Discrete Dirac systems on the semiaxis: rational reflection coefficients and Weyl functions

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ABSTRACT
We consider the cases of the self-adjoint and skew-self-adjoint discrete Dirac systems, obtain explicit expressions for reflection coefficients and show that rational reflection coefficients and Weyl functions coincide.

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1. Introduction
Discrete self-adjoint and skew-self-adjoint Dirac systems play an essential role in the study of Toeplitz matrices (and corresponding measures), of discrete integrable nonlinear equations (including isotropic Heisenberg magnet model) and of spectral theory of difference equations (see, e.g. [5, 6, 14, 20, 23] and references therein). Weyl–Titchmarsh theory of discrete systems is actively studied (see, e.g. [3, 13, 27, 28] and various references therein). In particular, Weyl–Titchmarsh theory of discrete self-adjoint and skew-self-adjoint Dirac systems was studied in [6–8, 14, 18, 24] (see also references therein). It is known that Weyl–Titchmarsh (or simply Weyl) functions of continuous Dirac systems on the semi-axis are closely related to the scattering data. Some particular results for the self-adjoint systems are contained, for instance, in [1, 11] and the general cases of continuous self-adjoint and skew-self-adjoint systems were treated in the recent paper [22]. The present article may be considered as the continuation of the paper [22], where the important discrete case is dealt with. We consider the cases of the self-adjoint and skew-self-adjoint discrete Dirac systems, obtain explicit expressions for reflection coefficients and show that rational reflection coefficients and Weyl functions coincide.
General-type discrete self-adjoint Dirac system has the form:

\[ y_{k+1}(z) = (I_m + izC_k)y_k(z) \quad (k \in \mathbb{N}_0), \tag{1} \]

where \( \mathbb{N}_0 \) stands for the set of non-negative integers, \( I_m \) is the \( m \times m \) identity matrix, ‘i’ is the imaginary unit (\( i^2 = -1 \)) and the \( m \times m \) matrices \( C_k \) are positive and \( j \)-unitary:

\[ C_k > 0, \quad C_kjC_k = j, \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix} \quad (m_1 + m_2 = m; \ m_1, m_2 > 0). \tag{2} \]

Below, the Jost solution and reflection coefficient of the system (1), (2) are introduced in a way, which is similar to the continuous case. Namely, the Jost solution \( \{F_k(z)\} (z \in \mathbb{R}) \) of the Dirac system (1), (2) is defined via its asymptotics:

\[ F_k(z) = (I_m + izj)^k (I_m + o(1)), \quad k \to \infty. \tag{3} \]

The reflection coefficient \( \mathcal{R}(z) \) is introduced via the blocks of \( F_0(z) \):

\[ \mathcal{R}(z) = \begin{bmatrix} I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} F_0(z) \begin{bmatrix} 0 & I_{m_2} \\ I_{m_1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}^{-1}. \tag{4} \]

Discrete skew-self-adjoint Dirac system (SkDDS) is given (see [8, 14]) by the formula:

\[ y_{k+1}(z) = \left( I_m + \frac{i}{z}C_k \right)y_k(z), \quad C_k = U_k^*jU_k \quad (k \in \mathbb{N}_0), \tag{5} \]

where the matrices \( U_k \) are unitary and \( j \) is defined in (2).

Direct and inverse problems (in terms of Weyl functions) were solved for systems (1), (2) in [7, 18] and for systems (5) in [6, 8]. In particular, in the case of rational Weyl matrix functions, direct and inverse problems were solved explicitly using our GBDT version [19, 21, 24] of the Bäcklund-Darboux transformation. For various versions of Bäcklund-Darboux transformations and related commutation methods see, for instance, [2, 4, 9, 12, 15–17] and references therein.

The results of the paper imply that the procedures to recover systems from the Weyl functions enable us to recover systems from the reflection coefficients as well. In the next section, we give some preliminary definitions and results in order to make the paper self-sufficient. Two subsections of Section 3 are dedicated to the reflection coefficients in the self-adjoint and skew-self-adjoint cases.

