Stability of Fixed Points and Chaos in Fractional Systems

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In this paper we propose a method to define the range of stability of fixed points for a variety of discrete fractional systems of the order $0 < \alpha < 2$. The method is tested on various forms of fractional generalizations of the standard and logistic maps. Based on our analysis we make a conjecture that chaos is impossible in the corresponding continuous fractional systems.
Many natural (biological, physical, etc.) and social systems possess power-law memory and can be described by the fractional differential/difference equations. Nonlinearity is an important property of these systems. Behavior of such systems can be very different from the behavior of the corresponding systems with no memory. Previous research on the issues of the first bifurcations and the stability of fractional systems mostly addressed the question of sufficient conditions. In this paper we propose the equations that allow calculations of the coordinates of the asymptotically stable period two sinks and the values of nonlinearity and memory parameters defining the first bifurcation form the stable fixed points to the $T = 2$ sinks.

I. INTRODUCTION

It is generally understood that socioeconomic and biological systems are systems with memory. Specific analysis showing that the memory in financial and socioeconomic systems obeys the power law can be found in papers [1,2,3] and sources cited in these papers. Power-law in human memory was investigated in [4,5]: the accuracy on memory tasks decays as a power law $\sim t^{-\beta}$, with $0 < \beta < 1$ and, with respect to human learning, it is shown in [6] that the reduction in reaction times that comes with practice is a power function of the number of training trials. Power-law adaptation has been used to describe the dynamics of biological systems in papers [7-15].

The impotence and origin of the memory in biological systems can be related to the presence of memory at the level of individual cells: it has been shown recently that processing of external stimuli by individual neurons can be described by fractional differentiation [16-18]. The orders of fractional derivatives $\alpha$ derived for different types of neurons fall within the interval $[0,1]$, which implies power-law memory $\sim t^\beta$ with power $\beta = 1 - \alpha$, $\beta \in [-1,0]$. For neocortical pyramidal neurons the order of the fractional derivative is quite small: $\alpha \approx 0.15$.

Viscoelastic properties of the human organ tissues are best described by fractional differential equations with time fractional derivatives, which implies the power-law memory (see, e.g., references in [19]). In most of the biological systems with the power-law behavior the power $\beta$ is between -1 and 1 ($0 < \alpha < 2$).

Among the fundamental scientific problems driving interest and research in fractional
dynamics are the origin of memory and a possibility of memory being present in the very basic equations of Physics. Could it be that the fundamental laws describing fields and particles are not memoryless and are governed by fractional differential/difference equations?

Because most of the social, biological, and physical systems are nonlinear, it is important to look for the fundamental differences in the behavior of nonlinear systems with and without memory. Let’s list some of the differences.

- Trajectories in continuous fractional systems of orders less than two may intersect (see, e.g., Fig. 2 from [19]) and chaotic attractors may overlap (see, e.g., Fig. 4 f from [20]).

- As a result of the previous statement, the Poincaré-Bendixson Theorem does not apply to fractional systems and even in continuous systems of the order $\alpha < 2$ non-existence of chaos is only a conjecture (see [19,21]).

- Periodic sinks may exist only in asymptotic sense and asymptotically attracting points may not belong to their own basins of attraction (see [20,22,24]). A trajectory starting from an asymptotically attracting point jumps out of this point and may end up in a different asymptotically attracting point.

- A way in which a trajectory is approaching an attracting point depends on its origin. Trajectories originating from the basin of attraction may converge faster (as $x_n \sim n^{-1-\alpha}$ for the fractional Riemann-Liouville standard map, see Fig. 1 from [20]) than trajectories originating from the chaotic sea (as $x_n \sim n^{-\alpha}$).

- Cascade of bifurcations type trajectories (CBTT) exist only in fractional systems. The periodicity of such trajectories is changing with time: they may start converging to the period $2^n$ sink, but then bifurcate and start converging to the period $2^{n+1}$ sink and so on. CBTT may end its evolutions converging to the period $2^{n+m}$ sink (Fig. 1(a)) or in chaos (Fig. 1(b)) [22,23].

- Continuous and discrete fractional systems may not have periodic solutions except fixed points (see, e.g., [25,31]). Instead they may have asymptotically periodic solutions.

- Fractional extensions of the volume preserving systems are not volume preserving. If the order of a fractional system is less than the order of the corresponding integer system, then behavior of the system is similar to the behavior of the corresponding
FIG. 1. Two examples of cascade of bifurcations type trajectories in the Caputo logistic $\alpha$-family of maps (Eq. (22) with $h = 1$ and $G_K(x) = x - Kx(1-x)$) with $\alpha = 0.1$ and $x_0 = 0.001$: (a) for the nonlinearity parameter $K = 22.37$ the last bifurcation from the period $T = 8$ to the period $T = 16$ occurs after approximately $5 \times 10^5$ iterations; (b) when $K = 22.423$ the trajectory becomes chaotic after approximately $5 \times 10^5$ iterations.

A particular problem related to the differentiation between fractional systems and integer ones, the first bifurcation on CBTT, and related problems of stability of fixed points in discrete fractional systems and transition to chaos in continuous fractional systems are considered in this paper.

