ON THE NON-HOMOGENEOUS NAVIER-STOKES SYSTEM WITH NAVIER FRICTION BOUNDARY CONDITIONS

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Abstract. We address the issue of existence of weak solutions for the non-homogeneous Navier-Stokes system with Navier friction boundary conditions allowing the presence of vacuum zones and assuming rough conditions on the data. We also study the convergence, as the viscosity goes to zero, of weak solutions for the non-homogeneous Navier-Stokes system with Navier friction boundary conditions to the strong solution of the Euler equations with variable density, provided that the initial data converge in $L^2$ to a smooth enough limit.

1. Introduction

We are concerned with the incompressible Navier-Stokes model with variable density in a bounded domain. The governing equations are given by the following system

$$
\begin{aligned}
\partial_t (\rho u) + \text{div} (\rho uu) - \nu \Delta u + \nabla \pi &= \rho f \text{ in } Q, \\
\text{div } u &= 0 \text{ in } Q, \\
\partial_t \rho + \text{div} (\rho u) &= 0 \text{ in } Q.
\end{aligned}
$$

(1.1)

Here, $Q \equiv \Omega \times (0,T)$, where $\Omega$ is a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial \Omega$, and $T > 0$. The unknowns are the velocity field $u$, the density $\rho$, and the pressure $\pi$ of the fluid. The parameter $\nu > 0$ is the viscosity coefficient of the fluid and $f$ is a given vector field driving the motion.

We supplement the system (1.1) with initial and Navier friction boundary conditions

$$
\begin{aligned}
u \cdot n &= 0 \text{ on } \Sigma, \\
[D(u)n + \alpha u]_{\text{tan}} &= 0 \text{ on } \Sigma, \\
\rho(0) &= \rho_0 \text{ in } \Omega, \\
(\rho u)(0) &= v_0 \text{ in } \Omega,
\end{aligned}
$$

(1.2)

where $\Sigma \equiv \partial \Omega \times (0,T)$, $n$ is the exterior normal vector to $\partial \Omega$, $\rho_0 \geq 0$ denotes the initial density and $v_0$ has to be at least such that $v_0(x) = 0$ whenever $\rho_0(x) = 0$. Moreover,
\( D(u) = \frac{1}{2} (\partial_i u_j + \partial_j u_i)_{1 \leq i, j \leq n} \) denotes the deformation tensor, \([\cdot]_{\tan}\) is the tangential component of a vector on \(\partial \Omega\), and \(\rho uu = \rho (u \otimes u)\). The constant \(\alpha \geq 0\) stands for the friction coefficient which measures the tendency of the fluid to slip on the boundary.

The goal of this paper is to study the convergence of solutions for (1.1)-(1.2), as the viscosity goes to zero, toward the solution of the Euler equations with variable density. Formally, when we drop the viscous term (i.e., taking \(\nu = 0\)) system (1.1)-(1.2) degenerates into the non-homogeneous Euler equations

\[
\begin{aligned}
\partial_t (\rho u) + \text{div} (\rho uu) + \nabla \pi &= \rho f \quad \text{in } Q, \\
\text{div } u &= 0 \quad \text{in } Q, \\
\partial_t \rho + \text{div} (\rho u) &= 0 \quad \text{in } Q, \\
u \cdot n &= 0 \quad \text{on } \Sigma, \\
\rho(0) &= \rho_0 \quad \text{in } \Omega, \\
(\rho u)(0) &= v_0 \quad \text{in } \Omega.
\end{aligned}
\]  

We aim at giving a justification of this formal procedure.

The issue of the vanishing viscosity limit or inviscid limit for the incompressible homogeneous Navier-Stokes equations is a classical problem in fluid mechanics. In the whole space and periodic cases, the inviscid limit was performed by several authors, see e.g. [4, 5, 18, 35]. In the case where there exist physical boundaries, the problem of convergence leads to the formation of a boundary layer if one supplements the Navier-Stokes equations with no-slip boundary conditions (which are the most often prescribed ones). This happens because there is a discrepancy between the no-slip boundary conditions for the Navier-Stokes equations and the tangential boundary conditions for the Euler equations.

There is no consensus on the boundary conditions to be prescribed for the Navier-Stokes equations, except for impermeable boundary which corresponds to the condition (1.2)1. Navier [28] claimed that the tangential component of the viscous stress at the boundary should be proportional to the tangential velocity, leading to the boundary condition (1.2)2. Conditions (1.2)1-(1.2)2 are called Navier friction boundary conditions, or simply Navier boundary conditions (another names have been used as well). These conditions were also derived by Maxwell [26] from the kinetic theory of gases and rigorously justified as a homogenization of the no-slip condition on a rough boundary (see [17]).

Recently, the inviscid limit for the Navier-Stokes equations with Navier boundary conditions was established, for which the reader is referred to [6, 13, 23, 25]. The situation in this case is thus very different from the case of no-slip boundary conditions and requires distinct involved arguments.

Concerning the non-homogeneous incompressible Navier-Stokes equations, it is worthwhile to remark that there exists a considerable number of papers devoted to their mathematical analysis, especially in the case where the equations are complemented with Dirichlet boundary conditions. Those results can be classified in two classes: on the one hand, there are existence results when the initial density is assumed to be positive and so there is no vacuum initially; and on the other hand, the case where the initial-vacuum is allowed. The first case has been addressed by several authors, see e.g. [1, 7, 19, 21, 29, 31, 3, 15], and references therein. In order to avoid vacuum, the basic assumption in the above-quoted
works is
\[ 0 < c_0 \leq \inf_{x \in \Omega} \rho_0(x) \leq \rho_0(x) \leq \sup_{x \in \Omega} \rho_0(x), \]
and so, in particular \( \rho_0, \rho \) have a positive lower bound. In the second one, when the initial-vacuum is allowed (\( \rho_0 \geq 0 \)), the problem (1.1)-(1.2) is more difficult to handle. Indeed, comparing with the first case, fewer results are available in the literature related to the existence of weak solutions (see \([32, 33, 20, 22]\)). In particular, to the best of our knowledge, existence of weak solutions has not been still treated for the non-homogeneous incompressible Navier-Stokes system with Navier boundary conditions. Let us mention the work \([16]\) where strong solutions to the system with slip boundary conditions are considered, however vacuum zones are not admitted. Thus, our first goal will be show the existence of weak solutions for (1.1)-(1.2), allowing vacuum and assuming rough conditions on the field \( f \) and initial data \( \rho_0 \) and \( v_0 \).

The vanishing viscosity limit for the non-homogeneous incompressible Navier-Stokes equations in the whole space, or with periodic boundary conditions, was addressed in \([14, 15, 8]\), as long as no vacuum states occur. They proved the convergence of local strong solutions in Hilbert and Sobolev spaces. One of the difficulties in this case is to show that the time existence is independent of the viscosity. As well as in the homogeneous case, it is expected that the issue of the inviscid limit in bounded domains presents boundary-layer phenomenon when one considers no-slip boundary condition.

Our second goal in this paper is to show that a weak solution of the non-homogeneous incompressible Navier-Stokes equations (1.1) with Navier boundary conditions (1.2) converges in the energy space toward the strong solution of the non-homogeneous incompressible Euler equations (1.3) in the inviscid limit. This extends some earlier results obtained by \([6, 23, 13, 30]\) in the homogeneous case. The strategy is to compare the smooth solution of the Euler equations and a weak solution of the Navier-Stokes equations, which is the leading idea of the proof of weak-strong uniqueness for the homogeneous incompressible Navier-Stokes equations. In our case, this approach arises suitably in view of the fact that only strong solutions are known to exist for the non-homogeneous Euler equations (1.3) (see e.g. \([34]\)). In fact, to best of our knowledge, there is no theory of weak solutions for (1.3).

This paper is structured as follows. In the next section, we establish a result of existence of weak solutions with finite energy for the problem (1.1)-(1.2) (see Theorem 2.3). The precise definition of weak solution with finite energy is given in Definition 2.1 below. In Section 3, we address the vanishing viscosity limit for the non-homogeneous Navier-Stokes equations with Navier friction boundary conditions (see Theorem 3.2). To this end, we first recall a result of local strong solution for the Euler equations with variable density (see Theorem 3.1). Finally, we proceed with the proof of the inviscid limit.

We finish this section by establishing some notations used throughout this manuscript. We denote by \( D(\Omega) \) and \( D'(\Omega) \) the space of functions of class \( C^\infty(\Omega) \) with compact support, and the space of distributions on \( \Omega \), respectively. We use standard notations for Lebesgue and Sobolev spaces. We denote by \( \| \cdot \|_p \) the norm in \( L^p(\Omega) \). Otherwise, the norm will be specified. For a Banach space \( X \), we indicate by \( \langle \cdot, \cdot \rangle_{X',X} \) the duality product between \( X' \) (the dual space of \( X \)) and \( X \). As usual, we will use the same notation for vector valued
and scalar valued spaces. There will be no danger of confusion since the difference will be clear in the context. Also, we denote by $H^1_\sigma(\Omega)$ the subspace of $H^1(\Omega)$ of divergence free vector fields tangent to the boundary. Finally, $C_w([0,T];X)$ represents the space of functions $u : [0,T] \to X$ which are continuous with respect to the weak topology.