In the paper, \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{R} \) denotes the real axis, \( \mathbb{C} \) stands for the complex plane, and \( \mathbb{C}_+ (\mathbb{C}_-) \) stands for the open upper (lower) half-plane. The spectrum of a square matrix \( A \) is denoted by \( \sigma(A) \).

2. Preliminaries

2.1. Self-adjoint case

1. Self-adjoint discrete Dirac system and stability of the explicit procedure to recover it from the Weyl function was studied in our recent paper [18]. We refer to [18] for the preliminary
definitions and results in this subsection. The fundamental \( m \times m \) solution \( \{W_k\} \) of (1) is normalized by

\[
W_0(z) = I_m.
\]

**Definition 2.1:** The Weyl function of the Dirac system (1) (which is given on the semi-axis \( 0 \leq k < \infty \) and satisfies (2)) is an \( m_1 \times m_2 \) matrix function \( \varphi(z) \) in the lower half-plane, such that the following inequalities hold:

\[
\sum_{k=0}^{\infty} q(z)^k \begin{bmatrix} \varphi(z)^* & I_{m_2} \end{bmatrix} W_k(z)^* C_k W_k(z) \begin{bmatrix} \varphi(z) \\ I_{m_2} \end{bmatrix} < \infty \quad (z \in \mathbb{C}_-),
\]

\[
q(z) := (1 + |z|^2)^{-1}.
\]

(For the case \( z \in \mathbb{C}_+ \), the definition of the Weyl function \( \varphi(z) \) of Dirac system (1), (2) was given in [7].)

**2.** In order to consider the case of rational Weyl functions, we introduce generalized Bäcklund-Darboux transformation (GBDT) of the discrete self-adjoint Dirac systems. Each GBDT of the initial discrete Dirac system is determined by a triple \( \{A, S_0, \Pi_0\} \) of parameter matrices. Here, we take a trivial initial system and choose \( n \in \mathbb{N} \), two \( n \times n \) parameter matrices \( A (\det A \neq 0) \) and \( S_0 > 0 \), and an \( n \times m \) parameter matrix \( \Pi_0 \) such that

\[
AS_0 - S_0 A^* = i\Pi_0 j \Pi_0^*.
\]

Define the sequences \( \{\Pi_r\} \) and \( \{S_r\} \) \((r \geq 0)\) using the triple \( \{A, S_0, \Pi_0\} \) and recursive relations

\[
\Pi_{k+1} = \Pi_k + iA^{-1}\Pi_k j \quad (k \geq 0),
\]

\[
S_{k+1} = S_k + A^{-1}S_k(A^*)^{-1} + A^{-1}\Pi_k \Pi_k^* (A^*)^{-1} \quad (k \geq 0).
\]

From (9)–(11), the validity of the matrix identity

\[
AS_r - S_r A^* = i\Pi_r j \Pi_r^* \quad (r \geq 0)
\]

follows by induction. In the self-adjoint case, we introduce admissible triples \( \{A, S_0, \Pi_0\} \) in the following way.

**Definition 2.2:** The triple \( \{A, S_0, \Pi_0\} \), where \( \det A \neq 0 \), \( S_0 > 0 \) and (9) holds, is called admissible.

In view of (11), for the admissible triple we have \( S_r > 0 \) \((r \geq 0)\). Thus, the sequence (potential) \( \{C_k\} (k \geq 0) \) is well-defined by the equality

\[
C_k := I_m + \Pi_k^* S_k^{-1} \Pi_k - \Pi_{k+1}^* S_{k+1}^{-1} \Pi_{k+1}.
\]

Moreover, from [18, Theorem 2.5] we see that the matrices \( C_k \) satisfy (2). We say that the potential \( \{C_k\} \) is determined by the admissible triple. The potential determined by an admissible triple \( \{A, S_0, \Pi_0\} \) is called pseudo-exponential. We note that the notion of
the pseudo-exponential (and strictly pseudo-exponential) potentials for the self-adjoint continuous case was introduced first in [10] (see also [11]). In the discrete case, some additional requirements on the admissible and strongly admissible triples (which determine pseudo-exponential and strictly pseudo-exponential, respectively, potentials) appear.

