VACUUM KERR-SCHILD METRICS
GENERATED BY NONTWISTING CONGRUENCES†
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ABSTRACT
The Kerr-Schild pencil of metrics $\tilde{g}_{ab} = g_{ab} + Vl_a l_b$, with $g_{ab}$ and $\tilde{g}_{ab}$ satisfying the vacuum Einstein equations, is investigated in the case when the null vector $l$ has vanishing twist. This class of Kerr-Schild metrics contains two solutions: the Kasner metric and a metric which can be obtained from the Kasner metric by a complex coordinate transformation. Both are limiting cases of the Kóta-Perjés metrics. The base space-time is a pp-wave.

Keywords: General Relativity, Kerr-Schild metrics, nontwisting null congruences

1. INTRODUCTION
The Kerr-Schild map
\begin{equation}
\tilde{g}_{ab} = g_{ab} + Vl_a l_b
\end{equation}
generates a pencil of space-time metrics $\tilde{g}_{ab}$ from the metric $g_{ab}$ where $l$ is a null vector and $V$ a function. The solution of the original problem, where a flat parent space-time is mapped to a vacuum space-time, has been known for some time. In the case of Kerr-Schild pencils with a non-twisting and divergence-free $l$, this is due to Trautman\textsuperscript{1}. For the congruences with nonvanishing divergence or twist, the general solution is given by Kerr and Schild\textsuperscript{2}.

Here we shall not impose any restriction on the parent metric $g$ apart from being Lorentzian and vacuum. Still, it follows from the vacuum Einstein equations for the pencil (0) that the null vector $l$ is tangent to geodesics. The general solution of this problem was given recently by us\textsuperscript{3,4,5}. However, the special case when $l$ is twist-free is not per se covered by the generic procedure, and hints only were given in Ref. 5 as to how to treat the twist-free fields. The full details are given in this paper.

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In section 2 we state a theorem characterizing the twist-free Kerr-Schild metrics by the value of the shear parameter \( \eta \). As in the generic case, the values \( \sin \eta = 0, \pm 1, \pm \frac{1}{\sqrt{2}} \) of the shear parameter are exceptional, and these metrics cannot be continuously extended to other values of \( \eta \). These exceptional cases are covered by the treatment in Ref. 5, letting the twist vanish. However, the generic treatment in Ref. 5 cannot be extended to the fields with arbitrary \( \eta \). Sections 3 and 4 are devoted to these fields. Nonetheless, the resulting metrics prove to be the twist-free limits of the generic metrics\(^5\).

2. SOLUTION OF THE \( \Psi_1 \) SYSTEM

The computations follow closely the pattern of the generic case where the Newman-Penrose formalism\(^6\) was used. The tetrad vectors are denoted \( l = D, n = \Delta, m = \delta, \) and \( \bar{m} = \bar{\delta} \). The expressions for the optical scalars, the curvature quantity \( \Psi_0 \), the Kerr-Schild potential and the null tetrad vector \( m \) in the twist-free case \((B = 0)\) become\(^5\):

\[
\rho = -\frac{1 + \cos \eta}{2r}, \quad \sigma = -\frac{\sin \eta}{2r}, \quad \Psi_0 = -\frac{\sin \eta \cos \eta}{2r^2},
\]

\[
V = \frac{V_0}{r^\cos \eta}, \quad m = r^{-\frac{\cos \eta + \sin \eta + 1}{4}(Q_1 + \imath r \sin \eta Q_2)}.
\]

(1)

where \( Q_{1,2} = Q_{1,2}^j \frac{\partial}{\partial x^j} \). The functions \( \eta, Q_1^j, Q_2^j \) and \( V_0 \) do not depend on \( r \), the affine parameter of the (geodesic) integral curves of \( l = D = \frac{\partial}{\partial r} \). The tetrad has been uniquely fixed by choosing

\[
\epsilon = 0, \quad \pi = \alpha + \beta, \quad m^0 = 0.
\]

(2)

The fields with \( \eta = 0 \) are algebraically special.

