Adaptive unknown input observer for nonlinear systems with application to fault diagnosis

Sharifuddin Mondal

1Department of Mechanical Engineering, National Institute of Technology Patna, Bihar, PIN-800005, India (e-mail: sharifuddin@nitp.ac.in).

Abstract. An adaptive unknown input observer for nonlinear systems whose nonlinear functions satisfy Lipschitz condition is presented. The main advantage of the proposed observer is that it is sufficient to satisfy the standard existence conditions of normal unknown input observer design. Another advantage is that it estimates both states and parameters simultaneously. So it can be very much useful for robust control and fault diagnosis. As a possible application, an actuator fault diagnosis scheme is developed for single link robot arm. Finally, numerical simulation is performed to validate the effectiveness of the proposed observer in estimating the parameter along with states.

1. Introduction

The importance of unknown input observer (UIO) is already well established for their diverse applications in robust control and fault diagnosis. Over the years, many types of unknown input observers have been proposed for both linear and nonlinear systems [1-6]. With the increasing demands for high precision control techniques, importance of unknown input observer that is robust and/or adaptive is increasing. In recent years, some researches have been carried out to design UIOs that are adaptive against certain parameters [7-12]. Most of these works are related to designing adaptive unknown input observer (AUIO) either for linear systems or application specific nonlinear systems. Mondal [12] developed an AUIO for state estimations of certain type of nonlinear systems. But there are no such AUIO that works for general types of nonlinear systems.

In this work, an adaptive unknown input observer for nonlinear systems has been developed. The observer is designed on the assumption that nonlinear functions of the system satisfies well know Lipschitz condition. The sufficient conditions of the observer design are also formulated. The main advantages of the proposed observer are that it is enough to satisfy the conditions that are required to design a conventional unknown input observer. Then the AUIO is reformulated considering a special scenario that is required to apply the observer to single link robot arm. This kind of observers finds wide applications in the fields of adaptive control, robust control and fault diagnosis [13-15]. As a possible application, an actuator fault diagnosis method for single link robot arm is outlined. Then numerical simulation is performed using MATLAB/SIMULINK software to demonstrate the ability of the observer in estimating states and parameters simultaneously. In this work, a fault is introduced in a parameter. By comparing estimated parameter with its nominal values, fault is detected and isolated.

2. Adaptive unknown input observer

2.1. Problem statement

Consider a nonlinear system with unknown inputs as

\[
\dot{x}(t) = Ax(t) + f(x(t), u(t)) + Bg(x(t), u(t))\theta + Ed(t)
\]

\[
y(t) = Cx(t)
\]

(1)

(2)
where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( y(t) \in \mathbb{R}^p \), \( d(t) \in \mathbb{R}^q \) are state, input, output and unknown input vectors. The matrices \( A, B, E, C \) are known matrices with appropriate dimensions. The functions \( f(x(t), u(t)) \) and \( g(x(t), u(t)) \) are continuous functions. Before defining the observer, following assumptions are stated.

**Assumption 1:** The unknown parameter \( \theta \) is bounded for a constant \( \sigma \) as
\[
|\theta|_2 < \sigma.
\]

**Assumption 2:** The nonlinear functions \( f(x(t), u(t)) \) and \( g(x(t), u(t)) \) are globally Lipschitz, i.e.,
\[
\|f(x(t), u(t)) - f(\hat{x}(t), u(t))\| \leq \gamma_1 (x(t) - \hat{x}(t))
\]
\[
\|g(x(t), u(t)) - g(\hat{x}(t), u(t))\| \leq \gamma_2 (x(t) - \hat{x}(t))
\]

where \( \gamma_1 \) and \( \gamma_2 \) are Lipschitz constants.

**Assumption 3:** For a symmetric positive definite matrix \( P_s \), a matrix \( S \) can be found out by satisfying
\[
(PB)^T P_s = SC.
\]

The assumptions 1-2 are standard assumptions considered in designing observers. Assumption 3 is also widely used in designing adaptive observers [10, 12].

