ANALYSIS VS. SYNTHESIS SPARSITY FOR $\alpha$-SHEARLETS

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Abstract. There are two notions of sparsity associated to a frame $\Psi = (\psi_i)_{i \in I}$: Analysis sparsity of $f$ means that the analysis coefficients $(f, \psi_i)_{i \in I}$ are sparse, while synthesis sparsity means that we can write $f = \sum_{i \in I} c_i \psi_i$ with sparse synthesis coefficients $(c_i)_{i \in I}$. Here, sparsity of a sequence $c = (c_i)_{i \in I}$ means $c \in \ell^p(I)$ for a given $p < 2$. We show that both notions of sparsity coincide if $\Psi = \text{SH} (\varphi, \psi; \delta) = (\psi_i)_{i \in I}$ is a discrete (cone-adapted) shearlet frame with sufficiently nice generators $\varphi$, $\psi$ and sufficiently small sampling density $\delta > 0$. The required ‘niceness’ of $\varphi, \psi$ is explicitly quantified in terms of Fourier-decay and vanishing moment conditions. In addition to $\ell^p$-sparsity, we even allow weighted $\ell^p$-spaces $\ell^p_w$, as a sparsity measure, with weights of the form $w^p = (2^j)^{(j, \ell, \delta, k)}$ where $j$ encodes the scale of the corresponding shearlet elements.

More precisely, we show that the shearlet smoothness spaces $\mathcal{A}^{p,q}_s (\mathbb{R}^2)$ introduced by Labate et al. simultaneously characterize analysis and synthesis sparsity with respect to a shearlet frame, in the sense that—for suitable $\varphi, \psi, \delta$—the following are equivalent: 1) $f \in \mathcal{A}^{p,q}_s (\mathbb{R}^2)$; 2) $(f, \psi_i)_{i \in I} \in \ell^p_w$; 3) $f = \sum_{i \in I} c_i \psi_i$ for suitable coefficients $c = (c_i)_{i \in I} \in \ell^p_w$.

As an application, we prove that shearlets yield (almost) optimal approximation rates for the class of cartoon-like functions: If $f$ is cartoon-like and $\varepsilon > 0$, then $\|f - f_N\|_{L^2} \lesssim N^{-(1-\varepsilon)}$ where $f_N$ is a linear combination of $N$ shearlets. This might appear to be a well-known statement, but an inspection of the existing proofs reveals that these only establish analysis sparsity of cartoon-like functions, which implies $\|f - f_N\|_{L^2} \lesssim N^{-1} \cdot (1 + \log N)^{3/2}$, where $g_N$ is a linear combination of $N$ elements of the dual frame $\hat{\Psi}$ to the shearlet frame $\Psi$. This is not completely satisfying, since only limited knowledge about the structure and properties of $\Psi$ is available.

In addition to classical shearlets, we also consider more general $\alpha$-shearlet systems. For these, the parabolic scaling is replaced by $\alpha$-parabolic scaling. The resulting systems range from ridgelet-like systems (for $\alpha = 0$) over classical shearlets ($\alpha = \frac{1}{2}$) to wavelet-like systems ($\alpha = 1$). In this more general case, the shearlet smoothness spaces $\mathcal{A}^{p,q}_s (\mathbb{R}^2)$ have to be replaced by the $\alpha$-shearlet smoothness spaces $\mathcal{A}^{p,q}_{\alpha s} (\mathbb{R}^2)$. We completely characterize the existence of these spaces for different values of $\alpha$. This allows us to decide whether sparsity with respect to $\alpha_1$-shearlets implies sparsity with respect to $\alpha_2$-shearlets, even for $\alpha_1 \neq \alpha_2$.

1. Introduction

A cone-adapted shearlet system is a directional multiscale system in $L^2 (\mathbb{R}^2)$ that is obtained by applying suitable translations, shearings and parabolic dilations to the generators $\varphi, \psi, \theta$. The shearings are utilized to obtain elements with different orientations; precisely, the number of different orientations on scale $j$ is approximately $2^{j/2}$, in stark contrast to wavelet-like systems which only employ a constant number of directions per scale. We refer to Definition 5.6 for a more precise description of shearlet systems.

One of the most celebrated properties of shearlets is their ability to provide “optimally sparse approximations” for functions that are governed by directional features like edges. This can be made more precise by introducing the class $\mathcal{E}^2 (\mathbb{R}^2)$ of $C^2$-cartoon-like functions; roughly, these are all compactly supported functions that are $C^2$ away from a $C^2$ edge. More rigorously, the class $\mathcal{E}^2 (\mathbb{R}^2)$ consists of all functions $f$ that can be written as $f = f_1 + f_2$ with $f_1, f_2 \in C^2_c (\mathbb{R}^2)$ and a compact set $B \subset \mathbb{R}^2$ whose boundary $\partial B$ is a $C^2$ Jordan curve; see also Definition 6.1 for a completely formal description of the class of cartoon-like functions. With this notion, the (almost) optimal sparse approximation of cartoon-like functions as understood in [44, 51] means that

$$
\|f - f_N\|_{L^2} \lesssim N^{-1} \cdot (1 + \log N)^{3/2} \quad \forall N \in \mathbb{N} \text{ and } f \in \mathcal{E}^2 (\mathbb{R}^2).
$$

Here, the $N$-term approximation $f_N$ is obtained by retaining only the $N$ largest coefficients in the expansion $f = \sum_{i \in I} (f, \psi_i) \tilde{\psi}_i$, where $\tilde{\Psi} = (\tilde{\psi}_i)_{i \in I}$ is a dual frame for the shearlet frame $\Psi = \text{SH} (\varphi, \psi; \delta) = (\psi_i)_{i \in I}$. Formally, this means $f_N = \sum_{i \in I_N} (f, \psi_i) \psi_i$, where the set $I_N \subset I$ satisfies $|I_N| = N$ and $|(f, \psi_i)| \geq |(f, \psi_j)|$ for all $i \in I_N$ and $j \in I \setminus I_N$.

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One can even show that the approximation rate in equation (1.1) is optimal up to log factors; i.e., up to log factors, no reasonable system \((\varnothing_n)_{n \in \mathbb{N}}\) can achieve a better approximation rate for the whole class \(\mathcal{E}^2(\mathbb{R}^2)\). The restriction to “reasonable” systems is made to exclude pathological cases like dense subsets of \(L^2(\mathbb{R}^2)\) and involves a restriction of the search depth: The \(N\)-term approximation \(f_N = \sum_{n \in J_N} c_n \varnothing_n\) has to satisfy \(|J_N| = N\) and furthermore \(J_N \subset \{1, \ldots, \pi(N)\}\) for a fixed polynomial \(\pi\). For more details on this restriction, we refer to \([38, Section 2.1.1]\).

The approximation rate achieved by shearlets is precisely the same as that obtained by (second generation) curvelets\([2]\). Note, however, that the construction of curvelets in \([2]\) uses bandlimited frame elements, while shearlet frames can be chosen to have compact support\([31, 46]\). A frame with compactly supported elements is potentially advantageous for implementations, but also for theoretical considerations, since localization arguments are highly simplified and since compactly supported frames can be adapted to frames on bounded domains, see e.g. \([40, 41]\). A further advantage of shearlets over curvelets is that curvelets are defined using rotations, while shearlets employ shearings to change the orientation; in contrast to rotations, these shearings leave the digital grid \(\mathbb{Z}^2\) invariant, which is beneficial for implementations.

1.1. Cartoon approximation by shearlets. Despite its great utility, the approximation result in equation (1.1) has one remaining feature: It yields a rapid approximation of \(f\) by a linear combination of \(N\) elements of the dual frame \(\Psi\) of the shearlet frame \(\Psi\), \(not\) by a linear combination of \(N\) elements of \(\Psi\) itself. If \(\Psi\) is a tight frame, this is no problem, but the only known construction of tight cone-adapted shearlet frames uses bandlimited generators. In case of a non-tight cone-adapted shearlet frame, the only knowledge about \(\Psi\) that is available is that \(\Psi\) is a frame with dual \(\Psi\); but nothing seems to be known\([36]\) about the support, the smoothness, the decay or the frequency localization of the elements of \(\Psi\). Thus, it is highly desirable to have an approximation result similar to equation (1.1), but with \(f_N\) being a linear combination of \(N\) elements of the shearlet frame \(\Psi = \text{SH}(\varphi, \psi, \theta, \delta)\) itself.

We will provide such a result by showing that analysis sparsity with respect to a (suitable) shearlet frame \(\text{SH}(\varphi, \psi, \theta, \delta)\) is equivalent to synthesis sparsity with respect to the same frame, cf. Theorem 6.3. Here, analysis sparsity with respect to a frame \(\Psi = (\psi_i)_{i \in I}\) means that the analysis coefficients \(A_\Psi f = (\langle f, \psi_i \rangle)_{i \in I}\) are sparse, i.e., they satisfy \(A_\Psi f \in \ell^p(I)\) for some fixed \(p \in (0, 2)\). Note that an arbitrary function \(f \in L^2(\mathbb{R}^2)\) always satisfies \(A_\Psi f \in \ell^2(I)\) by the frame property. Synthesis sparsity means that we can write \(f = S_\Psi c = \sum_{i \in I} c_i \psi_i\) for a sparse sequence \(c = (c_i)_{i \in I}\), i.e., \(c \in \ell^p(I)\). For general frames, these two properties need not be equivalent, as shown in Section A.

Note though that such an equivalence would indeed imply the desired result, since the proof of equation (1.1) given in \([31]\) proceeds by a careful analysis of the analysis coefficients \(A_\Psi f\) of a cartoon-like function \(f\): By counting how many shearlets intersect the “problematic” region \(\partial B\) where \(f = f_1 + 1_B \cdot f_2\) is not \(C^2\) and by then distinguishing whether the orientation of the shearlet is aligned with the boundary curve \(\partial B\) or not, the authors show \(\sum_{n > N} |\varnothing_n(f)|^2 \lesssim N^{-2} \cdot (1 + \log N)^3\), where \((\varnothing_n(f))_{n \in \mathbb{N}}\) is the nonincreasing rearrangement of the shearlet analysis coefficients \(A_\Psi f\). It is not too hard to see (see e.g. the proof of Theorem 6.3) that this implies \(A_\Psi f \in \ell^p(I)\) for all \(p > \frac{2}{3}\). Assuming that analysis sparsity with respect to the shearlet frame \(\Psi\) is indeed equivalent to synthesis sparsity, this implies \(f = \sum_{i \in I} c_i \psi_i\) for a sequence \(c = (c_i)_{i \in I} \in \ell^p(I)\). Then, simply by taking only the \(N\) largest coefficients of the sequence \(c\) and by using that the synthesis map \(S_\Psi : \ell^2(I) \to L^2(\mathbb{R}^2)\), \((c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \psi_i\) is bounded, it is not hard to see \(\|f - f_N\|_{L^2} \lesssim \|c - c \cdot I_{N}\|_\beta \lesssim N^{-(p^{-1} - 2)}\), where \(I_N \subset I\) is a set containing \(N\) largest coefficients of \(c\).

Thus, once we know that analysis sparsity with respect to a (suitable) shearlet frame is equivalent to synthesis sparsity, we only need to make the preceding argument completely rigorous.

1.2. Previous results concerning the equivalence of analysis and synthesis sparsity for shearlets. As noted above, analysis sparsity and synthesis sparsity need not be equivalent for general frames. To address this and other problems, Gröchenig\([35]\) and Gröchenig & Cordero\([3]\), as well as Gröchenig & Fornasier\([22]\) introduced the concept of (intrinsic) localized frames for which these two properties are indeed equivalent, cf. \([33, Proposition 2]\).

In contrast to Gabor- and wavelet frames, it is quite nontrivial, however, to verify that a shearlet or curvelet frame is intrinsically localized: To our knowledge, the only papers discussing a variant of this property are \([36, 42]\), where the results from \([36]\) about curvelets and shearlets are generalized in \([42]\) to the setting of \(\alpha\)-molecules; a generalization that we will discuss below in greater detail. For now, let us stick to the setting of \([36]\). In that paper, Grols considers a certain distance function \(\omega : \Lambda^3 \times \Lambda^2 \to [1, \infty)\) (cf. \([36, Definition 3.9]\) for the precise formula)
on the index set
\[ Λ^S := \left\{(j, ℓ, k, δ) ∈ \mathbb{N}_0 \times \mathbb{Z} × \mathbb{Z}^2 \times \{0, 1\} \mid -2^{j/2} ≤ ℓ < 2^{j/2}\right\}, \]
which is (a slightly modified version of) the index set that is used for shearlet frames. A shearlet frame \( Ψ = (ψ_λ)_{λ ∈ Λ^S} \) is called \( N \)-localized with respect to \( ω \) if the associated Gramian matrix \( A := A_Ψ := ((ψ_λ, ψ_\lambda'))_{λ, \lambda' ∈ Λ^S} \) satisfies
\[
|⟨ψ_λ, ψ_\lambda'| ≤ \|A\|_{B_N} · |ω(λ, \lambda')|^N \}
∀λ, \lambda' ∈ Λ^S, \tag{1.2}
\]
where \( \|A\|_{B_N} \) is chosen to be the optimal constant in the preceding inequality.

Then, if \( Ψ \) is a frame with frame bounds \( A, B > 0 \), i.e., if \( A^2 : \|f\|_{L^2}^2 ≤ \sum_{λ ∈ Λ^S} |⟨f, ψ_λ⟩|^2 ≤ B^2 : \|f\|_{L^2}^2 \) for all \( f \in L^2(\mathbb{R}^2) \). Lemma 3.3 shows that the infinite matrix \( A \) induces a bounded, positive semi-definite operator \( A : ℓ^2(Λ^S) → ℓ^2(Λ^S) \) that furthermore satisfies \( σ(A) \subset \{0\} \cup [A, B] \) and the Moore-Penrose pseudoinverse \( A^+ \) of \( A \) is the Gramian associated to the canonical dual frame \( Ψ \) of \( Ψ \). This is important, since [36] Theorem 3.11 now yields the following:

**Theorem.** Assume that \( Ψ = (ψ_λ)_{λ ∈ Λ^S} \) is a shearlet frame with sampling density \( δ > 0 \) and frame bounds \( A, B > 0 \). Furthermore, assume that \( Ψ \) is \( N + L \)-localized with respect to \( ω \), where
\[
N > 2 \quad \text{and} \quad L > 2 \frac{\ln(10)}{\ln(5/4)}. \tag{1.3}
\]

Then the canonical dual frame \( \tilde{Ψ} \) of \( Ψ \) is \( N^+ \)-localized with respect to \( ω \), with
\[
N^+ = N \cdot \left( 1 + \frac{\log \left( \frac{1 - 2^{-L/2}}{1 - 2^{-L} - 2^{-2L}} \right)^2}{\log \left( \frac{B^2 + A^2}{B^2 - A^2} \right)} \right)^{-1},
\]
where the constant \( C_δ > 0 \) only depends on the sampling density \( δ > 0 \).

To see how this theorem could in principle be used, note that the dual frame coefficients satisfy
\[
|⟨f, ψ_λ⟩|_{λ ∈ Λ^S} = A^+ |⟨f, ψ_λ⟩|_{λ ∈ Λ}. \tag{1.4}
\]
Consequently, if(!) the Gramian \( A^+ \) of the canonical dual frame \( \tilde{Ψ} \) of \( Ψ \) restricts to a well-defined and bounded operator \( A^+ : ℓ^p(Λ^S) → ℓ^p(Λ^S) \), then analysis sparsity with respect to \( Ψ \) would imply analysis sparsity with respect to \( \tilde{Ψ} \) and thus synthesis sparsity with respect to \( Ψ \), as desired. In fact, [36] Proposition 3.5 shows that if \( A^+ \) is \( N^+ \)-localized with respect to \( ω \), then \( A^+ : ℓ^p(Λ^S) → ℓ^p(Λ^S) \) is bounded as long as \( N^+ > 2p^{-1} \).

Thus, it seems that all is well, in particular since a combination of [39] Theorem 2.9 and Proposition 3.11 provides readily verifiable conditions on the generators \( ϕ, ψ, θ \) which ensure that the shearlet frame \( Ψ = SH(ϕ, ψ, θ; δ) \) is \( N \)-localized with respect to \( ω \).

There is, however, a well-hidden remaining problem which is also the reason why the equivalence of analysis and synthesis sparsity is not explicitly claimed in any of the papers [36] [3] [39] [37]: As seen above, we need \( N^+ > 2p^{-1} \), but it is not clear at all that this can be achieved with \( N^+ \) as in equation (1.3): There are strong interdependencies between the different quantities on the right-hand side of equation (1.3) which make it next to impossible to verify \( N^+ > 2p^{-1} \). Indeed, the results in [39] only yield \( \|A\|_{N^+L} < \infty \) under certain assumptions (which depend on \( N + L \)) concerning \( ϕ, ψ, θ \), but no explicit control over \( \|A\|_{N^+L} \) is given. Thus, it is not at all clear that increasing \( N \) (or \( L \)) will increase \( N^+ \). Likewise, the frame bounds \( A, B \) only depend on \( ϕ, ψ, θ \) (which are more or less fixed) and on the sampling density \( δ \). Thus, one could be tempted to change \( δ \) to influence \( A, B \) in equation (1.3) and thus to achieve \( N^+ > 2p^{-1} \). But the sampling density \( δ \) also influences \( C_δ \) and \( \|A\|_{N^+L} \), so that it is again not clear at all whether one can ensure \( N^+ > 2p^{-1} \) by modifying \( δ \).

A further framework for deriving the equivalence between analysis and synthesis sparsity for frames is provided by (generalized) coorbit theory [17] [18] [19] [53] [23] [24]. Here, one starts with a continuous frame \( Ψ = (ψ_λ)_{λ ∈ Χ} \) which is indexed by a locally compact measure space \((Χ, μ)\). In the case of classical, group-based coorbit theory [17] [18] [19], it is even required that \((ψ_λ)_{λ ∈ Χ} = (ψ(x)ψ_λ)_{λ ∈ Χ}\) arises from an integrable, irreducible unitary representation of a locally compact topological group \( G \), although one can weaken certain of these conditions [10] [11] [15] [8].

\footnote{Strictly speaking, [39] Definition 2.4] uses the index distance \( ω(λ, λ') = 2^{||λ - λ'||} (1 + 2^{min{||x, x'||}}d(λ, λ')) \) which is different from the distance \( ω(λ, λ') = 2^{||λ - λ'||} (1 + d(λ, λ')) \) used in [36] Definition 3.9. Luckily, this inconsistency is no serious problem, since the distance in [39] dominates the distance from [36], so that \( N \)-localization with respect to the [39]-distance implies \( N \)-localization with respect to the [36]-distance.}
Based on the continuous frame Ψ, one can then introduce so-called coorbit spaces Co(Y) which are defined in terms of decay conditions (specified by the function space Y) concerning the voice transform \( V_Ψ f (x) := (f, \psi_x) \) of a function or distribution \( f \). Coorbit theory then provides conditions under which one can sample the continuous frame \( Ψ \) to obtain a discrete frame \( Ψ_d = (\psi_{x_i})_{i \in I} \), but such that membership of a distribution \( f \) in \( \text{Co}(Y) \) is simultaneously equivalent to analysis sparsity and to synthesis sparsity of \( f \) with respect to \( Ψ_d \).

Thus, if one could find a continuous frame \( Ψ \) such that the prerequisites of coorbit theory are satisfied and such that the discretized frame \( Ψ_d \) coincides with a discrete, cone-adapted shearlet frame, one would obtain the desired equivalence between analysis sparsity and synthesis sparsity. There is, however, no known construction of such a frame \( Ψ \): Although there is a rich theory of shearlet coorbit spaces\(^{[9]} [13] [12] [7] [6] [14] [31] \) which fits into the more general framework of wavelet-type coorbit spaces\(^{[26] [30] [28] [29] [31] [32] [25]} \), the resulting discretized frames are not cone-adapted shearlet frames; instead, they are highly directionally biased (i.e., they treat the \( x \) and \( y \) direction in very different ways) and the number of directions per scale is infinite for each scale; therefore, these systems are unsuitable for most practical applications and for the approximation of cartoon-like functions, cf.\(^{[13]} \) Section 3.3. Hence—at least using the currently known constructions of continuous shearlet frames—coorbit theory can not be used to derive the desired equivalence of analysis and synthesis sparsity with respect to cone-adapted shearlet frames.

1.3. Our approach for proving the equivalence of analysis and synthesis sparsity for shearlets. In this paper, we use the recently introduced theory of structured Banach frame decompositions of decomposition spaces\(^{[62]} \) to obtain the desired equivalence between analysis and synthesis sparsity for (cone-adapted) shearlet frames. A more detailed and formal exposition of this theory will be given in Section 2; for this introduction, we restrict ourselves to the bare essentials.

The starting point in\(^{[62]} \) is a covering \( Q = (Q_k)_{k \in \mathbb{L}} \) of the frequency space \( \mathbb{R}^d \), where it is assumed that each \( Q_k \) is of the form \( Q_k = T_i Q + b_i \), for a fixed base set \( Q \subset \mathbb{R}^d \) and certain linear maps \( T_i \in \text{GL} (\mathbb{R}^d) \) and \( b_i \in \mathbb{R}^d \). Then, using a suitable partition of unity \( \Phi = (\varphi_i)_{i \in I} \) subordinate to \( Q \) and a suitable weight \( w = (w_i)_{i \in I} \) on the index set \( I \) of the covering \( Q \), one defines the associated decomposition space (quasi)-norm

\[
\|g\|_{\text{D}(Q, LP, \ell^q_w)} := \left\| (w_i \cdot \| \mathcal{F}^{-1} (\varphi_i \cdot \hat{g}) \|_{L^p})_{i \in I} \right\|_{\ell^q},
\]

while the associated decomposition space \( \text{D}(Q, LP, \ell^q_w) \) contains exactly those distributions \( g \) for which this quasi-norm is finite.

Roughly speaking, the decomposition space (quasi)-norm measures the size of the distribution \( g \) by frequency-localizing \( g \) to each of the sets \( Q_i \) (using the partition of unity \( \Phi \)), where each of these frequency-localized pieces is measured in \( L^p (\mathbb{R}^d) \), while the individual contributions are aggregated using a certain weighted \( \ell^q \)-norm. The underlying idea in\(^{[62]} \) is to ask whether the strict frequency localization using the compactly supported partition of unity \( \Phi \) can be replaced by a soft, qualitative frequency localization: Indeed, if \( \psi \in L^1 (\mathbb{R}^d) \) has essential frequency support in the base set \( Q \), then it is not hard to see that the function

\[
\psi^i := |\text{det} T_i|^{-1/2} \cdot \mathcal{F}^{-1} (L_{b_i} (\hat{\psi} \circ T_i^{-1})) = |\text{det} T_i|^{1/2} \cdot M_{b_i} (\hat{\psi} \circ T_i^T)
\]

has essential frequency support in \( Q_i = T_i Q + b_i \), for arbitrary \( i \in I \). Here, \( L_x \) and \( M_x \) denote the usual translation and modulation operators, cf. Section 1.3.

Using this notation, the theory developed in\(^{[62]} \) provides criteria pertaining to the generator \( \psi \) which guarantee that the generalized shift-invariant system

\[
\Psi_s := \left( L_{\delta \cdot T_i^{-1} \cdot k} \psi^i \right)_{i \in I, k \in \mathbb{Z}^d}
\]

forms, respectively, a Banach frame or an atomic decomposition for the decomposition space \( \text{D}(Q, LP, \ell^q_w) \), for sufficiently fine sampling density \( \delta > 0 \). The notions of Banach frames and atomic decompositions generalize the concept of frames for Hilbert spaces to the setting of (Quasi)-Banach spaces. The precise definitions of these two concepts, however, are outside the scope of this introduction; see e.g.\(^{[34]} \) for a lucid exposition.

For us, the most important conclusion is the following: If \( \Psi_s \) simultaneously forms a Banach space and an atomic decomposition for \( \text{D}(Q, LP, \ell^q_w) \), then there is an explicitly known (Quasi)-Banach space of sequences \( C^p_w \leq C^{I \times \mathbb{Z}^d} \), called the coefficient space, such that the following are equivalent for a distribution \( g \):

1. \( g \in \text{D}(Q, LP, \ell^q_w) \),
2. the analysis coefficients \( \left( \left( g, L_{\delta \cdot T_i^{-1} \cdot k} \psi^i \right) \right)_{i \in I, k \in \mathbb{Z}^d} \) belong to \( C^p_w \),
3. we can write \( g = \sum_{i \in I} \sum_{k \in \mathbb{Z}^d} (c^i_k \cdot \psi^i) \) for a sequence \( (c^i_k)_{i \in I, k \in \mathbb{Z}^d} \in C^p_w \).
One can even derive slightly stronger conclusions which make these purely qualitative statements quantitative. Now, if one chooses \( p = q \in (0, 2) \) and a suitable weight \( w = (w_i)_{i \in I} \) depending on \( p \), one can achieve \( C_{w,q}^p = \ell^p \left( I \times \mathbb{Z}^d \right) \).

Thus, in this case, the preceding equivalence can be summarized as follows:

If \( \psi \) is nice and \( \delta > 0 \) is small, then **analysis sparsity is equivalent to synthesis sparsity** w.r.t. \( \Psi_\delta \).

In fact, the theory developed in \([62]\) even allows the base set \( Q \) to vary with \( i \in I \), i.e., \( Q_i = T_i Q'_i + b_i \), at least as long as the family \( \{Q'_i \mid i \in I\} \) of different base sets remains finite. Similarly, the generator \( \psi \) is allowed to vary with \( i \in I \), so that \( \psi_i = |\det T_i|^{1/2} \cdot M_{b_i} [\psi_i \circ T_i^T] \), again with the provision that the set \( \{\psi_i \mid i \in I\} \) of generators is finite.

As we will see, one can choose a suitable covering \( Q = S \)—the so-called **shearlet covering** of the frequency space \( \mathbb{R}^2 \)—such that the system \( \Psi_\delta \) from above coincides with a shearlet frame. The resulting decomposition spaces \( D(S, L^p, \ell_q) \) are then (slight modifications of) the **shearlet smoothness spaces** as introduced by Labate et al. \([52]\).

In summary, the theory of **structured Banach frame decompositions of decomposition spaces** will imply the desired equivalence of analysis and synthesis sparsity with respect to cone-adapted shearlet frames. To this end, however, we first need to show that the technical conditions on the generators that are imposed in \([62]\) are indeed satisfied if the generators of the shearlet system are sufficiently smooth and satisfy certain vanishing moment conditions. As we will see, this is by no means trivial and requires a huge amount of technical estimates.

Finally, we remark that spaces similar to the shearlet smoothness spaces have also been considered by Vera: In \([58]\), he introduced so-called **shear anisotropic inhomogeneous Besov spaces**, which are essentially a generalization of the shearlet smoothness spaces to \( \mathbb{R}^d \). Vera then shows that the analysis and synthesis operators with respect to certain **bandlimited** shearlet systems are bounded between the shear anisotropic inhomogeneous Besov spaces and certain sequence spaces. Note that the assumption of bandlimited frame elements excludes the possibility of having compact support in space. Furthermore, boundedness of the analysis and synthesis operators alone does *not* imply that the **bandlimited** shearlet systems form Banach frames or atomic decompositions for the shear anisotropic Besov spaces, since this requires existence of a certain **reproducing formula**. In \([57]\), Vera also considers **Triebel-Lizorkin type** shearlet smoothness spaces and again derives similar boundedness results for the analysis and synthesis operators. Finally, in both papers \([58, 57]\), certain embedding results between the classical Besov or Triebel-Lizorkin spaces and the new “shearlet adapted” smoothness spaces are considered, similarly to our results in Section 7. Note though that we are able to completely characterize the existence of such embeddings, while \([58]\) only establishes certain necessary and certain sufficient conditions, without achieving a characterization.

### 1.4. \( \alpha \)-shearlets and cartoon-like functions of different regularity

The usual construction of shearlets employs the **parabolic dilations** \( \text{diag} \left( 2^j, 2^{j/2} \right) \) and (the dual frames of) the resulting shearlet systems turn out to be (almost) optimal for the approximation of functions that are \( C^2 \) away from a \( C^2 \) edge. Beginning with the paper \([49]\), it was realized that different regularities—i.e., “functions that are \( C^\beta \) away from a \( C^\beta \) edge”—can be handled by employing a different type of dilations, namely the \( \alpha \)-**parabolic dilations** \( \text{diag} \left( 2^j, 2^{\alpha j} \right) \), with the specific choice \( \alpha = \beta^{-1} \).

These modified shearlet systems were called **hybrid shearlets** in \([49]\), where they were introduced in the three-dimensional setting. In the Bachelor’s thesis \([45]\), precisely in \([45\text{ Section 4]}\), it was then shown also in the two-dimensional setting that shearlet systems employing \( \alpha \)-parabolic scaling—from now on called \( \alpha \)-**shearlet systems**—indeed yield (almost) optimal approximation rates for the model class of \( C^\beta \)-**cartoon-like functions**, if \( \alpha = \beta^{-1} \). Again, this comes with the caveat that the approximation is actually performed using the **dual frame** of the \( \alpha \)-shearlet frame.

Note, however, that the preceding result requires the regularity \( \beta \) of the \( C^\beta \)-cartoon-like functions to satisfy \( \beta \in (1, 2) \). Outside of this range, the arguments in \([45]\) are not applicable; in fact, it was shown in \([50]\) that the result concerning the optimal approximation rate fails for \( \beta > 2 \), at least for **\( \alpha \)-curvelets** \([58]\) instead of \( \alpha \)-shearlets.

These \( \alpha \)-curvelets are related to \( \alpha \)-shearlets in the same way that shearlets and curvelets are related \([39]\), in the sense that the associated coverings of the Fourier domain are equivalent and in that they agree with respect to **analysis sparsity**: If \( f \) is \( \ell^p \)-analysis sparse with respect to a (reasonable) \( \alpha \)-curvelet system, then the same holds with respect to any (reasonable) \( \alpha \)-shearlet system and vice versa. This was derived in \([37]\) as an application of the framework of \( \alpha \)-**molecules**, a common generalization of \( \alpha \)-shearlets and \( \alpha \)-curvelets; see also \([20]\) for a generalization to dimensions larger than two.

---

\(^2\)In fact, in \([39\text{ Section 4.1]}\) the three-dimensional counterparts of the scaling matrices \( \text{diag} \left( 2^{3\beta/2}, 2^{\beta/2} \right) \) are used, but the resulting hybrid shearlet systems have the same approximation properties as those defined using the \( \alpha \)-parabolic dilations \( \text{diag} \left( 2^j, 2^{\alpha j} \right) \) with \( \alpha = \beta^{-1} \); see Section \([13]\) for more details.
As we will see, one can modify the shearlet covering $S$ slightly to obtain the so-called $\alpha$-shearlet covering $S^{(\alpha)}$. The systems $\Psi_\delta$ (cf. equation (1.4)) that result from an application of the theory of structured Banach frame decompositions with the covering $S^{(\alpha)}$ then turn out to be $\alpha$-shearlet systems. Therefore, we will be able to establish the equivalence of analysis and synthesis sparsity not only for classical cone-adapted shearlet systems, but in fact for cone-adapted $\alpha$-shearlet systems for arbitrary $\alpha \in [0, 1]$, essentially without additional effort.

Even more, recall from above that the theory of structured Banach frame decompositions not only yields equivalence of analysis and synthesis sparsity, but also shows that each of these properties is equivalent to membership of the distribution $f$ under consideration in a suitable decomposition space $D(S^{(\alpha)}, L^p, ℓ^q_w)$. We will call these spaces $\alpha$-shearlet smoothness spaces and denote them by $\mathcal{S}^{p,q}_{\alpha,s}(\mathbb{R}^2)$, where the smoothness parameter $s$ determines the weight $w$. Using a recently developed theory for embeddings between decomposition spaces [6], we are then able to completely characterize the existence of embeddings between $\alpha$-shearlet smoothness spaces for different values of $\alpha$. Roughly, such an embedding $\mathcal{S}^{p,q}_{\alpha_1,s_1} \rightarrow \mathcal{S}^{p,q}_{\alpha_2,s_2}$ means that sparsity (in a certain sense) with respect to $\alpha_1$-shearlets implies sparsity (in a possibly different sense) with respect to $\alpha_2$-shearlets.

In a way, this extends the results of [37], where it is shown that analysis sparsity transfers from one $\alpha$-scaled system to another (e.g. from $\alpha$-curvelets to $\alpha$-shearlets); in contrast, our embedding theory characterizes the possibility of transferring such results from $\alpha_1$-shearlet systems to $\alpha_2$-shearlet systems, even for $\alpha_1 \neq \alpha_2$. It will turn out, however, that simple $ℓ^p$-sparsity with respect to $\alpha_1$-shearlets never yields a nontrivial $ℓ^p$-sparsity with respect to $\alpha_2$-shearlets, if $\alpha_1 \neq \alpha_2$. Luckily, one can remedy this situation by requiring $ℓ^p$-sparsity in conjunction with a certain decay of the coefficients with the scale. For more details, we refer to Section 7.

### 1.5. Structure of the paper.

Before we properly start the paper, we introduce several standard and non-standard notations in the next subsection.

In Section 2 we give an overview over the main aspects of the theory of structured Banach frame decompositions of decomposition spaces that was recently developed by one of the authors in [62].

The most important ingredient for the application of this theory is a suitable covering $Q = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ of the frequency space $\mathbb{R}^2$ such that the provided Banach frames and atomic decompositions are of the desired form; in our case we want to obtain cone-adapted $\alpha$-shearlet systems. Thus, in Section 3 we introduce the so-called $\alpha$-shearlet coverings $S^{(\alpha)}$ for $\alpha \in [0, 1]$ and we verify that these coverings fulfill the standing assumptions from [62]. The more technical parts of this verification are deferred to Section 4 in order to not disrupt the flow of the paper. Furthermore, Section 3 also contains the definition of the $\alpha$-shearlet smoothness spaces $\mathcal{S}^{p,q}_{\alpha,s}(\mathbb{R}^2) = D(S^{(\alpha)}, L^p, ℓ^q_w)$ and an analysis of their basic properties.

Section 4 contains the main results of the paper. Here, we provide readily verifiable conditions—smoothness, decay and vanishing moments—concerning the generators $\varphi, \psi$ of the $\alpha$-shearlet system $SH^{(\pm 1)}(\varphi, \psi; \delta)$ which ensure that this $\alpha$-shearlet system forms, respectively, a Banach frame or an atomic decomposition for the $\alpha$-shearlet smoothness space $\mathcal{S}^{p,q}_{\alpha,s}(\mathbb{R}^2)$. This is done by verifying the technical conditions of the theory of structured Banach frame decompositions. All of these results rely on one technical lemma whose proof is extremely lengthy and therefore deferred to Section 5.

For $\alpha$-shearlet systems, it is expected that $\frac{1}{2}$-shearlets are identical to the classical cone-adapted shearlet systems. This is not quite the case, however, for the shearlet systems $SH^{(\pm 1)}(\varphi, \psi; \delta)$ considered in Section 4. The reason for this is that the $\alpha$-shearlet covering $S^{(\alpha)}$ divides the frequency plane into four conic regions (the top, bottom, left, and right frequency cones) and a low-frequency region, while the usual definition of shearlets only divides the frequency plane into two cones (horizontal and vertical) and a low-frequency region. To remedy this fact, Section 5 introduces a slightly modified covering, the so-called unconnected $\alpha$-shearlet covering $S^{(\alpha)}_u$; the reason for this terminology being that the individual sets of the covering are not connected anymore. Essentially, $S^{(\alpha)}_u$ is obtained by combining each pair of opposing sets of the $\alpha$-shearlet covering $S^{(\alpha)}$ into one single set. We then verify that the associated decomposition spaces coincide with the previously defined $\alpha$-shearlet smoothness spaces. Finally, we show that the Banach frames and atomic decompositions obtained by applying the theory of structured Banach frame decompositions with the covering $S^{(\alpha)}_u$ indeed yield conventional cone-adapted shearlet systems.

In Section 6 we apply the equivalence of analysis and synthesis sparsity for $\alpha$-shearlets to prove that $\alpha$-shearlet frames with sufficiently nice generators indeed yield (almost) optimal $N$-term approximations for the class $E^\beta(\mathbb{R}^2)$ of $C^\beta$-cartoon-like functions, for $\beta \in (1, 2]$ and $\alpha = \beta^{-1}$. In case of usual shearlets (i.e., for $\alpha = \frac{1}{2}$), this is a straightforward application of the analysis sparsity of $C^\beta$-cartoon-like functions with respect to shearlet systems. But in case of $\alpha \neq \frac{1}{2}$, our $\alpha$-shearlet systems use the $\alpha$-parabolic scaling matrices $\delta(2^i, 2^{\alpha i})$, while analysis sparsity of $C^\beta$-cartoon-like functions is only known with respect to $\beta$-shearlet systems, which use the scaling matrices $\delta(2^{i/2}, 2^{(i/2)})$. Bridging the gap between these two different shearlet systems is not too hard, but cumbersome,
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so that part of the proof for α ≠ 1 is deferred to Section 4 since most readers are probably mainly interested in the (easier) case of classical shearlets (i.e., α = 1/2). The obtained approximation rate is almost optimal (cf. [28, Theorem 2.8]) if one restricts to systems where the N-term approximation is formed under a certain polynomial search depth restriction. But in the main text of the paper, we just construct some N-term approximation, which not necessarily fulfills this restriction concerning the search depth. In Section 5 we give a modified proof which shows that one can indeed retain the same approximation rate, even under a polynomial search depth restriction.

Finally, in Section 6 we completely characterize the existence of embeddings \( \mathcal{S}_{α,2}^p(\mathbb{R}^2) \hookrightarrow \mathcal{S}_{α,2}^{p,q}(\mathbb{R}^2) \) between α-shearlet smoothness spaces for different values of α. Effectively, this characterizes the cases in which one can obtain sparsity with respect to α2-shearlets when the only knowledge available is a certain sparsity with respect to α1-shearlets.

1.6. Notation. We write \( \mathbb{N} = \mathbb{Z}_{≥1} \) for the set of natural numbers and \( \mathbb{N}_0 = \mathbb{Z}_{≥0} \) for the set of natural numbers including 0. For a matrix \( A ∈ \mathbb{R}^{d×d} \), we denote by \( A^T \) the transpose of \( A \). The norm \( ||A|| \) of \( A \) is the usual operator norm of \( A \), acting on \( \mathbb{R}^d \) equipped with the usual euclidean norm \( ||·||_2 \). The open euclidean ball of radius \( r > 0 \) around \( x ∈ \mathbb{R}^d \) is denoted by \( B_r(x) \). For a linear (bounded) operator \( T : X → Y \) between (quasi)-normed spaces \( X,Y \), we denote the operator norm of \( T \) by

\[
||T|| := ||T||_{X → Y} := \sup_{||x||_X ≤ 1} ||Tx||_Y.
\]

For an arbitrary set \( M \), we let \( |M| ∈ \mathbb{N}_0 ∪ \{∞\} \) denote the number of elements of the set. For \( n ∈ \mathbb{N}_0 \), we write \( \underline{n} := \{1, \ldots, n\} \); in particular, \( \underline{0} = \emptyset \). For the closure of a subset \( M \) of some topological space, we write \( \overline{M} \).

The d-dimensional Lebesgue measure of a (measurable) set \( M ⊂ \mathbb{R}^d \) is denoted by \( λ(M) \) or by \( λ_d(M) \). Occasionally, we will also use the constant \( s_d := \mathcal{H}^{d−1}(S^{d−1}) \), the surface area of the euclidean unit-sphere \( S^{d−1} ⊂ \mathbb{R}^d \). The complex conjugate of \( z ∈ \mathbb{C} \) is denoted by \( \overline{z} \). We use the convention \( x^0 = 1 \) for all \( x ∈ [0,∞) \), even for \( x = 0 \).

For a subset \( M ⊂ B \) of a fixed base set \( B \) (which is usually implied by the context), we define the indicator function (or characteristic function) \( 1_M \) of the set \( M \) by

\[
1_M : B → \{0,1\}, x → \begin{cases} 
1, & \text{if } x ∈ M, \\
0, & \text{otherwise.}
\end{cases}
\]

The translation and modulation of a function \( f : \mathbb{R}^d → \mathbb{C}^k \) by \( x ∈ \mathbb{R}^d \) or \( ξ ∈ \mathbb{R}^d \) are, respectively, denoted by \( L_x f : \mathbb{R}^d → \mathbb{C}^k, y → f(y−x) \), and \( M_ξ f : \mathbb{R}^d → \mathbb{C}^k, y → e^{2πi(ξ·y)}f(y) \).

Furthermore, for \( g : \mathbb{R}^d → \mathbb{C}^k \), we use the notation \( \tilde{g} \) for the function \( \tilde{g} : \mathbb{R}^d → \mathbb{C}^k, x → g(−x) \).

For the Fourier transform, we use the convention \( \hat{f}(ξ) := (\mathcal{F}f)(ξ) := \int_{\mathbb{R}^d} f(x) e^{−2πi(ξ·x)}dx \) for \( f ∈ L^1(\mathbb{R}^d) \). It is well-known that the Fourier transform extends to a unitary automorphism \( \mathcal{F} : L^2(\mathbb{R}^d) → L^2(\mathbb{R}^d) \). The inverse of this map is the continuous extension of the inverse Fourier transform, given by \( (\mathcal{F}^{-1}f)(x) := \frac{1}{(2π)^d} \int_{\mathbb{R}^d} f(ξ) e^{2πi(ξ·x)}dξ \) for \( f ∈ L^1(\mathbb{R}^d) \). We will make frequent use of the space \( \mathcal{S}(\mathbb{R}^d) \) of Schwartz functions and its topological dual space \( \mathcal{S}'(\mathbb{R}^d) \), the space of tempered distributions. For more details on these spaces, we refer to [21, Section 9]; in particular, we note that the Fourier transform restricts to a linear homeomorphism \( \mathcal{F} : \mathcal{S}(\mathbb{R}^d) → \mathcal{S}(\mathbb{R}^d) \); by duality, we can thus define \( \mathcal{F} : \mathcal{S}'(\mathbb{R}^d) → \mathcal{S}'(\mathbb{R}^d) \) by \( \mathcal{F}\varphi = \varphi ∘ \mathcal{F} \) for \( \varphi ∈ \mathcal{S}'(\mathbb{R}^d) \).

Given an open subset \( U ⊂ \mathbb{R}^d \), we let \( \mathcal{D}'(U) \) denote the space of distributions on \( U \), i.e., the topological dual space of \( \mathcal{D}(U) := C_0^∞(U) \). For the precise definition of the topology on \( C_0^∞(U) \), we refer to [55, Chapter 6]. We remark that the dual pairings \( ⟨·,·⟩_{\mathcal{D}'×\mathcal{D}} \) and \( ⟨·,·⟩_{\mathcal{S}'×\mathcal{S}} \) are always taken to be bilinear instead of sesquilinear.

Occasionally, we will make use of the Sobolev space

\[
W^{N,p}(\mathbb{R}^d) = \{ f ∈ L^p(\mathbb{R}^d) \, | \, 0 ∈ \mathbb{N}_0 \text{ with } |α| ≤ N : \partial^α f ∈ L^p(\mathbb{R}^d) \}\, \text{ with } p ∈ [1,∞].
\]

Here, as usual for Sobolev spaces, the partial derivatives \( \partial^α f \) have to be understood in the distributional sense.

Furthermore, we will use the notations \( |x| := \min \{ k ∈ \mathbb{Z} : k ≥ x \} \) and \( |x| := \max \{ k ∈ \mathbb{Z} : k ≤ x \} \) for \( x ∈ \mathbb{R} \). We observe \( |x| ≤ x < |x| + 1 \) and \( |x| − 1 < x ≤ |x| \). Sometimes, we also write \( x_+ := \max\{0,x\} \) for \( x ∈ \mathbb{R} \).

Finally, we will frequently make use of the shearing matrices \( S_x \), the α-parabolic dilation matrices \( D_α^a \) and the involutive matrix \( \tilde{R} \), given by

\[
S_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \text{ and } \quad D_α^a := \begin{pmatrix} b & 0 \\ 0 & b^α \end{pmatrix}, \text{ as well as } \quad \tilde{R} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

for \( x ∈ \mathbb{R} \) and \( α, b ∈ [0,∞) \).
2. Structured Banach frame decompositions of decomposition spaces — A crash course

In this section, we give a brief introduction to the theory of structured Banach frames and atomic decompositions for decomposition spaces that was recently developed by one of the authors in [62].

We start with a crash course on decomposition spaces. These are defined using a suitable covering \( \mathcal{Q} = (Q_i)_{i \in I} \) of (a subset of) the frequency space \( \mathbb{R}^d \). For the decomposition spaces to be well-defined and for the theory in [62] to be applicable, the covering \( \mathcal{Q} \) needs to be a semi-structured covering for which a regular partition of unity exists. For this, it suffices if \( \mathcal{Q} \) is an almost structured covering. Since the notion of almost structured coverings is somewhat easier to understand than general semi-structured coverings, we will restrict ourselves to this concept.

**Definition 2.1.** Let \( \emptyset \neq \mathcal{O} \subset \mathbb{R}^d \) be open. A family \( \mathcal{Q} = (Q_i)_{i \in I} \) is called an almost structured covering of \( \mathcal{O} \), if for each \( i \in I \), there is an invertible matrix \( T_i \in \text{GL}(\mathbb{R}^d) \), a translation \( b_i \in \mathbb{R}^d \) and an open, bounded set \( Q_i' \subset \mathbb{R}^d \) such that the following conditions are fulfilled:

1. We have \( Q_i = T_i Q_i' + b_i \) for all \( i \in I \).
2. We have \( Q_i \subset \mathcal{O} \) for all \( i \in I \).
3. \( Q \) is admissible, i.e., there is some \( N_\mathcal{Q} \in \mathbb{N} \) satisfying \( |i^*| \leq N_\mathcal{Q} \) for all \( i \in I \), where the index-cluster \( i^* \) is defined as

\[
  i^* := \{ \ell \in I \mid Q_\ell \cap Q_i \neq \emptyset \} \quad \text{for } i \in I. \tag{2.1}
\]

4. There is a constant \( C_\mathcal{Q} > 0 \) satisfying \( \|T_i^{-1}T_j\| \leq C_\mathcal{Q} \) for all \( i \in I \) and all \( j \in i^* \).
5. For each \( i \in I \), there is an open set \( P_i' \subset \mathbb{R}^d \) with the following additional properties:
   a. \( P_i' \subset Q_i' \) for all \( i \in I \).
   b. The sets \( \{P_i' \mid i \in I\} \) and \( \{Q_i' \mid i \in I\} \) are finite.
   c. We have \( \mathcal{O} = \bigcup_{i \in I} (T_i P_i' + b_i) \).

**Remark.** • In the following, if we require \( \mathcal{Q} = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I} \) to be an almost structured covering of \( \mathcal{O} \), it is always implicitly understood that \( T_i, Q_i' \) and \( b_i \) are chosen in such a way that the conditions in Definition 2.1 are satisfied.

• Since each set \( Q_i' \) is bounded and since the set \( \{Q_i' \mid i \in I\} \) is finite, the family \( (Q_i')_{i \in I} \) is uniformly bounded, i.e., there is some \( R_\mathcal{Q} > 0 \) satisfying \( Q_i' \subset \overline{B_{R_\mathcal{Q}}}(0) \) for all \( i \in I \).

A crucial property of almost structured coverings is that these always admit a regular partition of unity, a notion which was originally introduced in [61] Definition 2.4.

**Definition 2.2.** Let \( \mathcal{Q} = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I} \) be an almost structured covering of the open set \( \emptyset \neq \mathcal{O} \subset \mathbb{R}^d \). We say that the family \( \Phi = (\varphi_i)_{i \in I} \) is a regular partition of unity subordinate to \( \mathcal{Q} \) if the following hold:

1. We have \( \varphi_i \in C_0^\infty(\mathcal{O}) \) with supp \( \varphi_i \subset Q_i \) for all \( i \in I \).
2. We have \( \sum_{i \in I} \varphi_i \equiv 1 \) on \( \mathcal{O} \).
3. For each \( \alpha \in \mathbb{N}_0^d \), the constant

\[
  C(\alpha) := \sup_{i \in I} \|\partial^\alpha \varphi_i^b\|_{\sup}
\]

is finite, where for each \( i \in I \), the normalized version \( \varphi_i^b \) of \( \varphi_i \) is defined as

\[
  \varphi_i^b : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto \varphi_i(T_i \xi + b_i). \tag*{\blacksquare}
\]

**Theorem 2.3.** (cf. [61] Theorem 2.8 and see [1] Proposition 1 for a similar statement)

Every almost structured covering \( \mathcal{Q} \) of an open subset \( \emptyset \neq \mathcal{O} \subset \mathbb{R}^d \) admits a regular partition of unity \( \Phi = (\varphi_i)_{i \in I} \) subordinate to \( \mathcal{Q} \).

Before we can give the formal definition of decomposition spaces, we need one further notion:

**Definition 2.4.** (cf. [10] Definition 3.1) Let \( \emptyset \neq \mathcal{O} \subset \mathbb{R}^d \) be open and assume that \( \mathcal{Q} = (Q_i)_{i \in I} \) is an almost structured covering of \( \mathcal{O} \). A weight \( w \) on the index set \( I \) is simply a sequence \( w = (w_i)_{i \in I} \) of positive numbers \( w_i > 0 \). The weight \( w \) is called \( \mathcal{Q} \)-moderate if there is a constant \( C_{\mathcal{Q},w} > 0 \) so that

\[
  w_j \leq C_{\mathcal{Q},w} \cdot w_i \quad \forall i \in I \text{ and all } j \in i^*, \tag{2.2}
\]

For an arbitrary weight \( w = (w_i)_{i \in I} \) on \( I \) and \( q \in \{0, \infty\} \) we define the weighted \( \ell^q \) space \( \ell^q_w(I) \) as

\[
  \ell^q_w(I) := \{ (c_i)_{i \in I} \in C^I \mid (w_i \cdot c_i)_{i \in I} \in \ell^q(I) \},
\]
equipped with the natural (quasi)-norm \( \| (c_i)_{i \in I} \|_{\mathcal{D}_F} := \| (w_i \cdot c_i)_{i \in I} \|_{L^p} \). We will also use the notation \( \| c \|_{\mathcal{D}_F} \) for arbitrary sequences \( c = (c_i)_{i \in I} \in [0, \infty]^I \) with the understanding that \( \| c \|_{\mathcal{D}_F} = \infty \) if \( c_i = \infty \) for some \( i \in I \) or if \( c \notin \ell^q_w(I) \).

Now, we can finally give a precise definition of decomposition spaces. We begin with the (easier) case of the so-called Fourier-side decomposition spaces.

**Definition 2.5.** Let \( Q = (Q_i)_{i \in I} \) be an almost structured covering of the open set \( \emptyset \neq O \subset \mathbb{R}^d \), let \( w = (w_i)_{i \in I} \) be a \( Q \)-moderate weight on \( I \) and let \( p, q \in (0, \infty] \). Finally, let \( \Phi = (\varphi_i)_{i \in I} \) be a regular partition of unity subordinate to \( Q \). We then define the associated Fourier-side decomposition space (quasi)-norm as

\[
\| g \|_{\mathcal{D}_F(Q, L^p, \ell^q_w)} := \left( \| \mathcal{F}^{-1}(\varphi_i \cdot g) \|_{L^p} \right)_{i \in I} \in [0, \infty) \quad \text{for each distribution } g \in \mathcal{D}'(O)
\]

The associated Fourier-side decomposition space is simply

\[
\mathcal{D}_F(Q, L^p, \ell^q_w) := \left\{ g \in \mathcal{D}'(O) \mid \| g \|_{\mathcal{D}_F(Q, L^p, \ell^q_w)} < \infty \right\}.
\]

**Remark.** Before we continue with the definition of the actual (space-side) decomposition spaces, a few remarks are in order:

- The expression \( \| \mathcal{F}^{-1}(\varphi_i \cdot g) \|_{L^p} \) makes sense for each \( i \in I \), since \( \varphi_i \in C^\infty_c(O) \), so that \( \varphi_i \cdot g \) is a compactly supported distribution on \( \mathbb{R}^d \) (and thus also a tempered distribution), so that the Paley-Wiener theorem (see e.g. [35] Theorem 7.23) shows that the tempered distribution \( \mathcal{F}^{-1}(\varphi_i \cdot g) \) is given by (integration against) a smooth function of which we can take the \( L^p \) quasi-norm.

- The notations \( \| g \|_{\mathcal{D}_F(Q, L^p, \ell^q_w)} \) and \( \mathcal{D}_F(Q, L^p, \ell^q_w) \) both suppress the specific regular partition of unity \( \Phi \) that was chosen. This is justified, since [60] Corollary 3.18 shows that any two \( L^p \)-BAPUs \( \Phi, \Psi \) yield equivalent quasi-norms and thus the same (Fourier-side) decomposition spaces. This suffices, since [60] Corollary 2.7 shows that every regular partition of unity is also a \( L^p \)-BAPU for \( Q \), for arbitrary \( p \in (0, \infty] \).

- Finally, [60] Theorem 3.21 shows that \( \mathcal{D}_F(Q, L^p, \ell^q_w) \) is a quasi-Banach space.

**Definition 2.6.** For an open set \( \emptyset \neq O \subset \mathbb{R}^d \), let \( Z(O) := \mathcal{F}^{-1}(C^\infty_c(O)) \subset \mathcal{S}(\mathbb{R}^d) \) and equip this space with the unique topology which makes the Fourier transform \( \mathcal{F} : C^\infty_c(O) \to Z(O), \varphi \mapsto \hat{\varphi} \) into a homeomorphism. The topological dual space of \( Z(O) \) is denoted by \( Z'(O) \). By duality, we define the Fourier transform on \( Z'(O) \) by \( \hat{g} := \mathcal{F} g := g \circ \mathcal{F} \in \mathcal{D}'(O) \) for \( g \in Z'(O) \).

Finally, under the assumptions of Definition 2.5, we define the (space-side) decomposition space associated to the parameters \( Q, p, q, w \) as

\[
\mathcal{D}(Q, L^p, \ell^q_w) := \left\{ g \in Z'(O) \mid \| g \|_{\mathcal{D}(Q, L^p, \ell^q_w)} := \| \hat{g} \|_{\mathcal{D}_F(Q, L^p, \ell^q_w)} < \infty \right\}.
\]

It is not hard to see that the Fourier transform \( \mathcal{F} : Z'(O) \to \mathcal{D}'(O) \) is an isomorphism which restricts to an isometric isomorphism \( \mathcal{F} : \mathcal{D}(Q, L^p, \ell^q_w) \to \mathcal{D}_F(Q, L^p, \ell^q_w) \).

**Remark.** For an explanation why the reservoirs \( \mathcal{D}'(O) \) and \( Z'(O) \) are the correct choices for defining \( \mathcal{D}_F(Q, L^p, \ell^q_w) \) and \( \mathcal{D}(Q, L^p, \ell^q_w) \), even in case of \( O = \mathbb{R}^d \), we refer to [60] Remark 3.13.

Now that we have formally introduced the notion of decomposition spaces, we present the framework developed in [62] for the construction of Banach frames and atomic decompositions for these spaces. To this end, we introduce the following set of notations and standing assumptions:

**Assumption 2.7.** We fix an almost structured covering \( Q = (T_i Q'_i + b_i)_{i \in I} \) with associated regular partition of unity \( \Phi = (\varphi_i)_{i \in I} \) for the remainder of the section. By definition of an almost structured covering, the set \( \{ Q'_i \mid i \in I \} \) is finite. Hence, we have \( \{ Q'_i \mid i \in I \} = \{ Q^{(1)}_0, \ldots, Q^{(n)}_0 \} \) for certain (not necessarily distinct) open, bounded subsets \( Q^{(1)}_0, \ldots, Q^{(n)}_0 \subset \mathbb{R}^d \). In particular, for each \( i \in I \), there is some \( k_i \in \mathbb{N} \) satisfying \( Q'_i = Q^{(k_i)}_0 \).

We fix the choice of \( n \in \mathbb{N} \), of the sets \( Q^{(1)}_0, \ldots, Q^{(n)}_0 \) and of the map \( I \to \mathbb{N}, i \mapsto k_i \) for the remainder of the section.

Finally, we need a suitable coefficient space for our Banach frames and atomic decompositions:

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3The exact definition of an \( L^p \)-BAPU is not important for us. The interested reader can find the definition in [60] Definition 3.5].
Definition 2.8. For given $p, q \in (0, \infty]$ and a given weight $w = (w_i)_{i \in I}$ on $I$, we define the associated coefficient space as
\[
C_{w}^{p,q} := \ell_{[\frac{1}{|\det T_i| \cdot \frac{\hat{w}}{w_i}]_{i \in I}}^{p} \left( \ell^{p}(\mathbb{Z}^{d}) \right)_{i \in I}
\]
\[
:= \left\{ c = (c^{(i)}_{k})_{i \in I, k \in \mathbb{Z}^{d}} \mid \|c\|_{C_{w}^{p,q}} := \left\| \left( \frac{|\det T_i|^{-\frac{1}{p}} \cdot w_{i} \cdot \|c^{(i)}_{k}\|_{p} \right)_{i \in I} \right\|_{\ell^{\infty}} < \infty \right\} \leq C^{1 \times \mathbb{Z}^{d}}.
\]
Remark. Observe that if $w_{i} = |\det T_i|^{-\frac{1}{p}} \cdot \frac{\hat{w}}{w_i}$ and if $p = q$, then $C_{w}^{p,q} = \ell^{p}(I \times \mathbb{Z}^{d})$, with equal (quasi)-norms.

Now that we have introduced the coefficient space $C_{w}^{p,q}$, we are in a position to discuss the existence criteria for Banach frames and atomic decompositions that were derived in [22]. We begin with the case of Banach frames.

Theorem 2.9. Let $w = (w_i)_{i \in I}$ be a $\mathcal{Q}$-moderate weight, let $\varepsilon, p_{0}, q_{0} \in (0, 1]$ and let $p, q \in (0, \infty]$ with $p \geq p_{0}$ and $q \geq q_{0}$. Define
\[
N := \left\lfloor \frac{d + \varepsilon}{\min\{1, p\}} \right\rfloor, \quad \tau := \min\{1, p, q\} \quad \text{and} \quad \sigma := \tau \cdot \left( \frac{d}{\min\{1, p\}} + N \right).
\]
Let $\gamma^{(0)}_{1}, \ldots, \gamma^{(0)}_{n} : \mathbb{R}^{d} \to \mathbb{C}$ be given and define $\gamma_{i} := \gamma^{(0)}_{i}$ for $i \in I$. Assume that the following conditions are satisfied:

1. We have $\gamma^{(0)}_{i} \in L^{1}(\mathbb{R}^{d})$ and $\mathcal{F}_{\gamma^{(0)}_{i}} \in C^\infty(\mathbb{R}^{d})$ for all $k \in \mathbb{N}$, where all partial derivatives of $\mathcal{F}_{\gamma^{(0)}_{i}}$ are polynomially bounded.
2. We have $\gamma^{(0)}_{k} \in C^{1}(\mathbb{R}^{d})$ and $\nabla \gamma^{(0)}_{k} \in L^{1}(\mathbb{R}^{d}) \cap L^{\infty}(\mathbb{R}^{d})$ for all $k \in \mathbb{N}$.
3. We have $\mathcal{F}_{\gamma^{(0)}_{k}}(\xi) \neq 0$ for all $\xi \in \mathcal{Q}^{(k)}$ and all $k \in \mathbb{N}$.
4. We have
\[
C_{1} := \sup_{i \in I} \sum_{j \in I} M_{j,i} < \infty \quad \text{and} \quad C_{2} := \sup_{i \in I} \sum_{j \in I} M_{j,i} < \infty,
\]
where
\[
M_{j,i} := \left( \frac{w_{j}^{(1)}}{w_{i}^{(1)}} \right)^{\tau} \cdot (1 + \|T_{j}^{-1}T_{i}\|)^{\sigma \max_{|\beta| \leq 1} \left( \frac{|\det T_{i}|^{-\frac{1}{p}} \cdot \frac{\hat{w}}{w_i} \cdot \|\nabla \gamma^{(0)}_{i} \cdot (T_{j}^{-1}(\xi - b_{j}))\|_{p} \right)_{i \in I}} ^{\tau}.
\]
Then there is some $\delta_{0} = \delta_{0}(p, q, w, \varepsilon, (\gamma_{i})_{i \in I}) > 0$ such that for arbitrary $0 < \delta \leq \delta_{0}$, the family
\[
\left( \mathcal{L}_{T_{i}, T_{j}^{-1}} \gamma^{[i]} \right)_{i \in I, k \in \mathbb{Z}^{d}} \quad \text{with} \quad \gamma^{[i]} := |\det T_{i}|^{1/2} \cdot M_{b} \left( \gamma_{i} \circ T_{i}^{-1} \right) \quad \text{and} \quad \gamma^{[i]}(x) = \gamma^{[i]}(-x)
\]
forms a Banach frame for $\mathcal{D}(\mathcal{Q}, L^{p}, \ell^{q}_{w})$. Precisely, this means the following:

- The analysis operator
\[
A^{(k)} : \mathcal{D}(\mathcal{Q}, L^{p}, \ell^{q}_{w}) \to C_{w}^{p,q}, f \mapsto \left( \gamma^{[i]} \ast f \right) \left( \delta \cdot T_{i}^{-1}Tk \right)_{i \in I, k \in \mathbb{Z}^{d}}
\]
is well-defined and bounded for each $\delta \in (0, 1]$. Here, the convolution $\gamma^{[i]} \ast f$ is defined as
\[
(\gamma^{[i]} \ast f)(x) = \sum_{i \in I} \mathcal{F}^{-1} \left( \gamma^{[i]} \cdot \varphi_{T_{i}^{-1}Tk} \hat{f} \right)(x) \quad \forall x \in \mathbb{R}^{d},
\]
where the series converges normally in $L^{\infty}(\mathbb{R}^{d})$ and thus absolutely and uniformly, for each $f \in \mathcal{D}(\mathcal{Q}, L^{p}, \ell^{q}_{w})$. For a more convenient expression of $(\gamma^{[i]} \ast f)(x)$, at least for $f \in L^{2}(\mathbb{R}^{d}) \subset L^{q}(\mathcal{O})$, see Lemma 5.12.

- For $0 < \delta \leq \delta_{0}$, there is a bounded linear reconstruction operator $R^{(k)} : C_{w}^{p,q} \to \mathcal{D}(\mathcal{Q}, L^{p}, \ell^{q}_{w})$ satisfying $R^{(k)} \circ A^{(k)} = \text{id}_{\mathcal{D}(\mathcal{Q}, L^{p}, \ell^{q}_{w})}$.

- We have the following consistency property: If $\mathcal{Q}$-moderate weights $w^{(1)} = (w_{i}^{(1)})_{i \in I}$ and $w^{(2)} = (w_{i}^{(2)})_{i \in I}$ and exponents $p_{1}, p_{2}, q_{1}, q_{2} \in (0, \infty]$ are chosen such that the assumptions of the current theorem are satisfied for $p_{1}, q_{1}, w^{(1)}$, as well as for $p_{2}, q_{2}, w^{(2)}$ and if $0 < \delta \leq \min \left\{ \delta_{0}(p_{1}, q_{1}, w^{(1)}, \varepsilon, (\gamma_{i})_{i \in I}), \delta_{0}(p_{2}, q_{2}, w^{(2)}, \varepsilon, (\gamma_{i})_{i \in I}) \right\}$ then we have the following equivalence:
\[
\forall f \in \mathcal{D}(\mathcal{Q}, L^{p_{1}}, \ell^{q_{1}}_{w^{(1)}}) : f \in \mathcal{D}(\mathcal{Q}, L^{p_{2}}, \ell^{q_{2}}_{w^{(2)}}) \iff \left( \gamma^{[i]} \ast f \right)(\delta \cdot T_{i}^{-1}Tk)_{i \in I, k \in \mathbb{Z}^{d}} \in C_{w}^{p_{1},q_{1}}.
\]
Finally, there is an estimate for the size of $\delta_{0}$ which is independent of the choice of $p \geq p_{0}$ and $q \geq q_{0}$: There is a constant $K = K(p_{0}, q_{0}, \varepsilon, d, \mathcal{Q}, \Phi, \gamma^{(0)}_{1}, \ldots, \gamma^{(0)}_{n}) > 0$ such that we can choose
\[
\delta_{0} = 1/\left[ 1 + K \cdot C^{4}_{\mathcal{Q}, w} \cdot \left( C_{1}^{1/2} + C_{2}^{1/2} \right)^{2} \right].
\]
Proof. This is a special case of Theorem 2.7 for $\Omega_0 = \Omega_1 = 1$, $K = 0$ and $v = v_0 \equiv 1$. □

Now, we provide criteria which ensure that a given family of prototypes generates atomic decompositions.

**Theorem 2.10.** Let $w = (w_i)_{i \in I}$ be a $Q$-moderate weight, let $\varepsilon, p_0, q_0 \in (0, 1]$ and let $p, q \in (0, \infty)$ with $p \geq p_0$ and $q \geq q_0$. Define

$$N := \left\lceil \frac{d + \varepsilon}{\min\{1, p\}} \right\rceil, \quad \tau := \min\{1, p, q\}, \quad \vartheta := \left(\frac{1}{p} - 1\right)_+, \quad \text{and} \quad \varrho := 1 + \frac{d}{\min\{1, p\}},$$

as well as

$$\sigma := \begin{cases} \tau \cdot (d + 1), & \text{if } p \in [1, \infty), \\ \tau \cdot (p^{-1} \cdot d + \lfloor p^{-1} \cdot (d + \varepsilon) \rfloor), & \text{if } p \in (0, 1). \end{cases}$$

Let $\gamma^{(0)}_1, \ldots, \gamma^{(0)}_n : \mathbb{R}^d \to \mathbb{C}$ be given and define $\gamma_i := \gamma^{(0)}_{k_i}$ for $i \in I$. Assume that there are functions $\gamma^{(0,j)}_1, \ldots, \gamma^{(0,j)}_n$ for $j \in \{1, 2\}$ such that the following conditions are satisfied:

1. We have $\gamma^{(0,1)}_k \in L^1(\mathbb{R}^d)$ for all $k \in \mathbb{N}$.
2. We have $\gamma^{(0,2)}_k \in C^1(\mathbb{R}^d)$ for all $k \in \mathbb{N}$.
3. We have

$$\Omega^{(p)} := \max_{k \in \mathbb{N}} \left\| \nabla \gamma^{(0,2)}_k \right\|_\varrho < \infty,$$

where $\left\| \cdot \right\|_\varrho = \sup_{x \in \mathbb{R}^d} (1 + |x|)^\tau \cdot \left\| f(x) \right\|$ for $f : \mathbb{R}^d \to \mathbb{C}^\ell$ and (arbitrary) $\ell \in \mathbb{N}$.
4. We have $\mathcal{F}\gamma^{(0,j)}_k \in C^\infty(\mathbb{R}^d)$ and all partial derivatives of $\mathcal{F}\gamma^{(0,j)}_k$ are polynomially bounded for all $k \in \mathbb{N}$ and $j \in \{1, 2\}$.
5. We have $\gamma^{(1)}_k = \gamma^{(0,1)}_k \ast \gamma^{(0,2)}_k$ for all $k \in \mathbb{N}$.
6. We have $\mathcal{F}\gamma^{(0)}_k(\xi) \neq 0$ for all $\xi \in Q^{(k)}_0$ and all $k \in \mathbb{N}$.
7. We have $\left\| \gamma^{(0)}_k \right\|_\varrho < \infty$ for all $k \in \mathbb{N}$.
8. We have

$$K_1 := \sup_{i \in I} \sum_{j \in I} N_{i,j} < \infty \quad \text{and} \quad K_2 := \sup_{j \in I} \sum_{i \in I} N_{i,j} < \infty,$$

where $N_{j,1} := \gamma^{(0,1)}_{k_j}$ for $j \in I$ and

$$N_{i,j} := \left( \frac{w_i}{w_j} \cdot (|\det T_j| / |\det T_i|)^\vartheta \cdot (1 + \left\| T_j^{-1} T_i \right\|) \cdot \left( |\det T_i|^{-1} \cdot \max_{|a| \leq N} \left\| \partial^\alpha \gamma^{(0)}_{k_j} \right\|_\varrho \cdot (T_j^{-1}|\xi - b_j|) \right)^\varrho \cdot \int_{Q_i} \left\| \partial^\alpha \gamma^{(0)}_{k_j} \right\|_\varrho \cdot (T_j^{-1} T_i) \right) \cdot d \xi.$$

Then there is some $\delta_0 \in (0, 1]$ such that the family

$$\Psi_\delta := \left( L_{\delta, T_i^{-T_k}} \gamma^{[i]}_i \right)_{i \in I, k \in \mathbb{Z}^d} \quad \text{with} \quad \gamma^{[i]}_i := |\det T_i|^{1/2} \cdot M_{b_i} \left[ \gamma_i \circ T_i^{-T_k} \right]$$

forms an atomic decomposition of $\mathcal{D}(Q, L^p, L^q)$, for all $\delta \in (0, \delta_0]$. Precisely, this means the following:

- The synthesis map

$$S^{(\delta)} : C^{p,q}_w \to \mathcal{D}(Q, L^p, L^q), (c_i^{(i)})_{i \in I, k \in \mathbb{Z}^d} \mapsto \sum_{i \in I} \sum_{k \in \mathbb{Z}^d} \left[ c_i^{(i)} \cdot L_{\delta, T_i^{-T_k}} \gamma^{[i]}_i \right]$$

is well-defined and bounded for every $\delta \in (0, 1]$.

- For $0 < \delta \leq \delta_0$, there is a bounded linear coefficient map $C^{(\delta)} : \mathcal{D}(Q, L^p, L^q) \to C^{p,q}_w$ satisfying

$$S^{(\delta)} \circ C^{(\delta)} = \text{id}_{\mathcal{D}(Q, L^p, L^q)}.$$

Finally, there is an estimate for the size of $\delta_0$ which is independent of $p \geq p_0$ and $q \geq q_0$: There is a constant $K = K(p_0, q_0, \varepsilon, d, Q, \Phi, \gamma^{(0)}_1, \ldots, \gamma^{(0)}_n) > 0$ such that we can choose

$$\delta_0 = \min \left\{ 1, \left[ K \cdot \Omega^{(p)} \cdot (K_1^{1/\varrho} + K_2^{1/\varrho}) \right]^{-1} \right\}. \quad \blacksquare$$
Remark. • Convergence of the series defining \( S^{(\delta)} \) has to be understood as follows: For each \( i \in I \), the series
\[
\sum_{k \in \mathbb{Z}^d} \left[ c_k^{(i)} \cdot L_{\delta \cdot T_{\xi}^{-\gamma} \psi} \right]
\]
converges pointwise absolutely to a function \( g_i \in L^1_{\text{loc}}(\mathbb{R}^d) \cap S'(\mathbb{R}^d) \) and the series \( \sum_{j \in I} g_j = S^{(\delta)}(c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} \) converges unconditionally in the weak-\( * \)-sense in \( Z'(\mathcal{O}) \), i.e., for every \( \phi \in Z(\mathcal{O}) = \mathcal{F}(C_c^\infty(\mathcal{O})) \), the series \( \sum_{j \in I} \langle g_j, \phi \rangle_{S', S} \) converges absolutely and the functional \( \phi \mapsto \sum_{j \in I} \langle g_j, \phi \rangle_{S', S} \) is continuous on \( Z(\mathcal{O}) \).

• The action of \( C^{(\delta)} \) on a given \( f \in D(Q, L^p, \ell^q_w) \) is independent of the precise choice of \( p, q, w \), as long as \( C^{(\delta)} f \) is defined at all.

\[ \Box \]

Proof of Theorem 2.10. This is a special case of Theorem 2.9 for \( \Omega = \Omega_1 = 1 \) and \( v = v_0 \equiv 1 \).

The main limitation of Theorem 2.10—in comparison to Theorem 2.9—is that we require each \( \gamma_k^{(0)} \) to be factorized as a convolution product \( \gamma_k^{(0)} = \gamma_k^{(0,1)} \ast \gamma_k^{(0,2)} \), which is tedious to verify. To simplify such verifications, the following result is helpful:

Proposition 2.11. (cf. [52, Lemma 6.9])

Let \( \varrho \in L^1 \left( \mathbb{R}^d \right) \) with \( \varrho \geq 0 \). Let \( N \in \mathbb{N} \) with \( N \geq d + 1 \) and assume that \( \gamma \in L^1 \left( \mathbb{R}^d \right) \) satisfies \( \gamma \in C^\infty(\mathbb{R}^d) \) with
\[
|\varrho^\gamma(\xi)| \leq \varrho(\xi) \cdot (1 + |\xi|)^{-(d+1+\epsilon)} \quad \forall \xi \in \mathbb{R}^d \quad \forall \alpha \in \mathbb{N}_0 \quad \text{with} \quad |\alpha| \leq N
\]
for some \( \epsilon \in (0, 1) \).

Then there are functions \( \gamma_1 \in C_0 \left( \mathbb{R}^d \right) \cap L^1 \left( \mathbb{R}^d \right) \) and \( \gamma_2 \in C^1 \left( \mathbb{R}^d \right) \cap W^{1,1} \left( \mathbb{R}^d \right) \) with \( \gamma = \gamma_1 \ast \gamma_2 \) and with the following additional properties:

1. We have \( \|\gamma_2\|_K \leq \varrho_d \cdot 2^{4+d+3K} \cdot |\gamma| \cdot |\gamma_2| \leq \varrho_{d+1+2K} \cdot 2^{4+d+3K} \cdot (1 + d)^{1+2K} \cdot (K + 1)! \) for all \( K \in \mathbb{N}_0 \), where \( \|\varrho\|_K := \sup_{x \in \mathbb{R}^d} \|\varrho(x)\|_K \).

2. We have \( \bar{\gamma}_2 \in C^\infty(\mathbb{R}^d) \) with all partial derivatives being polynomially bounded (even bounded).

3. If \( \bar{\gamma} \in C^\infty(\mathbb{R}^d) \) with all partial derivatives being polynomially bounded, the same also holds for \( \bar{\gamma}_1 \).

4. We have \( \|\gamma_1\|_N \leq (1 + d)^{1+2N} \cdot 2^{1+4+N} \cdot N! \cdot \|\varrho\|_{L^1} \) and \( \|\gamma\|_N \leq (1 + d)^{1+2N} \cdot \|\varrho\|_{L^1} \).

5. We have \( |\varrho^\gamma(\xi)| \leq 2^{1+d+4N} \cdot N! \cdot |\gamma(\xi)| \) for all \( \xi \in \mathbb{R}^d \) and \( \alpha \in \mathbb{N}_0^d \) with \( |\alpha| \leq N \).

\[ \Box \]

3. Definition and basic properties of \( \alpha \)-shearlet smoothness spaces

In this section, we introduce the class of \( \alpha \)-shearlet smoothness spaces. These spaces are a generalization of the “ordinary” shearlet smoothness spaces as introduced by Labate et al. [52]. Later on (cf. Theorem 3.13), it will turn out that these spaces simultaneously describe analysis and synthesis sparsity with respect to (suitable) \( \alpha \)-shearlet frames.

We will define the \( \alpha \)-shearlet smoothness spaces as certain decomposition spaces. Thus, we first have to define the associated covering and the weight for the sequence space \( l^q_{w_0}(I) \) that we will use.

Definition 3.1. Let \( \alpha \in [0, 1] \). The \( \alpha \)-shearlet covering \( S^{(\alpha)} \) is defined as
\[
S^{(\alpha)} := (S^{(\alpha)}_i)_{i \in I^{(\alpha)}} = (T_i Q_1')_{i \in I^{(\alpha)}} = (T_i Q_1' + b_i)_{i \in I^{(\alpha)}},
\]
where:

• The index set \( I^{(\alpha)} \) is given by \( I := I^{(\alpha)} := \{0\} \cup I_0 \), where
\[
I_0 := I_0^{\alpha} := \{(n, m, \epsilon, \delta) \in N_0 \times Z \times \{\pm 1\} \times \{0, 1\} \mid \|m\| \leq G_n \} \quad \text{with} \quad G_n := G_n^{\alpha} := \lfloor 2^{n(1-\alpha)} \rfloor.
\]

• The basic sets \( (Q')_i \in I^{(\alpha)} \) are given by \( Q'_0 := (-1, 1)^2 \) and by \( Q'_i := Q := U((-1, 1)) \) for \( i \in I_0^{(\alpha)} \), where we used the notation
\[
U_{(a, b)}(\gamma, \mu) := \left\{ \left( \frac{\xi}{\eta} \right) \in (\gamma, \mu) \times \mathbb{R} \mid \frac{\eta}{\xi} \in (a, b) \right\} \quad \text{for} \ a, b \in \mathbb{R} \quad \text{and} \ \gamma, \mu \in (0, \infty).
\]

• The matrices \( (T_i)_{i \in I^{(\alpha)}} \) are given by \( T_0 := I \) and by \( T_i := T_i^{(\alpha)} := R^\delta \cdot A_{n,m,\epsilon, \delta} \) with \( A_{n,m,\epsilon, \delta} := \epsilon \cdot D_2^{(\alpha)} \cdot S_{\mu}^T \) for \( i = (n, m, \epsilon, \delta) \in I_0^{(\alpha)} \). Here, the matrices \( R, S_k \) and \( D_2^{(\alpha)} \) are as in equation (1.5).

• The translations \( (b_i)_{i \in I^{(\alpha)}} \) are given by \( b_i := 0 \) for all \( i \in I^{(\alpha)} \).

Finally, we define the weight \( w = (w_i)_{i \in I} \) by \( w_0 := 1 \) and \( w_{n,m,\epsilon, \delta} := 2^n \) for \( (n, m, \epsilon, \delta) \in I_0 \).
Our first goal is to show that the covering $S^{(a)}$ is an almost structured covering of $\mathbb{R}^2$ (cf. Definition 2.1). To this end, we begin with the following auxiliary lemma:

**Lemma 3.2.** (1) Using the notation $U^{(\gamma, \delta)}_{(a, b)}$ from equation (3.2) and the shearing matrices $S_x$ from equation (1.3), we have for arbitrary $m, a, b \in \mathbb{R}$ and $\kappa, \lambda, \gamma, \mu > 0$ that

$$S_{m}^{T} U^{(\gamma, \mu)}_{(a, b)} = U^{(\gamma, \mu)}_{(m + a, m + b)} \quad \text{and} \quad \operatorname{diag}(\lambda, \kappa) U^{(\gamma, \mu)}_{(a, b)} = U^{(\lambda \gamma, \lambda \mu)}_{(\lambda a \gamma, \lambda b \mu)}.$$  

Consequently,

$$T_i^{(a)} U^{(\gamma, \mu)}_{(a, b)} = \varepsilon \cdot U^{(2^n \gamma, 2^n \mu)}_{(2n(\gamma - 1)(m + a), 2n(\gamma - 1)(m + b))} \quad \text{for all} \quad i = (n, m, \varepsilon), 0 \in I_0.$$  

In particular, $S^{(a)}_{n, m, \varepsilon, 0} = \varepsilon \cdot U^{(2^n / 3, 3 \cdot 2^n)}_{(2n(\gamma - 1)(m - 1), 2n(\gamma - 1)(m + 1))}.$

(2) Let $i = (n, m, \varepsilon, \delta) \in I_0$ and let $(\frac{\xi}{\eta}) \in S_i^{(a)}$ be arbitrary. Then the following hold:

(a) If $i = (n, m, \varepsilon, 0)$, we have $\|\eta\| < 3 \cdot |\xi|.$

(b) If $i = (n, m, \varepsilon, 1)$, we have $|\xi| < 3 \cdot |\eta|.$

(c) We have $2^{n-2} \leq \frac{\eta}{\xi} < \frac{2^n}{2} < |\frac{\xi}{\eta}| < 12 \cdot 2^n < 2^{n+4}.$ \hfill $\blacktriangleleft$

**Proof.** We establish the different claims individually:

(1) The following is essentially identical with the proof of [59] Lemma 6.3.4] and is only given here for the sake of completeness. We first observe the following equivalences:

$$\left( \begin{array}{c} \xi \\ \eta \end{array} \right) \in U_{(m + a, m + b)}^{(\gamma, \delta)} \iff \xi \in (\gamma, \delta) \quad \text{and} \quad m + a < \frac{\eta}{\xi} < m + b$$

$$\iff \xi \in (\gamma, \delta) \quad \text{and} \quad a < \frac{\eta - m\xi}{\xi} < b$$

$$\iff \left( \begin{array}{cc} 1 & 0 \\ m & 1 \end{array} \right)^{-1} \left( \begin{array}{c} \xi \\ \eta \end{array} \right) = \left( \begin{array}{c} \xi \\ \eta - m\xi \end{array} \right) \in U_{(a, b)}^{(\gamma, \delta)}$$

and

$$\left( \begin{array}{c} \xi \\ \eta \end{array} \right) \in U_{(a, b)}^{(\lambda \gamma, \lambda \mu)} \iff \xi \in (\lambda \gamma, \lambda \mu) \quad \text{and} \quad \frac{\kappa}{\lambda} \cdot a < \frac{\eta}{\xi} < \frac{\kappa}{\lambda} \cdot b$$

$$\iff \lambda^{-1} \xi \in (\gamma, \mu) \quad \text{and} \quad a < \frac{\kappa^{-1}\eta}{\lambda^{-1}\xi} < b$$

$$\iff \left( \begin{array}{cc} \lambda & 0 \\ 0 & \kappa \end{array} \right)^{-1} \left( \begin{array}{c} \xi \\ \eta \end{array} \right) = \left( \begin{array}{c} \lambda^{-1} \xi \\ \kappa^{-1}\eta \end{array} \right) \in U_{(a, b)}^{(\gamma, \mu)}.$$  

These equivalences show that $\operatorname{diag}(\lambda, \kappa) U^{(\gamma, \mu)}_{(a, b)} = U^{(\lambda \gamma, \lambda \mu)}_{(\lambda a \gamma, \lambda b \mu)}$ and $S_{m}^{T} U^{(\gamma, \mu)}_{(a, b)} = U^{(\gamma, \mu)}_{(m + a, m + b)}.$ But for $i = (n, m, \varepsilon, 0)$, we have $T_i^{(a)} = R^\theta \cdot A^{(a)}_{n, m, \varepsilon} = \varepsilon \cdot \operatorname{diag}(2^n, 2^n) \cdot S_{m}^{T}.$ This easily yields the claim.

