RANDOM WALKS AMONG TIME INCREASING CONDUCTANCES: HEAT KERNEL ESTIMATES

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ABSTRACT. For any graph having a suitable uniform Poincaré inequality and volume growth regularity, we establish two-sided Gaussian transition density estimates and parabolic Harnack inequality, for constant speed continuous time random walks evolving via time varying, uniformly elliptic conductances, provided the vertex conductances (i.e. reversing measures), increase in time. Such transition density upper bounds apply for discrete time uniformly lazy walks, with the matching lower bounds holding once the parabolic Harnack inequality is proved.

1. Introduction

One of the most studied models for random walks in disordered media is the (random) conductance model, based on a locally finite, connected graph $G = (V, E)$ equipped with a (random) collection of symmetric strictly positive, edge conductances $\Pi := \{\pi(x, y) > 0 : (x, y) \in E\}$ (for example, see [MB, Kum] and references therein). We consider here random walks among non-random but time-varying, edge-wise Borel measurable, conductances $\Pi_t$. In particular, taking $\Pi_t = \Pi_{[t]}$, the discrete time (simple) random walk (dtrw) $\{X_n\}$ on the corresponding sequence of weighted graphs $G_n = (G, \Pi_n)$ is constructed as a time-in-homogeneous $V$-valued Markov chain, having for $t \in \mathbb{N}$ the transition probabilities

$$K_t(x, y) := \frac{\pi_t(x, y)}{\pi_t(x)}, \quad (x, y) \in E, \ t \geq 0,$$

each of which is reversible with respect to the vertex conductances

$$\pi_t(x) := \sum_{\{y : (x, y) \in E\}} \pi_t(x, y).$$

Any such $\pi_t(\cdot)$ is a $\sigma$-finite measure on $V$ and we denote by $\mathcal{M}_+(V)$ those $\sigma$-finite measures which are bounded away from zero, namely functions $\mu : V \mapsto (0, \infty)$ with $\mu := \inf \mu(x) > 0$.

More generally, starting at $K_{n, n} = I$, any transition probabilities $\{K_t\}$ on $V$, induce the time-inhomogeneous transitions

$$K_{k-1, n}(x, z) := \mathbb{P}(X_n = z | X_{k-1} = x) = \sum_{y \in V} K_k(x, y) K_{k, n}(y, z), \ k \leq n.$$

Similarly, for $\{K_t\}$ of (1.1), the constant speed random walk (csrw), is the $V$-valued stochastic process of rcll sample path $t \mapsto Y_t$ that at the arrival times $\{T_n\}$ of an auxiliary unit rate Poisson
process, jumps to \( y \neq x \) according to
\[
\mathbb{P}(Y_{T_n} = y | Y_{T_n} = x) = K_{T_n}(x, y), \quad (x, y) \in E, \ n \in \mathbb{N}.
\] (1.4)

The CSRW is thus a time-inhomogeneous Markov process whose transition probabilities \( K_{s,t}(x, z) = \mathbb{P}(Y_t = z | Y_s = x) \) satisfy for \( u(t, x) = 1_{\{z=x\}} \), the backward equation
\[
u(s, x) := \mathbb{E}[u(t, Y_t) | Y_s = x] = e^{-(t-s)} u(t, x) + \int_s^t e^{-(\xi-s)} d\xi \sum_{y \in V} K_{\xi}(x, y) u(\xi, y).
\]

Any conductance model, having \( \Pi \) independent of \( t \), is a reversible, time-homogeneous network, of reversing measure \( \{\pi(x) : x \in V\} \). Time varying \( \Pi_t \) for which \( \pi_t(x) = \pi(x) \) and independent of \( t \), retain this reversibility (even though they form time-inhomogeneous transitions \( K_{s,t} \)). In contrast, as soon as \( \{\pi_t(x) : x \in V\} \) changes with time \( (t) \), the dynamics associated with \( (1.3) \) or with \( (1.5) \) become genuinely non-reversible. Nevertheless, it has been suggested in [ABGK] Conj. 7.1] that some universality applies for the recurrence versus transience of such dynamics. Specifically, [ABGK] Conj. 7.1] conjectures that if both conductance models corresponding to \( G_0 \) and \( G_\infty \) are recurrent, or alternatively, both \( G_0 \) and \( G_\infty \) are transient, then the same holds for the dynamic of non-decreasing \( \{n \mapsto G_n\} \), namely the dTRW evolving according to \( (1.3) \). Indeed, using flows to construct suitable sub or super-harmonic functions, such universality is established in [ABGK] Sec. 5] when \( G = \mathbb{T} \) is a tree, even allowing for conductances \( \Pi_n \) adapted to the path \( \{X_k, k \leq n\} \). In contrast, [ABGK] Sec. 6] shows that such universality fails for randomly adapted, increasing in time, conductances on \( G = \mathbb{Z}^2 \), whereas [ABGK] Ex. 3.5 and 3.6] demonstrates such failure in the non-adapted and non-monotone setting (even on the trivial tree \( G = \mathbb{Z} \)).

The intuition behind [ABGK] Conj. 7.1] owes to the equivalence between conductance models and electrical networks, yielding key comparisons such as Rayleigh monotonicity principle (due to which the random walk on any sub-graph \( G' \) of a recurrent \( G \) must also be recurrent). Lacking such comparisons for the time-varying conductances of \( (1.4) \), we instead seek alternative analytic tools, such as, establishing the relevant (two-sided) Gaussian transition density estimate under certain geometric assumptions on the underlying graphs. That is, to show that for some \( C \) finite, suitable \( \mu_{s,t}(\cdot) \), all \( x, y \in V \) and \( t - s \geq d(x, y) \),

\[
\begin{align*}
\text{(GHKU)} \quad K_{s,t}(x, y) &\leq \frac{C \mu_{s,t}(y)}{\pi_s(\mathbb{B}(y, \sqrt{t-s}))} \exp \left( - \frac{d(x, y)^2}{C(t-s)} \right), \\
\text{(GHKL)} \quad K_{s,t}(x, y) &\geq \frac{C^{-1} \mu_{s,t}(y)}{\pi_s(\mathbb{B}(y, \sqrt{t-s}))} \exp \left( - \frac{Cd(x, y)^2}{t-s} \right),
\end{align*}
\]

where \( \mathbb{B}(y, r) := \{ z \in V : d(y, z) \leq r \} \) denotes the corresponding \( G \)-ball.

We say that \( \{G_t\} \) satisfies the uniform volume doubling (VD) condition, if
\[
\sup_{t, r \geq 0} \sup_{x \in V} \left\{ \frac{\pi_t(\mathbb{B}(x, 2r))}{\pi_t(\mathbb{B}(x, r))} \right\} \leq C_D < \infty,
\]

and it is easy to see that for such \( \{G_t\} \) one essentially must take
\[
\mu_{s,t}(y) := (\pi_s K_{s,t})(y) = \sum_{x \in V} \pi_s(x) K_{s,t}(x, y) \quad t \geq s \geq 0,
\]

if hoping to get both \( (1.6) \) and \( (1.7) \) to hold for large \( t - s \). If \( \pi_t(\cdot) = \pi_0(\cdot) \) is independent of \( t \), then also \( \mu_{s,t}(y) = \pi_0(y) \) for all \( t \geq s \) (see Remark [1.7], and our Gaussian transition density estimates take the usual form of the time-homogeneous setting. More generally, the same applies whenever
\({\mu_{s,t}}\) of (1.9) are \(c\)-stable with respect to the strictly positive \(\sigma\)-finite measures \(\pi_s\) on \(V\), as in [SZ2] Defn 1.10. That is, whenever for some \(c\) finite
\[
e^{-1} \leq \frac{\mu_{s,t}(y)}{\pi_s(y)} \leq c, \quad \forall t \geq s \geq 0, \quad y \in V.
\] Subject to such \(c\)-stability, considering (1.6)–(1.7) for \(y = x\) yields that
\[
\int_s^\infty K_{s,t}(x,x)dt < \infty \iff \int_s^\infty \frac{dt}{\pi_s(B(x,\sqrt{t}))} < \infty,
\]
with the RHS of (1.11) implying transience for either CSRW or DTRW \(\{t \mapsto G_t\}\). If in addition \(\sup_s h_s > 0\) implies \(\inf_s h_s > 0\) for \(h_s := \mathbb{P}(X_t \neq x, \forall t > s|X_s = x)\), it thereby answers the transience versus recurrence question raised in [ABGK].

Aiming at a-priori Hölder continuity for solutions of the heat equation
\[
\partial_t u(t,x) = \mathcal{L}u(t,x), \quad t \geq 0, x \in \mathbb{M} \subseteq \mathbb{R}^d,
\]
on a Riemannian manifold \(\mathbb{M}\), with a divergence form operator
\[
\mathcal{L}u := \sum_{i,j=1}^d \partial_{x_i} \left( a_{ij}(t,x) \partial_{x_j} u \right)
\]
having symmetric, measurable, uniformly elliptic matrix of coefficients \(\{a_{ij}(\cdot)\}\), the study of heat kernel asymptotics for the corresponding diffusion on \(\mathbb{M}\) goes back at least to works of De Giorgi, Nash, Moser in mid-century. In particular, the characterization of two-sided Gaussian Heat Kernel Estimates (GHKE), for the solutions of (1.12) in terms of Poincaré Inequality (PI), plus the volume doubling (VD) property, and their equivalence to the Parabolic Harnack Inequality (PHI), are established independently by [Gr] [SC]. Such results have later been derived in [St1] [St2] for time-dependent, strongly local Dirichlet forms on metric measure spaces (subject to the existence of a time-invariant Radon measure in the underlying topological space). However, strongly local Dirichlet forms as in [St1] [St2] can have no jumps (nor killing). In particular, this assumption excludes the uniformly elliptic (and lazy) random walks on a (static) graph \(G_0\), for which such equivalence between GHKE, PHI and PI+VD is proved in [Del]. See also [BC], which proves a similar equivalence for CSRW on non-elliptic (static) graph \(G_0\), when the GHKE, PHI and PI+VD are suitably restricted (to large balls). One direction we pursue here, is to extend this graph part of the theory, by obtaining the GHKE (with \(\mu_{s,t}\) replaced by \(\pi_t\)), for both dynamics of (1.3) and (1.5), allowing for genuinely time-varying, non-decreasing \(\{\pi_t(x) : x \in V\}\). In a related context, the two-sided Gaussian heat kernel estimates are already provided in [DD] [GOS] for continuous-time symmetric rate random walks on \(\mathbb{Z}^d\) having time-dependent, uniformly elliptic jump rates \(c_i(x,y)\) (i.e. the so-called variable speed random walk VSRW; c.f. [MO] for the same in certain degenerate cases lacking uniform ellipticity). Indeed, the treatment of time-varying VSRW is much simpler than both DTRW and CSRW since any VSRW has the time-invariant reversing measure \(\{\pi_t(x) = 1 : x \in V, t \geq 0\}\). Similar reversible situation applies in [CGZ] where two-sided Gaussian heat kernel bounds are stated (without a detailed proof), for the DTRW of (1.3), provided \(\{\pi_n(x) = \pi_0(x)\}\) is constant in time and a uniform Sobolev inequality holds.

Evolving sets are used in [DHMP] for deriving the heat kernel on-diagonal upper bound (i.e. GHKE for \(x = y\)), from an \(L^1\)-isoperimetry property of \(\{G_n\}\) and uniformly lazy DTRW of non-decreasing \(n \mapsto \pi_n(x)\). This is already a delicate issue, for if \(n \mapsto \pi_n(x)\) are allowed to oscillate, then anomalous behavior may occur (cf. [ABGK] [HK] [SZ3]). General time-inhomogeneous transitions \(\{K_{k,n}(x,y)\}\) that satisfy (1.3) for some finite state space \(V\), are considered in [SZ1] [SZ2]. Aiming at merging...
Theorem 1.1. Suppose $K_t$ have reversible measures $\pi_t \in \mathcal{M}_+(V)$ with $t \mapsto \pi_t(x)$ non-decreasing for each $x \in V$ and the non-decreasing $s \mapsto N(s)$ is such that
\begin{equation}
N(s) \geq \sup_t \{N_{K_t^2,\pi_t}(s)\}.
\end{equation}
(a) Then, for the dynamics (1.3) and any $s \leq t$ one has the on-diagonal upper bound
\begin{equation}
\sup_{x,y \in V} \left\{ \frac{K_{s,t}(x,y)}{\pi_t(y)} \right\} \leq \psi \left( \frac{t-s}{3} \right),
\end{equation}
where $\psi(t) = 1/F^{-1}(t;c_*,N(\cdot))$ for $c_* = \pi_0$ and
\begin{equation}
F(u;a,N(\cdot)) := \int_a^u \frac{N(s)}{s} \, ds.
\end{equation}
(b) For the dynamics (1.5) replace $N_{K_t^2,\pi_t}(s)$ by $2N_{K_t,\pi_t}(s)$ in (1.15), with the RHS of (1.16) having the expectation over $\frac{1}{3} \text{Poisson}(2(t-s))$ law of the corresponding $\psi(\cdot)$. for such transitions, namely the suitable convergence to zero of $|K_{0,n}(x,y) - K_{0,n}(x',y)|$ as $n \to \infty$, \cite{SZ1} develop in this context analytic tools such as the Nash and log-Sobolev inequalities, where a key assumption of \cite{SZ2} is that the Markov transitions \{\{K_{0,n}\}\} yield $\mu_{0,n}$ as in (1.9) which are $c$-stable with respect to some $n_0 \in \mathcal{M}_+(V)$.

In the time homogeneous setting, Coulhon \cite{C1,C2,C3} systematically derives sharp upper bound on $\|K_{0,n}\|_{1} \to \infty$ out of Nash type inequalities, a technique which \cite{SZ2} then apply for time-inhomogenous finite Markov chains. Adapting this approach, we assume throughout that for some $n_0 \in \mathcal{M}_+(V)$, the reversing measures of \{\{K\}\} form a pointwise non-decreasing sequence $t \mapsto \pi_t(x)$, of positive functions on $V$. Our on-diagonal upper-bound, namely the RHS of (1.0) for $x = y$, is then a special case of the general framework of Section 2 where evolving reference measures such as $\mu_n := \mu_{0,n}$ of (1.3), are used to obtain these upper bounds (for DTRW), from the corresponding Nash profiles. Specifically, recall that for Markov operator $K$ on $V$ and its invariant $\sigma$-finite measure $\pi$,
the map
\begin{equation}
f \mapsto (Kf)(x) := \sum_{y \in V} K(x,y)f(y),
\end{equation}
satisfies $(Kf)^2 \leq Kf^2$ for bounded $f$, thereby extending to non-expanding map on $L^2(\pi)$ having the non-negative definite Dirichlet form
$$E_{K,\pi}(f,g) := \langle f - Kf, g \rangle_{\pi}$$
with $L^2(\pi) \subseteq \text{Dom}(E_{K,\pi})$. The Nash profile of such $(K, \pi)$ is
\begin{equation}
N_{K,\pi}(t) := \sup \left\{ \frac{\|f\|_{L^2(\pi)}^2}{E_{K,\pi}(f,f)} : 0 < \|f\|_{L^1(\pi)} \leq t \|f\|_{L^2(\pi)}^2 < \infty \right\}.
\end{equation}
Since $t \mapsto N_{K,\pi}(t)$ is non-decreasing, setting $N_{K,\pi}(\infty) = \infty$ yields Nash inequality
\begin{equation}
\|f\|_{L^2(\pi)}^2 \leq N_{K,\pi} \left( \frac{\|f\|_{L^1(\pi)}^2 / \|f\|_{L^2(\pi)}^2}{\pi(\text{supp} f)} \right) E_{K,\pi}(f,f),
\end{equation}
for any non-zero $f \in L^2(\pi)$. Further, by Cauchy-Schwarz inequality,
\begin{equation}
\|f\|_{L^1(\pi)}^2 \leq \pi(\text{supp} f) \|f\|_{L^2(\pi)}^2
\end{equation}
so $N_{K,\pi}(t)$ plays the role of $L^2$-isoperimetric profile, where $t$ acts as the volume. The application of Section 2 most relevant here is as follows.
Another application is given in Example 2.9 that strengthens the main result of [DHMP], producing their bound while assuming only that $N_{K^{-2}_m}(s) \leq \kappa_m^{-2}/d$ for some $\kappa_m > 0$.

Due to lack of reversibility for $K_{s,t}$ when $t > s$, the off-diagonal bound and the corresponding lower bound, require specializing to weighted graphs $\{G_t\}$ of uniform volume doubling, such that $K_t$ of (1.1) satisfies the following uniform Poincaré inequality.

**Definition 1.2.** For graph $G$ and $\pi_t : V \rightarrow \mathbb{R}_+$, we say that:

- The uniform volume growth with $v(r)$ doubling applies, if for some $C_V < \infty$,
  \[ C_V^{-1} \leq \frac{\pi_t(B(x,r))}{v(r)} \leq C_V, \quad \forall x \in V, \forall r \geq 0, \tag{1.18} \]
  where $v(\cdot)$ is non-decreasing, $v(2r) \leq C_V v(r)$ and $v(0) = v(1) = 1$. In particular, then $\pi_t(x) \in [C_V^{-1}, C_V]$ for all $t, x$ and the uniform $v\d$ condition holds (with $C_D = C_V^3$ in (1.18)).

- The uniform Poincaré inequality holds for $\pi_t$-reversible $K_t$, if
  \[ \inf_{f_r \in \mathbb{R}} \left\{ \sum_{x \in B(x_0,r)} |f(x) - f_r|^2 \pi_t(x) \right\} \leq C_P r^2 \sum_{x,y \in B(x_0,2r)} (f(x) - f(y))^2 K_t(x,y) \pi_t(x), \tag{1.19} \]
  for some $C_P < \infty$ and all $f : V \rightarrow \mathbb{R}$, $x_0 \in V$, $t, r \geq 0$.

- We call $\{K_t\}$ uniformly lazy and $\{G_t\}$ uniformly elliptic, if respectively,
  \[ \alpha_l := \inf_t \inf_{x \in V} \left\{ K_t(x,x) \right\} > 0, \tag{1.20} \]
  \[ \alpha_e := \inf_t \inf_{(x,y) \in E} \left\{ K_t(x,y) \right\} > 0, \tag{1.21} \]
  where if $\{G_t\}$ is uniformly lazy, as in (1.20), then in particular $(x, x) \in E$ for all $x \in V$. For uniformly elliptic and lazy $\{G_t\}$, set $\bar{\alpha} := \alpha_l \wedge \alpha_e$ (with $\bar{\alpha} = \alpha_e$ when CSRW concerned).

Next, we call $u(\cdot, \cdot) \geq 0$ on a time-space cylinder
\[ Q := Q(t_1, t_2; z, R) = [t_1, t_2] \times B(z, R) \tag{1.22} \]
a (non-negative) solution to the (backward) heat equation, if
\[ \partial_s u(s, x) = \sum_y K_s(x, y) u(s, y) - u(s, x), \quad \forall (s, x) \in Q, \tag{1.23} \]
for some non-negative boundary values (for $u$) outside $Q$. For bounded range $K_s$ we have that (1.23) holds on $B(z, R-r_0)$ even when restricting the sum to $B(z, R)$. Here $r_0 = 1$, so such solution is uniquely specified by $\{u(s, x) : d(z, x) = R \text{ or } s = t_2\}$. For discrete time we take $s \in \mathbb{N}$ and $\partial_s u(s, \cdot) := u(s-1, \cdot) - u(s, \cdot)$, whereas for CSRW we assume WLOG that $s \mapsto u(s, x)$ is absolutely continuous, so $\partial_s u$ exists a.e. and (1.23) interpreted as a distributional identity via integration by parts.

We say that the PHI holds for (1.23), if for any $0 < \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ some $\gamma = \gamma(\theta_i) \in (0, 1)$, any $T \geq (\theta_4 R)^2$ and solution $u$ of (1.23) on time-space cylinder $Q(T - (\theta_4 R)^2, T; z, 8R)$, we have
\[ (\theta_2 t_i - R)^2 \leq \gamma_1 \leq (\theta_2 t_i R)^2, x_1, x_2 \in B(z, R) \implies u(T - \tau_2, x_2) \geq \gamma u(T - \tau_1, x_1), \tag{1.24} \]
further restricting (1.24) in the discrete case to $\tau_2 \geq \tau_1 + d(x_1, x_2)$.