Our main theorem\(^4\) can be stated in the twist-free case again:

**Theorem on twist-free vacuum-vacuum Kerr-Schild maps:** Unless \( \sin \eta = 0, \pm 1, \pm \frac{1}{\sqrt{2}} \), the Kerr-Schild potential is restricted by the relation \( \delta V = 0 \).

The treatment of the metrics with \( \sin \eta = 0, \pm 1, \pm \frac{1}{\sqrt{2}} \) in Ref. 5 holds unchanged for curl-free fields. In what follows, we will consider the case where the shear is not restricted. The explicit \( r \) dependence of \( \delta V \) is known by (1). Hence we obtain:

\[
\delta V_0 = \delta \eta = 0.
\]

(3)
Though the forms of the spin coefficients (1) are simple under the curl-free condition, the integration functions $V_0$ and $\eta$ are not constant. (Unlike in the generic case $\bar{\rho} - \rho \neq 0$, it does not follow from the commutator $[\bar{\delta}, \delta]$ that their $\Delta$ derivatives vanish.)

The field equations are the Newman-Penrose equations and the vacuum Einstein-equations for Kerr-Schild metrics. The subsystem of equations containing $D$ derivatives of spin coefficients $\alpha, \beta, \tau, \pi$ and $\Psi_1$ is closed and has been referred to as the $\Psi_1$ system. The general solution for this system was obtained as a finite series in the complex phase factor $C = \frac{r \cos \eta - iB}{r \cos \eta + iB}$. For the present nontwisting case the function $B = 0$, thus $C = 1$.

When we take the limit $B \to 0$ for the four fundamental solutions of the $\Psi_1$ system, two of them vanish because they are proportional to $C^{-1}$. The fundamental solutions can be multiplied by arbitrary $r$-independent real functions. Thus we divide by $B$ and apply the l’Hospital rule, $\frac{C-1}{B} \to -\frac{2i}{r \cos \eta}$ to obtain the two missing fundamental solutions of the $\Psi_1$ system. The general solution is:

\[
\begin{align*}
\pi &= \frac{E_1 \pi^{(1)}}{r^{-\cos + \sin \eta + \frac{1}{2}}} + \frac{E_2 \pi^{(2)}}{r^{-\cos - \sin \eta + \frac{1}{2}}} + \frac{E_3 \pi^{(3)}}{r^{3 \cos + \sin \eta + \frac{1}{2}}} + \frac{E_4 \pi^{(4)}}{r^{3 \cos - \sin \eta + \frac{1}{2}}} \\
\tau &= \frac{E_1 \tau^{(1)}}{r^{-\cos + \sin \eta + \frac{1}{2}}} + \frac{E_2 \tau^{(2)}}{r^{-\cos - \sin \eta + \frac{1}{2}}} + \frac{E_3 \tau^{(3)}}{r^{3 \cos + \sin \eta + \frac{1}{2}}} + \frac{E_4 \tau^{(4)}}{r^{3 \cos - \sin \eta + \frac{1}{2}}} \\
\alpha &= \frac{E_1 \Omega^{(1)}}{r^{-\cos + \sin \eta + \frac{1}{2}}} + \frac{E_2 \Omega^{(2)}}{r^{-\cos - \sin \eta + \frac{1}{2}}} + \frac{E_3 \Omega^{(3)}}{r^{3 \cos + \sin \eta + \frac{1}{2}}} + \frac{E_4 \Omega^{(4)}}{r^{3 \cos - \sin \eta + \frac{1}{2}}} \\
\Psi_1 &= \frac{E_1 \Psi^{(1)}_1}{r^{-\cos + \sin \eta + \frac{1}{2}}} + \frac{E_2 \Psi^{(2)}_1}{r^{-\cos - \sin \eta + \frac{1}{2}}} + \frac{E_3 \Psi^{(3)}_1}{r^{3 \cos + \sin \eta + \frac{1}{2}}} + \frac{E_4 \Psi^{(4)}_1}{r^{3 \cos - \sin \eta + \frac{1}{2}}}
\end{align*}
\]

where $E_1, \ldots, E_4$ are arbitrary functions independent of $r$ and the coefficients $\pi^{(1)}, \ldots, \pi^{(4)}, \tau^{(1)}, \ldots, \tau^{(4)}, \alpha^{(1)}, \ldots, \alpha^{(4)}, \Psi^{(1)}_1, \ldots, \Psi^{(4)}_1$ enlisted in Table 1 are functions of $\eta$ only.

