Decoherence, Correlation, and Unstable Quantum States in Semiclassical Cosmology

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March 24, 2022

Abstract

It is demonstrated that almost any S-matrix of quantum field theory in curved spaces posses an infinite set of complex poles (or branch cuts). These poles can be transformed into complex eigenvalues, the corresponding eigenvectors being Gamow vectors. All this formalism, which is heuristic in ordinary Hilbert space, becomes a rigorous one within the framework of a properly chosen rigged Hilbert space. Then complex eigenvalues produce damping or growing factors. It is known that the growth of entropy, decoherence, and the appearance of correlations, occur in the universe evolution, but only under a restricted set of initial conditions. It is proved that the damping factors allow to enlarge this set up to almost any initial conditions.
1 Introduction

For many years unstable quantum states were represented by Gamow vectors \([1]\), i.e.,
eigenvectors corresponding to complex eigenvalues of the hamiltonian. But since the
hamiltonian is self-adjoint, if we use a Hilbert space as state space, eigenvalues must be
real. For this reason, Gamow vectors were deshonorabley excluded from ordinary quan-
tum mechanics and they were considered just as useful (but not rigorous) analogies or
approximations. Nevertheless some years ago it was proved that Gamow vectors belong
to an extension of Hilbert space, namely a rigged Hilbert space with a nuclear subspace
based in Hardy class functions (see e.g. \([2, 3, 4]\) and bibliography therein). Since then,
Gamow vectores are legal citizens of an extended version of quantum mechanics, where
complex eigenvalues greatly help the computation of survival probabilities, life times,
Liapunov variables, the evolution toward equilibrium, etc. \([5, 6]\). These eigenvalues also
allow the introduction of more refined physical concepts, e.g. the thermodynamical
arrow of time can be defined in a way which is free of the usual criticisms (Lochshmidt
objection, coarse-graining ambiguities, non-systematic approximations, etc.). Precisely,
studying this arrow of time we can see that all the problem of time asymmetry essentialy
has a cosmological origin \([5, 6]\). Therefore it is natural to demand if unstable quantum
states, considered as vectors of a rigged Hilbert space, can be used in quantum cos-

omology. This is, in fact, the case and the first examples are Refs \([7]\) and \([8]\) where a
simplified (toy model) version of the universe is studied as a Friedrichs model \([4]\).

In this paper we will present a complete (not only a toy model) semiclassical model of
the universe following the line of Refs. \([9]\) and \([10]\) and we will show how the presence
of unstable quantum states enlarges the set of cases where we can prove that the decoher-
ence phenomena appears. Also correlations appear (for unstable states) explaining the
outcome of a classical universe. Essentially, in this paper, we study, just one example,
but we will also especulate about eventual generalizations.

But first, let us briefly recall the guiding lines of the extension from Hilbert space
to rigged Hilbert space in usual quantum mechanics. The traditional set of states of a
quantum system is a Hilbert space, which leads, as it is well known, to time-reversibility.
It is precisely this property which changes drastically with the extension of the Hilbert
space \(\mathcal{H}\) to a rigged Hilbert space. This extension corresponds essentiaaly to the transition
from a space of square integrable functions to a space of distributions. This procedure is
not unique and different distribution spaces can be defined which are based on different
test function spaces. If we choose as the test function space \(\Phi^-\), generated by the
eigenfunctions of the energy \(\omega\) which are analytic in the lower complex halfplane, when
the real variable \(\omega\) is promoted to a complex variable \(z\) (precisely Hardy class functions),
we obtain the dual space \(\Phi^\times\), which is the required extension of the Hilbert space \(\mathcal{H}\).
The corresponding Gel’fand triplet is then

\[
\Phi^- \subset \mathcal{H} \subset \Phi^\times. \tag{1.1}
\]

If the same procedure is performed in the upper complex plane, the resulting triplet
As we will see the first of these choices, hence the space $\Phi_{-}$, corresponds to unstable decaying states while the second one, namely $\Phi_{+}$, corresponds to unstable growing states. In fact, the complex poles of the S-matrix are related, as it is well known, with unstable physical states. These poles can then be transformed into complex eigenvalues $z_n$ of the Hamiltonian using standard methods [3]. The essence of the rigged Hilbert space rather than the Hilbert space, as the framework for the quantum states of the system, is clearly exhibited precisely at this stage: the eigenvalues $z_n$ of a Hermitian operator are not real anymore in this extended space $\Phi$. If $\text{Im} \, z_n > 0$ then a growing prefactor appears in the time evolution of the corresponding eigenvector $|n_+\rangle$, giving rise to a growing state belonging to the rigged Hilbert space $\Phi_{+}$. On the contrary, if $\text{Im} \, z_n < 0$ the prefactor is a decaying one, the corresponding state $|n_-\rangle$ is decaying and belongs to another rigged Hilbert space, $\Phi_{-}$. Finally, $\text{Im} \, z_n = 0$ corresponds to an ordinary stable state belonging to the ordinary Hilbert space $\mathcal{H} = \Phi_+ \cap \Phi_{-}$ (more general models contain both, growing and decaying states [5,6]).

If $K$ is the Wigner time-reversal operator we have

$$K : \Phi_{-} \rightarrow \Phi_{+}, \quad K : \Phi_{+} \rightarrow \Phi_{-},$$

(1.3)

since decaying states are transformed into growing states (or vice-versa) by time-inversion. Then the choice of $\Phi_{-}$ (or $\Phi_{+}$) as our space of quantum states implies that $K$ is not defined inside $\Phi_{-}$ (or $\Phi_{+}$), so that irreversibility naturally appears and therefore the arrow of time also appears in the quantum regime.

It follows that the choice between $\Phi_{-}$ or $\Phi_{+}$ is irrelevant, since these two objects are identical (namely one can be obtained from the other by a mathematical symmetry transformation), and therefore the universes, that we will obtain with one choice or the other, are also identical and not distinguishable. Only the names past and future or decaying and growing will change but physics is the same and, e.g., we will always have equilibrium toward the future.