Stability of fractional systems was investigated in numerous papers based on various methods (Lyapunov’s direct and indirect methods, Lyapunov function, Routh-Hurwitz criterion, ...). Here we’ll list only some of the research papers, reviews, and books on the topic. Paper (37) is the most cited article on stability of linear fractional differential equations. In application to stability of nonlinear fractional differential equations, we’ll mention papers (38–44). Some of the results on stability of discrete fractional systems can be found in papers (45–50). The reviews on the topic include papers (51–53) and books (54,55). Almost all results obtained in the cited papers define sufficient conditions of stability and don’t allow to calculate the ranges of nonlinearity parameters and orders of derivatives for which fixed points are stable.

In this paper we derive the algebraic equations to calculate asymptotically period two sinks of discrete fractional systems, which define the conditions of their appearance, and conjecture that these equations define the values of nonlinearity parameters and orders of
derivatives for which fixed points become unstable. This conjecture is numerically verified for the fractional standard and logistic maps. This paper is a continuation of the research on general properties of fractional systems based on the properties of fractional maps. In Sec. II we review the most common forms of fractional maps. In Sec. III we derive the equations defining the ranges of nonlinearity parameters and orders of derivatives for which fixed points are stable. Sec. IV presents the summary of our results.

II. FRACTIONAL/FRACTIONAL DIFFERENCE MAPS

In this section some essential definitions and theorems are presented.

A. Fractional integrals and derivatives

In this paper we will use the definition of fractional integral introduced by Liouville, which is a generalization of the Cauchy formula for the n-fold integral

$$aI^p_t x(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{x(\tau)d\tau}{(t-\tau)^{1-p}},$$

where \( p \) is a real number, \( \Gamma() \) is the gamma function and we’ll assume \( a = 0 \).

The left-sided Riemann-Liouville fractional derivative \( _0^aD_t^\alpha x(t) \) is defined for \( t > \) as

$$_0^aD_t^\alpha x(t) = D_t^n _0^t\ I_t^{n-\alpha} x(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{x(\tau)d\tau}{(t-\tau)^{\alpha-n+1}},$$

where \( n-1 \leq \alpha < n, n \in \mathbb{Z}, D_t^n = d^n/dt^n \).

In the definition of the left-sided Caputo derivative, the order of integration and differentiation in Eq. (2) is switched:

$$^C_0^aD_t^\alpha x(t) = _0^a\ I_t^{n-\alpha} D_t^n x(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{D_t^n x(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \quad (n-1 < \alpha \leq n).$$
B. Fractional sums and differences

We’ll also use the proposed by Miller and Ross generalization of the forward sum/difference operator\(^6\)
\[
\Delta x(t) = x(t + 1) - x(t) \tag{4}
\]
(see below) and call it simply the fractional sum/difference operator. Nabla fractional difference, which is the generalization of the backward sum/difference operator \(\nabla x(t) = x(t) - x(t - 1)\)^7 is not considered in this paper.

The fractional sum \((\alpha > 0)/\)difference \((\alpha < 0)\) operator defined in\(^6\)
\[
a \Delta_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s) \tag{5}
\]
is a fractional generalization of the \(n\)-fold summation formula\(^6\,^{69}\)
\[
a \Delta_t^{-n} f(t) = \frac{1}{(n-1)!} \sum_{s=a}^{t-n} (t-s-1)^{(n-1)} f(s) = \sum_{s^n=a}^{s^{n-2}=a} \sum_{s^{n-1}=a} f(s^{n-1}), \tag{6}
\]
where \(n \in \mathbb{N}\). In Eq. \((5)\) \(f\) is defined on \(\mathbb{N}_a\) and \(a \Delta_t^{-\alpha}\) on \(\mathbb{N}_{a+\alpha}\), where \(\mathbb{N}_t = \{t, t+1, t+2, \ldots\}\).

The falling factorial \(t(\alpha)\) is defined as
\[
t(\alpha) = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}; \quad t \neq -1, -2, -3, \ldots \tag{7}
\]
and is asymptotically a power function:
\[
\lim_{t \to \infty} \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)t^\alpha} = 1, \quad \alpha \in \mathbb{R}. \tag{8}
\]

For \(\alpha > 0\) and \(m-1 < \alpha \leq m\) the fractional (left) Riemann-Liouville difference operator is defined (see\(^7\,^{11}\)) as
\[
a \Delta_t^\alpha x(t) = a \Delta_t^m \Delta_t^{-(m-\alpha)} x(t) \\
= \frac{1}{\Gamma(m-\alpha)} \Delta_t^m \sum_{s=a}^{t-(m-\alpha)} (t-s-1)^{(m-\alpha-1)} x(s) \tag{9}
\]
and the fractional (left) Caputo-like difference operator (see\(^7\)) as
\[
C_a \Delta_t^\alpha x(t) = C_a \Delta_t^{-(m-\alpha)} \Delta_t^m x(t) \\
= \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t-(m-\alpha)} (t-s-1)^{(m-\alpha-1)} \Delta_t^m x(s). \tag{10}
\]
Due to the fact that $a\Delta_t^\lambda$ in the limit $\lambda \to 0$ approaches the identity operator (see [58,68]), the definition Eq. (10) can be extended to all real $\alpha \geq 0$ with $\int^t_x \Delta^m x(t) = \Delta^m x(t)$ for $m \in \mathbb{N}_0$.