Once the affine-parameter dependence of the spin coefficients $\rho, \sigma, \alpha, \beta, \pi, \tau, \Psi_0, \Psi_1$ and of the Kerr-Schild potential is known, the question naturally arises whether these solutions are compatible with the rest of the equations? In the following section, we will address this question.
3. THE REMAINING FIELD EQUATIONS

First we find from the commutator \([\bar{\delta}, \delta]r = \mu - \bar{\mu}\) that the spin coefficient \(\mu\) is real. Then, writing the tetrad vector \(n\) in the form

\[
n = N \partial / \partial r + N^i \partial / \partial x^i
\]

where \(N, N^i\) are unknown functions of all of the coordinates, some \(\Delta\) derivatives can be written as:

\[
\Delta \rho = \frac{1 + \cos \eta}{2r^2} N + \frac{\sin \eta}{2r} \Delta \eta, \quad \Delta \sigma = \frac{1 + \cos \eta}{2r^2} N + \frac{\sin \eta}{2r} \Delta \eta, \\
\Delta \Psi_0 = \frac{\sin \eta \cos \eta}{r^3} N - \frac{\cos^2 \eta - \sin^2 \eta}{2r^2} \Delta \eta, \quad \Delta \ln V = \Delta \ln V_0 - \frac{\cos \eta}{r} N + \sin \eta \ln r \Delta \eta.
\]

The equations (NP 4.2.1,q) of Ref. 6, the complex conjugate of (NP 4.2.p), the fifth Bianchi relation (NP 4.5) and the last Kerr-Schild equation\(^3,5\) form a closed algebraic system in the real variables \(\mu, N, \Delta \eta, \Delta \ln V_0\) and the complex variables \(\lambda, \gamma, \Psi_2\) and their complex conjugates:

\[
\begin{align*}
\rho \mu - \sigma \lambda - \Psi_2 &= a_1 \\
\sigma \mu + \rho \lambda + \sigma (\gamma - 3\bar{\gamma}) + \frac{1 + \cos \eta}{2r^2} N + \frac{\sin \eta}{2r} \Delta \eta &= \bar{a}_2 \\
\rho \mu + \sigma \lambda - \rho (\gamma + \bar{\gamma}) + \Psi_2 + \frac{\sin \eta}{2r^2} N - \frac{\cos \eta}{2r} \Delta \eta &= a_3 \\
\Psi_0 \mu - 4\Psi_0 \gamma - 3\sigma \Psi_2 + \frac{\sin \eta \cos \eta}{r^3} N - \frac{\cos^2 \eta - \sin^2 \eta}{2r^2} \Delta \eta &= a_4 \\
\frac{\Psi_0}{\sigma} \mu - 2\rho (\gamma + \bar{\gamma}) + \Psi_2 + \bar{\Psi}_2 - \rho (\Delta \ln V_0 - \frac{\cos \eta}{r} N + \sin \eta \ln r \Delta \eta) &= a_5
\end{align*}
\]

where the source terms \(a_1, \ldots a_5\) are given in Table 2, and their \(r\) dependence is known.

This system corresponds to, but does not hold as a limiting case, of Eqs. (5.2) of Ref. 5.

