ON HINGE DOMINATION IN GRAPHS

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Abstract. A set $D_h$ of vertices in a graph $G = (V, E)$ is a hinge dset if every vertex $u$ in $V - D_h$ is adjacent to some vertex $v$ in $D_h$ and a vertex $w$ in $V - D_h$ such that $(v, w)$ is not an edge in $E$. The hinge domination number $γ_h(G)$ is the minimum size of a hinged dset. In this paper we determine hinge domination number $γ_h(G)$ for standard graphs and some shadow distance graphs.

Keywords: dominating set; hinge domination number; minimal dominating set.

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1. INTRODUCTION

A graph $G = (V, E)$, we mean a finite, nontrivial and undirected graph without loops and multiple edges. The concept of a dset is well known in graph theoretic literature and various domination parameters have been studied. A set $D_h$ of vertices in $G$ is called a hinge dominating set [1] if every $u \in V - D_h$ is adjacent to some vertex $v \in D_h$ and a vertex $w$ in $V - D_h$ such that
(v, w) is not an edge in E. The hinge domination number \( \gamma_h(G) \) [1] is the minimum size of a hinge dominating set. Throughout this paper we will denote \textit{dominating set} by \textit{dset}.

Let \( D \) be the set of all possible distances in \( G = (V, E) \) and let \( D_s \subset D \). The distance graph associated with \( G \) denoted by \( D(G, D_s) \) [7] is the graph with vertex set \( V \) and two vertices \( u \) and \( v \) are adjacent in it if \( d(u, v) \in D_s \). The shadow distance graph of \( G \), denoted by \( D^s_d(G, D_s) \) is obtained from \( G \) by considering two copies of \( G \) namely \( G \) itself and \( G' \) such that if \( u \in V(G) \) then the corresponding vertex \( u' \) is in \( V(G') \) and \( E(D^s_d(G, D_s)) = E(G) \cup E(G') \cup E_{DS} \) where \( E_{DS} \) consists of the set of all edges of the form \( e = (u, v') \) with the condition \( d(u, v) \in D_s \) in \( G \).

In this paper we determine the hinge domination number for some standard graphs and shadow distance graphs. We also show that the hinge domination number of the cycle graph provided in [1] is incorrect and provide the exact value.

2. \textbf{Main Results}

We begin this section with the following result which gives the condition for a minimal hinge dset.

\textbf{Theorem 2.1.} A hinge dset \( D_h \) is minimal if and only if for every \( v \in D_h \), one of the following condition holds:

\begin{enumerate}[(i)]
\item \( \text{deg}(v) = 0 \) in \( D_h \)
\item \( \exists \) a vertex \( u \) in \( V - D_h \) such that \( N(u) \cap D_h = \{v\} \).
\item \( <(V - D_h) \cup \{v\}> \) is connected
\end{enumerate}

\textbf{Proof.} For every \( u \in D_h \), if \( D_h - \{u\} \) is not a hinge dset in \( G \), it follows that either \( u \) is an isolated vertex of \( D_h \) or there exists a vertex \( v \in V - D_h \) such that \( N(v) \cap D_h = \{u\} \). Further, for \( v \in D_h \), it is clear that the induced graph of \( [(V - D_h) \cup \{v\}] \) is connected.

Conversely, if \( D_h \) is not minimal, there exist \( u \in D_h \) such that \( D_h - \{u\} \) is also a hinge dset. Thus, for at least one \( v \in D_h - \{u\} \) there is a path between \( u \) and \( v \) in \( G \). This contradicts condition (i). Also, If \( D_h - \{v\} \) is a hinge dset, then every \( u \in V - D_h \) is adjacent to at least one vertex in \( D_h - \{v\} \), so that condition (ii) also fails. Now, let us consider \( v \in D_h \) such that \( v \) does not satisfy conditions (i) and (ii). Then from conditions (i) and (ii), \( D_{h_1} = D_h - v \) is hinge
dset. Also by condition (iii), \( < V - D_h > \) is disconnected, so that \( D_{h_1} \) is a hinge dset of \( G \). This contradicts condition (iii). Hence the proof. □

