Horizon/matter systems near the extreme state

O.B. Zaslavskii

Department of Physics, Kharkov State University, Svobody Sq. 4, Kharkov
310077, Ukraine

e-mail: olegzas@aptm.kharkov.ua

It is shown that in the extreme limit with a zero surface gravity but nonzero local temperature the limiting metric of a generic static black hole is determined by a metric induced on a horizon and one function of two coordinates, stress-energy tensor of a source picking up its values from a horizon. The limiting procedure is extended to rotating black holes. If the extreme limit is due to merging a black hole horizon and cosmological one both horizons are always in thermal equilibrium in this limit. This is proved for a generic case of static or axially-symmetrical rotating spacetimes.

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I. INTRODUCTION

The significant part of physically relevant black hole and cosmological spacetimes (such as the Reissner-Nordström, Schwarzschild-de Sitter, Kerr metrics) possesses an extreme state. Either geometrical or thermodynamic properties of such a state differ from nonextreme one in an essential way: a surface gravity and a corresponding Hawking temperature are equal to zero, the proper distance from a horizon to any other point diverges, the entropy assigned to a horizon is zero [1]. Thus, the properties of the extreme state are, according to the prescription made in [1], highly nontrivial. On the other hand, the limiting transition from the topological sector of nonextreme black holes to the nonextreme one seemed, for a first glance, to be obvious. As, by definition, the surface gravity goes to zero as one approaches the state in question, one could expect that the limiting state is simply a state with the zero temperature and without profound thermodynamic properties and, apart from this, the
proper distance between a horizon and any other point tends to infinity in accordance with
known properties of the extreme black hole (the simplest example is the extreme Reissner-
Nordström black hole). It turned out, however, that this limit can be performed in a rather
nontrivial way, so the temperature and the above mentioned proper distance both remain
finite [2], [3], [4].

The main point here can be explained from the thermodynamic viewpoint and consists in
the crucial difference between the temperature $T_0$ measured at infinity (its role in black-hole
physics is played by the Hawking temperature $T_H$) and Tolman temperature $T$ in a static
spacetime measured by a local observer. In general relativity they are connected by the
relationship $T = T_0 / \sqrt{-g_{00}}$. Here $T$ is a local (Tolman) temperature, $T_0$ is that measured
at infinity. For nonextreme black hole $T_0$ coincides with the Hawking temperature $T_H$. In
the extreme limit $T_H \rightarrow 0$ by definition, so it would seem obvious that this limit is trivial
in the sense that $T \rightarrow 0$ everywhere. Meanwhile, it was shown in [2] that the limit under
discussion can be taken in such a way that simultaneously with $T_H$ the quantity $g_{00}$ goes to
zero as well, the local temperature $T$ remaining nonzero. It is worth stressing that both
temperatures have qualitatively different status in gravitational thermodynamics: it is the
quantity $T$ which determines the properties of the canonical or grand canonical ensemble,
thus having direct physical meaning whereas $T_0$ in the framework of these ensembles can
be rescaled in arbitrary way [3]. Therefore, it is essential that the limiting state is achieved
at a finite $T$. As was demonstrated in [2] for the Reissner-Nordström metric, the radial
coordinate in this limit becomes degenerate, so all points pick up the value of it equal to
that of the horizon. If, however, this coordinate is properly rescaled (along with a time
coordinate) the resulting geometry is well-defined and has an universal form [3]. It turns
out also that the limiting state possesses further interesting properties: although in the
limit under discussion temperature $T \neq 0$, the entropy of Hawking radiation surrounding a
horizon equals zero [4], quantum corrections to the entropy of a black hole itself have the
universal form [6].

In the present paper we generalize the treatment of the extreme limit in three directions.
First, we find the limiting form of a metric of a generic static distorted nonextreme black hole near the extreme state. Second, we include in consideration systems which possess simultaneously a black hole and cosmological horizons and show that in the limit when metrics induced on both horizons coincide (in particular, for spherically-symmetrical systems radii of both horizons merge) the state of thermal equilibrium arises irrespectively of details of a metric and matter distribution between horizons. Third, we apply our approach to rotating horizons and find the limiting metric near the state under discussion. Apart from general motivation connected with the role of the extreme state in black hole physics, the issue addressed in the present paper establishes relationship between different types of metrics in general relativity which are obtained one from another by changing parameters of solutions and, in fact, realizes the case when finding ”limits of spacetimes” is motivated physically from the viewpoint of black hole thermodynamics.

