RTT REALIZATION OF QUANTUM AFFINE SUPERALGEBRAS AND TENSOR PRODUCTS

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ABSTRACT. We use the RTT realization of the quantum affine superalgebra associated with the Lie superalgebra $\mathfrak{gl}(M,N)$ to study its finite-dimensional representations and their tensor products. In the case $\mathfrak{gl}(1,1)$, the cyclicity condition of tensor products of finite-dimensional simple modules is determined completely in terms of zeros and poles of rational functions. This in turn induces cyclicity of some particular tensor products of Kirillov-Reshetikhin modules related to $\mathfrak{gl}(M,N)$.

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1. INTRODUCTION

Let $q$ be a non-zero complex number which is not a root of unity. Let $\mathfrak{g} := \mathfrak{gl}(M,N)$ be the general linear Lie superalgebra. Let $U_q(\hat{\mathfrak{g}})$ be the associated quantum affine superalgebra. (We refer to [3.2] for the precise definition.) This is a Hopf superalgebra neither commutative nor co-commutative, and it can be seen as a deformation of the universal enveloping algebra of the following affine Lie superalgebra:

$$L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \bigoplus_{1 \leq i,j \leq M+N} E_{ij} \otimes \mathbb{C}[t, t^{-1}].$$

Here the $E_{ij}$ for $1 \leq i, j \leq M+N$ are the elementary matrices in $\mathfrak{g}$.

In this paper we are mainly concerned with the structure of tensor products of finite-dimensional simple $U_q(\hat{\mathfrak{g}})$-modules.

1.1. Backgrounds. Quantum superalgebras appear as the algebraic supersymmetries of some solvable models. For example, the quantum affine superalgebra $U_q(\hat{\mathfrak{sl}(M,N)})$ is the quantum supersymmetry analogue of the supersymmetric $t-J$ model (with or without a
boundary). A key problem is to diagonalize the commuting transfer matrices. In principle, this can be achieved \cite{Ko13} by constructing the bosonization of vertex operators, which are built over some highest weight Fock representations of $U_q(\widehat{\mathfrak{sl}(M, N)})$.

Another main interest in quantum superalgebras comes from the integrability structure in the context of the AdS/CFT correspondence \cite{Be12}. In this case, the underlying simple Lie superalgebra is $\mathfrak{psl}(2,2)$, which is the quotient of Lie superalgebra $\mathfrak{sl}(2,2)$ by its center, the line generated by the identity matrix. A striking feature differing $\mathfrak{psl}(2,2)$ from all the other simple Lie superalgebras (including simple Lie algebras) is that this simple Lie superalgebra admits a non-trivial three-fold central extension. Based on the Lie superalgebra $\mathfrak{psl}(2,2)$, several quantum superalgebras have been built as algebraic supersymmetries in AdS/CFT and the closely related Hubbard model: the quantum deformation of extended $\mathfrak{sl}(2,2)$ in \cite{BK08}, the quantum affine deformation of extended $\mathfrak{sl}(2,2)$ in \cite{BGM12}, and the conventional Yangian of extended $\mathfrak{sl}(2,2)$ in \cite{Be06, BDL}, to name a few. Representations of these superalgebras have been considered from different perspectives: \cite{Be07, MM14} for centrally extended $\mathfrak{sl}(2,2)$ and \cite{ADT10} for the conventional Yangian. For the quantum (affine) superalgebra of extended $\mathfrak{sl}(2,2)$, only 4-dimensional fundamental representations and $R$-matrices arising from their tensor products were discussed in \cite{BK08, BGM12}.

More closely related to our present paper is the work of Bazhanov-Tsuboi \cite{BT08} on Baxter’s $Q$-operators related to the quantum affine superalgebra $U_q(\widehat{\mathfrak{sl}(2,1)})$. In loc. cit they constructed the so-called oscillator representations of the upper Borel subalgebra $\mathfrak{B}_+$. These representations gave rise directly to the $Q$-operators and therefore found remarkable applications in spin chain models and in quantum field theory. Their oscillation construction has been generalized to the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}(\widetilde{M}, N)})$ in a recent paper of Tsuboi \cite{Ts12} by using RTT realization.

On the other hand, Hernandez-Jimbo \cite{HJ12} constructed similar oscillator representations of the upper Borel subalgebra $\mathfrak{B}_+$ of an arbitrary non-twisted quantum affine algebra. In their context, oscillator representations were realized as certain asymptotic limits of Kirillov-Reshetikhin modules over the quantum affine algebra, hence bearing the name asymptotic representations. The asymptotic construction enabled Frenkel-Hernandez \cite{FH13} to give a representation theoretic interpretation of Baxter’s $T-Q$ relations and to solve a conjecture of Frenkel-Reshetikhin on the spectra of quantum integrable systems \cite{FR99}.

Based on the above progress, it is natural to consider representation theory of the quantum affine superalgebra $U_q(\mathfrak{g})$, and more specifically the quantum superalgebras related to centrally extended $\mathfrak{sl}(2,2)$. In the present paper, $U_q(\mathfrak{g})$ is our main concern.

We are motivated by the following question: can the oscillator representations related to the quantum affine superalgebra $U_q(\mathfrak{g})$ in \cite{BT08, Ts12} be realized as asymptotic limits of Kirillov-Reshetikhin modules in the spirit of Hernandez-Jimbo?

In the super case, the representation theory of quantum affine superalgebras is still less developed, compared to the vast literature on representations of quantum affine algebras (see the two review papers \cite{CH10, Le11}).

\section{Representations of $U_q(\mathfrak{g})$}

In a recent paper \cite{Zh13}, we obtained a classification of finite-dimensional simple modules for the quantum affine superalgebra $U_q(\mathfrak{g})$. For the Kac-Moody superalgebra $\mathfrak{g}$, let $I_0 := \{1,2,\cdots, M+N-1\}$ be the set of vertices of the
distinguished Dynkin diagram. Hence \( i \in I_0 \) corresponds to the simple root \( \alpha_i \) and \( \alpha_M \) is an odd isotopic simple root. The main result in loc. cit can be stated as follows: up to tensor product by one-dimensional modules, finite-dimensional simple \( U_q(\widehat{\mathfrak{g}}) \)-modules are of the form \( S(f) \) where \( f = (f_i)_{i \in I_0} \) is an \( I_0 \)-tuple of rational functions \( f_i(z) \in \mathbb{C}(z) \) such that:

(a) if \( i \neq M \) then there exists a polynomial \( P_i(z) \) with constant term 1 such that
\[
f(z) = q_i^{\deg P_i} \frac{P_i(qz^{-1})}{P_i(q)}.
\]
Here \( q_i = q \) for \( i \leq M \) and \( q^{-1} \) otherwise;

(b) if \( i = M \), then \( f_i(z) \) as a meromorphic function is regular at \( z = 0 \) and \( z = \infty \).
Moreover, \( f_i(0)f_i(\infty) = 1 \).

We remark that (a) implies (b) but not vice versa. Hence this classification result is different from the case of quantum affine algebras [CP91].

In analogy with the non-graded case, Kirillov-Reshetikhin modules for \( U_q(\widehat{\mathfrak{g}}) \) will be those modules \( S(\varpi^{(i)}_{n,a}) \) where \( i \in I_0 \) is a Dynkin vertex, \( a \in \mathbb{C}^\times \) is a spectral parameter, \( n \in \mathbb{Z}_{>0} \) is a positive integer, and \( \varpi^{(i)}_{n,a} \) is the \( I_0 \)-tuple of rational functions whose \( i \)-th coordinate is \( q_i^n \frac{1-zaq^{-n}}{1-zaq} \) and whose other coordinates are 1. When \( n = 1 \), the Kirillov-Reshetikhin modules are also called fundamental modules.

### 1.2.1. Asymptotic limits

Let us fix a Dynkin vertex \( i \in I_0 \) and a spectral parameter \( a \in \mathbb{C}^\times \). For \( n \in \mathbb{Z}_{>0} \), the \( i \)-th coordinate for \( \varpi^{(i)}_{n,aq^n} \) has the asymptotic expression \( q_i^n \frac{1-zaq^{-n}}{1-zaq} \).

Informally, by taking asymptotic limit of the Kirillov-Reshetikhin modules \( S(\varpi^{(i)}_{n,aq^n}) \) we should get a “module” where the \( i \)-th coordinate is \( 1 - za \) (by first forgetting the constant term \( q_i^n \) and then taking the analysis limit \( \lim_{n \to \infty} q_i^n = 0 \)). This module should be an oscillator module.

The above intuitive argument was made mathematically rigorous in [HJ12], where inductive/projective systems of Kirillov-Reshetikhin modules were constructed and their inductive/projective limits were shown to be oscillator modules. One of the main ingredients for the construction of these systems is a cyclicity property of tensor products of Kirillov-Reshetikhin modules of a particular form. Also a result of [He10] Proposition 3.2 on \( q \)-characters of tensor products of simple modules was needed to establish stability and asymptotic properties of these systems.

### 1.2.2. Tensor products of Kirillov-Reshetikhin modules

Let us explain in detail the cyclicity result used in [HJ12]. In this paragraph let us replace the Lie superalgebra \( \mathfrak{g} \) by an arbitrary simple Lie algebra \( \mathfrak{g}'' \). The set \( J \) of Dynkin vertices, the numbers \( q_j \) for \( j \in J \), and the Kirillov-Reshetikhin modules \( S(\varpi^{(j)}_{n,a}) \) are similarly defined. Then

\[ (C) \text{ the tensor products } \bigotimes_{l=1}^k S(\varpi^{(j)}_{n,laq^{(j-2l)}}) \text{ for } k \geq 1 \text{ are cyclic}. \]

Here being cyclic means being of highest \( \ell \)-weight with respect to the Drinfeld type triangular decomposition of \( U_q(\widehat{\mathfrak{g}}) \).

Let us give a quick overview of cyclicity property of tensor products of finite-dimensional simple modules over \( U_q(\widehat{\mathfrak{g}}) \). We refer to [CH10; §5] for more historical comments. In [AK97; Conjecture 1] it was conjectured by Akasaka-Kashiwara that
(AK) let $V_1, V_2, \ldots, V_n$ be fundamental $U_q(\widehat{g}''')$-modules and let $x_1, x_2, \ldots, x_n \in \mathbb{Z}$. Then the tensor product $\otimes_{i=1}^n (V_i)_{q^{x_i}}$ is cyclic if $x_i \geq x_j$ for all $1 \leq i < j \leq n$.

This conjecture has been proved in the case of type $A_n^{(1)}$ and $C_n^{(1)}$ in loc. cit and later by Kashiwara [Ka02, Theorem 9.1] in full generality. Both proofs relied on crystal bases theory for modules over the quantum affine algebra. Now (C) is a direct consequence of (AK).

At the same time, a geometric proof of (AK) in the simply laced case was given by Varagnolo-Vasserot using quiver varieties [VV02, Corollary 7.17].

Also (AK) was generalized by Chari [Ch02, Theorem 4.4] with a more Lie theoretic and algebraic approach. By using the Weyl group action on the set of weights of a $U_q(\widehat{g}''')$-module, and the Braid group action on the affine Cartan subalgebra of $U_q(\widehat{g}''')$, Chari reduced the cyclicity problem on $U_q(\widehat{g}''')$-modules to a series of similar problems on $U_q(\widehat{sl}_2)$-modules corresponding to a fixed reduced expression of the longest element $w_0$ in the Weyl group. Eventually a sufficient condition for a tensor products of simple modules $S(f)$ to be cyclic was given in terms of Drinfeld polynomials defining these $f$ as in (a).

1.2.3. The super case. To construct asymptotic limits, we need inevitably such cyclicity property as (C) of tensor products of Kirillov-Reshetikhin modules over the quantum affine superalgebra $U_q(\widehat{g})$. However, the main techniques used in the non-graded case to deduce cyclicity results cannot be applied directly to the super case. For example, crystal base theory and quiver geometry for quantum affine superalgebras, or even for finite type quantum superalgebras, are still less developed [BKK00]. The main drawback comes from the fact that the Weyl group of $g$, being $S_M \times S_N$ instead of $S_{M+N}$, is too small to capture enough information on weights and linkage.

Nevertheless, we can prove a weak version of (AK) (yet stronger than (C)) for quantum affine superalgebras, by modifying the arguments of Chari in [Ch02]. Although our motivation of studying the cyclicity property of tensor products comes from the asymptotic construction, we think that cyclicity property is of independent interest, and a large part of the present paper is devoted to proving this weak version of (AK), Theorem 4.2.

For this purpose, we shall adopt the RTT realization of the quantum affine superalgebra $U_q(\widehat{g})$ instead of the Drinfeld realization. The main advantages are that: first of all, RTT generators are quantum analogues of such loop generators $E_{ij} \otimes t^n \in Lg$; secondly and more importantly, RTT generators have nice coproduct formulas. Our present work is inspired on the one hand by the work [MTZ04] of Molev, Tolstoy and Rui-Bin Zhang on simplicity of tensor products of evaluation modules for the quantum affine algebra $U_q(\widehat{gl}_N)$, where RTT realization and coproduct formulas for RTT generators made the relevant calculations transparent, and on the other hand by the work of Tsuboi [Ts12] on oscillation constructions using RTT realization as mentioned before.

In comparison to the non-graded case [AK97, Ch02, Ka02, VV02], our approach of studying cyclicity of tensor products differs from the perspective that we use (quantum analogue of) root vectors of the quantum affine superalgebra instead of Weyl groups. This is an idea already explored in our previous paper [Zh13] on classification of finite-dimensional simple modules, where Weyl group was replaced by Poincaré-Birkhoff-Witt linear generators in terms of Drinfeld generators for the quantum affine superalgebra. In particular, our approach applies also to quantum affine algebras of type $A$ (non-graded).
1.3. **Main results.** We study in full detail the RTT realization of the quantum affine superalgebra \( U_q(\hat{\mathfrak{g}}) \), including its definition, different kinds of grading, its degeneration to the finite type quantum superalgebra \( U_q(\mathfrak{g}) \), evaluation morphisms, its relationship with quantum double construction, and coproduct formulas for Drinfeld generators. Almost all the relevant results are proved in a uniform way. This makes the present paper longer than we have expected.

The first main result is an analogue of (AK) under the assumption that the fundamental modules \( V_i \) are the same (Theorem 4.2).

The idea of proof follows largely that of Chari [Ch02]. The RTT generators will replace the role of the Weyl group. The quantum analogues of \( E_{1,M+N} \otimes t^a, E_{M+N,1} \otimes t^a \) will be candidates for the longest element \( w_0 \) in the Weyl group. For the reduction argument, we will use representation theory of the \( q \)-Yangian \( Y_q(\mathfrak{gl}(1,1)) \) instead of \( U_q(\hat{\mathfrak{sl}}_2) \). Here \( q \)-Yangian \( Y_q(\mathfrak{g}) \) is a sub-Hopf-superalgebra of \( U_q(\hat{\mathfrak{g}}) \) generated by half of the RTT generators. It can be viewed as the upper Borel subalgebra.

Our second main result (Theorem 5.2) is on representation theory of \( Y_q(\mathfrak{gl}(1,1)) \).

1. There is a classification of finite-dimensional simple modules, up to tensor product by one-dimensional modules, in terms of highest \( \ell \)-weights parametrized by the set \( R \) of such rational functions \( f(z) \) that \( f(0) = 1 \) (hence regular at \( z = 0 \)). Let \( V(f) \) be the simple module of highest \( \ell \)-weight \( f \).

2. For \( f_1, \ldots, f_k \in R \), the tensor product \( \bigotimes_{j=1}^k V(f_j) \) is of highest \( \ell \)-weight (resp. of lowest \( \ell \)-weight) if and only if: for all \( 1 \leq i < j \leq k \) (resp. for all \( 1 \leq j < i \leq k \) ) the set of poles of \( f_i \) does not intersect with the set of zeros of \( f_j \).

3. The tensor product in (2) is simple if and only if it is of highest and lowest \( \ell \)-weight.

We can see (2) as a stronger improvement of (AK) for the \( q \)-Yangian \( Y_q(\mathfrak{gl}(1,1)) \) as the necessary condition of cyclicity is also described. However, (2) cannot be generalized to higher rank quantum affine superalgebras or \( q \)-Yangians. Indeed, (2) already fails if we replace \( Y_q(\mathfrak{gl}(1,1)) \) by the quantum affine algebra \( U_q(\mathfrak{sl}_2) \), as seen in [CP91, MY14] where the condition for a tensor product of simple \( U_q(\mathfrak{sl}_2) \)-modules to be cyclic was more sophisticated. Also, in the non-graded case due to the Weyl group action (more precisely the element \( w_0 \)) a tensor product of simple modules is of highest \( \ell \)-weight if and only if it is of lowest \( \ell \)-weight. Hence (3) is really a special feature in the super case.

Except Chari’s Lemma which requires coproduct formulas of Drinfeld generators, the proof of Theorem 5.2 is quite elementary and explicit. Eventually we arrive at a classical linear algebra problem on determining linear independence of some polynomials of a particular form (Lemma 5.4).

Surprisingly, reductions from \( U_q(\hat{\mathfrak{g}}) \)-modules to \( Y_q(\mathfrak{gl}(1,1)) \)-modules are already enough to prove Theorem 4.2. We believe that more general cyclicity results can be deduced in this way, although this needs extra efforts. See the end of §6.

In an upcoming paper [Zh], we shall use Theorem 4.2 to construct inductive systems of Kirillov-Reshetikhin modules and realize their limits as asymptotic modules over the \( q \)-Yangian \( Y_q(\mathfrak{g}) \), hence extending Hernandez-Jimbo’s asymptotic construction to the super case. Indeed, the \( Y_q(\mathfrak{gl}(1,1)) \)-modules \( V(1 - z) \) and \( V(\frac{1}{1-z}) \) can be viewed as prototypes of asymptotic modules.
At last, we would like to point out that nearly all the results in the present paper have direct analogues when replacing the quantum affine superalgebra $U_q(\hat{\mathfrak{g}})$ (or the $q$-Yangian) by the Yangian $Y(\mathfrak{g})$, a deformation of the universal enveloping algebra of the current Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}[t]$. The proofs of these results are essentially the same, as $Y(\mathfrak{g})$ admits a similar RTT realization [Zh96]. In [Zh95, Theorem 5], a similar criteria for a tensor product of finite-dimensional simple $Y(\mathfrak{gl}(1, 1))$-modules to be simple was given by Rui-Bin Zhang with a quite different approach from ours. Cyclicity of tensor products and Drinfeld realization for the Yangian were not considered there in full generality.

1.4. Outline. The paper is organized as follows. §2 collects some basic facts about highest weight representations of the finite type quantum superalgebra $U_q(\mathfrak{g})$. In §3 we study in detail the RTT realization of the quantum affine superalgebra $U_q(\mathfrak{g})$, review the Ding-Frenkel homomorphism between Drinfeld realization and RTT realization, and give an estimation for coproduct of Drinfeld generators (Proposition 3.13), which is needed to prove Chari’s lemma in the super case (Lemma 4.5). §4 studies highest $\ell$-weight representations for $U_q(\mathfrak{g})$ and states the first main result (Theorem 4.2) on tensor products of Kirillov-Reshetikhin modules. §5 discusses finite-dimensional representation theory for the $q$-Yangian $Y_q(\mathfrak{gl}(1, 1))$, which is used in §6 to conclude the proof of Theorem 4.2. In §A we give the complete proof of Proposition 3.13 on coproduct of Drinfeld generators.

Acknowledgments. The author is grateful to his supervisor David Hernandez for numerous discussions, and to Paul Zinn-Justin from whom he learned the notion of a Yang-Baxter algebra in his course “Intégrabilité Quantique” at Université Paris 6.

Part of the present work was done while the author was visiting Centre de Recherches Mathématiques in Montréal and was participating in the workshops “Combinatorial Representation Theory” in Montréal and “Yangians and Quantum Loop Algebras” in Austin. He is grateful to the organizers for hospitality and to Vyjayanthi Chari, Sachin Gautam and Valerio Toledano Laredo for stimulating discussions.

2. Preliminaries

In this section, we introduce the basic notations concerning the quantum superalgebra $U_q(\mathfrak{gl}(M,N))$ and its representations. Following Benkart-Kang- Kashiwara we review the character formula for fundamental representations.

2.1. Conventions. Throughout this paper, $q$ is fixed to be a non-zero complex number and not a root of unity. All the vector superspaces and superalgebras are defined over the base field $\mathbb{C}$. Recall that a vector superspace $V$ is a vector space $V$ with a $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$. We write $|x| = i$ for $i \in \mathbb{Z}_2$ and $x \in V_i$. It will happen usually that $V$ carries another grading by an abelian group $P$. In this case, we write $V = \oplus_{\alpha \in P} (V)_{\alpha}$ (we keep the parenthesis most of the time) and $|x|_P = \alpha$ for $\alpha \in P$ and $x \in (V)_{\alpha}$.

