We discuss deformation quantization of the covariant, light-cone and conformal gauge-fixed p-brane actions ($p > 1$) which are closely related to the structure of the classical and quantum Nambu brackets. It is known that deformation quantization of the Nambu bracket is not of the usual Moyal type. Yet the Nambu bracket can be quantized using the Zariski deformation quantization (discovered by Dito, Flato, Sternheimer and Takhtajan) which is based on factorization of polynomials in several real variables. We discuss a particular application of the Zariski deformed quantization in M-theory by considering the problem of a covariant formulation of Matrix theory. We propose that the problem of a covariant formulation of Matrix Theory can be solved using the formalism of Zariski deformation quantization of the triple Nambu bracket.
1 Introduction

Even though a fully background independent non-perturbative formulation of M/string theory is still largely mysterious, some notable progress toward a background dependent formulation of M/string theory has been recently made. At the moment, perhaps the most promising background dependent formulations of M/string theory are Matrix theory and AdS/CFT duality. The two approaches appear to be intimately related, as pointed out in [2].

Both formulations suggest a duality between maximally supersymmetric Yang-Mills theory (or appropriate conformally invariant field theory) and supergravity in a particular background. However, the underlying physical reason for such a duality is not fully understood. The flat space-time limit of this duality appears particularly puzzling at present.

It has also been recently uncovered that string and light-cone M-theory in the background of $B$ fields, taken in appropriate limits, are described by supersymmetric Yang-Mills theory on a non-commutative space. This non-commutative Yang-Mills theory is formulated using the Moyal bracket, a structure essential in the deformation quantization of the Poisson brackets.

In this article we suggest that a different kind of deformation quantization - called Zariski deformation quantization [5] - is the relevant mathematical structure for the formulation of a covariant Matrix theory.

We motivate our presentation by a discussion of deformation quantization of the covariant, light-cone and conformal gauge-fixed p-brane actions which are closely related to the structure of the classical and quantum Nambu brackets. It turns out that deformation quantization of Nambu brackets is not of the usual Moyal type. As shown by Dito, Flato, Sternheimer and Takhtajan the Nambu bracket can be quantized using the Zariski deformation quantization which is based on factorization of polynomials in several real variables.

In view of this fact we discuss the application of the Zariski deformed quantization to the problem of a covariant formulation of Matrix theory.

The paper is organized as follows: first, in section 2, we briefly review the connection between Nambu brackets and the covariant, light-cone and conformal gauge-fixed p-brane actions. Then in section 3, we discuss the fundamental properties of the classical Nambu bracket and the Zariski deformation quantization of the same. Then in section 4, we apply the results...
of section 3. to the problem of covariantization of Matrix theory in terms of the deformed Zariski product. In this section we state a precise proposal toward a covariant formulation of Matrix theory. In section 5, we compare the construction of quantum Nambu brackets in terms of square and cubic matrices [13] to the Zariski deformation quantization scheme and discuss a natural mathematical formulation of the space-time uncertainty principle of M-theory [14] in view of our proposal.

2 p-Branes and the Nambu Bracket

In this section we review the relation between the various forms of the bosonic p-brane actions \( p > 1 \) and the classical Nambu bracket. Let us start from the familiar covariant bosonic p-brane action \[ S = - \int d^{p+1}\xi \sqrt{-g}, \] where \( g = \det g_{ij} \) and the induced world-volume metric \( g_{ij} \) is \[ g_{ij} = \partial_i x^\mu \partial_j x^\nu \eta_{\mu\nu}. \] Here \( x^\mu (\mu = 0, 1, \ldots, d) \) denote the target space coordinates of a \( d+1 \) dimensional bosonic p-brane. \( x^\mu \)'s are functions of \( p+1 \) world-volume coordinates \( \xi^i, i = 0, 1, \ldots, p \). The equation of motion for \( x^\mu \) follows from (1): \[ \partial_i (\sqrt{-g} g^{ij} \partial_j x^\mu) = 0. \] The covariant p-brane action is reparametrization invariant \[ \delta x^\mu = \epsilon^i \partial_i x^\mu. \] The world-volume reparametrization invariance allows us to choose different gauges: we discuss the light-cone and conformal gauges in what follows, because of their relation to the structure of the classical Nambu bracket.

2.1 Volume Preserving Diffeormorphisms and the Classical Nambu Bracket

The light-cone gauge variables are denoted by \( x^a (a = 1, \ldots, d - 1) \). The light-cone coordinates are defined as \[ x^\pm = \frac{1}{\sqrt{2}} (x^d \pm x^0), \]
where in the light-cone gauge
\[ \partial_\tau x^+ \equiv p^+ \delta_{i0}. \]  
(6)

The world volume coordinates \( \xi^i \) split as
\[ (\xi^0, \xi^s) \rightarrow (t, \xi^s). \]  
(7)

The light-cone bosonic p-brane action is
\[ S_{lc} = \frac{1}{2} \int d^{p+1}\xi ((D_0 x^a)^2 - h). \]  
(8)

where \( h \equiv \det g_{rs} (r, s = 1, 2, ..., p) \) is the determinant of the induced p-dimensional metric and the covariant derivative
\[ D_0 x^a = (\partial_0 + u^s \partial_s) x^a. \]  
(9)

Note that the equations of motion imply
\[ \partial_s u^s = 0. \]  
(10)

Also, \( h \equiv \det g_{rs} (r, s = 1, 2, ..., p) \) is determined by the following expression
\[ h = (1/p!) \{ x^a_1, x^a_2, ..., x^a_p \}^2. \]  
(11)

The symbol \( \{ f, g, ...w \} \) denotes the classical Nambu bracket \[ 10 \] with respect to \( \xi_s \)
\[ \{ f, g, ..., w \} \equiv \varepsilon^{ij...k} \partial_{\xi^i} f \partial_{\xi^j} g ... \partial_{\xi^k} w, \]  
(12)

where \( f, g, ...w \) denote functions of \( \xi_s \). Thus the light-cone bosonic p-brane action can be rewritten as follows
\[ S_{lc} = \frac{1}{2} \int d^{p+1}\xi ((D_0 x^a)^2 - (1/p!) \{ x^a_1, x^a_2, ..., x^a_p \}^2). \]  
(13)

