Asymptotically Good Codes over non-Abelian Groups

Aria G. Sahebi and S. Sandeep Pradhan
Department of Electrical Engineering and Computer Science,
University of Michigan, Ann Arbor, MI 48109, USA.
Email: ariagh@umich.edu, pradhanv@umich.edu

Abstract—In this paper, we show that good structured codes over non-Abelian groups do exist. Specifically, we construct codes over the smallest non-Abelian group $D_6$ and show that the performance of these codes is superior to the performance of Abelian group codes of the same alphabet size. This promises the possibility of using non-Abelian codes for multi-terminal settings where the structure of the code can be exploited to gain performance.

Index Terms—group codes, structured codes, achievable rate, non-Abelian groups

I. INTRODUCTION

ALGEBRAICALLY structured codes are an important class of codes in coding/information theory and communications and evaluating the information-theoretic performance limits of such codes has been an area of significance [1]–[6]. It is well-known that linear codes achieve the symmetric capacity of $q$-ary channels where $q$ is a prime [7] [6]. Linear codes can also be used to compress a binary source losslessly down to its entropy [8]. Optimality of linear codes for certain communication problems motivates the study of algebraic-structured codes including Abelian and non-Abelian group codes.

In [8] it has been shown that for some multi-terminal communication settings, the average asymptotic performance of the ensemble of structured codes can be better than that of random codes. In recent years, such gains have been shown for a wide class of multi-terminal problems [5], [9], [10]. Thus, characterization of the information theoretic performance limits of these codes became important. However, the structure of the code restricts the encoder to abide by certain algebraic rules. This causes the performance of such codes to be inferior to random codes in some communication settings. Linear codes are highly structured and for some problems in information theory they cannot be optimal. Moreover, these codes can only be defined over alphabets of size a power of a prime.

Group codes are a generalization of linear codes which are algebraically structured and can be defined for any alphabet. These codes can outperform unstructured codes in certain communication problems [5]. Group codes were first studied by Slepian [11] for the Gaussian channel. In [12], the capacity of group codes for certain classes of channels has been computed. Further results on the capacity of group codes were established in [1], [13], [14].

II. PRELIMINARIES

1) Groups: A group is a set $G$ equipped with a binary operation “$\cdot$” to form an algebraic structure. The group operation “$\cdot$” must satisfy the group axioms (closure, associativity, identity and invertibility). A group is called Abelian if its operation is commutative and non-Abelian otherwise.

2) Group Codes: Given a group $G$, a group code $C$ over $G$ with block length $n$ is any subgroup of $G^n$ [3], [18]. A shifted group code over $G$, $C + v$ is a translation of a group code $C$ by a fixed vector $v \in G^n$.

3) Source and Channel Models: We consider discrete memoryless and stationary channels used without feedback. We associate two finite sets $\mathcal{X}$ and $\mathcal{Y}$ with the channel as the channel input and output alphabets. These channels can be characterized by a conditional probability law $W(y|x)$ for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. The set $\mathcal{X}$ admits the structure of a finite Abelian group $G$ of the same size. The channel is specified by $(G,\mathcal{Y},W)$. Assuming a perfect source coding block applied prior to the channel coding, the source of information generates messages over the set \{1, 2, \ldots, M\} uniformly.

It has been shown in [14] that Abelian group codes do not achieve the capacity of arbitrary channels. It has also been conjectured by several authors that non-Abelian group codes are inferior to Abelian group codes [15] [16] [17]. This motivates a loosening of the structure of the code yet further.

In this work, we focus on the point-to-point channel coding problem. We define a class of structured codes which includes the class of group codes and has less structure compared to group codes. We evaluate the performance of such codes over the smallest non-Abelian group $D_6$ and show that these codes have a strictly better performance compared to Abelian group codes. We use a combination of algebraic and information-theoretic tools for this task. This observation broadens our view to structured codes for possible use in multi-terminal settings.

The paper is organized as follows: In Section II, we introduce our notation and develop the required background. In Section III we define the ensemble of codes. We analyze the performance of these codes in Section IV where we solve an optimization problem and make several counting arguments. We compare the performance of the constructed codes to the performance of Abelian group codes in Section V and we conclude in Section VI.

This work was supported by NSF grants CCF-0915619 and CCF-1116021.
4) Achievability and Capacity: A transmission system with parameters \((n, M, \tau)\) for reliable communication over a given channel \((G, \mathcal{Y}, W)\) consists of an encoding mapping and a decoding mapping
\[
e : \{1, 2, \ldots, M\} \rightarrow G^n \\
f : \mathcal{Y}^n \rightarrow \{1, 2, \ldots, M\}
\]
such that for all \(m = 1, 2, \ldots, M\),
\[
\frac{1}{M} \sum_{m=1}^{M} W^n(f(Y^n) \neq m|X^n = e(m)) \leq \tau
\]
Given a channel \((G, \mathcal{Y}, W)\), the rate \(R\) is said to be achievable if for all \(\epsilon > 0\) and for all sufficiently large \(n\), there exists a transmission system for reliable communication with parameters \((n, M, \tau)\) such that
\[
\frac{1}{n} \log M \geq R - \epsilon, \ \tau \leq \epsilon
\]
The capacity of the channel is defined as the supremum of the set of all achievable rates.