Now an unknown input observer for the above system is designed as
\[
\dot{z}(t) = Nz(t) + Ly(t) + Pf(\hat{x}(t), u(t)) + PBg(\hat{x}(t), u(t))\theta
\]
\[
\dot{\hat{x}}(t) = z(t) - Hy(t)
\]
\[
\dot{\hat{\theta}}(t) = \frac{g(\hat{x}(t), u(t))^T S e_y(t)}{\rho}\quad \text{with} \quad \rho > 0.
\]
where \( e_y(t) = y(t) - \hat{y}(t) \) is output error, \( \hat{x}(t) \in \mathbb{R}^n \), \( \hat{y}(t) \in \mathbb{R}^p \) are estimated state and output vectors respectively. The matrices \( N, L, P, H, S \) are unknown matrices that need to be determined.

Consider the state error as
\[
e(t) = x(t) - \hat{x}(t)
\]
\[
e = x(t) - z(t) + HCx(t). \quad \text{using eq. (8)}
\]
\[
P = I + HC.
\]

The error dynamics \( \dot{e}(t) \) of the above observer (7)-(9) for the system (1) and (2) can be written as
\[
\dot{e}(t) = PAx(t) + Pf(x(t), u(t)) + PBg(x(t), u(t))\theta + PEd(t) - N\dot{x}(t) + H\dot{C}x(t) - LCx(t) - PB\dot{g}(\hat{x}(t), u(t))\theta
\]
Assuming \( \theta = \hat{\theta} - \hat{\theta} \) and simplifying the above equation, one can write as
\[
\dot{e}(t) = Ne(t) + (PA - LC - NP)x(t) + P\left(f(x(t), u(t)) - f(\hat{x}(t), u(t))\right) + PEd(t) + PB\left(g(x(t), u(t)) - g(\hat{x}(t), u(t))\right)\theta + PBg(\hat{x}(t), u(t))\hat{\theta}
\]

Now following conditions are applied to remove the unmeasurable variables:
\[
PA - LC - NP = 0
\]
\[
PE = 0.
\]

Now assume
\[
K = L + NH.
\]
From equations (11) and (13), one gets
\[
N = PA - KC
\]
Using equations (13), (14) and (16), one can write the equation (12) as
\[
\dot{e}(t) = (PA - KC)e(t) + P\left(f(x(t), u(t)) - f(\hat{x}(t), u(t))\right) + PB\left(g(x(t), u(t)) - g(\hat{x}(t), u(t))\right)\theta + PBg(\hat{x}(t), u(t))\hat{\theta}
\]

Now the problem in hand is that the observer gain matrices should be determined in such a way that the observer error dynamics (17) become stable i.e., \( e(t) \to 0 \) as \( t \to \infty \). The method of determining the observer matrices is presented in the next subsection.

### 2.2. Main results
In this subsection, the method of determining the observer gain matrices \((N, L, P, H, S)\) is discussed. The gain matrix \(H\) is calculated using equations (11) and (14) as
\[
H = -E(CE)^+ + Y(I_p - (CE)(CE)^+)
\] (18)

Once \(H\) is known, \(P\) is determined using equation (11). To determine the other matrices \((N, L)\), following theorem is proposed.

**Theorem 1:** For a given system, represented by equations (1)-(2), the system, defined by equations (7)-(9), will be the adaptive unknown input observer if there exists a symmetric matrix \(P_s\) and the LMI
\[
\begin{bmatrix}
P_s(PA) - ZC + (PA)^TP_s - C^TZ + 2I & P_s & P_sB \\
P_s & -\varepsilon_1 I & 0 \\
B^TP_s & 0 & -\varepsilon_2 I
\end{bmatrix} < 0
\]

where \(\varepsilon_1 = (\gamma_1^2 \sigma_m^2)^{-1}\) and \(\varepsilon_2 = (\gamma_2^2 \sigma_m^2 \sigma^2)^{-1}\) holds. The gain matrix \(K = (P_s)^{-1}Z\) also stabilizes the error dynamics (17), i.e., \(e(t) \to 0\) as \(t \to \infty\).

To prove this theorem, following lemma is required.

**Lemma 1:** If the nonlinear functions \(f(x(t), u(t))\) and \(g(x(t), u(t))\) satisfy conditions (4) and (5) respectively, then following inequalities hold.
\[
2e^T(t)P_s P \left(f(x(t), u(t)) - f(\hat{x}(t), u(t))\right) \leq \gamma_1^2 \sigma_m^2 e^T(t)P_s P e(t) + e^T(t)e(t)
\] (19)
\[
2e^T(t)P_sPB \left(g(x(t), u(t)) - g(\hat{x}(t), u(t))\right) \theta \leq \gamma_2^2 \sigma_m^2 \sigma^2 e^T(t)P_s BB^TP_s e(t) + e^T(t)e(t)
\] (20)

where \(\sigma_m\) is norm value of the matrix \(P\).

