§1. Introduction

The aging, dynamical behavior largely depending on the history of system, is one of the most striking phenomena in the complex systems such as spin glasses, glasses, polymers and proteins, and many theoretical models have been proposed to explain these phenomena. The Generalized Random Energy Model (GREM) is one of such models. Bouchaud and Dean calculated the correlation function $C(t + t_w, t_w)$ exactly and showed that $t/t_w$-scaling holds in this model, although the dynamics they introduced seems to be a little unnatural. We recently introduce more natural dynamics to this model, and carried out simulations on aging phenomena with temperature variations. As a result, we revealed that the fundamental features reported in these complex systems, such as the memory effect and the rejuvenation, were well reproduced along the hierarchical picture discussed in ref. 5, 6, 7.

In this paper, we calculate the exact correlation function for dynamics used in ref. 4 and we confirm that while the $t/t_w$-scaling holds as ref. 3 our result is different from theirs on exponents for power law relaxation. By introducing magnetizations to this model, we also calculate the ZFC magnetization $M_{ZFC}(t, t_w)$ which is measured by quenching the system from above $T_c$ to below and applying a weak magnetic field during $t$ after a waiting time $t_w$. We estimate $M_{ZFC}(t, t_w)$ by using a relation between the response function $R(t, t')$ and the correlation function $C(t, t')$,

$$R(t, t') = -\frac{1}{T} \frac{d}{dt} C(t, t'),$$

(1.1)
where $T$ is the temperature. This relation is valid when assuming that the external field $H$ is small enough and the transition probability from a state $\beta$ to $\alpha$ is modified by a factor $\exp(-\frac{H M_\beta}{T})$, where $M_\beta$ is the magnetization of a state $\beta$. This condition is satisfied in the present way of introducing magnetizations. Although this relation is very similar to the Fluctuation Dissipation Theorem (FDT) for equilibrium relaxation (actually it reduces to FDT if time homogeneity holds), it is valid even if the relaxation is non-equilibrium. These analytic results on the correlation function and the ZFC magnetization are checked by numerical simulations.

The organization of this manuscript is following. In section 2, we introduce the GREM. In section 3, we calculate the correlation function and the ZFC magnetization analytically and check these results by simulations. In section 4, we summarize this paper with some discussions.

§2. Model

The GREM is schematically shown in Fig. 1. The bottom points represent the accessible states of this system. This model is constituted by piling up $L$ layers hierarchically. For simplicity, we hereafter consider the case that each branching point has $N$ branches.

We say that $d(\alpha, \beta)$, the distance between two states $\alpha$ and $\beta$, is $k$ if the lowest common ancestor of these two states is in the $k$-th layer counted from the bottom, e.g., in Fig. 1 $d(\alpha, \beta) = 1$ and $d(\alpha, \gamma) = 2$. It is assumed that overlaps between states depend only on the distance and they decrease with increasing distance. Therefore $q_k$, the overlap between two states with $d(\alpha, \beta) = k$, satisfies the relation

$$q_0 > q_1 > \cdots > q_{L-1} > q_L = 0. \quad (2.1)$$

We hereafter denote as $h_k(B_{k-1})$ the barrier height from a branching point in the $k-1$-th layer, $B_{k-1}$, to its parent branching point. In the GREM, $h_k$ is given randomly and independently according to the distribution

$$\rho_k(h_k)dh_k = \frac{1}{T_c(k)} \exp\left[-\frac{h_k}{T_c(k)}\right]dh_k, \quad (2.2)$$

where $T_c(k)$ is the transition temperature of the $k$-th layer and satisfies the inequality

$$T_c(1) < T_c(2) < \cdots < T_c(L). \quad (2.3)$$

For dynamics, we employ Markov process similar to that used in ref. At first, the system is activated from a state $\beta$ to $\beta_k$ (the $k$-th ancestor of $\beta$) with probability $W(\beta, k)$ in unit time. After the activation, the system falls to one of all the states under $\beta_k$ with equal probability. We choose the uniform distribution for the initial condition to consider the situation that the system is quenched from an infinitely high temperature.