5) Typicality: Consider two random variables \(X\) and \(Y\) with joint probability density function \(p_{X,Y}(x, y)\) over \(\mathcal{X} \times \mathcal{Y}\). Let \(n\) be an integer and \(\epsilon\) be a positive real number. The sequence pair \((x^n, y^n)\) belonging to \(\mathcal{X}^n \times \mathcal{Y}^n\) is said to be jointly \(\epsilon\)-typical with respect to \(p_{X,Y}(x, y)\) if
\[
\forall a \in \mathcal{X}, \ \forall b \in \mathcal{Y} : \left| \frac{1}{n} N(a, b|x^n, y^n) - p_{X,Y}(a, b) \right| \leq \frac{\epsilon}{|\mathcal{X}||\mathcal{Y}|}
\]
and none of the pairs \((a, b)\) with \(p_{X,Y}(a, b) = 0\) occurs in \((x^n, y^n)\). Here, \(N(a, b|x^n, y^n)\) counts the number of occurrences of the pair \((a, b)\) in the sequence pair \((x^n, y^n)\). We denote the set of all jointly \(\epsilon\)-typical sequence pairs in \(\mathcal{X}^n \times \mathcal{Y}^n\) by \(A^n\ epsilon(X, Y)\).

Given a sequence \(x^n \in \mathcal{X}^n\), the set of conditionally \(\epsilon\)-typical sequences \(A^n\ epsilon(Y|x^n)\) is defined as
\[
A^n\ epsilon(Y|x^n) = \{y^n \in \mathcal{Y}^n | (x^n, y^n) \in A^n\ epsilon(X, Y)\}
\]

6) Dihedral Groups: A dihedral group of order 2\(p\) is the group of symmetries of a regular \(p\)-gon, including reflections and rotations and any combination of these operations. A dihedral group can be represented as a quotient of a free group as follows:
\[
D_{2p} = \langle x, y | x^p = 1, y^2 = 1, xyxy = 1 \rangle
\]
Dihedral groups are among the simplest non-Abelian groups.

7) Notation: In our notation, \(O(\epsilon)\) is any function of \(\epsilon\) such that \(\lim_{\epsilon \to 0} O(\epsilon) = 0\) and for a set \(A\), \(|A|\) denotes its size (cardinality).

III. A CLASS OF STRUCTURED CODES

Based on Forney’s analysis of group codes \([3]\), we construct a class of structured codes which we call pseudo-group codes.

In this note, we consider a class of pseudo-linear codes over \(D_6\) and find a lower bound on the capacity of such codes.

Let \(A_1, \ldots, A_n\) be subgroups of \(G\). We propose a method to construct a code whose output at time \(k\) forms the subgroup \(A_k\) of \(G\) and whose input is the subgroup \(F_k\) of \(A_k\):

- Choose the controllability index of the code \(\nu \in \mathbb{Z}^+\).
- For each \(k\), choose a normal series \(F_{k,0} \subset F_{k,1} \subset \ldots \subset F_{k,\nu} = F_k\) in \(F_k\).
- For each \(k\), define granules \(\Gamma_{[k,k]} = F_{k,0} \subset F_{k,1} \subset \ldots \subset F_{k,\nu} = F_k\) and for all \(1 \leq j \leq \nu\), define granules \(\Gamma_{[k,k+j]} = F_{k,0} \subset F_{k,1} \subset \ldots \subset F_{k,\nu} = F_k\).
- For each \(k\) and for all \(0 \leq j \leq \nu\) find \(C_{[k,k+j]}\) such that \(C_{[k,k+j]} \leq A_k\) and \(C_{[k,k+j]} \cap C_{[k,k+j-1]} \neq \emptyset\).

Definition III.1. Any code constructed using the above construction algorithm is called a pseudo-linear code over \(G\).

Theorem III.1. Any free group code is a pseudo-linear code.

Proof: It has been shown in \([3]\) that any free group code can be reconstructed using the above construction algorithm. This completes the proof.

For Abelian groups, the definition of pseudo-group codes coincides with the definition of group codes but for non-Abelian groups this class is larger than the class of group codes; i.e. it includes all group codes as well as some non-group codes.

In the following, we construct the ensemble of codes over \(D_6\). Consider codes with input group \(F_k = D_6\) and output group \(A_k = D_6\) for all \(k\). Let \(\nu\) be a nonnegative integer number. We choose the following normal series:

\[
F_{k,0} \subset F_{k,1} \subset \ldots \subset F_{k,\nu} = F_k
\]

(In the most general case we can have a chain isomorphic to:
\[
1 \subset 1 \subset \ldots \subset 1 \subset Z_2 \subset \ldots \subset Z_2 \subset D_6 \subset \ldots \subset D_6
\]
Since the input and output groups are not changing over time, the granules are time independent. Define $\Gamma_j = \Gamma_{[k,k+j]}$. Then,

\[
\begin{align*}
\Gamma_0 &= F_{k,0} = C_{[k,k]} = 1 \\
\Gamma_1 &= F_{k,1}/F_{k,0} = 1 \\
&\vdots
\Gamma_{\nu-1} &= F_{k,\nu-1}/F_{k,\nu-2} = 1 \\
\Gamma_{\nu} &= F_{k,\nu}/F_{k,\nu-1} = D_6
\end{align*}
\]

Next step is to find the projection of the code over finite intervals. Note that in this case we have $C_{[k,k+j]} \cong C_{[0,j]}$.