The unknown functions \(N, \Delta \eta\) and \(\Delta \ln V_0\) did not occur in the generic treatment. Now we cannot get rid of \(N\) by use of the commutator \([\delta, \bar{\delta}]r\) because \((\bar{\rho} - \rho)N\) vanishes, and we cannot prove property (ii) (that the \(\Delta\) derivatives vanish).
The imaginary parts of the first four equations of (7) yield:

\[
\begin{align*}
\lambda - \bar{\lambda} &= -3\sigma^2(a_1 - \bar{a}_1) + \sigma(a_4 - \bar{a}_4) - \Psi_0(a_2 - \bar{a}_2) \\
3\sigma^3 + \rho\Psi_0 \\
\gamma - \bar{\gamma} &= -\frac{a_2 - \bar{a}_2 + \rho(\lambda - \bar{\lambda})}{4\sigma} \\
\Psi_2 - \bar{\Psi}_2 &= -\sigma(\lambda - \bar{\lambda}) - (a_1 - \bar{a}_1) \\
C_1 &= a_1 + a_3 - \bar{a}_1 - \bar{a}_3 = 0
\end{align*}
\]  

(8)

The last relation is a constraint equation involving only expressions with known \( r \) dependence.

With algebraic manipulations on the first four equations of (7) one obtains:

\[
\begin{align*}
\mu &= \frac{1 + \cos\eta}{\sin\eta}\lambda + f_1 \\
r\Psi_2 &= -\frac{\cos\eta(1 + \cos\eta)}{\sin\eta}\lambda + f_2 \\
\frac{\gamma + \bar{\gamma} + \frac{N_r}{r}}{r} &= \frac{2(1 + \cos\eta)}{\sin\eta}\lambda + f_3 \\
\Delta\eta &= \frac{2r(a_1 + a_2) + (1 + \cos\eta)(2f_1 - f_3)}{\sin\eta} \equiv r C_2 + r^2 C_3
\end{align*}
\]  

(9)

where \( f_1, f_2, f_3 \) enlisted in Table 3 were formed from \( a_1, ...a_4 \). Note that the commutator \([\Delta, D]\eta\) implies that \( D\eta \) does not depend on \( r \), so \( \Delta\eta = 0 \) and \( C_2 = C_3 = 0 \) are constraints.

From the equations (NP 4.2.g) the \( r \) dependence of \( \lambda \) can be integrated:

\[
D(r^{1+\cos\eta}\lambda) = r^{1+\cos\eta}(\sigma f_1 + f_4)
\]

(10)

where the term \( f_4 \) with known \( r \) dependence is given in Table 3. After the elimination of \( \lambda \) from (10) and (NP 4.2.h), a fourth constraint \( C_4 = 0 \) arises. The solution of all these constraints allows one to express the \( Q_{1,2} \) derivatives of \( E_1, E_2 \) in terms of \( E_1, E_2 \), while \( E_3, E_4 \) is found to vanish (Table 4).

Finally, from \([\Delta, D]r\) one gets \((\gamma + \bar{\gamma} + \frac{N}{r}) = rD(-\frac{N}{r})\), and with a second integration one finds the \( r \) dependence of \( N \).

Using all these results in the last relation of (7), we express \( \Delta \ln V_0 \), which does not depend on \( r \) as follows from the commutator \([\Delta, D]\ln V_0\). This condition can be fulfilled
only if:
\[ E_1 = E_2 = E_3 = E_4 = 0 . \] (11)

We recover the same result as in the general case: only the trivial solution of the \( \Psi_1 \) system is compatible with the rest of the equations. We then have
\[
\begin{align*}
\kappa &= \epsilon = \pi = \tau = \alpha = \Psi_1 = \mu = \lambda = \Psi_2 = \Delta \eta = 0 \\
\gamma &= -\frac{H}{2} , \quad N = Hr , \quad \Delta lnV_0 = (\cos \eta + 2)H ,
\end{align*}
\] (12)

where \( H \) is an integration function restricted by the conditions \( Q_1 H = Q_2 H = 0 \) following from the commutator \([\delta, D]lnV_0\).

We complete the process of determining the \( r \) dependence of the spin coefficients by observing that
\[ \nu = \Psi_3 = \Psi_4 = 0 , \] (13)
from \([\delta, D]r\) and (NP 4.2.i,j) respectively. We conclude that the base space must be of type N.

