New soluble nonlinear models for scalar fields

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Abstract

We extend a deformation prescription recently introduced and present some new soluble nonlinear problems for kinks and lumps. In particular, we show how to generate models which present the basic ingredients needed to give rise to “dimension bubbles,” having different macroscopic space dimensions on the interior and the exterior of the bubble surface. Also, we show how to deform a model possessing lumplike solutions, relevant to the discussion of tachyonic excitations, to get a new one having topological solutions.

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I. INTRODUCTION

Nonlinear models in (1+1)-dimensional space-time play important roles both in field theory and in quantum mechanics. Some of such models possess defect solutions representing domain walls that appear, for example, in high energy physics \cite{1} and condensed matter \cite{2}.

In this work we extend the method introduced in Ref. \cite{3} to present new potentials bearing topological (kinklike) or non topological (lumplike) solutions. For example, we use the extended deformation procedure to build a semi-vacuumless model, and the corresponding domain wall which serves as seed for generation of “dimension bubbles,” as proposed in Refs. \cite{4,5}. We also show how to deform models having lumplike solutions, relevant to the discussion of tachyonic excitations, to generate new ones presenting topological solutions.

To begin, consider a theory of a single scalar real field in a (1+1)-dimensional space-time, described by the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi),
\]

(1)

We use the metric (+,−), and we work with dimensionless fields and coordinates, fields being defined in the whole space. The equation of motion for static fields is

\[
\frac{d^2 \phi}{dx^2} = V'(\phi),
\]

(2)

where the prime stands for the derivative with respect to the argument. Mathematically, this corresponds to consider two-point boundary value problems (with conditions imposed at \(x_1 = -\infty\) and \(x_2 = +\infty\)) for second-order ordinary differential equations \(6\).

Consider the broad class of potentials having at least one critical point \(\bar{\phi}\) (that is, \(V'(\bar{\phi}) = 0\)), for which \(V(\bar{\phi}) = 0\). In this case, solutions satisfying the conditions

\[
\lim_{x \to -\infty} \phi(x) = \bar{\phi}, \quad \lim_{x \to -\infty} \frac{d\phi}{dx} = 0, \quad \lim_{x \to +\infty} \phi(x) \to \infty,
\]

(3)

obey the first order equation (a first integral of (2))

\[
\frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 = V(\phi(x));
\]

(4)

thus, such solutions equally share their energy densities between gradient and potential parts.

Many important examples can be presented: the \(\phi^4\)-model, with \(V(\phi) = (1-\phi^2)^2/2\), is the prototype of theories having topological solitons (kinklike solutions) connecting two minima,
\( \phi(x) = \pm \tanh(x) \) in this case; a situation where non topological (lumplike) solutions exist is the “inverted \( \phi^4 \)-model”, with potential given by \( V(\phi) = \phi^2(1 - \phi^2)/2 \), lumplike defects being \( \phi(x) = \pm \text{sech}(x) \). One notice that the potential need not be nonnegative for all values of \( \phi \) but the solution must be such that \( V(\phi(x)) \geq 0 \) for the whole range \(-\infty < x < +\infty\).

II. THE DEFORMATION PROCEDURE

Both topological and non topological solutions can be deformed, according to the prescription introduced in [3], to generate infinitely many new soluble problems. This method can be stated in general form as the following theorem:

Let \( f = f(\phi) \) be a bijective function having continuous non-vanishing derivative. For each potential \( V(\phi) \), bearing solutions satisfying conditions [3] (or equivalently Eq. (4)), the \( f \)-deformed model, defined by

\[
\tilde{V}(\phi) = \frac{V[f(\phi)]}{[f'(\phi)]^2},
\]

possesses solution given by

\[
\tilde{\phi}(x) = f^{-1}(\phi(x)),
\]

where \( \phi(x) \) is a solution of the static equation of motion for the original potential \( V(\phi) \).

