Nonparametric Bayesian Structure Adaptation for Continual Learning

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Abstract

Continual Learning is a learning paradigm where machine learning models are trained with sequential or streaming tasks. Two notable directions among the recent advances in continual learning with neural networks are (i) variational Bayes based regularization by learning priors from previous tasks, and, (ii) learning the structure of deep networks to adapt to new tasks. So far, these two approaches have been orthogonal. We present a principled non-parametric Bayesian approach for learning the structure of feed-forward neural networks, addressing the shortcomings of both these approaches. In our model, the number of nodes in each hidden layer can automatically grow with the introduction of each new task, and inter-task transfer occurs through the overlapping of different sparse subsets of weights learned by different tasks. On benchmark datasets, our model performs comparably or better than the state-of-the-art approaches, while also being able to adaptively infer the evolving network structure in the continual learning setting.

1 Introduction

Continual learning (Ring 1997; Parisi et al. 2019) is the learning paradigm where a single model is subjected to a sequence of tasks. At any point of time, the model is expected to (i) make predictions for the tasks it has seen so far, (ii) if subjected to training data for a new task, adapt to the new task leveraging past experience if possible (forward transfer) and benefit the previous tasks if possible (backward transfer). While the desirable aspects of more mainstream transfer learning (sharing of bias between related tasks (Pan and Yang 2009)) might reasonably be expected here too, the principal challenge is to retain the predictive power for the older tasks even after learning new tasks, thus avoiding the so-called catastrophic forgetting. Real world applications in, for example, robotics or time-series forecasting, are rife with this challenging learning scenario, the ability to adapt to dynamically changing environments or evolving data distributions being essential in these domains. Also, as a skill innate to humans (Flesch et al. 2018), it is naturally an interesting scientific problem to reproduce the same capability in artificial predictive modelling systems.

With the versatility and expressive power of neural networks being well-established in the landscape of machine learning models, current approaches focus on endowing them with the above capability (Parisi et al. 2019).

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Almost all of these approaches are based on three foundational ideas. One of them is to constrain the parameter values to not deviate significantly from their previously learned value by using some kind of regularization or a trade off between previous and new learned weights as in (Schwarz et al. 2018; Kirkpatrick et al. 2017; Zenke et al. 2017; Smola et al. 2003; Lee et al. 2017). A natural way to accomplish this is to train a model using online (at task-level) Bayesian inference, whereby the posterior of the parameters learned from task \( t \) serve as the prior for task \( t + 1 \) (Nguyen et al. 2018; Zeno et al. 2018). This informed prior helps in forward transfer, and also prevents catastrophic forgetting by penalizing large deviations from itself. In particular Variational Continual Learning (Nguyen et al. 2018) (henceforth referred to as VCL) achieves state of the art results by applying this simple idea to Bayesian neural networks. The second idea is to perform incremental model selection for every new task. For neural networks, this is done by evolving the structure as newer tasks are encountered (Golkar et al. 2019; Li et al. 2019). The third idea is to invoke a form of 'replay’, whereby selected samples representative of previous tasks, are used to retrain the model after new tasks are learnt.

Distilling ideas from these approaches, we present a novel Bayesian approach to continual learning that seeks to incorporate the ability of structure learning into the simple yet effective framework of online Bayes. Our model can also takes advantage of a replay mechanism to improve predictive performance.

**Our contributions:** We leverage the fact that any particular task uses a sparse subset of the connections of a neural network, and different related tasks share different (albeit possibly overlapping) subsets. Thus, in the setting of continual learning, it would be more effective if the neural network could accommodate changes in its connections to dynamically adapt to a newly arriving task. Moreover, in our model, we perform automatic model selection by letting each task select the number of nodes in each hidden layer. All this is done under the principled framework of variational Bayes and a nonparametric Bayesian modeling paradigm.

