Asymptotic expansions of complete Kähler-Einstein metrics with finite volume on quasi-projective manifolds

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Abstract We give an elementary proof to the asymptotic expansion formula of Rochon and Zhang (2012) for the unique complete Kähler-Einstein metric of Cheng and Yau (1980), Kobayashi (1984), Tian and Yau (1987) and Bando (1990) on quasi-projective manifolds. The main tools are the solution formula for second-order ordinary differential equations (ODEs) with constant coefficients and spectral theory for the Laplacian operator on a closed manifold.

Keywords asymptotic expansions, Kähler-Einstein metric, quasi-projective manifolds, complex Monge-Ampère equations, second-order ODE, Schauder estimates, spectral theory

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1 Introduction

Complete non-compact Kähler-Einstein metrics play an important role in several complex variables and geometry as observed by Fefferman [6] and Cheng and Yau [5] in the 1970s and 1980s. The existence of such metrics in strictly pseudoconvex domains with smooth boundary in \( \mathbb{C}^n \) was proved by Cheng and Yau [4] extending Yau’s solution of Calabi’s conjecture [24]. In [4], the boundary regularity for the solution is also discussed. Later, a more precise boundary regularity theorem and an asymptotic expansion of the solution near boundary were obtained by Lee and Melrose [14] in 1982. Then, the coefficients of Lee-Melrose’s expansion were calculated by Lee [13] and Graham [7] (see also the recent work of Han and Jiang [9] for another proof for the asymptotic expansion formula).

If the manifold is not a Euclidean domain, up to now, all the known examples of complete Kähler-Einstein metrics with negative Einstein constants are quasi-projective. Let \( X \) be a smooth projective manifold of complex dimension \( n \), and \( D \subset X \) be a smooth hypersurface such that \( K_X + D \) is ample. In the 1980s and 1990s, in a series of papers Cheng and Yau [5], Kobayashi [12], Tian and Yau [22], and

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Bando [2] proved that the quasi-projective manifold \( X := X \setminus D \) admits a unique complete Kähler-Einstein metric \( \omega_{KE} \) with finite volume and \( \text{Ric}(\omega_{KE}) = -\omega_{KE} \). In fact, their results also allow \( D \) to be a simple normal crossing divisor, and \( K_{X} + D \) is only big and nef, and “ample modulo \( D \)”. For the Kähler-Ricci flow approach to the existence of such metrics, please refer to the work of Lott and Zhang [15].

The asymptotic expansion of these quasi-projective Kähler-Einstein metrics was first studied by Schumacher [19] in 1998. By the adjunction formula, \( K_{D} = (K_{X} + D) |_{D} \) is ample, so Yau’s theorem guarantees the existence of a unique Kähler-Einstein metric \( \omega_{D} \) satisfying \( \text{Ric}(\omega_{D}) = -\omega_{D} \). Schumacher [19] proved that the restriction of \( \omega_{KE} \) to directions parallel to \( D \) converges to \( \omega_{D} \). Later, a systematic study was made by Wu in his thesis [23] in 2006 by analyzing the mapping property of the linearized complex Monge-Ampère operator on weighted Cheng-Yau Hölder rings. Wu [23] obtained an asymptotic expansion of the solution \( u \) to the complex Monge-Ampère equation in terms of powers of \( \sigma = (\log \| s \|^{2})^{-1} \), where \( s \) is the defining section of \( D \). However, as observed by Rochon and Zhang [17] in 2012, the \( \sigma \log \sigma \) term should appear in general, depending on the normal bundle of \( D \). In [17], a more precise asymptotic expansion was obtained by using the so-called “\( b \)-calculus”, developed by Melrose’s school.

For asymptotic expansions of other types of canonical metrics, for example complete Calabi-Yau metrics or conic Kähler-Einstein metrics, we refer the reader to the work of Santoro [18], Jeffres et al. [10] and Yin and Zheng [25].

In this paper, we will give another proof of Rochon-Zhang’s theorem by elementary tools, i.e., besides rescaled interior Schauder estimates, the key tools are spectral decomposition for Laplacian operators on closed manifolds and the elementary theory of second-order ordinary differential equations with constant coefficients (see also Andersson et al. [1], Jian and Wang [11] and Han and Jiang [8, 9] for the ODE iteration method). Even though our result is not new, this elementary approach is interesting in itself, and the authors expect it to be useful in other geometric problems.

The main theorem of this paper is as follows.\footnote{In this paper, \( O(x^{N}) \), for any real number \( N > 0 \), denotes a function \( \psi \) such that for integers \( k, l \geq 0 \),}

\section*{Theorem 1.1.}

\begin{equation}
\text{Let } \omega \text{ be the Carlson-Griffiths metric defined in (2.1) according to the normalization of Lemma 2.2. Let } \omega_{KE} = \omega + \sqrt{-1} \partial \bar{\partial} u \text{ be the unique complete Kähler-Einstein metric with finite volume on } X = X \setminus D, \text{ and let } x = (-\log r^{2})^{-1}, \text{ where } r \text{ is the distance to } D \text{ with respect to some fixed Kähler metric on } X. \text{ Then we have a poly-homogeneous asymptotic expansion for } u:
\end{equation}

\begin{equation}
u \sim \sum_{i \in I} \sum_{j=0}^{N_{i}} c_{i,j} x^{j} (\log x)^{j}, \tag{2.1}
\end{equation}

where \( I \) is the index set determined by the eigenvalues of the Laplacian operator of the unique Kähler-Einstein metric on \( D \) and \( c_{i,j} \)'s are smooth functions on \( D \), regarded as functions in a neighborhood of \( D \) via the tubular neighborhood theorem. The precise meaning of the above expansion is that

\begin{equation}
u - \sum_{i \in I, k \leq k} \sum_{j=0}^{N_{i}} c_{i,j} x^{j} (\log x)^{j} = O(x^{k+}), \tag{2.2}
\end{equation}

where \( k_{+} \) is the next term of \( k \) in \( I \).

In this paper, \( O(x^{N}) \), for any real number \( N > 0 \), denotes a function \( \psi \) such that for integers \( k, l \geq 0 \),

\begin{equation}
| (x \partial_{x})^{k} \partial_{x}^{l} \psi | \leq C_{k,l,N} x^{N}.
\end{equation}

In view of the calculations in the appendix, we also have the following theorem.\footnote{In view of the calculations in the appendix, we also have the following theorem.}

\section*{Theorem 1.2.}

The conclusion of Theorem 1.1 still holds if we replace \( x \) by \( s := (-\log \| s \|^{2})^{-1} \):

\begin{equation}
u \sim \sum_{i} \sum_{j=0}^{N_{i}} c_{i,j} s^{j} (\log s)^{j},
\end{equation}

where \( s \) is the defining section of \( D \).
Usually, the coefficients of an asymptotic expansion formula in a geometric problem will carry important geometric information. For example, the famous heat kernel expansion and the Bergman kernel expansion play very important roles in Riemannian geometry and Kähler geometry. Let us also mention the boundary asymptotic expansion of conformally compact Einstein metrics, which is very useful in conformal geometry. It is expected that the coefficients of the asymptotic expansion in Theorem 1.1 will also carry interesting geometric information. Indeed, Rochon and Zhang [17] proved that $c_{1,1}$ is uniquely determined by the normal bundle of $D$. We will give a new proof using a different method and determine the next term $c_{1,0}$ up to a (yet to be computed) global constant.

Recently, Sun and Sun [21] studied the log K-stability of the polarized Riemann surface with standard cusp singularities. An important ingredient of their proof is a precise estimate of the Bergman kernel near the cusp singularity and in the neck region, which in turn requires better asymptotic behavior of the hyperbolic metric near the singularity. The result and method of this paper should be helpful to attack the higher-dimensional log K-stability problem.

The rest of this paper is organized as follows. In Section 2, we recall the basic facts concerning the construction of finite-volume complete quasi-projective Kähler-Einstein metrics, including Cheng-Yau’s quasi-coordinate map and their Hölder spaces. We shall derive some basic properties that will be used in later sections, and obtain the leading term of the solution via Cheng-Yau’s maximum principle on complete non-compact manifolds. Then in Section 3, we compute the linearization of the associated complex Monge-Ampère equation in local holomorphic charts. Since we need to work “semi-globally”, we shall need another set of coordinates that is not holomorphic in general, namely coordinates from the tubular neighborhood theorem. Since the holomorphic version of the tubular neighborhood theorem does not hold in general, this non-holomorphic coordinate transformation causes most of the complication of this paper. The detailed computation is included in the appendix for the convenience of the readers. Then we show in Section 4 that one can derive a series of formal approximate solutions. They can be viewed as a formal asymptotic expansion. The $x^p \log x$ term and the index set appear naturally in this process. In Section 5, we will finish the proof of the main theorem by induction, using the solution formula of second-order ODEs. Finally in Section 6, we compute $c_{1,1}$ and $c_{1,0}$ in the expansion.

2 Generalities on the complete Kähler-Einstein metrics

Let $D$ be a smooth hypersurface in $\mathbb{X}$. As is well known, $D$ determines a unique holomorphic line bundle $\mathcal{O}(D)$, and $D = (s)$ is the divisor of a (unique up to a non-zero constant factor) holomorphic section $s \in H^0(\mathbb{X}, \mathcal{O}(D))$. In the following, we always assume that $L := K_{\mathbb{X}} + D$ is ample. We choose a smooth Hermitian metric $h$ on $L$ such that the curvature form $\sqrt{-1} \Theta_L > 0$. We also choose a smooth metric on $\mathcal{O}(D)$, locally of the form $e^{-\varphi}$. Locally at some point $p \in D$, we choose coordinates such that $D$ is defined by $\{z_n = 0\}$. Then $\|s\|^2 = |z_n|^2 e^{-\varphi}$.

Now consider the following metric on $X = \mathbb{X} \setminus D$ (see [3]):

$$\omega := \sqrt{-1} \Theta_L - \sqrt{-1} \partial \bar{\partial} \log \left( \log \frac{1}{\epsilon \|s\|^2} \right)^2.$$  \hfill (2.1)

Direct computation shows that when $\epsilon \ll 1$, it is indeed a complete Kähler metric with finite volume. For simplicity, we rescale $s$ by $\sqrt{\epsilon}$, and from now on, we always assume $\epsilon = 1$. As observed by Kobayashi [12] and Tian and Yau [22], $(X, \omega)$ has bounded geometry of infinite order, which means that one can find a family of holomorphic maps of maximal rank from balls of definite size in $\mathbb{C}^n$ into $X$ (the so-called “quasi-coordinates”), whose images cover $X$ such that the pull-backs of $\omega$ to the pre-images are uniformly equivalent to the standard Euclidean metric and all the derivatives of the pull-back metric tensor are uniformly bounded. If we choose local holomorphic coordinates $(z_1, \ldots, z_n)$ such that $D$ is defined by $z_n = 0$, then typical quasi-coordinates $(z_1, \ldots, z_{n-1}, w)$ near $D$ can be defined by

$$z_n = \exp \left( \frac{1 + \eta}{1 - \eta} \cdot \frac{w + 1}{w - 1} \right).$$
Lemma 2.3. For the same constant \( k \), according to Cheng and Yau [4], one can define the global Hölder norm \( ||u||_{k, \alpha} \) to be the supremum of the Euclidean \( C^{k, \alpha} \) norms of the pull-back of \( u \) on quasi-coordinate charts. We define \( C^{k, \alpha}(X) \) to be the space of \( C^k \) functions \( u \) such that \( ||u||_{k, \alpha} < \infty \). Using Cheng-Yau’s method, Kobayashi [12], Tian and Yau [22] and Bando [2] proved the following existence theorem:

**Theorem 2.1** (See [2, 12, 22]). There exists a unique complete Kähler-Einstein metric \( \omega_{KE} = \omega + \sqrt{-1} \partial \bar{\partial} u \) on \( X \) satisfying \( \text{Ric}(\omega_{KE}) = -\omega_{KE} \). Moreover, \( u \in C^{k, \alpha}(X) \) for any \( k \in \mathbb{N} \) and \( 0 < \alpha < 1 \), and \( \omega_{KE} \) is equivalent to \( \omega \).

