Sieve functions in arithmetic bands, II

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Abstract. An arithmetic function \( f \) is called a sieve function of range \( Q \) if its Eratosthenes transform \( g = f \ast \mu \) is supported in \([1, Q] \cap \mathbb{N}\), where \( g(q) \ll_{\varepsilon} q^{\varepsilon} \) \( (\forall \varepsilon > 0) \). We continue our study of the distribution of such functions over short arithmetic bands, \( n \equiv ar + b \) (mod \( q \)), with \( 1 \leq a \leq H = o(N) \) and \( r, b \in \mathbb{Z} \) such that \( \gcd(r, q) = 1 \). In particular, we discuss the optimality of some results.

1. Introduction.

Given an arithmetic function \( f : \mathbb{N} \to \mathbb{C} \), for every \( N \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \) let us set

\[
\hat{f}_N(\alpha) \overset{\text{def}}{=} \sum_{n \sim N} f(n)e(n\alpha),
\]

where \( n \sim N \) means that \( n \in \mathbb{N} \), \( N \), \( 2N \] \( \cap \mathbb{N} \) and \( e(\alpha) \overset{\text{def}}{=} e^{2\pi i\alpha} \), as usual. If \( f \) is the convolution product of \( g \) and the constantly 1 function, i.e.

\[
f(n) = (g \ast 1)(n) = \sum_{d | n} g(d),
\]

we say, with Wintner [W], that \( g = f \ast \mu \) is the Eratosthenes transform of \( f \) (where \( \mu \) is the well-known Möbius function). We call \( f \) a sieve function of range \( Q \) if its Eratosthenes transform \( g \) is essentially bounded, namely \( g(d) \ll_{\varepsilon} d^{\varepsilon} \), \( \forall \varepsilon > 0 \), and vanishes outside \([1, Q] \) for some \( Q \in \mathbb{N} \), that is to say,

\[
f(n) = \sum_{d | n \atop d \leq Q} g(d).
\]

Note that \( f = g \ast 1 \) is essentially bounded if and only if so is \( g \).

As usual, \( \ll \) is Vinogradov’s notation, synonymous to Landau’s \( O \)-notation. In particular, \( \ll_{\varepsilon} \) means that the implicit constant might depend on an arbitrarily small and positive real number \( \varepsilon \), which might change at each occurrence. We also write \( g_Q \overset{\text{def}}{=} g \cdot 1_Q \) to mean that \( g \) vanishes outside \([1, Q] \), the function \( 1_Q \) being the indicator of \([1, Q] \cap \mathbb{N} \). In the above notation, we have set \( f_N \overset{\text{def}}{=} f_{2N} - f_N \).

In [CL2] we have established a formula that relates the so-called \( \ell \)th Ramanujan coefficient of a real sieve function \( f \) of range \( Q \), i.e.

\[
R_\ell(f) \overset{\text{def}}{=} \sum_{d \mid \ell \atop d \leq Q/\ell} \frac{g_Q(d)}{d},
\]

to the values of \( \hat{f}_N \) attained at rational numbers; hereafter we write \( n \equiv a \) (mod \( q \)) to abbreviate \( n \equiv a \) (mod \( q \)). More precisely, Lemma 3.1 [CL2] states that

(1) \[
\hat{f}_N(j/\ell) = R_\ell(f)N + O_{\varepsilon}(\ell Q^{\varepsilon}(Q + \ell)), \quad \forall \ell > 1, \forall j \in \mathbb{Z}^*_\ell,
\]

which holds uniformly in the complete set \( \mathbb{Z}^*_\ell \) of reduced residue classes modulo \( \ell \). Moreover, it easily seen that (compare also [C])

(2) \[
R_\ell(f) = \frac{1}{\ell} \sum_{m \leq Q/\ell} \frac{g(\ell m)}{m} \ll_{\varepsilon} Q^{\varepsilon}/\ell.
\]

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Here we continue our study of the distribution of a real sieve function $f$ over short arithmetic bands, i.e.

$$
\bigcup_{1 \leq a \leq H} \{ n \in (N, 2N] : n \equiv a \pmod{q} \}, \text{ with } H = o(N).
$$

