Combinatorial Properties of Mills’ Ratio.

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Abstract

We consider combinatorial properties of the Mills’ ratio, $R(t) = \int_t^\infty \phi(x) \frac{dx}{\phi(t)}$, where $\phi(t)$ is the standard normal density. We explore the interplay between a continued fraction expansion for the Mills’ ratio, the Laplace polynomials and a new family of combinatorial identities.

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1 Introduction

Combinatorial identities often have striking relations to the special functions. Let us consider the following two seemingly unrelated identities,

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{n} \frac{(n+m)!}{m!j!(m+2n+1-j)!} 2^{-n} = \sqrt{2\pi} e^2 (\Phi(2) - \Phi(1)), \quad (1)
\]

and

\[
\sum_{k=0}^{n} (-1)^k \frac{1}{2k+1} \binom{n}{k} = \frac{2^{2n} (n!)^2}{(2n+1)!}. \quad (2)
\]
where $\Phi(\cdot)$ is the standard Normal distribution function. In this paper we show that these identities actually have common combinatorial nature linked to the continued fraction for the Mills’ ratio,

$$R(t) = \frac{\bar{\Phi}(t)}{\varphi(t)},$$

where $\varphi(t)$ is the standard Normal density and $\bar{\Phi}(t) = \int_t^\infty \varphi(s) \, ds$ is the tail of the Normal distribution. The function $R(t)$ plays an important role in Probability Theory, Statistics, Stochastic Analysis and many applied areas including Queueing theory, Reliability and Mathematical Finance.

Despite that the function $R(t)$ was named after John Mills, who tabulated its values, the first known statements about $R(t)$ appeared, in fact, more than 200 years ago by Laplace who found the asymptotic expansion

$$R(t) \sim \frac{1}{t} - \frac{1}{t^3} + \frac{1 \cdot 3}{t^5} - \frac{1 \cdot 3 \cdot 5}{t^7} + \ldots$$

for $t > 0$. (3)

and the continued fraction expansion,

$$R(t) = \frac{1}{t + \frac{1}{t + \frac{2}{t + \frac{3}{t + \frac{4}{\ldots}}}}}. \quad (4)$$

Laplace also found the rational approximations, $R(t) \sim Q_{k-1}(t)/P_k(t)$ for the function $R(t)$, where $P_k(t)$ and $Q_k(t)$ are polynomials of degree $k$ and wrote down these polynomials for $k \leq 4$. Thirty years later, Jacobi gave a very short (and rigorous) derivation of the expansion (3). Jacobi also found recurrent equations for the polynomials in the rational approximation.

Since then, numerous papers and monographs discussing different properties of the Mill’s ratio were published. A modified proof of asymptotic (3) is discussed in [6]. More accurate asymptotic expansions for the Mill’s ratio were obtained in [15] and in [16]. The latter paper contains also a good account of the properties of the tail of a normal distribution.

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1 The first identity is new, to the best of our knowledge. The second identity is known (see [14]).

2 Examples can be found in [6, 11, 1, 8].
Recently, a few interesting papers on continued fractions and general approximation schemes for the Mills ratio were published. We mention here the papers [13], [3], [9] and [5] analyzing monotonicity properties of some functions involving $R(t)$. Based on these properties a series of irrational approximations for $R(t)$ were derived in [9] and [5].

In this paper, we discuss combinatorial identities connected to the continuous fractions for the Mills ratio. In Section 2 we introduce Laplace polynomials whose ratio form the continuous fractions for the Mills ratio and study their properties. To make this paper self-contained, we re-derive in Section 2 the rational approximations for the function $R(t)$.

Despite that the recurrent relations for the polynomials $P_k$ and $Q_k$ have been known for a very long time, their coefficients were not studied until recently. The first analysis of these coefficients, published in [9], appeared only in 2006, to the best of our knowledge.

It turned out that the properties of the polynomials $P_k$ closely resemble those of the Hermite polynomials (see also [2]). The combinatorial structure of the coefficients of the polynomials $Q_k$ is more complex. We find rather simple formula than the one in [9] in Section 3.

In Section 4 the double generating functions of these polynomials is computed. From the double generating function we find a series of combinatorial identities that includes (1) and (2). In Section 5 we discuss Laplace polynomials connections to the Hermite polynomials. The paper is closed with the relation between the Laplace polynomials and matching numbers in Section 6.