Fix $M, N \in \mathbb{Z}_{\geq 0}$. Set $I := \{1, 2, \cdots, M+N\}$. Define two maps as follows:

$| \cdot | : I \rightarrow \mathbb{Z}_2, i \mapsto |i| = \begin{cases} 0 & (i \leq M), \\ T & (i > M) \end{cases}$

$d : I \rightarrow \mathbb{Z}, i \mapsto d_i := \begin{cases} 1 & (i \leq M), \\ -1 & (i > M). \end{cases}$
Set $q_i := q^{d_i}$. Set $P := \oplus_{i \in I} \mathbb{Z} \epsilon_i$. Let $(,) : P \times P \rightarrow \mathbb{Z}$ be the bilinear form defined by $(\epsilon_i, \epsilon_j) = \delta_{ij} d_i$. Let $| , | : P \rightarrow \mathbb{Z}_2$ be the morphism of abelian groups such that $| \epsilon_i | = | i |$.

In the following, only three cases of $|x| \in \mathbb{Z}_2$ are admitted: $x \in I; x \in P, x \in \mathbb{Z}_2$-homogeneous vector of a vector superspace.

Unless otherwise stated, $g$ will always be the Lie superalgebra $\mathfrak{gl}(M, N)$, which is, the vector space $\oplus_{i,j \in I} \mathbb{C} E_{ij}$ with $\mathbb{Z}_2$-grading and Lie bracket:

$$[E_{ij}, |i| + |j|, [E_{ij}, E_{kk}] = \delta_{jk} E_{ii} - (-1)^{(|i|+|j|)(|k|+|l|)) \delta_{il} E_{kj}.$$  

Here, we view $\mathbb{Z}_2$ as a ring and $(-1)^r : \mathbb{Z}_2 \rightarrow \{1, -1\}$ as the sign map. Let $h = \oplus_{i \in I} \mathbb{C} E_{ii}$. Then $h$ is a Cartan subalgebra with respect to which $g$ has a root space decomposition:

$$g = h \oplus \bigoplus_{i,j \in I, i \neq j} (g)_{\epsilon_i - \epsilon_j}, \quad (g)_{\epsilon_i - \epsilon_j} = \{x \in g \mid [E_{kk}, x] = (\delta_{ik} - \delta_{jk})x \text{ for } k \in I\} = \mathbb{C} E_{ij}.$$  

Set $I_0 := I \setminus \{M + N\}$. For $i \in I_0$, set $\alpha_i := \epsilon_i - \epsilon_{i+1} \in P$. Introduce the root lattice $Q := \oplus_{i \in I_0} \mathbb{Z} \alpha_i \subset P$. Define $Q_{\geq 0} := \oplus_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i$.

2.2. The quantum superalgebra $U_q(g)$. This is the superalgebra defined by generators $e_i^+, t_j^+ (i \in I_0, j \in I)$ with $\mathbb{Z}_2$-degrees $|e_i^+| = |\alpha_i|$ and $|t_j^+| = 0$ and with relations

$$t_i t_j^{-1} t_j = 1, \quad t_i t_j = t_j t_i \quad (i, j \in I),$$

$$t_i e_{\alpha_i}^+ t_i^{-1} = q^{d_i (\epsilon_i, \epsilon_i)} e_{\alpha_i}^+ \quad (i \in I, j \in I_0),$$

$$[e_{\alpha_i}^+, e_{\alpha_j}^-] = \delta_{ij} \frac{t_i^{d_{ij}} t_{i+1}^{d_{ij}} - t_{i+1}^{d_{ij}} t_i^{d_{ij}}}{q_i - q_i^{-1}} \quad (i, j \in I_0),$$

together with Serre relations which we do not repeat (see [Zh13, §2.2] for example). $U_q(g)$ has a Hopf superalgebra structure with coproduct: for

$$\Delta(e_{\alpha_i}^+) = 1 \otimes e_{\alpha_i}^+ + e_{\alpha_i}^+ \otimes t_i^{d_{ij}} t_{i+1}^{d_{ij}}, \quad \Delta(e_{\alpha_i}^-) = t_i^{d_{ij}} t_{i+1}^{d_{ij}} \otimes e_{\alpha_i}^- + e_{\alpha_i}^- \otimes 1, \quad \Delta(t_j) = t_j \otimes t_j.$$  

There exists a $Q$-grading on $U_q(g)$ respecting the Hopf superalgebra structure:

$$|t_j|_Q = 0, \quad |t_i^+|_Q = \pm \alpha_i \quad (i \in I_0, j \in I).$$

In general, for $\alpha \in Q \subset P$, we have

$$(U_q(g))_\alpha = \{x \in U_q(g) \mid t_i x t_i^{-1} = q^{d_i (\epsilon_i, \alpha)} x \text{ for } i \in I\}.$$  

This $Q$-grading is compatible with the $\mathbb{Z}_2$-grading: $(U_q(g))_\alpha \subset U_q(g)_{|\alpha|}$ for $\alpha \in Q$.

2.3. Highest weight representations. Let $\lambda \in P$. Up to isomorphism, there exists a unique simple $U_q(g)$-module, denoted by $L(\lambda)$, which is generated by a vector $v_\lambda$ satisfying:

$$|v_\lambda| = |\lambda|, \quad e_{\alpha_i}^+ v_\lambda = 0, \quad t_j v_\lambda = q^{d_{ij}(\epsilon_j, \lambda)} v_\lambda \quad (i \in I_0, j \in I).$$

The action of the $t_j$ endows $L(\lambda)$ with the following $P$-grading:

$$(L(\lambda))_\alpha := \{x \in L(\lambda) \mid t_j x = q^{d_{ij}(\epsilon_j, \lambda)} x \text{ for } j \in I\}.$$  

Using the triangular decomposition for $U_q(g)$, one can show the following: $(L(\lambda))_\lambda = \mathbb{C} v_\lambda$; the $P$-grading on $L(\lambda)$ is compatible with the $\mathbb{Z}_2$-grading; for $\alpha \in P$, $(L(\lambda))_\alpha \neq 0$ only if $\lambda - \alpha \in Q_{\geq 0}$.  

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It was shown \[\text{Zr93}\] that for \(\lambda \in \mathbb{P}\), \(L(\lambda)\) is finite-dimensional if and only if \(d_i(\lambda, \alpha_i) > 0\) for all \(i \in I_0 \setminus \{M\}\).

**Example 1.** (Natural representation.) Let \(V = \oplus_{i \in I} C v_i\) be the vector superspace with \(\mathbb{Z}_2\)-grading \(|v_i| = |i|\). On \(V\) there is a natural representation \(\rho_0\) of \(U_q(\mathfrak{g})\) defined by:

\[
\rho_0(e_i^+) = E_{i,i+1}, \quad \rho_0(e_i^-) = E_{i+1,i}, \quad \rho_0(t_j) = \sum_{k \in I} q^{d_i(e_j, \epsilon_k)} E_{kk} \quad (i \in I_0, j \in I).
\]

Here the \(E_{ij} \in \text{End}(V)\) for \(i, j \in I\) are defined by \(E_{ij}v_k = \delta_{jk}v_i\). Clearly, \(V = L(\epsilon_1)\) as a \(U_q(\mathfrak{g})\)-module with \(v_1\) a highest weight vector, and \((V)_{\epsilon_i} = C v_i\) for \(i \in I\).

**Example 2.** Consider the tensor product \(V^\otimes 2\) as a \(U_q(\mathfrak{g})\)-module. Define subspaces

\[
(2.1) \quad V^+ := \bigoplus_{1 \leq i < j \leq M+N} C(q v_i \otimes v_j + (-1)^{|i||j|} v_j \otimes v_i) \oplus \bigoplus_{k=1}^M C(v_k \otimes v_k),
\]

\[
(2.2) \quad V^- := \bigoplus_{1 \leq i < j \leq M+N} C(q^{-1} v_i \otimes v_j - (-1)^{|i||j|} v_j \otimes v_i) \oplus \bigoplus_{k=1}^N C(v_{M+k} \otimes v_{M+k}).
\]

Then \(V^\otimes 2 = V^+ \oplus V^-\) is a decomposition of \(U_q(\mathfrak{g})\)-modules as follows:

\[
L(\epsilon_1)^\otimes 2 = L(2\epsilon_1) \oplus L(\epsilon_1 + \epsilon_2).
\]

In the following, the three vector superspaces \(V, V^+, V^-\) will be used frequently.

2.3.1. **Characters.** Let \(V\) be a \(U_q(\mathfrak{g})\)-module \(\mathbb{P}\)-graded via the action of the \(t_i\):

\[
(V)_\alpha = \{x \in V \mid t_ix = q^{d_i(\alpha, \epsilon_i)} x \text{ for } i \in I\}.
\]

Suppose that all the weight spaces \((V)_\alpha\) are finite-dimensional. Introduce **characters**

\[
\chi(V) := \sum_{\alpha \in \mathbb{P}} \dim(V)_\alpha[\alpha] \in \mathbb{Z}^\mathbb{P}.
\]

Here \(\mathbb{Z}^\mathbb{P}\) is the abelian group of functions \(\mathbb{P} \rightarrow \mathbb{Z}\) and \([\alpha] = \delta_{\alpha, \cdot}.. \) By definition, it is clear that \((U_q(\mathfrak{g}))_{\alpha}(V)_{\beta} \subseteq (V)_{\alpha + \beta}\) for \(\alpha, \beta \in \mathbb{P}\).

2.3.2. **Fundamental weights.** For \(r \in I_0\), let us define the \(r\)-th fundamental weight

\[
\varpi_r := \begin{cases} 
\sum_{i=1}^r \epsilon_i & (r \leq M), \\
-\sum_{i=r+1}^{M+N} \epsilon_i & (r > M).
\end{cases}
\]

Clearly, \(d_s(\varpi_r, \alpha_s) = \delta_{rs}\) for \(r, s \in I_0\).

For \(1 \leq r \leq M\) let \(\mathcal{B}_r\) be the set of functions \(f : \{1, 2, \ldots, r\} \rightarrow I\) such that: \(f(i) \leq f(i')\) for \(i \leq i'\); if \(1 \leq i < r\) and \(f(i) \leq M\), then \(f(i) < f(i + 1)\).

**Theorem 2.1.** \[\text{BKK00}\] For \(1 \leq r \leq M\) we have

\[
\chi(L(\varpi_r)) = \sum_{f \in \mathcal{B}_r} \left[\sum_{i=1}^r \epsilon_{f(i)}\right] \in \mathbb{Z}[\mathbb{P}].
\]

\(\square\)
Proof. This is obvious in view of the following:

It is enough to ensure that

End

requires lengthy calculations as done in \cite{DF93, Zy97}. For completeness and for later reference, we prove all of them in a uniform way, except the Ding-Frenkel homomorphism relating Drinfeld realization and RTT realization, which requires lengthy calculations as done in \cite{DF93, Zy97}.

3. Quantum affine superalgebra and q-Yangians

In this section, we introduce our central objects of study: the quantum affine superalgebra \( U_q(\mathfrak{g}) \) and the q-Yangian \( Y_q(\mathfrak{g}) \) within the framework of RTT. Most of the results in this section have appeared in the literature separately (for example \cite{FM02, MTZ04, Ts12}). For completeness and for later reference, we prove all of them in a uniform way, except the Yang-Baxter algebra with coproduct and counit

For an \( R \in \text{End}(V \otimes V) \) is an R-matrix if: \( R \) is invertible and of \( \mathbb{Z}_2 \)-degree \( \mathfrak{g} \); \( R \) satisfies the Yang-Baxter equation

(3.5) \( R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \in (\text{End}V)^{\otimes 3} \).

Here \( R_{12} = R \otimes \text{Id}_V, R_{23} = \text{Id}_V \otimes R \) and

\[
R_{13} = (\text{Id}_V \otimes c_{V,V}) R_{12} (\text{Id}_V \otimes c_{V,V}) = (c_{V,V} \otimes \text{Id}_V) R_{23} (c_{V,V} \otimes \text{Id}_V)
\]

with \( c_{V,V} : V \otimes V \mapsto V \otimes V, v_i \otimes v_j \mapsto (-1)^{|i||j|} v_j \otimes v_i \) the braiding.

Definition 3.1. For an R-matrix \( R \in \text{End}(V \otimes V) \), the Yang-Baxter algebra \( \mathfrak{YB}(R) \) is the superalgebra with

(YB1) generators \( l_{ij} \) for \( i, j \in I \) of \( \mathbb{Z}_2 \)-degrees \( |l_{ij}| = |i| + |j| \);

(YB2) relations \( R_{23} L_{12} L_{13} = L_{13} L_{12} R_{23} \in \mathfrak{YB}(R) \otimes \text{End}V \otimes \text{End}V \).

Here \( L = \sum_{i,j \in I} l_{ij} \otimes E_{ij} \in \mathfrak{YB}(R) \otimes \text{End}V \).

Already from the definition of R-matrix, one reads a natural representation of \( \mathfrak{YB}(R) \).

Lemma 3.2. There is a representation \( (\rho, V) \) of the superalgebra \( \mathfrak{YB}(R) \) on \( V \) defined by \( (\rho \otimes \text{Id}_{\text{End}V})(L) = R^{-1} \).

Proof. This is obvious in view of the following: \( S \mapsto S_{23} \) is a morphism of superalgebras \( (\text{End}V)^{\otimes 2} \mapsto (\text{End}V)^{\otimes 3} \); \( R^{-1} \) also satisfies the Yang-Baxter equation. \( \square \)

3.1. Super bialgebra structure. The Yang-Baxter algebra \( \mathfrak{YB}(R) \) can be made into a superbialgebra with coproduct and counit

(3.6) \( \Delta(l_{ij}) = \sum_{k \in I} (-1)^{|i||k|+|k||j|} l_{ik} \otimes l_{kj}, \quad \varepsilon(l_{ij}) = \delta_{ij} \).

To prove that \( \Delta \) is a well-defined superalgebra morphism, introduce

\[
T := \sum_{i,j,k \in I} (-1)^{|i||k|+|j||k|} l_{ik} \otimes l_{kj} \otimes E_{ij} = L_{13} L_{23} \in \mathfrak{YB}(R) \otimes \mathfrak{YB}(R) \otimes \text{End}V.
\]

It is enough to ensure that

\[
R_{34} T_{123} T_{124} = T_{124} T_{123} R_{34} \in \mathfrak{YB}(R)^{\otimes 2} \otimes (\text{End}V)^{\otimes 2}.
\]
Observe first of all that $|L| = \overline{3}$ implies
\[ L_{23} L_{14} = L_{14} L_{23}, \quad L_{13} L_{24} = L_{24} L_{13}. \]

It follows that
\[
R_{34} T_{123} T_{124} = R_{34} L_{13}(L_{23} L_{14}) L_{24} = (R_{34} L_{13} L_{14}) L_{23} L_{24}
\]
\[
\overset{\text{YB}}{=} L_{14} L_{13} (R_{34} L_{23} L_{24}) = L_{14} (L_{13} L_{24}) L_{23} R_{34}
\]
\[
= L_{14} L_{24} L_{13} L_{23} R_{34} = T_{124} T_{123} R_{34}.
\]

The co-associativity of $\Delta$ is clear from Equation (3.6).

Remark that in the above proof we do not need the Yang-Baxter equation for $R$.

3.1.2. Dependence on $R$-matrices. Given an $R$-matrix $R$, we will construct two new $R$-matrices $R', R''$ whose associated Yang-Baxter algebras are isomorphic.

The operator $R'$ is easy to define:
\[
R' := c_{\mathbf{V}, \mathbf{V}} R^{-1} c_{\mathbf{V}, \mathbf{V}} \in \text{End}(\mathbf{V} \otimes \mathbf{V}).
\]

Before giving the second operator $R''$, let us introduce a super version of transpose of matrices. This is the linear map $\tau : \text{End}\mathbf{V} \rightarrow \text{End}\mathbf{V}$ given by
\[
(3.7) \quad \tau(E_{ij}) := \epsilon_{ij} E_{ji}, \quad \epsilon_{ij} := (-1)^{|i||j|} = \begin{cases} 1 & (i \leq j), \\ (-1)^{|i|+|j|} & (i > j). \end{cases}
\]

Now $R''$ is given by
\[
R'' := (\tau \otimes \tau)(R^{-1}) \in \text{End}(\mathbf{V} \otimes \mathbf{V}).
\]

**Lemma 3.3.** $\tau : \text{End}\mathbf{V} \rightarrow \text{End}\mathbf{V}$ is an anti-automorphism of superalgebra.

**Proof.** We explain that $\tau$ is indeed a duality map. Let $\mathbf{V}^*$ be the dual of $\mathbf{V}$, with $(v^*_i : i \in I)$ the dual basis with respect to $(v_i : i \in I)$. Then $|v^*_i| = |v_i| = |i|$. Let $\epsilon_{ij} \in \text{End}\mathbf{V}^*$ be $v^*_i \mapsto \delta_{jk} v^*_j$. Then the linear isomorphism of vector superspaces $\mathbf{V} \rightarrow \mathbf{V}^*$, $v_i \mapsto v^*_i$ induces an isomorphism of superalgebras $a : \text{End}\mathbf{V} \rightarrow \text{End}\mathbf{V}^*$, $E_{ij} \mapsto \epsilon_{ij}$.

On the other hand, let $f : \mathbf{V} \rightarrow \mathbf{V}$ be a $\mathbb{Z}_2$-homogeneous linear map. Define its dual:
\[
f^* : \mathbf{V}^* \rightarrow \mathbf{V}^*, \quad l \mapsto (x \mapsto (-1)^{|l||f|} f(x))
\]
where $l \in \mathbf{V}^*$ is $\mathbb{Z}_2$-homogeneous and $x \in \mathbf{V}$. In this way, we construct an anti-isomorphism of superalgebras $b : \text{End}\mathbf{V} \rightarrow \text{End}\mathbf{V}^*$, $f \mapsto f^*$. It is straightforward to check that $\tau = a^{-1} \circ b$. Hence $\tau$ is an anti-automorphism of superalgebra.

Now we can state the main result of this paragraph.

**Proposition 3.4.** Let $R \in \text{End}(\mathbf{V} \otimes \mathbf{V})$ be an $R$-matrix. Then $R', R''$ are also $R$-matrices. Moreover: the assignment $l'_{ij} \mapsto l_{ij}$ extends to an isomorphism $\mathcal{YB}(R') \rightarrow \mathcal{YB}(R)$ of super bialgebras; the assignment $l''_{ij} \mapsto \epsilon_{ji} l_{ji}$ extends to an isomorphism $\Psi : \mathcal{YB}(R'') \rightarrow \mathcal{YB}(R)^\text{cop}$ of super bialgebras.
Proof. We only prove the part for \( R'' \). To show that \( R'' \) is an \( R \)-matrix, note that \( \tau^{\otimes 2}, \tau^{\otimes 3} \) are anti-automorphisms of superalgebras, from which the Yang-Baxter equation for \( R'' \) follows. Next, in order to show that the superalgebra morphism \( \Psi \) is well defined, we only need to ensure that

\[
R''_{23} T_{12} T_{13} = T_{13} T_{12} R''_{23} \in \mathcal{YB}(R) \otimes \text{End} V \otimes \text{End} V
\]

where \( T = (\text{Id}_{\mathcal{YB}(R)} \otimes \tau)(L) \in \mathcal{YB}(R) \otimes \text{End} V \).

By applying \( \text{Id}_{\mathcal{YB}(R)} \otimes \tau \otimes \tau \) to the Yang-Baxter equation \( R_{23} L_{12} L_{13} = L_{13} L_{12} R_{23} \) we get

\[
T_{12} T_{13} ((\tau \otimes \tau)(R))_{23} = ((\tau \otimes \tau)(R))_{23} T_{13} T_{12},
\]

from which comes the desired equation. The rest is clear. \( \square \)

Remark that the two operations \( R \mapsto R' \), \( R \mapsto R'' \) are involutions. We will sometimes be in the situation \( R' = R'' \). Proposition 3.4 then gives us an isomorphism \( \Psi : \mathcal{YB}(R) \rightarrow \mathcal{YB}(R)^{\text{op}} \) of super bialgebras.

3.2. The quantum affine superalgebra \( U_q(\mathfrak{g}) \). Recall the vector superspaces \( V^+, V^- \) in Example 2. Define the Perk-Schultz matrix [PSS81]

\[
R(z, w) = c_{V, V}((zq - wq^{-1})P^+ + (wq - zq^{-1})P^-) \in \text{End}(V \otimes V)[z, w]
\]

where \( P^+, P^- \) are projections with respect to the decomposition \( V^{\otimes 2} = V^+ \oplus V^- \).

