REGULARITY OF GEODESIC RAYS AND
MONGE-AMPERE EQUATIONS

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Abstract

It is shown that the geodesic rays constructed as limits of Bergman geodesics from a test configuration are always of class $C^{1,\alpha}$, $0 < \alpha < 1$. An essential step is to establish that the rays can be extended as solutions of a Dirichlet problem for a Monge-Ampère equation on a Kähler manifold which is compact.

1 Introduction

The purpose of this note is to establish the $C^{1,\alpha}$ regularity, $0 < \alpha < 1$, of the geodesic rays constructed in [PS07] from a test configuration by Bergman geodesic approximations. With the notations given in §2 below, our main result can be stated as follows:

Theorem 1 Let $L \to X$ be a positive holomorphic line bundle over a compact complex manifold $X$. Let $\rho$ be a test configuration for $L \to X$. Let $D^\times = \{0 < |w| \leq 1\}$ be the punctured unit disk, and $\pi_X$ the natural projection $X \times D^\times \to X$. For each metric $h_0$ on $L$ with positive curvature $\omega_0 \equiv -i/2 \partial \bar{\partial} \log h_0 > 0$, let $\Phi(z, w)$ be the $\pi_X^*(\omega_0)$-plurisubharmonic function on $X \times D^\times$ defined by

$$\Phi(z, w) = \lim_{k \to \infty} [\sup_{\ell \geq k} \Phi(\ell, z, w)]^*, \quad (z, w) \in X \times D^\times, \quad (1.1)$$

where $\Phi_k(z, w)$ are the functions defined by (2.12) below. Then for any $0 < \alpha < 1$, $\Phi(z, w)$ is a $C^{1,\alpha}$ generalized solution of the Dirichlet problem

$$(\pi_X^*(\omega_0) + i/2 \partial \bar{\partial} \Phi)^{n+1} = 0 \text{ on } X \times D^\times, \quad \Phi(z, w) = 0 \text{ when } |w| = 1. \quad (1.2)$$

The fact that $\Phi(z, w)$ is locally bounded and a solution of the Dirichlet problem was established in [PS07], so the new part of the theorem is the $C^{1,\alpha}$ regularity. In the case of toric varieties, the $C^{1,\alpha}$ regularity of geodesic rays was previously established by Song and Zelditch [SZ08], using an explicit analysis of orthonormal bases for $H^0(X, L^k)$ and the theory of large deviations. They also pointed out that, already for toric varieties, geodesic rays from test configurations can be at best $C^{1,1}$. The interpretation of the completely degenerate Monge-Ampère equation in (1.2) as the equation for geodesics in the space of Kähler potentials of class $c_1(L)$ on $X$ is well-known, and due to Donaldson [D99], Semmes [S], and Mabuchi [M].

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In [PS09], $C^{1,\alpha}$ geodesic rays were constructed in all generality from test configurations by a different approach, namely viscosity methods for the degenerate complex Monge-Ampère equation on a compactification $\tilde{X}_D$ of $X \times D^\times$. Thus our theorem can be established by showing that the above solution, more precisely $\Phi(z, w) - \Phi_1(z, w)$, can also be extended to $\tilde{X}_D$ and that such solutions must be unique. For this, it is essential to show that $\Phi(z, w) - \Phi_1(z, w)$ is uniformly bounded on $X \times D^\times$. We accomplish that with the help of a “lower-triangular” property of Donaldson’s equivariant imbeddings, relating $k$-th powers of sections of $H^0(X, L)$ to sections of $H^0(X, L^k)$, which may be of independent interest (c.f. Lemma 2 below).

The uniqueness follows from a comparison theorem for Monge-Ampère equations on Kähler manifolds with boundary, using the approximation theorems for plurisubharmonic functions obtained recently by Blocki and Kolodziej [BK] (see also Demailly and Paun [DP] for other approximation theorems). It is well-known that such approximation theorems would imply comparison theorems, by a straightforward adaptation to Kähler manifolds of the classic comparison theorem of Bedford and Taylor [BT82] for domains in $\mathbb{C}^m$. In fact such a comparison theorem was established in [BK] for Kähler manifolds without boundary. However, the particular version that we need does not seem available in the literature, and we have included a brief but complete derivation.

As has been stressed in [PS06], each test configuration defines a generalized vector field on the space of Kähler potentials, with the vector at each potential $h_0$ given by the tangent vector $\dot{\phi}$ to the geodesic at the initial time. This observation can now be given a precise formulation, using the measures recently introduced by Berndtsson [B09b]: for each generalized $C^{1,\alpha}$ geodesic $(-\infty, 0] \in t \to \phi(z,t) \equiv \Phi(z, e^t)$, the functional $\mu_\Phi : C^0_0(\mathbb{R}) \ni f \to \int_X f(\dot{\phi}) \omega^n_{\phi(\cdot, t)}$ defines a Borel measure on $\mathbb{R}$ which is independent of $t$. Taking $t = 0$, we can think of this measure as a way of characterizing $\dot{\phi}(0)$ by its moments. If $\Phi$ is the geodesic constructed in Theorem 1, the corresponding assignment $h_0 \to \mu_\Phi$ can be viewed as a precise realization of the generalized vector field defined by the test configuration $\rho$.

We note that Theorem 1 gives the regularity of the limiting function $\Phi(z, w)$, but it does not provide information on the precise rate of convergence of $\Phi_k$. For toric varieties, very precise rates of convergence have been provided by Song and Zelditch [SZ06, SZ08]. For general manifolds, in the case of geodesic segments, the precise rate of $C^0$ convergence has been obtained a few years ago by Berndtsson [B09a] with an additional twisting by $\frac{1}{k}K_X$, and very recently in [B09b] for the $\Phi_k$ themselves.

Finally, we would like to mention that geodesics have been constructed by Arezzo and Tian [AT], Chen [C00, C08], Chen and Tang [CT], Chen and Sun [CS], Blocki [B09] and others in various geometric situations. For geodesic segments, the $C^{1,\alpha}$ regularity has been established by Chen [C00]. Their construction by Bergman approximations is in [PS06]. This construction has also been extended by Rubinstein and Zelditch [RZ] to the construction of harmonic maps in the space of Kähler potentials, in the case of toric varieties.
2 The extension to a compact Kähler manifold

In this section, we show how the generalized geodesic rays constructed in [PS07], originally defined on $X \times \{0 < |w| \leq 1\}$, actually extend as bounded solutions of a complex Monge-Ampère equation over a compact Kähler manifold $\tilde{X}_D \supset X \times \{0 < |w| \leq 1\}$. We begin by introducing the notation and recalling the results of [PS07].

2.1 Test configurations

Let $L \to X$ be a positive line bundle over a compact complex manifold $X$. A test configuration $\rho$ for $L \to X$ [D02] is a homomorphism $\rho : \mathbb{C}^\times \to \text{Aut}(\mathcal{L} \to \mathcal{X} \to \mathbb{C})$, where $\mathcal{L}$ is a $\mathbb{C}^\times$ equivariant line bundle with ample fibers over a scheme $\mathcal{X}$, and $\pi : \mathcal{X} \to \mathbb{C}$ is a flat $\mathbb{C}^\times$ equivariant map of schemes, with $(\pi^{-1}(1), \mathcal{L}|_{\pi^{-1}(1)})$ isomorphic to $(X, L^r)$ for some fixed $r > 0$. After replacing $L$ by $L^r$ to a sufficiently high power, we may assume that $r = 1$.

It is convenient to denote $(\pi^{-1}(w), \mathcal{L}|_{\pi^{-1}(w)})$ by $(X_w, L_w)$. In particular, for each $\tau \neq 0$, $\rho(\tau)$ is an isomorphism between $(X_w, L_w)$ and $(X_{\tau w}, L_{\tau w})$.