2. Weak solutions for the non-homogeneous Navier-Stokes system

In this section we study the existence of global weak solutions for the non-homogeneous incompressible Navier-Stokes with Navier boundary conditions. We assume that the initial density $\rho_0$ belongs to $L^p(\Omega)$, $6 \leq p \leq \infty$, allowing to vanish, $v_0 \in L^{\frac{2p}{p+1}}(\Omega)$ with $\frac{|u_0|^2}{\rho_0} \in L^1(\Omega)$ and the external force $f \in L^1(0,T;L^{\frac{2p}{p+1}}(\Omega))$. The data $v_0$ and $\frac{|u_0|^2}{\rho_0}$ correspond formally to initial value for $\rho u$ and $\rho |u|^2$, respectively. In the case $p = \infty$, $L^r$-exponents depending on $p$ should be understood in the natural way, that is, as the limit when $p \to \infty$. For instance, $L^{\frac{2p}{p+1}}$ and $L^{\frac{2p}{p+1}}$ become $L^2$ in that case.

Now we introduce the definition of weak solution with finite energy for the system (1.1)-(1.2).

**Definition 2.1.** A weak solution for (1.1)-(1.2) is a pair of functions $(u, \rho)$ verifying the following items:

i) $u \in L^2(0,T;H^1_\sigma(\Omega))$, $\rho \in C([0,T];W^{-1, p}(\Omega)) \cap L^\infty(0,T;L^p(\Omega))$, $\rho \geq 0$ a.e. in $Q$, $\rho u \in L^\infty(0,T;L^{\frac{2p}{p+1}}(\Omega))$, $\sqrt{\rho u} \in L^\infty(0,T;L^2(\Omega))$, $\rho uu \in L^1(0,T;L^2(\Omega))$, such that equation $\partial_t \rho + \text{div} (\rho u) = 0$ is satisfied in $\mathcal{D}'(Q)$ and the momentum equation (1.1) is verified in the following sense:

$$\int_0^T \int_\Omega \rho u \cdot \partial_t \varphi + 2\alpha \int_0^T \int_\Omega u \cdot \varphi + 2\nu \int_0^T \int_\Omega D(u) : D(\varphi) - \int_0^T \int_\Omega \rho uu \cdot \nabla \varphi,$$

$$= \int_0^T \int_\Omega \rho f \cdot \varphi + \int_\Omega v_0 \varphi(0),$$

for all $\varphi \in C^1([0,T];H^1(\Omega))$, $\varphi(T,x) = 0$ a.e. in $\Omega$.

ii) The initial data (1.2) is verified in the following sense:

$$(\rho(0), \psi)_{W^{-1, p}(\Omega), W^{1, p'}(\Omega)} = \int_\Omega \rho_0 \psi dx, \ \forall \psi \in W^{1, p'}(\Omega).$$

iii) The following energy inequality

$$\frac{1}{2} \| \sqrt{\rho_0(t)}u(t) \|_2^2 + 2\alpha \int_0^t \int_\Omega |u|^2 + 2\nu \int_0^t \| Du \|_2^2 \leq \frac{1}{2} \| \sqrt{\rho_0} \|_2^2 + \int_0^t \int_\Omega \rho f \cdot u$$

holds for a.e. $t \in (0,T)$.

Let us make some commentaries about the previous definition. We first notice that the divergence-free boundary condition of the velocity field and the boundary condition (1.2) are given by the choice of the space $H^1_\sigma(\Omega)$. The weak formulation (2.1) also contains the boundary condition (1.2) in the sense that if $u$ is more regular, say $H^1_\sigma(\Omega) \cap H^2(\Omega)$,
it can be recovered. Indeed, first let us just recall, for the readers convenience, that the formulation (2.1) comes from following integration by parts:

**Lemma 2.2.** Let \( f \) and \( g \) be smooth vector fields such that \( g \) is divergence free and tangent to the boundary. Then

\[
- \int_{\Omega} \Delta f \cdot g = 2 \int_{\Omega} D(f) : D(g) - 2 \int_{\partial\Omega} [D(f)n]_{\text{tan}} \cdot g.
\]

Now, assuming that \( u \) is more regular and using the previous lemma, from (2.1), we obtain

\[
\int_{0}^{T} \int_{\partial\Omega} [D(u)n + \alpha u] \cdot \varphi = 0,
\]

for any test function \( \varphi \) satisfying \( \varphi \cdot n = 0 \) on \( \partial \Omega \); consequently, \([D(u)n + \alpha u]_{\text{tan}} = 0 \) on \( \partial \Omega \).

We also note that by taking test functions in \( D([0,T) \times \Omega) \) in the form \( \varphi_h \equiv \psi(x)\theta_h(z) \), where \( \psi \in D(\Omega) \) with divergence free, and \( \theta_h \in D([0,T)) \) such that \( \theta_h(z) = 1 \) for \( z \leq t \) and \( \theta_h(z) = 0 \) for \( z \geq t + h \), and taking the limit as \( h \to 0 \), we obtain an equivalent weak formulation for the momentum equation

\[
\int_{\Omega} (\rho u)(t) \psi + 2\alpha \nu \int_{0}^{t} \int_{\partial\Omega} u \cdot \psi + 2\nu \int_{0}^{t} \int_{\Omega} D(u) : D(\psi) - \int_{0}^{t} \int_{\Omega} \rho u u \cdot \nabla \psi
\]

\[
= \int_{0}^{t} \int_{\Omega} \rho f \cdot \psi + \int_{\Omega} v_{0} \psi,
\]

from which we deduce that \( \rho u \in C_{w}([0,T];L^{\frac{2p}{p+1}}(\Omega)) \). Hence, the initial data (1.2) is verified in the following sense: \( (\rho u)(t) \) converges weakly to \( v_{0} \) as \( t \to 0^{+} \).

The result of existence of weak solutions with finite energy is the following.

**Theorem 2.3.** Let \( 6 \leq p \leq \infty \), \( f \in L^{1}(0,T;L^{\frac{2p}{p+1}}(\Omega)) \), \( \rho_{0} \in L^{p}(\Omega) \), \( \rho_{0} \geq 0 \), a.e. in \( \Omega \), and \( v_{0} \in L^{\frac{2p}{p+1}}(\Omega), |v_{0}|^{2} \in L^{1}(\Omega) \). There exists a weak solution \( (\rho,u) \) of problem (1.1)-(1.2) in the sense of Definition 2.1.

**Remark 2.1.** Let us observe that the density \( \rho \) belongs to \( C([0,T];L^{p}(\Omega)) \). In fact, noting that \( u \in L^{2}(0,T;H^{1}_{0}(\Omega)) \subset L^{1}(0,T;L^{6}(\Omega)) \) and \( 1 \leq p' \leq 2 \leq 6 \), the desired claim follows from standard approximation and regularization arguments for transport equations (see e.g. [9]). On the other hand, when \( \rho_{0}(x) \geq c_{0} > 0 \), the weak solution of (1.1)-(1.2) given by Theorem 2.3 also verifies \( u \in L^{\infty}(0,T;L^{2}(\Omega)) \).

To prove the existence of a weak solution with finite energy, we first introduce a regularized problem, depending on a small positive parameter \( \epsilon \), which is constructed by a regularization of the continuity and momentum equations, as well as, a regularization of the data. More explicitly, fixed \( \epsilon > 0 \), we consider the following regularized problem related to (1.1)-(1.2): Find \((u_{\epsilon},\rho_{\epsilon})\) solution of system
where $\rho_{0,\varepsilon}$, similarly to \[11\] p.149 for $6 \leq p < \infty$, is such that $\rho_{0,\varepsilon} \in C^{2,r}(\Omega)$, $r \in (0,1)$ with

$$
\frac{\partial \rho_{0,\varepsilon}}{\partial n} = 0 \text{ on } \partial \Omega, \quad 0 < \varepsilon \leq \rho_{0,\varepsilon}(x), \quad x \in \Omega,
$$

$$
|\{x \in \Omega : \rho_{0,\varepsilon}(x) < \rho_0(x)\}| \to 0 \text{ as } \varepsilon \to 0,
$$

$$
\rho_{0,\varepsilon} \to \rho_0 \text{ in } L^p(\Omega) \text{ as } \varepsilon \to 0, \quad 6 \leq p < \infty,
$$

and the initial linear momentum $v_{0,\varepsilon}$ is defined as

$$
v_{0,\varepsilon}(x) = \begin{cases} 
v_0 & \text{if } \rho_{0,\varepsilon}(x) \geq \rho_0(x), \\
0 & \text{if } \rho_{0,\varepsilon}(x) < \rho_0(x). \end{cases}
$$

We introduce the concept of weak-strong solution to the previous regularized system.