II. EXTREME LIMIT FOR A GENERIC STATIC BLACK HOLE

Consider a generic static black hole without any assumptions about a spherical (or any other) symmetry. As in a general case there is no analog of radial coordinate $r$ we will use another approach. Let us choose the coordinate system which consists of equipotential surfaces $g_{00} = \text{const}$ and coordinates on each surface. (Such a system was used while proving the uniqueness theorem for static black holes in an asymptotically flat spacetimes. The latter assumption is not relevant for us.) Then a metric can be written in the form

$$ds^2 = -dt^2 + \chi^{-2}dV^2 + \gamma_{ab}dy^ady^b$$

Here $a, b = 1, 2$; components $\gamma_{ab}$ and $\chi$ depend, generally speaking, on all three spacelike coordinates. The quantity $\chi_0 = \lim_{V \to 0} \chi$ is a constant equal to the surface gravity which determines the Hawking temperature according to $T_H = (2\pi)^{-1}\chi_0$.

As before, we will consider such a limiting transition in the process of which a new manifold is obtained from the vicinity of a horizon. Similarly to we demand that
a local Tolman temperature $T = T_H/\sqrt{-g_{00}}$ remain finite in each point outside a horizon. Therefore, let simultaneously $V \to 0$ and $\chi_0 \to 0$ in such a way that the ratio $V/\chi_0$ be finite. It is convenient to introduce new coordinates according to

$$x = V/\chi_0, \quad t_1 = t\chi_0 \quad (2)$$

For small $\chi_0$ we can use the following expansion for $\chi$:

$$\chi^2(y^a, V) = \chi^2(y^a, x\chi_0) = \chi_0^2 + (x\chi_0)^2 f(y^1, y^2) + .... \quad (3)$$

In the limit under consideration linear terms in $\chi$ are missing since they would have destroyed the regularity of a metric when $\chi_0 \to 0$ as follows from substitution of $\chi$ into (1) (cf. [15] where power expansion for metric coefficients was used for the analysis of the regularity of spherically-symmetrical metrics).

As a result, we find the limiting form of a metric for $\chi_0 \to 0$:

$$ds^2 = -dt_1^2 x^2 + \frac{dx^2}{1 + x^2 f(y^1, y^2)} + \gamma^h_{ab} dy^a dy^b \quad (4)$$

It is supposed here that the function $f > 0$, $\gamma^h_{ab}(y^1, y^2) = \gamma_{ab}(y^1, y^2, V = 0)$. In so doing, a metric may be vacuum or have nonzero stress-energy tensor as a source. In the latter case the values of its components in the orthogonal frame can be obtained from the corresponding values for an original metric (1) by putting $V = 0$. In other words, this tensor picks up its values from a horizon of the metric (1).

It may happen that, depending on properties of the function $f$, the limiting metric may acquire elements of symmetry which are absent in an original metric (1). For instance, if a two-dimensional surface $x = \text{const}$ represents a sphere and $f = r^{-2}$ the metric (1) by virtue of the substitution can be cast into the form

$$ds^2 = r_+^2 (-dt_1^2 \sinh^2 x + dx^2 + d\omega^2) \quad (5)$$

If $f = 0$ the metric in the $(x, t_1)$ sector is the Rindler spacetime. These examples agree with results of [3]. In both cases the metric represents a direct product of two-dimensional spaces.
The latter property holds true for any (not necessarily spherically-symmetrical) metric with $f = \text{const.}$

Thus, we see that properties of limiting metrics are determined in fact by a single function $f$, a metric induced on an arbitrary surface $x = \text{const}$ being determined by the metric on a horizon of an original spacetime. As a rule, the metric of a four-dimensional black hole distorted by matter or external sources cannot be found exactly. However, formulae and reasonings listed above show that even though an original metric is not known one manages to obtained the limiting form of it near the extreme state. In other words, approaching the extreme state restricts the variety of metrics according to (4). To conclude this section, one reservation is in order. It was assumed above that a metric on a horizon is regular. Meanwhile, for extreme dilatonic black holes the horizon surface tends to zero [20], so this case is not encompassed by the approach developed above. Nevertheless, the limiting transition to a regular metric in the topological sector of nonextreme black holes in this case does exist [21].