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2 Polynomials $P_k(t), Q_k(t)$ and inequalities for $R(t)$

In this Section, we derive a series of inequalities for the Mills ratio expressed in terms of the Laplace polynomials, introduced below.
Our approach is based on the following simple idea. The function \( R(t) \) can be represented as a Laplace transform of a non-negative function on the positive semi-axis. According to the classical theorem\(^3\) proved by S. N. Bernstein [4], the Laplace transform, in this case, is a completely monotone function and, therefore, satisfies an infinite sequence of alternating inequalities for the derivatives of \( R(t) \). Suppose that \( R(t) \) satisfies the differential equation

\[
R'(t) = \alpha(t)R(t) + \beta(t).
\]

Then using the complete monotonicity property and relations derived from the differential equation one can find an infinite sequence of inequalities for the function \( R(t) \) expressed through the functions \( \alpha(t), \beta(t) \) and their derivatives.

Using this idea we obtain in Section 2 a sequence of “self-improving” inequalities

\[
\frac{Q_{k-2}(t)}{P_{k-1}(t)} \leq R(t) \leq \frac{Q_{k-1}(t)}{P_k(t)}, \quad k = 2, 4, 6, \ldots
\]

and show that \( Q_{k-1}(t)/P_k(t) \) represents the rational approximation of the continued fraction \( [4] \).

**Lemma 1.** \( R(t) \) is a completely monotone function.

**Proof.** We have for \( t \geq 0 \)

\[
\int_0^\infty e^{-tx} e^{-x^2/2} \, dx = \sqrt{2\pi} e^{t^2/2} \int_0^\infty \frac{e^{-(x+t)^2/2}}{\sqrt{2\pi}} \, dx = \Phi(t) = R(t).
\]

The statement of the lemma now follows from the Bernstein’s theorem (see [4]). \( \square \)

It follows from Lemma [4] that \( R(t) \) is infinitely differentiable on the half-line, \([0, \infty)\), and satisfies the inequality

\[
(-1)^n \frac{d^n R(t)}{dt^n} \geq 0, \quad n = 1, 2, \ldots, t \geq 0.
\] \(5\)

---

\(^3\)In fact, we use only a trivial part of this theorem: if a function is represented as a Laplace transform of a non-negative function of positive semi-axis, then it is completely monotone.
It is not difficult to verify that the function $R(t)$ satisfies the differential equation
\[ \frac{dR(t)}{dt} = t \cdot R(t) - 1. \] 
Since the function $R(t)$ is completely monotone we obtain from \([5]\) and \([6]\) for $t > 0$
\[ R(t) \leq \frac{1}{t}. \]
The latter inequality is equivalent to
\[ \Phi(t) \leq \frac{\varphi(t)}{t}, \quad t > 0. \] (7)
Differentiating $R(t)$ twice and using \([6]\) we obtain for $t > 0$
\[ \frac{d^2R(t)}{dt^2} = R(t) \left(t^2 + 1\right) - t. \]
Then from \([5]\), $n = 2$ we derive
\[ \Phi(t) > \frac{\varphi(t)}{t + t^{-1}}, \]
and together with \([7]\) delivers the well-known asymptotic relation
\[ \Phi(t) \sim \varphi(t) \cdot t^{-1} \quad \text{as } t \to +\infty. \]

Let us now consider the derivatives of the Mill’s ratio. From \([6]\) we find
\[ \frac{d^kR(t)}{dt^k} = t \cdot \frac{d^{k-1}R(t)}{dt^{k-1}} + (k - 1) \cdot \frac{d^{k-2}R(t)}{dt^{k-2}}, \quad k = 1, 2, \ldots. \] (8)
It follows from \([6]\) that the latter equation can be written
\[ \frac{d^kR(t)}{dt^k} = R(t) \cdot P_k(t) - Q_{k-1}(t), \quad k = 1, 2, \ldots, \] (9)
where $P_k(t)$ and $Q_k(t)$ are polynomials of degree $k$. We shall call $P_k(t)$ and $Q_k(t)$ the Laplace polynomials in what follows. The Laplace polynomials satisfy the following recurrent equations
\[ P_{k+1}(t) = tP_k(t) + P_k'(t), \quad (10) \]
\[ Q_k(t) = P_k(t) + Q_{k-1}(t), \quad (11) \]
where $P_0(t) = Q_0(t) = 1$. Using Equations \([10]\) and \([11]\) one can find $P_k(t)$ and $Q_k(t)$ for any integer $k$ (see Table 1).
Lemma 2. The Mill’s ratio satisfies the inequalities

\[ \frac{Q_{k-2}(t)}{P_{k-1}(t)} \leq R(t) \leq \frac{Q_{k-1}(t)}{P_k(t)}, \quad k = 2, 4, 6, \ldots \quad (12) \]

Proof. Inequalities (12) follow from the complete monotonicity of the function \( R(t) \), and Equations (5) and (9).