## 2 Preliminaries

Bayesian neural networks (Neal 2012) were introduced with the objective of estimating the full posterior over the weights of a neural network in order to alleviate overfitting issues and provide predictive uncertainty estimates for the parameters of the neural network. Hence, for data \( D \) and parameters \( w \), a prior \( p(w) \) is imposed on the parameters, and we seek to learn \( p(w|D) = \frac{p(D|w)p(w)}{p(D)} \), where \( p(D) = \int \frac{p(D|w)p(w)dw}{q(w)} \) over the weights. Note that \( q(w) \) is not restricted to be Gaussian. Reparameterized samples from this posterior are then used to approximate the lower bound via Monte Carlo sampling. Using \( S \) samples \( w^i \sim q(w) \) we thus have

\[
\mathcal{L} = \mathbb{E}_{q(w)} \ln \frac{p(D, w)}{p(w)} \\
\approx \frac{1}{S} \sum_{i=1}^{S} [\ln p(D|w^i) + p(w^i) - q(w^i)]
\]

Variational Continual Learning (VCL) (Nguyen et al. 2018) adopts this framework with the fundamental modification that \( p_{t+1}(w) = q_t(w) \), that is, a task reuses the previous task’s posterior as its prior. In order to generalize this construction to adaptively infer the model structure, that can potentially grow with each incoming task, we propose a nonparametric Bayesian model, as described next.

## 3 Nonparametric Bayesian Continual Learning

Our model operates by the following primitives for each hidden layer (Fig. 1 shows an illustration). Initially, assuming a single hidden layer for simplicity, the first task allocates as many hidden layer nodes as necessary, and learns the posterior over weights for a subset of the edges incident on each node. Each subsequent task reuses some of the edges learnt by the previous task and uses the posterior over the weights learnt by the previous task as the prior. Additionally, it may allocate new nodes and learn the posterior over some of their incident edges. It thus learns the posterior over (i) a subset of the weights used by the previous task, (ii) a subset of the
weights (incident on previously existing nodes) that were not used by the previous task, (iii) weights of a subset of the connections incident on the nodes newly allocated by itself. While making predictions, a task uses only the nodes/weights it has learnt.

More slack for later tasks in terms of model size (allowing it to create new nodes) indirectly lets the task learn better without deviating too much from the prior, which in this case is the posterior of the previous tasks. This reduces chances of catastrophic forgetting (Kirkpatrick et al. 2017).

3.1 Generative story

Consider a neural network having $L$ hidden layers. The weights in layer $l$, denoted as $W^l$, are given by a matrix of real-valued vectors $V^l$ point-wise multiplied by a binary matrix $B^l$. This ensures sparse connection weights between the layers. Note that both $V^l$ and $B^l$ are of the same size. Moreover, we model $B^l$ using the nonparametric Bayesian Indian Buffet Process (IBP) (Griffiths and Ghahramani 2011) prior which enables learning the size of $B^l$ (and consequently of $V^l$) from data. As a result, the number of node in the hidden layer is learned adaptively from data. The output layer weights are denoted as $W_{\text{out}}$. The generative story for our model, for $l = 1, \ldots, L$, is as follows:

$$B^l \sim \text{IBP}(\alpha)$$  \hspace{1cm} (2)
$$V^l_{d,k} \sim \mathcal{N}(0, \sigma^2_0)$$  \hspace{1cm} (3)
$$W^l = B^l \odot V^l$$  \hspace{1cm} (4)
$$W^\text{out}_{d,k} \sim \mathcal{N}(0, \sigma^2_0)$$  \hspace{1cm} (5)
$$y_n \sim \text{Lik}(W_{\text{out}}\phi_{NN}(x_n))$$  \hspace{1cm} (6)

for $n \in 1 \ldots N$, where '$\odot$' represents pointwise multiplication, $\phi_{NN}$ is the function computed (using parameter samples) up to the last hidden layer of the network thus formed, Lik denotes the likelihood distribution of the observed data.

Note that the above construction is somewhat reminiscent of the cascaded IBP based deep generative model for unsupervised learning proposed in (Adams et al. 2010). In contrast, our focus here is on supervised learning problems in the continual learning settings. Connection with the above model can also be traced to (Rai and Daumé III 2010) which uses an IBP prior on the shared latent basis of the weight vectors in a linear multitask learning model, allowing the size of the basis to be learned from data.