All Weyl functions \( \varphi(z) \) are contractive in \( \mathbb{C}_- \) and all the potentials \( \{C_k\} \), such that \( \varphi(z) \) (for the corresponding systems) are contractive and \( \varphi(-1/z) \) are strictly proper rational, are determined by some admissible triples [18]. Strongly admissible triples for the self-adjoint case are considered in Section 3.1.

We will need also the matrix function \( w_A \), which for each \( k \geq 0 \) is a so called transfer matrix function in Lev Sakhnovich form [24–26] and is defined by the relation

\[
w_A(k, \lambda) := I_n - ij\Pi_k^*S_k^{-1}(A - \lambda I_n)^{-1}\Pi_k \quad (\lambda \notin \sigma(A)). \tag{14}
\]

**Remark 2.3:** If \( w_A(k, \lambda) \) and \( w_A(k, \bar{\lambda}) \) are well-defined, that is

\[
\lambda \notin \left( \sigma(A) \cup \sigma(A^*) \right), \tag{15}
\]

according to [24, Corollary 1.13] we have

\[
w_A(k, \lambda)^{-1} = jw_A(k, \bar{\lambda})^*j. \tag{16}
\]

In particular, under condition (15) the matrix \( w_A(k, \lambda) \) is invertible.

The fundamental solution \( \{W_k\} \) of the Dirac system (1) admits the representation

\[
W_k(z) = w_A(k, -1/z) (I_n + iz)^k w_A(0, -1/z)^{-1} \quad (k \geq 0), \tag{17}
\]

where \( w_A \) is defined in (14). We note that in (17) (taking into account Remark 2.3) we assume that

\[
z \neq 0, \quad -1/z \notin \left( \sigma(A) \cup \sigma(A^*) \right). \tag{18}
\]

Now, we partition \( \Pi_k \) and write it down in the form

\[
\Pi_k = \left[ (I_n + iA^{-1})^k \vartheta_1 \quad (I_n - iA^{-1})^k \vartheta_2 \right], \tag{19}
\]

where \( \vartheta_1 \) and \( \vartheta_2 \) are \( n \times m_1 \) and \( n \times m_2 \), respectively, blocks of \( \Pi_0 \). Assume further in this subsection that

\[
\pm i \notin \sigma(A). \tag{20}
\]

In view of (12) and (19), setting

\[
R_r := (I_n + iA^{-1})^{-r}S_r (I_n - i(A^*)^{-1})^{-r} \tag{21}
\]

we have

\[
R_{k+1} - R_k = 2(I_n + iA^{-1})^{-k-1}A^{-1}(I_n - iA^{-1})^k \vartheta_2 \vartheta_2^* \left( (I_n - iA^{-1})^k \right)^* (A^{-1})^* \times \left( (I_n + iA^{-1})^{-k-1} \right)^* \geq 0. \tag{22}
\]
Since $R_0 = S_0 > 0$, relations (22) imply that there is a limit
\[
\lim_{k \to \infty} R_k^{-1} = \kappa_R \geq 0. \tag{23}
\]
In a similar way we introduce the matrices
\[
Q_r := (I_n - iA^{-1})^{-r}S_r \left( I_n + i(A^*)^{-1}\right)^{-r}, \tag{24}
\]
and show that
\[
Q_{k+1} - Q_k \geq 0. \tag{25}
\]
Since $Q_0 = S_0 > 0$, relations (25) imply that there is a limit
\[
\lim_{k \to \infty} Q_k^{-1} = \kappa_Q \geq 0. \tag{26}
\]

2.2. Skew-self-adjoint case

The preliminary definitions and results on the skew-self-adjoint discrete Dirac systems (SkDDS) (5) we take from [6] and sometimes from [8].

