COMPLETENESS OF SETS OF SHIFTS IN INVARIANT BANACH SPACES OF TEMPERED DISTRIBUTIONS VIA TAUBERIAN CONDITIONS

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Abstract. The main result of this paper is a far reaching generalization of the completeness result given by V. Katsnelson in a recent paper ([35]). Instead of just using a collection of dilated Gaussians it is shown that the key steps of an earlier paper ([27]) by the authors, combined with the use of Tauberian conditions (i.e. the non-vanishing of the Fourier transform) allow us to show that the linear span of the translates of a single function $g \in \mathcal{S}(\mathbb{R}^d)$ is a dense subspace of any Banach space satisfying certain double invariance properties.

In fact, a much stronger statement is presented: for a given compact subset $M$ in such a Banach space $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ one can construct a finite rank operator, whose range is contained in the linear span of finitely many translates of $g$, and which approximates the identity operator over $M$ up to a given level of precision.

The setting of tempered distributions allows to reduce the technical arguments to methods which are widely used in Fourier Analysis. The extension to non-quasi-analytic weights respectively locally compact Abelian groups is left to a forthcoming paper, which will be technically much more involved and uses different ingredients.

1. Introduction

This paper can be seen as an alternative approach to the question of completeness of sets of translates of a given test function for a large variety of Banach space of tempered distributions. The motivation for the current paper is the wish to demonstrate that the very specific results describing the density of the linear span of the set of all translates and dilates of the Gauss function as given in [35] can be generalized into several directions.

In the companion paper [27] we have shown that only the non-vanishing integral, i.e. without loss of generality the assumption $\int_{\mathbb{R}^d} g(x)dx = 1$ for the generating Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ is sufficient in order to guarantee the density for a large variety of Banach spaces with double module structure. As it turned out, the setting of the paper [9] appeared to be most appropriate, which is quite similar to the setting of so-called “standard spaces” as used in papers on compactness ([16]) or double module structures ([4]). As it is clear that there is only a chance for such a statement if $\mathcal{S}(\mathbb{R}^d)$ is a dense subspace of $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$, so we will make this minimality assumption throughout this paper.

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Observing that the Gauss function has another important property, namely a nowhere vanishing Fourier transform, Tauberian Theorems come to mind (see [43], [44], [39], [36], [18]). In the classical setting the non-vanishing of the Fourier transform \( \hat{g}(s) \neq 0 \) for all \( s \in \mathbb{R}^d \) for some \( g \in L^1(\mathbb{R}^d) \) implies that the linear span of its translates is dense in \( (L^1(\mathbb{R}^d), \| \cdot \|_1) \) (see [39], Chap.4.1). Similar results hold true for weighted \( L^1 \)-algebras, the so-called Beurling algebras \( (L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w}) \) (see e.g. [39], Chap.1.6). For this paper mostly polynomial weights equivalent to \( v_s(x) = (1 + |x|^2)^{s/2} \) are of interest. For them the Tauberian Theorem can be derived from the Tauberian condition going back to the work of A. Beurling (2) for details (see [39], Theorem 1.6.17).

The main result of this paper is a constructive realization of an approximation procedure which applies to all members of a family of translation and modulation invariant Banach spaces. Since the intersection of all these spaces under consideration is just the Schwartz space \( S(\mathbb{R}^d) \) it is natural to start from a Schwartz function \( g_0 \) with non-vanishing Fourier transform. We call this key assumption a Tauberian condition, because it is well-known from the Wiener’s classical Tauberian Theorem (see [39]).

The main advantage of the approach given here is the fact that it does not make use of dilations, so we do not have to generate Dirac sequences by compressing the given building block \( g \in S(\mathbb{R}^d) \) (which only has to satisfy the condition \( \hat{g}(0) \neq 0 \)). Thus at least the formulation of the main result of the current paper can easily be transferred to the setting of LCA (locally compact Abelian) groups without significant changes.

One may also argue that the restrictions to polynomially moderate weight functions and hence to stay within the world of \( S'(\mathbb{R}^d) \), the tempered distributions, is somewhat restrictive, and that one should look for such results also in the context of ultradistributions. In that context even more sensitive norms can be used for the space of test functions (so in a way the kind of approximation that has to be achieved is much more challenging), but also much bigger spaces are allowed which are not contained in \( S'(\mathbb{R}^d) \) anymore.

Overall we have chosen to present our results in the context of tempered distributions over \( G = \mathbb{R}^d \), in order to make the key steps more clear. One can formulate the same claim using the same principle ideas for general LCA groups, making use the Schwartz-Bruhat space, invoking Tauberian Theorems for Beurling algebras as found in the book of H. Reiter [39]. However, we will go for the most general setting, namely Banach spaces of ultra-distributions over LCA groups in a subsequent paper. In order to keep the paper readable we have avoided this outmost level of generality, mentioning once more that this is still a very far-reaching extension of the results in [35]. A list of examples of such Banach spaces is given in [27].

2. Translation and modulation invariant spaces

Throughout this paper we will work with the following standard assumptions, similar to the setting chosen in [9] or [27]:

**Definition 1.** A Banach space \( (B, \| \cdot \|_B) \) is called a minimal tempered standard space (abbreviated as MINTSTA, or equivalently a minimal TMIB in the sense of [8], [27]), if the following conditions are valid:

1. We assume the following chain of continuous embeddings:
   \[
   S(\mathbb{R}^d) \hookrightarrow (B, \| \cdot \|_B) \hookrightarrow S'(\mathbb{R}^d);
   \]

2. \( S(\mathbb{R}^d) \) is dense in \( (B, \| \cdot \|_B) \) (minimality);
(3) \((B, \| \cdot \|_B)\) is translation invariant, and for some \(s_1 \in \mathbb{N}\) and \(C_1 > 0\) one has
\[ \|T_x f\|_B \leq C_1(x)^{s_1} \|f\|_B \quad \forall x \in \mathbb{R}^d, f \in B. \tag{2} \]

(4) \((B, \| \cdot \|_B)\) is modulation invariant, and for some \(s_2 \in \mathbb{N}\) and \(C_2 > 0\) one has
\[ \|M_y f\|_B \leq C_2(y)^{s_2} \|f\|_B \quad \forall y \in \mathbb{R}^d, f \in B. \tag{3} \]

Here we use the Japanese bracket symbol \(\langle x \rangle\) respectively \(\langle y \rangle\) for the function \(v_s(z) = (1 + |z|^2)^{s_2}, z \in \mathbb{R}^d\), which is also known as Beurling weight of polynomial type, because it satisfies for \(s \geq 0\) the submultiplicativity property
\[ v_s(x + y) \leq v_s(x)v_s(y), \quad x, y \in \mathbb{R}^d. \tag{4} \]

For any such (polynomial) weight \(v\), the corresponding weighted \(L^1\)-space \(L^1_{v_s}(\mathbb{R}^d)\) is a Banach algebra with respect to convolution, continuously embedded into \((L^1(\mathbb{R}^d), \| \cdot \|_1)\) (since \(v_s(x) \geq 1\)) and having bounded approximate units (obtained by \(L^1\)-norm preserving compression, so-called Dirac sequences).