The central fiber $(X_0, L_0)$ is fixed under the action of $\rho$. Thus, for each $k$, $\rho$ induces a one-parameter subgroup of automorphisms

$$\rho_k(\tau) : H^0(X_0, L_0^k) \to H^0(X_0, L_0^k), \quad \tau \in \mathbb{C}^\times. \quad (2.1)$$

Since $\rho_k(\tau)$ is an algebraic one-parameter subgroup, there is a basis of $H^0(X_0, L_0^k)$ in which $\rho(\tau)$ is represented by a diagonal matrix with entries $\tau^{\eta^{(k)}_\alpha}$, where $\eta^{(k)}_\alpha$ are integers, $0 \leq \alpha \leq N_k \equiv \dim H^0(X_0, L_0^k) - 1$. Set

$$\lambda^{(k)}_\alpha = \eta^{(k)}_\alpha - \frac{1}{N_k + 1} \sum_{\beta=0}^{N_k} \eta^{(k)}_\beta,$$  

so that $(\lambda^{(k)}_\alpha)$ is the traceless component of $(\eta^{(k)}_\alpha)$. For a fixed $k$, we shall refer to $\eta^{(k)}_\alpha$ and $\lambda^{(k)}_\alpha$ respectively as the weights and the traceless weights of the test configuration $\rho$.

It is convenient to introduce an $(N_k + 1) \times (N_k + 1)$ diagonal matrix $B_k$ whose diagonal entries are given by the weights $\eta^{(k)}_\alpha$. Such a matrix is determined up to a permutation of the diagonal entries $\eta^{(k)}_\alpha$, and we fix one choice once for all. Then the traceless weights $\lambda^{(k)}_\alpha$ are the diagonal entries of the matrix $A_k$ defined by $A_k = B_k - (N_k + 1)^{-1}(\text{Tr } B_k)I$, and we have

$$\text{Tr } B_k = \sum_{\alpha=0}^{N_k} \eta^{(k)}_\alpha, \quad \text{Tr } A_k = 0. \quad (2.3)$$
For sufficiently large \( k \), the functions \( k(N_k + 1) \) and \( \text{Tr} B_k \) are polynomials in \( k \) of degree \( n + 1 \), so we have an asymptotic expansion

\[
\frac{\text{Tr} B_k}{k(N_k + 1)} \equiv F_0 + F_1 k^{-1} + F_2 k^{-2} + \cdots \quad (2.4)
\]

The Donaldson-Futaki invariant of \( \rho \) is defined to be the coefficient \( F_1 \).

### 2.2 Equivariant imbeddings of test configurations

An essential property of test configurations, due to Donaldson [D05], is that the entire configuration can be imbedded equivariantly in \( \mathbb{C}P^{N_k} \times \mathbb{C} \), in a way which respects a given \( L^2 \) metric on \( H^0(X, L^k) \). The following formulation [PS07] is most convenient for our purposes:

Let \( \mathcal{s}(k) = \{ s(k)_{\alpha} \}_{\alpha=0}^{N_k} \) be a basis for \( H^0(X, L^k) \). For all \( k \) sufficiently large, it defines a Kodaira imbedding

\[
l_{\mathcal{s}(k)} : X \ni z \rightarrow [s(k)_{0}(z) : s(k)_{1}(z) : \cdots : s(k)_{N_k}(z)] \in \mathbb{C}P^{N_k} \quad (2.5)
\]

of \( X \) into \( \mathbb{C}P^{N_k} \), with \( O(1) \) pulled back to \( L^k \). If \( h_0 \) is a fixed metric on \( L \) with \( \omega_0 \equiv -\frac{i}{2} \partial \bar{\partial} \log h_0 > 0 \), then \( H^0(X, L^k) \) can be equipped with the \( L^2 \) metric defined by the metric \( h_0^k \) on sections of \( L^k \) and the volume form \( \omega_0^n / n! \). For simplicity, we shall refer to this \( L^2 \) metric on \( H^0(X, L^k) \) as just the "\( L^2 \) metric defined by \( h_0 \)". Of particular importance are then the bases \( \mathcal{s}(k) \) which are orthonormal with respect to this \( L^2 \) metric.

**Lemma 1** Let \( \rho : C^\times \to \text{Aut}(\mathcal{L} \to \mathcal{X} \to C) \) be a test configuration, and fix a diagonal matrix \( B_k \) with the weights of \( \rho \) as diagonal entries as defined in §2.1. Fix a metric \( h_0 \) on \( L \) with positive curvature \( \omega_0 \), and corresponding \( L^2 \) metric on \( H^0(X, L^k) \). Then there is an orthonormal basis \( \mathcal{s}(k) \) of \( H^0(X, L^k) = H^0(X_1, L_1^k) \) with respect to the \( L^2 \) metric defined by \( h_0 \) and an imbedding

\[
I_{\mathcal{s}} : (\mathcal{L} \to \mathcal{X} \to C) \to (O(1) \times C \to \mathbb{C}P^{N_k} \times C \to C) \quad (2.6)
\]

satisfying

1. \( I_{\mathcal{s}(k)}|X = l_{\mathcal{s}(k)} \)
2. \( I_{\mathcal{s}(k)} \) intertwines \( \rho(\tau) \) and \( B_k \),

\[
I_{\mathcal{s}(k)}(\rho(\tau) \ell_w) = (\tau^{B_k} I_{\mathcal{s}(k)}(\ell_w), \tau w), \quad \ell_w \in L_w, \quad \tau \in C^\times. \quad (2.7)
\]

Let \( E_k = \pi_*(\mathcal{L}^k) \) be the direct images of the bundles \( \mathcal{L}^k \). Thus \( E_k \to C \) is a vector bundle over \( C \) of rank \( N_k + 1 \), and its sections \( S(w) \) are holomorphic sections of \( L_w \) for each \( w \in C \). The action of \( C^\times \) on the sections \( S \) is given by

\[
S^\tau(w) = \rho(\tau)^{-1} S(w\tau). \quad (2.8)
\]
Then a third key statement in the equivariant imbedding lemma is:

(3) The functions

\[ S_\alpha(w) \equiv w^{n_\alpha(k)} \rho(w) s_\alpha, \quad w \in \mathbb{C}^x \]  

(2.9)

extend to a basis for the free \( \mathbb{C}[w] \) module of all sections of \( E_k \rightarrow \mathbb{C} \) and they have the property: \( S_\alpha(1) = s_\alpha \). This extension still satisfies the relation

\[ \rho(\tau)^{-1} S_\alpha(w) = \tau^{n_\alpha(k)} S_\alpha(w), \quad w \in \mathbb{C}. \]  

(2.10)

2.3 The construction of geodesics

We come now to the construction of geodesics by Bergman approximations. Let \( \rho : \mathbb{C}^x \rightarrow \text{Aut}(\mathcal{L} \rightarrow X \rightarrow \mathbb{C}) \) be a test configuration for \( L \rightarrow X \), and fix a metric \( h_0 \) on \( L \) with positive curvature \( \omega_0 \). Let \( s(k) = \{s^{(k)}_\alpha(z)\} \) be an orthonormal basis for \( H^0(X, L^k) \) with respect to the \( L^2 \) metric defined by \( h_0 \) as in Lemma 1. Let

\[ D^x = \{w \in \mathbb{C}; 0 < |w| \leq 1\} \]  

(2.11)

be the punctured disk. Define the functions \( \Phi_k(z, w) \) by

\[ \Phi_k(z, w) = \frac{1}{k} \log \sum_{\alpha=0}^{N_k} |w|^{2n_\alpha(k)} |s^{(k)}_\alpha(z)|_{h_0}^2 - \frac{n}{k} \log k, \quad (z, w) \in X \times D^x. \]  

(2.12)

and \( \Phi(z, w) \) by

\[ \Phi(z, w) = \lim_{k \to \infty} \left[ \sup_{\ell \geq k} \Phi_\ell(z, w) \right]^* \]  

(2.13)

where \( \eta_\alpha^{(k)} \) are the weights of the test configuration \( \rho \), * denotes the upper semi-continuous envelope, i.e. \( f^*(z) = \lim_{\epsilon \to 0} \sup_{|w-z| < \epsilon} f(w) \), and \( |s_\alpha(z)|_{h_0}^2 \equiv s_\alpha(z)s_\alpha(z)h_0(z)k \) denotes the norm-squared of \( s_\alpha(z) \) with respect to the metric \( h_0 \). Then it is shown in [PS07] \(^2\) that \( \Phi(z, w) \) is a generalized geodesic ray in the sense that

