A double moving average control chart: Discussion

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ABSTRACT
A double moving average (DMA) control chart has been proposed in the literature for monitoring the process mean. Several studies have also been done based on this scheme. Unfortunately, the variance of DMA statistic that has been used in these studies is not correct. In this article, we provide the correct variance of the DMA chart and through a simulation study, we evaluate its performance. It is shown that the DMA chart is more effective than the moving average (MA) chart in detecting small shifts in the process mean and vice versa for moderate to large shifts. Moreover, the DMA chart, even with a small value of span \( w \), has similar, and in some cases better, detection ability than the EWMA and CUSUM charts, especially for moderate and large shifts.

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1. Introduction

Control charts are the most important tool of Statistical Process Control (SPC) and are used to monitor the location or dispersion parameter. Shewhart (1926) first introduced the control chart technique and since then, control charts have been used in many manufacturing processes. Although the Shewhart control charts are very easy to use and also very effective in detecting large shifts, they are ineffective in detecting small to moderate shifts. For this purpose, Page (1954) developed the cumulative sum (CUSUM) chart and Roberts (1959, 1966) proposed the moving average (MA) and the exponentially weighted moving average (EWMA) control charts. These charts are memory-type as their charting statistics are based not only on the current information, but also on the past.

The MA control chart with span \( w \) is based on monitoring the average of \( w \) most current observations. It is less sensitive than the EWMA and CUSUM charts in detecting small to moderate shifts, but more effective for large shifts. Many researchers have studied MA charts. Wong, Gan, and Chang (2004) proposed a single design procedure for the MA and the combined MA-Shewhart charts. Khoo (2004a, 2004b) studied the MA chart for monitoring binomial and Poisson distributed data. Khoo and Yap (2004) developed a MA chart for monitoring both upward and downward shifts in the mean and/or variability of a process. Khan, Aslam, and Jun (2016) proposed an EWMA chart for exponential distributed data using the MA statistic. Other works on the MA charts are those of Chen and Yang (2002), Zhang et al. (2004) and Areepong (2012).

Khoo and Wong (2008), motivated by the works of Shamma, Amin, and Shamma (1991) and Shamma and Shamma (1992), developed the double moving average (DMA) control chart for normally distributed data in an effort to improve the performance of MA chart in detecting small

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to moderate shifts. Many authors, such as Areepong and Sukparungsee (2011), Areepong (2016), Phant, Sukparungsee, and Areepong (2016), Bualuang, Areepong, and Sukparungsee (2017), Aslam et al. (2017), Phantu, Sukparungsee, and Areepong (2018) and Adeoti, Akomolafe, and Adebola (2019), have studied the DMA control chart. Unfortunately, the variance of the DMA statistic, computed by Khoo and Wong (2008), was not correct. Thus, the results of previous studies are unreliable.

In the present article, we compute the correct variance of the DMA statistic and we also study the DMA chart de novo. The rest of this article is organized as follows. In Sec. 2, we give a brief description of the DMA control chart as proposed by Khoo and Wong (2008) while in Sec. 3, we compute the correct variance of the DMA statistic in order to develop the DMA chart with correct control limits. In Sec. 4, we evaluate via Monte Carlo simulations the performance of the DMA chart and in Sec. 5, we compare it with the MA, EWMA and CUSUM charts. An illustrative example with a real dataset is provided in Sec. 6. Finally, conclusions are summarized in Sec. 7.

2. The double moving average control chart by Khoo and Wong (2008)

Suppose that \( X_{ij} \), with \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \) is the \( j \)th observation in the \( i \)th sample or subgroup of size \( n \geq 1 \). Moreover, it is assumed that the observations \( X_{ij} \) are independent and identically distributed random variables and the underlying process is normally distributed, i.e., \( X_{ij} \overset{iid}{\sim} N(\mu_0, \sigma^2) \), where \( \mu_0 \) and \( \sigma^2 \) are the in-control (IC) process mean and variance, respectively. It is well-known that the subgroup averages \( \bar{X}_1, \bar{X}_2, \ldots \) are computed by \( \bar{X}_i = (X_{i1} + X_{i2} + \ldots + X_{in})/n, i = 1, 2, \ldots, m \), and follow the normal distribution with mean and variance equal to \( \mu_0 \) and \( \sigma^2/n \), respectively.

The moving average statistic \( MA_i \) of span \( w \) at time \( i \) is defined as (Montgomery 2013)

\[
MA_i = \begin{cases} \sum_{j=1}^{i} X_j / i, & \text{for } i < w, \\ \sum_{j=w+1}^{i} X_j / w, & \text{for } i \geq w. \end{cases}
\]

(1)

The mean or expected value of the statistic \( MA_i \) is \( \mu_0 \) and its variance is given by

\[
Var(MA_i) = \begin{cases} \frac{\sigma^2}{m}, & \text{for } i < w, \\ \frac{\sigma^2}{nw}, & \text{for } i \geq w. \end{cases}
\]

(2)

Khoo and Wong (2008) proposed the double moving average (DMA) control chart which is based on computing the MA of subgroup averages twice. The calculation of the DMA, statistic of span \( w \) at time \( i \) is defined as

\[
DMA_i = \begin{cases} \sum_{j=1}^{i} MA_j / i, & \text{for } i < w, \\ \sum_{j=w+1}^{i} MA_j / w, & \text{for } i \geq w. \end{cases}
\]

(3)

They showed that the mean value of \( DMA_i \) statistic for \( i \geq w \) is equal to

\[
E(DMA_i) = \frac{1}{w} E \left( \sum_{j=i-w+1}^{i} MA_j \right) = \frac{1}{w} (w\mu_0) = \mu_0,
\]

(4)

which is the same as for periods \( i < w \). They also calculated the variance of \( DMA_i \) statistic and they found that for \( w > 2 \) is given by
\[
\text{Var}(\text{DMA}_i) = \begin{cases} 
\frac{\sigma^2}{n} \sum_{j=1}^{i} \frac{1}{j}, & \text{for } i \leq w, \\
\frac{\sigma^2}{nw^2} \sum_{j=i-w+1}^{w} \left( \frac{i-w+1}{i} \right), & \text{for } w < i < 2w - 1, \\
\frac{\sigma^2}{nw^2}, & \text{for } i \geq 2w - 1.
\end{cases}
\] (5)

For the case where \( w = 2 \), they noted that the variance of \( \text{DMA}_i \) statistic is obtained by using the first and third branches of Equation (5).