**Definition 2.4.** Let $6 \leq p \leq \infty$. A weak-strong solution of \((2.3)\) is a pair of functions $(u_\varepsilon, \rho_\varepsilon)$ satisfying $u_\varepsilon \in L^2(0,T;H^1(\Omega))$, $\rho_\varepsilon \in C([0,T];W^{-1,p}(\Omega)) \cap L^\infty(0,T;L^p(\Omega))$, $\rho_\varepsilon > 0$ a.e. in $Q$, $\rho_\varepsilon u_\varepsilon \in L^\infty(0,T;L^{2^*}(\Omega))$, $\rho_\varepsilon u_\varepsilon n \in L^1(0,T;L^2(\Omega))$, $\nabla \rho_\varepsilon \in L^2(0,T;L^2(\Omega))$, such that:

i) equation \((2.3)_3\) holds a.e. in $Q$. The boundary condition \((2.3)_0\) holds a.e. on $\Sigma$, and the initial condition \((2.3)_7\) holds a.e. in $\Omega$,

ii) the momentum equation \((2.3)_1\) is verified in the following sense

$$
- \int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \cdot \partial_t \varphi + 2\alpha \nu \int_0^T \int_\Omega \nabla u_\varepsilon \cdot \varphi + 2\nu \int_0^T \int_\Omega D(u_\varepsilon) : D(\varphi) \\
- \int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon u_\varepsilon \cdot \nabla \varphi + \varepsilon \int_0^T \int_\Omega \nabla \rho_\varepsilon \nabla (u_\varepsilon \cdot \varphi) = \int_0^T \int_\Omega \rho_\varepsilon f \cdot \varphi + \int_\Omega v_{0,\varepsilon} \varphi(0),
$$

for $\varphi \in C^1([0,T] \times \overline{\Omega})$, with $\text{div } \varphi = 0$, $\varphi \cdot n = 0$ on $\Sigma$, and $\varphi(T,x) = 0$ in $\Omega$.

In order to show the existence of weak-strong solution to the regularized problem we will consider the Galerkin approximations for the momentum equation and then will use a limiting procedure. The existence of solutions for this approximate problem will be obtained by linearization and the Schauder fixed point theorem. We can now state the result of existence weak-strong solution for the regularized system \((2.3)\).
Proposition 2.5. Let \( \rho_{0,\epsilon} \) be as in (2.4) and \( p, u_0, f \) as in Theorem 2.3. Then, there exists \((u_\epsilon, \rho_\epsilon)\) a weak-strong solution of (2.3), in the sense of Definition 2.4. Moreover, \((u_\epsilon, \rho_\epsilon)\) verifies \( \rho_\epsilon u_\epsilon \in L^2(0,T; L^{6/5}(\Omega)) \), \( \rho_\epsilon \in L^p(0,T; W^{2,\theta}(\Omega)) \) and \( \partial_t \rho_\epsilon \in L^q(0,T; L^q(\Omega)) \), for some \( q \geq \frac{3}{2} \).

**Proof.** We split the proof into five steps.

**Step 1:** Approximate problem. Let \( \{w^k\}_{k \in \mathbb{N}} \) be a smooth basis of \( H_0^1(\Omega) \), orthonormal in \( L^2(\Omega) \) and let \( \mathcal{Y}^m = \text{span} \{ w^1, ..., w^m \} \). Let us consider a sequence \( \{f^m\}_{m \in \mathbb{N}} \) in \( C([0,T]; L^\frac{2m}{m+1}(\Omega)) \) such that \( f^m \rightarrow f \) in \( L^1(0,T; L^\frac{2m}{m+1}(\Omega)) \).

For each \( m \in \mathbb{N} \) consider the problem of finding \( \rho^m \in C([0,T]; C^2(\Omega)) \) and \( u^m \in C^1([0,T]; Y^m) \) satisfying

\[
\begin{aligned}
\frac{\partial \rho^m}{\partial t} + \text{div} (\rho^m u^m) &= \epsilon \Delta \rho^m \text{ in } Q, \\
\frac{\partial \rho^m}{\partial n} &= 0 \text{ on } \Sigma, \\
\rho^m(0) &= \rho_{0,\epsilon} \text{ in } \Omega, \\
\end{aligned}
\]

\[
\begin{aligned}
&\int_{\Omega} \{ \partial_t (\rho^m u^m) \cdot v - \rho^m u^m \cdot \nabla v - \rho^m f^m \cdot v + 2\nu D\rho^m : Dv \} \, dx \\
&+ 2\alpha \nu \int_{\partial\Omega} u^m \cdot v = \epsilon \int_{\Omega} \nabla \rho^m \nabla (u^m \cdot v), \quad \forall v \in \mathcal{Y}^m, \\
&\phantom{=} u^m(0) = u^m_0,
\end{aligned}
\]  

where \( u^m_0 \in \mathcal{Y}^m \) is uniquely determined by

\[
\int_{\Omega} \rho_{0,\epsilon} u^m_0 \phi \, dx = \int_{\Omega} u^m_0 \phi \, dx \text{ for all } \phi \in \mathcal{Y}^m.
\]

For the sake of simplicity, we shall omit the dependence of \((u^m, \rho^m)\) on the parameter \( \epsilon \). Observe that \( u^m_0 \) is well-defined because the matrix with coefficients \( \int_{\Omega} \rho_{0,\epsilon} w^j w^i \, dx \) is invertible. Moreover, by using (2.4), the definition of \( v_{0,\epsilon} \) (see (2.5)) and the assumption \( \frac{|v_0|^2}{\rho_0} \in L^1(\Omega) \), the following estimate holds true:

\[
\int_{\Omega} v_{0,\epsilon} u^m_0 \, dx = \int_{\Omega} \rho_{0,\epsilon} |u^m_0|^2 \, dx \leq \int_{\Omega} \frac{|v_{0,\epsilon}|^2}{\rho_{0,\epsilon}} \, dx \leq \int_{\Omega} \frac{|v_0|^2}{\rho_0} \, dx \leq C.
\]

**Step 2:** Existence of solutions to the approximate problem. Fixed \( m \in \mathbb{N} \), the existence of approximate solutions \((u^m, \rho^m)\) of (2.7) - (2.11) is proved by linearization and the Schauder fixed point theorem. In fact, fixed \( m \in \mathbb{N} \) and given \( w \in C([0,T]; \mathcal{Y}^m) \), the following lemma gives the existence of \( \rho^m \in C([0,T]; C^2(\Omega)) \) such that

\[
\begin{aligned}
\frac{\partial \rho^m}{\partial t} + \text{div} (\rho^m w) &= \epsilon \Delta \rho^m \text{ in } Q, \\
\frac{\partial \rho^m}{\partial n} &= 0 \text{ on } \Sigma, \\
\rho^m(0) &= \rho_{0,\epsilon} \text{ in } \Omega.
\end{aligned}
\]
Lemma 2.6. ([12] Lemma 3.1) Let \( w \in C([0,T]; \mathcal{Y}^m) \) be a given vector field. Suppose that \( \rho_{0,\varepsilon} \in C^{2,r}(\Omega) \), \( r \in (0,1) \), \( \inf_{x \in \Omega} \rho_{0,\varepsilon}(x) > 0 \) and satisfies the compatibility condition

\[
\frac{\partial \rho}{\partial n} = 0 \text{ on } \partial \Omega.
\]

Then problem (2.11) possesses a unique classical solution

\[
\rho^m = \rho^m(w) \in W = \{ \rho^m \in C([0,T]; C^{2,r}(\bar{\Omega})), \ \partial_t \rho^m \in C([0,T]; C^{0,r}(\bar{\Omega})) \}.
\]

Moreover, the mapping \( w \to \rho^m(w) \) maps bounded sets in \( C([0,T]; \mathcal{Y}^m) \) into bounded sets in \( W \) and it is continuous with values in \( C^1([0,T] \times \bar{\Omega}) \).

Finally, as \( \text{div}(w) = 0 \), it holds

\[
\inf_{x \in \Omega} \rho_{0,\varepsilon}(x) \leq \rho^m(x,t) \leq \sup_{x \in \Omega} \rho_{0,\varepsilon}(x), \ t \in [0,T], \ x \in \Omega.
\] (2.12)

Posteriorly, given \( w \in C([0,T]; \mathcal{Y}^m) \) and \( \rho^m \in C([0,T]; C^2(\bar{\Omega})) \), we solve the following linear problem:

Find \( u^m(x,t) = \sum_{i=1}^{m} \psi_i(t)w^i(x) \) satisfying

\[
\int_{\Omega} \left\{ \partial_t (\rho^m u^m) \cdot v - \rho^m u^m w \cdot \nabla v - \rho^m f^m \cdot v + 2\nu Du^m : Dv \right\} dx
\]

\[
+ 2\alpha \nu \int_{\partial \Omega} u^m \cdot v + \frac{\varepsilon}{2} \int_{\Omega} \nabla \rho^m \nabla (u^m \cdot v) = 0, \ \forall v \in \mathcal{Y}^m,
\]

\[
u^m(0) = u_0^m,
\]

which is equivalent to

\[
\int_{\Omega} \left\{ \rho^m (\partial_t u^m + (w \cdot \nabla)u^m - f^m) \cdot v + 2\nu Du^m : Dv \right\} dx
\]

\[
+ 2\alpha \nu \int_{\partial \Omega} u^m \cdot v + \frac{\varepsilon}{2} \int_{\Omega} \nabla \rho^m \nabla (u^m \cdot v) = 0, \ \forall v \in \mathcal{Y}^m,
\]

\[
u^m(0) = u_0^m.
\] (2.13)

Notice that (2.13) leads us to a linear system of ordinary differential equations for \( \{\psi_j(t)\}_{j=1}^{m} \):

\[
\sum_{j=1}^{m} a_{ij}^1(t) \frac{d\psi_j}{dt} + \sum_{j=1}^{m} a_{ij}^2(t) \psi_j + a_{ij}^3(t) = 0, \ \text{in } (0,T), \ 1 \leq i \leq m,
\] (2.14)

where

\[
a_{ij}^1 = \int_{\Omega} \rho^m w^j w^i dx \in C^1([0,T]),
\]

\[
a_{ij}^2 = \int_{\Omega} \{ (\rho^m w \cdot \nabla w^j) \cdot w^i + 2\nu Dw^j : Dw^i \} dx + 2\alpha \nu \int_{\partial \Omega} \nabla w^j \cdot w^i + \frac{\varepsilon}{2} \int_{\Omega} \nabla \rho^m \nabla (w^j \cdot w^i) dx \in C^1([0,T]),
\]

\[
a_{ij}^3 = - \int_{\Omega} \rho^m f^m w^j dx \in C^1([0,T]).
\]
Since \( \rho^m \geq \epsilon \), it holds
\[
\sum_{i,j} a_{ij}^{1}(t)\xi_{ij} = \int_{\Omega} \rho^{m}(x,t)\left(\sum_{i=1}^{m} \xi_{i} w^{i}(x)\right)^{2} dx \geq \epsilon \sum_{i=1}^{m} |\xi_{i}|^{2}, \ \forall \xi \in \mathbb{R}^{m}.
\]

It follows that the matrix \( A = \{a_{ij}^{1}\}_{i,j} \) is symmetric and positive definite, and in particular \( A \) is invertible. From the classical theory of ordinary differential equations, system (2.11) has a unique solution \( \{\psi_{j}\}_{j=1}^{m} \in (C^{1}([0,T]))^{m} \); then, the solvability of the system (2.13) is guaranteed.