Remark 2.4: The notations here slightly differ from the notations in [6, 8]. In particular, we introduce the matrices $R_k$ and $Q_k$ in the both self-adjoint and skew-self-adjoint cases via formulas (21) and (24), respectively, but in [8] $R_k$ stands for $Q_k$ in the present notations and $Q_k$ stands for $R_k$.

Definition 2.5: The Weyl function of SkDDS is an $m_1 \times m_2$ matrix function $\varphi(z)$ in
\[
\mathbb{C}_M = \{ z \in \mathbb{C} : \Im(z) > M \} \quad \text{for some } M > 0,
\]
which satisfies the inequality
\[
\sum_{k=0}^{\infty} \begin{bmatrix} \varphi(z)^* & I_{m_2} \end{bmatrix} w_k(z)^* w_k(z) \begin{bmatrix} \varphi(z) \\ I_{m_2} \end{bmatrix} < \infty, \tag{27}
\]
where $w_k(z)$ is the fundamental solution of SkDDS normalized by $w_0(z) \equiv I_m$.

Let us fix again an integer $n > 0$, and consider an $n \times n$ matrix $A$ with det $A \neq 0$, an $n \times n$ matrix $S_0 > 0$ and an $n \times m$ matrix $\Pi_0$. These matrices should satisfy the identity
\[
AS_0 - S_0A^* = i\Pi_0\Pi_0^*. \tag{28}
\]
The sequences $\{\Pi_k\}$, $\{S_k\}$ and $\{C_k\}$ ($k \geq 0$) are introduced using the triple $\{A, S_0, \Pi_0\}$ and relations
\[
\Pi_{k+1} = \Pi_k + iA^{-1}\Pi_k j, \tag{29}
\]
\[
S_{k+1} = S_k + A^{-1}S_k(A^*)^{-1} + A^{-1}\Pi_k\Pi_k^*(A^*)^{-1}, \tag{30}
\]
\[
C_k = j + \Pi_k^*S_k^{-1}\Pi_k - \Pi_{k+1}^*S_{k+1}^{-1}\Pi_{k+1}. \tag{31}
\]
Similar to the self-adjoint case we write down $\Pi_k$ in the form
\[ \Pi_k = \left[ (I_n + iA^{-1})^k \vartheta_1 \ (I_n - iA^{-1})^k \vartheta_2 \right]. \]

Next, we need the notion of controllability from the system theory. Recall that the pair of matrices $\{A, \vartheta_1\}$ is called controllable if
\[ \text{span} \bigcup_{k=0}^{n-1} \text{Im} A^k \vartheta_1 = \mathbb{C}^n, \quad n = \text{ord}(A), \tag{32} \]
where Im stands for image and ord$(A)$ stands for the order of $A$. If $S_0 > 0$, the identity (28) holds and the pair $\{A, \vartheta_1\}$ is controllable, then according to [6, Lemma 3.2] and [6, Proposition 3.6] we have $A \neq 0, S_k > 0$ and the matrices $C_k$ admit representation $C_k = U_k^*jU_k$ from (5). That is, the sequence $\{C_k\}$ is well-defined and the corresponding system is a skew-self-adjoint Dirac system. In the skew-self-adjoint case, the triple $\{A, S_0, \Pi_0\}$, such that $S_0 > 0$, the identity (28) holds and the pair $\{A, \vartheta_1\}$ is controllable, is called admissible. The potential determined by this triple is called pseudo-exponential.

Moreover, if $\varphi(z)$ is a strictly proper rational $m_1 \times m_2$ matrix function then it is the Weyl function of some skew-self-adjoint Dirac system with the pseudo-exponential potential (see [8, Theorem 4.2]). We will require additionally that $i \not\in \sigma(A)$.

