Beyond Borel-amenability: scales and superamenable reducibilities

L. Motto Ros

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Straße 25, A-1090 Vienna, Austria

Abstract
We analyze the degree-structure induced by large reducibilities under the Axiom of Determinacy. This generalizes the analysis of Borel reducibilities given in [1], [6] and [5] e.g. to the projective levels.

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1. Introduction

Given a set of functions $\mathcal{F}$ from $\mathbb{R}$ into itself (also called a reducibility), we say that $A, B \subseteq \mathbb{R}$ are $\mathcal{F}$-equivalent if each of them is the $\mathcal{F}$-preimage of the other one, and call $\mathcal{F}$-degree of $A$ the collection of all sets $\mathcal{F}$-equivalent to $A$: our main goal is to study the structure of the $\mathcal{F}$-degrees for various $\mathcal{F}$. Building on the work of Andretta and Martin in [1] (where the case when $\mathcal{F}$ is the set of all Borel functions was considered), in [6] and [5] we have investigated various reducibility notions in the Borel context, but it is clear that there are also some natural sets of functions (such as projective functions) that can be used as reductions and which are strictly larger than the set of Borel functions. In this paper we will prove that, assuming $\text{AD} + \text{DC}$, structural results similar to those for the Borel context can be proved for larger and larger pointclasses. In particular, we will determine the degree-structure induced by the collection $\mathcal{F}_\Gamma$ of all $\Gamma$-functions (i.e. of those functions with the property that the preimage of a set in $\Gamma$ is still in $\Gamma$) in case $\Gamma$ is a boldface pointclass which is closed under projections, countably intersections and unions, and which has the scale and the uniformization property (under $\text{AD}$ these $\Gamma$’s coincide with the so-called tractable pointclasses — see Section 3).

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2\textsuperscript{As usual in Descriptive Set Theory we will always identify $\mathbb{R}$ with the Baire space $^{\omega}\mathbb{N}$.
The existence of such pointclasses is strictly related to the axioms one is willing to accept. For example, in $\mathbf{ZF}+\mathsf{AC}_\omega(\mathbb{R})$ the only known tractable pointclass is $\mathbf{\Sigma}_2^1$, but in general the stronger the axioms one is willing to adopt, the greater number of tractable pointclasses one gets (see [4], [3] and [8] for the results quoted below):

1. $\text{Det}(\Delta_{2n}^1)$ implies that there are at least $n+1$ tractable pointclasses, namely $\mathbf{\Sigma}_2^1, \ldots, \mathbf{\Sigma}_{2n+2}^1$. In particular, Projective Determinacy $\text{Det}(\bigcup_n \Delta_n^1)$ implies that each pointclass $\mathbf{\Sigma}_{2n+2}^1$ is tractable. A similar result holds for the even levels of the $\sigma$-projective pointclass;

2. if $\lambda$ is an ordinal of uncountable cofinality and $\langle \Gamma_\xi \mid \xi < \lambda \rangle$ is a chain of tractable pointclasses (i.e. $\Gamma_\xi \subseteq \Gamma_{\xi'}$ for $\xi < \xi'$), then the pointclass $\Gamma = \bigcup_{\xi < \lambda} \Gamma_\xi$ is tractable as well. In particular, $\sigma$-Projective Determinacy $\text{Det}(\bigcup_{\xi < \omega_1} \Delta_\xi^1)$ implies that the pointclass of all $\sigma$-projective sets is tractable (while the pointclass of all projective sets is not tractable);

3. Hyperprojective Determinacy $\text{Det}(\text{HYP})$ implies that the collection of all inductive sets is tractable;

4. if $\delta$ is limit of Woodin cardinals then $\Gamma_\xi^{\mu_\delta}$, the collection of all $\xi$-weakly homogeneous Suslin sets (for any $\xi < \delta$), is a tractable pointclass;

5. assuming $\text{AD} + \mathcal{V} = \mathbb{L}(\mathbb{R})$, the pointclass $\mathbf{\Sigma}_2^1$ is scaled (hence it has also the uniformization property by closure under coprojections), but if $\mathcal{V} = \mathbb{L}(\mathbb{R})$ and there is no wellordering of the reals then there is a $\mathbf{\Pi}_1^1$ subset of $\mathbb{R}^2$ that can not be uniformized (by any set in $\mathbb{R}^2$): thus, if we assume $\text{AD} + \mathcal{V} = \mathbb{L}(\mathbb{R})$, we get that $\mathbf{\Sigma}_2^1$ is the maximal tractable pointclass;

6. in contrast with the previous point, Woodin has shown that $\text{AD}_\mathcal{R}$ implies that every set of reals has a scale, and this in turn implies, by previous work of Martin, that there are nonselfdual scaled pointclasses with reasonable closure properties which lie arbitrarily high in the Wadge ordering: thus, in particular, under $\text{AD}_\mathcal{R}$ there are tractable pointclasses of arbitrarily high complexity.

All these examples show that our arguments allow to determine the degree-structures induced by larger and larger sets of reductions (assuming corresponding determinacy axioms). Nevertheless we have to point out that at the moment we are able to deal e.g. with $\mathbf{\Sigma}_1^{2n}$-reductions but not with $\mathbf{\Sigma}_2^{2n+1}$-reductions (for $n > 0$). This asymmetry arises from the zig-zag pattern of the regularity properties given by Moschovakis’ Periodicity Theorems, and reflects a phenomenon which is quite common in the $\text{AD}$ context: for instance, in [7] it was shown that the order type of the $\Delta_{2n}^1$ degrees is exactly $\delta_{2n+1}^1$, but no exact evaluation of the order type of $\Delta_{2n+1}^1$ has been given so far (apart from the inequalities $\delta_{2n+1}^1 < \text{order type of } \Delta_{2n+1}^1 < \delta_{2n+2}^1$).

Another important feature of large reductions is that to have our structural results we always need the full $\text{AD}$, as we have to use the Moschovakis’ Coding Lemmas: this should be contrasted with the Borel case, in which the determinacy axioms were used only in a local way. We finish this introduction by acknowledging our debt to A. Andretta and D. A. Martin for their [1] and for the
simple but crucial suggestion of using scales (instead of changes of topology) in
the present setup.