Henceforth we suppress the superscript denoting layer number from the notation for simplicity; the discussion will hold identically for all hidden layers.

When adapting to a new task, the posterior of $V$ learnt from previous tasks is used as the prior. A new $B$ is learnt afresh, to ensure that a task only learns the subset of weights relevant to it. As described before, to adaptively infer the number of nodes in each hidden layer, we use the IBP prior (Griffiths and Ghahramani 2011).
Doshi et al. 2009), whose truncated stick-breaking process (Doshi et al. 2009) construction is as follows

$$\nu_k \sim \text{Beta}(\alpha, 1)$$

(7)

$$\pi_k = \prod_{i=1}^{k} \nu_i$$

(8)

$$B_{d,k} \sim \text{Bern}(\pi_k)$$

(9)

for \(d \in 1, ..., D\) and \(k \in 1, 2, ..., K\), where \(K\) is the truncation level and \(\alpha\) controls the effective value of \(K\), i.e., the number of active nodes for the task. The intuitive explanation is that \(\pi_k\) decreases monotonically with \(k\) (being cumulatively multiplied by \(\nu_k, 0 < \nu_k < 1\)). Thus the (prior) probability of any connection to the \(k\)th node being active shrinks with \(k\), until, say, \(K\) nodes, after which no further nodes have any active incoming edges from the previous layer, which amounts to them being turned off from the structure.

Due to the cumulative product based construction of the \(\pi_k\)’s, an implicit ordering is imposed on the nodes being used. This ordering is preserved across tasks, and allocation of nodes to a task abides by this ordering, thus facilitating reuse of weights.

The truncated stick-breaking approximation is a practically plausible and intuitive solution for continual learning, since a fundamental tenet of continual learning is that the model complexity should not increase in an unbounded manner as more and more tasks are encountered. Suppose we fix a budget on the maximum allowed size of the network (say, the number of hidden nodes allowed in each layer) after it has seen, say, \(T\) tasks. This exactly corresponds to the truncation level for each layer. Then for each task, nodes are allocated conservatively from this total budget, in a fixed order, conveniently controlled by the \(\alpha\) hyperparameter.

3.2 Inference

Exact inference is intractable in this model due to non-conjugacy. Therefore, we resort to variational inference. We employ structured mean-field approximation (Hoffman and Blei 2015), which performs better than normally used mean-field approximation, as the former captures the dependencies in the approximate posterior distributions of \(B\) and \(\nu\). We use a variational posterior family of the following form

$$q(V, B, \nu) = q(V)q(B|\nu)q(\nu)$$

(10)

where, \(q(V) = \prod_{d=1}^{D} \prod_{k=1}^{K} \mathcal{N}(V_{d,k}|\mu_{d,k}, \sigma_{d,k}^2)\) is a mean field Gaussian approximation for the weights. Corresponding to the Beta-Bernoulli hierarchy of (7), we use the conditionally factorized variational posterior family, that is, \(q(B|\nu) = \prod_{d=1}^{D} \prod_{k=1}^{K} \text{Bern}(B_{d,k}|\theta_{d,k})\), where \(\theta_{d,k} = \sigma(\rho_{d,k} + \logit(\pi_k))\) and \(q(\nu) = \prod_{k=1}^{K} \text{Beta}(\nu_{k,1}, \nu_{k,2})\)

Thus we have \(\Theta = \{\nu_{k,1}, \nu_{k,2}, \{\mu_{d,k}, \sigma_{d,k}, \rho_{d,k}\}_{d=1}^{D}\}_{k=1}^{K}\) as the complete set of learnable variational parameters for our model.

Each column of \(B\) represents the binary mask for the weights incident to a particular node. Note that although these binary variables (in a single column of \(B\)) share a common prior, the posterior for each of these variables is different, thereby allowing a task to selectively choose a subset of the weights leading to an activation, with the common prior controlling the degree of sparsity.

