Consecutive square-free values of the form \([\alpha p], [\alpha p] + 1\)

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Abstract: In this short paper we shall prove that there exist infinitely many consecutive square-free numbers of the form \([\alpha p], [\alpha p] + 1\), where \(p\) is prime and \(\alpha > 0\) is irrational algebraic number. We also establish an asymptotic formula for the number of such square-free pairs when \(p\) does not exceed given sufficiently large positive integer.

Keywords: Consecutive square-free numbers, Asymptotic formula.
AMS Classification: 11L05 · 11N25 · 11N37.

1 Notations

Let \(N\) be a sufficiently large positive integer. The letter \(p\) will always denote prime number. By \(\varepsilon\) we denote an arbitrary small positive number, not necessarily the same in different occurrences. We denote by \(\mu(n)\) the Möbius function and by \(\tau(n)\) the number of positive divisors of \(n\). As usual \([t]\) and \(\{t\}\) denote the integer part, respectively, the fractional part of \(t\). Instead of \(m \equiv n \pmod{k}\) we write for simplicity \(m \equiv n \pmod{k}\). Moreover \(e(t) = \exp(2\pi it)\). Let \(\alpha > 0\) be irrational algebraic number. As usual \(\pi(N)\) is the prime-counting function.

Denote
\[
\sigma = \prod_p \left(1 - \frac{2}{p^2}\right). \tag{1}
\]

2 Introduction and statement of the result

In 1932 Carlitz \[3\] proved that there exist infinitely many consecutive square-free numbers. More precisely he established the asymptotic formula
\[
\sum_{n \leq N} \mu^2(n)\mu^2(n + 1) = \sigma N + \mathcal{O}(N^{\theta + \varepsilon}), \tag{2}
\]
where \(\sigma\) is denoted by \((1)\) and \(\theta = 2/3\).

Subsequently the reminder term of \((2)\) was improved by Mirsky \[9\] and Heath-Brown \[8\]. The best result up to now belongs to Reuss \[10\] with \(\theta = (26 + \sqrt{433})/81\).
In 2008 Gülöglu and Nevans [7] showed by asymptotic formula that the sequence
\[
\{[\alpha n]\}_{n=1}^{\infty}
\] (3)
contains infinitely many square-free numbers, where \(\alpha > 1\) is irrational number of finite type.

Recently Akbal [1] considered the sequence (3) with prime numbers and proved the when \(k \geq 2\) and \(\alpha > 0\) is of type \(\tau \geq 1\), then there exist infinitely many \(k\)-free numbers of the form \([\alpha p]\). Akbal also established an asymptotic formula for the number of such \(k\)-free numbers when \(p\) does not exceed given sufficiently large real number \(x\).

As a consequence of his result Akbal obtained the following

**Theorem 1.** Let \(\alpha > 0\) be an algebraic irrational number. Then
\[
\sum_{p \leq N} \mu^2([\alpha p]) = \frac{6}{\pi^2} \pi(N) + O\left(N^{\frac{9}{10}}\right).
\] (4)

**Proof.** See [1, Corollary 1].

In 2018 the author [4] showed that for any fixed \(1 < c < \frac{22}{13}\) there exist infinitely many consecutive square-free numbers of the form \([nc], [nc] + 1\).

Recently the author [5] proved that there exist infinitely many consecutive square-free numbers of the form \(x^2 + y^2 + 1, x^2 + y^2 + 2\).

Also recently the author [6] showed that there exist infinitely many consecutive square-free numbers of the form \([\alpha n], [\alpha n] + 1\), where \(\alpha > 1\) is irrational number with bounded partial quotient or irrational algebraic number.

Define
\[
\Sigma(N, \alpha) = \sum_{p \leq N} \mu^2([\alpha p])\mu^2([\alpha p] + 1).
\] (4)

Motivated by these results and following the method of Akbal [1] we shall prove the following theorem.

**Theorem 2.** Let \(\alpha > 0\) be irrational algebraic number. Then for the sum \(\Sigma(N, \alpha)\) defined by (4) the asymptotic formula
\[
\Sigma(N, \alpha) = \sigma\pi(N) + O\left(N^{\frac{9}{10}}\right)
\] (5)
holds. Here \(\sigma\) is defined by (1).