4. THE METRICS

The commutators among \( Q_1, Q_2, N = N^i \frac{\partial}{\partial r} \) and \( D \), namely
\[
\begin{align*}
[N, D] &= [Q_1, D] = [Q_2, D] = [Q_1, Q_2] = 0 \\
[Q_1, N] &= -\frac{\cos \eta + \sin \eta + 1}{2} HQ_1 , \quad [Q_2, N] = -\frac{\cos \eta - \sin \eta + 1}{2} HQ_2
\end{align*}
\] (14)

allow one to fix the coordinates \((r, x, y, u)\) in the following way:
\[
\begin{align*}
D &= \frac{\partial}{\partial r} , \quad Q_1 = \frac{\partial}{\partial x} , \quad Q_2 = \frac{\partial}{\partial y} , \\
N &= -\frac{\cos \eta + \sin \eta + 1}{2} H(u) \frac{\partial}{\partial x} - \frac{\cos \eta - \sin \eta + 1}{2} H(u) \frac{\partial}{\partial y} + F(u) \frac{\partial}{\partial u}
\end{align*}
\] (15)

where \( F(u) \) is an arbitrary function of \( u \). The tetrad vectors in the chosen coordinates are:
\[
\begin{align*}
l^a &= \delta^a_1 \\
\delta^a_2 &= r \frac{\cos \eta + \sin \eta + 1}{2} \delta^a_2 + r \frac{\cos \eta - \sin \eta + 1}{2} \delta^a_3 \\
n^a &= ur \delta^a_1 - \frac{\cos \eta + \sin \eta + 1}{2} u x \delta^a_2 - \frac{\cos \eta - \sin \eta + 1}{2} u y \delta^a_3 + F(u) \delta^a_4
\end{align*}
\] (16)
The Kerr-Schild potential is found by integration of $V_0$ from (12):

$$V = \Lambda e^{(\cos \eta + 2) \int \frac{du}{F(u)}}.$$  \hfill (17)

where $\Lambda$ is an arbitrary parameter.

We are now able to write down the base space metric using the completeness relation of the tetrad (16). With (17), the Kerr-Schild pencil has the form

$$ds^2 = - \frac{r^{\cos \eta + \sin \eta + 1}}{2} \left[ dx + \frac{(\cos \eta + \sin \eta + 1) xu}{2F(u)} du \right]^2$$

$$- \frac{r^{\cos \eta - \sin \eta + 1}}{2} \left[ dy + \frac{(\cos \eta - \sin \eta + 1) yu}{2F(u)} du \right]^2$$

$$- \frac{1}{2ru} \left[ dr - \frac{2ru}{F(u)} du \right]^2 + \frac{dr^2}{2ru}$$

$$ds^2 = ds^2 + \Lambda \frac{e^{(\cos \eta + 2) \int \frac{du}{F(u)}}}{r^{\cos \eta}} \frac{du^2}{F(u)^2}.$$  \hfill (18)

The curvature components are:

$$\tilde{\Psi}_0 = - \frac{\cos \eta \sin \eta}{2r^2}$$

$$\tilde{\Psi}_1 = \tilde{\Psi}_3 = 0$$

$$\tilde{\Psi}_2 = \Lambda \frac{\cos \eta (\cos \eta + 1) e^{(\cos \eta + 2) \int \frac{du}{F(u)}}}{4r^{\cos \eta + 2}}$$

$$\tilde{\Psi}_4 = -\Lambda^2 \frac{\cos \eta \sin \eta e^{2(\cos \eta + 2) \int \frac{du}{F(u)}}}{8r^{2\cos \eta + 2}}.$$  \hfill (19)

We conclude that the Kerr-Schild space-time is of type I in the Petrov classification.