(2) We again show the three claims individually:

(a) For $i = (n, m, \varepsilon, 0) \in I_0$, equation (3.3) yields for $(\frac{\xi}{\eta}) \in S_i^{(a)}$ that

$$\frac{\eta}{\xi} \in \left( 2^{n(a-1)}(m-1), 2^{n(a-1)}(m+1) \right) \subset \left( -2^{n(a-1)}(m+1), 2^{n(a-1)}(m+1) \right),$$

since $2^{n(a-1)}(m+1) \leq 2^{n(a-1)}(|m|+1)$ and

$$2^{n(a-1)}(m-1) \geq 2^{n(a-1)}(-|m|-1) = -2^{n(a-1)}(|m|+1).$$

Because of $|m| \leq G_n = [2^{n(1-\alpha)}] < 2^{n(1-\alpha)} + 1$ and $|\xi| > 0$, it follows that

$$|\eta| = |\xi| \cdot |\frac{\eta}{\xi}| \leq |\xi| \cdot 2^{n(a-1)} \cdot (|m|+1) < |\xi| \cdot 2^{-n(1-\alpha)} \cdot \left( 2^{n(1-\alpha)} + 2 \right)$$

$$\leq |\xi| \cdot \left( 1 + 2 \cdot 2^{-n(1-\alpha)} \right) \leq 3 \cdot |\xi|. $$

(b) For $i = (n, m, \varepsilon, 1) \in I_0$ we have

$$\left( \begin{array}{c} \eta \\ \xi \end{array} \right) = R^\theta \cdot \left( \begin{array}{c} \xi \\ \eta \end{array} \right) \in R S_{n, m, \varepsilon, 1}^{(a)} = R T_{n, m, \varepsilon, 1}^{(a)} Q = R R A_{n, m, \varepsilon}^{(a)} Q = A_{n, m, \varepsilon}^{(a)} Q = S_{n, m, \varepsilon, 0}^{(a)},$$

so that we get $|\xi| < 3 \cdot |\eta|$ from the previous case.
Lemma 3.3. The $\alpha$-shearlet covering $S^{(\alpha)}$ from Definition 3.3 is an almost structured covering of $\mathbb{R}^2$.

Since the proof of Lemma 3.3 is quite lengthy, although it does not yield too much insight, we postpone it to the appendix (Section B).

Finally, before we can formally define the $\alpha$-shearlet smoothness spaces, we still need to verify that the weight $w$ from Definition 3.1 is $S^{(\alpha)}$-moderate (cf. Definition 2.4).

Lemma 3.4. For arbitrary $s \in \mathbb{R}$, the weight $w^s = (w_i^s)_{i \in I}$, with $w = (w_i)_{i \in I}$ as in Definition 3.1 is $S^{(\alpha)}$-moderate (cf. equation (2.2)) with

$$C_{S^{(\alpha)}, w^s} \leq 39^{|s|}.$$ 

Furthermore, we have

$$\frac{1}{3} \cdot w_i \leq 1 + |\xi| \leq 13 \cdot w_i \quad \forall i \in I \text{ and all } \xi \in S^{(\alpha)}_i.$$ 

Proof. First, let $i = (n, m, \varepsilon, \delta) \in I_0$ be arbitrary. By Lemma 3.2, we get

$$\frac{1}{3} \cdot w_i = \frac{2^n}{3} \leq |\xi| \leq 1 + |\xi| \leq 1 + 12 \cdot 2^n \leq 13 \cdot 2^n = 13 \cdot w_i \quad \forall \xi \in S^{(\alpha)}_i.$$ 

Furthermore, for $i = 0$, we have $S^{(\alpha)}_0 = (1, 1)^2$ and thus

$$\frac{1}{3} \cdot w_i \leq w_i = 1 \leq 1 + |\xi| \leq 3 \cdot w_i \leq 13 \cdot w_i \quad \forall \xi \in S^{(\alpha)}_i.$$ 

This establishes the second part of the lemma.

Next, let $i, j \in I$ with $S^{(\alpha)}_i \cap S^{(\alpha)}_j \neq \emptyset$. Pick an arbitrary $\xi \in S^{(\alpha)}_i \cap S^{(\alpha)}_j$ and note as a consequence of the preceding estimates that

$$\frac{w_i}{w_j} \leq \frac{3 \cdot (1 + |\xi|)}{13 \cdot (1 + |\xi|)} = 39.$$ 

By symmetry, this implies $\frac{1}{39} \leq \frac{w_i}{w_j} \leq 39$ and thus also

$$\frac{w_i^s}{w_j^s} = \left(\frac{w_i}{w_j}\right)^s \leq 39^{|s|}.$$ 

Now, we can finally formally define the $\alpha$-shearlet smoothness spaces:

Definition 3.5. For $\alpha \in [0, 1]$, $p, q \in (0, \infty)$ and $s \in \mathbb{R}$, we define the $\alpha$-shearlet smoothness space $T^{p,q}_{\alpha,s} (\mathbb{R}^2)$ associated to these parameters as

$$T^{p,q}_{\alpha,s} (\mathbb{R}^2) := D \left( S^{(\alpha)}, L^p, \ell^q_{w^s} \right),$$

where the covering $S^{(\alpha)}$ and the weight $w^s$ are as in Definition 3.1 and Lemma 3.3 respectively.

Remark. Since $S^{(\alpha)}$ is an almost structured covering by Lemma 3.3 and since $w^s$ is $S^{(\alpha)}$-moderate by Lemma 3.4, Definition 2.5 and the associated remark show that $T^{p,q}_{\alpha,s} (\mathbb{R}^2)$ is indeed well-defined, i.e., independent of the chosen regular partition of unity subordinate to $S^{(\alpha)}$. The same remark also implies that $T^{p,q}_{\alpha,s} (\mathbb{R}^2)$ is a Quasi-Banach space.

Recall that with our definition of decomposition spaces, $T^{p,q}_{\alpha,s} (\mathbb{R}^2)$ is a subspace of $Z^\prime (\mathbb{R}^2) = \left[ F \left( C^\infty_c (\mathbb{R}^2) \right) \right]^\prime$. But as our next result shows, each $f \in T^{p,q}_{\alpha,s} (\mathbb{R}^2)$ actually extends to a tempered distribution:
Lemma 3.6. Let $\alpha \in [0, 1]$, $p, q \in (0, \infty]$ and $s \in \mathbb{R}$. Then

$$\mathcal{F}^{p,q}_{\alpha,s}(\mathbb{R}^2) \hookrightarrow S'((\mathbb{R}^2)^\ast),$$

in the sense that each $f \in \mathcal{F}^{p,q}_{\alpha,s}(\mathbb{R}^2)$ extends to a uniquely determined tempered distribution $f_S \in S'((\mathbb{R}^2)^\ast)$. Furthermore, the map $\mathcal{F}^{p,q}_{\alpha,s}(\mathbb{R}^2) \hookrightarrow S'((\mathbb{R}^2)^\ast), f \mapsto f_S$ is linear and continuous with respect to the weak-*$\ast$-topology on $S'((\mathbb{R}^2)^\ast)$.

Proof. It is well known (cf. [21, Proposition 9.9]) that $C_0^\infty(\mathbb{R}^2) \leq S((\mathbb{R}^2)^\ast)$ is dense. Since $\mathcal{F} : S((\mathbb{R}^2)^\ast) \rightarrow S((\mathbb{R}^2)^\ast)$ is a homeomorphism, we see that $Z(\mathbb{R}^2) = \mathcal{F}((C_0^\infty(\mathbb{R}^2))^\ast) \leq S((\mathbb{R}^2)^\ast)$ is dense, too. Hence, for arbitrary $f \in \mathcal{F}^{p,q}_{\alpha,s}(\mathbb{R}^2)$, if there is any extension $g \in S'((\mathbb{R}^2)^\ast)$ of $f \in Z((\mathbb{R}^2)^\ast)$, then $g$ is uniquely determined.

Next, by Lemma 3.3, $S^{(\alpha)}$ is almost structured, so that [6] Theorem 8.2] shows that $S^{(\alpha)}$ is a regular covering of $\mathbb{R}^2$. Thus, once we verify that there is some $N \in \mathbb{N}_0$ such that the sequence $w^{(N)}(\theta) = (w_i^{(N)})_{i \in I}$ defined by

$$w_i^{(N)} := |\det T_i^{(\alpha)}|^{1/p} \cdot \max \left\{ 1, \left\| T_i^{(\alpha)} \right\|^{i+1} \right\} \cdot \inf_{\xi \in S_i^{(\alpha)}} (1 + |\xi|)^{-N}$$

satisfies $w^{N}(\theta) \in \ell^{q'}_1(I)$ with $q' = \infty$ in case of $q \in (0, 1)$, then the claim of the present lemma is a consequence of [60, Theorem 8.3] and the associated remark. Here, $(S_i^{(\alpha)})^\ast = \bigcup_{j \in I} S_j^{(\alpha)}$.

Since $I = \{0\} \cup I_0$ and since the single (finite(!)) term $w_i^{(N)}(\theta)$ does not influence membership of $w^{(N)}(\theta)$ in $\ell^{q'}_1(I)$, we only need to show $w_i^{(N)}|_{I_0} \in \ell^{q'}_1(I_0)$. But for $i = (n, m, \varepsilon, \delta) \in I_0$, we have

$$\|T_i^{-1}\| = \left\| \begin{bmatrix} 2^{-n} & 0 \\ -2^{-n}m & 2^{-\varepsilon-\delta} \end{bmatrix} \right\| \leq 3.$$

Here, the last step used that $|2^{-n}| \leq 1$, $|2^{-\varepsilon-\delta}| \leq 1$ and that $|m| \leq G_n = \left[ 2^{n(1-\alpha)} \right] \leq 2^n = n$, so that $|2^{-\varepsilon-\delta}m| \leq 1$ as well.

Furthermore, Lemma 3.2 shows $\frac{2^n}{3} \leq |\xi| \leq 12 \cdot 2^n$ for all $\xi \in S_i^{(\alpha)}$. In particular, since we have $|\xi| \leq 2$ for arbitrary $\xi \in S_0^{(\alpha)} = (-1, 1)^2$, we have $i^* \subset I_0$ as soon as $\frac{2^n}{3} > 2$, i.e., for $n \geq 3$. Now, for $n \geq 3$ and $j = (\nu, \mu, \varepsilon, \delta) \in i^* \subset I_0$, there is some $\eta \in S_j^{(\alpha)} \cap S_i^{(\alpha)}$, so that Lemma 3.2 yields $\frac{2^n}{3} \leq |\eta| \leq 12 \cdot 2^n$. Another application of Lemma 3.2 then shows $|\eta| \geq \frac{2^n}{3} > \frac{1}{3} \cdot 12 \cdot 2^n = \frac{2^n}{108}$ for all $\xi \in S_j^{(\alpha)}$. All in all, we have shown $1 + |\xi| \geq |\xi| \geq \frac{2^n}{108}$ for all $\xi \in (S_i^{(\alpha)})^\ast$ for arbitrary $i = (n, m, \varepsilon, \delta) \in I_0$ with $n \geq 3$. But in case of $n \leq 2$, we simply have $1 + |\xi| \geq \frac{n}{2} \geq \frac{2^n}{108}$, so that this estimate holds for all $i = (n, m, \varepsilon, \delta) \in I_0$.

Overall, we conclude

$$w_i^{(N)} \leq 3^3 \cdot 2^{(1+\alpha)\frac{n}{p}} \cdot \left( \frac{2^n}{108} \right)^{-N} = 3^3 \cdot \frac{1}{108} \cdot 2^n \left( 1 + \frac{1}{p} \right) - s - N \quad \forall \; i = (n, m, \varepsilon, \delta) \in I_0.$$

For arbitrary $\theta \in (0, 1]$, this implies

$$\sum_{i = (n, m, \varepsilon, \delta) \in I_0} \left[ \frac{1}{w_i^{(N)}} \right]^\theta \leq 4 \cdot (3^3 \cdot \frac{1}{108})^\theta \cdot \sum_{n=0}^\infty \sum_{|m| \leq G_n} 2^{n\theta} \left( 1 + \frac{1}{p} \right) - s - N \quad \text{for} \; G_n \leq 2^n$$

as soon as $N > \frac{1}{\theta} + \frac{1+\alpha}{p} - s$, which can always be satisfied. Since we have $\ell^{q'}(I_0) \hookrightarrow \ell^{q'}(I_0)$ for $\theta \leq q'$, this shows that we always have $w^{(N)}(\theta) \in \ell^{q'}_1(I)$, for sufficiently large $N \in \mathbb{N}_0$. As explained above, we can thus invoke [60, Theorem 8.3] to complete the proof. □

Now that we have verified that the $\alpha$-shearlet smoothness spaces are indeed well-defined (Quasi)-Banach spaces, our next goal is to verify that the theory of structured Banach frame decompositions for decomposition spaces—as outlined in Section 2—applies to these spaces. This is the goal of the next section. As we will see (see e.g. Theorem 5.11), this implies that the $\alpha$-shearlet smoothness spaces simultaneously characterize analysis sparsity and synthesis sparsity with respect to (suitable) $\alpha$-shearlet systems.
4. Construction of Banach frame decompositions for \(\alpha\)-shearlet smoothness spaces

We now want to verify the pertinent conditions from Theorems 2.9 and 2.10 for the \(\alpha\)-shearlet smoothness spaces. To this end, first recall from Definition 3.1 that we have \(Q'_i = Q\) for all \(i \in I_0\) and furthermore \(Q_0^0 = (-1, 1)^2\). Consequently, in the notation of Assumption 2.7, we can choose \(n = 2\) and \(Q_0^{(1)} := Q = U_{(-1, 1)^2}\), as well as \(Q_0^{(2)} := (-1, 1)^2\).

We fix a low-pass filter \(\varphi \in W^{1,1}(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)\) and a mother shearlet \(\psi \in W^{1,1}(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)\). Then we set (again in the notation of Assumption 2.7) \(\gamma_i^{(0)} := \psi\) and \(\gamma_i^{(2)} := \varphi\), as well as \(k_0 := 2\) and \(k_i := 1\) for \(i \in I_0\). With these choices, the family \(\Gamma = (\gamma_i)_{i \in I_0}\) introduced in Theorems 2.9 and 2.10 satisfies \(\gamma_i = \gamma_i^{(0)} = \gamma_i^{(2)}\) for \(i \in I_0\) and \(\gamma_0 = \gamma_0^{(0)} = \gamma_0^{(2)}\), so that the family \(\Gamma\) is completely determined by \(\varphi\) and \(\psi\).

Our main goal in this section is to derive readily verifiable conditions on \(\sigma, \tau\). Our goal in the following is to derive conditions on \(|\xi| \leq 0\), \(|\xi| \leq 2\) for sufficiently small \(\delta > 0\).

Precisely, we assume \(\hat{\psi}, \hat{\varphi} \in C^\infty(\mathbb{R}^2)\), where all partial derivatives of these functions are assumed to be polynomially bounded. Furthermore, we assume (at least for the application Theorem 2.9) that

\[
\max_{|\beta| \leq 1} \max_{|\theta| \leq N} \left| \left( \partial^\beta \partial^{\beta'} \psi \right)(\xi) \right| \leq C \cdot \min \left\{ \left| \xi_1 \right|^{M_1}, (1 + |\xi_1|)^{-M_2} \right\} \cdot (1 + |\xi_2|)^{-K} = C \cdot \theta_1(\xi_1) \cdot \theta_2(\xi_2) = C \cdot \varphi(\xi),
\]

\[
\max_{|\beta| \leq 1} \max_{|\theta| \leq N} \left| \left( \partial^\beta \partial^{\beta'} \varphi \right)(\xi) \right| \leq C \cdot (1 + |\xi|)^{-H} = C \cdot \theta_0(\xi)
\]

for all \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\), a suitable constant \(C > 0\) and certain \(M_1, M_2, K, H \in [0, \infty)\) and \(N \in \mathbb{N}\). To be precise, we note that equation (4.1) employed the abbreviations

\[
\theta_1(\xi_1) := \min \left\{ \left| \xi_1 \right|^{M_1}, (1 + |\xi_1|)^{-M_2} \right\} \quad \text{and} \quad \theta_2(\xi_2) := (1 + |\xi_2|)^{-K}
\]

for \(\xi_1, \xi_2 \in \mathbb{R}\), as well as \(\varphi(\xi) := \theta_1(\xi_1) \cdot \theta_2(\xi_2)\) and \(\theta_0(\xi) := (1 + |\xi|)^{-H}\) for \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\).

Our goal in the following is to derive conditions on \(N, M_1, M_2, K, H\) (depending on \(p, q, s, \alpha\)) which ensure that the family \(\Psi_\delta\) indeed forms a Banach frame or an atomic decomposition for \(\mathcal{S}_\alpha^{p,q}(\mathbb{R}^2)\), for sufficiently small \(\delta > 0\).

To verify the conditions of Theorem 2.9 (recalling that \(b_j = 0\) for all \(j \in I\)), we need to estimate the quantity

\[
M_{j,i} := \left( \frac{w_j^{n_1}}{w_i^{n_2}} \right)^\tau \cdot \left( 1 + \|T_j^{-1}T_i\| \right)^\sigma \cdot \max_{|\beta| \leq 1} \left( \left| \det T_i \right|^{-1} \cdot \int_{S_i^{(\alpha)}} \max_{|\theta| \leq N} \left| \left( \partial^\beta \partial^{\beta'} \gamma_i \right)(T_j^{-1}T_i \xi) \right| d\xi \right)^\tau
\]

\[
(\text{eq. 4.1}) \leq C^\tau \cdot \left( \frac{w_j^{n_1}}{w_i^{n_2}} \right)^\tau \cdot \left( 1 + \|T_j^{-1}T_i\| \right)^\sigma \cdot \left( \left| \det T_i \right|^{-1} \cdot \int_{S_i^{(\alpha)}} \theta_0(\xi) \left( T_j^{-1}T_i \xi \right) d\xi \right)^\tau =: C^\tau \cdot M_{j,i}^{(0)}
\]

with \(\sigma, \tau > 0\) and \(N \in \mathbb{N}\) as in Theorem 2.9 and arbitrary \(i, j \in I\), where we defined \(\theta_j := \varphi\) for \(j \in I_0\), with \(\varphi\) and \(\theta_0\) as defined in equation (4.1).

In view of equation (4.2), the following—highly nontrivial—lemma is crucial:

**Lemma 4.1.** Let \(\alpha \in [0, 1]\) and \(\tau_0, \omega, c \in (0, \infty)\). Furthermore, let \(K, H, M_1, M_2 \in [0, \infty)\). Then there is a constant \(C_0 = C_0(\alpha, \tau_0, \omega, c, K, H, M_1, M_2) > 0\) with the following property:

If \(\sigma, \tau \in (0, \infty)\) and \(s \in \mathbb{R}\) satisfy \(\tau \geq \tau_0\) and \(\frac{s}{\tau} \leq \omega\) and if we have \(K \geq K_0 + c\), \(M_1 \geq M_1^{(0)} + c\), and \(M_2 \geq M_2^{(0)} + c\), as well as \(H \geq H_0 + c\) for

\[
K_0 := \begin{cases} 
\max \left\{ \frac{\omega - s}{\tau}, \frac{2 + \omega}{\tau} \right\}, & \text{if } \alpha = 1, \\
\max \left\{ \frac{1 - \alpha}{\tau} + \frac{2 + \omega}{\tau} - s, \frac{2 + \omega}{\tau} \right\}, & \text{if } \alpha \in [0, 1), 
\end{cases}
\]

\[
M_1^{(0)} := \begin{cases} 
\frac{1}{\tau} + s, & \text{if } \alpha = 1, \\
\frac{1}{\tau} + \max \left\{ s, 0 \right\}, & \text{if } \alpha \in [0, 1), 
\end{cases}
\]

\[
M_2^{(0)} := (1 + \alpha) \frac{\sigma}{\tau} - s,
\]

\[
H_0 := \frac{1 - \alpha}{\tau} + \frac{\sigma}{\tau} - s,
\]
then we have
\[
\max \left\{ \sup_{i \in I} \sum_{j \in I} M_{j,i}^{(0)}, \sup_{i \in I} \sum_{j \in I} M_{j,i}^{(0)} \right\} \leq C_0,
\]
where \(M_{j,i}^{(0)}\) is as in equation 12, i.e.,
\[
M_{j,i}^{(0)} := \left( \frac{w_j}{w_i} \right)^\gamma \cdot \left( 1 + \|T_j^{-1}T_i\| \right)^\sigma \cdot \left( |\det T_i|^{-1} \cdot \int_{S_i} \varphi_j (T_j^{-1}T_i \xi) d\xi \right)^\tau,
\]
with \(\varphi_0 (\xi) = (1 + |\xi|)^{-H}\) and \(\varphi_j (\xi) = \min \left\{ \{|\xi_1|^{M_1}, (1 + |\xi_1|)^{-M_2} \}, (1 + |\xi_2|)^{-K} \right\}\) for arbitrary \(j \in I_0\).

The proof of Lemma 4.1 is highly technical and very lengthy. In order to not disrupt the flow of the paper too severely, we deferred the proof to the appendix (Section C).

Using the general result of Lemma 4.1 we can now derive sufficient conditions concerning the low-pass filter \(\varphi\) and the mother shearlet \(\psi\) which ensure that \(\varphi, \psi\) generate a Banach frame for \(\mathcal{F}_{\alpha,s}^{p,q} (\mathbb{R}^2)\).

**Theorem 4.2.** Let \(\alpha \in (0, 1), \varepsilon, p_0, q_0 \in (0, 1)\) and \(s_0, s_1 \in \mathbb{R}\) with \(s_0 \leq s_1\). Assume that \(\varphi, \psi : \mathbb{R}^2 \rightarrow \mathbb{C}\) satisfy the following:

- \(\varphi, \psi \in L^1 (\mathbb{R}^2)\) and \(\hat{\varphi}, \hat{\psi} \in \mathcal{C}^\infty (\mathbb{R}^2)\), where all partial derivatives of \(\hat{\varphi}, \hat{\psi}\) have at most polynomial growth.
- \(\varphi, \psi \in C^1 (\mathbb{R}^2)\) and \(\nabla \varphi, \nabla \psi \in L^1 (\mathbb{R}^2) \cap L^\infty (\mathbb{R}^2)\).
- We have \(\hat{\psi} (\xi) \neq 0\) for all \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\) with \(\xi_1 \in [3^{-1}, 3]\) and \(|\xi_2| \leq |\xi_1|\), \(\hat{\varphi} (\xi) \neq 0\) for all \(\xi \in [-1, 1]^2\).

- There is some \(C > 0\) such that \(\hat{\psi} \) and \(\hat{\varphi} \) satisfy the estimates
\[
|\partial^\theta \hat{\psi} (\xi)| \leq C \cdot |\xi_1|^{M_1} (1 + |\xi_2|)^{-(1+K)} \quad \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \text{ with } |\xi_1| \leq 1, \\
|\partial^\theta \hat{\varphi} (\xi)| \leq C \cdot (1 + |\xi_1|)^{-M_2} (1 + |\xi_2|)^{-(K+1)} \quad \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \\
|\partial^\theta \varphi (\xi)| \leq C \cdot (1 + |\xi|)^{-H} \quad \forall \xi \in \mathbb{R}^2
\]
for all \(\theta \in \mathbb{N}_0^2\) with \(|\theta| \leq N_0\), where \(N_0 := \left[ p_0^{-1} \cdot (2 + \varepsilon) \right]\) and
\[
K := \varepsilon + \max \left\{ \frac{1 - \alpha}{\min \{p_0, q_0\}} + 2 \left( \frac{2}{p_0} + N_0 \right) - s_0, \frac{2}{\min \{p_0, q_0\}} + \frac{2}{p_0} + N_0 \right\}, \\
M_1 := \varepsilon + \frac{1}{\min \{p_0, q_0\}} + \max \{s_1, 0\}, \\
M_2 := \max \left\{ 0, \varepsilon + (1 + \alpha) \left( \frac{2}{p_0} + N_0 \right) - s_0 \right\}, \\
H := \max \left\{ 0, \varepsilon + \frac{1 - \alpha}{\min \{p_0, q_0\}} + \frac{2}{p_0} + N_0 - s_0 \right\}.
\]

Then there is some \(\delta_0 \in (0, 1)\) such that for \(0 < \delta \leq \delta_0\) and all \(p, q \in (0, \infty)\) and \(s \in \mathbb{R}\) with \(p \geq p_0, q \geq q_0\) and \(s_0 \leq s \leq s_1\), the following is true: The family
\[
\text{SH}^{(1, \pm)}_{\alpha, \varphi, \psi, \delta} := \left( L_{\delta \cdot T_i^{-1}} \varphi \right)_{i \in I, k \in \mathbb{Z}^2} \quad \text{with } \left( \left( \varphi \right)_{i} \right)(x) = \gamma[i](-x) \quad \text{and } \gamma[i] := \left| \langle \text{det } T_i \rangle \right|^{1/2} \cdot \left( \psi \circ T_i^{-1} \right), \quad \text{if } i \in I_0, \\
\psi, \quad \text{if } i = 0
\]
forms a Banach frame for \(\mathcal{F}_{\alpha,s}^{p,q} (\mathbb{R}^2) = D (S^{(\alpha)}, L^p, L^q)\). Precisely, this means the following:

1. The **analysis operator**
\[
A^{(\delta)} : \mathcal{F}_{\alpha,s}^{p,q} (\mathbb{R}^2) \rightarrow C_{w^*}^{p,q}, f \mapsto \left( \gamma[i] \ast f \right) \left( \delta \cdot T_i^{-1} \right)_{i \in I, k \in \mathbb{Z}^2}
\]
is well-defined and bounded for arbitrary \(\delta \in (0, 1]\), with the coefficient space \(C_{w^*}^{p,q}\) from Definition 2.8. The convolution \(\gamma[i] \ast f\) has to be understood as explained in equation 2.20; see Lemma 5.14 for a more convenient expression for this convolution, for \(f \in L^2 (\mathbb{R}^2)\).
(2) For $0 < \delta \leq \delta_0$, there is a bounded linear reconstruction operator

$$R^{(\delta)} : C^p_{\alpha,s} \rightarrow \mathcal{S}^p_{\alpha,s}(\mathbb{R}^2)$$

satisfying $R^{(\delta)} \circ A^{(\delta)} = \text{id}_{\mathcal{S}^p_{\alpha,s}(\mathbb{R}^2)}$.

(3) For $0 < \delta \leq \delta_0$, we have the following consistency statement: If $f \in \mathcal{S}^p_{\alpha,s}(\mathbb{R}^2)$ and if $p_0 \leq \hat{p} \leq \infty$, $q_0 \leq \hat{q} \leq \infty$ and so $s \leq \hat{s} \leq s_1$, then the following equivalence holds:

$$f \in \mathcal{S}^p_{\alpha,s} \iff [\gamma^{(\delta)} \ast f] \left( \delta \cdot T_k \right)_{i \in I, k \in \mathbb{Z}^2} \in C^p_{\alpha,s}(\mathbb{R}^2).$$

\textbf{Proof.} First, we show that there are constants $K_1, K_2 > 0$ such that

$$\max_{|\beta| \leq 1} \max_{|\theta| \leq N_0} \left( |(\partial^\theta \hat{\hat{\varphi}}(\psi))(\xi)| \leq K_1 \cdot \min \left\{ |\xi_1|^{M_1}, (1 + |\xi_1|)^{-M_2} \right\} \cdot (1 + |\xi_2|)^{-K} =: K_1 \cdot \varrho(\xi) \right)$$

and

$$\max_{|\beta| \leq 1} \max_{|\theta| \leq N_0} \left( |(\partial^\theta \hat{\hat{\varphi}}(\psi))(\xi)| \leq K_2 \cdot (1 + |\xi_2|)^{-H} =: K_2 \cdot \varrho_0(\xi) \right)$$

for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$.

To this end, we recall that $\varphi, \psi \in C^1(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2)$, so that standard properties of the Fourier transform show for $\beta = e_\ell$ (the $\ell$-th unit vector) that

$$\partial^\beta \psi(\xi) = 2\pi i \cdot \xi_\ell \cdot \hat{\varphi}(\xi) \quad \text{and} \quad \partial^\beta \varphi(\xi) = 2\pi i \cdot \xi_\ell \cdot \hat{\varphi}(\xi) \quad \forall \xi \in \mathbb{R}^2.$$  

Then, Leibniz’s rule yields for $\beta = e_\ell$ and arbitrary $\theta \in \mathbb{N}_0^2$ with $|\theta| \leq N_0$ that

$$\left| (\partial^\theta \partial^\beta \psi)(\xi) \right| = 2\pi \cdot \sum_{\nu \leq \theta} \left( \frac{\theta}{\nu} \right) \cdot (\partial^\nu \xi_\ell) \cdot (\partial^\beta \varphi)(\xi)$$

$$\leq 2^{N_0 + 1} \pi \cdot (1 + |\xi_\ell|) \cdot \max_{|\theta| \leq N_0} |(\partial^\nu \hat{\varphi})(\xi)|$$

$$\leq 2^{N_0 + 1} \pi \cdot (1 + |\xi_\ell|) \cdot C \cdot (1 + |\xi_1|)^{-M_2} \cdot (1 + |\xi_2|)^{-K}$$

$$\leq 2^{N_0 + 1} \pi C \cdot (1 + |\xi_1|)^{-M_2} \cdot (1 + |\xi_2|)^{-K}$$

since we have

$$|\partial^\nu \xi_\ell| = \begin{cases} |\xi_\ell|, & \text{if } \nu = 0 \\ 1, & \text{if } \nu = e_\ell \\ 0, & \text{otherwise} \end{cases}$$

and thus

$$|\partial^\nu \xi_\ell| \leq 1 + |\xi_\ell| \leq 1 + |\xi|.$$  

Above, we also used that $\sum_{\nu < \theta} \left( \frac{\theta}{\nu} \right) = (2, \ldots, 2^\theta) = 2^{|\theta|} \leq 2^{N_0}$, as a consequence of the $d$-dimensional binomial theorem (cf. [21] Section 8.1, Exercise 2.b]).

Likewise, we get

$$\left| (\partial^\theta \partial^\beta \varphi)(\xi) \right| = 2\pi \cdot \sum_{\nu \leq \theta} \left( \frac{\theta}{\nu} \right) \cdot (\partial^\nu \xi_\ell) \cdot (\partial^\beta \varphi)(\xi)$$

$$\leq 2^{N_0 + 1} \pi \cdot (1 + |\xi|) \cdot \max_{|\theta| \leq N_0} |(\partial^\nu \hat{\varphi})(\xi)|$$

$$\leq 2^{N_0 + 1} \pi C \cdot (1 + |\xi|)^{-H}$$

and, by assumption,

$$|\partial^\beta \varphi(\xi)| \leq C \cdot (1 + |\xi|)^{-(H + 1)} \leq C \cdot (1 + |\xi|)^{-H} = C \cdot \varrho_0(\xi).$$

With this, we have already established equation (4.4) with $K_2 := 2^{N_0 + 1} \pi C$.

To validate equation (4.5), we now distinguish the two cases $|\xi_1| > 1$ and $|\xi_1| \leq 1$:

\textbf{Case 1:} We have $|\xi_1| > 1$. In this case, $\vartheta(\xi) = (1 + |\xi_1|)^{-M_2} (1 + |\xi_2|)^{-K}$, so that equation (4.7) shows

$$\left| (\partial^\theta \hat{\hat{\varphi}}(\psi))(\xi) \right| \leq 2^{N_0 + 1} \pi C \cdot \varrho(\xi)$$

for $\beta = e_\ell$, $\ell \in \{1, 2\}$ and arbitrary $\theta \in \mathbb{N}_0^2$ with $|\theta| \leq N_0$. Finally, we also have

$$|\partial^\nu \hat{\varphi}(\xi)| \leq C \cdot (1 + |\xi_1|)^{-(1 + M_2)} (1 + |\xi_2|)^{-(1 + K)} \leq C \cdot (1 + |\xi_1|)^{-M_2} (1 + |\xi_2|)^{-K} = C \cdot \varrho(\xi)$$

and hence

$$\max_{|\beta| \leq 1} \max_{|\theta| \leq N_0} \left| (\partial^\theta \hat{\hat{\varphi}}(\psi))(\xi) \right| \leq 2^{N_0 + 1} \pi C \cdot \varrho(\xi)$$

for all $\xi \in \mathbb{R}^2$ with $|\xi_1| > 1$. 


Case 2: We have $|ξ_1| ≤ 1$. First note that this implies $(1 + |ξ_1|)^{-M_2} ≥ 2^{-M_2} ≥ 2^{-M_2} |ξ_1|^{M_1}$ and consequently $φ(ξ) ≥ 2^{-M_2} |ξ_1|^{M_1} · (1 + |ξ_2|)^{-K}$. Furthermore, we have for arbitrary $ℓ ∈ \{1, 2\}$ that

$$1 + |ξ_ℓ| ≤ \max \{1 + |ξ_1|, 1 + |ξ_2|\} ≤ \max \{2, 1 + |ξ_2|\} ≤ 2 · (1 + |ξ_2|).$$

In conjunction with equation (4.6), this shows for $β = ε_ℓ$, $ℓ ∈ \{1, 2\}$ and $θ ∈ \mathbb{N}_0$ with $|θ| ≤ N_0$ that

$$|\partial^\theta \partial^\gamma ψ| ≤ 2^{N_0+1} (1 + |ξ_ℓ|) · \max_{|θ| ≤ N_0} |\partial^\theta \hat{ψ}(ξ)| ≤ 2^{N_0+2} C \cdot (1 + |ξ_2|) · |ξ_1|^{M_1} · (1 + |ξ_2|)^{-K} ≤ 2^{M_2+2N_0} C · φ(ξ).$$

Finally, we also have

$$|\partial^\theta \hat{ψ}(ξ)| ≤ C · |ξ_1|^{M_1} · (1 + |ξ_2|)^{-K} ≤ C · |ξ_1|^{M_1} (1 + |ξ_2|)^{-K} ≤ 2M_2 C · φ(ξ).$$

All in all, we have shown $\max_{|θ| ≤ 1} \max_{|θ| ≤ N_0} |\partial^\theta \hat{ψ}(ξ)| ≤ 2^{2+M_2+N_0} C · φ(ξ)$ for all $ξ ∈ \mathbb{R}^2$ with $|ξ_1| ≤ 1$.

All together, we have thus established eq. (1.4) with $K_1 := 2^{2+M_2+N_0} C$. Now, define $C_0 := \max \{K_1, K_2\} = K_1$.

Now, for proving the current theorem, we want to apply Theorem 2.9 with $\gamma_0(0) := ψ$, $\gamma_0(2) := ϕ$ and $κ_i := 1$ for $ι ∈ I_0$ and $κ_0 := 2$, as well as $Q_0(1):= Q = U^{(3,1)}$ and $Q_0(2):= (-1,1)^2$, cf. Assumption 2.7 and Definition 3.1. In the notation of Theorem 2.9 we then have $\gamma_i := \gamma_i(0)$ for all $ι ∈ I$, i.e., $\gamma_i = ψ$ for $ι ∈ I_0$ and $\gamma_0 = ϕ$. Using this notation and setting furthermore $ν_i := φ$ for $ι ∈ I_0$, we have thus shown for arbitrary $N ∈ \mathbb{N}_0$ with $N ≤ N_0$ that

$$M_j,ι := \left(\frac{w_j^2}{w_ι^2}τ\right) · (1 + \|T_j^{-1}T_ι\|)^σ · \max_{|θ| ≤ 1} \left|\det T_ι\right|^{-1} · \max_{|θ| ≤ N} \left|\partial^\theta \hat{ψ}(ξ)\right| (T_j^{-1}τ)_j · dξ \leq C_0 \cdot \left(\frac{w_j^2}{w_ι^2}τ\right) · (1 + \|T_j^{-1}T_ι\|)^σ · \left|\det T_ι\right|^{-1} \cdot \left|\partial^\theta \hat{ψ}(ξ)\right| (T_j^{-1}τ)_j · dξ =: C_0 · M_j,ι(0)$$

for arbitrary $σ, τ > 0$, $s ∈ \mathbb{R}$ and the $S^{(α)}$-moderate weight $w^s$ (cf. Lemma 3.3).

In view of the assumptions of the current theorem, the prerequisites (1)-(3) of Theorem 2.9 are clearly fulfilled, but we still need to verify

$$C_1 := \sup_{ι ∈ I} \sum_{j ∈ I} M_j,ι < ∞ \quad \text{and} \quad C_2 := \sup_{j ∈ I} \sum_{ι ∈ I} M_j,ι < ∞,$$

with $M_j,ι$ as above, $τ := \min \{1, p, q\} ≥ \min \{p_0, q_0\} =: τ_0$, and

$$N := \left[\frac{2 + ε}{\min \{1, p\}}\right] ≤ \frac{2 + ε}{p_0} = N_0, \quad \text{as well as} \quad σ := \tau \cdot \left(\frac{2}{\min \{1, p\}} + N\right) ≤ \tau \cdot \left(\frac{2}{p_0} + N_0\right). \quad (4.8)$$

In particular, we have $\frac{s}{τ} ≤ \frac{2}{p_0} + N_0 = \frac{2}{p_0} + \left[\frac{2 + ε}{p_0}\right] =: ω$.

Hence, Lemma 4.1 (with $c = ε$) yields a constant $C_0 = C_0(α, τ_0, ω, ε, H, M_1, M_2)$ with $\max \{C_1, C_2\} ≤ C_0^2 C_0^2$, provided that we can show $H ≥ H_0 + ε$, $K ≥ K_0 + ε$ and $M_ℓ ≥ M_ℓ(0) + ε$ for $ℓ ∈ \{1, 2\}$, with $H_0, K_0, M_1(0), M_2(0)$ as defined in Lemma 4.1. But we have

$$H_0 = \frac{1}{τ} \cdot \frac{1}{ω} - s ≤ \frac{1}{τ} \cdot \frac{1}{ω} - s = \frac{1}{τ} \cdot \frac{1}{ω} + \frac{2}{p_0} + N_0 - s_0 \leq H + ε.$$

Furthermore,

$$M_2(0) = (1 + α)\frac{τ}{ω} - s ≤ (1 + α)ω - s_0 = (1 + α)\left(\frac{2}{p_0} + N_0\right) - s_0 \leq M_2 - ε.$$
and
\[ M_1^{(0)} \leq \frac{1}{\tau} + \max \{s, 0\} \leq \frac{1}{\min \{p_0, q_0\}} + \max \{s_1, 0\} = M_1 - \varepsilon, \]
as well as
\[ K_0 \leq \max \left\{ \frac{1 - \alpha}{\tau} + 2\frac{\sigma}{\tau} - s, \frac{2 + \sigma}{\tau} \right\} \]
\[ \leq \max \left\{ \frac{1 - \alpha}{\tau_0} + 2\omega - s_0, \frac{2}{\tau_0} + \omega \right\} \]
\[ = \max \left\{ \frac{1 - \alpha}{\min \{p_0, q_0\}} + 2\left(\frac{2}{p_0} + N_0\right) - s_0, \frac{2}{\min \{p_0, q_0\}} + \frac{2}{p_0} + N_0 \right\} \]
\[ = K - \varepsilon. \]
Thus, Lemma 4.1 is applicable, so that
\[ C_1^{1/\tau} = \left(\sup_{\omega \in \mathcal{L}} \sum_{j \in \mathcal{L}} M_{j, i} \right)^{1/\tau} \leq C_\diamond C_0, \]
where the right-hand side is independent of \( p, q \) and \( s \), since \( C_0 \) is independent of \( p, q \) and \( s \) and since
\[ C_\diamond = C_\diamond \left( \varepsilon, p_0, M_2, C \right) = 2^{2+M_2+N_0}= \pi C = 2^{2+M_2+\left[\frac{\Lambda}{\pi R} \right]} \pi C. \]
The exact same estimate holds for \( C_2 \).

We have shown that all prerequisites for Theorem 2.9 are fulfilled. Hence, the theorem implies that there is a constant \( K_0 = K_0 \left( p_0, q_0, \varepsilon, S^{(\alpha)}, \varphi, \psi \right) > 0 \) (independent of \( p, q, s \)) such that the family \( \hat{\mathcal{S}}_{\alpha, \varphi, \psi, \hat{\delta}} \) forms a Banach frame for \( \mathcal{S}_{p,q}^{(\alpha,s)} (\mathbb{R}^2) \), as soon as \( \delta \leq \delta_{00} \), where
\[ \delta_{00} := \left(1 + K_0 \cdot C_{S^{(\alpha), w^0}}^{\left(\pm 1\right)} \cdot \left(C_1^{1/\tau} + C_2^{1/\tau}\right)^{2} \right)^{-1}. \]
From Lemma 3.4 we know that \( C_{S^{(\alpha), w^0}} \leq 39^{|k|} \leq 39^{k_2} \) where \( s_2 := \max \{|s_0|, |s_1|\} \). Hence, choosing
\[ \delta_0 := \left(1 + 4 \cdot K_0 \cdot C_2^0 \cdot C_0^2 \cdot 39^{4k_2}\right)^{-1}, \]
we get \( \delta_0 \leq \delta_{00} \) and \( \delta_0 \) is independent of the precise choice of \( p, q, s \), as long as \( p \geq p_0, q \geq q_0 \) and \( s_0 \leq s \leq s_1 \).
Thus, for \( 0 < \delta \leq \delta_0 \) and arbitrary \( p, q, s \in (0, \infty), s \in \mathbb{R} \) with \( p \geq p_0, q \geq q_0 \) and \( s_0 \leq s \leq s_1 \), the family \( \hat{\mathcal{S}}_{\alpha, \varphi, \psi, \hat{\delta}} \) forms a Banach frame for \( \mathcal{S}_{p,q}^{(\alpha,s)} (\mathbb{R}^2). \)

Finally, we also come to verifiable sufficient conditions which ensure that the low-pass \( \varphi \) and the mother shearlet \( \psi \) generate atomic decompositions for \( \mathcal{S}_{p,q}^{(\alpha,s)} (\mathbb{R}^2) \).

**Theorem 4.3.** Let \( \alpha \in [0, 1], \varepsilon, p_0, q_0 \in (0, 1] \) and \( s_0, s_1 \in \mathbb{R} \), with \( s_0 \leq s_1 \). Assume that \( \varphi, \psi \in L^1 (\mathbb{R}^2) \) satisfy the following properties:
- We have \( \| \varphi \|_{1+\frac{2}{p_0}} < \infty \) and \( \| \psi \|_{1+\frac{2}{q_0}} < \infty \), where \( \|g\|_{\Lambda} = \sup_{x \in \mathbb{R}^2} (1 + |x|)^{\Lambda} |g(x)| \) for \( g : \mathbb{R}^2 \to C^\ell \) (with arbitrary \( \ell \in \mathbb{N} \) and \( \Lambda \geq 0 \)).
- We have \( \hat{\varphi}, \hat{\psi} \in C^\infty (\mathbb{R}^2) \), where all partial derivatives of \( \hat{\varphi}, \hat{\psi} \) are polynomially bounded.
- We have
  \[ \hat{\psi} \neq 0 \text{ for all } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \text{ with } \xi_1 \in [3^{-1}, 3] \text{ and } |\xi_2| \leq |\xi_1|, \]
  \[ \hat{\varphi} \neq 0 \text{ for all } \xi \in [-1, 1]^2. \]
- We have
  \[ |\partial^\beta \hat{\varphi}(\xi)| \lesssim (1 + |\xi|)^{-\Lambda_0}, \]
  \[ |\partial^\beta \hat{\psi}(\xi)| \lesssim \min \left\{ |\xi_1|^{\Lambda_1}, (1 + |\xi_1|)^{-\Lambda_2} \right\} \cdot (1 + |\xi_2|)^{-\Lambda_3} \cdot (1 + |\xi|)^{-(3+\varepsilon)} \] (4.9)
for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and all $\beta \in \mathbb{N}_0^2$ with $|\beta| \leq \lceil p_0^{-1} \cdot (2 + \varepsilon) \rceil$, where

$$
\Lambda_0 := \begin{cases} 
3 + 2\varepsilon + \max \left\{ \frac{1 - \alpha}{\min \{p_0, q_0\}} + 3 + s_1, 2 \right\}, & \text{if } p_0 = 1, \\
3 + 2\varepsilon + \max \left\{ \frac{1 - \alpha}{\min \{p_0, q_0\}} + \frac{2 + \varepsilon}{p_0} + 1 + \alpha, \frac{2 + \varepsilon}{p_0} + s_1, 2 \right\}, & \text{if } p_0 \in (0, 1), 
\end{cases}
$$

$$
\Lambda_1 := \varepsilon + \frac{1}{\min \{p_0, q_0\}} + \max \left\{ 0, (1 + \alpha) \left( \frac{1}{p_0} - 1 \right) - s_0 \right\},
$$

$$
\Lambda_2 := \begin{cases} 
\varepsilon + \max \{2, 3 (1 + \alpha) + s_1\}, & \text{if } p_0 = 1, \\
\varepsilon + \max \{2, (1 + \alpha) \left( 1 + \frac{1}{p_0} + \frac{2 + \varepsilon}{p_0} \right) \} + s_1, & \text{if } p_0 \in (0, 1), 
\end{cases}
$$

$$
\Lambda_3 := \begin{cases} 
\varepsilon + \max \{2, (1 + \alpha) \left( 1 + \frac{1}{p_0} + \frac{2 + \varepsilon}{p_0} \right) \} + 3 + s_1, & \text{if } p_0 = 1, \\
\varepsilon + \max \left\{ \frac{1 - \alpha}{\min \{p_0, q_0\}} + 6 + s_1, \frac{2}{\min \{p_0, q_0\}} + \frac{2 + \varepsilon}{p_0} \right\} + 1 + \alpha, & \text{if } p_0 \in (0, 1). 
\end{cases}
$$

Then there is some $\delta_0 \in (0, 1]$ such that for all $0 < \delta \leq \delta_0$ and all $p, q \in (0, \infty]$ and $s \in \mathbb{R}$ with $p \geq p_0, q \geq q_0$ and $s_0 \leq s \leq s_1$, the following is true: The family

$$
\mathcal{S}_{H}^{(\pm 1)}_{\alpha,s}(\mathbb{R}^2) \ni \left( L_{\delta} T_{\varepsilon}^{-1} r_{\varepsilon} [i] \right)_{i \in I, k \in \mathbb{Z}^2} \quad \text{with } \gamma_i := \begin{cases} 
\det T_{i}^{1/2} \cdot (\psi \circ T_{i}^{T}), & \text{if } i \in I_0, \\
\phi, & \text{if } i = 0
\end{cases}
$$

forms an atomic decomposition for $\mathcal{S}_{H}^{(\pm 1)}_{\alpha,s}(\mathbb{R}^2)$. Precisely, this means the following:

(1) The synthesis map

$$
S^{(\delta)} : C_{w}^{p,q} \rightarrow \mathcal{S}_{H}^{(\pm 1)}_{\alpha,s}(\mathbb{R}^2), (c_{k}(i))_{i \in I, k \in \mathbb{Z}^2} \mapsto \sum_{i \in I} \sum_{k \in \mathbb{Z}^2} \left( c_{k}(i) \cdot L_{\delta} T_{\varepsilon}^{-1} r_{\varepsilon} [i] \right)
$$

is well-defined and bounded for all $\delta \in (0, 1]$, where the coefficient space $C_{w}^{p,q}$ is as in Definition 2.8. Convergence of the series has to be understood as described in the remark in Theorem 2.10.

(2) For $0 < \delta \leq \delta_0$, there is a bounded linear coefficient map

$$
C^{(\delta)} : \mathcal{S}_{H}^{(\pm 1)}_{\alpha,s}(\mathbb{R}^2) \rightarrow C_{w}^{p,q}
$$

satisfying $S^{(\delta)} \circ C^{(\delta)} = \text{id}_{\mathcal{S}_{H}^{(\pm 1)}_{\alpha,s}(\mathbb{R}^2)}$.

Furthermore, the action of $C^{(\delta)}$ is independent of the precise choice of $p, q, s$. Precisely, if $p_1, p_2 \geq p_0, q_1, q_2 \geq q_0$ and $s(1), s(2) \in [s_0, s_1]$ and if $f \in \mathcal{S}_{H}^{p_1, q_1}_{\alpha,s(1)} \cap \mathcal{S}_{H}^{p_2, q_2}_{\alpha,s(2)}$, then $C^{(\delta)} f = C^{(\delta)} f$, where $C^{(\delta)} f$ denotes the coefficient operator for the choices $p = p_i, q = q_i$, and $s = s(i)$ for $i \in \{1, 2\}$.

Proof. Later in the proof, we will apply Theorem 2.10 to the decomposition space $\mathcal{S}_{H}^{p,q}_{\alpha,s}(\mathbb{R}^2) = \mathcal{D} (\mathcal{S}^{(\alpha)}, L^{p}, L_{w}^{q})$ with $w$ and $w^*$ as in Lemma 3.3 while Theorem 2.10 itself considers the decomposition space $\mathcal{D} (\mathcal{Q}, L^{p}, L_{w}^{q})$. To avoid confusion between these two different choices of the weight $w$, we will write $v$ for the weight defined in Lemma 3.3 so that we get $\mathcal{S}_{H}^{p,q}_{\alpha,s}(\mathbb{R}^2) = \mathcal{D} (\mathcal{S}^{(\alpha)}, L^{p}, L_{w}^{q})$. For the application of Theorem 2.10 we will thus choose $\mathcal{S} = \mathcal{S}^{(\alpha)}$ and $w = v^*$.

Our assumptions on $\phi$ show that there is a constant $C_1 > 0$ satisfying $|\partial^\beta \widehat{\phi}(\xi)| \leq C_1 \cdot (1 + |\xi|)^{-\Lambda_0}$ for all $\beta \in \mathbb{N}_0^2$ with $|\beta| \leq N_0 := \lceil p_0^{-1} \cdot (2 + \varepsilon) \rceil$, where $\Lambda_0 = 3 + \delta + 1 + \epsilon \geq 2 + \epsilon$, so that $\partial T_{i}^{1/2} \cdot \phi \in L^{1} (\mathbb{R}^2)$. We indeed have $|\partial^\beta \widehat{\phi}(\xi)| \leq \Lambda_0 - 3 + \delta + 1 + \epsilon \geq 2 + \epsilon$, so that $\partial T_{i}^{1/2} \cdot \phi \in L^{1} (\mathbb{R}^2)$. We indeed have $|\partial^\beta \widehat{\phi}(\xi)| \leq \Lambda_0 - 3 + \delta + 1 + \epsilon \geq 2 + \epsilon$, so that $\partial T_{i}^{1/2} \cdot \phi \in L^{1} (\mathbb{R}^2)$.

Consequently, Proposition 2.11 provides functions $\phi_1 \in C_0 (\mathbb{R}^2) \cap L^{1} (\mathbb{R}^2)$ and $\phi_2 \in C_1 (\mathbb{R}^2) \cap W^{1,1} (\mathbb{R}^2)$ with $\phi = \phi_1 + \phi_2$ and with the following additional properties:

(1) We have $\| \phi_2 \|_{\Lambda} < \infty$ and $\| \nabla \phi_2 \|_{\Lambda} < \infty$ for all $\Lambda \in \mathbb{N}_0$.
(2) We have $\phi_2 \in C^{\infty} (\mathbb{R}^2)$, where all partial derivatives of $\phi_2$ are polynomially bounded.
(3) We have $\phi_1 \in C^{\infty} (\mathbb{R}^2)$, where all partial derivatives of $\phi_1$ are polynomially bounded. This uses that $\phi \in C^{\infty} (\mathbb{R}^2)$ with all partial derivatives being polynomially bounded.
(4) We have

$$
|\partial^\beta \widehat{\phi_1}(\xi)| \leq \frac{C_2}{C_1} \cdot \phi_1(\xi) = C_2 \cdot (1 + |\xi|)^{3 + \epsilon - \Lambda_0} \quad \forall \xi \in \mathbb{R}^2 \text{ and } \beta \in \mathbb{N}_0^2 \text{ with } |\beta| \leq N_0.
$$

Here, $C_2$ is given by $C_2 := C_1 \cdot 2^{3 + 4N_0 \cdot N_0} \cdot N_0! \cdot 3^{N_0}$.
Likewise, our assumptions on $\psi$ show that there is a constant $C_3 > 0$ satisfying
\[
\left| \partial^\beta \hat{\psi} (\xi) \right| \leq C_3 \cdot \min \left\{ (\xi_1^{A_1}, (1 + |\xi_1|)^{-A_2}), (1 + |\xi_2|)^{-A_3} \right\} \cdot (1 + |\xi|)^{-3+\varepsilon} \quad \forall \xi \in \mathbb{R}^2 \forall \beta \in \mathbb{N}_0^2 \text{ with } |\beta| \leq N_0.
\]
Now, we again apply Proposition 2.11, but this time with $N = N_0 \geq d + 1$, with $\gamma = \psi$ and with $\varrho = \varrho_2$ for $\varrho_2 (\xi) := C_3 \cdot \min \left\{ (\xi_1^{A_1}, (1 + |\xi_1|)^{-A_2}), (1 + |\xi_2|)^{-A_3} \right\} \cdot (1 + |\xi|)^{-3+\varepsilon}$, where we note that $A_2 \geq 2 + \varepsilon$ and $A_3 \geq 3 \geq 2 + \varepsilon$, so that
\[
\varrho_2 (\xi) \leq C_3 \cdot (1 + |\xi_1|)^{-(2+\varepsilon)} \cdot (1 + |\xi_2|)^{-2+\varepsilon} \leq C_3 \cdot \max \left\{ (1 + |\xi_1|), 1 + |\xi_2| \right\} \cdot (1 + |\xi|)^{-2+\varepsilon} \leq C_3 \cdot (1 + ||\xi||_\infty)^{-2+\varepsilon} \in L^1 (\mathbb{R}^2).
\]
As we just saw, we indeed have $\left| \partial^\beta \hat{\psi} (\xi) \right| \leq \varrho_2 (\xi) \cdot (1 + |\xi|)^{-3+\varepsilon}$ for all $|\beta| \leq N_0$, since we are working in $\mathbb{R}^d = \mathbb{R}^2$. Consequently, Proposition 2.11 provides functions $\hat{\psi}_1 \in C_0 (\mathbb{R}^2) \cap L^1 (\mathbb{R}^2)$ and $\hat{\psi}_2 \in C_1 (\mathbb{R}^2) \cap W^{1,1} (\mathbb{R}^2)$ with $\psi = \hat{\psi}_1 \ast \hat{\psi}_2$ and with the following additional properties:

1. We have $\|\hat{\psi}_2\|_A < \infty$ and $\|\nabla \hat{\psi}_2\|_A < \infty$ for all $A \in \mathbb{N}_0$.
2. We have $\hat{\psi}_2 \in C^\infty (\mathbb{R}^2)$, where all partial derivatives of $\hat{\psi}_2$ are polynomially bounded.
3. We have $\hat{\psi}_1 \in C^\infty (\mathbb{R}^2)$, where all partial derivatives of $\hat{\psi}_1$ are polynomially bounded. This uses that $\hat{\psi} \in C^\infty (\mathbb{R}^2)$ with all partial derivatives being polynomially bounded.

We have
\[
\left| \partial^\beta \hat{\psi}_1 (\xi) \right| \leq \frac{C_4}{C_3} \cdot \varrho_2 (\xi) = C_4 \cdot \min \left\{ (\xi_1^{A_1}, (1 + |\xi_1|)^{-A_2}), (1 + |\xi_2|)^{-A_3} \right\} \cdot (1 + |\xi|)^{-3+\varepsilon} \quad \forall \xi \in \mathbb{R}^2 \text{ and } \beta \in \mathbb{N}_0^2 \text{ with } |\beta| \leq N_0.
\]
Here, $C_4$ is given by $C_4 := C_3 \cdot 2^{d+4N_0} \cdot N_0! \cdot 3^{N_0}$.

In summary, if we define $M_1 := A_1$, $M_2 := A_2$ and $K := A_3$, as well as $H := A_0 - 3 - \varepsilon$, then we have $M_1, M_2, K, H \geq 0$ and
\[
\max_{|\beta| \leq N_0} \left| \partial^\beta \hat{\psi}_1 (\xi) \right| \leq C_5 \cdot \min \left\{ (\xi_1^{M_1}, (1 + |\xi_1|)^{-M_2}), (1 + |\xi_2|)^{-M_3} \right\} \cdot (1 + |\xi|)^{-H} =: C_5 \cdot \varrho_0 (\xi),
\]
where we defined $C_5 := \max \{ C_2, C_4 \}$ for brevity. For consistency with Lemma [2.11] we define $\varrho_j := \varrho$ for arbitrary $j \in I_0$.

Now, define $n := 2$, $\gamma^{(0)} := \psi$ and $\gamma^{(2)} := \varphi$, as well as $\gamma^{(0,1)} := \psi_j$ and $\gamma^{(0,3)} := \varphi_j$ for $j \in \{1, 2\}$. We want to verify the assumptions of Theorem 2.10 for these choices and for $\mathcal{F}_{\varrho, \delta} (\mathbb{R}^2) = \mathcal{D} (\mathcal{S}^{(\alpha)} \cdot LP, \ell^{3}_\delta) \equiv \mathcal{D} (Q, LP, \ell^{3}_\delta)$. To this end, we recall from Definition 3.1 that $Q := \mathcal{S}^{(\alpha)} = (T_i Q'_i + b_i)_{i \in I}$, with $Q'_i = U^{(3-1,3)} (\cdot) = Q =: Q^{(1)}_0 = Q^{(k_0)}_0$ for all $i \in I_0$, where $k_i := 1$ for $i \in I_0$ and with $Q'_0 = (-1, 1)^2 =: Q^{(2)}_0 = Q^{(k_0)}_0$, where $k_0 := 2$ and $n := 2$, cf. Assumption [2.7].

Now, let us verify the list of prerequisites of Theorem 2.10:

1. We have $\gamma^{(0,1)} \in \{ \varphi_1, \psi_1 \} \subset L^1 (\mathbb{R}^2)$ for $k \in \{1, 2\}$ by the properties of $\varphi_1, \psi_1$ from above.
2. Likewise, we have $\gamma^{(0,2)} \in \{ \varphi_2, \psi_2 \} \subset C^1 (\mathbb{R}^2)$ by the properties of $\varphi_2, \psi_2$ from above.
3. Next, with $\gamma := y + \frac{d}{\min(1, p)}$ as in Theorem 2.10 we have $\gamma \leq 1 + \frac{2}{p_0} =: \gamma_0$ and thus, with $\Omega^{(p)}$ as in Theorem 2.10
\[
\Omega^{(p)} = \max_{k \in I_0} \left\{ \left| \gamma^{(0,2)} \right| \right\}_T + \max_{k \in I_0} \left| \nabla \gamma^{(0,2)} \right|_T
\]
\[
= \max \left\{ \left| \nabla \varphi_2 \right|_T, \left| \nabla \psi_2 \right|_T \right\} + \max \left\{ \left| \nabla \varphi_2 \right|_T, \left| \nabla \psi_2 \right|_T \right\}
\]
\[
\leq \max \left\{ \left| \nabla \varphi_2 \right|_{\gamma_0^1}, \left| \nabla \psi_2 \right|_{\gamma_0^1} \right\} + \max \left\{ \left| \nabla \varphi_2 \right|_{\gamma_0^1}, \left| \nabla \psi_2 \right|_{\gamma_0^1} \right\} =: C_6 < \infty
\]
by the properties of $\varphi_2, \psi_2$ from above.
4. We have $\mathcal{F}_{\varrho, \delta}^{(0,3)} \in \{ \varphi, \psi \} \subset C^\infty (\mathbb{R}^2)$ and all partial derivatives of these functions are polynomially bounded.
5. We have $\gamma^{(0,0)} := \psi = \psi_1 * \psi_2 := \gamma^{(0,1)} * \gamma^{(0,2)}$ and $\gamma^{(0,2)} := \varphi = \varphi_1 * \varphi_2 = \gamma^{(0,1)} * \gamma^{(0,2)}$. 

(6) By assumption, we have $\mathcal{F} \gamma_1^{(0)} (\xi) = \hat{\phi} (\xi) \neq 0$ for all $\xi \in \mathbb{T} = Q_0^{(1)}$. Likewise, we have $\mathcal{F} \gamma_2^{(0)} (\xi) = \hat{\psi} (\xi) \neq 0$ for all $\xi \in [-1, 1]^2 = (-1, 1)^2 = Q_0^{(2)}$.
(7) We have $\|\gamma_1^{(0)}\|_\gamma = \|\psi\|_T = \|\psi\|_T_0 = \|\psi\|_1 + \frac{\alpha}{p_0} < \infty$ and $\|\gamma_2^{(0)}\|_\gamma = \|\varphi\|_T = \|\varphi\|_1 + \frac{\alpha}{p_0} < \infty$, thanks to our assumptions on $\varphi, \psi$.

Thus, as the last prerequisite of Theorem 2.10 we have to verify

$$K_1 := \sup_{i \in I} \sum_{j \in I} N_{i,j} < \infty \quad \text{and} \quad K_2 := \sup_{j \in I} \sum_{i \in I} N_{i,j} < \infty,$$

where $\gamma_{j,1} := \gamma_{j,1}^{(0,1)}$ for $j \in I$ (i.e., $\gamma_{0,1} = \gamma_{1,1}^{(0,1)} = \varphi_1$ and $\gamma_{j,1} = \gamma_{j,1}^{(0,1)} = \psi_1$ for $j \in I_0$) and

$$N_{i,j} := \left( \frac{w_i}{w_j} \left[ \frac{\det T_j}{\det T_i} \right] \right)^{\tau} \cdot \left( 1 + \| T_j^{-1} T_i \| \right)^{\rho} \cdot \left( \int_{Q_i} \max_{|\beta| \leq N} \left[ \partial^\beta \gamma_{j,1} \right] (T_j^{-1} (\xi-b_j)) d\xi \right)^{\tau} \left( \text{since } b_j = 0 \text{ for all } j \in I \right) \quad \text{(since $b_j = 0$ for all $j \in I$)} \leq \left( \frac{w_j}{w_i} \right)^{-(1+\alpha)\rho - \sigma} \cdot \left( 1 + \| T_j^{-1} T_i \| \right)^{\rho} \cdot \left( \int_{\Omega_i^{(\alpha)}} \max_{|\beta| \leq N} \left[ \partial^\beta \gamma_{j,1} \right] (T_j^{-1} (\xi)) d\xi \right)^{\tau} \quad \text{(eq. 1.20 and } N \leq N_0 \leq C_5 \cdot M_{j,i}),$$

where the quantity $M_{j,i}^{(0)}$ is defined as in Lemma 1.1 but with $s^\sharp := (1+\alpha) \vartheta - s$ instead of $s$. At the step marked with $(*)$, we used that we have $w = w^\sharp$ and $|\det T_i| = v_i^{1+\alpha}$ for all $i \in I$.

To be precise, we recall from Theorem 2.10 that the quantities $N, \tau, \sigma, \vartheta$ from above are given (because of $d = 2$) by $\vartheta = (p^{-1} - 1) \tau$,

$$\tau = \min \{ 1, p, q \} \geq \min \{ p_0, q_0 \} =: \tau_0 \quad \text{and} \quad N = \left[ (d+\varepsilon) / \min \{ 1, p \} \right] \leq \left[ p_0^{-1} \cdot (2+\varepsilon) \right] = N_0,$$

as well as

$$\sigma = \left\{ \begin{array}{ll}
\tau \cdot (d+1) = 3 \cdot \tau, & \text{if } p \in [1, \infty], \\
\tau \cdot \left( \frac{d+\epsilon}{p} + \frac{d+\epsilon}{p} \right) = \tau \cdot \left( \frac{2}{p} + N_0 \right), & \text{if } p \in (0, 1].
\end{array} \right.$$ 

In particular, we have $\frac{2}{p} \leq \frac{2}{p_0} + N_0 =: \omega$, even in case of $p \in [1, \infty)$, since $\frac{2}{p_0} + N_0 \geq N_0 \geq 2+\varepsilon \geq 3$.

Now, Lemma 1.1 (with $\epsilon = \varepsilon$) yields a constant

$$C_0 = C_0 (\alpha, \tau_0, \omega, \varepsilon, K, H, M_1, M_2) = C_0 (\alpha, p_0, q_0, \varepsilon, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3) > 0$$

satisfying $\max \{ K_1, K_2 \} \leq C_0 \cdot C_0^\tau$, provided that we can show $H \geq H_0 + \varepsilon$, $K \geq K_0 + \varepsilon$ and $M_\ell \geq M_\ell^{(0)} + \varepsilon$ for $\ell \in \{ 1, 2 \}$, where

$$K_0 := \left\{ \begin{array}{ll}
\max \{ \frac{\alpha}{\tau} - s^\sharp, \frac{2+\alpha}{\tau} \}, & \text{if } \alpha = 1, \\
\max \{ \frac{1+\alpha}{\tau} - s^\sharp, \frac{2+\alpha}{\tau} \}, & \text{if } \alpha \in [0, 1),
\end{array} \right.$$

$$M_1^{(0)} := \left\{ \begin{array}{ll}
\frac{1}{\tau} + s^\sharp, & \text{if } \alpha = 1, \\
\frac{1}{\tau} + \max \{ s^\sharp, 0 \}, & \text{if } \alpha \in [0, 1),
\end{array} \right.$$

$$M_2^{(0)} := (1+\alpha) \frac{\sigma}{\tau} - s^\sharp,$$

$$H_0 := 1 - \frac{\alpha}{\tau} + \frac{\sigma}{\tau} - s^\sharp.$$

But we have

$$H_0 = \left\{ \begin{array}{ll}
\frac{1-\alpha}{\tau} + 3 + s, & \text{if } p \in [1, \infty], \\
\frac{1-\alpha}{\tau} + \frac{3}{p} + N - \left[ (1+\alpha) \left( \frac{1}{p} - 1 \right) - s \right], & \text{if } p \in (0, 1),
\end{array} \right.$$

$$= \left\{ \begin{array}{ll}
\frac{1-\alpha}{\tau} + 3 + s, & \text{if } p \in [1, \infty], \\
\frac{1-\alpha}{\tau} + \frac{1-\alpha}{p} + 1 + \alpha + \left[ \frac{2+\varepsilon}{p} \right] + s, & \text{if } p \in (0, 1),
\end{array} \right.$$

$$\leq \left\{ \begin{array}{ll}
\frac{1-\alpha}{\tau_0} + 3 + s_1, & \text{if } p \in [1, \infty], \\
\frac{1-\alpha}{\tau_0} + \frac{1-\alpha}{p_0} + 1 + \alpha + \left[ \frac{2+\varepsilon}{p_0} \right] + s_1, & \text{if } p \in (0, 1),
\end{array} \right.$$

$$\leq \Lambda_0 - 3 - 2\varepsilon = H - \varepsilon,$$
as an easy case distinction (using \( [p_0^{-1} \cdot (2 + \varepsilon)] \geq [2 + \varepsilon] \geq 3 \) and the observation that \( p \in (0, 1) \) entails \( p_0 \in (0, 1) \)) shows.

Furthermore,

\[
M_2^{(0)} = \begin{cases} 
3 \cdot (1 + \alpha) + s, & \text{if } p \in [1, \infty], \\
(1 + \alpha) \left( \frac{2}{p} + N \right) - \left[ (1 + \alpha) \left( \frac{1}{p} - 1 \right) - s \right], & \text{if } p \in (0, 1)
\end{cases}
\]

= \begin{cases} 
3 \cdot (1 + \alpha) + s, & \text{if } p \in [1, \infty], \\
(1 + \alpha) \left( 1 + \frac{1}{p} + N \right) + s, & \text{if } p \in (0, 1)
\end{cases}

\leq \begin{cases} 
3 \cdot (1 + \alpha) + s_1, & \text{if } p \in [1, \infty], \\
(1 + \alpha) \left( 1 + \frac{1}{p_0} + \left[ \frac{2 + \varepsilon}{p_0} \right] \right) + s_1, & \text{if } p \in (0, 1)
\end{cases}

\leq A_2 - \varepsilon = M_2 - \varepsilon,

as one can see again using an easy case distinction, since \([p_0^{-1} \cdot (2 + \varepsilon)] \geq [2 + \varepsilon] \geq 3\).

Likewise,

\[
M_1^{(0)} \leq \frac{1}{\tau} + \max \{ s^4, 0 \} \leq \frac{1}{\tau_0} + \max \left\{ \frac{1 - \alpha}{\tau} + 2 \left( \frac{\sigma}{\tau} - s^4 \right), \frac{2 + \sigma}{\tau} \right\}
\]

\[
\leq \frac{1}{\tau_0} + \max \left\{ 0, \left( 1 + \alpha \right) \left( \frac{1}{p} - 1 \right) - s \right\}
\]

\[
\leq \frac{1}{\tau_0} + \max \left\{ 0, \left( 1 + \alpha \right) \left( \frac{1}{p_0} - 1 \right) - s_0 \right\}
\]

= \Lambda_1 - \varepsilon = M_1 - \varepsilon.

Finally, we also have

\[
K_0 \leq \max \left\{ \frac{1 - \alpha}{\tau} + 2 \frac{\sigma}{\tau} - s^4, \frac{2 + \sigma}{\tau} \right\}
\]

\[
= \max \left\{ \frac{1 - \alpha}{\tau} + 6 + s, \frac{2}{\tau} + 3 \right\}, \quad \text{if } p \in [1, \infty],
\]

\[
\leq \max \left\{ \frac{1 - \alpha}{\tau} + 6 + s, \frac{2}{\tau} + 3 \right\}, \quad \text{if } p \in [1, \infty],
\]

\[
\leq \max \left\{ \frac{1 - \alpha}{\tau_0} + 6 + s_1, \frac{2}{\tau_0} + 3 \right\}, \quad \text{if } p \in [1, \infty],
\]

\[
\leq \max \left\{ \frac{1 - \alpha}{\tau_0} + \frac{2 + \varepsilon}{p_0} + 2 \left[ \frac{2 + \varepsilon}{p_0} \right], \frac{1 - \alpha}{\tau_0} + 2 + \frac{2 + \varepsilon}{p_0} + 2 \left[ \frac{2 + \varepsilon}{p_0} \right], \frac{2 + \varepsilon}{p_0} + \frac{2 + \varepsilon}{p_0} \right\}, \quad \text{if } p \in (0, 1)
\]

\leq \Lambda_3 - \varepsilon = K - \varepsilon,

as one can see again using an easy case distinction and the estimate \([p_0^{-1} \cdot (2 + \varepsilon)] \geq [2 + \varepsilon] \geq 3\).

Consequently, Lemma 1.1 is indeed applicable and yields \( \max \{ K_1, K_2 \} \leq C_5^2 C_6^2 \). We have thus verified all assumptions of Theorem 2.10, which yields a constant

\[
K = K \left( p_0, q_0, \varepsilon, d, \mathcal{Q}, \Phi, \gamma_1^{(0)}, \ldots, \gamma_0^{(0)} \right) = K \left( p_0, q_0, \varepsilon, \alpha, \varphi, \psi \right) > 0
\]

such that the family \( \mathcal{S}_{\alpha, \varphi, \psi, \delta}^{(1)} \) from the statement of the current theorem yields an atomic decomposition of the \( \alpha \)-shearlet smoothness space \( \mathcal{J}^{p,q}_{\alpha,\delta}(\mathbb{R}^2) = \mathcal{D}(\mathcal{Q}, L^p, \ell^q_{\varphi,\psi}) \), as soon as

\[
0 < \delta \leq \delta_0 := \min \left\{ 1, \left( \frac{K \cdot \Omega^{(p)} \cdot \left( K_1^{1/\tau} + K_2^{1/\tau} \right) \right)^{-1} \right\}
\]

But in equation (4.13) we saw \( \Omega^{(p)} \leq C_6 \) independently of \( p \geq p_0, q \geq q_0 \) and of \( s \in [s_0, s_1] \), so that

\[
\delta_0 \geq \delta_0 := \min \left\{ 1, \left[ 2K \cdot C_6 C_6 \right]^{-1} \right\}
\]

where \( \delta_0 > 0 \) is independent of the precise choice of \( p, q, s \), as long as \( p \geq p_0, q \geq q_0 \) and \( s \in [s_0, s_1] \). The claims concerning the notion of convergence for the series defining \( \mathcal{S}^{(\delta)} \) and concerning the independence of the action of \( C^{(\delta)} \) from the choice of \( p, q, s \) are consequences of the remark after Theorem 2.10. \( \square \)
If $\varphi, \psi$ are compactly supported and if the mother shearlet $\psi$ is a tensor product, the preceding conditions can be simplified significantly:

**Corollary 4.4.** Let $\alpha \in [0, 1]$, $\varepsilon, p_0, q_0 \in (0, 1]$ and $s_0, s_1 \in \mathbb{R}$ with $s_0 \leq s_1$. Let $\Lambda_0, \ldots, \Lambda_3$ as in Theorem 4.3 and set $N_0 := \left[ p_0^{-1} \cdot (2 + \varepsilon) \right]$.

Assume that the mother shearlet $\psi$ can be written as $\psi = \psi_1 \otimes \psi_2$ and that $\varphi, \psi_1, \psi_2$ satisfy the following:

1. We have $\varphi \in C^0_c(\Lambda_0) (\mathbb{R}^2)$, $\psi_1 \in C^0_c(\Lambda_2 + 3 + \varepsilon) (\mathbb{R})$, and $\psi_2 \in C^0_c(\Lambda_3 + 3 + \varepsilon) (\mathbb{R})$.
2. We have $\frac{\partial^\ell \psi_1 (0)}{\partial x_\ell} = 0$ for $\ell = 0, \ldots, N_0 + [\Lambda_1] - 1$.
3. We have $\hat{\varphi} (\xi) \neq 0$ for all $\xi \in [-1, 1]^2$.
4. We have $\hat{\psi}_1 (\xi) \neq 0$ for all $\xi \in [3^{-1}, 3]$ and $\hat{\psi}_2 (\xi) \neq 0$ for all $\xi \in [-3, 3]$.

Then, $\varphi, \psi$ satisfy all assumptions of Theorem 4.3.

**Proof.** Since $\varphi, \psi \in L^1 (\mathbb{R}^2)$ are compactly supported, it is well known that $\hat{\varphi}, \hat{\psi} \in C^\infty (\mathbb{R}^2)$ with all partial derivatives being polynomially bounded (in fact bounded). Thanks to the compact support and boundedness of $\varphi, \psi$, we also clearly have $\|\varphi\|_{1 + \frac{\beta}{p_0}} < \infty$ and $\|\hat{\psi}\|_{1 + \frac{\beta}{p_0}} < \infty$.

Next, if $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ satisfies $\xi_1 \in [3^{-1}, 3]$ and $|\xi_2| \leq |\xi_1|$, then $|\xi_2| \leq |\xi_1| \leq 3$, i.e., $\xi_2 \in [-3, 3]$. Thus $\hat{\psi} (\xi) = \hat{\psi}_1 (\xi_1) \cdot \hat{\psi}_2 (\xi_2) \neq 0$, as required in Theorem 4.3.

Hence, it only remains to verify

$$
|\partial^\beta \hat{\varphi} (\xi)| \lesssim (1 + |\xi|)^{-\Lambda_0} \quad \text{and} \quad |\partial^\beta \hat{\psi} (\xi)| \lesssim \min \left\{ |\xi_1|^\Lambda_1, (1 + |\xi_1|)^{-\Lambda_2}, (1 + |\xi_2|)^{-\Lambda_3}, (1 + |\xi|)^{-(3 + \varepsilon)} \right\}
$$

for all $\xi \in \mathbb{R}^2$ and all $\beta \in \mathbb{N}_0^2$ with $|\beta| \leq N_0$. To this end, we first recall that differentiation under the integral shows for $g \in C_c(\mathbb{R}^d)$ that $\hat{g} \in C^\infty (\mathbb{R}^d)$, where the derivatives are given by

$$
\partial^\beta \hat{g} (\xi) = \int_{\mathbb{R}^d} g(x) \cdot \partial^\beta \xi e^{-2\pi i \xi \cdot x} \, dx = \int_{\mathbb{R}^d} (-2\pi i)^\beta g(x) \cdot e^{-2\pi i \xi \cdot x} \, dx = \left( \mathcal{F} [x \mapsto (-2\pi i)^\beta g(x)] \right) (\xi). \quad (4.14)
$$

Furthermore, the usual mantra that “smoothness of $f$ implies decay of $\hat{f}$” shows that every $g \in W^{N,1} (\mathbb{R}^d)$ satisfies $|\hat{g} (\xi)| \lesssim (1 + |\xi|)^{-N}$, see e.g. [62] Lemma 6.3.

Now, because of $\varphi \in C^0_c(\Lambda_0) (\mathbb{R}^2)$, we also have $[x \mapsto (-2\pi i)^\beta \varphi (x)] \in C^0_c(\Lambda_0) (\mathbb{R}^2) \hookrightarrow W^{[\Lambda_0],1} (\mathbb{R}^2)$ and thus

$$
|\partial^\beta \hat{\varphi} (\xi)| = \left| \left( \mathcal{F} [x \mapsto (-2\pi i)^\beta \varphi (x)] \right) (\xi) \right| \lesssim (1 + |\xi|)^{-[\Lambda_0]} \leq (1 + |\xi|)^{-\Lambda_0},
$$

as desired.

For the estimate concerning $\hat{\psi}$, we have to work slightly harder: With the same arguments as for $\varphi$, we get

$$
|\partial^\beta \hat{\psi}_1 (\xi)| \lesssim (1 + |\xi|)^{-(\Lambda_2 + 3 + \varepsilon)} \quad \text{and} \quad |\partial^\beta \hat{\psi}_2 (\xi)| \lesssim (1 + |\xi|)^{-(\Lambda_3 + 3 + \varepsilon)}
$$

for all $|\beta| \leq N_0$. Now, in case of $|\xi_1| \geq 1$, we have $|\xi_1|^\Lambda_1 \geq 1 \geq (1 + |\xi_1|)^{-\Lambda_2}$ and thus

$$
|\partial^\beta \hat{\psi}_1 (\xi)| = \left| \left( \partial^\beta \hat{\psi}_1 \right) (\xi_1) \cdot \left( \partial^\beta \hat{\psi}_2 \right) (\xi_2) \right| \lesssim (1 + |\xi_1|)^{-(\Lambda_2 + 3 + \varepsilon)} \cdot (1 + |\xi_2|)^{-(\Lambda_3 + 3 + \varepsilon)} = \min \left\{ |\xi_1|^\Lambda_1, (1 + |\xi_1|)^{-\Lambda_2}, (1 + |\xi_2|)^{-\Lambda_3} \right\} \cdot \left( (1 + |\xi_1|) (1 + |\xi_2|) \right)^{-(3 + \varepsilon)} \leq \min \left\{ |\xi_1|^\Lambda_1, (1 + |\xi_1|)^{-\Lambda_2}, (1 + |\xi_2|)^{-\Lambda_3}, (1 + |\xi|)^{-(3 + \varepsilon)} \right\},
$$

as desired. Here, the last step used that $(1 + |\xi_1|) (1 + |\xi_2|) \geq 1 + |\xi_1| + |\xi_2| \geq 1 + |\xi|$.

It remains to consider the case $|\xi_1| \leq 1$. But for arbitrary $\beta_1 \in N_0$ with $\beta_1 \leq N_0$, our assumptions on $\hat{\psi}_1$ ensure

$$
\partial^\beta \hat{\psi}_1 (0) = 0 \quad \text{for all } \theta \in \{0, \ldots, [\Lambda_1] - 1\}, \text{ where we note } \Lambda_1 > 0, \text{ so that } [\Lambda_1] - 1 \geq 0.
$$

But as the Fourier transform of a compactly supported function, $\hat{\psi}_1$ (and thus also $\partial^\beta \hat{\psi}_1$) can be extended to an entire function on
\[ \frac{\partial^{\beta_1} \hat{\psi}_1}{\partial \xi^\ell} \left( \xi_1 \right) = \sum_{\theta = 0}^{\infty} \frac{\partial^\theta}{\theta!} \left[ \frac{\partial^{\beta_1} \hat{\psi}_1}{\partial \xi^\ell} \right] (0) \cdot \xi_1^\theta = \sum_{\theta = [\Lambda_1]}^{\infty} \frac{\partial^\theta}{\theta!} \left[ \frac{\partial^{\beta_1} \hat{\psi}_1}{\partial \xi^\ell} \right] (0) \cdot \xi_1^\theta \quad (4.15) \]

for all \( \xi \in \mathbb{R} \), where the power series in the last line converges absolutely on all of \( \mathbb{R} \). In particular, the (continuous(!)) function defined by the power series is bounded on \([-1, 1] \), so that we get \( \left| \frac{\partial^{\beta_1} \hat{\psi}_1}{\partial \xi^\ell} \left( \xi_1 \right) \right| \lesssim |\xi_1|^{[\Lambda_1]} \leq |\xi_1|^{\Lambda_1} \) for \( \xi_1 \in [-1, 1] \). Furthermore, note \( (1 + |\xi_1|)^{-\Lambda_2} \geq 2^{-\Lambda_2} \geq 2^{-\Lambda_2} \cdot |\xi_1|^{\Lambda_1} \), so that

\[
\left| \frac{\partial^\beta \hat{\psi}}{\partial \xi} \left( \xi \right) \right| = \left| \left( \frac{\partial^{\beta_1} \hat{\psi}_1}{\partial \xi} \right) (\xi_1) \cdot \left( \frac{\partial^{\beta_2} \hat{\psi}_2}{\partial \xi} \right) (\xi_2) \right|
\lesssim |\xi_1|^{\Lambda_1} \cdot (1 + |\xi_2|)^{-\left(\Lambda_1 + 2 + \varepsilon\right)}
\lesssim 2^{\Lambda_2} \cdot \min \left\{ |\xi_1|^{\Lambda_1}, (1 + |\xi_2|)^{-\Lambda_2} \right\} \cdot (1 + |\xi_2|)^{\Lambda_3} \cdot (1 + |\xi_1|)^{3 + \varepsilon} \cdot (1 + |\xi_2|)^{(-3 + \varepsilon)}.
\]

Finally, we provide an analogous simplification of the conditions of Theorem 4.2.

