ON THE ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF NONLINEAR ELLIPTIC PROBLEMS IN DOMAINS BECOMING UNBOUNDED

LUCA ESPOSITO\(^1\), PROSENJIT ROY\(^2\) AND FIROJ SK\(^2\)

1 DipMat, University of Salerno, Italy.
E-mail: luesposi@unisa.it
2 Indian Institute of Technology, Kanpur, India.
E-mail: prosenjit@iitk.ac.in and firoj@iitk.ac.in

Abstract. We analyze the asymptotic behavior of the eigenvalues of nonlinear elliptic problems under Dirichlet boundary conditions and mixed (Dirichlet, Neumann) boundary conditions on domains becoming unbounded. We make intensive use of Picone identity to overcome nonlinearity complications. Altogether the use of Picone identity makes the proof easier with respect to the known proof in the linear case. Surprisingly the asymptotic behavior under mixed boundary conditions critically differs from the case of pure Dirichlet boundary conditions for some class of problems.

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1. Introduction

In this paper we study nonlinear elliptic eigenvalue problems on domains which become unbounded in one or several directions. We have basically focused on operators related to the \(p\)-Laplacian. To be more precise let us introduce some notations that we will use in the rest of the paper.

Let \(1 \leq m < n\) and let \(\omega_1, \omega_2\) be two open bounded sets in \(\mathbb{R}^m\) and \(\mathbb{R}^{n-m}\) respectively. For every \(\ell > 0\) let us define \(\Omega_\ell:=\ell \omega_1 \times \omega_2\). We will denote, for every \(x \in \mathbb{R}^n\)
\[
  x = (X_1, X_2),
\]
with
\[
  X_1 = (x_1, \ldots, x_m), \quad X_2 = (x_{m+1}, \ldots, x_n).
\]
\(\nabla, \nabla X_1\) and \(\nabla X_2\) will denote gradient vectors in \(\mathbb{R}^n, \mathbb{R}^m\) and \(\mathbb{R}^{n-m}\) respectively. Let \(A = A(X_1, X_2)\) be an \(n \times n\)-symmetric matrix of the type
\[
  A = \begin{pmatrix}
    A_{11}(x) & A_{12}(X_2) \\
    A_{12}^T(X_2) & A_{22}(X_2)
  \end{pmatrix},
\]
where \(A_{22}\) is an \((n-m) \times (n-m)\) matrix. We will assume that \(A\) is an uniformly bounded and uniformly positive definite matrix on \(\mathbb{R}^m \times \omega_2\). Precise conditions on the matrix \(A\) will be clarified in section 3. We start considering the following eigenvalue problem with Dirichlet boundary condition for any \(p \geq 2\),
We are interested in the study of the asymptotic behaviour of the first eigenvalue \( \lambda_{1,\ell}^D(\Omega_\ell) \) of the above problem as \( \ell \to \infty \). In the linear case \( p = 2 \), in a seminal paper of Chipot and Rougirel (see \([12]\)), it was proved that the k-th eigenvalue of (1.1) (see \([12]\) for the definition of k-th eigenvalue) converges to the first eigenvalue of the corresponding cross section problem that we now introduce in the general case \( p \geq 2 \).

We will denote by \( \mu_1(\omega_2) \) and \( W \) respectively the first eigenvalue and the first normalized \((\|W\|_{L^p(\omega_2)} = 1)\) eigenfunction of (1.2). In the first part of the paper we are able to prove that also in the nonlinear case \( p > 2 \) the first eigenvalue of (1.1) converge to the first eigenvalue of the problem (1.2) on cross section. Moreover, we would like to introduce the following minimization problem on the infinite strip \( \Omega_\infty := \mathbb{R}^m \times \omega_2 \) in order to get a deeper insight of the asymptotic behaviour of \( \lambda_{1,\ell}^D(\Omega_\ell) \):

\[
\Lambda_\infty = \inf_{\substack{u \in W_0^{1,p}(\Omega_\infty), u \neq 0}} \frac{\int_{\Omega_\infty} |A \nabla u \cdot \nabla u|_2^p}{\int_{\Omega_\infty} |u|^p}.
\]

Precisely we prove the following theorem.

**Theorem 1.1.** Let \( p \geq 2 \) and \( \mu_1(\omega_2) \) denote the first eigenvalue of (1.2), then there exists a constant \( C \) depending only on \( A, \omega_2, p \), such that

\[
\mu_1(\omega_2) \leq \lambda_{1,\ell}^D(\Omega_\ell) \leq \mu_1(\omega_2) + \frac{C}{\ell},
\]

for every \( \ell > 0 \). Moreover \( \Lambda_\infty = \mu_1(\omega_2) \).

The lower bound in Theorem 1.1 was proved by Chipot and Rougirel (see \([12]\)), in the linear case \( p = 2 \), using an approximation argument for the matrix \( A \) that relies on the linearity of the equation and it is not clear if the same argument can be employed in the nonlinear case. In this paper we present a complete different argument that relies on a clever use of Picone identity (see Theorem 2.2). In spite of the difficulty of nonlinearity our approach turns out to be simpler and shorter than \([12]\). The upper bound in Theorem 1.1 is obtained, in a similar manner as in (\([12]\)), by constructing a suitable test function on the truncated domains \( \Omega_\ell = \frac{\ell}{2} \omega_1 \times \omega_2 \) and then letting \( \ell \) tending to infinity.

The second part of the paper concerns the eigenvalue problem for the same operator as in (1.1), but with mixed boundary conditions. For technical reasons (which precisely the construction of test function “\( \phi_\ell \)” in the proof of Theorem 1.2 ), we can only allow the
domain $\Omega_\ell$ to become unbounded in one direction, i.e. we assume that $\omega_1 = (-1,1)$ and $A_{11}(x) = a_{11}(x_2)$. Namely we consider the following eigenvalue problem on $\Omega_\ell = (-\ell, \ell) \times \omega_2$:

$$
\begin{cases}
-\text{div}(\overline{|A(X_2)\nabla u_\ell \cdot \nabla u_\ell|}^{\frac{p-2}{2}} A(X_2)\nabla u_\ell) = \lambda_M(\Omega_\ell)|u_\ell|^{p-2}u_\ell & \text{in } \Omega_\ell, \\
u_\ell = 0 & \text{on } \gamma_\ell := (-\ell, \ell) \times \partial \omega_2, \\
(A(X_2)\nabla u_\ell) \cdot \nu = 0 & \text{on } \Gamma_\ell := \{-\ell, \ell\} \times \omega_2,
\end{cases}
$$

(1.3)

where $\nu$ denotes the outward unit normal to $\Gamma_\ell$. For the case $p = 2$, in Chipot, Roy and Shafrir [10] it was proved that when $\ell$ goes to plus infinity the limit of the first eigenvalue $\lambda_M(\Omega_\ell)$ exists. In addition this limit is strictly smaller than $\mu_2(\omega_2)$ if and only if $A_{12} \cdot \nabla X_2 W \neq 0$ a.e. on $\omega_2$. In the nonlinear case ($p \geq 2$), we prove that this gap phenomenon still holds under the same condition $A_{12} \cdot \nabla X_2 W \neq 0$ a.e. on $\omega_2$. In particular we prove the following theorem.