The original world-volume diffeomorphisms \( \delta x^\mu = \epsilon^i \partial_i x^\mu \) reduce in the light-cone gauge to the p-dimensional volume preserving diffeomorphisms described by the action of the classical Nambu bracket (see section 3. for details)
\[ \delta x^a = \{ u, v, ..., x^a \}. \]  
(14)

The longitudinal coordinate \( x^- \) is determined from the primary constraint
\[ p^a \partial_\tau x^a + p^+ \partial_\tau x^- \approx 0 \] and the requirement, implied by the equation of motion, that the longitudinal momentum is time-independent \( \partial_0 p_+ = 0. \)
2.2 Conformal Diffeomorphisms

The conformal gauge is defined as follows

\[ g_{00} = -h, \quad g_{0a} = 0, \quad g_{ab} = h_{ab}. \] (15)

If \( a = 1 \) we get the usual conformal gauge of perturbative string theory.

The parameters \( \epsilon^0 \) and \( \epsilon^a \) from the equation \( \delta x^\mu = \epsilon^i \partial_i x^\mu \) satisfy the following relations

\[ \partial_0 \epsilon^0 = \partial_a \epsilon^a, \quad \partial_0 \epsilon^a = hh^{ab} \partial_b \epsilon^0. \] (16)

The conformal gauge action is

\[ S_c = \frac{1}{2} \int d^{p+1}\xi ((\partial_0 x^\mu)^2 - (1/p!)\{x^{\mu_1}, ..., x^{\mu_p}\}^2). \] (17)

Note the similarity between the structures of \( S_{lc} \) and \( S_c \). The obvious difference is that the form of \( S_c \) is covariant with respect to target space indices, unlike the form of \( S_{lc} \).

This action is invariant under \( \delta x^\mu = \epsilon^i \partial_i x^\mu \) with \( \epsilon^0 \) and \( \epsilon^a \) given above. One can also check that the constraints

\[ T_{00} = 1/2(\partial_0 x^\mu)^2 + 1/(2p!)\{x^{\mu_1}, ..., x^{\mu_p}\}^2 \approx 0 \]
\[ T_{0a} = \partial_0 x^\mu \partial_a x^\mu \approx 0 \] (18)

transform into each other under this residual symmetry.

2.3 Discretization of p-Branes and the Quantum Nambu Bracket

Given the expressions for \( S, S_{lc} \) and \( S_c \) there obviously exists a natural relation between the structure of the classical Nambu bracket and the covariant, light-cone and conformal gauge-fixed p-brane actions. Thus, by quantizing the p-dimensional Nambu bracket we can naturally discretize either covariant or light-cone or conformal p-brane action.

What do we mean by ”discretize”? Recall that the light-cone membrane action (p=2) can be regularized (or discretized) by applying the Goldstone-Hoppe map between representation theories of the algebra of the area preserving diffeomorphisms (APD) and the \( N = \infty \) limit of Lie algebras.
In particular, the Goldstone-Hoppe prescription instructs us to perform the following translation of the transverse spatial coordinates

\[
\{ x^a, x^b \} \rightarrow [X^a, X^b],
\]

where now \( X^a \) denote large \( N \times N \) hermitian matrices. Moreover, integration is translated into tracing

\[
\int d\xi^1 d\xi^2 \cdots \rightarrow Tr.
\]

This dictionary translates the light-cone membrane action \([7], [8]\) into the action of Matrix theory \([1]\).

The obvious generalization of this prescription for the case of p-branes is \([13], [15]\)

\[
\{ x^{b_1}, x^{b_2}, \ldots, x^{b_l} \} \rightarrow [X^{b_1}, X^{b_2}, \ldots, X^{b_l}],
\]

where for the moment the nature of \( X^{b_k} \) is not precisely defined (see sections 3. and 4.). Note that in this prescription index \( b \) is either the transverse space index \( a \) in the light-cone gauge, or the covariant index \( \mu \) in the conformal gauge; in both cases \( l = p \). The object on the right-hand side is what we call the quantum Nambu bracket (see section 3. for a precise definition). Also in this case

\[
\int d\xi^1 d\xi^2 \cdots d\xi^{p+1} \rightarrow TR
\]

where \( TR \) is a suitable generalization of the operation of tracing.

Given the expressions for the light-cone and conformal gauge-fixed p-brane actions, we can obviously apply this dictionary. The classical Nambu bracket in the expressions for \( S_{lc} \) and \( S_c \) is replaced by its quantum counterpart and integration by tracing.

The covariant p-brane action is of course also invariant under the subgroup of world-volume diffeomorphism. The covariant action \( S \) can be rewritten as

\[
S = -\int d^{p+1}\xi \sqrt{-det g_{ij}} = -\int d^{p+1}\xi \sqrt{\frac{1}{(p+1)!} \{x^{a_1}, x^{a_2}, \ldots, x^{a_{p+1}}\}^2},
\]

or equivalently, in the polynomial form

\[
S = 1/2 \int d^{p+1}\xi (\frac{1}{(p+1)!} \{x^{a_1}, x^{a_2}, \ldots, x^{a_{p+1}}\}^2 - \epsilon).
\]
Therefore the above dictionary can be directly applied at the level of the covariant action (with $b$ denoting the covariant index $\mu$ and $l = p + 1$). This procedure gives an appropriately discretized version of the covariant p-brane action. We will use this formal observation when we discuss the issue of covariantization of Matrix theory in section 4.

Note that the volume preserving diffeomorphisms are not related to the gauge transformations of the Yang-Mills type. In fact it was conjectured in [6] that p-dimensional volume preserving diffeomorphisms are related to an infinite-dimensional non-Abelian antisymmetric tensor gauge theory.

