Diagrams of Noncommutative $\Phi^3$ Theory from String Theory

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Abstract

Starting from tree and one-loop tachyon amplitudes of open string theory in the presence of a constant $B$-field, we explore two problems. First we show that in the noncommutative field theory limit the amplitudes reduce to tree and one-loop diagrams of the noncommutative $\Phi^3$ theory. Next, we check factorization of the one-loop amplitudes in the long cylinder limit.
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1 Introduction

The fact that string theory can not only be used as a candidate for unified theories but as a powerful tool for computing perturbative field theory amplitudes is known since the early seventies. In the simplest case of a scalar field Scherk showed how to derive tree and one-loop diagrams of the $\Phi^3$ theory from the dual model (pre-string theory) [1, 2]. Later this approach was extensively used in the framework of string theory by many people (see, e.g., [3-8], and [9] for a review). Recently, the idea that the spacetime coordinates do not commute draw much attention (see [10] and a list of references therein). On the one hand, scalar noncommutative field theories were studied in [11-18]. On the other hand, it was realized that noncommutative geometry naturally appears in the framework of open string theory in the presence of a constant $B$-field. The purpose of this paper is to show that tree and one-loop diagrams of the noncommutative $\Phi^3$ theory can be also derived from string amplitudes.

Before starting our discussion of one loop diagrams of the noncommutative $\Phi^3$ theory, we will make a detour and discuss open strings in the presence of a constant $B$-field at the tree level (see, e.g., [10] and references therein).

In this case the world-sheet action is given by

$$S = \frac{1}{4\pi\alpha'} \int_D d^2z \left( g_{ij} \partial_a X^i \partial^a X^j - 2i\pi\alpha' B_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j \right) + \varphi,$$

where $D$ means the string world-sheet, namely a disk. $g_{ij}, B_{ij}, \varphi$ are the constant metric, antisymmetric tensor and dilaton fields, respectively. $X^i$ map the world-sheet to the target space ($Dp$-brane) and $i, j = 1, \ldots, d = p + 1$. The world-sheet indices are denoted by $a, b$. The disk can be, of course,

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conformally mapped to the upper half plane i.e., the region \( \text{Im} \, w \geq 0 \) on the complex plane whose coordinate is \( w \).

To analyze open string theory defined by the world-sheet action (1.1), one first has to determine the propagator. To do so, it is necessary to define the boundary conditions. They are

\[
g(\partial - \bar{\partial}) \mathcal{G}(w, w') + 2\pi \alpha' B(\partial + \bar{\partial}) \mathcal{G}(w, w') = 0 \quad \text{for} \quad \text{Im} \, w = 0.
\]

(1.2)

Here \( \mathcal{G}^{ij}(w, w') = \langle X^i(w) X^j(w') \rangle \) and \( \partial = \partial/\partial w, \bar{\partial} = \partial/\partial \bar{w} \).

Evaluated at boundary points, the propagator with these boundary conditions is \( [19, 10] \)

\[
\mathcal{G}(s, s') = -2\alpha' G^{-1} \ln |s - s'| + \frac{i}{2} \theta \epsilon(s - s') \, ,
\]

(1.3)

where

\[
G = (g - 2\pi \alpha' B) g^{-1} (g + 2\pi \alpha' B) \quad , \quad \theta = -(2\pi \alpha')^2 (g + 2\pi \alpha' B)^{-1} B (g - 2\pi \alpha' B)^{-1} .
\]

(1.4)

\( s = \text{Re} \, w \) and \( \epsilon(s) \) is the step function that is 1 or \(-1\) for positive or negative \( s \). Following [10], we will refer to \( g, B \) as closed string parameters (variables) and \( G, \theta \) as open string parameters. There is an interesting point that we should mention about the tree level. The propagator between boundary points depends only on the open string parameters.

Let us now define the open string tachyonic vertex operator \( [3] \)

\[
V(k) = \int ds \, e^{ik \cdot X} ,
\]

(1.5)

where \( A \cdot B \equiv A_i B^i \). A simple analysis shows that the vertex operator (its integrand) is a primary field of conformal dimension one as long as \( \alpha' k G^{-1} k = 1 \).