2. Basic facts and superamenability

We will firstly recall some definitions and basic facts for the reader’s conve-
nience. For all undefined symbols, terminology, and for the proofs omitted here
we refer the reader to [2], [4] and [6]. If \( \Gamma \subseteq \mathcal{P}(\mathbb{R}) \) is any boldface pointclass,
we say that the surjection \( \varphi: P \rightarrow \lambda \) is a \( \Gamma \)-norm if there are relations \( \leq_\Gamma \) and
\( \leq_{\tilde{\Gamma}} \) in \( \Gamma \) and \( \tilde{\Gamma} \), respectively, such that for every \( y \in P \) and every \( x \in \mathbb{R} \)
\[
 x \in P \land \varphi(x) \leq \varphi(y) \iff x \leq_{\tilde{\Gamma}} y \iff x \leq_\Gamma y.
\]
To each norm \( \varphi \) we can associate the prewellordering (i.e. the transitive, reflex-
ive, connected and well-founded relation) \( \leq^\varphi \) defined by
\[
 x \leq^\varphi y \iff \varphi(x) \leq^\varphi \varphi(y) \quad (\text{for every } x,y \in P).
\]
A pointclass \( \Gamma \) is said to be normed if every \( P \in \Gamma \) admits a \( \Gamma \)-norm. In this case, if \( \Gamma \) is closed under finite
intersections and unions, then \( \Gamma \) has the reduction property while \( \tilde{\Gamma} \) has the
separation property, and if \( \varphi \) is a \( \Gamma \)-norm on a set \( D \in \Delta_\Gamma \) then \( \leq^\varphi \) is in \( \Delta_{\tilde{\Gamma}} \).
Moreover, we can define
\[
 \delta_\Gamma = \sup \{ \xi \mid \xi \text{ is the length of a prewellordering of } \mathbb{R} \text{ which is in } \Delta_\Gamma \}
\]
(clearly \( \delta_\Gamma < (2^{\aleph_0})^+ \) for every \( \Gamma \subseteq \mathcal{P}(\mathbb{R}) \)). If \( \Gamma \) is closed under coprojections,
countable intersections and countable unions, then there is a regular \( \Gamma \)-norm
with length \( \delta_\Gamma \): this implies that for every pointclass \( \Lambda \)
\[
 \Gamma \cup \tilde{\Gamma} \subseteq \Lambda \Rightarrow \delta_\Gamma < \delta_\Lambda. \quad (1)
\]
A \( \Gamma \)-scale on \( P \subseteq \mathbb{R} \) is a sequence \( \vec{\varphi} = (\varphi_n \mid n \in \omega) \) of norms on \( P \) such that
1. if \( x_0,x_1,\ldots \in P \), \( \lim_i x_i = x \) for some \( x \), and for each \( n \) we have \( \lim_i \varphi_n(x_i) = \lambda_n \) for
some ordinal \( \lambda_n \), then \( x \in P \) and \( \varphi_n(x) \leq \lambda_n \) for each \( n \);
2. there are relations \( S_\Gamma(n,x,y) \) and \( S_{\tilde{\Gamma}}(n,x,y) \) in \( \Gamma \) and \( \tilde{\Gamma} \) respectively such
that for every \( y \in P \), every \( n \in \omega \) and every \( x \in \mathbb{R} \)
\[
 x \in P \land \varphi_n(x) \leq \varphi_n(y) \iff S_\Gamma(n,x,y) \iff S_{\tilde{\Gamma}}(n,x,y).
\]
If every set in \( \Gamma \) admits a \( \Gamma \)-scale we say that the pointclass \( \Gamma \) is scaled,
and in this case if \( \Gamma \) is closed under finite intersections and unions we can also
require that on each \( P \in \Gamma \) there is a \( \Gamma \)-scale such that
if \( x_0,x_1,\ldots \in P \) and for each \( n \) we have \( \lim_i \varphi_n(x_i) = \lambda_n \) for some \( \lambda_n \), then
there exists some \( x \in P \) for which \( \lim_i x_i = x \).
\[(\star)\]
If \( \Gamma \) is scaled and closed under coprojections then \( \Gamma \) has the uniformization
property, i.e. for every \( P \subseteq \mathbb{R} \times \mathbb{R} \) which is in \( \Gamma \) there is some \( P^* \subseteq P \) such
that for every $x$ in the projection of $P$ there is a unique $y \in \mathbb{R}$ that satisfies $(x, y) \in P^*$ (and in this case we will say that $P^*$ uniformizes $P$). The same is true also for the pointclass $\exists \Gamma = \{ A \subseteq \mathbb{R} \mid A$ is the projection of a set in $\Gamma \}$, that is $\exists \Gamma$ is scaled and has the uniformization property.

Finally, we want to recall some results which are consequences of the full AD.

**Lemma 2.1** (First Coding Lemma). Assume AD and let $<$ be a strict well-founded relation on some $S \subseteq \mathbb{R}$ with rank function $\rho : S \to \lambda$. Moreover, let $\Gamma \supseteq \Delta^0_1$ be a pointclass closed under projections, countable unions and countable intersections, and assume that $< \in \Gamma$. Then for every function $f : \lambda \to \mathcal{P}(\mathbb{R})$ there is a choice set $C \in \Gamma$, that is a set $C \subseteq \mathbb{R} \times \mathbb{R}$ such that

i) $(x, y) \in C \Rightarrow x \in S \land y \in f(\rho(x))$,

ii) $f(\xi) \neq \emptyset \Rightarrow \exists x \exists y(\rho(x) = \xi \land (x, y) \in C)$.

Note that our formulation of Lemma 2.1 is slightly different from the original one (due to Moschovakis): nevertheless, one can easily check that our statement is a particular case (and hence a consequence) of the Moschovakis’ one. Using a similar reformulation of the Second Coding Lemma, we get that if $\Gamma \supseteq \Delta^0_1$ is closed under projections, countable unions and countable intersections, then AD implies that $\delta_\Gamma$ is a cardinal of uncountable cofinality. Thus, in particular, if $\varphi = (\varphi_n \mid n \in \omega)$ is a scale on a set $D \in \Delta_\Gamma$ then there is $\lambda < \delta_\Gamma$ such that $\varphi_n : D \to \lambda$ for every $n \in \omega$. Moreover we have that $\bigcup_{\xi < \lambda} A_\xi \in \Gamma$ for every $\lambda < \delta_\Gamma$ and every family $\{ A_\xi \mid \xi < \lambda \} \subseteq \Gamma$.