\[
\begin{align*}
C_{[0,0]} &\cong \Gamma_0 = 1, C_{[0,0]} \leq D_6 \Rightarrow C_{[0,0]} = 1 \\
C_{[1,1]} &\cong C_{[0,0]} = 1 \\
&\Rightarrow C_{[1,1]} \cdot C_{[0,0]} = 1 \\
C_{[0,1]}/C_{[0,0]} \cdot C_{[1,1]} &\cong \Gamma_1 = 1, C_{[0,1]} \leq D_6 \Rightarrow C_{[1,1]} = 1 \\
&\vdots \\
C_{[1,\nu-1]} &\cong C_{[0,\nu-2]} = 1 \\
&\Rightarrow C_{[0,\nu-2]} \cdot C_{[1,\nu-1]} = 1 \\
C_{[0,\nu-1]}/C_{[0,\nu-2]} \cdot C_{[1,\nu-1]} &\cong \Gamma_{\nu-1} = 1, C_{[0,\nu-1]} \leq D_6 \\
&\Rightarrow C_{[1,\nu-1]} = 1 \\
C_{[1,\nu]} &\cong C_{[0,\nu-1]} = 1 \\
&\Rightarrow C_{[0,\nu-1]} \cdot C_{[1,\nu]} = 1 \\
C_{[0,\nu]}/C_{[0,\nu-1]} \cdot C_{[1,\nu]} &\cong \Gamma_{\nu} = D_6, C_{[0,\nu]} \leq D_6^{\nu+1}
\end{align*}
\]

Therefore,

\[
C_{[0,\nu]} = \langle g^0, h^0 \rangle = \langle g^0, h^0 \rangle^3 = 1, (h^0)^2 = 1, g^0 h^0 g^0 h^0 = 1 \tag{2}
\]

where $g^0, h^0 \in D_6^{\nu+1}$.

It can be shown that if we take $g^0 = g_{00}g_{01}\ldots g_{0\nu}$ and $h^0 = h_{00}h_{01}\ldots h_{0\nu}$ where $g_{0i}$ and $h_{0i}$ are chosen jointly according to Table 1 then the third condition in Equation 2 will also be satisfied. Therefore for any such $g^0$ and $h^0$, the group $C_{[0,\nu]} = \langle g^0, h^0 \rangle = \langle (g^0)^3 = 1, (h^0)^2 = 1, g^0 h^0 g^0 h^0 = 1 \rangle$ is a subgroup of $D_6$. As tough our input group is only restricted to be a subgroup of $D_6$ and not necessarily $D_6$ itself.

Similarly, we get $C_{[k,k+j]} = \langle g^k, h^k \rangle = \langle (g^k)^3 = 1, (h^k)^2 = 1, g^k h^k g^k h^k = 1 \rangle$ where $g^k = g_{k0}g_{k1}\ldots g_{k\nu}$ and $h^k = h_{k0}h_{k1}\ldots h_{k\nu}$ and $g_{0i}$’s and $h_{0i}$’s are chosen according to Table 1.

Note that any element in $D_6$ can be uniquely written as $x^\alpha y^\beta$ for some $\alpha \in Z_6$ and $\beta \in Z_2$. Define the transversal functions as:

\[
T_k(x^\alpha y^\beta) = (g^k)^\alpha (h^k)^\beta
\]

Let $\ldots u_0 u_1 u_2 \ldots$ be the information digits where $u_i = x^{a_i} y^{b_i}$, then the output of the code is $\ldots c_0 c_1 c_2 \ldots$ where $c_i = B_{i-1} u_i g_{k-1} u_{i-k} g_{k-1} h_{k-1} u_{i-k} \ldots g_{k-1} h_{k-1}$.

Assume the input is fed circularly to the code (tail biting). We also add a dither to the code.

Here we give a summary of the resulting ensemble of codes. Each code in this ensemble has a rate of $R = \frac{1}{n} \log 6$.

- For $i = 1, \ldots, n$ and $j = 1, \ldots, k$ choose $g_{ij}$ and $h_{ij}$ randomly according to Table 1 for $(i,j) \neq (i',j')$, $(g_{ij}, h_{ij})$ is chosen independently from $(g_{i'j'}, h_{i'j'})$.

- For $i = 1, \ldots, n$, choose the dither $B_i$ uniformly randomly from $D_6$.

- Given the input sequence $u = (u_1, \ldots, u_k)$ where $u_i = x^{a_i} y^{b_i}, a_i \in Z_6, b_i \in Z_2$ for $i = 1, \ldots, k$, the output sequence is equal to $c = (c_1, \ldots, c_n)$ where

\[
c_1 = g_{11} h_{11} g_{12} h_{12} \cdots g_{1k} h_{1k} \cdot B_1 \\
c_2 = g_{21} h_{21} g_{22} h_{22} \cdots g_{2k} h_{2k} \cdot B_2 \\
\vdots \\
c_n = g_{n1} h_{n1} g_{n2} h_{n2} \cdots g_{nk} h_{nk} \cdot B_n
\]

We denote this by $c = G(u) \cdot B$.

![Fig. 1: $g_{ij}$ is chosen from $\{1, x, x^2\}$ and $h_{ij}$ is chosen from $\{y, xy, x^2y\}$]
Then the rate $R^*$ is achievable using pseudo-group codes over $D_6$ where

$$R^* = \min \left( \log_2 \frac{6}{H(X|Y)} \right)$$

The rest of this section is devoted to give a sketch of the proof of this theorem. Consider the class of pseudo-group codes over $D_6$ of the form used for the channel $(D_6, Y, W)$. The set of messages is $u \in D_6^k$ and for each message $u \in D_6^k$ the encoder maps it to $c \in D_6^3$ where $c = G(u) \cdot B$. At the receiver, after receiving the channel output $y \in \mathbb{Z}^m$, the decoder looks for a message $\hat{u} \in D_6^k$ such that $\hat{c} = \hat{G}(\hat{u}) \cdot B$ is jointly $c$-typical with $y$ with respect to $P_X W_{Y|X}$ where $P_X$ is uniform over $D_6$ and $\epsilon > 0$ is arbitrary. If it finds a unique such $\hat{c}$, it decodes $y$ to $\hat{u}$, otherwise it declares error.