The difference between the present dynamics and the one by Bouchaud and Dean exists in the
hopping probability \( W(\beta, k) \). Inspired by the model proposed by Yoshino\cite{11}, we give \( W(\beta, k) \) as

\[
W(\beta, k) = \frac{1}{\tau_0 \prod_{i=1}^{k} \tilde{\tau}_i(\beta_{i-1})} \left[ 1 - \frac{1}{\tilde{\tau}_{k+1}(\beta_k)} \right],
\]

where \( \tau_0 \) is a microscopic time scale, \( \tilde{\tau}_k \equiv \exp(h_k/T) \) and \( \tilde{\tau}_{L+1}(\beta_L) \equiv \infty \). Note that the first factor in the right hand represents the probability that the system can be activated from \( \beta \) to \( \beta_k \) and the second one represents that the system can not be activated from \( \beta_k \) to \( \beta_{k+1} \). Hereafter \( \tau_0 \) is used as the unit of time and set to 1. From eq. (2.2), the distribution of \( \tilde{\tau}_k \) can be written as

\[
p_k(\tilde{\tau}_k) d\tilde{\tau}_k = \frac{x_k}{\tilde{\tau}_k^{k+1}} d\tilde{\tau}_k \quad (\tilde{\tau}_k \geq 1),
\]

where \( x_k \equiv T/T_c(\beta) \).

On the other hand, Bouchaud and Dean give the hopping probability as

\[
W(\beta, k) = \frac{1}{\tau_0 \tilde{\tau}_k(\beta_{k-1})}.
\]

But this hopping probability is not so realistic because, e.g., \( W(\beta, k+1) > W(\beta, k) \) if \( h_{k+1}(\beta_k) < h_k(\beta_{k-1}) \).

Now let us introduce the following two functions. The first one \( \Pi(k, \alpha, t) \), is the probability that the system which initially stays at \( \alpha \) reaches a state with the distance \( k \) at \( t \). This function is defined by the Green function \( G_{\beta\alpha}(t) \), i.e., the probability that the system which initially stays at \( \alpha \) reaches \( \beta \) at \( t \), as

\[
\Pi(k, \alpha, t) \equiv \sum_{\beta: d(\alpha, \beta) = k} G_{\beta\alpha}(t),
\]

where the sum of this equation is taken over all the states \( \beta \) which satisfy the condition \( d(\alpha, \beta) = k \).

The next function, \( P(\alpha, B, t_w) \), is the probability that we find the system at a state \( \alpha \) at time \( t_w \) when the initial condition is located anywhere under \( B \), and is defined by the Green function as

\[
P(\alpha, B, t_w) = \frac{1}{N(B)} \sum_{\beta \in B} G_{\alpha\beta}(t_w),
\]

where \( N(B) \) is the number of states under \( B \). Using these two functions, we can represent \( C(t + t_w, t_w) \), the correlation between \( t_w \) and \( t + t_w \), as

\[
C(t + t_w, t_w) = \sum_{k=0}^{L-1} q_k \sum_{\alpha} \Pi(k, \alpha, t) P(\alpha, B_{\text{top}}, t_w).
\]

We hereafter calculate the correlation function along the following strategy:

(i) Calculate the Laplace transform of \( P(\alpha, B_{\text{top}}, t_w) \)

\[
\hat{P}(\alpha, B_{\text{top}}, E) \equiv \int_0^{\infty} dt_w \exp[-Et_w] P(\alpha, B_{\text{top}}, t_w).
\]

(2.10)
(ii) Calculate the Green function and estimate

\[ \tilde{\Pi}(k, \alpha, E') \equiv \int_0^\infty dt \Pi(k, \alpha, t) \exp[-E't]. \] (2.11)

(iii) By taking the sum for \( \alpha \), calculate

\[ \hat{C}(E' + E, E) \equiv \sum_{k=0}^{L-1} q_k \sum_{\alpha} \tilde{\Pi}(k, \alpha, E') \hat{P}(\alpha, B_{\text{top}}, E). \] (2.12)

(iv) Carry out the inverse Laplace transformation of \( \hat{C}(E' + E, E) \) to obtain \( C(t + t_w, t_w) \).