To prove this assertion, notice that the static equation of motion of the new theory is written in terms of the old potential as

\[
\frac{d^2 \phi}{dx^2} = \frac{1}{f'(\phi)} V'[f(\phi)] - 2V[f(\phi)] \frac{f''(\phi)}{[f'(\phi)]^3}.
\]

On the other hand, taking the second derivative with respect to \( x \) of Eq. (6), one finds

\[
\frac{d^2 \tilde{\phi}}{dx^2} = \frac{1}{f'(\tilde{\phi})} \frac{d^2 \phi}{dx^2} - \frac{f''(\tilde{\phi})}{[f'(\tilde{\phi})]^3} \left( \frac{d\phi}{dx} \right)^2.
\]

It follows from (2), (4) and (6) that \( d^2 \phi/dx^2 = V'[f(\phi)] \) and \( (d\phi/dx)^2 = 2V[f(\phi)] \) so that \( \tilde{\phi} \) satisfies (7), as stated. The energy density of the solution (6) of the \( f \)-deformed potential (5) is given by

\[
\tilde{\varepsilon}(x) = \left( \frac{df^{-1}}{d\phi} \right)^2 \left( \frac{d\phi}{dx} \right)^2 \bigg|_{\phi = \phi_{\pm}(x)}.
\]
Naturally, the deformation procedure heavily depends on the deformation function $f(\phi)$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is bijective and has no critical points. In this case, the $f$-deformation (and the deformation implemented by its inverse $f^{-1}$) can be applied successively and one can define equivalence classes of potentials related to each other by repeated applications of the $f$- (or the $f^{-1}$-) deformation. Each of such classes possesses an enumerable number of elements which correspond to smooth deformations of a representative one, all having the same topological characteristics. The generation sequence of new theories is depicted in the diagram below.

As an example not considered before, take the $\phi^6$-model. This model, for which the potential $V(\phi) = \phi^2(1-\phi^2)^2/2$ has three degenerated minima at 0 and $\pm 1$, is important since it allows the discussion of first-order transitions. It possesses kinklike solutions, $\phi_\pm^+(x) = \pm \sqrt{[1 + \tanh(x)]/2}$, $\phi_\pm^-(x) = \pm \sqrt{[1 - \tanh(x)]/2}$, connecting the central vacuum with the lateral ones. Take $f(\phi) = \sinh(\phi)$ as the deforming function. The sinh-deformed $\phi^6$-potential is

$$V(\phi) = \frac{1}{2} \tan^2(\phi) \left[ 1 - \sinh^2(\phi) \right]^2$$

and the sinh-deformed defects are

$$\tilde{\phi}_\pm^+(x) = \pm \arcsinh \left( \sqrt{\frac{1}{2} [1 + \tanh(x)]} \right),$$

$$\tilde{\phi}_\pm^-(x) = \pm \arcsinh \left( \sqrt{\frac{1}{2} [1 - \tanh(x)]} \right).$$

Notice that, since $f'(\phi) > 1$ for the sinh-deformation, the energy of the deformed solutions is diminished with respect to the undeformed kinks. The reverse situation emerges if one takes the inverse deformation implemented with $f^{-1}(\phi) = \arcsinh(\phi)$.

Interesting situations arise if one takes polynomial functions implementing the deformations. Consider

$$p_{2n+1}(\phi) = \sum_{j=0}^{n} c_j \phi^{2j+1},$$

$$\cdot \cdot \cdot \quad \hat{V} \quad f^{-1} \quad \hat{V} \quad f^{-1} \quad V \quad f \quad \hat{V} \quad f \quad \tilde{V} \quad \cdot \cdot \cdot$$

$$\cdot \cdot \cdot \quad \hat{\phi}_d \quad f \quad \hat{\phi}_d \quad f \quad \phi_d \quad f^{-1} \quad \tilde{\phi}_d \quad f^{-1} \quad \tilde{\phi}_d \quad \cdot \cdot \cdot$$
with $c_j > 0$ for all $0 \leq j \leq n$. These are bijective functions from $\mathbb{R}$ into $\mathbb{R}$ possessing positive derivatives. Fixing $n = 0$ corresponds to a trivial rescaling of the field. For $n = 1$, taking $c_0 = c_1 = 1$, one has $f(\phi) = p_3(\phi) = \phi + \phi^3$ with inverse given by $f^{-1}(\phi) = (2/\sqrt{3}) \sinh[\arcsinh(3\sqrt{3}\phi/2)/3]$. Thus, the $p_3$-deformed $\phi^4$ model, for which the potential has the form

$$\tilde{V}(\phi) = \frac{1}{2} \left( \frac{1 - \phi^2 - 2\phi^4 - \phi^6}{1 + 3\phi^2} \right)^2,$$

supports topological solitons given by

$$\tilde{\phi}_\pm(x) = \pm \frac{2}{\sqrt{3}} \sinh \left[ \frac{1}{3} \arcsinh \left( \frac{3\sqrt{3}}{2} \tanh(x) \right) \right].$$