The expected value of the average probability of error for this coding scheme is given by

$$E\{P_{avg}(err)\} = \sum_{u \in D_6^k} \frac{1}{6^k} \sum_{\hat{c} \in D_6^3} P(G(u) \cdot B = c) \sum_{\hat{u} \neq u \in A_\hat{c}(X|y)} P(G(\hat{u}) \cdot B = \hat{c}|G(u) \cdot B = c) W(y|c) + O(\epsilon)$$

We need to evaluate the conditional probability $P(G(\hat{u}) \cdot B = \hat{c}|G(u) \cdot B = c)$ to proceed. For $u, \hat{u} \in D_6^k$ and $x, \hat{x} \in D_6^n$, let $u = (u_1, \ldots, u_k)$ where $u_i = x^{g_i} y^{b_i}$ for $i = 1, \ldots, k$ and $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_k)$ where $\hat{u}_i = x^{g_i} y^{b_i}$ for $i = 1, \ldots, k$.

Also let $c = (c_1, \ldots, c_n)$ and $\hat{c} = (\hat{c}_1, \ldots, \hat{c}_n)$ and define $\theta = c \hat{c}^{-1} = (\theta_1, \ldots, \theta_n)$ where $\theta_i = x^{a_i} y^{\beta_i}$. Define the following:

$N_1(c, \hat{c}) = \{ i \in [1, \ldots, n] | \beta_i = 1 \}$

$N_2(c, \hat{c}) = \{ i \in [1, \ldots, n] | \beta_i = 0, \alpha_i \neq 0 \}$

$N_3(c, \hat{c}) = \{ i \in [1, \ldots, n] | \beta_i = 0, \alpha_i = 0 \} = n - n_1 - n_2$

$M_1(u, \hat{u}) = \{ i \in [1, \ldots, k] | b_i \neq \hat{b}_i \}$

$M_2(u, \hat{u}) = \{ i \in [1, \ldots, k] | b_i = \hat{b}_i, a_i \neq \hat{a}_i \}$

$M_3(u, \hat{u}) = \{ i \in [1, \ldots, k] | b_i = \hat{b}_i, a_i = \hat{a}_i \} = k - m_1 - m_2$

Also define $n_1(c, \hat{c}) = |N_1(c, \hat{c})|$, $n_2(c, \hat{c}) = |N_2(c, \hat{c})|$, $n_3(c, \hat{c}) = |N_3(c, \hat{c})|$, $m_1(u, \hat{u}) = |M_1(u, \hat{u})|$, $m_2(u, \hat{u}) = |M_2(u, \hat{u})|$, $m_3(u, \hat{u}) = |M_3(u, \hat{u})|$.

**Lemma IV.1.** For $u, \hat{u} \in D_6^k$ and $c, \hat{c} \in D_6^3$, if $\hat{u} \neq u$, then

$$P(G(\hat{u}) \cdot B = \hat{c}|G(u) \cdot B = c) = \frac{1}{10^{km}} \cdot \left[ \frac{10^{k-m_1+1}}{3} \left( \sum_{l=1}^{m_1} \binom{m_1}{l} g_l^{l-1} \right)^{n_1} \right] \cdot \left[ \frac{10^{k-m_2} - 2}{3} + \frac{10^{k-m_1+k}}{3} \left( \sum_{l=2}^{m_1} \binom{m_1}{l} g_l^{l-1} \right)^{n_2} \right] \cdot \left[ \frac{10^{k-m_2-1}}{3} + \frac{10^{k-m_1+k}}{3} \left( \sum_{l=2}^{m_1} \binom{m_1}{l} g_l^{l-1} \right)^{n_3} \right]$$

Moreover, for a fixed $u$, let $T_{m_1, m_2}(u)$ be the set of all $\hat{u}$ with $m_1(u, \hat{u}) = m_1$, $m_2(u, \hat{u}) = m_2$, then

$$|T_{m_1, m_2}(u)| = \left( \begin{array}{c} k \\ m_1 \end{array} \right) \cdot \left( \begin{array}{c} k - m_1 \\ m_2 \end{array} \right) \cdot 3^{m_1} \cdot 2^{m_2}$$

**Proof:** First Note that

$$P(G(\hat{u}) \cdot B = \hat{c}|G(u) \cdot B = c) =$$

Hence, we first find the $j$th probability in this expression for some arbitrary $i \in 1, 2, \ldots, n$. Consider the case where $\beta_i = 1$. Since the ensemble has a uniform distribution, we need to count the number of $g_j$’s and $h_{ij}$’s such that the equality

$$g_1^{a_1} h_1^{b_1} \cdots g_k^{a_k} h_k^{b_k} \cdots = x^{a_i} y^{\beta_i}$$

is satisfied and divide this number to the total number of choices. Use the equality $yx = x^2 y$ to argue that the power of $y$ on the left hand side of this expression adds, i.e. the power of $y$ on the left hand side is equal to the sum of the powers of $y$ terms appearing in the expression. This is equal to

$$\sum_{j=1}^{k} (b_j + \hat{b}_j)$$

Where the addition is done mod-2. This can be written as

$$\sum_{j \in M_1(c, \hat{c})} (b_j + \hat{b}_j) = \left( [j \in M_1]_{h_{ij} \in \{y, xy, x^2 y\}} \right)$$