Fractional h-difference operators, which are generalizations of the fractional difference operators, were introduced in [73,74]. The h-sum operator is defined as

$$(a\Delta_h^\alpha f)(t) = \frac{h}{\Gamma(\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\alpha} (t - (s + 1)h)^{\alpha-1} f(sh),$$

where $\alpha \geq 0$, $(a\Delta_h^0 f)(t) = f(t)$, $f$ is defined on $(h\mathbb{N})_a$, and $a\Delta_h^{-\alpha}$ on $(h\mathbb{N})_{a+ah}$. $(h\mathbb{N})_t = \{t, t+h, t+2h, \ldots\}$. The h-factorial $t_h^{(\alpha)}$ is defined as

$$t_h^{(\alpha)} = h^\alpha \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \alpha)} = h^\alpha \left(\frac{t}{h}\right)^{\alpha},$$

where $t/h \neq -1, -2, -3, \ldots$. With $m = \lceil \alpha \rceil$ the Riemann-Liouville (left) h-difference is defined as

$$(a\Delta_h^a x)(t) = (\Delta_h^m (a\Delta_h^{(m-\alpha)} x))(t) = \frac{h}{\Gamma(m - \alpha)}$$

$$\times \Delta_h^{m} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(m-\alpha)} (t - (s + 1)h)^{(m-\alpha)-1} x(sh) \quad (13)$$

and the Caputo (left) h-difference is defined as

$$(a\Delta_h^{\alpha,*} x)(t) = (a\Delta_h^{-(m-\alpha)} (\Delta_h^m x))(t) = \frac{h}{\Gamma(m - \alpha)}$$

$$\times \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(m-\alpha)} (t - (s + 1)h)^{(m-\alpha)-1} (\Delta_h^m x)(sh), \quad (14)$$

where $(\Delta_h^m x)(t)$ is the mth power of the forward h-difference operator

$$(\Delta_h x)(t) = \frac{x(t + h) - x(t)}{h}. \quad (15)$$

As it has been noted in [73,74], due to the convergence of solutions of fractional Riemann-Liouville h-difference equations when $h \to 0$ to solutions of the corresponding differential equations, they can be used to solve fractional Riemann-Liouville differential equations numerically.
C. Fractional maps

Maps with power-law memory can be introduced directly as a particular form of maps with memory (see papers \[19,45\] which contain references and discussions on the topic). The most general form of the convolution-type map with power-law memory introduced in \[19\] can be written as

\[x_n = \sum_{k=1}^{[\alpha]-1} \frac{c_k}{\Gamma(\alpha-k+1)} (nh)^{\alpha-k} + \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{n-1} (n-k)^{\alpha-1} G_K(x_k),\]  

(16)

where \(\alpha \geq 0\), \(K\) is a parameter, and \(h\) is a constant time step between the time instants \(t_n = a + nh\) and \(t_{n+1}\). For a physical interpretation of this formula we consider a system which state is defined by the variable \(x(t)\) and evolution by the continuous function \(G_K(x)\). The value of the state variable at the time \(t_n\), \(x_n = x(t_n)\), is a weighted total of the functions \(G_K(x_k)\) from the values of this variable at the past time instants \(t_k = a + kh\), \(0 \leq k < n\), \(t_k = kh\). The weights are the times between the time instants \(t_n\) and \(t_k\) to the fractional power \(\alpha - 1\). Eq. (16) in the limit \(h \to 0^+\) yields the Volterra integral equation of the second kind

\[x(t) = \sum_{k=1}^{[\alpha]-1} \frac{c_k}{\Gamma(\alpha-k+1)} (t-a)^{\alpha-k} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{G_K(\tau, x(\tau))d\tau}{(t-\tau)^{1-\alpha}}, \quad (t > a),\]  

(17)

This equation is equivalent to the fractional differential equation with the Riemann-Liouville or Grünwald-Letnikov fractional derivative\[19,70/\]

\[a_{RL/GL}^\alpha D_t^\alpha x(t) = G_k(t, x(t)), \quad 0 < \alpha\]  

(18)

with the initial conditions

\[a_{RL/GL}^\alpha D_t^{\alpha-k} x(a^+) = c_k, \quad k = 1, 2, ..., [\alpha].\]  

(19)

For \(\alpha \notin \mathbb{N}\) we assume \(c_{[\alpha]} = 0\), which corresponds to a finite value of \(x(a)\).

The same map, Eq. (16), called the universal map, represents the solution of the fractional generalization of the differential equation of a periodically (with the period \(h\)) kicked system (see \[23,33,60,62\] for the fractional universal maps and \[77\] in regular dynamics).
To derive the equations of the fractional universal map, which we’ll call the universal 
α-family of maps (α-FM) for α ≥ 0, we start with the differential equation
\[ \frac{d^\alpha x}{dt^\alpha} + G_K(x(t - \Delta h)) \sum_{k=-\infty}^{\infty} \delta\left(\frac{t}{h} - (k + \varepsilon)\right) = 0, \quad (20) \]
where ε > Δ > 0, α ∈ ℝ, α > 0, and consider it as ε → 0. The initial conditions should correspond to the type of the fractional derivative used in Eq. (20). The case α = 2, Δ = 0, and \( G_K(x) = KG(x) \) corresponds to the equation whose integration yields the regular universal map.