Naturally, the inverse deformation can be implemented leading to another new soluble problem. But if one takes $n \geq 2$, the inverse of $p_{2n+1}$ can not be in general expressed analytically in terms of known functions. This leads to circumstances where one knows analytically solutions of potentials which can not be expressed in term of known functions and, conversely, one has well-established potentials for which solitonic solutions exist but are not expressible in terms of known functions. For example, take $f(\phi) = p_5(\phi) = \phi + \phi^3 + \phi^5$. The $p_5$-deformed $\phi^4$ model has potential given by

$$\tilde{V}(\phi) = \frac{1}{2} \left( \frac{1 - \phi^2 - 2\phi^4 - 3\phi^6 - 2\phi^8 - \phi^{10}}{1 + 3\phi^2 + 5\phi^4} \right)^2,$$

but its solutions $\tilde{\phi}_\pm(x) = \pm p_5^{-1}(\tanh x)$ are not known analytically. On the other hand,

$$\hat{\phi}_\pm(x) = \pm \left[ \tanh(x) + \tanh^3(x) + \tanh^5(x) \right]$$

are topological solutions of the potential $\hat{V}(\phi) = (1 - [p_5^{-1}(\phi)]^2)/(2[p_5^{-1}(\phi)]^2)$ which can not be written in terms of known functions.

The procedure can also be applied to potentials presenting non topological, lumplike, solutions which are of direct interest to tachyions in String Theory [7, 8, 9]. Take, for example, the Lorentzian lump

$$\phi_l(x) = \frac{1}{x^2 + 1}\]$$

which solves Eq. (2) for the potential

$$V(\phi) = 2(\phi^3 - \phi^4).\]"
and satisfies conditions (3). Distinctly of the topological solitons, this kind of solution is not stable. In fact, the ‘secondary potential’, that appears in the linearized Schrödiger-like equation satisfied by the small perturbations around $\phi_l(x)$ is given by

$$U(x) = V''(\phi_l(x)) = 12 \frac{x^2 - 1}{(x^2 + 1)^2}.$$  
(20)

This potential, a symmetric volcano-shaped potential, has zero mode given by $\eta_0(x) \approx \phi'_l(x) = -2x/(x^2 + 1)^2$, which does not correspond to the lowest energy state since it has a node. Deforming the potential (19) with $f(\phi) = \sinh \phi$ leads to the potential $\tilde{V}(\phi) = 2 \tanh^2(\phi) [\sinh(\phi) - \sinh^2(\phi)]$ which possesses the lumplike solution $\tilde{\phi}_l(x) = \arcsinh [(x^2 + 1)^{-1}]$.

### III. THE EXTENDED DEFORMATION PRESCRIPTION

The deformation prescription is powerful. The conditions under which our theorem holds are maintained if we consider a function for which the contra-domain is an interval of $\mathbb{R}$, that is, if we take $f : \mathbb{R} \rightarrow I \subset \mathbb{R}$. In this case, however, the inverse transformation (engendered by $f^{-1} : I \rightarrow \mathbb{R}$) can only be applied for models where the values of $\phi$ are restricted to $I \subset \mathbb{R}$.

We illustrate this possibility by asking for a deformation that leads to a model of the form needed in Ref. [5], described by a “semi-vacuumless” potential, in contrast with the vacuumless potential studied in Ref. [11, 12]. Consider the new deformation function $f(\phi) = 1 - 1/\sinh(e^\phi)$, acting on the potential $V(\phi) = (1 - \phi^2)^2/2$. The deformed potential is

$$\tilde{V}(\phi) = \frac{1}{2} e^{-2\phi} \text{sech}^2(\phi) \left(2 \sinh(e^\phi) - 1\right)^2,$$