Let $L \subseteq M_1$ be the set of indices $j \in M_1$ where $h_{ij}, j \in \{y, xy, x^2 y\}$. Since the power of $y$ on the right hand side is equal to one, the cardinality of $L$ must be odd. We count the number of solutions of (4) as follows: Let $L \subseteq M_1$ be arbitrary with an odd cardinality. For $j \notin M_1$ choose $g_{ij}$ and $h_{ij}$ arbitrarily $(10^{k-m_2})$ choices. For $j \in M_1 \setminus L$ let $g_{ij} = 1$ and $h_{ij} = 1$ (1 choice). Since the cardinality of $L$ is assumed to be odd, it should have at least one element (say $j^*$). For $j \in L \setminus \{j^*\}$ choose $g_{ij}$ from $\{1, x, x^2\}$ and $h_{ij}$ from $\{y, xy, x^2 y\}$ arbitarily $(y^{l-1})$ choices where $l = |L|$. Also choose $g_{ij^*}$ arbitrarily from $\{1, x, x^2\}$ (3 choices). After moving terms to the other side we will get the following expression

$$h_{ij^*} = x^{\text{some power, some power}}$$

Note that, by construction, the power of $y$ on the right hand side has to be equal to $y$ and hence $h_{ij^*}$ has a unique solution
in \( \{y, xy, x^2y\} \). Therefore, it turns out that the number of choices for the case \( \beta_i = 1 \) is equal to
\[
10^{k-m_1} \cdot 3 \cdot \sum_{\substack{l=1 \\text{odd}}}^{m_1} \binom{m_1}{l} g^{l-1}
\]
Note that if \( m_1 = 0 \) the above expression is defined to be zero.

Now consider the case where \( \beta_i = 0 \). In this case it is possible to have \( m_1 = 0 \). We will first consider this case. Since \( m_1 = 0 \), for all \( j = 1, \ldots, k \) we have \( \tilde{b}_j = b_j \). On the other hand, since \( \tilde{u} \neq u \), there must exist an index \( j \in 1, \ldots, k \) such that \( \tilde{a}_j \neq a_j \). It can be shown that the total number of choices in this case is equal to
\[
10^{k-1} \cdot 3 \left( 10^{m_2} - 1 \right)
\]
if \( \theta_i \neq 1 \) and it is equal to
\[
10^{k-1} \cdot 3 \left( 10^{m_2} + 1 \right)
\]
if \( \theta_i = 1 \). The next and the last case is where \( \beta_i = 0 \) and \( m_1 \neq 0 \). In this case the argument is similar to the case where \( \beta_i = 1 \) (and hence \( m_1 \neq 0 \)). The difference here is that we need to choose a subset \( L \) of \( M_1 \) with an even cardinality \( l \).

For \( l > 0 \) the argument is similar to the case with \( \beta_i = 1 \) and for \( l = 0 \), the argument is similar to the case with \( \beta_i = 0 \) and \( m_1 = 0 \). Therefore the number of choices in this case is equal to
\[
10^{k-m_2} \left( 10^{m_2} - 1 \right)
\]
if \( \theta_i \neq 1 \) and it is equal to
\[
10^{k-m_2} \left( 10^{m_2} + 1 \right)
\]
if \( \theta_i = 1 \).

Since the total number of choices is equal to \( 10^{kn} \) the assertion about the conditional probability in the lemma follows. For a fixed \( u \in \mathbb{D}_d^k \), the number of \( \tilde{u} \)'s in \( \mathbb{D}_d^k \) such that \( m_1(u, \tilde{u}) = m_1 \) and \( m_2(u, \tilde{u}) = m_2 \) is calculated as follows.

Fix \( m_1 \) positions out of \( k \) positions (\( \binom{k}{m_1} \) choices) and in these positions let \( \tilde{b}_j = b_j + 1 \) and \( \tilde{a}_j \) arbitrary (\( 3^{m_1} \) choices). Fix \( m_2 \) positions among the remaining \( k - m_1 \) positions (\( \binom{k-m_1}{m_2} \) choices) and in these positions let \( \tilde{b}_j = b_j \) and choose \( \tilde{a}_j \neq a_j \) (\( 2^{m_2} \) choices). It follows that the total number of choices for \( \tilde{u} \) is equal to
\[
|T_{m_1, m_2}(u)| = \binom{k}{m_1} \cdot 3^{m_1} \cdot 2^{m_2}
\]

**Lemma IV.2.** Let \( y \in \mathbb{D}_d^n \) be an arbitrary channel output sequence. For any \( x \in A^n_m(X|y) \), we have
\[
|\{ x : \{y, xy, x^2y\}^{n_1} \times \{x, x^2\}^{n_2} \times \{1\}^{n_1-n_2} \cap A^n_m(X|y) \}|
\]
and in turn we have the following lemma:

Define
\[
A(m_1) = \sum_{\substack{l=1 \\text{odd}}}^{m_1} \binom{m_1}{l} g^{l-1}
\]
\[
B(m_1, m_2) = \frac{(10^{m_2} + 2)}{10^{m_2}} + \sum_{\substack{l=2 \\text{even}}}^{m_1} \binom{m_1}{l} g^{l-1}
\]
\[
C(m_1, m_2) = \frac{(10^{m_2} - 1)}{10^{m_2}} + \sum_{\substack{l=2 \\text{even}}}^{m_1} \binom{m_1}{l} g^{l-1}
\]
Using the above lemma and definitions, the expected value of the average probability of error can be upper bounded by:
\[
E[\{P_{\text{avg}}(err)\}]
\]
\[
\leq \sum_{m_1=0}^{k} \sum_{m_2=0}^{k-m_1} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1-1} \left( \binom{k}{m_1} \binom{k-m_1}{m_2} \cdot 3^{m_1} \cdot 2^{m_2} \frac{1}{10^{kn}} \right)
\]
\[
10^{(kn-m_1)} \cdot \frac{1}{3^n} A(m_1) B(m_1, m_2)^{n_1-n_2} C(m_1, m_2)^{n_2}
\]
\[
|\{ x : \{y, xy, x^2y\}^{n_1} \times \{x, x^2\}^{n_2} \times \{1\}^{n_1-n_2} \cap A^n_m(X|y) \}|
\]

