PARTIAL REGULARITY FOR SYMMETRIC QUASICONVEX FUNCTIONALS ON BD

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ABSTRACT. We establish the first partial regularity results for (strongly) symmetric quasiconvex functionals of linear growth on BD, the space of functions of bounded deformation. By Rindler’s foundational work [61], symmetric quasiconvexity is the foremost notion as to sequential weak*-lower semicontinuity of functionals on BD. The overarching main difficulty here is Ornstein’s Non-Inequality, hereby implying that the BD-case is genuinely different from the study of variational integrals on BV. In particular, this paper extends the recent work of Kristensen and the author [42] from the BV- to the BD-situation. Alongside, we establish partial regularity results for strongly quasiconvex functionals of superlinear growth by reduction to the full gradient case, which might be of independent interest.

1. INTRODUCTION

1.1. Aims and scope. Let \( n \geq 2 \) and \( \Omega \) be an open and bounded subset of \( \mathbb{R}^n \) with Lipschitz boundary. A vast class of variational problems connected to plasticity is set by virtue of linear growth functionals depending on the symmetric gradient, cf. [10, 37, 68, 21]. Possibly allowing for non-convex energies, a unifying perspective on the topic as considered in variants in [13, 61] is given by the canonical variational principle

\[
\hat{F}[v] := \int_{\Omega} f(\varepsilon(v)) \, dx \quad \text{to minimise over a Dirichlet class } D_{u_0},
\]

where \( u_0 : \Omega \to \mathbb{R}^n \) is a suitable Dirichlet datum and \( \varepsilon(v) := \frac{1}{2} (Dv + Dv^\top) \) denotes the symmetric gradient of a map \( v : \mathbb{R}^n \to \mathbb{R}^n \). Most crucially, \( f : \mathbb{R}^{n \times n}_{\text{sym}} \to \mathbb{R} \) is assumed to be a continuous integrand of linear growth. By this we understand that there exists a constant \( L > 0 \) such that

\[
|f(z)| \leq L(1 + |z|) \quad \text{for all } z \in \mathbb{R}^{n \times n}_{\text{sym}}.
\]

Following the foundational work of Rindler [61], a necessary and sufficient condition for the associated relaxed functionals to be suitably lower semicontinuous is that of symmetric quasiconvexity, cf. Section 1.2 below. In view of the direct method of the calculus of variations, symmetric quasi-convexity thus plays the central rôle for functionals of the form (1.1). In this respect, it is the aim of the present paper to provide the first regularity theory for such functionals and to thereby complement the recently available existence theory from a regularity viewpoint.

To elaborate more on these matters, we start by noting that the growth bound (LG) suggests to consider (1.1) on Dirichlet classes \( W_{u_0}^{1,1}(\Omega; \mathbb{R}^n) := u_0 + W_0^{1,1}(\Omega; \mathbb{R}^n) \) for \( u_0 \in W^{1,1}(\Omega; \mathbb{R}^n) \). However, by Ornstein’s Non-Inequality [59], it is not possible to bound the \( L^1 \)-norm of \( Du \) against that of \( \varepsilon(u) \). In fact, for every \( n \geq 2 \) there exists a sequence \( (\varphi_j) \subset C^\infty_0(B(0,1); \mathbb{R}^n) \) for which \( (\varepsilon(\varphi_j)) \) remains bounded in \( L^1(\Omega; \mathbb{R}^{n \times n}_{\text{sym}}) \) whereas \( \|D\varphi_j\|_{L^1(\Omega; \mathbb{R}^{n \times n})} \to \infty \) as \( j \to \infty \), cf. [5, 22, 44, 45] for instance. This is in stark contrast with the situation when \( L^1 \) is replaced by \( L^p \), \( 1 < p < \infty \). Indeed, in the latter case the corresponding result can be reduced to standard
singular integral estimates known as KORN-type inequalities. In consequence, $F$ is not coercive on $W^{1,1}(\Omega; \mathbb{R}^n)$ but on

$$\text{LD}(\Omega) := \{ u \in L^1(\Omega; \mathbb{R}^n) : \varepsilon(u) \in L^1(\Omega; \mathbb{R}^{n \times n}_{\text{sym}}) \}$$

endowed with the LD-norm $\|u\|_{\text{LD}} := \|u\|_{L^1} + \|\varepsilon(u)\|_{L^1}$. It is thus natural to let the Dirichlet datum $u_0$ belong to $\text{LD}(\Omega)$ and consider the variational principle (1.1) over the affine class $\mathcal{P}_m := \text{LD}_0(\Omega) := u_0 + \text{LD}_0(\Omega)$, where $\text{LD}_0(\Omega)$ is the closure of $C^1_c(\Omega; \mathbb{R}^n)$ for the LD-norm.

Still, LD is an $L^1$-based space and hence fails to be reflexive; as a consequence, it lacks an appropriate version of the Banach-Alaoglu theorem concerning weak convergence. Thus it is required to relax $F$ to the space $\text{BD}(\Omega)$ given by

$$\text{BD}(\Omega) := \{ u \in L^1(\Omega; \mathbb{R}^n) : E\varepsilon(u) \in \mathcal{M}(\Omega; \mathbb{R}^{n \times n}_{\text{sym}}) \}.$$  

Here, $\mathcal{M}(\Omega; \mathbb{R}^{n \times n}_{\text{sym}})$ are the finite, $\mathbb{R}^{n \times n}_{\text{sym}}$-valued Radon measures on $\Omega$, and we use the notation $E\varepsilon(u)$ instead of $\varepsilon(u)$ to indicate that $E\varepsilon(u)$ is a measure. The relaxation here is taken with respect to weak*-convergence in $\text{BD}(\Omega)$, and we refer the reader to Sections 1.2 and 2.2 for the requisite background terminology. This space – which contains $\text{BV}(\Omega; \mathbb{R}^n)$ as a proper subspace – takes a prominent role in plasticity, and has been studied from various perspectives by a notable plenty of authors, see [68, 21, 46, 47, 5, 9, 66, 12] among others.

From a calculus of variations and hereafter lower semicontinuity perspective, the central notion for functionals $F$ is a variant of MORREY’s quasiconvexity [57], namely the aforementioned symmetric quasiconvexity. As we shall recap below, this notion has been proven necessary and sufficient for the weak*-relaxation $\mathcal{F}$ of $F$ to be (sequentially weak*) lower semicontinuous on $\text{BD}$ relatively recently by RINDLER [61], thereby extending the classical works of AMBROSIO & DAL MASO [7] as well as FONSECA & MÜLLER [35] from the BV- to the BD-situation. Yet, for such symmetric quasiconvex functionals the properties of minima are far from being understood – in particular, a regularity theory is still missing – and hence the objective of this paper is to make a first step in this direction and to close this gap.

1.2. Symmetric quasiconvexity and relaxation. Before embarking on the regularity issue raised above in detail, we briefly pause and discuss the relevant relaxed functionals $\mathcal{F}$ that are required for defining the notion of (local) minimality in the sequel. Already appearing in variants in [23, 36], we start by recalling the following definition as given, e.g., in [13, 61].

**Definition 1.1** (Symmetric quasiconvexity). A continuous integrand $f: \mathbb{R}^{n \times n}_{\text{sym}} \to \mathbb{R}$ is said to be symmetric quasiconvex provided for any open and bounded Lipschitz subset $\Omega$ of $\mathbb{R}^n$ there holds

$$f(z) \leq \frac{1}{\Omega} \int_{\Omega} f(z + \varepsilon(\varphi)) \, dx \quad \text{for all } \varphi \in C^\infty_c(\Omega; \mathbb{R}^n) \text{ and } z \in \mathbb{R}^{n \times n}_{\text{sym}}.$$  

Returning to the functional $F$ defined in terms of $f: \mathbb{R}^{n \times n}_{\text{sym}} \to \mathbb{R}$ by (1.1), let $u_0 \in \text{LD}(\Omega)$ be a given Dirichlet datum. Essentially solely subject to the additional linear growth assumption (LG), RINDLER [61] established that symmetric quasiconvexity is a necessary and sufficient condition for the Lebesgue-Serrin extension

$$\mathcal{F}_{u_0}[u] := \inf \left\{ \liminf_{k \to \infty} F[u_k] : (u_k) \subset u_0 + \text{LD}_0(\Omega), u_k \rightharpoonup u \text{ in } \text{BD}(\Omega) \right\}$$

for $u \in \text{BD}(\Omega)$ to possess an integral representation as follows. Letting $u \in \text{BD}(\Omega)$, we denote $E\varepsilon(u) = E\varepsilon' + E\varepsilon'' = \varepsilon(u) + \mathcal{E}(u) + \mathcal{D}(u)$ with $\mathcal{E}(u)$ the Lebesgue-Radon-Nikodým decomposition of $E\varepsilon(u)$; cf. Section 2.2 for this and the subsequent terminology. Given a symmetric quasiconvex integrand $f$ satisfying (LG), the main result of [61] asserts that $\mathcal{F}_{u_0}[u]$ can be represented as

$$\mathcal{F}_{u_0}[u] = \int_{\Omega} f(\varepsilon(u)) \, dx + \int_{\partial \Omega} f^\infty(\frac{dE\varepsilon(u)}{d\varepsilon(u)}) \, d|E\varepsilon'(u)|$$

$$+ \int_{\partial \Omega} f^\infty(\text{Tr}_{\partial\Omega}(v - u) \circ \nu_{\partial\Omega}) \, d\mathcal{H}^{n-1}, \quad u \in \text{BD}(\Omega).$$  

(1.2)
Here, \( f^\infty(z) := \limsup_{\lambda \to 0} \frac{f(\lambda z)}{\lambda} \) denotes the strong recession function of \( f \) at \( z \in \mathbb{R}_{\text{sym}}^{n \times n} \) capturing the integrand’s behaviour at infinity, and \( \text{Tr}_{\Omega} \) displays the boundary trace operator on \( \text{BD}(\Omega) \).

Most notably, the integral representation (1.2) was established in [61] without relying on the BD-variant of ALBERTI’s rank-one theorem [6]. By now, the latter has been proved by DE PHILIPPIS & RINDLER in the seminal work [25] in a much more general context, allowing for a simplified proof of (1.2) (cf. [26, 11, 18]) but had not been available at the time of [61].

Hence, in particular, the integral functional on the right-hand side of (1.2) is sequentially weak*-lower semicontinuous on \( \text{BD}(\Omega) \). With this notation, we say that \( u \in \text{BD}(\Omega) \) is a BD-minimiser (or generalised minimiser) for \( F \) subject to the boundary datum \( u_0 \) provided

\[
F_{u_0}[u] \leq F_{u_0}[u + \varphi]
\]

holds for all \( \varphi \in \text{BD}_0(\Omega) = \{ v \in \text{BD}(\Omega) : \text{Tr}_{\Omega}(v) = 0 \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \partial \Omega \} \). A similar notion of local BD-minima can be introduced for \( u \in \text{BD}_0(\Omega) \) which, for \( u \in \text{BD}(\Omega) \), reduces to validity of (1.3) for all competitor maps \( \varphi \in \text{BD}(\Omega) \) with compact support in \( \Omega \), cf. Section 2.3 for more detail. In consequence, augmenting the linear growth assumption (LG) with a suitable coerciveness condition on the symmetric quasiconvex integrand \( f \), existence of BD-minima for \( F \) follows at ease. As will be discussed below in Section 1.4, such a coerciveness criterion goes hand in hand with the partial regularity of BD-minima. Toward the latter, it is thus natural to contextualise the partial regularity for BD-minima with available results in the literature and thereby outline the main obstructions first.

1.3. Contextualisation and overview. In the common language of regularity theory, the minimisation of functionals (1.1) displays a purely vectorial problem, thereby leading to a system of Euler-Lagrange equations satisfied by minimisers rather than a single equation. Even in the case where the symmetric gradient in (1.1) is replaced by the full gradient, it is a well-known feature of such multiple integrals to produce minima which are not everywhere \( C^{1,\alpha} \)-Hölder continuous but only on a large set. This phenomenon is referred to as partial regularity.

In the superlinear growth regime with full gradients, the study of partial regularity for minima has a long and rich history, starting with the seminal work of E. DE GIUSEPPE & FUSCO [2]; see also [22, 29, 30, 31, 51, 28, 4] and MINGIONE’s survey article [56] for a non-exhaustive list of other contributions. However, until recently, for full gradient linear growth functionals the only contribution had been the local-in-phase-space result due to ANZELLOTTI & GIAQUINTA [10] and its adaptation to the model integrands \( z \mapsto (1 + |z|^p)^{1/p}, \ p \neq 2 \), by SCHMIDT [62]. This approach, which crucially relies on comparison with mollifications and thus works well for convex integrands by Jensen’s inequality, has been extended to the BD-setting by the author [40]. Yet, due to the very method of proof, it seems to be restricted to convex integrands and a generalisation of the strategy to the quasiconvex case seems difficult; also see the discussion in [10, 62] and [40].

At present, in the full gradient, quasiconvex linear growth case on BV, the only partial regularity result up to date has been given by KRISTENSEN and the author [42]. In this work, a direct comparison with suitably \( \mathcal{A} \)-harmonic maps is implemented that overcomes any indirect argument as is found e.g. in the blow-up method or, quite implicitly, in the proof of the \( \mathcal{A} \)-harmonic approximation lemma due to DUAZAAR et al. [29, 30, 31]. Let us note that similarly to [10, 62, 40], the sole use of direct arguments is somewhat dictated here by the comparatively weak compactness properties of BV and BD. In fact, examplarily pursuing the blow-up method for linear growth functionals, it is necessary to establish that the weak*-limit of a blow-up sequence satisfies a strongly elliptic Legendre-Hadamard system. However, by possible concentration effects, this conclusion seems unreachable since there are no general compactness improvements for the relevant blow-up sequences: Such compactness boosts would require some uniform local integrability enhancements, usually provided by GEHRING’S lemma in reliance on Caccioppoli-type inequalities, or higher (fractional) differentiability estimates a la MINGIONE [54, 55]. Whereas the former cannot be implemented in the linear growth situation – essentially
due to the non-availability of a sublinear Sobolev-Poincaré-type inequality, cf. Buckley & Koskela [19] and the discussion in Section 5.5 –, higher fractional differentiability results on minima such as in [54, 55] are confined to the convex situation. On the one hand, the latter approach is centered around the Euler-Lagrange system satisfied by minima, crucially utilising the positive definiteness of the integrands’ second derivatives. Such a procedure is ruled out in the (strongly) quasiconvex situation not only because of the integrands’ Hessians not being positive definite: By the foundational work of Müller & Sverák [58], the Euler-Lagrange system for minima of strongly quasiconvex functionals cannot yield regularity results in itself. Similar issues already arise in the full gradient situation, equally for other techniques such as the A-harmonic approximation, and we refer the reader to [41] for a further discussion thereof.

1.4. Main Results. After these preparations, we now pass to the description of the main results of the present paper. To begin with, symmetric quasiconvexity and the linear growth hypothesis (LG) together are easily seen not to be sufficient for $F$ given by (1.1) to produce bounded minimising sequences in LD($\Omega$). With the latter being a necessary condition to make the direct method of the calculus of variations work, we instead consider the following strengthening: We say that $f \in C(\mathbb{R}^{n,n})$ is strongly symmetric quasiconvex provided there exists $\ell > 0$ such that the function

\begin{equation}
\mathbb{R}^{n,n}_{\text{sym}} \ni z \mapsto f(z) - \ell V(z)
\end{equation}

is symmetric quasiconvex, where $V(z) := \sqrt{1 + |z|^2} - 1$ shall be referred to as the reduced area integrand or simply the V-function. Section 3 provides sample integrands satisfying (1.4) and demonstrates their ubiquity. This condition particularly yields the existence of BD-minima – a fact that is addressed in detail in the appendix, cf. Section 6.1 – but also proves instrumental for establishing their partial regularity provided $f$ is sufficiently smooth.

As to partial regularity, let us note that the approaches mentioned and sketched in Section 1.3 above can be modified when $f$: $\mathbb{R}^{n,n}_{\text{sym}} \to \mathbb{R}$ is of $p$-growth, $1 < p < \infty$ and satisfies the canonical modification of (1.4) to this setup by subsequent use of the standard $L^p$-Korn inequalities (see Section 3 for more detail). However, this is not even necessary: As a consequence, there exists an open and bounded set with Lipschitz boundary and let $u_0 \in LD(\Omega)$ be a given Dirichlet datum. Moreover, suppose that $f$: $\mathbb{R}^{n,n}_{\text{sym}} \to \mathbb{R}$ is of class $C^{2,1}_{\text{loc}}(\mathbb{R}^{n,n})$, linear growth in the sense of (LG) and strongly symmetric quasiconvex in the sense of (1.4).

Then for each $M > 0$ there exists $\varepsilon_M = \varepsilon_M(\ell, L, f^n) > 0$ such that for every BD-minimiser $u \in BD(\Omega)$ of $F$ (in the sense of (1.3)) the following holds: If $x_0 \in \Omega$ and $R > 0$ with $B(x_0, R) \Subset \Omega$ are such that

\[ \frac{\text{E}_{\text{int}}(B(x_0, R))}{\mathcal{L}^n(B(x_0, R))} < M, \quad \text{and} \quad \int_{B(x_0, R)} \left| \mathcal{E} u - \frac{\text{E}_{\text{int}}(B(x_0, R))}{\mathcal{L}^n(B(x_0, R))} \right| \, dx + \frac{1}{\mathcal{L}^n(B(x_0, R))} \left| \text{E}_{\text{int}}(B(x_0, R)) \right| < \varepsilon_M, \]

then $u$ is of class $C^{1,\alpha}$ on $B(x_0, R)$ for any $0 < \alpha < 1$. As a consequence, there exists an open subset $\Sigma_u$ of $\Omega$ with $\mathcal{L}^n(\Omega \setminus \Sigma_u) = 0$ such that for every $x_0 \in \Sigma_u$, $u$ is of class $C^{1,\alpha}$ in a neighbourhood of $x_0$ for any $0 < \alpha < 1$. In particular, denoting $\Sigma_u = \Omega \setminus \Sigma_u$, we have
\[ \Sigma_u = \Sigma_u^1 \cup \Sigma_u^2 \]

\[
\begin{aligned}
\forall x_0 \in \Omega : \liminf_{R \searrow 0} & \left( \int_{B(x_0,R)} |\mathcal{E}u - (\mathcal{E}u)_{B(x_0,R)}| \, dx + \frac{|E^u|_{\mathcal{B}(x_0,R)}}{\mathcal{L}^n(B(x_0,R))} \right) > 0 \\
\cup \left\{ x_0 \in \Omega : \limsup_{R \searrow 0} \frac{E_u(B(x_0,R))}{\mathcal{L}^n(B(x_0,R))} = \infty \right\}
\end{aligned}
\]

Theorem 1.2 will be established in Section 5. Before we highlight some aspects of the proof, let us comment on the hypotheses of Theorem 1.2. Condition (a) is rather of technical than instrumental nature and can be relaxed. To keep our exposition at a reasonable length, however, we stick to this assumption throughout. Similarly, the theorem can be formulated for \(x\)-dependent integrands, but we believe that this is standard and thus prefer to stick to the autonomous case throughout. Subject to (a)–(c) from above, it is moreover not too difficult to show that BD-minima are actually \(C^{1,\alpha}\)-partially regular once the \(C^{1,\alpha}\)-regularity of Theorem 1.2 is established. Such extensions shall be addressed in Section 5.5.

In proving Theorem 1.2 we rely in an essential way on an improved estimate of the BD-minimisers’ distances from suitable \(\lambda\)-harmonic approximants in terms of a superlinear power of the excess. To the best of our knowledge, an estimate of this form has only been derived recently in the BV-full gradient setup in [42], strongly relying on the full gradients of minima belonging to \(\mathcal{M}\). The aforementioned superlinear excess power, in turn, stems from the higher regularity properties of the \(\lambda\)-harmonic approximants on good balls. To define the latter notion appropriately, we remark that the \(\lambda\)-harmonic approximants on generic balls receive their higher Sobolev regularity up to the boundary from the higher regularity of their prescribed Dirichlet data; the precise correspondence is displayed in Proposition 5.3. For arbitrary balls \(B \subset \subset \Omega\) and \(u \in BD(\Omega)\), we can only assert that \(\mathcal{E}u(B) \subset L^1(\partial B; \mathbb{R}^n)\). This motivates the Fubini-type Theorem 4.1, implying that on sufficiently many spheres, BD-maps have interior traces with some additional differentiability and integrability beyond \(L^1\). We wish to stress that by Ornstein’s Non-Inequality this step does not follow as for BV, where the tangential traces \(\partial u\) can be shown to belong to \(L^1(\partial B)\) on sufficiently many balls \(B\) (see Remark 4.2). Here we crucially use the embedding \(BD \hookrightarrow W^{s,\infty} \) for \(n \geq 2, 0 < s < 1\) together with novel Poincaré-type inequalities to be proved in Section 2. Up from here, it is then the overall aim of the proof to show that Ornstein’s Non-Inequality essentially becomes invisible throughout the comparison estimates, simultaneously keeping track of the enlarged nullspace of the symmetric gradient in comparison with that of the full gradient. This comes along with both further conceptual and technical difficulties, and Section 5 is devoted to their precise discussion and resolution. Finally, let us mention that the approach as developed here should also give a streamlining and unifying treatment for the BV-case in the dimensions \(n = 2\) and \(n \geq 3\); cf. Remark 5.9.

1.5. Structure of the Paper. In Section 2, we fix notation, prove and collect miscellaneous background results. In Section 3, we examine the classes of symmetric quasiconvex more closely and establish partial regularity in the superlinear growth regime. Section 4 then serves to prove a Fubini-type theorem for BD that is instrumental in the proof of Theorem 1.2, and Section 5 is devoted to the proof of the latter. We conclude with an appendix in Section 6.