**Remark 1.3.** If $u(\cdot, \cdot)$ satisfies (1.23) then so does $au(\cdot, \cdot) + b$. Considering $b \downarrow 0$ we deduce that it suffices to prove the PHI only for strictly positive solutions.
Recall [Del] that for CSRW on time-invariant conductances, the PHI is equivalent to uniformly elliptic conductances satisfying both the PI and VD. Our next result extends this to time-varying, non-decreasing vertex conductances $t \mapsto \pi_t(\cdot)$.

**Theorem 1.4.** [parabolic Harnack inequality]
Suppose $\{G_t\}$ of non-decreasing $t \mapsto \pi_t(x) \in M_+(V)$ and $C_0 := \sup_{t,x} \left\{ \frac{\pi_t(x)}{\pi_0(x)} \right\}$ finite, is uniformly elliptic, satisfying the uniform volume doubling condition and the uniform Poincaré inequality. Then, the PHI holds for the continuous time heat equation (1.23) and some $\gamma = \gamma(C_P, C_D, \alpha, C_0)$ positive.

The approach to establish PHI from volume doubling and Poincaré inequalities, as in [Gr] and SC (that we adapt for proving Theorem 1.4), relies on taking the time derivative of the logarithm of the heat kernel. Having a discrete-time version of such a step, is a well known open challenge. This difficulty can be circumvented by first deriving the HKE-s and then deducing the PHI from them, see [FS]. Indeed, for time-invariant conductances, as in [Del], one compares the transition probabilities of DTRW to those of the CSRW, thereby obtaining the Gaussian estimates for the DTRW, which in turn yield the PHI. However, such a comparison with the CSRW is not available in our time-varying setting. Alternatively, in [HS2], Hebisch and Saloff-Coste prove the PHI for discrete-time dynamic, directly, from a scale invariant elliptic Harnack inequality and local Sobolev inequalities. Unfortunately, it is unclear what should be the analogous elliptic objects to study in the time-varying setting.

For time-invariant conductances the PHI implies Hölder regularity of (non-negative) solutions of the heat equation (see [Del, Pages 227-228]). This extends to our setting, yielding the Hölder regularity of $(s, x) \mapsto K_{s,t}(x, z)$ under the conditions of Theorem 1.4.

**Proposition 1.5.** The PHI implies existence of $h(\gamma) > 0$ such that for any $z \in V$, $R \geq 1$, $T \geq 4R^2$ and solution $u \geq 0$ of (1.23) on $Q = Q(T - 4R^2, T; z, 8R)$, if $y_j \in B(z, R)$ and $(T - s_j) \in [R^2, 4R^2]$, $j = 1, 2$, then

$$|u(s_2, y_2) - u(s_1, y_1)| \leq (4/R)^h |s_2 - s_1|^{1/2} \vee d(y_1, y_2)^h \sup_Q \{u\}.$$  \hfill (1.25)

As mentioned before, in this context we derive a GHKL for both CSRW and uniformly lazy DTRW, with the matching GHKL as a consequence of the PHI.

**Theorem 1.6.** [two-sided Gaussian HKE]
Consider either CSRW or a uniformly lazy DTRW associated with (1.1), for non-decreasing $t \mapsto \pi_t(x)$ and $\{G_t\}$ of uniform volume growth $v(r)$ and $v_\alpha(r)$ doubling.

(a) Let $I(r) = r^2$ on $[0, 1]$, and for $r > 1$ set $I(r) = r(\log r + 1)$ for the CSRW, while $I(r) = \infty$ for the DTRW. Then, the uniform Poincaré inequality yields that for some finite $C = C(C_P, C_V, \alpha)$,

$$K_{s,t}(x, y) \leq \frac{C}{v(\sqrt{t-s})} \exp \left\{ - \frac{t-s}{C} I(\frac{d(x,y)}{t-s}) \right\}, \quad \forall x, y \in V, \ t \geq s \geq 0. \hfill (1.26)$$

(b) Suppose (1.26) and the PHI hold (so for DTRW, the graphs $\{G_t\}$ are uniformly elliptic and lazy). Then, for $\mu_{s,t}$ of (1.9), some $C_* = C_*(C, \alpha)$ finite and all $t - s \geq d(x,y)$,

$$\frac{C_*^{-1} \mu_{s,t}(y)}{v(\sqrt{t-s})} \exp \left( - \frac{C_* d(x,y)^2}{t-s} \right) \leq K_{s,t}(x, y) \leq \frac{C_* \mu_{s,t}(y)}{v(\sqrt{t-s})} \exp \left( - \frac{d(x,y)^2}{C_* (t-s)} \right). \hfill (1.27)$$

**Remark 1.7.** In case of DTRW, if $t \mapsto \pi_t(x)$ is non-decreasing at each $x \in V$, we find inductively that

$$\mu_t(x) := \mu_0, t(x) \leq \mu_s, t(x) \leq \mu_{s,t}(x) \leq \mu_{t, t}(x) = \pi_t(x), \quad \forall t \geq s \geq s' \geq 0, \ x \in V \hfill (1.28)$$
Recall (1.34) that \( \mu_{s,t} \) of the CSRW is the expected value over \( N \sim \text{Poisson}(t - s) \) and jump times \( s < T'_1 < \cdots < T'_N \leq t \) of the value \( \mu_{0,N}^{(\omega)} \) for a DTRW starting at \( \pi_s \) and using \( \{K_{T_m}\} \) in (1.3).

With \( s' \in (T'_L, T'_{L+1}) \cap (s, t) \) for some \( 0 \leq L \leq N \), clearly \( \mu_{s',t} \) exceeds the expected value of \( \mu_{L,N}^{(\omega)} \) for the same DTRW, so (1.28) applies also for any CSRW with non-decreasing \( t \mapsto \pi_t(\cdot) \). Likewise, for both DTRW and CSRW, if \( t \mapsto \pi_t(\cdot) \) is non-increasing, then so is \( s \mapsto \mu_{s,t}(x) \). In the special case of \( \pi_t(x) = \pi(x) \) independent of \( t \) we have that \( \mu_{s,t} = \pi \) and Theorem 1.8 recovers (under uniform Poincaré inequality and uniform volume growth \( v(r) \) with \( v(r) \) doubling), the Gaussian upper bound for DTRW stated in [CGZ, Sec. 7].

We believe that \( \{\mu_{s,t}\} \) of Theorem 1.6 are all within a uniform constant of \( \pi_0 \). That is,

**Conjecture 1.8.** If \( \{G_t\} \), of non-decreasing \( t \mapsto \pi_t(\cdot) \), is uniformly elliptic, of uniform volume growth \( v(r) \) with \( v(r) \) doubling, and satisfies the uniform Poincaré inequality, then for the corresponding CSRW or uniformly lazy DTRW, \( \inf_{t,x} \{\mu_0(t,x)\} \geq 1/c(C_P, C_V, \bar{\alpha}) > 0 \).

In view of (1.28), upon verifying Conjecture 1.8 the RHS of (1.11) should provide a criterion for transience/recurrence of CSRW in terms of the volume growth of \( G_0 \) (and upon proving the discrete time PHI, the same would apply for uniformly elliptic and lazy DTRW).

**Remark 1.9.** Without monotonicity of \( n \mapsto \pi_n \), even for \( \{\pi_n\} \) that are c-stable wrt the function \( \nu_0(x) \equiv 1 \) on \( \mathbb{G} = \mathbb{Z}^2_{\geq 0} \), the reference \( \mu_n = \mu_{0,n} \) may be non-comparable with \( \pi_n \). For example, fixing \( \eta, \epsilon > 0 \) let \( \pi_n(x, x + 1) = 1 + (-1)^{n+x} \eta \) with \( \pi_n(x, x) = 1 + \epsilon 1_{n+x \text{ odd}} \) when \( x > 0 \) and \( \pi_n(0,0) = \pi_n(2,2) \pi_n(0,1)/2 \) (to assure that \( K_n(0,0) = K_n(2x, 2x) \) for any \( n, x \)). Classifying states into types \( A \) or \( B \) according to \( n + X_n \) being even or odd, respectively, yields an \( \{A,B\} \)-valued homogeneous Markov chain of invariant measure \( [m_A, m_B] = [3(1 + \epsilon), 3 + \epsilon] \). The process \( \{X_n\} \) has drift \( \eta \) at the \( A \mapsto A \) moves with opposite drift at \( B \mapsto B \) moves. Consequently, \( \{X_n\} \) has asymptotic speed \( v = \eta/(3 + 2\epsilon) \) to the right. In particular, for some \( C \) finite and any \( y \in \mathbb{Z}_{\geq 0} \) we have for all \( n \geq Cy/v \), the fast decay

\[
\mu_n(y) = \sum_{x \geq 0} K_{0,n}(x, y) \leq C \sqrt{n} e^{-(nv-y)^2/(4n)}.
\]

In Section 4 we show that the GHKU of Theorem 1.1 and Theorem 1.6(a) apply as soon as \( t \mapsto e^{\alpha t} \pi_t(\cdot) \) is non-decreasing for some non-decreasing \( t \mapsto a_t \) such that

\[
A := \sup_{t \geq 0} \{a_{2t+1} - a_t\} < \infty.
\]  

Further, we get the matching GHKL if (1.29) applies for

\[
a_t = \sup_{t} \sup_{0 \leq s_0 < \cdots < s_t = t} \left\{ \sum_{i=0}^{t-1} \rho_{\pi}(s_i, s_{i+1}) \right\}, \quad \rho_{\pi}(s, s') := \sup_{x \in V} \left| \log \frac{\pi_s(x)}{\pi_{s'}(x)} \right|\]

(considering for DTRW only \( s_i \in \mathbb{N} \)). In particular, for CSRW with absolutely continuous \( s \mapsto \pi_s(\cdot) \) we have in (1.30) absolutely continuous \( a_t \) such that \( \alpha \partial_t a_t = \sup_{x \in V} |\partial_t \log \pi_t(x)| \).

**Proposition 1.10.** Suppose \( \{G_t\} \) uniformly elliptic of uniform volume growth \( v(r) \) with \( v(r) \) doubling, has the uniform Poincaré inequality and some \( \{a_t\} \) satisfies (1.29)–(1.30).

(a) The GHKU holds for either CSRW or uniformly lazy DTRW, without \( \mu_{s,t}(\cdot) \), in (1.6), and with some \( C = C(A, C_P, C_V, \bar{\alpha}) \) finite.

(b) The matching GHKL holds for CSRW, and subject to the discrete time PHI, also for DTRW.
Remark 1.11. Starting at a uniformly elliptic $G_0$ of volume growth $v(r)$ with $v(r)$ doubling that satisfies the Poincaré inequality, Proposition 1.10 yields the matching GEKE for the CSRW on $\pi_t(x, y) = \pi_0(x, y)e^{h_t(x,y)}$, whenever $\limsup_t\{\|h_t\|_\infty\}$ and $\limsup_t\{(t+1)\|\partial_t h_t\|_\infty\}$ are finite. In particular, this setting allows us to have forever oscillating $t \mapsto \pi_t(x)$.

While in Proposition 1.10 we have $a_n = O(\log n)$, we next show that no such Gaussian estimates hold universally when $a_n$ grows as $O(n^{1/2+i})$ for some $i > 0$. It is interesting to find a sharp threshold in the context of Proposition 1.10 and in particular to determine whether $a_n \leq O(n^{1/2})$ suffices for such Gaussian density bounds.

Proposition 1.12. For any $i > 0$, there exist uniformly bounded, uniformly elliptic, and uniformly lazy, time-varying edge-conductances on $\mathbb{Z}$, with

$$\limsup_{n \to \infty} \{n^{-i/2}a_n\} < \infty, \quad (1.31)$$

such that neither (1.6) nor (1.7) hold for the corresponding DTRW $\{X_n\}$.

Section 2 explores a general framework using evolving reference measures for obtaining on-diagonal transition probability upper bounds from Nash profiles of underlying graphs and can be read independently of the rest of the paper. Section 3 adapts to our time-inhomogeneous setting the perturbation-interpolation technique of [HS] Section 2 for deriving off-diagonal upper bounds (from a given on-diagonal upper bound), concluding with the GHKU of Theorem 1.6(a). We establish in Section 4 the PII of Theorem 1.3 and the regularity estimate of Proposition 1.5. Section 5 then complete the derivation of Theorem 1.6 whereas Section 6 deals with the perturbative regime of Propositions 1.10 and 1.12.

2. Nash Inequalities

Given a dynamic (1.3) for Markov kernels $\{K_n\}$ the Nash method relies on finding auxiliary Markov kernels $Q_n$ reversible for some $\nu_n \in \mathcal{M}_+(V)$, having useful Nash profiles $\mathcal{N}_{Q_n,\nu_n}(t)$ as well as the following contraction properties.

Assumption 2.1. For any $f \in L^2(\nu_n)$

$$\|K_n f\|_{L^2(\nu_n)}^2 + \mathcal{E}_{Q_n,\nu_n}(f, f) \leq \|f\|_{L^2(\nu_n)}^2. \quad (2.1)$$

For any $f \in L^1(\nu_n)$

$$\|K_n f\|_{L^1(\nu_n)} \leq \|f\|_{L^1(\nu_n)}. \quad (2.2)$$

In particular $K_n$ must be a bounded operator from $L^p(\nu_n)$ to $L^p(\nu_n-1)$ for $p = 1, 2$.

We proceed to provide two canonical examples in which Assumption 2.1 holds.

Example 2.2. If $\nu \in \mathcal{M}_+(V)$ and Markov kernel $K$ are such that $\mu = (\nu K) \in \mathcal{M}_+(V)$, then $f \mapsto Kf$ of (1.3) extends uniquely to the non-negative, bounded linear map $K_{\mu \to \nu} : L^2(\mu) \to L^2(\nu)$. Its dual $K_{\nu \to \mu}^* : L^2(\nu) \to L^2(\mu)$ then satisfies

$$\langle h, K_{\nu \to \mu}^* g \rangle_\mu = \langle Kh, g \rangle_\nu, \quad \forall g \in L^2(\nu), h \in L^2(\mu), \quad (2.3)$$

with the self-adjointed non-negative operator $Q = K^* K$ such that

$$\langle h, Qf \rangle_\mu = \langle Kh, Kf \rangle_\nu, \quad \forall f, h \in L^2(\mu). \quad (2.4)$$
Taking \( h = \delta_x \) and \( f = \delta_y \) in (2.4), we further see that
\[
Q(x, y) := \frac{1}{\mu(x)} \sum_z \nu(z)K(z, x)K(z, y),
\]
is a \( \mu \)-reversible, Markov transition kernel. Further, for \( f \in L^2(\mu) \) we have from (2.3) that
\[
\mathcal{E}_{Q, \mu}(f, f) = \|f\|^2_{L^2(\mu)} - \|Kf\|^2_{L^2(\mu)}.
\]
Since \( \mu = (\nu K) \) we also have that
\[
\|Kf\|_{L^1(\nu)} = \sum_x \nu(x) \sum_y K(x, y)f(y) \leq \sum_{x,y} \nu(x)K(x, y)|f(y)| = \sum_y (\nu K)(y)|f(y)| = \|f\|_{L^1(\mu)}.
\]
Thus, Assumption 2.1 holds for \( \mu_n := \mu_{0,n} \) of (1.9) and the corresponding \( \mu_n \)-reversible Markov kernels \( Q_n := K_{\mu_{n-1} \rightarrow \mu_n}^* K_n \), provided
\[
\mu_n \in \mathcal{M}_+(V), \quad n = 0, 1, 2, \ldots
\]
Starting at any \( \mu_0 \in \mathcal{M}_+(V) \), one has (2.8) for uniformly lazy walks, where \( \mu_n \geq \hat{\alpha} \mu_{n-1} \) are strictly positive, since
\[
\hat{\alpha} := \inf_{n,y} \sum_x K_n(x, y) \geq \alpha_l
\]
of (1.20), and by induction having per \( n, y \) only finitely many \( x \in V \) for which \( K_n(x, y) > 0 \) guarantees the finiteness of \( \mu_n(y) \).

If \( \{\mu_{0,n}\} \) are \( c \)-stable (see (1.10)), then similarly to the considerations of [SZ2], one may estimate the Nash profile \( N_{Q_n, \mu_n}(t) \) in terms of say \( N_{Q_1, \mu_1}(\cdot) \). However, not withstanding Conjecture 1.8 we have no systematic way towards such \( c \)-stability, without which we have little control on \( N_{Q_n, \mu_n}(t) \).

Example 2.3. If the Markov kernel \( K \) has an invariant measure \( \pi \in \mathcal{M}_+(V) \) then considering Example 2.2 for \( \nu = \pi \) results with \( \mu = (\nu K) = \pi \). Suppose now that \( K_n \) have invariant measures \( \pi_n \in \mathcal{M}_+(V) \) such that \( n \mapsto \pi_n(x) \) are non-decreasing. Then \( n \mapsto \|Kf\|_{L^p(\pi_n)} \) are non-decreasing for \( p = 1, 2 \), hence from (2.4)-(2.7) we deduce that Assumption 2.1 holds for \( \nu_n = \pi_n \). If further \( K_n \) is \( \pi_n \)-reversible, as in Theorem 1.1 then from (2.5) we see that \( Q_n = K_n^* \).

In view of Example 2.3 part (a) of Theorem 1.1 is a special case of our next proposition dealing with the more general setting of Assumption 2.1.

**Proposition 2.4.** Suppose in addition to Assumption 2.1 that for non-decreasing \( s \mapsto N(s) \)
\[
N(s) \geq \sup_k \{-N_{Q_k, \nu_k}(s)\}, \quad \inf_k \{\nu_k\} \geq c_* > 0.
\]
Then, for \( \psi(t) = 1/F^{-1}(t; c_*, N(\cdot)) \) of (1.17), the bound (1.16) holds with \( \nu_n \) instead of \( \pi_n \).

