AN INFINITESIMALLY NONRIGID POLYHEDRON WITH NONSTATIONARY VOLUME IN THE LOBACHEVSKY 3-SPACE

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Abstract

We give an example of an infinitesimally nonrigid polyhedron in the Lobachevsky 3-space and construct an infinitesimal flex of that polyhedron such that the volume of the polyhedron isn’t stationary under the flex.

Keywords

Infinitesimally nonrigid polyhedron, Lobachevsky space, hyperbolic space, volume, total mean curvature, infinitesimal flex, Schlaefli formula.

1 Introduction

The Bellows Conjecture states that every flexible polyhedron preserves its oriented volume during the flex. In 1996 I. Kh. Sabitov [1] gave an affirmative answer to the Bellows Conjecture in the Euclidean 3-space. In 1997 V. A. Alexandrov [2] has built a flexible polyhedron in the spherical 3-space which changes its volume during the flex. The question whether the Bellows Conjecture holds true in the Lobachevsky 3-space is still open.

In the note of the editor of the Russian translation of [3] I. Kh. Sabitov proposed to consider the Bellows Conjecture at the level of infinitesimal flexes. Roughly, we can formulate I. Kh. Sabitov’s question as follows: is it true that, for every infinitesimally nonrigid polyhedron, the volume it bounds is stationary under its infinitesimal flex? In case the answer to I. Kh. Sabitov’s question were positive, we would automatically validate the Bellows Conjecture for the flexible polyhedra. Of course, we can always additionally triangulate any initial face of a polyhedron so that there exists a new vertex $A$ of the triangulation which is an internal point of the initial face, then attach to $A$ a nonzero velocity vector orthogonal to the initial face, leave all other vertices of the polyhedron fixed and thus construct an infinitesimal flex of the new polyhedron based on the movements of all its vertices. The volume of the polyhedron with the “false” vertex under the constructed infinitesimal flex is nonstationary, but this trivial example is of a little interest to study the Bellows Conjecture.

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Having constructed a nontrivial counterexample in [4], V. A. Alexandrov gave a negative answer to I. Kh. Sabitov’s question for infinitesimally nonrigid polyhedra in the Euclidean 3-space. An example of a flexible polyhedron in the spherical 3-space, constructed in [2], which changes its volume during the flex, yields that the answer to this question is also negative for infinitesimally nonrigid polyhedra in the spherical 3-space. The main result of this paper reads as follows.

**Theorem** In the Lobachevsky 3-space there is a sphere-homeomorphic intersection-free polyhedron and its infinitesimal flex such that the volume it bounds isn’t stationary under the flex.

The polyhedron mentioned in the theorem is built explicitly. It’s similar to a polyhedron in the Euclidean 3-space which was first constructed by A. D. Alexandrov and S. M. Vladimirova [5] and later studied by A. D. Milka [6].

## 2 Constructing $S$

Throughout this paper we call a polyhedral surface a polyhedron.

Consider a regular pyramid $\mathcal{P}$ in the Lobachevsky 3-space with a regular concave star with $n$ petals as the base. We denote vertices of the star by $A_i, B_i, i = 1, \ldots, n$, and we note that the orthogonal projection of the vertex $N$ of $\mathcal{P}$ onto its base coincides with the center $C$ of the star, see Fig. 1. We reflect $\mathcal{P}$ in the plane that contains its base and denote by $S$ a suspension which consists of both initial and reflected pyramids without their common base. We denote by $S$ the vertex of $S$ symmetric to $N$ with respect to the plane containing the base of $\mathcal{P}$. A cycle formed by the edges of the base of $\mathcal{P}$ is called the equator of the suspension $S$.

Note that the lengths of all edges of the equator of $S$ are equal to each other by construction. Moreover, the lengths of all edges $SA_i, NA_i, i = 1, \ldots, n$, are equal to each other, and also the lengths of all edges $NB_i, SB_i, i = 1, \ldots, n$, are equal to each other too.
By construction, $S$ possesses multiple symmetries and the spatial body bounded by $S$ consists of identical tetrahedral “bricks”. Consider one of these tetrahedra, see Fig. 2. Denote its surface by $T$, and its vertices by $N, A, B, C$. Note that $\angle ACN = \angle BCN = \pi/2$ by construction. Let’s use the following notations for the lengths of the edges and for the plane angles of $T$: $|CN| = h$, $|CA| = p$, $|CB| = q$, $|AB| = a$, $|NA| = b$, $|NB| = c$, $\angle ACB = \alpha$, $\angle CAN = \beta$, $\angle BAN = \gamma$, $\angle CAB = \delta$, $\angle CBN = \varphi$, $\angle CBA = \psi$, $\angle ABN = \theta$, $\angle ANB = \lambda$, $\angle CNA = \mu$, $\angle CNB = \nu$. Denote the dihedral angles of $T$ at the edge $AB$ by $\angle AB$, at the edge $NA$ by $\angle NA$, and at the edge $NB$ by $\angle NB$.