Definition 3.5. The quantum affine superalgebra \( U_q(\mathfrak{g}) \) is the superalgebra defined by

1. \( \mathbb{Z}_2 \)-grading \( |s^{(n)}_{ij}| = |t^{(n)}_{ij}| = |i| + |j| \);
2. \( R_{23}(z, w)T_{12}(z)T_{13}(w) = T_{13}(w)T_{12}(z)R_{23}(z, w) \);
3. \( R_{23}(z, w)S_{12}(z)S_{13}(w) = S_{13}(w)S_{12}(z)R_{23}(z, w) \);
4. \( t^{(0)}_{ij} = s^{(0)}_{ji} = 0 \) for \( 1 \leq i < j \leq M + N \);
5. \( t^{(0)}_{ii} s^{(0)}_{ii} = 1 = s^{(0)}_{ii} t^{(0)}_{ii} \) for \( i \in I \).

Here \( T(z) = \sum_{i,j \in I} t_{ij}(z) \otimes E_{ij} \in (U_q(\mathfrak{g}) \otimes \text{End} V)[[z^{-1}]] \) and \( t_{ij}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} t^{(n)}_{ij} z^{-n} \in U_q(\mathfrak{g})[[z^{-1}]] \) (similar convention for \( S(z) \) with the \( z^{-n} \) replaced by the \( z^n \)).

The \( q \)-Yangian \( Y_q(\mathfrak{g}) \) is the subalgebra of \( U_q(\mathfrak{g}) \) generated by the \( (s^{(0)}_{ii})^{-1}, s^{(n)}_{ij} \) for \( i, j \in I \) and \( n \in \mathbb{Z}_{\geq 0} \).

As in [3.11] \( U_q(\mathfrak{g}) \) is endowed with a super bialgebra structure:

\[
\Delta(s^{(n)}_{ij}) = \sum_{a=0}^{n} \sum_{k \in I} (-1)^{(|i|+|k|)(|k|+|j|)} s^{(a)}_{ik} \otimes s^{(n-a)}_{kj},
\]

\[
\Delta(t^{(n)}_{ij}) = \sum_{a=0}^{n} \sum_{k \in I} (-1)^{(|i|+|k|)(|k|+|j|)} t^{(a)}_{ik} \otimes t^{(n-a)}_{kj}.
\]
Indeed, the antipode $S : U_q(\hat{g}) \rightarrow U_q(\hat{g})$ exists:

\begin{align}
(3.16) \quad \sum_{i,j \in I} S(s_{ij}(z)) \otimes E_{ij} &= (\sum_{i,j \in I} s_{ij}(z) \otimes E_{ij})^{-1} \in (U_q(\hat{g}) \otimes \text{End}V)[z], \\
(3.17) \quad \sum_{i,j \in I} S(t_{ij}(z)) \otimes E_{ij} &= (\sum_{i,j \in I} t_{ij}(z) \otimes E_{ij})^{-1} \in (U_q(\hat{g}) \otimes \text{End}V)[z^{-1}].
\end{align}

Here the RHS of the above formulas are well defined thanks to Relation (3.12). It follows that $U_q(\hat{g})$ is a Hopf superalgebra, with $Y_q(\hat{g})$ a sub-Hopf-superalgebra.

3.3. Structure of the Quantum affine superalgebra. We study the weight grading, the $\mathbb{Z}$-grading and evaluation morphisms for $U_q(\hat{g})$. Moreover, we explain that $U_q(\hat{g})$ is indeed a quantum double associated with a Hopf pairing.

3.3.1. The Perk-Schultz R-matrix. The exact form of $R(z, w) \in (\text{End}V \otimes \text{End}V)[z, w]$ is:

\begin{equation}
(3.18) \quad R(z, w) = \sum_{i \in I} (zq_i - wq_i^{-1}) E_{ii} \otimes E_{ii} + (z - w) \sum_{i \neq j} E_{ii} \otimes E_{jj} + z \sum_{i < j} (q_i - q_i^{-1}) E_{ji} \otimes E_{ij} + w \sum_{i < j} (q_j - q_j^{-1}) E_{ij} \otimes E_{ji}.
\end{equation}

Let us gather the following fundamental properties of the Perk-Schultz R-matrix $R(z, w)$.

**Proposition 3.6.** The Perk-Schultz R-matrix $R(z, w)$ verifies the following relations.

- (PS1) Yang-Baxter: $R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2)$.
- (PS2) Unitarity relation: $R(z, w)c_{V,V}R(w, z)c_{V,V} = (zq - wq^{-1})(wq - zq^{-1}) \text{Id}_{V \otimes V}$.
- (PS3) Ice rule: $R_{ab,cd}(z, w) \neq 0 \iff \epsilon_a + \epsilon_b = \epsilon_c + \epsilon_d \in \mathbb{P}$ for $a, b, c, d \in I$.
- (PS4) $c_{V,V}R(z, w)c_{V,V} = (\tau \otimes \tau)(R(z, w)) \in (\text{End}V) \otimes 2[z, w]$.
- (PS5) Let $R = R(1, 0), R' = c_{V,V}R^{-1}c_{V,V}$. Then $R(z, w) = zR - wR'$.
- (PS6) Hecke relation: $R' = R - (q - q^{-1})c_{V,V}$.

Here $R_{ab,cd}(z, w) \in \mathbb{C}[z, w]$ for $a, b, c, d \in I$ are the matrix elements defined by

$$
R(z, w)(v_c \otimes v_d) = \sum_{a \in I} R_{ab,cd}(z, w)(v_a \otimes v_b) \in V^\otimes 2[z, w].
$$

**Proof.** (PS1)-(PS3) have been observed in [PS5]. (PS4) comes from Formula (3.18). (PS5) follows from (PS2) and Formula (3.18). For (PS6), note that

$$
c_{V,V}R = qP^+ - q^{-1}P^-
$$

gives a diagonal decomposition for the matrix $c_{V,V}R \in \text{End}(V^\otimes 2)$. Henceforth

$$(c_{V,V}R)^2 = (q - q^{-1})c_{V,V}R + \text{Id}_{V \otimes V}, \quad (c_{V,V}R)^{-1} = c_{V,V}R - (q - q^{-1})\text{Id}_{V \otimes V},$$

from which the Hecke relation follows. \hfill \Box

**Remark 3.7.** (1) Later we shall care about the parameter $q$. In this case, write $R(z, w) = R_q(z, w), R = R_q, R' = R'_q$. Define

$$
\mathcal{R}(z, w) = R_q(z, w) := \frac{R(z, w)}{zq - wq^{-1}}.
$$

Then the inverses of these matrices $R_q, R'_q, R_q(z, w)$ have the following simple expression:
(PS7) \((R_q)^{-1} = R_{q^{-1}}, \quad (R'_q)^{-1} = R'_{q^{-1}}, \quad R_q(z, w)^{-1} = R_{q^{-1}}(z, w)\).

(2) Remark that in Definition 3.3.5 of \(U_q(\mathfrak{g})\), one can replace \(R(z, w)\) by \(R_q(z, w)\) everywhere. As for Relation (3.11), \(R_q(z, w)\) should be viewed as a formal power series in \(z\).

(3) From Proposition 3.3.4 and (PS4) follows an isomorphism of Hopf superalgebras

\[
(3.19) \quad \Psi : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})^{\text{cop}}, \quad s_{ij}^{(n)} \mapsto \varepsilon_j t_{ij}^{(n)}, \quad t_{ij}^{(n)} \mapsto \varepsilon_j s_{ij}^{(n)}.
\]

3.3.2. Z-grading. There exists a \(\mathbb{Z}\)-grading on \(U_q(\mathfrak{g})\):

\[
(3.20) \quad |s_{ij}^{(n)}|_z = n, \quad |t_{ij}^{(n)}|_z = -n.
\]

This \(\mathbb{Z}\)-grading is compatible with the Hopf superalgebra structure. In particular, we get an one-parameter family \(\Phi_a : a \in \mathbb{C}^\times\) of Hopf superalgebra automorphisms:

\[
(3.21) \quad \Phi_a : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}), \quad s_{ij}^{(n)} \mapsto a^n s_{ij}^{(n)}, \quad t_{ij}^{(n)} \mapsto a^{-n} t_{ij}^{(n)}.
\]

The main reason behind this \(\mathbb{Z}\)-grading is that \(R(az, aw) = R(z, w)\) for all \(a \in \mathbb{C}^\times\).

3.3.3. Automorphisms given by power series. Let \(f(z) \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]\) and \(g(z) \in 1 + z\mathbb{C}[[z]]\). There exists an automorphism of superalgebra:

\[
(3.22) \quad \phi_{(f(z), g(z))} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}), \quad t_{ij}(z) \mapsto f(z) t_{ij}(z), \quad s_{ij}(z) \mapsto g(z) s_{ij}(z).
\]

These automorphisms behave well under coproduct in the following way:

\[
(\phi_{(f_1(z), g_1(z))} \otimes \phi_{(f_2(z), g_2(z))}) \circ \Delta = \Delta \circ \phi_{(f_1(z) f_2(z), g_1(z) g_2(z))} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})^{\otimes 2}
\]

for \(f_1(z), f_2(z) \in 1 + z^{-1}\mathbb{C}[[z^{-1}]],\ g_1(z), g_2(z) \in 1 + z\mathbb{C}[[z]]\).

Note that \(\phi_{(f(z), g(z))}\) restricts to an automorphism of \(q\)-Yangian: \(\phi_q : Y_q(\mathfrak{g}) \rightarrow Y_q(\mathfrak{g})\).

3.3.4. Evaluation morphisms. Recall the \(R\)-matrix \(R \in (\text{End} V)^{\otimes 2}\) in Proposition 3.6. As in Definition 3.5, let us define \(U_q(\mathfrak{g})\) to be the superalgebra generated by \(s_{ij}, t_{ji}\) for \(1 \leq i \leq j \leq M + N\), with \(\mathbb{Z}_2\)-degrees

\[
|s_{ij}| = |t_{ji}| = |i| + |j|
\]

and with RTT relations (FRT90)

\[
R_{23} T_{12} T_{13} = T_{13} T_{12} R_{23}, \quad R_{23} S_{12} S_{13} = S_{13} S_{12} R_{23},
\]

\[
R_{23} T_{12} S_{13} = S_{13} T_{12} R_{23}, \quad s_{ii} t_{ii} = 1 = t_{ii} s_{ii}.
\]

Here, as usual, \(T = \sum_{i \leq j} t_{ij} \otimes E_{ji}\), \(S = \sum_{i \leq j} s_{ij} \otimes E_{ij} \in U_q(\mathfrak{g}) \otimes \text{End} V\). \(U_q(\mathfrak{g})\) is endowed with a Hopf superalgebra structure with similar coproduct as in formulas (3.14)-(3.15).

**Proposition 3.8.** (1) The assignment \(s_{ij} \mapsto s_{ij}^{(0)}, \ t_{ji} \mapsto t_{ji}^{(0)}\) extends uniquely to a Hopf superalgebra morphism \(\iota : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})\).

(2) The assignment \(s_{ij}(z) \mapsto s_{ij} - z t_{ij}, \ t_{ij}(z) \mapsto t_{ij} - z^{-1} s_{ij}\) extends uniquely to a superalgebra morphism \(\text{ev} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})\).

We understand that \(s_{ji} = t_{ij} = 0\) in the superalgebra \(U_q(\mathfrak{g})\) for \(1 \leq i < j \leq M + N\). The morphism \(\text{ev}\) is called an evaluation morphism. It is clear that \(\text{ev} \circ \iota = \text{Id}_{U_q(\mathfrak{g})}\).
Proof. (1) To show that \( \ell \) is well defined, we only need to check that in \( U_q(\mathfrak{g}) \otimes \text{EndV} \),
\[
R_{23} T_{12}(\infty) T_{13}(\infty) = T_{13}(\infty) T_{12}(\infty) R_{23}, \quad R_{23} S_{12}(0) T_{13}(\infty) = T_{13}(\infty) S_{12}(0) R_{23}, \quad R_{23} S_{12}(0) S_{13}(0) = S_{13}(0) S_{12}(0) R_{23}.
\]
By comparing the coefficients of \( z \) in both sides of Relations (3.9), (3.11), (3.10), we get the above three equations respectively.

(2) As before, we only need to check that \( \text{ev} \) respects Relations (3.9)-(3.11) (the other relations are obvious). We do this for Relation (3.11), the other two being analogous. In other words, we need to show that in the superalgebra \( (U_q(\mathfrak{g}) \otimes (\text{EndV})^{\otimes 2})[z, z^{-1}, w] \),
\[
(z R_{23} - w R'_{23})(T_{12} - z^{-1} S_{12})(S_{13} - w T_{13}) = (S_{13} - w T_{13})(T_{12} - z^{-1} S_{12})(z R_{23} - w R'_{23}).
\]
Note that Proposition 3.3 says that in \( U_q(\mathfrak{g}) \otimes (\text{EndV})^{\otimes 2} \),
\[
R'_{23} S_{12} S_{13} = S_{13} S_{12} R'_{23}, \quad R'_{23} T_{12} T_{13} = T_{13} T_{12} R'_{23}, \quad R'_{23} S_{12} T_{13} = T_{13} S_{12} R'_{23}.
\]
By comparing the coefficients of both sides, we are left to verify the following:
\[
R_{23} S_{12} T_{13} - T_{13} S_{12} R_{23} = R'_{23} T_{12} S_{13} - S_{13} T_{12} R'_{23}.
\]
By using the Hecke relation (PS6) in Proposition 3.6 we get (writing \( c = c_{V,V} \))
\[
\text{RHS} = (R_{23} - (q - q^{-1}) c_{23}) T_{13} S_{12} - S_{13} T_{12} (R_{23} - (q - q^{-1}) c_{23})
\]
\[
= (R_{23} T_{12} S_{13} - S_{13} T_{12} R_{23}) - (q - q^{-1}) c_{23} T_{12} S_{13} + (q - q^{-1}) S_{13} T_{12} c_{23}
\]
\[
= LHS + T_{13} S_{12} R_{23} - (q - q^{-1}) c_{23} T_{12} S_{13} - R_{23} S_{12} T_{13} + (q - q^{-1}) S_{13} T_{12} c_{23}
\]
\[
= LHS + T_{13} S_{12} (R_{23} - (q - q^{-1}) c_{23}) - (R_{23} - (q - q^{-1}) c_{23}) S_{12} T_{13}
\]
\[
= LHS + T_{13} S_{12} R_{23} - R'_{23} S_{12} T_{13} = LHS.
\]
In the above, we used that \( c_{23} T_{1i} = T_{1j} c_{23} \) for \( \{i, j\} = \{2, 3\} \). This concludes the proof. \( \square \)

In the above proof, the quadratic Hecke relation (PS6) has been used repeatedly in an essential way. This may give an explanation of the fact: for \( \mathfrak{g}' \) a complex simple finite-dimensional Lie algebra, evaluation morphisms \( \text{ev} : U_q(\mathfrak{g}') \rightarrow U_q(\mathfrak{g}) \) exist only in type A. (Even in type A, the image of \( \text{ev} \) is contained in certain enlargement of \( U_q(\mathfrak{g}') \).)

3.3.5. Isomorphisms between \( U_q(\mathfrak{g}) \) and \( U_q(\mathfrak{g}) \). As their notations suggest, the two Hopf superalgebras \( U_q(\mathfrak{g}) \) and \( U_q(\mathfrak{g}) \) should be isomorphic.

**Proposition 3.9.** There is an isomorphism of Hopf superalgebras \( DF : U_q(\mathfrak{g}) 
\rightarrow U_q(\mathfrak{g}) \)
\[
e_i^+ \mapsto \frac{s_{i,i+1}^{a-1}s_{i,i+1}}{1 - q_i^{a-2}}, \quad e_i^- \mapsto t_{i+1,i} t_{i,i}^{-1} - 1, \quad t_{jj}^a \mapsto s_{jj} = t_{jj}^{-1} \quad (i \in I_0, j \in I).
\]

We postpone the proof of this proposition to \textbf{[3.3.2]}

In the above formulas, the scalars are chosen in such a way that the natural representation of \( U_q(\mathfrak{g}) \) on \( V \) defined later will match perfectly with that of \( U_q(\mathfrak{g}) \) (see Example 1 and Example 4).

We list the following relations in the superalgebra \( U_q(\mathfrak{g}) \) to be used later:
\[
[s_{i,i+1}, t_{j+1,j}] = \delta_{ij} (q_i - q_i^{-1}) (t_{ii} s_{i+1,i+1} - s_{ii} t_{i+1,i+1}) \quad i, j \in I_0, \quad [t_{ji}, t_{kj}] = (q_j - q_j^{-1}) t_{jj} t_{ki} \quad i, j, k \in I, i < j < k.
\]
The second equation above is deduced in the proof of Lemma 3.1. See Equation (A.29).

The first equation will follow from Theorem 3.12.

3.3.6. Quantum double construction. We reformulate the RTT definition of $U_q(\mathfrak{g})$ as a quantum double construction, as in the non-graded case [FRT90, Theorem 16]. This will in turn give a RTT presentation of the $q$-Yangian $Y_q(\mathfrak{g})$ in Definition 3.5.

Let $A, B$ be two Hopf superalgebras. Call a bilinear form $\varphi : A \times B \to \mathbb{C}$ a Hopf pairing if $\varphi$ is of $\mathbb{Z}_2$-degree $0$, and if $\varphi$ satisfies

$$
\varphi(a, b') = (-1)^{|b||b'|} \varphi(a(1), b) \varphi(a(2), b'), \quad \varphi(a, 1) = \varepsilon_A(a);
$$

$$
\varphi(aa', b) = \varphi(a'(1), b(1)) \varphi(a(2), b(2)), \quad \varphi(1, b) = \varepsilon_B(b)
$$

for $\mathbb{Z}_2$-homogeneous $a, a' \in A$ and $b, b' \in B$. Here we adapt the Sweedler notation $\Delta(x) = x(1) \otimes x(2)$. Given such a Hopf pairing, one can endow the vector superspace $A \otimes B$ with a unique Hopf superalgebra structure satisfying [KRT97, Theorem 3.2]

(QD1) $a \mapsto a \otimes 1$, $b \mapsto 1 \otimes b$ are morphisms of Hopf superalgebras respectively;

(QD2) for $\mathbb{Z}_2$-homogeneous $a \in A, b \in B$, we have $(a \otimes 1)(1 \otimes b) = a \otimes b$ and $(1 \otimes b)(a \otimes 1) = (-1)^{|a(1)||b|+(|b(2)|+|b(3)|)|a(2)|+|a(3)||b(3)|} \varphi(a(1), S_B(b(1))) \varphi(a(3), b(3)) a(2) \otimes b(2).

Let $D_\varphi(A, B)$ be the Hopf superalgebra thus obtained.

In our context, $A$ (resp. $B$) is the superalgebra generated by the $s_{ij}^{(n)}(s_{ij}^{(0)})^{-1}$ (resp. the $i_j^{(n)}(i_j^{(0)})^{-1}$) with $\mathbb{Z}_2$-gradings and with defining relations as in Definition 3.3 (without Relation 3.11 which makes no sense). Clearly $A$ and $B$ are Hopf superalgebras with coproducts defined by formulas (3.1)- (3.5).

Proposition 3.10. There exists uniquely a Hopf pairing $\hat{\varphi} : A \times B \to \mathbb{C}$ such that

(3.23) \[ \sum_{i,j,a,b \in I} E_{ab} \otimes E_{ij} \sum_{m,n \in \mathbb{Z}_{\geq 0}} z^{-m} w^n \hat{\varphi}(s_{ij}^{(n)}, t_{ab}^{(m)}) = \mathcal{R}(z, w) \in (End V)^{\otimes 2}[[z^{-1}, w]]. \]

The assignment $s_{ij}^{(n)} \otimes 1 \mapsto s_{ij}^{(n)}, \quad 1 \otimes i^{(n)} \mapsto i^{(n)}$ extends uniquely to a surjective morphism of Hopf superalgebras $D_\hat{\varphi}(A, B) \to U_q(\mathfrak{g})$ whose kernel is the ideal generated by the $s_{ii}^{(0)} \otimes 1 - 1 \otimes (i^{(0)})^{-1}, \quad 1 \otimes t_{ii}^{(0)} - (s_{ii}^{(0)})^{-1} \otimes 1 \quad (i \in I)$.

Moreover, $D$ restricts to a Hopf superalgebra isomorphism $D|_A : A \to Y_q(\mathfrak{g})$.