**Theorem 2.2.** For any graph \( G \), \( \gamma_h(G) \geq \frac{n+1}{\Delta(G)+1} \).

**Proof.** Let \( D_h \) be a minimum hinge dset of \( G \) and the number of edges \( t \) in \( G \) having one \( v \in D_h \) and the other in \( V - D_h \). Since \( \Delta(G) \geq degv \ \forall \ v \in D_h \). For every \( v \in D_h \) has at least one unique neighbor in \( D_h \), \( t \leq \gamma_h(G) \cdot \Delta(G) - 1 \). Also \( t \geq |V - D_h| = n - \gamma_h(G) \). Hence \( n - \gamma_h(G) \leq \Delta(G) \cdot \gamma_h(G) - 1 \). This gives \( \gamma_h(G) \geq \frac{n+1}{\Delta(G)+1} \). □

**Theorem 2.3.** For any graph \( G = (V, E) \) such that \( |V| = p \) and \( |E| = q \), \( p - q \leq \gamma(G) \leq \gamma_h(G) \).

**Proof.** Suppose \( q \geq p - 1 \), then \( 1 \geq p - q \) since \( \gamma_h(G) \geq 1 \), \( \gamma_h(G) \geq p - q \). So assume \( q \leq p \) then \( G \) has at least \( p - q \) components. At least one vertex per component is required in any hinge dset. Therefore \( p - q \leq \gamma(G) \leq \gamma_h(G) \). □

**Theorem 2.4.** For any graph \( G \), \( \left\lfloor \frac{p}{1+\Delta(G)} \right\rfloor \leq \gamma(G) \leq \gamma_h(G) \).

**Proof.** Let \( D_h \) be a hinge dset of \( G \). Each vertex dominates at most itself and \( \Delta(G) \) other vertices. From the proof of the theorem, it follows that \( \gamma(G) = \frac{p}{1+\Delta(G)} = \gamma_h(G) \) if and only if \( \gamma_h \) set \( D_h \) such that \( N[u] \cap N[v] = \phi \) for all \( u, v \in D_h \) and \( |N(v)| = \Delta(G) \) for all \( v \in D_h \). For example, the cycle \( C_6 \) has \( \gamma(G) = 2 = \gamma_h(G) \) and \( \frac{p}{1+\Delta(G)} = 2 \). □

**Theorem 2.5.** Let \( D_h \) be a hinge dset of \( G \) such that \( |D_h| = \gamma_h(G) \). Then \( |V(G) - D| \leq deg(v) \).

**Proof.** Let \( D_h \) be a hinge dset of \( G \), then \( |degv - degu| \leq 1 \ \forall \ v \in D_h, u \in V - D_h \) and every vertex \( v \in V - D_h \) is adjacent to one vertex in \( D_h \). Hence each vertex in \( V - D_h \) contributes at least one to the sum of degrees of the vertex of \( D_h \). Hence \( |V(G) - D| \leq deg(v) \). □

The following result is from [1] related to the cycle graph \( C_n \).

**Proposition 2.2** For \( n \geq 3 \), \( \gamma_h(C_n) = \begin{cases} k & \text{if} \ n = 3k \\ k+1 & \text{if} \ n = 3k+1 \\ k+2 & \text{if} \ n = 3k+2 \end{cases} \).

From this result, it is clear that \( \gamma(C_3) = 1 \). As a counter example we observe that the graph \( C_3 \) illustrated in figure 1 has hinge domination number 3.
We now provide the correct value of $\gamma_h(C_n)$ in our next result.

**Theorem 2.6.** If $n \geq 3$, then

$$
\gamma_h(C_n) = \begin{cases} 
2 & n = 4, 6 \\
3 & n = 3, 5 \\
\lfloor \frac{n}{3} \rfloor, & n \equiv 0 \mod 3 \\
\lfloor \frac{n}{3} \rfloor + 1, & n \equiv 1 \mod 3 \\
\end{cases}
$$

**Proof.** Let $V(C_n) = \{v_i | 1 \leq i \leq n\}$ and $E(C_n) = \{e_i | 1 \leq i \leq n\}$ where $e_i = (v_i, v_{i+1})$, $i = 1, 2, ..., n$, where computation is under modulo $n$.

