AFFINE TODA SYSTEMS
COUPLED TO MATTER FIELDS

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ABSTRACT

We investigate higher grading integrable generalizations of the affine Toda systems, where the flat connections defining the models take values in eigensubspaces of an integral gradation of an affine Kac-Moody algebra, with grades varying from \( l \) to \(-l\) \((l > 1)\). The corresponding target space possesses nontrivial vacua and soliton configurations, which can be interpreted as particles of the theory, on the same footing as those associated to fundamental fields. The models can also be formulated by a Hamiltonian reduction procedure from the so called two–loop WZNW models. We construct the general solution and show the classes corresponding to the solitons. Some of the particles and solitons become massive when the conformal symmetry is spontaneously broken by a mechanism with an intriguing topological character and leading to a very simple mass formula. The massive fields associated to non zero grade generators obey field equations of the Dirac type and may be regarded as matter fields. A special class of models is remarkable. These theories possess a \( U(1) \) Noether current which, after a special gauge fixing of the conformal symmetry, is proportional to a topological current. This leads to the confinement of the matter field inside the solitons, which can be regarded as a one dimensional bag model for QCD. These models are also relevant to the study of electron self–localization in (quasi)-one-dimensional electron–phonon systems.

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1 Introduction

Integrable theories in low dimensions, besides their intrinsic beauty, have become a very important tool in the understanding of basic non-perturbative aspects of physical theories. They constitute always a laboratory to test ideas on confinement, quantum physics of solitons and many others. In some cases they provide realistic models for very interesting phenomena in condensed matter physics, statistical mechanics, and in high energy physics under special kinematical conditions. More recently, they unexpectedly reappeared as describing the dependence upon coupling constants of the low-energy effective actions for supersymmetric Yang-Mills theories in four dimensions (see e.g. ref. [1]).

In the present paper we introduce a new class of integrable theories in $1 + 1$ dimensions, presenting very interesting physical properties, and which we hope, will help understanding the role of solitons in quantum field theories. The models generalize the abelian and non-abelian affine Toda theories, in the sense that they contain matter fields coupled to the (gauge) Toda fields. The arising equations are affine specialisations of some general system in two dimensions [3] which becomes integrable when associated with a Lie algebra of finite growth; see [3] and references therein. They represent affine (non-abelian) extensions of the corresponding finite system [4], which, as compared with [5], contain matter fields.

We introduce the models through a zero curvature condition, where the flat connections take values on a affine Kac-Moody algebra $\hat{G}$ endowed with an integral gradation. The connection has components not only on the 0, ±1 grades, as for the usual Toda fields, but also on eigensubspaces of grades varying from $l$ to $-l$, with $l$ being a positive integer greater than unity. The components of the connection with grades ±$l$, denoted by $E_{\pm l}$, are constant (field independent) and play a crucial role in specifying the physical properties of the theory. Following [6, 7], the models are made conformally invariant by the introduction of fields in the direction of the central term and grading operator of $\hat{G}$. They can also be obtained by Hamiltonian reduction from the so-called two–loop WZNW models [6, 5].

An initial physical motivation for studying such dynamical systems is the same as the one for finite systems [5]. Namely, one describes a nontrivial, not necessarily Riemannian target space created by the Toda type fields in the presence of some additional matter fields. The latter are related with higher flows of the corresponding flat connection in the trivial holomorphic principal fibre bundle $\mathcal{M} \times \hat{G} \rightarrow \mathcal{M}$, where $\mathcal{M}$ is a two–dimensional manifold and $\hat{G}$ is an exponential mapping of an affine Lie algebra $\hat{G}$. In the same way as for the finite systems, using a relevant specialisation of the Inönü–Wigner contraction [8], it is possible, for certain models, to eliminate the back reaction of some matter fields to the Toda type fields. As a result, when such a procedure is applied to all matter fields, the latter will simply propagate in the field of a given Toda solution.

Here, in contrast with the finite Lie algebra case, where nonabelian Toda systems lead to the exactly solvable conformal systems in the presence of a black hole, the corresponding target space possesses nontrivial vacua and soliton configurations, which can be interpreted as particles of the theory on the same footing as those associated to the fundamental fields.

The conformal symmetry is spontaneously broken like in the usual abelian and non-abelian conformal affine Toda systems [5, 6], generating masses for some particles and solitons through a Higgs like mechanism. The masses of the fundamental particles are determined by
the eigenvalues of the constant operators $E_{\pm l}$, appearing in the flat connection. The masses of the solitons have a topological character in the sense that they are determined by the asymptotic value of the field in the direction of the central term of $\hat{G}$. It turns out that this is also related to the eigenvalues of $E_{\pm l}$, and for these reasons we are able to obtain a very simple and suggestive mass formula for solitons and particles.

The massive fields associated to non zero grade generators will be found to satisfy two dimensional Dirac type equations. They are interpreted as matter fields, and at first sight they are $c$–number fields, so that they would be dubbed as bosons. However, the issue of their statistics can only been solved by considering the corresponding quantum field theory. It is well known that in two dimensions the statistics of fields depends upon the coupling constant, and perhaps that could be quantized such these matter fields become anticommuting operators. An argument in this direction will indeed be given for a special class of models.

The general solution of the system is constructed following the methods of reference [3], based on representation theory of affine Lie algebras. Some new features appear here, due to the higher grade fields, which require more delicate techniques to obtain the expression of their general solution. We also use the dressing tranformations [3, 10, 11, 12], as an alternative, to construct the solutions in the orbit of the vacuum. The soliton solutions are obtained through the so–called solitonic specialization [13, 14], see also [14] for the nonabelian case. According to that, the one–soliton solutions are determined by choosing the constant group element, parametrizing the solutions in the orbit of the vacuu m, as an exponentiation of an eigenvector of the operators $E_{\pm l}$, and the multi–solitons, by taking it as a product of such exponentials.

There is a special class of models which present some remarkable physical properties. Any integral gradation of an affine Kac-Moody algebra [15] possesses a period such that the eigensubspaces, with grades differing by a multiple of that period, have the same structure. For the principal gradation, for instance, that period is equal to the Coxeter number. In addition, subspaces of grade $n$ and $-n$ are always isomorphic. By choosing the operators $E_l$ and $E_{-l}$, such that one is the image of the other under such isomorphism, and in addition, taking $l$ to be equal to the period associated to the gradation, one obtains models possessing a special $U(1)$ Noether current depending only on the matter fields. It is then possible, under some circumstances, to choose one solution in each orbit of the conformal group, such that for these solutions, that $U(1)$ current is equal to a topological current depending only on the (gauge) zero grade fields. The submodel obtained by such special gauge fixing of the conformal symmetry, presents some very interesting properties due to this equivalence. We show for instance, in the case of a model associated to $sl(2)$, that the matter fields get confined inside the solitons. In addition, the masses of solitons and particles are shown to be proportional to their $U(1)$ charges, in a manner very similar to what occurs in four dimensional gauge theories with Higgs in the adjoint representation and in the BPS limit. We believe that this equality between topolgical and Noether currents will play an important role in the understanding of the quantum theory of the solitons. We also point out that such type of models are related to several interesting phenomena in (quasi)-one-dimensional electron-phonon physical systems.

The paper is organized as follows. In Section 2 we define the models, presenting their
equations of motion and discussing their symmetries. The general solution, in terms of highest weight representations is worked out in detail in Section 3 and illustrated with many examples. The holomorphic factorizable representation of the general solution given in Section 3, admits a remarkable specialization [13], which is used in Section 4 for the calculation of the soliton solutions. In Section 5, we perform the related dressing procedure, which gives all solutions in the orbit of the vacuum, including the solitonic ones. The dressing method gives further insight into the soliton properties and it is very useful for the applications which follow. In Section 6, an alternative formulation of our system as two–loop WZNW model, provides an improved energy–momentum tensor obtained via the Sugawara construction [16, 17]. The masses of solitons and particles are calculated in Section 7 with the help of that tensor, through the spontaneous breakdown of the conformal symmetry by a Higgs like mechanism. In Section 8 we discuss the physical properties of the higher grading fields establishing that the massive ones satisfy Dirac like equations and discussing some of their peculiarities, related to parity and complex conjugation. Section 9 is devoted to a specially interesting class of models possessing the $U(1)$ Noether current mentioned above. In Section 10, we treat in great detail two models associated to the principal gradation of $sl(2)^{(1)}$, discussing their physical applications. Section 11 is devoted to the conclusions and perspectives for future investigations.

2 Formulation of the System

Consider an untwisted affine Kac-Moody algebra $\hat{G}$ endowed with an integral gradation $\hat{G} = \bigoplus_{n \in \mathbb{Z}} \hat{G}_n$, and denote

$$\hat{G}_+ = \bigoplus_{n > 0} \hat{G}_n, \quad \hat{G}_- = \bigoplus_{n > 0} \hat{G}_{-n}. \quad (2.1)$$

Notice that by an affine Lie algebra we mean a loop algebra corresponding to a finite dimensional simple Lie algebra $G$ of rank $r$, extended by the center $C$ and the derivation $D$. According to [15], integral gradations of $\hat{G}$ are labelled by a set of co-prime integers $s = (s_0, s_1, \ldots, s_r)$, and the grading operators are given by

$$Q_s \equiv H_s + N_s D - \frac{1}{2N_s} \text{Tr} (H_s)^2 \quad C. \quad (2.2)$$

Here

$$H_s \equiv \sum_{a=1}^r s_a \lambda^v_a \cdot H^0, \quad N_s \equiv \sum_{i=0}^r s_i m^\psi_i, \quad \psi = \sum_{a=1}^r m^\psi_a \alpha_a, \quad m^\psi_0 = 1; \quad (2.3)$$

$H^0$ is an element of the Cartan subalgebra of $G$; $\alpha_a, a = 1, 2, \ldots, r$, are its simple roots; $\psi$ is its maximal root; $m^\psi_a$ the integers in expansion $\psi = \sum_{a=1}^r m^\psi_a \alpha_a$; and $\lambda^v_a$ are the fundamental co–weights satisfying the relation $\alpha_a \cdot \lambda^v_b = \delta_{ab}$.

Let $\mathcal{M}$ be a two dimensional manifold with local coordinates $x_+$ and $x_-$; $\hat{G}$ be an affine Lie algebra corresponding to a finite dimensional complex simple Lie algebra $G$ with the Lie group $G$; $\mathcal{A}$ be a flat connection in the trivial holomorphic principal fibre bundle $\mathcal{M} \times \hat{G} \rightarrow \mathcal{M}$. Specify the connection in such a way that its $(1,0)$-component takes values in the subspaces
\( \bigoplus_{n=0}^{l} \hat{\mathcal{G}}_{++n} \), and \((0,1)\)-component takes values in \( \bigoplus_{n=-l}^{l} \hat{\mathcal{G}}_{-n} \), with \( l \) being a fixed positive integer. In other words, up to a relevant gauge transformation, these components, satisfying the zero curvature condition

\[
\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0,
\]

are of the form

\[
A_+ = -B \ F^+ \ B^{-1}, \quad A_- = -\partial_- B \ B^{-1} + F^-.
\]

Here \( B \) is a mapping from \( \mathcal{M} \) to the Lie group \( \hat{\mathcal{G}}_0 \) with the Lie algebra \( \hat{\mathcal{G}}_0 \); \( F^\pm \) are mappings to \( \bigoplus_{n=1}^{l} \hat{\mathcal{G}}_{\pm n} \) of the form

\[
F^+ = E_l + \sum_{m=1}^{l-1} F^+_m, \quad F^- = E_{-l} + \sum_{m=1}^{l-1} F^-_m,
\]

with \( E_{\pm l} \) being some fixed elements of \( \hat{\mathcal{G}}_{\pm l} \); and \( F^\pm_1 \), \( 1 \leq m \leq l - 1 \), take values in \( \hat{\mathcal{G}}_{\pm m} \).