By construction, the dihedral angle of $T$ at the edge $CN$ is equal to $\alpha$, the dihedral angles of $S$ at the edges of its equator are equal to $2\angle AB$, at the edges $NA_i$ and $SA_i$, $i = 1, \ldots, n$, are equal to $2\angle NA$, and at the edges $NB_i$ and $SB_i$, $i = 1, \ldots, n$, are equal to $2\angle NB$.

Further we show that the suspension $S$ constructed above can be taken as a polyhedron whose existence is proclaimed by the theorem.

3 A condition for infinitesimal nonrigidity

A deformation of a polyhedral surface $S$ is a family of surfaces $S(t)$, $t \in (-1, 1)$, which depends analytically on the parameter $t$, preserves the combinatorial structure of $S$, and is such that $S(0) = S$.

A deformation of a polyhedral surface $S$ with triangular faces is called its infinitesimal flex if the lengths of all edges of $S(t)$ are stationary at $t = 0$.

An infinitesimal flex is called nontrivial if there exist two vertices of $S(t)$ which are not connected by an edge of $S(t)$ and are such that the spatial distance between them is not stationary.

A polyhedron is called infinitesimally nonrigid if it possesses a nontrivial infinitesimal flex.

Determine a deformation of the suspension $S$ constructed in the previous section as follows. The point $C$ is fixed. At the moment $t$, the point $N$ goes to the point $N(t)$ lying on the ray $\overrightarrow{CN}$ at the distance from $C$ determined by the formula

$$h(t) = h + tu,$$

where $u$ is a real number which has a meaning of velocity and which will be specified below. The point $S$ goes to the point $S(t)$ lying on the ray $\overrightarrow{CS}$ at the distance from $C$ determined by the formula $p(t) = p + tv$, where $v$ is a real number which has a meaning of velocity. The point $B_i$, $i = 1, \ldots, n$, goes to the point $B_i(t)$ lying on the ray $\overrightarrow{CB_i}$ at the distance from $C$ determined by the formula $q(t) = q + tw$, where $w$ is a real number which has a meaning of velocity and which will be specified below.

In order to determine the movements of other points of the suspension $S(t)$ let’s use the statement of Ceva’s theorem in the Lobachevsky space \cite{7}:

*Given a triangle $\triangle ABC$ and points $\hat{A}, \hat{B}$, and $\hat{C}$ that lie on sides $BC, CA$, and $AB$ of $\triangle ABC$. Then the segments $\overrightarrow{A\hat{A}}, \overrightarrow{B\hat{B}},$ and $\overrightarrow{C\hat{C}}$ intersect at one point if and only if one of*
the following equivalent relations holds:

\[
\frac{\sin \angle AC \tilde{C} \sin \angle BA \tilde{A} \sin \angle CB \tilde{B}}{\sin \angle \tilde{C} \tilde{C} \tilde{B} \sin \angle A \tilde{A} \tilde{C} \sin \angle \tilde{B} \tilde{B} \tilde{A}} = 1;
\]

\[
\frac{\sinh A \tilde{C} \sinh B \tilde{A} \sinh C \tilde{B}}{\sinh C \tilde{B} \sinh A \tilde{C} \sinh B \tilde{A}} = 1.
\]

(2)

In terms of the statement of Ceva’s theorem, let’s take the point \(P(t)\) of the segment \(A(t)B(t)\) for which the equality

\[
\frac{\sinh A(t)P(t)}{\sinh P(t)B(t)} = \frac{\sinh AP}{\sinh PB}
\]

holds true, as a new position of any point \(P\) of the edge \(AB\) at the moment \(t\).