For \( M \) open string tachyons, the tree amplitude is given by

\[
A_M = A(k_1, \ldots, k_M) = N_0 (\alpha')^\Delta G_s^M \text{Tr}(\lambda_1 \ldots \lambda_M) \langle V(k_1) \ldots V(k_M) \rangle
\]

\[
+ \text{noncyclic permutations} \quad , \quad \text{where} \quad \langle \ldots \rangle = \int \mathcal{D}X' \, e^{-S} .
\]

(1.6)

Here \( \Delta = \frac{d - 2}{4} M - \frac{d}{2} \). We split the integral into the integral over the zero mode \( X \) and the integral over nonzero modes. We use the following measure for the integral over the zero mode \( [3] \)

\[
\mathcal{V}_d = \int d^{p+1}X \sqrt{G} .
\]

(1.7)

This assumes that the dilaton field is redefined as \( [10] \)

\[
\hat{\varphi} = \varphi + \frac{1}{2} \ln \det \left( G (g + 2\pi \alpha' B)^{-1} \right) .
\]

(1.8)

However, we have defined the open string coupling as \( G_s^2 = e^{\hat{\varphi}} \) rather than \( G_s = e^{\varphi} \) as it was done in [10].

Moreover, by dividing the invariant measure of the Möbius group in (1.6), three vertex operators are fixed to arbitrary positions on the boundary. Each vertex operator is related to a factor \( G_s \) together

\[\text{1 For the sake of simplicity, we use the matrix notations here and below.}\]

\[\text{2 We will give some motivations for such a definition in the next section.}\]

\[\text{3 The zero mode integration includes the X-dependence in } \langle \ldots \rangle \text{ and gives a factor } \delta (\sum k_i).\]
with the Chan-Paton degrees of freedom. Thus the amplitude has the appropriate trace factor. \( \mathcal{N}_0 \) is a
normalization constant (see, e.g., [8]).

In fact, there are two possibilities in taking the limit \( \alpha' \to 0 \), [10]. The first is to do so while keeping the closed string parameters \( g, B \) fixed. As a result, in this case one expects the ordinary \( \Phi^3 \) theory. This
is the standard field theory limit. The second one is to keep the open string theory parameters \( G, \theta \) fixed. The expected result now is the noncommutative \( \Phi^3 \) theory. This is the noncommutative field theory
limit or Seiberg-Witten limit. Since we are interested in the noncommutative field theory we will mainly
discuss the noncommutative field theory limit.

The tree-tachyon amplitudes already show that the noncommutative field theory limit of string am-
plitudes corresponds to tree-diagrams of the noncommutative \( \Phi^3 \) theory with colour indices

\[
\hat{S} = Tr \int d^{p+1}X \sqrt{\text{det} G} \left( \frac{1}{2} \partial_i \Phi \partial^i \Phi + \frac{1}{2} m^2 \Phi^2 + \frac{1}{6} g \Phi \Phi \Phi \right), \quad (1.9)
\]

where the coupling constant \( g \) depends on the open string coupling and string parameter as \( G_s (\alpha')^{(d-6)/4} \).
The \( \ast \)-product is defined by

\[
f(x) \ast \psi(x) = e^{\frac{i}{2} \theta_{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j}} f(x + y) \psi(x + z) \quad . \quad (1.10)
\]

2 Open string one-loop amplitudes in the presence of constant \( B \)-field

2.1 General analysis

The world-sheet action is now given by

\[
S = \frac{1}{4 \alpha'} \int_{C_2} d^2 w \left( g_{ij} \partial_a X^i \partial^a X^j - 2i \pi \alpha' B_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j \right) , \quad (2.1)
\]

where \( C_2 \) denotes the string world-sheet for the one-loop orientable open string i.e., a cylinder (annulus). Note that there is no a constant dilaton field as the Euler characteristic of the cylinder is zero. We
describe \( C_2 \) as the region

\[
0 \leq \text{Re} w \leq 1 \quad , \quad w \equiv w + 2i \tau
\]
on the complex plane whose metric is \( ds^2 = dw d\bar{w} \). A flat annulus with inner radius \( a \) and outer radius \( b \) can be obtained from the cylinder by

\[
z = a \exp (-w \ln q) \quad ,
\]

where the modular parameter of the annulus is given by \( q = a/b = \exp (-\frac{\pi}{\tau}) \).