III. DISTORTED BLACK-HOLE AND COSMOLOGICAL HORIZONS IN THERMAL EQUILIBRIUM

It was assumed in the preceding section that a system possesses only a black hole horizon. The extreme limit implying that $T_H \to 0$ could be realized due to merging a black hole and inner horizons as it takes place, for instance, in the Reissner-Nordström metric. Meanwhile, there exists also another class of examples of physical interest in which, along with a black hole horizon, a cosmological one is present and the extreme limit is achieved due to merging both horizons. The interesting point here consists in that, although radii of horizons coincide, the proper distance between them remain finite, the Hawking temperatures associated with both horizons being equal, so a black hole and cosmological horizons attain the thermal equilibrium in the extreme limit. The corresponding observation were made, among static metrics, for a spherically-symmetrical spacetimes only and, moreover, only for the several
particular classes of them - such as the Schwarzschild - de Sitter and Reissner-Nordström -
de Sitter metrics [9], [10], [11], [12] (plus the Kerr- de Sitter metric for the case with rotation
[16] - see next section). The aim of the present section is to generalize the corresponding
results to generic static black hole - cosmological systems with distorted horizons.

We will use the approach similar to that of the preceding section. Consider the metric

$$ds^2 = -dt^2 f(V) + \chi^2 dV^2 + \gamma_{ab} dy^a dy^b$$

with $$f(V) = V^2(V_0^2 - V^2)g(V)$$ where the function $$g$$ is regular and nonzero near a black
hole horizon $$V = 0$$ and a cosmological one $$V = V_0$$. For the metric to be regular near
$$V = V_0$$, the function $$\chi$$ must have the form $$\chi^2 = (V_0^2 - V^2)h(V)$$ where $$h$$ is regular near
$$V_0$$. In general, the Hawking temperatures associated with both horizon are different, so the
thermal equilibrium is impossible.

Consider now the limit $$V_0 \to 0$$ in which both horizons seemingly coincide. Introducing
the new variables according to $$V = V_0 z$$ and $$t = t_1 V_0^{-2} [h(0)g(0)]^{-1/2}$$ we obtain

$$ds^2 = h(0)^{-1} [-dt_1^2 z^2 (1 - z^2) + \frac{d^2 z}{1 - z^2}] + \gamma_{ab} dy^a dy^b$$

where $$\gamma_{ab} = \lim_{V \to 0} \gamma_{ab}$$. By substitution $$z = \sin \frac{x}{2}$$ this metric can be cast to the form

$$ds^2 = [4h(0)]^{-1} [-dt_1^2 \sin^2 x + dx^2] + \gamma_{ab} dy^a dy^b$$

It is easily seen from (8) that the horizons are located at $$x = 0$$ and $$x = \pi$$, so the proper
distance between them remains finite. For the Euclidean version of this metric the choice of
the period equal to $$2\pi$$ makes the metric regular simultaneously near both horizons, so both
horizon have the same temperature and are in the state of thermal equilibrium.

It is worth stressing two crucial points. First, the conclusion derived above is valid
irrespective of the presence of matter between both horizons and the particular properties
of its stress-energy tensor. It means, in particular, that the backreaction of quantum field
on the Schwarzschild - de Sitter or Reissner-Nordström - de Sitter metric does not destroy
the state of thermal equilibrium between both horizons in the extreme limit. In so doing,
this limit is to be understood as the coalescence of radii $r_+$ and $r_-$ where these quantities themselves are related to the system "dressed" by Hawking radiation. Second, the general structure \((8)\) follows from the regularity of the manifold and does not use field equations, so it is model-independent. Thus, self-consistent quantum-corrected geometries do exist in the extreme limit under discussion. Whereas their general form is determined by eq.\((8)\) the other details such as the coefficient $h(0)$ cannot be found without taking into account field equations. Carrying out backreaction program in such a situation is beyond the scope of the present paper and deserves separate treatment. Here we only mention that the self-consistent solutions of field equations similar to the $(x, t_1)$ part of the \((8)\) exist in two-dimensional dilaton gravity \cite{18}. The existence and properties of self-consistent solutions with backreaction of quantum radiation can be, in particular, of interest from the viewpoint of investigating dynamics of (anti-)evaporation of Schwarzschild-de Sitter black holes \cite{19}. The obtained form of the limiting geometry may be also of interest for the effect of production of black hole pairs in cosmology \cite{12}.

