ON SPHERICAL SLANT HELICES IN EUCLIDEAN 3-SPACE

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ABSTRACT. In this paper we consider the spherical slant helices in $\mathbb{R}^3$. Moreover, we show how could be obtained to a spherical slant helix and we give some spherical slant helix examples in Euclidean 3-space.

Key Words: Slant helix, spherical curve, geodesic curvature.

1. Introduction

In [2], A slant helix in Euclidean space $\mathbb{R}^3$ was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that $\alpha$ is a slant helix in $\mathbb{R}^3$ if and only if the geodesic curvature of the principal normal of a space curve $\alpha$ is a constant function.

In [6], Kula and Yayli have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helix.

In [5], Kula, Ekmekci, Yayli and Ilarslan have studied the relationship between the plane curves and slant helices in $\mathbb{R}^3$. They obtained that the differential equations which are characterizations of a slant helix.

In this paper we consider the spherical slant helices in $\mathbb{R}^3$. We also present the parametric slant helices, their curvatures and torsions. Moreover, we give some slant helix examples in Euclidean 3-space.

2. Preliminaries

We now recall some basic concept on classical geometry of space curves and the definition of slant helix in $\mathbb{R}^3$. A curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ is a space curve. $T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|}$ is a unit tangent vector of $\alpha$ at $s$. If $\kappa(s) \neq 0$, then the unit principal binormal vector $B(s)$ of the curve $\alpha$ at $s$ is given by $\frac{\alpha'(s) \wedge \alpha''(s)}{\|\alpha'(s) \wedge \alpha''(s)\|}$. The unit vector $N(s) = B(s) \wedge T(s)$ is called the unit normal vector of $\alpha$ at $s$. For the derivatives of the Frenet frame the Serret-Frenet formula hold:

\[
\begin{align*}
T'(s) &= \nu \kappa(s) N(s), \\
N'(s) &= \nu (-\kappa(s) T(s) + \tau(s) B(s)), \\
B'(s) &= -\nu \tau(s) N(s),
\end{align*}
\]

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where $\kappa$ is the curvature of the curve $\alpha$ at $s$, $\tau$ is the torsion of the curve $\alpha$ at $s$ and $\nu = \|\alpha'\|$. For a general parameter $s$ of a space curve $\alpha$, we can calculate $\kappa$ the curvature and the torsion $\tau$ as follows:

$$\kappa(s) = \frac{\|\alpha'(s) \wedge \alpha''(s)\|}{\|\alpha'(s)\|^3}, \quad \tau(s) = \frac{\det \langle \alpha'(s), \alpha''(s), \alpha'''(s) \rangle}{\|\alpha'(s) \wedge \alpha''(s)\|^2}.$$  

**Definition 2.1.** A curve $\alpha$ with $\kappa(s) \neq 0$ is called a slant helix if the principal normal vector line of $\alpha$ make a constant angle with a fixed direction [2].

**Theorem 2.1.** $\alpha$ is a slant helix if and only if the geodesic curvature of the spherical image of the principal normal indicatrix $(N)$ of $\alpha$

$$(2.2) \quad \sigma(s) = \left( \frac{\kappa^2}{\nu (\kappa^2 + \tau^2)} \right)^{\frac{1}{2}} \left( \frac{\tau}{\kappa} \right)'(s)$$

is a constant function [2].

**Definition 2.2.** Specially, if $\kappa(s) = 1$, a slant helix is called a Salkowski curve and if $\tau(s) = 1$, a slant helix is called an anti-Salkowski curve [4].

**Theorem 2.2.** Let $\alpha$ be a space curve in $\mathbb{R}^3$. The following statements are equivalent:

1. $\alpha$ is a spherical, i.e., it is contained in a sphere of radius $r$
2. $\left( \frac{1}{\kappa} \right)^2 + \left( \frac{1}{\nu \tau} \right)^2 = r^2$.
3. $\frac{\kappa}{\nu} \left( \frac{1}{\kappa} \right)' + \frac{\tau}{\kappa} = 0$.
4. $\frac{\kappa}{\nu} = A \cos \left( \int \nu \tau ds \right) + B \sin \left( \int \nu \tau ds \right)$

where, $A, B$ are constant and $\sqrt{A^2 + B^2} = r$ [8].