As described in section 2, Bayes-by-backprop is an obvious candidate for inference in this context. The Evidence Lower Bound (ELBO) may be expressed as in terms of data-dependent likelihood and data-independent KL terms as

$$\mathcal{L} = \mathbb{E}_{q(V, B, \nu)}[\ln p(Y|V, B, \nu)] - KL(q(V, B, \nu)||p(V, B, \nu))$$

(11)

Using the factorization of the joint prior as \(p(V, B, \nu) = p(V)p(B|\nu)p(\nu)\) and the mean-field factorization of the posterior, the KL divergence term of (11) can be decomposed into

$$KL[q(V)||p(V)] + \mathbb{E}_{q(\nu)}[KL[q(B|\nu)||p(B|\nu)] + KL[q(\nu)||p(\nu)]$$

(12)
All the KL divergence terms in the above expression have closed form expressions; hence using them directly rather than estimating them from Monte Carlo samples alleviates the approximation error as well as the computational overhead due to sampling, to some extent. The expectation terms are optimized by unbiased gradients from the respective posteriors. Analogous to (1) we thus have

$$\mathcal{L} = \frac{1}{S} \sum_{i=1}^{S} \left[ f_i(V^i, B^i, \nu^i) - KL[q(B|\nu^i)||p(B|\nu^i)] - KL[q(V)||p(V)] - KL[q(\nu)||p(\nu)] \right]$$  \hspace{1cm} (13)

The log-likelihood term is decomposed as

$$f_i(V, B, \nu) = \log \text{Lik}(Y|V, B, \nu)$$

$$= \log \text{Lik}(Y|W_{\text{out}}\phi_{NN}(X; V, B, \nu))$$

where \((X, Y)\) is the training data. For regression, Lik can be a Normal distribution with some noise variance, while for classification it can be Bernoulli with a probit or logistic link function.

**Sampling details** We obtain unbiased reparameterized gradients for all the parameters of the variational posterior distributions. For the Bernoulli distributed variables, we employ the Gumbel-softmax trick (Jang et al. 2017), also known as CONCRETE (Maddison et al. 2017). For Beta distributed \(\nu\)'s, the Implicit Reparameterization Gradient technique (Figurnov et al. 2018) is used. For the real-valued weights, the standard location-scale trick of Gaussians is used.

### 4 Other Key Considerations

The model described in the previous section is almost completely specified for a single task. As such, it is by itself interesting as a Bayesian neural net model that adapts its architecture, driven by the task at hand. However, as evidenced by recent works (see Section 5), it is the setting of continual learning that can benefit tremendously from such ability to learn model structure. In this section, we therefore focus on applying the model in this setting.

#### 4.1 Prior Masking

To reduce the catastrophic forgetting while learning the tasks sequentially, using the unlearnt variance (i.e the weights which are not used till now) in calculation of KL divergence hinders the learning and promotes forgetting. This is because, while training, these weights were updated partially according to previous tasks but not used in the end.

To overcome this issue, we mask the new prior for next task with initial prior as follows:

$$p_t(V_{d,k}) = B_{d,k}^{\sigma} q_{t-1}(V_{d,k}) + (1 - B_{d,k}^{\sigma}) p_0(V_{d,k})$$  \hspace{1cm} (15)

where \(B^{\sigma}\) is the overlapped mask of all previously learnt tasks i.e \((B_1 \cup B_2 \ldots \cup B^{t-1})\), \(q_{t-1}, p_t\) are the previous posterior and current prior, respectively, and \(p_0\) is the prior used for first task. This makes sense as the partially trained weights will cause undesirable regularization for next task as it does not help retaining the previous tasks performance.

Standard choice of the initial prior \(p_0\), such as a zero mean normal distribution, significantly reduces the catastrophic forgetting by promoting the use of new weights or weights with higher variance in previously learnt tasks.