From Theorem 2 it follows that there exist infinitely many consecutive square-free numbers of the form \([\alpha p], [\alpha p] + 1\), where \(p\) is prime and \(\alpha > 0\) is irrational algebraic number.
3 Lemmas

Lemma 1. (Erdős-Turán inequality) Let \( \{t_k\}_{k=1}^K \) be a sequence of real numbers. Suppose that \( I \subset [0, 1) \) is an interval. Then

\[
\left| \# \{k \leq K : \{t_k\} \in I \} - K|I| \right| \ll \frac{K}{H} + \sum_{h \leq H} \frac{1}{h} \left| \sum_{k \leq K} e(ht_k) \right|
\]

for any \( H \gg 1 \). The constant in the \( O \)-term is absolute.

Proof. See (2), Theorem 2.1.

Lemma 2. Suppose that \( H, D, T, N \geq 1 \). Let \( \alpha > 0 \) be irrational algebraic number. Then

\[
\sum_{H < h \leq 2H} \sum_{D < d \leq 2D} \sum_{T < t \leq 2T} \left| \sum_{p \leq N} e \left( \frac{\alpha hp}{d^2t^2} \right) \right| \ll (HDTN)^\varepsilon \left( H^{1/2} D^{2/5} T^{1/2} N^{1/2} + H^{3/5} DT N^{4/5} + HDTN^{3/4} + H^{3/4} D^{3/2} T^{3/2} N^{3/4} \right)
\]

(6)

Proof. This lemma is very similar to result of Akbal (1). Inspecting the arguments presented in (1), Lemma 3, the reader will easily see that the proof of Lemma 2 can be obtained by the same manner.

4 Proof of the Theorem

Assume

\[
2 \leq z \leq (\alpha N)^{2/3}.
\]

(7)

We use (4) and the well-known identity \( \mu^2(n) = \sum_{d \mid n} \mu(d) \) to write

\[
\Sigma(N, \alpha) = \sum_{p \leq N} \mu^2([\alpha p]) \mu^2([\alpha p] + 1) = \sum_{p \leq N} \sum_{d \mid [\alpha p]} \mu(d) \sum_{t \mid [\alpha p] + 1} \mu(t)
\]

\[
= \sum_{d, t \mid [\alpha p]} \mu(d) \mu(t) \sum_{p \leq N \mid [\alpha p] \equiv 0 (d^2) \mid [\alpha p] + 1 \equiv 0 (t^2)} 1 = \Sigma_1(N) + \Sigma_2(N),
\]

(8)

where

\[
\Sigma_1(N) = \sum_{dt \leq z \mid (d, t) = 1} \mu(d) \mu(t) \sum_{p \leq N \mid [\alpha p] \equiv 0 (d^2) \mid [\alpha p] + 1 \equiv 0 (t^2)} 1,
\]

(9)

\[
\Sigma_2(N) = \sum_{dt > z \mid (d, t) = 1} \mu(d) \mu(t) \sum_{p \leq N \mid [\alpha p] \equiv 0 (d^2) \mid [\alpha p] + 1 \equiv 0 (t^2)} 1.
\]

(10)
Estimation of $\Sigma_1(N)$

From (10) and Chinese remainder theorem we obtain

$$\Sigma_1(N) = \sum_{d,t \leq z \atop (d,t) = 1} \mu(d)\mu(t) \sum_{p \leq N \atop [\alpha p] = q \left(\frac{d^2t^2}{N}\right)} 1,$$

(11)

where $1 \leq q \leq d^2t^2 - 1$.

It is easy to see that the congruence $[\alpha p] \equiv q \left(\frac{d^2t^2}{N}\right)$ is tantamount to

$$\frac{q}{d^2t^2} < \left\{ \frac{\alpha p}{d^2t^2} \right\} < \frac{q + 1}{d^2t^2}.$$

(12)

Bearing in mind (11), (12) and Lemma 1 we get

$$\Sigma_1(N) = \pi(N) \sum_{d,t \leq z \atop (d,t) = 1} \frac{\mu(d)\mu(t)}{d^2t^2} + O \left( \frac{N}{H} \sum_{d \leq z} 1 \right) + O \left( \sum_{d \leq z} \frac{1}{h} \sum_{p \leq N} e \left( \frac{\alpha hp}{d^2t^2} \right) \right).$$

(13)

It is well-known that

$$\sum_{d,t \leq z \atop (d,t) = 1} \frac{\mu(d)\mu(t)}{d^2t^2} = \prod_p \left( 1 - \frac{2}{p^2} \right).$$

(14)

On the other hand

$$\sum_{d \leq z} \frac{1}{d^2t^2} \ll \sum_{d \leq z} \frac{\tau(n)}{n^2} \ll z^{1+\varepsilon}.\quad \text{(15)}$$

By the same way

$$\sum_{d \leq z} \frac{1}{d^2t^2} \ll z^{1+\varepsilon}.\quad \text{(16)}$$

From (13) – (16) it follows

$$\Sigma_1(N) = \sigma \pi(N) + O \left( \frac{\pi(N)z^{1+\varepsilon}}{N} \right) + O \left( \frac{N}{H} \sum_{d \leq z} \frac{1}{h} \sum_{p \leq N} e \left( \frac{\alpha hp}{d^2t^2} \right) \right).$$

(17)

where $\sigma$ is denoted by (1).