In order to explore these non-Yang-Mills transformations it is natural to study the discretization of p-branes by quantizing the Nambu bracket. The solution to the quantization problem is given by the Zariski deformation quantization discovered in [5] which we discuss in the next section.

3 Deformation Quantization and the Nambu Bracket

For simplicity, in this section we consider only the triple Nambu bracket. Our treatment can be easily extended to the general case.

3.1 Classical Nambu Bracket

We start this section with a short review of the fundamental properties of the classical Nambu bracket following [13]. Consider a three-dimensional space parametrized by $\{x^i\}$. The three-dimensional volume preserving diffeomorphisms (VPD) on this space are described by a differentiable map

$$x^i \rightarrow y^i(x)$$

such that

$$\{y^1, y^2, y^3\} = 1$$

where, by definition, as in section 2.

$$\{A, B, C\} \equiv \epsilon^{ijk}\partial_i A \partial_j B \partial_k C$$

is the Nambu-Poisson bracket, or Nambu bracket, or Nambu triple bracket, which satisfies [4], [12], [16], [17]
1. Skew-symmetry

\[ \{A_1, A_2, A_3\} = (-1)^{\epsilon(p)} \{A_{p(1)}, A_{p(2)}, A_{p(3)}\}, \quad (28) \]

where \( p(i) \) is the permutation of indices and \( \epsilon(p) \) is the parity of the permutation.

2. Derivation

\[ \{A_1A_2, A_3, A_4\} = A_1\{A_2, A_3, A_4\} + \{A_1, A_3, A_4\}A_2, \quad (29) \]

3. Fundamental Identity (FI-1) \[12], \[17]\]

\[
\begin{align*}
\{\{A_1, A_2, A_3\}, A_4, A_5\} &+ \{A_3, \{A_1, A_2, A_4\}, A_5\} \\
+ \{A_3, A_4, \{A_1, A_2, A_5\}\} & = \{A_1, A_2, \{A_3, A_4, A_5\}\}. \quad (30)
\end{align*}
\]

The three-dimensional VPD involve two independent functions. Let these functions be denoted by \( f \) and \( g \). The infinitesimal three-dimensional VPD generator is then given as

\[
D(f, g) \equiv \epsilon^{ijk}\partial_i f \partial_j g \partial_k \equiv D_k(f, g) \partial_k. \quad (31, 32)
\]

The volume-preserving property is nothing but the identity

\[
\partial_i D^i(f, g) = \partial_k (\epsilon^{ijk} \partial_i f \partial_j g) = 0. \quad (33)
\]

Given an arbitrary scalar function \( X(x^i) \), the three-dimensional VPD act as

\[
D(f, g)X = \{f, g, X\}. \quad (34)
\]

Apart from the issue of global definition of the functions \( f \) and \( g \), we can represent an arbitrary infinitesimal volume-preserving diffeomorphism in this form.

On the other hand, if the base three-dimensional space \( \{x^i\} \) is mapped into a target space of dimension \( d + 1 \) whose coordinates are \( X^\alpha (\alpha = 0, 1, 2, \ldots, d) \), the induced infinitesimal volume element is

\[
d\sigma \equiv \sqrt{|X^\alpha, X^\beta, X^\gamma|^2} dx^1 dx^2 dx^3, \quad (35)
\]
provided the target space is a flat Euclidean space. The volume element is of course invariant under the general three-dimensional diffeomorphisms.

The triple product \( \{X^\alpha, X^\beta, X^\gamma\} \) is also “invariant” under the VPD. More precisely, it transforms as a scalar. Namely,

\[
\{Y^\alpha, Y^\beta, Y^\gamma\} - \{X^\alpha, X^\beta, X^\gamma\} = \epsilon D(f, g) \{X^\alpha, X^\beta, X^\gamma\} + O(\epsilon^2)
\]  

for

\[
Y = X + \epsilon D(f, g) X.
\]  

This is due to the Fundamental Identity FI-1 which shows that the operator \( D(f, g) \) acts as a derivation within the Nambu bracket. For fixed \( f \) and \( g \), we can define a finite transformation by

\[
X(t) \equiv \exp(tD(f, g)) \to X = \sum_{n=0}^{\infty} \frac{t^n}{n!} \{f, g, \{f, g, \{f, g, \{f, g, X\}, \ldots\}\}, \ldots\}
\]

which satisfies the Nambu “equation of motion” [10]

\[
\frac{d}{dt} X(t) = \{f, g, X(t)\}.
\]

The Nambu-Poisson structure is preserved under this evolution equation.

Notice that in the case of the usual Poisson structure, the algebra of two-dimensional area preserving diffeomorphisms is given by

\[
[D(f_1), D(f_2)] = D(f_3)
\]

where

\[
f_3 = \{f_1, f_2\}
\]

\[
D(f) X = \{f, X\}.
\]

It turns out that the three-dimensional analogue of the commutator algebra [13]

\[
D(A_1)D(A_2) = D(\{A_1, A_2\})
\]

can be written using the quantum triple Nambu commutator [10]

\[
[A, B, C]_N \equiv ABC - ACB + BCA - BAC + CAB - CBA
\]
as follows
\[ D(A_1, A_2)D(A_3, B) = 2D(A_1, A_2, A_3, B), \] (45)
or equivalently
\[ D(B_1, B_2)D(A_1, A_2)D(A_3, B_3) = 4D(A_1, A_2, A_3, B_1, B_2, B_3). \] (46)
Both relations are equivalent to the Fundamental Identity
\[ \{A_1, A_2, \{A_3, B, C\}\} + \{A_2, A_3, \{A_1, B, C\}\} + \{A_3, A_1, \{A_2, B, C\}\} = \{\{A_1, A_2, A_3\}, B, C\}. \] (47)
This result suggests that there is a new kind of symmetry based on a new composition law whose infinitesimal algebra is given by the triple commutator (14). It was conjectured in [13] that this symmetry is related to the gauge transformations that are not of the Yang-Mills type [6].