As a result, they defined the control limits of the DMA chart for \( w > 2 \) as follows

\[
\text{UCL} / \text{LCL} = \begin{cases} 
\mu_0 + \frac{L \sigma}{\sqrt{n}} \sqrt{\sum_{j=1}^{i} \frac{1}{j}}, & \text{for } i \leq w, \\
\mu_0 + \frac{L \sigma}{n^{1/2}} \sqrt{\sum_{j=i-w+1}^{w} \left( \frac{i-w+1}{i} \right)}, & \text{for } w < i < 2w - 1, \\
\mu_0 + \frac{L \sigma}{n^{1/2}}, & \text{for } i \geq 2w - 1,
\end{cases}
\] (6)

where \( L > 0 \) is the width of control limits. When \( w = 2 \), the control limits of the DMA chart are given by the first and third branch of Equation (6). The centerline (CL) of the DMA chart is the IC value of process mean \( \mu_0 \). The DMA chart is constructed by plotting the \( \text{DMA}_i \) statistics versus the sample number or time \( i \). If a charting statistic exceeds the control limits, then the process is considered to be out-of-control (OOC). Otherwise, the process is IC.

### 3. The corrected control limits of the DMA chart

Unfortunately, the mathematical expression of the variance of \( \text{DMA}_i \) statistic reported in Equation (5) is not correct as the terms of covariance between \( \text{MA}_i \) statistics have been ignored.

For example, assume that \( w = 4 \) and \( i = 3 \). Then, as per definition

\[
\begin{align*}
\text{DMA}_3 &= \sum_{j=1}^{3} \text{MA}_j = \text{MA}_1 + \text{MA}_2 + \text{MA}_3 = \bar{X}_1 + \frac{\bar{X}_2 + \bar{X}_3}{2} = \frac{\bar{X}_1 + \bar{X}_2 + \bar{X}_3}{3} \\
&= \frac{11}{18} \bar{X}_1 + \frac{5}{18} \bar{X}_2 + \frac{2}{18} \bar{X}_3
\end{align*}
\]

Therefore,

\[
\text{Var}(\text{DMA}_3) = \frac{25 \sigma^2}{54n}
\]

On the other hand, according to the formula given in the first branch of Equation (5), we have \( \text{Var}(\text{DMA}_3) = \frac{11 \sigma^2}{54n} \). As we will prove later, the added term of \( \frac{14 \sigma^2}{54n} \) corresponds to the covariances between \( \text{MA}_i \) statistics.

In the following lines, we compute the mean and variance of \( \text{DMA}_i \) statistic.

- For \( i < w \), we have

\[
\sum_{j=1}^{i} \text{MA}_j = \sum_{j=1}^{i} \left( \sum_{l=1}^{j} \frac{1}{j} \right) = \sum_{l=1}^{i} \left( \sum_{j=l}^{i} \frac{1}{j} \right) = \sum_{l=1}^{i} a_l \bar{X}_l,
\]

where \( a_l = \sum_{j=1}^{l} 1/j \). Therefore, for \( i < w \), we have

\[
\text{DMA}_i = \frac{\sum_{l=1}^{i} a_l \bar{X}_l}{i}.
\]

Now, for \( i < w \), the mean value of \( \text{DMA}_i \) statistic is
\[ E(DMA_i) = E\left(\sum_{l=1}^{i} a_l \bar{X}_l \right) = \left( \sum_{l=1}^{i} a_l \right) \mu_0 = \mu_0, \]

as

\[ \sum_{l=1}^{i} a_l = \sum_{l=1}^{i} \sum_{j=1}^{i} \frac{1}{j} = \sum_{j=1}^{i} \sum_{l=1}^{j} \frac{1}{j} = i. \]

Also, the variance of \( DMA_i \) statistic is

\[ \text{Var}(DMA_i) = \sum_{l=1}^{i} a_l^2 \sigma^2 / ni^2. \]

We can simplify \( \sum_{l=1}^{i} a_l^2 \) as follows

\[
\sum_{l=1}^{i} a_l^2 = \sum_{l=1}^{i} \left( \sum_{j=1}^{i} \frac{1}{j} \right)^2 = \sum_{l=1}^{i} \left[ \sum_{j=1}^{i} \frac{1}{j^2} + 2 \sum_{1 \leq j < l \leq i} \frac{1}{jl} \right] \\
= \sum_{l=1}^{i} \sum_{j=1}^{l} \frac{1}{l^2} + 2 \sum_{j=1}^{i} \sum_{l=1}^{j} \frac{1}{jl} = \sum_{j=1}^{i} \frac{1}{j} + 2 \sum_{l=1}^{i} \sum_{j=1}^{l} \frac{1}{jl}. \]

So, an alternative derivation of \( \text{Var}(DMA_i) \) is given by

\[ i^2 \text{Var}(DMA_i) = \text{Var} \left( \sum_{j=1}^{i} MA_j \right) = \sum_{j=1}^{i} \text{Var}(MA_j) + 2 \sum_{1 \leq j < l \leq i} \text{Cov}(MA_j, MA_l). \]

But, for \( 1 \leq j \leq i \),

\[ \text{Var}(MA_j) = \frac{\sigma^2}{nj}, \]

and, for \( 1 \leq j_1 < j_2 \leq i \),

\[ \text{Cov}(MA_{j_1}, MA_{j_2}) = \text{Cov} \left( \frac{1}{j_1} \sum_{h=1}^{j_1} \bar{X}_h, \frac{1}{j_2} \sum_{h=1}^{j_2} \bar{X}_h \right) = \frac{1}{j_1 j_2} \sum_{h=1}^{j_1} \sum_{h=1}^{j_2} \frac{\sigma^2}{n} = \frac{\sigma^2}{nj_1 j_2}. \]