If we assume also DC we get that for every $n \in \omega$

$$A \in \Sigma^1_{2n+2} \iff A = \righttext{the union of } \delta^1_{2n+1}-many \text{ sets in } \Delta^1_{2n+1},$$

(2) where for every $n$ we put $\delta^1_1 = \delta^2_1 = \delta^\Pi_1$. On the other hand, assuming AD + DC we have that $\Delta^1_{2n+1} = B^*_{2n+1}$, where $B^*_{2n+1}$ is the least boldface pointclass which contains all the open sets and is closed under complementation and unions of length less then $\delta^1_{2n+1}$ (i.e. it is the least $\delta^1_{2n+1}$-complete algebra of sets which contains all the open sets). Thus under AD + DC the projective sets are completely determined by the projective ordinals $\delta^1_1$ and the operations of complementation and “well-ordered” union. All these facts together allow us to prove the following simple lemma.

**Lemma 2.2.** Assume $\text{AD} + \text{DC}$ and let $n \neq 0$ be an even number. Then $D \in \Delta^1_n$ if and only if there is a $\Delta^1_{n-1}$-partition $\langle D_\xi \mid \xi < \delta^1_{n-1} \rangle$ of $\mathbb{R}$ such that $D = \bigcup_{\xi \in S} D_\xi$ for some $S \subseteq \delta^1_{n-1}$.

**Proof.** By (2), $A \in \Sigma^1_n$ if and only if $A$ is the union of a family $\mathcal{B} = \{ B_\xi : R \mid \xi < \delta^1_{n-1} \} \subseteq \Delta^1_{n-1}$. Since $\Delta^1_{n-1} = B_{n-1}$, we can refine $\mathcal{B}$ to a pairwise disjoint family with the same properties by defining $B'_\xi = B_\xi \setminus \bigcup_{\eta < \xi} B_\eta$ for every $\xi < \delta^1_{n-1}$. Applying this argument to both $D$ and $\neg D$ we get the result. The converse is obvious since $\Sigma^1_n$ is closed under well-ordered unions of length smaller than $\delta^1_n$. \qed
A set of functions $F$ from $\mathbb{R}$ into itself is called a set of reductions if it is closed under composition, contains $L$ (the collection of all Lipschitz functions with constant $\leq 1$), and admits a surjection $j: \mathbb{R} \to F$. Given such an $F$, we put $A \leq_F B$ if and only if $A = f^{-1}(B)$ for some $f \in F$. Since $\leq_F$ is a preorder, we can consider the associated equivalence relation $\equiv_F$ and the corresponding $F$-degrees $[A]_F = \{B \subseteq \mathbb{R} \mid A \equiv_F B\}$. Our main goal is to determine the structure of the $F$-degrees with respect to the preorder induced on them by $\leq_F$. Notice that under AD we have the Semi-linear Ordering Principle for $F$:

$$\forall A, B \subseteq \mathbb{R}(A \leq_F B \lor B \leq_F \neg A).$$

(SLO$^F$)

As already pointed out in [6], the arguments used to determine the degree-structures induced by Borel reducibilities (namely changes of topology) cannot be applied outside the Borel context without losing the crucial property that the new topology is still Polish. Moreover, one can see that the dichotomy countable/uncountable is inadequate when dealing with large reductions, and new ordinals must be involved. The natural choice is to consider the characteristic ordinal of $F$

$$\delta_F = \sup\{\xi \mid \xi \text{ is the length of a prewellordering of } \mathbb{R} \text{ which is in } \Delta_F\},$$

where $\Delta_F = \{A \subseteq \mathbb{R} \mid A \leq_F N_{(0)}\}$ is the characteristic set of $F$.

It is clear that if $F \subseteq \mathcal{G}$ (or even just if $\Delta_F \subseteq \Delta_{\mathcal{G}}$) then $\delta_F \leq \delta_{\mathcal{G}}$. In general the converse is not true — see the observation below. Using this ordinal we can give the following definition.

**Definition 1.** A set of reductions $F$ is called superamenable if

i) $\text{Lip} \subseteq F$, where $\text{Lip}$ is the set of all Lipschitz functions (irrespective of their constant);

ii) for every $\eta < \delta_F$, every $\Delta_F$-partition $^3\langle D_\xi \mid \xi < \eta \rangle$ of $\mathbb{R}$ and every sequence of functions $\langle f_\xi \mid \xi < \eta \rangle$ we have that

$$f = \bigcup_{\xi < \eta}(f_\xi \restriction D_\xi) \in F.$$

Superamenability is clearly a natural extension of Borel-amenability as presented in [6], since any set of reductions $F \subseteq \text{Bor}$ is Borel-amenable if and only if it is superamenable. (This is because $\delta_{\text{Lip}} = \delta_{\text{Bor}} = \delta_1 = \omega_1$. To see this, it is clearly enough to show that $\delta_{\text{Lip}} \geq \omega_1$: let $\alpha < \omega_1$ and $z \in \text{WO}_\alpha = \{w \in \mathbb{R} \mid w \text{ codes a wellordering } \leq_w \omega \text{ of length } \alpha\}$. Then for every $x, y \in \mathbb{R}$ put

$$x \leq y \iff z(<(x(0), y(0))) = 1 \iff x(0) \leq_z y(0).$$

It is clear that this is a prewellordering on $\mathbb{R}$ of length $\alpha$, and one can easily check that its image under the canonical homeomorphism between $\mathbb{R}^2$ and $\mathbb{R}$ is

$^3\Gamma$-partition of a set $A \subseteq \mathbb{R}$ is simply a sequence $\langle A_\xi \mid \xi < \lambda \rangle$ of pairwise disjoint sets in $\Gamma$ such that $\lambda$ is an ordinal and $A = \bigcup_{\xi < \lambda} A_\xi$. 

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in \([N_{(0,0)}]_\ell \subseteq \Delta_{\text{Lip}}\). Thus if \(\text{Lip} \subseteq \mathcal{F} \subseteq \text{Bor}\) then \(\delta_\mathcal{F} = \omega_1\): in particular, all the Borel-amenable sets of reductions \(\mathcal{F}\) give rise to the same characteristic ordinal \(\delta_\mathcal{F} = \omega_1\), and from this easily follows that in this case the two definitions coincide.)

If we assume \(\text{AD} + \text{DC}\), a particular place among the superamenable sets of reductions which are subsets of the projective functions is occupied by the \(\Delta^1_{2n+2}\)-functions — see also the next section.

**Proposition 2.3.** Assume \(\text{AD} + \text{DC}\) and let \(\mathcal{F}\) be a superamenable set of reductions such that \(\Delta_\mathcal{F}\) is a proper subset of the collection of the projective sets. Let \(n\) be the smallest natural number such that \(\Delta_\mathcal{F} \subseteq \Delta^1_{n+1}\). If \(n \leq 3\) then \(\Delta_\mathcal{F} \subseteq \Delta^1_{n+1}\) and after a selfdual degree. Moreover, we have that \(\Delta_\mathcal{F} \subseteq \Delta^1_{n+1}\) and \(\Delta_\mathcal{F} \subseteq \Delta^1_{n+1}\), hence in particular, \(\Delta_\mathcal{F} \subseteq \Delta^1_{n+1}\).

