( \tilde{V} = \{1, \ldots, k\} \) such that \( \tilde{G}_{\tilde{H},z}(\tilde{f}) = \tilde{G}_{\tilde{H},z}(\tilde{f}) \) holds for any \( f \in U_\rho \). Indeed, let us prepare the following notations:

\[
\pi : V = \{1, \ldots, m\} \to \tilde{V} := \{1, \ldots, k\}, \quad \pi(x) := \begin{cases}
  x & (x < k) \\
  k & (x \geq k),
\end{cases}
\]

\[
\pi : E \to \tilde{E} := \{\tilde{e} \mid e \in E\}, \quad \pi(e) = \tilde{e} := \{\pi(x) \mid x \in e\} \subset \tilde{V},
\]

\[
\tilde{\omega} : \tilde{E} \to \mathbb{R}_{>0}, \quad \tilde{\omega}_{\tilde{e}} := \sum_{e \in \pi^{-1}(\tilde{e})} \omega_e,
\]

\[
\tilde{d} : \tilde{V} \to \mathbb{R}_{>0}, \quad \tilde{d}_x := \begin{cases}
  d_x & (x < k) \\
  \sum_{y=k}^m d_y & (x \geq k).
\end{cases}
\]

Since each matrix \( S_{xy} \) with \( x < y \) satisfies

\[
\tilde{S}_{xy} = \begin{cases}
  S_{xy} & (x < y < k) \\
  S_{xk} & (x < k \leq y) \\
  0 & (k \leq x < y),
\end{cases}
\]

\( \tilde{M}_{e,\rho} := \{\tilde{M} \mid M \in M_{e,\rho}\} \) satisfies \( \tilde{M}_{e,\rho} = M_{\tilde{e},\tilde{\rho}} \) for any \( e \in E \). As \( \tilde{D} = \text{diag}(\tilde{d}_1, \ldots, \tilde{d}_k) \), we have

\[
\tilde{G}_{\tilde{H},z}(\tilde{f}) = (\tilde{D} + z \sum_{e \in E} \omega_e \tilde{M}_{e,\rho})(\tilde{f}) = (\tilde{D} + z \sum_{\tilde{e} \in \tilde{E}} \tilde{\omega}_{\tilde{e}} \tilde{M}_{\tilde{e},\tilde{\rho}})(\tilde{f}) = \tilde{G}_{\tilde{H},z}(\tilde{f})
\]

for any \( f \in U_\rho \).

By our assumption of the induction, there exists a symmetric matrix \( N_{\tilde{\rho},z} = (c_{ij}) \in M_k(K(z)) \) such that \( N_{\tilde{\rho},\lambda} \) has non-negative entries for any \( \lambda > 0 \) and \( (N_{\tilde{\rho},\lambda} \circ \tilde{G}_{\tilde{H},\lambda})(\tilde{f}) = \tilde{f} \) holds for any \( f \in U_\rho \). Since \( \tilde{G}_{\tilde{H},\lambda}(\tilde{f}) = \tilde{G}_{\tilde{H},\lambda}(\tilde{f}) = \tilde{G}_{\tilde{H},\lambda}(\tilde{f}) \),
we have \((N_{\rho,\lambda} \circ G_{H,\lambda})(f) = f\) for any \(f \in U_{\rho}\), where

\[
N_{\rho,z} := \begin{pmatrix}
c_{11} & \cdots & c_{1k} & \cdots & c_{1k} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
c_{k1} & \cdots & c_{kk} & \cdots & c_{kk} \\
\vdots & \cdots & \vdots & \ddots & \vdots \\
c_{k1} & \cdots & c_{kk} & \cdots & c_{kk}
\end{pmatrix} \in M_{m}(K(z))
\]

is also a symmetric matrix with \(N_{\rho,\lambda}\) having non-negative entries for \(\lambda > 0\). The proposition is established.

\[\square\]

A.2. Linear programming. In order to prove Theorem [A.1] we prepare the following lemma.