**Corollary 4.5.** Let \( \alpha \in [0, 1], \varepsilon, p_0, q_0 \in (0, 1] \) and \( s_0, s_1 \in \mathbb{R} \) with \( s_0 \leq s_1 \). Let \( K, M_1, M_2, H \) as in Theorem 4.2 and set \( N_0 := \left[ p_0^{-1} \cdot (2 + \varepsilon) \right] \).

The functions \( \varphi, \psi \) fulfill all assumption of Theorem 4.2 if the mother shearlet \( \psi \) can be written as \( \hat{\psi} = \hat{\psi}_1 \otimes \hat{\psi}_2 \), where \( \varphi, \hat{\psi}_1, \hat{\psi}_2 \) satisfy the following:

1. We have \( \varphi \in C^{[H+1]}_c(\mathbb{R}^2), \hat{\psi}_1 \in C^{[M_2+1]}_c(\mathbb{R}), \) and \( \hat{\psi}_2 \in C^{[K+1]}_c(\mathbb{R}) \).
2. We have \( \frac{\partial^\ell \hat{\psi}_1}{\partial \xi} (0) = 0 \) for \( \ell = 0, \ldots, N_0 + [M_1] - 1 \).
3. We have \( \hat{\varphi}(\xi) \neq 0 \) for all \( \xi \in [-1, 1]^2 \).
4. We have \( \hat{\psi}_2(\xi) \neq 0 \) for all \( \xi \in [3^{-1}, 3] \) and \( \hat{\psi}_2(\xi) \neq 0 \) for all \( \xi \in [-3, 3] \).

**Proof.** Observe \( \varphi, \psi \in C^2_c(\mathbb{R}^2) \) \( \subseteq L^1(\mathbb{R}^2) \) and note \( \hat{\varphi}, \hat{\psi} \in C^\infty(\mathbb{R}^2) \), where all partial derivatives of these functions are bounded (and thus polynomially bounded), since \( \varphi, \psi \) are compactly supported. Next, since \( K, H, M_1, M_2 \geq 0 \), our assumptions clearly entail \( \varphi, \psi \in C^2_c(\mathbb{R}^2) \), so that \( \nabla \varphi, \nabla \psi \in L^1(\mathbb{R}^2) \) \( \cap L^\infty(\mathbb{R}^2) \). Furthermore, we see exactly as in the proof of Corollary 4.3 that \( \hat{\psi}(\xi) \neq 0 \) for all \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \) with \( \xi_1 \in [3^{-1}, 3] \) and \( |\xi_2| \leq |\xi_1| \).

Thus, it remains to verify that \( \hat{\varphi}, \hat{\psi} \) satisfy the decay conditions in equation (4.13). But we see exactly as in the proof of Corollary 4.3 (cf. the argument around equation (4.14)) that \( \left| \frac{\partial^\ell \hat{\varphi}}{\partial \xi}(\xi) \right| \lesssim (1 + |\xi|)^{-(H+1)} \leq (1 + |\xi|)^{-(H+1)} \), as well as \( \left| \frac{\partial^{\beta_1} \hat{\psi}_1}{\partial \xi}(\xi_1) \right| \lesssim (1 + |\xi_1|)^{-[M_1+1]} \leq (1 + |\xi_1|)^{-[M_1+1]} \) and \( \left| \frac{\partial^{\beta_2} \hat{\psi}_2}{\partial \xi}(\xi_2) \right| \lesssim (1 + |\xi_2|)^{-(K+1)} \leq (1 + |\xi_2|)^{-(K+1)} \) for all \( \beta \in \mathbb{N}^2_0 \) and \( \beta_1, \beta_2 \in \mathbb{N}_0 \). This establishes the last two lines of equation (4.13).

For the first line of equation (4.13), we see as in the proof of Corollary 4.3 (cf. the argument around equation (4.14)) that
\[
\left| \frac{\partial^{\beta_1} \hat{\psi}_1}{\partial \xi}(\xi_1) \right| \lesssim |\xi_1|^{[M_1]} \leq |\xi_1|^{M_1} \quad \text{for all } \xi_1 \in [-1, 1].
\]
Since we saw above that \( \left| \frac{\partial^{\beta_2} \hat{\psi}_2}{\partial \xi}(\xi_2) \right| \leq (1 + |\xi_2|)^{-(K+1)} \) for all \( \xi_2 \in \mathbb{R} \), we have thus also established the first line of equation (4.13).

**5. The unconnected α-shearlet covering**

The \( \alpha \)-shearlet covering as introduced in Definition 3.1 divides the frequency space \( \mathbb{R}^2 \) into a low-frequency part and into four different frequency cones: the top, bottom, left and right cones. But for real-valued functions, the absolute value of the Fourier transform is symmetric. Consequently, there is no non-zero real-valued function with Fourier transform essentially supported in the top (or left, ...) cone.

For this reason, it is customary to divide the frequency plane into a low-frequency part and two different frequency cones: the horizontal and the vertical frequency cone. In this section, we account for this slightly different partition of the frequency plane, by introducing the so-called unconnected \( \alpha \)-shearlet covering. The reason for this nomenclature is that the connected base set \( Q = U_{(3^{-1}, 3)} \) from Definition 3.1 is replaced by the unconnected set \( Q \cup (-Q) \). We then show that all results from the preceding two sections remain true for this modified covering, essentially since the associated decomposition spaces are identical, cf. Lemma 5.3.

\[\text{(4.15)}\]
Definition 5.1. Let $\alpha \in [0,1]$. The unconnected $\alpha$-shearlet covering $S_u^{(\alpha)}$ is defined as
\[
S_u^{(\alpha)} := (W_v^{(\alpha)})_{v \in V^{(\alpha)}} := (W_v)_{v \in V^{(\alpha)}} := (B_v W'_v + b_v)_{v \in V^{(\alpha)}},
\]
where:

- The index set $V^{(\alpha)}$ is given by $V := V^{(\alpha)} := \{0\} \cup V_0$, where $V_0 := V^{(\alpha)}_0 := \{(n,m,\delta) \in \mathbb{N}_0 \times \mathbb{Z} \times \{0,1\} \mid |m| \leq G_n\}$ with $G_n := G^{(\alpha)}_n := \lfloor 2^{n(1-\alpha)} \rfloor$.

- The basic sets $(W_v)_{v \in V^{(\alpha)}}$ are given by $W'_v := (-1,1)^2$ and by $W''_v := Q_v := U^{3,-1,3}_{(-1,1)} \cup [-U^{3,-1,3}_{(-1,1)}]$ for $v \in V^{(\alpha)}$. The notation $U^{(\gamma,\mu)}_{(a,b)}$ used here is as defined in equation (3.1).

- The matrices $(B_v)_{v \in V^{(\alpha)}}$ are given by $B_0 := B^{(\alpha)}_0 := \text{id}$ and by $B_v := B^{(\alpha)}_v := R^T \cdot A^{(\alpha)}_{n,m}$, where we define $A^{(\alpha)}_{n,m} := 2^{3\alpha} \cdot S^T_m$ for $v = (n,m,\delta) \in V_0$. Here, the matrices $R, S, S^T$ and $D^{(\alpha)}_b$ are as in equation (1.5).

- The translations $(b_v)_{v \in V^{(\alpha)}}$ are given by $b_v := 0$ for all $v \in V^{(\alpha)}$.

Finally, we define the weight $u = (u_v)_{v \in V}$ by $u_0 := 1$ and $u_{n,m,\delta} := 2^n$ for $(n,m,\delta) \in V_0$.

The unconnected $\alpha$-shearlet covering $S_u^{(\alpha)}$ is highly similar to the (connected) $\alpha$-shearlet covering $S^{(\alpha)}$ from Definition 5.1. In particular, we have $Q_v = Q \cup (-Q)$ with $Q = U^{3,-1,3}_{(-1,1)}$ as in Definition 5.1. To further exploit this connection between the two coverings, we define the projection map
\[
\pi: I^{(\alpha)} \to V^{(\alpha)}, \quad i \mapsto \begin{cases} 0, & \text{if } i = 0, \\ (n,m,\delta), & \text{if } i = (n,m,\varepsilon,\delta) \in I^{(\alpha)}. \end{cases}
\]
Likewise, for $\varepsilon \in \{\pm 1\}$, we define the $\varepsilon$-injection
\[
t_{\varepsilon}: V^{(\alpha)} \to I^{(\alpha)}, \quad v \mapsto \begin{cases} 0, & \text{if } v = 0, \\ (n,m,\varepsilon,\delta), & \text{if } v = (n,m,\delta) \in V^{(\alpha)}. \end{cases}
\]
Note that $B_v^{(\alpha)} = \varepsilon \cdot T^{(\alpha)}_{t_{\varepsilon}(v)}$ for all $v \in V_0^{(\alpha)}$, so that
\[
W_v^{(\alpha)} = S_{t_{\varepsilon}(v)}^{(\alpha)} \cup S_{t_{-\varepsilon}(v)}^{(\alpha)} = \bigcup_{\varepsilon \in \{\pm 1\}} S_{t_{\varepsilon}(v)}^{(\alpha)} \quad \forall v \in V_0^{(\alpha)},
\]
(5.1)
since $B_v^{(\alpha)} [-U^{3,-1,3}_{(-1,1)}] = -B_v^{(\alpha)} U^{3,-1,3}_{(-1,1)} = T^{(\alpha)}_{t_{\varepsilon}(v)} Q = S_{t_{\varepsilon}(v)}^{(\alpha)}$. Because of $W_v^{(\alpha)} = (-1,1)^2 = S_0^{(\alpha)}$, equation (5.1) remains valid for $v = 0$. Using these observations, we can now prove the following lemma:

**Lemma 5.2.** The unconnected $\alpha$-shearlet covering $S_u^{(\alpha)}$ is an almost structured covering of $\mathbb{R}^2$.

**Proof.** In Lemma 3.3 we showed that the (connected) $\alpha$-shearlet covering $S^{(\alpha)}$ is almost structured. Thus, for the proof of the present lemma, we will frequently refer to the proof of Lemma 3.3.

First of all, recall from the proof of Lemma 3.3, the notation $P'_{(n,m,\varepsilon,\delta)} = U^{3,1/2,1/2}_{(-3/4,3/4)}$ for arbitrary $(n,m,\varepsilon,\delta) \in I_0$.

Then, for $v = (n,m,\delta) \in V_0$ let us define $R'_{(n,m,\delta)} := P'_{(n,m,1,\delta)} \cup \left(-P'_{(n,m,1,\delta)}\right)$. Furthermore, set $R_0' := R_0'$, again with $P_0' = (-\frac{3}{4}, \frac{3}{4})^2$ as in the proof of Lemma 3.3. Then it is not hard to verify $R_0' \subset W_v'$ for all $v \in V$.

Furthermore, in the proof of Lemma 3.3 we showed $\bigcup_{i \in I} I_iP_i' = \mathbb{R}^2$. But this implies
\[
\bigcup_{v \in V} \left( (B_v R_v' + b_v) = R_0' \cup \bigcup_{(n,m,\delta) \in V_0} B_{(n,m,\delta)} R_{(n,m,\delta)}' \right)
\]
\[
= P_0' \cup \bigcup_{(n,m,\delta) \in V_0} \left( B_{(n,m,\delta)} P_{(n,m,1,\delta)}' \cup -B_{(n,m,\delta)} P_{(n,m,1,\delta)}' \right)
\]
(since $P_{(n,m,1,\delta)} = P_{(n,m,-1,\delta)}'$)
\[
= P_0' \cup \bigcup_{(n,m,\delta) \in V_0} \left( T_{(n,m,\delta)} P_{(n,m,1,\delta)}' \cup T_{(n,m,-1,\delta)} P_{(n,m,-1,\delta)}' \right)
\]
\[
= \bigcup_{i \in I} (T_i P_i' + b_i) = \mathbb{R}^2.
\]
Next, if $W_t^{(a)} \cap W_s^{(a)} \neq \emptyset$, then equation (5.1) yields certain $\varepsilon, \beta \in \{\pm 1\}$ such that $S_t^{(a)} \cap S_s^{(a)} \neq \emptyset$. But this implies $(k, \ell, \gamma) = \pi((k, \ell, \beta, \gamma))$, where $(k, \ell, \beta, \gamma) \in I_0 \cap (n, m, \varepsilon, \delta)^*$ and where the index cluster is formed with respect to the covering $S^{(a)}$. Consequently, we have shown
\[(n, m, \delta)^* \subset \{0\} \cup \bigcup_{\varepsilon \in \{\pm 1\}} \pi(I_0 \cap (n, m, \varepsilon, \delta)^*). \quad (5.2)
\]
But since $S^{(a)}$ is admissible, the constant $N := \sup_{I \in I^*} |I^*|$ is finite. But by what we just showed, we have $|(n, m, \delta)^*| \leq 1 + 2N$ for all $(n, m, \delta) \in V_0$. Finally, using a very similar argument one can show
\[0^* S^{(a)} \subset \{0\} \cup \pi(I_0 \cap 0^* S^{(a)}),\]
where the index-cluster is taken with respect to $S_u^{(a)}$ on the left-hand side and with respect to $S^{(a)}$ on the right-hand side. Thus, $|0^* S^{(a)}| \leq 1 + N$, so that $\sup_{v \in V} |S_u^{(a)}|$ is finite, since $S^{(a)}$ is an almost structured covering. Now, let $v \in V$ and $r \in v^*$ be arbitrary. We distinguish several cases:

**Case 1:** We have $v = (n, m, \delta) \in V_0$ and $r = (k, \ell, \gamma) \in V_0$.
As above, there are thus certain $\varepsilon, \beta \in \{\pm 1\}$ such that $(k, \ell, \beta, \gamma) \in (n, m, \varepsilon, \delta)^*$. Hence,
\[\|B_v^{-1} B_r\| = \left\| (\varepsilon \cdot T_{n,m,\varepsilon,\delta})^{-1} \cdot \beta \cdot T_{k,\ell,\beta,\gamma} \right\| = \left\| (T_{n,m,\varepsilon,\delta})^{-1} \cdot T_{k,\ell,\beta,\gamma} \right\| \leq C.\]

**Case 2:** We have $v = 0$ and $r = (k, \ell, \gamma) \in V_0$. There is then some $\beta \in \{\pm 1\}$ satisfying $(k, \ell, \beta, \gamma) \in 0^*$, where the index-cluster is taken with respect to $S^{(a)}$. Hence, we get again that
\[\|B_v^{-1} B_r\| = \|T_0^{-1} \cdot \beta \cdot T_{k,\ell,\beta,\gamma} \| = \|T_0^{-1} \cdot T_{k,\ell,\beta,\gamma} \| \leq C.\]

**Case 3:** We have $v = (n, m, \delta) \in V_0$ and $r = 0$. Hence, $0 \in (n, m, \varepsilon, \delta)^*$ for some $\varepsilon \in \{\pm 1\}$, so that
\[\|B_v^{-1} B_r\| = \left\| (\varepsilon \cdot T_{n,m,\varepsilon,\delta})^{-1} \cdot T_0 \right\| = \left\| T_{n,m,\varepsilon,\delta}^{-1} \cdot T_0 \right\| \leq C.\]

**Case 4:** We have $v = r = 0$. In this case, $\|B_v^{-1} B_r\| = 1 \leq C$.
Hence, we have verified $\sup_{v \in V} \sup_{r \in v^*} \|B_v^{-1} B_r\| < \infty$. Since the sets $\{W_v^* \mid v \in V\}$ and $\{R^*_v \mid v \in V\}$ are finite families of bounded, open sets (in fact, each of these families only has two elements), we have shown that $S_u^{(a)}$ is an almost structured covering of $\mathbb{R}^2$.

Before we can define the decomposition spaces associated to the unconnected $\alpha$-shearlet covering $S_u^{(a)}$, we need to verify that the weights that we want to use are $S_u^{(a)}$-moderate.

**Lemma 5.3.** Let $u = (u_v)_{v \in V}$ as in Definition 5.1. Then $u^* = (u_v^*)_{v \in V}$ is $S_u^{(a)}$-moderate with $C_{S_u^{(a)}, u^*} \leq 39|s|$. \hfill \blacktriangleleft

**Proof.** As seen in equation (5.1), we have $W_v^{(a)} = \bigcup_{\varepsilon \in \{\pm 1\}} S_{t_{\varepsilon,v}}^{(a)}$ for arbitrary $v \in V$ (also for $v = 0$). Furthermore, it is easy to see $u_v = w_{t_{\varepsilon,v}}(v)$ for arbitrary $\varepsilon \in \{\pm 1\}$ and $v \in V$.

Thus, if $W_v^{(a)} \cap W_r^{(a)} \neq \emptyset$ for certain $v, r \in V$, there are $\varepsilon, \beta \in \{\pm 1\}$ such that $S_{t_{\varepsilon,v}}^{(a)} \cap S_{t_{\beta,r}}^{(a)} \neq \emptyset$. But Lemma 5.3 shows that $u^*$ is $S^{(a)}$-moderate with $C_{S^{(a)}, u^*} \leq 39|s|$. Hence,
\[u_v^*/u_r^* = w_{t_{\varepsilon,v}^*(v)}/w_{t_{\beta,r}^*(r)} \leq 39|s|. \hfill \blacktriangleleft\]

Since we now know that $S_u^{(a)}$ is an almost structured covering of $\mathbb{R}^2$ and since $u^*$ is $S_u^{(a)}$-moderate, we see precisely as in the remark after Definition 5.5 that the unconnected $\alpha$-shearlet smoothness spaces that we now define are well-defined Quasi-Banach spaces. We emphasize that the following definition will only be of transitory relevance, since we will immediately show that the newly defined unconnected $\alpha$-shearlet smoothness spaces are identical with the previously defined $\alpha$-shearlet smoothness spaces.

**Definition 5.4.** For $\alpha \in [0, 1]$, $p, q \in (0, \infty]$ and $s \in \mathbb{R}$, we define the unconnected $\alpha$-shearlet smoothness space $\mathcal{G}_{\alpha, s}^p (\mathbb{R}^2)$ associated to these parameters as
\[\mathcal{G}_{\alpha, s}^p (\mathbb{R}^2) := \mathcal{D} \left( S_u^{(a)}, \mathcal{L}^p, \ell^q_{u^*} \right),\]
where the covering $S_u^{(a)}$ and the weight $u^*$ are as in Definition 5.1 and Lemma 5.3 respectively. \hfill \blacktriangleleft
Lemma 5.5. We have
\[ S_{\alpha}^p(q)(\mathbb{R}^2) = \mathcal{P}_{\alpha}^{p,q}(\mathbb{R}^2) \quad \forall \alpha \in [0,1], \quad p, q \in (0, \infty) \quad \text{and} \quad s \in \mathbb{R}, \]
with equivalent quasi-norms. ▶

Proof. We will derive the claim from [60, Lemma 6.11, part (2)], with the choice \( Q := \tilde{S}_{\alpha}^1 \) and \( P := S_{\alpha}^1 \), recalling that \( S_{\alpha}^{p,q}(\mathbb{R}^2) = D(S_{\alpha}^p, L^p, L^q) \) and likewise \( \mathcal{P}_{\alpha}^{p,q}(\mathbb{R}^2) = F^{-1} \left[ D_F \left( S_{\alpha}^p, L^p, L^q \right) \right] \).

To this end, we first have to verify that the coverings \( S_{\alpha}^1 \) and \( S_{\alpha}^1 \) are weakly equivalent. This means that
\[ \sup_{t \in T} \left\{ v \in V \left| W_{i}^\alpha \cap S^1_{\alpha} \not= \emptyset \right. \right\} < \infty \quad \text{and} \quad \sup_{v \in V} \left\{ i \in T \left| S^1_{\alpha} \cap W_{i}^\alpha \not= \emptyset \right. \right\} < \infty. \]

We begin with the first claim and thus let \( i \in T \) be arbitrary. It is easy to see \( S^1_{\alpha} \subset W^\alpha(i) \). Consequently, if \( v \in V \) satisfies \( W_{i}^\alpha \cap S^1_{\alpha} \not= \emptyset \), then \( \emptyset \subset W_{i}^\alpha \cap S^1_{\alpha} \subset W_{i}^\alpha \cap W^\alpha(i) \) and thus \( v \in [\pi(i)]^* \), where the index-cluster is formed with respect to \( S_{\alpha}^1 \). On the one hand, this implies
\[ w_{i}^\alpha = w_{\pi(i)}^\alpha \geq w_{i}^\alpha \quad \text{if} \quad S^1_{\alpha} \cap W^\alpha(i) \neq \emptyset, \quad \text{for arbitrary} \ t \in \mathbb{R}, \quad (5.3) \]
since \( u_t \) is \( S_{\alpha}^1 \)-moderate by Lemma 5.3. On the other hand, we get
\[ \sup_{i \in T} \left\{ v \in V \left| W_{i}^\alpha \cap S^1_{\alpha} \not= \emptyset \right. \right\} \leq \sup_{i \in T} \| \pi(i) \|^{*} \leq \sup_{v \in V} \| v \|^{*} < \infty, \]

since we know that \( S^1_{\alpha} \) is admissible (cf. Lemma 5.2).

Now, let us verify the second claim. To this end, let \( v \in V \) be arbitrary. For \( i \in T \) with \( S^1_{\alpha} \cap W^\alpha(i) \neq \emptyset \), equation 5.3 shows \( \emptyset \not= \bigcup_{i \in \{ \pm 1\}} \left( S^1_{\alpha} \cup \left( S^1_{\alpha} \cap W^\alpha(i) \right) \right) \) and thus \( i \in \bigcup_{i \in \{ \pm 1\}} [\pi(i)]^\ast \), where the index-cluster is formed with respect to \( S^1 \). As above, this yields
\[ \sup_{v \in V} \left\{ i \in T \left| S^1_{\alpha} \cap W_{i}^\alpha \not= \emptyset \right. \right\} \leq \sup_{v \in V} \| \pi \|^{*} + \| \pi \|^{*} \leq 2 \cdot \sup_{i \in T} \| v \|^{*} < \infty, \]
since \( S^1 \) is admissible (cf. Lemma 5.3).

We have thus verified the two main assumptions of [60, Lemma 6.11], namely that \( Q, P \) are weakly equivalent and that \( u_t \not\sim u_t \) if \( W^\alpha(i) \cap S^1_{\alpha} \not= \emptyset \), thanks to equation 5.3. But since we also want to get the claim for \( p \in (0,1) \), we have to verify the additional condition (2) of [60, Lemma 6.11], i.e., that \( P = S^1 = (S^1_{\alpha})_{i \in T} = (T_j Q^j)_{j \in I} \) is almost subordinate to \( Q = S^1 \) and that \( |\det \left( T_{j}^{-1} B_j \right) | \leq 1 \) if \( W \cap S^1 \not= \emptyset \). But we saw in equation 5.3 that if \( W^\alpha(i) \cap S^1_{\alpha} \not= \emptyset \), then
\[ |\det \left( T_{j}^{-1} B_j \right) | = \left( w_{j}^{\alpha+1} \right)^{j^{-1}} \cdot \left( w_{j}^{\alpha+1} \right)^{\alpha} \leq 1, \]

Furthermore, \( S^1_{\alpha} \subset W^\alpha(i) \) for all \( j \in I \), so that \( P = S^1 \) is subordinate (and thus also almost subordinate, cf. [60, Definition 2.10]) to \( Q = S^1 \), as required. The claim is now an immediate consequence of [60, Lemma 6.11]. □

In order to allow for a more succinct formulation of our results about Banach frames and atomic decompositions in the setting of the unconnected \( \alpha \)-shearlet covering, we now introduce the notion of cone-adapted \( \alpha \)-shearlet systems. As we will see in Section 6.1 these systems are different, but intimately connected to the cone-adapted \( \beta \)-shearlet systems (with \( \beta \in (1,\infty) \)) as introduced in [71, Definition 3.10]. There are three main reasons why we think that the new definition is preferable to the old one:

1. With the new definition, a family \( (L \circ k) \cup \left( \psi_j, l, \delta, k \right)_{j, l, \delta, k} \) of \( \alpha \)-shearlets has the property that the shearlets \( \psi_j, l, \delta, k \) of scale \( \delta \) have essential frequency support in the dyadic corona \( \{ \xi \in \mathbb{R}^2 \mid 2^{j-c} < |\xi| < 2^{j+c} \} \) for suitable \( c > 0 \). In contrast, for \( \beta \)-shearlets, the shearlets of scale \( j \) have essential frequency support in \( \left\{ \xi \in \mathbb{R}^2 \mid 2^{j-c} < |\xi| < 2^{j+c} \right\} \), cf. Lemma 1.2.

2. With the new definition, a family of cone-adapted \( \alpha \)-shearlets is also a family of \( \alpha \)-molecules, if the generators are chosen suitably. In contrast, for \( \beta \)-shearlets, one has the slightly inconvenient fact that a family of cone-adapted \( \beta \)-shearlets is a family of \( \beta^{-1} \)-molecules, cf. [71, Proposition 3.11].

3. The new definition includes the two boundary values \( \alpha \in \{0,1\} \) which correspond to ridgelet-like systems and to wavelet-like systems, respectively. In contrast, for \( \beta \)-shearlets, the boundary values \( \beta \in \{1,\infty\} \) are excluded from the definition.
We remark that a very similar definition to the one given here is already introduced in [20] Definition 5.1, even generally in \( \mathbb{R}^d \) for \( d \geq 2 \).

**Definition 5.6.** Let \( \alpha \in [0,1] \). For generators \( \varphi, \psi \in L^1(\mathbb{R}^2) + L^2(\mathbb{R}^2) \) and a given sampling density \( \delta > 0 \), we define the cone-adapted \( \alpha \)-shearlet system with sampling density \( \delta \) generated by \( \varphi, \psi \) as

\[
SH_\alpha(\varphi, \psi; \delta) := \left\{ \gamma[v,k] \right\}_{v \in V, k \in \mathbb{Z}^2} := \left\{ (L_{\delta, B_{v+k}^\varphi} \gamma[v])_{v \in V, k \in \mathbb{Z}^2} \right\}
\]

with \( \gamma[v] := \begin{cases} \left| \det B_v \right|^{1/2} \cdot \left( \psi \circ B_v^T \right), & \text{if } v \in V_0, \\ \varphi, & \text{if } v = 0, \end{cases} \)

where \( V, V_0 \) and \( B_v \) are as in Definition 5.1. Note that the notation \( \gamma[v,k] \) suppresses the sampling density \( \delta > 0 \). If we want to emphasize this sampling density, we write \( \gamma[v,k,\delta] \) instead of \( \gamma[v,k] \).

**Remark 5.7.** In case of \( \alpha = \frac{1}{2} \), the preceding definition yields special cone-adapted shearlet systems: As defined in [21] Definition 1.2, the cone-adapted shearlet system \( \text{SH}(\varphi, \psi; \theta; \delta) \) with sampling density \( \delta > 0 \) generated by \( \varphi, \psi, \theta \in L^2(\mathbb{R}^2) \) is \( \text{SH}(\varphi, \psi; \theta; \delta) = \Phi(\varphi; \psi; \theta; \delta) \cup \Psi(\psi; \psi; \theta; \delta) \cup \Theta(\theta; \theta) \), where

\[
\begin{align*}
\Phi(\varphi; \psi; \delta) & := \left\{ \varphi_k := \varphi(k - \delta k) \mid k \in \mathbb{Z}^2 \right\}, \\
\Psi(\psi; \delta) & := \left\{ \psi_{j,\ell,k} := 2^{\frac{3}{2}j} \cdot \psi(2^j \cdot S_{\ell} A_{2^j} \cdot -\delta k) \mid j \in \mathbb{N}_0, \ell \in \mathbb{Z} \text{ with } |\ell| \leq \left[ 2^{j/2} \right] \text{ and } k \in \mathbb{Z}^2 \right\}, \\
\Theta(\theta; \delta) & := \left\{ \theta_{j,\ell,k} := 2^{\frac{3}{2}j} \cdot \theta(2^j \cdot S_{\ell} A_{2^j} \cdot -\delta k) \mid j \in \mathbb{N}_0, \ell \in \mathbb{Z} \text{ with } |\ell| \leq \left[ 2^{j/2} \right] \text{ and } k \in \mathbb{Z}^2 \right\},
\end{align*}
\]

with \( S_k = \left( \begin{smallmatrix} 1 & -k_1 \\ 0 & 1 \end{smallmatrix} \right), A_{2^j} = \text{diag}(2^j, 2^j) \) and \( \tilde{A}_{2^j} = \text{diag}(2^j, 2^j) \).

Now, the most common choice for \( \theta = \psi \circ R \) for \( R = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \). With this choice, we observe in the notation of Definitions 5.6 and 5.7 that

\[
\gamma[0,k] = L_{\delta, B_{v+k}^\varphi} \gamma[0] = L_{\delta} \varphi = \varphi(k - \delta k) = \varphi_k \quad \forall k \in \mathbb{Z}^2.
\]

Furthermore, we note because of \( \alpha = \frac{1}{2} \) that

\[
B_{j,\ell,0}^T = \left[ \begin{array}{cc} 2^j & 0 \\ 0 & 2^{j/2} \end{array} \right] \cdot \left[ \begin{array}{cc} 1 & 0 \\ \ell & 1 \end{array} \right] = S_{\ell} \cdot A_{2^j},
\]

with \( |\det B_{j,\ell,0}| = 2^{2j} \), so that

\[
\gamma[j,\ell,0,k] = L_{\delta, [S_{\ell} A_{2^j}]^{-1} k} \gamma[j,\ell,0] = 2^{2j} \cdot \psi(S_{\ell} \cdot A_{2^j} \cdot -\delta k) = \psi_{j,\ell,k} \quad \forall (j, \ell, 0) \in V_0 \text{ and } k \in \mathbb{Z}^2.
\]

Finally, we observe \( \Theta(S_{\ell} \tilde{A}_{2^j} \cdot -\delta Rk) \), as well as

\[
R \cdot S_{\ell} \cdot \tilde{A}_{2^j} = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left( \begin{array}{cc} 1 & 0 \\ \ell & 1 \end{array} \right) \cdot \left( \begin{array}{cc} 2^{j/2} & 0 \\ 0 & 2^j \end{array} \right) = \left( \begin{array}{cc} 2^{j/2} & 2^j \\ 2^j & 2^{j/2} \end{array} \right)
\]

and

\[
B_{j,\ell,1}^T = \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 2^{j/2} \end{array} \right] \cdot \left( \begin{array}{cc} 1 & 0 \\ \ell & 1 \end{array} \right) = \left[ \begin{array}{cc} 2^{j/2} & 0 \\ 2^j & 2^{j/2} \end{array} \right] = (R \cdot S_{\ell} \cdot \tilde{A}_{2^j}).
\]

Consequently, we also get

\[
\gamma[j,\ell,1,k] = L_{\delta, [R \cdot S_{\ell} \cdot \tilde{A}_{2^j}]^{-1} k} \gamma[j,\ell,1] = 2^{2j} \cdot \psi(R \cdot S_{\ell} \cdot \tilde{A}_{2^j} \cdot -\delta R k) = 2^{2j} \cdot \psi(R \cdot S_{\ell} \cdot \tilde{A}_{2^j} \cdot -\delta RR k) = \theta_{j,\ell,k}
\]

for arbitrary \((j, \ell, 1) \in V_0 \) and \( k \in \mathbb{Z}^2 \). Since \( \mathbb{Z}^2 \rightarrow \mathbb{Z}^2, k \mapsto Rk \) is bijective, this implies

\[
\text{SH}(\varphi, \psi; \theta; \delta) = \text{SH}_{1/2}(\varphi, \psi; \delta) \text{ up to a reordering in the translation variable } k \quad \text{if } \theta = \psi \circ R.
\]

We now want to transfer Theorems 4.2 and 4.3 to the setting of the unconnected \( \alpha \)-shearlet covering. The link between the connected and the unconnected setting is provided by the following lemma:

**Lemma 5.8.** With \( \varrho, \varrho_0 \) as in equation 4.1, set \( \tilde{\varrho}_0 := \varrho_0 \), as well as \( \tilde{\varrho}_v := \varrho \) for \( v \in V_0 \). Moreover, set

\[
\tilde{M}_{r,v}^{(0)} := \left( \begin{array}{c} u_r^v \\ u_r^{-v} \end{array} \right)^\tau \left( 1 + \|B^{-1} B_v\| \right)^\tau \left( |\det B_v|^{-1} \cdot \int_{W_v^{(0)}} \tilde{\varrho}_v(B^{-1} \xi) \, d\xi \right)^\tau
\]

for \( v, r \in V \). Then we have

\[
\tilde{M}_{r,v}^{(0)} \leq 2^\tau \cdot M_{s(r),s(v)}^{(0)}
\]

for all \( v, r \in V \), where \( M_{s(r),s(v)}^{(0)} \) is as in Lemma 4.7.
Proof. First of all, recall

\[
W'_v = \begin{cases} 
U^{(3^{-1},3)}_{(1,1)} \cup [ - U^{(3^{-1},3)}_{(1,1)} ] = Q'_{\xi_1(v)} \cup [ - Q'_{\xi_1(v)} ], & \text{if } v \in V_0, \\
(1,1)^2 = (-1,1)^2 \cup [ -(-1,1)^2 ] = Q'_{\xi_1(v)} \cup [ - Q'_{\xi_1(v)} ], & \text{if } v = 0
\end{cases}
\]

and \(B_v = T_{\xi_1(v)}\), as well as \(u_v = w_{\xi_1(v)}\) and \(\tilde{\varphi}_v = \varphi_{\xi_1(v)}\) for all \(v \in V\). Thus,

\[
\tilde{M}_{r,v}^{(0)} = \left( \frac{u_v^r}{u_v} \right)^{\tau} \cdot \left( 1 + \| B_v^{-1} B_v \|^\tau \right) \cdot \left( \left| \det B_v \right|^{1/\tau} \int_{W'_{v}} \tilde{\varphi}_v (B_v^{-1} B_v) \right)^{\tau}
\]

(with \(\zeta = B_v^{-1} \xi\))

\[
= \left( \frac{w_{\xi_1(v)}^r}{w_{\xi_1(v)}} \right)^{\tau} \cdot \left( 1 + \| T_{\xi_1(v)}^{-1} T_{\xi_1(v)} \|^\tau \right) \cdot \left( \left| \det T_{\xi_1(v)} \right|^{1/\tau} \int_{Q'_{\xi_1(v)}} \varphi_{\xi_1} \left( T_{\xi_1(v)}^{-1} T_{\xi_1(v)} \right) \right)^{\tau}
\]

(since \(\varphi_{\xi_1}(r, \zeta) = \varphi_{\xi_1}(r, \xi)\))

\[
= \left( \frac{w_{\xi_1(v)}^s}{w_{\xi_1(v)}} \right)^{\tau} \cdot \left( 1 + \| T_{\xi_1(v)}^{-1} T_{\xi_1(v)} \|^\tau \right) \cdot \left( \left| \det T_{\xi_1(v)} \right|^{1/\tau} \int_{Q'_{\xi_1(v)}} \varphi_{\xi_1} \left( T_{\xi_1(v)}^{-1} T_{\xi_1(v)} \right) \right)^{\tau}
\]

(with \(\xi = T_{\xi_1(v)} \zeta\))

\[
= 2^{\tau} \cdot \left( \frac{w_{\xi_1(v)}^s}{w_{\xi_1(v)}} \right)^{\tau} \cdot \left( 1 + \| T_{\xi_1(v)}^{-1} T_{\xi_1(v)} \|^\tau \right) \cdot \left( \left| \det T_{\xi_1(v)} \right|^{1/\tau} \int_{Q'_{\xi_1(v)}} \varphi_{\xi_1} \left( T_{\xi_1(v)}^{-1} \right) \right)^{\tau}
\]

\[
= 2^{\tau} \cdot M_{r,v}^{(0)}.
\]

Since the map \(\iota_1 : V \to I\) is injective, Lemma 5.3 implies

\[
\max \left\{ \left( \sup_{v \in V} \sum_{r \in \mathbb{C}} \tilde{M}_{r,v}^{(0)} \right)^{1/\tau}, \left( \sup_{r \in \mathbb{C}} \sum_{v \in V} \tilde{M}_{r,v}^{(0)} \right)^{1/\tau} \right\} \leq 2 \cdot \max \left\{ \sup_{i \in I} \sum_{j \in \mathbb{C}} M_{j,i}^{(0)} + \sup_{i \in I} \sum_{j \in \mathbb{C}} M_{j,i}^{(0)} \right\}.
\]

Then, recalling Lemma 5.5 and using precisely the same arguments as for proving Theorems 4.2 and 4.3, one can prove the following two theorems:

Theorem 5.9. Theorem 4.2 remains essentially valid if the family \(\tilde{\mathcal{SH}}_{\alpha, \varphi, \psi, \delta}\) is replaced by the \(\alpha\)-shearlet system

\[
\mathcal{SH}_\alpha (\tilde{\varphi}, \tilde{\psi}; \delta) = (L_{\delta, B_v^{-1}}^T) \tilde{\gamma}^{[v]}(v, k) \in \mathbb{C}^2 \text{ with } \gamma^{[v]} := \begin{cases} \left| \det B_v \right|^{1/2} \cdot (\psi \circ B_v^T), & \text{if } v \in V_0, \\
\varphi, & \text{if } v = 0,
\end{cases}
\]

where \(\tilde{\varphi}(x) = \varphi(-x)\) and \(\tilde{\psi}(x) = \psi(-x)\). The only two necessary changes are the following:

1. The assumption \(\tilde{\psi}(\xi) \neq 0\) for \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\) with \(\xi_1 \in [3^{-1}, 3]\) and \(|\xi_2| \leq |\xi_1|\) has to be replaced by

\[
\tilde{\psi}(\xi) \neq 0 \text{ for } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \text{ with } \frac{1}{3} \leq |\xi_1| \leq 3 \text{ and } |\xi_2| \leq |\xi_1|.
\]

2. For the definition of the analysis operator \(A^{(\delta)}\), the convolution \(\gamma^{[v]} \ast f\) has to be defined as in equation (2.3), but using a regular partition of unity \(\langle \varphi_v \rangle_{v \in V}\) for \(\mathcal{S}^{(\alpha)}_{\varphi}\), i.e.,

\[
(\gamma^{[v]} \ast f)(x) = \sum_{\ell \in V} \mathcal{F}^{-1} \left( \gamma^{[v]} \cdot \varphi_{\ell} \cdot \hat{f} \right)(x) \quad \forall x \in \mathbb{R}^d,
\]

where the series converges normally in \(L^\infty(\mathbb{R}^2)\) and thus absolutely and uniformly, for all \(f \in \mathcal{S}^{(\alpha, \varphi)}_{\varphi} (\mathbb{R}^2)\). For a more convenient expression for this convolution—at least for \(f \in L^2(\mathbb{R}^2)\)—see Lemma 5.12 below. △
Theorem 5.10. Theorem \[4.3\] remains essentially valid if the family \(\mathcal{SH}(\varphi, \psi, \delta)\) is replaced by the \(\alpha\)-shearlet system

\[
\mathcal{SH}_\alpha (\varphi, \psi; \delta) = \left( L_{\delta, B_\varphi^\alpha, k} \gamma[v] \right)_{v \in \mathbb{V}, k \in \mathbb{Z}^2}
\]

with \(\gamma[v] := \begin{cases} \left| \text{det } B_v \right|^{1/2} \left( \psi \circ B_v^T \right), & \text{if } v \in V_0, \\ \varphi, & \text{if } v = 0. \end{cases}\)

The only necessary change is that the assumption \(\hat{\psi}(\xi) \neq 0\) for \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\) with \(\xi_1 \in [3^{-1}, 3]\) and \(|\xi_2| \leq |\xi_1|\) has to be replaced by

\[\hat{\psi}(\xi) \neq 0\] for \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\) with \(\frac{1}{3} \leq |\xi_1| \leq 3\) and \(|\xi_2| \leq |\xi_1|\).

Remark 5.11. With the exact same reasoning, one can also show that Corollaries \[4.4\] and \[5.5\] remain valid with the obvious changes. Again, one now has to require

\[\hat{\psi}_1(\xi) \neq 0\] for \(\frac{1}{3} \leq |\xi| \leq 3\).

instead of \(\hat{\psi}_1(\xi) \neq 0\) for \(\xi \in [3^{-1}, 3]\).

The one remaining limitation of Theorems \[4.2\] and \[5.9\] is their somewhat strange definition of the convolution \((\gamma^{[i]} * f)(x)\). The following lemma makes this definition more concrete, under the assumption that we already know \(f \in L^2(\mathbb{R}^2)\). For general \(f \in \mathcal{S}'(\mathbb{R}^2)\), this need not be the case, but for suitable values of \(p, q, s\), we have \(\mathcal{S}^p_q(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)\), as we will see in Theorem \[5.13\]

Lemma 5.12. Let \((\varphi_\ell)_{\ell \in I}\) be a regular partition of unity subordinate to some almost structured covering \(Q = (Q_i)_{i \in I}\) of \(\mathbb{R}^d\). Assume that \(\gamma \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\) with \(\hat{\gamma} \in C^\infty(\mathbb{R}^d)\), where all partial derivatives of \(\hat{\gamma}\) are polynomially bounded. Let \(f \in L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow Z' (\mathbb{R}^d)\) be arbitrary. Then we have

\[
\sum_{\ell \in I} \mathcal{F}^{-1} \left( \hat{\gamma} \cdot \varphi_\ell \cdot \hat{f} \right)(x) = (f, L_\gamma \hat{\varphi}_\ell, \hat{\varphi}_\ell)(x) \quad \forall x \in \mathbb{R}^d,
\]

where \(\hat{\gamma}(x) = \gamma(-x)\) and where \((f, g) = \int_{\mathbb{R}^d} f(x) \cdot g(x) \, dx\).

Proof. In the expression \(\mathcal{F}^{-1} \left( \hat{\gamma} \cdot \varphi_\ell \cdot \hat{f} \right)(x)\), the inverse Fourier transform is the inverse Fourier transform of the compactly supported, tempered distribution \(\hat{\gamma} \cdot \varphi_\ell \cdot \hat{f} \in \mathcal{S}'(\mathbb{R}^d)\). But by the Paley-Wiener theorem (see e.g. [55] Theorem 7.23]), the tempered distribution \(\mathcal{F}^{-1} \left( \hat{\gamma} \cdot \varphi_\ell \cdot \hat{f} \right)\) is given by (integration against) a (uniquely determined) smooth function, whose value at \(x \in \mathbb{R}^d\) we denote by \(\mathcal{F}^{-1} \left( \hat{\gamma} \cdot \varphi_\ell \cdot \hat{f} \right)(x)\). Precisely, we have

\[
\mathcal{F}^{-1} \left( \hat{\gamma} \cdot \varphi_\ell \cdot \hat{f} \right)(x) = \left\langle \hat{\gamma} \cdot \hat{f}, \varphi_\ell \cdot e^{2\pi i(x, \xi)} \right\rangle_{\mathcal{D}'(\mathbb{R}^d) \cap C_0^\infty(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \hat{\gamma}(\xi) \cdot \hat{f}(\xi) \cdot e^{2\pi i(x, \xi)} \cdot \varphi_\ell(\xi) \, d\xi.
\]

But since \(Q\) is an admissible covering of \(\mathbb{R}^d\) and since \((\varphi_\ell)_{\ell \in I}\) is a regular partition of unity subordinate to \(Q\), we have

\[
\sum_{\ell \in I} \left| \hat{\gamma}(\xi) \cdot \hat{f}(\xi) \cdot e^{2\pi i(x, \xi)} \cdot \varphi_\ell(\xi) \right| \leq \left| \hat{\gamma}(\xi) \cdot \hat{f}(\xi) \right| \cdot \sum_{\ell \in I} |\varphi_\ell(\xi)|
\]

\[
\leq \sup_{\ell \in I} \|\varphi_\ell\|_{\sup} \cdot \left| \hat{\gamma}(\xi) \cdot \hat{f}(\xi) \right| \cdot \sum_{\ell \in I} I_{Q_i}(\xi)
\]

\[
\leq N_Q \cdot \sup_{\ell \in I} \|\varphi_\ell\|_{\sup} \cdot \left| \hat{\gamma}(\xi) \cdot \hat{f}(\xi) \right| \in L^1(\mathbb{R}^d),
\]

since \(\hat{\gamma}, \hat{f} \in L^2(\mathbb{R}^d)\). Since we also have \(\sum_{\ell \in I} \varphi_\ell \equiv 1\) on \(\mathbb{R}^d\), we get by the dominated convergence theorem that

\[
\sum_{\ell \in I} \mathcal{F}^{-1} \left( \hat{\gamma} \cdot \varphi_\ell \cdot \hat{f} \right)(x) = \int_{\mathbb{R}^d} \hat{\gamma}(\xi) \cdot \hat{f}(\xi) \cdot e^{2\pi i(x, \xi)} \cdot \varphi_\ell(\xi) \, d\xi
\]

\[
= \int_{\mathbb{R}^d} \hat{\gamma}(\xi) \cdot \hat{f}(\xi) \cdot e^{2\pi i(x, \xi)} \, d\xi = \mathcal{F}^{-1} (\hat{\gamma} \cdot \hat{f})(x),
\]

where \(\mathcal{F}^{-1} (\hat{\gamma} \cdot \hat{f}) \in L^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)\) by the Riemann-Lebesgue Lemma and Plancherel’s theorem, because of \(\hat{\gamma} \cdot \hat{f} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\). But Young’s inequality shows \(\gamma \ast f \in L^2(\mathbb{R}^d)\), while the convolution theorem yields \(\mathcal{F}(\gamma \ast f) = \hat{\gamma} \cdot \hat{f}\). Hence, \(\gamma \ast f = \mathcal{F}^{-1} (\hat{\gamma} \cdot \hat{f})\) almost everywhere. But both sides of the identity are continuous.
functions, since the convolution of two $L^2$ functions is continuous. Thus, the equality holds everywhere, so that we finally get
\[
\sum_{\ell \in \mathbb{Z}} F^{-1} \left( \hat{\gamma} \cdot \varphi_\ell \cdot \hat{f} \right)(x) = F^{-1} \left( \hat{\gamma} \cdot \hat{f} \right)(x) = (\gamma * f)(x) = \int_{\mathbb{R}^d} f(y) \cdot \gamma(x-y) \, dy = (f, L_x \hat{\gamma}).
\]

We close this section with a theorem that justifies the title of the paper: It formally encodes the fact that analysis sparsity is equivalent to synthesis sparsity for $(\alpha)$-shearlet systems.

**Theorem 5.13.** Let $\alpha \in [0,1]$, $\varepsilon, p_0 \in (0,1)$ and $s^{(0)} \geq 0$ be arbitrary. Assume that $\varphi, \psi \in L^1(\mathbb{R}^2)$ satisfy the assumptions of Theorems 5.9 and 5.10 with $q_0 = p_0$ and $s_0 = 0$, as well as $s_1 = s^{(0)} + (1 + \alpha) \left( p_0^{-1} - 2^{-1} \right)$. For $\delta > 0$, denote by $\text{SH}_\alpha(\varphi, \psi; \delta) = \left( \gamma^{[v,k,\delta]} \right)_{v \in V, k \in \mathbb{Z}^2}$ the $\alpha$-shearlet system generated by $\varphi, \psi$, as in Definition 5.6.

Then there is some $\delta_0 \in (0,1)$ with the following property: For all $p \in [p_0, 2]$ and all $s \in [0,s^{(0)}]$, we have
\[
\mathcal{J}^{p,p}_{\alpha,s+1(\alpha)(p^{-1}-1-1)}(\mathbb{R}^2) = \left\{ f \in L^2(\mathbb{R}^2) \mid \left( u_v^s \cdot \langle f, \gamma^{[v,k,\delta]} \rangle_{L^2} \right)_{v \in V, k \in \mathbb{Z}^2} \in \ell^p(V \times \mathbb{Z}^2) \right\}
\]
\[
= \left\{ \sum_{(v,k) \in V \times \mathbb{Z}^2} c_{\delta}^{(v)} \cdot \gamma^{[v,k,\delta]} \in \ell^p(V \times \mathbb{Z}^2) \right\},
\]
as long as $0 < \delta \leq \delta_0$. Here, the weight $u = (u_v)_{v \in V}$ is as in Definition 5.1, i.e., $u_{n,m,\delta} = 2^n$ and $u_0 = 1$.

In fact, for $f \in \mathcal{J}^{p,p}_{\alpha,s+1(\alpha)(p^{-1}-1-1)}(\mathbb{R}^2)$, we even have a (quasi-)norm equivalence
\[
\|f\|_{\mathcal{J}^{p,p}_{\alpha,s+1(\alpha)(p^{-1}-1-1)}(\mathbb{R}^2)} \leq \left\| \left( u_v^s \cdot \langle f, \gamma^{[v,k,\delta]} \rangle_{L^2} \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p(V \times \mathbb{Z}^2)} \approx \inf \left\{ \left\| \left( u_v^s \cdot c_{k}^{(v)} \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p(V \times \mathbb{Z}^2)} \mid f = \sum_{(v,k) \in V \times \mathbb{Z}^2} c_{\delta}^{(v)} \cdot \gamma^{[v,k,\delta]} \text{ with uncond. conv. in } L^2(\mathbb{R}^2) \right\}.
\]

In particular, $\mathcal{J}^{p,p}_{\alpha,s+1(\alpha)(p^{-1}-1-1)}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$ and $\text{SH}_\alpha(\varphi, \psi; \delta)$ is a frame for $L^2(\mathbb{R}^2)$.

**Remark.** As one advantage of the decomposition space point of view, we observe that $\mathcal{J}^{p,q}_{\alpha,s}(\mathbb{R}^2)$ is easily seen to be translation invariant, while this is not so easy to see in the characterization via analysis or synthesis sparsity in terms of a discrete $\alpha$-shearlet system.

**Proof.** We start with a few preparatory definitions and observations. For brevity, we set
\[
\|f\|_{\ell^p, p,s, \alpha} := \inf \left\{ \left\| \left( u_v^s \cdot c_k^{(v)} \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p(V \times \mathbb{Z}^2)} \mid f = \sum_{(v,k) \in V \times \mathbb{Z}^2} c_{k}^{(v)} \cdot \gamma^{[v,k,\delta]} \text{ with uncond. conv. in } L^2(\mathbb{R}^2) \right\}
\]
for $f \in \mathcal{J}^{p,p}_{\alpha,s+1(\alpha)(p^{-1}-1-1)}(\mathbb{R}^2)$ and $s \in [0,s^{(0)}]$, as well as $p \in [p_0, 2]$.

Next, our arguments entail that $\varphi, \psi$ satisfy the assumptions of Theorem 5.24 (and thus equation 4.13) for $s_0 = 0$ and $s_1 = s^{(0)} + (1 + \alpha) \left( p_0^{-1} - 2^{-1} \right) \geq 0$. But this implies (in the notation of Theorem 4.2) that $K, H, M_2 \geq 2 + \varepsilon$. Hence,
\[
(1 + |\xi_1|)^{-\left(M_2 + 1\right)} (1 + |\xi_2|)^{-\left(K+1\right)} \leq (1 + |\xi_1|) (1 + |\xi_2|)^{-\left(2 + \varepsilon\right)} \leq (1 + |\xi|)^{-\left(2 + \varepsilon\right)} \in L^1(\mathbb{R}^2).
\]

Therefore, equation 5.14 entails $\hat{\varphi}, \hat{\psi} \in L^1(\mathbb{R}^2)$, so that Fourier inversion yields $\varphi, \psi \in L^1(\mathbb{R}^2) \cap C_0(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$. Consequently, $\gamma^{[v]} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for all $v \in V$, which will be important for our application of Lemma 5.12 later in the proof.

Finally, for $g : \mathbb{R}^2 \rightarrow \mathbb{C}$, set $g^* : \mathbb{R}^2 \rightarrow \mathbb{C}, x \mapsto g(-x)$. For $g \in L^1(\mathbb{R}^2)$, we then have $\hat{g^*}(\xi) = \overline{\hat{g}(\xi)}$ for all $\xi \in \mathbb{R}^2$. Therefore, in case of $g \in C^1(\mathbb{R}^2)$ with $g, \nabla g \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and with $\hat{g} \in C^\infty(\mathbb{R}^2)$, this implies that $g^*$ satisfies the same properties and that $|\partial^\theta \hat{g^*}| = |\hat{g^*} \theta|$ for all $\theta \in \mathbb{N}_0^2$. These considerations easily show that since $\varphi, \psi$ satisfy the assumptions of Theorem 5.3 (with $q_0 = p_0$ and $s_0 = 0$, as well as $s_1 = s^{(0)} + (1 + \alpha) \left( p_0^{-1} - 2^{-1} \right)$), so do $\varphi^*, \psi^*$.

Thus, Theorem 5.9 yields a constant $\delta_1 \in (0,1]$ such that the $\alpha$-shearlet system $\text{SH}_\alpha(\varphi^*, \psi^*; \delta) = \text{SH}_\alpha(\varphi^*, \psi^*; \delta)$ forms a Banach frame for $\mathcal{J}^{p,q}_{\alpha,s}(\mathbb{R}^2)$ for all $p, q \in [p_0, \infty]$ and all $s \in \mathbb{R}$ with $0 \leq s \leq s^{(0)} + (1 + \alpha) \left( p_0^{-1} - 2^{-1} \right)$, as long as $0 < \delta \leq \delta_1$. Likewise, Theorem 5.10 yields a constant $\delta_2 \in (0,1]$ such that $\text{SH}_\alpha(\varphi, \psi; \delta)$ yields an atomic decomposition of $\mathcal{J}^{p,q}_{\alpha,s}(\mathbb{R}^2)$ for the same range of parameters, as long as $0 < \delta \leq \delta_2$. Now, let us set $\delta_0 := \min \{\delta_1, \delta_2\} \in (0,1]$. 

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Let $p \in [p_0, 2]$ and $s \in \left[0, s^{(0)}\right]$ be arbitrary and set $s^3 := s + (1 + \alpha) \left( p^{-1} - 2^{-1} \right)$. It is not hard to see directly from Definition 14.8—and because of $|\det B_v| = u_v^{1+\alpha}$ for all $v \in V$—that the quasi-norm of the coefficient space $C^{p,p}_{u^3\cdot}$ satisfies
\[
\left\| \left( c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{C^{p,p}_{u^3\cdot}} = \left\| \left( |\det B_v|^{-1/2} \cdot u_v^{s^3} \cdot \left\| \left( c^{(v)}_k \right)_{k \in \mathbb{Z}^2} \right\|_{\ell^p} \right)_{v \in V} \right\|_{\ell^p} = \left\| \left( u_v^{s} \cdot c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p} \in [0, \infty]
\]
for arbitrary sequences $\left( c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2}$, and $C^{p,p}_{u^3\cdot}$ contains exactly those sequences for which this (quasi)-norm is finite. Now, note because of $s \geq 0$ and $p \leq 2$ that $C^{p,p}_{u^3\cdot} \rightarrow \ell^2 \left( V \times \mathbb{Z}^2 \right)$, since $u_v \geq 1$ for all $v \in V$ and since $\ell^p \rightarrow \ell^2$.

Next, note that we have
\[
s_0 = 0 \leq s \leq s^3 \leq s^{(0)} + (1 + \alpha) \left( p_0^{-1} - 2^{-1} \right) = s_1,
\]
so that $SH_s (\varphi, \psi; \delta)$ forms an atomic decomposition of $\mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right)$ for all $0 < \delta \leq \delta_0$. This means that the synthesis operator
\[
S^{(\delta)} : C^{p,p}_{u^3\cdot} \rightarrow \mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right), \left( c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2} \mapsto \sum_{(v,k) \in V \times \mathbb{Z}^2} c^{(v)}_k \cdot \varphi |[v,k,\delta] is well-defined and bounded with unconditional convergence of the series in $\mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right)$. This implicitly uses that the synthesis operator $S^{(\delta)}$ as defined in Theorem 14.9 is bounded and satisfies $S^{(\delta)} (\delta_{v,k}) = \varphi [v,k,\delta]$ for all $(v,k) \in V \times \mathbb{Z}^2$ and that we have $c = \left( c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2} = \sum_{(v,k) \in V \times \mathbb{Z}^2} c^{(v)}_k \cdot \delta_{v,k}$ for all $c \in C^{p,p}_{u^3\cdot}$, with unconditional convergence in $C^{p,p}_{u^3\cdot}$, since $p \leq 2 < \infty$. This immediately yields
\[
\Omega_1 := \left\{ \sum_{(v,k) \in V \times \mathbb{Z}^2} c^{(v)}_k \cdot \varphi [v,k,\delta] \left| \left( u_v^{s} \cdot c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2} \in \ell^p \left( V \times \mathbb{Z}^2 \right) \right. \right\} = \text{range} \left( S^{(\delta)} \right) \subset \mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right).
\]
Further, if $f \in \mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right)$ and if $c = \left( c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2}$ is an arbitrary sequence satisfying $f = \sum_{(v,k) \in V \times \mathbb{Z}^2} c^{(v)}_k \cdot \varphi [v,k,\delta]$ with unconditional convergence in $L^2 \left( \mathbb{R}^2 \right)$, there are two cases:

\textbf{Case 1.} We have \( \left\| \left( u_v^{s} \cdot c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p} = \infty \). In this case, $\| f \|_{\mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right)} \leq \| S^{(\delta)} \| \cdot \| f \|_{C^{p,p}_{u^3\cdot}}$ is trivial.

\textbf{Case 2.} We have \( \left\| \left( u_v^{s} \cdot c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p} < \infty \). In this case, we get $c \in C^{p,p}_{u^3\cdot}$ and $f = S^{(\delta)} c$. Therefore, we see
\[
\| f \|_{\mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right)} \leq \| S^{(\delta)} \| \cdot \| c \|_{C^{p,p}_{u^3\cdot}} = \| S^{(\delta)} \| \cdot \left\| \left( u_v^{s} \cdot c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p}.
\]
All in all, we have thus established
\[
\| f \|_{\mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right)} \leq \| S^{(\delta)} \| \cdot \| f \|_{C^{p,p}_{u^3\cdot}} \quad \forall f \in \mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right).
\]

Next, note that the considerations from the preceding paragraph with the choice $p = 2$ and $s = 0$ also show that $S^{(\delta)} : \ell^2 \left( V \times \mathbb{Z}^2 \right) \rightarrow \mathcal{A}^{2,2}_{\alpha,0} \left( \mathbb{R}^2 \right)$ is well-defined and bounded. But \cite{Voigtlaender2012} Lemma 6.10 yields $\mathcal{A}^{2,2}_{\alpha,0} \left( \mathbb{R}^2 \right) = L^2 \left( \mathbb{R}^2 \right)$ with equivalent norms. Since we saw above that $C^{p,p}_{u^3\cdot} \rightarrow \ell^2 \left( V \times \mathbb{Z}^2 \right)$ for all $p \leq 2$ and $s \geq 0$, this implies in particular that the series defining $S^{(\delta)} c$ converges unconditionally in $L^2 \left( \mathbb{R}^2 \right)$ for arbitrary $c \in C^{p,p}_{u^3\cdot}$, for arbitrary $s \in \left[0, s^{(0)}\right]$ and $p \in [p_0, 2]$.

But from the atomic decomposition property of $SH_s (\varphi, \psi; \delta)$, we also know that there is a bounded coefficient operator $C^{(\delta)} : \mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right) \rightarrow C^{p,p}_{u^3\cdot}$ satisfying $S^{(\delta)} \circ C^{(\delta)} = \text{id}_{\mathcal{A}^{p,p}_{\alpha,s^3}}$. Thus, for arbitrary $f \in \mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right)$ and $c = (c^{(v)}_k)_{v \in V, k \in \mathbb{Z}^2} := C^{(\delta)} f \in C^{p,p}_{u^3\cdot}$, we have $f = S^{(\delta)} c = \sum_{(v,k) \in V \times \mathbb{Z}^2} c^{(v)}_k \cdot \varphi [v,k,\delta] \in \Omega_1$, where the series converges unconditionally in $L^2 \left( \mathbb{R}^2 \right)$ (and in $\mathcal{A}^{p,p}_{\alpha,s^3} \left( \mathbb{R}^2 \right)$). In particular, we get
\[
\| f \|_{C^{p,p}_{u^3\cdot}} \leq \left\| \left( u_v^{s} \cdot c^{(v)}_k \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p} = \| e \|_{C^{p,p}_{u^3\cdot}} \leq \| C^{(\delta)} \| \cdot \| f \|_{\mathcal{A}^{p,p}_{\alpha,s^3}} < \infty,
\]
as well as
\[
\| f \|_{L^2 \left( \mathbb{R}^2 \right)} \leq \| f \|_{\mathcal{A}^{2,2}_{\alpha,0}} \leq \| S^{(\delta)} \| \cdot \| e \|_{\mathcal{A}^{2,2}_{\alpha,0}} \cdot \| e \|_{\mathcal{A}^{2,2}_{\alpha,0}} = \| S^{(\delta)} \| \cdot \| e \|_{\mathcal{A}^{2,2}_{\alpha,0}} \cdot \| e \|_{\mathcal{A}^{2,2}_{\alpha,0}} \leq \| S^{(\delta)} \| \cdot \| e \|_{\mathcal{A}^{2,2}_{\alpha,0}} \cdot \| e \|_{\mathcal{A}^{2,2}_{\alpha,0}} < \infty
\]
for all \( f \in \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \). Up to now, we have thus shown \( \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) = \Omega_1 \) (with \( \Omega_1 \) as in equation (5.5)) and \( \|f\|_{\Star,p,s,\delta} \simeq \|f\|_{\mathcal{S}_{\alpha,s}^{p,p}} \) for all \( f \in \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \), with \( \|f\|_{\Star,p,s,\delta} \) as in equation (5.4). Finally, we have also shown \( \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2) \).

Thus, it remains to show
\[
\Omega_2 := \left\{ f \in L^2(\mathbb{R}^2) \mid \left( u_v^* \cdot \langle f, \gamma[v,k,\delta] \rangle_{L^2} \right)_{v \in V, k \in \mathbb{Z}^2} \in \ell^p \left( V \times \mathbb{Z}^2 \right) \right\} \subseteq \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2),
\]
as well as \( \|f\|_{\mathcal{S}_{\alpha,s}^{p,p}} \simeq \left\| \left( u_v^* \cdot \langle f, \gamma[v,k,\delta] \rangle_{L^2} \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p} \) for \( f \in \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \). But Theorem 5.3 (applied with \( \varphi^*, \psi^* \) instead of \( \varphi, \psi \)) shows that the analysis operator
\[
A^{(\delta)} : \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \to C_{\alpha,s}^{p,p}, f \mapsto \left( \langle \varphi[v] * f \rangle (\delta \cdot B_v^{-T} k) \right)_{v \in V, k \in \mathbb{Z}^2}
\]
is well-defined and bounded, where (cf. Theorem 4.2), the family \( \{\varphi[v]\}_{v \in V} \) is given by \( \varphi[v] = |\det B_v|^{1/2} \cdot (\psi^* \circ B_v^T) \) for \( v \in V_0 \) and by \( \varphi[0] = \varphi^* \). Note that this yields \( \varphi[v] = \gamma[v] \), where the family \( \{\gamma[v]\}_{v \in V} \) is as in Definition 6.6.

Now, since we already showed \( \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2) \) and since \( \varphi[v] \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \) for all \( v \in V \), as we saw at the start of the proof, Lemma 5.12 yields
\[
\left( \langle \varphi[v] * f \rangle (\delta \cdot B_v^{-T} k) \right)_{v \in V, k \in \mathbb{Z}^2} = \left( f, L_{\delta \cdot B_v^{-T} k} \gamma[v] \right)_{L^2} = \left( f, \gamma[v,k,\delta] \right)_{L^2}
\]
for all \( f \in \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \) and \( (v,k) \in V \times \mathbb{Z}^2 \). We thus see \( \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \subseteq \Omega_2 \) and
\[
\left\| \left( u_v^* \cdot \langle f, \gamma[v,k,\delta] \rangle_{L^2} \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p} = \|A^{(\delta)}\|_{C_{\alpha,s}^{p,p}} \leq \|A^{(\delta)}\| \cdot \|f\|_{\mathcal{S}_{\alpha,s}^{p,p}} \quad \forall f \in \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2).
\]

Conversely, let \( f \in \Omega_2 \) be arbitrary, i.e., \( f \in L^2(\mathbb{R}^2) \) with \( \left( u_v^* \cdot \langle f, \gamma[v,k,\delta] \rangle_{L^2} \right)_{v \in V, k \in \mathbb{Z}^2} \in \ell^p (V \times \mathbb{Z}^2) \). This means \( f \in L^2(\mathbb{R}^2) = \mathcal{S}_{\alpha,0}^{2,2}(\mathbb{R}^2) \) and \( \left( \langle \varphi[v] * f \rangle (\delta \cdot B_v^{-T} k) \right)_{v \in V, k \in \mathbb{Z}^2} \in C_{\alpha,s}^{p,p} \), again by Lemma 5.12. Thus, the consistency statement of Theorem 4.2 shows \( f \in \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \). Therefore, \( f = R^{(\delta)} A^{(\delta)} f \) for the reconstruction operator \( R^{(\delta)} : C_{\alpha,s}^{p,p} \to \mathcal{S}_{\alpha,s}^{p,p}(\mathbb{R}^2) \) that is provided by Theorem 5.4 (applied with \( \varphi^*, \psi^* \) instead of \( \varphi, \psi \)). Thus,
\[
\|f\|_{\mathcal{S}_{\alpha,s}^{p,p}} \leq \|R^{(\delta)}\| \cdot \|A^{(\delta)}\|_{C_{\alpha,s}^{p,p}} \|f\|_{C_{\alpha,s}^{p,p}} = \|R^{(\delta)}\| \cdot \left\| \left( u_v^* \cdot \langle f, \gamma[v,k,\delta] \rangle_{L^2} \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{\ell^p}.
\]

If we apply the preceding considerations for \( s = 0 \) and \( p = 2 \), we in particular get
\[
\|f\|_{L^2} \simeq \|f\|_{\mathcal{S}_{\alpha,s}^{2,2}} \simeq \left\| \left( f, \gamma[v,k,\delta] \right)_{L^2} \right\|_{\ell^2} \quad \forall f \in L^2(\mathbb{R}^2) = \mathcal{S}_{\alpha,0}^{2,2}(\mathbb{R}^2),
\]
which implies that the \( \alpha \)-shearlet system \( \text{SH}_{\alpha}(\varphi, \psi; \delta) = \{\gamma[v,k,\delta]\}_{v \in V, k \in \mathbb{Z}^2} \) is a frame for \( L^2(\mathbb{R}^2) \).

6. APPROXIMATION OF CARTOON-LIKE FUNCTIONS USING \( \alpha \)-SHEARLETS

One of the most celebrated properties of shearlet systems is that they provide (almost) optimal approximation rates for the model class \( \mathcal{E}^2(\mathbb{R}^2; \nu) \) of cartoon-like functions, which we introduce formally in Definition 6.1 below. More precisely, this means (cf. [51] Theorem 1.3) for the case of compactly supported shearlets that
\[
\|f - f(N)\|_{L^2} \leq C \cdot N^{-1} \cdot (1 + \log N)^{3/2} \quad \forall N \in \mathbb{N} \text{ and } f \in \mathcal{E}^2(\mathbb{R}^2; \nu),
\]
(6.1)
where \( f(N) \) is the so-called \( N \)-term approximation of \( f \).

The exact interpretation of this \( N \)-term approximation, however, requires some explanation, as was briefly discussed in the introduction: In general, given a dictionary \( \Psi = (\psi_i)_{i \in I} \) in a Hilbert space \( \mathcal{H} \) (which is assumed to satisfy \( \text{span} \{\psi_i \mid i \in I\} = \mathcal{H} \)), we let
\[
\mathcal{H}_{\Psi}^{(N)} := \left\{ \sum_{i \in J} \alpha_i \psi_i \right\mid J \subset I \text{ with } |J| \leq N \text{ and } (\alpha_i)_{i \in J} \in \mathbb{C}^J \}
\]
(6.2)
denote the subset (which is in general not a subspace) of \( \mathcal{H} \) consisting of linear combinations of (at most) \( N \) elements of \( \Psi \). The usual definition of a (general non-unique) best \( N \)-term approximation to \( f \in \mathcal{H} \) is any \( f_{\Psi}^{(N)} \in \mathcal{H}_{\Psi}^{(N)} \) satisfying
\[
\|f - f_{\Psi}^{(N)}\| = \inf_{g \in \mathcal{H}_{\Psi}^{(N)}} \|f - g\|.
\]
This definition is given for example in [50] Section 3.1. Note, however, that in general, it is not clear whether such a best \( N \)-term approximation exists. But regardless of whether a best \( N \)-term approximation exists or not, we can always define the \( N \)-term approximation error as

\[
\alpha_{\Psi}^{(N)}(f) := \inf_{g \in \mathcal{H}_{\Psi}^{(N)}} \|f - g\|. \tag{6.3}
\]

All in all, the goal of (nonlinear) \( N \)-term approximations is to approximate an element \( f \in \mathcal{H} \) using only a fixed number of elements from the dictionary \( \Psi \). Thus, when one reads the usual statement that shearlets provide (almost) optimal \( N \)-term approximation rates for cartoon-like functions, one could be tempted to think that equation (6.1) has to be understood as

\[
\alpha_{\Psi}^{(N)}(f) \leq C \cdot N^{-1} \cdot (1 + \log N)^{3/2} \quad \forall N \in \mathbb{N} \text{ and } f \in \mathcal{E}^2(\mathbb{R}^2; \nu) , \tag{6.4}
\]

where the dictionary \( \Psi \) is a (suitable) shearlet system. This, however, is not what is shown e.g. in [50]. What is shown there, instead, is that if \( \tilde{\Psi} = (\tilde{\psi}_i)_{i \in I} \) denotes the (canonical) dual frame (in fact, any dual frame will do) of a suitable shearlet system \( \Psi \), then we have

\[
\alpha_{\tilde{\Psi}}^{(N)}(f) \leq C \cdot N^{-1} \cdot (1 + \log N)^{3/2} \quad \forall N \in \mathbb{N} \text{ and } f \in \mathcal{E}^2(\mathbb{R}^2; \nu) .
\]

This approximation rate using the dual frame \( \tilde{\Psi} \) is not completely satisfactory, since for non-tight shearlet systems \( \Psi \), the properties of \( \tilde{\Psi} \) (like smoothness, decay, etc) are largely unknown. Note that there is no known construction of a tight, compactly supported cone-adapted shearlet frame. Furthermore, to our knowledge, there is—up to now—nothing nontrivial known about \( \alpha_{\Psi}^{(N)}(f) \) for \( f \in \mathcal{E}^2(\mathbb{R}^2) \) in the case that \( \Psi \) is itself a shearlet system, unless \( \Psi \) is a tight shearlet frame.

This difference between approximation using the primal and the dual frame is essentially a difference between analysis and synthesis sparsity: The usual proof strategy to obtain the approximation rate with respect to the dual frame is to show that the analysis coefficients \( \{(f, \psi_i)\}_{i \in I} \) are sparse in the sense that they lie in some (weak) \( \ell^p \) space. Then one uses the reconstruction formula

\[
f = \sum_{i \in I} \langle f, \psi_i \rangle \tilde{\psi}_i \quad \text{using the dual frame } \tilde{\Psi} = (\tilde{\psi}_i)_{i \in I}
\]

and truncates this series to the \( N \) terms with the largest coefficients \( |\langle f, \psi_i \rangle| \). Using the sparsity of the coefficients, one then obtains the claim. In other words, since the analysis coefficients with respect to \( \Psi = (\psi_i)_{i \in I} \) are the synthesis coefficients with respect to \( \tilde{\Psi} \), analysis sparsity with respect to \( \Psi \) yields synthesis sparsity with respect to \( \tilde{\Psi} \). Conversely, analysis sparsity with respect to \( \tilde{\Psi} \) yields synthesis sparsity with respect to \( \Psi \) itself. But since only limited knowledge about \( \tilde{\Psi} \) is available, this fact is essentially impossible to apply.

But our preceding results concerning Banach frames and atomic decompositions for \((\alpha)\)-shearlet smoothness spaces show that analysis sparsity is equivalent to synthesis sparsity (cf. Theorem 5.13) for sufficiently nice and sufficiently densely sampled \( \alpha \)-shearlet frames. Using this fact, we will show in this section that we indeed have

\[
\alpha_{\Psi}^{(N)}(f) \leq C_{\varepsilon} \cdot N^{-(1-\varepsilon)} \quad \forall N \in \mathbb{N} \text{ and } f \in \mathcal{E}^2(\mathbb{R}^2; \nu) ,
\]

where \( \varepsilon \in (0,1) \) can be chosen arbitrarily and where \( \Psi \) is a (suitable) shearlet frame. In fact, we will also obtain a corresponding statement for \( \alpha \)-shearlet frames. Note though that the approximation rate \( N^{-(1-\varepsilon)} \) is slightly inferior to the rate of decay in equation (6.4). Nevertheless—to the best of our knowledge—this is still the best result on approximating cartoon-like functions by shearlets (instead of using the dual frame of a shearlet frame) which is known.

Our proof strategy is straightforward: The known analysis-sparsity results, in conjunction with our results about Banach frames for shearlet smoothness spaces, show that \( \mathcal{E}^2(\mathbb{R}^2; \nu) \) is a bounded subset of a certain range of shearlet smoothness spaces. Thus, using our results about atomic decompositions for these shearlet smoothness spaces, we get synthesis sparsity with respect to the (primal(!)) shearlet frame. We then truncate this (quickly decaying) series to obtain a good \( N \)-term approximation.

We begin our considerations by recalling the notion of \( C^\beta \)-cartoon-like functions, which were originally introduced (in a preliminary form) in [15].

\[4\]Of course, one knows \( \alpha_{\Psi}^{(N)}(f) \to 0 \) as \( N \to \infty \), but this holds for every \( f \in L^2(\mathbb{R}^2) \) and every frame \( \Psi \) of \( L^2(\mathbb{R}^2) \).
Definition 6.1. Fix parameters $0 < q_0 < q_1 < 1$ once and for all.

- For $\nu > 0$ and $\beta \in (1, 2]$, the set $\text{STAR}^\beta(\nu)$ is the family of all subsets $B \subset [0,1]^2$ for which there is some $x_0 \in \mathbb{R}^2$ and a $2\pi$-periodic function $\phi : \mathbb{R} \to [0, q_1]$ with $\phi \in C^\beta(\mathbb{R})$ such that

$$B - x_0 = \left\{ r \cdot \begin{cases} \cos \phi \\ \sin \phi \end{cases} : \phi \in [0, 2\pi] \text{ and } 0 \leq r \leq \phi(\phi) \right\}$$

and such that the $\beta - 1$ Hölder semi-norm $|g|_{\beta - 1} = \sup_{\phi, \phi' \in B, \phi \neq \phi'} \frac{|g(\phi) - g(\phi')|}{|\phi - \phi'|^{1/\beta}}$ satisfies $|g|_{\beta - 1} \leq \nu$.

- For $\nu > 0$ and $\beta \in (1, 2]$, the class $\mathcal{E}^\beta(\mathbb{R}^2; \nu)$ of cartoon-like functions with regularity $\beta$ is defined as

$$\mathcal{E}^\beta(\mathbb{R}^2; \nu) := \{ f_1 + 1_B \cdot f_2 : B \in \text{STAR}^\beta(\nu) \text{ and } f_i \in C^\beta_c([0,1]^2) \text{ with } \|f_i\|_{C^\beta} \leq \min\{1, \nu\} \text{ for } i \in \{2\} \}$$

where $\|f\|_{C^\beta} = \|f\|_{C^\beta}[\sup \| \nabla f \|_{C^\beta} - 1 \text{ and } |g|_{\beta - 1} = \sup_{x,y \in \mathbb{R}^2, x \neq y} \frac{|g(x) - g(y)|}{|x - y|^{1/\beta}}}$ for $g : \mathbb{R}^2 \to \mathbb{C}$, as well as

$$C^\beta_c([0,1]^2) = \{ f \in C^{[\beta]}(\mathbb{R}^2) : \supp f \subset [0,1]^2 \text{ and } \|f\|_{C^\beta} < \infty \}.$$

Finally, we set $\mathcal{E}^\beta(\mathbb{R}^2) := \bigcup_{\nu > 0} \mathcal{E}^\beta(\mathbb{R}^2; \nu)$.

**Remark.** The definition of $\text{STAR}^\beta(\nu)$ given here is slightly more conservative than in [38] Definition 2.5], where it is only assumed that $\phi : \mathbb{R} \to [0, q_1]$ with $0 < q_1 < 1$, instead of $\phi : \mathbb{R} \to [0, q_1]$. We also note that $|g|_{\beta - 1} = \|g\|_{C^\beta}$ in case of $\beta = 2$. This is a simple consequence of the definition of the derivative and of the mean-value theorem. Hence, in case of $\beta = 2$, the definition given here is consistent with (in fact, slightly stronger than) the one used in [39] Definition 1.1).

Further, we note that in [37] Definition 5.9], the class $\mathcal{E}^\beta(\mathbb{R}^2)$ is simply defined as

$$\{ f_1 + 1_B \cdot f_2 : f_1, f_2 \in C^\beta_c([0,1]^2), B \subset [0,1]^2 \text{ Jordan dom. with regular closed piecewise } C^\beta \text{ boundary curve} \}.$$

Even for this—much more general—definition, the authors of [37] then invoke the results which are derived in [38] under the more restrictive assumptions.

This is somewhat unpleasant, but does not need to concern us: In fact, in the following, we will frequently use the notation $\mathcal{E}^\beta(\mathbb{R}^2; \nu)$, but the precise definition of this space is not really used; all that we need to know is that $\phi, \psi$ are suitable shearlet generators, then the $\beta$-shearlet coefficients $c = (c_j, k, \xi, m)_{j,k,\xi,m}$ of $f \in \mathcal{E}^\beta(\mathbb{R}^2; \nu)$ satisfy $c \in L^{\frac{1}{1+\beta}}(\mathbb{R}^2)$ for all $\varepsilon > 0$, with $\|f\|_{L^{\frac{1}{1+\beta}}} \leq C_{\varepsilon,\beta,\phi,\psi}$. Below, we will derive this by combining [38] Theorem 4.2 with [37] Theorem 5.6, where [37] Theorem 5.6 does not use the notion of cartoon-like functions at all.

As our first main technical result in this section, we show that the $C^\beta$-cartoon-like functions are bounded subsets of suitably chosen $\alpha$-shearlet smoothness spaces. Once we have developed this property, we obtain the claimed approximation rate by invoking the atomic decomposition results from Theorem 5.10.

**Proposition 6.2.** Let $\nu > 0$ and $\beta \in (1, 2]$ be arbitrary and let $p \in (2/(1+\beta), 2)$. Then

$$\mathcal{E}^\beta(\mathbb{R}^2; \nu) \text{ is a bounded subset of } \mathcal{S}^{p,p}_{\beta - 1, (1+\beta - 1)(p - 1 - 2 - 1)}(\mathbb{R}^2).$$

**Proof.** Here, we only give the proof for the case $\beta = 2$. For $\beta \in (1, 2)$, the proof is more involved and thus postponed to the appendix (Section D). The main reason for the additional complications in case of $\beta \in (1, 2)$ is that our proof essentially requires that we already know that there is some sufficiently nice, cone-adapted $\alpha$-shearlet system with respect to which the $C^\beta$-cartoon-like functions are analysis sparse (in a suitable “almost $L^{2/(1+\beta)}$” sense). In case of $\beta = 2$, this is known, since we then have $\alpha = \beta^{-1} = 1$, so that the $\alpha$-shearlet systems from Definition 4.6 coincide with the usual cone-adapted shearlets, cf. Remark 5.7. But in case of $\beta \in (1, 2)$, it is only known (cf. [37] Theorem 5.6) that $C^\beta$-cartoon-like functions are analysis sparse with respect to suitable $\beta$-shearlet systems (cf. Definition 4.7, and note $\beta \notin [0,1]$). Thus, the notion of $\beta$-shearlets does not collide with our notion of $\alpha$-shearlets for $\alpha \in (0,1)$ which are different, but closely related to the $\beta^{-1}$-shearlet systems from Definition 5.3. Making this close connection precise is what mainly makes the proof in case of $\beta \in (1, 2)$ more involved, cf. Section D.

Thus, let us consider the case $\beta = 2$. Choose $\phi_0 \in C^\infty_c(\mathbb{R})$ with $\phi_0 > 0$ and $\phi_0 \not\equiv 0$, so that $\hat{\phi}_0(0) = \|\phi_0\|_{L^1} > 0$. By continuity of $\hat{\phi}_0$, there is thus some $\nu > 0$ with $\hat{\phi}_0(\xi) \neq 0$ on $[-\nu, \nu]$. Now, define $\phi_1 := \phi_0(3 \cdot / \nu)$ and note that $\phi_1 \in C^\infty_c(\mathbb{R})$ with $\hat{\phi}_1(\xi) = \frac{1}{3 \nu} \hat{\phi}_0(\nu \xi/3) / \nu$ for $\xi \in [-3,3]$. Now, set $\psi := \phi_1 \otimes \phi_1 \in C^\infty_c(\mathbb{R}^2)$ and $\psi_2 := \phi_1$, as well as $\psi_1 := \phi_1^{(8)}$, the 8-th derivative of $\phi_1$. By differentiating under the integral and by performing partial integration, we get for $0 \leq k \leq 7$ that

$$\frac{d^k}{d\xi^k} |_{\xi = 0} \psi_1 = \frac{d^k}{d\xi^k} \bigg|_{\xi = 0} \phi_1^{(8)} = \int_{\mathbb{R}} \phi_1^{(8)}(x) \cdot (-2\pi i \xi)^k \, dx = (-1)^k \cdot \int_{\mathbb{R}} \phi_1(x) \cdot \frac{d^k (-2\pi i \xi)^k}{d\xi^k} \, dx = 0,$$  

(6.5)
since $\frac{d^{|\theta|}}{d\xi^{|\theta|}} = 0$ for $0 \leq k \leq 7$. Next, observe $\hat{\varphi}(\xi) = \hat{\varphi}_1(\xi_1) \cdot \hat{\varphi}_2(\xi_2) \neq 0$ for $\xi \in [-3, 3]^2 \supset [-1, 1]^2$, as well as $\hat{\psi}_2(\xi) \neq 0$ for $\xi \in [-3, 3]$ and finally

$$\hat{\varphi}_1(\xi) = (2\pi i \xi)^8 \cdot \hat{\varphi}_1(\xi) = (2\pi)^8 \cdot \xi^8 \cdot \hat{\varphi}_1(\xi) \neq 0 \text{ for } \xi \in [-3, 3] \setminus \{0\},$$

which in particular implies $\hat{\psi}_1(\xi) \neq 0$ for $\frac{4}{5} \leq |\xi| \leq 3$.