**Proof.** First we prove that \(\delta^1_{n-1} < \delta_\mathcal{F} \leq \delta^1_n\). Since \(\Delta_\mathcal{F}\) is closed under \(\text{Lip}\)-preimages, by \(\text{SLO}\) (which is a consequence of \(\text{AD}\)) either \(\Delta_\mathcal{F} \subseteq \Delta^1_{n-1}\) or else \(\Delta^1_{n-1} \subseteq \Delta_\mathcal{F}\). The minimality of \(n\) implies the second possibility, and since \(\Delta^1_{n-1} \subseteq \Delta_\mathcal{F}\) \(\Rightarrow \sum^1_{n-1} \cup \Pi^1_{n-1} \subseteq \Delta_\mathcal{F}\) by \(\text{SLO}\) again, applying (1) with \(\Gamma = \Pi^1_{n-1}\) we get \(\delta^1_{n-1} < \delta_\mathcal{F}\). Finally, \(\delta_\mathcal{F} \leq \delta^1_n\) by the choice of \(n\).

Now let \(D \in \Delta^1_n\) and \(\langle D_\xi \mid \xi < \delta^1_{n-1}\rangle\) be a \(\Delta^1_{n-1}\)-partition of \(\mathbb{R}\) such that \(D = \bigcup_{\xi \in S} D_\xi\) for some \(S \subseteq \delta^1_{n-1}\) (such a partition exists by Lemma 2.2).

Moreover, let \(f_i\) be the constant function with value \(\vec{i} = \langle i, i, i, \ldots \rangle\), and for every \(\xi < \delta^1_{n-1}\) put \(f_\xi = f_0\) if \(\xi \in S\) and \(f_\xi = f_1\) otherwise. It is clear that \(f_0, f_1 \in L \subseteq \mathcal{F}\) and that since \(\delta^1_{n-1} < \delta_\mathcal{F}\)

\[
f = \bigcup_{\xi < \delta^1_{n-1}} \langle f_\xi \mid D_\xi \rangle \in \mathcal{F}
\]

by superamenability. But \(f\) reduces \(D\) to \(N_{(0,0)}\), hence \(D \in \Delta_\mathcal{F}\).

Recall that by Theorem 3.1 of [6], the structure of the \(\mathcal{F}\)-degrees is completely determined whenever we can establish what happens at limit levels (of uncountable cofinality) and after a selfdual degree. Moreover, we have that Lemma 4.4 of [6] holds in our new context (hence, in particular, \(D \cap A \leq_\mathcal{F} A\) for every \(D \in \Delta_\mathcal{F}\) and every \(A \neq \mathbb{R}\)), but the definition of the decomposition property given in that paper must be adapted to the new setup.

**Definition 2.** Let \(\mathcal{F}\) be a superamenable set of reductions. A set \(A \subseteq \mathbb{R}\) has the decomposition property with respect to \(\mathcal{F}\) if there is some \(\eta < \delta_\mathcal{F}\) and a \(\Delta_\mathcal{F}\)-partition \(\langle D_\xi \mid \xi < \eta\rangle\) of \(\mathbb{R}\) such that \(D_\xi \cap A \leq_\mathcal{F} A\) for every \(\xi < \eta\).

The set of reductions \(\mathcal{F}\) has the decomposition property (DP for short) if every \(\mathcal{F}\)-selfdual \(A \subseteq \mathbb{R}\) such that \(A \notin \Delta_\mathcal{F}\) has the decomposition property with respect to \(\mathcal{F}\).

Note that the new definition of the decomposition property is coherent (i.e. coincide) with the original one whenever \(\mathcal{F} \subseteq \text{Bor}\), as this implies \(\delta_\mathcal{F} = \omega_1\).
3. Tractable pointclasses and large reducibilities

We call existential pointclass any boldface pointclass \( \Delta^0_1 \subseteq \Gamma \neq \mathcal{P}(\mathbb{R}) \) which is closed under projections, countable unions and countable intersections. For instance, the projective pointclasses \( \Sigma^1_n \) and the collections \( S(\kappa) \) of all \( \kappa \)-Suslin sets (where \( \kappa \) is some infinite cardinal) are existential pointclasses.

Moreover we will call tractable pointclasses those existential pointclasses \( \Gamma \)'s which have the uniformization property and such that either \( \Gamma \) or \( \tilde{\Gamma} \) is scaled. Notice that not all the existential pointclasses are tractable, as e.g. \( \Sigma^1_{2n+1} \) does not have the uniformization property. Note also that if \( \Gamma \) has a universal set then \( \Gamma \) is tractable if and only if \( \Gamma \) is a scaled existential pointclass with the uniformization property. (Assume towards a contradiction that \( \Gamma \) is an existential pointclass with the uniformization property and that \( \tilde{\Gamma} \) is scaled: since \( \tilde{\Gamma} \) is also closed under coprojections, we would have that \( \tilde{\Gamma} \) has the uniformization property as well, and this would in turn imply that both \( \Gamma \) and \( \tilde{\Gamma} \) have the reduction property. But this contradicts a standard fact in Descriptive Set Theory, see e.g. Proposition 22.15 in [2].) In particular, this equivalence is true under AD (as \( \text{SLO}^\text{b} \) implies that any nonselfdual boldface pointclass has a universal set).

**Proposition 3.1.** Let \( \Gamma \) be an existential pointclass, and let \( f: \mathbb{R} \to \mathbb{R} \) be any function. Then the following are equivalent:

i) \( f \) is a \( \Gamma \)-function (equivalently, a \( \tilde{\Gamma} \)-function);
ii) \( f \) is a \( \Delta^0 \Gamma \)-function;
iii) \( f \) is \( \Delta^0 \Gamma \)-measurable (equivalently, \( \Gamma \)-measurable);
iv) \( \text{graph}(f) \in \Delta^0 \Gamma \), where \( \text{graph}(f) = \{(x,y) \in \mathbb{R}^2 \mid f(x) = y\} \);
v) \( \text{graph}(f) \in \Gamma \).