Splitting the range of $h, d$ and $t$ of the exponential sum in (17) into dyadic subintervals of the form $H < h \leq 2H$, $D < d \leq 2D$, $T < t \leq 2T$, where $DT < z$ and applying Lemma 2 we find

$$\sum_{d \leq z} \sum_{h \leq H} \frac{1}{h} \sum_{p \leq N} e \left( \frac{\alpha hp}{d^2t^2} \right) \ll (HDTN)^\varepsilon \left( D^2T^2N^{1/2} + DTN^{4/5} + D^{3/2}T^{3/2}N^{3/4} \right)$$

$$\ll (HzN)^\varepsilon \left( zN^{1/2} + zN^{4/5} + z^{3/2}N^{3/4} \right).$$

(18)
Taking into account (7), (17), (18) and choosing $H = N^{1/5}$ we obtain
\[
\Sigma_1(N) = \sigma\pi(N) + O\left(N^{\varepsilon} \left(z^2 N^{1/2} + zN^{4/5} + z^{3/2} N^{3/4} + Nz^{-1}\right)\right). \tag{19}
\]

**Estimation of $\Sigma_2(N)$**

By (7), (10), (15) and Chinese remainder theorem we get
\[
\Sigma_2(N) \ll \sum_{dt > z} \sum_{\substack{n \leq N \\ \alpha n \equiv 0 (d^2)}} 1 = \sum_{dt > z} \sum_{\substack{n \leq N \\ \alpha n \equiv 1 (d^2)}} 1 \ll \sum_{dt > z} \sum_{\substack{m \leq \alpha N \\ \alpha n \equiv l (d^2 t^2)}} 1 
\ll N \sum_{dt > z} \frac{1}{d^2 t^2} \ll N^{1+\varepsilon} z^{-1}. \tag{20}
\]

**The end of the proof**

Bearing in mind (8), (19), (20) and choosing $z = N^{1/10}$ we establish the asymptotic formula (5).

The theorem is proved.

**References**

[1] Y. Akbal, *A short note on some arithmetical properties of the integer part of $\alpha p$*, Turkish Journal of Mathematics, **43**(3), (2019), 1253 – 1262.

[2] R. C. Baker, *Diophantine Inequalities (London Mathematical Society Monographs)*, New York, NY, USA: Clarendon Press, (1986).

[3] L. Carlitz, *On a problem in additive arithmetic II*, Quart. J. Math., **3**, (1932), 273 – 290.

[4] S. I. Dimitrov, *Consecutive square-free numbers of the form $[n^{c}], [n^{c}] + 1$*, JP Journal of Algebra, Number Theory and Applications, **40**, 6, (2018), 945 – 956.

[5] S. I. Dimitrov, *On the number of pairs of positive integers $x, y \leq H$ such that $x^2 + y^2 + 1, x^2 + y^2 + 2$ are square-free*, arXiv:1901.04838 [math.NT] 5 Jan 2019.

[6] S. I. Dimitrov, *On the distribution of consecutive square-free numbers of the form $[\alpha n], [\alpha n] + 1$, arXiv:1903.04545v2 [math.NT] 22 Mar 2019.

[7] A. M. Güloğlu, C. W. Nevans, *Sums of multiplicative functions over a Beatty sequence*, Bull. Austral. Math. Soc., **78**, (2008), 327 – 334.

[8] D. R. Heath-Brown, *The Square-Sieve and Consecutive Square-Free Numbers*, Math. Ann., **266**, (1984), 251 – 259.

[9] L. Mirsky, *On the frequency of pairs of square-free numbers with a given difference*, Bull. Amer. Math. Soc., **55**, (1949), 936 – 939.

[10] T. Reuss, *Pairs of $k$-free Numbers, consecutive square-full Numbers*, arXiv:1212.3150v2 [math.NT] 19 Mar 2014.