II. Calculation of large mass hierarchy from number of extra dimensions

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Abstract

The higher-dimensional generalization of Randall-Sundrum approach with additional positive curvature $n$-dimensional and Ricci-flat $m$-dimensional compact subspaces is considered in pure gravity theory with metric of space-time and $(p + 1)$-form potential as basic fields. Introduction of mass term of $(p+1)$-form potential into the action of co-dimension one brane permits to stabilize brane’s position and hence to calculate the value of Planck/electroweek scales ratio. There are no ad hoc too large or too small parameters in the theory; calculated mass hierarchy strongly depends on dimensionalities $m$, $n$ of additional subspaces and its observed large value in 4 dimensions (i.e. for $p = 3$) is received in particular in $D13$ ($m = 1$, $n = 7$) or $D16$ ($m = 2$, $n = 9$) space-times.

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1 Introduction

In previous Paper I [1] the mass term of the antisymmetric \((p+1)\)-differential form \(A_{p+1}\) was introduced into the brane action in scalar-gravity modification of Randall-Sundrum (RS) model [2] and it was shown that this permits to stabilize brane’s position and to calculate the observed large number of Planck/electroweek scales ratio through the moderate value of the dimensionless dilaton-antisymmetric tensor field coupling constant \(\alpha\). The theory however is not too predictive since in this model mass hierarchy is quite sensitive to the choice of \(\alpha\) (it depends non-analytically on \(\alpha^2\)) and physical grounds for \(\alpha\) to be of necessary value are unclear. The values of \(\alpha\) known from string theory or in \((p+2)\)-dimensional scalar-gravity theory received by compactification of a certain number of dimensions in higher dimensional pure gravity theory are far from the value of \(\alpha \approx 0,3\) which gives observed mass hierarchy in Paper I.

Situation however improves drastically if higher dimensional space-time contains subspace of non-zero curvature. In this paper we consider \(D\)-dimensional \((D = (p + 1) + n + m + 1)\) space-time with one extra dimension of the RS theory and two additional Euclidian subspaces: of the constant curvature one \((n\)-dimensional sphere \(S^n\) below) and \(m\)-dimensional compact Ricci flat one, e.g. \(m\)-dimensional commutative torus \(T^m\). For \(m = 1\) i.e. when \(T^m = S^1\) the observed value of mass hierarchy is received in D13 space-time \((p = 3, n = 7, m = 1)\). And we take the theory without dilaton; the basic fields are metric of \(D\)-dimensional space-time and \((p + 1)\)-differential form \(A_{p+1}\) (potential of antisymmetric field strength \(F_{p+2}\)).

The bulk space-time considered in the paper must be limited in the ”Randall-Sundrum” direction \(z\) \((0 < z_{\text{min}} < z < z_{br})\). The most natural way to limit space-time regularly ”from below”, i.e. at \(z = z_{\text{min}}\), is perhaps to consider the generalized solution with ”bolt” at this point (see in the concluding Sec. 5). However, just like in the model considered in [1], the choice of \(z_{\text{min}}\) does not influence essentially the calculated value of mass hierarchy. Crucially important for this calculation is the location of the brane limiting space-time ”from above” at \(z = z_{br}\) with \(Z_2\)-symmetry imposed at the brane. For the particular choice of additional subspaces pointed out above this co-dimension one brane has a structure of product of Lorentzian space-time \(M^{1,p}\), sphere \(S^n\) and commutative torus \(T^m\):

\[
M_{br}^{(D-1)} = M^{1,p} \times S^n \times T^m.
\]
Correspondingly there are three different Israel jump conditions - at every brane’s subspace each. To make these conditions consistent we are enforced to introduce three terms in the brane action: (1) mass term of the \((p + 1)\)-form (the introduction of such a term is a novel idea of this paper and of the paper [1]); (2) brane’s tension term (already used in [1] to make Israel conditions consistent); (3) brane’s curvature term (non-zero at the \(S^n\) subspace). These terms must be fine-tuned to each other. It is important to note however that this fine-tuning (see (24) below) does not contain too large or too small numbers. Thus dimensional constants characterizing all three terms of brane action are of one and the same scales; in the paper we consider electroweek scale as a fundamental one.

Sec. 2 presents the primary bulk and brane’s action, dynamical equations and bulk solution of the model. In Sec. 3 Israel jump conditions and \((p + 2)\)-form ”screening” condition at the brane are written down, mechanism stabilizing brane’s position is demonstrated and brane’s location is calculated. In Sec. 4 the Planck/electroweek scales ratio as a function of dimensionalities \(m, n\) is calculated. The possible generalizations of the model are discussed in Sec. 5.