### 3.2 Deformation Quantization of the Nambu bracket

Armed with this background we are ready to discuss the deformation quantization of the Nambu bracket.

The basic philosophy of deformation quantization [18] is to view quantization as the procedure of replacing the algebra of observables in classical mechanics, such as functions defined over classical phase space, with the usual product of functions, by the algebra defined by a deformed product of functions (called star or Moyal product).

The Moyal (star) product [18] of two functions on a 2-dimensional phase space \((q, p)\) is defined as a deformation of the ordinary product of functions
\[ (f * g)(q, p) = \exp[i\hbar/2(\partial_q \partial_p - \partial_p \partial_q)](f, g) \equiv \exp[i\hbar/2\mathcal{P}](f, g) \] (48)
and is known to be associative
\[ (f * g) * h = f * (g * h). \] (49)
The Moyal bracket is defined as
\[ \{f, g\} \equiv \frac{1}{i\hbar}(f * g - g * f). \] (50)
The properties of the Moyal bracket are the same as the properties of the Poisson bracket or the commutator of operators in quantum mechanics. This enables one in principle to perform quantum-mechanical calculations within the classical framework without using Hilbert-space formalism. There are other formal advantages of this viewpoint.

For example, the Feynman path integral can be viewed as the Fourier transform over the momentum variable \( p \) of \( \exp^*(-iH/\hbar)(q, p) \) where

\[
\exp^*(f)(q, p) = 1 + f(q, p) + \frac{1}{2!}(f * f)(q, p) + \ldots \quad (51)
\]

In the symplectic case the star product is given by the Feynman path integral. The general solution for the deformation quantization of the algebra of functions on a Poisson manifold, as formulated by Kontsevich [20], can also be understood from this point of view.

It is interesting that the same set-up of Moyal deformation quantization does not work for the Nambu bracket. The basic obstruction is provided by the property of the Fundamental Identity.

For example, the exponentiated Nambu bracket

\[
[f_1, f_2, f_3] = \frac{1}{6} \sum_p \epsilon(p)(f_{p1}, f_{p2}, f_{p3}) \quad (52)
\]

where the deformed product is defined as

\[
(f_1, f_2, f_3)_h = \exp[i\hbar/2(\{\partial_x, \partial_y, \partial_z\})](f_1, f_2, f_3) \quad (53)
\]

does not satisfy the property of the Fundamental Identity of the classical Nambu bracket [5].

The solution of the deformation quantization problem is rather non-trivial in this situation and has been discovered in [5]. The solution involves the Zariski product which is based on the factorization of polynomials in several real variables.

Before describing the formalism we have to specify what we mean by a quantum triple Nambu bracket. In general we want an object \([F, G, W]\) which satisfies the properties analogous to the classical Nambu bracket \(\{f, g, w\}\) as listed in the previous section. (Here \(f, g, w\) are functions of three variables, and the nature of \(F, G, W\) is left open for the moment.) Thus \([F, G, W]\) is expected to satisfy [5], [12], [16], [17].
1. Skew-symmetry

\[ [A_1, A_2, A_3] = (-1)^{\epsilon(p)} [A_{p(1)}, A_{p(2)}, A_{p(3)}], \quad (54) \]

where again \( p(i) \) is the permutation of indices and \( \epsilon(p) \) is the parity of the permutation.

2. Derivation

\[ [A_1 A_2, A_3, A_4] = A_1 [A_2, A_3, A_4] + [A_1, A_3, A_4] A_2, \quad (55) \]

3. Fundamental Identity (F.I.)

\[
[ [A_1, A_2, A_3], A_4, A_5 ] + [ A_3, [ A_1, A_2, A_4 ], A_5 ] \\
+ [ A_3, A_4, [ A_1, A_2, A_5 ] ] = [ A_1, A_2, [ A_3, A_4, A_5 ] ]. \quad (56)
\]

(Note that the two-dimensional quantum Nambu bracket which satisfies above properties is just the usual commutator of matrices \([A, B] \equiv AB - BA\). In this case the F.I. reduces to the Jacobi identity.)

Now we describe the Zariski quantization scheme following the original work of [3].

The basic idea of [3] is to replace the usual product of functions in the classical Nambu bracket which is Abelian, associative, distributive and respects the Leibnitz rule, by another (deformed) product with the same properties. Then the modified Nambu bracket defined by such a product will satisfy the same properties of the classical Nambu bracket.

The construction is based on a couple of preliminary definitions [3]:

1) Let \( N^{irr} \) be the set of real irreducible normalized polynomials of three variables \( x^1, x^2, x^3 \). Let \( Z_0 \) be a real vector space having a basis indexed by products of elements of \( N^{irr} \). Denote the basis of \( Z_0 \) by \( Z_{u_1...u_m} \) where \( u_1...u_m \) are elements of \( N^{irr} \). The vector space \( Z_0 \) becomes an algebra by defining a product \( \bullet \)

\[
Z_{u_1...u_m} \bullet Z_{v_1...v_n} = Z_{u_1...u_m v_1...v_n} \quad (57)
\]
called the Zariski product.

Let \( Z_{\hbar} \) be the vector space of polynomials in \( \hbar \) (the deformation parameter) with coefficients in \( Z_0 \). Then let

\[
\zeta(\sum r \hbar^r u_r) = \sum r \hbar^r Z_{u_r}, \quad (58)
\]
The deformed Zariski product is defined as

\[ Z_{u_1 \ldots u_m} \circ h Z_{v_1 \ldots v_n} = \zeta((u_1, \ldots, u_m) \times \alpha (v_1 \ldots v_n)) \] (59)

where the operation \( \times_\alpha \) for two irreducible polynomials \( u_1 \) and \( v_1 \) is given by the following formula

\[ u_1 \times_\alpha v_1 = \frac{1}{2} (u_1 \ast v_1 + v_1 \ast u_1) \] (60)

and where \( \ast \) is the usual Moyal product with respect to two variables \( x_1 \) and \( x_2 \). In general, one has to fully symmetrize the \( \ast \) product of irreducible polynomials. Also, the deformed product is extended from \( Z_0 \) to \( Z_\hbar \) by requiring that it annihilates the non-zero powers of the deformation parameter.

