Scaling of the Random-Field Ising Model at Zero Temperature

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Abstract

The exact determination of ground states of small systems is used in a scaling study of
the random-field Ising model. While three variants of the model are found to be in the same
universality class in 3 dimensions, the Gaussian and bimodal models behave distinctly in 4
dimensions with the latter apparently having a discontinuous jump in the magnetization. A
finite-size scaling analysis is presented for this transition.

PACS numbers:05.50.+q, 64.60.cn, 75.10.Hk
The random-field Ising model (RFIM) has been a subject of theoretical and experimental interest for many years. Yet, there remain many unresolved questions: 1) Is there a difference in the behaviors of the RFIM on varying the distribution of random fields from, say, Gaussian to bimodal? 2) Does the transition in three dimensions (3D) remain continuous down to zero temperature ($T=0$)? 3) Are there variants of the RFIM that are in the same universality class?

In this note, we summarize the results of a scaling analysis of the RFIM at $T=0$. We have studied three versions of the RFIM in 3D and the bimodal and Gaussian RFIM in 4D. System sizes up to $16 \times 16 \times 16$ and $10 \times 10 \times 10 \times 10$ have been considered with the averaging carried out over up to 10000 realizations (samples) of the random-field distribution. For each sample, we obtained the ground state exactly. We find that all three variants of the RFIM are probably in the same universality class in 3D and estimate two of the exponents. In 4D, our results suggest that the Gaussian model has a continuous transition whereas the bimodal model has a first order transition at zero temperature. A novel finite-size scaling analysis is presented for this latter case. We discuss the physical ramifications of our results at the end of this paper.

The RFIM is described by the Hamiltonian

$$\mathcal{H}_{RFIM} = -J \sum_{<ij>} S_i S_j - \sum_i h_i S_i \ ,$$

(1)

where $S_i = \pm 1$ and for the bimodal model $h_i = \pm h_r$ randomly whereas the Gaussian model is characterized by a Gaussian distribution of $h_i$ with a width that we will call $h_r$. A variant of the RFIM that we have studied in 3D considers an Ising ferromagnet in which a fraction
of spins, \( p/2 \), is frozen to be \( S_i = +1 \) (in a quenched random manner) and an equal fraction \( (p/2) \) of spins is frozen to be \( S_i = -1 \). We will denote this model as the random-pinned (RP) model. All three models are identical and correspond to the Ising ferromagnet when \( h_r \) and \( p \) are set equal to zero. Thus at \( T=0 \), for \( D=3 \) and 4, the system is in a broken symmetry state and has a magnetization equal to, say, +1. Our studies are restricted to \( T=0 \), at which the magnetization goes to zero at a threshold value of \( h_r \) or \( p \). Our focus is on studying this transition by determining the ground states of the models exactly for various values of the parameters.

We note that the RP model is a cousin of the spin one Ising model introduced by Pereyra et al.\(^{14}\) for a system of \( N_2 \) molecules within a molecular crystal of CO molecules. It is also related to the model of a dilute antiferromagnet in a uniform field\(^{16}\) with \( S_i = \pm 1 \), but with a fixed fraction of the sites having no spin (due to the random, quenched dilution).

The three models that we have studied are convenient in the sense that just one parameter needs to be tuned at \( T=0 \) in order to reach the phase transition point. This is in contrast to the recently introduced asymmetric generalization of the random-field model for the description of the liquid-vapor phase transition in porous media.\(^{13}\) In that model, a fraction \( p \) of the sites have a field \( +h' \), whereas the rest of the sites have a field \( -h'' \), with the values of both \( h' \) and \( h'' \) needing fine tuning for a given \( p \neq \frac{1}{2} \). Our earlier studies have shown that a value of \( p \) away from \( \frac{1}{2} \) or correlations in the locations of the random field lead to a tendency for the critical point at zero temperature to be supplanted by a triple point at which three first order lines intersect. The location of a zero-temperature critical
point (fixed point, in general) plays a key role in a scaling description of the random-field phase transition since it suggests that the important competition is between the ordering tendency of the exchange and the disordering tendency of the random field with the role of temperature being secondary. The zero-temperature fixed point scenario leads to measurable predictions of a violation of conventional hyperscaling and sluggish activated dynamics. It is important to note, however, that the absence of a zero-temperature critical point does not automatically exclude the zero-temperature fixed point off in some point in interaction space.

Within this context, it is interesting to note that within mean-field theory, the bimodal model and Gaussian model of the RFIM behave distinctly with the former having no \( T = 0 \) critical point unlike the latter. Our own studies of a bimodal model on a Bethe lattice show that on varying the coordination number of the lattice, both types of behavior are observed.

The simplest expectation in dimensions below the upper critical dimension (ucd) is that the bimodal model and the Gaussian model, for which the ucd is 6, are both in the same universality class and have a zero-temperature critical point. Our results suggest that while this is possibly true in 3D, differences arise even in 4D. We have used a polynomial-time flow algorithm first implemented in the study of the RFIM by Ogielski to determine the exact ground state.