If $n = 4$ and 6, the sets $D_h = \{v_1, v_2\}$ and $D_h = \{v_2, v_5\}$ are minimal so that $\gamma_h(G) = 2$. Also, for $n = 3$ and 5, the set $D_h = \{v_1, v_2, v_3\}$ is minimal so that $\gamma_h(G) = 3$. Let $n \geq 7$. Then, for

**case(i):** $n = 3i + 4$, $i = 1, 2, 3, ...$, we consider the set $D_h = \{v_{3s-2}\}$, $1 \leq s \leq \lfloor \frac{n}{3} \rfloor$.

**case(ii):** $n = 3j + 5$, $j = 1, 2, 3, ...$, we consider the set $D_h = \{v_{3t-2}\} \cup \{v_{n-1}\} \cup \{v_n\}$, $1 \leq t \leq \lfloor \frac{n}{3} \rfloor - 1$.

and for

**case(iii):** $n = 3k + 6$, $k = 1, 2, 3, ...$, we consider the set $D_h = \{v_{3r-2}\}$, $1 \leq r \leq \frac{n}{3}$.

It is clear that the sets $D_h$ in cases (i), (ii) and (iii) are minimal hinge sets. Thus, some vertex $v \in D_h$ is adjacent to only one vertex $u \in V - D_h$ and not to any other vertex.
Therefore, since $|D_h| = \begin{cases} \frac{n}{3}, & n \equiv 0 \pmod{3} \\ \left\lceil \frac{n}{3} \right\rceil, & n \equiv 1 \pmod{3} \\ \left\lceil \frac{n}{3} \right\rceil + 1, & n \equiv 2 \pmod{3} \end{cases}$

we immediately obtain $\gamma_h(C_n) = \begin{cases} \frac{n}{3}, & n \equiv 0 \pmod{3} \\ \left\lceil \frac{n}{3} \right\rceil, & n \equiv 1 \pmod{3} \\ \left\lceil \frac{n}{3} \right\rceil + 1, & n \equiv 2 \pmod{3} \end{cases}$

Hence the proof. \hfill \Box

For the path graph $P_n$, the following result can be found in [1].

**Proposition 2.2** $\gamma_h(P_n) = \begin{cases} 2, & n = 2 \\ k + 2, & n = 3k \\ \left\lceil \frac{n-1}{3} \right\rceil + 1, & n \neq 3k \end{cases}$

In the next theorem, a modified version of this result is provided.

**Theorem 2.7.** If $n \geq 3$, then $\gamma_h(P_n) = \begin{cases} \frac{n}{3} + 2, & n \equiv 0 \pmod{3} \\ \left\lceil \frac{n}{3} \right\rceil, & n \equiv 1 \pmod{3} \\ \left\lceil \frac{n}{3} \right\rceil + 1, & n \equiv 2 \pmod{3} \end{cases}$

*Proof.* Let $V(P_n) = \{v_i/1 \leq i \leq n\}$ and $E(P_n) = \{e_i/1 \leq i \leq n-1\}$ where $e_i = (v_i, v_{i+1})$, $i = 1, 2, \ldots, n-1$, where computation is under modulo $n$.

If $n = 3$ and $4$, the sets $D_h = \{v_1, v_2, v_3\}$ and $D_h = \{v_1, v_4\}$ are minimal so that $\gamma_h(P_3) = 3$ and $\gamma_h(P_4) = 2$ respectively. Let $n \geq 5$. Then, for

**case(i):** $n = 3i + 2, i = 1, 2, 3 \ldots$, we consider the set $D_h = \{v_{3s-2}\} \cup \{v_n\}, 1 \leq s \leq \left\lceil \frac{n}{3} \right\rceil$. 

case(ii): $n = 3j + 3$, $j = 1, 2, 3 \ldots$, we consider the set $D_h = \{v_{3t-2}\} \cup \{v_n\} \cup \{v_t\}$, $1 \leq t \leq n$. 

and for

case(iii): $n = 3k + 4$, $k = 1, 2, 3 \ldots$, we consider the set $D_h = \{v_{3r-2}\}$, $1 \leq r \leq \lceil \frac{n}{3} \rceil$.

It is clear that the sets $D_h$ in cases $(i), (ii)$ and $(iii)$ are minimal hinge dsets. Thus, some vertex $v \in D_h$ is adjacent to only one vertex $u \in V - D_h$ and not to any other vertex.

