Approximated integrability of the Dicke model

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Abstract – A very approximate second integral of motion of the Dicke model is identified within a broad energy region above the ground state, and for a wide range of values of the external parameters. This second integral, obtained from a Born-Oppenheimer approximation, classifies the whole regular part of the spectrum in bands, coming from different semi-classical energy surfaces, and labelled by its corresponding eigenvalues. Results obtained from this approximation are compared with exact numerical diagonalization for finite systems in the superradiant phase, obtaining a remarkable accord. The region of validity of our approach in the parameter space, which includes the resonant case, is unveiled. The energy range of validity goes from the ground state up to a certain upper energy where chaos sets in, and extends far beyond the range of applicability of a simple harmonic approximation around the minimal energy configuration. The upper energy validity limit increases for larger values of the coupling constant and the ratio between the level splitting and the frequency of the field. These results show that the Dicke model behaves like a two-degree-of-freedom integrable model for a wide range of energies and values of the external parameters.

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Introduction. – Despite it was formulated more than 60 years ago, the Dicke model [1] has been object of intense research during the last years. One of the reasons is that it shows a very rich and complex behavior, in spite of being quite simple. It is characterized by thermal [2,3], quantum (QPT) [4] and excited-state quantum (ESQPT) [5] phase transitions, giving rise to superradiance. These phase transitions are also extended to non-equilibrium driven systems [6]. Besides, the Dicke model shows a transition from regularity to chaos, observable both in the normal and in the superradiant phases [7]. Theoretical studies on this model are enhanced by recent experimental results. In cavity QED, the superradiant phase transition is forbidden by a no-go theorem [8]. However, this kind of transition has been observed in several systems [9], and the Dicke model itself has been simulated by means of a Bose-Einstein condensate in an optical cavity [10]. A dynamical non-equilibrium superradiant phase transition has been also observed [11]. The same model with just one atom, the Rabi model, has been explored by means of superconducting QED [12]. Finally, similar techniques have been applied to few-atoms Dicke models [13].

For all these reasons, a precise knowledge of the spectrum of the Dicke model is a theoretical challenge and a necessity. The low-lying spectrum can be approximated by two independent harmonic oscillators through a Holstein-Primakoff approximation in the thermodynamical limit [4]. However, specially in the superradiant phase, this method only works in a very narrow region around the ground state, leaving unexplained the whole non-chaotic or regular energy region that extends far beyond the validity of the quadratic approximation. This regular energy region goes from the ground to an upper energy (above or below the ESQPT critical energy) that depends strongly on the level splitting and frequency of the field [14]. On
the other hand, it has been recently shown that the Rabi model (included the driven one) can be considered as integrable [15], though there is some controversy about this fact [16]. A method similar to the one used in [15] has been recently applied to the Dicke model [17,18], but it does not generate a closed formula for the spectrum. This should be expected as the Hamiltonian does not have as many integrals of motion as degrees of freedom, and thus in that sense it is non-integrable. Therefore, a general concise analytic solution of the Dicke model is far to be obtained.

In this letter we derive explicitly a second integral of motion of the Dicke model, very approximately holding within its whole non-chaotic energy regime and a wide range of values of the external parameters, specially in the superradiant region. Its range of applicability is estimated analytically, and unveiled by means of stringent numerical calculations.

Our derivation is based on a Born-Oppenheimer approximation (BOA) [19] where the fast and slow variables are, respectively, the atomic and bosonic ones. Similar approaches have already been reported for the Dicke and related models. Regarding the Dicke model, the BOA has been used to study its ground-state properties. In ref. [20] the finite-size scaling of the entanglement between the components of the system and other physical observables is reported, and in ref. [21] the BOA is employed to study the finite-size dependence of the tunnelling-driven ground-state energy splitting in the superradiant phase. The BOA has, likewise, been applied to the one-atom Dicke model, the Rabi, as well as its rotating-wave approximated Jaynes-Cummings model. In ref. [22] the BOA is used to determine the entanglement of the single atom to the bosonic variable, whereas in ref. [23] it is used to unveil some previously unnoticed aspects of the rotating-wave approximation. Finally, a different and complementary full quantum BOA valid for an arbitrary number of atoms, where the fast variables are the bosonic ones, is reported in [24].

