Note on a conjecture of Bateman and Diamond concerning the abstract PNT with Malliavin-type remainder

Frederik Broucke 1

Received: 11 December 2020 / Accepted: 22 June 2021 / Published online: 30 June 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH Austria, part of Springer Nature 2021

Abstract

Given $\beta \in (0, 1)$, we show the existence of a Beurling generalized number system whose integer counting satisfies $N(x) = ax + O(x \exp(-c \log^{\beta} x))$ for some $a > 0$ and $c > 0$, and whose prime counting function satisfies $\pi(x) = \text{Li}(x) + \Omega(x \exp(-c'(\log x)^{\frac{\beta}{\beta + 1}}))$ for some $c' > 0$. This is done by generalizing a construction of Diamond, Montgomery, and Vorhauer. This Beurling system serves as additional motivation for a conjecture of Bateman and Diamond from 1969, concerning the prime number theorem with Malliavin-type remainder.

Keywords Beurling generalized prime numbers · Bateman–Diamond conjecture · PNT with Malliavin remainder · Diamond–Montgomery–Vorhauer probabilistic method

Mathematics Subject Classification Primary 11M41 · 11N80; Secondary 11M26 · 11N05

1 Introduction

A. Beurling [2] defined a system of generalized primes $\mathcal{P}$ as an unbounded sequence $(p_j)_{j \geq 1}$ satisfying $1 < p_1 \leq p_2 \leq \ldots$. The corresponding system of generalized integers $\mathcal{N}$ is the multiplicative semigroup generated by 1 and $\mathcal{P}$. With these systems one associates functions $\pi(x)$ and $N(x)$ counting the number of primes, respectively.

---

1 Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Krijgslaan 281, 9000 Ghent, Belgium
integers, not exceeding \(x\), taking multiplicities into account for those integers having multiple representations as a product of generalized primes.

A substantial part of the theory of Beurling generalized primes focuses on the asymptotic behavior of the counting functions \(\pi\) and \(N\) and how one is related to the other. This was in fact Beurling’s main motivation; in his seminal paper [2], he showed the following abstract form of the prime number theorem (PNT). Suppose that for some \(a > 0\) and \(\gamma > 3/2\), \(N(x) = ax + O\left(x^{\gamma/2}\right)\). Then \(\pi(x) \sim x/\log x\) holds. In the same article, it was also shown that the condition \(\gamma > 3/2\) is sharp (see also [5]). The first abstract PNT with remainder actually predates Beurling’s paper in some sense, since the methods of Landau’s work on the prime ideal theorem [11], cast into the language of Beurling generalized primes, yield the following theorem (PNT with de la Vallée Poussin remainder). Suppose that \(N(x) = ax + O\left(x^{\theta}\right)\) holds for some \(a > 0\) and \(\theta < 1\). Then there is a constant \(c > 0\) such that \(\pi(x) = \text{Li}(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right)\).

Here \(\text{Li}\) represents the logarithmic integral, which we choose to define in this article as

\[
\text{Li}(x) := \int_{1}^{x} \frac{1-u^{-1}}{\log u} \, du.
\]

Landau’s PNT was shown to be optimal by Diamond, Montgomery, and Vorhauer [6] in the case \(\theta > 1/2\). Namely, they showed the existence of a system of Beurling generalized primes satisfying \(N(x) = ax + O\left(x^{\theta}\right)\) and \(\pi(x) = \text{Li}(x) + \Omega\left(x \exp\left(-c\sqrt{\log x}\right)\right)\). (The relation \(f = g + \Omega(h)\) means that \((f - g)/h \not\to 0\).)

More general remainders (which we shall refer to as Malliavin remainders) were considered by Malliavin, namely

\[
\pi(x) = \text{Li}(x) + O\left(x \exp(-c \log^\alpha x)\right) \quad \text{for some } c > 0,
\]

and

\[
N(x) = ax + O\left(x \exp(-c' \log^\beta x)\right) \quad \text{for some } a, c' > 0,
\]

with \(0 < \alpha, \beta \leq 1\). Malliavin showed [12] that \((P_\alpha)\) implies \((N_\beta)\) with \(\beta = \alpha/(\alpha + 2)\), and that \((N_\beta)\) implies \((P_\alpha)\) with \(\alpha = \beta/10\). The first result was improved by Diamond [4], who showed that \((P_\alpha)\) implies \((N_\beta)\) with \(\beta = \alpha/(\alpha + 1)\) and furthermore with \((\log x \log \log x)^\beta\) instead of \((\log x)^\beta\) in the exponential, see also [9] for the case \(\alpha = 1\). Diamond’s improved result has recently been shown to be optimal for \(\alpha = 1\) by the author together with Debruyne and Vindas in [3]. In an upcoming article, the same authors will show that the theorem of Diamond is optimal for every \(0 < \alpha \leq 1\). In the converse direction, the value for \(\alpha\) was improved to \(\alpha = 7.91\) by Hall in [8]. (A slight refinement of his argument actually yields \(\alpha = \beta/(\beta + 6.91)\), se e.g. [7, Section 16.4].) Hall’s proof consists of a Tauberian argument combined with bounds on the zeta function which are obtained via a generalization of the familiar “3-4-1-
inequality”. The value 6.91 arises from a specific choice of a positive trigonometric polynomial.¹