#### 4.2 Segregating the head

It has been shown in prior work (Zeno et al. 2018) that using separate last layers (commonly referred to as “heads”) for different tasks dramatically improves performance in continual learning. The individual tasks in continual learning are typically multiclass classification problems, with each task classifying between 2 classes (see section 6). Thus we have a generalized linear model that uses the embeddings from the last hidden layer,
Algorithm 1: Nonparametric Bayesian CL

Input: Initial Prior \( p_0(\Theta) \)

Initialize the network parameters and coresets
\( p_{new} \leftarrow p_0(\Theta) \)

for \( t = 1:T \) do

Observe current task data \( D_t \);

Update coresets \( \text{(Nguyen et al. 2018)} \);

Masked Training:
\( \mathcal{L}_t \leftarrow \text{ELBO (eq. 11) with prior } p_{new}; \)
\( \Theta_t \leftarrow \arg \min \mathcal{L}_t; \)

Selective Finetuning:
Fix the IBP parameters and learned mask;
\( \Theta_t \leftarrow \arg \min \mathcal{L}_t; \)
\( p_{new} \leftarrow q_t(\Theta); \)
\( p_{new} \leftarrow \text{Mask}(p_{new}) \text{ using eq } 15; \)

Perform prediction for given test set.

end

with the parameters upto the last layer involved in transfer and adaptation. Contrary to this is the situation where the difference between the tasks arises solely due to difference in the input distributions, with the desired outputs being the same across tasks, e.g. the Permuted MNIST experiment. In such cases, the head must be same for all tasks.

4.3 Prediction-driven training with coresets

Proposed in \( \text{(Nguyen et al. 2018)} \) as a method for cleverly sidestepping the issue of catastrophic forgetting, the coreset comprises representative training data samples from all tasks. Let \( M^{(t-1)} \) denote the posterior state of the model before learning task \( t \). With the \( t \)-th task’s arrival having data \( D_t \), a coreset \( C_t \) is created comprising choicest examples from tasks \( 1 \ldots t \). Using data \( D_t \setminus C_t \) and having prior \( M^{(t-1)} \), new model posterior \( M^t \) is learnt. For predictive purposes at this stage (the test data comes from tasks \( 1 \ldots t \)), a new posterior \( M^t_{\text{pred}} \) is learnt with \( M^t \) as prior and with data \( C_t \). Note that \( M^t_{\text{pred}} \) is used only for predictions at this stage, and does not have any role in the subsequent learning of, say, \( M^{(t+1)} \). Such a predictive model is learnt after every new task, and discarded thereafter. Intuitively it masks sense as some new learnt weights for future tasks can help the older task to perform better (backward transfer) at testing time.

For our model, during this coreset-based training phase after task \( t \), only the weights are updated for the tasks \( 1, \ldots, t-1 \) with (and using) the binary mask fixed at its previously learned value, i.e., a task refines only its own subset of weights.

4.4 The IBP hyperparameter \( \alpha \)

Although we found using a sufficiently large value of \( \alpha \) without tuning to perform reasonably, several considerations can be made regarding its choice.

Scheduling across tasks \ Intuitively, \( \alpha \) should be incremented for every new task according to some schedule. Information about task relatedness can play a decisive role in formulating the schedule. Smaller increments of \( \alpha \) discourage creation of new nodes and encourage more sharing of already existing connections across tasks.

Learning \( \alpha \) \ Although not investigated in this work, one viable alternative to choosing \( \alpha \) by cross-validation could be to learn it. This can be easily accommodated into our variational framework by imposing a gamma prior on \( \alpha \) and using a suitably parameterized gamma variational posterior. The only difference in the objective would be in the KL terms: the KL divergence of \( \nu \) will then also have to estimated by Monte Carlo approximation (because of dependency on \( \alpha \) in the prior). Also, since gamma distribution does not have an analytic closed form KL divergence, the Weibull distribution can be a suitable alternative \( \text{(Zhang et al. 2018)} \).
4.5 Other Practical issues

**Space complexity** The proposed scheme entails storing a binary matrix for each layer of each task which results into 1 bit per weight parameter, which is not very prohibitive. Apart from the fact that mature technologies exist to compress and manipulate sparse matrices, it must be noted that the initial tasks make use of only a limited number of the first few columns of the weight vector, and hence does not pose any significant overhead.

**Adjusting bias terms** The IBP selection acts on the weight matrix only. For the hidden nodes not selected in a task, their corresponding biases need to be removed as well. In principle, the bias vector for a hidden layer should be multiplied by the binary vector $u$, with $u_i = I[\exists d : B_{d,i} = 1]$. In practice, we simply scale each bias component by the maximum reparameterized Bernoulli value in that column.

