CONCENTRATION RESULTS FOR SOLUTIONS OF A SINGULARLY
Perturbed Elliptic System with Variable Coefficients

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Abstract. In this article we shall study the following elliptic system with coefficients:

\[
\begin{align*}
\epsilon^2 \Delta u + c(x)u &= b(x) |v|^{q-1} v, \quad \text{and} \quad -\epsilon^2 \Delta v + c(x)v = a(x) |u|^{p-1} u \quad \text{in} \ \Omega \\
u > 0, \ v > 0 \ \text{in} \ \Omega, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \ \partial \Omega
\end{align*}
\]

where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^n, n \geq 3\). The coefficients \(a(x), b(x)\) and \(c(x)\) are positive bounded smooth functions. We shall study the existence of point concentrating solutions and discuss the role of the coefficients to determine the concentration profile of the solutions. We have also discussed some applications of our main theorem towards the existence of solutions concentrating on higher-dimensional orbits.

1. Introduction

Consider the following singularly perturbed coupled elliptic system:

\[
\begin{align*}
-\epsilon^2 \Delta u + c(x)u &= b(x) |v|^{q-1} v, \quad \text{and} \quad -\epsilon^2 \Delta v + c(x)v = a(x) |u|^{p-1} u \\
u > 0, \ v > 0 \ \text{in} \ \Omega, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \ \partial \Omega
\end{align*}
\]

where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^n\), \(\epsilon\) is a small positive parameter and \(n \geq 3\). The exponents \(p, q\) satisfy \(p, q > 1\) and

\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}
\]

i.e \(p\) and \(q\) lies below the critical hyperbola (for reference look at [10]). The weights \(a, b, c\) are smooth functions with

\[
K_1 \leq a(x), b(x), c(x) \leq K_2 \quad \text{in} \quad \Omega
\]

for some real constants \(K_1, K_2 > 0\).

In this work, we shall study the point concentration behavior for some least energy solution of the above system. We will also describe the role of the coefficients to determine the location of the concentration.

For the scalar case, the equation with nonconstant weights appears in many cases where the authors want to study the existence of higher-dimensional concentrating solutions (see [6, 11, 13, 20, 21, 22]). In [2, 3], Ambrosetti, Malchiodi, and Ni studied the existence of spherical concentrating solutions. Later in [6, 7, 13], the authors proved the existence of solutions concentrating on higher dimensional (\(S^1, S^3, \text{and} \ S^7\) orbits. In these works (6, 7, 13), the orbits of concentration produced by a reduction process (Hopf Fibration), which leads to an anisotropic problem in lower dimensions. Some more results related to...
anisotropic equations, showing the existence of solutions concentrating in higher dimensional orbits can be found in [12] [21] [22].

The singularly perturbed elliptic system, with Neumann boundary condition, was first studied by Avila and Yang in [5]. They proved the existence of nontrivial positive solutions whose point of maximum approaches to a common point on the boundary as ε goes to 0. This result was generalized by Ramos and Yang for some general convex nonlinearities in [15]. Then in [4], Pistoia and Ramos proved the concentration happens at the point of maximum of the mean curvature of the boundary. All these results are related to the existence of point concentrating solutions and the profile of the concentration. In [8], the authors studied the existence of radial solutions for a Neumann system with radial weights.

To the best of our knowledge, this is the first study for the Neumann elliptic system with non-trivial weights. We have considered general weight functions, which help us to find many other higher dimensional layered solutions for different singularly perturbed systems. We shall show the concentration profile mainly depends upon the coefficients a, b and c and, in a special case, on the mean curvature of the boundary too. We define a function Λ on the boundary of the domain. (see (3.10) for more details)

$$
\Lambda(x) = \left( \frac{b(x)}{c(x)} \right)^{\frac{q+1}{q-pq}} \left( \frac{a(x)}{c(x)} \right)^{\frac{q+1}{pq}} (c(x))^{1-\frac{n}{2}}
$$

The main result in this paper is the following,

**Theorem 1.1.** Under assumption (1.2), (1.3) ∃ an ε0 such that for 0 < ε < ε0 the equation (1.1) has non constant positive solutions uε, vε ∈ C^2(Ω). Moreover, both solutions concentrate for ε → 0 on a common point P_ε ∈ ∂Ω, with P_ε satisfying:

(i) \lim_{ε→0} P_ε = \inf_{x∈∂Ω} Λ(x) if Λ is not constant.

(ii) \lim_{ε→0} P_ε = \sup_{x∈∂Ω} H(x)γ(x) + η(x) if Λ is constant.

Where H is the mean curvature of the boundary and γ(x) = \frac{C_1}{\sqrt{c(x)}}, η(x) = \frac{C_2}{\sqrt{c(x)}} \frac{∂}{∂n}(C_3 \ln a(x) + C_4 \ln b(x) + C_5 \ln c(x)) are functions on ∂Ω. And the constants are given in Proposition 4.1.

Note that the concentration profile of the solution depends on the weight functions only unless the function Λ becomes constant. The following system in \mathbb{R}^n plays an important role in finding the asymptotic profile of the solutions.

(1.4) \quad -Δu + u = |v|^{q-1}v and -Δv + v = |u|^{p-1}u, \quad u, v > 0 in \mathbb{R}^n.

The energy functional I_ε(u, v) : H^1(Ω) × H^1(Ω) → \mathbb{R} associated to equation (1.1) is given by

(1.5) \quad I_ε(u, v) = \int_Ω |ε^2(∇u, ∇v) + c(x)uv - a(x)F(u) - b(x)G(v)| dx

with F(s) = \frac{s^{p+1}}{p+1} and G(s) = \frac{s^{q+1}}{q+1}. Let \mathcal{H} := H^1(Ω) × H^1(Ω) equipped with the norm \| (u, v) \| = \| u \| + \| v \|. Note that under the assumption (1.2) one can have q + 1 > 2^* > p + 1 and J_ε may not be well defined over \mathcal{H}. In [4] and [19] authors proved the existence result using Dual variation method and Fraction setting respectively. But for a modified problem as in [15], we shall see that the solutions are uniformly bounded and hence the integrals are well defined (One can see from [14] also). So for the moment, we
assume $2 < p \leq q < \frac{N+2}{N-2}$. Then $J_\varepsilon$ is well defined and belongs to $C^2(\mathcal{H}, \mathbb{R})$. Furthermore
\begin{equation}
\label{eq:1.6}
DL_\varepsilon(u, v)(\phi, \psi) = \langle u, \psi \rangle_\varepsilon + \langle \phi, v \rangle_\varepsilon - \int_{\Omega} [a(x)f(u)\phi + b(x)g(v)\psi]dx
\end{equation}
where $f(u) = |u|^{p-1}u$, $g(v) = |v|^{q-1}v$ and the quadratic term of the energy $\langle \cdot, \cdot \rangle_\varepsilon$ is defined as
\begin{equation}
\label{eq:1.7}
\langle u, v \rangle_\varepsilon = \int_{\Omega} [\varepsilon^2 \langle \nabla u, \nabla v \rangle + c(x)uv]dx
\end{equation}
is positive definite (negative definite) on $E^+(E^-)$ where $E^\pm = \{(\phi, \pm \phi) : \phi \in H^1(\Omega)\}$. Note $\mathcal{H} = E^+ \oplus E^-$. 

The paper is organized as follows. In the second section, using of Benci-Rabinowitz’s theorem, we have proved the existence of solutions $(u_\varepsilon, v_\varepsilon)$ for $\varepsilon$ small enough. In the third section, we have described some behavior of the solutions and the solutions of the limit problem. In the fourth and fifth section, we have shown the upper and lower energy estimate of the energy functional giving the proof of main theorem. Finally, in section 6 we shall discuss some applications, as mentioned earlier.

2. Existence of Mountain pass-type solution and known results

In this section, our goal is to find solution to the problem (1.1) from truncated system. Without loss of generality, one can assume $q + 1 \geq p + 1 > 2$, and $p + 1 < \frac{2N}{N-2}$. Then for the sequence $j = 1, 2, 3, \ldots$ define
\begin{equation}
g_j(s) = \begin{cases} 
A_j |s|^{p-1} s + B_j & \text{for } s \geq j, \\
|s|^q & \text{for } |s| \leq j, \\
A_j |s|^{p-1} s + \tilde{B}_j & \text{for } s \leq -j.
\end{cases}
\end{equation}
The coefficients are
\begin{equation}
A_j = \left(\frac{q}{p} + o(1)\right) j^{q-p} - A_j \quad \text{and} \quad B_j = \left(\frac{p-q}{p} + o(1)\right) j^q - \tilde{B}_j,
\end{equation}
chosen in such a way that the functions $g_j$ become $C^1$ (See [15]). And we consider the modified problem
\begin{equation}
\label{eq:2.1}
\begin{cases} 
-\varepsilon^2 \Delta u + c(x)u = b(x)g_j(v), & \text{and} \quad -\varepsilon^2 \Delta v + c(x)v = a(x)f(u) \quad \text{in } \Omega \\
u > 0, \ v > 0 \quad \text{in } \Omega, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \quad \text{on } \partial \Omega
\end{cases}
\end{equation}
The energy functional is
\begin{equation}
\label{eq:2.2}
I_\varepsilon(u, v) = \int_{\Omega} [\varepsilon^2 \langle \nabla u, \nabla v \rangle + c(x)uv - a(x)F(u) - b(x)G_j(v)]dx
\end{equation}
Hence, the energy functional (2.2) is $C^2$ over the Hilbert space $\mathcal{H}$. Furthermore, if $(u_{\varepsilon_j}, v_{\varepsilon_j})$ solves the system (2.1) and are uniformly bounded, then $(u_{\varepsilon_j}, v_{\varepsilon_j})$ solves the problem (1.1) for a large $j$. The proof of uniform boundedness is closely related to the proof given in [15].
We recall the following Liouville-type result from [15]. Suppose $f_\infty$ and $g_\infty$ are $C^1$ functions satisfy, for some positive constants $c_1, c_2, \forall s \in \mathbb{R}$, the following:

(a) $c_1 |s|^{q+1} \leq g_\infty(s) s \leq c_2 |s|^{q+1}$
(b) $g_\infty(s) s \leq (q+1) G_\infty(s)$
(c) $p g_\infty(s) s \leq g'_{\infty}(s)s^2$

Let $(u, v)$ be $C^2$ solution of $-\Delta u = g_\infty(v), -\Delta v = f_\infty(u)$ in $\mathbb{R}^n$ or with Neumann data in $\mathbb{R}^n_+$ (up to rotation and translation). We say $(u, v)$ has finite index if $\exists R_0 > 0$ with property that for all $\phi \in H^1$ such that support of $\phi$ in $\mathcal{B}(0, R_0)$ with $2 \int |\nabla \phi|^2 - f'(u) \phi^2 - g'(v) \phi^2 \geq 0$.

**Proposition 2.1** (proposition 1.4 in [15]). Let $g_\infty \in C^1(\mathbb{R})$ and $f_\infty = C_0 |s|^{p-1} s$, with $C_0 > 0$, $p, q$ satisfies critical hyperbola condition and suppose $(u, v)$ has finite index, that $u = v$.

(i) if $g_\infty = 0$, then $u = 0$

(ii) if $g_\infty$ satisfies (a), (b), (c) then $u = 0 = v$.

And we have

**Proposition 2.2.** For any given sequence $\varepsilon_j$, let $u_j, v_j$ be solutions to the problem (2.1) with $\varepsilon = \varepsilon_j$. If there exists $k \in \mathbb{N}$ such that $m(u_j, v_j) \leq k$ for every $j$, then $\exists K > 0$ such that

$$\|u_j\|_\infty + \|v_j\|_\infty \leq K \quad \forall j.$$

**Proof.** Suppose $\|u_j\|_\infty + \|v_j\|_\infty \to \infty$. Define

$$M_j = \sup_{x \in \bar{\Omega}} \{ \max \{|u_j(x)|^{1/p}, |v_j(x)|^{1/q}\} \}$$

Clearly $M_j \to \infty$. Let $\lim_{j \to \infty} \frac{j}{M_j^{q/p}} = l \in [0, \infty)$ (upto a subsequence).