To analyze open string theory defined by the world-sheet action \( (2.1) \), we take a slight modification of the propagator found in [19]. So, we define the boundary conditions as

\[
g \partial_x \Phi(w, w') - 2 \pi \alpha' i B \partial_y \Phi(w, w') = \left\{ \begin{array}{l}
\frac{\pi \alpha'}{2} \quad \text{for} \quad x = 0 , \\
-\frac{\pi \alpha'}{2} \quad \text{for} \quad x = 1 .
\end{array} \right. \quad (2.2)
\]

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4Both limits assume that the corresponding tachyon mass is treated as a free, but fixed parameter.
Here \( w = x + iy \). The propagator with these boundary conditions is then

\[
\mathcal{G}(w, w') = -\alpha' g^{-1} \ln \left| q^\frac{1}{2}(w' - w) - q^\frac{1}{2}(w - w') \right| - 2\alpha' G^{-1} \sum_{n=1}^\infty \ln \left| 1 - q^2 n - 2 + w + \bar{w}' \right| \left| 1 - q^2 n - w - w' \right| \]

\[
- \alpha' g^{-1} \sum_{n=1}^\infty \ln \left| 1 - q^2 n + w - w' \right| \left| 1 - q^2 n + w + \bar{w}' \right| \left| 1 - q^2 n - 2 + w + \bar{w}' \right| \left| 1 - q^2 n - w - w' \right| \left| 1 - q^2 n - 2 + w + \bar{w}' \right|^{-1} \left| 1 - q^2 n - w - w' \right|^{-1} \]

\[
- \frac{1}{2\pi} \theta \sum_{n=1}^\infty \ln \left( 1 - q^{2n-2+w+\bar{w}'} \right) \left( 1 - q^{2n-w-\bar{w}'} \right) \left( 1 - q^{2n-2+w+\bar{w}'} \right)^{-1} \left( 1 - q^{2n-2+w+\bar{w}'} \right)^{-1} \]  

(2.3)

Since we are interested in open string vertex operators that are inserted on the boundaries of the cylinder, we need to restrict the above propagator to the boundaries to get the relevant propagator. Evaluated at boundary points, it is

\[
\mathcal{G}(y, y') = \frac{1}{2} \alpha' g^{-1} \ln q - 2\alpha' G^{-1} \ln \left[ q^\frac{1}{2} \vartheta_4 \left( \frac{|y - y'|}{2\tau}, \frac{i}{\tau} \right) / D(\tau) \right] \quad \text{for } x \neq x' ,
\]

\[
\mathcal{G}(y, y') = \pm \frac{1}{2} i \theta_\perp (y - y') - 2\alpha' G^{-1} \ln \left[ \vartheta_1 \left( \frac{|y - y'|}{2\tau}, \frac{i}{\tau} \right) / D(\tau) \right] \quad \text{for } x = x' ,
\]

where \( \pm \) correspond to \( x = 1 \) and \( x = 0 \), respectively. \( \vartheta_1 \) and \( \vartheta_4 \) are the Jacobi theta functions. We have also added constants to the propagators. They will be set to a convenient value in the next section. The \( \epsilon_\perp \) function excludes the zero mode contribution. Explicitly, it is given by

\[
\epsilon_\perp(y) = \frac{i}{\pi} \ln \left( \frac{1 - q^{-iy}}{1 - q^{iy}} \right) = \epsilon(y) - \frac{y}{\tau} .
\]

(2.6)

In fact, what we have found above considerably differs from what we had in section 1 where the theory (its open string sector) is completely described in terms of the open string parameters. In our present discussion, we have obtained that the closed string metric \( g \) no longer decouples. This fact has important consequences that we will discuss in the next section.

One basic question we should ask now is how to determine the modular measure \( |d\tau|_B \) of open bosonic string with a constant \( B \)-field in arbitrary dimension. For this it is crucial that the measure factorises as \( f(B)|d\tau|_0 \), where the \( B \)-dependent factor is given by

\[
f(B) = \det(1 + 2\pi \alpha' g^{-1} B) .
\]

It was found by direct calculation in \([9]\). From Eq. (1.4) we get \( \sqrt{\det g f(B)} = \sqrt{\det G} \) (see also \([21]\)). Moreover, it was suggested in \([21]\) that the original theory whose action is given by (1.1) or (2.1) can be also described by a simpler action

\[
S = \frac{1}{4\pi \alpha'} \int_{C^2} d^2 z \, g_{ij} \partial_a X^i \partial^a X^j
\]