Substituting the gauge potentials (2.5) into (2.4), one gets the equations of motion

\[
\partial_+ \left( \partial_- B \ B^{-1} \right) = [E_{-l}, B E_l \ B^{-1}] + \sum_{n=1}^{l-1} [F^-_n, B \ F^+_n \ B^{-1}],
\]

\[
\partial_- F^+_m = [E_l, B^{-1} \ F^-_{l-m} \ B] + \sum_{n=1}^{l-m-1} [F^+_m, B^{-1} \ F^-_{n+m} \ B],
\]

\[
\partial_+ F^-_m = -[E_{-l}, B \ F^+_m \ B^{-1}] - \sum_{n=1}^{l-m-1} [F^-_m, B \ F^+_m \ B^{-1}].
\]

Note that a consideration of the systems generated by the flat connection with the components \( A_\pm \) taking values in the subspaces \( \bigoplus_{n=0}^{l} \hat{\mathcal{G}}_{\pm n} \) where \( l_\pm \) can be different positive integers, follows completely the same line. The systems of that type, even without appealing to their explicit formulation, are integrable, whenever the corresponding Lie algebra is finite dimensional or affine; see [2], and also [18], [3]. We restrict ourselves here only to the case with \( l_+ = l_- = l \); however, other (asymmetric) possibilities can be attractive as well, and their investigation follows the same arguments as presented here [4].

Since \( Q_s \) and \( C \) are in \( \hat{\mathcal{G}}_0 \), we parametrise \( B \) as

\[
B = b e^\eta Q_s \ e^\nu C,
\]

where \( b \) is a mapping to \( G_0 \), the subgroup of \( \hat{\mathcal{G}}_0 \) generated by all elements of \( \hat{\mathcal{G}}_0 \) other than \( Q_s \) and \( C \). The fields \( \eta \) and \( \nu \) correspond to the extension of the loop algebra, and, as we

Footnote: Finishing the present paper, we were acquainted with paper [19] where the authors have studied some special non–left–right symmetric, as they called heterotic conformal Toda system, corresponding to the case of the series \( A_r \), endowed with the principal gradation, and a choice when \( l_+ = 2, l_- = 1 \). (In fact, their consideration is pretty valid for an arbitrary finite dimensional Lie algebra, and, with a minor modification, for an affine Lie algebra.) They also used some additional restrictions, in particular that, in our notations, \( E_2 = \sum_{\alpha} [E^{0}_{\alpha}, E^{0}_{(\alpha+1)}] \), \( F^+_1 = [E_2, F^+_1] \), \( E_{-2} = 0 \), \( F^-_1 = \sum_{\alpha} E^{0}_{-\alpha} \), where \( F^-_1 \) is a mapping to \( \mathcal{G}_{-1} \).
will show below, are responsible for making the system conformally invariant [3]. Clearly, the order of the three factors in (2.10) is irrelevant, since they commute. In addition, we will use a special basis for the generators of $\mathcal{G}_0$ such that they are all orthogonal to $Q_s$ and $C$. From (2.2) one observes that the generators of $\mathcal{G}_0$ are, besides $C$ and $Q_s$, the elements $H^0_a$, $a = 1, 2, \ldots, r$, of the Cartan subalgebra, and step operators $E_{\pm \alpha}^0$ and $E_{\pm \beta}^{\pm 1}$, such that $\sum_{a=1}^r s_a \lambda^0_a \cdot \alpha = 0$, and $\sum_{a=1}^r s_a \lambda^0_a \cdot \beta = N_s$. There can be no step operators $E_n^0$, with $|n| > 1$, as explained in appendix C of ref. [4]. Therefore, shifting the Cartan elements as

$$\tilde{H}_a^0 = H_a^0 - \frac{1}{N_s} \text{Tr} \left( H_s H_a^0 \right) C = H_a^0 - \frac{2}{\alpha^2} s_a C,$$

(2.11)

one gets

$$\text{Tr} \left( C^2 \right) = \text{Tr} \left( C \tilde{H}_a^0 \right) = \text{Tr} \left( Q_s \tilde{H}_a^0 \right) = 0, \quad \text{Tr} \left( Q_s C \right) = N_s,$$

$$\text{Tr} \left( \tilde{H}_a^0 \tilde{H}_b^0 \right) = \text{Tr} \left( H_a^0 H_b^0 \right) = 4 \alpha_a \alpha_b / \alpha^2 \equiv \eta_{ab},$$

(2.12)

for all $a, b = 1, \ldots, r$. Here we have used $H_a^0 = 2 \alpha_a \cdot H^0 / \alpha^2_a$, $\text{Tr} \left( x \cdot H^0 y \cdot H^0 \right) = x \cdot y$, and $\text{Tr} \left( C D \right) = 1$. For more detail of such a special basis, see appendix C of ref. [4].

Substituting (2.10) into the equations of motion (2.7)–(2.9), one has

$$\partial_+ \left( \partial_- b^{-1} \right) + \partial_+ \partial_- \nu C = e^{\nu \left[ E_{-l}, b F_L b^{-1} \right] + \sum_{n=1}^{l-1} e^{\eta \left[ F_n^-, b F_n^+ b^{-1} \right]}},$$

(2.13)

$$\partial_- F^+_m = e^{(l-m)\eta \left[ E_L, b^{-1} F_{l-m}^- b \right] + \sum_{n=1}^{l-m-1} e^{\eta \left[ F^+_m, b^{-1} F^-_n \right]}},$$

(2.14)

$$\partial_+ F^-_m = - e^{(l-m)\eta \left[ E_L, b F^+_m b^{-1} \right] - \sum_{n=1}^{l-m-1} e^{\eta \left[ F^-_m, b F^+_n \right]}},$$

(2.15)

$$\partial_+ \partial_- \eta Q_s = 0,$$

(2.16)

where the last equation is a consequence of the fact that $D$, and hence $Q_s$, can not be obtained as the Lie bracket of any two elements of $\mathcal{G}$.

Let us discuss briefly the meaning of system (2.13)–(2.16). First, it is clear that for $l = 1$, when all the mappings $F^\pm_n$, $n > 1$, are absent in the game, these equations coincide with the standard conformally affine Toda system. Second, let us conventionally call the fields, parametrising the mapping $b$, the Toda type fields; and the fields, entering a parametrisation of the mappings $F^\pm_n$, the matter fields coupled, in accordance with equations (2.13)–(2.16), to the Toda type fields; the reason for that will be clear from the following observation. Namely, using a relevant specialisation of the Inönü–Wigner contraction [4], one can bring to zero the back reaction to the Toda type fields for some or all of the matter fields. Recall that under the Inönü–Wigner contraction of a simple Lie algebra $\mathcal{G}$, its elements are multiplied by constant parameters, some of which tend to zero in a consistent way. Roughly speaking, with this procedure one multiplies by a contraction parameter, say $\kappa$, the elements of a subspace $\mathcal{P} \subset \mathcal{G}$, complementary to some subalgebra $\mathcal{H}$ of $\mathcal{G}$; $\mathcal{P} \rightarrow \kappa \mathcal{P} = \mathcal{P}^{(\kappa)}$, $\mathcal{H} \rightarrow \mathcal{H}$. Then it is clear that there exists a limit $\kappa \rightarrow 0$, when the corresponding algebra $\mathcal{G}^{(0)}$ contains $\mathcal{P}^{(0)}$ as an ideal; in other words, $\mathcal{G}$ becomes the semi–direct sum $\mathcal{G}^{(0)}$ of $\mathcal{H}$ and $\mathcal{P}^{(0)}$, and, hence, we end
up with a non–semisimple Lie algebra. In particular, if a simple Lie algebra $G$ is endowed with a $\mathbb{Z}$-gradation, supply the grading subspaces $G_n$, corresponding to the subspace $P$, with the parameter $\kappa$, and then tend it to zero in a consistent way, having in mind the grading property $[G_m, G_n] \subset G_{m+n}$. The same scheme takes place for the affine algebras. Then, we can eliminate contributions coming from some appropriate mappings $F^\pm_m$ in (2.13)–(2.15), and, of course, have a possibility to arrive at the case when equation (2.13) does not contain the sum of the last $l-1$ terms in the r.h.s. As a result, we come to an equation which looks similar to those for the standard affine Toda system, however, in general, with a different meaning of the elements $E_{\pm l}$ which belong here to the subspaces $\hat{G}_{\pm l}$. Evidently, there are many other meaningful possibilities. Note that an analogous Inönü–Wigner procedure, albeit for the corresponding representations of the algebras, can be applied to obtain the solution of the contracted systems, starting from the general solution to system (2.13)–(2.16) obtained in the next section.

The structure of the vacuum of the system (2.13)–(2.16) is rather complicated. We will discuss some aspects of it below. However, there is a simple condition that guarantees the existence of static (vacuum) solutions. If the elements $E_{\pm l}$ satisfy the relation

$$[E_l, E_{-l}] = \beta C,$$

where

$$\beta = \frac{l}{N_s} \text{Tr} (E_l E_{-l}),$$

then

$$b = 1, \quad F^\pm_m = 0, \quad \eta = 0, \quad \nu = -\beta x_+ x_-,$$

is a (vacuum) solution of (2.13)–(2.16).

Another possibility for vacuum solutions arises when $E_{\pm l}$, $l > 1$, belong to a Heisenberg subalgebra of $\hat{G}$, see [15, 20],

$$[E_M, E_N] = \text{Tr} (E_M E_{-N}) M \delta_{M+N,0} C,$$

where $M, N$ belong to some (infinite) subset $\mathbb{Z}_E$ of the integer numbers $\mathbb{Z}$. In such cases one has that

$$b = 1, \quad \eta = 0, \quad F^\pm_M = c^\pm_M E_{\pm M}, \quad F^\pm_m = 0, \quad \text{if } m \notin \mathbb{Z}_E, \quad \nu = -\Omega x_+ x_-,$$

is a solution of (2.13)–(2.16) with $c^\pm_M$ being constants, and

$$\Omega \equiv \beta + \sum_{M=1}^{l-1} \text{Tr} (E_M E_{-M}) M c^+_M c^-_M.$$  

(2.21)

Obviously, the system (2.13)–(2.16) may have many more vacuum solutions besides (2.18) and (2.20). However, the condition (2.17) guarantees the existence of at least one vacuum solution. Such a fact, as we will see below, favors the existence of soliton solutions.