Let us summarize the organization of this paper. Section 2 introduce the model that we will study. In Section 3 it will be demostrated that the S-matrix of the model has an infinite set of complex poles (or branch cuts), and how these poles are transformed in complex eigenvalues that originate, in turn, damping factors of unstable decaying states. In Section 4 it is shown how these damping factors enlarge the set of initial conditions where decoherence occurs. In Section 5 we will prove that there is correlation in all unstable states. Finally, we briefly state our conclusions in Section 6. Two Appendixes complement this work.
2 The Model

Let us consider the model of Sec. 3 of Ref. [10] where a Robertson-Walker metric is studied (that we will mainly consider in the flat case), with a total action $S = S_g + S_f$, being $S_g$ the gravitational action and $S_f$ the matter action (the usual action of a spinless massive field $\Phi$). The gravitational action is given by

$$S_g = M^2 \int d\eta \left[ -\frac{1}{2} \dot{a}^2 - V(a) \right], \quad (2.1)$$

where $M$ is the Planck mass, $\eta$ is the conformal time, $a$ is the Robertson-Walker scale factor, $\dot{a} = da/d\eta$, and $V(a)$ is the potential function that arises from the spatial curvature, a possible cosmological constant and eventually a classical matter field. As this last field is arbitrary, for the sake of simplicity, let us study the case where the classical matter field is such that $V(a) = B^2/2(1 - A^2/a^2)$ where $A$ and $B$ are arbitrary constants.

This case is the simplest of all, but we believe that the main features that we will find will also be presented in more general cases, as we will argue below. The role played by the classical field is completely natural, in the context of this paper. In fact, we will essentially work using some results of quantum field theory in curved space-time, where the geometry of space-time is fixed “a priori” (namely there is no back-reaction). The classical field is, precisely, the agency that do this job, fixing a class of possible geometries (but the properties that we will find will be the same for almost all classical field).

The Wheeler-DeWitt equation for our Robertson-Walker model is

$$\left[ \frac{1}{2M^2} \partial_a^2 + M^2 V(a) - \frac{1}{2} \int_k \left( \partial \phi_k^2 - \Omega_k^2 \phi_k^2 \right) \right] \Psi(a, \Phi) = 0. \quad (2.2)$$

Thus after making the WKB ansatz, the Hamilton-Jacobi equation appears as [10]

$$(\frac{dS}{da})^2 = B^2(1 - \frac{A^2}{a^2}), \quad (2.3)$$

where $S$ is the principal Jacobi function. Thus the (semi) classical time parameter or WKB time $\eta$ is given by

$$\frac{d}{d\eta} = \frac{dS}{da} \frac{d}{da}. \quad (2.4)$$

Then in our simplified model we have the following class of geometries, in terms of this conformal time $d\eta = a^{-1} dt$ [12]

$$a = \pm \left( A^2 + B^2 \eta^2 \right)^{\frac{1}{2}} + C \quad (2.5)$$

where $C$ is an arbitrary constant. Using different values for this constant and different choices for the $\pm$ sign we obtain different classical solutions (in a more general case many
constants would be necessary). Going now to Ref. [13] (eq. (3.113)) we can see that the semiclassical (or quantum field theory in curved space-time) problem is solved for all four dimensional, spatially flat, cosmological models with scale factor

\[ C(\eta) = a^2 = A^2 + B^2 \eta^2 \quad -\infty < \eta < \infty \]  

(2.6)

where \( A \) and \( B \) are constants. Then if we consider a massive, conformally coupled scalar field, the energy function \( \Omega_k^2 \) reads

\[ \Omega_k^2 = m^2 a^2 + k^2 = k^2 + m^2 (A^2 + B^2 \eta^2) \]  

(2.7)

where \( m \) is the mass of the quantum matter field and \( k = |\vec{k}|/a \) is the linear momentum of this field, in the case of flat space Robertson-Walker universe (or a function of this momentum in the two other cases, namely open and close, being \( k \) a discrete variable in the close case). Then (2.7) coincides with the last equation of page 70 of Ref. [13].

If we ideally consider the evolution of the universe from \( \eta \to -\infty \) to \( \eta \to +\infty \) (even if really we would like to have only an expanding universe and therefore \( \eta \geq 0 \), we will discuss this issue below) and we define the corresponding adiabatic vacua \( |0,\text{in}\rangle \) for \( \eta \to -\infty \) and \( |0,\text{out}\rangle \) for \( \eta \to +\infty \) the Bogolyubov coefficients are (Ref. [13], eq. (3.124))

\[ \alpha_{kj} = i (2\pi)^{\frac{1}{2}} \frac{\exp(-\frac{\pi}{2} \lambda_k)}{\Gamma[\frac{1}{2}(1-i\lambda_k)]} \delta_{kj} = \alpha_k \delta_{kj}, \]  

(2.8)

\[ \beta_{kj} = -i \exp(-\frac{\pi}{2} \lambda_k) \delta_{kj} = \beta_k \delta_{kj}, \]  

(2.9)

where \( \lambda_k = k^2/Bm + A^2m/B \) and \( \delta_{kj} \) is the Kronecker \( \delta \), for the discrete case, and the Dirac \( \delta \), for the continuous one.

Let us comment now on the choice of the vacua since really we would like to study only the evolution \( \eta \geq 0 \). The \( |0,\text{out}\rangle \) vacuum is the adiabatic physical vacuum for \( a \to +\infty \), where the classical regime must naturally appear, therefore it is a completely reasonable vacuum. Let us suppose that the vacuum at \( \eta = 0 \) is just \( |0,\text{in}\rangle \). This is, of course, a completely arbitrary choice, that will be discussed in the next section, where we will introduce a general vacuum at \( \eta = 0 \) that we will call \( |0,0\rangle \). Anyhow with this arbitrary choice (2.8) and (2.9) are corrects.

### 3 The poles of the S-matrix and the unstable quantum states

From (3.46) and (3.47) of Ref. [13], or more generally from Sec. 2 of Ref. [14] it can be seen that there is a pole in the S-matrix (between the “in” and the “out” Fock spaces) where the function \( \Lambda_{ji} = -i \sum_k \beta_{kj} \alpha_{ik}^{-1} \) has a pole, namely where \( \alpha_{kj} = 0 \) (or \( \beta_{kj} \) has a
pole. Using (2.8) it must be \( \alpha_k = 0 \) or, which is the same thing, that \( \Gamma[1/2(1 - i\lambda_k)] \)
would have a pole. \( \Gamma(z) \) has a pole if \( z = -n \) \((n = 0, 1, 2, \ldots)\), see, e.g., [13] or [16] (no
poles are produced by the \( \beta \)'s given by (2.9)). Therefore \( S \) has a pole if
\[
k^2 = mB[-\frac{mA^2}{B} - 2i(n + \frac{1}{2})],
\]
and the squared energy, for each pole, reads
\[
\Omega_k^2 = m^2a^2 + mB[-\frac{mA^2}{B} - 2i(n + \frac{1}{2})].
\]