§3. Analytic Calculations and Simulations on the GREM

3.1 Derivation of the formal solution

Now let us consider the event that the system initially stays at \( \beta \) and reaches \( \alpha \) at time \( t_w \). The probability for this event is \( P(\alpha, \beta, t_w) = G_{\alpha\beta}(t_w) \). There are two possibilities for this event to happen:

(i) \( \alpha = \beta \) and the system is never activated during \( t_w \).

(ii) The system is activated to \( \beta_k \) at \( t' \)\(< t_w \) and reaches \( \alpha \) after that.

In the case (ii), the probability that the system reaches \( \alpha \) after the activation is \( P(\alpha, \beta_k, t_w - t') \) because the next state is chosen randomly from all the states under \( \beta_k \) (recall the definition of \( P(\alpha, B, t_w) \)). Therefore, we obtain the following integral equation

\[ P(\alpha, \beta, t_w) = \delta_{\alpha\beta} \exp[-\frac{t_w}{\tilde{\tau}_1(\beta)}] + \sum_{i=1}^L \int_0^{t_w} dt' W(\beta, i) \exp[-\frac{t'}{\tilde{\tau}_1(\beta)}] P(\alpha, \beta_i, t_w - t'). \] (3.1)

The Laplace transformation of this equation leads us to

\[ \hat{P}(\alpha, \beta, E) = \frac{\delta_{\alpha\beta} \tilde{\tau}_1(\beta)}{E \tilde{\tau}_1(\beta) + 1} + \sum_{i=1}^L \frac{W(\beta, i) \tilde{\tau}_1(\beta)}{E \tilde{\tau}_1(\beta) + 1} \hat{P}(\alpha, \beta_i, E). \] (3.2)

If we substitute the Laplace transformation of eq. (2.8) into eq. (3.2), it becomes linear equations of the Green functions, so that we can calculate the Green function \( \hat{P}(\alpha, \beta, E) \) itself and \( \hat{P}(\alpha, B_{\text{top}}, E) \). The calculation is described in appendix and the results are

\[ \hat{P}(\alpha, B_{\text{top}}, E) = \frac{\tilde{\tau}_1(\alpha)}{N \{ E \tilde{\tau}_1(\alpha) + 1 \} \prod_{i=1}^L Z_i(\alpha)}, \] (3.3)

and

\[ \hat{G}_{\alpha\beta}(E) = \frac{\tilde{\tau}_1(\beta) \delta_{\alpha\beta}}{E \tilde{\tau}_1(\beta) + 1} + \sum_{k=1}^L \frac{1}{N^k} \frac{\tilde{\tau}_1(\alpha) \tilde{\tau}_1(\beta) W(\beta, k) \delta_{\alpha_k\beta_k}}{\{ E \tilde{\tau}_1(\alpha) + 1 \} \{ E \tilde{\tau}_1(\beta) + 1 \} Z_k(\beta_k) \prod_{l=1}^{k-1} Z_l(\alpha_l) Z_l(\beta_l)}, \] (3.4)

where the function \( Z_k \) is defined as

\[ Z_k(B_k) \equiv 1 - [1 - \{ \tilde{\tau}_{k+1}(B_k) \}^{-1}] X_k(B_k) \quad (0 \leq k \leq L), \] (3.5)
and $X_k$ is defined by the following recursive equations,

$$X_0(B_0) \equiv \frac{1}{E + 1} \quad (3.6a)$$

$$X_{k+1}(B_{k+1}) \equiv \frac{1}{N} \sum_{B_k \in B_{k+1}} \frac{X_k(B_k)}{(1 - X_k(B_k))\tilde{\tau}_{k+1}(B_k) + X_k(B_k)}. \quad (3.6b)$$