**Selective Finetuning** While training with reparameterization (Gumbel-softmax), the learnt masks are close to binary but not completely binary which affects task performance a bit. So we fine-tune the network with fixed structure (i.e Beta-Bernoulli distributions parameters are fixed) after it has been learned, to restore the accuracy of the task. Algorithm 1 summarizes our models training procedure. The method for update of coresets that we used are similar to as it was proposed in (Nguyen et al. 2018)

5 Related Work

One of the most important challenges in continual learning (henceforth referred to as CL) is to prevent catastrophic forgetting, which is typically addressed through regularization of the parameter updates, preventing them from drastically changing from the value learnt from the previous task(s). Notable methods based on this strategy include Elastic Weight Consolidation (EwC) (Kirkpatrick et al. 2017), Synaptic Intelligence (SI) (Zenke et al. 2017), Laplace approximation (Smola et al. 2003), etc. Superceding these methods is the Bayesian approach, a natural remedy of catastrophic forgetting in that, for any task, the posterior of the model learnt from the previous task serves as the prior for the current task, which is the canonical online Bayes. This approach is utilized in a number of works like variational continual learning (VCL)(Nguyen et al. 2018) and task agnostic variational Bayes(Zeno et al. 2018) for learning Bayesian neural networks in the CL setting. Our work is most similar in spirit to and builds upon this body of work.

Another key aspect in CL methods is replay, where some samples from previous tasks (selected randomly or by some heuristic) are used to fine-tune the model after learning a new task (thus refreshing its memory in some sense and avoiding catastrophic forgetting). Some of the works using this idea include (Lopez-Paz et al. 2017), which solves a constrained optimization problem at each task, the constraint being that the loss should decrease monotonically on a heuristically selected replay buffer; (Hu et al. 2018), which uses a partially shared parameter space for inter-task transfer and generates the replay samples through a data-generative module; and (Titsias et al. 2019), which learns a Gaussian process for each task, with a shared mean function in the form a feedforward neural network, the replay buffer being the set of inducing points typically used to speed up GP inference. For VCL(Nguyen et al. 2018) and our work, the coreset (section 4.3) serves as a replay buffer; but we emphasize that it is not the primary mechanism to overcome catastrophic forgetting in these cases, but rather an additional mechanism to preventing it.

Recent work in CL has investigated allowing the structure of the model to dynamically change with newly arriving tasks. Among these, strong evidence in support of our assumptions can be found in (Golkar et al. 2019), which also learns different sparse subsets of the weights of each layer of the network for different tasks. The sparsity is enforced by a combination of weighted $L_1$ regularization and threshold-based pruning. There are also methods that do not learn subset of weights to be used but rather learn the subset of hidden layer node outputs to be used for each task; such a strategy is adopted by either using Evolutionary Algorithms to select the node subsets (Fernando et al. 2017) or by training the network with task embedding based attention masks (Serrà et al. 2018).

Some recently proposed non-Bayesian models (Li et al. 2019, Yoon et al. 2018) either reuse the parameters of a layer, dynamically grows the size of the hidden layer, or adapt them, or spawn a new set of parameters (the model complexity being bounded through regularization terms). In comparison, our model already has a fixed budget in advance, which is motivated by the desideratum of CL that the model complexity should not increase in an unbounded manner as more and more tasks are encountered.
6 Experiments

We evaluate our model on similar benchmarks as VCL (Nguyen et al. 2018). We experimented with different existing approaches such as, Pure Rehearsal (Robins 1995), EwC (Kirkpatrick et al. 2017), IMM (Lee et al. 2017) etc. We use Tensorflow (Abadi et al. 2015) for optimizing the variational lower bound through automatic differentiation; particularly, the gradients of log-gamma and digamma functions, and also the implicit reparameterized gradients of the Beta distribution are used as made available.

