Existence of ground state solutions of Nehari-Pankov type to Schrödinger systems

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Abstract This paper is dedicated to studying the following elliptic system of Hamiltonian type:

$$
\begin{align*}
-\varepsilon^2 \Delta u + u + V(x)v &= Q(x)F_u(u,v), \quad x \in \mathbb{R}^N, \\
-\varepsilon^2 \Delta v + v + V(x)u &= Q(x)F_v(u,v), \quad x \in \mathbb{R}^N, \\
|u(x)| + |v(x)| &\to 0, \quad \text{as} \ |x| \to \infty,
\end{align*}
$$

where $N \geq 3$, $V, Q \in C(\mathbb{R}^N, \mathbb{R})$, $V(x)$ is allowed to be sign-changing and $\inf Q > 0$, and $F \in C^1(\mathbb{R}^2, \mathbb{R})$ is superquadratic at both 0 and infinity but subcritical. Instead of the reduction approach used in Ding et al. (2014), we develop a more direct approach—non-Nehari manifold approach to obtain stronger conclusions but under weaker assumptions than those in Ding et al. (2014). We can find an $\varepsilon_0 > 0$ which is determined by terms of $N, V, Q$ and $F$, and then we prove the existence of a ground state solution of Nehari-Pankov type to the coupled system for all $\varepsilon \in (0, \varepsilon_0]$.

Keywords Hamiltonian elliptic system, ground state solutions of Nehari-Pankov type, strongly indefinite functionals

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1 Introduction

In this paper, we study standing waves for the following system of time-dependent nonlinear Schrödinger equations:

$$
\begin{align*}
\imath \hbar \frac{\partial \phi_1}{\partial t} + \frac{\hbar^2}{2m} \Delta \phi_1 + \phi_1 + f(x, \phi) \phi_2 &= 0, \\
\imath \hbar \frac{\partial \phi_2}{\partial t} + \frac{\hbar^2}{2m} \Delta \phi_2 + \phi_2 + f(x, \phi) \phi_1 &= 0,
\end{align*}
$$

(1.1)

where $m$ is the mass of a particle, $\hbar$ is the Planck constant, $\phi = (\phi_1, \phi_2)$, $\phi_1(t,x)$ and $\phi_2(t,x)$ are the complex valued envelope functions. Suppose that $f(x, e^{i\theta} \phi) = f(x, \phi)$ for $\theta \in [0, 2\pi]$. We will look for standing waves of the form

$$
\phi_1(t,x) = e^{i\omega t}u(x), \quad \phi_2(t,x) = e^{i\omega t}v(x),
$$

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which propagate without changing their shape and thus have a soliton-like behavior. System (1.1)
aris quite naturally in nonlinear optics and Bose-Einstein condensates (see [3,17,24] and the refer-
ces therein). In general, the above coupled nonlinear Schrödinger system leads to the elliptic system of Hamiltonian form
\[
\begin{cases}
-\varepsilon^2 \Delta u + u = H_v(x,u,v), & x \in \mathbb{R}^N, \\
-\varepsilon^2 \Delta v + v = H_u(x,u,v), & x \in \mathbb{R}^N, \\
|u(x)| + |v(x)| \to 0, & \text{as } |x| \to \infty,
\end{cases}
\]
where \(N \geq 3, H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})\) and \(\varepsilon > 0\) is a small parameter. The study of the systems similar to (1.2) has only been quite recently. When \(\varepsilon = 1\), it was considered recently in some works [6,10,15,16,34–39]. For a similar problem on a bounded domain we refer the reader to [5,8,10,14] and the references therein. For a survey on this direction see [9,23].

For the case \(\varepsilon > 0\) is a small parameter, there are some recent works considering the existence of solutions (see, for example, [2,13,24] and the references therein). In contrast with the case \(\varepsilon = 1\), except for the difficulties that the lack of the compactness of the Sobolev embedding and the energy functional is strongly indefinite, no uniqueness results seem to be known for the “limit problem” and this is in some cases a crucial assumption in the single equation case. So asymptotic analysis of solutions with respect to small \(\varepsilon > 0\) has been very recently performed (see, for example, [2,4,12,13,19–22,24] and their references). Except for [12,13,24], most of the above works considered the case that
\[
H(x,u,v) = F(u) + G(v).
\]
In particular, in [4], Ávila and Yang obtained the existence of positive solutions which concentrate on the boundary of \(\Omega\) for an elliptic system with zero Neumann boundary condition on a bounded domain \(\Omega\) (see also [19]). In [20], Ramos and Soares considered the following problem:
\[
\begin{cases}
-\varepsilon^2 \Delta u + V(x)u = g(v), & x \in \Omega, \\
-\varepsilon^2 \Delta v + V(x)v = f(u), & x \in \Omega, \\
u,v \in H^1(\mathbb{R}^N),
\end{cases}
\]
where \(\Omega\) is a domain of \(\mathbb{R}^N, V \in C(\mathbb{R}^N, \mathbb{R})\) satisfies
\[
0 < V(0) = \min_{x \to \infty} V(x) < \liminf_{x \to \infty} V(x) \in (0, \infty),
\]
\(f(u)\) and \(g(v)\) are power type functions, superlinear but subcritical at infinity. The authors established the existence of positive solutions which concentrate, as \(\varepsilon \to 0\), at a prescribed finite number of local minimum points (possibly degenerate) of the potential \(V\). Different from those discussed in [2,20,21], Ding et al. [12] dealt with existence and concentration phenomena of the ground state solutions to the following subcritical problem:
\[
\begin{cases}
-\varepsilon^2 \Delta u + u + V(x)u = Q(x)g(|z|)u, & x \in \mathbb{R}^N, \\
-\varepsilon^2 \Delta v + v + V(x)v = Q(x)g(|z|)v, & x \in \mathbb{R}^N, \\
|u(x)| + |v(x)| \to 0, & \text{as } |x| \to \infty,
\end{cases}
\]
where \(z := (u,v), V,Q \in C^1(\mathbb{R}^N, \mathbb{R})\) and \(g \in C^1(\mathbb{R}^+, \mathbb{R}^+)\). Since the energy functional \(\Phi_{\varepsilon}\) associated with (1.4) is strongly indefinite, to overcome this difficulty, as in [1] (see also [20,21]), the authors constructed a reduced functional \(R_{\varepsilon}\) whose critical points are in one to one to critical points of \(\Phi_{\varepsilon}\), which was first proposed in [1]. With the help of the Nehari manifold of \(\Phi_{\varepsilon}\), an important information of the least energy \(c_{\varepsilon}\) was obtained. By estimating the asymptotic behavior of \(c_{\varepsilon}\) as \(\varepsilon \to 0\), they proved \(c_{\varepsilon}\) is attained for sufficiently small \(\varepsilon > 0\). In order to state their results, some notation and assumptions are required. Set
\[
V_{\min} := \min_{x \in \mathbb{R}^N} V(x), \quad V_{\max} := \max_{x \in \mathbb{R}^N} V(x), \quad V := \{x \in \mathbb{R}^N : V(x) = V_{\min}\}.
\]
Theorem 1.1

**Remark.** Solutions of Nehari-Pankov type to (1.5) for all small $\varepsilon > 0$ are valid when finding a ground state solution of Nehari-Pankov type. This approach is different from that of Szulkin and Weth [25]. This approach is to construct a more direct approach—non-Nehari manifold approach which was first proposed in [26] for a single Schrödinger equation (see also [7, 27, 28, 30–32]). The main idea of this approach is to construct a reduction method, which is completely different from that of Szulkin and Weth. Instead of the reduction method used in [12], we will use a more direct approach—non-Nehari manifold approach which was first proposed in [26] for a single Schrödinger equation (see also [7, 27, 28, 30–32]). The main idea of this approach is to construct a minimizing Cerami sequence for the energy functional outside Nehari-Pankov manifold by using the diagonal method, which is a different approach from that of Szulkin and Weth [25]. This approach is valid when finding a ground state solution of Nehari-Pankov type.

In [12], Ding et al. proved the following theorem.

**Theorem 1.1** (See [12, Theorem 1]). Let (A0), (G1), (G2) and (G3) be satisfied. Suppose that (A1) or (A2) is satisfied. Then for sufficiently small $\varepsilon > 0$, (1.4) has a least energy solution $\hat{\varepsilon}_\varepsilon = (\hat{u}_\varepsilon, \hat{v}_\varepsilon)$.

Theorem 1.1 is very interesting. In its proof, many new tricks were used to overcome the difficulties caused by the strong indefinity of the energy functional $\Phi_\varepsilon$ associated with (1.4). We point that the regularity assumptions $V, Q \in C^1$ and $g \in C^1$ are very crucial in [12], which seem to be necessary when the reduction method is used. Motivated by the works [12], in this paper, we further study the existence of the ground state solutions of Nehari-Pankov type to the following more general problem:

\begin{align}
\begin{cases}
-\varepsilon^2 \Delta u + u + V(x)u = Q(x)F_v(u,v), & x \in \mathbb{R}^N, \\
-\varepsilon^2 \Delta v + v + V(x)u = Q(x)F_u(u,v), & x \in \mathbb{R}^N,
\end{cases}
\end{align}

(1.5)

where $V, Q \in C(\mathbb{R}^N, \mathbb{R})$ and $F \in C^1(\mathbb{R}^2, \mathbb{R})$. Instead of the reduction method used in [12], we will use a more direct approach—non-Nehari manifold approach which was first proposed in [26] for a single Schrödinger equation (see also [7, 27, 28, 30–32]). The main idea of this approach is to construct a minimizing Cerami sequence for the energy functional outside Nehari-Pankov manifold by using the diagonal method, which is different from that of Szulkin and Weth. This approach is valid when finding a ground state solution of Nehari-Pankov type.

We will obtain stronger conclusions on existence of the ground state solutions of Nehari-Pankov type to (1.5) for small $\varepsilon > 0$ but under weaker assumptions than those in [12]. Roughly speaking, we can find an $\varepsilon_0 > 0$ which is determined by terms of $N, V, Q$ and $F$, and then we prove the existence of a ground state solutions of Nehari-Pankov type to (1.5) for all $\varepsilon \in (0, \varepsilon_0]$. In particular, we only need $V, Q \in C(\mathbb{R}^N, \mathbb{R})$ and $F \in C^1(\mathbb{R}^2, \mathbb{R})$. To the best of our knowledge, there seems to be no similar results in the literature.

To state our theorems accurately, we set

\begin{align}
\mathcal{N}D_0 = \left\{ h \in C(\mathbb{R}^+, \mathbb{R}^+): \begin{array}{l}
h(0) = 0 \text{ and } h(s) \text{ is nondecreasing on } \mathbb{R}^+, \\
\text{there exist constants } p \in (2, 2^*) \text{ and } c_0 > 0 \text{ such that } \\
|h(s)| \leq c_0(1 + |s|^{p-2}), \quad \forall s \geq 0.
\end{array} \right\}.
\end{align}

(1.6)
Furthermore, we make the following assumptions:

(V0) \( V, Q \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}) \), \( \|V\|_{\infty} \leq \frac{2ab}{a^2+\beta} \) and \( 0 < Q_{\min} \leq Q_{\max} < \infty \), where \( a, b > 0 \) and \( \eta \in (0, 1) \);  
(V1) \( V_{\min} < V_{\infty} \), and there exist \( x_v \in V \) and \( R > 0 \) such that  
\[
Q(x_v) \geq Q(x), \quad \forall |x| \geq R;
\]

(V2) \( Q_{\max} > Q_{\infty} \), and there exist \( x_q \in Q \) and \( R > 0 \) such that  
\[
V(x_q) \leq V(x), \quad \forall |x| \geq R;
\]

(F1) there exist \( g_i, h_j \in \mathcal{N}D_0 \), \( \alpha_i, \beta_i, \alpha_j^*, \beta_j^* \in \mathbb{R} \) with \( \alpha_i^2 + \beta_i^2 \neq 0 \) and \( \alpha_j^* > \beta_j^2 \), \( i = 1, 2, \ldots, k; j = 1, 2, \ldots, l \), such that  
\[
F(u, v) = \sum_{i=1}^{k} \int_{0}^{1} g_i(s)ds + \sum_{j=1}^{l} \int_{0}^{1} h_j(s)ds;
\]

(F2) \( \lim_{|au+bv| \to \infty} \frac{F(u, v)}{|au+bv|^2} = \infty \);  
(F3) there exist \( \mathcal{C}_0 > 0 \), \( \mathcal{T}_0 > 0 \), \( \mu > 2 \) and \( F_0 \in \mathcal{C}(\mathbb{R}^2, \mathbb{R}) \) with \( F_0(u, v) > 0 \) if \( au + bv \neq 0 \), such that  
\[
F(tz) \geq \mathcal{C}_0 t^\mu F_0(z), \quad \forall z \in \mathbb{R}^2, \quad t \geq \mathcal{T}_0.
\]

Remark 1.2. It is clear that

(i) (F3) is weaker than (AR)-condition: there exists \( \mu > 2 \) such that \( F(z) \cdot z \geq \mu F(z) > 0 \) for \( z \neq 0 \);  
(ii) (F2) is also weaker than the common super-quadratic condition (SQ): \( \lim_{|z| \to \infty} \frac{|F(z)|}{|z|^2} = \infty \);  
(iii) let \( F(z) = \int_{0}^{1} g(s)ds \). Then (G1)–(G3) imply (F1)–(F3);  
(iv) let \( Q(x) \equiv 1 \) and  
\[
F(z) = \int_{0}^{1} g(s)ds + \int_{0}^{1} h(s)ds.
\]

Then (1.5) reduces to (1.3), which was studied in [20]. Moreover, the assumptions in [20, (H)] also imply (F1)–(F3).

