Application of Bayesian networks and Petri nets apparatus for the study of projects implementation calendar plans

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Abstract. Time schedules reflect the main aspects of the projects implementation related to the work completion time, its financing, resource provision and risk distribution. This article proposes mechanisms for studying the time schedules of projects implemented in conditions of risk and uncertainty. The mechanisms developed in the framework of the study are based on the use of colored Petri and Bayes nets apparatus. The use of these tools allows to simulate the execution of project work under certain evidence, restrictions or rules for the allocation of resources between them. Colored Petri nets with predicative transitions allow simulating the overall dynamics of the project. Bayesian networks are used to assess the risks of individual project activities based on available evidence. Based on the results of the simulation modelling, the time parameters of the events and activities of the project are calculated, and the possibility of the project being completed within the scheduled time is assessed. The work examines different options for the distribution of project resources, taking into account and not taking into account competition between works and a possible shortage of resources.

1. Introduction

Scheduling is one of the most important stages of project preparation. The time schedule reflects the approach, duration, material and production resources used at each stage [1]. When drawing up schedules, the following are considered known: the sequence of work, the characteristics of each work, the relationship and restrictions on their implementation; deadlines for the start and / or end of work on the complex as a whole and / or on individual objects; the need in resources in general and by periods of work execution; the ability to use specific resources at various sites; productivity of available labor and technical resources. Scheduling aims to produce a plan that meets all requirements and is best in relation to a certain performance criterion.

One of the well-tested and proven methods of scheduling is network planning, based on the concept of a network model (network schedule). The network model is a weighted directed graph, the nodes of which correspond to events, and the arcs correspond to the project work for which the schedule is drawn up. An event is the end of one or more project activities for which the event-node is final. An example of a network diagram is shown in Figure 1. This network diagram depicts a project containing 21 jobs and 13 events associated with the completion of one or more jobs.
To study the temporal characteristics of projects represented by network models, the critical path method is most often used, the main task of which is to find the critical path in the graph. The critical path of a network graph means the path from the initial work start to the final end of all work that has the longest duration of all full paths (a full path is any path from the start to the end). The jobs that are on the critical path are called critical. The estimate of the total critical path duration is determined by the smallest total project activities duration. Reducing this estimate is possible only by reducing the critical path duration. Consequently, any delay in the work execution on the critical path will increase the duration of the entire project.

For each work, the concept of a time reserve is introduced, by which the difference between the moments (time is measured from the beginning of the project in certain units of measurement) of the late and early finish (beginning) time of work is understood. The existence of a time reserve at the work allows to delay this work within the reserve and this will not affect the entire project completion date. Work on the critical path has zero time reserves. In some works, a complete path is called critical if it has a minimum total (over all works of a given path) time reserve.

Here is an algorithm for finding the critical path in a network graph.

1. For each event $i (i = 1, \ldots, m)$ calculate the early finish time using the following formula:

$$\gamma_p(i) = \max_{j \in \Gamma_i} (\gamma_p(j) + \gamma(j, i)),$$

where $\gamma(j, i)$ — is the work execution time ($j, i$).

2. For an end event, the late finish time $\gamma_n(m)$ is considered to be equal to the early finish time or the specified scheduled time. For each event except the end one, calculate the late time of the event using the following formula:

$$\gamma_n(i) = \min_{j \in \Gamma_i} (\gamma_n(j) - \gamma(i, j)).$$

3. For each job $(i, j)$ calculate the early and late start and finish time of work:

$$\gamma_{es}(i, j) = \gamma_p(i);$$
$$\gamma_{ef}(i, j) = \gamma_p(i) + \gamma(i, j);$$
$$\gamma_{ls}(i, j) = \gamma_n(j);$$
$$\gamma_{lf}(i, j) = \gamma_n(j) - \gamma(i, j).$$
4. For each event, calculate the time reserve using the formula:

\[ R(i) = \gamma_n(i) - \gamma_p(i); \]