The models introduced above are completely characterised by the data $\{\hat{G}, Q_s, l, E_{\pm l}\}$; and we have a quite large class of systems with physical properties crucially depending on a choice of those data.

Equations (2.13) – (2.16) are invariant under the conformal transformation

$$x_+ \to f(x_+), \quad x_- \to g(x_-),$$

(2.22)
with \(f\) and \(g\) being analytic functions; and with the fields transforming as

\[
\begin{align*}
    b(x_+, x_-) &\rightarrow \tilde{b}(\tilde{x}_+, \tilde{x}_-) = b(x_+, x_-), \\
    e^{-\nu(x_+, x_-)} &\rightarrow e^{-\tilde{\nu}(\tilde{x}_+, \tilde{x}_-)} = (f')^\delta (g')^\delta e^{-\nu(x_+, x_-)}, \\
    e^{-\eta(x_+, x_-)} &\rightarrow e^{-\tilde{\eta}(\tilde{x}_+, \tilde{x}_-)} = (f')^{1/l} (g')^{1/l} e^{-\eta(x_+, x_-)}, \\
    F_m^+(x_+, x_-) &\rightarrow \tilde{F}_m^+(\tilde{x}_+, \tilde{x}_-) = (f')^{-1+m/l} F_m^+(x_+, x_-), \\
    F_m^-(x_+, x_-) &\rightarrow \tilde{F}_m^-(\tilde{x}_+, \tilde{x}_-) = (g')^{-1+m/l} F_m^-(x_+, x_-),
\end{align*}
\]

(2.23–2.27)

where the conformal weight \(\delta\), associated to \(e^{-\nu}\), is arbitrary.

Notice that the Lorentz transformation \(x_\pm \rightarrow \lambda^\pm x_\pm\) is obtained from (2.22) by taking \(f(x_+) = x_+/\lambda\) and \(g(x_-) = \lambda x_-\). Therefore, from (2.23)–(2.27) we get the fields transforming as

\[
B \rightarrow \tilde{B}, \quad F_m^+ \rightarrow \lambda^{1-m/l} F_m^+, \quad F_m^- \rightarrow \lambda^{-1+m/l} F_m^-.
\]

(2.28)

Equations (2.13)–(2.16) are also invariant under the transformations

\[
\begin{align*}
    b(x_+, x_-) &\rightarrow h_L(x_-) b(x_+, x_-) h_R(x_+), \\
    F_m^+(x_+, x_-) &\rightarrow h_R^{-1}(x_+) F_m^+(x_+, x_-) h_R(x_+), \\
    F_m^-(x_+, x_-) &\rightarrow h_L(x_-) F_m^-(x_+, x_-) h_L^{-1}(x_-),
\end{align*}
\]

(2.29–2.31)

where \(h_L(x_-)\) and \(h_R(x_+)\) are elements of subgroups \(\mathcal{H}^L_0\) and \(\mathcal{H}^R_0\) of \(G_0\), respectively, satisfying the conditions

\[
h_R(x_+) E_l h_R^{-1}(x_+) = E_l, \quad h_L^{-1}(x_-) E_{-l} h_L(x_-) = E_{-l}.
\]

(2.32)

The left and right gauge transformations commute, and so the gauge group is \(\mathcal{H}^L_0 \otimes \mathcal{H}^R_0\). Whenever \(\mathcal{H}^L_0\) and \(\mathcal{H}^R_0\) have a set of common generators, we get an important subgroup of the gauge group, namely \(\mathcal{H} \equiv \mathcal{H}^L_0 \cap \mathcal{H}^R_0\). These are global gauge transformations, where the fields are transformed under conjugation (\(h_L = h_R^{-1} \equiv \mathcal{H} = \text{const.}\)),

\[
b \rightarrow h_D b h_D^{-1}, \quad F_m^\pm \rightarrow h_D F_m^\pm h_D^{-1},
\]

(2.33)

and \(E_{\pm t} = h_D E_{\pm t} h_D^{-1}\). We discuss the relevance of these transformations below.

### 3 General Solution

The flat connection (2.23) and the equations of motion themselves are written in a completely analogous way as those for the corresponding finite dimensional systems [1]; and, as we have already mentioned in the Introduction, they represent a specialisation of a more general scheme discussed in [3] on the basis of some original papers referred therein. To obtain the general solution, we have to look at the problem of a similarity between the structure of the highest weight representation spaces of \(\mathcal{G}\) and \(\mathcal{G}\), annihilated by the action of the subspaces \(\mathcal{G}_{+m}\) and \(\mathcal{G}_{+m}\), respectively. To clarify this point, let us recall, following [4], some results...
concerning the corresponding finite dimensional case, however, on some more general level, applicable for the affine systems.

First, give some definitions and notations. The flat connection $A$ in question, is represented in the gradient form,

$$A_{\pm} = g^{-1} \partial_{\pm} g,$$

with $g : \mathcal{M} \mapsto G$; and we write $A_{\pm}$ with the help of the modified Gauss decomposition for $g$,

$$g = \mu_{-} \nu_{+} \gamma_{0-} \quad \text{and} \quad g = \mu_{+} \nu_{-} \gamma_{0+},$$

respectively, in accordance with the Lie algebra decomposition $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+$; $\gamma_{0\pm} : \mathcal{M} \mapsto G_0$, $\mu_{\pm}, \nu_{\pm} : \mathcal{M} \mapsto G_{\pm}$. The grading conditions provide the holomorphic property of $\mu_{\pm}$, namely that they satisfy the initial value problem

$$\partial_{\pm} \mu_{\pm}(z_{\pm}) = \mu_{\pm}(z_{\pm}) \mathcal{E}_{\pm}(z_{\pm}),$$

where

$$\mathcal{E}_{\pm}(z_{\pm}) = \sum_{m=1}^{M} \mathcal{E}_m^{\pm}(\Phi^\pm); \quad \mathcal{E}_m^{\pm}(\Phi^\pm) = \sum_{\alpha \in \Delta_+^m} \Phi_{\alpha m}^{\pm}(z_{\pm}) X_{\alpha},$$

with arbitrary functions $\Phi_{\alpha m}^{\pm}(z_{\pm})$ determining the general solution to the system under consideration; $\Delta_+^m$ is the set of the positive roots of $\mathcal{G} = \sum_{m \in \mathbb{Z}} \mathcal{G}_m$ corresponding to the root vectors $X_\alpha$ in the subspace $\mathcal{G}_m$. Denote by $|h >^{(m)}$ and the dual, $<^{(m)} < h'|$, to $|h >^{(m)}$, the basis vectors in a representation space, annihilated by all the subspaces $\mathcal{G}_{\pm n}$ and $\mathcal{G}_{-n}$, $n \geq m$, respectively. Then the general solution for the Toda type fields contained in $B$, is expressed in terms of

$$(1) < h'|(\gamma_{0-}^+)^{-1} \mu_{+} \gamma_{0-} |h >^{(1)} = (1) < h' |B^{-1} |h >^{(1)};$$

cf. [3], while the others, say matter fields from $V^{\pm}_m$, are determined via the matrix elements

$$<^{(m)} < h'|\hat{g}_0^{-1} \mu_{-} |h >^{(m)}, \quad \text{and} \quad <^{(m)} <^{(m)} < h'|\mu_{+} \hat{g}_0^{-1} |h >^{(1)}.$$

Here $\hat{g}_0 \equiv \gamma_{0+}, \gamma_{0}^{-}, \gamma_{\pm}^{x} \equiv \gamma_{\pm}^{x}(x_{\pm}) : \mathcal{M} \mapsto G_0$ are arbitrary mappings; and $B^{-1} = (\gamma_{0+}^+(x_{+}))^{-1} \hat{g}_0^{-1} \gamma_{0-}^-(x_{-})$. The whole set of arbitrary functions which determine the general solution to the problem, consists of those parametrising the mappings $\gamma_{0}^{-}$, and the functions $\Phi_{\alpha}^{\pm m}, \alpha \in \Delta_+^m, 1 \leq m \leq M - 1; \text{while} \Phi_{\alpha}^{\pm M}$ are expressed in terms of corresponding parameters of $\gamma_{\pm}^{x}$. The mappings $\mathcal{F}_m^{\pm}$ are determined via the matrix elements (3.4) in a similar way to the construction discussed very briefly in [4]. Here we give this construction with more details.

Namely, comparing decompositions (2.3) with those obtained by substitution of (3.2) in (3.1) with account of the holomorphic property of $\mu_{\pm}$, one sees that

$$F^{\pm} = \mp (\gamma_{0}^{-})^{-1} B^{\mp 1} (\nu_{-}^{-1} \partial_{\pm} \nu_{-}) B^{\mp 1} \gamma_{0}^{+}.$$  

Consider, for example, the first matrix element in (3.4), which can be identically rewritten as

$$<^{(1)} < h'|\hat{g}_0^{-1} \mu_{+}^- |h >^{(m)} = (1) < h' |(\hat{g}_0^{-1} \nu_{-} \hat{g}_0) \nu_{+}^{-1} |h >^{(m)} = (1) < h' |\nu_{+}^{-1} |h >^{(m)}.$$
Then, differentiating (3.8) over \(x_+\) with the help of (3.7), we have
\[
\partial_+^{(1)} \left< h' \hat{g}_0^{-1} \mu_+^{-1} \mu_- | h >^{(m)} \right> = \left< h' (\hat{B} F^+ \hat{B}^{-1}) \nu_+^{-1} | h >^{(m)} \right>,
\] (3.9)
where \(\hat{B} \equiv B \gamma_0^+\). Now, with \(m = 2\), relation (3.9) takes the form
\[
\partial_+^{(1)} \left< h' \hat{g}_0^{-1} \mu_+^{-1} \mu_- | h >^{(2)} \right> = \left< h' (\hat{B} F^+ \hat{B}^{-1}) \right. | h >^{(2)},
\] (3.10)
which determines the mapping \(F_1^+\). To make the next step and calculate \(F_2^+\), recall that the mapping \(\nu_+^{-1}\) entering equation (3.7), is expressed in terms of \(\mathcal{L} \equiv \hat{B} F^+ \hat{B}^{-1}\) as the Volterra type decomposition
\[
\nu_+^{-1} = \sum_{n=0}^{\infty} \int_{\Omega^n(x_+)} dy \mathcal{L}(y^1) \mathcal{L}(y^2) \cdots \mathcal{L}(y^n),
\]
(3.11)
where \(\Omega^n(x_+)\) is the simplex in \(\mathbb{R}^n\) given by \(\Omega^n(x_+) = \{ y \in \mathbb{R}^n : a \leq y^1 \leq y^{n-1} \leq \cdots \leq y^n \leq y_+ \}; a\) is some constant determining the initial value problem; \(\nu_+(a)\) is taken to be unity, with an appropriate choice of the initial value problem for \(\mu_\pm\). (Note that here, of course, the mappings \(\mathcal{L}(y)\) depend also on the variable \(x_-\).) Then, with account of formula (3.11), it follows from (3.9) that for \(m = 3\) there is a relation
\[
\partial_+^{(1)} \left< h' \hat{g}_0^{-1} \mu_+^{-1} \mu_- | h >^{(3)} \right> = \left< h' (\hat{B} F_1^+ \hat{B}^{-1}) \int x_+ dy \mathcal{L}(y^1) \mathcal{L}(y^2) \cdots \mathcal{L}(y^n) \right. | h >^{(3)} + \left(1 < h' (\hat{B} F_2^+ \hat{B}^{-1}) | h >^{(3)},
\]
(3.12)
and hence, knowing already \(F_1^+\), we obtain an expression for \(F_2^+\). Continuing the procedure, we determine all the mappings \(F_m^+\) by the recurrent formula
\[
\partial_+^{(1)} \left< h' \hat{g}_0^{-1} \mu_+^{-1} \mu_- | h >^{(m+1)} \right> = \partial_+^{(1)} \left< h' \hat{g}_0^{-1} \mu_+^{-1} \mu_- | h >^{(m+1)} \right>,
\]
(3.13)
where \(\mathcal{L}_{k_i} \equiv \hat{B} F_{k_i} \hat{B}^{-1}\) with integers \(k_i \geq 1\) entering the sum in (3.13) only for \(\sum_{i=1}^{s} k_i = m\) and only once if \(k_i = k_{i-1}\). An analogous scheme allows to calculate the mappings \(F_m^-\). Note in this context that, in comparison with the method for constructing the Toda type fields containing in \(B\), which is, of course, a direct repetition of those in [3], the proposed construction of the matter fields \(F_m^\pm\) represents a novel feature of the integration scheme.