This letter is organized as follows. After presenting the BOA and the analysis of its regions of applicability in the model parameter space, we give numerical evidence showing that the whole regular part of the Dicke dynamics can be understood from the adiabatic invariant coming from the BOA. Finally, we show the ability of the BOA to reproduce the regular exact Dicke spectrum.

**The Dicke model and its second integral of motion.** – The Dicke model depicts the interaction of $N$ two-level atoms with a single bosonic mode. The atoms can be described by means of collective pseudospin operators, giving rise to ($\hbar = 1$)

\[
\hat{H} = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{J}_z + \frac{2\gamma}{\sqrt{2J}} \hat{J}_x (\hat{a}^\dagger + \hat{a}),
\]

where $\omega$ is the frequency of the bosonic mode, $\omega_0$ the level splitting, $\gamma$ the coupling constant, and $j$ the total pseudospin $j = N/2$. This model can be numerically solved for tens or even few hundreds [25] of atoms, depending on $\gamma$. The main difficulty is that the bosonic Hilbert space is infinite and has to be truncated for an exact diagonalization; the larger the coupling constant, the more bosons are required for convergence.

Here we develop a very approximated solution using a Born-Oppenheimer approximation, valid when a fast system is coupled to a much slower one. When this condition is fulfilled, the fast motion can be solved for frozen (adiabatically changing) values of the slow coordinates; and then solved for the slow motion, by considering temporal averages of the fast coordinates. We apply this idea to the Dicke model, considering that the atomic are the fast coordinates, whereas the bosonic are the slow ones. To implement this approach, we consider a semi-classical approximation for the slow variables, $\hat{a} = (1/\sqrt{2})(\hat{q} + i\hat{p})$, from which we obtain

\[
\hat{H}(p,q) = \frac{\omega}{2} (p^2 + q^2) + \omega_0 \hat{J}_z + \frac{2\gamma}{\sqrt{J}} q \hat{J}_x.
\]

If we consider $q$ and $p$ as fixed parameters, this Hamiltonian can be exactly diagonalized by means of a rotation around the y-axis,

\[
\omega_0 \hat{J}_z + \frac{2\gamma}{\sqrt{J}} q \hat{J}_x = \sqrt{\omega^2 + \left(\frac{2\gamma q}{\sqrt{J}}\right)^2} \hat{J}_z' \equiv \omega p(q) \hat{J}_z'.
\]

Considering that the critical coupling of the quantum phase transition is $\gamma_c = \sqrt{\omega_0 \omega}/2$, this rotation entails

\[
H(p,q) = \frac{\omega}{2} (p^2 + q^2) + \omega_0 \sqrt{1 + f^2 \frac{\omega^2 q^2}{\omega_0^2}} \hat{J}_z',
\]

where $f = \gamma/\gamma_c$. This Hamiltonian describes the Larmor precession, with frequency $\omega p(q)$, of the pseudospin around an adiabatically changing $q$-dependent axis. By applying the Hamiltonian to the eigenvector of $\hat{J}_z'$, $\hat{J}_z'[j,m';q] = m'[j,m';q]$, we obtain the eigenvalues

\[
E_m'(p,q) = \frac{\omega}{2} (p^2 + q^2) + \omega_0 \sqrt{1 + f^2 \frac{\omega^2 q^2}{\omega_0^2}} m'.
\]

These eigenvalues define effective semi-classical Hamiltonians for the slow bosonic variables, with a number of important consequences:

i) In the region where the approximation is valid, the spectrum of the Dicke model is divided in bands, each one corresponding to a semi-classical energy surface $E_m'(p,q)$, labelled by the quantum number $m'$. Each one has just a degree of freedom and hence is classically integrable. The ground state belongs to the band with $m' = -j$.

ii) The expectation value of any physical observable $\hat{O}(\hat{a}^\dagger, \hat{a}; \hat{J}_x, \hat{J}_y, \hat{J}_z)$, in stationary states at given energy $E$, can be evaluated semi-classically

\[
\langle \hat{O} \rangle = \frac{\int dp dq O(p,q) \delta [E - E_m'(p,q)]}{\int dp dq \delta [E - E_m'(p,q)]},
\]