Proof. (Sketch) By abuse of language, let $\mathcal{F}_A$ (resp. $\mathcal{F}_B$) be the superalgebra generated by the $s_{ij}^{(n)}$ (resp. the $t_{ij}^{(n)}$) for $i, j \in I, n \in \mathbb{Z}_{\geq 0}$, and with $\mathbb{Z}_2$-gradings $|s_{ij}^{(n)}| = |t_{ij}^{(n)}| = |i|+|j|$. Then $\mathcal{F}_A$ and $\mathcal{F}_B$ are super bialgebras with coproduct given by Equations (3.11)-(3.15).

Now Formula (3.23) above determines a bilinear map $\varphi : \mathcal{F}_A \otimes \mathcal{F}_B \to \mathbb{C}$ satisfying all the properties of a Hopf pairing. According to [KRT97, Chapter 3] it is enough to show that $\varphi$ respects Relations (3.9)-(3.10), (3.12)-(3.13), and that (QD2) is equivalent to Relation (3.11). We only check Relation (3.10). (The other relations can be done in the same way.)

For this, define the bilinear map

\[
\varphi_3 : \mathcal{F}_A \otimes (End V)^{\otimes 2} \otimes \mathcal{F}_B \otimes End V \to (End V)^{\otimes 3}
\]

\[
(a \otimes a' \otimes x \otimes y, b \otimes b' \otimes z) \mapsto (-1)^{|(x+y)|(|b|+|b'|+|z|)+|a'||b|} \varphi(a, b) \varphi(a', b') z \otimes x \otimes y.
\]
for $\mathbb{Z}_2$-homogeneous vectors $a, a', x, y, z, b, b'$. Then Relation \eqref{eq:10} amounts to:

$$
\varphi_3(R_{34}(z, w)S_{13}(z)S_{24}(w) - S_{14}(w)S_{23}(z)R_{34}(z, w), T_{23}(u)T_{13}(u)) = 0.
$$

From the definitions of $\varphi$ and $\varphi_3$, we see that the LHS of the above equation becomes:

$$
R_{23}(z, w)R_{13}(u, w)R_{12}(u, z) - R_{12}(u, z)R_{13}(u, w)R_{23}(z, w),
$$

which is zero because of the Yang-Baxter Equation \eqref{eq:7} (PS1). \hfill \Box

3.3.7. Weight grading. The following relations hold in $U_q(\mathfrak{g})$:

\begin{equation}
\label{eq:24}
\begin{array}{l}
\displaystyle s_{ij}^{(0)} s_{jk}^{(n)} = q^{(\epsilon_i, \epsilon_j - \epsilon_k)} s_{ik}^{(n)} s_{ji}^{(0)}, \\
\displaystyle s_{ij}^{(0)} t_{jk}^{(n)} = q^{(\epsilon_i, \epsilon_j - \epsilon_k)} t_{jk}^{(n)} s_{ij}^{(0)}.
\end{array}
\end{equation}

As the $s_{ij}^{(0)}$ are invertible, $U_q(\mathfrak{g})$ is endowed with a $\mathbb{Q}$-grading: for $\alpha \in \mathbb{Q}$,

$$
(U_q(\mathfrak{g}))_\alpha = \{ x \in U_q(\mathfrak{g}) | s_{ii}^{(0)} x (s_{ii}^{(0)})^{-1} = q^{(\epsilon_i, \alpha)} x \quad \text{for} \ i \in I \}.
$$

The $\mathbb{Q}$-grading is compatible with the Hopf superalgebra structure and with the $\mathbb{Z}_2$-grading:

\begin{equation}
\label{eq:25}
|s_{ij}^{(n)}|_\mathbb{Q} = |t_{ij}^{(n)}|_\mathbb{Q} = \epsilon_i - \epsilon_j \quad (i, j \in I).
\end{equation}

Let us prove Equation \eqref{eq:24}. To begin with, for $n \in \mathbb{Z}_{\geq 0}$, by taking the coefficients of $z^{n+1}$ (resp. $z^{1-n}$) in Relation \eqref{eq:10} (resp. Relation \eqref{eq:11}), we observe that

$$
\begin{align*}
R_{23}S_{12}^{(0)} S_{13}^{(n)} &= S_{13}^{(0)} S_{12}^{(n)} R_{23}, \\
R_{23}T_{12}^{(n)} S_{13}^{(0)} &= S_{13}^{(0)} T_{12}^{(n)} R_{23}.
\end{align*}
$$

Here $S^{(n)} := \sum_{ij} s_{ij}^{(n)} \otimes E_{ij} \in U_q(\mathfrak{g}) \otimes E_{ij}$ (similar for $T^{(n)}$). Now Equation \eqref{eq:24} comes from the following lemma and from the automorphism defined by Equation \eqref{eq:19}.

**Lemma 3.11.** Let $U$ be a superalgebra. For $i, j \in I$ let $a_{ij}, b_{ij} \in U$ be elements of $\mathbb{Z}_2$-degree $|i| + |j|$. Assume that $b_{ij} = 0$ if $i > j$. Introduce

$$
A := \sum_{i,j \in I} a_{ij} \otimes E_{ij}, \quad B := \sum_{i,j \in I} b_{ij} \otimes E_{ij} \in U \otimes \operatorname{End} V.
$$

Suppose that $R_{23}A_{12}B_{13} = B_{13}A_{12}R_{23}$. Then

$$
b_{kk}a_{ij} = q^{(\epsilon_k, \epsilon_j - \epsilon_i)} a_{ij} b_{kk} \quad \text{for} \ i, j, k \in I.
$$

**Proof.** We shall prove only the case $k \neq i, j$. The idea is to compare the matrix coefficients of $v_j \otimes v_k \mapsto v_i \otimes v_k$ for the operator equation

$$
A_{12}B_{13} = R_{23}^{-1} B_{13}A_{12} R_{23} \in U \otimes \operatorname{End} V^\otimes 2.
$$

For a statement $P$ define $\delta(P) = 1$ if $P$ is true and 0 otherwise. The matrix coefficient for the LHS is $a_{ij} b_{kk}$. For the RHS, in view of the ice rule for $R$, we see that the corresponding coefficient $c$ should be the coefficient $c_1$ of $v_i \otimes v_k$ in

$$
R_{23}^{-1} B_{13} A_{12} (v_j \otimes v_k + \delta(j < k) (q_j - q_j^{-1}) v_k \otimes v_j).
$$

To determine $c_1$, it is enough to consider the part $u_1$ containing $v_i \otimes v_k, v_k \otimes v_i$ in the vector

$$
B_{13} A_{12} (v_j \otimes v_k + \delta(j < k) (q_j - q_j^{-1}) v_k \otimes v_j)
$$

A straightforward calculation shows that

$$
u_1 = (b_{kk} a_{ij} \pm \delta(j < k)(q_j - q_j^{-1})b_{kj}a_{ik}) v_i \otimes v_k + (\delta(j < k)(q_j - q_j^{-1})b_{ij}a_{kk} \pm b_{ik}a_{kj}) v_k \otimes v_i.
$$
Theorem 3.12. KU [DF93, pp.286], following Yao-Zhong Zhang [Zy97].

Ding-Frenkel homomorphism.

3.4. Drinfeld realization and coproduct formulas. We explain that as Hopf superalgebras the quantum affine superalgebra $U_q(\mathfrak{g})$ is not far from the quantum loop superalgebra $U_q(Lg)$ defined by Drinfeld generators (such quantum loop superalgebra with $g$ replaced by $\mathfrak{sl}(M,N)$ has been used in [Zh13] to study finite-dimensional representations). Also some coproduct estimations for the Drinfeld generators are given.

3.4.1. Ding-Frenkel homomorphism. We review a super analogue of Ding-Frenkel homomorphism between Drinfeld and RTT realizations of the quantum affine algebra $U_q(\mathfrak{g}_n)$ [DF93, pp.286], following Yao-Zhong Zhang [Zy97].

The Gauss decomposition gives uniquely

$$e_{ij}^\pm(z), f_{ji}^\pm(z), K_i^\pm(z) \in U_q(\mathfrak{g})([z,\pm z]) \quad \text{for} \quad 1 \leq i < j \leq M, 1 \leq i \leq M + N$$

such that in the superalgebra $(U_q(\mathfrak{g}) \otimes \text{End}V)([z,\pm z])$

$$\left\{\begin{array}{l}
S(z) = (\sum_{i<j} f_{ji}^\pm(z) \otimes E_{ji} + 1 \otimes \text{Id}_V)(\sum_i K_i^+(z) \otimes E_{ii})(\sum_{i<j} e_{ij}^\pm(z) \otimes E_{ij} + 1 \otimes \text{Id}_V), \\
T(z) = (\sum_{i<j} f_{ji}^\pm(z) \otimes E_{ji} + 1 \otimes \text{Id}_V)(\sum_i K_i^-(z) \otimes E_{ii})(\sum_{i<j} e_{ij}^\pm(z) \otimes E_{ij} + 1 \otimes \text{Id}_V).
\end{array}\right.$$ 

For example, $K_i^+(z) = s_{11}(z)$ and $K_i^-(z) = t_{11}(z)$. For $i \in I_0 = I \setminus \{M + N\}$, define

$$X_i^+(z) = e_{i,i+1}^+(z) - e_{i, i+1}^-(z) = \sum_n X_{i,n}^+ z^n, \quad X_i^-(z) = f_{i,i+1}^+(z) - f_{i, i+1}^-(z) = \sum_n X_{i,n}^- z^n.$$ 

Theorem 3.12. [DF93, Zy97] The superalgebra $U_q(\mathfrak{g})$ is generated by the coefficients of $X_i^\pm(z), K_i^\pm(z)$ with $i \in I_0, j \in I$. Moreover,

$$K_i^+(z)K_j^\prime(z)(w) = K_j^\prime(z)(w)K_i^+(z),$$

(Cartan)

$$\left\{\begin{array}{l}
K_i^+(z)X_j^\pm(w) = (q_i - q_i^{-1}w)\frac{1}{z-w}X_j^\pm(w)K_i^+(z), \\
K_i^-(z)X_j^\pm(w) = (q_i - q_i^{-1}w)\frac{1}{z-w}X_j^\pm(w)K_i^-(z),
\end{array}\right.$$ 

(Drinfeld)

$$\left\{\begin{array}{l}
(q_i - q_i^{-1}w)X_i^\pm(z)X_j^\pm(w) = (q_i^\pm z - q_i^{-1}w)X_j^\pm(w)X_i^\pm(z) \quad \text{if} \quad i \neq M, \\
(q_i^\pm z - q_i^{-1}w)X_i^\pm(z)X_{i+1}^\pm(w) = (z - w)X_{i+1}^\pm(w)X_i^\pm(z), \\
(z - w)X_i^\pm(z)X_{i+1}^\pm(w) = (q_i + 1z - q_i^{-1}w)X_{i+1}^\pm(w)X_i^\pm(z), \\
X_i^\pm(z)X_j^\pm(w) = \delta_{ij}(q_i - q_i^{-1})\delta (\frac{z}{w})(K_i^\prime(z))^{-1} - K_i^+(z)K_i^-(z) - K_i^+(z)K_i^-(z)^{-1},
\end{array}\right.$$ 

Observe that the last three terms in $u_1$ do not contribute to $v_i \otimes v_k$ when applying $R_{23}^{-1}$. It follows that the coefficient $c = c_1$ of $v_i \otimes v_k$ in $R_{23}^{-1}u_1$ is exactly $b_{kk}a_{ij}$. In other words, $a_{ij}b_{kk} = b_{kk}a_{ij}$, as desired. □
(Serre) \[
\begin{aligned}
[X_i^*(z_1), X_j^*(z_2), X_j^*(w)]_{q^{-1}} + \{z_1 \leftrightarrow z_2\} &= 0 \quad \text{if } (i \neq M, |j - i| = 1), \\
[[[X_{M-1}(u), X_M^*(z_1)], X_M^*(v)]_{q^{-1}}, X_M^*(z_2)] + \{z_1 \leftrightarrow z_2\} &= 0 \quad \text{if } (M, N > 1).
\end{aligned}
\]

3.4.2. Proof of Proposition 3.9. As an application of Theorem 3.12 above, let us give a proof of Proposition 3.9 which says that the Ding-Frenkel homomorphism restricted to finite type quantum superalgebras is indeed an isomorphism.

First of all, \(DF : U_q(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})\) is a well-defined Hopf superalgebra homomorphism thanks to Theorem 3.12 Proposition 3.8 and [Ya94, Theorem 10.5.1] on defining relations of the quantum superalgebra \(U_q(\mathfrak{g})\) of type A. It is easy to prove that \(DF\) is surjective in view of the relations preceding §3.3.6.

Next, let \(A\) (resp. \(B\)) be the subalgebra of \(U_q(\mathfrak{g})\) generated by the \(e_i^+, t_j^{\pm1}\) (resp. the \(e_i^-\) and \(t_j^{\pm 1}\)) for \(i \in I_0, j \in I\). Then, \(A, B\) are sub-Hopf-superalgebras. Moreover, according to [Ya94, §2.4, Prop.10.4.1], there exists a non-degenerate Hopf pairing \(\varphi_1 : A \times B \longrightarrow \mathbb{C}\) defined by:

\[
\varphi_1 (t_i, t_j) = q^{-\langle e_i^+, e_j^- \rangle}, \quad \varphi_1 (e_i^+, e_j^-) = \frac{\delta_{ij}}{q_i - q_i^{-1}}.
\]

Furthermore, \(U_q(\mathfrak{g})\) is the quotient of \(\mathcal{D}_{\varphi_1}(A, B)\) by the ideal generated by \(1 \otimes t_i^{\pm 1} - t_i^{\pm 1} \otimes 1\).

On the other hand, let \(A'\) (resp. \(B'\)) the subalgebra of \(\mathcal{U}(\mathfrak{g})\) generated by the \(s_{ij}, s_{kk}^{-1}\) (resp. the \(t_{ij}, t_{jk}^{-1}\)) for \(1 \leq i \leq j \leq M + N\) and \(k \in I\). Clearly \(A'\) and \(B'\) are sub-Hopf-superalgebras. Moreover, Propositions 3.8 and 3.10 say that there exists a Hopf pairing \(\varphi_2 : A' \times B' \longrightarrow \mathbb{C}\) given by

\[
\sum_{a, b, i, j \in I} \varphi_2 (s_{ij}, t_{ab}) E_{ab} \otimes E_{ij} = R \in \text{End} \mathbb{V} \otimes 2.
\]

Similarly, \(\mathcal{U}_q(\mathfrak{g})\) is the quotient of \(\mathcal{D}_{\varphi_2}(A', B')\) by the ideal generated by \(1 \otimes s_{ii}^{\pm 1} - s_{ii}^{\pm 1} \otimes 1\).

It is straightforward to show that \(DF(A) = A'\) and \(DF(B) = B'\). Moreover,

\[
\varphi_2 (DF(a), DF(b)) = \varphi_1 (a, b) \quad \text{for } a \in A, b \in B.
\]

Let \(f : A \longrightarrow A'\) (resp. \(g : B \longrightarrow B'\)) be the Hopf superalgebra morphism induced by \(DF\). Then \(f, g\) are surjective. Moreover, \(DF\) is induced by the Hopf superalgebra morphism

\[
\mathcal{D}F := f \otimes g : \mathcal{D}_{\varphi_1}(A, B) \longrightarrow \mathcal{D}_{\varphi_2}(A', B'), \quad a \otimes b \mapsto f(a) \otimes g(b).
\]

As \(DF(t_i) = s_{ii}^{d_{ii}}\) for \(i \in I\), we are reduced to show that \(\mathcal{D}F\) is injective. Note that

\[
\ker \mathcal{D}F = \ker (f \otimes g) = \ker f \otimes B + A \otimes \ker g.
\]

The non-degeneracy of \(\varphi_1\) implies that the RHS above is zero. \(\square\)

We remark that \(\varphi_2\) defined above is non-degenerate. Hence we can write down the universal \(R\)-matrix of \(\mathcal{U}_q(\mathfrak{g})\) [Ya94, Theorem 10.6.1] in terms of the RTT generators. Similar arguments should apply to the affine case, which however requires additional information on some central elements of \(U_q(\mathfrak{g})\), the so-called quantum Berezinians, and their behaviour under the Hopf pairing \(\hat{\varphi}\). We hope to return to these issues in future works.
3.4.3. \textit{Coproduct formulas}. Let us define the Drinfeld generators $K^\pm_{i,\pm s}$ for $s \in \mathbb{Z}_{\geq 0}$ by

$$K^\pm_i(z) = \sum_{s \in \mathbb{Z}_{\geq 0}} K^\pm_{i,\pm s} z^\pm s \in U_q(\hat{g})[[z^\pm 1]].$$

Then $K^\pm_{i,0} = (s^{(0)}_i)^\pm 1$. Moreover Cartan relations in Theorem 3.12 imply that

$$|X^\pm_{i,n}|_Q = \pm \alpha_i, \quad |K^\pm_{j,\pm s}|_Q = 0 \quad \text{for } i \in I_0, j \in I, n \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0}.$$

\textbf{Proposition 3.13.} Let $i \in I_0, j \in I, n \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0}$. Then

\begin{align*}
(3.26) \quad & \Delta(K^\pm_{j,\pm s}) - \sum_{a=0}^{s} K^\pm_{j,\pm a} \otimes K^\pm_{j,\pm (s-a)} \in \sum_{a \in Q_{\geq 0} \setminus \{0\}} (U_q(\hat{g}))(\alpha) \otimes (U_q(\hat{g}))(-\alpha), \\
(3.27) \quad & \Delta(X^+_{i,n}) - 1 \otimes X^+_{i,n} \in \sum_{a \in Q_{\geq 0} \setminus \{0\}} (U_q(\hat{g}))(\alpha) \otimes (U_q(\hat{g}))(\alpha), \\
(3.28) \quad & \Delta(X^-_{i,n}) - X^-_{i,n} \otimes 1 \in \sum_{a \in Q_{\geq 0} \setminus \{0\}} (U_q(\hat{g}))(\alpha) \otimes (U_q(\hat{g}))(\alpha).
\end{align*}

The proof of this proposition is given in Appendix A.

4. \textbf{Highest weight representations}

In this section, we state one of the main results in this paper: some tensor products of Kirillov-Reshetikhin modules over $U_q(\hat{g})$ are highest $\ell$-weight modules.

4.1. \textbf{Highest $\ell$-weight modules}. Let $V$ be a $U_q(\hat{g})$-module. A non-zero vector $v \in V \setminus \{0\}$ is called a \textit{highest $\ell$-weight vector} if $v$ is $\mathbb{Z}_2$-homogeneous and

$$s^{(n)}_{ij} v = 0 = l^{(n)}_{ij} v, \quad s^{(n)}_{kk} v = l^{(n)}_{kk} v \in \mathbb{C} v \quad (n \in \mathbb{Z}_{\geq 0}, i, j, k \in I, i < j).$$

$V$ is called a \textit{highest $\ell$-weight module} if $V = U_q(\hat{g})v$ for some highest $\ell$-weight vector. Similarly, the notions of lowest $\ell$-weight vector and lowest $\ell$-weight module are defined by replacing the above condition ($i < j$) with ($i > j$). Similarly, one can define the notions of highest/lowest $\ell$-weight modules/vectors for representations of the $q$-Yangian $Y_q(\hat{g})$ by dropping the $l^{(n)}_{ij}$ above.

Thanks to Proposition 3.9 there is a highest weight representation theory for the quantum superalgebra $\mathcal{U}_q(\mathfrak{g})$. For example, let $V$ be a $\mathcal{U}_q(\mathfrak{g})$-module. A non-zero vector $v \in V$ is called a \textit{highest weight vector} if $v$ is $\mathbb{Z}_2$-homogeneous and $s_{ij} v = 0, s_{kk} v \in \mathbb{C} v$ for $i, j, k \in I, i < j$. In particular, for $\lambda \in \mathbf{P}$, we have simple $\mathcal{U}_q(\mathfrak{g})$-module $(DF^{-1})^\lambda L(\lambda)$ which will be written as $L(\lambda)$ by abuse of language. More explicitly, $L(\lambda)$ is the simple $\mathcal{U}_q(\mathfrak{g})$-module generated by a vector $v_\lambda$ such that

$$|v_\lambda| = |\lambda|, \quad s_{ij} v_\lambda = 0, \quad s_{kk} v_\lambda = q^{(e_k, \lambda)}_{(i, j)} v_\lambda \quad (i, j, k \in I, i < j).$$
4.1.1. Highest $\ell$-weights and tensor product. Let $V, V'$ be $U_q(\mathfrak{g})$-modules of highest $\ell$-weights with $v, v'$ highest $\ell$-weight vectors respectively. Then $v \otimes v'$ is also a highest $\ell$-weight vector. By definition, there exist $f_i^\pm(z), g_i^\pm(z) \in (\mathbb{C}[[z^\pm 1]])^\times$ for $i \in I$ such that

$$s_{ii}(z)v = f_i^+(z)v, \quad t_{ii}(z)v = f_i^-(z)v, \quad s_{ii}(z)v' = g_i^+(z)v', \quad t_{ii}(z)v' = g_i^-(z)v'.$$

From the Gauss decomposition in (3.4.1) we see that

$$K_i^+(z)v = f_i^+(z)v, \quad K_i^-(z)v' = g_i^+(z)v'.$$

On the other hand, from the coproduct formulas of $s_{ii}(z), t_{ii}(z)$ it follows

$$s_{ii}(z)(v \otimes v') = f_i^+(z)g_i^+(z)(v \otimes v'), \quad t_{ii}(z)(v \otimes v') = f_i^-(z)g_i^-(z)(v \otimes v').$$

Henceforth, similar formulas hold for $K_i^+(z)(v \otimes v')$. This observation will be used in (A.3) to conclude the proof of Proposition 3.13.