**Case 1.** $l > 0$ Now choose a sequence $\lambda_j \in \mathbb{R}^+$ and $x_j \in \bar{\Omega}$ such that $\lambda_j^2 M_{j+1}^{p-1} = 1$, $M_j = \{ \max \{|u_j(x_j)|^{1/p}, |v_j(x_j)|^{1/q}\} \}$ respectively. Consider the blow-up scheme

$$\tilde{u}_j(x) = \frac{1}{M_j^{q/p}} u_j(\lambda_j \varepsilon_j x + x_j), \quad \tilde{v}_j(x) = \frac{1}{M_j^{p/q}} v_j(\lambda_j \varepsilon_j x + x_j) \quad \text{in } \bar{\Omega}_j = \frac{\Omega - x_j}{\lambda_j \varepsilon_j}$$

As $\|\tilde{u}_j\|_\infty, \|\tilde{v}_j\|_\infty \leq 1$, so $(\tilde{u}_j, \tilde{v}_j)$ converges in $C^2_{\text{loc}}$ to $(u, v)$, solves the limit problem $-\Delta u = g_\infty(v), -\Delta v = f_\infty(u)$, where $f_\infty(s) = a(x_0)|s|^p$ and

$$g_\infty(s) = b(x_0) \begin{cases} \frac{2|q-p|s^p + b-p}{p} & \text{for } s \geq 1, \\ \frac{2|q-p|s^q - b-p}{q} & \text{for } |s| < 1, \\ \frac{2|q-p|s^q - b-p}{q} & \text{for } s \leq 1, \end{cases}$$

Since $\|u\|_\infty, \|v\|_\infty \leq 1$, one can easily check condition (a), (b) and (c) and Proposition 2.1 tells that $u = v = 0$ contradicting the fact that one of $\tilde{u}_j(0), \tilde{v}_j(0)$ has to be one.$\square$

**Lemma 2.1.** $I_\varepsilon$ satisfies the Pales-Smale condition.
Proof. For a Palais-Smale sequence \((u_n, v_n)\) of \(I_\varepsilon\) we have

\[
I'_\varepsilon(u_n, v_n)(u_n, v_n) = 2\langle u_n, v_n \rangle - \int_{\Omega} a(x)f(u_n)u_n dx - \int_{\Omega} b(x)g_j(v_n) v_n dx
\]

Noting that \(g_j(s) \geq (p+1)G_j(s)\), and \(f(s) = (p+1)F(s)\), for all \(s\), we have

\[
2I_\varepsilon(u_n, v_n) - I'_\varepsilon(u_n, v_n)(u_n, v_n) = \int_{\Omega} a(x)(f(u_n)u_n - 2F(u_n)) + \int_{\Omega} b(x)(g_j(v_n) v_n - 2G_j(v_n)) \geq \mu_n \langle u_n \rangle - \int_{\Omega} f(u_n)u_n dx - \int_{\Omega} b(x)g_j(v_n) v_n dx
\]

and hence

\[
\int_{\Omega} [f(u_n)u_n + g_j(v_n) v_n] dx \leq C_1 + C_2 \mu_n \langle u_n \rangle + \langle v_n \rangle.
\]

Now note that \(ap^{p+1} + bp^{p+1} - abp - abp = (a - b)(ap^{p-1} - bp^{p-1}) \geq 0\), and there exists a constant \(C \equiv C(j, A, B)\) such that \(\int_{\Omega} g_j(v_n) u_n dx \leq C \int_{\Omega} v_n^p u_n dx\). Then we can estimate

\[
\|u_n\|^2 + \|v_n\|^2 \leq I'_\varepsilon(u_n, v_n)(u_n, v_n) + C \int_{\Omega} a(x)u_n^p + b(x)v_n^p u_n dx
\]

\[
\leq \mu_n \langle u_n \rangle + \langle v_n \rangle + C_1 \int_{\Omega} \left[ u_n^{p+1} + v_n^{p+1} \right] dx
\]

\[
\leq \mu_n \langle u_n \rangle + \langle v_n \rangle + C_2 \int_{\Omega} \left[ f(u_n)u_n + g_j(v_n) v_n \right] dx
\]

\[
\leq C_3 + C_4 \mu_n \langle u_n \rangle + \langle v_n \rangle
\]

So we have \(\|u_n\| + \|v_n\| \leq C\). Hence every Palais-Smale sequence \((u_n, v_n)\) is bounded. Then we have up to subsequence \(u_n \rightharpoonup u\) and \(v_n \rightharpoonup v\) in \(H\), and clearly \(I'_\varepsilon(u, v) = 0\). Now from (2.4), we have for large \(j\)

\[
2I_\varepsilon(u, v) = \int_{\Omega} a(x)(u^+)^{p+1} dx + \int_{\Omega \setminus \{v \leq j\}} b(x)(u^+)^{q+1} dx + \int_{\Omega \cap \{v \geq j\}} b(x)(v^+)^{q+1} dx \\
\geq 0
\]

Thanks to compact embedding,

\[
\int_{\Omega} F(u_n - u) dx = \int_{\Omega} F(u_n) dx - \int_{\Omega} F(u) dx + o(1),
\]

and the same is true for \(G_j(u_n - u)\). Hence we have for \((\phi, \psi) \in H\)

\[
\int_{\Omega} f(u_n - u) \phi dx - \int_{\Omega} f(u_n) \phi dx + \int_{\Omega} f(u) \phi dx \leq o(1) \|\phi\|,
\]

and the same for \(g_j, \psi\).

Plugging \(\bar{u}_n = u_n - u\) and \(\bar{v}_n = v_n - v\) into (2.4) and from (2.7), (2.8),

\[
L_\varepsilon(\bar{u}_n, \bar{v}_n) = \int_{\Omega} \varepsilon^2 \left[ \langle \bar{u}_n, \bar{v}_n \rangle + c(x)\bar{u}_n \bar{v}_n - a(x)F(\bar{u}) - b(x)G_j(\bar{v}) \right] dx
\]

\[
= I_\varepsilon(u_n, v_n) - I_\varepsilon(u, v) + o(1)
\]

It is easy to see

\[
I'_\varepsilon(\bar{u}_n, \bar{v}_n)(\phi, \psi) = I'_\varepsilon(u_n, v_n)(\phi, \psi) - I'_\varepsilon(u, v)(\phi, \psi) + o(1) = o(1)
\]
Let us fix a sequence \( u \). Then with \( 0 \varepsilon H \) such that \( \|u\|_H^2 \leq \varepsilon \) by \( \|u\|_H^2 = \varepsilon \leq \varepsilon \). As \( \varepsilon \) to Hopf's lemma, \( \varepsilon \) exists. We may assume \( m(u_\varepsilon, v_\varepsilon) \leq 1 \). Thanks to Hopf's lemma, \( (u_\varepsilon, v_\varepsilon) > 0 \). Again, relative Morse index gives that solutions are non-constants.

3. Preliminary estimates

Let \( u_\varepsilon, v_\varepsilon \in H \) be any ground-state solutions for system (2.1). Then \( u_\varepsilon > 0, v_\varepsilon > 0 \). As \( u_\varepsilon, v_\varepsilon \in H \) uniformly bounded, from now on we shall denote \( g_j \) for \( g \) and \( G_j \) by \( G \). Let \( x_\varepsilon \in \Omega \) be such that

\[
\max_{\Omega} u_\varepsilon = u_\varepsilon(x_\varepsilon).
\]

Let us fix a sequence \( \varepsilon_j \) in such a way that \( x_j := x_{\varepsilon_j} \rightarrow x_0 \in \Omega \) and \( z_j \rightarrow z_0 \in \partial \Omega \) where \( z_j \in \partial \Omega \) such that

\[
d_j := dist(x_j, \partial \Omega) = |x_j - z_j|
\]

We denote \( u_j := u_{\varepsilon_j} \) and \( v_j := v_{\varepsilon_j} \). The re-scaled solutions

\[
\bar{u}_j(x) := u_j(\varepsilon_j x + x_j), \quad \bar{v}_j(x) := v_j(\varepsilon_j x + x_j), \quad x \in \Omega_j := \frac{1}{\varepsilon_j}(\Omega - x_j)
\]

solve the system

\[
\begin{aligned}
-\Delta \bar{u}_j + c(\varepsilon_j x + x_j)\bar{u}_j &= b(\varepsilon_j x + x_j)g(\bar{v}_j), \\
-\Delta \bar{v}_j + c(\varepsilon_j x + x_j)\bar{v}_j &= a(\varepsilon_j x + x_j)f(\bar{u}_j) \quad \text{in } \Omega_j \\
u_j > 0, \quad \bar{v}_j > 0 \quad \text{in } \Omega_j, \quad \text{and} \quad \frac{\partial \bar{u}_j}{\partial \nu} = 0 = \frac{\partial \bar{v}_j}{\partial \nu} \quad \text{on } \partial \Omega_j
\end{aligned}
\]
Furthermore, the corresponding energy functional of (3.2) in Cartesian coordinates takes the form

\[ I_j(u_j, v_j) = \int_{\Omega_j} (\nabla u_j, \nabla v_j) + c(x_j + \varepsilon_j x) u_j v_j - a(x_j + \varepsilon_j x) F(u_j) - b(x_j + \varepsilon_j x) G(v_j) dx \]

(3.3)

where \( F(s) = \int_0^s f(t) dt \) and \( G(s) = \int_0^s g(t) dt \). According to proposition 2.2 (see [4]), we may assume \( p = q \). The critical points of the functional (3.3) are now the solutions of the system (3.2). By \( L^p \) and the Schauder estimate, we have the convergence in \( H^1(\mathbb{R}^n) \) and in \( C_c^2(\mathbb{R}^n) \) to a nonzero solution of the limit system

\[
\begin{align*}
-\Delta u + c(x_0) u &= b(x_0) g(v), \quad \text{and} \quad -\Delta v + c(x_0) v = a(x_0) f(u) \quad \text{in } U \\
& \quad u > 0, \ v > 0 \ \text{in } U, \ \text{and} \ \frac{\partial u}{\partial v} = 0 = \frac{\partial v}{\partial v} \ \text{on } \partial U
\end{align*}
\]

(3.4)

where \( U = \{ x \in \mathbb{R}^n : \langle x, n(x_0) \rangle < \rho_0 \} \), where

\[ \rho_0 = \lim_{j \to \infty} \rho_j, \ \rho_j := d_j / \varepsilon_j \]

The corresponding energy functional is

\[ I_{[x_0]}(u, v) = \int_U \left[ (\nabla u, \nabla v) + c(x_0) u v - a(x_0) F(u) - b(x_0) G(v) \right] dx \]

(3.5)

Since \( \rho_j \to 0 \) ((a), lemma 3.1), so without loss of generality we can take \( U = \mathbb{R}^n_+ \).

**Remark 3.1.**

Suppose \( I''_\phi(u, v)(\phi u, \phi u)(\phi v, \phi v) \geq 0 \) and \( I''_\phi(u, v)(\phi v, \phi v)(\phi v, \phi v) \geq 0 \) for all test function in \( U \). Multiplying both sides of \(-\Delta u + c(x_0) u = b(x_0) v^p \) and \(-\Delta v + c(x_0) v = a(x_0) u^p \) with \( \phi^2 u \) and \( \phi^2 v \) respectively we get

\[
\int 2(u^2 + v^2)|\nabla \phi|^2 \geq \int \phi^2 \left[ (u^2 + v^2)(a(x_0) f'(u) + b(x_0) g'(v)) - (a(x_0) f(u) v + b(x_0) g(v) u) \right] \\
\quad \geq \int \phi^2 (a(x_0) f(u) v + b(x_0) g(v) u)
\]

Last step follows from the fact \( p, q > 1 \) and \( u^2 + v^2 \geq 2uv \).