2) Let \( E = Z_0[y^1, y^2, y^3] \) be the algebra of polynomials in three variables \( y^1, y^2, y^3 \) with coefficients in \( Z_0 \). Let \( A_0 \) be a subalgebra of \( E \) generated by the following “Taylor series” in \( E \)

\[ J(Z_u) = Z_u + \sum_i y^i Z_{\partial_i u} + \frac{1}{2} \sum_{i,j} y^i y^j Z_{\partial_i \partial_j u} + \ldots \equiv \sum_n \frac{1}{n!} (\sum_i y^i \partial_i)^n (Z_u) \] (61)

where \( \partial_i Z_u \equiv Z_{\partial_i u} \) and \( u \) are normalized polynomials in three variables \( x^1, x^2, x^3 \) that can be uniquely factorized in \( u_1, \ldots, u_m \), and \( \partial_i \) is a derivative with respect to \( x^i \). Define the following generalized derivative

\[ \Delta_a(J(Z_u)) \equiv J(Z_{\partial_a u}) \] (62)

where \( a = 1, 2, 3 \).

Then the classical Nambu bracket on \( A_0 \) is defined as

\[ [A, B, C] \circ = \sum_p \epsilon(p) \Delta_p A \circ \Delta_{p_1} B \circ \Delta_{p_3} C \] (63)

for three elements \( A, B, C \) from \( A_0 \).

It can be shown that there exists a deformation of this Nambu bracket in the sense of deformation quantization.

Let \( E_\hbar \) be the algebra of polynomials of \( \hbar \) (the deformation parameter) with coefficients in \( E \) defined above. Consider a subspace \( A_\hbar \) of \( E_\hbar \) consisting of polynomials in \( \hbar \) with coefficients in \( A_0 \). Then the authors of [3] define a deformed product

\[ J(Z_u) \circ h J(Z_v) = Z_u \circ h Z_v + \sum_i y^i (Z_{\partial_i u} \circ h Z_u + Z_u \circ h Z_{\partial_i u} Z_u) + \ldots \] (64)
where the deformed Zariski product of basis elements is defined as before.

Then one can define the quantum Nambu bracket on $A_{\hbar}$ as follows

$$[A, B, C]_{\hbar} = \sum_p \epsilon(p) \Delta_{p_1} A \cdot_{\hbar} \Delta_{p_2} B \cdot_{\hbar} \Delta_{p_3} C$$  \hspace{1cm} (65)

for three elements $A, B, C$ from $A_{\hbar}$.

It can be shown \cite{5} that this quantum Nambu bracket satisfies all three fundamental properties: it is totally antisymmetric, it satisfies the derivation property and the Fundamental Identity.

Therefore, this construction solves the quantization problem.

Obviously the same construction can be applied to the usual Poisson bracket. In that case the Zariski quantized Poisson bracket is not the skew-symmetrized form of an associative (star) product.

Nevertheless, one can define a Moyal-Zariski product by replacing \cite{5} \newcommand{\hbar}{\hbar}

$$\exp[i\hbar/2\mathcal{P}] \to \exp[i\hbar/2\mathcal{P}_{\hbar}]$$  \hspace{1cm} (66)

in the defining formula for the Moyal bracket. This procedure gives another associative deformation of the usual product of functions. The skew-symmetrized form of this associative deformed product will provide a Lie algebra deformation. Note that the leading term in the expansion of such a bracket is given by the Zariski quantized Poisson bracket!

This fact is of crucial importance for our proposal concerning the link between covariant Matrix theory and Zariski deformation quantization (see next section).

We conclude this section with the following comment:

The star product relevant for the general deformation quantization of the Poisson bracket, as discovered by Kontsevich, can be understood from the path integral point of view \cite{20}. It is natural to ask whether the star product (such as the deformed Zariski product) relevant for the quantization of the triple Nambu bracket can be understood from the path integral point of view. One obvious idea that comes to mind is that such a path integral has to be over loop variables, which appear naturally in the Hamiltonian formulation of the Nambu mechanics \cite{12}.
4 Covariant Matrix Theory and Zariski Quantization

In this section we propose that the problem of covariantization of Matrix theory can be solved using the formalism of Zariski quantization.

Recall that Matrix theory uses \( N \times N \) hermitian matrices to represent the transverse coordinates (and their super-partners) of \( N \) D0-branes in the (discretized) light-cone frame \( X^i \) \( (i = 1, 2, \ldots, 9) \). The statistics of D0-branes are encoded in the \( U(N) \) gauge symmetry \( t = X^+ = \text{light-cone time} \) \( X^i \rightarrow UX^i U^{-1} \). The number \( N \) of D0-branes is connected to the longitudinal momentum \( P^+ \) in the light-like direction by \( P^+ = \frac{N}{R} \) where \( R = g_s \ell_s \) is the compactified radius in the \( X^- \) direction. Also \( \alpha' = l_p^3/R \), where \( l_p \) is the 11-dimensional Planck length.

What should be the structure of a covariant Matrix theory? We expect at least

1. Generalization of the matrix algebra and the emergence of higher symmetry

2. \( R \) (and perhaps \( N \)) should appear as dynamical variables.

At present, the only guiding principle in trying to search for such a formulation is that the formalism should reduce in the light-cone gauge to Matrix theory after appropriately fixing the gauge using the higher symmetry, and that it should have 11-dimensional (super) Poincare invariance in the limit \( R \rightarrow \infty \).