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2. Preliminaries

2.1. General Notation. We now briefly gather notation. Unless otherwise stated, \(\Omega\) always denotes an open and bounded Lipschitz subset of \(\mathbb{R}^n\). We denote \(B(x_0,r) := \{ x \in \mathbb{R}^n : |x-x_0| <\)
the open ball of radius \( r > 0 \) centered at \( x_0 \) and, to avoid ambiguities, use the symbol \( \mathbb{R}^{n \times n}_\text{sym} \) to denote the closed unit ball in \( \mathbb{R}^{n \times n}_\text{sym} \) with respect to the Frobenius norm \( |A| := (\sum_{j=1}^n |a_{ij}|^2)^{1/2} \).\( A = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n} \). Whenever \( X \) is a finite dimensional real vector space, the symbol \( \langle \cdot, \cdot \rangle \) is used to denote the usual inner product on \( X \) and \( \mathcal{S}(X) \) is the space of symmetric bilinear forms on \( X \). To avoid ambiguities, note that duality pairings are exclusively used with subscripts, so e.g. \( \langle \cdot, \cdot \rangle_{r \times \mathcal{D}} \) for the pairing between distributions and test functions. Also, for two given vectors \( a, b \in \mathbb{R}^n \), \( a \circ b := \frac{1}{2} (ab^T + ba^T) \) denotes their symmetric tensor product, and we record that

\[
\frac{1}{\sqrt{2}} |a| |b| \leq |a \circ b| \leq |a| |b| \quad \text{for all} \ a, b \in \mathbb{R}^n.
\]

The symbol \( \mathcal{L}(V; W) \) denotes the bounded linear operators between two normed linear spaces \( V \) and \( W \). As usual, \( \mathcal{L}^n \) and \( \mathcal{H}^{n-1} \) denote the \( n \)-dimensional Lebesgue and the \((n-1)\)-dimensional Hausdorff measure, respectively, and put \( \partial_b := \mathcal{L}^n(B(0,1)) \). For notational brevity, we shall also sometimes write \( d\mathcal{H}^{n-1}(x) = d\sigma_n \), but this will be clear from the context. Moreover, we denote \( \mathcal{M}(\Omega; \mathcal{H}^m) \) the \( \mathcal{H}^m \)-valued finite Radon measures on \( \Omega \). Given \( \mu \in \mathcal{M}(\Omega; \mathcal{H}^m) \) and \( A \in \mathcal{B}(\Omega) \) (the Borel \( \sigma \)-algebra on \( \Omega \)), then we recall that \( \mu \ll A := \mu(- \cap A) \) is the restriction of \( \mu \) to \( A \). We will also employ the usual notation of dashed integrals for average or mean integrals, but in our context these are always understood with respect to \( \mathcal{H}^m \). So, when \( u \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m) \) or \( \mu \in \mathcal{M}(\mathbb{R}^n; \mathcal{H}^m) \), we denote for a bounded set \( A \in \mathcal{B}(\mathbb{R}^n) \) with \( \mathcal{L}^n(A) \in (0, \infty) \)

\[
(u)_A := \int_A u d\mathcal{L}^n := \int_A u dx \quad \text{and} \quad (\mu)_A := \int_A d\mu := \frac{\mu(A)}{\mathcal{L}^n(A)}.
\]

If \( A = B(x, r) \) is a ball, we write \( (u)_{B(x,r)} := (u)_{B(|x|, r)} \) or \( (\mu)_{B(x,r)} := (\mu)_{B(|x|, r)} \) for brevity. Lastly, we denote by \( c, C > 0 \) generic constants that might change from line to line and shall only be specified provided their precise dependence on foregoing parameters is required. Similarly, we write \( a \asymp b \) if there exist two constants \( c, C > 0 \) such that \( ca \leq b \leq Ca \); in particular, \( c, C > 0 \) do not depend on \( a \) or \( b \).

2.2. The space BD. In the following we recall the definition and record the properties of BD-maps as shall be required in the upcoming sections; for more detail, the reader is referred to \([66, 12, 9, 5]\) and the references therein. We say that a measurable map \( u: \Omega \to \mathbb{R}^n \) belongs to \( \text{BD}(\Omega) \) (and is then said to be of bounded deformation) if and only if \( u \in L^1(\Omega; \mathbb{R}^n) \) and

\[
|Du|/|\Omega| := \sup \left\{ \int_\Omega \langle u, \nabla (\varphi) \rangle \, dx : \varphi \in C^1_c(\Omega; \mathbb{R}^{n \times n}) \right\} < \infty.
\]

The space \( \text{BD}_{\text{loc}}(\Omega) \) then is defined in the obvious manner. Given \( u \in \text{BD}(\Omega) \), the Lebesgue-Radon-Nikodým decomposition of \( Eu \) yields

\[
\text{Eu} = \text{E}^\circ u + E^\circ u = \mathcal{E}u \mathcal{L}^n \ll \Omega + \frac{d|Eu|}{d|E^\circ u|}|E^\circ u|, \tag{2.3}
\]

where \( E^\circ u \ll \mathcal{L}^n \) and \( E^\circ u \perp \mathcal{L}^n \) are the absolutely continuous or singular parts of \( Eu \) with respect to \( \mathcal{L}^n \), respectively; in particular, we have \( u \in \text{LD}(\Omega) \) if and only if \( u \in \text{BD}(\Omega) \) and \( Eu \ll \mathcal{L}^n \). Moreover, \( \mathcal{E}u \) is the density of \( E^\circ u \) with respect to \( \mathcal{L}^n \) and coincides with the symmetric part of the approximative gradient of \( u \), cf. \([5]\). If \( \Sigma \subset \Omega \) is a \( C^1 \)-manifold oriented by \( \nu: \Sigma \to \mathbb{S}^{n-1} \), then \( \text{Eu} \ll \Sigma \) is given by Kohn’s formula \([46]\)

\[
\text{Eu} \ll \Sigma = (u^+ - u^-) \circ \nu \mathcal{H}^{n-1} \ll \Sigma, \tag{2.4}
\]

where \( u^+ \) and \( u^- \) are the right and left traces of \( u \) along \( \Sigma \). These, in turn, are well-defined upon the orientation of \( \nu \), and can be computed for \( \mathcal{H}^{n-1} \)-a.e. \( x \in \Sigma \) by virtue of

\[
u(x) = \lim_{r \to 0} \int_{\Sigma(x,r)} u(y) \, dy.
\]
where \( \Sigma(x,r) := B(x,r) \cap \{y \in \mathbb{R}^n : \langle y-x, v(x) \rangle \geq 0\} \) for \( r > 0 \); in fact, one even has
\[
\lim_{r \downarrow 0} \int_{\Sigma^\pm(x,r)} |u(y) - u^\pm(x)| \, dy = 0
\]
for \( \mathcal{H}^{n-1} \)-a.e. \( x \in \Sigma \). Throughout, we will work with the following modes of convergence: Let \( u, u_1, u_2, \ldots \in BD(\Omega) \). We say that \( (u_k) \) converges to \( u \) in the norm topology provided \( \|u_k - u\|_{BD(\Omega)} \to 0 \), where \( \|v\|_{BD(\Omega)} := \|v\|_{L^1(\Omega; \mathbb{R}^n)} + |Ev(\Omega)| \) for \( v \in BD(\Omega) \). On the other hand, we say that \( (u_k) \) converges to \( u \) in the weak*-sense if \( u_k \to u \) strongly in \( L^1(\Omega; \mathbb{R}^n) \) and \( Ev_k \rightharpoonup Ev \) in the sense of weak*-convergence of \( \mathbb{R}^{n\times n} \)-valued Radon measures on \( \Omega \), and in the strict sense (or strictly) if \( u_k \to u \) strongly in \( L^1(\Omega; \mathbb{R}^n) \) and \( |Ev_k|(*) \to |Ev|(*) \) as \( k \to \infty \). Lastly, if \( u_k \to u \) strongly in \( L^1(\Omega; \mathbb{R}^n) \)
\[
\int_\Omega \sqrt{1 + |\varepsilon(u_k)|^2} \, dx + |E'u_k|_\Omega \to \int_\Omega \sqrt{1 + |\varepsilon u|^2} \, dx + |E'u|_\Omega, \quad k \to \infty,
\]
then we shall say that \( (u_k) \) converges to \( u \) in the area-strict sense. Note that, if we put \( \langle \cdot \rangle := \sqrt{1 + |\cdot|^2} \), then area-strict convergence amounts to \( \langle Ev_k \rangle(\Omega) \to \langle Ev \rangle(\Omega) \) in the sense of functions of measures to be recalled in Section 2.3 below. It is then routine to show that norm implies area-strict, area-strict implies strict and strict implies weak*-convergence. When working with \( u \in LD(\Omega) \), we usually work with the norm \( \|u\|_{LD(\Omega)} := \|u\|_{L^1(\Omega; \mathbb{R}^n)} + \|\varepsilon(u)\|_{L^1(\Omega; \mathbb{R}^{n\times n})} \) (recall that we write \( \varepsilon(u) \) for \( Eu \) provided \( Eu \ll \mathcal{L}^n \)).

As is by now well-known (cf. \([66, 12, 18]\)), Lipschitz regularity of \( \partial \Omega \) implies the existence of a linear, bounded, surjective trace operator \( \text{Tr} : BD(\Omega) \to L^1(\partial \Omega; \mathbb{R}^n) \), where boundedness is understood with respect to the respective norm topologies. Crucially, this operator is already surjective when acting on \( LD(\Omega) \). Moreover, it is also continuous for strict convergence in \( BD(\Omega) \) (and hence area-strict convergence, too) but not for weak*-convergence as specified above. As a consequence, there exists a bounded linear extension operator \( \mathcal{E}_{LD} : LD(\Omega) \to LD(\mathbb{R}^n) \). We can now collect some refined results on smooth approximation, cf. \([41, \text{Sec. 4}]\):

**Lemma 2.1** (Area-strict smooth approximation). Let \( \Omega \subset \mathbb{R}^n \) be an open and bounded with Lipschitz boundary and let \( u_0 \in LD(\Omega) \). Then for each \( u \in BD(\Omega) \) there exists a sequence \( (u_j) \subset u_0 + C_c^\infty(\Omega; \mathbb{R}^n) \) such that \( \|u_j - u\|_{L^1(\Omega; \mathbb{R}^n)} \to 0 \) and
\[
\int_\Omega \sqrt{1 + \varepsilon(u_j)^2} \, dx \to \int_\Omega \sqrt{1 + \varepsilon u^2} \, dx + |E'u|_\Omega + \int_{\partial \Omega} |u - u_0| \, d\mathcal{H}^{n-1} \quad \text{as} \quad j \to \infty.
\]

We will also need a fractional embedding theorem for \( BD \) as one of the main ingredients in the partial regularity proof. Hence we recall that for any measurable subset \( U \) of \( \mathbb{R}^n \) with \( \mathcal{L}^n(U) > 0 \), \( 0 < \theta < 1 \) and \( 1 \leq p < \infty \) the Sobolev-Slobodeckij-space \( W^{0,p}(U; \mathbb{R}^m) \) is defined as the linear space of all \( u \in L^p(U; \mathbb{R}^m) \) such that the Gagliardo seminorm
\[
[u]_{W^{0,p}(U; \mathbb{R}^m)} := \left( \int_U \frac{|u(x) - u(y)|^p}{|x-y|^{n+\theta p}} \, dx \, dy \right)^{\frac{1}{p}}
\]
is finite. The full norm on \( W^{0,p}(U; \mathbb{R}^m) \) then is given by \( \| \cdot \|_{W^{0,p}(U; \mathbb{R}^m)} := \|u\|_{L^p(U; \mathbb{R}^m)} + [u]_{W^{0,p}(U; \mathbb{R}^m)} \). By now, it is well-known that \( BV(\mathbb{R}^n) \hookrightarrow W^{0,n/(n-1+\theta)}(\mathbb{R}^n) \) for \( n \geq 2 \) and \( \theta \in (0,1) \). This embedding is due to KOLYADA \([48]\) (also see \([16]\)), but the result corresponding to \( BD \) has been obtained only recently and is essentially due to VAN SCHAFUTGEN \([71]\). Since we will need this framework and a refinement thereof in a different context as well, we briefly recall the requisite notions.

Let \( A(D) := \sum_{k=1}^n A_k \partial_k \) be a linear constant-coefficient differential operator, where \( V \equiv \mathbb{R}^N, W \equiv \mathbb{R}^M \) are finite dimensional \( \mathbb{R} \)-vector spaces and for each \( k \in \{1, \ldots, n\} \), \( A_k \in \mathcal{L}(V; W) \). Here the partial derivatives \( \partial_k \) act componentwisely on \( u = (u_1, \ldots, u_N) : \mathbb{R}^n \to \mathbb{R}^N \). Then \( A(D) \) has symbol \( A(\xi) = \sum_{k=1}^n \xi_k A_k : V \to W \) for \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). This operator now is called elliptic.
provided for each $\xi \in \mathbb{R}^n \setminus \{0\}$, $A[\xi] : V \to W$ is injective, and cancelling provided the symbol mapping $\xi \mapsto A[\xi]$ is sufficiently spread in the sense of
\begin{equation}
\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} A[\xi](V) = \{0\}.
\end{equation}
By [71, Prop. 6.3], the symmetric gradient operator is elliptic and cancelling if $n \geq 2$. Since specialising to the symmetric gradient does not simplify the proof, we give the more general

**Lemma 2.2.** Let $A[D]$ be an elliptic operator of the above form and suppose that $u \in L^1_{\text{loc}}(\mathbb{R}^n; V)$. If, for some $0 < \alpha < 1$, $A[D]u$ is of class $C^{0,\alpha}$ in an open neighbourhood of some $x_0 \in \mathbb{R}^n$, then the full distributional gradient $Du$ exists as a $C^{0,\alpha}$-regular map in a neighbourhood of $x_0$.

**Proof.** Since the result is local, we may assume that $u$ is compactly supported and that $A[D]u \in C^{0,\alpha}(\mathbb{R}^n; W)$. For each $k \in \{1, \ldots, n\}$, ellipticity of $A[D]$ implies that the map $m_k : \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(W; V)$ given by $m_k(\xi) := \xi_k (A^* [\xi] A[\xi])^{-1} A^* [\xi]$ for $\xi = (\xi_1, \ldots, \xi_n)$ is well-defined, of class $C^\infty$ and homogeneous of degree zero on $\mathbb{R}^n \setminus \{0\}$; here, $A^* [\xi]$ is the adjoint symbol of $A[\xi]$. Hence, by the Hörmander-Mihlin multiplier theorem [65, Chpt. VI, Prop. 4], the Fourier multiplication operator $M_k(\varphi) := \mathcal{F}^{-1}(m_k \varphi)$, originally defined for $\varphi \in C^\infty_c(\mathbb{R}^n; W)$, extends to a bounded linear operator $M_k : L^p(\mathbb{R}^n; W) \to L^p(\mathbb{R}^n; V)$ for each $1 < p < \infty$. Since $\partial_\xi \varphi = M_k(A[D]\varphi)$ for $\varphi \in C^\infty_c(\mathbb{R}^n; V)$, we obtain from $A[D]u \in L^p(1/(1-\alpha)(\mathbb{R}^n; W)$ by routine means that $\partial_\xi u \in L^{p/(1-\alpha)}(\mathbb{R}^n; V)$. By Morrey’s embedding it then follows that $u$ is of class $C^{0,\alpha}$, and the proof is complete. \[\square\]

To state the next proposition, we remind the reader that on connected, open subsets of $\mathbb{R}^n$, the nullspace of the symmetric gradient operator in $\mathcal{D}'(\Omega; \mathbb{R}^n)$ is given by the rigid deformations
\begin{equation}
\mathcal{R}(\Omega) := \{ x \mapsto Ax + b : A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n \}.
\end{equation}
Then, for each open, bounded and connected $\Omega \subset \mathbb{R}^n$, we have for all $u \in \text{BD}(\Omega)$
\begin{equation}
\min_{a \in \mathcal{R}(\Omega)} \| u - a \|_{L^1(\Omega; \mathbb{R}^n)} = \inf_{a \in \mathcal{R}(\Omega)} \| u - a \|_{L^1(\Omega; \mathbb{R}^n)} \leq c \| u \|_{\text{Ed}(\Omega)}
\end{equation}
with $c = c(\Omega, n) > 0$. We refer to (2.9) as the Poincaré inequality on $\text{BD}(\Omega)$. Now we have

**Proposition 2.3.** Let $n \geq 2$ and $0 < \theta < 1$. Moreover, let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with Lipschitz boundary. Then there holds
\[
\text{BD}(\Omega) \hookrightarrow W^{\theta, \frac{n}{n-\theta}}(\Omega; \mathbb{R}^n),
\]
continuity of the embedding being understood with respect to the norm topologies. Moreover, the following holds:

(a) If $\Omega$ is connected, then for each $u \in \text{BD}(\Omega)$, there exists $a \in \mathcal{R}(\Omega)$ such that
\[
\| u - a \|_{W^{\theta, \frac{n}{n-\theta}}(\Omega; \mathbb{R}^n)} \leq c \| u \|_{\text{Ed}(\Omega)}
\]
where $c > 0$ is a constant that only depends on $\Omega, n)$.

(b) There exists a constant $c = c(n, \theta) > 0$ such that for every $x_0 \in \mathbb{R}^n$, $R > 0$ and every $u \in \text{BD}(\mathbb{R}^n)$ there exists $a \in \mathcal{R}(\mathbb{R}^n)$ such that
\begin{equation}
\left( \int_{B_R} \int_{B_R} \frac{|u(u(x)) - u(y)|^p}{|x-y|^{n+\frac{np}{\theta}}} \, dx \, dy \right)^{\frac{1}{p}} \leq CR^{1-\theta} \int_{B_R} |u|, \label{eq:2.10}
\end{equation}
where $u_0 := u - a$.

**Proof.** We argue for $\text{LD}(\Omega)$ first and hence let $u \in \text{LD}(\Omega)$. Since $\partial \Omega$ is Lipschitz, we record from above that there exists a bounded linear extension operator $\mathcal{E}_{\text{LD}} : \text{LD}(U) \to \text{LD}(\mathbb{R}^n)$. As $C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$ is norm-dense in $\text{LD}(\mathbb{R}^n)$, we find a sequence $\{v_k\} \subset C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$ with $v_k \to \mathcal{E}_{\text{LD}} u$ as $k \to \infty$ in $\text{LD}(\mathbb{R}^n)$. Then we obtain for $u_k := v_k|_{\Omega}$ that $u_k \to u$ in $\text{LD}(\Omega)$. By [71, Prop. 6.4],
the symmetric gradient operator $A[D] = \varepsilon$ is elliptic and cancelling as an operator from $V = \mathbb{R}^n$ to $W = \mathbb{R}^{n \times n}_\text{sym}$, and thus [71, Thm. 1.3 and Thm. 8.1] yield the inequality\footnote{Note that the embedding $BD \to L^{\infty}$ is originally due to Strauss [67].}
\[
\|v\|_{L^\infty(R^n;\mathbb{R}^n)} + \|\varepsilon(v)\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})} \leq C\|\varepsilon(v)\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})} \quad \text{for all } v \in C_c^0(\mathbb{R}^n;\mathbb{R}^n)
\]
with some finite constant $C' = C'(n, \theta) > 0$. This consequently yields
\[
\|u_k\|_{W^{\theta,\infty}((1+\varepsilon)(\Omega,\mathbb{R}^n))} \leq C\|\varepsilon(u_k)\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})} + \|\varepsilon(u_k)\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})} \leq C\|\varepsilon(u_k)\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})} + \|\varepsilon\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})} \leq C\|\varepsilon\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})} + C\|\varepsilon\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})} + C\|\varepsilon\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})}
\]
where $C = C(n, \theta) > 0$ is a constant. Since $v_k \to \varepsilon_{LD}u$ in $LD(\mathbb{R}^n)$ as $k \to \infty$ and by boundedness of $\mathcal{E}_{LD}$, $\liminf_{k \to \infty} \|u_k\|_{W^{\theta,\infty}((1+\varepsilon)(\Omega,\mathbb{R}^n))} \leq \|\varepsilon\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})}$. Also, since $u_k \to u$ in $L^1(\Omega;\mathbb{R}^n)$, we achieve $u_k \to u \in \mathcal{L}^n\text{-a.e.}$ for a non-relabeled subsequence. Therefore, by Fatou’s lemma, $\|u_k\|_{W^{\theta,\infty}((1+\varepsilon)(\Omega,\mathbb{R}^n))} \leq \|\varepsilon\|_{L^1(\mathbb{R}^n;\mathbb{R}^{n \times n}_\text{sym})}$. Now, for $u \in BD(\Omega)$, the statement follows from the inequality just proved by approximation with respect to the strict topology, passage to a suitable subsequence $\mathcal{L}^n\text{-a.e.}$ and Fatou’s lemma. We come to (a), for which we use Poincaré’s inequality on $BD(\Omega)$. Letting $\Omega \subset \mathbb{R}^n$ be open, bounded and connected, for each $u \in BD(\Omega)$ there exists $a \in \mathcal{A}(\Omega)$ such that, for some constant $C = C(\Omega) > 0$, $\|u - a\|_{L^1(\Omega;\mathbb{R}^n)} \leq C E u(\Omega)$. With this choice of $a$ and by the foregoing part of the proposition,
\[
\|u - a\|_{W^{\theta,\infty}((1+\varepsilon)(\Omega,\mathbb{R}^n))} \leq C\|u - a\|_{L^1(\Omega;\mathbb{R}^n)} + \|E u(\Omega)\| \leq C E u(\Omega),
\]
where $C = C(\Omega, n, \theta) > 0$ is constant. Ad (b). We may assume that $x_0 = 0$, and shall write $B_r := B(0,r)$ for $r > 0$. Letting $u \in LD(\mathbb{R}^n)$, we first determine an element $b \in \mathcal{A}(\mathbb{R}^n)$ such that (\*) in the following inequality holds on the unit ball, due to part (a) with $\Omega = B(0,1)$:
\[
\left( \int_{B_r} \int_{B_r} \frac{|u_b(x) - u_b(y)|^p}{|x - y|^{n+\theta p}} \, dx \, dy \right)^{\frac{1}{p}} \leq \frac{R^{\frac{n-\theta p}{p}}}{R^{\frac{n-\theta p}{p}}} \left( \int_{B_1} \int_{B_1} \frac{|u_b(Rx) - u_b(Ry)|^p}{|x - y|^{n+\theta p}} \, dx \, dy \right)^{\frac{1}{p}} \leq \frac{R^{\frac{n-\theta p}{p}}}{R^{\frac{n-\theta p}{p}}} R \int_{B_1} |(\varepsilon(u_b))(Rx)| \, dx = R \frac{1+\theta}{\theta} \int_{B_r} |\varepsilon(a)| \, dx.
\]
This in turn determines $a \in \mathcal{A}(\mathbb{R}^n)$. We divide the last inequality by $R^\theta$ to consequently deduce (2.10) for LD-maps $u$, and the BD-case follows by smooth approximation in the strict topology by routine means. The proof is hereby complete.