The following statement on the Laplace polynomials is known for a very long time.

Lemma 3 (Jacobi, Pinelis, Kouba). The polynomials \( P_k(t) \) and \( Q_k(t) \) satisfy the relation

\[
P_{k+1}(t) = tP_k(t) + kP_{k-1}(t),
\]
\[
Q_{k+1}(t) = tQ_k(t) + (k + 1)Q_{k-1}(t)
\]

Proof. This Lemma is proved by induction.

Lemma 3 immediately implies the continued fraction representation (1). Indeed,

\[ Q_k(t) = tQ_{k-1}(t) + kQ_{k-2}(t). \]

Then from the Stiltjes property of the continued fractions (see [2], Lemma 5.5.2, pg. 256) we find

\[
\frac{Q_{k-1}(t)}{P_k(t)} = (t + 1 \cdot (t + 2 \left( t + \ldots (t + (k-1)t^{-1} \ldots)^{-1} \right)^{-1})^{-1})^{-1}.
\]

Lemma 3 also implies that the following sequences are monotone:

\[ \frac{Q_{2n-1}(t)}{P_{2n}(t)} \text{ is increasing and } \frac{Q_{2n}(t)}{P_{2n+1}(t)} \text{ is decreasing for } n = 1, 2, \ldots, t > 0. \]

In the next section we will find explicit formulae for their coefficients.

3 Coefficients of Laplace polynomials

Let us denote by \( p_{k,m} \) \( m = 0, 1, \ldots \), the coefficients of the polynomial \( P_k(t) \) and by \( q_{k,m} \) the coefficients of \( Q_k(t) \). For the sake of convenience, we introduce the polynomial \( P_0(t) = 1. \)
Theorem 4. Denote \( n = \frac{k-m}{2} \). The coefficients \( p_{k,m} \) and \( q_{k,m} \) satisfy the following relations. If \( m > k \) or \( k - m \equiv 1 \pmod{2} \) then

\[
p_{k,m} = q_{k,m} = 0 \quad k, m = 0, 1, \ldots
\]  

(13)

If \( k \equiv m \pmod{2} \) and \( k \geq m \) then

\[
p_{k,m} = \frac{k!}{m! \cdot 2^n \cdot n!}
\]

(14)

and

\[
q_{k,m} = \frac{(k+m)!!}{m!} \cdot 2^{-n} \sum_{j=0}^{n} \binom{k+1}{j}
\]

(15)

Proof. At first, we establish Equations (13) and (14). We have

\[
P_k(t) = \sum_{m=0}^{k} p_{k,m} t^m, \quad k = 0, 1, \ldots
\]

\[
Q_k(t) = \sum_{m=0}^{k} q_{k,m} t^m, \quad k = 0, 1, \ldots
\]

From (10) and (11) we find that for \( k = 1, 2, \ldots \) and \( m = 1, 2, \ldots, k \)

\[
p_{k+1,m} = p_{k,m-1} + (m+1) \cdot p_{k,m+1},
\]

(16)

\[
q_{k,m} = p_{k,m} + (m+1) \cdot q_{k-1,m+1}.
\]

(17)

If \( m = 0 \) then Formula (16) is understood as

\[
p_{k+1,0} = p_{k,1}, \quad k = 2, 4, \ldots
\]

(18)

Probably, the most convenient way to find the general formula for the coefficient \( p_{k,m} \) is to look at the diagonals \( k - m = \text{const} \) of the matrix \( p \). The following lemma proves the statements of Theorem 4 related to the coefficients \( p_{k,m} \).

Lemma 5. If \( k - m \equiv 1 \pmod{2} \), the coefficients \( p_{k,m} = 0 \). If \( k - m = 2n, n \in \mathbb{Z} \),

\[
p_{k,m} = \frac{k!}{m! \cdot 2^n \cdot n!}
\]

