Top quark production near threshold:

on resummation of $O\left((\beta_0 \alpha_s)^n\right)$ QCD corrections

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Abstract

We discuss the resummation of the potentially large $O\left((\beta_0 \alpha_s)^n\right)$ QCD corrections to the total cross section of the process $e^+e^- \rightarrow t\bar{t}$ near the threshold. In this approximation, the cross section factorization into the short- and long-distance parts is valid. The short-distance correction is reduced to the production vertex renormalization. It amounts to $-9.5\%$ and is well under control. The long distance corrections are accounted for as the effect of the coupling’s running in the QCD potential. We argue that the accuracy of present predictions for the cross section in the scheme with the running QCD coupling is about 10%.

PACS numbers: 12.38.Bx, 12.38.Cy, 13.85.Lg, 14.65.Ha

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1. The theoretical studies of the process $e^+e^- \rightarrow t\bar{t}$ near the threshold are of interest for many applications. First of all, the top quark will be studied in the near threshold kinematics at the Next Linear Collider, and precise measurements of its mass and width are among the most important experimental issues [1]. From this standpoint, reliable estimations of the QCD perturbative effects are mandatory.

The general approach for the problem of the top quark production near the threshold was suggested in [2], where it was demonstrated that the top quark width plays the role of an infrared cutoff which suppresses all nonperturbative long-distance effects. The size of the nonperturbative corrections was estimated in [3,4] to amount to less than one per cent. Therefore, one could hope to predict the total cross section by perturbative calculations with such an accuracy. One-loop QCD corrections were discussed in [3,5,6] and Higgs-induced electroweak corrections, which have the same order as the QCD ones, were discussed in [7]. Recently, the $\mathcal{O}(\alpha_s^2)$ QCD perturbative corrections to the $t\bar{t}$ threshold cross section were calculated in [8,9] and were found to be comparable in size with the first order correction. This result can cast some doubts upon the applicability of QCD perturbation theory to the problem under discussion.

In this note we make an attempt to answer the question: “How can higher order perturbative corrections modify the NNLO result?”. Surely, an exact answer to this question deserves an exact knowledge of those higher order corrections which is far beyond our reach at the moment. Nevertheless, there exists a method to estimate a contribution of higher orders, which is known to work well at different problems (see for example [10,11,12,13,14,15,16] and references therein). This method exploits the large value of $\beta_0 = 11 - \frac{2}{3}N_F$, the lowest order coefficient from the QCD beta-function, and consists in an approximation of the complete result for the $(n+1)$th order, by its counterpart of the $n$th order in $\beta_0$. The latter is found by the naive non-abelianization (NNA) prescription: in the (gauge-invariant) result for a set of Feynman diagrams containing $n$ light quark vacuum polarization bubbles, one replaces $N_F^n$ by $(-3\beta_0/2)^n$. In other words, taking into account only the running of the QCD coupling, one assumes it to be a reasonable approximation for all radiative corrections.

A nice feature of this approach with respect to the threshold cross section analysis is a complete factorization of hard and soft corrections, both of which reside on its own scale.

In what follows, we first discuss the intermediate kinematic region $\alpha_s \ll v \ll 1$. We resum $\mathcal{O}((\beta_0\alpha_s)^n)$ terms to the first two leading coefficients in the expansion in $v$. Then, we consider the cross section near the threshold, where short-distance effects are factorized to the normalization of the cross section and the long-distance effects are absorbed to the static potential with the running coupling. We perform a numerical evaluation with such the potentials to test the influence of the higher order QCD corrections on the cross section.

2. Let us first consider the kinematic region $\alpha_s \ll v \ll 1$, where $v$ is the relative velocity of quark and antiquark. In this region, we have an extra small parameter, $\alpha_s/v$, so that the interaction between the slowly moving quark and antiquark can be treated perturbatively. In the lowest order, this interaction is caused by the one–gluon exchange between two particles. Corresponding one–loop correction to the pair production ampli-
A_0(v), can be easily obtained using the QED result by Schwinger \[17\]:

$$A_0(v) = \frac{C_F\alpha_s}{\pi} \left\{ \frac{\pi^2}{2v} - 2 \right\}. \quad (1)$$

Here the former term is determined by the low–energy scale, so that all powers of $C_F\alpha_s/v$ should be resummed in the vicinity of the threshold. The latter term has its origin at the relativistic scale.