**Theorem 1.2.** For $p \geq 2$, we have

$$
\lim_{\ell \to \infty} \sup \lambda_M^1(\Omega_\ell) < \mu_1(\omega_2),
$$

provided $A_{12} \cdot \nabla X_2 W \neq 0$ a.e. on $\omega_2$, otherwise $\lambda_M^1(\Omega_\ell) = \mu_1(\omega_2)$ for all $\ell > 0$.

The main steps to prove the above theorem uses the same argument as in [10]. The first step is to study a “dimension reduction” problem, namely we let $\ell$ go to zero in (1.3). As a matter of fact it turns out that $\lim_{\ell \to 0} \lambda_M^1(\Omega_\ell) < \mu_1(\omega_2)$ if and only if $A_{12} \cdot \nabla X_2 W \neq 0$ a.e. on $\omega_2$. This provides us the main tool to construct test functions on $\Omega_\ell$ in order to prove the gap phenomenon for large values of $\ell$. We address the readers to [1], [4], [21] and the references there in, for the general study of problems on “dimension reduction”.

Asymptotic behavior when the parameter $\ell \to \infty$ for different type of problems subject to different boundary conditions were studied in past. We refer to [13] and [14] for the study of Stokes problem and elliptic equations with Neuman boundary conditions. Asymptotic behaviour for the minimizers of purely variational problem is done in [9], [25]. We refer to [2], [5], [6], [7], [8], [11], [15], [16], [17], [18], [19], [20], [22], [23], [29] and the reference mentioned there in for other related work in this direction.

### 2. Some Preliminary results

In this section we will summarize some standard features about eigenvalues and eigenfunctions of p-Laplacian type operators. Let $\Omega$ be a bounded open regular subset of $\mathbb{R}^n$, $1 < p < \infty$, we will denote by $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ the usual spaces of functions defined by

$$
W^{1,p}(\Omega) = \{ v \in L^p(\Omega) : \partial_x v \in L^p(\Omega), i = 1, 2, \ldots, n \},
$$

and

$$
W_0^{1,p}(\Omega) = \{ v \in W^{1,p}(\Omega) : v = 0 \text{ on } \partial \Omega \}.
$$
Thanks to the classical Poincaré inequality, we will always assume that the space $W^{1,p}_0(\Omega)$ is equipped with the norm

\begin{equation}
||v||^p_{p,\Omega} = \int_{\Omega} |\nabla v|^p.
\end{equation}

Let $A(x)$ be a symmetric, uniformly positive definite and uniformly bounded matrix in $\mathbb{R}^n$. We assume that $A(x)$ is $C^1$ i.e. each component is $C^1$. Therefore we consider the following Dirichlet eigenvalue problem:

\begin{equation}
\begin{cases}
\begin{aligned}
-\text{div} \left( |A \nabla u|^{\frac{p-2}{2}} A \nabla u \right) &= \lambda(\Omega) |u|^{p-2} u & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega.
\end{aligned}
\end{cases}
\end{equation}

For reader convenience we will refer to the weak formulation of \((2.2)\), which consists in the following:

\begin{equation}
\begin{cases}
\begin{aligned}
u &\in W^{1,p}_0(\Omega), \\
\int_{\Omega} |A \nabla u \cdot \nabla v|^{\frac{p-2}{2}} A \nabla u \cdot \nabla v &= \lambda(\Omega) \int_{\Omega} |u|^{p-2} v & \text{for any } v \in W^{1,p}_0(\Omega).
\end{aligned}
\end{cases}
\end{equation}

We denote by $\lambda^1_D(\Omega)$ and $u_0$ the first eigenvalue and the first eigenfunction of \((2.2)\) respectively. Now we collect some properties of the first eigenpair $(\lambda^1_D(\Omega), u_0)$.

**Proposition 2.1.** The following properties verified by $\lambda^1_D(\Omega)$ and $u_0$

- $\lambda^1_D(\Omega)$ is finite and strictly positive.
- $\lambda^1_D(\Omega)$ fulfill the following variational characterization by means of the Rayleigh quotient,

\begin{equation}
\lambda^1_D(\Omega) = \inf_{u \in W^{1,p}_0(\Omega), \ u \neq 0} \frac{\int_{\Omega} |A \nabla u \cdot \nabla u|^{\frac{p}{2}}}{\int_{\Omega} |u|^p} = \frac{\int_{\Omega} |A \nabla u_0 \cdot \nabla u_0|^{\frac{p}{2}}}{\int_{\Omega} |u_0|^p}.
\end{equation}

- $u_0$ is bounded and is in $C^{1,\gamma}(\Omega)$ for some $\gamma > 0$.
- $\lambda^1_D(\Omega)$ is simple and the function $u_0$ does not change sign in $\Omega$.

In the case of pure p-Laplacian eigenvalue problem the properties listed above are well known. The reader is addressed to [3], [24], [26], [27], [28] for the proof. In the general case of equation \((2.2)\) the same results can be proved with obvious slight modifications.

One of the main tools for the proof of the main result of this paper is the following Picone’s identity. For the sake of completeness we give here the proof of this fundamental inequality (see also [3]).

**Theorem 2.2.** (Picone’s identity) Suppose $A$ is a symmetric positive definite matrix on $\mathbb{R}^n$ and $u, v$ two differentiable functions with $u \geq 0$ and $v > 0$. Define

\[
L(u, v) = |A \nabla u \cdot \nabla u|^{\frac{p}{2}} - p u^{p-1} |A \nabla v \cdot \nabla v|^{\frac{p-2}{2}} (A \nabla v \cdot \nabla u) + (p-1) u^p |A \nabla v \cdot \nabla v|^{\frac{p-2}{2}} (A \nabla v \cdot \nabla v),
\]
and
\[ R(u, v) = |A \nabla u \cdot \nabla u|^{\frac{p}{2}} - \nabla \left( \frac{u^p}{v^{p-1}} \right) |A \nabla v \cdot \nabla v|^{\frac{p-2}{2}} A \nabla v. \]

Then \( L(u, v) = R(u, v) \geq 0. \) Moreover \( L(u, v) = 0 \) a.e. in \( \Omega \) if and only if \( \nabla (\frac{u}{v}) = 0 \) a.e. in \( \Omega \).