Let us denote by $\beta^*(\alpha)$ the supremum of all $\beta$ for which $(N_\beta)$ follows from $(P_\alpha)$, and similarly $\alpha^*(\beta)$ as the supremum of all $\alpha$ for which $(P_\alpha)$ follows from $(N_\beta)$. The aforementioned results imply that $\beta^*(1) = 1/2$, $\beta^*(\alpha) \geq \alpha/(\alpha + 1)$ for $0 < \alpha < 1$, $\alpha^*(1) = 1/2$ and $\alpha^*(\beta) \geq \beta/(\beta + 6.91)$ for $0 < \beta < 1$. Based on Landau’s PNT and the exponent $\beta = \alpha/(\alpha + 1)$ in the converse problem, it was conjectured² by Bateman and Diamond [1] that $\alpha^*(\beta) \geq \beta/(\beta + 1)$.

In this note, we will show that $\alpha^*(\beta) \leq \beta/(\beta + 1)$. This will be done by showing the existence of a Beurling prime system satisfying

$$N(x) = ax + O\left(x \exp(-c' \log^\beta x)\right) \text{ and } \pi(x) = \text{Li}(x) + O\left(x \exp(-c (\log x)^{\frac{\beta}{\beta+1}})\right),$$

for some $a > 0$ and some $c, c' > 0$, but for which also

$$\pi(x) = \text{Li}(x) + \Omega\left(x \exp(-c'' (\log x)^{\frac{\beta}{\beta+1}})\right),$$

for some $c'' > c$, showing that the exponent $\beta/(\beta + 1)$ cannot be improved. The example is found by generalizing the construction by Diamond, Montgomery, and Vorhauer, which corresponds to the case $\beta = 1$. Since their example yields the optimal exponent $\alpha^*(1) = 1/2$ in the case $\beta = 1$, it is not unreasonable to imagine that the exponent $\beta/(\beta + 1)$ occurring in a natural generalization of their example would also be optimal.

## 2 Preliminaries

The construction of the example consists of two parts. First a continuous generalized number system is provided by specifying its zeta function. This continuous system will satisfy the desired asymptotic relations. Then, using a probabilistic method due to Diamond, Montgomery, and Vorhauer, it is shown that there exists a discrete system which approximates the continuous one in a suitable sense.

For a Beurling prime system $\mathcal{P}$, one defines its zeta function $\zeta_{\mathcal{P}}$ as $\zeta_{\mathcal{P}}(s) = \sum_{n \in \mathcal{N}} n^{-s}$, wherever this series converges. Here $s = \sigma + it$ is a complex variable. The function $\zeta_{\mathcal{P}}$ satisfies the identity

$$\zeta_{\mathcal{P}}(s) = \exp\left(\int_1^{\infty} x^{-s} d\Pi(x)\right).$$

¹ The corresponding optimization problem for positive trigonometric polynomials dates back to Landau and is well-studied, see e.g. [13]. In particular it is known that the smallest value which can be obtained via this method is strictly above 6.87.

² They expressed this with the careful wording: “There is quite likely room for improvement here [from $\alpha = \beta/7.91$], possibly to the value $\alpha = \beta/(\beta + 1)$.”
where $\Pi(x) := \sum_{\nu \geq 1} \pi(x^{1/\nu})/\nu$ (which is sometimes referred to as Riemann’s prime counting function). The above identity is an immediate consequence of the defining property of $\mathcal{N}$.

It is useful to extend the notion of generalized number system to include not necessarily discrete systems. In a broader sense [2,7], a Beurling generalized number system is defined as a pair $(\Pi, N)$ of non-decreasing, right-continuous, unbounded functions supported on $[1, \infty)$ and satisfying $\Pi(1) = 0, N(1) = 1,$ and

$$\zeta(s) := \int_1^\infty x^{-s} dN(x) = \exp\left(\int_1^\infty x^{-s} d\Pi(x)\right).$$

A system arising from a sequence $(p_j)_{j \geq 1}$, as defined in the introduction, will be called a discrete generalized number system. When constructing examples of discrete systems, it is often easier to first construct a system (in the extended sense) with the desired properties, and to then “discretize” this system, rather than to come up with a discrete system right away. One possible way of discretizing systems is by a procedure used in [6], and later refined in [14]. This procedure yields the following

**Lemma 1** (Diamond et al. [6], Zhang [14]) Let $f$ be a non-negative function supported on $[1, \infty)$ and satisfying

$$f(u) \ll \frac{1}{\log u} \text{ and } \int_1^\infty f(u) du = \infty.$$

Then there exists an increasing sequence of numbers $(p_j)_{j \geq 1}, p_1 > 1$ and $p_j \to \infty$ such that for any $t$ and any $x \geq 1$

$$\left| \sum_{p_j \leq x} p_j^{-it} - \int_1^x u^{-it} f(u) du \right| \ll \sqrt{x} + \sqrt{\frac{x \log(|t| + 1)}{\log(x + 1)}}.$$  \hspace{1cm} (1)

Note in particular that setting $t = 0$ yields $\pi(x) = \sum_{p_j \leq x} 1 = \int_1^x f(u) du + O(\sqrt{x})$.