Multiplying equation (2.11) by \( \frac{1}{2}|u^{m}|^{2} \), integrating on \( \Omega \) and adding the result to (2.13) with \( v = u^{m}(t) \), after some calculations, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho^{m}} u^{m}(t)\|_{2}^{2} + 2\nu \|D u^{m}(t)\|_{2}^{2} + 2\nu \epsilon \int_{\Omega} |u^{m}|^{2} = \int_{\Omega} \rho^{m} f^{m} \cdot u^{m}.
\]

Thus, the Hölder inequality implies
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho^{m}} u^{m}(t)\|_{2}^{2} \leq \|\sqrt{\rho^{m}} u^{m}(t)\|_{2}\|\sqrt{\rho^{m}} f^{m}(t)\|_{2}.
\]

Using a generalized Gronwall lemma (52 Lemma 5), from last inequality we can obtain
\[
\|\sqrt{\rho^{m}} u^{m}(t)\|_{2} \leq \|\sqrt{\rho^{0}} u^{m}_{0}\|_{2} + \int_{0}^{T} \|\sqrt{\rho^{m}} f^{m}\|_{2} dt.
\]

From the previous inequality, (2.12) and (2.10), we conclude that \( u^{m} \) is bounded in \( C([0,T]; L^{2}(\Omega)) \) (independent of \( m \) and \( w \)). Thus, as \( u^{m}(x,t) = \sum_{j=1}^{m} \psi_{j}(t) w^{j}(x) \) and \( \sum_{j=1}^{m} |\psi_{j}(t)|^{2} = \|u^{m}\|_{2}^{2} \), then \( \{\psi_{j}\} \) is bounded in \( C([0,T]) \) which implies that \( u^{m} \) is bounded (independent of \( w \)) in \( C([0,T]; Y^{m}) \). Moreover, if \( w \) is bounded in \( C([0,T]; Y^{m}) \) from (2.11) and the symmetry of \( A = \{a_{ij}^{1}\}_{i,j} \), the set \( \{\partial_{t} \psi_{j} : 1 \leq j \leq m\} \) is bounded in \( C([0,T]; Y^{m}) \) which implies that \( \partial_{t} u^{m} \) is bounded in \( C([0,T]; Y^{m}) \). Thus, we conclude that \( u^{m} \) is bounded in \( C^{1}([0,T]; Y^{m}) \) provided \( w \) is bounded in \( C([0,T]; Y^{m}) \).

Given \( w \in C([0,T]; Y^{m}) \), from Lemma 2.4 there exists a unique \( \rho^{m} \in C([0,T]; C^{2,\alpha}(\Omega)) \) solution of (2.11). Knowing \( w \) and \( \rho^{m} \) there exists a unique solution \( u^{m} \) of (2.13) in \( C^{1}([0,T]; Y^{m}) \) which is bounded (independent of \( m \)) in \( C^{1}([0,T]; Y^{m}) \), provided \( w \) is bounded in \( C([0,T]; Y^{m}) \). Thus, there are \( M_{1}, M_{2} > 0 \) such that \( \|u^{m}\|_{C^{1}([0,T]; Y^{m})} \leq M_{2} \) if \( \|w\|_{C([0,T]; Y^{m})} \leq M_{1} \). Denote by \( B_{1} \) the closed ball in \( C([0,T]; Y^{m}) \) of radio \( M_{1} \), and \( B_{2} \) the closed ball in \( C^{1}([0,T]; Y^{m}) \) of radio \( M_{2} \). Then, the mapping
\[
\mathcal{T} : B_{1} \to B_{2}
\]
\[
w \mapsto u^{m}
\]
is continuous. The Arzelà-Ascoli theorem implies that \( B_{2} \subset C([0,T]; Y^{m}) \) compactly and therefore the mapping \( \mathcal{T} \) is continuous and compact from \( B_{1} \) into \( B_{1} \). Then, the Schauder fixed point theorem implies the existence of a fixed point \( u^{m} \) for a given \( T > 0 \). Taking \( \rho^{m} \) the corresponding solution of (2.11), we obtain the existence of an approximate solution \((u^{m}, \rho^{m})\) of (2.7)–(2.8).
Step 3: Estimates for \((u^m, \rho^m)\). We will obtain several estimates for the approximate solution \((u^m, \rho^m)\) which are independent of \(m\) and, in general, are also independent of \(\epsilon > 0\). In the sequel, \(C\) will denote a constant independent of \(m\) and \(\epsilon\) that may change from an estimate to another. If \(C\) depends on \(\epsilon\) we shall indicate this dependence by \(C(\epsilon)\).

The first estimate comes from Lemma 2.6

\[ 0 < \inf_{x \in \Omega} \rho_{0, \epsilon}(x) \leq \rho^m(x, t). \]

Next, we multiply (2.7) by \(\rho^m|\rho^m|^{p-2}\), where \(p > 2\), integrate by parts and use the incompressibility of the flow \(u^m\) to obtain

\[ \frac{1}{p} \frac{d}{dt} \|\rho^m(t)\|^p_p + \epsilon(p-1) \int_{\Omega} \|\nabla \rho^m\| \|\rho^m\|^{p-2} \| \nabla \rho^m \|^2 = 0, \]

then

\[ \|\rho^m\|_{L^\infty(0,T;L^p(\Omega))} \leq \|\rho_{0, \epsilon}\|_{L^p(\Omega)} \leq C, \] (2.15)

because (2.4)3 and (2.4)4. Also, we have that

\[ \|\rho^m(t)\|_2^2 + 2\epsilon \int_0^t \|\nabla \rho^m(s)\|_2^2 ds = \|\rho_{0, \epsilon}\|_2^2, \] (2.16)

so, it follows that

\[ (\sqrt{\epsilon} \nabla \rho^m, \sqrt{\epsilon} \Delta \rho^m) \] is uniformly bounded in \(L^2(0,T;L^2(\Omega) \times W^{-1,2}(\Omega)).\) (2.17)

Integrating equation (2.7)1 with respect to the space variable and using that \(\partial \rho^m / \partial n| \Sigma = 0\), yield the total mass conservation

\[ \int_{\Omega} \rho^m(t) dx = \int_{\Omega} \rho_{0, \epsilon} dx, \quad t \in [0,T]. \] (2.18)

Notice that equation (2.8) is equivalently to

\[ \int_{\Omega} \{ \rho^m (\partial_t u^m + (u^m \cdot \nabla) u^m - f^m) \cdot v + 2\nu D u^m : D v \} dx \\
+ 2\alpha \nu \int_{\partial \Omega} u^m \cdot v - \frac{\epsilon}{2} \int_{\Omega} \nabla \rho^m \nabla (u^m \cdot v) = 0 \quad \forall v \in \mathcal{Y}^m. \] (2.19)

Multiplying equation (2.7)1 by \(\frac{1}{2} |u^m|^2\), integrating on \(\Omega\) and adding the result with (2.19) for \(v = u^m(t)\), we find after some calculations

\[ \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho^m} u^m(t)\|_2^2 + 2\nu \|D u^m(t)\|_2^2 + 2\nu \alpha \int_{\partial \Omega} |u^m|^2 = \int_{\Omega} \rho^m f^m \cdot u^m. \] (2.20)

Thus, the Hölder inequality implies

\[ \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho^m} u^m(t)\|_2^2 \leq \|\sqrt{\rho^m} u^m(t)\|_2 \|\sqrt{\rho^m} f^m(t)\|_2. \]

Applying a generalized Gronwall lemma ([32] Lemma 5)), it follows that

\[ \|\sqrt{\rho^m} u^m(t)\|_2 \leq \|\sqrt{\rho_{0, \epsilon}} u^m_0\|_2 + \int_0^T \|\sqrt{\rho^m} f^m\|_2, \]
henceforth, by applying again Hölder’s inequality, and using (2.10) and (2.15), we obtain
\[ \|\sqrt{\rho^{m}}u^{m}(t)\|_{2} \leq \|\sqrt{\rho_{0}^{m}}u_{0}^{m}\|_{2} + \|\rho^{m}\|_{L^{\infty}(0,T;L^{p}((\Omega))}\|f^{m}\|_{L^{1}(0,T;L^{2p/(1+2p)}((\Omega))} \leq C, \]
because \((f^{m})\) is bounded in \(L^{1}(0,T;L^{2p/(1+2p)}((\Omega)))\).