Now we shall discuss some results similar to the scalar case as in Boyen and Park [9].
Proposition 3.1. Let \( u, v \) be any positive radially symmetric solution to the problem (3.4), then

\[
(i). \quad \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial z_n} \frac{\partial v}{\partial z_i} z_n dz = \frac{2}{n+1} \int_{\mathbb{R}^n_+} (\nabla u \cdot \nabla v) z_n dz
\]

\[
(ii). \quad \int_{\mathbb{R}^n_+} [a(x_0)\left( \frac{1}{2} f(u) - F(u) \right) + b(x_0)\left( \frac{1}{2} g(v) - G(v) \right)] z_n
\]

\[
= \int_{\mathbb{R}^n_+} [\langle \nabla u, \nabla v \rangle + c(x_0)uv - a(x_0)F(u) - b(x_0)G(v)] z_n dz + \frac{1}{2} \int_{\partial \mathbb{R}^n_+} uv d\sigma
\]

\[
(iii). \quad \int_{\mathbb{R}^n_+} [\langle \nabla u, \nabla v \rangle + c(x_0)uv - a(x_0)F(u) - b(x_0)G(v)] z_n dz = 2 \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial z_n} \frac{\partial v}{\partial z_n} z_n dz
\]

\[
(iv). \quad \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial z_i} \frac{\partial v}{\partial z_i} z_n = \frac{1}{n+1} \int_{\mathbb{R}^n_+} (\nabla u \cdot \nabla v) z_n dz, \quad i = 1, 2, ..., n - 1.
\]

Proof. We start with the polar coordinate system as in [17]. Since \( u, v \) are radial, using integration in polar coordinates (i), (iv) follows. The second estimation can be seen by integration by-parts formula. Multiplying \( z_n^2 \frac{\partial u}{\partial z_n} \) and \( z_n^2 \frac{\partial v}{\partial z_n} \) with \(-\Delta v + c(x_0)v = a(x_0)u^p \) and \(-\Delta u + c(x_0)u = b(x_0)v^q \) respectively, followed by integration gives the required estimate. \( \square \)

Towards the goal i.e. to characterize the location of spikes, we consider the following change of variable.

\[
U(x) = \left( \frac{b(x_0)}{c(x_0)} \right)^{\alpha_1} \left( \frac{a(x_0)}{c(x_0)} \right)^{\beta_1} u(\frac{x}{\sqrt{c(x_0)}})
\]

\[
V(x) = \left( \frac{b(x_0)}{c(x_0)} \right)^{\alpha_2} \left( \frac{a(x_0)}{c(x_0)} \right)^{\beta_2} v(\frac{x}{\sqrt{c(x_0)}})
\]

where \( \alpha_1 = \frac{1}{(pq-1)} \), \( \alpha_2 = \frac{p}{(pq-1)} \), \( \beta_1 = \frac{q}{(pq-1)} \) and \( \beta_2 = \frac{1}{(pq-1)} \).

Under the above transformations, system (3.4) modified into following system.

\[
\begin{cases}
-\Delta U + U = V^q, & \text{in } \mathbb{R}^n_+ \\
-\Delta V + V = U^p & \text{in } \mathbb{R}^n_+ \\
U > 0, & \text{in } \mathbb{R}^n_+, \quad \frac{\partial U}{\partial v} = 0 = \frac{\partial V}{\partial v} & \text{on } \partial \mathbb{R}^n_+.
\end{cases}
\]

We denote the energy

\[
I_\infty(u, v) = \int_{\mathbb{R}^n_+} [\langle \nabla u, \nabla v \rangle + uv - F(u) - G(v)] dx
\]

and

\[
\Lambda(x) = \left( \frac{b(x)}{c(x)} \right)^{-\alpha_1 - \alpha_2} \left( \frac{a(x)}{c(x)} \right)^{-\beta_1 - \beta_2} (c(x))^{1 - \frac{q}{2}}
\]

Under the above change of variable, One can easily see the energy takes the form

\[
I_{|x_0|}(u, v) = \left( \frac{b(x_0)}{c(x_0)} \right)^{-\alpha_1 - \alpha_2} \left( \frac{a(x_0)}{c(x_0)} \right)^{-\beta_1 - \beta_2} (c(x_0))^{1 - \frac{q}{2}} I_\infty(U, V)
\]

\[
= \Lambda(x_0) I_\infty(U, V)
\]
It is well known (see [11, 19]) that all strong positive solutions of (3.8) are radially symmetric, and there exists a ground state radially symmetric solution $U, V$ of (3.8) such that $U(x) = U(|x|)$ and $V(x) = V(|x|)$, satisfying the decay estimates

$$|D^a U(x)|, |D^a V(x)| \leq C \exp(-\delta|x|),$$

for some $c, \delta > 0$ and for all $|a| \leq 2$.

**Lemma 3.1.** Let $u_j, v_j$ be the solutions of (1.1). Then

(a) There exists $C, \varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, $u_j$ has a maximum point $x_j$ satisfying

$$\text{dist}(x_j, \partial \Omega) \leq C \varepsilon$$

(b) $x_j \in \partial \Omega$ for $j$ sufficiently large.

(c) $x_j$ is also the unique max point of $v_j$ for $j$ sufficiently large.

**Proof.** Using the method of contradiction, result has been proved in [15]. Here we will only sketch the difference arising due to positive weight coefficients. First we will prove (b), (c) under assumption of (a) i.e. $x_j \to x_0 \in \partial \Omega$ and argument based on Theorem 2.1 of [15]. Using reflection through hyperplane and unique maximum point of limit problem with change of variable (3.6) (3.7), one can see

$$\frac{\text{dist}(x_j, \partial \Omega)}{\varepsilon_j} \to 0 \quad \text{as} \quad j \to \infty$$

From Blow-up argument (3.1), $((\bar{u}_j, \bar{v}_j)) \to (u, v)$ in $C^2_{loc}$, where $(u, v)$ solves the limit problem (3.1) and we have seen $(u, v)$ has exponential decay at infinity. Already we observed that (see Remark (3.1)) for a test function, either $I''_{\varepsilon}(u, v)(\phi u, \phi v) < 0$ or $I''_{\varepsilon}(u, v)(\phi v, \phi v) < 0$ must hold. So we may assume that, there exist a test function $\phi_1$ with support in $B(0, R_0)$ and $I''_{\varepsilon}(u, v)(\phi_1 u, \phi_1 v) < 0$. Again, $C^2_{loc}$ convergence allows to see that $I''_{\varepsilon}(\bar{u}_j, \bar{v}_j)(\phi_1 u, \phi_1 v) < 0$ for large enough $j$. Since $m(u_j, v_j) \leq 1, I''_{\varepsilon}(\bar{u}_j, \bar{v}_j)(h_j, h_j) \geq 0$ for all $h_j \in H^1(\Omega_j)$ with $h_j = 0$ in $B(0, R_0)$.

Since limit solution decays exponentially, given $\delta > 0$, $\exists R > 0$ such that

$$\int_{U \cap (B(0, 2R) \setminus B(0, R))} a(x_0)f(u)v + b(x_0)g(v)u < \delta$$

so, from the fact $C^2_{loc}$ convergence, for a large $j$

$$\int_{\Omega_j \cap (B(0, 2R) \setminus B(0, R))} a(\varepsilon x + x_j)f(\bar{u}_j)\bar{v}_j + b(\varepsilon x + x_j)g(\bar{v}_j)\bar{u}_j < \delta$$

Fix a smooth cut-off function $\psi$ such that $\psi = 0$ in $B(0, R)$ and $\psi = 1$ in $\overline{CB(0, 2R)}$. So $I''_{\varepsilon}(\bar{u}_j, \bar{v}_j)(h_j, h_j)(h_j, h_j) \geq 0$ for $h_j = \psi \bar{u}_j$ as well as $\psi \bar{v}_j$. Now remark 3.1 reads

$$\int_{\Omega_j \cap C B(0, 2R)} (a(\varepsilon x + x_j)f(\bar{u}_j)\bar{v}_j + b(\varepsilon x + x_j)g(\bar{v}_j)\bar{u}_j) < \delta$$
Hence together (3.14) and (3.15) proves that for any $\delta > 0$ there exist $R > 0$ (large enough) such that

$$\int_{\Omega_j \cap \mathbb{C}B(0,R)} (a(\varepsilon_j x + x_j)f(\tilde{u}_j)\tilde{v}_j + b(\varepsilon_j x + x_j)g(\tilde{v}_j)\tilde{u}_j) < \delta \quad (3.16)$$

Let $y_j$ be any sequence of maximum points of $v_j$ in $\tilde{\Omega}$. First we claim that there exist a constant $C > 0$ such that for $j$ very large,

$$\operatorname{dist}(x_j, y_j) \leq C \varepsilon_j$$

Define $\tilde{u}_j(x) := u_j(\varepsilon_j x + y_j)$, $\tilde{v}_j(x) := v_j(\varepsilon_j x + y_j)$, $x \in \tilde{\Omega}_j := \frac{1}{\varepsilon_j}(\Omega - y_j)$. Since $\tilde{u}_j \to \tilde{u}$, $\tilde{v}_j \to \tilde{v}$ as in blow-up scheme defined for $\tilde{u}_j, \tilde{v}_j$ and $\tilde{v}(0) \neq 0$, inequality (3.16) reads as

$$\int_{\tilde{\Omega}_j \cap \mathbb{C}B(0,R)} (a(\varepsilon_j x + y_j)f(\tilde{u}_j)\tilde{v}_j + b(\varepsilon_j x + y_j)g(\tilde{v}_j)\tilde{u}_j) < \delta \quad (3.17)$$

Denote $\varepsilon_j z_j := y_j - x_j$, if $|z_j| \to \infty$ then both inequalities (3.16) and (3.17) with (1.3) concludes

$$\int_{\Omega_j} (f(\tilde{u}_j)\tilde{v}_j + g(\tilde{v}_j)\tilde{u}_j) < 2\delta \quad (3.18)$$

Since $\delta$ is arbitrary, it contradicts the fact $\tilde{v}(0) \neq 0$. This proves the claim. Remaining estimations follow directly by lifting (see (3.6) and (3.7)) ODE condition and non degenerate maximum points from $U, V$, radial solutions to the problem (3.8) to limit solution $u, v$.

Note that same proof (3.13)-(3.16) can be carried to achieve

$$\int_{\Omega_j \cap \mathbb{C}B(0,R)} (a(\varepsilon_j x + y_j)\tilde{u}_j^2 + b(\varepsilon_j x + y_j)\tilde{v}_j^2) < \delta \quad (3.19)$$

Similarly, again by contradiction argument we can observed that for any $\delta > 0$ there exists $R > 0, j_0 \in \mathbb{N}$ such that

$$|\tilde{u}_j| + |\tilde{v}_j| \leq \delta, x \in \Omega_j \cap \mathbb{C}B(0, R) \text{ for all } j > j_0$$

To see the proof, let there exist a $\rho > 0$ and $z_n \in \Omega_j$ such that $|z_n| \to \infty$ and $|\tilde{u}_j(z_j)| > \rho$ for all $j$. Define $\tilde{\Omega}_j := \Omega_j - z_j$ and $\tilde{x}_j = \varepsilon_j z_j + x_j \in \Omega$.

Now set $\tilde{u}_j(x) := \tilde{u}_j(x + z_j) = u_j(\varepsilon_j x + \tilde{x}_j)$, $\tilde{v}_j(x) := \tilde{v}_j(x + \tilde{x}_j) = v_j(\varepsilon_j x + \tilde{x}_j)$.

From the way of choosing the sequence $z_j$, we have $\tilde{u}_j(0) = u_j(\tilde{x}_j) = \tilde{u}_j(z_j) > \rho$ for all $j$.

Now (3.16) reads as

$$\int_{\tilde{\Omega}_j \cap \mathbb{C}B(0,R)} (a(\varepsilon_j x + \tilde{x}_j)f(\tilde{u}_j)\tilde{v}_j + b(\varepsilon_j x + \tilde{x}_j)g(\tilde{v}_j)\tilde{u}_j) < \delta \quad (3.21)$$

$$\Rightarrow \int_{\Omega_j \cap \mathbb{C}B(z_j,R)} (a(\varepsilon_j x + x_j)f(\tilde{u}_j)\tilde{v}_j + b(\varepsilon_j x + x_j)g(\tilde{v}_j)\tilde{u}_j) < \delta$$
Since $z_j \to \infty$, we have

$$\int_{\Omega_j} (f(\tilde{u}_j)\tilde{v}_j + g(\tilde{v}_j)\tilde{u}_j) < 2\delta$$

Which imply $C^1_{loc}$ limit of $\tilde{u}_j, \tilde{v}_j$ i.e. $u = v = 0$. Which is a contradiction to the existence of the sequence $z_j$. This proves (3.20).