(2.7)

while correlation functions of the vertex operators include the build in star products (\( \theta \)-dependence). It is easy to see that it indeed works at the tree-level where the ansatz is simply

\[
\langle V_1 \ast V_2 \ast \cdots \ast V_N \rangle .
\]

(2.8)
In fact, it follows because the corresponding Chan-Paton factor is $Tr(\lambda_1 \ldots \lambda_N)$. After this is understood, it becomes clear that the $\theta$-dependence is more involved on higher loop levels where there are products of traces. For example, in the case of interest the Chan-Paton factor is $Tr(\lambda_1 \ldots \lambda_N)Tr(\lambda_{N+1} \ldots \lambda_M)$. For $N = 0$ or $M - N = 0$ that corresponds to planar diagrams, the ansatz reduces to the above one. This also follows from the corresponding propagator (2.5). It is clear that for $N \neq 0$, $M - N \neq 0$ that corresponds to non-planar diagrams, the ansatz (2.8) has to be modified. In this case the propagators (2.4)-(2.5) say us what to do. However, the measure is completely defined by the action (2.7).

Thus the problem reduced to the old one namely, how to extend the modular measure of open string to arbitrary dimension. This has been much studied to compute perturbative field theory amplitudes via string theory in the $\alpha' \to 0$ limit (see, e.g., [9]). In making our further analysis, we will adopt the proposal in [23] according to which the interpretation of each factor is transparent. The measure is thus

$$\int [d\tau]_0 = \int_0^\infty \frac{d\tau}{\tau} \gamma^2 \eta(i\tau)^{2-d}$$

where $\eta$ is the Dedekind eta function.

The question that immediately arises is whether this measure could be obtained directly from a calculation, for instance, along the lines of [23]. At the tree-level, the commutation relations for the modes of $X^i$ that follow from the action (2.7) are as usual namely,

$$[X^i, p^j] = 2i\alpha' G^{ij} \quad , \quad [\alpha^i_n, \alpha^j_m] = 2\alpha' \delta_{n+m,0} G^{ij}$$

where $p^i \equiv \alpha_0^i$. Moreover, it turns out that the Virasoro generators don’t depend on the zero modes $X$. So, one can formally repeat the standard analysis to build the physical states via states in the Fock space. Moreover, it also means that the tachyonic vertex operator is given by (1.5). $d - 2$ in (2.9) assumes that the reparametrization ghosts are included.

The scattering amplitudes can be defined in two ways. One can modify the ansatz (2.8) to do it consistent with the $\theta$ dependence that follows from the propagators or it can be done by

$$A_{N,M} = A(k_1, \ldots, k_N; k_{N+1}, \ldots, k_M)$$

$$= N_1 (\alpha')^2 G^{M} Tr(\lambda_1 \ldots \lambda_N)Tr(\lambda_{N+1} \ldots \lambda_M) \mathcal{V}_d\langle V(k_1) \ldots V(k_M) \rangle$$

$$+ \text{noncyclic permutations} \quad , \quad \langle \ldots \rangle = \int [d\tau]_0 \int D\mathcal{X}' e^{-S}$$

Here $N$ vertex operators are attached to one boundary and $M - N$ vertex operators to the other one as in figure 1. $N_1$ is a proper normalization constant (see [8]). The correlation functions of exponential operators are simply computed by using the explicit form of the propagator.

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3 A motivation for this analogy is due to matrix (reduced) models where a map from a matrix to a function $A \to a(x)$ leads to $AB \to a(x) * b(x)$ (see, e.g., [23] and references therein).

4 It is well-known that string theory imposes restrictions on possible gauge groups introduced via the Chan-Paton method (see, e.g., [23] and references therein). In the problem at hand the $B$-field may lead to a potential clash with the Chan-Paton method. So the question arises whether such a method is still consistent. We will show in the appendix that this is the case. However there exists the only allowed gauge group namely, $U(N)$.

