**EIGENVALUE MONOTONICITY FOR THE RICCI-HAMILTON FLOW**

LI MA

_In Memory of S.S.Chern_

**Abstract.** In this short note, we discuss the monotonicity of the eigen-values of the Laplacian operator to the Ricci-Hamilton flow on a compact or a complete non-compact Riemannian manifold. We show that the eigenvalue of the Laplacian operator on a compact domain associated with the evolving Ricci flow is non-decreasing provided the scalar curvature having a non-negative lower bound and Einstein tensor being not too negative. This result will be useful in the study of blow-up models of the Ricci-Hamilton flow.

We study the monotonicity property of the eigenvalues of the Laplacian operator \( \Delta := \Delta_{g(t)} \) of the evolving metric \((g(t))\) along the Ricci-Hamilton flow. Our main result is stated in the Main Theorem below.

We recall the definition of Ricci-Hamilton flow. Let \((M, g_{0})\) be a compact or a complete non-compact Riemannian manifold. We are given a family of Riemannian metrics \(\{g(t)\}\) with \(g(0) = g_{0}, 0 \leq t < T\). Let \(g = g(t)\). In the local coordinates \((x^{i})\), we write

\[
g = g_{ij}dx^{i}dx^{j}.\]

Then the Ricci-Hamilton flow is defined by the evolution equation for Riemannian metrics:

\[
\partial_{t}g_{ij} = -2R_{ij}, \quad \text{on } M_{T} := M \times [0, T)
\]

where \(R_{ij}\) is the Ricci tensor of the metric \(g := g(t)\) and \(T\) is the maximal existing time for the flow. In [2], R.Hamilton proved the local existence of the flow for the compact manifold case. His argument is much more simplified by De Turck [1]. When \((M, g_{0})\) is a complete non-compact Riemannian manifold with bounded geometry, W.X.Shi [5] obtained the local existence result for the flow. In [4], G.Perelman introduced two important entropy functionals which are monotone along the Ricci-Hamilton flow coupled with back-ward heat flows.

We consider the change rate of first eigenvalue of Laplacian operator associated with the Ricci flow \(g(t)\) in a compact domain in \(M\). More precisely, let \(D\) be a compact domain with smooth boundary in the manifold \(M\). Let \(f := f(x, t)\) be the first eigen-function of \(\Delta\). Then we have

\[
-\Delta f = \mu f, \quad \text{in } D,
\]

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with Dirichlet boundary condition
\[ f = 0, \quad \text{on } \partial D \]
where \( \mu := \lambda_1(D) > 0 \) is the corresponding first eigenvalue of \( \Delta \). We can normalize \( f \) such that
\[ \int_D f^2 dv = 1. \tag{2} \]
Here \( dv \) is the volume form of the metric \( g = g(t) \).

We will use the following conventions: For \( F = F(t) \) being a quantity on \( D \), we let \( F' \) be the derivative with respect to the time variable \( t \) and let \( F_i, F_{ij}, F_{ij,k}, \) etc, be the covariant derivatives of \( F \) with respect to the Levi-Civita connection of the metric \( g \). We also assume that when \( D = M \) is compact, we impose the natural condition that \( \int_M f dv = 0 \). We abuse the upper or lower indexes in this paper when their meanings are clear. We write by \( D_T = D \times [0, T) \).

Differentiating (1), we get
\[ -\Delta' f - \Delta f' = \mu' f + \mu f', \quad \text{on } D, \tag{3} \]

Multiplying both sides of (3) by \( f \) and integrating over \( D \), we have
\[ -\int \Delta' f \cdot f - \int \Delta f' \cdot f = \mu' + \mu \int f' f. \tag{4} \]

Note that
\[ -f' \Delta f = \mu' f f, \quad \text{on } D. \]
Then we have
\[ \mu' = -\int \Delta' f f \]
Hence using (7) in the appendix we obtain that
\[ \mu' = -2 \int R_{ij} f_{ij} f. \]
Hereafter, we abuse the use of upper and lower indices by using orthonormal moving frame. If \( n = 2 \), then we have
\[ R_{ij} = \frac{1}{2} R g_{ij}, \]
and
\[ \mu' = -\int R \Delta f \cdot f = \mu \int f^2 R dv \geq 0, \]
provided \( R > 0 \).