Now, as $p, q > 1$ from (3.19) and (3.20), for all $j > j_0$ we have

$$(3.22) \int_{\Omega_j \cap B(0,R)} (a(\varepsilon_j x + x_j)f(\tilde{u}_j)\tilde{u}_j + b(\varepsilon_j x + x_j)g(\tilde{v}_j)\tilde{v}_j) < \delta$$

The estimation (a) proved on basis of contradiction argument. Suppose there is no such $C$ such that $\text{dist}(x_j, \partial \Omega) \leq C\varepsilon$ holds. let $\tilde{u}_j(x) := u_j(\varepsilon_j x + x_j)$, $\tilde{v}_j(x) := v_j(\varepsilon_j x + x_j), x \in \tilde{\Omega}_j := \varepsilon_j^{-1}(\Omega - x_j)$ be the blow-up scheme with limit problem in $\mathbb{R}^n$, $u, v$ be the limit solution and both are readily symmetric with respect to origin. The decay property of $\tilde{u}_j, \tilde{v}_j$ i.e. (3.22) leads to the fact $I_j(\tilde{u}_j, \tilde{v}_j) + o(1) = I_{|x_0|}(u, v)$. Note that the energy $I_{|x_0|}(u, v)$ is over $\mathbb{R}^n$. The radial symmetry of limit solution $(u, v)$ gives,

$$I_j(\tilde{u}_j, \tilde{v}_j) + o(1) = I_{|x_0|}(u, v) = 2\int_{\mathbb{R}^n} \left[ (\nabla u, \nabla v) + c(x_0)uv - a(x_0)F(u) - b(x_0)G(v) \right] dx$$

Now we will follow Theorem 3.1 and Theorem 3.5 in [15] with Appendix A to get a contradiction. The key idea of these theorems is about finding maximum points along an energy curve $\alpha(t) = I(t(u,v) + (1 - t)(\phi-\phi), \phi \in H^1$. The maximum point $t = 1$ exists and the existence is proved by finding another curve $\beta(t)$ which intersect $\alpha(t)$ at critical points. □

4. UPPER ENERGY ESTIMATION

**Theorem 4.1.** Let $(u_j, v_j)$ be a minimal energy solution of system (4.1) with the corresponding functional (3.5). Then we have

$$(4.1) c\varepsilon_j \leq \varepsilon_j^n \left( \Lambda(x_0) I_{\infty}(u,v) - \varepsilon_j[(n-1)H(x_0)\gamma + \eta] + o(\varepsilon_j) \right)$$

for $j$ sufficiently large.

Let $x_j \in \partial \Omega$ is the point of maximum of $(u_j, v_j)$ the solution of (4.1) with limit $x_0 \in \partial \Omega$. Without loss of generality we can assume $x_0 = 0$ and the inner normal at $x_0$ to $\partial \Omega$ directed towards positive $x_n$-axis.

Then there exists neighborhood $B_{\delta_1}(0)$ of 0 in $\mathbb{R}^n$, $\tilde{B}_{\delta_2}(0)$ of 0 in $\mathbb{R}^{n-1}$ and a map $\psi : \tilde{B}_{\delta_2}(0) \to \mathbb{R}$ such that $\partial \Omega \cap B_{\delta_1}(0)$ is the graph of $\psi$. Now define a map $\Phi : \tilde{B}_{\delta_2}(0) \times \mathbb{R} \to \mathbb{R}^n$ by

$$\Phi_j(y) = \begin{cases} y_j - y_n \frac{\partial \psi}{\partial x_j}(y') & \text{for } j = 1, 2, \ldots, n-1 \\ y_n + \psi(y') & \text{for } j = n \end{cases}$$

Since $\nabla \psi(0) = 0$, we note that $D\Phi(0) = Id$ and hence $\Phi$ is locally invertible at 0 i.e. there exists balls $B_\delta(0)$, $B_{\delta'}(0)$ and a map $\Psi : B_{\delta'}(0) \to B_\delta(0)$ such that $\Psi = \Phi^{-1}$ and $D\Psi(0) = D\Phi(0)^{-1}$.

Now let us take $\Omega_j := \frac{\Omega - x_0}{\varepsilon_j}$ and for $x \in \Omega_j$ define $y = \varepsilon_j x + x_0 \in \Omega$ and $w = \Psi(y) = \Psi(\varepsilon_j x + x_0) \in \mathbb{R}^n$ for $y \in \text{domain of } \Psi$ and $z = \frac{w}{\varepsilon_j} = \frac{\Psi(y)}{\varepsilon_j} = \frac{\Psi(\varepsilon_j x + x_0)}{\varepsilon_j}$. Then note that $y = \Phi(\varepsilon_j z)$

let $\chi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is radially symmetric, $\chi = 1$ in $B(0, \delta)$ and $\chi = 0$ in $\mathbb{R}^n \setminus B(0, 2\delta)$. For a small $\delta > 0$, such that $B = B_+ (0, 2\delta)$ subset of domain of $\Phi$. Define for $x \in \Omega_j \cap B_+ (0, 2\delta)$

$$u_j(x) = u \left( \frac{\Psi(\varepsilon_j x + x_0)}{\varepsilon_j} \right) \chi(\varepsilon_j x + x_0)$$

$$v_j(x) = v \left( \frac{\Psi(\varepsilon_j x + x_0)}{\varepsilon_j} \right) \chi(\varepsilon_j x + x_0)$$

Where $(u, v)$ solves the limit problem \[3.3\] over $\mathbb{R}^n_+$. It has been shown in appendix (A) that $u_j, v_j \in H^1(\Omega_j)$ and they solves following PDE over $\Omega_j$

\[
\begin{cases}
  - \Delta u_j + c(x_0)u_j = b(x_0)v_j^q + \mu_j(x), \\
  - \Delta v_j + c(x_0)v_j = a(x_0)u_j^p + \nu_j(x) & \text{in } \Omega_j \\
  u > 0, v > 0 \text{ in } \Omega_j, \text{ and } \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \text{ on } \partial \Omega_j
\end{cases}
\]

Where $\mu_j$ and $\nu_j$ are given in \[7.3\] and \[7.4\]

Let $H(p)$ denotes the mean curvature of $\partial \Omega$ at the point $p$, then we have $\Delta \psi(0) = (n - 1)H(0)$ and

$$(4.4) \quad |D(\Phi(y))| = 1 - (n - 1)H(0)y_n + O(|y|^2).$$

We denote the energy $I_j(u_j, v_j)$

$$= \int_{\Omega_j} \left[ \nabla u_j, \nabla v_j \right] + c(x_0 + \varepsilon_j x)u_j v_j - a(x_0 + \varepsilon_j x)F(u_j) - b(x_0 + \varepsilon_j x)G(v_j) \right] \, dx$$

$$= \int_{\Omega_j} \left[ \frac{1}{2} b(x_0)g(v_j)v_j - G(v_j) b(x_0 + \varepsilon_j x) \right] + \frac{1}{2} \left[ \mu_j v_j + \nu_j u_j \right]$$

$$+ \left[ \frac{1}{2} a(x_0) f(u_j) u_j - F(u_j) a(x_0 + \varepsilon_j x) \right] + \left[ u_j v_j c(\varepsilon_j x + x_0) - c(x_0) u_j v_j \right]$$

Let $(u, v)$ and $(U, V)$ are solutions to the problems \[3.3\] and \[3.8\] respectively. We fix some notations. In proposition \[4.4\] we will establish a more compact form of the boundary
functions.
\[ \gamma = \gamma(f) + \gamma(g) + \xi - \tau, \quad \eta = \eta(f) + \eta(g) - \Theta \]
\[ \gamma(f) := a(x_0) \int_{\mathbb{R}^n_+} \left( \frac{1}{2} f(u)u - F(u) \right) x_n dx \]
\[ \gamma(g) := b(x_0) \int_{\mathbb{R}^n_+} \left( \frac{1}{2} g(v)v - G(v) \right) x_n dx \]
\[ \Theta := \int_{\mathbb{R}^n_+} udv, x > dx \]
\[ \tau := \frac{1}{2} \int_{\partial \mathbb{R}^n_+} uv d\sigma \]

**Proposition 4.1.**

(i) \[ \gamma = \frac{5}{n+1} \frac{\Lambda(x_0)}{\sqrt{c(x_0)}} \int_{\mathbb{R}^n_+} \langle \nabla U, \nabla V \rangle z_n dz \]

(ii) \[ \eta = \frac{\Lambda(x_0)}{\sqrt{c(x_0)}} \left[ \frac{\partial_n a(x_0)}{a(x_0)} \int_{\mathbb{R}^n_+} F(U)x_n + \frac{\partial_n b(x_0)}{b(x_0)} \int_{\mathbb{R}^n_+} G(V)x_n - \frac{\partial_n c(x_0)}{c(x_0)} \int_{\mathbb{R}^n_+} UVx_n \right] \]

**Proof.** The proof follows easily from Proposition 3.1 along with change of variable (3.6)-(3.7).

\[ \gamma = \gamma(f) + \gamma(g) + \xi - \tau \]
\[ = \int_{\mathbb{R}^n_+} \left[ (\nabla u, \nabla v) + c(x_0)uv - a(x_0)F(u) - b(x_0)G(v) \right] z_n + \frac{1}{2} \int_{\partial \mathbb{R}^n_+} uv d\sigma + \xi - \tau \]
\[ = 2 \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial z_n} \frac{\partial v}{\partial z_n} z_n dz + \frac{1}{2} \int_{\partial \mathbb{R}^n_+} uv d\sigma + \xi - \tau \]
\[ = \frac{4}{n+1} \int_{\mathbb{R}^n_+} \langle \nabla u, \nabla v \rangle z_n + \frac{1}{2} \int_{\partial \mathbb{R}^n_+} uv d\sigma + \frac{1}{n+1} \int_{\mathbb{R}^n_+} \langle \nabla u, \nabla v \rangle z_n dz + \frac{1}{2} \int_{\partial \mathbb{R}^n_+} uv d\sigma \]
\[ = \frac{5}{n+1} \int_{\mathbb{R}^n_+} \langle \nabla u, \nabla v \rangle z_n dz \]
\[ = \frac{5}{n+1} \frac{\Lambda(x_0)}{\sqrt{c(x_0)}} \int_{\mathbb{R}^n_+} \langle \nabla U, \nabla V \rangle z_n dz \]

Note that, radial symmetry of \( u, v \) implies that both \( u(x', x_n), v(x', x_n) \) are even in \( x' \). Thus, under symmetry the integral can be reduced to a simpler form.

\[ \eta = \eta(f) + \eta(g) - \Theta \]
\[ = \int_{\mathbb{R}^n_+} F(u)(a'(x_0), x) + G(v)(b'(x_0), x) - uv(c'(x_0), x) \]
\[ = \partial_n a(x_0) \int_{\mathbb{R}^n_+} F(U)x_n + \partial_n b(x_0) \int_{\mathbb{R}^n_+} G(V)x_n - \partial_n c(x_0) \int_{\mathbb{R}^n_+} UVx_n \]
\[ = \frac{\Lambda(x_0)}{\sqrt{c(x_0)}} \left[ \frac{\partial_n a(x_0)}{a(x_0)} \int_{\mathbb{R}^n_+} F(U)x_n + \frac{\partial_n b(x_0)}{b(x_0)} \int_{\mathbb{R}^n_+} G(V)x_n - \frac{\partial_n c(x_0)}{c(x_0)} \int_{\mathbb{R}^n_+} UVx_n \right] \]
In lemma 4.1 we prove the asymptotic expansion of the energy $I_j(u_j, v_j)$ and the proof follows from Lemma 4.2 and Lemma 4.3.

