Collision, explosion and collapse of homoclinic classes

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Abstract

Homoclinic classes of generic $C^1$-diffeomorphisms are maximal transitive sets and pairwise disjoint. We here present a model explaining how two different homoclinic classes may intersect, failing to be disjoint. For that we construct a one-parameter family of diffeomorphisms $(g_s)_{s \in [-1,1]}$ with hyperbolic points $P$ and $Q$ having nontrivial homoclinic classes, such that, for $s > 0$, the classes of $P$ and $Q$ are disjoint, for $s < 0$, they are equal, and, for $s = 0$, their intersection is a saddle-node.

Introduction

In this paper we study the collision of non-trivial homoclinic classes via saddle-node bifurcations and the dynamics before and after this collision. The main motivation of this paper comes from recent results about maximal transitive sets: for generic $C^1$-diffeomorphisms, the homoclinic classes are either disjoint or coincide, see [Ar] and [CMP].

Let us start by recalling some basic definitions. Given a diffeomorphism $f$, an $f$-invariant set $\Lambda$ is transitive if there is an $x \in \Lambda$ whose forward orbit is dense in $\Lambda$, i.e., $\Lambda = \bigcup_{i \in \mathbb{N}} f^i(x)$. A transitive set is maximal if it is a maximal element of the family of all transitive sets partially ordered by inclusion. Observe that any transitive set is contained in a maximal one. A transitive set $\Lambda$ is saturated if it contains every transitive set $\Sigma$ such that $\Lambda \cap \Sigma \neq \emptyset$. Clearly, every saturated transitive set is also maximal. The homoclinic class of a saddle $P$ of $f$, denoted by $H(P,f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of $P$. Every homoclinic class is a transitive set, not necessarily maximal nor saturated.

The problem of characterizing and describing (for a large class of systems) maximal and saturated transitive sets is a key problem in dynamics. In fact, these saturated transitive sets are the natural candidates for playing the role of the elementary pieces of dynamics (similar to the role of the basic sets in the Smale hyperbolic theory, [Sm]). Recently, [Ab] states that for generic $C^1$-diffeomorphisms $f$ having finitely many different homoclinic classes the non-wandering set of $f$, $\Omega(f)$, is the disjoint union of such classes. Moreover, these classes verify a weak form of hyperbolicity (existence of a dominated splitting, see [BDP]) and are the maximal invariant sets of a fixed filtration (see Section 6.3) independent of the generic diffeomorphism in a neighborhood of $f$.

Consider a closed manifold $M$ and denote by $\text{Diff}^1(M)$ the space of $C^1$-diffeomorphisms endowed with the usual uniform topology. In [Ar], it is proved that homoclinic classes of generic

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by a generic diffeomorphism we mean a diffeomorphism in a residual subset $R$ of $\text{Diff}^1(M)$. 

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diffeomorphisms are maximal transitive sets. [CMP] generalizes this result by proving that homoclinic classes of generic diffeomorphisms are saturated transitive sets. Thus homoclinic classes of generic diffeomorphisms are either equal or disjoint. Let us observe that there are locally generic diffeomorphisms having saturated transitive sets without periodic orbits (so which are not homoclinic classes), see [BD2].

The goal of this paper is to give examples of homoclinic classes which are not saturated transitive sets, presenting an explanation of how this pathology arises. In fact, we exhibit homoclinic classes which are not contained in any saturated transitive set. For simplicity, we consider diffeomorphisms defined on three manifolds, but our constructions can be carried out to higher dimensions after straightforward modifications.

The homoclinic classes in this paper simultaneously contain (in a stable way) hyperbolic fixed points of Morse index (dimension of the unstable bundle) one and two, hence not hyperbolic. We construct a diffeomorphism $f$ with saddles $P$ and $Q$ of Morse indices one and two, such that their homoclinic classes are nontrivial, maximal transitive, whose intersection is just a saddle-node (so the classes are not saturated transitive sets).

**Theorem A** There exist an open set $W$ and a family of diffeomorphisms $(g_s)_{s \in [-1,1]}$ such that, for every $s$, the diffeomorphism $g_s$ has hyperbolic fixed points $P$ and $Q$ of Morse indices 1 and 2 such that the maximal invariant set of $g_s$ in $W$, denoted by $\Lambda_s$, verifies the following:

- For every small $s < 0$, the set $\Lambda_s \cap \Omega(g_s)$ is the disjoint union of $H(P,g_s)$ and $H(Q,g_s)$, where $H(P,g_s)$ and $H(Q,g_s)$ are non-hyperbolic and locally maximal.
- For $s = 0$, $\Lambda_s = H(P,g_s) \cup H(Q,g_s)$, where $H(P,g_s)$ and $H(Q,g_s)$ are locally maximal and $H(P,g_s) \cap H(Q,g_s) = \{S\}$, where $S$ is a saddle-node fixed point.
- For every small $s > 0$, $\Lambda_s = H(P,g_s) = H(Q,g_s)$.

This result means that the homoclinic classes of $P$ and $Q$ collide at $s = 0$ and thereafter explode (the point $P$ that does not belong to $H(Q,g_0)$ is in $H(Q,g_s)$ for positive $s$, and the same holds for the point $Q$ and $H(P,g_s)$). Finally, these two homoclinic classes also collapse ($H(Q,g_s) = H(P,g_s)$) for positive $s$.

Taking the set $W$ to be a level of a filtration, (see Section 1.3), one gets the following:

**Theorem B** The homoclinic classes $H(P,g_0)$ and $H(Q,g_0)$ are not saturated and they are not contained in any saturated transitive set.

Our construction involves saddle-node bifurcations and heterodimensional cycles. In fact, we introduce a codimension-two bifurcation, the saddle-node heterodimensional cycles, and study the lateral homoclinic classes of a saddle-node. Let us explain all that in details.

Consider a diffeomorphism $f$ having two hyperbolic fixed points $P$ and $Q$ with Morse indices 1 and 2, respectively. Then, $f$ has a heterodimensional cycle associated to $P$ and $Q$ if the stable manifold of $P$, denoted by $W^s(P,f)$, and the unstable manifold of $Q$, $W^u(Q,f)$, have a non-empty transverse intersection, and the unstable manifold of $P$, $W^u(P,f)$, and the stable one of $Q$, $W^s(Q,f)$, have a quasi-transverse intersection throughout the orbit of a point $x_0$, i.e., $T_{x_0} W^s(Q,f) + T_{x_0} W^u(P,f) = T_{x_0} W^s(Q,f) \oplus T_{x_0} W^u(P,f)$, thus $\dim(T_{x_0} W^s(Q,f) +$
A saddle-node $S$ of a diffeomorphism $f$ is a periodic point (we here assume to be fixed) such that the derivative of $f$ at $S$ has 1 as its only eigenvalue in the unitary circle. We consider saddle-nodes of saddle type (i.e., the derivative of $f$ at $S$ simultaneously has eigenvalues inside and outside the unitary circle). This means that the tangent bundle of $M$ at $S$ has a $Df$-invariant splitting $E^{ss} \oplus E^c \oplus E^{uu}$, where $E^{ss}$ (resp. $E^{uu}$) is the bundle spanned by the eigenvectors associated to the contracting (resp. expanding) eigenvalues, and $E^c$ is the eigenspace associated to the eigenvalue 1 (in our context, all these spaces have dimension 1). According to the theory of invariant manifolds, see [HPS], there are defined the strong stable and unstable manifolds of the saddle-node, defined as the unique $f$ invariant manifolds tangent at $S$ to $E^{ss}$ and to $E^{uu}$ and denoted by $W^{ss}(S,f)$ and $W^{uu}(S,f)$, respectively.

Motivated by the fact (generic) saddle-nodes of saddle type simultaneously behave as points of index two and one (the stable and unstable manifolds of the saddle-node have both dimension 2), we define saddle-node heterodimensional cycles. A diffeomorphism $f$ has a saddle-node heterodimensional cycle associated to a saddle-node $S$ and the saddle $P$ of Morse index one if the (two-dimensional) unstable manifold of $S$ and stable manifold of $P$ have nonempty transverse intersection and the (one-dimensional) invariant manifolds $W^{ss}(S,f)$ and $W^{uu}(P,f)$ have a quasi-transverse intersection along the orbit of some point. One similarly defines saddle-node heterodimensional cycles associated to a saddle-node $S$ and a saddle $Q$ of Morse index two.

Roughly speaking, in our construction we consider a diffeomorphism $f$ simultaneously having two saddle-node heterodimensional cycles. We consider a two parameter family $(f_{t,s})_{t, s \in [-1,1]}$ of diffeomorphisms such that $f_{0,0}$ has a pair of saddle-node heterodimensional cycles, one associated to a saddle-node $S$ and a saddle $P$ of Morse index one and other one associated to a saddle $Q$ of index one and the saddle-node $S$. The parameter $t$ describes the unfolding of the cycles (relative motion between compact parts of $W^u(P, f_{t,0})$ and $W^{ss}(S, f_{t,0})$ and of $W^s(Q, f_{t,0})$ and $W^{uu}(S, f_{t,0})$). The parameter $s$ describes the unfolding of the saddle-node: for positive $s$ there are two saddles $S^+_s$ and $S^-_s$ of indices 2 and 1, colliding at $s = 0$ to the saddle-node $S$ and disappearing for negative $s$. We see that, fixed a small $t > 0$, for $s > 0$ (before the collapse of the saddles), $H(P, f_{t,s}) = H(S^+_s, f_{t,s})$ and $H(Q, f_{t,s}) = H(S^-_s, f_{t,s})$ for all small positive $s$. Moreover, $H(P, f_{t,s}) \cap H(Q, f_{t,s}) = \emptyset$. At the saddle-node bifurcation we have $H(P, f_{1,0}) \cap H(Q, f_{1,0}) = \{S\}$. Finally, for $s < 0$, after the disappearing of the saddles, $H(P, f_{t,s}) = H(Q, f_{t,s})$. See the results in Section 4. Theorem A follows by considering the arc $g_s = f_{t,-s}$. To deduce Theorem B from Theorem A we consider a filtration having the open set $W$ as a level and analyze the orbits of recurrent points of $\Lambda_s$.

In forthcoming papers, we will illustrate how this type of bifurcation naturally appear as secondary bifurcations in the unfolding of heterodimensional cycles and give the model for the collision, explosion, and collapse of (nontrivial) hyperbolic homoclinic classes, see [DRd].

Let us say a few words about our constructions. As mentioned, our setting necessarily corresponds to a non-generic situation, so we focus our attention on an example (we have not done any effort for generality). We begin by presenting (in Section 1) a model for the unfolding of a heterodimensional cycle. This model (motivated by [D1], [BD1], [DR4], and [DR5]) allows us to give a rather transparent explanation of the dynamics in the unfolding of a cycle by reducing it to the study of the dynamics of an iterated system of functions defined on an interval, this is done in Section 2. Recall that the dynamics of a (linear) Smale horseshoe is given by two affine expanding maps of the interval (say $I = [0, 1]$) whose domains of definition are two disjoint closed subintervals of $I$.
(say $[0, 1/3]$ and $[1/3, 1]$). The interval $(1/3, 2/3)$ is the main gap of the horseshoe and corresponds to points in the basin of attraction of a sink. The affine model associated to heterodimensional cycles is given by infinitely many expanding affine maps $F_i$ defined on subintervals $I_i$ of $I$ which are non-disjoint (the interior of the intervals $I_i$ are pairwise disjoint, but $I_i$ and $I_{i+1}$ have a common extreme). Thus in this model there are no gaps and there are no escaping points.

In Section 3, we prove that, after unfolding of the cycle, the dynamics of the model family is non-hyperbolic: the point of index 1 in the cycle belongs to the homoclinic class of the point of index 2 in the cycle. In fact, in this section, using the one-dimensional reduction, we give a shorter and clearer proof of the results in $[D_1]$. Since all the constructions in this paper rely heavily on this proof and there is not any written version of this approach, we have decided to include a short description of them.

In Section 4, we introduce the lateral homoclinic classes of a saddle-node $S$ of saddle type of a diffeomorphism $f$ as above, $H^+(S, f)$ and $H^-(S, f)$, respectively defined as the closure of the transverse intersections $W^u(S, f) \cap W^s(S, f)$ and $W^s(S, f) \cap W^u(S, f)$. These lateral homoclinic classes essentially behave as the usual ones. We see that for arcs $f_t$ unfolding at $t = 0$ the saddle-node heterodimensional cycle (associated to the saddle of index one $P$ and the saddle-node $S$) one has $H(P, f_t) \subset H^+(S, f_t)$ for all small positive $t$. Moreover, under mild conditions, one gets $H^+(S, f_t) = H(P, f_t)$ for all small $t > 0$. The inclusion $H(P, f_t) \subset H^+(S, f_t)$ follows adapting (in a rather straightforward way) the results for the model family in Section 3. For the inclusion $H^+(S, f_t) \subset H(P, f_t)$ we need new ingredients that we borrow from $[D_2]$.

Using the results in Sections 3 and 4, we get a complete description of the homoclinic classes $H(P, f_{t,s})$ and $H(Q, f_{t,s})$ before the collapse of the saddles $S^+_t$ and $S^-_t$ to the saddle-node. Finally, to study $H(P, f_{t,s})$ and $H(Q, f_{t,s})$ after the collision, we introduce new systems of iterated (one dimensional) functions and analyze their dynamics.

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## 1 Heterodimensional cycles: a model family

In this section, we construct a model one-parameter family $(f_t)_{t \in [-1, 1]}$ of diffeomorphisms unfolding a heterodimensional cycle. The study of the semi-local dynamics of $f_t$ will be reduced to the analysis of a one-parameter family of endomorphisms with infinitely many discontinuities which describe the dynamics of $f_t$ in the central direction, see Section 2.