Whenever we use generic, submultiplicative \textit{Beurling weights} (not necessarily of polynomial type) we will make use of the standard notation \(w\). The corresponding weighted \(L^1\)-spaces \((L^1_{w}(\mathbb{R}^d), \| \cdot \|_{L^1_{w}(\mathbb{R}^d)})\) are Banach algebras with respect to convolution (we use the symbol \(f \ast g\), called \textit{Beurling algebras}. The natural norm to be used is (cf. [33])
\[ \|f\|_{L^1_{w}(\mathbb{R}^d)} = \|f\|_{1,w} := \|f \cdot w\|_{L^1}, \quad f \in L^1_{w}(\mathbb{R}^d). \]

For the formulation of our arguments the terminology of the theory of Banach modules will be convenient (see [14, 16, 40]):

**Definition 2.** A Banach space \((B, \| \cdot \|_B)\) is called a Banach module over a Banach algebra \((A, \| \cdot \|_A)\) if there is a (usually natural) embedding of \((A, \| \cdot \|_A)\) into the operator algebra over \((B, \| \cdot \|_B)\), written as \((a, b) \mapsto a \bullet b\), such that this bilinear (and associative) mapping satisfies
\[ \|a \bullet b\|_B \leq \|a\|_A \|b\|_B, \quad a \in A, b \in B. \tag{5} \]

If, in addition, \((B, \| \cdot \|_B) \hookrightarrow (A, \| \cdot \|_A)\) and the module operation is just the internal multiplication of the algebra \((A, \| \cdot \|_A)\), we call \((B, \| \cdot \|_B)\) a Banach ideal.

A Banach module is called \textit{essential} if \(A \bullet B\) generates a dense subspace of \((B, \| \cdot \|_B)\).

For our applications the Banach algebras \((A, \| \cdot \|_A)\) have bounded approximate units, i.e. there are bounded sequences or nets \(\{e_\alpha\}_{\alpha \in I}\) of elements in \((A, \| \cdot \|_A)\) such that
\[ \lim_{\alpha \to \infty} e_\alpha \bullet a = a, \quad a \in A. \]

In such a case the Cohen-Hewitt Factorization Theorem can be applied and gives
\[ B = A \bullet B = \{a \bullet b \mid a \in A, b \in B\}. \]

We only consider Beurling algebras as Banach convolution algebras (in this case we write \(f \ast g\) for the abstract multiplication \(f \bullet g\) acting on a MINTSTA via convolution (we will speak of \textit{Banach convolution modules}) or Banach algebras inside of \((C_0(\mathbb{R}^d), \| \cdot \|_\infty)\) acting on \((B, \| \cdot \|_B)\) via pointwise multiplication (we write \(f \cdot g\), or simply \(fg\)).

Essential Banach ideals in the Banach convolution algebra \((L^1(\mathbb{R}^d), \| \cdot \|_1)\) are exactly the \textit{Segal algebras} in the sense of H. Reiter ([33]) (also called \textit{normed ideals} in [3]), see [11]. Further references to \textit{abstract Segal algebras} are given in papers of J.T. Burnham [5], see also [12].
The density of \( S(\mathbb{R}^d) \) in \( (B, \| \cdot \|_B) \) and the conditions (2) and (3) imply a double module structure for such Banach spaces:

**Proposition 1.** Any MINTSTA \((B, \| \cdot \|_B)\) has a double module structure:

1. \((B, \| \cdot \|_B)\) is an essential Banach convolution module over the Beurling algebra \( L^1_{v_1} : \)
   \[\|g * f\|_B \leq \|g\|_{L^1_{v_1}} \|f\|_B, \quad g \in L^1_{v_1}, f \in B,\]
   and for any bounded, approximate unit \((e_\alpha)_{\alpha \in I} \) in \((L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})\) one has:
   \[\lim_{\alpha \to \infty} \|e_\alpha * f - f\|_B = 0, \quad \forall f \in B.\]

2. \((B, \| \cdot \|_B)\) is an essential Banach module with respect to pointwise multiplication over the Fourier-Beurling algebra \( A_{s_2} := \mathcal{F}L^1_{v_2}, \) with the corresponding norm estimate. Correspondingly bounded approximate units for pointwise multiplication act accordingly.

3. The Wiener amalgam space \((W(A_{s_2}, \ell^1_{v_2}), \| \cdot \|_{W(A_{s_2}, \ell^1_{v_2})})\) with local component \( A_{s_2} \) and global component \( \ell^1_{v_2} \) is continuously and densely embedded into \((B, \| \cdot \|_B)\).

The statement in this proposition only summarize results which may be considered as folklore. Integrated action appears in a similar form in \([4, 16]\) or \([9]\), for example. For the last (minimality) statement see \([14]\), or more explicitly \([17, \text{Proposition 6}]\). Also recall that the Wiener amalgam space \((W(A_{s_2}, \ell^1_{v_2}), \| \cdot \|_{W(A_{s_2}, \ell^1_{v_2})})\) is the subspace of continuous functions on \( \mathbb{R}^d \) which belong locally to the Banach algebra \( A_{s_2} \), which is supposed to contain \( \mathcal{D}(\mathbb{R}^d) \). These spaces can be characterized with the help of smooth BUPUs (bounded partitions of unity) in \( \mathcal{D}(\mathbb{R}^d) \), i.e. \( \psi \in \mathcal{D}(\mathbb{R}^d) \), with \( \sum_{k \in \mathbb{Z}^d} T_k \psi \equiv 1 \), as follows:

\[\|f\|_{W(A_{s_2}, \ell^1_{v_2})} := \sum_{k \in \mathbb{Z}^d} \|f \cdot T_k \psi\|_{A_{s_2} v_2}(k) < \infty.\]

Since this fact was partially motivating our approach and because we think that its statement may be interesting for the reader we have included it here. However, we will not make use of the third statement and thus we do not go into technical details here.

The following lemma is inspired by the investigations in \([12]\):

**Lemma 1.** Let \((B, \| \cdot \|_B) \subset S'(\mathbb{R}^d)\) be a translation invariant Banach space with

\[\|T_x f\|_B \leq w(x) \|f\|_B, \quad \forall f \in B, x \in \mathbb{R}^d,\]

for some submultiplicative function \( w \) on \( \mathbb{R}^d \).