5 In a general case one also has to add a constant dilaton field as $\chi \hat{\phi}$. Here $\chi$ is the Euler characteristic.
So, the amplitudes are

\[ A_{N,M} = N \left( 2\pi \right)^d \left( \alpha' \right)^k \Delta G_s^M \left[ \left( \lambda_1 \ldots \lambda_N \right) \lambda_{N+1} \ldots \lambda_M \right] \delta \left( \sum_{i=1}^M k_i \right) \int_0^\infty \frac{d\tau}{\tau} \frac{q}{2} \left[ \eta \left( i \tau \right) \right]^{2-d} q^{ \frac{1}{2} \alpha' k g^{-1} k} \]

\[ \times \prod_{i=1}^M \int_0^{y_{i-1}} dy_i \prod_{i=1}^N \prod_{j=N+1}^M \left[ q^{-1} \frac{\delta^1 \left( \frac{y_{i+1}}{2\tau}, i \tau \right)}{D(\tau)} \right]^{2\alpha' k i G^{-1} k_j} \]

\[ \times \prod_{N+1}^M e^{ \frac{1}{2} \iota \epsilon \left( y_{ij} \right) k_i k_j} \left[ \frac{\delta^1 \left( \frac{y_{i+1}}{2\tau}, i \tau \right)}{D(\tau)} \right]^{2\alpha' k i G^{-1} k_j} + \text{noncyclic permutations} \]

(2.12)

where \( y_{ij} = y_i - y_j \) and \( k = \sum_{i=1}^N k_i \).

\[ t = 2\pi \alpha' \tau \quad , \quad \nu_i = y_i / 2\tau \quad (2.13) \]

finite.

\[ ^8 \text{We disregard the pinching configurations that result in the one-particle reducible diagrams.} \]
In this limit, the propagators become

\[ \mathcal{G}(\nu, \nu') = \frac{1}{4t} \theta G\theta + G^{-1}t \left( (\nu - \nu')^2 - |\nu - \nu'| \right) \quad \text{for} \quad x \neq x' \quad , \quad \text{(2.14)} \]

\[ \mathcal{G}(\nu, \nu') = \pm \frac{1}{2} i \theta e_\perp (\nu - \nu') + G^{-1}t \left( (\nu - \nu')^2 - |\nu - \nu'| \right) \quad \text{for} \quad x = x' \quad . \quad \text{(2.15)} \]

At this point a couple of comments are in order.

(i) The \( \epsilon_\perp \) function defined by Eq. (2.16) is related via \( \frac{\partial}{\partial y} \epsilon_\perp (y) = 2\delta_\perp (y) \) to the delta function that excludes the zero mode i.e., \( \delta_\perp (y) = \delta(y) - \frac{1}{2}\tau \). Naively, the difference between these \( \epsilon \) functions disappears in the \( \tau \to \infty \) limit. In fact, this is not the case because the last term survives as far as the rescaling (2.13) is taken into account. Thus

\[ \epsilon_\perp (\nu) = \epsilon (\nu) - 2\nu \quad . \quad \text{(2.16)} \]

(ii) The difference between \( \epsilon \) and \( \epsilon_\perp \) indeed disappears for all planar amplitudes as well as tree level amplitudes. In this case the zero mode contribution to the amplitudes \( \sum_{i < j} p_i \theta p_j \nu_{ij} \) vanishes due to the momentum conservation \( \sum p_i = 0 \).

Thus, in the field theory limit the amplitudes (2.12) become

\[ A_{N,M} = N!^g M^g \text{Tr}(\lambda_1 \cdots \lambda_N) \text{Tr}(\lambda_{N+1} \cdots \lambda_M) \delta \left( \sum_{i=1}^M k_i \right) \prod_{i < j}^N e^{-\frac{i}{2}k_i \theta k_j} \prod_{N+1}^M e^{\frac{i}{2}k_i \theta k_j} \int_0^\infty \frac{dt}{t} t^{M-\frac{d}{2}} e^{-m^2 t - k_1 \theta k} \]

\[ \times \prod_i^M \int_0^{\nu_1-1} d\nu_i \prod_{i < j}^M e^{i(\nu_i^2 - \nu^2_j) k_i G^{-1} k_j} \prod_{i < j}^N e^{i \nu_i k_i \theta k_j} \prod_{N+1}^M e^{-i \nu_i k_i \theta k_j} + \text{noncyclic permutations} \quad . \quad \text{(2.17)} \]

where \( k \circ k = -\frac{i}{2} k \theta G \theta k \). To get this form we introduced the mass for the open string tachyon as \( m^2 = (2 - d)/24\alpha^3 \) and used the relation between the open string coupling constant and the \( \Phi^3 \) theory coupling constant.