Consider a diffeomorphism $f$ with a heterodimensional cycle having the following dynamical configuration. In local coordinates in $\mathbb{R}^3$, the cycle is associated to saddle fixed points $Q = (0, 0, 0)$ and $P = (0, 1, 0)$ of indices 2 and 1, respectively, verifying the following conditions:

**Partially hyperbolic (semi-local) dynamics of the cycle:**

- In the set $[-1, 1] \times [-1, 2] \times [-1, 1]$ the diffeomorphism has the form
  \[ f(x, y, z) = (\lambda_s x, F(y), \lambda_u z), \]
  where $F: [-1, 2] \to (-1, 2)$ is a strictly increasing monotone map with exactly two fixed points, a source at 0 and a sink at 1, and $0 < \lambda_s < d_m < d_M < \lambda_u$, where $0 < d_m < F'(x) < d_M$ for all $x \in [0, 1]$. 

• There is \( \delta > 0 \) such that \( F \) is linear in \([-\delta, \delta]\) and affine in \([1-\delta, 1+\delta]\). We denote by \( \beta > 1 \) and \( 0 < \lambda < 1 \), the eigenvalues of \( F \) at 0 and 1, respectively.

Observe that \([-1, 1] \times \{(0, 0)\} \subset W^s(Q), \{0\} \times [0, 1] \times [-1, 1] \subset W^u(Q), \{(0, 0)\} \times [-1, 1] \subset W^u(P) ,\) and \([-1, 1] \times (0, 1) \times \{0\} \subset W^s(Q). \) Thus \( \gamma = \{0\} \times (0,1) \times \{0\} \) is a normally hyperbolic curve contained in \( W^u(Q) \cap W^s(P). \)

Existence and unfolding of the cycle:

• The cycle: There exist \( k_0 \in \mathbb{N} \) and a small neighborhood \( U \) of \((0,1,-1/2) \in W^u(P) \) such that the restriction of \( f^{k_0} \) to \( U \) is a translation,

\[
f^{k_0}(x,y,z) = (x-1/2,y-1,z+1/2).
\]

In particular, \( f^{k_0}(0,1,-1/2) = (-1/2,0,0) \in W^u(Q) \). Thus \( W^s(Q) \) and \( W^u(P) \) meet throughout the orbit of the heteroclinic point \((-1/2,0,0)\). By construction, \((-1/2,0,0)\) is a quasi-transverse heteroclinic point.

• The unfolding of the cycle: Consider the one parameter family \((f_t)\) of diffeomorphisms coinciding with \( f \) in \([-1,1] \times [-1,2] \times [-1,1] \) and such that the restriction of \( f_t^{k_0} \) to \( U \) is of the form

\[
f_t^{k_0}(x,y,z) = (x-1/2,y-1+t,z+1/2) = f^{k_0}(x,y,z) + (0,t,0).
\]

Therefore, for small \( t > 0 \), \((-1/2,t)\) \times [-1,1] \subset W^u(P,f_t). \) Thus \( x_t = (-1/2,t,0) \) is a transverse homoclinic point of \( P \) (for \( f_t \)). Similarly, \( y_t = (-1/2,0,0) \) is a transverse homoclinic point of \( Q \) (for \( f_t \)).

Consider a small neighborhood of the heterodimensional cycle associated with \( f_0 \), that is, an open set \( W \) containing the connexion curve \( \gamma = \{0\} \times [0,1] \times \{0\} \) and the \( f_0 \)-orbit of the heteroclinic point \((-1/2,0,0)\). For small \( t \), let \( \Lambda_t \) be the maximal \( f_t \)-invariant set in \( W \), \( \Lambda_t = \cap_{n \in \mathbb{Z}} f_t^n(W) \). We also consider the forward and backward invariant sets in \( W \), \( \Lambda_t^+ = \cap_{n \geq 0} f_t^{-n}(W) \) and \( \Lambda_t^- = \cap_{n \geq 0} f_t^n(W) \).

Fix a small positive \( \rho \) and consider the fundamental domains of \( F \) given by \( D^+ = [\beta^{-1}\rho, \rho] \) and \( D^- = [1-\rho, \lambda(1-\rho)] \) contained in the neighborhoods of 0 and 1 where \( F \) is affine. We can choose \( \rho \) such that \( F^N(D^+) = D^- \) for some \( N \in \mathbb{N} \) (for notational simplicity let us put \( N = 1 \)). The map \( F^N \) is the transition from 0 to 1\(^2\).

Suppose that that the eigenvalues \( \lambda \) and \( \beta \) and the map \( F \) verify the following conditions:

\[
(T1) \quad F'(x) \geq \frac{1}{2} \frac{1-\lambda}{1-\beta^{-1}}, \text{ for all } x \in D^+,
\]

\[
(T2) \quad (1-\lambda) < \beta^{-1}, \text{ and}
\]

\[
(T3) \quad \frac{(1-\lambda)\lambda}{2(1-\beta^{-1})\beta} = \ell > 1.
\]

\(^2\)This transition plays a key role for determining the dynamics after unfolding the cycle and it is determined by the Mather invariant of \( F \), [Ma]. in a neighborhood of 0, the map \( F \) is the exponential of the vector field \( X(y) = (\log \beta) \cdot \frac{\partial}{\partial y} \), whose flow is \( y \mapsto \beta^t y \). Similarly, in a neighborhood of 1, \( F \) is the time-one of \( Y(y) = (\log \lambda) \cdot (y-1) \cdot \frac{\partial}{\partial y} \). Consider, for \( y \) close to \( 0^+ \), an \( n \) large such that \( F^n(x) \) is close to \( 1^- \), and write \( D F^n(y)(X(y)) = \mu(y) Y(F^n(y)) \). Using the local \( F \)-invariance of \( X \) and \( Y \) (near 0 and 1), one has that \( \mu(x) \) does not depend on \( n \) and that \( \mu(x) = \mu(F(x)) \). The function \( \mu \) is the Mather invariant of \( F \), which describes its distortion. For instance, if \( \mu \) is identically 1, then \( F \) is exactly the exponential of a vector field.
To get condition (T1) it is enough to consider $F$ with small distortion. To have conditions (T2) and (T3) it is enough to take $\beta$ close enough to $1^+$. In fact, later we will consider the case where $Q$ is a saddle-node (saddle-node heterodimensional cycles), see Section 4. Our first result is:

**Theorem 1.1** For every $t > 0$ sufficiently small, $H(P, f_t) \subset H(Q, f_t)$ and $\Lambda_t \subset H(Q, f_t)$.

This theorem was stated in [D1]. Here we give a more conceptual prove of it, which enables us to introduce some technical tools to be used systematically later on. First, in Section 2 we will introduce the system of iterated functions associated to the cycle (this approach is motivated by [DR3]). In Section 3 we deduce the theorem from the results in Section 2.

## 2 Expanding one-dimensional dynamics associated to the cycle

For each small $t > 0$, consider the scaled fundamental domains $D_t^\pm$ defined as follows: let $D_t^- = [1 - t, \lambda(1 - t)]$ and define $k_t$ as the smaller $k \in \mathbb{N}$ with $F^{-k}(D_t^-) \subset [0, t]$. We define

$$D_t^+ = [a_t, b_t] = [\beta^{-1}(b_t), b_t] = F^{-k_t}(D_t^-),$$

where $\beta^{-2} t < a_t < b_t \leq t$.

We next define an expanding map $R_t, R_t: D_t^+ \to D_t^+$, with discontinuities that will describe the dynamics (in the central direction) of the return map of $f_t$ defined on $[-1, 1] \times D_t^+ \times [-1, 1]$. For each small $t > 0$ define the transition map $T_t$ from $D_t^+$ to $D_t^-$ by

$$T_t: D_t^+ \to D_t^-, \quad x \mapsto T_t(x) = F^{k_t}(x).$$

**Lemma 2.1** The map $T_t$ verifies $T_t(x) > \ell > 1$ for all $x \in D_t^+$, where $\ell$ is as in condition (T3).

**Proof:** Given $x \in D_t^+$ let $k_t = n_t(x) + 1 + m_t(x)$, where $F^{m_t(x)}(x) \in D^+$ and $F^i(x) \notin D^+$ for all $1 \leq i < n_t(x)$. We claim that

$$\frac{1}{\beta t} \leq \beta^{m_t(x)} \leq \frac{\beta^2}{t} \quad \text{and} \quad \lambda t \leq \lambda^{m_t(x)} \leq \frac{t}{\lambda}.$$  \hspace{1cm} (1)

For first pair of inequalities just note $x \in D_t^+ \subset (\beta^{-2} t, \ell]$ and $\beta^{m_t(x)} x \in [\beta^{-1}, 1]$. The other ones follow analogously.

Since $T_t(x) = \lambda^{m_t(x)} F(\beta^{m_t(x)} x)$, hypotheses (T1) and (T3) and the estimates in (1) give

$$|T_t'(x)| = \lambda^{m_t(x)} |F'(x)||\beta^{m_t(x)}| \geq (\lambda t) \left(\frac{1}{2}\frac{1-\lambda}{1-\beta^{-1}}\right) \left(\frac{1}{\beta t}\right) = \frac{1-\lambda}{\beta(1-\beta^{-1})} = \ell > 1,$$

as we claimed. \hfill \Box

Since $T_t(x) \in D_t^- = [1 - t, 1 - \lambda t]$ for all $x \in D_t^+$, we can define the map $G_t$ by $G_t: D_t^+ \to [0, t]$, $x \mapsto G_t(x) = T_t(x) + (t - 1)$.

**Remark 2.2** The map $G_t$ is monotone increasing and $G_t(D_t^+) = [0, t(1 - \lambda)].$
Claim 2.3 Let \((a_t, b_t) = \tilde{D}_t^+ \subset D_t^+.\) Given \(x \in \tilde{D}_t^+\) let \(i(x) \in \mathbb{Z}\) be the minimum \(i\) with \(\beta^i(G_t(x)) \in D_t^+\). Then there is \(i_0 > 0\) (maximum with such property) such that \(i(x) \geq i_0\) for all \(x \in \tilde{D}_t^+\).

Proof: Observe first that, since \(\tilde{D}_t^+ \subset [\beta^{-2}t, t]\), \(b_t \in (\beta^{-1}t, t]\). On the other hand, from Remark 2.2 and \((1 - \lambda) < \beta^{-1}\) (condition (T2)), \(G_t(\tilde{D}_t^+) = (0, t(1 - \lambda)] \subset (0, \beta^{-1}t)\). Thus the right extreme of \(G_t(\tilde{D}_t^+)\) is less than the left extreme of \(D_t^+\), hence \(i(x) > 0\) for all \(x \in \tilde{D}_t^+\), ending the proof of the claim.

Finally, the return map \(R_t\) is defined by
\[
R_t: \tilde{D}_t^+ \rightarrow D_t^+, \quad R_t(x) = \beta^i(G_t(x)) = \beta^i(T_t(x) + (t - 1)).
\]

Next we study the dynamics of \(R_t\): the map \(R_t\) is uniformly expanding and has (infinitely many) discontinuities where the lateral derivatives are well defined. These discontinuities will play a key role in our constructions. The definition of \(i_0 \in \mathbb{N}\) in Claim 2.3 implies that \(\beta^{i_0}(a_t) \in G_t(D_t^+)\). For each \(i \geq i_0\) define \(d_i \in \tilde{D}_t^+\) by \(G_t(d_i) = \beta^{-i}(a_t)\). By construction, the sequence \((d_i)_{i \geq i_0}\) corresponds to the discontinuities of \(R_t\) and verifies the following:

- \(d_{i+1} < d_i\) and \(d_i \rightarrow a_t\),
- Let \([d_{i+1}, d_i] = I_i, i > i_0,\) and \(I_{i_0} = [d_{i_0}, b_t]\). The map \(R_t\) is continuous and strictly increasing in the interior of each interval \(I_i\). We continuously extend \(R_t\) to the whole \(I_i\), obtaining a bi-valued return map \(R_t\) with \(R_t(d_i) = \{a_t, F(a_t) = b_t\}\) for all \(i > i_0\). In particular, the restriction of \(R_t\) to any \(I_i, i > i_0\), is onto. We let \(R_t(b_t) = c_t \leq b_t\).

The main properties of \(R_t\) are summarized in the next lemma.

Lemma 2.4 The restriction of \(R_t\) to each interval \(I_i, i > i_0\), is onto and \(R_t'(x) > \ell > 1\) for all \(x \in (a_t, b_t)\) (if \(x = d_i\) this means that the lateral derivatives of \(R_t\) at \(x\) are greater than \(\ell\)). Moreover, \(G_t(R_t(d_i)) = 0\) follows from \(R_t(d_i) = a_t\) and \(G_t(a_t) = 0\).

The expansiveness of \(R_t\) follows from Lemma 2.1 and Claim 2.3 Condition \(G_t(R_t(d_i)) = 0\) follows from \(R_t(d_i) = a_t\) and \(G_t(a_t) = 0\).

Lemma 2.5 Consider a small \(t > 0\) and an open subinterval \(J\) of \(\tilde{D}_t^+\). Then there is a first \(k \in \mathbb{N} \cup \{0\}\) such that \(R^k_t(J)\) contains a discontinuity of \(R_t\). In particular, there is \(x \in J\) such that \(G_t(R^k_t(x)) = 0\).

Proof: If the interval \(J\) contains a discontinuity we are done. Otherwise, let \(i > 0\) be such that the intervals \(J, R_t(J), \ldots, R^i_t(J)\) do not contain discontinuities. Thus, for each \(k \in \{0, \ldots, i\}\), there is \(i_k \geq i_0\) such that \(R^k_t(J) \subset I_{i_k}\). Lemma 2.4 implies that \(|R^k_t(J)| \geq \ell^k|J|, \ell > 1\), for all \(k \in \{0, \ldots, i\}\). Since the size of the intervals \(I_i\) is upper bounded, this inequality implies that there is a first \(m \in \mathbb{N}\) such that \(R^m_t(J)\) is not contained in any \(I_i\), thus it intersects the set of discontinuities of \(R_t\).

3 The maximal invariant set: Proof of Theorem 1.1

Next proposition is the main technical result of this section. Heuristically, it means that the one-dimensional stable manifold of \(Q\) topologically behaves as a two-dimensional manifold.