As said before, our construction is a natural generalization of the construction in [6]. The results in [6] were later sharpened by Zhang in [14], and are also provided in the monograph [7] of Diamond and Zhang. In this article, we shall roughly follow the structure of [7, Sections 17.4–17.9]. All necessary definitions and lemmas will be given; however if the proof of a statement is identical or very similar to a proof in [7], we will omit it and refer to [7] instead. In fact, most of the arguments employed there can be carried over to the case $0 < \beta < 1$. However, a new difficulty arises as the considered zeta function does not seem to have meromorphic continuation beyond $\text{Re } s = 1$. We overcome this difficulty by considering natural approximations of it by more regular zeta functions.

The main idea in [6] is to construct a zeta function which has infinitely many zeros on the curve $\sigma = 1 - 1/\log |t|$, and none to the right of it, and which is of not too large growth. The zeros are “responsible” for the de la Vallée Poussin remainder in the PNT, while the moderate growth of the zeta function allows one to deduce the desired
asymptotics of $N$ via Perron inversion. We will modify the construction so that the zeros will lie on the curve\footnote{This curve is suggested by a result of Ingham, which shows what error term in the classical PNT would follow from a general zero-free region $\sigma > 1 - \eta(t)$ for the Riemann zeta function, see [10, pages 60–65].} $\sigma = 1 - 1/\left(\log|t|\right)^{1/\beta}$. The zeros are obtained by taking products of rescaled and translated versions of the function $G$ defined as

$$G(z) := 1 - \frac{e^{-z} - e^{-2z}}{z}, \quad G(0) := 0.$$ 

The function $G$ has zeros $z_0 = 0$ and $z_{\pm n} = x_n \pm iy_n, n \in \mathbb{N}_0$, with

$$-b \log \frac{n\pi}{2} \leq x_n < -\frac{1}{2} \log \frac{n\pi}{2}, \quad \text{and} \quad n\pi < y_n < (n + 1)\pi$$

for some constant $b > 1/2$. It has no other zeros. For $z \neq 0$ we have the simple approximation

$$G(x + iy) = 1 + \theta \frac{e^{-x} + e^{-2x}}{|x + iy|}, \quad |\theta| \leq 1; \quad (2)$$

while for $x \geq -1$,

$$|G(x + iy)| \leq 1 + e^2 - e, \quad (3)$$

which follows from the identity $G(z) = 1 - \int_1^2 e^{-zu}du$. The logarithm of $G$ can also be expressed as a Mellin transform:

**Lemma 2** The function $\log G(z)$ is well-defined for $x = \text{Re} z > 0$ and has the representation

$$\log G(z) = -\int_1^\infty g(u)u^{-z-1}du,$$

where

$$g(u) := \sum_{n=1}^\infty \frac{1}{n}\chi^{*n}(u).$$

Here, $\chi$ is the indicator function of $[e, e^2]$ and $*$ denotes the multiplicative convolution of functions supported on $[1, \infty)$: $(f \ast h)(x) := \int_1^x f(x/u)h(u)du/u. The function g is non-negative, supported on $[e, \infty)$, and on intervals $(e^m, e^{m+1})$ it equals a polynomial in $\log u$ of degree at most $m - 1$.

The function $g(u)$ gets close to $1/\log u$ for large $u$. We have the following estimates.

**Lemma 3** For $u > e^2$,

$$g(u) \log u = 1 + O\left(u^{-(1/2)\log(\pi/2)}\right),$$
and for \( u \geq e^5 \), \( g \) is differentiable and satisfies

\[
(g(u) \log u)' = O\left(u^{-1-(1/2)\log(\pi/2)}\right).
\]

For proofs of the above statements and lemmas, we refer to [7, Sections 17.5, 17.6].

3 The example

We will now define the continuous example by specifying its zeta function. Let \( \beta \in (0, 1) \) and set

\[
l_k = 4^k, \quad \gamma_k = \exp((l_k)^\beta) = e^{4^\beta k}, \quad \rho_k = 1 - \frac{1}{l_k} + i\gamma_k.
\]

These are the same parameters as in [7, Section 17.7], except for \( \gamma_k \), which we have set to be \( \exp((l_k)^\beta) \) instead of \( \exp(l_k) \). The points \( \rho_k \) now lie on the curve \( \sigma = 1 - 1/(\log t)^{1/\beta} \) instead of \( \sigma = 1 - 1/\log t \). Next we set

\[
\zeta_C(s) := \frac{s}{s-1} \prod_{k=1}^{\infty} G(l_k(s-\rho_k))G(l_k(s-\overline{\rho_k})).
\]

Using (2), we see that the product converges uniformly in the half plane \( \sigma \geq 1 \), so this zeta function is holomorphic in the open half plane \( \sigma > 1 \). For \( \beta < 1 \), this zeta function does not seem to have analytic continuation to a larger half plane, unlike the case \( \beta = 1 \). The factor \( s/(s-1) \) corresponds to the main term \( \text{Li}(x) \) in the PNT, while the factors of the infinite product will produce the desired oscillation. That \( \zeta_C \) is indeed the zeta function of a Beurling system is a consequence of the following lemma.