We perform our evaluations on three CL benchmark datasets: SplitMNIST, Split notMNIST(small)\(^2\) and Permutated MNIST. For Split MNIST, the tasks are 5 binary classifications, the splits being 0/1, 2/3, 4/5, 6/7, 8/9. Similarly for Split notMNIST, we have the 5 binary classification splits as A/B, C/D, E/F, G/H, I/J. For Permutated MNIST, each task is a multiclass classification problem. However, for each task, a fixed random permutation is applied to the pixels of the images of all classes. We generated 5 such tasks for our experiments. For SplitMNIST and notMNIST, the heads (section 4.2) are separate for different tasks, while for permuted MNIST, the head is shared.

6.1 Performance evaluation

Consider we have \(T\) tasks arriving sequentially. To gauge the effectiveness of our model towards preventing catastrophic forgetting, we report (following standard protocol) (i) the test accuracy of task \(t\) after learning each of its subsequent tasks \((t, t + 1, t + 2, \ldots T)\); and (ii) the average test accuracy over all previous tasks \(1, 2, \ldots t\) after learning each task \(t\).

We use a feedforward network (having ReLU activations) with a single hidden layer having total budget of 200. For fairness of comparison, we use the same architecture for each of the baselines that we are comparing with. We also report results on some additional CL metrics Díaz-Rodríguez et al. 2018 in the supplementary material.

| Method  | mnist | notMnist | permuted |
|---------|-------|----------|----------|
| Naïve   | 79.615| 72.339   | 94.490   |
| Rehearsal | 99.102| 95.203   | 97.565   |
| EwC     | 78.530| 90.297   | 95.392   |
| IMM (mode) | 92.206| 84.442   | 96.433   |
| VCL     | 99.002| 93.732   | 97.353   |
| VCL(coreset) | 98.731| 94.993   | 97.464   |
| Ours    | 99.793| 97.06    | 97.731   |
| Ours(coreset) | 99.807| 95.901   | 97.897   |

Table 1: Comparison of final mean accuracies obtained using different approaches

Hyperparameter settings We adopted Adam optimizer for our model keeping a learning rate of 0.01 for normal and 0.001 for selective training for all tasks. The temperature hyperparameter of the Gumbel-softmax reparameterization for Bernoulli was annealed from 10.0 to a minimum limit of 0.25. The value of \(\alpha\) was initialized at 80.0 for all tasks. Similar to VCL, we initialized our models with maximum-likelihood training for first task. For all datasets, our models were trained for 5 epochs. Selective Finetuning was run for 5 epochs as well. Coreset size used was 100 for MNIST and notMNIST, and 200 for PermutatedMNIST. Coresets were selected using random and k-center methods (Nguyen et al. 2018).

\(^2\)Available at http://yaroslavvb.com/upload/notMNIST/

Figure 2: Mean test accuracies of tasks seen so far as new tasks are observed, for (a)splitMNIST, (b)notMNIST and (c)Permutated MNIST, here (right) side plots are scaled version of (left) ones for better insight.
Quantitative Results: Fig. 2 shows the mean test accuracies on splitMNIST, notMNIST and permuted MNIST as new tasks are observed, and Table 1 shows the final mean accuracy. As Fig. 2 shows, the average test accuracy of our method (without as well as with coresets) is better than VCL (here, we have used random point selection method for coresets). Moreover, the accuracy drops much more slowly than VCL on both datasets, which shows that our method suffers from catastrophic forgetting much less than VCL.

We observe performance gains on all 3 benchmark datasets. The notMNIST and permuted MNIST tasks are widely considered to be more challenging than MNIST. In Fig. 3, we show the accuracy on individual tasks (for tasks 1-4) as newer tasks arrive. In this case too, we observe that our method does yield similar individual task accuracies and has much less forgetting as compared to VCL and other baselines. Also, as with VCL, using coreset was found to improve performance a bit. We also note that some of the old tasks’ accuracies increases with training of new tasks which shows the presence of backward transfer, which is another desideratum of CL.

One further observation is that, for VCL, while the individual test accuracies improve with more training on each task, the overall performance across all tasks drops gradually, possibly since more training on a single task adapts the model more specifically for that task, leading to forgetting of the previous tasks. Our model, on the other hand, was found to be immune to overtraining, since each task learns its own sparse subset of parameters.
6.2 Structural observations

An important aspect of our work is that the results reported above, which are competitive with the state of the art, are achieved with a very sparse neural network structures learnt by the model, which we analyze qualitatively here.