### 2 Description of the model and bulk solution

Theory used in the paper is described by the action in \(D\)-dimensional space-time which includes two bulk and three (“hatted”) brane terms:

\[
S_{(D)} = M^{D-2} \int \left\{ R_{(D)}^{(D)} - \frac{1}{2(p+2)!} F_{p+2}^2 \right\} \sqrt{-g^{(D)}} \, d^Dx + GH,
\]

where ”Planck mass” \(M\) in \(D\) dimensions is supposed to be of the electroweek scale; \(R_{(D)}^{(D)}\) is scalar curvature in \(D\) dimensions; \(g_{AB}, F, A\) denote metric, \((p + 2)\)-form tensor field strength and its potential correspondingly; \(R_{br}^{(D-1)}\) is scalar curvature of the brane described by the brane’s induced metric; GH - Gibbons-Hawking surface term.

\(\hat{\mu}, \hat{\sigma}\) and \(\hat{\kappa}\) in (2) are densities located at the brane:

\[
\hat{\mu} = \frac{\mu \delta(z-z_{br})}{N}, \quad \hat{\sigma} = \frac{\sigma \delta(z-z_{br})}{N}, \quad \hat{\kappa} = \frac{\kappa \delta(z-z_{br})}{N},
\]
where \( N \) is a lapse function of \( z \)-coordinate transverse to the brane; dimensional constants \( \mu, \sigma, \kappa \) characterize \((p+1)\)-form mass term, brane’s tension and strength of brane’s gravity correspondingly.

We take the following anzats for the metric of \( D \)-dimensional space-time:

\[
\text{ds}^2_{(D)} = b^2 \text{ds}^2_{(p+1)} + a^2 d\Omega_{(n)} + c^2 \text{ds}^2_{(m)} + N^2 dz^2, \tag{4}
\]

where \( D = p + n + m + 2, \) \( \text{ds}^2_{(p+1)} \) is metric of \((p+1)\)-dimensional Lorentzian space-time \( M^{1,p}, d\Omega_{(n)} \) is metric of unit \( n \)-dimensional sphere and \( \text{ds}^2_{(m)} \) is metric of \( m \)-dimensional compact Euclidian Ricci-flat manifold. The anti-symmetric \((p+1)\)-form potential possesses non-zero component on \( M^{1,p} \):

\[
A_{p+1} = f(z) \epsilon_{\mu_1 \cdots \mu_{p+1}}, \tag{5}
\]

wherefrom

\[
F_{p+2} = f'(z) \epsilon_{\mu_1 \cdots \mu_{p+1} z} = Q \frac{b^{p+1} N}{a^n c^m} \epsilon_{\mu_1 \cdots \mu_{p+1} z}, \tag{6}
\]

prime means derivation over \( z \), the last equality in (6) is the solution of bulk ”Maxwell” equation \( (14) \) below, ”charge” \( Q \) is an arbitrary constant of the solution. From (5), (6) with account of (4) and negative signature of the Lorentzian manifold \( M^{1,p} \) it follows:

\[
\frac{A^2_{p+1}}{(p+1)!} = - \frac{f^2}{b^{2p+2}}, \tag{7}
\]

\[
\frac{F^2_{p+2}}{(p + 2)!} = - \frac{f'^2}{b^{2p+2} N^2} = - \frac{Q^2}{a^{2n} c^{2m}}. \tag{8}
\]

From three brane’s subspaces in (1) only sphere \( S^n \) gives contribution to the brane’s curvature. Hence:

\[
R^{(D-1)}_{br} = \frac{n(n - 1)}{a^2}. \tag{9}
\]

After these preliminaries we shall write down dynamical equations following from the action (2). Those equations are: constraint (10), three second order Einstein equations (11)-(13) for metric’s scale functions \( b(z), a(z), c(z) \) and equation (14) for potential \( A_{p+1} \) defined in (5). Expressions (7), (8), (9) for \( A^2, F^2 \) and brane’s curvature are taken into account in writing down Eqs.
we remind that $D = 2 = p + n + m$, prime means derivation over $z$:

$$-\frac{n(n-1)}{2a^2} + \frac{1}{2N^2} \left[ p(p+1)\frac{b^2}{b} + n(n-1)\frac{a'^2}{a^2} + m(m-1)\frac{c'^2}{c^2} 
+ 2n(p+1)\frac{b'a'}{ba} + 2m(p+1)\frac{b'c'}{bc} + 2mn\frac{a'c'}{ac} \right] = -\frac{Q^2}{4a^2c^2}; \quad (10)$$