On the other hand, by integrating (2.20) on \((0,t)\) and proceeding similarly as before, we arrive at
\[ \frac{1}{2}\|\sqrt{\rho^{m}}u_{m}(t)\|_{2}^{2} + 2\nu\int_{0}^{t}\|Du^{m}\|_{2}^{2} + 2\nu\alpha\int_{0}^{t}\int_{\Omega}|u^{m}|^{2} \leq \frac{1}{2}\|\sqrt{\rho_{0}^{m}}u_{0}^{m}\|_{2}^{2} + C\|\sqrt{\rho^{m}}u^{m}\|_{L^{\infty}(0,T;L^{2}((\Omega)))}\|\rho^{m}\|_{L^{\infty}(0,T;L^{p}((\Omega)))}\|f^{m}\|_{L^{1}(0,T;L^{2p/(1+2p)}((\Omega))}. \]
Therefore, \((Du^{m})\) is bounded in \(L^{2}(0,T;L^{2}((\Omega)))\). To estimate \((u_{m})\) in \(L^{2}(0,T;H^{1}((\Omega)))\) we shall apply the following generalized Korn inequality (see [12], Th. 10.17)
\[ \|v\|_{H^{1}((\Omega))} \leq C(K,M,p) (\|D(v)\|_{2}^{2} + \|Rv\|_{2}^{2}) \leq C(K,M,p) (\|D(v)\|_{2}^{2} + \|R\|_{1}\|Rv\|_{2}^{2}), \quad (2.21) \]
for \(v \in H^{1}_{0}(\Omega)\) and any function \(R \geq 0\) such that \(0 < M \leq \int_{\Omega}Rdx, \|R\|_{p} \leq K\), for some \(p > 1\). Without loss of generality we can assume that \(\rho_{0} \geq 0\) and \(\rho_{0} \neq 0\). It follows from (2.18) and \(\rho_{0,\epsilon} \rightarrow \rho_{0}\) in \(L^{p}(\Omega), 6 \leq p < \infty\), that
\[ M = \frac{1}{2}\|\rho_{0}\|_{1} \leq \int_{\Omega}\rho_{0}^{m}dx = \int_{\Omega}\rho_{0,\epsilon}dx \leq 2\|\rho_{0}\|_{1}, \quad \text{for small } \epsilon > 0, \]
so we can take \(R = \rho^{m}\) in (2.21) and use previous estimates in order to infer that \((u_{m})\) is bounded in \(L^{2}(0,T;H^{1}((\Omega))) \subset L^{2}(0,T;L^{6}((\Omega)))\), independently of \(m\) and \(\epsilon\). In the case \(p = \infty\), we also can change the same \(M\) since \(\rho_{0,\epsilon} \rightarrow \rho_{0}\) and the weak-* convergence \(\rho_{0,\epsilon} \rightarrow \rho_{0}\) in \(L^{\infty}\) implies that \(\int_{\Omega}\rho_{0,\epsilon}dx \rightarrow \int_{\Omega}\rho_{0}dx\).

Next, by applying Hölder’s inequality, we estimate
\[ \|\sqrt{\rho^{m}}u^{m}\|_{L^{2}(0,T;L^{2p/(1+2p)}((\Omega)))} \leq \|\rho^{m}\|_{L^{\infty}(0,T;L^{p}((\Omega)))}\|u^{m}\|_{L^{2}(0,T;L^{6}((\Omega)))} \leq C. \]

Since \(p \geq 6\), we have that \(\frac{12p}{6+2p} \geq 4\), hence we can use interpolation to obtain
\[ \|\sqrt{\rho^{m}}u^{m}\|_{4} \leq \|\sqrt{\rho^{m}}u^{m}\|_{2}^{(p-6)/(4p-6)}\|\sqrt{\rho^{m}}u^{m}\|_{\frac{12p}{6+2p}}^{3p/(4p-6)}. \]
Thus, taking \(\zeta = \frac{2(4p-6)}{3p}\), it follows that \(\zeta \geq 2\) and
\[ \|\sqrt{\rho^{m}}u^{m}\|_{4} \leq \|\sqrt{\rho^{m}}u^{m}\|_{2}^{(p-6)/3p}\|\sqrt{\rho^{m}}u^{m}\|_{\frac{12p}{6+2p}}^{2}. \]
Consequently, \((\sqrt{\rho^{m}}u^{m})\) is bounded in \(L^{5}(0,T;L^{4}((\Omega)))\), which implies that
\[ \|\rho^{m}u^{m}u^{m}\|_{L^{5/2}(0,T;L^{2}((\Omega)))} \leq C. \quad (2.22) \]
By using that \((\rho^m), (\sqrt{\rho^m} u^m)\) and \((u^m)\) are bounded in \(L^\infty(0, T; L^p(\Omega))\), \(L^\infty(0, T; L^2(\Omega))\) and \(L^2(0, T; L^6(\Omega))\) respectively, we deduce the following bounds for \((\rho^m u^m)\),

\[
\|\rho^m u^m\|_{L^\infty(0, T; L^{2p}(\Omega))} \leq \left\| \rho^m \right\|_{L^\infty(0, T; L^p(\Omega))} \left\| \sqrt{\rho^m} u^m \right\|_{L^\infty(0, T; L^2(\Omega))} \leq C, \tag{2.23}
\]

\[
\|\rho^m u^m\|_{L^2(0, T; L^{2p}(\Omega))} \leq \left\| \rho^m \right\|_{L^\infty(0, T; L^p(\Omega))} \|u^m\|_{L^2(0, T; L^6(\Omega))} \leq C. \tag{2.24}
\]

By interpolation it follows that

\[
\|\rho^m u^m\|_{L^q(0, T; L^2(\Omega))} \leq C, \quad \text{with } q = \frac{2(2p - 3)}{3} \geq 6.
\]

By using the maximal regularity for parabolic equations, we obtain that \((\rho^m)\) is bounded in \(L^q(0, T; H^1(\Omega))\), which implies that \((\nabla \rho^m)\) is bounded in \(L^q(0, T; L^2(\Omega))\). This bound together the estimate for \(u^m\) in \(L^2(0, T; L^6(\Omega))\) allow to apply the classical \(L^q\)– theory of parabolic equations (see \cite{12}, Th.10.22) to conclude that

\[
\|\partial_t \rho^m\|_{L^q(0, T; L^2(\Omega))} + \|\rho^m\|_{L^q(0, T; W^{2,q}(\Omega))} \leq C(\epsilon), \quad \text{with } q = \frac{2g}{q + 2} \geq 3, \tag{2.25}
\]

Finally, we are going to estimate the derivative in time of \(\rho^m\) and \(\rho^m u^m\). To this end, consider the equation \(\partial_t \rho^m = -\text{div} (\rho^m u^m) + \epsilon \Delta \rho^m\) in \(Q\), and notice that \(W^{-1,2}(\Omega) \subset W^{-1,\frac{2p}{p+1}}(\Omega)\), then, by using (2.17) and (2.23), we conclude that

\[
\|\partial_t \rho^m\|_{L^2(0, T; W^{-1,\frac{2p}{p+1}}(\Omega))} \leq C\left(\|\rho^m u^m\|_{L^\infty(0, T; L^{\frac{2p}{p+1}}(\Omega))} + \sqrt{\epsilon} \|\sqrt{\epsilon} \Delta \rho^m\|_{L^2(0, T; W^{-1,2}(\Omega))}\right) \leq C, \tag{2.26}
\]

for small \(\epsilon > 0\).

From the momentum equation in (2.18) together with (2.22), the Hölder inequality and Sobolev imbedding, for all \(v \in \mathcal{D}(\Omega)\) and \(s > 3\), we have

\[
\left| \frac{d}{dt} \int_{\Omega} \rho^m u^m v \right| \leq \|\rho^m u^m\|_2 \|\nabla v\|_2 + 2\nu \|Du^m\|_2 \|Dv\|_2 + \|\rho^m\|_p \|f^m\|_2 \|v\|_2 + 2\alpha \nu \|u^m\|_{L^2(\partial \Omega)} \|v\|_{L^2(\partial \Omega)} + \frac{\epsilon}{2} \|\nabla \rho^m\|_2 \|u^m\|_6 \|\nabla v\|_3 + \frac{\epsilon}{2} \|\nabla \rho^m\|_2 \|\nabla u^m\|_2 \|v\|_\infty \\
\leq C\left(\|\rho^m u^m\|_2 + \|\rho^m\|_p \|f^m\|_2 + \|u^m\|_{H^1(\Omega)} + \frac{\epsilon}{2} \|\nabla \rho^m\|_2 \|u^m\|_{H^1(\Omega)}\right) \|v\|_{W^{1,s}(\Omega)}.
\]

Therefore, by using (2.17), we obtain

\[
\left| \frac{d}{dt} \int_{\Omega} \rho^m u^m v \right| \leq h_m \|v\|_{W^{1,s}(\Omega)}, \quad \forall v \in \mathcal{D}(\Omega),
\]

for some \(h_m \in L^1(0, T)\), which implies that

\[
\|\partial_t (\rho^m u^m)\|_{L^1(0, T; W^{-1,s'}(\Omega))} \leq C. \tag{2.27}
\]
Step 4: Convergence properties. From the uniform estimates obtained in the previous step, we will deduce some convergences for the approximate solution. We first observe that from (2.16)-(2.17), (2.26), and since $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,\frac{2p}{p+1}}(\Omega)$, by applying Lemma 4 of [32], we conclude that $(\rho^m)$ is relatively compact in $L^2(0,T;L^2(\Omega))$.