Therefore, since $|D_h| = \begin{cases} \frac{n}{3} + 2, & n \equiv 0 \pmod{3} \\ \lceil \frac{n}{3} \rceil, & n \equiv 1 \pmod{3} \\ \lceil \frac{n}{3} \rceil + 1, & n \equiv 2 \pmod{3} \end{cases}$

we immediately obtain $\gamma_h(P_n) = \begin{cases} \frac{n}{3} + 2, & n \equiv 0 \pmod{3} \\ \lceil \frac{n}{3} \rceil, & n \equiv 1 \pmod{3} \\ \lceil \frac{n}{3} \rceil + 1, & n \equiv 2 \pmod{3} \end{cases}$

Hence the proof. \hfill \Box

We now determine the hinge domination number for some shadow distance graphs.

**Theorem 2.8.** If $n \geq 2$, then $\gamma_h(D_2\{P_n\}) = \begin{cases} 2 & n = 2 \\ n - 1 & n \geq 3 \end{cases}$

**Proof.** Let $V(P_n) = \{v_i/1 \leq i \leq n\}$ and $V(P_n^{'}) = \{v_i'/1 \leq i \leq n\}$. Let $E(P_n) = \{e_i/1 \leq i \leq n - 1\}$ and $E(P_n^{'}) = \{e_i'/1 \leq i \leq n - 1\}$, where $e_i = (v_i, v_{i+1})$, $e_i' = (v_i', v_{i+1}')$ for $i = 1, 2, \ldots, n - 1$.

Let $G = (D_2\{P_n\})$.

If $n = 2$, $D_h = \{v_1, v_2\}$ is minimal so that $\gamma_h(G) = 2$.

Let $n \geq 3$

Consider $D_h = \{v_{2j-1}\} \cup \{v_{2k}'\}$, where $1 \leq j \leq \lceil \frac{n}{2} \rceil$, $1 \leq k \leq \frac{n}{2} - 1$, when $n$ is even and $1 \leq k \leq \lceil \frac{n}{2} \rceil$ when $n$ is odd

If $D_h$ is not a hinge dset of $G$, there exists a vertex $v \in D_h$ such that $D_{h_1} = D_h - \{v\}$ is a hinge dset of $G$ and also, $\langle V - D_h \rangle$ is disconnected. This implies that $D_h$ is a hinge dset of $G$, which contradicts condition $(iii)$. Therefore, $D_h$ is minimal and since

$|D_h| = \begin{cases} 2, & n = 2 \\ n - 1, & n \geq 3 \end{cases}$, so that $\gamma_h(D_2\{P_n\}) = \begin{cases} 2, & n = 2 \\ n - 1, & n \geq 3 \end{cases}$
Hence the proof.

□

Theorem 2.9. If \( n \geq 3 \), then \( \gamma_h(D_2\{C_n\}) = \begin{cases} \frac{2n}{3}, & n \equiv 0 \text{(mod3)} \\ \frac{2(n-1)}{3} + 2, & n \equiv 1 \text{(mod3)} \\ \frac{2(n-2)}{3} + 4, & n \equiv 2 \text{(mod3)} \end{cases} \)

Proof. Let \( V(C_n) = \{v_i/1 \leq i \leq n\} \) and \( V(C'_n) = \{v'_i/1 \leq i \leq n\} \). Let \( E(C_n) = \{e_i/1 \leq i \leq n\} \) and \( E(C'_n) = \{e'_i/1 \leq i \leq n\} \), where \( e_i = (v_i, v_{i+1}) \) and \( e'_i = (v'_i, v'_{i+1}) \) for \( i = 1, 2, \ldots, n \), where computation is under modulo \( n \).

Let \( G = (D_2\{C_n\}) \).

Let \( n \geq 3 \). Then, for

\textbf{case(i):} \( n = 3a, a = 1, 2, 3 \ldots \), we consider the set \( D_h = \{v_{3j-2}\} \cup \{v'_{3k-2}\}, 1 \leq j, k \leq \frac{n}{3} \).