Proposition 3.1 For every small \(t > 0\) and every two-disk \(\chi\) with \(W^s(P, f_t) \cap \chi \neq \emptyset, W^s(Q, f_t) \cap \chi \neq \emptyset\). In particular, \(W^s(P, f_t)\) is contained in the closure of \(W^s(Q, f_t)\).
Proof of the inclusion $H(P, f_t) \subset H(Q, f_t)$ in Theorem 1.1. This inclusion follows from Proposition 3.1. By the definition of $H(P, f_t)$, it suffices to see that any $x \in W^s(P, f_t) \cap W^u(P, f_t)$ is accumulated by homoclinic points of $Q$. The geometric configuration of the cycle implies that $W^u(P, f_t) \subset \text{closure}(W^u(Q, f_t))$. Thus, given any $x \in H(P, f_t)$ and any $n > 0$, there is a disk $\Delta_n$ simultaneously contained in $W^u(Q, f_t)$ and in the ball of radius $1/n$ centered at $x$ whose interior meets transversely $W^s(P, f_t)$. By Proposition 3.1 $\Delta_n \cap W^s(Q, f_t) \neq \emptyset$. Thus there is $y_n \in \Delta_n \cap H(Q, f_t)$. By construction, $y_n \to x$, proving the inclusion $H(P, f_t) \subset H(Q, f_t)$.

3.1 Homoclinic classes - Proof of Proposition 3.1

We now go into the details of the proof of Proposition 3.1. We first introduce some definitions.

- A set $\Delta \subset [-1, 1] \times [-1, 2] \times [-1, 1]$ is a vertical strip if $\Delta = \{x\} \times [l_1, l_2] \times [r_1, r_2]$, where $l_1 < l_2$ and $r_1 < 0 < r_2$. The segment $\{x\} \times [l_1, l_2] \times \{0\}$ is the basis of $\Delta$. The width and the height of $\Delta$ are $w(\Delta) = (l_2 - l_1)$ and $h(\Delta) = (r_2 - r_1)$. The strip $\Delta$ is complete if $r_2 = 1$ and $r_1 = -1$, well located if $[l_1, l_2]$ is contained in the interior of $D^+_1$, and perfect if it simultaneously is complete and well located.

- A subset $J \subset [-1, 1] \times [-1, 2] \times [-1, 1]$ is a vertical segment if $J = \{x\} \times \{l_1\} \times [r_1, r_2]$, where $r_1 < 0 < r_2$. The height of $J$ is $h(J) = (r_2 - r_1)$. As above, the segment $J$ is complete if $r_1 = -1$ and $r_2 = 1$, well located if $l_1$ is in the interior of $D^+_1$, and perfect if it simultaneously is complete and well located.

- A vertical segment $J$ (resp. strip $\Delta$) is at the right of $Q$ if $l_1 \in (0, 1]$.

Given an interval $\alpha \subset [-1, 2]$, denote by $\Delta(\{x\} \times \alpha \times \{0\})$ (resp. $J(x, y, 0)$) the unique complete vertical strip (resp. segment) with basis $\{x\} \times \alpha \times \{0\}$ (resp (x, y, 0)). The following algorithm associates to perfect segments and strips their successors:

**Algorithm 3.2** Let $\Delta = \Delta(\{x\} \times \alpha \times \{0\})$ be a perfect strip and define $G_t(\Delta)$ as the perfect strip such that:

- the basis of $G_t(\Delta)$ is of the form $\{x'\} \times G_t(\alpha) \times \{0\}$, where $x' = \lambda^k \cdot x - 1/2$,
- $G_t(\Delta)$ is contained in $f^k_t(\Delta)$ (where $F^k(D^+_1) = D^-_1$).

Suppose now that $\alpha$ does not contain discontinuities, i.e. $\alpha \subset (d_i, d_{i+1})$ for some $i$. Define $R_t(\Delta)$ as the perfect strip such that:

- the basis of $R_t(\Delta)$ is of the form $\{x'\} \times R_t(\alpha) \times \{0\}$, where $x' = \lambda^k \cdot (\lambda^k \cdot x - 1/2)$,
- $R_t(\Delta)$ is contained in $f^k_t(\Delta)$.

Similarly, to a perfect segment $J = J(x, y, 0)$ we associate perfect segments $G_t(J)$ and $R_t(J)$ (provided $y \neq d_i$ for all $i$).
The strips $G_t(\Delta)$ and $R_t(\Delta)$ in Algorithm 3.2 are obtained as follows. Given a set $A$ and a point $x \in A$, denote by $C(x, A)$ the connected component of $A$ containing $x$. Consider a small tubular neighborhood $V$ of $f_0^m((0, 1) \times [-1, 1]) \subset W^u(P, f_0)$, $k_0$ as in the definition of the cycle in Section 11 then

$$G_t(\Delta) = C(f_t^{k_1}(x, y, 0), f_t^{k_1}(\Delta) \cap V) \cap [-1, 1]^3), \quad R_t(\Delta) = C(f_t^i(x', y', 0), f_t^{k_1+i}(\Delta) \cap V) \cap [-1, 1]^3),$$

where $(x, y, 0)$ is any point in the basis of $\Delta$, $x' = (\lambda_t^{k_1} x - 1/2)$, and $y' \in G_t(\alpha)$. The construction for the successors of the segments is analogous.

**Lemma 3.3** The manifold $W^u(P, f_t)$ contains a perfect segment for all small $t > 0$.

**Proof:** Consider the transverse homoclinic point $x_t = (-1/2, t, 0)$ of $P$ (for $f_t$). Recall that $t \geq b_t$ and $\beta^{-1} t \in D_t^+$ = $[a_t, b_t]$. Let us assume that $t > b_t$, and thus $\beta^{-1} t \in (a_t, b_t)$ (the case $t = b_t$ follows similarly, so it will be omitted). Consider the complete vertical segment $R_t(J)$, where $J = J(-\lambda_{s_0}^{-1}/2, \beta^{-1} t, 0) \subset W^u(P, f_t)$. If $R_t(\beta^{-1} t)$ belongs to the interior of $D_t^+$, then $R_t(J) \subset W^u(P, f_t)$ is the announced segment. Otherwise, $R_t(\beta^{-1} t) = b_t$ and there is a homoclinic point of $P$ of the form $(x', b_t, 0)$. Using the $\lambda$-lemma and the product structure of the cycle, one gets homoclinic points $(x_n, y_n, 0)$ of $P$ and complete segments $J_n = J(x_n, y_n, 0) \subset W^u(P, f_t)$ such that $x_n \rightarrow x'$, $y_n \rightarrow b_t$, and $y_n$ is increasing. Thus $y_n$ belongs to the interior of $D_t^+$ for every big $n$ and $J_n \subset W^u(P, f_t)$ is perfect.

For cleanness we first prove Proposition 3.4 in the following special case:

**Proposition 3.4** Let $\chi \subset [-1, 1] \times [-1, 2] \times [-1, 1]$ be a set of the form $\{x\} \times A$, where $A$ is a disk of $\mathbb{R}^2$ whose interior contains a point of the form $(y, 0)$ with $y \in (0, 2)$. Then $\chi$ intersects transversely $W^s(Q, f_t)$.

We claim that is enough to prove the Proposition 3.4 for perfect strips:

**Lemma 3.5** Let $\Delta$ be a perfect strip. Then there is $k \in \mathbb{N}$ such that $f_t^k(\Delta) \cap W^s(Q, f_t) \neq \emptyset$.

**Proof:** Suppose that $\Delta = \Delta(\{x_0\} \times \alpha \times \{0\})$, $\alpha$ in the interior of $D_t^+$. By Lemma 2.5 there exist $y_0$ in the interior of $\alpha$ and $k \in \mathbb{N}$ such that $G_t(R_t^k(y_0)) = 0$. Thus the vertical strip $G_t(R_t^k(\Delta))$ (contained in the forward orbit of $\Delta$) intersects transversely $[-1, 1] \times \{(0, 0)\} \subset W^s(Q, f_t)$. □

**Proof of Proposition 3.4** By Lemma 3.3 $W^u(P, f_t)$ contains a perfect vertical segment $J$. By definition, $\chi$ meets transversely $W^s(P, f_t)$, which implies, by the $\lambda$-lemma, that the forward orbit of $\chi$ contains a sequence of complete strips $\chi_n$ accumulating to $J$. Thus $\chi_n$ contains a perfect strip for all $n$ large. Lemma 3.5 now implies that $\chi_n$ (and thus $\chi$) transversely meets $W^s(Q, f_t)$, ending the proof of the proposition. □

**Proof of Proposition 3.1** We can assume that $\chi$ is transverse to $W^s_{loc}(P, f_t)$ and contained in $[-1, 1] \times [-1, 2] \times [-1, 1]$. If $\chi$ contains a subset of the form $\{x\} \times A$, where $A$ is an open subset of $\mathbb{R}^2$ containing a point $(0, y)$ with $y \in (0, 2)$, Proposition 3.4 implies the result. For the general case, consider a point $(x_0, y_0, 0)$, $y_0 \in (0, 2)$, in the interior of $\chi \cap W^s_{loc}(P, f_t)$ and for every big $n$ the vertical strip

$$\Sigma_n = \{x_0\} \times [y_0 - 1/n, y_0 + 1/n] \times [-1/n, 1/n].$$

The strips $\Sigma_n$ verify the hypotheses of Proposition 3.4, hence there is $(x_0, y_n, z_n) \in \Sigma_n \cap W^s(Q, f_t)$ such that $H_n = [-1, 1] \times \{(y_n, z_n)\} \subset W^s(Q, f_t)$. Since $(x_0, y_n, z_n) \rightarrow (x_0, y_0, 0)$, it is immediate that $H_n$ meets transversely $\chi$ for all large $n$, ending the proof of the proposition. □
3.2 The maximal $f_i$-invariant set

To prove the second part of Theorem 1.1 ($\Lambda_t \subset H(Q, f_i)$) let $V_0$ be the connected component of the neighborhood of the cycle $W$ containing the heteroclinic point $(-1/2, 0, 0)$. There are two types of points of $\Lambda_t$: (a) those points whose orbit does not meet $V_0$ (i.e., the set $\{0\} \times [0, 1] \times \{0\}$) and (b) those having an iterate in $V_0$.

We claim that every point of type (a) belongs to $H(Q, f_i)$: given any $(0, x, 0)$, $x \in (0, 1)$, consider the disk $\Delta_n = \{0\} \times [x - 1/n, x + 1/n] \times [-1/n, +1/n] \subset W^u(Q, f_i)$ satisfying the hypothesis of Proposition 3.1. Hence $\Delta_n \cap W(Q, f_i) \neq \emptyset$ and thus $\Delta_n \cap H(Q, f_i) \neq \emptyset$. Since this holds for all $n \in \mathbb{N}$, $(0, x, 0) \in H(Q, f_i)$.

For points $w \in \Lambda_t$ of type (b), after replacing $w$ by some iterate of it, we can assume that $w \in V_0$. Consider the sequence $(n_i(w))_{i \in \mathbb{Z}}$ associated to $w$, where $\mathcal{I}(w) \subset \mathbb{Z}$ is an interval in $\mathbb{Z}$, inductively defined as follows: let $n_0(w) = 0$ and, assuming defined $n_j(w)$, $j \geq 0$, we define $n_{j+1}(w)$ as the first integer $k > n_j(w)$ such that $f_k^i(w) \in V_0$. We argue analogously for negative $j$, assuming defined $n_j(x)$, $j \leq 0$, $n_{j-1}(w)$ is the first negative integer $k < n_j(w)$ with $f_k^i(w) \in V_0$.

Define $\mathcal{I}_i^+(b)$ as the subset of $\Lambda_t \cap V_0$ of points $w$ such that $\mathcal{I}(w)$ is upper bounded. The subset $\mathcal{I}_i^-(b)$ is defined similarly. We let $\mathcal{I}_i^+(b) = \mathcal{I}_i^+(b) \cap \mathcal{I}_i^-(b)$. We borrow from [DR2, Lemma 4.1] the following lemma whose proof is straightforward:

**Lemma 3.6** For every small $t > 0$, $\mathcal{I}_i^+(b) \subset W^s(P, f_i) \cup W^s(Q, f_i)$ and $\mathcal{I}_i^-(b) \subset W^u(P, f_i) \cup W^u(Q, f_i)$.

Next result immediately follows by observing that $f_t$ (resp. $f_t^{-1}$) exponentially expands the vertical (resp. horizontal) segments:

**Remark 3.7** Let $w = (x, y, z) \in \mathcal{I}_i^+(\infty)$ (resp. $w \in \mathcal{I}_i^-(\infty)$). Then $\{(x, y)\} \times [z - \varepsilon, z + \varepsilon] \cap W^s(P, f_i) \neq \emptyset$ (resp. $[x - \varepsilon, x + \varepsilon] \times \{(y, z)\} \cap W^u(Q, f_i) \neq \emptyset$) for every $\varepsilon > 0$.

To prove the theorem we consider the following four cases.

**Case (i):** $w = (x, y, z) \in \mathcal{I}_i^-(b) \setminus \mathcal{I}_i^+(b)$.

By Remark 3.7, there is a sequence $w_n = (x_n, y_n, z_n) \in W^s(P, f_i)$ with $w_n \to w$. We claim that $w_n \in H(Q, f_i)$ for all large $n$. Thus $w \in H(Q, f_i)$. To prove the claim, note that the distances between the backward iterates of $w_n$ and $w$ exponentially decrease, we get that $w_n \in \Lambda_t$. Moreover, since $w \in \mathcal{I}_i^-(b)$, Lemma 3.6 implies that $w, w_n \in W^u(P, f_i) \cup W^u(Q, f_i)$. If $w_n \in W^u(P, f_i)$, then $w_n \in H(P, f_i) \subset H(Q, f_i)$ (recall the first part of Theorem 1.1 proved above) and we are done. Otherwise, $w_n \in W^u(Q, f_i)$ and for each $k$ large, there is a small vertical strip $\Delta_k$ of diameter less than $1/k$, whose interior is contained in $W^u(Q, f_i)$ and contains $w_n$. Since $w_n \in W^u(P, f_i)$, $\Delta_k \cap W^s(P, f_i)$. Thus, by Proposition 3.1, $W^s(Q, f_i)$ intersects transversely the interior of $\Delta_k$. Hence, since the interior of $\Delta_k$ is contained in $W^u(Q, f_i)$, $\Delta_k$ contains a homoclinic point $y_k$ of $Q$. From $\text{diam}(\Delta_k) \to 0$, we get $y_k \to w_n$, which implies $w_n \in H(Q, f_i)$.