Then \((B_{1,w}, \| \cdot \|_{B_{1,w}})\), the vector space \( B_{1,w} = L^1_w \cap B \) is a dense, essential Banach ideal inside the Banach convolution algebra \((L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})\) with respect to the norm

\[\|f\|_{B_{1,w}} = \|f\|_{L^1_w} + \|f\|_B, \quad f \in B_{1,w}.\]

Thus it is a Banach space with respect to the norm \((7)\) and \(L^1_w * B_{1,w} \subset B_{1,w}\) with

\[\|g * f\|_{B_{1,w}} \leq \|g\|_{L^1_w} \|f\|_{B_{1,w}}, \quad g \in L^1_w, f \in B_{1,w},\]

and that for any bounded, approximate unit \((e_\alpha)_{\alpha \in I} \) in \((L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})\):

\[\lim_{\alpha \to \infty} \|e_\alpha * f - f\|_{B_{1,w}} = 0, \quad \forall f \in B_{1,w},\]

and consequently (by the Cohen-Hewitt Factorization Theorem):

\[L^1_w * B_{1,w} = B_{1,w}.\]
If \( w \) is a Beurling weight of polynomial growth, then \( \mathcal{S}(\mathbb{R}^d) \) is dense in \( \mathcal{B}_{1,w}(\mathbb{R}^d) \), hence \( \mathcal{D}(\mathbb{R}^d) = \mathcal{S} \cap C_c(\mathbb{R}^d) \) is a dense subspace of \( (\mathcal{B}_{1,w}, \| \cdot \|_{\mathcal{B}_{1,w}}) \).

**Proof.** The facts collected in the above lemma are essentially based on more detailed considerations published in [12]. The first step is to verify that \( \mathcal{B}_{1,w} \) is a dense subspace of \( L^1_w(\mathbb{R}^d) \). For this purpose it is enough to note that compactly supported elements are dense in any Beurling algebra, i.e. for given \( \varepsilon > 0 \) one finds \( \varphi \in C_c(\mathbb{R}^d) \) with

\[
\| f - \varphi \|_{L^1_w} < \varepsilon / 4. \tag{13}
\]

By smoothing \( \varphi \in C_c(\mathbb{R}^d) \) with a sufficiently small supported test function \( \psi \in \mathcal{D}(\mathbb{R}^d) \) one has

\[
\| \psi - \varphi * \varphi \|_{L^1_w} < \varepsilon / 4. \tag{14}
\]

Observing that \( \psi * \varphi \in \mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset \mathcal{B} \) we note that \( \psi * \varphi \in \mathcal{B}_{1,w} \) and satisfies

\[
\| f - \psi * \varphi \|_{L^1_w} < \varepsilon / 2. \tag{15}
\]

Since both \( (L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w}) \) and \( (\mathcal{B}, \| \cdot \|_\mathcal{B}) \) are translation invariant and have continuous translation the same is true for \( (\mathcal{B}_{1,w}, \| \cdot \|_{\mathcal{B}_{1,w}}) \).

The remaining consequences are just stated for easier reference but are standard results. The last step is a consequence of the Cohen-Hewitt Factorization Theorem ( [31], Chap.32).

Next let us recall the important Beurling-Domar condition, which ensures the existence of band-limited elements (i.e. functions with a compactly supported Fourier transform) in general Beurling algebras. According to [38] (and referring to the important paper by Y. Domar, [10]) a Beurling weight \( w \) satisfies the Beurling-Domar non-quasianalyticity condition if one has

\[
(BD) \quad \sum_{n \geq 1} \frac{w(nx)}{n^2} < \infty \quad \forall x \in \mathbb{R}^d. \tag{16}
\]

It is an easy exercise to check that any polynomial weight satisfies the (BD)-condition. One of the advantages of increasing, radial symmetric weights, such as \( v_s \) for \( s \geq 0 \) is the fact that one can create easily bounded approximate units simply by applying to an arbitrary \( g \in \mathcal{S}(\mathbb{R}^d) \subset L^1_{v_s} \) with \( \hat{g}(0) = 1 \) the \( L^1 \)-norm preserving compression operator

\[
\text{St}_\rho g(x) = \rho^{-d} g(x/\rho), \quad \rho \to 0. \tag{17}
\]

The next result is just a reminder concerning Fourier-Beurling algebras, i.e. pointwise Banach algebras obtained as \( (\mathcal{F}L_w^1(\mathbb{R}^d), \| \cdot \|_{\mathcal{F}L^1_w}) \), with the norm \( \| f \|_{\mathcal{F}L^1_w} = \| f \|_{L^1_w} \). The statements of the following lemma are essentially restatements of facts found in [38].

**Lemma 2.** Let \( w \) be a weight function satisfying the (BD)-condition. Then

i) the Beurling algebra \( (L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w}) \) has bounded approximate units consisting of band-limited elements.

ii) the Banach algebra \( (\mathcal{F}L^1_w(\mathbb{R}^d), \| \cdot \|_{\mathcal{F}L^1_w}) \) is a Wiener algebra in the sense of H. Reiter ([33]), meaning that it is a regular Banach algebra of continuous functions, which allows (among others) to separate compact sets from open neighborhoods (see Chap.2.4 of [33] resp. [39]). In particular, the compactly supported elements are dense.

iii) for weight functions of polynomial growth one has in addition:

\[
\mathcal{S}(\mathbb{R}^d) \hookrightarrow (L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w}) \quad \text{and for any } g \in \mathcal{S}(\mathbb{R}^d) \quad \text{with } \int_{\mathbb{R}^d} g(x)dx = 1 \quad \text{one has}
\]

\[
\| \text{St}_\rho f - f \|_{L^1_w(\mathbb{R}^d)} \to 0, \quad \text{for } \rho \to 0.
\]
The third claim is essentially Lemma 1 of [27], see also Corollary 1 of [9], see also [39], Proposition 1.6.14.

**Lemma 3.** For any MINTSTA \((B, \| \cdot \|_B)\) the band-limited elements form a dense subspace of \((B, \| \cdot \|_B)\). The same is true for \((B_{1,w}, \| \cdot \|_{B_{1,w}})\), which is in fact also a MINTSTA itself.

**Proof.** Since \(S(\mathbb{R}^d)\) is dense in \(L^1_w(\mathbb{R}^d)\) for the (polynomial) weights appearing in Proposition 1 the (BD)-condition is clearly satisfied. Hence there are (even bounded) approximate units in \((L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})\) which are band-limited (i.e. compactly supported on the Fourier transform side), we write \((e_\alpha)_{\alpha \in I}\).

Since \(e_\alpha * f\) approximates \(f \in B\) according to (11) and

\[
\text{supp}(\mathcal{F}(e_\alpha * f)) \subset \text{supp}(\hat{e}_\alpha)
\]

we find that the band-limited elements are dense in \((B, \| \cdot \|_B)\). \(\square\)

For the proof of the main result we will make use of the following key observations already explained in detail in [27], using so-called BUPUs, i.e. (bounded) uniform partitions of unity of size \(|\Psi| = \delta > 0\), i.e. of countable collections of continuous functions \(\psi_i, i \in I\), with \(0 \leq \psi_i(x)\), \(\text{supp}(\psi_i) \subset B_\delta(x_i)\) and \(\sum_{i \in I} \psi_i(x) \equiv 1\).