(21)

which is depicted in Fig. 1. The kinklike solution is

$$\tilde{\phi}(x) = \ln \left[\text{arcsinh} \left(\frac{1}{1 - \tanh(x)}\right)\right].$$

(22)

The deformed potential (21) engenders the required profile: it has a minimum at $\tilde{\phi} = \ln[\text{arcsinh}(1/2)]$ and another one at $\phi \rightarrow \infty$. It is similar to the potential required in Ref. [5] for the existence of dimension bubbles. The bubble can be generated from the above (deformed) model, after removing the degeneracy between $\tilde{\phi}$ and $\phi \rightarrow \infty$, in a way similar to the standard situation, which is usually implemented with the $\phi^4$ potential, the
undeformed potential that we have used to generate (21). A key issue is that such bubble is unstable against collapse, unless a mechanism to balance the inward pressure due to the surface tension in the bubble is found. In Ref. [5], the mechanism used to stabilize the bubble requires another scalar field, in a way similar to the case of non topological solitons previously proposed in Ref. [13]. This naturally leads to another scenario, which involves at least two real scalar fields.

![Graph of the deformed potential](image)

**FIG. 1:** The deformed potential $\tilde{V}(\phi)$ of Eq. (21), plotted as a function of the scalar field $\phi$; the dashed line shows the potential of the undeformed $\phi^4$ model.

The deformation procedure can be extended even further, by relaxing the requirement of $f$ being a bijective function, under certain conditions. Suppose that $f$ is not bijective but it is such that its inverse $f^{-1}$ (which exists in the context of binary relations) is a multi-valued function with all branches defined in the same interval $I \subset \mathbb{R}$. If the domain of definition of $f^{-1}$ contains the interval where the values of the solutions $\phi(x)$ of the original potential vary, then $\tilde{\phi}(x) = f^{-1}(\phi(x))$ are solutions of the new model obtained by implementing the deformation with $f$. However, one has to check out whether the deformed potential $\tilde{V}(\phi) = V[f(\phi)]/(f'(\phi))^2$ is well defined on the critical points of $f$. In fact, this does not happen in general but occurs for some interesting cases.

Consider, for example, the function $f(\phi) = 2\phi^2 - 1$; it is defined for all values of $\phi$ and its inverse is the double valued real function $f^{-1}(\phi) = \pm\sqrt{(1+\phi)/2}$, defined in the interval $[-1, \infty)$. If we deform the $\phi^4$ model with this function we end up with the potential $\tilde{V}(\phi) = \phi^2(1 - \phi^2)^2/2$. The deformed kink solutions are given by $\tilde{\phi}(x) = \pm\sqrt{(1+\phi(x))/2}$ with $\phi(x)$ replaced by the solutions ($\pm \tanh(x)$) of the $\phi^4$ model, which reproduce the known
solutions of the $\phi^6$ theory. The important aspect, in the present case, is that the tanh-kink corresponds to field values restricted to the interval $(-1, +1)$ which is contained within the domain of definition of the two branches of $f^{-1}(\phi)$. The fact that the $\phi^6$ model can be obtained from the $\phi^4$ potential in this way is interesting, since these models have distinct characteristics. Notice that the critical point of $f$ at $\phi = 0$ does not disturb the deformation in this case; this always occur for potentials having a factor $(1 - \phi^2)$, since the denominator of $\tilde{V}(\phi)$ (Eq. 5) is cancelled out. One can go on and apply this deformation to the $\phi^6$ model; now, one finds the deformed potential

$$\tilde{V}(\phi) = \frac{1}{2} \phi^2(1 - \phi^2)^2(1 - 2\phi^2)^2,$$

(23)

with solutions given by

$$\tilde{\phi}(x) = \pm \sqrt{\frac{1}{2} \left( 1 \pm \frac{1}{2} [1 \pm \tanh(x)] \right)},$$

(24)

corresponding to kinks connecting neighbouring minima (located at $-1, -1/\sqrt{2}, 0, 1/\sqrt{2}$ and 1) of the potential (23). This potential is illustrated in Fig. 2. Repeating the procedure for the potential (23), the deformed potential is a polynomial function of degree 18, having sixteen kink solutions connecting adjacent minima of the set \( \{0, \pm 1, \pm \sqrt{2}/2, \pm \sqrt{2 + \sqrt{2}}/2, \pm \sqrt{2 - \sqrt{2}}/2\} \), and so on and so forth.