5. For each job, calculate the total time reserve using the formula:

\[ R_{n(i,j)} = \gamma_n(j) - \gamma_p(i) - \gamma(i,j); \]

6. Determine the critical path of work.

For the network graph shown in Figure 1, the value of the critical execution time for jobs lying on the critical path is 301, and the critical path itself consists of jobs (0,1), (1,5), (5,6), (6,7), (7,11), (11,12), (12,13). The time reserve for all activities on the critical path is 0. Finding the critical path allows you to manage project resources. In particular, it is possible to move resources in a directed manner from jobs that are not on the critical path to jobs that are on the critical path in order to increase the productivity of the latter and reduce project implementation time.

In jobs [2–4], algorithms of the critical path are investigated for various types of data set-up (ratio scales, interval scales, fuzzy and linguistic scales) about jobs, deterministic and stochastic time parameters of project execution.

Additional opportunities for studying network graphs are opened by the Petri nets [5] and Bayesian nets apparatus [6]. In particular, using these tools, one can model the stochastic characteristics of jobs and events, take into account the evidence relating to individual jobs, simulate the situations of the cycles presence in the performance of jobs, take into account the quality of the job, and model the resources redistribution between jobs.

2. The network plan execution process simulation with the colored Petri and Bayesian networks

Petri net is an illustrative formalized model for describing the behavior of parallel systems with asynchronous interactions [5]. In a compact form it reflects the relationships structure between the elements of the system and the dynamics of changes in its states for given initial conditions. Graphically, a Petri net is a directed bipartite graph, the nodes of which are places and transitions, and each arc of the graph leads from a place to a transition or from a transition to a place. Places are depicted as circles, and transitions as two parallel lines.

Classical Petri nets are specified as a tuple:

\[ SP = \langle P, T, E, M_0 \rangle, \]

where \( P \) — set of places (we denote the number of positions as \( N \)), \( T \) — is a set of transitions (we denote the number of transitions as \( H \)), \( E \) — is a set of arcs leading either from places to transitions or from transitions to places, \( M_0 = (m_0^1, \ldots, m_0^N) \) — is an initial marking, describing in the simplest case the number of marks or chips in each place at the initial moment of time. For each transition \( t_j \), you can enter a set of input and output places, which we denote as \( I^I(t_j) \) and \( I^O(t_j) \) respectively. Similarly, for each place \( p_i \), you can enter a set of input and output transitions \( J^I(p_i) \) and \( J^O(p_i) \).

Dynamic changes in the Petri net are carried out due to the mechanism for changing the marking (starting from the initial marking) and the mechanism for triggering transitions. The transition is triggered at the moment when it comes into an active state. The active transition condition is described by the requirements for marking at all input places \( I^I(t_j) \). Transition triggering in different concepts of Petri nets is carried out in different ways. In some concepts, the transition firing consists in moving one marker from each input place to the output, in some concepts in moving the number of markers equal to the multiplicity of the input arcs leading
from the input place to the transition. Thus, the dynamics of the Petri net is described by transitions from one admissible marking to another:

\[ M^l = (m^l_1, \ldots, m^l_N) \rightarrow M^d = (m^d_1, \ldots, m^d_N). \]

Along with classical Petri nets, colored, predicative, temporary, stochastic Petri nets are also considered. When using colored Petri nets, markers are set by variables (colors), and the multiplicities of arcs are functions of these variables. Predicative networks are a kind of colored networks; in predicative networks, each transition is associated with a predicate from the values of colors or functions assigned to arcs, the truth of which leads to the triggering of the transition. Temporary networks are also a kind of colored networks, in which either the time of the marker staying in them is attributed to the places or the duration of the firing is attributed to the transitions. In stochastic networks, each transition is characterized by the probability of its activation in a certain time.

