the Relative Entropy as an Increasing Function of Time in Cosmology

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Abstract
In this paper, it is shown that the relative entropy is an increasing function of time in both the linear regime and the non-linear regime during the large scale structure formation in cosmology.

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1 Introduction
In [1] the authors have proposed a measure which quantifies the distinguishability of the actual mass distribution from its spatial average, borrowing a well-known concept in standard information, the relative entropy,

\[ S\{p||q\} = \sum_i p_i \ln \frac{p_i}{q_i}, \] (1)
where $\{q_i\}$ is the probability distribution and $\{p_i\}$ is the actual one. For a continuum the relevant quantity is

$$
\frac{S\{\rho||\langle \rho \rangle_D\}}{V_D} = \langle \rho \ln \frac{\rho}{\langle \rho \rangle_D} \rangle_D,
$$

(2)

where $\rho$ is the actual distribution and $\langle \cdot \cdot \cdot \rangle_D$ is its spatial average in the volume $V_D$ on the compact domain $D$ of the manifold $\Sigma$. For scalar functions $\Psi(t, X^i)$, the averaging operation, in term of Riemannian volume integration, is defined as

$$
\langle \Psi(t, X^i) \rangle_D := \frac{1}{V_D} \int_D \sqrt{g} d^3X \, \Psi(t, X^i),
$$

(3)

with $g := det(g_{ij})$ and the volume of an arbitrary compact domain, $V_D(t) := \int_D \sqrt{g} d^3X$; $X^i$ are coordinates in a $t = \text{const.}$ hypersurface (with 3-metric $g_{ij}$) that are comoving with fluid elements of dust:

$$
ds^2 = -dt^2 + g_{ij} dX^i dX^j.
$$

(4)

The derivative of the relative entropy with respect to the time is given in [1] by

$$
\dot{S}\{\rho||\langle \rho \rangle_D\} = \frac{d}{dt} \frac{1}{V_D} \int_D \sqrt{g} d^3X \, \Psi(t, X^i),
$$

(5)

where $\theta$ denotes the local expansion rate (as the trace of the extrinsic curvature of the hypersurfaces $t = \text{const.}$). Hereafter, a “dot” denotes the derivative respect the physical time, $\dot{\Psi} = d\Psi/dt$ and $\delta$ denotes the derivative of a local field from its spatial average, $\delta \Psi = \Psi - \langle \Psi \rangle_D$.

In [1], a conjecture is given: The Relative Information Entropy of a dust matter model $S\{\rho||\langle \rho \rangle_D\}$ is, for sufficiently large times, globally (i.e. averaged over the whole compact manifold $\Sigma$) an increasing function of time.

In the cosmology, the ‘sufficiently large times’ means after the beginning of the non-linear regime during the structure formation. In this regime, the overdense elements ($\delta \rho > 0$) stop expanding ($\theta \leq 0$) and the underdense elements ($\delta \rho < 0$) expand ($\theta > 0$). So we see the conjecture is plausible.

In this paper it will be shown that, in cosmology, the relative entropy is globally an increasing function of time even in the linear regime during the structure formation. It should be noted that this point can not be concluded from Eq.(5), because during the linear regime, the overdense elements ($\delta \rho > 0$) are also expanding ($\theta > 0$).
Below, firstly an proof of the positivity of the entropy, \( S\{\rho||\langle\rho\rangle_D\} \) is shown. Then it is shown that the entropy is globally an increasing function of time even in the linear regime during the structure formation.

2 the Positivity of the Relativity Entropy

We first express the relative entropy in term of the integral,

\[
S\{\rho||\langle\rho\rangle_D\} = \int_D \sqrt{g} d^3X \rho \ln \frac{\rho}{\langle\rho\rangle_D}.
\]

For an overdense volume element, its density may be taken as \( \rho = \langle\rho\rangle_D + \delta\rho, \delta\rho > 0 \). In order to ensure the average density to be \( \langle\rho\rangle_D \), we may divide the overdense elements into three types. For an element of the first type with \( \rho = \langle\rho\rangle_D + \delta\rho \), there must exist one or several underdense elements that their densities can be expressed as \( \rho = \langle\rho\rangle_D - h_i\delta\rho \), with \( h_i > 0, \sum_i h_i = 1, i = 1, 2, \cdots \). The region where the elements of the first type are localized is denoted by \( \tilde{D}_1 \).