We will call this energy \( \Omega_k \) simply \( \Omega_n \). Thus, we have an infinite set of unstable
states with mean life
\[
\tau_n = \frac{2^{\frac{3}{2}}\{m^2(a^2 - A^2) + m^4(a^2 - A^2) + 4B^2m^2(n + \frac{1}{2})^2\}^{\frac{1}{2}}}{2Bm(n + \frac{1}{2})}.
\]

Let us observe that the energy and mean life are \( a \)-dependent. Therefore we have
two possibilities:

1) either we can consider that the in and out states corresponds to \( a >> 1 \), where
these mean lifes are big but still finite, or

2) we transform all the equations to the non-rescaled case, where the physical real
values are the physical time \( t = \int ad\eta \), the physical energy \( \Omega_k/a \) and the physical
momentum \( k/a \).

We follow the first alternative, and sketch the second one in the Appendix A.

Therefore the universe evolution creates unstable particles as well as stable ones.
Using the standard method explained in Refs. [2] and [3] we can promote these unstable
states to vectors of an adequate rigged Hilbert space and build a basis of this space with
stable modes with real energies \( \Omega_k \) plus unstable modes with complex “energy” \( \Omega_n \) given
by (3.2) (in the open case this procedure is direct, since we have a continuous spectrum
to begin with, in the close case we must use assumption 3 of Ref. [7]).

This would be the state of affairs if we use the (quite arbitrary) vacuum \( |0, in\rangle \) of
section 2. In this case we have found an infinite discrete set of unstable states. What
happens if we use a generic (i.e., almost any) vacuum \( |0, 0\rangle \) at \( \eta = 0 \)? This generic
vacuum will be related to \( |0, in\rangle \) by some Bogolyubov coefficients \( \tilde{\alpha}_{kj}, \tilde{\beta}_{kj} \). Then the \( \tilde{\alpha} \)
coefficient relating \( |0, 0\rangle \) to \( |0, out\rangle \) reads
\[
\tilde{\alpha}_{ik} = \sum_j \tilde{\alpha}_{ij}\alpha_{jk} + \tilde{\beta}_{ij}\beta_{jk}^* = \frac{\tilde{\alpha}_{ik}i(2\pi)^{\frac{3}{2}}\exp(-\frac{\pi}{2}\lambda_k)}{\Gamma[\frac{1}{2}(1 - i\lambda_k)]} + \tilde{\beta}_{ik}i\exp(-\frac{\pi}{2}\lambda_k).
\]

The poles are now located where this alpha vanishes. The roots in \( k \) of the corre-
sponding equation, \( \tilde{\alpha}_{ik} = 0 \) can be found only if we fix the arbitrary coefficients \( \tilde{\alpha}_{ik}, \tilde{\beta}_{ik} \).
Of course if these coefficients are fixed in a very particular way the equation will have
no roots. But if the functions \( \tilde{\alpha}_{ik}, \tilde{\beta}_{ik} \) are fixed in a generic (i.e. in almost any) way, the
equation will have a set of complex roots, that correspond to unstable particles created by the universe evolution. This statement is equivalent to claim that a generic S-matrix, for our problem, has infinite numbers of poles or cuts. Let us say an infinite set of poles to precise the ideas (cuts will be studied in Appendix A). Even if we have not, by now, a rigorous mathematical proof of this theorem, we think that we can sketch a reasonable convincing demonstration.

Let \( \{ |n, \text{in} \rangle \} \), \( \{ |0, 0 \rangle \} \), and \( \{ |m, \text{out} \rangle \} \) be the basis of the Fock spaces corresponding to vacua \( |0, \text{in} \rangle \), \( |0, 0 \rangle \), and \( |0, \text{out} \rangle \). The in-out S-matrix, with an infinite set of poles, reads

\[
S_{nm} = \langle n, \text{in} | m, \text{out} \rangle = \sum_l \langle n, \text{in} | l, 0 \rangle \langle l, 0 | m, \text{out} \rangle \tag{3.5}
\]

where the states \( |l, 0 \rangle \) are a complete set. From some (infinite) values of \( n \) and \( m \) we know that \( S_{nm} \) has poles. Let us consider one of these values, then the l.h.s. of (3.4) has also a pole, and therefore one of its terms has a pole. Then, either:

i) one of the factors inside the summatory of (3.4) has a finite number of poles and the other one an infinite number of poles, or

ii) both factors have an infinite number of poles.

But (i) must be excluded since, in this case, time evolution \((-\infty < \eta < 0)\) would be qualitatively different to evolution \((0 < \eta < +\infty)\) and this fact would break the time symmetry, which is impossible since evolution equations, time evolution of \( a \), and boundary conditions are time symmetric with respect \( \eta = 0 \). Then, the \( 0-\text{out} \) matrix \( \langle l, 0 | m, \text{out} \rangle \), corresponding to evolution \( \eta \geq 0 \), has an infinite number of poles.

We give an alternative demonstration in Appendix B, which is valid for every evolution and every spatial geometry.

Even if a rigorous proof of these facts would be welcomed we believe that the reasonings above, and the ones in Appendix B, are quite convincing. Essentially the periodical nature of \( \bar{\alpha}_{ik} \) is inherited from its definition; \( \bar{\alpha}_{ik} = (\bar{u}_i, u_k) \) (eq. (3.36), Ref. [13]) where \( \bar{u}_i \) and \( u_k \) are two different negative frequency solutions of the corresponding Klein-Gordon equation. As in flat space-time, these solutions are functions like \( \exp(-ikt) \), they somehow must keep the periodicity in \( k \) in curved space-time. The \( \alpha \) coefficients always have a periodic behaviour in the complex plane as it is shown in equations (3.91), (3.124), (4.60), (4.61), (4.95), (5.41), (5.110), and (5.111) of Ref. [13]. Therefore the S-matrix has an infinite and discrete set of complex eigenvalues for almost any initial condition \( |0, 0 \rangle \).

Now that we know that the 0-out S-matrix has an infinite set of complex poles, we can find the complex eigenvalues [2,3].