$$\frac{1}{N^2} \left[ \frac{b''}{b} + \frac{b'}{b} \left( -\frac{N'}{N} + p\frac{b'}{b} + (n-1)\frac{a'}{a} + m\frac{c'}{c} \right) \right] =$$

$$\frac{1}{2(D-2)} \left[ -\frac{Q^2}{a^2c^2m} (n + m - 1) 
-\hat{\mu} \frac{f^2}{2b^2p+2} (2n + 2m - 1) + \hat{\sigma} + \hat{\kappa} \frac{n(n-1)}{a^2} \right]; \quad (11)$$

$$\frac{n-1}{a^2} - \frac{1}{N^2} \left[ \frac{a''}{a} + \frac{a'}{a} \left( -\frac{N'}{N} + (p+1)\frac{b'}{b} + (n-1)\frac{a'}{a} + m\frac{c'}{c} \right) \right] =$$

$$\frac{1}{2(D-2)} \left[ \frac{Q^2}{a^2c^2m} (p + 1) 
+\hat{\mu} \frac{f^2}{2b^2p+2} (2p + 1) + \hat{\sigma} - \hat{\kappa} \frac{n(n-1)}{a^2} \frac{2p + 2m + n}{n} \right]; \quad (12)$$

$$\frac{1}{N^2} \left[ \frac{c''}{c} + \frac{c'}{c} \left( -\frac{N'}{N} + (p+1)\frac{b'}{b} + (n-1)\frac{a'}{a} + (m-1)\frac{c'}{c} \right) \right] =$$

$$\frac{1}{2(D-2)} \left[ \frac{Q^2}{a^2c^2m} (p + 1) 
+\hat{\mu} \frac{f^2}{2b^2p+2} (2p + 1) + \hat{\sigma} + \hat{\kappa} \frac{n(n-1)}{a^2} \right]; \quad (13)$$
The bulk solution of these equations is given below for the choice of $z$ to be a proper distance i.e. for the choice $N = 1$ in the metric (4):

$$b = \left(\frac{z}{l}\right)^u, \quad a = \frac{m}{n + m - 1} z, \quad c = \left(\frac{z}{l}\right)^{- (n - 1)/m},$$

$$f = \left[\frac{2(p + n + m)}{(n + m - 1)(p + 1)}\right]^{1/2} \left(\frac{z}{l}\right)^{(p + 1)u} + \text{const},$$

where

$$u = \frac{(n + m - 1)(n - 1)}{(p + 1)m}$$

and length $l$ is an arbitrary constant of the solution determined by ”charge” $Q$ of the $(p + 2)$-form by the relation:

$$Q^2 = \frac{2m^{2n-2}(n - 1)^2(p + n + m)}{(n + m - 1)^{2n-1}(1)}.$$  

The important feature of this solution is the essential anisotropy in dependence on $z$ of scale factors of different subspaces of metric (4). In case $m = 0$ solution (15), (16) must come to the well known supergravity solution where $a = \text{const}$, $F_{p+2}^2 = \text{const}$ and $b$ depends on $z$ in the AdS-type exponential way. This limit is not immediately seen in (15), (16). However it becomes transparent if this solution is rewritten in the lapse-gauge $N = b^{-1};$ the generalized version of bulk solution in this gauge is written down in Sec. 5. Space-time described by the metric (15) posseses singularity at $z = 0$, hence it must be limited at some $z = z_{\text{min}} > 0$ - see discussion in Sec. 5.

### 3 Jump conditions and stabilization of brane’s position

Now we shall consider the role of ”hatted” $\delta$-function brane terms in the RHS of equations (11)-(14). Integration of these equations over brane’s po-
sition with account of definitions (3) and \( Z_2 \)-symmetry imposed at the brane gives immediately three Israel jump conditions (19)-(21) for the first derivatives of scale factors \( b, a, c \) of three brane’s subspaces and ”screening” condition (22) for \((p + 2)\)-form (cf. (5), (6)) which is a crucial one for brane’s stabilization:

\[
\frac{2}{N^2} b' = \frac{1}{2(D-2)} \left[ - \frac{\mu}{N} \frac{f^2}{2b^{2p+2}} (2n + 2m - 1) + \frac{\sigma}{N} + \frac{\kappa n(n-1)}{a^2} \right],
\]

(19)

\[
\frac{2}{N^2} a' = \frac{1}{2(D-2)} \left[ \frac{\mu}{N} \frac{f^2}{2b^{2p+2}} (2p + 1) + \frac{\sigma}{N} - \frac{\kappa n(n-1)}{N} \frac{2p + 2m + n}{a^2} \right],
\]