**Case (ii):** $w = (x, y, z) \notin \mathcal{I}_i^+(b) \cup \mathcal{I}_i^-(b)$.

We claim that $w$ is accumulated by points $w_n \in \mathcal{I}_i^+(\infty) \cap \mathcal{I}_i^-(b)$, and the result follows from Case (i). To prove the claim observe that, by Remark 3.7, there is a sequence $w_n = (x_n, y, z) \in W^u(Q, f_i)$ with $w_n \to w$. Since the distances between the forward iterates of $w_n$ and $w$ exponentially decrease, $w_n \in \Lambda_t$. This also implies that $w_n \in \mathcal{I}_i^+(\infty)$. Finally, $w_n \in W^u(Q, f_i)$ implies $w_n \in \mathcal{I}_i^-(b)$, ending the proof of the claim.
Case (iii): $w = (x, y, z) \in \mathcal{I}^+_t(b) \setminus \mathcal{I}^-_t(b)$.

By Lemma 3.6, $w \in W^s(P, f_t) \cup W^u(Q, f_t)$ and, by replacing $w$ by a forward iterate, we can assume that $w = (x, y, 0)$, $y \geq 0$. Remark 3.7 gives a sequence $w_n = (x_n, y_n, 0) \in W^u(Q, f_t)$ with $w_n \to w$. For each $n$, there is a vertical disk $H_n \subset W^u(Q, f_t)$ centered at $w_n$, of diameter less than $1/n$. Clearly, $H_n$ intersects transversely $W^s(P, f_t)$. Thus, by Proposition 3.1, $H_n \cap W^s(Q, f_t) \neq \emptyset$. As in the previous cases, this implies that $H_n \cap H(Q, f_t) \neq \emptyset$ for all $n$ large, thus $w \in H(P, f_t)$.

Case (iv): $w \in \mathcal{I}^+_t(b)$.

By Lemma 3.6 there are four possibilities: (1) $w \in W^s(Q, f_t) \cap W^u(Q, f_t)$, (2) $w \in W^s(P, f_t) \cap W^u(P, f_t)$, (3) $w \in W^s(P, f_t) \cap W^u(Q, f_t)$, and (4) $w \in W^u(P, f_t) \cap W^s(Q, f_t)$. Recall that the intersections above are transverse or quasi-transverse, depending on the case. Hence, in case (1), $w \in H(Q, f_t)$ and, in case (2), $w \in H(P, f_t) \subset H(Q, f_t)$. In case (3), the same proof of $\{0\} \times [0, 1] \times \{0\} \subset H(Q, f_t)$ implies that $w \in H(Q, f_t)$: just observe that for every disk $\Delta \subset W^u(Q, f_t)$ containing $w$, $W^s(P, f_t) \cap \Delta \neq \emptyset$, thus $\Delta \cap H(Q, f_t) \neq \emptyset$. It still remains the case $w \in W^s(Q, f_t) \cap W^u(P, f_t)$. By replacing $w$ by a forward iterate, we can assume that $w = (x, 0, 0)$, $x \in [-1, 1]$, and the following lemma and Cases (i) and (ii) easily imply case (4):

Lemma 3.8 Let $w = (x, 0, 0) \in V_0 \cap (W^s(Q, f_t) \cap W^u(P, f_t))$. Then there is a sequence $w_n \to w$ with $w_n \in \mathcal{I}^+_t(\infty)$.

Proof: For each $n \in \mathbb{N}$, consider the rectangle $R_n(x) = \{x\} \times [0, 1/n] \times [-1/n, 1/n]$.

Claim 3.9 There exists $\kappa_n$ in $R_n(x) \in \Lambda^+_t$ whose forward orbit returns to $V_0$ infinitely many times.

Assuming this claim, we now finish the proof of the lemma: similarly as in Remark 3.7, but now considering points in $\Lambda^+_t$, we have that the point $\kappa_n = (x_n, y_n, z_n)$ is accumulated by points $\kappa^m_n \in W^u(Q, f_t)$ of the form $(x^m_n, y_n, z_n)$. Since the distances between the forward iterates of $\kappa_n$ and $\kappa^m_n$ decrease, the forward orbit of $\kappa^m_n$ is contained in $W$ and returns infinitely many times to $V_0$. On the other hand, since $\kappa^m_n \in W^u(Q, f_t)$, its backward orbits also is in $W$. Thus the whole orbit of $\kappa^m_n$ is in $W$, so $\kappa^m_n \in \Lambda_t$ and $\kappa^m_n \in \mathcal{I}^+_t(\infty)$. By Cases (i) and (ii) above, $\kappa^m_n \in H(Q, f_t)$, thus $\kappa_n \in H(Q, f_t)$. To prove the claim, we need the following fact:

Fact 3.10 Let $R = R_n(x)$, $1/n < t$. Then there is $i = i(R) \in \mathbb{N}$ such that, for every $j \geq i$, $f^j_t(R)$ contains a rectangle $\Gamma(R, j)$ of the form $\{a\} \times [0, 1/n] \times [-1/n, 1/n]$.

Proof: Let $N_i$ be the smaller $i \in \mathbb{N}$ such that $F^i(1/n) \in (1 - t + 1/n, 1)$ and write

$e = (1 - t + 1/n + g) = F^{N_i}(1/n), \quad g \in (0, t - 1/n).$

The definition of the unfolding of the cycle implies that, for each $j \geq 0$, $f^{N_i+j}_t(R_n(x))$ contains a rectangle of the form

$\{\lambda^{N_i+j}_n(x) - 1/2\} \times [0, t + \lambda^j(g + 1/n - t)] \times [-1, 1] \supset \{\lambda^{N_i+j}_n(x) - 1/2\} \times [0, 1/n] \times [-1, 1],$

where the inclusion follows from $t + \lambda^j(g + 1/n - t) \geq t + \lambda^j(1/n - t) \geq 1/n$. This finishes the proof of the fact.

To prove Claim 3.9 consider $R_n(x) = R(0)$ and, using Fact 3.10 let $R(1) = \Gamma(R(0), i(R(0)))$. Write $i_0 = i(R_0)$ and $R^1 = f^{-i_0}_t(R(1)) \subset R(0)$. Assume inductively defined numbers $i_{k-1}$ and rectangles $R(k)$ and $R^k$ for every $k \in \{0, \ldots, j\}$ as follows:
• $R(k) = \Gamma(R(k-1), i(R(k-1)))$ and $i_{k-1} = i(R(k-1))$, in particular, $R(k)$ satisfies the hypotheses of Fact 3.10.

• $R^k \subset R^{k-1} \subset \cdots \subset R^1 \subset R(0) = R_{\alpha}(x)$ and $R^k = f_t^{-i_0-\cdots-i_{k-1}}(R(k))$.

We define $i_k = i(R(k))$, $R(k+1) = \Gamma(R(k), i_k)$ and $R^{k+1} = f_t^{-i_0-\cdots-i_k}(R(k+1))$, completing the inductive process. Now it suffices to take any point in the non-empty intersection $\cap_{k\in\mathbb{N}} R^k$. □

The proof of Theorem 1.1 is now complete.

4 Saddle-node heterodimensional cycles

In this section, we consider saddle-node heterodimensional cycles. For that, in the definition of the heterodimensional cycle in Section 1, we replace the function $F$ (defining the central dynamics) by a one parameter family of maps $\Phi_s: [-1,2] \to \mathbb{R}$ such that:

• For every $s$, the point 1 is an attracting hyperbolic point of $\Phi_s$ and $\Phi_s$ is linearizable in a neighborhood of 1 (independent of $s$). We denote by $0 < \lambda < 1$ the eigenvalue of $\Phi_s$ at 1.

• Locally in 0, the map $\Phi_s$ is of the form $\Phi_s(x) = x + x^2 - s$. Thus, for $s > 0$, $\Phi_s$ has two hyperbolic fixed points $\pm \sqrt{s}$ (an attractor and a repellor) collapsing at $s = 0$. Moreover, for every $s < 0$, $\Phi_s$ has no fixed points close to 0.

• Every $\Phi_s$ is strictly increasing and has no fixed points different from 1 and $\pm \sqrt{s}$.

We now define, as in Section 1, a two parameter family of diffeomorphisms $f_{t,s}$: the parameters $t$ and $s$ describing the motion of the unstable manifold of $P$ and the unfolding of the saddle-node (i.e., $f_{t,s}(0,y,0) = (0, \Phi_s(y), 0)$), respectively. Observe that $P = (0,1,0)$, $S^- = (0, -\sqrt{s}, 0)$ and $S^+ = (0, \sqrt{s}, 0)$ ($s \geq 0$) are fixed points of $f_{t,s}$.

We let $f_t = f_{t,0}$. For the saddle-node $S = (0,0,0)$ of $f_t$, there are defined the stable and unstable manifolds (denoted $W^s(S, f_t)$ and $W^u(S, f_t)$) and the strong stable and unstable manifolds (denoted by $W^{ss}(S, f_t)$ and $W^{uu}(S, f_t)$). Observe that $W^s(S, f_t)$ and $W^u(S, f_t)$ are two-manifolds with boundary and $W^{ss}(S, f_t)$ and $W^{uu}(S, f_t)$ have both dimension one. Notice that

\[
\{0\} \times [0,1] \times [-1,1] \subset W^u(S, f_t), \quad [-1,1] \times [-2,0] \times \{0\} \subset W^s(S, f_t),
\]

\[
[-1,1] \times \{(0,0)\} \subset W^{ss}(S, f_t), \quad \{(0,0)\} \times [-1,1] \subset W^{uu}(S, f_t).
\]

Keeping in mind these relations, we have that,

• for all $t$, $W^u(S, f_t)$ meets transversely $W^s(P, f_t)$ throughout the segment $\{0\} \times (0,1) \times \{0\}$,

• for $t = 0$, $W^u(P, f_0)$ meets quasi-transversely $W^{ss}(S, f_0)$ along the orbit of $(-1/2,0,0)$,

• for $t > 0$, the point $(-1/2,t,0)$ is a transverse homoclinic point of $P$ and $(-1/2, 0, 0)$ is a point of transverse intersection between $W^{ss}(S, f_t)$ and $W^u(S, f_t)$.

In this case, we say that the arc $f_t = f_{t,0}$ has a saddle-node heterodimensional cycle associated to $P$ and $S$ at $t = 0$. This cycle can be thought as a limit case of the heterodimensional cycles in Section 1 where the derivative of the point of index two $Q$ is $1^+$. 12
The two-fold behavior of the saddle-node $S$, as a point of index two and one simultaneously, leads us to consider, for small positive $t$, the lateral homoclinic classes of $S$ defined by

$$H^+(S, f_t) = W^u(S, f_t) \cap W^{ss}(S, f_t) \quad \text{and} \quad H^-(S, f_t) = W^s(S, f_t) \cap W^{au}(S, f_t).$$

As in the case of the usual homoclinic classes, we have that:

**Proposition 4.1** For every small $t > 0$, $H^+(S, f_t)$ (resp. $H^-(S, f_t)$) is transitive and the periodic points of index two (resp. one) form a dense subset of it.

Consider a neighborhood $W$ of the saddle-node heterodimensional cycle defined as in Section 3 and denote by $\Upsilon_t$ the maximal invariant set of $f_t$ in $W$.

**Theorem 4.2** For every small $t > 0$, one has that $H(P, f_t) \subset H^+(S, f_t)$ and $\Upsilon_t \subset H(S^+, f_t)$.

The proof of Theorem 4.2 follows as the one of Theorem 1.1, the only difficulty being to redefine appropriately the one-dimensional dynamics associated to the cycle (recall Section 2). This will be briefly done in the next section. To get the inclusion $H^+(S, f_t) \subset H(P, f_t)$ we need the following distortion property for the saddle-node map $\Phi = \Phi_0$.

(SN) Let $K > 0$ be the maximum of $|\Phi''(x)|/|\Phi'(x)|$, $x \in [0, 1]$, then $\frac{4e^K (1 - \lambda)}{\lambda^6} < \frac{1}{2}$, where $\lambda \in (2/3, 1)$.

**Theorem 4.3** Under the assumption (SN), $H^+(S, f_t) \subset H(P, f_t)$ holds for all small positive $t > 0$.

To prove this theorem we need new ingredients that will be introduced in Section 4.3.

### 4.1 One-dimensional dynamics for the saddle-node cycle

We now adapt the definitions of scaled fundamental domains, transitions and returns for saddle-node cycles. As in Section 2 for each $t > 0$, define the fundamental domains $D_t^- = [1 - t, 1 - \lambda t]$ and $D_t^+ = [a_t, b_t]$, $a_t = \Phi^{-1}(b_t)$, where $D_t^+$ is the first backward iterate of $D_t^-$ by $\Phi$ contained in $[0, t]$. We have $\Phi^{k_t}(D_t^+) = D_t^-$, for some $k_t \in \mathbb{N}$. Observe that $|D_t^-| = t (1 - \lambda)$ and, since $b_t \in (0, t]$, $|D_t^+| \leq t^2$. For small $t > 0$, define the transition $\mathcal{T}_t$ and the map $\mathcal{G}_t$ by

$$\mathcal{T}_t: D_t^+ \to D_t^-, \quad x \mapsto \mathcal{T}_t(x) = \Phi^{k_t}(x) \quad \text{and} \quad \mathcal{G}_t: [0, t(1 - \lambda)], \quad x \mapsto \mathcal{G}_t(x) = \mathcal{T}_t(x) + t.$$

**Lemma 4.4** The maps $\mathcal{T}_t$ and $\mathcal{G}_t$ are uniformly expanding for all small $t > 0$.

**Proof:** It suffices to see that $(\Phi^{k_t})'(z) > 1$ for all $z \in D_t^+$. We use the following standard lemma (whose proof is omitted here):

**Bounded Distortion Lemma 4.5** Let $K > 0$ be as above. Then, for every pair of points $z, y \in D_t^+$ and every small $t > 0$, it holds

$$e^{-K} \leq \frac{(\Phi^{k_t})'(z)}{(\Phi^{k_t})'(y)} \leq e^K.$$
The lemma now follows by the mean value theorem, taking $y$ with $(\Phi^{k_t})'(y) = |D_t^-|/|D_t^+| \geq (1-\lambda)/t$. Thus, if $t$ is small, $(\Phi^{k_t})'(z) \geq (e^{-K} (1-\lambda))/t > 1$, for all $z \in D_t^+$.