The uniqueness follows from Yau’s Schwarz lemma. To prove the existence, one solves the complex Monge-Ampère equation

\[
\log (\omega + \sqrt{-1} \partial \bar{\partial} u)^n = u,
\]

where \( f \) is a smooth function on \( X \) such that \( \text{Ric}(\omega) + \omega = \sqrt{-1} \partial \bar{\partial} f \). As in [12, 22], \( f \in C^{k, \alpha}(X) \) for any \( k \in \mathbb{N} \) and \( 0 < \alpha < 1 \). In fact, if we write the bundle metric \( h \) locally as \( e^{-\varphi} / \phi \), where \( \Phi = (\sqrt{-1})^n \phi dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \) is a smooth volume form on \( X \), then \( f \) can be chosen as \( \log \phi \), where \( \Psi = ||s||^2 (\log ||s||^2)^2 \omega^n \). Since \( D \) is smooth and locally \( s = z_n \), direct computation shows that \( \Psi \) extends to a continuous volume form on \( X \), and hence \( f \) extends to a continuous function on \( X \). In fact, \( f \big|_D \) is a smooth function on \( D \).

**Lemma 2.2.** If we choose the bundle metric on \( L \) such that \( \sqrt{-1} \Theta_L \big|_D \) is the canonical Kähler-Einstein metric \( \omega_D \) satisfying \( \text{Ric}(\omega_D) = -\omega_D \)\(^1\) and define \( \sigma := -\log(||s||^2) \), then there is a constant \( c_0 \) such that \( f = -c_0 + O(\sigma^{-1}) \) in a neighborhood of \( D \).

**Proof.** We do calculations in local coordinates. First, it is easy to see that \( f - f \big|_D = O(\sigma^{-1}) \). So it suffices to show \( f \big|_D \equiv -c_0 \) for some constant \( c_0 \).

By direct computation, we have

\[
\Psi(z',0) = 2ne^{-\varphi(z',0)}(\sqrt{-1} \Theta_L \big|_D)^{n-1} \wedge (\sqrt{-1} dz_n \wedge d\bar{z}_n) = 2ne^{-\varphi(z',0)} \omega_D^{n-1} \wedge (\sqrt{-1} dz_n \wedge d\bar{z}_n) = \Phi(z',0) = \phi(z',0)(\sqrt{-1})^n dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.
\]

So we have \( f \big|_D(z',0) = \log \phi(z',0) + \varphi(z',0) - \log \det(\partial \bar{\partial} \varphi(z',0) + c_n. \) Consequently,

\[
\sqrt{-1} \partial \bar{\partial} \varphi f \big|_D = (\sqrt{-1} \Theta_L \big|_D + \text{Ric}(\omega_D) = \omega_D + \text{Ric}(\omega_D) = 0.
\]

This implies \( f \big|_D \equiv -c_0 \) for some constant \( c_0 \).

Using this lemma, we can find the leading-order behavior of \( u \) near \( D \).

**Lemma 2.3.** For the same constant \( c_0 \) as above, we have \( u = c_0 + O(\sigma^{-1} \log \sigma) \).

**Proof.** The main tool of our proof is the following version of Cheng-Yau’s maximum principle in [22]. Note that since \( \omega \) has bounded geometry, its curvature is bounded.

**Lemma 2.4 (Cheng-Yau’s maximum principle).** Let \((M^n, g)\) be a complete Riemannian manifold with sectional curvature bounded from below. Let \( \varphi \) be a smooth function on \( M \) such that \( \sup_M \varphi < \infty \). Then there exists a sequence of points \( \{p_i\} \subset M \) such that

\[
\lim_i \varphi(p_i) = \sup_M \varphi, \quad \lim_i |\nabla \varphi|_g(p_i) = 0, \quad \lim_i \sup_i \text{Hess} \varphi(p_i) \leq 0.
\]

Take a sufficiently small neighborhood \( U \) of \( D \) such that \( \sigma^{-1} \log \sigma \) is strictly positive on \( \partial U \). We shall find large positive constants \( a \) and \( b \) such that \( -\sigma^{-1} \log \sigma \leq u - c_0 \leq b \sigma^{-1} \log \sigma \) in \( U \). For this, we use the test functions \( M_+ := u - c_0 + a \sigma^{-1} \log \sigma \) and \( M_- := u - c_0 - b \sigma^{-1} \log \sigma \) with constants \( a, b > 0 \) to be determined later. We take \( M_- \) as an example, and the discussion for \( M_+ \) is similar.

\(^1\) This is always possible by [19].
First, we assume $M_{-} < 0$ on $\partial U$. This is true if $b$ is large enough, since $u$ is bounded. If $\sup_{U \setminus D} M_{-} \geq 0$, by Cheng-Yau’s maximum principle, we can find a sequence $p_{i} \in U \setminus D$ such that $M_{-}(p_{i}) \to \sup_{U \setminus D} M_{-}$ and $\limsup_{i} \text{Hess} M_{-}(p_{i}) \leq 0$. To make life easier, we assume $p_{i} \to D$ at present. So we have

$$\sup M_{-} = \lim(f + u)(p_{i}) + \lim(-c_{0} - f - b\sigma^{-1}\log \sigma)(p_{i})$$

$$= \lim \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^{n}}{\omega^{n}}(p_{i}) + \lim(-c_{0} - f - b\sigma^{-1}\log \sigma)(p_{i})$$

$$\leq \lim \log \frac{(\omega + b\sqrt{-1}\partial\bar{\partial}\sigma^{-1}\log \sigma)^{n}}{\omega^{n}}(p_{i}) + \lim(-c_{0} - f - b\sigma^{-1}\log \sigma)(p_{i})$$

$$\leq \lim \left(-c_{0} - f(p_{i}) - \left(\frac{3}{2}b\sigma^{-1} + o(\sigma^{-1})\right)(p_{i})\right).$$

Since $f + c_{0} = O(\sigma^{-1})$, for $b$ large enough, we must have $M_{-} \leq 0$.

In general, $p_{i}$ may converge to an interior point of $U \setminus D$, i.e., $M_{-}$ achieves its maximum somewhere. We first need to use Wu’s isomorphism theorem in [23] to get $u - c_{0} = O(\sigma^{-r})$ for any $0 < r < 1$. Then for any sufficiently small $\delta > 0$, let $U_{\delta}$ be the open subset on which $\sigma^{-1} < \delta$. Take $b_{\delta} := \max\{\delta^{-1}(-\log \delta)^{-1}\sup_{U_{\delta}}(u - c_{0}), sup_{U_{\delta}}|f + c_{0}|\}$ and set $M_{-} := u - c_{0} - b_{\delta}\sigma^{-1}\log \sigma$. Essentially the same computation shows that $\max M_{-} > 0$ will lead to a contradiction for sufficiently small $\delta > 0$. 

For the higher derivatives of $u$, we have the following lemma.

**Lemma 2.5.** Under the same assumption as in Lemma 2.2, for a solution $u$ of (2.2), we have

$$|\nabla^{k}u|_{\sigma} \leq C_{k}\sigma^{-1}\log \sigma$$

(2.3) for any integer $k \geq 1$.

**Proof.** Define $v = u - c_{0}$. By Lemma 2.3, $v = O(\sigma^{-1}\log \sigma)$. In addition, $v$ satisfies the equation

$$\log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}v)^{n}}{\omega^{n}} - v = f + c_{0}.$$ 

In quasi-coordinates, we can rewrite the equation as $A^{j_{i}}\partial_{j}v - v = f + c_{0}$, where

$$A^{j_{i}} = \int_{0}^{1} ((g_{ki} + tu_{ki})^{-1})^{i}dt.$$ 

By Theorem 2.1, we can view this as a uniformly elliptic linear equation on $v$ with smooth coefficients. Since by Lemma 2.3, $v = O(\sigma^{-1}\log \sigma)$, the lemma follows from classical interior Schauder estimates if we have the following claim.

**Claim.** For any integer $k \geq 0$, we have $|\nabla^{k}(f + c_{0})|_{\sigma} = O(\sigma^{-1})$.

We shall prove this by mathematical induction. The case $k = 0$ is proved in Lemma 2.2. Now we assume $|\nabla^{i}(f + c_{0})|_{\sigma} = O(\sigma^{-1})$ for $i = 0, \ldots, k - 1$. In quasi-coordinates $(z_{1}, \ldots, z_{n-1}, w) = (z', w)$, we have $\partial^{\alpha}z_{i}z_{j}w_{i}\partial_{i,j}(f + c_{0}) = O(\sigma^{-1})$ for any multi-index $\alpha$ and integer $j$ such that $|\alpha| + j \leq k - 1$. Now for any $\alpha, j$ with $|\alpha| + j = k - 1$, we have

$$\partial^{\alpha}_{z_{i}z_{j}}w_{i}\partial_{i,j}(f + c_{0}) = (\log |z_{n}|^{2})^{-1}a(z', w).$$ 

Since $f$ is in fact smooth with respect to $\hat{x} := (-\log |z_{n}|^{2})^{-1}$, we conclude that any derivatives of $a(z', w)$ with respect to the coordinates $(z', w)$ are still bounded. If we take another derivative in the $z'$ or $z'$ derivative, then obviously we still have $O(\sigma^{-1})$. On the other hand, it is direct to check that

$$\frac{\partial}{\partial w}(-\log |z_{n}|^{2})^{-1} = O((-\log |z_{n}|^{2})^{-1}).$$

So we have $\partial^{\alpha}_{z_{i}z_{j}}w_{i}\partial_{i,j}(f + c_{0}) = O(\sigma^{-1})$ for any multi-index $\alpha$ and integer $j$ such that $|\alpha| + j \leq k$. 

$\square$
Choose local holomorphic coordinates \((z_1, \ldots, z_n)\) such that locally \(D = \{ z_n = 0 \}\). Write \(z_n := re^{i\theta}\).
At this moment, we define \(x = (-\log r^2)^{-1}\). Another observation about \(f\) and \(u\), which is of crucial importance for our later discussions, is that they are essentially independent of \(\theta\).