Let us recall that in [CL2] we have proved that the inequality (hereafter, we omit $a \geq 1$ in sums like $\sum_{a \leq H}$)

$$(3) \quad T_f(q, N, H) \overset{\Delta}{=} \sum_{a \leq H} \sum_{n \sim N} f(n) - \frac{H}{q} \hat{f}_N(0) \ll \varepsilon N^\varepsilon (N/q + q + Q)
$$

holds for every real sieve function $f$ of range $Q \ll N$, after assuming that $H = o(q)$, as $q \to \infty$, and $q = o(N)$, as $N \to \infty$. Such conditions are required in order to avoid overlapping and sporadicity of the arithmetic bands, respectively. By a straightforward application of (1) and (2) here we generalize the previous inequality for

$$
T_f(q, r, b, N, H) \overset{\Delta}{=} \sum_{a \leq H} \sum_{n \sim N, n \equiv a + b (q)} f(n) - \frac{H}{q} \hat{f}_N(0),
$$

where $r, b \in \mathbb{Z}$ are such that $(r, q) = 1$ (hereafter $(r, q) = g.c.d.(r, q)$, as usual). In particular, note that $T_f(q, 1, 0, N, H) = T_f(q, N, H)$.

**Theorem.** Let $q, N, H, Q \in \mathbb{N}$ be such that $Q \ll N, H = o(q)$ and $q = o(N)$. For every sieve function $f : \mathbb{N} \to \mathbb{R}$ of range $Q$ and all $r, b \in \mathbb{Z}$ such that $(r, q) = 1$, one has

$$(4) \quad T_f(q, r, b, N, H) \ll \varepsilon N^\varepsilon (N/q + q + Q).$$

**Proof.** By the orthogonality of additive characters $e_q(t) \overset{\Delta}{=} e(t/q), (q \in \mathbb{N}, t \in \mathbb{Z})$, we get

$$
T_f(q, r, b, N, H) = \frac{1}{q} \sum_{a \leq H} \sum_{n \sim N} f(n) \sum_{k \leq q} e_q(k(ar + b - n)) - \frac{H}{q} \hat{f}_N(0) = \frac{1}{q} \sum_{k < q} \sum_{a \leq H} e_q(k(ar + b)) \hat{f}_N(-k/q)
$$

$$
= \frac{1}{q} \sum_{k \leq q} \sum_{j \in \mathbb{Z}_\ell} \hat{f}_N(-j/\ell) \sum_{a \leq H} e_\ell(j(ar + b)),
$$

where we have set $\ell = q/(k, q)$ and $j = k/(k, q)$. Since $(r, q) = 1$, for any $\ell | q$ there exists an integer $\tau$ such that $r\tau \equiv 1 \pmod{\ell}$. Therefore we write

$$
T_f(q, r, b, N, H) = \frac{1}{q} \sum_{\ell \mid q} \sum_{j \in \mathbb{Z}_\ell} \hat{f}_N \left(-\frac{j\tau}{\ell}\right) e_\ell(j\tau b) \sum_{a \leq H} e_\ell(ja).
$$

By applying (1), (2) and the well-known inequality (see [Da], Ch.25)

$$
\sum_{V_1 < v \leq V_2} e(v\alpha) \ll \min \left(V_2 - V_1, \frac{1}{\|\alpha\|} \right),
$$

we conclude that

$$
T_f(q, r, b, N, H) \ll \frac{1}{q} \sum_{\ell \mid q} \sum_{j \in \mathbb{Z}_\ell} \left( |R_\ell(f)| N + (\ell Q)^\varepsilon (Q + \ell) \right) \sum_{j \in \mathbb{Z}_\ell} \frac{1}{\|j/\ell\|}
$$

$$
\ll \varepsilon \sum_{\ell \mid q} \sum_{j \in \mathbb{Z}_\ell} \left( \frac{N}{\ell} + Q + \ell \right) \ll \varepsilon N^\varepsilon \left( \frac{N}{q} + Q + q \right),
$$

that is (4). 

\[ \square \]

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Remark 1. Besides (4), from the previous proof it transpires that also the upper bound of

\[ T_f(q, r, b, N, H) = \frac{1}{q} \sum_{j, \ell \in \mathbb{Z}_0^+} \hat{f}_N^{-1}\left( \frac{j \ell}{q} \right) e_q(j \ell r) \sum_{n \leq H} e_q(n a) \leq \frac{1}{q} \sum_{j, \ell \in \mathbb{Z}_0^+} \sum_{j, \ell \in \mathbb{Z}_0^+} 1 \max \left| \hat{f}_N^{-1}(j/\ell) \right| \]

is independent of both \( b = 1 \) and \( r \).