Therefore, the variance of \( DMA_i \) statistic is computed by

\[ \text{Var}(DMA_i) = \frac{\sigma^2}{ni^2} \left[ \sum_{j=1}^{i} \frac{1}{j} + 2 \sum_{l=1}^{i} \sum_{j=l+1}^{l} \frac{1}{jl} \right] = \sum_{l=1}^{i} a_l^2 \sigma^2 / ni^2. \]

Back to the previous example, as per our formula, we have \( a_1 = 11/6, a_2 = 5/6, a_3 = 1/3 \) and as a result

\[ \text{Var}(DMA_3) = \frac{\sum_{l=1}^{3} a_l^2 \sigma^2}{9n} = \frac{25\sigma^2}{54n}. \]

- For \( w \leq i < 2w - 1 \), the \( DMA_i \) statistic is defined as

\[ DMA_i = \frac{\sum_{j=i-w+1}^{i} MA_j}{w}. \]
Its mean value is,

\[ E(DMA_i) = \frac{\sum_{j = i-w+1}^{i} E(MA_j)}{w} = \mu_0, \]

and its variance is computed by

\[ Var(DMA_i) = \frac{1}{w^2} \left[ Var \left( \sum_{j = i-w+1}^{i} MA_j \right) \right] = \frac{1}{w^2} \left[ Var \left( \sum_{j_1 = i-w+1}^{w-1} MA_{j_1} + \sum_{j_2 = w}^{i} MA_{j_2} \right) \right]. \]

Now,

\[ MA_{j_1} = \frac{1}{j_1} \sum_{l_1 = 1}^{j_1} X_{l_1} \quad \text{and} \quad MA_{j_2} = \frac{1}{w} \sum_{l_2 = j_2-w+1}^{j_2} X_{l_2}. \]

For \( i - w + 1 \leq j_1 \leq w - 1 \),

\[ Var(MA_{j_1}) = Var \left( \frac{1}{j_1} \sum_{l_1 = 1}^{j_1} X_{l_1} \right) = \frac{\sigma^2}{j_1 n}. \]

For \( i - w + 1 \leq j_{11} < j_{12} \leq w - 1 \),

\[ Cov(MA_{j_{11}}, MA_{j_{12}}) = Cov \left( \frac{1}{j_{11}} \sum_{l_{11} = 1}^{j_{11}} X_{l_{11}}, \frac{1}{j_{12}} \sum_{l_{12} = 1}^{j_{12}} X_{l_{12}} \right) = \frac{\sigma^2}{j_{11} j_{12} n}. \]

For \( i - w + 1 \leq j_1 \leq w - 1, w \leq j_2 \leq i \),

\[ Cov(MA_{j_1}, MA_{j_2}) = Cov \left( \frac{1}{w} \sum_{l_2 = j_2-w+1}^{j_2} X_{l_2} \right) = \frac{(j_1 - j_2 + w)\sigma^2}{j_1 wn}. \]

For \( w \leq j_2 \leq i \),

\[ Var(MA_{j_2}) = Var \left( \frac{1}{w} \sum_{l_2 = j_2-w+1}^{j_2} X_{l_2} \right) = \frac{\sigma^2}{wn}. \]

For \( w \leq j_{21} < j_{22} \leq i \),

\[ Cov(MA_{j_1}, MA_{j_2}) = Cov \left( \frac{1}{w} \sum_{l_2 = j_{21}-w+1}^{j_{21}} X_{l_2}, \frac{1}{w} \sum_{l_2 = j_{21}-w+1}^{j_{21}} X_{l_2} \right) = \frac{(j_{21} - j_{22} + w)\sigma^2}{w^2 n}. \]

Therefore, we have

\[ w^2 Var(DMA_i) = Var \left( \sum_{j_1 = i-w+1}^{w-1} MA_{j_1} + \sum_{j_2 = w}^{i} MA_{j_2} \right) \]

\[ = \sum_{j_1 = i-w+1}^{w-1} Var(MA_{j_1}) + 2 \sum_{i-w+1 \leq j_{11} < j_{12} \leq w-1} Cov(MA_{j_{11}}, MA_{j_{12}}) \]

\[ + 2 \sum_{i-w+1 \leq j_{21} < j_{22} \leq i} Cov(MA_{j_{21}}, MA_{j_{22}}) + \sum_{j_2 = w}^{i} Var(MA_{j_2}) \]

\[ + 2 \sum_{w \leq j_{21} < j_{22} \leq i} Cov(MA_{j_{21}}, MA_{j_{22}}). \]
Thus, for \( w \leq i < 2w - 1 \), the variance of \( D_{MA_i} \) statistic is given by

\[
\text{Var}(D_{MA_i}) = \left(\frac{\sigma^2}{nw^2}\right) \left[ \sum_{j=w}^{i-1} \frac{1}{j_1} + \sum_{i-w+1}^{w-1} \frac{2}{j_1} + \sum_{j_1=i-w+1}^{w-1} \sum_{j_2=i}^{w-1} \frac{2(j_1 - j_2 + w)}{j_1 j_2} \right]
\]

- For \( i \geq 2w - 1 \), the \( D_{MA_i} \) statistic is defined as

\[
D_{MA_i} = \sum_{j=i-w+1}^{i} MA_j
\]

The mean value of \( D_{MA_i} \) statistic is

\[
E(D_{MA_i}) = E\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w}\right) = \frac{1}{w} \sum_{j=i-w+1}^{i} E(MA_j) = \mu_0
\]

and its variance is computed by

\[
w^2 \text{Var}(D_{MA_i}) = \text{Var}\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w}\right) = \sum_{j=i-w+1}^{i} \text{Var}(MA_j) + 2 \sum_{i-w+1 \leq j_1 < j_2 \leq i} \text{Cov}(MA_{j_1}, MA_{j_2}).
\]

Now, for \( i - w + 1 \leq j \leq i \),

\[
\text{Var}(MA_j) = \text{Var}\left(\frac{1}{w} \sum_{l=j-w+1}^{j} \bar{X}_{l}\right) = \frac{\sigma^2}{wn},
\]

and, for \( i - w + 1 \leq j_1 < j_2 \leq i \),

\[
\text{Cov}(MA_{j_1}, MA_{j_2}) = \text{Cov}\left(\frac{1}{w} \sum_{l=j_1-w+1}^{j_1} \bar{X}_{l}, \frac{1}{w} \sum_{l=j_2-w+1}^{j_2} \bar{X}_{l}\right) = \frac{(j_1 - j_2 + w)\sigma^2}{w^2 n}.
\]