Before presenting our results, we give three nonlinear examples to illustrate the above assumptions.

Example 1.3. Let \( F(z) = |au+fv|^\mu \), where \( \mu \in (2, 2^*) \) and \( a, b > 0 \) with \( \|V\|_{\infty} < \frac{2ab}{a^2+\beta} \). It is easy to see that \( F(z) \) satisfies (F1)–(F3) with \( F_0 = F \), but not (AR).

Example 1.4. Let  
\[
F(z) = |au+fv|^\mu + (u^2 + uv + v^2) \ln(1 + u^2 + uv + v^2),
\]
where \( \mu \in (2, 2^*) \) and \( a, b > 0 \) with \( \|V\|_{\infty} < \frac{2ab}{a^2+\beta} \). It is easy to see that \( F(z) \) satisfies (F1)–(F3) with \( F_0(u, v) = |au+fv|^\mu \), but not (AR).

Example 1.5. Let \( F(z) = |2u + v|^\mu + |u + 2v|^\mu \), where \( \mu \in (2, 2^*) \). It is easy to see that \( F(z) \) satisfies (F1)–(F3) with \( a = b = 1 \) and \( F_0 = F \).

Let \( x_m = x_v \) if (V1) holds, or \( x_m = x_q \) if (V2) holds. Replacing \( u(\varepsilon x + x_m) \) and \( v(\varepsilon x + x_m) \) by \( u(x) \) and \( v(x) \), respectively, we have that (1.5) is equivalent to  
\[
\begin{cases}
-\Delta u + u + V(\varepsilon x + x_m)v = Q(\varepsilon x + x_m)F_v(u, v), & x \in \mathbb{R}^N, \\
-\Delta v + v + V(\varepsilon x + x_m)u = Q(\varepsilon x + x_m)F_u(u, v), & x \in \mathbb{R}^N, \\
|u(x)| + |v(x)| \to 0, & |x| \to \infty.
\end{cases}
\]

Let \( E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \). Then \( E \) is a Hilbert space with the standard inner product  
\[
(z_1, z_2)_{H^1(\mathbb{R}^N)} = (u_1, u_2)_{H^1(\mathbb{R}^N)} + (v_1, v_2)_{H^1(\mathbb{R}^N)}, \quad \forall z_i = (u_i, v_i) \in E, \quad i = 1, 2,
\]
and the corresponding norm
\[ \|z\|_{H^1(\mathbb{R}^N)} = (\|u\|_{H^1(\mathbb{R}^N)}^2 + \|v\|_{H^1(\mathbb{R}^N)}^2)^{1/2}, \quad \forall z = (u, v) \in E. \]

Let \( E = E^- \oplus E^+ \) be an orthogonal decomposition (see Section 2). Define a functional
\[ \Phi_\varepsilon(z) = \int_{\mathbb{R}^N} \left[ \nabla u \cdot \nabla v + uv + \frac{1}{2} V(\varepsilon x + x_m)|z|^2 \right] dx - \int_{\mathbb{R}^N} Q(\varepsilon x + x_m)F(z)dx \tag{1.8} \]
for all \( z = (u, v) \in E. \) Under assumptions (V0), (F1) and (F2), \( \Phi_\varepsilon \in C^1(E, \mathbb{R}) \) and
\[ \langle \Phi'_\varepsilon(z), \varphi \rangle = \int_{\mathbb{R}^N} \left[ \nabla u \cdot \nabla \psi + \nabla v \cdot \nabla \phi + (uv + \psi \phi) + V(\varepsilon x + x_m)z \cdot \varphi \right] dx 
- \int_{\mathbb{R}^N} Q(\varepsilon x + x_m)F(z) \cdot \varphi dx, \quad \forall z = (u, v), \ \varphi = (\phi, \psi) \in E. \tag{1.9} \]

Let
\[ \mathcal{N}_\varepsilon^- = \{ z \in E \setminus E^- : \langle \Phi'_\varepsilon(z), z \rangle = \langle \Phi'_\varepsilon(z), \zeta \rangle = 0, \ \forall \zeta \in E^- \}. \tag{1.10} \]
\( \mathcal{N}_\varepsilon^- \) was first introduced by Pankov [18], which is a subset of the Nehari manifold
\[ \mathcal{N}_\varepsilon = \{ z \in E \setminus \{0\} : \langle \Phi'_\varepsilon(z), z \rangle = 0 \}. \tag{1.11} \]

We are now in a position to state the first main result of this paper.

**Theorem 1.6.** Assume that \( V, Q \) and \( F \) satisfy (V0), (V1) and (F1)–(F3). Then there exists an \( \varepsilon_0 > 0 \) such that (1.5) has a nontrivial solution \( \tilde{z}_\varepsilon = (\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \in \mathcal{N}_\varepsilon^- \) and \( \Phi_\varepsilon(\tilde{z}_\varepsilon) = \inf_{\mathcal{N}_\varepsilon^-} \Phi_\varepsilon > 0 \) for \( \varepsilon \in (0, \varepsilon_0] \), where \( z_\varepsilon(x) = \tilde{z}_\varepsilon(\varepsilon x + x_v). \) If (V1) is replaced by (V2), then the above conclusion remains true by replacing \( x_v \) with \( x_q. \)

The “limit problem” associated with (1.7) is an autonomous system
\[ \begin{cases} 
-\Delta u + u + V(x_m)v = Q(x_m)F_u(u, v), & x \in \mathbb{R}^N, \\
-\Delta v + v + V(x_m)u = Q(x_m)F_u(u, v), & x \in \mathbb{R}^N. 
\end{cases} \tag{1.12} \]

We will prove that the least energy \( c_\varepsilon := \inf_{\mathcal{N}_\varepsilon^-} \Phi_\varepsilon \) is attained for \( \varepsilon \in (0, \varepsilon_0] \) by comparing with \( c_\varepsilon \) and the least energy \( c_0 \) associated with “limit problem” (1.12). Therefore, it is very crucial if \( c_0 \) can be attained, i.e., if (1.12) has a solution at which \( \Phi_0 \) has the least energy \( c_0 \) on \( \mathcal{N}_0^- \). Prior to this, we consider the following more general periodic system
\[ \begin{cases} 
-\Delta u + V_1(x)u + V_2(x)v = W_1(x, u, v), & x \in \mathbb{R}^N, \\
-\Delta v + V_1(x)v + V_2(x)u = W_2(x, u, v), & x \in \mathbb{R}^N, \\
u, v \in H^1(\mathbb{R}^N), 
\end{cases} \tag{1.13} \]
where \( N \geq 3, V_1, V_2 : \mathbb{R}^N \to \mathbb{R} \) and \( W : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R} \). More precisely, we make the following assumptions:

(V0’) \( V_1, V_2 \in C(\mathbb{R}^N) \) and satisfy
\[ |V_2(x)| \leq \frac{2\eta ab}{\alpha^2 + \beta^2} V_1(x), \quad 0 < \inf_{x \in \mathbb{R}^N} V_1(x) \leq \sup_{x \in \mathbb{R}^N} V_1(x) < \infty, \tag{1.14} \]
where \( a, b > 0 \) and \( \eta \in (0, 1) \);

(V1’) \( V_1(x) \) and \( V_2(x) \) are 1-periodic in each of \( x_1, x_2, \ldots, x_N \);

(W0) \( W \in C(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^+) \), \( W(x, z) \) is continuously differentiable on \( z \in \mathbb{R}^2 \) for every \( x \in \mathbb{R}^N \), and there exist constants \( p \in (2, 2^*) \) and \( C_0 > 0 \) such that
\[ |W_z(x, z)| \leq C_0(1 + |z|^{p-1}), \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2; \tag{1.15} \]
Let (1.13). Section 3 is devoted to the proof of Theorem 1.7. In Section 4, we discuss the existence of

Then \( z \) for all \( \alpha \), \( \beta \), \( \gamma \). Prior to this, we define one set as follows:

Assume that \( E \). Then \( \Phi(z) = \int_{\mathbb{R}^N} \left[ \nabla u \cdot \nabla v + V_1(x)uv + \frac{1}{2} V_2(x)|v|^2 \right] dx - \int_{\mathbb{R}^N} W(x,z) dx, \) \( \Phi(z) \) for all \( z = (u,v) \in E \). Furthermore, under the assumptions \( (V0)', (W0) \) and \( (W1) \), \( \Phi \in C^1(E, \mathbb{R}) \) and

\[
\langle \Phi(z), \varphi \rangle = \int_{\mathbb{R}^N} \left[ \nabla u \cdot \nabla \psi + \nabla v \cdot \nabla \phi + V_1(x)(w\psi + v\phi) + V_2(x) z \cdot \varphi \right] dx
\]

\[
- \int_{\mathbb{R}^N} W_z(x,z) \cdot \varphi dx, \quad \forall z = (u,v), \quad \varphi = (\phi, \psi) \in E.
\]

Let

\[
N^- = \{ z \in E \setminus E^- : \langle \Phi(z), z \rangle = \langle \Phi(z), \zeta \rangle = 0, \forall \zeta \in E^- \}.
\]

For (1.13), we obtain the following existence theorem on the ground state solutions of Nehari-Pankov type.

**Theorem 1.7.** Assume that \( V \) and \( W \) satisfy \( (V0)', (V1') \) and \( (W0)-(W4) \). Then (1.13) has a solution \( z^* \in N^- \) such that \( \Phi(z^*) = \inf_{N^-} \Phi > 0 \).

However, it is not easy to check Assumption (W4). Next, we give several class functions satisfying (W4).

Prior to this, we define one set as follows:

\[
\mathcal{ND} = \left\{ h \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R}^+) : \begin{array}{l}
\text{h is 1-periodic in each of } x_1, x_2, \ldots, x_N \\
\text{and is nondecreasing in } t \in [0, \infty) \text{ for every } x \in \mathbb{R}^N; \\
\text{h(0,0) = 0 for } x \in \mathbb{R}^N; \\
\text{there exist constants } p \in (2, 2^*) \text{ and } C_1 > 0 \text{ such that } \\
|h(x,t)| \leq C_1(1 + |t|^{p-2}), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.
\end{array} \right\}.
\]

**Corollary 1.8.** Assume that \( V \) and \( W \) satisfy \( (V0)', (V1') \) and \( (W2) \), and that

\[
W(x,u,v) = \sum_{i=1}^k \int_{0}^{\alpha_i u + \beta_i v} g_i(x,s) ds + \sum_{j=1}^l \int_{0}^{\sqrt{u^2 + 2\beta_j' uv + \alpha_j' v^2}} h_j(x,s) ds,
\]

where \( \alpha_i, \beta_i, \alpha_j', \beta_j' \in \mathbb{R} \) with \( \alpha_i^2 + \beta_i^2 \neq 0 \) and \( \beta_j' > 0 \), \( g_i, h_j \in \mathcal{ND}, i = 1, 2, \ldots, k; j = 1, 2, \ldots, l. \) Then (1.13) has a solution \( z^* \in N^- \) such that \( \Phi(z^*) = \inf_{N^-} \Phi > 0 \).

The paper is organized as follows. In the next section, we develop a functional setting to deal with (1.7) and (1.13). Section 3 is devoted to the proof of Theorem 1.7. In Section 4, we discuss the existence of ground state solutions of Nehari-Pankov type to (1.7).