In the case when \( R \geq C > 0 \) in \( D \times [t_0, t) \) for some uniform constant \( C \), we have
\[ (\log \mu)' \geq C, \]
and \( \mu(t) \geq \mu(t_0) e^{C(t-t_0)} \) for \( t > t_0 \).

It is also clear that when \( R \leq 0 \), we have
\[ \mu' = -\int R \Delta f \cdot f = \mu \int f^2 R dv \leq 0. \]
If we further have that $R \leq -C \leq 0$ in $D \times [t_0, t)$ for some uniform constant $C > 0$, we have

$$(\log \mu)' \leq -C,$$

and $\mu(t) \leq \mu(t_0)e^{-C(t-t_0)}$ for $t > t_0$. So we have the following result

**Proposition 1.** Assume $n = 2$.

1. If $R \geq C > 0$ in $D \times [t_0, t)$ for some uniform constant $C > 0$ along the Ricci flow, we have

$$(\log \mu)' \geq C,$$

and $\mu(t) \geq \mu(t_0)e^{C(t-t_0)}$ for $t > t_0$.

2. If we have that $R \leq -C \leq 0$ in $D \times [t_0, t)$ for some uniform constant $C > 0$ along the Ricci flow, we have

$$(\log \mu)' \leq -C,$$

and $\mu(t) \leq \mu(t_0)e^{-C(t-t_0)}$ for $t > t_0$.

We now consider the higher dimensional case when $n \geq 3$. We denote by $E_{ij}$ the Einstein tensor

$$E_{ij} := R_{ij} - \frac{R}{2}g_{ij}$$

of the metric $g = \{g_{ij}\}$.

When $n \geq 3$, we notice that

$$\mu'/2 = -\int R_{ij}f_if_j = \int (R_{ij}f)_{ij}f_i = \int R_{ij}f_if_i - \int R_{ij}f_if_j$$

$$= \frac{1}{2} \int R_{ij}f_if_j + \int R_{ij}f_if_j$$

$$= -\frac{1}{2} \int R(f_if_i) + \int R_{ij}f_if_j$$

$$= -\frac{1}{2} \int R\Delta f - \frac{1}{2} R|\nabla f|^2 + \int R_{ij}f_if_j$$

$$= \frac{\mu}{2} \int Rf^2 + \int (R_{ij} - \frac{R}{2}g_{ij})f_if_j$$

$$= \frac{\mu}{2} \int Rf^2 + \int E_{ij}f_if_j.$$

Here we have used again the contracted second Bianchi identity

$$2R_{ij;} = R_i.$$

Assume that $R - 2a \geq 0$ on $D \times \{t\}$ and

$$E_{ij} \geq -ag_{ij}, \text{ in } D \times \{t\}$$
for some constant $a$. Then we have
\[
\mu'/2 = \frac{\mu}{2} \int R f^2 + \int E_{ij} f_i f_j \\
\geq \frac{\mu}{2} \int R f^2 - a \int |\nabla f|^2 \\
= \mu \int (\frac{R}{2} - a) f^2 \geq 0.
\]

From the proof, one can see that the same result is also true for higher order eigenvalues. Therefore, we conclude the following result:

**Main Theorem.** Let $g = g(t)$ be the evolving metric along the Ricci-Hamilton flow with $g(0) = g_0$ being the initial metric in $M$. Let $D$ be a smooth bounded domain in $(M, g_0)$. Let $\mu > 0$ be the first eigenvalue of the Laplacian operator of the metric $g(t)$. If there is a constant $a$ such that the scalar curvature $R \geq 2a$ in $D \times \{t\}$ and the Einstein tensor
\[
E_{ij} \geq -a g_{ij}, \text{ in } D \times \{t\},
\]
then we have $\mu' \geq 0$, that is $\mu$ is non-increasing in $t$, furthermore, $\mu'(t) > 0$ for the scalar curvature $R$ not being the constant $2a$. The same result is also true for higher order eigenvalues.