(a) \( \pi_X(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi \geq 0 \) on \( X \times D^x \), where \( \pi_X \) is the projection \( X \times D^x \rightarrow X \) on the first factor;

\(^2\)Actually, in [PS07], the weights \( \eta_\alpha^{(k)} \) in the definition of \( \Phi_k(z, w) \) were replaced by the traceless weights \( \lambda_\alpha^{(k)} \). If we denote by \( \Phi^\#_k(z, w) \) the functions obtained in this manner with the traceless weights, then we have

\[ \Phi_k(z, w) = \Phi^\#_k(z, w) + \frac{\text{Tr} B_k}{k(N_k + 1)} \log |w|^2. \]  

(2.14)

It follows that the complex Hessians of \( \Phi_k(z, w) \) and \( \Phi^\#_k(z, w) \) are identical. However, the behaviors near \( |w| = 0 \) of \( \Phi_k(z, w) \) and \( \Phi^\#_k(z, w) \) are different, and for our purposes, it is important to work with \( \Phi_k(z, w) \).
(b) For each finite \( T > 0 \), we have
\[
\sup_k |\Phi_k(z, w)|, |\Phi(z, w)| \leq C_T \quad \text{for } (z, w) \in X \times \{e^{-T} < |w| \leq 1\}
\] (2.15)
with \( C_T \) a constant independent of \( z, w \) and \( k \), but possibly depending on \( T \);

(c) \( \Phi(z, w) \) is continuous when \( |w| = 1 \), and is a solution in the sense of pluripotential theory of the following Dirichlet problem for the completely degenerate Monge-Ampère equation
\[
(\pi^*_X(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi)^{n+1} = 0 \quad \text{on } X \times D^x, \quad \Phi(z, w) = 0 \quad \text{when } |w| = 1.
\] (2.16)

The geodesic \( \Phi(z, w) \) is non-constant if the test configuration is non-trivial, that is, not holomorphically equivalent to a product test configuration. We note that in the boundary value problem (2.16), the behavior of \( \Phi(z, w) \) near \( w = 0 \) is not specifically assigned.

### 2.4 Formulation in terms of equivariant imbeddings

We come now to the main task in this chapter, which is to identify the solution (2.16) with the restriction to \( X \times D^x \) of the solution of a standard Dirichlet problem on a compact Kähler manifold \( \tilde{X}_D \) with boundary.

Let \( \pi : X \to \mathbb{C} \) be the projection map, and \( D = \{ w \in \mathbb{C} : |w| \leq 1 \} \). Let \( \mathcal{X}_D = \pi^{-1}(D) \), \( \mathcal{X}^x_D = \pi^{-1}(D^x) \). The space \( \mathcal{X}^x_D \) is isomorphic to \( X \times D^x \) under the correspondence
\[
X \times D^x \ni (z, w) \to \rho(w)(z) \in X_w,
\] (2.17)
where \( z \in X \) is viewed as a point in \( X_1 \). This correspondence lifts to a correspondence between \( L \times D^x \) and the restriction \( \mathcal{L}^x_D \) of \( \mathcal{L} \) over \( \mathcal{X}^x_D \).

Let \( p : \tilde{X} \to X \to \mathbb{C} \) be an \( S^1 \) equivariant smooth resolution and \( \tilde{\mathcal{L}} = p^*\mathcal{L} \). The first step is to show that the functions \( \Phi_k(z, w) - \Phi_1(z, w) \) of (2.12), which are defined on \( X \times D^x \), may be extended to plurisubharmonic functions on all of \( \mathcal{X}_D = p^{-1}(\mathcal{X}_D) \).

Let us fix a metric \( h_0 \) on \( L \) with positive curvature \( \omega_0 \). Let \( \underline{z}(k) \) be the orthonormal basis for \( H^0(X, L^k) \) with respect to \( h_0 \) provided by Lemma 1, and let \( I_{\underline{z}(k)} \) be a corresponding equivariant imbedding of the test configuration. Let \( \Phi_k(z, w) \) be defined by (2.12). Define a closed \((1,1)\)-form \( \Omega_k \) on \( \mathcal{X}_D \) by
\[
\Omega_k = \frac{1}{k} (I_{\underline{z}(k)} \circ p)^* \omega_{FS}
\] (2.18)
where \( \omega_{FS} \) is the Fubini-Study metric on \( \mathbb{CP}^{N_k} \). Define as well a hermitian metric \( H_k \) on \( \tilde{\mathcal{L}} \) by \( H_k = (I_{\underline{z}(k)} \circ p)^* (h_{FS})^{1/k} \), where \( h_{FS} \) is the Fubini-Study metric on the hyperplane bundle \( O(1) \) over \( \mathbb{CP}^{N_k} \). Thus \( \Omega_k \) is the curvature of \( H_k \). The restriction of \( \Omega_k \) to \( \mathcal{X}_D^x \) can
be readily worked out explicitly in terms of the coordinates \((z, w)\). Using the intertwining property of the equivariant imbedding,

\[
X \times D^\times \ni (z, w) \rightarrow \rho(w)z \rightarrow I_{\mathcal{I}(k)}(\rho(w)z) = w^{B_k}I_{\mathcal{I}(k)}(z) = (w^{B_k}I_{\mathcal{I}(k)})(z), w)
\]

we find that \(I_{\mathcal{I}(k)}\) is given by

\[
I_{\mathcal{I}(k)} : X \times D^\times \ni (z, w) \rightarrow ([w^{\eta_0^{(k)}} s_0^{(k)}(z) : w^{\eta_1^{(k)}} s_1^{(k)}(z) : \cdots : w^{\eta_{N_k}^{(k)}} s_{N_k}^{(k)}(z)], w). \tag{2.20}
\]

Since the Fubini-Study metric \(h_{FS}\) on \(O(1)\) at \([s_0 : s_1 : \cdots : s_{N_k}] \in \mathbb{CP}^{N_k}\) is given by \(h_{FS} = (|s_0|^2 + \cdots + |s_{N_k}|^2)^{-1}\), we obtain the following expression for \(\Omega_k\),

\[
\Omega_k|_{X \times D^\times} = \frac{1}{k} \partial \bar{\partial} \log \sum_{\alpha=0}^{N_k} |w|^{2\eta_{\alpha}^{(k)}} |s_{\alpha}^{(k)}(z)|^2. \tag{2.21}
\]

Recalling that the norm with respect to \(h_0^k\) of a section \(s(z)\) of \(L^k\) is given by \(|s(z)|^2_{h_0^k} = |s(z)|^2_{h_0^k}\), we find the following key relation between the \((1, 1)\)-forms \(\Omega_k\) and the potentials \(\Phi_k(z, w)\) defined earlier in (2.12),

\[
\Omega_k|_{X \times D^\times} = \pi_X^*(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi_k(z, w). \tag{2.22}
\]

### 2.5 The extension of \(\Psi_k\) to the total space \(\tilde{X}_D\)

The relation (2.22) that we have just obtained shows that the form \(\pi_X^*(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi_k(z, w)\), defined originally on \(X_D^*\), admits the natural extension \(\Omega_k\) to the whole of \(\tilde{X}_D\).