Integration of Eq. (20) with the Riemann-Liouville fractional derivative \( 0D_t^\alpha x(t) \) and the initial conditions
\[ (0D_t^{\alpha-k} x)(0+) = c_k, \quad (21) \]
where \( k = 1, \ldots, N \) and \( N = \lceil \alpha \rceil \), yields the Riemann-Liouville universal α-FM Eq. (16).

Integration of Eq. (20) with the Caputo fractional derivative \( C_0^\alpha D_t^\alpha x(t) \) and the initial conditions \( (D_k^\alpha x)(0+) = b_k, \quad k = 0, \ldots, N - 1 \) yields the Caputo universal α-FM
\[ x_{n+1} = \sum_{k=0}^{N-1} \frac{b_k}{k!} h^k (n + 1)^k \]
\[ - \frac{h^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{n} G_K(x_k)(n - k + 1)^{\alpha-1}. \quad (22) \]

In this paper we’ll refer to the map Eqs. (16), the RL universal α-FM, as the Riemann-Liouville universal map with power-law memory or the Riemann-Liouville universal fractional map; we’ll call the Caputo universal α-FM, Eq. (22), the Caputo universal map with power-law memory or the Caputo universal fractional map.

In the case of integer α the universal map converges to \( x_n = 0 \) for α = 0 and \( x_{n+1} = x_n - hG_K(x_n) \) for α = 1. and for α = N = 2 with \( p_{n+1} = (x_{n+1} - x_n)/h \)
\[ \begin{cases} p_{n+1} = p_n - hG_K(x_n), & n \geq 0, \\ x_{n+1} = x_n + hp_{n+1}, & n \geq 0. \end{cases} \quad (23) \]

N-dimensional, with \( N \geq 2 \), universal maps are investigated in [23], where it is shown that they are volume preserving.
D. Universal fractional difference map

In what follows we will consider fractional Caputo difference maps - the only fractional difference maps which behavior has been investigated. The following theorem is essential to derive the universal fractional difference map

**Theorem 1** For $\alpha \in \mathbb{R}, \alpha \geq 0$ the Caputo-like $h$-difference equation

$$(0\Delta_h^\alpha x)(t) = -G_K(x(t + (\alpha - 1)h)), \quad (24)$$

where $t \in (h\mathbb{N})^m$, with the initial conditions

$$(0\Delta_h^k x)(0) = c_k, \quad k = 0, 1, ..., m - 1, \quad m = \lceil \alpha \rceil \quad (25)$$

is equivalent to the map with $h$-factorial-law memory

$$x_{n+1} = \sum_{k=0}^{m-1} \frac{c_k}{k!} ((n+1)h)^{(k)}_h - \frac{h^\alpha}{\Gamma(\alpha)} \sum_{s=0}^{n+1-m} (n - s - m + \alpha)^{(\alpha-1)} G_K(x_{s+m-1}), \quad (26)$$

where $x_k = x(kh)$, which is called the $h$-difference Caputo universal $\alpha$-family of maps.

In the case of integer $\alpha$ the fractional difference universal map converges to $x_{n+1} = -G_K(x_n)$ for $\alpha = 0$, $x_{n+1} = x_n - hG_K(x_n)$ for $\alpha = 1$, and for $\alpha = N = 2$ with $p_{n+1} = (x_{n+1} - x_n)/h$

$$\begin{cases}
p_{n+1} = p_n - hG_K(x_n), \quad n \geq 1, \quad p_1 = p_0, \\
x_{n+1} = x_n + hp_{n+1}, \quad n \geq 0.
\end{cases} \quad (27)$$

N-dimensional, with $N \geq 2$, difference universal maps are volume preserving.

All above considered universal maps in the case $\alpha = 2$ yield the standard map if $G_K(x) = K \sin(x)$ (harmonic nonlinearity) and we’ll call them the standard $\alpha$-families of maps. When $G_K(x) = x - Kx(1 - x)$ (quadratic nonlinearity) in the one-dimensional case all maps yield the regular logistic map and we’ll call them the logistic $\alpha$-families of maps.

**III. PERIOD TWO SINKS AND STABILITY OF FIXED POINTS**

In fractional systems not only the speed of convergence of trajectories to the periodic sinks but also the way in which convergence occurs depends on the initial conditions. As
FIG. 2. Asymptotically period two trajectories for the Caputo logistic $\alpha$-family of maps with $\alpha = 0.1$ and $K = 15.5$: (a) nine trajectories with the initial conditions $x_0 = 0.29 + 0.04i$, $i = 0, 1, ..., 8$ ($i = 0$ corresponds to the rightmost bifurcation); (b) $x_0 = 0.61 + 0.06i$, $i = 1, 2, 3$; (c) $x_0 = 0.95 + 0.04i$, $i = 1, 2, 3$. As $n \to \infty$ all trajectories converge to the limiting values $x_{lim1} = 0.80629$ and $x_{lim2} = 1.036030$ (see Eq (61)). The unstable fixed point is $x_{lim0} = (K - 1)/K = 0.93548$.

$n \to \infty$, all trajectories in Fig. 2 converge to the same period two ($T = 2$) sink (as in Fig. 2 c), but for small values of the initial conditions $x_0$ all trajectories first converge to the $T = 1$ trajectory which then bifurcates and turns into the $T = 2$ sink converging to its limiting value. As $x_0$ increases, the bifurcation point $n_{bif}$ gradually evolves from the right to the left (Fig. III(a)). Ignoring this feature may result (as in other papers) in very messy bifurcation diagrams.