Now, setting $\psi := \varphi_1 \otimes \varphi_2$, we want to verify that $\varphi, \psi$ satisfy the assumptions of Theorem 5.13 with the choices $\varepsilon = \frac{4}{5}$, $p_0 = \frac{7}{4}$, $s(0) = 0$ and $\alpha = \frac{4}{5}$. Since we have $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R})$ and since $\hat{\psi}_2(\xi) \neq 0$ for $\xi \in [-3, 3]$ and $\hat{\varphi}_1(\xi) \neq 0$ for $\frac{4}{5} \leq |\xi| \leq 3$ and since finally $\hat{\varphi}(\xi) \neq 0$ for $\xi \in [-1, 1]^2$, Remark 5.11 and Corollaries 4.4 and 4.3 show that all we need to check is $\hat{\varphi}_1 = 0$ for all $\ell = 0, \ldots, N_0 + [A_1] - 1$ and all $\ell = 0, \ldots, N_0 + [M_1] - 1$, where $N_0 = \left[ p_0^{-1} \cdot (2 + \varepsilon) \right] = [27/8] = 4,$

$$A_1 = \varepsilon + p_0^{-1} + \max \left\{ 0, (1 + \alpha) \left( p_0^{-1} - 1 \right) \right\} = \frac{5}{2} \leq 3,$$

and $M_1 = \varepsilon + p_0^{-1} + \max \left\{ 0, (1 + \alpha) \left( p_0^{-1} - 2^{-1} \right) \right\} = \frac{1}{4} + 3 \leq 4$.

cf. Theorems 4.12 and 4.3. Hence, $N_0 + [A_1] - 1 \leq 7$ and $N_0 + [M_1] - 1 \leq 7$, so that equation (6.5) shows that $\varphi, \psi$ indeed satisfy the assumptions of Theorem 5.13. That theorem yields because of $\alpha = \frac{4}{5}$ some $\delta_0 \in (0, 1]$ such that the following hold for all $0 < \delta \leq \delta_0$:

- The shearlet system $SH_{1/2}^{\varphi, \varphi, \psi; \delta} \equiv \{ \gamma^{v,k,\delta} \}_{v \in V, k \in \mathbb{Z}^2}$ is a frame for $L^2(\mathbb{R}^2)$.

- Since $p \in (2 / (1 + \beta), 2) \subset \left[ \frac{7}{4}, 2 \right] = [p_0, 2]$, we have

$$\mathcal{J}_{\alpha, p}^{\varphi, \psi, \delta}(\xi) \equiv \mathcal{J}_{\alpha, p}^{\varphi, \psi, \delta}(\xi) \in L^2(\mathbb{R}^2) = \left\{ f \in L^2(\mathbb{R}^2) \mid \left( f, \gamma^{v,k,\delta}_L \right)_{v \in V, k \in \mathbb{Z}^2} \in L^p(V \times \mathbb{Z}^2) \right\}$$

and there is a constant $C_p = C_p(\varphi, \psi, \delta) > 0$ such that

$$\| f \|_{\mathcal{J}_{\alpha, p}^{\varphi, \psi, \delta}(\xi)} \leq C_p \cdot \left\| \left( f, \gamma^{v,k,\delta}_L \right)_{v \in V, k \in \mathbb{Z}^2} \right\|_{L^p(\mathbb{R}^2)} \quad \forall f \in \mathcal{J}_{\alpha, p}^{\varphi, \psi, \delta}(\xi) \in L^2(\mathbb{R}^2).$$

Thus, since we clearly have $L^2(R^2; \nu) \subset L^2(E)$, it suffices to show that there is a constant $C = C(\nu, \delta, \varphi, \psi) > 0$ such that $\| A_{\varphi} f \|_{L^p} \leq C < \infty$ for all $f \in L^2(\mathbb{R}^2; \nu)$, where $A_{\varphi} \equiv \left( \left( f, \gamma^{v,k,\delta}_L \right)_{v \in V, k \in \mathbb{Z}^2} \right)$. Here, we note that the sequence $A_{\varphi} f$ just consists of the shearlet coefficients of $f$ (up to a trivial reordering in the translation variable $k$) with respect to the shearlet frame $SH_{1/2}^{\varphi, \varphi, \psi; \delta}$ with $\theta(x, y) = \psi(y, x)$, cf. Remark 5.7. Hence, there is hope to derive the estimate $\| A_{\varphi} f \|_{L^p} \leq C$ as a consequence of 5.11. equation (3)], which states that

$$\sum_{n > N} |\lambda(f)|^2 \leq C \cdot N^{-2} \cdot (1 + \log N)^3 \quad \forall N \in \mathbb{N} \text{ and } f \in L^2(\mathbb{R}^2; \nu),$$

(6.6)

where $(|\lambda(f)|)_{n \in \mathbb{N}}$ are the absolute values of the shearlet coefficients of $f$ with respect to the shearlet frame $SH_{1/2}^{\varphi, \varphi, \psi; \delta}$, ordered nonincreasingly. In particular, $\| A_{\varphi} f \|_{L^p} = \| |\lambda(f)| \|_{L^p}.$

Note that in order for 5.11. equation (3)] to be applicable, we need to verify that $\varphi, \psi, \theta$ satisfy the assumptions of 5.13. Theorem 1.3, i.e., $\varphi, \psi, \theta$ need to be compactly supported (which is satisfied) and

$$\begin{align*}
(1) & \quad \left| \hat{\varphi}(\xi) \right| \leq \min \{ 1, |\xi_1|^\sigma \} \cdot \min \left\{ 1, |\xi_1|^{-\gamma} \right\} \cdot \min \{ 1, |\xi_2|^{-\gamma} \} \\
(2) & \quad \left| \frac{\partial}{\partial \xi_1} \hat{\varphi}(\xi) \right| \leq \left| h(\xi_1) \right| \cdot \left( 1 + \frac{|\xi_2|}{|\xi_1|} \right)^{-\gamma} \quad \text{for some } h \in L^1(\mathbb{R})
\end{align*}$$

certain (arbitrary) $\sigma > 5$ and $\tau \geq 4$. Furthermore, $\theta$ needs to satisfy the same estimate with interchanged roles of $\xi_1, \xi_2$. But in view of $\theta(x, y) = \psi(y, x)$, it suffices to establish the estimates for $\psi$. To this end, recall from above that $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ is analytic with $\frac{d^{\left| |\xi| \right|}}{d\xi^{|\xi|}} \hat{\psi}_1 = 0$ for $0 \leq k \leq 7$. This easily implies $\left| \hat{\psi}_1(\xi) \right| \leq |\xi|^\gamma$ for $|\xi| \leq 1$, see e.g., the proof of Corollary 4.4.13 in particular equation 4.4.15. Furthermore, since $\hat{\psi}_1, \hat{\psi}_2 \in C^\infty(\mathbb{R})$, we get for arbitrary $K \in \mathbb{N}$ that $\left| \hat{\psi}_i(\xi) \right| \leq (1 + |\xi|)^{-K}$ for $i \in \{1, 2\}$. Altogether, we conclude $\left| \hat{\psi}_1(\xi) \right| \leq \min \{ 1, |\xi|^\gamma \} \cdot (1 + |\xi|)^{-8}$ and likewise $\left| \hat{\psi}_2(\xi) \right| \leq (1 + |\xi|)^{-8}$ for all $\xi \in \mathbb{R}$, so that the first estimate is fulfilled for $\sigma := 8 > 5$ and $\tau := 8 \geq 4$.

Next, we observe for $\xi \in \mathbb{R}^2$ with $\xi_1 \neq 0$ that

$$1 + \frac{|\xi_2|}{|\xi_1|} \leq (1 + |\xi_2|) \cdot \left( 1 + |\xi_1|^{-1} \right) \leq 2 \cdot (1 + |\xi_2|) \cdot \max \{ 1, |\xi_1|^{-1} \}$$

and thus 
\[(1 + |\xi_2|/|\xi_1|)^{-8} \geq 2^{-8} \cdot (1 + |\xi_2|)^{-8} \cdot \min \left\{ 1, |\xi_1|^8 \right\}.
\]
But since we have \(\hat{\psi}_2 \in S(\mathbb{R})\) and thus \(\left| \frac{\partial}{\partial \xi_2} \hat{\psi}(\xi) \right| \lesssim (1 + |\xi|)^{-8}\), this implies
\[
\left| \frac{\partial}{\partial \xi_2} \hat{\psi}(\xi) \right| = \left| \hat{\psi}_1(\xi_1) \cdot \hat{\psi}_2(\xi_2) \right| \lesssim (1 + |\xi_1|)^{-8} \cdot (1 + |\xi_2|)^{-8} \cdot \min \left\{ 1, |\xi_1|^8 \right\}
\]
\[
\lesssim (1 + |\xi_1|)^{-8} \cdot (1 + |\xi_2|/|\xi_1|)^{-8},
\]
so that the second condition from above is satisfied for our choice \(\tau = 8\), with \(h(\xi_1) = (1 + |\xi_1|)^{-8}\).

Consequently, we conclude from [38, equation (3)] that equation \(6.10\) is satisfied. Now, for arbitrary \(M \in \mathbb{N}_{>4}\), we apply equation \(6.10\) with \(N = \left\lfloor \frac{M}{2} \right\rfloor \geq 2\), noting that \(\left\lfloor \frac{M}{2} \right\rfloor \leq M/2 + 1 \leq \frac{M}{2} + \frac{M}{4} = \frac{3}{4}M \leq M\) to deduce
\[
\frac{1}{4}M \cdot |\lambda(f)|^2_{M} \leq |\lambda(f)|^2_{M} \cdot (M - [M/2]) \leq \sum_{[M/2] < n < M} |\lambda(f)|^2_n \leq \sum_{n > [M/2]} |\lambda(f)|^2_n
\]
\[
\leq C \cdot [M/2]^{-2} \cdot (1 + \log [M/2])^3
\]
\[
\leq 4C \cdot M^{-2} \cdot (1 + \log M)^3,
\]
which implies \(|\lambda(f)|_M \lesssim \sqrt{16C} \cdot [M^{-1} \cdot (1 + \log M)]^{3/2}\) for \(M \in \mathbb{N}_{>4}\). But since \(E^2(\mathbb{R}^2; \nu) \subset L^2(\mathbb{R}^2)\) is bounded and since the elements of the shearlet frame \(SH(\varphi, \psi, \theta; \delta)\) are \(L^2\)-bounded, we have \(\|\lambda(f)\|_{n \in \mathbb{N}} \leq 1\), so that we get \(|\lambda(f)|_M \lesssim [M^{-1} \cdot (1 + \log M)]^{3/2}\) for all \(M \in \mathbb{N}\) and all \(f \in E^2(\mathbb{R}^2; \nu)\), where the implied constant is independent of the precise choice of \(f\). But this easily yields \(\|\lambda(f)\|_{M \in \mathbb{N}} \leq 1\), since \(p \in (\frac{3}{2}, 2) = (2/2 + 1)\), \(2\).

Here, the implied constant might depend on \(\varphi, \psi, \delta, p, \nu\), but not on \(f \in E^2(\mathbb{R}^2; \nu)\).

We can now easily derive the claimed statement about the approximation rate of functions \(f \in E^\beta(\mathbb{R}^2; \nu)\) with respect to \(\beta^{-1}\)-shearlet systems.

**Theorem 6.3.** Let \(\beta \in (1, 2]\) be arbitrary. Assume that \(\varphi, \psi \in L^4(\mathbb{R}^2)\) satisfy the conditions of Theorem \(6.10\) for \(\alpha = \beta^{-1}\), \(p_0 = q_0 = 0 = s_0 = 0 = \frac{1}{2}\) and some \(\varepsilon \in (0, 1]\) (see Remark \(6.14\) for simplified conditions which ensure that these assumptions are satisfied).

Then there is some \(\delta_0 = \delta_0(\varepsilon, \beta, \varphi, \psi) > 0\) such that for all \(0 < \delta \leq \delta_0\) and arbitrary \(f \in E^\beta(\mathbb{R}^2)\) and \(N \in \mathbb{N}\), there is a function \(f^{(N)} \in L^2(\mathbb{R}^2)\) which is a linear combination of \(N\) elements of the \(\beta^{-1}\)-shearlet frame \(\Psi = SH_{\beta^{-1}}(\varphi, \psi, \delta) = \left\{ \gamma^{(k_0, k_\delta)} \right\} \in V \times k_\mathbb{Z}^2\) such that the following holds:

For arbitrary \(\sigma > 0\), there is a constant \(C = C(\beta, \delta, \nu, \sigma, \varphi, \psi) > 0\) satisfying
\[
\|f - f^{(N)}\|_{L^2} \leq C \cdot N^{-\sigma} \forall f \in E^\beta(\mathbb{R}^2; \nu) \text{ and } N \in \mathbb{N}.
\]

**Remark.** It was shown in [38, Theorem 2.8] that no dictionary \(\Phi\) can achieve an error \(a_0^{(N)}(f) \leq C \cdot N^{-\theta}\) for all \(N \in \mathbb{N}\) and \(f \in E^\beta(\mathbb{R}^2; \nu)\) with \(\theta > \frac{1}{2}\), as long as one insists on a polynomial depth restriction for forming the \(N\)-term approximation. In this sense, the resulting approximation rate is almost optimal. We remark, however, that it is not immediately clear whether the \(N\)-term approximation whose existence is claimed by the theorem above can be chosen to satisfy the polynomial depth search restriction. There is a long-standing tradition [2, 38, 44, 51] to omit further considerations concerning this question; therefore, we deferred to Section \(E\) the proof that the above approximation rate can also be achieved using a polynomially restricted search depth.

For more details on the technical assumption of polynomial depth restriction in \(N\)-term approximations, we refer to [38, Section 2.1.1].

**Proof.** Set \(\alpha := \beta^{-1}\). Under the given assumptions, Theorem \(6.10\) ensures that \(SH_{\alpha}(\varphi, \psi, \delta)\) forms an atomic decomposition for \(\mathcal{F}_{\alpha, \delta}^{2, 2}(\mathbb{R}^2)\) for all \(p \geq p_0, q \geq q_0\) and \(s_0 \leq s \leq s_1\), for arbitrary \(0 < \delta \leq \delta_0\), where the constant \(\delta_0 = \delta_0(\varepsilon, \beta, \varphi, \psi, \delta) > 0\) is provided by Theorem \(6.10\). Fix some \(0 < \delta \leq \delta_0\).

Let \(S(\delta) : C_{\alpha, 2}^{2, 2}(\mathbb{R}^2) \to C_{\alpha, 0}^{2, 2}(\mathbb{R}^2)\) and \(C(\delta) : C_{\alpha, 0}^{2, 2}(\mathbb{R}^2) \to C_{\alpha, 0}^{2, 2}(\mathbb{R}^2)\) be the synthesis map and the coefficient map whose existence and boundedness is guaranteed by Theorem \(6.10\) since \(2 \geq p_0 = q_0\) and since \(s_0 \leq s \leq s_1\). Note directly from Definition \(2.8\) that \(C_{\alpha, 2}^{2, 2}(\mathbb{R}^2) = \ell^2(V \times \mathbb{Z}^2)\) and that \(C_{\alpha, 0}^{2, 2}(\mathbb{R}^2) = L^2(\mathbb{R}^2)\) (cf. [60, Lemma 6.10]). Now, for arbitrary \(f \in E^\beta(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)\), let \((c_j^{(f)})_{j \in V \times \mathbb{Z}^2} := c^{(f)} : E^\delta(\mathbb{R}^2 \times \mathbb{Z}^2)\).
Furthermore, for \( f \in \mathcal{E}^\beta (\mathbb{R}^2) \) and \( N \in \mathbb{N} \), choose a set \( J_N^f \subset V \times \mathbb{Z}^2 \) with \( |J_N^f| = N \) and such that \( |c_j^f| \geq |c_j^{(1)}| \) for all \( j \in J_N^f \) and all \( i \in (V \times \mathbb{Z}^2) \setminus J_N^f \). For a general sequence, such a set need not exist, but since we have \( c_j^f \in \ell^2 \), a moment’s thought shows that it does, since for each \( \varepsilon > 0 \), there are only finitely many indices \( i \in V \times \mathbb{Z}^2 \) satisfying \( |c_i^f| \geq \varepsilon \).

Finally, set \( f^{(N)} := S(\beta) \left[ \mathbb{1}_{J_N^f} \cdot c(f) \right] \) \( \in \ell^2 (\mathbb{R}^2) \) and note that \( f^{(N)} \) is indeed a linear combination of (at most) \( N \) elements of \( \mathcal{SH}_{\beta^{-1}} (\varphi, \psi; \delta) = (\gamma_{[v,k])})_{v \in V, k \in \mathbb{Z}^2} \), by definition of \( S(\delta) \). Moreover, note that the so-called Stechkin lemma (see e.g. [17], Lemma 3.3) shows

\[
\left\| c(f) - \mathbb{1}_{J_N^f} \cdot c(f) \right\|_{\ell^2} \leq N^{-\left( \frac{\beta}{2} - \frac{\varepsilon}{2} \right)} \cdot \left\| c(f) \right\|_{\ell^p} \quad \forall N \in \mathbb{N} \text{ and } p \in (0, 2),
\]

for which \( \left\| c(f) \right\|_{\ell^p} < \infty \). \( \text{(6.7)} \)

It remains to verify that the \( f^{(N)} \) satisfy the stated approximation rate. To show this, let \( \sigma, \nu > 0 \) be arbitrary. Because of \( \frac{1}{p} + \frac{1}{q} = 1 \) as \( p \downarrow 2 / (1 + \beta) \), there is some \( p \in (2 / (1 + \beta), 2) \) satisfying \( \frac{1}{p} - \frac{1}{q} \geq \frac{\beta}{2} - \sigma \). Set \( s := (1 + \alpha) \left( p^{-1} - 2^{-1} \right) \leq (1 + \alpha) \left( \frac{1}{2} - \frac{1}{2} \right) = \beta \leq (1 + \beta) = s_1 \).

Now, observe \( |\delta_{v}^{(\alpha,s)}| = u_{V}^{(\alpha,s)} \) for all \( v \in V \), so that the remark after Definition 2.8 shows that the coefficient space \( C^{\nu,s}_{V} \) satisfies \( C^{\nu,s}_{V} = \ell^p (V \times \mathbb{Z}^2) = C^{2,2}_{V} \). Therefore, Theorem 5.10 and the associated remark (and the inclusion \( \mathcal{J}^{\nu,s}_{V} (\mathbb{R}^2 ) \to \ell^2 (\mathbb{R}^2 ) \) from Theorem 5.13) show that the synthesis map and the coefficient map from above restriction to bounded linear operators

\[
S^{(\beta)} : \ell^p (V \times \mathbb{Z}^2) \to \mathcal{J}^{\nu,s}_{V} (\mathbb{R}^2) \quad \text{and} \quad C^{(\beta)} : \mathcal{J}^{\nu,s}_{V} (\mathbb{R}^2) \to \ell^p (V \times \mathbb{Z}^2).
\]

Next, Proposition 4.2 shows \( \mathcal{E}^{\beta} (\mathbb{R}^2) \subset \mathcal{J}^{\nu,s}_{V} (\mathbb{R}^2) \) and even yields a constant \( C_1 = C_1 (\beta, \nu, p) > 0 \) satisfying \( \left\| f \right\|_{\mathcal{J}^{\nu,s}_{V}} \leq C_1 \) for all \( f \in \mathcal{E}^{\beta} (\mathbb{R}^2) \). This implies

\[
\left\| c(f) \right\|_{\ell^p} = \left\| C^{(\beta)} f \right\|_{\ell^p} \leq \left\| C^{(\beta)} \right\|_{\mathcal{J}^{\nu,s}_{V} \to \ell^p} \cdot \left\| f \right\|_{\mathcal{J}^{\nu,s}_{V}} \leq C_1 \cdot \left\| C^{(\beta)} \right\|_{\mathcal{J}^{\nu,s}_{V} \to \ell^p} < \infty \quad \forall f \in \mathcal{E}^{\beta} (\mathbb{R}^2) \).
\]

By putting everything together and recalling \( S^{(\beta)} \circ C^{(\beta)} = \text{id}_{\mathcal{J}^{\nu,s}_{V}} = \text{id}_{L^2} \), we finally arrive at

\[
\left\| f - f^{(N)} \right\|_{L^2} \leq \left\| S^{(\beta)} [c(f) - \mathbb{1}_{J_N^f} \cdot c(f)] \right\|_{L^2} \leq \left\| S^{(\beta)} \right\|_{\ell^2 \to \mathcal{J}^{\nu,s}_{V}} \cdot \left\| c(f) - \mathbb{1}_{J_N^f} \cdot c(f) \right\|_{\ell^2} (\text{since } \mathcal{J}^{\nu,s}_{V} (\mathbb{R}^2) = L^2 (\mathbb{R}^2) \text{ with equivalent norms}) \leq \left\| S^{(\beta)} \right\|_{\ell^2 \to \mathcal{J}^{\nu,s}_{V}} \cdot \left\| c(f) - \mathbb{1}_{J_N^f} \cdot c(f) \right\|_{\ell^2} \leq \left\| S^{(\beta)} \right\|_{\ell^2 \to \mathcal{J}^{\nu,s}_{V}} \cdot \left\| c(f) - \mathbb{1}_{J_N^f} \cdot c(f) \right\|_{\ell^2}
\]

\[
\leq \left\| S^{(\beta)} \right\|_{\ell^2 \to \mathcal{J}^{\nu,s}_{V}} \cdot \left\| c(f) - \mathbb{1}_{J_N^f} \cdot c(f) \right\|_{\ell^2} \leq \left\| S^{(\beta)} \right\|_{\ell^2 \to \mathcal{J}^{\nu,s}_{V}} \cdot \left\| c(f) - \mathbb{1}_{J_N^f} \cdot c(f) \right\|_{\ell^2} \leq C_1 \cdot \left\| C^{(\beta)} \right\|_{\mathcal{J}^{\nu,s}_{V} \to \ell^p} \cdot \left\| S^{(\beta)} \right\|_{\ell^2 \to \mathcal{J}^{\nu,s}_{V}} \cdot \left\| c(f) - \mathbb{1}_{J_N^f} \cdot c(f) \right\|_{\ell^2} (\text{since } \frac{1}{p} \geq \frac{\beta}{2} - \sigma \geq \frac{\beta}{2} - \sigma)
\]

for all \( N \in \mathbb{N} \) and \( f \in \mathcal{E}^{\beta} (\mathbb{R}^2) \). Since \( p \) only depends on \( \alpha, \beta \), this easily yields the desired claim.

We close this section by making the assumptions of Theorem 6.3 more transparent:

**Remark 6.4.** With the choices of \( \alpha, p_0, q_0, s_0, s_1 \) from Theorem 6.3, one can choose \( \varepsilon = \varepsilon (\beta) \in (0, 1) \) such that the constants \( \left[ p_0^{-1} \cdot (2 + \varepsilon) \right] \) and \( \lambda_0, \ldots, \lambda_4 \) from Theorem 4.3 satisfy \( \lambda_1 \leq 3 \), as well as

\[
\left[ \frac{2 + \varepsilon}{p_0} \right] = \begin{cases} 
3, & \text{if } \beta < 2, \\
4, & \text{if } \beta = 2, \\
11, & \text{if } \beta < 2, \\
12, & \text{if } \beta = 2, \\
11, & \text{if } \beta < 2, \\
12, & \text{if } \beta = 2.
\end{cases}
\]

Thus, in view of Remark 5.11 (which refers to Corollary 4.3), it suffices in every case to have \( \varphi \in \mathcal{C}^{15}_{12} (\mathbb{R}^2) \) and \( \psi = \psi_1 \otimes \psi_2 \) with \( \psi_1 \in \mathcal{C}^{15} (\mathbb{R}) \) and \( \psi_2 \in \mathcal{C}^{15} (\mathbb{R}) \) and with the following additional properties:

\begin{enumerate}
\item \( \varphi (\xi) \neq 0 \) for all \( \xi \in [-1, 1]^2 \),
\item \( \varphi (\xi) \neq 0 \) for \( \frac{1}{3} \leq |\xi| \leq 3 \) and \( \psi_2 (\xi) \neq 0 \) for all \( \xi \in [-3, 3] \),
\item We have \( \frac{d}{dx} \varphi \big|_{\xi=0} \psi_1 = 0 \) for \( 0 \leq \ell \leq 5 \). In case of \( \beta < 2 \), it even suffices to have this for \( 0 \leq \ell \leq 5 \).
\end{enumerate}
Proof. We have $p_0^{-1} = \frac{1+\beta}{2}$ and thus $\frac{2}{p_0} = 1 + \beta \in (2, 3)$ in case of $\beta < 2$. Hence, $\frac{2+\varepsilon}{p_0} \in (2, 3)$ for $\varepsilon = \varepsilon (\beta)$ sufficiently small. In case of $\beta = 2$, we get $\frac{2+\varepsilon}{p_0} = 3 + \frac{\varepsilon}{p_0} \in (3, 4)$ for $\varepsilon > 0$ sufficiently small. This establishes the claimed identity for $N_0 := \lceil p_0^{-1} \cdot (2 + \varepsilon) \rceil$. For the remainder of the proof, we always assume that $\varepsilon$ is chosen small enough for this identity to hold.

Next, the constant $\Lambda_1$ from Theorem 4.3 satisfies because of $\alpha = \beta^{-1}$ that

$$\Lambda_1 = \varepsilon + \frac{1}{\min \{p_0, q_0\}} + \max \left\{ 0, (1 + \alpha) \left( \frac{1}{p_0} - 1 \right) - s_0 \right\}$$

$$= \varepsilon + \frac{1 + \beta}{2} + (1 + \beta^{-1}) \left( \frac{1 + \beta}{2} - 1 \right)$$

$$= \varepsilon + \frac{1 + \beta}{2} - \beta^{-1}$$

which is strictly increasing with respect to $\beta > 0$. Therefore, we always have $\Lambda_1 \leq \varepsilon + \frac{1}{2} + 2 - \frac{2}{\beta^2} = 2 + \frac{1}{\beta} + \varepsilon \leq 3$ for $\varepsilon \leq \frac{1}{4}$.

Furthermore, the constant $\Lambda_0$ from Theorem 4.3 is—because of $p_0 = \frac{2}{1+\beta} < 1$—given by

$$\Lambda_0 = 2\varepsilon + 3 + \max \left\{ \frac{1 - \alpha}{\min \{p_0, q_0\}} + \frac{2 + \varepsilon}{p_0} + 1 + \alpha + \frac{\beta}{p_0}, 2 \right\}$$

$$= 2\varepsilon + 3 + \max \left\{ \frac{1}{2} - \beta^{-1}, 2 \right\}$$

$$= 2\varepsilon + 3 + \max \left\{ \frac{3}{2} + \frac{\beta}{2} + N_0, 2 \right\}$$

$$\leq 7 + N_0 + \frac{1}{2} + 2\varepsilon.$$ 

Since $N_0 = 3$ for $\beta < 2$, this easily yields $\Lambda_0 \leq 11$ for $\varepsilon \leq \frac{1}{4}$. Similarly, we get $\Lambda_0 \leq 12$ for $\beta = 2$ and $\varepsilon \leq \frac{1}{4}$.

Likewise, the constant $\Lambda_2$ from Theorem 4.3 satisfies

$$\Lambda_2 = \varepsilon + \max \left\{ 2, (1 + \alpha) \left( 1 + \frac{1}{p_0} + \frac{2 + \varepsilon}{p_0} \right) + s_1 \right\}$$

$$= \varepsilon + \max \left\{ 2, (1 + \beta^{-1}) \left( 1 + \frac{1}{2} + N_0 \right) + \frac{1 + \beta}{2} \right\}$$

$$= \varepsilon + \max \left\{ 2, \frac{5}{2} + \beta + N_0 + \frac{3}{2} \beta^{-1} + \beta^{-1} N_0 \right\}$$

$$= \varepsilon + \frac{5}{2} + \beta + N_0 + \frac{3}{2} \beta^{-1} + \beta^{-1} N_0.$$ 

Hence, in case of $\beta < 2$, we thus get $\Lambda_2 = \varepsilon + \frac{1}{2} + \frac{3}{2} \beta^{-1} =: \varepsilon + g (\beta)$, where $g : (0, \infty) \to \mathbb{R}$ is strictly convex with $g (1) = 11$ and $g (2) = \frac{39}{4} < 11$, so that $g (\beta) < 11$ for all $\beta \in (1, 2)$. Thus, $\Lambda_2 < 11$ for sufficiently small $\varepsilon = \varepsilon (\beta) > 0$. Finally, for $\beta = 2$, we get $\Lambda_2 = 11 + \frac{1}{\beta} + \varepsilon < 12$ for $0 < \varepsilon < \frac{1}{4}$.

As the final constant, we consider

$$\Lambda_3 = \varepsilon + \max \left\{ \frac{1 - \alpha}{\min \{p_0, q_0\}} + \frac{3 - \alpha}{p_0} + 2 + \frac{2 + \varepsilon}{p_0} + 1 + \alpha + s_1, \frac{2 + \varepsilon}{p_0} + \frac{2}{p_0} + \frac{2 + \varepsilon}{p_0} \right\}$$

$$= \varepsilon + \max \left\{ \frac{1}{2} - \beta^{-1}, \frac{3 - \beta^{-1}}{2}, 4 N_0 + 1 + \beta^{-1} + \frac{1 + \beta}{2}, 1 + \beta + 1 + \beta + N_0 \right\}$$

$$= \varepsilon + \max \left\{ \frac{5}{2} \beta + \frac{5}{2} + 2 N_0, 2 + 2 \beta + N_0 \right\}$$

$$= \varepsilon + \frac{5}{2} \beta + \frac{5}{2} + 2 N_0.$$ 

In case of $\beta = 2$, this means $\Lambda_3 = \varepsilon + 15 + \frac{1}{\beta} < 16$ for $0 < \varepsilon < \frac{1}{4}$. Finally, for $\beta < 2$, we get $\Lambda_3 < 13 + \frac{1}{\beta} + \varepsilon < 14$ for $0 < \varepsilon < \frac{1}{4}$. \qed
7. Embeddings between α-shearlet smoothness spaces

In the preceding sections, we saw that the α-shearlet smoothness spaces \( \mathcal{S}^{p,q}_{\alpha_1,\alpha_2} (\mathbb{R}^2) \) simultaneously characterize analysis and synthesis sparsity with respect to (sufficiently nice) α-shearlet systems; see in particular Theorem 5.13. Since we have a whole family of α-shearlet systems, parametrized by \( \alpha \in [0,1] \), it is natural to ask if the different systems are related in some way, e.g. if \( \ell^{p,q} \)-sparsity, \( p \in (0,2) \), with respect to α1-shearlet systems implies \( \ell^{p,q} \)-sparsity with respect to α2-shearlet systems, for some \( q \in (0,2) \).

In view of Theorem 5.13, this is equivalent to asking whether there is an embedding

\[
\mathcal{S}^{p,q}_{\alpha_1,\alpha_2} (\mathbb{R}^2) \hookrightarrow \mathcal{S}^{q,q}_{\alpha_2,\alpha_2} (\mathbb{R}^2). 
\]

Note, however, that equation (7.1) is equivalent to asking whether one can deduce \( \ell^{p,q} \)-sparsity with respect to \( \alpha_2 \)-shearlets from \( \ell^{p,q} \)-sparsity with respect to \( \alpha_1 \)-shearlets without any additional information. If one does have additional information, e.g., if one is only interested in functions \( f \) with \( \supp f \subset \Omega \), where \( \Omega \subset \mathbb{R}^2 \) is fixed and bounded, then the embedding in equation (7.1) is a sufficient, but in general not a necessary criterion for guaranteeing that \( f \) is \( \ell^{p,q} \)-sparse with respect to \( \alpha_2 \)-shearlets if it is \( \ell^{p,q} \)-sparse with respect to \( \alpha_1 \)-shearlets.

More generally, we will completely characterize the existence of the embedding

\[
\mathcal{S}^{p,q}_{\alpha_1,\alpha_2} (\mathbb{R}^2) \hookrightarrow \mathcal{S}^{q,q}_{\alpha_2,\alpha_2} (\mathbb{R}^2) 
\]

for arbitrary \( p_1, p_2, q_1, q_2 \in (0,\infty) \), \( \alpha_1, \alpha_2 \in [0,1] \) and \( s_1, s_2 \in \mathbb{R} \). As an application, we will then see that the embedding (7.1) is never fulfilled for \( p, q \in (0,2) \), but that if one replaces the left-hand side of the embedding (7.1) by \( \mathcal{S}^{p,p}_{\alpha_1,\alpha_2} (\mathbb{R}^2) \) for some \( \alpha > 0 \), then the embedding holds for suitable \( p, q \in (0,2) \). Without further information, \( \ell^{p,q} \)-sparsity with respect to \( \alpha_1 \)-shearlets never implies nontrivial \( \ell^{p,q} \)-sparsity with respect to \( \alpha_2 \)-shearlets; but one can still transfer sparsity in some sense if one has \( \ell^{p,q} \)-sparsity with respect to \( \alpha_1 \)-shearlets, together with a certain decay of the \( \alpha_1 \)-shearlet coefficients with the scale.

We remark that the results in this section can be seen as a continuation of the work in [27]. In that paper, the authors develop the framework of \( \alpha \)-molecules which allows one to transfer (analysis) sparsity results between different systems that employ \( \alpha \)-parabolic scaling; for example between \( \alpha \)-shearlets and \( \alpha \)-curvelets. Before Theorem 4.2, the authors note that “it might though be very interesting for future research to also let \( \alpha \)-molecules for different \( \alpha \)’s interact.” In this way, this is precisely what we are doing in this section, although we focus on the special case of \( \alpha \)-shearlets instead of (more general) \( \alpha \)-molecules.

In order to characterize the embedding (7.2), we will invoke the embedding theory for decomposition spaces [60] that was developed by one of the authors; this will greatly simplify the proof, since we do not need to start from scratch. In order for the theory in [60] to be applicable to an embedding \( \mathcal{D}(Q, L^{p_1}_q, \ell^{p_2}_q) \hookrightarrow \mathcal{D}(P, L^{p_2}_q, \ell^{p_2}_q) \), the two coverings \( Q = (Q_i)_{i \in I} \) and \( P = (P_i)_{i \in J} \) need to be compatible in a certain sense. For this, it suffices if \( Q \) is almost subordinate to \( P \) (or vice versa); roughly speaking, this means that the covering \( Q \) is finer than \( P \). Precisely, it means that each set \( Q_i \) is contained in \( P'_j \) for some \( j \in J \), where \( n \in \mathbb{N} \) is fixed and where \( P'_j = \bigcup_{t \in J^n} P_t \). Here, the sets \( j^n \) are defined inductively, via \( L^* := \bigcap_{t \in L} L_t^* \) (with \( L^* \) as in Definition 2.1) and with \( L^{(n+1)*} := (L^*)^* \) for \( L \subset J \). The following lemma establishes this compatibility between different \( \alpha \)-shearlet coverings.

**Lemma 7.1.** Let \( 0 \leq \alpha_1 \leq \alpha_2 \leq 1 \). Then \( S^{(\alpha_1)} = (S^{(\alpha_1)}_i)_{i \in I^{(\alpha_1)}} \) is almost subordinate to \( S^{(\alpha_2)} = (S^{(\alpha_2)}_j)_{j \in I^{(\alpha_2)}} \). ▶

**Proof.** Since we have \( \bigcup_{i \in I^{(\alpha_1)}} S^{(\alpha_1)}_i = \mathbb{R}^2 = \bigcup_{j \in I^{(\alpha_2)}} S^{(\alpha_2)}_j \) and since all of the sets \( S^{(\alpha_1)}_i \) and \( S^{(\alpha_2)}_j \) are open and path-connected, [60] Corollary 2.13 shows that it suffices to show that \( S^{(\alpha_1)} \) is weakly subordinate to \( S^{(\alpha_2)} \). This means that we have \( \sup_{i \in I^{(\alpha_1)}} |L_i| < \infty \), with

\[
L_i := \left\{ j \in I^{(\alpha_2)} \left| S^{(\alpha_2)}_j \cap S^{(\alpha_1)}_i \neq \emptyset \right. \right\} \quad \text{for } i \in I^{(\alpha_1)}.
\]

To show this, we first consider only the case \( i = (n,m,\varepsilon,0) \in I^{(\alpha_1)}_0 \) and let \( j \in L_i \) be arbitrary. We now distinguish several cases regarding \( j \):

**Case 1:** We have \( j = (k,\ell,\beta,0) \in I^{(\alpha_2)}_0 \). Let \( (\xi,\eta) \in S^{(\alpha_1)}_i \cap S^{(\alpha_2)}_j \). In view of equation (5.3), this implies \( \xi \in (2^n/3, 2 \cdot 2^n) \cap (2^k/3, 2 \cdot 2^k) \), so that in particular \( \varepsilon = \beta \). Furthermore, we see \( 2^k/3 < |\xi| < 3 \cdot 2^n \), which yields \( 2^n - k > \frac{1}{3} > 2^{-4} \). Analogously, we get \( 2^n/3 < |\eta| < 3 \cdot 2^k \) and thus \( 2^n - k < 3 < 2^4 \). Together, these considerations imply \( |n - k| \leq 3 \).

Furthermore, since \( (\xi,\eta) \in S^{(\alpha_1)}_i \cap S^{(\alpha_2)}_j \), equation (6.3) also shows

\[
\frac{\xi}{\eta} = \frac{\xi}{\varepsilon} = \frac{\beta \eta}{\beta \xi} \in 2^{n(\alpha_1-1)} (m-1, m+1) \cap 2^{k(\alpha_2-1)} (\ell - 1, \ell + 1).
\]
Hence, we get the two inequalities
\[ 2^{n(\alpha_1-1)}(m+1) > 2^{k(\alpha_2-1)}(\ell - 1) \quad \text{and} \quad 2^{n(\alpha_1-1)}(m-1) < 2^{k(\alpha_2-1)}(\ell + 1) \]
and thus
\[ \ell < (m+1)2^{n(\alpha_1-k\alpha_2+k-n)} + 1 \quad \text{and} \quad \ell > (m-1)2^{n(\alpha_1-k\alpha_2+k-n)} - 1. \]
In other words,
\[ \ell \in ((m-1)2^{n(\alpha_1-k\alpha_2+k-n)} - 1, (m+1)2^{n(\alpha_1-k\alpha_2+k-n)} + 1) \cap \mathbb{Z} =: s_{m}^{(n,k)}. \]
But since any interval \( I = (A, B) \) with \( A \leq B \) satisfies \( |I \cap \mathbb{Z}| \leq B - A + 1 \), the cardinality of \( s_{m}^{(n,k)} \) can be estimated by
\[
\left| s_{m}^{(n,k)} \right| \leq (m + 1)2^{n\alpha_1-k\alpha_2+k-n} + 1 - (m-1)2^{n\alpha_1-k\alpha_2+k-n} + 1 + 1
= 3 + 2 \cdot 2^{n\alpha_1-k\alpha_2+k-n}
\]
(since \( |n-k|\leq3 \))
\[
= 3 + 2^3 \cdot 2^{n(\alpha_1-\alpha_2)} \cdot 2^{3\alpha_2}
\]
(since \( \alpha_1-\alpha_2\leq0 \) and \( \alpha_2\leq1 \))
\[
= 3 + 2^7 = 131.
\]
Thus,
\[ L_i^{(0)} := \left\{ j = (k, \ell, \beta, 0) \in I_0^{(\alpha_2)} \left| S_j^{(\alpha_1)} \cap S_j^{(\alpha_2)} \neq \emptyset \right. \right\} \subset \bigcup_{t=n-3}^{n+3} \left( \{t\} \times s_{m}^{(n,t)} \times \{\pm 1\} \times \{0\} \right),
\]
which is a finite set, with at most \( 7 \cdot 131 \cdot 2 = 1834 \) elements.

**Case 2:** We have \( j = (k, \ell, \beta, 1) \in I_0^{(\alpha_2)}. \) Let \( (\xi, \eta) \in S_j^{(\alpha_1)} \cap S_j^{(\alpha_2)}. \) With similar arguments as in the previous case, this implies \( \xi \in \varepsilon(2^n/3, 3 \cdot 2^n), \eta \in \beta(2^k/3, 3 \cdot 2^k) \) and \( \frac{\xi}{\eta} \in 2^{n(\alpha_1-1)}(\ell - 1, \ell + 1). \) Furthermore since \( (\xi, \eta) \in S_j^{(\alpha_1)} \cap S_j^{(\alpha_2)} \), we know from Lemma 3.2 that \( |\eta| < 3|\xi| \) and \( |\xi| < 3|\eta| \).

Thus, \( 2^k/3 < |\eta| < 3 \cdot |\xi| < 3 \cdot 3 \cdot 2^n \) and hence \( 2^{k-n} \leq 2^7 \leq 2^5 \). Likewise, \( 2^n/3 < |\xi| < 3 \cdot |\eta| < 3 \cdot 3 \cdot 2^k \) and hence \( 2^{n-k} \leq 2^5 \), so that we get \( |n - k| \leq 4 \). Now, we distinguish two subcases regarding \( |\eta|/|\xi| \):

1. We have \( |\eta|/|\xi| > 1. \) Because of \( |m| \leq 2^{n(1-\alpha_1)} \leq 1 + 2^{n(1-\alpha_1)} \), this implies
\[
1 < \frac{|\eta|}{|\xi|} < 2^{n(\alpha_1-1)}(|m| + 1) \leq 2^{n(\alpha_1-1)} \left( 2^{n(1-\alpha_1)} + 1 + 1 \right) = 1 + 2 \cdot 2^{n(\alpha_1-1)}
\]
and hence
\[
\frac{1}{1 + 2 \cdot 2^{n(\alpha_1-1)}} < \left| \frac{\xi}{\eta} \right| < 1.
\]
Furthermore, we know \( |\xi/\eta| < 2^{k(\alpha_2-1)}(|\ell| + 1) \), so that we get
\[
\frac{1}{1 + 2 \cdot 2^{n(\alpha_1-1)}} < \left| \frac{\xi}{\eta} \right| < 2^{k(\alpha_2-1)}(|\ell| + 1) \quad \text{and hence} \quad |\ell| > \frac{2^{k(1-\alpha_2)}}{1 + 2 \cdot 2^{n(\alpha_1-1)}} - 1.
\]
Thus, we have
\[
|\ell| \in \mathbb{Z} \cap \left( \frac{2^{k(1-\alpha_2)}}{1 + 2 \cdot 2^{n(\alpha_1-1)}} - 1, \left[ 2^{k(1-\alpha_2)} \right] \right) \subset \mathbb{Z} \cap \left( \frac{2^{k(1-\alpha_2)}}{1 + 2 \cdot 2^{n(\alpha_1-1)}} - 1, 2^{k(1-\alpha_2)} + 1 \right) =: s_{m}^{(n,k)},
\]
where as above
\[
\left| s_{m}^{(n,k)} \right| \leq 2^{k(1-\alpha_2)} + 1 - \frac{2^{k(1-\alpha_2)}}{1 + 2 \cdot 2^{n(\alpha_1-1)}} + 1 + 1 = 3 + 2^{k(1-\alpha_2)} \left( 1 - \frac{1}{1 + 2 \cdot 2^{n(\alpha_1-1)}} \right)
\]
\[
= 3 + 2^{k(1-\alpha_2)} \frac{2 \cdot 2^{n(\alpha_1-1)}}{1 + 2 \cdot 2^{n(\alpha_1-1)}}
\]
\[
\leq 3 + 2 \cdot 2^{k(1-\alpha_2)-n(1-\alpha_1)}
\]
(since \( 1-\alpha_2\geq0 \) and \( |n-k| \leq 4 \))
\[
= 3 + 2 \cdot 2^{4(1-\alpha_2)}2^{n(\alpha_1-\alpha_2)}
\]
(since \( \alpha_1-\alpha_2\leq0 \) and \( \alpha_2\geq0 \))
\[
\leq 3 + 2 \cdot 2^4 = 35.
\]
Finally, note that \( |\ell| \in s_{m}^{(n,k)} \) implies \( \ell \in \pm s_{m}^{(n,k)} \), with \( |\pm s_{m}^{(n,k)}| \leq 70. \)
(2) We have $|\eta/\xi| \leq 1$. This yields $1 \leq |\ell/\eta| < 2^{k(\alpha_2-1)}(|\ell|+1)$ and hence $|\ell| > 2^{k(1-\alpha_2)} - 1$. Thus, we have

$$|\ell| \in \mathbb{Z} \cap \left(2^{k(1-\alpha_2)} - 1, 2^{k(1-\alpha_2)} \right] \subset \mathbb{Z} \cap \left(2^{k(1-\alpha_2)} - 1, 2^{k(1-\alpha_2)} + 1 \right) =: \mathfrak{s}^{(n,k)},$$

where one easily sees $|\mathfrak{s}^{(n,k)}| \leq 3$ and then $\ell \in \pm \mathfrak{s}^{(n,k)}$ with $|\pm \mathfrak{s}^{(n,k)}| \leq 6$.

All in all, we see

$$L_i^{(1)} := \left\{ j = (k, \ell, \beta, 1) \in J_0^{(a_2)} \middle| S_{i}^{(a_2)} \cap S_{i}^{(a_1)} \neq \emptyset \right\} \subset \bigcup_{t=n-4}^{n+4} \left[ \{t \} \times ([\pm \mathfrak{s}^{(n,t)}] \cup [\pm \mathfrak{s}^{(n,t)}]) \times \{ \pm 1 \} \times \{1 \} \right]$$

and hence $|L_i^{(1)}| \leq 9 \cdot (70 + 6) \cdot 2 = 1368$.

In total, Cases 1 and 2 show because of $L_i \subset L_i^{(0)} \cup L_i^{(1)} \cup \{0\}$ that $|L_i| \leq |L_i^{(0)}| + |L_i^{(1)}| + |\{0\}| \leq 3203$ for all $i = (n, m, \varepsilon, 0) \in I_0^{(a_1)}$.

But in case of $i = (n, m, \varepsilon, 1) \in I_0^{(a_1)}$, we get the same result. Indeed, if we set $\tilde{\gamma} := 1 - \gamma$ for $\gamma \in \{0, 1\}$, then

$$I_0^{(a_2)} \cap L_{(n,m,\varepsilon,1)} = \left\{ (k, \ell, \beta, \gamma) \in I_0^{(a_2)} \middle| S_{k,\ell,\beta,\gamma}^{(a_2)} \cap S_{n,m,\varepsilon,1}^{(a_1)} \neq \emptyset \right\}$$

$$= \left\{ (k, \ell, \beta, \gamma) \in I_0^{(a_2)} \middle| R S_{k,\ell,\beta,\gamma}^{(a_2)} \cap R S_{n,m,\varepsilon,0}^{(a_1)} \neq \emptyset \right\}$$

$$= \left\{ (k, \ell, \beta, \tilde{\gamma}) \in I_0^{(a_2)} \middle| S_{k,\ell,\beta,\tilde{\gamma}}^{(a_2)} \cap S_{n,m,\varepsilon,0}^{(a_1)} \neq \emptyset \right\}$$

and thus $|I_0^{(a_2)} \cap L_{(n,m,\varepsilon,1)}| \leq 3202$, so that $|L_{(n,m,\varepsilon,0)}| \leq 3203$.

It remains to consider the case $i = 0$. But for $\xi \in S_{0}^{(a_1)} = (-1,1)^2$, we have $1 + |\xi| \leq 3$. Conversely, Lemma 3.4 shows $1 + |\xi| \geq 1/2 \cdot w_j = 2^k/3$ for all $\xi \in S_j^{(a_2)}$ and all $j = (k, \ell, \beta, \gamma) \in I_0^{(a_2)}$. Hence, $j \in L_0$ can only hold if $2^k/3 \leq 3$, i.e., if $k \leq 3$. Since we also have $|\ell| \leq 2^{k(1-\alpha_2)} \leq 2^k \leq 2^3 = 8$, this implies $L_0 \subset \{0\} \cup \{0, 1, 2, 3\} \times \{-8, \ldots, 8\} \times \{\pm 1\} \times \{0, 1\}$

and hence $|L_0| \leq 1 + 4 \cdot 17 \cdot 2 \cdot 2 = 273 \leq 3203$.

In total, we have shown $\sup_{i \in I^{(a_1)}} |L_i| \leq 3203 < \infty$, so that $S^{(a_1)}$ is weakly subordinate to $S^{(a_2)}$. As seen at the beginning of the proof, this suffices.

Now that we have seen that $S^{(a_1)}$ is almost subordinate to $S^{(a_2)}$ for $\alpha_1 \leq \alpha_2$, the theory from [60] is applicable. But the resulting conditions simplify greatly, in addition to the coverings, also the employed weights are compatible in a certain sense. Precisely, for two coverings $Q = (Q_i)_{i \in I}$ and $P = (P_j)_{j \in J}$ and for a weight $w = (w_i)_{i \in I}$ on the index set of $Q$, we say that $w$ is relatively $\mathcal{P}$-moderate, if there is a constant $C > 0$ with

$$w_i \leq C \cdot w_\ell$$

for all $i, \ell \in I$ with $Q_i \cap P_\ell \neq \emptyset \neq Q_\ell \cap P_i$ for some $j \in J$.

Likewise, the covering $Q = (T_i, Q_i' + b_i)_{i \in I}$ is called relatively $\mathcal{P}$-moderate, if the weight $|(\det T_i)|_{i \in I}$ is relatively $\mathcal{P}$-moderate. Our next lemma shows that these two conditions are satisfied if $Q$ and $P$ are two $\alpha$-shearlet coverings.

**Lemma 7.2.** Let $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and let $S^{(a_1)}$ and $S^{(a_2)}$ be the associated $\alpha$-shearlet coverings. Then the following hold:

1. $S^{(a_1)}$ is relatively $S^{(a_2)}$-moderate.
2. For arbitrary $s \in \mathbb{R}$, the weight $w^s = (w_i^s)_{i \in I^{(a_1)}}$ with $w = (w_i)_{i \in I^{(a_1)}}$ as in Definition 3.1 (considered as a weight for $S^{(a_1)}$) is relatively $S^{(a_2)}$-moderate. More precisely, we have $39^{-|s|} \cdot w_j^{s} \leq w_i^s \leq 39^{|s|} \cdot w_j^s$ for all $i \in I^{(a_1)}$ and $j \in I^{(a_2)}$ with $S_i^{(a_1)} \cap S_j^{(a_2)} \neq \emptyset$.

**Proof.** It is not hard to see $|\det T_i^{(a_1)}| = w_i^{1+\alpha_1}$ for all $i \in I^{(a_1)}$. Thus, the second claim implies the first one.

To prove the second one, let $i \in I^{(a_1)}$ and $j \in I^{(a_2)}$ with $S_i^{(a_1)} \cap S_j^{(a_2)} \neq \emptyset$. Thus, there is some $\xi \in S_i^{(a_1)} \cap S_j^{(a_2)}$.

In view of Lemma 3.4 this implies

$$w_j \leq 3 \cdot (1 + |\xi|) \leq 39 \cdot w_j,$$

from which it easily follows that $39^{-|s|} \cdot w_j^{s} \leq w_i^s \leq 39^{|s|} \cdot w_j^s$. This establishes the second part of the second claim of the lemma.
But this easily implies that the weight \( w^s \) is relatively \( S^{(\alpha_2)} \)-moderate: Indeed, let \( i, \ell \in I^{(\alpha_1)} \) be arbitrary with 
\[ S_i^{(\alpha_1)} \cap S_j^{(\alpha_2)} \neq \emptyset = S_{\ell}^{(\alpha_1)} \cap S_j^{(\alpha_2)} \] 
for some \( j \in I^{(\alpha_2)} \). This implies \( w^s_i \leq 39^{|s|} \cdot w^s_j \leq (39^2)^{|s|} \cdot w^s \), as desired. □

Now that we have established the strong compatibility between the \( \alpha \)-shearlet coverings \( S^{(\alpha_1)} \) and \( S^{(\alpha_2)} \) and of the associated weights, we can easily characterize the existence of embeddings between the \( \alpha \)-shearlet smoothness.

**Theorem 7.3.** Let \( \alpha_1, \alpha_2 \in [0, 1] \) with \( \alpha_1 \leq \alpha_2 \). For \( s, r \in \mathbb{R} \) and \( p_1, p_2, q_1, q_2 \in (0, \infty) \), the map
\[
\mathcal{A}^{p_1, q_1}_{\alpha_1} (\mathbb{R}^2) \to \mathcal{A}^{p_2, q_2}_{\alpha_2} (\mathbb{R}^2), \quad f \mapsto f
\]
is well-defined and bounded if and only if we have \( p_1 \leq p_2 \) and
\[
\begin{align*}
\left\{ \begin{array}{ll}
r > s + (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) + (\alpha_2 - \alpha_1) \left( \frac{1}{q_2} - \frac{1}{p_1} \right) + (1 - \alpha_2) \left( \frac{1}{q_2} - \frac{1}{q_1} \right), & \text{if } q_2 < q_1, \\
r \geq s + (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) + (\alpha_2 - \alpha_1) \left( \frac{1}{q_2} - \frac{1}{p_1} \right) + (1 - \alpha_2) \left( \frac{1}{q_2} - \frac{1}{q_1} \right), & \text{if } q_2 \geq q_1.
\end{array} \right.
\end{align*}
\]
Likewise, the map
\[
\mathcal{A}^{p_1, q_1}_{\alpha_1} (\mathbb{R}^2) \to \mathcal{A}^{p_2, q_2}_{\alpha_2} (\mathbb{R}^2), \quad f \mapsto f
\]
is well-defined and bounded if and only if we have \( p_1 \leq p_2 \) and
\[
\begin{align*}
\left\{ \begin{array}{ll}
s > r + (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) + (\alpha_2 - \alpha_1) \left( \frac{1}{q_2} - \frac{1}{p_1} \right) + (1 - \alpha_2) \left( \frac{1}{q_2} - \frac{1}{q_1} \right), & \text{if } q_2 < q_1, \\
s \geq r + (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) + (\alpha_2 - \alpha_1) \left( \frac{1}{q_2} - \frac{1}{p_1} \right) + (1 - \alpha_2) \left( \frac{1}{q_2} - \frac{1}{q_1} \right), & \text{if } q_2 \geq q_1.
\end{array} \right.
\end{align*}
\]
Here, we used the notations
\[
p^\prime := \min \{p, p'\}, \quad \text{and} \quad \frac{1}{p_{\pm \alpha}} := \min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\},
\]
where the conjugate exponent \( p' \) is defined as usual for \( p \in [1, \infty) \) and as \( p' := \infty \) for \( p \in (0, 1) \).

**Proof.** For the first part, we want to invoke part (4) of [60] Theorem 7.2, with \( Q = S^{(\alpha_2)} = (T^{(\alpha_2)}_i Q'_i)_{i \in I^{(\alpha_2)}} \) and \( \mathcal{P} = S^{(\alpha_1)} = (T^{(\alpha_1)}_i Q'_i)_{i \in I^{(\alpha_1)}} \) and with \( w = (w^+_i)_{i \in I^{(\alpha_2)}} \) and \( v = (w^-_i)_{i \in I^{(\alpha_1)}} \). To this end, we first have to verify that \( Q, \mathcal{P}, w, v \) satisfy [60] Assumption 7.1. But we saw in Lemma 3.1 that \( w \) and \( v \) are \( Q \)-moderate and \( \mathcal{P} \)-moderate, respectively. Furthermore, \( Q, \mathcal{P} \) are almost structured coverings (cf. Lemma 3.3) and thus also semi-structured coverings (cf. [60] Definition 2.5) of \( \mathcal{O} = \mathcal{O}' = \mathbb{R}^2 \). Furthermore, since \( \{Q'_i \mid i \in I^{(\alpha)}\} \) is a finite family of nonempty open sets (for arbitrary \( \alpha \in [0, 1] \)), it is not hard to see that \( S^{(\alpha)} \) is an open covering of \( \mathbb{R}^2 \) and that there is some \( \varepsilon > 0 \) and for each \( i \in I^{(\alpha)} \) some \( \eta_i \in \mathbb{R}^2 \) with \( B_{\varepsilon} (\eta_i) \subset Q'_i \). Thus, \( S^{(\alpha)} \) is a tight, open semi-structured covering of \( \mathbb{R}^2 \) for all \( \alpha \in [0, 1] \). Hence, so are \( Q, \mathcal{P} \). Finally, [61] Corollary 2.7 shows that if \( \Phi = (\varphi_i)_{i \in I^{(\alpha)}} \) and \( \Psi = (\psi_j)_{j \in I^{(\alpha)}} \) are regular partitions of unity for \( Q, \mathcal{P} \), respectively, then \( \Phi, \Psi \) are \( L^p \)-BAPUs (cf. [60] Definitions 3.5 and 3.6) for \( Q, \mathcal{P} \), simultaneously for all \( p \in (0, \infty) \). Hence, all assumptions of [60] Assumption 7.1 are satisfied.

Next, Lemma 7.2 shows that \( \mathcal{P} = S^{(\alpha_1)} \) is almost subordinate to \( Q = S^{(\alpha_2)} \) and Lemma 7.2 shows that \( \mathcal{P} \) and \( v \) are relatively \( Q \)-moderate, so that all assumptions of [60] Theorem 7.2, part (4) are satisfied.

Now, let us choose, for each \( j \in I^{(\alpha_2)} \), an arbitrary index \( i_j \in I^{(\alpha_1)} \) with \( S_{i_j}^{(\alpha_1)} \cap S_j^{(\alpha_2)} \neq \emptyset \). Then [60] Theorem 7.2, part (4) shows that the embedding
\[
\mathcal{A}^{p_1, q_1}_{\alpha_1} (\mathbb{R}^2) \hookrightarrow \mathcal{A}^{p_2, q_2}_{\alpha_2} (\mathbb{R}^2)
\]
holds if and only if we have \( p_1 \leq p_2 \) and if furthermore, the following expression (then a constant) is finite:
\[
K := \left\| \begin{pmatrix} w^s_i & \det T^{(\alpha_2)}_j \left( \frac{1}{q_2} - \frac{1}{p_1} \right) + \det T^{(\alpha_1)}_{i_j} \left( \frac{1}{p_1} - \frac{1}{p_2} \right) - \left( \frac{1}{q_2} - \frac{1}{p_1} \right) \end{pmatrix} \right\|_{\ell^{p_2} (q_1/q_2)}.
\]
[Lemma 7.2]
\[
\begin{align*}
\left( \frac{2k}{q^2} \right)^{\frac{1}{2}} & \cdot \frac{k (1 + \alpha_2) \left( \frac{1}{q_2} - \frac{1}{p_1} \right) + k (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} - \left( \frac{1}{q_2} - \frac{1}{p_1} \right) \right)}{(k, \ell, \beta, \gamma) \in I^{(\alpha_2)}} \left\| \begin{pmatrix} \frac{1}{p_2} & \frac{1}{p_1} \end{pmatrix} \right\|_{\ell^{p_2} (q_1/q_2)} \\
& = \left( \frac{2k}{q^2} \right)^{\frac{1}{2}} \cdot \frac{k (s-r + (1 + \alpha_2) \left( \frac{1}{q_2} - \frac{1}{p_1} \right) + (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} - \left( \frac{1}{q_2} - \frac{1}{p_1} \right) \right))}{(k, \ell, \beta, \gamma) \in I^{(\alpha_2)}} \left\| \begin{pmatrix} \frac{1}{p_2} & \frac{1}{p_1} \end{pmatrix} \right\|_{\ell^{p_2} (q_1/q_2)}.
\end{align*}
\]
Note that we only took the norm of the sequence with \( j \in I^{0\{(\alpha_2)\}}_0 \), omitting the term for \( j = 0 \), in contrast to the definition of \( K \) in \[60\] Theorem 7.2. This is justified, since we are only interested in finiteness of the norm, for which the single (finite!) term for \( j = 0 \) is irrelevant.

Now, we distinguish two different cases regarding \( q_1 \) and \( q_2 \):

**Case 1:** We have \( q_2 < q_1 \). This implies \( q := q_2 \cdot (q_1/q_2)' < \infty \), cf. \[60\] Equation (4.3)]. For brevity, let us define \( \Theta := s - r + (\alpha_2 - \alpha_1) \left( \frac{1}{q_2} - \frac{1}{p_2} \right) + (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \). Then, we get

\[
K \prec \| (2^{k\theta})_{(k,\ell,\beta,\gamma)\in I^{0\{(\alpha_2)\}}_0} \|_{\ell^q} \leq 2^{k\varphi} = \sum_{k=0}^{\infty} 2^{k\varphi} = \sum_{k=0}^{\infty} \sum_{|\ell| \leq 2^{k(\alpha_2)}} \sum_{\beta \in \{1\}^{\gamma \in \{0,1\}}} 1 = 4 \cdot \sum_{k=0}^{\infty} 2^{k\varphi} (1 + 2 \cdot 2^{k(1-\alpha_2)}) \times \sum_{k=0}^{\infty} 2^{k\varphi(1+\alpha_2)}.
\]

Now, note from the remark to \[60\] Lemma 4.8 that \( \frac{1}{p\cdot q} = \left( \frac{1}{p} - \frac{1}{q} \right)_+ \) for arbitrary \( p, q \in (0, \infty) \). Hence, in the present case, we have \( q^{-1} = (q_2^{-1} - q_1^{-1})_+ + q_2^{-1} - q_1^{-1} \). Therefore, we see that the last sum from above—and therefore \( K \)—is finite if and only if \( q \cdot \Theta + 1 - \alpha_2 < 0 \). But this is equivalent to

\[
s - r + (\alpha_2 - \alpha_1) \left( \frac{1}{q_2} - \frac{1}{p_1} \right) + (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) = \Theta < (\alpha_2 - 1) \cdot (q_2^{-1} - q_1^{-1}),
\]

from which it easily follows that the claimed equivalence from the first part of the theorem holds in case of \( q_2 < q_1 \).

**Case 2:** We have \( q_2 \geq q_1 \). This implies \( q_2 \cdot (q_1/q_2)' = \infty \), cf. \[60\] Equation (4.3)]. Thus, with \( \Theta \) as in the previous case, we have

\[
K \prec \sup_{(k,\ell,\beta,\gamma)\in I^{0\{(\alpha_2)\}}_0} 2^{k\Theta},
\]

so that \( K \) is finite if and only if \( \Theta \leq 0 \), which is equivalent to

\[
s \geq r + (\alpha_2 - \alpha_1) \left( \frac{1}{q_2} - \frac{1}{p_1} \right) + (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right).
\]

As in the previous case, this shows for \( q_2 \geq q_1 \) that the claimed equivalence from the first part of the theorem holds.

For the second part of the theorem, we make use of part (4) of \[60\] Theorem 7.4, with \( \mathcal{Q} = S^{(\alpha_1)} = (T^{(\alpha_1)}_{i})_{i\in I^{(\alpha_1)}} \) and \( \mathcal{P} = S^{(\alpha_2)} = (T^{(\alpha_2)}_{i})_{i\in I^{(\alpha_2)}} \) and with \( w = (w_i)_{i\in I^{(\alpha_1)}} \) and \( v = (v_i)_{i\in I^{(\alpha_2)}} \). As above, one sees that the corresponding assumptions are fulfilled.

Thus, \[60\] Theorem 7.4, part (4)] shows that the embedding \( \mathcal{H}_{\alpha_1}^{\mathcal{Q}^0, \mathcal{P}^0} (\mathbb{R}^2) \hookrightarrow \mathcal{H}_{\alpha_2}^{\mathcal{Q}^0', \mathcal{P}^0'} (\mathbb{R}^2) \) holds if and only if we have \( p_1 \leq p_2 \) and if furthermore the following expression (then a constant) is finite:

\[
C := \left\| \frac{w_j}{w^{\mathcal{Q}^0, \mathcal{P}^0}} \cdot \det T_{i}^{(\alpha_1)} \left( \frac{1}{p_2} - \frac{1}{q_1} \right) + \cdot \det T_{i}^{(\alpha_2)} \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \left( \frac{1}{q_2} - \frac{1}{q_1} \right) \right\|_{\ell^p(\mathcal{Q}^0, \mathcal{P}^0')},
\]

where for each \( j \in I^{(\alpha_2)} \) an arbitrary index \( i_j \in I^{(\alpha_1)} \) with \( S^{(\alpha_1)}_{i_j} \cap S^{(\alpha_2)}_{j} \neq \emptyset \) is chosen.

But in view of Lemma \[7.2\], it is not hard to see that \( C \) satisfies

\[
C \prec \left\| \frac{g_{kr}}{2^{k\alpha_2}} \cdot 2^{k(1+\alpha_2)} \left( \frac{1}{p_2} - \frac{1}{q_1} \right) + \cdot 2^{k(1+\alpha_1)} \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \left( \frac{1}{q_2} - \frac{1}{q_1} \right) \right\|_{\ell^p(\mathcal{Q}^0, \mathcal{P}^0')},
\]

\[
= \left\| \frac{k}{2} \left( \left( \frac{1}{p_1} - \frac{1}{p_2} \right) - \left( \frac{1}{p_2} - \frac{1}{q_1} \right) \right) + \left( 1 + \alpha_2 \right) \left( \frac{1}{p_1} - \frac{1}{q_1} \right) - s + r \right\|_{\ell^p(\mathcal{Q}^0, \mathcal{P}^0')},
\]

\[
= \left\| \frac{k}{2} \left( \left( \frac{1}{p_1} - \frac{1}{p_2} \right) + (\alpha_2 - \alpha_1) \left( \frac{1}{p_2} - \frac{1}{q_1} \right) - s + r \right) \right\|_{\ell^p(\mathcal{Q}^0, \mathcal{P}^0')}.
\]

As above, we distinguish two cases regarding \( q_1 \) and \( q_2 \):
Case 1: We have $q_2 < q_1$, so that $q := q_2 \cdot (q_1/q_2)' < \infty$. But setting
\[
\Gamma := (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) + (\alpha_2 - \alpha_1) \left( \frac{1}{p_2} - \frac{1}{q_1} \right) - s + r,
\]
we have
\[
C^e \asymp \left\| 2^{k \Gamma} \left( k, \ell, \beta, \gamma \right) \in I^{(\alpha_2)} \right\|_{L^p} = \sum_{(k, \ell, \beta, \gamma) \in I^{(\alpha_2)}} 2^{k \cdot \Gamma}
\]
\[
= \sum_{k=0}^{\infty} 2^{k \cdot \Gamma} \sum_{|\ell| \leq 2^{(1 - \alpha_2)}} \sum_{\beta \in \{\pm 1\}} \sum_{\gamma \in \{0, 1\}} 1 \asymp \sum_{k=0}^{\infty} 2^{k(\Gamma + 1 - \alpha_2)}.
\]
As above, we have $q^{-1} = (q_2^{-1} - q_1^{-1})_+ = q_2^{-1} - q_1^{-1}$ and we see that the last sum—and thus $C$—is finite if and only if we have $q \cdot \Gamma + 1 - \alpha_2 < 0$, which is equivalent to
\[
(1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) + (\alpha_2 - \alpha_1) \left( \frac{1}{p_2} - \frac{1}{q_1} \right) - s + r = \Gamma < (\alpha_2 - 1) \cdot (q_2^{-1} - q_1^{-1}).
\]
Based on this, it is not hard to see that the equivalence stated in the second part of the theorem is valid for $q_2 < q_1$.

Case 2: We have $q_2 \geq q_1$, so that $q_2 \cdot (q_1/q_2)' = \infty$. In this case, we have—with $\Gamma$ as above—that
\[
C \asymp \sup_{(k, \ell, \beta, \gamma) \in I^{(\alpha_2)}} 2^{k \Gamma},
\]
which is finite if and only if $\Gamma \leq 0$, which is equivalent to
\[
s \geq r + (1 + \alpha_1) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) + (\alpha_2 - \alpha_1) \left( \frac{1}{p_2} - \frac{1}{q_1} \right).
\]
This easily shows that the claimed equivalence from the second part of the theorem also holds for $q_2 \geq q_1$. \hfill \Box

With Theorem 7.3, we have established the characterization of the general embedding from equation (7.2). Our main application, however, was to determine under which conditions $\ell^p$-sparsity of $f$ with respect to $\alpha_1$-shearlet systems implies $\ell^q$-sparsity of $f$ with respect to $\alpha_2$-shearlet systems, if one has no additional information. As discussed around equation (7.1), this amounts to an embedding $\mathcal{S}_{\alpha_1,(1+\alpha_1)(p^{-1} - 2^{-1})}^{\alpha_2,(1+\alpha_2)(q^{-1} - 2^{-1})}(\mathbb{R}^2) \hookrightarrow \mathcal{S}_{\alpha_2,(1+\alpha_2)(q^{-1} - 2^{-1})}^{\alpha_1,(1+\alpha_1)(p^{-1} - 2^{-1})}(\mathbb{R}^2)$.

Since we are only interested in nontrivial sparsity, and since arbitrary $L^2$ functions have $\alpha$-shearlet coefficients in $\ell^2$, the only interesting case is for $p, q \leq 2$. This setting is considered in our next lemma:

Lemma 7.4. Let $\alpha_1, \alpha_2 \in [0, 1)$ with $\alpha_1 \neq \alpha_2$, let $p, q \in (0, 2)$ and let $\varepsilon \in [0, \infty)$. The embedding
\[
\mathcal{S}_{\alpha_1,(1+\alpha_1)(p^{-1} - 2^{-1})}^{\alpha_2,(1+\alpha_2)(q^{-1} - 2^{-1})}(\mathbb{R}^2) \hookrightarrow \mathcal{S}_{\alpha_2,(1+\alpha_2)(q^{-1} - 2^{-1})}^{\alpha_1,(1+\alpha_1)(p^{-1} - 2^{-1})}(\mathbb{R}^2)
\]
holds if and only if we have $p \leq q$ and $q \geq \left( \frac{1}{2} + \frac{\varepsilon}{|\alpha_1 - \alpha_2|} \right)^{-1}$.

Remark. The case $\varepsilon = 0$ corresponds to the embedding which is considered in equation (7.1). Here, the preceding lemma shows that the embedding can only hold if $q \geq 2$. Since the $\alpha_2$-shearlet coefficients of every $L^2$ function are $\ell^2$-sparse, we see that $\ell^p$-sparsity with respect to $\alpha_1$-shearlets does not imply any nontrivial $\ell^q$-sparsity with respect to $\alpha_2$-shearlets for $\alpha_1 \neq \alpha_2$, if no additional information than the $\ell^p$-sparsity with respect to $\alpha_1$-shearlets is given.

But in conjunction with Theorem 7.3, we see that if the $\alpha_1$-shearlet coefficients \((f, \psi_{[j,\ell,i],k})_{j,\ell,i} \in I^{(\alpha_1)}, k \in \mathbb{Z}^2\) satisfy
\[
\left\| 2^{\varepsilon j} \left( f, \psi_{[j,\ell,i],k} \right)_{j,\ell,i} \in I^{(\alpha_1)}, k \in \mathbb{Z}^2 \right\|_{L^p} < \infty \tag{7.3}
\]
for some $\varepsilon > 0$, then one can derive $\ell^q$-sparsity with respect to $\alpha_2$-shearlets for $q \geq \max \left\{ p, \left( \frac{1}{2} + \frac{\varepsilon}{|\alpha_1 - \alpha_2|} \right)^{-1} \right\}$.

Observe that equation (7.3) combines an $\ell^p$-estimate with a decay of the coefficients with the scale parameter $j \in \mathbb{N}_0$.

Proof. Theorem 7.3 shows that the embedding can only hold if $p \leq q$. Thus, we only need to show for $0 < p \leq q \leq 2$ that the stated embedding holds if and only if we have $q \geq \left( \frac{1}{2} + \frac{\varepsilon}{|\alpha_1 - \alpha_2|} \right)^{-1}$.
For brevity, let \( s := \varepsilon + (1 + \alpha_1) \left( p^{-1} - 2^{-1} \right) \) and \( r := (1 + \alpha_2) \left( q^{-1} - 2^{-1} \right) \). We start with a few auxiliary observations: Because of \( p \leq q \leq 2 \), we have \( q^p = \min \{q, q'\} = q \) and \( \frac{1}{p+1} = \min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\} = 1 - \frac{1}{p} \), as well as
\[
\frac{1}{q^p} - \frac{1}{p} = \frac{1}{q} - \frac{1}{p} \leq 0 \text{ and } \frac{1}{p} + \frac{1}{q} \geq 1, \text{ so that } \frac{1}{q} - \frac{1}{p} = \frac{1}{q} - 1 + \frac{1}{p} \geq 0.
\]

Now, let us first consider the case \( \alpha_1 < \alpha_2 \). Since we assume \( p \leq q \), Theorem 7.3 shows that the embedding holds if and only if
\[
s \geq r + (1 + \alpha_1) \left( \frac{1}{p} - \frac{1}{q} \right) + (\alpha_2 - \alpha_1) \left( \frac{1}{q^p} - \frac{1}{p} \right) + \epsilon \geq (1 + \alpha_2) \left( q^{-1} - 2^{-1} \right) + (1 + \alpha_1) \left( p^{-1} - q^{-1} \right)
\]
\[
\iff \epsilon \geq (1 + \alpha_2) \left( q^{-1} - 2^{-1} \right) + (1 + \alpha_1) \left( 2^{-1} - q^{-1} \right) = (\alpha_2 - \alpha_1) \left( q^{-1} - 2^{-1} \right)
\]
\[
\iff q \geq \left( 1 + \frac{\epsilon}{\alpha_2 - \alpha_1} \right)^{-1} = \left( \frac{1}{2} + \frac{\epsilon}{|\alpha_2 - \alpha_1|} \right)^{-1}.
\]

Finally, we consider the case \( \alpha_1 > \alpha_2 \). Again, since \( p \leq q \), Theorem 7.3 (with interchanged roles of \( \alpha_1, \alpha_2 \) and \( r, s \)) shows that the desired embedding holds if and only if
\[
s \geq r + (1 + \alpha_2) \left( \frac{1}{p} - \frac{1}{q} \right) + (\alpha_1 - \alpha_2) \left( \frac{1}{q} - \frac{1}{p+1} \right) + \epsilon \geq (1 + \alpha_2) \left( \frac{1}{q} - \frac{1}{2} \right) + (1 + \alpha_2) \left( \frac{1}{p} - \frac{1}{2} \right) + (\alpha_1 - \alpha_2) \left( \frac{1}{q} - 1 + \frac{1}{p} \right)
\]
\[
\iff \epsilon \geq (1 + \alpha_2) \left( p^{-1} - 2^{-1} \right) + (1 + \alpha_1) \left( 2^{-1} - p^{-1} \right) + (\alpha_1 - \alpha_2) \left( q^{-1} - 1 + p^{-1} \right)
\]
\[
\iff \epsilon \geq (\alpha_1 - \alpha_2) \left( 2^{-1} - p^{-1} + q^{-1} - 1 + p^{-1} \right) = (\alpha_1 - \alpha_2) \left( q^{-1} - 2^{-1} \right)
\]
\[
\iff q \geq \left( 1 + \frac{\epsilon}{\alpha_1 - \alpha_2} \right)^{-1} = \left( \frac{1}{2} + \frac{\epsilon}{|\alpha_1 - \alpha_2|} \right)^{-1}.
\]

This completes the proof. \( \square \)

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Appendix A. Nonequivalence of analysis and synthesis sparsity for general frames

In this section, we present two examples which show that for general frames, neither does analysis sparsity imply synthesis sparsity, nor vice versa. We begin with the (easier) case that synthesis sparsity does not imply analysis sparsity:

Example A.1. We consider the Hilbert space \( \ell^2(\mathbb{N}) \) with the standard orthonormal basis given by \((\delta_n)_{n \in \mathbb{N}}\). The family \( \Psi := (\psi_n)_{n \in \mathbb{N}_0} \) given by \( \psi_n := \delta_n \) for \( n \in \mathbb{N} \) and by \( \psi_0 := \frac{1}{\sqrt{2}} \) clearly forms a frame in \( \ell^2(\mathbb{N}) \).

Furthermore, \( f := \psi_0 \) is clearly \( \ell^p \)-synthesis sparse with respect to \( \Psi \) for arbitrary \( p \in (0,2) \), since we have \( f = \sum_{n \in \mathbb{N}_0} c_n \psi_n \) with \((c_n)_{n \in \mathbb{N}} = \delta_0 \in \ell^p(\mathbb{N}_0) \) for all \( p \in (0,2) \). But the analysis coefficients are given by \( A_{\Psi} f = \langle (f, \psi_n) \rangle_{n \in \mathbb{N}_0} \) with \( (f, \psi_n) = \frac{1}{\sqrt{2}} \) for \( n \in \mathbb{N} \). Hence, \( A_{\Psi} f \notin \ell^p(\mathbb{N}_0) \) for \( p \in (0,1) \).

Thus, for general frames, it is not true that \( \ell^p \)-synthesis sparsity implies \( \ell^p \)-analysis sparsity.

Finally, we give a counterexample to the reverse implication. We remark that the counterexample constructed below is in fact a Riesz basis, not simply a frame.

Example A.2. We again consider the Hilbert space \( \ell^2(\mathbb{N}) \) with the standard orthonormal basis given by \((\delta_n)_{n \in \mathbb{N}}\).

Choose some \( N \in \mathbb{N} \) with \( N > \sum_{n=1}^{\infty} \frac{1}{n^2} \) (i.e., \( N > \frac{\pi^2}{6} \approx 1.60 \)) and set

\[
\psi_n := \delta_n - \frac{1}{N \cdot n^2} \sum_{\ell=1}^{N \cdot n^2} \delta_{2n+\ell} \quad \text{for } n \in \mathbb{N}.
\]

Note that \( \psi_n \in \ell^1(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N}) \) with \( \|\psi_n\|_{\ell^1} \leq 1 + \frac{1}{N^2} \sum_{\ell=1}^{N^2} 1 = 2 \). We now want to show that the analysis operator \( A_{\Psi} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \), \( x \mapsto \langle (x, \psi_n) \rangle_{n \in \mathbb{N}} \) associated to the family \( \Psi = (\psi_n)_{n \in \mathbb{N}} \) is well-defined, bounded and invertible. For this, it suffices by a Neumann series argument to show \( \text{sup}_{\|x\|_{\ell^2} \leq 1} \|x - A_{\Psi} x\|_{\ell^2} < 1 \).

But for arbitrary \( x = (x_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}) \), we have

\[
\|x - A_{\Psi} x\|_{\ell^2}^2 = \| (x_n)_{n \in \mathbb{N}} - \langle (x, \psi_n) \rangle_{n \in \mathbb{N}} \|_{\ell^2}^2 = \sum_{n=1}^{\infty} \left( \frac{1}{N \cdot n^2} \sum_{\ell=1}^{N \cdot n^2} x_{2n+\ell} \right)^2
\]

\[
\leq \sum_{n=1}^{\infty} \left( \frac{1}{N \cdot n^2} \sum_{\ell=1}^{N \cdot n^2} \left| x_{2n+\ell} \right|^2 \right)^2
\]

\[
(Cauchy-Schwarz) \leq \sum_{n=1}^{\infty} \left[ \frac{1}{N \cdot n^2} \sum_{\ell=1}^{N \cdot n^2} \left| x_{2n+\ell} \right|^2 \right] \left( \sum_{\ell=1}^{N \cdot n^2} 1 \right)^2
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{N \cdot n^2} \sum_{\ell=1}^{N \cdot n^2} \left| x_{2n+\ell} \right|^2 \leq \sum_{n=1}^{\infty} |x_n|^2 \frac{1}{N} \sum_{n=1}^{\infty} \frac{1}{n^2},
\]

so that we get \( \text{sup}_{\|x\|_{\ell^2} \leq 1} \|x - A_{\Psi} x\|_{\ell^2} \leq \sqrt{\frac{1}{N} \cdot \sum_{n=1}^{\infty} n^{-2}} < 1 \), as desired.

As seen above, this implies that \( A_{\Psi} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) is well-defined, bounded and boundedly invertible. Hence, so is the synthesis operator \( S_{\Psi} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \), \( (c_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} c_n \psi_n \), since \( S_{\Psi} = A_{\Psi}^* \). Therefore, the family \( \Psi = (\psi_n)_{n \in \mathbb{N}} = (S_{\Psi} \delta_n)_{n \in \mathbb{N}} \) is the image of an orthonormal basis under an invertible linear operator, so that \( \Psi \) is a Riesz-basis and in particular a frame for \( \ell^2(\mathbb{N}) \), see [3] Definition 3.6.1, Proposition 3.6.4 and Theorem 3.6.6).

Now, set \( f := \delta_1 \in \ell^p(\mathbb{N}) \) and note \( \text{supp} \psi_n \subset \{ n, n+1, \ldots \} \) for every \( n \in \mathbb{N} \), so that \( \langle f, \psi_n \rangle = 0 \) for all \( n \geq 2 \). Hence, \( A_{\Psi} f = \delta_1 \in \ell^p(\mathbb{N}) \) for all \( p \in (0,2) \), so that \( f \) is analysis sparse with respect to \( \Psi \).

But \( f \) is not \( \ell^p \)-synthesis sparse with respect to \( \Psi \) for \( p \leq 1 \): If \( f = S_{\Psi} c \) for \( c = (c_n)_{n \in \mathbb{N}} \in \ell^p(\mathbb{N}) \hookrightarrow \ell^1(\mathbb{N}) \) with \( p \leq 1 \), then the uniform boundedness \( \|\psi_n\|_{\ell^1} \leq 2 \) ensures that the series \( f = \sum_{n \in \mathbb{N}} c_n \psi_n \) converges unconditionally in \( \ell^1(\mathbb{N}) \). In particular, with the continuous linear functional \( \varphi : \ell^1(\mathbb{N}) \to C, (x_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} x_n \), we would have \( 1 = \varphi(f) = \sum_{n \in \mathbb{N}} c_n \varphi(\psi_n) = 0 \), since \( \varphi(\psi_n) = 0 \) for all \( n \in \mathbb{N} \).

This contradiction shows that \( f \) is not \( \ell^p \)-synthesis sparse with respect to \( \Psi \) for \( p \leq 1 \), even though \( f \) is \( \ell^p \)-analysis sparse.

\[\Diamond\]
APPENDIX B. THE \(\alpha\)-SHEARLET COVERING IS ALMOST STRUCTURED

In this section, we provide the proof of Lemma 3.3, whose statement we repeat here for the sake of convenience:

**Lemma.** The \(\alpha\)-shearlet covering \(S^{(\alpha)}\) from Definition 2.2 is an almost structured covering of \(\mathbb{R}^2\).