Lemma 4 For \( \sigma > 1 \),

\[
\zeta_C(s) = \exp\left(\int_1^{\infty} v^{-s} f_C(v)dv\right),
\]

with

\[
f_C(v) := \frac{1 - v^{-1}}{\log v} - 2 \sum_{k \geq 1} \frac{g(v^{1/l_k})}{l_k} v^{-1/l_k} \cos(\gamma_k \log v), \quad v \geq 1.
\]

We have \( f_C(v) > 0 \) for \( v > 1 \) and \( f_C \) satisfies the Chebyshev estimates for some \( \delta \in (0, 1) \)

\[
(1 - \delta) \frac{1 - v^{-1}}{\log v} \leq f_C(v) \leq (1 + \delta) \frac{1 - v^{-1}}{\log v}, \quad v \geq e^4.
\]
The proof is identical to the one in [7], since in that proof, the cosine is bounded trivially by 1, and this is the only place where the altered parameter $\gamma_k$ occurs.

The lemma implies that $\zeta_C$ is the zeta function of the Beurling number system with Riemann prime counting function $\Pi_C$ given by $\Pi_C(x) = \int_1^x f_C(v) dv$. The “integer counting function” $N_C$ is uniquely determined by $\Pi_C$ (explicitly $dN_C = \exp^{*}(d\Pi_C)$, where the exponential is the exponential with respect to multiplicative convolution of measures on $[1, \infty)$ (see e.g. [7, Chapter 3]).

Next, we use Lemma 1 with $f = f_C$ to obtain a sequence $P = (p_j)_{j \geq 1}$ of Beurling primes which satisfies (1) with $f = f_C$. Denote the prime and integer counting function of $P$ by $\pi$ and $N$ respectively. We also consider its Chebyshev prime counting function $\psi$, defined as the summatory function of $\Lambda$, with $\Lambda(n) = \log p_j$ if $n = p_j^\nu$ for some $p_j \in P$, $\nu \geq 1$, and zero otherwise. In the next two sections, we will show the following relations:

\[ N(x) = ax + O\left(x \exp\left(-c \log^\beta x\right)\right), \quad \text{for some } a > 0 \text{ and } c > 0; \quad (5) \]

and

\[ \limsup_{x \to \infty} \frac{\psi(x) - x}{x \exp\left(-\beta^{-\frac{1}{\beta+1}}(\beta + 1)(\log x)^{\frac{\beta}{\beta+1}}\right)} = 2, \]

\[ \liminf_{x \to \infty} \frac{\psi(x) - x}{x \exp\left(-\beta^{-\frac{1}{\beta+1}}(\beta + 1)(\log x)^{\frac{\beta}{\beta+1}}\right)} = -2. \quad (6) \]

From these two relations it then follows that $\alpha^*(\beta) \leq \beta/(\beta + 1)$, since $\pi(x) = \int_1^x (1/\log u) d\psi(u) + O(\sqrt{x})$.

4 Asymptotics of $N$

In order to deduce asymptotic information of $N$, we will use a Perron inversion formula. We will bypass the problem of the apparent absence of analytic continuation of $\zeta_C$ beyond $\sigma = 1$ by considering for $K \geq 1$

\[ \zeta_{C,K}(s) := \frac{s}{s-1} \prod_{k=1}^K G(l_k(s - \rho_k)) G(l_k(s - \overline{\rho_k})) = \exp\left(\int_1^\infty v^{-s} f_{C,K}(v) dv\right), \]

where $f_{C,K}$ is defined as in (4), but with the summation ranging only up to $K$. Then $\zeta_{C,K}$ has analytic continuation to the whole complex plane, with the exception of a simple pole at $s = 1$. Note that also $f_{C,K} > 0$, since by the non-negativity of $g$,

\[ \left| 2 \sum_{k=1}^K \frac{g(v^{1/l_k})}{l_k} v^{-1/l_k} \cos(\gamma_k \log v) \right| \leq 2 \sum_{k \geq 1} \frac{g(v^{1/l_k})}{l_k} v^{-1/l_k} \leq \begin{cases} 0 & \text{if } v < e^4; \\ \delta \frac{1-v^{-1}}{\log v} & \text{if } v \geq e^4. \end{cases} \]

(The last inequality is proved in [7, Lemma 17.20] and is also used in the proof of Lemma 4). We furthermore have that $f_C(v) = f_{C,K}(v)$ whenever $v < e^{4K+1}$, since
Next we set
\[ d\Pi_K(v) = \chi_{[1,e^{4k+1}]}(v) d\Pi(v) + \chi_{[e^{4k+1},\infty)}(v) f_{C,K}(v) dv. \tag{7} \]

Here, \( \Pi \) is the Riemann prime counting function of the discrete system \( \mathcal{D} \), and \( \chi_I \) denotes the indicator function of a set \( I \). With \( \Pi_K \) we associate the “integer counting function” \( N_K \) (i.e. \( dN_K = \exp^s(d\Pi_K) \)) and the zeta function \( \zeta \). One might view the Beurling system \((\Pi_K, N_K)\) as an intermediate system between the discrete system \( \mathcal{D} \) and the continuous one given by \( f_{C,K} \). Since \( \Pi_K = \Pi \) on \([1, e^{4k+1})\), also \( N_K = N \) on \([1, e^{4k+1})\).