(20)

\[
\frac{2}{N^2} c' = \frac{1}{2(D-2)} \left[ \frac{\mu}{N} \frac{f^2}{2b^{2p+2}} (2p + 1) + \frac{\sigma}{N} + \frac{\kappa n(n-1)}{N} \frac{2p + 2m + n}{a^2} \right] - \frac{2}{N^2} f' = \frac{\mu}{N} f.
\]

(21)

Eqs. (19)-(22) are valid at the brane’s position \( z_{br} \) which is determined from (22) and (16) through dimensional constant \( \mu \) and dimensionalities \( m, n \):

\[
z_{br} = -\frac{2(n + m - 1)(n - 1)}{m\mu}.
\]

(23)

It is seen that we must put \( \mu < 0 \) in the action (2). In determining \( z_{br} \) we put equal to zero the arbitrary \( const \) in the solution for \( f \) in (16). In some future investigations this \( const \) may be considered non-zero (e.g. to make potential \( A_{p+1} \) equal to zero at \( z = z_{min} \) i.e. at the ”bolt” of generalized solution written down in Sec. 5). In any case because of growth of \( f \) with \( z \) in (16) this will give just a small corrections to the brane’s position (23) and hence to the calculated value of the Planck/electroweek scales ratio.

Substitution of the expression (23) for \( z_{br} \) and solution (16) (with \( const = 0 \) in this expression for \( f \)) into Israel jump conditions (19)-(21) shows that only two of them are independent and give the following consistency fine-tuning conditions for brane action parameters \( \mu, \sigma, \kappa \) in (2):

\[
\kappa \mu = 4, \quad \frac{\sigma}{\mu} = -\frac{n}{n-1} + \frac{(p + n + m)}{(p + 1)(n + m - 1)},
\]

(24)

where \( \mu < 0, \sigma > 0, \kappa < 0 \).
4 Calculation of mass hierarchy in 4 dimensions

To calculate from the action (2) the Planck/electroweek scales ratio (further on in this Section we shall put \((p + 1) = 4\)) we must integrate over \(z\) the "4-dimensional" term of the curvature \(R^{(D)}(D = 5 + n + m)\) in the action (2) which is equal to \(\tilde{R}^{(4)}/b^2\), where \(\tilde{R}^{(4)}\) is a scalar curvature in 4-dimensional space-time. Using metric (1) specified in (15) we get from (2):

\[
M_{Pl}^2 = M^{3+n+m}V_m\Omega_n \int_{z_{min}}^{z_{br}} b^2a^n c^m \, dz = M^{3+n+m}l^n V_m \Omega_n \left( \frac{m}{n + m - 1} \right)^n \int_{z_{min}}^{z_{br}} \left( \frac{z}{l} \right)^{2u+1} \, dz,
\]

(25)

where \(u\) is determined in (17), \(l\) - in (18), \(V_m\) is dimensional volume of compact Ricci-flat subspace, \(\Omega_n\) - volume of \(n\)-dimensional unit sphere. The choice of the lower limit \(z_{min}\) does not effect essentially the value of integral in (25) and we shall put \(z_{min} = 0\) below. Upper limit \(z_{br}\) is determined in (23). With this we receive finally from (25) the Planck/electroweek scales ratio in 4 dimensions as a function of dimensionalities \(m, n\) and dimensional constants of the theory \(M, \mu, l, V_m\):

\[
\frac{M_{Pl}}{M} = \left[ (Ml)^{(n+1)} M^n V_m \Omega_n \right]^{1/2} \left( \frac{m}{n + m - 1} \right)^{n/2} \left[ \frac{2(n + m - 1)(n - 1)}{m} \right]^{\delta},
\]

\[
\delta \equiv 1 + \frac{(n + m - 1)(n - 1)}{4m}.
\]

(26)

Following the general approach of having all arbitrary "input" dimensional constants of the primary action and of the solution to be of one and the same order and with a goal to present a quantitative estimation of the result (26) we put in (26)

\[-\mu = M, \quad l = M^{-1}, \quad V_m \Omega_n = M^{-m} \]

(27)

which gives in particular for two cases of minimal values of \(m = 1\) and \(m = 2\)
and for certain values of $n$ in each case:

1) $m = 1$:

$$\frac{M_{\text{Pl}}}{M} = 1, 5 \cdot 10^{12} \quad (n = 6); \quad 3, 1 \cdot 10^{18} \quad (n = 7); \quad 2, 4 \cdot 10^{26} \quad (n = 8). \quad (28)$$

2) $m = 2$:

$$\frac{M_{\text{Pl}}}{M} = 5, 4 \cdot 10^{12} \quad (n = 8); \quad 1, 3 \cdot 10^{17} \quad (n = 9); \quad 1, 9 \cdot 10^{22} \quad (n = 10). \quad (29)$$

Very strong dependence of mass hierarchy on dimensionalities of subspaces is perhaps the most interesting feature of the proposed approach. As it is seen from (28), (29) the value of mass hierarchy close to the observed one is received in particular in $D13 \; (m = 1, \; n = 7)$ or in $D16 \; (m = 2, \; n = 9)$ space-times.