\section*{2.3. Functions of measures.} Here we briefly record the most important features of functions being applied to measures. Hence let $f : \mathbb{R}^{n \times n}_\text{sym} \to \mathbb{R}_\geq 0$ be convex and of linear growth, by which we understand (LG) in this subsection. We recall that the recession function $f^\infty : \mathbb{R}^{n \times n}_\text{sym} \to \mathbb{R}$ is given by
\[
f^\infty(z) := \lim_{t \to 0} f\left(\frac{z}{t}\right), \quad z \in \mathbb{R}^{n \times n}_\text{sym},
\]
and by convexity and the linear growth hypothesis, \( f^\circ \) is well-defined. Given \( \mu \in \mathcal{M}(\Omega; \mathbb{R}^{n \times n}_{\text{sym}}) \), we denote its Lebesgue-Radon-Nikodým decomposition \( \mu = \mu^s + \mu^t \) and then define the measure \( f(\mu) \) by

\[
(2.11) \quad f(\mu)(A) := \int_A f \left( \frac{d\mu^a}{d\mathcal{L}^n} \right) \, d\mathcal{L}^n + \int_A f^\circ \left( \frac{d\mu^t}{d|\mu^t|} \right) \, d|\mu^t| \quad \text{for all } A \in \mathcal{R}(\Omega).
\]

If \( f \) is merely assumed symmetric-rank-one convex (so is convex with respect to directions in the symmetric rank-one cone \( \mathbb{R}^n \circ \mathbb{R}^n : = \{ a \circ b : a, b \in \mathbb{R}^n \} \)) and of linear growth, then (2.11) still is a valid definition provided the density \( \frac{d\mu^t}{d|\mu^t|} \) takes values in the symmetric-rank-one cone \( |\mu^t|\)-a.e.. In fact, in this situation, \( f \) is convex along directions contained in \( \mathbb{R}^n \circ \mathbb{R}^n \) and so, by the linear growth assumption, \( f^\circ \) exists provided \( z \in \mathbb{R}^n \circ \mathbb{R}^n \). When applying such integrands \( f \) to \( E\mu \) for \( \mu \in \text{BD}(\Omega) \), then the recent work of DE PHILIPPIS & RINDLER [25] yields \( \frac{d\mu^a}{d\mathcal{L}^n} \in \mathbb{R}^n \circ \mathbb{R}^n \) \( |E\mu|\)-a.e.. Hence

\[
\int_A f(\mu)(A) = \int_A f(\mu)(A) := \int_A f(\mu)(A) = \int_A f(\mu)(A) = \int_A f(\mu)(A) + \int_A f^\circ(u) \, d|E\mu|
\]

for \( \mu \in \text{BD}(\Omega) \) is in fact a well-posed definition. Working from the previous ideas and upon the method of proof for signed variants given in [50], the fundamental background fact result that we shall rely on in the sequel now is essentially due to RINDLER [61]:

**Theorem 2.5.** Let \( \Omega \subset \mathbb{R}^n \) be an open and bounded Lipschitz domain and let \( f \in C(\mathbb{R}^{n \times n}) \) be a symmetric quasiconvex integrand which, in addition, satisfies (LG). Then, with the notation of (2.3), the functional

\[
\mathcal{F}[\mu; \Omega] := \int_\Omega f(\mu) \, dx + \int_\Omega f^\circ \left( \frac{dE\mu}{d|E\mu|} \right) \, d|E\mu|
\]

for \( \mu \in \text{BD}(\Omega) \) is lower semicontinuous with respect to weak*-convergence in the space \( \text{BD}(\Omega) \), where \( \text{Tr}_{\partial \Omega}(u) \) is the trace of \( u \) along \( \partial \Omega \).

Finally, a lemma on the continuity of symmetric rank-one-convex functions for the area-strict metric that we shall frequently employ in conjunction with smooth approximation; in effect, it appears as a generalisation of the classical convex RESHETNYAK (semi-)continuity theory [60]:

**Lemma 2.6.** (Symmetric rank-one-convexity and area-strict continuity). Let \( f \in C(\mathbb{R}^{n \times n}) \) be symmetric rank-one convex with (LG) and let \( \Omega \subset \mathbb{R}^n \) be an open and bounded set. Then \( \text{BD}(\Omega) \ni u \mapsto f(\mu)(\Omega) \) is continuous with respect to area-strict convergence.

The lemma follows from [18, Prop. 5.1] upon specifying to the symmetric gradient.

### 2.4. V-function estimates and Korn-type inequalities

For future applications in Section 3, we record some non-standard forms of Korn-type inequalities and gather here the relevant background results from BREIT & DIENING [17]. Note that, alternatively, the specifically required forms could also be tracked back to ACERBI & MINGIONE [4] but then would follow only by inspection of the proof of [4, Thm. 3.1].

A convex, left-continuous function \( \psi : \mathbb{R}_{\geq 0} \to [0, \infty] \) is called a \( \Phi \)-function provided

\[
(2.12) \quad \psi(0) = 0, \quad \lim_{t \downarrow 0} \psi(t) = 0, \quad \lim_{t \to \infty} \psi(t) = \infty.
\]

We now say that a \( \Phi \)-function \( \psi \) is of class \( \Delta_2 \) provided there exists \( K > 0 \) such that \( \psi(2t) \leq K \psi(t) \) for all \( t \geq 0 \), and the infimum over all possible such constants is denoted \( \Delta_2(\psi) \). Similarly, we say that a \( \Phi \)-function \( \psi \) is class \( \mathcal{V}_2 \) provided the Fenchel conjugate \( \psi^\circ(t) := \sup_{s \geq 0}(st - \psi(s)) \) is of class \( \Delta_2 \); we put \( \mathcal{V}_2(\phi) := \Delta_2(\psi^\circ) \). We then have

**Proposition 2.7** ([17, Thm. 1]). Let \( \psi : \mathbb{R}_{\geq 0} \to [0, \infty] \) be a \( \Phi \)-function. Then the following are equivalent:

(a) \( \psi \) is both of class \( \Delta_2 \) and \( \mathcal{V}_2 \), abbreviated by \( \psi \in \Delta_2 \cap \mathcal{V}_2 \).
(b) There exists a constant \( A = A(\Delta_2(\psi), \nabla_2(\psi)) > 0 \) such that for all \( u \in C^0_c(\mathbb{R}^n; \mathbb{R}^n) \) there holds
\[
\int_{\mathbb{R}^n} \psi(|Du|) \, dx \leq \int_{\mathbb{R}^n} \psi(A|\varepsilon(u)|) \, dx.
\]

We next collect some facts about shifted \( \Phi \)-functions from [27, 28]. Letting \( \psi \) be as in in (2.12), we put
\[
\eta(t) := \int_0^t \phi(t+s) \, ds, \quad t \geq 0.
\]

The following lemma compactly gathers the for us most relevant results on shifted \( \Phi \)-functions:

**Lemma 2.8.** Let \( \phi \in C^2([\mathbb{R}_0; \mathbb{R}_0]) \) such that \( c_1 t \phi'(t) \leq \phi(t) \leq c_2 t \phi''(t) \) for some \( c_1, c_2 > 0 \) and all \( t \geq 0 \). Given \( a \geq 0 \), define \( \phi_a \) by (2.13). Then both \( \Delta_2(\phi_a) \) and \( \nabla_2(\phi_a) \) are finite and independent of \( a \).

We come to the requisite estimates of \( V \)-functions, which we define for \( 1 \leq p < \infty \) by
\[
V_p(z) := (1 + |z|^2)^{\frac{1}{p}} - 1, \quad z \in \mathbb{R}^m,
\]
so that, with the terminology of (1.4)ff., \( V = V_1 \); note that \( V_p \in \Delta_2 \nabla_2 \) if and only if \( 1 < p < \infty \).

**Lemma 2.9 ([42, Sec. 2.4], [24, Lem. 2.4]).** Let \( m \in \mathbb{N} \). Then there exist constants \( c > 0 \) merely depending on \( m \) such that for all \( \lambda \geq 1 \),
\[
\begin{align*}
V_\lambda(z) &\leq \lambda^2 V(z) \quad \text{for all } \lambda \geq 1, \\
V(z+w) &\leq 2(V(z) + V(w)) \quad \text{for all } z, w \in \mathbb{R}^m.
\end{align*}
\]

Moreover, for each \( 1 < p < \infty \) there exist two constants \( 0 < \Theta_p \leq \Theta < \infty \) such that for all \( z, z' \in \mathbb{R}^m \) there holds
\[
\begin{align*}
\theta_p(1 + |z|^2 + |z'|^2)^{\frac{p-2}{2}} |z|^2 &\leq V_p(z + z') - V_p(z) - (V_p'(z), z') \\
&\leq \Theta_p(1 + |z|^2 + |z'|^2)^{\frac{p-2}{2}} |z'|^2.
\end{align*}
\]

2.5. **Miscellaneous auxiliary results.** In this final subsection we gather some mixed technical results. We begin with the Ekeland Variational Principle [32], helping us to obtain good approximating sequences of certain BD-maps later on, in a form given in [39, Thm. 5.6, Rem. 5.5]:

**Lemma 2.10 (Ekeland Variational Principle).** Let \( (X,d) \) be a complete metric space and let \( \mathcal{F} : X \to \mathbb{R} \cup \{+\infty\} \) be a lower semicontinuous function for the metric topology, bounded from below and taking a finite value at some point. Assume that for some \( u \in X \) and some \( \varepsilon > 0 \) we have \( \mathcal{F}[u] \leq \inf_X \mathcal{F} + \varepsilon \). Then there exists \( v \in X \) such that
\[
\begin{align*}
(a) \quad d(u,v) &\leq \sqrt{\varepsilon}, \\
(b) \quad \mathcal{F}[v] &\leq \mathcal{F}[u], \\
(c) \quad \mathcal{F}[w] &\leq \mathcal{F}[v] + \sqrt{\varepsilon}d(v,w) \quad \text{for all } w \in X.
\end{align*}
\]

For the following, let us recall that a symmetric bilinear form \( \mathcal{A} \in \mathcal{S}(\mathbb{R}^{n \times n}) \) is called strongly rank-one convex or strongly Legendre-Hadamard provided there exists \( \lambda > 0 \) such that for all \( a, b \in \mathbb{R}^n \) there holds \( \mathcal{A}[a \otimes b, a \otimes b] \geq \lambda |a \otimes b|^2 \). For such bilinear forms, we have the following:
is a topologically linear isomorphism. Here, $W^{-1,p}(B; \mathbb{R}^N) := (W^{1,p}_0(B; \mathbb{R}^N))^*$ as usual. Moreover, if $-\text{div}(\lambda N u) = 0$ in $\mathcal{D}'(\Omega; \mathbb{R}^N)$, then there holds $u \in C^\infty(\Omega; \mathbb{R}^N)$ together with
\[
\sup_{B(x_0, \frac{R}{2})} \|u - A\|_R + \sup_{B(x_0, \frac{R}{2})} \|V^2 u\|_R \leq C \int_{B(x_0, R)} |\nabla u - A| \, dx \quad \text{for all } A \in \mathbb{R}^{N \times N}
\]
for all $B(x_0, R) \subseteq \Omega$, where $C = C(n, N, \lambda, \Lambda) > 0$ is a constant.

Note that this lemma appears in the spirit of MAZ’YA & SHAPOSHNIKOVA [53, Lem. 15.2.1] or TRIEBEL [70, Sec. 5.7.2], but these references do not give the precise form as required here. Finally, a standard iteration result:

**Lemma 2.12** ([42, Lem. 4.4]). Let $\theta \in (0, 1)$ and $R > 0$. Suppose that $\Phi, \Psi : (0, R] \to \mathbb{R}$ are non-negative functions such that $\Phi$ is bounded and $\Psi$ is decreasing together with $\Psi(\sigma t) \leq \sigma^{-2}\Psi(t)$ for all $t \in (0, R]$ and $\sigma \in (0, 1]$. Moreover, suppose that there holds
\[
\Phi(r) \leq \theta \Phi(s) + \Psi(s - r)
\]
for all $r, s \in [\frac{R}{2}, R]$ with $r < s$. Then there exists a constant $C = C(\theta) > 0$ such that
\[
\Phi\left(\frac{R}{2}\right) \leq C\Psi(R).
\]

### 3. Strong Symmetric Quasiconvexity and Coerciveness

In this intermediate section we give justification of the strong symmetric quasiconvexity condition as it appears in Theorem 1.2 and compare it with the corresponding variants for superlinear – i.e., $1 < p < \infty$ – growth functionals. As we shall elaborate on in detail below, $p$-strong symmetric quasiconvexity reflects a coerciveness property of the associated multiple integrals and thus is related to $p$-strong quasiconvexity by virtue of Korn’s inequality. Specifying this in Theorem 3.2, we directly obtain a partial regularity result from the full gradient case. In turn, the failure of Korn’s inequality in the $L^1$-framework does not allow to conclude that symmetric quasiconvex, linear growth functionals depending on the symmetric gradients are coercive on $BV$. This underlying obstruction in reducing Theorem 1.2 to the corresponding variant for strongly quasiconvex full gradient functionals on $BV$ thus motivates the need of an independent proof of Theorem 1.2 and hence the theme of the paper at all.

Rather than reproducing the proof of 42, Prop. 3.1] with the relevant but easy modifications, we confine to stating the following equivalence between strong symmetric quasiconvexity and coerceness; throughout, $\Omega$ is assumed to be an open and bounded Lipschitz domain in $\mathbb{R}^n$.

**Lemma 3.1.** Let $f \in C(\mathbb{R}^{n \times n})$ satisfy (LG) and let $u_0 \in W^{1,1}(\Omega; \mathbb{R}^n)$ be a given Dirichlet datum. Then all minimising sequences of the variational problem
\[
(3.1) \quad \text{to minimise } \int_{\Omega} f(\varepsilon(v)) \, dx \quad \text{over } LD_{u_0}(\Omega)
\]
are bounded in $LD(\Omega)$ if and only if there exists $z_0 \in \mathbb{R}^{n \times n}_{sym}$ and $\ell > 0$ such that the function $h : z \mapsto f(z) - \ell V(z)$ is symmetric quasiconvex at $z_0$; this is, for all $\phi \in C^1_b(\Omega; \mathbb{R}^n)$ there holds
\[
h(z_0) \leq \int_{\Omega} h(z_0 + \varepsilon(\phi)) \, dx.
\]

Clearly, if there exists $\ell > 0$ such that $f - \ell V$ is symmetric quasiconvex at every $z_0 \in \mathbb{R}^{n \times n}$ then $f$ is strongly symmetric quasiconvex. As can be seen from [42, Prop. 2.14], strongly (symmetric) quasiconvex integrands can be obtained by approximation from essentially any quasiconvex $p$-growth integrand. As to the above lemma, let us note that even in the slightly more accessible case of convex integrands $f : \mathbb{R}^{n \times n} \to \mathbb{R}$, there is no result available ensuring the boundedness of minimising sequences for (3.1) in $W^{1,1}(\Omega; \mathbb{R}^n)$; the only result available in this direction for convex integrands achieves such a boundedness assertion locally, for particular minimising sequences and a very restricted ellipticity range strictly included in that implied by
Theorem 1.2. See Remark 3.3 below for a discussion. To explain how Korn’s inequality changes the situation in the superlinear growth regime, we adopt a more general viewpoint. Henceforth, let $1 \leq p < \infty$ and suppose that $G \in C(\mathbb{R}^{n \times n})$ is of $p$-growth in the sense that there exists $c > 0$ such that

\begin{equation}
(3.2) \quad |G(z)| \leq c(1 + |z|^p)
\end{equation}

for all $z \in \mathbb{R}^{n \times n}$. Recalling the function $V_p$ from (2.14), we say that a function $G: \mathbb{R}^{n \times n} \to \mathbb{R}$ is $p$-strongly quasiconvex if and only if there exists $\lambda > 0$ such that

\begin{equation}
(3.3) \quad \mathbb{R}^{n \times n} \ni z \mapsto G(z) - \lambda V_p(z) \in \mathbb{R} \quad \text{is quasiconvex.}
\end{equation}

In a similar manner, we say that a function $g: \mathbb{R}^{n \times n} \to \mathbb{R}$ is $p$-strongly symmetric quasiconvex if and only if

\begin{equation}
(3.4) \quad \mathbb{R}^{n \times n} \ni z \mapsto g(z) - \lambda V_p(z) \in \mathbb{R} \quad \text{is symmetric quasiconvex.}
\end{equation}

As a consequence of the last part of Lemma 2.9, if $1 < p < \infty$, then $p$-strong quasiconvexity of $G \in C^2(\mathbb{R}^{n \times n})$ is equivalent to the existence of a constant $v_1 > 0$ such that

\begin{equation}
(3.5) \quad v_1 \int_Q (1 + |\xi|^2 + |D\phi|^2)^{\frac{p-2}{2}} |D\phi|^2 \, dx \leq \int_Q G(z + D\phi) - G(z) \, dx
\end{equation}

holds for all $z \in \mathbb{R}^{n \times n}$ and all $\phi \in C^2_0(\mathcal{Q}; \mathbb{R}^n)$; the $p$-strong symmetric quasiconvexity can be characterised analogously. For completeness, however, we note that this is not the case for $p = 1$; see Remark 3.3 below. The main purpose of the remaining section is to establish the following theorem, which follows in a relatively easy way from known partial regularity results for full gradient functionals:

**Theorem 3.2.** Let $1 < p < \infty$ and suppose that $g \in C^2(\mathbb{R}^{n \times n})$ is an integrand which

(a) is of $p$-growth, i.e., satisfies (3.2) for all $z \in \mathbb{R}^{n \times n}$ and

(b) is $p$-strongly symmetric quasiconvex in the sense of (3.4).

Then for any local minimiser $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^n)$ of the corresponding integral functional

\begin{equation}
(3.6) \quad v \mapsto \int_\Omega g(\xi(v)) \, dx
\end{equation}

there exists an open subset $\Omega_\alpha$ of $\Omega$ such that $\mathcal{L}^n(\Omega \setminus \Omega_\alpha) = 0$ and $u$ is of class $C^{1,\alpha}$ for each $0 < \alpha < 1$ in a neighbourhood of any of the elements of $\Omega_\alpha$.

**Remark 3.3.** When linear growth integrands are concerned, setting $p = 1$ in (3.5) does not give rise to an equivalent notion of (1-)strong quasiconvexity in the sense of (3.3) with $p = 1$. This can be even seen for strongly convex linear growth integrands such as the area integrand $m: z \mapsto \sqrt{1 + |z|^2} (= V(z) + 1)$, compare (5.2) from below. The underlying reason for this is that convex, linear growth $C^2$-integrands typically exhibit $(p,q)$-type growth behaviour on the level of the second derivatives in the following sense: There exist $1 < a < \infty$ and constants $\Lambda_1, \Lambda_2 > 0$ such that

\begin{equation}
(3.7) \quad \Lambda_1 \frac{|z|^2}{(1 + |\xi|^2)^2} \leq \langle m''(\xi)z, z \rangle \leq \Lambda_2 \frac{|z|^2}{(1 + |\xi|^2)^2} \quad \text{for all } z, \xi \in \mathbb{R}^{n \times n},
\end{equation}

see [41] and [15, Ex. 4.17] for a discussion. Connecting with the theme from above, if $f \in C^2(\mathbb{R}^{n \times n})$ is convex, satisfies (LG) and (3.7) with $1 < a < \frac{n+1}{\pi}$, then specific minimising sequences obtained by a vanishing viscosity approach are locally uniformly bounded in $W^{1,1}$, cf. [41]. However, this neither implies their global boundedness in $W^{1,1}$ nor the global boundedness in $W^{1,1}$ for all minimising sequences. On the other hand, the area integrand $m$ satisfies (3.7) with $a = 3 > \frac{n+1}{\pi}$ and so neither the local uniform boundedness of suitable minimising sequences in $W^{1,1}$ can be asserted.
3.1. Proof of Theorem 3.2, $1 < p < 2$. Toward the proof of Theorem 3.2 in the growth regime $1 < p < 2$, we record the following result due to Carozza, Fusco and Mingione [22]:

**Proposition 3.4** ([22, Thm. 3.2]). Let $1 < p < 2$ and suppose that $G \in C^2(\mathbb{R}^{n \times n})$ satisfies (3.2) for all $z \in \mathbb{R}^{n \times n}$ together with (3.3). Then for any local minimiser $u \in W^{1,p}_0(\Omega; \mathbb{R}^n)$ of the integral functional $v \mapsto \int_{\Omega} G(Dv) \, dx$ there exists an open set $\Omega_u \subset \Omega$ with $\mathcal{L}^n(\Omega \setminus \Omega_u) = 0$ such that $u$ is of class $C^{1,\alpha}$ in a neighbourhood of any of the elements of $\Omega_u$.

Working from Proposition 3.4, let $g \in C^2(\mathbb{R}^{n \times n}_{\text{sym}})$ satisfy the assumptions of Theorem 3.2. We then define a new integrand $G_g : \mathbb{R}^{n \times n} \to \mathbb{R}$ by

$$
G_g(z) := g(z_{\text{sym}}), \quad z \in \mathbb{R}^{n \times n}.
$$

Our aim is to establish that $G_g$ satisfies the assumptions of Proposition 3.4. Clearly, $G_g = g \circ \Pi_{\text{sym}}$, where $\Pi_{\text{sym}} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}_{\text{sym}}$ is the orthogonal projection onto the symmetric matrices, and hence $G_g \in C^2(\mathbb{R}^{n \times n})$. Moreover, since $|z_{\text{sym}}| \leq |z|$ for all $z \in \mathbb{R}^{n \times n}$, $|G_g(z)| \leq c(1 + |z_{\text{sym}}|^p) \leq c(1 + |z|^p)$, and so $G_g$ satisfies (3.2) for all $z \in \mathbb{R}^{n \times n}$. It thus remains to show that $G_g$ is $p$-strongly quasiconvex. As an instrumental ingredient, we claim that there exists a constant $c = c(p,n) > 0$ such that

$$
(3.9) \quad \int_Q (1 + |z|^2 + |D\phi|^2)^{\frac{p-2}{2}} |D\phi|^2 \, dx \leq c \int_Q (1 + |z_{\text{sym}}|^2 + |\epsilon(\phi)|^2)^{\frac{p-2}{2}} |\epsilon(\phi)|^2 \, dx
$$

holds for all $z \in \mathbb{R}^{n \times n}$ and all $\phi \in C_c^\infty(Q; \mathbb{R}^n)$.