**Lemma 2.6.** For any function \(v\) defined in a neighborhood of \(D\) and belonging to \(\bigcap_{k \geq 1} C^{k,\alpha}\), the intersection of all the Cheng-Yau’s Hölder spaces, we have \(\partial^k_\theta v = O(x^\infty)\), where \(O(x^\infty)\) means a function \(\psi\) satisfying \(|\partial_{z^i, z^j}^k \partial_{\bar{z}^i} \psi| \leq C_{k,l,N} x^N\) for any \(k \geq 0, l \geq 0\) and \(N \in \mathbb{N}\). In particular, we have
\[
\partial^k_\theta f = O(x^\infty), \quad \partial^k_\theta u = O(x^\infty).
\]

**Proof.** We only need to prove the case \(l = 0\). Recall that all the derivatives of \(f\) and \(u\) with respect to quasi-local coordinates are uniformly bounded. If we choose local holomorphic coordinates \((z_1, \ldots, z_n)\) such that \(D\) is defined by \(z_n = 0\), then the quasi-coordinates \((z_1, \ldots, z_{n-1}, w)\) can be defined by
\[
z_n = \exp\left(\frac{1 + \eta}{1 - \eta} \frac{w + 1}{w - 1}\right),
\]
where \(0 < \eta < 1\) and \(|w| \leq \frac{2}{3}\). Then we have
\[
\frac{\partial}{\partial \theta} = i\left(\frac{z_n}{\partial z_n} - \frac{z_n}{\partial \bar{z}_n}\right) = -i\frac{1 - \eta}{2(1 + \eta)} \left(1 - w\right)^2 \frac{\partial}{\partial w} - \left(1 - w\right)^2 \frac{\partial}{\partial \bar{w}}.
\]
On the other hand, we have
\[
x^{-1} \sim \frac{1 + \eta - |w|^2}{1 - \eta |1 - w|^2}.
\]
Direct computation shows that
\[
x^2 \frac{\partial}{\partial x} = \text{Re}\left(z_n \frac{\partial}{\partial z_n}\right) = \frac{\eta - 1}{2(1 + \eta)} \text{Re}\left(\left(1 - w\right)^2 \frac{\partial}{\partial w}\right).
\]
This implies that the coefficients of \(x^{-1} \partial_\theta\) and \(x \partial_x\) are bounded and smooth. So if \(\psi\) is in Cheng-Yau’s Hölder space \(C^{N,\alpha}(X, g_\theta)\), then \(\partial_{\bar{z}} \partial_\theta \psi = O(x^k)\) for \(k + l \leq N\). Since any such function \(\partial_{\bar{z}} \partial_\theta \psi\) must be periodic in \(\theta\), we have \(\int_0^{2\pi} \partial_{\bar{z}} \partial_\theta \psi d\theta = 0\) when \(k + 1 \geq 1\). This means that we can find (for fixed \(z'\) and \(x\)) \(\theta_0\) such that \(\partial_{\bar{z}} \partial_\theta \psi(\theta_0) = 0\). So we can integrate the \(\theta\) variable from \(\theta_0\) to get \(\partial_{\bar{z}} \partial_\theta \psi = O(x^{k'})\) for any \(1 \leq k' \leq k\).

Finally, since both \(f\) and \(u\) are in all \(C^{k,\alpha}(X)\), we get the result. \(\square\)

## 3 The linearized operator under local coordinates

In this section, we shall compute the linearized complex Monge-Ampère operator in local coordinate charts. This will be used in the next two sections to derive the asymptotic expansion. Choose local holomorphic coordinates \((z_1, \ldots, z_n)\) as before such that locally \(D = \{ z_n = 0 \}\). Recall that \(z_n := re^{i\theta}\) and \(x = (-\log r^2)^{-1}\). Also recall that
\[
\sigma = -\log \|s\|^2 = -\log r^2 + \varphi = \frac{1}{x} + \varphi = \frac{1 + x\varphi}{x}.
\]
If locally \(\Theta_L = \partial_{ij} dz_i \wedge d\bar{z}_j\), and write \(\omega := \sqrt{-1} g_{ij} dz_i \wedge d\bar{z}_j\), then we have
\[
g_{ij} = \partial_{ij} - \partial_{i} \partial_{j} \log(-\log r^2 + \varphi)^2
= \partial_{ij} - \frac{2x \varphi_{ij}}{1 + x\varphi} + V_i V_j.
\]
where
\[ V_i := \sqrt{2} \left( \frac{x^{-1}}{x-1} + \varphi \right) = \frac{\sqrt{2} x}{r} \left( -\delta_{in} \varphi - \sqrt{-\theta} + r \varphi \right) = \frac{\sqrt{2} x}{r} V_i. \]

To compute \( g^{ij} \), set
\[ g_{ij} = g^{ij} = \frac{2x \varphi_i j}{1 + x \varphi}, \]
which is positive definite near \( D \). Then direct computation shows that
\[ g^{ij} = \frac{g^{ij} p_i p_j}{1 + g^{p_i p_j}} = g^{ij} - \frac{g^{ij} p_i p_j}{r^2} + \frac{g^{ij} p_i p_j}{r^2}, \]
so we derive (with \( \alpha, \beta = 1, \ldots, n - 1 \))
\[
\begin{align*}
g^{\beta \alpha} &= \frac{g^{\beta \alpha} - \frac{g^{\beta \alpha} g^{\beta \alpha}}{g^{\alpha \alpha}} + O(r)}{g^{\alpha \alpha}} + O(r), \\
g^{\alpha \alpha} &= \left( g^{\beta \alpha} - \frac{g^{\beta \alpha} g^{\beta \alpha}}{g^{\alpha \alpha}} \right) \varphi_{\beta n} + \frac{g^{\beta \alpha}}{g^{\alpha \alpha} r^2} (1 + 2 \varphi) + O(r), \\
g^{\beta \beta} &= \left( g^{\beta \beta} - \frac{g^{\beta \beta} g^{\beta \beta}}{g^{\beta \beta}} \right) \varphi_{\beta n} + \frac{g^{\beta \beta}}{g^{\beta \beta} r^2} (1 + 2 \varphi) + O(r). \tag{3.1}
\end{align*}
\]

Notice that the \((n - 1) \times (n - 1)\) matrix \( (g^{\beta \alpha} - \frac{g^{\beta \alpha} g^{\beta \alpha}}{g^{\alpha \alpha}}) |_D \) on \( D \) is exactly the inverse of \( (g_{\alpha \beta}) |_D \). We write it as \( \eta^{\beta \alpha} \).

For an operator \( T_0 \), defined as
\[ T_0 v = \eta^{\beta \alpha} v_{\alpha \beta} + x^2 \text{Re} \left( C_{\alpha \beta} \frac{\partial v_{\alpha}}{\partial x} \right) + \frac{1}{2} x^2 v_{xx} + x v, \tag{3.2}
\]
we say \( T = T_0 + O(x) \), if \( T = T_1 + T_\infty \), where \( T_1 \) has the same form (3.2) as \( T_0 \), i.e.,
\[ T_1 v = c_1^\alpha \eta^{\beta \alpha} v_{\alpha \beta} + x^2 \text{Re} \left( c_2 C_{\alpha \beta} \frac{\partial v_{\alpha}}{\partial x} \right) + \frac{c_3}{2} x^2 v_{xx} + c_4 x v, \]
and the factors \( c_1^{\alpha \beta}, c_2, c_3 \) and \( c_4 \) are all of the form \( 1 + O(x) \). \( T_\infty \) is called an \( O(x^\infty) \) operator, acting on general functions, and satisfying \( T_\infty(u) = O(x^\infty) \), where \( u \) is a solution of (2.2). Generally, for a linear differential operator \( T_0 \) as above, we can similarly define \( T_0 + O(x) \) to be a linear differential operator of the form \( T_1 + T_\infty \), where \( T_1 \) has the same partial derivatives as \( T_0 \), but the coefficients are \( 1 + O(x) \) times the corresponding coefficients of \( T_0 \) and \( T_\infty(u) = O(x^\infty) \) for any solution \( u \) of (2.2).

**Proposition 3.1.** When acting on functions independent of \( \theta \) or functions solving (2.2), the Laplacian operator can be written as
\[ g^{\beta \alpha} \partial_{\alpha} \partial_{\beta} = \eta^{\beta \alpha} \partial_{\alpha} \partial_{\beta} + x^2 \text{Re} \left( C_{\alpha \beta} \frac{\partial^2 v_{\alpha}}{\partial x^2} \right) + \frac{1}{2} x^2 \partial_x^2 + x \partial_x + O(x), \]
where \( C_{\alpha \beta} = C_{\alpha \beta} (\cos \theta - \sqrt{-1} \sin \theta) \). Here, \( C_{\alpha \beta} \) are bounded and smooth in \( x, \theta \) and other complex coordinates, and \( (\eta^{\beta \alpha})_{1 \leq \alpha, \beta \leq n-1} \) is the inverse of the \((n - 1) \times (n - 1)\) matrix \( (\eta_{\alpha \beta}) |_D \). Note that \( \eta^{\beta \alpha} u_{\alpha \beta} \) is just the Laplacian operator on \( D \) with respect to the canonical KE (Kähler-Einstein) metric. For simplicity, we set \( \Delta_D := \eta^{\beta \alpha} \partial_{\alpha} \partial_{\beta} \).
Proof. By (3.1),

\[ g^{\alpha \beta} = \frac{r^2}{2x^2} + O \left( \frac{r^2}{x} \right), \]
\[ g^{\alpha \beta} = \left( \frac{\partial \alpha}{\partial \beta} - \frac{\partial \bar{\alpha}}{\partial \bar{\beta}} \right) \bigg|_D + O(r), \]
\[ g^{\alpha \beta} = O(r), \]
\[ g^{\beta \alpha} = O(r). \]

Since the matrix \((\frac{\partial \alpha}{\partial \beta} - \frac{\partial \bar{\alpha}}{\partial \bar{\beta}}) \bigg|_D\) is precisely the inverse of \((\theta_{\alpha \beta}) \bigg|_D\), the proposition follows easily from the observation that for mixed derivatives, we have

\[ ru_{\alpha \beta} = \frac{1}{2} \left( r (\cos \theta - \sqrt{-1} \sin \theta) \frac{\partial u_{\alpha \beta}}{\partial r} - (\sin \theta + \sqrt{-1} \cos \theta) \frac{\partial u_{\alpha \beta}}{\partial \theta} \right) \]
\[ = (\cos \theta - \sqrt{-1} \sin \theta) \left( x^2 \frac{\partial u_{\alpha \beta}}{\partial x} - \sqrt{-1} \frac{\partial u_{\alpha \beta}}{\partial \theta} \right). \]

This completes the proof. \( \square \)

What is most relevant to us is another set of non-holomorphic coordinates. By the tubular neighborhood theorem, we can find a neighborhood \( U_\delta \) of \( D \) diffeomorphic to the normal bundle of \( D \). Even though the normal bundle can be made to a complex manifold, this diffeomorphism is in general not holomorphic. The coordinates we use are bundle coordinates: locally \( z^*_\alpha (\alpha = 1, \ldots, n-1) \) are coordinates of \( D \), and \( \xi \) and \( \eta \) are fiber coordinates. We also need polar coordinates with respect to \((\xi, \eta)\), i.e., \( \xi = r^* \cos \theta^* \) and \( \eta = r^* \sin \theta^* \). Again, we set \( x^* = 1/(-\log (r^*)^2) \). Note that \( x^* \) is globally defined on \( U_\delta \), but neither \( \xi \) and \( \eta \) nor \( \theta^* \) is global. However, \( \frac{\partial}{\partial \theta^*} \) is globally defined on \( U_\delta \setminus D \).