Bayesian networks are used to estimate the posterior probabilities of various outcomes and are directed acyclic graphs with a set of nodes \( X \), which are discrete or continuous random variables, and with many arcs \( G \) reflecting the parent-child relationship. The set of parent nodes for the set of nodes \( Y \) is considered as the set:

\[ \text{Parents}(Y) = \{ x : \exists y \in Y, \exists (x, y) \in G \}. \]

The set of children for a set of nodes \( Y \) is defined as:

\[ \text{Children}(Y) = \{ x : \exists y \in Y, \exists (y, x) \in G \}. \]

Each node of a Bayesian network with discrete random variables is characterized by a table of conditional probabilities. Each node of a Bayesian network with continuous random variables is characterized by a distribution of conditional probabilities. An example of the simplest network with discrete random variables is shown in Figure 2.

![Bayesian Network Example](image)

| C  | P(S=t) |
|----|--------|
| t  | 0.1    |
| f  | 0.5    |

| C  | P(R=t) |
|----|--------|
| t  | 0.8    |
| f  | 0.2    |

| C | R | P(W=t) |
|---|---|--------|
| t | t | 0.99   |
| t | f | 0.9    |
| f | t | 0.9    |
| f | f | 0.0    |

**Figure 2.** An example of the structure of a Bayesian network with tables of conditional probabilities \((t = \text{true}, f = \text{false})\).
When modeling processes using Bayesian networks, the hypothesis of conditional independence is considered fulfilled, which is the assumption that each host \( y \), for known values of the parents \( \text{Parents}(y) \), does not depend on any set such that \( x \notin Y \) and \( X \not\subseteq Y \).

In Bayesian networks, the joint probability of the distribution of all nodes is the product of the conditional probabilities of each host for the known values of the parents:

\[
P(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | \text{Parents}(X_i)).
\]

The main task that Bayesian networks can solve is the problem of probabilistic inference. The essence of this problem is to calculate the distribution of posterior probabilities for a set of query variables \( X = \{X_1, X_2, \ldots, X_n\} \), if an event is given in which certain values are assigned to a set of evidence variables \( E = \{E_1, E_2, \ldots, E_m\} \), \( E_1 = e_1, E_2 = e_2, \ldots, E_n = e_n \) (the set of all nodes of a Bayesian network can be represented as a sum of three sets \( X \cup E \cup Y \), where \( Y = \{Y_1, Y_2, \ldots, Y_p\} \) is a set of hidden variables). The desired posterior probability distribution \( P(X | E) \) can be calculated \( P(X | E) = \alpha \sum_{Y} P(X, E, Y) \), where \( \alpha \) — is the normalizing factor. Exact search, variable elimination, and clustering algorithms can be used to calculate \( P(X | E) \). For multiply connected Bayesian networks, it is very difficult to implement accurate probabilistic inference algorithms. One of the well-proven approximation algorithms is the likelihood weighting algorithm. The likelihood weighting algorithm is performed by generating samples of Bayesian network state variable values that are weighted according to the likelihood of their correspondence to the observed evidence variables. Each sample is formed separately, passing sequentially in topological order from one state variable to the next along the entire network. When applying the likelihood weighting algorithm to a complex network, problems arise, which are connected to the temporal and spatial requirements increase with each update. In the direction of solving resource problems, a modification of the weighing algorithm is used, taking into account the likelihood — the particle filtering algorithm.

In this paper, an algorithm is proposed that allows to apply the Petri and Bayesian nets apparatus to analyze the critical path in the case of the presence of risk and uncertainty factors in the project implementation.

When depicting a Petri net for net graphs, the events are depicted as transitions, and works as Petri net places. A place that characterizes the start of the project work is used additionally. In the simplest version, places can be assigned a certain time resource characterizing the time required to complete the corresponding work. Transitions are triggered only when work is done in all places from which arcs lead to this transition.

The simplest Petri net for the net graph shown in Figure 1 is shown in Figure 3.