For simplicity we only consider the regular case, i.e. BUPUs which are obtained as translates of a single function:

**Definition 3.** A (countable) family of translates \(\Psi = (T_\lambda \psi)_{\lambda \in \Lambda}\), where \(\psi\) is a compactly supported function (i.e. \(\psi \in C_c(\mathbb{R}^d)\)), and \(\Lambda = A(\mathbb{Z}^d)\) a lattice in \(\mathbb{R}^d\) (for some nonsingular \(d \times d\)-matrix \(A\)) is called a regular BUPU (or more precisely a \(\Lambda\)-regular BUPU, or a \(\Lambda\)-invariant BUPU) if

\[
\sum_{\lambda \in \Lambda} \psi(x - \lambda) \equiv 1. \tag{18}
\]

We will write \(\text{diam}(\Psi) \leq \gamma\) if \(\text{supp}(\psi) \subset B_\gamma(0)\) for some \(\gamma \to 0\).

We will be interested in the case of “finer and finer” BUPUs, i.e. the case \(\text{diam}(\Psi) \to 0\). Sometimes we will simply write \(|\Psi|\) instead of \(\text{diam}(\Psi)\). The use of fine partitions of unity is also well established in the context of usual distribution theory, see e.g. [12].

The following results are required in the sequel, we refer [27] for details of the proof. The method can even be used to introduce convolution of measures (see [22]) and goes back to the discretization introduced in [19].

**Proposition 2.** There exists \(C_1 > 0\) such that for any BUPU \(\Psi = (\psi_i)_{i \in I}\) with \(|\Psi| \leq \delta \leq 1\) the mapping \(f \mapsto D_\Psi f\), given by

\[
D_\Psi f = \sum_{i \in I} c_i \delta_{x_i} \quad \text{with} \quad c_i = \int_{\mathbb{R}^d} f(x) \psi_i(x) dx, \tag{19}
\]

satisfies

\[
\sum_{i \in I} |c_i| w(x_i) \leq C_1 \|f\|_{L^1_w}, \quad f \in L^1_w(\mathbb{R}^d). \tag{20}
\]

Moreover, given \(g \in L^1_w\) and \(\varepsilon > 0\) there exists \(\delta \in (0, 1]\) such that \(|\Psi| \leq \delta\) implies

\[
\|g * f - g * D_\Psi f\|_{L^1_w} < \varepsilon. \tag{21}
\]
**Proposition 3.** Since any MINTSTA \((B, \| \cdot \|_B)\) is an essential Banach module over some Beurling algebra \((L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})\) one has for any \(h \in L^1_w(\mathbb{R}^d)\): Given \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(|\Psi| \leq \delta\) implies:

\[
\|g * h - g * D\Psi h\|_B \leq \varepsilon \|h\|_{L^1_w}.
\]

**Remark 1.** It is important to note (for the proof of our main theorem) that the convergence is uniform for bounded subsets of \(L^1_w(\mathbb{R}^d)\), because it depends only on the continuous translation property in \((B, \| \cdot \|_B)\), and thus on the quality of \(g \in B\).

### 3. Preparing the Ground

Preparing for the main result of this note we start by recalling the characterization of compact subsets in spaces with a double module structure as described in \([10]\). We will rephrase the result given there in a form which is more suitable for the current setting. It just makes use of the fact that approximate units can always be chosen from any dense subspace of \(\mathcal{S}(\mathbb{R}^d)\), e.g. to choose pointwise approximate units (which help to localize functions).

**Theorem 1.** A closed and bounded subset \(M\) of a MINTSTA is compact in \((B, \| \cdot \|_B)\) if and only it is tight and equicontinuous, which means, that for any \(\varepsilon > 0\) there exist a band-limited function \(g \in \mathcal{S}(\mathbb{R}^d)\) and a compactly supported function \(h \in \mathcal{S}(\mathbb{R}^d)\) such that \(f \mapsto g * f\) and \(f \mapsto h \cdot f\) are bounded operators on \((B, \| \cdot \|_B)\) and satisfy

\[
\|h \cdot f - f\|_B \leq \varepsilon, \quad \forall f \in M,
\]

and

\[
\|g * f - f\|_B \leq \varepsilon, \quad \forall f \in M.
\]

**Remark 2.** The assumptions boil down to choose appropriate elements from a Dirac sequence \((g_n)_{n \geq 1}\), e.g. to choose \(g\) as a suitable compression of a band-limited function \(g_0\) with \(\widehat{g_0}(0) = 1\) and \(\text{supp}(\hat{g_0})\) compact. Correspondingly one can choose for pointwise multiplication from a sequence of (inverse) Fourier transforms of another Dirac sequence, i.e. \(h_n := \mathcal{F}^{-1}(g_n)\), but now with compact support on the time side.

Note that approximate units can be chosen from any dense subspace of \((L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})\).

The most useful examples are band-limited Dirac sequences and compactly supported pointwise approximate units (which help to localize functions).

For the proof of our main result we will also need the following useful lemma:

**Lemma 4.** Given a double Banach module \((B, \| \cdot \|_B)\), over a Beurling algebra \(L^1_v(\mathbb{R}^d)\) with respect to convolution and over \(\mathcal{J}L^1_v(\mathbb{R}^d)\) under pointwise multiplication. Assume that \(M\) is a bounded and tight subset of \((B, \| \cdot \|_B)\) and \(S\) a bounded, tight subset of \((L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})\). Then the set \(S * M\) is a bounded and tight subset of \((B, \| \cdot \|_B)\).

**Proof.** Without loss of generality suppose \(\|g\|_{L^1_w(\mathbb{R}^d)} \leq 1\) and \(\sup_{f \in M} \|f\|_B \leq 1\). The estimate \(\|g * f\|_B \leq \|g\|_{L^1_w} \cdot \|f\|_B \leq 1\) shows that \(S * M\) is bounded in \((B, \| \cdot \|_B)\).

Since \((B, \| \cdot \|_B)\) is a Banach module over the Fourier-Beurling algebra \(\mathcal{J}L^1_v(\mathbb{R}^d)\) \(\text{for some weight of polynomial growth}\) we may assume that there is a compactly supported (pointwise) approximation to the identity \((k_\alpha)_{\alpha \in I}\), with \(\|k_\alpha\|_{L^1_v} \leq C_v < \infty\) \((\text{usually} C_v = 1)\). Since \(L^1_w(\mathbb{R}^d)\) contains \(\mathcal{S}(\mathbb{R}^d)\) it is clear that \(\mathcal{J}L^1_v(\mathbb{R}^d)\) also contains compactly supported functions \(h \in \mathcal{S}(\mathbb{R}^d)\) which are plateau-like, i.e. which satisfy \(h(x) \equiv 1\) on a
Hence we get the final estimate by combining estimates (27) and (29):  

\[ \|k_1 \cdot g - g\|_{L^1_v} \leq \epsilon/(2 \cdot C_v), \quad g \in S, \]

and  

\[ \|k_2 \cdot f - f\|_B \leq \epsilon/2, \quad f \in M. \]