**Proof.** It is not hard to see that i) implies ii), and since \( \Delta^0 \Gamma \) is closed under countable unions and intersections, we have that ii) implies iii). Moreover, iii) implies iv) since \( \Delta^0 \Gamma \) is closed under countable intersections and

\[
(x, y) \in \text{graph}(f) \iff \forall n(x \in f^{-1}(\mathbb{N}y|n))).
\]

Clearly iv) implies v) and, finally, v) implies i) since if \( A \in \Gamma \) then

\[
f^{-1}(A) = \{x \in \mathbb{R} \mid \exists y((x, y) \in \text{graph}(f) \land y \in A)\} \in \Gamma.
\]

by closure of \( \Gamma \) under projections and finite intersections. \( \square \)

Given an existential pointclass \( \Gamma \), we can define the set of functions

\[
\mathcal{F}_\Gamma = \{f: \mathbb{R} \to \mathbb{R} \mid f \text{ is a } \Delta^0 \Gamma \text{-function}\}
\]

(equivalently, \( \mathcal{F}_\Gamma \) is the collection of all functions which satisfy any of the conditions in Proposition 3.1), and it is immediate to check that \( \Delta^0 \mathcal{F}_\Gamma = \Delta^0 \Gamma \) and \( \delta \mathcal{F}_\Gamma = \delta \Gamma \) is a cardinal of uncountable cofinality — see Section 2. A set of reductions \( \mathcal{F} \) will be called tractable if \( \mathcal{F} = \mathcal{F}_\Gamma \) for some tractable pointclass \( \Gamma \).
Now assume AD and let $\Gamma$ be an existential pointclass. Using the fact that $\Gamma \neq \mathcal{P}(\mathbb{R})$, by Remark 3.2 of [6] we have that there is a surjection of $\mathbb{R}$ onto $\mathcal{F}_\Gamma$, and thus $\mathcal{F}_\Gamma$ is automatically a set of reductions since it is trivially closed under composition. Moreover, Lip $\subseteq \text{Bor} \subseteq \mathcal{F}_\Gamma$ and, using the fact that $\Gamma$ is closed under unions of length less than $\delta_\Gamma$, we have that $\mathcal{F}_\Gamma$ is a superamenable set of reductions: in fact, if $A \in \Gamma$ and $f = \bigcup_{\xi < \lambda}(f_\xi | D_\xi)$ (with $\lambda < \delta_\Gamma$, $f_\xi \in \mathcal{F}_\Gamma$, and $\langle D_\xi | \xi < \lambda \rangle$ a $\Delta_\Gamma$-partition of $\mathbb{R}$) we have

$$f^{-1}(A) = \bigcup_{\xi < \lambda}(f_\xi^{-1}(A) \cap D_\xi) \in \Gamma,$$

hence $f \in \mathcal{F}_\Gamma$. In particular, the set of all $\Delta_{n}^E$-functions (for each $n$) is superamenable.

We will now try to determine, under AD + DC, the structure of degrees induced by $\mathcal{F}_\Gamma$. For the sake of simplicity, we will systematically use the symbol $\Gamma$ instead of $\mathcal{F}_\Gamma$ in all the notations related to reductions: for example, we will write $\leq_\Gamma$, $[A]_\Gamma$, SLO$^\Gamma$ instead of $\leq_{\mathcal{F}_\Gamma}$, $[A]_{\mathcal{F}_\Gamma}$, SLO$^{\mathcal{F}_\Gamma}$, and so on.

The first step is to prove that $\mathcal{F}_\Gamma$ has the decomposition property, but to have this result we must assume that either $\Gamma$ or $\bar{\Gamma}$ has the scale property. In both cases, we have that every $D \in \Delta_\Gamma$ admits a $\Gamma$-scale $\psi = \langle \psi_n^D | n \in \omega \rangle$ on $D$ with the property $(\ast)$ and such that $\psi_n^D : D \to \eta_D$ (for some $\eta_D < \delta_\Gamma$). Similarly, if $f \in \mathcal{F}_\Gamma$ then there is a $\Gamma$-scale $\bar{\psi} = \langle \bar{\psi}_n^f | n \in \omega \rangle$ on graph($f$) and an ordinal $\eta_f < \delta_\Gamma$ such that $\bar{\psi}$ has the property $(\ast)$ and $\bar{\psi}_n^f : \text{graph}(f) \to \eta_f$ for every $n \in \omega$.

**Theorem 3.2 (AD).** Let $\Gamma$ be an existential pointclass such that either $\Gamma$ or $\bar{\Gamma}$ is scaled. Then every $\mathcal{F}_\Gamma$-selfdual $A \subseteq \mathbb{R}$ such that $A \notin \Delta_{\mathcal{F}_\Gamma} = \Delta_\Gamma$ has the decomposition property with respect to $\mathcal{F}$.

**Proof.** Towards a contradiction with AD, assume that for every $\eta < \delta_\Gamma$ and every $\Delta_\Gamma$-partition $\langle D_\xi | \xi < \eta \rangle$ of $\mathbb{R}$ there is some $\xi_\eta < \eta$ such that $A \cap D_{\xi_\eta} \equiv_\Gamma A$: we will construct a flip-set, that is a subset $F$ of the Cantor space $2^\omega$ with the property that $z \in F \iff w \notin F$ whenever $z, w \in 2^\omega$ and $\exists n(z(n) \neq w(n))$. Since every flip-set can not have the Baire property, this will give the desired contradiction.

The ideas involved in the present proof are not far from those used for Theorem 5.3 of [6], but in this case we will use $\Gamma$-scales instead of changes of topology. Let us say that $A$ is not decomposable in $D \in \Delta_\Gamma$ if there is no $\eta < \delta_\Gamma$ and no $\Delta_\Gamma$-partition $\langle D_\xi | \xi < \eta \rangle$ of $D$ such that $A \cap D_\xi \prec_\Gamma A$ for each $\xi < \eta$. Arguing as in the original proof, one can prove that if $A$ is not decomposable in some $D \in \Delta_\Gamma$ then there is some $f \in \mathcal{F}_\Gamma$ such that range($f$) $\subseteq D$ and

$$\forall x \in D(x \in A \cap D \iff f(x) \notin \neg A \cap D).$$

We will construct a countable sequence of nonempty $\Delta_\Gamma$-sets

$$\ldots \subseteq D_2 \subseteq D_1 \subseteq D_0 = \mathbb{R},$$
a sequence of $\Delta_\tau$-functions $f_n : \mathbb{R} \to D_n$ and, for every $z \in \omega^2$, a sequence $\{\alpha_k^m(z) \mid k, m \in \omega\}$ of ordinals strictly smaller than $\delta_\tau$ such that for every $n \in \omega$

$$\forall m \leq n \forall x, y \in D_{n+1} \forall k < n(\psi_k^m(g_{m+1} \circ \ldots \circ g_n(x), g_m \circ \ldots \circ g_n(x)) = \alpha_k^m(z)),$$

(3)

where $g_i = f_i$ if $z(i) = 1$ and $g_i = id$ otherwise\(^4\) (when $m = n$, the expression $\psi_k^m(g_{m+1} \circ \ldots \circ g_n(x), g_m \circ \ldots \circ g_n(x)) = \alpha_k^m(z)$ in the equation above must be simply understood as $\psi_k^n(x, g_n(x)) = \alpha_k^n(z)$).