In this work, it is proposed to use a more complex version, variable markers (colors) are assigned to the Petri net nodes, among which there is a special activity flag marker (performing work in a given place) and two special markers, the first of which indicates the time when the place comes into the active state — the initial time marker, the second is the real time from the beginning of the project execution — the real time marker. The active flag marker, the initial time marker and the real-time marker of a certain place activate their values at the moment the transition is triggered leading to this place. When the transition is triggered, the initial time marker is set to the real time marker. During the work execution, the real-time marker increases in real time, counted from the beginning of the project. A special marker is created for each job based on modeling the influence of uncertainties on the process of performing the job based on a Bayesian network. Using the evidence available by the time the place comes into the active state by the probabilistic inference method, implemented using the weighing algorithm, taking into account the likelihood, the work risk level is calculated. The Bayesian network nodes are various random factors that directly or indirectly affect the process of performing work.
Figure 3. One of the variants of the Petri net for the net graph shown in Figure 1.

are special risk banks for various types of projects, on the basis of which the structure of the Bayesian network can be formed.

In the framework of our study, we will assume that the remaining markers are some functions of the amount of each type of resources allocated for the work execution \( r_1, \ldots, r_L \). Suppose that in the process of the initial planning of resource allocation by work, each resource is considered as a random variable distributed according to the normal law with the distribution density:

\[
    f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x - m)^2}{2\sigma^2}}.
\]

For places intervals for each type of resource are set for places \( r_1, \ldots, r_L \):

\[
    (r_1^0 - \Delta r_1, r_1^0 + \Delta r_1), \ldots, (r_L^0 - \Delta r_L, r_L^0 + \Delta r_L),
\]

where \( r_1^0, \ldots, r_L^0 \) — average values of the analyzed quantities. These intervals characterize the range within which the allocation of resources for the work is planned.

We will take three functions as mandatory markers for each position: the time of work \( T(r_1, \ldots, r_L) \) and the quality of work \( K(r_1, \ldots, r_L) \), which are functions \( L \) of random resources \( r_1, \ldots, r_L \). Let us dwell a little on the methods of constructing these functions. The function \( T(r_1, \ldots, r_L) \) can be built either on the basis of certain regulatory documents (in particular, in construction) or on the basis of a retrospective base, using methods of statistical data analysis to build regression models or machine learning methods. Regarding the type of functions that describe the execution time, two statements can be formulated.

Statement 1. If there is no synergetic effect in the production system in the course of work execution, then the dependence of the work execution time — the number of resources used \( n \) — leads to a reduction in the execution time by the same amount.
**Statement 2.** In production systems, the dependence of the work execution time — the amount of resources used function has a concave character in the case when the first differences increase with an increase in the amount of the distributed resource, and a convex character — when they decrease.

These statements can be used when choosing the type of function that approximates the dependence of the work execution time on the amount of resources used.

Let us dwell in more detail on the methods of constructing the work performance quality function $K(r_1, \ldots, r_L)$. The work and the principles of its assessment are fairly well stated in ISO standards. The following definition is used as the main definition of quality: “Quality is the degree to which a set of its own distinctive properties (characteristics) fulfills the requirements (needs or expectations) of interested parties that are established, usually assumed or obligatory”. In accordance with this definition, the work performance quality can be considered as a certain function with values from the segment $[0, 1]$. In the process of constructing the quality function, two tasks are solved in the work: determining the quality for each individual resource and determining the quality for a set of resources and, if necessary, for a group of works.

Currently, there is a significant number of methods in which analytical formulas are used to calculate the quality assessment of a certain resource, in the construction of which it is assumed that the quality assessment depends on the ratio of the available resource with the basic, ideal, expected amount:

$$K_j = \varphi(r_j, r_j^{\text{basic}}), \quad K_j = \varphi(r_j, r_j^{\text{expected}}).$$

Most often, when assessing the quality of individual resources, the following types of dependencies are used:

$$K_j = \frac{r_j}{r_j^{\text{basic}}}, \quad K_j = \frac{r_j - r_j^{\text{min}}}{r_j^{\text{basic}} - r_j^{\text{min}}}, \quad K_j = e^{-(r_j^0)^{m_j}},$$

where $0 < m_j \leq M$ — positive constant,

$$r^0 = \frac{2r_j - (r_j^{\text{max}} + r_j^{\text{min}})}{r_j^{\text{max}} - r_j^{\text{min}}}$$

— linear function of $r_j$.