**Lemma 4.1.**

$$I_j(u_j, v_j) = I_{|x_0|}(u, v) - \varepsilon_j[(n - 1)H(x_0)\gamma + \eta] + o(\varepsilon_j)$$

**Lemma 4.2.**

$$\int_{\Omega_j} \frac{1}{2} a(x_0) f(u_j) u_j - F(u_j) a(x_0 + \varepsilon_j x) dx$$

$$= \int_{\mathbb{R}^n} a(x_0) \left( \frac{1}{2} f(u) u - F(u) \right) dx - \varepsilon_j \left( (n - 1)H(x_0)\gamma(f) + \eta(f) \right) + o(\varepsilon_j).$$

**Proof.** For $\varepsilon$ very small, we have

$$\int_{\Omega_j} \frac{1}{2} a(x_0) f(u_j) u_j - F(u_j) a(x_0 + \varepsilon_j x) dx$$

$$= \int_{\Omega_j} \left( \frac{1}{2} f(u_j) u_j - F(u_j) \right) a(x_0) - F(u_j) \varepsilon_j < a'(x_0), x > + o(\varepsilon_j) \right) dx.$$

Consider the first part of above integral. Note that for the change of variable $z = \frac{\Psi(\varepsilon_j x + x_0)}{\varepsilon_j}$ equivalent to $x = \frac{\phi(\varepsilon_j z) - x_0}{\varepsilon_j}$ one has

$$dx = [1 - \varepsilon_j(n - 1)H(x_0)\varepsilon_n + O(\|\varepsilon_j z\|^2)]dz$$

set $K(x_0) := (n - 1)H(x_0)$ and we have

$$\int_{\Omega_j} \left[ \frac{1}{2} f(u_j(x)) u_j(x) - F(u_j(x)) \right] dx$$

$$= \frac{(p - 1)}{2(p + 1)} \int_{B_+(0, \delta_j/\varepsilon_j)} \left( u(z) \chi(\varepsilon_j z) \right)^{p+1} (1 - \varepsilon_j(n - 1)H(x_0)\varepsilon_n + O(\|\varepsilon_j z\|^2)) \right) dz$$

As $(u, v)$ has exponential decay. By straightforward calculation, we have

$$\int_{\Omega_j} \left[ \frac{1}{2} f(u_j(x)) u_j(x) - F(u_j(x)) \right] dx$$

$$= \frac{(p - 1)}{2(p + 1)} \int_{\mathbb{R}^n} u^{p+1}(z)[1 - \varepsilon_j(n - 1)H(x_0)\varepsilon_n] dz + o(\varepsilon_j)$$

(4.5)
Similarly, for the second term
\[
\int_{\Omega_j} u_j^{p+1}(x) < a'(x_0), x > dx \\
= \int_{B_+^{0}(0,2\delta/\varepsilon_j)} (u(z)\chi(z))^p+1(1 - \varepsilon_j K(x_0)z_n) < a'(x_0), \frac{\Phi(x_0z) - x_0}{\varepsilon_j} > dz \\
= \int_{B_+^{0}(0,2\delta/\varepsilon_j)} (u(z))^p+1(1 - K(x_0)z_n) < a'(x_0), \varepsilon_j z > dz + o(\varepsilon_j) \\
= \int_{B_+^{0}(0,2\delta/\varepsilon_j)} (u(z))^p+1(1 - \varepsilon_j K(x_0)z_n) < a'(x_0), \varepsilon_j x > dz + o(\varepsilon_j)
\]
(4.6)
\[= \varepsilon_j \int_{\mathbb{R}^n_+} (u(z))^p+1 < a'(x_0), x > dz + o(\varepsilon_j) \]
Hence lemma follows from 4.5 and 4.6.

Lemma 4.3.

1. \[\int_{\Omega_j} \left[ \frac{1}{2} b(x_0)g(v_j)v_j - G(v_j)b(x_0 + \varepsilon_j x) \right] dx \]
   \[= \int_{\mathbb{R}^n_+} b(x_0) \left( \frac{1}{2} g(v) - G(v) \right) dx - \varepsilon_j \left( (n - 1)H(x_0)\gamma + \eta(g) \right) + o(\varepsilon_j) \]
2. \[\int_{\Omega_j} [u_jv_j - c(x_0)u_j v_j] dx = \varepsilon_j \Theta + o(\varepsilon_j) \]
3. \[\frac{1}{2} \int_{\Omega_j} [\mu_jv_j + v_j u_j] dx = -\varepsilon_j(n - 1)H(x_0)(\xi - \tau) + o(\varepsilon_j) \]

Where the constants \( \xi = \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_k}(z) \frac{\partial u}{\partial x_l}(z) z_n dz \) for \( 1 \leq k \leq n - 1 \) and \( \tau = \frac{1}{2} \int_{\partial \mathbb{R}^n_+} u\omega d\sigma \)

Proof of statement (3) follows from (7.3) and (7.4).

Lemma 4.4.

\[\sup_{E^- \oplus \mathbb{R}^n_+} I_j = I_j(u_j, v_j) + o(\varepsilon_j)\]

Proof. We will show that the same argument [4], lemma 3.3 can be adopted for our purpose with proper care of positive coefficients. The assumption (4.3) will take care of all the issues due to coefficients in energy estimations. The proof based on the exponential decay of limit solution and integration by parts i.e. (4.3), with decay (7.3), (7.4). The whole idea is to study asymptotic behaviour of maximum points \((s_j, t_j)\) of the energy \( \chi_j(s, t) := I_j(s(u_j, v_j) + t(\phi_j, -\phi_j)) \).

Moreover we can prove \( \chi_j(s_j, t_j) = \chi_j(1, 0) + o(\varepsilon_j) \)

Proof of Theorem 4.1. From the definition of least energy solution(see the way of construction), we have

\[I_j(\tilde{u}_j, \tilde{v}_j) \leq \sup_{E^- \oplus \mathbb{R}^n_+} I_j \]
Where
\[ \tilde{u}_j(x) := u_{\epsilon_j}(\epsilon_j x + x_0), \tilde{v}_j(x) := v_{\epsilon_j}(\epsilon_j x + x_0), x \in \Omega_j := \frac{1}{\epsilon_j} (\Omega - x_0) \]
with least energy solution \((u_{\epsilon_j}, v_{\epsilon_j})\) to the problem (1.1). It is easy to see,
\[ c_\epsilon = \epsilon_j I_j(\tilde{u}_j, \tilde{v}_j) \]
Hence the theorem follows. \(\square\)

5. LOWER ENERGY ESTIMATION

**Theorem 5.1.** Let \((u_\epsilon, v_\epsilon)\) be a minimal energy solution of system (1.1) with the corresponding functional (1.5). Then we have
\[ c_\epsilon \geq \epsilon_j \left( A(x_j) I_\infty(u, v) - \epsilon_j [(n - 1) H(x_j) \gamma + \eta] + o(\epsilon_j) \right) \]
for \(\epsilon\) sufficiently small.

As in the previous lemma, we can take a particular coordinate system such that \(x_j = 0\) and the inner normal at \(x_j\) to \(\partial \Omega\) directed towards the positive \(x_n\) axis. And we define \(\psi_j\), \(\Phi_j\) and \(\Psi_j\) in a similar way such that
I. \(\nabla \psi_j(0) = 0\)
II. \(D\Phi_j(0) = Id\) and
III. \((\Psi_j)^{-1} = \Phi_j\)
Define,
\[ \bar{u}_j(z) = \tilde{u}_j \left( \Phi_j(\epsilon_j z) - x_j \right) \chi(\Phi_j(\epsilon_j z) - x_j) \]
\[ \bar{v}_j(x) = \tilde{v}_j \left( \Phi_j(\epsilon_j z) - x_j \right) \chi(\Phi_j(\epsilon_j z) - x_j) , z \in \mathbb{R}_+^n. \]

Where \((\bar{u}, \bar{v})\) solves the problem (5.2) Similarly as in Proposition 5.1 of [5] we get positive constants \(c, \theta\) such that
\[ (5.1) \quad \bar{u}_j(x) \leq ce^{-\theta|x|} \text{ and } \bar{v}_j(x) \leq ce^{-\theta|x|} \]
Now from Lemma 8.3 we see that \(\bar{u}_j, \bar{v}_j \in H^1(\mathbb{R}_+^n)\) and they solve following PDE over \(\mathbb{R}_+^n\) with neumann boundary condition.
\[ \left\{ \begin{array}{ll}
-\Delta \bar{u}_j(z) + c(\Phi_j(\epsilon_j z))\bar{u}_j(z) = b(\Phi_j(\epsilon_j z))g(\bar{v}_j) + \bar{\mu}_j(z) \\
-\Delta \bar{v}_j(z) + c(\Phi_j(\epsilon_j z))\bar{v}_j(z) = a(\Phi_j(\epsilon_j z))f(\bar{u}_j) + \bar{\nu}_j(z)
\end{array} \right. \]
where \(x = \frac{\Phi_j(\epsilon_j z) - x_j}{\epsilon_j}\) and \(\bar{\mu}_j(z) , \bar{\nu}_j(z)\) is given in Lemma 8.3.
Note that
ON A SINGULARLY PERTURBED ANISOTROPIC ELLIPTIC SYSTEM

\[
I_{[x_j]}(\bar{u}_j, \bar{v}_j) = \int_{\mathbb{R}^n} \left[ (\nabla \bar{u}_j, \nabla \bar{v}_j) + c(x_j) \bar{u}_j \bar{v}_j - a(x_j) F(\bar{u}_j) - b(x_j) G(\bar{v}_j) \right] dx
\]

\[
= \int_{\mathbb{R}^n} b(\Psi(x_0)) \left( \frac{1}{2} g(\bar{v}_j) \bar{v}_j - G(\bar{v}_j) \right) + a(\Phi(x_0)) \left( \frac{1}{2} f(\bar{u}_j) \bar{u}_j - F(\bar{u}_j) \right) + \frac{1}{2} \left( \mu_j \bar{v}_j + \nu_j \bar{u}_j \right)
\]

\[
\quad + \frac{1}{2} \left( c(x_j) - c(\Psi(x_0)) \right) \bar{u}_j \bar{v}_j - \left( a(x_j) - a(\Phi(x_0)) \right) F(\bar{u}_j) - \left( b(x_j) - b(\Phi(x_0)) \right) G(\bar{v}_j) + o(\varepsilon)
\]

Lemma 5.1.

\[
\int_{\mathbb{R}^n} \left[ \frac{1}{2} g(\bar{v}_j) \bar{v}_j b(\Phi(x_j)) - b(\Phi(x_j)) G(\bar{v}_j) \right] dz
\]

\[
= \int_{\Omega_j} \left( \frac{1}{2} g(\bar{v}_j) \bar{v}_j - G(\bar{v}_j) \right) b((x_j + x_j) dz
\]

\[
+ \varepsilon_j (n-1) H(x_j) b(x_j) \int_{\mathbb{R}^n} \left( \frac{1}{2} g(v) v - G(v) \right) z_n dz + o(\varepsilon)
\]

Proof.

\[
\int_{\mathbb{R}^n} \left[ \frac{1}{2} b(\Phi(x_j)) g(\bar{v}_j) \bar{v}_j - b(\Phi(x_j)) G(\bar{v}_j) \right] dz
\]

\[
= \left[ \frac{1}{2} - \frac{1}{q+1} \right] \int_{\mathbb{R}^n} b(\Phi(x_j)) \bar{v}_j^{q+1} dz
\]

\[
= \left[ \frac{1}{2} - \frac{1}{q+1} \right] \int_{\mathbb{R}^n} b(\Phi(x_j)) \bar{v}_j^{q+1} \left( \frac{\Phi(x_j) - x_j}{\varepsilon_j} \right) \chi^{q+1} (\Phi(x_j) - x_j) dx
\]

Under the change of variable \( \Phi(x_j) - x_j = \varepsilon_j x \), \( dz = (1+y_n K(x_j) + o(|y|) dx \), \( K(x_j) := (n-1) H(x_j) \) we get
= \int_{\Omega_j \cap B(0, \frac{2\Delta}{\varepsilon_j})} b(\varepsilon_j x + x_j) \tilde{v}_j^{q+1}(x) \chi^{q+1}(\varepsilon_j x)[1 + \varepsilon_j x_n K(x_j) + O(\varepsilon_j x_n^2)]dx

= \int_{\Omega_j \cap B(0, \frac{2\Delta}{\varepsilon_j})} b(\varepsilon_j x + x_j) \tilde{v}_j^{q+1}(x) \chi^{q+1}(\varepsilon_j x)[1 + \varepsilon_j x_n K(x_j)]dx