The matrix elements (3.3) and (3.4) realise, in general, a matrix version of the tau–function for the standard Toda system. As it was already noted in [21], from the point of view of the Lie algebra representation theory, they are closely related to the Shapovalov form defined on a Lie algebra \(G\). Such a relationship seems to be quite general and natural; it takes place for a wide class of nonlinear integrable systems, including, in particular, abelian and nonabelian (conformal and affine) Toda systems associated with the simple Lie algebras \(G\). First recall some known definitions, see e.g. [22], concerning the Shapovalov form.

Let \(\varphi\) be a Cartan subalgebra of \(G\), \(\varphi^*\) be an algebra dual to \(\varphi\); \(U(G)\) be the universal enveloping algebra for \(G\). The Shapovalov form defines the linear mapping \(U(G) \otimes_C U(G) \rightarrow U(\varphi)\) and is realised as a bilinear form \(\langle x^\vee y \rangle_0\) for any two elements \(x, y \in U(G)\). Here
\( x \rightarrow x^\vee \) is the Chevalley involution for \( x^\vee = x' \), and the hermitean Chevalley involution for \( x^\vee = x^* \); the subscript “0” means the projection of \( U(\mathcal{G}) \) on \( U(\varphi) \) which is parallel to \( \mathcal{G}_0 U(\mathcal{G}) + U(\mathcal{G}) \mathcal{G}_+ \). Note that the given definition is naturally extended for the case of the algebra \( U'(\mathcal{G}) = U(\mathcal{G}) \otimes_{U(\varphi)} R(\varphi^*) \), where \( R(\varphi^*) \) is the algebra of rational functions over \( \varphi^* \). It is very important that the form \((x^\vee y)_0\) is degenerated on the left ideal \( U'(\mathcal{G}) \mathcal{G}_+ \), and is not degenerated on the subalgebra \( U(\mathcal{G}_-) \); and hence is not degenerated on the space \( U'(\mathcal{G})/U'(\mathcal{G}) \mathcal{G}_+ \) which is a rational span of the corresponding Verma module.

One can get convinced that the matrix elements \((3.3)\) and \((3.6)\) are nothing but the Shapovalov type forms \((x^\vee y)_0\) for some special two elements \( x, y \in U_h(\mathcal{G}_-) \) of the Lie algebra \( \mathcal{G} \), with the coefficients being holomorphic functions. And, moreover, since this consideration is valid for the affine case as well, the general solution to equations \((2.7) - (2.9)\) which are the conformally affine analogues of the corresponding conformal system \([4]\), also can be written in terms of these forms. The reason is that, in accordance with the general construction, see \([3]\) and references therein, the grading conditions, realised in the form of decomposition \((2.3)\) in arbitrary functions \( \Phi^\pm \), provide the holomorphic factorisability of the solution; a differential geometry formulation of this fact is given in \([23]\).

An explicit formulation of solutions like \((3.3)\) and \((3.6)\) as the series (infinite for an affine case, while absolutely convergent in accordance with the arguments given in \([3]\) and references therein) in arbitrary functions \( \Phi^\pm_\alpha \) can be done, when it is needed, by the following purely algebraic scheme, standard for the Lie algebra representations theory. Construct a basis in \( U(\mathcal{G}) \) with the help of the monomials \( X_m = X_{m_1} \cdots X_{m_1} \) in the basis elements \( X_m = X_m(\Phi_m) \) of \( \mathcal{G} \). In particular, let \( \hat{X}^\pm = X^\pm \), be such a basis in \( U(\mathcal{G}_\pm) \) with the weight \( \mu_m \). Then the elements \( \hat{X}^+ \hat{X}^- \) generate a basis of \( U'(\mathcal{G}) \) over \( R(\varphi^*) \), and this procedure gives for any weight \( \mu \in \varphi^* \) a vector space \( F_\mu(\mathcal{G}) \) of all formal series \( \sum_{m,n} c_{m,n} \hat{X}^+_m \hat{X}^-_n \) with \( c_{m,n} \in R(\varphi^*) \), where the sum runs over all monomials of the weight \( \mu = \mu_m - \mu_n \). The subspaces \( F_\mu(\mathcal{G}) \) are in turn the subspaces of the algebra \( F(\mathcal{G}) \) graded by the weights \( \mu \). Finally, one needs to transform the elements \( \hat{X}^+_m \hat{X}^-_n \) entering \((3.3)\) and \((3.6)\) to the series of the monomials \( \hat{X}^- \hat{X}_0 \hat{X}^+ \), with \( \hat{X}_0 \in U(\varphi) \), by using the commutation relations of the algebra \( \mathcal{G} \). Note that for the abelian systems corresponding to the principal gradation of \( \mathcal{G} \), the most suitable choice is the Verma type basis when the monomials \( \hat{X}^+_m \) are constructed in terms of the Chevalley generators satisfying the defining relations. For other gradations, an adequate basis has not been discovered yet, and this is why the known explicit expressions for the general solutions of the nonabelian Toda type systems \([3]\) are written in a rather complicated form, though their holomorphic factorisability in terms of the form like \((3.3)\), related in this case with the generalised Verma modules \([24]\), is quite clear.

Let us now show how the structure of the matrix elements \((3.3)\) and \((3.6)\) is made concrete for the case of the principal gradation. Here it is suitable to take as the basis vectors \(|h >| (1)\) the highest weight vectors \(|i >, 1 \leq i \leq r, \) of the fundamental representations of \( \hat{\mathcal{G}} \). It seems more illustrative to discuss the formulation in question for the finite Toda systems; for the corresponding affine deformations, due to the similarity between the structure of the highest weight representation spaces of \( \mathcal{G} \) and \( \hat{\mathcal{G}} \), it is practically the same.

Since for the principal gradation of \( \mathcal{G} \), the subalgebra \( \mathcal{G}_0 \) is abelian, parametrise the
mapping $b(x_+, x_-)$ and $\gamma_0^\pm(x_\pm)$ as

$$b = e^{(\phi(x_+, x_-), H^0)}, \quad \gamma_0^\pm(x_\pm) = e^{(\phi^\pm(x_\pm), H^0)},$$

where $(\phi, H^0) \equiv \sum_{i=1}^r \phi_i H_i^0$.

To write down expressions (3.5) and (3.6) in an explicit way, we use the Verma type basis for the spaces of the fundamental representations of $G$. There, the highest vector $|i\rangle$ of the $i$-th fundamental representation satisfies the relations

$$E_{-j}^0|j_m\ldots j_1; i\rangle = 0; \quad H_j^0|j_m\ldots j_1; i\rangle = \delta_{ij}|j_m\ldots j_1; i\rangle; \quad E_{-j}^0|j_m\ldots j_1; i\rangle = 0, \quad i \neq j; \quad (3.14)$$

and the basis we are looking for, are constructed only with the help of the Chevalley generators $E_{-j}^0$; namely we choose as the basis vectors those of the set

$$|j_m\ldots j_1; i\rangle \equiv E_{-j}^0|j_m\ldots j_1; i\rangle, \quad 1 \leq j \leq r, \quad 0 \leq m \leq N_i - 1; \quad (3.15)$$

with nonzero norm; $N_i$ is the dimension of the representation. There take place the following useful formulas:

$$E_{-j}^0|j_m\ldots j_1; i\rangle = |jj_m\ldots j_1; i\rangle,$$

$$H_j^0|j_m\ldots j_1; i\rangle = (\delta_{ij} - \sum_{q=1}^m k_{jq})|j_m\ldots j_1; i\rangle,$$

$$E_{+j}^0|j_m\ldots j_1; i\rangle = \sum_{q=1}^m \delta_{jq}(\delta_{ij} - \sum_{n=1}^{q-1} k_{jn})|j_m\ldots j_q\ldots j_1; i\rangle,$$  

(3.16)

which are evident in virtue of the defining relations for the Cartan and Chevalley elements, namely,

$$[H_i^0, H_j^0] = 0, \quad [H_i^0, E_{+j}^0] = \pm k_{ji} E_{+j}^0, \quad [E_{+j}^0, E_{-j}^0] = \delta_{ij} H_i^0,$$  

(3.17)

with $k$ being the Cartan matrix of $G$. Here $\hat{q}$ means the absence of the root vector $E_{-j}^0$ in the corresponding formula for the basis vector $|jj_m\ldots j_1; i\rangle$. If the norm of the vector $|jj_m\ldots j_1; i\rangle$ or $|j_m\ldots j_q\ldots j_1; i\rangle$ is equal to zero, the corresponding term in the r.h.s. of formulas (3.16) is naturally absent.