Therefore, for \( i \geq 2w - 1 \), the variance of \( D_{MA_i} \) statistic is given by

\[
\text{Var}(D_{MA_i}) = \frac{1}{w^2} \left[ \sum_{j=i-w+1}^{i} \text{Var}(MA_j) + 2 \sum_{i-w+1 \leq j_1 < j_2 \leq i} \text{Cov}(MA_{j_1}, MA_{j_2}) \right]
= \frac{\sigma^2}{nw^2} \left[ 1 + \sum_{i-w+1 \leq j_1 < j_2 \leq i} \frac{2(j_1 - j_2 + w)}{w^2} \right].
\]

Thus, we may conclude that, for any \( i \), the mean value of \( D_{MA_i} \) statistic is equal to \( \mu_0 \) and about its variance, we have

for \( i < w \)

\[
\text{Var}(D_{MA_i}) = \frac{\sigma^2}{n \mu_0^2} \left[ \sum_{j=1}^{i} \frac{1}{j} + 2 \sum_{j_1=1}^{i-1} \sum_{j_2=j_1+1}^{i} \frac{1}{j_1 j_2} \right] = \frac{\sum_{j=1}^{i} a_j^2 \sigma^2}{n \mu_0^2}, \tag{7}
\]
for \( w \leq i < 2w - 1 \),

\[
\text{Var}(DMA_i) = \left( \frac{\sigma^2}{w^2 n} \right) \left[ \sum_{j=1}^{w-1} \frac{1}{j} + \sum_{i-w+1 \leq j \leq i} \frac{2}{j/i} + \sum_{i-w+1 \leq j \leq w} 2(i/j - j/w + w) \right].
\]

and for \( i \geq 2w - 1 \)

\[
\text{Var}(DMA_i) = \left( \frac{\sigma^2}{w^2 n} \right) \left[ 1 + \sum_{i-w+1 \leq j \leq w} 2(i/j - j/w + w) \right].
\]

In the “Online Supplement”, we compute the variance of \( DMA_i \) statistic for span \( w = 5 \).

The corrected control limits of the DMA chart are given by

\[
\text{UCL}/\text{LCL} = \mu_0 \pm L \sqrt{\text{Var}(DMA_i)},
\]

where \( \text{Var}(DMA_i) \) is given by Equation (7) if \( i < w \), Equation (8) if \( w \leq i < 2w - 1 \) and Equation (9) if \( i \geq 2w - 1 \). When \( w = 2 \), the control limits of the DMA chart are computed based on Equations (7) and (9).

4. Performance evaluation of DMA chart

The performance of a control chart is usually measured in terms of its run-length distribution. The run-length is the number of statistics that must be plotted in a control chart before initiating an OOC signal. The average run-length (ARL) is defined as the expected number of charting statistics before initiating an OOC signal. The average run-length (ARL) is defined as the expected number of charting statistics before initiating an OOC signal. The average run-length (ARL) is defined as the expected number of charting statistics before initiating an OOC signal. The average run-length (ARL) is defined as the expected number of charting statistics before initiating an OOC signal.

The run-length distribution of the DMA chart is calculated performing Monte Carlo simulations in R with 10,000 repetitions. Without loss of generality, we consider that the underlying process for the IC condition follows the \( N(0, 1) \) distribution. Table 1 presents the \( L \) values of DMA charts with \( w = 2, 3, 4, 5, 8, 10, 12 \) and 15 for different \( \text{ARL}_0 \) values. It can be seen that as the value of \( w \) increases, the value of \( L \) decreases in order to obtain the desired \( \text{ARL}_0 \) value.

To study the performance of the DMA chart, we use some of \( L \) values shown in Table 1. We point out that the zero-state ARL performance is studied, as it is assumed that the shift occurs at the start. As the normal distribution is symmetric, we consider only positive shifts (in units of standard deviation) of the process mean; these are \( \delta = 0.2, 0.4, 0.6, 0.8, 1.0, 1.25, 1.5, 2.0 \) and 3.0. The results are similar for negative shifts. Moreover, the performance is evaluated for the case of individual measurements (\( n = 1 \)) and subgrouped data of size \( n = 5 \).

| \( \text{ARL}_0 \) | \( w = 2 \) | \( w = 3 \) | \( w = 4 \) | \( w = 5 \) | \( w = 8 \) | \( w = 10 \) | \( w = 12 \) | \( w = 15 \) |
|------------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 100              | 2.509      | 2.390      | 2.291      | 2.201      | 2.010      | 1.915      | 1.830      | 1.720      |
| 150              | 2.650      | 2.545      | 2.448      | 2.365      | 2.180      | 2.085      | 2.010      | 1.912      |
| 200              | 2.751      | 2.650      | 2.566      | 2.481      | 2.300      | 2.210      | 2.131      | 2.050      |
| 250              | 2.822      | 2.728      | 2.641      | 2.569      | 2.396      | 2.307      | 2.233      | 2.145      |
| 300              | 2.881      | 2.790      | 2.710      | 2.634      | 2.465      | 2.380      | 2.305      | 2.210      |
| 370              | 2.951      | 2.864      | 2.780      | 2.709      | 2.547      | 2.464      | 2.388      | 2.301      |
| 500              | 3.045      | 2.968      | 2.886      | 2.820      | 2.650      | 2.580      | 2.510      | 2.428      |
Tables 2 and 3 display the ARL and SDRL (in the parenthesis) values given an \( ARL_0 \approx 200 \) and 370, respectively. The results clearly show that the performance of the DMA chart is better as the value of \( w \) increases. It is also noted that the new results based on the correct control limits indicate that the DMA chart is less effective in detecting small shifts, but more effective in detecting moderate to large shifts than the initial study of DMA chart (Khoo and Wong 2008).

5. Comparison study

To compare the performance of control charts, it is recommended to have a similar desired value of \( ARL_0 \). The chart with the smaller \( ARL_1 \) value in a specific shift can detect it more quickly than the other charts. In this section, we compare the performance of the DMA chart with the MA, EWMA and CUSUM charts using the zero-state ARL measure. A brief description of these charts is given in the following lines.