(19)
| \( k \) | \( P_k(t) \) | \( Q_{k-1}(t) \) |
|---|---|---|
| 1 | \( t \) | 1 |
| 2 | \( t^2 + 1 \) | \( t \) |
| 3 | \( t^3 + 3t \) | \( t^2 + 2 \) |
| 4 | \( t^4 + 6t^2 + 3 \) | \( t^4 + 5t \) |
| 5 | \( t^5 + 10t^3 + 15t \) | \( t^5 + 9t^2 + 8 \) |
| 6 | \( t^6 + 15t^4 + 45t^2 + 15 \) | \( t^6 + 14t^3 + 33t \) |
| 7 | \( t^7 + 21t^5 + 105t^3 + 105t \) | \( t^7 + 20t^4 + 87t^2 + 48 \) |
| 8 | \( t^8 + 28t^6 + 210t^4 + 420t^2 + 105 \) | \( t^8 + 27t^6 + 185t^4 + 279t \) |

Table 1: Laplace polynomials \( P_k(t) \) and \( Q_{k-1}(t) \).

| \( k \) | \( m \) |
|---|---|
| 0 | 1 1 0 0 ... 0 |
| 1 | 0 1 0 ... 0 |
| 2 | 1 0 1 0 ... 0 |
| 3 | 0 3 0 1 0 ... 0 |
| 4 | 3 0 6 0 1 0 ... 0 |
| 5 | 0 15 0 10 0 1 0 ... 0 |
| 6 | 15 0 45 0 15 0 1 0 0 |
| 7 | 0 105 0 105 0 21 0 1 0 |
| 8 | 105 0 420 0 210 0 28 0 1 |

Table 2: Matrix \( p = \|p_{k,m}\| \).
Proof. Note that from the relation $p_{1,0} = 0$ and Equation (16) it follows that $p_{k,m} = 0$ if $k - m$ is odd. Consider the case $k - m$ is an even number. The proof is carried out by double induction along the even diagonals of the matrix $p$ (see Table 2). If $m = k$ then, obviously, $p_{k,m} = 1$. Consider the diagonal $k - m = 2$. If $k = 3$ then $p_{3,1} = 3$. Suppose

$$p_{k,k-2} = \binom{k}{2} \quad \text{for } k = 3, 4, \ldots, K - 1.$$ 

Then, using recurrent relation (16), we obtain

$$\left(\binom{K-1}{2}\right) + (K-1) \cdot 1 = \binom{K}{2}.$$ 

Thus, we proved (19) for $k - m = 2$.

Let us now prove (19) in the general case. Suppose we already verified this relation for $k - m = 2n$, $n = 0, 1, \ldots, N$. We shall prove (19) for $k - m = 2(N + 1)$. From (18) we obtain

$$p_{2N+2,0} = p_{2N+1,1} = \frac{(2N + 1)!}{2^N N!}.$$ 

Taking into account the relation

$$\frac{(2N + 1)!}{2^N N!} = \frac{(2N + 2)!}{2^{N+1} (N + 1)!},$$

we obtain that $p_{2N+2,0}$ satisfies (19); the induction base is verified.

Suppose for $l = 0, 1, \ldots, L - 1$

$$p_{2N+2+l,l} = \frac{(2N + 2 + l)!}{l! 2^{N+1} (N + 1)!}.$$ 

Then from (16) we find

$$p_{2N+2+L,L} = p_{2N+1+L,L-1} + (L + 1)p_{2N+1+L,L+1}.$$ 

The last element belongs to 2Nth diagonal. Therefore

$$p_{2N+2+L,L} = \frac{(2N + 1 + L)!}{(L - 1)! \cdot 2^{N+1}(N + 1)!} + (L + 1) \cdot \frac{(2N + 1 + L)!}{(L + 1)! \cdot 2^{N} N!}.$$ 

Finally, we derive

$$p_{2N+2+L,L} = \frac{(2N + 2 + L)!}{L! \cdot 2^{N+1}(N + 1)!}$$

as was to be proved. $\square$
From (19) one can easily obtain

**Lemma 6.** The polynomials \( P_k(t) \) satisfy the relation

\[
P'_k(t) = kP_{k-1}(t), \quad k = 1, 2, \ldots ,
\]

where \( P_0(t) = 1 \).

**Proof.** Equation (20) follows from the relation

\[
p_{k,m} \cdot m = p_{k-1,m-1} \cdot k, \quad m = 1, 2, \ldots , k,
\]

that follows directly from Lemma 5.

\[
\begin{array}{|c|cccccccc|}
\hline
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
0 & 1 & 0 & 0 & \ldots & & & & & 0 \\
1 & 0 & 1 & 0 & \ldots & & & & & 0 \\
2 & 2 & 0 & 1 & 0 & \ldots & & & & 0 \\
3 & 0 & 5 & 0 & 1 & 0 & \ldots & & & 0 \\
4 & 8 & 0 & 9 & 0 & 1 & 0 & \ldots & 0 \\
5 & 0 & 33 & 0 & 14 & 0 & 1 & 0 & \ldots & 0 \\
6 & 48 & 0 & 87 & 0 & 20 & 0 & 1 & 0 & 0 \\
7 & 0 & 279 & 0 & 185 & 0 & 27 & 0 & 1 & 0 \\
\hline
\end{array}
\]

Table 3: Coefficients \( q_{k,m} \).