2 Variational setting

Let \( V_\varepsilon(x) := V(\varepsilon x + x_m) \) and \( Q_\varepsilon(x) := Q(\varepsilon x + x_m) \). Then we can rewrite (1.7) as

\[
\begin{align*}
-\Delta u + u + V_\varepsilon(x) v &= Q_\varepsilon(x) F_u(u,v), & x \in \mathbb{R}^N, \\
-\Delta v + v + V_\varepsilon(x) u &= Q_\varepsilon(x) F_v(u,v), & x \in \mathbb{R}^N, \\
|u(x)| + |v(x)| &\to 0, & \text{as } |x| \to \infty.
\end{align*}
\]
We will mainly deal with (2.1) instead of (1.7). Let
\[ E^- = \left\{ \left( -\frac{u}{a}, \frac{u}{b} \right) : u \in H^1(\mathbb{R}^N) \right\}, \quad E^+ = \left\{ \left( \frac{u}{a}, \frac{u}{b} \right) : u \in H^1(\mathbb{R}^N) \right\}. \]
For any \( z = (u, v) \in E \), set
\[ z^- = \left( \frac{au - bv}{2a}, \frac{bv - au}{2b} \right), \quad z^+ = \left( \frac{au + bv}{2a}, \frac{au + bv}{2b} \right). \]
It is obvious that \( z = z^- + z^+ \). Now we define two new inner products on \( E \):
\[ (z_1, z_2) = \int_{\mathbb{R}^N} \left[ (\nabla z_1^+ \cdot \nabla z_2^+ + \nabla z_1^- \cdot \nabla z_2^-) + (z_1^+ \cdot z_2^+ + z_1^- \cdot z_2^-) \right] dx, \quad \forall z_i = (u_i, v_i) \in E, \quad i = 1, 2 \]
and
\[ (z_1, z_2)_i = \int_{\mathbb{R}^N} \left[ (\nabla z_1^+ \cdot \nabla z_2^+ + \nabla z_1^- \cdot \nabla z_2^-) + V_1(x)(z_1^+ \cdot z_2^+ + z_1^- \cdot z_2^-) \right] dx, \quad \forall z_i = (u_i, v_i) \in E, \quad i = 1, 2. \]
The corresponding norms are
\[ \|z\| = \sqrt{(z, z)}, \quad \|z\|_i = \sqrt{(z, z)_i}, \quad \forall z = (u, v) \in E. \]
By virtue of (V0) and (V0'), it is easy to check that the norms \( \| \cdot \|, \| \cdot \|_i \) and \( \| \cdot \|_{H^1(\mathbb{R}^N)} \) are equivalent on \( E \). It is easy to see that \( z^- \) and \( z^+ \) are orthogonal with respect to the inner products \( (\cdot, \cdot) \) and \( (\cdot, \cdot)_i \). Thus we have \( E = E^- \oplus E^+ \). By a simple calculation, one can get that
\[ \|z\|_i^2 = \int_{\mathbb{R}^N} \left[ \left( |\nabla z^+_1|^2 + |\nabla z^-_1|^2 \right) + \left( |z^+_1|^2 + |z^-_1|^2 \right) \right] dx, \quad \forall z \in E, \quad (2.2) \]
\[ \|z\|_i^2 = \int_{\mathbb{R}^N} \left[ \left( |\nabla z^+_1|^2 + |\nabla z^-_1|^2 \right) + V_1(x)(|z^+_1|^2 + |z^-_1|^2) \right] dx, \quad \forall z \in E, \quad (2.3) \]
and
\[ \frac{ab}{a^2 + b^2} \left( \|z^+_1\|_i^2 - \|z^-_1\|_i^2 \right) = \int_{\mathbb{R}^N} \left[ \nabla u \cdot \nabla v + V_1(x)uv \right] dx, \quad \forall z = (u, v) \in E. \]
Therefore, the functionals \( \Phi_z \) defined by (1.8) and \( \Phi \) by (1.16) can be rewritten as
\[ \Phi_z(z) = \frac{ab}{a^2 + b^2} \left( \|z^+_1\|_i^2 - \|z^-_1\|_i^2 \right) + \frac{1}{2} \int_{\mathbb{R}^N} V_1(x)|z|^2 dx - \int_{\mathbb{R}^N} Q_1(x)F(z) dx, \quad \forall z \in E, \quad (2.4) \]
and
\[ \Phi(z) = \frac{ab}{a^2 + b^2} \left( \|z^+_1\|_i^2 - \|z^-_1\|_i^2 \right) + \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|z|^2 dx - \int_{\mathbb{R}^N} W(x, z) dx, \quad \forall z \in E, \quad (2.5) \]
respectively. Our hypotheses imply that \( \Phi_z, \Phi \in C^1(E, \mathbb{R}) \), and a standard argument shows that the critical points of \( \Phi_z \) and \( \Phi \) are solutions of (1.5) and (1.13), respectively. Moreover, by (1.9) and (1.17), it holds that
\[ \langle \Phi'_z(z), \varphi \rangle = \frac{2ab}{a^2 + b^2} \left[ (z^+, \varphi^+) - (z^-, \varphi^-) \right] + \int_{\mathbb{R}^N} V_1(x)z \cdot \varphi dx \]
and
\[ \langle \Phi'_z(z), z \rangle = \frac{2ab}{a^2 + b^2} \left[ (z^+_1)^2 - (z^-_1)^2 \right] + \int_{\mathbb{R}^N} V_1(x)|z|^2 dx - \int_{\mathbb{R}^N} Q_1(x)F(z) \cdot z dx, \quad \forall z \in E, \quad (2.6) \]
\[ \langle \Phi'(z), \varphi \rangle = \frac{2ab}{a^2 + b^2} \left[ (z^+, \varphi^+) - (z^-, \varphi^-) \right] + \int_{\mathbb{R}^N} V_2(x)z \cdot \varphi dx \]
and
\[ \langle \Phi'(z), z \rangle = \frac{2ab}{a^2 + b^2} \left[ (z^+_1)^2 - (z^-_1)^2 \right] + \int_{\mathbb{R}^N} V_2(x)|z|^2 dx - \int_{\mathbb{R}^N} W(x, z) \cdot z dx, \quad \forall z \in E, \quad (2.7) \]
Then there exist a constant $c$ where

$$
\langle \Phi'(z), z \rangle = \frac{2ab}{a^2 + b^2} (\|z^+\|_T^2 - \|z^-\|_T^2) + \int_{\mathbb{R}^N} V_2(x)|z|^2 \, dx - \langle \Phi'(z), z \rangle, \quad \forall z \in E.
$$

(2.9)

Lemma 2.1. Suppose that (V0), (F1)–(F2) are satisfied. If $z = (u, v)$ is a critical point of $\Phi_z$, then $|z(x)| \to 0$ as $|x| \to \infty$. In a word, $z$ is a solution to (2.1).

The proof is almost standard (see [11, Lemma 2.1 and Theorem 2.1]).

3 Ground state solutions of Nehari-Pankov type for periodic systems

Let $X = X^- \oplus X^+$ be a real Hilbert space with $X^- \perp X^+$ and $X^-$ be separable. On $X$ we define a new norm

$$
\|u\|_\tau := \max \left\{ \|u^+\|, \sum_{k=1}^{\infty} 1/\sqrt{2k+1} \|u^-, e_k\| \right\}, \quad \forall u = u^- + u^+ \in X,
$$

(3.1)

where $\{e_k\}_{k=1}^{\infty}$ is a total orthonormal basis of $X^-$. The topology generated by $\| \cdot \|_\tau$ will be denoted by $\tau$ and all topological notions related to it will include the symbol. It is clear that

$$
\|u^+\| \leq \|u\|_\tau \leq \|u\|, \quad \forall u \in X.
$$

(3.2)

For a functional $\phi \in C^1(X, \mathbb{R})$, $\phi$ is said to be $\tau$-upper semi-continuous if

$$
u_n, u \in X, \quad \|u_n - u\|_\tau \to 0 \Rightarrow \phi(u) \geq \limsup_{n \to \infty} \phi(u_n);
$$

(3.3)

weakly sequentially lower semi-continuous if

$$
u_n \rightharpoonup u \quad \text{in} \quad X \Rightarrow \phi(u) \leq \liminf_{n \to \infty} \phi(u_n);
$$

and $\phi'$ is said to be weakly sequentially continuous if

$$
u_n \rightharpoonup u \quad \text{in} \quad X \Rightarrow \lim_{n \to \infty} \langle \phi'(u_n), v \rangle = \langle \phi'(u), v \rangle, \quad \forall v \in X.
$$

It is easy to see that (3.3) holds if and only if

$$
u_n, u \in X, \quad \|u_n - u\|_\tau \to 0 \Rightarrow \phi(u) \geq \liminf_{n \to \infty} \phi(u_n).
$$

(3.4)

Lemma 3.1 (See [26, Theorem 2.4]). Let $X = X^- \oplus X^+$ be a real Hilbert space with $X^- \perp X^+$ and $X^-$ be separable. Suppose that $\phi \in C^1(X, \mathbb{R})$ satisfies the following assumptions:

- (H1) $\phi$ is $\tau$-upper semi-continuous;
- (H2) $\phi'$ is weakly sequentially continuous;
- (H3) there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that

$$
\kappa := \inf \phi(S^+_\rho) > \sup \phi(\partial{Q}_r),
$$

where

$$
S^+_\rho = \{u \in X^+ : \|u\| = \rho\}, \quad Q_r = \{v + se : v \in X^-, \ s \geq 0, \ \|v + se\| \leq r\}.
$$

Then there exist a constant $c \in [\kappa, \sup \phi(Q_r)]$ and a sequence $\{u_n\} \subset X$ satisfying

$$
\phi(u_n) \to c, \quad \|\phi'(u_n)|(1 + \|u_n\|) \to 0.
$$

(3.5)

Let

$$
\Psi(z) = \int_{\mathbb{R}^N} W(x, z) \, dx.
$$
Employing a standard argument, one can check easily the following lemma.

**Lemma 3.2.** Suppose that \( (V0') \), \( (W0) \) and \( (W1) \) are satisfied. Then \( \Psi \) is non-negative, weakly sequentially lower semi-continuous, and \( \Psi' \) is weakly sequentially continuous.

**Lemma 3.3.** Suppose that \( (V0') \), \( (W0) \), \( (W1) \) and \( (W4) \) are satisfied. Then there holds

\[
\Phi(z) \geq \Phi(\theta z + \zeta) + \frac{ab}{a^2 + b^2} \|\zeta\|^2_1 - \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|\zeta|^2 dx
+ \frac{1 - \theta^2}{2}(\Phi'(z), z) - \theta(\Phi'(z), \zeta), \quad \forall \theta \geq 0, \ z \in E, \ \zeta \in E^-.
\]

**Proof.** By (2.5), (2.8), (2.9) and (W4), one has

\[
\Phi(z) - \Phi(\theta z + \zeta) = \frac{ab}{a^2 + b^2} \|\zeta\|^2_1 - \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|\zeta|^2 dx
+ \frac{1 - \theta^2}{2}(\Phi'(z), z) - \theta(\Phi'(z), \zeta)
\geq \frac{ab}{a^2 + b^2} \|\zeta\|^2_1 - \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|\zeta|^2 dx
+ \frac{1 - \theta^2}{2}(\Phi'(z), z) - \theta(\Phi'(z), \zeta), \quad \forall \theta \geq 0, \ z \in E, \ \zeta \in E^-.
\]

This shows that (3.6) holds.

From Lemma 3.3, we have the following two corollaries.

**Corollary 3.4.** Suppose that \( (V0') \), \( (W0) \), \( (W1) \) and \( (W4) \) are satisfied. Then for \( z \in N^- \),

\[
\Phi(z) \geq \Phi(\theta z + \zeta) + \frac{ab}{a^2 + b^2} \|\zeta\|^2_1 - \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|\zeta|^2 dx, \quad \forall \theta \geq 0, \ \zeta \in E^-.
\]

**Corollary 3.5.** Suppose that \( (V0') \), \( (W0) \), \( (W1) \) and \( (W4) \) are satisfied. Then

\[
\Phi(z) \geq \frac{ab\theta^2}{a^2 + b^2} \|z\|^2_1 + \frac{\theta^2}{2} \int_{\mathbb{R}^N} V_2(x)|(z^+)^2 - |z^-|^2| dx - \int_{\mathbb{R}^N} W(x, \theta z^+) dx
+ \frac{1 - \theta^2}{2}(\Phi'(z), z) + \theta^2(\Phi'(z), z^-), \quad \forall z \in E, \ \theta \geq 0.
\]

**Lemma 3.6.** Suppose that \( (V0') \), \( (W0) \), \( (W1) \) and \( (W4) \) are satisfied. Then

(i) there exists \( \rho > 0 \) such that

\[
m := \inf_{N^-} \Phi \geq \kappa := \inf\{\Phi(z) : z \in E^+, \|z\|_1 = \rho\} > 0;\]

(ii) \( \|z^+\|_1^2 \geq \max\left\{\frac{(1 - \eta^2)}{(\frac{1}{a+1} + \eta^2)} \|z^-\|^2_1, \frac{(1 - \eta)(a^2 + b^2)}{2ab} m\right\} \) for all \( z \in N^- \).