We may transform these metrics to a simpler form by the coordinate transformations

$$t = e^{\int \frac{du}{F(u)}} , \quad v = \frac{1}{\sqrt{2}} xt \frac{\cos \eta + \sin \eta + 1}{2r^2} , \quad w = \frac{1}{\sqrt{2}} yt \frac{\cos \eta - \sin \eta + 1}{2r^2} , \quad p = rt.$$  \hfill (20)

Noting that $t$ is function only of $u$, we define a new coordinate $q$ in place of $t$ by $dq = \frac{dt}{u}$. The old coordinates $(r, x, y, u)$ can be expressed in terms of the new ones $(v, w, p, q)$ provided that the first relation of (20) is invertible. The base and Kerr-Schild metrics in these new coordinates are:

$$ds^2 = -p^{\cos \eta + \sin \eta + 1} dv^2 - p^{\cos \eta - \sin \eta + 1} dw^2 + 2dpdq$$

$$d\tilde{s}^2 = ds^2 + \Lambda p^{-\cos \eta} dq^2.$$  \hfill (21)
The parent space metric has been obtained by Bilge\textsuperscript{7}. There has been some controversy in the literature (cf. Kupeli\textsuperscript{8} and Bilge\textsuperscript{7}) whether or not this pp-wave is unique in the considered class of Kerr-Schild maps. Here we succeeded in demonstrating the uniqueness.

The vectors \( \frac{\partial}{\partial v}, \frac{\partial}{\partial w} \) and \( \frac{\partial}{\partial q} \) are (commuting) Killing vectors of both metrics, moreover the last of them is covariantly constant. In the shear-free limit \( \sin \eta \to 0 \) both metrics admit a fourth Killing vector \( v \frac{\partial}{\partial w} - w \frac{\partial}{\partial v} \), thus they are plane symmetric metrics (13.9) of Ref. 9.

The metric \( d\tilde{s}^2 \) in (21) can be put into a more familiar form as follows,

(a) in case \( \Lambda < 0 \) we introduce the new coordinates \((T, X, Y, Z)\):

\[
T = (-\Lambda)^{-1} \left( p_{1} \frac{\cos \eta}{\cos \eta + 1} \right)^{\frac{\cos \eta - \sin \eta + 1}{\cos \eta + 1}} v
\]

\[
X = \left( (-\Lambda)^{1/2} \left( \frac{\cos \eta}{2} + 1 \right) \right) \left( \frac{\cos \eta + \sin \eta + 1}{\cos \eta + 1} \right)^{\frac{1}{2}} w
\]

\[
Y = \left( (-\Lambda)^{1/2} \left( \frac{\cos \eta}{2} + 1 \right) \right) \left( \frac{-\cos \eta}{\cos \eta + 1} \right)^{\frac{1}{2}} \left( -\Lambda \right)^{1/2} p_{1}^{1 + \cos \eta} \]

\[
Z = \left( (-\Lambda)^{1/2} \left( \frac{\cos \eta}{2} + 1 \right) \right) \left( \frac{-\cos \eta}{\cos \eta + 1} \right)^{\frac{1}{2}} \left( -\Lambda \right)^{1/2} p_{1}^{1 + \cos \eta} \]

The Kerr-Schild metric is the Kasner metric\textsuperscript{10}:

\[
d\tilde{s}^2 = dT^2 - T^{2p_1} dX^2 - T^{2p_2} dY^2 - T^{2p_3} dZ^2,
\]

where the powers

\[
p_1 = \frac{\cos \eta + \sin \eta + 1}{\cos \eta + 2}, \quad p_2 = \frac{\cos \eta - \sin \eta + 1}{\cos \eta + 2}, \quad p_3 = \frac{-\cos \eta}{\cos \eta + 2}
\]

satisfy the required relations \( p_1 + p_2 + p_3 = 1 \) and \( (p_1)^2 + (p_2)^2 + (p_3)^2 = 1 \).