For an overdense element of the second type, such one or several underdense elements does not exist. However, corresponding to several elements of the second type, there must exist one underdense element. Writing the density of this underdense element as \( \rho = \langle\rho\rangle_D - \delta\rho \), the densities of the several overdense elements can be expressed as \( \rho = \langle\rho\rangle_D + g_i\delta\rho \), with \( g_i > 0, \sum_i g_i = 1, i = 2, 3, \cdots \). The region where these underdense elements are localized is denoted by \( \tilde{D}_2 \). In fact, here we have given an correspondence between the overdense elements and the underdense elements by giving \( h_i \) and \( g_i \).

Then Eq.(6) can be written as

\[
S\{\rho||\langle\rho\rangle_D\} = \int_{\tilde{D}_1} \sqrt{g} d^3X f_1(\langle\rho(t, X^i)\rangle_D, \delta\rho(t, X^i)) \\
+ \int_{\tilde{D}_2} \sqrt{g} d^3X f_2(\langle\rho(t, X^i)\rangle_D, \delta\rho(t, X^i)),
\]

\[
f_1(\rho_0, \delta\rho) = (\rho_0 + \delta\rho) \ln \frac{\rho_0 + \delta\rho}{\rho_0} + \sum_i (\rho_0 - h_i\delta\rho) \ln \frac{\rho_0 - h_i\delta\rho}{\rho_0},
\]

\[
f_2(\rho_0, \delta\rho) = \sum_i (\rho_0 + g_i\delta\rho) \ln \frac{\rho_0 + g_i\delta\rho}{\rho_0} + (\rho_0 - \delta\rho) \ln \frac{\rho_0 - \delta\rho}{\rho_0}.
\]
where \( \rho_0 \equiv \langle \rho \rangle_D \). The derivatives of the two functionals, \( f_1(\langle \rho \rangle_D, \delta \rho) \) and \( f_2(\langle \rho \rangle_D, \delta \rho) \), with respect to \( \delta \rho \), are

\[
\frac{\delta f_1}{\delta(\delta \rho)} = \ln \frac{\langle \rho \rangle_D + \delta \rho}{\langle \rho \rangle_D} + \sum_i (-f_i) \ln \frac{\langle \rho \rangle_D - f_i \delta \rho}{\langle \rho \rangle_D} > 0, 
\]

\[
\frac{\delta f_2}{\delta(\delta \rho)} = \sum_i g_i \ln \frac{\langle \rho \rangle_D + g_i \delta \rho}{\langle \rho \rangle_D} - \ln \frac{\langle \rho \rangle_D - \delta \rho}{\langle \rho \rangle_D} > 0.
\]

We have used the conditions \( \sum_i h_i = 1 \) and \( \sum_i g_i = 1 \). Then two inequalities are obtained: \( f_1(\langle \rho \rangle_D, \delta \rho) \geq f_1(\langle \rho \rangle_D, 0) = 0 \) and \( f_2(\langle \rho \rangle_D, \delta \rho) \geq f_2(\langle \rho \rangle_D, 0) = 0 \). Now, due to Eq.(7), it can be concluded that the relative entropy, \( S\{\rho||\langle \rho \rangle_D\} \), is positive.