We will conclude this subsection with a couple of examples to illustrate the use of the string amplitudes in practical calculations of one loop Feynman diagrams of the noncommutative \( \Phi^3 \) theory.

**Example 1.** As a warmup, let us consider the simplest nonplanar amplitude and, as by-product, reproduce a result of [13]. The amplitude (modulo a normalization constant) is given by

\[ A_{1,2} = g^2 \text{Tr}(\lambda_1) \text{Tr}(\lambda_2) \delta (k_1 + k_2) \int_0^\infty dt t^{1-\frac{d}{2}} e^{-m^2 t - k_1 \circ k_1 / t} \int_0^1 d\nu_1 e^{i(\nu_1^2 - \nu^2_1) k_1 G^{-1} k_1} \quad . \quad \text{(2.18)} \]

Here we fix the translational invariance on the cylinder by setting the second vertex operator at the origin i.e., \( \nu_2 = 0 \).

The next step in finding the correspondence with the field theory diagram is to introduce new integration variables which correspond to the Schwinger parameters

\[ \alpha_1 = t \nu_1 \quad , \quad \alpha_2 = t (1 - \nu_1) \quad . \quad \text{(2.19)} \]

In terms of these variables, the amplitude is written as

\[ A_{1,2} = g^2 \text{Tr}(\lambda_1) \text{Tr}(\lambda_2) \delta (k_1 + k_2) \int_0^\infty \int_0^\infty \frac{d\alpha_1 d\alpha_2}{(\alpha_1 + \alpha_2)^2} e^{-m^2(\alpha_1 + \alpha_2)} e^{\frac{-k_1 \circ k_1}{(\alpha_1 + \alpha_2)}} e^{\frac{-\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)}} k_1 G^{-1} k_1 \quad . \quad \text{(2.20)} \]

It is now clear that what we have found is exactly the simplest nonplanar Feynman diagram of the noncommutative \( \Phi^3 \) theory as shown in Fig.2.

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\[^9\text{We have set } D(\tau) = \tau^{-1} [\eta(i/\tau)]^2.\]
So, the expression (2.21) becomes

\[ g^4 \text{Tr}(\lambda_1 \lambda_2) \text{Tr}(\lambda_3 \lambda_4) \delta \left( \sum_{i=1}^{4} k_i \right) e^{-\frac{i}{2} k_1 \theta k_2 + \frac{i}{2} k_3 \theta k_4} \int_{0}^{\infty} dt \frac{d}{dt} e^{-\frac{m^2}{2} t - k\theta k/t} \]

\times \int_{0}^{1} d\nu_1 \int_{0}^{\nu_2} d\nu_2 \int_{0}^{\nu_2} d\nu_3 e^{i \nu_1 k_1 \theta k_2 - i \nu_2 k_3 \theta k_4} \prod_{i=1}^{3} e^{|i(\nu_i^2 - \nu_i)k_i G^{-1}k_i|} \prod_{i<j}^{3} e^{2i(\nu_i - \nu_j)k_i G^{-1}k_j} .

The Schwinger parameters can be introduced as

\[ \alpha_1 = t \nu_1 \quad , \quad \alpha_2 = t \nu_2 \quad , \quad \alpha_3 = t \nu_3 \quad , \quad \alpha_4 = t(1 - \nu_1) . \]

So, the expression (2.21) becomes

\[ g^4 \text{Tr}(\lambda_1 \lambda_2) \text{Tr}(\lambda_3 \lambda_4) \delta \left( \sum_{i=1}^{4} k_i \right) e^{-\frac{i}{2} k_1 \theta k_2 + \frac{i}{2} k_3 \theta k_4} \prod_{i=1}^{4} \int_{0}^{\infty} d\alpha_i \alpha^\frac{d}{2} e^{-\frac{m^2}{2} \alpha} e^{-\frac{i}{2} \alpha k \theta k} e_0^\alpha(\alpha_1 \theta k_2 - \alpha_3 \theta k_4) \]