As the \( \{ |k, \text{out} \rangle \} \) basis is complete we have

\[
\int |k, \text{out} \rangle \langle k, \text{out} | dk = 1, \tag{3.6}
\]

where \( |k, \text{out} \rangle \in \mathcal{H} \) and the integral means that we must integrate over the continuous
spectrum of energies and other quantum numbers. Using the standard techniques of Ref. [2,3] we can transform the last equation in

$$\sum_{n} |n, out-\rangle\langle n, out+| + \int_{k} |k, out-\rangle\langle k, out+| dk = 1, \quad (3.7)$$

where $|n, out-\rangle$, $|k, out-\rangle \in \Phi^-\times$ and the first summatory corresponds to the discrete unstable modes and the integral to the stable continuous ones.

We choose a $\Phi_-$ test function based in Hardy functions for below, all the poles will have negative imaginary part and all the unstable states are decaying ones, and they will belong to $\Phi^-\times$ (as we already know we can also make a symmetric choice).

According to Wheeler-DeWitt equation (2.2), the field hamiltonian reads

$$h = \frac{1}{2} \int_{k} \left( -\partial^2_{\phi_k} + \Omega^2_{k} \phi^2_k \right) dk = \int_{k} \Omega_k a^\dagger_k a_k dk, \quad (3.8)$$

where $a_k$ and $a^\dagger_k$ are the usual creation and anihilation operators. From now, we will always refer to the out case with $a >> 1$ and $h, \Omega_k, a_k, and a^\dagger_k$ will be $h^{out}, \Omega^{out}, a^{out}_k,$ and $a^{out\dagger}_k$. There are new creation and anihilation operators for the discrete spectrum: $\bar{a}^{out}_n, \bar{a}^{out\dagger}_n$ and for the continuous ones $a^{\star out}_k, \bar{a}^{\star out}_k$ (the definition of $\star$ is given in Appendix B). Vectors $|n, out-\rangle$ will be created by the repeated action of $\bar{a}^{out}_n$ on $|0, out\rangle$, and vectors $|k, out-\rangle$ will be created by $\bar{a}^{\star out}_k$ analogously.

Therefore $h^{out}$ now reads

$$h^{out} = \sum_{n} \bar{a}^{\star out}_n \bar{a}^{out}_n + \int_{k} \Omega_k \bar{a}^{\star out}_k \bar{a}^{out}_k dk, \quad (3.9)$$

and we will have

$$h^{out}|n, out\rangle = \Omega_n n|n, out\rangle, \quad (3.10)$$

so the evolution of $|n, out\rangle \in \Phi^-\times$ has a damping prefactor $\exp(-n\eta/\tau_n)$ since $\Omega_n$ has an imaginary component.

Thus, as we now have damping factors $\exp(-n\eta/\tau_n)$ in the evolution equations, it will be very easy to find Lyapunov variables, and in particular a growing entropy for almost any initial condition as in Refs. [5][6][17]. This result must be compared with the one of Ref. [17] where a Lyapunov variable was found for the universe evolution using the standard methods [18][19] based in an arbitrary coarse-graining and a particular (generalized molecular chaos) initial condition. The new result is much more satisfactory than the old one, since now we have a growing entropy for almost any initial conditions solving Lochsmidt criticisms.

### 4 Decoherence

Decoherence is a dissipative process, and we know [20] that it is closely related to another dissipative phenomenon, that is particle creation from the gravitational field...
Particle creation has been studied in the quantum field theory in curved spaces as the semiclassical limit of quantum cosmology. We will restrict ourselves to the semiclassical approximation to study the decoherence phenomenon.

Decoherence naturally appears in systems where the Hamiltonian has complex eigenvalues, as it is proved in Ref. [25]. Let us consider the formalism developed in Refs. [9] and [10] to see that, this is also the case, in the system we are studying. We labelled the three-geometry with the scalar factor \( a \) (the indices \( \alpha \) and \( \beta \) symbolizes the choice of the sign and constant in (2.5)), and \( \Phi_N \) is the mode \( N \) of the matter field; precisely, we have used \( n \) for the discrete unstable states and \( k \) for the continuous stable states; when we will be referring to both kinds of states, we will call the index \( N \).

The WKB solution of the Wheeler-DeWitt equation reads (9 eq. (2.8))

\[
\Psi(a, [\Phi_N]) = e^{i MS(a)} \chi(a, [\Phi_N]),
\]

where \( S \) is the principal Jacobi function of (2.3) and \( \chi(a, [\Phi_N]) \) can always be written as

\[
\chi(a, [\Phi_N]) = \prod_N \chi_N(\eta, \Phi_N).
\]

We can obtain \( \chi_N(\eta, \Phi_N) \) by a Gaussian approximation

\[
\chi_N(\eta, \Phi_N) = A_N(\eta) e^{i \alpha(\eta)} e^{-B_N(\eta) \Phi_N^2},
\]

Functions \( A_N(\eta) \) and \( \alpha_N(\eta) \) are real while \( B_N(\eta) \) is complex, precisely \( B_N(\eta) = B_{NR}(\eta) + i B_{NI}(\eta) \) and can be obtained solving the system:

\[
\dot{A}_N(\eta) = \pi^{-\frac{1}{4}} (2B_{NR}(\eta))^{\frac{1}{2}},
\]

\[
\dot{\alpha}_N(\eta) = -B_{NR}(\eta),
\]

\[
\dot{B}_N(\eta) = -2i B_N^2(\eta) + \frac{i}{2} \Omega_N^2(\eta).
\]

From [10] or [20] we can learn the conditions for the occurrence of decoherence if we use only the real \( \Omega_N \), namely the ones of the continuous spectrum. In Ref. [20] the computation is made only through a linear approximation of \( B(a) \) as a function of \( a \). In Ref. [20] decoherence takes place only in the case where the Bogolyubov coefficients \( \beta_n \) are small and imaginary. In Ref. [10] decoherence takes place unless the environment is very ordered and fine tuned. Thus we cannot say that there is decoherence for almost any initial condition if the \( \Omega_N \) are all real, like in these works. Let us see, what happens in our model if we use a basis with infinite complex modes \( \Omega_n \) as well as real modes \( \Omega_k \).