4.1.2. Kirillov-Reshetikhin modules. For $a \in \mathbb{C}^\times$, define the evaluation morphism $ev_a := ev \circ \Phi_a : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$, here $ev$ and $\Phi_a$ are given by Proposition 3.13 and by Formula (3.24) respectively. We can pull back $U_q(\mathfrak{g})$-modules $V$ to get $U_q(\mathfrak{g})$-modules $ev_a^*V$. When there is no confusion, we simply write $v = ev_a^*v$ for $v \in V$.

For example, take $\lambda \in P$. Consider $ev_a^*L(\lambda)$. Let $v_\lambda$ be a highest weight vector for the $U_q(\mathfrak{g})$-module $L(\lambda)$, then $v_\lambda$ is a highest $\ell$-weight vector

$$|v_\lambda| = |\lambda|, \quad s_{ii}(z)v_\lambda = (q^{(\lambda,e_i)} - zaq^{-(\lambda,e_i)})v_\lambda, \quad t_{ii}(z)v_\lambda = (q^{-(\lambda,e_i)} - z^{-1}a^{-1}q^{(\lambda,e_i)})v_\lambda.$$

**Definition 4.1.** The $U_q(\mathfrak{g})$-modules $ev_a^*L(k\mathbb{Z}_r \otimes \mathbb{C}[k\mathbb{Z}_r])$ for $a \in \mathbb{C}^\times, r \in I_0, k \in \mathbb{Z}_{\geq 0}$, are called Kirillov-Reshetikhin modules, denoted by $W_{k,a}^{(r)}$.

In the above definition, the tensor product by an one-dimensional module is needed to ensure that the highest $\ell$-weight vectors are of $\mathbb{Z}_2$-degree $\mathfrak{g}$. The main result of this section is the following.

**Theorem 4.2.** Let $k \in \mathbb{Z}_{>0}$, $r \in I_0$ and $a \in \mathbb{C}^\times$. Let $x_j \in \mathbb{Z}$ for $1 \leq j \leq k$. Assume $x_i \geq x_j$ for all $1 \leq i < j \leq k$. Then the $U_q(\mathfrak{g})$-module $\bigotimes_{j=1}^k W_{1,aQ_{x_j}}^{(r)}$ is of highest $\ell$-weight.

Large part of the rest of the paper is devoted to the proof of the theorem. The outline is as follows: first we reduce to the case $1 \leq r \leq M$ with the help of a Hopf superalgebra isomorphism (Remark 4.4)

$$U_q(\mathfrak{g}(\overline{N}, M))^{\text{cop}} \longrightarrow U_q(\mathfrak{g}(\overline{M}, N));$$

next we study in (5) in detail the case $\mathfrak{g} = \mathfrak{gl}(1, 1)$; finally we conclude in (6) the proof by restriction arguments from $\mathfrak{g}$ to $\mathfrak{gl}(1, 1)$. Throughout the proof, a cyclicity result of Chari (Lemma 4.5) is used repeatedly.
4.2. Reduction to the case \( r \leq M \). Let \( \mathfrak{g}' = \mathfrak{gl}(N, M) \). We shall define the quantum affine superalgebra \( U_q(\mathfrak{g}') \). Recall that \( U_q(\mathfrak{g}) \) is constructed from the vector superspace \( V \) and the Perk-Schultz matrix \( R_q(z, w) \). We begin with the index set \( I \) endowed with \( \mathbb{Z}_2 \)-partition

\[
I = \{1 < 2 < \cdots < M + N\} = I_\Gamma \sqcup I_T, \quad I_\Gamma = \{1 < 2 < \cdots < M\}.
\]

\( V = \bigoplus_{i \in I} C v_i \) and \( I = I_\Gamma \sqcup I_T \) are linked in the following way: \(|v_i| = |i|\). The Perk-Schultz matrix \( R_q(z, w) \) is determined by \( I \) as seen from Formula (3.18), in which the summation is over \( I \), and \( q_i = q^{(-1)^{|i|}} \).

Now introduce

\[
J = \{1 < 2 < \cdots < M + N\} = J_\Gamma \sqcup J_T, \quad J_\Gamma = \{1 < 2 < \cdots < N\}.
\]

For \( s \in \mathbb{Z}_2 \) and \( j \in J_s \), write \(|j|^J = s\). Let \( V^J = \bigoplus_{j \in J} C w_j \) be the vector superspace

\[
|w_j| = |j|^J \quad (j \in J).
\]

Let \( e_{ij} \in \text{End} V^J \) be \( w_{jk} \rightarrow \delta_{jk} w_i \) for \( i, j, k \in J \). Let \( R^J(z, w) = R_q^J(z, w) \) be the Perk-Schultz matrix defined by Formula (3.18) with summation over \( J \), with the \( q^J_i \) for \( i \in I \) replaced by the \( q^J_i = q^{(-1)^{|i|}} \) for \( j \in J \), and with the \( E_{ij} \) for \( i, j \in I \) replaced by the \( e_{ij} \) for \( i, j \in J \).

Define the quantum affine superalgebra \( U_q(\mathfrak{g}') \) in exactly the same way as in Definition 3.5. For distinction, write the defining generators as \( s_{ij; J}^{(n)} \), \( t_{ij; J}^{(n)} \).

Finally, define \( \varepsilon^J_{ij} \) for \( i, j \in J \) in the same way as in Formula (3.7), with the \(|i|\) for \( i \in I \) replaced by the \(|i|^J\) for \( i \in J \). For \( i \in J \), let \( \tau = N + M + 1 - i \in I \).

**Proposition 4.3.** The assignment \( s_{ij; J}^{(n)} \rightarrow \varepsilon^J_{ij} s_{ij; J}^{(n)} \), \( t_{ij; J}^{(n)} \rightarrow \varepsilon^J_{ji} t_{ji; J}^{(n)} \) extends uniquely to a Hopf superalgebra isomorphism \( f_{J, I} : U_q(\mathfrak{g}')^{\text{cop}} \rightarrow U_q(\mathfrak{g}) \).

**Proof.** Introduce the linear isomorphism \( f : V \rightarrow V^J, \quad v_i \mapsto w_i \). Let \( f_s : \text{End} V \rightarrow \text{End} V^J, \quad h \mapsto fhf^{-1} \) be the induced map. Then

\[
f_s : \text{End} V \rightarrow \text{End} V^J, \quad E_{ij} \mapsto e^J_{ij}
\]

is an isomorphism of superalgebras. Moreover, we have

\[
f_s \otimes f_s(R_q(z, w)) = c_{V^J, V^J} R_{q^{-1}}^J(z, w)c_{V^J, V^J} = (\tau_J \otimes \tau_J)(R_q^J(z, w)^{-1}).
\]

Here \( \tau_J : \text{End} V^J \rightarrow \text{End} V^J, \quad e_{ij} \mapsto \varepsilon^J_{ij} e_{ji} \), and the last equation comes from Proposition 3.6 (PS4)-(PS5). Applying \( \text{Id}_{U_q(\mathfrak{g})} \otimes f_s \otimes f_s \) to Relation (3.9), we get

\[
((\tau_J \otimes \tau_J)(R_q^J(z, w)^{-1}))_{23} \widehat{T}_{12}(z) \widehat{T}_{13}(w) = \widehat{T}_{13}(w) \widehat{T}_{12}(z) ((\tau_J \otimes \tau_J)(R_q^J(z, w)^{-1}))_{23}.
\]

Here \( \widehat{T}(z) = (\text{Id}_{U_q(\mathfrak{g})} \otimes f_s)(T(z)) \). Next applying \( \text{Id}_{U_q(\mathfrak{g})} \otimes \tau_J \otimes \tau_J \) to the above equation,

\[
\widehat{T}_{12}(z) \widehat{T}_{13}(w)(R_q^J(z, w)^{-1})_{23} = (R_q^J(z, w)^{-1})_{23} \widehat{T}_{13}(w) \widehat{T}_{12}(z).
\]

Here \( \widehat{T}(z) = (\text{Id}_{U_q(\mathfrak{g})} \otimes \tau_J)(\widehat{T}(z)) \). In other words,

\[
R_{23}^J(z, w) \widehat{T}_{12}(z) \widehat{T}_{13}(w) = \widehat{T}_{13}(w) \widehat{T}_{12}(z) R_{23}^J(z, w) \in (U_q(\mathfrak{g}) \otimes \text{End} V^J)((z^{-1}, w^{-1})).
\]
Theorem 4.2 is of highest weight. In other words, let \( U \) be an affine superalgebra and \( \tau_f \) be the highest weight vector. Clearly, \( \tau_f \) is a highest weight vector, as Chari did in the non-graded case [Ch02, Lemma 4.2].

**Remark 4.4.** Let \( M + 1 \leq r < M + N \). Let \( v \) be a highest \( \ell \)-weight vector for the \( U_q(\widehat{g}) \)-module \( W^{(r)}_{k,a} \). Then \( f^* \) is a highest \( \ell \)-weight vector in \( f^* \) with:

\[
\begin{align*}
 s_{ii,j}(z)f^*_{i,j}v &= f^*_{i,j}v \begin{cases} 
 q^k - zq^{-k} & \text{(if } i \leq N + M - r), \\
 1 - za & \text{(if } i > N + M - r), 
\end{cases} \\
t_{ii,j}(z)f^*_{i,j}v &= f^*_{i,j}v \begin{cases} 
 q^{-k} - z^{-1}a^{-1}q^k & \text{(if } i \leq N + M - r), \\
 1 - za & \text{(if } i > N + M - r). 
\end{cases}
\end{align*}
\]

In other words, \( f^*_{i,j}W^{(r)}_{k,a} \) is a Kirillov-Reshetikhin module for the quantum affine superalgebra \( U_q(\widehat{g}(N, M)) \), corresponding to the fundamental weight \( \varpi_{N+M-r} \). Clearly, \( 1 \leq N + M - r \leq N - 1 \). Thus, to prove Theorem 4.2 we can assume \( 1 \leq r \leq M \).

### 4.3 A cyclicity result of Chari

To prove that a tensor product of \( U_q(\widehat{g}) \)-modules as in Theorem 4.2 is of highest \( \ell \)-weight, it is enough to prove that a certain vector is generated by the highest \( \ell \)-weight vector, as Chari did in the non-graded case [Ch02, Lemma 4.2].

**Lemma 4.5.** Let \( V_+ \) (resp. \( V_- \)) be a \( U_q(\widehat{g}) \)-module of highest \( \ell \)-weight (resp. of lowest \( \ell \)-weight). Let \( v_+ \in V_+ \) (resp. \( v_- \in V_- \)) be a highest \( \ell \)-weight vector (resp. lowest \( \ell \)-weight vector). Then the \( U_q(\widehat{g}) \)-module \( V_+ \otimes V_- \) (resp. \( V_- \otimes V_+ \)) is generated by \( v_+ \otimes v_- \) (resp. \( v_- \otimes v_+ \)).

**Proof.** If \( v \in V \) is a highest/lowest \( \ell \)-weight vector for a \( U_q(\widehat{g}) \)-module \( V \), then according to Proposition 4.2 \( f^*_{i,j}v \) is a highest/lowest \( \ell \)-weight vector for the \( U_q(\widehat{g}) \)-module \( f^*_{i,j}V \). Hence, it is enough to prove the first part: \( V_+ \otimes V_- = U_q(\widehat{g})(v_+ \otimes v_-) \).

As \( V_- \) is a lowest \( \ell \)-weight \( U_q(\widehat{g}) \)-module with lowest \( \ell \)-weight vector \( v_- \), \( V_- \) is spanned by the above vectors with \( \alpha = \alpha_i + \cdots + \alpha_s \). In particular, with respect to the action of the \( s^{(0)}_{i_1} \), \( V_- \) is endowed with a \( \mathbb{Q}_{\geq 0} \)-grading such that \( (V_-)_\alpha \) is spanned by the above vectors with \( \alpha = \alpha_i + \cdots + \alpha_s \). This \( \mathbb{Q}_{\geq 0} \)-grading in turn endows \( V_- \) with a \( \mathbb{Z}_{\geq 0} \)-grading such that \( (V_-)_n \) is spanned by the above vectors with \( n = s \). We prove by induction on \( n \in \mathbb{Z}_{\geq 0} \) that

\[
(P_n) : V_+ \otimes (V_-)_n \subseteq U_q(\widehat{g})(v_+ \otimes v_-).
\]

When \( n = 0 \), \( (V_-)_0 = \mathbb{C}v_- \). For all \( v \in V_+ \), we have

\[
X^+_{i,n}(v \otimes v_-) = X^+_{i,n}v \otimes v_-
\]

since \( (U_q(\widehat{g}))_{-\alpha}v_- = 0 \) for \( \alpha \in \mathbb{Q}_{\geq 0} \setminus \{0\} \). As \( V_+ \) is of highest \( \ell \)-weight generated by the highest \( \ell \)-weight vector \( v_+ \), we get \( V_+ \otimes v_- \subseteq U_q(\widehat{g})(v_+ \otimes v_-) \). Now assume \( (P_k) \) for \( k \leq n \).
Let us prove \((P_{n+1})\). Take \(\mathbb{Z}_2\)-homogeneous vectors \(v_1 \in V_+\) and \(v_2 \in (V_-)_\beta \subseteq (V_-)_n\). We have
\[
X^+_{i,n}(v_1 \otimes v_2) \in (-1)^{|i||v_1|}v_1 \otimes X^+_{i,n}v_2 + \sum_{\alpha \in \mathbb{Q}_{\geq 0} \setminus \{0\}} (U_q(\hat{\mathfrak{g}}))_\alpha v_1 \otimes (U_q(\hat{\mathfrak{g}}))_{\alpha - \alpha}v_2.
\]
On the other hand, for \(\alpha \in \mathbb{Q}_{\geq 0} \setminus \{0\}\), by definition
\[
(U_q(\hat{\mathfrak{g}}))_{\alpha - \alpha}v_2 \subseteq (V_-)_{\beta + \alpha - \alpha} \subseteq \sum_{k \leq n} (V_-)_k.
\]
It follows that \(v_1 \otimes X^+_{i,n}v_2 \in U_q(\hat{\mathfrak{g}})(v_+ \otimes v_-)\). As \((V_-)_{n+1}\) is spanned by the \(X^+_{i,n}v_2\) with \(v_2 \in (V_-)_n\), we conclude. 

Our proof is slightly different from that of Chari [Ch02, Lemma 4.2] in the sense that we do not use the Drinfeld-Jimbo generators (see the end of [A.3]).

4.4. Natural representations. From Lemma 3.2, Propositions 3.4 and 3.6 (PS4)-(PS5) together with Remark 3.7 (PS7) follows a representation \(\rho(1)\) of the quantum superalgebra \(\mathcal{U}_q(\mathfrak{g})\) on \(V\):
\[
(\rho(1) \otimes \text{Id}_{\text{End}V})(T) = (\text{Id}_{\text{End}V} \otimes \tau)(R^{-1}), \quad (\rho(1) \otimes \text{Id}_{\text{End}V})(S) = (\text{Id}_{\text{End}V} \otimes \tau)((R')^{-1}).
\]
To be more precise,
\[
\rho(1)(s_{ii}) = q_i E_{ii} + \sum_{j \neq i} E_{jj} = \rho(1)(t_{ii}^{-1}) \quad (\text{for } i \in I),
\]
\[
\rho(1)(s_{ij}) = (q_i - q_i^{-1})E_{ij}, \quad \rho(1)(t_{ji}) = (q_i^{-1} - q_i)E_{ji} \quad (\text{for } 1 \leq i < j \leq M + N).
\]
From Proposition 3.9 and Example 1 it follows that \(\rho(0) = \rho(1) \circ DF\). In other words, the Ding-Frenkel isomorphism \(DF : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\hat{\mathfrak{g}})\) respects the natural representations. We can therefore write \(V \cong L(e_1)\) as \(\mathcal{U}_q(\hat{\mathfrak{g}})\)-modules.

For \(a \in \mathbb{C}^\times\), define \(\rho_a := \rho(1) \circ \text{ev}_a\). The representations \((V, \rho_a)\) are called natural representations of the quantum affine superalgebra \(\mathcal{U}_q(\hat{\mathfrak{g}})\). For simplicity, let \(V(a)\) be the \(\mathcal{U}_q(\hat{\mathfrak{g}})\)-module corresponding to \((V, \rho_a)\). It is clear that \(V(a) \cong W^{(1)}_{1,a}\) as \(\mathcal{U}_q(\hat{\mathfrak{g}})\)-modules (assuming \(M \neq 0\)).

The following lemma says that Perk-Schultz \(R\)-matrices can be interpreted as intertwining operators, from which comes naturally the Yang-Baxter equation Proposition 3.6 (PS1).

**Lemma 4.6.** Let \(a, b \in \mathbb{C}^\times\). Then \(c_{V,V} R(z,w)|_{(z,w)=(a,b)} : V(a) \otimes V(b) \to V(b) \otimes V(a)\) is a morphism of \(\mathcal{U}_q(\hat{\mathfrak{g}})\)-modules. 

The proof is direct, using properties of the Perk-Schultz \(R\)-matrix in Proposition 3.6. We shall not use this result in the sequel.

For natural representations, it is possible to determine completely the cyclicity condition.

**Proposition 4.7.** Let \(k \in \mathbb{Z}_{\geq 0}\) and \(a_i \in \mathbb{C}^\times\) for \(1 \leq i \leq k\). The \(\mathcal{U}_q(\hat{\mathfrak{g}})\)-module \(\bigotimes_{i=1}^k V(a_i)\) is of highest \(\ell\)-weight if and only if \(a_i \neq a_j q_i^{-2}\) for \(1 \leq i < j \leq k\). It is of lowest \(\ell\)-weight if and only if \(a_i \neq a_j q_i^{-2} q_{M+N}^{-1}\) for \(1 \leq i < j \leq k\).

The proof of this proposition is postponed to 6.2.
5. REPRESENTATIONS OF THE $q$-YANGIAN $Y_q(\mathfrak{g}(1,1))$

Fix $M = N = 1$ and $\mathfrak{g} = \mathfrak{gl}(1,1)$. We study the category $\mathcal{F}$ finite-dimensional representations of the $q$-Yangian $Y_q(\mathfrak{g})$. Up to tensor product by one-dimensional modules, simple objects in $\mathcal{F}$ are parametrized by rational functions as in [HJ12, Theorem 3.11]. Also, an explicit condition for a tensor product of simple objects to be of highest $\ell$-weight is given in terms of poles and zeros of rational functions (Theorem 5.2).

5.1. Simple objects in $\mathcal{F}$. Let us first construct some obvious $Y_q(\mathfrak{g})$-modules.

5.1.1. One-dimensional $Y_q(\mathfrak{g})$-modules. Let $D = C v$ be an one-dimensional $Y_q(\mathfrak{g})$-module. As $v$ is a highest/lowest $\ell$-weight vector, there exist $s \in \mathbb{Z}_2, a, b \in \mathbb{C}^\times$ and $f(z), g(z) \in 1 + z \mathbb{C}[[z]]$ such that

$|v| = s, \quad s_{11}(z)v = af(z)v, \quad s_{22}(z)v = bg(z)v, \quad s_{12}(z)v = s_{21}(z)v = 0.$

It follows from Theorem 3.12 that $X^{+}_{1,n}v = 0 = X^{-}_{1,n+1}v$. Henceforth $K^+_{1}(z)(K^+_{2}(z))^{-1}v \in \mathbb{C}^x v$. In other words, $f(z) = g(z)$. In summary, there are three types of one-dimensional $Y_q(\mathfrak{g})$-modules: $\mathbb{C}_s, \mathbb{C}_{(a,b)}, \mathbb{C}_f$ where $s \in \mathbb{Z}_2, (a,b) \in (\mathbb{C}^\times)^2$ and $f \in 1 + z \mathbb{C}[[z]]$. All one-dimensional $Y_q(\mathfrak{g})$-modules factorize uniquely into tensor products $\mathbb{C}_s \otimes \mathbb{C}_{(a,b)} \otimes \mathbb{C}_f$.