3. Slant Helix and Its Projection

In this section, we investigate curvature of slant helix and projection of slant helix in the plane.

**Theorem 3.1.** Let $\alpha$ be a space curve in $\mathbb{R}^3$. The following statements are equivalent:

1. $\alpha$ is a slant helix.
2. $\kappa = \frac{1}{\nu \tau} \theta' \sin \theta$
3. $\tau = \frac{1}{\nu \tau} \theta' \cos \theta$.

$$(3.1) \quad \vec{U} = \frac{\tau}{a \sqrt{\kappa^2 + \tau^2}} T + N + \frac{\kappa}{a \sqrt{\kappa^2 + \tau^2}} B = \text{constant}.$$  

Where $a \neq 0$ is a constant and $\theta$ is a function of $s$ [3].

**Theorem 3.2.** Let $\alpha$ be a slant helix with Frenet frame $\{T, N, B\}$, curvature $\kappa$, torsion $\tau$ and $\vec{a} = \cos \theta_1 T + \cos \theta_2 N + \cos \theta_3 B$ be axis of slant helix.

$$(3.1) \quad \alpha_x(s) = \alpha(s) - \langle \alpha(s), \vec{a} \rangle \vec{a}$$
is a plane curve and its curvature $\kappa_\pi$ is

$$
(3.2) \quad \kappa_\pi = \frac{(1 + a^2) \sin \theta}{(a^2 + \sin^2 \theta)^{\frac{3}{2}}} \kappa.
$$

**Proof.** Differentiating the eq. (3.1), we get

$$
(3.3) \quad \alpha'_\pi(s) = \nu(T - \cos \theta_1 \vec{a}).
$$

Therefore

$$
\nu_\pi = \frac{\left\| \alpha'_\pi(s) \right\|}{ds_\pi} = \nu \|T - \cos \theta_1 \vec{a}\| = \nu \sin \theta_1.
$$

If we derive eq. (3.3) again, we obtain

$$
\alpha''_\pi(s) = \nu' T + \nu^2 \kappa N - \nu' \cos \theta_1 \vec{a} - \nu^2 \kappa \cos \theta_2 \vec{a}.
$$

Thus

$$
\alpha'_\pi \times \alpha''_\pi = \nu^3 \kappa (B - \cos \theta_2 T \times \vec{a} - \cos \theta_1 \vec{a} \times N)
$$

$$
= \nu^3 \kappa \cos \theta_3 \vec{a}.
$$

It follows

$$
(3.4) \quad \kappa_\pi = \frac{\|\alpha'_\pi \times \alpha''_\pi\|}{\nu_\pi^3} = \frac{\cos \theta_3}{\sin^3 \theta_1} \kappa
$$

and $\tau_\pi = 0$. We can easily see that

$$
\cos \theta_1 = -\frac{\cos \theta}{\sqrt{1 + a^2}}, \quad \cos \theta_2 = \frac{a}{\sqrt{1 + a^2}}, \quad \cos \theta_3 = \frac{\sin \theta}{\sqrt{1 + a^2}}.
$$

In this case,

$$
\sin \theta_1 = \frac{\sqrt{a^2 + \sin^2 \theta}}{\sqrt{1 + a^2}},
$$

$$
\frac{ds_\pi}{ds} = \nu \frac{\sqrt{a^2 + \sin^2 \theta}}{\sqrt{1 + a^2}} \kappa.
$$

$$
(3.5) \quad \kappa_\pi = \frac{(1 + a^2) \sin \theta}{(a^2 + \sin^2 \theta)^{\frac{3}{2}}} \kappa.
$$

$$
(3.6) \quad \kappa_\pi = \frac{\left\| \alpha'_\pi(s) \right\|}{ds_\pi} = \nu \sin \theta_1.
$$

$$\square$$

4. Spherical slant helices

In this section, we investigate parametric equation of spherical slant helix and we give some theorem.