As in Section 2, given $x \in (a_t, b_t] = \tilde D_t^+$, let $i(x) \in \mathbb{Z}$ be the first $i$ with $\Phi^i(\mathcal{G}_t(x)) \in D_t^+$.

**Lemma 4.6** There exists $i_0 > 0$ such that $i(x) \geq i_0$ for all $x \in \tilde D_t^+$.

**Proof:** To prove the lemma it is enough to see that $\mathcal{G}_t(D_t^+) \subset (0, a_t)$. By definition, $\mathcal{G}_t(D_t^+) = [0, (1-\lambda) t]$. Observe that, if $t$ is small enough,

$$\Phi^2((1-\lambda)t) = \Phi((1-\lambda)t + (1-\lambda)^2 t^2) = (1-\lambda)t + 2(1-\lambda)^2 t^2 + \text{h.o.t.} < t.$$

Thus, the right extreme $\Phi^2((1-\lambda)t)$ of $\Phi^2(\mathcal{G}_t(D_t^+))$ is less than $t$. In particular, the right extreme of $\mathcal{G}_t(D_t^+)$ is less than $\Phi^{-2}(t)$, and the the lemma follows from $D_t^+ \subset (\Phi^{-2}(t), t]$.

The return map $\mathcal{R}_t$ is now defined by

$$\mathcal{R}_t : \tilde D_t^+ \to D_t^+, \quad \mathcal{R}_t(x) = \Phi^i(x)(\mathcal{G}_t(x)) = \Phi^i(x)(\mathcal{Z}_t(x) + t).$$

As in the case of the map $R_t$ in Section 2 for each $i \geq i_0$, there is $\delta_i \in \tilde D_t^+$ with $\mathcal{G}_t(\delta_i) = \Phi^{-i}(a_t)$. The points $\delta_i$ are the discontinuities of $\mathcal{R}_t$. In this way, we get a decreasing sequence $(\delta_i)_{i \geq i_0}$ with $\delta_i \to a_t$, and intervals $J_i = [\delta_{i+1}, \delta_i]$, $i > i_0$, and $J_{i_0} = [\delta_{i_0}, b_t]$ such that $\mathcal{R}_t$ is continuous and increasing in the interior of each $J_i$. Extending $\mathcal{R}_t$ continuously to the whole $J_i$ we get a bi-valuated map with $\mathcal{R}_t(\delta_i) = \{a_t, b_t\}$ for all $i > i_0$.

**Lemma 4.7** The restriction of $\mathcal{R}_t$ to each interval $J_i, i > i_0$, is onto. Moreover, there is $\ell > 1$ such that $\mathcal{R}_t(\delta_i) > \ell$ for all $x \in (a_t, b_t]$ (if $x = \delta_i$ this means that the lateral derivatives of $\mathcal{R}_t$ at $x$ are greater than $\ell$). Finally, $\mathcal{G}_t(\mathcal{R}_t(\delta_i)) = 0$ for all $i \geq i_0$.

**Proof:** The lemma follows as Lemma 4.7 observing that $i_0 > 0$ (Lemma 4.6), $\mathcal{G}_t$ is expanding (Lemma 4.4), and that the derivative of $\Phi$ in $(0, t]$ is bigger than one.

Arguing as in Section 2 one gets the following lemma (corresponding to Lemma 2.5):

**Lemma 4.8** Given any subinterval $I$ of $D_t^+$ there are $x \in I$ and $i \geq 0$ with $\mathcal{G}_t(\mathcal{R}_t^i(x)) = 0$.

### 4.2 Lateral Homoclinic classes. Proof of Theorem 4.2

To prove Theorem 4.2 we proceed as in Section 3. After redefining vertical strips and segments and using Lemma 4.8 one gets that, for any small $t > 0$ and any disk $\chi$ with $W^s(P, f_t) \cap \chi \neq \emptyset$, $W^{ss}(S, f_t) \cap \chi \neq \emptyset$ (recall Proposition 3.1). The inclusion $(H(P, f_t) \cup \Upsilon_t) \subset H^+(S, f_t)$ follows exactly as $(H(P, f_t) \cup \Lambda_t) \subset H(Q, f_t)$ in the case of heterodimensional cycles.

#### 4.3 Proof of Theorem 4.3

Consider the homoclinic point $x_t = (-1/2, t, 0)$ of $P$ for $f_t$ and the fundamental domains $\Delta_t^+(i) = \Phi^{-i}(\Delta_t^+(0))$, $i \geq 0$, where $\Delta_t^+(0) = [\Phi^{-1}(t), t]$. Let $\kappa_t$ be the first $k \in \mathbb{N}$ such that $\Phi^k(\Delta_t^+(0)) \subset \cdots$
iterate of it, we can assume that 

\[ |\Phi^{k_2}(\Delta_+^t(0))| \leq t (1 - \lambda) \text{ and } |\Delta_+^t(0)| \geq \lambda t^2, \]

we get, using the Bounded Distortion Lemma 4.3

\[
(\Phi^{k_2})'(x) < \frac{1 - \lambda}{\lambda t} e^K, \quad \text{for all } x \in \Delta_+^t(0). \tag{2}
\]

Denote by \( \delta_i^t \) the length of \( \Delta_+^t(i) \). Since the derivative of \( \Phi \) near 0 is close to 1 and strictly bigger than 1 in \((0, t)\), for small \( t \), we have that

\[
\delta_0^t \geq \delta_i^t \geq \frac{9 \delta_0}{10}, \quad i = 1, \ldots, 4. \quad \text{In particular, } \sum_{i=0}^{4} \delta_i^t \in [4\delta_0^0, 5\delta_0]. \tag{3}
\]

We now construct a family \( \mathcal{H}_t \) of homoclinic points of \( P \) for \( f_t \) such that the set \( \{y: (x, y, 0) \in \mathcal{H}_t\} \) is dense in the fundamental domain \( \Delta_+^+(0) \):

**Proposition 4.9** For every small \( t > 0 \) there are sequences of homoclinic points of \( P \) of the form 

\[
(b_{i_1, i_2, \ldots, i_m, k}, x_{i_1, i_2, \ldots, i_m, k}, 0)_{k \in \mathbb{N}^*}, \quad b_{i_1, i_2, \ldots, i_m, k} \in [-1, 0],
\]

such that

- \((H1)\) \( x_{i_1, i_2, \ldots, i_m, k} \in \bigcup_{i_0=0}^{4} \Delta_+^t(i) = \Delta_t \),
- \((H2)\) \( x_{i_1, i_2, \ldots, i_m, k} \to x_{i_1, i_2, \ldots, i_m} \) as \( k \to \infty \),
- \((H3)\) \( x_{i_1, i_2, \ldots, i_m, 0} < x_{i_1, i_2, \ldots, (i_m-1)} \) for every \( i_m \geq 1 \),
- \((H4)\) \( \text{diam}(x_{i_1, i_2, \ldots, i_m, k}) \to 0 \) as \( m \to \infty \),
- \((H5)\) \( (x_i) \) is increasing and \( x_i \to t^- \) as \( i \to \infty \),
- \((H6)\) \( x_0 \in \Delta_+^t(1) \) and \( x_0 \notin \Delta_+^t(0) \).

This proposition will be proved in Section 4.3.2. From the proposition one gets the following:

**Corollary 4.10** The set \( \mathcal{H}_t = \bigcup_{n, k \in \mathbb{N}^*} (x_{i_1, i_2, \ldots, i_k, n}) \) contains a dense subset of \( \Delta_+^t(0) \).

**Proof:** The proof of is identical to \([\text{D}_2, \text{Lemma } 4.1]\), but we repeat it here for completeness. Take any point \( x \in \Delta_+^t(0) \). If \( x \in \mathcal{H}_t \) there is nothing to prove. Otherwise, by \((H5)\) and \((H6)\), there is \( i_1 > 1 \) with \( x_{(i_1-1)} < x < x_{i_1} \). Analogously, by \((H2)\) and \((H3)\), there is \( i_2 > 1 \) with \( x_{i_1, i_2-1} < x < x_{i_1, i_2} \). Inductively, using \((H2)\) and \((H3)\) as above, we get a sequence \( \{i_k\} \), \( i_k > 1 \), such that, for all \( k \), \( x_{i_1, \ldots, (i_k-1)} < x < x_{i_1, \ldots, i_k} \). Finally, from \((H4)\), \( \lim_{k \to \infty} x_{i_1, \ldots, i_k} = x \), ending the proof of the lemma. \( \square \)

### 4.3.1 Proof of Theorem 4.3

The deduction of Theorem 4.3 from Corollary 4.10 follows as in \([\text{D}_2, \text{Section } 5]\). For completeness, we sketch here this proof. Consider any \( w \in W^u(S, f_t) \cap W^s(S, f_t) \). By replacing \( w \) by some iterate of it, we can assume that \( w = (x, 0, 0) \), \( |x| \) small. We prove that, for every \( \varepsilon > 0 \), the square \( S(\varepsilon) = (x - \varepsilon, x + \varepsilon) \times (0, \varepsilon) \times \{0\} \) contained in \( W^s(P, f_t) \) transversely intersects \( W^u(P, f_t) \). This
immediately implies that \( w \in H(P, f_t) \). The configuration of the cycle and the \( \lambda \)-lemma imply that there is \( n(\varepsilon) > 0 \) such that \( f_t^{-n(\varepsilon)}(S(\varepsilon)) \) contains a disk \( S'(\varepsilon) \) of the form

\[
S'(\varepsilon) = [-1, 1] \times (\bar{y} - \xi, \bar{y} + \xi) \times \{ \bar{z} \}; \quad \bar{y} \in (1 - t, 1), \bar{z} \in [-1, 1] \text{ and small } \xi > 0.
\]

Let \( m \in \mathbb{N} \) be such that \( \Phi^{-m}(\bar{y}) \in \Delta_t^+(0) \). Thus \( f_t^{-m}(S'(\varepsilon)) \) contains the strip

\[
\hat{S}(\varepsilon) = [-1, 1] \times (\Phi^{-m}(\bar{y} - \xi), \Phi^{-m}(\bar{y} + \xi)) \times \{ \lambda u^{-m} \bar{z} \} \subset W^s(P, f_t).
\]

Since \( \Phi^{-m}(\bar{y}) \) belongs to \( \Delta_t^+(0) \), Corollary 4.10 implies that \( \hat{S}(\varepsilon) \) meets \( W^u(P, f_t) \). Thus \( \hat{S}(\varepsilon) \) contains a homoclinic point of \( P \) and the same holds for \( S(\varepsilon) \).

\[\square\]

4.3.2 Proof of Proposition 4.9 Sequences of homoclinic points:

Consider the interval \([1 - \eta_t, 1]\), where \( \eta_t = \delta_t^1 + \delta_t^0 \). Define \( \alpha_t \) as the first natural number \( \alpha \) with

\[
\Phi^{\kappa_t + \alpha}(\Delta_t^+(0)) \subset [1 - \eta_t, 1].
\]

Observe that \(|\Delta_t^+(0)| = \delta_t^0 < 4t^2\) and \( \delta_t^1 < \delta_t^0 \). Thus, for small \( t > 0 \), \( \eta_t < 2 \delta_t^0 < 2t^2 < \lambda t \). Since, by definition, \( \Phi^{\kappa_t}(\Delta_t^+(0)) \subset [1 - t, 1 - \lambda t] \) and \( (1 - \lambda t) < (1 - \eta_t) \), we get that \( \alpha_t \geq 1 \) for every small \( t \). Observe also that \( t^2 < \eta_t < 2t^2 \), where the first inequality follows from (3) and \( \delta_t^0 > 3t^2/4 \) if \( t \) is small enough.

Lemma 4.11 For every small \( t > 0 \) it holds \( \lambda^{\alpha_t} \leq (2t)/\lambda \).

Proof: By definitions of \( \kappa_t \) and \( \alpha_t \), \( \Phi^{\alpha_t}(\Delta_t^+(0)) = [1 - e_t^-, 1 - e_t^+] \), where \( e_t^- \in [\lambda t, t] \), and \( \Phi^{\alpha_t}(1 - e_t^-) \in [1 - \eta_t, 1 - \lambda \eta_t] \). Thus, since \( \Phi \) is affine near 1, \( \lambda^{\alpha_t}(e_t^-) \in (0, \eta_t] \). Thus, from \( t^2 < \eta_t < 2t^2 \) and \( \lambda t \leq e_t^- \leq t \),

\[
\lambda^{\alpha_t} \leq \frac{\eta_t}{e_t^-} \leq \frac{2t}{\lambda},
\]

ending the proof of the lemma. \[\square\]

Next contraction lemma is necessary for getting (H4) in Proposition 4.9 and along the inductive definition of the sequences \((x_{i_1, i_2, \ldots, i_n, k})_k\).

Lemma 4.12 \( L = \max\{(\Phi^{\kappa_t + \alpha_t + j})(x); x \in \bigcup_{i=0}^j \Delta_t^+(i) \text{ and } j \geq 0\} < \frac{1}{2} \).

Proof: Since \( \Phi \) is a contraction near 1, it is enough to compute the estimate when \( j = 0 \). We split the trajectory of a point \( x \in \Delta_t^+(i) \) going from \( \Delta_t^+(i) \) to \([1 - \eta_t, 1]\) as follows: (i) \( i \) iterates, \( i \leq 4 \), for \( x \) going from \( \Delta_t^+(i) \) to \( \Delta_t^+(0) \); (ii) \( \kappa_t \) iterates for \( \Phi^i(x) \) going from \( \Delta_t^+(0) \) to \( \Phi^{\kappa_t}(\Delta_t^+(0)) \); and (iii) \( \alpha_t \) iterates for \( \Phi^{\kappa_t + \alpha_t}(x) \) going from \( \Phi^{\kappa_t}(\Delta_t^+(0)) \) to \([1 - \eta_t, 1]\). This construction involves \((i + \kappa_t + \alpha_t)\) iterations of \( x \) by \( \Phi \), that is, we need to remove the last \( i \) iterations, corresponding to a contraction by \( \lambda^i \). We claim that

\[
L \leq \left( (2t + 1)^i \right) \left( \frac{e_t^R (1 - \lambda)}{\lambda t} \right) \frac{1}{\lambda^j},
\]

(4)
Proof: The sequence 

Lemma 4.14

observe that, by construction, \( \alpha \).