Each of the horizons contributes the term $A/4$ ($A$ is the surface area of the horizon) into the entropy which is equal to their sum, both contributions being equal. It is instructive to note here the difference with the situation when the event horizon and the inner one merge whereas the event horizon and the cosmological one do not. Let for simplicity the system is a spherically-symmetrical and $r_-, r_+$ and $r_c$ correspond to the inner, event and cosmological horizon, respectively. Let $r_- \to r_+ \neq r_c$. As is shown in \cite{2}, \cite{3}, the limiting metric has only one horizon. As an original metric (with arbitrary $r_- < r_+ < r_c$) had either the black hole or cosmological horizon one may wonder where the entropy associated with a cosmological horizon got to. The point, however, is that all points of the Euclidean manifold have $r = r_+$, so the cosmological horizon as well as any points with $r > r_+$ do not belong to the manifold at all. As a result, only the horizon $r = r_+$ contributes to the entropy. The zero-loop entropy of the original spacetime associated with horizons is $S_0 = \pi(r_+^2 + r_c^2)$. The entropy of the final state after the limiting transition is performed is not $S_0$ with $r_+$ and $r_c$ replaced by their limiting values, but is equal to $\pi r_+^2$ instead. On the other hand, for the case described
by eq. (8) \( r_+ = r_c \) and the entropy does equal \( \pi (r_+^2 + r_c^2) = 2\pi r_+^2 \).

IV. EXTREME LIMIT FOR NONEXTREME ROTATING BLACK HOLES

Discuss now the case of four-dimensional rotating black holes. It is easy to carry out the corresponding limiting procedure to the extreme state for an axially-metric of a general form:

\[
ds^2 = -dt^2 f + (N^\phi dt + d\phi)^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2
\]

(9)

where all metric coefficients depend on \( r \) and \( \theta \) only. Let they have the form

\[
f = (r - r_+)(r - r_-)\mu(r, \theta), \quad g_{rr} = \left[(r - r_+)(r - r_-)\lambda(r, \theta)\right]^{-1}, \quad N^\phi = (r - r_+)\eta(r, \theta)
\]

(10)

where \( r_+ > r_- \) corresponds to the event horizon and functions \( \mu, \lambda, \eta \) are finite near \( r_+ \). In particular, all properties indicated above are inherent to the Kerr and Kerr-Newman metrics in the frame rotating with respect to a distant observer with the angular velocity equal to that of a black hole. The Hawking temperature for a black hole spacetime described by the metric (9) is equal to \( T_H = (4\pi)^{-1}(r_+ - r_-)\alpha \) where \( \alpha = \sqrt{\lambda h \mu h} \), the index "h" indicates that the corresponding quantities are to be taken at the horizon. The quantity \( \alpha \) is constant on the horizon surface due to the constancy of a surface gravity.

Consider what is happening in the limit \( r_+ \to r_- \). Let us make the substitution

\[
r - r_+ = (r_+ - r_-) \sinh^2 \frac{x}{2}, \quad t = 2t_1[(r_+ - r_-)\alpha]^{-1}
\]

(11)

Then the metric in the limit at hand reads

\[
ds^2 = \lambda_h^{-1}(-dt^2 \sinh^2 x + dx^2) + g_{\phi\phi}(d\phi + dt_1 \sinh^2 \frac{x}{2} \eta h)^2 + g_{\theta\theta} d\theta^2
\]

(12)

In particular, substituting into (9), (12) the explicit values of coefficients for the Kerr metric we obtain

\[
ds^2/a^2 = (1 + \cos^2 \theta)(-dt_1^2 \sinh^2 x + dx^2 + d\theta^2) + 4 \frac{\sin^2 \theta}{(1 + \cos^2 \theta)}(d\phi + 2\sin^2 x/2dt_1)^2
\]