In view of (3.9), let $\phi \in C_c^\infty(Q; \mathbb{R}^n)$ and $z \in \mathbb{R}^{n \times n}$ be arbitrary. Since $1 < p < 2$, the function $s \mapsto (1 + |s|^2 + |D\phi(s)|^2)^{\frac{p-2}{2}}$ is decreasing in $s$ for every $x \in Q$. Thus, as $|z_{\text{sym}}| \leq |z|$ for all $z \in \mathbb{R}^{n \times n}$,

$$
(3.10) \quad \int_Q (1 + |z|^2 + |D\phi|^2)^{\frac{p-2}{2}} |D\phi|^2 \, dx \leq \int_Q (1 + |z_{\text{sym}}|^2 + |D\phi|^2)^{\frac{p-2}{2}} |D\phi|^2 \, dx =: (\ast).
$$

Now, define a function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$
(3.11) \quad \psi(t) = (1 + t)^{p-2}t^2, \quad t \geq 0.
$$

Then we have, with the correspondingly shifted function $\psi_a$, being defined for $a \geq 0$ by (2.13),

$$
(3.12) \quad \psi_a(t) \simeq \psi''(a + t)^2 \simeq (1 + a + t)^{p-2}t^2 \simeq (1 + |a|^2 + |t|^2)^{\frac{p-2}{2}}t^2,
$$

and the constants implicit in ‘$\simeq$’ are independent of $a$; the lengthy yet elementary verification of this fact is deferred to the appendix, Section 6.2. As a consequence of Lemma 2.8 and $p > 1$, $\psi_a$ belongs to $\Delta_2 \cap V_2$ and, most importantly, $\Delta_2(\psi_a)$ and $V_2(\psi_a)$ are independent of $a$. Hence, by Proposition 2.7, there exists a constant $A = A(\Delta_2(\psi_a), V_2(\psi_a)) > 0$ – which, since $\Delta_2(\psi_a)$ and $V_2(\psi_a)$ do not depend on $a$, is actually independent of $a$: $A = A(\Delta_2(\psi), V_2(\psi)) > 0$ – such that for all $\phi \in C_c^\infty(Q; \mathbb{R}^n)$ there holds

$$
(3.13) \quad \int_Q \psi_a(|D\phi|) \, dx \leq \int_Q \psi_a(A|\epsilon(\phi)|) \, dx.
$$

Clearly, since $\psi$ and each $\psi_a$ are monotonically increasing, we may assume that $A > 1$. Applying the previous inequality to the particular choice $a = |z_{\text{sym}}|$, we therefore obtain

$$
(\ast) \quad (3.12) \quad \int_Q \psi_a(|D\phi|) \, dx \leq c \int_Q \psi_a(|\epsilon(\phi)|) \, dx \leq c \int_Q \psi_a(|D\phi|^2)^{\frac{p-2}{2}} |\epsilon(\phi)|^2 \, dx \leq cA^2 \int_Q (1 + |z_{\text{sym}}|^2 + |\epsilon(\phi)|^2)^{\frac{p-2}{2}} |\epsilon(\phi)|^2 \, dx,
$$

for all $z \in \mathbb{R}^{n \times n}$.
the last inequality being valid by \( A > 1 \) and \( p - 2 < 0 \). Then, combining (3.10) and (3.14), we arrive at (3.9). We can then proceed in showing that \( G_\delta \) defined by (3.8) is strongly \( p \)-quasiconvex. To this end, let \( \phi \in C_c^0(\Omega; \mathbb{R}^n) \) and \( z \in \mathbb{R}^{n \times n} \) be arbitrary. Then we find
\[
\int_O G_\delta(z + D\phi) - G_\delta(z) \, dx = \int_O g(z^{\text{sym}} + \epsilon(\phi)) - g(z^{\text{sym}}) \, dx \\
\geq \lambda \int_O (1 + |z^{\text{sym}}|^2 + |\epsilon(\phi)|^2)^{\frac{p-2}{2}} |\epsilon(\phi)|^2 \, dx \\
\geq \nu \int_O (1 + |\phi|^2 + |D\phi|^2)^{\frac{p-2}{2}} |D\phi|^2 \, dx,
\]
where \( \nu = \frac{\lambda}{\lambda} \) with \( \lambda > 0 \) from (3.4) and the constant \( c > 0 \) from (3.9). To conclude the proof of Theorem 3.2 for \( 1 < p < 2 \), let \( u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^n) \) be a local minimiser of the functional given by (3.6). Then we have for all compactly supported Sobolev maps \( \psi \in W^{1,p}(\Omega; \mathbb{R}^n) \)
\[
\int_{\Omega} G_\delta(D(u + \psi)) \, dx = \int_{\Omega} g(\epsilon(u + \psi)) \, dx \geq \int_{\Omega} g(\epsilon(u)) \, dx = \int_{\Omega} G_\delta(Du) \, dx.
\]
Hence similarly we obtain a local minimiser of the functional \( \nu \mapsto \int_{\Omega} G_\delta(D\nu) \, dx \), and \( G_\delta \) satisfies the assumptions of Proposition 3.4. Thus Theorem 3.2 follows for the growth range \( 1 < p < 2 \).

3.2. Proof of Theorem 3.2, 2 \( \leq \) \( p \) \( < \) \( \infty \). Aiming to imitate the preceding proof for \( 2 \leq p < \infty \), we note that (3.10) cannot be derived similarly for \( p > 2 \). In fact, the relevant map \( s \mapsto (1 + s^2 + |D\phi(x)|^2)^{\frac{p-2}{2}} \) is not decreasing in \( s \) anymore. To obtain Theorem 3.2 for this growth range though, we require a refined partial regularity result for full gradient functionals. If \( G \in C^2(\mathbb{R}^{n \times n}) \) satisfies (3.3) from above, then it equally satisfies the slightly weaker condition
\[
(3.15) \quad \mu \int_O |D\phi|^2 + |D\phi|^p \, dx \leq \int_O G(z + D\phi) - G(z) \, dx
\]
for all \( z \in \mathbb{R}^{n \times n} \) and all \( \phi \in C_c^0(\Omega; \mathbb{R}^n) \), where \( \mu > 0 \) is a constant. Subject to the weaker condition (3.15), KRISTENSEN & TAHERT [51] established the following regularity result:

**Proposition 3.5 ([51, Thm. 4.1]).** Let \( p \geq 2 \) and suppose that \( G \in C^2(\mathbb{R}^{n \times n}) \) is an integrand that satisfies the \( p \)-growth condition (i) from above together with (3.15). Then any local minimiser of the integral functional
\[
\mathcal{G} : u \mapsto \int_{\Omega} G(Du) \, dx
\]
is \( C^{1,\alpha} \)-partially regular: There exists an open subset \( \Omega_\alpha \subset \Omega \) with \( \mathcal{H}^n(\Omega \setminus \Omega_\alpha) = 0 \) such that \( u \) is of class \( C^{1,\alpha} \) for any \( 0 < \alpha < 1 \) in a neighbourhood of any of the elements of \( \Omega_\alpha \).

Actually, the preceding proposition is stated in [51] quite differently, in the context of \( W^{1,q} \)-local minimisers: here, given \( 1 \leq q < \infty \), \( u \in (W^{1,q}_\text{loc} \cap W^{1,p})(\Omega; \mathbb{R}^n) \) is called a \( W^{1,q} \)-local minimiser provided there exists \( \delta > 0 \) such that, for all \( \phi \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^n) \) with \( \|D\phi\|_{L^{q}(\Omega;\mathbb{R}^{n \times n})} \leq \delta \) there holds \( \mathcal{G}[\phi] \leq \mathcal{G}[\phi + \phi] \). Clearly, any local minimiser \( u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^n) \) of \( \mathcal{G} \) is a \( W^{1,p} \)-local minimiser, and so Proposition 3.5 is in action for all such maps \( u \).

In view of Theorem 3.2, we define \( G_\delta \) analogously as in (3.8). Then similarly as above, \( G_\delta \in C^2(\mathbb{R}^{n \times n}) \) and \( G_\delta \) satisfies (3.2) for all \( z \in \mathbb{R}^{n \times n} \). Since \( g : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \) satisfies (3.4), we obtain similarly to (3.5) that there exists \( \nu_2 > 0 \) such that
\[
\nu_2 \int_Q (1 + |z^{\text{sym}}|^2 + |\epsilon(\phi)|^2)^{\frac{p-2}{2}} |\epsilon(\phi)|^2 \, dx \leq \int_Q g(z^{\text{sym}} + \epsilon(\phi)) - g(z^{\text{sym}}) \, dx
\]
for all \( z \in \mathbb{R}^{n \times n} \) and all \( \phi \in C_c^0(\Omega; \mathbb{R}^n) \). Therefore, as \( p \geq 2 \), there exists \( \tilde{\nu}_2 > 0 \) such that
\[
\tilde{\nu}_2 \int_Q |\epsilon(\phi)|^2 + |\epsilon(\phi)|^2 \, dx \leq \int_Q g(z^{\text{sym}} + \epsilon(\phi)) - g(z) \, dx
\]
for all \( z \in \mathbb{R}^{n \times n} \) and all \( \varphi \in C^0_c(Q; \mathbb{R}^n) \). By the usual \( L^q \)-Korn inequalities (i.e., considering the \( \Phi \)-function \( t \mapsto r^q \) for \( q > 1 \) in Proposition 2.7), there exists a constant \( c = c(p) > 0 \) such that
\[
\int_Q G_\varepsilon(z + D\varphi) - G_\varepsilon(z) \, dx = \int_Q g(\sym + \varepsilon(\varphi)) - g(\sym) \, dx \\
\geq \tilde{\varepsilon}_2 \int_Q |\varepsilon(\varphi)|^p + |\varepsilon(\varphi)|^p \, dx \geq c\tilde{\varepsilon}_2 \int_Q |D\varphi|^p + |D\varphi|^p \, dx
\]
for all \( \varphi \in C^0_c(Q; \mathbb{R}^n) \) and \( z \in \mathbb{R}^{n \times n} \). Hence \( G_\varepsilon \) satisfies (3.15) with \( \mu = c\tilde{\varepsilon}_2 \); the claimed partial regularity assertion of Theorem 3.2 for \( 2 \leq p < \infty \) then follows from Proposition 3.5, and the proof is complete. □

We conclude this section by justifying the particular use of Proposition 3.5 instead of perhaps more classical partial regularity results and giving examples of \( p \)-strongly symmetric quasiconvex integrands.

Remark 3.6 (Proposition 3.5 and Evans’ result from [33]). The first partial regularity theorem for strongly quasiconvex integrands in the growth regime \( 2 \leq p < \infty \) is due to Evans [33], also see Acerbi & Fusco [2]. These results are stated for \( p \)-strongly quasiconvex full gradient functionals in the spirit of (3.5). If we do not take the detour via the weaker condition (3.15) (which is sufficient for Proposition 3.5 but unclear to suffice for the partial regularity conclusions of [33, 2]), then we are bound to establish the Korn-type inequality (3.9) for \( p \geq 2 \). Whereas the proof of the latter is trivial for \( p = 2 \), it is not obvious to us how to approach it for \( p > 2 \).

4. A Fubini-type theorem for BD–maps

As one of the main tools in the proof of Theorem 1.2, we now give a Fubini-type result for functions of bounded deformation. In effect, this establishes that on \( \mathcal{L}^1 \)-a.e. sphere with fixed center, BD-maps possess additional fractional differentiability and integrability; on arbitrary spheres, we can only expect \( L^1 \)-integrability of interior traces. Aiming to linearise later on, suitable competitor maps attaining these more regular boundary values then will equally belong to better Sobolev spaces and so the results of Lemma 2.11 become accessible.

Theorem 4.1. Let \( n \geq 2 \) and \( 0 < \theta < 1 \) be arbitrary. Let \( x_0 \in \mathbb{R}^n, R > 0 \) and \( u \in \operatorname{BD}_\operatorname{loc}(\mathbb{R}^n) \). Then for \( \mathcal{L}^1 \)-almost all radii \( 0 < r < R \), the restrictions \( u|_{\partial B(x_0, r)} \) are well–defined and belong to the space \( W^{0,n/(n-1-\theta)}(\partial B(x_0, r); \mathbb{R}^n) \).

Moreover, there exists a constant \( C = C(n, \theta) > 0 \) (which, in particular, is independent of \( x_0, R \) and \( u \)), such that for all \( 0 < s < r \leq R \) there exists \( t \in (s, r) \) with
\[
(4.1) \quad \left( \int_{\partial B(x_0, t)} \int_{\partial B(x_0, t)} \frac{|u_\alpha(x) - u_\alpha(y)|^\frac{n-1}{\theta}}{|x - y|^{n-1+\frac{n-1}{\theta}}} \, d\sigma_x \, d\sigma_y \right)^{\frac{n-1+\theta}{n}} \leq C \frac{r^n}{t^{n-1(n-1+\theta)}(s - r)^{\frac{n-1+\theta}{\theta}}} \times \frac{1}{t} \int_{\partial B(x_0, r)} |\varepsilon u|,
\]
where \( \alpha \in \mathcal{A}(\mathbb{R}^n) \) is a suitable rigid deformation. Especially, \( C > 0 \) does not depend on \( u, s, t, r, R \) or \( x_0 \).

Proof. It is no loss of generality to assume \( x_0 = 0 \), and hence we write \( B_r := B(0, r) \) in the sequel. For clarity, we divide the proof into three parts.

Step 1. A general Fubini-type theorem for \( W^{0,p} \)-maps. In a first step, we let \( 0 < \theta < 1 \), \( 1 \leq p < \infty \) and let \( u \in (W^{0,p} \cap C)(\mathbb{R}^n; \mathbb{R}^n) \). The aim of this step is to show the inequality
\[
(4.2) \quad \int_0^R \int_{\partial B_r \times \partial B_r} |u(x) - u(y)|^p \frac{|x - y|^{n+\theta p}}{|x - y|^{n+\theta p}} \, d\sigma_x \, d\sigma_y \, dr \leq C \int_{B_R \times B_R} |u(x) - u(y)|^p \frac{|x - y|^{n+\theta p}}{|x - y|^{n+\theta p}} \, dx \, dy
\]
for all $R > 0$, where $C = C(n, \vartheta, p) > 0$ is a constant. Denoting the integral on the left by $(*)$, we change variables to the unit ball and put $\tilde{x} = rx$, $\tilde{y} = ry$. We thereby obtain, with $S^{n-1} := \partial B(0, 1)$,

$$
(*) := \int_0^R \int_{\partial B_r} \frac{|u(\tilde{x}) - u(\tilde{y})|^p}{|x - y|^{n + \vartheta p - 1}} \, d\sigma_x \, d\sigma_y \, dr
$$

In comparison with the right-hand side of (4.2), the ultimate integral only contains one integral with respect to the radii at the cost of a lower power in the integrand's denominator. We thus must argue for the appearance of the second such integral while rising the power of the relevant integrand by 1. To do so, let $x \in S^{n-1}$ and $0 < t < R$ and be given. We put

$$
\pi_t(x, y) := t \frac{x + y}{|x + y|}, \quad y \in S^{n-1} \setminus \{ -x \},
$$

which is the projection of the mid point of the line segment $[x, y]$ onto $\partial B_r$, cf. Figure 4. Hence, the mapping $\Pi_{t, x} : S^{n-1} \setminus \{ -x \} \to S^{n-1}$ given by $\Pi_{t, x}(y) := \pi_t(x, y)$ is well-defined. We now estimate for arbitrary $x \in S^{n-1}$ and $y \in S^{n-1} \setminus \{ -x \}$

$$
|u(rx) - u(ry)|^p \leq C(|u(rx) - u(\pi_t(x, y))|^p + |u(ry) - u(\pi_t(x, y))|^p).
$$

Hence for all $0 < a(x, y) \leq b(x, y) \leq R$, an integration with respect to $t \in [a(x, y), b(x, y)]$ yields

$$
|u(rx) - u(ry)|^p \leq C \int_{a(x, y)}^{b(x, y)} |u(rx) - u(\pi_t(x, y))|^p \, dt + C \int_{a(x, y)}^{b(x, y)} |u(ry) - u(\pi_t(x, y))|^p \, dt.
$$

At this point, fix $0 < r \leq R$. We then choose $a(x, y) := r(1 - \frac{|x - y|}{4})$ and $b(x, y) := r$. This particularly implies by $|x - y| \leq 2$ for all $x, y \in S^{n-1}$

$$
|b(x, y) - a(x, y)| = r \frac{|x - y|}{4} \quad \text{and} \quad \frac{r}{2} \leq a(x, y) \leq b(x, y) = r.
$$

![Figure 1. The geometric situation in the proof of Theorem 4.1 in two dimensions for selected points $y = y_i$. Excluding the $\mathcal{H}^{n-1}$-nullset $(-x)$, we project the midpoints of the line segment of $x$ and $y_i$ onto $\partial B_r$. This gives rise to the projections $z_i = \pi_t(x, y_i)$, and we consequently integrate with respect to $t$ to have the second radius integral emerging.](image)
Now, for all \(x \in S^{n-1}\) and \(y \in S^{n-1} \setminus \{ -x \}\) there holds
\[
| x - \frac{x+y}{|x+y|} | \leq |x-y|,
\]
an elementary inequality which is proved in the appendix. We thus have for all \(0 < t \leq r \leq R\)
\[
|rx - \pi_1(x,y)| = |r - \frac{x+y}{|x+y|}| \leq r \left| x - \frac{x+y}{|x+y|} \right| + r(1 - \frac{t}{r}) \leq |x-y| + (r-t).
\]
Combining (4.3), (4.4) and (4.5), we then obtain
\[
(*) \leq C \int_{S^{n-1} \times S^{n-1}} \int_0^R (r^{n-1})^2 \int_{r(1-\frac{t}{r})}^r \frac{|u(rx) - u(\pi_1(x,y))|^p}{|rx - \pi_1(x,y)|^{n+\sigma p-1}} \, dr \, d\sigma_x \, d\sigma_y
\]
\[
+ C \int_{S^{n-1} \times S^{n-1}} \int_0^R (r^{n-1})^2 \int_{r(1-\frac{t}{r})}^r \frac{|u(ry) - u(\pi_1(x,y))|^p}{|rx - \pi_1(x,y)|^{n+\sigma p-1}} \, dr \, d\sigma_y \, d\sigma_x =: I + II,
\]
where we have used that for each \(x \in S^{n-1}\), \(\{ -x \}\) is a nullset for \(\mathcal{H}^{n-1}\). The two integrals are symmetric in \(x\) and \(y\) (also note that \(\pi_1(x,y) = \pi_1(y,x)\)), and so it suffices to employ the desired estimate for one of these two integrals. We first estimate by virtue of the first part of (4.5)
\[
I \leq C \int_{S^{n-1} \times S^{n-1}} \int_0^R (r^{n-1})^2 \int_{r(1-\frac{t}{r})}^r \frac{|u(rx) - u(\pi_1(x,y))|^p}{|rx - \pi_1(x,y)|^{n+\sigma p}} \, dr \, d\sigma_x \, d\sigma_y =: J,
\]
so that the desired second radius integral has emerged. To estimate \(J\), note that if \(r(1 - \frac{|x-y|}{4}) \leq t \leq r\), then
\[
-t \leq r \left( \frac{|x-y|}{4} - 1 \right) \quad \Rightarrow \quad t - r \leq r \left( \frac{|x-y|}{4} \right) \quad \Rightarrow \quad |x-y| \leq 5 \frac{r}{4} |x-y|.
\]
Moreover, we note that for such \(t\), we have
\[
r(1 - \frac{|x-y|}{4}) \leq t \leq r \Rightarrow (1 - \frac{|x-y|}{4}) \leq \frac{t}{r} \leq 1 \Rightarrow \frac{t}{r} \leq \frac{1}{1 - \frac{|x-y|}{4}} \leq 2.
\]
We then estimate
\[
J \leq C \int_{S^{n-1} \times S^{n-1}} \int_0^R (r^{n-1})^2 \int_{r(1-\frac{t}{r})}^r \frac{|u(rx) - u(\pi_1(x,y))|^p}{|rx - \pi_1(x,y)|^{n+\sigma p}} \, dr \, d\sigma_x \, d\sigma_y
\]
\[
= C \int_{S^{n-1} \times S^{n-1}} \int_0^R (\pi_1(x,y))^{n+\sigma p-1} \int_{r(1-\frac{t}{r})}^r \frac{|u(rx) - u(\pi_1(x,y))|^p}{|rx - \pi_1(x,y)|^{n+\sigma p}} \left( \frac{r}{t} \right)^n r^{n-1} \, dr \, d\sigma_x \, d\sigma_y
\]
\[
\leq C \int_{S^{n-1} \times S^{n-1}} \int_0^R r^{n-1} \int_{r(1-\frac{|t|}{r})}^r \frac{|u(rx) - u(\pi_1(x,y))|^p}{|rx - \pi_1(x,y)|^{n+\sigma p}} r^{n-1} \, dr \, d\sigma_x \, d\sigma_y =: J'.
\]
At this point, we change variables and put \(z := (x+y)/|x+y|\). By the geometry of the map \(\Pi_{1,x}\) and the fact that for any \(y \in S^{n-1}\) there holds \(S^{n-1} \setminus \{ -x \} \ni y \mapsto (x+y)/|x+y| \in S^{n-1}\), a routine estimation then yields
\[
J' \leq C \int_{S^{n-1} \times S^{n-1}} \int_0^R \int_{r(1-\frac{|t|}{r})}^r \frac{|u(rx) - u(\pi_1(x,y))|^p}{|rx - \pi_1(x,y)|^{n+\sigma p}} r^{n-1} \, dr \, d\sigma_x \, d\sigma_y
\]
\[
\leq C \int_0^R \int_0^R \int_{S^{n-1}} \frac{|u(rx) - u(\pi_1(x,y))|^p}{|rx - \pi_1(x,y)|^{n+\sigma p}} r^{n-1} \, dr \, d\sigma_x \, d\sigma_y
\]
\[
\leq C \int_{B(0,R) \times B(0,R)} \frac{|u(\bar{x}) - u(\bar{y})|^p}{|\bar{x} - \bar{y}|^{n+\sigma p}} \, d\bar{x} \, d\bar{y},
\]
the ultimate inequality being a direct consequence of a passage to polar coordinates; here, \(C > 0\) still only depends on \(n, \sigma\) and \(p\). This establishes (4.2) and concludes step 1.

**Step 2. Existence of sufficiently many Lebesgue points.** Since we finally aim to apply step 1 for the particular choice \(\sigma = \theta = \varphi = \frac{n}{n+\sigma}\), we record that \(\sigma > 1\) so that the traces of \(W^{\theta,p}\)-maps are a priori not well-defined along \(\partial B_1\); thus we assumed \(u \in (W^{\theta,p} \cap C)(\mathbb{R}^n; \mathbb{R}^n)\) in step 1
so that this issue did not arise. In order to make use of step 1 for BD-maps \( u \) by Proposition 2.3, we start off by ensuring the explicit pointwise evaualizability of \( u, H^{n-1}\text{-a.e.} \) on \( L^1\text{-a.e.} \) sphere centered at the origin. Toward this aim, let \( u \in BD(\mathbb{R}^n) \) and \( 0 < R_1 < R_2 < \infty \) be arbitrary. Since \( E_u \) is a Radon measure, so is \( |E_u| \) and hence the set \( I := \{ t \in (R_1, R_2) : |E_u(\partial B_t) > 0 \} \) is at most countable. Hence \( L^1((R_1, R_2) \setminus I) = L^1((R_1, R_2)) = R_2 - R_1 \). Let \( t \in (R_1, R_2) \setminus I. \) Since \( \partial B_t \) is a \( C^1\text{-} \) hypersurface, (2.4) yields

\[
(4.10) \quad E_u(\partial B_t - u^+ - u^-) \cap \nu_{\partial B_t} \ll H^{n-1} \ll \partial B_t
\]

with the one-sided Lebesgue limits \( u^+ \) and the outer unit normal \( \nu_{\partial B_t} \) to \( \partial B_t \). Therefore,

\[
(4.11) \quad \int_{\partial B_t} |u^+ - u^-| \cap \nu_{\partial B_t} \ll H^{n-1} = |E_u(\partial B_t)| \ll \ll \frac{t}{t - 0} = 0.
\]

This implies \( |u^+ - u^-| \cap \nu_{\partial B_t} = 0 \ll H^{n-1}\text{-a.e} \) on \( \partial B_t \), and since \( |a| |b| \leq \sqrt{2}|a \cap b| \) by (2.1) for all \( a, b \in \mathbb{R}^n \), we conclude that \( \tilde{u}(x) := u^+(x) = u^-(x) \) holds for \( H^{n-1}\text{-a.e} \) \( x \in \partial B_t \). Then, by (2.5), we have for \( H^{n-1}\text{-a.e} \) such \( x \in \partial B_t \)

\[
(4.12) \quad \lim_{r \searrow 0} \int_{B(x,r) \cap B_t} |u - \tilde{u}(x)| \, dL^2 = \lim_{r \searrow 0} \int_{B(x,r) \cap B_t} |u - \tilde{u}(x)| \, dL^2 = 0.
\]

As a consequence, we obtain with \( \omega_t := L^2(\mathbb{B}(0, 1)) \)

\[
(4.13) \quad \lim_{r \searrow 0} \int_{B(x,r) \cap B_t} |u - \tilde{u}(x)| \, dL^2 = \lim_{r \searrow 0} \left( \int_{B(x,r) \cap B_t} \frac{L^2(\mathbb{B}(x,r) \cap B_t)}{\omega_t} \right) \int_{B(x,r) \cap B_t} |u - \tilde{u}(x)| \, dL^2
\]

\[
= \int_{B(x,r) \cap B_t} |u - \tilde{u}(x)| \, dL^2 = 0.
\]

Hence, \( H^{n-1}\text{-a.e} \) \( x \in \partial B_t \) is a Lebesgue point of \( u \). In conclusion, \( H^{n-1}\text{-a.e} \) \( x \in \partial B_t \) is a Lebesgue point for \( u \) for \( L^1\text{-a.e} \) radius \( t \in (R_1, R_2) \). Let us call a sphere \( \partial B_t \) with this property a \textit{Lebesgue sphere} for \( u \).