We will apply the following Perron formula for \( \int N_K \) (to guarantee absolute convergence):
\[ \int_1^\infty N_K(u) du = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \zeta(s) x^{s+1} \frac{ds}{s(s+1)}. \tag{8} \]

Here \( \kappa \) is a number larger than 1. We will shift the contour to a contour to the left of \( \sigma = 1 \), picking up a residue at \( s = 1 \) which will provide the main term in (5). The integral over the shifted contour will be estimated by comparing \( \zeta_K \) with \( \zeta_{C,K} \), and by applying some bounds for \( \zeta_{C,K} \) which we will derive shortly. We have
\[
\log \zeta_K(s) - \log \zeta_{C,K}(s) = \int_1^{e^{4K+1}} u^{-s} (d\Pi_K(u) - f_{C,K}(u) du) = \int_1^{e^{4K+1}} u^{-s} (d\Pi(u) - d\pi(u)) + \int_1^{e^{4K+1}} u^{-\sigma} u^{-it} (d\pi(u) - f_{C}(u) du).
\]

Here we used that \( f_{C,K} = f_C \) on \([1, e^{4k+1})\). Note that both of these integrals are entire functions of \( s \). From this it follows that also \( \zeta_K \) has meromorphic continuations to \( \mathbb{C} \), with a sole simple pole at \( s = 1 \). Since \( d\Pi - d\pi \) is a positive measure, and since \( \Pi(x) - \pi(x) = O(\sqrt{x}) \) (this follows immediately from the definition of \( \Pi \) and the bound \( \pi(x) \ll x \)), the first integral is uniformly bounded (independent of \( K \)) in the half plane \( \sigma \geq 3/4 \) say. For the second integral, we integrate by parts and use the bound (1) to see that it is uniformly bounded by a constant times \( \sqrt{\log(2 + |r|)} \) in the half plane \( \sigma \geq 3/4 \). For the remainder of this section we fix positive constants \( A \) and \( B \), independent of \( K \), so that
\[
|\log \zeta_K(s) - \log \zeta_{C,K}(s)| \leq \begin{cases} A & \text{if } \sigma \geq 3/4, \ |r| \leq 2, \\ A + B \sqrt{\log |r|} & \text{if } \sigma \geq 3/4, \ |r| \geq 2. \end{cases} \tag{9}
\]

Let now \( x \geq e^4 \) be fixed, and let \( K \) be such that \( e^{4K} \leq x < e^{4K+1} \). Then \( N(x) = N_K(x) \). Set \( \sigma_1 = 1 - (1/2)(\log x)^{\beta-1}, \sigma(t) = 1 - (1/4)\log |t| / \log x, \) and let \( k(\beta) \) be such that \( (3/2)\gamma_k < (1/2)\gamma_{k+1} \) for \( k \geq k(\beta) \).
Lemma 5  The following bounds hold uniformly (with implicit constants independent of $K$):

for $\sigma_1 \leq \sigma \leq 2$:

1. if $0 \leq t \leq 2$, then $\zeta_K(\sigma + it) \ll 1/|\sigma - 1|$;

2. if $t \geq 2$ and $|t - \gamma_K| \geq \gamma_K/2$, for every $k \in \{k(\beta), k(\beta) + 1, \ldots, K\}$, then

$$\zeta_K(\sigma + it) \ll \exp(B\log|t|);$$

3. if $t \geq 2$ and $|t - \gamma_{k_0}| < \gamma_{k_0}/2$ for some $k_0 \in \{k(\beta), k(\beta) + 1, \ldots, K\}$, then

$$\zeta_K(\sigma + it) \ll \begin{cases} \exp(B\log|t|)\left(1 + \frac{|t|}{4^{k_0}|\sigma + it - \rho_{k_0}|}\right) & \text{if } 4^{k_0}|\sigma + it - \rho_{k_0}| \geq 1, \\ \exp(B\log|t|) & \text{if } 4^{k_0}|\sigma + it - \rho_{k_0}| \leq 1; \end{cases}$$

for $|t| \geq 2\gamma_K$ and $\sigma(t) \leq \sigma \leq 2$:

$$\zeta_K(\sigma + it) \ll \exp(B\log|t|).$$

The proof is essentially the same as that of [7, Lemma 17.22]. First we use (9) to compare with $\zeta_{C,K}$. Then we use (2) to approximate the factors in the product. The main point is that $\exp(2\kappa_k(1-\sigma)) \ll \gamma_k$ for $k \leq K$ for $\sigma \geq \sigma_1$, while for $\sigma \geq \sigma(t)$, we have $\exp(2\kappa_k(1-\sigma)) \leq \sqrt{|t|}$. By definition of $k(\beta)$, for each fixed $t$ there is at most one $k_0 \in \{k(\beta), \ldots, K\}$ with $|t - \gamma_{k_0}| < \gamma_{k_0}/2$. For the terms with $k < k(\beta)$, we just employ some uniform bound in the half plane $\sigma \geq 3/4$ say. We omit the details.