+ \varepsilon^{-n} \int_{\Omega_j \cap B(0, \frac{\Delta}{\varepsilon_j}) \setminus B(0, \frac{\Delta}{\varepsilon_j})} b(\varepsilon_j x + x_j) \tilde{v}_j^{q+1}(x) \chi^{q+1}(\varepsilon_j x)[1 + \varepsilon_j x_n K(x_j)]dx

= \int_{\Omega_j} b(\varepsilon_j x + x_j) \tilde{v}_j^{q+1}(x)\chi^{q+1}(\varepsilon_j x)[1 + \varepsilon_j x_n K(x_j) + O(\varepsilon_j x_n^2)]dx

= \int_{\Omega_j} b(\varepsilon_j x + x_j) \tilde{v}_j^{q+1}(x)\chi^{q+1}(\varepsilon_j x)[1 + \varepsilon_j x_n K(x_j) + o(\varepsilon_j)]dx

= \int_{\Omega_j} b(\varepsilon_j x + x_j) \tilde{v}_j^{q+1}(x)dx + \varepsilon_j(n - 1)H(x_j) \int_{\Omega_j} b(\varepsilon_j x + x_j) \tilde{v}_j^{q+1}(x)x_n + o(\varepsilon_j)

Considering the second term

\int_{\Omega_j} b(\varepsilon_j x + x_j) \tilde{v}_j^{q+1}(x)x_n dx

= \int_{\Omega_j \cap B(0, \frac{\Delta}{\varepsilon_j})} b(\varepsilon_j x + x_j) \tilde{v}_j^{q+1}(x)\chi(x\varepsilon_j)x_n dx + o(\varepsilon_j)

= \int_{\mathbb{R}^n \cap B(0, \frac{\Delta}{\varepsilon_j})} b(\Phi_j^j(\varepsilon_j z)) \tilde{v}_j^{q+1}(z) \left( \frac{\Phi_j^j(\varepsilon_j z) - z_j}{\varepsilon_j} \right) [\Phi_j^j(\varepsilon_j z) - (z_j)_n] |D\Phi_j^j(\varepsilon_j z)| dz

We shall use the following expansions:

|D\Phi_j^j(\varepsilon_j z)| = [1 - K(x_j)z_n\varepsilon_j + o(\varepsilon_j)]

b(\Phi_j^j(\varepsilon_j x)) = b(x_j) + \varepsilon_j < B'(x_j), z > + o(\varepsilon_j)

[\Phi_j^j(\varepsilon_j z) - (x_j)_n] = z_n\varepsilon_j + o(\varepsilon_j).

= \int_{\mathbb{R}^n \cap B(0, \frac{\Delta}{\varepsilon_j})} b(x_j) \tilde{v}_j^{q+1} \left( \frac{\Phi_j^j(\varepsilon_j z) - x_j}{\varepsilon_j} \right) z_n dz + o(\varepsilon_j)

= \int_{\mathbb{R}^n \cap B(0, \frac{\Delta}{\varepsilon_j})} b(x_j) \tilde{v}_j^{q+1} \left( \frac{\Phi_j^j(\varepsilon_j z) - x_j}{\varepsilon_j} \right) \chi(\Phi_j^j(\varepsilon_j z)x_n dx + o(\varepsilon_j)

= \int_{\mathbb{R}^n} b(x_j) v^{q+1}(x)z_n dx + o(\varepsilon_j) = \int_{\mathbb{R}^n} b(x_j) v^{q+1}(x)z_n dx + o(\varepsilon_j)
In a similar way, we have following results.

**Lemma 5.2.**

\[
\int_{\mathbb{R}^n_+} \left[ \frac{1}{2} f(\bar{u}_j) \bar{u}_j a(\Phi^j(\varepsilon_j z)) - a(\Phi^j(\varepsilon_j z)) F(\bar{u}_j) \right] dz \\
= \int_{\Omega_j} \left( \frac{1}{2} - \frac{1}{p+1} \right) u^{p+1}_j a((\varepsilon_j z + x_j)) dz \\
+ \varepsilon_j K(x_j) a(x_j) \int_{\mathbb{R}^n_+} \left( \frac{1}{2} - \frac{1}{p+1} \right) u^{p+1} z_n dz + o(\varepsilon_j)
\]

**Lemma 5.3.**

(i) \( \int_{\mathbb{R}^n_+} \bar{u}_j \bar{v}_j \left[ c(x_j) - c(\Phi^j(\varepsilon_j z)) \right] dz = -\varepsilon_j \int_{\mathbb{R}^n_+} u v < \nabla c(x_j), z > dz + o(\varepsilon_j) \)

(ii) \( \int_{\mathbb{R}^n_+} G(\bar{v}_j) \left[ b(\Phi^j(\varepsilon_j z)) - b(x_j) \right] dz = \varepsilon_j \int_{\mathbb{R}^n_+} G(v) < b'(x_j), z > dz \)

(iii) \( \int_{\mathbb{R}^n_+} F(\bar{u}_j) \left[ a(\Phi^j(\varepsilon_j z)) - a(x_j) \right] dz = \varepsilon_j \int_{\mathbb{R}^n_+} F(u) < a'(x_j), z > dz \)

And from lemma 8.4 and Lemma 8.5 we get:

**Lemma 5.4.**

\[
\int_{\mathbb{R}^n_+} \frac{1}{2} \left[ \bar{u}_j \bar{v}_j + \bar{v}_j \bar{u}_j \right] dx = \varepsilon_j K(x_j)(\xi - \tau) + o(\varepsilon_j)
\]

where \( \xi := \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial z_k}(z) \frac{\partial v}{\partial z_k}(z) dz \) for \( 1 \leq k \leq n - 1 \) and \( \tau := \frac{1}{2} \int_{\partial \mathbb{R}^n_+} uv d\sigma \)

Considering all above estimations, The energy \( I_{|x_j|}(\bar{u}_j, \bar{v}_j) \) reads as

\[
I_{|x_j|}(\bar{u}_j, \bar{v}_j) = I_j(\bar{u}_j, \bar{v}_j) + \varepsilon_j [K(x_j)(\gamma(f) + \gamma(g) + \xi - \tau) + (\eta(f) + \eta(g) - \Theta)] + o(\varepsilon_j)
\]

\[
= I_j(\bar{u}_j, \bar{v}_j) + \varepsilon_j [K(x_j)(\gamma + \eta)] + o(\varepsilon_j)
\]

At the end of this section, We recall lemma 4.1 from [4] to conclude the theorem.

**Lemma 5.5.**

\( (5.3) \)

\[
I_{|x_j|}(u, v) \leq I_{|x_j|}(\bar{u}_j, \bar{v}_j) + o(\varepsilon_j).
\]

**Proof.** It can be shown that, for any \( e = (e_1, e_2) \) such that \( e_i \in H^1(\mathbb{R}^n_+) \) and \( e_1 \neq -e_2 \),

\[
I_{|x_j|}(u, v) \leq \sup_{\mathcal{H}_-} \mathcal{H}_+ \sup_{\mathbb{R}^n_+} I_{|x_j|}
\]

where \( \mathcal{H}_- := \{(\phi, -\phi) : \phi \in H^1(\mathbb{R}^n_+)\} \). The proof follows by similar argument as in Lemma 4.4 on the function \( I_{|x_j|} \).

\( \square \)
Proof of Lemma 5.1. From Lemma 5.5,

\[ I_{|x_j|}(u,v) \leq I_j(\tilde{u}_j, \tilde{v}_j) + \varepsilon_j[K(x_j)\gamma + \eta] + o(\varepsilon_j) \]

\[ I_j(\tilde{u}_j, \tilde{v}_j) \geq I_{|x_j|}(u,v) - \varepsilon_j[K(x_j)\gamma + \eta] + o(\varepsilon_j) \]

\[ c_\varepsilon \geq \varepsilon^n[I_{|x_j|}(u,v) - \varepsilon_j[K(x_j)\gamma + \eta] + o(\varepsilon_j)] \]

Hence we have lower estimate.

Proof of Theorem 1.1.

Case (I): Suppose \( \Lambda \) is non constant. Since \( x_j \) converges to \( x_0 \) we get from lemma 4.1 and lemma 5.1 that the point of concentration is at the minimum point of the function \( \Lambda \).

Case (II): If \( \Lambda \) is constant then from the first order approximation of the energy \( c_\varepsilon \) and using the fact that \( x_j \to x_0 \), we get the point of concentration \( x_0 \), which minimizes the function \( H(x)\gamma(x) + \eta(x) \) in \( \partial \Omega \).

\[ \square \]

6. Applications: Concentration on higher dimensional orbits

In this section we shall show some applications of the previous result.

First let us consider the following equation:

\[
\begin{align*}
-\varepsilon^2 \Delta u + u &= |v|^{q-1} v, \text{ and } -\varepsilon^2 \Delta v + v = |u|^{p-1} u \quad \text{in } A \\
u &> 0, v > 0 \text{ in } \Omega, \text{ and } \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \quad \text{on } \partial A
\end{align*}
\]

where \( A = \{x \in \mathbb{R}^n : a < |x| < b\} \), with \( 0 < a < b \). Recall the only spheres which have group structure are \( S^0, S^1, S^3, S^7, S^{15} \) (Hurwitz, 1898). And using the group structure one has the following classical Hopf fibration:

\[
\begin{align*}
S^0 &\hookrightarrow S^1 \rightarrow \mathbb{R}P^1 \\
S^1 &\hookrightarrow S^3 \rightarrow \mathbb{C}P^1 \equiv S^2 \\
S^3 &\hookrightarrow S^7 \rightarrow \mathbb{H}P^1 \equiv S^4 \\
S^7 &\hookrightarrow S^{15} \rightarrow \mathbb{O}P^1 \equiv S^8
\end{align*}
\]

where \( \mathbb{R}P, \mathbb{C}P, \mathbb{H}P, \mathbb{O}P \) are real, complex, quaternionic and octonionic projective spaces respectively. If \( \pi \) is the corresponding Hopf maps in the above then \( \pi \) is Harmonic Morphism, i.e. (J. C. Wood [18])

\[
\begin{align*}
\Delta_{S^3} &\rightarrow \Delta_{S^2} \\
\Delta_{S^7} &\rightarrow \Delta_{S^4} \\
\Delta_{S^{15}} &\rightarrow \Delta_{S^8}
\end{align*}
\]

And one can easily determine the map \( \pi \) reduces the system [6.1] to the following
where $\Omega = (a^2, b^2) \times S^{m-1}$, $m = 3, 5, 9$ (for details see [7]). And we have the theorem:

**Theorem 6.1.**

(i) for $n = 4$ and $\frac{1}{p+1} + \frac{1}{q+1} > 1/3$ the equation (6.1) has a solution which concentrates on a $S^1$ orbit in the inner boundary.

(ii) for $n = 8$ and $\frac{1}{p+1} + \frac{1}{q+1} > 3/5$ the equation (6.1) has a solution which concentrates on a $S^3$ orbit in the inner boundary.

(iii) for $n = 16$ and $\frac{1}{p+1} + \frac{1}{q+1} > 7/9$ the equation (6.1) has a solution which concentrates on a $S^7$ orbit in the inner boundary.

Proof.

We observe that the equation (6.2) is an anisotropic system with coefficients $a(x) = b(x) = c(x) = \frac{|x|^2}{2|x|}$ which gives $\Lambda(x) = (2|x|)^{n/2-1}$ for $n = 3, 5, 9$. As $\Lambda(x)$ attains infimum on the inner boundary of the annulus we have from the main result the point of concentration for the reduced problem is on the inner boundary. And hence the corresponding solution for the original problem concentrates on the corresponding 1, 3, and 7-dimensional spheres lying on the inner boundary of $A$ respectively. □

Now consider the following equation with weights:

$$
\begin{cases}
-\varepsilon^2 \Delta u + \frac{1}{2|x|} u = \frac{1}{2|x|} v^q, & \text{in } A \\
-\varepsilon^2 \Delta v + \frac{1}{2|x|} v = \frac{1}{2|x|} w^p & \text{in } A \\
u > 0, \ v > 0 & \text{in } A,
\end{cases}
$$

where $A = \{x \in \mathbb{R}^4 : 0 < a < |x| < b\}$, $\alpha, \beta$ any real number. The exponents $p, q > 1$ and

$$
\frac{1}{p+1} + \frac{1}{q+1} > 1/3
$$

We look for the solutions $(u_\varepsilon, v_\varepsilon)$ which are invariant under $S^1$ action i.e. in the space:

$$
H^1_+ (A) = \{u \in H^1(A) : u(T_\tau(z)) = u(z), \forall \tau \in [0, 2\pi]\},
$$

where $T_\tau$ is the following fixed point free one parameter group action on $A$

$$
T_\tau(z) = z(r, t, \theta_1 + \tau, \theta_2 + \tau)
$$

for $\tau \in [0, 2\pi)$. And the co-ordinate system we consider here is $A = I \times S^3$.