For example, let us illustrate the structure of the fundamental representation space by an example of the simple Lie algebras of rank 2, i.e., the algebras $A_2 \equiv sl(3, C)$, $B_2 \equiv o(5, C)$, $C_2 \equiv sp(4, C)$, and $G_2$. Namely, the basis vectors of the 1-st and 2-nd fundamental representations of these algebras respectively are

$A_2$:

$$|1\rangle, \quad |1; 1\rangle, \quad |21; 1\rangle;$$

$$|2\rangle, \quad |2; 2\rangle, \quad |12; 2\rangle;$$

$B_2$:

$$|1\rangle, \quad |1; 1\rangle, \quad |21; 1\rangle, \quad |221; 1\rangle, \quad |1221; 1\rangle;$$

$$|2\rangle, \quad |2; 2\rangle, \quad |12; 2\rangle, \quad |212; 2\rangle;$$

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the standard Toda systems, see [3].

and the properties (3.14)–(3.15) of the basis vectors. Technically, it is the same task as for can be calculated in the explicit form of finite polynomials, using the defining relations (3.17)

Here, with the decomposition

Finally, the recurrent formula (3.13) is reduced to the expression

In the case under consideration here, one can take \( |j_{m-1} \ldots j_1; i> \) as the basis vectors \( |h>^{(m)} \). Then, using the relation \( H_j^0|i> = \delta_{ij}|i> \), expression (3.15) is written in the form

while the first formula in (3.16) reads as

Finally, the recurrent formula (3.13) is reduced to the expression

Here, with the decomposition \( F_m^+ = \sum_{\alpha \in \Delta_m^+} f_\alpha^+ E_{+\alpha}^0 \), the mappings \( L_{k_i} \) read as

where the product in the exponential is defined by the commutation relation \( [(\phi, H^0), E_{+\alpha}^0] = (\alpha, \phi) E_{+\alpha}^0 \). All the matrix elements of type

can be calculated in the explicit form of finite polynomials, using the defining relations (3.17) and the properties (3.14)–(3.13) of the basis vectors. Technically, it is the same task as for the standard Toda systems, see [3].

In particular, for \( m = 1 \), since \( F_1^+ = \sum_j f_j^+ E_j^0 \), one has

|1>, |1; 1>, |21; 1>, |121; 1>
|2>, |2; 2>, |12; 2>, |112; 2>, |2112; 2>

G_2:

|1>, |1; 1>, |21; 1>, |121; 1>, |1121; 1>, |21121; 1>, |121121; 1>
|2>, |2; 2>, |12; 2>, |112; 2>, |2112; 2>, |21112; 2>, |221112; 2>, |2121112; 2>, |1221112; 2>, |12121112; 2>, |212121112; 2>.
For \( m = 2 \) one can represent the mapping \( F_2^+ \) as

\[
F_2^+ = \sum_{j_1j_2} A_{j_1j_2} E_{j_1}^0 E_{j_2}^0,
\]

where the matrix \( A_{j_1j_2} \) is obtained from \( 2\delta_{j_1j_2} - k_{j_1j_2} \) by replacing all nonzero entries by \( f_{j_1j_2}^+(x_+, x_-) \equiv -f_{j_2j_1}^+(x_+, x_-) \), so that

\[
<i|F_2^+|j_2j_1; i > = \delta_{ij_1} (2\delta_{j_1j_2} - k_{ij_2}) A_{ij_2}.
\]

Remembering that \( \mathcal{L}_1 = \sum_j \epsilon^{(k, \phi - \phi^*)} f_j^+ E_j^0 \), formula (3.19) for \( m = 2 \) is read as

\[
-k_{ij} f_{ij}^+ = \epsilon^{(k, \phi - \phi^*)} \{ \partial_+ \frac{<i|\mu_+^- \mu_-|j; i >}{<i|\mu_+^+ \mu_-|i >} + k_{ij} \epsilon^{(k, \phi - \phi^*)} \int x^+ dy \epsilon^{(k, \phi - \phi^*)} (y) f_{ij}^+(y) \} = \epsilon^{(k, \phi - \phi^*)} \{ \partial_+ \frac{<i|\mu_+^- \mu_-|j; i >}{<i|\mu_+^+ \mu_-|i >} + k_{ij} \frac{<j|\mu_+^- \mu_-|j; j >}{<j|\mu_+^+ \mu_-|j >} \partial_+ \frac{<i|\mu_+^- \mu_-|i >}{<i|\mu_+^+ \mu_-|i >} \}, \quad i \neq j.
\]

Note that the matrix element \( <i|\mu_+^- \mu_-|j; i > \) also vanishes when \( k_{ij} = 0 \).

To obtain explicit solution for the coefficient functions \( f_\alpha^+ \) entering the mappings \( F_m^+ \) for higher values of \( m \), rewrite the above given decomposition as

\[
F_m^+ = \sum_{i_1 \cdots i_m} f_{i_1 \cdots i_m}^+ E_{i_1}^0 \cdots E_{i_m}^0 \sum_{s=1}^m \alpha_s \in \Delta_m^+,
\]

with \( \alpha_i \) being the simple roots of \( \mathcal{G} \); thereof

\[
\mathcal{L}_{kj} (y_j) = \sum_{i_1^{(j)} \cdots k_j^{(j)}} \epsilon^{\sum_{s=1}^{k_j} (k, \phi - \phi^*)} (y_j) f_{i_1^{(j)} \cdots k_j^{(j)}}^+ (y_j) E_{i_1}^0 \cdots E_{i_m}^0 \sum_{s=1}^{k_j} \alpha_{i_s} \in \Delta_k^+,
\]

where \( \sum_{s=1}^{k_j} \alpha_{i_s} \in \Delta_k^+ \). Introduce now the notation

\[
\mathcal{D}_{i_1 \cdots i_m; j_1 \cdots j_m}^{(i)} \equiv <i|E_{i_1}^0 \cdots E_{i_m}^0 \sum_{s=1}^{k_j} \alpha_{i_s} |j_m \cdots j_1; i >; \quad (3.24)
\]

with which

\[
<i|F_m^+|j_m \cdots j_1; i > = \sum_{\alpha \in \Delta_m^+} f_\alpha^+ <i|E_{+\alpha}^0 |j_m \cdots j_1; i > \equiv \sum_{\alpha \in \Delta_m^+} \mathcal{D}_{\alpha; j_1 \cdots j_m}^{(i)} f_\alpha^+ \delta_{\alpha} \sum_{s=1}^{k_j} \alpha_{i_s} \equiv \mathcal{D}_{i_m; j_1 \cdots j_m}^{(i)} f_{j_1 \cdots j_m}^+;
\]

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and

\[
<i | \mathcal{L}_{k_1}(y^1) \mathcal{L}_{k_2}(y^2) \cdots \mathcal{L}_{k_s}(y^s) | j_m \ldots j_1; i > = \sum_{I_{k_1} \ldots I_{k_s}} \prod_{j=1}^{s} \sum_{l=1}^{k_j} e^{(k_1,\phi^-)(l)(y^j)} f^+_{i_1 \ldots i_{k_j}}(y^j) \mathcal{D}^{(i]}_{I_{k_1} \ldots I_{k_s}; j_m}.
\]

Here the sum over \( I_{k_i} \) means the sum over all indices \( i_1^{(j)}, \ldots, i_{k_j}^{(j)} \). Hence, the recurrent formula (3.18) takes the form

\[
\mathcal{D}^{(i]}_{l_m; j_m} f^+_{j_1 \ldots j_m} = e^{\sum_{s=2}^{m} (k,\phi^-)(s)} \partial_x \left\{ i | \mu_+^{-1} \mu_- | j_m \ldots j_1; i > - i | \mu_+^{-1} \mu_- | i > \right\}
\]

\[
\sum_{s=2}^{m} \sum_{k_1 \ldots k_s} \int_{\Omega^s(x_+)} dy \sum_{I_{k_1} \ldots I_{k_s}} \prod_{j=1}^{s} \sum_{l=1}^{k_j} e^{(k_1,\phi^-)(l)(y^j)} f^+_{i_1 \ldots i_{k_j}}(y^j) \mathcal{D}^{(i]}_{I_{k_1} \ldots I_{k_s}; j_m}
\]

Here the matrix elements (3.24) can be calculated explicitly. For this goal one needs to rewrite the root vectors \( E_0^\pm \) as monomials over the Chevalley generators, and then use the corresponding formula from [3] for the matrix elements of type \( i | E^0_{+1} \cdots E^0_{+1} E^0_{-1} \cdots E^0_{-1} | i > \) written in terms of the Cartan matrix.

### 4 Solitons from the general solution

To obtain soliton solutions to the system (2.7)–(2.9) from the general solutions described in the previous section, we use the method suggested in [13] for the abelian affine (periodic) Toda system, and then discussed in [14], [25] and [5] for their nonabelian generalisations. In the cases when \( E_{\pm l}, l > 1 \), live in a Heisenberg subalgebra of \( \mathcal{G} \), see (2.19), consider the initial value problem (3.3) with \( \tilde{E}_\pm \) in (3.4) taken to be \( \mathcal{E}_\pm \), which are

\[
\mathcal{E}_\pm \equiv E_{\pm l} + \sum_{N=1}^{l-1} c^\pm_N E_{\pm N}, \quad \text{and so} \quad [\mathcal{E}_+, \mathcal{E}_-] = \Omega C;
\]

where \( c^\pm_N \) and \( \Omega \) were introduced in (2.20) and (2.21), respectively. In other words, choose all arbitrary functions \( \Phi^\pm_{\alpha} \) there to be zero except those standing in the direction of the Heisenberg subalgebra generators \( E_{\pm M} \) and \( E_{\pm l} \). If \( E_{\pm l} \) do not lie in a Heisenberg subalgebra, take \( \tilde{E}_\pm \) in (3.4) to be just \( \mathcal{E}_{\pm l} \) (\( c^\pm_0 = 0 \)). Then the mappings \( \mu_\pm \) are

\[
\mu_\pm = \mu^0_\pm = \mu^0_\pm e^{x_+ \mathcal{E}_\pm},
\]

where \( \mu^0_\pm \) are some fixed mappings independent on the local coordinates \( x_\pm \). It is quite clear that then one ends up with the following expressions for the matrix elements (3.5), (3.6) determining, with such a choice of the mappings, a particular solution to system (2.7)–(2.9):

\[
\tau_{mn}(\mu^0) = (m) < h | e^{-x_+ \mathcal{E}_+} \mu^0 e^{x_+ \mathcal{E}_-} | h > (n),
\]
with $\mu^0 \equiv (\mu^0_+)^{-1}\mu_0^0$ being a fixed mapping $\mathcal{M} \mapsto \hat{G}$. Next step in the calculation is to remove the dependence on the elements $\mathcal{E}_\pm$ in (4.3), which can be easily done by replacing the exponential with $\mathcal{E}_+ (\mathcal{E}_-)$ to the extreme right (left) position where it gives unity under the action on the corresponding basis state which is annihilated by $\mathcal{E}_+ (\mathcal{E}_-)$. Note that this procedure is, of course, as it was mentioned in the previous section, common for all methods for obtaining an explicit series for the matrix elements under consideration, and is traditional in the representation theory, see e.g. [22]. Now we make use of the second ingredient of the scheme which consists in an assumption that the mapping $\mu^0$ is an eigenvector with respect to $\mathcal{E}_+$ and $\mathcal{E}_-$. Finally, following [13], we write $\mu^0$ as a product of the exponentials, $\mu^0 = \prod_{i=1}^{N} \exp \mathcal{V}_i$, where $\mathcal{V}_i$ are eigenvectors under the adjoint action of the elements $\mathcal{E}_\pm$, namely $[\mathcal{E}_\pm, \mathcal{V}_i] = \omega^{(i)} \mathcal{V}_i$. Note that different orders of the exponentials in the product, in general, can give different soliton solutions. These two assumptions concerning a choice of the fixed mapping $\mu^0$, clearly are equivalent to imposing some initial conditions on (3.3), which are relevant for obtaining $N$-soliton solutions of our system. The solutions are characterised by soliton parameters $\omega^{(i)} \equiv \omega^{(i)}_+ - \omega^{(i)}_-$ and $v^{(i)} \equiv -\frac{\omega^{(i)}_+ + \omega^{(i)}_+}{\omega^{(i)}_+ - \omega^{(i)}_-}$, and, if one interpretes $v^{(i)}$ as velocity of the $i$-th soliton, $\omega^{(i)}_+ \omega^{(i)}_-$ must be negative, otherwise $|v^{(i)}| > 1$; moreover, velocities $v^{(i)}$ can be taken to be equal one to each other. To be more precise,

$$\tau_{mn}(\mu^0) = \langle m | h^{N} \prod_{i=1}^{N} e^{\mathcal{V}_i \exp[\omega^{(i)}(x-v^{(i)}t)]} | h \rangle \langle h | \rangle^{(n)}, \quad (4.4)$$

where we are back to the spatial and time variables, $x_\pm = t \pm x$. The final step in the construction of the soliton solutions is to use an appropriate basis vectors $| h \rangle \langle h |$ in the representation space, annihilated by all the subspaces $\mathcal{G}_{+s}$ and $\mathcal{G}_{-t}$, $s \geq n$, $t \geq m$, respectively.