To determine the movement of an internal point \(Q\) of the face \(\triangle ABC\), at first we construct points \(\tilde{A}, \tilde{B},\) and \(\tilde{C}\), as the intersections of the edges \(BC, CA,\) and \(AB\) with the rays \(AQ, BQ,\) and \(CQ\), and then determine their positions \(\tilde{A}(t), \tilde{B}(t),\) and \(\tilde{C}(t)\) at the moment \(t\) by the method described above. By Ceva’s theorem, the segments \(A(t)\tilde{A}(t), B(t)\tilde{B}(t),\) and \(C(t)\tilde{C}(t)\) intersect at one point (the relation (2) remains true at every moment \(t\)). Consider this point of intersection as a new position \(Q(t)\) of the point \(Q\) at the moment \(t\).

The deformation of \(S\) described above, naturally produces a deformation of the tetrahedron \(T\) which we denote by \(T(t)\). The lengths of all edges as well as the values of all plane and dihedral angles of \(T\) are functions in \(t\) and their notations naturally succeed from the notations for the corresponding entities of \(T\). For example, we denote the length of the edge \(N(t)A(t)\) by \(b(t)\), the value of the plane angle \(\angle CA(t)N(t)\) by \(\beta(t)\), and the value of the dihedral angle of \(T(t)\) at the edge \(N(t)A(t)\) by \(\angle N(t)A(t)\), etc.

Let’s find a relation between \(u, v,\) and \(w\) implying that the deformation \(S(t)\) is an infinitesimal flex. We only need to study the deformation of the face \(ABN\) in \(T\) because all faces of \(S\) move in the same way.

Apply the Pythagorean theorem for the Lobachevsky space \(\mathbb{S}\) to the triangle \(\triangle N(t)CA(t)\):

\[
\cosh b(t) = \cosh(h + tu) \cosh(p + tv)
\]

and to the triangle \(\triangle N(t)CB(t)\):

\[
\cosh c(t) = \cosh(h + tu) \cosh(q + tw)
\]

of \(T(t)\).

Using the Cosine Law for the Lobachevsky space \(\mathbb{S}\) applied to the triangle \(\triangle A(t)CB(t)\), and taking it into account that the angle \(\alpha\) remains constant during the deformation (and is equal to \(\frac{\pi}{n}\)), we get:

\[
\cosh a(t) = \cosh(p + tv) \cosh(q + tw) - \sinh(p + tv) \sinh(q + tw) \cos \alpha.
\]

(5)

Further it will be useful for us to study stationarity of the function \(f(t) = \cosh l(t)\) instead of stationarity of the length \(l(t)\) of any edge of \(S(t)\), because \(f'(0) = l'(0) \sinh l(0)\) and \(l(0) > 0\), and thus \(f'(0) = 0\) if and only if \(l'(0) = 0\).
Let’s differentiate (3): \( (\cosh b(t))' = u \sinh(h+tu) \cosh(p+tv) + v \cosh(h+tu) \sinh(p+tv) \).

Thus, stationarity of the length \( b(t) \) of the edge \( N(t)A(t) \) is equivalent to the condition \( (\cosh b(t))'|_{t=0} = u \sinh h \cosh p + v \cosh h \sinh p = 0 \), or

\[
v = -\frac{\tanh h}{\tanh p} u. \tag{6}
\]

Similarly, stationarity of the length \( c(t) \) of the edge \( N(t)B(t) \) is equivalent to the condition

\[
w = -\frac{\tanh h}{\tanh q} u. \tag{7}
\]

Differentiating (5), we find the condition for stationarity of the length \( A(t)B(t) \):

\[
(cosh a(t))'|_{t=0} = v \sinh p \cosh q + w \cosh p \sinh q - \cos \alpha \{ v \cosh \sinh q + w \sinh p \cosh q \} = 0. \tag{8}
\]

Substituting (6) and (7) into (8), we get:

\[
u \tanh h \left[ \cos \alpha \left\{ \frac{\cosh p \sinh q}{\tanh p} + \frac{\sinh p \cosh q}{\tanh q} \right\} - \frac{\sinh p \cosh q}{\tanh p} - \frac{\cosh p \sinh q}{\tanh q} \right] = 0.
\]

Thus, the deformation under consideration of \( S \) is an infinitesimal flex if and only if (6), (7) and

\[
\cos \alpha \left\{ \frac{\cosh p \sinh q}{\tanh p} + \frac{\sinh p \cosh q}{\tanh q} \right\} = 2 \cosh p \cosh q
\]

hold true. Hence, \( S \) allows the infinitesimal flex of the form described in the beginning of this section if and only if \( p, q \), and \( \alpha \) satisfy the following relation:

\[
\frac{\tanh p}{\tanh q} = \frac{1 \pm \sin \alpha}{\cos \alpha}. \tag{9}
\]

The so-constructed infinitesimal flex is nontrivial because the distance between the poles \( N(t) \) and \( S(t) \) is not stationary.