Since the form \(\pi_X^*(\omega_0)\) does not extend by itself to \(\tilde{\mathcal{I}}\), we re-write \(\Omega_k\) as

\[
\Omega_k = \Omega_1 + \frac{i}{2} \partial \bar{\partial} (\Phi_k - \Phi_1) \equiv \Omega_1 + \frac{i}{2} \partial \bar{\partial} \Psi_k. \tag{2.23}
\]

The function \(\Psi_k = \Phi_k - \Phi_1\) has a simple interpretation that shows that it extends as a smooth function to the whole of \(\tilde{X}_D\): as we saw earlier in §2.2, under the maps \(I_{\mathcal{I}(k)}\) and \(I_{\mathcal{I}(1)}\) of the test configuration \(\rho\), the Fubini-Study metric \(h_{FS}\) pulls back respectively to \(\frac{H_k}{k} = (\sum_{\alpha} |w|^{2\eta_{(k)}^{(k)}} |s_{(k)}^{(k)}(z)|^2)^{-1}\) and \(H_1 = (\sum_{\alpha} |w|^{2\eta_{(1)}^{(1)}} |s_{(1)}^{(1)}(z)|^2)^{-1}\) on \(L \times D^\times\). Thus we may write

\[
\Psi_k = \log \frac{H_1}{H_k} - \frac{n}{k} \log k. \tag{2.24}
\]

The right hand side is a well-defined, smooth scalar function over the whole of \(\tilde{X}_D\), since it is the logarithm of the ratio of two smooth metrics on the same line bundle \(\tilde{\mathcal{L}} \to \tilde{X}_D\).
Since $\Omega_k$ is non-negative as the pull-back of a non-negative form, the function $\Psi_k$ is $\Omega_1$-plurisubharmonic. We also define

$$\Psi = \lim_{k \to \infty} [\sup_{\ell \geq k} \Psi_\ell]^*$$

which is an extension of $\Phi - \Phi_1$ to $\tilde{X}_D$.

### 2.6 Uniform estimates for $\Psi_k$

Recall that in [PS07], as quoted in (2.15) above, we only have bounds for the functions $\Phi_k(z, w)$ when $|w| > e^{-T}$, for some fixed finite $T > 0$. Since the function $\Psi_k$ extends to a smooth function on $\tilde{X}_D$, it follows that it is bounded on $\tilde{X}_D$. However, the bound may a priori depend on $k$. The most important step in the extension to $\tilde{X}_D$ is to show that this bound can actually be made uniform in $k$.

We carry this out with several lemmas. The first is the following essential “lower triangular lemma”:

**Lemma 2** Fix a test configuration $\rho$, and a metric $h_0$ on $L$ with positive curvature $\omega_0$. For each $k$, let $s(k) = \{s_\alpha^{(k)}\}_{\alpha=0}^{N_k}$ be an orthonormal basis for $H^0(X, L_k)$ as in Lemma 1. Then for any $s_\beta^{(1)}$ in $s(1)$, we can write

$$(s_\beta^{(1)})^k = \sum_{\lambda_\alpha^{(1)} \leq k\lambda_\beta^{(1)}} a_{\beta\alpha} s_\alpha^{(k)}$$

where $a_{\beta\alpha} \in \mathbb{C}$ and the subindex indicates the range of indices $\alpha$ which are allowed. Furthermore, the coefficients $a_{\beta\alpha}$ satisfy the bound

$$|a_{\beta\alpha}| \leq V^\frac{1}{2} M^k$$

where we have set $M = \sup_{0 \leq \beta \leq N_1} \sup_X |s_\beta^{(1)}|_{h_0}$ and $V = f_X \omega_0^n$.

**Proof of Lemma 2**: For each $k$, let $E_k = \pi_* (L^{k}) \to \mathbb{C}$, and let $S_0(w), \ldots, S_{N_k}(w)$ be a basis for the free $\mathbb{C}[w]$ module of sections of $E_k \to \mathbb{C}$, as provided in Lemma 1. Now let $S_\beta$ be an element of this basis for $E_1 \to \mathbb{C}$, and some $\beta$ with $0 \leq \beta \leq N_1$. Then $\rho(\tau)^{-1} S_\beta(w) = \tau^{k\beta^{(1)}} S_\beta(w)$ which implies

$$\rho(\tau)^{-1} S_\beta^{(k)}(w \tau) = \tau^{k\beta^{(1)}} S_\beta^{(k)}(w)$$

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3In general, given a non-negative smooth, closed $(1,1)$-form $\Omega$ on a complex manifold $X$, we say that a scalar function $\Phi$ is $\Omega$-plurisubharmonic if $f_\alpha + \Phi$ is plurisubharmonic on $U_\alpha$ for each $\alpha$, if $X = \cup_\alpha U_\alpha$ is a covering of $X$ by coordinate charts $U_\alpha$ with $\Omega = \frac{i}{2} \partial \bar{\partial} f_\alpha$ on $U_\alpha$.  

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On the other hand, $S^k_\beta$ is a section of $E_k$ so we may write

$$S^k_\beta(w) = \sum_{\alpha=0}^{N_k} a_\alpha(w) S_\alpha(w) \quad (2.29)$$

for certain uniquely defined polynomials $a_\alpha(w) \in \mathbb{C}[w]$. Applying the $C^\times \times$ action to both sides of (2.29) we obtain

$$\sum_{\alpha=0}^{N_k} \tau^{k\eta^{(1)}_\beta} a_\alpha(w) S_\alpha(w) = \tau^{k\eta^{(1)}_\beta} S^k_\beta(w) = \rho(\tau)^{-1} S^k_\beta(w \tau) = \sum_{\alpha=0}^{N_k} a_\alpha(w \tau) \tau^{\eta^{(k)}_\alpha} S_\alpha(w) \quad (2.30)$$

Comparing coefficients we obtain

$$\tau^{k\eta^{(1)}_\beta} a_\alpha(w) = a_\alpha(w \tau) \tau^{\eta^{(k)}_\alpha} \quad (2.31)$$

Setting $w = 1$ we see that $a_\alpha(\tau) = a_{\beta\alpha} \tau^{r_\alpha}$ for some integer $r_\alpha$ and some $a_{\beta\alpha} \in \mathbb{C}$. But $a_\alpha(w)$ is a polynomial. Thus $r_\alpha \geq 0$ and $a_\alpha(w) = a_{\beta\alpha} w^{r_\alpha}$ for all $w \in \mathbb{C}$. The equation (2.31) implies that if $a_{\beta\alpha} \neq 0$ we have $k\eta^{(1)}_\beta = r_\alpha + \eta^{(k)}_\alpha$ and thus $\eta^{(k)}_\alpha \leq k\eta^{(1)}_\beta$. Evaluating (2.29) at $w = 1$ we obtain the first part of the lemma.

Finally, the orthonormality of the sections $s^{(k)}_\alpha$ implies

$$|a_{\beta\alpha}| = |\langle (s^{(1)}_\beta), s^{(k)}_\alpha \rangle_{h^{k}_0 \omega^0} | \leq \int |s^{(1)}_\beta|_{h_0}^k \cdot |s^{(k)}_\alpha|_{h_0^k \omega^0} \leq M^k V \frac{1}{2} \quad (2.32)$$

The lemma is proved.

*Remark:* It may happen that $\eta^{(k)}_\alpha < k\eta^{(1)}_\beta$ for all $\alpha$ with $a_{\beta\alpha} \neq 0$, that is, it may happen that $a_\alpha(w)$ vanishes at $w = 0$ for all $\alpha$. This would mean that $S^k_\beta(0)$ is a non-zero section of $H^0(X_0, L_0)$ but that $S^k_\beta(0) = 0 \in H^0(X_0, L^k_0)$, in other words, the section $S^k_\beta(0)$ is nilpotent (which is possible if $X_0$ is a non-reduced scheme, that is, if $X_0$ has nilpotent elements in its structure sheaf).

Next, we also need

**Lemma 3** The complex manifold $\tilde{X}_D$ always admits a Kähler metric.

This lemma was proved in [PS07a]. In fact, it is proved there that there exists a line bundle $\mathcal{M}$ on $\tilde{X}_D$ which is trivial on $\tilde{X}^{\times}$, and such that $\mathcal{L}^m \otimes \mathcal{M}$ is positive for some fixed positive power $m$. The desired Kähler metric on $\tilde{X}_D$ can then be taken to be the ratio of the curvature of $\mathcal{L}^m \otimes \mathcal{M}$ by $m$. Q.E.D.