We have the following relations between \((z_1, \ldots, z_n)\) and \((z_1^*, \ldots, z_{n-1}^*, \xi, \eta)\):

1. \( z_\alpha^* \big|_D = z_\alpha \big|_D \) for \( \alpha = 1, \ldots, n-1 \).
2. \( z_n = 0 \) if and only if \( \xi = \eta = 0 \).
3. \( r^* \) equals the distance to \( D \) with respect to some fixed good Riemannian metric, and hence is a globally defined function on \( U_\delta \).

We shall need the following three technical lemmas, whose proofs are contained in the appendix.

**Lemma 3.2.** All the derivatives of the solution \( u \) to (2.2) with respect to \( \theta^* \) are of order \( O(x^\infty) \).

**Lemma 3.3.** The Laplacian operator can also be written as

\[ \frac{\partial^2}{\partial x^2} + x^* \frac{\partial}{\partial x^*} + (x^*)^2 \left( C_{\alpha \beta} \frac{\partial^2}{\partial x \partial z^*_\alpha} \right) + \Delta_D + O(x^*), \]

where acting on \( u \) or \( \psi \) which is independent of \( \theta^* \).

**Lemma 3.4.** For any non-negative \( i \in \mathbb{R} \) and integer \( j \geq 0 \), and smooth function \( c_{i,j} \) on \( D \), we have

\[ |\nabla \nabla (c_{i,j} (x^*)^i (\log x^*)^j)|_\omega = O((x^*)^i (-\log x^*)^j). \]

In the rest of this paper, locally we always use coordinate charts like \( z^*_\alpha, x^* \) and \( \theta^* \). For simplicity, we still denote them by \( z_\alpha, x \) and \( \theta \).

To simplify notations, in the following sections of the paper, we simply write \((x^*, \theta^*)\) as \((x, \theta)\). Hopefully this will cause no confusion.

Given a function \( \psi \) in \( U_\delta \), we can average the \( \theta \)-direction by integration\(^2\). To be precise, for any fixed point \( p \in D \) with the coordinate \( z^* \), and fixed \( x \), we derive a function

\[ \tilde{\psi}(z^*, x) = \frac{1}{2\pi} \int_0^{2\pi} \psi(z^*, x, \theta) d\theta \]

\(^2\) This kind of construction has already appeared in [17].
Lemma 4.1. For any smooth function $\psi$ defined near $D$ such that $\omega + \sqrt{-1} \partial \bar{\partial} \psi$ is positive and equivalent to $\omega$, we have
\[
\left| \log \frac{(\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n}{\omega^n} - g^{ij} \psi_{ij} \right| \leq C \|
abla \nabla \psi\|^2.
\]

Proof. Write $g_t$ for the metric associated with $\omega + t \sqrt{-1} \partial \bar{\partial} \psi$. Then we have
\[
\log \frac{(\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n}{\omega^n} = \int_0^1 \frac{\partial}{\partial t} \log \det (g_{ij} + t \psi_{ij}) dt = \left( \int_0^1 g_t^{ij} dt \right) \psi_{ij}
\]
\[
\quad = g^{ij} \psi_{ij} + \left( \int_0^1 (g_t^{ij} - g^{ij}) dt \right) \psi_{ij}
\]
\[
\quad = g^{ij} \psi_{ij} + \int_0^1 \int_0^1 \frac{\partial}{\partial \tau} \psi^{ij}_\tau d\tau d\tau
\]
\[
\quad = g^{ij} \psi_{ij} - \int_0^1 t \left( \int_0^1 g^{ij}_\tau g^{pq}_\tau d\tau d\tau \right) \psi_{ij} \psi_{pq}.
\]

By our assumption, the metrics $g_{\tau \tau}$ are uniformly equivalent to $g$, so we get the conclusion from the above identity.

### 4 Constructing a formal expansion

Lemma 2.6 suggests that we should find approximate solutions of the form
\[
\psi_k = \sum_{i \in I, j \leq k} \sum_{j=0}^{N_i} c_{i,j} x^i (\log x)^j,
\]
where $I$ is an index set defined below, and $c_{i,j}$ are functions on $D$ such that
\[
Q(\psi_k) := \log \frac{(\omega + \sqrt{-1} \partial \bar{\partial} \psi_k)^n}{\omega^n} - \psi_k - f = O(x^{k+1} (\log x)^{N_k+1}).
\]

Here, integers $N_i$ can be defined inductively and explicitly if $I$ is known. $k_+$ is the first element in $I$ larger than $k$. Note that by Lemma 3.4, $\omega + \sqrt{-1} \partial \bar{\partial} \psi_k$ is uniformly equivalent to $\omega$ when $x$ is small.

The index set $I$ is defined as follows: first, we assume that $\{\lambda_k\}$’s are increasing eigenvalues of $-\Delta_D = -\eta^{0 \alpha} \partial_{\alpha} \partial_{\beta}$ on $D$. Denote the two zeros of $\frac{1}{2} k^2 + \frac{1}{2} k - 1 - \lambda_k$ by $m_k$ and $M_k$, where $m_k \geq 1$, $M_k \leq -2$ and
\[
m_k \sim \sqrt{2\lambda_k}, \quad M_k \sim -\sqrt{2\lambda_k}
\]
as $k \to \infty$. Then we define the index set $I$ as the monoid generated by $\{1\} \cup \{m_k\}_{k=1}^\infty$, and align its elements in the ascending order. The reason for using the monoid instead of just adding these numbers to $\mathbb{Z}$ is the nonlinearity of the problem. For example, when we derive the expansion recursively, each time we want to get a new term, we need to solve an equation of the form $L\psi = \eta$, where $L$ is the linearization operator and the nonlinear term $\eta$ contains multiplications of the terms we obtained earlier.

We denote by $E^\perp$ the eigenfunction space of $-\Delta_D$ with respect to the eigenvalue $\lambda$, and $E^\perp_\kappa$ its perpendicular space. We need to approximate the operator $Q(\psi)$ by its linearization and estimate its error. The following calculation is well-known.

Lemma 4.1. For any smooth function $\psi$ defined near $D$ such that $\omega + \sqrt{-1} \partial \bar{\partial} \psi$ is positive and equivalent to $\omega$, we have
\[
|\log \frac{(\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n}{\omega^n} - g^{ij} \psi_{ij}| \leq C \|
abla \nabla \psi\|^2.
\]

Proof. Write $g_t$ for the metric associated with $\omega + t \sqrt{-1} \partial \bar{\partial} \psi$. Then we have
\[
\log \frac{(\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n}{\omega^n} = \int_0^1 \frac{\partial}{\partial t} \log \det (g_{ij} + t \psi_{ij}) dt = \left( \int_0^1 g_t^{ij} dt \right) \psi_{ij}
\]
\[
\quad = g^{ij} \psi_{ij} + \left( \int_0^1 (g_t^{ij} - g^{ij}) dt \right) \psi_{ij}
\]
\[
\quad = g^{ij} \psi_{ij} + \int_0^1 \int_0^1 \frac{\partial}{\partial \tau} g^{ij}_\tau d\tau d\tau
\]
\[
\quad = g^{ij} \psi_{ij} - \int_0^1 t \left( \int_0^1 g^{ij}_\tau g^{pq}_\tau d\tau d\tau \right) \psi_{ij} \psi_{pq}.
\]

By our assumption, the metrics $g_{\tau \tau}$ are uniformly equivalent to $g$, so we get the conclusion from the above identity.
As the 0th-order approximation, we choose $\psi_0 = c_0$. Then by Lemma 2.2, we have $Q(\psi_0) = O(x)$. To find higher-order approximations, we define
\[
\mathcal{N} := \frac{1}{2} x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - 1. \]

If $\psi_1 := c_0 + c_{1,0} x$, then
\[
Q(\psi_1) = g^{ij} \partial_i \partial_j \psi_1 + O(|\nabla \nabla \psi_1|^2) - \psi_1 - f
= \Delta_D \psi_1 + \mathcal{N} \psi_1 - f + O(x^2)
= (\Delta_D c_{1,0} - \tilde{f}_1)x + O(x^2),
\]
where $\tilde{f}_1$ is defined in (3.4). However, $\Delta_D c_{1,0} = \tilde{f}_1$ is solvable if and only if $\tilde{f}_1$ is orthogonal to the eigenfunctions associated with the 0 eigenvalue of $\Delta_D$, i.e., $\int_D f d\nu_D = 0$. If this is not true, we shall need a log-term correction: set instead $\psi_1 := c_0 + c_{1,0} x + c_{1,1} x \log x$. Then
\[
Q(\psi_1) = g^{ij} \partial_i \partial_j \psi_1 + O(|\nabla \nabla \psi_1|^2) - \psi_1 - f
= \Delta_D \psi_1 + \mathcal{N} \psi_1 - f + O(x^2 \log x^2)
= \left( \Delta_D c_{1,0} + \frac{3}{2} c_{1,1} - \tilde{f}_1 \right) x + (\Delta_D c_{1,1}) x \log x + O(x^2 \log x^2).
\]

If we require $Q(\psi_1) = O(x^2 \log x^2)$, then we have
\[
\Delta_D c_{1,0} + \frac{3}{2} c_{1,1} - \tilde{f}_1 = 0, \quad \Delta_D c_{1,1} = 0. \tag{4.2}
\]
So $c_{1,1}$ must be a constant such that $\int_D \left( \frac{3}{2} c_{1,1} - \tilde{f}_1 \right) d\nu_D = 0$. Then $c_{1,0}$ is solvable and unique up to a constant. So $c_{1,0}$ cannot be determined locally, and hence we call it “the first global term”.

Now we proceed to higher-order approximations: suppose that we have already found $\psi_-$ such that $Q(\psi_-) = d_{i,j} x^i (\log x)^j + O(x^i (\log x)^{j-1})$,

where $i \in I$ and $d_{i,j}$ is a smooth function on $D$. At present we assume $j > 0$. We want to find $c_{i,j} \in C^\infty(D)$ such that for $\psi := \psi_- + c_{i,j} x^i (\log x)^j$, we have
\[
Q(\psi) = O(x^i (\log x)^{j-1}).
\]

Now we have
\[
Q(\psi) = Q(\psi_-) + \log \frac{\omega_\psi + \sqrt{-1} \partial \bar{\partial} (c_{i,j} x^i (\log x)^j)}{\omega_\psi_-} - c_{i,j} x^i (\log x)^j
= (d_{i,j} - c_{i,j}) x^i (\log x)^j + g^{qp}_{\psi_-} \partial_p \partial_q (c_{i,j} x^i (\log x)^j) + O(x^i (\log x)^{j-1}),
\]
where we use Lemma 4.1 in the second equality. Since $^{(3)}$
\[
g^{qp}_{\psi_-} - g^{qp} = - \left( \int_0^1 g^{qk}_{\psi_-} g^{lp}_{\psi_-} dt \right) \psi_{kl}
\]
and $|\nabla \nabla \psi|_\omega = O(x \log x)$, by Lemma 3.4 we have
\[
Q(\psi) = (d_{i,j} - c_{i,j}) x^i (\log x)^j + g^{qp} \partial_p \partial_q (c_{i,j} x^i (\log x)^j) + O(x^i (\log x)^{j-1})
= \left( d_{i,j} + \left( \Delta_D + \frac{1}{2} \right) c_{i,j} \right) x^i (\log x)^j + O(x^i (\log x)^{j-1}).
\]

3) Here, $g_\psi$ will mean the metric associated with $\omega_\psi + \sqrt{-1} \partial \bar{\partial} \psi$. 