\section{Discussion}

One of the natural ways to generalize the bulk metric \cite{1}, \cite{15} is to introduce the additional compact space-like direction with a role of Euclidian "time" like it was done e.g. in \cite{3}, \cite{4}. In Sec. 5 of Paper I \cite{11} this sort of solution was written down for scalar-gravity theory, here we shall do it for space-time with additional subspaces in a theory without dilaton considered above. Thus let us view metric $ds^2_{(p)}$ of the Lorentzian subspace of the space-time \cite{11} as a product of $p$-dimensional Lorentzian space-time $M^{1,(p-1)}$ and a circle $S^1$ with coordinate $y$. The generalization of bulk solution \cite{15} (written down in "Schwarzschild" lapse-gauge) is described by the metric:

$$ds^2_{(D)} = b^2 ds^2_{(p)} + \Delta dy^2 + \frac{dr^2}{\Delta} + a^2 d\Omega_{(n)} + c^2 ds^2_{(m)}, \quad (30)$$

where

$$b \sim r^\xi, \quad a \sim r^{1-\xi}, \quad c \sim r^{-(1-\xi)(n-1)/m}, \quad \Delta(r) = C_1 r^{2\xi} + C_2 r^{(1-p)\xi}. \quad (31)$$
In (31) \( \xi = u/(1 + u) \) and \( u \) is a function of dimensionalities \( p, n, m \) determined in (17).

Term "\( C_1 \)" in \( \Delta \) in (31) is generated by the bulk potential \( \sim F_{p+2}^2 \); in case it is the only term in \( \Delta \) Eqs. (30), (31) give the bulk solution (15) rewritten in the lapse-gauge \( N = b^{-1} \). It is easily seen from (17) that for \( m = 0 \) we have \( \xi \equiv u/(1 + u) = 1 \) and metric (30), (31) comes to the well known Schwarzschild-AdS solution:

\[
\begin{align*}
ds_{(D)}^2 &= b^2 ds_{(p)}^2 + \Delta dy^2 + \frac{dr^2}{\Delta} + a^2 d\Omega_{(n)},
\end{align*}
\]

where

\[
\begin{align*}
b \sim r, \quad a = \text{const}, \quad \Delta = C_1 r^2 + C_2 r^{1-p}.
\end{align*}
\]

Term "\( C_2 \)" in \( \Delta \) in (31) is a generalization of conventional Schwarzschild term. With a choice of arbitrary constant \( C_2 \) it is always possible to make \( \Delta(r) = 0 \) at some \( r = r_0 \) called "bolt" which, as it was said above, may be used to limit space-time regularly "from below" in the "Randall-Sundrum" direction \( r \) (also named \( z \) in the bulk of the paper).

There are only two terms in \( \Delta \) in (31) or (33) contrary to the solution given e.g. in [3] where there are four terms in \( \Delta \) - one additional term is generated by the Maxwell field and another one (which is a constant) is generated by non-zero cosmological term of the Lorentzian \( M^{1,(p-1)} \) space-time. In our case however because of presence of additional subspaces of the \( D \)-dimensional space-time the corresponding "Maxwell field" generalization of metrics (30), (31) or (32), (33) is not immediately seen. The non-zero cosmological term of the Lorentzian space-time \( M^{1,(p-1)} \) will result in the well known additional constant term in \( \Delta \) in the conventional solution (32), (33) (case \( m = 0 \)). However when there is additional subspace of dimensionality \( m > 0 \) we did not manage to find generalization of the solution (30), (31) to the case of non-zero curvature of the Lorentzian subspace \( M^{1,(p-1)} \). To find such a solution would be especially interesting in view of the discussed in [5] possibility to receive the small observed value of the positive cosmological constant from adjustment of Israel conditions in the "brane-bolt" model.

And of course it would be very interesting if some physical grounds were pointed out for appearance in the brane action of the antisymmetric tensor field mass term used in [1] and in the present paper.
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