In this paper we consider the asymptotic stability of periodic points. A periodic point is asymptotically stable if there exists an open set such that all trajectories with initial conditions from this set converge to this point as $t \to \infty$. It is known from the study of the ordinary nonlinear dynamical systems that as a nonlinearity parameter increases the system bifurcates. This means that at the point (value of the parameter) of birth of the $T = 2^{n+1}$ sink, the $T = 2^n$ sink becomes unstable. In this section we will investigate the $T = 2$ sinks of discrete fractional systems and apply our results to analyze stability of the systems’ fixed points.
A. $0 < \alpha < 1$

When $0 < \alpha < 1$, all introduced in this paper forms of the universal $\alpha$-family of maps, Eqs. (16), (22), (26), can be written in the form

$$x_{n+1} = x_0 - \sum_{k=0}^{n} \tilde{G}(x_k)U(n - k + 1).$$

(28)

In this formula $\tilde{G}(x) = h^\alpha G_K(x)/\Gamma(\alpha)$ and $x_0$ is the initial condition ($x_0 = 0$ in Eq. (16)).

In fractional maps, Eqs. (16) and (22),

$$U_\alpha(n) = n^{\alpha-1}, \quad U_\alpha(1) = 1$$

(29)

and in fractional difference maps, Eq. (26),

$$U_\alpha(n) = (n + \alpha - 2)^{(\alpha-1)},$$

$$U_\alpha(1) = (\alpha - 1)^{(\alpha-1)} = \Gamma(\alpha).$$

(30)

For $n = 2N$ Eq. (28) can be written (after subtracting $x_{2N}$) as

$$x_{2N+1} = x_{2N} - \tilde{G}(x_{2N})U_\alpha(1)$$

$$+ \sum_{n=1}^{N} \tilde{G}(x_{2N-2n+1}) \left( U_\alpha(2n - 1) - U_\alpha(2n) \right)$$

$$+ \sum_{n=1}^{N} \tilde{G}(x_{2N-2n}) \left( U_\alpha(2n) - U_\alpha(2n + 1) \right).$$

(31)

The terms $U_\alpha(2n - 1) - U_\alpha(2n)$ are of the order $n^{\alpha-2}$. If we assume that in the limit $n \to \infty$ period $T = 2$ sink exists,

$$x_o = \lim_{n \to \infty} x_{2n+1}, \quad x_e = \lim_{n \to \infty} x_{2n},$$

(32)

then the series in Eq. (31) converge absolutely. In the limit $n \to \infty$ Eq.(31) converges to

$$x_o - x_e = \left[ \tilde{G}(x_o) - \tilde{G}(x_e) \right] W_\alpha,$$

(33)

where $W_\alpha$ is a converging series

$$W_\alpha = \sum_{n=1}^{\infty} \left[ U_\alpha(2n - 1) - U_\alpha(2n) \right],$$

(34)
which can be computed numerically with \( U_\alpha(n) \) defined either by Eq. (29) or by Eq. (30).

Now, instead of subtracting, let’s add \( x_{2N} \) to \( x_{2N+1} \):

\[
x_{2N+1} + x_{2N} = 2x_0 - \sum_{n=1}^{2N} \left[ \tilde{G}(x_{2N-n+1}) + \tilde{G}(x_{2N-n}) \right] U_\alpha(n) - \tilde{G}(x_0) U_\alpha(2N + 1).
\]

If \( T = 2 \) sink exists, then, in the limit \( n \to \infty \), the left-hand side (LHS) of Eq. (35), as well as the first term on the right-hand side (RHS) and the last term of this equation, is finite. Expressions in the brackets in Eq. (35) tend to the limit \( \tilde{G}(x_o) + \tilde{G}(x_e) \). Because the series \( \sum_{n=1}^{\infty} U_\alpha(n) \) is diverging, the only case in which Eq. (35) can be true is when

\[
\tilde{G}(x_o) + \tilde{G}(x_e) = 0.
\]

Equations which define the existence and value of the asymptotic \( T = 2 \) sink can be written as

\[
\begin{align*}
G_K(x_o) + G_K(x_e) &= 0, \\
x_o - x_e &= \frac{W_\alpha}{\Gamma(\alpha)} h^\alpha \left[ G_K(x_o) - G_K(x_e) \right].
\end{align*}
\]

- It is easy to see that the fixed point, defined by the equation \( G_K(x_o) = 0 \) is a solution of the system Eq. (37).