Using the $S^1$ action we get the following reduced equation (similarly as in [6])

$$
\begin{cases}
-\varepsilon^2 \Delta u + \frac{u}{(2|x|)^{1-\varepsilon^2}} = \frac{v^q}{(2|x|)^{1-\varepsilon^2}}, & \text{in } \Omega \\
-\varepsilon^2 \Delta v + \frac{v}{(2|x|)^{1-\varepsilon^2}} = \frac{w^p}{(2|x|)^{1-\varepsilon^2}} & \text{in } \Omega \\
u > 0, \ v > 0 & \text{in } \Omega,
\end{cases}
$$

where $\Omega = \{x \in \mathbb{R}^3 : a^2 < |x| < b^2\}$. 

Note that the equation [6.7] is the anisotropic problem with $a(x) = (2|x|)^{1-\frac{2}{n}}$, $b(x) = (2|x|)^{1-\frac{2}{n}}$ and $c(x) = (2|x|)^{-1}$. And hence we get $\Lambda(x) = |x|^{\frac{pq-1}{pq-1} - \alpha(q+1) - \beta(p+1)}$.

Again using the Theorem [1.1] we have:

**Theorem 6.2.** Under assumptions [6.4] there is an $\varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_0$ the equation [6.3] has non constant positive solutions $u_\varepsilon$, $v_\varepsilon \in C^1(\bar{A})$. Moreover, both solutions concentrate for $\varepsilon \to 0$ on a common $S^1$-orbit $S(r_\varepsilon)$, where $r_\varepsilon$ denotes the radius of the circular orbit, and satisfies:

1. $r_\varepsilon \to b$ for $2\alpha(p+1) + 2\beta(q+1) > pq - 1$
2. $r_\varepsilon \to a$ for $2\alpha(p+1) + 2\beta(q+1) < pq - 1$
3. $\frac{r_\varepsilon^2}{2} \to |x_0|$ for $2\alpha(p+1) + 2\beta(q+1) = pq - 1$

where $x_0$ maximizes the function $C_1H(x)|x|^\frac{1}{n} + C_2x_n|x|^{-\frac{2}{n}}$ on $\partial\Omega$ for some constant $C_1$ and $C_2$.

**Proof.**

Case (i): Under the assumption $2\alpha(p+1) + 2\beta(q+1) > pq - 1$ the minimum of $\Lambda(x)$ occurs at the outer boundary of $\Omega$, i.e at $r = \frac{a^2}{2}$.

Case (ii): Under the assumption $2\alpha(p+1) + 2\beta(q+1) < pq - 1$ the minimum of $\Lambda(x)$ occurs at the inner boundary of $\Omega$, i.e at $r = \frac{a^2}{2}$.

Case (iii): For the exponents satisfying $\alpha(q+1) + \beta(p+1) = pq - 1$, note that $\Lambda$ becomes constant and we go for the first order approximation to locate the concentration. Then, we get the point of concentration for the reduced problem [6.7] maximizes $C_1H(x)|x|^\frac{1}{n} + C_2x_n|x|^{-\frac{2}{n}}$ in $\partial\Omega$, where

\[
C_1 = \frac{5\sqrt{2}}{n+1} \int_{\mathbb{R}^n_+} <\nabla U, \nabla V > z_n dz
\]

\[
C_2 = \sqrt{2} \left[ \int_{\mathbb{R}^n_+} \left( \frac{\beta}{2} - 1 \right) F(U) + \left( \frac{\beta}{2} - 1 \right) G(V) + UV \right] z_n dz \right].
\]

Accordingly we get the circle of concentration as the orbit of this point. 

**7. Appendix A**

**7.1. Calculation of $\mu_j$ and $\nu_j$**

Let $y = \varepsilon_j x + x_0$ and $w = \Psi(y)$ and $z = \frac{w}{\varepsilon_j}$,

\[
\frac{\partial u_j(x)}{\partial x_i} = \left[ \frac{\partial u}{\partial z_l}(z) \frac{\partial \Psi_i}{\partial y_l}(y) \chi(w) + \varepsilon_j u(z) \left[ \frac{\partial \chi}{\partial w_i}(w) \frac{\partial \Psi_i}{\partial y_l}(y) \right] \right]
\]

And hence

\[
\frac{\partial^2 u_j(x)}{\partial x_i^2} = \left[ \frac{\partial^2 u}{\partial z_l \partial z_k}(z) \frac{\partial \Psi_i}{\partial y_l}(y) \frac{\partial \Psi_i}{\partial y_l}(y) + \varepsilon_j \frac{\partial u}{\partial z_l}(z) \frac{\partial^2 \Psi_i}{\partial y_l}(y) \right] \chi(w)
\]

\[
+ 2\varepsilon_j \left[ \frac{\partial u}{\partial z_l}(z) \frac{\partial \Psi_i}{\partial y_l}(y) \left[ \frac{\partial \chi}{\partial w_k}(w) \frac{\partial \Psi_k}{\partial y_l}(y) \right] \right]
\]

\[
+ \varepsilon_j^2 u(z) \left[ \frac{\partial^2 \chi}{\partial w_l \partial z_k}(w) \frac{\partial \Psi_k}{\partial y_l}(y) f \partial \Psi_i \partial y_l(y) + \frac{\partial \chi}{\partial w_l}(w) \frac{\partial^2 \Psi_i}{\partial y_l^2}(y) \right]
\]
So we get

\[ \Delta u_j = \left[ \frac{\partial^2 u}{\partial y_i \partial y_j} + \frac{\partial^2 \Psi_k}{\partial y_i \partial y_j} \right] (w) + \varepsilon_j \left[ \frac{\partial u}{\partial y_i} + \frac{\partial^2 \Psi_k}{\partial y_i \partial y_j} \right] (w) + 2 \varepsilon_j \left[ \frac{\partial u}{\partial y_i} \frac{\partial^2 \Psi_k}{\partial y_i \partial y_j} \right] \]

(7.1)

\[ = (A_1 + \varepsilon_j A_2) \chi(w) + \varepsilon_j A_3 + \varepsilon_j^2 A_4 \]

**Lemma 7.1.** \( A_1 = \Delta u(z) + 2\varepsilon_j \sum_{k,l=1}^{n-1} \frac{\partial^2 u}{\partial y_i \partial y_j} \frac{\partial^2 \chi}{\partial y_i \partial y_j} (0) z_n + O(|z|^2) \)

**Proof.** From [16, 17] we get:

For \( 1 \leq k, l \leq n - 1 \)

\[ \sum_{i=1}^{n} \frac{\partial \Psi_k}{\partial y_i} \frac{\partial \Psi_l}{\partial y_i} = \sum_{i=1}^{n-1} \left( \delta_{ki} + \frac{\partial^2 \psi}{\partial x_i \partial x_k} (w) w_n \right) \left( \delta_{li} + \frac{\partial^2 \psi}{\partial x_i \partial x_l} (w) w_n \right) + \frac{\partial \psi}{\partial x_k} \frac{\partial \psi}{\partial x_l} (w) + O(|w|^2) \]

\[ = \delta_{kl} + 2 \frac{\partial^2 \psi}{\partial x_k \partial x_l} (w) w_n + O(|w|^2) \]

Similarly for \( 1 \leq j \leq n - 1 \) and \( k = n \), using \( \nabla \psi(0) = 0 \), we get

\[ \sum_{i=1}^{n} \frac{\partial \Psi_n}{\partial y_i} \frac{\partial \Psi_l}{\partial y_i} = O(|w|^2) \]

and for \( k = l = n \) we have

\[ \sum_{i=1}^{n} \left( \frac{\partial \Psi_n}{\partial y_i} \right)^2 = \sum_{i=1}^{n-1} \left( - \frac{\partial \psi}{\partial x_i} (w) \right)^2 + 1 + O(|w|^2) = 1 + O(|w|^2) \]

Hence from [32] we get

\[ A_1 = \Delta u(z) + 2\varepsilon_j \sum_{k,l=1}^{n-1} \frac{\partial^2 u}{\partial y_i \partial y_j} \frac{\partial^2 \psi}{\partial y_i \partial y_j} (0) z_n + O(|z|^2 e^{-\delta|z|}) \]

\[ \square \]

Using the fact \( \Delta \Psi_k(0) = 0 \) and \( \Delta \Psi_n(0) = -\Delta \psi(0) \) we get

**Lemma 7.2.** \( A_2 = -\frac{\partial u}{\partial x_n} (z) \Delta \psi(0) + O(|\varepsilon_j z| e^{-\delta|z|}) \)

**Lemma 7.3.** \( u_j, v_j \) satisfies

\[ \begin{cases} -\Delta u_j + c(x_0) u_j = b(x_0) v_j^q + \mu_j(x), \\ -\Delta v_j + c(x_0) v_j = a(x_0) u_j^q + v_j(x) \quad \text{in } \Omega_j \end{cases} \]

\[ \begin{cases} u > 0, \ v > 0 \ \text{in } \Omega_j, \ \text{and} \ \frac{\partial u}{\partial v} = 0 = \frac{\partial v}{\partial u} \ \text{on } \partial \Omega_j \end{cases} \]

(7.2)
where (with Einstein summation, $1 \leq k, l \leq (n - 1)$)

\begin{equation}
\mu_j(x) = -2\varepsilon_j \frac{\partial^2 u}{\partial z_l \partial z_k}(z) \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0) z_n + \varepsilon_j \frac{\partial u}{\partial z_n}(z) \Delta \psi(0) + \varepsilon_j^2 O(|z|^2 e^{-\delta |z|})
\end{equation}

\begin{equation}
\nu_j(x) = -2\varepsilon_j \frac{\partial^2 v}{\partial z_l \partial z_k}(z) \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0) z_n + \varepsilon_j \frac{\partial v}{\partial z_n}(z) \Delta \psi(0) + \varepsilon_j^2 O(|z|^2 e^{-\delta |z|})
\end{equation}

Proof. Using the fact $u$ and $v$ decays exponentially, $1 - \chi$ vanishes in $B(0, \delta)$ we get the result easily from Lemma 7.1, Lemma 7.1 and Lemma 7.2.

\begin{lemma}
\end{lemma}

\begin{align*}
\xi \Delta \psi(0) + O(\varepsilon_j) &= \sum_{k,l=1}^{n-1} \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0) \int_{\Omega_j} \frac{\partial^2 u}{\partial z_l \partial z_k}(z) z_n v_j(z) dx \\
&= \sum_{k,l=1}^{n-1} \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0) \int_{\Omega_j} \frac{\partial^2 v}{\partial z_l \partial z_k}(z) z_n u_j(z) dx
\end{align*}

where $\xi = \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_k}(z) \frac{\partial u}{\partial x_k}(z) z_n dz$ for $1 \leq k \leq n - 1$