The axis $\vec{a}$ of slant helix could chosen axis Oz without loss of generality. Now since the tangent and normal are orthogonal unit vectors, we may write $T_\pi(s_\pi) = (\cos \phi_\pi, \sin \phi_\pi)$ and $N_\pi(s_\pi) = (\sin \phi_\pi, -\cos \phi_\pi)$, $\phi_\pi(s_\pi)$ being the angle between the $x$-axis and tangent. We observe that the curvature has the interpretation

$$
\kappa_\pi = \phi'_\pi
$$

i.e., it is derivative of the tangent-angle $\phi_\pi$ with respect to the arc length $s_\pi$. 
The function $\kappa_\pi(s_\pi)$ specifying the curvature in terms of arc length along a plane curve is called the intrinsic equation of that curve. It uniquely defines the curve. We have the explicit representation

$$x(s) = \int \cos \phi_\pi ds_\pi, \quad y(s) = \int \sin \phi_\pi ds_\pi$$

of the curve $\alpha(s) = (x(s), y(s), z(s))$, where

$$\phi_\pi = \int \kappa_\pi ds_\pi.$$

Therefore, from eq. (4.2) and eq. (3.6)

$$\phi_\pi = -\arctan \left( \frac{\sqrt{1 + a^2} \tan \theta}{a} \right) + \sqrt{1 + a^2} \theta.$$

and by using eq. (4.1), eq. (3.5),

$$x(s) = \int \cos \phi_\pi ds_\pi = \nu \int \left( \frac{a}{\sqrt{1 + a^2}} \cos \theta \cos \frac{\sqrt{1 + a^2} \theta}{a} + \frac{\sqrt{1 + a^2}}{\sqrt{\sin^2 \theta + a^2}} \sin \theta \sin \frac{\sqrt{1 + a^2} \theta}{a} \right) ds.$$

Then from the fourth equation of Theorem 2.2 and the second equation of Theorem 3.1, we have

$$ds = \frac{1}{a \nu} \left( A \sin \theta \cos \left( \frac{\sin \theta}{a} \right) + B \sin \theta \sin \left( \frac{\sin \theta}{a} \right) \right) d\theta.$$

Therefore, by using eq. (4.4),

$$x(s) = \frac{1}{\sqrt{1 + a^2}} \left( \sqrt{1 + a^2} \cos \theta \cos \frac{\sqrt{1 + a^2} \theta}{a} \left( B \cos \left( \frac{\sin \theta}{a} \right) - A \sin \left( \frac{\sin \theta}{a} \right) \right) - \cos \frac{\sqrt{1 + a^2} \theta}{a} \left( \cos \left( \frac{\sin \theta}{a} \right) (A + aB \sin \theta) + (B - aA \sin \theta) \sin \left( \frac{\sin \theta}{a} \right) \right) \right),$$

And similarly,

$$y(s) = -\frac{1}{\sqrt{1 + a^2}} \left( \sqrt{1 + a^2} \cos \theta \cos \frac{\sqrt{1 + a^2} \theta}{a} \left( B \cos \left( \frac{\sin \theta}{a} \right) - A \sin \left( \frac{\sin \theta}{a} \right) \right) + \sin \frac{\sqrt{1 + a^2} \theta}{a} \left( \cos \left( \frac{\sin \theta}{a} \right) (A + aB \sin \theta) + (B - aA \sin \theta) \sin \left( \frac{\sin \theta}{a} \right) \right) \right),$$

Moreover, since

$$\langle \alpha''(s), \vec{a} \rangle = z''(s) = -\nu' \frac{\cos \theta}{\sqrt{1 + a^2}} + \nu^2 \frac{a}{\sqrt{1 + a^2}} \kappa,$$
we find that
\[ z(s) = \frac{1}{\sqrt{1 + a^2}} \cos \left[ \frac{\sin \theta}{a} \right] (-aA + B \sin \theta) - \sin \left[ \frac{\sin \theta}{a} \right] (aB + A \sin \theta). \]

where \( A^2 + B^2 = 1 \), \( a \) is constant and \( \theta = \theta(s) \).

As a result of the above findings, we can give the following theorem.

**Theorem 4.1.** Under the above notation, \( \alpha \) is a spherical slant helix. Moreover, all spherical slant helix can be constructed by above method.