- As it was mentioned above, when \( h \to 0 \), fractional difference equations converge to the corresponding fractional differential equations. As \( h \to 0 \), the second equation from the system Eq. (37) leads to \( x_o - x_e \to 0 \). This implies that in fractional differential equations of the order \( 0 < \alpha < 1 \) transition from a fixed point to periodic trajectories will never happen. A strict proof of the impossibility of periodic trajectories (except fixed points) in autonomous fractional systems described by the fractional differential equation

\[
\frac{d^\alpha x}{dt^\alpha} = G_K(x(t)), \quad 0 < \alpha < 1
\]

with the Caputo or Riemann-Liouville fractional derivative was given in \cite{27} (Theorem 9 there). Nonexistence of periodic trajectories and the fact that in regular dynamics transition to chaos occurs through cascades of the period doubling bifurcations, leads us to the following conjecture

**Conjecture 2** Chaos does not exist in continuous fractional systems of the orders \( 0 < \alpha < 1 \).
B. \textbf{1 < \alpha < 2}

For \(1 < \alpha < 2\) map equations Eqs. (16), (22), (26), can be written in the form

\[
x_{n+1} = x_0 + f(\alpha)[h(n+1)]^\beta p_0
\]

\[
-\hbar \sum_{k=0}^{n} \tilde{G}(x_k)U_\alpha(n-k+1) + hf_1(n).
\]  

(39)

Here \(\tilde{G}(x) = h^{\alpha-1}G_K(x)/\Gamma(\alpha)\), \(x_0\) and \(U(n)\) are defined the same way as in Eqs. (28), (29), and (30), \(p_0\) is the initial momentum (\(b_k\) or \(c_k\) in corresponding formulae), \(\beta\) is equal to 1 in Eqs. (22), (26) and \(\alpha - 1\) in Eq. (16) \(f(\alpha)\) is 1 in Eqs. (22), (26) and \(1/\Gamma(\alpha)\) in Eq. (16), and \(f_1(n) = 0\) in Eqs. (16), (22) and \(f_1(n) = h^{\alpha-1}G(x_0)(n-1+\alpha)^{(\alpha-1)}/\Gamma(\alpha) \sim n^{\alpha-1}\) in Eq. (26).

If we define

\[
p_{n+1} = \frac{x_{n+1} - x_n}{\hbar},
\]

(40)

then, taking into account that \(U_\alpha(0) = 0\), from Eq. (39) follows

\[
p_{n+1} = \tilde{f}(n)p_0
\]

\[
-\sum_{k=0}^{n} \tilde{G}(x_k)\tilde{U}_\alpha(n-k+1) + f_1(n) - f_1(n-1),
\]  

(41)

where

\[
\tilde{U}_\alpha(n) = U_\alpha(n) - U_\alpha(n-1)
\]

\[
= \begin{cases} 
    n^{\alpha-1} - (n - 1)^{\alpha-1} & \sim n^{\alpha-2} \\
    \text{and } \tilde{U}_\alpha(1) = 1 \text{ in Eqs. (16), (22)}; \\
    (n + \alpha - 2)^{(\alpha-1)} - (n + \alpha - 3)^{(\alpha-1)} \\
    = (\alpha-1)(n + \alpha - 3)^{(\alpha-2)} \\
    = (\alpha-1)U_{\alpha-1}(n) \sim n^{\alpha-2} \\
    \text{and } \tilde{U}_\alpha(1) = \Gamma(\alpha) \text{ in Eq. (26)},
\end{cases}
\]

(42)

\[f_1(n) - f_1(n-1) = 0\] in Eqs. (16), (22) and \(f_1(n) - f_1(n-1) \sim n^{\alpha-1}\) in Eq. (26), \(\tilde{f}(n) = 1\) in Eqs. (22), (26) and \(\tilde{f}(n) \sim n^{\alpha-2}\) in Eq. (16). Note, that the definition of \(\tilde{U}_\alpha(1)\) in Eq. (42) and \(U_\alpha(1)\) in Eqs. (29), (30) are identical.
Assuming existence of the $T = 2$ sink and limits $x_o$ and $x_e$ are defined by Eq. (32), the limiting values for $p$ are defined by

$$p_o = \lim_{n \to \infty} p_{2n+1} = \lim_{n \to \infty} \frac{x_{2n+1} - x_{2n}}{h} = \frac{x_o - x_e}{h}$$

and

$$p_e = \lim_{n \to \infty} p_{2n} = -p_o.$$  

(43)

As in the derivation of Eqs. (33) and (36), if we add and subtract expressions for $p_{2N+1}$ and $p_{2N}$, we’ll arrive at relations

$$p_o - p_e = \left[ \tilde{G}(x_o) - \tilde{G}(x_e) \right] \tilde{W}_\alpha$$

(44)

and

$$\tilde{G}(x_o) + \tilde{G}(x_e) = 0,$$

(45)

where $\tilde{W}_\alpha$ is a converging series

$$\tilde{W}_\alpha = \sum_{n=1}^{\infty} \left[ \tilde{U}_\alpha(2n - 1) - \tilde{U}_\alpha(2n) \right].$$

(46)

Let’s note that with $U_\alpha(n) = n^{\alpha-1}$, as defined in Eq. (29), $\tilde{W}$ is identical to the introduced in\(^{23}\) $V_{\alpha l}$ defined as

$$\tilde{W}_\alpha = V_{\alpha l} = \sum_{n=1}^{\infty} (-1)^{n+1} \left[ n^{\alpha-1} - (n - 1)^{\alpha-1} \right].$$

(47)

High accuracy algorithm for calculating $V_{\alpha l}$ is presented in Appendix to\(^{23}\). For $U(n)$ defined by Eq. (30) $\tilde{W}$ was calculated in\(^{56}\). Taking into account that converging series Eq. (46) can be written as

$$\tilde{W}_\alpha = \tilde{U}_{\alpha 1} - \sum_{n=1}^{\infty} \left[ \tilde{U}_\alpha(2n) - \tilde{U}_\alpha(2n + 1) \right],$$