Combining these estimates we find that  

\[ \|(k_1 \cdot g) \ast (k_2 \cdot f) - g \ast f\|_B \leq \|(k_1 \cdot g - g) \ast (k_2 \cdot f)\|_B + \|g \ast (k_2 \cdot f - f)\|_B. \]  

(25)

The first term can be estimated as  

\[ \leq \|k_1 \cdot g - g\|_{L^1_v} \|k_2 \cdot f\|_B \leq (\epsilon/(2 \cdot C_v)) \|k_2\|_{\mathcal{J}L^1_v} \|f\|_B \leq \epsilon/2, \]  

(26)

and the second term in a similar way. Combining these estimates we find that  

\[ \|(k_1 \cdot g) \ast (k_2 \cdot f) - g \ast f\|_B \leq \epsilon, \quad g \in S, f \in M. \]  

(27)

Now it is obvious that the elements in \( B \) of the form  

\[ \{k_1 \cdot g \ast (k_2 \cdot f), g \in S, f \in M\} \]

have common compact support inside the compact set \( Q = \text{supp}(k_1) + \text{supp}(k_2) \). As explained above we can find now some \( k_3 \in \mathcal{J}L^1_v(\mathbb{R}^d) \), with  

\[ \|k_3\|_{\mathcal{J}L^1_v(\mathbb{R}^d)} = \|k_3\|_{L^1_v} \leq C_v, \quad \text{and} \quad k_3(x) \equiv 1 \text{ on } Q. \]

Since \( (B, \| \cdot \|_B) \) is a pointwise Banach module over \( \mathcal{J}L^1_v(\mathbb{R}^d) \) we also have  

\[ \|k \cdot h\|_B \leq \|k\|_{\mathcal{J}L^1_v(\mathbb{R}^d)} \|h\|_B, \quad k \in \mathcal{J}L^1_v(\mathbb{R}^d), h \in B. \]  

(28)

Multiplying the terms in estimate (27) by \( k_3 \) (the first one is unchanged!), we obtain  

\[ \|(k_1 \cdot g) \ast (k_2 \cdot f) - k_3 \cdot (g \ast f)\|_B \leq \|k_3\|_{L^1_v} \cdot \|(k_1 \cdot g) \ast (k_2 \cdot f) - g \ast f\|_B \leq C_v \epsilon. \]  

(29)

Hence we get the final estimate by combining estimates (27) and (29):  

\[ \|k_3 \cdot (g \ast f) - g \ast f\|_B \leq (1 + C_v) \epsilon, \quad g \in S, f \in M. \]  

(30)

□

Another estimate which we will need, but which might be useful as a separate result elsewhere, is formulated in the following lemma.

In the course of the proof of our main result we will also make use of the following observation:

**Lemma 5.** If \( (B, \| \cdot \|_B) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \) is a translation invariant Banach space with  

\[ \|T_xf\|_B \leq w(x)\|f\|_B, \quad \forall f \in B. \]

Then for any pair of functions \( g_3 \in \mathcal{S}(\mathbb{R}^d) \) and \( k \in C_c(\mathbb{R}^d) \) the product-convolution operator \( f \mapsto k \cdot (g_3 \ast f) \) defines a bounded operator from \( (B, \| \cdot \|_B) \) into \( L^1_v(\mathbb{R}^d) \).
Lemma 6. For any family $\{g_3\} \subseteq B \subset \mathcal{S}'(\mathbb{R}^d)$, which (at least for $f \in \mathcal{S}(\mathbb{R}^d)$) allows to interpret the convolution for any $f \in B$ in a pointwise sense:

$$g_3 \ast f(x) = \int_{\mathbb{R}^d} f(z + x)g_3(-z)dz = \sigma_3(T_x f),$$

(31)

where $\sigma_3$ means the tempered distribution induced by $g_3 \in B$ (using $g^*(x) := g(-x)$).

By the continuity of both sides we can thus use the identity

$$g_3 \ast f(x) = \sigma_3(T_x f), \quad f \in B, \; x \in \mathbb{R}^d.$$ 

(32)

Since translation is continuous, this implies that $g_3 \ast f$ is a continuous function (of polynomial growth). As a pointwise estimate we have

$$|g_3 \ast f(x)| \leq \|g_3\|_{B'} \|T_x f\|_B \leq \|g_3\|_{B'} \|T_x\|_B \|f\|_B.$$ 

(33)

Since $K := \text{supp}(k)$ is a compact set, the weight function is bounded on $-K$, or

$$\sup_{x \in K} w(-x) = C_k < \infty$$

which implies for any $x \in \text{supp}(k)$ the pointwise estimate

$$|k(x)||g_3 \ast f(x)| \leq w(-x)\|g_3\|_{B'} \|f\|_B \leq C_k \|g_3\|_{B'} \|f\|_B.$$ 

(34)

This in turn provides us with the required estimate:

$$\| |k| \cdot |g_3 \ast f| \|_{L^1_{\dot{B}'}} = \int_{\mathbb{R}^d} |k(x)||g_3 \ast f(x)|w(x)dx \leq [C_k \|k\|_{L^1_{\dot{B}'}}, \|g_3\|_{B'}] \|f\|_B.$$ 

(35)

$$\Box$$

For the proof of the main result we will make use of the following key observations already explained in detail in [27], using so-called BUPUs, i.e. \( \text{(bounded) uniform partitions of unity of size} \ |\Psi| = \delta > 0 \), i.e. of countable collections of continuous functions $\psi_i, i \in I$, with $0 \leq \psi_i(x), \text{supp}(\psi_i) \subset B_\delta(x_i)$ and $\sum_{i \in I} \psi_i(x) \equiv 1$.

The following technical lemma will be useful for our considerations. It requires the notion of well-spread sets, playing a big role for the use of BUPUs for the characterization of Wiener amalgam space (see [13]) or in coorbit theory (see [20]).

Definition 4. A discrete family $X = \{x_i\}_{i \in I}$ is \textit{relatively separated} if for some (and thus for any) $r > 0$ the number of points from $X$ is uniformly bounded, i.e. if

$$\sup_{x \in \mathbb{R}^d} \#\{i \in I \mid x_i \in B_r(x)\} \leq C_r < \infty.$$ 

Lemma 6. For any family $\{x_j\} \subseteq J$ which is $\delta/3$ dense, there exists a subfamily $\{x_i\}_{i \in I}$ which is well-spread and $\delta$-dense. Moreover, there exists corresponding BUPUs (in the sense of [13]) $(\psi_i)_{i \in I}$ of size $\delta$, with $\text{supp}(\psi_i) \subseteq B_{\delta}(x_i), i \in I$.