(b) in case \( \Lambda > 0 \) the new coordinates \((T, X, Y, Z)\) are introduced in the following
manner:

\[
T = \left(\frac{\Lambda}{\Lambda} - \frac{1}{2}\right) p \cos \eta + 1 \\
X = \left[\left(\frac{\Lambda}{\Lambda} + 1\right) \left(\frac{\cos \eta}{2} + 1\right)\right]^{\frac{\cos \eta + \sin \eta + 1}{2}} v \\
Y = \left[\left(\frac{\Lambda}{\Lambda} + 1\right) \left(\frac{\cos \eta}{2} + 1\right)\right]^{\frac{\cos \eta - \sin \eta + 1}{2}} w \\
Z = \left[\left(\frac{\Lambda}{\Lambda} + 1\right) \left(\frac{\cos \eta}{2} + 1\right)\right]^{\frac{-\cos \eta}{2}} \left[\left(\frac{\Lambda}{\Lambda} + 1\right) \left(\frac{p^{1 + \cos \eta}}{1 + \cos \eta}\right)\right].
\] (25)

The Kerr-Schild metric is a sign-flipped version of the Kasner metric as described by McIntosh\textsuperscript{11}:

\[
ds^2 = -dT^2 - T^2 p_1 dX^2 - T^2 p_2 dY^2 + T^2 p_3 dZ^2.
\] (26)

This metric arises by way of a complex coordinate transformation on the Kasner metric.

5. CONCLUDING REMARKS

Both solutions (23) and (26) are the special cases of the Kóta-Perjé metric (5.11) of Ref. 12 for \( B = 0 \).

Lemma. The Kasner and sign-flipped Kasner metrics (24) and (26), respectively, are the exhaustive members of the class of vacuum Kerr-Schild metrics generated by non-twisting congruences.

Bilge\textsuperscript{7} has considered vacuum metrics with a non-twisting geodesic congruence and subject to the conditions that \( \sigma = a \rho \) and \( Da = 0 \). He has proven that \( a \) is a constant. The relation of our work with Bilge’s can be established by noting that his \( a = \frac{\sin \eta}{1 + \cos \eta} \). Bilge’s solution involves the exceptional values \( \sin \eta = 0, \pm 1, \pm 4/5 \) which are different from the ones given in our Theorem. Apparently, the reason is that the Kerr-Schild property has not been imposed on the metric \( \tilde{g}_{ab} \) by Bilge.
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\(\pi^{(1)} = -2(3\cos\eta + 4\sin\eta)(\cos\eta - 1)\cos\eta\)

\(\pi^{(2)} = -2(4\cos^2\eta + 3\cos\eta\sin\eta - 3\cos\eta + 4\sin\eta - 4)(\cos\eta - 1)\)

\(\pi^{(3)} = 4 \left[(\sin\eta + 3)\cos\eta + (\sin\eta + 1) + (\sin\eta - 1)\cos^2\eta - 3\cos^3\eta\right]\)

\(\pi^{(4)} = -4 \left[(\sin\eta + 1)\cos^2\eta + (\sin\eta - 1) + (\sin\eta - 3)\cos\eta + 3\cos^3\eta\right]\)

\(\tau^{(1)} = 2(\cos^2\eta - 2\cos\eta\sin\eta + 2\cos\eta + 2\sin\eta - 2)(\cos\eta - 1)\)

\(\tau^{(2)} = 2(2\cos\eta - \sin\eta - 1)(\cos\eta - 1)\cos\eta\)

\(\tau^{(3)} = \pi^{(3)}\)

\(\tau^{(4)} = -\pi^{(4)}\)

\(\alpha^{(1)} = [4(\sin\eta + 1) - (\sin\eta - 3)\cos\eta - 7\cos^2\eta] \cos\eta\)

\(\alpha^{(2)} = 2(\cos\eta + 4)(\sin\eta - 1) - (7\sin\eta - 11)\cos^2\eta - \cos^3\eta\)

\(\alpha^{(3)} = \frac{\pi^{(3)}}{2}\)

\(\alpha^{(4)} = \frac{\pi^{(4)}}{2}\)

\(\Psi_1^{(1)} = (\cos^2\eta - 7\cos\eta\sin\eta - 3\cos\eta - 4\sin\eta - 4)(\cos\eta - 1)\cos\eta\)

\(\Psi_1^{(2)} = (7\cos^3\eta - \cos^2\eta\sin\eta + 5\cos^2\eta + 10\cos\eta\sin\eta - 10\cos\eta + 8\sin\eta - 8)(\cos\eta - 1)\)