Similarly, as $(\rho^m)$ is bounded in $L^\infty(0,T;L^p(\Omega))$ and $L^p(\Omega) \hookrightarrow W^{-1,p}(\Omega) \hookrightarrow W^{-1,\frac{2p}{p+1}}(\Omega)$, we have that $(\rho^m)$ is relatively compact in $C([0,T];W^{-1,p}(\Omega))$.

Moreover, as $s > \frac{6p}{5p-6}$, it holds $L^{\frac{6p}{5p-6}}(\Omega) \hookrightarrow W^{-1,\frac{6p}{5p-6}}(\Omega) \hookrightarrow W^{-1,s'}(\Omega)$; so, from (2.24) and (2.27), we get $(\rho^m u^m)$ is relatively compact in $L^2(0,T;W^{-1,\frac{6p}{5p-6}}(\Omega))$.

Thus, in view of the uniform bounds obtained in the previous step, we have that the sequence $(u^m, \rho^m, \rho^m u^m, \rho^m u^m u^m)$ converges (up to subsequences) to some $(u_\epsilon, \rho_\epsilon, \xi_1, \xi_2)$, as $m \to \infty$, in the following sense:

\begin{align*}
  &\rho^m \to \rho_\epsilon \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ and a.e. in } Q, \\
  &\rho^m \to \rho_\epsilon \text{ weakly-* in } L^\infty(0,T;L^p(\Omega)), \\
  &\rho^m \to \rho_\epsilon \text{ strongly in } C([0,T];W^{-1,p}(\Omega)), \\
  &u^m \to u_\epsilon \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)), \\
  &\rho^m u^m \to \xi_1 \text{ weakly in } L^2(0,T;L^{\frac{6p}{5p-6}}(\Omega)), \\
  &\rho^m u^m \to \xi_1 \text{ weakly-* in } L^\infty(0,T;L^{\frac{6p}{5p-6}}(\Omega)), \\
  &\rho^m u^m \to \xi_2 \text{ strongly in } L^2(0,T;W^{-1,\frac{6p}{5p-6}}(\Omega)).
\end{align*}

Next, we identify the limits $\xi_1$ and $\xi_2$. To this end, notice that the product mapping from $H^1(\Omega) \times W^{-1,p}(\Omega)$ to $W^{-1,\frac{6p}{5p-6}}(\Omega)$ is continuous (see [32] Lemma 3). Therefore, (2.30) and (2.31) lead us to

\begin{align*}
  &\rho^m u^m \to \rho_\epsilon u_\epsilon \text{ weakly in } L^2(0,T;W^{-1,\frac{6p}{5p-6}}(\Omega)).
\end{align*}

Convergence (2.34) together with (2.32) implies that $\xi_1 = \rho_\epsilon u_\epsilon$. In analogous way, from (2.31) and (2.32), we obtain that

\begin{align*}
  &\rho^m u^m \to \rho_\epsilon u_\epsilon u_\epsilon \text{ weakly in } L^1(0,T;W^{-1,\frac{6p}{5p-6}}(\Omega)).
\end{align*}

The uniqueness of the limit in the sense of distributions implies that $\xi_2 = \rho_\epsilon u_\epsilon u_\epsilon$. 

Step 5: Passing to the limit as \( m \to \infty \). Finally, we will prove the existence of a weak-strong solution to problem (2.3) by passing to the limit as \( m \to \infty \) in the approximate problem (2.7)-(2.8).

Since \( \rho^m \to \rho_\epsilon \), \( \rho^m u^m \to \rho_\epsilon u_\epsilon \) and \( \Delta \rho^m \to \Delta \rho_\epsilon \) in \( \mathcal{D}'(Q) \), we can pass to the limit in (2.7) in the sense of distributions. Therefore

\[
\partial_t \rho_\epsilon + \text{div} (\rho_\epsilon u_\epsilon) = \epsilon \Delta \rho_\epsilon \text{ in } \mathcal{D}'(Q).
\]

In view of the estimate (2.25), and since \( \rho_\epsilon \) inherits regularity from \( \rho^m \), we have in particular that

\[
\partial_t \rho_\epsilon + \text{div} (\rho_\epsilon u_\epsilon) = \epsilon \Delta \rho_\epsilon \text{ a.e. in } Q.
\]

From (2.30) we have that \( \rho^m(0) \to \rho_\epsilon(0) \) in \( W^{-1,p}(\Omega) \). But, the initial condition (2.7)3 says that \( \rho^m(0) = \rho_{0,\epsilon} \in C^{2,1}(\Omega) \), so that \( \rho_\epsilon(0) = \rho_{0,\epsilon} \). It is not difficult to see that the boundary condition is also satisfied.

In order to pass to the limit in (2.8)1 as \( m \to \infty \), we notice that by using the energy identity (2.16) for \( \rho^m \) and the corresponding one for \( \rho^\epsilon \), we can prove that \( \nabla \rho^m \) converges strongly to \( \nabla \rho^\epsilon \) in \( L^2(0,T;L^2(\Omega)) \), (see [10] for a similar argument). Hence, from the convergences obtained in step 4, we can classically pass to the limit in (2.8)1 as \( m \to \infty \), and obtain that (2.6) holds true. The proof is complete. \( \square \)

Proof Theorem 2.3

Proof. Let \((\rho_\epsilon, u_\epsilon)\) be the weak-strong solution of (2.3) given by Proposition 2.5. As we mentioned in step 3 of the proof of Proposition 2.5, most of the obtained estimates are also independent of \( \epsilon > 0 \). So, proceeding as in step 4, we can conclude the existence of \((\rho_\epsilon, u_\epsilon, \beta_1, \beta_2)\) and a subsequence of \((\rho_\epsilon, u_\epsilon, \rho_\epsilon u_\epsilon, \rho_\epsilon u_\epsilon u_\epsilon)\), still indexed by \( \epsilon \), such that the following convergences hold true as \( \epsilon \to 0 \):

\[
\begin{align*}
\rho_\epsilon & \to \rho \text{ weakly-* in } L^\infty(0,T;L^p(\Omega)), \\
\rho_\epsilon & \to \rho \text{ strongly in } C([0,T];W^{-1,p}(\Omega)), \\
u_\epsilon & \to u \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)), \\
\rho_\epsilon u_\epsilon & \to \beta_1 \text{ weakly in } L^2(0,T;L^\frac{6p}{\sigma+3p}(\Omega)), \\
\rho_\epsilon u_\epsilon & \to \beta_1 \text{ weakly-* in } L^\infty(0,T;L^\frac{6p}{\sigma+3p}(\Omega)), \\
\rho_\epsilon u_\epsilon u_\epsilon & \to \beta_2 \text{ weakly in } L^{\frac{3}{2}}(0,T;L^2(\Omega)).
\end{align*}
\]

Moreover, using (2.17) and the continuity of the product mapping from \( H^1(\Omega) \times W^{-1,2}(\Omega) \) to \( W^{-1,\frac{2}{3}}(\Omega) \), we obtain

\[
\begin{align*}
\epsilon \nabla \rho_\epsilon & \to 0 \text{ strongly in } L^2(0,T;L^2(\Omega)), \\
\epsilon u_\epsilon \Delta \rho_\epsilon & \to 0 \text{ weakly in } L^1(0,T;W^{-1,\frac{2}{3}}(\Omega)).
\end{align*}
\]

Working as in the end of step 3, one can prove that \( \beta_1 = \rho u \) and \( \beta_2 = \rho uu \). Moreover, from above convergences, in analogous way as we did in step 5, we can pass to the limit
in the regularized problem (2.3). The only difference here is that the two terms involving the parameter $\epsilon$ vanish.

Thenceforth, we obtain that equation $\partial_t \rho + \text{div} (\rho u) = 0$ is verified in $D'(Q)$. From (2.36) we have that $\rho_{0,\epsilon} = \rho_0(0) \rightarrow \rho(0)$ in $W^{-1,p}(\Omega)$. However, as $\rho_{0,\epsilon} \rightarrow \rho_0$ in $L^p(\Omega)$, when $6 \leq p < \infty$, and $\rho_{0,\epsilon} \rightharpoonup \rho_0$ weakly-* in $L^\infty(\Omega)$, when $p = \infty$, then $\rho(0) = \rho_0$ at least in $W^{-1,p}(\Omega)$.