In an intermediate step, we claim the following: Let \( -\infty < a < b < \infty \) and let \( J \subset (a, b) \) be a measurable subset of full Lebesgue measure, i.e., \( L^1((a, b) \setminus J) = 0 \). Then for every \( g \in L^1((a, b); \mathbb{R}_{\geq 0}) \) there exists \( \xi_0 \in J \) which is a Lebesgue point for \( g \) and satisfies

\[
(4.14) \quad g^*(\xi_0) = \lim_{r \searrow 0} \int_{(\xi_0 - r, \xi_0 + r)} \frac{g(x) \, dx}{b - a} \leq \frac{2}{b - a} \int_{(a, b)} g(x) \, dx,
\]

where \( g^* \) is the precise representative of \( g \). To see this, we note that \( L^1\text{-a.e.} \) element of \( J \) is a Lebesgue point for \( g \), and hence the first equality in (4.14) holds for \( L^1\text{-a.e.} \) \( \xi_0 \in J \). Assume towards a contradiction that the overall claim is wrong. Then we find \( g \in L^1((a, b); \mathbb{R}_{\geq 0}) \) such that for all \( \xi_0 \in J \) which are Lebesgue points for \( g \) there holds

\[
(4.15) \quad g^*(\xi_0) > \frac{2}{b - a} \int_{(a, b)} g(x) \, dx.
\]

Since this holds for \( L^1\text{-a.e.} \) \( \xi_0 \in (a, b) \), we infer by integrating with respect to \( \xi_0 \in J \)

\[
2 \int_{(a, b)} g(y) \, dy \leq \frac{2}{b - a} \int_{(a, b)} \int_{(a, b)} g(y) \, dy \, dx \leq \int_{(a, b)} g(y) \, dy.
\]

By non-negativity of \( g \), this implies \( g \equiv 0 \text{ } L^1\text{-a.e.} \) in \( (a, b) \). This contradicts (4.15) and the proof of the intermediate claim is complete.

Step 3. Conclusion. Let now \( 0 < \theta < 1 \) be arbitrary and put \( p := n/(n - 1 + \theta) \). Given \( u \in BD(\mathbb{R}^n) \), we consider for \( \varepsilon > 0 \) the smooth approximations \( u^\varepsilon(x) := \rho_{\varepsilon} * u(x) \), where \( \rho \in ...
\( C^\infty_c(B(0,1);[0,1]) \) is a radial mollifier with \( \|\rho\|_{L^1(B(0,1))} = 1 \), and \( \rho_\varepsilon(x) := \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right) \) is the \( \varepsilon \)-rescaled variant. We record that for each Lebesgue point \( x \in \mathbb{R}^n \) of \( u \), there holds \( u^\varepsilon(x) \to u^*(x) \) with the precise representative \( u^* \) of \( u \) as \( \varepsilon \searrow 0 \). Moreover, based on Proposition 2.3 (b), we choose a rigid deformation \( \alpha \in \mathcal{R}(\mathbb{R}^n) \) such that, with \( u_\alpha := u - \alpha \),

\[
(4.16) \quad \left( \int_{B_r} \int_{B_r} \frac{|u_\alpha(x) - u_\alpha(y)|^p}{|x - y|^{n+\theta p}} \, dx \, dy \right)^\frac{1}{p} \leq C r^{1 - \theta} \frac{\int_{\mathbb{R}^n} |Eu|}{r^p},
\]

and analogously put \( u^\varepsilon_\alpha = (u - \alpha)^\varepsilon \). Now consider the set

\[
J := \{ t \in (s, r) : \partial B_t \text{ is a Lebesgue sphere for } u \}.
\]

Since \( \alpha \) is a rigid deformation and thus continuous, by step 2, for every \( t \in J \) and \( \mathcal{H}^{n-1} \)-a.e. \( x \in \partial B_t, u^\varepsilon_\alpha(x) \to u^*_\alpha(x) \) as \( \varepsilon \searrow 0 \). Hence, by Fatou’s lemma and \( \mathcal{L}^1((s, r) \setminus J) = 0 \),

\[
\int_s^r \int_{\partial B_t \times \partial B_t} \frac{|u^\varepsilon_\alpha(x) - u^\varepsilon_\alpha(y)|^p}{|x - y|^{n+\theta p}} \, d\sigma_x \, d\sigma_y \leq \liminf_{\varepsilon \searrow 0} \int_s^r \int_{\partial B_t \times \partial B_t} \frac{|u^\varepsilon_\alpha(x) - u^\varepsilon_\alpha(y)|^p}{|x - y|^{n+\theta p}} \, d\sigma_x \, d\sigma_y \leq \liminf_{\varepsilon \searrow 0} \int_s^r \int_{\partial B_t \times \partial B_t} \frac{|u^\varepsilon_\alpha(x) - u^\varepsilon_\alpha(y)|^p}{|x - y|^{n+\theta p}} \, d\sigma_x \, d\sigma_y \leq C \liminf_{\varepsilon \searrow 0} \int_{B_t \times B_t} \frac{|u^\varepsilon_\alpha(x) - u^\varepsilon_\alpha(y)|^p}{|x - y|^{n+\theta p}} \, dx \, dy \leq C r^{n} \left( r^{1 - \theta} \int_{\mathbb{R}^n} |Eu| \right)^p,
\]

additionally having employed Jensen’s inequality in the ultimate step. We then define a function \( \lambda : (0, R) \to \mathbb{R}_{\geq 0} \) for \( \mathcal{L}^1 \)-a.e. \( 0 < r < R \) by

\[
\lambda(r) := \int_{\partial B_r \times \partial B_r} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\theta p}} \, d\sigma_x \, d\sigma_y.
\]

With \( J \) from above, the last part of step 2 implies the existence of some \( t \in J \) such that

\[
\lambda^*(t) \leq \frac{2}{r - s} \int_{(s,r)} \lambda(t) \, dt \leq C \frac{r^n}{r - s} \left( r^{1 - \theta} \int_{\mathbb{R}^n} |Eu| \right)^p
\]

which, upon rewriting the left-hand side of the previous inequality in terms of \( u^*_\alpha \), yields

\[
\left( \int_{\partial B(0,J)} \int_{\partial B(0,J)} \frac{|u_\alpha(x) - u_\alpha(y)|^p}{|x - y|^{n+\theta p}} \, d\sigma_x \, d\sigma_y \right)^\frac{1}{p} \leq C \frac{r^{1 - \theta} |Eu|}{r^n} \frac{1}{r^n}.
\]

It is clear that \( C > 0 \) does not depend on \( u \) nor \( R \), and so we arrive at (4.1). The proof is complete.

\[ \square \]

**Remark 4.2.** In the BV-case, a Fubini-type property can be established by noting that for \( u \in BV(\mathbb{R}^n;\mathbb{R}^N) \), the tangential derivative \( \partial_t u \) on \( \mathcal{L}^1 \)-almost every sphere \( \partial B(0,t) \) is a finite Radon measure, too. This is discussed and utilised in [8] and [42]. By ORNSTEIN’s Non-Inequality, we see no argument to ensure that for generic maps \( u \in BD(\Omega), \partial_t u \) should be a Radon measure on even sufficiently many spheres. Also note that, by the very nature of the objects considered, any sort of ‘symmetric tangential derivative’ does not make sense. As to step 1 in the above proof, Fubini-type theorems for maps \( u \in B^{p}_r \) and \( u \in F^{p}_r \) have been given by TRIEBEL in the case where spheres are replaced by affine subspaces of \( \mathbb{R}^n \), cf. [69, Chpt. 2.5.13]. To reduce to this setting by local coordinate transformations, transforming the left hand side of (4.2) gives rise to additional localisation terms on the right hand side. It is not clear to us how to control these to obtain the requisite form of the estimate, an issue which does not arise in the above proof.
5. Strong Symmetric Quasiconvexity: Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2, the corresponding first part being a consequence of a similar $\varepsilon$-regularity result, cf. Proposition 5.7 below. Toward this objective, we aim to compare the given generalised minimiser with a suitable $A$-harmonic approximations via linearisation. Since linear elliptic problems subject to $L^1$-boundary data are, in general, ill posed, this can only be achieved on good balls where the boundary traces of $u$ share higher fractional differentiability. In this way, the corresponding $A$-harmonic approximation will be well-defined; note that this unclear for general balls on whose boundaries a given BD-minimiser $u$ is only known to possess traces in $L^1$. Consequently, this is where the Fubini-type property of BD-maps as given in the last section enters. To arrive at the desired excess decay, we shall estimate a $V$-function-type distance of $u$ to its $A$-harmonic approximation in terms of a superlinear power of the excess, cf. Proposition 5.4. Postponing the precise discussion to Remark 5.5, a linear instead of superlinear power of the excess $-$ which would come out by easier means $-$ is not sufficient to conclude the excess decay. In conjunction with the Caccioppoli inequality of the second kind to be proved in Section 5.1, we will then show in Section 5.3 that the estimates gathered so far for good balls are in fact sufficient to conclude a preliminary excess for all relevant balls, i.e., those on which the excess does not exceed a certain constant.

In order to implement the linearisation strategy in the main part of the partial regularity proof, we introduce for $f: \mathbb{R}^{n\times n}_{\text{sym}} \rightarrow \mathbb{R}$ satisfying (a)$--$(c) from Theorem 1.2 and $w \in \mathbb{R}^{n\times n}_{\text{sym}}$ the integrands

$$f_w(\xi) := f(\xi + w) - f(w) - \langle f'(w), \xi \rangle, \quad \xi \in \mathbb{R}^{n\times n}_{\text{sym}},$$

and remind the reader of the function $V: \mathbb{R}^{n\times n}_{\text{sym}} \rightarrow \mathbb{R}$ given by $V(\xi) := \sqrt{1 + |\xi|^2} - 1$.

Lemma 5.1. For all $w, z \in \mathbb{R}^{n\times n}_{\text{sym}}$ we have (with an obvious interpretation for $w = 0$ or $z = 0$)

$$\langle V''(w)z, z \rangle = \frac{1}{(1 + |w|^2)^{\frac{3}{2}}} \quad \text{and} \quad V_w(z) \geq \frac{1}{16} \frac{V(z)}{(1 + |w|^2)^{\frac{3}{2}}}.$$

Moreover, for each $m > 0$ there exists a constant $c = c(m) \in [1, \infty)$ with the following properties:

If $f: \mathbb{R}^{n\times n}_{\text{sym}} \rightarrow \mathbb{R}$ satisfies hypotheses (a)$--$(c) from Theorem 1.2, then for all $z \in \mathbb{R}^{n\times n}_{\text{sym}}$ and all $w \in \mathbb{R}^{n\times n}_{\text{sym}}$ with $|w| \leq m$ there holds

(i) $|f_w(z)| \leq cLV(z),$

(ii) $|f_w(z)| \leq c\min\{|z|, 1\},$

(iii) $|f_w''(0)z - f_w'(z)| \leq cLV(z).$

and for all $w \in \mathbb{R}^{n\times n}_{\text{sym}}$ and open balls $B \subset \mathbb{R}^n$ we have

$$\frac{c}{\ell} \int_B V(\varepsilon(\varphi)) \, dx \leq \int_B f_w(\varepsilon(\varphi)) \, dx \quad \text{for all } \varphi \in \text{LD}_0(B).$$

Proof. All assertions apart from (5.3) are taken from [42, Lems. 4.1, 4.2]. To see (5.3), let $B \subset \mathbb{R}^n$ be an open ball and let $\varphi \in \text{LD}_0(B), w \in \mathbb{R}^{n\times n}_{\text{sym}}$ with $|w| \leq m$ be arbitrary. With condition (c) from Theorem 1.2 in the third step we deduce

$$\int_B \frac{V(\varepsilon(\varphi)) \, dx}{(1 + |w|^2)^{\frac{3}{2}}} \leq 16 \int_B V_w(\varepsilon(\varphi)) \, dx = 16 \left( \int_B V(w + \varepsilon(\varphi)) - V(w) \, dx - \int_B \langle f'(w), \varepsilon(\varphi) \rangle \, dx \right)_{\varepsilon=0}$$

$$\leq \frac{16}{\ell} \int_B f(w + \varepsilon(\varphi)) - f(w) \, dx - \frac{16}{\ell} \int_B \langle f'(w), \varepsilon(\varphi) \rangle \, dx = \frac{16}{\ell} \int_B f_w(\varepsilon(\varphi)) \, dx.$$
5.1. Caccioppoli inequality of the second kind. In this section we give the requisite form of the Caccioppoli inequality of the second kind, and it is here where the BD-minimality crucially enters. However, different from other proof schemes, let us emphasize that this inequality will not be used to deduce higher integrability of generalised minima; in fact, Gehrings’s lemma does not quite seem to fit into the linear growth framework, cf. Section 5.5 below for a discussion. From now on, we tacitly suppose that \( f : \mathbb{R}^{n \times n} \to \mathbb{R} \) satisfies (a)-(c) from Theorem 1.2 without further mentioning unless it is explicitly stated otherwise.

**Proposition 5.2** (of Caccioppoli-type). Let \( m > 0 \). Then there exists a constant \( c = c(m, n, \frac{d}{\rho}) \in [1, \infty) \) such that if \( a : \mathbb{R}^n \to \mathbb{R}^n \) is an affine-linear mapping with |\( Ea | \leq m \) and \( B = B(x_0, R) \subseteq \Omega \) a ball, then there holds

\[
\int_{\overline{B(x_0, \frac{R}{2})}} V(E(u - a)) \leq c \int_{\overline{B(x_0, R)}} V\left( \frac{u - a}{R} \right) \, dx
\]

(5.4)

for every local BD-minimiser \( u \in \text{BD} (\Omega) \).

*Proof.* The proof evolves around a scheme for establishing Caccioppoli-type inequalities in the quasiconvex setting originally due to Evans [33, Lem. 3.1]. Recalling the definition of the shifted integrands, cf. (5.1), we put \( \hat{f} := f_\ell (a) \) and \( \hat{u} := u - a \). We then record that \( \hat{u} \) is a local minimiser the functional

\[
\mathcal{F}[v] := \int_\Omega \hat{f}(Ev)
\]

over \( \text{BD}(\Omega) \). Let \( \frac{d}{\rho} < r < s < R \) be arbitrary and choose a cut-off function \( \rho \in C^0_c (B(x_0, s); [0, 1]) \) with \( 1_{B(x_0, r)} \leq \rho \leq 1_{B(x_0, s)} \) and \( |\nabla \rho| \leq \frac{2}{r-s} \). We then define \( \varphi := \rho \hat{u} \) and \( \psi := (1 - \rho) \hat{u} \), so that \( \tilde{u} = u - a = \varphi + \psi \). Before we continue, let us remark that with \( \ell > 0 \) from hypothesis (c) of Theorem 1.2 and \( c = c(m) > 0 \),

\[
\frac{\ell}{c} \int_{B(x_0, s)} V(\varphi) \leq \int_{B(x_0, s)} \hat{f}(E \varphi).
\]

To see this inequality, note \( \varphi|_{\partial B(x_0, s)} = 0 \) and hence we find an approximating sequence \( (\varphi_k) \subset C^\infty_c (B(x_0, s); \mathbb{R}^n) \) which converges in the (symmetric) area–strict sense on \( B(x_0, s) \) to \( \varphi \) as \( k \to \infty \). From Lemma 5.1, cf. (5.3), we then deduce (5.5) with \( \varphi \) replaced by \( \varphi_k \). In the resulting inequality, by definition of (symmetric) area–strict convergence, the left-hand side converges to \( \frac{\ell}{c} \int_{B(x_0, s)} V(\varphi) \). For the right-hand side we invoke the continuity result for symmetric rank–1 convex functionals with respect to area–strict convergence, cf. Lemma 2.6. By area-strict convergence and the fact that symmetric quasiconvexity implies symmetric rank–1–convexity, we hereby obtain (5.5).

Consequently, using (generalised) minimality of \( \hat{u} \) with respect to its own boundary values and \( \partial I_{|\partial B(x_0, r)} = \psi|_{\partial B(x_0, r)} = \psi|_{\partial B(x_0, r)} \) in the second step, we obtain

\[
\frac{\ell}{c} \int_{B(x_0, r)} V(\tilde{E}\hat{u}) \leq \frac{\ell}{c} \int_{B(x_0, s)} V(\varphi) \leq \int_{B(x_0, s)} \hat{f}(E\varphi) + \int_{B(x_0, r)} \left( \hat{f}(E\varphi) - \hat{f}(E\hat{u}) \right) \quad \text{(by (5.5))}
\]

\[
\leq \int_{B(x_0, s)} \hat{f}(E\psi) + \int_{B(x_0, r)} \left( \hat{f}(E\varphi) - \hat{f}(E\hat{u}) \right)
\]

\[
\leq \int_{B(x_0, s) \setminus B(x_0, r)} \hat{f}(E\psi) + \int_{B(x_0, r) \setminus B(x_0, s)} \left( \hat{f}(E\varphi) - \hat{f}(E\hat{u}) \right),
\]

\[
=: I + II,
\]

where the last inequality holds as \( \varphi, \hat{u} \) coincide on \( B(x_0, r) \). Then, by Lemmas 5.1(i) and 2.9,

\[
I \leq cL \int_{\overline{B(x_0, s) \setminus B(x_0, r)}} V(E\psi) = cL \int_{\overline{B(x_0, s) \setminus B(x_0, r)}} V\left( \left( 1 - \rho \right) \frac{dE\hat{u}}{d|E\hat{u}|} |E\hat{u}| - (\nabla \rho \circ \hat{u}) \right),
\]

\[
II \leq cL \int_{\overline{B(x_0, r) \setminus B(x_0, s)}} V(\hat{f}(E\varphi) - \hat{f}(E\hat{u})).
\]
\[ \leq 2cL \int_{B(x_0, s) \setminus B(x_0, r)} V(E\tilde{u}) + 8cL \int_{B(x_0, s)} V\left(\frac{\tilde{u}}{s-r}\right) \, dx \]

On the other hand, we similarly find
\[ H \leq \int_{B(x_0, s) \setminus B(x_0, r)} f\left(\rho \frac{dE\tilde{u}}{d|E\tilde{u}|}\right) |E\tilde{u}| + \nabla \rho \circ \tilde{u} \cdot \mathcal{L}^n \setminus \Omega - f(E\tilde{u}) \]
\[ \leq 3cL \int_{B(x_0, s) \setminus B(x_0, r)} V(E\tilde{u}) + 8cL \int_{B(x_0, s)} V\left(\frac{\tilde{u}}{s-r}\right) \, dx. \]

Therefore, gathering estimates, we find
\[ -\frac{\ell}{c} \int_{B(x_0, r)} V(E\tilde{u}) \leq 16cL \int_{B(x_0, s) \setminus B(x_0, r)} V(E\tilde{u}) + 16cL \int_{B(x_0, s)} V\left(\frac{\tilde{u}}{s-r}\right) \, dx. \]

We now apply Widman’s hole-filling trick and hence add \( 16cL \int_{B(x_0, r)} V(E\tilde{u}) \) to both sides of the previous inequality and divide the resulting inequality by \( (\frac{s}{2} + 16cL) \). In consequence, letting \( \theta := 16cL/(\frac{s}{2} + 16cL), \) we have \( 0 < \theta < 1 \) and get
\[ \int_{B(x_0, r)} V(E\tilde{u}) \leq \theta \int_{B(x_0, s) \setminus B(x_0, r)} V(E\tilde{u}) + \theta \int_{B(x_0, r)} V\left(\frac{\tilde{u}}{s-r}\right) \, dx. \]

From here the conclusion is immediate by Lemma 2.12. The proof is complete. \( \square \)

5.2. Estimating the distance to the \( \Delta \)-harmonic approximation. In this section we present the key result that allows to deduce the requisite excess decay needed in the proof of Theorem 1.2. Here our strategy is as follows: Letting \( m > 0 \) be a given number and \( \alpha : \mathbb{R}^n \to \mathbb{R}^n \) an affine-linear map with \( |D\alpha| \leq m \), we first establish an improved estimate for the \( \Delta \)-function type distance of \( \tilde{u} := u - \alpha \) to a suitable \( \Delta \)-harmonic approximation on good balls \( B(x_0, R_0) \subset \Omega \). Here goodness refers to balls on whose boundaries \( \partial B(x_0, R_0) \) the map \( \tilde{u} \) is of class \( W^{1, \infty}(\mathbb{R}^n) \). This is accomplished in Proposition 5.4. By the Fubini-type property of BD-maps, it is then clear that whenever \( x_0 \in \Omega \) is fixed, then \( \mathcal{L}^n \)-a.e. radius \( R_0 \in (x_0, \frac{3}{2} \text{dist}(x_0, \partial \Omega)) \) will qualify as a good radius. It shall then be the aim of the subsequent section to justify to have the relevant estimates on good balls to conclude a preliminary excess decay. We begin with the following proposition, making Lemma 2.11 available for the sequel.

Proposition 5.3. Let \( \mathcal{A} \in \mathcal{S}(\mathbb{R}^{n \times n}) \) be a strongly symmetric rank-one convex bilinear form, i.e., \( \mathcal{A} \) satisfies for two constants \( \nu_1, \nu_2 > 0 \) and all \( a, b \in \mathbb{R}^n, z_1, z_2 \in \mathbb{R}^{n}_{\text{sym}} \)
\[ (5.6) \quad \nu_1 |a \odot b|^2 \leq \mathcal{A}[a \odot b, a \odot b] \quad \text{and} \quad |\mathcal{A}[z_1, z_2]| \leq \nu_2 |z_1| |z_2|. \]

Let \( L\nu := -\text{div}(\mathcal{A} \nu) \), where \( \mathcal{A} \) is identified with its representing matrix in \( \mathbb{R}^{(n \times n) \times (n \times n)} \). Then for each \( k \in \mathbb{N}, 1 < q < \infty \) and any open ball \( B \subset \mathbb{R}^n \), the mapping
\[ \Phi : W^{k,q}(B; \mathbb{R}^n) \ni u \mapsto (L(u), \mathcal{T}_B u) \in W^{k-2,q}(B; \mathbb{R}^n) \times W^{k-\frac{1}{2},q}(\partial B; \mathbb{R}^n) \]

is a topologically linear isomorphism. Moreover, if \( u \in LD(\Omega) \) satisfies \( Lu = 0 \) in \( \mathcal{D}'(\Omega; \mathbb{R}^n) \), then there holds \( u \in C^\infty(\Omega; \mathbb{R}^n) \) and
\[ (5.8) \quad \sup_{B(x_0, R)} |\nabla u - A| + R \sup_{B(x_0, R)} |\nabla^2 u| \leq C \int_{B(x_0, R)} |\nabla u - A| \, dx \]

for all \( A \in \mathbb{R}^{n \times n} \) and balls \( B(x_0, R) \subset \Omega \), where \( C = C(n, \nu_1, \nu_2) > 0 \) is a constant.