**Proof.** By \( (V0') \), we have

\[
\|V_2(x)\| \leq \frac{2ab\eta}{a^2 + b^2} V_1(x).
\]

It follows from (2.3) that

\[
\frac{2ab}{a^2 + b^2} \|z^+\|_1^2 + \int_{\mathbb{R}^N} V_2(x)|z^+|^2 dx
= \int_{\mathbb{R}^N} \left[ \frac{2ab}{a^2 + b^2} |\nabla z^+|^2 + \left( \frac{2ab}{a^2 + b^2} V_1(x) + V_2(x) \right) |z^+|^2 \right] dx
\geq \frac{2ab}{a^2 + b^2} \int_{\mathbb{R}^N} ||\nabla z^+|^2 + (1 - \eta)V_1(x)|z^+|^2| dx
\]
and
\[
\frac{ab}{a^2 + b^2} (\|z^+\|^2 - \|z^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|z|^2\,dx
\]
\[
= \frac{ab}{a^2 + b^2} (\|z^+\|^2 - \|z^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)(|z^+|^2 + |z^-|^2)\,dx + \int_{\mathbb{R}^N} V_2(x)z^+ \cdot z^-\,dx
\]
\[
\leq \frac{ab}{a^2 + b^2} (\|z^+\|^2 - \|z^-\|^2) + \frac{1 + \eta}{2(1 - \eta)} \int_{\mathbb{R}^N} |V_2(x)||z^+|^2\,dx + \frac{1 + \eta}{4\eta} \int_{\mathbb{R}^N} |V_2(x)||z^-|^2\,dx
\]
\[
\leq \frac{ab}{a^2 + b^2} \int_{\mathbb{R}^N} \left[ \nabla z^+|^2 + \frac{1 + \eta^2}{1 - \eta} V_1(x)|z^+|^2 \right]\,dx - \frac{ab}{a^2 + b^2} \int_{\mathbb{R}^N} \left[ \nabla z^-|^2 + \frac{1 - \eta}{2} V_1(x)|z^-|^2 \right]\,dx
\]
\[
\leq \frac{(1 + \eta^2)ab}{(1 - \eta)(a^2 + b^2)} \|z^+\|^2 - \frac{(1 - \eta)ab}{2(a^2 + b^2)} \|z^-\|^2; \quad \forall z \in E. \tag{3.10}
\]

The rest of the proof is standard, so we omit it. \(\square\)

**Lemma 3.7.** Suppose that (V0’), (W0), (W1) and (W2) are satisfied. Let \(e \in E^+\) with \(\|e\|_1 = 1\). Then there is \(r_0 > \rho\) such that \(\sup \Phi(\partial \Omega_r) \leq 0\) for \(r \geq r_0\), where
\[
\Omega_r = \{ \zeta + se : \zeta \in E^-, s \geq 0, \|\zeta + se\|_1 \leq r \}. \tag{3.11}
\]

**Proof.** (2.5) and (3.10) imply \(\Phi(z) \leq 0\) for \(z \in E^-\). Next, it is sufficient to show that \(\Phi(z) \to -\infty\) as \(z \in E^- \oplus \mathbb{R}^+ e\) and \(\|z\|_1 \to \infty\). Arguing indirectly, assume that for some sequence \(\{\zeta_n + s_n e\} \subset E^- \oplus \mathbb{R}^+ e\) with \(\|\zeta_n + s_n e\|_1 \to \infty\), there is an \(M > 0\) such that \(\Phi(\zeta_n + s_n e) \geq -M\) for all \(n \in \mathbb{N}\). Set \(e = (\frac{\tilde{w}_n}{a}, \frac{\tilde{w}_n}{b})\), \(\zeta_n = (-\frac{\tilde{w}_n}{a}, \frac{\tilde{w}_n}{b})\) and
\[
\xi_n = (\zeta_n + s_n e)/\|\zeta_n + s_n e\|_1 = \xi^- + t_n e.
\]

Then \(\|\xi^- + t_n e\|_1 = 1\). Passing to a subsequence, we may assume that \(t_n \to \bar{t}\) and \(\xi_n \to \xi\) in \(E\), and then \(\xi_n \to \xi\) a.e. on \(\mathbb{R}^N\),
\[
\xi_n := \left( -\frac{\tilde{w}_n}{a}, \frac{\tilde{w}_n}{b} \right) \to \xi^- := \left( -\frac{\tilde{w}}{a}, \frac{\tilde{w}}{b} \right) \text{ in } E.
\]

Hence, by (2.5) and (3.10), one has
\[
- \frac{M}{\|\zeta_n + s_n e\|^2_1} \leq \Phi(\zeta_n + s_n e)
\]
\[
= \frac{ab}{a^2 + b^2} \frac{2\bar{t}_n^2}{a^2 + b^2} - \frac{ab}{a^2 + b^2} \frac{1}{\|\xi^-\|_1^2} + \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|\xi^- + t_n e|^2\,dx
\]
\[
- \int_{\mathbb{R}^N} W(x, \frac{-w_n + s_n e_0}{a}, \frac{w_n + s_n e_0}{b}) \frac{\|\bar{t}_n e\|^2_1}{\|\zeta_n + s_n e\|^2_1}\,dx
\]
\[
\leq \frac{(1 + \eta^2)ab \bar{t}_n^2}{(1 - \eta)(a^2 + b^2)} - \frac{(1 - \eta)ab}{2(a^2 + b^2)} \frac{\|\xi^-\|_1^2}{\|\zeta_n + s_n e\|^2_1}
\]
\[
- \int_{\mathbb{R}^N} W(x, \frac{-w_n + s_n e_0}{a}, \frac{w_n + s_n e_0}{b}) \frac{\|\bar{t}_n e\|^2_1}{\|\zeta_n + s_n e\|^2_1}\,dx. \tag{3.12}
\]

If \(\bar{t} = 0\), then it follows from (3.12) that
\[
0 \leq \frac{(1 - \eta)ab}{2(a^2 + b^2)} \frac{\|\xi^-\|_1^2}{\|\zeta_n + s_n e\|^2_1} + \int_{\mathbb{R}^N} W(x, \frac{-w_n + s_n e_0}{a}, \frac{w_n + s_n e_0}{b}) \frac{\|\bar{t}_n e\|^2_1}{\|\zeta_n + s_n e\|^2_1}\,dx
\]
\[
\leq \frac{(1 + \eta^2)ab \bar{t}_n^2}{(1 - \eta)(a^2 + b^2)} + \frac{M}{\|\zeta_n + s_n e\|^2_1} \to 0,
\]

which yields \(\|\xi^-\|_1 \to 0\), and so \(1 = \|\xi_n\|_1 \to 0\), which leads to a contradiction.
Lemma 3.8. Suppose that $\tilde{t} \neq 0$, then $s_n \to \infty$. Hence, it follows from (3.12), (W2) and Fatou's lemma that

$$0 \leq \limsup_{n \to \infty} \left[ \frac{(1 + \eta^2)ab^2}{(1 - \eta)(a^2 + b^2)} \frac{(1 - \eta)ab}{2(a^2 + b^2)} \|z_n^-\|^2 - \int_{\mathbb{R}^N} W(x, -w_n + s_n c_0, w_n + s_n c_0) \frac{d\xi_n}{\|\xi_n + s_n e\|^2} dx \right]$$

$$= \limsup_{n \to \infty} \left[ \frac{(1 + \eta^2)ab^2}{(1 - \eta)(a^2 + b^2)} \frac{(1 - \eta)ab}{2(a^2 + b^2)} \|z_n^-\|^2 - \int_{\mathbb{R}^N} W(x, -w_n + s_n c_0, w_n + s_n c_0) \frac{|t_n e_0|^2 dx}{|s_n e_0|^2} \right]$$

$$\leq \frac{(1 + \eta^2)ab}{(1 - \eta)(a^2 + b^2)} \lim_{n \to \infty} t_n^2 - \liminf_{n \to \infty} \int_{\mathbb{R}^N} W(x, -w_n + s_n c_0, w_n + s_n c_0) \frac{|t_n e_0|^2 dx}{|s_n e_0|^2}$$

$$\leq \frac{(1 + \eta^2)ab^2}{(1 - \eta)(a^2 + b^2)} - \int_{\mathbb{R}^N} \liminf_{n \to \infty} \left[ \frac{W(x, -w_n + s_n c_0, w_n + s_n c_0)}{|s_n e_0|^2} |t_n e_0|^2 \right] dx$$

$$= -\infty,$$

which leads to a contradiction.

Since $E^-$ is separable, let $\{e_k\}_{k=1}^{\infty}$ be a total orthonormal basis of $E^-$. On $E$ we define the $\tau$-norm

$$\|z\|_\tau := \max \left\{ \|z^+\|_1, \sum_{k=1}^{\infty} \frac{1}{2k+1} (z^-, e_k) \right\}, \quad \forall z \in E.$$

(3.13)

It is clear that

$$\|z^+\|_1 \leq \|z\|_\tau \leq \|z\|_1, \quad \forall z \in E.$$ (3.14)

Lemma 3.8. Suppose that (V0'), (W0), (W1) and (W2) are satisfied. Then $\Phi \in C^1(E, \mathbb{R})$ is $\tau$-upper semi-continuous and $\Phi'$ is weakly sequentially continuous.

Proof. It is clear that $\Phi \in C^1(E, \mathbb{R})$. First, we prove that $\Phi$ is $\tau$-upper semi-continuous. Let $z_n \rightharpoonup z$ in $E$ and $\Phi(z_n) \geq c$. It follows from (2.5), (3.10), (3.14) and (W0) that $z_n^+ \to z^+$ in $E$ and

$$C_1 \geq \frac{(1 + \eta^2)ab}{(1 - \eta)(a^2 + b^2)} \|z_n^+\|^2 \geq c + \frac{(1 - \eta)ab}{2(a^2 + b^2)} \|z_n^-\|^2.$$

This shows that $\{z_n^+\} \subset E^-$ is bounded. It is easy to show that $z_n^- \rightharpoonup z^- \iff z_n^- \to z^-$, and so, $z_n \to z$ a.e. on $\mathbb{R}^N$. Note that

$$\left| \int_{\mathbb{R}^N} V_2(x)(z^+_n \cdot z^-_n - z^+ \cdot z^-) dx \right|$$

$$\leq \int_{\mathbb{R}^N} |V_2(x)||z^+_n - z^+||z^-_n| dx + \int_{\mathbb{R}^N} V_2(x)z^+ \cdot (z^-_n - z^-) dx = o(1)$$

(3.15)

and

$$\liminf_{n \to \infty} \int_{\mathbb{R}^N} \left[ |\nabla z^-_n|^2 + \left( V_1(x) - \frac{a^2 + b^2}{2ab} V_2(x) \right) |z^-_n|^2 \right] dx$$

$$\geq \int_{\mathbb{R}^N} \left[ |\nabla z^-|^2 + \left( V_1(x) - \frac{a^2 + b^2}{2ab} V_2(x) \right) |z^-|^2 \right] dx.$$ (3.16)

Hence, it follows from (W2), (2.5), (3.15), (3.16) and Fatou's lemma that

$$-\Phi(z) = \frac{ab}{a^2 + b^2} \left( \|z^-\|^2_1 - \|z^+\|^2_1 \right) - \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|z|^2 dx + \int_{\mathbb{R}^N} W(x,z) dx$$

$$= \frac{ab}{a^2 + b^2} \int_{\mathbb{R}^N} \left[ |\nabla z^-|^2 + \left( V_1(x) - \frac{a^2 + b^2}{2ab} V_2(x) \right) |z^-|^2 \right] dx.$$
Lemma 3.12. By virtue of (W0) and (W1), for

\[ -\frac{ab}{a^2 + b^2} \int_{\mathbb{R}^N} \left[ \nabla z^+ \right]^2 + \left( V_1(x) + \frac{a^2 + b^2}{2ab} V_2(x) \right) |z^+|^2 \, dx \]

\[ -\int_{\mathbb{R}^N} V_2(x) z^+ \cdot z^- \, dx + \int_{\mathbb{R}^N} W(x, z) \, dx \]

\[ \leq \liminf_{n \to \infty} \left( \frac{ab}{a^2 + b^2} \int_{\mathbb{R}^N} \left[ \nabla z_n^- \right]^2 + \left( V_1(x) - \frac{a^2 + b^2}{2ab} V_2(x) \right) |z_n^-|^2 \, dx \right) \]

\[ -\frac{ab}{a^2 + b^2} \int_{\mathbb{R}^N} \left[ \nabla z_n^+ \right]^2 + \left( V_1(x) + \frac{a^2 + b^2}{2ab} V_2(x) \right) |z_n^+|^2 \, dx \]

\[ -\int_{\mathbb{R}^N} V_2(x) z_n^+ \cdot z_n^- \, dx + \int_{\mathbb{R}^N} W(x, z_n) \, dx \]

\[ = \liminf_{n \to \infty} \left[ -\Phi(z_n) \right] = -\limsup_{n \to \infty} \Phi(z_n). \]

This shows that \( \Phi \) is \( \tau \)-upper semi-continuous.