\times \prod_{i=1}^{4} e^{\frac{1}{2} \alpha(\alpha_1 \theta k_1 G^{-1}k_1 - \alpha_1 \theta k_2 G^{-1}k_2 - \alpha_2 \theta k_3 G^{-1}k_3 - \alpha_3 \theta k_4 G^{-1}k_4 - \alpha_1 \theta k_2 G^{-1}(k_2 + k_3) - \alpha_2 \theta k_3 G^{-1}k_3)} ,

where \( \alpha = \sum_{i=1}^{4} \alpha_i \). It is straightforward to get the Schwinger representation for the other terms. As a result, the amplitude \( A_{2,4} \) reduces to a sum of three nonplanar Feynman diagrams of the noncommutative \( \Phi^4 \) theory as shown in fig.3.

\[ ^{10}H \text{ means the Heaviside step function.} \]
2.3 $\tau \to 0$ limit

Now we consider another limit: $\tau \to 0$ while keeping all other parameters fixed. It is well-known that for $B = 0$ this is the open string UV limit. Moreover, it is interpreted as a long-distance effect because the leading asymptotics are given by the closed string tachyon as well as the lightest closed string states. So, our purpose is to analyze what happens in the presence of the $B$-field.

In this limit, the propagators (2.4)-(2.5) become

$$G(y,y') = -\frac{\pi}{2} i \theta_{\perp} (y - y') + 2 i \alpha' G^{-1} P(y-y')$$

for $x \neq x'$.

$$G(y,y') = \pm \frac{1}{2} i \theta_{\perp} (y - y') + 2 i \alpha' G^{-1} P(y-y')$$

for $x = x'$.

Here $P(y) = \sum_{n=1}^{\infty} \frac{1}{n} \cos \frac{\pi}{\tau} n y$. As in the $\tau \to \infty$ case that we discussed first, it is useful to define new variables

$$t = \frac{\pi}{2} \alpha', \quad \nu_i = y_i/2 \tau.$$

Then we get the amplitudes (modulo a normalization constant)

$$A_{N,M} = (\alpha')^{\Delta - 1} \delta \left( \sum_{i=1}^{M} k_i \right) \left[ G_s^N \text{Tr}(\lambda_1 \ldots \lambda_N) \prod_{i=1}^{M-1} \int_{\nu_i=0}^{\nu_i-1} d\nu_i \prod_{i<j} e^{-\frac{1}{2} \theta_{\perp} (\nu_i \theta \nu_j)} e^{-2 \alpha' k_i k_j} P(\nu_i) \right] \nu_N = 0$$

$$\times \frac{1}{k g^{-1} k + m^2} \left[ G_s^{M-N} \text{Tr}(\lambda_{N+1} \ldots \lambda_M) \prod_{i=N+1}^{M-1} \int_{\nu_i=0}^{\nu_i-1} d\nu_i \prod_{N+1}^{M} e^{\frac{1}{2} \theta_{\perp} (\nu_i \theta \nu_j)} e^{-2 \alpha' k_i k_j} P(\nu_i) \right] \nu_M = 0 + \text{noncyclic permutations}.$$

The translational invariance is now fixed by setting $\nu_N = \nu_M = 0$. We also introduce the mass for the closed string tachyon as $m^2 = (2 - d)/6 \alpha'$.

What we see from the above is that in the $\tau \to 0$ limit for $B \neq 0$ the amplitudes factorize as in the case $B = 0$. There is no new effect here. So, the interpretation is the standard one as a long-distance effect with the asymptotics due to the closed string states.

3 Concluding Comments

What we have learned is that in the noncommutative field theory limit of open string theory with the $B$-field at the one loop level there is a signal of closed string sector. To be more precise, we found that
\theta appears not only via the \(*\)-product as it does at the tree level but via \(-\frac{1}{4}\theta G \theta\) that corresponds to the closed string metric \(g\). A recent analysis of noncommutative field theories assumes that some closed string modes already appear there \([18]\). We do not see this explicitly in our analysis of the noncommutative field theory limit where we found only the closed string parameters rather than the modes. However, to be cautious, we should mention that we did not discuss singularities and their regularization.

It is important to emphasize that the factor \(q^{\frac{1}{2} \alpha' k_g^{-1} k}\), or equivalently the \(g\)-dependence of amplitudes, which is crucial for the noncommutative field theory limit of nonplanar diagrams is universal. It does not depend on the kind of vertex operators. This is clear because the effect is due to the first term in the propagator \((2.4)\). It is a constant, so contributions come only from exponents.