When assessing the resources quality and, in general, the work quality, you can use the I. B. Russman approach [7, 8], based on calculating the difficulty of achieving goals. The idea of assessing the difficulty of achieving a goal arose from intuitive considerations that when other things being equal, the lower the quality of resources and the higher the requirements for them are, the more difficult it is to obtain a result of a certain quality.

Before introducing the difficulty of achieving goals, we introduce a value $0 < \mu_j < 1$ — a quality estimate of the resource — $j$. From the point of view of achieving goals, not all resources quality values are achievable, therefore, the concept of resource quality requirements is introduced — $\varepsilon_j$, which satisfies the conditions: $0 < \varepsilon_j < 1, \varepsilon_j < \mu_j$.

The difficulty of achieving the goal $d_j = d_j(\mu_j, \varepsilon_j) = d(\mu, \varepsilon)$ as the resources quality and the requirements for them function should have the following properties [8]:

a) the difficulty of achieving the result is maximum with the minimum resource quality $d(\mu, \varepsilon) = 1$, where $\varepsilon = \mu$.

b) the difficulty of achieving the result is minimal with the extremely high resource quality $d(\mu, \varepsilon) = 0$, where $\mu = 1, \mu > \varepsilon$. 

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c) \( d(\mu, \varepsilon) = 0 \), where \( \mu > 0, \varepsilon = 0 \).

As a function satisfying these properties, the following function can be used [8]:

\[
d = \frac{\varepsilon(1 - \mu)}{\mu - \varepsilon}, \quad d(0, 0) = 0, \quad d(1, 1) = 1.
\]

Since in the performance of any work several resources are used, the quality assessment should be an integral function of the particular difficulties for each resource. Within the framework of this study, the following function will be used:

\[
D = 1 - \prod_{j=1}^{L} (1 - d_j).
\]

The different resources importance can be taken into account by introducing importance factors into the formula:

\[
D = 1 - \prod_{j=1}^{L} (1 - d_j)^{\beta_j}.
\]

The difficulty of achieving a result is understood as a certain measure of poor quality or a risk assessment of low quality resources use, therefore the work quality can be assessed as: \( K = 1 - d \).

In the future, we need to assess the quality of a group of works. To define it, let us introduce into consideration operations on difficulties:

1. a generalized addition operation:

\[
d_1 + d_2 = d_1 + d_2 - d_1 \cdot d_2;
\]

2. a generalized multiplication operation by a non-negative number \( \lambda \):

\[
\lambda \otimes d = 1 - (1 - d)^\lambda.
\]

3. a generalized exponentiation operation:

\[
d^\lambda = 1 - e^{-\left(\ln\frac{1}{1-d}\right)^\lambda}.
\]

Using the introduced operations, it is possible to construct the quality functions of the work groups:

\[
\Phi(D_1, \ldots, D_G) = \max \left( \frac{1}{\lambda_1} \otimes D_1, \ldots, \frac{1}{\lambda_G} \otimes D_G \right);
\]

\[
\Phi(D_1, \ldots, D_G) = \lambda_1 D_1 \oplus \cdots \oplus \lambda_G D_G;
\]

\[
\Phi(D_1, \ldots, D_G) = D_0 \otimes D_1^{\lambda_1} \otimes \cdots \otimes D_G^{\lambda_G};
\]

\[
\Phi(D_1, \ldots, D_G) = \left( \lambda_1 \otimes D_1^{-\alpha} \oplus \cdots \oplus \lambda_G \otimes D_G^{-\alpha} \right)^{-\frac{1}{\pi}}.
\]