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If $\frac{1}{2}i^2 + \frac{1}{2}i - 1$ is not an eigenvalue of $-\Delta_D$ or equivalently $i$ is not one of $\pi_k$’s, then we can find a unique $c_{i,j}$ such that $Q(\psi) = O(x^i(\log x)^{j-1})$.

If $i = m_l$ for some $l \in \mathbb{N}$, write $d_{i,j} = d_{i,j}^0 + d_{i,j}^1$, the orthogonal decomposition with respect to $E_{\lambda_i}$. We need to first modify $\psi_-$.

**Claim.** There is a smooth function $\rho \in C^\infty(D)$ such that

$$Q(\psi_- + \rho x^i(\log x)^{j+1}) = d_{i,j}^1 x^i(\log x)^j + O(x^i(\log x)^{j-1}).$$

In fact, the same computation as above gives

$$Q(\psi_- + \rho x^i(\log x)^{j+1}) = d_{i,j}^1 x^i(\log x)^j + \rho x^i(\log x)^{j+1} + O(\log x)^{j-1})
- \rho x^i(\log x)^{j+1} + O(x^i(\log x)^{j-1})
= \left(d_{i,j}^0 + (j + 1)\left(i + \frac{1}{2}\right)\rho\right)x^i(\log x)^j + \rho x^i(\log x)^{j+1} + O(x^i(\log x)^{j-1}).$$

We can choose $\rho = -\frac{d_{i,j}^0}{(j + 1)(i + \frac{1}{2})}$. Then we can find $c_{i,j}$ such that

$$\psi := \psi_- + c_{i,j} x^i(\log x)^j + \rho x^i(\log x)^{j+1}$$

satisfies $Q(\psi) = O(x^i(\log x)^{j-1})$. Note that in this case, $c_{i,j}$ is unique up to an element of $E_{\lambda_i}$.

When $j = 0$, initially we have $Q(\psi_-) = d_{i,0} x^i + O(x^i(\log x)^m)$, where $i_*$ is the next term of $i$ in $I$ and $m$ depends on the choice of $\psi_-$. We try to find $\psi = \psi_- + c_{i,0} x^i + \rho x^i \log x$ such that $Q(\psi) = O(x^i(\log x)^m)$. The discussion is the same as above.

**Remark 4.2.** From the above discussion, we see that $c_{\pi_l^*,0}$’s are all independent global terms. For any $l \in \mathbb{N}$, $c_{\pi_l^*,0}$ is unique up to an element in $E_{\lambda_i}$. So we have infinitely many formal solutions. There exist special formal solutions such that $x^\pi_l(\log x)^i$ appears in the formal solutions only if $\pi_l \in \mathbb{N}$. However, from our proof in the next section, other non-integer $\pi_l$’s also appear in the expansion in general.

## 5 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by induction. We shall prove that if we have an asymptotic expansion of a certain order, we can obtain a higher-order expansion. The main tools are the solution formula for second-order ODEs with constant coefficients and the method of “separation of variables”. We shall use the fact that if a function $\psi$ on $D$ has better regularity, then the “generalized Fourier series” of $\psi$ with respect to the eigenfunctions of $\Delta_D$ has better convergence properties.

We write $u$ as $u = c_0 + \tilde{u} + R$, where

$$R(x, z', \theta) := u(x, z', \theta) - \frac{1}{2\pi} \int_{S^1} u(x, z', \theta_1) d\theta_1
= \frac{1}{2\pi} \int_{S^1} \left( \int_0^1 \partial_{\theta} u(x, z', t\theta + (1 - t)\theta_1) dt \right)(\theta - \theta_1) d\theta_1,$$

which is $O(x^\infty)$ by Lemma 2.6, and

$$\tilde{u}(x, z') = \frac{1}{2\pi} \int_{S^1} u(x, z', \theta) d\theta - c_0.$$

Define a nonlinear differential operator $F(\psi) := \mathcal{N} \psi + \Delta_D \psi - Q(\psi)$. Then the equation (2.2) becomes $\mathcal{N} u + \Delta_D u = F(u)$. Taking averages with respect to $\theta$ on both sides, we have

$$\mathcal{N} \tilde{u} + \Delta_D \tilde{u} = \bar{F}(\tilde{u}),$$

(5.1)
where

\[ \tilde{F}(\tilde{v}) = \frac{1}{2\pi} \int_0^{2\pi} (F(c_0 + \tilde{v} + R) + c_0) d\theta = \frac{1}{2\pi} \int_0^{2\pi} (F(c_0 + \tilde{v}) + c_0) d\theta + O(x^\infty). \]

We can view \( \tilde{v} \) as a function on \( D \times [0, \delta] \). Even though \( F \) depends on the choice of local coordinates, \( \tilde{F}(\tilde{v}) \) is globally defined since the left-hand side of (5.1) is globally defined on \( D \times [0, \delta] \). To simplify the exposition, we introduce the following definition.

**Definition 5.1.** We say that \( \tilde{v} \) has an expansion of order \( O(x^k) \) for some \( k \in I \), if there are smooth coefficients \( c_{i,j} \) defined on \( D \) such that

\[ \tilde{v} = \psi_k + R_k = \sum_{i,l \in I, j \geq 0} c_{i,j} x^i (\log x)^j + R_k, \]

where under the local coordinate system, the remainder \( R_k \) satisfies that for some \( \epsilon \in (0, k_+ - k) \),

\[ |x^l \partial_x^m \partial_{x^2} R_k| \leq C(k, l, m, \epsilon) x^{k+\epsilon} \]

for any integers \( l \) and \( m \). Equivalently, \( R_k = O(x^{k+\epsilon}) \).

**Proof of Theorem 1.1.** It suffices to prove that \( \tilde{v} \) has an asymptotic expansion. According to Lemmas 2.3 and 2.5, \( \tilde{v} = 0 + R_0 \), where \( R_0 = \tilde{v} \) such that \( |R_0|_{C_0} \leq C_1 x(- \log x) \) for any \( l \). So we say that \( \tilde{v} \) has an expansion of order \( O(x^0) \).

Inductively, we assume that \( \tilde{v} \) has an expansion of order \( k \), and the goal is to prove the case \( k_+ \).

We adjust \( \epsilon \) to be smaller if necessary such that \( \mathfrak{m}_l > k_+ \) implies that \( \mathfrak{m}_l > k_+ + \epsilon \). Indeed, we do not have a uniform \( \epsilon \) for all \( l \).

**Lemma 5.2.** If \( \tilde{v} \) has an expansion of order \( O(x^k) \), say, \( \tilde{v} = \psi_k + R_k \), then \( \psi_k \) coincides with one of the formal approximate solutions constructed in Section 4, and \( \tilde{F}(\tilde{v}) \) has an expansion of order \( O(x^{k+\epsilon}) \), i.e.,

\[ \tilde{F}(\tilde{v}) = \sum_{i \in I, i \leq k_+, j = 0} \tilde{F}_{i,j} x^i (\log x)^j + \tilde{R}_{F,k+} \]

with \( \tilde{R}_{F,k+} = O(x^{k_++\epsilon}) \).

**Proof.** We first prove that if \( \tilde{v} = \psi_k + R_k \), then \( \psi_k \) coincides with one of the formal approximate solutions. By the discussion in Section 4, it is easy to see that we only need to prove

\[ Q(c_0 + \psi_k) = O(x^{k_++\epsilon}), \]

and then by the induction argument in Section 4, \( \psi_k \) must coincide with one of the formal approximate solutions for any \( k \in \mathbb{Z}_{\geq 0} \).

In fact, since \( u = c_0 + \psi_k + R_k + R \) satisfies \( Q(u) = 0 \), we must have

\[ 0 = Q(c_0 + \psi_k + R_k + R) = Q(c_0 + \psi_k) + \int_0^1 \frac{d}{dt} Q(c_0 + \psi_k + t(R_k + R)) dt \]

\[ = Q(c_0 + \psi_k) + \int_0^1 g^{\psi_k+t(R_k+R)} \partial_{\psi_k+t(R_k+R)} dt - R_k - R. \]

Since \( R_k + R = O(x^{k_++\epsilon}) \), from the above equation we get \( Q(c_0 + \psi_k) = O(x^{k_++\epsilon}) \).

Now we prove that \( \tilde{F}(\tilde{v}) \) has an expansion of order \( O(x^{k_+}) \) if \( \tilde{v} \) has an expansion of order \( O(x^k) \). For this, by definition, we only need to check the expansion for \( F(c_0 + \tilde{v}) \).
If \( k = 0 \), \( F(c_0 + \tilde{v}) = F(c_0 + R_0) = f + c_0 + O(x^2(\log x)^2) \), which confirms the claim. If \( k \geq 1 \),

\[
F(c_0 + \psi_k + R_k) = \Delta_D(\psi_k + R_k) + N(c_0 + \psi_k + R_k)
- Q(c_0 + \psi_k) - (Q(c_0 + \psi_k + R_k) - Q(c_0 + \psi_k))
\]

\[
= F(c_0 + \psi_k) + (\Delta_D + N + 1)R_k - \int_0^1 g_{\psi_k + 1R_k} \partial_i \partial_j R_k dt
\]

\[
= F(c_0 + \psi_k) + (\Delta_D + N + 1)R_k - g_{\tilde{\psi}} \partial_i \partial_j R_k + O(x^{k+\epsilon})
\]

\[
= F(c_0 + \psi_k) + O(x^{k+\epsilon}),
\]

where the last equality comes from Lemma 3.3. After averaged in \( \theta \), \( \tilde{F}(\psi_k) \) has an explicit expansion of any order. From the above equality, we have \( \tilde{F}(\psi_k + R_k) - \tilde{F}(\psi_k) = O(x^{k+\epsilon}) \). So we conclude that \( \tilde{F}(\tilde{v}) \) has an expansion of order \( O(x^{k+\epsilon}) \). \( \square \)

Assume \( -\Delta_D \varphi_l = \lambda_l \varphi_l \) for analytic functions \( \varphi_l \) on \( D \) with \( \int_D \varphi_l^2 dv_D = 1 \), where \( dv_D \) is the volume form associated with the canonical Kähler-Einstein metric \( \omega_D \). Then \( \{\varphi_l\}_{l=0}^{\infty} \) is an orthonormal basis of \( L^2(D, dv_D) \). As for any fixed \( x > 0 \), \( \tilde{v} \) is a smooth function on \( D \), and we write \( \tilde{v} = \sum_i \tilde{v}_l(x)\varphi_l \), where

\[
\tilde{v}_l = \int_D (\tilde{v}\varphi_l) dv_D.
\]

Then

\[
-\lambda_l \tilde{v}_l + \lambda_l \tilde{v}_l = (\tilde{F}(\tilde{v}))_l,
\]

where \( (\tilde{F}(\tilde{v}))_l(x) = \int_D \tilde{F}(\tilde{v})\varphi_l dv_D \). We view (5.3) as a non-homogeneous second-order ordinary differential equation with respect to \( x \), and then by choosing a fixed small \( x_0 > 0 \) we find that

\[
\tilde{v}_l = C_1 x_{0}^m + C_2 x_{0}^{m_1} - \frac{2x_{0}^{m_1}}{m_l - m_l} \int_x^{x_0} (\tilde{F}(\tilde{v}))_l y^{1-m_l} dy \\
+ \frac{2x_{0}^{m_1}}{m_l - m_l} \int_x^{x_0} (\tilde{F}(\tilde{v}))_l y^{1-m_l} dy.
\]