The MA chart is based on plotting the MA statistic given by Equation (1) and its control limits are given by

\[
L = 2.751, 2.650, 2.566, 2.481, 2.300, 2.210, 2.131, 2.050
\]

\[
\delta = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.25, 1.5, 2.0, 3.0
\]

\[
w = 2, 3, 4, 5, 8, 10, 12, 15
\]

\[
\begin{array}{cccccccccccc}
    n & \delta & L = 2.751 & 2.650 & 2.566 & 2.481 & 2.300 & 2.210 & 2.131 & 2.050 \\
    \hline
    1 & 0.0 & 201.7 & 199.8 & 200.3 & 200.4 & 200.4 & 200.6 & 199.1 & 199.9 \\
    & (202.7) & (197.6) & (199.3) & (200.2) & (200.3) & (200.3) & (202.5) & (202.2) & (200.3) \\
    & 0.2 & 143.7 & 128.3 & 119.2 & 112.2 & 95.7 & 90.9 & 85.6 & 80.9 \\
    & (143.7) & (125.9) & (116.1) & (110.6) & (92.7) & (88.1) & (83.3) & (78.1) & (73.7) \\
    & 0.4 & 74.9 & 58.3 & 50.2 & 45.0 & 37.0 & 34.3 & 32.6 & 31.6 \\
    & (72.8) & (55.9) & (47.6) & (42.5) & (33.4) & (29.5) & (27.3) & (25.7) & (24.3) \\
    & 0.6 & 38.7 & 28.4 & 24.6 & 21.5 & 18.9 & 18.0 & 17.6 & 17.6 \\
    & (37.7) & (24.0) & (22.2) & (18.6) & (14.5) & (13.2) & (12.3) & (12.2) & (12.2) \\
    & 0.8 & 21.2 & 15.8 & 14.1 & 12.9 & 11.8 & 11.7 & 11.6 & 12.0 \\
    & (19.6) & (14.0) & (11.3) & (9.8) & (7.8) & (7.6) & (7.6) & (7.6) & (7.6) \\
    & 1.0 & 13.0 & 10.1 & 9.1 & 8.6 & 8.4 & 8.4 & 8.3 & 8.3 \\
    & (11.6) & (8.2) & (6.6) & (5.9) & (5.2) & (5.4) & (5.7) & (6.2) & (6.2) \\
    & 1.25 & 7.8 & 6.4 & 6.1 & 6.0 & 6.1 & 6.1 & 5.9 & 5.7 \\
    & (6.4) & (4.6) & (3.9) & (3.6) & (3.8) & (4.2) & (4.4) & (4.7) & (4.7) \\
    & 1.5 & 5.3 & 4.6 & 4.5 & 4.6 & 4.5 & 4.3 & 4.1 & 4.0 \\
    & (3.9) & (2.9) & (2.6) & (2.7) & (3.1) & (3.3) & (3.3) & (3.4) & (3.4) \\
    & 2.0 & 3.0 & 2.9 & 2.9 & 2.9 & 2.7 & 2.5 & 2.4 & 2.4 \\
    & (1.8) & (1.6) & (1.8) & (1.9) & (2.0) & (1.9) & (1.8) & (1.8) & (1.8) \\
    & 3.0 & 1.6 & 1.5 & 1.5 & 1.4 & 1.3 & 1.3 & 1.2 & 1.2 \\
    & (0.8) & (0.8) & (0.8) & (0.8) & (0.7) & (0.6) & (0.6) & (0.5) & (0.5) \\
    5 & 0.0 & 200.2 & 200.3 & 200.8 & 200.7 & 200.4 & 200.2 & 200.6 & 201.2 \\
    & (196.1) & (196.3) & (198.9) & (197.2) & (205.3) & (199.6) & (204.4) & (205.0) & (205.0) \\
    & 0.2 & 62.6 & 49.1 & 42.5 & 37.5 & 30.6 & 28.7 & 27.2 & 26.8 \\
    & (62.2) & (48.1) & (40.2) & (34.5) & (26.5) & (23.8) & (21.9) & (20.5) & (20.5) \\
    & 0.4 & 16.4 & 12.5 & 11.3 & 10.5 & 10.0 & 9.9 & 9.9 & 10.0 \\
    & (15.1) & (10.5) & (8.7) & (7.6) & (6.3) & (6.5) & (7.0) & (7.0) & (7.0) \\
    & 0.6 & 6.5 & 5.6 & 5.5 & 5.4 & 5.4 & 5.3 & 5.2 & 4.9 \\
    & (5.1) & (3.9) & (3.3) & (3.2) & (3.5) & (3.8) & (4.0) & (4.0) & (4.0) \\
    & 0.8 & 3.6 & 3.4 & 3.5 & 3.4 & 3.3 & 3.2 & 3.0 & 2.8 \\
    & (2.4) & (2.0) & (2.0) & (2.1) & (2.4) & (2.4) & (2.4) & (2.4) & (2.3) \\
    & 1.0 & 2.4 & 2.4 & 2.4 & 2.4 & 2.2 & 2.0 & 1.9 & 1.8 \\
    & (1.4) & (1.4) & (1.5) & (1.5) & (1.5) & (1.4) & (1.4) & (1.4) & (1.3) \\
    & 1.25 & 1.7 & 1.7 & 1.6 & 1.6 & 1.5 & 1.4 & 1.4 & 1.3 \\
    & (0.9) & (1.0) & (1.0) & (0.9) & (0.9) & (0.8) & (0.7) & (0.7) & (0.7) \\
    & 1.5 & 1.3 & 1.3 & 1.3 & 1.2 & 1.2 & 1.2 & 1.1 & 1.1 \\
    & (0.6) & (0.6) & (0.6) & (0.6) & (0.5) & (0.4) & (0.4) & (0.4) & (0.4) \\
    & 2.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\
    & (0.2) & (0.2) & (0.2) & (0.2) & (0.1) & (0.1) & (0.1) & (0.1) & (0.1) \\
    & 3.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\
    & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0)
\]

To compare the performance of control charts, it is recommended to have a similar desired value of \( ARL_0 \). The chart with the smaller \( ARL_1 \) value in a specific shift can detect it more quickly than the other charts. In this section, we compare the performance of the DMA chart with the MA, EWMA and CUSUM charts using the zero-state ARL measure. A brief description of these charts is given in the following lines.