**Proof:** This can be proved in the same way that is presented in [10] for similar cases. Proof is ignored here due to space limitation.

**Proof of the theorem 1:** Consider Lyapunov-Krasovskii functional \(V(t)\) as
\[
V(t) = e^T(t)P_s e(t) + \rho \theta^T \hat{\theta}
\] (21)
where \(\rho > 0\) and \(P_s\) is symmetric positive definite matrix

Differentiating (21) with respect to time, one gets
\[
\dot{V}(t) = 2e^T(t)P_s \dot{e}(t) + 2\rho \hat{\theta}^T \hat{\theta}
\] (22)
Replacing (17) in (22), one can write
\[
\dot{V}(t) = 2e^T(t)P_s \dot{e}(t) + 2\rho \hat{\theta}^T \hat{\theta}
\] (23)
Now after rearrangement and applying lemma 1, one can write
\[
\dot{V}(t) \leq e^T(t)\left\{P_s(PA - KC) + (PA - KC)^TP_s\right\} e(t) + 2e^T(t)P_s P \left(f(x(t), u(t)) - f(\hat{x}(t), u(t))\right) + 2e^T(t)P_sPB \left(g(x(t), u(t)) \theta - g(\hat{x}(t), u(t)) \hat{\theta}\right) + 2\rho \hat{\theta}^T \hat{\theta}
\] (24)
To discard the effect of unknown parameter from equation (24), following condition is applied.
\[
2e^T(t)P_s PB g(\hat{x}(t), u(t)) \hat{\theta} + 2\rho \hat{\theta}^T \hat{\theta} = 0
\] (25)
This relation gives the form required for parameter estimation given in (9) as \(\theta\) is generally constant.

Now the relation (24) is written as
\[
\dot{V}(t) = e^T(t)\left(P_s(PA - KC) + (PA - KC)^TP_s\right) e(t) + 2e^T(t)P_s P \left(f(x(t), u(t)) - f(\hat{x}(t), u(t))\right) + 2e^T(t)P_sPB \left(g(x(t), u(t)) \theta - g(\hat{x}(t), u(t)) \hat{\theta}\right) + 2\rho \hat{\theta}^T \hat{\theta}
\] (26)
It is well established that if \(\dot{V}(t) < 0\), then \(e(t) \to 0\) as \(t \to \infty\). Now from equation (26), \(\dot{V}(t) < 0\) is possible only when following condition is satisfied.
\[
P_s(PA - KC) + (PA - KC)^TP_s + \gamma_1^2 \sigma_m^2 P_s P_s + \gamma_2^2 \sigma_m^2 \sigma^2 P_s BB^TP_s + 2I e(t)
\] (27)
Finally applying Schur’s complement [2, 10, 16] in (27), one gets the form presented in theorem 1. This completes the proof.

Now solving the LMI presented in theorem 1 using LMI toolbox provided with MATLAB software package [17], \(P_s\) and \(Z\) are determined. Using these values, \(K\) is then calculated as \(K = (P_s)^{-1}Z\). The matrix \(N\) is determined using the relation (16). The matrix \(L\) is found out from (15). Finally the matrix \(S\) is chosen arbitrarily satisfying the condition (6). This completes the observer design. In the next section, the above result will be reformulated for a special type of nonlinear system.

3. AUIO when \(g(.)\) depends only on inputs
In this section, the adaptive observer is reformulated for special type of nonlinear system. The system equation is written as follows.

\[ \dot{x}(t) = Ax(t) + f(x(t),u(t)) + Bg(u(t))\theta + Ed(t) \]  

(28)

Now the problem is to design an adaptive UIO for the system represented by equations (28) and (2). Here the nonlinear function \( g(.) \) depends only on input \( u(t) \). The assumptions (1) and (3) will remain the same but little change will occur in assumption (2) as only one nonlinear function will be there instead of two. Now the observer for the system (28) and (2) can be written as

\[ \dot{z}(t) = Nz(t) + Ly(t) + Pf(x(t),u(t)) + PBg(u(t))\hat{\theta} \]  

(29)

\[ \dot{x}(t) = z(t) - Hy(t) \]  

(8)

\[ \dot{\hat{\theta}}(t) = \frac{g(u(t))^T e_y(t)}{\rho} \]  

(30)

with \( \rho > 0 \).

where \( e_y(t) = y(t) - \hat{y}(t) \) and equation (8) is rewritten for better understanding.