Turning to the proof of Proposition 2.4, note that fixing hereafter \( \{\nu_n\} \subseteq \mathcal{M}_+(V) \), part of Assumption 2.1 is having bounded operators \( K_n : L^2(\nu_n) \rightarrow L^2(\nu_{n-1}) \), and hence the dual (adjoint) non-negative operators
\[
K_n^* : L^2(\nu_{n-1}) \rightarrow L^2(\nu_n)
\]
defined as in \((K_n)^* \) of (2.3). Clearly for any \( 0 \leq m < n \),
\[
K_{m,n}^* := K_n^* K_{n-1}^* \cdots K_{m+1}^*,
\]
is the adjoint of \( K_{m,n} : L^2(\nu_m) \rightarrow L^2(\nu_n) \) of (1.3), about which we have the following bound.
Lemma 2.5. Under Assumption 2.1 if $N_k(t) \geq N_{Q_n, \nu_n}(t)$ is non-decreasing in $t$, then for any $0 \leq m \leq n,$
\[ \psi_n(n-m)^{1/2} \geq \|K_{m,n}\|_{L^1(\nu_n) \to L^2(\nu_n)} = \|K_{m,n}^*\|_{L^2(\nu_n) \to L^\infty(\nu_n)}, \tag{2.9} \]
provided $\psi_n(0) \geq 1/\nu_n$ and for $F(\cdot)$ of (1.17),
\[ \frac{1}{\psi_n(j+1)} = F^{-1}\left(1; \frac{1}{\psi_n(j)}, N_{n-j}(\cdot)\right), \quad j = 0, \ldots, n-1. \tag{2.10} \]

Proof. Fix $g : V \to \mathbb{R}$ such that $\|g\|_{L^1(\nu_n)} = 1$ and let
\[ J_n(j) := \|K_{m,n}g\|^2_{L^2(\nu_n)} \quad \text{with} \quad j = n-m. \]
Since $K_{n,n} = I$ we have that
\[ J_n(0) = \|g\|^2_{L^2(\nu_n)} \leq \frac{1}{\nu_n} \sum_j [\nu_n(x)|g(x)|]^2 \leq \frac{1}{\nu_n} \leq \psi_n(0), \tag{2.11} \]
is finite (since $\nu_n \in M_+(V)$). In particular, here $L^1(\nu_n) \subseteq L^2(\nu_n) \subseteq L^\infty(\nu_n)$ and the identity on the RHS of (2.10) follows by duality between $L^1(\nu_n)$ and $L^\infty(\nu_n)$. Turning to prove inductively for $j = 1, 2, \ldots$ the inequality on its LHS, recall that $K_{m-1,n} = K_{m}K_{m,n}$ hence considering (2.11) for $f_m := K_{m,n}g$ in $L^2(\nu_m)$ and the definition (1.11) of $N_{Q_n, \nu_m} \leq N_m$ we find that
\[ J_n(j) - J_n(j+1) = \|f_m\|^2_{L^2(\nu_m)} - \|K_m f_m\|^2_{L^2(\nu_m-1)} \geq \mathcal{E}_{Q_m, \nu_m}(f_m, f_m) \geq \frac{\|f_m\|^2_{L^2(\nu_m)}}{N_m(t)}, \]
provided $t$ is such that
\[ \|f_m\|^2_{L^1(\nu_m)} \leq t\|f_m\|^2_{L^2(\nu_m)}. \]
Next, iterative consideration of (2.2) down from $n$ to $m$ yields that
\[ \|f_m\|_{L^1(\nu_m)} \leq \|g\|_{L^1(\nu_n)} = 1, \]
hence by the definition of $J_n(j)$ we can use $t = 1/J_n(j)$ to deduce that
\[ J_n(j) - J_n(j+1) \geq H_{n-j}(J_n(j)), \quad j = 0, \ldots, n-1, \]
where $H_m(u) := u/N_m(1/u)$. Since $t \mapsto N_m(t)$ is non-decreasing, so is the positive $u \mapsto H_m(u)$ and consequently the piece-wise linear interpolation of $J_n(\cdot)$ to $[0, n]$ satisfies
\[ -\frac{d}{du}\left\{J_n(u)\right\} \geq H_{n-j}(J_n(u)), \quad \forall u \in (j, j+1). \]
It is easy to verify that
\[ -\frac{d}{du}\left\{\psi_n(u)\right\} = H_{n-j}(\psi_n(u)), \quad \forall u \in (j, j+1) \]
for the continuous $\psi_n(\cdot)$ of (2.10), which by (2.11) starts at $\psi_n(0) \geq J_n(0)$. Thus, the continuous $\psi_n(u) - J_n(u)$ is non-negative on $[0, n]$ and in particular $J_n(j) \leq \psi_n(j)$ for all $0 \leq j \leq n$. This holds whenever $\|g\|_{L^1(\nu_n)} = 1$, so $\psi_n(j)$ control the relevant operator norms. \hfill $\square$

With $\nu_n > 0$, the stated bound of Proposition 2.4 is equivalent to the operator norm bound
\[ \psi\left(\frac{n-m}{3}\right) \geq \|K_{m,n}\|_{L^1(\nu_n) \to L^\infty(\nu_m)} = \|K_{m,n}^*\|_{L^1(\nu_m) \to L^\infty(\nu_n)} \tag{2.12} \]
To handle the latter norm, we next adapt the argument of [SZ2] proof of Theorem 2.3].
Lemma 2.6. In the setting of Lemma 2.5 if $A_n \geq 0$ are such that for $n \in [m,N]$  
\[ A_n^2 \leq \sup_{m \leq \ell \leq n} \left\{ \frac{A_\ell}{\psi_n(n-\ell)} \right\}, \]  
(2.12)  
then
\[ M_N := \sup_{n \in [m,N]} \left\{ A_n \|K_{m,n}\|_{L^1(\nu_n) \to L^\infty(\nu_n)} \right\} \leq 1. \]

Proof. Considering (2.9) at some $\ell \in [m,n]$ we have by Lemma 2.5 that
\[ \|K_{m,\ell}^*\|_{L^1(\nu_n) \to L^\infty(\nu_n)} \leq \|K_{m,\ell}\|_{L^1(\nu_n) \to L^2(\nu_n)} \|K_{\ell,n}^*\|_{L^2(\nu_n) \to L^\infty(\nu_n)} \]
\[ \leq \|K_{m,\ell}\|_{L^1(\nu_n) \to L^2(\nu_n)} \psi_n(n-\ell)^{1/2}. \]

Recall that a Markov kernel $K_{m,\ell}$ is a contraction from $L^\infty(\nu_\ell)$ to $L^\infty(\nu_m)$ (as $\nu_\ell$ is strictly positive on $V$). By duality, the adjoint $K_{m,\ell}^*$ is thus a contraction from $L^1(\nu_m)$ to $L^1(\nu_\ell)$. Moreover, for any $\ell$ and $f_\ell$
\[ \|f_\ell\|_{L^2(\nu_\ell)}^2 \leq \|f_\ell\|_{L^\infty(\nu_\ell)} \|f_\ell\|_{L^1(\nu_\ell)} \]
and taking $f_\ell = K_{m,\ell}^* g$ for arbitrary $g \in L^1(\nu_m)$ we get that
\[ \|K_{m,\ell}^*\|_{L^1(\nu_m) \to L^2(\nu_\ell)}^2 \leq \|K_{m,\ell}^*\|_{L^1(\nu_m) \to L^\infty(\nu_\ell)} \|K_{m,\ell}^*\|_{L^1(\nu_m) \to L^1(\nu_\ell)} \leq \|K_{m,\ell}^*\|_{L^1(\nu_m) \to L^\infty(\nu_\ell)}. \]

By the definition of $M_N$, we thus deduce upon choosing the optimal $\ell$ from (2.12), that for any $n \in [m,N]$,
\[ A_n^2 \|K_{m,n}^*\|_{L^1(\mu_m) \to L^\infty(\mu_n)}^2 \leq A_n^2 \|K_{m,\ell}^*\|_{L^1(\mu_m) \to L^\infty(\mu_n)} \psi_n(n-\ell) \]
\[ \leq A_n^2 M_N A_\ell^{-1} \psi_n(n-\ell) \leq M_N. \]

Finally, taking the supremum over $n$ in the l.h.s, we find that $M_N \leq 1$, as claimed. \hfill \Box

Starting with an upper bound $N_\ell(s)$ on the Nash profile functions, one merely applies Lemmas 2.5 and 2.6. That is, first solving the map (2.10) corresponding to $N_\ell(s)$ in order to get an upper bound on $\psi_n(\cdot)$, from which by (2.12) one deduces the diagonal upper bound $1/A_n$. This is much simplified in the presence of uniform bounds, as in Proposition 2.4.

Proof of Proposition 2.4. Considering Lemma 2.5 for $N_\ell(\cdot) = N(\cdot)$ and $\psi_n(0) = 1/c_\ell$, we have the bound (2.9) for the non-increasing $\psi_n(\cdot) = \psi(\cdot)$ of Proposition 2.4. In particular, by its definition via (1.11),
\[ \frac{d}{dt} \log (1/\psi(t)) = N(1/\psi(t)) \]
is non-decreasing. Integrating this over intervals of length $n/6$ we thus deduce that
\[ \frac{\psi(n/6)}{\psi(n/3)} \leq \frac{\psi(n/3)}{\psi(n/2)}, \quad n = 0,1,\ldots \]
Thus, $A_n = 1/\psi(n/3)$ and $\ell = [n/2]$ satisfies (2.12), so Lemma 2.6 completes the proof. \hfill \Box

Coupling the dynamics of (1.3) and (1.5) we deduce part (b) out of part (a) of Theorem 1.1.

Proof of Theorem 1.1(b). From the arrival times $\{T_n^\ell\}$ of an auxiliary Poisson process of rate 2, we construct $\{Y_t\}$ obeying (1.5) by independently censoring the jump at each time $T_n^\ell$ with probability $1/2$ and proceeding at the non-censored $\{T_n^\ell\} \subseteq \{T_n^\ell\}$ as in (1.3). Fixing a realization $\omega = \{T_n^\ell\}$, at most $N_t - N_s$ jump attempts are made in $[s,t]$ by the corresponding dynamics $\{X_n\}$ of (1.3)
having the 1/2-uniformly lazy transitions \( K_n^{(\omega)} := \frac{1}{2}K_{T_n'} + \frac{1}{2}I \). Recall that the Dirichlet form of any \( \pi \)-reversible \( K \) has the symmetric form

\[
\mathcal{E}_{K,\pi}(f, g) = \frac{1}{2} \sum_{x,y \in V} (f(x) - f(y))(g(x) - g(y))\pi(x)K(x, y)
\]  

(2.13)

and under (1.20) we have that \( Q(x, y) \geq \alpha_l K(x, y) \) for \( Q = K^2 \) and all \( x, y \in V \). Thus, in the reversible case \( \mathcal{E}_{K^2,\pi}(f, f) \geq \alpha_l \mathcal{E}_{K,\pi}(f, f) \) for all \( f \) and consequently,

\[
\mathcal{N}_{K,\pi}(s) \geq \alpha_l \mathcal{N}_{K^2,\pi}(s), \quad \forall s.
\]  

(2.14)

This applies for \( K = K_n^{(\omega)} \), \( \alpha_l = 1/2 \) and the non-decreasing \( n \mapsto \pi_{T_n'}(x) \), hence by part (a) of Theorem 1.1, the bound (1.16) holds for \( \{X_n\} \) with \( \psi(.) \) as stated in part (b). That is,

\[
\frac{K_n^{(\omega)}(x, y)}{\pi_t(y)} \leq \frac{K_{N_s,\omega}(x, y)}{\pi_t'(y)} \leq \psi\left(\frac{N_t - N_s}{3}\right).
\]

To complete the proof, note that \( K_{s,t}(x, y) \) is the expected value of \( K_{N_s,\omega}(x, y) \) over \( \omega \), whereas \( N_t - N_s \sim \text{Poisson}(2(t - s)) \).

Recalling that the Poincaré inequality together with uniform volume growth for \( v(r) \) doubling, provide an upper bound on the Nash profile, we deduce from Theorem 1.1 that in the context of Theorem 1.6 the on-diagonal upper bound (1.16) holds for \( \psi(t) = C'/\sqrt{t} \).

Lemma 2.7. Suppose \( G \) and non-decreasing \( t \mapsto \pi_t(x) \) satisfy uniform volume growth \( v(r) \) as in (1.18), for \( v(r) \) doubling. Further, the \( \pi \)-reversible \( K_t \) satisfy the uniform point (1.19), and in case of (1.3) such \( \{K_t\} \) are also \( \alpha_l \)-uniformly lazy. Then, for \( C'(C_P, C_V, \alpha_l) \) finite,

\[
\|K_{s,t}\|_{L^1(\pi_t) \to L^\infty(\pi_s)} = \sup_{x,y \in V} \frac{K_{s,t}(x, y)}{\pi_t(y)} \leq \frac{C'}{v(\sqrt{t - s})} \quad \forall s \in [0, t].
\]  

(2.15)

Proof. In case of the dynamic (1.3), with (2.14) applicable for \( K_t \), by Theorem 1.1 a) it suffices to show that if \( \pi \)-reversible \( K \) satisfies the Poincaré inequality (1.19) and the uniform growth assumption (1.18) for volume doubling \( v(r) \), then for \( v^{-1}(s) := \inf\{r \geq 1 : v(r) \geq s\} \) and some \( C(C_P, C_V) \) finite,

\[
\alpha_l N(s) := C\left(v^{-1}(C)s)^2 \geq \mathcal{N}_{K,\pi}(s), \quad \forall s.
\]  

(2.16)

Indeed, \( v^{-1}(s/C_V^k) \leq 2^{-k}v^{-1}(s) \) so splitting the integral (1.17) for such non-decreasing \( s \mapsto N(s) \), to intervals \([u/C_V^{k+1}, u/C_V^k] \), \( k \geq 0 \) shows that the RHS of (1.17) is dominated by the largest intervals and hence for some \( C' = C'(C/\alpha_l, C_V) \) finite and any \( t \geq 1/3, \)

\[
\psi(t) = \frac{1}{F^{-1}(t; \bar{z}_0, N(\cdot))} \leq \frac{C'}{v(\sqrt{3}t)},
\]

with (2.15) thus a consequence of (1.16). Further, recall Theorem 1.1 b) that for the dynamic (1.3) the preceding bound on \( \psi(t) \) always hold (with \( \alpha_l = 1/2 \)), and in this context we arrive for \( N_\lambda \sim \text{Poisson}(\lambda) \) and the non-decreasing \( v(r) \geq 1 \), at

\[
\|K_{s,t}\|_{L^1(\pi_2) \to L^\infty(\pi_1)} \leq \mathbb{E}\left[\frac{C'}{v(\sqrt{2(t-s)})}\right] \leq \frac{C'}{v(\sqrt{t - s})} + C'e^{-(t-s)/\kappa}
\]  

(2.17)

(as \( \mathbb{P}(N_{2\lambda} \leq \lambda) \leq e^{-\lambda/\kappa} \) for some \( \kappa \) finite and all \( \lambda \geq 0 \)). With \( v(r)e^{r^2/\kappa} \) uniformly bounded, upon increasing \( C' \) we thus get (2.15) out of (2.17).
Turning to establish (2.16), recall that from (2.13) and the covering argument in the proof of [SC, Lemma 2.4], it follows that for some \( c = c(C_p, C_V) \) finite, any \( f \in L^2(\pi) \) and \( r > 0 \)
\[
\| f - f_r \|_{L^2(\pi)}^2 \leq c r^2 \mathcal{E}_{K,\pi}(f, f),
\]
where
\[
f_r(x) := \frac{1}{\pi(B(x, r))} \sum_{y \in B(x, r)} f(y) \pi(y).
\]
Further, by the uniform volume growth assumption of (1.18),
\[
\| f \|_{L^\infty(\pi)} \leq \frac{1}{\inf_x \pi(B(x, r))} \| f \|_{L^1(\pi)} \leq \frac{C_V}{v(r)} \| f \|_{L^1(\pi)},
\]
\[
\| f \|_{L^1(\pi)} \leq \frac{C V}{r^2} \| f \|_{L^1(\pi)}.
\]
Consequently
\[
\| f \|_{L^2(\pi)}^2 \leq 2 \| f \|_{L^2(\pi)}^2 + \| f_r \|_{L^2(\pi)}^2 \leq 2 c r^2 \mathcal{E}_{K,\pi}(f, f) + \frac{2 C^3}{v(r)} \| f \|_{L^1(\pi)}^2.
\] (2.18)
Setting \( C := 2c + 2C^3 \) and \( r = v^{-1}(C) \), it follows from (2.18) that if \( \| f \|_{L^1(\pi)}^2 \leq s \) and \( \| f \|_{L^2(\pi)}^2 = 1 \), then
\[
\frac{1}{\mathcal{E}_{K,\pi}(f, f)} \leq C r^2 = a_l N(s).
\]
To complete the proof, recall that \( N_{K,\pi}(s) \) is the maximum of \( 1/\mathcal{E}_{K,\pi}(f, f) \) over such \( f \).

We next review the connection between the Nash and isoperimetric profiles of a Markov chain \( Q \) having invariant measure \( \pi \). To this end, first recall the \( L^2 \)-isoperimetric (or spectral) profile of such a chain \((Q, \pi)\) on infinite state space \( V \), defined as the non-increasing
\[
\Lambda_{Q,\pi}(u) = \inf\{ \lambda_Q(\Omega) : \Omega \subseteq V, \pi(\Omega) \leq u \}
\] (2.19)
where
\[
\lambda_Q(\Omega) = \inf \left\{ \mathcal{E}_{Q,\pi}(f, f) : \text{support}(f) \subseteq \Omega, \| f \|_{L^2(\pi)} = 1 \right\}.
\] (2.20)
In words, \( \lambda_Q(\Omega) \) is the smallest eigenvalue of the operator \( I - Q \) with Dirichlet boundary condition in \( \Omega \). Also recall the \( L^1 \)-isoperimetric (or conductance) profile
\[
\Phi_{Q,\pi}(u) = \inf\{ \frac{1}{\pi(\Omega)} \sum_{x \in \Omega} \pi(x) Q(x, \Omega^c) : \Omega \subseteq V, \pi(\Omega) \leq u \}.
\]
The \( L^2 \) and \( L^1 \) profiles are related via Cheeger’s inequality (see [LS]),
\[
\frac{1}{2} \Phi_{Q,\pi}^2(u) \leq \Lambda_{Q,\pi}(u) \leq \Phi_{Q,\pi}(u).
\] (2.21)
As shown next, the Nash profile \( N_{Q,\pi}(\cdot) \) contains the same information as the \( L^2 \)-isoperimetric profile (see [GMT, Lemma 2.1] for such a result in case of finite Markov chains; the proof for \( V \) infinite is provided here for the reader’s convenience).

**Lemma 2.8.** For Markov operator \( Q \), its \( \sigma \)-finite invariant measure \( \pi \), and any \( u > 0 \),
\[
\frac{1}{\Lambda_{Q,\pi}(u)} \leq N_{Q,\pi}(u) \leq \frac{2}{\Lambda_{Q,\pi}(4u)}.
\]
Proof. By the Cauchy-Schwarz inequality $\|f\|_{L^1(\pi)}^2 \leq \pi(\Omega)\|f\|_{L^2(\pi)}^2$ for any $f$ supported within $\Omega$, yielding that $\Lambda_{Q,\pi}(u) \geq 1/N_{Q,\pi}(u)$ via the definitions (1.14) and (2.19). We proceed to show that

$$\Lambda_{Q,\pi}(4u) \leq 2/N_{Q,\pi}(u).$$

Namely, that for any $f \in L^2(\pi)$

$$\|f\|_{L^2(\pi)}^2 = u\|f\|_{L^2(\pi)}^2 \implies \frac{1}{2}\|f\|_{L^2(\pi)}^2 \Lambda_{Q,\pi}(4u) \leq \mathcal{E}_{Q,\pi}(f, f). \tag{2.22}$$

Indeed, recall from (2.19) that for $f \in L^2(\pi)$,

$$\|f\|_{L^2(\pi)}^2 \Lambda_{Q,\pi}(\pi(\text{support}(f))) \leq \mathcal{E}_{Q,\pi}(f, f).$$

Next, for $t \geq 0$ set $f_t := (f - t)_+ \text{ supported on } \Omega_t := \{f > t\}$. Obviously $f_t \in L^2(\pi)$, with $f^2 - 2t|f| \leq f_t^2$ and $\mathcal{E}_{Q,\pi}(f_t, f_t) \leq \mathcal{E}_{Q,\pi}(f, f)$. Hence,

$$\left[\|f\|_{L^2(\pi)}^2 - 2t\|f\|_{L^1(\pi)} \right] \Lambda_{Q,\pi}(\pi(\Omega_t)) \leq \|f_t\|_{L^2(\pi)}^2 \Lambda_{Q,\pi}(\pi(\Omega_t)) \leq \mathcal{E}_{Q,\pi}(f_t, f_t) \leq \mathcal{E}_{Q,\pi}(f, f).$$

Since $\pi(\Omega_t) \leq t^{-1}\|f\|_{L^1(\pi)}$, if $4t\|f\|_{L^1(\pi)} = \|f\|_{L^2(\pi)}$ for $t = \|f\|_{L^1(\pi)}/(4u)$ finite, then

$$\frac{1}{2}\|f\|_{L^2(\pi)}^2 \Lambda_{Q,\pi}(4u) \leq \frac{1}{2}\|f\|_{L^2(\pi)}^2 \Lambda_{Q,\pi}(\pi(\Omega_t)) \leq \mathcal{E}_{Q,\pi}(f, f),$$

as claimed in (2.22).  \hfill \Box

By Lemma 2.8, any lower bound on the $L^2$ or $L^1$-isoperimetric profile can be turned into an upper bound on the Nash profile.