\textbf{case(ii):} \( n = 3b + 1, b = 1, 2, 3 \ldots \), we consider the set \( D_h = \{v_{3j-2}\} \cup \{v'_{3k-2}\}, 1 \leq j, k \leq \left\lceil \frac{n}{3} \right\rceil \) and for

\textbf{case(iii):} \( n = 3c + 2, c = 1, 2, 3 \ldots \), we consider the set \( D_h = \{v_{3j-2}\} \cup \{v'_{3k-2}\} \cup \{v_{n}\} \cup \{v'_{n}\}, 1 \leq j, k \leq \left\lceil \frac{n}{3} \right\rceil \).

If \( D_h \) is not a hinge dset of \( G \), there exists a vertex \( v \in D_h \) such that \( D_{h_1} = D_h - \{v\} \) is a hinge dset of \( G \) and also, \( <V - D_h>\) is disconnected. This implies that \( D_{h_1} \) is a hinge dset of \( G \), which contradicts condition (iii). Therefore, \( D_h \) is minimal and since

\[ |D_h| = \begin{cases} \frac{2n}{3}, & n \equiv 0 \text{(mod3)} \\ \frac{2(n-1)}{3} + 2, & n \equiv 1 \text{(mod3)} \\ \frac{2(n-2)}{3} + 4, & n \equiv 2 \text{(mod3)} \end{cases} \]

so that \( \gamma_h(D_2\{C_n\}) = \begin{cases} \frac{2n}{3}, & n \equiv 0 \text{(mod3)} \\ \frac{2(n-1)}{3} + 2, & n \equiv 1 \text{(mod3)} \\ \frac{2(n-2)}{3} + 4, & n \equiv 2 \text{(mod3)} \end{cases} \)

Hence the proof.

□
Theorem 2.10. If \( n \geq 3 \), then \( \gamma_h(D_h\{P_n, \{2\}\}) = \begin{cases} 
 2 & n = 3 \\
 3 & n = 4 \\
 n - 1 & n \geq 5 
\end{cases} \)

Proof. Let \( V(P_n) = \{v_i/1 \leq i \leq n\} \) and \( V(P'_n) = \{v'_i/1 \leq i \leq n\} \). Let \( E(P_n) = \{e_i/1 \leq i \leq n - 1\} \) and \( E(P'_n) = \{e'_i/1 \leq i \leq n - 1\} \), where \( e_i = (v_i,v_{i+1}) \), \( e'_i = (v'_i,v'_{i+1}) \) for \( i = 1,2,.....n-1 \).

Let \( G = (D_{sd}\{P_n, \{2\}\}) \).

If \( n = 3,4 \), the sets \( D_h = \{v_1,v'_1\} \) and \( D_h = \{v_1,v_4,v'_2\} \) are minimal so that \( \gamma_h(G) = 2 \) and 3 respectively.

Let \( n \geq 5 \)

Consider \( D_h = \{v_{2j-1}\} \cup \{v'_{2k+1}\} \), where \( 1 \leq j \leq \lfloor \frac{n}{2} \rfloor \), \( 1 \leq k \leq \frac{n}{2} - 1 \) where \( n \) is even, \( 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \) where \( n \) is odd.

Let \( D_h \) is not hinge dset of \( G \), there exists a vertex \( v \in D_h \), then \( D_{h_1} = D_h - v \) is dset of \( G \), also \( <V - D_h > \) is disconnected. This implies that \( D_{h_1} \) is a hinge dset of \( G \), This contradicts condition (iii).

Therefore, \( D_h \) is minimum and

\[
|D_h| = \begin{cases} 
 2 & n = 3 \\
 3 & n = 4 \\
 n - 1 & n \geq 5 
\end{cases}, \text{ so that } \gamma_h(D_h\{P_n, \{2\}\}) = \begin{cases} 
 2 & n = 3 \\
 3 & n = 4 \\
 n - 1 & n \geq 5 
\end{cases}
\]

Hence the proof. \( \square \)

Theorem 2.11. if \( n \geq 4 \), then \( \gamma_h(D_h\{C_n, \{2\}\}) = \begin{cases} 
 \frac{2n}{3}, & n \equiv 0(\text{mod}3) \\
 \frac{2(n-1)}{3} + 2, & n \equiv 1(\text{mod}3) \\
 \frac{2(n-2)}{3} + 2, & n \equiv 2(\text{mod}3) 
\end{cases} \)

Proof. Let \( V(C_n) = \{v_i/1 \leq i \leq n\} \) and \( V(C'_n) = \{v'_i/1 \leq i \leq n\} \). Let \( E(C_n) = \{e_i/1 \leq i \leq n\} \) and \( E(C'_n) = \{e'_i/1 \leq i \leq n\} \), where \( e_i = (v_i,v_{i+1}) \) and \( e'_i = (v'_i,v'_{i+1}) \) for \( i = 1,2,.....n \), where computation is under modulo \( n \).