Remark 2. Recalling that here trivial bound means \( \ll N^{1+\varepsilon} H/q \), both (3) and (4) are non-trivial once the width \( \theta \defeq \frac{\log H}{\log N} \) is positive, \( q \ll \sqrt{N^{1-\delta} H} \) and \( q \ll N^{1-\delta} H \), for a suitable \( \delta > 0 \). Since it is assumed that \( Q \ll q \ll Q \) hereafter, we get a bound \( Q \ll N^{1-\delta} H \) and thus go beyond the square-root of \( N \) (for \( \theta > 0 \)). Consistently with the terminology introduced in [CL2], we stop at the level 1/2.

In the final section of [CL1] we studied the so-called level 1/2. In fact, our present results, insofar they generalize our previous ones, already go beyond level 1/2.

Here we discuss the possibility of going beyond \( 1/2 + \varepsilon \) in the present context of the sieve functions in arithmetic bands, namely by taking \( Q > \sqrt{N^{1+\delta} H} \) for a certain small and fixed \( \delta > 0 \), so that \( N^{1+\delta} H/Q = o(Q) \), provided \( \delta > \varepsilon \). Indeed, in section 3 we exhibit a particular sieve function whose behavior in arithmetic bands confirms the optimality of such level.

2. Length inertia in arithmetic bands.

In [CL1] we studied the so-called length inertia property for weighted Selberg integrals (see [CL2] for the link between such integrals and the distribution of a sieve function in arithmetic bands). Such a property permits transfer of non-trivial bounds in short intervals of length \( h \), say, to similar bounds in longer intervals of length \( H = \infty(h) \) (that is \( h = o(H) \), as \( H \to \infty \)). Here we show that a length inertia property holds also for the distribution of a sieve function in arithmetic bands. Indeed, we have

\[ T_f(q, N, [H/h]h) \defeq \sum_{a \leq [H/h]h} \sum_{n \in \{a(q) \}} f(n) - \frac{H/h}{q} \hat{f}_N^{-1}(0) = \sum_{j \leq [H/h]} \left( \sum_{(j-1)h < a \leq jh} \sum_{n \in \{a(q) \}} f(n) - \frac{h}{q} \hat{f}_N^{-1}(0) \right) \]

\[ = \sum_{j \leq [H/h]} \left( \sum_{a \leq jh} \sum_{n \in \{a(j-1)h \}} f(n) - \frac{h}{q} \hat{f}_N^{-1}(0) \right) = \sum_{j \leq [H/h]} T_f(q, 1, (j-1)h, N, h). \]

3. Optimality of the level.

Let us set \( \text{sgn}(x) \defeq x/|x| \) for all \( x \in \mathbb{R} \setminus \{0\} \) and \( \text{sgn}(0) \defeq 0 \). Then, for any fixed \( q \in \mathbb{N} \cap (Q, 2Q) \), we define the arithmetic function \( g(d) \) as

\[ g(d) = g(d, q, N, H) \defeq \text{sgn} \left( \sum_{a \leq H} \left( \sum_{m \sim N/d} \frac{1}{q} \sum_{m \sim N/d} 1 \right) \right), \]

if \( d \in \mathbb{N} \cap (Q, 2Q) \) and \( g(d) \defeq 0 \) otherwise. It is plain that \( g \) is the Eratosthenes transform of the sieve function \( f(n) = f(n, q, N, H) \defeq \sum_{d|n} g(d) \) of range \( 2Q \). By noting that \( \text{sgn}(x)x = |x| \) for all \( x \in \mathbb{R} \), we write

\[ |T_f(q, N, H)| = \left| \sum_{a \leq H} \left( \sum_{n \in \{a(q) \}} f(n) - \frac{1}{q} \sum_{n \sim N} f(n) \right) \right| = \left| \sum_{d \sim Q} \sum_{a \leq H} \left( \sum_{m \sim N/d} 1 - \frac{1}{q} \sum_{m \sim N/d} 1 \right) \right| \]
\[
\sum_{d \sim Q} \left( \sum_{m \sim N/d} \left( \frac{1}{q} \sum_{m \sim N/d} 1 \right) \right).
\]