Assuming the state error like previous case, the error dynamics for the presented system can be determined as

\[ \dot{e}(t) = (PA - KC)e(t) + P(f(x(t),u(t)) - f(\hat{x}(t),u(t)) + PBg(u(t))\hat{\theta} \]  

(31)

In the same way as previous section, following theorem is presented to determine the observer matrices.

**Theorem 2**: For a given system, represented by equations (28) and (2), the system, defined by equations (29), (8) and (30), will be the adaptive unknown input observer if there exists a symmetric matrix \( P_s \) and the LMI

\[
\begin{bmatrix}
P_s(PA) - ZC + (PA)^TP_s - C^TZ + I & P_s \\
-\varepsilon_1 I & P_s
\end{bmatrix} < 0
\]

where \( \varepsilon_1 = (\gamma_1^2 \sigma_m^2)^{-1} \) holds. The gain matrix \( K = (P_s)^{-1}Z \) also stabilizes the observer error dynamics (31), i.e., \( e(t) \to 0 \) as \( t \to \infty \).

**Proof of theorem 2**: Considering the Lyapunov-Krasovskii functional as given in (21) and following the steps mentioned in the proof of theorem 1, one can easily prove the theorem 2. So detailed proof is omitted here.

Other observer matrices are determined in the same way that are presented in the previous section. A numerical example is presented in the next section to show the effectiveness of the observer.

4. Numerical example-AUIO in fault diagnosis

In this section, a single link robot arm that is widely used for validating different estimation and control algorithms [2-3, 10, 12] is considered. First, the actuator fault diagnosis method is presented. Then simulation results are given.

4.1 Actuator Fault Diagnosis

In state space model, any actuator fault can be accommodated into input matrix (\( B \)). Let the matrix \( B \) becomes \( B_u \) when a fault occurs. In the present problem (single link robot arm), single actuator is present, so a constant \( \delta \) that relates these two matrices can be considered as

\[ B_u = B\delta \]  

(32)

In ideal situation, \( \delta = 1 \) (no fault). When fault occurs, it will be changed to another value depending on level of fault. The idea of fault diagnosis using adaptive observer is that this \( \delta \) is assumed as adaptive parameter i.e., \( \theta = \delta \) and the system is remodelled. Then an adaptive observer is designed to estimate the states and parameters. Other uncertain parameters are incorporated as unknown inputs so their effects will not appear in estimated states and parameters. When a fault occurs, the estimated parameter will settle to a value other than 1. So observing the estimated parameter, fault can be detected. Magnitude of the fault can also be determined using the nominal values of input matrix and using relation (32).

4.2 Numerical results
It is assumed that the value of the viscous friction constant $F_2$ is uncertain and unknown. The effect of this parameter in the system is considered to have a known part ($F_2 \dot{\theta}_2(t)$) and an unknown part which is modelled as an unknown input $d(t)$. It is assumed that a fault occurs in the actuator. So the value of $B$ becomes unknown ($B_u$). It is remodelled as $B_u = B\theta$. In ideal situation, $\|\theta\| = 1$, which gives $\sigma = 1$. With this arrangement, the system matrices and vectors are given as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -48.6486 & 48.6486 & -1.2432 & 0 \\ 19.3548 & -19.3548 & 0 & -0.6882 \\ \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 2.1622 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad d(t) = -\frac{\Delta F_2 \dot{\theta}_2(t)}{J_2},$$

$$g(x(t), u(t)) = \begin{bmatrix} 0 \\ 0 \\ -33.2274 \cos(x_2) \end{bmatrix}$$

and $x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$.

The measurement matrix is taken as $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, which satisfies the existence conditions of adaptive unknown input observer. The observer matrices $P, H$ are determined with a suitable $Y$. Then the LMI technique is used to determine other matrices.