The proof that \( \Phi' \) is weakly sequentially continuous is standard, so we omit it.

Lemma 3.9. Suppose that (V0'), (W0), (W1), (W2) and (W4) are satisfied. Then there exist a constant \( c \in [\kappa, \sup \Phi(\Omega_r)] \) for \( r \geq r_0 \) and a sequence \( \{z_n\} \subset E \) satisfying

\[ \Phi(z_n) \to c, \quad \|\Phi'(z_n)\|_{E^*} (1 + \|z_n\|_1) \to 0, \quad (3.17) \]

where \( \Omega_r \) is defined by (3.11).

Proof. Lemma 3.9 is a direct corollary of Lemmas 3.1, 3.2, 3.6(i), 3.7 and 3.8.

Applying Corollary 3.4, Lemmas 3.6(i), 3.7 and 3.9, we can prove the following lemma in a similar way to [26, Lemma 3.8].

Lemma 3.10 (See [29, Lemma 3.9]). Suppose that (V0'), (W0), (W1), (W2) and (W4) are satisfied. Then there exist a constant \( c_* \in [\kappa, m] \) and a sequence \( \{z_n\} = \{(u_n, v_n)\} \subset E \) satisfying

\[ \Phi(z_n) \to c_*, \quad \|\Phi'(z_n)\| (1 + \|z_n\|_1) \to 0. \quad (3.18) \]

Lemma 3.11. Suppose that (V0'), (W0), (W1), (W2) and (W4) are satisfied. Then for any \( z \in E \setminus E^-, \mathcal{N} \cap (E^- \oplus \mathbb{R}^+ z) \neq \emptyset \), i.e., there exist \( t(z) > 0 \) and \( \zeta(z) \in E^- \) such that \( t(z) + \zeta(z) \in \mathcal{N}^- \).

The proof is the same as one of [25, Lemma 2.6].

Lemma 3.12. Suppose that (V0'), (W0), (W1), (W2), (W3) and (W4) are satisfied. Then any sequence \( \{z_n\} = \{(u_n, v_n)\} \subset E \) satisfying

\[ \Phi(z_n) \to c \geq 0, \quad \langle \Phi'(z_n), z_n \rangle \to 0, \quad \langle \Phi'(z_n), z_n^- \rangle \to 0 \quad (3.19) \]

is bounded in \( E \).

Proof. To prove the boundedness of \( \{z_n\} \), arguing by contradiction, suppose that \( \|z_n\|_1 \to \infty \). Let

\[ \tilde{z}_n = (\tilde{u}_n, \tilde{v}_n) := z_n / \|z_n\|_1. \]

Then \( \|\tilde{z}_n\|_1 = 1 \). By the Sobolev embedding theorem, there exists a constant \( C_2 > 0 \) such that \( \|\tilde{z}_n^+\|_2 \leq C_2 \). If

\[ \delta := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\tilde{z}_n^+|^2 \, dx = 0, \]

then by Lions’ concentration compactness principle [33, Lemma 1.21], \( \tilde{z}_n^+ \to 0 \) in \( L^p(\mathbb{R}^N) \). Fix

\[ \varphi = [(a^2 + b^2)(1 + c)/(1 - \eta)ab]^{1/2}. \]

By virtue of (W0) and (W1), for \( \epsilon = 1/4 \sqrt{ab} > 0 \), there exists \( C_\epsilon > 0 \) such that

\[ W(x, z) \leq \epsilon |z|^2 + C_\epsilon |z|^p, \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2. \]
Hence, it follows that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} W(x, \theta \hat{z}_n^+ \parallel x) \, dx \leq \epsilon \theta^2 \limsup_{n \to \infty} \int_{\mathbb{R}^N} |\hat{z}_n^+ \parallel x|^2 \, dx + C, \theta^p \limsup_{n \to \infty} \int_{\mathbb{R}^N} |\hat{z}_n^+ \parallel x|^p \, dx \\
\leq \epsilon (\theta C_2)^2 = \frac{1}{4}.
\] (3.20)

Let \( \theta_n = \theta / ||z_n||_1 \). Hence, by virtue of (V0'), (2.3), (3.8), (3.19) and (3.20), one can get
\[
c + o(1) = \Phi(z_n)
\geq \frac{ab\theta_n^2}{a^2 + b^2} ||z_n||_1^2 + \frac{\theta_n^2}{2} \int_{\mathbb{R}^N} V_2(x)(||z_n^+ \parallel x^2 - ||z_n^- \parallel x^2) \, dx - \int_{\mathbb{R}^N} W(x, \theta_n z_n^+ \parallel x) \, dx \\
+ \frac{1 - \theta_n^2}{2} \langle \Phi'(z_n), z_n \rangle + \theta_n^2 \langle \Phi'(z_n), z_n^- \rangle \\
= \frac{ab\theta_n^2}{a^2 + b^2} ||\hat{z}_n||_1^2 + \frac{\theta_n^2}{2} \int_{\mathbb{R}^N} V_2(x)(||\hat{z}_n^+ \parallel x^2 - ||\hat{z}_n^- \parallel x^2) \, dx - \int_{\mathbb{R}^N} W(x, \theta \hat{z}_n^+ \parallel x) \, dx \\
+ \left( \frac{1 - \eta}{2 \|z_n\|^2_1} \right) \langle \Phi'(z_n), z_n \rangle + \frac{\theta_n^2}{\|z_n\|^2_1} \langle \Phi'(z_n), z_n^- \rangle \\
\geq \frac{(1 - \eta)ab\theta_n^2}{a^2 + b^2} - \int_{\mathbb{R}^N} W(x, \theta \hat{z}_n^+ \parallel x) \, dx + o(1) \\
\geq \frac{(1 - \eta)ab\theta_n^2}{a^2 + b^2} - \frac{1}{4} + o(1) \\
\geq \frac{a + b}{4} + c + o(1).
\]

This contradiction shows that \( \delta > 0 \). Going if necessary to a subsequence, we may assume the existence of \( k_n \in \mathbb{Z}^N \) such that
\[
\int_{B_{r+\sqrt{2}}(k_n)} |\hat{z}_n^+ \parallel x|^2 \, dx > \frac{\delta}{2}.
\]

Let \( \zeta_n(x) := \hat{z}_n(x + k_n) \). Then
\[
\int_{B_{r+\sqrt{2}}(0)} |\zeta_n^+ \parallel x|^2 \, dx > \frac{\delta}{2}.
\] (3.21)

Now we define
\[
z_n^{k_n}(x) := (u_n^{k_n}(x), v_n^{k_n}(x)) = z_n(x + k_n).
\]
Then \( z_n^{k_n} / ||z_n||_1 = \zeta_n \) and \( ||\zeta_n||^2_{H^1(\mathbb{R}^N)} = ||\hat{z}_n||^2_{H^1(\mathbb{R}^N)} \). Passing to a subsequence, we have \( \zeta_n^+ \rightarrow \zeta^+ \) in \( H^1(\mathbb{R}^N) \), \( \zeta_n^- \rightarrow \zeta^- \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \), \( 2 \leq s <- 2^* \) and \( \zeta_n \rightarrow \zeta^+ \) a.e. on \( \mathbb{R}^N \). Obviously, (3.21) implies that \( \zeta^+ \neq 0 \). For a.e. \( x \in \{ y \in \mathbb{R}^N : \zeta^+(y) \neq 0 \} =: \Omega \), we have
\[
\lim_{n \to \infty} |au_n^{k_n}(x) + bv_n^{k_n}(x)| = \infty.
\]

Hence, it follows from (2.5), (3.10), (3.19), (W2), (W3) and Fatou’s lemma that
\[
0 = \lim_{n \to \infty} \frac{c + o(1)}{||z_n||_1^2} = \lim_{n \to \infty} \frac{\Phi(z_n)}{||z_n||_1^2} \\
= \lim_{n \to \infty} \frac{ab}{a^2 + b^2} (||\hat{z}_n||_1^2 - ||\hat{z}_n^-||_1^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)(||\hat{z}_n||_1^2 \, dx - \int_{\mathbb{R}^N} W(x, u_n, v_n) \, dx \\
\leq \lim_{n \to \infty} \frac{(1 - \eta^2)ab}{(1 - \eta)(a^2 + b^2)} ||\hat{z}_n||_1^2 - \frac{(1 - \eta)ab}{2(a^2 + b^2)} ||\hat{z}_n^-||_1^2 - \int_{\Omega} W(x + k_n, u_n^{k_n}, v_n^{k_n}) \, dx \\
= \lim_{n \to \infty} \frac{(1 - \eta^2)ab}{(1 - \eta)(a^2 + b^2)} ||\hat{z}_n||_1^2 - \frac{(1 - \eta)ab}{2(a^2 + b^2)} ||\hat{z}_n^-||_1^2 - \frac{4a^2b^2}{a^2 + b^2} \int_{\Omega} W(x, u_n^{k_n}, v_n^{k_n}) \, dx \\
\leq \frac{(1 + \eta^2)ab}{(1 - \eta)(a^2 + b^2)} - \frac{4a^2b^2}{a^2 + b^2} \int_{\Omega} \liminf_{n \to \infty} \frac{W(x, u_n^{k_n}, v_n^{k_n})}{|au_n^{k_n} + bv_n^{k_n}||\zeta_n^+||_1^2} \, dx = -\infty.
\]
This contradiction shows that \( \{||u_n||_1\} \) is bounded.

In the last part of the proof of Lemma 3.12, we make use of the periodicity of \( W(x, z) \) on \( x \), which is still valid by using \( W_2' \) instead of \( W_2 \) and \( W_3 \). Therefore, we have the following lemma.

**Lemma 3.13.** Suppose that \( (V') \), \( (W_0) \), \( (W_1) \), \( (W_2') \) and \( (W_4) \) are satisfied. Then any sequence \( \{z_n\} = \{(u_n, v_n)\} \subset E \) satisfying (3.19) is bounded.

**Lemma 3.14** (See [28, Lemma 2.3]). Suppose that \( t \mapsto h(x, t) \) is nondecreasing on \( \mathbb{R} \) and \( h(x, 0) = 0 \) for any \( x \in \mathbb{N}^N \). Then there holds

\[
\left( 1 - \frac{\theta^2}{2} \tau - \theta \sigma \right) h(x, \tau)|\tau| \geq \int_{\theta \tau + \sigma}^\tau h(x, s)|s|ds, \quad \forall \theta \geq 0, \tau, \sigma \in \mathbb{R}. \tag{3.22}
\]

**Lemma 3.15** (See [29]). Suppose that \( W(x, u, v) = \int_0^{\alpha u + \beta v} g(x, s)ds, \) where \( \alpha, \beta \in \mathbb{R} \) with \( \alpha^2 + \beta^2 \neq 0 \) and \( g \in \mathcal{N} \). Then \( W \) satisfies \( (W_0), (W_1) \) and \( (W_4) \).

**Lemma 3.16** (See [29]). Suppose that \( W(x, u, v) = \int_0^{\alpha u + \beta v} h(x, s)ds, \) where \( \alpha, \beta \in \mathbb{R} \) with \( \alpha > \beta^2 \) and \( h \in \mathcal{N} \). Then \( W \) satisfies \( (W_0), (W_1) \) and \( (W_4) \).