\(\Psi_1^{(3)} = 2(2\cos^2\eta - 4\cos\eta\sin\eta + \cos\eta + \sin\eta - 3)(\cos\eta + 1)\cos\eta\)

\(\Psi_1^{(4)} = -2(2\cos^2\eta + 4\cos\eta\sin\eta + \cos\eta - \sin\eta - 3)(\cos\eta + 1)\cos\eta\)

**Table 1. The coefficients in the general solution of the \(\Psi_1\) system**

\(a_1 = \delta\alpha - \delta\beta - \alpha\tilde{\alpha} - \beta\tilde{\beta} + 2\alpha\beta\), \(a_2 = \delta\tau - \tau(\tau + \beta - \tilde{\alpha})\), \(a_3 = \tilde{\delta}\tau + \tau(\tilde{\beta} - \alpha - \tilde{\tau})\), \(a_4 = \delta\Psi_1 - \Psi_1(4\tau + 2\beta)\), \(a_5 = \delta(\tilde{\tau} - \pi) + \tilde{\delta}(\tau - \tilde{\pi}) - 6\pi\tilde{\pi} + 2\pi\tilde{\alpha} + 2\tilde{\pi}\alpha - 2\tau\tilde{\pi} - 2\alpha\tau - 2\tilde{\alpha}\tau + 3\tau\pi + 3\tilde{\tau}\pi\)

**Table 2. The source terms of eq. (7)**
\[f_1 = \frac{-2ra_1 - 2b_1}{1+c} - \frac{2b_1}{s(1+c)} - \frac{2[sb_2 + 2cr(a_1 + a_2)]}{(1+c)^2}\]

\[f_2 = \frac{b_1}{s} + \frac{[sb_2 + 2cr(a_1 + a_2)]}{1+c}\]

\[f_3 = -2ra_1 + \frac{b_1}{s} - \frac{[sb_2 + 2cr(a_1 + a_2)]}{1+c}\]

\[f_4 = \bar{\delta} \pi + \bar{\sigma} \mu + \pi^2 + (\alpha - \bar{\beta}) \pi\]

where: \(s = \sin \eta\), \(c = \cos \eta\),

\[b_1 = 2r^2a_4 - 2r(c\bar{a}_2 + sa_3) - 4sc(\gamma - \bar{\gamma})\], \(b_2 = 2r\bar{a}_2 + 2s(\gamma - \bar{\gamma})\)

**Table 3. The inhomogeneous terms in eq. (9,10)**

\[
(30c^5 + 40c^4s - 12c^4 - 16c^3s + 2c^3 - 59c^2s + 4c^2 + 18cs - 34c - 6s + 6) \cdot Q_1(E_1) = \\
2E_1^2(110c^8 - 20c^7s - 174c^7 + 168c^6s - 184c^6 - 347c^5s + 546c^5 + 127c^4s \\
- 290c^4 + 296c^3s - 220c^3 - 352c^2s + 340c^2 + 152cs - 152c - 24s + 24)
\]

\[
(2c+s-1) \cdot Q_2(E_1) = -[Q_1(E_2)(c+2s-2) + (12cs^2 - 12cs + 40s^4 - 40s^3 - 24s^2 + 24s)E_1E_2]
\]

\[
(7c^2 - 24cs - 40c + 30s + 34) \cdot Q_1(E_2) = \\
2E_1E_2(82c^5 + 76c^4s - 9c^4 - 237c^3s - 289c^3 + 136c^2s + 244c^2 \\
+ 36cs - 36c - 8s + 8)
\]

\[
(8c^4 - 6c^3s + 2c^3 - 4c^2s - 5c^2 + 6cs + s - 1) \cdot Q_2(E_2) = \\
2c^2E_2^2(10c^5 - 20c^4s - 18c^4 + 16c^3s + 8c^3 + 25c^2s \\
+ 10c^2 - 25cs - 18c + 4s + 8)
\]

\[E_3 = E_4 = 0\]

**Table 4. The solution of constraints C_1 - C_4 for Q_i(E_k), E_3 and E_4**