Let \( G = (D_{sd}\{C_n, \{2\}\}) \).

Let \( n \geq 4 \). Then for
case(i): $n = 3a + 1$, $a = 1, 2, 3 \ldots$, we consider the set $D_h = \{v_{3j-2}\} \cup \{v'_{3k}\} \cup \{v_{n}\}$, $1 \leq j \leq \lfloor \frac{n}{3} \rfloor$, $1 \leq k \leq \lfloor \frac{n}{3} \rfloor$

case(ii): $n = 3b + 2$, $b = 1, 2, 3 \ldots$, we consider the set $D_h = \{v_{3j-2}\} \cup \{v'_{3k-2}\}$, $1 \leq j, k \leq \lfloor \frac{n}{3} \rfloor$

and for

case(iii): $n = 3c + 3$, $c = 1, 2, 3 \ldots$, we consider the set $D_h = \{v_{3j-2}\} \cup \{v'_{3k-2}\}$, $1 \leq j, k \leq \lfloor \frac{n}{3} \rfloor$

Let $D_h$ is not hinge set of $G$, there exists a vertex $v \in D_h$, then $D_{h_1} = D_h - v$ is set of $G$, also $<V - D_h>$ is disconnected. This implies that $D_{h_1}$ is a hinge set of $G$. This contradicts condition (iii).

Therefore, $D_h$ is minimal and

$$|D_h| = \begin{cases} \frac{2n}{3}, & n \equiv 0(\text{mod}3) \\ \frac{2(n-1)}{3} + 2, & n \equiv 1(\text{mod}3) \\ \frac{2(n-2)}{3} + 2, & n \equiv 2(\text{mod}3) \end{cases}$$

so that $\gamma_h(D_h\{C_n, \{2\}\}) = \begin{cases} \frac{2n}{3}, & n \equiv 0(\text{mod}3) \\ \frac{2(n-1)}{3} + 2, & n \equiv 1(\text{mod}3) \\ \frac{2(n-2)}{3} + 2, & n \equiv 2(\text{mod}3) \end{cases}$

Hence the proof. \qed

\textbf{Theorem 2.12.} If $n \geq 4$, then $\gamma_h(D_h\{P_n, \{3\}\}) = \begin{cases} 4, & n = 4, 5 \\ \frac{2n}{3}, & n \equiv 0(\text{mod}3) \\ \frac{2(n-1)}{3} + 1, & n \equiv 1(\text{mod}3) \\ \frac{2(n-2)}{3} + 2, & n \equiv 2(\text{mod}3) \end{cases}$

\textbf{Proof.} Let $G = (D_{sd}\{P_n, \{3\}\})$.

If $n = 4, 5$, the set $D_h = \{v_1, v_4, v'_1, v'_4\}$ is minimal so that $\gamma_h(G) = 4$.

Let $n \geq 6$. Then for

case(i): $n = 3a + 1$, $a = 1, 2, 3 \ldots$, we consider $D_h = \{v_{3j-2}\} \cup \{v'_{3k}\} \cup \{v_{n}\}$, $1 \leq j \leq \lfloor \frac{n}{3} \rfloor$, $1 \leq k \leq \lfloor \frac{n}{3} \rfloor$

case(ii): $n = 3b + 2$, $b = 1, 2, 3 \ldots$, we consider $D_h = \{v_{3j}\} \cup \{v'_{3k-2}\}$, $1 \leq j \leq \lfloor \frac{n}{3} \rfloor$, $1 \leq k \leq \lfloor \frac{n}{3} \rfloor$

and for

case(iii): $n = 3c + 3$, $c = 1, 2, 3 \ldots$, we consider $D_h = \{v_{3j}\} \cup \{v'_{3k-2}\}$, $1 \leq j, k \leq \frac{n}{3}$
Let $D_h$ is not hinge dset of $G$, there exists a vertex $v \in D_h$, then $D_{h_1} = D_h - v$ is dset of $G$, also $< V - D_h >$ is disconnected. This implies that $D_{h_1}$ is a hinge dset of $G$, This contradicts condition (iii).