**FIG. 2:** The deformed potential $\tilde{V}(\phi)$ of Eq. (23), plotted as a function of the scalar field $\phi$; the dashed line shows the potential of the undeformed $\phi^6$ model.

The deformation implemented by the function $f(\phi) = 2\phi^2 - 1$ can also be applied to a potential possessing lumplike solutions. Consider the inverted $\phi^4$ potential $V(\phi) = \phi^2(1 -$
\(\phi^2)/2\), which has the lump solutions \(\phi(x) = \pm \text{sech}(x)\). The deformed potential, in this case, is given by

\[
\tilde{V}(\phi) = \frac{1}{8}(2\phi^2 - 1)^2(1 - \phi^2). \tag{25}
\]

This potential, which is also unbounded from below, vanishes for \(\phi = \pm \sqrt{2}/2, \pm 1\), has an absolute maximum at \(\phi = 0\) and local minima and maxima for \(\pm \sqrt{2}/2\) and \(\pm \sqrt{3}/2\), respectively. Figure 3 shows a plot of this potential. If we take \(\phi(x) = + \text{sech}(x)\) as the original lump, then the deformed solutions obtained are given by

\[
\tilde{\phi}(x) = \pm \sqrt{\frac{1}{2}[1 + \text{sech}(x)]}, \tag{26}
\]

corresponding to lumps which start in the local minima (for \(x = -\infty\)), go to the lateral zeros (when \(x = 0\)) and come back to the same minima (for \(x = +\infty\)) of the potential (25). On the other hand, if the undeformed lump is \(\phi(x) = - \text{sech}(x)\), taking naively \(f^{-1}(- \text{sech}(x))\) leads to functions that do not have derivative at \(x = 0\) and, therefore, are not acceptable solutions of the equations of motion. In fact, the deformation of the \(- \text{sech}(x)\) lump results in deformed kinks given by

\[
\tilde{\phi}(x) = \pm \begin{cases} 
- \sqrt{\frac{1}{2}[1 - \text{sech}(x)]} & x \leq 0 \\
+ \sqrt{\frac{1}{2}[1 - \text{sech}(x)]} & x \geq 0 
\end{cases}. \tag{27}
\]

Each of such solutions (e.g., the (+) one) starts at a minimum (\(\phi = -\sqrt{2}/2\)) when \(x = -\infty\), goes to the absolute maximum (\(\phi = 0\)) at \(x = 0\) running along one of the branches of \(f^{-1}\) (the lower branch) and, passing continuously to the other branch (the upper one), reaches the other local minimum (\(\phi = +\sqrt{2}/2\)) when \(x = +\infty\). One notice that, again, the number of solutions duplicates using such a deformation, whose inverse is a double-valued function. But, in this case, novel topological solutions emerge as deformations of a non topological one.

Potentials which have a factor \((1 - \phi^2)\) can also be deformed using the function \(f(\phi) = \sin(\phi)\), producing many interesting situations. In fact, suppose the potential can be written in the form

\[
V(\phi) = (1 - \phi^2)U(\phi); \tag{28}
\]

this is always possible for all well-behaved potentials that vanish at both values \(\phi = \pm 1\), as shown by Taylor expansion. Then, the sin-deformation leads to the potential

\[
\tilde{V}(\phi) = U[\sin(\phi)], \tag{29}
\]
FIG. 3: The deformed potential $\tilde{V}(\phi)$ of Eq. (25), plotted as a function of the scalar field $\phi$; the dashed line shows the potential of the undeformed inverted $\phi^4$ model.

which is a periodic potential, the critical points of $\sin(\phi)$ not causing any problem to the deformation process. The inverse of the sine function is the infinitely valued function $f^{-1}(\phi) = (-1)^k \text{Arcsin}(\phi) + k\pi$, with $k \in \mathbb{Z}$ and $\text{Arcsin}(\phi)$ being the first determination of $\arcsin(\phi)$ (which varies from $-\pi/2$, for $\phi = -1$, to $+\pi/2$, when $\phi = +1$), defined in the interval $[-1, +1]$. So to each solution of the original potential, whose field values range in the interval $[-1, +1]$, one finds infinitely many solutions of the deformed, periodic, potential.