As discussed above, we would like to consider the effects of vacuum polarization only. The most appropriate way to do that is to substitute $1/(k^2 - \lambda^2)$ for $1/k^2$ in the gluon propagator, and to calculate the amplitude $A_\lambda(v)$ as a function of the “gluon mass” $\lambda$. Then the first–order in $\alpha_s$ correction to the production amplitude, which is valid to all orders in $\beta_0\alpha_s$, equals $[12, 13, 14, 15, 11]$

$$A(v) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda^2}{\lambda^2} [A_\lambda(v) - A_0(v)] \text{Im} \frac{1}{1 + \Pi(\lambda^2)}, \quad (2)$$

where

$$\Pi(k^2) = -\beta_0\alpha_s \frac{4\pi}{\lambda^2} \ln \left( -\frac{k^2}{\mu^2} e^C \right) \quad (3)$$

is the one–loop polarization operator, $\mu$ is the normalization point for the coupling constant, $\alpha_s \equiv \alpha_s(\mu)$, while $C$ is a scheme–dependent constant. In what follows, we use $C_{\text{MS}} = -5/3$. The result for $A_\lambda(v)$ reads $[18, 19]$

$$A_\lambda(v) = \frac{C_F\alpha_s}{v} \text{arctg} \frac{v}{z} - \frac{2C_F\alpha_s}{3\pi} \left\{ \frac{3\pi}{2z} + \frac{z^6 - 2z^4 - 2z^2 - 6}{z\sqrt{z^2 - 4}} \text{arch} \frac{z}{2} - z^4 \ln z + z^2 + \frac{3}{2} \right\} \quad (4)$$

$$= A_s^\lambda(v) + A_h^\lambda. \quad (5)$$

Here $z = \lambda/m$, $m$ is the mass of the $t$–quark. Inserting the difference $A_\lambda(v) - A_0(v)$ into (4), we obtain the amplitude in the intermediate kinematic region as a sum of the soft and hard terms,

$$A(v) = A_s(v) + A^h. \quad (6)$$

3. Now we would like to find the hard correction, $A^h$, due to (3). Inserting (7) into the integral (2), one can obtain its expansion in power series over $a = \beta_0\alpha_s(\mu)/(4\pi)$:

$$A^h = \frac{C_F\alpha_s(m)}{\pi} \sum_{n=0} r_n a^n, \quad (8)$$

where the lower order coefficients at $\mu = m$ are:

$$r_0 = 2, \quad r_1 = \frac{11}{6}, \quad r_2 = \frac{4\pi^2}{3} + \frac{163}{18}, \quad r_3 \approx 57.05, \quad r_4 \approx 1131. \quad (9)$$

The zeroth order coefficient coincides with that from Schwinger’s result (1), the first order one is in accord with the result obtained in $[19]$, while $r_2$, $r_3$ and $r_4$ are new. Due to the
Figure 1: $A_{\text{exact}}^{h}(a)$ (solid line); $A^{h}$ (dashed and dotted lines) expanded in $a^{n}$, as a function of $a$. Vertical lines mark the actual values of $a$ at the threshold for top ($m=175$ GeV), bottom ($m=4.8$ GeV), and charm ($m=1.4$ GeV) quarks.

fact that the vector current’s anomalous dimension is zero, one can easily restore the $\mu$–dependence of the results. In Fig.1, we compare the results of the lower orders inclusion with the exact (in NNA approximation) result. The actual values of $a$ for $c$, $b$, and $t$–quarks are indicated in order to demonstrate how the perturbation theory works for the current renormalization at the corresponding thresholds.

Taking $\alpha_{s}(m_{t}) = 0.1$, so that $a = 0.061$, we can calculate an “exact” value of the hard correction to the amplitude of $t\bar{t}$ production:

$$A_{\text{exact}}^{h}(a)|_{a \to 0.061} = -2.236 \frac{\alpha_{s}C_{F}}{\pi} \approx (-2 - 1.33a - 41.7a^{2})\frac{\alpha_{s}C_{F}}{\pi}. \quad (10)$$

The second equation is an extrapolation formula which works in the region $a = (0, 0.1)$ and can be used to obtain $A_{\text{exact}}^{h}(a)$ at different $\alpha_{s}(m_{t})$. Adding up this result to unity, we obtain the renormalization of the vertex $\gamma^{*}t\bar{t}$ at the threshold, by the hard QCD correction (in NNA approximation). The $O(\alpha_{s}^{2}\beta_{0})$ correction amounts to about 6% of the one–loop $O(\alpha_{s})$ result $A^{h}_{0}$, the $O(\alpha_{s}^{3}\beta_{0}^{2})$ correction gives about 4% and the rest in the sum gives 2%. The sum of higher order corrections constitutes 12% of $A^{h}_{0}$. The net QCD correction to the renormalization of the $\gamma^{*}t\bar{t}$ vertex amounts to $-9.5\%$, in the NNA approximation.