Proof. The generic approach to this task is to find a maximal subfamily $I \subset J$ such that one has a pavement of size $\delta/3$ in $\mathbb{R}^d$, i.e. such that the balls $B_{\delta/3}(x_i)$ are pairwise disjoint for $i \in I \subset J$. The maximality implies that for any $j \in J \setminus I$ the corresponding ball $B_{\delta/3}(x_j)$ has nontrivial intersection with one of the balls centered at $x_i$, for some $i \in I$. As the original family (running through the index set $J$) covers all of $\mathbb{R}^d$ it is an immediate consequence of the triangle inequality that the family $B_{2\delta/3}(x_i), i \in I$ covers all of $\mathbb{R}^d$, and the same is true for $(B_{\delta}(x_i))_{i \in I}$.
The proof requires a couple of steps. Let us fix the sets constituting the pavement and the uniform volume estimate for balls of radius \( r + \delta \) (independent from \( x \)) then provides the required counting estimate, uniformly with respect to \( x \).

In order to create a BUPU of size \( \delta \) with \( \text{supp}(\psi_i) \subset B_\delta(x_i) \) we proceed as follows: First we observe that the selected family of balls has finite overlap. Moreover, starting from some function \( \varphi \in C_c^0(\mathbb{R}^d) \) with \( \varphi(z) = 1 \) on \( B_{2\delta/3}(0) \) and \( \text{supp}(\varphi) \subset B_\delta(0) \), we set \( \Phi = \sum_{i \in I} T_{x_i} \varphi \) and put \( \psi_i \equiv T_{x_i} \varphi / \Phi \). This implies \( \sum_{i \in I} \psi_i(x) \equiv 1 \) and
\[
\text{supp}(\psi_i) \subset \text{supp}(T_{x_i} \varphi) = x_i + \text{supp}(\varphi) \subset B_\delta(x_i).
\]

Remark 3. For the Euclidian set there is an alternative method to select a well-separated set from a given \( \delta/3 \)-dense family \( (x_j)_{j \in J} \). Starting from a partition of \( \mathbb{R}^d \) with cubes of size \( \alpha > 0 \) (small enough), centered at the lattice \( \alpha \mathbb{Z}^d \), one can pick one point within each such cube. We leave the details to the interested reader.

4. The Main Result

Theorem 2. Given a relatively compact set \( M \) in any MINTSTA \( (B, \| \cdot \|_B) \) and some \( g_0 \in S(\mathbb{R}^d) \) with \( \hat{g}_0(y) \neq 0 \) for all \( y \in \mathbb{R}^d \), we can show the following:

For any given \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that for any \( \delta \)-dense set \( (x_i)_{i \in I} \) one can pick a finite subset \( F \subset I \) and construct a finite rank operator \( T \) with range in the linear span of \( S(g_0) := \{ T_{x_i}g_0, i \in F \} \) with
\[
\| T(f) - f \|_B \leq \varepsilon, \quad \forall f \in M.
\]

Remark 4. We will give the proof for compact sets. In fact, since all the estimates allow the transition from \( M \) to the closure \( \overline{M} \) of the set \( M \) in \( (B, \| \cdot \|_B) \) (and obviously the validity for \( \overline{M} \) implies the validity for \( M \)) this implies the general case.

Proof. The proof requires a couple of steps. Let us fix \( \varepsilon > 0 \).

1. First of all one has to check that \( S(g_0) \) is a subset of \( B \). In fact, we have \( g_0 \in S(\mathbb{R}^d) \subset B \cap L^1_w(\mathbb{R}^d) \subset B \), and all the involved spaces are translation invariant. Thus any finite rank operator with range in a finite-dimensional subspace of the linear span of \( S(g_0) \) will be bounded on \( (B, \| \cdot \|_B) \).

2. For the given \( \varepsilon > 0 \) we choose first (according to (24) in Thm. 1) a band-limited function \( g \in S(\mathbb{R}^d) \subset B_{1,w} \) such that
\[
\| f - g * f \|_B < \varepsilon/4, \quad \forall f \in M.
\]

3. Invoking now the Tauberian condition and the smoothness of \( \hat{g}_0 \) we observe that for such a band-limited \( g \in S(\mathbb{R}^d) \subset B_{1,w}(\mathbb{R}^d) \subset L^1_w(\mathbb{R}^d) \) with \( \text{spec}(g) := \text{supp}(\hat{g}) = Q_1 \) (some compact set) we can find another band-limited function \( h_1 \in S(\mathbb{R}^d) \) with compact spectrum \( Q_2 = \text{supp}(\hat{h}_1) \), such that \( g = g * h_1 \) (or \( \hat{h}_1(s) = 1 \) on \( Q_1 \)). Since \( \hat{g}_0(y) \neq 0 \) for \( y \in Q_2 \) it follows that \( 1/f_\hat{g}_0(y) \) is a smooth function as well, and consequently the pointwise product \( h_1(y)/\hat{g}_0(y) \) defines

\[1\text{We cannot expect that there is a uniform bound on the operators constructed below!}\]
Next we observe that the tightness of the set $M \subset B$ implies that also $g_1 * g * M$ is a tight set in $(B, \| \cdot \|_B)$ (by choosing $S = \{ g_1 * g \}$ in Lemma 4). Thus we can find some compactly supported function $k \in C_c(\mathbb{R}^d)$ such that
\[
\| k \cdot (g_1 * g * f) - (g_1 * g * f) \|_B \leq \varepsilon / 4 \| g_0 \|_{L^1_B}, \quad \forall f \in M.
\]
We also note that by Lemma 5 (with $g_3 = g * g_1$)
\[
S_k(M) := \{ h := k \cdot (g_1 * g * f), f \in M \}
\]
is a bounded subset of $L^1_B(\mathbb{R}^d)$, so that Proposition 3 can be invoked. Using Lemma 3 we can apply the discretization operator for any BUPU (and we can take $\delta > 0$ only depending on $g_0 \in \mathcal{S}(\mathbb{R}^d) \subset B$) and will get:
\[
\| g_0 * h - g_0 * D_q h \|_B \leq \varepsilon / 4, \quad \forall h \in S_k(M).
\]
Recalling identity (38) and the estimate (37) we note that
\[
\| f - g_0 * (g_1 * g * f) \|_B = \| f - g * f \|_B \leq \varepsilon / 4, \quad f \in M,
\]
and furthermore, using the estimate (39), one obtains for $f \in M$:
\[
\| f - g_0 * (g_1 * g * f) \|_B \leq \| g_0 \|_{L^1_B} \| h - g_1 * g * f \|_B \leq \varepsilon / 4.
\]
Applying the triangular equation to the last two estimates, i.e. using
\[
\| f - g_0 * h \|_B \leq \| f - g_0 * (g_1 * g * f) \|_B + \| g_0 * (g_1 * g * f - h) \|_B
\]
we arrive at the estimate
\[
\| f - g_0 * h \|_B \leq \varepsilon / 4 + \varepsilon / 4 = \varepsilon / 2, \quad f \in M.
\]
So finally combining the estimates (45) and (11) we arrive at our final estimate:
\[
\| f - g_0 * D_q h \|_B = \| f - g_0 * h \|_B + \| g_0 * h - g_0 * D_q h \|_B \leq \varepsilon, \quad f \in M.
\]
Writing the approximation operator $T$ in explicit form, we have
\[
T f = g_0 \ast D_q h = g_0 \ast D_q (k \cdot (g_1 * g * f)) = \sum_{i \in F} c_i T_\psi_i g_0,
\]
with the coefficients $(c_i)_{i \in F}$ depending in a linear way on the function $f \in B$ via
\[
c_i := \int_{\mathbb{R}^d} h(x) \psi_i(x) dx = \int_{\mathbb{R}^d} (k \cdot (g_1 * g * f))(x) \psi_i(x) dx, \quad i \in F.
\]
Here, due to the fact that the centers are well-spread and that $\text{supp}(k)$ is compact the following set $F$ is a finite:
\[
F := \{ i \in I \mid \psi_i \not= 0 \}.
\]
Remark 5. A careful analysis of the proof given above shows that the assumption \( g \in S(\mathbb{R}^d) \) is convenient in the given setting, but in fact it is only required to assume \( g \in B_{1,w}(\mathbb{R}^d) = B \cap L_w^1(\mathbb{R}^d) \). For such a case the pointwise inversion argument based on smoothness used in step (3) of the above has to replaced by Wiener’s inversion Theorem for Beurling algebras which has been already mentioned earlier (see [39], Chap.1.6.5). Since in this case the possible choices of \( g \) depend on \( B \) (and thus is not universal with respect to the family of Banach spaces under consideration) we have chosen to present our results in the form of Theorem 2.