Our analysis is based on studying the scaling behavior of the Binder parameter \( g = \frac{1}{2}[3 - \frac{<m^4>}{<m^2>^2}] \), where \( m \) is the magnetization per site of a single realization and \( < ... > \)
denotes a configurational averaging:

\[ g_L(x) = g(L^{1/\nu}t) , \]  

(2)

with \( t = x - x_c \), where \( x \) is \( h_r \) for the bimodal and Gaussian models and \( p \) for the RP model. The transition value \( x_c \) is first determined by requiring that \( g_L(x_c) \) is a constant independent of \( L \). We then collapse the \( g_L(x) \) curves using the correlation length exponent \( \nu \) as an adjustable parameter. In order to determine the order-parameter exponent \( \beta \), we use

\[ \langle m^2 \rangle = L^{-2\beta/\nu} \psi(L^{1/\nu}t) , \]  

(3)

with \( \beta \) as an adjustable parameter.

For the smaller sizes considered we studied 10000 independent samples, while for the larger systems we averaged over a few 1000 independent realisations. We note that our method does not involve problems of equilibration encountered in Monte Carlo simulations, that the number of samples we have considered easily exceeds the usual numbers that are studied in simulations and that the zero-temperature analysis accesses the random-field transition directly.

Typical scaling plots are shown in Figures 1-3. The scaling relations are expected to hold for \( L \to \infty, t \to 0, \) and \( \langle m^2 \rangle \to 0 \). However, for the RFIM, \( \beta \) is close to zero so that for the sizes that we are able to study, \( \langle m^2 \rangle \) is rather large. This is the principal reason for the scaling not being better than it is. The same fact precludes us from determining \( \eta \). A measurement of \( \eta \) entails, e.g., a scaling collapse of \( \langle m^2 \rangle \) at \( t=0 \) in the presence of a uniform field \( H \) of varying strength. However, the large value of \( \langle m^2 \rangle \) at \( t = H = 0 \)
makes such a calculation inherently imprecise.

While moderately good scaling is obtained for $g$, we have found that the scaling of $<m^2>$ with predetermined values of $x_c$ and $\nu$ does not work well in both the $x < x_c$ and $x > x_c$ regimes simultaneously. The best scaling plots are obtained on ignoring the $x < x_c$ region (since that is where $|m|$ is very large) and choosing the optimal value of $\beta$ that collapses the largest size systems best for $x > x_c$. Our results are summarized in Table I.

For $D = 3$ the results (i.e. the values of $g_L(x_c)$ and the exponents $\beta$, $\nu$) are consistent with all models being in the same universality class. The extreme closeness of $g_c$ to unity implies that the distribution of $|m|$ at criticality is sharply peaked at a non-zero value. This is confirmed by inspecting this distribution directly (Figure 4). However, despite the smallness of $\beta$, there is no evidence that the transition is first order for $D = 3$: the weight in $P(|m|)$ is concentrated around values of $|m|$ close to unity, without the additional peak at $|m| = 0$ which would be expected if ferromagnetic and paramagnetic phases were to coexist at $h_r = h_c$.

In $D = 4$, the situation is very different. The values of the exponent $\nu$ seem to be different for Gaussian and bimodal models; the exponent $\beta$ is demonstrably non-zero for the Gaussian model. The values of $g_c$ are different for the two field distributions and an inspection of the whole distribution $P(|m|)$ (Figure 4) shows clearly that the Gaussian and bimodal models behave distinctly. While $P(|m|)$ for the Gaussian model has a single peak at $|m| > 0$, characteristic of a continuous transition, the bimodal model exhibits a double peaked distribution, with peaks at $|m| = 1$ (not shown - see the caption to Figure 4) and
$|m| = 0$, suggestive of a discontinuous jump in $|m|$ at the transition (as predicted by mean field theory$^3$).

Finite-size scaling at a first-order transition requires the introduction of certain exponents analogous to critical exponents of continuous transitions.$^{22}$ We discuss briefly how this scaling analysis goes for a first-order random-field transition at $T = 0$. For this transition, of course, $t = h_r - h_c$ plays the role of the ‘thermal’ variable, while the ground-state energy $\langle E \rangle$ and its derivatives $d\langle E \rangle/dt$ and $d^2\langle E \rangle/dt^2$ play the roles of the free energy, the entropy and the specific heat respectively. Therefore, we anticipate that $d\langle E \rangle/dt$ as well as the magnetization $\langle m \rangle = -d\langle E \rangle/dH$ will be discontinuous at $h_c$. Now it is easily shown that $d\langle E \rangle/d(\ln h_r) = -\sum_i h_i S_i$. If one plots the distribution over samples of this quantity for $h_r = h_c$, one again finds a double-peaked distribution for the 4D bimodal model (but single peaked for the other models). A peak at the origin corresponds to the samples with most of the spins aligned, and there is a broad second peak corresponding to the non-zero weight at small $|m|$ in Figure 4. We infer that $d\langle E \rangle/dt$ is discontinuous at $t = 0$ as anticipated.