Thus, given the known resources allocated to perform the work located in a specific place, time \( T(r_1, \ldots, r_L) \) and quality \( K(r_1, \ldots, r_L) \) markers can be calculated. We will assume that these functions \( F(r_1, \ldots, r_L) \) depending on the resources \( r_1, \ldots, r_L \) are distributed according to the normal law. It is necessary to determine the parameters of the corresponding distribution law based on the quantities characteristics \( r_1, \ldots, r_L \). If the intervals for values changes are specified \( r_1, \ldots, r_L \):

\[
((r_1^0 - \Delta r_1, r_1^0 + \Delta r_1), \ldots, (r_L^0 - \Delta r_L, r_L^0 + \Delta r_L)),
\]
where \( r_1^0, \ldots, r_L^0 \) — the average values of the analyzed quantities, then to obtain the range of function variation \( F(r_1, \ldots, r_L) \) the following estimate can be used:

\[
\Delta F \leq \sum_{i=1}^{L} \left| \frac{\partial F(r_1^0, \ldots, r_L^0)}{\partial r_i} \right| \Delta r_i.
\]

After having found the range of value variation \( F(r_1, \ldots, r_L) \):

\[
(F(r_1^0, \ldots, r_L^0) - \Delta F, F(r_1^0, \ldots, r_L^0) + \Delta F),
\]

the three sigma rule can be used and the value \( \sigma_F = \frac{\Delta F}{3} \) can be taken as the standard deviation of the quantity \( F(r_1, \ldots, r_L) \). Accordingly, the value \( F(r_1^0, \ldots, r_L^0) \) is taken as the mathematical expectation \( F(r_1, \ldots, r_L) \). Provided that the analyzed value \( F(r_1, \ldots, r_L) \) is distributed according to the normal law, it is possible to determine the upper \( F_u \) and lower \( F_l \) bounds of the value change \( F(r_1, \ldots, r_L) \) at a certain level of significance \( \alpha \). Thus, work execution time and work execution quality are random variables that are generated according to the distribution law, the characteristics of which are described above.

Based on the algorithm for finding the critical path for the initial network graph, the critical path, time reserves for each event and time reserves for each work are determined. Moreover, in the algorithm of the shortest path, when assessing the early start time of the events, it is proposed to use the upper bound \( F_u \), and when assessing the late start time of the events — the lower bound \( F_L \). This approach can be interpreted as follows: the worst case of early start time and the maximum speed of reaching later start time are considered. The early and late start times will be used to build predictive transitions.

The dynamics of the considered Petri net is formed according to the following rules. When the activity marker comes to a certain place, based on the available evidence and using the likelihood weighting method, based on the Bayesian random factors influence model, the work implementation risk \( R_i \) is calculated, a value of a normally distributed random variable is generated \( T(r_1, \ldots, r_L) \) — the work time and a normally distributed value \( K(r_1, \ldots, r_L) \) — the work performance and the initial time marker becomes equal to the real time of the input transition and does not change further, and the real-time marker also becomes equal to the real time of the input transition and further increases in real time.

When arcs go from several places \( I^I(t_j) \) to the transition \( t_j \), the transition is triggered if the following predicative conditions are met:

1. Initial time markers \( \tau(p_i) \) and real-time markers \( v(p_i) \) of all places from the set \( I^I(t_j) \) satisfy the condition: \( v(p_j) - \tau(p_i) \geq T(r_1, \ldots, r_L) \) and \( \forall p_i \in I^I(t_j) \);
2. \( \max_{p_i \in I^I(t_j)} v(p_i) \leq \gamma_n(t_j) \);
3. \( R(p_i) \leq \psi(p_i) \) and \( \forall p_i \in I^I(t_j) \) where \( \psi(p_i) \) — place \( p_i \) in \( I^I(t_j) \) risk threshold.
4. \( K(r_1, \ldots, r_L) \geq K(p_i) \), \( \forall p_i \in I^I(t_j) \) or \( \Phi(D_1, \ldots, D_G) \leq d \), where \( K(p_i) \) — place \( p_i \) in \( I^I(t_j) \) quality threshold, \( d \) — difficulty of achieving goals threshold for a group of work for places \( p_i \) in \( I^I(t_j) \).