**Proof.** The equality case \( L(u, v) = R(u, v) \) trivially follows by expanding \( R(u, v) \). Now by hypothesis on the matrix \( A \), we can write \( |A \nabla u \cdot \nabla u| = |A^{\frac{1}{2}} \nabla u|^2 \). Thus
\[
L(u, v) = |A \nabla u \cdot \nabla u|^{\frac{p}{2}} - \frac{pu^{p-1} |A \nabla v \cdot \nabla v|^{\frac{p-2}{2}} (A \nabla v \cdot \nabla u)}{v^{p-1}} + \left( p-1 \right) u^p |A \nabla v \cdot \nabla v|^{\frac{p-2}{2}} (A \nabla v \cdot \nabla u)
\]
\[
= \left| A^{\frac{1}{2}} \nabla u \right|^p + \left( p-1 \right) \frac{u^p |A^{\frac{1}{2}} \nabla v|^p}{v^{p-1}} \left( |A^{\frac{1}{2}} \nabla v|^{p-2} (A \nabla v \cdot \nabla u) \right)
\]
\[
= p \left( \left| A^{\frac{1}{2}} \nabla u \right|^p + \frac{u^p |A^{\frac{1}{2}} \nabla v|^p}{v^{p-1}} \right) - \frac{pu^{p-1} |A^{\frac{1}{2}} \nabla v|^{p-2} (A \nabla v \cdot \nabla u)}{v^{p-1}}
\]
\[
= p \left( \left| A^{\frac{1}{2}} \nabla u \right|^p + \left( u \left| A^{\frac{1}{2}} \nabla v \right| \right)^{(p-1)q} \right) \left( |A^{\frac{1}{2}} \nabla u| \left| A^{\frac{1}{2}} \nabla v \right| - A \nabla u \cdot \nabla v \right).
\]

Using Young’s inequality we have,
\[
\frac{\left| A^{\frac{1}{2}} \nabla u \right|^p}{p} + \left( u \left| A^{\frac{1}{2}} \nabla v \right| \right)^{(p-1)q} \geq \frac{u^{p-1} |A^{\frac{1}{2}} \nabla u| |A^{\frac{1}{2}} \nabla v|^{p-1}}{v^{p-1}}
\]
where \( 1/p + 1/q = 1 \). Equality holds when
\[
\left| A^{\frac{1}{2}} \nabla u \right| = \frac{u}{v} \left| A^{\frac{1}{2}} \nabla v \right|.
\]

Therefore we get \( L(u, v) \geq 0. \) So when, \( L(u, v)(x_0) = 0 \) and \( u(x_0) \neq 0 \) we must have \( \left| A^{\frac{1}{2}} \nabla u \right| = \frac{u}{v} \left| A^{\frac{1}{2}} \nabla v \right| \) and \( \left| A^{\frac{1}{2}} \nabla u \right| \left| A^{\frac{1}{2}} \nabla v \right| = A \nabla u \cdot \nabla v \) and thus we obtain \( A^{\frac{1}{2}} \nabla u = \frac{u}{v} A^{\frac{1}{2}} \nabla v \). Therefore, \( \nabla \left( \frac{u}{v} \right)(x_0) = 0. \) On the other hand if, \( u(x_0) = 0 \) then \( \nabla u = 0 \) a.e. on \( \{ u(x) = 0 \} \) and thus \( \nabla \left( \frac{u}{v} \right) = 0 \) a.e. on \( \{ u(x) = 0 \} \). Therefore we conclude that \( \nabla \left( \frac{u}{v} \right) = 0 \) a.e. in \( \Omega \).

### 3. Convergence of the First Eigenvalue for Dirichlet Case.

Resuming the notation used in the introduction we denote \( \Omega_\ell = \ell \omega_1 \times \omega_2 \) to be an open subset of \( \mathbb{R}^n \), where \( \omega_1, \omega_2 \) are two open bounded sets in \( \mathbb{R}^m \), and \( \mathbb{R}^{n-m} \) respectively and \( \ell > 0 \). The variables in \( \omega_1 \), and \( \omega_2 \) are denoted by \( X_1 \) and \( X_2 \) respectively. We will write \( x = (X_1, X_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \) accordingly.

The matrix
\[
A = \begin{pmatrix}
A_{11}(x) & A_{12}(X_2) \\
A_{12}^t(X_2) & A_{22}(X_2)
\end{pmatrix},
\]
is an \( n \times n \)-symmetric matrix, and assume that the block matrix \( A_{22} \) is \( C^1 \) regularity.
We will assume that $A$ is uniformly bounded and uniformly positive definite matrix on $\mathbb{R}^m \times \omega_2$; namely there exists two positive constants $M$, $\lambda$ such that

$$||A(x)|| \leq M \quad \text{a.e. } x \in \mathbb{R}^m \times \omega_2,$$

$$A(x)\xi \cdot \xi \geq \lambda |\xi|^2 \quad \text{a.e. } x \in \mathbb{R}^m \times \omega_2, \quad \forall \xi \in \mathbb{R}^n.$$  

In the following $|| \cdot ||$ will denote the norm of matrices, $| \cdot |$ the euclidean norm, and “.” the usual euclidean scalar product.

In this section we investigate the asymptotic behaviour of the first eigenvalue $\lambda^1_D(\Omega_\ell)$ of the problem (1.1) for $\ell \to \infty$. Indeed we prove Theorem 1.1 which claim that $\lambda^1_D(\Omega_\ell)$ converges to the first eigenvalue $\mu_1(\omega_2)$ of the problem (1.2). For the reader convenience we quote the weak formulation of the problem (1.2).

(3.1) \[
\begin{cases}
  u \in W^{1,p}_0(\omega_2), \\
  \int_{\omega_2} |A_{22} \nabla X_2 u \cdot \nabla X_2 u|^\frac{p-2}{2} A_{22} \nabla X_2 u \cdot \nabla X_2 v = \mu(\omega_2) \int_{\omega_2} |u|^{p-2} u v & \forall v \in W^{1,p}_0(\omega_2).
\end{cases}
\]

Remember that $\mu_1(\omega_2)$ and $W$ denote the first eigenvalue and the first normalized ($||W||_{L^p(\omega_2)} = 1$) eigenfunction of the problem (3.1) respectively.

As observed in the introduction, the first eigenvalue $\mu_1(\omega_2)$ has a variational characterization by the Rayleigh quotient:

$$\mu_1(\omega_2) = \inf \left\{ \int_{\omega_2} |A_{22}(X_2) \nabla X_2 u \cdot \nabla X_2 u|^{\frac{2}{p}} : u \in W^{1,p}_0(\omega_2), \int_{\omega_2} |u|^p = 1 \right\}$$

$$= \inf_{u \in W^{1,p}_0(\omega_2) \setminus \{0\}} \frac{\int_{\omega_2} |A_{22}(X_2) \nabla X_2 u \cdot \nabla X_2 u|^{\frac{2}{p}}}{\int_{\omega_2} |u|^p}.$$

Moreover, $\mu_1(\omega_2)$ is simple and the eigenfunction $W$ is differentiable and has constant sign in the domain, that we should fix as the positive sign in the sequel.