To determine \( C_1 \) and \( C_2 \), we first take \( x = x_0 \) to get \( \tilde{v}_l(x_0) = C_1 x_0^{m_1} + C_2 x_0^{m_2} \). Since \( \tilde{v}_l \in L^\infty \), we multiply (5.4) by \( x^{-m_1} \) and let \( x \to 0 \), and then

\[
0 = C_2 + \frac{2}{m_l - m_l} \int_0^{x_0} (\tilde{F}(\tilde{v}))_l y^{1-m_l} dy.
\]

We conclude

\[
\tilde{v}_l = \left( \tilde{v}_l(x_0) \cdot x_0^{m_1} + \frac{2x_{0}^{m_1}}{m_l - m_l} \int_0^{x_0} (\tilde{F}(\tilde{v}))_l y^{1-m_l} dy \right) \cdot x_0^{m_l}
\]

\[
- \frac{2x_{0}^{m_1}}{m_l - m_l} \int_x^{x_0} (\tilde{F}(\tilde{v}))_l y^{1-m_l} dy
\]

\[
- \frac{2x_{0}^{m_1}}{m_l - m_l} \int_0^{x} (\tilde{F}(\tilde{v}))_l y^{1-m_l} dy.
\]

Then

\[
\tilde{v} = \sum_l \tilde{v}_l(x_0) x_0^{m_1} \varphi_l + \sum_l \frac{2x_{0}^{m_1}}{m_l - m_l} x_0^{m_1} \varphi_l \int_0^{x_0} (\tilde{F}(\tilde{v}))_l y^{1-m_l} dy
\]

\[
- \sum_l \frac{2x_{0}^{m_1}}{m_l - m_l} \int_x^{x_0} (\tilde{F}(\tilde{v}))_l y^{1-m_l} dy
\]

\[
- \sum_l \frac{2x_{0}^{m_1}}{m_l - m_l} \int_0^{x} (\tilde{F}(\tilde{v}))_l y^{1-m_l} dy.
\]
Fix an index $A > 0$. The first summation in (5.6) is easy to treat, i.e.,

$$
\sum_{l} \tilde{v}_{l}(x_{0}) \frac{x^{m_{l}}}{x_{0}^{m_{l}}} = \sum_{m_{l} < A} \left( \frac{\tilde{v}_{l}(x_{0})}{x_{0}^{m_{l}}} \right) \cdot x^{m_{l}} + H_{A,0}x^{A},
$$

(5.7)

where

$$
H_{A,0} = \sum_{m_{l} \geq A} \frac{\tilde{v}_{l}(x_{0})}{x_{0}^{m_{l}}} x^{m_{l} - A} \varphi_{l}.
$$

This is an expansion with respect to $x$, it only has finitely many terms with $m_{l} < A$, and the corresponding coefficients only depend on $z'$. Our goal is to estimate the derivatives of the remainder of the form

$$(x \partial_{z})^{p} \partial_{z'}^{q} H_{A,0}.$$

To this end, notice that for any $N \in \mathbb{N},$

$$
|\tilde{v}_{l}(x_{0})| = \frac{1}{\lambda_{l}^{N}} \left| \int_{D} \tilde{v}(\cdot, x_{0}) \cdot \Delta_{D}^{N} \varphi_{l}(\cdot) d\nu_{D} \right| = \frac{1}{\lambda_{l}^{N}} \left| \int_{D} (\Delta_{D}^{N} \tilde{v}(\cdot, x_{0}) \cdot \varphi_{l}(\cdot)) d\nu_{D} \right| \leq C(\tilde{v}, x_{0}, N) \left| \lambda_{l}^{N} \right|.
$$

(5.8)

Here, $\Delta_{D}^{N} \tilde{v}(\cdot, x_{0})$ is evaluated at $x = x_{0}$, so bounded by the interior estimates of $\tilde{v}$. Then for any $p, q \in \mathbb{N}$, by (5.8),

$$
|(x \partial_{z})^{p} \Delta_{D}^{q} H_{A,0}| = \left| (x \partial_{z})^{p} \Delta_{D}^{q} \sum_{l} \frac{\tilde{v}_{l}(x_{0})}{x_{0}^{m_{l}}} x^{m_{l} - A} \varphi_{l} \right| = \sum_{l} \left| (m_{l} - A)^{p} \lambda_{l}^{q} \frac{\tilde{v}_{l}(x_{0})}{x_{0}^{m_{l}}} \varphi_{l} \right| \leq C(\tilde{v}, x_{0}, N) \sum_{l} \left| \lambda_{l}^{p + q - N} \varphi_{l} \right|.
$$

By the standard estimates of eigenvalues and eigenfunctions of $\Delta_{D}$ (see, for example, [20, Corollary 5.1.2]), as $\dim_{\mathbb{R}} D = 2n - 2$, $\lambda_{l} \sim l^{1+\alpha}$ as $l \to \infty$ and $|\lambda_{l}^{n-\alpha} \varphi_{l}| \leq C(\omega_{D})$. So we can set $N > \frac{p}{2} + q + \frac{2n-3}{4} + n$ so that

$$
\sum_{l} \left| \lambda_{l}^{p + q - N} \varphi_{l} \right| \leq C(N, \omega_{D}).
$$

Note that the interchange of $\Delta_{D}$ with the infinite summation is justified by the above estimate. Now $(x \partial_{z})^{p} \Delta_{D}^{q} H_{A,0}$ is bounded, which further implies that $(x \partial_{z})^{p} \Delta_{D}^{q-1} H_{A,0}$ is $C^{1,\alpha}_{w}(D)$ in $z'$ and $z$ by standard elliptic estimates. Eventually, it implies (probably with a different $q$) that

$$
|(x \partial_{z})^{p} \partial_{z'}^{q} H_{A,0}| \leq C(v, x_{0}, p, q, \omega_{D}).
$$

(5.9)

Now we need the following technical result.

**Proposition 5.3.** Fix an index $A > 0$. Assume that on $D \times [0, x_{0}]$, we have a function $F(x, z') = x^{i}(\log x)^{j} \cdot w(x, z')$ for some $i \in I$ and $j \in \mathbb{Z}$ such that $i \leq A$, $0 \leq j \leq N_{i}$ and

$$
|x^{i} \partial_{z}^{j} \partial_{z'}^{m_{l}} w| \leq C_{l, m}
$$

under the local coordinate system. In addition, we assume that $w$ only depends on $z'$ when $i < A$. 

Define
\[ F_l = \int_D F \cdot \varphi_l dv_D. \]

Then the following terms
\[
\begin{align*}
H_1 &= \sum_{i=1}^{\infty} \frac{x_i^{m_l}}{m_l!} \int_0^{x_0} y^{1-m_l} F dy,
H_2 &= \sum_{i=1}^{\infty} \frac{x_i^{m_l}}{m_l!} \int_0^{x} y^{1-m_l} F dy,
H_3 &= \sum_{i=1}^{\infty} \frac{x_i^{m_l}}{m_l!} \int_0^{x_0} y^{1-m_l} F dy,
\end{align*}
\]

have an expansion of the form:

1. if \( i = A \),
\[
\sum_{l,l < A} H_{i,0}(z') x^l + \sum_{m=0}^{j+1} H_{A,m}(x, z') x^A (\log x)^m;
\]

2. if \( i < A \),
\[
\begin{align*}
&\sum_{l,l < A} H_{i,0}(z') x^l + \sum_{m=0}^{j+1} H_{i,m}(z') x^l (\log x)^m \\
&\quad + \sum_{l,l < A} H_{i,0}(z') x^l + \sum_{m=0}^{j+1} H_{A,m}(x, z') x^A,
\end{align*}
\]

where all the coefficients \( H_{i,m} \)'s (\( l \leq A \)) satisfy that for any \( q \in \mathbb{N} \),
\[
|\partial_{x,z}^q H_{i,m}| \leq C(l, m, i, j, q).
\]

Here, \( H_{i,j+1} \) is not a zero function only if \( i = \overline{m}_l \) for some \( l \in \mathbb{N} \).

In addition, for any \( p, q \in \mathbb{N} \), we have that on \( D \times (0, x_0) \),
\[
|x^p \partial_{x,z}^q H_{i,m}| \leq C(l, m, i, j, p, q).
\]

Note that when we apply Proposition 5.3, \( x^A \) terms play the role of error terms. We first finish the proof of Theorem 1.1 assuming Proposition 5.3.

Recall that \( \tilde{F}(\tilde{v}) \) has the expansion (5.2) by induction, and \( \tilde{v} \) can be solved from \( \tilde{F}(\tilde{v}) \) by (5.6).

First apply (5.7) and (5.9) with \( A = k_+ + \epsilon \) to derive that \( \sum_i \frac{\tilde{v}(x_0) x_i^{\epsilon}}{\epsilon!} \) has a boundary expansion of order \( k_+ \). Here, \( \epsilon \) is well set such that \( \overline{m}_l < A = k_+ + \epsilon \) implies that \( \overline{m}_l \leq k_+ \).

For each term \( \tilde{F}_{i,j} x^l (\log x)^j \) or \( \tilde{R}_{F,k_+} \) in the expansion of (5.6), applying Proposition 5.3 with \( A = k_+ + \epsilon \) and \( F = \tilde{F}_{i,j} x^l (\log x)^j \), \( w = \tilde{F}_{i,j} \) or \( F = \tilde{R}_{F,k_+} \), \( w = x^{-k_+ - \epsilon} \tilde{R}_{F,k_+} \), respectively, we derive finitely many expansions in the forms of (5.10) and (5.11) with (5.13) holding.

By summing up these finitely many expansions, we see that \( v \) has a boundary expansion of order \( O(x^{k_+}) \) in the sense of Definition 5.1. Then we complete the induction.

Finally, we prove Proposition 5.3.

Proof of Proposition 5.3. First, we show the expansion and (5.12). Notice that \( F_l \)'s only depend on \( x \).