**Example 2.9.** Consider Example 2.3 with $\nu_n = \pi_n$ invariant for $K_n$ such that $n \mapsto \pi_n(x)$ are non-decreasing (and $Q_n = K_n^* K_n$). Suppose also that for some $d > 0$ and positive $\kappa_n$

$$\mathcal{N}_{Q_n,\pi_n}(s) \leq 4\alpha_n^{-1}\kappa_n^{-2}(4s)^{2/d}, \quad \forall s, n, \tag{2.23}$$

where $\gamma_n := \sum_{m=1}^n \kappa_n^2$ are such that for some $c_0 \geq 2$,

$$\gamma_n \geq c_0 \implies \exists \ell(n) \quad 1 + \gamma_{\ell(n)} \leq \frac{2}{3}. \tag{2.24}$$

It is easy to check that for some $c = c(\alpha, d, \pi_0)$ finite,

$$\psi_n(j) = c\left(1 + \gamma_n - \gamma_{n-j}\right)^{-d/2},$$

satisfies (2.11) and consequently the bound (2.9). The condition (2.24) allows for taking $A_n = c_1^{-1}(1 + \gamma_n)^{d/2}$ with $c_1 = c(1 + c_0)^d$ in Lemma 2.0, thereby concluding that

$$\sup_{x, y \in V} \left\{\frac{K_{0,n}(x, y)}{\pi_n(y)}\right\} \leq c_1 \left(1 + \sum_{m=1}^n \kappa_m^2\right)^{-d/2}. \tag{2.25}$$

In particular, the bound (2.25) recovers [DHMP Theorem 1.2], proved before for the dynamic (1.3) with $(K_t, \pi_t)$ of (1.11)-(1.12) via evolving sets techniques. More precisely, [DHMP] assume that the uniform lazy property (1.20) holds and $L^1$-isoperimetric profiles such that for some $d > 1$ and positive $\kappa_n$

$$\Phi_{K_n,\pi_n}(s) \geq \kappa_n s^{-1/d}. \tag{2.26}$$

As noted by [DHMP], if $\sup_{x, n}\{\pi_n(x)\} \leq C^d$ then $\kappa_n \leq \left(\frac{\pi_0}{2}\right)^{1/d} \leq C$, so (2.24) holds (for $c_0 = 2 + 3C^2$). Recall the left inequality of (2.16) that for $Q_n = K_n^2$ in the reversible case

$$\mathcal{N}_{Q_n,\pi_n}(s) \leq c_1^{-1}\mathcal{N}_{K_n,\pi_n}(s), \quad \forall s, n.$$
Finally, by Lemma 2.2 and Cheeger’s inequality (2.21), the isoperimetric bound (2.20) implies
\[ N_{K_n}(s) \leq 2\Lambda_n(4s)^{-1} \leq 4\Phi_n(4s)^{-2} \leq 4\kappa_n^{-2}(4s)^{2/d}, \quad \forall s, n. \]
So, the assumptions of [DHMP] imply both (2.23) and (2.24), thereby yielding (2.25).

3. Gaussian Upper Bounds

We adapt the technique of [HS, Section 2] for deriving off-diagonal Gaussian upper bounds via complex interpolation techniques. Specifically, in this section we work with \( L^p \) spaces of \( \mathbb{C} \)-valued functions, with \( C_0(V) \) denoting the dense linear subspace of finitely supported \( \mathbb{C} \)-valued functions. Considering \( \rho : V \to \mathbb{R} \) such that the non-negative linear operators on \( C_0(V) \)
\[ K_{\theta,t}^\rho f(x) := w_{-\theta}K_{s,t}(w_{\theta})f(x), \quad w_{\theta}(x) := e^{\theta\rho(x)}, \quad 0 \leq s \leq t \leq T, \quad \theta \in \mathbb{R}, \quad (3.1) \]
have bounded \( L^2(\nu_\theta) \to L^2(\nu_\theta) \) norms, we study the unique continuous extension of \( K_{\theta,t}^\rho \) for both continuous and discrete time (where \( s, t \in \mathbb{Z}_+ \)). Our main example is \( \rho(x) = d(x, x_0) \) for the graph distance \( d(x, y) = d_G(x, y) \) in a locally finite, connected graph \( G \) and a fixed vertex \( x_0 \in V \).

For completeness we first prove the following proposition, which summarizes the interpolation method of [HS].

**Proposition 3.1.** (see [HS, Lemma 2.2]) Suppose \( \{K_{\theta,t}^\rho\} \) are as in \( (3.1) \), with non-negative linear operators \( K_{\theta,t} = K_{\theta,\xi\xi}K_{\xi,t} \), \( s \leq \xi \leq t \) (with \( K_{\xi\xi} = I \)), such that
\[ \|K_{\theta,t}\|_{L^\infty(\nu_0) \to L^\infty(\nu_0)} \leq 1, \quad \forall 0 \leq s \leq t \leq T, \quad (3.2) \]
for strictly positive \( \sigma \)-finite measures \( \{\nu_\theta\} \), with \( \nu_0 \in \mathcal{M}(V) \). Suppose that for \( t \geq s \geq 0\), \( T \] :
(a) For a non-decreasing \( t \mapsto a_t \) satisfying \( (1.29) \), the Gaffney bound
\[ \left\| \int_0^t K_{\theta,s}^{\rho} d\tau \right\|_{L^2(\nu_\theta) \to L^2(\nu_\theta)} \leq \exp(a_t - a_s + \chi(\theta)(t - s)), \quad \forall \theta \in \mathbb{R}, \quad (3.3) \]
with \( c_1^{-1} \theta^2 \leq \chi(\theta) \leq c_1 \theta^2, \quad \forall |\theta| \leq \delta, \quad (3.4) \)
holds on \( C_0(V) \), for \( \theta \mapsto \chi(\theta) \) non-decreasing, some \( \delta > 0 \) and \( c_1 < \infty \).
(b) The \( 2 \to \infty \) bound
\[ \|K_{\theta,t}\|_{L^2(\nu_\theta) \to L^\infty(\nu_\theta)} \leq \varphi(t - s), \quad (3.5) \]
holds with \( \tau \mapsto \varphi(\tau) \tau^\beta \) non-decreasing on \( [0, T] \) for some \( \beta > 0 \). Then, for some finite \( C_1(\alpha, \beta, \delta) \) and \( C_2 = C_{2}(A, \alpha, \nu_0, \varphi(1)), \)
\[ \left\| K_{\theta,0,T}^\rho \right\|_{L^2(\nu_\theta) \to L^\infty(\nu_0)} \leq C_2 \varphi(T) \exp(C_1 \chi(\theta)T), \quad \forall \theta \in \mathbb{R}, \quad T \geq 1. \quad (3.6) \]

**Remark 3.2.** Clearly, \( (3.2) \) holds for Markov transition probabilities \( \{K_{\theta,t}\} \) and strictly positive \( \{\nu_s\} \).

**Proof.** Since \( (\nu)^1/q \|f\|_{L^\infty(\nu)} \leq \|f\|_{L^q(\nu)} \) for any \( f \) and \( \nu \in \mathcal{M}(V) \), necessarily
\[ \left\| K_{\theta,t}^\rho \right\|_{L^p(\nu_\theta) \to L^\infty(\nu_\theta)} \leq (\nu_0)^{-1/(2q)} \left\| K_{\theta,t}^\rho \right\|_{L^2(\nu_\theta) \to L^2(\nu_\theta)}, \quad \forall p, q, t > 0. \quad (3.7) \]
Considering \( (3.3) \) and \( (3.7) \) at \( s = 0, t = T, p = q = 1 \), our assumption \( (3.4) \) and having \( \varphi(T) \geq \varphi(1)T^{-\beta} \) for \( T \geq 1 \), yield \( (3.6) \) for \( C_1 = 1 + c_1 \kappa \beta/\delta^* \), \( C_2 = (\nu_0)^{-1/2} \varphi(1)^{-1} \) and \( \|\theta\| \geq \delta^*/(1 + \log T) \), where
\[ \kappa := \sup_{t \geq 1} \left\{ \frac{\log t}{t} (1 + \log t)^2 \right\} < \infty. \]
We proceed to derive (3.6) when $|\theta| < \delta_*/(1 + \log T)$ by closely following [SC2, Subsection 4.2.2]. To this end, (3.2) and (3.5) are invariant under the re-scaling $\nu_{s,t} := e^{-2(a_t - a_s)} \nu_s$ and yield by Riesz-Thorin interpolation theorem that
\[
\|K_{s,t}\|_{L^{2p}(\nu_t) \to L^\infty(\nu_{s,t})} \leq \varphi(t - s)^{1/p}, \quad \forall p \geq 1.
\] (3.8)

To apply Stein’s interpolation theorem, consider the $\mathbb{C}$-valued weights $w_{yz}(x) = w_\theta(x)^z$ indexed on the strip $\mathbb{S} := \{z = u + ib : u = \Re(z) \in [0, 1]\}$. For fixed $\theta \in \mathbb{R}$ and $m \leq \ell$ the associated map $z \mapsto K_{s,t}^{\theta z} = w_{-a_z}K_{m,\ell} \nu_{s,t}$ forms an $\mathbb{S}$-analytic collection of linear operators on $C_0(V)$ such that $K_{m,\ell}^{\theta(u+ib)} = w_{-ib}K_{m,\ell}^{\theta u}$. With $|w_{\pm ib}(x)| = 1$ we thus have for $u = 0$ the $L^\infty$-contraction
\[
\|K_{s,t}^{\theta ib}\|_{L^\infty(\nu_t) \to L^\infty(\nu_{s,t})} \leq \|K_{s,t}\|_{L^\infty(\nu_t) \to L^\infty(\nu_{s,t})} \leq 1.
\] (3.9)

Moving to $\nu_{s,t}$ eliminates the term $a_t - a_s$ in (3.3), so we get for $u = 1$ the $L^2$-norm bound
\[
\|K_{s,t}^{\theta(1+ib)}\|_{L^2(\nu_t) \to L^2(\nu_{s,t})} \leq \|K_{s,t}^{\theta}\|_{L^2(\nu_t) \to L^2(\nu_{s,t})} \leq \exp(\chi(\theta)(t - s)).
\] (3.10)

By Stein’s interpolation (see [SW]), from (3.9) and (3.10) we have for $\lambda = 1/p \in [0, 1]$,
\[
\|K_{s,t}^{\theta \lambda}\|_{L^{2p}(\nu_t) \to L^{2p}(\nu_{s,t})} \leq \exp(\lambda(\chi(\theta)(t - s))
\] and upon replacing $\theta$ by $\theta/\lambda$, deduce that for all $\theta \in \mathbb{R}$,
\[
\|K_{s,t}^{\theta(1+ib)}\|_{L^{2p}(\nu_t) \to L^{2p}(\nu_{s,t})} \leq \|K_{s,t}^{\theta}\|_{L^{2p}(\nu_t) \to L^{2p}(\nu_{s,t})} \leq \exp\left(p^{-1}(\chi(\theta)(t - s))\right).
\] (3.11)

Next, employing (3.8) gives that
\[
\|K_{s,t}^{\theta ib}\|_{L^{2p}(\nu_t) \to L^{\infty}(\nu_{s,t})} \leq \|K_{s,t}\|_{L^{2p}(\nu_t) \to L^{\infty}(\nu_{s,t})} \leq \varphi(t - s)^{1/p}.
\] (3.12)

With $\nu_t = \nu_{t,T}$, from (3.11) and (3.12) we conclude by yet another application of Stein’s interpolation theorem, that for $\lambda = p/q \in [0, 1]$ and any $q \geq p \geq 1$,
\[
\|K_{s,t}^{\theta \lambda}\|_{L^{2p}(\nu_{t,T}) \to L^{2q}(\nu_{s,T})} \leq \varphi(t - s)^{\frac{1}{p} - \frac{1}{q}} \exp\left(\lambda p^{-1}(\chi(\theta)(t - s))\right).
\]

Considering $\hat{\nu}_t = \nu_{t,T}$ and replacing once more $\theta$ by $\theta/\lambda$, we get that
\[
\|K_{s,t}^{\theta \lambda}\|_{L^{2p}(\hat{\nu}_t) \to L^{2q}(\hat{\nu}_s)} \leq \left(\exp(\chi(\theta)(t - s))\right)^{\frac{1}{p} - \frac{1}{q}} \exp\left(q^{-1}(\chi(\theta)(t - s))\right).
\] (3.13)

Next, proceeding similarly to the proof of [HS, Lemma 2.2], set $\eta_j = c_0 j^{-2}$ such that $\sum_{j \geq 2} \eta_j = 1$ and partition $T$ into (non-increasing) blocks $\ell_j = \lfloor \eta_j T \rfloor \geq 1$ for $2 \leq j \leq m \leq \sqrt{c_0 T}$ and $\ell_1 = T - \sum_{j=2}^m \ell_j$. We further set the corresponding strictly decreasing
\[
t_j = \sum_{k=j+1}^m \ell_k, \quad 0 \leq j \leq m.
\] (3.14)
With $\tilde{\nu}_0 = e^{-2at} \nu_0$, in view of (3.17) we get for any non-decreasing $q_j \geq q_0 = 1$,
\[
\|K_{0,T}^\theta\|_{L^2(\nu_T) \to L^\infty(\nu_0)} \leq \kappa_0 e^{a_T/q_m} \prod_{j=1}^m \|K_{t_j,t_j-1}^\theta\|_{L^{2q_j} (\tilde{\nu}_{t_j-1}) \to L^{2q_j} (\tilde{\nu}_{t_j})} \quad \forall \theta \in \mathbb{R},
\] (3.15)
where $\kappa_0 := (1 \land \nu_0)^{-1}$ is finite. Further, from (1.29) we have that for $A_o = A / \log 2$,
\[
a_T - a_t \leq A + A_o \log \left( \frac{T + 1}{t + 1} \right) \leq A + A_o \log (T/t), \quad \forall t \in [0, T].
\] (3.16)
Recall (3.14) that $t_{j-1} \geq \ell_j$, hence by our interpolation bound (3.13) and (3.16),
\[
\|K_{t_j,t_j-1}^\theta\|_{L^{2q_j} (\tilde{\nu}_{t_j-1}) \to L^{2q_j} (\tilde{\nu}_{t_j})} \leq \left( e^{A \log \left( \frac{T}{\ell_j} \right)^{A_o} \varphi(\ell_j)} \right)^{(1/q_j - 1/q_j)} \exp \left( q_j^{-1} \kappa \varphi(\theta \ell_j) \right).
\]
Upon plugging these bounds into (3.15), recalling (3.4), that $a_T \leq 2A + A_o \log T$ for $T \geq 1$, and our assumption that $\tau \mapsto \varphi(\tau) \tau^\beta$ is non-decreasing on $[0, T]$, we find that for $|\theta| \leq \delta_*/q_m$ and $\beta_* := \beta + A_o$,
\[
\|K_{0,T}^\theta\|_{L^2(\nu_T) \to L^\infty(\nu_0)} \leq \kappa_0 e^{2A_T \gamma T / q_m} \prod_{j=1}^m \left( \frac{T}{\ell_j} \right)^{A_o \varphi(\ell_j)} \exp \left( c_1 q_j \ell_j \theta^2 \right)
\]
\[\leq \zeta_T \varphi(T) \exp \left( \beta_* \gamma T + c_1^2 b_T \chi(T)T \right)
\]
where
\[
b_T := \sum_{j=1}^m q_j \frac{\ell_j}{T}, \quad \gamma_T := \sum_{j=1}^m \left( \frac{1}{q_j - 1} - \frac{1}{q_j} \right) \log (T/\ell_j), \quad \zeta_T := \kappa_0 e^{2A \varphi(1)^{-1/q_m} T^{\beta_*/q_m}}.
\]
Set $q_j := 1 + (\log j)^2$ and maximal $m(T) \geq 1$ such that $q_m < 1 + \kappa^{-1} \log T$, with $\kappa \geq 2/\log 2$ implying $c_0 T \geq T \geq m^2$. It yields $q_m \geq 1$ such that $\log T/q_m \leq 2\kappa_1$, hence $\zeta_T$ is uniformly bounded. Further, both $\gamma_T$ and $b_T$ are bounded by some universal constant since the series $\sum_j (\log j) / (q_j - 1 - 1/q_j)$ and $\sum_j q_j j^{-2}$ converge. Combined, these facts imply that (3.6) holds for some $C_1(c_1), C_2(A, \beta, \nu_0, \varphi(1))$ finite and all $|\theta| < \delta_*/(1 + \log T)$, as claimed. □

Applying Proposition 3.3. Consider the setting of Proposition 3.1 for Markov transition probabilities $\{K_s,t\}$ such that $\nu_s K_{s,t} \leq \nu_t$ whenever $T \leq s \leq t \leq 2T$, the Gaffney bound (3.3)–(3.4) holds for $\rho(\cdot) = d(\cdot, x)$, and the $1 \to \infty$ bound
\[
\|K_{s,t}\|_{L^1(\nu_s) \to L^\infty(\nu_s)} \leq \psi(t-s), \quad 0 \leq s \leq t \leq 2T,
\] (3.17)
holds with $\tau \mapsto \psi(\tau) \tau^\beta$ non-decreasing. For $C_2(A, \beta, \nu_0, \varphi(1))$ finite and $\kappa \geq 4C_1(c_1, \beta, \delta_*)$,
\[
d(x, y) \leq \kappa \delta_* T, \quad T \geq 1 \quad \Longrightarrow \quad \frac{K_{0,2T}(x,y)}{\nu_{2T}(y)} \leq C_2 \psi(T) \exp \left( \frac{-d^2(x,y)}{2\kappa T} \right).
\] (3.18)

Proof. By Riesz-Thorin interpolation the $1 \to \infty$ bound (3.17) implies the $2 \to \infty$ bound (3.5) with $\varphi(\tau) = \psi(\tau)^{1/2}$. Hence, for $\rho(\cdot) = d(\cdot, x)$, we have from Proposition 3.3 that
\[
\|K_{0,T}^\theta\|_{L^2(\nu_T) \to L^\infty(\nu_0)} \leq C_2 \psi(T)^{1/2} \exp \left( C_1 \chi(\theta)T \right), \quad \forall T \geq 1, \quad \theta \in \mathbb{R}.
\] (3.19)
Considering the adjoint $K_{s,t}^*$ of $K_{s,t} : L^2(\nu_t) \to L^2(\nu_s)$, we have by duality that
\[
\left\| K_{T,2T}^\theta \right\|_{L^1(\nu_T) \to L^2(\nu_T)} = \left\| w_T K_{T,2T}^\theta w_{-\theta} \right\|_{L^2(\nu_T) \to L^\infty(\nu_T)}.
\] (3.20)

Setting $(K_{s,t}^*)^{-\theta} = w_T K_{s,t}^\theta w_{-\theta}$, we further have by duality that
\[
\left\| K_{s,t}^* \right\|_{L^1(\nu_s) \to L^\infty(\nu_t)} = \left\| K_{s,t} \right\|_{L^1(\nu_t) \to L^\infty(\nu_s)} \leq \psi(t-s),
\] (3.21)
\[
\left\| (K_{s,t}^*)^{-\theta} \right\|_{L^2(\nu_s) \to L^2(\nu_t)} = \left\| K_{s,t}^\theta \right\|_{L^2(\nu_t) \to L^2(\nu_s)} \leq \exp ((t-s)(\chi(\theta))),
\] (3.22)
where the identity in (3.22) holds since $(K_{s,t}^*)^{-\theta}$ is the adjoint of $K_{s,t}^\theta : L^2(\nu_t) \to L^2(\nu_s)$. Recall that the adjoint $K_{s,t}^*$ of each Markov operator $K_{s,t}$ is a non-negative linear operator. Further, our assumption that $(\nu_s K_{s,t}) \leq \nu_t$ for strictly positive $\{\nu_s\}$ when $T \leq s \leq t \leq 2T$, yields for $f_{t,y} := (1/\nu_t(y))\delta_y$ any $y \in V$ and $[-1,1]$-valued $g \in C_0(V)$,
\[
|\langle K_{s,t}^* g \rangle(y) | = |\langle f_{t,y}, K_{s,t}^* g \rangle_{\nu_t} | = | \langle g, K_{s,t} f_{t,y} \rangle_{\nu_s} | \leq \langle K_{s,t}, f_{t,y} \rangle_{\nu_s} \leq \frac{\nu_s K_{s,t}(y)}{\nu_t(y)} \leq 1.
\] (3.23)

Thus, $\left\| K_{s,t}^* \right\|_{L^\infty(\nu_s) \to L^\infty(\nu_t)} \leq 1$, with (3.21) and (3.22) allowing us to apply Proposition 5.1 with $\varphi(\tau) = \psi(\tau)^{1/2}$ for the adjoint operators on time interval $[T,2T]$, to get that
\[
\left\| (K_{s,t}^* T)^{-\theta} \right\|_{L^2(\nu_T) \to L^\infty(\nu_T)} \leq C_2 \psi(T)^{1/2} \exp (C_1 \chi(\theta)T).
\] (3.24)

With $C_2' = C_2^2$, upon combining the latter bound with (3.19) and (3.20) we deduce that
\[
\left\| K_{0,2T}^\theta \right\|_{L^1(\nu_T) \to L^\infty(\nu_0)} \leq C_2' \psi(T) \exp (2C_1 \chi(\theta)T).
\] (3.25)

Since $\rho(y) - \rho(x) = d(y,x)$, specializing this operator bound to test function $f(y) = \delta_y$ yields
\[
K_{0,2T}(x,y)e^{\theta d(x,y)} \leq C_2' \psi(T) \exp (2C_1 \chi(\theta)T) \nu_{2T}(y) \quad \forall y \in V.
\] (3.26)

In view of (3.4), taking $\theta = d(x,y)/(\kappa T) \leq \delta_x$ in (3.26), establishes the bound (3.18). □

The next lemma, is part of the (discrete) integral maximum principle of [CGZ, Theorem 2.2, Proposition 2.3] and key to our proof of the Gaffney bound (3.3).