**Proof of Theorem 1.1.** By an abuse of notation we still denote with $W$ the extension of $W$ on $\mathbb{R}^m \times \omega_2$ defined by setting $W(X_1, X_2) = W(X_2)$. Then we have

$$- \text{div} \left(|A \nabla W \cdot \nabla W|^{\frac{p-2}{2}} A \nabla W\right) = \mu_1(\omega_2) |W|^{p-2} W \quad \text{in } \Omega_\infty = \mathbb{R}^m \times \omega_2.$$

Let $\phi$ be any function in $C^\infty_c(\Omega_\ell)$. We are now in position to use Picone’s identity 2.2 because $W$ is $C^1$ and $W > 0$, then we get

$$|A \nabla (|\phi|) \cdot \nabla (|\phi|)|^\frac{2}{p} - \nabla \left( \frac{|\phi|^p}{W^{p-1}} \right) |A \nabla W \cdot \nabla W|^{\frac{p-2}{2}} A \nabla W \geq 0.$$

Integrating over $\Omega_\ell$ and using Green’s theorem we deduce

(3.3) \[
\int_{\Omega_\ell} |A \nabla \phi \cdot \nabla \phi|^\frac{2}{p} - \int_{\Omega_\ell} \nabla \left( \frac{|\phi|^p}{W^{p-1}} \right) |A \nabla W \cdot \nabla W|^{\frac{p-2}{2}} A \nabla W \geq 0,
\]

$$\int_{\Omega_\ell} |A \nabla \phi \cdot \nabla \phi|^\frac{2}{p} \geq \int_{\Omega_\ell} -\text{div} \left(|A \nabla W \cdot \nabla W|^{\frac{p-2}{2}} A \nabla W\right) \frac{|\phi|^p}{W^{p-1}}.$$
Then using (3.2) we acquire
\[ \int_{\Omega_\ell} |A \nabla \phi \cdot \nabla \phi|^{\frac{p}{2}} \geq \mu_1(\omega_2) \int_{\Omega_\ell} W^{p-1} \frac{|\phi|^p}{W^{p-1}} = \mu_1(\omega_2) \int_{\Omega_\ell} |\phi|^p. \]
Since the last inequality holds true for any \( \phi \in C^\infty_0(\Omega_\ell) \) we deduce by density in \( W^{1,p}_0(\Omega_\ell) \)
\[ \mu_1(\omega_2) \leq \inf_{\phi \neq 0} \frac{\int_{\Omega_\ell} |A \nabla \phi \cdot \nabla \phi|^{\frac{p}{2}}}{\int_{\Omega_\ell} |\phi|^p} = \lambda_D^1(\Omega_\ell). \]

The estimate from below for the eigenvalue \( \lambda_D^1(\Omega_\ell) \) quoted in Theorem 1.1 is then proved. To prove the estimate from above we use a suitable test function in the Rayleigh quotient characterizing \( \lambda_D^1(\Omega_\ell) \). Let us choose \( v_\ell \) be a smooth function in \( W^{1,p}_0(\ell \omega_1) \) such that

- \( v_\ell = 1 \) in \( \frac{\ell}{2} \omega_1 \);
- \( 0 \leq v_\ell \leq 1, |\nabla v_\ell| \leq \frac{1}{p} \) everywhere.

Let \( W \) be the first eigenfunction of the section problem as above. The function
\[ u_\ell(x) = v_\ell(x_1) W(x_2) \in W^{1,p}_0(\ell \omega_1) \]
is a good test function in (2.3). Thus, using Minkowski inequality and structure condition of the matrix \( A \) we have

\[
\lambda_D^1(\Omega_\ell) \leq \frac{\int_{\Omega_\ell} |A \nabla (v_\ell W) \cdot \nabla (v_\ell W)|^{\frac{p}{2}}}{\int_{\Omega_\ell} |u_\ell|^p} = \frac{\int_{\Omega_\ell} |(A_{11} \nabla X_1 v_\ell \cdot \nabla X_1 v_\ell) W^2 + (2A_{12} \nabla X_2 W \cdot \nabla X_1 v_\ell) (v_\ell W) + (A_{22} \nabla X_2 W \cdot \nabla X_2 W) v_\ell^2|^{\frac{p}{2}}}{\int_{\ell \omega_1} |v_\ell|^p} \leq \frac{\left( \int_{\ell \omega_1} |(A_{11} \nabla X_1 v_\ell \cdot \nabla X_1 v_\ell) W^2|^{\frac{p}{2}} \right)^{2/p} + \left( \int_{\ell \omega_1} |(A_{11} \nabla X_1 v_\ell \cdot \nabla X_1 v_\ell) W^2|^{\frac{p}{2}} \right)^{2/p} + \left( \int_{\ell \omega_1} |(2A_{12} \nabla X_2 W \cdot \nabla X_1 v_\ell) (v_\ell W)|^{\frac{p}{2}} \right)^{2/p}}{\int_{\ell \omega_1} |v_\ell|^p}. \]

Where we also used the fact that \( ||W||_p = 1 \). Recalling that \( W \) is an eigenfunction we deduce

\[
\lambda_D^1(\Omega_\ell) \leq \mu_1(\omega_2)^{2/p} \left( \int_{\ell \omega_1} |v_\ell|^p \right)^{\frac{2}{p}} + ||A_{11}||_{\infty} \left( \int_{\ell \omega_1} |\nabla X_1 v_\ell|^p \int_{\omega_2} |W|^p \right)^{\frac{2}{p}} + 2 ||A_{12}||_{\infty} \left( \int_{\ell \omega_1} |\nabla X_1 v_\ell|^p \right)^{\frac{2}{p}} \left( \int_{\omega_2} |\nabla X_2 W|^p |W|^{\frac{p}{2}} \right)^{\frac{2}{p}} \int_{\ell \omega_1} |v_\ell|^p. \]
In the last estimate we also used the fact that $|v_\ell| \leq 1$. Now we observe that, since $W$ is an eigenfunction and thanks to the ellipticity condition on the matrix $A$, the following implication holds true
\[
\int_{\omega_2} |\nabla X_2 W|^p \leq \frac{\mu_1(\omega_2)}{\lambda} \Rightarrow \int_{\omega_2} |\nabla X_2 W|^\frac{p}{2} \leq C.
\]