We remark that these result may be useful in the study of blow-up models of Ricci-Hamilton flow on a complete Riemannian manifold $(M, g_0)$.

**Appendix**

Given a Riemannian manifold $(M^n, g)$. In the local coordinates $(x^i)$, we write
\[
g = g_{ij} dx^i dx^j.
\]
Let $u$ be a smooth function on $M$. Then the Laplacian of $u$ is defined by
\[
\Delta u = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j u)
\]
where
\[
(g^{ij}) = (g_{ij})^{-1}
\]
is the inverse of the matrix $(g_{ij})$ and $|g| = det(g_{ij})$.

For convenient of readers, we review some facts about Ricci-Hamilton flow. Recall that the metric $g = g(t)$ satisfies the Ricci-Hamilton flow:
\[
\partial_t g_{ij} = -2R_{ij}, \text{ on } M_T.
\]
Along this flow, we have that
\[
\partial_t g^{ij} = 2R^{ij}, \text{ on } M_T.
\]
and
\[
\partial_t dv = -R dv, \quad (5)
\]
where \(dv\) is the volume element and \(R\) is the scalar curvature of the metric \(g(t)\) respectively. For smooth functions \(u\) and \(v\) with compact support in \(M\), we have by a use of the divergence theorem that
\[
\int g^{ij} u_i v_j dv = - \int \Delta uv dv.
\]
We write by \(u^i = g^{ij} u_j\). Differentiating both sides of this equation and using (5) we have
\[
2 \int R^{ij} u_i v_j - \int R u^i v_i = - \int \Delta' uv + \int \Delta u \cdot v R. \tag{6}
\]
Since
\[
- \int R u^i v_i = \int (Ru^i)_i v = \int R u^i v + \int R v \Delta u,
\]
we have that
\[
2 \int R^{ij} u_i v_j + \int (\nabla R, \nabla u) v = - \int \Delta' uv.
\]
As before, using integration by part and using the contracted second Bianchi identity, we obtain that
\[
2 \int R^{ij} u_i v_j = - \int (\nabla R, \nabla u) v - 2 \int R^{ij} u_{ij} v.
\]
Combining this with (6) we get
\[
\int \Delta' uv = 2 \int R^{ij} u_{ij} v.
\]
Hence we have the variation formula for the Laplacian operator to the Ricci-Hamilton flow:
\[
\Delta' u = 2 R^{ij} u_{ij}. \tag{7}
\]
In particular when \(n = 2\), we have that
\[
R_{ij} = \frac{1}{2} R g_{ij},
\]
and
\[
\Delta' u = R \Delta u.
\]
For more material about Ricci-Hamilton flow, one may see \[4\] and \[3\]. An earlier version of this paper has some misprint in sign, and it is in arxiv.math.DG/0403065. Using our idea, D.Kokotov and D.Korotkin [Normalized Ricci Flow on Riemann Surfaces and Determinant of Laplacian, Letters in Mathematical Physics, 71(3)(2005)241-242] can give a simple proof of the fact that the determinant of Laplace operator in a smooth metric over compact Riemann surfaces of an arbitrary genus \(g\) monotonously grows under the normalized Ricci flow. Together with results of Hamilton and B.Chow that under the action of the normalized Ricci flow a smooth metric tends asymptotically to the metric of constant curvature, this leads to a simple proof of the Osgood-Phillips-Sarnak theorem stating that within the class of smooth metrics with fixed conformal class and fixed volume the
determinant of the Laplace operator is maximal on the metric of constant curvature.

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Li Ma: Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
E-mail address: lma@math.tsinghua.edu.cn