From the wave function (4.1), and after the integration on modes of the scalar field (considered here as the “environment”), we obtain the following reduced density matrix...
\[ \bar{\rho}_r(a, a') = \exp[-iMS_\alpha(a) + iMS_\alpha(a')]\bar{\rho}^{\alpha\alpha}(a, a') + \exp[-iMS_\alpha(a) + iMS_\beta(a')]\bar{\rho}^{\alpha\beta}(a, a') + \exp[-iMS_\beta(a) + iMS_\alpha(a')]\bar{\rho}^{\beta\alpha}(a, a') + \exp[-iMS_\beta(a) + iMS_\beta(a')]\bar{\rho}^{\beta\beta}(a, a'), \]  

(4.7)

where, as we have said, \( \alpha \) and \( \beta \) symbolize two different classical solutions, namely two different choices of the sign \( \pm \) and the constant \( C \) of (2.5), and

\[ \bar{\rho}^{\alpha\beta}(a, a') = \prod_N \bar{\rho}^{\alpha\beta}_r(a, a') = \prod_N \int d\Phi_N \chi_\alpha^*(\eta, \Phi_N)\chi_\beta^*(\eta', \Phi_N). \]  

(4.8)

From (3.20) of Ref. [10], it is

\[ B_N = -\frac{i}{2} \dot{g}_N, \]  

(4.9)

where \( g_N \) is the wave function that represents the quantum state of the universe being also the solution of the differential equation

\[ \ddot{g}_N + \Omega_N^2 g_N = 0, \]  

(4.10)

\( \Omega_N \) can be the complex energy \( \Omega_n \) in our treatment. From (2.6), (2.7), and (3.1) we know that if the initial state is \( |0, in \rangle \), the complex energies are

\[ \Omega_n^2 = m^2a^2 - m[2iB(n + \frac{1}{2}) + mA^2]. \]  

(4.11)

In the more general case we use an arbitrary initial state \( |0, 0 \rangle \), instead of \( |0, in \rangle \). From the discussion of Sec. 3, we know that, in a generic case, an infinite set of complex poles does exist. Then we must change (3.1) by \( k^2 = k_n^2 \) (\( n = 0, 1, 2, \ldots \)) where this are the points where the infinite poles are located in complex plane \( k^2 \); (3.2) now reads

\[ \Omega_n^2 = m^2a^2 + k_n^2. \]  

(4.12)

Let us now see that decoherence takes place if there is an infinite set of complex modes (even in a more general case than the one of the time evolution fixed by (2.3), the only one we have studied in great detail above, if we use the theorem of Appendix B).

Let us consider the asymptotic (or adiabatic) expansion of function \( g_N \) when \( a \to +\infty \) in the basis of the out modes. \( g_N \) is the wave function that represent the state of the universe, corresponding to the arbitrary initial state \( |0, 0 \rangle \), and its expansion reads
\[ g_N = \frac{P_N}{\sqrt{2\Omega_N}} \exp[-i \int_0^\eta \Omega_N d\eta] + \frac{Q_N}{\sqrt{2\Omega_N}} \exp[i \int_0^\eta \Omega_N d\eta], \quad (4.13) \]

where \( P_N \) and \( Q_N \) are arbitrary coefficients showing that \( |0, 0\rangle \) is really arbitrary.

It is obvious that if all the \( \Omega_N \) are real, like in the case of the \( \Omega_k \), (4.13) will have an oscillatory nature, as well as its derivative. This will also be the behaviour of \( B_k \) in (4.9). Therefore the limit of \( B_k \) when \( \eta \to +\infty \) will be not well defined even if \( B_k \) itself is bounded.

But if \( \Omega_N \) is complex the second term of (4.13) will have a damping factor and the first a growing one. In fact, the complex extension of eq. (4.13) (with \( N = k \)) reads

\[ g_n = \frac{P_n}{\sqrt{2\Omega_n}} \exp[-i \int_0^\eta \Omega_n d\eta] + \frac{Q_n}{\sqrt{2\Omega_n}} \exp[i \int_0^\eta \Omega_n d\eta], \quad (4.14) \]

Therefore when \( \eta \to +\infty \) we have

\[ B_n \approx -\frac{i}{2 g_N} \frac{\dot{g}_N}{g_N} = \frac{1}{2} \Omega_n. \quad (4.15) \]

Then we have two cases:

i) \( \Omega_N = \Omega_k \in \mathcal{R}^+ \) for the real factors. Then we see that when \( \eta \to +\infty \), the r.h.s. of (4.8) is an oscillatory function with no limit in general. We only have a good limit for some particular initial conditions (as \( Q_N = 0 \) or \( P_N = 0 \) \cite{[9, 10, 20, 26]}).

ii) \( \Omega_N = \Omega_n = E_n - \frac{i}{2} \tau_n^{-1} \in \mathcal{C} \) for the complex factors. If we choose the lower Hardy class space \( \Phi_- \) to define our rigged Hilbert space we will have a positive imaginary part, and there will be a growing factor in the first term of (4.13) and a damping factor in the second one. In this case, for \( a \to +\infty \), we have a definite limit: \( B_n = 1/2 \Omega_n \).

So we can say nothing about the limit of the real factors (and therefore nothing in general for the product of these real factors) while the complex factors have definite limits for every initial conditions, namely for every \( |0, 0\rangle \) state.

Therefore let us compute the \( \tilde{\rho}_{r\alpha\beta} \) for the complex factor, since these are the only quantities whose limits we know for sure. So let us compute these matrix elements using eq. (2.29) of Ref. [9] or (2.24) of [10], namely

\[ \tilde{\rho}_{r\alpha\beta}^{\alpha\beta}(a, a') = \left( \frac{4B_{nR}(\eta, \alpha)B_{nR}(\eta', \beta)}{|B_n^*(\eta, \alpha) + B_n(\eta', \beta)|^2} \right)^{\frac{1}{2}} \exp[-i\alpha_n(\eta, \alpha) + i\alpha_n(\eta', \beta)], \quad (4.16) \]

where \( \alpha \) and \( \beta \) mean that the \( B \) refers to these classical solutions. \( \tilde{\rho}_{r\alpha\beta}^{\alpha\beta}(a, a') \) can be obtained using (4.8). Let us first compute

\[ \log|\tilde{\rho}_r^{\alpha\beta}(a, a')| = \sum_n \log|\tilde{\rho}_{r\alpha\beta}^{\alpha\beta}(a, a')|. \quad (4.17) \]

Now it can be proved that if \( \Im B_n \approx \Im \frac{1}{2} \Omega_n \neq 0 \),
\[ |\tilde{\rho}_{r}^{\alpha\beta}(a, a')| < 1. \quad (4.18) \]

In fact, calling \( B_{n}^{*}(\eta, \alpha) = z = x + iy \) and \( B_{n}(\eta', \beta') = \zeta = \xi + iy \), we can compute
\[
\frac{4x\xi}{(x + \xi)^{2} + (y + \eta)^{2}} \leq \frac{4x\xi}{|x|^2} \leq 1,
\]
(4.19)
since from \(|x + \xi|^2 \geq 0\) it follows that \(4x\xi \leq |x + \xi|^2\). Then all terms of the r.h.s. of (4.17) are negative if \( \text{Im} \ B_{n} \neq 0 \).