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\]
where $O(p, q) = \langle j, m' | \hat{O}(\hat{a}^\dagger, \hat{a}; \hat{J}_x, \hat{J}_y, \hat{J}_z) | j, m' \rangle$, and the bosonic operators ($\hat{a}^\dagger, \hat{a}$) are written in terms of their classical limit ($p, q$).

iii) The energy levels of the Dicke model are labelled by two quantum numbers, $E_{m', n}$. The first one, $m'$, identifies the energy surface. The second one, $n$, determines the position of the energy level inside the corresponding band. This quantum number can be obtained from the Einstein-Brillouin-Keller (EBK) action quantization rules [26],

$$\int_{C_{m'}} pdq = 2\pi (n + \alpha_n),$$  \hspace{1cm} (7)

where $C_{m'}$ is a closed curve for the $m'$-th effective Hamiltonian, and $n$ an integer number. The numbers $\alpha_n$, related to the Maslov index, are equal to 1/4 for each turning point, giving rise to $\alpha_n = 1/2$. Finally, since the vacuum energy of the harmonic oscillator, $E_0 = \omega/2$, has been removed in the Dicke model, the energy values obtained from eq. (7) have to be shifted by the same amount. We shall see later that this procedure constitutes a very good approximation to the energy spectrum of the Dicke model, in a wide energy range above the ground state.

iv) $\hat{J}_z$, is an adiabatic invariant, semi-classical integral of motion. Thus, its quantum version

$$\hat{J}_z = \frac{\hat{J}_z + \sqrt{f^2 \omega^2 + \frac{1}{2} z^2} (\hat{a} + \hat{a}^\dagger) \hat{J}_x}{\sqrt{1 + f^2 \omega^2 + \frac{1}{2} z^2} (\hat{a} + \hat{a}^\dagger)^2},$$ \hspace{1cm} (8)

is expected to be a very approximated Hamiltonian commuting operator, which is a consequence of the fact that in the BOA, the Hamiltonian eigenfunctions are of the form $\Psi(q)|j, m'; q\rangle$ (see below).

As long as the method of freezing the slow bosonic variables holds, the Dicke model can be separated in a set of semi-classical one-degree-of-freedom integrable models. A bound for this approximation can be obtained calculating the non-adiabatic coefficients (NACs). For this purpose, we express the eigenfunctions of the Dicke model as a superposition of different eigenvectors of $\hat{J}_z$,

$$|\Psi\rangle = \sum_m \Psi_m(q) |jm; q\rangle,$$

yielding to the following eigenvalue equation:

$$\omega \sum_m \hat{p} \Psi_m(q) |jm; q\rangle |jm; q\rangle + \Psi_m(q) |jm; q\rangle |p^2; jm; q\rangle + \left(\frac{1}{2} \omega q^2 + V_{nm}(q) - E\right) \Psi_m(q) = 0,$$

where $V_{nm}(q) = \omega q^2/2 + \omega P(q)m'$ is the adiabatic potential. If the first and second terms, which couple different $m'$ sectors, are neglected, we obtain the BOA and the eigenfunctions are of the form $\Psi_m(q)|jm; m'; q\rangle$. Hence, corrections to the BOA come from $|jm'; q|p^2; jm; q\rangle$, since $|jm'; q|p^2; jm; q\rangle$ can be directly obtained from the former.

These NACs can be calculated exactly

$$\langle jm'; q|p|jm; q\rangle = -i \left\langle jm'; q \frac{d}{dq} jm; q\right\rangle = -i \left\langle jm'; q \frac{d}{dq} \Psi(q) |jm; q\rangle - \Psi(q) |jm; q\rangle \frac{d}{dq} \right\rangle,$$

where $\beta(q)$ is the rotation angle given $\cos \beta(q) = \frac{\omega q}{\omega P(q)}$, and $(jm)$ are eigenvectors of $\hat{J}_z$. Since the rotation connecting $\hat{J}_z$ with $\hat{J}_z'$ is around the $y$-axis, the operator $\hat{J}_y = \hat{J}_y'$ and the matrix elements can be calculated straightforwardly. After calculating the derivative of $\beta(q)$, the NACs are