5.1.2. Evaluation modules in $\mathcal{F}$. Following [Ts12], let us define $\hat{\mathcal{U}}_q(\mathfrak{g})$ to be the superalgebra generated by $\hat{s}_{ij}, \hat{t}_{ji}, \hat{s}_{ii}^{-1}$ for $1 \leq i \leq j \leq 2$, with $\mathbb{Z}_2$-degrees and defining relations the same as those for $\mathcal{U}_q(\mathfrak{g})$ in 3.3.4 except the last relation which is replaced by $\hat{s}_{ii}\hat{s}_{ii}^{-1} = 1 = \hat{s}_{ii}^{-1}\hat{s}_{ii}$.

In particular, the $\hat{t}_{ii}$ are not required to be invertible. From the proof of Proposition 3.8 we see that there are well-defined evaluation morphisms $ev_a : Y_q(\mathfrak{g}) \rightarrow \hat{\mathcal{U}}_q(\mathfrak{g})$, $s_{ij}(z) \mapsto \hat{s}_{ij} - za\hat{t}_{ij}$. As usual, we understand that $\hat{s}_{ji} = 0 = \hat{t}_{ij}$ when $i < j$. Clearly $\hat{\mathcal{U}}_q(\mathfrak{g})$ is $\mathbb{Q}$-graded with respect to the conjugate actions of the $\hat{s}_{ii}$. Let us write down the defining relations of $\hat{\mathcal{U}}_q(\mathfrak{g})$:

$|\hat{s}_{ii}|_\mathbb{Q} = |\hat{t}_{ii}|_\mathbb{Q} = 0, \quad |\hat{s}_{ij}|_\mathbb{Q} = \epsilon_i - \epsilon_j = -|\hat{t}_{ji}|_\mathbb{Q},$

$\hat{s}_{12}^2 = 0 = \hat{t}_{21}^2, \quad \hat{s}_{12}\hat{t}_{21} + \hat{t}_{21}\hat{s}_{12} = (q-q^{-1})(\hat{t}_{11}\hat{s}_{22} - \hat{s}_{11}\hat{t}_{22}).$

From the above presentation of $\mathcal{U}_q(\mathfrak{g})$ and from the evaluation morphisms, it is easy to build up explicit representations for $Y_q(\mathfrak{g})$.

Let $a \in \mathbb{C}^\times$. We shall define two evaluation representations $\rho^\pm_a$ of $Y_q(\mathfrak{g})$ on the vector superspace $V = \mathbb{C}v_1 \oplus \mathbb{C}v_2$. It is enough to give their generating matrices $[\rho^\pm_a] := (\rho^\pm_a(s_{ij}(z)))_{1 \leq i, j \leq 2}$ with respect to the standard basis $(v_1, v_2)$. More precisely,

$[\rho^+_a] = \begin{pmatrix} (1-Za)E_{11} + (q^{-1} - za)E_{22} & (q-q^{-1})E_{12} \\ -zaE_{21} & E_{11} + q^{-1}E_{22} \end{pmatrix},$

$[\rho^-_a] = \begin{pmatrix} E_{11} + q^{-1}E_{22} & (q^{-1} - q)E_{12} \\ -zaE_{21} & (1-Za)E_{11} + (q^{-1} - za)E_{22} \end{pmatrix}.$

Let $L^\pm_{1,a}$ be the $Y_q(\mathfrak{g})$-modules associated with the representations $\rho^\pm_a$. 

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5.1.3. **Classification of simple objects in \( \mathcal{F} \).** Finite-dimensional simple \( Y_q(\mathfrak{g}) \)-modules are classified in terms of highest \( \ell \)-weights in the following way.

**Lemma 5.1.**  
1. A finite-dimensional simple \( Y_q(\mathfrak{g}) \)-module must be of highest \( \ell \)-weight.  
2. Let \( S \) be a simple \( Y_q(\mathfrak{g}) \)-module generated by a highest \( \ell \)-weight vector \( v \) with 
\[
|v| = 0, \quad s_{ii}(z)v = f_i(z)v, \quad f_i(z) \in (\mathbb{C}[\!(z)\!])^\times \quad \text{for} \ i = 1, 2.
\]
Then \( S \) is finite-dimensional if and only if 
\[
\frac{f_i(z)}{f_j(z)} = \frac{P(z)}{Q(z)} \quad \text{for some polynomials} \quad P(z), Q(z) \in \mathbb{C}[z] \quad \text{with non-zero constant terms.}
\]

**Proof.** The proof of Part (1) is the same as that of [Zh13, Lemma 4.12], by considering the action of the \( s_{ii}^{(0)}, (s_{ii}^{(0)})^{-1} \). For Part (2), “only if” comes from Theorem 3.12 and [HJ12, Lemma 3.9]. For the “if” part, write
\[
P(z) = a \prod_{i=1}^{m} (1 - zc_i), \quad Q(z) = b \prod_{j=1}^{n} (1 - zd_j), \quad c_i, d_j, a, b \in \mathbb{C}^\times.
\]
Then \( S \) is a sub-quotient of the tensor product
\[
(\bigotimes_{i=1}^{m} L_{1,c_i}^+) \otimes (\bigotimes_{j=1}^{n} L_{d_j}^-) \otimes \mathbb{C}(f_1(0),f_2(0)) \otimes \mathbb{C} f',
\]
where \( f'(z) = f_1(z)f_1(0)^{-1} \prod_{i=1}^{m}(1 - zc_i)^{-1} \). As the \( L_{i,a}^\pm \) are always two-dimensional, \( S \) must be finite-dimensional. \( \square \)

Let us define \( \mathbb{R} \) to be the subset of \( (\mathbb{C}[\!(z)\!])^\times \) consisting of power series of the form \( P(z)Q(z)^{-1} \) with \( P(z), Q(z) \in 1 + z\mathbb{C}[z] \). Identically, \( \mathbb{R} \) is the set of rational functions \( f(z) \in \mathbb{C}(z) \) such that \( f(0) = 1 \). Here we view a rational function as a meromorphic function \( \mathbb{C} \cup \{ \infty \} \rightarrow \mathbb{C} \cup \{ \infty \} \). For such \( f \), let \( V(f) \) be the simple \( Y_q(\mathfrak{g}) \)-module generated by a highest \( \ell \)-weight vector \( v \) satisfying
\[
|v| = 0, \quad s_{11}(z)v = f(z)v, \quad s_{22}(z)v = v.
\]
For example, when \( a \in \mathbb{C}^\times \),
\[
V(1 - za) \cong L_{1,a}^+, \quad V(\frac{1}{1 - za}) \cong L_{1,a}^- \otimes \mathbb{C} \frac{1}{1 - za}.
\]
According to Lemma 5.1, \( V(f) \) is finite-dimensional. Moreover, all finite-dimensional simple \( Y_q(\mathfrak{g}) \)-modules can be factorized uniquely into \( V(f) \otimes D \) with \( D \) one-dimensional and \( f \in \mathbb{R} \).

5.2. **Tensor product of simple modules.** For \( f \in \mathbb{R} \), let \( Z(f) \) (resp. \( P(f) \)) be the set of zeros (resp. poles) of the meromorphic function \( f \). It is possible that \( \infty \in Z(f) \cup P(f) \). The main result of this section can be stated as follows.

**Theorem 5.2.** Let \( f_1, f_2, \ldots, f_s \in \mathbb{R} \). For \( 1 \leq i \leq s \), let \( v_i \) be a highest \( \ell \)-weight vector in the simple \( Y_q(\mathfrak{g}) \)-module \( V(f_i) \). Let \( V := \bigotimes_{i=1}^{s} V(f_i) \) and \( v := \bigotimes_{i=1}^{s} v_i \in V \). Then
\begin{itemize}
  \item[(a)] \( V = Y_q(\mathfrak{g})v \) if and only if \( P(f_i) \cap Z(f_j) = \emptyset \) for all \( 1 \leq i < j \leq s \);
  \item[(b)] \( Y_q(\mathfrak{g})v \) is the unique simple sub-\( Y_q(\mathfrak{g}) \)-module of \( V \) if and only if \( Z(f_i) \cap P(f_j) = \emptyset \) for all \( 1 \leq i < j \leq s \);
  \item[(c)] \( V \) is simple if and only if \( P(f_i) \cap Z(f_j) = \emptyset \) for all \( 1 \leq i \neq j \leq s \).
\end{itemize}
Remark 5.3. (1) The theorem above can be viewed as a super analogue of [CP94] Theorems 3.4.4.8 on classification and construction of finite-dimensional simple $U_q(\hat{\mathfrak{sl}}_2)$-modules in terms of Drinfeld polynomials. See [MY14, Theorem 4.6] for a closer statement involving rational functions instead of Drinfeld polynomials.

(2) Let $a_n \in \mathbb{C}^\times$ be given for $n \in \mathbb{Z}_{\geq 0}$. Suppose $a_n \neq a_m$ whenever $n \neq m$. Then for $n \in \mathbb{Z}_{> 0}$ the tensor product of $Y_q(g)$-modules

$$W_n := (\bigotimes_{i=0}^{n-1} V(1 - za_{i+1}) \otimes V(1 - za_n))$$

is of highest $\ell$-weight, and its simple quotient is isomorphic to $V(1 - za_0)$. Hence given a finite-dimensional simple module $S$, we find infinitely many finite-dimensional highest $\ell$-weight modules whose simple quotients are isomorphic to $S$, and the dimensions of these modules can be arbitrarily large. This gives a clue on the Weyl modules defined in [Zh13, §4.1] for the quantum loop superalgebra $U_q(L\mathfrak{sl}(M,N))$.

The proof of Theorem 5.2 will be given in §5.3.

5.2.1. Factorization into prime simple modules. Let $f \in \mathbb{R}$. Write $f(z) = \frac{N(z)}{D(z)}$ where

$$N(z) = \prod_{i=1}^{s} (1 - za_i), \quad D(z) = \prod_{i=1}^{t} (1 - zb_i)$$

such that $a_i, b_i \in \mathbb{C}^\times$ and $a_i \neq b_j$ for $1 \leq i \leq s, 1 \leq j \leq t$. Then

$$V(f) \cong \bigotimes_{i=1}^{s} V\left(\frac{1 - za_i}{1 - zb_i}\right) \quad \text{if } s = t,$$

$$V(f) \cong \left(\bigotimes_{i=1}^{t} V\left(\frac{1 - za_i}{1 - zb_i}\right) \otimes \bigotimes_{j=t+1}^{s} V(1 - za_j)\right) \quad \text{if } s > t,$$

$$V(f) \cong \left(\bigotimes_{i=1}^{s} V\left(\frac{1 - za_i}{1 - zb_i}\right) \otimes \bigotimes_{j=s+1}^{t} V\left(\frac{1}{1 - zb_j}\right)\right) \quad \text{if } s < t.$$}

According to Theorem 5.2, these are factorizations of simple modules into prime simple modules. Here by a **prime** simple module we mean a simple module $S$ which cannot be written as $S_1 \otimes S_2$ with $S_i$ being modules of dimension $> 1$ [HL10, §2.2].

5.2.2. Constructions of prime simple modules. We have seen in §5.1.2 the explicit formulas for $V(1 - za)$ and $V\left(\frac{1}{1 - za}\right)$. There still remains the third kind of prime simple modules, namely $V\left(\frac{1 - za}{1 - zb}\right)$ for $a, b \in \mathbb{C}^\times$ and $a \neq b$. Indeed, it is easy to check the following without using Theorem 5.2(2): the tensor product of highest $\ell$-weight vectors in $V(1 - za) \otimes V\left(\frac{1}{1 - zb}\right)$ generates the unique simple sub-$Y_q(g)$-module, which is two-dimensional and isomorphic to $V\left(\frac{1 - za}{1 - zb}\right)$. Let $\rho_{a,b}$ be the corresponding representation of $Y_q(g)$ on $V$. After some base...
change the generating matrix becomes

\[
[\rho_{a,b}] = \begin{pmatrix}
\frac{1-za}{1-zb}E_{11} + q^{-1}q^{-1} \frac{a-b}{1-zb} E_{22} & \frac{(a-b)(1-b)}{1-zb} E_{12} \\
\frac{z}{1-zb}E_{21} & \frac{1}{1-zb}E_{11} + q^{-1}q^{-1} \frac{a-b}{1-zb} E_{22}
\end{pmatrix}.
\]

Remark that the matrix \([\rho_{a,b}]\) is well-defined even if \(ab = 0\). In particular, for \(a \in \mathbb{C}^x\), \([\rho_{a,0}]\) (resp. \([\rho_{0,a}]\)) is a generating matrix for the representation associated to \(V(1 - za)\) (resp. to \(V(\frac{1}{1-z}1)\)). Hence all the prime simple modules are built upon the vector superspace \(V\).

5.2.3. **Duals of prime simple modules.** Fix \(a, b \in \mathbb{C}\) such that \(a \neq b\). Let \(\rho_{a,b}\) be the representation of \(Y_q(\mathfrak{g})\) on \(V\) as in the preceding paragraph. Let \([\rho_{a,b}]\) be its generating matrix with respect to the standard basis \((v_1, v_2)\). Since \(Y_q(\mathfrak{g})\) is a Hopf superalgebra, there exists naturally a representation \(\rho_{a,b}^*\) of \(Y_q(\mathfrak{g})\) on \(V^*\) defined by:

\[
\rho_{a,b}^*(x) := (\rho_{a,b}(\mathbb{S}x))^* \quad \text{for } x \in Y_q(\mathfrak{g}).
\]

Here we adopt the notations in the proof of Lemma 3.3. Let \((v_1^*, v_2^*)\) be the dual basis of \(V^*\) with respect to \((v_1, v_2)\). Let \(e_{ij} \in \text{End}V^*\) be such that \(e_{ij}v_k^* = \delta_{jk}v_i^*\). Then

\[
E_{ii}^* = e_{ii}, \quad E_{12}^* = e_{21}, \quad E_{21}^* = -e_{12}.
\]

Let us compute the generating matrix of \(\rho_{a,b}^*\) with respect to the basis \((v_1^*, v_2^*)\). By definition, \([\rho_{a,b}^*]_{ij} = \rho_{a,b}(\mathbb{S}(s_{ij}(z)))^*\). On the other hand, in view of Equation (3.16),

\[
[\rho_{a,b}(\mathbb{S}(s_{ij}(z)))]_{1 \leq i, j \leq 2} = [\rho_{a,b}]^{-1}
\]

The matrices above should be seen as matrices over the superalgebra \(\text{End}V\). A direct calculation indicates:

\[
[\rho_{a,b}]^{-1} = \begin{pmatrix}
q^{-1}q^{-1} & \frac{(a-b)(1-b)}{q^{-1}1} \\
\frac{z}{q^{-1}1} & \frac{1}{1-z} + q^{-1}q^{-1} \frac{a-b}{q^{-1}1}
\end{pmatrix},
\]

from which we obtain the generating matrix of \(\rho_{a,b}^*\) with respect to the basis \((v_1^*, v_2^*)\) of \(V^*:\)

\[
[\rho_{a,b}]^* = \frac{1 - za}{q^{-1}1} \begin{pmatrix}
\frac{q^{-1}q^{-1}1}{1-z} & \frac{(a-b)(1-b)}{1-z} \\
\frac{z}{1-z} & \frac{1}{1-z} + q^{-1}q^{-1} \frac{a-b}{1-z}
\end{pmatrix} \approx \frac{1 - za}{q^{-1}1} \quad \text{[\rho_{a,b}].}
\]

In the above equation, \(\approx\) means that the two matrices on both sides are of the same form. They are by no means in the same superalgebra. In conclusion, as \(Y_q(\mathfrak{g})\)-modules:

\[
V(\frac{1 - za}{1 - zb})^* \cong \mathbb{C}_\mathfrak{g} \otimes \mathbb{C}(q, q) \otimes \mathbb{C}_\mathfrak{g} \otimes V(\frac{1 - zb}{1 - za}),
\]

5.3. **Proof of Theorem 5.2** Note that (c) follows directly from (a) and (b).

5.3.1. **Tensor products of prime simple modules.** Let us prove (a) and (b) under the condition that the \(f_i \in \mathbb{R}\) are of the form \(f_i(z) = \frac{1 - za}{1 - zb}\), where \(a_i, b_i \in \mathbb{C}\) and \(a_i \neq b_i\). In this case, \(P(f_i) \cap Z(f_j) = \emptyset\) if and only if \(b_i \neq a_j\). Moreover, the \(V(f_i)\) are always two-dimensional, and \(V(f_i)^* \cong V(f_i^{-1}) \otimes D_i\) for some one-dimensional module \(D_i\). By definition of the dual modules, (b) is equivalent to the following statement:

(b1) The tensor product \(\bigotimes_{i=1}^{s} V(f_i)\) is of lowest \(\ell\)-weight if and only if \(a_i \neq b_j\) for \(1 \leq i < j \leq s\).
Let us prove (a). For \(1 \leq i \leq s\), let \(u_i^+\) (resp. \(u_i^-\)) be a highest (resp. lowest) \(\ell\)-weight vector in \(V(f_i)\). Then from the explicit realization of \(V(f_i)\) we see that

\[
|u_i^+| = \overline{0}, \quad s_{11}(z)u_i^+ = \frac{1-za_i}{1-zb_i}u_i^+, \quad s_{22}(z)u_i^+ = u_i^+, \quad s_{21}(z)u_i^- = \frac{z\lambda_i}{1-zb_i}u_i^-.
\]

where \(\lambda_i \in \mathbb{C}^\times\). We remark that Lemma 4.3 still holds when replacing \(U_q(\mathfrak{g})\)-modules by \(Y_q(\mathfrak{g})\)-modules. Indeed, if \(W\) is a highest \(\ell\)-weight \(Y_q(\mathfrak{g})\)-module with \(w\) a highest \(\ell\)-weight vector, then by Theorem 3.12 we see that \(W\) is spanned by vectors of the form \(X_{-1}^r \cdot \ldots \cdot X_{-1}^r u\) where \(r \in \mathbb{Z}_{\geq 0}\) and \(n_i \in \mathbb{Z}_{\geq 1}\) for \(1 \leq i \leq r\). Hence the proof of Lemma 4.3 goes perfectly for \(Y_q(\mathfrak{g})\)-modules.

Let \(V := \bigotimes_{i=1}^s V(f_i)\) and \(u := \bigotimes_{i=1}^s u_i^+\). Via the action of the \(s_{ii}^{(0)}\), \(V\) and the \(V(f_i)\) are \(\mathbb{Q}\)-graded:

\[
(V)_\lambda := \{x \in V|s_{ii}^{(0)} x = q^{|(e_i, \lambda)} x \text{ for } i = 1, 2\}.
\]

As \(|u_i^+|_Q = 0, |u_i^-|_Q = -\alpha_1\), we see that: \(|u|_Q = 0; (V)_\lambda \neq 0\) if and only if \(\lambda = -t\alpha_1\) for some \(0 \leq t \leq s\); \(\dim(V)_{-\alpha_1} = \binom{s}{t}\). In particular, \((V)_{-\alpha_1}\) is generated by the vectors \(w_j := \bigotimes_{i=1}^{j-1} u_i^+ \otimes u_i^- \otimes \bigotimes_{j=i+1}^s u_j^+\) for \(1 \leq j \leq s\).

On the other hand, set \(V' := Y_q(\mathfrak{g})u\). As a highest \(\ell\)-weight module, \(V'\) is \(\mathbb{Q}\)-homogeneous. Moreover, from Theorem 3.12 we see that

\[
(V')_{-\alpha_1} = \sum_{n \in \mathbb{Z}_{\geq 1}} \mathbb{C} X_{-1}^n u = \sum_{n \in \mathbb{Z}_{\geq 1}} \mathbb{C} s_{21}^{(n)} u.
\]

In other words, \((V')_{-\alpha_1}\) is generated by the coefficients of \(s_{21}(z)u \in V[[z]]\).