### 4 Calculating metric elements of \( \mathcal{T}(t) \)

Let’s obtain formulae for the dihedral angles \( \angle A(t)B(t) \), \( \angle N(t)A(t) \), and \( \angle N(t)B(t) \) of the tetrahedron \( \mathcal{T}(t) \), which will be used in a proof of the theorem.

First we calculate the sines and cosines of the plane angles of \( \mathcal{T}(t) \).

Apply the Cosine Law for the Lobachevsky space to the triangle \( \triangle CA(t)N(t) \) to calculate the cosine of the angle \( \beta(t) \): \( \cosh(h+tu) = \cosh(p+tv) \cosh b(t) - \sinh(p+tv) \sinh \cosh b(t) \cos \beta(t) \).

Thus, taking into account (4) and formulae of hyperbolic trigonometry, we get:

\[
\cos \beta(t) = \frac{\sinh(p+tv) \cosh(h+tu)}{\sinh b(t)} = \frac{\sinh(p+tv) \cosh(h+tu)}{\sqrt{\cosh^2(h+tu) \cosh^2(p+tv) - 1}}. \tag{10}
\]

(Here and below \( \sqrt{s} \) stands for a branch of the square root that takes a positive real value for a positive real \( s \).) To calculate the sine of \( \beta(t) \) we apply the Sine Law for the Lobachevsky space \( \mathcal{S} \) to \( \triangle CA(t)N(t) \):

\[
\frac{\sin \beta(t)}{\sinh(h+tu)} = \frac{\sin \pi/2}{\sinh b(t)} = \frac{1}{\sqrt{\cosh^2(h+tu) \cosh^2(p+tv) - 1}}.
\]
and therefore,
\[ \sin \beta(t) = \frac{\sinh(h + tu)}{\sinh b(t)} = \frac{\sinh(h + tu)}{\sqrt{\cosh^2(h + tu)\cosh^2(p + t v) - 1}}. \] (11)

Similarly, we obtain the formulae for the cosine and sine of the angle \( \varphi(t) \) in \( \triangle CB(t)N(t) \):
\[ \cos \varphi(t) = \frac{\sinh(q + t w) \cosh(h + tu)}{\sinh c(t)} = \frac{\sinh(q + t w) \cosh(h + tu)}{\sqrt{\cosh^2(h + tu)\cosh^2(q + t w) - 1}}, \] (12)
\[ \sin \varphi(t) = \frac{\sinh(h + tu)}{\sinh c(t)} = \frac{\sinh(h + tu)}{\sqrt{\cosh^2(h + tu)\cosh^2(q + t w) - 1}}. \] (13)

for the cosine and sine of the angle \( \mu(t) \) in \( \triangle CA(t)N(t) \):
\[ \cos \mu(t) = \frac{\sinh(h + tu) \cosh(p + t v)}{\sinh b(t)} = \frac{\sinh(h + tu) \cosh(p + t v)}{\sqrt{\cosh^2(h + tu)\cosh^2(p + t v) - 1}}, \] (14)
\[ \sin \mu(t) = \frac{\sinh(p + t v)}{\sinh b(t)} = \frac{\sinh(p + t v)}{\sqrt{\cosh^2(h + tu)\cosh^2(p + t v) - 1}}, \] (15)

and for the cosine and sine of the angle \( \nu(t) \) in \( \triangle CB(t)N(t) \):
\[ \cos \nu(t) = \frac{\sinh(h + tu) \cosh(q + t w)}{\sinh c(t)} = \frac{\sinh(h + tu) \cosh(q + t w)}{\sqrt{\cosh^2(h + tu)\cosh^2(q + t w) - 1}}, \] (16)
\[ \sin \nu(t) = \frac{\sinh(q + t w)}{\sinh c(t)} = \frac{\sinh(q + t w)}{\sqrt{\cosh^2(h + tu)\cosh^2(q + t w) - 1}}. \] (17)