Now from (4.11) or (4.12) we can see that the \( B_{n}(\eta, \alpha) \approx 1/2 \Omega_{n}(\eta, \alpha) \), corresponding to the discrete complex modes, have all almost the same asymptotic value when \( a \to +\infty \), therefore all the terms of the r.h.s. of (4.17) have almost the same asymptotic value. As they are all negative and almost equal, the summatory of (4.17) has an asymptotic value \(-\infty\) and therefore \(\tilde{\rho}_{r}^{\alpha\beta}(a, a')\) of (4.8) vanishes only considering the discrete complex modes factors.

Then we have decoherence if \( B_{n}^{*}(\eta, \alpha) \neq B_{n}(\eta', \beta) \) namely if \( \Omega_{n}^{*}(\eta, \alpha) \neq \Omega_{n}(\eta', \beta) \), or using (4.12), if (for an infinite set of \( n \)) we have
\[
m^2[\pm(A^2 + B^2\eta^2)^{\frac{1}{2}} + C_{\alpha}]^2 \neq m^2[\pm(A^2 + B^2\eta'^2)^{\frac{1}{2}} + C_{\beta}]^2. \quad (4.20)
\]
So we necessarily have decoherence:

i) for different classical solutions, i.e. for different constants \( C_{\alpha} \neq C_{\beta} \), or different \( \pm \) signs, even if the time is the same \( \eta = \eta' \).

ii) for the same classical solutions \( (C_{\alpha} = C_{\beta}, \text{and same} \pm \text{sign}) \) if the times \( \eta \) and \( \eta' \) are different.

Now we can discuss the choice of either \( \Phi_{-} \) or \( \Phi_{+} \) for the test function spaces. \( \Phi_{-} \) produces the space \( \Phi_{-}^{\times} \) of the decaying states, while \( \Phi_{+} \) will produce the space \( \Phi_{+}^{\times} \) or growing states. Of course one choice becomes the other if we change the \( \pm \) sign in (2.5) and also in all other equations that are a consequence of (2.5). To choose \( \Phi_{-} \) or \( \Phi_{+} \) corresponds to the choice of the arrow of time in the quantum regime as explained in the introduction. Choosing the Gel’fand triplet \( \Phi_{-} \subset \mathcal{H} \subset \Phi_{+} \) is equivalent to say that the unstable created particles produced by the universe expansion will decay. This is the motivation of the choice of the arrow of time in the quantum regime.

Nevertheless we can conceive more complex models where growing and decaying particles could coexist at the same time. Then, we can adopt a more conservative attitude. We can consider that the space \( \Phi_{+} \oplus \Phi_{-} \) as the space of the test functions with the corresponding space \( \Phi_{+}^{\times} \oplus \Phi_{-}^{\times} \) where there are mixed decaying and growing states. This choice is possible in the quantum regime, and therefore there will be no arrow of time in this regime. But in the classical limit, solution in space \( \Phi_{+}^{\times} \) will decohere with the solutions in \( \Phi_{-}^{\times} \) since they correspond to different choices of the \( \pm \) sign [26]. Therefore for a classical universe either we have a state in \( \Phi_{+}^{\times} \) or \( \Phi_{-}^{\times} \) (which, on the other hand, it is an irrelevant choice). Under this perspective, we will not have an arrow of time in the quantum regime but this arrow will appear naturally in the classical regime.
5 Correlation

From Ref. [10] we can also learn the conditions for the existence of correlation. But, for the same reason used in Sec. 4, we cannot say that there is correlation for almost any initial condition. This correlation depends on the initial conditions, and it can be easily obtained from system (4.4)-(4.6). In fact for certain initial condition, if the $\Omega_N^2$ are real, it turns out that $B_{NR} = 0$ and all the conditions for the existence of correlations of Ref. [10] are not fulfilled and therefore there is no correlation. Precisely, if $B_{NR} = 0$ and $B_N = iB_{NI}$ and all energies $\Omega_N$ are real (it would be better to call it $\Omega_k$), (4.6) reads

$$\dot{B}_{NI} = \frac{1}{2} \Omega_N^2,$$

(5.1)

where all the variables are real. Therefore if $B_{NR} = 0$ at $a = 0$, $B_{NR} = 0$ at every time and there is no correlation. Thus correlation, as decoherence, depends crucially on the initial condition and there is no correlation for the above initial condition. Correlation takes place inside each classical solution and it therefore can be computed using the Wigner function, associated with $\tilde{\rho}^{\alpha\alpha}_{rn}(\eta, \eta')$ [10, 27]

$$F^{\alpha\alpha}(n)(a, P) \approx C^2(T) \sqrt{\frac{\pi}{\sigma^2}} \exp \left[ - \frac{(P - M \dot{S} + \dot{\alpha} - \frac{B_{nR}}{4B_{nR}})^2}{\sigma^2} \right].$$

(5.2)

where $a$, $a' = a \pm \frac{\Delta}{M}$, and $P$ is the canonical momentum.

Nothing new can be said about the real continuous modes, all was already said in Ref. [10]. We must only study the complex discrete unstable modes. This is nevertheless important since, most likely, the universe is in an unstable mode or more generally in a linear combination of unstables modes (see Ref. [7] and [28, 29], where the universe is in a “tunneling” unstable state, i.e. a typical Gamow vector).