Remark 6. The proof of the main theorem demonstrates a couple of facts: While ideally one would like to replace the given functions \( f \in M \) by functions which are both band-limited (hence smooth) and compactly supported, the impossibility of getting this done at the same time requires to carry out the corresponding modifications of the given functions \( f \in M \) stepwise, in order to come up with the final approximation, obtained via a discrete convolution.

In order to make things work one has to make use of the fact that convolution typically improves \emph{local} properties (such as continuity or smoothness), while preserving global decay conditions. In a similar way pointwise multiplication by smooth, compactly supported functions will preserve the local properties, while improving the \emph{global} ones (e.g. by creating a compactly supported function).

The structure of the proof would not really simplify if we had restricted our attention to the unweighted case. Even for simple special cases such as \( B = L^p(\mathbb{R}^d) \) the key steps would be the same. A crucial property used is the uniformity of the corresponding approximation arguments over (relatively) compact sets \( M \) in \( (B, \| \cdot \|_B) \).

5. Applications

Although we have indicated that there is room for further generalization we have to point out that the list of examples to which the above result applies is endless. The first author has tried to collect the construction principles widely (or occasionally) used in Fourier Analysis in a systematic way. It appears to be harder to identify space which are generally important, containing \( S(\mathbb{R}^d) \) as a dense subspace, but not satisfying the assumptions formulated for this paper.

In order to mention at least some of the most important cases which are playing a major role in the literature let us mention:

1. Weighted \( L^p \)-spaces, for \( 1 \leq p < \infty \) with polynomial weights (see [13], [30]);
2. Wiener amalgam spaces of the form \( W(L^p, \ell^1_v) \), for \( 1 \leq p, q < \infty \) (see [33]), but also weighted Wiener amalgam spaces, as treated in [31];
3. The Besov-Triebel-Lizorkin spaces \( (B^s_{p,q}(\mathbb{R}^d), \| \cdot \|_{B^s_{p,q}}) \) resp. \( (F^s_{p,q}(\mathbb{R}^d), \| \cdot \|_{F^s_{p,q}}) \), for \( 1 \leq p, q < \infty \) (see [37] or the books of H. Triebel [41]);
4. Modulation spaces in general (see [20]), but specifically the classical modulation spaces \( (M^s_{p,q}(\mathbb{R}^d), \| \cdot \|_{M^s_{p,q}}) \), also for the case \( 1 \leq p, q < \infty \). See [3] and [7];
5. More general decomposition spaces, as discussed in the literature, based on the approach given in [25];
6. Tauberian Theorems have been derived directly in the context of functions of bounded means in [18], where they have been a crucial step in order to extend Wiener’s Third Tauberian Theorem to the full range \( 1 < p \leq \infty \), and for \( \mathbb{R}^d \) instead of just \( p = 2 \) and \( d = 1 \) (see [43],[44]).
(7) The atomic space \( \mathcal{H}(q, p, \alpha) \) appearing in \([23]\) as well as many other atomic spaces, including the exotic case discussed in \([28]\), are MINTSTAs;

(8) General construction principles for a further variety of function spaces as well as a long list of references are provided in \([21]\).

The fact that the Schwartz space can be characterized as the intersection of modulation spaces (see \([29]\), Proposition 11.3.1) implies that the arguments used for the proof of our main result also provide a constructive way to verify that the set of translates of \( g \) generates a dense subspace of the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \).

\[ \text{Corollary 1.} \] Assume that \( g \in \mathcal{S}(\mathbb{R}^d) \) satisfies \( \hat{g}(y) \neq 0 \) for all \( y \in \mathbb{R}^d \). Then for any finite set \( M \subset \mathcal{S}(\mathbb{R}^d) \) there is a sequence of finite rank operators \( T_n \) with range in the linear span of \( S(g_0) := \{ T_x g_0, i \in F \} \), such that \( T_n(f) \to f \) in \( \mathcal{S}(\mathbb{R}^d) \) for \( n \to \infty \), for each \( f \in M \). In particular, this linear span is dense in \( \mathcal{S}(\mathbb{R}^d) \).

\[ \text{Proof.} \] We only have to observe that the family of norms for the spaces \( \mathcal{M}^\infty_s(\mathbb{R}^d) \), for \( s \geq 0 \) define a topology which is equivalent to the usual topology on \( \mathcal{S}(\mathbb{R}^d) \) (see also \([32]\)). Also, \( \mathcal{S}(\mathbb{R}^d) \) is not only the intersection of all these spaces, but one can also replace them by the closure of \( \mathcal{S}(\mathbb{R}^d) \) in each of these space (for each fixed \( s \geq 0 \)). Since such spaces are typical examples for the setting of our main result we can guarantee convergence of a suitable sequence of finite rank operators for each of these norms, even uniformly with respect to compact subsets of such a space. \( \square \)

The extension of the statement to relatively compact subsets of \( \mathcal{S}(\mathbb{R}^d) \) is just a matter of technical arguments which are beyond the focus of our paper. Also, it is true that one can find easier, non-constructive arguments for the last statement, making use of standard Fourier transform methods for the space of tempered distributions.

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\[ \text{References} \]

[1] G. F. Bachelis, W. A. Parker, and K. A. Ross. Local units in \( L^1(G) \). \textit{Proc. Amer. Math. Soc.}, 31:312–313, 1972.

[2] A. Beurling: Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionnelle. \textit{IX. Congr. Math. Scand., pp.345-366, Helsingfors, 1938}.

[3] A. Benyi and K. A. Okoudjou. \textit{Modulation Spaces}. Springer, Birkhäuser, New York, 2020.