**Lemma A.3.** Let \(H\) be a generalized hypergraph. For any \(\rho \in R_{n}\), there exist vectors \(v_{1}(z), \ldots, v_{k}(z) \in K(z)^{n}\) such that for any \(\lambda > 0\), the closure of the image \(G_{\lambda}(U_{\rho})\) is expressed as

\[
\overline{G_{\lambda}(U_{\rho})} = \text{ConvConv}_{\mathbb{R}}(\{v_{1}(\lambda), \ldots, v_{k}(\lambda)\}) := \sum_{i=1}^{k} \mathbb{R}_{\geq 0} \cdot v_{i}(\lambda),
\]

where \(G_{\lambda}\) is given in [A.2].

**Proof.** We use the notations in the proof of Proposition [A.2]. For each element \(f = (f_{1}, \ldots, f_{n})^{\top} \in U_{\rho}\), let \(g_{i}^{f} > \cdots > g_{l}^{f}\) be given by \(\{f_{1}, \ldots, f_{n}\} = \{g_{i}^{f}, \ldots, g_{l}^{f}\}\) and put \(h_{i}^{f} := g_{i}^{f} - g_{i+1}^{f}\) for \(1 \leq i \leq l - 1\) and \(h_{l}^{f} := g_{l}^{f}\), which gives a one-to-one correspondence between the sets \(U_{\rho}\) and \(\{(h_{1}, \ldots, h_{l})^{\top}; h_{1} > 0, \ldots, h_{l-1} > 0, h_{l} \in \mathbb{R}\}\). Since \(f_{i} = g_{i}^{f} = \sum_{j=k}^{l} h_{j}^{f}\) for some \(1 \leq k \leq l\), [A.3] and [A.4] show that

\[
G_{\nu} f = Df + z \sum_{e \in E} \omega_{e}(f_{e}^{+} - f_{e}^{-}) \text{Conv}(\{\delta_{x} - \delta_{y}; x \in e \cap I_{e,\nu}^{+}, y \in e \cap I_{e,\nu}^{-}\})
\]

\[
= \sum_{j=1}^{l} h_{j}^{f} \left\{ \eta_{j} + z \sum_{J_{e} \ni j} \omega_{e} \text{Conv}(\{\delta_{x} - \delta_{y}; x \in e \cap I_{e,\nu}^{+}, y \in e \cap I_{e,\nu}^{-}\}) \right\}
\]

for \(f \in U_{\rho}\), where \(J_{e} = \{k^{+}, k^{+} + 1, \ldots, k - 1\} \subset \{1, \ldots, l - 1\}\) is given by \(f_{e}^{+} = g_{k}^{f}\), and \(\eta_{j} = (\eta_{ij_{1}}, \ldots, \eta_{ij_{n}})^{\top} \in \mathbb{Q}(d)^{n}\) is given by

\[
\eta_{ij} = \begin{cases} d_{i} \quad (f_{i} \geq g_{i}^{f}) \\ 0 \quad (f_{i} < g_{i}^{f}). \end{cases}
\]

Note that the definitions of \(J_{e}\) and \(\eta_{j}\) are independent of the choice of \(f \in U_{\rho}\). Thus \(G_{\nu}(U_{\rho}) \subset \mathbb{R}^{n}\) is the set of points

\[
(A.5) \quad \sum_{j=1}^{l} h_{j} \left\{ \eta_{j} + \lambda \sum_{J_{e} \ni j} \omega_{e} \text{Conv}(\{\delta_{x} - \delta_{y}; x \in e \cap I_{e,\nu}^{+}, y \in e \cap I_{e,\nu}^{-}\}) \right\},
\]

where \((h_{1}, \ldots, h_{l})^{\top}\) varies over \(\{h_{1} \geq 0, \ldots, h_{l-1} \geq 0, h_{l} \in \mathbb{R}\}\). Moreover we also notice that
the multiplication of a convex set \( \text{Conv}(\{u_i\}) \) by a scalar \( c \in \mathbb{R} \) is also a convex set: \( c \cdot \text{Conv}(\{u_i\}) = \text{Conv}(\{c \cdot u_i\}) \),