### 3.2 Calculations of the correlation function

Let us calculate $X_k(B_k)$ in the explicit form. From eq. (3.6), we can expect that $X_k(B_k)$ does not depend on $B_k$ in the limit $N \to \infty$ (we hereafter simply denote as $X_k$) and is calculated recursively as

$$X_k = x_k I(V_k, x_k) \quad (1 \leq k \leq L), \quad (3.7)$$

$$I(V, x) \equiv \int_1^\infty \frac{du}{u} \frac{u^{-x}}{V u + 1}, \quad (3.8)$$

where

$$V_k = \frac{1 - X_{k-1}}{X_{k-1}}. \quad (3.9)$$

In the following calculations, we consider the case $x_L < \cdots < x_1 < 1$. In the assumption $E \ll 1$ (equivalent to $t_w \gg 1$), we find

$$X_k \approx 1 - C_k E^{\gamma_k} \quad (1 \leq k \leq L), \quad (3.10)$$

where

$$\gamma_k = \prod_{l=1}^k x_l, \quad (3.11a)$$

$$C_k \equiv \begin{cases} \Gamma(1 - x_1)\Gamma(1 + x_1) & (k = 1), \\ \Gamma(1 - x_k)\Gamma(1 + x_k)C_{k-1}^{x_k} & (k \geq 2). \end{cases} \quad (3.11b)$$

By using this result to eqs. (3.3) and (3.3), $\hat{P}(\alpha, B_{\text{top}}, E)$ is explicitly given as

$$\hat{P}(\alpha, B_{\text{top}}, E) \approx \frac{E^{-\gamma_L} \prod_{m=1}^{L-1} \tilde{\tau}_m(\alpha_{m-1})}{C_L N^L \{E\tilde{\tau}_1(\alpha) + 1\} \prod_{l=1}^{L-1} \{C_l E^{\gamma_l} \tilde{\tau}_{l+1}(\alpha_l) + 1\}}, \quad (3.12)$$

where we have used the fact $\tilde{\tau}_{L+1} = \infty$.

Next we estimate $\tilde{P}(k, \alpha, E')$ defined by eqs. (2.7) and (2.11). By taking the leading order of $\frac{1}{N}$ in eq. (3.4), we find

$$\tilde{P}(0, \alpha, E') \approx \frac{\tilde{\tau}_1(\alpha)}{E'\tilde{\tau}_1(\alpha) + 1}, \quad (3.13a)$$

$$\tilde{P}(k, \alpha, E') \approx \frac{\tilde{\tau}_1(\alpha)W(\alpha, k)}{\{E'\tilde{\tau}_1(\alpha) + 1\} \prod_{k=1}^{L} Z_l(\alpha_l)} \frac{1}{N_k} \sum_{\beta: d(\alpha, \beta) = k} \frac{\tilde{\tau}_1(\beta)}{\{E'\tilde{\tau}_1(\beta) + 1\} \prod_{l=1}^{k-1} Z_l(\beta_l)} \quad \begin{align*}
&\approx \frac{\tilde{\tau}_1(\alpha)W(\alpha, k)}{\{E'\tilde{\tau}_1(\alpha) + 1\} \prod_{k=1}^{L} Z_l(\alpha_l)} x_1 I(E', x_1 - 1) \prod_{l=1}^{k-1} x_{l+1} I(C_l E^{\gamma_l}, x_{l+1} - 1) \\
&= C_k \tilde{\tau}_1(\alpha) W(\alpha, k) E^{\gamma_k - 1} / \{E'\tilde{\tau}_1(\alpha) + 1\} \prod_{l=1}^{L} Z_l(\alpha_l) \quad (1 \leq k \leq L). \quad (3.13b)
\end{align*}$$
We are now in the position to calculate eq. \((2.12)\). At first, let us define a function
\[
\hat{C}_k(E' + E, E) \equiv \sum_\alpha \hat{P}(\alpha, B_{\text{top}}, E)\hat{\Pi}(k, \alpha, E').
\] (3.14)