**Lemma 4** There exists a finite constant $C$ so that

$$\sup_{k \geq 1} \sup_{\tilde{X}_D} |\Psi_k| \leq C < \infty. \quad (2.33)$$

In particular,

$$\sup_{\tilde{X}_D} |\Psi| \leq C < \infty. \quad (2.34)$$
Proof of Lemma 4: Let $H$ be a Kähler metric on $\tilde{X}_D$, which exists by Lemma 3. Since $\Psi_k$ is $\Omega_1$-plurisubharmonic, it follows that
$$\Delta_H \Psi_k \geq -C_1,$$
where $\Delta_H$ is the Laplacian with respect to $H$, and $C_1$ is an upper bound for the trace of $\Omega_1$ with respect to the metric $H$. On the other hand, $\Psi_k|_{\partial\tilde{X}_D} \to -\Phi_1$ uniformly as $k \to \infty$, and thus $\Psi_k|_{\partial\tilde{X}_D} \leq C_2$. Let $u$ be the smooth function on $\tilde{X}_D$ which is the solution of the Dirichlet problem
$$\Delta_H u = -C_1 \text{ on } \tilde{X}_D, \quad u = C_2 \text{ on } \partial\tilde{X}_D.$$ 
By the maximum principle, we have $\Psi_k \leq u$ for all $k$, and this gives the upper bound.

To establish the lower bound, it suffices to prove that
$$\Psi_k \geq -C \text{ on } X^\times_D$$
where $C$ is a constant independent of $k$, since each function $\Psi_k$ is smooth on $\tilde{X}_D$. On $X^\times$, we can use the explicit expressions for $X \times D^\times$ and write
$$\Psi_k = \log \frac{(\sum_{\alpha=0}^{N_k} |w|^{2n(k)} \cdot |s_{\alpha}^{(k)}|_{h_0}^{2})^{\frac{1}{k}}}{\sum_{\beta=0}^{N_1} |w|^{2\eta_0^{(1)}} \cdot |s_{\beta}^{(1)}|_{h_0}^{2}} - \frac{n}{k} \log k.$$ 
(2.35)
Now fix $w$ with $0 < |w| \leq 1$, fix $z \in X$, and choose $\beta_0$ so that
$$|w|^{2\eta_0^{(1)}} \cdot |s_{\beta_0}^{(1)}(z)|_{h_0}^{2} = \sup_{0 \leq \beta \leq N_1} |w|^{2\eta_0^{(1)}} \cdot |s_{\beta}^{(1)}(z)|_{h_0}^{2}. $$
(2.36)
In view of Lemma 2, we can write
$$|(s_{\beta_0}^{(1)})^k|_{h_0}^{2} \leq M^k V^\frac{1}{k} \sum_{k\eta_0^{(1)} \geq \eta_0^{(k)}} |s_{\alpha}^{(k)}|_{h_0}^{2}. $$
(2.37)
Since $|w| \leq 1$, we have then
$$|w|^{2k\eta_0^{(1)}} \cdot |s_{\beta_0}^{(k)}(z)|_{h_0}^{2} \leq M^k V \sum_{k\eta_0^{(1)} \geq \eta_0^{(k)}} |w|^{\eta_0^{(k)} \cdot |s_{\alpha}^{(k)}(z)|_{h_0}^{2}} \leq M^k V (N_k + 1) \sum_{k\eta_0^{(1)} \geq \eta_0^{(k)}} |w|^{2\eta_0^{(1)} \cdot |s_{\alpha}^{(1)}(z)|_{h_0}^{2}}. $$
(2.38)
Returning to $\Psi_k$, we can now write
$$\Psi_k(z, w) \geq \log \frac{(\sum_{\alpha=0}^{N_k} |w|^{2n(k)} \cdot |s_{\alpha}^{(k)}(z)|_{h_0}^{2})^{\frac{1}{k}}}{(N_1 + 1)|w|^{2\eta_0^{(1)} \cdot |s_{\alpha}^{(1)}(z)|_{h_0}^{2}}} - \frac{n}{k} \log k \geq -\frac{1}{k} \log (V(N_k + 1)) - 2 \log M - \frac{n}{k} \log k - \log (N_1 + 1)$$
(2.39)
in view of the preceding inequality. This establishes Lemma 4 since $N_k \leq C k^n$. 

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2.7 The Monge-Ampère equation on the whole of $\tilde{X}_D$

With the uniform estimates provided by Lemma 4, it follows readily that the function $\Psi$ defined by (2.25) is a bounded, $\Omega_1$-plurisubharmonic function on $\tilde{X}_D$. Since it satisfies a completely degenerate Monge-Ampère equation on $X^*_D$, and since the singular set $X_0$ is an analytic subvariety, it follows from general pluripotential theory that it satisfies the same completely degenerate equation on $\tilde{X}_D$. We give now a direct proof of this fact, since we already have at hand all the necessary ingredients. It suffices to observe that $\Psi_k$ satisfies the following properties:

Lemma 5 The functions $\Psi_k$ satisfy

(a) $\sup_k \sup_{\tilde{X}_D} |\Psi_k| \leq C < \infty$;

(b) $\int_{\tilde{X}_D} (\Omega_1 + i/2 \partial\bar{\partial}\Psi_k)^{n+1} \leq C_k^1$;

(c) Let $T$ be the vector field $T = \partial/\partial t$ defined in a neighborhood of the boundary $|w| = 1$ on $\tilde{X}_D$, where $t = \log |w|$. Then $\sup_U |T \Psi_k| \leq C$, where $C$ is a constant, and $U$ is a neighborhood of the boundary $|w| = 1$, independent of $k$.

(d) $\sup_{\partial\tilde{X}_D} |\Psi_k + \Phi_1| \leq a_k$, with $a_k$ decreasing to 0 and $\sum_{k=1}^{\infty} a_k < \infty$.

Proof of Lemma 5: Part (a) is just the statement of Lemma 4. Part (b) follows from the fact that the form $\Omega_1 + i/2 \partial\bar{\partial}\Psi_k$ is smooth on $X^*_D$, and that its Monge-Ampère mass on $X^*_D$ coincides with the Monge-Ampère mass of $\Omega_k = (\pi^*_X(\omega_0) + i/2 \partial\bar{\partial}\Phi_k)$ on $X \times D^\times$. As we already observed in the footnote 1, $\Phi_k$ and $\Phi_k^\#$ have the same complex Hessian. So the desired estimate follows from the analogous estimate for the Monge-Ampère mass of $(\pi^*_X(\omega_0) + i/2 \partial\bar{\partial}\Phi_k^\#)$ established in Lemma 4.3 of [PS07]. Part (c) follows from the bound $|\eta_k^{(k)}| \leq C \dot{k}$, established in Lemma 3.1 of [PS07]. Finally, Part (d), with $a_k = C k^{-2}$, follows from the Tian-Yau-Zelditch theorem [T90, Y93, Z] (see also Catlin [Ca] and Lu [L]) as shown in the case of geodesic segments in [PS06]. Q.E.D.

We can now formulate the main theorem of this chapter:

Theorem 2 Let $L \to X$ be a positive line bundle over a compact complex manifold, let $\rho$ be a test configuration, and let $h_0$ be a metric on $L$ with positive curvature $\omega_0$. Let $\tilde{X}$ be an $\mathbb{S}^1$ invariant resolution $p : \tilde{X} \to X \to \mathbb{C}$ of $X$, and $\tilde{X}_D = (\pi \circ p)^{-1}(D)$. Let $\Phi_k, \Phi$ be defined as in (2.12) and (2.13). Set

$$\Psi = \Phi - \Phi_1$$ on $X \times D^\times$. \hfill (2.43)

Then the function $\Psi$ extends to a bounded, $\Omega_1$-plurisubharmonic function on $\tilde{X}_D$, which is a generalized solution of the following Dirichlet problem on $\tilde{X}_D$,

$$(\Omega_1 + i/2 \partial\bar{\partial}\Psi)^{n+1} = 0 \text{ on } \tilde{X}_D, \quad \Psi = -\Phi_1 \text{ on } \partial\tilde{X}_D. \hfill (2.44)$$

Here $\Omega_1$ is the pull-back to $\tilde{X}_D$ of the Fubini-Study metric by $I_{\omega(1)} \circ p$. 