Proof. Using the exponential decay estimate 3.12 we get

\begin{align*}
\int_{\Omega_j} \frac{\partial^2 u}{\partial z_l \partial z_k}(z) z_n v_j(z) dx \\
&= \int_{\Omega_j} \frac{\partial^2 u}{\partial z_l \partial z_k}(\frac{\Psi(\varepsilon_j x + x_0)}{\varepsilon_j}) \frac{\Psi_n(\varepsilon_j x + x_0)}{\varepsilon_j} v(\frac{\Psi(\varepsilon_j x + x_0)}{\varepsilon_j}) \chi(\Psi(\varepsilon_j x + x_0)) dx \\
&= \varepsilon_j^{-n} \int_{\Omega} \frac{\partial^2 u}{\partial z_l \partial z_k}(\frac{\Psi(y)}{\varepsilon_j}) \frac{\Psi_n(y)}{\varepsilon_j} v(\frac{\Psi(y)}{\varepsilon_j}) \chi(\Psi(y)) dy \\
&= \varepsilon_j^{-n} \int_{\mathbb{R}^n_+} \frac{\partial^2 u}{\partial z_l \partial z_k}(\frac{w}{\varepsilon_j}) \frac{w_n v}{\varepsilon_j} \chi(w) |det(D\Phi(w))| dw \\
&= \int_{\mathbb{R}^n_+} \frac{\partial^2 u}{\partial z_l \partial z_k}(z) z_n v(z) \chi(\varepsilon_j z) |det(D\Phi(\varepsilon_j z))| dz \\
&= \int_{\mathbb{R}^n_+} \frac{\partial^2 u}{\partial z_l \partial z_k}(z) z_n v(z) (1 - (n - 1)H(0) \varepsilon_j z_n + O(|\varepsilon_j z|^2)) dz \\
&= \int_{\mathbb{R}^n_+} \frac{\partial^2 u}{\partial z_l \partial z_k}(z) z_n v(z) (1 - (n - 1)H(0) \varepsilon_j z_n + O(|\varepsilon_j z|^2)) dz \\
&- \int_{\mathbb{R}^n_+ \setminus B(0, \delta/\varepsilon_j)} \frac{\partial^2 u}{\partial z_l \partial z_k}(z) z_n v(z) (1 - (n - 1)H(0) \varepsilon_j z_n + O(|\varepsilon_j z|^2)) dz \\
&+ \int_{\mathbb{R}^n_+ \setminus B(0, \delta/\varepsilon_j)} \frac{\partial^2 u}{\partial z_l \partial z_k}(z) z_n v(z) \chi(\varepsilon_j z) (1 - (n - 1)H(0) \varepsilon_j z_n + O(|\varepsilon_j z|^2)) dz \\
&= \int_{\mathbb{R}^n_+} \frac{\partial^2 u}{\partial z_l \partial z_k}(z) z_n v(z) (1 - \varepsilon_j (n - 1)H(0) z_n) dz + o(\varepsilon_j)
\end{align*}

Now for $1 \leq k, l \leq n - 1$ we have
Lemma 7.5. \[ \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial z_l \partial z_k}(z) z_n v(z) dz = \int_{\mathbb{R}^n} \frac{\partial u}{\partial z_l}(z) \frac{\partial v}{\partial z_k}(z) z_n dz \]
hence using the symmetry in the integral we get
\begin{align*}
\sum_{k,l=1}^{n-1} \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0) \int_{\Omega_j} \frac{\partial^2 u}{\partial z_l \partial z_k} z_n v_j(z) dx &= \sum_{k=1}^{n-1} \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0) \int_{\mathbb{R}^n} \frac{\partial u}{\partial z_k}(z) \frac{\partial v}{\partial z_l}(z) z_n dz \\
&= \xi \Delta \psi(0)
\end{align*}

Similar calculation shows the other equality. \(\Box\)

**Lemma 7.5.**
\[ \int_{\mathbb{R}^n} \left[ \frac{\partial u}{\partial z_n} v + u \frac{\partial v}{\partial z_n} \right] dz = \int_{\partial \mathbb{R}^n} uv d\sigma \]
**Proof.** Proof is obvious from integration by parts formula. \(\Box\)

8. **Appendix B**

8.1. **Calculation of \(\bar{\mu}_j\) and \(\bar{\nu}_j\):** let \(w = \varepsilon_j z\) and \(y = \Phi(w)\) and \(x = \frac{y-x_j}{\varepsilon_j}\),
\[ \frac{\partial \bar{u}_j(z)}{\partial z_i} = \left[ \frac{\partial \bar{u}}{\partial x_l}(x) \frac{\partial \Phi_l}{\partial y_i}(w) \right] \chi(\varepsilon_j x) + \varepsilon_j \bar{u}(x) \left[ \frac{\partial \chi}{\partial w_l}(\varepsilon_j x) \frac{\partial \Phi_l}{\partial y_i}(w) \right] \]
Then
\[ \frac{\partial^2 \bar{u}_j(z)}{\partial x_i^2} = \left[ \frac{\partial^2 \bar{u}}{\partial x_l \partial x_l}(x) \frac{\partial \Phi_l}{\partial w_i}(w) + \varepsilon_j \frac{\partial \bar{u}}{\partial x_l}(x) \frac{\partial^2 \Phi_l}{\partial w_i^2}(w) \right] \chi(\varepsilon_j x) + 2\varepsilon_j \left[ \frac{\partial \bar{u}}{\partial x_l}(x) \frac{\partial \Phi_l}{\partial y_i}(w) \right] \left[ \frac{\partial \chi}{\partial w_l}(\varepsilon_j x) \frac{\partial \Phi_l}{\partial y_i}(w) \right] + \varepsilon_j^2 \bar{u}(x) \left[ \frac{\partial^2 \chi}{\partial x_i \partial x_l}(\varepsilon_j x) \frac{\partial \Phi_k}{\partial w_l}(w) + \frac{\partial \chi}{\partial x_l}(\varepsilon_j x) \frac{\partial^2 \Phi_k}{\partial w_l^2}(w) \right] \]
Hence
\begin{align*}
\Delta \bar{u}_j &= \left[ \frac{\partial^2 \bar{u}}{\partial x_l \partial x_l}(x) \frac{\partial \Phi_k}{\partial w_i}(w) \frac{\partial \Phi_l}{\partial w_i}(w) \right] \chi(\varepsilon_j x) + \varepsilon_j^2 \bar{u} \left[ \frac{\partial^2 \chi}{\partial x_l \partial x_l}(x) \frac{\partial \Phi_k}{\partial w_i}(w) + \frac{\partial \chi}{\partial x_l}(\varepsilon_j x) \frac{\partial^2 \Phi_k}{\partial w_i^2}(w) \right] \\
&+ \varepsilon_j \left[ \frac{\partial \bar{u}}{\partial x_l}(x) \frac{\partial^2 \Phi_k}{\partial w_i}(w) \frac{\partial \Phi_l}{\partial w_i}(w) + \frac{\partial \chi}{\partial x_l}(\varepsilon_j x) \frac{\partial \Phi_k}{\partial w_i}(w) \frac{\partial \Phi_l}{\partial w_i}(w) \right] \\
&= (A_1 + \varepsilon_j A_2) \chi(w) + \varepsilon_j A_3 + \varepsilon_j^2 A_4 \quad (\text{say})
\end{align*}
**Lemma 8.1.** \(A_1 = \Delta \bar{u}_j(x) - 2\varepsilon_j \sum_{k,l=1}^{n-1} \frac{\partial^2 \bar{u}_j}{\partial x_l \partial x_k} \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0) z_n + O(|\varepsilon_j|^2)\)
Lemma 8.3.

\[ u \]

Lemma 8.2.

\[ (8.2) \]

where (with Einstein summation, \( \sum_{i=1}^{n} \))

\[ \delta_{li} - \frac{\partial^2 \psi}{\partial x_i \partial x_l}(y')y_n \left( \delta_{li} - \frac{\partial^2 \psi}{\partial x_i \partial x_l}(y')y_n \right) + \frac{\partial \psi}{\partial x_k}(y') \frac{\partial \psi}{\partial x_l}(y') + O(|y|^2) \]

\[ = \delta_{kl} - 2 \frac{\partial^2 \psi}{\partial x_k \partial x_l}(y')y_n + O(|y|^2) \]

\[ = \delta_{kl} - 2 \frac{\partial^2 \psi}{\partial x_k \partial x_l}(y')y_n + O(|y|^2) \]

Similarly for \( 1 \leq j \leq n - 1 \) and \( k = n \), using \( \nabla \psi(0) = 0 \), we get

\[ \sum_{i=1}^{n} \frac{\partial \Phi_{\nu_i}}{\partial y_i}(y) = O(|y|^2) \]

and for \( k = l = n \) we have

\[ \sum_{i=1}^{n} \left( \frac{\partial \psi}{\partial x_i}(y) \right)^2 = \sum_{i=1}^{n} \left( \frac{\partial \psi}{\partial x_i}(y) \right)^2 + 1 + O(|y|^2) = 1 + O(|y|^2) \]

Hence from \[ 8 \] we get

\[ A_1 = \Delta \tilde{u}_j(x) - 2 \epsilon_j \sum_{k,l=1}^{n-1} \frac{\partial^2 \tilde{u}_j}{\partial x_k \partial x_l} \frac{\partial^2 \psi}{\partial z_k \partial z_l}(0)z_n + O(|\epsilon_j z|^2 e^{-\theta|z|}) \]

\[ \square \]

Lemma 8.2. \( A_2 = \frac{\partial \Phi_{\nu_i}}{\partial z_n}(z) \Delta \psi(0) + O(|\epsilon_j z| e^{-\theta|z|}) \)

Proof.

\[ \Delta \Phi_k(0) = 0 \text{ and } \Delta \Phi_n(0) = \Delta \psi(0). \]

A direct calculation shows that: \( A_2 = \frac{\partial \nu_j}{\partial z_n}(z) \Delta \psi(0) + O(|\epsilon_j z| e^{-\theta|z|}) \)

\[ \square \]

Lemma 8.3. \( u_j, v_j \) satisfies

\[ (8.2) \]

\[ \begin{cases} & \Delta \overline{u}_j + c(\epsilon_j x + x_j) \overline{u} = b(\epsilon_j x + x_j) \overline{\nu}_j + \overline{\nu}_j(z), \quad \text{in } \Omega_j \\ & \Delta \overline{v}_j + c(\epsilon_j x + x_j) \overline{v}_j = a(\epsilon_j x + x_j) \overline{\psi}_j + \overline{\psi}_j(z), \quad \text{in } \Omega_j \\ & u > 0, v > 0 \text{ in } \Omega_j, \text{ and } \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \text{ on } \partial \Omega_j \end{cases} \]

where (with Einstein summation, \( 1 \leq k, l \leq (n-1) \))

\[ (8.3) \]

\[ \mu_j(z) = 2 \epsilon_j \frac{\partial^2 \overline{u}}{\partial x_k \partial x_l}(x) \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0)z_n - \epsilon_j \frac{\partial \overline{u}}{\partial x_n}(x) \Delta \psi(0) + \epsilon_j^2 O(|z|^2 e^{-\theta|z|}) \]

\[ (8.4) \]

\[ \nu_j(z) = 2 \epsilon_j \frac{\partial^2 \overline{v}}{\partial x_k \partial x_l}(x) \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0)z_n - \epsilon_j \frac{\partial \overline{v}}{\partial x_n}(x) \Delta \psi(0) + \epsilon_j^2 O(|z|^2 e^{-\theta|z|}) \]
Proof. Using the fact $\tilde{u}$ and $\tilde{v}$ decays exponentially, $1 - \chi$ vanishes in $B(0, \delta)$ we get the result. □

Also using the $C^2_{loc}$ convergence and by dominated convergence theorem we get the following

Lemma 8.4.

$$\xi \Delta \psi(0) + O(\varepsilon_j) = \sum_{k,l=1}^{n-1} \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0) \int_{\mathbb{R}^n_+} \frac{\partial^2 \tilde{u}}{\partial x_k \partial x_l}(x) z_n \tilde{v}_j(z) dz$$

$$= \sum_{k,l=1}^{n-1} \frac{\partial^2 \psi}{\partial x_k \partial x_l}(0) \int_{\mathbb{R}^n_+} \frac{\partial^2 \tilde{v}}{\partial x_k \partial x_l}(x) z_n \tilde{u}_j(z) dz$$

where $\xi = \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_k}(z) \frac{\partial v}{\partial x_k}(z) z_n dz$ for $1 \leq k \leq n - 1$

Lemma 8.5.

$$\int_{\mathbb{R}^n_+} \left[ \frac{\partial \tilde{u}_j}{\partial x_n}(x) \tilde{v}_j(z) + \tilde{u}_j(z) \frac{\partial \tilde{v}_j}{\partial x_n} \right] dz = \int_{\partial \mathbb{R}^n_+} uv d\sigma + O(\varepsilon_j)$$

Proof. Proof is obvious from integration by parts formula. □

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