**Proof.** First of all, we define the family \((T_iP'_i + b_i)_{i \in I}\), with \(P'_i := U^{(1/2,5/2)}_{(-3/4,3/4)}\) for \(i \in I_0\) and \(P'_0 := (- \frac{3}{4}, \frac{3}{4})^2\); all being open sets. It is not hard to see \(P'_i \subset Q'_i\) for all \(i \in I\). We now show that \((T_iP'_i + b_i)_{i \in I}\) covers \(\mathbb{R}^2\). First, we note

\[
\bigcup_{m=-[2^{n(1-\alpha)}]}^{[2^{n(1-\alpha)}]} \left(2^{n(\alpha-1)} \left(m - \frac{3}{4}\right), 2^{n(\alpha-1)} \left(m + \frac{3}{4}\right)\right) = \bigcup_{m=-[2^{n(1-\alpha)}]}^{[2^{n(1-\alpha)}]} \left(2^{n(\alpha-1)} \left(m - \frac{3}{4}, m + \frac{3}{4}\right)\right)
\]

\[
= 2^{n(\alpha-1)} \left(- [2^{n(1-\alpha)}] - \frac{3}{4}, [2^{n(1-\alpha)}] + \frac{3}{4}\right)
\]

\[
\sup \frac{\alpha}{\infty} \bigg(\begin{array}{c}
\frac{\alpha}{2} - \frac{\alpha}{2} \cdot 2^n \\
\frac{\alpha}{2}
\end{array}\bigg) \times \mathbb{R} \bigg| \frac{\alpha}{\xi} \in [-1, 1] \bigg)
\]

Using this inclusion, as well as equation (3.3), and recalling \(G_n = \left[2^{n(1-\alpha)}\right]\), we conclude

\[
\bigcup_{n=0}^{\infty} \bigg(\begin{array}{c}
\frac{\alpha}{2} - \frac{\alpha}{2} \cdot 2^n \\
\frac{\alpha}{2}
\end{array}\bigg) \times \mathbb{R} \bigg| \frac{\alpha}{\xi} \in [-1, 1] \bigg)
\]

Furthermore, since \(T_{j,\ell,-1,0} = -T_{j,\ell,1,0}\), we have

\[
\bigcup_{n=0}^{\infty} \bigg(\begin{array}{c}
\frac{\alpha}{2} - \frac{\alpha}{2} \cdot 2^n \\
\frac{\alpha}{2}
\end{array}\bigg) \times \mathbb{R} \bigg| \frac{\alpha}{\xi} \in [-1, 1] \bigg)
\]

and since \(R \left(\frac{\alpha}{\xi}\right) = \left(\frac{\alpha}{\xi}\right)\), we finally get

\[
\bigcup_{i \in I_0} T_iP'_i \supset \left\{ \left(\frac{\alpha}{\xi}, \frac{\alpha}{\eta}\right) \in \mathbb{R}^2 \bigg| \frac{\alpha}{\xi} > \frac{1}{2} \text{ and } |\eta| \leq |\xi| \right\} \cup \left\{ \left(\frac{\alpha}{\xi}, \frac{\alpha}{\eta}\right) \in \mathbb{R}^2 \bigg| \frac{\alpha}{\xi} > \frac{1}{2} \text{ and } |\xi| \leq |\eta| \right\} =: M.
\]

Since we clearly have \(T_0P'_0 + b_0 = (- \frac{3}{4}, \frac{3}{4})^2 \supset \left[\frac{-1}{2}, \frac{1}{2}\right]^2\), it suffices to show that each \(\left(\frac{\alpha}{\xi}\right) \in \mathbb{R}^2 \setminus \left[\frac{-1}{2}, \frac{1}{2}\right]^2\) satisfies \(\left(\frac{\alpha}{\xi}\right) \in M\), in order to prove that \((T_iP'_i + b_i)_{i \in I}\) covers all of \(\mathbb{R}^2\). To see this, we distinguish two cases for \(\left(\frac{\alpha}{\xi}\right) \in \mathbb{R}^2 \setminus \left[\frac{-1}{2}, \frac{1}{2}\right]^2\):

**Case 1.** \(|\eta| \geq |\xi|\). Then \(|\eta| > \frac{1}{2}\), since otherwise we would have \(|\xi| \leq |\eta| \leq \frac{1}{2}\), contradicting \(\left(\frac{\alpha}{\xi}\right) \in \mathbb{R}^2 \setminus \left[\frac{-1}{2}, \frac{1}{2}\right]^2\). Hence, \(\left(\frac{\alpha}{\xi}\right) \in M\).

**Case 2.** \(|\eta| \leq |\xi|\). Then \(|\xi| > \frac{1}{2}\), since otherwise we would have \(|\eta| \leq |\xi| \leq \frac{1}{2}\), contradicting \(\left(\frac{\alpha}{\xi}\right) \in \mathbb{R}^2 \setminus \left[\frac{-1}{2}, \frac{1}{2}\right]^2\). Hence, \(\left(\frac{\alpha}{\xi}\right) \in M\).

All in all, we have shown that \((T_iP'_i + b_i)_{i \in I}\) is a covering of \(\mathbb{R}^2\); because of \(Q_i = T_iQ'_i + b_i \subset T_iP'_i + b_i\) for all \(i \in I\), we also see that \(S^{(\alpha)}\) covers all of \(\mathbb{R}^2\). Moreover, the sets \(\{P'_i | i \in I\}\) and \(\{Q'_i | i \in I\}\) are finite; in fact, each of these sets only has two elements. Furthermore, we clearly have \(Q_i = T_iQ'_i + b_i \subset \mathbb{R}^2\) for all \(i \in I\).

Thus, to verify that \(S^{(\alpha)}\) is an almost structured covering of \(\mathbb{R}^2\), we only have to verify that \(S^{(\alpha)}\) is admissible and that \(\sup_{i \in I} \sup_{j \in I_0} \left\|T_i^{-1}T_j\right\|\) is finite, cf. Definition 2.1. To this end, we define

\[
M_i := i^* \cap I_0 \quad \text{and} \quad M_i^{(\nu)} := \{(k, \ell, \beta, \gamma) \in M_i | \gamma = \nu\}, \quad \text{as well as} \quad C_i^{(\nu)} := \sup_{j \in M_i^{(\nu)}} \left\|T_i^{-1}T_j\right\|
\]
for $i \in I_0$ and $\nu \in \{0, 1\}$. Note $M_i = M_i^{(0)} \cup M_i^{(1)}$. Next, for $(k, \ell, \beta, \gamma) \in I_0$, we define
\[
(k, \ell, \beta, \gamma)' := \begin{cases} (k, \ell, \beta, 1), & \text{if } \gamma = 0, \\ (k, \ell, \beta, 0), & \text{if } \gamma = 1. \end{cases}
\]

It is not hard to see $T_i' = RT_i$ and $Q_i' = Q_i = Q$ for all $i \in I_0$. Hence, we have the following equivalence for $i, j \in I_0$:
\[
\emptyset \neq S_i^{(\alpha)} \cap S_j^{(\alpha)} \iff \emptyset \neq T_i Q \cap T_j Q \\
\iff \emptyset \neq R [T_i Q \cap T_j Q] = T_i' Q \cap T_j' Q \\
\iff \emptyset \neq S_i^{(\alpha)} \cap S_j^{(\alpha)}'.
\]

Furthermore,
\[
T_i^{-1} T_j = (RT_i)^{-1} RT_j = T_{i'}^{-1} T_j'.
\]

Hence, $M_i' = \{ j' | j \in M_i \}$ and $C_i^{(\nu)} = C_j^{(\nu)}$ for $\nu \in \{0, 1\}$ and all $i \in I_0$, so that it suffices to consider the case $i = (n, m, \varepsilon, 0) \in I_0$ from now on. We distinguish two cases regarding $j \in M_i$:

**Case 1:** $j = (k, \ell, \beta, 0) \in M_i^{(0)}$. We have \( \emptyset \neq S_i^{(\alpha)} \cap S_j^{(\alpha)} \). Since $S_i^{(\alpha)} \subset \varepsilon (0, \infty) \times \mathbb{R}$ and $S_j^{(\alpha)} \subset \beta (0, \infty) \times \mathbb{R}$, this implies $\varepsilon = \beta$, so that equation (B.3) yields
\[
\emptyset \neq \varepsilon \cdot \left( S_i^{(\alpha)} \cap S_j^{(\alpha)} \right) = S_{n,m,1,0}^{(\alpha)} \cap S_{k,\ell,1,0}^{(\alpha)} = U(2^{n/3}, 2^{n/2}) \cap U(2^{k/3}, 2^{k/2}) \subset (0, \infty) \times \mathbb{R}.
\]

Now, we consider the diffeomorphism $\Phi : (0, \infty) \times \mathbb{R} \to (0, \infty) \times \mathbb{R}, (\xi, \eta) \mapsto \left( \frac{\eta}{\xi}, \frac{9}{\xi} \right)$ and observe the easily verifiable identity $\Phi \left( U(\gamma, \mu) \right) = (\gamma, \mu) \times (a, b)$. Consequently, we get
\[
\emptyset \neq \left[ \left( \frac{a}{3}, \frac{b}{3} \right) \cap \left( \frac{a}{3}, \frac{b}{3} \right) \right] \times \left[ \left( 2^{n(a-1)} (m+1), 2^{n(a-1)} (m+1) \right) \cap \left( 2^{k(a-1)} (\ell+1), 2^{k(a-1)} (\ell+1) \right) \right]
\]

In particular, $\frac{a}{3} < 3 \cdot 2^n$ and $\frac{b}{3} < 3 \cdot 2^k$, which yields $2^{k-n} < 9 \cdot 2^4$ and $2^n-k < 9 \cdot 2^4$. Thus, $|k-n| < 4$ and hence $|k-n| \leq 3$, since $k-n \in \mathbb{Z}$.

Furthermore, we get
\[
2^{k(a-1)} (\ell+1) < 2^{n(a-1)} (m+1) \quad \text{and} \quad 2^{n(a-1)} (m+1) < 2^{k(a-1)} (\ell+1),
\]

which implies
\[
\ell - 1 < 2^{n-k(a-1)} (m+1) \quad \text{and} \quad \ell + 1 > 2^{n-k(a-1)} (m+1).
\]

Because of $0 \leq 1 - \alpha \leq 1$, and $|k-n| \leq 3$, we have $2^{n-k(a-1)} = 2^{1-a(k-n)} \leq 2^3$ and thus
\[
2^{1-a(k-n)} m - 9 \leq -1 - 2^{1-a(k-n)} + 2^{1-a(k-n)} m \leq \ell < 1 + 2^{k-n} - 2^{k-n} 2^{1-a(k-n)} m \leq 2^{1-a(k-n)} m + 9.
\]

Thus, with $M_{n,m,\lambda} := \mathbb{Z} \cap [2^{1-a}(\lambda-n) m - 9, 2^{1-a}(\lambda-n) m + 9]$, we have shown
\[
\{ j = (k, \ell, \beta, 0) | n+3 \leq \lambda = n+3 \} \times \times \{ \varepsilon \} \times \times \{ 0 \}.
\]

Because of $|M_{n,m,\lambda}| \leq 19$, the set on the right-hand side has at most $7 \cdot 19 = 133$ elements, so that we get $|M_i^{(0)}| \leq 133 = N$. Finally, we note
\[
\left\| T_i^{-1} T_j \right\| = \left\| \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & 2^{-k} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 2^{k-n} \\ 2^{n-k} \end{pmatrix} \right\|.
\]

Now, since $|k-n| \leq 3$, we have $0 \leq 2^{k-n} \leq 2^3$ and $0 \leq 2^{n-k} \leq 2^3$. Furthermore, we saw above that $|\ell - 2^{1-a(k-n)} m| \leq 9 \cdot 2^{a(k-n)} \leq 9 \cdot 2^3$. All in all, this implies $\left\| T_i^{-1} T_j \right\| \leq 11 \cdot 2^3 \leq 2^7 = 128$. Since $j \in M_i^{(0)}$ was arbitrary, we conclude $C_i^{(0)} \leq 128 = K$.

**Case 2:** $j = (k, \ell, \beta, 1) \in M_i^{(1)}$. By definition of $M_i$, there is some $(\xi, \eta) \in S_i^{(\alpha)} \cap S_j^{(\alpha)}$. Lemma 5.2 implies $2^{n-2} < |(\xi, \eta)| < 2^{n+4}$, as well as $2^{k-2} < |(\xi, \eta)| < 2^{k+4}$ and thus $2^{n-2} < 2^{k+4}$ as well as $2^{k-2} < 2^{n+4}$. Consequently, $|n-k| < 6$ and thus $|n-k| \leq 5$, since $n-k \in \mathbb{Z}$.
Next, we explicitly compute the transition matrix $T_i^{-1}T_j$:

$$T_i^{-1}T_j = (A_{n,m}^{(a)})^{-1} R_A^{(a)}_{k,\ell,\beta} = \varepsilon \beta \begin{pmatrix} 1 & 0 & 2^{-n} & 0 & 2^{k} & 0 \\ -m & 1 & 0 & 2^{-n} & 2^{\alpha} & 2^{\alpha} \\ 1 & 0 & 0 & 2^{-n} & 2^{k} & 2^{k} \\ \end{pmatrix}.$$

Now we distinguish three different subcases regarding $\alpha \in \{0,1\}$ and $n \in \mathbb{N}_0$:

**Case 2(a):** $\alpha \neq 1$ and $n < \frac{12}{1-\alpha}$. Since $|n-k| < 5$ this implies $k \leq 5 + n < 5 + \frac{12}{1-\alpha} = \frac{17-5\alpha}{1-\alpha} \leq \frac{17}{1-\alpha}$. We thus have

$$M^{(1)}_i \subseteq \bigcup_{k=0}^{[17/(1-\alpha)]} \{ \{k\} \times \{\lceil [2(1-\alpha)]\rceil, \ldots, [\lceil 2(1-\alpha)\rceil]\} \times \{\pm 1\} \times \{0,1\} \} =: M,$$

and hence $|M^{(1)}_i| \leq |M| \leq N_0$, for some absolute constant $N_0 = N_0 (\alpha) \in \mathbb{N}$, since $M$ is a finite set. Note also that $i \in M$, since $n < \frac{12}{1-\alpha} \leq \frac{17}{1-\alpha}$. Consequently,

$$C^{(1)}_i = \sup_{j \in M^{(1)}_i} \|T_i^{-1}T_j\| \leq \max_{\gamma, \lambda \in M} \|T^{-1}_\lambda T\| := K_0.$$

**Case 2(b):** $\alpha \neq 1$ and $n \geq \frac{12}{1-\alpha}$. Since $|n-k| < 5$ this implies $k \geq n - 5 \geq \frac{12}{1-\alpha} - 5 = \frac{7+5\alpha}{1-\alpha}$. We know from Lemma 3.2 that if $0 < \frac{\eta}{\xi} \leq 3$ or $0 < \frac{1}{\xi} < 3 \frac{\eta}{\xi}$, i.e., $\frac{1}{\xi} < \frac{3}{\xi} < 3$.

Now, we claim $|m| \geq \frac{64}{3} - 1$. To see this, assume towards a contradiction that $|m| < \frac{64}{3} - 1$. This implies because of $n \geq \frac{12}{1-\alpha}$, because of equation (3.3) and because of $(\frac{\eta}{\xi}) \in S_{n,m,\varepsilon,0}^{(a)} = S_{n,m,\varepsilon,0}^{\alpha}$ that

$$\frac{\eta}{\xi} \in \left(2^{-n(1-\alpha)} (m-1), 2^{-n(1-\alpha)} (m+1) \right) \cap \left(-\frac{64/3}{2^{n(1-\alpha)}}, \frac{64/3}{2^{n(1-\alpha)}} \right) \subseteq \left(-\frac{64/3}{2^{n(1-\alpha)}}, \frac{64/3}{2^{n(1-\alpha)}} \right) \subseteq \left(-\frac{1}{3}, \frac{1}{3} \right),$$

in contradiction to $\frac{\eta}{\xi} > \frac{1}{3}$. Thus we must have $|m| \geq \frac{64}{3} - 1$.

Likewise, we have $|\ell| \geq \frac{64}{3} - 1$. Indeed, since we have $k \geq \frac{7+5\alpha}{1-\alpha}$ and $1 \leq \frac{\xi}{\eta} < 3 \frac{\eta}{\xi}$, the assumption $|\ell| < \frac{64}{3} - 1$ yields the contradiction

$$\frac{\xi}{\eta} \in \left(2^{-(k(1-\alpha)} (\ell-1), 2^{-(k(1-\alpha)} (\ell+1) \right) \cap \left(-\frac{64/3}{2^{k(1-\alpha)}}}, \frac{64/3}{2^{k(1-\alpha)}} \right) \subseteq \left(-\frac{64/3}{2^{k(1-\alpha)}}, \frac{64/3}{2^{k(1-\alpha)}} \right) \subseteq \left(-\frac{1}{3}, \frac{1}{3} \right).$$

Consequently, we must have $|\ell| \geq \frac{64}{3} - 1$.

Now, since $|m| \geq \frac{64}{3} - 1$, we either have $m \geq \frac{64}{3} - 1 > 0$ or $m \leq 1 - \frac{64}{3} < 0$. Let us distinguish these two cases:

**Case 2(b)(i):** $m \geq \frac{64}{3} - 1$. Since $(\frac{\eta}{\xi}) \in S_{n,m,\varepsilon,0}^{(a)} \cap S_{k,\ell,\beta,0}^{(a)} = S_{n,m,\varepsilon,0}^{(a)} \cap S_{k,\ell,\beta,0}^{(a)}$ and using equation (3.3), we see $\frac{\eta}{\xi} > 2^{-n(1-\alpha)} (m-1) > 0$ and $0 < \frac{\xi}{\eta} < 2^{-k(1-\alpha)} (\ell+1)$. Hence, $\ell > -1$ and since $|\ell| \geq \frac{64}{3} - 1$, we have $\ell \geq \frac{64}{3} - 1$.

First, we want to show $m \geq 2^{n(1-\alpha)} - 65$. Thus, assume towards a contradiction that $m < 2^{n(1-\alpha)} - 65$ and note that $2^{n(1-\alpha)} - 2^{6} - 1 = 2^{n(1-\alpha)} - 65 > 2^{2^{6}} - 65 > 0$, since $n \geq \frac{12}{1-\alpha}$. Now, we get

$$\frac{\xi}{\eta} < 2^{-k(1-\alpha)} (\ell+1) \leq 2^{-k(1-\alpha)} (2^{k(1-\alpha)} + 1) < 2^{-k(1-\alpha)} (2^{k(1-\alpha)} + 1 + 1) = 1 + 2^{-k(1-\alpha)+1}$$

and

$$\frac{\xi}{\eta} = \left(\frac{\eta}{\xi}\right)^{-1} > \left(2^{-n(1-\alpha)} (m+1) \right)^{-1} = \frac{2^{n(1-\alpha)}}{m+1} > \frac{2^{n(1-\alpha)}}{2^{n(1-\alpha)} - 2^{6} - 1} > 1 + \frac{2^{6}}{2^{n(1-\alpha)} - 2^{6}}.$$

Thus $2^{n(1-\alpha)+1} > \frac{2^{6}}{2^{n(1-\alpha)} - 2^{6}}$ and hence $2^{n-k(1-\alpha)} > 2^{5}$ in contradiction to $2^{n-k(1-\alpha)} \leq 2^{n-k(1-\alpha)} \leq 2^{n-k(1-\alpha)} \leq 2^{5}$. Thus, $m \geq 2^{n(1-\alpha)} - 65$.

Next, we similarly show $\ell \geq 2^{k(1-\alpha)} - 65$. Again, we assume towards a contradiction that $\ell < 2^{k(1-\alpha)} - 2^{6} - 1$ and note $2^{k(1-\alpha)} - 2^{6} - 1 \geq 2^{7+5\alpha} - 2^{6} - 1 > 0$. Now, on the one hand we get

$$\left(\frac{\xi}{\eta}\right)^{-1} > \left(2^{-k(1-\alpha)} (\ell+1) \right)^{-1} = \frac{2^{k(1-\alpha)}}{\ell+1} > \frac{2^{k(1-\alpha)}}{2^{k(1-\alpha)} - 2^{6} - 1} > 1 + \frac{2^{6}}{2^{k(1-\alpha)} - 2^{6}}.$$
but on the other hand
\[
\left( \frac{k}{\eta} \right)^{-1} = \frac{\eta}{k} < 2^{-n(1-\alpha)} (m+1) \leq 2^{-n(1-\alpha)} \left( \left[2^{n(1-\alpha)}\right] + 1 \right) < 2^{-n(1-\alpha)} \left(2^{n(1-\alpha)} + 2\right) = 1 + 2^{-n(1-\alpha)+1},
\]
i.e., \(2^{(k-n)(1-\alpha)} > 2^5\) in contradiction to \(|n-k| \leq 5\). Thus, \(\ell \geq 2^{k(1-\alpha)} - 2^6 - 1 = 2^{k(1-\alpha)} - 65\).

Using these estimates for \(m\) and \(\ell\), we can now bound the entries of \(T_i^{-1}T_j\) (cf. eq. (B.2)): We have
\[
|2^{k\alpha-n}\ell| \leq 2^{k\alpha - n \left[2^{k(1-\alpha)}\right]} < 2^{k\alpha - n \left(2^{k(1-\alpha)} + 1\right)} = 2^{k-n} + 2^{k\alpha-n} \leq 2^5 + 2^{k-n} \leq 2 \cdot 2^5
\]
and furthermore \(|2^{k\alpha-n}| \leq 2^{k-n} \leq 2^5\), as well as
\[
|2^{k\alpha-n}m| = 2^{k\alpha-n}|m| \leq 2^{k\alpha-n} \left[2^{n(1-\alpha)}\right] < 2^{k\alpha-n} \left(2^{n(1-\alpha)} + 1\right) = 2^{(k-n)\alpha} + 2^{k\alpha-n} \leq 2^5 + 2^{k-n} \leq 2^5 + 2^5.
\]
Finally, having in mind
\[
0 \leq \ell m \leq \left(2^{k(1-\alpha)} + 1\right) \left(2^{n(1-\alpha)} + 1\right) = 2^{n(1-\alpha)}2^{k(1-\alpha)} + 2^{k(1-\alpha)} + 2^{n(1-\alpha)} + 1,
\]
as well as \(\ell \geq 2^{k(1-\alpha)} - 65 > 0\) and \(m \geq 2^{n(1-\alpha)} - 65 > 0\), we get
\[
|2^{k\alpha-n} - 2^{k\alpha-n}\ell m| = 2^{k\alpha-n} \left|2^{n(1-\alpha)+k(1-\alpha)} - \ell m\right| \leq 2^{k\alpha-n} \left(2^{n(1-\alpha)+k(1-\alpha)} + 2^{k(1-\alpha)} + 2^{n(1-\alpha)} + 1 - \ell m\right) + \left|-2^{k(1-\alpha)} - 2^{n(1-\alpha)} - 1\right|
\]
\[
= 2^{k\alpha-n} \left(2^{n(1-\alpha)+k(1-\alpha)} + 2^{k(1-\alpha)} + 2^{n(1-\alpha)} + 1 - \ell m\right) \leq 2^{k\alpha-n} \left(2^{n(1-\alpha)+k(1-\alpha)} - \left(2^{k(1-\alpha)} - 65\right) \left(2^{n(1-\alpha)} - 65\right) + 2 \cdot \left(2^{k(1-\alpha)} + 2^{n(1-\alpha)} + 1\right)\right)
\]
\[
\leq 2^{k\alpha-n} \left(67 \cdot 2^{k(1-\alpha)} + 67 \cdot 2^{n(1-\alpha)} - 65 + 2 \cdot \left(2^{k(1-\alpha)} + 2^{n(1-\alpha)} + 1\right)\right)
\]
\[
\leq 2^5 \left(67 + 67 \cdot 2^{n-k(1-\alpha)} + 2 \cdot 2^{k(1-\alpha)}\right)
\]
\[
\leq 2^5 \left(67 + 67 \cdot 2^5 + 2\right) = 70816.
\]
Thus, we have \(\|T_i^{-1}T_j\| \leq 2^5 + 2^5 + 2^5 + 70816 = 70976 =: K_1\) for all \(j \in M_1^{(1)}\), as long as \(\alpha \neq 1\) and \(i = (n, m, \varepsilon, 0) \in I_0\) with \(n \geq \frac{12}{4\alpha}\) and \(m \geq \frac{44}{1\alpha} - 1\).

**Case 2(b)(ii):** \(m \leq -\frac{44}{1\alpha} + 1\). Then we have \(\frac{\xi}{\eta} < 2^{-n(1-\alpha)} (m+1) < 0\) and \(2^{-k(1-\alpha)} (\ell - 1) - \frac{\xi}{\eta} < 0\). Hence, \(\ell < 1\) and since \(|\ell| \geq \frac{64}{1\alpha} - 1\), we have \(\ell \leq -\frac{64}{1\alpha} + 1\). Setting \(\tilde{m} := -m\) and \(\tilde{\ell} := -\ell\) and using \(-\frac{\xi}{\eta}, -\frac{\xi}{\eta}\) instead of \(\frac{\xi}{\eta}, \frac{\xi}{\eta}\) we get, with the same arguments as in the previous case, that \(\tilde{m} \geq 2^{n(1-\alpha)} - 65\) and \(\tilde{\ell} \geq 2^{k(1-\alpha)} - 65\), i.e. \(m \leq -2^{n(1-\alpha)} + 65\) and \(\ell \leq -2^{k(1-\alpha)} + 65\). Consequently, since \(\tilde{m} \tilde{\ell} = m\ell\) and \(|m| = |\tilde{m}|\), as well as \(|\ell| = |\tilde{\ell}|\), we get the same bounds for the matrix entries as in the previous case. Thus, \(\|T_i^{-1}T_j\| \leq K_1\).

All in all, since the cases 2(b)(i) and 2(b)(ii) are the only ones possible—assuming that we are in case 2(b)—we get \(C_n^{(1)} \leq K_1\) if \(\alpha \neq 1\) and if \(i = (n, m, \varepsilon, 0)\) satisfies \(n \geq \frac{12}{4\alpha}\). Finally, in both of the cases from above, we saw that \(\ell \leq -2^{k(1-\alpha)} + 65 \leq -\left[2^{k(1-\alpha)}\right] + 66\) or that \(\ell \geq 2^{k(1-\alpha)} - 65 \geq \left[2^{k(1-\alpha)}\right] - 66\). Consequently, we get for the whole case 2(b) that \(M_1^{(1)} \subset M\)

\[
\tilde{M} := \bigcup_{\lambda = n-5}^{n+5} \{\lambda\} \times \{\left[2^\lambda(1-\alpha)\right] - 66, \ldots, [2^\lambda(1-\alpha)]\} \cup \{-\left[2^\lambda(1-\alpha)\right], \ldots, -\left[2^\lambda(1-\alpha)\right] + 66\} \times \{\pm 1\} \times \{1\}
\]
and thus \(|M_1^{(1)}| \leq |\tilde{M}| \leq 11 \cdot 2 \cdot 67 \cdot 2 = 2948 =: N_1\), independent of \(i = (n, m, \varepsilon, 0) \in I_0\), as long as \(\alpha \neq 1\) and \(n \geq \frac{12}{4\alpha}\).

**Case 2(c):** \(\alpha = 1\). In this case, the matrix \(T_i^{-1}T_j\) from equation (B.2) reduces to
\[
T_i^{-1}T_j = \varepsilon \beta \left( \begin{array}{cc} 2^{k-n}\ell & 2^{k-n} \ell m \\ 2^{k-n} - 2^{k-n}\ell m & -2^{k-n}m \end{array} \right)
\]
and we have $|m| \leq G_n = 1$, as well as $|\ell| \leq G_k = 1$. Thus, recalling $|n - k| \leq 5$, we can easily bound all matrix elements uniformly: We have $|2k-\ell| = 2k-\ell \leq 2^5$ and $|2k-n| \leq 2^5$, as well as $-2k-n \leq 2^k-n \leq 2^5$ and finally

$$|2k-n - 2k-\ell m| = 2k-n|1-\ell m| \leq 2^k-n (1 + |\ell| \cdot |m|) \leq 2^5 \cdot 2$$

and thus $\|T_i^{-1}T_j\| \leq 2^5 + 2^5 + 2^5 + 2 \cdot 2^5 = 160 =: K_2$, independent of $i = (n,m,\varepsilon,0) \in I_0$, as long as $\alpha = 1$.

Furthermore, since we saw above that $|k-n| \leq 5$ for $j = (k,\ell,1) \in M_i^{(1)}$, we get

$$M_i^{(1)} \subset \bigcup_{\lambda = n-5}^{n+5} \{|\lambda\} \times \{-1,0,1\} \times \{\pm 1\} \times \{1\}$$

and thus $|M_i^{(1)}| \leq 11 \cdot 3 \cdot 2 = 66 =: N_2$.

All in all, the cases $2(a)$, $2(b)$ and $2(c)$ entail for $i = (n,m,\varepsilon,0) \in I_0$ that

$$C_i^{(1)} \leq K_3 := \begin{cases} K_2, & \text{if } \alpha = 1, \\ \max\{K_0,K_1\}, & \text{if } \alpha \neq 1 \end{cases} \quad \text{and also} \quad |M_i^{(1)}| \leq N_3 := \begin{cases} N_2, & \text{if } \alpha = 1, \\ \max\{N_0,N_1\}, & \text{if } \alpha \neq 1. \end{cases}$$

Furthermore, putting cases 1 and 2 together yields for arbitrary $i = (n,m,\varepsilon,0) \in I_0$ that

$$C_i := \sup_{j \in M_i} \|T_i^{-1}T_j\| = \max\{C_i^{(0)},C_i^{(1)}\} \leq \max\{K,K_3\} =: K_4$$

and

$$|M_i| = |M_i^{(0)} \cup M_i^{(1)}| \leq |M_i^{(0)}| + |M_i^{(1)}| \leq N + N_3 =: N_4.$$
APPENDIX C. THE PROOF OF LEMMA C.1

In this section, we provide the (highly technical and lengthy) proof of Lemma [4.1] For this proof, the following lemma will turn out to be extremely useful.

**Lemma C.1.** For \( f : \mathbb{R}^d \to \mathbb{C} \) and \( \theta \in [0, \infty) \), define \( \|f\|_{\theta} := \sup_{x \in \mathbb{R}^d} (1 + |x|)^{\theta} |f(x)| \in [0, \infty] \).

Then, for each \( N \in [0, \infty) \) and \( p \in (0, \infty) \), arbitrary \( \beta, L > 0 \) and \( M \in \mathbb{R} \) and all measurable \( f : \mathbb{R}^d \to \mathbb{C} \) we have

\[
\sum_{k \in \mathbb{Z}} |\beta k + M|^N \left( \int_{\beta k + M - L}^{\beta k + M + L} |f(x)| \, dx \right)^p \leq 2^{1+p} \cdot 10^{N+3} \cdot \|f\|_{\frac{p}{p(\lambda + 2)}}^p \cdot (1 + L)^{N} \cdot \left(1 + \frac{L+1}{\beta}\right). \]

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**Remark.** Note that \((1+\theta)^N \leq [2 \cdot \max\{1, \theta\}]^N \leq 2^N \cdot \max\{1, \theta^N\} \leq 2^N \cdot (1 + \theta^N)\) for arbitrary \( \theta \geq 0 \), so that an application of the preceding lemma with \( N = \lambda \) for \( \lambda \in \{0, N\} \) yields

\[
\sum_{k \in \mathbb{Z}} (1 + |\beta k + M|)^N \left( \int_{\beta k + M - L}^{\beta k + M + L} |f(x)| \, dx \right)^p \leq 2^N \cdot \sum_{\lambda \in \{0, N\}} \sum_{k \in \mathbb{Z}} |\beta k + M|^{\lambda} \left( \int_{\beta k + M - L}^{\beta k + M + L} |f(x)| \, dx \right)^p \leq 2^{3+p+N} \cdot 10^{N+3} \cdot \|f\|_{\frac{p}{p(\lambda + 2)}}^p \cdot (1 + L)^{N} \cdot \left(1 + \frac{L+1}{\beta}\right). \]

Here, the last step used that we have \( \lambda \leq N \) and hence \( \|f\|_{\frac{p}{p(\lambda + 2)}} \leq \|f\|_{\frac{p}{p(N+2)}} \) for \( \lambda \in \{0, N\} \) and furthermore that \( 1 + L^\lambda = 2 \leq 2 \cdot (1 + L^N) \) for \( \lambda = 0 \) and trivially \( 1 + L^\lambda \leq 2 \cdot (1 + L^N) \) for \( \lambda = N \).

**Proof.** Since otherwise the claim is trivial, we can assume \( \|f\|_{\frac{p}{p(N+2)}} < \infty \). We distinguish three cases for \( k \in \mathbb{Z} \):

**Case 1:** We have \( \beta k + M \geq 10 \cdot L > 0 \). This implies \( x \geq \beta k + M - L \geq \frac{9}{10} (\beta k + M) > 0 \) for arbitrary \( x \in [\beta k + M - L, \beta k + M + L] \) and hence

\[
|f(x)| \leq \|f\|_{\frac{p}{p(N+2)}} \cdot (1 + |x|)^{-\frac{p}{p(N+2)}} \leq \|f\|_{\frac{p}{p(N+2)}} \cdot \left(1 + \frac{9}{10} (\beta k + M)\right)^{-\frac{p}{p(N+2)}} \leq \left(\frac{10}{9}\right)^{\frac{p}{p(N+2)}} \cdot \|f\|_{\frac{p}{p(N+2)}} \cdot (1 + |\beta k + M|)^{-\frac{p}{p(N+2)}}.
\]

This yields

\[
|\beta k + M|^N \left( \int_{\beta k + M - L}^{\beta k + M + L} |f(x)| \, dx \right)^p \leq \left(\frac{10}{9}\right)^{\frac{p}{p(N+2)}} \cdot (2L)^p \cdot (1 + |\beta k + M|)^{-2}. \quad (C.2)
\]

**Case 2:** We have \( \beta k + M \leq -10 \cdot L < 0 \). This implies \( x \leq \beta k + M + L \leq \frac{9}{10} (\beta k + M) < 0 \) and hence \( |x| \geq \frac{9}{10} |\beta k + M| \) for arbitrary \( x \in [\beta k + M - L, \beta k + M + L] \). This easily implies that estimate (C.2) also holds in this case.

**Case 3:** We have \( |\beta k + M| \leq 10 \cdot L \). In this case, we have \( -10 \cdot L \leq \beta k + M \leq 10 \cdot L \) and hence

\[
-10L - M \leq \beta k \leq \frac{10L - M}{\beta},
\]
which implies \( k \in \mathbb{Z} \cap \left[ -\frac{M}{\beta} - \frac{10L}{\beta}, \frac{M}{\beta} + \frac{10L}{\beta} \right] \). But every (closed) interval \( I \) of length \( R \geq 0 \) satisfies \( |I \cap \mathbb{Z}| \leq 1 + R \), so that there are at most \( 1 + \frac{20L}{\beta} \) possible values of \( k \) for which the present case is satisfied. Hence,

\[
\sum_{k \in \mathbb{Z} \atop \beta k + M \leq (10L)^N} |\beta k + M|^N \left( \int_{\beta k + M - L}^{\beta k + M + L} |f(x)| \, dx \right)^p \leq 10^N \| f \|_{L^\infty}^p \cdot \left( 1 + 20 \frac{L}{\beta} \right) \cdot L^N \cdot (2L)^p \\
\leq 2^p \cdot 10^N \cdot \| f \|_{P\cdot(\mathbb{N}+2)}^p \cdot L^{N+p} \cdot \left( 1 + 20 \frac{L}{\beta} \right) .
\]

All in all, we arrive at

\[
\sum_{k \in \mathbb{Z} \atop \beta k + M \leq (10L)^N} \left( \int_{\beta k + M - L}^{\beta k + M + L} |f(x)| \, dx \right)^p \\
\leq 2^p 10^N \| f \|_{P\cdot(\mathbb{N}+2)}^p L^{N+p} \cdot \left( \frac{10}{9} \right)^N + 2^p \left( \frac{1}{9} \right) \| f \|_{P\cdot(\mathbb{N}+2)}^p L^p \cdot \sum_{k \in \mathbb{Z} \atop |\beta k + M| \geq (10L)^N} (1 + |\beta k + M|)^{-2} .
\]

(C.3)

Now, define \( g : \mathbb{R} \to [0, \infty], x \mapsto \sum_{k \in \mathbb{Z}} (1 + |\beta (k + x)|)^{-2} \) and note that \( g \) is 1-periodic and also that

\[
\sum_{k \in \mathbb{Z} \atop |\beta k + M| \geq (10L)^N} (1 + |\beta k + M|)^{-2} \leq \sum_{k \in \mathbb{Z}} (1 + |\beta k + M|)^{-2} = \sum_{k \in \mathbb{Z}} \left( 1 + \left| \beta \left( k + \frac{M}{\beta} \right) \right| \right)^{-2} = g \left( \frac{M}{\beta} \right) .
\]

Our next goal is to show \( g(x) \leq 2 + \frac{10}{\beta} \) for all \( x \in \mathbb{R} \). Since \( g \) is 1-periodic, it suffices to consider \( x \in [0, 1] \). Now, we again distinguish three cases regarding \( k \in \mathbb{Z} \):

**Case 1:** We have \( k \geq \frac{1}{\beta} \) and hence \( \beta (k + x) \geq \beta k \geq 1 \). This implies

\[
\sum_{k \geq \frac{1}{\beta}} (1 + |\beta (k + x)|)^{-2} \leq \sum_{k \geq \frac{1}{\beta}} (\beta k)^{-2} = \beta^{-2} \cdot \sum_{k \geq \frac{1}{\beta}} k^{-2} .
\]

Now, note for arbitrary \( y > 0 \) that for \( n \in \mathbb{Z} \), we have \( n \geq y > 0 \) and hence \( n \geq 1 \), which implies \( n + 1 \leq 2n \), so that we get for \( z \in [n, n+1] \) the estimate \( z^{-2} \geq (n+1)^{-2} \geq (2n)^{-2} = n^{-2}/4 \) and hence

\[
\sum_{n \in \mathbb{Z} \geq y} n^{-2} = \sum_{n \geq y} \int_{n-1}^{n+1} n^{-2} \, dz \leq 4 \sum_{n \geq y} \int_{n-1}^{n+1} z^{-2} \, dz \leq 4 \cdot \int_{y}^{\infty} z^{-2} \, dz = 4 \cdot \frac{z^{-1}}{|z|_1} \bigg|_{z=y}^{z=\infty} = \frac{4}{y} .
\]

Thus, \( \sum_{k \geq \frac{1}{\beta}} (1 + |\beta (k + x)|)^{-2} \leq \beta^{-2} \cdot \sum_{k \geq \frac{1}{\beta}} k^{-2} \leq \beta^{-2} \cdot \frac{4}{y} = \frac{4}{\beta} \).

**Case 2:** We have \( k \leq -\frac{1}{\beta} - 1 \), which entails \(- (k + 1) \geq \frac{1}{\beta} \). For \( x \in [0, 1] \), this implies

\[
\beta (k + x) \leq \beta (k + 1) \leq \beta \left( -\frac{1}{\beta} \right) = -1 < 0 \quad \text{and hence} \quad |\beta (k + x)| = -\beta (k + x) \geq -\beta (k + 1) > 0 ,
\]

so that we get

\[
\sum_{k \leq \frac{1}{\beta} - 1} (1 + |\beta (k + x)|)^{-2} \leq \sum_{k \in \mathbb{Z} \leq \frac{1}{\beta} - 1} (-\beta (k + 1))^{-2} \\
\quad \text{(with} \ t = -(k+1)\text{)} = \sum_{\ell \in \mathbb{Z} \geq 1/\beta} (\beta \ell)^{-2} \\
\quad \text{(as above)} \leq \beta^{-2} \cdot \frac{4}{\beta} = \frac{4}{\beta} .
\]

**Case 3:** We have \(-\frac{1}{\beta} - 1 \leq k \leq \frac{1}{\beta} \) and hence \( k \in \mathbb{Z} \cap \left[ -\frac{1}{\beta} - 1, \frac{1}{\beta} \right] \), so that there are at most \( 2 + \frac{2}{\beta} \) possible values of \( k \) for which this case holds. Hence,

\[
\sum_{k \in \mathbb{Z} \atop -\frac{1}{\beta} - 1 \leq k \leq \frac{1}{\beta}} (1 + |\beta (k + x)|)^{-2} \leq 2 \left( 1 + \frac{1}{\beta} \right) ,
\]

Summarizing all three cases, we easily see \( g(x) \leq 2 + \frac{4}{\beta} + 2 \left( 1 + \frac{1}{\beta} \right) = 2 + \frac{10}{\beta} \) for all \( x \in \mathbb{R} \), as claimed.
Returning to the proof of the claim of the lemma, we recall from equation \((C.3)\) (and the displayed equation after that) that we have

\[
\sum_{k \in \mathbb{Z}} |\beta k + M|^N \left( \int |f(x)|^p \, dx \right)^p \\
\leq 2^p \cdot 10^N \|f\|^{p^+}_{p/(N+2)} \cdot L^{N+p} \cdot \left(1 + \frac{20L}{\beta}\right) + 2^p \left( \frac{10}{9} \right)^{N+2} \|f\|^{p^+}_{p/(N+2)} \cdot L^p \cdot g\left(\frac{M}{\beta}\right) \\
\leq 2^p \cdot 10^{N+2} \cdot \|f\|^{p^+}_{p/(N+2)} \cdot \left[ \left(1 + 20\frac{L}{\beta}\right) + L^p \cdot g\left(\frac{M}{\beta}\right) \right] \\
\leq 2^{1+p} \cdot 10^{N+3} \cdot \|f\|^{p^+}_{p/(N+2)} \cdot L^p \cdot \left[ L^N \cdot \left(1 + \frac{L}{\beta}\right) + \left(1 + \frac{1}{\beta}\right) \right] \\
\leq 2^{1+p} \cdot 10^{N+3} \cdot \|f\|^{p^+}_{p/(N+2)} \cdot L^p \cdot \left(1 + L^N\right) \cdot \max \left\{1 + \frac{L}{\beta}, 1 + \frac{1}{\beta}\right\} \\
\leq 2^{1+p} \cdot 10^{N+3} \cdot \|f\|^{p^+}_{p/(N+2)} \cdot L^p \cdot \left(1 + \frac{L}{\beta}\right),
\]

which completes the proof. \(\Box\)

The proof of Lemma 4.1 will occupy the whole remainder of this section. In fact, we divide the remainder of this section into several subsections, each of which handles a certain subset of the whole set of pairs \((i, j) \in I^2\). Precisely, we define for \((e, d) \in \{\pm 1\} \times \{0, 1\}\) the set

\[
I^{(e,d)} := \{(n, m, \varepsilon, \delta) \in I_0 \mid \varepsilon = e \text{ and } \delta = d\}.
\]

Furthermore, we set \(I^{(0)} := \{0\}\) and \(L := \{0\} \cup \{(\pm 1) \times \{0, 1\}\}\). Then \(I = \bigcup_{\ell \in L} I^{(\ell)}\), so that

\[
\sup_{i \in I} \sum_{j \in I} M_{j,i}^{(0)} \leq \sum_{\ell_1 \in L} \sup_{\ell_2 \in I^{(\ell_1)}} \sum_{j \in I^{(\ell_2)}} M_{j,i}^{(0)} \leq \sum_{\ell_1, \ell_2 \in L} \sup_{i \in I^{(\ell_1)}} \sum_{j \in I^{(\ell_2)}} M_{j,i}^{(0)} \tag{C.4}
\]

and likewise

\[
\sup_{j \in I} \sum_{i \in I} M_{j,i}^{(0)} \leq \sum_{\ell_1, \ell_2 \in L} \sup_{j \in I^{(\ell_2)}} \sum_{i \in I^{(\ell_1)}} M_{j,i}^{(0)}. \tag{C.5}
\]

Now, each of the subsections of this section handles a specific choice of \(\ell_1, \ell_2 \in L\), which in principle are 25 cases. Luckily, it will turn out that many of these cases can be handled completely analogously, so that the actual number of subsections is smaller.

We first only consider the case \(\ell_1, \ell_2 \in \{\pm 1\} \times \{0, 1\}\). Then, \(I^{(\ell_1)}, I^{(\ell_2)} \subset I_0\), so that \(g_j = g_1 = g\) and so that \(i \in I^{(\ell_1)}\) and \(j \in I^{(\ell_2)}\) are of the form \(i = (n, m, \varepsilon, \delta)\) and \(j = (\nu, \mu, e, d)\) for certain \(n, \nu \in \mathbb{N}_0, m, \mu \in \mathbb{Z}\) with \(|m| \leq G_n\) and \(|\mu| \leq G_\nu\) and certain \(\varepsilon, e \in \{\pm 1\}\) and \(\delta, d \in \{0, 1\}\). We will keep this convention throughout the section, without mentioning it explicitly.

In the remainder of the proof, the notation \(x_+ := (x)_+ := \max \{0, x\}\) for \(x \in \mathbb{R}\) will be frequently useful. We immediately observe \(2^{x_+} = \max \{1, 2^x\}\) and \(\min \{1, 2^x\} = 2^{-(x)_+}\).

Next, we collect two estimates concerning \(\theta_1, \theta_2\) that will frequently be useful: First, if \(C^{-1} \leq \eta \leq C\) for some \(C \geq 1\), then \(1 + |\eta\xi| \geq 1 + C^{-1} |\xi| \geq C^{-1} \cdot (1 + |\xi|)\) and thus

\[
\theta_1 (\eta\xi) = \min \left\{|\eta\xi|^{M_1}, (1 + |\eta\xi|)^{-M_2}\right\} \\
\leq \min \left\{ C^{M_1} \cdot |\xi|^{M_1}, C^{M_2} \cdot (1 + |\xi|)^{-M_2}\right\} \leq C^{M_3} \cdot \theta_1 (\xi) \tag{C.6}
\]

for arbitrary \(\xi \in \mathbb{R}\) and \(M_3 := \max \{M_1, M_2\}\).

Finally, if \(\eta \geq C\) for some \(C \in (0, 1]\), then \(1 + |\eta\xi| \geq 1 + C \cdot |\xi| \geq C \cdot (1 + |\xi|)\), so that

\[
\theta_2 (\eta\xi) = (1 + |\eta\xi|)^{-K} \leq C^{-K} \cdot (1 + |\xi|)^{-K} = C^{-K} \cdot \theta_2 (\xi) \quad \forall \xi \in \mathbb{R}. \tag{C.7}
\]

Now, we properly start the proof of Lemma 4.1 by distinguishing the different values of \(\ell_1, \ell_2 \in L\).
C.1. We have $\ell_1 = \ell_2 = (1,0)$. For brevity, let $\ell := (1,0)$. Geometrically, the present case means that $i, j \in I(\ell)$ both belong to the right cone, i.e., $\varepsilon = c = 1$ and $\delta = d = 0$. Thus, we have

$$T_j^{-1}T_i = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix} \begin{pmatrix} 2^{-\nu} & 0 \\ 0 & 2^{-\alpha \nu} \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 2^{\nu} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 2^{\alpha(n-\nu)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 2^{\alpha(n-\nu)} \end{pmatrix}$$

and hence, since $2^{\alpha(n-\nu)} \leq 2^{\alpha(n-\nu)_+} \leq 2^{(n-\nu)_+}$ and $2^{-\nu} \leq 2^{(n-\nu)_+}$,

$$\|T_j^{-1}T_i\| \leq 2 \cdot \left(2^{(n-\nu)_+} + \omega_{n,m,\nu,\mu}\right) \leq 2 \cdot 2^{(n-\nu)_+} \cdot (1 + \omega_{n,m,\nu,\mu}) \quad \text{for} \quad \omega_{n,m,\nu,\mu} := \left|2^{\alpha(n-\nu)} m - 2^{n-\nu}\right|,$$

which finally yields

$$(1 + \|T_j^{-1}T_i\|)^\sigma \leq 3^\sigma \cdot 2^{\sigma(n-\nu)_+} \cdot (1 + \omega_{n,m,\nu,\mu})^\sigma.$$  \hfill (C.8)

On the other hand, with $\varrho, \theta_1, \theta_2$ as in equation (11), we have because of $\theta_j = \varrho$ that

$$|\det T_i|^{-1} \cdot \int_{I(\ell)} \varrho \left(T_j^{-1} \xi\right) \, d\xi = |\det T_i|^{-1} \cdot \int_{T_i Q} \varrho \left(T_j^{-1} \xi\right) \, d\xi$$

$$(\xi \in \Omega) = \int_{\overline{Q}} \varrho \left(T_j^{-1} \xi\right) \, d\eta$$

$$(\xi = \varrho, \text{cf. Def. 3.1}) = \int_{1/3} \int_{\mathbb{R}} \left(\frac{\eta \varrho}{\eta_1}\right) \cdot \varrho \left(2^{\alpha(n-\nu)} m - 2^{n-\nu} \eta_1 + 2^{\alpha(n-\nu)} \eta_2\right) \, d\eta_2 \, d\eta_1$$

$$(\xi = \varrho) \text{in inner integral} = \int_{1/3} \eta_1 \left(2^{\alpha(n-\nu)} \eta_1\right) \cdot \theta_2 \left(2^{\alpha(n-\nu)} m - 2^{n-\nu} \eta_1 + 2^{\alpha(n-\nu)} \eta_1\right) \, d\eta_1 \, d\eta_1$$

$$\leq 3 \cdot \int_{1/3} \theta_1 \left(2^{\alpha(n-\nu)} \eta_1\right) \cdot \int_{-1}^{1} \left(1 + \eta_1\right) \cdot \left(2^{\alpha(n-\nu)} m - 2^{n-\nu} \mu + 2^{\alpha(n-\nu)} \xi\right)^{-K} \, d\xi \, d\eta_1$$

$$(\eta_1 = 2^{\alpha(n-\nu)} m - 2^{n-\nu} \mu + 2^{\alpha(n-\nu)} \xi) = 3^{K+1} \cdot 2^{\alpha(n-\nu)} \cdot \int_{1/3} \theta_1 \left(2^{\alpha(n-\nu)} \eta_1\right) \, d\eta_1 \cdot \int_{-1}^{1} \left(1 + \left(2^{\alpha(n-\nu)} m - 2^{n-\nu} \mu + 2^{\alpha(n-\nu)} \xi\right)^{-K} \, d\xi \right.$$  \hfill (C.9)

Now, since the assumptions of Lemma 4.1 ensure $K \geq \frac{1}{3} (\sigma + 2)$, an application of Lemma C.1 and of the associated remark (with $\varphi = \tau \in (0, \infty)$, $\beta = 2^{\alpha(n-\nu)} > 0$, $N = \sigma \geq 0$, $M = -2^{n-\nu} \mu \in \mathbb{R}$ and $L = 2^{\alpha(n-\nu)} > 0$) yields

$$\sum_{m \in \mathbb{Z}} \left(1 + \left|2^{\alpha(n-\nu)} m + (2^{n-\nu})\right|\right)^\sigma \left[\int_{2^{\alpha(n-\nu)} m + (-2^{n-\nu} \mu) + 2^{\alpha(n-\nu)}}^{2^{\alpha(n-\nu)} m + (+2^{n-\nu} \mu) + 2^{\alpha(n-\nu)}} \left(1 + \|\eta_1\|_2\right)^{-K} \, d\eta_1 \right]$$

$$\leq 2^{3+3+1} \cdot 2^{\alpha(n-\nu)} \cdot \left(1 + 2^{\alpha(n-\nu)} \right) \cdot \left(1 + 1 + \frac{2^{\alpha(n-\nu)} m}{2^{n-\nu}} \right)$$

$$\left(\left\|1 + \|\cdot\|_2\right\|^p_{p/(N+2)} \leq 1 \text{ since } K \geq 2^{+\varepsilon}\right) \leq 2^{3+\tau + \sigma} \cdot 2^{\alpha(n-\nu)} \cdot \left(1 + \frac{2^{\alpha(n-\nu)} \sigma}{2^{n-\nu}}\right) \cdot \left(1 + \frac{1 + 2^{\alpha(n-\nu)} m}{2^{n-\nu}}\right)$$

$$\leq 2^{5+\tau + \sigma} \cdot 2^{\alpha(n-\nu)} \cdot \left(1 + \frac{2^{\alpha(n-\nu)} \sigma}{2^{n-\nu}}\right) \cdot \left(1 + \frac{1 + 2^{\alpha(n-\nu)} m}{2^{n-\nu}}\right)$$

$$\leq 2^{5+\tau + \sigma} \cdot 2^{\alpha(n-\nu)} \cdot \left(1 + \frac{2^{\alpha(n-\nu)} \sigma}{2^{n-\nu}}\right) \cdot \left(1 + \frac{1 + 2^{\alpha(n-\nu)} m}{2^{n-\nu}}\right)$$

$$\leq 2^{18+\tau + 5\sigma} \cdot 2^{\alpha(n-\nu)} + \alpha(n-\nu)_+ + \alpha(n-\nu)_+.$$  \hfill (C.10)
Consequently, we get for arbitrary $j = (\nu, \mu, 0, 1) \in I^{(\ell)}$ the estimate

\[
\sum_{i \in I^{(\ell)}} \left[ \left( \frac{w^i}{w_1^i} \right)^\tau \left( 1 + \| T_j^{-1} T_i \| \right)^\sigma \left( \left| \det T_i \right|^{-1} \int_{S(n)} \theta_j \left( T_j^{-1} \xi \right) d\xi \right)^\tau \right]
\leq \sum_{n \in \mathbb{N}_0} \left[ \left( 1 + \left| \sum_{2^{\alpha(n-\nu), 2^{-n+\mu}}} \sigma \right| \right)^\tau \left( \int_{2^{(n-\nu)} m - 2^{n-\nu} \mu + 2^{n-\nu}} \left( 1 + \left| \eta_2 \right| \right)^{-K} d\eta_2 \right)^\tau \right].
\]

(eqs. (C.8), (C.9)) \leq 2^{18+7\sigma+\tau(5+2K+2M_3)} \cdot \sum_{n \in \mathbb{N}_0} \left[ \left( 1 + \left| \sum_{2^{(n-\nu)} m - 2^{n-\nu} \mu + 2^{n-\nu}} \sigma \right| \right)^\tau \left( \int_{2^{(n-\nu)} m - 2^{n-\nu} \mu + 2^{n-\nu}} \left( 1 + \left| \eta_2 \right| \right)^{-K} d\eta_2 \right)^\tau \right].
\]

(eq. (C.10)) \leq 2^{18+7\sigma+\tau(5+2K+2M_3)} \cdot \sum_{n \in \mathbb{N}_0} \left[ \left( 1 + \left| \sum_{2^{(n-\nu)} m - 2^{n-\nu} \mu + 2^{n-\nu}} \sigma \right| \right)^\tau \left( \int_{2^{(n-\nu)} m - 2^{n-\nu} \mu + 2^{n-\nu}} \left( 1 + \left| \eta_2 \right| \right)^{-K} d\eta_2 \right)^\tau \right].
\]

Now, observe $\theta_1 \left( 2^{n-\nu} \right) = \min \left\{ 2^{(n-\nu)M_1}, (1 + 2^{n-\nu})^{-M_2} \right\}$ and hence

\[
2^{\tau s(\nu-n) + \alpha(\nu-n)_+ + \sigma(1+\alpha)(n-\nu)_+} \cdot \left[ \theta_1 \left( 2^{n-\nu} \right) \right]^\tau \leq \left\{ \begin{array}{ll}
2^{-|\nu-n|(\tau M_1 - \tau s - \alpha)} & \text{if } \nu > n,
2^{-|\nu-n|(\tau M_2 + \tau s - \sigma(1+\alpha))} & \text{if } \nu \leq n.
\end{array} \right.
\]

Here, we used that $M_2 \geq M_2^{(0)} + c \geq (1 + \alpha) \frac{\sigma}{\tau} - s + c$, as well as $M_1 \geq M_1^{(0)} + c \geq s + \frac{\sigma}{\tau} + c$ by the assumptions of Lemma 4.1. Thus, all in all, we arrive at

\[
\sum_{\ell \in \mathbb{Z}} \left[ \left( \frac{w^\ell}{w_1^\ell} \right)^\tau \cdot \left( 1 + \| T_j^{-1} T_i \| \right)^\sigma \left( \left| \det T_i \right|^{-1} \int_{S(n)} \theta_j \left( T_j^{-1} \xi \right) d\xi \right)^\tau \right]
\leq 2^{18+7\sigma+\tau(5+2K+2M_3)} \cdot \sum_{n \in \mathbb{N}_0} 2^{-\tau|\nu-n|}
\leq 2^{18+7\sigma+\tau(5+2K+2M_3)} \cdot \sum_{\ell \in \mathbb{Z}} \left( \sum_{\lambda \in \mathbb{Z}} \left( 1 + \left| \sum_{2^{\alpha(n-\nu), 2^{-n+\mu}}} \sigma \right| \right)^\tau \left( \left| \det T_i \right|^{-1} \int_{S(n)} \theta_j \left( T_j^{-1} \xi \right) d\xi \right)^\tau \right)
\leq 2^{18+7\sigma+\tau(5+2K+2M_3)} \cdot \left( 1 - 2^{-\tau s} \right).
\]

Likewise, for the summation over $j$ instead of $i$, we apply Lemma 6.1 and the associated remark (using the choices $p = \tau \in (0, \infty)$, $\beta = 2^{n-\nu} > 0$, $N = \sigma \geq 0$, $M = 2^{\alpha(n-\nu)} m \in \mathbb{R}$ and $L = 2^{\alpha(n-\nu)} > 0$) to get

\[
\sum_{\mu \in \mathbb{Z}} \left( 1 + \left| \sum_{2^{\alpha(n-\nu), 2^{-n+\mu}}} \sigma \right| \right)^\tau \left( \left| \det T_i \right|^{-1} \int_{S(n)} \theta_j \left( T_j^{-1} \xi \right) d\xi \right)^\tau \leq 2^{3p+\frac{pN}{\beta} + N^3 + \left\| (1 + |\bullet|^K)^{-K} \right\|_p^{(N+2)} \cdot L \cdot \left( 1 + \frac{L + 1}{\beta} \right)}
\]

\[
\left( \frac{1 + |\bullet|^K}{\frac{1}{p} (N+2)} \right)^p \leq 1, \text{ since } K \geq \frac{2+\sigma}{\tau} \text{ by the assumptions of Lemma 6.1}
\]

\[
\left( 1 + \left| \sum_{2^{\alpha(n-\nu), 2^{-n+\mu}}} \sigma \right| \right)^\tau \left( \left| \det T_i \right|^{-1} \int_{S(n)} \theta_j \left( T_j^{-1} \xi \right) d\xi \right)^\tau \leq 2^{18+7\sigma+\tau(5+2K+2M_3)} \cdot \left( 1 + 2^{\sigma \alpha(n-\nu)} \right) \cdot \left( 1 + 2^{(1-\alpha)(\nu-n)} + 2^{\nu-n} \right)
\]

\[
\left( \frac{1 + |\bullet|^K}{\frac{1}{p} (N+2)} \right)^p \leq 1, \text{ since } K \geq \frac{2+\sigma}{\tau} \text{ by the assumptions of Lemma 6.1}
\]
Now we get as above for arbitrary $i = (n, m, 1, 0) \in I^{(0)}$ that
\[
\sum_{j \in I^{(i)}} \left[ \left( \frac{w_j}{\tilde{w}_j} \right)^T \left( 1 + \|T_j^{-1} T_i\| \right)^\sigma \left( |\det T_i|^{-1} \int_{S_i^{(\alpha)}} \vartheta_j (T_j^{-1} \xi) \, d\xi \right)^T \right]^{(2)} \leq 2^{18+7\sigma + (5+2K+2M_3)} \cdot \sum_{\nu \in \mathbb{N}_0} 2^{\tau s(\nu-n)+\sigma (1+\alpha)(n)_{+}} \cdot \left[ \vartheta_1 (2^{n-\nu}) \right]^T . \]

Here, the step marked with $(*)$ is justified by equations (C.8), (C.9), and (C.11).

As above, we observe
\[
2^{\tau s(\nu-n)+\sigma (1+\alpha)(n)_{+}} \cdot \left[ \vartheta_1 (2^{n-\nu}) \right]^T \leq \begin{cases} 2^{-|\nu-n|} (\tau M_1 - \tau s - 1) \leq 2^{-\tau c |\nu-n|}, & \text{if } \nu \geq n, \\
2^{-|\nu-n|} (\tau M_2 + \tau s - \sigma (1+\alpha)) \leq 2^{-\tau c |\nu-n|}, & \text{if } \nu \leq n,
\end{cases}
\]

where we used that we have $M_1 \geq M_1^{(0)} + c \geq \frac{1}{2} s + c$ and $M_2 \geq M_2^{(0)} + c \geq (1+\alpha) \frac{2}{\tau} s + c$ by the assumptions of Lemma 4.1. Consequently, we conclude
\[
\sum_{j \in I^{(i)}} \left[ \left( \frac{w_j}{\tilde{w}_j} \right)^T \left( 1 + \|T_j^{-1} T_i\| \right)^\sigma \left( |\det T_i|^{-1} \int_{S_i^{(\alpha)}} \vartheta_j (T_j^{-1} \xi) \, d\xi \right)^T \right] \leq 2^{19+7\sigma + (5+2K+2M_3)} \sum_{\ell \in \mathbb{Z}} 2^{-\tau c |\nu-n|} \leq 2^{19+7\sigma + (5+2K+2M_3) / (1 - 2^{-\tau c})} .
\]

In summary, in this subsection, we have shown for $C_0^{(1)} := 2^{19+7\sigma + (5+2K+2M_3) / (1 - 2^{-\tau c})}$ that
\[
\sup_{i \in I^{(0)}} \sum_{j \in I^{(1)}} M_{j,i}^{(0)} \leq C_0^{(1)} \quad \text{and} \quad \sup_{j \in I^{(1)}} \sum_{i \in I^{(0)}} M_{j,i}^{(0)} \leq C_0^{(1)} .
\]

C.2. We have $\ell_1 = (1, 1)$ and $\ell_2 = (1, 0)$. Geometrically, the present case means that $i$ belongs to the top cone, while $j$ belongs to the right cone, i.e., $e_1 = e = 1$, $\delta = 1$ and $d = 0$. In this case, we have
\[
T_j^{-1} T_i = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix} \begin{pmatrix} 2^{-\nu} & 0 & 0 \\ 0 & 2^{\nu} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 2^m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} = \begin{pmatrix} 2^{-\nu} & 0 \\ -\mu & 2^{\nu} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2^n & 0 \end{pmatrix} \begin{pmatrix} 2^m & 0 \\ m & 1 \end{pmatrix} = \begin{pmatrix} 2^{\nu} & 0 \\ -\mu & 2^{\nu} \end{pmatrix} \begin{pmatrix} 2^\alpha & 0 \\ 2^n & 0 \end{pmatrix} = \begin{pmatrix} 2^{\nu} & 0 \\ -\mu & 2^{\nu} \end{pmatrix} \begin{pmatrix} 2^{\alpha} & 0 \\ 2^n & 0 \end{pmatrix} = \begin{pmatrix} 2^{\nu} & 0 \\ -\mu & 2^{\nu} \end{pmatrix} \begin{pmatrix} 2^{\alpha} & 0 \\ 2^n & 0 \end{pmatrix} .
\]

As our first step, we want to obtain an estimate for $\|T_j^{-1} T_i\|$. To this end, recall $|m| \leq G_n = 2^{2(1-\alpha)}$ and $2^{\nu (1-\alpha)}$; hence $|2^{\nu \alpha} m| \leq 2^{\nu \alpha} + 2^{\nu} \leq 2 \cdot 2^{\nu} \leq 2 \cdot 2^{(n-\nu) \cdot 2}$. Likewise, $2^{\alpha} \leq 2 \cdot 2^{\alpha \nu} \leq 2 \cdot 2^{\alpha \nu} \leq 2 \cdot 2^{(n-\nu) \cdot 2} \leq 2 \cdot 2^{(n-\nu) \cdot 2}$ and $2^{\nu \alpha} \leq 2^{\nu} \leq 2^{(n-\nu) \cdot 2}$.

Finally, setting
\[
\kappa := \frac{\mu}{2^{\nu (1-\alpha)}} \quad \text{and} \quad \ell := \frac{m}{2^{2(1-\alpha)}} ,
\]

we have
\[
2^{\nu \alpha} - 2^{\nu \alpha} \mu m = 2^{\nu \alpha} \cdot \left( 1 - \frac{\mu}{2^{\nu (1-\alpha)}} \frac{m}{2^{2(1-\alpha)}} \right) = 2^{\nu \alpha} \cdot (1 - \kappa \ell) =; 2^{\nu \alpha} \cdot \lambda_{n,m,\nu,\mu} ,
\]

and also
\[
|\kappa| = \frac{|\mu|}{2^{\nu (1-\alpha)}} \leq \frac{1 + 2^{\nu (1-\alpha)}}{2^{\nu (1-\alpha)}} = 1 + 2^{-\nu (1-\alpha)} \leq 2 \quad \text{and} \quad |\ell| = \frac{|m|}{2^{2(1-\alpha)}} \leq 1 + 2^{-n (1-\alpha)} \leq 2 .
\]
All in all, we have shown
\[
(1 + \|T_j^{-1}T_i\|)^\sigma \leq \left(1 + 5 \cdot 2^{(n-\nu)+} + 2^{n-\nu} \cdot |\lambda_{n,m,\nu,\mu}|\right)^\sigma
\leq \left(1 + 5 \cdot 2^{(n-\nu)+}\right)^\sigma \cdot \left(1 + 2^{n-\nu} \cdot |\lambda_{n,m,\nu,\mu}|\right)^\sigma
\leq 6^\sigma \cdot 2^{\sigma(n-\nu)+} \cdot \left(1 + 2^{n-\nu} \cdot |\lambda_{n,m,\nu,\mu}|\right)^\sigma.
\]

Next, we consider the integral term occurring in \(M_{j,i}^{(0)}\). Precisely, with \(\varrho_1\) and \(\theta_1\) as in equation (14), we observe
\[
|\text{det} \, T_i|^{-1} \int_{T_j^{-1}\xi} \varrho_j \, (T_j^{-1}T_i) \, d\xi = \int_{T_j^{-1}\xi} \varrho \, (T_j^{-1}T_i) \, d\eta
= \int_{\eta^{-1}}^{1} \varrho \left(2^{n-\nu} \cdot \lambda_{n,m,\nu,\mu} \cdot \eta - 2^{n-\nu} \cdot \mu \xi\right) \, d\eta \, d\eta
\]

(\text{with } \xi = \eta/\eta_1) = \int_{\eta^{-1}}^{1} \eta_1 \int_{\eta_1^{-1}}^{1} \varrho \left(2^{n-\nu} \cdot \lambda_{n,m,\nu,\mu} \cdot \eta - 2^{n-\nu} \cdot \mu \xi\right) \, d\xi \, d\eta
\]

(since \(\eta_1 \leq 1\)) \leq 3 \int_{\eta_1^{-1}}^{1} \theta_1 \left(2^{n-\nu} \cdot \lambda_{n,m,\nu,\mu} \cdot \eta - 2^{n-\nu} \cdot \mu \xi\right) \, d\xi \, d\eta
\]

(\text{eq. (14a)}) \leq 3^{2+K+M_3} \cdot \int_{\eta_1^{-1}}^{1} \theta_1 \left(2^{n-\nu} \cdot \lambda_{n,m,\nu,\mu} \cdot \eta - 2^{n-\nu} \cdot \mu \xi\right) \, d\xi \, d\eta
\]

As our next step, we derive several basic estimates for the quantities appearing in equation (15):

(1) We have
\[
\theta_1 \left(2^{n-\nu} \cdot (m + \xi)\right) \leq 4^{M_3} \cdot \min \left\{1, 2^{M_1(n-\nu)}\right\} = 4^{M_3} \cdot 2^{-M_1(n-\nu)}, \quad \forall \xi \in [-1, 1]. \quad \text{(C.16)}
\]

To see this, we consider the cases \(|m| \leq 1\) and \(|m| \geq 2\). In case of \(|m| \leq 1\), we have \(|m + \xi| \leq |m| + |\xi| \leq 2\) and thus
\[
\theta_1 \left(2^{n-\nu} \cdot (m + \xi)\right) \leq \min \left\{1, 2^{n-\nu} \cdot (m + \xi)\right\}^{M_1}
\leq 2^{M_1} \cdot \min \left\{1, 2^{M_1(n-\nu)}\right\},
\]

(since \(n_0 \leq \alpha\) and \(M_1 \geq 0\), as well as \(M_1 \leq M_3\)) \leq 2^{M_3} \cdot \min \left\{1, 2^{M_1(n-\nu)}\right\},
\]

which is even slightly better than the estimate (C.16). Next, in case of \(|m| \geq 2\), we have
\[
\frac{|m|}{2} \leq |m| - 1 \leq |m| - |\xi| \leq |m + \xi| \leq |m| + |\xi| \leq 1 + |m| \leq 2 \, |m|, \quad \forall \xi \in [-1, 1], \quad \text{(C.17)}
\]

so that equation (C.16) yields
\[
\theta_1 \left(2^{n-\nu} \cdot (m + \xi)\right) \leq 2^{M_3} \cdot \theta_1 \left(2^{n-\nu} \cdot m\right)
\]

(cf. eq. (14a)) = \[2^{M_3} \cdot \theta_1 \left(2^{n-\nu} \cdot l\right]
\leq 2^{M_3} \cdot \min \left\{1, 2^{M_1(n-\nu)}\right\},
\]

(since \(|l| \leq 2\)) \leq 2^{M_3} \cdot \min \left\{1, 2^{M_1(n-\nu)}\right\},
\]

(since \(M_1 \leq M_3\)) \leq 2^{M_3} \cdot \min \left\{1, 2^{M_1(n-\nu)}\right\}, \quad \forall \xi \in [-1, 1].
\]

We have thus established equation (16) in both cases.
(2) Next, in case of $|\kappa| \leq \frac{1}{2}$, we have
\[
2^{-(n-\nu)\lambda_{n,m,\nu,\mu} - 2^{\alpha-n}\nu \mu \xi} = 2^{-(n-\nu)\lambda} (1 + n - \nu) - 2^{(n-\nu)\lambda \xi} \\
= 2^{-(n-\nu)\lambda} \cdot \left(1 - |\kappa| - 2^{-(n-\nu)\lambda \xi} \right) \\
\geq 2^{-(n-\nu)\lambda} \cdot \left(1 - |\kappa| - 2^{-(n-\nu)\lambda \xi} \right) \\
\geq 2^{-(n-\nu)\lambda} \cdot \left(1 - \frac{1}{2} - \frac{1}{4} \right) \\
= \frac{2^{n-\nu\lambda}}{4} \quad \forall \xi \in [-1, 1]. \tag{C.18}
\]

(3) Finally, we want to obtain an estimate similar to equation (C.18) also if $|\kappa| < \frac{1}{3}$. To this end, we additionally assume $\alpha < 1$ and $n \geq \frac{3}{1-\alpha}$, since this ensures $-n(1-\alpha) \leq -3$ and thus $2^{-(n-\nu)\lambda} \leq \frac{1}{8}$. Consequently,
\[
2^{-(n-\nu)\lambda_{n,m,\nu,\mu} - 2^{\alpha-n}\nu \mu \xi} = 2^{-(n-\nu)\lambda} (1 + n - \nu) - 2^{(n-\nu)\lambda \xi} \\
= 2^{-(n-\nu)\lambda} \cdot \left(1 - |\kappa| - 2^{-(n-\nu)\lambda \xi} \right) \\
\geq 2^{-(n-\nu)\lambda} \cdot \left(1 - \frac{1}{2} - \frac{1}{8} \cdot 2 \right) \\
= \frac{2^{n-\nu\lambda}}{4} \quad \forall \xi \in [-1, 1]. \tag{C.19}
\]

For the last estimate above, we needed to assume $\alpha < 1$. To avoid cumbersome case distinctions later on, we now consider the special case $\alpha = 1$, so that we can then assume $\alpha < 1$ for the remainder of the subsection.

C.2.1. The special case $\alpha = 1$. Because of $\alpha = 1$, we simply have $\kappa = \mu$ and $\nu = m$. Further, $G_n = \left[2^{n(1-\alpha)}\right] = 1$ for all $n \in \mathbb{N}_0$, i.e., $m, \mu \in \{-1, 0, 1\}$. Consequently, we also get $\lambda_{n,m,\nu,\mu} = 1 - \mu m \in \{0, 1, 2\}$ and estimate (C.15) takes the form
\[
|\det T_i|^{-1} \cdot \int_{S_i^{(n)}} \theta_j \left(T_j^{-1} \xi \right) d\xi \leq 3^{2K+M_3} \cdot \int_{-1}^{1} \theta \left(2^{-\nu} (m + \xi) \right) \cdot (1 + |2^{-\nu} (1 - \mu (m + \xi))|)^{-K} d\xi. \tag{C.20}
\]

Finally, we get because of $\alpha = 1$ and $\lambda_{n,m,\nu,\mu} \in \{0, 1, 2\}$ from equation (C.14) that
\[
\left(1 + \left|T_j^{-1} \xi \right| \right)^{\sigma} \leq \left(1 + 5 \cdot 2^{(1-n)} + 2^{n-\nu} \cdot |\lambda_{n,m,\nu,\mu}| \right)^{\sigma} \\
\leq \left(1 + 5 \cdot 2^{(1-n)} + 2^{n-\nu} \right)^{\sigma} \\
\leq 8^{\sigma} \cdot 2^{2(1-n)} \cdot \sigma. \tag{C.21}
\]

Next, we distinguish two subcases:

1. If $n - \nu \leq 0$, then $|2^{-\nu} (m + \xi)| \leq 2 \cdot 2^{-\nu}$ since $|m| \leq 1$ and $|\xi| \leq 1$. Hence
\[
\theta_j \left(2^{-\nu} (m + \xi) \right) \leq \left|2^{-\nu} (m + \xi)\right|_{M_1} \leq M_1 \cdot 2^{M_1(n-\nu)} \leq M_1 \cdot 2^{M_1(n-\nu)} \leq 2^{M_1} \cdot 2^{-M_1(n-\nu)}.
\]

2. Otherwise, $n - \nu \geq 0$, so that there are again two subcases:

(a) If $|m + \xi| \geq \frac{1}{2}$, then $\frac{1}{2} \leq |m + \xi| \leq 2$, so that equation (C.6) yields
\[
\theta_j \left(2^{-\nu} (m + \xi) \right) \leq M_1 \cdot \theta_j \left(2^{-\nu} (m + \xi) \right) \leq M_1 \cdot \left(1 + |2^{-\nu} (m + \xi)|\right)^{-M_2} \leq M_1 \cdot 2^{-M_2(n-\nu)}.
\]

(b) Otherwise, $|m + \xi| \leq \frac{1}{2}$ and hence $|1 - \mu (m + \xi)| \geq 1 - |\mu (m + \xi)| \geq 1 - |m + \xi| \geq \frac{1}{2}$, which implies
\[
(1 + |2^{-\nu} (1 - \mu (m + \xi))|)^{-K} \leq \left(\frac{1}{2} \cdot 2^{-\nu}\right)^{-K} \leq 2^K \cdot 2^{-K(n-\nu)}.
\]

All in all, we have for all $\xi \in [-1, 1]$ that
\[
\theta_j \left(2^{-\nu} (m + \xi) \right) \cdot (1 + |2^{-\nu} (1 - \mu (m + \xi))|)^{-K} \leq \begin{cases} 
2^{M_1 + K} \cdot 2^{-M_1(n-\nu)}, & \text{if } n \leq \nu \\
2^{M_1 + K} \cdot 2^{-\min\{M_1, K\}(n-\nu)}, & \text{if } n \geq \nu 
\end{cases}
\]
\[
= 2^{M_1 + K} \cdot 2^{-M_1(n-\nu)} + 2^{-\min\{M_1, K\}(n-\nu)}.
\]
and thus

\[ M_{j,i}^{(0)} = \left( \frac{\mu_j^i}{\mu_j^i} \right)^\tau \left( 1 + \| T_j^{-1} T_i \| \right)^\sigma \left( \det T_i \right)^{-1} \left( \int_{S^{(\alpha)}} \Theta_j (T_j^{-1} \xi) d \xi \right)^\tau \]

(\text{eqs. } 1 \text{ and } 4) \leq 2^{\tau(n+\sigma)} \cdot 8^{\sigma} \cdot 2^{\sigma(n+\nu+\delta)} \cdot \left[ 3^{2+K+M_3} + 2^{1+M_3 + K} \cdot 2^{\omega-M_1(n-\nu)} \cdot 2^{-\min\{M_2,K\}(n-\nu)} \right]^\tau

\leq 8^\sigma \cdot 6^{\tau(2+K+M_3)} \cdot \left\{ \begin{array}{ll}
2^{-\left[\left(\tau \min\{M_2,K\} + s\tau - \sigma \right)\right]} & \text{if } n - \nu \geq 0, \\
2^{-\left[\left(\tau M_1 + s\tau \right)\right]} & \text{if } n - \nu \leq 0.
\end{array} \right.

But the assumptions of Lemma 4.1 ensure that \( M_1 \geq M_1^{(0)} + c \geq s + c \) and \( M_2, K \geq \frac{\sigma}{\tau} - s + c \), which entails \( \tau \min\{M_2,K\} + s\tau - \sigma \geq \tau c \), as well as \( \tau M_1 + s\tau \geq \tau c \), so that \( M_{j,i}^{(0)} \leq 8^\sigma \cdot 6^{\tau(2+K+M_3)} \cdot 2^{-\tau c|n-\nu|} \) for all \( i \in I^{(\ell_1)} \) and \( j \in I^{(\ell_2)} \). Consequently, we get because of \( G_n = G_\nu \) for all \( n, \nu \in \mathbb{N}_0 \) that

\[ \sum_{i \in I^{(\ell_1)}} M_{j,i}^{(0)} \leq 8^\sigma \cdot 6^{\tau(2+K+M_3)} \cdot \sum_{n=0}^{\infty} 2^{-\tau c|n-\nu|} \cdot 3 \cdot G_n \]

\[ \leq 3 \cdot 8^\sigma \cdot 6^{\tau(2+K+M_3)} \cdot \sum_{\ell \in \mathbb{Z}} 2^{-\tau c|\ell|} \]

\[ \leq 3 \cdot 8^\sigma \cdot 6^{\tau(2+K+M_3)} \cdot \frac{2}{1 - 2^{-\tau c}} =: C_1 \]

for arbitrary \( j = (\nu, \mu, c, d) \in I^{(\ell_2)} \). Exactly the same estimate also yields \( \sum_{j \in I^{(\ell_2)}} M_{j,i}^{(0)} \leq C_1 \) for arbitrary \( i = (n, m, \varepsilon, \delta) \in I^{(\ell_1)} \), as long as \( \alpha = 1 \).

C.2.2. The general case \( \alpha \in [0,1) \). In this subsection, we first consider two special cases and then the remaining general case.

**Case 1:** \( n \leq \frac{3}{1-\alpha} \). In this case, equation (C.16) yields

\[ \theta_1 \left( 2^{n\alpha - \nu} \cdot (m + \xi) \right) \leq 4^{M_3} \cdot \min \left\{ 1, 2^{M_1(n-\nu)} \right\} \leq 4^{M_3} \cdot 2^{\frac{\nu}{1-\alpha}} \cdot 2^{-M_1 \nu} \quad \forall \xi \in [-1,1]. \]

Furthermore, equation (C.14) entails, because of \( |\lambda_{n,m,\nu,\mu}| = |1 - \kappa| \leq 5 \), that

\[ \left( 1 + \| T_j^{-1} T_i \| \right)^\sigma \leq 6^\sigma \cdot 2^{\sigma(n-\nu) + \left( 1 + 5 \cdot 2^{n-\nu} \right)^\sigma} \]

\[ \leq 6^\sigma \cdot 2^{\sigma(n-\nu) + \left( 1 + 5 \cdot 2^{\frac{n}{1-\alpha}} \right)^\sigma} \]

(since \( (n-\nu)_+ = n-\nu - \frac{3}{1-\alpha} \) for \( n-\nu \geq 0 \) and \( (n-\nu)_+ = 0 \leq \frac{3}{1-\alpha} \) otherwise)

\[ \leq 6^\sigma \cdot 2^{\frac{6n}{1-\alpha}} \cdot \left( 1 + 5 \cdot 2^{\frac{n}{1-\alpha}} \right)^\sigma \]

\[ \leq 6^\sigma \cdot 2^{\frac{6n}{1-\alpha}} =: C_2. \]

In combination with equation (C.15), we conclude

\[ M_{j,i}^{(0)} = \left( \frac{w_j^i}{w_j^i} \right)^\tau \left( 1 + \| T_j^{-1} T_i \| \right)^\sigma \left( \det T_i \right)^{-1} \left( \int_{S^{(\alpha)}} \Theta_j (T_j^{-1} \xi) d \xi \right)^\tau \]

\[ \leq C_2 \cdot 2^{\nu(n-\nu)} \cdot \left[ 3^{2+K+M_3} + \int_{1}^{\infty} \Theta_1 \left( 2^{n-\nu} \cdot (m + \xi) \right) \left( 1 + \left| 2^{n-\nu} \lambda_{n,m,\nu,\mu} - 2^{n-\nu} \mu_1 \xi \right| - K \right) d \xi \right]^\tau \]

\[ \leq C_2 \cdot 2^{\nu(n-\nu)} \cdot \left[ 2 \cdot 3^{2+K+M_3} + 4^{M_3} \cdot 2^{\frac{3M_3}{1-\alpha}} \cdot 2^{-M_1 \nu} \right]^\tau \]

\[ = C_2 \cdot 2^{\nu(n-\nu)} \cdot \left[ 2 \cdot 3^{2+K+M_3} + 4^{M_3} \cdot 2^{\frac{3M_3}{1-\alpha}} \right]^\tau \]

\[ = 2^{\nu(s-M_1)} \cdot C_3. \]
Since our assumptions imply $M_1 \geq M_1^{(0)} + c \geq s + \frac{1}{\tau} + c \geq s + \frac{1-\alpha}{\tau} + c$, we get $1 - \alpha + \tau s - \tau M_1 \leq -\tau c$ and hence

$$
\sup_{i=(n,m,\epsilon,\delta) \in I^{(1)}} \sum_{j \in I^{(2)}} M_{j,i}^{(0)} \leq C_3 \cdot \sum_{\nu=0}^{\infty} \sum_{|\nu| \leq G_\nu} 2^{\tau \nu (s-M_1)}
$$

(since $G_n = [2^{(1-\alpha)n}] \leq 1+2^{(1-\alpha)n} \leq 2 \cdot 2^{(1-\alpha)n}$)

$$
\leq 6C_3 \cdot \sum_{\nu=0}^{\infty} 2^{\nu (1-\alpha + \tau s - \tau M_1)} \leq 6C_3 \cdot \sum_{\nu=0}^{\infty} 2^{\nu (1-\alpha + \tau s - \tau M_1)}
$$

(C.23)

Furthermore, since $\tau (s - M_1) \leq 1 - \alpha + \tau s - \tau M_1 \leq -\tau c < 0$, we also have

$$
\sup_{j \in I^{(2)}, i=(n,m,\epsilon,\delta) \in I^{(1)}} M_{j,i}^{(0)} \leq C_3 \cdot \sup_{\nu \in \mathbb{N}_0} 2^{\tau \nu (s-M_1)} \cdot \sum_{n \leq \frac{3}{1-\alpha}} \sum_{|\nu| \leq G_n} 1
$$

(since $G_n = [2^{(1-\alpha)n}] \leq [2^n] = s$)

$$
\leq C_3 \cdot \left( 1 + \frac{3}{1-\alpha} \right) \cdot (1 + 2 \cdot 8) \leq \frac{68 \cdot C_3}{1 - \frac{3}{1-\alpha}}.
$$

(C.24)

This completes our considerations for the special case $n \leq \frac{3}{1-\alpha}$. In the remainder of this subsection, we can (and will) thus assume $n \geq \frac{3}{1-\alpha}$. 