Using the simulation mechanisms of the proposed Petri nets, it is possible to estimate the probability of triggering various transitions, and, consequently, the probability of various project events start times. If the probability of failure of transitions under conditions 2-4 is high, above a certain threshold, then it is possible to take measures to manage risk and redistribute resources between works. As measures for risk management, one can consider: reduction, diversification, avoidance and transfer of risks. The reallocation consists of establishing new planned resource allocation intervals:

\[
((r_1^0 - \Delta r_1, r_1^0 + \Delta r_1), \ldots, (r_L^0 - \Delta r_L, r_L^0 + \Delta r_L)).
\]
Resources can be reallocated from jobs that are not on the critical path to jobs that are on the critical path. The study does not set the task of automatically determining these changes.

Let us consider another option for working with resources in Petri nets. Suppose that the resources distribution between projects is carried out from a single center. Resources are divided into two groups, resources that are returned to the center after each use and resources that are initially allocated for each work separately, for example, financial resources, materials, etc. The resources that are allocated for each job are considered initially known and deterministic. Due to the resources that return to the center, the lower boundaries are known at which work can begin in each specific place \( r_{hi} \) and the optimal resource value for performing this work \( \bar{r}_{hi} \) is known. If there is an optimal resource value in the center, then it is the optimal amount of this resource that is taken to perform work in this place.

The resources that return to the center are proposed to be modeled using Petri nets. Several more places are added to the Petri net, each of which is responsible for the resource returned to the center after being used at a certain job. Each source network place is modeled as a subnetwork of the form shown in Figure 4.

**Figure 4.** Representation of a place as a subnet of a Petri net.

Transition \( t_{j1} \) — is a transition that activates place \( p_i \). Place \( p_{i1} \) — is a place that duplicates place \( p_i \) within a subnet. From place \( p_{i1} \), the activity marker instantly jumps to a transition \( t_{j2} \). The places \( p_{r1}, p_{r2}, \ldots, p_{rk} \) correspond to the places that manage the redistributed resources (common to all jobs). This transition becomes active if the conditions for all resources for the place \( p_{r1} \) are met. The amount of each resource required to complete a given job must be greater than or equal to a certain minimum amount of resources \( r_{hi} \) required to get started. In this case, the multiplicity of arcs leading from the resource potentials to the analyzed potential \( t_{j2} \) is equal to \( \min(r_{hi}, \bar{r}_{hi}) \). The considered transition comes into an active state if the predicate takes a true value:

\[
r_h \geq r_{hi}, \quad \forall h.
\]

The multiplicity of outgoing from the transition \( t_{j2} \) arcs leading to resource places coincides with the multiplicity of incoming arcs, the entire resource that was used when performing work in the place \( p_i \) comes back. The activity marker comes to place \( p_{i2} \), a work time marker \( T(r_1, \ldots, r_L) \) and a work quality marker \( K(r_1, \ldots, r_L) \). The procedure for further development of events is similar to the procedure described above for the case of a Petri net with a stochastic distribution of resources. Petri net with resource places simulation modeling is aimed mainly
at assessing the amount of redistributed resources. Having found the critical path, in this case, estimates of the work execution time with a minimum amount of all resources are used to determine the early start times of the events. In order to estimate the late start times, estimates of the work completion time with the optimal amount of resources are used.

3. Conclusion
Within the framework of this paper, several variants of Petri nets are proposed for the study of scheduling problems. The colored Petri nets with multiple arcs and predicative transitions are used. Petri nets tools allow to simulate the dynamics of the project execution system behaviour under certain procedures or rules for the allocation of system resources intended for the work execution. The resources allocation has a direct impact on the time and work quality, which in turn determine the ability to comply with the project deadlines. To assess the risk of performing individual jobs, a probabilistic inference algorithm based on Bayesian networks is used.

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