Proof. We reduce to Lemma 2.11 and define \( \mathcal{A} \in \mathcal{S}(\mathbb{R}^{n \times n}) \) by \( \mathcal{A}[z_1, z_2] := \frac{1}{\nu_1} \mathcal{A}[(z_1, z_2)] \), \( z_1, z_2 \in \mathbb{R}^{n}_{\text{sym}} \). Then (5.6) in conjunction with (2.1) yields
\[ |a \odot b|^2 \leq |a|^2 |b|^2 \leq 2 |a \odot b|^2 \leq \frac{2}{\nu_1} \mathcal{A}[a \odot b, a \odot b] = \frac{2}{\nu_1} \mathcal{A}[a \odot b, a \odot b]. \]
Hence \( \mathcal{A} \in \mathcal{S}(\mathbb{R}^{n+1}) \) satisfies the hypotheses of Lemma 2.11 with \( \lambda = \frac{1}{2} \). With the above terminology, we then have \( \text{div}(\mathcal{A}\varepsilon(v)) = \text{div}(\mathcal{A}\nabla v) \) and so \( \Phi \) given by (5.7) is a toplinear isomorphism by Lemma 2.11. The additional estimate (5.8) then follows similarly, now invoking the second part of Lemma 2.11. The proof is complete. \( \square \)

We now come to the \( \lambda \)-harmonic approximation. Recalling that the number \( m > 0 \) and the affine-linear map \( a: \mathbb{R}^n \to \mathbb{R}^n \) with \( |Ea| \leq m \) are assumed fixed throughout, we put

\[
\tilde{u} := u - a.
\]

Given a ball \( B = B(x_0, R) \Subset \Omega \) and \( u \in \text{BD}(\Omega) \) with \( u|_{\partial B} \in W^{1, \frac{n+1}{n}}(\partial B; \mathbb{R}^n) \), we consider the strongly symmetric rank-one system

\[
\begin{cases}
-\text{div}(\mathcal{A}\varepsilon(h)) = 0 & \text{in } B, \\
\mathcal{H} = \tilde{u} & \text{on } \partial B,
\end{cases}
\]

where \( \mathcal{A} := \tilde{f}'(0) \) with \( \tilde{f} := f_{\varepsilon(a)} \), cf. (5.1); note that, if \( f \) satisfies hypothesis (c) from Theorem 1.2, it is routine to check that \( \mathcal{A} \) is a strongly symmetric rank-one bilinear form. Put \( k = 1 \) and \( q := 1 + \frac{1}{n} \). Then \( k - \frac{1}{q} = \frac{1}{n+1} \), and in this situation Theorem 5.3 yields that there exists a unique \( h \in W^{1,1/n}(B(x_0, R); \mathbb{R}^n) \) solving (5.9). We now have the following

**Proposition 5.4.** Suppose that \( f \in C(\overline{B}^{n+1}_{\text{sym}}) \) satisfies (a)–(c) from Theorem 1.2 and let \( 1 < q < \frac{n+1}{n} \), \( m > 0 \) be given. Then there exists a constant \( C = C(m, n, q, L, \ell) > 0 \) with the following property: Suppose that \( u \in \text{BD}(\Omega) \) is a local BD-minimiser for \( F \) and \( B = B(x_0, R) \Subset \Omega \) is an open ball such that \( u|_{\partial B} \in W^{1, \frac{n+1}{n}}(\partial B; \mathbb{R}^n) \). Moreover, let \( a: \mathbb{R}^n \to \mathbb{R}^n \) be an affine-linear mapping with \( |Ea| \leq m \) and denote \( h \) the unique solution of the linear system (5.9) with \( \tilde{u} := u - a \). Then there holds

\[
\int_{B(x_0, R)} V\left(\frac{\tilde{u} - h}{R}\right) \, dx \leq C \left( \int_{B(x_0, R)} V(E\tilde{u}) \right)^q.
\]

**Proof.** We fix a ball \( B(x_0, R) \Subset \Omega \) such that the hypotheses of the proposition are in action. The proof then evolves in three steps:

**Step 1. Ekeland approximation.** To avoid manipulations on measures, we first employ an approximation procedure that allows us to work with LD- instead of BD-maps. To this end, let \( \delta > 0 \) be arbitrary but fixed. Then we apply the area-strict approximation of Lemma 2.1 to find \( \tilde{w} \in \text{LD}_2(B(x_0, R)) := \tilde{u} + \text{LD}_0(B(x_0, R)) \) such that

\[
\begin{aligned}
\int_{B(x_0, R)} \left| \frac{\tilde{u} - \tilde{w}_\delta}{R} \right| \, dx &+ \int_{B(x_0, R)} V(E\tilde{u}) - \int_{B(x_0, R)} V(E\tilde{w}_\delta) \, dx \leq \delta^2, \\
\int_{B(x_0, R)} \tilde{f}(E\tilde{w}_\delta) \, dx &\leq \int_{B(x_0, R)} \tilde{f}(E\tilde{u}) + \delta^2,
\end{aligned}
\]

where the dash is understood with respect to the Lebesgue measure \( \mathcal{L}^n \). Note that we can assume without loss of generality that \( \tilde{w}_\delta \in \text{LD}(B(x_0, R)) \) since \( \tilde{u} \) only enters in the definition of \( \text{LD}_2(B(x_0, R)) \) through prescribing the traces. However, as \( \text{LD}(B(x_0, R)) \) and \( \text{BD}(B(x_0, R)) \) have the same trace space on \( \partial B(x_0, R) \), we can find a LD-map that has the same boundary traces on \( \partial B(x_0, R) \) and then proceed as before. Crucially, \( (\text{LD}_2(B(x_0, R)), d_{\text{sym}}) \) is a complete metric space, where \( d_{\text{sym}}(v_1, v_2) := \|\varepsilon(v_1 - v_2)\|_{1,1}(B(x_0, R); \mathcal{S}_{\text{sym}}^{n+1}) \) is the symmetric gradient-L1 metric. It is then routine to check that all the requirements for the Ekeland variational principle, Lemma 2.10, are satisfied; in particular, by (6.1) from the appendix, the local BD-minimality of \( \tilde{u} \) gives

\[
\int_{B(x_0, R)} \tilde{f}(E\tilde{w}_\delta) \, dx \leq \inf_{w \in \tilde{u} + \text{LD}_0(B(x_0, R))} \int_{B(x_0, R)} f(E(w)) + \delta^2,
\]
We deduce that there exists a mapping \( \tilde{v} \in LD_{2}(B(x_0, R)) \) which satisfies
\[
\int_{B(x_0, R)} \tilde{f}(\tilde{v}(x)) \, dx \leq \int_{B(x_0, R)} \tilde{f}(\tilde{w}(x)) \, dx,
\]
\[
\int_{B(x_0, R)} \left| \tilde{v} - \tilde{w} \right| \, dx + \int_{B(x_0, R)} |\tilde{v}(x) - \tilde{w}(x)| \, dx \leq (1 + c_{\text{Poinc}}) \delta,
\]
\[
\int_{B(x_0, R)} \tilde{f}(\tilde{v}(x)) \, dx \leq \int_{B(x_0, R)} \tilde{f}(\tilde{w}(x)) \, dx + \delta \int_{B(x_0, R)} |\tilde{v}(x) - \tilde{\phi}| \, dx
\]
for all \( \tilde{\phi} \in LD_{2}(B(x_0, R)) \), where \( c_{\text{Poinc}} > 0 \) is an arbitrary but fixed constant for the Poincaré inequality in \( LD_{2}(B(x_0, R)) \); note that the above inequality scales correctly and hence \( c_{\text{Poinc}} > 0 \) is in fact independent of \( R \). Working from here, we obtain
\[
\int_{B(x_0, R)} \left( \tilde{f}'(\tilde{v}(x)), \tilde{e}(\tilde{v}) \right) \, dx \leq \delta \int_{B(x_0, R)} |\tilde{e}(\tilde{v})| \, dx
\]
for all \( \tilde{\psi} \in LD_{2}(B(x_0, R)) \) and
\[
\int_{B(x_0, R)} \left( \tilde{f}'(0)\tilde{e}(\tilde{v}), \tilde{e}(\tilde{v}) \right) \, dx \leq \int_{B(x_0, R)} (CLV(\tilde{e}(\tilde{v})) + \delta) |\tilde{e}(\tilde{v})| \, dx
\]
for all \( \tilde{\psi} \in W^{1, \infty}_{0}(B(x_0, R); \mathbb{R}^{N}) \). Indeed, for every \( \theta \in \mathbb{R} \setminus \{0\} \), \( \tilde{\phi}_{\theta}^{\pm} := \tilde{v} \pm \theta \tilde{\phi} \) qualifies as a competitor in (5.12). Hence,
\[
\int_{B(x_0, R)} \left( \tilde{f}(\tilde{v}(x)) - \tilde{f}(\tilde{v}(\pm \theta \tilde{\phi})) \right) \, dx \leq \delta \int_{B(x_0, R)} |\tilde{e}(\tilde{v})| \, dx.
\]
In this situation, sending \(|\theta| \searrow 0 \) yields (5.13). We then consequently find
\[
\int_{B(x_0, R)} \left( \tilde{f}'(0)\tilde{e}(\tilde{v}), \tilde{e}(\tilde{v}) \right) \, dx \leq \int_{B(x_0, R)} \left( \tilde{f}'(0)\tilde{e}(\tilde{v}) - \tilde{f}'(\tilde{v}(x)), \tilde{e}(\tilde{v}) \right) \, dx
\]
\[
+ \int_{B(x_0, R)} \left( \tilde{f}'(\tilde{v}(x)), \tilde{e}(\tilde{v}) \right) \, dx \leq \int_{B(x_0, R)} (cLV(\tilde{e}(\tilde{v})) + \delta) |\tilde{e}(\tilde{v})| \, dx
\]
by Lemma 5.1(iii) and (5.13); note that now \( c \) depends on \( m \). The same obviously is valid for \( -\tilde{\phi} \) instead of \( \tilde{\phi} \). This establishes (5.14). In effect, (5.13) provides perturbed Euler-Lagrange equations as a substitute for the ANZELLOTTI-type Euler-Lagrange equations for measures.

**Step 2. Truncations and improved regularity for the comparison maps.** Starting from (5.14), we let \( \phi \in W^{1, \infty}_{0}(B(x_0, R); \mathbb{R}^{N}) \) be arbitrary and put \( \psi := \tilde{v} - h \). We scale back to the unit ball and therefore put, for \( x \in B(0, 1) \),
\[
\Psi(x) := \frac{1}{R} \psi(x_{0} + Rx), \quad \Phi(x) := \frac{1}{R} \phi(x_{0} + Rx), \quad U(x) := \frac{1}{R} \tilde{v}(x_{0} + Rx).
\]
Since \( h \) satisfies (5.9), we conclude from (5.14) with \( \lambda := \tilde{f}''(0) \)
\[
\int_{B(0, 1)} \langle \lambda e(\Psi), e(\Phi) \rangle \, dx \leq CL \int_{B(0, 1)} V(\epsilon(U)) \epsilon(\Phi) \, dx + \delta \int_{B(0, 1)} |\epsilon(\Phi)| \, dx.
\]
We then define a truncation operator \( T : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \) by
\[
T(y) := \begin{cases} 
\frac{y}{|y|} & \text{if } |y| \leq 1, \\
\frac{y}{|y|} & \text{if } |y| > 1,
\end{cases} \quad y \in \mathbb{R}^{n},
\]
and note that \( T(\Psi) \in L^{\infty}(B(0, 1); \mathbb{R}^{n}) \). Let us now consider the linear system
\[
\left\{ \begin{array}{ll}
- \text{div}(\lambda e(T)) &= T(\Psi) & \text{in } B(0, 1), \\
T &= 0 & \text{on } \partial B(0, 1)
\end{array} \right.
\]
with its corresponding weak formulation

\begin{align}
(5.17) \quad \int_{B(0,1)} \langle \mathcal{A} \mathcal{E}(\mathbf{T}), \varepsilon(\rho) \rangle \, dx &= \int_{B(0,1)} \langle T(\Psi), \rho \rangle \, dx \quad \text{for all } \rho \in C^\infty_c(B(0,1); \mathbb{R}^n).
\end{align}

Since \( f \) is assumed strongly symmetric quasiconvex, it is strongly symmetric rank-one convex. Fix \( p > n + 1 \). Then, by Proposition 5.3, there exists a unique solution \( T \in W^{2,p}(B(0,1); \mathbb{R}^n) \) of (5.16) with \( u|_{\partial B(0,1)} = 0 \). Thus there exists a constant \( C = C(m,n,p,L,\ell) > 0 \) with \( \ell, L > 0 \) being the ellipticity constants for \( \mathcal{A} \) (which, in turn, only depend on \( f \) and \( a \) and thus \( \ell, L, n \) and \( m \)) such that

\begin{align}
(5.18) \quad \int_{B(0,1)} |T'|^p \, dx + \int_{B(0,1)} |DT|^p \, dx + \int_{B(0,1)} |D^2 T|^p \, dx \leq C \int_{B(0,1)} |T(\Psi)|^p \, dx.
\end{align}

In this situation, we invoke Morrey’s embedding \( W^{1,p}(B) \hookrightarrow L^\infty(B) \) to find that \( T \) is Lipschitz together with the corresponding bound

\begin{align}
(5.19) \quad \|DT\|_{L^\infty(B; \mathbb{R}^{n \times n})} \leq C \left( \|DT\|_{L^p(B; \mathbb{R}^{n \times n})} + \|D^2 T\|_{L^p(B; \mathbb{R}^{n \times n \times n})} \right)^{\frac{1}{p}} \quad \text{for all } \rho \in LD_0(B(0,1)).
\end{align}

As \( T|_{\partial B(0,1)} = 0 \), from here we deduce \( T \in W^{1,\infty}_0(B(0,1); \mathbb{R}^n) \). Approximating a generic map \( \rho \in LD_0(B(0,1)) \) by elements from \( C^\infty_c(B(0,1); \mathbb{R}^n) \) in the LD-norm topology, we obtain

\begin{align}
(5.20) \quad \int_{B(0,1)} \langle \mathcal{A} \mathcal{E}(\mathbf{T}), \varepsilon(\rho) \rangle \, dx = \int_{B(0,1)} \langle T(\Psi), \rho \rangle \, dx \quad \text{for all } \rho \in LD_0(B(0,1)).
\end{align}

Now, because of \( 2 \leq n < p < \infty \), we have \( |T(y)|^p \leq |\Psi(y)|^p \leq |y|^2 \) for if \( |y| \leq 1 \) and thus there holds

\begin{align}
(5.21) \quad \|T(\Psi)\|_{L^p}^p = \int_{B(0,1)} |T(\Psi)|^p \, dx \leq c \int_{B(0,1)} |V(\Psi)| \, dx
\end{align}

by Lemma 2.9. Combining (5.21) with (5.19) consequently yields

\begin{align}
(5.22) \quad \|\varepsilon(T)\|_{L^\infty} \leq \|DT\|_{L^\infty} \leq c \left( \int_{B(0,1)} |V(\Psi)| \, dx \right)^{\frac{1}{2}},
\end{align}

and here \( c > 0 \) only depends on \( \ell, L, n \) and \( p \).

**Step 3. Conclusion for the approximating maps \( \tilde{\psi} \):** We now combine the estimates gathered to far to obtain inequality (5.10) in a perturbed form. Recalling (2.15), we successively obtain

\begin{align*}
\int_{B(0,1)} V(\Psi) \, dx &\leq \int_{B(0,1)} \min\{ |\Psi|, |\Psi|^2 \} \, dx \\
&= \int_{B(0,1)} \langle T(\Psi), \Psi \rangle \, dx \quad \text{(by definition of } T \text{)} \\
&= \int_{B(0,1)} \langle \mathcal{A} \mathcal{E}(\mathbf{T}), \varepsilon(\Psi) \rangle \, dx \quad \text{(by testing (5.20) with } \rho = \Psi \text{)} \\
&= \int_{B(0,1)} \langle \mathcal{A} \mathcal{E}(\mathbf{T}), \varepsilon(\mathbf{T}) \rangle \, dx \quad \text{(as } \mathcal{A} \in \mathcal{S}^{\text{sym}}(\mathbb{R}^{n \times n})) \\
&\leq CL \int_{B(0,1)} (V(\varepsilon(U)) + \delta) |\varepsilon(T)| \, dx \quad \text{(by testing (5.15) with } \Phi = \mathbf{T} \text{)} \\
&\leq CL \int_{B(0,1)} (V(\varepsilon(U)) + \delta) \, dx \|\varepsilon(T)\|_{L^\infty} \\
&\leq CL \left( \int_{B(0,1)} (V(\varepsilon(U)) + \delta) \, dx \right) \left( \int_{B(0,1)} |V(\Psi)| \, dx \right)^{\frac{1}{2}} \quad \text{(by (5.22))}.
\end{align*}

We therefore obtain

\begin{align}
(5.23) \quad \left( \int_{B(0,1)} V(\Psi) \, dx \right)^{\frac{1}{p}} \leq CL \left( \int_{B(0,1)} (V(\varepsilon(U)) + \delta) \, dx \right).
\end{align}
At this stage recall that our choice of $p$ was only restricted to $p > n + 1$. For $1 < q < \frac{n+1}{n}$ as in the proposition, we thus find $n + 1 < p < \infty$ such that $p' = \frac{p}{p-1} = q$ and thus

$$\int_{B(0,1)} V(\nabla f) \, dx \leq C \left( \int_{B(0,1)} V(\epsilon'(U)) \, dx \right)^q + C\delta^n L^m(B(0,1))^q. \tag{5.24}$$

At this stage we scale back to the original ball to find

$$\int_{B(x_0,R)} V\left(\frac{\nabla \tilde{u} - h}{R}\right) \, dx \leq C \left( \int_{B(x_0,R)} V(\epsilon(U)) \, dx \right)^q + C\delta^n L^m(B(0,1))^q, \tag{5.25}$$

and we note that the constant $C > 0$ only depends on $m, n, q, L$ and $\ell$.

**Remark 5.5 (On the exponent $q$ in the previous proposition).** It is important to remark that the exponent $q$ as it appears in the previous proposition can be chosen **strictly larger** than one. In the classical works on $A$–harmonic approximation (cf. [29]–[31] or the exposition of the method in the recent monograph [14]), this corresponds to a suitable linear growth version of approximate $A$–harmonicity. From a technical perspective, the importance of $q > 1$ is given by (5.45) from below, where the smallness assumption on the excess gives smallness of the critical quantity

$$\left( \frac{E(x_0, R_0)}{R_0^{n+1}} \right)^{q-1} \cdot$$

If we could not use $q > 1$ and only had $q = 1$ at our disposal, this critical term would equal one and thus destroy the excess decay later on in Proposition 5.7.

In the preceding Proposition 5.4 we have estimated a $V$–function type distance of $\tilde{u} = u - a$ to its $A$–harmonic approximation $h$, where $A = \bar{f}'(0) = f''(a)(0)$. We conclude this subsection by showing how suitable Lebesgue norms of $Dh$ can be controlled by virtue of $\tilde{u}$:

**Lemma 5.6.** In the situation of Proposition 5.4 there exists a constant $C = C(n, \ell, L) > 0$ such that for each $b \in A(R^n)$ the map $h := h - b$ satisfies, with $\tilde{u} := \tilde{u} - b$,

$$\left( \int_{B_R} |\tilde{u}|^{\frac{n+1}{n}} \, dx \right)^{\frac{n}{n+1}} \leq CR^{\frac{n}{n+1}} \left( \int_{\partial B_R} \frac{\tilde{u}}{|x-y|^{\frac{n+1}{n}}} \, d\sigma_{\partial B_R} \right)^{\frac{n}{n+1}}. \tag{5.26}$$

**Proof.** It is no loss of generality to assume $x_0 = 0$. By the choice of the radius $R > 0$, $\tilde{u}|_{\partial B(0,R)} \in W^{1, \frac{n+1}{n}}(\partial B(0,R); R^n)$. We focus on the case $R = 1$ first, the statement will then follow at the end of the proof by scaling. With this choice of $x_0$ and $R$, and adopting the terminology of
Proposition 5.3, denote \( S := \Phi^{-1}(0, \cdot) \). Given \( b \in \mathcal{D}(\mathbb{R}^n) \), we set \( \tilde{u}_b := \tilde{u} - b (= u - a - b) \) and define

\[
u_b := (\tilde{u}_b)_{\partial B(0,1)} := \int_{\partial B(0,1)} \tilde{u}_b \nu \mathcal{H}^{n-1}(\partial \mathbb{R}^n),
\]

where the dash is now understood with respect to \( \mathcal{H}^{n-1} \mid \partial B(0,1) \). Since \( h \) solves (5.9), \( h := \tilde{h} - u_b := h - b - u_b \) is the unique solution of

\[
\left\{ \begin{array}{l}
-\text{div}(A\xi(h)) = 0 \quad \text{in } B(0,1), \\
h = \tilde{u}_b - u_b \quad \text{on } \partial B(0,1).
\end{array} \right.
\]

Hence we have \( h = S(\tilde{u}_b - u_b) \) so that, by Proposition 5.3 with some \( C = C(n, L, \ell) > 0 \),

\[
\|Dh\|_{L^{\frac{n+1}{n}}(B(0,1);\mathbb{R}^{n \times n})} \leq \|h\|_{W^{1,\frac{n+1}{n}}(B(0,1);\mathbb{R}^n)} \leq C\|\tilde{u}_b - u_b\|_{W^{1,\frac{n+1}{n}}(\partial B(0,1);\mathbb{R}^n)}.
\]

On the other hand, if \( x, y \in \partial B(0,1) \), then \( |x - y| \leq 2 \) and so \( \mathcal{H}^{n-1}(\partial B(0,1)) \leq n|b|^{\frac{2}{n-1}} \). Thus,

\[
\left( \int_{\partial B(0,1)} |\tilde{u}_b(x) - u_b|^{\frac{n+1}{n}} \, d\sigma_x \right)^{\frac{n}{n+1}} = \left( \int_{\partial B(0,1)} \int_{\partial B(0,1)} |\tilde{u}_b(x) - \tilde{u}_b(y)|^{\frac{n+1}{n}} \, d\sigma_x \, d\sigma_y \right)^{\frac{n}{n+1}} \leq C(n) \left( \int_{\partial B(0,1)} \int_{\partial B(0,1)} \frac{|\tilde{u}_b(x) - \tilde{u}_b(y)|^{\frac{n+1}{n}} \, d\sigma_x \, d\sigma_y}{|x - y|^{n-1 + \frac{2}{n}}} \right)^{\frac{n}{n+1}}.
\]

As a consequence, we obtain in conjunction with (5.28) and a constant \( C = C(n, L, \ell) > 0 \)

\[
\|D\tilde{h}\|_{L^{\frac{n+1}{n}}(B(0,1);\mathbb{R}^{n \times n})} = \|Dh\|_{L^{\frac{n+1}{n}}(B(0,1);\mathbb{R}^{n \times n})} \leq C\|\tilde{u}_b\|_{W^{1,\frac{n+1}{n}}(\partial B(0,1);\mathbb{R}^n)}.
\]