- the Minkowski sum of convex sets \( \text{Conv}(\{u_i\}) \) and \( \text{Conv}(\{v_j\}) \) is also a convex set: \( \text{Conv}(\{u_i\}) + \text{Conv}(\{v_j\}) = \text{Conv}(\{u_i + v_j\}) \),

- the multiplication of a convex set \( \text{Conv}(\{u_i\}) \) by the (non-negative) real numbers \( \mathbb{R} (\mathbb{R}_{\geq 0}) \) is a convex cone: \( \mathbb{R} \text{Conv}(\{u_i\}) = \text{ConvCone}(\{u_i \cup \{-u_i\}\} \text{Conv}(\{u_i\}) = \text{ConvCone}(\{u_i\}), \) and

- the Minkowski sum of convex cones \( \text{ConvCone}(\{u_i\}) \) and \( \text{ConvCone}(\{v_j\}) \) is a convex cone: \( \text{ConvCone}(\{u_i\}) + \text{ConvCone}(\{v_j\}) = \text{Conv}(\{u_i \cup \{v_j\}\}). \)

Hence the proposition follows from the expression (A.3). □

From now on, we assume that \( H \) is a hypergraph. As \( G_{\lambda}f = (I + \lambda L) \circ D(f) \), one has

\[(A.6) \quad N_{\rho,\lambda}(g) = D^{-1} \circ J_{\lambda}(g)\]

for \( g \in G_{\lambda}(U_\rho) \). Since \( J_{\lambda} \) is a (single-valued) continuous function by Lemma 2.1, the relation (A.6) holds for \( g \in G_{\lambda}(U_\rho) \). Here it follows from (the proof of) Proposition 3.3 that the \( \lambda \)-nonlinear Kantorovich difference \( \text{KD}_{\lambda}(x_0, y_0) \) of \( x_0 \in V \) and \( y_0 \in V \) is given by

\[\text{KD}_{\lambda}(x_0, y_0) = \sup \{\langle J_{\lambda}g, \delta_{x_0} - \delta_{y_0} \rangle : g \in F \},\]

where \( F \) is the set of functions \( g \in \mathbb{R}^n \) satisfying the conditions

- (a) \( 0 \leq g_x \leq d_x \cdot \text{diam}(H) \) \( (x \in V) \),  
- (b) \( \langle g, \delta_x - \delta_y \rangle \leq d(x, y) \) \( (x, y \in V) \).

Since \( N_{\rho,\lambda} \) and \( D \) are symmetric matrices and \( \{G_{\lambda}(U_\rho)\}_{\rho \in \mathbb{R}} \) covers the whole space \( \mathbb{R}^n \), we have

\[(A.7) \quad \text{KD}_{\lambda}(x_0, y_0) = \max_{\rho \in \mathbb{R}_+} \sup \left\{ \langle J_{\lambda}g, \delta_{x_0} - \delta_{y_0} \rangle : g \in F \cap \overline{G_{\lambda}(U_\rho)} \right\} \]

\[= \max_{\rho \in \mathbb{R}_+} \sup \left\{ \langle g, D \circ N_{\rho,\lambda}(\delta_{x_0} - \delta_{y_0}) \rangle : g \in F \cap \overline{G_{\lambda}(U_\rho)} \right\}.\]

Now we consider an order \( < \) on \( K(z) \), given so that \( p(z), q(z) \in K(z) \) satisfy \( p(z) < q(z) \) if and only if \( q(z) - p(z) = z^k r(z) \) for some \( k \in \mathbb{Z} \) and \( r(z) \in K(z) \) with \( 0 < r(0) < \infty \). In other words, \( p(z), q(z) \in K(z) \) satisfy \( p(z) < q(z) \) if and only if there exists \( \lambda_0 > 0 \) such that \( p(\lambda) < q(\lambda) \) for any \( \lambda \in \mathbb{R} \) with \( 0 < \lambda < \lambda_0 \). Then \( < \) becomes a total order on \( K(z) \) and thus \( (K(z), <) \) is an ordered field.