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2.8 Positivity of the background form away from the central fiber

The equation (2.44) provides an extension of the degenerate complex Monge-Ampère equation to the compact manifold with boundary $\tilde{\mathcal{X}}_D$. It is however written with respect to a background $(1,1)$-form $\Omega_1$ which may be degenerate. In preparation for applications of uniqueness theorems for the complex Monge-Ampère equation, we rewrite it now with a background $(1,1)$-form which is non-negative everywhere, and strictly positive away from the central fiber $p^{-1}(X_0)$.

For this, we make use of Lemma 1 of [PS09], which asserts the existence of a $S^1$ invariant metric $H_0$ on $\tilde{\mathcal{L}}$ with the following properties:

\[
\begin{align*}
\Omega_0 &\equiv -\frac{i}{2} \partial \bar{\partial} \log H_0 \geq 0 \quad \text{on } \tilde{\mathcal{X}}_D, \quad \Omega_0 > 0 \quad \text{on } \mathcal{X}_D^* \\
H_0|_{\partial \tilde{\mathcal{X}}_D} &= h_0.
\end{align*}
\] (2.45)

Let $\Psi_0$ be defined by

\[
\Psi_0 = \log \frac{H_0}{(I^{(1)}_s \circ p)'(h_{FS})} = \log \frac{H_0}{H_1},
\] (2.46)

which is a smooth function on $\tilde{\mathcal{X}}_D$, since it is the logarithm of the ratio of two metrics on the same line bundle $\tilde{\mathcal{L}}$. Restricted to $\partial \tilde{\mathcal{X}}_D$, it is given by

\[
\Psi_0|_{\partial \tilde{\mathcal{X}}_D} = \log \frac{h_0}{(\sum_{\alpha=0}^{N_1} |s^{(1)}_\alpha|^2 h_0)^{-1}} = \log \sum_{\alpha=0}^{N_1} |s^{(1)}_\alpha|^2 h_0 = \Phi_1|_{\partial \tilde{\mathcal{X}}_D}.
\] (2.47)

Let $\Psi$ be the solution on $\tilde{\mathcal{X}}_D$ of the completely degenerate Monge-Ampère equation with background form $\Omega_1$ as given in Theorem 2. Define the function $\hat{\Phi}$ on $\tilde{\mathcal{X}}_D$ by

\[
\hat{\Phi} = \Psi + \Psi_0.
\] (2.48)

Clearly $(\Omega_0 + \frac{i}{2} \partial \bar{\partial} \hat{\Phi})^{n+1} = 0$ on $\tilde{\mathcal{X}}_D$. Furthermore, restricted to the boundary $\partial \tilde{\mathcal{X}}_D$, we have

\[
\hat{\Phi}|_{\partial \tilde{\mathcal{X}}_D} = \Psi|_{\partial \tilde{\mathcal{X}}_D} + \Psi_0|_{\partial \tilde{\mathcal{X}}_D} = -\Phi_1|_{\partial \tilde{\mathcal{X}}_D} + \Phi_1|_{\partial \tilde{\mathcal{X}}_D} = 0.
\] (2.49)

In summary, we have obtained the following alternative formulation of Theorem 2:

**Theorem 3** Let the setting be the same as in Theorem 2, and let $H_0$ be a metric on $\mathcal{L}$ as in (2.45), $\Psi_0$ be defined as in (2.46), and $\hat{\Psi} \equiv \Phi - \Phi_1 + \Psi_0$. Then the function $\hat{\Psi}$ is a bounded, $\Omega_0$-plurisubharmonic generalized solution of the following Dirichlet problem,

\[
(\Omega_0 + \frac{i}{2} \partial \bar{\partial} \hat{\Psi})^{n+1} = 0 \quad \text{on } \tilde{\mathcal{X}}_D, \quad \hat{\Psi}|_{\partial \tilde{\mathcal{X}}_D} = 0.
\] (2.50)
3 The comparison principle on Kähler manifolds

We derive now the uniqueness theorem that we need. We note that there has been considerable progress recently on uniqueness theorems for the complex Monge-Ampère equation, and in particular for certain broad classes of possibly unbounded solutions (see e.g. Blocki [B03], Blocki and Kolodziej [BK], Dinew [D], and references therein). However, there does not appear to be a version that would apply directly to our situation, namely to the Dirichlet problem on Kähler manifolds with boundary, for $\Omega_0$-plurisubharmonic functions where the closed $(1,1)$-form $\Omega_0$ is non-negative, but may be degenerate. We provide such a version below, just by following the original arguments of Bedford and Taylor [BT82] in $\mathbb{C}^m$.

3.1 The comparison principle

**Theorem 4** Let $(M, \Omega)$ be a compact Kähler manifold with smooth boundary $\partial M$ and dimension $m$, and let $\Omega_0$ be a smooth, non-negative, closed $(1,1)$-form. Then we have

$$\int_{\{u<v\}} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} v)^m \leq \int_{\{u<v\}} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} u)^m. \quad (3.1)$$

for all $u, v$ in $L^\infty$, $\Omega_0$-plurisubharmonic, and satisfying $\liminf_{z \to \partial M} (u(z) - v(z)) \geq 0$.

We adapt the original proof of Bedford-Taylor [BT82] to our setting. The main steps are as follows. The first step is a version of the theorem, in the special case of smooth data:

**Lemma 6** Let $u, v \in C^\infty(M)$ be $\Omega_0$-plurisubharmonic functions satisfying $u(z) - v(z) \geq 0$ for $z \in \partial M$, and $\Omega_0$ a smooth, closed, non-negative $(1,1)$-form. Assume that $\{u < v\}$ has smooth boundary. Then

$$\int_{\{u<v\}} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} v)^m \leq \int_{\{u<v\}} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} u)^m. \quad (3.2)$$

The proof is identical to [BT76], Proposition 4.1. Next, we need a notion of capacity adapted to Kähler manifolds, as e.g. in [PS06]:

Let $M'$ be any open subset with compact closure in $M$, and let $M' \subseteq \bigcup_{\alpha=1}^N U_\alpha$ be a finite cover of $M'$ by a fixed system of coordinate neighborhoods $U_\alpha$. We say that $E \subseteq M'$ has capacity $c(E, N) < \varepsilon$ if we can write $E = \bigcup_{\alpha=1}^N E_\alpha$, with $E_\alpha \subseteq U_\alpha$ Borel subsets and $\Sigma_{\alpha=1}^N c(E_\alpha, U_\alpha) < \varepsilon$, where $c(A, B)$ is the capacity for subsets of $\mathbb{C}^m$,

$$c(A, B) = \sup \{ \int_A (\frac{i}{2} \partial \bar{\partial} v)^m; \ v \text{ plurisubharmonic, } 0 \leq v \leq 1 \}. \quad (3.3)$$
We say that \( c(E, M) = 0 \) if \( c(E, M) < \varepsilon \) for every \( \varepsilon > 0 \). With this definition, it is easy to extend the quasi-continuity theorem of Bedford-Taylor [BT82] to Kähler manifolds: any \( \Omega_0 \)-plurisubharmonic function \( u \) on \( M \) is “quasi-continuous”, i.e., for any open subset \( M' \) with compact closure and any \( \varepsilon > 0 \), there is an open set \( G \subset M' \) so that \( c(G, M') < \varepsilon \) and \( u \) is continuous on \( M' \setminus G \). With this notion of capacity, we can axiomatize the limiting processes in [BT82]:

**Lemma 7** Let \( M' \) be a fixed open subset with compact closure of the complex manifold \( M \). Assume that \( u, v, u_k, v_j \) are Borel measurable functions on an open neighborhood of \( M' \), and \( d\mu_k, dv_j, d\mu, dv \) are non-negative Borel measures on \( M' \) with the following properties:

(a) \( u, v \) are upper semi-continuous and quasi-continuous;

(b) \( u_j, v_j \in C^\infty(M') \), \( u_j, v_j \) decrease to \( u \) and \( v \) respectively. Furthermore, there exists a \( \delta > 0 \) and a neighborhood \( W \) of \( M \setminus M' \) so that, for any \( k \), there exists \( J_k \) satisfying

\[
  u_k \geq v_j + \delta \text{ on } W, \quad \text{for } j \geq J_k. 
\]

(c) \( d\mu_k \to d\mu, dv_j \to dv \) weakly as \( k \) and \( j \) tend to \( \infty \);

