Certain \( t \)-partite Graphs*

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Abstract: By making use of the generalized concept of orthogonality in Latin squares, certain \( t \)-partite graphs have been constructed and a suggestion for a net work system has been made.

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1. Introduction

There is much work that has been turned out on \( t \)-partite graphs and especially Liu et al [3,4,5] and Jones, Pullman and Rees [2] have studied these graphs and their chromatic numbers. In the present paper we give the construction of a \( t \)-partite graph, which is of use in communication and information systems.

A graph \( G \) is defined to be a pair \( V(G), E(G) \), where \( V(G) \) is a non-empty finite set of elements called vertices, and \( E(G) \) is a finite family of unordered pairs of (not necessarily distinct) elements of \( V(G) \), called edges. Note that the use of word family permits the existence of multiple edges. We shall call \( V(G) \), the vertex set and \( E(G) \) the edge family of \( G \). Suppose that the vertex set of graph \( G \) can be divided into two distinct sets \( V_1 \) and \( V_2 \) in such a way that every edge of \( G \) joins a vertex of \( V_1 \) to a vertex set of \( V_2 \). Then the \( G \) is said to be bipartite graph (some times denoted by \( G(V_1, V_2) \)). An alternate way of thinking of a bipartite graph is in terms of coloring its vertices with two colors say red and blue. A graph is bipartite graph if we can color each vertex red or blue in such a way that every edge has a red end and a blue end. If in a bipartite graph \( G(V_1, V_2) \) if every vertex of \( V_1 \) is joined to every vertex of \( V_2 \) then \( G \) is called a complete bipartite graph, usually denoted by \( K_{r,s} \), where \( r \) and \( s \) are the numbers of vertices in \( V_1 \) and \( V_2 \) respectively. Note that \( K_{r,s} \) has \( r+s \) vertices and \( rs \) edges. A \( k \)-partite graph is one whose vertices set can be partitioned onto \( k \) subsets so that no edge has both end in any one subset.

A complete \( k \)-partite graph is one that is simple in which each vertex is joined to every vertex that is not in the same subset. The complete \( m \)-partite graph has \( n \) vertices in which each part has either \( [n/m] \) or \( \{n/m\} \) vertices, which is denoted by \( T_{m,n} \).

We quote some of the properties of these graphs from Bondy and Murthy [1], now we quote from it

\[
\varepsilon(T_{m,n}) = \left( \binom{n-k}{2} + (m-1) \binom{k+1}{2} \right), \text{ where } k = \lfloor n/m \rfloor.
\]

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If \( G \) is a complete \( m \)-partite graph on \( n \) vertices, then \( \varepsilon(G) \subseteq \varepsilon(T_{m,n}) \) with equality only if \( G \cong T_{m,n} \).

A bipartite graph, which is having a unique bipartition, is connected.
A graph \( G \) is bipartite if and only if every circuit in \( G \) has even length.

A Latin square arrangement is an arrangement of \( n \) symbols in \( n^2 \) cells arranged in \( n \) rows and \( n \) columns such that every symbol occurs once in each row and in each column. Then \( n \) is called the order of the Latin square.

Two Latin squares \( A \) and \( B \) of the same order \( n \) are called orthogonal if the \( n^2 \) ordered pairs \((a_{i,j}, b_{i,j})\), \ldots, the pairs formed by superimposing one square on the other\ldots, are all different.

In the given set of \( N \) Latin squares, if any two Latin squares are orthogonal then that set of \( N \) squares is called set of Mutually Orthogonal Latin Squares (MOLS) of order \( n \).

Recently Mohan [6] proposed a new concept of \( t \)-orthogonality, on a set of Latin squares.

**Definition 1.1:** If \( t \)-Latin squares from a set of Latin squares of the same order \( s \), \( 2 \leq t \leq s \) are superimposed on one another and in each cell, the ordered \( t \)-tuple occurs once and only once in the resultant array, then they are \( t \)-orthogonal and the set is called the set of Mutually \( t \)-Orthogonal Latin Squares, denoted by \( M \((t-O)\) LS \).

The classical orthogonality is called as 2-orthogonality.

**Proposition 1.1:** There exists a set of \( M \((t-O)\) LS \) of side \( n \) when \( n \) is prime or \((n+1)\) is prime.

For other technical terminology refer to Wilson and Watkins [7], for applications of graphs refer to Bondy and Murthy [1], for other details refer to Liu et al [3,4,5] and for chromaticity Jones et al [2].

2. Method of construction

Let there be a set of mutually \( t \)-orthogonal Latin squares. Then, form \( t \) sets of vertices say \( A,B,C,D,E, \ldots \), such that each set is having \( (a_1, a_2, \ldots, a_n) \). After superimposing some \( t \) Latin squares of \( M \((t-O)\) LS \), each cell in the resultant array is a \( t \)-tuple, and each \( t \)-tuple comes only once in the array. Consider that each \( t \)-tuple is a chain of edges, as \( a_1 \rightarrow a_2 \rightarrow a_3\ldots \rightarrow a_t \). This forms a communication channel. And in the ordered \( t \)-tuple, the first co-ordinate belongs to the first Latin squares, the second co-ordinate belongs to the second Latin squares and so on of the set chosen. Thus we get a network system. For each ordered \( t \)-tuple; we have to consider only \( n^2 \) tuples leaving \( \left( \binom{n}{t} - n^2 \right) \)-tuples aside such that \( t \)-tuple should come only once.