Finally, it remains to verify the energy inequality (2.2). As usual, integrating (2.20) over $(0,t)$, multiplying the result by $\phi \in D(0,T)$, $\phi \geq 0$, integrating over $(0,T)$, and passing to the limit using convergences (2.28)-(2.33), we arrive at the following inequality

$$
\int_0^T \left( \frac{1}{2} \|\phi_\epsilon u_\epsilon(t)\|^2 + \int_0^t [2\nu\|D_\epsilon u_\epsilon(s)\|^2 + 2\nu\alpha \int_{\partial\Omega} |u_\epsilon|^2] ds \right) \phi(t) dt \\
\leq \frac{1}{2} \int_\Omega \frac{|v_\epsilon|^2}{\rho_\epsilon} dx \int_0^T \phi(t) dt + \int_0^T \left( \int_0^t \int_\Omega \rho_\epsilon u_\epsilon \right) \phi(t) dt.
$$

As $\epsilon \rightarrow 0$, we have $\int_\Omega \frac{|v_\epsilon|^2}{\rho_\epsilon} dx \rightarrow \int_\Omega \frac{|v_0|^2}{\rho_0} dx \leq C$. From convergences (2.35)-(2.39), as $\epsilon \rightarrow 0$ we find

$$
\int_0^T \left( \frac{1}{2} \|u(t)\|^2 + \int_0^t [2\nu\|u(s)\|^2 + 2\nu\alpha \int_{\partial\Omega} |u|^2] ds \right) \phi(t) dt \\
\leq \frac{1}{2} \int_\Omega \frac{|v_0|^2}{\rho_0} dx \int_0^T \phi(t) dt + \int_0^T \left( \int_0^t \int_\Omega \rho_0 u \right) \phi(t) dt,
$$

for any $\phi \in D(0,T)$, $\phi \geq 0$. This yields the energy inequality (2.2) which implies in particular that $\sqrt{\rho u} \in L^\infty(0,T;L^2(\Omega))$. \(\Box\)

**Remark 2.2.** Observe that, from the regularity of $\rho$ and $\rho u$, equation $\partial_t \rho + \text{div} (\rho u) = 0$ holds in $W^{-1,\infty}(0,T;L^p(\Omega)) \cap L^\infty(0,T;W^{-1,\frac{2p}{p+1}}(\Omega))$.

**Remark 2.3.** Concerning the pressure, observe that

$$
\rho u \in L^\infty(0,T;L^\frac{2p}{p+1}(\Omega)) \Rightarrow \partial_t (\rho u) \in W^{-1,\infty}(0,T;L^\frac{2p}{p+1}(\Omega)),
$$

$$
\rho uu \in L^{\frac{p}{2}}(0,T;L^2(\Omega)) \Rightarrow \text{div}(\rho uu) \in L^{\frac{p}{2}}(0,T;W^{-1,2}(\Omega)),
$$

$$
u u \in L^2(0,T;H^1(\Omega)) \Rightarrow \Delta u \in L^2(0,T;W^{-1,2}(\Omega)),
$$

$$
\rho f \in L^1(0,T;L^\frac{2p}{p+1}(\Omega)) \Rightarrow \rho f \in L^1(0,T;W^{-1,\frac{2p}{p+1}}(\Omega)).
$$

From the De Rham theorem there exists a distribution $\pi \in W^{-1,\infty}(0,T;L^\frac{2p}{p+1}(\Omega))$ such that

$$
\partial_t (\rho u) + \text{div}(\rho uu) - \nu \Delta u + \nabla \pi = \rho f \text{ in } W^{-1,\infty}(0,T;W^{-1,\frac{2p}{p+1}}(\Omega)).$$
3. Vanishing viscosity limit

In this section we establish the convergence of a weak solution of the non-homogeneous Navier-Stokes equations with Navier boundary conditions to the strong solution of non-homogeneous Euler equations when the viscosity coefficient goes to zero. To this end, consider the following limiting problem

\begin{align*}
\begin{cases}
\partial_t(\rho u) + \text{div} (\rho uu) + \nabla \pi = \rho f & \text{in } Q, \\
\text{div } u = 0 & \text{in } Q, \\
\partial_t \rho + \text{div} (\rho u) = 0 & \text{in } Q, \\
u \cdot n = 0 & \text{on } \Sigma, \\
\rho(0) = \rho_0 & \text{in } \Omega, \\
u(0) = u_0 & \text{in } \Omega.
\end{cases}
\end{align*}

(3.1)

We recall a result of existence of strong solutions to the non homogeneous Euler system (3.1). For another results on this subject see [15, 2, 24].

**Theorem 3.1.** (see [34]) Let $\Omega$ with boundary $\partial \Omega$ smooth enough, and $p > 3$. Assume that $\rho_0 \in W^{2,p}(\Omega)$, $0 < \rho_\ast \leq \rho_0 \leq \rho^* < \infty$, for some constants $\rho_\ast, \rho^*$; moreover, assume that $u_0 \in W^{2,p}(\Omega)$, $u_0 \cdot n = 0$ on $\partial \Omega$, $\text{div}(u_0) = 0$, $f \in L^1(0,T;W^{2,p}(\Omega))$. Then there exists a time $T_\ast \in (0,T]$ such that problem (3.1) has a unique solution $(\rho,u,\pi)$ which satisfies

a) $\rho \in C([0,T_\ast];W^{2,p}(\Omega)) \cap C^1([0,T_\ast];W^{1,p}(\Omega))$, $0 < \rho_\ast \leq \rho \leq \rho^* < \infty$,

b) $u \in C([0,T_\ast];W^{2,p}(\Omega)) \cap W^{1,1}(0,T_\ast;W^{1,p}(\Omega))$,

c) $\pi \in L^1(0,T_\ast;W^{3,p}(\Omega))$.

Moreover, if $f \in C([0,T];W^{2,p}(\Omega))$ then $u \in C^1([0,T_\ast];W^{1,p}(\Omega))$ and $\pi \in C([0,T_\ast];W^{3,p}(\Omega))$, hence $(\rho,u,\pi)$ is a classical solution.

**Remark 3.1.** Observe that, since $p > 3$, by the Sobolev embedding we have that $\nabla \rho \in L^\infty(\Omega \times (0,T_\ast))$, $\nabla u \in L^\infty(\Omega \times (0,T_\ast))$, $u_t \in L^1(0,T_\ast;L^\infty(\Omega))$ and $\nabla \pi \in L^1(0,T_\ast;L^\infty(\Omega))$.

Let $(u^\nu, \rho^\nu)$ be a weak solution of

\begin{align*}
\begin{cases}
\partial_t(\rho^\nu u^\nu) + \text{div} (\rho^\nu u^\nu u^\nu) - \nu \Delta u^\nu + \nabla \pi^\nu = \rho^\nu f^\nu & \text{in } Q, \\
\text{div } u^\nu = 0 & \text{in } Q, \\
\partial_t \rho^\nu + \text{div} (\rho^\nu u^\nu) = 0 & \text{in } Q, \\
u^\nu \cdot n = 0 & \text{on } \Sigma, \\
[D(u^\nu)n + \alpha u^\nu]_\text{tan} = 0 & \text{on } \Sigma, \\
\rho^\nu(0) = \rho_0^\nu & \text{in } \Omega, \\
(u^\nu(0)) = u_0^\nu & \text{in } \Omega,
\end{cases}
\end{align*}

(3.2)

with $0 < \rho_0^\nu \in L^p(\Omega)$ given by Theorem 2.3. This weak solution satisfies

\[
\frac{1}{2}\|\rho^\nu(t)u^\nu(t)\|_2^2 + 2\nu \int_0^t \int_{\partial \Omega} |u^\nu|^2 + 2\nu \int_0^t \|D(u^\nu)\|_2^2 \\
\leq \frac{1}{2}\|\rho_0^\nu\|_2^2 + \int_0^t \int_{\Omega} \rho^\nu f^\nu \cdot u^\nu,
\]

(3.3)
and
\[ \frac{1}{2} \| \rho'(t) \|_2^2 \leq \frac{1}{2} \| \rho_0' \|_2^2. \] (3.4)

We now state the main result of this section.

**Theorem 3.2.** Under the hypotheses of Theorems 3.1 and 2.3 let \((u, \rho)\) be the solution of (3.1) obtained in Theorem 3.1 and let \((\nu', \rho')\) be the one of (3.2) given in Theorem 2.3. Assume further that \(0 < \rho_* \leq \rho'_0 \leq \rho^*, \) for all \(\nu > 0,\) where \(\rho_* \) and \(\rho^*\) are the same constants of Theorem 3.1. Then there exists \(C > 0\) independent of \(\nu > 0\) such that, for all \(t \in (0, T_s),\) the following inequality holds:

\[
\|u(t) - \nu'(t)\|_2^2 + \|\rho(t) - \rho'(t)\|_2^2 \\
\leq C \left( \left\| \sqrt{\rho_0'} u_0 - \frac{\nu_0'}{\sqrt{\rho_0'}} \right\|_2^2 + \| \rho_0 - \rho'_0 \|_2^2 + \nu \int_0^{T_*} \| u \|_{H^1(\Omega)}^2 + \int_0^{T_*} \| f - f' \|_{L^2} \right). 
\]

In particular, if \(\left\| \sqrt{\rho_0'} u_0 - \frac{\nu_0'}{\sqrt{\rho_0'}} \right\|_2^2 + \| \rho_0 - \rho'_0 \|_2 \to 0\) as \(\nu \to 0,\) and \(f'\) converges to \(f\) in \(L^1(0, T_*; L^{\frac{2p}{p-1}}(\Omega)),\) then
\[
\sup_{0 < s < t} (\|u(s) - \nu'(s)\|_2 + \|\rho(s) - \rho'(s)\|_2) \to 0 \text{ as } \nu \to 0. \] (3.5)

**Remark 3.2.** We stress that (3.5) implies in particular that
\[
\sup_{0 < s < t} \| \sqrt{\rho} u - \sqrt{\rho'} \nu' \|_2 \to 0 \text{ as } \nu \to 0,
\]

since we can write
\[
\sqrt{\rho} u - \sqrt{\rho'} \nu' = (\sqrt{\rho} - \sqrt{\rho'}) u + \sqrt{\rho'} (u - \nu').
\]
Moreover, by interpolation, we conclude that, for \(p \geq 2,\)
\[
\sup_{0 < s < t} \| \rho - \nu' \|_p \to 0 \text{ as } \nu \to 0.
\]

Let us proceed with the proof of Theorem 3.2.