Let us now focus our attention on the Perron integral. Since $N_K$ is non-decreasing, $\int_{x-1}^x N_K(u)du \leq N_K(x) \leq \int_x^{x+1} N_K(u)du$. Combining this with the Perron formula (8), we have

$$N_K(x) \leq \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \zeta_K(s) \frac{(x+1)^s - x^{s+1}}{s(s+1)}ds.$$

We shift the contour to a contour $\Gamma$ to the left of $\sigma = 1$. Set $J = \lfloor (\log x)^{\min(3\beta/2, 1)} \rfloor$ and

$$\Gamma_1 := [\sigma_1, \sigma_1 + i2^{J}] ;$$

$$\Gamma_2 := [\sigma_1 + i2^{J}, \sigma_2(2^{J}) + i2^{J}] \cup \{\sigma(t) + it : 2^{J} \leq t \leq x\};$$

$$\Gamma_3 := [3/4 + ix, 3/4 + i\infty).$$

Note that $\sigma_1 > \sigma(2^{J})$. We let $\Gamma$ be the union of the $\Gamma_i$ and their complex conjugates. Moving from $\sigma = \kappa$ to $\Gamma$ is allowed, since the contribution of the connecting piece $[3/4 + iT, \kappa + iT]$ tends to 0 as $T \to \infty$, by the last bound of Lemma 5. By the residue theorem we have

$$N_K(x) \leq a_K(x + 1/2) + O(I_1 + I_2 + I_3),$$
Lastly, we bound \( I_1 \) the integral over \( I_1 \). For \( I_1 \) we perform a dyadic splitting, and write

\[
I_1 = I_{1,0} + \sum_{j=1}^{J-1} I_{1,j},
\]

where \( I_{1,0} \) is the part of the integral where \( 0 \leq t \leq 2 \), and \( I_{1,j} \) the part of the integral where \( 2^j \leq t \leq 2^{j+1} \). By bounding \((x + 1)^{s+1} - x^{s+1}\) by \(|s + 1| x^\sigma\), and using the bounds from Lemma 5, we see that

\[
I_{1,0} \ll x \exp(-(1/2) \log^\beta x)(\log x)^{1-\beta}, \quad I_{1,j} \ll x \exp(-(1/2) \log^\beta x) \exp(B \sqrt{\log 2^{j+1}}).
\]

Indeed, if some \( t \) satisfies \(|t - \gamma_{k_0}| < \gamma_{k_0}/2\) for some \( k_0 \geq k(\beta) \), we use the third bound of Lemma 5 and the estimate

\[
\int_{|t-\gamma_{k_0}|<\gamma_{k_0}/2, |\sigma_1+it-\rho_{k_0}| \geq 4^{-k_0}} \frac{dt}{4^{k_0} |\sigma_1+it-\rho_{k_0}|} \ll \frac{1}{4^{k_0}} (\log \gamma_{k_0} + \log 4^{k_0}) \ll 1.
\]

Also, by definition of \( k(\beta) \), there are at most two values of \( k \geq k(\beta) \) such that \(|t - \gamma_k| < \gamma_k/2\) for some \( t \) in a dyadic interval \([2^j, 2^{j+1}]\). Using the estimate \( \sum_{j \leq J} e^{D\sqrt{J}} \ll \sqrt{J} e^{D\sqrt{J}} \) we get

\[
I_1 \ll x \exp(-(1/2) \log^\beta x)(\log x) \exp\left(O\left((\log x)^{3\beta/4}\right)\right) \ll x \exp(-c \log^\beta x),
\]

for any \( c < 1/2 \).

For \( I_2 \), we again bound \((x + 1)^{s+1} - x^{s+1}\) by \(|s + 1| x^\sigma\) and use the last bound of Lemma 5 (note that \( 2^J \geq 2 \gamma_K \)). We get

\[
I_2 \ll x \exp(-(1/2) \log^\beta x) + x \int_{2^j}^{x} \exp(B \sqrt{\log t} - (5/4) \log t) \, dt
\]

\[
\ll x \exp(-(1/2) \log^\beta x) + x \exp\left(-(\log 2)/8 (\log x)^{\min(3\beta/2, 1)}\right) \ll x \exp(-(1/2) \log^\beta x).
\]

Lastly, we bound \((x + 1)^{s+1} - x^{s+1}\) by \(x^{\sigma+1}\) and use the last bound from Lemma 5 to get

\[
I_3 \ll x^{7/4} \int_{x}^{\infty} \exp(B \sqrt{\log t} - 2 \log t) \, dt \ll x^{7/8}.
\]

Concluding the above calculations, when \( K \) is such that \( e^{4K} \leq x < e^{4K+1} \) we have for any \( c < 1/2 \)

\[
N_K(x) = a_K x + O\left(x \exp(-c \log^\beta x)\right).
\]
The inequality \( \geq \) can be shown in a completely analogous way. It is important to note that the implicit big-\( O \) constant is independent of \( K \).