Then we can repeat the reasonings of [10] from (2.24) to (2.28) and, with the same assumptions we will arrive to this last equation, that now reads

$$F^{\alpha\alpha}(n)(T, P) \approx C^2(T) \sqrt{\frac{\pi}{\sigma^2}} \exp \left[ - \frac{(P - M \dot{S} + \dot{\alpha} - \frac{B_{nR}}{4B_{nR}})^2}{\sigma^2} \right].$$

(5.3)

In the case of the our unstable states we have, for $a(\eta) \to +\infty$

$$\dot{\alpha} = -B_{nR} = -\frac{1}{\sqrt{2}} \left[ m^2 B^2 \eta^2 + (m^4 B^4 \eta^4 + 4m^2 B^2(n + \frac{1}{2})^2) \right]^\frac{1}{2},$$

(5.4)

$$\dot{B}_{nI} = 2\sqrt{2} \frac{m^3 B^3 \eta}{\left[ m^2 B^2 \eta^2 + \left( m^4 B^4 \eta^4 + 4m^2 B^2(n + \frac{1}{2})^2 \right)^\frac{1}{2} \right]^2 + 4m^2 B^2(n + \frac{1}{2})^2},$$

(5.5)

and the inverse of the correlation width of the reduced density matrix is
\[ \sigma^2 = \frac{1}{2} \left[ \frac{m^2 B^2 \eta^2 + (m^4 B^4 \eta^4 + 4m^2 B^2 (n + \frac{1}{2}))^{\frac{1}{2}}}{m^2 B^2 \eta^2 + (m^4 B^4 \eta^4 + 4m^2 B^2 (n + \frac{1}{2}))^{\frac{1}{2}}} + 4m^2 B^2 (n + \frac{1}{2}) \right], \]

if \( \eta \) is big: \(|\dot{B}_R| > |\dot{B}_I|\).

We can see that when \( \eta \to +\infty \), \( \sigma^2 \to 0 \), and there is a good correlation and the Wigner function is a Gaussian function, of width \( \sigma \), peaked about

\[ P = M \dot{S} - \dot{\alpha} + \frac{\dot{B}_{nI}}{4B_{nR}}, \]

where the first term of the r.h.s. gives the classical result and the last two are the quantum correlation to the classical trajectory.

On the other hand, in this state we can predict strong correlations between coordinates and momenta [10], because

\[ \left( M \dot{S} - \dot{\alpha} + \frac{\dot{B}_{nI}}{4B_{nR}} \right)^2 \gg \sigma^2, \]

(in the preceding equation we can see that, in the limit of large \( \eta \), the l.h.s. is proportional to \( \eta^2 \), while \( \sigma^2 \sim \eta^{-1} \)).

For the generic initial condition \(|0, 0\rangle\) we can use (4.12) and we will reach to the same conclusion. Therefore there is a perfect correlation for the unstable states of our model. Decoherence and correlation will produce the outcome of an asymptotic classical regime in the far future.

6 Conclusions

We have demonstrated that, the S-matrix of almost any quantum field theory in curved spaces model, has an infinite set of poles (or cuts). The presence of this singularities produces the appearance of unstable states (with complex eigenvalues) in the universe evolution. The corresponding eigenvectors are Gamow vectors and produce exponentially decaying terms as in the Friedrichs model of resonances. But the best feature of these decaying terms is that they simplify and clarify calculations.

E. Calzetta and F. Mazzitelli [20] have demonstrated that, under suitable conditions, the expansion of the universe leads to decoherence if this expansion produces particle creation as well. Our unstable states enlarge the set of initial conditions where decoherence occurs. In fact, the damping factors (related to the imaginary part of S-matrix's poles), allow that the interference elements of the reduced density matrix, dissapear for almost any initial conditions.

Following the reasonings of Ref. [10], we also demonstrate that the unstable states satisfy the correlation conditions, which, with the decoherence phenomenon, are the origin of the semiclassical Einstein equations.
For simplicity, we assume (as usual) that the state of the environment can be described by a Gaussian wave function (eq. (4.3)). This is indeed a restricted class of states [10], but general states could also be implemented in our formalism. The arbitrary election of the coefficients $P_N$ and $Q_N$, in eq. (4.13), shows that the set of initial conditions is really arbitrary.

Finally, we can say that the existence of unstable states in the universe evolution (coming from singularities in the Riemann second sheet of the analytical extension of the S-matrix) can help us to understand the quantum to classical transition and other dissipative aspects of the universe evolution.

**Acknowledgments**

We would like to thank I. Prigogine, L. Bombelli, E. Gunzig and F.D. Mazzitelli for discussions. This work was partially supported by the Directorate-General for Science, Research and Development of the Commission of the European Communities under contract ECRU002 (DG-III), by the Institute Internationaux de Physique et de Chimie Solvay, and the University of Buenos Aires.

**Appendix A**

Equation (2.7) reads:

$$\Omega_k^2 = k^2 + m^2 a^2,$$

so the relation between the physical energy and momentum is

$$\left( \frac{\Omega_k}{a} \right)^2 = m^2 + \left( \frac{k}{a} \right)^2.$$  

(A.1)

From (3.1) we know that we will have a resonance if

$$\left( \frac{\Omega_n}{a} \right)^2 = m^2 + \frac{mB}{a^2} \left[ - \frac{mA^2}{B} - 2i(n + \frac{1}{2}) \right].$$

(A.3)

Since $a \to +\infty$ the only way to have resonance for energies different from $m$, is that at the same time $n \to +\infty$, so let us define

$$N(n, a) = \frac{n}{a^2},$$

(A.4)

when both $a \to +\infty$ and $n \to +\infty$. If $n$ grows in the proper way, $N(n, a)$ remains finite and we have
\[
\left( \frac{\Omega_k}{a} \right)^2 = m^2 - 2imBN. \quad (A.5)
\]

Then
\[
\frac{\Omega_k}{a} = \frac{1}{\sqrt{2}} \left[ m^2 + \sqrt{m^4 + 4m^2B^2N^2} \right]^{\frac{1}{2}} - \frac{i}{\sqrt{2}} \frac{2mBN}{\left[ m^2 + \sqrt{m^4 + 4m^2B^2N^2} \right]^{\frac{1}{2}}}. \quad (A.6)
\]

So the physical decaying time is
\[
\tau = \sqrt{2} \frac{a^2}{2mBN} \left[ m^2 + \sqrt{m^4 + 4m^2B^2n^2/a^4} \right], \quad (A.7)
\]
where \( a \gg 1 \) and therefore also \( n \gg 1 \) to get a finite result.
Let us now compute the difference between the square of two subsequent physical energies
\[
\left( \frac{\Omega_{n+1}}{a} \right)^2 - \left( \frac{\Omega_n}{a} \right)^2 = -i \frac{2mB}{a^2}. \quad (A.8)
\]
When \( a \to +\infty \) this difference vanishes and therefore, most likely, we have a cut singularity located at the physical energies
\[
\frac{\Omega_k}{a} = (m^2 - 2imBN)^{\frac{1}{2}}. \quad (A.9)
\]

It is easier to deal with a set of infinite poles that to work with cuts. Therefore we postpone, further discussion of cuts for future papers.