The MA chart is based on plotting the MA statistic given by Equation (1) and its control limits are given by
The process is considered to be OOC if any plotted point $MA_i$ exceeds the control limits.

The EWMA chart in based on plotting the statistic

$$Z_i = \lambda X_i + (1 - \lambda)Z_{t-1},$$

where $0 < \lambda \leq 1$ is the smoothing constant and $Z_0 = \mu_0$. The control limits of the EWMA chart are given by

$$UCL/LCL = \begin{cases} 
\mu_0 \pm L\frac{\sigma}{\sqrt{ni}} , & \text{for } i < w, \\
\mu_0 \pm L\frac{\sigma}{\sqrt{nw}} , & \text{for } i \geq w. 
\end{cases}$$

For large values of $i$, the control limits become

Table 3. ARL and SDRL values of DMA control charts when $ARL_0 \approx 370$.

| $n$ | $\delta$ | $w=2$ | $w=3$ | $w=4$ | $w=5$ | $w=8$ | $w=10$ | $w=12$ | $w=15$ |
|-----|---------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1   | 0.0     | 371.5 | 370.3 | 369.6 | 369.7 | 371.0 | 369.2 | 370.1 | 370.7 | (370.7) | (370.7) | (367.2) | (369.5) | (371.6) | (363.7) | (370.8) | (369.3) | (370.2) |
| 0.2 | 250.2   | 218.0 | 198.9 | 186.0 | 155.5 | 142.8 | 131.7 | 120.4 |       | (250.5) | (219.9) | (198.9) | (186.7) | (151.7) | (138.3) | (124.8) | (114.7) |
| 0.4 | 119.4   | 90.5  | 75.3  | 65.5  | 51.7  | 47.2  | 43.9  | 40.9  |       | (117.6) | (86.6)  | (71.2)  | (62.1)  | (47.7)  | (41.5)  | (37.5)  | (33.8)  |
| 0.6 | 57.7    | 40.8  | 33.6  | 29.6  | 24.1  | 22.2  | 21.2  |       |       | (55.9)  | (37.6)  | (30.6)  | (26.5)  | (19.3)  | (16.3)  | (14.8)  | (14.2)  |
| 0.8 | 29.7    | 21.6  | 18.1  | 15.9  | 14.1  | 13.9  | 13.7  |       |       | (27.8)  | (19.5)  | (15.1)  | (12.6)  | (9.4)   | (8.6)   | (8.2)   | (8.5)   |
| 1.0 | 17.4    | 12.9  | 10.9  | 10.2  | 9.8   | 9.9   | 9.9   |       |       | (16.0)  | (10.7)  | (8.0)   | (7.1)   | (7.5)   | (5.9)   | (5.9)   | (6.5)   |
| 1.25| 9.7     | 7.7   | 7.1   | 7.0   | 7.0   | 7.1   | 7.0   |       |       | (8.2)   | (5.6)   | (4.5)   | (4.1)   | (4.1)   | (4.4)   | (4.7)   | (5.0)   |
| 1.5 | 6.3     | 5.4   | 5.1   | 5.2   | 5.3   | 5.2   | 5.0   |       |       | (5.0)   | (3.4)   | (3.0)   | (3.0)   | (3.3)   | (3.5)   | (3.7)   | (3.8)   |
| 2.0 | 3.4     | 3.3   | 3.3   | 3.3   | 3.1   | 3.0   | 2.8   |       |       | (2.1)   | (1.8)   | (1.8)   | (1.9)   | (2.2)   | (2.2)   | (2.1)   | (2.0)   |
| 3.0 | 1.7     | 1.7   | 1.6   | 1.6   | 1.5   | 1.4   | 1.4   |       |       | (0.9)   | (0.9)   | (0.9)   | (0.8)   | (0.8)   | (0.7)   | (0.7)   |       |
| 5   | 0.0     | 370.7 | 370.3 | 370.9 | 369.1 | 370.9 | 369.3 | 369.7 | 369.1 | (362.7) | (368.7) | (375.0) | (371.4) | (368.4) | (370.0) | (370.5) | (370.8) |
| 0.2 | 100.8   | 75.4  | 60.9  | 53.5  | 41.8  | 38.8  | 35.3  | 33.4  |       | (96.8)  | (73.2)  | (57.9)  | (50.4)  | (36.9)  | (33.4)  | (29.2)  | (25.6)  |
| 0.4 | 22.9    | 16.5  | 14.0  | 12.8  | 11.7  | 11.5  | 11.6  | 11.8  |       | (21.7)  | (14.6)  | (11.2)  | (9.7)   | (7.4)   | (6.9)   | (6.9)   | (7.4)   |
| 0.6 | 8.4     | 6.7   | 6.2   | 6.1   | 6.3   | 6.3   | 6.2   |       |       | (6.9)   | (4.6)   | (3.8)   | (3.5)   | (3.7)   | (4.0)   | (4.3)   | (4.5)   |
| 0.8 | 4.2     | 3.9   | 3.9   | 3.9   | 3.9   | 3.7   | 3.6   |       |       | (2.8)   | (2.2)   | (2.1)   | (2.3)   | (2.6)   | (2.7)   | (2.7)   | (2.7)   |
| 1.0 | 2.8     | 2.7   | 2.7   | 2.7   | 2.5   | 2.4   | 2.3   |       |       | (1.5)   | (1.5)   | (1.6)   | (1.7)   | (1.7)   | (1.7)   | (1.7)   | (1.6)   |
| 1.25| 1.9     | 1.9   | 1.8   | 1.8   | 1.7   | 1.6   | 1.5   |       |       | (1.0)   | (1.0)   | (1.1)   | (1.1)   | (1.0)   | (0.9)   | (0.9)   | (0.8)   |
| 1.5 | 1.4     | 1.4   | 1.4   | 1.3   | 1.3   | 1.2   | 1.2   |       |       | (0.7)   | (0.7)   | (0.7)   | (0.6)   | (0.6)   | (0.5)   | (0.5)   | (0.5)   |
| 2.0 | 1.1     | 1.1   | 1.0   | 1.0   | 1.0   | 1.0   | 1.0   |       |       | (0.3)   | (0.2)   | (0.2)   | (0.2)   | (0.2)   | (0.1)   | (0.1)   | (0.1)   |
| 3.0 | 1.0     | 1.0   | 1.0   | 1.0   | 1.0   | 1.0   | 1.0   |       |       | (0.0)   | (0.0)   | (0.0)   | (0.0)   | (0.0)   | (0.0)   | (0.0)   | (0.0)   |