**Proof.** Let \( \alpha \) be a space curve with Frenet frame \( \{ T, N, B \} \), curvature \( \kappa \) and torsion \( \tau \).

In this case, we will show that \( \sigma \) is constant and \( \alpha \in S^2 \).

By simple calculation, spherical indicatrices \( T(s), N(s), B(s) \) of the curve \( \alpha \), respectively, are

\[
T(s) = \left( \frac{a \cos \theta \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right]}{\sqrt{1 + a^2}} + \sin \theta \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right], \right.
\]

\[
\left. \frac{a \cos \theta \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right]}{\sqrt{1 + a^2}} - \sin \theta \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right], -\frac{\cos \theta}{\sqrt{1 + a^2}} \right),
\]

\[
N(s) = \left( \frac{\cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right]}{\sqrt{1 + a^2}}, \frac{\sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right]}{\sqrt{1 + a^2}}, \frac{a}{\sqrt{1 + a^2}} \right),
\]

\[
B(s) = \left( -\frac{a \sin \theta \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right]}{\sqrt{1 + a^2}} + \cos \theta \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right], \right.
\]

\[
\left. -\frac{a \sin \theta \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right]}{\sqrt{1 + a^2}} - \cos \theta \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right], \frac{\sin \theta}{\sqrt{1 + a^2}} \right).
\]

And also, by the formulae of the curvature and the torsion for a general parameter, we can calculate that

\[
\kappa(s) = \frac{1}{|A \cos \left[ \frac{\sin \theta}{a} \right] + B \sin \left[ \frac{\sin \theta}{a} \right]|},
\]

\[
\tau(s) = \frac{\cot \theta}{A \cos \left[ \frac{\sin \theta}{a} \right] + B \sin \left[ \frac{\sin \theta}{a} \right]}.
\]

Therefore, by using eq. \( 2.2 \)

\[
\sigma(s) = -a = constant
\]

which means that \( \alpha \) is a slant helix.

Finally,

\[ \| \alpha \| = A^2 + B^2 = 1. \]

Then, \( \alpha \) is spherical slant helix. Thus the proof of theorem is completed. \( \square \)
Proposition 4.1. If \( \frac{\sqrt{1+a^2}}{a} \) is a rational number, then spherical slant helix \( \alpha \) is closed curve. If \( \frac{\sqrt{1+a^2}}{a} \) is not a rational number, then the curve spherical slant helix \( \alpha \) never closes.

Definition 4.1. Let \( \alpha \) be a spherical curve. Let us denote \( T(s) = \nu(s)\alpha'(s) \), and we call \( T \) a unit tangent vector of \( \alpha \) at \( s \). We now set a vector \( Y(s) = \alpha(s) \wedge T(s) \). By definition, we have an orthonormal frame \( \{ \alpha(s), T(s), Y(s) \} \). This frame is called the Sabban frame of \( \alpha \) \[7\].

Theorem 4.2. The indicatrix \( Y \) of the spherical slant helix \( \alpha \) is a spherical slant helix.

Proof. Let \( Y \) be curve with Frenet frame \( \{ T^Y, N^Y, B^Y \} \), curvature \( \kappa^Y \), torsion \( \tau^Y \) and the geodesic curvature of the spherical image of the principal normal indicatrix \( (N^Y) \) of \( Y \) be \( \sigma^Y \). Then we have

\[
Y'(s) = \nu \kappa \alpha \wedge N,
\]
\[
Y''(s) = (\nu \kappa') \alpha \wedge N + \nu' \kappa B - \nu' \kappa \alpha \wedge T + \nu^2 \kappa' \alpha \wedge N + \nu^2 \kappa^2 \alpha \wedge T,
\]
\[
Y''(s) = (\nu \kappa')' \alpha \wedge N + (\nu \kappa')' \nu B + \nu \tau(\nu \kappa') \alpha \wedge B - \nu \kappa (\nu \kappa') \alpha \wedge T
+ (\nu^2 \kappa')' \alpha \wedge N - \nu^2 \kappa^2 \alpha \wedge T
+ (\nu^2 \kappa^2) \alpha \wedge B - \nu^3 \kappa T N - \nu^3 \kappa^2 \alpha \wedge N.
\]

By the formulae the curvature and the torsion for a general parameter, we can calculate that

\[
\kappa^Y = \frac{\kappa}{\sqrt{\kappa^2 - 1}},
\]
\[
\tau^Y = \frac{\tau}{\sqrt{\kappa^2 - 1}}.
\]

Moreover, \( \sigma^Y(s) = a \), so that \( Y(s) \) is a spherical slant helix. \(\square\)

Remark 4.1. There are not spherical Salkowski and anti-Salkowski curve.