Also, by the definitions of \( \text{Observe that, for each } i \), the form \( \Phi \).

Need the following algorithm about the creation of homoclinic points, which is a consequence of the

definition of the unfolding of the heterodimensional cycle.

Consider the homoclinic point \((-1/2, t, 0)\) of \( P \) for \( f_t \) (satisfying Algorithm 4.13) and the sequences \((y_i)_{i \in \mathbb{N}^*}\), and \((x_i)_{i \in \mathbb{N}^*}\) defined by

\[
y_i = \Phi^\kappa_i + \alpha_i + i \delta_i (t) \quad \text{and} \quad x_i = (t - 1) + y_i, \quad y_i \to 1 \quad \text{and} \quad x_i \to t.
\]

Observe that, for each \( i \geq 0 \), there is a homoclinic point \((b_i, x_i, 0)\) of \( P \) verifying Algorithm 4.13. Also, by the definitions of \( \alpha_i \) and \( \kappa_i \), \( y_i \in [1 - \eta_i, 1] \) for all \( i \geq 0 \). Thus, since \( \eta_i = \delta_i^0 + \delta_i^1 \), one has

\[
x_i \in [t - \eta_i, t] = [t - (\delta_i^0 + \delta_i^1), t] = \Delta_i^+(1) \cup \Delta_i^+(0).
\]

Lemma 4.14

The sequence \((x_i)_{i \in \mathbb{N}^*}\) verifies (H5) and (H6).

Proof: Condition (H5) follows by definition. To get (H6), i.e. \( x_0 = y_0 + (1 - t) \in (\Delta_i^+(1) \setminus \Delta_i^+(0)) \), observe that, by construction, \( x_0 \in [t - \eta_i, t - \lambda \eta_i] \) and \( \Delta_i^+(1) = [t - \eta_i, t - \delta_i^0] \). Thus, by [55], it is enough to check that \( \lambda \eta_i = \lambda (\delta_i^0 + \delta_i^1) > \delta_i^0 \). This inequality follows from \( \delta_i^1 > 2 \delta_i^0 \), (3) and (SN), observing that

\[
\frac{\delta_i^0}{\delta_i^0 + \delta_i^1} < \frac{\delta_i^0}{2 \delta_i^1} < \frac{\delta_i^0}{2 (9/10) \delta_i^0} = \frac{10}{18} < \frac{2}{3} < \lambda.
\]

The proof of the lemma is now complete.

We now proceed with the construction of the sequences in Proposition 4.13. For each \( j \in \mathbb{N}^* \), define the sequences \((y_j, i)_{i \in \mathbb{N}^*}\) and \((x_{j, i})_{i \in \mathbb{N}^*}\) as follows,

\[
y_{j, i} = \Phi^\kappa_i + \alpha_i + j \quad \text{and} \quad x_{j, i} = (t - 1) + y_{j, i}.
\]

We claim that \( y_{j, i} \to y_j \) and, consequently, \( x_{j, i} \to x_j \), as \( i \to \infty \). For that just observe that \( \lim_{i \to \infty} x_i = t \), thus, by continuity, \( \lim_{i \to \infty} y_{j, i} = \lim_{i \to \infty} \Phi^\kappa_i + \alpha_i + j (x_i) = \Phi^\kappa_i + \alpha_i + j (t) = y_j \).

Lemma 4.15

The points \((x_{j, i})_{i}\) belong to \( \bigcup_{i=0}^t \Delta_i^+(i) \) for all \( i, j \in \mathbb{N} \cup \{0\} \).
Proof: By construction, the sequences \((x_{i,j})_i\) are increasing, thus it is enough to prove that \(x_{0,0} \in \bigcup_{i=0}^4 \Delta^+_i(1)\). Consider the diameter \(d_0 = (t - x_0)\) of \((x_i)_{i \in \mathbb{N}^+}\). By Lemma 4.13, \(d_0 < \delta^0 + \delta^1\). Let \(d_1 = |x_0 - x_{0,0}|\) be the diameter of the sequence \((x_{0,i} = \Phi^{\delta^0 + \delta^1}(x_i) + (t - 1))_i\), which is equal to the diameter of \((\Phi^{\delta^0 + \delta^1}(x_i))_i\). Therefore, since \((x_i)_i \subset \Delta^+_i(0) \cup \Delta^+_i(1)\), by Lemma 4.12 the diameter \(d_1\) is bounded by

\[
d_1 \leq L d_0 < d_0/2 < (\delta^0 + \delta^1)/2 < (2 \delta^0)/2 < \delta^0.
\]

Since \(x_0 \in \Delta^+_i(1)\) (Lemma 4.14), to prove that \(x_{0,0} \in \bigcup_{i=0}^4 \Delta^+_i(1)\), it is enough to see that

\[
x_{0,0} = x_0 - d_1 > x_0 - (\delta^2 + \delta^3 + \delta^4) \iff d_1 < (\delta^2 + \delta^3 + \delta^4),
\]

which immediately follows from \(\delta^2 + \delta^3 + \delta^4 > \delta^0 > d_1\), the first inequality being consequence of \(\delta^i > (9\delta^0)/(10)\), see Lemma 4.14, and the last from Lemma 4.12. This ends the proof of the lemma.

Suppose now inductively defined sequences \((y_{i_1,i_2,\ldots,i_m,i})_{i \in \mathbb{N}^+}\) and \((x_{i_1,i_2,\ldots,i_m,i})_{i \in \mathbb{N}^+}\) by

\[
y_{i_1,i_2,\ldots,i_m,i} = \Phi^{\delta^0 + \delta^1 + \delta^2 + \delta^3}(x_{i_2,\ldots,i_m,i}) \quad \text{and} \quad x_{i_1,i_2,\ldots,i_m,i} = (t-1) + y_{i_1,i_2,\ldots,i_m,i},
\]

satisfying conditions (H1), (H2), (H3) and (H4)

(H2b) \((y_{i_1,i_2,\ldots,i_m,i})_i \rightarrow y_{i_1,i_2,\ldots,i_m}\) as \(i \rightarrow \infty\),

(H3b) \(y_{i_1,i_2,\ldots,i_m,0} < y_{i_1,i_2,\ldots,i_m,i-1}\) for all \(i \geq 1\).

(H4b) Let \(d_m, m \geq 0\), be the diameter of the sequence \((x_{0,\ldots,0,i})_i\). Then \(d_m \leq (d_{m-1})/2\).

Observe that (H2) and (H2b) (resp. (H3) and (H3b)) are equivalent. Notice that, for \(m = 1\), (H1) follows from Lemma 4.15 (H2) and (H2b) from definition, and (H4b) from (6). To check (H3b), \(y_{i_1,i_2,\ldots,i_m,0} < y_{i_1,i_2,\ldots,i_m,i-1}\) for every \(i \geq 1\), recall that, by Lemma 4.14, \(x_0 < \Phi^{-1}(t) < t\), thus

\[
y_{i-1} = \Phi^{\delta^0 + \delta^1 + \delta^2 + \delta^3 + \delta^4}(t) = \Phi^{\delta^0 + \delta^1 + \delta^2 + \delta^3}(\Phi^{-1}(t)) \geq \Phi^{\delta^0 + \delta^1 + \delta^2 + \delta^3}(x_0) = y_{i-1}.
\]

For simplicity we say that the sequences \((z_{i_1,i_2,\ldots,i_m,i})_{i \in \mathbb{N}^+}, z = x, y, \) are of generation \(m\).

Lemma 4.16 Property (H4b) implies (H4) in Proposition 4.10

Proof: By construction, the diameters of the sequence of generation \(m\) are bounded by the diameter \(d_m\) of \((x_{0,\ldots,0,i})_i\). Thus, inductively, \(d_m \leq (1/2) d_{m-1} \leq (1/2)^m d_0\), so \(d_m \rightarrow 0\).

Keeping in mind Lemmas 4.14 and 4.15 in order to prove Proposition 4.10 it suffices to see that the sequences above verify (H1), (H2b), (H3b) and (H4b). We argue inductively on the generation of the sequences and assume satisfied these conditions for sequences of generation less than or equal to \(m\). To verify (H2b) for the sequences of generation \(m + 1\) note that, by induction, \((y_{i_1,i_2,\ldots,i_m,i})_i \rightarrow y_{i_1,i_2,\ldots,i_m}\). Thus, by continuity of \(\Phi\) and by definition,

\[
(y_{i_1,i_2,\ldots,i_m,i})_i = (\Phi^{\delta^0 + \delta^1 + \delta^2 + \delta^3}(x_{i_2,\ldots,i_m,i}))_i \rightarrow \Phi^{\delta^0 + \delta^1 + \delta^2 + \delta^3}(y_{i_1,i_2,\ldots,i_m}) = y_{i_1,i_2,\ldots,i_m}.
\]

To prove (H3b) observe that, by induction, \(x_{i_1,i_2,\ldots,i_m,0} < x_{i_1,i_2,\ldots,i_m,i-1}\). Thus, since \(\Phi\) is increasing,

\[
y_{j_1,j_2,\ldots,j_m,0} = \Phi^{\delta^0 + \delta^1 + \delta^2 + \delta^3}(x_{j_1,j_2,\ldots,j_m,0}) < \Phi^{\delta^0 + \delta^1 + \delta^2 + \delta^3}(x_{j_1,j_2,\ldots,j_m,i-1}) = y_{j_1,j_2,\ldots,j_m-1}.
\]
To check (H4b) observe that, by the induction hypotheses (H1), \( x_{0,\ldots,0,i} \in \bigcup_{i=0}^{s} \Delta_i^+(i) \), Lemma 4.12 and the fact that the sequences \((y_{0,\ldots,0,i})\) and \((x_{0,\ldots,0,i})\) have the same diameter imply that

\[
d_{m+1} = \text{diam}(\Phi^{\alpha_1+\alpha_2}(x_{0,\ldots,0,i})) = L \text{diam}(x_{0,\ldots,0,i}) = L d_m \leq \frac{d_m}{2},
\]

which ends the proof of (H4b).

Finally, to get (H1) for sequences of generation \((m+1)\) it is enough to see that, for every \( m, \sum_{i=0}^{m} d_m < 4 \delta_i^0 < \sum_{i=0}^{m} \delta_i^1 \), where the last inequality follows from \([3]\). By induction and Lemma 4.13 which implies that \( d_0 \leq \delta_i^0 + \delta_i^1 \), we have

\[
\sum_{i=0}^{m} d_m \leq \sum_{i=0}^{m-1} (1/2)^i d_0 \leq \sum_{i=0}^{m-1} (1/2)^i (\delta_i^0 + \delta_i^1) \leq \frac{1}{1-2} (\delta_i^0 + \delta_i^1) < 2 (\delta_i^0 + \delta_i^1) < 4 \delta_i^0,
\]

ending the proof of our claim.

The construction of the sequences \((x_{i_1,i_2,\ldots,i_m})_{i \in \mathbb{N}^*}\) of Proposition 4.9 is now complete.

5 Homoclinic classes before collapsing the saddles \( S^+_s \) and \( S^-_s \)

We now return to the two parameter family \( f_{t,s} \) in Section 4. Observe that \( S^+_s = (0, \sqrt{s}, 0), s > 0, \) is a fixed point of index two of \( f_{t,s} \) (any \( t > 0 \)) and that \( f_{\sqrt{s},s} \) has a heterodimensional cycle associated to \( S^+_s \) and \( P \):

- \( W^u(S^+_s, f_{\sqrt{s},s}) \) meets transversely \( W^s(P, f_{\sqrt{s},s}) \) throughout the segment \( \{0\} \times (\sqrt{s}, 1) \times \{0\} \),
- \( W^u(P, f_{\sqrt{s},s}) \) meets quasi-transversely \( W^s(S^+_s, f_{\sqrt{s},s}) \) along the orbit of \((-1/2, \sqrt{s}, 0)\) (just observe that \([-1, 1] \times \{(\sqrt{s}, 0)\} \subset W^s(S^+_s, f_{\sqrt{s},s}) \) and that \((-1/2, \sqrt{s}, 0) \in W^u(P, f_{\sqrt{s},s})\).

In what follows we assume that the saddle-node arc \( \Phi_s \) verifies condition (SN) in Section 4.

**Theorem 5.1** There exist a small \( s_0 > 0 \) and a strictly positive map \( \tau \) defined on \((0, s_0)\) such that, for every \( s \in (0, s_0) \) and \( t \in (\sqrt{s}, \sqrt{s} + \tau(s)) \),

- \( H(P, f_{t,s}) = H(S^+_s, f_{t,s}), \) and
- there is a neighborhood \( W_s \) of the cycle of \( f_{\sqrt{s},s} \) (associated to \( P \) and \( S^+_s \)) such that the maximal invariant set \( \Lambda_{t,s} \) of \( f_{t,s} \) in \( W_s \) is equal to \( H(S^+_s, f_{t,s}) \).