It remains to see that \( a_K \) is close to \( a \), the density of \( N \). We have that

\[
a_K = \exp \left( \int_1^\infty \frac{1}{u} \left( d\Pi_K(u) - d\Li(u) \right) \right), \quad a = \exp \left( \int_1^\infty \frac{1}{u} \left( d\Pi(u) - d\Li(u) \right) \right),
\]

where we used that \( \exp \int_1^\infty u^{-\sigma} d\Li(u) = s/(s-1) \) to compute the residues. Indeed, if a system of Beurling integers has a density, it is equal to the right hand residue of its zeta function, \( \lim_{\sigma \to 1^+} (\sigma - 1) \zeta(\sigma) \), see e.g. [7, Proposition 5.1]. We then write \( (\sigma - 1)\zeta_K(\sigma) = \sigma \exp \int_1^\infty u^{-\sigma} \left( d\Pi_K(u) - d\Li(u) \right) \), and taking the limit \( \sigma \to 1^+ \) yields \( a_K \). Similarly for \( a \). The fact that both integrals converge follows from the estimates \( \Pi_K(x) - \Li(x), \Pi(x) - \Li(x) \ll x \exp(-c' \log^\alpha x) \) for some \( \alpha > 0 \) and \( c' > 0 \), cfr. Section 5. By (7),

\[
a_K = a \exp \left( \int_{e^K+1}^\infty \frac{1}{u} \left( f_{C,K}(u) du - d\Pi(u) \right) \right)
\]

\[
= a \exp \left( \int_{e^K+1}^\infty \frac{1}{u} \left( f_{C,K}(u) - f_C(u) \right) du + \int_{e^K+1}^\infty \frac{1}{u} \left( f_C(u) du - d\Pi(u) \right) \right).
\]

The first integral equals

\[
- \sum_{k=K+1}^\infty \log(G(l_k(1-\rho_k))G(l_k(1-\bar{\rho}_k))) \ll \sum_{k=K+1}^\infty \frac{1}{k} \ll e^{-d\beta(K+1)} \ll \exp(-\log^\beta x),
\]

where we used (2) and \( x < e^{4K+1} \). The second integral above is bounded by \( 1/\sqrt{e^{4K+1}} \ll 1/\sqrt{x} \) by (1). This gives that \( a_K = a \{1 + O(\exp(-\log^\beta x))\} \). We conclude that for any \( x \geq e^4 \) and any \( c < 1/2 \), upon selecting \( K \) such that \( e^4 \leq x < e^{4K+1} \),

\[
N(x) = N_K(x) = a_K x + O(x \exp(-c \log^\beta x)) = ax + O(x \exp(-c \log^\beta x)),
\]

which shows (5).

## 5 Asymptotics of \( \psi \)

The analysis of the prime counting function of the Diamond–Montgomery–Vorhauer example (corresponding to \( \beta = 1 \)) can be readily adapted to the case of general \( \beta \); no new technical difficulties arise. We give a summary of the analysis, but refer to [7, Section 17.9] for the details.

Given a fixed \( x \), let \( K \) again be such that \( e^{4K} \leq x < e^{4K+1} \). We shall analyze the Chebyshev prime counting function \( \psi(x) = \int_1^x \log u d\Pi(u) \). First note that \( \psi(x) = \psi_C(x) + O(\sqrt{x} \log x) \), which follows from Lemma 1. Here \( \psi_C(x) = \)
$$\int_1^x \log u d\Pi_C(u) = \int_1^x (\log u) f_C(u) du.$$ It suffices to analyze $\psi_C$. Using the same notations as in [7], we have

$$\psi_C(x) = x - 1 - \log x - 2F(x),$$

with

$$F(x) = \sum_{k=1}^K \int_{e^k}^x (\log v) 4^{-k} g(v) v^{-4-k} \cos(\gamma_k \log v) dv =: \sum_{k=1}^K I_k(x).$$

Transforming the integrals with the substitution $u = v^{4-k}$, splitting the integration range in $[e, e^5]$ and $[e^5, x^{4-k}]$, integrating by parts, and using the bounds from Lemma 3, one shows that

$$I_k(x) = \frac{x^{1-4^{-k}}}{\gamma_k} \sin(\gamma_k \log x) + O \left\{ x^{5/16} + \frac{1}{\gamma_k} \left( x^{1-4^{-k}(1+\frac{1}{2} \log(\pi/2))} + \frac{x^{1-4^{-k}}}{\gamma_k} \right) \right\}, \quad k \leq K - 2.$$

To estimate the integrals $I_{K-1}$, $I_K$, we transform again to the variable $u = v^{4-k}$ and split the integration range in intervals $[e^m, e^{m+1}), m < 16$. On $[e^m, e^{m+1})$, we write $g(u)$ as a polynomial in $\log u$ of degree at most $m-1$, and integrate by parts. This yields

$$I_{K-1}(x) \ll x \exp(-4^{-2\beta} \log x), \quad I_K(x) \ll x \exp(-4^{-\beta} \log x),$$

which is ok, since $\beta > \beta/(\beta + 1)$. One then proceeds by showing that $F(x)$ is dominated by at most two terms $I_{k_0}(x)$ and $I_{k_0+1}(x)$, with $k_0$ close to $\frac{1}{\beta+1}(K + \log(1/\beta))$. Consider

$$\frac{x^{1-4^{-k}}}{\gamma_k} = x \exp(-4^{-k} \log x - 4^\beta k) = x \exp \left( -\frac{\log x}{\lambda} - \lambda^\beta \right),$$

where we have written $\lambda = 4^k$. The function $-\lambda^{-1} \log x - \lambda^\beta$ reaches its maximum at $\lambda_{\text{max}}$,

$$\lambda_{\text{max}} = \left( \frac{\log x}{\beta} \right)^{\frac{1}{\beta+1}}, \quad \text{and} \quad -\frac{\log x}{\lambda_{\text{max}}} - (\lambda_{\text{max}})^\beta = -\beta^{-\frac{\beta}{\beta+1}} (\beta + 1) (\log x)^{\frac{\beta}{\beta+1}}.$$