(13)
where \(a\) is the angular momentum parameter of the Kerr black hole. It is nothing else than
the rotating analog of the Bertotti-Robinson metric \([17]\). In a similar manner for the extreme
limit of the nonextreme Kerr-Newman black holes the metric is

\[
\begin{align*}
\ds = (r_+^2 + a^2 \cos^2 \theta) (-dt_1^2 \sinh^2 x + dx^2 + d\theta^2) + \frac{(r_+^2 + a^2)^2}{r_+^2 + a^2 \cos^2 \theta} \sin^2 \theta [d\phi + 4(\frac{r_a}{r_+^2 + a^2}) \sinh^2 \frac{x}{2} dt_1]^2
\end{align*}
\]

(14)

For uncharged extreme black holes \(r_+ = a\) and \((14)\) turns into \((13)\). For static holes
\(a = 0\) and \((14)\) turns into the finite temperature version of the Bertotti-Robinson spacetime
in agreement with the previous result \([3]\).

The electromagnetic field for the spacetime \((14)\) is obtained in the limit under discussion
from that of the Kerr-Newman in a straightforward manner. The result reads

\[
\begin{align*}
\mathbf{F} &= \frac{2Qa r_+ \cos \theta \sin \theta (r_+^2 + a^2)}{(r_+^2 + a^2 \cos^2 \theta)^2} d\theta \wedge (d\phi + \frac{4a r_+}{r_+^2 + a^2} \sinh^2 \frac{x}{2} dt_1) + \frac{Q \sinh x (r_+^2 - a^2 \cos^2 \theta)}{(r_+^2 + a^2 \cos^2 \theta)^2} dx \wedge dt_1
\end{align*}
\]

(15)

The next example is a class of metric for which either effect of rotation or a cosmological
horizon is present. It means that the metric we start with has now the form \((9)\) but instead
of \((10)\) the metric coefficients obey the conditions

\[
\begin{align*}
f = (r - r_1)(r_2 - r) \mu, \quad g_{rr} = (r - r_1)(r_2 - r) \lambda, \quad N^\phi = (r - r_1) \eta
\end{align*}
\]

(16)

Then, repeating calculations step by step we arrive at

\[
\begin{align*}
\ds = \lambda_h^{-1} (-dt_1^2 \sin^2 x + dx^2) + g_{\phi\phi}^h (d\phi + \eta^h \sin^2 \frac{x}{2} dt_1) + g_{\theta\theta}^h d\theta^2
\end{align*}
\]

(17)

Similarly to the case of non-rotational metrics, both horizon are situated at the finite proper
distance and have (in the limit in question) equal temperatures. In particular, the form \((17)\)
includes the extreme limit of the Kerr-de Sitter metric obtained earlier in \([16]\).

V. SUMMARY AND CONCLUSION

We have developed rather general unified approach to the treatment of the extreme state
(state with a zero surface gravity) within the nonextreme topological sector. It is based on
the physical demand that a local Tolman temperature be finite at any point outside a horizon. The corresponding limiting procedure is well-defined and carrying it out shows that a thin layer adjoining an event horizon develops into a new manifold. In so doing, a proper distance between a horizon and any other point outside it remains finite. If a metric is non-vacuum the stress-energy tensor of the resulting metric picks up its values from a horizon of an original metric, so it does not depend on a distance from a horizon. The approach presented in this paper enabled us to handle on equal footing the cases when the extreme limit arises either due to coalescing an event and inner horizons of a black hole or when it is due to merging of a black hole and cosmological horizons. We have manage to relax the condition of a spherical symmetry and obtain a general form of the limiting metric of a static black hole which is determined by two entities. The first one is a two-dimensional metric induced on a horizon of an original metric which plays a role of a metric induced on an equipotential surface $g_{00}$ of the limiting metric, component of it being the same for any such a surface. The second one is a single function of two variables (coordinates on a surface $g_{00} = \text{const}$) only. Thus, even without complete knowledge of a solution which possesses the extreme limit one may gain information about the limiting form of it. For rotating black holes the limiting metric obtained from the Kerr and Kerr-Newman solutions is obtained explicitly and has a rather simple form.

While black holes themselves in a sense possess universality due to uniqueness and no-hair theorems (strong gravitational field deletes information about details of a system structure, so only a few parameters characterize black hole spacetimes) the extreme limit turns out to be even ”more universal” as compared to other black hole configurations. Owing this a number of general conclusion have been made above irrespectively of details of a system.

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