The final observer matrices are $P = \begin{bmatrix} 1 & 0.8147 & 0 & 0 \\ 0 & 1.9058 & 0 & 0 \\ 0 & 0.1270 & 1 & 0 \\ 0 & 0.9134 & -1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 0.8147 & 0 \\ 0.9058 & 0 \\ 0.1270 & 0 \\ 0.9134 & -1 \end{bmatrix}$, $N = \begin{bmatrix} 0 & -33.7675 & -13.4126 & -13.5979 \\ 0 & -38.6494 & -9.3419 & -7.4361 \\ -48.6486 & 342.1144 & 103.0181 & 104.3883 \\ 48.6486 & -351.0777 & -123.0050 & 123.3348 \end{bmatrix}$, $L = \begin{bmatrix} 89.6360 & 0.8147 \\ 81.6360 & 1.9058 \\ -672.1426 & 0.1270 \\ 709.0683 & 0.9134 \end{bmatrix}$, $S = \begin{bmatrix} 4.4295 & 1.2090 \end{bmatrix}$, $\rho = 0.04$.

The input signal for the system is chosen as follows: $u = 1.0 \sin(wt)$ with $w = 1 \text{ rad/s}$. The fault is introduced in such a way that $\theta$ changes to 0.5 at 10 seconds from 1. Now the adaptive observer is designed. The true & estimated states and the estimated errors are plotted in Fig 1 and Fig 2 respectively.
Nominal and estimated parameters are plotted in Fig. 3. Actual and estimated parameters are shown in Fig. 4. Fig. 1 and Fig. 2 show that the observer is estimating the states efficiently. Fig. 3 confirms that the estimated parameter is settling at 0.5 other than 1 that is confirming the fault in the actuator. Fig. 4 confirms the efficacy of the observer in estimating the parameters correctly. As the observer is designed for constant parameters, it is not able to track the changes from 1 to 0.5.

5. Conclusion
An adaptive unknown input observer has been presented. The observer is designed for nonlinear systems that satisfy Lipschitz condition. This type of observer is very much useful in adaptive and robust controls. It can estimate both states and parameters simultaneously. So it is very useful for fault diagnosis of nonlinear systems. As a possible application, an actuator fault diagnosis method has been devised for single link robot arm. Using numerical simulation, effectiveness of the proposed observer is validated.

6. References
[1] Ahmadizadeh S, Zarei J and Karimi HR 2014 Asian J. Contr. 16 (4) 1009-1019.
[2] Mondal S 2015 Int. J. Dyn. Contr. 3(4) 448-456.
[3] Mondal S, Chakraborty G and Bhattacharyya K 2007 Proc. Int. Conf. Adv. on Contr. Optim. Dyn. Sys. (Bangalore) 117-120.
[4] Ali AA 2013 Ph.D. dissertation (University of Michigan, USA).
[5] Zhu F 2007 IEEE Int. Conf. on Contr.Autom. (Gangzhou) 529–534.
[6] Na J, Chen AS, Herrmann G, Burke R and Brace C 2018 IEEE Trans. on Veh. Tech. 67(1) 409-421.
[7] Bowong S and Tewa JJ 2008 Math. Comp. Modell. 48(11-12) 1826-1839.
[8] Heydari M and Demetriou MA 2017 Inter. J. Syst. Sci. 48(1) 182-189.
[9] Lin S-Y, Yen J-Y, Chen M-S, Chang S-H and Kao C-Y 2017 IEEE Trans. on Autom. Contr. 62(8) 4073–4078.
[10] Mondal S and Chung WK 2013 Inter. J. Adapt. Contr. Sig. Proc. 27(7) 610-619.
[11] Salgado I and Chirez I 2018 IEEE Trans. on Neu. Netw. Learn. Syst. 29(8) 3499-3509.
[12] Mondal S 2018 Inter. Conf. on Recent Adv. in Mat. Manuf. Tech. (Hyderabad).
[13] Guo R, Guo K, Gen Q, Zhang J, Dong J and Bai L 2016 Math. Prob. Enggg. 2016 1-12.
[14] Lee H, Snyder S and Hovakimyan N 2014 AIAA Guid. Nav. Contr. Conf. (National Harbor, Maryland).
[15] Wang D and Lum K-Y 2007 Int. J. Adapt. Contr. Sig. Proc. 21(1) 31–48.
[16] Boyd S, El Ghaoui L, Feron E and Balakrishnan V 1994 *Linear Matrix Inequalities in Systems and Control Theory* (SIAM: Philadelphia, PA).

[17] Gahinet P, Nemirovsky A, Laum AJ and Chilali M 1995 *LMI Control Toolbox: For Use with MATLAB* (The Mathworks Inc.: USA).