$$UCL/LCL = \begin{cases} 
\mu_0 \pm L\frac{\sigma}{\sqrt{ni}}, & \text{for } i < w, \\
\mu_0 \pm L\frac{\sigma}{\sqrt{nw}}, & \text{for } i \geq w. 
\end{cases}$$
\[ \text{UCL} / \text{LCL} = \mu_0 \pm L \sigma \sqrt{\frac{\lambda}{n(2 - \lambda)}}. \]

The classical Shewhart chart is a special case of the EWMA chart for \( \lambda = 1 \). Small values of \( \lambda \) are recommended for detecting small shifts while larger values are more appropriate for detecting larger shifts (Montgomery 2013). A process is considered to be OOC if a plotted point \( Z_i \) lies outside the control limits.

The charting statistics of a CUSUM chart are computed by

\[ C_i^+ = \max \left[ 0, \tilde{X}_i - (\mu_0 + K) + C_{i-1}^+ \right] \]
\[ C_i^- = \max \left[ 0, (\mu_0 + K) - \tilde{X}_i + C_{i-1}^- \right], \]

where \( C_0^+ = C_0^- = 0 \) and \( K \) is the reference value, usually computed by

\[ K = \frac{\delta \sigma}{2 \sqrt{n}}. \]
The CUSUM chart that has been designed to detect quickly a shift $\delta$ is very effective for the specific shift. A process is considered to be OOC if a charting statistic exceeds the decision interval $H = h \sigma / \sqrt{n}$, where $h > 0$. Otherwise, the process is considered to be IC.

To make valid conclusions, the ARL$_0$ value of the competing control charts is set equal to 370. For the MA chart, we use the same values of $w$ with those of DMA chart while for the EWMA chart, we use the asymptotic control limits and values of $k = 0.05, 0.10, 0.25, 0.50, 0.75$ and 1.00, where the latter value corresponds to the Shewhart chart. Finally, the CUSUM chart is designed to detect quickly shifts of $\delta = 0.2, 0.8$ and 1.5. The OOC performance of MA, EWMA and CUSUM charts is presented in Tables 4–6, respectively.

Comparing the DMA and MA charts for individual measurements, we conclude that the range of shifts where the DMA chart outperforms the MA chart decreases as the value of $w$ increases. For example, the DMA chart is more effective than the MA chart in detecting shifts of $\delta / C_2 = 0.0$ when $w = 2$, $\delta < 1.0$ when $w = 5$ and $\delta < 0.6$ when $w = 10, 12$ or 15. For the rest range of shifts, i.e., for larger shifts, the MA chart is more effective than the DMA chart. On the other hand, when $n = 5$, the DMA chart is still more sensitive in detecting small shifts, but its superiority

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**Table 5. ARL and SDRL values of EWMA control charts when $ARL_0 \approx 370$.**

| $n$ | $\lambda = 0.05$ | 0.10 | 0.25 | 0.50 | 0.75 | 1.00 |
|-----|------------------|------|------|------|------|------|
| 1   |                  |      |      |      |      |      |
|     | 3.704            | 3.709| 3.702| 3.702| 3.703| 3.692|
|     | (3.614)          | (3.729)| (3.634)| (3.628)| (3.749)| (3.628)|
| 0.2 | 2.991            | 2.875| 2.735| 2.799| 2.801| 2.806|
|     | (1.707)          | (1.745)| (2.345)| (2.809)| (3.035)| (3.035)|
| 0.4 | 3.666            | 4.121| 63.1 | 105.7| 152.4| 197.0|
|     | (23.5)           | (32.1)| (58.5)| (104.0)| (149.9)| (194.9)|
| 0.6 | 2.089            | 2.100| 2.282| 4.862| 80.8 | 118.0|
|     | (10.7)           | (13.3)| (23.8)| (46.5)| (79.5)| (119.7)|
| 0.8 | 1.419            | 1.344| 1.578| 26.0 | 43.4 | 70.5 |
|     | (6.2)            | (7.2)| (11.8)| (23.8)| (43.0)| (70.0)|
| 1.0 | 1.088            | 0.973| 1.029| 15.1 | 25.5 | 44.0 |
|     | (4.1)            | (4.5)| (6.8)| (13.0)| (24.0)| (43.5)|
| 1.25| 0.825            | 0.779| 0.693| 9.1  | 14.2 | 24.6 |
|     | (2.7)            | (2.9)| (3.8)| (7.1)| (13.0)| (24.1)|
| 1.5 | 0.672            | 0.587| 0.52  | 6.0  | 8.7  | 14.8 |
|     | (2.0)            | (2.1)| (2.4)| (4.2)| (7.5)| (14.2)|
| 2.0 | 0.502            | 0.420| 0.35  | 3.4  | 4.1  | 6.3  |
|     | (1.2)            | (1.2)| (1.3)| (1.9)| (3.0)| (5.7)|
| 3.0 | 0.332            | 0.279| 0.22  | 1.9  | 1.8  | 2.0  |
|     | (0.7)            | (0.7)| (0.6)| (0.8)| (1.0)| (1.4)|

The CUSUM chart that has been designed to detect quickly a shift $\delta$ is very effective for the specific shift. A process is considered to be OOC if a charting statistic exceeds the decision interval $H = h \sigma / \sqrt{n}$, where $h > 0$. Otherwise, the process is considered to be IC.
gradually decreases in detecting moderate shifts (0.4 ≤ \( \delta \) ≤ 1.0), as the value of \( w \) increases. The two charts are comparable for larger shifts (\( \delta > 1.0 \)).