The proofs of the inclusion \( H(P, f_{t,s}) \subset H(S_s, f_{t,s}) \) and the second part of the theorem follow as in Theorem 4.1. So we just sketch these proofs. For a fixed \( s > 0 \) and \( t > \sqrt{s}, t \) close to \( \sqrt{s} \), \( t = \sqrt{s} + \tau \), consider the scaled fundamental domains \( D_{t,s}^\pm \) of \( \Phi_s \),

\[
D_{t,s}^- = [1 - (t - \sqrt{s}), 1 - \lambda(t - \sqrt{s})] = [1 - \tau, 1 - \lambda \tau]
\]

and \( D_{t,s}^+ \) defined as the first backward iterate of \( D_{t,s}^- \) in \([\sqrt{s}, \sqrt{s} + \tau]\). Let \( \Phi_{k_{t,s}}(D_{t,s}^+) = D_{t,s}^- \), where \( k_{t,s} \in \mathbb{N} \). These domains play the role of \( D_{t,s}^\pm \) in Section 2. Observe that

\[
\ell(t, s) = \frac{|D_{t,s}^+|}{|D_{t,s}^-|} \geq \frac{\tau (1 - \lambda)}{\tau (2\sqrt{s} + \tau)} \geq \frac{\tau (1 - \lambda)}{\tau (2\sqrt{s} + \tau)} \geq \frac{1 - \lambda}{2\sqrt{s} + \tau}.
\]
By shrinking \( s \), we can assume that \(|\Psi_s''(x)|/|\Psi_s'(x)| < K\) for all \( x \in [-1, 2] \) (\( K \) as in Lemma 4.5). Thus, there is \( s_0 > 0 \) and a map \( \tau: (0, s_0) \rightarrow \mathbb{R}^+ \) such that

\[
\ell(t, s) e^{-K} > 2, \quad \text{for all } s \in (0, s_0) \text{ and } t \in (\sqrt{s}, \sqrt{s + \tau(s)}).
\]

Exactly as in Section 2 for \( s \in (0, s_0) \) and \( t \in (\sqrt{s}, \sqrt{s + \tau(s)}) \), we define maps

\[
T_{t,s}: D_{t,s}^+ \rightarrow D_{t,s}^-, \quad G_{t,s}: D_{t,s}^- \rightarrow [\sqrt{s}, \sqrt{s + \tau(1 - \lambda)]}, \quad G_{t,s}(x) = T_{t,s}(x) + (t - 1),
\]

\[
R_{t,s}: D_{t,s}^+ \rightarrow D_{t,s}^+, \quad R_{t,s}(x) = \Phi_s^i(x)(G_{t,s}(x)),
\]

where, as in Section 2 \( i(x) \) is the first forward iterate of \( G_{t,s}(x) \) by \( \Phi_s \) in \( D_{t,s}^+ \).

As in the proof of Lemma 4.1, the Bounded Distortion Lemma 4.5 and equation (7) imply that the maps \( T_{t,s}, G_{t,s} \) and \( R_{t,s} \) are all uniformly expanding. The inclusions \( H(P, f_{t,s}) \subset H(S_s^+, f_{t,s}) \) and \( \Lambda_{t,s} = H(S_s^+, f_{t,s}) \) now follow as in the proof of Theorem 1.1.

To prove \( H(P, f_{t,s}) \subset H(S_s^+, f_{t,s}) \), recall that Theorem 4.3 gives small \( \bar{\tau} > 0 \) with \( H^+ (S_s, f_{t,0}) \subset H(P, f_{t,0}) \) for all \( t \in (0, \bar{\tau}) \). The proof of this inclusion follows after constructing the sequences of homoclinic points of \( P \) in Proposition 4.9. The proof of such a proposition only involves distortion control of the saddle-node map in \([0, 1]\) and the definition of contracting itineraries, (Lemma 4.12). Clearly, these properties also hold after replacing, for small positive \( s \), the saddle-node \( S \) by the hyperbolic point \( S_s^+ \) and considering the restriction of the saddle-node map to \([\sqrt{s}, 1]\). The sketch of the proof of Theorem 5.1 is now complete.

### 6 Collision, explosion and collapse of homoclinic classes

In this section we prove Theorems A and B. Consider a two parameter family of diffeomorphism \( f_{t,s} \) locally defined as follows:

**Partially hyperbolic local dynamics:**

- In the set \( C = [-1, 1] \times [-2, 2] \times [-1, 1], f_{t,s}(x, y, z) = (\lambda^s x, \Psi_s(y), \lambda^u z) \), where \( 0 < \lambda^s < 1 < \lambda^u \) and \( \Psi_s: [-2, 2] \rightarrow (-3, 2) \) is a strictly increasing map such that \( \lambda^s < d_m < \Psi'(x) < d_M < \lambda^u \) for all \( x \in [0, 1] \) and small \( |s| \).

- \( \Psi_s(1) = 1 \) and \( \Psi_s(-1) = -1 \) for all \( s \). Moreover, there exists \( \delta > 0 \) such that, in the intervals \([-1 - \delta, -1 + \delta] \) and \([-1 + \delta, 1 - \delta] \), \( \Psi_s \) is affine and independent of \( s \). Furthermore, \( \Psi_s(-1) = \beta \) \( \Psi'(1) = \lambda \), where \( 0 < \lambda < 1 < \beta \).

- There is \( \delta > 0 \) such that the restriction of \( \Psi_s \) to \([-\delta, \delta]\) is of the form \( \Psi_s(x) = x + x^2 - s \).

- For \( s < 0 \), \( \Psi_s \) has (exactly) two fixed points in \([-2, 2]\) (the points \( \pm 1 \)), for \( s = 0 \), \( \Psi_0 \) has (exactly) three fixed points \( (\pm 1, 0) \), and, for \( s > 0 \), \( \Phi_s \) has (exactly) four fixed points \( (\pm 1, \pm \sqrt{s}) \).

We let \( P = (0, 1, 0), Q = (0, -1, 0), S = (0, 0, 0) \) and, for positive \( s \), \( S_s^+ = (0, \pm \sqrt{s}, 0) \), the fixed points of \( f_{t,s} \) in \( C \).

**Existence and unfolding of cycles:**
• Clearly, these properties also hold after replacing, for small positive $s(\ell)$, the translation $f^{k_o}_{t,s}(x,y,z) = (x - 1/2, y - 1 + t, z + 1/2)$, recall the definition of $f_t$ in Section [1].

As in previous sections, we have:

• for $(t,s) = (0,0)$, $f_{0,0}$ exhibits a pair of saddle-node heterodimensional cycles, associated to $P$ and $S$ ($W^u(P,f_{0,0})$ meets transversely $W^{ss}(S,f_{0,0})$) and to $Q$ and $S$ (here $W^s(Q,f_{0,0})$ intersects $W^{uu}(S,f_{0,0})$). Just observe that $[-1,1] \times \{(0,0)\} \subset W^{ss}(S,f_{0,0})$ and $\{(0,0)\} \times [-1,1] \subset W^{uu}(S,f_{0,0})$.

• For small $|t|$ and $s < 0$, the homoclinic classes of $P$ and $Q$ are both nontrivial: notice that, for negative $s$, $[-1,1] \times (-1,2) \times \{0\} \subset W^s(P,f_{t,s})$ and $\{0\} \times (-1,2) \times [-1,1] \subset W^u(Q,f_{t,s})$, thus $(-1/2,t,0)$ and $(-1/2,-1,0)$ are homoclinic points of $P$ and $Q$, respectively.

• $f_{\sqrt{s},s}$ has a pair of heterodimensional cycles associated to $P$ and $S^+_s$ and to $Q$ and $S^-_s$ (this is obtained exactly as in Section [5]).

As in Section [4] we assume the following distortion property (similar to (SN)) involving the saddle-node $S$ and the saddles $P$ and $Q$: Let $K$ be an upper bound for $|\Psi_s'(x)|/|\Psi_s'(x)|$, for small $|s|$ and $x \in [-1,1]$,

$$\text{(DS)} \max\left\{ \frac{4e^K(1-\lambda)}{\lambda^6}, \frac{4e^K(1-\beta^{-1})}{\beta^{-6}} \right\} < \frac{1}{2}, \text{ where } 2/3 < \lambda < 1 < \beta < 3/2.$$

### 6.1 Dynamics before collapsing the saddles $S^+_s$ and $S^-_s$

Theorems [4.2] and [4.3] give the existence of a small positive $\tilde{t}$ such that $H(P,f_{\tilde{t},0}) = H^+(S,f_{\tilde{t},0})$ and $H(Q,f_{\tilde{t},0}) = H^-(S,f_{\tilde{t},0})$. The proof of these identities only involve the following ingredients:

• The ratio between the lengths of the scaled fundamental domains at the hyperbolic point and at the saddle-node, and that such a ratio is arbitrarily large (for the inclusions $H(P,f_{t,0}) \subset H^+(S,f_{t,0})$ and $H(Q,f_{t,0}) \subset H^-(S,f_{t,0})$).

• The inclusion $H^+(S,f_{\tilde{t},0}) \subset H(P,f_{\tilde{t},0})$ (resp. $H^-(S,f_{\tilde{t},0}) \subset H(Q,f_{\tilde{t},0})$) is obtained by constructing sequences of homoclinic points of $P$ (resp. $Q$) verifying Proposition [4.9]. The proof of such a proposition only involves distortion control of the saddle-node map in $[0,1]$ (resp. $[-1,0]$) and construction of contracting itineraries (Lemma [4.12]).

Clearly, these properties also hold after replacing, for small positive $s$, the saddle-node $S$ by the hyperbolic point $S^+_s$ (considering the restriction of $\Psi_s$ to $[\sqrt{s},1]$) and and the saddle-node by the point $S^-_s$ (considering the restriction of $\Psi_s$ to $[-1,-\sqrt{s}]$). In this way, we get:

**Theorem 6.1** If $t > 0$ is small, there is a small $s(t) > 0$ such that $H(P,f_{t,s}) = H(S^+_s,f_{t,s})$ and $H(Q,f_{t,s}) = H(S^-_s,f_{t,s})$ for all $s \in (0,s(t))$. 

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6.2 Dynamics after collapsing the saddles $S^+_s$ and $S^-_s$

**Theorem 6.2** For every small $t > 0$ and $s < 0$, $H(P, f_{t,s}) = H(Q, f_{t,s})$.

This theorem follows by using some of the ideas in Theorem 1.1. We will prove the inclusion $H(P, f_{t,s}) \subset H(Q, f_{t,s})$, $(H(Q, f_{t,s}) \subset H(P, f_{t,s})$ similarly follows by considering $f^{-1}_{t,s}$). As in Sections 2 and 4.1, we define transitions $T_{t,s}$ and return maps $R_{t,s}$ as follows. Consider the fundamental domain $D^-_t = [1 - 3t, 1 - 3\lambda t]$ of $\Psi_s$ and define $k_{t,s}$ as the first $k \in \mathbb{N}$ such that $\Psi_s(-t) \in \Psi_s^{-k}(D^-_t)$. We let $D^+_{t,s} = \Psi_s^{-k_{t,s}}(D^-_t)$ and define the maps

$$T_{t,s}: D^+_{t,s} \to D^-_t, \quad G_{t,s}(x) = \Psi_s^{k_{t,s}}(x), \quad G_{t,s}: D^+_{t,s} \to [-2t, -t], \quad G_{t,s}(x) = T_{t,s}(x) + t - 1.$$ 

Observe that, by definition, $G_{t,s}(D^+_{t,s}) \subset [-2t, t(1 - 3\lambda)] \subset [-2t, -t]$ (recall $\lambda > 2/3$).

Observing that, by definition of $\Psi_s$, $|D^+_{t,s}| \leq t^2 + s$, the Bounded Distortion Lemma 4.5 and $|D^-_t| = 3\lambda(1 - t)$ immediately give that

$$T'_{t,s}(x) \geq (e^{-K2}) \frac{|D^-_t|}{|D^+_{t,s}|} \geq (e^{-K2}) \frac{3\lambda(1 - t)}{t^2 + s}.$$ 

This inequality immediately implies the following:

**Lemma 6.3** The maps $T_{t,s}$ and $G_{t,s}$ are 61-expanding for all small $t > 0$ and $s < 0$.

Since, by definition of $D^+_{t,s}$, $(-t)$ is at the left of $D^+_{t,s}$, then, for each $x \in D^+_{t,s}$, there is a first $i(x) \geq 0$ such that $\Psi_s^{i(x)}(G_{t,s}(x)) \in D^+_{t,s}$. We now define the return map $R_{t,s}$ by

$$R_{t,s}: D^+_{t,s} \to D^+_{t,s}, \quad R_{t,s}(x) = \Psi_s^{i(x)}G_{t,s}(x).$$

**Lemma 6.4** The map $R_{t,s}$ is 3-expanding for all small $t > 0$ and $s < 0$.

Observe that, contrary what happens with the return maps $R_t$ and $R_t$ in Sections 2 and 4.1, the expansion for $R_{t,s}$ does not follow immediately from the expansion of $T_{t,s}$: the $i(x)$ iterates by $\Psi_s$ at the left of 0 contribute with a small contraction. **Proof:** By Lemma 6.3 it is enough to verify that the contraction introduced by $\Psi_s^{i(x)}$ is at most 1/20. For that recall that $G_{t,s}(D^+_{t,s}) \subset [-2t, -t]$ and observe that $i(x) \leq i_0$, where $i_0$ is the first natural number with $\Psi_s^{i_0}(-2t) \in D^+_{t,s}$. Write $D^+_{t,s}(j) = \Psi_s^{-j}(D^+_{t,s})$ and note that, for every $x \in D^+_{t,s}$, $|D^+_{t,s}(i(x))| \leq |D^+_{t,s}(i_0)| < 9t^2 + s$. For each $i \in \{0, \ldots, i_0\}$, there is $z_i \in D^+_{t,s}(i)$ such that

$$(\Psi^i_s)'(z_i) = \frac{|D^+_{t,s}|}{|D^+_{t,s}(i)|} \geq \frac{|D^+_{t,s}|}{|D^+_{t,s}(i_0)|} \geq \frac{|D^+_{t,s}|}{9t^2 + s}.$$ 

Using the arguments in the Bounded Distortion Lemma 4.5, we have that, for every $x \in D^+_{t,s}(i)$,

$$\Psi^i_s(x) \geq e^{-K L_t} \frac{|D^+_{t,s}|}{9t^2 + s},$$

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where $K > 0$ is an upper bound for $|\Psi''_s(x)|/|\Psi'_s(x)|$, $x \in [-1, 1]$ and $|s|$ small, and $L_t = 4t$ is the length of $[-3t, t]$. Finally, observing that $|D^+_{t,s}| > (t^2)/2$,

$$\left(\Psi^i_s(x)\right)'(x) \geq \frac{e^{-K4t} |D^+_{t,s}|^2}{9t^2 + s} > \frac{e^{-K4t} t^2}{18t^2 + 2s} > \frac{1}{20},$$

for every small $t$ and $s$, ending the proof of the lemma.