(d) The measures \( d\mu_k, dv_k, d\mu, dv \) are uniformly bounded with respect to capacity, in the following sense: there exists a constant \( C \) such that for any \( \varepsilon, 0 < \varepsilon < 1 \) and any Borel subset \( G \subset M' \) with \( c(G, M') \) less than \( \varepsilon \),

\[
  \int_G (d\mu_k + dv_j + d\mu + dv) \leq C \varepsilon \quad \text{for all } k. 
\]

(e) For each \( k \), there exists \( J_k \) so that

\[
  \int_{\{u_k < v_j\}} dv_j \leq \int_{\{u_k < v_j\}} d\mu_k \quad \text{for } j \geq J_k. 
\]

Then we can conclude that

\[
  \int_{\{u < v\}} dv \leq \int_{\{u \leq v\}} d\mu. 
\]

To establish this lemma, recall that if \( \mu_j, \mu \) are non-negative Borel measures on a compact topological measure space, with uniformly bounded total measures, and \( d\mu_j \to d\mu \) weakly, then for any open subset \( \mathcal{O} \) and any compact subset \( K \), we have (see e.g. [EG])

\[
  \int_{\mathcal{O}} d\mu \leq \liminf_{j \to \infty} \int_{\mathcal{O}} d\mu_j, \quad \limsup_{j \to \infty} \int_{K} d\mu_k \leq \int_{K} d\mu. 
\]

If the measures \( d\mu_j \) are uniformly bounded in capacity, then the first inequality extends to all sets \( E \) which are “quasi-open”, in the sense that for any \( \varepsilon > 0 \), there exist an open set \( \mathcal{O} \), and Borel sets \( G_\varepsilon \subset M' \) and \( G'_\varepsilon \subset M' \) with capacities less than \( \varepsilon \) so that

\[
  E \subset \mathcal{O} \cup G_\varepsilon, \quad \mathcal{O} \subset E \cup G'_\varepsilon 
\]
Similarly, the second inequality extends to all sets $E$ which are “quasi-compact”, in the sense that for any $\varepsilon > 0$, there exist a compact set $K$, and Borel sets $G_\varepsilon \subset M'$ and $G'_\varepsilon \subset M'$ with capacities less than $\varepsilon$ so that $E \subset K \cup G_\varepsilon$, $K \subset E \cup G'_\varepsilon$.

Proof of Lemma 7: We take limits in (3.6) successively as $j \to \infty$ and then as $k \to \infty$.

First, consider the left hand side of (3.6). Both integrand and domain of integration depend on $j$, so we change first to a domain of integration independent of $j$ by writing

$$\int_{\{u_k < v_j\}} d\nu_j \geq \int_{\{u_k < v\}} d\nu_j \quad (3.10)$$

since $v_j \geq v$. Now the set $\{u_k < v\}$ is not necessarily open, but it is quasi-open in the sense defined above. Indeed, by the quasi-continuity of $v$, for each $\varepsilon > 0$, $v = V_\varepsilon$ for a function $V_\varepsilon$ continuous on $M$, outside a set $G_\varepsilon$ of capacity less than $\varepsilon$. Thus $\{u_k < V_\varepsilon\} \subset \{u_k < v\} \cup G_\varepsilon$ and $\{u_k < v\} \subset \{u_k < V_\varepsilon\} \cup G_\varepsilon$, and $\{u_k < V_\varepsilon\}$ is open. Applying the inequality (3.8) for quasi-open sets, we get

$$\liminf_{j \to \infty} \int_{\{u_k < v_j\}} d\nu_j \geq \int_{\{u_k < v\}} d\nu. \quad (3.11)$$

Next, the limit as $j \to \infty$ of the right hand side of (3.6) can be bounded in a straightforward way by

$$\lim_{j \to \infty} \int_{\{u_k < v\}} d\mu_k \leq \int_{\{u_k \leq v\}} d\mu_k. \quad (3.12)$$

Altogether, for each $k$, the limit as $j \to \infty$ of the inequality (3.6) produces

$$\int_{\{u_k < v\}} d\nu \leq \int_{\{u_k \leq v\}} d\mu_k. \quad (3.13)$$

The second step is to take the limit of (3.13) as $k \to +\infty$. The left hand side gives

$$\lim_{k \to \infty} \int_{\{u_k < v\}} d\nu = \int_{\{u < v\}} d\nu. \quad (3.14)$$

For the right hand side, where integrand and domain of integration both depend on $k$, we argue in complete analogy with the preceding case and begin by writing write

$$\int_{\{u_k \leq v\}} d\mu_k \leq \int_{\{u \leq v\}} d\mu_k, \quad (3.15)$$

since $u \leq u_k$. The set $\{u \leq v\}$ is quasi-compact, since for each $\varepsilon$, by the quasi-continuity of $u$, we can write $u = U_\varepsilon$ outside a set of capacity less than $\varepsilon$, with $U_\varepsilon$ a continuous function on $M$. The weak convergence $d\mu_k \to d\mu$ implies, by (3.8),

$$\limsup_{k \to \infty} \int_{\{u_k \leq v\}} d\mu_k \leq \limsup_{k \to \infty} \int_{\{u \leq v\}} d\mu_k \leq \int_{\{u \leq v\}} d\mu. \quad (3.16)$$
The lemma is proved.

We would like to apply Lemma 7 to our context. Let $\Omega_0$ be a smooth, non-negative closed $(1, 1)$-form on $\bar{M}$, and let $u, v$ be $\Omega_0$-plurisubharmonic and bounded on the Kähler manifold $(M, \Omega)$. By the theorem of Blocki and Kolodziej [BK], for any open subset $M'$ of $M$ with compact closure, there exists a decreasing sequence $\varepsilon_j \downarrow 0$, and sequences $u_j, v_j$ of smooth functions with

$$u_j \downarrow u, \quad v_j \downarrow v$$

in an open neighborhood of $M'$, and $u_j$ and $v_j$ are $(\Omega_0 + \varepsilon_j \Omega)$-plurisubharmonic.

**Lemma 8** Let $u, v \in L^\infty$ be $\Omega_0$-plurisubharmonic functions satisfying

$$\liminf_{z \to \partial M} (u(z) - v(z)) \geq 3\delta$$

for some fixed constant $\delta > 0$. Let $M'$ be any open subset of $M$ with compact closure, with $u - v > 2\delta$ in a neighborhood $K$ of $\partial M'$. Let $u_j, v_j$ be the decreasing sequences approximating $u$ and $v$ as given by the theorem of Blocki and Kolodziej, and let

$$d\mu_k = \Omega_0 + \varepsilon_k \Omega + \frac{i}{2} \partial \bar{\partial} u_k, \quad d\nu_j = \Omega_0 + \varepsilon_j \Omega + \frac{i}{2} \partial \bar{\partial} v_j,$$

$$d\mu = \Omega_0 + \frac{i}{2} \partial \bar{\partial} u, \quad d\nu = \Omega_0 + \frac{i}{2} \partial \bar{\partial} v.$$  

(3.19)

Then all five conditions (a-e) of Lemma 7 are satisfied.

**Proof of Lemma 8:** The condition (a) follows directly from the $\Omega_0$-plurisubharmonicity of $u, v$, and the Bedford-Taylor Theorem on the quasi-continuity of plurisubharmonic functions on $\mathbb{C}^m$, applied to each coordinate chart of $M$.

To prove (b), fix an index $k$. For each point $z_0 \in K$, choose $j_{z_0}$ so that $v_{j_0}(z_0) < u_k(z_0) - \delta$, which is possible, since $v_j(z_0)$ converges to $v(z_0)$ and $v < u - 2\delta \leq u_k - 2\delta$. By the upper-semicontinuity of the function $v_{j_0} - u_k$, it follows that there is a neighborhood $\mathcal{O}_{z_0}$ with $v_{j_0} < u_k - \delta$ on $\mathcal{O}_{z_0}$. Let $\bigcup_{\alpha=1}^N \mathcal{O}_{z_0}$ be a finite cover of the compact set $K$, and let $J_k = \max_{1 \leq \alpha \leq N} j_{z_0}$. Then for any $j \geq J_k$ and any $z \in K$, pick $z_\alpha$ with $z \in \mathcal{O}_{z_\alpha}$. Since $v_j$ is a decreasing sequence, we have

$$v_j(z) \leq v_{j_\alpha}(z) < u_k(z) - \delta,$$

(3.20)

which is the desired statement.