Using the elementary inequality $(a + b)^q \leq a^q + q^{q-1}(b^q + a^{q-1}b)$ for $a, b \geq 0$, $q \geq 1$ and denoting with $L_m$ the Lebesgue measure in $\mathbb{R}^m$ we get
\[
\lambda_j^j(\Omega_\ell) \leq \left\{ \mu_1(\omega_2)^{2/p} + \frac{\|A_{11}\|_\infty (\int_{\omega_1} |\nabla X_1 v_\ell|^p)^{\frac{2}{p}} + 2C_2^2 \|A_{12}\|_\infty (\int_{\omega_1} |\nabla X_1 v_\ell|^2)^{\frac{2}{p}}}{\|v_\ell\|_{p,\ell_\omega_1}^2} \right\}^{\frac{p}{2}}
\leq \left\{ \mu_1(\omega_2)^{2/p} + \frac{\|A_{11}\|_\infty (L_m(\ell_\omega_2)^{p/2})^{2/p} + 2C_2^2 \|A_{12}\|_\infty (L_m(\ell_\omega_2)^{p/2})^{2/p}}{(\ell_\omega_2)^{2/p}} \right\}^{\frac{p}{2}}
= \left\{ \mu_1(\omega_2)^{2/p} + \frac{2m\|A_{11}\|_\infty^{2/p}}{\ell} + \frac{2m+1 C_2^2 \|A_{12}\|_\infty^{2/p}}{\ell} \right\}^{\frac{p}{2}}
\leq \left( \mu_1(\omega_2)^{2/p} + \frac{C_1}{\ell} \right)^{\frac{p}{2}} \leq \mu_1(\omega_2) + p2^{\frac{p-4}{p}} \left( \frac{C_1^2}{\ell} + \frac{C_1(\mu_1(\omega_2))^{p-2}}{\ell} \right).
\]

Hence the estimate from above is then proved.

Clearly, for any $\ell > 0$, we have $\Lambda_\infty \leq \lambda_j^j(\Omega_\ell)$ and for the lower bound of $\Lambda_\infty$ we proceed exactly in the same way as it is done in (3.3) where $\Omega_\ell$ is replaced by $\Omega_\infty$. Then letting $\ell \to \infty$ we conclude that $\Lambda_\infty = \mu_1(\omega_2$. \hfill $\Box$

4. The Gap Phenomenon for Mixed Boundary Conditions

In this section we are concerned about the mixed boundary eigenvalue problem. Let us discuss some results that would be required to the main proof of the Theorem 1.2. As we mentioned in the introduction, first we study the asymptotic behavior of $\lambda_j^j(\Omega_\ell)$ as $\ell \to 0$, which is a key ingredient to proof of the Theorem 1.2.

An appropriate space for mixed boundary eigenvalue problem is
\[
V(\Omega_\ell) = \{ v \in W^{1,\beta}(\Omega_\ell) : v = 0 \text{ on } \gamma_\ell \},
\]
where $\gamma_\ell = (-\ell, \ell) \times \partial \omega_2$ and the boundary value is defined in the sense of trace. Thanks to the classical Poincaré inequality, the space $V(\Omega_\ell)$ becomes a Banach space with respect to the norm (2.1).

The weak formulation of the eigenvalue problem (1.3) is given by
\[
\begin{cases}
u \in V(\Omega_\ell), \\
\int_{\Omega_\ell} |A\nabla u \cdot \nabla v|^{\frac{p}{2}} A\nabla u \cdot \nabla v = \lambda_M(\Omega_\ell) \int_{\Omega_\ell} |u|^{p-2} uv \text{ for any } v \in V(\Omega_\ell).
\end{cases}
\]
The first eigenvalue $\lambda_1^M(\Omega_\ell)$ for (4.1) is associated with a variational characterization

$$\lambda_1^M(\Omega_\ell) = \inf \left\{ \int_{\Omega_\ell} |A(X_\ell) \nabla u \cdot \nabla u|^\frac{p}{2} : u \in V(\Omega_\ell), \int_{\Omega_\ell} |u|^p = 1 \right\}$$

(4.2)

Moreover, the first eigenvalue is simple and the corresponding eigenfunction has constant sign in the domain.

**Theorem 4.1 (Dimension Reduction).** For $p \geq 2$, we have $\lim_{\ell \to 0} \lambda_1^M(\Omega_\ell) = \Lambda$ where

$$\Lambda = \inf \left\{ \int_{\omega_2} \left( A_{22}(X_2) \nabla u \cdot \nabla u - \frac{|A_{12}(X_2) \cdot \nabla u|^2}{a_{11}(X_2)} \right)^\frac{p}{2} : u \in W_0^{1,p}(\omega_2), \int_{\omega_2} |u|^p = 1 \right\}.$$

**Proof.** The reason why we find $\Lambda$ as the limiting value will be clarified by the following observation. Let

$$B = \begin{pmatrix} b_{11} & b_{12} \\ B_{12}^t & B_{22} \end{pmatrix}$$

be a positive definite $n \times n$ matrix in $\mathbb{R}^n$ and we write any vector $z = (z_1, Z_2) \in \mathbb{R}^n$ with $Z_2 \in \mathbb{R}^{n-1}$. Then it is easy to see by using elementary calculus that for any fixed $Z_2$ we have

$$\min_{z_1 \in \mathbb{R}} (Bz \cdot z)^\frac{p}{2} = \left( B_{22}Z_2 \cdot Z_2 - \frac{|B_{12}Z_2|^2}{b_{11}} \right)^\frac{p}{2}$$

(4.3)

and the minimum in (4.3) is attained for $z_1 = -\frac{B_{12}Z_2}{b_{11}}$. Applying (4.3) with $B = A(X_2)$ we obtain, for any $\ell > 0$,

$$\int_{\Omega_\ell} |A(X_2) \nabla u_\ell \cdot \nabla u_\ell|^\frac{p}{2} \geq \int_{\Omega_\ell} \left( A_{22}(X_2) \nabla X_2 u_\ell \cdot \nabla X_2 u_\ell - \frac{|A_{12}(X_2) \cdot \nabla X_2 u_\ell|^2}{a_{11}(X_2)} \right)^\frac{p}{2}$$

(4.4)

$$\geq \Lambda \int_{\Omega_\ell} |u_\ell|^p.$$

It is clear by (4.4) the lower bound

(4.5) $\Lambda \leq \lim_{\ell \to 0} \lambda_1^M(\Omega_\ell)$.