**Definition 2.6:** In the skew-self-adjoint case, the triple $\{A, S_0, \Pi_0\}$, where $S_0 > 0$, the identity (28) is valid, the pair $\{A, \vartheta_1\}$ is controllable and $i \not\in \sigma(A)$, is called strongly admissible. The potentials determined by the strongly admissible triples are called strictly pseudo-exponential.

Note that [8, Proposition 4.8] implies that if $S_0 > 0$, (28) holds and 0, $i \not\in \sigma(A)$ then $S_k > 0$, the matrices $C_k$ are well-defined and there is a strongly admissible triple which determines the same potential $\{C_k\}$ as $\{A, S_0, \Pi_0\}$. The fundamental solution $w_k$ of SkDDS determined by the strongly admissible triple $\{A, S_0, \Pi_0\}$ has the form
\[ w_k(z) = w_A(k, -z) \left( I_m + \frac{i}{z} \right)^k w_A(0, -z)^{-1}, \tag{33} \]
whereas $w_A$ in the skew-self-adjoint case is given by
\[ w_A(k, \lambda) := I_m - i\Pi_k S_k^{-1}(A - \lambda I_n)^{-1}\Pi_k \quad (\lambda \not\in \sigma(A)). \tag{34} \]

**Remark 2.7:** We consider (33) (and (33) holds) for those $z$, where the right-hand side of (33) is well-defined. In the skew-self-adjoint case, according to [24, Corollary 1.13] we have $w_A(k, \lambda)^{-1} = w_A(k, \lambda)^*$. Hence, the right-hand side of (33) is well-defined for $z$ such that
\[ z \neq 0, \quad -z \not\in \left( \sigma(A) \cup \sigma(A^*) \right). \tag{35} \]

Taking into account Remark 2.4, we see that [8, Proposition 4.10] and [8, (4.34)] imply that
\[ \lim_{k \to \infty} Q_k^{-1} = 0; \quad \lim_{k \to \infty} Q_k^{-1}\tilde{G}(A)^k \vartheta_1 = 0, \quad \tilde{G}(A) := (A - iI_n)^{-1}(A + iI_n) \tag{36} \]
in the case of a strongly admissible triple $\{A, S_0, \Pi_0\}$. 
3. Reflection coefficients

3.1. Reflection coefficients: self-adjoint case

In this subsection, we express (via the triple \( \{A, S_0, \Pi_0\} \)) the Jost solution and reflection coefficient, which are the analogues of the corresponding functions in the continuous case.

Uniqueness of the solution of the inverse problem to recover system from the Weyl function (see [18, Theorem 2.3]) together with Theorems 2.6 and 2.8 and Proposition 2.7 (all from [18]) imply that without loss of generality one can require that \( \sigma(A) \subset (\mathbb{C}_+ \cup \mathbb{R}) \). Further we use a stronger requirement

\[
\sigma(A) \subset \mathbb{C}_+, \quad i \notin \sigma(A).
\] (37)

Following [6, 10], we call the admissible triple satisfying (37) strongly admissible and we introduce the class of the strictly pseudo-exponential potentials \( \{C_k\} \).

**Definition 3.1:** The potentials \( \{C_k\} \) of the Dirac systems (1), (2), which are determined by the strongly admissible triples, are called strictly pseudo-exponential.

In view of (14), (19), (21) and (24), we have a representation

\[
w_A(k, -1/z) = I_m - izj \begin{bmatrix} \partial_1^* R_k^{-1}(I_n + zA)^{-1} \vartheta_1 & \partial_1^* R_k^{-1}(I_n + zA)^{-1} G(A) \vartheta_2 \\ \partial_2^* (G(A)k)^* R_k^{-1}(I_n + zA)^{-1} \vartheta_1 & \partial_2^* Q_k^{-1}(I_n + zA)^{-1} \vartheta_2 \end{bmatrix},
\] (38)

where we assume that \( z \neq 0, \quad -1/z \notin \sigma(A) \) and

\[
G(A) := (I_n + iA^{-1})^{-1}(I_n - iA^{-1}).
\] (39)