Suppose first that \(V = V'\) is of highest \(\ell\)-weight. Then the coefficients of \(s_{21}(z)u\) generate an \(s\)-dimensional subspace.

\[
s_{21}(z)u = \sum_{i=1}^s \prod_{j=1}^{i-1} s_{22}(z)u_j^+ \otimes s_{21}(z)u_i^+ \otimes \bigotimes_{j=i+1}^s s_{11}(z)u_j^+ = \sum_{i=1}^s \frac{z\lambda_i}{1-zb_i} \prod_{j=i+1}^s \frac{1-za_j}{1-zb_j} \bigotimes_{j=1}^{i-1} u_j^+ \otimes u_i^- \otimes \bigotimes_{j=i+1}^s u_j^+ = \frac{z}{\prod_{i=1}^s (1-zb_i)} \sum_{i=1}^s \lambda_i g_i(z) w_i.
\]

Here the \(g_i(z) \in \mathbb{C}[z]\) are defined by

\[
g_i(z) = \prod_{j=1}^{i-1} (1-zb_j) \prod_{j=i+1}^s (1-za_j).
\]

It follows that the polynomials \(g_i(z) \in \mathbb{C}[z]\) must be linearly independent. In view of Lemma 5.4 below, we must have \(b_i \neq a_j\) for \(1 \leq i < j \leq s\), as desired.
Next suppose that $b_i \neq a_j$ for $1 \leq i < j \leq s$. We show by induction on $s$ that $V$ is of highest $\ell$-weight. For $s = 1$ this is evident. Assume $s > 1$. Then we can assume furthermore that $\otimes_{i=2}^s V(f_i)$ is of highest $\ell$-weight. Now Lemma 4.5 says that

$$V = Y_q(\mathfrak{g})w_1.$$ 

Since $b_i \neq a_j$ for $1 \leq i < j \leq s$, the polynomials $g_i(z)$ are linearly independent (Lemma 5.4). Hence the coefficients of $s_{21}(z)u$ generate an $s$-dimensional subspace. It follows that $w_1 \in V'$. Hence $V = V'$ is of highest $\ell$-weight.

**Lemma 5.4.** Let $k \in \mathbb{Z}_{>0}$. Let $a_i, a'_i \in \mathbb{C}$ be given for $1 \leq i \leq k$. For $1 \leq j \leq k$, define

$$f_j(z) := \left(\prod_{i=1}^{j-1}(1 - za_i)\right) \left(\prod_{i=j+1}^k(1 - za'_i)\right) \in \mathbb{C}[z].$$

Then the $f_j(z)$ are linearly independent if and only if $a_i \neq a'_i$ for all $1 \leq i < j \leq k$.

**Proof.** The $k$ polynomials $f_j(z)$ are of degree $\leq k - 1$. Introduce

$$f_1(z) \wedge f_2(z) \wedge \cdots \wedge f_k(z) = \Delta(1 \wedge z \wedge \cdots \wedge z^{k-1}) \in \wedge^k \mathbb{C}[z].$$

Then the $f_j(z)$ are linearly independent if and only if $\Delta \neq 0$. For $j + s \leq k$, take

$$f_j^{(s)}(z) = \left(\prod_{i=1}^{j-1}(1 - za_i)\right) \left(\prod_{i=j+s+1}^k(1 - za'_i)\right).$$

Then $f_j^{(0)}(z) = f_j(z)$ and

$$f_j^{(s)}(z) - f_{j+1}^{(s)}(z) = (a_i - a'_i)zf_j^{(s+1)}(z)$$

for $i + s + 1 \leq k$. Take $\omega = 1 \wedge z \wedge \cdots \wedge z^{k-1}$. We have

$$\Delta \omega = \bigwedge_{i=1}^k f_i^{(0)}(z) = \left(\bigwedge_{i=1}^k f_i^{(0)}(z) - f_i^{(0)}_{i+1}(z)\right) \wedge f_k(z) = \left(\bigwedge_{i=1}^k (a_i - a'_i)zf_j^{(1)}(z)\right) \wedge f_k^{(0)}(z)$$

$$= \left(\prod_{i=1}^{k-1} (a_i - a'_i + 1)\right) \left(\bigwedge_{i=1}^{k-1} zf_i^{(1)}(z)\right) \wedge 1$$

$$= \left(\prod_{i=1}^{k-1} (a'_i - a_i)\right) \left(\bigwedge_{i=1}^{k-1} (a_i - a'_i + 1)\right) \left(\bigwedge_{i=1}^{k-2} zf_i^{(1)}(z)\right) \wedge zf_{k-1}^{(1)}(z)$$

$$= \left(\prod_{i=1}^{k-2} (a'_i - a_i)\right) \left(\bigwedge_{i=1}^{k-2} (a_i - a'_i + 1)\right) \left(\bigwedge_{i=1}^{k-2} zf_i^{(2)}(z)\right) \wedge z$$

$$= \left(\prod_{i=1}^{k-2} (a'_i - a_i)\right) \left(\bigwedge_{i=1}^{k-2} (a'_i - a_i + 1)\right) \left(\bigwedge_{i=1}^{k-2} z^2 f_i^{(2)}(z)\right) \wedge z$$

$$= \cdots = \prod_{1 \leq i < j \leq k} (a'_j - a_i)\omega.$$ 

Clearly $\Delta \neq 0$ if and only if $a_i \neq a'_j$ for $1 \leq i < j \leq k$. \qed
This ends the proof of Theorem \[5.2\] (a) in the case where the \(f_i(z)\) are of the form \(\frac{1-z^a}{1-z^b}\) with \(a_i, b_i \in \mathbb{C}\) and \(a_i \neq b_i\). Similar arguments lead to (b) by considering lowest \(\ell\)-weight vectors and by developing the series \(s_{12}(z)(\bigotimes_{i=1}^r u_i^-) \in V[[z]]\).

5.3.2. **End of proof.** In general, given \(f \in \mathbb{R}\), one can find a decomposition (not necessarily unique) \(f = \prod_{i=1}^d f^{(i)}\) such that: 
\[
f^{(i)} = \frac{1-z^a}{1-z^b} \quad \text{where } a_i, b_i \in \mathbb{C} \text{ and } a_i \neq b_i; \quad a_i \neq b_j \text{ for } 1 \leq i \neq j \leq d.
\]
It follows that (5.3.1)
\[
V(f) \cong \bigotimes_{i=1}^d V(f^{(i)}), \quad P(f) = \bigcup_{i=1}^d P(f^{(i)}), \quad Z(f) = \bigcup_{i=1}^d Z(f^{(i)}).
\]
Hence \(V(f)^* \cong V(f^{-1}) \otimes D_f\) with \(D_f\) a one-dimensional module. (b) is equivalent to

(b2) The tensor product \(\bigotimes_{i=1}^s V(f_i)\) is of lowest \(\ell\)-weight if and only if \(Z(f_i) \cap P(f_j) = \emptyset\) for all \(1 \leq i < j \leq s\).

Now (a), (b2) follow easily from the factorization of simple modules and from the special case (5.3.1) where the \(f_i\) are of the form \(\frac{1-z^a}{1-z^b}\). \(\square\)

As we see in Theorem \[5.2\] the conditions for a tensor product of finite-dimensional simple \(Y_q(g)-\)modules to be of highest \(\ell\)-weight and to be of lowest \(\ell\)-weight respectively are in general different, which is quite contrary to the non-graded case, where these two conditions are the same due to the Weyl group action.

6. PROOF OF THEOREM \[4.2\]

The whole section is devoted to the proof of Theorem \[4.2\] The outline is as follows. In view of Remark \[4.3\] one can assume \(1 \leq r \leq M\). In particular, \(M > 0\). Next, by the following induction argument from \(U_q(gl(M, N))\)-modules to \(U_q(gl(M, N+1))\)-modules, we can assume furthermore \(N > 0\). Then we shall prove the theorem by induction on \(r\). For the initial step \(r = 1\), Theorem \[4.2\] is a special case of Proposition \[4.7\].

Throughout the proof, we use the following convention. Let \(f: A \to B\) be a morphism of superalgebras. Let \(V\) be a \(B\)-module. Suppose that \(W\) is a sub-vector-superspace of \(V\) stable under the action of \(f(A)\). We write \(f^*W\) as the sub-\(A\)-module of \(f^*V\) induced by the action of \(f(A)\) on \(W\). (\(f^*W\) has no sense!)

6.1. **Induction.** Let \(g' = gl(M, N)\) and \(g'' = gl(M, N+1)\). Let \(h: U_q(g') \to U_q(g'')\) be the superalgebra morphism defined by \(s_{ij}(z) \mapsto s_{ij}(z), t_{ij}(z) \mapsto t_{ij}(z)\) for \(1 \leq i, j \leq M+N\). Let \(1 \leq r \leq M, k \in \mathbb{Z}_{>0}\) and \(a_j \in \mathbb{C}^x\) for \(1 \leq j \leq k\). \(v_j\) is a highest \(\ell\)-weight vector in \(ev_{a_j}^* L(\varpi_r; g'')\). Here we view \(\varpi_r\) as a weight associated to the Lie superalgebra \(g''\) and \(L(\varpi_r; g'')\) as a simple highest weight \(U_q(g'')\)-module of highest weight \(\varpi_r\). Define
\[
K(a_j) := h(U_q(g'))v_j \subseteq ev_{a_j}^* L(\varpi_r; g'').
\]

Then \(\bigotimes_{j=1}^k K(a_j)\) is a sub-\(U_q(g')\)-module of \(h^*(\bigotimes_{j=1}^k ev_{a_j}^* L(\varpi_r; g''))\). Moreover
\[
h^*(\bigotimes_{j=1}^k K(a_j)) \cong \bigotimes_{j=1}^k ev_{a_j}^* L(\varpi_r; g')
\]
as $U_q(\hat{g}^0)$-modules. If the tensor product $\bigotimes_{j=1}^k \ev_{a_j}^k \otimes L(\pi_{\tau}; g^0)$ of $U_q(\hat{g}^0)$-modules is of highest $\ell$-weight, then so is the corresponding tensor product of $U_q(\hat{g}^0)$-modules.

Assume in the rest of the section $N > 0$. Let $U$ be the $q$-Yangian $Y_q(\mathfrak{gl}(1,1))$ as in \[5\]

6.2. **Proof of Proposition 4.7.** The idea is the same as that of the proof of Theorem 5.2. As before, we prove only the highest $\ell$-weight part.

We adopt the notations of \[4.4\]. Let $V := \bigotimes_{j=1}^k V(a_j)$. Let $v := v_1^{\otimes k} \in V$. As in \[5.3.1\], $V$ is $P$-graded via the action of the $s^{(0)}_{ii}$.

First suppose that $V$ is of highest $\ell$-weight. Then from Theorem 3.12 it follows that

$$(V)_{k \ell_1 - \alpha_1} = \sum_{n \geq 1} C s_{21}^{(n)} v.$$

As in the proof of Theorem 5.2 we get an explicit expression of $s_{21}(z)v$, which implies that the following polynomials

$$f_j(z) = \left( \prod_{i=1}^{j-1} (1 - za_i) \right) \left( \prod_{i=j+1}^{k} (1 - za_i q^{-2}) \right)$$

for $1 \leq j \leq k$ are linearly independent. In view of Lemma 5.3 this says that $a_i \neq a_j q^{-2}$ for $1 \leq i < j \leq k$.

Next, assume that $a_i \neq a_j q^{-2}$ for $1 \leq i < j \leq k$. By induction on $k$, one can suppose that $\bigotimes_{j=2}^k V(a_i)$ is of highest $\ell$-weight. Note that $V(a_1)$ is a lowest $\ell$-weight $U_q(\hat{g}^0)$-module with $v_{M+N}$ a lowest $\ell$-weight vector. It is enough to verify that (Lemma 4.5)

$$v_{M+N} \otimes v_1^{\otimes k-1} \in U_q(\hat{g})v.$$

For $1 \leq j \leq k$, let $K(a_j)$ be the subspace of $V(a_j)$ spanned by $v_1, v_{M+N}$. According to the ice rule Proposition 3.10 (PS3), there exists a morphism of superalgebras $f : U \to U_q(\hat{g})$

$$s_{11}(z) \mapsto s_{11}(z), \quad s_{12}(z) \mapsto s_{1,M+N}(z), \quad s_{21}(z) \mapsto s_{M+N,1}(z), \quad s_{22}(z) \mapsto s_{M+N,M+N}(z),$$

From weight gradings on $V(a_j)$ and $V$, it follows that: the $K(a_j)$ are stable by $U$; the tensor product $\bigotimes_{j=1}^k K(a_j)$ is stable by $U$; as $U$-modules

$$f^\bullet(\bigotimes_{j=1}^k K(a_j)) \cong \bigotimes_{j=1}^k f^\bullet(K(a_j)).$$

Here the RHS should be understood as a tensor product of $U$-modules. From the explicit formula of the action of the $s_{ij}(z)$ on $V(a)$ defined in \[4.4\] we see that

$$f^\bullet(K(a_j)) \cong C_{(q,1)} \otimes C_{1-za_j} \otimes V(\frac{1 - za_j q^{-2}}{1 - za_j})$$

as $U$-modules. Now Theorem 5.2 (a) implies immediately that $\bigotimes_{j=1}^k f^\bullet(K(a_j))$ is of highest $\ell$-weight. In particular,

$$v_{M+N} \otimes v_1^{\otimes k-1} \in U f^\bullet(v_1^{\otimes k}) \subseteq U_q(\hat{g})v.$$

Hence, $V$ is of highest $\ell$-weight, as desired. $\square$
The initial step \( r = 1 \) for the induction argument on \( 1 \leq r \leq M \) has been established. Now suppose that \( r > 1 \). Let us consider the \( U_q(\mathfrak{g}) \)-module \( W_{1,a}^{(r)} \).

6.3. Weight grading on \( W_{1,a}^{(r)} \). Fix \( a \in \mathbb{C}^* \). The \( U_q(\mathfrak{g}) \)-module \( W_{1,a}^{(r)} = \text{ev}_q^*L(\varpi_r) \) is \( \mathbf{P} \)-graded under the action of the \( s_{ii}^{(0)} \). By Theorem 2.21, \( (W_{1,a}^{(r)})_\lambda \) is non-zero if and only if

\[
\lambda = \epsilon_i + \epsilon_{i_2} + \cdots + \epsilon_{i_r}
\]

where \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq M + N \) and \( i_s < i_{s+1} \) if \( i_s \leq M \). Moreover, for such \( \lambda \), \( (W_{1,a}^{(r)})_\lambda \) is always one-dimensional, and for \( x \in (W_{1,a}^{(r)})_\lambda \),

\[
s_{ii}(z)x = (q^{\lambda \epsilon_i} - zaq^{-\lambda \epsilon_i}) x, \quad t_{ii}(z)x = (q^{-\lambda \epsilon_i} - z^{-1}a^{-1}q^{\lambda \epsilon_i}) x.
\]

Let \( v_a^+ \) (resp. \( v_a^- \)) be a highest (resp. lowest) \( \ell \)-weight vector in \( W_{1,a}^{(r)} \). Then

\[
v_a^+ \in (W_{1,a}^{(r)})_{\varpi_r}, \quad v_a^- \in (W_{1,a}^{(r)})_{rM+N}.
\]

Introduce \( u_a^\pm \in W_{1,a}^{(r)} \)

\[
u_a^+ = s_{1,M+N}^{(0)} v_a^-, \quad u_a^- = t_{M+N,1}^{(0)} v_a^+.
\]

Then from the following Chevalley relation we get \( u_a^+ \neq 0 \),

\[
s_{1,M+N}^{(0)} t_{M+N,1}^{(0)} + t_{M+N,1}^{(0)} s_{1,M+N}^{(0)} = (q - q^{-1}) (t_{11}^{(0)} s_{M+N,M+N}^{(0)} - s_{11}^{(0)} t_{M+N,M+N}^{(0)}).
\]

Here we used the assumption that \( N > 0 \). In particular,

\[
\mathbb{C} u_a^+ = (W_{1,a}^{(r)})_{\epsilon_1 + (r-1)\epsilon_M + N}, \quad \mathbb{C} u_a^- = (W_{1,a}^{(r)})_{\epsilon_2 + \cdots + \epsilon_r + \epsilon_M + N}.
\]

Introduce vector subspaces \( K^+(a) = \mathbb{C} v_a^+ \cup \mathbb{C} u_a^- \), \( K^-(a) = \mathbb{C} v_a^- \cup \mathbb{C} u_a^+ \subseteq W_{1,a}^{(r)} \). The \( \mathbf{P} \)-grading on \( W_{1,a}^{(r)} \) says that the subspaces \( K^\pm(a) \) are both sub-\( U \)-module of \( f^*W_{1,a}^{(r)} \). Let \( f^*K^\pm(a) \) be the \( U \)-modules thus obtained.

Claim. Let \( k \in \mathbb{Z}_{>1} \) and \( a_j \in \mathbb{C}^* \) for \( 1 \leq j \leq k \). Then we have the following:

1. \( \bigotimes_{j=2}^{k} K^+(a_j) \) is a sub-\( U \)-module of \( f^*\bigotimes_{j=2}^{k} W_{1,a_j}^{(r)} \);
2. \( K^-(a_1) \otimes \bigotimes_{j=2}^{k} K^+(a_j) \) is a sub-\( U \)-module of \( f^*\bigotimes_{j=1}^{k} W_{1,a_j}^{(r)} \);
3. as \( U \)-modules, there exists a canonical isomorphism

\[
f^*(K^-(a_1) \otimes \bigotimes_{j=2}^{k} K^+(a_j)) \cong f^*K^-(a_1) \otimes \bigotimes_{j=2}^{k} f^*K^+(a_j).
\]

The proof of the claim relies on the following facts:

4. if \( s_{ii}(z)K^+(a) \neq 0 \) and \( i \in \{1, M + N\}, 1 < l < M + N \), then \( r < l < M + N \);
5. if \( i \neq l \) and \( r < l < M + N \), then \( s_{ii}(z)K^\pm(a) = 0 \).

These are checked directly using the \( \mathbf{P} \)-grading on \( W_{1,a}^{(r)} \).

Next, as \( U \)-modules, using notations in [5,1] we get

\[
f^*K^-(a) \cong \mathbb{C}_{(r-1)i} \otimes \mathbb{C}_{(q,q^1-r)} \otimes \mathbb{C}_{1-zaq^{2r-2}} \otimes V\left(\frac{1-zaq^{-2}}{1-zaq^{2r-2}}\right),
\]
6.4. **End of proof.** Let us be in the situation of Theorem 4.2 with $1 \leq r \leq M$. Write $a_j = a q^{r_j}$. We prove by induction on $k$ that the tensor product $V = \bigotimes_{j=1}^{k} W_{1,a_j}^{(r)}$ is of highest $\ell$-weight. Assume that the $U_q(\mathfrak{g})$-module $\bigotimes_{j=2}^{k} W_{1,a_j}^{(r)}$ is of highest $\ell$-weight. Then it is enough to ensure (Lemma 4.5)

\[ x := v_{a_1}^{-} \otimes \bigotimes_{j=2}^{k} v_{a_j}^{+} \in U_q(\mathfrak{g})(\bigotimes_{j=1}^{k} v_{a_j}^{+}). \]

Remark that by definition

\[ x = v_{a_1}^{-} \otimes \bigotimes_{j=2}^{k} v_{a_j}^{+} \in K^{-}(a_1) \otimes \bigotimes_{j=2}^{k} K^{+}(a_j) =: L_1. \]

The claim above says that $L_1$ is a sub-$U$-module of $V$. Moreover, as $U$-modules,

\[ f^{*} L_1 \cong D \otimes V \left( \frac{1 - za_1 q^{-2}}{1 - za_1 q^{2r_2 - 2}} \right) \otimes \bigotimes_{j=2}^{k} V \left( \frac{1 - za_j q^{-2}}{1 - za_j} \right), \]

\[ D \cong C_{(r-1)} \otimes C(q^k, q^{1-r}) \otimes C(1 - za_1 q^{2r_2 - 2}) \prod_{i=2}^{k} C(1 - za_1). \]

The RHS of the first equation above is of highest $\ell$-weight in view of Theorem 5.2 as $a_1 q^{2r_2 - 2} \neq a_j q^{-2}$ for $2 \leq j \leq k$ and $a_j \neq a_l q^{-2}$ for $2 \leq j < l \leq k$. It follows that

\[ x \in f(U)(u_{a_1}^{+} \otimes \bigotimes_{j=2}^{k} v_{a_j}^{+}) \subseteq U_q(\mathfrak{g})(u_{a_1}^{+} \otimes \bigotimes_{j=2}^{k} v_{a_j}^{+}). \]

We are left to verify in turn that

\[ y := u_{a_1}^{+} \otimes \bigotimes_{j=2}^{k} v_{a_j}^{+} \in U_q(\mathfrak{g})(\bigotimes_{j=1}^{k} v_{a_j}^{+}). \]

Take $U'$ to be the quantum affine superalgebra $U_q(\mathfrak{gl}(M-1, N))$. From the ice rule, we get a superalgebra morphism $g : U' \longrightarrow U_q(\mathfrak{g})$ defined by

\[ s_{ij}(z) \mapsto s_{i+1,j+1}(z), \quad t_{ij}(z) \mapsto t_{i+1,j+1}(z). \]

For $b \in \mathbb{C}^{\times}$, let $K(b) = g(U') v_{b}^{+} \subseteq W_{1,b}^{(r)}$. Clearly, $K(b)$ is a sub-$U'$-module of $g^{*} W_{1,b}^{(r)}$. Moreover,

\[ g^{*} K(b) \cong \text{ev}^{*}_{b} L(\varpi_{r-1}; \mathfrak{gl}(M-1, N)), \quad u_{b}^{+} \in K(b). \]

Now it is straightforward to verify: $\bigotimes_{j=1}^{k} K(a_j)$ is a sub-$U'$-module of $g^{*}(\bigotimes_{j=1}^{k} W_{1,a_j}^{(r)})$; there exist canonical isomorphisms of $U'$-modules

\[ g^{*}(\bigotimes_{j=1}^{k} K(a_j)) \cong \bigotimes_{j=1}^{k} g^{*} K(a_j) \cong \bigotimes_{j=1}^{k} \text{ev}_{a_j}^{*} L(\varpi_{r-1}; \mathfrak{gl}(M-1, N)). \]
The induction hypothesis on \( r \) (which keeps \( N \) unchanged) shows that the RHS above is of highest \( \ell \)-weight. Hence

\[
y = u_{a_1}^+ \otimes (\bigotimes_{j=2}^{k} v_{a_j}^+) \in g(U^r) (\bigotimes_{j=1}^{k} v_{a_j}^+) \subseteq U_q(\mathfrak{g}) \bigotimes_{j=1}^{k} v_{a_j}^+.
\]

This concludes the proof of Theorem 4.2. \( \Box \)

**Remark 6.1.** Let \( 1 \leq r \leq M, k \in \mathbb{Z}_{>0} \) and \( a_j \in \mathbb{C}^\times \) for \( 1 \leq j \leq k \). From the proof of Theorem 4.2 we see that the \( U_q(\mathfrak{g}) \)-module \( \bigotimes_{j=1}^{k} W_{l,a_j}^{(r_j)} \) is of highest \( \ell \)-weight provided that \( \frac{a_i}{a_j} \notin \{ q^{-2l} : 1 \leq l \leq r \} \) for all \( 1 \leq i < j \leq k \).