With the notation in Lemma A.3 we consider the convex cone

\[W_\rho := \text{ConvCone}_{K(z)}(\{v_1(z), \ldots, v_k(z)\}) = \sum_{i=1}^k K(z)_{\geq 0} \cdot v_i(z) \subset K(z)^n.\]

It should be noted that some of the concepts of linear programming over the real numbers, such as Farkas-Minkowski-Weyl theorem and the simplex method, can
be easily extended to that over an arbitrary ordered field (see \cite{21, 22}). Farkas-Minkowski-Weyl theorem says that there exist vectors $w_1(z), \ldots, w_m(z) \in K(z)^n$ such that the convex cone $W_\rho$ is expressed as

$$W_\rho = \{g(z) \in K(z)^n : \langle g(z), w_i(z) \rangle \leq 0 \ (1 \leq i \leq m)\}.$$  

In viewing (A.7), we consider the linear program $LP(z)$:

$$\text{maximize } \langle g(z), D \circ N_{\rho,z}(\delta x_0 - \delta y_0) \rangle$$
subject to
(a) $0 \leq g(z)_x \leq d_x \cdot \text{diam}(H) \ (x \in V)$
(b) $\langle g(z), \delta_x - \delta_y \rangle \leq d(x, y) \ (x, y \in V)$
(c) $\langle g(z), w_i(z) \rangle \leq 0 \ (1 \leq i \leq m)$.

As the range of $g(z)$ is bounded, the simplex method guarantees that there exists an optimal solution $g(\rho) \in K(\rho)^n$ to the linear program $LP(z)$ with optimal value $h(\rho) \in K(\rho)$. Moreover the following proposition holds (see \cite{21, 2.3, 22, Corollary 2}).

**Proposition A.4.** Under the above notations, there exists $\lambda_\rho \in \mathbb{R}_{>0}$ such that for every $0 < \lambda < \lambda_\rho$, $g(\rho)(\lambda) \in K(\lambda)^n$ is an optimal solution to the linear program $LP(\lambda)$ with optimal value $h(\rho)(\lambda) \in K(\lambda)$.

**Proof of Theorem A.1.** As $\#R_n < \infty$, $h^*(z) := \max\{h(\rho)(z) : \rho \in R_n\}$ and $\lambda_* := \min\{\lambda_\rho : \rho \in R_n\}$ satisfy $h^*(z) \in K(z)$ and $\lambda_* > 0$. Thanks to (A.4) and Proposition A.4 the Kantorovich difference $KD_\lambda(x_0, y_0)$ is expressed as $KD_\lambda(x_0, y_0) = h^*(\lambda)$ for any $0 < \lambda < \lambda_*$. Since $h^*(z)$ is a rational function of $z$, the limit in (A.4) exists, which establishes the theorem. \hfill \Box

**Appendix B. More general settings**

Our arguments so far are applicable to more general settings for submodular transformations. Here, submodular transformation is a vector valued set function consisting of submodular functions. In this section, we review about submodular functions, submodular transformations, and these Laplacian and show some examples. We also give a sufficient condition for a submodular transformation to be able to straightforwardly generalize the curvature notions in Section 3 and theorems in Section 5. For more details about submodular transformations, see \cite{45}.

**B.1. Submodular function.** Let $V$ be a nonempty finite set. A function $F : 2^V \to \mathbb{R}$ is a submodular function if for any $S, T \subset V$, $F$ satisfies

$$F(S) + F(T) \geq F(S \cup T) + F(S \cap T).$$
An element $v \in V$ is relevant in $F : 2^V \to \mathbb{R}$ if there is a $S \subset V$ such that $F(S) \neq F(S \cup \{v\})$. We say that $v$ is irrelevant in $F$ if $v$ is not relevant in $F$. We define the support $\text{supp}(F)$ of $F$ as the set of elements which are relevant in $F$. A set function $F : 2^V \to \mathbb{R}$ is symmetric if $F(S) = F(V \setminus S)$ holds for any $S$. We say $F$ is normalized if $F(V) = 0$.