Let $D^+_{t,s} = [e^{-t}, e^+_{t,s}]$. Observe that there exist $i_1$ and $i_2 \in \mathbb{N}$ such that $i(x) \in [i_1, i_2]$ for all $x \in D^+_{t,s}$ and, for each $i \in [i_1, i_2 - 1]$, there is $d_i \in D^+_{t,s}$ with $\Psi_s(G_{t,s}(d_i)) = e^{-t}_{i,s}$. As in Section 2 and by definition, $d_{i_2 - 1} < d_{i_2 - 2} < \cdots < d_{i_1}$ and the points $d_i$ are the discontinuities of $\mathcal{R}_{t,s}$. Moreover, for each $i_1 \leq i < i_2$, $G_{t,s}(d_i) = \text{int}(D^+_{t,s})$ and $G_{t,s}$ is increasing in $(d_{i + 1}, d_i)$. Write $[d_{i_1}, d_i] = I_i$, $i_1 \geq i \geq i_2 - 2$, $I_{i_2 - 1} = [e_{i_2 - 2}, d_{i_2 - 1}]$, and $I_{i_1} = [d_{i_1}, e^+_{t,s}]$. We continuously extend $\mathcal{R}_{t,s}$ to the closed intervals $I_i$ (so $\mathcal{R}_{t,s}$ is bivaluated at any $d_i$).

**Lemma 6.5** Given any subinterval $J$ of $D^+_{t,s}$, there is $m \geq 0$ such that $\mathcal{R}^m_{t,s}(J) = D^+_{t,s}$.

**Proof:** It is enough to see $\mathcal{R}^m_{t,s}(J)$ contains an interval $[d_{i_1}, d_i]$, $i_1 \leq i \leq i_2 - 2$, for some $m \in \mathbb{N}^*$. Write $J = J_0$. If $\mathcal{R}_{t,s}(J_0)$ contains two discontinuities we are done. Otherwise, $\mathcal{R}_{t,s}(J_0) \subset I_i$ for some $i$ and we let $J_1 = \mathcal{R}_{t,s}(J_0)$. By Lemma 6.4, $|J_1| > 3|J_0|$. Finally, if $\mathcal{R}_{t,s}(J_0)$ just contains one discontinuity, say $d_i$, then $\mathcal{R}_{t,s}(J_0)$ only meets $I_i$ and $I_{i-1}$. Write $J^+_1 = \mathcal{R}_{t,s}(J_0) \cap I_i$ and $J^-_1 = \mathcal{R}_{t,s}(J_0) \cap I_{i-1}$, and let $J_1$ be the biggest $J^+_1$. By Lemma 6.3, $|J_1| > (3/2)|J_0|$. We now inductively define intervals $J_i$ contained in the forward orbit of $J_0$ such that either $J_{i+1}$ contains two discontinuities or $|J_{i+1}| \geq (3/2)|J_i|$. Since the size of $D^+_{t,s}$ is finite, at some step we get a first interval $J_m$ containing two discontinuities.

Lemma 6.5 is the main step to prove Theorem 6.2 whose proof follows arguing as in the proof of Theorem 1.1 after the following considerations:

**Proposition 6.6** Let $\chi \subset C = [-1, 1] \times [-1, 2] \times [-1, 1]$, $x \in [-1, 1]$ be a set of the form $\chi = \{x\} \times A$, where $A$ is a disk of $\mathbb{R}^2$ whose interior contains a point of the form $(y, 0)$, with $y \in (-1, 2)$. Then $\chi$ intersects transversely $W^s(Q, f_{t,s})$.

This result exactly corresponds to Proposition 3.4. After proving it, Theorem 6.2 is deduced as follows. As in Section 3 Proposition 6.6 implies that $W^s(Q, f_{t,s})$ meets transversely every two-disk $\chi$ with $W^s(P, f_{t,s}) \cap \chi \neq \emptyset$. Finally, arguing exactly as in Section 3.1, we get $H(P, f_{t,s}) \subset H(Q, f_{t,s})$.

We now go to the details of the proof of Proposition 6.6. The first step is the following (corresponding to Lemma 3.3 in the proof of Propositions 3.3 and 3.4).

**Lemma 6.7** The manifold $W^u(P, f_{t,s})$ contains a vertical segment of the form $\{(x, y)\} \times [-1, 1]$, $x \in [-1, 1]$ and $y \in D^+_{t,s}$.

**Proof:** The proof follows as in Lemma 3.3 so we just sketch it. Take the homoclinic point $x_t = (-1/2, t, 0)$ of $P$ for $f_{t,s}$ and let $r \in \mathbb{N}$ be such that $\Psi_r^s(t) \in D^+_{r,s}$. By construction,

$$\{(x_t + \lambda^r_s(-1/2), \Psi^i_s(t + t - 1)) \times [-1, 1] \subset W^u(P, f_{t,s}) \}.$$

Let $i = i(\Psi^r_s(t) + t - 1)$. Thus

$$\{(\lambda^i_s(-1/2 + \lambda^r_s(-1/2)), \Psi^i_s(t + t - 1)) \times [-1, 1] \subset W^u(P, f_{t,s}) \}.$$
If $\Psi^i_s(\Psi^i_s(t) + t - 1)$ is in the interior of $D^+_0$, we are done. Otherwise, $\Psi^{i+1}_s(\Psi^i_s(t) + t - 1)$ is the right extreme of $D^+_0$. In this case the result follows using the $\lambda$-lemma: $W^u(P, f_{t,s})$ accumulates the previous point at the right, recall the proof of Lemma 3.3. □

**Lemma 6.8** For every $x \in [-1, 1]$, $W^s(Q, f_{t,s})$ meets transversely $\{x\} \times D^+_0 \times [-1, 1]$.

**Proof:** Observe that, by construction, the point $(0, 0, -1/2)$ belongs to $W^u(Q, f_{t,s})$. Define $j > 0$ by $\Psi^j_s(0) \in D^+_0$ (where $j \to \infty$ as $s \to 0$). It is now immediate that

$$H = [-1, 1] \times \{(\Psi^{-j}_s(0), -\lambda^{-j}_s(1/2))\} \subset W^s(Q, f_{t,s}) \quad \text{and} \quad H \cap \{(x) \times D^+_0 \times [-1, 1]\} \neq \emptyset,$$

ending the proof of the lemma. □

We are now ready to prove Proposition 6.6. By the $\lambda$-lemma and Lemma 6.7, the forward orbit of the disk $\chi$ contains a strip $\Delta$ of the form $\{x\} \times [a, b] \times [-1, 1]$, $x \in [-1, 1]$, where $a_0 = [a, b] \subset D^+_0$. Using the map $\mathcal{R}_{t,s}$ and exactly arguing as in Section 3.1 we inductively define strips $\Delta_k = \{x_k\} \times \alpha_k \times [-1, 1]$, $x_k \in [-1, 1]$ and $\alpha_k \subset D^+_0$, such that $\Delta_{k+1} \subset f_{i_{t,s}}(\Delta_k)$ and $\alpha_{k+1} = \mathcal{R}_{t,s}(\alpha_k)$. By Lemma 6.5 there is a first $k \in \mathbb{N}$ such that $\alpha_{k+1} = \mathcal{R}_{t,s}(\alpha_k)$ contains $D^+_0$. By Lemma 6.8, $W^s(Q, f_{t,s}) \cap \Delta_{k+1} \neq \emptyset$, thus $W^s(Q, f_{t,s}) \cap \Delta \neq \emptyset$, ending the proof of the proposition.

### 6.3 End of the proof of Theorem A

We now construct a one-parameter family of diffeomorphisms $(g_s)$ satisfying Theorem A. For that consider the arc $f_{t,s}$ defined as in the beginning of Section 6. We fix small $\ell > 0$ and consider the arc $g_s = f_{\ell t, -s}$. The results in the previous section imply that

- for every $s < 0$, $H(P, g_s)$ and $H(Q, g_s)$ are non-hyperbolic and disjoint (Theorem 6.1),
- $s = 0$, $H(P, g_0) = H^+(S, g_0)$ and $H(Q, f_0) = H^-(S, g_0)$, (Theorems 4.2 and 4.3 see the beginning of Section 6.1),
- for every $s > 0$, $H(P, g_s) = H(Q, g_s)$, (Theorem 6.2).

To finish the proof of Theorem A we need to see that $\{s\} \subset H(P, g_s) \cap H(Q, g_s)$ and to describe the maximal invariant set of $g_s$ in the neighborhood $W$.

We assume that the neighborhood of the cycle $W$ is a level of a filtration of $f_{0,0}$ (thus, by continuity and compactness, it is also a level of a filtration for $f_{t,s}$ for every small $s$ and $t$). This means that there are compact sets $M_2$ and $M_1$, $M_1$ contained in the interior of $M_2$, such that $M_2 \setminus M_1 = W$ and $f_{0,0}(M_i)$ is contained in the interior of $M_i$, $i = 1, 2$. Hence, if $x \in W$ and $f_{i,0}(x) \notin W$ for some $i$, then $x$ is wandering: suppose, for instance, that $f_{i_0,0}(x) \in M_1$, where $i_0 > 0$. Then, there is a neighborhood $U_x \subset W$ of $x$ such that $f_{i_0,0}(U_x) \subset M_1$. The definition of the filtration implies that $f_{i_0,0+j}(U_x) \subset M_1$ for all $j \geq 0$. Thus $f_{i,0+j}(U_x) \cap U_x = \emptyset$ for all $j \geq 0$. By shrinking $U_x$, we can assume that $U_x, f_{i_0,0}(U_x), \ldots, f_{i_0,0}(U_x)$ are pairwise disjoint. Therefore $f_{i,0+j}(U_x) \cap U_x = \emptyset$ for all $j > 0$, and $x$ is wandering.

Using the definition of the arc $f_{t,s}$, it is immediate to check the following: let $\Lambda_{t,s}$ be the maximal invariant set of $f_{t,s}$ in $W$. 

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Remark 6.9 For small positive $s$,

- Every point $(x, y, s) \in C = [-1, 1] \times [-2, 2] \times [-1, 1]$ with $y \in (-\sqrt{s}, \sqrt{s})$ (resp., $y \in [2, -1]$ or $y \in (1, 2)$) is wandering.

- Consider $w = (x, y, z) \in C \cap \Lambda_{t,s}$ and, for $i \in \mathbb{Z}$, let $w_i = g_i^+(w) = f_{i,t,s}^+(w)$. If $w_i \in C$ we let $w_i = (x_i, y_i, z_i)$. Suppose that $y_i \in [\sqrt{s}, 1]$ and $y_j \in [-2, \sqrt{s})$ for some $j > 0$. Then, for every $w_n \in C$ with $n \geq j$, one has $y_n \in [-2, \sqrt{s})$.

For $s = 0$,

- Every point $(x, y, s) \in C$ with $y \in [-2, -1]$ or $y \in (1, 2)$ is wandering.

- Consider $w = (x, y, z) \in C \cap \Lambda_{t,0}$ and, for $i \in \mathbb{Z}$, let $w_i = g_i^0(w) = f_{i,0}^+(w)$. If $w_i \in C$ we let $w_i = (x_i, y_i, z_i)$. Suppose that $y_i \in (0, 1]$ and $y_j \in [-2, 0)$ for some $j > 0$. Then, for every $w_n \in C$ with $n \geq j$, one has $y_n \in [-2, 0)$.

The remark immediately implies that, for small $s > 0$,

$$\Lambda_{t,s} \cap \Omega(f_{t,s}) = \Lambda_{t,s}^+ \cup \Lambda_{t,s}^-,$$

where $\Lambda_{t,s}^+$ (resp., $\Lambda_{t,s}^-$) is the set of points $w \in \Lambda_{t,s} \cap \Omega(f_{t,s})$ such that, for every $i \in \mathbb{Z}$ with $w_i = (x_i, y_i, z_i) \in C$ one has $y_i \in [\sqrt{s}, 1]$ (resp., $y_i \in [-1, \sqrt{s})$). As in Section 3, $H(P, f_{t,s}) = \Lambda_{t,s}^+$ and $H(Q, f_{t,s}) = \Lambda_{t,s}^-$. Observe that we need to exclude the segment $\{0\} \times (-\sqrt{s}, \sqrt{s}) \times \{0\}$ contained in $\Lambda_{t,s}$ and consisting of wandering points.

For the saddle-node parameter $s = 0$, one argues similarly. First, as above, one has that

$$\Lambda_{t,0} = (\Lambda_{t,0} \cap \Omega(f_{t,0})) = \Lambda_{t,0}^+ \cup \Lambda_{t,0}^-,$$

where $\Lambda_{t,0}^+$ (resp., $\Lambda_{t,0}^-$) is the set of points $w \in \Lambda_{t,0}$ such that, for every $i \in \mathbb{Z}$ with $w_i = (x_i, y_i, z_i) \in C$, one has $y_i \in [0, 1]$ (resp., $y_i \in [-1, 0]$). As in the case $s > 0$, $H(P, f_{t,0}) = \Lambda_{t,0}^+$ and $H(Q, f_{t,0}) = \Lambda_{t,0}^-$. It is immediate that $\Lambda_{t,0}^+ \cap \Lambda_{t,0}^- = \{S\}$.

For parameters $s < 0$ the result follows similarly (but now the situation is much more simple).

6.4 Proof of Theorem B

Clearly, the homoclinic classes $H(P, f_{t,0})$ and $H(Q, f_{t,0})$ are not saturated. We claim that there is not any transitive saturated set $\Sigma$ containing $H(P, f_{t,0})$. Otherwise, the set $\Sigma$ must also contain $H(Q, f_{t,0})$. Thus $\Sigma$ contains the whole $\Lambda_{t,0}$. Using the filtration one has $\Sigma = \Lambda_{t,0}$. But this set is not transitive: Remark 6.9 implies that there is no orbit going from a small neighborhood $U_Q$ of $Q$ to a small neighborhood $U_P$ of $P$ and thereafter reuturning to $U_Q$. This contradiciton ends the proof of the theorem.
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