**Case 2:** We have $|\nu| \geq \frac{1}{4}$ and $|m| \geq 2 \geq 2 |\nu| \vee (|\nu| \geq \frac{1}{4})$, as well as $n \geq \frac{3}{1-\alpha}$. We first show that these conditions imply

$$
\theta_1 (2^n \alpha - \nu, (m + \xi)) \leq 2^{5M_3} \cdot |\nu|^{M_1} \cdot 2^{-M_1 (n-\nu)+} \cdot 2^{-M_2 (n-\nu)+}
$$

$$
= \begin{cases} 
2^{5M_3} \cdot |\nu|^{M_1} \cdot 2^{-M_2 (n-\nu)}, & \text{if } \nu \leq n \\
2^{5M_3} \cdot |\nu|^{M_1} \cdot 2^{-M_2 (n-\nu)+}, & \text{if } \nu > n
\end{cases}
$$

\forall \xi \in [-1,1].

(C.25)

To establish equation ([C.25]), we first note $\frac{|m|}{2} \leq |m + \xi| \leq 2 |m|$, (cf. equation ([C.17])) since $|m| \geq 2$. Hence, equations ([C.6]) and ([C.13]) yield

$$
\theta_1 (2^n \alpha - \nu, (m + \xi)) \leq 2^{M_3} \cdot \theta_1 (2^n \alpha - \nu, |m|) = 2^{M_3} \cdot \theta_1 (2^n \alpha - \nu, |\xi|).
$$

Now, we distinguish the two cases that are suggested by equation ([C.25]):

1. In case of $n \leq \nu$, we get $n - \nu = -|n - \nu|$ and thus

$$
\theta_1 (2^n \alpha - \nu, (m + \xi)) \leq 2^{M_3} \cdot \theta_1 (2^n \alpha - \nu, |\xi|) \leq 2^{M_3} \cdot 2^{-M_2 (n-\nu)+} \cdot 2^{M_1 (|\xi|^{M_1})}
$$

which is even slightly better than equation ([C.25]).

2. In case of $n > \nu$, we have $|\xi| \geq \frac{1}{4}$, since we assume $(n \leq \nu) \vee (|\nu| \geq \frac{1}{4})$. Consequently, $\frac{1}{4} \leq |\xi| \leq 2 \leq 4$, so that equation ([C.6]) yields

$$
\theta_1 (2^n \alpha - \nu, (m + \xi)) \leq 2^{M_3} \cdot \theta_1 (2^n \alpha - \nu, |\xi|) \leq 2^{M_3} \cdot 2^{M_1 (|\nu|^{M_1})}
$$

$$
\leq 8^{M_3} \cdot (1 + 2^{-n-\nu} - M_2)
$$

$$
\leq 8^{M_3} \cdot 2^{-M_2 (n-\nu)+}
$$

(since $|\xi| \geq \frac{1}{4}$)

$$
\leq 8^{M_3} \cdot 4^{M_1} \cdot |\nu|^{M_1} \cdot 2^{-M_2 (n-\nu)}
$$

$$
\leq 2^{5M_3} \cdot |\nu|^{M_1} \cdot 2^{-M_2 (n-\nu)+} = 2^{5M_3} \cdot |\nu|^{M_1} \cdot 2^{-M_2 (n-\nu)+},
$$

which establishes equation ([C.25]) also in this case.

We now properly start the proof: First, note that $|\xi|^{M_1} \leq 2^{M_1} \leq 2^{M_3}$, so that equation ([C.25]) yields the estimate

$$
\theta_1 (2^n \alpha - \nu, (m + \xi)) \leq 2^{5M_3} \cdot 2^{-M_2 (n-\nu)+} \cdot 2^{-M_2 (n-\nu)+} \forall \xi \in [-1,1].
$$

In combination with equation ([C.16]), we...
conclude

$$\left| \det T_i \right|^{-1} \int_{S(n)} \theta_j \left( T_j^{-1} \right) d \xi \leq 3^{2+K+M_3} \cdot \int_{-1}^{1} \theta_1 \left( 2^{n-\nu} (m+\xi) \right) \cdot \left( 1 + \left| 2^{n-\nu} (1-\kappa l) - 2^{\alpha (n-\nu) \xi} \right|^{-K} d \xi \right)^{-K} \int_{-2n(n-\nu) |\kappa|}^{2n(n-\nu) |\kappa|} \left( 1 + \left| 2^{n-\nu} - 2^{n-\nu} \kappa \ell \kappa l - \eta \right|^{-K} d \eta \right)$$

(with $\eta = 2^{\alpha (n-\nu) \kappa l} \xi$)

$$\leq 3^{2+K+5M_3} \cdot 2^{-M_1 (\nu-n)_+} - 2^{\alpha (n-\nu) \xi} \left| \int_{-2n(n-\nu) |\kappa|}^{2n(n-\nu) |\kappa|} \left( 1 + \left| 2^{n-\nu} - 2^{n-\nu} \kappa \ell \kappa l - \eta \right|^{-K} d \eta \right)$$

(since $\varepsilon = m/2^{n(1-\alpha)}$ and $|\kappa| \geq 4$)

$$\leq 3^{4+K+5M_3} \cdot 2^{\alpha (n-\nu) |\kappa|} \left( 1 + \left| 2^{n-\nu} - 2^{\alpha (n-\nu) \kappa m - \eta} \right|^{-K} d \eta \right)$$

(with $\xi = 2^{n-\nu} - 2^{\alpha (n-\nu) \kappa m - \eta}$)

$$= 3^{4+K+5M_3} \cdot 2^{\alpha (n-\nu) |\kappa|} \left( 1 + \left| 2^{n-\nu} - 2^{\alpha (n-\nu) \kappa m - \eta} \right|^{-K} d \eta \right)$$

For brevity, let us set $L := 2^{\alpha (n-\nu) |\kappa|}$ (which is independent of $m$) and $C_4 := 6^\sigma \cdot 3^{4+K+5M_3}$, as well as

$$A_{n,m,\nu,\mu} := \left( 1 + 2^{n-\nu} |\lambda_{n,m,\nu,\mu}| \right)^{\sigma} \left( 1 + 2^{n-\nu} |1 - \kappa l| \right)^{\sigma}$$

$$\left( \text{eq. (C.14)} \right) = \left( 1 + 2^{n-\nu} \left| 1 - 2^{\alpha (n-1) \kappa m} \right| \right)^{\sigma}$$

In combination with equation (C.14), the preceding estimate yields

$$\sum_{|m| \leq G_n} M_{j,i}^{(0)} \leq \sum_{|m| \leq G_n} \left( \frac{\| u \|}{\| w \|} \right)^{\tau} \left( 1 + \left\| T_j^{-1} T_i \right\| \right)^{\sigma} \left( \left| \det T_i \right|^{-1} \cdot \int_{S(n)} \theta_j \left( T_j^{-1} \xi \right) d \xi \right)^{\tau}$$

$$\left( \text{eq. (C.14)} \right) \leq C_4 \cdot 2^{\tau (n-\nu) + \sigma (n-\nu) |\kappa|} \left( 2^{-\kappa M_1 (\nu-n)_+} - 2^{-\kappa M_2 (\nu-n)_+} - 2^{\alpha (n-\nu) |\kappa|} \right)^{\tau}$$

$$= \sum_{m \in Z} \left| A_{n,m,\nu,\mu} \left( \int_{-2^{\alpha (n-\nu) \kappa m} - 2^{\alpha (n-\nu) |\kappa|} (1 + |\xi|)^{-K} d \xi \right)^{\tau} \right.$$}

$$\left. \left( \xi = -\text{sign}(\kappa) \cdot m \text{ and eq. (C.26)} \right) \right) = C_4 \cdot 2^{\tau (n+\nu) (\nu-n) + \sigma (n-\nu)_+ - M_1 (\nu-n)_+ - M_2 (\nu-n)_+}$$

$$\sum_{|\ell| \in Z} \left( 1 + 2^{n-\nu} - 2^{\alpha (n-\nu) |\kappa| \ell} \right)^{\sigma} \left( \int_{-2^{\alpha (n-\nu) |\kappa| \ell} + 2^{\alpha (n-\nu) \kappa m - L}}^{2^{\alpha (n-\nu) |\kappa| \ell} + 2^{\alpha (n-\nu) \kappa m + L}} (1 + |\xi|)^{-K} d \xi \right)^{\tau}.$$}

Now, an application of Lemma (C.1) and of the associated remark (with $p = \tau$, $N = \sigma$, $\beta = 2^{\alpha (n-\nu) |\kappa|} > 0$ and $L = 2^{\alpha (n-\nu) |\kappa|}$, as well as $M = 2^{n-\nu}$) yields

$$= \sum_{|m| \leq G_n} \left( 1 + 2^{n-\nu} - 2^{\alpha (n-\nu) |\kappa| \ell} \right)^{\sigma} \left( \int_{2^{\alpha (n-\nu) |\kappa| \ell} + 2^{\alpha (n-\nu) \kappa m - L}}^{2^{\alpha (n-\nu) |\kappa| \ell} + 2^{\alpha (n-\nu) \kappa m + L}} (1 + |\xi|)^{-K} d \xi \right)^{\tau}$$

$$\leq 2^{3+\tau \sigma \cdot 10^3 \sigma} \cdot \left( 1 + |\bullet| \right)^{-K} \left( 2^{n-\nu} |\kappa| \right)^{\tau} \cdot \left( 1 + 2^{\alpha (n-\nu) |\kappa|} \right)^{\sigma} \cdot \left( 1 + 2^{\alpha (n-\nu) |\kappa|} \right)^{\tau}$$

(since $K \geq 2^{4+\tau \sigma} \cdot 10^3 \sigma$ and $4 \leq |\kappa| \leq 2$)

$$\leq 2^{3+\tau \sigma \cdot 10^3 \sigma} \cdot 2^{\alpha (n-\nu) (\nu-n) + \sigma (n-\nu)_+ - M_1 (\nu-n)_+ - M_2 (\nu-n)_+} \leq 2^{3+\tau \sigma \cdot 10^3 \sigma} \cdot 2^{\alpha (n-\nu) (\nu-n) + \sigma (n-\nu)_+ - M_1 (\nu-n)_+ - M_2 (\nu-n)_+}$$

All in all, we get for $C_5 := C_4 \cdot 2^{\tau^2 + 2\tau + \sigma \cdot 10^3 \sigma}$ that

$$\sum_{|m| \leq G_n} M_{j,i}^{(0)} \leq C_5 \cdot 2^{\alpha (n-\nu) (\nu-n) + \sigma (n-\nu)_+ + 2^{n-\nu} (\nu-n)_+ + 2^{\tau (n+\nu) (\nu-n) + \sigma (n-\nu)_+ + \tau M_1 (\nu-n)_+ - \tau M_2 (\nu-n)_+}$$

$$= C_5 \cdot 2^{\tau (n+\nu) (\nu-n) + \sigma (n-\nu)_+ + (1+\alpha)\sigma (n-\nu)_+ - \tau M_1 (\nu-n)_+ - \tau M_2 (\nu-n)_+}$$

$$= C_5 \cdot \left\{ \begin{array}{ll} 2^{\tau (n+\nu) |\ell| \phi M_1 - s \phi M_1}, & \text{if } n \geq \nu, \\
2^{\tau |n| |\nu|-s \phi M_1}, & \text{if } n \leq \nu \end{array} \right.$$}
As usual, this implies
\[
\sup_{j \in I^*} \sum_{i = (m, n, s, \ell) \in I^*} M_{j, i}^{(0)} \leq C_5 \cdot \sum_{\ell \in \mathbb{Z}} 2^{-\tau c|\ell|} \leq \frac{2C_5}{1 - 2^{-\tau c}}. \tag{C.27}
\]

In addition to the preceding inequality, we also need to estimate the corresponding expression where the sum is taken over \(j\) instead of over \(i\). To this end, we set \(L := 2^{1+\alpha(n-\nu)}\) for brevity and estimate similar to the preceding case
\[
\int_{-1}^1 \left(1 + \left|2^{n-\nu \alpha (1-K\ell)} - 2^n (n-\nu) K\xi\right| \right)^{-K} d \xi \leq \frac{2^{\alpha(n-\nu)}}{|\xi|} \cdot \int_{-1}^1 \left(1 + \left|2^{n-\nu \alpha (1-K\ell)} - 2^n (n-\nu) K\xi\right| \right)^{-K} d \xi
\]
(since \(\frac{4}{3} \leq |\xi| \leq 2\) \(\leq 4 \cdot 2^{\alpha(n-\nu)} \)).

Now, a combination of equations (C.15) and (C.25) yields
\[
|\text{det} T_i|^{-1} \int_{S_i^{(a)}} \theta_j \left(T_j^{-1} \xi\right) d \xi \leq 3^{2+K+5M_3} \cdot |\xi|^{M_1} \cdot 2^{-M_4(n-\nu)_+ - M_2(n-\nu)_+} \int_{-1}^1 \left(1 + \left|2^{n-\nu \alpha (1-K\ell)} - 2^n (n-\nu) K\xi\right| \right)^{-K} d \xi
\]
\[
\leq 3^{4+K+5M_3} \cdot |\xi|^{M_1} \cdot 2^{\alpha(n-\nu)n+M_2(n-\nu)_+} \cdot \int_{-1}^1 \left(1 + \left|2^{n-\nu \alpha (1-K\ell)} - 2^n (n-\nu) K\xi\right| \right)^{-K} d \xi.
\]

In conjunction with equations (C.14) and (C.26), this entails
\[
M_{j, i}^{(0)} = \left(\frac{w^s}{w^s}\right)^\tau (1 + \left\|T_j^{-1} T_i\right\|) \cdot (|\text{det} T_i|^{-1} \int_{S_i^{(a)}} \theta_j \left(T_j^{-1} \xi\right) d \xi)^\tau
\]
\[
\leq 6^\sigma \cdot 3^{(4+K+5M_3)} \cdot 2^\tau (s + a) (n - \nu) + (n - \nu)_+ - \tau M_1 (n - \nu)_+ - \tau M_2 (n - \nu)_+ \cdot |\xi|^{M_1}_\tau.
\]

For brevity, set \(C_6 := 6^\sigma \cdot 3^{(4+K+5M_3)} \) and \(C_7 := 2^{5+2\tau+2\sigma} \cdot 10^3 + \sigma \cdot C_6 \) and recall from equations (C.20) and (C.13) that
\[
A_{n, m, \nu, \mu} = (1 + 2^{n-\alpha \nu} |1 - K\ell|)^{\sigma} = (1 + 2^{n-\alpha \nu} - 2^n |\nu| \cdot \text{sign}(\nu) \cdot \mu)^{\sigma}. \tag{C.28}
\]

We now invoke Lemma (C.1) and the associated remark (with \(L = 2^{1+\alpha(n-\nu)}, N = \sigma, p = \tau, M = 2^{-\nu \alpha}\) and \(\beta = 2^{n-\nu} |\nu|\)) to justify the following estimate:
\[
\sum_{|\nu| \leq G_{n_{\nu}}} M_{j, i}^{(0)} \leq C_6 \cdot 2^{\tau (s + a) (n - \nu) + (n - \nu)_+ - \tau M_1 (n - \nu)_+ - \tau M_2 (n - \nu)_+} \cdot |\xi|^{M_1}_\tau.
\]
(\text{eq. (C.28) and \(\ell = -\text{sign}(\nu)\mu\)) \(= C_6 \cdot 2^{\tau (s + a) (n - \nu) + (n - \nu)_+ - \tau M_1 (n - \nu)_+ - \tau M_2 (n - \nu)_+} \cdot |\xi|^{M_1}_\tau.
\]
(\text{Lem. (C.1) and remark}) \(\leq 2^{3+\tau + \sigma} \cdot C_6 \cdot 2^{\tau (s + a) (n - \nu) + (n - \nu)_+ - \tau M_1 (n - \nu)_+ - \tau M_2 (n - \nu)_+} \cdot |\xi|^{M_1}_\tau.
\]
\[
\left\| (1 + |\nu|)^{-K} \right\| \leq 2^{\tau (1+\alpha(n-\nu))} \cdot \left(1 + 2^{\sigma (1+\alpha(n-\nu))} \cdot \left(1 + 2^{1+\alpha(n-\nu)} \cdot \left(1 + \frac{2^{1+\alpha(n-\nu)}}{2^{n-\nu} |\nu|} \right) + \frac{2^{n-\nu} |\nu|}{|\nu|}\right)\right).
\]
Here, we note that we indeed have $\beta > 0$, since $|m| \geq 2 > 0$, so that $\iota \neq 0$. Now, recall $|\iota| \leq 2$ and $M_1 \geq \frac{1}{4}$, so that $|\iota|^\tau M_1 \leq 2^\tau M_1 \leq 2^\tau M_3$ and furthermore

$$\frac{|\iota|^\tau M_1}{|\iota|} = |\iota|^\tau M_1 - 1 \leq 2^\tau M_1 - 1 \leq 2^\tau M_3.$$ 

Hence, we can continue the estimate from above as follows:

$$\sum_{s.t. \ Case \ 2 \ holds} M_{j,i}^{(0)} \leq 2^\tau M_3 C_7 \cdot 2^\tau s(n-\nu) + (1+\alpha)\sigma(n-\nu)_{+} \cdot \tau M_1(n-\nu)_{+} \cdot (1+2(1-\alpha)(n-\nu)+2^\nu-n) \leq 2^{2+\tau} M_3 C_7 \cdot 2^\tau s(n-\nu) + (1+\alpha)\sigma(n-\nu)_{+} \cdot \tau M_1(n-\nu)_{+} \cdot 2^\nu(n-\nu).$$

Now, set $C_8 := 2^\tau M_3 C_7$ and observe

$$2^\tau s(n-\nu) + (1+\alpha)\sigma(n-\nu)_{+} - \tau M_1(n-\nu)_{+} \cdot 2^\nu(n-\nu) \leq \begin{cases} 2^\tau s(n-\nu) - \tau M_1(n-\nu)_{+} \cdot 2^\nu(n-\nu) = 2^{\nu-n}(|\tau M_1 - \tau s| - 1), & \text{if } n \leq \nu, \\ 2^\tau s(n-\nu) + (1+\alpha)\sigma(n-\nu) - \tau M_2(n-\nu)_{+} \cdot 2^\nu(n-\nu) = 2^{\nu-n}(|\tau s(1+\alpha)\sigma M_2|), & \text{if } n \geq \nu \end{cases} \leq 2^{-\tau c|\nu-n|},$$

since the assumptions of Lemma 4.1 ensure $M_1 \geq M_1^{(0)} + c \geq s + \frac{1}{4} + c$, as well as $M_2 \geq M_2^{(0)} + c \geq (1+\alpha)\frac{3}{4} - s + c$.

All in all, we finally conclude

$$\sup_{i=(n,m,\nu,\delta) \in I^{(1)}_{\iota,j} \ s.t. \ Case \ 2 \ holds} M_{j,i} \cdot \sum_{n \in \mathbb{N}_0} 2^{-\tau c|\nu-n|} \leq C_8 \cdot \sum_{n \in \mathbb{N}_0} 2^{-\tau c|\nu-n|} \leq \frac{2C_8}{1 - 2^{-\tau c}}, \quad (C.29)$$

which completes our considerations in the present case.

**Case 3:** The remaining case, i.e., $[|\iota| < \frac{1}{4}] \lor [|m| \leq 1] \lor [(n > \nu) \land (|\iota| < \frac{1}{4})]$, as well as $n \geq \frac{3}{4}$.

Our first step is to show

$$M_{j,i}^{(0)} \leq C_9 \cdot 2^\tau s(n-\nu) \cdot 2^\sigma(n-\nu)_{+} \cdot (1+2^{\nu-\nu} |\lambda_{n,m,\nu,\mu}|)^\sigma \cdot \min \left\{ 1, 2^\tau M_1(n-\nu) \right\} \cdot \min \left\{ 1, 2^\tau K(n-\nu) \right\} \leq C_{10} \cdot 2^\tau s(n-\nu) \cdot 2^\sigma(n-\nu)_{+} \cdot 2^\tau M_1(n-\nu)_{+} \cdot 2(\sigma-K)(n-\nu)_{+}$$

for $C_9 := 6^\sigma \cdot (3^3 + K + 3M_3 \cdot 4^K)^\tau$ and $C_{10} := 6^\sigma \cdot C_9$. Furthermore, as an intermediate result of independent interest, we also show

$$|2^{\nu-\nu^\alpha} \lambda_{n,m,\nu,\mu} - 2^{\nu-\nu^\alpha} \mu \xi| \geq \frac{2^{\nu-\nu^\alpha}}{4} \quad \forall \xi \in [-1,1]. \quad (C.31)$$

Here, the step marked with $(\ast)$ in equation (C.30) used that $|\lambda_{n,m,\nu,\mu}| \leq 5$, so that

$$\lambda_{n,m,\nu,\mu}(1 + 2^{\nu-\nu^\alpha} |\lambda_{n,m,\nu,\mu}|)^\sigma \leq 6^\sigma \cdot 2^{\sigma(n-\nu)_{+}}. \quad (C.32)$$

To prove equations (C.30) and (C.31), we distinguish three subcases:

1. We have $|m| \leq 1$. Because of $n \geq \frac{3}{4}$, this implies

$$|\iota| = 2^{-(1-a)n} |m| \leq 2^{-(1-a)n} \leq 2^{-3} = \frac{1}{8} \leq \frac{1}{4},$$

so that equation (C.19) yields $|2^{\nu-\nu^\alpha} \lambda_{n,m,\nu,\mu} - 2^{\nu-\nu^\alpha} \mu \xi| \geq \frac{2^{\nu-\nu^\alpha}}{4}$ for all $\xi \in [-1,1]$, i.e., equation (C.31) holds. Hence, a combination of equations (C.14), (C.15), (C.31), and (C.16) yields

$$M_{j,i}^{(0)} = \left( \frac{w_j^\tau}{w_i^\tau} \right)^\tau \cdot \left( 1 + \| T_j^{-1} T_i \| \right)^\sigma \cdot (\det T_i)^\tau \cdot \left( \int_{\mathcal{S}_1} \Theta_j \left( T_j^{-1} \xi \right) d\xi \right)^\tau \leq 6^\sigma \cdot 2^\tau s(n-\nu) \cdot 2^\sigma(n-\nu)_{+} \cdot (1 + 2^{\nu-\nu^\alpha} |\lambda_{n,m,\nu,\mu}|)^\sigma \cdot \left[ 3^3 + K + 3M_3 \cdot \min \left\{ 1, 2^\tau M_1(n-\nu) \right\} \cdot \left( 1 + \frac{2^{\nu-\nu^\alpha}}{4} \right)^{-K} \right]^\tau \leq C_9 \cdot 2^\tau s(n-\nu) \cdot 2^\sigma(n-\nu)_{+} \cdot (1 + 2^{\nu-\nu^\alpha} |\lambda_{n,m,\nu,\mu}|)^\sigma \cdot \min \left\{ 1, 2^\tau M_1(n-\nu) \right\} \cdot \min \left\{ 1, 2^\tau K(n-\nu) \right\}.$$ 

Thus, equations (C.30) and (C.31) are valid in this case.

2. We have $|\iota| \leq \frac{1}{4}$. In this case, equation (C.15) yields $|2^{\nu-\nu^\alpha} \lambda_{n,m,\nu,\mu} - 2^{\nu-\nu^\alpha} \mu \xi| \geq \frac{2^{\nu-\nu^\alpha}}{4}$ for all $\xi \in [-1,1]$. Then, validity of equations (C.30) and (C.31) follows just as in the previous case.
(3) The remaining case, i.e., $|\kappa| \geq \frac{1}{4}$ and $|m| \geq 2$. Since we are in Case 3, this entails $n > \nu$ and $|\kappa| < \frac{1}{4}$.

Since we also have $\alpha < 1$ and $n \geq \frac{3 - \alpha}{\alpha}$, equation (C.19) yields $2^{n - \nu \alpha} \lambda_{n, m, \nu, \mu} - 2^{\nu - \nu \mu} \mu |\kappa| \leq 2^{n - \nu \alpha}$ for all $\xi \in [-1, 1]$, so that the desired estimates follow just as in the previous two cases.

Now, we observe that $n \geq \nu$ implies $(n - \nu)_+ = n - \nu$, as well as $(\nu - n)_+ = 0$ and finally $(n - \nu)_+ = n - \nu$, since $n \geq \nu \geq \nu$. Consequently, equation (C.30) yields

$$
\sum_{i=(n,m,\nu,\mu) \in I^{(1)}} M_{j,i}^{(0)} \leq C_0 \cdot \sum_{n=\nu}^{\infty} \sum_{|m| \leq G_n} \left[ 2^{r(n-n)} \cdot 2^{r(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)} \right]
$$

for all $n \geq \nu$ and Case 3 holds

$$
\begin{align*}
(\text{since } G_n = & \left[ 2^{n(1-\alpha)} \right] \leq 1 + 2^{n(1-\alpha)} \leq 2^{n(1-\alpha)} \right) \leq 6 C_{10} \cdot \sum_{n=\nu}^{\infty} 2^{r(n-\nu)} \cdot 2^{r(n-n)} \cdot 2^{r(\sigma - \tau K)(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)} \\
& = 6 C_{10} \cdot 2^{2\tau(\sigma - \tau \sigma K - \sigma \alpha)} \cdot \sum_{n=\nu}^{\infty} 2^{r(n-\nu)} \cdot 2^{r(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)}
\end{align*}
$$

(C.33)

Here, the last step used that $K > 2 + \sigma \geq \frac{1}{1+\sigma}$ by the assumptions of Lemma [4] so that $1 + \sigma - \tau K \leq 0$. Furthermore, the step marked with $(*)$ used that the assumptions of Lemma [4] imply $K \geq K_0 + c \geq \frac{1}{1+\sigma} + 2^{\frac{1}{2}} - \nu$ and thus $1 - \sigma - \tau s + 2 \sigma - \tau K \leq -\tau c < 0$ and finally that

$$
\sum_{n=\nu}^{\infty} 2^{r(n-\nu)} \cdot \sum_{i=0}^{\infty} 2^{r(\sigma - \tau K)(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)}
$$

for arbitrary $\phi \in (-\infty, 0)$.

To estimate the sum over $j$ instead of over $i$, we observe again that $n \geq \nu$ implies $n \geq \nu \geq \nu$. In combination with equation (C.30), this implies

$$
\sum_{j=(\nu,\mu,\nu,\mu) \in I^{(2)}} M_{j,i}^{(0)} \leq C_0 \cdot \sum_{n=\nu}^{\infty} \sum_{|m| \leq G_n} 2^{r(n-\nu)} \cdot 2^{r(n-n)} \cdot 2^{r(\sigma - \tau K)(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)} \cdot 2^{r(\sigma - \tau K)(n-\nu)}
$$

(C.35)

To further estimate the right-hand side of this expression, we first observe that $g : [0, \infty) \to [0, \infty), x \mapsto x \cdot 2^{-x}$ is differentiable with derivative $g'(x) = 2^{-x} \cdot (1 - x \cdot \ln 2)$. Hence, $g'(x) > 0$ for $0 \leq x < \frac{1}{\ln 2}$ and $g'(x) < 0$ for $x > \frac{1}{\ln 2}$. Consequently, $g$ attains its unique global maximum at $x = \frac{1}{\ln 2}$. But we have $\ln 2 = \frac{1}{2} \ln 2^2 = \frac{1}{2} \ln e = \frac{1}{2}$ and thus $g(x) \leq g \left( \frac{1}{\ln 2} \right) = \frac{1}{\ln 2} \cdot 2^{-\frac{1}{\ln 2}} \leq \frac{2}{e} \leq 1$ for all $x \in [0, \infty)$. For arbitrary $n \in \mathbb{N}_0$ and $\phi > 0$, this implies $n \cdot 2^{-\phi n} \leq \frac{1}{\phi} \cdot (\phi n \cdot 2^{-\phi n}) = \frac{1}{\phi} \cdot g(\phi n) \leq \frac{1}{\phi}$ and thus

$$
(n+1) \cdot 2^{-\phi n} \leq 1 + n \cdot 2^{-\phi n} \leq 1 + \frac{1}{\phi} \quad \forall n \in \mathbb{N}_0 \text{ and } \phi > 0.
$$

Now, set $\beta := 1 - \alpha + \tau s - (1 + \alpha) \sigma + \tau K$ for brevity and note

$$
\begin{align*}
2^{\beta(n-\nu)} \sum_{n=\nu}^{\infty} 2^{2\beta n} & \leq \begin{cases}
2^{\beta n} & \text{if } \beta \geq 0, \\
2^{\beta(n-\nu)} & \text{if } \beta < 0
\end{cases} \\
& \leq \begin{cases}
(n+1) \cdot 2^{-n(1-\alpha)\tau c} & \text{if } \beta \geq 0, \\
(n+1) \cdot 2^{-n\tau c} & \text{if } \beta < 0
\end{cases}
\end{align*}
$$

(C.36)

Now, set $\beta := 1 - \alpha + \tau s - (1 + \alpha) \sigma + \tau K$ for brevity and note

$$
\begin{align*}
2^{\beta(n-\nu)} \sum_{n=\nu}^{\infty} 2^{2\beta n} & \leq \begin{cases}
2^{\beta n} & \text{if } \beta \geq 0, \\
2^{\beta(n-\nu)} & \text{if } \beta < 0
\end{cases} \\
& \leq \begin{cases}
(n+1) \cdot 2^{-n(1-\alpha)\tau c} & \text{if } \beta \geq 0, \\
(n+1) \cdot 2^{-n\tau c} & \text{if } \beta < 0
\end{cases}
\end{align*}
$$

(C.36)
where we recall that we assume $\alpha < 1$ in the present case. Furthermore, the step marked with $(\ast)$ used that the assumptions of Lemma 4.1 ensure $K \geq K_0 + c \geq \frac{1 - \alpha}{\tau} + 2\sigma - s + c \geq 2\sigma - s + c$ and thus $2\sigma - \tau s - \tau K \leq -\tau c$, as well as $K \geq K_0 + c \geq \frac{1 - \alpha}{\tau} + c$, so that $\sigma + 1 - \tau K \leq -\tau c$.

By plugging this into equation (C.35), we obtain

$$
\sum_{j = (\nu, \mu, \varepsilon, \delta) \in I^\ell_2} M_{j}^{(0)} \leq 6C_{10} \cdot \left(1 + \frac{1}{(1 - \alpha)\tau c}\right) \quad \forall i = (n, m, \varepsilon, \delta) \in I^\ell_i. \quad (C.36)
$$

Together, equations (C.33) and (C.36) take care of the case $\nu \leq n$, under the general assumptions of the current case. Hence, we only need to further consider the case $\nu > n$, which we now do.

Using estimate (C.30), we get

$$
\sum_{i = (n, m, \varepsilon, \delta) \in I^\ell_1} M_{j}^{(0)} \leq C_{10} \cdot \sum_{n = 0}^{\nu} \sum_{\left|m\right| \leq G_n} 2^{\tau s(n - \nu)} \cdot 2^{\tau s(n - \nu)} \cdot 2^{(\sigma K)(n - \nu)} \cdot 2^{n(1 - \alpha)}.
$$

We now divide the sum into the two parts were we know the sign of $n - \nu \alpha$. First, we observe that the assumptions of Lemma 4.1 entail $\sigma - \tau K \leq 0$, so that $2^{(\sigma K)(n - \nu \alpha) +} \leq 1$. Consequently,

$$
\sum_{0 \leq n \leq \nu \alpha} 2^{(\tau s - \tau M_1)(n - \nu \alpha)} \cdot 2^{\tau s(1 - \alpha)} \cdot 2^{n(1 - \alpha)} \leq 2^{\nu(\tau s - \tau M_1)} \cdot \sum_{n = 0}^{\nu \alpha} 2^{(\tau s - \tau M_1 + 1 - \alpha) n}
$$

(eq. and $\tau M_1 - \tau s + 1 - \alpha \geq \tau M_1 - \tau s > 0$) \hspace{1cm} (\ast)

$$
\left(\text{since } \tau M_1 - \tau s + 1 - \alpha \geq \tau M_1 - \tau s > 0 \text{ and } |\nu \alpha| \leq \nu \alpha\right) \leq \frac{2^{\tau M_1 - \tau s + 1 - \alpha}}{2^{\tau M_1 - \tau s + 1 - \alpha} - 1} \cdot 2^{\nu(\tau s - \tau M_1)} \cdot 2^{\nu(\tau M_1 - \tau s + 1 - \alpha)}
$$

$$
\left(\text{since } \tau M_1 - \tau s + 1 - \alpha \geq \tau M_1 - \tau s \geq 0 \text{ and } |\nu \alpha| \leq \nu \alpha\right) \leq \frac{2^{\tau M_1 - \tau s + 1 - \alpha}}{2^{\tau M_1 - \tau s + 1 - \alpha} - 1} \cdot 2^{\nu(\tau s - \tau M_1)} \cdot 2^{\nu(\tau M_1 - \tau s + 1 - \alpha)}
$$

$$
\leq \frac{1}{1 - 2^{\nu(\tau s - \tau M_1)}} = 2.
$$

Here, the step marked with $(\ast)$ used that the geometric sum formula shows

$$
\sum_{\ell = 0}^{n} 2^{\phi \ell} = \frac{2^{(n + 1)\phi} - 1}{2^\phi - 1} \leq \frac{2^{(n + 1)\phi} - 1}{2^\phi - 1} = \frac{2^\phi}{2^\phi - 1} \cdot 2^n \phi = \frac{1}{1 - 2^{-\phi}} \cdot 2^n \phi \quad \text{for arbitrary } \phi > 0. \quad (C.37)
$$

Now, we consider the remaining part of the sum. To this end, we first observe that the assumptions of Lemma 4.1 entail $M_1 \geq M_0 := s + \frac{c}{\tau}$. In conjunction with $n \leq \nu$, this implies $-\tau M_1 (\nu - n) \leq -\tau M_0 (\nu - n)$ and thus
we conclude that the assumptions of Lemma 4.1 hold. Consequently,

$$\sum_{\nu_0 < n \leq \nu} 2^{(\tau s - \tau M_1)(\nu - n)} 2^{(\sigma - \tau K)(n - \nu)} + 2^n(1 - \alpha) = \sum_{n = 1\Leftrightarrow |\nu_0|}^{\nu} 2^{(\tau s - \tau M_1)(\nu - n)} 2^{(\sigma - \tau K)(n - \nu)} + 2^n(1 - \alpha) \leq \sum_{n = 1\Leftrightarrow |\nu_0|}^{\nu} 2^{(\tau s - \tau M_0)(\nu - n)} 2^{(\sigma - \tau K)(n - \nu)} + 2^n(1 - \alpha) = 2^{\nu(\tau s - \tau M_0 - \alpha(\sigma - \tau K))} \sum_{n = 1\Leftrightarrow |\nu_0|}^{\nu} 2^n(1 - \alpha + \tau M_0 - \tau s + \sigma - \tau K) \leq 2^{\nu(-\alpha - \alpha(\sigma - \tau K))} \sum_{n = 1\Leftrightarrow |\nu_0|}^{\infty} 2^n(1 - \alpha + \tau M_0 - \tau s + \sigma - \tau K) \leq \frac{1}{1 - 2^{1 + \sigma - \tau K}}, \quad 2^{\nu(\nu K - \sigma - 1)} 2^{\nu(n - |\nu_0|)(1 + \sigma - \tau K)} \leq \frac{1}{1 - 2^{1 + \sigma - \tau K}} \leq 2.
$$

Altogether, the preceding four displayed equations show

$$\sup_{j = (\nu, \mu, c, d) \in I^{(\ell_2)}} \sum_{n < \nu \text{ and Case 3 holds}} M_{j, i}^{(0)} \leq 6 C_{10} \cdot (2 + 2) = 24 \cdot C_{10}. \quad (C.38)$$

It remains to consider the sum over $j \in I^{(\ell_2)}$ instead of over $i \in I^{(\ell_1)}$. To this end, we first consider the special case $\alpha = 0$. In this case we have $G_\nu = \left\lfloor 2^{(1 - \alpha)\nu} \right\rfloor = 2^\nu$ for all $\nu \in \mathbb{N}_0$, as well as $(n - \nu_0)_+ = n_+ = n$, so that equation (C.30) implies

$$\sum_{j = (\nu, \mu, c, d) \in I^{(\ell_2)}} M_{j, i}^{(0)} \leq C_{10} \cdot \sum_{\nu = n}^{\infty} \sum_{|\mu| \leq G_\nu} 2^{(\tau s - \tau M_1)(\nu - n)} 2^{(\sigma - \tau K)(n - \nu_0)} \leq 3 C_{10} \cdot 2^n(\sigma - \tau K + \tau M_1 - \tau s) \sum_{\nu = n}^{\infty} 2^{\nu(1 - \tau s - \tau M_1)} \leq \frac{3 C_{10}}{1 - 2^{1 + \tau s - \tau M_1}} \cdot 2^n(\sigma - \tau K + \tau M_1 - \tau s + 1 + \tau s - \tau M_1) \leq \frac{3 C_{10}}{1 - 2^{1 - \tau c}} \cdot 2^n(1 + \sigma - \tau K) \leq \frac{3 C_{10}}{1 - 2^{1 - \tau c}} \text{ in case of } \alpha = 0. \quad (C.39)$$

Having taken care of the case $\alpha = 0$, we can now assume $\alpha > 0$. With another application of equation (C.30), we conclude

$$\sum_{j = (\nu, \mu, c, d) \in I^{(\ell_2)}} M_{j, i}^{(0)} \leq C_{10} \cdot \sum_{\nu = n}^{\infty} \sum_{|\mu| \leq G_\nu} 2^{(\tau s - \tau M_1)(\nu - n)} 2^{(\sigma - \tau K)(n - \nu_0)} \leq 6 C_{10} \cdot \sum_{\nu = n}^{\nu} \sum_{\nu_0}^{\infty} 2^{\nu(1 - \alpha)} 2^{(\tau s - \tau M_1)(\nu - n)} 2^{(\sigma - \tau K)(n - \nu_0)} \leq 2^{(\sigma - \tau K)(n - \nu_0)} + 2^{(\sigma - \tau M_0)(n - \nu_0)} \leq 2^{\nu(1 - \alpha)} + 2^{(\sigma - \tau M_0)(n - \nu_0)} + 2^{\nu(1 - \alpha)} \leq 2^{C_{10}(1 - \alpha)}. \quad (C.40)$$

As in the previous case, we now split the series into two parts, according to the sign of $n - \nu_0$. But first, we observe by the assumptions of Lemma 4.1 that $K \geq K_0 := \frac{1 + \tau c}{\tau}$ and hence $2^{(\sigma - \tau K)(n - \nu_0)} \leq 2^{(\sigma - \tau K_0)(n - \nu_0)} \leq 2^{-\nu_0}$. 


Now, for \( n \leq \nu \leq \left\lfloor \frac{n}{\alpha} \right\rfloor \), we have \( n - \nu \alpha \geq 0 \) and thus

\[
\sum_{n \leq \nu \leq \left\lfloor \frac{n}{\alpha} \right\rfloor} \left( 2^{\nu(1-\alpha)} \cdot 2^{\nu(\sigma K)} \cdot 2^{\nu-n} \right) \leq \sum_{n \leq \nu \leq \left\lfloor \frac{n}{\alpha} \right\rfloor} \left( 2^{\nu(1-\alpha)} \cdot 2^{(\sigma K)\nu} \cdot 2^{-(n-\nu\alpha)} \right)
\]

\[
= 2^{n(\alpha M_1 - \nu s - 1)} \cdot \sum_{n \leq \nu \leq \left\lfloor \frac{n}{\alpha} \right\rfloor} 2^{(\alpha M_1 - \nu s)}
\]

\[
\leq 2^{n(\alpha M_1 - \nu s - 1)} \cdot \sum_{\nu = n}^{\infty} 2^{(\alpha M_1 - \nu s)}
\]

(eq. C.34) and \( 1+\alpha + \sigma K \nu \leq 0 \) by the assump. of Lem. 4.1

\[
\leq \frac{1}{1 - 2^\alpha M_1} \cdot 2^{n(\alpha M_1 - \nu s)}
\]

Finally, for the second part of the series, we have

\[
\sum_{\nu > \left\lfloor \frac{n}{\alpha} \right\rfloor} \left( 2^{\nu(1-\alpha)} \cdot 2^{\nu(\sigma K)} \cdot 2^{\nu-n} \right) \leq 2^{n(\alpha M_1 - \nu s)} \cdot \sum_{\nu = 1 + \left\lfloor \frac{n}{\alpha} \right\rfloor}^{\infty} 2^{(\alpha M_1 - \nu s)}
\]

(eq. C.34) and \( 1 - \alpha + \sigma K \nu \leq 0 \) by the assump. of Lem. 4.1

\[
\leq \frac{1}{1 - 2^{-\alpha M_1}} \cdot 2^{n(\alpha M_1 - \nu s)} \cdot \sum_{\nu = 1 + \left\lfloor \frac{n}{\alpha} \right\rfloor}^{\infty} 2^{\nu(\sigma K)}
\]

(since \( 1 - \frac{n}{\alpha} \) \( \geq 0 \) and \( 1 - \alpha + \sigma K \nu \leq 0 \) by assump. of Lem. 4.1)

\[
\leq \frac{1}{1 - 2^{-\alpha M_1}} \cdot 2^{n(\alpha M_1 - \nu s)} \cdot 2^{\nu(\sigma K)}
\]

(since \( 1 + \alpha + \sigma K \nu \leq 0 \) by the assump. of Lem. 4.1)

All in all, the preceding three displayed equations show for \( \alpha > 0 \) that

\[
\sup_{i = (n,m,e,d)} \sum_{j \in I_{c}^{(1)}} M_{j,i}^{(0)} \leq \frac{12 \cdot C_{10}}{1 - 2^{-\alpha M_1}}
\]

(C.40)

and in view of equation (C.39), this estimate also holds in case of \( \alpha = 0 \).

Overall, our considerations in this subsection have established the bound

\[
\sup_{i \in I_{c}^{(1)}} \sum_{j \in I_{c}^{(2)}} M_{j,i}^{(0)} \leq C_{0} \quad \text{if} \quad \alpha = 1,
\]

cf. equation (C.22). Furthermore, in case of \( \alpha \in [0, 1) \), we have shown

\[
\sup_{i \in I_{c}^{(1)}} \sum_{j \in I_{c}^{(2)}} M_{j,i}^{(0)} \leq \sup_{i \in I_{c}^{(1)}} \left[ \sum_{j \in I_{c}^{(2)}} M_{j,i}^{(0)} \right] + \frac{6C_{3}}{1 - 2^{-\tau C}} + \frac{2C_{8}}{1 - 2^{-\tau C}} + \left( 6C_{10} \cdot \left( 1 + \frac{1}{(1 - \alpha) C_{11}} \right) \right) \frac{12 \cdot C_{10}}{1 - 2^{-\alpha M_1}}
\]

(C.40)

\[
=: C_{0}^{(2)}
\]

Note that the constant \( C_{0}^{(2)} \) has a different value depending on whether \( \alpha = 1 \) or \( \alpha < 1 \).

Likewise, we have shown

\[
\sup_{j \in I_{c}^{(2)}} \sum_{i \in I_{c}^{(1)}} M_{j,i}^{(0)} \leq C_{1} \quad \text{if} \quad \alpha = 1,
\]

cf. equation (C.22). Furthermore, in case of \( \alpha \in [0, 1) \), we have shown

\[
\sup_{i \in I_{c}^{(1)}} \sum_{j \in I_{c}^{(2)}} M_{j,i}^{(0)} \leq \sup_{i \in I_{c}^{(1)}} \left[ \sum_{j \in I_{c}^{(2)}} M_{j,i}^{(0)} \right] + \frac{6C_{3}}{1 - 2^{-\tau C}} + \frac{2C_{8}}{1 - 2^{-\tau C}} + \left( 6C_{10} \cdot \left( 1 + \frac{1}{(1 - \alpha) C_{11}} \right) \right) \frac{12 \cdot C_{10}}{1 - 2^{-\alpha M_1}}
\]

(C.40)

\[
=: C_{0}^{(2)}
\]
see again equation (C.22). In case of \( \alpha \in [0, 1) \), we have also shown

\[
\left( \sup_{j \in I^{(1)}} \sum_{i \in I^{(1)}} M_{j,i}^{(0)} \right) \leq \left( \sup_{j \in I^{(1)}} \sum_{i \in I^{(1)}} M_{j,i}^{(0)} \right)_{\text{s.t. Case 1 holds}} + 2C_0 \cdot \frac{2C_5}{1 - 2^{-\tau c}} + \left( 24 \cdot C_{10} + 6C_{10} \right) \cdot \frac{1}{1 - 2^{-\tau c}} =: C_0^{(3)}.
\]

Note as above that \( C_0^{(3)} \) has a different value depending on whether \( \alpha = 1 \) or \( \alpha < 1 \).

Finally, we observe that \( C_0^{(1)} \) and \( C_0^{(2)} \) can be estimated solely in terms of \( \alpha, \tau_0, \omega, c, K, H, M_1, M_2 \). Indeed, for arbitrary \( C \geq 0 \), we have because of \( \tau \geq \tau_0 \) that

\[
C^{1/\tau} \leq \left( \max \{1, C\} \right)^{1/\tau} \leq \left( \max \{1, C\} \right)^{1/\tau_0} = \max \left\{ 1, C^{1/\tau_0} \right\}.
\]

Thus, if a constant \( C \geq 0 \) can be bounded only in terms of \( \alpha, \tau_0, \omega, c, K, H, M_1, M_2 \), then so can \( C^{1/\tau} \). In particular, \( \left( \frac{1}{1 - 2^{-\tau c}} \right)^{1/\tau} \leq \max \left\{ 1, \left( \tau_0 c \right)^{-1/\tau_0} \right\} \). Moreover, using again that \( \tau \geq \tau_0 \), we get \( \ell^{\omega} \ni \ell^{\tau} \), where the embedding does not increase the norm. Hence,

\[
\left( \frac{1}{1 - 2^{-\tau c}} \right)^{1/\tau} = \left( \sum_{n=0}^{\infty} 2^{-\tau_0 c} \right)^{1/\tau} \leq \left( \sum_{n=0}^{\infty} 2^{-\tau_0 c} \right)^{1/\tau_0} = \left( \frac{1}{1 - 2^{-\tau_0 c}} \right)^{1/\tau_0} =: \Omega_2.
\]

Similarly, since \( \frac{1}{1 - \alpha} \geq 1 \) for \( \alpha < 1 \), we have \( \left( \frac{1}{1 - \alpha} \right)^{1/\tau_0} =: \Omega_3 \).

Finally, using once more that \( \tau \geq \tau_0 \), we see

\[
\left( \sum_{i=1}^{n} a_i \right)^{1/\tau} \leq \left( n \cdot \max \{a_1, \ldots, a_n\} \right)^{1/\tau_0} \leq n^{1/\tau_0} \cdot \max \left\{ a_1^{1/\tau_0}, \ldots, a_n^{1/\tau_0} \right\}
\]

for arbitrary \( a_1, \ldots, a_n \geq 0 \). Thus, if \( a_1^{1/\tau_0}, \ldots, a_n^{1/\tau_0} \) can be estimated only in terms of \( \alpha, \tau_0, \omega, c, K, H, M_1, M_2 \), then so can \( \left( \sum_{i=1}^{n} a_i \right)^{1/\tau} \). All in all, we have shown that the set of all expressions/coefficients \( C \geq 0 \) for which \( C^{1/\tau} \) can be estimated only in terms of \( \alpha, \tau_0, \omega, c, K, H, M_1, M_2 \) is closed under multiplication and addition. Hence, it suffices to show \( C_i^{1/\tau} \leq L_i \) for \( i \in \{1, \ldots, 10\} \), where \( L_i \) only depends on \( \alpha, \tau_0, \omega, c, K, H, M_1, M_2 \).

In this end, recall that \( \omega \leq \omega \) and that \( M_3 = \max \{M_1, M_2\} \) only depends on \( M_1, M_2 \). Hence, recalling that the constants \( C_2, \ldots, C_{10} \) are only needed in case of \( \alpha \in [0, 1) \), we get

\[
C_1^{1/\tau} = 3 \cdot 8^{\alpha} \cdot 6^{\tau_0 + K + M_3} \cdot 2 \cdot 1^{-\tau_0} \cdot 2^{1/\tau_0} \cdot 2^{6\omega} \cdot 2^{6\omega} =: L_1,
\]

\[
C_2^{1/\tau} = 6^{2\alpha} \cdot 2^{6\omega} \cdot 2^{6\omega} \leq 6^{2\omega} \cdot 2^{6\omega} =: L_2,
\]

\[
C_3^{1/\tau} = \left( \frac{4^{\alpha}}{2^{3\tau_0 + K + M_3}} \cdot 2 \cdot 3^{2\omega} \cdot 2^{3\omega} \right) \leq L_2 \cdot 2 \cdot 3^{2\tau_0 + K + M_3} \cdot 4^{\omega} \cdot 2^{3\omega} =: L_3,
\]

\[
C_4^{1/\tau} = 6^{2\alpha} \cdot 3^{4\omega + K + M_3} \leq 6^{2\omega} \cdot 3^{4\omega + K + M_3} =: L_4,
\]

\[
C_5^{1/\tau} = \left( \frac{4^{\alpha}}{2^{3\tau_0 + K + M_3}} \cdot 2^{3\omega + 6\omega} \right) \leq L_4 \cdot 2^{\frac{1}{\tau_0}} \cdot 2^{3\omega + 6\omega} =: L_5,
\]

\[
C_6^{1/\tau} = 6^{2\alpha} \cdot 3^{4\omega + K + M_3} \leq 6^{2\omega} \cdot 3^{4\omega + K + M_3} =: L_6,
\]

\[
C_7^{1/\tau} = \left( \frac{4^{\alpha}}{2^{3\tau_0 + K + M_3}} \cdot 2^{3\omega + 6\omega} \right) \leq L_6 \cdot 2^{\frac{1}{\tau_0}} \cdot 2^{3\omega + 6\omega} =: L_7,
\]

\[
C_8^{1/\tau} = 2^{3\omega} \cdot M_3 \cdot 4^{K} \leq 6^{\omega} \cdot 3^{4\omega + K + M_3} \cdot 4^{K} =: L_8,
\]

\[
C_9^{1/\tau} = 6^{2\alpha} \cdot 3^{4\omega + K + M_3} \cdot 4^{K} \leq 6^{\omega} \cdot 3^{4\omega + K + M_3} \cdot 4^{K} =: L_9,
\]

\[
C_{10}^{1/\tau} = 6^{2\alpha} \cdot 3^{4\omega + K + M_3} \cdot 4^{K} \leq 6^{\omega} \cdot 3^{4\omega + K + M_3} \cdot 4^{K} =: L_{10},
\]

where the constants \( L_1, \ldots, L_{10} \) only depend on \( \alpha, \tau_0, \omega, c, K, H, M_1, M_2 \). Taken together, these considerations easily imply \( C_0^{(2)} \leq \left[ C_0^{(2)} \right]^7 \) and \( C_0^{(3)} \leq \left[ C_0^{(3)} \right]^7 \), where \( C_0^{(2)} \) and \( C_0^{(3)} \) only depend on \( \alpha, \tau_0, \omega, c, K, H, M_1, M_2 \).
Likewise, the constant $C_0^{(1)} = 2^{19+7\sigma+r(5+2K+2M_3)} / (1-2-rc)$ from Subsection C.1 can be estimated by

$$C_0^{(1)} = \Omega_2 \cdot 2^{2\frac{2S}{\varepsilon}+7\sqrt{s}+5+2K+2M_3} \leq \Omega_2 \cdot 2^{\frac{12S}{\varepsilon}+7\sqrt{s}+5+2K+2M_3} =: C_0^{(1)}$$

where $C_0^{(1)}$ only depends on $\alpha$, $\tau_0$, $\omega$, $c$, $K$, $H$, $M_1$, $M_2$.

C.3. We have $\ell_1 = (1,0)$ and $\ell_2 = (1,1)$. Geometrically, this case means that $i$ belongs to the right cone, while $j$ belongs to the upper cone. In this case, we have $i = (n,m,1)$ and $j = (\nu, \mu, 1)$ and hence—because of $R = R_{10}$—that

$$T_{\nu, \mu, 1, 0}^{-1} T_{n,m,1,0} = \left( R \cdot A_{\nu, \mu, 1}^{(\alpha)} \right)^{-1} A_{n,m,1}^{(\alpha)} = \left( R \cdot A_{\nu, \mu, 1}^{(\alpha)} \right)^{-1} \cdot \left( R \cdot A_{n,m,1}^{(\alpha)} \right) = T_{\nu, \mu, 1, 0}^{-1} T_{n,m,1,0}\,$$

Since furthermore $\omega_{(\nu, \mu, 1, 0)} = \omega$ and since the weight $w_{(n,m,\varepsilon, \delta)} = 2^n$ is independent of $\varepsilon, \delta$, we get

$$M_{(\nu, \mu, 1, 0), (n,m,1,0)}^{(0)} = \left( \frac{w_{\nu, \mu, 1}}{w_{n,m,1}} \right)^\tau \left( \sum_{\tau}^{(1)} \sum_{\tau}^{(1)} \int_{S^{(\alpha)}_{(n,m,1)}} \omega_{(\nu, \mu, 1)} T_{(\nu, \mu, 1), 0}^{-1} T_{(n,m,1), 0} \right)^\tau$$

But Subsection C.2 shows under the assumptions of Lemma 4.1 that

$$\sup_{i \in I^{(1)}} \sum_{j \in I^{(1)}} M_{j,i}^{(0)} = \sup_{n \in \mathbb{N}_0} \sum_{\nu \in \mathbb{N}_0} \sum_{\mu \in \mathbb{N}_0} M_{(\nu, \mu, 1, 0), (n,m,1,0)}^{(0)} \leq C_0^{(2)} \leq \left[ C_0^{(2)} \right]^\tau$$

and

$$\sup_{j \in I^{(1)}} \sum_{i \in I^{(1)}} M_{j,i}^{(0)} = \sup_{\nu \in \mathbb{N}_0} \sum_{\mu \in \mathbb{N}_0} \sum_{n \in \mathbb{N}_0} M_{(\nu, \mu, 1, 0), (n,m,1,0)}^{(0)} \leq C_0^{(3)} \leq \left[ C_0^{(3)} \right]^\tau$$

C.4. We have $\ell_1 = (1,0)$ and $\ell_2 = (1,1)$. Geometrically, this case means that both $i$ and $j$ belong to the upper cone. In this case, we have $i = (n,m,1)$ and $j = (\nu, \mu, 1, 1)$ and hence

$$T_{\nu, \mu, 1, 1}^{-1} T_{n,m,1,1} = \left( R \cdot A_{\nu, \mu, 1}^{(\alpha)} \right)^{-1} \cdot \left( R \cdot A_{n,m,1}^{(\alpha)} \right) = \left( R \cdot A_{\nu, \mu, 1}^{(\alpha)} \right)^{-1} \cdot \left( R \cdot A_{n,m,1}^{(\alpha)} \right) = T_{\nu, \mu, 1, 0}^{-1} T_{n,m,1,0},$$

as well as $\omega_{(\nu, \mu, 1, 1)} = \omega_{(\nu, \mu, 1, 0)} = \omega$. This implies precisely as in the preceding subsection that

$$M_{(\nu, \mu, 1, 1), (n,m,1,1), 0}^{(0)} = M_{(\nu, \mu, 1, 0), (n,m,1,0)}^{(0)} \cdot$$

Then, we use that Subsection C.4 shows under the assumptions of Lemma 4.1 that

$$\sup_{i \in I^{(1)}} \sum_{j \in I^{(1)}} M_{j,i}^{(0)} \leq C_0^{(1)} \leq \left[ C_0^{(1)} \right]^\tau$$

and

$$\sup_{j \in I^{(1)}} \sum_{i \in I^{(1)}} M_{j,i}^{(0)} \leq C_0^{(1)} \leq \left[ C_0^{(1)} \right]^\tau.$$
C.5. We have \( \ell_1, \ell_2 \in \{-1\} \times \{0,1\} \). This case comprises all the cases considered in Subsections C.1–C.4, with the only difference that geometrically the lower and left cones are considered instead of the upper and right cones. In this case, we have \( i = (n, m, -1, \delta) \) and \( j = (\nu, \mu, -1, d) \) and hence
\[
T_{\nu, \mu, -1, d}^{-1} T_{n, m, -1, \delta} = (-1) \cdot (-1) \cdot T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta} = T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta},
\]
as well as \( \varrho_{\nu, \mu, -1, d} = \varrho_{\nu, \mu, 1, d} = \varrho \). As in Subsection C.2, this implies that
\[
M_{(\nu, \mu, -1, d), (n, m, -1, \delta)}^{(0)} = M_{(\nu, \mu, 1, d), (n, m, 1, \delta)}^{(0)}.
\]
Hence, depending on \( \delta \) and \( d \) we get the same estimates as in Subsections C.2.

C.6. We have \( \ell_1 \in \{-1\} \times \{0,1\} \) and \( \ell_2 \in \{1\} \times \{0,1\} \). Geometrically this means that \( i \) belongs to the left or lower cone and \( j \) belongs to the right or upper cone. In this case, we have \( i = (n, m, -1, \delta) \) and \( j = (\nu, \mu, 1, d) \) and hence
\[
T_{\nu, \mu, 1, d}^{-1} T_{n, m, -1, \delta} = (-1) \cdot T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta}.
\]
Consequently, we get \( \| T_{\nu, \mu, 1, d}^{-1} T_{n, m, -1, \delta} \| = \| T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta} \| \). Now, since we have \( \varrho (-\xi) = \varrho (\xi) \) for all \( \xi \in \mathbb{R}^2 \) and \( \varrho_{(\nu, \mu, 1, d)} = \varrho \), we finally see
\[
\int_Q \varrho_{(\nu, \mu, 1, d)} \left( T_{\nu, \mu, 1, d}^{-1} T_{n, m, -1, \delta} \xi \right) d\xi = \int_Q \varrho_{(\nu, \mu, 1, d)} \left( -T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta} \xi \right) d\xi = \int_Q \varrho_{(\nu, \mu, 1, d)} \left( T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta} \xi \right) d\xi.
\]
As before this implies \( M_{(\nu, \mu, 1, d), (n, m, -1, \delta)}^{(0)} = M_{(\nu, \mu, 1, d), (n, m, 1, \delta)}^{(0)} \) and depending on \( \delta \) and \( d \) we get the same estimates as in Subsections C.1.

C.7. We have \( \ell_1 \in \{1\} \times \{0,1\} \) and \( \ell_2 \in \{-1\} \times \{0,1\} \). Geometrically this means that \( i \) belongs to the right or upper cone and \( j \) belongs to the left or lower cone. In this case, we have \( i = (n, m, 1, \delta) \) and \( j = (\nu, \mu, -1, d) \) and hence
\[
T_{\nu, \mu, -1, d}^{-1} T_{n, m, 1, \delta} = (-1) \cdot T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta}.
\]
Consequently, \( \| T_{\nu, \mu, -1, d}^{-1} T_{n, m, 1, \delta} \| = \| T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta} \| \). Now, since \( \varrho_{(\nu, \mu, -1, d)} = \varrho_{(\nu, \mu, 1, d)} = \varrho \) and since \( \varrho (-\xi) = \varrho (\xi) \) for all \( \xi \in \mathbb{R}^2 \), we get
\[
\int_Q \varrho_{(\nu, \mu, -1, d)} \left( T_{\nu, \mu, -1, d}^{-1} T_{n, m, 1, \delta} \xi \right) d\xi = \int_Q \varrho_{(\nu, \mu, 1, d)} \left( -T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta} \xi \right) d\xi = \int_Q \varrho_{(\nu, \mu, 1, d)} \left( T_{\nu, \mu, 1, d}^{-1} T_{n, m, 1, \delta} \xi \right) d\xi.
\]
As before this implies \( M_{(\nu, \mu, -1, d), (n, m, 1, \delta)}^{(0)} = M_{(\nu, \mu, 1, d), (n, m, 1, \delta)}^{(0)} \) and depending on \( \delta \) and \( d \) we get the same estimates as in Subsections C.1.

C.8. We have \( \ell_1 = 0 \) and \( \ell_2 \in \{\pm 1\} \times \{0,1\} \). In this case, we have for \( j = (\nu, \mu, e, d) \in I^{(\ell_2)} \subset I_0 \) and \( i \in I^{(\ell_1)} = I^{(0)} = \{0\} \) that
\[
\| T_{\nu, \mu, 1, d}^{-1} T_i \| = \| T_{\nu, \mu, 1, d}^{-1} \| = \| e \cdot \left( A_{\nu, \mu, 1}^{(0)} \right)^{-1} \cdot R^{-d} \| = \| \left( A_{\nu, \mu, 1}^{(0)} \right)^{-1} \| \leq \left\| \begin{pmatrix} 2^{-\nu} & 0 \\ -2^{-\nu} & 2^{-\alpha} \end{pmatrix} \right\| \leq 3 \cdot \max \{1, 2^{-\nu} | \mu | \}.
\]
But because of \( |\mu| \leq G_\nu = \frac{\rho}{2(1-\alpha)\nu} \leq \frac{\rho}{2\nu} = 2\nu \), we have \( 2^{-\nu} |\mu| \leq 1 \), which yields \( \| T_{\nu, \mu, 1, d}^{-1} T_i \| \leq 3 \).

Next, recall that \( S_{(\nu, \mu, 1)}^{(0)} = S_0^{(0)} = (-1,1)^2 \). Because of \( (-1, 1)^2 = (-1, 1)^2 \), this implies in case of \( d = 0 \) that
\[
T_{\nu, \mu, 1, d}^{-1} S_{(\nu, \mu, 1)}^{(0)} = \left( A_{\nu, \mu, 1}^{(0)} \right)^{-1} (-1,1)^2 = \left\{ \begin{pmatrix} 2^{-\nu} \\ -2^{-\nu} \mu \end{pmatrix} \cdot \xi \right\} \{ \eta_1, \eta_2 \} \in \mathbb{R}^2 \mid \{ \eta_1, \eta_2 \} \in \mathbb{R}^2 \right\}
\]

Likewise, since \( R = R^{-1} \) and since \( (1,1)^2 = (-1,1)^2 \), we also get in case of \( d = 1 \) that
\[
T_{\nu, \mu, 1, d}^{-1} S_{(\nu, \mu, 1)}^{(0)} = \left( A_{\nu, \mu, 1}^{(0)} \right)^{-1} R (-1,1)^2 = \left( A_{\nu, \mu, 1}^{(0)} \right)^{-1} (-1,1)^2
\]

\[
= \left\{ \eta_1, \eta_2 \in \mathbb{R}^2 \mid \eta_1 \in (-2^{-\nu}, 2^{-\nu}), \eta_2 \in (-2^{-\alpha} \nu, 2^{-\alpha} \nu) - \mu \eta_1 \right\}.
\]
Consequently, we get in all cases that

$$|\det T_0|^{-1} \cdot \int_{S_j^{(s)}} \varrho_j (T_j^{-1} \xi) \, d \xi = |\det T_j| \cdot \int_{S_j^{(s)}} \varrho (\eta) \, d \eta$$

$$\leq 2^{(1+\alpha)\nu} \cdot \int_{-2^{-\nu}}^{2^{-\nu}} \theta_1 (\eta_1) \cdot \int_{-\mu n_1 - 2^{-\alpha\nu}}^{\mu n_1 + 2^{-\alpha\nu}} (1 + |\eta_2|)^{-K} \, d \eta_2 \, d \eta_1.$$ 

But for $|\eta_1| \leq 2^{-\nu}$, we have $\theta_1 (\eta_1) \leq |\eta_1|^{M_1} \leq 2^{-M_1 \nu}$, so that

$$|\det T_0|^{-1} \cdot \int_{S_j^{(s)}} \varrho (T_j^{-1} \xi) \, d \xi \leq 2^{-M_1 \nu} \cdot 2^{(1+\alpha)\nu} \cdot \int_{-2^{-\nu}}^{2^{-\nu}} \int_{-\mu n_1 - 2^{-\alpha\nu}}^{\mu n_1 + 2^{-\alpha\nu}} d \eta_2 \, d \eta_1 \leq 4 \cdot 2^{-M_1 \nu}.$$ 

All in all, this implies

$$M_{j,0}^{(0)} = \left( \frac{w_3}{w_0} \right)^{\tau} \cdot (1 + \|T_j^{-1} T_0\|)^{\tau} \cdot \left( |\det T_0|^{-1} \int_{S_j^{(s)}} \varrho_j (T_j^{-1} \xi) \, d \xi \right)^{\tau} \leq 4^{\tau} \cdot 2^{\tau \sigma \nu} \cdot 4^{\tau} \cdot 2^{-M_1 \tau \nu},$$

which yields

$$\sum_{j \in I(\ell_2)} M_j^{(0)} = \sum_{\nu=0}^{\infty} 2^{\tau \nu (s - M_1)} \sum_{\nu=0}^{\infty} 2^{\nu (\tau - M_1)} \sum_{\nu=0}^{\infty} 2^{\nu \tau c} \leq \frac{2^{3\tau} + 2^\tau + 2^\tau}{1 - 2^{-\tau c}} \leq C_0^{(4)},$$

since the assumptions of Lemma 41 entail $M_1 \geq M_1^{(0)} + c \geq s + \frac{1}{\tau} + c \geq s + \frac{1-\alpha}{\tau} + c$.

Likewise, we get

$$\sup_{j \in I(\ell_2)} \sum_{i \in I(0)} M_{j,i}^{(0)} = \sup_{j \in I(\ell_2)} M_{j,0}^{(0)} \leq \sum_{j \in I(\ell_2)} M_j^{(0)} \leq C_0^{(4)}.$$ 

Finally, we see as at the end of Subsection C.2 that

$$C_0^{(4)} \leq \left( \frac{1}{1 - 2^{-\tau c}} \right)^{1/\tau} \cdot 2^{\frac{1}{\tau} + 2\frac{\tau}{\tau c} - 2\frac{\tau}{\tau c}} \leq \left( \frac{1}{1 - 2^{-\tau c}} \right)^{1/\tau} \cdot 2^{\frac{1}{\tau} + 2\frac{\tau}{\tau c} - 2\frac{\tau}{\tau c}} =: C_0^{(4)},$$

where $C_0^{(4)}$ only depends on $\alpha, \tau, \omega, c, K, H, M_1, M_2$.

C.9. We have $\ell_2 = 0$ and $\ell_1 \in \{\pm 1\} \times \{0, 1\}$. In this case, we have for $i = (n, m, \varepsilon, \delta) \in I(\ell_1) \subseteq I_0$ and $j \in I(\ell_2) = I(0) = \{0\}$ that

$$1 + \|T_j^{-1} T_i\| = 1 + \|T_i\| = 1 + \left\| \begin{pmatrix} 2^n & 0 \\ 0 & 2^{\alpha m} \end{pmatrix} \right\| \leq 5 \cdot 2^n,$$

since $|2^{\alpha m}| \leq 2^{\alpha |G_\nu|} \leq 2^{\alpha (2^{(1-\alpha)n} + 1)} \leq 2 \cdot 2^n$.

Furthermore, we note $\lambda_\delta (Q) \leq 18$, since $Q = Q' = V^{(3-1,5)}_{(-1, 3)} \subset (\frac{1}{3}, 3) \times (-3, 3)$. Thus,

$$|\det T|^{-1} \cdot \int_{S_j^{(s)}} \varrho_j (T_j^{-1} \xi) \, d \xi = \int_{Q} \varrho_0 (T_i \eta) \, d \eta \leq 18 \cdot \sup_{\eta \in Q} \varrho_0 (T_i \eta).$$ 

(C.41)

Now, we distinguish the cases $\delta = 0$ and $\delta = 1$:

1. For $\delta = 0$, we have

$$T_i \eta = \left( \varepsilon \cdot \begin{pmatrix} 2^n \eta_1 \\ 2^n \eta_2 \end{pmatrix} \right) \quad \text{for} \quad \eta = (\eta_1, \eta_2) \in \mathbb{R}^2.$$ 

But for $\eta \in Q$, we have $\frac{1}{3} < \eta_1 < 3$ and hence $2^n \eta_1 \geq 2^n / 3$, so that we get

$$\varrho_0 (T_i \eta) \leq (1 + |\varepsilon \cdot 2^n \eta_1|)^{-H} \leq 3^H \cdot 2^{-H n},$$
which yields \(|\det T_0|^{-1} \cdot \int_{S_0^{(0)}} \varrho_j (T_{j}^{-1} \xi) \, d\xi \leq 18 \cdot 3^H \cdot 2^{-Hn}\) by virtue of equation (C.41).

(2) For \(\delta = 1\), we have

\[
T_i \eta = \left( \varepsilon \cdot (2^{n0} \eta_1 + 2^{n0} \eta_2) \right) \varepsilon \cdot 2^{n0} \eta_1
\]

for \(\eta = (\eta_1, \eta_2) \in \mathbb{R}^2\).

Again, for \(\eta \in Q\), we have \(2^n \eta_1 \geq 2^n/3\) and hence

\[
\varrho_0 (T_i \eta) \leq (1 + |\varepsilon \cdot 2^n \eta_1|)^{-H} \leq 3^H \cdot 2^{-Hn},
\]

which as above yields \(|\det T_0|^{-1} \cdot \int_{S_0^{(0)}} \varrho_0 (T_{0}^{-1} \xi) \, d\xi \leq 18 \cdot 3^H \cdot 2^{-Hn}\).

In total, we get for each case the estimate

\[
M_{0,j}^{(0)} = \left( \frac{w_0^j}{w_0^j} \right)^\tau \cdot (1 + \|T_0^{-1} T_j\|)^\sigma \cdot \left( |\det T_0|^{-1} \cdot \int_{S_0^{(0)}} \varrho_0 (T_{0}^{-1} \xi) \, d\xi \right)^\tau \\
\leq 2^{-\sigma r n} \cdot 5^\sigma \cdot 2^{n\sigma} \cdot 18^\sigma \cdot 3^H \cdot 2^{-Hn} \\
\leq 2^{3\sigma + 5\sigma + 2 H \tau} \cdot 2^{n\tau} (\frac{3}{2} - s - H)
\]

Thus, we get on the one hand

\[
\sum_{i \in I^{(1)}} M_{0,j}^{(0)} \leq 2^{3\sigma + 5\sigma + 2 H \tau} \cdot \sum_{n=0}^{\infty} \sum_{|m| \leq G_n} 2^{n\tau} (\frac{3}{2} - s - H)
\]

(since \(G_n = \left[ 2^{n(1-\alpha)} \right] \leq 1 + 2^{n(1-\alpha)} \leq 2 \cdot 2^{n(1-\alpha)} \) \(\leq 2^{3 + 3\sigma + 5\sigma + 2 H \tau} \cdot \sum_{n=0}^{\infty} 2^{n(1-\alpha)} 2^\sigma (\frac{3}{2} - s - H)
\]

\[
\leq 2^{3 + 3\sigma + 5\sigma + 2 H \tau} \cdot \frac{1}{1 - 2^{-c \sigma}} =: C_0^{(5)},
\]

since the assumptions of Lemma 4.1 imply \(H \geq H_0 + c = \frac{1}{\tau} + \frac{\varepsilon}{\tau} - s + c\).

Likewise, the summation over \(j\) yields

\[
\sup_{i \in I^{(1)}} \sum_{j \in I^{(0)}} M_{0,j}^{(0)} = \sup_{i \in I^{(1)}} M_{0,i}^{(0)} \leq \sum_{i \in I^{(1)}} M_{0,i}^{(0)} \leq C_0^{(5)}.
\]

Finally, we get as at the end of Subsection C.2 that

\[
\left[ C_0^{(5)} \right]^{1/\tau} \leq \left( \frac{1}{1 - 2^{-c \sigma_0}} \right)^{1/\tau} \cdot 2^{\frac{3}{2} + \frac{3}{2} + \frac{5}{2} + 2 H} \leq \left( \frac{1}{1 - 2^{-c \sigma_0}} \right)^{1/\tau} \cdot 2^{\frac{3}{2} + \frac{3}{2} + \frac{5}{2} + 2 H} =: C_0^{(6)},
\]

where \(C_0^{(6)}\) only depends on \(\alpha, \tau_0, \omega, c, K, H, M_1, M_2\).

C.10. We have \(\ell_1 = \ell_2 = 0\). Here, the sum and the supremum reduce to a single term, namely to

\[
M_{0,0}^{(0)} = \left( \frac{w_0^0}{w_0^0} \right)^\tau \cdot (1 + \|T_0^{-1} T_0\|)^\sigma \cdot \left( |\det T_0|^{-1} \cdot \int_{S_0^{(0)}} \varrho_0 (T_{0}^{-1} \xi) \, d\xi \right)^\tau
\]

(since \(Q_0^{(0)} = (-1,1)^2\)) \(\leq 2^\sigma \cdot \left( \int_{Q_0^{(0)}} (1 + |\xi|)^{-H} \, d\xi \right)^\tau \leq 2^\sigma \cdot [\lambda_d (Q_0^{(0)})]^\tau \leq 2^\sigma \cdot 4^\tau =: C_0^{(6)},
\]

where \(\left[ C_0^{(6)} \right]^{1/\tau} \leq 2^\sigma \cdot 4 \leq 4 \cdot 2^\omega =: C_0^{(6)}\).

C.11. Completing the proof of Lemma 4.1 By recalling equations (C.3) and (C.5) and by collecting our results from Subsections C.1, C.10 we finally conclude that

\[
\max \left\{ \sup_{j \in I} \sum_{i \in I} M_{0,j}^{(0)} : \sup_{i \in I} \sum_{j \in I} M_{0,j}^{(0)} \right\} \leq 25 \cdot \left( \max \left\{ C_0^{(1)}, C_0^{(2)}, C_0^{(3)}(1), C_0^{(4)}, C_0^{(5)}, C_0^{(6)} \right\} \right)^\tau,
\]

given that the assumptions of Lemma 4.1 are fulfilled. This easily yields the claim of Lemma 4.1

\[\square\]
APPENDIX D. THE PROOF OF PROPOSITION D.2 IN THE GENERAL CASE

Recall that the parameter $\alpha$ for the definition of the $\beta$-shearlet smoothness spaces $\mathcal{S}^{p,q}_{\beta,\delta}(\mathbb{R}^2)$ satisfies $\alpha \in [0,1]$, as for the theory of $\alpha$-molecules developed in [37] or as for $\alpha$-curvelets [38]. In contrast, there is a definition of cone-adapted $\beta$-shearlets (cf. [37, Definition 3.10]) for $\beta \in (1,\infty)$.

In this section we introduce so-called **reciprocal $\beta$-shearlet smoothness spaces** $\mathcal{S}^{p,q}_{\beta,\delta}(\mathbb{R}^2)$ which will turn out to be the smoothness spaces associated to $\beta$-shearlets. Our main goal is to show $\mathcal{S}^{p,q}_{\beta,\delta}(\mathbb{R}^2) = \mathcal{S}^{p,q}_{\beta^{-1},\delta}(\mathbb{R}^2)$ for $\beta \in (1,\infty)$, i.e., the reciprocal $\beta$-shearlet smoothness spaces coincide with the usual $\alpha$-shearlet smoothness spaces for $\alpha = \beta^{-1}$. This will allow us to transfer approximation results that are known for $\beta$-shearlets to approximation results for $\alpha$-shearlets, which is not entirely trivial, since the two definitions differ quite heavily for $\beta \neq 2$, see also the discussion before Definition 5.6. Once this property from $\beta$-shearlets to $\alpha$-shearlets is established, we use it to prove Proposition D.2 for $\beta \in (1,2)$.

We begin with the definition of the reciprocal $\beta$-shearlet covering:

**Definition D.1.** For $\beta \in (1,\infty)$, define

$$J_0 := J_0^{(\beta)} := \{ (j,\ell,\delta) \in \mathbb{N}_0 \times \mathbb{Z} \times \{0,1\} \mid |\ell| \leq H_j \} \quad \text{with} \quad H_j := H_j^{(\beta)} := \left[ 2^{\frac{j}{\beta}(\beta-1)} \right].$$

Furthermore, recall the matrices $S_{\alpha}, D^{(\alpha)}_{\mu}$ and $R$ from equation (1.5), and define

$$Y^{(\beta)}_{j,\ell,\delta} := Y^{(\beta)}_{j,\ell,\delta} := R^\delta \cdot D^{(1/\beta)}_{\mu_j}(2^{\ell/\beta}) \cdot S_{\beta}^T \quad \text{for} \quad (j,\ell,\delta) \in J_0$$

and $P_{j} := P := U_{(-3,3)}^{(n_0^{-1},\mu_0)} \cup (-U_{(-3,3)}^{(n_0^{-1},\mu_0)})$ for $j \in J_0$ with $U_{(a,b)}^{(\gamma,\mu)}$ as in equation (3.1) and with $\mu_0 := \mu_0^{(\beta)} := 3 \cdot 2^{1/2}$.

Finally, define $J := J^{(\beta)} := \{ 0 \} \cup J_0$, set $c_j := 0$ for all $j \in J$ and $Y_0 := Y^{(\beta)}_0 := \text{id}$, as well as $P_0 := (-1,1)^2$.

Then, the **reciprocal $\beta$-shearlet covering** is defined as

$$S^{(\beta)} := (S^{(\beta)}_j)_{j \in J} := (Y^{(\beta)}_j P_{j})_{j \in J} = (Y^{(\beta)}_j P_{j} + c_j)_{j \in J}. \quad \blacksquare$$

**Remark.** The notation $S^{(\beta)}$ for the reciprocal $\beta$-shearlet covering might appear to be ambiguous with the notation $S^{(\alpha)}$ for the $\alpha$-shearlet covering introduced in Definition 5.1, but this is no real ambiguity: The parameter $\beta$ in the preceding definition always satisfies $\beta \in (1,\infty)$, while the parameter $\alpha$ from Definition 5.1 satisfies $\alpha \in [0,1]$, so that no ambiguity is possible. ♦

As for the usual $\alpha$-shearlet covering, our first goal is to show that $S^{(\beta)}$ is an almost structured covering of $\mathbb{R}^2$. In this case, however, it will turn out to be useful to show the following slightly more general result:

**Lemma D.2.** Let $\beta \in (1,\infty)$, $a, b \in \mathbb{R}$ and $\gamma, \mu, A \in (0,\infty)$ be arbitrary and let $U := U_{(a,b)}^{(\gamma,\mu)} \cup (-U_{(a,b)}^{(\gamma,\mu)})$, as well as $U_0 := (-A,A)^2$. Define $U_j := U$ for $j \in J_0$ and consider the family

$$U := (U_j)_{j \in J} := (Y^{(\beta)}_j U_j')_{j \in J}. \quad \blacksquare$$

Then there are constants $N \in \mathbb{N}$ and $C, L \geq 1$ (depending on $\beta, a, b, \gamma, \mu, A$) such that the following are true:

1. We have $L^{-1} \cdot 2^{n/2} \leq |\xi| \leq L \cdot 2^{n/2}$ for all $\xi \in U_{n,m,\varepsilon}$ and arbitrary $(n,m,\varepsilon) \in J_0$.
2. We have $|\ell| \leq N$ for all $\ell \in J$ and $i^* := \{ j \in J \mid U_j \cap U_i \neq \emptyset \}$.
3. We have $\|Y^{(\beta)}_i - Y^{(\beta)}_j\| \leq C$ for all $i \in J$ and $j \in i^*$. ♦

**Proof.** The proof uses the same ideas as that of Lemma 5.3 and is only provided here for completeness.

Set $c := \max \{|a|, |b|\}$ and note $U_{(a,b)}^{(\gamma,\mu)} \subseteq U_{(a,b)}^{(\gamma,\mu)}(-c,c)$, so that we can assume $a = -c$ and $b = c$, since the claim of the lemma is stronger the larger the set $U_{(a,b)}^{(\gamma,\mu)}$ is. By even further enlarging this set, we can also assume $c \geq 1$. With the same reasoning, we can assume $A \geq 1$.