Let $T_\ell = \{ x \in \omega_2 : \text{dist}(x, \partial \omega_2) \leq \ell \}$ be a neighbourhood of $\partial \omega_2$ for $\ell > 0$. Fix for any $\beta \in (0, 1)$ and let $\rho_\ell$ be an approximation of the characteristic function of $\omega_2$, as $\ell \to 0$:

$$\rho_\ell \in C_0^\infty(\omega_2), \quad 0 \leq \rho_\ell \leq 1, \quad \rho_\ell = 1 \text{ in } \omega_2 \setminus T_\ell, \quad |\nabla \rho_\ell| \leq \frac{1}{\ell^\beta} \text{ in } T_\ell.$$

(4.6)

Hence for $\ell \to 0$ one has $\rho_\ell \to 1$ pointwise. Let us define a function $u_\ell$ on $\Omega_\ell$ by

$$u_\ell(x) = W(X_2) - \frac{x_1 \rho_\ell(X_2) A_{12}(X_2) \cdot \nabla W}{a_{11}(X_2)}.$$
Then by properties (4.6), and using the fact that (4.7), we emphasize that the function \( u_\ell \) defined above does not necessarily belong to the space \( W^{1,p} \), since the first eigenfunction \( W \) is almost in \( C^{1,\gamma} \). To resolve this difficulty, we provide a smooth approximation argument, motivated by [14, Ch.14]. Now define a family of functions \( \{ F_\epsilon \}_{\epsilon > 0} \) in \( C^\infty \) by using standard mollification which satisfies the following

\[
\lim_{\epsilon \to 0} F_\epsilon(X_2) = \frac{A_{12}(X_2) \cdot \nabla W}{a_{11}(X_2)} \text{ in } L^p(\omega_2).
\]

Then we define

\[
(4.7) \quad u_\epsilon^\ell(x) = W(X_2) - x_1 \rho_\ell(X_2) F_\epsilon(X_2).
\]

By using simple elementary inequality for the vectors \( a, b \) and \( q \geq 1 \)

\[
|b|^q \geq |a|^q + q \langle |a|^{q-2} a, b - a \rangle,
\]

and using the fact that \( x_1 \) is an odd function in \( (-\ell, \ell) \) we infer that

\[
(4.8) \quad \int_{\Omega_\ell} |A\nabla u_\epsilon^\ell \cdot \nabla u_\epsilon^\ell|^{\frac{q}{2}} = \int_{\Omega_\ell} |W(X_2) - x_1 \rho_\ell(X_2) F_\epsilon(X_2)|^{p} \geq \int_{\Omega_\ell} |W(X_2)|^{p} = 2\ell \int_{\omega_2} |W|^p.
\]

Now

\[
\int_{\Omega_\ell} |A\nabla u_\epsilon^\ell \cdot \nabla u_\epsilon^\ell|^{\frac{q}{2}}
= \int_{\Omega_\ell} |a_{11}(\partial_{x_1} u_\epsilon^\ell)^2 + 2(A_{12} \cdot \nabla X_2 u_\epsilon^\ell)\partial_{x_1} u_\epsilon^\ell + A_{22} \nabla X_2 u_\epsilon^\ell \cdot \nabla X_2 u_\epsilon^\ell|^{\frac{q}{2}}
= \int_{\Omega_\ell} |a_{11}\rho_\ell^2 F_\epsilon^2 - 2\rho_\ell F_\epsilon(A_{12} \cdot \nabla W) - 2x_1 \rho_\ell F_\epsilon(F_\epsilon A_{12} \cdot \nabla \rho_\ell + \rho_\ell A_{12} \cdot \nabla F_\epsilon)
+ (A_{22} \nabla W - x_1(F_\epsilon A_{22} \cdot \nabla \rho_\ell + \rho_\ell A_{22} \cdot \nabla F_\epsilon)) \cdot (\nabla W - x_1(F_\epsilon \nabla \rho_\ell + \rho_\ell \nabla F_\epsilon))|^{\frac{p}{2}}.
\]

Hence by using Minkowski inequality we have

\[
(4.9) \quad \int_{\Omega_\ell} |A\nabla u_\epsilon^\ell \cdot \nabla u_\epsilon^\ell|^{\frac{q}{2}} \leq (I_1 + I_2)^{\frac{q}{2}},
\]

where

\[
I_1 := \left( \int_{\Omega_\ell} |a_{11} F_\epsilon^2 - 2(A_{12} \cdot \nabla W) F_\epsilon + A_{22} \nabla W \cdot \nabla W|^{\frac{p}{2}} \right)^{\frac{2}{p}}
\]

and

\[
I_2 := \left( \int_{\Omega_\ell} |a_{11} F_\epsilon^2 (\rho_\ell^2 - 1) + 2(1 - \rho_\ell)(A_{12} \cdot \nabla W) F_\epsilon - 2x_1 H_\epsilon^\ell(X_2) + x_1^2 G_\epsilon^\ell(X_2)|^{\frac{p}{2}} \right)^{\frac{2}{p}},
\]

where

\[
H_\epsilon^\ell(X_2) := \rho_\ell F_\epsilon^2 (A_{12} \cdot \nabla \rho_\ell) + \rho_\ell F_\epsilon(A_{12} \cdot \nabla F_\epsilon) + F_\epsilon(A_{22} \nabla W \cdot \nabla \rho_\ell) + \rho_\ell(A_{22} \nabla W \cdot \nabla F_\epsilon),
\]

\[
G_\epsilon^\ell(X_2) := (F_\epsilon A_{22} \cdot \nabla \rho_\ell + \rho_\ell A_{22} \cdot \nabla F_\epsilon) \cdot (F_\epsilon \nabla \rho_\ell + \rho_\ell \nabla F_\epsilon).
\]

Then by properties (4.6) of the function \( \rho_\ell \), and for fixed \( \epsilon > 0 \) we have

\[
(4.10) \quad |H_\epsilon^\ell(X_2)| \leq C_1 + \frac{C_2}{\ell^2} \quad \text{and} \quad |G_\epsilon^\ell(X_2)| \leq C_3 + \frac{C_4}{\ell^{2\gamma}}.
\]
where $C_i$ ($i = 1, 2, 3, 4$) are positive constants independent of $\ell$, and we define

$$K_\ell^r = \int_{\Omega_2} |a_{11} F_\ell^2 (\rho_\ell^2 - 1) + 2(1 - \rho_\ell)(A_{12} \cdot \nabla W) F_\ell|^\frac{p}{2}.$$

Since $\rho_\ell \to 1$ pointwise as $\ell \to 0$ and then by dominated convergence theorem we conclude that $K_\ell^r \to 0$ as $\ell \to 0$. Now we estimate the above integrals $I_1, I_2$ in the following:

**Estimate for $I_1$**:

$$I_1 \leq (2\ell)^\frac{2}{p} \left( \int_{\Omega_2} |a_{11} F_\ell^2 - 2(A_{12} \cdot \nabla W) F_\ell + A_{22} \nabla W \cdot \nabla W|^\frac{p}{2} \right)^\frac{2}{p}.$$