The light-cone degrees of freedom (D0-branes) carry the quantum number of 11-dimensional (super)gravitons. Thus we expect that the covariant version of Matrix theory will describe a many-body quantum theory of interacting 11-dimensional (super)gravitons.

The covariant theory is also expected to be a quantum mechanical theory invariant under world-line reparametrizations, simply because in the light-cone gauge we have a quantum mechanical theory with a globally defined time.

Of course it is tempting to try to quantize the world-volume membrane theory for which we know a classical covariant action principle in view of the results in sections 2. In particular, as we have explicitly said in section
2., the expression for the world volume of a membrane is invariant under the classical 3D volume preserving diffeomorphisms. Thus it is natural to expect that a discretized membrane theory should be formulated in terms of discretized volume preserving diffeomorphisms. (A preliminary attempt at discretization of the world-volume membrane theory was reported in [21].)

This idea appears natural as a generalization of the following pictorial correspondence

\[
\begin{align*}
\text{matrices/commutators} & \leftrightarrow \text{2D surface} \\
U(N) & \leftrightarrow \text{2D APD} \\
\downarrow & \\
\text{triple Nambu bracket} & \leftrightarrow \text{3D volume} \\
? & \leftrightarrow \text{3D diffeos}
\end{align*}
\]

since the two indices of the square matrices in Matrix theory just correspond to the discretized Fourier indices on the membrane.

In the following we propose a precise algebraic structure that replaces the question mark in the above diagram.

### 4.1 The Proposal

Here we state a concrete proposal for the covariantization of Matrix theory. For simplicity, we discuss the bosonic part only. Our proposal can be extended to include maximal supersymmetry.

We propose that the following action defines a covariant Matrix theory

\[
S_M = -\int d^3\xi \sqrt{\frac{1}{6}[X^\mu, X^\nu, X^\rho]^2} \frac{1}{N}.
\]

(67)

Here \(\mu = 0, 1, ..., 10\) and \(X^\mu\) are elements of \(A_{1/N}\) as defined in section 3. -in other words they are polynomials in \(1/N\) with coefficients in \(A_0\) (recall that \(A_0\) is the algebra of "shifted" polynomials of three variables, in this case, \(\xi_1, \xi_2, \xi_3\)). We take \(1/N\) to be the deformation parameter of the Zariski quantization.

\(N\) is interpreted as the number of 11-dimensional (super)gravitons.
The action $S_M$ is invariant under Zariski deformed world-volume diffeomorphisms $\delta X^\mu = \epsilon^i \bullet \Delta_i X^\mu$, with $\Delta$ from section 3.

Due to the usual properties of the product $\bullet_\bar{\Delta}$ (recall that the deformed product is Abelian, associative and distributive and the formal derivative $\Delta_a$ respects linearity, the Leibnitz rule and the commutativity of the derivatives in many variables) we can formally repeat the same steps of fixing the light-cone gauge for this theory, as in section 2.

Thus, the light cone action is given by a Zariski deformed version of the usual-light cone action for the M-theory membrane, with a deformation parameter $1/N$

$$S_{lc} = 1/2 \int d^3 \xi ((\Delta_0 X^a)^2 - (1/2)[X^a, X^b]_\bar{\Delta}^2) \cdot 1_N.$$  

(68)

Now we use the crucial observation from section 3. The skew-symmetrized form of the Moyal-Zariski associative deformed product provides a Lie algebra deformation, and is thus in the same class as the commutator of matrices. Moreover, the leading term (in $N$) in the expansion of such a bracket is given by the Zariski quantized Poisson bracket.

In view of this fact, it is natural to propose that this Zariski deformed light-cone action is in the same universality class as the Matrix theory action [1], where $N$ is the number of D0-branes.

Thus the above covariant action can be viewed as a description of a large $N$ quantum mechanical theory of $N$ interacting 11-dimensional (super)gravitons.

Notice that our proposal can be formulated for the Polyakov type action as well, due to the properties of the deformed Zariski product. Because of the usual properties of the deformed product $\bullet_\bar{\Delta}$ it is possible to formulate our proposal for the supersymmetric case as well.

Finally we comment on the $N \rightarrow \infty$ limit of our proposal.

The $N \rightarrow \infty$ of the action $S_M$ reduces to the bosonic part of the usual M-theory membrane action with space-time coordinates $x^\mu$ being the elements of the algebra $A_0$, where $\bullet$ defines the two product and $\Delta$ the formal derivation with respect to the world-volume coordinates $\xi$. If our proposal is correct this $N \rightarrow \infty$ limit theory should be in the same universality class as the ordinary bosonic membrane [7]. The same comments apply to the supersymmetric case.
As in Matrix theory, the membrane is just one sector of the theory. Other physical sectors should exist as well.

We conclude this subsection with a list of a few obvious questions:
- is there an explicit matrix version of our conjecture?
- what is the physical interpretation of anti-branes in our proposal?
- does our proposal include the M-theory five-brane in the large N limit?

At present we do not have definitive answers to these important questions, so we postpone their discussion for the future.

5 Discussion

In this, concluding section, we compare the construction of quantum Nambu brackets in terms of square and cubic matrices \cite{13} to the Zariski deformation quantization scheme and discuss a natural mathematical formulation of the space-time uncertainty principle of M-theory \cite{14} in view of our proposal.

5.1 Square and Cubic Matrices vs. Zariski Quantization

An explicit matrix realization of the quantum Nambu bracket, which is skew-symmetric and obeys the Fundamental Identity was given in \cite{13}. This realization was contrasted to the example constructed in \cite{16}.

Define a totally antisymmetric triple bracket of three matrices \(A, B, C\) as

\[
[A, B, C] \equiv (\text{tr}A)B, C + (\text{tr}B)C, A + (\text{tr}C)A, B.
\]

(69)

Then \(\text{tr}[A, B, C] = 0\), and if \(C = 1\), \([A, B, 1] = N[A, B]\), where \(N\) is the rank of square matrices. This bracket is obviously skew-symmetric and it can be shown to obey the Fundamental Identity \cite{13}.