$$\langle jm'; q|p|jm; q\rangle = \frac{C_{nm}(j, m)}{\sqrt{J}} (C_{+}(j, m)\delta_{m', m+1} - C_{-}(j, m)\delta_{m', m-1}),$$

where $C_{\pm}(j, m) = \sqrt{j(j+1) - m(m \pm 1)}$. These NACs couple adjacent sectors of $m'$ and have to be small in order for the BOA to work adequately. Their absolute value can be expressed as

$$|C_{nm}(j, m)| \sim \frac{f^3}{\sqrt{J}} \frac{\omega}{\omega_0} \Psi_{nm}(j, m).$$ \hspace{1cm} (9)

These NACs are small if $\omega_0 \gg \omega$, and, except in $q = 0$, they can be made arbitrary small for large enough coupling $f \gg 1$. A more precise estimate can be done in the ground-state region. Here, the value of $q$ can be obtained from the semi-classical approximation [4] and $C_{\pm}(j, -j) = \sqrt{27}$, since the ground state always belong to the $m' = -j$ energy surface. These results lead to

$$|C_{nm}(j, m)| \sim \frac{1}{f^3} \frac{\omega}{\omega_0}.$$ \hspace{1cm} (10)

Therefore, we conclude that the BOA is valid in the ground-state region either if $f \gg 1$ or $\omega_0 \gg \omega$. In particular, it is worth noting that the BOA is valid in resonance, provided that $f$ is large enough. Above this region, we can draw a generic picture stating that the larger the coupling constant $\gamma$ and/or the smaller the ratio $\omega/\omega_0$, the wider the region of energies for which this scenario is expected to work. This is specially interesting because it has been recently shown that the limit $\omega/\omega_0 \rightarrow 0$ is equivalent to the thermodynamic limit $j \rightarrow \infty$, giving rise to QPTs and ESQPTs in the Rabi model [27]. The results we show in this letter suggest that the same behavior is expected for the complete Dicke model, with any finite number of atoms.

To complement this analysis, we show in fig. 1 the adiabatic potentials, $V_{nm}(q)$, and the non-adiabatic coefficients $C_{nm}(q)$ for two paradigmatic cases: in panels (a) and (b) we show the off-resonant case, $\omega = 1 + \omega_0 = 5$, with $f = 3$, and in panels (c) and (d) we show a resonant case, $\omega = \omega_0 = 1$, with $f = 5$; both are obtained with $j = 15$. 

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Red potentials have a double-well structure, and correspond for \( q = 0 \). Therefore, for a given adiabatic potential \( m' \), the worst performance or even break down of the BOA is expected at energies allowing to explore this region. For the adiabatic potentials with spontaneous symmetry breaking, \(-j \leq m' < -j/f^2\) (red in fig. 1), this happens at energies close and above \( E \gtrsim \omega_0 m' \), whereas for single-well potentials, \(-j/f^2 < m' \leq j\) (blue in fig. 1), this happens for any allowed energy \( E \geq \omega_0 m' \). Note, however, that even in \( q = 0 \), since \( C_{\text{NAC}}(q = 0) \propto f \sqrt{\omega/\omega_0} \), the BOA can work properly for arbitrary \( m'\) if the ratio \( \omega/\omega_0 \) is small enough (\( \sqrt{\omega/\omega_0} \ll 1/f \)).

**Numerical tests.** – Besides the analytical argument, a stringent test is required to show that the regular energy

As each energy surface starts with values of \( q \) corresponding to the minima of \( V_m(q) \), we see that the NACs are in both cases small in the low-lying energy region of a number of bands, and that this number is larger for the off-resonant case. Hence, we expect that the BOA remains valid in a wide region above the ground state, and that this region is larger in the non-resonant case. As is well known, the NACs are larger in the region where the adiabatic potentials are closer. In our case this region is located around \( q = 0 \). Therefore, for a given adiabatic potential \( m' \), the worst performance or even break down of the BOA is expected at energies allowing to explore this region. For the adiabatic potentials with spontaneous symmetry breaking, \(-j \leq m' < -j/f^2\) (red in fig. 1), this happens at energies close and above \( E \gtrsim \omega_0 m' \), whereas for single-well potentials, \(-j/f^2 < m' \leq j\) (blue in fig. 1), this happens for any allowed energy \( E \geq \omega_0 m' \). Note, however, that even in \( q = 0 \), since \( C_{\text{NAC}}(q = 0) \propto f \sqrt{\omega/\omega_0} \), the BOA can work properly for arbitrary \( m'\) if the ratio \( \omega/\omega_0 \) is small enough (\( \sqrt{\omega/\omega_0} \ll 1/f \)).