Having constructed all these sequences, we can finish the proof in the following way: first fix $y_{n+1} \in D_{n+1}$ (for every $n \in \omega$). For every $z \in \omega^2$, every $n \in \omega$ and every $m \leq n$ define $x_m^n = g_m \circ \ldots \circ g_n(y_{n+1})$, and note that $x_m^n \in D_m$. If we fix $m$ and let vary the parameter $n$ we get

$$\lim_n \psi_k^m(x_{m+1}^n, x_m^n) = \alpha_k^m(z) < \eta_{g_m} < \delta_\tau$$

for every $k \in \omega$. This implies, by the property (⋆) of the scales involved, that the sequence $\{(x_{m+1}^n, x_m^n) \mid n \in \omega\}$ converges to some $(x_{m+1}, x_m) \in \text{graph}(g_m)$, that is to some pair of points such that $x_m \in D_m$ and $g_m(x_{m+1}) = x_m$. Observe also that the sequence $(x_m \mid m \in \omega)$ is well defined since $(\{x_n, y_n\} \mid n \in \omega)$ converges to $(x, y)$ if and only if $(x_n \mid n \in \omega)$ converges to $x$ and $(y_n \mid n \in \omega)$ converges to $y$, and the limit of a converging sequence is unique.

Clearly, the points $x_m$ really depend on the choice of $z \in \omega^2$, hence we should have written $x_m = x_m(z)$. If $z, w \in \omega^2$ and $n_0 \in \omega$ are such that $\forall n > n_0(z(n) = w(n))$ then $\forall n > n_0(x_n(z) = x_n(w))$, and if $z(n_0) \neq w(n_0)$ then $x_n(z), x_n(w) \in D_{n_0}$ but $x_n(z) \in A \cap D_{n_0} \iff x_n(w) \notin A \cap D_{n_0}$.

Therefore we get that $\{z \in \omega^2 \mid x_0(z) \in A\}$ is a flip-set, a contradiction!

Now we will construct by induction the $D_n$’s, the $f_n$’s and the $\alpha_k^m$’s, granting inductively that $A$ is not decomposable in $D_n$. First put $D_0 = \mathbb{R}$ and let $f_0$ be any reduction of $A$ to $\neg A$. Suppose to have constructed $D_j$, $f_j$ and $\alpha_k^m(z)$ for every $j, m \leq n, k < n$ and $z \in \omega^2$. Moreover fix $s \in n^{+1}_2$ and define $\alpha_k^m(z) = g_i$ for every $i \leq n$ by letting $g_i = f_i$ if $s(i) = 1$ and $g_i = id$ otherwise. For every $\tau \in n^{+1}(\eta_{g_0})$ consider the set

$$D_\tau^0 = \{x \in D_n \mid \forall i < n + 1(\psi_i^{g_0}(g_1 \circ \ldots \circ g_n(x), g_0 \circ \ldots \circ g_n(x)) = \tau(i))\}$$

(where if $n = 0$ by $g_1 \circ \ldots \circ g_n(x)$ we simply mean the point $x$). Observe also that if $D_\tau^0 \neq \emptyset$ then $\forall j < n(\tau(j) = \alpha_k^m(z))$ for every $m \supseteq s$. Since $A$ is not decomposable in $D_n$ by inductive hypothesis, the fact that $D_\tau^0 \mid \tau \in n^{+1}(\eta_{g_0})$ is a $\Delta_\tau$-partition in less than $\eta_{g_0} < \delta_\tau$ pieces of $D_n$ implies that there must be some $\tau \in n^{+1}(\eta_{g_0})$ such that $A$ is still not decomposable in $D_\tau^0$. Hence we can put $D^0 = D_\tau^0 \subseteq D_n$, and for every $z \in \omega^2$ such that $z \supseteq s$ and every $k < n + 1$

---

\(^4\)The ordinals $\alpha_k^m(z)$ will really depend only on $z \upharpoonright \max\{m, k\} + 1$, and we will also have that $\alpha_k^m(z) < \eta_{g_m}$ for every $m, k \in \omega$.  

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we can also define $\alpha^0_\xi (z) = \tau_0 (k)$ (observe that since $A$ is not decomposable in $D^0_\tau$, then $D^0_\tau \neq \emptyset$, and hence the definition of the $\alpha^0_\xi (z)$ is well given).

Inductively, for every $m + 1 < n + 1$ we can repeat the above construction defining for every $\tau \in n + 1(\eta_{g_{m+1}})$ the set

$$D^m_\tau = \{ x \in D^n \mid \forall i < n + 1 (\psi^{g_{m+1}}_i (g_{m+2} \circ \ldots \circ g_n (x), g_{m+1} \circ \ldots \circ g_n (x)) = \tau (i)) \}$$

(where if $m + 1 = n$, as usual, $g_{m+2} \circ \ldots \circ g_n (x)$ simply denotes the point $x$). The sequence $\langle D^m_\tau \mid \tau \in n + 1(\eta_{g_{m+1}}) \rangle$ forms a $\Delta_\Gamma$-partition in less than $\eta_{g_{m+1}}$ pieces of $D^m$, and since by inductive hypothesis $A$ is not decomposable in $D^m$, there must be some $\tau_{m+1}$ such that $A$ is still not decomposable in $D^m_{\tau_{m+1}}$.

Moreover, for such $\tau_{m+1}$ we have that $\alpha^{m+1}_k (z) = \tau_{m+1} (k)$ for every $k < n$ and every $s \subseteq z \in \omega^2$ (since $\emptyset \neq D^m_{\tau_{m+1}} \subseteq D^m$). Hence we can coherently define $D_{m+1}^m = D_{\tau_{m+1}}^m$ and $\alpha^{m+1}_k (z) = \tau_{m+1} (k)$ for every $k < n + 1$ and every $z \in \omega^2$ such that $z \supseteq s$.