The rest of the proof, i.e., for general \( R > 0 \), follows by standard scaling as follows. At this stage, we pass to general radii \( R > 0 \). For this, we use linearity of the problem (5.27). Since \( h \) solves (5.27), the function \( w : B(0,1) \to \mathbb{R}^n \) given by \( w(x) := \tilde{h}(Rx) \) is seen to solve

\[
\left\{ \begin{array}{l}
-\text{div}(A\xi(w)) = 0 \quad \text{in } B(0,1), \\
w = \psi \quad \text{on } \partial B(0,1),
\end{array} \right.
\]

where \( \psi : \partial B(0,1) \to \mathbb{R}^n \) is given by \( \psi(x) = \tilde{u}_b(Rx) \) for \( x \in \partial B(0,1) \). Employing (5.29) in the first and changing variables \( x' = Rx \) and \( y' = Ry \) in the second step, we then see with \( p = \frac{n+1}{n} \) and \( \kappa = \frac{1}{n+1} \) that

\[
\left( \int_{B(0,1)} |Dw|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{\partial B(0,1) \times \partial B(0,1)} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{n-1 + \kappa p}} \, d\sigma_x \, d\sigma_y \right)^{\frac{1}{p}}
\]

\[
= CR^{-\frac{n-1+\kappa p}{n}} \left( \int_{\partial B(0,1) \times \partial B(0,1)} \frac{|\tilde{u}_b(x') - \tilde{u}_b(y')|^p}{|x' - y'|^{n-1 + \kappa p}} \, d\sigma_{x'} \, d\sigma_{y'} \right)^{\frac{1}{p}}.
\]

On the other hand, \( R^{\frac{1}{n-1}} \|D\tilde{h}\|_{L^p(B(0,R);\mathbb{R}^{n \times n})} = \|Dw\|_{L^p(B(0,1);\mathbb{R}^{n \times n})} \). Hence the previous inequality implies by regrouping with \( C = C(n) > 0 \)

\[
\left( \int_{B(0,R)} |D\tilde{h}(x')|^p \, dx' \right)^{\frac{1}{p}} \leq CR^\kappa R^{-1} \left( \int_{\partial B(0,R) \times \partial B(0,R)} \frac{|\tilde{u}_b(x') - \tilde{u}_b(y')|^p}{|x' - y'|^{n-1 + \kappa p}} \, d\sigma_{x'} \, d\sigma_{y'} \right)^{\frac{1}{p}}.
\]

We then note that \( \kappa - 1 = \frac{1}{n+1} - 1 = -\frac{n}{n+1} \), and this concludes the proof. \( \Box \)
5.3. Excess decay. The objective of the present subsection is to establish the excess decay that will eventually lead to the desired partial regularity assertion of Theorem 1.2 by virtue of an iteration scheme. To this end, let \( u \in \text{BD}(\Omega) \) be a generalised minimiser of \( F \), where the integrand satisfies (a)–(b) from Theorem 1.2, and let \( M_0 > 0 \) be a given number. Our strategy then runs in four steps: In a first step, we choose a ball for which both the mean value \( u \) and a certain excess quantity of \( Eu \) is small. Then, in a second step, we slightly diminish the radius of the given ball to obtain a ball on whose boundary we may apply the Fubini-type theorem for BD-maps. This makes the \( \hat{h} \)-harmonic approximation available. Defining suitable comparison maps in step 3, we then combine Propositions 5.2 and 5.4 in step 4 to conclude a preliminary excess decay. In doing so, we define for \( z \in \Omega \) and \( 0 < r < \text{dist}(z, \partial \Omega) \) two excess quantities by

\[
E(u; z, r) := \int_{B(z, r)} V(Eu - (Eu)_{B(z, r)}) \quad \text{and} \quad \tilde{E}(u; z, r) := \frac{E(z, r)}{\partial B B(n)},
\]

and we will often write \( E(z, r) := E(u; z, r) \), assuming that \( u \) is fixed. Here, as usual, \( (Eu)_{B(z, r)} = Eu(B(z, r)) / \mathcal{L}^n(B(z, r)) \).

**Step 1. Smallness Assumptions.** Let \( M_0 > 0 \) be given and fix a ball \( B_{R_0} = B(x_0, R_0) \in \Omega \) such that

\[
| (Eu)_{B_{R_0}} | < M_0.
\]

and

\[
\int_{B_{R_0}} | Eu - (Eu)_{B_{R_0}} | \leq 1.
\]

We write \( B_r := B(x_0, r) \) in all of what follows.

**Step 2. Selection of a good radius.** In a second step, we fix an affine–linear map \( a : \mathbb{R}^n \to \mathbb{R}^n \) with \( \varepsilon(a) = (Eu)_{B_{R_0}} \). We then put \( \tilde{a} := u - a \) and \( f := f_{\varepsilon(a)} \), cf. (5.1). Starting from \( R_0 > 0 \) as given above, we now apply Theorem 4.1. Consequently, we find \( R \in (\frac{2}{10} R_0, R_0) \) such that \( \tilde{a} |_{\partial B_R} \in \text{W}^{1, 1 + \frac{1}{n}}(\partial B_R; \mathbb{R}^n) \) and a rigid deformation \( \alpha \in \mathcal{A}(\mathbb{R}^n) \) together with the corresponding estimate (with \( \theta = \frac{1}{n+1} \)) and accordingly \( p = \frac{n+1}{n} \) in Theorem 4.1

\[
\left( \int_{\partial B_{(x_0, R)}} \int_{\partial B_{(x_0, R)}} \frac{|\tilde{a}(x) - \tilde{a}(y)|^{1 + \frac{1}{n}}}{|x - y|^{(\alpha - 1) + \frac{1}{n}}} \, d\sigma_x \, d\sigma_y \right)^{\frac{1}{n+1}} \leq C \frac{R_0^{\frac{n+1}{n}}}{\left( \frac{2}{10} R_0 \right)^{\frac{n+1}{n+1}}} \frac{(1 - \alpha)^{\frac{n+1}{n}}}{\frac{n+1}{n} \int_{B(x_0, R_0)} |Eu|}
\]

\[
\leq C \frac{R_0^{\frac{n+1}{n}}}{R_0^{\frac{n+1}{n}}} \int_{B(x_0, R_0)} |Eu - (Eu)_{B(x_0, R_0)}|.
\]

where we recall \( \tilde{u}_a := \tilde{u} - \alpha (u - a - \alpha) \), and \( C = C(n) > 0 \) is a constant. We now put \( b := \alpha \) and shall consider the map \( \tilde{u}_b := \tilde{u} - b = u - a - b \) in the sequel.

**Step 3. Definition of comparison maps.** We put \( \hat{A} := f''((Eu)_{B_R}) \) and pick the \( \hat{A} \)-harmonic mapping \( \hat{h} : B_R \to \mathbb{R}^n \) solving

\[
\begin{cases}
- \text{div} (f''((Eu)_{B_R}) \varepsilon(\hat{h})) = 0 & \text{in } B_R, \\
\hat{h} = \tilde{u}_b & \text{on } \partial B_R.
\end{cases}
\]

We are thus in the setting of (5.9) and Lemma 5.6 from above; by Proposition 5.3, \( \hat{h} \in C^\infty(B_R; \mathbb{R}^n) \). Then we define

\[
A(x) := \hat{h}(x) + D\hat{h}(x_0)(x - x_0) \quad \text{and} \quad a_0(x) := a(x) + A(x), \quad x \in B_R.
\]

We then obtain

\[
|Ea_0| = |Eu + E\hat{h}(x_0)| = |(Eu)_{B_{R_0}} + (E\hat{h})(x_0)|
\]
\[ \leq M_0 + \sup_{\mathcal{B}_{R/2}} |D\tilde{h}| \] (as \(|E_{\mathcal{V}}| \leq |D\mathcal{V}|\))
\[ \leq M_0 + c \int_{\mathcal{B}_R} |D\tilde{h}| \, dx \] (by (5.8))
\[ \leq M_0 + c \left( \int_{\mathcal{B}_R} |D\tilde{h}| \frac{n+1}{n} \, dx \right)^{\frac{n}{n+1}} \] (by Jensen),

and thus
\[ |E_{\mathcal{V}0}| \leq M_0 + cR \frac{C}{n+1} \left( \int_{\partial\mathcal{B}_R} \int_{\partial\mathcal{B}_R} \frac{\left|\tilde{u}_b(x) - \tilde{u}_b(y)\right|^{\frac{n+1}{n}}}{|x-y|^{n-1+\frac{1}{n}}} \, d\sigma_x \, d\sigma_y \right)^{\frac{n}{n+1}} \] (by Lemma 5.6)
\[ \leq 2M_0 + \frac{c}{R^2} \int_{\mathcal{B}_{(\sigma_0, R_0)}} |E_{U} - (E_{\mathcal{V}})_{\mathcal{B}_{\sigma R_0}}| \] (by (5.33))
\[ \leq 2M_0 + c, \]

where the last estimates holds because of (5.32). Here, \( c = c(n, L, \ell) > 0 \) is a constant that we fix now. In particular, the constants appearing here do not depend on \( R \) or \( R_0 \). Summarising, if we put \( m := 2M_0 + c \) as on the right side of the previous chain of inequalities, then we obtain
\[ |E_{\mathcal{V}_0}| \leq m. \] (5.37)

**Step 4. Comparison estimates.** Let \( 0 < \sigma < \frac{1}{2} \) be arbitrary. We note, as a consequence of Lemma 2.9 and Jensen’s inequality,
\[ \int_{\mathcal{B}_{\sigma R_0}} V(Eu - (E_{\mathcal{V}})_{\mathcal{B}_{\sigma R_0}}) \leq \int_{\mathcal{B}_{\sigma R_0}} V(Eu - E_{\mathcal{V}0} + E_{\mathcal{V}0} - (E_{\mathcal{V}})_{\mathcal{B}_{\sigma R_0}}) \]
\[ \leq C \int_{\mathcal{B}_{\sigma R_0}} V(Eu - E_{\mathcal{V}0}) + C \int_{\mathcal{B}_{\sigma R_0}} V(E(U - a_0))_{\mathcal{B}_{\sigma R_0}} \]
\[ \leq C \int_{\mathcal{B}_{\sigma R_0}} V(Eu - a_0) \]
\[ = C \int_{\mathcal{B}_{\sigma R_0}} V(E(u - a_0)). \] (5.38)

Our next objective is to apply the Caccioppoli–type inequality, Proposition 5.2. Having chosen \( m > 0 \) as it appears in Proposition 5.2 by (5.37), we find \( c = c(m, n, \ell) > 0 \) such that (5.4) holds; note that \( b + a_0 \) is affine-linear, too, with \(|E(b + a_0)| = |E_{\mathcal{V}0}| \leq m\). We then estimate, using (5.38) and the Caccioppoli–type inequality in the first step,
\[ \int_{\mathcal{B}_{\sigma R_0}} V(Eu - (E_{\mathcal{V}})_{\mathcal{B}_{\sigma R_0}}) \leq C \int_{\mathcal{B}_{2\sigma R_0}} V\left( \frac{\tilde{u}(x) - b(x) - A(x)}{\sigma R_0} \right) \, dx \]
\[ \leq C \int_{\mathcal{B}_{2\sigma R_0}} V\left( \frac{\tilde{u}(x) - b(x) - \tilde{h}(x) - A(x) + \tilde{h}(x)}{\sigma R_0} \right) \, dx \]
\[ \leq C \int_{\mathcal{B}_{2\sigma R_0}} V\left( \frac{\tilde{u} - b - \tilde{h}}{\sigma R_0} \right) \, dx + C \int_{\mathcal{B}_{2\sigma R_0}} V\left( \frac{\tilde{h} - A}{\sigma R_0} \right) \, dx \]
\[ \leq \frac{C}{\sigma^2} \int_{\mathcal{B}_R} V\left( \frac{\tilde{u}_b - \tilde{h}}{R} \right) \, dx + C \int_{\mathcal{B}_{2\sigma R_0}} V\left( \frac{\tilde{h} - A}{\sigma R_0} \right) \, dx \]
\[ =: I + II, \]

where \( C = C(m, n, \frac{\ell}{2}) > 0 \) is a constant. Here we have used \( B_{2\sigma R_0} \subset B_R \), uniform comparability of \( R \) and \( R_0 \) and the fact that \( V(\lambda z) \leq c\lambda^2 V(z) \) for a constant \( c > 0 \), all \( z \in \mathbb{R}_{\text{sym}} \) and \( |\lambda| \geq 1 \).
Similarly as in the estimation given in (5.36), we again employ Lemma 5.6 to further obtain

\begin{equation}
(5.39) \quad I = \frac{C}{\sigma} \int_{B_R} V \left( \frac{\tilde{u} - \tilde{h}}{R} \right) \, dx = \frac{CR_0^q}{\sigma^q} \int_{B_R} V \left( \frac{\tilde{u} - \tilde{h}}{R} \right) \, dx \leq C \frac{R_0^q}{\sigma^q} \left( \int_{B_R} V(\tilde{E}u) \right)^q,
\end{equation}estinal comparability of the strongly symmetric rank-one convex system (5.34) with boundary datum \( \tilde{u} = u - a \). As to II, let \( x \in B_{2\sigma R_0} \). We employ a pointwise estimate to find by use of Taylor’s formula

\begin{align*}
\frac{\tilde{h}(x) - A(x)}{\sigma R_0} &\leq C \left( \sup_{B_{R/2}} |D^2 \tilde{h}(x)| \right) \frac{|x - x_0|^2}{\sigma R_0} \\
&\leq C \left( \sup_{B_{R/2}} |D^2 \tilde{h}(x)| \right) \frac{(2\sigma R_0)^2}{\sigma R_0} \quad (\text{since } x \in B(x_0, 2\sigma R_0)) \\
&\leq C \sigma R \left( \sup_{B_{R/2}} |D^2 \tilde{h}(x)| \right) \quad (\text{since } R \leq \frac{10}{\sigma^2} R) \\
&\leq C \sigma \int_{B_R} |D \tilde{h}| \, dx \quad (\text{by Proposition 5.3}) \\
&\leq C \sigma \left( \int_{B_R} |D \tilde{h}| \right)^{\frac{q}{q+1}} = : \text{III} \quad (\text{by Jensen}).
\end{align*}

Similarly as in the estimation given in (5.36), we again employ Lemma 5.6 to further obtain

\begin{align*}
\text{III} &\leq C \sigma \int_{B_R} |E \tilde{u}| \overset{\text{Def}}{=} C \sigma \int_{B_R} |E(u - a)| \\
&= C \sigma \int_{B_R} |E u - (E u)_{B_{R_0}}| \quad (\text{since } E u = (E u)_{B_{R_0}}) \\
&= C \sigma \left( \int_{B_R} |E u - (E u)_{B(0, R_0)}| \right)^{\frac{q}{2}} \\
&\leq C \sigma \left( \int_{B_{R_0}} V \left( \left| E u - (E u)_{B_{R_0}} \right| \right) \right)^{\frac{1}{2}} \quad (\text{by (5.32) and (2.15)}_1) \\
&\leq C \sigma \left( \int_{B_{R_0}} V \left( \left| E u - (E u)_{B(R_0)} \right| \right) \right)^{\frac{1}{2}} \quad (\text{by Jensen and } \frac{\sigma}{R} R_0 < R < R_0).
\end{align*}

Collecting estimates, we obtain with a constant \( C = C(m, n, L, \ell) > 0 \) and for all \( x \in B_{2\sigma R_0} \)

\begin{equation}
(5.40) \quad V \left( \frac{\tilde{h}(x) - A(x)}{\sigma R_0} \right) \leq CV \left( \sigma \left( \int_{B_{R_0}} V \left( \left| E u - (E u)_{B_{R_0}} \right| \right) \right)^{\frac{1}{2}} \right) =: CV(Y),
\end{equation}

where \( Y \) is defined in the obvious manner. Now, since \( V(r) \leq |r|, 0 < \sigma < \frac{1}{4} \) and by (5.32), \( Y \leq 1 \). Consequently, integrating (5.40) over \( B_{2\sigma R_0} \), we obtain with \( C = C(m, n, L, \ell) > 0 \)

\begin{equation}
(5.41) \quad \text{II} = C \int_{B_{2\sigma R_0}} V \left( \frac{\tilde{h}(x) - A(x)}{\sigma R_0} \right) \, dx \leq C(\sigma R_0)^n V(Y) \\
\overset{(2.15)_1}{\leq} C(\sigma R_0)^n \min \{ Y, Y^2 \} \leq C(\sigma R_0)^{n+2} R_0^{n} \int_{B_{R_0}} V \left( \left| E u - (E u)_{B(x_0, R_0)} \right| \right).$
\]
Combining estimates (5.39) and (5.41), we then find with a constant $C = C(n, m, L, \ell, q) > 0$ that

$$E(x_0, \sigma R_0) \leq \frac{C}{\sigma^2} \left( \int_{B_R} V(Eu) \right)^q + C\sigma^{n+2} R_0^q \sigma \int_{B_R} V(|Eu|)$$

$$= \left( \frac{C}{\sigma^2} \frac{E(x_0, R_0)}{R_0^q} \right)^{q-1} E(x_0, R_0) + C\sigma^{n+2} E(x_0, R_0)$$

(5.42)

We will now use the previous inequality to deduce a preliminary excess decay.

**Proposition 5.7.** Let $f: \mathbb{R}^{n \times n}_{\text{sym}} \to \mathbb{R}$ satisfy (a)–(c) from Theorem 1.2. Given $0 < \alpha < 1$, $M_0 > 0$ and $1 < q < \frac{n+1}{\alpha}$, there exist two parameters $\sigma = \sigma(n, L, \ell, \alpha, M_0, q) \in (0, \frac{1}{\alpha})$ as well as $\tilde{e} = \tilde{e}(n, L, \ell, \alpha, M_0, q) \in (0, 1)$ such that every local BD-minimizer $u \in \text{BD}(\Omega)$ of the functional $F$ satisfies the following: If $B(x_0, R_0) \Subset \Omega$ is an open ball with $0 < R_0 \leq 1$ together with

$$\tilde{e}(u, x_0, R_0) \leq \tilde{e}_0^2$$

then there holds

$$\tilde{E}(u, x_0, \sigma R_0) \leq \sigma^{1+\alpha} \tilde{E}(u, x_0, R_0).$$

(5.44)

**Proof.** Let $\alpha \in (0, 1)$ and $M_0 > 0$ be given. We start by choosing a preliminary $\tilde{e}_0 > 0$ in a way such that (5.43) implies (5.31) and (5.32). We estimate with $H := Eu - (Eu)|_{B(x_0, R_0)}$, Lemma 2.9 and the shorthands $A_{\tilde{e}_0}^\circ := B(x_0, R_0) \cap \{|H| \leq 1\}$

$$\int_{B(x_0, R_0)} |H| \leq \frac{1}{\omega_n R_0} \int_{A_{\tilde{e}_0}} |H| \left. + \frac{1}{\omega_n R_0} \int_{\Omega_{\tilde{e}_0}} |H| \right. = \frac{\mathcal{L}^n(A_{\tilde{e}_0})}{\omega_n R_0^q} \int_{A_{\tilde{e}_0}} |H| + \frac{C}{R_0^q} \int_{A_{\tilde{e}_0}} V(|H|)$$

$$\leq C \frac{\mathcal{L}^n(A_{\tilde{e}_0})}{R_0^q} \left( \int_{B(x_0, R_0)} |H| \right)^{\frac{1}{2}} + C \int_{B(x_0, R_0)} V(|H|) \overset{(5.43)}{\leq} C \tilde{e}_0 + \tilde{e}_0^2,$$

where $C = C(n) > 0$. We now choose $\tilde{e}_0 > 0$ so small such that the very right-hand side of the preceding inequality is smaller or equal to 1. At this stage, for $1 < q < \frac{n+1}{\alpha}$, (5.42) is available and therefore yields for $0 < \sigma \leq \frac{1}{2}$

$$\tilde{E}(u, x_0, \sigma R_0) \leq \left( \frac{C}{\sigma^{n+2}} \left( \tilde{E}(u, x_0, R_0) \right)^{q-1} + C\sigma^2 \right) \tilde{E}(u, x_0, R_0)$$

(5.45)

where now $C = C(n, M_0, L, \ell, q) > 0$. We subsequently choose $\sigma = \sigma(n, M_0, L, \ell, q, \alpha) > 0$ so small such that with the constant $C > 0$ from (5.45) there holds $2C\sigma^2 \leq \sigma^{1+\alpha}$. We then put $\tilde{e} := \min\{\sigma^{n+2}, \tilde{e}_0\}$. In turn, inserting these choices into (5.45) gives

$$\tilde{E}(u, x_0, \sigma R_0) \leq (2C\sigma^2) \tilde{E}(u, x_0, R_0) \leq \sigma^{1+\alpha} \tilde{E}(u, x_0, R_0),$$

and this is precisely (5.47). The proof is complete. 

\[\square\]

\[2\]Note that the constant $C > 0$ in (5.42) depends on $n, m, L, \ell$ and $q$, but by (5.36), $m$ depends on $n$ and $M_0$ only.
5.4. Iteration and Proof of Theorem 1.2. To conclude the proof of Theorem 1.2, we need to iterate Proposition 5.7.

**Corollary 5.8 (Iteration).** Let \( f: \mathbb{R}^m_{\text{sym}} \to \mathbb{R} \) satisfy (a)–(c) from Theorem 1.2. Given \( 0 < \alpha < 1 \) and \( M_0 > 0 \), there exist \( \varepsilon = \varepsilon(n, L, \alpha, M_0) \in (0, 1) \) and \( R_0 = R_0(n, L, \alpha, M_0, \alpha) \in (0, 1) \) such that every generalised local minimiser \( u \in \text{BD}(\Omega) \) of the functional \( F \) satisfies the following: If \( x_0 \in \Omega \) and \( 0 < R < R_0 \) are such that \( B(x_0, R) \subset \Omega \) and

\[
\sup_{B(x_0, R)} |u| \leq M_0/2,
\]

then there holds

\[
\mathcal{E}(u; x_0, R) \leq \varepsilon^2 \quad \text{and} \quad |(Eu)_{B(x_0, R)}| \leq M_0/2,
\]

and

\[
\mathcal{E}(u; x_0, R) \leq C \left( \frac{R}{R_0} \right)^{2n+1} \mathcal{E}(u; x_0, R) \quad \text{for all} \quad 0 < r \leq R.
\]

Here, \( C = C(n, L, \alpha, M_0) > 0 \) is a constant.