[4] W. Braun and H. G. Feichtinger. Banach spaces of distributions having two module structures. \textit{J. Funct. Anal.}, 51:174–212, 1983.
[5] J. T. Burnham. Closed ideals in subalgebras of Banach algebras I. *Proc. Amer. Math. Soc.*, 32:551–555, 1972.
[6] J. Cigler. Normed ideals in $L^1(G)$. *Nederl. Akad. Wetensch. Proc. Ser. A*, Ser. A(74):273–282, 1969.
[7] E. Cordero and L. Rodino. *Time-frequency Analysis of Operators*. De Gruyter Studies in Mathematics, Berlin, 2020.
[8] P. Dimovski, S. Pilipovic, B. Prangoski, and J. Vindas. Translation–modulation invariant Banach spaces of ultradistributions. *J. Fourier Anal. Appl.*, 25(3):819–841, 2019.
[9] P. Dimovski, S. Pilipovic, and J. Vindas. New distribution spaces associated to translation-invariant Banach spaces. *Monatsh. Math.*, 177(4):495–515, 2015.
[10] Y. Domar. Harmonic analysis based on certain commutative Banach algebras. *Acta Math.*, 96:1–66, 1956.
[11] D. H. Dunford. Segal algebras and left normed ideals. *J. Lond. Math. Soc. (2)*, 8:514–516, 1974.
[12] H. G. Feichtinger. Results on Banach ideals and spaces of multipliers. *Math. Scand.*, 41(2):315–324, 1977.
[13] H. G. Feichtinger. Gewichtsfunktionen auf lokalkompakten Gruppen. *Sitzber. d. österr. Akad. Wiss.*, 188:451–471, 1979.
[14] H. G. Feichtinger. A characterization of minimal homogeneous Banach spaces. *Proc. Amer. Math. Soc.*, 81(1):55–61, 1981.
[15] H. G. Feichtinger. Banach convolution algebras of Wiener type. In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, p.509–524, Vol.35, North-Holland, Colloq. Math. Soc. Janos Bolyai, [Amsterdam] 1983.
[16] H. G. Feichtinger. Compactness in translation invariant Banach spaces of distributions and compact multipliers. *J. Math. Anal. Appl.*, 102:289–327, 1984.
[17] H. G. Feichtinger. Minimal Banach spaces and atomic representations. *Publ. Math. Debrecen*, 34(3-4):231–240, 1987.
[18] H. G. Feichtinger. An elementary approach to Wiener’s third Tauberian theorem for the Euclidean $n$-space. In *Symposia Math.*, Vol. XXIX of *Analisa Armonica*, pages 267–301, Cortona, 1988.
[19] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.*, 86(2):307–340, 1989.
[20] H. G. Feichtinger. Discretization of convolution and reconstruction of band-limited functions from irregular sampling. pages 333–345. Academic Press, Boston, MA, 1991.
[21] H. G. Feichtinger. Modulation spaces on locally compact Abelian groups. In R. Radha, M. Krishna, and S. Thangavelu, editors, *Proc. Internat. Conf. on Wavelets and Applications*, pages 1–56, Chennai, January 2002, 2003. New Delhi Allied Publishers.
[22] H. G. Feichtinger. Choosing Function Spaces in Harmonic Analysis, volume 4 of *The February Fourier Talks at the Norbert Wiener Center*, *Appl. Numer. Harmon. Anal.*, pages 65–101. Birkhäuser/Springer, Cham, 2015.
[23] H. G. Feichtinger. A novel mathematical approach to the theory of translation invariant linear systems. In Peter J. Bentley and I. Pesenson, editors, *Novel Methods in Harmonic Analysis with Applications to Numerical Analysis and Data Processing*, pages 483–516. Birkhäuser, Cham, 2017.
[24] H. G. Feichtinger and J. Feuto. Predual of Fofana’s spaces. *Mathematics (MDPI)*, 7(6):528, 2019.
[25] H. G. Feichtinger. Homogeneous Banach spaces are Banach convolution modules over $M(G)$. *MDPI, Mathematics*. Vol. 10 No.3, 2022, 1-22.
[26] H. G. Feichtinger and P. Gröbner. Banach spaces of distributions defined by decomposition methods. I. *Math. Nachr.*, 123:97–120, 1985.
[27] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.*, 86(2):307–340, 1989.
[28] H. G. Feichtinger and A. Gumber. Completeness of shifted dilates in invariant Banach spaces of tempered distributions. *Proc. Amer. Math. Soc.*, 149(12):5195–5210, 08 2021.
[29] H. G. Feichtinger and G. Zimmermann. An exotic minimal Banach space of functions. *Math. Nachr.*, 239-240:42–61, 2002.
[30] K. Gröchenig. *Foundations of Time-Frequency Analysis*. *Appl. Numer. Harmon. Anal. Birkhäuser*, Boston, MA, 2001.
[30] K. Gröchenig. Weight functions in time-frequency analysis. In L. Rodino and et al., editors, *Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis*, volume 52 of *Fields Inst. Commun.*, pages 343–366. Amer. Math. Soc., Providence, RI, 2007.

[31] K. Gröchenig, C. Heil, and K. Okoudjou. Gabor analysis in weighted amalgam spaces. *Sampl. Theory Signal Image Process.*, 1(3):225–259, 2002.

[32] K. Gröchenig and G. Zimmermann. Spaces of test functions via the STFT. *J. Funct. Spaces Appl.*, 2(1):25–53, 2004.

[33] C. Heil. An introduction to weighted Wiener amalgams. In M. Krishna, R. Radha, and S. Thangavelu, editors, *Wavelets and their Applications (Chennai, January 2002)*, pages 183–216. Allied Publishers, New Delhi, 2003.

[34] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups*. Springer, Berlin, Heidelberg, New York, 1970.

[35] V. Katsnelson. On the completeness of Gaussians in a Hilbert functional space. *Complex Anal. Oper. Theory*, 13(3):637–658, 2019.

[36] J. Korevaar. *Tauberian Theory. A Century of Developments*. Springer, Berlin, 2004.

[37] J. Peetre. *New Thoughts on Besov Spaces*. Duke University Mathematics Series, No. 1, Duke University, 1976. vi+305 pp.

[38] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups*. Clarendon Press, Oxford, 1968.

[39] H. Reiter and J. D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.* Clarendon Press, Oxford, 2000.

[40] M. A. Rieffel. Multipliers and tensor products of $L^p$-spaces of locally compact groups. *Studia Math.*, 33:71–82, 1969.

[41] H. Triebel. *Theory of Function Spaces. Vol. 78 of Monographs in Mathematics*. Birkhäuser, Basel, 1983.

[42] W. Walter. *Einführung in die Theorie der Distributionen*. B.I. Taschenbuch, 1974.

[43] N. Wiener. Tauberian theorems. *Ann. of Math. (2)*, 33(1):1–100, 1932.

[44] N. Wiener. *The Fourier Integral and Certain of its Applications*. Cambridge University Press, Cambridge, 1933.