We would like also to compare this number with the two-loop QCD correction to the vector current normalization constant, $A_{\text{two–loop}}^{h}(\mu_{\text{fact}}^{2})$, derived in [20, 9, 8], which
depends on a factorization scale. We take eq. (22) from [8] (or eq. (39) from [4]), to obtain
\[ A_{\text{two-loop}}(m^2) = -2.34 \frac{\alpha_s C_F}{\pi} \] and \( A_{\text{two-loop}}(m^2/2) = -2.14 \frac{\alpha_s C_F}{\pi} \). One can see that our result (10) is rather close to these numbers.

We conclude that the discussed corrections to the short–distance part of the production amplitude are well under control.

As for the long–distance part in the intermediate region, we obtain
\[ A_s(v) = C_F \alpha_s(\mu) \pi v \sum_{n=0} t_n a^n, \]
where the lower order coefficients are:
\[ t_0 = \frac{1}{2}, \quad t_1 = -\frac{D}{2}, \quad t_2 = \frac{D^2}{2} + \frac{\pi^2}{3}, \]
\[ t_3 = -\frac{D^3}{2} - \pi^2 D, \quad t_4 = \frac{D^4}{2} + 2D^2 \pi^2 + \frac{8\pi^4}{5}, \]
and \( D = C - 2 \ln(\frac{\mu}{m}) \). The zeroth order coefficient coincides with that from Schwinger’s result (1), the first order one coincides with the result obtained in [19], while \( t_2, t_3 \) and \( t_4 \) are new.

It is worthy to stress that the intermediate region has only academic interest, since at \( \alpha_s = 0.1 \) there is no room for the strong inequality \( \alpha_s \ll v \ll 1 \). Either \( O(v^n) \) relativistic corrections become equally important, or the ratio \( \frac{\alpha_s}{v} \) proves to be of order unity. On the other hand, one can use the results for the short distance correction \( A \) directly at the threshold region, after subtraction of the resummed Coulomb singularities. In what follows we discuss such a resummation.

4. For the sake of completeness, we would like to discuss how the running of the QCD coupling modifies the cross section near the threshold. This issue was a subject of many papers. It was discussed in [3, 8], with the NLO accuracy and in [22], with the NNLO accuracy. Our main task here is to analyze how the perturbative expansion works in the scheme with the running QCD coupling and how accurate are the predictions, made in this scheme. We perform a numerical analysis in the position space and compare its results with those obtained in the fixed normalization point scheme. Our analysis is complementary to that made in [22], where a comparison of the potentials in the momentum and position spaces was discussed.

Near the threshold, the ratio \( \frac{\alpha_s}{v} \) is not small and one has to resum all \( O\left((\frac{\alpha_s}{v})^n\right) \) terms [21]. For the \( t\bar{t} \) production, we should also include the finite width of the \( t \)–quark, \( \Gamma_t \). The cross section of the process \( e^+e^- \rightarrow t\bar{t} \) near the threshold normalized to the cross section for \( e^+e^- \rightarrow \mu^+\mu^- \), is [2]:
\[ R = N_{c\bar{c}} \frac{24\pi}{s} \text{Im} G(E + i\Gamma_t; 0, 0), \]
where \( G(E + i\Gamma_t; r, r') \) is the Green function for the Schrödinger equation:
\[ (H - E - i\Gamma_t)G(E + i\Gamma_t; r, r') = \delta(r - r'), \quad H = \frac{P^2}{m} + V(r). \]

\footnote{1In what follows, we disregard relativistic corrections.}
Thus, the problem of the soft corrections resummation to all orders in $\alpha_s$, reduces to a calculation of $G(E + i\Gamma_t; 0, 0)$ for the Schrödinger equation with a heavy quark–antiquark potential $V(r)$. In the NNA approximation, the short–distance correction $A^b$ derived at the previous section, enters into $R$ through the factor $1 + A^b$.

It is well known, that at LO [2] and NLO [3] no choice exists of the normalization point for the coupling constant, which would result in an appropriately similar behavior of the threshold cross sections calculated in the fixed point perturbative potential and the “running” one, respectively. This means, that an account of running is actually important, since it allows one to resum large logarithmic corrections which arise in the fixed point perturbation theory. However, an account of the NNLO corrections makes two cross sections much closer to each other, if the high normalization point is used, $\mu \sim m_t$.

The difference becomes an effect of $O(\alpha_s(\mu)^3)$ and appears to be not very large, at the level of 5-10%.