For any integer \( N > 0 \), \( |\Delta^N_F| \leq C(F, N) x^l (\log x)^j \). So we have the estimates of generalized Fourier coefficients, if \( \lambda_i \neq 0 \) and
\[
|F_l| = \frac{1}{\lambda_i} \left| \int_D F \cdot \Delta^N_D \varphi_l dv_D \right|
\]
\[
= \frac{1}{\lambda_i^N} \left| \int_D (\Delta_D^N F) \varphi l \, dv_D \right|
\leq \frac{C(F, N)}{\lambda_i^N} x^l (- \log x)^j. \tag{5.14}
\]

Now we look into the three integrals with \( H_1 \) first. Formally \( H_1 \) is already in the form of (5.10) or (5.11), as

\[
H_1 = \sum_{m_i < A} \left( \frac{x^{m_i - A} m_i - m_j}{m_i - m_j} \int_0^{y_0} y^{-1 - m} F_l \, dy \right) \cdot x^{m_i} + H_{A,0}(z', x) x^A,
\]

where

\[
H_{A,0} = \sum_{m_i \geq A} \frac{x^{m_i - A} m_i - m_j}{m_i - m_j} \int_0^{y_0} y^{-1 - m} F_l \, dy.
\]

Here, \( H_{A,0} \) is an infinite summation. Again, as \( \dim D = 2n - 2 \), we have \( \lambda_i \sim l \frac{n}{l} \), as \( l \to \infty \) and \( |\lambda_i^{2n-3} \varphi l| \leq C(\omega D) \). Then if \( m_i \geq A \), applying (5.14), we have

\[
\left| \frac{x^{m_i - A} m_i - m_j}{m_i - m_j} \varphi l \int_0^{y_0} y^{-1 - m} F_l \, dy \right| \leq \lambda_i^{-N + 2n-3} x^{m_i - A} C(F) \cdot |\lambda_i^{2n-3} \varphi l|
\leq C(F, N, \omega D, x_0) \lambda_i^{-N + 2n-3}.
\]

We set \( N = \frac{5}{2} n - 3 \). Then

\[
\sum_{m_i \geq A} \lambda_i^{-N + 2n-3} = \sum_{m_i \geq A} \lambda_i^{-2n} \leq \sum_{m_i \geq A} l^{-\frac{2n}{3}},
\]

which is bounded by a constant only depending on \( n \). Hence the sum of terms in \( H_{A,0} \) is convergent. In addition, for any \( q \in \mathbb{N} \),

\[
|\Delta_D^q H_{A,0}| = \left| \sum_{m_i \geq A} \frac{x^{m_i - A} m_i - m_j}{m_i - m_j} \lambda_i^q \varphi l \int_0^{y_0} y^{-1 - m} F_l \, dy \right|
\leq C(F, N, \omega D, x_0) \sum_{m_i \geq A} \lambda_i^{q-N + 2n-3}.
\]

We can set \( N \) much larger than \( q \) such that \( \Delta_D^q H_{A,0} \) is bounded, which further implies that \( \Delta_D^{q-1} H_{A,0} \) is \( C^{1, \alpha} \) in \( z' \) and \( z' \). This eventually implies (5.12).

The discussion of \( H_2 \) is similar. The difference here is that all the terms are of order \( x^i (- \log x)^m \) for some \( m \in [0, j] \).

• If \( i = A \), we write \( H_2 \) as \( H_{A,j} x^A (\log x)^j \). We estimate \( \Delta_D^q H_{A,j} \). By (5.14),

\[
|\Delta_D^q \sum_{l=1}^{\infty} \frac{x^{m_i - A} (\log x)^{-j} \varphi l}{m_i - m_j} \int_0^{y_0} y^{-1 - m} F_l \, dy |
\leq \sum_{l=1}^{\infty} \frac{C(F, N) \lambda_i^{N+q} \varphi l \cdot x^{m_i - A} (\log x)^{-j}}{m_i - m_j} \int_0^{y_0} y^{A-1-m} (\log y)^l \, dy
\leq \sum_{l=1}^{\infty} C(F, N, q, j) \lambda_i^{-N+q} |\varphi l|,
\]

which converges if \( N \) is large compared with \( q \) and \( n \).
Lemma 6.1. Proof. Integrating the first equation of (4.2) over $\mathbb{C}$ can be written as

$$\int_0^x y^{-1-\overline{m}_l} F_l dy = \int_D w \varphi d\nu_D \cdot \int_0^x y^{-1-\overline{m}_l} (\log y)^j dy,$$

which generates terms like $w_l \cdot x^i (\log x)^m$ for $0 \leq m \leq j$. Hence we have the expansion as (5.11).

For the estimates of coefficients, we can proceed in a similar way as in the case $i = A$ to derive (5.12).

For $H_3$, notice that when $\overline{m}_l = i$,

$$\int x^{-1-\overline{m}_l} \cdot x^i (\log x)^j dx = \frac{1}{j+1} (\log x)^{j+1}.$$

So we may have a term of order $x^i (\log x)^{j+1}$ in the expansion of $H_3$. In $H_3$, for terms with $\overline{m}_l \geq A$, we just apply (5.14) to show that

$$\sum_{\overline{m}_l \geq A} \frac{x^{\overline{m}_l}}{\overline{m}_l - \overline{m}_l} \int_x^{x_0} y^{-1-\overline{m}_l} F_l dy$$

(5.15)

can be written as

- $H_{A,m}(z', x) x A (\log x)^m$, if $i = A$,

where $m = j + 1$ if $A = \overline{m}_l$ for some $l \in \mathbb{N}$ and otherwise $m = j$.

- $\sum_{m = 0}^j H_{l,m}(z') x^i (\log x)^m + H_{A,0}(z', x) x A$, if $i < A$,

as in this case $w$ only depends on $z'$. All the coefficients $H_{l,m}$ satisfy (5.12). For the finite terms with $\overline{m}_l < A$ (essentially we do not worry about the finite summation),

- if $i = A$,

$$\sum_{\overline{m}_l < A} \frac{x^{\overline{m}_l}}{\overline{m}_l - \overline{m}_l} \int_x^{x_0} y^{-1-\overline{m}_l} F_l dy = \sum_{\overline{m}_l < A} \frac{x^{\overline{m}_l}}{\overline{m}_l - \overline{m}_l} \int_x^{x_0} y^{-1-\overline{m}_l} F_l dy - \sum_{\overline{m}_l < A} \frac{x^{\overline{m}_l}}{\overline{m}_l - \overline{m}_l} \int_0^x y^{-1-\overline{m}_l} F_l dy,$$

which can be dealt with in the same way as for $H_1$ and $H_2$;

- if $i < A$, $w$ only depends on $z'$, and we still derive (5.12) by applying the explicit integral formula of $x^{-1-\overline{m}_l} \cdot x^i (\log x)^j$ and (5.14).

Secondly, we prove (5.13). As $H_{l,m}$ is independent of $x$ if $l < A$, so we only need to consider $(x \partial_x) P H_{A,m}$. The only trouble is that $x \partial_x (x^{\overline{m}_l}) = \overline{m}_l x^{\overline{m}_l}$, which produces an extra factor $\overline{m}_l \sim \lambda^j l$. So we simply increase $N$ to deal with this factor. \qed

6 Computation of coefficients $c_{1,1}$ and $c_{1,0}$

Now we compute the coefficients of our expansion. Note that $c_{1,1}$ should be regarded as the first coefficient, since $x|\log x|$ is much larger than $x$ near the divisor $D$.

Lemma 6.1. The coefficient $c_{1,1}$ is a topological number, only depending on $D$ and its normal bundle.

Proof. Integrating the first equation of (4.2) over $D$, we have

$$c_{1,1} = \frac{2}{3 \text{vol}(D)} \int_D f_1 |_D d\nu_D.$$

Recall that

$$f = \log \frac{\Phi}{\|s\|^2 (\log \|s\|^2)^2 \omega^n}.$$
In local holomorphic coordinates, we have
\[
\text{det}(g_{ij}) = \text{det}(\partial_{ij} + V_i V_j) = \text{det}(\partial_{ij})(1 + \tilde{\omega}^2 V_i V_j).
\]

Direct computation shows that
\[
\tilde{f}_1 |_D = (\partial_x f) |_D = (\partial_x f) |_D \\
= \partial_x |_{x=0} \log \frac{\phi}{\|s\|^2 (\log \|s\|^2)^2} \text{det}(g_{ij}) \\
= 2 \left( \eta^{\alpha \beta} \frac{\partial^{\alpha \beta} \phi}{\partial^{\alpha \beta} s} \right) |_D \cdot \varphi_{pq} |_D \\
= 2 \left( \eta^{\alpha \beta} \frac{\partial^{\alpha \beta} \phi}{\partial^{\alpha \beta} s} \right) |_D \cdot \varphi_{\alpha \beta} |_D \\
= 2 \eta^{\beta \alpha} \varphi_{\alpha \beta} |_D.
\]

It is essentially the trace of \((\sqrt{1 - \bar{\partial} \partial} \varphi) |_D\) with respect to \(\omega_D\), up to a constant factor. Since the restriction of the line bundle \(\mathcal{O}(D)\) to \(D\) is just the normal bundle of \(D\) in \(\mathbb{X}\), denoted by \(N_D\), and \([\omega_D] = 2 \pi c_1 (K_D)\), we have
\[
c_{1,1} = \frac{4(n-1)}{3} \cdot \frac{K_{n-2}^D \cdot N_D}{K_{n-1}^D}.
\]

Here, we use the fact that the normal bundle \(N_D \cong \mathcal{O}(D) |_D\) and \(K_D \cong \mathcal{O}(K_X + D) |_D\).

\[
\text{Note that the appearance of the } x \log x \text{ term and its coefficients has already been pointed out by Rochon and Zhang [17]}.^4
\]

Now we try to compute \(c_{1,0}\): recall that \(c_{1,0}\) satisfies the equation (4.2), i.e.,
\[
\eta^{\beta \alpha} \partial_\alpha \partial_\beta c_{1,0} = \tilde{f}_1 - \frac{3}{2} c_{1,1} = \eta^{\beta \alpha} \left( 2 \bar{\varphi}_{\alpha \beta} - \frac{3}{2(n-1)} c_{1,1} \eta_{\alpha \beta} \right).
\]

If we denote the Green function of \(\Delta_D\) on \(D\) by \(G_D(x, y)\), then we have
\[
c_{1,0} = c + \int_D G_D(x, y) \left( \tilde{f}_1 - \frac{3}{2} c_{1,1} \right) (y) d\nu_D(y).
\]

When \(D\) is a pluri-canonical or pluri-anticanonical divisor, \(2 \sqrt{-\bar{\partial} \partial} \varphi |_D = -\frac{3}{2(n-1)} c_{1,1} \omega_D\) is a coboundary, and in that case, we can write down a precise local formula for the convolution part up to a constant.

In any case, the determination of the constant \(c\) is non-trivial, since it depends on global geometry instead of local quantities. When \(n = 1\), using modular functions, Qian [16] managed to obtain rather precise information for the coefficients. However, her method does not work when \(n \geq 2\). We shall leave the calculation of this constant to a future work.

**Remark 6.2.** Since the choice of the Carlson-Griffiths reference metric is not unique, we can rescale the metric on \(\mathcal{O}(D)\) by \(e^F\) and rescale the metric on \(K_X\) by \(e^{-F}\). This keeps the metric on \(L\) unchanged, but the Carlson-Griffiths metric changes to \(\omega + \sqrt{-\bar{\partial} \partial} (2xF |_D + O(x^2))\) and hence \(c_{1,0}\) changes to \(c_{1,0} - 2F |_D\). In this way, from the above calculation we can actually choose an \(F\) such that \(c_{1,0}\) is a constant. The positivity of the new reference metric can be guaranteed by rescaling \(s\), or equivalently, rescaling the bundle metric of \(\mathcal{O}(D)\) by a sufficiently small constant.

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^4 Since Rochon-Zhang’s \(x\) is 2-times our \(x\), their result is \(\frac{2(n-1)}{3} \cdot \frac{K_{n-2}^D \cdot N_D}{K_{n-1}^D}\).

^5 In particular, the observation in the final remark in Section 6 is due to him.
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Appendix A Proofs of Lemmas 3.2–3.4

Proof of Lemma 3.2. By the chain rule, we have

$$\frac{\partial}{\partial \theta} = \frac{\partial z^*_\alpha}{\partial \theta} \frac{\partial}{\partial z^*_\alpha} + \frac{\partial z^*_\alpha}{\partial \theta} \frac{\partial}{\partial z^*_\alpha} + \frac{\partial x^*}{\partial \theta} \frac{\partial}{\partial x^*} + \frac{\partial \theta^*}{\partial \theta} \frac{\partial}{\partial \theta^*}.$$ 

It is easy to see that the coefficients of the first 3 terms are all of order $O(r)$, while $\frac{\partial \theta^*}{\partial \theta}$ is of order $O(1)$ and non-vanishing near $D$. So we can prove by induction that derivatives with respect to $\theta^*$ are also of order $O(x^\infty)$.