Consider firstly the $\phi^4$ model. Applying the sin-deformation to it, one gets $\tilde{V}(\phi) = \cos^2(\phi)/2$ which is one of the forms of the sine-Gordon potential. The deformed solutions thus obtained is given by $\tilde{\phi}(x) = (-1)^k \text{Arcsin} [\pm \tanh(x)] + k\pi$, which correspond to all the kink solutions (connecting neighbouring minima) of this sine-Gordon model. For example, the kink solutions $\pm \tanh x$, which connect the minima $\phi = \pm 1$ of the $\phi^4$ model in both directions, are deformed into the kinks $\pm \text{Arcsin} [\tanh(x)] = 2\text{Arctan}(e^{\pm x}) - \pi/2$ (which runs between $-\pi/2$ and $\pi/2$) if one takes $k = 0$ while, for $k = 1$, the resulting solutions connect the minima $\pi/2$ and $3\pi/2$ of the deformed potential.

This example can be readily extended to other polynomial potentials, leading to a large class of sine-Gordon type of potentials. For instance, the $\phi^6$ model, $V(\phi) = \phi^2(1 - \phi^2)^2/2$, deformed by the sine function, becomes the potential

$$\tilde{V}(\phi) = \frac{1}{2} \cos^2(\phi)(1 - \cos^2(\phi)), \quad (30)$$
having kink solutions given by

$$\tilde{\phi}(x) = \pm (-1)^k \text{Arcsin} \left( \sqrt{\frac{1}{2} [1 \pm \tanh(x)]} \right) + k\pi .$$

(31)

If, on the other hand, one considers the potential $V(\phi) = (1 - \phi^2)^{3/2}$, which is unbounded below and supports kinklike solutions connecting the two inflection points at $\pm 1$, one gets the potential

$$\tilde{V}(\phi) = \frac{1}{2} \cos^4(\phi) ,$$

(32)

with solutions given by

$$\tilde{\phi}(x) = \pm (-1)^k \text{Arcsin} \left( \frac{x}{\sqrt{1 + x^2}} \right) + k\pi .$$

(33)

A particularly interesting situation appears if one consider the inverted $\phi^4$ model, which presents lumplike solutions. The sin-deformation of the potential $V(\phi) = \phi^2(1 - \phi^2)/2$ leads to the potential $\tilde{V}(\phi) = \sin^2(\phi)/2$. In this case, the lump solutions of $V(\phi)$, namely $\phi(x) = \pm \text{sech}(x)$, are deformed into $\tilde{\phi}(x) = \pm (-1)^k \text{Arcsin} [\text{sech}(x)] + k\pi$. Consider the $(+)$-solution and take initially $k = 0$. As $x$ varies from $-\infty$ to 0, sech($x$) goes from 0 to 1, and Arcsin [sech($x$)] = $2\text{Arctan}(e^x)$ changes from 0 to $\pi/2$. If one continuously makes $x$ goes from 0 to $+\infty$, then the deformed solution passes to the $k = 1$ branch of arcsin($\phi$), $-\text{Arcsin} [\text{sech}(x)] + \pi$ ($= 2\text{Arctan}(e^x)$ for $0 \leq x < +\infty$), which varies from $\pi/2$ to $\pi$ as $x$ goes from 0 to $+\infty$. Thus, in this case, the lump solution $+\text{sech}(x)$ of the inverted $\phi^4$ model is deformed in the kink of the sine-Gordon model connecting the minima $\phi = 0$ and $\phi = \pi$. Under reversed conditions (taking the $k = 1$ branch before the $k = 0$ one), the lump solution $-\text{sech}(x)$ leads to the anti-kink solution of the sine-Gordon model running from the minimum $\phi = \pi$ to 0. The other topological solutions of the sine-Gordon model are obtained considering the other adjacent branches of arcsin($\phi$). This is another remarkable example since one has non topological solutions being deformed in topological ones.

Finally, we mention that many other soluble models can be construct following the procedure presented in this paper. Also, the discussion raised here might be extended to consider situations where the fields are constrained to intervals of $\mathbb{R}$, thus representing two-point boundary value problems defined in finite or semi-infinite intervals of the $x$-axis. Such study is left for future work.
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