Let us now discuss a scheme with the running QCD coupling. Usually a starting point for the analysis is a perturbative potential in the scheme with a fixed normalization point, in the momentum space. The “running” potential is obtained by resumming all logarithmic terms to the running coupling:

$$V(r) = - C_F \frac{\alpha_s(1/r')}{r} \left\{ 1 + \frac{\alpha_s(1/r')}{4\pi} a_1 + \left( \frac{\alpha_s(1/r')}{4\pi} \right)^2 \left[ \frac{\beta_0^2 \pi^2}{3} + a_2 \right] \right\} (16) + \left( \frac{\alpha_s(1/r')}{4\pi} \right)^3 \left[ 16 \beta_0^3 \zeta(3) + \beta_0 \left( 3\beta_0 a_1 + \frac{5}{2} \beta_1 \right) \left( \frac{\pi^2}{3} + a_3 \right) \right].$$

Here $r' = re^{\gamma_E}$, $\gamma_E$ is the Euler constant. The coefficients $a_1, a_2$ are known [23, 24], while $a_3$ is still unknown. The coupling $\alpha_s(1/r')$ suffers from the Landau pole, appearing as a consequence of extrapolation into the strong coupling domain of the perturbation theory result. The usual way to handle this singularity is an introduction of some model potential, mimicking non–perturbative effects, at the large distances $r > r_0 \sim 1/\Lambda_{\text{QCD}}$ [3, 6]. However, the contribution of the “non-perturbative” region to the resulting cross section proves to be extremely small, less than $0.1 - 0.3\%$, due to the large width of the top quark. Therefore this cross section can be considered as model–independent.

To find the cross section for various potentials, we solve the Schrödinger equation numerically. Our result for the running NNLO potential agrees with the corresponding result of [22], obtained in coordinate space, when the same input parameters are chosen. In Fig.2, we plot the cross section $R$ as a function of the energy $E$. We have chosen $m_t = 175$ GeV, $\alpha_s(m_Z) = 0.118$, $\Gamma_t = 1.43$ GeV. For a conservative estimate we have chosen $a_3 = 100a_2$ (recall that $a_2 \approx 70a_1$).

Unfortunately, the resummation of the logarithms does not place the higher order corrections under control. One can see from Fig.2, that the corrections to the cross section are not small in the vicinity of the peak. Moreover, these corrections do not show a decrease in their values with increasing order. The weak convergence of the perturbative series is caused by the enormously large non–logarithmic coefficients, $a_n$, entering the potential (17). We see that the $1S$ peak is shifted by about 0.5 GeV upon inclusion of each new order into the potential. At least partially, this effect can be due to the use of the pole mass. It is known that the latter suffers from the renormalon ambiguities and
thus cannot be determined with accuracy better than $\Lambda_{\text{QCD}}$\textsuperscript{[25, 26, 12]}. The same order ambiguities are present in the interaction potential, $V(r)$. As was shown in \textsuperscript{[27, 28, 29, 30]}, the leading order $O(\Lambda_{\text{QCD}}r)$ ambiguities cancel in the combination $V(r) + 2m$.

It is rather natural to anticipate that a more safe way to construct an expansion is to use a running mass at the low normalization point $\mu$, but the issue requires a more detailed study.

Let us now comment on the accuracy of results for the cross section. Fig.2 shows, that the conservative estimate of $a_3$ suggests the deviation of the NNLO curve by about 10% at the peak. We have also calculated the cross section with the NNLO potential in the scheme with the fixed normalization point and have compared it with the “running” NNLO result from Fig.2. The difference is again about 10%. Our estimate of the accuracy confirms the conclusion of the paper \textsuperscript{[22]}.

5. We have studied QCD radiative corrections, originating from the running of the coupling $\alpha_s$, to the total cross section of the process $e^+e^- \rightarrow t\bar{t}$ near the threshold. We have resummed potentially large $O((\beta_0\alpha_s)^n)$ short-distance QCD corrections to the production vertex $\gamma^*t\bar{t}$. The exact in $\beta_0\alpha_s$ short–distance correction renormalizes the current to $-9.5\%$ with respect to the Born result and is well under control. The long distance corrections are accounted for by using a potential with the running QCD coupling and are more significant. We demonstrate that the accuracy of present predictions for the cross section in the scheme with the running QCD coupling is about 10%.
Acknowledgments

We are grateful to A.Hoang, V.Fadin, K.Melnikov and N.Uraltsev for useful advices. The work of O.Ya. was supported by the German Federal Ministry for Research and Technology (BMBF) under contract number 05 7WZ91P (0). A.Y. acknowledges the financial support from the Russian Foundation for Fundamental Research, grant 98-02-17913.

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