**Proof of Theorem 1.7.** Applying Lemmas 3.10 and 3.12, we deduce that there exists a bounded sequence \( \{z_n\} = \{(u_n, v_n)\} \subset E \) satisfying (3.18). The rest of the proof is standard.

Employing Theorem 1.7, the conclusion of Corollary 1.8 follows by Lemmas 3.14–3.16.

### 4 Ground state solutions of Nehari-Pankov type for (2.1)

Without loss of generality, from now on, we assume that \( x_0 = 0 \in \mathcal{V}. \) We only consider the case when \( (V_1) \) is satisfied, since the arguments are similar when \( (V_2) \) is satisfied. Then

\[
V(0) = V_{\min}, \quad Q(x) \leq Q(0), \quad \forall |x| \geq R. \tag{4.1}
\]

Let \( V_1 = 1 \) and \( V_2 = V_\varepsilon \) (or \( \tilde{V}, V_{\min} \) and \( V_{\max} \)), \( W(x, z) = Q_\varepsilon(x)F(z) \) (or \( Q(0)F(z) \) and \( Q_{\min}F(z) \)). Then \( (V_0), (F_1) \) and \( (F_2) \) imply \( (V_0'), (W_0), (W_1), (W_2') \) and \( (W_4) \), respectively. Let

\[
\tilde{V} := \frac{1}{2}(V_\infty + V_{\min}) = \frac{1}{2}(V_\infty + V(0)).
\]

We define three auxiliary functionals as follows:

\[
\Phi(z) = \frac{ab}{a^2 + b^2} (\|z^+\|^2 - \|z^-\|^2) + \frac{\tilde{V}}{2} \int_{\mathbb{R}^N} |z|^2dx - Q(0) \int_{\mathbb{R}^N} F(z)dx, \quad \forall z \in E, \tag{4.2}
\]

\[
\Phi_0(z) = \frac{ab}{a^2 + b^2} (\|z^+\|^2 - \|z^-\|^2) + \frac{V(0)}{2} \int_{\mathbb{R}^N} |z|^2dx - Q(0) \int_{\mathbb{R}^N} F(z)dx, \quad \forall z \in E, \tag{4.3}
\]

and

\[
\Phi_\varepsilon(z) = \frac{ab}{a^2 + b^2} (\|z^+\|^2 - \|z^-\|^2) + \frac{V_{\max}}{2} \int_{\mathbb{R}^N} |z|^2dx - Q_{\min} \int_{\mathbb{R}^N} F(z)dx, \quad \forall z \in E. \tag{4.4}
\]

Let

\[
\mathcal{N}^- = \{z \in E \setminus E^- : \langle \Phi'(z), z \rangle = \langle \Phi'(z), \zeta \rangle = 0, \forall \zeta \in E^- \} \tag{4.5}
\]

and

\[
\mathcal{N}_0^- = \{z \in E \setminus E^- : \langle \Phi_0'(z), z \rangle = \langle \Phi_0'(z), \zeta \rangle = 0, \forall \zeta \in E^- \} \tag{4.6}
\]

be the Nehari-Pankov “manifolds” of the functionals \( \Phi \) and \( \Phi_0 \), respectively. Let

\[
c_\varepsilon = \inf_{\mathcal{N}^-} \Phi_\varepsilon, \quad c = \inf_{\mathcal{N}^-} \Phi, \quad c_0 = \inf_{\mathcal{N}_0^-} \Phi_0. \tag{4.7}
\]

Applying Lemma 3.3 and Corollary 3.5 to \( \Phi_\varepsilon \), we have the following two lemmas.
Lemma 4.1. Suppose that (V0) and (F1) are satisfied. Then

\[
\Phi_\varepsilon(z) \geq \Phi_\varepsilon(\theta z + \zeta) + \frac{ab}{a^2 + b^2} \|z\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} V_\varepsilon(x) |z|^2 \, dx + \frac{1 - \theta^2}{2} (\Phi'_\varepsilon(z), z)
\]
\[
- \theta (\Phi'_\varepsilon(z), \zeta), \quad \forall \theta > 0, \quad z \in E, \quad \zeta \in E^-.
\]  

Lemma 4.3. Suppose that (V0) and (F1) are satisfied. Then \(\hat{c} = \Phi(\hat{z})\). In view of Lemma 3.11, there exist \(\hat{t} > 0\) and \(\hat{\zeta} \in E^-\) such that \(\hat{t} \hat{z} + \hat{\zeta} \in N_0^-\), and so \(\Phi_0(\hat{t} \hat{z} + \hat{\zeta}) \geq c_0\).

Lemma 4.2. Suppose that (V0) and (F1) are satisfied. Then

\[
\Phi_\varepsilon(z) \geq \frac{ab \theta^2}{a^2 + b^2} \|z\|^2 + \frac{\theta^2}{2} \int_{\mathbb{R}^N} V_\varepsilon(x) (|z^+|^2 - |z^-|^2) \, dx - \int_{\mathbb{R}^N} Q_\varepsilon(x) F(\theta z^+) \, dx
\]
\[
+ \frac{1 - \theta^2}{2} (\Phi'_\varepsilon(z), z) + \theta^2 (\Phi'_\varepsilon(z), z^-), \quad \forall \theta > 0, \quad z \in E.
\]

By virtue of Corollary 1.8, under Assumptions (V0), (F1) and (F2), there exists a \(\hat{z} \in \mathcal{N}^-\) such that \(\hat{c} = \Phi(\hat{z})\). In view of Lemma 3.11, there exist \(\hat{t} > 0\) and \(\hat{\zeta} \in E^-\) such that \(\hat{t} \hat{z} + \hat{\zeta} \in N_0^-\), and so \(\Phi_0(\hat{t} \hat{z} + \hat{\zeta}) \geq c_0\).

Lemma 4.4. Suppose that (V0), (V1), (F1) and (F2) are satisfied. Then \(\hat{c} \geq c_0 + \hat{\delta}\), where

\[
\hat{\delta} := \frac{V_{\infty} - V_{\min}}{4} \int_{\mathbb{R}^N} |\hat{t} \hat{z} + \hat{\zeta}|^2 \, dx > 0
\]

is independent of \(\varepsilon > 0\).

Proof. Applying Lemma 3.3 to \(\Phi(z)\), one has

\[
\hat{c} = \Phi(\hat{z}) \geq \Phi(\hat{t} \hat{z} + \hat{\zeta})
\]
\[
= \Phi_0(\hat{t} \hat{z} + \hat{\zeta}) + \frac{\hat{V} - V_{\min}}{2} \int_{\mathbb{R}^N} |\hat{t} \hat{z} + \hat{\zeta}|^2 \, dx
\]
\[
\geq c_0 + \frac{V_{\infty} - V_{\min}}{4} \int_{\mathbb{R}^N} |\hat{t} \hat{z} + \hat{\zeta}|^2 \, dx
\]
\[
= c_0 + \hat{\delta}.
\]

This completes the proof. \(\square\)

By virtue of Corollary 1.8, under Assumptions (V0), (F1) and (F2), there exists a \(z_0 \in \mathcal{N}_0^-\) such that \(c_0 = \Phi_0(z_0)\). Then

\[
\Phi_0(z_0) \geq \Phi_0(tz_0 + \zeta), \quad (\Phi'_0(z_0), z_0) = (\Phi'_0(z_0), \zeta) = 0, \quad \forall t \geq 0, \quad \zeta \in E^-.
\]

In view of Lemma 3.11, for any \(\varepsilon > 0\), there exist \(t_\varepsilon > 0\) and \(\zeta_\varepsilon \in E^-\) such that \(t_\varepsilon z_0 + \zeta_\varepsilon \in N_\varepsilon^-\), and so \(\Phi_\varepsilon(t_\varepsilon z_0 + \zeta_\varepsilon) \geq c_\varepsilon\) and \(\Phi_\varepsilon(t_\varepsilon z_0 + \zeta_\varepsilon) \geq \Phi_\varepsilon(t z_0 + \zeta), \forall t \geq 0, \quad \zeta \in E^-\). Set

\[
\alpha_\varepsilon := \frac{ab}{a^2 + b^2} (\|z_0^+\|^2 - \|z_0^-\|^2) + \frac{V_{\min}}{2} \int_{\mathbb{R}^N} |z_0|^2 \, dx - Q_{\max} \int_{\mathbb{R}^N} F(z_0) \, dx.
\]

Clearly, \(\alpha_\varepsilon\) is independent of \(\varepsilon > 0\). Analogous to the proof of Lemma 3.7, one can demonstrate the following lemma.
Lemma 4.4. Suppose that (V0), (V1), (F1) and (F2) are satisfied. Then there is an $M_0 > 0$ independent of $\varepsilon > 0$ such that

$$
\Phi_\varepsilon(\zeta + sz_0) \leq \alpha_0 - 1, \quad \forall \zeta \in E^-, \ s \geq 0, \ \|\zeta + sz_0\| \geq M_0.
$$

(4.15)

Lemma 4.5. Suppose that (V0), (V1), (F1) and (F2) are satisfied. Then

$$
M_1 := \sup_{\varepsilon > 0} |t_\varepsilon| \leq M_0 \|z_0^+\|^{-1}, \quad M_2 := \sup_{\varepsilon > 0} \|\zeta_\varepsilon\| \leq M_0 (1 + \|z_0\| \|z_0^+\|^{-1}).
$$

(4.16)

Proof. Note that

$$
\alpha_0 = \frac{ab}{a^2 + b^2} (\|z_0^+\|^2 - \|z_0^-\|^2) + \frac{V_{\min}}{2} \int_{\mathbb{R}^N} |z_0|^2 \, dx - Q_{\max} \int_{\mathbb{R}^N} F(z_0) \, dx
\leq \Phi_\varepsilon(z_0) \leq \Phi_\varepsilon(t_\varepsilon z_0 + \zeta_\varepsilon) \leq \Phi_\varepsilon(t_\varepsilon z_0 + \zeta_\varepsilon).
$$

(4.17)

By Lemma 4.4, one obtains that

$$
\sup_{\varepsilon > 0} \|t_\varepsilon \varepsilon + \zeta_\varepsilon\| \leq M_0.
$$

(4.18)

Since

$$
\|t_\varepsilon z_0 + \zeta_\varepsilon\|^2 = \|t_\varepsilon z_0^+\|^2 + \|t_\varepsilon z_0^- + \zeta_\varepsilon\|^2 \geq t_\varepsilon^2 \|z_0^+\|^2,
$$

it follows from (4.18) and the above that

$$
M_1 := \sup_{\varepsilon > 0} |t_\varepsilon| \leq M_0 \|z_0^+\|^{-1}, \quad M_2 := \sup_{\varepsilon > 0} \|\zeta_\varepsilon\| \leq M_0 (1 + \|z_0\| \|z_0^+\|^{-1}).
$$

This completes the proof. \qed

In view of (F1), there exists a constant $\beta_0 > 0$ such that

$$
|F_\varepsilon(z)| \leq \beta_0 (|z| + |z|^{p-1}), \quad \forall z \in \mathbb{R}^2.
$$

(4.19)

Now, we can choose $R_0 > R$ sufficiently large such that

$$
V(x) \geq \hat{V}, \quad Q(x) \leq Q(0), \quad \forall |x| \geq R_0
$$

(4.20)

and

$$
M_1^2 \int_{|x| > R_0} |z_0|^2 \, dx - (1 + M_1^2) Q_{\max} \int_{|x| > R_0} F_\varepsilon(z_0) \cdot z_0 \, dx
+ 2(1 + \beta_0 Q_{\max}) M_1 M_2 \left( \int_{|x| > R_0} |z_0|^2 \, dx \right)^{1/2}
+ 2\beta_0 \gamma_p Q_{\max} M_1 M_2 \left( \int_{|x| > R_0} |z_0|^p \, dx \right)^{(p-1)/p} \leq \frac{\delta}{2}.
$$

(4.21)

where $\gamma_p$ is the embedding constant with $\|\cdot\|_p \leq \gamma_p \|\cdot\|$. For the $R_0 > 0$ given above, we can choose an $\varepsilon_0 > 0$ such that

$$
\sqrt{2 + 2\beta_0 Q_{\max}} M_1 M_2 \|z_0\|_2 \left\{ \sup_{|x| \leq R_0} \|V_\varepsilon(x) - V(0)\| + \beta_0 |Q_\varepsilon(x) - Q(0)| \right\}^{1/2}
+ M_1^2 \sup_{|x| \leq R_0} \|V_\varepsilon(x) - V(0)\| \|z_0\|_2 \left( \frac{1 + M_1^2}{2} \right)
\sup_{|x| \leq R_0} \|Q_\varepsilon(x) - Q(0)\| \int_{|x| \leq R_0} F_\varepsilon(z_0) \cdot z_0 \, dx
\leq \frac{\delta}{4}, \quad \forall \varepsilon \in [0, \varepsilon_0].
$$