For \(k = 0\), this function is estimated as
\[
\hat{C}_0(E' + E, E) \approx \frac{E^{-\gamma_L}}{C_L}x_1J(E, E'; x_1 - 2) \prod_{l=1}^{L-1} x_{l+1}I(C_lE^\gamma_l, x_{l+1} - 1)
\]
\[
= \hat{g}(E', E; \gamma_1) + \frac{1}{E(E' - E)},
\] (3.15)

where
\[
J(A, B; \nu) \equiv \int_1^{\infty} \frac{ds}{s} (As + 1)(Bs + 1),
\] (3.16)

and
\[
\hat{g}(E', E; \alpha) \equiv \frac{E^{\alpha-1}E^{-\alpha}}{E - E'}.
\] (3.17)

Similarly, for \(k \geq 1\)
\[
\hat{C}_k(E' + E, E) \approx \frac{E^{-\gamma_L}E^{\gamma_k} - 1}{C_L}x_1J(E, E'; x_1 - 1) \prod_{l=1}^{k-1} x_{l+1}J(C_lE^\gamma_l, C_lE^{\gamma_l}; x_{l+1} - 1)
\]
\[
\times x_{k+1}\left\{ J(C_kE^{\gamma_k}, C_kE^{\gamma_k}; x_{k+1} - 1) - J(C_kE^{\gamma_k}, C_kE^{\gamma_k}(); x_{k+1} - 1) \right\}
\]
\[
\times \prod_{m=k+1}^{L-1} x_{m+1}I(C_mE^{\gamma_m}, x_{m+1} - 1)
\]
\[
\approx \hat{g}(E', E; \gamma_{k+1}) - \hat{g}(E', E; \gamma_k),
\] (3.18)

where we have used the fact \(J(C_kE^{\gamma_k}, C_kE^{\gamma_k}; x_{k+1} - 1)\) is negligible to \(J(C_kE^{\gamma_k}, C_kE^{\gamma_k}; x_{k+1} - 2)\).

Now let us carry out the inverse Laplace transformation of eqs.\((3.15)\) and \((3.18)\). At first, \(\frac{1}{E(E' - E)}\) is transformed to 1. As for \(\hat{g}(E', E; \alpha)\), the transformation from \(E'\) to \(t\) leads us to
\[
g(t, E; \alpha) = \frac{-1}{\Gamma(1 - \alpha)} \int_0^t ds (t - s)^{-\alpha}E^{-\alpha}e^{Es}.
\] (3.19)

Then by transforming from \(E\) to \(t_w\), we find
\[
g(t, t_w; \alpha) = f\left(\frac{t}{t + t_w}; \alpha\right) - 1,
\] (3.20)

where
\[
f(x; \alpha) \equiv \frac{1}{\Gamma(\alpha)\Gamma(1 - \alpha)} \int_x^1 du (1 - u)^{\alpha-1}u^{-\alpha}.
\] (3.21)

From these results, we finally obtain
\[
C(t + t_w, t_w) = \sum_{k=0}^{L-1} q_k C_k(t + t_w, t_w)
\]
\[
= q_0 f\left(\frac{t}{t + t_w}; \gamma_1\right) + \sum_{k=1}^{L-1} q_k \left\{ f\left(\frac{t}{t + t_w}; \gamma_{k+1}\right) - f\left(\frac{t}{t + t_w}; \gamma_k\right) \right\}.
\] (3.22)
The asymptotic behavior in regimes \( t \ll t_w \) and \( t \gg t_w \) is

\[
C_0(t + t_w, t_w) \sim \begin{cases} 
1 - (t/t_w)^{1-\gamma_1} & (t \ll t_w), \\
(t/t_w)^{-\gamma_1} & (t \gg t_w),
\end{cases}
\]

(3.23a)

\[
C_k(t + t_w, t_w) \sim \begin{cases} 
(t/t_w)^{1-\gamma_k} & (t \ll t_w), \\
(t/t_w)^{-\gamma_{k+1}} & (t \gg t_w) \quad (1 \leq k \leq L - 1).
\end{cases}
\]

(3.23b)

As a result,

\[
C(t + t_w, t_w) \sim \begin{cases} 
q_0 - (q_0 - q_1)(t/t_w)^{1-\gamma_1} & (t \ll t_w), \\
q_{L-1}(t/t_w)^{-\gamma_L} & (t \gg t_w).
\end{cases}
\]