**Lemma 3.4.** Suppose $\pi$ is $\sigma$-finite measure and $K$ is a $\pi$-reversible, bounded range Markov transition on $V$. Then, for $f$ strictly positive and $u \in C_0(V)$,
\[
2\langle fu(Ku-u) \rangle_{\pi} \leq \frac{1}{4} \langle u^2 f^{-1} K |\nabla f|^2 \rangle_{\pi},
\] (3.27)
and for $\alpha_{1}$-uniformly lazy $K$, also
\[
\langle f(Ku)^2 - fu^2 \rangle_{\pi} \leq \frac{1}{4\alpha_{1}} \langle u^2 f^{-1} K |\nabla f|^2 \rangle_{\pi}.
\] (3.28)

**Proof.** For bounded range $K(x,y)$, any $u \in C_0(V)$ and $f$, the function
\[
(K(\nabla u \nabla f))(x) := \sum_{y \in V} K(x,y) \langle u(y) - u(x) \rangle (f(y) - f(x))
\] (3.29)
is in $C_0(V)$. Following the algebra of [CGZ, Eq. (2.7)-(2.8)], if such $K(x,y)$ is $\pi$-reversible then
\[
2\langle fu(Ku-u) \rangle_{\pi} = -\langle K(\nabla f u)(\nabla u) \rangle_{\pi} = -\langle f K(\nabla u)^2 \rangle_{\pi} - \langle u K(\nabla u \nabla f) \rangle_{\pi}.
\] (3.30)
Further, as in [CGZ] proof of Theorem 2.2, for strictly positive $\alpha f$,
\[-\alpha \langle fK|\nabla u|^2 \rangle_\pi - \langle uK(\nabla u \nabla f) \rangle_\pi \leq \frac{1}{4\alpha} \langle u^2 f^{-1} K|\nabla f|^2 \rangle_\pi , \tag{3.31}\]
which for $\alpha = 1$ yields (3.27) when combined with (3.30). Next recall as in [CGZ, Eq. (2.9)], that for $\alpha_t$-uniformly lazy Markov transition $K$ and any $u \in C_0(V)$, by Cauchy-Schwarz
\[(Ku - u)^2(x) \leq (1 - \alpha_t)(K|\nabla u|^2)(x), \quad \forall x \in V . \]

Multiplying by $f(\cdot) \geq 0$ and integrating over the $\sigma$-finite measure $\pi$, results with
\[\langle f(Ku - u)^2 \rangle_\pi \leq (1 - \alpha_t)\langle fK|\nabla u|^2 \rangle_\pi . \tag{3.32}\]

For bounded range $K(\cdot, \cdot)$ all functions are in $C_0(V)$, so combining (3.30), (3.32) and (3.31) we have
\[\langle f(Ku)^2 \rangle_\pi - \langle fu^2 \rangle_\pi = \langle f(Ku - u)^2 \rangle_\pi + 2\langle fu(Ku - u) \rangle_\pi \leq -\alpha_t\langle fK|\nabla u|^2 \rangle_\pi - \langle uK(\nabla u \nabla f) \rangle_\pi \leq \frac{1}{4\alpha_t} \langle u^2 f^{-1} K|\nabla f|^2 \rangle_\pi , \tag{3.33}\]
as stated in (3.28).

We proceed to establish the Gaffney bound (3.3) for non-decreasing $t \mapsto \pi_t \in \mathcal{M}_+(V)$ and bounded range $\pi_t$-reversible Markov operators $K_t$.

**Lemma 3.5.** [The Gaffney lemma] Suppose that Markov operators $K_t$:

(a) have reversible measures $\pi_t \in \mathcal{M}_+(V)$ with $t \mapsto \pi_t(x)$ non-decreasing for any $x \in V$.

(b) have uniformly bounded range. That is, for some $r_0 < \infty$
\[
\{y \in V : K_t(x,y) > 0\} \subset B(x,r_0), \quad \forall x \in V, t \in \mathbb{N} . \tag{3.33}\]

(c) in case of discrete time, also $\inf_{t,x}\{K_t(x,x)\} \geq \alpha > 0$.

Then, the $2 \to 2$ bound (3.33) holds for $K^\theta_{s,t} : L^2(\pi_t) \to L^2(\pi_s)$ provided
\[L_\rho := \sup_{x \neq y \in V} \left\{ \frac{|\rho(x) - \rho(y)|}{d(x,y)} \right\} < \infty , \tag{3.34}\]
where $a_t \equiv 0$, for the dynamics (1.5) we have $\delta_* = 1$, $\alpha = 1$ and $\chi(\cdot) = \zeta(\cdot)$ for
\[
\zeta(\theta) := \frac{1}{8\alpha} (e^{2\alpha_0 L_\rho |\theta|} - 1)^2, \tag{3.34}\]
whereas for (1.3) set $\delta_* = \infty$ and $\chi(\theta) = c_1 \theta^2$ with $c_1 := \frac{1}{2} \sup_{\theta} \{\theta^{-2} \log(1 + 2\zeta(\theta))\}$ finite.

**Proof.** For the $L^2(\pi_t)$-closure of non-negative $K^\theta_{s,t}$ it suffices to get (3.33) for $0 \leq g \in C_0(V)$, namely
\[e^{2\chi(\theta)s} \|K^\theta_{s,t} g\|_{L^2(\pi_s)}^2 \leq e^{2\chi(\theta)t} \|K^\theta_{t,t} g\|_{L^2(\pi_t)}^2 , \quad \forall s \in [0,t] , \quad 0 \leq g \in C_0(V) . \tag{3.35}\]

For $f_s(x) := w_{-2\theta}(x)e^{2\chi(\theta)s}$ and $u \geq 0$ solving (1.23) on $[0,t] \times V$, with $u_t(\cdot) = u(t,\cdot) \in C_0(V)$, let
\[E_s(u) := \sum_{x \in V} f_s(x)u_s^2(x)\pi_s(x) , \quad s \in [0,t] . \tag{3.36}\]

In particular, (3.35) amounts to $E_s(u(\infty)) \leq E_t(u(\infty))$ for $u(\infty) = w_{\theta}g \in C_0(V)$ (which for CSRW is absolutely continuous, see (1.5)). For large enough $R_k \uparrow \infty$ consider the solution $u^{(k)} \geq 0$ of (1.23) on $Q := Q(0,t;z,R_k)$ with $u^{(k)}(t) = u^{(k)}(\infty)$ and $u^{(k)} \equiv 0$ outside $Q$ (which correspond to the transition probabilities (1.3) or (1.5), killed at exiting $B(z,R_k)$). By monotone convergence $E_s(u^{(k)}) \uparrow E_s(u(\infty))$ with equality at $s = t$ and $k$ large (c.f. [FG] (3.10)) for such argument),
and we thus proceed to show more generally that \( s \mapsto E_s(u) \) is non-decreasing on \([0, t]\) for any solution \( u \) of (1.23) on finite time-space cylinder \( Q \), with zero boundary conditions (hence with \( u_s \in C_0(V) \) at any \( s \leq t \)). To this end, with \( f_s \geq 0 \) and non-decreasing \( s \mapsto \pi_s(x) \) bounded on \( Q \), clearly \( f_s u_s^2 \pi_s \leq f_s u_s^2 \pi + \int_{f_s} \pi \partial_{-\xi}(\xi f_s u_s^2) d\xi \) at each \((s, x) \in Q\). Thus, it suffices to show that (as a distribution in case of CSRW),

\[
\Delta E_s(u) := \langle \partial_{-s}(f_s u_s^2) \rangle_{\pi_s} \leq 0, \quad \text{for a.e. } s \in (0, t]. \tag{3.37}
\]

With \( f_s' \) strictly positive and \( \rho \) Lipschitz, by the uniform bounded range assumption (3.3),

\[
\frac{1}{8a} f_s^{-1} K_s(\nabla f_s')^2 \leq \zeta(\theta) f_s', \quad \forall s, s' \geq 0. \tag{3.38}
\]

For the dynamics of (1.3), since \( u_{s'} = K_s u_s \) and \( f_s = e^{2\chi(\theta)} f_s' \) for \( s' = s - 1 \), we have that

\[
\Delta E_s(u) = \langle f_s' u_s^2 \rangle_{\pi_s} - f_s u_s^2 \pi_s = \langle f_s' (K_s u_s)^2 \rangle_{\pi_s} - \langle f_s u_s^2 \pi_s \rangle_{\pi_s}, \tag{3.39}
\]

where \( h_s := (e^{2\chi(\theta)} - 1) f_s' \). Similarly, for the dynamics of (1.5), since \( \partial_{-s} f_s = -2\chi(\theta) f_s \) and a.e. \( \partial_{-s} u_s = (K_s - I) u_s \) (unless \( u_s = 0 \) by our zero boundary condition), we find that a.e.

\[
\Delta E_s(u) = \langle 2 u_s f_s \partial_{-s} u_s + u_s^2 \partial_{-s} \pi_s \rangle_{\pi_s} = 2 \langle u_s f_s (K_s - I) u_s \rangle_{\pi_s} - \langle h_s u_s^2 \rangle_{\pi_s}, \tag{3.40}
\]

now with \( h_s := 2\chi(\theta) f_s \). In view of (3.33), taking \( \alpha = 1 \), \( \chi_c(\cdot) = \zeta(\cdot) \) and \( s' = s \) in the continuous time setting, while \( \chi_d(\cdot) = c_1 \theta^2 \geq \frac{1}{2} \log (1 + 2\zeta(\theta)) \) in discrete time, yields that in both cases

\[
\frac{1}{4a} u_s^2 f_s^{-1} K_s(\nabla f_s')^2 \langle \pi_s \rangle \leq \langle u_s^2 h_s \rangle_{\pi_s}.
\]

Thus, having \( \pi_s \)-reversible \( K = K_s \), strictly positive \( f = f_s' \) and \( u = u_s \in C_0(V) \), upon combining (3.27) and (3.40), or (3.28) and (3.39), we get (3.37) for both the continuous and discrete time dynamics. To complete the proof of the lemma, just confirm that (3.41) holds for \( \chi_c(\cdot) = \zeta(\cdot) \), \( \delta_s = 1 \) and some \( c_1 \) finite.

**Proof of Theorem 1.6(a):** In case of lazy DTRW it suffices to consider \( d(x, y) \leq t - s \) where the bound of (1.26) is merely the conclusion of Proposition 3.3 for \( \psi(k) = C' / \sqrt{2k} \), \( \delta_s = \infty \), the dynamics (1.3) for \( \{ (K_{t+s}, \pi_{t+s}) : r \in (0, t - s) \} \) and \( \nu_r = \pi_r \). Indeed, the required \( 1 \rightarrow \infty \) bound (3.17) is provided by Lemma 2.7 whereas the \( 2 \rightarrow 2 \) Gaffney bound of (3.3) is proved for \( \alpha t \equiv 0 \) and \( \rho(\cdot) = d(\cdot, x) \), in Lemma 3.5. The same applies for the CSRW, except that now \( \delta_s = 1 \) in the \( 2 \rightarrow 2 \) Gaffney bound, hence also in (3.18). Nevertheless, in this case (3.26) holds with \( \chi(\theta) \leq \frac{1}{4} \log (d(x, y)/(2C_1 T)) \) for \( T = (t - s)/2 \) yields the stated bound (1.26). □

Further use of the integral maximum principle as in [CZGZ] Prop. 2.5] yields the following lemma that we shall use in the sequel to strengthen the preceding GHKU.

**Lemma 3.6.** Let \( I(r) = r^2 \) for the dynamics (1.3) and \( I(\cdot) \) as in Theorem 1.6(a) for the dynamics (1.5). Then, in the setting of Lemma 3.5 for

\[
f_s(x) := \exp \left( - \eta(s + 1) I(\rho(x)/(s + 1)) \right), \quad x \in V, s \in \mathbb{Z}_+, \tag{3.41}
\]

the function \( s \mapsto E_s(u) \) of (3.36) is non-decreasing provided \( \inf_x \rho(x) \geq 1 \), \( u_t \in C_0(V) \), and \( \eta \in [0, c_2^{-1}] \) for some \( c_2 (L_{\rho}, r_0, \alpha) \) finite.

**Proof.** Following the proof of Lemma 3.5, consider first the discrete dynamic (1.3). Then, by (3.28) and (3.39) it suffices to find \( c_2 < \infty \) such that for \( c_2 \eta \in [0, 1] \) and \( s' = s - 1 \geq 0 \)

\[
\frac{1}{4a} f_s^{-1}(K_s|\nabla f_s'|^2) \leq h_s, \tag{3.42}
\]
where for the strictly positive $f_s$ of (3.41), at $x \in V$,
\[
h_s(x) := f_s(x) - f_s'(x) = f_s(x)(e^{\frac{\eta_c(x)^2}{(s+1)^2}} - 1).
\]

Next, if $d(x, y) \leq r_0$ then $|\rho(x) - \rho(y)| \leq L_\rho r_0$ and as $\rho(x) \geq 1$ also $|\rho(x)^2 - \rho(y)^2| \leq c_3 \rho(x)$ for $c_3 := L_\rho r_0 (2 + L_\rho r_0)$ finite. In this case, the inequality $|e^w - 1| \leq e^{|w|} - 1$ yields
\[
|f_s'(y) - f_s'(x)| \leq f_s'(x)(e^{\frac{\eta_c(x)^2 - \eta_c(y)^2}{(s+1)^2}} - 1) \leq f_s'(x)(e^{\frac{c_3 \eta_c(x)}{(s+1)^2}} - 1).
\]

Next, recall that for any $\alpha > 0$ there exists $b(\alpha)$ finite, such that $(4\alpha)^{-1}(e^w - 1)^2 \leq e^{b(\alpha)w^2} - 1$ for any $w \geq 0$. Thus, by assumption (3.33), the LHS of (3.42) is bounded above by
\[
f_s'(x)(e^{\frac{b(\alpha)\eta_c(x)^2}{(s+1)^2}} - 1) \leq h_s(x),
\]
provided non-negative $\eta \leq (2b(\alpha)c_3^2)^{-1}$ is chosen in (3.41). Turning to the dynamic (1.25), by (3.27) and (3.30) it similarly suffices to show that
\[
\sup_{y \in B(x, r_0)} |f_s(x) - f_s(y)|^2 \leq 2\partial_s f_s^2(x),
\]
which for $f_s(\cdot)$ of (3.41), $c_3 = L_\rho r_0$ and an $L_\rho$-Lipshitz function $\rho(\cdot) \geq 1$, follows from
\[
\sup_{\delta \in (0, r]} \sup_{|r' - r| \leq c_3 \delta} \left\{ e^{\eta[rI'(r') - I(r)]/\delta} - 1 \right\}^2 \leq 4\eta[rI'(r') - I(r)]
\]
(take $\delta = (s+1)$, $r = \rho(x)\delta$ and $r' = \rho(y)\delta$). Further, with $rI'(r) - I(r) = r(r \wedge 1)$ and $I'(r) = 2(r \wedge 1) + (\log r)_+$ non-decreasing on $\mathbb{R}_+$, it suffices in turn to verify that
\[
g(r) := \exp(\eta c_3 I'((1 + c_3) r)) - 1 - 2\sqrt{\eta r(r \wedge 1)} \leq 0, \quad \forall r > 0.
\]
To this end, note that $g(0) = 0$ and it is not hard to check that $g'(r) \leq 0$ whenever $2\eta c_3 \leq 1$ and $\sqrt{\eta c_3 (1 + c_3)} e \leq 1$. That is, for any non-negative $\eta \leq 1/c_2(c_3)$, as claimed. □

4. PARABOLIC HARNACK INEQUALITY

We adapt here Grigor’yan’s approach [Gr] to proving PHI to the case of continuous time heat equation (1.23) on graphs associated with the CSRW, when $t \mapsto \pi_t(x)$ are non-decreasing and uniformly bounded. Building on weighted Poincaré and $L^2$-mean-value inequalities, the crucial element of the proof is a first growth lemma (here Lemma 4.1). Combining such first growth lemma with the uniform volume doubling condition, one then derives the second growth lemma (here Lemma 4.6), which yields the Harnack inequality by a quite intricate, but by now classical, argument. We thus proceed with the weighted Poincaré inequality of [Del] Prop. 2.2.

Proposition 4.1. [Weighted Poincaré inequality]
Suppose $\pi$ has VD property with constant $C_D$ and the Poincaré inequality with constant $C_P$ holds for uniformly elliptic, $\pi$-reversible, Markov transition $K(\cdot, \cdot)$ on $E$. Then, there exist $C'_P(C_P, C_D, \alpha_e)$ finite, such that for $B := B(z, 2r)$ and $\eta(\cdot) := \{1 - d(\cdot, z)/(2r)\}^2$,
\[
C'_P r^2 (\eta \wedge 1)(\nabla f)^2 \geq \frac{\pi(\mathbb{H}_f)}{\pi(B)} (\eta f^2) \pi \quad \forall r > 0, z \in V, f : V \to \mathbb{R}_+
\]
where $\mathbb{H}_f := B(z, r) \cap f^{-1}\{0\}$, $\eta(x, y) := \eta(x) \wedge \eta(y)$ and $(\eta \wedge 1)(\nabla f)^2$ is as in (3.29).
Proof. From [Del Prop. 2.2] we have the weighted Poincaré inequality
\[ C_P' r^2 \langle (\eta, K)|\nabla f|^2 \rangle \geq \langle \eta f^2 \rangle - \langle \eta f \rangle \frac{\langle \eta f \rangle}{\langle \eta \rangle} \langle \eta f \rangle , \]
where the right-inequality is merely Cauchy-Schwarz for \( f \in L^1 \). Since \([0,1]\)-valued \( \eta(\cdot) \) is supported on \( \mathbb{B} \) and exceeds \( 1/4 \) throughout \( \mathbb{B}(z,r) \), we arrive at (4.1).

The next ingredient is \( L^2 \)-mean value inequality (denoted \( \text{ML}^2 \)), analogous to the one in [CG Sect. 4.1] for uniformly lazy DTRW on time-invariant graph. To this end, recall first that a \( \pi \)-reversible Markov transition \( K(x,y) \) satisfies a relative Faber-Krahn (FK) inequality if there exist positive \( a, \nu \) such that
\[ \lambda_K(\Omega) \geq \frac{a}{r^2} \left( \frac{\pi(\mathbb{B})}{\pi(\Omega)} \right) \nu, \quad \forall r > 0, z \in V, \Omega \subseteq \mathbb{B}, |\Omega| \geq 1, \tag{4.2} \]
where \( \lambda_K(\Omega) \) of [CG Prop. 2.3] is the smallest eigenvalue of \( I - K \) with Dirichlet boundary condition in \( \Omega \). By [CG Prop. 2.3], the FK inequality (4.2) follows from the \( \text{vd} \) property and Poincaré inequality, with constants \( a, \nu \) that depend only on \( C_D \) and \( C_P \). Proceeding to adapt the relevant part of [CG Sect. 4] to our continuous time-varying setting, for Markov kernels \( \{K_t\} \) of uniformly bounded range (as in (3.33)), we denote by \( \pi(\cdot) \) the \( \sigma \)-finite measure on \([0,\infty) \times V\) such that
\[ \pi(S) = \int_0^\infty \left[ \sum_{s \in V} 1_{\{(t,x) \in S\} \pi_t(x)} \right] dt \]
and call \( u : Q \to \mathbb{R}_+ \) a super-solution (of the heat equation) on \( Q = Q(t_1,t_2;z,R) \) of (4.22) if
\[ \partial_s u(s,x) \geq \sum_y K_s(x,y)u(s,y) - u(s,x), \quad \forall (s,x) \in Q, \tag{4.3} \]
for some non-negative boundary values outside \( Q \) (restricting to \( s \in \mathbb{N} \) in discrete time, while for CSRW the inequality is between distributions and holds a.e.). Similarly, \( u \geq 0 \) is called a sub-solution on \( Q \) when the reversed inequality (4.3) holds (see [Del Sec. 2.2]).

Remark 4.2. If \( u \geq 0 \) is a solution of (4.23) on \( Q \), it must satisfy there (1.5) for CSRW stopped upon exiting \( \mathbb{B}(z,R) \). For any \( \Phi(\cdot) \) convex, \( v = \Phi(u) \) is then absolutely continuous on \( Q \), and by Jensen’s inequality has \( \text{LHS} \leq \text{RHS} \) in (1.5) (throughout \( Q \)). Taking \( s \uparrow t \) we deduce that \( v \) is a sub-solution on \( Q \). Likewise, \( v = \Phi(f) \) is a super-solution on \( Q \) whenever \( \Phi(\cdot) \) is concave.