Note that $\lambda_{\text{max}} < 4^{K-2}$ for $x$ sufficiently large. Now set $\mu = \log \lambda_{\text{max}} / \log 4$, $k_0 = \lfloor \mu \rfloor$, and write

$$E(x) = x \exp \left( -\beta^{-\frac{\beta}{\beta+1}} (\beta + 1) (\log x)^{\frac{\beta}{\beta+1}} \right).$$
We have
\[
\left| I_{k_0}(x) \right| \leq E(x)\{1 + o(1)\}, \quad I_{k_0+1}(x) = o(E(x)) \quad \text{if } \mu - k_0 \in [0, 1/3];
\]
\[
I_{k_0}(x) = o(E(x)), \quad I_{k_0+1}(x) = o(E(x)) \quad \text{if } \mu - k_0 \in (1/3, 2/3);
\]
\[
I_{k_0}(x) = o(E(x)), \quad \left| I_{k_0+1}(x) \right| \leq E(x)\{1 + o(1)\} \quad \text{if } \mu - k_0 \in [2/3, 1).
\]

Also in every case, the terms \( I_k(x), k \neq k_0, k_0 + 1 \) are \( O\left( x \exp\left( -d \left( \log x \right) \frac{\beta}{\beta + 1} \right) \right) \) for some \( d > \beta - \frac{\beta}{\beta + 1} (\beta + 1) \), and there are \( K - 2 = O(\log \log x) \) such terms. Combining all these estimates shows that
\[
\limsup_{x \to \infty} \frac{\psi(x) - x}{E(x)} \leq 2, \quad \liminf_{x \to \infty} \frac{\psi(x) - x}{E(x)} \geq -2.
\]

In order to show equality and hence prove (6), one considers an increasing sequence of values for \( x \), so that \( (\log x/\beta) \frac{1}{\beta + 1} \) gets arbitrarily close to perfect fourth powers \( 4^{k_0} \), for some \( k_0 \leq K - 2 \) (\( k_0 \) and \( K \) of course depending on \( x \)), and where also \( \sin(\gamma_k \log x) \) gets arbitrarily close to \( -1 \) (for the lim sup) or \( 1 \) (for the lim inf).

Acknowledgements The author thanks Dr. Gregory Debruyne and Prof. Jasson Vindas for their kind advice and useful remarks.

Funding: The author was funded by the Ghent University Bijzonder Onderzoeksfonds (BOF) with Grant Number 01J04017.

Availability of data and material Not applicable.

Declarations

Conflict of interest Not applicable.

Code availability Not applicable.

References

1. Bateman, P.T., Diamond, H.G.: Asymptotic distribution of Beurling’s generalized prime numbers. In: LeVeque, W.J. (ed.) Studies in Number Theory, pp. 152–210. Mathematical Association of America, Washington, DC (1969)
2. Beurling, A.: Analyse de la loi asymptotique de la distribution des nombres premiers généralisés. Acta Math. 68, 255–291 (1937)
3. Broucke, F., Debruyne, G., Vindas, J.: Beurling integers with RH and large oscillation. Adv. Math. 370, 107240 (2020)
4. Diamond, H.G.: Asymptotic distribution of Beurling’s generalized integers. Ill. J. Math. 14, 12–28 (1970)
5. Diamond, H.G.: A set of generalized numbers showing Beurling’s theorem to be sharp. Ill. J. Math. 14, 29–34 (1970)
6. Diamond, H.G., Montgomery, H.L., Vorhauer, U.M.A.: Beurling primes with large oscillation. Math. Ann. 334, 1–36 (2006)
7. Diamond, H.G., Zhang, W.-B.: Beurling Generalized Numbers. Mathematical Surveys and Monographs Series. American Mathematical Society, Providence (2016)

8. Hall, R.S.: The prime number theorem for generalized primes. J. Number Theory 4, 313–320 (1972)

9. Hilberdink, T.W., Lapidus, M.L.: Beurling zeta functions, generalised primes, and fractal membranes. Acta Appl. Math. 94, 21–48 (2006)

10. Ingham, A.E.: The Distribution of Prime Numbers. Cambridge Tracts in Mathematics and Mathematical Physics, vol. 30. Cambridge University Press, Cambridge (1932)

11. Landau, E.: Neuer Beweis des Primzahlsatzes und Beweis des Primidealsatzes. Math. Ann. 56, 645–670 (1903)

12. Malliavin, P.: Sur le reste de la loi asymptotique de répartition des nombres premiers généralisés de Beurling. Acta Math. 106, 281–298 (1961)

13. Révész, S.G.: On some extremal problems of Landau. Serdica Math. J. 33, 125–162 (2007)

14. Zhang, W.-B.: Beurling primes with RH and Beurling primes with large oscillation. Math. Ann. 337, 671–704 (2007)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.