**Proof.** The differences \(\omega = u - \nu', \) \(\sigma = \rho - \rho', \) \(q = \pi - \pi'\) satisfy

\[
\begin{align*}
\rho' [\omega_t + \nu' \cdot \nabla \omega] + \nabla q &= -\rho' \omega \cdot \nabla u - \sigma (u_t + u \cdot \nabla u) - \nu \Delta u' + \sigma f + \rho'(f - f') \quad \text{in } Q, \\
\text{div } \omega &= 0 \quad \text{in } Q, \\
\partial_t \sigma + u' \cdot \nabla \sigma &= -\omega \cdot \nabla \rho \quad \text{in } Q, \\
\omega \cdot n &= 0 \quad \text{on } \Sigma, \\
\sigma(0) &= \rho_0 - \rho_0' \quad \text{in } \Omega, \\
\omega(0) &= u_0 - u_0' \quad \text{in } \Omega.
\end{align*}
\] (3.6)

Formally, by multiplying third equation in (3.6) by \(\sigma\) and integrating in space and time we obtain
\[
\frac{1}{2} \| \sigma(t) \|_2^2 \leq \frac{1}{2} \| \sigma(0) \|_2^2 - \int_0^t \int_\Omega \omega \cdot \nabla \rho \sigma, \quad (3.7)
\]

where we have used that \( \text{div}(u^\nu) = 0 \) and \( u^\nu \cdot n = 0 \) on \( \partial \Omega \) to deduce that \( \int_0^t \int_\Omega (u^\nu \cdot \nabla \sigma) \sigma = 0 \). In fact, to justify the previous inequality we proceed in this way: consider the energy inequality (3.4) and obtain other three inequalities: \( J_1 \) by multiplying the continuity equation (3.1) for \( \rho \) by \( \rho \), \( J_2 \) by multiplying the same equation (3.1) by \( \rho^\nu \) and \( J_3 \) by multiplying the equation (3.2) for \( \rho^\nu \) by \( \rho \). One obtains (3.7) by doing \( (3.4) + J_1 - J_2 - J_3 \).

From (3.7), by using the H"{o}lder and Young inequalities we arrive at

\[
\| \sigma(t) \|_2^2 \leq \| \sigma(0) \|_2^2 + \int_0^t \| \nabla \rho \|_\infty (\| \omega \|_2^2 + \| \sigma \|_2^2). \quad (3.8)
\]

Formally, by multiplying the first equation in (3.6) by \( \omega \), and integrating in space and time, and finally by using integration by parts (Lemma 2.2) in the Laplacian term, one gets

\[
\frac{1}{2} \| \sqrt{\rho^\nu(t)} \omega(t) \|_2^2 + 2 \nu \alpha \int_0^t \int_{\partial \Omega} u^\nu \cdot w + 2 \nu \int_0^t \int_\Omega D(u^\nu) \cdot D(w) \\
\leq \frac{1}{2} \| \sqrt{\rho_0^\nu} u_0 - \frac{v^\nu}{\sqrt{\rho_0^\nu}} \|_2^2 - \int_0^t \int_\Omega \rho^\nu \omega \cdot \nabla u \cdot \omega + \int_0^t \int_\Omega \sigma (u_t + u \cdot \nabla u) \cdot \omega \\
+ \int_0^t \int_\Omega \sigma f \cdot \omega + \int_0^t \int_\Omega \rho^\nu (f - f^\nu) \cdot \omega. \quad (3.9)
\]

To justify the previous inequality, we first obtain three inequalities: \( I_1 \) by multiplying the momentum equation (3.1) by \( u \), \( I_2 \) by multiplying the same equation (3.1) by \( u^\nu \) and \( I_3 \) by multiplying the momentum equation (3.2) for \( u^\nu \) by \( u \). Next, we consider the energy inequality (3.3) and do (3.3) + \( I_1 - I_2 - I_3 \) to arrive at (3.9).

Proceeding as in [13], we rewrite

\[
2 \nu \alpha \int_0^t \int_{\partial \Omega} u^\nu \cdot w + 2 \nu \int_0^t \int_\Omega D(u^\nu) \cdot D(w) \\
= 2 \nu \alpha \int_0^t \int_{\partial \Omega} \left| w + \frac{u}{2} \right|^2 + 2 \nu \int_0^t \int_\Omega \left| D(w + \frac{u}{2}) \right|^2 - \frac{\nu \alpha}{2} \int_0^t \int_{\partial \Omega} \left| u \right|^2 - \frac{\nu}{2} \int_0^t \int_\Omega \left| D(u) \right|^2.
\]
Thus, by using Hölder and Young inequalities, we have
\[
\frac{1}{2} \| \sqrt{\rho'} \omega \|_2^2 \leq \frac{1}{2} \left( \| \sqrt{\rho_0'} u_0 - \frac{v_0^{\nu}}{\sqrt{\rho_0'}} \|_2^2 + \int_0^t \| u \|_\infty \| \sqrt{\rho'} \omega \|_2^2 + \int_0^t \| u_t \|_\infty \| \sigma \|_2 \| \omega \|_2 \\
+ \int_0^t \| u \cdot \nabla u \|_\infty \| \sigma \|_2 \| \omega \|_2 + \frac{\nu \alpha}{2} \int_0^t \int_{\partial \Omega} |u|^2 + \frac{\nu}{2} \int_0^t \| D(u) \|_2^2 \\
+ \int_0^t \| f \|_\infty \| \sigma \|_2 \| \omega \|_2 + \int_0^t \| \sqrt{\rho'} \|_{2p} \| f - f' \|_{\frac{2p}{p-1}} \| \sqrt{\rho'} \omega \|_2 \\
\leq \frac{1}{2} \left( \| \sqrt{\rho_0'} u_0 - \frac{v_0^{\nu}}{\sqrt{\rho_0'}} \|_2^2 + \int_0^t \| u \|_\infty \| \sqrt{\rho'} \omega \|_2^2 \\
+ \int_0^t (\| u_t \|_\infty + \| u \cdot \nabla u \|_\infty + \| f \|_\infty)(\| \omega \|_2^2 + \| \sigma \|_2^2) \\
+ C \nu \int_0^t \| u \|_{H^1(\Omega)} + C \| \sqrt{\rho'} \|_{L^\infty(0,T_*,L^{2p}(\Omega))} \int_0^t \| f - f' \|_{\frac{2p}{p-1}} \| \sqrt{\rho'} \omega \|_2 \right)
\]
where \( C = C(\rho^*, \alpha, \Omega) \) is independent of \( \nu \) and we have used that \( \int_{\partial \Omega} |u|^2 \leq C(\Omega) \| u \|_{H^1(\Omega)} \).

Notice also that
\[
\| \sqrt{\rho'} \|_{L^\infty(0,T_*,L^\infty(\Omega))} = \| \rho^{\nu} \|_{L^\infty(0,T_*,L^\infty(\Omega))} \leq \| \rho_0^{\nu} \|_{\infty}^{1/2} \leq C(\rho^*)^{1/2}.
\]
(3.10)

Adding (3.8) and using (3.10), we obtain
\[
\sqrt{\rho_0'} \| \omega(t) \|_2^2 + \| \sigma(t) \|_2^2
\leq \left( \| \sqrt{\rho_0'} u_0 - \frac{v_0^{\nu}}{\sqrt{\rho_0'}} \|_2^2 + \| \sigma(0) \|_2^2 + C \nu \int_0^t \| u \|_{H^1(\Omega)}^2 + C \int_0^t \| f - f' \|_{\frac{2p}{p-1}} \| \omega \|_2 \\
+ C \int_0^t (\| \nabla \rho \|_\infty + \| \nabla u \|_\infty + \| u_t \|_\infty + \| u \cdot \nabla u \|_\infty + \| f \|_\infty)(\| \omega \|_2^2 + \| \sigma \|_2^2) \right)
\]

Now we apply a Gronwall type inequality [27, p.360] to obtain
\[
\| u(t) - u^{\nu}(t) \|_2^2 + \| \rho(t) - \rho^{\nu}(t) \|_2^2
\leq C \left( \left\| \sqrt{\rho'} u_0 - \frac{v_0^{\nu}}{\sqrt{\rho'}} \right\|_2^2 + \| \rho_0 - \rho_0^{\nu} \|_2^2 + \nu \int_0^{T_*} \| u \|_{H^1(\Omega)}^2 + \int_0^{T_*} \| f - f^{\nu} \|_{\frac{2p}{p-1}} \right)
\]
where \( C \) is a positive constant depending on
\[
\rho^*, \rho_0, \alpha, \Omega, \| \nabla \rho \|_{L^1(0,T_*,L^\infty(\Omega))}, \| \nabla u \|_{L^1(0,T_*,L^\infty(\Omega))}, \\
\| u_t \|_{L^1(0,T_*,L^\infty(\Omega))}, \| u \|_{L^\infty(\Omega \times (0,T_*))}, \| f \|_{L^1(0,T_*,L^\infty(\Omega))}.
\]
Due to regularity of \( \rho, u \) and \( f \) in Theorem 3.1 (see Remark 3.1), all the above norms are finite. The proof is then complete. \( \square \)
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