Let us now express the coefficients \( q_{k,m} \) through the elements of the matrix \( p \).

**Lemma 7.** The coefficients \( q_{k,m} \) satisfy the relations

\[
q_{k,m} = 0 \quad \text{for } k - m \equiv 1 \pmod{2},
\]

\[
m! \cdot q_{k,m} = \sum_{j=0}^{n} (m+j)! \cdot p_{k-j,m+j}, \quad \text{for } k \equiv m \pmod{2}.
\]

(21)
Proof. Let \( \hat{q}_{k,m} = m! \cdot q_{k,m} \). Then from (17) we obtain

\[
\hat{q}_{k,m} = m! \cdot p_{k,m} + \hat{q}_{k-1,m+1}.
\]

Equation (21) follows from (22).

Now we are in the position to prove Formula (15). From (21) we have

\[
\hat{q}_{k,m} = \sum_{j=0}^{n} \frac{(k - j)!}{2^{n-j} (n - j)!}
\]

\[
\quad = (k - n)! \cdot \sum_{j=0}^{n} \binom{k - j}{n - j} 2^{j-n}
\]

\[
\quad = (k - n)! \cdot \sum_{i=0}^{n} \binom{k - n + i}{i} 2^{-i},
\]

where \( 2n = k - m \). Notice that \( k - n = \frac{k+m}{2} \). Further simplification of the equation for \( \hat{q}_{k,m} \) is based on the Cauchy integral representation for the binomial coefficients

\[
\binom{k}{n} = \frac{1}{2\pi i} \oint_{\gamma} \frac{(1 + z)^k}{z^{n+1}} \, dz,
\]

where \( \gamma \), the contour of integration, is a circle \( \{z : |z| = r_*\} \) of a sufficiently small radius, \( r_* \) (say, \( r_* = 1/2 \)). Then we find

\[
\sum_{i=0}^{n} \binom{k - n + i}{i} 2^{-i} = \sum_{i=0}^{n} 2^{-i} \frac{1}{2\pi i} \oint_{\gamma} \frac{(1 + z)^{k-n+i}}{z^{i+1}} \, dz
\]

\[
\quad = \frac{1}{2\pi i} \oint_{\gamma} \frac{(1 + z)^{m+n}}{z} \cdot \sum_{j=0}^{n} \frac{(1 + z)^j}{2^j} \, dz
\]

\[
\quad = 2 \cdot \frac{1}{2\pi i} \oint_{\gamma} (1 + z)^k \cdot \frac{1 - (\frac{1+z}{2z})^{n+1}}{z - 1} \, dz
\]

\[
\quad = 2 \cdot \frac{1}{2\pi i} \oint_{\gamma} (1 + z)^k \cdot (2z)^{n+1} - (1 + z)^{k+n+1} \, dz
\]

\[
\quad = 2 \cdot \frac{1}{2\pi i} \oint_{\gamma} \frac{(1 + z)^{m+n}}{z - 1} \, dz + 2^{-n} \cdot \frac{1}{2\pi i} \oint_{\gamma} \frac{(1 + z)^{m+2n+1}}{z^{n+1}(1 - z)} \, dz.
\]
The first integral
\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{(1 + z)^{m+n}}{z - 1} \, dz = 0.
\]
The second integral can be computed as follows. The integrand
\[
\frac{(1 + z)^{m+2n+1}}{1 - z} = (1 + z)^{k+1} \sum_{j=0}^{\infty} z^j
\]
\[= \sum_{i=0}^{k+1} \binom{k+1}{i} z^i \cdot \sum_{j=0}^{\infty} z^j
\]
\[= \sum_{l=0}^{\infty} \alpha_l z^l,
\]
where
\[\alpha_l = \sum_{i=0}^{\min(k+1,l)} \binom{k+1}{i}.
\]
Therefore
\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{(1 + z)^{m+2n+1}}{z^{n+1} (1 - z)} \, dz = \alpha_n.
\]
Since \(n < k\), we have
\[\alpha_n = \sum_{i=0}^{n} \binom{k+1}{i}.
\]
Finally, we obtain
\[
\hat{q}_{k,m} = (k - n)! \cdot 2^{-n} \sum_{i=0}^{n} \binom{k+1}{i},
\]
as was to be proved.