**Estimate for $I_2$**: Again applying Minkowski inequality and by (4.10) we obtain

$$I_2 \leq \left( \int_{\Omega_2} |a_{11} F_\ell^2 (\rho_\ell^2 - 1) + 2(1 - \rho_\ell)(A_{12} \cdot \nabla W) F_\ell|^\frac{p}{2} \right) + \left( \int_{\Omega_2} |x^2 G_\ell^2(X_2) - 2x_1 H_\ell^2(X_2)|^\frac{p}{2} \right)^\frac{2}{p}.$$

$$\leq (2\ell)^\frac{2}{p} (K_\ell^r)^\frac{2}{p} + \left( \int_{\Omega_2} |x^2 G_\ell^2(X_2)|^\frac{p}{2} \right)^\frac{2}{p} + \left( \int_{\Omega_2} |2x_1 H_\ell^2(X_2)|^\frac{p}{2} \right)^\frac{2}{p}.$$

$$\leq (2\ell)^\frac{2}{p} (K_\ell^r)^\frac{2}{p} + \ell^2 \left( \int_{\Omega_2} |G_\ell^2(X_2)|^\frac{p}{2} \right)^\frac{2}{p} + 2\ell \left( \int_{\Omega_2} |H_\ell^2(X_2)|^\frac{p}{2} \right)^\frac{2}{p}.$$

$$= (2\ell)^\frac{2}{p} \left( (K_\ell^r)^\frac{2}{p} + C_4^2 + C_2 \ell^2 + 2C_3 \ell + 2C_4 \ell^{1-\beta} \right) = (2\ell)^\frac{2}{p} \left( (K_\ell^r)^\frac{2}{p} + C(\ell) \right).$$

Now plugging the estimates (4.11), (4.12) into (4.9) we obtain

$$\int_{\Omega_2} |A \nabla u_\ell^r \cdot \nabla u_\ell^r|^\frac{p}{2} \leq 2\ell \left[ \left( \int_{\Omega_2} |a_{11} F_\ell^2 - 2(A_{12} \cdot \nabla W) F_\ell + A_{22} \nabla W \cdot \nabla W|^\frac{p}{2} \right)^\frac{2}{p} + (K_\ell^r)^\frac{2}{p} + C(\ell) \right]^\frac{p}{2}.$$

Therefore combining (4.8) and (4.13) we have

$$\lim_{\ell \to 0} \lambda_1^r(\Omega_\ell)$$

$$\leq \lim_{\ell \to 0} \frac{\int_{\Omega_2} |A \nabla u_\ell^r \cdot \nabla u_\ell^r|^\frac{p}{2}}{\int_{\Omega_2} |u_\ell^r|^p}$$

$$\leq \lim_{\ell \to 0} \left[ \left( \int_{\Omega_2} |a_{11} F_\ell^2 - 2(A_{12} \cdot \nabla W) F_\ell + A_{22} \nabla W \cdot \nabla W|^\frac{p}{2} \right)^\frac{2}{p} + (K_\ell^r)^\frac{2}{p} + C(\ell) \right]^\frac{p}{2}$$

$$= \int_{\Omega_2} |a_{11} F_\ell^2 - 2(A_{12} \cdot \nabla W) F_\ell + A_{22} \nabla W \cdot \nabla W|^\frac{p}{2}.$$
Letting $\epsilon \to 0$ and using the fact $F_\epsilon^2 \to \frac{A_{12}(X_2) \cdot \nabla W}{a_{11}(X_2)^{2}}$ in $L^\frac{p}{2}(\omega_2)$, we infer that
\[
\limsup_{\ell \to 0} \lambda_{1}^M(\Omega_{\ell}) \leq \int_{\omega_2} A_{22}(X_2) \nabla W \cdot \nabla W - \frac{|A_{12}(X_2) \cdot \nabla W|^2}{a_{11}(X_2)^{2}} \frac{n}{2} = \Lambda,
\]
which together with (4.5) gives the desired result.

\textbf{Proof of Theorem 1.2:} Case 1: Suppose the condition holds first i.e. $A_{12} \cdot \nabla W \neq 0$ a.e. on $\omega_2$. Then we obtain
\[
\Lambda \leq \int_{\omega_2} |A_{22}(X_2) \nabla W \cdot \nabla W - \frac{|A_{12}(X_2) \cdot \nabla W|^2}{a_{11}(X_2)^{2}}|^{\frac{p}{2}} < \int_{\omega_2} |A_{22}(X_2) \nabla W \cdot \nabla W|^\frac{p}{2} = \mu_1(\omega_2).
\]
By the proof of the above theorem there exists $\ell_0 > 0$ and $\epsilon_0 > 0$ such that the function $u_{\ell_0}^{\epsilon_0}$ defined by (4.7) satisfies
\[
(4.14) \quad \int_{\Omega_{\ell_0}} |A \nabla u_{\ell_0}^{\epsilon_0} \cdot \nabla u_{\ell_0}^{\epsilon_0}|^{\frac{p}{2}} < \mu_1(\omega_2) \int_{\Omega_{\ell_0}} |u_{\ell_0}^{\epsilon_0}|^p.
\]
Let $\alpha > 1$ be a constant whose value will be choose later. For $\ell > \ell_0 + \alpha$ we define a function $\phi_\ell$ as follows,
\[
\phi_\ell(x_1, X_2) = \begin{cases} 
{u_{\ell_0}^{\epsilon_0}(x_1 - \ell + \ell_0, X_2)} & \text{in } (\ell - \ell_0, \ell) \times \omega_2, \\
\frac{\xi(x_1)}{\alpha} W(X_2) & \text{in } (\ell - \ell_0 - \alpha, \ell - \ell_0) \times \omega_2, \\
0 & \text{in } \Omega_{\ell_0 - \alpha}, \\
\frac{\xi(x_1)}{\alpha} W(X_2) & \text{in } (\ell_0 - \ell_0 - (\ell - \ell_0 - \alpha)) \times \omega_2, \\
u_{\ell_0}^{\epsilon_0}(x_1 + \ell - \ell_0, X_2) & \text{in } (-\ell, \ell_0 - \ell) \times \omega_2,
\end{cases}
\]
where
\[
\xi(x_1) = \begin{cases} 
x_1 - \ell + \ell_0 + \alpha & \text{if } x_1 \in (\ell - \ell_0 - \alpha, \ell - \ell_0), \\
-x_1 - \ell + \ell_0 + \alpha & \text{if } x_1 \in (\ell - \ell_0, -\ell + \ell_0 + \alpha).
\end{cases}
\]
By simple change of variable we get
\[
\int_{\Omega_{\ell} \setminus \Omega_{\ell_0}} |\phi_\ell|^p = \int_{\Omega_{\ell_0}} |u_{\ell_0}^{\epsilon_0}|^p
\]
and thus we have
\[
(4.15) \quad \int_{\Omega_{\ell}} |\phi_\ell|^p = \int_{\Omega_{\ell_0}} |u_{\ell_0}^{\epsilon_0}|^p + \frac{2\alpha}{p + 1}.
\]
Similarly we have
\[
(4.16) \quad \int_{\Omega_{\ell}} |A \nabla \phi_\ell \cdot \nabla \phi_\ell|^\frac{p}{2} = \int_{\Omega_{\ell_0}} |A \nabla u_{\ell_0}^{\epsilon_0} \cdot \nabla u_{\ell_0}^{\epsilon_0}|^\frac{p}{2} + \int_{\Omega_{\ell_0} \setminus \Omega_{\ell_0}} |A \nabla \phi_\ell \cdot \nabla \phi_\ell|^\frac{p}{2}.
\]
Let $S = \Omega_{\ell-\ell_0} \setminus \Omega_{\ell-\ell_0-\alpha}$ and $S^+ = (\ell - \ell_0 - \alpha, \ell - \ell_0) \times \omega_2$. Using Minkowski inequality and the fact that $\phi_\ell$ is an even function of $x_1$ on $S$. We estimate the above last integral as follows