### 5 Dressing transformations

Now we present an alternative way of constructing solutions given by the solitonic specialisation of the general solution discussed above. They are obtained by the dressing transformations which are non local gauge transformations acting on the gauge potentials $A_\mu$ and leaving their grading structure invariant [4, 10, 11, 12]. The dressing procedure requires existence of two gauge tranformations mapping a given $A_\mu$ to the potential $A^\rho_\mu$ with the same grading spectrum. Namely, there must exist $\Theta_+$ and $\Theta_-$ such that

$$A_\mu \rightarrow A^\rho_\mu \equiv \Theta_+ A_\mu \Theta_- - \partial_\mu \Theta_+ \Theta_-^{-1}, \quad (5.1)$$

with $A_\mu$ and $A^\rho_\mu$ having the same grading structure as (2.3). Here $\Theta_\pm$ are mappings from $\mathcal{M}$ to $\hat{G}$. Since $A_\mu$ satisfy the zero curvature condition, they are of the form

$$A_\mu = -\partial_\mu TT^{-1}. \quad (5.2)$$

\(^5\)By convention, here and in what follows derivatives act only on the first term of products, unless parentheses indicate otherwise.
Equivalence of these two transformations implies that

\[ T^\rho \equiv \Theta_+ T = \Theta_- T \rho, \; \text{or, equivalently,} \; \Theta_-^{-1} \Theta_+ = T \rho T^{-1}, \quad (5.3) \]

with \( \rho \) being a constant mapping from \( \mathcal{M} \) to \( \hat{G} \), and \( A_\mu^\rho = -\partial_\mu T^\rho (T^\rho)^{-1} \).

The dressing transformations are defined, in fact, in terms of a modified Gauss decomposition of the element \( T \rho T^{-1} \),

\[ T \rho T^{-1} = \left( T \rho T^{-1} \right)_{<0} \left( T \rho T^{-1} \right)_{0} \left( T \rho T^{-1} \right)_{>0} \quad (5.4) \]

with \( (T \rho T^{-1})_{0}, \; (T \rho T^{-1})_{>0} \) and \( (T \rho T^{-1})_{<0} \) belonging to the subgroups of \( \hat{G} \), whose Lie algebras, defined in (2.1), are \( \hat{G}_0, \; \hat{G}_+ \; \text{and} \; \hat{G}_- \), respectively. Then, we define the element \( T^\rho \) as

\[ T^\rho \equiv \Theta^{(0)}_+ \left( T \rho T^{-1} \right)_{>0} T = \Theta^{(0)}_- \left( T \rho T^{-1} \right)_{<0} T \rho, \quad (5.5) \]

where \( \Theta^{(0)}_+ \) and \( \Theta^{(0)}_- \) are introduced in such a way that

\[ \Theta^{(0)}_-^{-1} \Theta^{(0)}_+ = \left( T \rho T^{-1} \right)_0. \quad (5.6) \]

Comparing with (5.3), we write the elements \( \Theta \pm \) in (5.1) as

\[ \Theta_+ = \Theta^{(0)}_+ \Theta^{\gamma}_+, \quad \Theta_- = \Theta^{(0)}_- \Theta^{\gamma}_-; \quad (5.7) \]

with

\[ \Theta^{\gamma}_+ = \left( T \rho T^{-1} \right)_{>0}, \quad \Theta^{\gamma}_- = \left( T \rho T^{-1} \right)_{<0}^{-1}. \quad (5.8) \]

Therefore, the dressing transformations provide a mapping between solutions. Hence, the space of solutions to the system can be splitted into orbits of such transformations. The knowledge of a given simple solution is then enough to generate a whole orbit of solutions. In the case of the models under consideration, interesting solutions, namely solitons, fortunately are on the orbit of a vacuum solution. That is the fact we will explore; it is related to the solitonic specialisation discussed above.

Perform now the dressing transformation by taking as an initial configuration a vacuum solution of (2.13)–(2.16). As we have said, the model under consideration may have several type of vacuum solutions. However, here we will deal with the solutions of type (2.18) or (2.20).

For the vacuum solutions (2.20), the gauge potentials (2.5) become

\[ A^{(0)}_+ = -\mathcal{E}_+, \quad A^{(0)}_- = \mathcal{E}_- + \Omega x_+ C, \quad (5.9) \]

with \( \mathcal{E}_\pm \) given by (4.1). They can be written as

\[ A^{(0)}_\pm = -\partial_\pm T_0 T_0^{-1}, \quad (5.10) \]

with

\[ T_0 = e^{x_+ \mathcal{E}_+} e^{-x_- \mathcal{E}_-}. \quad (5.11) \]
The gauge potentials for the vacuum solution (2.18) are obtained from (5.9) by taking $c_{n}^{\pm} = 0$. In fact, they are connected by the gauge transformation

$$A_{\pm}^{(0)} = \tilde{T}_{0} A_{\pm}^{(0)} |_{c_{n}^{\pm} = 0} \tilde{T}_{0}^{-1} = \partial_{\pm} \tilde{T}_{0} T_{0}^{-1},$$

with $\tilde{T}_{0} = \exp[\pi_{+} (E_{+} - E_{0})] \exp[-x_{-} (E_{-} - E_{-0})]$.

However, in general, the vacuum solutions (2.18) and (2.20) may not be connected by any dressing transformation, and, in such a case, the existence of two elements of form (2.4), is not always possible. Consequently, one can have soliton solutions lying on different orbits under the dressing transformations.

In order to perform the dressing procedure, we take (5.9) as initial gauge potentials. Then, we obtain, under the dressing procedure, the solutions on the orbit of vacuum (2.20), under the dressing transformations.

$$b E_{l} b^{-1} + \sum_{m=1}^{l-1} b F_{m}^{+} b^{-1} = \Theta_{\pm} E_{l} \Theta_{\pm}^{-1} + \sum_{n=1}^{l-1} c_{n}^{+} \Theta_{\pm} E_{n} \Theta_{\pm}^{-1} + \partial_{\pm} \Theta_{\pm} \Theta_{\pm}^{-1},$$

$$-\partial_{\pm} b b^{-1} - \partial_{\pm} (\nu + \Omega x_{+} x_{-}) C + E_{-l} + \sum_{m=1}^{l-1} F_{m}^{-} = \Theta_{\pm} E_{-l} \Theta_{\pm}^{-1} + \sum_{m=1}^{l-1} c_{m}^{-} \Theta_{\pm} E_{-m} \Theta_{\pm}^{-1} - \partial_{\pm} \Theta_{\pm} \Theta_{\pm}^{-1}.$$  

(5.12)

Note that in the above relations, the fields $b$, $\nu$ and $F_{m}^{\pm}$ stand for the solutions on the orbit of the vacuum solution (2.20). The procedure to construct the solution requires to split the above equations into the eigensubspaces of the grading operator (2.2). It is convenient to write

$$\Theta_{+}^{>} = \exp \left( \sum_{s > 0} t^{(s)} \right), \quad \Theta_{-}^{<} = \exp \left( \sum_{s > 0} t^{(-s)} \right),$$

where $t^{(s)} \in \hat{G}_{\pm s}$.

The mappings $t^{(s)}$, for each choice of $p$, are determined from (5.8) with $T$ being $T_{0}$ given in (5.11). Then, the components of (5.12) and (5.13) in each eigensubspace, give an equation connecting the fields with $t^{(s)}$. Thus the solutions for the fields $b$, $\nu$ and $F_{m}^{\pm}$ are determined from $t^{(s)}$. Such a procedure is rather cumbersome, but fortunately, one needs to know very few $t^{(s)}$’s to get the solution. For instance, taking relations (5.12) and (5.13) for $\Theta_{+}$ ($\Theta_{-}$) with grade components 0 and $-l$ ($l$ and 0), one gets

$$\Theta_{+}^{(0)} = h_{L}^{-1} (x_{-}), \quad \Theta_{-}^{(0)} = b e^{(\nu + \Omega x_{+} x_{-})} C h_{R} (x_{+})$$

with $h_{L} (x_{-})$ and $h_{R} (x_{+})$ defined in (2.32).

From (5.3), (5.11) and (5.15) it follows that

$$\Theta_{-}^{<}^{-1} \left( h_{L} (x_{-}) b e^{(\nu + \Omega x_{+} x_{-})} C h_{R} (x_{+}) \right)^{-1} \Theta_{+}^{>} = e_{x_{+}} e_{x_{-}} e_{-x_{-}} e_{-x_{+}} e_{x_{+}} e_{x_{-}}.$$  

(5.16)
The space–time dependence of the r.h.s. of the above relation is given explicitly. One can extract the solutions out of (5.16) by taking the expectation value of its both sides between suitable states of a given representation of \( \hat{G} \), in a similar way to that one explained in section 3.

The solitons solutions are obtained from (5.16) by choosing the fixed group element \( \rho \), characterising the dressing transformation, as the exponential of an eigenvector of \( E_{\pm} \), i.e.

\[
\rho = e^V. \tag{5.17}
\]

That is the solitonic specialization discussed in section 4. Indeed, if \( V \) satisfies the relations

\[
[E_{\pm}, V] = \omega_\pm V, \tag{5.18}
\]

then (5.16) reads as

\[
\exp \left( e^{x_+ \omega_+ - x_- \omega_-} V \right) \equiv \exp \left( e^{\gamma(x - vt)} V \right), \tag{5.19}
\]

with \( \gamma = \omega_+ + \omega_- \), and \( v = (\omega_- - \omega_+) / (\omega_+ + \omega_-) \), since \( x_\pm = t \pm x \).