Proposition 4.3. \([L^2 \text{-mean value inequality}]\)
Suppose \( t \mapsto \pi_t(x) \) is non-decreasing with \( C_0 := \sup_{t,x} \{ \pi_t(x) \} \) finite and the \( \pi_t \)-reversible, Markov operators \( K_t \) satisfy (3.33) and the relative FK inequality with same positive \( a, \nu \). Then, for \( \vartheta(t) := \max\{t, t^{-1/\nu}\} \), some \( C = C(a, \nu, C_0) < \infty \) any \( T \geq 2t \geq 4, R > r_0, z \in V \) and sub-solution \( u(\cdot, \cdot) \) on \( Q := Q(T - 2t; T; z, R) \) of (4.3),
\[ \text{ML}^2 : \quad u^2(T - t, z) \leq \frac{C\vartheta(t/R^2)}{\pi(Q)} \int_Q u^2 \, d\pi. \tag{4.4} \]

Proof. We follow closely the argument in [CG Sect. 4], starting with the analogue of [CG Corollary 4.7]. To this end, for any functions \( u, g \) on \( V \),
\[ |\nabla(gu)|^2 - (\nabla g^2 u)(\nabla u) = u(x)u(y)|\nabla g|^2 \leq \frac{1}{2} u^2(x)|\nabla g|^2 + \frac{1}{2} u^2(y)|\nabla g|^2. \]
Hence, for any $\pi$-reversible operator $K(x, y)$ on $V$ and $g \in \mathcal{C}_0(V)$,
\[ 2\mathcal{E}_{K,\pi}(gu, gu) + 2\langle g^2u(Ku - u) \rangle_{\pi} \leq \langle u^2K|\nabla g|^2 \rangle_{\pi} \]  
(4.5)
(recall (2.13) and the LHS of (3.30)). Fix any $\eta(s, x)$ supported on finite time-space region $[T - 2t, T] \times \Omega$ with $\eta(T, \cdot) \equiv 0$ and $\|\nabla \eta\|_\infty + \|\partial_s \eta\|_\infty \leq M$. Since $s \mapsto \pi_s(x)$ are non-decreasing, from (4.5) for $g = \eta_s = \eta(s, \cdot)$ differentiable in $s$, any sub-solution $u_s = u(s, \cdot)$ and the $\pi_s$-reversible $K_s$, we get that at a.e. $s \in [T - 2t, T]$,
\[ 2\mathcal{E}_{K,\pi_s}(\eta su_s, \eta su_s) + \partial_{s-s}\langle \eta_s^2u_s \rangle_{\pi_s} \leq \langle \eta_s^2K_s|\nabla \eta_s|^2 \rangle_{\pi_s} + \langle \eta_s^2 \partial_{s-s} \rangle_{\pi_s}. \]
(Integrating both sides over $[T - \tau, T]$ yields the analogue of [CG Eq. (4.15)]. That is, for $\tilde{\Omega} = \{ \z \in V : d(z, \Omega) \leq r_0 \}$ and any $\tau \leq 2t$,
\[ \|\eta u\|^2_{L^2(\pi_{T-\tau})} + 2 \int_{T-\tau}^T \mathcal{E}_{K,\pi_s}(\eta u, \eta u) \, ds \leq 2M \int_{[T-\tau, T] \times \tilde{\Omega}} u^2 \, d\pi. \]  
(4.6)
We proceed to adapt the proof from [CG Sect. 4.4 & 4.5] of the $\text{ML}^2$. Indeed, by the assumed monotonicity of $s \mapsto \pi_s$ and uniformity of $u, \nu$, here the relative Faber-Krahn inequality (4.2) yields that for any $s \geq 0, z \in V, r > 0$ and non-empty $\Omega \subseteq \mathbb{B}(z, r)$,
\[ \lambda_{\pi_s}(\Omega) \geq \Lambda(\pi_s(\Omega)), \quad \Lambda(\xi) := \frac{\alpha}{r^2}\pi_0(\mathbb{B}(z, r))^\nu \xi^{-\nu}. \]  
(4.7)
With (4.6) and (4.7), taking the roles of [CG Eq. (4.15)] and [CG Eq. (4.19)], respectively, the proof of [CG Eq. (4.20)] applies verbatim, upon changing on [CG page 681] to $I := \int_\Psi u^2 \, d\pi$, $I' := \int_{\Psi'} (u - \theta)^2_d \, d\pi$, for a solution $u(\cdot, \cdot)$, constant $\theta > 0$ and invoking hereafter the time inversion $s \mapsto (T - s)$ on $\Psi' \subset \Psi$ and all other time-space cylinders from [CG]. We proceed as in [CG pages 685-687] to compare via [CG Eq. (4.20)] the values of $I_{n-1}$ and $I_n := \int_{\Psi_n} (u - \theta_n^2_d) \, d\pi$, for $\theta_n = \theta(2 - 2^{-n})$ and decreasing cylinders $\Psi_n := Q(T - 2t + n, T - t + n; z, R_n)$, with $t_n = t2^{-n}$ and $R_n = [R_{n-1}/2]$, starting at $\Psi_0 = Q$. Iterating to $N = \max\{n : R_n \geq r_0 + 1, t_n \geq 2 \}$, we arrive at [CG Eq. (4.36)] where $\beta := \Lambda(1) \leq \pi_0(z)\nu$ by (4.2) for $\Omega = \{ \z \}$ (as $\lambda_{\pi_0}(\{\z\}) \leq 1$).

Setting $M = 2$, $\hat{\tau} = 2$ and $\hat{T} - \hat{\tau} = T - t$, consider (4.6) for the sub-solution $\hat{u} := (u - 2\theta_N)^+$ (recall Remark (4.2), and $[0, 1]$-valued $\hat{\eta}$ supported on $[\hat{T} - 2\hat{\tau}, \hat{T}] \times \Omega$, such that $\hat{\eta}(T - t, z) = 1$ and $\|\nabla \hat{\eta}\|_\infty + \|\partial_s \hat{\eta}\|_\infty \leq M$. Since $\hat{Q} = [\hat{T} - \hat{\tau}, \hat{T}] \times \Omega \subset \Psi_N$ and $\theta_N \leq 2 \theta$, we have that
\[ (u(T - t, z) - 2\theta)^2 \pi_0(z) \leq 2M \int_{\hat{Q}} \hat{u}^2 \, d\pi \leq 2MI_N. \]  
(4.8)
Continuing as in [CG], we cancel the common power of $m(z) = \pi_0(z)$ from both sides of [CG Eq. (4.38)], en-route to [CG Eq. (4.39)] and thereby to $\text{ML}^2$ by taking $\theta = \frac{1}{4}u(T - t, z)$.

Having the key ingredients of Prop. 4.1 and Prop. 4.3, we now establish the first growth lemma.

Lemma 4.4. [First growth lemma] Suppose $\{G_t\}$ are as in Theorem 7.4.
For any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, C_P, C_{D, \alpha}, \alpha, \mathcal{C}_0) > 0$ such that for all $T \geq 6R^2$, $z \in V$ and any positive solution $u(\cdot)$ of (1.22) on $Q := Q(T - 4R^2; T; z, 2R)$,
\[ \frac{\pi(Q(T - R^2, T; z, R) \cap u^{-1}(1, \infty))}{\pi(Q(T - R^2, T; z, R))} \geq \delta \implies \inf_{Q(T - 3R^2, T - 2R^2; z, R)} u \geq \varepsilon. \]  
(4.9)
Proof. Fixing $z \in V$ and $T \geq 4R^2$ set $\mathbb{B} := \mathbb{B}(z, 2R)$ and $\eta(\cdot) := \{[1 - d(\cdot, z)/(2R)]^+\}^2$ as in Prop. 4.1. Recall Remark 1.3 that any solution $u > 0$ of (1.23) on $Q$ can be replaced by bounded away from zero solutions $u_b = (1 - b)u + b$, without altering the LHS of (4.9). Hence, by Remark 4.2
WLOG we have the associated super-solution $\tilde{u} := 1 \land u \geq b$ for some $b > 0$, and uniformly bounded sub-solution $v := -\log \tilde{u}$. Consider the functions $F(s) := \langle (\eta v_s) \rangle_{\pi_s}$ on $T - 4R^2, T]$, $\tilde{u}_s := \tilde{u}(s, \cdot)$ and $v_s := v(s, \cdot)$ on $V$ and the subset $\mathbb{H}_{v_s} = \mathbb{B}(z, R) \cap u(s, \cdot)^{-1}([1, \infty))$ of $\mathbb{B}$. Having $s \mapsto \pi_s(\cdot)$ non-decreasing and $\tilde{u} \geq b$ a super-solution, it follows from (4.34) and the LHS of (4.33) that as distributions, for a.e. $s$,

$$
\partial_s F(s) \geq \langle \eta \partial_s \tilde{u}_s \rangle_{\pi_s} \geq \langle \frac{\eta}{u_s} K_s \nabla \tilde{u}_s \rangle_{\pi_s} = -\frac{1}{2} \langle K_s (\nabla (\frac{\eta}{u_s}) \nabla \tilde{u}_s) \rangle_{\pi_s}. \tag{4.10}
$$

Setting $\psi(c) = \frac{1}{2c}$ for $c > 0$ and $\psi(0) = 1$, recall that for any $a, b > 0$ and $c, d \geq 0$,

$$
-(\frac{d}{b} - \frac{c}{a})(b - a) \geq \frac{1}{2}(c \land d)(\log b - \log a)^2 - |d - c| \psi\left(\frac{c \land d}{|d - c|}\right),
$$

(see [SZh] Inequality (1.23))). For $(x, y) \in E$ and $k \in \mathbb{Z}_+$, if $\eta_h = k^2(2R)^{-2}$ then necessarily $|\nabla \eta| \leq (2k + 1)(2R)^{-2}$, so $|\nabla \eta|\psi(\eta_h/|\nabla \eta|) \leq \frac{9}{8} R^{-2}$ (or zero, whenever $d(x, z) \land d(z, y) > 2R$). Hence, upon summing over $\pi_s K_s$ we get as in [BSA] proof of (5.7), that

$$
-s \langle K_s (\nabla (\frac{\eta}{u_s}) \nabla \tilde{u}_s) \rangle_{\pi_s} \geq \frac{1}{2} \langle (\eta K_s) \nabla v_s \rangle_{\pi_s}^2 - \frac{9}{4} R^{-2} \pi_s(\mathbb{B}). \tag{4.11}
$$

Next, by Prop. 4.1 for $v_s \geq 0$ and $r = R$, followed by Cauchy-Schwartz,

$$
C' p R^2 \langle (\eta K_s) \nabla v_s \rangle_{\pi_s}^2 \geq \frac{\pi_s(\mathbb{H}_{v_s})}{\pi_s(\mathbb{B})} \langle \eta v_s^2 \rangle_{\pi_s} \geq \frac{\pi_s(\mathbb{H}_{v_s})}{\pi_s(\mathbb{B})^2} F(s)^2.
$$

Plugging this into (4.10) \& (4.11) yields

$$
\partial_s F(s) \geq L(s) F(s)^2 - D(s), \quad L(s) := \frac{R^{-2} \pi_s(\mathbb{H}_{v_s})}{C'_p (2C_0)^2 \pi_0(\mathbb{B})^2}, \quad D(s) := \frac{9}{8} R^{-2} \pi_s(\mathbb{B}).
$$

Following [Gr] pg. 67], let $J(t) = F(t) - \int_t^T D(s) ds$ and $t_* := \sup\{t \leq T : J(t) \leq 0\}$. With $F$, $L$, $D$ non-negative and $J(T) \geq 0$, we have on $[t_*, T]$ that $\partial_s J \geq LF^2 \geq L J^2$ and consequently $J(t) \leq (\int_{t_*}^t L(s) ds)^{-1}$. Further $t \mapsto J(t)$ is non-decreasing and $J(t_*) = 0$, so this bound extends to all $t$. Thus, on $[T - 4R^2, T - R^2],\n
$$
F(t) \leq \left(\int_{T - R^2}^T L(s) ds \right)^{-1} + \int_t^T D(s) ds. \tag{4.12}
$$

From the LHS of (4.9) and definition of $\mathbb{H}_{v_s}$ we have $R^{-2} \int_{T - 4R^2}^T \pi_s(\mathbb{H}_{v_s}) ds \geq \delta C_D^{-1} \pi_0(\mathbb{B})$, hence the first term on the RHS of (4.12) is at most $\delta^{-1} C_D C'_p (2C_0)^2 \pi_0(\mathbb{B})$. The other term is at most $7C_0 \pi_0(\mathbb{B})$, so for some $C_1(C'_p, C_D, C_0)$ finite,

$$
F(t) \leq C_1 \pi_0(\mathbb{B}) \delta^{-1}, \quad \forall t \in [T - 4R^2, T - R^2]. \tag{4.13}
$$

For $t \in [T - 3R^2, T - 2R^2]$, integrating (4.10) on $I_t := [t - R^2, t + R^2]$, yields by (4.11) and (4.13) that

$$
\int_{I_t} \langle (\eta K_s) \nabla v_s \rangle_{\pi_s} ds \leq 4F(t + R^2) + \frac{9}{2} R^{-2} \pi(I_t \times \mathbb{B}) \leq C_2 \pi_0(\mathbb{B}) \delta^{-1}, \tag{4.14}\n$$

where $C_2 = 4C_1 + 9C_0$. Since $\zeta := 1_{\mathbb{B}^c} \leq 16\eta$ for $\mathbb{B}^c := \mathbb{B}(z, 3R/2)$, it follows that

$$
\bar{v}_s := \langle v_s \zeta \rangle_{\pi_s} \leq \frac{16F(s)}{\pi_0(\mathbb{B})}, \quad \mathcal{E}_{K_s, \pi_s}(v_s \zeta, v_s \zeta) \leq 8((\eta K_s) \nabla v_s \rangle_{\pi_s}^2.\n$$
Recall that under uniform ellipticity, the $\phi$ property and (weak) Poincaré inequality ($\Pi$) of (1.19) implies the strong-$\Pi$ where $\mathbb{B}(x_0, r)$ replaces $\mathbb{B}(x_0, 2r)$ on the RHS of (1.19) (see [B2 Cor. A.51] or [Kim Prop. 3.3.2]). From the strong-$\Pi$ on $\mathbb{B}'$ and the preceding bounds,

$$
(v_s^2 \zeta)_{\pi_s} = \bar{v}_s^2 \langle \zeta \rangle_{\pi_s} + \langle (v_s - \bar{v}_s)^2 \rangle_{\pi_s} \leq \frac{16^2 \mathcal{F}(s)^2}{\pi_0(\mathbb{B}')^2} + 16C_P R^2 \langle (\eta_t K_s) | \nabla v_s \rangle_{\pi_s}^2.
$$

(4.15)

Combining (4.13)-(4.15), we get for some $C_3(C_1, C_2, C_D, C_P)$ finite and all $t \in [T - 3R^2, T - 2R^2],$

$$
\int_{I_t \times \mathbb{B}'} v^2 d\pi \leq C_3 R^2 \pi_0(\mathbb{B}) \delta^{-2}.
$$

(4.16)

Applying the $L^2$-mean value of Prop. 4.3 to the sub-solution $v$ on $I_t \times \mathbb{B}(x, R/2)$ together with the $\phi$ property of $\pi_0,$ we get for $C_4 = C_1(C_P, C_D, \alpha_e, C_0)$ finite and all $(t, x) \in Q(T - 3R^2, T - 2R^2; z, R),$

$$
v^2(t, x) \leq \frac{C_4 R^{-2}}{\pi_0(\mathbb{B}(x, R/2))} \int_{I_t \times \mathbb{B}(x, R/2)} v^2 d\pi \leq \frac{C_5 R^{-2}}{\pi_0(\mathbb{B})} \int_{I_t \times \mathbb{B}'} v^2 d\pi \leq C_6^2 \delta^{-2},
$$

using (4.16) in the last step. That is, (4.9) holds with $\varepsilon = \exp(-C_6/\delta) > 0.$

\[\square\]

\textbf{Remark 4.5.} An alternative and quicker approach by Fabes-Stroock utilizes the weighted Poincaré inequality in a different way (e.g. [FS, Ba, BK]). It relies on having a-priori that

$$
K_{t-s, t}(x, x) \leq \frac{C}{\pi_0(\mathbb{B}(x, \sqrt{s}))} \text{ and } \inf_{s, t, y} \sum_{x \in \mathbb{B}(y, C \sqrt{s})} K_{t-s, t}(x, y) > 0
$$

(which take the role of (4.9) in proving the first growth lemma). However, lacking a uniform in $y$ lower bound on $\sum_{x \in \mathbb{B}(y, C \sqrt{s})} K_{t-s, t}(x, y),$ prevents using this approach in our time-varying setting.

Under uniform volume doubling condition, the first growth lemma implies second growth lemma, following the same proof as [Gr Lemma 4.2].

\textbf{Lemma 4.6.} [Second growth lemma] For $\{G_t\}$ of Theorem 1.4, some $\theta = \theta(C_P, C_D, \alpha_e, C_0)$ finite, $c = c(\delta, C_P, C_D, \alpha_e, C_0) > 0,$ $u(\cdot)$ as in Lemma 4.4 and $T' \in [T - (R/2)^2 + r^2, T],$

$$
\frac{\pi(Q(T' - r^2, T'; z, R) \cap u^{-1}(1, \infty))}{\pi(Q(T' - r^2, T'; z, R))} \geq \delta \Rightarrow u(z, T - 4R^2) \geq c \left(\frac{\pi(Q(T' - r^2, T'; z, R))}{\pi(Q(T - R^2, T; z, R))}\right)^\theta.
$$

The proof of the PHI as in Theorem 1.4 from the first and second growth lemmas is standard in the literature (e.g. see the argument provided near the end of [Gr Section 4]).

\textbf{Proof of Proposition 4.5.} Fixing $y_j \in \mathbb{B}(z, R)$ and $(T - s_j) \in [R^2, 4R^2]$ such that $s_2 \geq s_1,$ we consider nested time-space cylinders $Q(i) := [s_1, s_1 + R_i^2] \times \mathbb{B}(y_1, R_i)$ for $R_i = 2^i,$ $i \geq 0$ and the corresponding $M(i) := \sup_{Q(i)} \{u\},$ $m(i) := \inf_{Q(i)} \{u\}$ and $w(i) := M(i) - m(i).$ With $i_2 := \sup \{i : Q(i) \subseteq Q\}$ we have that $w(i_2) \leq M(i_2) \leq \sup Q \{u\}$. Similarly, setting $r := |s_2 - s_1|^{1/2} \vee d(y_1, y_2)$ and $i_1 := \inf \{i : r \leq R_i\}$ we have that $s_2, y_j) \in Q(i), j = 1, 2,$ hence $\|w(s_2, y_2) - w(s_1, y_1)\| \leq w(i_2).$ Clearly $R_{i_1} \leq 2r,$ while $R \leq 2R_{i_2}$ (since $(T - s_1) \geq R^2$ and $\mathbb{B}(y_1, R) \subseteq \mathbb{B}(z, 2R)).$ The inequality (1.24) is trivial unless $r \leq R/4$ and thereby $i_1 \leq i_2.$ It thus suffices to show that $w(i - 1) \leq (1 - \gamma)w(i)$ for some $\gamma(\theta_j)$ from (1.24) and all $i \leq i_2$ (as then $w(i) \leq (1 - \gamma)^{i_2 - i}w(i_2),$ yielding (1.25) for $2^{-h} = 1 - \gamma.$ To this end, consider for (non-negative) solutions $u - m(i)$ and $M(i) - u,$ the PHI in $Q(i) \subseteq Q$ with $\theta_1 = \theta_2 = \frac{\theta_3}{\sqrt{2}},$ $\theta_3 = \frac{\sqrt{\theta_1}}{2}$ and $\theta_4 = 1,$ to compare the solution at $(s_1 + 2R_{i_2}^2, y_1)$
with its infimum over $Q(i - 1)$. Setting $v(i) = u(s_1 + 2R_{i-1}^2, y_i)$, these comparisons yield by the PHI that
\[
\gamma(v(i) - m(i)) \leq m(i - 1) - m(i),
\]
\[
\gamma(M(i) - v(i)) \leq M(i) - M(i - 1),
\]
and hence $w(i - 1) \leq (1 - \gamma)w(i)$, as claimed. \hfill \Box

5. From ghke and phi to Gaussian lower bounds

In this section we establish the matching ghke of (1.27) out of the (weaker) upper bound (1.26) and the phi (1.24). To this end, we start with the following elementary fact.