Given this example of a three-dimensional quantum Nambu bracket, consider the following "gauge transformation"

\[
\delta A \equiv i[X, Y, A],
\]

(70)

where the factor \(i\) is introduced for Hermitian matrices. This transformation represents an obvious quantum form of the three-dimensional volume
preserving diffeomorphisms. By the definition of the triple bracket, the generalized gauge transformation takes the following explicit form

\[ \delta A = i \left( \left( \text{tr}X \right) Y - \left( \text{tr}Y \right) X, A \right) + \left( \text{tr}A \right) \left[ X, Y \right] \]. \tag{71} \]

It is shown in [13] that if \( \text{tr}A_i = 0, i = 1, \ldots, n \), then

\[ \text{tr}(A_1 A_2 \ldots A_n) \] \tag{72} is gauge invariant. Note that the gauge transformation of a commutator does not satisfy the usual composition rule, namely

\[ \left[ X, Y, \left[ A, B \right] \right] \neq \left[ \left[ X, Y, A \right], B \right] + \left[ A, \left[ X, Y, B \right] \right] \]

Similar comments apply to \( \left[ \left[ X^i, X^j \right], \left[ X^l, X^k \right] \right] \).

Notice that the form of the gauge transformation (71) indicates that a bosonic Hermitian matrix \( A \) can be transformed into a form proportional to the unit \( N \times N \) matrix as long as \( \text{tr}A \neq 0 \). In other words, since the gauge transformation is traceless, one can show that a Hermitian matrix can be brought to the following form

\[ A \rightarrow \frac{1}{N} \text{tr}A 1_N. \]

This matrix realization of the triple quantum Nambu bracket can be generalized to a representation in terms of three-index objects or cubic matrices [13].

To this end, introduce the following generalization of the trace

\[ \langle A \rangle \equiv \sum_{pm} A_{pmp}, \quad \langle AB \rangle \equiv \sum_{pqm} A_{pqm} B_{qmp}, \quad \langle ABC \rangle \equiv \sum_{pqrm} A_{pqm} B_{qmr} C_{rmp}, \tag{73} \]

which satisfy \( \langle AB \rangle = \langle BA \rangle \) and \( \langle ABC \rangle = \langle BCA \rangle = \langle CBA \rangle = \langle CAB \rangle \). Furthermore, define a triple-product

\[ \langle ABC \rangle_{ijk} \equiv \sum_{p} A_{ijp} \langle B \rangle C_{pjk} = \sum_{pqm} A_{ijp} B_{qmp} C_{pjk}. \tag{74} \]

and the following skew-symmetric quantum Nambu bracket

\[ [A, B, C] \equiv \langle ABC \rangle + \langle BCA \rangle + \langle CAB \rangle - \langle CBA \rangle - \langle ACB \rangle - \langle BAC \rangle. \tag{75} \]
The middle index \( j \) of \( A_{ijk} \) can be treated as an internal index for the matrix realization of the triple quantum Nambu bracket. Note also that \( \langle (ABC) \rangle = \langle B \rangle \langle AC \rangle \neq \langle ABC \rangle \) and \( \langle (ABC)D \rangle = \langle B \rangle \langle ACD \rangle \).

Then by using the following relations

\[
((ABC)DE) = ((ADC)BE) = (AB(CDE)) = (AD(CBE)),
\]

\[
(A(BCD)E) = (A(DCB)E),
\]

one can directly prove that the skew-symmetric Nambu bracket \( (75) \) with the triple-product \( (74) \) obeys the Fundamental Identity \( [13] \).

The “trace” \( \langle AB \rangle \) has the property

\[
\langle [X, Y, A]B \rangle + \langle A[X, Y, B] \rangle = 0,
\]

provided \( \langle A \rangle = \langle B \rangle = 0 \). Therefore, since \( \langle [A, B, C] \rangle = 0 \) for any three-index objects \( A \), \( B \) and \( C \), the trace of the product of Nambu brackets \( \langle [A, B, C][D, E, F] \rangle \) is gauge invariant. Notice that if one generalizes the trace \( (73) \) as

\[
\langle A_1 A_2 \cdots A_n \rangle \equiv \sum_{p_1, p_2, \ldots, m} A_{i_1}^{(1)} A_{i_2}^{(2)} \cdots A_{i_n}^{(n)} p_1 p_2 \cdots p_n, \quad n = 1, 2, \ldots, (77)
\]

one can also demonstrate that the trace of any product of Nambu brackets \( \langle [A, B, C][D, E, F] \cdots [X, Y, Z] \rangle \) is gauge invariant \( [13] \).

Define \( I_{ijk} \equiv \delta_{ik}^{(j)} \), where \( \delta_{ik}^{(j)} = 0 \), if \( i \neq k \), for any \( j \) and \( \delta_{ik}^{(j)} = 1 \), if \( i = k \), for any \( j \). Then

\[
(AIB) = \langle I \rangle \sum_p A_{ijp} B_{pj}, \quad (IAB) = \langle BA \rangle = \langle A \rangle B,
\]

\[
(IAI) = \langle A \rangle \delta_{ik}^{(j)}, \quad (IIA) = \langle IA \rangle = \langle I \rangle A,
\]

and \( [A, I, B] = \sum_p (A_{ijp} B_{pj} - B_{ijp} A_{pj}) \). Hence for any middle index \( j \), \([A, I, B]\) reduces to the usual commutator \( [A^{(j)}, B^{(j)}] \) for the matrices \( A^{(j)}_{ik} \equiv A_{ijk} \) and \( B^{(j)}_{ik} \equiv B_{ijk} \).