In certain network system, we do require certain channel only and certain other channels are to be prohibited.

In such situations this type of \( t \)-partite graphs are more useful.
3. Some illustrations:

Example 3.1. For an example we construct Latin squares of order 4, following the method given in [6],

\[(a_{ij}) = (i \times j) \mod 5, \quad i = 1, 2, 3, 4, \quad j = 1, 2, 3, 4,\]

then we have the Latin square as follows:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3 \\
3 & 1 & 4 & 2 \\
4 & 3 & 2 & 1 \\
\end{array}
\]

By applying \(\pi A_i = A_{i+1} \mod n\) on this we get three other Latin squares as

\[
\begin{array}{cccccccc}
2 & 4 & 1 & 3 & 3 & 1 & 4 & 2 & 4 & 3 & 2 & 1 \\
3 & 1 & 4 & 2 & 4 & 3 & 2 & 1 & 1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1 & 1 & 2 & 3 & 4 & 2 & 4 & 1 & 3 \\
1 & 2 & 3 & 4 & 2 & 4 & 1 & 3 & 3 & 1 & 4 & 2 \\
\end{array}
\]

Then these 4-Latin squares form M (t-O) LS, where t=3, 4.

Now consider t=3 and form the array as follows

\[
\begin{array}{cccccccc}
(1 & 2 & 3) & (2 & 4 & 1) & (3 & 1 & 4) & (4 & 3 & 2) \\
(2 & 3 & 4) & (4 & 1 & 3) & (1 & 4 & 2) & (3 & 2 & 1) \\
(3 & 4 & 1) & (1 & 3 & 2) & (4 & 2 & 3) & (2 & 1 & 4) \\
(4 & 1 & 2) & (3 & 2 & 4) & (2 & 3 & 1) & (1 & 4 & 3) \\
\end{array}
\]

And now, the corresponding network system (with multiple communications) is as follows:
Where the first, second, and third co-ordinates of the 3-tuples belongs to $A$, $B$, and $C$ respectively.

**Note 3.1.** If $(n+1)$ is prime, then in the $t$-partite graphs they have multiple edges, where $t = 3, 4, \ldots, n$.

**Note 3.2.** If $n$, is prime, we will get these $t$-partite graphs without multiple edges, where $t = 2, 3, \ldots, n-1$.

**Note 3.3.** When 2-orthogonal, we get bipartite graphs.

**Example 3.2:** For $n = 5$, since it is prime, we get 2, 3, 4, 5-orthogonal. If we take 3-orthogonal, by adopting $(i+hj) \mod 5$, we have

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $h=1$ | 2 | 3 | 4 | 5 | 1 |
| $i=1,2,3,4,5$ | 3 | 4 | 5 | 1 | 2 |
| $j=1,2,3,4,5$ | 4 | 5 | 1 | 2 | 3 |

The 4-orthogonal array is given by

|   | 2 | 3 | 4 | 5 | 1 |
|---|---|---|---|---|---|
| $h=2$ | 3 | 5 | 2 | 4 | 1 |
| $i=1,2,3,4,5$ | 4 | 1 | 3 | 5 | 2 |
| $j=2,4,1,3,5$ | 5 | 2 | 4 | 1 | 3 |

The 4-orthogonal array is given by
(2345) (3524) (4253) (5432) (1111)  
(3451) (4135) (5314) (1543) (2222)  
(4512) (5241) (1425) (2154) (3333)  
(5123) (1352) (2531) (3215) (4444)  
(1234) (2413) (3142) (4321) (5555)

Now the corresponding network system (with single communication) is as follows:

3. In application perspective we have:

The relational structure among families is very much like a complete graph and the same holds for friendship communities (take for instance the scholars of a school-class, everybody knows the others). Considering a large network of social relationships it is therefore natural to decompose the graph into a (normally not disjoint) union of complete graph. The linkage structure between the complete graphs is then a natural quantity to measure the overlap between the family-like communities (respectively complete graphs). The search for the largest complete subgraph in a given graph is a classical problem in algorithmic complexity and known to be NP-complete. But in many applications it turns out the search can be efficiently be done since the complete subgraphs are not so large. Another nice thing about complete graphs is the relative simple analysis of processes taking place on such graphs, e.g. stochastic processes can usually in this case well described by the dynamics of their expectation values. There is of course much more to say
(also about graphs which are close to complete graphs but still almost complete).

4. Conclusion

In this method, we get distinct \( t \)-tuples in the resultant array constructed, which give out distinct communication channels in total but having multiple communications or single communication from peripheral to peripheral and that are of much use in many network systems.

Further work in this direction can be seen in a sequel to this paper to appear shortly.

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