Corollary 8.
\[q_{2n,0} = 2^n n! \quad n = 0, 1, 2, \ldots \quad (23)
\]

Proof. In the case \(m = 0, k = 2n\) we have
\[q_{k,m} = \hat{q}_{k,m} = n! \sum_{i=0}^{n} \binom{2n+1}{i}.
\]
Equation (23) then follows from the identity
\[ \sum_{i=0}^{n} \binom{2n + 1}{i} = 2^{2n}. \]
The corollary is thus proved.

4 Generating functions

In this section we compute the double generating functions of the Laplace polynomials. Denote
\[ P(s, t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} p_{k,m} \cdot t^m s^k. \] (24)

Lemma 9. The series (24) converges for all complex numbers \( t \) and \( s \) such that \( |t| < \infty \) and \( |s| < \infty \). The generating function, \( P(s, t) \), is
\[ P(s, t) = \exp \left( st + \frac{s^2}{2} \right). \] (25)

Proof. Equation (25) is known (see [9]). We shall prove (25) for the sake of completeness. The series
\[ G_m(s) = \sum_{k=0}^{\infty} p_{k,m} \frac{s^k}{k!} \]
converges for all \( m \in \mathbb{Z}_+ \). Denote \( n = (k - m)/2 \). We have
\[ P(s, t) = \sum_{m=0}^{\infty} G_m(t) \cdot t^m \]
and the latter series converges for all $t, \ |s| < \infty$. Therefore,

$$
\mathcal{P}(s, t) = \sum_{m=0}^{\infty} t^m \sum_{k=0}^{\infty} \frac{k!}{m! 2^n n!} \frac{s^k}{k!} \\
= \sum_{m=0}^{\infty} t^m \sum_{n=0}^{\infty} \frac{s^{m+2n}}{m! 2^n n!} \\
= \sum_{m=0}^{\infty} \frac{(st)^m}{m!} \sum_{n=0}^{\infty} \frac{s^{2n}}{2^n n!} \\
= \exp \left( st + \frac{s^2}{2} \right).
$$

Formula (25) is thus proved. \[\square\]

Let us now compute the generating function of the Laplace polynomials $Q_k(t)$. Denote\footnote{Notice that the pair of polynomials $P_k$ and $Q_{k-1}$ determine the $k$th approximation of the Mill’s ratio.}

$$Q(s, t) := \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} q_{k,m} t^m \frac{s^{k+1}}{(k+1)!}.$$

**Lemma 10.** The generating function $Q(s, t)$ is

$$Q(s, t) = \sqrt{2\pi} \ e^{(s+t)^2/2} \cdot (\Phi(s+t) - \Phi(t)). \quad (26)$$

**Proof.** We have,

$$Q(s, t) = \sum_{k=0}^{\infty} Q_k(t) \frac{s^{k+1}}{(k+1)!} \quad \text{and} \quad \mathcal{P}(s, t) = \sum_{k=0}^{\infty} P_k(t) \frac{s^k}{k!}.$$

The Taylor series expansion for the function $R(t)$ can be written as

$$R(s + t) = \sum_{k=0}^{\infty} \frac{d^k R(t)}{dt^k} \frac{s^k}{k!}.$$

The derivatives of the Mills ratio satisfy the equation

$$\frac{d^k R(t)}{dt^k} = P_k(t) R(t) - Q_{k-1}(t).$$
Then we have
\[ R(s + t) = \sqrt{2\pi} e^{(s+t)^2/2} \Phi(s + t) \]
and therefore
\[
\sqrt{2\pi} e^{(s+t)^2/2} \Phi(s + t) = \sum_{k=0}^\infty \frac{\frac{d^k R(t)}{dt^k}}{k!} s^k
\]
\[
= R(t) \sum_{k=0}^\infty \frac{P_k(t)}{k!} s^k - \sum_{k=0}^\infty \frac{Q_{k-1}(t)}{k!} \cdot s^k
\]
\[
= R(t) \cdot e^{st+s^2/2} - Q(s, t)
\]
\[
= \sqrt{2\pi} e^{t^2/2} \Phi(t) \cdot e^{st+s^2/2} - Q(s, t).
\]
Therefore,
\[ Q(s, t) = \sqrt{2\pi} e^{(s+t)^2/2} \cdot (\Phi(t) - \Phi(s + t)). \]
The latter relation implies Equation \(26\). \qed

Now we are in a position to derive a series of identities from the double generating function \(Q(s, t)\).