Therefore, $D_h$ is minimal and

$$|D_h| = \begin{cases} 
4, & n = 4, 5 \\
\frac{2n}{3}, & n \equiv 0(\text{mod}3) \\
\frac{2(n-1)}{3} + 1, & n \equiv 1(\text{mod}3) \\
\frac{2(n-2)}{3} + 2, & n \equiv 2(\text{mod}3)
\end{cases}$$

so that $\gamma_h(D_h\{P_n, \{3\}\}) = \begin{cases} 
4, & n = 4, 5 \\
\frac{2n}{3}, & n \equiv 0(\text{mod}3) \\
\frac{2(n-1)}{3} + 1, & n \equiv 1(\text{mod}3) \\
\frac{2(n-2)}{3} + 2, & n \equiv 2(\text{mod}3)
\end{cases}$

Hence the proof $\square$

Theorem 2.13. If $n \geq 6$, then $\gamma_h(D_h\{C_n, \{3\}\}) = \begin{cases} 
4, & n = 6 \\
\frac{2n}{3}, & n \equiv 0(\text{mod}3) \\
\frac{2(n-1)}{3} + 2, & n \equiv 1(\text{mod}3) \\
\frac{2(n-2)}{3} + 2, & n \equiv 2(\text{mod}3)
\end{cases}$

Proof: Let $G = (D_{sd}\{C_n, \{3\}\})$.

If $n = 6$, $D_h = \{v_1, v_4, v'_1, v'_4\}$ is minimal so that $\gamma_h(G) = 4$.

Let $n \geq 7$. Then for

**case(i):** $n = 3a + 4$, $a = 1, 2, 3 \ldots$, we consider $D_h = \{v_{3j-2}\} \cup \{v'_{3k}\} \cup \{v'_n\}$, $1 \leq j \leq \left\lceil \frac{n}{3} \right\rceil$, $1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$

**case(ii):** $n = 3b + 5$, $b = 1, 2, 3 \ldots$, we consider $D_h = \{v_{3j-2}\} \cup \{v'_{3k-2}\}$, $1 \leq j, k \leq \left\lfloor \frac{n}{3} \right\rfloor$

and for

**case(iii):** Let $n = 3c + 6$, $c = 1, 2, 3 \ldots$, we consider $D_h = \{v_{3j-2}\} \cup \{v'_{3k-2}\}$, $1 \leq j, k \leq \frac{n}{3}$

Let $D_h$ is not hinge dset of $G$, there exists a vertex $v \in D_h$, then $D_{h_1} = D_h - v$ is dset of $G$, also $< V - D_h >$ is disconnected. This implies that $D_{h_1}$ is a hinge dset of $G$, This contradicts condition (iii).
Therefore, $D_h$ is minimal and

$$|D_h| = \begin{cases} 4, & n = 6 \\ \frac{2n}{3}, & n \equiv 0 \,(\text{mod} \, 3) \\ \frac{2(n-1)}{3} + 2, & n \equiv 1 \,(\text{mod} \, 3) \\ \frac{2(n-2)}{3} + 2, & n \equiv 2 \,(\text{mod} \, 3) \end{cases},$$

so that $\gamma_h(D_h(C_n, \{3\})) = \begin{cases} 4, & n = 6 \\ \frac{2n}{3}, & n \equiv 0 \,(\text{mod} \, 3) \\ \frac{2(n-1)}{3} + 2, & n \equiv 1 \,(\text{mod} \, 3) \\ \frac{2(n-2)}{3} + 2, & n \equiv 2 \,(\text{mod} \, 3) \end{cases}$

□

3. Conclusion

In this paper, the hinge domination number of some standard graphs and shadow distance graphs related to the path and cycle graphs is determined. The hinge domination number related to the cycle $C_n$ which was provided in [1] is corrected and, a more generalized result for the hinge domination number of the path $P_n$ is provided.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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