The Cosine Law for the Lobachevsky space applied twice to the triangle \( \triangle A(t)CB(t) \) leads us to the formulae:
\[ \cos \delta(t) = \frac{\cosh(p + t v) \cosh a(t) - \cosh(q + t w)}{\sinh(p + t v) \sinh a(t)}, \] (18)
\[ \cos \psi(t) = \frac{\cosh(q + t w) \cosh a(t) - \cosh(p + t v)}{\sinh(q + t w) \sinh a(t)}. \] (19)

From the Sine Law for the Lobachevsky space applied to \( \triangle A(t)CB(t) \), it follows that:
\[ \frac{\sin \delta(t)}{\sinh(q + t w)} = \frac{\sin \alpha}{\sinh a(t)} = \frac{\sin \psi(t)}{\sinh(p + t v)}, \]
and thus the formulae
\[ \sin \delta(t) = \frac{\sin \alpha \sinh(q + t w)}{\sinh a(t)}, \] (20)
\[ \sin \psi(t) = \frac{\sin \alpha \sinh(p + t v)}{\sinh a(t)} \] (21)
hold true.
The Cosine Law for the Lobachevsky space three times applied to the triangle $\triangle A(t)N(t)B(t)$ leads us to the formulae:

\[
\begin{align*}
\cos \theta(t) &= \frac{\cosh a(t) \cosh c(t) - \cosh b(t)}{\sinh a(t) \sinh c(t)}, \\
\cos \gamma(t) &= \frac{\cosh a(t) \cosh b(t) - \cosh c(t)}{\sinh a(t) \sinh b(t)}, \\
\cos \lambda(t) &= \frac{\cosh b(t) \cosh c(t) - \cosh a(t)}{\sinh b(t) \sinh c(t)}.
\end{align*}
\] (22–24)

Taking into account (3)–(5), we calculate $\sinh a(t)$, $\sinh b(t)$, and $\sinh c(t)$ from (23)–(24):

\[
\begin{align*}
\sinh a(t) &= \sqrt{\cosh^2 a(t) - 1} = \sqrt{(\cosh(p + tv) \cosh(q + tw) - \sinh(p + tv) \sinh(q + tw) \cos \alpha)^2 - 1}, \\
\sinh b(t) &= \sqrt{\cosh^2 b(t) - 1} = \sqrt{(\cosh(h + tu) \cosh(p + tv))^2 - 1}, \\
\sinh c(t) &= \sqrt{\cosh^2 c(t) - 1} = \sqrt{(\cosh(h + tu) \cosh(q + tw))^2 - 1}.
\end{align*}
\]

The fact that the values of the angles in a hyperbolic triangle are greater than 0 and less than $\pi$ yields that the sines of the angles of a hyperbolic triangle are nonnegative. Hence, $\sin \theta(t) = \sqrt{1 - \cos^2 \theta(t)}$, $\sin \gamma(t) = \sqrt{1 - \cos^2 \gamma(t)}$, $\sin \lambda(t) = \sqrt{1 - \cos^2 \lambda(t)}$.

Consider the unit sphere $\Sigma$ centered at the vertex $A(t)$ of $\mathcal{T}(t)$. Denote the points of the intersection of $\Sigma$ and the rays $\overline{A(t)C}$, $\overline{A(t)N(t)}$, and $\overline{A(t)B(t)}$ by $C_A(t)$, $N_A(t)$, and $B_A(t)$ correspondingly. They determine a triangle $\triangle C_A(t)N_A(t)B_A(t)$ which consists of the points of the intersection of $\Sigma$ and the rays emitted from $A(t)$ and passing through the points of the face $\triangle CB(t)N(t)$ of $\mathcal{T}(t)$. By construction, the angle of the spherical triangle $\triangle C_A(t)N_A(t)B_A(t)$ at the vertex $C_A(t)$ is equal to $\pi/2$, the angle at $N_A(t)$ is equal to $\angle N(t)A(t)$, the angle at $B_A(t)$ is equal to $\angle A(t)B(t)$, the length of the side $C_A(t)N_A(t)$ is equal to $\beta(t)$, the length of $N_A(t)B_A(t)$ is equal to $\gamma(t)$, and the length of $C_A(t)B_A(t)$ is equal to $\delta(t)$.