**Lemma 5.1.** Suppose $\Gamma : V \mapsto \mathbb{R}_+$ are such that
\[
\Gamma_\tau(z) \leq \frac{C}{v(\sqrt{\tau})}e^{-\rho_\tau(x,z)/C},
\]
for some $C < \infty$, $x \in V$, a doubling function $v(r) \geq C^{-2\|B(x,r)\|}$ and
\[
\rho_\tau(x,z) := d(x,z)\left(\frac{(d(x,z)}{\ell} \wedge 1\right), \quad x,z \in V, \ t \geq 0. \tag{5.2}
\]
Then, for some $c'(C_\tau, C)$ finite,
\[
\Gamma_\tau(B(x,R)^c) \leq c'e^{-R/(2C)}, \quad \forall R \geq 1, \ \forall x \in V. \tag{5.3}
\]
Further, if $\Gamma$ are probability measures, then for $b > 0$ and some $\kappa(C_\tau, C, b) < \infty$,
\[
\inf_{\tau \geq b, x \in V} \{ \Gamma_\tau(B(x, \kappa\sqrt{\tau})) \} \geq \frac{1}{2}. \tag{5.4}
\]

**Proof.** Note that $1 - e^{-(1+2(\ell \wedge \tau))/(C\tau)} \leq 3\ell/(C\tau)$ for $\ell \geq 1$. From (5.1) we thus get after summation by parts that
\[
\Gamma_\tau(B(x,R)^c) \leq \frac{C}{v(\sqrt{\tau})} \sum_{\ell > R} |\partial B(x,\ell)|e^{-\ell(\ell \wedge \tau)/(C\tau)} \leq 3C^2 \sum_{\ell > R} \frac{v(\ell) e^{-\ell(\ell \wedge \tau)/(C\tau)}}{v(\sqrt{\tau})}. \tag{5.6}
\]
With $v(\ell)$ doubling, one has $v(\ell) \leq v(\sqrt{\tau})(2\ell/\sqrt{\tau})^{c_\tau}$ for $c_\tau := \log_2 C_\tau$ and any $\ell \geq \sqrt{\tau}$. Hence, for some $c'(C_\tau, C)$ and any $R \geq \tau \geq 1$,
\[
\Gamma_\tau(B(x,R)^c) \leq 3C^2 \int_{R}^{\infty} (4\ell)^{c_\tau + 1} e^{-u/C} du \leq c'e^{-R/(2C)}
\]
as claimed in (5.3). Similarly, for $\kappa = \kappa(C_\tau, C, b)$ large enough and all $\tau \geq b$,
\[
\Gamma_\tau(B(x,\kappa\sqrt{\tau})^c) \leq 3C^2 \int_{\kappa}^{\infty} (4\ell)^{c_\tau + 1} e^{-u(\kappa\sqrt{\tau})/C} du \leq \frac{1}{2},
\]
yielding (5.4) in case $\Gamma(V) = 1$. \hfill \Box

We next utilize the fact that for $\{K_t\}$ of (1.1), the transition probabilities $K_{\cdot \cdot, \cdot}(\cdot , y)$ of the CSRW and DTRW, are solutions of (1.28).

**Lemma 5.2.** Fixing Borel measurable $\{\Pi_x\}$, $t \geq 0$ and $z_\ast \in V$, the functions $u(s, x) := K_{s,t}(x, z_\ast)$ for the CSRW and DTRW associated with (1.1), solve the corresponding heat equation (1.28) on the time-space cylinder $Q(0, t ; z, R)$ for any $R \geq 1$ and $z \in V$. The phi then implies that:
(a). For some $\gamma(\varphi, \delta) \in (0, 1)$, any $\delta \in (0, 1)$, $\varphi \geq 1/(1 - \delta)$, all $t \geq \tau \geq 1$, and $x_1, x_2, z_\ast \in V$,
\[
d(x_1, x_2) \leq 2[2\varphi\sqrt{\tau}] \implies K_{t-\tau,t}(x_2, z_\ast) \geq \gamma K_{t-\tau,t}(x_1, z_\ast), \tag{5.5}
\]
where for DTRW we further assume that $\tau \in \mathbb{N}$ and $d(x_1, x_2) \leq (1 - \delta)\tau \in \mathbb{N}$.

(b) Suppose in addition that $C_0 := \sup_{s, x} \left\{ \frac{\pi_s(x)}{\pi_0(x)} \right\}$ is finite, $s \mapsto \pi_s(\cdot)$ is non-decreasing and the DTRW is uniformly lazy. Then,

$$ s \mapsto D_{s,t}(y) := \sum_{x \in V} \pi_s(x)K_{s,t}(x,y)^2 \tag{5.6} $$

is non-decreasing on $[0, t]$. Further, for $C_1 = C_1(\gamma, \alpha_1, C_0)$ finite,

$$ C_1D_{t-2s,t}(y) \geq D_{t-s,t}(y), \quad \forall y \in V, \ s \geq 0. \tag{5.7} $$

**Remark 5.3.** From part (b) we have that the non-decreasing $\tilde{D}(s) := \pi_t(y)/D_{t-s,t}(y)$ is a doubling function, with $\tilde{D}(0) = 1$ such that $\tilde{D}(2s) \leq C_1\tilde{D}(s)$ for all $y \in V$ and $2s \leq t$.

**Proof.** (a). From [1.3] it immediately follows that for DTRW the non-negative $u_k(x) = K_{k,t}(x, z_*)$ satisfies (1.22) on $Q(0, t; z, R)$ of (1.22) (with $\partial_s u(s, \cdot) = u(s - 1, \cdot) - u(s, \cdot)$). Similarly, in case of CSRW, it follows from (1.5) that $s \mapsto u(s, x) = K_{s,t}(x, z_*)$ is an absolutely continuous, solution of (1.22) on $Q(0, t; z, R)$. Next, if $d(x_1, x_2) \leq 2R$ for integer $R := \lfloor 2\sqrt{\tau} \rfloor \geq 1$, then $x_1, x_2 \in \mathbb{B}(z, R)$ for some $z \in V$. Further, for $\tau \geq 1$ we have $R + 1 \geq 2\sqrt{\tau} \geq 2(1 - \delta)$, whence the PHI applies for $T = t$, $\tau = \tau$ and $\gamma = \delta\tau$, with $\varphi = 1/\tau$, $\varphi = \sqrt{\delta}/(1 + \delta)$ and the corresponding $\gamma = \gamma(\tau) \in (0, 1)$.

(b). While proving Lemma 3.5, we showed that $s \mapsto E_{s}(u)$ of (3.36) is non-decreasing whenever $u_s(x) \geq 0$ solves (1.24) for some $u_t \in C(0,V)$. By part (a) this applies to $s \mapsto D_{s,t}(y)$ which corresponds to $u_s(x) = K_{s,t}(x, y)$ and $f_s(x) \equiv 1$. Next, from (5.5) for $x_1 = x_2$ we have that $K_t \cdot 2s, t(x, y) \geq \gamma K_{t-1,s}(x, y)$ for $\gamma(2, 1/2) > 0$, all $x, y \in V$, $2s \in [1, t]$, yielding that $C_1D_{t-2s,t}(y) \geq D_{t-s,t}(y)$ for $C_1 = C_0\gamma^{-2}$ finite. Lowering $\gamma \leq \alpha_1$ for lazy DTRW, or $\gamma \leq e^{-1}$ for CSRW, we further have that $K_t \cdot 2s, t(x, y) \geq \gamma$, hence $C_1D_{t-1,t}(y) \geq D_{t,t}(y)$, extending (5.7) to all $2s \in [0, t]$.

Utilizing the GKHE of Theorem 1.6(a) as well as Lemmas 3.6, 5.1 and 5.2(b), we next establish a moment generating bound which is key for getting matching GKHE.

**Lemma 5.4.** In the setting of Theorem 1.6(b), for dynamics (1.3) and (1.5), the functions $\rho_t(\cdot)$ of (5.2) and $D_{s,t}(\cdot)$ of (5.3), some $\theta_0(\gamma, C_V, \alpha_1) > 0$ and $C(\gamma, C_V, \alpha_1)$ finite,

$$ D_{\tau,2\tau}(y; \theta_0) := \sum_{z \in V} \pi_{\tau}(z)K_{\tau,2\tau}(z, y)^2e^{\theta_0\rho_{\tau}(z,y)} \leq C_2D_{\tau,2\tau}(y), \quad \forall y \in V, \ \tau \geq 1. \tag{5.8} $$

**Proof.** Fixing $y \in V$, $\tau \geq 1$ and $t = 2\tau$, let

$$ J(\ell, s) := \sum_{z \in V} \pi_{\tau-s}(z)K_{t-s,t}(z, y)^21_{\{d(z,y)\geq \ell\}}, \quad \ell \geq 0, \ s \in [0, t]. \tag{5.9} $$

Since $\partial_{-\ell}\{\ell(\frac{\ell}{\tau} \wedge 1)\} \geq -2\ell/\tau$ and $1 - e^{-w} \leq -w$, it follows after summation by parts that

$$ D_{\tau,2\tau}(y; \theta) := \sum_{\ell=0}^{\infty} e^{\theta(\frac{\ell}{\tau} \wedge 1)}(J(\ell, \tau) - J(\ell + 1, \tau)) \leq J(0, \tau) + 2\theta \sum_{\ell=1}^{\infty} J(\ell, \tau)\frac{\ell}{\tau}e^{\theta(\frac{\ell}{\tau} \wedge 1)}. $$

With $J(0, \tau) = D_{\tau,2\tau}(y)$ we thus get (5.8) upon showing that

$$ J(\ell, \tau) \leq C_2J(0, \tau)e^{-2\theta_0\ell(\frac{\ell}{\tau} \wedge 1)}, \tag{5.10} $$

for some $\theta_0(\gamma, C_V, \alpha_1)$ positive, $C_2(\gamma, C_V, \alpha_1)$ finite and all $\ell \geq \ell_0(\gamma, C_V, \alpha_1)$ finite, whereupon taking $C_2 \geq e^{\theta_0}\theta_0'$ it suffices to show (5.10) for $\ell > \sqrt{\tau}$. To this end, we proceed by adapting the proof of [CGZ] Prop. 5.4] to handle both time-varying $\pi_1(\cdot)$ and the CSRW (see also [FG], Lemma 4.1] for CSRW with constant conductances). First apply Lemma 3.6 for $u_s(\cdot) := K_{t-s', t}(\cdot, y)$, with
\(u_{s'} \in \mathcal{C}_0(V)\), and the Lipschitz function \(\rho(\cdot; \ell') := \ell' + 1 - d(\cdot, y) \wedge \ell'\), to deduce that for some \(\eta = \eta(\alpha) > 0\), any \(s' \in [0, \ell]\), \(\ell' \geq 0\) and \(y \in V\),

\[
s \mapsto E_s(\ell', s') := \sum_{z \in V} \pi_{t-s'+s}(z)K_{t-s'+s,t}(z, y)^2 e^{-\eta(s'+1)/2} \left(\rho(z; \ell')/(s'+1)\right),
\]

is non-decreasing on \([0, s']\). Further, \(I(1) = 1\) and \(\rho(z; \ell') = 1\) whenever \(d(z, y) \geq \ell'\) while \(\rho(z; \ell') \geq \ell' - \ell\) whenever \(d(z, y) < \ell \leq \ell'\). Hence, for any \(s \in [0, s']\) and \(\ell \leq \ell'\),

\[
e^{-\eta}J(\ell', s') \leq E_0(\ell', s') \leq E_{s-s}(\ell', s') \leq J(\ell, s) + J(0, s) e^{-\eta(s'-s+1)/2} \left(\frac{\ell'-\ell}{s'+1}\right).
\]

We then get (5.10) upon recalling Remark 5.3 that due to the Phi the function \(s \mapsto \tilde{D}(s) = J(0, 0)/J(0, s)\) is non-decreasing and \(C_1\)-doubling, hence regular in the sense of [CGZ Def. 5.1]. Indeed, [CGZ (5.11)] is derived from [CGZ (5.10)] by iterating (5.11) for consecutive terms of the sequence \(j_0 = \min\{j \geq 1 : j_0 \geq j\}\) (since in their case, of dTRW, one has that \(J(R, s) = 0\) whenever \(R > s\)). For such parameters \((\ell' - \ell)/s' - s + 1 \leq \ell/s' \leq 1\), hence \(I(r) = r^2\) (even for CSRW), and taking \(\kappa < \sqrt{\log(2C)} + \eta\) makes [CGZ CASE 1] hold here as well. Thus, the only difference is that for the CSRW we still have to bound the last term of the iteration, \(e^{\eta}J(\ell_{j_0}, s_{j_0})\), by the RHS of (5.10). For this task apply Lemma 5.1 to \(\Gamma(s) := \pi_{t-s}(z)K_{t-s,t}(z, y)^2\), in which case (5.3) amounts to

\[
J(R, s) \leq e^{-R/C}, \quad \forall R \geq s \geq 1
\]

(for \(C \ll 1.26\) and \(c'(<C, C)\) finite). Next, recall (5.7) that \(J(0, \tau) \geq (C_1C)^{-1} \tau^{-c_1}\) for some \(c_1\) finite and all \(\tau \geq 1\). With \(\ell_{j_0} \geq s_{j_0} + \ell/t\) and \(e^{\eta}J(\ell_{j_0}, s_{j_0}) \leq c'J(0, \tau) \tau^{-c_1} e^{-\ell/t} C\),

which for \(\ell \geq \sqrt{\tau}\) and \(\theta_0 \leq (5C)^{-1}\), is further bounded by the RHS of (5.10).

**Proof of Theorem 1.6 (b).** Proceeding to derive the matching gHKE of (1.27), since our assumptions apply for \((K_{s+r}, \pi_{s+r}) : r \in [0, t - s]\), it suffices to do so for \(s = 0\) and fixed \(x, y \in V\) such that \(d(x, y) \leq t\).

**Step 1: Improved gHKE.** Recall the gHKE (1.26) implying that (5.11) holds for \(K_{s+r}(x, \cdot)\) with \(C < \infty\) independent of \(s, \tau \geq 0\) and \(x \in V\). Further, by the triangle inequality, we have for the non-increasing \(t \mapsto \rho_t(x, z)\) of (5.2) that

\[
\frac{1}{2} \rho_t(x, y) \leq \rho_t(x, z) + \rho_t(y, z), \quad \forall z \in V, t \geq 0.
\]

Hence, setting \(t = 2\tau\), \(\tau \geq 1\) and \(\theta_1 = \frac{1}{2}((\theta_0 \wedge C^{-1})\), by Chapman-Kolmogorov and (5.12), followed by Cauchy-Schwartz, (5.8), Lemma 5.4 and the inequality (5.7), we arrive at

\[
K_{0, t}(x, y)^2 e^{\theta_1 \rho_t(x, y)} \leq \left[\sum_{z \in V} K_{0, t}(x, z) e^{\theta_1 \rho_t(x, z)} K_{t, t}(z, y) e^{\theta_1 \rho_t(y, z)}\right]^2 \leq C V D_{t, t}(y) 2\theta_1 \sum_{z \in V} K_{0, t}(x, z)^2 e^{2\theta_1 \rho_t(x, z)} \leq C V C_2 D_{t, t}(y) \frac{C}{v(\sqrt{r})} \sum_{z \in V} K_{0, t}(x, z) \leq \frac{C'}{v(\sqrt{r})} D_{0, t}(y),
\]
where \( C' = C_V^2 C_2 C_1 C \). Applying the same argument on \([\tau, 2\tau]\) instead of \([0, t]\), yields that for any \( y \in V \) and \( \tau \geq 1 \),

\[
\Gamma_\tau(z) := \frac{\pi_\tau(z) K_{\tau,2\tau}(z,y)^2}{D_{\tau,2\tau}(y)} \leq \frac{C_V C'}{v(\sqrt{\tau})} e^{-\theta_1 \rho_{\tau}(z,y)}.
\]

In view of (5.6), these \( \Gamma_\tau(\cdot) \) are probability measures on \( V \), hence by Lemma 5.1 there exists \( \kappa(C_V, C_V C' \vee \theta_1^{-1}, 1) \) finite such that for \( R = \kappa \sqrt{\tau} \),

\[
\Gamma_\tau(B(y, R)) D_{\tau,t}(y) \geq \frac{1}{2} D_{\tau,t}(y) \geq \frac{1}{2C_1} D_{0,t}(y)
\]

(5.14)

(5.14)

(using the RHS of (5.7) for the last inequality). By the definition (1.9) and Lemma 5.2 (a) (at \( x_1 = z, x_2 = x, z_* = y \)), we have for \( \gamma(\kappa/2, 1/2) > 0 \) and \( R \) as above,

\[
\left[ \frac{\mu_{0,t}(y)}{\pi_0(B(y, R))} \right]^2 \geq \inf_{x \in B(y, R)} \{ K_{0,t}(x,y)^2 \} \geq \gamma^2 D_{\tau,t}(y) \frac{\Gamma_\tau(B(y, R))}{\pi_\tau(B(y, R))}
\]

(5.15)

(5.15)

(where for DTRW we restrict to \( \tau \geq 2R \)). In view of the assumed uniform volume growth with \( v(r) \) doubling, from (5.14) and (5.15) it follows that for some \( C_3(C_V, C_1, \kappa) \) finite,

\[
\mu_{0,t}(y)^2 \geq \frac{\gamma^2}{2C_1 C_V} v(R) D_{0,t}(y) \geq C_3^{-1} v(\sqrt{t}) D_{0,t}(y).
\]

(5.16)

(5.16)

For DTRW, our derivation of (5.16) required \( t \geq 2(2\kappa)^2 \), but with \( K_t(\cdot, \cdot) \) uniformly elliptic, one easily extends (5.16) to all \( t \geq 0 \) upon increasing \( C_3 \) to some \( C_3(\alpha) \) finite. Finally, combining (5.13) and (5.16) we have that for \( C_* := \sqrt{C_3 C'} \vee (2/\theta_1) \) finite,

\[
K_{0,t}(x,y) \leq C_* \mu_{0,t}(y) e^{-\rho_t(x,y)/C_*},
\]

(5.17)

(5.17)

as stated in the RHS of (1.27).

• **Step II: matching GHKL.** With (5.17) holding for \( K_{s,s+t}(\cdot, \cdot), s \geq 0 \), it yields the bound (5.4) for \( b = 1/2 \), the probability measures \( K_{s,s+t}(x, \cdot) \) and \( \Gamma_{s+t}(\cdot) := \pi_s(\cdot) K_{s,s+t}(\cdot,y)/\mu_{s,s+t}(y) \), some \( \kappa(C_V, C, b) \geq 2 \), all \( x, y \in V \), \( s \geq 0 \) and \( \tau \geq b \). Fixing \( \varphi \geq 2(1+\kappa^2) \), \( \delta = 1/2 \) and \( \gamma \in (0, 1) \) as in (5.5), we further have that for all \( x, y \in V \), \( t \geq \tau \geq 1 \) and \( r \leq 2\varphi \sqrt{\tau} \) (with \( 4r \leq \tau \in \mathbb{N} \) in case of DTRW),

\[
K_{t-\tau,t}(x,y) \geq \gamma \sup_{z \in B(x,2\tau)} \{ K_{t-\delta\tau,t}(z,y) \}.
\]

(5.18)

(5.18)

Setting \( n_* = 1 \) for CSRW and \( n_* = \lceil \sqrt{8\varphi^2} \rceil \) for DTRW, (5.18) applies when \( r = 2\varphi \sqrt{\tau} \) and \( \tau \geq n_* \). Further, if \( d(x,y) \leq r \) then \( B(y, r) \subseteq B(x, 2\tau) \), so \( r = 2\varphi \sqrt{\tau} \geq \kappa \sqrt{\tau} \) yields by (5.4)

\[
K_{t-\tau,t}(x,y) \geq \frac{\gamma}{\pi_{t-\delta\tau}(B(y,r))} \sum_{z \in B(y,r)} \pi_{t-\delta\tau}(z) K_{t-\delta\tau,t}(z,y) \geq \frac{\gamma \mu_{t-\delta\tau,t}(\cdot)}{2C_V v(2\varphi \sqrt{\tau})}.
\]

(5.19)

(5.19)

With \( v(\cdot) \) volume doubling and \( s \mapsto \mu_{s,t}(y) \) is non-decreasing, taking \( t \geq \tau \geq n_* \) in (5.19) yields the GHKL for near-diagonal \( d(x,y) \leq 2\varphi \sqrt{\tau} \). It extends to all \( d(x,y) \leq t < n_* \) since only \( y = x \) is relevant for \( t < 1 \) (and the GHKL then trivially holds), and for DTRW having uniformly elliptic conductances implies that \( K_{0,t}(x,y) \geq (\alpha e)^{n_*} \) whenever \( d(x,y) \leq t < n_* \).