Next, note with $U_{(\kappa,\lambda)}^{(\nu,\xi)}_{(B,C)}$ as in equation (3.1) that

$$V_{(\kappa,\lambda)}^{(\nu,\xi)} := U_{(\kappa,\lambda)}^{(\nu,\xi)} \cup (-U_{(\kappa,\lambda)}^{(\nu,\xi)}) = \left\{ \left( \begin{array}{c} \xi \\ \eta \end{array} \right) \in \mathbb{R}^2 \times \mathbb{R} \mid |\xi| \in (\kappa,\lambda) \text{ and } \frac{\eta}{\xi} \in (B,C) \right\}$$

for arbitrary $B, C \in \mathbb{R}$ and $\kappa, \lambda > 0$. It is now an easy consequence of equation (3.2) and of $a = -c$ and $b = c$ that

$$U_{n,m,0} = V_{(\kappa,\lambda)}^{(n/2,\mu/2)} \cap (2^{n-1/2} \cdot 2^{m-1/2}) \quad \forall (n,m,0) \in J_0. \quad (D.2)$$
Now, since we have $m + c \leq |m| + c$ and $m - c \geq -|m| - c = -(|m| + c)$, we get for arbitrary $\left(\frac{\xi}{\eta}\right) \in U_{(n,m,0)}$ because of $|m| \leq \left[2^{\frac{\alpha}{2}}(\beta-1)\right] \leq 2^{\frac{\alpha}{2}}(\beta-1) + 1$ that

$$\left|\frac{\eta}{\xi}\right| < 2^{\frac{\alpha}{2}(1-\beta)} (|m| + c) \leq 2^{\frac{\alpha}{2}(1-\beta)} \left(2^{\frac{\alpha}{2}(\beta-1)} + 1 + c\right) \leq c + 2 \leq 3c. \quad (D.3)$$

Here, we used that $2^{\frac{\alpha}{2}(1-\beta)} \leq 1$, since $\beta > 1$. Consequently, we get

$$\gamma \cdot 2^{\frac{\alpha}{2}}n \leq |\xi| \leq \left|\frac{\xi}{\eta}\right| \leq |\xi| + |\eta| \leq (1 + 3c) \cdot |\xi| < 2^{\frac{\alpha}{2}}n \cdot 4\mu_c.$$  

This establishes the first part of the lemma for $L := \max \{\gamma^{-1}, 4\mu_c, 1\}$, since we have $U_{n,m,1} = R \cdot U_{n,m,0}$ and $|R\xi| = |\xi|$ for all $\xi \in \mathbb{R}^2$.

Now, let $i = (n, m, \delta) \in J_0$ be fixed and let $(j, \ell, \varepsilon) \in J_0$ such that there is some $\left(\frac{\xi}{\eta}\right) \in U_{n,m,\delta} \cap U_{j,\ell,\varepsilon} \neq \emptyset$. In the following, we want to derive conditions on $(j, \ell, \varepsilon)$ which allow us to estimate the set $i^n$, as well as the norm $\|Y_{n,0}^{-1}Y_j\|$.  

First of all, set $M := \left\lceil \frac{\alpha}{2} \cdot \log_2 (L^2) \right\rceil \in \mathbb{N}_0$, so that $2^M \geq 2^{\frac{\alpha}{2} \cdot \log_2 (L^2)}$ and thus $2^{\frac{\alpha}{2}M} \geq \log_2 (L^2) = L^2$. Consequently, the first part of the lemma implies $L^{-1} \cdot 2^{\frac{\alpha}{2}j} \leq \left|\left(\frac{\xi}{\eta}\right)\right| \leq L \cdot 2^{\frac{\alpha}{2}n}$ and thus $2^{\frac{\alpha}{2}(j-n)} \leq L^2 \leq 2^{\frac{\alpha}{2}M}$, which entails $j - n \leq M$. By symmetry, we in fact get $|j - n| \leq M$ and thus $j \in \{n - M, \ldots, n + M\}$.

In order to establish further conditions on $(j, \ell, \varepsilon)$, we distinguish several cases depending on $\varepsilon, \delta$:

**Case 1:** We have $\varepsilon = \delta = 0$. In this case, equation (D.2) shows

$$2^{\frac{\alpha}{2}n} (m - c) < \frac{\eta}{\xi} < 2^{\frac{\alpha}{2}n} (m + c) \quad \text{and} \quad 2^{\frac{\alpha}{2}j} \ (\ell - c) < \frac{\eta}{\xi} < 2^{\frac{\alpha}{2}j} \ (\ell + c).$$

By rearranging, this implies for $C_1 := \left(2^{\frac{\alpha}{2}M} + 1\right) \cdot c$ that

$$\ell \leq 2^{\frac{\alpha}{2}j-n} (m+c) + 2^{\frac{\alpha}{2}j-n} m + C_1, \quad \text{as well as} \quad \ell \geq 2^{\frac{\alpha}{2}j-n} (m-c) - c \geq 2^{\frac{\alpha}{2}j-n} m - C_1.$$ 

Consequently, with

$$\Gamma_{n,m,t} := Z \cap \left[2^{\frac{\alpha}{2}(t-n)} m - C_1, 2^{\frac{\alpha}{2}(t-n)} m + C_1 \right],$$

we have established $(j, \ell, \varepsilon) \in \bigcup_{t=n-M}^{n+M} \{t\} \times \Gamma_{n,m,t} \times \{0\}$. But since every (closed) interval $I = [B, D]$ satisfies $|I \cap Z| \leq 1 + D - B$, we have $|\Gamma_{n,m,t}| \leq 1 + 2C_1$ and thus

$$|\{j, \ell, 0\} \in J_0 \ |U_{j,\ell,0} \cap U_{n,m,0} \neq \emptyset\} \leq \sum_{t=n-M}^{n+M} |\{t\} \times \Gamma_{n,m,t} \times \{0\}| \leq (1 + 2M) \cdot (1 + 2C_1). \quad (D.4)$$

Furthermore, a direct computation shows

$$Y_{n,0}^{-1} Y_{j,0} = \left(\begin{array}{ccc}
2^{\frac{\alpha}{2}(j-n)} m & 0 \\
2^{\frac{\alpha}{2}n} \ell - 2^{\frac{\alpha}{2}(j-n)} m & 2^{\frac{\alpha}{2}n}
\end{array}\right).$$

But thanks to $|j - n| \leq M$, we have $0 \leq 2^{\frac{\alpha}{2}(j-n)} \leq 2^{\frac{\alpha}{2}M}$ and $0 \leq 2^{\frac{\alpha}{2}n} \leq 2^{\frac{\alpha}{2}M}$. Finally, we saw above that $\left|\ell - 2^{\frac{\alpha}{2}(j-n)} m\right| \leq C_1$, so that

$$\left|2^{\frac{\alpha}{2}n} \ell - 2^{\frac{\alpha}{2}(j-n)} m\right| = 2^{\frac{\alpha}{2}n} \left|\ell - 2^{\frac{\alpha}{2}(j-n)} m\right| \leq 2^{\frac{\alpha}{2}M} C_1.$$ 

All in all, this implies $\|Y_{n,0}^{-1} \cdot Y_{j,0}\| \leq 2^{\frac{\alpha}{2}M} + 2^{\frac{\alpha}{2}M} + 2^{\frac{\alpha}{2}M} C_1$ and thus concludes our considerations for the present case.

**Case 2:** We have $\varepsilon = 1$ and $\delta = 0$. In this case, a direct calculation shows

$$Y_{n,0}^{-1} Y_{j,1} = \left(\begin{array}{ccc}
2^{\frac{\alpha}{2}(j-\beta_n)} \ell & 2^{\frac{\alpha}{2}(j-\beta_n)} m \ell \\
2^{\frac{\alpha}{2}(\beta_j-n)} - 2^{\frac{\alpha}{2}(j-\beta_n)} m \ell & -2^{\frac{\alpha}{2}(j-\beta_n)} m \ell
\end{array}\right). \quad (D.5)$$
We immediately recall that \(|m| \leq \left[ \frac{2^\beta}{\beta-1} \right] \leq 1 + 2 \frac{2^\beta}{\beta-1} \leq 2 \cdot 2^\frac{2}{\beta-1} \) and likewise \(|\ell| \leq 2 \cdot 2^\frac{2}{\beta-1} \). In conjunction with \(|n - j| \leq M \) and \(\beta > 1\), this implies
\[
\begin{align*}
|2^\beta(j-\beta)\ell| &\leq 2 \cdot 2^\beta(j-\beta)n \cdot 2^\beta(j-\beta-1) = 2 \cdot 2^\beta(j-n) \leq 2 \cdot 2^\beta M, \\
|2^\beta(j-\beta)| &\leq 2 \cdot 2^\beta(j-n) \cdot 2\beta(j-\beta) \leq 2 \cdot 2^\beta \leq 2^\beta, \\
|2^\beta(j-\beta)n| &\leq 2 \cdot 2^\beta(j-\beta)n \cdot 2^\beta(j-\beta-1) = 2 \cdot 2^\beta(j-n) \leq 2 \cdot 2^\beta.
\end{align*}
\]
(D.6)

In order to estimate the remaining entry of \(Y_{n,m,0}^{-1}Y_{j,\ell,1}\) and to obtain an estimate similar to equation [D.4], we have to work harder. To this end, define
\[
K := \min \left\{ \left( 12 \cdot c^2 \right)^{-1}, 2(3M \cdot 3c) \right\} \in (0, 1) \quad \text{and} \quad n_0 := \frac{2}{\beta-1} \cdot \log_2 \left( K^{-1} \right) \in (0, \infty).
\]
(D.7)

Based on these quantities, we now distinguish two subcases:

**Case 2(a):** We have \(n \geq M + n_0\). First note that this implies \(j \geq n - M \geq n_0\). Furthermore, we have \(2^\beta(j-\beta) = 2\log_2(K^{-1}) = K^{-1}\) and thus \(2^\beta(j-\beta) \geq K^{-1}\) and \(2^\beta(j-\beta) \geq K^{-1}\). Next, note that equation \(\text{[D.3]}\) implies because of \((\xi/\eta) \in U_{n,m,0}\) that \(|\eta/\xi| < 3c\). Likewise, since
\[
\left( \frac{\eta}{\xi} \right) = R \left( \frac{\xi}{\eta} \right) \in R \cdot U_{j,\ell,1} = RR \cdot U_{j,\ell,1} = U_{j,\ell,0},
\]
(D.8)

another application of equation \(\text{[D.3]}\) shows \(\eta \neq 0\) and \(|\xi/\eta| < 3c\), so that \((3c)^{-1} < |\eta/\xi| < 3c\).

We now claim that this implies \(|m| > c\). Indeed, if this was false, we would get from equation \(\text{[D.2]}\) because of \(2^\beta(1-\beta) \leq K\) that
\[
(3c)^{-1} < \left| \frac{\eta}{\xi} \right| < 2^\beta(1-\beta) \cdot (|m| + c) \leq 2c \cdot 2^\beta(1-\beta) \leq 2cK \leq \frac{2c}{12c^2} = \frac{1}{2} < 3c^2 < (3c)^{-1},
\]
a contradiction. Because of \(|m| > c\) we either have \(m > c\) or \(m < -c\). Let us now set \(C_2 := 2M \cdot 3c\) and distinguish these two subcases:

**Case 2(a)(i):** We have \(m > c\). We first claim that this implies \(m \geq 2^\beta(j-\beta) - C_2\). To see this, assume towards a contradiction that \(m < 2^\beta(j-\beta) - C_2\). But equation \(\text{[D.2]}\) shows because of \((\xi/\eta) \in U_{n,m,0}\) that
\[
0 < 2^\beta(j-\beta) \cdot (m - c) < \frac{\eta}{\xi} < 2^\beta(1-\beta) \cdot (m + c) < 2^\beta(1-\beta) \cdot \left( 2^\beta(j-\beta) - C_2 + c \right).
\]
By taking reciprocals and noting \(C_2 \geq 3c > c\), we arrive at
\[
\frac{\xi}{\eta} > \frac{2^\beta(j-\beta)}{2^\beta(1-\beta) - C_2 + c} = 1 + \frac{C_2 - c}{2^\beta(1-\beta) - C_2 + c} > 1 + \frac{C_2 - c}{2^\beta(j-\beta)}.
\]
But another application of equations \(\text{[D.2]}\) and \(\text{[D.8]}\) shows because of \(|\ell| \leq \left[ 2^{j(\beta-1)/2} \right] \leq 1 + 2^{j(\beta-1)/2}\) that
\[
\frac{\xi}{\eta} < 2^\beta(j-\beta) \cdot (\ell + c) \leq 2^\beta(1-\beta) \left( 2^\beta(j-\beta) + 1 + c \right) \leq 1 + \frac{1 + c}{2^\beta(j-\beta)}.
\]
A combination of the last two displayed equations finally yields
\[
\frac{C_2 - c}{2^\beta(j-\beta)} < \frac{1 + c}{2^\beta(j-\beta)} \quad \text{and thus} \quad C_2 < c + 2^\frac{j-\beta+1}{j-\beta} \cdot (1 + c) \leq c + 2^\frac{\beta-1}{\beta} \cdot 2c \leq c + 2^\beta \cdot 2c \leq 3c \cdot 2M = C_2,
\]
a contradiction. Here, we used that \(|n - j| \leq M\) and that \(c \geq 1\). This contradiction shows \(m \geq 2^\beta(j-\beta) - C_2\).

Now, we claim similarly that \(\ell \geq 2^\beta(j-\beta) - C_2\). To see this, assume towards a contradiction that \(\ell < 2^\beta(j-\beta) - C_2\). Recall from equation \(\text{[D.2]}\) and because of \(m > c\) that \(\frac{\xi}{\eta} \geq 2^\beta(j-\beta) \cdot (m - c) > 0\), so that also \(\frac{\xi}{\eta} > 0\). Now, an application of equations \(\text{[D.2]}\) and \(\text{[D.8]}\) shows
\[
0 < \frac{\xi}{\eta} < 2^\beta(j-\beta) \cdot (\ell + c) < 2^\beta(j-\beta) \cdot \left( 2^\beta(j-\beta) - C_2 + c \right).
\]
By taking reciprocals, we get as above because of \(C_2 \geq 3c > c\) that
\[
\frac{\eta}{\xi} > \frac{2^\beta(j-\beta)}{2^\beta(j-\beta) - C_2 + c} = 1 + \frac{C_2 - c}{2^\beta(j-\beta) - C_2 + c} > 1 + \frac{C_2 - c}{2^\beta(j-\beta)}.
\]
But equation (D.10) shows because of $\left(\xi \right) \in U_{n,m,0}$ and since $|m| \leq \left[2^{n(\beta-1)/2}\right] \leq 1 + 2^{n(\beta-1)/2}$ that
\[
\eta \xi < 2^{\frac{1}{2}(1-\beta)} (m + c) \leq 2^{\frac{1}{2}(1-\beta)} \left(2^{\frac{1}{2}(\beta-1)} + 1 + c \right) \leq 1 + \frac{1 + c}{2^{\beta(\beta-1)}}.
\]
Again, by combining the preceding two displayed equations, we obtain a contradiction.

We have thus shown $\ell \geq 2^{\frac{1}{2}(\beta-1)} - C_2 \geq 2^{\frac{1}{2}(\beta-1)} - \left(1 + C_2\right) \geq \left[2^{\frac{1}{2}(\beta-1)}\right] - (1 + [C_2])$. Hence, setting $C_3 := 1 + [C_2]$, we have shown for $n \geq M + n_0$ and $m \geq 0$ (which entails $m > c$) that
\[
\left|\{(j, \ell, 1) \in J_0 | U_{j,\ell,1} \cap U_{n,m,0} \neq \emptyset\}\right| \leq \sum_{t=-n-M}^{n+M} |t| \times \left\{|2^{\frac{1}{2}(\beta-1)} - C_3, \ldots, [2^{\frac{1}{2}(\beta-1)}]\} \times \{1\}\right|
\leq (1 + 2M) \cdot (1 + C_3).
\]
Now, we can finally also estimate the remaining entry of the transition matrix $Y_{n,m,0}^{-1}Y_{j,\ell,1}$ (cf. equation (D.5)): Recall from the beginning of Case 2(a) and from equation (D.7) that $2^{\frac{1}{2}(\beta-1)} \geq K^{-1} \geq 2^{\beta M} \cdot 3c = C_2$ and likewise that $2^{\frac{1}{2}(\beta-1)} \geq C_2$. Hence, $\ell \geq 2^{\frac{1}{2}(\beta-1)} - C_2 \geq 0$ and similarly $m \geq 0$, so that
\[
0 \leq \ell m \leq \left[2^{\frac{1}{2}(\beta-1)}\right] \cdot \left[2^{\frac{1}{2}(\beta-1)}\right] \leq \left(1 + 2^{\frac{1}{2}(\beta-1)}\right) \cdot \left(1 + 2^{\frac{1}{2}(\beta-1)}\right) = 2^{\beta(\beta-1)} 2^{\beta(\beta-1)} + 2^{\beta(\beta-1)} + 2^{\beta(\beta-1)} + 1.
\]
Consequently, we get because of $\ell \geq 2^{\frac{1}{2}(\beta-1)} - C_2 \geq 0$ and $\ell \geq 2^{\frac{1}{2}(\beta-1)} - C_2 \geq 0$
\[
\left|2^{\frac{1}{2}(\beta-j-n)} - 2^{\frac{1}{2}(j-\beta n)} m \ell\right| = 2^{\frac{1}{2}(\beta-j-n)} 2^{\beta(\beta-1)} - m \ell
\leq 2^{\frac{1}{2}(\beta-j-n)} \left(2^{\beta(\beta-1)} 2^{\beta(\beta-1)} + 2^{\beta(\beta-1)} + 2^{\beta(\beta-1)} + 1 - m \ell\right) + 2^{\beta(\beta-1)} + 2^{\beta(\beta-1)} + 1
\leq 2^{\frac{1}{2}(\beta-j-n)} \left(2^{\beta(\beta-1)} 2^{\beta(\beta-1)} + 2^{\beta(\beta-1)} + 2^{\beta(\beta-1)} + 2 - m \ell\right)
\leq 2^{\beta-j-n} \left(2^{\beta(\beta-1)} + 2 \cdot 2^{\beta(\beta-1)} + 2 \cdot 2^{\beta(\beta-1)} + 2 - \left(2^{\beta(\beta-1)} - C_2\right) \left(2^{\beta(\beta-1)} - C_2\right)\right)
\leq 2^{\beta-j-n} \cdot \left(2 + C_2\right) \cdot 2^{\beta(\beta-1)} + 2 + C_2 + 2 \cdot 2^{\beta(\beta-1)} + 2
\leq 2^{\beta-j-n} \cdot \left(2 + C_2\right) \cdot 2^{\beta(\beta-1)} + 2 + C_2 + 2 \cdot 2^{\beta(\beta-1)}
\]
(since $|j-n| \leq M$ and $\beta > 1$) \leq 2^{\beta-M} \cdot \left(2 + C_2\right) \cdot 2^{\beta(\beta-1)} M + 3 \cdot 2^{\beta-M} + C_4.

In conjunction with equation (D.6), this implies $\|Y_{n,m,0}^{-1}Y_{j,\ell,1}\| \leq 2 \cdot 2^{\beta M} + 3 \cdot 2^{\beta M} + C_4$.

Case 2(a)(ii): We have $m < -c$. Here, we set $\bar{m} := -m$ and $\bar{\ell} := -\ell$ and note that
\[
2^{\frac{1}{2}(1-\beta)} (m - c) < \frac{\eta}{\xi} < 2^{\frac{1}{2}(1-\beta)} (m + c),
\]
so that $\left(\xi, -\eta\right) \in U_{n,-m,0} = U_{n,\bar{m},0}$. Likewise, it is not hard to see $\left(\xi, -\eta\right) \in U_{j,-\ell,1} = U_{j,\bar{\ell},1}$, so that Case 2(a)(i) shows (because of $\bar{m} > c$) that $\bar{m} \geq 2^{\frac{1}{2}(\beta-1)} - C_2$ and $\bar{\ell} \geq 2^{\frac{1}{2}(\beta-1)} - C_2 \geq \left[2^{\frac{1}{2}(\beta-1)}\right] - C_3$, which entails $\ell \leq -\left[2^{\frac{1}{2}(\beta-1)}\right] + C_3$. Hence, we have shown for $n \geq M + n_0$ and $m < 0$ (which entails $m < -c$) that
\[
\left|\{(j, \ell, 1) \in J_0 | U_{j,\ell,1} \cap U_{n,m,0} \neq \emptyset\}\right| \leq \sum_{t=-n-M}^{n+M} |t| \times \left\{-\left[2^{\frac{1}{2}(\beta-1)}\right], \ldots, -\left[2^{\frac{1}{2}(\beta-1)}\right] + C_3\right\} \times \{1\}\right|
\leq (1 + 2M) \cdot (1 + C_3),
\]
as in the preceding case.

Finally, because of $\ell m = \bar{\ell} \cdot \bar{m}$, we get $\left|2^{\frac{1}{2}(\beta-j-n)} - 2^{\frac{1}{2}(j-\beta n)} m \ell\right| = \left|2^{\frac{1}{2}(\beta-j-n)} - 2^{\frac{1}{2}(j-\beta n)} \bar{m} \bar{\ell}\right| \leq C_4$ from equation (D.10) and thus $\|Y_{n,m,0}^{-1}Y_{j,\ell,1}\| \leq 2 \cdot 2^{\beta M} + 3 \cdot 2^{\beta M} + C_4$ as in the previous case.

Case 2(b): We have $n \leq n_0 + M$. This implies $j \leq n_0 + 2M$ and $|\ell| \leq 2^{\frac{1}{2}(\beta-1)} \leq 2^{\frac{1}{2}} \leq [2^{\beta(n_0+2M)}]$, because of $|n - j| \leq M$. On the one hand, this implies
\[
\left|\{(j, \ell, 1) \in J_0 | U_{j,\ell,1} \cap U_{n,m,0} \neq \emptyset\}\right| \leq \left\{|0, \ldots, n_0 + 2M| \times \left\{-\left[2^{\beta(n_0+2M)}\right], \ldots, -\left[2^{\beta(n_0+2M)}\right]\right\} \times \{1\}\right|
\leq (n_0 + 2M + 1) \cdot (1 + 2 \cdot 2^{\beta(n_0+2M)}))
\]
and on the other hand
\[
\left\| Y_{n,m,0}^{-1} \right\| \leq \max_{n' \leq M + n_0} \max_{|m'| \leq \left(2^{n'} - 1\right)/2} \max_{j' \leq n_0 + 2M} \max_{|\epsilon'| \leq \left(2^{j'} - 1\right)/2} \left\| Y_{n',m',0}^{-1} \cdot Y_{j',\ell',1} \right\| =: C_5.
\]

**Case 3:** We have \( \varepsilon = \delta = 1 \). Here, we observe that \( U_{n,m,1} \cap U_{j,\ell,1} = R \cdot (U_{n,m,0} \cap U_{j,\ell,0}) \), so that \( U_{n,m,1} \cap U_{j,\ell,1} \neq \emptyset \) if and only if \( U_{n,m,0} \cap U_{j,\ell,0} \neq \emptyset \). Consequently, we get from Case 1, equation (D.4) that
\[
\left| \{(j, \ell, 1) \in J_0 | U_{j,\ell,1} \cap U_{n,m,1} \neq \emptyset \} \right| = \left| \{(j, \ell, 1) \in J_0 | U_{j,\ell,1} \cap U_{n,m,0} \neq \emptyset \} \right| \leq (1 + 2M) \cdot (1 + 2C_1).
\]
Likewise, since \( Y_{n,m,1}^{-1} \cdot Y_{j,\ell,1} = Y_{n,m,0}^{-1} \cdot R^{-1} \cdot R \cdot Y_{j,\ell,0} = Y_{n,m,0}^{-1} \cdot Y_{j,\ell,0} \), we get in case of \( U_{n,m,1} \cap U_{j,\ell,1} \neq \emptyset \) that
\[
\left\| Y_{n,m,1}^{-1} \cdot Y_{j,\ell,1} \right\| \leq 2^{\frac{M}{2}} + 2^M + 2^M C_1,
\]
since \( U_{n,m,0} \cap U_{j,\ell,0} \neq \emptyset \), cf. Case 1.

**Case 4:** We have \( \varepsilon = 0 \) and \( \delta = 1 \). As in the previous case, we observe \( U_{n,m,1} \cap U_{j,\ell,0} = R \cdot (U_{n,m,0} \cap U_{j,\ell,1}) \), so that we can reduce the present case to the setting of Case 2, similar to what was done in Case 3. In view of equations (D.9), (D.11) and (D.12), this implies
\[
\left| \{(j, \ell, 0) \in J_0 | U_{j,\ell,0} \cap U_{n,m,1} \neq \emptyset \} \right| = \left| \{(j, \ell, 1) \in J_0 | U_{j,\ell,1} \cap U_{n,m,0} \neq \emptyset \} \right| \leq \max \{(1 + 2M) \cdot (1 + C_3), (n_0 + 2M + 1) \cdot (1 + 2 \cdot 2^{\beta(2M + 2n)})\},
\]
as well as
\[
\left\| Y_{n,m,1}^{-1} \cdot Y_{j,\ell,0} \right\| \leq \max \left\{ 2 \cdot 2^M + 3 \cdot 2^M + C_4, C_5 \right\},
\]
provided that \( U_{n,m,1} \cap U_{j,\ell,0} \neq \emptyset \).

It remains to consider the case \( i = 0 \) or \( j = 0 \). Recall from the first part of the lemma that \( |\xi| \geq L^{-1} \cdot 2^\beta \) for all \( \xi \in U_{n,m,\varepsilon} \). Conversely, for \( \xi \in U_0 = U_0' \), we have \( |\xi| \leq 2A \), so that \( U_0 \cap U_{n,m,\varepsilon} \neq \emptyset \) can only hold if \( 2^{\frac{\beta}{2}} \leq 2AL \), i.e., \( n \leq \frac{\beta}{2} \cdot \log_2(2AL) =: n_1 \in \mathbb{N}_0 \). On the one hand, this implies because of \( |\ell| \leq 2^{\beta(\beta - 1)} \leq |\beta \ell| \) for \((j, \ell, \varepsilon) \in J_0\) that
\[
|\{j \in J | U_j \cap U_0 \neq \emptyset \}| \leq |\{0\} \cup \{0, \ldots, n_1\} \times \{-[2^{\beta n}], \ldots, [2^{\beta n}]\} \times \{\pm 1\}| \leq 1 + 2 \cdot (1 + n_1) \cdot (1 + 2 \cdot 2^{\beta n_1}).
\]
On the other hand, we get in case of \( U_0 \cap U_{n,m,\varepsilon} \neq \emptyset \) for some \((n, m, \varepsilon) \in J_0\) that
\[
\left\| Y_0^{-1} Y_{n,m,\varepsilon} \right\| = \left\| \left( \begin{array}{c} 2^{\frac{\beta n}{2}} \ 0 \\ 0 \ 2^{\frac{\beta n}{2}} \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right\| \leq \left\| \left( \begin{array}{c} 2^{\frac{\beta n}{2}} \\ 0 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right\| \leq \max \left\{ 2^{\beta n}, 2^{\beta n} \right\} \cdot (2 + |m|) \leq 2^{\beta n_1} \cdot (2 + 2^{\beta n_1}),
\]
as well as
\[
\left\| Y_{n,m,\varepsilon}^{-1} Y_0 \right\| = \left\| \left( \begin{array}{c} 1 \\ -m \end{array} \right) \cdot \left( \begin{array}{c} 2^{\frac{\beta n}{2}} \\ 0 \\ 2^{\frac{\beta n}{2}} \end{array} \right) \right\| \leq \left\| \left( \begin{array}{c} 1 \\ -m \end{array} \right) \cdot \left( \begin{array}{c} 2^{\frac{\beta n}{2}} \\ 0 \end{array} \right) \right\| \leq 2 + |m| \leq 2 + 2^{\beta n_1}.
\]
Taken together, the preceding cases easily yield the claim of the lemma.

As a corollary of the preceding lemma, we can now easily show that the reciprocal \( \beta \)-shearlet covering is indeed an almost structured covering of \( \mathbb{R}^2 \).

**Corollary D.3.** For every \( \beta \in (1, \infty) \), the family \( S^{(\beta)} \) from Definition (D.1) is an almost structured covering of \( \mathbb{R}^2 \). Furthermore, if we set \( v_{n,m,\varepsilon} := 2^{\beta n} \) for \((n, m, \varepsilon) \in J_0 \) and \( v_0 := 1 \), then the weight \( v^s = \left( v_j \right)_{j \in J} \) is \( S^{(\beta)} \)-moderate for arbitrary \( s \in \mathbb{R} \).

Precisely, we have \( C_{S^{(\beta)},v^s} \leq K^{2|s|} \) for some absolute constant \( K = K(\beta) \geq 1 \) which also satisfies
\[
K^{-1} \cdot v_j \leq 1 + |\xi| \leq K \cdot v_j \quad \forall \xi \in S_j^{(\beta)} \quad \text{and all} \ j \in J.
\]

**Proof.** First of all, note that an application of Lemma (D.2) with \( a = -3, b = 3, \mu = \mu_0(\beta) = 3 \cdot 2^{\beta/2} \) and \( \gamma = \mu_1^{-1} \), as well as \( A = 1 \) yields constants \( L, N, \gamma \) satisfying \( L^{-1} \cdot 2^\beta \leq |\xi| \leq L \cdot 2^\beta \) for all \((n, m, \varepsilon) \in J_0^{(\beta)} \) and all \( \xi \in S^{(\beta)}_{n,m,\varepsilon} \), as well as \( |j^*| \leq N \) for all \( j \in J^{(\beta)} \) and finally \( \| Y_{j^*}^{-1} Y_j \| \leq C \) for all \( j \in J^{(\beta)} \) and \( i \in j^* \).

Thus, since we have \( S^{(\beta)} = \{ Y_j P_j^* + c_j \}_{j \in J} \) with \( \{ P_j^* \}_{j \in J} \) having only two elements, in order to establish that \( S^{(\beta)} \) is an almost structured covering of \( \mathbb{R}^2 \) it suffices to prove \( \mathbb{R}^2 \) and
\[
\mathbb{R}^2 = \bigcup_{j \in J} T_j R_j \quad \text{for} \ R_0 := \left( -\frac{a}{4} \frac{b}{4} \right)^2 \text{and} \]
\( R'_j := U_{(\varepsilon, 1)}^{2(-\beta/2, 3/2)} \cup \left[-U_{(\varepsilon, 1)}^{2(-\beta/2, 3/2)}\right], \) since clearly each \( R'_j \) is open with \( \overline{R'_j} \subset P'_j \) and since \( \{R'_j \mid j \in J\} \) is finite.

But an analog of equation \((3.2)\) (see equations \((12.1)\) and \((12.2)\) for more details) shows

\[
Y_{n,m,0}R'_{n,m,0} = V_{2(n(1-\beta)/2)(m-1),2(n(1-\beta)/2)(m+1)}^{2(\beta(n-1)/2,3(\beta(n+1)/2))} = \left\{ \left( \frac{\zeta}{\eta} \right) \in \mathbb{R}^* \times \mathbb{R} \mid |\xi| \in \left(2^{\frac{\beta}{2}}(n-1), 2^{\frac{\beta}{2}}(n+1)\right) \text{ and } \frac{\eta}{\zeta} \in \left(2^{\frac{\beta}{2}}(1-\beta)(m-1), 2^{\frac{\beta}{2}}(1-\beta)(m+1)\right) \right\}
\]

for all \((n, m, 0) \in J_0^{(\beta)}\). But recalling the notation \( H_n = H_n^{(\beta)} = \left[2^n(\beta-1)/2\right] \), we see

\[
\bigcup_{m=-H_n}^{H_n} \left(2^{\frac{\beta}{2}}(1-\beta)(m-1), 2^{\frac{\beta}{2}}(1-\beta)(m+1)\right) = 2^{\frac{\beta}{2}}(1-\beta), \bigcup_{m=-H_n}^{H_n} \left(m-1, m+1\right)
\]

\[
\supset 2^{\frac{\beta}{2}}(1-\beta), \left(-\left[2^n(\beta-1)/2\right] - 1, \left[2^n(\beta-1)/2\right] + 1\right)
\]

\[
\supset 2^{\frac{\beta}{2}}(1-\beta), \left[-2^n(\beta-1)/2, 2^n(\beta-1)/2\right] = [-1, 1]
\]

and because of \( \beta > 1 \) and since \( (2^{-1/2})^2 = 1/2 < \frac{9}{16} = (\frac{3}{4})^2 \), we also get

\[
\bigcup_{n=0}^{\infty} \left(2^{\frac{\beta}{2}}(n-1), 2^{\frac{\beta}{2}}(n+1)\right) \supset \left(2^{-\frac{\beta}{2}}, \infty\right) \supset \left(2^{-\frac{\beta}{2}}, \infty\right) \supset [3/4, \infty).
\]

Taken together, this implies

\[
\bigcup_{n=0}^{\infty} \bigcup_{m=-H_n}^{H_n} Y_{n,m,0}R'_{n,m,0} \supset \bigcup_{n=0}^{\infty} \left\{ \left( \frac{\zeta}{\eta} \right) \in \mathbb{R}^* \times \mathbb{R} \mid |\xi| \in \left(2^{\frac{\beta}{2}}(n-1), 2^{\frac{\beta}{2}}(n+1)\right) \text{ and } \frac{\eta}{\zeta} \in [-1, 1] \right\}
\]

\[
\supset \{(\xi, \eta) \in \mathbb{R}^* \times \mathbb{R} \mid |\xi| \in [3/4, \infty) \text{ and } |\eta| \leq |\xi|\} =: M_1
\]

and therefore also

\[
\bigcup_{n=0}^{\infty} \bigcup_{m=-H_n}^{H_n} Y_{n,m,1}R'_{n,m,1} = R \cdot \left[ \bigcup_{n=0}^{\infty} \bigcup_{m=-H_n}^{H_n} Y_{n,m,0}R'_{n,m,0} \right]
\]

\[
= \{(\xi, \eta) \in \mathbb{R}^* \times \mathbb{R} \mid |\eta| \in [3/4, \infty) \text{ and } |\xi| \leq |\eta|\} =: M_2.
\]

Altogether, we see \( \mathbb{R}^2 = \bigcup_{j \in J} Y_j R'_j \), since for \( \left( \frac{\xi}{\eta} \right) \in \mathbb{R}^2 \setminus \left(\frac{3}{4}, \frac{3}{4}\right)^2 = \mathbb{R}^2 \setminus [Y_0 R'_0] \), there are only two cases:

**Case 1.** We have \( |\xi| \leq |\eta|\). This implies \( |\eta| \geq \frac{3}{4} \) and thus \( \left( \frac{\xi}{\eta} \right) \in M_2 \), since otherwise \( \left( \frac{\xi}{\eta} \right) \in \left(\frac{-3}{4}, \frac{3}{4}\right)^2 \).  

**Case 2.** We have \( |\eta| \leq |\xi|\). This yields \( |\xi| \geq \frac{3}{4} \) and thus \( \left( \frac{\xi}{\eta} \right) \in M_1 \), since otherwise \( \left( \frac{\xi}{\eta} \right) \in \left(\frac{-3}{4}, \frac{3}{4}\right)^2 \).

We have thus shown that \( S^{(\beta)} \) is an almost structured covering of \( \mathbb{R}^2 \), so that it remains to verify the part of the lemma related to the weight \( v \).

But for \( j = 0 \) and \( \xi \in S^{(\beta)}_0 = (-1, 1)^2 \), we simply have \( (2 + L)^{-1} \cdot v_j \leq v_j = 1 \leq 1 + |\xi| \leq 3 \leq (2 + L) \cdot v_j \) since \( L \geq 1 \). Furthermore, for \( j = (n, m, \varepsilon) \in J^{(\beta)}_0 \), we have

\[
(2 + L)^{-1} \cdot v_j \leq L^{-1} \cdot 2^{\frac{\beta}{n}} \leq |\xi| \leq 1 + |\xi| \leq 1 + L \cdot 2^{\frac{\beta}{n}} \leq (1 + L) \cdot 2^{\frac{\beta}{n}} \leq (2 + L) \cdot v_j
\]

for all \( \xi \in S^{(\beta)}_j \). Therefore, we have shown \( K^{-1} \cdot v_j \leq 1 + |\xi| \leq K \cdot v_j \) for all \( j \in J^{(\beta)} \) and \( \xi \in S^{(\beta)}_j \) with \( K := 2 + L \), as claimed in the last part of the lemma.

Finally, assume \( S_j^{(\beta)} \cap S_j^{(\beta)} = \emptyset \). For an arbitrary \( \xi \in S_j^{(\beta)} \cap S_j^{(\beta)} \), this implies \( v_i \leq K \cdot (1 + |\xi|) \leq K^2 \cdot v_j \) and thus \( K^{-2} \cdot v_j \leq K^2 \) by symmetry. This easily yields \( \frac{v_i}{v_j} \leq K^2 |\xi| \), so that \( v^* \) is \( S^{(\beta)} \)-moderate, with \( CS_{S^{(\beta)}, v^*} \leq K^2 |\xi| \), as claimed.

Since we now know that \( S^{(\beta)} \) is an almost structured covering of \( \mathbb{R}^2 \) and that \( v^* \) is \( S^{(\beta)} \)-moderate, we see precisely as in the remark after Definition \((3.6)\) that the reciprocal \( \beta \)-shearlet smoothness spaces that we now define are well-defined quasi-Banach spaces. As for the unconnected \( \alpha \)-shearlet smoothness spaces, the following definition will only be of transitory relevance, since we will immediately show that the newly defined reciprocal \( \beta \)-shearlet smoothness spaces are identical with the previously defined \( \alpha \)-shearlet smoothness spaces, for \( \alpha = \beta^{-1} \).
Definition D.4. For $\beta \in (1, \infty)$, $p, q \in (0, \infty]$ and $s \in \mathbb{R}$, we define the reciprocal $\beta$-shearlet smoothness space $\mathcal{F}_{\beta,s}^{p,q}(\mathbb{R}^2)$ associated to these parameters as

$$\mathcal{F}_{\beta,s}^{p,q}(\mathbb{R}^2) := D(\mathcal{S}(\beta), L^p, \ell^q),$$

where the covering $\mathcal{S}(\beta)$ and the weight $\nu^s$ are defined as in Definition D.1 and Corollary D.3, respectively.

Next, we want to show $\mathcal{F}_{\beta,s}^{p,q}(\mathbb{R}^2) = \mathcal{F}_{\beta,s}^{p,q}(\mathbb{R}^2)$. To this end, we will utilize the general theory of embeddings between decomposition spaces that was developed in [60]. The main prerequisite for an application of this theory is to have a certain compatibility between the two relevant coverings. This compatibility is established in the next lemma:

Lemma D.5. Let $\beta \in (1, \infty)$ and set $\alpha := \beta^{-1} \in (0, 1)$. Then, for each $i \in I^{(\alpha)}$, there is some $j = j_i \in J^{(\beta)}$ satisfying $S_i^{(\alpha)} \subset S_j^{(\beta)}$.

Remark. The set $P$ in Definition D.4 is chosen precisely to make the preceding lemma true. In general, one could have chosen $P$ to be smaller.

Proof. For $i = 0$, we clearly have $S_0^{(\alpha)} = (-1,1)^2 = S_0^{(\beta)}$, so that we can assume $i = (n,m,\varepsilon,\delta) \in I_0^{(\alpha)}$ in the following. Let us first consider the case $\varepsilon = 1$ and $\delta = 0$. Define $j := [2\alpha n] \in N_0$ and observe $2\alpha n - 1 < j \leq 2\alpha n$. Recall the notation $\mu_0 = 3 \cdot 2^{\beta/2}$ from Definition D.1 and note for arbitrary $\ell \in \mathbb{Z}$ with $|\ell| \leq H_j$ that

$$S_{j,\ell,0}^{(\beta)} \supset \text{diag}(2^{j/2}, 2^{\beta}) \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] U_{(-3,3)}(\mu_0^{-1}, \mu_0) \supset U_{(-1,1)}^{(2\beta/2,\mu_0^{-1}, 2\beta/2, \mu_0)} \left( 2^{1-j/2} (\omega-3), 2^{1-j/2} (\omega+3) \right) \quad \text{and} \quad S_j^{(\alpha)} = U_{(2n/3, 2n/3)}(2n^{-1}(m-1), 2n^{-1}(m+1)),
$$

thanks to equation (32). Consequently, it suffices to show that we have $(2^{\beta/2}, \mu_0^{-1}, 2\beta/2, \mu_0) \supset (2n/3, 3 \cdot 2^n)$ and that one can choose $\ell \in \mathbb{Z}$ with $|\ell| \leq H_j$ such that

$$\left( 2^{j/2} (\omega-1/2) (\ell-3), 2^{j/2} (\omega-1/2) (\ell+3) \right) \supset \left( 2^n (m-1), 2^n (m+1) \right). \quad (D.13)$$

The first of these inclusions is straightforward to verify: We have $\mu_0 = 3 \cdot 2^{\beta/2} \geq 3$ and $j \leq 2\alpha n = 2n/3$, so that $2^{\beta/2} \mu_0^{-1} \leq 1/3 \cdot 2^{\beta/2} \leq 1/3 \cdot 2^n$. Furthermore, since $j > 2\alpha n - 1$,

$$\frac{2\beta/2}{2^n/3} \cdot \mu_0 \geq 2^n \cdot 2^{-\beta} \cdot 3 \cdot 2^{\beta/2} = 3 \cdot 2^n.$$

Thus, all that remains is to show that one can choose $\ell$ suitably. To this end, let $\ell_0 := \left( 2^n (m-1) \right) \in \mathbb{Z}$ and observe

$$\ell_0 \leq 2^n (m-1) - 2^n (m-1) = 2^n (m-1) \leq 2^n (m-1) = H_j^{(\beta)}.$$

We now distinguish two cases:

Case 1: We have $\ell_0 \geq -H_j^{(\beta)}$. In this case, we set $\ell := \ell_0$ and note $|\ell| \leq H_j^{(\beta)}$. Furthermore, we observe

$$2^{j/2} (\omega-1/2) (\ell-3) < 2^{j/2} (\omega-1/2) \leq 2^n (m-1).$$

Finally, since we have $\ell + 1 > 2n^{-1}(m-1) + (\beta-1)/2$, we get

$$2^{j/2} (\omega-1/2) (\ell+3) > 2^{j/2} (\omega-1/2) \left[ 2 + 2n^{-1}(m-1) \right] = 2 \cdot 2^{j/2} (\omega-1/2) \leq 2^{n^{-1}(m-1)} (m-1).$$

The last two displayed equations establish the desired inclusion (D.13), so that indeed $S_i^{(\alpha)} \subset S_j^{(\beta)}$.

Case 2: We have $\ell_0 < -H_j^{(\beta)}$. This implies $2^n (m-1) \leq -1$, since we would otherwise have

$$\ell_0 = \left[ 2^n (m-1) \right] \geq -2^{(\beta-1)/2} = -H_j^{(\beta)}.$$
Consequently, we get for \( \ell := -H_j^{(\beta)} \in \mathbb{Z} \) that
\[
2^{n(\alpha-1)} (m+1) = 2^{n(\alpha-1)} (m - 1) + 2 \cdot 2^n(\alpha-1) \\
\leq -1 + 2 \cdot 2^{n(\alpha-1)} \\
\leq -1 + 2 \cdot 2^\beta (1-\beta) - 2 \\
\leq 2^\beta (1-\beta) \cdot (\ell - 3).
\]
Finally, recall \( |m| \leq 2^{n(1-\alpha)} \leq 1 + 2^{n(1-\alpha)} \), so that
\[
2^{n(\alpha-1)} (m - 1) \geq -2^{n(\alpha-1)} (|m| + 1) \geq -2^{n(\alpha-1)} \left( 2^{n(1-\alpha)} + 2 \right) \\
\geq -1 - 2 \cdot 2^n(\alpha-1) \\
\geq -1 - 2 \cdot 2^\beta (1-\beta) - 2 \\
\geq 2^\beta (1-\beta) \cdot (\ell - 3).
\]
We have thus again established the inclusion \( \text{[D.13]} \), so that \( S_i^{(\alpha)} \subset S_j^{(\beta)} \).

Up to now, we have constructed for \( i = (n, m, \varepsilon, \delta) \in J_0^{(\alpha)} \) with \( \varepsilon = 1 \) and \( \delta = 0 \) some \( (j, \ell, 0) \in J_0^{(\beta)} \) with \( S_i^{(\alpha)} \subset S_j^{(\beta)} \), so that it remains to consider the general case \( \varepsilon \in \{ \pm 1 \} \) and \( \delta \in \{ 0, 1 \} \). But since the base-set \( P \) from Definition \( \text{[D.1]} \) satisfies \( P = -P \), we have \( S_j^{(\beta)} = -S_j^{(\beta)} \), so that \( S_n^{(\alpha)} = -S_n^{(\beta)} \), \( -S_{n,m,1,0}^{(\beta)} \subset -S_{j,\ell,0}^{(\beta)} \) assuming \( S_{n,m,1,0}^{(\alpha)} \subset S_{j,\ell,0}^{(\beta)} \). Finally, assuming that \( S_{n,m,\varepsilon,0}^{(\alpha)} \subset S_{j,\ell,0}^{(\beta)} \), we get
\[
S_{n,m,\varepsilon,1}^{(\alpha)} = R \cdot S_{n,m,\varepsilon,0}^{(\alpha)} \subset R \cdot S_j^{(\beta)} = S_j^{(\beta)}.
\]
This completes the proof. \( \Box \)

Now, we can finally show that the reciprocal \( \beta \)-shearlet smoothness spaces are identical to the \( \alpha \)-shearlet smoothness spaces from Section 3.

**Lemma D.6.** Let \( \beta \in (1, \infty) \), \( s \in \mathbb{R} \) and \( p, q \in (0, \infty) \). Then
\[
\mathcal{F}_{(\beta)}^p \left( \mathbb{R}^2 \right) = \mathcal{F}_{(\beta^{-1})}^q \left( \mathbb{R}^2 \right).
\]

**Proof.** Set \( \alpha := \beta^{-1} \in (0, 1) \) for brevity. As in the proof of Lemma \( \text{[5.4]} \) we want to invoke \( \text{[6.1]} \), Lemma 6.11, part (2)], with the choice \( \mathcal{P} := S^{(\alpha)} \) and \( \mathcal{Q} := S^{(\beta)} \), recalling that \( \mathcal{F}_{(\beta^{-1})}^q \left( \mathbb{R}^2 \right) = \mathcal{D} \left( S^{(\alpha)} \right) \subseteq L^p \left( \mathbb{R}^2 \right) = \mathcal{F}^{-1} \left[ \mathcal{D}_\mathcal{F} \left( S^{(\alpha)} \right) \subseteq L^p \left( \mathbb{R}^2 \right) \right] \)
and likewise \( \mathcal{F}_{(\beta)}^p \left( \mathbb{R}^2 \right) = \mathcal{F}^{-1} \left[ \mathcal{D}_\mathcal{F} \left( S^{(\beta)} \right) \subseteq L^p \left( \mathbb{R}^2 \right) \right] \).

To this end, we first have to verify that we have \( \nu_i^s \asymp w_i^s \) if \( S_i^{(\alpha)} \cap S_j^{(\beta)} \neq \emptyset \) and that the coverings \( S^{(\alpha)} \) and \( S^{(\beta)} \) are weakly equivalent. This means that
\[
\sup_{i \in I^{(\alpha)}} \left\{ j \in J^{(\beta)} \mid S_j^{(\beta)} \cap S_i^{(\alpha)} \neq \emptyset \right\} < \infty \quad \text{and} \quad \sup_{j \in J^{(\beta)}} \left\{ i \in I^{(\alpha)} \mid S_i^{(\alpha)} \cap S_j^{(\beta)} \neq \emptyset \right\} < \infty.
\]

For the first point, let \( K \geq 1 \) as in Corollary \( \text{[5.3]} \) i.e., such that \( K^{-1} \cdot \nu_j \leq 1 + |\xi| \leq K \cdot \nu_j \) for all \( j \in J^{(\beta)} \) and all \( \xi \in S_j^{(\beta)} \). Likewise, Lemma \( \text{[5.4]} \) shows \( \frac{1}{13} \cdot w_i \leq 1 + |\xi| \leq 13 \cdot w_i \) for all \( i \in I^{(\alpha)} \) and \( \xi \in S_i^{(\alpha)} \). Consequently, if \( S_i^{(\alpha)} \cap S_j^{(\beta)} \neq \emptyset \), we can choose some \( \xi \in S_i^{(\alpha)} \cap S_j^{(\beta)} \), so that
\[
(13K)^{-1} \cdot w_i \leq K^{-1} \cdot (1 + |\xi|) \leq \nu_j \leq K \cdot (1 + |\xi|) \leq 13K \cdot w_i.
\]
Consequently, we get
\[
(13K)^{-1} |t| \leq \frac{\nu_i^s}{w_i^s} \leq (13K)^{|t|} \quad \forall t \in \mathbb{R} \quad \text{if} \quad i \in I^{(\alpha)} \quad \text{and} \quad j \in J^{(\beta)} \quad \text{with} \quad S_i^{(\alpha)} \cap S_j^{(\beta)} \neq \emptyset.
\]
It remains to show that \( S^{(\alpha)} \) and \( S^{(\beta)} \) are weakly equivalent. To this end, let \( i \in I^{(\alpha)} \) be arbitrary and note from Lemma D.5 that \( S^{(\alpha)}_i \subset S^{(\beta)}_j \) for some \( j \in J^{(\beta)} \). Thus, for arbitrary \( j \in J^{(\beta)} \) with \( \emptyset \neq S^{(\beta)}_j \cap S^{(\alpha)}_i \neq \emptyset \), we get \( \emptyset \subset S^{(\alpha)}_j \cap S^{(\beta)}_j \subset S^{(\beta)}_j \) and thus \( j \in j^*_i \). This implies
\[
\sup_{i \in I^{(\alpha)}} \left\| j \in J^{(\beta)} \left| S^{(\beta)}_j \cap S^{(\alpha)}_i \neq \emptyset \right. \right\| \leq \sup_{i \in I^{(\alpha)}} |j^*_i| \leq \sup_{j \in J^{(\beta)}} |j^*| = N_{S^{(\beta)}} < \infty,
\]
since \( S^{(\beta)} \) is an almost structured covering of \( \mathbb{R}^2 \).

For the second part of weak equivalence, we have to work harder: Let \( j \in J^{(\beta)} \) be arbitrary. For each \( i \in I^{(\alpha)} \) with \( S^{(\alpha)}_i \cap S^{(\beta)}_j \neq \emptyset \), Lemma D.5 yields some \( j_i \in J^{(\beta)} \) satisfying \( S^{(\alpha)}_i \subset S^{(\beta)}_{j_i} \). Hence, \( \emptyset \subset S^{(\alpha)}_i \cap S^{(\beta)}_j \subset S^{(\beta)}_j \cap S^{(\beta)}_{j_i} \), so that \( j_i \in j^* \). Thus, with \( [S^{(\beta)}_j]^* := \bigcup_{i \in J^*} S^{(\beta)}_i \), we have shown \( S^{(\alpha)}_i \subset S^{(\beta)}_{j_i} \subset [S^{(\beta)}_j]^* \) for arbitrary \( i \in I^{(\alpha)} \) with \( S^{(\alpha)}_i \cap S^{(\beta)}_j \neq \emptyset \).

Now, we will need the easily verifiable identities \( |\det T^{(\alpha)}_i| = w^{1+\alpha}_i \) and \( |\det Y^{(\beta)}_j| = v^{1+\beta - 1}_j = v^{1+\alpha}_j \) for \( i \in I^{(\alpha)} \) and \( j \in J^{(\beta)} \). To use these identities, set \( M_j := \left\{ i \in I^{(\alpha)} \, \big| \, S^{(\alpha)}_i \cap S^{(\beta)}_j \neq \emptyset \right\} \), as well as
\[
C_1 := \min \left\{ \lambda_2 \left( \left. \left| \left( -1,1 \right)^2 \right. \right|, \lambda_2 \left( U^{(3+1),3} \right) \right) \right\} > 0 \quad \text{and} \quad C_2 := \max \left\{ \lambda_2 \left( \left. \left| \left( -1,1 \right)^2 \right. \right|, \lambda_2 \left( P \right) \right) \right\} > 0,
\]
with \( P \) as in Definition D.1. We clearly have \( \sum_{i \in M_j} 1_{S^{(\alpha)}_i} \leq \sum_{i \in I^{(\alpha)}} 1_{S^{(\alpha)}_i} \leq N_{S^{(\alpha)}} \). But because of \( S^{(\alpha)}_i \subset [S^{(\beta)}_j]^* \) for \( i \in M_j \), this implies
\[
0 < C_1 \cdot \sum_{i \in M_j} w^{1+\alpha}_i = C_1 \cdot \sum_{i \in M_j} |\det T^{(\alpha)}_i| \leq \sum_{i \in M_j} \lambda_2 (S^{(\alpha)}_i) = \int_{\mathbb{R}^2} \sum_{i \in M_j} 1_{S^{(\alpha)}_i} (\xi) \, d\xi \\
\leq N_{S^{(\alpha)}} \cdot \lambda_2 \left( [S^{(\beta)}_j]^* \right) \leq N_{S^{(\alpha)}} \cdot \sum_{\ell \in J^*} \lambda_2 (S^{(\beta)}_{\ell}) \\
\leq C_2 N_{S^{(\alpha)}} \cdot \sum_{\ell \in J^*} |\det Y^{(\beta)}_{\ell}| = C_2 N_{S^{(\alpha)}} \cdot \sum_{\ell \in J^*} v^{1+\alpha}_{\ell} \\
\text{(Corollary D.3)} \leq C_2 N_{S^{(\alpha)}} \cdot |j^*| \cdot K^{2(1+\alpha)} \cdot v^{1+\alpha}_{j^*} \quad \text{(eq. D.14 and \( S^{(\alpha)}_i \cap S^{(\beta)}_j \neq \emptyset \) for \( i \in M_j \))} \\
\text{and thus almost subordinate, cf. Definition 2.10)} \to Q = S^{(\beta)} = (Y^{(\beta)}_j P^{\ell}_j)_{j \in J^{(\beta)}} \text{ and that we have}
\]
\[
|\det \left( T^{(\alpha)}_i \right)^{-1} Y^{(\beta)}_j | \lesssim 1 \quad \text{if} \quad S^{(\beta)}_j \cap S^{(\alpha)}_i \neq \emptyset
\]
But Lemma D.5 shows that \( P = S^{(\alpha)} \) is subordinate (and thus also almost subordinate, cf. Definition 2.10)) to \( Q = S^{(\beta)} \). Furthermore, in case of \( S^{(\beta)}_j \cap S^{(\alpha)}_i \neq \emptyset \), equation D.14 yields
\[
|\det \left( \left( T^{(\alpha)}_i \right)^{-1} Y^{(\beta)}_j \right) | = \left( w^{1+\alpha}_i \right)^{-1} v^{1+\alpha}_j \lesssim 1,
\]
as desired. The claim now follows from [60, Lemma 6.11].

Now, we show that a suitable \( \beta \)-shearlet system generated by bandlimited functions yields a Banach frame for the reciprocal shearlet smoothness spaces. We restrict ourselves to bandlimited functions, since this simplifies the proof.

But first, we review the precise definition of a \( \beta \)-shearlet system from [37, Definition 3.10].

**Definition D.7.** For \( c \in (0, \infty) \) and \( \beta \in (1, \infty) \) and given generators \( \varphi, \psi, \theta \in L^2 (\mathbb{R}^2) \), the cone-adapted \( \beta \)-shearlet system \( \text{SH (\varphi, \psi, \theta; c, \beta)} \) with sampling density \( c \) generated by \( \varphi, \psi, \theta \) is defined as
\[
\text{SH (\varphi, \psi, \theta; c, \beta)} := \Phi (\varphi; c, \beta) \cup \Psi (\psi; c, \beta) \cup \Theta (\theta; c, \beta),
\]
where
\[\Phi(\varphi; c, \beta) = \{ \varphi(\cdot - ck) \mid k \in \mathbb{Z}^2 \},\]
\[\Psi(\psi; c, \beta) = \{ 2^{(\beta+1)/4} \cdot \psi \left( S_T A_{\beta - 1,2^j/2} \cdot \cdot - ck \right) \mid j \in \mathbb{N}_0, k \in \mathbb{Z}^2 \text{ and } \ell \in \mathbb{Z} \text{ with } |\ell| \leq \left[ 2^{(\beta-1)/2} \right] \},\]
\[\Theta(\theta; c, \beta) = \{ 2^{(\beta+1)/4} \cdot \theta \left( S_T A_{\beta - 1,2^j/2} \cdot \cdot - ck \right) \mid j \in \mathbb{N}_0, k \in \mathbb{Z}^2 \text{ and } \ell \in \mathbb{Z} \text{ with } |\ell| \leq \left[ 2^{(\beta-1)/2} \right] \},\]
where \( A_{\alpha,s} = \text{diag} (s, s^\alpha) \) and \( \tilde{A}_{\alpha,s} = \text{diag} (s^\alpha, s) \), as well as \( S_\ell = (\frac{1}{2} \ell) \) for \( \alpha \in [0, 1] \), \( s \in (0, \infty) \) and \( \ell \in \mathbb{R} \).

**Proposition D.8.** Let \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^2) \) with \( \hat{\varphi}, \hat{\psi} \in C_c^\infty(\mathbb{R}^2) \) and the following additional properties:

1. We have \( \hat{\varphi}(\xi) \neq 0 \) for all \( \xi \in [-1,1]^2 \).
2. We have \( \hat{\psi}(\xi) \neq 0 \) for all \( \xi \in \mathcal{P} \) with \( P \) as in Definition D.7.
3. We have \( \sup \| \hat{\psi} \|_p \in \mathbb{R}^* \times \mathbb{R} \).

Then, for \( \beta \in (1, \infty), p_0, q_0 \in [0, 1] \) and \( s_0, s_1 \in \mathbb{R} \) with \( s_0 \leq s_1 \), there is some \( \delta_0 = \delta_0(\beta, p_0, q_0, s_0, s_1, \varphi, \psi) > 0 \) such that for every \( 0 < \delta \leq \delta_0 \), all \( p \in [p_0, \infty] \), all \( q \in [q_0, \infty] \) and all \( s \in \mathbb{R} \) with \( s_0 \leq s \leq s_1 \), the family \( \text{SH}(\varphi, \psi; \delta, \beta) \) forms a Banach frame for \( \mathcal{A}^{p,q}_{\beta,s}(\mathbb{R}^2) \), where \( \delta := \| \psi \|_P \), i.e., \( \delta(x, y) = \psi(y, x) \).

Precisely, this means the coefficient space \( C^{p,q}_{\beta,s} \) as in Definition D.8 (with \( Q = \mathcal{S}(\beta) \) and \( w = v^* \)) and with
\[\gamma_{[i,k,\delta]} := \begin{cases} \varphi(\cdot - \delta k), & \text{if } i = 0, \\ \frac{1}{2}^{(\beta+1)/4} \cdot \psi \left( S_T A_{\beta - 1,2^j/2} \cdot \cdot - \delta k \right), & \text{if } i = (j, \ell, 0), \\ \frac{1}{2}^{(\beta+1)/4} \cdot \theta \left( S_T A_{\beta - 1,2^j/2} \cdot \cdot - \delta k \right), & \text{if } i = (j, \ell, 1) \end{cases}
\]
for \( i \in J^{(\beta)} \) and \( k \in \mathbb{Z}^2 \) that the following hold:

1. For each \( 0 < \delta \leq 1 \), the analysis map
\[A^{(\delta)} : \mathcal{A}^{p,q}_{\beta,s}(\mathbb{R}^2) \to C^{p,q}_{\beta,s}, f \mapsto \left( \left\langle f, \gamma_{[i,k,\delta]} \right\rangle_{Z^\prime(\mathbb{R}^2), Z(\mathbb{R}^2)} \right)_{i \in J^{(\beta)}, k \in \mathbb{Z}^2}
\]
is well-defined and bounded.

2. For all \( 0 < \delta \leq \delta_0 \), there is a bounded linear reconstruction map \( R^{(\delta)} : C^{p,q}_{\beta,s} \to \mathcal{A}^{p,q}_{\beta,s}(\mathbb{R}^2) \) satisfying \( R^{(\delta)} \circ A^{(\delta)} = \text{id}_{C^{p,q}_{\beta,s}(\mathbb{R}^2)} \).

3. We have the following consistency statement: If \( f \in \mathcal{A}^{p,q}_{\beta,s}(\mathbb{R}^2) \) and if \( p_1 \in [p_0, \infty) \) and \( q_1 \in [q_0, \infty] \) and \( s_0 \leq r \leq s_1 \), then we have the following equivalence:
\[f \in \mathcal{A}^{p,q}_{\beta,s}(\mathbb{R}^2) \iff A^{(\delta)} f \in C^{p,q}_{\beta,s}.
\]

**Proof.** We want to verify that Theorem D.9 applies in the current setting, i.e., with \( Q = \mathcal{S}(\beta) = (Y_j^{(\beta)} P_j^\prime)_{j \in J^{(\beta)}} \). To this end, we first recall the notation introduced in Assumption 2.7. If we set \( n := 2 \) and \( Q_0^{(1)} := P \) with \( \mu_0 = 3 \cdot 2^{\beta/2} \) and \( P = \bigcup_{(3)} U_1^{(3,0)} \cup \bigcup_{(3)} - U_2^{(3,0)} \) as in Definition D.1, as well as \( Q_0^{(2)} := (-1,1)^2 \) and finally \( k_j := 1 \) for \( j \in J_0^{(\beta)} \) and \( k_0 := 2 \), then we have \( P_j^\prime = Q_0^{(k_j)} \) for all \( j \in J^{(\beta)} \).

Now, we set \( \gamma_{[0]}^{(0)} := \psi \) and \( \gamma_{[0]}^{(1)} := \varphi \), as well as \( \varepsilon := 1 \). With these choices, we want to verify the prerequisites of Theorem D.9. We clearly have \( \gamma_{[0]}^{(0)}(\xi) = \hat{\psi}(\xi) \neq 0 \) for all \( \xi \in \mathcal{P} = Q_0^{(1)} \) and likewise that \( \gamma_{[0]}^{(0)}(\xi) = \hat{\varphi}(\xi) \neq 0 \) for all \( \xi \in [-1,1]^2 = Q_0^{(2)} \).

Consequently, since we are interested in the decomposition space \( \mathcal{A}^{p,q}_{\beta,s}(\mathbb{R}^2) = \mathcal{D} \left( \mathcal{S}(\beta), L^p, \ell^{q'}_{v^*} \right) \) in \( \mathbb{R}^d = \mathbb{R}^2 \), it remains to verify
\[C_1 := \sup_{i \in J^{(\beta)}} \sum_{j \in J^{(\beta)}} M_{j,i} < \infty \quad \text{and} \quad C_2 := \sup_{j \in J^{(\beta)}} \sum_{i \in J^{(\beta)}} M_{j,i} < \infty,
\]
where
\[M_{j,i} := \left( \frac{\psi_i^\top}{v_i^\top} \right)^\tau (1 + \| Y^{-1} Y_i \|) \cdot \max_{|\nu| \leq 1} \left( \left| \det Y_i \right|^{-1} \cdot \int_{\tilde{q}_i^{(\gamma)}} \max_{|\alpha| \leq N} \left( \left| \frac{\partial \gamma_{[j]}^\gamma}{\partial \gamma_{[\gamma]}^j} \right| (Y^{-1} Y_i) \right) \cdot d\xi \right)^\tau,
\]
with
\[N := \left\lceil \frac{d + \varepsilon}{\min\{1, p\}} \right\rceil =: N_0, \quad \tau := \min\{1, p, q\} \geq \tau_0 := \min\{p_0, q_0\} \quad \text{and} \quad \sigma := \tau \cdot \left( \frac{d}{\min\{1, p\}} + N_0 \right).
\]
and where $\gamma_j := \gamma_{kj}^{(0)}$ for $j \in J^{(\beta)}$, i.e., $\gamma_j = \psi$ for $j \in J_0^{(\beta)}$ and $\gamma_0 = \varphi$.

Now, since $\hat{\varphi} \in C_c^\infty (\mathbb{R}^2)$, there is some $A > 1$ satisfying $\text{supp } \hat{\varphi} \subset (-A, A)^2$. Furthermore, since $\text{supp } \hat{\psi} \subset \mathbb{R}^* \times \mathbb{R}$ is compact, there are $0 < \lambda \leq \mu$ and $B > 0$ with

$$\text{supp } \hat{\psi} \subset \{ (\xi, \eta) \in \mathbb{R}^2 \mid \lambda < |\xi| < \mu \text{ and } |\eta| < \lambda B \}$$

$$\subset \{ (\xi, \eta) \in \mathbb{R}^2 \mid \lambda < |\xi| < \mu \text{ and } -B < \eta/|\xi| < B \} = U_{(-B,B)} \cup \left[ -U_{(-B,B)} \right] =: U.$$}

By possibly shrinking $\lambda$ and enlarging $\mu$ and $B$, we can assume $\lambda \leq \mu_0^{-1}$, $\mu \geq \mu_0$ and $B \geq 3$, so that $U \supset P = Q_0^{(1)}$. Setting $U_0' := (-A, A)^2$ and $U_j' := U$ for $j \in J^{(\beta)}$, we have just shown supp $\hat{\gamma}_j \subset U_j'$ for all $j \in J^{(\beta)}$. But standard properties of the Fourier transform (see e.g., Theorem 8.22) show $\hat{\varphi}^\nu \gamma_j (\xi) = (2\pi \xi)^\nu \cdot \hat{\gamma}_j (\xi)$ and thus again supp $\partial^\nu \hat{\varphi} \gamma_j \subset U_j'$ for all $j \in J^{(\beta)}$ and arbitrary $\nu, \alpha, \beta \in \mathbb{N}_0$. Therefore, we get

$$\max_{|\nu| \leq 1} \max_{|\alpha| \leq N} \left| \left( \partial^\nu \hat{\varphi} \gamma_j \right) (Y_j^{-1} \xi) \right| \leq \left( \sup_{j \in J^{(\beta)}} \max_{|\nu| \leq 1} \max_{|\alpha| \leq N} \left| \partial^\nu \hat{\varphi} \gamma_j \right| \right) \cdot \left| \text{supp } \hat{\varphi} \gamma_j \right| (Y_j^{-1} \xi) =: K_1 \cdot \hat{1}_{U_j'} (Y_j^{-1} \xi) = K_1 \cdot \hat{1}_{Y_j U_j'} (\xi)$$

for all $\xi \in \mathbb{R}^2$ and $j \in J^{(\beta)}$. Here, we emphasize that the constant $K_1$ is finite since $\{ \gamma_j \mid j \in J^{(\beta)} \} = \{ \varphi, \psi \} \subset S (\mathbb{R}^2)$ is a finite set.

Next, if we set $U_j := Y_j U_j'$ for $j \in J^{(\beta)}$, then Lemma 2.2 yields constants $L_1, C > 0$ and $M \in \mathbb{N}$ (depending only on $\lambda, \mu, A, B, \beta$) such that

$$T_j := \left\{ i \in J^{(\beta)} \mid U_i \cap U_j \neq \emptyset \right\} \text{ satisfies } \left| T_j \right| \leq M \quad \forall j \in J^{(\beta)},$$

$$\left| \text{supp } \hat{\varphi} \gamma_j \right| \leq C \quad \forall i, j \in J^{(\beta)} \text{ with } U_i \cap U_j \neq \emptyset,$$

$$L_1 \cdot v_j \leq |\xi| \leq L_1 \cdot v_j \quad \forall \xi \in U_j \text{ and } v_j \in U_j.$$

As a slight modification, the last estimate yields because of $v_j \geq 1$ that $(1 + L_1)^{-1} \cdot v_j \leq |\xi| \leq 1 + |\xi| \leq (1 + L_1) \cdot v_j$ for all $\xi \in U_j$ and $j \in J_0^{(\beta)}$. Likewise, for $\xi \in U_0 = (-A, A)^2$, we have

$$(1 + 2A)^{-1} \cdot v_j \leq 1 \leq 1 + |\xi| \leq 1 + 2A = (1 + 2A) \cdot v_j,$$

so that there is a constant $L_2 = L_2 (A, \lambda, \mu, \beta) > 0$ satisfying $L_2 \cdot v_j \leq 1 + |\xi| \leq L_2 \cdot v_j$ for all $\xi \in U_j$ and $j \in J^{(\beta)}$. In particular, for $i \in T_j$ there is some $\xi \in U_i \cap U_i \neq \emptyset$, so that $v_i \leq L_2 \cdot (1 + |\xi|) \leq L_2 \cdot v_j$. By symmetry, we also get $v_j \leq L_2 \cdot v_j$ and thus $(v_j^\nu/v_i^\nu) \leq L_2^{|\nu|}$ for all $j \in J^{(\beta)}$ and $i \in T_j$.

Putting everything together and recalling $P_j \subset U_j'$ for all $j \in J^{(\beta)}$, we thus see

$$M_j,i = \left( \frac{v_j^\nu}{v_i^\nu} \right)^\tau \cdot (1 + \|Y_j^{-1} Y_i\|)^\sigma \cdot \max_{|\alpha| \leq 1} \left( \left| \text{det } Y_i \right|^{-1} \cdot \int_{S^1} \max_{|\alpha| \leq N} \left| \left( \partial^\nu \hat{\varphi} \gamma_j \right) (Y_j^{-1} \xi) \right| d \xi \right)^\tau$$

$$\leq \left( \frac{v_j^\nu}{v_i^\nu} \right)^\tau \cdot (1 + \|Y_j^{-1} Y_i\|)^\sigma \cdot \left( K_1 \cdot |\text{det } Y_i|^{-1} \cdot \int_{U_j} \hat{1}_{Y_j U_j'} (\xi) d \xi \right)^\tau$$

$$= \left( \frac{v_j^\nu}{v_i^\nu} \right)^\tau \cdot (1 + \|Y_j^{-1} Y_i\|)^\sigma \cdot \left( K_1 \cdot |\text{det } Y_i|^{-1} \cdot \lambda_2 (U_i \cap U_j) \right)^\tau$$

$$\leq 1 \cdot T_j \cdot \frac{v_j^\nu}{v_i^\nu} \cdot (1 + \|Y_j^{-1} Y_i\|)^\sigma \cdot \left( K_1 \cdot \sup_{i \in J^{(\beta)}} \lambda_2 (U_i) \right)^\tau$$

$$\leq 1 \cdot T_j \cdot L_2^{|\nu|} \cdot (1 + C)^\sigma \cdot \left( K_1 \cdot \lambda_2 (U_i) \right)^\tau$$

$$= K_2$$

for $K_2 := \max \{ \lambda_2 ((-A, A)^2) \cdot \lambda_2 (U) \}$. Hence, using $1 \cdot T_j (i) = 1 \cdot T_j (j)$, we finally get

$$C_1^{1/\tau} = \sup_{i \in J^{(\beta)}} \left( \sum_{j \in J^{(\beta)}} M_j,i \right)^{1/\tau} \leq L_2^{|\nu|} \cdot (1 + C)^\frac{\nu}{\tau} \cdot K_2 \cdot \sup_{i \in J^{(\beta)}} \left| T_j \right|^{1/\tau}$$

$$\leq L_2^{|\nu|} \cdot (1 + C)^\frac{\nu}{\tau} \cdot M_1^{1/\tau}$$

$$\leq L_2^{|\nu|} \cdot (1 + C)^\frac{\nu}{\tau} \cdot M_1^{1/\tau} =: K_3,$$
where the last step used \( \frac{d}{\tau} \leq \frac{2}{p_0} + N_0 \). Precisely the same arguments also show \( C_{1/2}^{1/\tau} \leq K_3 < \infty \). Observe that \( K_3 \) is independent of \( p, q, \) and \( s \), as long as \( p \geq p_0, q \geq q_0 \) and \( s_0 \leq s \leq s_1 \).

Consequently, all assumptions of Theorem 2.9 are satisfied. Furthermore, since the sets \( \mathbb{R}^* \times \mathbb{R} \), \( P \) and \((-1, 1)^2\) are symmetric, we see analogously that all assumptions of Theorem 2.9 are still satisfied (possibly with a slightly different constant \( K_3 \)) if \( \varphi \) is replaced by \( \tilde{\varphi} \) and \( \psi \) by \( \tilde{\psi} \), where \( \tilde{f}(x) = f(-x) \). Thus, \( \gamma_j = \tilde{\psi} \) for \( j \in J_0^{(\beta)} \) and \( \gamma_0 = \tilde{\varphi} \). Consequently, with a fixed regular partition of unity \( \Phi = (\varphi_{\ell})_{\ell \in \mathcal{J}(\beta)} \) for \( \mathcal{S}^{(\beta)} \), Theorem 2.9 yields a constant \( K = K \left( p_0, q_0, S^{(\beta)}, \Phi, \tilde{\varphi}, \tilde{\psi} \right) = K \left( p_0, q_0, \beta, \varphi, \psi \right) > 0 \), such that for arbitrary \( 0 < \delta \leq \delta_0 \) the family
\[
\left( L_{\delta, Y_i \cdot \varphi_{\ell}} \gamma_{[i]} \right)_{i \in \mathcal{J}^{(\beta)}, k \in \mathbb{Z}^2}
\]
with \( \gamma_{[i]} = |\det Y_i|^{1/2} \cdot M_{ci} \left[ \gamma_i \circ Y_i \right] \) and \( \gamma_{[i]}(x) = \gamma_{[i]}(-x) \)
yields a Banach frame for \( \mathcal{F}^{P, q}_{\beta, s}(\mathbb{R}^2) = \mathcal{D} \left( \mathcal{S}^{(\beta)}, L^p_{\nu^q}, \right) \), as precisely described in Theorem 2.9. But with what we just saw and thanks to Corollary D.3 we have
\[
\delta_0 = \left( 1 + K \cdot C_{m}^{1/\tau} \cdot \left( C_{1/2}^{1} + C_{1/2}^{1/\tau} \right)^2 \right)^{-1} \geq \left( 1 + 4 \cdot K \cdot K_4^{\max \{|s_0|, |s_1|\}} \cdot K_3 \right)^{-1} =: \delta_0
\]
for a suitable constant \( K_4 = K_4(\beta) \geq 1 \) which is provided by Corollary D.3.

Finally, note that the coefficient map \( A^{(\beta)} f = \left( \gamma_{[i]} \ast f \left( \delta \cdot Y_i \cdot \gamma \right) \right)_{i \in \mathcal{J}^{(\beta)}, k \in \mathbb{Z}^2} \) from Theorem 2.9 uses a somewhat peculiar definition of the convolution \( \gamma_{[i]} \ast f \), cf. equation (2.3). Precisely, with the regular partition of unity \( \Phi = (\varphi_{\ell})_{\ell \in \mathcal{J}(\beta)} \) from above, we have
\[
\gamma_{[i]} \ast f = \sum_{\ell \in \mathcal{J}} F^{-1} \left( \gamma_{[i]} \cdot \varphi_{\ell} \cdot \hat{f} \right)(x)
\]
(series is finite sum, since \( \Phi \) is a locally finite and \( \gamma_{[i]} \in C_{\infty}^{\infty}(\mathbb{R}^2) \))

\[
\left( \sum_{i \in \mathcal{J}, \varphi_{\ell} \equiv 1 \text{ on } \mathbb{R}^2 \right) = \left( \sum_{\ell \in \mathcal{J}} F^{-1} \left( \gamma_{[i]} \cdot \varphi_{\ell} \cdot \hat{f} \right) \right)(x)
= \left( \sum_{i \in \mathcal{J}} \langle \gamma_{[i]} \cdot \hat{f}, e^{2\pi i \langle x, \cdot \rangle} \rangle_{D'(\mathbb{R}^2), C_{\infty}^{\infty}(\mathbb{R}^2)}
= \langle f, L_x \gamma_{[i]} \rangle_{Z'(\mathbb{R}^2), Z(\mathbb{R}^2)}
= \langle f, L_x \gamma_{[i]} \rangle_{Z'(\mathbb{R}^2), Z(\mathbb{R}^2)}
\]
(Fourier inversion)

for all \( x \in \mathbb{R}^2 \) and \( i \in \mathcal{J}^{(\beta)} \).

It remains to verify that the family \( \left( L_{\delta, Y_i \cdot \varphi_{\ell}} \gamma_{[i]} \right)_{i \in \mathcal{J}^{(\beta)}, k \in \mathbb{Z}^2} \) is (almost) identical to the family \( \left( \gamma_{[i,k]} \right)_{i \in \mathcal{J}^{(\beta)}, k \in \mathbb{Z}^2} \) from the statement of the theorem. Recall that \( c_i = 0 \) for all \( i \in \mathcal{J}^{(\beta)} \). Now, for \( i = 0 \), \( Y_i = Y_0 = \text{id} \) and thus
\[
L_{\delta, Y_i \cdot \varphi_{\ell}} \gamma_{[i,k]} = L_{\delta, \varphi_{\ell}} \gamma_{[i,k]} = L_{\delta, \varphi_{\ell}} \varphi = \varphi (\bullet - \delta k) = \gamma_{[i,k,d]}.
\]
Next, in case of \( i = (j, \ell, 0) \in J^{(\beta)}_0 \), recall from Definition D.1 that \( Y_i^T = S_{\ell} \cdot \text{diag} \left( 2^{j/2}, 2^{j/2} \right) = S_{\ell} \cdot A_{\beta-1,2^{j+1/2}} \) and \( |\det Y_i| = 2^{j+1/2} \), so that
\[
L_{\delta, Y_i \cdot \varphi_{\ell}} \gamma_{[i]} = L_{\delta} \left[ S_{\ell} A_{\beta-1,2^{j+1/2}} \right]^{-1} \gamma_{[i]} = 2^{j+1/2} \cdot \psi \left( S_{\ell} \cdot A_{\beta-1,2^{j+1/2}} \bullet - \delta k \right) = \gamma_{[i,k,d]}.
\]
Finally, in case of \( i = (j, \ell, 1) \in J^{(\beta)}_0 \), a direct calculation shows
\[
Y_i^T = \begin{pmatrix} 2^{j+1/2} & 2^{j+3/2} & 0 \\ 2^{j+3/2} & 2^{j+5/2} & 0 \end{pmatrix} = R \cdot S_{\ell}^T \cdot \tilde{A}_{\beta-1, 2^{j+3/2}}.
\]
so that

\[
L_{δ Y^{-T} k} \gamma^i [i] = 2^{2(1+β)} \cdot \psi \left( Y^{-T} \left[ \bullet - δ Y^{-T} k \right] \right)
= 2^{2(1+β)} \cdot \psi \left( R \left[ S_{k}^T \cdot \tilde{A}_{β-1,2β/2} \cdot \bullet - δ R k \right] \right)
= 2^{2(1+β)} \cdot \theta \left( S_{k}^T \cdot \tilde{A}_{β-1,2β/2} \cdot \bullet - δ R k \right)
= γ^i [i,Rk,β].
\]

But since \( \mathbb{Z}^2 \to \mathbb{Z}^2, k \mapsto R k \) is bijective, it is not hard to see directly from the definition of the coefficient space \( C^p_{v^q} \) (cf. Definition 2.3) that if we set \( δ_i := 1 \) for \( i \) of the form \( i = (j, ℓ) \) and \( δ_i := 0 \) otherwise, then

\[
Ω : C^p_{v^q} \to C^p_{v^q}, \left( c^{(i)}_k \right)_{i \in J, k \in \mathbb{Z}^2} \mapsto \left( c^{(i)}_{R^i k} \right)_{i \in J, k \in \mathbb{Z}^2}
\]
is an isometric isomorphism. All in all, we have shown

\[
\text{supp } Q,W ≤ M_0 \times \mathbb{Z}^2 \quad \text{for a suitable constant } M_0 \text{ constructed in } [38, \text{Section 3}]; \text{ see also } [37, \text{Definition 2.2}]. \text{ Then, } [38, \text{Theorem 4.2}] \text{ yields a constant } β \in (1,2] \text{ such that we have}
\]

\[
|θ^i_N (f)| ≤ C \cdot \left[ N^{-1} \cdot (1 + \log N) \right]^{1+β/2} \quad \forall N \in \mathbb{N} \text{ and } f ∈ \mathcal{E}^β (\mathbb{R}^2; ν)
\]

where \( θ^i_N (f) \) denotes the \( N \)-th largest (in absolute value) \( α \)-curvelet coefficient of \( f \) with respect to the \( α \)-curvelet frame \( \{ψ^i μ \}_{μ ∈ M} \). This easily implies

\[
\| (f, ψ^i μ)_{μ ∈ M} \|_{L^p} = \| (θ^i_N (f))_{N ∈ \mathbb{N}} \|_{L^p} = C_{1}^{(p)} \quad \forall f ∈ \mathcal{E}^β (\mathbb{R}^2; ν) \text{ and } p > \frac{2}{1+β},
\]

for a suitable constant \( C_{1}^{(p)} = C_{1}^{(p)} (β, ν) \).

Now, let \( ϕ, ψ \) be real-valued functions satisfying the requirements of Proposition D.8 and let \( θ := ψ \circ R \). Let \( p_0 = q_0 = 1, s_0 = 0 \) and \( s_1 = 1/2 \) and choose \( δ_0 = δ_0 (β, p_0, q_0, s_0, s_1, ϕ, ψ) > 0 \) as provided by Proposition D.8 so that the \( \alpha \)-adapted \( β \)-shearlet system \( \text{SH} (ϕ, ψ, θ; δ, β) \) forms a Banach frame for \( \mathcal{F}^{p,q}_{β,s} (\mathbb{R}^2) \) for all \( 0 < δ ≤ δ_0, p ≥ p_0, q ≥ q_0 \) and \( s_0 ≤ s ≤ s_1 \), in the sense of Proposition D.8.

From this point on, the proof heavily uses the results and terminology of [37]. Since \( ϕ, ψ, θ \) are bandlimited, [37]

Proposition 3.11(ii) shows that \( \text{SH} (ϕ, ψ, θ; δ, β) \) is a system of \( β^{-1} \)-molecules of order \( (α, β, ∞, ∞, ∞, ∞) \) with respect to the parametrization \( (Λ^1, Φ^1) \) with \( τ = δ, σ = 2β/2, η = σ^{-j(1-α)} \) and \( L_j = [σ^j(1-α)] \), cf. [37] Definitions 3.7 and 3.8 for details of this parametrization. Furthermore, [37] Proposition 3.3(iii) shows that the \( α \)-curvelet frame \( \{ψ^i μ \}_{μ ∈ M} \) from above is a system of \( α \)-molecules of order \( (α, ∞, ∞, ∞, ∞) \) with respect to the parametrization \( (Λ^1, Φ^1) \) given in [37] Definition 3.2, with parameters \( σ = 2, τ = 1 \) and \( L_j = 2j(1-α) \) as above.