Similarly, we build a spherical triangle $\triangle C_B(t)N_B(t)A_B(t)$. Its angle at the vertex $C_B(t)$ is equal to $\pi/2$, the angle at $N_B(t)$ is equal to $\angle N(t)B(t)$, the angle at $A_B(t)$ is equal to $\angle A(t)B(t)$, the length of the side $C_B(t)N_B(t)$ is equal to $\varphi(t)$, the length of $N_B(t)A_B(t)$ is equal to $\theta(t)$, and the length of $C_B(t)A_B(t)$ is equal to $\psi(t)$.

Applying the Cosine Law for the spherical space (2) twice to $\triangle C_A(t)N_A(t)B_A(t)$, we obtain the formulae:

\[
\begin{align*}
\cos \angle A(t)B(t) &= \frac{\cos \beta(t) - \cos \gamma(t) \cos \delta(t)}{\sin \gamma(t) \sin \delta(t)}, \\
\cos \angle N(t)A(t) &= \frac{\cos \delta(t) - \cos \gamma(t) \cos \beta(t)}{\sin \gamma(t) \sin \beta(t)}.
\end{align*}
\]

Again, applying the Cosine Law for the spherical space to $\triangle C_B(t)N_B(t)A_B(t)$, we get:

\[
\begin{align*}
\cos \angle N(t)B(t) &= \frac{\cos \psi(t) - \cos \varphi(t) \cos \theta(t)}{7 \sin \varphi(t) \sin \theta(t)}.
\end{align*}
\]
Now apply the Sine Law for the spherical space \[8\] to \(\triangle C A(t)N A(t)B A(t)\):

\[
\frac{\sin \angle N(t)A(t)}{\sin \delta(t)} = \frac{\sin \angle A(t)B(t)}{\sin \beta(t)} = \frac{\sin \pi/2}{\sin \gamma(t)}.
\]

Hence,

\[
\sin \angle A(t)B(t) = \frac{\sin \beta(t)}{\sin \gamma(t)} \quad \text{and} \quad \sin \angle N(t)A(t) = \frac{\sin \delta(t)}{\sin \gamma(t)}.
\]

Again, apply the Sine Law for the spherical space to \(\triangle C B(t)N B(t)A B(t)\):

\[
\frac{\sin \angle N(t)A(t)}{\sin \nu(t)} = \frac{\sin \angle N(t)B(t)}{\sin \mu(t)} = \frac{\sin \alpha}{\sin \lambda(t)}.
\]

Thus,

\[
\sin \angle N(t)B(t) = \sin \alpha \frac{\sin \mu(t)}{\sin \lambda(t)}.
\]

In the proof of the theorem given below we use also the following three evident relations:

\[
\frac{d\angle N(t)A(t)}{dt} = -\frac{d}{dt}(\cos \angle N(t)A(t)) \frac{1}{\sin \angle N(t)A(t)},
\]

\[
\frac{d\angle N(t)B(t)}{dt} = -\frac{d}{dt}(\cos \angle N(t)B(t)) \frac{1}{\sin \angle N(t)B(t)},
\]

and

\[
\frac{d\angle A(t)B(t)}{dt} = -\frac{d}{dt}(\cos \angle A(t)B(t)) \frac{1}{\sin \angle A(t)B(t)}.
\]

5 Proof of the theorem

Remind that, according to the Schl"{a}fli formula for polyhedra in the Lobachevsky 3-space \[8\] of the curvature \(-1\), the equality

\[
dV = -\frac{1}{2} \sum_{e} l_e d\theta_e
\]

holds true, where \(dV\) stands for the variation of the volume of the polyhedron, \(l_e\) stands for the length of an edge \(e\) of the polyhedron, \(d\theta_e\) stands for the variation of the dihedral angle of the polyhedron attached to the edge \(e\), and summation is taken over all edges \(e\) of the polyhedron.

Show that the polyhedron \(S(0)\) from the family of suspensions \(S(t), t \in (-1, 1)\), constructed in Section \[2\] with parameters of the tetrahedron \(\mathcal{T}\)

\[
p = \text{artanh} \frac{1}{2}, \quad q = \text{artanh} \frac{\sqrt{3}}{2}, \quad h = \text{artanh} \frac{1}{2}, \quad \alpha = \frac{\pi}{6} \quad (\text{i. e. } n = 6)
\]

and the velocities of deformation

\[
u = \frac{\sqrt{3}}{4}, \quad v = -\frac{\sqrt{3}}{4}, \quad w = -\frac{1}{4},
\]

can be taken as a polyhedron whose existence is asserted in the theorem.