The finite-size scaling analysis starts from the scaling form for the singular part of the configuration-averaged energy density, $\langle E(t, H) \rangle = L^{-D} f(t L^{1/\nu_t}, H L^{1/\nu_H})$, where $H$ is a uniform magnetic field, and $y$ is the scaling dimension of the Hamiltonian at the transition.$^8$ Discontinuities in the derivatives with respect to $t$ and $H$ in the limit $L \to \infty$ imply $1/\nu_t = 1/\nu_H = D - y$. This generalizes the result $1/\nu_t = 1/\nu_H = D$ of conventional first-order transitions at $T_c > 0$. At the transition, the (connected) susceptibility $\chi_{con} = \partial \langle m \rangle / \partial H \sim L^{D-y} = L^{2-\eta}$, giving $\eta = 2 - D + y$. The exponent $\bar{\eta}$ is defined through the ‘disconnected’
susceptibility $\chi_{\text{dis}} = L^D \langle m^2 \rangle \sim L^{4-\tilde{\eta}}$, giving $\tilde{\eta} = 4-D$, because $\langle m^2 \rangle$ jumps at the transition. Finally one can readily derive a Schwartz-Soffer inequality $\tilde{\eta} \leq 2\eta$, following the method used for continuous transitions. Together with the scaling relations derived above, this implies $\eta \geq D/2$ and $\nu_t = \nu_H \geq 2/D$, reminiscent of a general inequality for random systems. For $D = 4$ we obtain $\nu_t \geq 1/2$, consistent with our numerical result $\nu = 0.6 \pm 0.1$. If, as has been suggested, the Schwartz-Soffer inequality is saturated this becomes $\nu_t = 1/2$, still consistent with the numerical result.

Our results have implications for experimental realizations of the RFIM. It is likely that for situations such as the liquid vapor transition in aerogel, the correlations in the strands as well as the large porosity cause the $T=0$ critical point to be supplanted by a first order transition. The key question then is what the exponents are for the continuous transition at non-zero temperature when the random-field strength is weak. The simplest scenario is that the governing fixed point is still at $T=0$ and is characterized by conventional random-field exponents but this has not been explicitly demonstrated yet. A more intriguing scenario would correspond to an entirely new universality class. In the former case, the as yet unexplained results of Wong and Chan could perhaps be attributed to the experiments not being carried out sufficiently close to the critical point to see the crossover from bulk-like behavior to the “true” random-field behavior.

We are indebted to Moses Chan for stimulating discussions. This work was supported by grants from EPSRC, KBN (grant number 2P302-127, Poland), INFN (Italy), NASA, NATO, a NSF MRG grant, and the Petroleum Research Fund administered by the American
Chemical Society. MRS acknowledges the E. U., contract ERB CHB GTC 940636, for financial support.
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| MODEL     | $h_c$ or $p_c$ | $g_c$    | $\beta$  | $\nu$  |
|-----------|----------------|---------|---------|--------|
| Bimodal 3D | $2.25 \pm 0.02$ | $0.996 \pm 0.002$ | $0.025 \pm 0.015$ | $1.2 \pm 0.15$ |
| Gaussian 3D | $2.33 \pm 0.03$ | $0.997 \pm 0.002$ | $0.031 \pm 0.015$ | $1.2 \pm 0.15$ |
| RP 3D     | $0.19 \pm 0.01$ | $0.997 \pm 0.002$ | $0.025 \pm 0.02$  | $1.2 \pm 0.2$  |
| Bimodal 4D | $3.71 \pm 0.05$ | $0.80 \pm 0.05$  | $0(\leq 0.01)$    | $0.6 \pm 0.1$  |
| Gaussian 4D | $4.17 \pm 0.05$ | $0.96 \pm 0.05$  | $0.13 \pm 0.02$    | $0.8 \pm 0.1$  |

Table Caption: Summary of the results obtained in this paper.
FIGURE CAPTIONS

1. The scaling plots for the $D=3$ Gaussian model for the parameters shown in Table I. The values of $L$ studied are 6, 8, 12, and 16. The symbols with more sides correspond to larger values of $L$.

2. Same as Figure 1 but for the 4D Gaussian system. Here $L=4$, 5, 6 and 10.

3. Same as Figure 2 but for the 4D bimodal system with $L=5$, 6, 8 and 10.

4. Probability distributions for the absolute value of magnetization for the Gaussian and bimodal 3- and 4-D systems (for $L=16$ and 10 respectively) at the phase transition (Table I). In the bimodal case, the bin with $|m|=1$ is not shown. 80% of the samples in 3D and 64% in 4D had $|m|=1$. On increasing the system size, the peaks become more pronounced while retaining the same structure.\textsuperscript{15}