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Fig. 3: (Color online) Peres lattices for $a^\dagger a_j$, with $j = 15$. Red circles show the exact results for the Dicke model. Solid lines represent the semi-classical description eq. (6). Panel (a) shows the case $f = 3$ with $\omega = 1$ and $\omega_0 = 5$; panel (b), $f = 5$ with $\omega = \omega_0 = 1$.

Approximately an integer number $m'$. In the insets of both panels we show the distance between the numerical results for the expectation value of $J_z$, and their closest integer number, $m'$. We can see that these distances are very small in a wide region above the ground state —up to $E/(j\omega_0) \sim -2$ for the off-resonant case, and up to $E/(j\omega_0) \sim -6$ in the resonant case. These results suggest that, at those energies, $\hat{J}_z$ commutes with the corresponding projector $|E_0 \rangle \langle E_0 |$, and thus $m'$ is a good quantum number. To prove this fact, we plot in panels (a) and (c) the number of principal components (NPC) of each Hamiltonian eigenstate in the eigenbasis $|j, m'; q \rangle$, which is defined as the (first) Participation Ratio, NPC = $1/\sum_{m'=1}^{m'} p_{m'}$, where $p_{m'}$ is the probability that an eigenstate lays in the eigenspace generated by $|j, m'; q \rangle$, that is the square modulus of the corresponding eigenstate projected over the space spanned by $|j, m'; q \rangle$. We can see that NPC ≈ 1 in the low-energy region, showing that each eigenstate within this region belongs to the subspace generated by $|j, m'; q \rangle$, being $m'$ the corresponding fixed quantum number. We can also see that this region extends until the ESQPT critical energy, $E/(j\omega_0) = -1$, for the case with $\omega_0 = 5$, except for a small number of energy levels located around $E/(j\omega_0) \sim -2$. In resonance, NPC ≈ 1 up to $E/(j\omega_0) \sim -4$. The former results are fully compatible with the analysis presented in [7,14], where it is shown that in the case $\omega < \omega_0$, the onset of chaos occurs at energies above the ESQPT, whereas in the resonant case it occurs at energies below $E/(j\omega_0) = -1$. Furthermore, our results state that a necessary condition for the onset of chaos is that $\hat{J}_z$ ceases to be an integral of motion, and thus the eigenstates cannot be labelled by two quantum numbers anymore. These results provide a simple explanation for both the approximated conservation rules observed in quenches leaving the system in the superradiant phase, very recently reported in [29], and also the transition from a Poisson to Wigner statistics in the nearest-neighbour energy level distributions as energy is increased, reported in [7,30].

In fig. 3 we show the Peres lattices for the number of bosons $a^\dagger a$, together with the semi-classical description given by eq. (6). We can see that the region of applicability of eq. (6) coincides with the region in which $J_z$ is a good integral of motion. This fact confirms our conclusion saying that the spectrum of the Dicke model is divided in independent bands, each one having one semi-classical degree of freedom, within the regular region. This is specially interesting for the case with $\omega_0 = 5$, shown in panel (a). In the upper band we can see a dip at $E/(j\omega_0) = -1$, which is properly reproduced by the semi-classical description; similar features are seen in the second and third bands, though in a less clear way. This is a neat signature of a one-degree-of-freedom ESQPT, characterized by logarithmic singularities in the density of states and expected values of representative observables [5,31]; the same kind of ESQPT has been recently reported in the Rabi model [27].

So, results shown in fig. 3 suggest that every band has its own ESQPT, provided that the integrable region extends up to the corresponding energy value. This issue will be treated in detail elsewhere.