Now put $D(s) = D^n$ and repeat the whole construction for every $s \in n + 1$; let $\langle s_i \mid 1 \leq i \leq 2^{n+1} \rangle$ be an enumeration without repetitions of $n+1$, and define $D(s_1)$ as above, $D(s_2)$ with the same construction but using $D(s_1)$ instead of $D_n$ in the first stage, and so on. Finally, put $D_{n+1} = D(s_{2n+1})$, and let $f_{n+1} \in \mathcal{F}_\Gamma$ be obtained as at the beginning of this proof. Clearly we have that $A$ is not decomposable in $D_{n+1}$, and it is straightforward to inductively verify that condition (3) holds for the sequences constructed. \qed

Now we want to prove the natural restatement of Lemma 4.5 of [6] in this new context\footnote{Note that Lemma 4.5 of [6] is still true when we consider superamenable sets of reductions $\mathcal{F}$, but if $\mathcal{Bor} \subseteq \mathcal{F}$ this is not enough to determine the corresponding degree-structure.}, i.e. considering $\mathcal{S}_\Gamma$-partitions instead of countable partitions. The fundamental key to prove this result (and thus to determine the whole degree-structure induced by $\mathcal{F}_\Gamma$) is the following lemma, which unfortunately (till the moment) can be proved only if $\Gamma$ is an existential pointclass with the uniformization property.

**Lemma 3.3 (AD)**. Let $\Gamma$ be an existential pointclass with the uniformization property. For every $\eta < \delta_\Gamma$, every $\Delta_\Gamma$-partition $\langle D_\xi \mid \xi < \eta \rangle$ of $\mathbb{R}$, and every family of non trivial (i.e. different from $\mathbb{R}$) sets $\{ A_\xi \mid \xi < \eta \}$, we have that $A_\xi \leq_\Gamma B$ for every $\xi < \eta$ then $\bigcup_{\xi < \eta} (A_\xi \cap D_\xi) \leq_\Gamma B$ (for every $B \subseteq \mathbb{R}$).

**Proof.** Since $A_\xi \cap D_\xi \subseteq A_\xi$ we can clearly assume that $A_\xi \subseteq D_\xi$ for every $\xi < \eta$ and prove that if $A_\xi \leq_\Gamma B$ for every $\xi < \eta$ then $\bigcup_{\xi < \eta} A_\xi \leq_\Gamma B$.

Let $G \subseteq \mathbb{R}^3$ be a universal set for $\Gamma$, i.e., a set in $\Gamma$ such that the sets $A \subseteq \mathbb{R}^2$ which are in $\Gamma$ are exactly those of the form $G_x = \{(y, z) \in \mathbb{R}^2 \mid (x, y, z) \in G\}$ for some $x \in \mathbb{R}$. For every $\xi < \eta$, let

$$F_\xi = \{ x \in \mathbb{R} \mid G_x \text{ is the graph of a } \Delta_\Gamma \text{-function which reduces } A_\xi \text{ to } B \},$$

and observe that each $F_\xi$ is nonempty by our hypotheses. Let now $\leq \in \Delta_\Gamma$ be a prewellordering of length $\eta$ (which exists since $\eta < \delta_\Gamma$), consider its strict
Proof. For part \( A \) of degrees such that
\[
\text{and we are done.}
\]
and for \( \forall \) partition of \( \xi < \eta \) and for \( \text{preordering must exist by definition of } R \). In fact, let \( \leq \) be a prewellordering in \( R \) and every \( \xi < \eta \) be (the unique ordinal) such that \( x \in D_\xi \), so that \( x \in \bigcup_{\xi < \eta} A_\xi \iff x \in A_\xi \).

Now we have that \( (x,f(x)) \in f^* \subseteq \hat{f} \), and thus \( (x,f(x)) \) is in the graph of some \( \Delta_\Gamma \)-function that was a reduction of \( A_\xi \) to \( B \). Hence
\[
x \in \bigcup_{\xi < \eta} A_\xi \iff x \in A_\xi \iff f(x) \in B
\]
as required if we define \( D_\xi \) for every \( \xi < \eta \). It is straightforward to check that \( \tilde{f} \) is in \( \Gamma \) and hence admits a uniformization \( f^* \) which is again in \( \Gamma \). Thus \( f^* \) is the graph of a \( \Gamma \)-function \( f \) (see Proposition 3.1), and we claim that \( f \) reduces \( \bigcup_{\xi < \eta} A_\xi \) to \( B \). Fix some \( x \in R \) and let \( \xi < \eta \) be (the unique ordinal) such that \( x \in D_\xi \), so that \( x \in \bigcup_{\xi < \eta} A_\xi \iff x \in A_\xi \).

Now we have that \( (x,f(x)) \in f^* \subseteq \hat{f} \), and thus \( (x,f(x)) \) is in the graph of some \( \Delta_\Gamma \)-function that was a reduction of \( A_\xi \) to \( B \). Hence
\[
x \in \bigcup_{\xi < \eta} A_\xi \iff x \in A_\xi \iff f(x) \in B
\]
as required if we define \( D_\xi \) for every \( \xi < \eta \). It is straightforward to check that \( \tilde{f} \) is in \( \Gamma \) and hence admits a uniformization \( f^* \) which is again in \( \Gamma \). Thus \( f^* \) is the graph of a \( \Gamma \)-function \( f \) (see Proposition 3.1), and we claim that \( f \) reduces \( \bigcup_{\xi < \eta} A_\xi \) to \( B \). Fix some \( x \in R \) and let \( \xi < \eta \) be (the unique ordinal) such that \( x \in D_\xi \), so that \( x \in \bigcup_{\xi < \eta} A_\xi \iff x \in A_\xi \).

Now observe that for every \( \eta < \delta_\Gamma \) there is a \( \Delta_\Gamma \)-partition of \( R \) into \( \eta \) many pieces. In fact, let \( \leq \) be a prewellordering in \( \Delta_\Gamma \) of length \( \eta + 1 < \delta_\Gamma \) (such a preordering must exist by definition of \( \delta_\Gamma \)). Let \( \rho : S \to \eta + 1 \) be its rank function.

If \( \eta = \mu + 1 \), put \( D = \{ x \in S \mid \rho(x) < \mu \} \) and check that \( \langle D_\xi \mid \xi < \eta \rangle \) is the partition required if we define \( D_\xi = \emptyset \) for every \( \xi < \eta \).