Considering hereafter \( d(x,y) \in [2\varphi \sqrt{\tau}, t] \) and \( t \geq n_* \), fix integers \( R = [2\varphi / d(x,y)] \geq 4 \) and \( \ell = [d(x,y)/R] \geq 4 \). We further find \( x_i \in V \) with \( x_0 = x \) and \( x_\ell = y \) such that \( d(x_i, x_{i+1}) \leq R \) for \( 0 \leq i \leq \ell - 1 \). Setting \( \tau = t/\ell \geq 2 \) (or its integer part for the DTRW), let \( l_0 = 0, t_{2\ell} = t \) and \( t_{2i-1} := (t_{2i} + t_{2(i-1)})/2 \), with \( t_{2i} - t_{2(i-1)} = \tau \) for CSRW, or in \{\( \tau, \tau + 1 \)} for DTRW (as needed). It is easy to check that \( \kappa \sqrt{\tau} \leq R \leq 2\varphi \sqrt{\tau} \), and further that the extra requirement \( 4R \leq \tau \in \mathbb{N} \)
which we need in case of the DTRW, holds whenever \( d(x, y) \leq t/6 \). For such \( x, y, t \) we get by Chapman-Kolmogorov and (5.18) followed by (5.4), that

\[
K_{0,t}(x, y) \geq \sum_{z \in \mathbb{B}(x, 2R)} K_{0,t_2}(x, z_1) \prod_{i=2}^{\ell-1} \left[ \sum_{z_i \in \mathbb{B}(x, R)} K_{t_2(i-1), t_2i}(z_i, z) \right] K_{t_2(\ell-1), t}(z_{\ell-1}, y)
\]

\[
\geq \gamma^{\ell-1} \left[ \sum_{z \in \mathbb{B}(x, R)} K_{0,t_2}(x, z) \prod_{i=2}^{\ell-1} \left[ \sum_{z_i \in \mathbb{B}(x, R)} K_{t_2(i-1), t_2i}(x_i-1, z) \right] K_{t_2(\ell-1), t}(x_{\ell-1}, y) \right]
\]

\[
\geq \left( \frac{\gamma}{2} \right)^{\ell-1} K_{t_2(\ell-1), t}(x_{\ell-1}, y) \geq \left( \frac{\gamma}{2} \right)^{\ell} \frac{\mu_0, t(y)}{C_V V(2C V)} \exp \left\{ -C d(x, y)^2 / t \right\},
\]

where \( d(x_{\ell-1}, y) \leq R \) so the last inequality is merely (5.19) for \( \tau' := t - t_{2\ell-1} \in [\tau / 2, \tau] \) and \( \kappa \sqrt{\tau'} \leq R \leq 2 \varphi \sqrt{\tau'} \). Consequently, with \( \tau \leq t, v(\cdot) \) volume doubling and \( \ell \geq d(x, y)^2 / (2\varphi t) \),

\[ K_{0,t}(x, y) \geq \left( \frac{\gamma}{2} \right)^{\ell} \frac{\mu_0, t(y)}{C_V V(2C V)} \exp \left\{ -C d(x, y)^2 / t \right\}, \tag{5.20} \]

for some \( C = C(C_V, \gamma, \varphi) \) finite. Note that for \( d(x, y) / t \in [1/6, 1] \) and uniformly elliptic DTRW we have that \( K_{0,t}(x, y) \geq (\alpha_0)^t \geq e^{-Cd(x,y)^2/t} \), where \( C = 1 \vee 36 \log(1/\alpha_0) \). Recalling that \( \mu_0, t(y) \leq 1 \leq C V(\sqrt{t}) \) this extends the validity of (5.20) to all \( d(x, y) \leq t \).

\section{The Perturbative Regime}

For \( \{K_t\} \) of (1.1) the transition kernels \( \{K_{s,t}\} \) are unchanged by re-scaling the conductances

\[
\pi_{u,v}(x, y) = e^{a_u - a_v} \pi_u(x, y) \quad \forall (x, y) \in E
\]

In particular, for \( \{a_t\} \) of (1.30) one has that \( a_{u'} - a_u \geq \rho \pi(u', u) \) for all \( u' \geq u \) and hence \( u \mapsto \pi_{u,v}(x) \) is non-decreasing (for each \( x \in V \)). More generally, working under the framework of Example 2.3, the Nash profiles re-scale as

\[
\hat{N}_{Q_u, \pi_u}(s) = N_{Q_u, \pi_u}(e^{a_u - a_u} s),
\]

yielding an on-diagonal transition density upper bound when \( u \mapsto \pi_{u,v}(x) \) are non-decreasing and (1.24) holds. Also, if (1.29) applies for \( a_t \) of (1.30), then \( \mu_{s,t}(\cdot) \) of (1.9) are uniformly bounded below provided \( s/t \) is.

\begin{lemma}
\label{lem:6.1}
Consider \( \{K_t\} \) of (1.1) with non-decreasing \( u \mapsto \pi_{u,v}(x) \) of (6.1), such that the non-decreasing \( t \mapsto a_t \) satisfies (1.29).
\begin{enumerate}[(a)]
\item Suppose \( \pi_\star := \inf_{t,x} \{\pi_t(x)\} > 0 \) and for some non-decreasing \( N(\cdot) \),
\[
N(s) \geq \sup_t \{N_{K^2_t, \pi_t}(s)\}.
\]
\item Then, for the dynamic (1.3), \( \psi(t) = 1/F^{-1}(t; \pi_\star, N(\cdot)) \) and \( F(\cdot) \) of (1.11),
\[
\sup_{x, y \in V} \left\{ \frac{K_{s,t}(x, y)}{\pi_t(y)} \right\} \leq e^{2A \psi \left( \frac{t-s}{6} \right)}, \quad \forall s \in [0, t]. \tag{6.3}
\]
\end{enumerate}
\end{lemma}

For the dynamic (1.5), replace \( N_{K^2_t, \pi_t}(s) \) by \( 2N_{K_t, \pi_t}(s) \) in (6.2) and \( \psi(\frac{t-s}{6}) \) on the RHS of (6.3) by \( \mathbb{E}[\psi(Z)] \), where \( Z \sim \frac{1}{2} \text{Poisson}(t - s) \).

\begin{enumerate}[(b)]
\item If \( \{a_t\} \) of (1.30) satisfies (1.28), then for \( \mu_{s,t} \) of (1.9), under either (1.3) or (1.5),
\[
\mu_{s,t}(y) \geq e^{-\gamma A \pi_t(y)}, \quad \forall y \in V, \quad \gamma \in \mathbb{N}, \quad (t + 1) \leq 2^\gamma (s + 1). \tag{6.4}
\]
\end{enumerate}
Proof. (a) Fixing $s \in [0, t]$, set $v = (t + s)/2$, or its integer part in case of DTRW. Note that $t - v \geq (t - s)/2$ and $(t + 1) \leq 4(v + 1)$. Further, with $v \in [s, t]$, we have that $K_{s,t} = K_{s,v}K_{v,t}$ and since the Markov kernel $K_{s,t}$ is an $L^\infty$-contraction, the LHS of (6.3) is bounded above by $\|K_{v,t}\|L^1(\pi_v) \to L^\infty(\pi_v)$. To bound the latter quantity, consider for $u \geq v$ the non-decreasing $u \mapsto \widehat{\pi}_{u,v}$. Since $\widehat{\pi}_{u,v} \geq \pi_u \geq c_*$ and

$$N_{K_\infty}^2(\pi ; \widehat{\pi}_{u,v})(s) = N_{K_\infty}^2(\pi_u e^{\alpha_u - \alpha_v} s) \leq N(s),$$

applying Theorem 1.1(a) for the dynamics (1.3) and $\{(K_u, \pi_{u,v}), u \in [v, t]\}$, we have that

$$\|K_{v,t}\|L^1(\pi_v) \to L^\infty(\pi_v) = e^{\alpha_t - \alpha_v} \|K_{v,t}\|L^1(\widehat{\pi}_{v,u}) \to L^\infty(\widehat{\pi}_{v,u}) \leq e^{2A} \psi\left(\frac{t - v}{3}\right)$$

(6.5) yielding (6.3). Replacing Theorem 1.1(a) by Theorem 1.1(b), the analogous argument applies for the dynamic (1.5).

(b) Fixing $x_s = x$ and $x_t = y$, we have from (1.1) that for the dynamic (1.3),

$$K_{s,t}(x, y) = \sum_{\{x_{s+1}, \ldots, x_{t-1}\}} \prod_{r=s+1}^{t} \frac{\pi_r(x_r)}{\pi_s(x)} K_r(x_r, x_{r-1}) \geq \eta_s(t) \frac{\pi_t(y)}{\pi_s(x)} \left[K_t \cdots K_{s+1}\right](y, x),$$

(6.6) where under (1.30),

$$\eta_s(t) := \inf_{z \in V} \left\{ \frac{\pi_r(z)}{\pi_r(z) + 1}\right\} \geq \prod_{r=s}^{t-1} e^{-(\alpha_t - \alpha_r)} = e^{-(\alpha_t - \alpha_s)}.$$ 

Multiplying (6.6) by $\pi_s(x)$ and summing over $x$ we see that $\mu_{s,t}(y)/\pi_t(y) \geq e^{-(\alpha_t - \alpha_s)}$, which by (1.29) is further bounded below by $e^{-\gamma A}$ whenever $(t + 1) \leq 2^k(s + 1)$ (see (5.16)). Next, recall Remark 1.7 that $\mu_{s,t}$ of the CSRW is the expected value over $L^{\sim}$-Poisson$(t - s)$ and jump times $s = T_1' < T_2' < \cdots < T_L' \leq t$ of the value $\mu_{0,L}$ for the DTRW using $\{(K_m, \pi_{T_m})\}$. Since $\alpha_{T_{m+1} - T_m} \geq \rho_{\pi}(T_{m+1}, T_m)$ for all $m \in \mathbb{N}$, due to (1.30), by the preceding argument, for each $\omega$,

$$\mu_{0,L}(y) \geq e^{-(\alpha_{T_L'} - \alpha_{T_1'})} \pi_{T_L'}(y) \geq e^{-(\alpha_t - \alpha_s)} \pi_t(y).$$

Thus, having (1.29) for $t \rightarrow a_t$, implies that (6.4) holds also for the dynamic (1.5). \qed

Proof of Proposition 1.10 (a) Recall from the proof of Lemma 2.7 that for CSRW or uniformly lazy DTRW on $G_t$, the assumed uniform Poincaré inequality and volume growth $v(r)$ with $v(r)$ doubling yield the Nash profile bound for $N(\cdot)$ of (2.16). Following the rest of the proof of Lemma 2.7 while applying Lemma 6.1(a) instead of Theorem 1.1 we deduce that the on-diagonal GHUK bound (2.15) also holds here, for some $C'(A, C_P, C_V, \alpha_I)$ finite. Next, adapting the proof of Proposition 3.3 we proceed to deduce for $T \geq 1$ the $1 \rightarrow \infty$ norm bound similar to (6.25) for the operator $K_{0,2T}$.

To this end, we use here the dual $\hat{K}_{s,t}$ of $K_{s,t}: L^2(\pi_{s,T}) \rightarrow L^2(\pi_{T,T})$ for the re-scaled non-decreasing conductances $\pi_{s,T}$ of (6.1), and $r \geq T$. Replacing (3.20), we have

$$\|K_{r,2T}^\theta\|L^1(\pi_{2T}) \rightarrow L^\infty(\pi_{2T}) = e^{\alpha_{2T} - \alpha_T} \|K_{r,2T}^\theta\|L^1(\pi_{r,2T}) \rightarrow L^\infty(\pi_{2T}) \leq e^A \|\hat{K}_{T,2T}^\theta\|^{-\theta}\|L^2(\pi_{T,T}) \rightarrow L^\infty(\pi_{T,T})$$

so it suffices to bound via Proposition 1.1 the $L^2(\pi_{T,T}) \rightarrow L^\infty(\pi_{T,T})$ norm of $\hat{K}_{T,2T}^\theta$ as in (3.24) and the $L^2(\pi_{T,T}) \rightarrow L^\infty(\pi_{T,T})$ norm of $K_{0,2T}$ as in (3.19). For the $2 \rightarrow \infty$ bound on $\|\hat{K}_{T,2T}^\theta\|^{-\theta}$, recall (3.23) that since $r \mapsto \pi_{s,T}$ are non-decreasing, $\|\hat{K}_{s,t}\|L^2(\pi_{s,T}) \rightarrow L^\infty(\pi_{T,T}) \leq 1$ and Lemma 3.5 provides the $2 \rightarrow 2$ analog of (3.22) for $\|\hat{K}_{s,t}\|^{-\theta}$. Further, the $1 \rightarrow \infty$ bound of (2.19) on $K_{s,t}$
yields the $2 \to \infty$ bound \((3.35)\) with \(\varphi(\tau) = C'v(\sqrt{T})^{-1/2}\) and in conjunction with \((3.21)\) for \(\hat{\tau}_{\nu:T}\), allows us to apply Proposition \((3.1)\) with this choice of \(\varphi(\tau)\) for the adjoint operators \(\hat{K}_{s,t}^*\). Similarly to \((3.24)\) it yields that for some \(C_2(A)\) and \(C_1\) finite,
\[
\left\| (\hat{K}_{T,2T}^*)^{-\theta} \right\|_{L^2(\hat{\tau}_{\nu:T}) \to L^\infty(\hat{\tau}_{2T:T})} \leq C_2 v(\sqrt{T})^{-1/2} \exp(C_1 \chi(\theta)T).
\]
Turning next to the $2 \to \infty$ bound on \(K_0^{\theta,T}\), utilizing the non-decreasing measures \(u \mapsto \hat{\tau}_{u:v}\) Lemma \((3.5)\) provides the $2 \to 2$ bound on \(K_v^\theta\) with \(\hat{\tau}_{u:v}\) replacing \(\pi_u\) (and \(a_u \equiv 0\)). Thus,
\[
\left\| K_v^\theta \right\|_{L^2(\pi_u) \to L^2(\pi_v)} = e^{(a_u - a_v)/2} \left\| K_v^\theta \right\|_{L^2(\pi_u) \to L^2(\pi_v)} \leq \exp(a_u - a_v + \chi(\theta)(u - v)).
\]
The non-decreasing \(t \mapsto a_t\) satisfies \((1.29)\), so Proposition \((3.1)\) establishes the bound \((3.19)\) which in turn yields the heat kernel upper bound \((1.26)\) (see our proof of Theorem \((1.6)\)(a)).

(b). As in the proof of Theorem \((1.6)\)(b) it suffices to establish the relevant GHKL for \(K_{0,2T}\). To this end, thanks to \((6.4)\) and the assumed uniform volume growth with \(v(\cdot)\) doubling, we have that \(\{\mu_{s,t}(\cdot) : s, t \in [T, 2T]\}\) are \(c\)-stable with respect to \(\pi(x) \equiv 1\) for \(c = c(C_V, A)\) and all \(T\). Moreover, in part (a) we established the bound \((1.26)\), therefore \(\{K_{s,s+t} : s, s + t \in [T, 2T]\}\) satisfy the improved GHKL of \((3.17)\). The \(\Phi\) is invariant to the re-scaling \((6.1)\) and \(a_t \leq a_T + A\) for \(t \in [T, 2T]\). Hence, considering Theorem \((1.4)\) for \((K_{s,T}, \pi_{s,t})\), yields for the crskw that \((K_t, \pi_t)\) satisfy the \(\Phi\) on \([T, 2T]\), while for DTRW we assumed such \(\Phi\), as well as uniformly elliptic conductances. Proceeding as in Step II of the proof of Theorem \((1.6)\)(b), yields the GHKL of \((1.27)\) for some \(C_* = C_*(C, \gamma, C_V, A, \alpha)\) finite and \(K_{T,2T}\) (omitting wlog the bounded factor \(\mu_{T,2T}\)). From the upper bound \((1.26)\) it further follows, as in \((5.4)\), that for some \(\kappa(C_P, C_V, \alpha_t, A)\) finite,
\[
\sum_{z \in B(x, \kappa \sqrt{T})} K_{0,T}(x, z) \geq \frac{1}{2}, \quad \forall x \in V, \ T \geq 1.
\]
With \(d(z, y)^2 \leq 2d(x, z)^2 + 2d(x, y)^2\), we get by combining \((1.27)\) and \((6.7)\), that
\[
K_{0,2T}(x, y) \geq \frac{1}{2} \inf_{z \in B(x, \kappa \sqrt{T})} \left\{ K_{T,2T}(z, y) \right\} \geq \frac{e^{-2C_*\kappa^2/2} - 2C_*d(x,y)^2/T}{2C_*v(\sqrt{T})},
\]
whenever \(d(x,y) \leq T - \kappa \sqrt{T}\). With \(G_t\) uniformly elliptic, increasing \(C_*\) (in terms of \(\alpha_t\)), such GHKL extends to all \(d(x,y) \in [T - \kappa \sqrt{T}, 2T]\) (as we have seen already after \((5.20)\)).

Proof of Proposition \((1.12)\). We refine the counter-example provided in \((HK)\) Proposition 1.4] for \(G = \mathbb{Z}\), by fixing \(\eta, \delta_n \in (0, 1/2), \) \(\delta_0 = 0\) and setting the uniformly bounded
\[
\pi_n(x, x+1) = 1 + (-1)^{n+x} \eta, \quad \pi_n(x, x) = 1 - (-1)^{n+x} \delta_n.
\]
Then, \(\pi_n(x) = 3 - (-1)^{n+x} \delta_n\) satisfy \((1.30)\) with \(a_{n+1} - a_n \leq \frac{2}{3} (\delta_n + \delta_{n+1})\), and \(K_n(x, y)\) of \((1.1)\) satisfies \((1.20)\), \((1.21)\) with \(\alpha = 1/7\). The process \(\{X_n\}\) induces on types \(A\) and \(B\) that correspond to \(n + X_n\) being even and odd, respectively, the in-homogeneous \(\{A, B\}\)-valued Markov chain of transition probabilities:
\[
q_n(A, B) = \frac{1 - \delta_n}{3 - \delta_n}, \quad q_n(A, A) = \frac{2}{3 - \delta_n}, \\
q_n(B, A) = \frac{1 + \delta_n}{3 + \delta_n}, \quad q_n(B, B) = \frac{2}{3 + \delta_n}.
\]
The uniformly bounded increments $X_{n+1} - X_n$ are zero on transitions between types $A$ and $B$ and otherwise they are $\pm 1$-valued of mean $\Delta_n(A) = \frac{2q_n}{3-\delta_n}$ and $\Delta_n(B) = -\frac{2q_n}{3+\delta_n}$, when at time $n$ the type is $A$ or $B$, respectively. Note that

$$v_n := \frac{1}{2} \sum_{i=1}^{n} (\Delta_i(A) + \Delta_i(B)) \geq \frac{2\eta}{9} a_n = O(n^{1/2+\epsilon}),$$

and starting at $X_0 = 0$ (i.e. at type $A$), since $q_n(B,A) > 1/3 > q_n(A,B)$ for all $n$, such $\{A,B\}$-valued Markov chain induces the drift $\mathbb{E}X_n \geq v_n$. Thus, from the concentration of the pair-empirical $\{A,B\}$-valued measure around its limit $[2/6, 1/6, 1/6, 2/6]$, we deduce that for some $C > 0$,

$$p_n := \mathbb{P}(|X_n| \leq Cn^{(1+\epsilon)/2}) \leq C^{-1} \exp(-Cn^{\epsilon}).$$  \hfill (6.8)

It is easy to check that if the lower bound of (1.7) held for the uniformly bounded above and below $\pi_n(\cdot)$, then necessarily $\inf_n \{p_n\} > 0$ in contradiction with (6.8). Further, even if only the upper bound of (1.6) held for $\pi_n(\cdot)$, then since $\nu > 0$ necessarily $\mathbb{P}(|X_n| > Cn^{(1+\epsilon)/2}) \to 0$, again contradicting (6.8).

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