More general cyclicity results on tensor products of Kirillov-Reshetikhin modules of the form \( \bigotimes_{j=1}^{k} W_{l,a_j}^{(r_j)} \) can be hopefully obtained in this way. For this purpose, it is necessary to determine first of all the zeros and poles of \( R \)-matrices between \( W_{l_1,a_1}^{(r_1)} \) and \( W_{l_2,a_2}^{(r_2)} \), in view of Kashiwara’s cyclicity results in the non-graded case [Ka02]. In type A, this should be possible after a fusion procedure [DO94, Po13].

**Appendix A. Proof of the coproduct formulas**

Proposition 3.13 is proved in essentially the same way as [Zh13 Prop.5.4]. However, it should be noted that the coproduct estimations in *loc. cit* are not enough as seen from the proof of Chari’s Lemma 4.5.

Without loss of generality, we shall prove the coproduct formulas for \( K_{j,s}^+, X_{i,n}^\pm \) for \( i \in I, j \in I \) and \( s,n \in \mathbb{Z}_{>0} \). Proof of other cases is parallel.

For simplicity, let \( U := U_q(\mathfrak{g}) \). From the Gauss decomposition in 3.4.11 we see that

\[
X_{i,0}^- = t_{i+1,i}^{(0)} (t_{ii}^{(0)})^{-1}, \quad X_{1,1}^- = -s_{21}^{-1} (s_{11}^{(0)})^{-1}, \quad K_{i,0}^+ = s_{ii}^{(0)} = (K_{i,0}^-)^{-1}.
\]

In the following, for two vectors \( x, y \) in a vector space, we write \( x \equiv y \) if \( x \in \mathbb{C}^\times y \).

**A.1. Quantum brackets.** Let \( x \in U_\alpha, y \in U_\beta \) be \( \mathbb{Q} \)-homogeneous. Define

\[
[x, y] := xy - (-1)^{|\alpha||\beta|} q^{(\alpha, \beta)} yx.
\]

Given \( x_s \in U_\beta \) for \( 1 \leq s \leq r \), define *iterated quantum brackets*

\[
[x_1, x_2, \ldots, x_r]_L := [x_1, x_2, \ldots, x_{r-1}]_L, x_r], \quad [x_1, x_2, \ldots, x_r]_R := [x_1, x_2, \ldots, x_{r-1}]_R.
\]

**Lemma A.1.** \( [X_{1,1}^-, X_{2,0}^-, X_{3,0}^-, \ldots, X_{M+N-1,0}^-]_L = s_{M+N,1}^{(1)} (s_{11}^{(0)})^{-1} \).

**Proof.** Fix \( i,j,k \in I \) such that \( i < j < k \). By taking the matrix coefficients of \( v_j \otimes v_i \mapsto v_k \otimes v_j \) for the operator equation:

\[
R_{23}(z,w) T_{12}(z) S_{13}(w) = S_{13}(w) T_{12}(z) R_{23}(z,w)
\]

we see that

\[
(-1)^{|i|+|j|} (z-w) t_{kj}(z) s_{ji}(w) + z (q_i - q_i^{-1}) t_{ji}(z) s_{ki}(w) = (z-w)(-1)^{|i|+|j|} s_{ji}(w) t_{kj}(z) + w (q_i - q_i^{-1}) s_{jj}(w) t_{ki}(z).
\]
Next by comparing the coefficients of \( zw \) we get
\[
(-1)^{\lfloor i \rfloor + \lfloor j \rfloor} (t_{kj}^{(0)} s_{ji}^{(1)} - s_{ji}^{(0)} t_{kj}^{(0)}) + (q_i - q_j^{-1}) t_{jj}^{(0)} s_{ki}^{(1)} = 0.
\]
In other words,
\[
(A.29) \quad [s_{ji}^{(1)}, t_{kj}^{(0)}] = (q_j - q_j^{-1}) t_{jj}^{(0)} s_{ki}^{(1)}.
\]
Note that for \( 1 \leq j \leq M + N - 1 \) we have
\[
X_{1,1}^- = -s_{21}^{(1)} (s_{11}^{(0)})^{-1}, \quad X_{j,0}^- = t_{j+1,j}^{(0)} (t_{jj}^{(0)})^{-1}.
\]
By repeatedly applying Equation \((A.29)\) we find the desired quantum bracket. \(\square\)

A.2. Relations on Drinfeld generators. Let us introduce the \( H_i,s \) for \( i \in I \) and \( s \in \mathbb{Z}_{>0} \) by the following functional equations:
\[
K_i^+(z) = s_{ii}^{(0)} \exp((q_i - q_i^{-1}) \sum_{s \in \mathbb{Z}_{>0}} H_i,s z^s) \in U[[z]].
\]
Clearly the \( H_i,s \) commute with each other as the \( K_i^+,n \) do. Moreover,
\[
[X_{i,m}, X_{j,n}^+] = \delta_{ij}(q_i - q_i^{-1})(\Psi_{i,m+n} - \delta_{m+n,0}s_{ii}^{(0)} (s_{i+1,i+1}^{(0)})^{-1}) \quad \text{for } i \in I_0, m + n \geq 0,
\]
\[
\sum_{k \geq 0} \Psi_{i,k} z^k = (s_{ii}^{(0)})^{-1} s_{i+1,i+1}^{(0)} \exp((q - q^{-1}) \sum_{s \in \mathbb{Z}_{>0}} (d_{i+1} H_{i+1,s} - d_i H_i,s) z^s).
\]
Next from the relations between \( K_i^+(z) \) and \( X_j^+(w) \) we deduce that for \( i \in I, j \in I_0 \)
\[
[H_i,s, X_{j,n}^+] = 0 \quad \text{if } i \neq j, j + 1,
\]
\[
[H_i,s, X_{j,n}^+] = \pm q_i^{s} s X_{i,n+s}^+,
\]
\[
[H_i,s, X_{i-1,n}^+] = \mp q_i^{-s} s X_{i,n+s}^+.
\]
Set \( h_{i,s} := d_i H_i,s - d_{i+1} H_{i+1,s} \) for \( i \in I_0, s \in \mathbb{Z}_{>0} \). Then for \( 1 \leq i \leq M + N - 2 \) one can find \( c_{i+1} \in \mathbb{C}^\times \) such that
\[
[h_{i,1}, X_{i+1,n}^+] = \pm c_{i+1} X_{i+1,n+1}^+.
\]
Let \( c_1 \in \mathbb{C}^\times \) be such that \( [H_{1,1}, X_{1,n}^+] = \pm c_1 X_{1,n+1}^+ \).

A.2.1. \( h_{i,1} \) as quantum brackets. To distinguish with \( K \) which we have used before, let us introduce \( L_i := s_{ii}^{(0)} (s_{i+1,i+1})^{-1} \) for \( i \in I_0 \). Introduce also
\[
E_0 := s_{M+N,1}^{(1)} (s_{M+N,M+N})^{-1}, \quad E_i := X_{i,0}^+, \quad L_0 := (L_1 L_2 \cdots L_{M+N-1})^{-1}.
\]
Then for \( 0 \leq i \leq M + N - 1 \), we have
\[
\Delta(E_i) = 1 \otimes E_i + E_i \otimes L_i^{-1}.
\]
Lemma A.2. \( h_{i,1} \mid \{ E_i, E_{i-1}, E_{i-2}, \cdots, E_1, E_{i+1}, E_{i+2}, \cdots, E_{M+N-1}, E_0 \} R \) for \( i \in I_0 \).
Proof. First, let us first compute $h_{1,1}$. Note that
\[ E_0 \doteq \{X_{1,1}^-, X_{2,0}^-, X_{3,0}^-, \cdots, X_{M+N-1,0}^-\}L_1L_2 \cdots L_{M+N-1}. \]
Now by induction on $2 \leq i \leq M + N - 1$ it is easy to see that
\[ [E_i, E_{i+1}, \cdots, E_{M+N-1}, E_0]_R \doteq [X_{1,1}^-, X_{2,0}^-, \cdots, X_{i-1,0}^-]L_1L_2 \cdots L_{i-1}. \]
In particular, when $i = 2$ we obtain
\[ [E_2, E_3, \cdots, E_{M+N-1}, E_0]_R \doteq X_{1,1}^- L_1. \]
Using $[E_1, X_{1,1}^-] \doteq L_1^{-1} h_{1,1}$ we conclude that
\[ [E_1, E_2, \cdots, E_{M+N-1}]_R \doteq h_{1,1}. \]
Next, we have $[h_{1,1}, X_{2,0}^-] \doteq X_{2,1}^-$. In this way we compute $X_{2,1}^-$ as
\[ X_{2,1}^- \doteq [h_{1,1}, X_{2,0}^-] = [[E_1, E_2, \cdots, E_{M+N-1}, E_0]_R, X_{2,0}^-] \doteq [E_1, [L_2 - L_2^{-1}], [E_3, \cdots, E_{M+N-1}, E_0]_R]_{q^{-\alpha_1 \alpha_2}} \doteq [E_1, [E_3, \cdots, E_{M+N-1}, E_0]_R L_2^{-1}]_{q^{-\alpha_1 \alpha_2}} \doteq [E_1, [E_3, \cdots, E_{M+N-1}, E_0]_R]_{q^{-\alpha_1 \alpha_2}} L_2^{-1} = [E_1, E_3, \cdots, E_{M+N-1}, E_0]_R L_2^{-1}. \]
It follows that
\[ h_{2,1} \doteq [E_2, X_{2,1}^-] L_2 \doteq [E_2, [E_1, E_3, \cdots, E_{M+N-1}, E_0]_R L_2^{-1}] L_2 \doteq [E_2, [E_1, E_3, \cdots, E_{M+N-1}, E_0]_R]_{q^{-\alpha_2 \alpha_2}} = [E_2, E_1, E_3, \cdots, E_{M+N-1}, E_0]_R. \]
In general, by an induction argument on $2 \leq i \leq M + N - 1$, one can show that
\[ X_{i,1}^- \doteq [E_{i-1}, E_{i-2}, \cdots, E_i, E_{i+1}, E_{i+2}, \cdots, E_{M+N-1}, E_0]_R L_i^{-1}, \]
\[ h_{i,1} \doteq [E_i, E_{i-1}, \cdots, E_1, E_{i+1}, E_{i+2}, \cdots, E_{M+N-1}, E_0]_R. \]
This concludes the proof. \hfill \square

A.2.2. Coproduct for $h_{i,1}$. Introduce the length function $\ell : Q_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$ by
\[ \ell \left( \sum_{i \in I_0} n_i \alpha_i \right) = \sum_{i \in I_0} n_i. \]
In the following, when we write $\ell(\alpha)$, it should be understood implicitly that $\alpha \in Q_{\geq 0}$. For $i \in I_0$, let $U_i$ be the subalgebra of $U$ generated by the $E_j$ with $j \in I_0 \setminus \{i\}$. Clearly $U_i$ is a $Q$-graded subalgebra.

Let us first consider $h_{1,1}$:
\[ h_{1,1} \doteq [E_1, E_2, \cdots, E_{M+N-1}, E_0]_R. \]
To compute $\Delta(h_{1,1})$, notice first that
\[ [\Delta E_1, \Delta E_2, \cdots, \Delta E_{M+N-1}, E_0] \otimes L_0^{-1} \doteq [E_1, E_2, \cdots, E_{M+N-1}, E_0]_R \otimes 1. \]
Note that $[E_0]_Q = -(\alpha_1 + \alpha_2 + \cdots + \alpha_{M+N-1})$. It follows that
\[ \Delta(h_{1,1}) \in 1 \otimes h_{1,1} + h_{1,1} \otimes 1 \]
Lemma A.3 applied to \( n \)

Assume that the coproduct formula (A.30) is true for

\[
\Delta(X_{i,n}^+) - 1 \otimes X_{i,n}^+ + \sum_{m=0}^{n} X_{i,m}^+ \otimes A_i + \sum_{\ell(\alpha)>1} (U_1)_\alpha \otimes U_{\alpha} - \sum_{\ell(\alpha_1)>0} U_\alpha \otimes U_{\alpha_1}.
\]

Proof. Let us assume first \( 2 \leq i \leq M + N - 1 \). Then \( [h_{i-1,1}, X_{i,n}^+] = c_i X_{i,n+1}^+ \) for \( n \in \mathbb{Z}_{\geq 0} \).

We prove the above coproduct formula by induction on \( n \in \mathbb{Z}_{\geq 0} \). Clearly,

\[
\Delta(X_{i,0}^+) = 1 \otimes X_{i,0}^+ + X_{i,0}^+ \otimes L_{i}^{-1} \in 1 \otimes X_{i,0}^+ + X_{i,0}^+ \otimes A_i.
\]

Assume that the coproduct formula (A.30) is true for \( n \). Remark that for \( j \in I_0 \setminus \{ i \} \)

\[
[E_j \otimes X_{j,-1}^-, 1 \otimes X_{i,0}^+] = 0.
\]

Lemma A.3 applied to \( h_{i-1,1} \) with \( s = i \)

\[
\Delta(c_i X_{i,n+1}^+) \in c_i 1 \otimes X_{i,n+1}^+ + \mathbb{C}^x E_i \otimes [X_{i,n}^+, X_{i,1}^-]
\]
\[
+ \sum_{m=0}^{n} [h_{i-1,1}, X_{i,m}^+] \otimes A_i + \sum_{\ell(\alpha) > 1} (U_\alpha) \otimes U_{\alpha - \alpha} + \sum_{\ell(\alpha - \alpha_i) > 0} U_\alpha \otimes U_{\alpha_i - \alpha} \\
\subseteq c_i 1 \otimes X_{i,n+1}^+ + \sum_{m=0}^{n+1} X_{i,m}^+ \otimes A_i + \sum_{\ell(\alpha) > 1} (U_\alpha) \otimes U_{\alpha - \alpha} + \sum_{\ell(\alpha - \alpha_i) > 0} U_\alpha \otimes U_{\alpha_i - \alpha}.
\]

This establishes Equation (A.30).

Next, when \(i = 1\), we use the relation \([H_{i,1}, X_{1,n}^\pm] = \pm c_1 X_{1,n+1}^\pm\) and the coproduct formula \(\Delta(H_{1,1})\) in Lemma A.3. The rest is parallel as in the case \(i > 1\).

Lemma A.4 can be viewed as a refinement of Equation (3.27). In a similar way, it is not difficult to prove Equation (3.28) by using Lemma A.3

Corollary A.5. For \(i \in I_0\) and \(n \in \mathbb{Z}_{\geq 0}\) we have

\[
(A.31) \quad \Delta(\Psi_{i,n}) \in A_i \otimes A_i + \sum_{\ell(\alpha) > 0} U_\alpha \otimes U_{-\alpha}.
\]

Proof. For \(n = 0\), this is clear since \(\Psi_{i,0} = L_i\). For \(n > 0\), we have

\[
\Psi_{i,n} \doteq [X_{i,n}^+, X_{i,0}^-].
\]

It is enough to consider the bracket \([\Delta(X_{i,n}^+), \Delta(X_{i,0}^-)]\). Remark that by definition

\[
[X_{i,0}^-, x] = 0 \quad \text{for} \quad x \in U_i.
\]

Now Equation (A.31) follows from Equation (A.30) and from the fact that \(\Delta(X_{i,0}^-) = L_i \otimes X_{i,0}^- + X_{i,0}^- \otimes 1\).

It is due to the proof of the above corollary that we introduce the subalgebras \(U_i\).

A.3. Proof of Proposition 3.13. It is enough to prove Equation (3.20). Observe

\[
\Delta(K_1^+(z)) = \Delta(s_{11}(z)) \in s_{11}(z) \otimes s_{11}(z) + \sum_{\ell(\alpha) > 0} U_\alpha \otimes U_{-\alpha}[[z]].
\]

It is therefore enough to show that: for \(i \in I_0\) and \(n \in \mathbb{Z}_{\geq 0}\)

\[
\Delta(\Psi_{i,n}) \in \sum_{m=0}^{n} A_{i,m} \otimes A_{i,n-m} + \sum_{\ell(\alpha) > 0} U_\alpha \otimes U_{-\alpha}.
\]

Clearly, \(\Delta(\Psi_{i,0}) = \Psi_{i,0} \otimes \Psi_{i,0}\). In view of Corollary A.5 let us define \(\Delta_i(\Psi_{i,n}) \in A_i \otimes A_i\) to be such that \(\Delta(\Psi_{i,n}) - \Delta_i(\Psi_{i,n}) \in \sum_{\ell(\alpha) > 0} U_\alpha \otimes U_{-\alpha}\).

Fix \(i \in I_0\). From the highest \(\ell\)-weight representation theory ([4.1.1]) of the quantum affine superalgebra \(U\) we observe that the subalgebra \(A_i\) is an algebra of Laurent polynomials:

\[
A_i = \mathbb{C}[\Psi_{i,n} : n \in \mathbb{Z}_{\geq 0}][\Psi_{i,0}, \Psi_{i,0}^{-1}].
\]

So is the tensor algebra \(A_i \otimes A_i\). It follows that an element \(x \in A_i \otimes A_i\) is completely determined by the data \(\chi \times \mu(x)\) where \(\chi, \mu\) are algebra homomorphisms \(A_i \rightarrow \mathbb{C}\). Let us
fix \( n \in \mathbb{Z}_{>0} \). Since \( \sum_{\ell(a)>0} U_\alpha \otimes U_{-\alpha} \) always kills the tensor product of two highest \( \ell \)-weight vectors, we conclude that
\[
\chi \times \mu(\Delta_i(\Psi_{i,n})) = \sum_{m=0}^{n} \chi(\Psi_{i,m}) \mu(\Psi_{i,n-m}) = \chi \times \mu(\sum_{m=0}^{n} \Psi_{i,m} \otimes \Psi_{i,n-m}).
\]
It follows that \( \Delta_i(\Psi_{i,n}) = \sum_{m=0}^{n} \Psi_{i,m} \otimes \Psi_{i,n-m} \).

To conclude this section, we remark that the elements \( E_i, L_{\pm}^i \) with \( 0 \leq i \leq M+N-1 \) introduced in §A.2.1 satisfy all the relations of the Borel subalgebra of \( U_q(\hat{\mathfrak{sl}}(M,N)) \) defined by Drinfeld-Jimbo generators (See [Ya99, Proposition 6.7.1]). More excitingly, a \( q \)-character theory for finite-dimensional representations of \( U_q(\mathfrak{g}) \) can be similarly developed as in [FR99], based on the above coproduct formulas.

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