(4.22)

Lemma 4.6. Suppose that (V0), (V1), (F1) and (F2) are satisfied. Then

$$
c_\varepsilon \geq c_\varepsilon - 3\delta/4, \quad \forall \varepsilon \in [0, \varepsilon_0].
$$

(4.23)
Proof. From (F1), (2.4), (4.3), (4.9), (4.13), (4.16), (4.19), (4.21), (4.22) and Hölder's inequality, we have
\[
c_0 = \Phi_0(z_0) = \Phi_\varepsilon(t_\varepsilon z_0 + \zeta_\varepsilon) + (1 - \eta) \left\{ \frac{|\zeta_\varepsilon|^2}{a^2 + b^2} + \frac{1 - t_\varepsilon^2}{2} \right\} \Phi_\varepsilon(z_0) - t_\varepsilon \Phi_\varepsilon(z_0, \zeta_\varepsilon)
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^N} |V_\varepsilon(x) - V_\varepsilon(0)| |z_0|^2 \, dx + \int_{\mathbb{R}^N} (Q_\varepsilon(x) - Q(0)) F(z_0) \, dx
\]
\[
\geq c_\varepsilon + \frac{1}{2} \int_{\mathbb{R}^N} |V_\varepsilon(x) - V_\varepsilon(0)| |z_0|^2 \, dx + \int_{\mathbb{R}^N} |Q_\varepsilon(x) - Q(0)| F(z_0) \, dx
\]
\[
+ \frac{1 - t_\varepsilon^2}{2} \left\{ \int_{\mathbb{R}^N} |V_\varepsilon(x) - V(0)| |z_0|^2 \, dx + \int_{\mathbb{R}^N} (Q(0) - Q_\varepsilon(x)) F(z_0) \cdot z_0 \, dx \right\}
\]
\[
- t_\varepsilon \left\{ \int_{\mathbb{R}^N} |V_\varepsilon(x) - V(0)| z_0 \cdot \zeta_\varepsilon \, dx + \int_{\mathbb{R}^N} (Q(0) - Q_\varepsilon(x)) F(z_0) \cdot \zeta_\varepsilon \, dx \right\}
\]
\[
= c_\varepsilon + \frac{1}{2} \int_{\mathbb{R}^N} |Q_\varepsilon(x) - Q(0)| F(z_0) \cdot z_0 - F(z_0) \cdot z_0 \, dx
\]
\[
- \frac{M_1^2}{2} \int_{\mathbb{R}^N} |V_\varepsilon(x) - V(0)| |z_0|^2 \, dx - \frac{M_1^2}{2} \int_{\mathbb{R}^N} |Q_\varepsilon(x) - Q(0)| F(z_0) \cdot z_0 \, dx
\]
\[
- M_1 \int_{\mathbb{R}^N} |V_\varepsilon(x) - V(0)| |\zeta_\varepsilon| \, dx - M_1 \int_{\mathbb{R}^N} |Q_\varepsilon(x) - Q(0)| F(z_0) |\zeta_\varepsilon| \, dx
\]
\[
\geq c_\varepsilon - \frac{M_1^2}{2} \int_{|x| \leq R_0} |V_\varepsilon(x) - V(0)| |z_0|^2 \, dx - \frac{1}{2} \int_{|x| \leq R_0} |Q_\varepsilon(x) - Q(0)| F(z_0) \cdot z_0 \, dx
\]
\[
- \sqrt{2 + 2\beta_0 Q_{\text{max}} M_1 M_2} \left\{ \int_{\mathbb{R}^N} |V_\varepsilon(x) - V(0)| |z_0|^2 \, dx \right\}^{1/2}
\]
\[
- \beta_0 \gamma_p (2 Q_{\text{max}})^{1/p} M_1 M_2 \left\{ \int_{\mathbb{R}^N} |Q_\varepsilon(x) - Q(0)| |z_0|^p \, dx \right\}^{(p-1)/p}
\]
\[
\geq c_\varepsilon - \frac{M_1^2}{2} \int_{|x| > R_0} |z_0|^2 \, dx - (1 + M_1^2) Q_{\text{max}} \int_{|x| > R_0} F(z_0) \cdot z_0 \, dx
\]
\[
- 2(1 + \beta_0 Q_{\text{max}}) M_1 M_2 \left\{ \int_{|x| > R_0} |z_0|^2 \, dx \right\}^{1/2}
\]
\[
- 2 \beta_0 \gamma_p Q_{\text{max}} M_1 M_2 \left\{ \int_{|x| > R_0} |z_0|^p \, dx \right\}^{(p-1)/p}
\]
\[
\geq c_\varepsilon - \sqrt{2 + 2\beta_0 Q_{\text{max}} M_1 M_2} \left\{ \sup_{|x| \leq R_0} |V_\varepsilon(x) - V(0)| + \beta_0 |Q_\varepsilon(x) - Q(0)| \right\}^{1/2}
\]
\[
- \frac{M_1^2}{2} \sup_{|x| \leq R_0} |V_\varepsilon(x) - V(0)| |z_0|^2 \, dx - \frac{1 + M_1^2}{2} \sup_{|x| \leq R_0} |Q_\varepsilon(x) - Q(0)| \int_{|x| \leq R_0} F(z_0) \cdot z_0 \, dx
\]
\[
- \beta_0 \gamma_p (2Q_{\text{max}})^{1/p} M_1 M_2 \|z_0\|_p^{p-1} \left\{ \sup_{|x| \leq R_0} |Q_\varepsilon(x) - Q(0)| \right\}^{(p-1)/p} \frac{1}{2} \delta_n^2 \\
\geq c_\varepsilon - \frac{3\delta_n}{4}.
\]

This completes the proof. \(\square\)

Similar to Lemma 3.6, we can demonstrate that for any \(\varepsilon > 0\), there exists a \(\rho_\varepsilon > 0\) such that
\[
c_\varepsilon = \inf_{N_\varepsilon} \Phi_\varepsilon \geq \kappa_\varepsilon := \inf \{ \Phi_\varepsilon(z) : z \in E^+, \|z\| = \rho_\varepsilon \} > 0.
\] (4.24)

Applying Lemmas 3.10 and 3.13 to \(\Phi_\varepsilon\), we have the following two lemmas.

**Lemma 4.7.** Suppose that (V0), (F1) and (F2) are satisfied. Then there exist a constant \(\bar{c}_\varepsilon \in [\kappa_\varepsilon, c_\varepsilon]\) and a sequence \(\{z_\varepsilon^n\} = \{(u_\varepsilon^n, v_\varepsilon^n)\} \subset E\) satisfying
\[
\Phi_\varepsilon(z_\varepsilon^n) \to \bar{c}_\varepsilon, \quad \|\Phi'_\varepsilon(z_\varepsilon^n)(1 + \|z_\varepsilon^n\|)\| \to 0.
\] (4.25)

**Lemma 4.8.** Suppose that (V0), (F1) and (F2) are satisfied. Then the sequence \(\{z_\varepsilon^n\} = \{(u_\varepsilon^n, v_\varepsilon^n)\} \subset E\) satisfying (4.25) is bounded in \(E\).

Similar to (3.10), one has
\[
\frac{ab}{a^2 + b^2} (\|z_\varepsilon^n^+\|^2 - \|z_\varepsilon^n^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_\varepsilon(x)|z_\varepsilon^n|^2 \, dx \\
\leq \frac{(1 + \eta^2)ab}{(1 - \eta)(a^2 + b^2)} \|z_\varepsilon^n^+\|^2 - \frac{(1 - \eta)ab}{2(a^2 + b^2)} \|z_\varepsilon^n^-\|^2, \quad \forall z \in E.
\] (4.26)

By (F1), (2.4), (4.25) and (4.26), one has
\[
\bar{c}_\varepsilon + o(1) = \Phi_\varepsilon(z_\varepsilon^n) \\
= \frac{ab}{a^2 + b^2} (\|z_\varepsilon^n^+\|^2 - \|z_\varepsilon^n^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_\varepsilon(x)|z_\varepsilon^n|^2 \, dx - \int_{\mathbb{R}^N} Q_\varepsilon(x)F(z_\varepsilon^n) \, dx \\
\leq \frac{(1 + \eta^2)ab}{(1 - \eta)(a^2 + b^2)} \|z_\varepsilon^n^+\|^2 - \frac{(1 - \eta)ab}{2(a^2 + b^2)} \|z_\varepsilon^n^-\|^2 \\
\leq \frac{(1 + \eta^2)ab}{(1 - \eta)(a^2 + b^2)} \|z_\varepsilon^n^+\|^2.
\] (4.27)

Therefore, it follows from Lemma 3.11 that there exist \(t_\varepsilon^n > 0\) and \(\zeta_\varepsilon^n \in E^-\) such that \(t_\varepsilon^n z_\varepsilon^n + \zeta_\varepsilon^n \in \hat{N}^-\), and so
\[
\hat{\Phi}(t_\varepsilon^n z_\varepsilon^n + \zeta_\varepsilon^n) \geq \bar{c}_\varepsilon, \quad \langle \hat{\Phi}'(t_\varepsilon^n z_\varepsilon^n + \zeta_\varepsilon^n), t_\varepsilon^n z_\varepsilon^n + \zeta_\varepsilon^n \rangle = \langle \hat{\Phi}'(t_\varepsilon^n z_\varepsilon^n + \zeta_\varepsilon^n), \zeta \rangle = 0, \quad \forall \zeta \in E^-.
\] (4.28)

**Lemma 4.9.** Suppose that (V0), (V1) and (F1)–(F3) are satisfied. Then for any \(\varepsilon > 0\), there exist \(K_1(\varepsilon) > 0\) and \(K_2(\varepsilon) > 0\) such that
\[
0 \leq t_\varepsilon^n \leq K_1(\varepsilon), \quad \|\zeta_\varepsilon^n\| \leq K_2(\varepsilon), \quad \forall \varepsilon > 0.
\] (4.29)