(3.24)

### 3.3 Calculations of the ZFC magnetization

Now let us explain how to introduce magnetizations to the GREM. We assign the value of magnetization to a state \( \alpha \) as

\[
M_\alpha = \mathcal{M}_0(\alpha) + \mathcal{M}_1(\alpha_1) + \cdots + \mathcal{M}_{L-1}(\alpha_{L-1}),
\]

(3.25)

where \( \mathcal{M}_k(\alpha_k) \) is a contribution from the branching point \( \alpha_k \), and is given from a distribution function \( \mathcal{D}_k(\mathcal{M}_k) \) as an independent random variable with zero mean. Note that \( \mathcal{M}_n(n = k, k + 1, \cdots) \) are common between two states \( \alpha \) and \( \beta \) if \( d(\alpha, \beta) = k \). This means that there is a correlation of magnetizations between two states, whose averaged value \( \overline{M_\alpha M_\beta} \) is

\[
\overline{M_\alpha M_\beta} = \sum_{n=k}^{L-1} \overline{\mathcal{M}_n^2} \quad (d(\alpha, \beta) = k),
\]

(3.26)

where \( \overline{\mathcal{M}_n^2} \) denotes the variance of \( \mathcal{D}_n(\mathcal{M}_n) \). Therefore we can obtain the correlation function of magnetization by setting \( q_k \) in eq. (3.25) to \( \sum_{n=k}^{L-1} \overline{\mathcal{M}_n^2} \).

The effect of magnetic field \( H \) is considered by changing \( h_1(\beta) \) into \( h_1(\beta) + HM_\beta \). As a result, the hopping rate \( W(\beta, k) \) and the transition probability from \( \beta \) to any states are modified by a factor \( \exp(-HM_\beta) \). Therefore, as mentioned in § 3, we can use the relation eq. (3.13) to calculate \( M_{ZFC}(t, t_w) \) as

\[
M_{ZFC}(t, t_w) \equiv H \int_{t_w}^{t+t_w} dt' R(t + t_w, t') = \sum_{k=0}^{L-1} \frac{\beta H \overline{M_k^2}}{\Gamma(\gamma_k) \Gamma(1 - \gamma_k)} \int_{\frac{t}{t_w}}^1 du u^{\gamma_k} (1 - u)^{-\gamma_k}. \]

(3.27)

### 3.4 Simulations

To check the validity of analytic results obtained above, we carry out simulations on the GREM with \( L = 3 \), \( x_1 = 0.6 \), \( x_2 = 0.4 \) and \( x_3 = 0.3 \) (see ref. 4 for the details of the simulation). We
first observe $C_k(t + t_w, t_w)$, the probability that the distance between the states at $t_w$ and $t + t_w$ is $k$, for $t_w = 10^2, 10^3, \ldots, 10^7$. We take a random average over $10^6$ samples. In Fig. 2, we plot $C_k(t + t_w, t_w)$ as a function of $t/t_w$ with the analytic result shown in eq. (3.22). We can see that the data are rather consistent with the analytic result.

Next we observe the ZFC magnetization $M_{ZFC}(t, t_w)$ in the case that $H/T_c(3) = 0.1$ and $M_k = 1$ $(k = 0, 1, 2)$ for $t_w = 10^2, 3 \times 10^2, 10^3, \ldots, 10^5$. The number of samples used for a random average is $10^6$. In Fig. 3, the scaling plot of $M_{ZFC}(t, t_w)$ is shown with the analytic result, eq. (3.27). The validity of the analytic result is well confirmed.