$$
\int_{\Omega_{\ell-\ell_0}} |A \nabla \phi_\ell \cdot \nabla \phi_\ell|^{\frac{p}{2}}
= \int_S |a_{11}(X_2) (\partial_{x_1} \phi_\ell)^2 + A_{22} \nabla X_2 \phi_\ell \cdot \nabla X_2 \phi_\ell + 2(A_{12} \cdot \nabla X_2 \phi_\ell) (\partial_{x_1} \phi_\ell)|^{\frac{p}{2}}
= \frac{1}{\alpha^p} \int_S |(A_{22} \nabla W \cdot \nabla W) \xi^2 + a_{11}(X_2) W^2 + 2(A_{12} \cdot \nabla W)\xi \xi'|^{\frac{p}{2}}
\leq \frac{1}{\alpha^p} \left\{ \left( \int_S |A_{22} \nabla W \cdot \nabla W|^{\frac{p}{2}} |\xi|^{p} \right)^{\frac{2}{p}} + \left( \int_S |a_{11}(X_2) W^2 + 2(A_{12} \cdot \nabla W)\xi \xi'|^{\frac{p}{2}} \right)^{\frac{2}{p}} \right\}^{\frac{p}{2}}
(4.17)
= \frac{1}{\alpha^p} (I_1 + I_2)^{\frac{p}{2}},
$$

where

$$
I_1 = \left( \int_S |A_{22} \nabla W \cdot \nabla W|^{\frac{p}{2}} |\xi|^{p} \right)^{\frac{2}{p}} = \left( 2 \int_{S^+} |A_{22} \nabla W \cdot \nabla W|^{\frac{p}{2}} \xi^{p} \right)^{\frac{2}{p}} = \left( 2\mu_1(\omega_2) \int_{\ell-\ell_0-\alpha}^{\ell-\ell_0} (x_1 - \ell + \ell_0 + \alpha)^{p} \, dx_1 \right)^{\frac{2}{p}} = \left( \frac{2\alpha^{p+1} \mu_1(\omega_2)}{p+1} \right)^{\frac{2}{p}},
$$

and

$$
I_2 = \left( \int_S |a_{11}(X_2) W^2 + 2(A_{12} \cdot \nabla W)\xi \xi'|^{\frac{p}{2}} \right)^{\frac{2}{p}}, \text{ since } ||a_{11}(X_2) W^2 + 2(A_{12} \cdot \nabla W)\xi \xi'||_{\infty} \leq M \text{ for some } M > 0, \text{ then we have } I_2 \leq C \frac{\alpha^2}{\alpha^p}.
$$

Now plugging the above estimates into (4.17) and then using the elementary inequality which stated in section 3 we obtain

$$
(4.18) \int_{\Omega_{\ell-\ell_0}} |A \nabla \phi_\ell \cdot \nabla \phi_\ell|^{\frac{p}{2}} \leq \frac{1}{\alpha^p} \left( \left( \frac{2\alpha^{p+1} \mu_1(\omega_2)}{p+1} \right)^{\frac{2}{p}} + C \alpha^{\frac{2}{p}} \right) \leq \frac{2\alpha \mu_1(\omega_2)}{p+1} + \frac{C_1}{\alpha} + \frac{C_2}{\alpha^{p-1}}.
$$

Combining (4.15), (4.16) and (4.18) we get

$$
\lambda^1_M(\Omega_\ell) \leq \frac{\int_{\Omega_\ell} |A \nabla \phi_\ell \cdot \nabla \phi_\ell|^{\frac{p}{2}}}{\int_{\Omega_\ell} |\phi_\ell|^{p}} \leq \frac{\int_{\Omega_\ell} |A \nabla u_{\ell_0}^{\alpha_0} \cdot \nabla u_{\ell_0}^{\alpha_0}|^{\frac{p}{2}} + \mu_1(\omega_2) \int_{\Omega_\ell} |u_{\ell_0}^{\alpha_0}|^{p} + C \frac{\alpha}{p+1}}{\int_{\Omega_\ell} |u_{\ell_0}^{\alpha_0}|^{p} + \frac{2\alpha}{p+1}}.
$$

Therefore,

$$
\lambda^1_M(\Omega_\ell) - \mu_1(\omega_2) \leq \frac{\int_{\Omega_\ell} |A \nabla u_{\ell_0}^{\alpha_0} \cdot \nabla u_{\ell_0}^{\alpha_0}|^{\frac{p}{2}} + \mu_1(\omega_2) \int_{\Omega_\ell} |u_{\ell_0}^{\alpha_0}|^{p} + C \frac{\alpha}{p+1}}{\int_{\Omega_\ell} |u_{\ell_0}^{\alpha_0}|^{p} + \frac{2\alpha}{p+1}}.
$$

By (4.14) it is clear that for a fixed large enough $\alpha$ such that the RHS of the above is negative and get the desired result.
Case 2: Suppose the condition of the Theorem doesn’t hold i.e. $A_{12} \cdot \nabla W = 0$ in $\omega_2$. Then $\Lambda$ becomes $\mu_1(\omega_2)$ and by (4.4) we conclude that $\mu_1(\omega_2) \leq \lambda^1_M(\Omega_\ell) \forall \ell > 0$. Now by choosing $u(x) = W(X_2)$ as a test function in (4.2) then we get $\lambda^1_M(\Omega_\ell) \leq \mu_1(\omega_2)$. This completes the proof of the theorem. □

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