**Corollary 11.**
\[
\sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{j=0}^n \frac{(n + m)!}{m! n! (m + 2n + 1 - j)!} 2^{-n} = \sqrt{2\pi} e^2 (\Phi(2) - \Phi(1)). \tag{27}
\]

*Proof.* Indeed, substituting \(s = r = 1\) into \(26\), we obtain
\[ Q(1, 1) = \sqrt{2\pi} e^2 \cdot (\Phi(2) - \Phi(1)). \]
On the other hand, by definition of the generating function, \(Q(r, s)\)
\[ Q(1, 1) = \sum_{k=0}^\infty \sum_{m=0}^\infty q_{k,m} \frac{1}{(k + 1)!}. \]
Using substitution, \(k = m + 2n\), from \(15\) and the latter equation we obtain the identity \(27\). Corollary \(11\) is thus proved. \qed

Let us now establish the connection between the second identity, \(26\) and the generating function \(Q(s, t)\).

\[ \text{That can be found in } [14]. \]
Corollary 12. The coefficients $q_{2n,0}$, of the Taylor series expansion for the function $Q(s,0)$ are

$$q_{2n,0} = \sum_{k=0}^{n} (-1)^k \frac{1}{2k+1} \binom{n}{k}.$$ 

Remark 13. The latter formula for the coefficients $q_{2n,0}$ implies the identity (2)

Proof. Let us substitute $t = 0$ in (26). Then we obtain

$$Q(s,0) = \sqrt{2\pi} \exp \left( \frac{s^2}{2} \right) \left( \Phi(s) - \frac{1}{2} \right)$$

$$= \sum_{k=0}^{\infty} Q_k(0) \frac{s^{k+1}}{(k+1)!}$$

$$= \sum_{n=0}^{\infty} q_{2n,0} \frac{s^{2n+1}}{(2n+1)!}.$$

From (23) we find

$$\sum_{n=0}^{\infty} q_{2n,0} \frac{s^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} s^{2n+1} \frac{2^n \cdot n!}{(2n+1)!}.$$

Therefore

$$Q(s,0) = \sum_{n=0}^{\infty} s^{2n+1} \frac{2^n \cdot n!}{(2n+1)!}.$$ 

The standard normal cdf satisfies the relation (see [1])

$$\Phi(s) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n s^{2n+1}}{2^n n!(2n+1)}.$$
Then we derive

\[
Q(s, 0) = \sqrt{2\pi}\exp\left(\frac{s^2}{2}\right) \cdot \left(\Phi(s) - \frac{1}{2}\right)
\]

\[
= \sum_{j=0}^{\infty} \frac{s^{2j}}{2^j j!} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2n + 1)}
\]

\[
= \sum_{n=0}^{\infty} s^{2n+1} \sum_{j=0}^{n} \frac{(-1)^j}{2^j j! 2^n j(2j + 1)}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{2^n n!} \sum_{j=0}^{n} \frac{(-1)^j}{j(2j + 1)}.
\]

Comparing the latter equation with (28) we derive the identity (2).

\[\square\]

5 Laplace and Hermite polynomials

Neither polynomials \(P_k(t)\) nor \(Q_k(t)\) form a system of orthogonal polynomials in the real domain. Nevertheless, the polynomials \(P_k(t)\) are intimately connected to the Hermite polynomials of the complex argument. Indeed, the Hermite and the Laplace polynomials can be defined as

\[
H_k(t) := (-1)^k e^{t^2/2} \frac{d^k \exp(-t^2/2)}{dt^k}; \quad P_k(t) := e^{-t^2/2} \frac{d^k \exp(t^2/2)}{dt^k}.
\]

Then we obtain

\[
P_k(t) = (-i)^k H_k(it).
\]

This relation (see also [9]) allows us to reformulate the classical results obtained for the Hermite polynomials in terms of the Laplace polynomials. In particular, one can easily derive the generating function, \(P(s, t)\), from the generating function

\[
H(t, s) := \sum_{k=0}^{\infty} H_k(t) \frac{s^k}{k!} = e^{st-s^2/2}
\]

for the Hermite polynomials.