The suspension \(S(0)\) is not infinitesimally rigid because \(p, q, \text{ and } \alpha\) from \(\text{(26)}\) satisfy \(\text{(9)}\).
Let’s verify that the nontrivial infinitesimal flex from Section 3 with the coefficients (27) can be taken as an infinitesimal flex whose existence is stated in the theorem.

Using the Schl"afli formula (25) and taking into account notations and remarks of Section 2, we see that the variation of the volume of \( S \) can be taken as an infinitesimal flex whose existence is stated in the theorem.

Substituting the values of parameters from (26) and (27) into the formulae of Sections 3 and 4, we sequentially find the hyperbolic sines and cosines of the lengths of the edges and the variations of the dihedral angles of the tetrahedron \( T(t) \) at \( t = 0 \):

\[
cosh a(t) = \cosh \left( -\text{artanh} \frac{1}{2} \right) \cosh \left( -\text{artanh} \frac{\sqrt{3}}{2} \right) - \frac{\sqrt{3}}{2} \sinh \left( -\text{artanh} \frac{1}{2} \right) \sinh \left( -\text{artanh} \frac{\sqrt{3}}{2} \right),
\]
\[
cosh b(0) = \cosh \left( -\text{artanh} \frac{1}{2} \right) \cosh \left( \text{artanh} \frac{1}{2} \right), \quad \cosh c(t) = \cosh \left( \text{artanh} \frac{1}{2} \right) \cosh \left( -\text{artanh} \frac{\sqrt{3}}{2} \right),
\]
\[
\frac{d\angle A(t)B(t)}{dt}(0) = \frac{\sqrt{13}}{4}, \quad \frac{d\angle N(t)A(t)}{dt}(0) = \frac{\sqrt{7}}{4}, \quad \frac{d\angle N(t)B(t)}{dt}(0) = -\frac{\sqrt{13}}{4},
\]

and thus, by (28),

\[
\frac{dV_S(0)}{dt} = -12 \left[ \frac{\sqrt{13}}{4} \text{arcosh} \left( \cosh \left( -\text{artanh} \frac{1}{2} \right) \cosh \left( -\text{artanh} \frac{\sqrt{3}}{2} \right) \right) -
\frac{\sqrt{3}}{2} \sinh \left( -\text{artanh} \frac{1}{2} \right) \sinh \left( -\text{artanh} \frac{\sqrt{3}}{2} \right) + \frac{\sqrt{7}}{4} \text{arcosh} \left( \cosh \left( -\text{artanh} \frac{1}{2} \right) \cosh \left( \text{artanh} \frac{1}{2} \right) \right) -
\frac{\sqrt{13}}{4} \text{arcosh} \left( \cosh \left( \text{artanh} \frac{1}{2} \right) \cosh \left( -\text{artanh} \frac{\sqrt{3}}{2} \right) \right) \right] =

\frac{3\sqrt{7}}{8} \left[ 8 \ln \frac{4 + \sqrt{7}}{3} + 11 \ln \frac{7 - \sqrt{13}}{6} \right] < -3\sqrt{7} \ln \left[ \left( \frac{4 + \sqrt{7}}{3} \right)^8 \left( \frac{7 - \sqrt{13}}{6} \right)^{11} \right] < 0. \quad \square
\]

6 Concluding remarks

Using notations of Section 5, we determine the integral mean curvature of a polyhedron \( S(t) \) in the 3-space as follows:

\[
M(S(t)) = \frac{1}{2} \sum_{e} l_e(t)(\pi - \theta_e(t)).
\]

R. Alexander [9] proved that the integral mean curvature of any polyhedron in the Euclidean 3-space is stationary under every its infinitesimal flex.

The lengths of the edges of the suspension \( S(t) \) are stationary under the infinitesimal flex of \( S(t) \) from Section 3. Hence, the variation of the integral mean curvature of \( S(t) \) at \( t = 0 \) is equal to the variation of the volume \( dV_S(0) \). Therefore, the proof of our theorem automatically implies that the variation of the integral mean curvature for the infinitesimal flex of \( S(t) \) constructed above is not equal to zero. Thus, the integral mean curvature of
an infinitesimally nonrigid polyhedron is not always stationary in the Lobachevsky space as well as in the spherical space but is always stationary in the Euclidean space.

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