In order to show that \( |T_f(q, N, H)| \gg NH/q \), we set

\[
S = S(q, Q, H, N) \overset{\text{def}}{=} \left\{ d \in \mathbb{N} \cap (Q, 2Q] : \sum_{a \leq H} \sum_{m \sim N/d} 1 \geq 1 \right\},
\]

\[
E \overset{\text{def}}{=} (\mathbb{N} \cap (Q, 2Q]) \setminus S
\]

and prove that \( |S| = o(Q) \), which in turns yields \( |E| \gg Q \). Indeed, from the latter inequality we observe that

\[
|T_f(q, N, H)| = \sum_{d \in S} \left| \sum_{a \leq H} \left( \sum_{m \sim N/d} 1 - \frac{1}{q} \sum_{m \sim N/d} 1 \right) \right| + \sum_{d \in E} H q \sum_{m \sim N/d} 1
\]

\[
\gg |E| \frac{NH}{qQ} \gg NH/q.
\]

Now let us prove that \( |S| = o(Q) \). To this end, after recalling that the divisor function \( d(n) \overset{\text{def}}{=} \sum_{d \mid n} 1 \) is essentially bounded and \( q > Q \), we note that

\[
|S| \leq \sum_{d \in S} \sum_{a \leq H} \sum_{m \sim N/d} 1 \leq \sum_{a \leq H} \sum_{n \sim N} d(n) \ll \varepsilon \frac{N^{1+\varepsilon}H}{Q}.
\]

If \( Q > \sqrt{N^{1+\delta}H} \) for a certain \( \delta > 0 \), then \( N^{1+\varepsilon}H/Q = o(Q) \) once \( \varepsilon < \delta \), that yields the desired conclusion.

### 4. Concluding remarks.

Sieve functions are ubiquitous in analytic number theory. Besides the truncated divisor sum \( \Lambda_R \) (see [G]), that is a linear combination of sieve functions of range \( R \) (see [CL2] for our remarks on \( \Lambda_R \)), we quote the so-called restricted divisor function

\[
\tau_Q(n) \overset{\text{def}}{=} (1_Q * 1)(n) = \sum_{d \mid n, d \leq Q} 1,
\]

whose Eratosthenes transform is the indicator \( 1_Q \) of \([1, Q] \cap \mathbb{N} \). We refer the reader to [T] for results on the distribution of \( \tau_Q \) in short arithmetic progressions. Here we wish to stress that in [T] one finds also conjectures and average results concerning the distribution in arithmetic bands of the more complicated function

\[
\tau_{Q,R}(n) \overset{\text{def}}{=} (1_Q * 1_R)(n) = \sum_{(d,t)} 1_{Q}(d)1_{R}(t).
\]

Such an essentially bounded function is linked to the pair correlation problem for fractional parts of the quadratic function \( \alpha k^2 \), with \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \) (compare also [S]). While in [T], Conjecture 1.2, it is pursued the research of an upper bound for

\[
\sum_{a \leq H} \sum_{n \equiv ar (q)} \tau_{Q,R}(n) - \frac{HQR}{q}, \text{ with } (r,q) = 1,
\]

under suitable conditions on \( H, Q, R \), here our Theorem leads to an asymptotic formula for the inverse Eratosthenes transform of \( \tau_{Q,R} \) in arithmetic bands, namely for

\[
\sum_{a \leq H} \sum_{n \equiv ar (q)} (\tau_{Q,R} * \mu)(n).
\]
In [CL3] we established an asymptotic inequality for the exponential sum associated to the localized divisor functions, a family of functions including the aforementioned $\tau_Q$, $\tau_{Q,R}$, and the standard divisor function $d_k$, $k \geq 2$ (recall that $d_k(n)$ is the number of ways to write $n$ as a product of $k$ positive integers). The particular instance of such an inequality for $d_k$ is

$$\sum_{n\sim N} d_k(n)e(n\alpha) \ll_{k,\varepsilon} (Nq)^{\varepsilon}(N/q + q + N^{1-1/k}),$$

for all $\alpha \in [a/q - 1/q^2, a/q + 1/q^2]$, with $(a,q) = 1, q > 1$. Somehow, this can be regarded as being analogous to the inequality which follows by combining (1) with (2). Such a circumstance is remarkable even in view of the fact that any $d_k$ falls short of being a sieve function, with a sort of Eratosthenes transform which turns out to be the sum of $d_{k-1}$ plus some restricted divisor functions (see the last section of [CL1]).

Finally, since the function proposed in section 3 seems to be rather artificial in that it depends on a fixed modulus $q \in \mathbb{N} \cap (Q,2Q]$, an intriguing open question to ask is how many standard sieve functions might support the optimality of the level accomplished by our Theorem and the results of [CL2].

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