As expected by proposed model (and as shown in the Fig. 4(a-e)) the IBP prior enforces the weights to be concentrated mainly on the first few nodes, and the structure becomes sparser with every epoch. For both notMNIST and Permuted MNIST datasets, a maximum of around 40% incoming connections are used at most for the first task.

Another important observation (as shown in Fig. 5) is the percentage of weights that are being shared between different tasks and how the number of active weights vary across different tasks. Qualitatively, it appears that most newer tasks tend to allocate fewer weights and yet perform well, implying effective forward transfer. One can also easily observe that the weight sharing between similar tasks like those in splitMNIST is a much higher than that of non-similar tasks such as permuted MNIST. This seemingly leads to the hypothesis that a single binary mask common for all tasks is sufficient. Experimental observations, however, dispel such a belief by showing drastic degradation in performance.

We therefore conclude that although a new task tend to share weights learnt by old tasks, the new connections that it creates are indispensable for its performance. Intuitively, the more unrelated a task is to previously seen ones, the more new connections it will make, thus reducing negative transfer (an unrelated task adversely affecting other tasks) between tasks.

7 Conclusion

We have successfully unified structure learning in neural networks with their variational inference in the setting of continual learning, demonstrating competitive performance with the state-of-the-art model. Here we have concentrated on discriminative models; it will be interesting to investigate how similar ideas can be applied to deep generative models, as was done in (Nguyen et al. 2018). It may also be interesting to extend this idea to more complicated network architectures such as convolutional or residual networks, possibly by simplifying the inference via small-variance asymptotics, leading to principled thresholding schemes for pruning the structure. Adapting other sparse Bayesian structure learning methods, e.g. (Ghosh et al. 2018) to the continual learning setting is also a promising avenue. Incorporating task similarity information, if available (e.g. to tune α for different tasks), and conversely inferring it if not (using the extent of overlap in structure between tasks) is also another aspect that remains to be explored. Adapting the depth of the network is a more challenging endeavour that might also be undertaken. We leave these extensions for future work.

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A Additional Experimental Results on Permuted MNIST

We quantified and observed the forward and backward transfer of our and VCL model, using the three metrics given in (Díaz-Rodríguez et al. 2018) on Permuted MNIST dataset as follows:

A.1 Accuracy

It’s defined as the over all model performance averaging over all the task pairs as follows:

$$\text{Acc} = \frac{\sum_{i \geq j} R_{i,j}}{N(N-1)/2}$$

where, $R_{i,j}$ is the obtained test classification accuracy of the model on task $t_j$ after observing the last sample from task $t_i$.

A.2 Forward Transfer

The ability of previously learnt task to perform on new task better is termed as forward transfer and can be represented as:

$$\text{FWT} = \frac{\sum_{i<j} R_{i,j}}{N(N-1)/2}$$

A.3 Backward Transfer

The ability of newly learned task to affect the performance of previous tasks is termed as backward transfer. It can be defined as:

$$\text{BWT} = \frac{\sum_{i=2}^{N-1} \sum_{j=1}^{i-1} (R_{i,j} - R_{j,j})}{N(N-1)/2}$$

We compare VCL model and Our model over these three metrics.

| Method   | Acc  | FWT  | BWT  |
|----------|------|------|------|
| Naive    | 93.866 | 0.1  | -0.036 |
| EwC      | 95.293 | 0.1  | -0.018 |
| Rehearsal| 97.402 | 0.1  | 0.0013 |
| VCL      | 97.392 | 0.1  | -0.004 |
| Ours     | 97.812 | 0.1  | 0.0012 |

Table 2: Comaprrison on other metrics

The Forward transfer in this case almost same for all models since the task does not pose any kind of similarity between them and so without training the accuracy remains to that of a random output model.

On the other hand we can see that our models, backward transfer is almost as good as Pure Rehearsal based model and is the best among all in terms of accuracy.