**Example B.1.** Let $H = (V, E)$ be a hypergraph, and $e \in E$ a hyperedge. Then, the cut function $F_e$ of $e$ defined as follows is a submodular function:

$$F_e(S) = \begin{cases} 1 & \text{if } e \cap S \neq \emptyset \text{ and } e \cap (V \setminus S) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that a vertex $v \in V$ is relevant in $F_e$ if and only if $v \in e$. Furthermore, $F_e$ is symmetric and normalized.

For a submodular function $F : 2^V \to \mathbb{R}$, we define

$$P(F) = \left\{ g \in \mathbb{R}^V : \sum_{x \in S} g(x) \leq F(S) \text{ for any } S \subset V \right\}$$

and

$$B(F) = \left\{ g \in P(F) : \sum_{x \in V} g(x) = F(V) \right\}$$

called the submodular polyhedron and the base polytope respectively. Then, it is known that $B(F)$ is a bounded polytope.

The Lovász extension $f : \mathbb{R}^V \to \mathbb{R}$ of a submodular function $F : 2^V \to \mathbb{R}$ is defined by

$$f(g) = \max_{b \in B(F)} b^\top g.$$ 

It is known that $f(\chi_S) = F(S)$ for any $S \subset V$. Here, $\chi_S$ is the characteristic function of $S$. In particular, $f$ is indeed an extension of $F$. It is also known that the Lovász extension $f$ of a submodular function $F$ is convex.

For the Lovász extension $f$ of a submodular function $F$, we set

$$\partial f(g) = \arg\max_{b \in B(F)} b^\top g.$$ 

Then, it is known that $\partial f(g)$ is the sub-differential of $f$ at $g$.

**B.2. Submodular transformation and submodular Laplacian.** Let $V, E$ be nonempty finite sets. A function $F : 2^V \to \mathbb{R}^E ; S \mapsto (F_e(S))_{e \in E}$ is a submodular transformation if each $F_e$ is a submodular function. A submodular transformation $F$ is symmetric (resp. normalized) if any $F_e$ is symmetric (resp. normalized).

The Lovász extension $f : \mathbb{R}^V \to \mathbb{R}^E$ of a submodular transformation $F$ is defined by $f = (f_e)$ such that $f_e$ is the Lovász extension of $F_e$.

For a submodular transformation $F : \mathbb{R}^V \to \mathbb{R}^E$, we consider a weight function $\omega : E \to \mathbb{R}_{>0}$. Then, we call the quadruple $(V, E, F, \omega)$ a weighted submodular transformation. We stand for the quadruple as $F$. We define the degree $d_e$ for
Let \( F = (V, E, F, \omega) \) be a submodular transformation. Then, we define the submodular Laplacian \( L: \mathbb{R}^V \rightarrow 2^{\mathbb{R}^V} \) by
\[
L(g) := \left\{ \sum_{e \in E} \omega(e)b_eb_e^T g; \ b_e \in \partial f_e(g) \right\} \subset \mathbb{R}^V.
\]
We call \( \mathcal{L} := L \circ D^{-1} \) the normalized Laplacian. We set the inner product \( \langle f, g \rangle := f^T D^{-1} g \) and consider \( (\mathbb{R}^V, \langle \cdot, \cdot \rangle) \) as a Hilbert space. Then, by a similar argument as in \([20, \text{Lemma 14, Lemma 15}]\), the following holds:

**Proposition B.2.** The normalized Laplacian \( \mathcal{L} \) is a maximal monotone operator on the Hilbert space \((\mathbb{R}^V, \langle \cdot, \cdot \rangle)\).