§4. Summary and Discussions

In this manuscript, we have introduced more realistic dynamics than that used in ref. 3 to the GREM, and calculated the time correlation function exactly. The results obtained by Bouchaud and Dean and ours are very similar in the sense that $t/t_w$ scaling holds in both calculations and the asymptotic behavior shown in eqs. (3.23) and (3.24) is the same. But in ref. 3 all the exponents $\gamma_k (k = 1, \ldots, L)$ in eqs. (3.23) and (3.24) are replaced with $x_k$. The reason for this difference on the exponents is that, as seen from eqs. (2.4) and (2.6), the hopping rate $W(\beta, k)$ depends on $\tilde{\tau}_i(\beta)$ $(i = 1, \ldots, k)$ in our calculation, but only on $\tilde{\tau}_k(\beta)$ in ref. 3.

We have also calculated the ZFC magnetization with the relation eq. (1.1) and shown the validity by simulation. From this result, we have confirmed that eq. (1.1) is really valid in the GREM. But, as mentioned in §1, this relation holds only on the specific models and kinetics. Actually, Cugliandolo and Kurchan proposed a different relation between the correlation function and the response function on the SK model and confirmed it numerically. It is very interesting subject to investigate how these two functions are related in other systems in non-equilibrium.

Now let us comment on two main reasons which seem to make this model solvable. The first one is that the number of branches in each layer is very large. For example, as seen from eqs. (3.3), (3.5) and (3.6), $\hat{P}(\alpha, B_{top}, E)$ depends on the full structure of the tree ($\tilde{\tau}_k$ of all the branches, to be specific). But this quantity is self-averaging and only $\tilde{\tau}_k(\alpha_{k-1})$ $(k = 1, 2, \ldots, L)$ dependence remains in the limit $N \to \infty$. The second one is the choice of the hopping rate $W(\beta, k)$. From eq. (3.12), we can see that $\hat{P}(\alpha, B_{top}, E)$ depends on $\tilde{\tau}_k(\alpha_{k-1})$ as the following form

$$
\hat{P}(\alpha, B_{top}, E) = \prod_{k=1}^{L} F_k \{ E, \tilde{\tau}_k(\alpha_{k-1}) \}.
$$

(4.1)

This means that $\hat{P}(\alpha, B_{top}, E)$ depends on $\tilde{\tau}_k(\alpha_{k-1})$ independently. As seen above, this property makes our calculation simple. We confirmed that this property relies on the choice of $W(\beta, k)$ and does not hold if, for example, we set it as

$$
W(\beta, k) = \frac{1}{\prod_{i=1}^{k-1} \tilde{\tau}_i(\beta_{i-1})}.
$$

(4.2)
In this manuscript, we have succeeded to calculate the Green function of this model. By using it directly, we may be able to carry out simulations as done in ref. 4 in longer time regime.

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Appendix: Derivation of eqs. (3.3) and (3.4)

We first prove the following equations:

\[
\hat{P}(\alpha, B_k, E) = \frac{Y_k(\alpha)}{Z_k(B_k)} \delta_{B_k, \alpha} + \sum_{i=k+1}^{L} b(B_k, i) \hat{P}(\alpha, A_i(B_k), E) \quad (0 \leq k \leq L), \tag{A.1a}
\]

\[
b(B_k, i) = \frac{[1 - \{\hat{\tau}_{i+1}[A_i(B_k)]\}^{-1}]X_k(B_k)}{Z_k(B_k) \prod_{l=k+1}^{\infty} \hat{\tau}_l[A_l-1(B_k)]}, \quad (A.1b)
\]

where \(\alpha_0 = \alpha\). The notation \(A_m(B_n) (m \geq n)\) stands for the branching point which is the ancestor of \(B_n\) and is in the \(m\)-th layer. The functions \(X_k\) and \(Z_k\) are defined by eqs. (3.6) and (3.5), and \(Y_k\) is defined as

\[
Y_0(B_0) \equiv \frac{1}{E + 1}, \quad (A.2a)
\]

\[
Y_k(\alpha) \equiv \frac{Y_0(\alpha)}{N^k \prod_{l=0}^{k-1} Z_l(\alpha_l)} \quad (1 \leq k \leq L). \tag{A.2b}
\]

We can easily check that eq. (A-1) is reduced to eq. (1.2) for \(k = 0\). Next we can prove eq. (A-1) in terms of mathematical induction by substituting eq. (A-1) for \(k = n\) into the relation

\[
\hat{P}(\alpha, B_{n+1}, E) = \frac{1}{N} \sum_{B_a \in B_{n+1}} \hat{P}(\alpha, B_n, E) \quad (1 \leq n \leq L - 1). \tag{A.3}
\]

Equation (3.3) is derived from eq. (A-1) for \(k = L\).