The results from Tables 3 and 5 indicate that for the case of individual measurements, the EWMA chart with \( \lambda = 0.05 \) is more sensitive in detecting small shifts (\( \delta \leq 0.6 \)) while the DMA chart has comparable performance with the best-performing EWMA chart for shifts of 0.6 ≤ \( \delta \) ≤ 1.5. Additionally, as the size of shift in the specific range increases, a DMA chart with a small value of \( w \) performs similarly with the best-performing EWMA chart. For example, a DMA chart with \( w \geq 10 \) has similar performance with the EWMA chart with \( \lambda = 0.10 \) for \( \delta = 0.8 \) while a DMA chart with \( w \geq 3 \) has similar detection ability with the EWMA chart with \( \lambda = 0.25 \) for \( \delta = 1.5 \). Finally, the DMA chart is more sensitive than the EWMA chart for large shifts (\( \delta \geq 2.0 \)). For the case of subgrouped data, the differences between the ARL values of the DMA chart with \( w = 15 \) and the best-performing EWMA chart are negligible for small shifts (\( \delta \leq 0.4 \)) while a DMA chart with \( w \geq 10 \) performs a slightly better than the best-performing EWMA chart for moderate shifts (0.8 ≤ \( \delta \) ≤ 1.5). For large shifts (\( \delta \geq 2.0 \)), the DMA chart performs similarly with EWMA charts with a large value of \( \lambda \). We point out that the DMA chart outperforms the Shewhart chart over the entire range of shifts.

A performance comparison between DMA and CUSUM charts when \( n = 1 \) indicates that the CUSUM chart optimal designed to detect quickly a small shift is more effective than the DMA chart only for a small range of shifts around it while for moderate to large shifts, the DMA chart is more effective. A DMA chart with a large value of \( w \) is more sensitive than a CUSUM chart optimal designed to detect quickly a moderate or large shift for the entire range of shifts. For example, the CUSUM chart optimal designed to detect a shift of \( \delta = 0.2 \) outperforms the DMA chart at this shift. For shifts 0.4 ≤ \( \delta \) ≤ 1.0, a DMA chart even with a small value of \( w \) is more effective than the specific CUSUM chart while for shifts \( \delta > 1.0 \), all DMA charts outperform the CUSUM chart. On the other side, a CUSUM chart optimal designed to detect quickly a shift of \( \delta = 0.8 \) or 1.5 has the same or worse detection ability than a DMA chart with a large value of \( w \) at the specific shifts. For the case of subgrouped data, the DMA chart performs better even for shifts where the CUSUM chart is optimal designed. For example, a DMA chart with \( w \geq 12 \) outperforms the CUSUM chart optimal designed for a shift of \( \delta = 0.2 \) at the specific shift while all DMA charts are more sensitive for the rest range of shifts.

### Table 6. ARL and SDRL values of CUSUM control charts when ARL\( _{20} \approx 370.0 \)

| \( \delta \) | \( n = 1 \) | \( n = 5 \) |
|---|---|---|
| \( h = 13.4749 \) | \( 5.7033 \) | \( 3.3412 \) | \( 13.4749 \) | \( 5.7033 \) | \( 3.3412 \) |
| 0.0 | 370.5 | 370.0 | 370.6 | 370.6 | 370.4 | 370.1 |
| 0.2 | 98.3 | 145.2 | 202.8 | 37.8 | 38.4 | 63.9 |
| 0.4 | 43.1 | 47.2 | 78.4 | 17.6 | 11.8 | 13.5 |
| 0.6 | 27.4 | 22.6 | 33.6 | 11.5 | 6.9 | 6.2 |
| 0.8 | 19.8 | 14.1 | 17.5 | 8.6 | 4.8 | 4.0 |
| 1.0 | 15.7 | 10.2 | 10.9 | 6.9 | 3.8 | 2.9 |
| 1.25 | 12.5 | 7.5 | 7.1 | 5.6 | 3.0 | 2.3 |
| 1.5 | 10.3 | 5.9 | 5.2 | 4.7 | 2.5 | 1.9 |
| 2.0 | 7.7 | 4.2 | 3.4 | 3.6 | 2.0 | 1.4 |
| 3.0 | 5.2 | 2.8 | 2.1 | 2.6 | 1.3 | 1.0 |

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In order to demonstrate the practical application of the DMA chart, we use the inside diameter measurements (in mm) on automobile engine piston rings, provided Montgomery (2013). The dataset consists of 40 samples, each of size $n = 5$ and is presented in Table 7. The first 25 samples represent the phase I observations. The IC values of process mean and standard deviation are $\mu_0 = 74.0012$ mm and $\sigma = 0.01$ mm, respectively. The last 15 samples represent the OOC observations. Setting an $\text{ARL}_0=370$, we construct a DMA chart with $w = 4$ and $L = 2.780$ (see Table 1). The charting statistics and the control limits are also presented in Table 7. The control chart is displayed in Figure 1. Moreover, we construct the MA chart with $w = 4$, shown in Figure 2. From these Figures, we conclude that both charts detect a first OOC signal at the 27th sample.

### 7. Conclusion

The DMA control chart was proposed by Khoo and Wong (2008) in order to improve the performance of the MA control chart for the detection of small to moderate shifts. Unfortunately, the computed variance of the DMA statistic was not correct as the covariance between MA
statistics was ignored. In this article, we calculate the correct variance of the DMA statistic and we study the performance of DMA chart performing numerical simulations. It is shown that the detection ability of DMA chart improves as the value of the span $w$ increases. A comparison study with the MA chart demonstrates that the DMA chart is more effective for small to moderate shifts while this range of shifts becomes narrower as the value of $w$ increases. Furthermore, for individual measurements, the DMA chart has similar performance with the EWMA and CUSUM charts for moderate shifts, but it is more (less) effective in detecting large (small) shifts. The superiority of the DMA chart versus the EWMA and CUSUM charts enlarges for subgrouped data.

In terms of future work, the researchers who have studied the DMA scheme can update their results using the correct control limits.

Figure 1. The DMA control chart with $w = 4$ for piston-ring data.

Figure 2. The MA control chart with $w = 4$ for piston-ring data.
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