More strongly, the normalized Laplacian \( \mathcal{L} \) is the sub-differential of the convex function \( Q: \mathbb{R}^V \rightarrow \mathbb{R} \) defined by
\[
Q(g) = \frac{1}{2} \sum_{e \in E} \omega(e)f_e(g)^2,
\]
where \( g = D^{-1} g \).

By Proposition [13.2], we can define the resolvent \( J_\lambda \), the canonical restriction \( \mathcal{L}_0 \), and the heat semigroup \( h_t \) for the Laplacian \( \mathcal{L} \). Then, the straight extension of Lemma 2.2 holds.

We define \( \pi \in \mathbb{R}^V \) as \( \pi(x) = d_x/\text{vol}(V) \). Then, the following holds:

**Lemma B.3 ([45, Lemma 3.1]).** We assume that \( F \) is normalized, i.e., \( F_e(V) = 0 \) for any \( e \in E \). Then, \( \mathcal{L}(\pi) = 0 \) holds.

By Lemma B.3, the similar lemmas as Lemma 2.3 and Lemma 2.4 hold for the normalized submodular Laplacian \( \mathcal{L} \). This implies that by similar arguments, we can obtain the straightforward extensions of definitions and theorems in Section 3 and Section 5 for any normalized submodular transformation \( F \) with the normalized submodular Laplacian \( \mathcal{L} \) for \( F \).

**B.3. Examples.** In [45], Yoshida gave many examples of submodular transformations such as undirected graph (Example 1.1, 1.2, and 1.4), directed graph (Example 1.5), hypergraph (Example 1.6), submodular hypergraph (Example 1.7), mutual information (Example 1.8), and directed information (Example 1.9). We here show another example:
Example B.4 (directed hypergraph). A weighted directed hypergraph $H$ is defined as the triple $H = (V, E, \omega)$ of a set of vertices $V$, a set of hyperarcs $E \subset 2^V \times 2^V$, and a weight function $\omega: E \to \mathbb{R}_{>0}$. Here, a hyperarc $e \in E$ is an ordered pair $(t_e, h_e)$ of a set of tails $t_e$ and a set of heads $h_e$. If $|t_e| = |h_e| = 1$ holds for any $e \in E$, then $H$ is a usual directed graph. If $t_e = h_e$ holds for any $e \in E$, then $H$ can be regarded as an undirected hypergraph. Hence, directed hypergraph is a generalization of directed graph and hypergraph.

We define the set function $F_e: 2^V \to \mathbb{R}$ as the cut function for $e = (t_e, h_e)$, i.e.,

$$F_e(S) := \begin{cases} 1 & \text{if } S \cap t_e \neq \emptyset \text{ and } (V \setminus S) \cap h_e \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is easy to show that $F_e$ is submodular. Hence, the quadruple $F = (V, E, F = (F_e)_e, \omega)$ becomes a submodular transformation. We remark that $F$ is normalized and not symmetric.

For this submodular transformation, by a simple calculation from definition of $B(F_e)$, we have

$$(B.1) \quad B(F_e) = \text{Conv}(\{\delta_x - \delta_y; x \in t_e, y \in h_e\} \cup \{0\}).$$

The base polytope for hypergraph (2.2) is a realization of this for $t_e = h_e$. By the representation (B.1), the Lovász extension $f_e$ of $F_e$ is written as

$$f_e(g) = \max\{\max\{g(x) - g(y); x \in t_e, y \in h_e\}, 0\}.$$ 

Because any examples introduced in this subsection, $F$ is normalized, i.e., $F(V) = 0$. Hence, the similar definitions of coarse Ricci curvatures for $F$ as in Section 3 and the similar theorems as in Section 5 hold. Comparing properties of curvatures for these examples with them of other curvatures introduced by [35], [15] is an interesting problem. We leave it for a future work.

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1At first glance, this specialization seems to be strange. However, from the point of view of submodular transformation, this looks natural. Indeed, under the assumption $t_e = h_e$, the cut function is same as this for undirected hypergraphs
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