**Proof.** If along a subsequence \(t_\varepsilon^n < T_0\), we are through. So we may assume that \(t_\varepsilon^n \geq T_0\). In view of Lemma 4.8, there exists a constant \(C_3 > 0\) such that \(\|z_\varepsilon^n^+\|_2 \leq C_3\). If
\[
\delta_\varepsilon := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |z_\varepsilon^n|^2 \, dx = 0,
\]
then by Lions’ concentration compactness principle [33, Lemma 1.21], \(z_\varepsilon^n^+ \to 0\) in \(L^p(\mathbb{R}^N)\). Fix \(\vartheta = [(1 + \eta^2)(1 + \bar{c}_\varepsilon)/(1 - \eta)^2\bar{c}_\varepsilon]^{1/2}\). By virtue of (F1), for \(\epsilon = 1/4Q_{\text{max}}(\partial C_3)^2 > 0\), there exists a \(C_\varepsilon > 0\) such that
\[
F(z) \leq \epsilon |z|^2 + C_\varepsilon |z|^p, \quad \forall z \in \mathbb{R}^2.
\]
Hence, it follows from (4.2), (4.28), (F1) and (F3) that
\[
\int_{B_1} |(\tilde{z}^\varepsilon_n)^+|^2 \, dx > \frac{\delta_\varepsilon}{2}.
\]
From (4.11), (4.25), (4.27) and (4.30), one has
\[
\tilde{c}_\varepsilon + o(1) = \Phi_\varepsilon(\tilde{z}^\varepsilon_n) \geq \left(1 - \frac{1}{2}\right) \frac{\|z_n^+\|^2}{a^2 + b^2} - \int_{\mathbb{R}^N} Q_\varepsilon(x) F(\vartheta z^\varepsilon_n) \, dx
\]
\[
+ \frac{1}{2} (\Phi'_\varepsilon(\tilde{z}^\varepsilon_n), z^\varepsilon_n) + \vartheta^2 (\Phi'_\varepsilon(\tilde{z}^\varepsilon_n), z^-) + \frac{1}{2} \frac{\|z_n^+\|^2 - \frac{1}{4} + o(1)}{1 + \eta^2}
\]
\[
\geq \frac{3}{4} + \tilde{c}_\varepsilon + o(1).
\]
This contradiction shows that \(\delta_\varepsilon > 0\). Going if necessary to a subsequence, we may assume the existence of \(k_n \in \mathbb{Z}^N\) such that \(\int_{B_1 + \sqrt{\eta} \pi(k_n)} |z^\varepsilon_{n+1}^+| \, dx > \frac{\delta_\varepsilon}{2}\). Let \(\tilde{z}^\varepsilon_n(x) = z^\varepsilon_n(x + k_n)\) and \(\tilde{\zeta}^\varepsilon_n(x) = \zeta^\varepsilon_n(x + k_n)\). Then \(\|\tilde{z}^\varepsilon_n\| = \|z^\varepsilon_n\|\) and
\[
\int_{B_1 + \sqrt{\eta} \pi(0)} |(\tilde{z}^\varepsilon_n)^+|^2 \, dx > \frac{\delta_\varepsilon}{2}.
\]
Passing to a subsequence, we have \(\tilde{z}^\varepsilon_n \to \tilde{z}^\varepsilon\) and \((\tilde{z}^\varepsilon_n)^+ \to (\tilde{z}^\varepsilon)^+\) in \(E\), \((\tilde{z}^\varepsilon)^+ \to (\tilde{z}^\varepsilon)^+\) in \(L^2_{\text{loc}}(\mathbb{R}^N)\), \(2 < s < 2^*\) and \((\tilde{z}^\varepsilon)^+ \to (\tilde{z}^\varepsilon)^+\) a.e. on \(\mathbb{R}^N\). Obviously, (4.31) implies that \((\tilde{z}^\varepsilon)^+ \neq 0\). Let \(\eta^\varepsilon_n := \tilde{\zeta}^\varepsilon_n / t_n^\varepsilon\). Hence, it follows from (4.2), (4.28), (F1) and (F3) that
\[
0 = \langle \hat{\Phi}'(t^\varepsilon_n z^\varepsilon_n + \tilde{\zeta}^\varepsilon_n), t^\varepsilon_n z^\varepsilon_n + \tilde{\zeta}^\varepsilon_n \rangle
\]
\[
= \langle \hat{\Phi}'(t^\varepsilon_n z^\varepsilon_n + \tilde{\zeta}^\varepsilon_n), t^\varepsilon_n z^\varepsilon_n + \tilde{\zeta}^\varepsilon_n \rangle
\]
\[
= \langle \hat{\Phi}'(t^\varepsilon_n (z^\varepsilon_n + \eta^\varepsilon_n)), t^\varepsilon_n (z^\varepsilon_n + \eta^\varepsilon_n) \rangle
\]
\[
= \left[ \frac{2ab}{a^2 + b^2} (\|z^\varepsilon_n^+\|^2 - \|z^\varepsilon_n^-\|^2 + \eta^\varepsilon_n^2) + \hat{V} \int_{\mathbb{R}^2} |z^\varepsilon_n + \eta^\varepsilon_n|^2 \, dx \right] (t^\varepsilon_n)^2
\]
\[
- Q(0) \int_{\mathbb{R}^N} F_z(t^\varepsilon_n (z^\varepsilon_n + \eta^\varepsilon_n)) \cdot t^\varepsilon_n (z^\varepsilon_n + \eta^\varepsilon_n) \, dx
\]
\[
\leq \left[ \frac{2ab}{a^2 + b^2} (\|z^\varepsilon_n^+\|^2 - \|z^\varepsilon_n^-\|^2 + \eta^\varepsilon_n^2) + \hat{V} \int_{\mathbb{R}^2} |z^\varepsilon_n + \eta^\varepsilon_n|^2 \, dx \right] (t^\varepsilon_n)^2
\]
\[
- 2Q(0) \int_{\mathbb{R}^N} F(t^\varepsilon_n (z^\varepsilon_n + \eta^\varepsilon_n)) \, dx
\]
\[
\leq \left[ \frac{2ab}{a^2 + b^2} (\|z^\varepsilon_n^+\|^2 - \|z^\varepsilon_n^-\|^2 + \eta^\varepsilon_n^2) + \hat{V} \int_{\mathbb{R}^2} |z^\varepsilon_n + \eta^\varepsilon_n|^2 \, dx \right] (t^\varepsilon_n)^2
\]
\[
- 2C_0 Q(0)(t^\varepsilon_n)^n \int_{\mathbb{R}^N} F_0(z^\varepsilon_n + \eta^\varepsilon_n) \, dx,
\]
which, together with (4.26), implies that
\[
0 \leq \frac{2ab}{a^2 + b^2} (\|z^\varepsilon_n^+\|^2 - \|z^\varepsilon_n^-\|^2 + \eta^\varepsilon_n^2) + \hat{V} \int_{\mathbb{R}^2} |z^\varepsilon_n + \eta^\varepsilon_n|^2 \, dx
\]
This shows that $\{\|\eta_{n,\varepsilon}\|\}_{n=1}^{\infty}$ is bounded in $E^{-}$. Passing to a subsequence, we have $\eta_{n,\varepsilon} \rightharpoonup \eta^{\varepsilon}$ in $E^{-}$, $\eta_{n,\varepsilon} \rightarrow \eta^{\varepsilon}$ in $L^{p}_{loc}(\mathbb{R}^{N})$, $2 \leq s < 2^*$ and $\eta_{n,\varepsilon} \rightharpoonup \eta^{\varepsilon}$ a.e. on $\mathbb{R}^{N}$.

Since $(\tilde{z}^{\varepsilon} + \eta^{\varepsilon})^{+} = (z^{\varepsilon})^{+} \neq 0$, it follows from (F3) that $\int_{\mathbb{R}^{N}} F_{0}(\tilde{z}^{\varepsilon} + \eta^{\varepsilon})dx > 0$, which, together with (4.32), implies that $\{t^{\varepsilon}_{n}\}_{n=1}^{\infty}$ is bounded, and so $\{\|\zeta_{n}^{\varepsilon}\|\}_{n=1}^{\infty}$ is also bounded. Therefore, there exist $K_{1}(\varepsilon) > 0$ and $K_{2}(\varepsilon) > 0$ such that (4.29) holds.

**Theorem 4.10.** Assume that $V$, $Q$ and $F$ satisfy (V0), (V1) and (F1)–(F3). Then for $\varepsilon \in (0, \varepsilon_{0}]$, (2.1) has a solution $z_{\varepsilon} \in E$ such that $\Phi_{\varepsilon}(z_{\varepsilon}) = \inf_{N_{\varepsilon}^{+}} \Phi_{\varepsilon} > 0$.

**Proof.** By Lemmas 4.7 and 4.8, there exists a bounded sequence $\{z_{n}^{\varepsilon}\} = \{(u_{n}^{\varepsilon}, v_{n}^{\varepsilon})\} \subset E$ satisfying (4.25). Thus there exists $z^{\varepsilon} = (u^{\varepsilon}, v^{\varepsilon}) \in E$ such that $z_{n}^{\varepsilon} \rightharpoonup z^{\varepsilon}$. Next, we prove that $z^{\varepsilon} \neq 0$ for all $\varepsilon \in (0, \varepsilon_{0}]$.

Arguing by contradiction, suppose that $z^{\varepsilon} = 0$ for some $\varepsilon \in (0, \varepsilon_{0})$, i.e., $z_{n}^{\varepsilon} \rightarrow 0$ in $E$, and so $z_{n}^{\varepsilon} \rightarrow 0$ in $L^{p}_{loc}(\mathbb{R}^{N})$, $2 \leq s < 2^*$ and $z^{\varepsilon} \rightarrow 0$ a.e. on $\mathbb{R}^{N}$.

We first prove that $\zeta_{n}^{\varepsilon} \rightarrow 0$ in $E^{-}$. Since $\{\|\zeta_{n}^{\varepsilon}\|\}_{n=1}^{\infty}$ is bounded, passing to a subsequence we may assume that $\zeta_{n}^{\varepsilon} \rightarrow \zeta^{\varepsilon}$ in $E^{-}$. By Lemma 4.9, $\{t_{n}^{\varepsilon}\}$ is bounded, and then $t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon} \rightarrow \zeta^{\varepsilon}$. By Brezis-Lieb’s lemma (see [33, Lemma 1.32]), one can demonstrate that

$$
\hat{\Phi}(t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}) - \hat{\Phi}(t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon} - \zeta^{\varepsilon}) - \hat{\Phi}(\zeta^{\varepsilon}) = o(1),
$$

which, together with (4.9), (V0), and (V1), yields

$$
o(1) \geq -\hat{\Phi}(\zeta^{\varepsilon}) \geq \frac{ab}{a^2 + b^2} \|\zeta^{\varepsilon}\|^2 - \frac{\hat{V}}{2} \int_{\mathbb{R}^{N}} |\zeta^{\varepsilon}|^2 dx \geq \frac{(1 - \eta)ab}{a^2 + b^2} \|\zeta^{\varepsilon}\|^2.
$$

This shows that $\zeta^{\varepsilon} = 0$, i.e., $\zeta_{n}^{\varepsilon} \rightarrow 0$ in $E^{-}$. By (4.2), (4.9), (4.20), (4.25), (4.28) and (F1), we have

$$
c_{\varepsilon} + o(1) \geq \hat{c}_{\varepsilon} + o(1) = \Phi_{\varepsilon}(z_{n}^{\varepsilon})
\geq \Phi_{\varepsilon}(t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}) + \frac{1 - (t_{n}^{\varepsilon})^2}{2} \langle \Phi_{\varepsilon}'(z_{n}^{\varepsilon}), z_{n}^{\varepsilon} \rangle - t_{n}^{\varepsilon} \langle \Phi_{\varepsilon}'(z_{n}^{\varepsilon}), \zeta_{n}^{\varepsilon} \rangle
= \Phi_{\varepsilon}(t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}) + o(1)
= \hat{\Phi}(t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}) + \frac{1}{2} \int_{\mathbb{R}^{N}} (V_{2}(x) - \hat{V}) |t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}|^2 dx
+ \int_{\mathbb{R}^{N}} [Q(0) - Q_{2}(x)] F(t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}) dx + o(1)
\geq \hat{c} + \frac{1}{2} \int_{|x| \leq R_{0}/\varepsilon} (V_{2}(x) - \hat{V}) |t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}|^2 dx
+ \int_{|x| \leq R_{0}/\varepsilon} |Q(0) - Q_{2}(x)| F(t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}) dx + o(1)
\geq \hat{c} - 2V_{\max} - V_{\infty} - V(0) \int_{|x| \leq R_{0}/\varepsilon} |t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}|^2 dx
+ |Q(0) - Q_{\max}| \int_{|x| \leq R_{0}/\varepsilon} F(t_{n}^{\varepsilon}z_{n}^{\varepsilon} + \zeta_{n}^{\varepsilon}) dx + o(1)
= \hat{c} + o(1).
$$

On the other hand, by Lemmas 4.3 and 4.6, one has

$$
\hat{c} \geq c_{0} + \hat{\delta} \geq c_{\varepsilon} + \frac{1}{4} \delta,
$$

which contradicts (4.36). Therefore, $z^{\varepsilon} \neq 0$ for all $\varepsilon \in (0, \varepsilon_{0}]$. In a standard way, we can certify that $\Phi_{\varepsilon}'(z^{\varepsilon}) = 0$ and $\Phi_{\varepsilon}(z^{\varepsilon}) = c_{\varepsilon} = \inf_{N_{\varepsilon}^{+}} \Phi_{\varepsilon}$. This shows that $z^{\varepsilon} \in E$ is a solution for (2.1) with $\Phi_{\varepsilon}(z^{\varepsilon}) = \inf_{N_{\varepsilon}^{+}} \Phi_{\varepsilon} > 0$. 

\[\blacksquare\]
Proof of Theorem 1.6. For \( \varepsilon \in (0, \varepsilon_0] \), Theorem 4.10 implies that (2.1) has a solution \( z_\varepsilon \in E \) such that
\[
\Phi_\varepsilon(z_\varepsilon) = \inf_{N^-} \Phi_\varepsilon > 0.
\]
Then
\[
\hat{z}_\varepsilon(x) = (\hat{u}_\varepsilon(x), \hat{v}_\varepsilon(x)) := z_\varepsilon(\varepsilon^{-1}(x - x_v)) = (u_\varepsilon(\varepsilon^{-1}(x - x_v)), v_\varepsilon(\varepsilon^{-1}(x - x_v)))
\]
is a nontrivial solution of (1.5).

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