From the physical point of view it is more interesting to consider the noncommutative gauge theory. In fact, at the one-loop level it can be done along the lines of the present paper by considering the vertex operator for a gauge field

\[
V(\xi, k) = \int ds \xi \cdot \partial_s X e^{ik \cdot X}
\]

instead of the tachyon operator \((1.5)\). Then, the corresponding amplitudes are computed by using the propagators \((2.4)-(2.5)\) within the point splitting renormalization scheme \([10]\). We hope to return to this important problem in the near future \([24]\).

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Appendix

The purpose of this appendix is to show how restrictions on gauge groups arise within open string theory in the presence of the \(B\)-field. In general, there are a variety of ways to do this. Here we will focus on the classical recipe \([25]\) which is based on simple factorization properties of string amplitudes. So, let us take the tree amplitude \((1.6)\) and look at its factorization \(A_M \rightarrow A_P A_{M-P}\). It is well-known that the only novelty due to the \(B\)-field is a phase factor

\[
P^M_{1} = \prod_{i<j}^{M} e^{-\frac{i}{2} \theta k_i \theta k_j}
\]  

that appears in the expression for the amplitude. The crucial fact for what follows is that \(P^M_{1}\) due to momentum conservation obeys the factorization relation

\[
P^M_{1} = P^P_{1} P^{M}_{P+1}
\]

With above formula the generalization of the classical analysis is straightforward. As a result, Eq. (6.1.11)
of [25] becomes

\[
Tr \left[ (\lambda_1 \ldots \lambda_P \mathcal{P}^P_1 - (-)^P \lambda_P \ldots \lambda_1 (\mathcal{P}^P_1)^{-1}) \left( \lambda_{P+1} \ldots \lambda_M \mathcal{P}^M_{P+1} - (-)^{M-P} \lambda_M \ldots \lambda_{P+1} (\mathcal{P}^M_{P+1})^{-1} \right) \right] \\
= \sum_{\alpha} Tr \left[ (\lambda_1 \ldots \lambda_P \mathcal{P}^P_1 - (-)^P \lambda_P \ldots \lambda_1 (\mathcal{P}^P_1)^{-1}) \lambda_\alpha \right] \\
\times Tr \left[ \lambda_\alpha^T \left( \lambda_{P+1} \ldots \lambda_M \mathcal{P}^M_{P+1} - (-)^{M-P} \lambda_M \ldots \lambda_{P+1} (\mathcal{P}^M_{P+1})^{-1} \right) \right].
\]

(A.3)

This equation is satisfied if a matrix

\[
\lambda = \lambda_1 \ldots \lambda_P \mathcal{P}^P_1 - (-)^P \lambda_P \ldots \lambda_1 (\mathcal{P}^P_1)^{-1}
\]

belongs to the algebra of the matrices \( \lambda_i \). It is known that in the case of \( \theta = 0 \) the allowed gauge groups are \( U(n) \), \( SO(n) \) and \( USp(2n) \). To see what survives for nonzero \( \theta \) let us specialize to \( P = 2 \). Then the following direct algebra

\[
\lambda^\dagger = \left( \lambda_1 \lambda_2 e^{-\frac{i}{2} k_1 \theta k_2} - \lambda_2 \lambda_1 e^{\frac{i}{2} k_1 \theta k_2} \right) = -\lambda \quad \text{for} \quad \lambda_i \in u(n),
\]

(A.5)

\[
\lambda^T = - \left( \lambda_1 \lambda_2 e^{\frac{i}{2} k_1 \theta k_2} - \lambda_2 \lambda_1 e^{-\frac{i}{2} k_1 \theta k_2} \right) \neq -\lambda \quad \text{for} \quad \lambda_i \in so(n),
\]

(A.6)

\[
\lambda^T = s^{-1} \left( \lambda_2 \lambda_1 e^{-\frac{i}{2} k_1 \theta k_2} - \lambda_1 \lambda_2 e^{\frac{i}{2} k_1 \theta k_2} \right) s \neq -s^{-1} \lambda s \quad \text{for} \quad \lambda_i \in usp(n), \text{ where } \lambda^T_i = -s^{-1} \lambda_i s
\]

(A.7)

shows that the only survivor is \( U(n) \). The latter is in harmony with the result obtained within noncommutative gauge theory claiming that the noncommutative gauge transformation is consistent only for the unitary group \( U(n) \) (see e.g., [26]).

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