This quantity can be upper bounded by
\[
|\{ x : \{y, xy, x^2y\}^{n_1} \times \{1, x, x^2\}^{n_1-n_2} \cap A^n_m(X|y) \}|
\]

**Proof:** First we prove the following:
\[
|\{ x : \{y, xy, x^2y\}^{n_1} \times \{1, x, x^2\}^{n_1-n_2} \cap A^n_m(X) \}|
\]
\[
\leq \left( \frac{n}{n_1} \right) 2^{n[H(X|X)]+O(\epsilon)}
\]

Where the random variable \( X \) takes value from the set of cosets of \( \{1, x, x^2\} \) in \( \mathbb{D}_d \).

The cardinality in question is related to the conditional entropy of a random variable \( W \) jointly distributed with \( X \) which
satisfies the constraints of the following optimization problem:

$$\min_{p(g,w)} -H(X,W) = \sum_{g \in D_6} \sum_{w \in D_6} p(g,w) \log p(g,w)$$

s.t.

$$\sum_{w \in D_6} p(g,w) = P_X(g)$$

$$\sum_{w \in D_6} p(g \cdot w^{-1},w) = P_X(g)$$

$$\sum_{g \in D_6, w \in \{1, x, x^2\}} p(x,w) = \alpha$$

$$\sum_{g \in D_6, w \in \{y, xy, x^2y\}} p(x,w) = 1 - \alpha$$

where $\alpha = \frac{n-m_1}{m}$ and the minimization is over all probability mass functions on $D_6 \times D_6$. Let $p = P_X(\{1, x, x^2\})$. It can be shown that if $\alpha \geq |1 - 2p|$, the following distribution satisfies the KKT conditions for this optimization problem.

$$p_{XW}(x,w) = \begin{cases} 
  \frac{2p-1+\alpha}{2(1-p)^2} p_X(x) P_X(x \cdot w) & x, w \in \{1, x, x^2\} \\
  \frac{1-p+\alpha}{2p(1-p)^2} p_X(x) P_X(x \cdot w) & x \in \{y, xy, x^2y\}, w \in \{1, x, x^2\} \\
  \frac{1-\alpha}{2p(1-p)} p_X(x) P_X(x \cdot w) & x \in \{y, xy, x^2y\}, w \in \{y, xy, x^2y\} 
\end{cases}$$

The entropy of this joint pmf can be found to be equal to

$$H(X,W) = 2H(X) + h(\frac{2p-1+\alpha}{2}, \frac{1-2p+\alpha}{2}, \frac{1-\alpha}{2}, \frac{1-\alpha}{2})$$

$$-2h(p)$$

where $p = P_X(\{1, x, x^2\})$ and $h$ is the entropy function. Next we prove that for $\alpha \geq |1 - 2p|$, $h(\frac{2p-1+\alpha}{2}, \frac{1-2p+\alpha}{2}, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}) \leq h(\alpha) + h(p)$

For $\alpha = 0$ this statement is trivial. Assume $\alpha \neq 0$. Note that

$$h(\frac{2p-1+\alpha}{2}, \frac{1-2p+\alpha}{2}, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}) =$$

$$-2p - 1 + \alpha \log \frac{2p - 1 + \alpha}{2\alpha} - 1 - 2p + \alpha \log \frac{1 - 2p + \alpha}{2\alpha} + (1 - \alpha) \log \frac{1 - \alpha}{2\alpha}$$

$$= -2p + \frac{1 - \alpha}{2\alpha} + \log \frac{2p - 1 + \alpha}{2\alpha} + \frac{1 - 2p + \alpha}{2\alpha}$$

$$= \alpha \left[ h(\frac{2p-1+\alpha}{2\alpha}) - h(\frac{2p-1+\alpha}{2}) + (1 - \alpha) \log(1 - \alpha) + (1 - \alpha) \right]$$

$$= \alpha \left[ h(\frac{2p-1+\alpha}{2\alpha}) + (1 - \alpha) + h(\frac{2p-1+\alpha}{2}) + (1 - \alpha) + h(\frac{2p-1+\alpha}{2}) + (1 - \alpha) + h(\alpha) \right]$$

Since the function $h(\cdot)$ is convex, for two points $x_1, x_2 \in [0, 1]$ and a number $\alpha \in [0, 1]$, we have

$$\alpha h(x_1) + (1 - \alpha) h(x_2) \leq h(\alpha x_1 + (1 - \alpha) x_2)$$

Let $x_1 = \frac{2p-1+\alpha}{2\alpha}$ and $x_2 = \frac{1 - \alpha}{2}$ to get

$$\alpha h(\frac{2p-1+\alpha}{2\alpha}) + (1 - \alpha) h(\frac{2p - 1 + \alpha}{2} + \frac{1 - \alpha}{2}) = h(p)$$

Therefore

$$\frac{2p-1+\alpha}{2}, \frac{1-2p+\alpha}{2}, \frac{1-\alpha}{2}, \frac{1-\alpha}{2} \leq h(\alpha) + h(p)$$

Using the equality $H([X]) = h(p)$ we get $H(W|X) \leq H(X||X) + h(\alpha)$. Finally we use Stirling’s approximation to get

$$[c \cdot \{y, xy, x^2y\}^n \times \{1, x, x^2\}^{n-n_1}] \cap A^n(X)$$

$$\leq \left( \frac{n}{n_1} \right)^{2n[H(X||X)] + O(e)}$$

The generalization to the statement of the lemma is relatively straight forward and is omitted.