Proof. We choose that \( \alpha \) is unit speed Salkowski curve without loss of generality. Since \( \kappa(s) = 1 \) and by using

\[
\left( \frac{1}{\tau} \left( \frac{1}{\kappa} \right)' \right) + \frac{\tau}{\kappa} = 0,
\]

we have \( \tau(s) = 0 \). Therefore, \( \alpha \) is plane curve and is not spherical slant helix.

Similarly, we choose that \( \alpha \) is unit speed Anti-Salkowski curve without loss of generality. Then, by simple calculation, we have \( \sigma(s) \) is not constant which means that \( \alpha \) is not spherical slant curve. \(\square\)

5. Example

In this section we give two example of spherical slant helices in Euclidean 3-space and draw its pictures and its tangent indicatrix, normal indicatrix, and binormal indicatrix by using Mathematica.
Example 5.1. For $a = 1$, $A = 1$, $B = 0$, we consider a spherical slant helix $\alpha$ defined by

\[ x(s) = -\cos \theta \sin(\sqrt{2}\theta) \sin(\sin \theta) \]
\[ + \frac{1}{\sqrt{2}} \cos(\sqrt{2}\theta)(-\cos(\sin \theta) + \sin \theta \sin(\sin \theta)) \]
\[ y(s) = \cos \theta \cos(\sqrt{2}\theta) \sin(\sin \theta) \]
\[ + \frac{1}{\sqrt{2}} \sin(\sqrt{2}\theta)(-\cos(\sin \theta) + \sin \theta \sin(\sin \theta)) \]
\[ z(s) = \frac{1}{\sqrt{2}}(-\cos(\sin \theta) + \sin \theta \sin(\sin \theta)). \]

The picture of the curve $\alpha$ is rendered in Figure 1.

The parametrization of the tangent indicatrix $T = (T_1, T_2, T_3)$ of the spherical slant helix $\alpha$ is

\[ T_1(s) = \frac{1}{\sqrt{2}} \cos \theta \cos(\sqrt{2}\theta) + \sin(\sqrt{2}\theta) \sin \theta, \]
\[ T_2(s) = \frac{1}{\sqrt{2}} \sin \theta \cos(\sqrt{2}\theta) - \sin(\sqrt{2}\theta) \cos \theta, \]
\[ T_3(s) = -\frac{1}{\sqrt{2}} \cos \theta. \]

The picture of the tangent indicatrix is rendered in Figure 2 (a).

The parametrization of the normal indicatrix $N = (N_1, N_2, N_3)$ of the spherical slant helix $\alpha$ is

\[ N_1(s) = \frac{1}{\sqrt{2}} \cos(\sqrt{2}\theta), \]
\[ N_2(s) = \frac{1}{\sqrt{2}} \sin(\sqrt{2}\theta), \]
\[ N_3(s) = \frac{1}{\sqrt{2}}. \]

The picture of the normal indicatrix is rendered in Figure 2 (b).

The parametrization of the binormal indicatrix $B = (B_1, B_2, B_3)$ of the spherical slant helix $\alpha$ is
\[ B_1(s) = -\frac{1}{\sqrt{2}} \sin \theta \cos(\sqrt{2} \theta) + \sin(\sqrt{2} \theta) \cos \theta, \]
\[ B_2(s) = -\frac{1}{\sqrt{2}} \sin \theta \sin(\sqrt{2} \theta) - \cos(\sqrt{2} \theta) \cos \theta, \]
\[ B_3(s) = \frac{1}{\sqrt{2}} \sin \theta. \]

The picture of the binormal indicatrix is rendered in Figure 2 (c).