(48)

where

$$\tilde{U}_{\alpha 1} = \begin{cases} 1 & \text{in Eqs. (16), (22),} \\ \Gamma(\alpha) & \text{in Eq. (26),} \end{cases}$$

(49)

and using absolute convergence of series Eq. (34) (and, correspondingly, the series on the
first line of Eq. (51) below), for $0 < \alpha < 1$ we can write

\[
\tilde{W}_\alpha = \tilde{U}_{\alpha 1} - \sum_{n=1}^{\infty} \left\{ \left[ U_{\alpha}(2n) - U_{\alpha}(2n - 1) \right] - \\
- \left[ U_{\alpha}(2n + 1) - U_{\alpha}(2n) \right] \right\}
= \tilde{U}_{\alpha 1} + \sum_{n=1}^{\infty} \left[ U_{\alpha}(2n - 1) - U_{\alpha}(2n) \right]
- \sum_{n=1}^{\infty} \left[ U_{\alpha}(2n) - U_{\alpha}(2n + 1) \right] = W_{\alpha} + \tilde{U}_{\alpha 1}
- U_{\alpha}(2) + U_{\alpha}(3) - U_{\alpha}(4) + U_{\alpha}(5) - \ldots = 2W_{\alpha}.
\]

Let us notice that in fractional difference maps Eq. (48) can be written as

\[
\tilde{W}_\alpha = (\alpha - 1)\Gamma(\alpha - 1) - (\alpha - 1) \sum_{n=1}^{\infty} \left[ U_{\alpha-1}(2n) - \\
U_{\alpha-1}(2n + 1) \right] = (\alpha - 1)W_{\alpha-1} = \frac{\alpha - 1}{2} \tilde{W}_{\alpha-1}.
\]

Finally, the equations which define the existence and value of the asymptotic $T = 2$ sink for $0 < \alpha < 2$ can be written as

\[
\begin{align*}
\begin{cases}
G_{K}(x_o) + G_{K}(x_e) = 0, \\
x_o - x_e = \frac{W_{\alpha}}{2(\alpha)h^{\alpha}} \left[ G_{K}(x_o) - G_{K}(x_e) \right],
\end{cases}
\end{align*}
\]

where $\tilde{W}_\alpha$ is defined by Eqs. (48), (49). Notice that according to Eq (48) $\tilde{W}_1 = 1$.

- As in the case $0 < \alpha < 1$, for $1 < \alpha < 2$ the fixed point, defined by the equation $G_{K}(x_o) = 0$ is a solution of the system Eq. (52).
- As $h \to 0$, fractional difference equations converge to the corresponding fractional differential equations and $x_o - x_e \to 0$, which implies that in fractional differential equations of the order $1 < \alpha < 2$ transition from a fixed point to periodic trajectories will never happen. Now we may formulate a stronger conjecture:

**Conjecture 3** *Chaos does not exist in continuous fractional systems of the orders $0 < \alpha < 2$.***
C. Examples

Now we’ll consider application of the results from this section to the introduced at the end of Section II fractional and fractional difference standard \((G_K(x) = K \sin(x))\) and logistic \((G_K(x) = x - Kx(1 - x))\) \(\alpha\)-families of maps.

1. Standard \(\alpha\)-families of maps

With \(G_K(x) = K \sin(x)\) all above considered forms of the universal map for \(\alpha = 2\) converge to the regular standard map and they are called the standard \(\alpha\)-families of maps. These families of maps are usually considered on a torus (mod \(2\pi\)). The first equation of the system Eq. (52) yields

\[
\sin \frac{x_o + x_e}{2} \cos \frac{x_o - x_e}{2} = 0,
\]

which on \(x \in [-\pi, \pi]\) yields two solutions

symmetric point \(x_{osy} = -x_{esy}\) and

shift - \(\pi\) point \(x_{osh} = x_{esh} - \pi\).

Then, the second equation of Eq. (52) yields the equation which together with Eq. (54) defines two \(T = 2\) sinks for \(0 < \alpha < 2\)

\[
\sin x_{osy} = \frac{2\Gamma(\alpha)}{W_\alpha h_\alpha K} x_{osy},
\]

and

\[
\sin x_{osh} = \frac{\pi\Gamma(\alpha)}{W_\alpha h_\alpha K}.
\]

The symmetric \(T = 2\) sink appears when

\[
h_\alpha |K| > h_\alpha |K_{C1s}| = \frac{2\Gamma(\alpha)}{W_\alpha}
\]