$\square$
Proof of Lemma 3.3. By Proposition 3.1, we first check
\[
\frac{\partial}{\partial x} = \frac{\partial z^*_\alpha}{\partial x} \frac{\partial}{\partial z^*_\alpha} + \frac{\partial z^*_\alpha}{\partial x} \frac{\partial}{\partial z^*_\alpha} + \frac{\partial z^*}{\partial x} \frac{\partial}{\partial z^*} + \frac{\partial \theta^*}{\partial x} \frac{\partial}{\partial \theta^*}
\]
term by term, i.e.,
\[
\frac{\partial z^*_\alpha}{\partial x} = \frac{\partial z^*_\alpha}{\partial r} \frac{\partial r}{\partial x} = \frac{r}{2} \frac{\partial z^*_\alpha}{\partial r} = O(x^\infty).
\]
Similar to \(\frac{\partial z^*}{\partial x}\),
\[
\frac{\partial \theta^*}{\partial x} = \frac{\partial \theta^*}{\partial r} \frac{r}{2} = \frac{r}{2} \frac{\partial z^*_n}{\partial r} = O(x^{-2}).
\]
By the previous lemma, we can ignore all the derivatives with respect to \(\theta^*\), too. So we only need to compute \(\frac{\partial x^*}{\partial x}\). Note that we have the Taylor expansion near \(D\):
\[
r^* = Ar + O(r^2),
\]
where \(A > 0\) is a locally defined function on \(D\). Then we have
\[
\frac{\partial x^*}{\partial x} = \frac{dx^*}{dr^*} \frac{dr^*}{\partial x} \frac{r(x^*)^2}{x^* x^*} \frac{\partial r^*}{\partial r} = A + O(r) + 1 + 4 \log A x + O(x^2),
\]
so
\[
\frac{\partial^2 x^*}{\partial x^2} = 4 \log A + O(x) + 4 \log A + O(x^2).
\]
Then we conclude that
\[
\left(\frac{\partial}{\partial x^2} + \frac{x}{\partial x}\right) v = \left(\frac{\partial}{\partial x^2} + x^* \frac{\partial}{\partial x^*}\right) v + O(x^2).
\]
Secondly, we check tangential directions, i.e.,
\[
\frac{\partial}{\partial z^\alpha} = \frac{\partial z^*_{\alpha}}{\partial z^\alpha} \frac{\partial}{\partial z^*_{\alpha}} + \frac{\partial z^*_{\alpha}}{\partial z^\alpha} \frac{\partial}{\partial z^*_{\alpha}} + \frac{\partial z^*}{\partial z^\alpha} \frac{\partial}{\partial z^*} + \frac{\partial \theta^*}{\partial z^\alpha} \frac{\partial}{\partial \theta^*}.
\]
We compute it term by term.
First, we can write \(z^*_\beta = z_\beta + a_\beta(z)\), where \(a_\beta\) is a local smooth function of \(z\) such that \(a_\beta(z_1, \ldots, z_{n-1}, 0) \equiv 0\). This implies
\[
\frac{\partial a_\beta}{\partial z^\alpha} = O(r), \quad \frac{\partial a_\beta}{\partial z^\alpha} = O(r),
\]
so
\[
\frac{\partial z^*_\beta}{\partial z^\alpha} = \delta_{\alpha \beta} + O(r), \quad \frac{\partial z^*}{\partial z^\alpha} = O(r).
\]
For the similar reason, all the higher-order purely tangential derivatives of \(z^*_\alpha\) are of order \(O(r)\). Second, we have
\[
\frac{\partial x^*}{\partial z^\alpha} = \frac{2(x^*)^2}{r^*} \left( \cos \theta^* \frac{\partial \xi}{\partial z^\alpha} + \sin \theta^* \frac{\partial \eta}{\partial z^\alpha} \right).
\]
Recalling that \(\xi |_D = 0\), we conclude that \(\frac{\partial \xi}{\partial z^\alpha}\) (and in fact all the tangential derivatives) is of order \(O(x^*)\), from which we conclude that
\[
\frac{\partial x^*}{\partial z^\alpha} = O((x^*)^2).
\]
Similarly,
\[
\frac{\partial \theta^*}{\partial z^\alpha} = \frac{1}{r^*} \left( -\sin \theta^* \frac{\partial \xi}{\partial z^\alpha} + \cos \theta^* \frac{\partial \eta}{\partial z^\alpha} \right) = O(1).
\]
From the expression of \(\frac{\partial x^*}{\partial z^\alpha}\), we can further calculate \(\frac{\partial^2 x^*}{\partial z^\alpha \partial z^\beta}\) by the chain rule, and it is easy to see that all but one term are of order \(O((x^*)^2)\). The remaining term is
\[
- \frac{\partial x^*}{\partial z^\alpha} \frac{1}{r^*} \frac{\partial r^*}{\partial z^\beta} = - \frac{\partial x^*}{\partial z^\alpha} \frac{1}{r^*} \left( \cos \theta^* \frac{\partial \xi}{\partial z^\beta} + \sin \theta^* \frac{\partial \eta}{\partial z^\beta} \right).
\]
This is again of order \(O((x^*)^2)\). For the same reason,

\[
\frac{\partial^2 \theta^*}{\partial z_\alpha \partial z_\beta} = O(1).
\]

So we get

\[
\frac{\partial}{\partial z_\alpha} = \frac{\partial}{\partial x^*} + O((x^*)^2) \frac{\partial}{\partial x^*} + O((x^*)^\infty) \text{ operators.}
\]

For second-order derivatives, we have

\[
\frac{\partial^2}{\partial z_\alpha \partial z_\beta} = \left(\frac{\partial^2}{\partial z_\alpha \partial z_\beta} + \frac{\partial}{\partial z_\alpha} \frac{\partial}{\partial z_\beta} + \frac{\partial x^*}{\partial z_\alpha} \frac{\partial x^*}{\partial z_\beta} + \frac{\partial \theta^*}{\partial z_\alpha} \frac{\partial \theta^*}{\partial z_\beta}\right)
\]

\[
= \frac{\partial^2}{\partial z_\alpha \partial z_\beta} + \frac{\partial}{\partial z_\alpha} \frac{\partial}{\partial z_\beta} + \frac{\partial x^*}{\partial z_\alpha} \frac{\partial x^*}{\partial z_\beta} + \frac{\partial \theta^*}{\partial z_\alpha} \frac{\partial \theta^*}{\partial z_\beta}.
\]

\[
+ O((x^*)^2) \left(\frac{\partial^2}{\partial x^* \partial x^*} + \frac{\partial}{\partial x^*} + \frac{\partial^2}{\partial x^* \partial x^*} + \frac{\partial^2}{\partial x^* \partial x^*}\right)
\]

\[
+ O((x^*)^\infty) \text{ operators.}
\]

Next, we compute mixed derivatives, i.e.,

\[
\frac{\partial}{\partial z_\alpha} = \frac{\partial^2}{\partial z_\alpha \partial z_\beta} + \frac{\partial^2}{\partial z_\alpha \partial z_\beta} + \frac{\partial x^*}{\partial z_\alpha} \frac{\partial x^*}{\partial z_\beta} + \frac{\partial \theta^*}{\partial z_\alpha} \frac{\partial \theta^*}{\partial z_\beta}.
\]

It is obvious that

\[
\frac{\partial z_\beta}{\partial z_\alpha} = O(1), \quad \frac{\partial z_\beta}{\partial z_\alpha} = O(1),
\]

\[
\frac{\partial x^*}{\partial z_\alpha} = \frac{2(x^*)^2}{r^*} \left(\cos \theta^* \frac{\partial \xi}{\partial z_\alpha} + \sin \theta^* \frac{\partial \eta}{\partial z_\alpha}\right) = O((x^*)^2(r^*)^{-1}),
\]

\[
\frac{\partial \theta^*}{\partial z_\alpha} = \frac{1}{r^*} \left(\frac{\partial \xi}{\partial z_\alpha} + \cos \theta^* \frac{\partial \eta}{\partial z_\alpha}\right) = O((r^*)^{-1}).
\]

By the chain rule, we can see that

\[
z_n \frac{\partial^2 x^*}{\partial z_\alpha \partial z_\beta} = z_n \frac{2(x^*)^2}{r^*} \left(\cos \theta^* \frac{\partial^2 \xi}{\partial z_\alpha \partial z_\beta} + \sin \theta^* \frac{\partial^2 \eta}{\partial z_\alpha \partial z_\beta}\right)
\]

\[
+ \frac{2z_n}{r^*} \left(\cos \theta^* \frac{\partial \xi}{\partial z_\alpha} + \sin \theta^* \frac{\partial \eta}{\partial z_\alpha}\right) \frac{\partial \theta^*}{\partial z_\beta}
\]

\[
+ \frac{2z_n}{r^*} \frac{\partial x^*}{\partial z_\alpha} \frac{\partial x^*}{\partial z_\beta} = O((x^*)^2) + O((x^*)^3) + O((x^*)^2) = O((x^*)^2).
\]

Similarly, we have

\[
z_n \frac{\partial^2 \theta^*}{\partial z_\alpha \partial z_\beta} = O(1).
\]
Since in the mixed second-order derivatives, $\frac{\partial}{\partial z_n}$ always goes with $z_n$, and all the derivatives involving $\theta^*$ can be ignored, we have

$$z_n \frac{\partial^2}{\partial z_n \partial z_{\beta}} = z_n \left( \frac{\partial z_n^*}{\partial z_n} \frac{\partial}{\partial z_n} + \frac{\partial z_n^*}{\partial z_n} \frac{\partial}{\partial z_n} + \frac{\partial x^*}{\partial z_n} \frac{\partial}{\partial z_n} + \frac{\partial \theta^*}{\partial z_n} \frac{\partial}{\partial z_n} \right)$$

Then the lemma follows from (3.1).

Finally, we have

$$\text{Proof of Lemma 3.4.} \quad \text{We use the local holomorphic coordinates to check this. First, we have}$$

$$\frac{\partial^2}{\partial z_n \partial z_{\beta}} (c_{i,j}(z_1^*, \ldots, z_{n-1}^*) (x*)^i (\log x*)^j) = \frac{\partial^2}{\partial z_n \partial z_{\beta}} (x*)^i (\log x*)^j + O((x*)^i (\log x*)^j).$$

Recall that

$$\frac{\partial x^*}{\partial z_n} = O((x*)^2), \quad \frac{\partial}{\partial z_n} = \frac{\partial}{\partial z_n} + O((x*)^2) \frac{\partial}{\partial x^*} + O((x*)^\infty) \text{ operators},$$

so we have

$$\frac{\partial^2}{\partial z_n \partial z_{\beta}} (c_{i,j}(z_1^*, \ldots, z_{n-1}^*) (x*)^i (\log x*)^j) = O((r*)^{-1} (x*)^i+1 (\log x*)^j).$$

Finally, we have

$$\frac{\partial^2}{\partial z_n \partial z_{\beta}} (c_{i,j}(z_1^*, \ldots, z_{n-1}^*) (x*)^i (\log x*)^j) = O((x*)^i (\log x*)^j).$$

Then the lemma follows from (3.1).