For other branching points, \(\hat{P}(\alpha, B_k, E)\) is given as

\[
\hat{P}(\alpha, B_k, E) = \sum_{i=k}^{L} a(B_k, i) Y_i(\alpha) \delta_{A_i(B_k), \alpha} \quad (0 \leq k \leq L), \tag{A.4}
\]

\[
a(B_k, i) = \begin{cases} 
\frac{1}{Z_k(B_k)} & (i = k), \\
\frac{X_k(B_k) [1 - \{\hat{\tau}_{i+1}[A_i(B_k)]\}^{-1}]}{Z_k(B_k) \prod_{l=k+1}^{\infty} \hat{\tau}_l[A_l-1(B_k)]} & (i > k).
\end{cases} \tag{A.5}
\]

From these equations for \(k = 0\), we can derive eq. (3.4). We hereafter prove these equations recursively. For \(k = L\), they are obviously valid from eq. (3.3). Next we show that eqs. (A-4) and (A-5) are valid for \(k = n - 1\) if they hold for \(k \geq n\). By substituting eq. (A-4) into the right hand of eq. (A-1a) for \(k = n - 1\), we find

\[
\hat{P}(\alpha, B_{n-1}, E) = \frac{Y_{n-1}(\alpha)}{Z_{n-1}(B_{n-1})} \delta_{B_{n-1}, \alpha_{n-1}} + \sum_{i=n}^{L} S(B_{n-1}, i) Y_i(\alpha) \delta_{A_i(B_{n-1}), \alpha_i}, \tag{A.6}
\]

where

\[
S(B_{n-1}, i) = \sum_{k=n}^{i} b(B_{n-1}, k) a[A_k(B_{n-1}), i], \tag{A.7}
\]
Therefore eqs. (A.4) and (A.5) are proved for $k = n - 1$ if $S(B_{n-1}, i) = a(B_{n-1}, i)$ ($i \geq n$). By substituting eqs. (A.1b) and (A.5) into eq. (A.7), it is shown that

$$S(B_{n-1}, i) = \frac{X_{n-1}(B_{n-1})[1 - \{\bar{\tau}_{i+1}[A_i(B_{n-1})]\}^{-1}]}{Z_{n-1}(B_{n-1}) \prod_{m=n}^i \bar{\tau}_m[A_m(B_{n-1})]}$$

$$\times \left\{ \frac{1}{Z_i[A_i(B_{n-1})]} + \sum_{k=n}^{i-1} \frac{1 - Z_k[A_k(B_{n-1})]}{\prod_{l=k}^i Z_l[A_l(B_{n-1})]} \right\}$$

$$= a(B_{n-1}, i), \quad (A.8)$$

where we have used eq. (3.3).

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FIGURE CAPTIONS

Fig. 1 The structure of the Generalized Random Energy Model with $L = 2$ and $N = 5$.

Fig. 2 $C_{k}(t + t_w, t_w)$ versus $t/t_w$ for $t_w = 10^2, 10^3, \ldots, 10^7$. The solid line indicates the analytic result obtained in eq. (3.22).

Fig. 3 $M_{ZFC}(t, t_w)$ versus $t/t_w$ for $t_w = 10^3, 3 \times 10^3, 10^4, \ldots, 10^6$. The solid line indicates the analytic result obtained in eq. (3.27).
Fig. 1
$C_0(t+t_w, t_w)$

Fig. 2(a)
$C_1(t + t_w, t_w)$

![Graph showing $C_1(t + t_w, t_w)$ over $t/t_w$ values ranging from $1e-6$ to $1e+6$.]

**Fig. 2(b)**
Fig. 2(c)
\( \frac{M_{\text{ZFC}}(t, t_w)}{H} \)