**Appendix B**

Let us demonstrate that there are an infinite set of poles or cuts in the general case.
Ordinary anihilation and creation operators are related by the usual canonical commutation relations
\[
[a_\omega, a^\dagger_{\omega'}] = \delta(\omega - \omega'). \quad (B.1)
\]
If we promote the real variable \( \omega \) to a complex variable \( z \), it is demonstrated in Ref. [5] that these relations must be substituted by
\[
[a_z, a^\dagger_{z'}] = \delta(z - z'), \quad (B.2)
\]
where \( z, z' \in \Gamma \), \( \Gamma \) is a contour in the complex plane (for details see Ref. [5]) and
\[
a_z^* = Ka_z^\dagger K^\dagger, \quad (B.3)
\]
where $K$ is the Wigner or time-inversion operator. In Refs. [5] and [6] it is proved that this operator changes $z$ into $z^*$ so really $a_z^*$ can be considered as a function of $z^*$ rather than $z$.

Now if we make a Bogolyubov transformation among operators labelled by $z$, the canonical commutation relations (3.6) must be kept invariant. Therefore the usual relations among Bogolyubov coefficients, such that $a_{ij} = \alpha_{ij} \delta_{ij}$, $\beta_{ij} = \beta_{ij} \delta_{ij}$, namely

$$|\alpha_\omega|^2 - |\beta_\omega|^2 = 1, \quad (B.4)$$

in the complex case it reads

$$\alpha_z a_{z^*} - \beta_z b_{z^*} = 1. \quad (B.5)$$

Now let us study the functions $\Lambda_z = \beta_z / \alpha_z$. The poles of these functions originate the poles of the S-matrix. Let us compute:

$$\frac{|\beta_z b_{z^*}|}{|\alpha_z a_{z^*}|} = \frac{|\beta_z b_{z^*}|}{|1 + \beta_z b_{z^*}|} = \frac{|\zeta|}{|1 + \zeta|}$$

$$= \frac{|\xi + i\eta|}{|1 + \xi + i\eta|} = \frac{(\xi^2 + \eta^2)^{\frac{1}{2}}}{[(1 + \xi)^2 + \eta^2]^\frac{1}{2}} < 1, \quad (B.6)$$

where $\beta_z b_{z^*} = \zeta = \xi + i\eta$. Then function $\beta_z b_{z^*} / \alpha_z a_{z^*}$ has a bounded modulus and therefore it cannot be analytic in all the complex plane if it is not a constant. But a constant is not a generic function. Therefore function $\Lambda_z \Lambda_{z^*}$ must have some singularities some poles and/or some cuts. If it is a cut, either $\Lambda_z$ or $\Lambda_{z^*}$ has a cut and therefore also the S-matrix, but a cut can be considered (in all the demonstrations of this paper) like an infinite set of poles and the process is finished and the theorem is demonstrated. On the other hand let us suppose that the function $\Lambda_z \Lambda_{z^*}$ has just a pole at $z_0$. Therefore either $\Lambda_z$ has a pole at $z_0$ or $\Lambda_{z^*}$ has a pole at this point. Thus either $S$ has a pole at $z_0$ or at $z_0^*$, thus in any case $S$ has at least one pole.

Now as function $\Lambda_z$ has a pole let us say at $z_0$, we can define a function

$$\Lambda_z^{(1)} = \Lambda_z - \frac{|\text{Res} \Lambda_z|_{z_0}}{z - z_0}. \quad (B.7)$$

Let us suppose that this function has no poles. We can compute the modulus $|\Lambda_z^{(1)} \Lambda_{z^*}^{(1)}|$ which obviously is bounded since

$$|\Lambda_z^{(1)} \Lambda_{z^*}^{(1)}| < 1, \quad (B.8)$$

if $z$ and $z^*$ are far enough of $z_0$, and also near to $z_0$ since $\Lambda_z^{(1)}$ has no pole there, because the only pole of $\Lambda_z$ has at $z_0$ and it has been eliminated via the second term of the r.h.s. of (B.7). Then function $\Lambda_z^{(1)} \Lambda_{z^*}^{(1)}$ is analytic and bounded in all the complex plane and therefore it is a constant. But again a constant is not a generic function so $\Lambda_z^{(1)} \Lambda_{z^*}^{(1)}$
have some singularities, either a cut, in which case the proof is finished, or it has a pole at \( z_0^{(1)} \). Now we can define a new function

\[
\Lambda_z^{(2)} = \Lambda_z^{(1)} - \frac{|\text{Res}\Lambda_z^{(1)}|_{z_0^{(1)}}}{z - z_0^{(1)}},
\]

and so forth and then prove that \( \Lambda_z \), and therefore \( S \), have both an infinite set of poles, or a cut, which can also be considered as an infinite continuous set of point-like singularities, for all the issues considered in this paper. We can also introduce another way to foresee the presence of this infinite set of poles.

According to what we have just said the equation \( \bar{\alpha}_{ik} = 0 \) of (3.4) has at least a root (since we cannot say that \( \bar{\beta}_{ik} \) would have any pole in a generic case). But this equation has a “periodical” nature (in the complex plane) as we can see if we try to solve a similar equation like

\[
\frac{1}{\Gamma(1-z)} = \xi.
\]

Using (1.2.2) of Ref. [16] this equation can be written as

\[
\frac{1}{\pi} \Gamma(z) \sin \pi z = \xi.
\]

Let us only consider the case \( \xi = y \in \mathbb{R}, z = x \in \mathbb{R}, x > 0 \) then

\[
\frac{1}{\pi} \Gamma(x) \sin \pi x = y.
\]

The function \( \Gamma(x) \) is always \( > 0 \) if \( x > 0 \) and it is a growing function. Then the l.h.s. of (B.12) is a periodical function modulated by a growing one, then we can see the periodical nature of the problem. Then the function (B.12) intersects at an infinite set of points the straight line \( y = \text{const} \). Therefore (B.12) has an infinite number of roots. As \( \bar{\alpha}_{ik} = 0 \) is a generalization of (B.12) and it has at least a roots, we can conclude that it has an infinite number of root owing its periodical nature.

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