If instead \( \eta \) is limit, put \( D = \{ x \in S \mid \rho(x) < \eta \} \) and check that \( \langle D_\xi \mid \xi < \eta \rangle \) is again as required if we define \( D_0 = \emptyset \) and \( D_{\xi + 1} = \{ x \in D \mid \rho(x) = \xi \} \) for every \( \xi < \eta \).

**Theorem 3.4 (AD).** Let \( \Gamma \) be a tractable pointclass. Then we have that:

i) if \([A]_\Gamma\) is limit of cofinality strictly less than \( \delta_\Gamma \) then \( A \leq_\Gamma \neg A \);

ii) for every \( \eta < \delta_\Gamma \), every \( \Delta_\Gamma \)-partition \( \langle D_\xi \mid \xi < \eta \rangle \) of \( R \) and every \( A,B \subseteq R \),

\[
A \cap D_\xi \leq_\Gamma B \iff \forall \xi < \eta (A_\xi \leq_\Gamma B_\xi);
\]

iii) if \([A]_\Gamma\) is limit of cofinality greater then \( \delta_\Gamma \) then \( A \not\leq_\Gamma \neg A \);

iv) after an \( \mathcal{F}_\Gamma \)-seldual degree there is a nonseldual pair.

**Proof.** For part i), let \( \text{cof}_\Gamma(A) = \eta < \delta_\Gamma \) and let \( \{ [A_\xi]_\Gamma \mid \xi < \eta \} \) be any family of degrees such that \( A_\xi \leq_\Gamma A \) for every \( \xi < \eta \) and such that for every \( B \) for which \( \forall \xi < \eta (A_\xi \leq_\Gamma B) \) we have that \( B \not\leq_\Gamma A \). Let \( \langle D'_\xi \mid \xi < \eta \rangle \) be any \( \Delta_\Gamma \)-partition of \( R \) (which must exists by the observation preceding this theorem), and for \( \xi < \eta \) define \( D_\xi = \{ x + y \in R \mid y \in D'_\xi \} \), \( C_\xi = \{ x + y \in D_\xi \mid x \in A_\xi \} \), and \( C = \bigcup_{\xi < \eta} C_\xi \). Note that \( C_\xi \subseteq D_\xi \) and \( C_\xi \equiv_\Gamma A_\xi \) for every \( \xi < \eta \). It is clear that we can assume \( C_\xi \not\leq_\Gamma \) for every \( \xi < \eta \) and apply Lemma 3.3 to the \( \Delta_\Gamma \)-partition \( \langle D_\xi \mid \xi < \eta \rangle \) and to the \( C_\xi \)'s to get \( C \leq_\Gamma A \). Conversely, for every \( \xi < \eta \)
\[
A_\xi \equiv_\Gamma C_\xi = C \cap D_\xi \leq_\Gamma C,
\]
hence $A \leq \Gamma C$ by our hypotheses (since otherwise $C <_{\Gamma} A$). Thus it is enough to show that $C$ is $\Gamma$-selfdual. To see this, observe that since $C_\xi \equiv_{\Gamma} A_\xi <_{\Gamma} A \equiv_{\Gamma} C$ we have also $C_\xi <_{\Gamma} \neg C$ by SLO$^\Gamma$ (which follows from AD): therefore we can apply Lemma 3.3 again with $B = \neg C$ to get $C \leq_{\Gamma} \neg C$.

For part ii) simply apply Lemma 3.3 with $A_\xi = A \cap D_\xi$ (for every $\xi < \eta$).

For part iii), assume that $A_\Gamma$ is limit (in particular, $A \notin \Delta_{\Gamma}$) and $A \leq_{\Gamma} \neg A$. By Theorem 3.2 there is some $\eta < \delta_\Gamma$ and a $\Delta_{\Gamma}$-partition $\langle D_\xi \mid \xi < \eta \rangle$ of $\mathbb{R}$ such that $A \cap D_\xi <_{\Gamma} A$ for every $\xi < \eta$. By part ii), $[A]_{\Gamma}$ is the supremum of the family $\mathcal{A} = \{ [A \cap D_\xi]_{\Gamma} \mid \xi < \eta \}$, and hence $\mathcal{A}$ witnesses that $[A]_{\Gamma}$ is cofinality strictly less than $\delta_\Gamma$. Therefore, if $[A]_{\Gamma}$ is limit of cofinality greater than $\delta_\Gamma$ then $A \not\approx_{\Gamma} \neg A$.

Finally, for part iv) it is enough to prove that if $A$ and $B$ are two $\mathcal{F}_{\Gamma}$-selfdual sets such that $A <_{\Gamma} B$ (which implies $B \notin \Delta_{\Gamma}$), then there is some $C$ such that $A <_{\Gamma} C <_{\Gamma} B$. By Theorem 3.2 again, there must be some $\eta < \delta_\Gamma$ and some $\Delta_{\Gamma}$-partition $\langle D_\xi \mid \xi < \eta \rangle$ of $\mathbb{R}$ such that $B \cap D_\xi <_{\Gamma} A$ for every $\xi < \eta$. If $B \cap D_\xi \leq_{\Gamma} A$ for every $\xi < \eta$, then we would have $B \leq_{\Gamma} A$ by part ii), a contradiction! Hence there must be some $\xi_0 < \eta$ such that $B \cap D_{\xi_0} \not\subseteq_{\Gamma} A$, and by SLO$^\Gamma$ and $\mathcal{F}_{\Gamma}$-selfduality of $A$, we get $A <_{\Gamma} B \cap D_{\xi_0} <_{\Gamma} B$.

The previous theorem shows that if $\Gamma$ is tractable we can completely describe the hierarchy of the $\mathcal{F}_{\Gamma}$-degrees using Theorem 3.4 and Theorem 3.1 of [6]: it is a well-founded preorder of length $\Theta$, nonselfdual pairs and selfdual degrees alternate, at limit levels of cofinality strictly less than $\delta_\Gamma$ there is a selfdual degree, while at limit levels of cofinality equal or greater than $\delta_\Gamma$ there is a nonselfdual pair. Thus the degree-structure induced by $\mathcal{F}_{\Gamma}$ looks like this:

\[
\begin{array}{cccccccccccc}
\bullet & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \cdots & \bullet & \bullet \\
\end{array}
\]

\[\text{cof} < \delta_{\mathcal{F}} \quad \text{cof} \geq \delta_{\mathcal{F}}\]

Note that the previous picture is coherent with the description of the structure of the $\mathcal{F}$-degrees when $\mathcal{F}$ is Borel-amenable: in fact, as already observed, in that case we have $\delta_{\mathcal{F}} = \omega_1$, and therefore picture (4) coincides with the usual one.

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