Other examples of triple-products \( (ABC)_{ijk} \) which also satisfy the same relations as eq. \( (76) \) and hence lead to the F.I. for the skew-symmetric Nambu bracket \( (75) \) were listed in \( [13] \)

\[
\sum_{pq} A_{ijp} B_{iqj} C_{pjk}, \quad \sum_{pqmn} A_{ijp} B_{qmn} C_{pnm}, \quad \sum_{pqmn} A_{iqp} B_{qmn} C_{pjk}. \quad (79)
\]
The explicit examples of the quantum Nambu bracket presented above do not satisfy the derivation property. Also, the form of the above quantum Nambu brackets does not completely parallel the form of the classical Nambu bracket.

The classical triple Nambu bracket of three functions $f, g, h$ of three variables $\tau, \sigma_1, \sigma_2$ can be obviously rewritten as

$$\{f, g, w\} = \dot{f}\{g, w\} + \dot{g}\{w, f\} + \dot{w}\{f, g\}$$

(80)

where $\dot{f} = \partial_\tau f$ and $\{f, g\} = \partial_{\sigma_1} f \partial_{\sigma_2} g - \partial_{\sigma_2} f \partial_{\sigma_1} g$.

If we try to extrapolate our previous definition of the quantum Nambu bracket in terms of $\tau$-dependent square matrices to

$$[F, G, W] = \dot{F}[G, W] + \dot{G}[W, F] + \dot{W}[F, G]$$

(81)

where $\dot{F} = \partial_\tau F$ and the usual matrix multiplication is assumed, one can show that such $[F, G, W]$ does not satisfy the Fundamental Identity $^3$. From the analogy with the covariant membrane and our discussion in section 4., we interpret $\tau$ as a potential world-line parameter.

Obviously the Zariski deformation quantization provides us with a quantum Nambu bracket which has the same properties as its classical counterpart and therefore can be used for the discretization of $p$-brane actions as discussed in subsection 2.3. That is why our proposal toward a covariant formulation of Matrix theory from section 4. was formulated in terms of the algebraic structures needed in the Zariski deformation quantization.

It is not clear at present whether the Zariski deformation quantization has an explicit representation in terms of matrices (either square or cubic).

### 5.2 Space-Time Uncertainty Relation in M-theory

The main signature of the space-time uncertainty relation $^{14}$ is the opposite scaling of the transverse and longitudinal directions with respect to the fundamental string (in perturbative string theory) or with respect to a $Dp$-brane in non-perturbative string/M theory (see also $^{22}$). In perturbative string theory the space-time uncertainty relation incorporates the effects of conformal symmetry. In non-perturbative string/M theory, the space-time

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$^2$This was explicitly shown by H. Awata - private communication.
uncertainty relation captures the essential features of the physics of D-branes as well as the property of holographic behavior \[23\]. Thus it is reasonable to expect that the space-time uncertainty relation says something important about the underlying physical foundation of M-theory. Here we want to comment on the mathematical structure underlying the space-time uncertainty relation in view of our proposal from section 4.

Perhaps the simplest way of characterizing the space-time uncertainty relation is by writing

$$\delta T \delta X \sim \alpha', \quad (82)$$

where $\delta T$ and $\delta X$ measure the effective longitudinal (along the world-volume of a p-brane) and transverse space-time distances. This equation is true in Matrix theory \[1\]

$$X^a \rightarrow \lambda X^a, t \rightarrow \lambda^{-1} t, \quad (83)$$

provided the longitudinal distance is identified with the global time of Matrix theory. The effective super Yang-Mills theory which describes the dynamics of $D0$-branes is invariant under (83) if the string coupling constant is simultaneously rescaled

$$g_s \rightarrow \lambda^3 g_s. \quad (84)$$

The space-time uncertainty relation leads readily to the well known characteristic space-time scales in M-theory \[14\]. Note that eq. (83) can also be applied to $Dp$-branes, provided

$$g_s \rightarrow \lambda^{3-p} g_s. \quad (85)$$

What is the natural mathematical set-up for the space-time uncertainty relation in string theory (82)? It was proposed by Yoneya and Li and Yoneya \[14\] that the right point of view for discussing the space-time uncertainty principle in string theory, is to treat all space-time coordinates as infinite dimensional matrices.

Remarkably, the string theory space-time uncertainty relation can be understood as a limit of the space-time uncertainty relation in M-theory as noticed by Li and Yoneya \[14\]. In Matrix theory eq. (82) can be rewritten as

$$\delta T \delta X_T \sim l_p^3/R, \quad (86)$$

where $\delta X_T$ and $\delta T$ respectively measure transverse spatial and time directions.
Li and Yoneya have proposed a more general relation \cite{14} by observing that the uncertainty for the longitudinal direction in physical processes that involve individual D0-branes is $\delta X_L \sim R$. Then \cite{84} reads
\begin{equation}
\delta T \delta X_T \delta X_L \sim l_p^3.
\end{equation}
This is the space-time uncertainty relation in M-theory. Note that only the fundamental length scale of M-theory ($l_p$) figures in this relation.

The point we want to make is that the form of the space-time uncertainty relation in M-theory is very reminiscent of the form of the quantum triple Nambu bracket we have discussed in the previous section \cite{13}. In particular it is natural to extend the proposal of Yoneya and Li and Yoneya \cite{14} for the case of Matrix theory, and suggest a covariant version of the space-time uncertainty relation in Matrix theory which is consistent with (87)
\begin{equation}
[X^\mu, X^{\nu}, X^{\lambda}]_{\mathcal{H}}^2 \sim l_p^6
\end{equation}
Here $\mu, \nu, \lambda = 0, 1, ..., 10$. The triple bracket in the above formula is the quantum triple Nambu bracket in the sense of Zariski deformation quantization with the deformation parameter $\frac{1}{N}$ and $X^{\mu}$'s are elements of $A_{\mathcal{H}}$ as in the previous section.

In view of the relation between the space-time uncertainty principle \cite{14} and the UV/IR relation \cite{22} and the holographic principle \cite{23} it is natural to expect that this mathematical formulation of the space-time uncertainty principle captures the mathematical content of the holographic principle in the flat space-time limit.

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