Relations (37) and (39) yield

\[
\sigma(G(A)) \subset \{ \lambda : |\lambda| < 1 \}.
\] (40)

Hence, from (23), (26) and (38) we derive

\[
\lim_{k \to \infty} w_A(k, -1/z) = \begin{bmatrix} \chi_1(z) & 0 \\ 0 & \chi_2(z) \end{bmatrix},
\] (41)

\[
\chi_1(z) := I_{m_1} - iz \partial_1^* \kappa_R(I_n + zA)^{-1} \vartheta_1, \quad \chi_2(z) := I_{m_2} + iz \partial_2^* \kappa_Q(I_n + zA)^{-1} \vartheta_2.
\] (42)

Rewrite (17) in the form

\[
W_k(z)w_A(0, -1/z) = w_A(k, -1/z) \left( I_m + iz \right)^k,
\] (43)

We note that the rational matrix functions \( \chi_1(z) \) and \( \chi_2(z) \) are invertible excluding a finite number of points. Thus, one can rewrite (43) in the form

\[
W_k(z)w_A(0, -1/z) \text{diag} \{\chi_1(z)^{-1}, \chi_2(z)^{-1}\} = w_A(k, -1/z) \text{diag} \{\chi_1(z)^{-1}, \chi_2(z)^{-1}\} \left( I_m + iz \right)^k,
\] (44)
where diag stands for the block diagonal matrix. For all real-valued \( z \) (excluding, possibly, a finite number of points) relations (41) and (44) imply that

\[
W_k(z)w_A(0, -1/z) \text{diag}(\chi_1(z)^{-1}, \chi_2(z)^{-1}) = (I_m + izj)^k (I_m + o(1)),
\]

where \( k \to \infty \). Compare (3) with (45) in order to see that the solution of the system (1), (2) given on the left-hand side of (45) is the Jost solution. In other words, the Jost solution \( \{F_k\} \) is given by the equalities

\[
F_k(z) = W_k(z)w_A(0, -1/z) \begin{bmatrix} \chi_1(z)^{-1} & 0 \\ 0 & \chi_2(z)^{-1} \end{bmatrix}.
\]

Let us partition the matrix function \( w_A(0, \lambda) \) into the blocks corresponding to the blocks of \( j \) in the formula (2):

\[
w_A(0, \lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}.
\]

In other words, we partition \( w_A(0, \lambda) \) so that \( a(\lambda) \) above is an \( m_1 \times m_1 \) matrix function. It was shown in the proof of [18, Theorem 2.6] (see [18, (2.29)]) that the Weyl function \( \varphi(z) \) of the system (1), (2) (in \( \mathbb{C}_- \)) is given by the formula

\[
\varphi(z) = b(-1/z)d(-1/z)^{-1}.
\]

Relations (4), (6) and (46)–(48) imply the following theorem.

**Theorem 3.2:** Let Dirac system (1), (2) be a system with the strictly pseudo-exponential potential \( \{C_k\} \). Then the Weyl function \( \varphi(z) \) is the unique analytic continuation of the reflection coefficient \( \mathcal{R}(z) \) of this system. That is, the reflection coefficient and the Weyl function are given by the same rational matrix function.

From [18, Theorem 2.6] and Theorem 3.2 we derive the following corollary.

**Corollary 3.3:** Let the potential \( \{C_k\} \) be determined by a strongly admissible triple \( \{A, S_0, \Pi_0\} \). Then, the reflection coefficient of the Dirac system (1), (2) is given by the formula

\[
\mathcal{R}(z) = -iz\vartheta_1 S_0^{-1}(I_n + zA^\times)^{-1}\vartheta_2, \quad A^\times = A + i\vartheta_2 \vartheta_2^* S_0^{-1}.
\]