For (c), it suffices to establish the weak convergence on each compact subset of $M$. Covering the compact set by a finite number of coordinate charts $U_\alpha$, it suffices to establish the weak convergence on each chart $U_\alpha$. We may assume that on $U_\alpha$,

$$\Omega_0 = \frac{i}{2} \partial \bar{\partial} f_{0,\alpha}, \quad \Omega = \frac{i}{2} \partial \bar{\partial} f_\alpha,$$

(3.21)

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where \( f_\alpha \) may be assumed \( > 0 \) by adding a suitable large constant. Then
\[
0 \leq \Omega_0 + \varepsilon_j \Omega + \frac{i}{2} \partial \overline{\partial} u_j = \frac{i}{2} \partial \overline{\partial} (f_{0,\alpha} + \varepsilon_j f_\alpha + u_j) \tag{3.22}
\]
Thus the functions \( f_{0,\alpha} + \varepsilon_j f_\alpha + u_j \) are plurisubharmonic and decreasing to \( f_{0,\alpha} + u \). The Bedford-Taylor monotonicity theorem implies the weak convergence on \( M_\alpha \),
\[
(\Omega_0 + \varepsilon_j \Omega + \frac{i}{2} \partial \overline{\partial} u_j)^m \rightarrow (\Omega_0 + \frac{i}{2} \partial \overline{\partial} u)^m. \tag{3.23}
\]
The case of \( v_j \) is similar, so this establishes (c).

The statement (d) is a consequence of the fact that \( u_k, u, v_j, \) and \( v \) can be assumed to be all uniformly bounded in absolute value by the same constant \( C \). By the definition of capacity, it follows for example that for each \( E_\alpha \subset U_\alpha \), \( U_\alpha \) coordinate chart, we have
\[
\int_{E_\alpha} d\mu \leq \|f_{0,\alpha} + u\|_{L^\infty} c(E_\alpha, U_\alpha). \tag{3.24}
\]
Finally, the statement (e) follows by applying the smooth version Lemma 6, to the level sets \( \{u_k + \lambda < v_j\} \), which have compact closure in \( M' \) and smooth boundary for generic \( \lambda > 0 \). Letting \( \lambda \downarrow 0 \) gives the desired inequality. Q.E.D.

**Proof of Theorem 4:** If we replace \( u \) by \( u + 3\delta \) with \( \delta > 0 \), then the condition (3.18) is satisfied. Choosing \( M' \) as in Lemma 8, we can apply Lemma 7, and obtain
\[
\int_{\{u+3\delta < v\}} (\Omega_0 + \frac{i}{2} \partial \overline{\partial} v)^m \leq \int_{\{u+3\delta \leq v\}} (\Omega_0 + \frac{i}{2} \partial \overline{\partial} u)^m. \tag{3.25}
\]
The theorem follows by letting \( \delta \downarrow 0 \).

### 3.2 A uniqueness theorem for completely degenerate complex Monge-Ampère equations

The comparison theorem implies the following uniqueness theorem, for \( \Omega_0 \)-plurisubharmonic solutions of a completely degenerate Monge-Ampère equations, where the form \( \Omega_0 \) is allowed to be degenerate along an analytic subvariety:

**Theorem 5** Let \((M, \Omega)\) be a Kähler manifold with smooth boundary \( \partial M \) and dimension \( m \), and let \( u, v \in L^\infty \) be \( \Omega_0 \)-plurisubharmonic functions satisfying
\[
(\Omega_0 + \frac{i}{2} \partial \overline{\partial} u)^m = (\Omega_0 + \frac{i}{2} \partial \overline{\partial} v)^m = 0, \quad \limsup_{z \to \partial M} (u(z) - v(z)) = 0. \tag{3.26}
\]
If \( \Omega_0 \) is \( \geq 0 \) everywhere, and \( > 0 \) away from an analytic subvariety of strictly positive codimension which does not intersect \( \partial M \), then we must have \( u = v \) on \( M \).
Proof: By adding the same large constant to both $u$ and $v$, we may assume that $u, v > 0$. We argue by contradiction, and thus begin by assuming that $S = \{ u < v \} \neq \emptyset$. Since $u, v$ are $\Omega_0$-plurisubharmonic, the set $S$ must have strictly positive measure (it suffices to work in local coordinates, and apply the corresponding well-known property of plurisubharmonic functions on $\mathbb{C}^m$). Furthermore, since we can write

$$ S = \bigcup_{\varepsilon > 0} \{ u < (1 - \varepsilon)v \} \equiv \bigcup_{\varepsilon > 0} S_\varepsilon, \quad (3.27) $$

it follows that $S_\varepsilon$ must have strictly positive measure for some $\varepsilon > 0$. Fix one such value of $\varepsilon$. Since $u \geq v \geq (1 - \varepsilon)v$ on $\partial M$, we may apply the comparison principle for Kähler manifolds and obtain

$$ 0 \geq \int_{S_\varepsilon} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} u)^m \geq \int_{S_\varepsilon} (\Omega_0 + (1 - \varepsilon)\frac{i}{2} \partial \bar{\partial} v)^m $$

$$ = \int_{S_\varepsilon} \{(1 - \varepsilon)(\Omega_0 + \frac{i}{2} \partial \bar{\partial} v) + \varepsilon \Omega_0\}^m $$

$$ \geq \varepsilon^m \int_{S_\varepsilon} \Omega_0^m $$

$$ \geq \varepsilon^m \int_{S_\varepsilon} \Omega_0^m \quad (3.28) $$

since the form $\Omega_0 + \frac{i}{2} \partial \bar{\partial} v$ is non-negative. Let now $V_\delta$ be the complement of a neighborhood of the divisor $D$, with $\Omega_0^m \geq \delta \Omega^m$ for each $\delta > 0$ small enough. Clearly for each $\delta > 0$

$$ \int_{S_\varepsilon} \Omega_0^m \geq \int_{S_\varepsilon \cap V_\delta} \Omega_0^m \geq \delta \int_{S_\varepsilon \cap V_\delta} \Omega^m. \quad (3.29) $$

Since $M \setminus D = \bigcup_{\delta > 0} V_\delta$ and $D$ has measure 0 with respect to the volume form $\Omega^m$, we have

$$ 0 < \int_{S_\varepsilon} \Omega^m = \lim_{\delta \to 0} \int_{S_\varepsilon \cap V_\delta} \Omega^m \quad (3.30) $$

which implies that $\int_{S_\varepsilon \cap V_\delta} \Omega^m > 0$ for some $\delta > 0$. Altogether, we obtain a contradiction. Thus $\{ u < v \}$ must be empty. Interchanging the roles of $u$ and $v$ completes the proof of the theorem.

4 Proof of Theorem 1

We can now prove Theorem 1. In Theorem 3, we have shown that the function $\hat{\Phi}$ is a bounded, $\Omega_0$-plurisubharmonic solution of the Dirichlet problem (2.50) on $\mathcal{X}_D$. On the other hand, in [PS09] (Theorem 3), it was shown that the same Dirichlet problem admits a bounded, $\Omega_0$-plurisubharmonic solution which is $C^{1,\alpha}$ for any $0 < \alpha < 1$ on $\mathcal{X}_D^\times$. By Theorem 5, it follows that the two solutions must coincide. Thus $\hat{\Phi}$ is $C^{1,\alpha}$ on $\mathcal{X}_D^\times$.

Since $\hat{\Phi} = \Phi - \Phi_1 + \Psi_0$ and both $\Phi_1$ and $\Psi_0$ are smooth on $\mathcal{X}_D^\times$, it follows that $\Phi$ is $C^{1,\alpha}$ on $\mathcal{X}_D^\times = X \times D^\times$. Q.E.D.
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