Another useful fact about polynomials \(P_k(t)\) is formulated in the following
Proposition 14. The polynomial $P_k(t)$ satisfies the differential equation

$$y'' + ty' - ky = 0. \tag{30}$$

Proof. This result can be derived from the corresponding differential equation for the Hermite polynomials. One can also derive Equation (30) from (16) and Lemma 6.\hfill \square

Polynomials $P_k(t)$ form basis in the space of polynomials on the real line. In particular, the polynomial $Q_k(t)$ can be represented as a linear combination

$$Q_k(t) = \sum_{j \in J(k)} \beta_{k,j} P_j(t), \tag{31}$$

where the set of indices $J(k) = \{j : 0 \leq j \leq k, k \equiv j \mod 2\}$.

Proposition 15. For every $k = 0, 1, 2, \ldots$, there is a unique representation (31) for the Laplace polynomials $Q_k(t)$; the coefficients $\beta_{k,j}$ are

$$\beta_{k,j} = \frac{n!}{j!}, \quad \text{where } n = (k + j)/2, \ j \in J(k). \tag{32}$$

Proof. Equation (32) can be proved by induction. For $k = 1$ $\beta_{1,1} = 1$. Suppose the Proposition is proved for $k \leq k_*$. Let us prove that $Q_{k_*+1}$

\[
\begin{array}{cccccccccc}
| k | & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | \ldots & 0 | & & & & \\
| 1 | 0 | 1 | 0 | \ldots & 0 | & & & & \\
| 2 | 1 | 0 | 1 | 0 | \ldots & 0 | & & & & \\
| 3 | 0 | 2 | 0 | 1 | 0 | \ldots & 0 | & & & & \\
| 4 | 2 | 0 | 3 | 0 | 1 | 0 | \ldots & 0 | & & & & \\
| 5 | 0 | 6 | 0 | 4 | 0 | 1 | 0 | \ldots & 0 | & & & & \\
| 6 | 6 | 0 | 12 | 0 | 5 | 0 | 1 | 0 | 0 | & & & & \\
| 7 | 0 | 24 | 0 | 20 | 0 | 6 | 0 | 1 | 0 | & & & & \\
\end{array}
\]

Table 4: Coefficients $\beta_{k,j}$. satisfies Equation (31) with $\beta_{k_*+1,j}$ defined by (32).
Proof. We have from Lemma 3

\[ Q_{k+1}(t) = tQ_k(t) + (k_1 + 1)Q_{k-1}(t) \]

\[ = t \sum_{j \in J(k_1)} \beta_{k_1,j}P_j(t) + (k_1 + 1) \sum_{j \in J(k_1-1)} \beta_{k_1-1,j}P_j(t) \]

\[ = \sum_{j \in J(k_1)} \beta_{k_1,j} (P_{j+1}(t) - P_j(t)) + (k_1 + 1) \sum_{j \in J(k_1-1)} \beta_{k_1-1,j}P_j(t). \]

Thus the induction step will be proved if we show that the coefficients \( \beta_{k,j} \) satisfy the relation

\[ \beta_{k+1,j} = \beta_{k,j-1} - (j + 1)\beta_{k,j+1} + (k + 1)\beta_{k-1,j}. \] (33)

But it is easy to verify that the coefficients \( \beta_{k,j} \) defined by (32) satisfy (33).

\[ \square \]

6 Laplace polynomials and matching numbers

Consider a complete graph \( G_k \), i.e. the graph with \( k \) vertices such that every two vertices are connected by a single edge. Recall (see [2]) that a set of edges sharing no common vertices is called matching. Let \( M(k, m) \) be a number of matchings with \( m \) edges in the complete graph, \( G_k \). The following relation between \( M(k, m) \) and the coefficients of the Laplace polynomials, \( p_{k,j} \), follows from the very well known formula for the matching number (see [2], [14]):

Lemma 16.

\[ M(k, m) = p_{k,k-2m}. \] (34)

Proof. We have

\[ M(k, m) = \binom{k}{2m} (2m - 1) \cdot (2m - 3) \cdot \cdots \cdot 1 = \frac{k!}{(k - 2m)! \cdot 2^m \cdot m!}. \]

Then from (14) we obtain (34). The lemma is thus proved. \( \square \)

Denote \( k_1 = \lfloor k/2 \rfloor \) the number of edges in the maximal matching. Let

\[ \mathfrak{M}_k(t) := \sum_{m=0}^{k_1} M(k, m)t^m, \]
be the generating function of the matching numbers of the graph $G_k$. Then from Lemma 16 we obtain

**Corollary 17.** For $k \geq 2$

$$M_k(t) = t^{k/2} P_k \left( t^{-1/2} \right).$$

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