**Remark 3.4:** The Weyl functions \( \varphi(z) \) and reflection coefficients \( \mathcal{R}(z) \) considered in Theorem 3.2 and Corollary 3.3 are rational and contractive on \( \mathbb{C}_- \) and \( \mathbb{R} \), respectively. Moreover, \( \varphi(-1/z) \) and \( \mathcal{R}(-1/z) \) are strictly proper rational. It would be very interesting to find conditions on the potential \( \{C_k\} \) under which these properties hold (similar to the case of rational Weyl functions of Jacobi matrices). So far only necessary conditions

\[
\lim_{k \to \infty} C_k = I_m, \quad \lim_{k \to \infty} \rho_k = 0
\]

(under assumption \( -i \not\in \sigma(A) \)) are known (see [18, Theorem 3.1]). Here \( \rho_k \) are so called Verblunsky-type coefficients.
3.2. Reflection coefficients: skew-self-adjoint case

In the skew-self-adjoint case, we define the reflection coefficient $R(z)$ in a slightly more general way than in the self-adjoint case. That is, we consider the matrix valued $m_2 \times m$ solution $Y$ of the system (5):

$$Y_k(z) = \left(1 - \frac{i}{z}\right)^k \left[\begin{array}{c} 0 \\ I_{m_2} \end{array}\right] + o(1), \quad k \to \infty,$$

and set

$$R(z) = \left[I_{m_1} \quad 0\right] Y_0(z) \left[\begin{array}{c} 0 \\ I_{m_2} \end{array}\right] Y_0(z)^{-1}. \quad (51)$$

In order to express $R(z)$ via a strongly admissible triple $\{A, S_0, \Pi_0\}$, we derive from (19), (21), (24) and (34) the representation

$$w_A(k, -z) \left[\begin{array}{c} 0 \\ I_{m_2} \end{array}\right] = \left[\begin{array}{c} 0 \\ I_{m_2} \end{array}\right] + i \left[\begin{array}{c} \partial_1^* \left(\tilde{G}(A)^k\right)^* Q_k^{-1}(zI_n + A)^{-1} \partial_2 \\ \partial_2^* Q_k^{-1}(zI_n + A)^{-1} \partial_2 \end{array}\right], \quad (52)$$

where $z \not\in \sigma(-A)$ (i.e. the matrix $zI_n + A$ is invertible) and $\tilde{G}$ is introduced in (36).

Formulas (36) and (52) imply that

$$\lim_{k \to \infty} w_A(k, -z) \left[\begin{array}{c} 0 \\ I_{m_2} \end{array}\right] = \left[\begin{array}{c} 0 \\ I_{m_2} \end{array}\right]. \quad (53)$$

It follows from (33), (50) and (53) that

$$Y_k(z) = \left(1 - \frac{i}{z}\right)^k w_A(k, -z) \left[\begin{array}{c} 0 \\ I_{m_2} \end{array}\right]. \quad (54)$$

Hence, after we take into account (51) and (similar to the self-adjoint case) partition $w_A$ (as in (47)), we obtain

$$R(z) = b(-z) d(-z)^{-1}. \quad (55)$$

On the other hand, according to [6, (3.24)] the Weyl function $\varphi(z)$ of the system (5) is also given by the right-hand side of (55). Thus, the following theorem is proved.

**Theorem 3.5:** Let Dirac system (5) be a system with the strictly pseudo-exponential potential $\{C_k\}$. Then the Weyl function $\varphi(z)$ is the analytic continuation of the reflection coefficient $R(z)$ of this system. More precisely, the reflection coefficient and the Weyl function are given by the same rational matrix function.

The next corollary follows from [6, Theorem 3.8] and Theorem 3.5.

**Corollary 3.6:** Let the potential $\{C_k\}$ be determined by a strongly admissible triple $\{A, S_0, \Pi_0\}$. Then, the reflection coefficient of the skew-self-adjoint Dirac system (5) is given by the formula

$$R(z) = -i \partial_1^* S_0^{-1}(zI_n + A^\times)^{-1} \partial_2, \quad A^\times = A - i \partial_2^* \partial_2 S_0^{-1}. \quad (56)$$
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