and the shift-\(\pi\) \(T = 2\) sink appears when

\[
h_\alpha |K| > \frac{\pi}{2} h_\alpha |K_{C1s}|.
\]
FIG. 3. 2D bifurcation diagrams for fractional (solid thing lines) and fractional difference (bold and dashed lines) Caputo standard (a) and $h = 1$ logistic (b) maps. First bifurcation, transition from the stable fixed point to the stable period two ($T = 2$) sink, occurs on the bottom curves. $T = 2$ sink (in the case of standard $\alpha$-families of maps antisymmetric $T=2$ sink with $x_{n+1} = -x_n$) is stable between the bottom and the middle curves. Transition to chaos occurs on the top curves. For the standard fractional map transition from $T = 2$ to $T = 4$ sink occurs on the line below the top line (the third from the bottom line). Period doubling bifurcations leading to chaos occur in the narrow band between the two top curves. All bottom curves, as well as the next to the bottom in (a), are obtained using formulae Eqs. (57), (58), and (62). Two dashed lines for $1 < \alpha < 2$ in (b) are obtained by interpolation. The remaining lines are results of the direct numerical simulations. Stability of the fixed point for the fractional difference logistic $\alpha$-family of maps is calculated using both, Eq. (62) (bold solid line) and the direct numerical simulations (a dashed line branching from the solid line). The difference is due to the slow, as $n^{-\alpha}$ (see 50), convergence of trajectories to the $T = 2$ sink for small $\alpha$ ($x$ vs. $K$, fixed $\alpha$, bifurcation diagrams used to find the first bifurcation were calculated on trajectories after 5000 iterations).

2. Logistic $\alpha$-families of maps

With $G_K(x) = x - Kx(1 - x)$ all above considered forms of the universal map for $\alpha = 1$ converge to the regular logistic map and they are called the logistic $\alpha$-families of maps. Th
system Eq. (52) becomes
\[
\begin{cases}
(1 - K)(x_o + x_e) + K(x_o^2 + x_e^2) = 0, \\
x_o - x_e = \frac{\tilde{W}h^\alpha}{2\Gamma(\alpha)}(x_o - x_e)[1 - K + (x_o + x_e)]
\end{cases}
\]
(59)

Two fixed point solutions with \( x_o = x_e \) are \( x_o = 0 \), stable for \( K < 1 \), and \( x_o = (K - 1)/K \).

The \( T = 2 \) sink is defined by the equation
\[
x_o^2 - \left( \frac{2\Gamma(\alpha)}{\tilde{W}K h^\alpha} + \frac{K - 1}{K} \right)x_o + \frac{2\Gamma^2(\alpha)}{(\tilde{W}K h^\alpha)^2} + \frac{(K - 1)\Gamma(\alpha)}{\tilde{W}K^2 h^\alpha} = 0,
\]
(60)
which has solutions
\[
x_o = \frac{K_{C1s} + K - 1 \pm \sqrt{(K - 1)^2 - K_{C1s}^2}}{2K}
\]
(61)
defined when
\[
K \geq 1 + \frac{2\Gamma(\alpha)}{\tilde{W} h^\alpha} = 1 + K_{C1s} \quad \text{or} \quad K \leq 1 - \frac{2\Gamma(\alpha)}{\tilde{W} h^\alpha} = 1 - K_{C1s}
\]
(62)
The first inequality of Eq. (62) was derived in $^{24}$ for \( h = 1 \) and \( 1 < \alpha < 2 \). In this paper we consider \( K > 0 \) and \( h \leq 1 \). It follows from the definition, Eq. (48), that \( \tilde{W} < \tilde{U}_1 \), which is either 1 or \( \Gamma(\alpha) \), and it is known that \( \Gamma(\alpha) > 0.885 \) for \( \alpha > 0 \). Then, \( 2\Gamma(\alpha)/(\tilde{W} h^\alpha) > 1 \) and we may ignore the second of the inqualities in Eq. (62). We may also note that the fixed point \( x = (K - 1)/K \) is stable when
\[
1 \leq K < K_{C1l} = 1 + \frac{2\Gamma(\alpha)}{\tilde{W} h^\alpha} = 1 + K_{C1s}.
\]
(63)

IV. CONCLUSION

Figs. 3 a and b, the two-dimensional bifurcation diagrams, present results of the computer simulations of the fractional and fractional difference standard and logistic maps. Low curves on these diagrams are obtained using Eqs. (57), (58), and (62). They are in good agreement with the results (also used to calculate all other curves) obtained by the direct numerical simulations by calculating \( x \) vs. \( K \) bifurcation diagrams for various \( \alpha \in (0, 2) \) after 5000 iterations. Slight difference in Fig. 3 b for the fractional difference logistic map for \( \alpha < 0.2 \) is probably due to the slow, as \( \sim n^{-\alpha} \) convergence of trajectories to the fixed points. This confirms the validity of Eq. (52) to calculate the coordinates of the asymptotic \( T = 2 \) sinks and the points of the first bifurcations for the discrete fractional/fractional difference maps. The continuous limits of the considered in this paper discrete maps are fractional differential
equations and from the consideration presented in this paper we may conclude that chaos is impossible in systems described by equations

\[
\frac{d^\alpha x}{dt^\alpha} = f(x)
\]  

(64)

with \( 0 < \alpha < 2 \).

There are still many unanswered questions related to the behavior of fractional systems. They include:

- What is the nature and the corresponding analytic description of the bifurcations on a single trajectory of a fractional system?
- What kind of self-similarity can be found in CBTT?
- How to describe a self-similar behavior corresponding to the bifurcation diagrams of fractional systems? Can constants, similar to the Feigenbaum constants be found?
- Can cascade of bifurcations type trajectories be found in continuous systems?

This paper is a small step in investigation of the fractional dynamical systems and we hope that the following works will lead to more complete description of fractional (with power-law memory) systems which have many applications in biological, social, and physical systems.

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