The corollary is proved in a standard manner, the proof following, e.g., [42, Prop. 4.8] or [14, Lem. 5.8]; note that the dependence on \( q \) in Proposition 5.7 is removed by specialising, e.g., to \( q = \frac{2n+1}{n+1} \in (1, \frac{2n+1}{n+1}) \). Working from here, we can proceed to the

**Proof of Theorem 1.2.** Under the conditions of Theorem 1.2, let \( x_0 \in \Omega_u \) so that

\[
\lim_{r \to 0} \int_{B(x_0, r)} |\mathcal{E} u - (Eu)_{B(x_0, r)}| \, d\mathcal{L}^n + \frac{|E' u|(B(x_0, r))}{\alpha n} = 0.
\]

Since \( V(\cdot) \leq |\cdot| \), this yields

\[
\lim_{r \to 0} \int_{B(x_0, r)} V(|\mathcal{E} u - (Eu)_{B(x_0, r)}|) \, d\mathcal{L}^n + \frac{|E' u|(B(x_0, r))}{\alpha n} = 0.
\]

Our aim is to show that the conditions of (5.46) remain valid for all points in a neighbourhood of \( x_0 \). We start by noting that for \( 0 < \delta < 1 \) which we assume sufficiently small but fixed,

\[
\sup_{0 < r < \delta} |(Eu)_{B(x_0, r)}| =: M < \infty
\]

as a second consequence of \( x_0 \in \Omega_u \). We then define \( M_0 \) as above. In consequence, Theorem 5.8 provides us with a radius \( R_0 > 0 \) and a threshold \( \varepsilon > 0 \) such that (5.46) implies (5.47) provided \( 0 < R < R_0 \). We then choose \( 0 < R < R_0 \) in a way such that

\[
\mathcal{E}(u; x_0, R) \leq \frac{\varepsilon^2}{4^{n+1}} \quad \text{and} \quad |(Eu)_{B(x_0, R)}| \leq M(\leq \frac{1}{4} M_0),
\]

being possible by the definition of \( M > 0 \) and (5.49). Our aim is to show that with \( R' := \frac{1}{4} R \) there holds \( \mathcal{E}(u; x, R') \leq \varepsilon^2 \) and \( |(Eu)_{B(x, R')}| \leq \frac{1}{4} M_0 \) for all \( x \in B(x_0, R') \). We have

\[
\int_{B(x, R')} \left( |\mathcal{E} u - (Eu)_{B(x, R')}| \right) \, d\mathcal{L}^n + \frac{|E' u|(B(x, R'))}{\alpha n (R')^n} \leq \frac{2}{\omega_n} \int_{B(x_0, R')} \int_{B(x_0, R)} V(|\mathcal{E} u(y) - \mathcal{E} u(z)|) \, dy \, dz + \frac{4^{n+1}}{\omega_n} \frac{|E' u|(B(x_0, R))}{(R')^n} 
\]

\[
\leq \frac{8}{\omega_n} \int_{B(x_0, R')} \int_{B(x_0, R)} V(|\mathcal{E} u(y) - (Eu)_{B(x_0, R)}|) \, dy \, dz + \frac{4^{n+1}}{\omega_n} \frac{|E' u|(B(x_0, R))}{(R')^n} 
\]

\[
\leq \frac{8}{\omega_n} \frac{2^{4n}}{4^{n+3}} \varepsilon^2 \leq \varepsilon^2.
\]

On the other hand, we have

\[
|\text{sup}_{B(x, R')} (Eu)| \leq \int_{B(x, R')} |\mathcal{E} u| \, d\mathcal{L}^n + \frac{|E' u|(B(x, R'))}{\alpha n (R')^n}.
\]
the relevant embeddings. This forces to argue via Besov-Nikolski˘ı spaces in the full gradient,
Remarks and Extensions.

In this concluding section, we discuss some aspects, extensions
and limitations of the results presented so far.

We begin by noting that, under the assumptions of Theorem 1.2, we can actually establish $C^2$-partial regularity of generalised minima. Namely, letting $x_0 \in \Omega_\alpha$, we have $u \in C^{1,\alpha}(B(x_0, r); \mathbb{R}^n)$ for some $r > 0$ and all $0 < \alpha < 1$. This is a consequence of Schauder estimates based on the $C^{1,\alpha}$-regularity of $u$ in a neighbourhood of $x_0$. Namely, choosing $|h|$ small enough, we consider the finite differences $\tau_{\alpha}u(x) := \varepsilon(u)(x + h\varepsilon) - \varepsilon(u)(x)$, where $x$ belongs to a suitable neighbourhood of $x_0$ and $\varepsilon$ is the $s$-th unit vector. Then, following [42, Thm. 4.9], we set

$$\mathcal{D}(x)[\xi, \eta] := \int_0^1 \left( f''(\varepsilon(u))(x + t\tau_{\alpha}u(x)) \xi, \eta \right) dt, \quad \xi, \eta \in \mathbb{R}^{n\times n}_{sym}.$$ 

By condition (a) from Theorem 1.2, $\mathcal{D} \in C^{0,1/2}(U; S(\mathbb{R}^{n\times n}_{sym}))$ for some open neighbourhood $U$ of $x_0$. As we can assume that $u$ is of class $C^{1,\alpha}(U; \mathbb{R}^n)$ by occasionally making $U$ smaller, we infer similarly as to (5.9) that $\mathcal{D}$ is uniformly strongly symmetric rank-one convex on $U$. In particular, there exists a constant $\lambda > 0$ such that $\mathcal{D}(x)[\xi, \xi] \geq \lambda\|\xi\|^2$ for all $\xi \in \mathbb{R}^n \odot \mathbb{R}^n$. Working from
here, it is not too difficult to establish an inequality of Garding type for some $r > 0$ suitably small: There exist $\gamma_1, \gamma_2 > 0$ such that there holds
\[
\int_{B(x_0,r)} \frac{\partial^2(x)}{\partial (\nabla \phi)^2} \, dx \geq \int_{B(x_0,r)} \gamma_1 |\nabla \phi|^2 - \gamma_2 |\phi|^2 \, dx \quad \text{for all } \phi \in C^0_c(B(x_0,r); \mathbb{R}^n),
\]
and $|\partial^2(x)| \leq C$ for some constant $C = C(f, x_0, r) > 0$. On the other hand, since $u$ is a minimiser and of class $C^{1, \alpha}(B(x_0,r); \mathbb{R}^n)$ for any $0 < \alpha < 1$, we deduce the Euler-Lagrange equation
\[
\int_{B(x_0,r)} \partial^2(x) (\nabla u, \nabla \phi) \, dx = 0 \quad \text{for all } \phi \in C^0_c(B(x_0,r); \mathbb{R}^n).
\]
At this stage, picking an arbitrary localisation function $\rho \in C^\infty_c(B(x_0,r); [0, 1])$, we may test the preceding equation with $\phi = \tau_{e^{-\rho} v} (\rho \tau_{e^{-\rho} u})$ for $|h|$ suitably small. Here $\tau_{e^{-\rho} v}(x) := v(x - he_\rho)$ denotes the backward finite difference. As a consequence, we obtain that
\[
\int_{B(x_0,r)} \langle f'(u) \partial_x u, \nabla \phi \rangle \, dx = 0 \quad \text{for all } \phi \in C^1_c(\Omega; \mathbb{R}^n)
\]
holds for any $s \in \{1, \ldots, n\}$. At this stage, we invoke Proposition 5.3 and reduce to [38, Thm. 3.2] to find that $\partial_x u$ is of class $C^{1,1/2}$ in a neighbourhood of $x_0$. This is not quite the asserted statement, and to derive it, we note that if $u$ is of class $C^{2,1/2}$ in a neighbourhood of $x_0$, then $f''(u)$ is locally Lipschitz as a consequence of $f \in C^{2,1}_\text{loc}(\mathbb{R}^n)$ and the aforementioned regularity of $u$. The Hölder regularity then is a consequence of a subsequent application of the Schauder estimates as given in [38, Thm. 3.2].

An analogous regularity theory can be set up when $x$-dependent integrands $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ are considered, and we refer the reader to the corresponding statements in [42, Sec. 6]; these follow in an analogous way once the regularity results from Theorem 1.2 are available. However, the case of fully non-autonomous integrands $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ comes along with two major difficulties. First, to the best of the author’s knowledge, there is no integral representation available at present; the arguments of RINDLER [61] do not seem to easily generalise to this situation. In contrast to (1.3), the definition of generalised minima then must be given directly by the Lebesgue-Serrin-type extension. To then access the Euler-Lagrange equations satisfied by the respective BD-minimisers, it is necessary to employ a careful approximation procedure. This in principle being possible, we would still need a higher integrability result on the gradients of BD-minima as it is usually required (cf. [39, Thm. 9.5 ff.]). In the quasiconvex, superlinear growth context, the latter is obtained as a consequence of the Caccioppoli inequality of the second kind in conjunction with the Sobolev inequality. In this situation, the Gehring lemma then boosts the so derived reverse Hölder inequality with increasing supports to the higher integrability of the gradients. In the linear growth situation, working from the Caccioppoli-type inequality strictly requires a sublinear Sobolev inequality, the unconditional availability of which being ruled out by a counterexample due to BUCKLEY & KOSKELA [19]. This is an important issue, as otherwise we would immediately obtain that BD-minimiser belongs to $W^{1, q}_{\text{loc}}$ for some $q > 1$, a fact which would simplify several stages of the above proof. A similar issue had been identified by ANZELLOTTI & GIAQUINTA [10, Sec. 6] within the framework of convex full gradient functionals. However, note that if $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ satisfies a splitting condition $f(x, y, z) = f_1(x, z) + f_2(x, y)$ for some strongly symmetric quasiconvex integrand $f_1 : \Omega \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ of linear growth and $f_2 : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ being convex and of at most $\frac{a}{n-1}$ growth in the second variable, then suitable regularity results can be formulated. Also, SCHMIDT [64] provides an interesting alternative of a partial regularity proof for convex, fully non-autonomous integrands of (super)quadratic growth that does not utilise Gehring’s lemma. The drawback here is that does not seem to generalise easily to the (super)quadratic quasiconvex situation; even if it would, it needed to be compatible with the above proof scheme.
6. APPENDIX

6.1. Existence of BD-minima. Implicitly used in the main part, we now briefly justify the existence of generalised minima for the Dirichlet problem (1.1) in the sense of (1.3), now being subject to the strong symmetric quasiconvexity of \( f \in C(\mathbb{R}^{n \times n}_{\text{sym}}) \), and gather some consequences. This program is somewhat analogous to [18, Thm. 5.3] where, however, a different coerciveness condition was employed. We hereafter let \( u_0 \in LD(\Omega) \) be a given Dirichlet datum and \( f \in C(\mathbb{R}^{n \times n}_{\text{sym}}) \) a strongly symmetric quasiconvex integrand satisfying both (1.4) and the linear growth assumption (LG). Our objective is to establish (with the notation of (1.1) ff.)

\[
(6.1) \quad \inf_{L D_{u_0}(\Omega)} F = \min_{B D(\Omega)} F_{u_0},
\]

particulary asserting the existence of BD-minimisers. Toward the latter, we note that because \( F \) is strongly symmetric quasiconvex, we have for all \( \varphi \in C^c_c(\Omega; \mathbb{R}^n) \)

\[
f(0) = f(0) - \ell V(0) \leq \int_{\Omega} f(\varepsilon(\varphi)) - \ell V(\varepsilon(\varphi)) \, dx,
\]
as follows easily by passing from \( Q = (0,1)^n \) to general open domains \( \Omega \), cf. DACOROGNA [23]. Thus, by Lemma 5.1(ii) with \( w = 0 \)

\[
f(0) - \ell v + \int_{\Omega} V(\varepsilon(\varphi)) \, dx \leq \int_{\Omega} f(\varepsilon(\varphi)) - f(\varepsilon(u_0 + \varphi)) \, dx + \int_{\Omega} f(\varepsilon(u_0 + \varphi)) \, dx
\]

\[
\leq c(L) \int_{\Omega} |u_0| \, dx + \int_{\Omega} f(\varepsilon(u_0 + \varphi)) \, dx.
\]

At this stage, we pick an open and bounded Lipschitz set \( \tilde{\Omega} \subset \mathbb{R}^n \) such that \( \text{dist}(\partial \Omega, \partial \tilde{\Omega}) > 0 \) and find, following the discussion in Section 2.2, \( \bar{\pi}_0 \in LD_0(\tilde{\Omega}) \) such that \( \bar{\pi}_0|_\Omega = u_0 \). We then put, for \( v \in BD(\Omega) \)

\[
(6.2) \quad \bar{\nu} := \begin{cases} v & \text{in } \Omega, \\ \bar{\pi}_0 & \text{in } \tilde{\Omega} \setminus \Omega. \end{cases}
\]

By Lipschitz regularity of \( \partial \Omega \) and since \( \bar{\pi}_0 \in LD(\tilde{\Omega}), \bar{\nu} \in BD(\tilde{\Omega}) \). Hence, we have for all \( \varphi \in C^c_c(\Omega; \mathbb{R}^n) \),

\[
\ell \int_{\tilde{\Omega}} V(\varepsilon(\bar{\pi}_0 + \varphi)) \, dx - \ell \int_{\tilde{\Omega}} V(\varepsilon(\bar{\pi}_0)) \, dx = C(u_0, L, \Omega) = \ell \int_{\tilde{\Omega}} V(\varepsilon(u_0 + \varphi)) \, dx - C(u_0, L, \Omega)
\]

\[
= \ell \int_{\tilde{\Omega}} V(\varepsilon(u_0 + \varphi)) - V(\varepsilon(\varphi)) \, dx + \ell \int_{\tilde{\Omega}} V(\varepsilon(\varphi)) \, dx - C(u_0, L, \Omega)
\]

\[
\leq C(\ell) \int_{\Omega} |\varepsilon(u_0)| + c(L) \int_{\Omega} |u_0| \, dx - f(0) - \ell v(\varepsilon(u_0 + \varphi)) \, dx
\]

\[
+ \ell \int_{\tilde{\Omega}} f(\varepsilon(u_0 + \varphi)) \, dx
\]

\[
= C(u_0, \ell, L, f(0), \Omega) + \ell \int_{\tilde{\Omega}} f(\varepsilon(\bar{\pi}_0 + \varphi)) \, dx - \int_{\tilde{\Omega}} f(\bar{\pi}_0) \, dx.
\]

At this stage, let \( v \in BD(\Omega) \) be arbitrary and pick, due to Lemma 2.1, a sequence \( (v_j) \subset u_0 + C^c_c(\Omega; \mathbb{R}^n) \) such that \( \tilde{v}_j \to \tilde{\nu} \) symmetric area-strictly on \( BD(\tilde{\Omega}) \). Since for every \( j \in \mathbb{N} \), \( \tilde{v}_j \) is of the form \( \tilde{\pi}_0 + \varphi_j \) with some \( \varphi_j \in C^c_c(\Omega; \mathbb{R}^n) \), we obtain

\[
\ell \int_{\Omega} V(\varepsilon(\tilde{v}_j)) \, dx - \ell \int_{\tilde{\Omega}} V(\varepsilon(\bar{\pi}_0)) \, dx \leq C(u_0, \ell, L, f(0), \Omega) + \ell \int_{\tilde{\Omega}} f(\varepsilon(\tilde{v}_j)) \, dx - \int_{\tilde{\Omega}} f(\bar{\pi}_0) \, dx.
\]

Since \( f \) is symmetric quasiconvex, it is symmetric rank-one convex in the sense as specified in Section 2.3. Therefore, Lemma 2.6 (which precisely yields continuity of the associated integral
functions for symmetric rank-one convex integrands) and the very definition of \( V \) yield by passing \( j \to \infty \)
\[
\ell \int_{\Omega} V(\overline{E} v) - \ell \int_{\Omega} V(c(p_0) \lambda) dx - C(u_0, \ell, L, f(0), \Omega) \leq \int_{\Omega} f(\overline{E} v) - \int_{\Omega} f(c(p_0) \lambda) dx.
\]
By definition of \( \overline{E} \), we thereby obtain
\[
\ell \int_{\Omega} V(\overline{E} v) + \ell \int_{\Omega} |\text{Tr}_{\partial \Omega}(u_0 - v)| \, d\mathcal{H}^{n-1} - C(u_0, \ell, L, f(0), \Omega) \leq \int_{\Omega} f(\overline{E} v) + \int_{\partial \Omega} f'(\text{Tr}_{\partial \Omega}(u_0 - v) \oplus v_{\partial \Omega}) \, d\mathcal{H}^{n-1} = \mathcal{F}_{u_0}[v].
\]
This proves that \( \mathcal{F}_{u_0} \) is bounded below on \( \text{BD}(\Omega) \). Let \( \langle u_j \rangle \subset \text{BD}(\Omega) \) be a minimising sequence for \( \mathcal{F}_{u_0} \), i.e., \( \mathcal{F}_{u_0}[u_j] \to \inf_{\text{BD}(\Omega)} \mathcal{F}_{u_0} \). By (6.3) and the estimate \( V(\cdot) + 1 \geq |\cdot| \), \( \langle u_j \rangle \) is bounded in \( \text{BD}(\Omega) \). Thus, we may extract a non-relabeled subsequence and find some \( u \in \text{BD}(\Omega) \) such that \( u_j \rightharpoonup u \) in \( \text{BD}(\Omega) \). As a consequence of Theorem 2.5, \( \mathcal{F}_{u_0}[u] \leq \liminf_{j \to \infty} \mathcal{F}_{u_0}[u_j] = \inf_{\text{BD}(\Omega)} \mathcal{F}_{u_0} \). Hence, \( u \) is a BD-minimiser in the sense of (1.3).

We come to (6.1). Since there holds \( L_{\text{D} u_0}(\Omega) \subset \text{BD}(\Omega) \) and \( \mathcal{F}_{u_0} | L_{\text{D} u_0}(\Omega) = F \) on \( L_{\text{D} u_0}(\Omega) \), we obtain \( \geq \) in (6.1). For the other direction, pick a BD-minimiser \( u \in \text{BD}(\Omega) \) for \( F \), its existence having been proved above. Choosing an extension \( p_0 \) of the Dirichlet datum \( u_0 \) as above and defining \( \tilde{u} \) via (6.2), we invoke Lemma 2.1 to obtain a sequence \( \langle u_j \rangle \subset u_0 + C^p_{\Omega}^e(\Omega; \mathbb{R}^n) \) such that \( \tilde{u}_j \to \tilde{u} \) area-strictly in \( \text{BD}(\Omega) \). Then, again by Lemma 2.6, \( \mathcal{F}_{u_0}[u_j] | \Omega \to \mathcal{F}_{u_0}[u] \) as \( j \to \infty \). Thus, since \( \tilde{u}_j | \Omega \in L_{\text{D} u_0}(\Omega) \) and \( \mathcal{F}_{u_0}[u_j] = F[u_j] \) for all \( j \in \mathbb{N} \),
\[
\inf_{L_{\text{D} u_0}(\Omega)} F \leq \liminf_{j \to \infty} \mathcal{F}_{u_0}[u_j] = \mathcal{F}_{u_0}[u] = \min_{\text{BD}(\Omega)} \mathcal{F}_{u_0}[u].
\]
Since we already established that \( \min_{\text{BD}(\Omega)} \mathcal{F}_{u_0} \leq \inf_{L_{\text{D} u_0}(\Omega)} F \), the proof of (6.1) is complete.

6.2. Auxiliary Estimates on the \( V_p \)-functions. In this section we provide the proof of the auxiliary estimation (3.12) that helped to establish a particular form of a Korn-type inequality. The first uniform comparability assertion of (3.12) is a basic property of shifted \( N \)-functions, cf. [28, Def. 2 and Sec. 2]. We thus begin by showing that \( \psi \) given by (3.11) satisfies the conditions of Lemma 2.8 together with the second uniform comparability asserted in (3.12). This means that
\[
(1 + a + t)^p - 2t^2 \leq \psi''(a + t)^2 \leq C(1 + a + t)^{p - 2} t^2
\]
for some \( 0 < c < C < \infty \) independent of \( a \geq 0 \) and \( t > 0 \). We start with (6.4)2, and recall that 1 < \( p < 2 \) throughout this section. To this end, note that for \( t > 0 \)
\[
\frac{d}{dt} \psi(t) = (p - 2)(1 + t)^p - 3 t^2 + 2(1 + t)^p - 2 t,
\]
\[
\frac{d^2}{dt^2} \psi(t) = (p - 2)(p - 3)(1 + t)^p - 4 t^2 + 4(p - 2)(1 + t)^p - 3 t + 2(1 + t)^p - 2.
\]
Since 1 < \( p < 2 \), the second term on the right-hand side of the ultimate inequality is negative. Therefore,
\[
\psi''(a + t)^2 \leq [(p - 2)(p - 3)(1 + a + t)^p - 4(a + t)^2 t^2 + 2(1 + a + t)^p - 2 t^2,
\]
establishing the upper bound asserted by (6.4). The lower bound requires a refined argument. Since \( p > 1 \), \( c_p := p^2 - p \) is strictly positive. We write for \( t > 0 \)
\[
\frac{\psi''(a + t)^2}{(1 + a + t)^{p - 2} t^2} = \frac{(p - 2)(p - 3)(1 + a + t)^{p - 4}(a + t)^2 t^2 + 4(p - 2)(1 + a + t)^{p - 3}(a + t)^2}{(1 + a + t)^{p - 2} t^2}
\]
We claim that the term on the very right-hand side of the previous equation is larger or equal than \( c_p \). To see this, note that with this choice of \( c_p \), we have
\[
\frac{p-3}{4} + 1 = \frac{p+1}{4} \leq \frac{2}{4(2-p)} - c_p.
\]
In consequence,
\[
\left[ \frac{p-3}{4} \left( \frac{a+t}{1+a+t} \right)^2 + \frac{a+t}{1+a+t} \right] \leq \frac{p-3}{4} + 1 \leq \frac{c_p-2}{4(p-2)}
\]
and therefore, by \( p < 2 \), establishes
\[
4(p-2) \left[ \frac{p-3}{4} \left( \frac{a+t}{1+a+t} \right)^2 + \frac{a+t}{1+a+t} \right] \geq c_p - 2.
\]

The claimed lower bound follows. We turn to the third uniform comparability assertion of (3.12), which is equivalent to the existence of constants \( 0 < c \leq C < \infty \) such that
\[
(6.6) \quad c(1+t^2+a^2)^{\frac{p-2}{2}} t^2 \leq (1+t+a)^{p-2} t^2 \leq C(1+t^2+a^2)^{\frac{p-2}{2}} t^2.
\]
holds for all \( a, t \geq 0 \). First note that
\[
\sqrt{1+t^2+a^2} \leq (1+t+a)^{\frac{p}{2}} = 1 + t + a
\]
so that, because of \( 1 < p < 2 \), \( (1+t+a)^{p-2} \leq (1+t^2+a^2)^{\frac{p-2}{2}} \), and so the upper bound in (6.6) follows. For the lower bound note that, because of Young’s inequality
\[
1 + t + a = \sqrt{(1+t+a)^2} \leq \sqrt{8 + 8t^2 + 8a^2} \leq \sqrt{8} \sqrt{1+t^2+a^2},
\]
thereby establishing the lower bound in (6.6); for the latter estimate, we could have alternatively argued by virtue of Lemma 2.9, cf. (2.15). We now turn to (6.4). Setting \( a = 0 \) in (6.4)\(_1\), (6.4)\(_1\) is obviously equivalent to
\[
(6.7) \quad \psi'(t) \simeq (1+t)^{p-2} t.
\]
By (6.5)\(_1\) and \( 1 < p < 2 \), we have \( \psi'(t) \leq 2(1+t)^{p-2} t \) for all \( t > 0 \). On the other hand, for \( t > 0 \),
\[
\frac{\psi'(t)}{(1+t)^{p-2} t} = \frac{(p-2)(1+t)^{p-3} t^2 + 2(1+t)^{p-2} t}{(1+t)^{p-2} t} = (p-2) \frac{t}{1+t} + 2 \geq p.
\]
The proof of (3.12) is complete.

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