In fig. 4 we display the results of the EBK quantization rules, eq. (7). In the first two panels, we show level diagrams for the off-resonant, panel (a), and the resonant, panel (b), cases, both with $j = 10$. We display the excitation energy, $\Delta E/(j\omega_0) = (E - E_0)/(j\omega_0)$, being $E_0$ the energy of the ground state, as a function of $f = \gamma/\gamma_c$. We plot the exact eigenvalues with symbols, and the EBK energies, obtained from eq. (7), with solid lines. For both the resonant and the off-resonant cases we have restricted ourselves to a small $f$-window, and to a small number of levels with $m' = -10$, $-9$, and $-8$. The distance between the exact expectation value for $J_z$ and the closest integer $m'$ for all the numeric energy levels is less than $10^{-4}$. These panels show two remarkable results. First, eq. (7) reproduces almost perfectly the exact numeric eigenvalues. Second, true level crossings between same parity states are observed, at least for the EBK energy levels. This latter fact could be related to the possibility of real crossings within parity sectors in the Dicke model, discussed in ref. [17]. For the exact numeric eigenvalues it is very difficult to determine if the crossings are avoided or real. Since the BOA is such a good approximation, the minimum distance between two anticrossing levels (if any) could be smaller than the precision of our computational codes (an issue that will be treated elsewhere). In the main part of panel (c) we plot the error of the EBK quantized energy, $\Delta E/(j\omega_0) = |E_{\text{EBK}} - E_{\text{exact}}|/(j\omega_0)$, as a function of the energy, $E/(j\omega_0)$ (solid red line), for the resonant case. As a reference, we also plot the same calculation for the two harmonic oscillators approximation [4] (solid green line). In the very low-energy region both
approximations give comparable accuracy, but the harmonic approximation breaks down rapidly as energy increases \((E/\omega_0)^2 \sim -11\), contrary to the BOA results which give an accurate description of the exact spectrum until very large excitation energies. To determine the size of the EBK quantization errors compared with the spacings of the energy levels, in the inset we show the errors averaged over a window of 10 consecutive levels, and normalized by the mean level spacing, \(s_n = E_{n+1} - E_n\), in the same window. We can see that this error is less than \((s)/100 up to \(E/(j\omega_0) \sim -4.5\). This result, together with the evidences obtained from the NACs, the Peres lattices and the NPCs, allows us to conclude that the BOA provides a very accurate approximation for the exact spectrum at least up to this energy value.

**Conclusions.** – By using a slow-fast motion, Born-Oppenheimer approximation, we have been able to describe the regular (non-chaotic) energy regime of the Dicke model in the superradiant phase, and to establish the region in parameter space where the approximation is valid. The very approximate second integral of motion, \(J_r\), was identified, allowing to label with two quantum numbers, \(E_{m',n}\), all the energy levels within the regular energy region. By means of stringent numerical tests, we have tested the accuracy of this result. We have shown that the corresponding semi-classical picture provides a very good estimate for the expectation values of representative physical observables. And the same for the energy levels \(E_{m',n}\), obtained with the EBK quantization rules, which do not require the numerical diagonalization of the Hamiltonian. Since the non-adiabatic coefficients decrease as a function of \(f = \gamma/\gamma_c\) and \(\omega_0/\omega\), making the BOA a better approximation, it is expected that the regular energy interval of the Dicke model increases accordingly to these same variables. The relevant resonant case is included in the region of applicability of the BOA in the superradiant phase, contrary to the naive expectation that is only valid if \(\omega_0/\omega \gg 1\). The BOA applies at energies far above the region in which the double harmonic oscillator approximation holds [4], even reaching the critical energy of the ESQPT when the Dicke model is out of resonance. It is important to stress that even if for many systems the low-energy spectrum is regular and commonly disregarded for the spectral statistics, the approximated integrals of motion are not always identified as in the present case. We think these results gives new insights into the wide regular region of the Dicke model, by extending the theoretical tools to describe what the quadratic approximation, obtained in turn from a Holstein-Primakoff one, leaves unexplained. What is more, the knowledge of this second integral of motion is necessary to study properly the model, and can be very useful for the research of non-equilibrium dynamics and the route to chaos, both theoretically and experimentally.

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