3 the Relative Entropy as an Increasing Function of Time in Linear Regime

In linear regime, both \( \delta \rho \) and \( \langle \rho \rangle_D \) are decreasing functions of time. Then, due to the results in the last section, it cannot be concluded whether \( S\{\rho||\langle \rho \rangle_D\} \) is an increasing function or not. However, it is well known that the density contrast on the overdense region: \( \delta \equiv \frac{\delta \rho}{\langle \rho \rangle_D} \), is an increasing function of time in linear regime. Although the density contrast on the underdense region is an decreasing function of time in linear regime, let’s express the density contrast on the underdense region as \( -\delta = -\frac{\delta \rho}{\langle \rho \rangle_D} \) with \( \delta \rho > 0 \) and \( \delta > 0 \). Then \( \delta \) on the underdense region is also an increasing function of time in linear regime. Now, rewrite the relative entropy in terms of the density contrast as

\[
S\{\rho||\langle \rho \rangle_D\} = \int_{D_1} d^3X \, \sqrt{g} \langle \rho(t, X^i) \rangle_D \tilde{f}_1(\delta(t, X^i)) \\
+ \int_{D_2} d^3X \, \sqrt{g} \langle \rho(t, X^i) \rangle_D \tilde{f}_2(\delta(t, X^i)),
\]

\[
\tilde{f}_1(\delta) \equiv (1 + \delta) \ln(1 + \delta) + \sum_i (1 - h_i \delta) \ln(1 - h_i \delta).
\]

\[
\tilde{f}_2(\delta) \equiv \sum_i (1 + g_i \delta) \ln(1 + g_i \delta) + (1 - \delta) \ln(1 - \delta).
\]

The derivatives of the two functionals, \( \tilde{f}_1(\delta) \) and \( \tilde{f}_2(\delta) \), with respect to \( \delta \), are

\[
\tilde{f}_1'(\delta) = \ln(1 + \delta) + \sum_i (-h_i) \ln(1 - h_i \delta) > 0,
\]

\[
\tilde{f}_2'(\delta) = \sum_i (1 + g_i \delta) \ln(1 + g_i \delta) + (1 - \delta) \ln(1 - \delta).
\]
\[ \tilde{f}_2'(\delta) = \sum_i g_i \ln(1 + g_i \delta) + \ln(1 - \delta) > 0, \]  

We have used the conditions \( \sum_i h_i = 1 \) and \( \sum_i g_i = 1 \). Additional, we have also used the equations \( \frac{\delta h_i}{\delta \delta} = \frac{\delta g_i}{\delta \delta} = 0 \). This implies we assumes that \( h_i \) and \( g_i \) are independent of \( \delta \). We believe this is reasonable. The reason is that the changes in \( h_i \) or \( g_i \) just give another different correspondence between the overdense elements and the underdense elements. However, this does not change the relative entropy if the distribution of the density is unchanged. Now we know \( \tilde{f}_1(\delta) \) and \( \tilde{f}_2(\delta) \) are increasing functionals of \( \delta \). At the same time, for a dust matter model, the variable \( \sqrt{g} \langle \rho(t, X_i) \rangle_D \) is an constant of time (This point has been indicated in [1]). Now, according to \( \delta \) as an increasing function of time and Eq. (12), it can be concluded that, in cosmology, the Relative Entropy is globally an increasing function of time in linear regime during the structure formation.

4 Conjecture and Discussion

Above, the relative entropy as an increasing function of time in linear regime in cosmology has been shown. The process is not same as in [1]. The derivative of the relative entropy with respect to time is not given, but express the relative entropy in the density contrast. The key step in this process is the validity of Eq. (7). This step is believed to be correct. The analysis above is applicable to the non-linear regime, too. The results are summarized as follows. The equations (12) and (15),(16) show that the relative entropy, \( S\{\rho||\langle\rho\rangle_D\} \), is an increasing functional of the density contrast, \( \delta \). And the density contrast is an increasing function of time in linear regime and the non-linear regime. So it is concluded that the relative entropy is globally an increasing function of time in both the linear regime and the non-linear regime.

Now a conjecture is given: The Relative Information Entropy of a dust matter model \( S\{\rho||\langle\rho\rangle_\Sigma\} \) is, globally (i.e. averaged over the whole compact manifold \( \Sigma \)) an increasing function of time.

Compared with the conjecture in [1], in the conjecture here, the condition, “for sufficiently large times”, is taken off.

This paper just give an illustration of the conjecture during the large scale structure formation in cosmology. A strict proof need be explored further.
References

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