Therefore, for each eigenvector \( V \), expression (5.13) corresponds to a solution that travels with a constant velocity \( v \) without dispersion. Depending upon the properties of \( V \), as we will see below in the examples, such solutions correspond to one–soliton solutions.

The multi–soliton solutions are obtained by taking \( \rho \) to be the product of several one–soliton \( \rho \)'s, i.e.,

\[
\rho = e^{V_1} e^{V_2} e^{V_3} \ldots e^{V_N}, \tag{5.20}
\]

with each \( V_i \) satisfying \([E_{\pm}, V_i] = \omega^i_\pm V_i\).

Notice that, under the global gauge transformations (2.33), the gauge potentials (2.5) are transformed as \( A_{\pm} \rightarrow h_D A_{\pm} h_D^{-1} \). Therefore, from (5.2) one has \( T \rightarrow h_D T \), and consequently (5.8) implies \( \Theta^+ \rightarrow h_D \Theta^+ h_D^{-1} \) and \( \Theta^- \rightarrow h_D \Theta^- h_D^{-1} \). Hence, with solution (5.16) corresponding to a fixed element \( \rho \), a solution, obtained from that by a global gauge transformation (2.33), is given by (5.16) with the replacement

\[
\rho \rightarrow h_D \rho h_D^{-1}, \tag{5.21}
\]

if the condition \( h_D E_{\pm} h_D^{-1} = E_{\pm} \) is satisfied. For the solutions on the orbit of the vacuum (2.18), that is indeed true, since \( E_{\pm} = E_{\pm l} \); see (1.1). For the solitonic case, one then obtains for each eigenvector \( V \) of \( E_{\pm} \), an orbit of equivalent one–soliton (or multi–soliton) solutions generated by \( h_D V h_D^{-1} \).

6 Stress tensor and Hamiltonian reduction

The two–loop WZNW model was introduced in [6]; it leads, under the Hamiltonian reduction procedure, to the conformally affine abelian Toda systems. The structure of the two–loop WZNW model was further studied in [26]; in [6] the Hamiltonian reduction was extended to non abelian affine Toda models. The procedure we discuss here is, in fact, simpler than those in [6], since, due to the extra higher grade fields of the reduced model, all the constraints are of the first class.
We use the Hamiltonian reduction to obtain, from the Sugawara construction for the two–loop WZNW model, the energy–momentum tensor for the systems defined in section 2. For such systems it is not easy to get the canonical energy–momentum tensor in a direct way, because, in general, the expression for the Lagrangian is not known yet; and, in fact, in terms of the fields $F_m^\pm$ introduced in (2.6), it is non local.

The action for the two–loop WZNW model is the same as that for the ordinary WZNW theory, with the fields $\hat{g}$ being mappings from $\mathcal{M}$ to $\hat{G}$. Therefore, the equations of motion for the two–loop WZNW model [6] are

\[
\partial_+ \left( \partial_- \hat{g} \hat{g}^{-1} \right) = 0, \quad \partial_- \left( \hat{g}^{-1} \partial_+ \hat{g} \right) = 0.
\]  

(6.1)

We are interested in those mappings which can be represented in the form

\[
\hat{g} = NBM,
\]  

(6.2)

where $N$, $B$, and $M$ are generated by the subalgebras $\hat{G}_+$, $\hat{G}_0$ and $\hat{G}_-$, respectively, defined in (2.1). Introduce the mappings $K_{L/R}$ as

\[
\hat{g}^{-1} \partial_+ \hat{g} = M^{-1}K_RM = M^{-1} \left( B^{-1}N^{-1} \partial_+ NB + B^{-1} \partial_+ B + \partial_+ MM^{-1} \right) M, \quad (6.3)
\]

and so, from (6.1), one obtains

\[
\begin{align*}
\partial_- K_R &= -[K_R, \partial_- MM^{-1}], \\
\partial_+ K_L &= [K_L, N^{-1} \partial_+ N];
\end{align*}
\]  

(6.4)

cf. those in [3, 6].

We show now that the system (2.7) – (2.9) can be obtained from the Hamiltonian reduction of the two–loop WZNW model by imposing the constraints

\[
\begin{align*}
\left( \partial_- \hat{g} \hat{g}^{-1} \right)_{<l} &= E_{<l}, \\
\left( \partial_- \hat{g} \hat{g}^{-1} \right)_{\geq l} &= 0, \\
\hat{g}^{-1} \partial_+ \hat{g} \big|_l &= E_l, \\
\left( \hat{g}^{-1} \partial_+ \hat{g} \right)_{>l} &= 0.
\end{align*}
\]  

(6.5)

From (6.3) one sees that constraints (6.4) are equivalent to the following ones:

\[
\begin{align*}
\left( \partial_- MM^{-1} \right)_{<l} &= B^{-1}E_{<l}B, \\
\left( \partial_- MM^{-1} \right)_{\geq l} &= 0, \\
\left( N^{-1} \partial_+ N \right)_{l} &= BE_lB^{-1}, \\
\left( N^{-1} \partial_+ N \right)_{>l} &= 0.
\end{align*}
\]  

(6.6)

Therefore, the mappings $\partial_- MM^{-1}$ and $N^{-1} \partial_+ N$ have components in the subspaces $\hat{G}_{<n}$ and $\hat{G}_{n}$, respectively, with $1 \leq n \leq l-1$. To provide a relation to system (2.7) – (2.9), denote

\[
B \partial_- MM^{-1} B^{-1} = E_{<l} + \sum_{m=1}^{l-1} F_m^- \equiv F^-, \quad B^{-1} N^{-1} \partial_+ N \quad B = E_l + \sum_{m=1}^{l-1} F_m^+ \equiv F^+.
\]  

(6.7)
Substituting the mappings $K_{L/R}$, defined in (3.3), into (3.4), and using constraints (3.6), one can easily check that $B$, $F^+$ and $F^-$ satisfy (2.7)–(2.9). Hence, the system can indeed be obtained by the Hamiltonian reduction from the two–loop WZNW model.

Similarly to the finite dimensional case of the standard Toda system [27] and its higher grading generalizations [4], one can prove that the mappings $N$, $M$ and $B$ entering the decomposition $\hat{g} = NBM$, are determined by the holomorphic mappings $\mu_{\pm}$, see (3.3), via the formulas

$$N = \sigma_+(x_-)(\gamma_0^-)^{-1}\nu_+\gamma_0^-, \quad M = \sigma_-(x_+)(\gamma_0^+)^{-1}\nu_-\gamma_0^+, \quad \text{(6.8)}$$

and the equality

$$\mu_{+}^{-1}\mu_{-} = \nu_{-}\hat{g}0\nu_{+}^{-1}.$$  

So, the WZNW field $\hat{g}$ is represented in a holomorphic factorisable form

$$\hat{g}^{-1} = \hat{g}_L(x_+)\hat{g}_R(x_-), \quad \text{(6.9)}$$

with

$$\hat{g}_R = \mu_-(x_-)\gamma_0^-(x_-)^{-1}\sigma_-(x_-), \quad \hat{g}_L = \mu_+(x_+)(\gamma_0^+(x_+)^{-1}\sigma_-(x_+). \quad \text{(6.10)}$$

Here $\gamma_0^\pm$ are arbitrary mappings determining the general solution to our system; $\sigma_\pm$ are additional arbitrary mappings generated by the subspaces $\sum_{m>0}\mathcal{G}_{\pm m}$, specific for the WZNW theory, and they become absent (or, in a sense, hidden) after the Hamiltonian reduction to the Toda type theories.

It follows from equations (6.1) that the corresponding chiral currents,

$$J_R(X; x_+) \equiv -k \, \text{Tr} \left( X\hat{g}^{-1}\partial_+ \hat{g} \right), \quad J_L(X; x_-) \equiv k \, \text{Tr} \left( X\partial_- \hat{g} \hat{g}^{-1} \right), \quad \text{(6.11)}$$

with $X \in \hat{G}$, satisfy the equations $\partial_- J_R = \partial_+ J_L = 0$.

The two–loop WZNW model is conformally invariant, and its energy–momentum tensor is given by the Sugawara construction [16, 17]. On the classical level, we have its components given by [3]

$$T_{++} = \frac{k}{2} \text{Tr} \left( \hat{g}^{-1} \partial_+ \hat{g} \right)^2, \quad T_{--} = \frac{k}{2} \text{Tr} \left( \partial_- \hat{g} \hat{g}^{-1} \right)^2, \quad \text{(6.12)}$$

and $T_{+-} = T_{-+} = 0$. Here $k$ is the central term of the two–loop current algebra,

$$[J(X, x), J(X', y)]_{pb} = J([X, X'], x)\delta(x - y) + k \text{Tr} (X X')\delta'(x - y); \quad \text{(6.13)}$$

$X$ and $X'$ are elements of an affine Kac-Moody algebra $\hat{G}$. The above relation is satisfied by the left and right currents (6.11); and currents of different chiralities commute.

The components $T_{++}$ and $T_{--}$ have vanishing Poisson bracket denoted $[*,*]_{pb}$, and each of them generates a copy of the centerless Virasoro algebra,

$$[T(x), T(y)]_{pb} = 2T(y)\delta'(x - y) - T'(y)\delta(x - y). \quad \text{(6.14)}$$

The left (right) currents have vanishing Poisson bracket with $T_{++}$ ($T_{--}$), and are transformed under $T_{--}$ ($T_{++}$) as primary fields of conformal weight 1,

$$[T(x), J(X;y)]_{pb} = J(X;y)\delta'(x - y) - J'(X;y)\delta(x - y). \quad \text{(6.15)}$$
Under constraints (6.5), the currents $J_R(X^{(-l)}; x)$ and $J_L(X^{(l)}; x)$, such that
$\text{Tr} \left( X^{(-l)} E_l \right) \neq 0$, and $\text{Tr} \left( X^{(l)} E_{-l} \right) \neq 0$, take fixed non vanishing values. Therefore, the Virasoro generators do not weakly commute with such currents. In other words, since the currents are not scalars under the conformal transformations generated by $T_{++}$ and $T_{--}$, it means that constraints (6.5) break the conformal invariance. However, one can use the currents in the direction of the grading operator $Q_s$ given in (2.2), to improve the Virasoro generators. Define

$$L_{++}(x_+) \equiv T_{++}(x_+) + \frac{1}{l} J_R'(Q_s, x_+), \quad L_{--}(x_-) \equiv T_{--}(x_-) - \frac{1}{l} J_L'(Q_s, x_-). \quad (6.16)$$