It can be shown that for all $\delta > 0$, there exists an integer $M_1(\delta)$ such that for $m_1 \geq M_1(\delta)$, $A(m_1), B(m_1, m_2), C(m_1, m_2) < \frac{10^{-n_1}}{2(1-\delta)}$. It can also be shown that for all $\delta' > 0$, there exists an integer $M_2(\delta')$ such that for $m_2 \geq M_2(\delta')$, $A(m_1) < \frac{10^{-n_1-s_1\delta}}{2(1-\delta')}$ and $B(m_1, m_2), C(m_1, m_2) < \frac{10^{-n_1-s_1\delta}}{2(1-\delta')}$.

For arbitrary $\delta, \delta' > 0$, we break the expected error probability into several terms as follows:

$$\mathbb{E}[P_{\text{avg}}(\text{err})] = P_1(\delta) + \sum_{m_1=0}^{M_1(\delta)-1} P_2(m_1, \delta') + P_3(\delta, \delta')$$

where

$$P_1(\delta) = \sum_{m_1=M_1(\delta)}^{k} \sum_{m_2=0}^{k-m_1} \sum_{n_1=0}^{n-m_1} \sum_{n_2=0}^{n-n_1} \left( \frac{k}{m_1} - \frac{1}{m_2} \right) \cdot 3^{m_1} \cdot 2^{m_2}$$

$$P_2(m_1, \delta') = \sum_{m_2=M_2(\delta')}^{k-m_1} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \left( \frac{k}{m_1} - \frac{1}{m_2} \right) \cdot 3^{m_1} \cdot 2^{m_2}$$

$$P_3(\delta, \delta') = \sum_{m_1=M_1(\delta)}^{k} \sum_{m_2=M_2(\delta')}^{k-m_1} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \left( \frac{k}{m_1} - \frac{1}{m_2} \right) \cdot 3^{m_1} \cdot 2^{m_2}$$

and $P_3(\delta, \delta')$ is defined similar to $P_1(\delta)$ except that the first summation runs from 0 to $M_1(\delta) - 1$ and the second summation runs from 0 to $M_2(\delta') - 1$. Next we show that

$$P_1(\delta) \leq \exp_2 \left\{ -n \left[ \log_2(6(1-\delta)) - \frac{k}{n} \log_2 6 - H(X|Y) \right] \right\}$$

and

$$P_2(m_1, \delta') \leq \exp_2 \left\{ -n \left[ \log_2(3(1-\delta)) - \frac{k}{n} \log_2 3 - H(X|X|Y) + O(e) \right] \right\}$$

and note that $P_3(\delta, \delta')$ is independent of the rate $R$ (It can be shown that this term goes to zero as $n$ increases regardless of
the value of \( R \). We have 

\[ P_1(\delta) = \sum_{m_1 = M(\delta)}^{k} \sum_{m_2 = 0}^{k-m_1} \sum_{n_1 = 0}^{n-m_1} \sum_{n_2 = 0}^{n-m_2} \binom{k}{m_1} \binom{k-m_1}{m_2} \cdot 3^{m_1} \cdot 2^{m_2}. \]

For a fixed \( m_1 \), consider 

\[ P_2(m_1, \delta') = \sum_{m_2 = M(\delta')}^{k} \sum_{n_1 = 0}^{n-m_1} \sum_{n_2 = 0}^{n-m_2} \binom{k}{m_1} \binom{k-m_1}{m_2} \cdot 3^{m_1} \cdot 2^{m_2}. \]

For a fixed \( m_1 \), consider 

\[ P_1(\delta) \leq \sum_{m_1 = M(\delta)}^{k} \sum_{m_2 = 0}^{k-m_1} \binom{k}{m_1} \binom{k-m_1}{m_2} \cdot 3^{m_1} \cdot 2^{m_2}. \]

Note that the result of the last two summations is simply equal to 

\[ |A^\beta(X|y)| = 2^{n[H(X|Y) + O(\epsilon)]} \]

Therefore, 

\[ P_1(\delta) \leq \sum_{m_1 = M(\delta)}^{k} \binom{k}{m_1} \binom{k-m_1}{m_2} \cdot 3^{m_1} \cdot 2^{m_2}. \]

Note that the result of the last summation is equal to \( 3^{k-m_1} \) and hence the result of the last two summations is equal to \( 6^k \). Therefore, 

\[ P_1(\delta) \leq \exp \left\{ -n \left[ \log_2 6 (1 - \delta) - \frac{k}{n} \log_2 6 - H(X|Y) \right] \right\} \]
Note that for a fixed $m_1$, $(\frac{k}{m_1})^{3m_1}$ is a polynomial in $k$. Hence,
\[
P_2(m_1, \delta') \leq \exp_2 \left\{-n \left[ \log_2[3(1-\delta)] - \frac{k}{n} \log_2 3 - H(X|Y) + O(\epsilon) \right] \right\}
\]
We observe that if $R < \log_2[6(1-\delta)] - H(X|Y)$ then $P_1(\delta)$ goes to zero as $n$ increases and if $R < \frac{\log_2 6}{\log_2 3} \left\{ \log_2 [3(1-\delta')] - H(X|[X|Y]) \right\}$ then $P_2(m_1, \delta')$ vanishes as the block length increases. Therefore, if both conditions are satisfied the expected value of the block error probability goes to zero. In conclusion, The rate $R$ is achievable if
\[
\begin{cases}
R < \log_2[6(1-\delta)] - H(X|Y) \\
R < \frac{\log_2 6}{\log_2 3} \left\{ \log_2 [3(1-\delta')] - H(X|[X|Y]) \right\}
\end{cases}
\]
Since $\delta$ and $\delta'$ are arbitrary, we conclude that the rate $R^*$ is achievable where
\[
R^* = \min \left( \log_2 6 - H(X|Y), \frac{\log_2 6}{\log_2 3} \left\{ \log_2 3 - H(X|[X|Y]) \right\}, \log_2 6 [1 - H(X|[X_2]|Y)] \right)
\]
where $[X|3]$ takes values from cosets of $\{0, 2, 4\}$ and $[X|2]$ takes values from cosets of $\{0, 3\}$. In the following example we show that the achievable rate using the new code can be strictly larger than the rate achievable using Abelian group codes.