Next, [37] Theorem 5.7 shows that the \( α \)-curvelet parametrization \( (Λ^1, Φ^1) \) (defined in [37] Definition 3.2) and the \( α \)-shearlet parametrization \( (Λ^1, Φ^1) \) are \((α,k)\)-consistent for all \( k > 2 \); cf. [37] Definition 5.5 for the definition of \((α,k)\)-consistency. Now, for arbitrary \( p ∈ [2/(1+β), 1] \), [37] Theorem 5.6 shows that \( \{ψ^i μ \}_{μ ∈ M} \) and \( \text{SH} (ϕ, ψ, θ; δ, β) = (γ^i [i,k,δ])_{(i,k) ∈ J(δ) × \mathbb{Z}^2} \) are sparsity equivalent in \( L^p \), which means (cf. [37] Definition 5.3))

5Before [37] Proposition 3.11, it is required that the generators \( ϕ, ψ, θ \) of a \( band-limited \) \( β \)-shearlet system satisfy \( \text{supp } ϕ ⊂ Q, \text{supp } ψ ⊂ W \) and \( \text{supp } θ ⊂ W \), where \( Q ⊂ \mathbb{R}^2 \) is a cube centered at the origin and \( W, W ⊂ \mathbb{R}^2 \) satisfy \( W ⊂ [-a,a] × [-b,b] \) and \( W ⊂ (-a,-b] × [b,c] \) for certain \( a > 0 \) and \( 0 < b < c \). This is of course impossible, since \( ϕ, ψ, θ \) would then need to be simultaneously bandlimited and compactly supported. What is actually meant is \( \text{supp } ϕ ⊂ Q, \text{supp } ψ ⊂ W \) and \( \text{supp } θ ⊂ W \), with \( Q, W, W \) as above. Note the interchange of the sets \( W \) and \( W \) compared to the condition in [37]. It is not hard to see that our generators \( ϕ, ψ, θ \) satisfy these corrected assumptions, since \( \text{supp } ψ ⊂ \mathbb{R}^2 × \mathbb{R} \).
that the operator \( A : \ell^p(M) \to \ell^p(J(\beta) \times \mathbb{Z}^2) \) given by the infinite matrix \( (\langle \psi_{\mu}, \gamma_{[i,k,\delta]}^{[k,i,\delta]} \rangle_{L^2})_{\mu \in M, (i,k) \in J(\beta) \times \mathbb{Z}^2} \) is well-defined and bounded. Now, since \( (\psi_{\mu})_{\mu \in M} \) is a tight frame, we get

\[
f = \sum_{\mu \in M} \langle f, \psi_{\mu} \rangle_{L^2} \cdot \psi_{\mu}
\]

and thus

\[
\langle f, \gamma_{[i,k,\delta]}^{[k,i,\delta]} \rangle_{L^2} = \sum_{\mu \in M} \langle \psi_{\mu}, \gamma_{[i,k,\delta]}^{[k,i,\delta]} \rangle_{L^2} \langle f, \psi_{\mu} \rangle_{L^2} = A \left( \langle f, \psi_{\mu} \rangle_{L^2} \right)_{\mu \in M}.
\]

Consequently, since \( \varphi, \psi \) and thus also \( \gamma_{[i,k,\delta]}^{[k,i,\delta]} \) are real-valued, we get from equation (D.15) that

\[
\left\| \left( \langle f, \gamma_{[i,k,\delta]}^{[k,i,\delta]} \rangle_{Z^2(\mathbb{R}^2)}, Z(\mathbb{R}^2) \right)_{(i,k) \in J(\beta) \times \mathbb{Z}^2} \right\|_{\ell^p} = \left\| \left( \langle f, \gamma_{[i,k,\delta]}^{[k,i,\delta]} \rangle_{L^2} \right)_{(i,k) \in J(\beta) \times \mathbb{Z}^2} \right\|_{\ell^p} = A \left( \langle \psi_{\mu}, \gamma_{[i,k,\delta]}^{[k,i,\delta]} \rangle_{L^2} \right)_{\mu \in M} \leq \| A \|_{\ell^p \to \ell^p} C_1^{(p)} = C_2^{(p)} < \infty
\]

for all \( f \in \mathcal{E}^\beta(\mathbb{R}^2; \nu) \subset L^2(\mathbb{R}^2) = \mathcal{S}^{[2]}(\mathbb{R}^2) \) (cf. [37, Lemma 6.10]) and \( p \in (2/(1 + \beta), 1] \). But since \( \ell^1 \to \ell^p \) for \( p \geq 1 \), this estimate in fact holds for all \( p \in (2/(1 + \beta), \infty) \), with \( C_2^{(p)} := C_2^{(1)} \) for \( p \geq 1 \).

In view of the consistency in Proposition D.8 and since the remark after Definition 2.8 shows \( C_3^{(p)} \) is the reconstruction operator provided by Proposition D.8. This uses that we indeed have \( p \geq p_0 = q_0 = 2/(1 + \beta) \) and

\[
s_0 = 0 \leq (1 + \beta)^{-1} \left( \frac{1}{p} - \frac{1}{2} \right) \leq (1 + \beta)^{-1} \left( \frac{1 + \beta}{2} - \frac{1}{2} \right) = \frac{1}{2} (1 + \beta) = s_1.
\]

so that Proposition D.8 applies. Since Lemma D.3 shows \( \mathcal{S}^{[p]}(\mathbb{R}^2) \cong \mathcal{S}^{[p]}(\mathbb{R}^2) \), the proof is complete. \( \square \)

**Appendix E. A slight twist for achieving polynomial search depth**

In Theorem 6.3, we saw for \( \beta \in (1, 2] \) and \( \alpha = \beta^{-1} \) that suitable \( \alpha \)-shearlet systems achieve the approximation rate \( \| f - f_N \|_{L^2} \lesssim N^{-\left( \frac{1}{\beta} - \epsilon \right)} \) for arbitrary \( \epsilon > 0 \) and \( C^{[\beta]} \)-cartoon-like functions \( f \in \mathcal{E}^\beta(\mathbb{R}^2) \). Furthermore, we recalled from [37] that this approximation rate is essentially optimal, in the sense that no system \( \Phi = \{ \varphi_n \}_{n \in \mathbb{N}} \) can achieve an approximation rate better than \( N^{-\beta/2} \) for the whole class \( \mathcal{E}^\beta(\mathbb{R}^2; \nu) \), if one imposes a polynomial search depth for forming the \( N \)-term approximation \( f_N \). This means that \( f_N \) is assumed to be a linear combination of \( N \) elements of \( \{ \varphi_1, \ldots, \varphi_{\pi(N)} \} \), where \( \pi \) is a fixed polynomial, independent of \( f \). We did not show, however, that the \( N \)-term approximations \( f_N \) constructed in Theorem 6.3 satisfy such a polynomial search depth restriction. The goal of this section is precisely to show that this is possible for a suitable enumeration \( (\psi_n)_{n \in \mathbb{N}} \) of the \( \alpha \)-shearlet system under consideration.

The proof, however, is surprisingly nontrivial: In the proof of Theorem 6.3, we used that

\[
f = \sum_{i \in V \times \mathbb{Z}^2} c_i \psi_i
\]

for a sequence \( c = (c_i)_{i \in V \times \mathbb{Z}^2} \) with \( c \in \bigcap_{p > 2/(1 + \beta)} \ell^p(V \times \mathbb{Z}^2) \) and then truncated \( c \) to \( c \cdot 1_{J_N} \) to form \( f_N = \sum_{i \in J_N} c_i \psi_i \), where \( J_N \subset V \times \mathbb{Z}^2 \) contains the indices of the \( N \) largest entries of \( c \). But the positions of these indices depend heavily on \( c(e(f)) \) and thus on \( f \), while the polynomial search depth restriction requires us to use only indices in \( \{ 1, \ldots, \pi(N) \} \), where \( \pi \) is independent of \( f \).

Thus, what we essentially need is a certain (weak) decay of the coefficients, uniformly over the whole class \( \mathcal{E}^\beta(\mathbb{R}^2; \nu) \). But with our present decomposition space formalism, we cannot express such a decay, cf. Theorem 6.13. By choosing the exponent \( s \) for the weight \( u^s \) suitably, we can enforce a decay of the coefficients with the scale. But since the weight is independent of the translation variable \( k \in \mathbb{Z}^2 \) and since the space \( \ell^p(\mathbb{Z}^2) \) is permutation invariant, the current formalism cannot impose a decay of the coefficients as \( |k| \to \infty \).

Ultimately, this is caused by the definition of the decomposition spaces: It is not hard to see that the spaces \( \mathcal{D}(Q, L^p, L^w) \) are isometrically translation invariant. What we need, therefore, is a modified type of decomposition
spaces which does not have this property. Luckily, such a type of decomposition spaces already exists. In fact, the theory of structured Banach frame decompositions in [62] was developed for the spaces \( D(\mathcal{Q}, L^p, \mathcal{E}_w) \), where the Lebesgue spaces \( L^p(\mathbb{R}^d) \) are replaced by the weighted Lebesgue spaces \( L^p_w(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C} \mid v \cdot f \in L^p(\mathbb{R}^d) \} \) with \( \| f \|_{L^p_w} = \| v \cdot f \|_{L^p} \), where \( v : \mathbb{R}^d \to (0, \infty) \) is measurable. This theory is briefly discussed in the next subsection.

E.1. Structured Banach frame decompositions of weighted decomposition spaces. The weight \( v \) from above needs to satisfy certain regularity properties to ensure that the spaces \( D(\mathcal{Q}, L^p_w, \mathcal{E}_w) \) are well-defined. Precisely, we say that a measurable weight \( v : \mathbb{R}^d \to (0, \infty) \) is \( v_0\)-moderate for some weight \( v_0 : \mathbb{R}^d \to (0, \infty) \) if we have

\[
v(x + y) \leq v(x) \cdot v_0(y) \quad \forall x, y \in \mathbb{R}^d.
\]

(E.1)

Now, as in Section 2 let us fix an almost structured covering \( \mathcal{Q} = (T_i Q_i + b_i)_{i \in I} \) of an open set \( \varnothing \neq \mathcal{O} \subset \mathbb{R}^d \) with associated regular partition of unity \( \Phi = (\varphi_i)_{i \in I} \) for the remainder of the subsection and assume that \( \mathcal{Q} \) satisfies Assumption 2.7. The weight \( v_0 \) is called \( (\mathcal{Q}, \Omega_0, \Omega_1, K)\)-regular, for \( \Omega_0, \Omega_1 \subset [1, \infty) \) and \( K \subset [0, \infty) \), if it satisfies the following:

(1) \( v_0 \) is measurable and symmetric, i.e., \( v_0(-x) = v_0(x) \) for all \( x \in \mathbb{R}^d \).

(2) \( v_0 \) is submultiplicative, i.e., \( v_0(x + y) \leq v_0(x) \cdot v_0(y) \) for all \( x, y \in \mathbb{R}^d \).

(3) We have \( v_0(x) \leq \Omega_1 \cdot (1 + |x|)^K \) for all \( x \in \mathbb{R}^d \).

(4) We have \( K = 0 \), or \( \| \mathcal{T}_i^{-1} \| \leq \Omega_0 \) for all \( i \in I \).

We note that the preceding assumptions imply \( v_0(x) \geq 1 \) for all \( x \in \mathbb{R}^d \). Indeed, \( v_0(0) = v_0(x + (-x)) \leq |v_0(x)|^2 \) for all \( x \in \mathbb{R}^d \) by symmetry and submultiplicativity. For \( x = 0 \), this yields \( v_0(0) \geq 1 \), since \( v_0(0) > 0 \). Finally, we then see \( 1 \leq v_0(0) \leq |v_0(x)|^2 \) and hence \( v_0(x) \geq 1 \) for all \( x \in \mathbb{R}^d \).

The following example introduces the class of weights in which we will be mainly interested.

Example E.1. The standard weight \( \omega_0 \) is given by \( \omega_0 : \mathbb{R}^d \to (0, \infty), x \mapsto 1 + |x| \). It is submultiplicative, since

\[ 1 + |x + y| \leq 1 + |x| + |y| \leq (1 + |x|) \cdot (1 + |y|) \quad \forall x, y \in \mathbb{R}^d. \]

Hence, if we have \( K = 0 \) and \( \Omega_1 = 1 \), or if \( K > 0 \) and \( \| \mathcal{T}_i^{-1} \| \leq \Omega_0 \) for all \( i \in I \), then \( \omega_0 \) is \( (\mathcal{Q}, \Omega_0, 1, K) \)-regular.

Furthermore, if \( L \in \mathbb{R} \) with \( |L| \leq K \), then \( \omega_0^L \) is \( \omega_0 \)-moderate for \( L \geq 0 \), this follows from submultiplicativity of \( \omega_0 \), since \( \omega_0^L(x + y) \leq \omega_0^L(x) \omega_0^L(y) \leq \omega_0^L(x) \omega_0^L(y) \). If \( L < 0 \), then our considerations for \( L \geq 0 \) show \( \omega_0^{-L}(x) = \omega_0^L([x + y] - y) \leq \omega_0^{-L}(x + y) \omega_0^{-L}(-y) \). Rearranging again yields the claim.

Finally, in case of the unconnected \( \alpha \)-sheaf covering \( \mathcal{Q} = \mathcal{S}_u^{(\alpha)} = (B_i W_i')_{v \in V^{(\alpha)}} \), we have \( \| B_{\ell, \delta}^{-1} \| \leq 3 =: \Omega_0 \) for all \( v \in V^{(\alpha)} \). Indeed, for \( v = 0 \), this is trivial and for \( v = (j, \ell, \delta) \in V^{(\alpha)} \), we have

\[
\| B_{\ell, \delta}^{-1} \| = \left\| \begin{pmatrix} 2^{-j} & 0 \\ -2^{-j} & 2^{-\alpha j} \end{pmatrix} \right\| \leq 2^{-j} + 2^{-\alpha j} + |2^{-j}| \leq 3. \]

Here, the last step used that \( |\ell| \leq 2(1 - \alpha j) \leq 2^{j} \). Therefore, \( \omega_0^K \) is \( (S_u^{(\alpha)}, 3, 1, K) \)-regular for \( K \geq 0 \).

Now, we can define the modified, weighted decomposition spaces.

Definition E.2. Let \( p, q \in (0, \infty) \) and let \( w = (w_i)_{i \in I} \) be \( \mathcal{Q} \)-moderate. Further, let \( v_0 \) be \( (\mathcal{Q}, \Omega_0, \Omega_1, K) \)-regular and let \( v \) be \( v_0 \)-moderate.

Then, the \( (\text{weighted}) \) decomposition space \( (\text{quasi})\)-norm of \( g \in Z'(\mathcal{O}) \) is defined as

\[
\| g \|_{D(\mathcal{Q}, L^p_w, \mathcal{E}_w)} := \left\| (F^{-1}(\varphi_i \cdot \tilde{g}))_{i \in I} \right\| \mathcal{E}_w \in [0, \infty]
\]

and the associated \( (\text{weighted}) \) decomposition space is \( D(\mathcal{Q}, L^p_w, \mathcal{E}_w) := \left\{ g \in Z'(\mathcal{O}) \left| \| g \|_{D(\mathcal{Q}, L^p_w, \mathcal{E}_w)} < \infty \right. \right\} \).

Remark. It is a consequence of [62] Proposition 2.24, Lemma 5.5, and Corollary 6.5] that the resulting space is a well-defined Quasi-Banach space, with equivalent (quasi-)norms for different choices of \( \Phi \). Indeed, [62] Proposition 2.24 shows that the definition is independent of the \( Q_0 \)-BAPU \( \Phi \), while [62] Corollary 6.5] ensures that every regular partition of unity is a \( Q_0 \)-BAPU. Finally, [62] Lemma 5.5] establishes completeness of \( D(\mathcal{Q}, L^p_w, \mathcal{E}_w). \)

Recall from Section 3 that the Banach frame and atomic decomposition results for \( D(\mathcal{Q}, L^p, \mathcal{E}_w) \) were formulated in terms of the coefficient space \( C_{w}\mathcal{Q} \) from Definition 2.5. This coefficient space needs to be slightly adjusted in the present case.
Definition E.3. Under the assumptions of Definition E.2 and for \( \delta \in (0, \infty) \), define the weighted coefficient space \( C^{p,q}_{w,v,\delta} \) as
\[
C^{p,q}_{w,v,\delta} := \left\{ c = (c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} \mid \|c\|_{C^{p,q}_{w,v,\delta}} := \left\| \left( |\det T_i|^{\frac{1}{p} - \frac{1}{q}} \cdot w_i \cdot \left[ v(\delta \cdot T_i^{-1} k) \cdot c_k^{(i)} \right]_{k \in \mathbb{Z}^d} \right)_{i \in I} \right\|_\ell_p \right\} < \infty \right\}.
\]

The corresponding “weighted version” of Theorem 2.9 on the existence of Banach frames for decomposition spaces reads as follows:

**Theorem E.4.** Assume that \( Q \) satisfies Assumption 2.7. Let \( \Omega_0, \Omega_1 \in [1, \infty) \), \( K \in [0, \infty) \) and \( \varepsilon, p_0, q_0 \in (0, 1] \). Let \( v_0 \) be \((Q, \Omega_0, \Omega_1, K)\)-regular. Let \( w = (w_i)_{i \in I} \) be a \( Q \)-moderate weight and let \( v \) be \( v_0 \)-moderate. Finally, let \( p, q \in (0, \infty) \) with \( p \geq p_0 \) and \( q \geq q_0 \).

Define
\[
N := \left( K + \frac{d + \varepsilon}{\min\{1, p\}} \right), \quad \tau := \min\{1, p, q\} \quad \text{and} \quad \sigma := \tau \cdot \frac{d + \varepsilon}{\min\{1, p\}} + K + N.
\]

Let \( \gamma_1^{(0)}, \ldots, \gamma_n^{(0)} : \mathbb{R}^d \to \mathbb{C} \) be given and define \( \gamma_i := \gamma_i^{(0)} \) for \( i \in I \). Assume that the following conditions are satisfied:

1. We have \( \gamma_k^{(0)} \in L^1_{\Omega_{\mathbb{Z}^d}^1}(\mathbb{R}^d) \) and \( \mathcal{F} \gamma_k^{(0)} \in C^\infty(\mathbb{R}^d) \) for all \( k \in \mathbb{N} \), where all partial derivatives of \( \mathcal{F} \gamma_k^{(0)} \) are polynomially bounded.
2. We have \( \mathcal{F} \gamma_k^{(0)}(\xi) \neq 0 \) for all \( \xi \in Q_0^{(k)} \) and all \( k \in \mathbb{N} \).
3. We have \( \gamma_k^{(0)} \in C^1(\mathbb{R}^d) \) and \( \nabla \gamma_k^{(0)} \in L^1_{\Omega_0}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) for all \( k \in \mathbb{N} \).
4. We have
\[
C_1 := \sup_{i \in I} \sum_{j \in J} M_{j,i} < \infty \quad \text{and} \quad C_2 := \sup_{j \in J} \sum_{i \in I} M_{j,i} < \infty,
\]

where
\[
M_{j,i} := \left( \frac{w_j}{w_i} \right)^{\tau} \cdot \left( 1 + \left| T_j^{-1} T_i \right| \right)^{\sigma} \cdot \max_{|\beta| \leq 1} \left( |\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\alpha| \leq N} \left( |\partial^\alpha \tilde{T}^{\beta} \gamma_j| \right) \left( T_j^{-1} (\xi - b_j) \right) \, d\xi \right)^{\tau}.
\]

Then there is some \( \delta_0 = \delta_0(p, q, w, v, v_0, \varepsilon, (\gamma_i)_{i \in I}) > 0 \) such that for arbitrary \( 0 < \delta \leq \delta_0 \), the family
\[
\left( L_{\delta \cdot T_i^{-1} k}^{\gamma[i]} \right)_{i \in I, k \in \mathbb{Z}^d}
\]
forms a Banach frame for \( \mathcal{D}(Q, L^p_{v,w}, \ell^q_{w}) \). Precisely, this means the following:

- **The analysis operator**
\[
A(\delta) : \mathcal{D}(Q, L^p_{v,w}, \ell^q_{w}) \to C^{p,q}_{w,v,\delta} ; f \mapsto \left( [\gamma[i] \ast f] (\delta \cdot T_i^{-1} k) \right)_{i \in I, k \in \mathbb{Z}^d}
\]

is well-defined and bounded for each \( \delta \in (0, 1] \). Here, the convolution \( [\gamma[i] \ast f] \) is defined as in equation (2.20), where now the series converges normally in \( L^\infty_{\Omega_{\mathbb{Z}^d}^1}(\mathbb{R}^d) \) and thus absolutely and locally uniformly, for each \( f \in \mathcal{D}(Q, L^p_{v,w}, \ell^q_{w}) \). Of course, the simplified expression from Lemma 5.12 still holds if \( f \in L^2(\mathbb{R}^d) \subset \mathcal{D}'(\mathcal{O}) \).

- For \( 0 < \delta \leq \delta_0 \), there is a bounded linear **reconstruction operator** \( R(\delta) : C^{p,q}_{w,v,\delta} \to \mathcal{D}(Q, L^p_{v,w}, \ell^q_{w}) \) satisfying \( R(\delta) \circ A(\delta) = \text{id}_{\mathcal{D}(Q, L^p_{v,w}, \ell^q_{w})} \).

- We have the following consistency property: If \( Q \)-moderate weights \( w^{(1)} = (w_i^{(1)})_{i \in I} \) and \( w^{(2)} = (w_i^{(2)})_{i \in I} \) and exponents \( p_1, p_2, q_1, q_2 \in (0, \infty) \), as well as two \( v_0 \)-moderate weights \( v_1, v_2 : \mathbb{R}^d \to (0, \infty) \) are chosen such that the current assumptions are satisfied for \( p_1, v_1, w^{(1)} \), as well as for \( p_2, q_2, w^{(2)} \) and if \( 0 < \delta \leq \min\{\delta_0(p_1, q_1, v_1, v_0, \varepsilon, (\gamma_i)_{i \in I}), \delta_0(p_2, q_2, w^{(2)}, v_2, v_0, \varepsilon, (\gamma_i)_{i \in I})\} \), then we have the equivalence:
\[
\forall f \in \mathcal{D}(Q, L^p_{v_2, w^{(2)}}, \ell^q_{w^{(2)}}) ; \quad f \in \mathcal{D}(Q, L^p_{v_1, w^{(1)}}, \ell^q_{w^{(1)}}) \iff \left( [\gamma[i] \ast f] (\delta \cdot T_i^{-1} k) \right)_{i \in I, k \in \mathbb{Z}^d} \in C^{p,q}_{w^{(1)},v_1,\delta}.
\]

Finally, there is an estimate for the size of \( \delta_0 \) which is independent of the choice of \( p \geq p_0 \) and \( q \geq q_0 \) and of \( v, v_0 \):

There is a constant \( L = L(p_0, q_0, K, \varepsilon, d, Q, \Phi, \Omega_0, \Omega_1, \gamma_1^{(0)}, \ldots, \gamma_n^{(0)}) > 0 \) such that we can choose
\[
\delta_0 = \left( 1 + L \cdot C^4_{Q,w} \cdot \left( C_{1}^{1/\tau} + C_{2}^{1/\tau} \right)^2 \right)^{-1}.
\]
Proof. For brevity, set \( N_0 := \left[ K + p_0^{-1} \cdot (d + \varepsilon) \right] \) and note \( N \leq N_0 \).

First of all, we verify that the family \( \Gamma = (\gamma_i)_{i \in I} \) satisfies \([62, \text{Assumption 3.6}]\). To this end, we want to apply \([62, \text{Lemma 3.7}]\) (with \( N = n \)). Recall that \( \gamma_i = \gamma_{k_i} \) and from Assumption \(2.7\) that \( Q'_i = Q^{(k_i)}_0 \) for all \( i \in I \). Thus, in the notation of \([62, \text{Lemma 3.7}]\), we have for \( \ell \in \mathbb{N} \) that

\[
Q^{(k)} = \bigcup \{ Q'_i \mid i \in I \text{ and } k_i = k \} \subset Q^{(k)}_0.
\]

But by our assumption, by continuity of \( \mathcal{F} \mathcal{K} \gamma_k^{(0)} \) and by compactness of the sets \( Q^{(k)}_0 \), there is some \( c > 0 \) satisfying \( |(\mathcal{F} \mathcal{K} \gamma_k^{(0)})(\xi)| \geq c \) for all \( \xi \in Q^{(k)}_0 \) and all \( \ell \in \mathbb{N} \). Consequently, \([62, \text{Lemma 3.7}]\) shows that \( \Gamma \) satisfies \([62, \text{Assumption 3.6}]\) and also yields the estimate \( \Omega^2_{(p,K)} \leq \Omega_3 \) for a constant \( \Omega_3 = \Omega_3 \left( Q, \gamma^{(0)}, \ldots, \gamma^{(0)}_n, p_0, K, d \right) > 0 \). Here, \( \Omega^{(p,K)}_{2} \) is a constant defined in \([62, \text{Assumption 3.6}]\). To obtain this estimate, we used that \( p \geq p_0 \).

Now, since the family \( \Gamma \) satisfies \([62, \text{Assumption 3.6}]\), the assumptions of the present theorem easily imply that all assumptions of \([62, \text{Corollary 6.6}]\) are satisfied. This uses the special structure of the family \( \Gamma = (\gamma_i)_{i \in I} \), i.e., that \( \gamma_i = \gamma_{k_i}^{(0)} \) for each \( i \in I \).

In particular, \([62, \text{Corollary 6.6}]\) shows that the operators \( \vec{A} \) and \( \vec{B} \) from \([62, \text{Assumption 3.1 and Assumption 4.1}]\) are well-defined and bounded with \( \| \vec{A} \|_{\max\{1,\frac{1}{p}\}} \leq L_1^{(0)} \left( C_1^{1/\tau} + C_2^{1/\tau} \right) \) and \( \| \vec{B} \|_{\max\{1,\frac{1}{p}\}} \leq L_0^{(0)} \left( C_1^{1/\tau} + C_2^{1/\tau} \right) \)

for

\[
L_1^{(0)} = \Omega^{-\alpha}_{0} \Omega_{1} \cdot d^{1/\min\{1,p\}} \cdot (4d)^{1+2N} \cdot (\varepsilon^{-1} \cdot s_d)^{1/\min\{1,p\}} \cdot \max_{|\alpha| \leq N} C^{(\alpha)}
\]

\[
\leq \Omega^{-\alpha}_{0} \Omega_{1} \cdot d^{1/p_0} \cdot (4d)^{1+2N_0} \cdot (1 + \varepsilon^{-1} \cdot s_d)^{1/p_0} \cdot \max_{|\alpha| \leq N_0} C^{(\alpha)} =: L_1.
\]

Note that \( L_1 = L_1 (d, \varepsilon, Q, \Phi, p_0, \Omega_0, \Omega_1, K) \), since the constants \( C^{(\alpha)} \) from Definition \([62, \text{Definition 2.2}]\) only depend on \( \alpha, Q, \Phi \).

Since \([62, \text{Corollary 6.6}]\) is applicable to \( \Gamma \), we see that \( \Gamma \) satisfies \([62, \text{Assumption 4.1}]\). Therefore, \([62, \text{Lemma 4.3}]\) shows that the series in equation \([2.3]\) converges normally in \( L^2_{(1+\varepsilon) \cdot K} (\mathbb{R}^d) \) for all \( f \in \mathcal{D} (Q, L_{p_0}, L_{p_0}) \). Since each of the summands of the series is a continuous function, this yields absolute and locally uniform convergence of the series.

Next, since \( \vec{A} \) and \( \vec{B} \) are bounded, \([62, \text{Theorem 4.7}]\) is applicable. This shows that the family \( \left( L^{(0)}_{\delta_i - \varepsilon_k, \gamma_i^{(0)}} \right)_{i \in I, k \in \mathbb{Z}^d} \) yields a Banach frame for \( \mathcal{D} (Q, L_{p_0}, L_{p_0}) \) as in the statement of the current theorem, as soon as \( 0 < \delta \leq \delta_0 \) for \( \delta_0 = 1/(1 + 2 \cdot \| F_0 \|_2) \), where the operator \( F_0 \) is defined in \([62, \text{Lemma 4.6}]\). That lemma also yields the estimate

\[
\| F_0 \| \leq 2^{\frac{d}{p_0}} C_{Q, \Phi, p_0} \cdot \| \Gamma_Q \|_2 \cdot \left( \| \vec{A} \|_{\max\{1,\frac{1}{p}\}} + \| \vec{B} \|_{\max\{1,\frac{1}{p}\}} \right) \cdot L_2^{(0)}
\]

\[
\leq 2^{\frac{d}{p_0}} C_{Q, \Phi, p_0} \cdot \| \Gamma_Q \|_2 \cdot \left( C_1^{1/\tau} + C_2^{1/\tau} \right) \cdot 2 L_1 \cdot L_2^{(0)},
\]

where

\[
C_{Q, \Phi, p_0} = \sup_{i \in I} \left[ \| F_i \|_2 \right]^{\frac{1}{\min\{1,p\}}}.
\]

and where \( \Gamma_Q : \ell_{p_0}^{\infty} (I) \to \ell_{p_0}^{\infty} (I) \) is the \( Q \)-clustering map given by \( \Gamma_Q (c_i)_{i \in I} = (c_i^*)_{i \in I} \), where \( c_i^* = \sum_{\ell \in I} c_{i \ell} \).

Further, with \( M := \left[ K + \frac{d+1}{\min\{1,p\}} \right] \), the constant \( L_2^{(0)} \) is given by

\[
L_2^{(0)} = \begin{cases} 
\left( \frac{2^{6d} \cdot 768 \cdot d^d}{2^{2d} \cdot 12^d \cdot d^{d}} \cdot M^{M + 1} \cdot N_{\Omega^2_{(p,K)}} \cdot (1 + R \cdot Q \cdot C_{\Omega^2_{(p,K)}}) \cdot d_{\Omega^2_{(p,K)}}^{d-1} \cdot \Omega_{0}^{3} \Omega_{1}^{3} \Omega_{2}^{(p,K)} 
, \text{ if } p < 1, \\
\left( \frac{2^{2d} \cdot [K] \cdot \sqrt{d}}{2^{12} \cdot M} \cdot N_{\Omega^2_{(p,K)}} \cdot (1 + R \cdot Q \cdot C_{\Omega^2_{(p,K)}}) \cdot d_{\Omega^2_{(p,K)}}^{d-1} \cdot \Omega_{0}^{3} \Omega_{1}^{3} \Omega_{2}^{(p,K)} 
, \text{ if } p \geq 1 \\
\end{cases}
\]

\[
\leq \begin{cases} 
\left( \frac{2^{2d/p_0} \cdot (2^{2d} \cdot d^{33} \cdot M)^{M_0} + N_{\Omega^2_{(p,K)}} \cdot (1 + R \cdot Q \cdot C_{\Omega^2_{(p,K)}}) \cdot d_{\Omega^2_{(p,K)}}^{d-1} \cdot \Omega_{0}^{3} \Omega_{1}^{3} \Omega_{2}^{(p,K)} 
, \text{ if } p < 1, \\
\left( \frac{2^{2d/p_0} \cdot (2^{2d} \cdot d^{33} \cdot M)^{M_0} + N_{\Omega^2_{(p,K)}} \cdot (1 + R \cdot Q \cdot C_{\Omega^2_{(p,K)}}) \cdot d_{\Omega^2_{(p,K)}}^{d-1} \cdot \Omega_{0}^{3} \Omega_{1}^{3} \Omega_{2}^{(p,K)} 
, \text{ if } p \geq 1 \\
\end{cases}
\]

(since \( C \geq 1 \)).

Here, the last step used that \( M_0 = \left[ K + p_0^{-1} \cdot (d + 1) \right] \geq \left[ K + d + 1 \right] = [K] + d + 1, \) as well as \( \Omega_0, \Omega_1 \geq 1 \). Note as above that \( L_2 = L_2 (d, p_0, Q, \Omega_0, \Omega_1, \Omega_3, K, M_0) = K_2 (d, p_0, Q, \Omega_0, \Omega_1, K, \gamma^{(0)}_1, \ldots, \gamma^{(0)}_n) \).
As seen in [60] Lemma 4.13, we have \( \| \Gamma_{\mathcal{Q}} \| \leq C_{\mathcal{Q},w} N^{1+\frac{1}{k}} \) \( \cdot C_{\mathcal{Q},w} N^{1+\frac{1}{k}} \). Furthermore, [62] Corollary 6.5 shows that there is a function \( \varrho \in C_c(\mathbb{R}^d) \) (which only depends on \( \mathcal{Q} \)) such that

\[
C_{\mathcal{Q},\Phi,\nu_0,p} \leq \Omega_0^K \Omega_1 \cdot (4d)^{1/2N} \cdot \left( \frac{s_d}{\varepsilon} \right)^{1/\min\{1,\rho\}} \cdot 2^N \cdot \lambda_d \left( \bigcup_{i \in \mathcal{I}} Q_i' \right) \cdot \max_{|\alpha| \leq N} \| \partial^\alpha \varrho \|_{\sup} \cdot \max_{|\alpha| \leq N} C(\alpha)
\]

\[
\leq \Omega_0^K \Omega_1 \cdot (8d)^{1/2N_0} \cdot \left( 1 + \frac{8d}{\varepsilon} \right)^{1/p_0} \cdot (2R_{Q})^{d} \cdot \max_{|\alpha| \leq N_0} \| \partial^\alpha \varrho \|_{\sup} \cdot \max_{|\alpha| \leq N} C(\alpha) =: L_3.
\]

Here, we used that \( Q_i' \subset B_{R_{Q}} (0) \subset [-R_{Q}, R_{Q}] \) for all \( i \in \mathcal{I} \). Since the constants \( C(\alpha) = C(\alpha) (\Phi, \mathcal{Q}) \) from Definition 2.2 only depend on \( \alpha, \Phi, \mathcal{Q} \) and since \( \varrho \) only depends on \( \mathcal{Q} \), we see \( L_3 = L_3 (d, \varepsilon, p_0, \mathcal{Q}, \Phi, \Omega_0, \Omega_1, K) \).

All in all, we arrive at

\[
\| F_0 \| \leq 2^{1+\frac{d}{\varepsilon}} N^{2+\frac{d}{\varepsilon}} \cdot L_1 L_2 L_3^2 \cdot C_{\mathcal{Q},w} \cdot \left( C_{1/\tau} + C_{2/\tau} \right) = L_4 \cdot C_{\mathcal{Q},w} \cdot \left( C_{1/\tau} + C_{2/\tau} \right)
\]

for a suitable constant \( L_4 = L_4 \left( d, \varepsilon, p_0, q_0, \mathcal{Q}, \Phi, \gamma_{1}^{(0)}, \ldots, \gamma_{n}^{(0)}, \Omega_0, \Omega_1, K \right) \), so that the family \( \left( L_{i,d,-\tau,\gamma_{[i]}^{(\nu)}} \right)_{i \in \mathcal{I}, k \in \mathbb{Z}^d} \) yields a Banach frame for \( \mathcal{D}(\mathcal{Q}, L^p, \mathcal{U}_w) \) as soon as \( 0 < \delta \leq \delta_0 \) for \( \delta_0 := \left( 1 + 2 \left[ L_4 \cdot C_{\mathcal{Q},w} \cdot \left( C_{1/\tau} + C_{2/\tau} \right) \right]^{2} \right)^{\frac{1}{2}} \), since \( \delta_0 \leq \delta_0 \). Now, setting \( L := 2 \cdot L_4 \) yields the claim.

Finally, we present a "weighted version" of Theorem 2.10 concerning the existence of atomic decompositions for decomposition spaces.

**Theorem E.5.** Assume that \( \mathcal{Q} \) satisfies Assumption 2.7. Let \( \Omega_0, \Omega_1 \in [1, \infty) \), \( K \in [0, \infty) \) and \( \varepsilon, p_0, q_0 \in (0, 1] \). Let \( \nu_0 \) be \( (\mathcal{Q}, \Omega_0, \Omega_1, K) \)-regular. Let \( w = (w_i)_{i \in \mathcal{I}} \) be a \( \mathcal{Q} \)-moderate weight and let \( \nu \) be \( \nu_0 \)-moderate. Finally, let \( p, q \in (0, \infty] \) with \( p \geq p_0 \) and \( q \geq q_0 \).

Define

\[
N := \left[ K + \frac{d + \varepsilon}{\min\{1, p\}} \right], \quad \tau := \min\{1, p, q\}, \quad \vartheta := \left( \frac{1}{p-1} \right)^{+}, \quad \text{and} \quad T := K + 1 + \frac{d}{\min\{1, p\}}
\]

as well as

\[
\sigma := \begin{cases} \tau \cdot N, & \text{if } p \in [1, \infty], \\ \tau \cdot \left( p^{-1} \cdot d + K + N \right), & \text{if } p \in (0, 1). \end{cases}
\]

Let \( \gamma_{1}^{(0)}, \ldots, \gamma_{n}^{(0)} : \mathbb{R}^d \rightarrow \mathbb{C} \) be given and define \( \gamma_i := \gamma_{[i]}^{(\nu)} \) for \( i \in \mathcal{I} \). Assume that there are functions \( \gamma_{1}^{(0)}, \ldots, \gamma_{n}^{(0)} \) for \( j \in \{1, 2\} \) such that the following conditions are satisfied:

1. We have \( \gamma_{k}^{(0,1)} \in L_{1+\frac{1}{|\cdot|}}^{1} (\mathbb{R}^d) \) for all \( k \in \mathbb{N} \).
2. We have \( \gamma_{k}^{(0,2)} \in C^1 (\mathbb{R}^d) \) for all \( k \in \mathbb{N} \).
3. We have

\[
\Omega^{(p)} := \max_{k \in \mathbb{N}} \left\| \gamma_{k}^{(0,2)} \right\| \tau + \max_{k \in \mathbb{N}} \left\| \nabla \gamma_{k}^{(0,2)} \right\| _\tau < \infty,
\]

where \( \| f \|_\tau = \sup_{x \in \mathbb{R}^d} (1 + |x|)^{\tau} \cdot | f (x) | \) for \( f : \mathbb{R}^d \rightarrow \mathbb{C}^\ell \) and (arbitrary) \( \ell \in \mathbb{N} \).
4. We have \( \mathcal{F} \gamma_{k}^{(0,3)} \in C^\infty (\mathbb{R}^d) \) and all partial derivatives of \( \mathcal{F} \gamma_{k}^{(0,3)} \) are polynomially bounded for all \( k \in \mathbb{N} \) and \( j \in \{1, 2\} \).
5. We have \( \gamma_{k}^{(0)} = \gamma_{k}^{(0,1)} * \gamma_{k}^{(0,2)} \) for all \( k \in \mathbb{N} \).
6. We have \( \left\| \gamma_{k}^{(0)} \right\| _\tau < \infty \) for all \( k \in \mathbb{N} \).
7. We have \( \left\| \mathcal{F} \gamma_{k}^{(0)} \right\| (\xi) \neq 0 \) for all \( \xi \in Q_{0}^{(k)} \) and all \( k \in \mathbb{N} \).
8. We have

\[
K_1 := \sup_{i \in \mathcal{I}} \sum_{j \in I} N_{i,j} < \infty \quad \text{and} \quad K_2 := \sup_{j \in \mathcal{I}} \sum_{i \in \mathcal{I}} N_{i,j} < \infty,
\]

where \( \gamma_{j,1} := \gamma_{k_j}^{(0,1)} \) for \( j \in \mathcal{I} \) and

\[
N_{i,j} := \left( \frac{w_i}{w_j} \cdot | \det T_{i} | / | \det T_{j} | \right)^{\tau} \cdot (1 + | T_{j}^{-1} T_{i} | )^{\sigma} \cdot | \det T_{i} |^{-1} \cdot \int_{Q_{i,j}} \max_{|\alpha| \leq N} \left| \partial^\alpha \gamma_{j,1} \right| (T_{j}^{-1} (\xi - \tau_{j})) \cdot d \xi \right)^{\tau}.
\]
Then there is some $\delta_0 \in (0, 1]$ such that the family
\[ \Psi_\delta := \left( L_{T_{i, T_{i-k}}} \right)_{i \in I, k \in \mathbb{Z}^d} \] with $\gamma_i := |\det T_i|^{1/2} \cdot M_{ik} \cdot \left[ T_i \right]_T$
forms an atomic decomposition of $D(Q, L^p_v, \ell^2_w)$, for all $\delta \in (0, \delta_0]$. Precisely, this means the following:

- **The synthesis map** $S^{(\delta)} : C^{p,q}_{w,v,\delta} \to D(Q, L^p_v, \ell^2_w)$, $(c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} \mapsto \sum_{i \in I} \sum_{k \in \mathbb{Z}^d} \left[ c_k^{(i)} : L_{T_{i-k}} \right] \gamma_i$

is well-defined and bounded for every $\delta \in (0, 1]$.

- For $0 < \delta \leq \delta_0$, there is a bounded linear coefficient map $C^{(\delta)} : D(Q, L^p_v, \ell^2_w) \to C^{p,q}_{w,v,\delta}$ satisfying
\[ S^{(\delta)} \circ C^{(\delta)} = \text{id}_{D(Q, L^p_v, \ell^2_w)}. \]

Finally, there is an estimate for the size of $\delta_0$ which is independent of $p \geq p_0$, $q \geq q_0$ and of $v, \psi_0$: There is a constant
\[ L = L \left( p_0, q_0, \varepsilon, d, Q, \Phi, \gamma^{(0)}_1, \ldots, \gamma^{(0)}_n, \Omega_0, \Omega_1, K \right) > 0 \] such that we can choose
\[ \delta_0 = \min \left\{ 1, 1/\left[ L \cdot \Omega(p) \cdot \left( K_{1/\tau}^{1/\tau} + K_{2/\tau}^{1/\tau} \right) \right] \right\}. \]

Remark. Convergence of the series defining $S^{(\delta)}c$ has to be understood as in the remark to Theorem 2.10. Also as in that remark, the action of the coefficient map $C^{(\delta)}$ on a given $f \in D(Q, L^p_v, \ell^2_w)$ is independent of the precise choice of $p, q, v, w$, as long as $C^{(\delta)}f$ is defined at all.

Proof. For brevity, set $N_0 := \left[ K + p_0^{-1} \cdot (d + \varepsilon) \right]$. As in the proof of Theorem 5.3, we see as a consequence of [62] Lemma 3.7 that $\Gamma = (\gamma_i)_{i \in I}$ satisfies [62] Assumption 3.6, with $\Omega_2^{(p,K)} \leq \Omega_3$ for a suitable constant $\Omega_3 = \Omega_3 \left[ \Phi, \gamma^{(0)}_1, \ldots, \gamma^{(0)}_n, \Omega_0, \Omega_1, K \right] > 0$. For brevity, set $\Omega_5 := \Omega_0^{16K} \Omega_1^{16L}$.

Now, since we have $\gamma_i = \gamma_i^{(0)} = \gamma_1^{(0)} \cdot \ldots \cdot \gamma_n^{(0)}$ for all $i \in I$, it is easy to see that all assumptions of [62] Corollary 6.7 are satisfied. Consequently, [62] Corollary 6.7 shows that the operator $\overline{C}$ defined in [62] Assumption 5.1 is well-defined and bounded, with
\[ \| \overline{C} \|_{\text{max} \{ 1, p \}} \leq L_1^{(0)} \left( K_{1/\tau}^{1/\tau} + K_{2/\tau}^{1/\tau} \right), \]
where
\[ L_1^{(0)} = \Omega_0^{K} \Omega_1 \cdot (4d)^{1+2N} \cdot \frac{(s_d + d)^{1/2}}{\varepsilon} \cdot \max_{|\alpha| \leq N} C^{(\alpha)} \leq \Omega_0^{K} \Omega_1 \cdot (4d)^{1+2N} \cdot \frac{(s_d + d)^{1/2}}{\varepsilon}, \]

where the constants $C^{(\alpha)} = C^{(\alpha)}(Q, \Phi)$ are as in Definition 2.2. Thus, $L_1 = L_1 \left( d, p_0, \varepsilon, Q, \Phi, \Omega_0, \Omega_1, K \right)$.

Finally, [62] Corollary 6.7 shows that $\Gamma$ satisfies all assumptions of [62] Theorem 5.6, so that the family $\Psi_\delta$ defined in the statement of the theorem yields an atomic decomposition of $D(Q, L^p_v, \ell^2_w)$ as soon as $0 < \delta \leq \min \{ 1, \delta_0 \}$, where $\delta_0 > 0$ is defined by
\[ \delta_0^{-1} := \left( 2^{2d} \cdot \left( 2^{17} \cdot d^2 \cdot (K + 2 + d) \right)^{K+d+3} \cdot \left( 1 + R_2 \right)^{d+1} \cdot \Omega_0^{4K} \Omega_1^{16} \Omega_2^{(p,K)} \frac{\| \overline{C} \|}{\varepsilon}, \right) \]

if $p \geq 1,$

\[ \left( 2^{2d} \cdot \left( 2^{17} \cdot d^2 \cdot (K + 2 + d) \right)^{K+d+3} \cdot \left( 1 + R_2 \right)^{d+1} \cdot \Omega_0^{4K} \Omega_1^{16} \Omega_2^{(p,K)} \frac{\| \overline{C} \|}{\varepsilon}, \right) \leq \left( 2^{14d/p_0} \left( 1 + \frac{s_d}{p_0} \right)^{K+d+3} \cdot \left( 2^{14d/p_0} \left( 1 + \frac{s_d}{p_0} \right)^{K+d+3} \right)^{K+2+\frac{4d}{p_0}} \cdot \left( 1 + R_2 \right)^{4K} \cdot \Omega_5 \Omega(p) \cdot L_1 \cdot \left( K_{1/\tau}^{1/\tau} + K_{2/\tau}^{1/\tau} \right), \right) \]

if $p < 1.$

for
\[ L_2 := \left( 2^{14d} \cdot \left( 1 + p_0^{-1} \cdot s_d \right)^{1/p_0} \cdot \left( 2^{14d} \cdot (K + 1 + (d+1) \cdot p_0^{-1})^{3} \right)^{K+2+\frac{4d}{p_0}} (1 + R_2)^{4K} \Omega_5 \cdot L_1. \]

Here, our application of [62] Theorem 5.6 implicitly used that the constant $\Omega(p)$ from the statement of Theorem 5.6 satisfies $\Omega(p) = \Omega_2^{(p,K)}$ with $\Omega_2^{(p,K)}$ as in [62] Assumption 5.1.

Note that $L := L_2 \left( d, \varepsilon, p_0, Q, \Phi, \gamma^{(0)}_1, \ldots, \gamma^{(0)}_n, \Omega_0, \Omega_1, K \right)$ and finally observe that if
\[ \delta_0 = \min \left\{ 1, \left[ L \cdot \Omega(p) \cdot \left( K_{1/\tau}^{1/\tau} + K_{2/\tau}^{1/\tau} \right) \right]^{-1} \right\} \]
is defined as in the statement of Theorem 4.5, then \( \delta_0 \leq \min \{ 1, \delta_0 \} \), so that the family \( \Psi_{\delta} \) indeed yields an atomic decomposition of \( \mathcal{D}(Q, L^p_w, \ell^q_w) \) as soon as \( \delta \in (0, \delta_0) \). Finally, the remark associated to \([\Psi] \) Theorem 5.6] shows that convergence of the series in the definition of \( S^{(\delta)} \) occurs as claimed in the remark after Theorem 2.10 and that the action of \( C^{(\delta)} \) on a given \( f \in \mathcal{D}(Q, L^p_w, \ell^q_w) \) is independent of the precise choice of \( p, q, w, v \), as claimed in the remark to Theorem E.5.

\( \square \)

E.2. Cartoon approximation with \( \alpha \)-shearlets and polynomial search depth. In view of the results in the preceding subsection, we first define a new variant of the \( \alpha \)-shearlet smoothness spaces:

**Definition E.6.** Let \( \alpha \in [0, 1] \), let \( \omega_0 \) be the standard weight from Example 5.1 and let \( u = (u_w)_w \in \mathcal{V}(\alpha) \) be as in Definition 5.1. For \( p, q \in (0, \infty) \) and \( s, \kappa \in \mathbb{R} \), we define the (weighted) \( \alpha \)-shearlet smoothness space as

\[
\mathcal{F}^{p,q}_{\alpha,s,\kappa}(\mathbb{R}^2) := \mathcal{D}\left( S^{(\alpha)}_u, L^p_{\omega_0}, \ell^q_u \right).
\]

In this section, we will only consider exponents \( \kappa \geq 0 \), for which clearly \( \mathcal{F}^{p,q}_{\alpha,s,\kappa}(\mathbb{R}^2) \hookrightarrow \mathcal{F}^{p,q}_{\alpha,s}(\mathbb{R}^2) \hookrightarrow \mathcal{S}'(\mathbb{R}^2) \), cf. Lemma 5.6. Now, for \( 0 \leq \kappa \leq \kappa_0 \), Example 5.1 shows that the weight \( \omega_0 \) used above is \( \omega_0^{(p,q)} \)-moderate and that \( \omega_0^{(p,q)} \) is \((S^{(\alpha)}_u, 3, 1, \kappa_0)\)-regular. Then, by repeating the proofs of Theorems 4.2 and 4.3 for the modified values of \( N, \sigma, \tau \) or \( N, \kappa, \tau, \gamma \), one easily sees that Theorems E.7 and E.8 remain valid (with the proper modifications) for the more general spaces \( \mathcal{F}^{p,q}_{\alpha,s,\kappa}(\mathbb{R}^2) \), cf. Theorems E.7 and E.8 below.

The only nontrivial modification in the proof is the following: In the proof of Theorem 4.3, Proposition 2.11 \((N = N_0)\) is used to obtain factorizations \( \varphi = \varphi_1 \ast \varphi_2 \) and \( \psi = \psi_1 \ast \psi_2 \), where one still has a certain control over \( \varphi_1, \varphi_2, \psi_1, \psi_2 \). Indeed, Proposition 2.11 ensures that \( \varphi_2, \psi_2, \nabla \varphi_2, \nabla \psi_2 \) decay faster than any polynomial, so that the constant \( \Omega^{(p)} \) from Theorem 5.5] is finite. But Theorem 5.5] requires \( \varphi_1, \psi_1 \in L^1_{(1+|\bullet|)^\kappa_0}(\mathbb{R}^2) \), whereas Theorem 2.11 only required \( \varphi_1, \psi_1 \in L^1(\mathbb{R}^2) \). But this is still guaranteed by Proposition 2.11 since it implies \( \|\varphi_1\|_{N_0} \|\psi_1\|_{N_0} < \infty \), where now \( N_0 = \left[ \kappa_0 + p_0^{-1} \cdot (2 + \varepsilon) \right] \geq \kappa_0 + 2 + \varepsilon > \kappa_0 + 2 \), from which we easily get \( \varphi_1, \psi_1 \in L^1_{(1+|\bullet|)^\kappa_0}(\mathbb{R}^2) \).

The precise statements of the “weighted” versions of Theorems 5.9 and 5.10 are as follows:

**Theorem E.7.** Let \( \alpha \in [0, 1] \), \( \varepsilon, p_0, q_0 \in (0, 1] \), \( \kappa_0 \in (0, \infty) \) and \( s_0, s_1 \in \mathbb{R} \) with \( s_0 \leq s_1 \). Assume that \( \varphi, \psi : \mathbb{R} \rightarrow \mathbb{C} \) satisfy the following:

- \( \varphi, \psi \in L^1_{(1+|\bullet|)^\kappa_0}(\mathbb{R}^2) \) and \( \hat{\varphi}, \hat{\psi} \in C^\infty(\mathbb{R}^2) \), where all partial derivatives of \( \hat{\varphi}, \hat{\psi} \) have at most polynomial growth.
- \( \varphi, \psi \in C^1(\mathbb{R}^2) \) and \( \nabla \varphi, \nabla \psi \in L^1_{(1+|\bullet|)^\kappa_0}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \).

We have

\[
\hat{\psi}(\xi) \neq 0 \text{ for all } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \text{ with } |\xi_1| \in [3^{-1}, 3] \text{ and } |\xi_2| \leq |\xi_1|,
\]

\[
\hat{\varphi}(\xi) \neq 0 \text{ for all } \xi \in [-1, 1]^2.
\]

- \( \varphi, \psi \) satisfy equation (1.3) for all \( \theta \in \mathbb{N}_0^2 \) with \( |\theta| = N_0 \), where \( N_0 := \left[ \kappa_0 + p_0^{-1} \cdot (2 + \varepsilon) \right] \) and

\[
K := \varepsilon + \max \left\{ \frac{1 - \alpha}{\min \{ p_0, q_0 \}} + 2 \left( \frac{2}{p_0} + \kappa_0 + N_0 \right), \frac{2}{\min \{ p_0, q_0 \}} + \frac{2}{p_0} + \kappa_0 + N_0 \right\},
\]

\[
M_1 := \varepsilon + \frac{1}{\min \{ p_0, q_0 \}} + \max \{ 0, s_1 \},
\]

\[
M_2 := \max \left\{ 0, \varepsilon + (1 + \alpha) \left( \frac{2}{p_0} + \kappa_0 + N_0 \right) - s_0 \right\},
\]

\[
H := \max \left\{ 0, \varepsilon + \frac{1 - \alpha}{\min \{ p_0, q_0 \}} + \frac{2}{p_0} + \kappa_0 + N_0 - s_0 \right\}.
\]

Then there is some \( \delta_0 \in (0, 1] \) such that for \( 0 < \delta \leq \delta_0 \) and all \( p, q \in (0, \infty) \) and \( \kappa, s \in \mathbb{R} \) with \( p \geq p_0, q \geq q_0 \) and \( s_0 \leq s \leq s_1 \), as well as \( 0 \leq \kappa \leq \kappa_0 \), the following is true: The family

\[
\text{SH}_{\alpha}(\hat{\varphi}, \hat{\psi}; \delta) = \left( L_{\delta} B_{\gamma} \hat{\gamma}^{(v)} \right)_{v \in \mathcal{V}(\alpha), k \in \mathbb{Z}^2}
\]

with \( \gamma^{(v)}(x) = \gamma^{(v)}(-x) \) and \( \gamma^{(v)} := \left\{ \begin{array}{ll} |\det B_v|^{\frac{1}{2}} \cdot (\psi \circ B_T), & \text{if } v \in \mathcal{V}^{(\alpha)}_0, \\ \varphi, & \text{if } v = 0 \end{array} \right. \)

forms a Banach frame for \( \mathcal{F}^{p,q}_{\alpha,s,\kappa}(\mathbb{R}^2) \).

The precise interpretation of this statement is as in Theorem 4.2] with the obvious changes. In particular, the coefficient space \( C^{p,q}_{u,w} \) needs to be replaced by \( C^{p,q}_{w,u} \).

\( \square \)
Theorem E.8. Let $\alpha \in [0, 1], \varepsilon, p_0, q_0 \in (0, 1], \kappa_0 \in [0, \infty)$ and $s_0, s_1 \in \mathbb{R}$ with $s_0 \leq s_1$. Assume that $\varphi, \psi : \mathbb{R}^2 \to \mathbb{C}$ satisfy the following:

- We have $\|\varphi\|_{\infty} + \frac{1}{p_0} < \infty$ and $\|\psi\|_{\infty} + \frac{1}{q_0} < \infty$, where $\|g\|_\lambda = \sup_{x \in \mathbb{R}^2} (1 + |x|)^\lambda |g(x)|$ for $g : \mathbb{R}^2 \to \mathbb{C}^\ell$ (with arbitrary $\ell \in \mathbb{N}$) and $\Lambda \geq 0$.
- We have $\hat{\varphi}, \hat{\psi} \in C^\infty (\mathbb{R}^2)$, where all partial derivatives of $\hat{\varphi}, \hat{\psi}$ are polynomially bounded.
- We have $\hat{\psi} (\xi) \neq 0$ for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ with $|\xi_1| \in [3^{-1}, 3]$ and $|\xi_2| \leq |\xi_1|$, $\hat{\varphi} (\xi) \neq 0$ for all $\xi \in [-1, 1]^2$.

- $\varphi, \psi$ satisfy equation (1.3) for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and all $\beta \in \mathbb{N}_0^2$ with $|\beta| \leq N_0 := \left[ \kappa_0 + p_0^{-1} \cdot (2 + \varepsilon) \right]$, where

$$A_0 := \left\{ 3 + 2\varepsilon + \max \left\{ 2, \frac{1 - \alpha}{\min \{p_0, q_0\}} \cdot N_0 + s_1 \right\}, \quad \text{if } p_0 = 1, \right.$$  

$$3 + 2\varepsilon + \max \left\{ 2, \frac{1 - \alpha}{\min \{p_0, q_0\}} + 1 + \alpha + s_1 \right\}, \quad \text{if } p_0 \in (0, 1),$$

$$A_1 := \varepsilon + \frac{1}{\min \{p_0, q_0\}} + \max \left\{ 0, (1 + \alpha) \left( \frac{1}{p_0} - 1 \right) - s_0 \right\},$$

$$A_2 := \left\{ \varepsilon + \max \left\{ 2, (1 + \alpha) \cdot N_0 + s_1 \right\}, \quad \text{if } p_0 = 1, \right.$$  

$$\varepsilon + \max \left\{ 2, (1 + \alpha) \left( \frac{1}{p_0} + \kappa_0 + N_0 \right) + s_1 \right\}, \quad \text{if } p_0 \in (0, 1),$$

$$A_3 := \left\{ \varepsilon + \max \left\{ 2, \frac{1 - \alpha}{\min \{p_0, q_0\}} + 2 N_0 + s_1, \frac{1 - \alpha}{\min \{p_0, q_0\}} \right\}, \quad \text{if } p_0 = 1, \right.$$  

$$\varepsilon + \max \left\{ 2, \frac{1 - \alpha}{\min \{p_0, q_0\}} + 2 + \kappa_0 + 2 N_0 + 2 \alpha + s_1, \frac{2}{\min \{p_0, q_0\}} \right\}, \quad \text{if } p_0 \in (0, 1).$$

Then there is some $\delta_0 \in (0, 1]$ such that for all $0 < \delta \leq \delta_0$ and all $p, q \in (0, \infty)$ and $\kappa, s \in \mathbb{R}$ with $p \geq p_0, q \geq q_0$ and $s_0 \leq s \leq s_1$, as well as $0 \leq \kappa \leq \kappa_0$, the following is true: The family

$$\text{SH}_\alpha (\varphi, \psi; \delta) = \left\{ L_{\delta} B_{\varphi}^{T} v \left| v \in V(\alpha), k \in \mathbb{Z}^2 \right. \right\}$$

with $\gamma[v] := \left\{ \begin{array}{ll} |\det B_{\varphi}|^{1/2} \cdot (\psi \circ B_{\psi}^T), & \text{if } v \in V(\alpha), \\ \varphi, & \text{if } v = 0 \end{array} \right.$ forms an atomic decomposition for $\mathcal{A}^p,q,\kappa,\sigma(\mathbb{R}^2)$. Precisely, this has to be understood as in Theorem E.8 with the obvious changes. In particular, the coefficient space $C_{\alpha,q}^p,\kappa,\sigma$ needs to be replaced by $C_{\alpha,q}^{p,\kappa,\sigma}$.

Remark E.9. Of course, Remark 5.11 (cf. Corollaries 4.14 and 4.15) also applies in the current setting; one simply needs to replace the old values of $N_0$ and $K, M_1, M_2, H$ or $A_0, \ldots, A_3$ with the modified ones.

We can now finally show that the approximation rate stated in Theorem 6.3 can also be achieved when restricting to polynomial search depth:

Theorem E.10. Let $\beta \in (1, 2]$ be arbitrary and set $\alpha := \beta^{-1} \in [0, 1]$. Let $\varepsilon \in (0, 1]$ be arbitrary and set $\pi(x) := 40000 \cdot x^{14+4[1/\varepsilon]}$ for $x \in \mathbb{R}$. There is an enumeration $g : N \to V(\alpha) \times \mathbb{Z}^2$, with the index set $V(\alpha)$ from Definition 5.1, such that the following is true:

Assume that $\varphi, \psi$ satisfy the assumptions of Theorem E.8 for the choices $p_0 = q_0 = \frac{2}{1+\beta}$, $\kappa_0 = \varepsilon$ and $s_0 = 0$, as well as $s_1 := \frac{1}{2} (1 + \beta)$ and for $\varepsilon$ as above. Then there is some $\delta_0 \in (0, 1]$ such that every $0 < \delta \leq \delta_0$ satisfies the following: If $\gamma[v]_{v \in V(\alpha), k \in \mathbb{Z}^2} := \text{SH}_\alpha (\varphi, \psi; \delta)$ denotes the $\alpha$-shearlet system generated by $\varphi, \psi$, then there is for each $f \in \mathcal{E}^\beta (\mathbb{R}^2)$ and each $N \in \mathbb{N}$ a function $f_N$ which is a linear combination of $N$ elements of the set $\{ \gamma[v]\}_{n=1, \ldots, \pi(N)}$ and such that for all $\sigma, \nu > 0$ there is a constant $C = C (\varphi, \psi, \delta, \varepsilon, \sigma, \nu, \beta) > 0$ (independent of $f, N$) satisfying

$$\|f - f_N\|_{L^2} \leq C \cdot N^{-\delta} \quad \forall f \in \mathcal{E}^\beta (\mathbb{R}^2; \nu) \text{ and all } N \in \mathbb{N}.$$  

Remark. Using Remark E.2, one can show similarly to Remark 6.3 that the above theorem is applicable (with a suitable choice of $\varepsilon$), if $\varphi, \psi$ satisfy the assumptions stated in Remark 6.4.

Proof. Let $N \in \mathbb{N}$ be arbitrary and choose $n \in \mathbb{N}_0$ with $2^n < N < 2^{n+1}$, i.e., $n = \lfloor \log_2 N \rfloor$. For $v \in V(\alpha)$, we denote by $s(v)$ the scale encoded by $v$, i.e., $s(0) := -1$ and $s(j, m, i) := j$ for $(j, m, i) \in V_0(\alpha)$. Then, we define

$$W_N := \left\{(v, k) \in V(\alpha) \times \mathbb{Z}^2 \mid s(v) \leq 4n \text{ and } |B_v^{T} k| \leq 2^{2[n/\varepsilon]} \right\}. \tag{E.2}$$
Now, note that if $(v, k) = ((j, m, \nu), k) \in (V^{(\alpha)} \times Z^2) \cap W_N$, then $j \leq 4n$ and $|m| \leq 2^{(1-\alpha)j}$, so that we get

$$|k| = |B_{v}^{T}B_{v}^{-T}k| \leq \|B_{v}^{T}\| \cdot 2^{2[n/\varepsilon]} \leq 2^{2(1+n/\varepsilon)} \cdot \left\| \left( \begin{array}{cc} 0 & \nu \\ \nu & \varepsilon \end{array} \right) \right\|$$

$$\leq 4 \cdot 2^{2n/\varepsilon} \cdot (2^{2n} + 2^{\alpha n}) \cdot |m| \leq 16 \cdot 2^{2n/\varepsilon} \cdot 2^{j} \leq 16 \cdot 2^{(4+2[1/\varepsilon])n},$$

and thus $k \in \{-16 \cdot 2^{\alpha n}, \ldots, 16 \cdot 2^{\alpha n}\}$, where we defined $n_0 := 4 + 2[1/\varepsilon] \in \mathbb{N}$ for brevity. Furthermore, clearly $|m| \leq 2^{(1-\alpha)j}$, thus $m \in \{-2^{4n}, \ldots, 2^{4n}\}$. Finally, in case of $(v, k) = (0, k) \in V^{(\alpha)} \times Z^2$, we get $|k| = |B_{v}^{T}k| \leq 2^{2[n/\varepsilon]} \leq 2^{4n}$ and hence $k \in \{-2^{4n}, \ldots, 2^{4n}\}$. All in all, we have shown

$$W_N \subset \left\{ 0 \times \{-2^{4n}, \ldots, 2^{4n}\} \right\} \cup \bigcup_{j=0}^{4n} \left\{ j \times \{-2^{4n}, \ldots, 2^{4n}\} \right\} \times \{ 0, 1 \} \times \{-16 \cdot 2^{\alpha n}, \ldots, 16 \cdot 2^{\alpha n}\},$$

and thus

$$|W_N| \leq (1 + 2 \cdot 2^{4n})^2 + (1 + 4n) \cdot (1 + 2 \cdot 2^{4n}) \cdot (1 + 32 \cdot 2^{4n})^2 \leq (3 \cdot 2^{4n})^2 + 5n \cdot 3 \cdot 2^{4n} \cdot 2 \cdot (33 \cdot 2^{4n})^2 \leq 9 \cdot 2^{2n+30} \cdot 33^2 \cdot 2^{(5+2\alpha)n} \leq 40000 \cdot 2^{(5+2\alpha)n} \leq 40000 \cdot N^{5+2\alpha}.$$

Next, for arbitrary $(v, k) \in V^{(\alpha)} \times Z^2$ that there is some $n \in \mathbb{N}_0$ with $s(v) \leq 4n$ and $|B_{v}^{T}k| \leq 2^{2[n/\varepsilon]}$, so that $(v, k) \in W_N$ for $N = 2^n$. Hence, $W := V^{(\alpha)} \times Z^2 = \bigcup_{n \in \mathbb{N}} W_N$. Now, choose the enumeration $\theta : \mathbb{N} \to W$ such that if first enumerates $W_1$ (in an arbitrary way), then $W_2 \backslash W_1$ (arbitrarily), then $W_3 \backslash (W_1 \cup W_2)$, and so on. Formally, if we define $M_0 := 0$ and $M_N := W_N \cup \bigcup_{n=1}^{N} W_n \in \mathbb{N}$, then it satisfies $\theta \left( \sum_{n=1}^{N} M_n \right) = \bigcup_{n=1}^{N} W_n$ for all $N \in \mathbb{N}$. Because of $\sum_{n=1}^{N} M_n \leq \sum_{n=1}^{N} |W_n| \leq 40000 \cdot 2^{N+2\alpha} \leq 40000 \cdot N^{6+2\alpha} = \pi(N)$, we thus have $\theta\left( \pi(N) \right) \supset \bigcup_{n=1}^{N} W_n \supset W_N$ for all $N \in \mathbb{N}$. For brevity, let us set $Z_N := \theta\left( \pi(N) \right) \subset W$ for $N \in \mathbb{N}$.

We thus constructed the enumeration $\theta : \mathbb{N} \to V^{(\alpha)} \times Z^2$ from the statement of the theorem. Now, let $\alpha, \beta, \psi$ be arbitrary. By the assumptions of the theorem. Then Theorem [10] yields some $\delta_0 \in (0, 1)$ such that if $0 < \delta < \delta_0$, then the system $\mathcal{S}^{\alpha, \beta}_N$ of $\mathcal{S}^{\alpha, \beta}_N$ decomposes simultaneously for all $\alpha$-shearlet-smoothness spaces $\mathcal{S}^{\alpha, \beta}_N(\mathbb{R}^2)$ for $p, q \geq p_0, 0, s_0 \leq s \leq s_1$ and $0 \leq \kappa \leq \kappa_0 = \varepsilon$. Let $0 < \delta < \delta_0$ be arbitrary and let $\mathcal{S}_0^{(\delta)} \subset \mathcal{S}^{(\delta)}$ be the associated synthesis and coefficient operators. As noted in Theorem [3], the domain and codomain of these operators strictly speaking depend on the choice of $\alpha, \beta, s_0, s_1, \beta$, but the action of these operators does not. Hence, we commit the weak notational crime of not indicating this dependence.

For $n \in \mathbb{Z}^{\geq 0}$ and $\mathcal{C}_0^{(\delta)} \subset \mathcal{S}^{(\delta)}(\mathbb{R}^2) \subseteq \mathcal{S}^{2, \alpha, \beta}_0(\mathbb{R}^2)$, let $e^{(f)} := \sum_{i=1}^{c(f)} e_{\gamma} := C^{(f)} \in C^{(\delta)}f \in C^{(\delta)}w^{(\delta)}p_{\kappa} \subseteq \ell^2(\mathbb{W})$ and choose a subset $J^{(f)} \subset Z_N$ satisfying $|J^{(f)}| = \pi(N)$ and $|e^{(f)}_{\gamma} | \geq | \gamma_{\delta} |$ for all $j \in J^{(f)}$ and all $i \in Z_N \setminus J^{(f)}$. Such a choice is possible, since $Z_N$ is finite with $|Z_N| = \pi(N) \geq N$. Finally, set

$$f_{\delta} := \mathcal{S}_0^{(\delta)} \left( \mathbb{I}_{J^{(f)}} \cdot e^{(f)} \right).$$

By definition of $\mathcal{S}_0^{(\delta)}, f_{\delta}$ is then a linear combination of $N$ elements of the set $\{ \gamma_{\delta} | \ell = 1, \ldots, \pi(N) \}$, as desired. It remains to verify the claimed approximation rate. Thus, let $\sigma, \nu > 0$ be arbitrary.

We start with some preliminary considerations: In view of Remark [9], we see that there are symmetric, real-valued functions $\varphi_0, \bar{\psi}_0 \in C_{c}(\mathbb{R}^2)$ which satisfy the assumptions of Theorem [10] for the choices of $\nu_0, q_0, \kappa_0, s_0, s_1, \varepsilon$ from the current theorem. Hence, there is $\tau > 0$ such that the $\alpha$-shearlet system $\mathcal{S}^{(\delta)}(\mathbb{R}^2)$ forms a Banach frame for all $\alpha$-shearlet smoothness spaces $\mathcal{S}^{(\alpha, \beta)}(\mathbb{R}^2)$, for the same range of parameters as above. Note that the distinction between $\mathcal{S}^{(\delta)}$ and $\mathcal{S}^{(\delta)}_0$ does not matter by symmetry of $\varphi_0, \bar{\psi}_0$. As a consequence of the stability and real-valuedness of $\varphi_0, \bar{\psi}_0$, we then see that the analysis operator $A^{(\delta)}$ from Theorem [10] satisfies $A^{(\delta)}f := \left( (f, \theta^{(\delta)})_{L^2} \right) \in \mathcal{S}^{(\delta)}(\mathbb{R}^2)$ and thus in particular for $f \in \mathcal{S}^{(\delta)}(\mathbb{R}^2)$.

Now, for $v \in V^{(\alpha)}$ and $f \in \mathcal{S}^{(\alpha, \beta)}(\mathbb{R}^2; \nu)$, we have $\|f\|_{L^\infty} \leq C_1$ and thus

$$\|f\|_{L^\infty} \leq C_1 \cdot C_2 \cdot \|e^{(f)}_{\gamma} \|_{L^\infty} \leq C_1 \cdot \|e^{(f)}_{\gamma} \|_{L^\infty} \leq C_1 \cdot \|e^{(f)}_{\gamma} \|_{L^\infty} \leq \mathcal{S}_0^{(\delta)} \left( \mathbb{I}_{J^{(f)}} \cdot e^{(f)} \right),$$

with $C_2 := \max \{ \|\varphi_0\|_{L^1}, \|\psi_0\|_{L^1} \}$. By the consistency statement of Theorem [10] (see Theorem [12]), this shows $f \in \mathcal{S}^{(\delta)}(\mathbb{R}^2)$ with $\|f\|_{\mathcal{S}^{(\delta)}} \leq C_3 \cdot A^{(\delta)}f \leq C_3 \cdot \|A^{(\delta)}f\|_{\mathcal{S}^{(\delta)}} \leq C_4 \cdot \|A^{(\delta)}f\|_{\mathcal{S}^{(\delta)}} \leq C_4 \cdot \|A^{(\delta)}f\|_{\mathcal{S}^{(\delta)}}$, with the reconstruction operator $\mathcal{R}^{(\delta)}$ provided by Theorem [10] (for $\varphi_0, \bar{\psi}_0$). Here, we used the easily verifiable identity
$C_{v_0}^{\infty,\infty} = \ell^\infty_{\text{odd}} (V^{(\alpha)} \times \mathbb{Z}^2)$, where $u = (u_v^{(\alpha)/2})_v^{(\alpha)}$ is interpreted as a weight on $V^{(\alpha)} \times \mathbb{Z}^2$ in the obvious way.

Now, choose $A \geq 1$ with $\text{supp } \varphi_0, \text{supp } \psi_0 \subset (-A, A)^2$. Further, note that $R = R^{-1} = RT = (\frac{1}{1 \ 0})$ preserves the $\ell^\infty$-norm, so that every $(j, m, l) \in V^{(\alpha)}$ satisfies

$$\|B_{j, m,l} \|_{\ell^\infty} = \| \left( \left( \frac{2^{-j} - 2^{-j + m}}{2^{-j} - 2^{-j + m}} \right) \right) \|_{\ell^\infty} \leq 2^{-j} + 2^{-\alpha j} + |2^{-j} m| \leq 3,$$

since $|m| \leq \left[ 2^{1-\alpha} j \right] \leq 2j$. Further, clearly $\|B_0 \|_{\ell^\infty} = \| \| \|_\infty = 1 \leq 3$. Now, since each $f \in E^\beta (\mathbb{R}^2)$ satisfies $\text{supp } f \subset [-1, 1]^2$, we see that $\langle f, \theta^{[v,k]} \rangle_{L^2_2} \neq 0$ can only hold if

$$\emptyset \subseteq [-1, 1]^2 \cap \text{supp } \theta^{[v,k]} \quad \left\{ \begin{array}{l} (\text{with } \theta_0 = \psi \text{ for } v \in V^{(\alpha)} \text{ and } \theta_0 = \varphi) = [-1, 1]^2 \cap \text{supp } L_{\tau' , v, \tau} \theta \circ B_{v, l}^T \end{array} \right\}$$

$$\subset [-1, 1]^2 \cap \left[ \tau \cdot B_{v, l}^T k + B_{v, l}^T (-A, A)^2 \right] \text{,}$$

which implies $\tau \cdot B_{v, l}^T k \subset [-1, 1]^2 - B_{v, l}^T (-A, A)^2 \subset [-1, 1]^2 + 3(-A, A)^2 \subset [-4A, 4A]^2$, since $A \geq 1$. But Proposition 6.2 shows because of $1 \in (2/(1 + \beta), 2)$ that $E^\beta (\mathbb{R}^2; \nu) \in L^{1,1}_{\alpha, (1 + \alpha)(1-2\epsilon - 1)} (\mathbb{R}^2)$ is bounded, i.e., $\| A(f) \|_{L^1_{\alpha, (1 + \alpha)(1-2\epsilon - 1)} (\mathbb{R}^2)} \leq C_4 = C_4 (\beta, \nu)$. Since the associated coefficient space is $C_{\alpha, (1 + \alpha/2)} = \ell^1 (W)$, this implies $\| A(f) \|_{L^1} \leq C_3 = C_3 (\beta, \nu)$. But since we just saw that $\omega^{\alpha}_0 (\tau \cdot B_{v, l}^T k) \leq 9A$ for those $(v, k) \in W$ for which $(A(f) \nu, k) \neq 0$, this implies $\| A(f) \|_{L^{1,1}_{\alpha, (1 + \alpha/2 - \epsilon) \nu} (\mathbb{R}^2)} \leq 9A \cdot C_3$ for all $f \in E^\beta (\mathbb{R}^2; \nu)$, as one can see directly from Definition 3.3. By the consistency statement of Theorem 3.7 (see Theorem 3.2), this shows as above that $f \in J^{\beta}_{\alpha, (1 + \alpha/2 - \epsilon) \nu} (\mathbb{R}^2)$ with $\| f \|_{J^{\beta}_{\alpha, (1 + \alpha/2 - \epsilon) \nu}} \leq C_6 = C_6 (\beta, \nu, \kappa_0, \varphi_0, \psi_0, \tau)$, for all $f \in E^\beta (\mathbb{R}^2; \nu)$. Here, we used that $s_0 = 0 \leq \frac{1}{2} \leq \frac{2}{1 + \beta} = s_1$, since $\alpha, \beta \leq 1$.

Now, we continue with the proof of the approximation rate. Since we have $p^{-1} - 2^{-1} \rightarrow \beta/2$ as $p \rightarrow \frac{2}{1 + \beta}$ and $\frac{2}{1 + \beta - 2} = \frac{2}{2 - \sigma}$, there is some $p = p(\beta, \sigma) \in (2/(1 + \beta), 2]$ with $p^{-1} - 2^{-1} > \frac{2}{2 - \sigma}$. By Proposition 6.2 $E^\beta (\mathbb{R}^2; \nu) \in J^{\beta}_{\alpha, (1 + \alpha)(p-1-2\epsilon)} (\mathbb{R}^2)$ is bounded and the associated coefficient space to this $\alpha$-shearlet smoothness space is $C_{\alpha, (1 + \alpha)(p-1-2\epsilon)} = \ell^p (W)$, so that we get $\| c(f) \|_p = \| C(f) \|_{C_{\alpha, (1 + \alpha)(p-1-2\epsilon)} (W)} \leq C_7 = C_7 (\varphi, \beta, \delta, \rho, \nu, \nu)$. Here, we used that $s_0 = 0 \leq \left( 1 + \alpha \right) (p^{-1} - 2^{-1}) \leq (1 + \alpha) \left( \frac{2}{2 - \sigma} - \frac{1}{2} \right) = \frac{2 + \alpha}{2} = s_1$ and $p \geq p_0 = \frac{2}{2 - \sigma}$, so that $J^{\beta}_{\alpha, (1 + \alpha)(p-1-2\epsilon)} (\mathbb{R}^2)$ is in the “allowed” range.

Likewise, our considerations from above showed that $E^\beta (\mathbb{R}^2; \nu)$ is a bounded subset of $J^{\beta}_{\alpha, 0} (\mathbb{R}^2)$, and of $J^{\beta}_{\alpha, 0} (\mathbb{R}^2)$, so that there are constants $C_8, C_9$ (only dependent on $\varphi, \psi, \delta, \beta, \nu, \epsilon$) with $\| c(f) \|_{C_{\alpha, 0} (\mathbb{R}^2)} \leq C_8$ and $\| c(f) \|_{\infty_{\alpha, 0} (\mathbb{R}^2)} \leq C_9$, since $C_{\alpha, 0} = \ell^\infty_{\text{odd}} (V^{(\alpha)} \times \mathbb{Z}^2)$. Finally, set $C_{10} := \| S(\delta) \|_{L^2 \rightarrow L^2}$.

Because of $S(\delta) \circ \tau = \text{id}_{\mathbb{R}^2}$ and $c(f) = C(f) f$, we have

$$\| f - f_N \|_{L^2} = \| S(\delta) (c(f) - \text{1}_{J_{\epsilon}^N(f)} \cdot c(f)) \|_{L^2} \leq C_{10} : \| c(f) - 1_{J_{\epsilon}^N(f)} \cdot c(f) \|_{L^2 (W)} \quad \left( \text{since } J_{\epsilon}^N(f) \subset Z_N \right) \leq C_{10} : \left( \| c(f) \|_{L^2 (W \setminus Z_N)} + \| c(f) - 1_{J_{\epsilon}^N(f)} \cdot c(f) \|_{L^2 (Z_N)} \right).$$

(E.3)

Now, our choice of the set $J_{\epsilon}^N(f)$, together with Stechkin’s estimate (see e.g. [24] Proposition 2.3), shows

$$\| c(f) - 1_{J_{\epsilon}^N(f)} \cdot c(f) \|_{L^2 (Z_N)} \leq N^{-\frac{1}{2} - \frac{1}{2}} \cdot \| c(f) \|_{L^2 (Z_N)} \leq C_7 \cdot N^{-\frac{1}{2} - \frac{1}{2}} \leq C_7 \cdot N^{-\frac{1}{2} - \frac{1}{2}},$$

since $p^{-1} - 2^{-1} \geq \frac{1}{2} - \sigma$, so that it suffices to further estimate the first term in equation (E.3).

But for $(v, k) \in W \setminus Z_N \subset W \setminus W_N$, we have $s(v) \geq 4n$ (and thus in particular $v \in V^{(\alpha)}$), or $|B_{v, l}^T k| > 2^{n/\epsilon}$, where we recall that $2^n \leq N < 2^{n+1}$. In the first case, we have $\| c_{v,k}(f) \| \leq C_9 \cdot u^{-1+\alpha/2}_v \leq C_9 \cdot u^{-1+\alpha/2}_v \leq C_9 \cdot 2^{-2n}$ and in the second case, we get $\omega^{\alpha}_0 (\delta \cdot B_{v, l}^T k) = (1 + \| \delta \cdot B_{v, l}^T k \|)^\beta \geq \delta^\beta \cdot 2^{2n} \geq \delta \cdot 2^{2n}$ and thus

$$\left( \frac{f}{c_{v,k}} \right)^2 \leq C_9 \cdot \left( \frac{f}{c_{v,k}} \right)^2 \leq C_9 \cdot \frac{2^{-2n}}{\delta} \cdot \omega^{\alpha}_0 (\delta \cdot B_{v, l}^T k) \cdot \left( \frac{f}{c_{v,k}} \right)^2.$$
Therefore,
\[
\left\| c(f) \right\|^2_{l^2(W, Z_N)} \leq \sum_{v \in V^{(a)}} \sum_{k \in \mathbb{Z}^2} |c(f)_{v,k}|^2 + \sum_{v \in V^{(a)}} \sum_{k \in \mathbb{Z}^2} |c(f)_{v,k}|^2 \leq C_9 \cdot 2^{-2n} \sum_{v \in V^{(a)}} \sum_{k \in \mathbb{Z}^2} \left| \frac{c(f)_{v,k}}{C_9 \cdot 2^{-2n}} \sum_{v \in V^{(a)}} \sum_{k \in \mathbb{Z}^2} \omega^{\alpha \sigma}_0 (\delta \cdot B^{-T}_v k) \right|_{l^1(V^{(a)})}^2
\]

(since $\omega^{\alpha \sigma}_0 \geq 1$)
\[
= C_9 \cdot (1 + \delta^{-1}) \cdot 2^{-2n} \left\| c(f) \right\|_{C^{1,1}_{\omega^{2+\sigma}_0, \omega^{\sigma}_0, \delta}} \leq C_9 C_9 \cdot (1 + \delta^{-1}) \cdot 2^{-2n}
\]

(since $N \leq 2^{n+1}$ and $2^{-\sigma} - \sigma \leq 2^{-\sigma} \leq 1$)
\[
\leq 4C_8 C_9 \cdot (1 + \delta^{-1}) \cdot N^{-2} \leq 4C_8 C_9 \cdot (1 + \delta^{-1}) \cdot N^{-2} \cdot 2^{-\sigma}.
\]

Taking the square root and recalling equation $\text{(E3)}$ finishes the proof. \qed

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