**Example 5.2.** For \( A = 1, B = 0 \) and \( a = 2, a = 3, a = 4 \), the picture of the spherical slant helix \( \alpha \), respectively, is rendered in Figure 4.

**Example 5.3.** For \( a = 1, A = 1, B = 0 \), We consider a spherical slant helix \( Y \) is defined by

\[ x(s) = -\cos \theta \sin(\sqrt{2} \theta) \cos(\sin \theta) \]
\[ + \frac{1}{\sqrt{2}} \cos(\sqrt{2} \theta)(\cos(\sin \theta)) \sin \theta + \sin(\sin \theta)) \]
\[ y(s) = \cos \theta \cos(\sqrt{2} \theta) \cos(\sin \theta) \]
\[ + \frac{1}{\sqrt{2}} \sin(\sqrt{2} \theta)(\cos(\sin \theta)) \sin \theta + \sin(\sin \theta)) \]
\[ z(s) = \frac{1}{\sqrt{2}} (-\cos(\sin \theta) \sin \theta + \sin(\sin \theta)). \]

The picture of the curve \( \alpha \) is rendered in Figure 4.

The parametrization of the tangent indicatrix \( T^Y = (T_1^Y, T_2^Y, T_3^Y) \) of the spherical slant helix \( Y \) is

\[ T_1^Y(s) = -\frac{1}{\sqrt{2}} \cos \theta \cos(\sqrt{2} \theta) - \sin(\sqrt{2} \theta) \sin \theta, \]
\[ T_2^Y(s) = -\frac{1}{\sqrt{2}} \cos \theta \sin(\sqrt{2} \theta) + \cos(\sqrt{2} \theta) \sin \theta, \]
\[ T_3^Y(s) = \frac{1}{\sqrt{2}} \cos \theta. \]

The picture of the tangent indicatrix is rendered in Figure 5 (a).

The parametrization of the normal indicatrix \( N^Y = (N_1^Y, N_2^Y, N_3^Y) \) of the spherical slant helix \( Y \) is
\[ N_1^Y (s) = -\frac{1}{\sqrt{2}} \cos(\sqrt{2} \theta), \]
\[ N_2^Y (s) = -\frac{1}{\sqrt{2}} \sin(\sqrt{2} \theta), \]
\[ N_3^Y (s) = -\frac{1}{\sqrt{2}}. \]

The picture of the normal indicatrix is rendered in Figure 5 (b).

The parametrization of the binormal indicatrix \( B^Y = (B_1^Y, B_2^Y, B_3^Y) \) of the spherical slant helix \( Y \) is
\[ B_1^Y (s) = -\frac{1}{\sqrt{2}} \sin \theta \cos(\sqrt{2} \theta) + \sin(\sqrt{2} \theta) \cos \theta, \]
\[ B_2^Y (s) = -\frac{1}{\sqrt{2}} \sin \theta \sin(\sqrt{2} \theta) - \cos(\sqrt{2} \theta) \cos \theta, \]
\[ B_3^Y (s) = \frac{1}{\sqrt{2}} \sin \theta. \]

The picture of the binormal indicatrix is rendered in Figure 5 (c).

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Figure 1. For $a = 1$, $A = 1$, $B = 0$, spherical slant helix $\alpha$.

Figure 2. For $a = 1$, $A = 1$, $B = 0$, tangent indicatrix of the spherical slant helix $\alpha$ (a), normal indicatrix of the spherical slant helix $\alpha$ (b) and binormal indicatrix of the spherical slant helix $\alpha$ (c).
For $A = 1$, $B = 0$ and $a = 2$, the spherical slant helix $\alpha$ (a), For $A = 1$, $B = 0$ and $a = 3$, the spherical slant helix $\alpha$ (b) and For $A = 1$, $B = 0$ and $a = 4$, the spherical slant helix $\alpha$ (c).

For $a = 1$, $A = 1$, $B = 0$, spherical slant helix $Y$. 
Figure 5. For \( a = 1, A = 1, B = 0 \), tangent indicatrix of the spherical slant helix \( \alpha \) (a), normal indicatrix of the spherical slant helix \( \alpha \) (b) and binormal indicatrix of the spherical slant helix \( \alpha \) (c).