A. An Example

We give an example where the capacity of group codes is zero whereas the constructed code achieves a strictly positive rate. Consider the channel depicted in the figure below where $\epsilon_1 = 0.1$, $\epsilon_1 = 0.2$ and $\epsilon_1 = 0.15$. If we maximize over all possible labellings of the channel input alphabet, it can be shown that both coding schemes achieve the symmetric capacity of the channel which is equal to 0.0139 bits per channel use. However, if the labels are assumed to be fixed, the achievable rate using pseudo-group codes is equal to $R^* = \min(0.0139, 0.0227) = 0.0139$ and the achievable rate using Abelian group codes is equal to $R = \min(0.0139, 0.0227, 0) = 0$. Indeed using the converse provided in [14] we can show that the capacity of Abelian group codes over this channel is equal to zero. We observe that for this channel, the codes over $D_6$ outperforms the code over $Z_6$.

B. Comparison

If we compare the two achievable rate regions, we observe that for the case of abelian group codes there is an additional term in the minimization which can be explained by the additional structure of the abelian group codes. Indeed, the pseudo-group code over $D_6$ is additive (homomorphic) with respect to the $y$ generator and is not homomorphic with respect to the $x$ generator whereas Abelian group codes are homomorphic with respect to both of their generators. This means compared to Abelian Group codes, the constructed codes gain a higher rate by reducing the structure.

VI. Conclusion

We have shown that good structured codes over non-Abelian groups do exist. We constructed codes over the smallest non-Abelian group $D_6$ and showed that the performance of these codes is superior to the performance of Abelian group codes of the same alphabet size.

REFERENCES

[1] R. Ahlswede and J. Gemma, “Bounds on algebraic code capacities for noisy channels I,” Information and Control, vol. 19, no. 2, pp. 124–145, 1971.
[2] S. S. Pradhan and K. Ramchandran, “Distributed source coding using syndromes (DISCUS): Design and construction,” IEEE Transactions on Information Theory, vol. 49, no. 3, pp. 626–643, 2003.
[3] G. D. F. Jr and M. Trott, “The dynamics of group codes: State spaces, trellis diagrams, and canonical encoders,” IEEE Transactions on Information Theory, vol. 39, no. 9, pp. 1491–1513, 1993.
[4] G. Como, and F. Fagnani, “The capacity of finite abelian group codes over symmetric memoryless channels,” IEEE Transactions on Information Theory, vol. 55, no. 5, pp. 2037–2054, 2009.
[5] D. Krithivasan and S. S. Pradhan, “Distributed source coding using abelian group codes,” 2011, IEEE Transactions on Information Theory(57):1495-1519.
[6] R. L. Dobrushin, “Asymptotic optimality of group and systematic codes for some channels,” Theor. Probab. Appl., vol. 8, pp. 47–59, 1963.
[7] P. Elias, “Coding for noisy channels,” IRE Conv. Record, vol. part. 4, pp. 37–46, 1955.
[8] J. Korner and K. Marton, “How to encode the modulo-two sum of binary sources,” IEEE Transactions on Information Theory, vol. IT-25, pp. 219–221, Mar. 1979.
[9] T. Philosof, A. Kishy, U. Erez, and R. Zamir, “Lattice strategies for the dirty multiple access channel,” Proceedings of IEEE International Symposium on Information Theory, July 2007, nice, France.
[10] B. A. Nazer and M. Gastpar, “Computation over multiple-access channels,” IEEE Transactions on Information Theory, vol. 53, no. 10 pages 3079-3090, Oct. 2007.
[11] D. Slepian, “Group codes for the Gaussian channel,” Bell Syst. Tech. Journal, 1968.
[12] R. Ahlswede, “Group codes do not achieve Shannons’s channel capacity for general discrete channels,” The annals of Mathematical Statistics, vol. 42, no. 1, pp. 224–240, Feb. 1971.
[13] R. Ahlswede and J. Gemma, “Bounds on algebraic code capacities for noisy channels II,” Information and Control, vol. 19, no. 2, pp. 146–158, 1971.
[14] A. G. Sahebi and S. S. Pradhan, “On the Capacity of Abelian Group Codes Over Discrete Memoryless Channels,” July 2011, petersburg, Russia.
[15] D. Forney, “On the Hamming Distance Properties of Group Codes.”
[16] J. Interlando, R. Palazzo, and M. Elia, “Group block codes over nonabelian groups are asymptotically bad,” IEEE Transactions on Information Theory, vol. 42, pp. 1277–1280, 1996.
[17] P. Massey, “Many Non-Abelian Groups Support Only Group Codes That Are Conformant To Abelian Group Codes.”
[18] N. J. Bloch, Abstract Algebra With Applications. Englewood Cliffs, New Jersey: Prentice-Hall, Inc, 1987.