One can easily verify that these components generate two commuting copies of the Virasoro algebra, which are centerless because $\text{Tr}(Q_s)^2 = 0$, for the particular grading operator in (2.2). In addition, these generators weakly commute with constraints (6.5), and, therefore, the reduced model is conformally invariant. Substituting constraints (6.5) in (6.16), we obtain the energy–momentum tensor for the reduced model, which generate two commuting copies of the Virasoro algebra under the Dirac bracket. Indeed, one can easily check, using (6.3), (6.12), (6.11) and (6.5), that

$$\frac{1}{k} L_{++}^{\text{red}}(x_+) = \frac{1}{2} \text{Tr} \left( B^{-1} \partial_+ B \right)^2 - \frac{1}{l} \text{Tr} \left( Q_s \partial_+ \left( B^{-1} \partial_+ B \right) \right) - \frac{1}{l} \sum_{n=1}^{l-1} \text{Tr} \left( \left( \partial_+ F^+ \right)_n \left( M Q_s M^{-1} \right)_{-n} \right) + \sum_{n=1}^{l-1} \left( 1 - \frac{n}{l} \right) \text{Tr} \left( \left( M^{-1} \partial_+ M \right)_{-n} \left( M^{-1} F^+ M \right)_n \right), \quad (6.17)$$

and

$$\frac{1}{k} L_{--}^{\text{red}}(x_-) = \frac{1}{2} \text{Tr} \left( \partial_- B B^{-1} \right)^2 - \frac{1}{l} \text{Tr} \left( Q_s \partial_- \left( \partial_- B B^{-1} \right) \right) - \frac{1}{l} \sum_{n=1}^{l-1} \text{Tr} \left( \left( \partial_- F^- \right)_{-n} \left( N^{-1} Q_s N \right)_n \right) + \sum_{n=1}^{l-1} \left( 1 - \frac{n}{l} \right) \text{Tr} \left( \left( \partial_- N N^{-1} \right)_n \left( N F^- N^{-1} \right)_{-n} \right). \quad (6.18)$$

Representing

$$N = \exp \left( \sum_{s>0} \zeta_s \right), \quad M = \exp \left( \sum_{s>0} \xi_{-s} \right), \quad (6.19)$$

one sees that the mappings $F^+$ and $F^-$ defined in (5.7), depend on $\zeta_s$ and $\xi_{-s}$, respectively, for $1 \leq s \leq l - 1$. In addition, due to the grading structure of the terms entering (6.17) and (6.18), the fields $\zeta_s$ and $\xi_{-s}$, for $s > l$, do not contribute to $L_{++}^{\text{red}}$ and $L_{--}^{\text{red}}$. Therefore, the components (6.17) and (6.18) of the energy–momentum tensor, are local functions of the fields $B$, $\zeta_s$ and $\xi_{-s}$, $1 \leq s \leq l - 1$, of the reduced model. If one wishes to express those components in terms of $F^\pm$, one gets a non local expression. That is, in fact, a difficulty for
obtaining the canonical energy–momentum tensor for system (2.7)–(2.9). The corresponding Lagrangian, written in terms of $F^\pm$, is also non local.

The canonical energy–momentum tensor (6.17)–(6.18) is conserved and traceless as a consequence of the fact that it is the reduced two–loop WZNW stress tensor, i.e.,

$$\partial_+ L^\text{red}_{++} = 0, \quad \partial_+ L^\text{red}_{--} = 0, \quad L^\text{red}_{+-} = L^\text{red}_{-+} = 0.$$  \hfill (6.20)

Notice that such a tensor has a part which is a total derivative, namely

$$\text{Tr} \left( Q_s \partial_+ (B^{-1} \partial_+ B) \right) \text{TR} \left( Q_s \partial_+ (\partial_+ B B^{-1}) \right)$$

in (6.17), and

$$\text{Tr} \left( Q_s \partial_- (\partial_- B B^{-1}) \right) \text{TR} \left( Q_s \partial_- (B^{-1} \partial_- B) \right)$$

in (6.18). They do not correspond to the full reduced improvement terms $J^R_{ij}(Q_s)$, but only to part of it. We can use them to construct a trivially conserved tensor, namely

$$S^\mu_\nu \equiv -\frac{k}{l} \text{Tr} \left( Q_s \left( \partial_\mu \left( B^{-1} \partial_\nu B \right) - g_{\mu\nu} \partial_\rho \left( B^{-1} \partial^\rho B \right) \right) \right) = -\frac{k N_s}{l} \left( \partial_\mu \partial_\nu - g_{\mu\nu} \partial^2 \right) \nu,$$  \hfill (6.21)

where, in the last equality, we have made use of the fact that, thanks to the basis chosen to parametrise $B$, see (2.10)–(2.12), $Q_s$ is orthogonal to all generators of $B$, except the central term $C$. So, such a tensor is symmetric and conserved

$$\partial^\mu S^\mu_\nu = 0.$$  \hfill (6.22)

Then, introduce an energy–momentum tensor $\Theta^\mu_\nu$ as

$$\Theta^\mu_\nu = I^\text{red}_{\mu\nu} - S^\mu_\nu.$$  \hfill (6.23)

Due to (6.20) and (6.22), it is also symmetric and conserved,

$$\partial^\mu \Theta^\mu_\nu = 0,$$  \hfill (6.24)

but it is not traceless.

## 7 Masses of fundamental particles and solitons

As we have seen above, the system under consideration is conformally invariant. Therefore, since we do not have a continuum mass spectrum, its fundamental particles have to be massless. However, such a symmetry can be spontaneously broken by choosing a particular constant solution for the field $\eta$, say $\eta = \eta_0$. The resulting theory is then massive. Representing the mapping $B$ as $B \equiv \exp T$, and considering only the linear field approximation, i.e., the free part of the equations of motion (2.7)–(2.9), one gets

$$\partial_+ \partial_- T = -v_\eta [E_{-1}, [E_1, T]],$$  \hfill (7.1)

$$\partial_+ \partial_- F^+_m = -v_\eta [E_{-1}, [E_1, F^+_m]],$$  \hfill (7.2)

$$\partial_+ \partial_- F^-_m = -v_\eta [E_{-1}, [E_1, F^-_m]].$$  \hfill (7.3)

---

6The metric is $g_{++} = g_{--} = 0$, $g_{+-} = g_{-+} = 1/2$, and since $B$ commutes with $Q_s$, one has $\partial_+ \text{TR} (Q_s B^{-1} \partial_+ B) = \partial_+ \text{TR} (Q_s \partial_+ B B^{-1})$.

7Due to the fact that $B$ commutes with $Q_s$, it follows that $S^\mu_\nu$ is symmetric and conserved independently of the basis we use.
where \( v_\eta = e^{i\eta_0} \).

Therefore, the masses of fundamental particles in such a theory are given by the eigenvalues of the operator \([E_{-l}, [E_l, \star]]\) in the subspaces \( \mathcal{G}_n, n = 0, \pm 1, \pm 2, \ldots \pm (l - 1) \), i.e.,

\[
[E_{-l}, [E_l, X]] = \lambda X. \tag{7.4}
\]

Since \( \partial_+ \partial_- = \frac{1}{4} (\partial_x^2 - \partial_z^2) \), we obtain the masses from the Klein–Gordon type equations (7.1) – (7.3) as

\[
m_\lambda^2 = 4 \lambda v_\eta. \tag{7.5}
\]

That result constitute a generalization of the arguments used in the abelian and non abelian affine Toda models [28, 5]. Of course, we are interested in those cases where the eigenvalues of the operator \([E_{-l}, [E_l, \star]]\) are real and positive on the subspaces under consideration. That will be, in fact, one of the conditions we use to select the data \( \{\mathcal{G}, Q_s, l, E_{\pm l}\} \) for defining physical models through (2.5).

Notice that the field \( e^{i\eta} \) plays the role of a Higgs field, since it not only spontaneously breaks the conformal symmetry, but also because its vacuum expectation value sets the mass scale of the theory. We have here the same mechanism as in non abelian affine Toda theories [29, 3].

The masses of the fields \( \zeta_m \) and \( \xi_{-m} \), introduced in (6.19) for \( 1 \leq m \leq l - 1 \), are the same as those of the fields \( F_{\pm m} \). Indeed, substituting (6.7) in (2.14)–(2.15), and taking the linear field approximation with \( \eta = \eta_0 \), we get

\[
\partial_+ \partial_- \zeta_m = v_\eta [E_l, [E_{-l}, \zeta_m]],
\]

\[
\partial_+ \partial_- \xi_{-m} = v_\eta [E_l, [E_{-l}, \xi_{-m}]]. \tag{7.6}
\]

where a trivial integration is performed to eliminate one derivative.

Let us explain now, following the reasonings of [29] and [3], that the masses of solitons are also generated by the spontaneous breakdown of the conformal symmetry. The energy of classical solutions are given by the space integral of the \((0, 0)\) component of energy–momentum tensor \( L_{\mu\nu}^{\text{red}} \) defined in (6.17)–(6.18). In the Lorentz frame where the classical soliton solution is static, the energy should be interpreted as the mass of the soliton. However, since the theory is conformally invariant, it has no mass scale, and the soliton mass should vanish. When the conformal symmetry is spontaneously broken by choosing a particular constant solution for the field \( \eta \), we obtain a massive theory. Construct the energy–momentum tensor of such a theory as follows. Clearly, the tensor \( \Theta_{\mu\nu} \), introduced in (6.23) and evaluated at any classical solution, satisfies (6.24). Therefore, the tensor defined by

\[
\Theta_{\mu\nu}^{\text{broken}} \equiv \Theta_{\mu\nu} |_{\eta=\text{constant}}, \tag{7.7}
\]

is symmetric and conserved,

\[
\partial^\mu \Theta_{\mu\nu}^{\text{broken}} = 0, \tag{7.8}
\]

since \( \eta = \text{constant} \) is a solution of the equations of motion. Then, let the energy in the massive theory be proportional to the space integral of \( \Theta_{00}^{\text{broken}} \). Using (5.24) and (6.23), we obtain the soliton mass in the form

\[
\frac{M}{\sqrt{1 - v^2}} = - \left( \int_{-\infty}^{\infty} dx \Theta_{00}^{\text{broken}} - E_{\text{vac}} \right) = - \frac{k N_s}{l} \partial_x (\nu + \Omega x^+ x^-) |_{-\infty}, \tag{7.9}
\]

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because the integral of $L_{00}^{\text{red}}$ vanishes by the above arguments. Here $v$ is the soliton velocity in the units of the speed of the light. Notice that we have subtracted the energy $E_{\text{vac}}$ of the vacuum solution which is, in fact, divergent. Of course, the vacuum solution is not unique, and it is not clear which one provides the absolute minimum of the energy. We will use the following prescription for the soliton mass formula. For the soliton solutions lying, under the dressing transformations, on the orbit of the vacuum solution (2.20), we take $\Omega$ in (7.9) to be that one given in (2.21). However, for those soliton solutions lying on the orbit of the vacuum (2.18), we take $\Omega$ in (7.9) to be equal to the parameter $\beta$ introduced in (2.17). Such a prescription guarantees the finiteness of the soliton masses.

The soliton masses are determined solely by the behaviour at $x = \pm \infty$ of the space derivative of the field $\nu$. That is quite a remarkable fact. In addition, as we now explain, it is very easy to obtain such a behaviour in the general case from the solitonic solutions (4.3) or (5.19).

Consider the integrable highest weight representation of $\hat{G}$ with highest weight state $|\lambda_s\rangle$, satisfying the relations [15, 5]

$$H^0_a \mid \lambda_s\rangle = s_a \mid \lambda_s\rangle, \quad C \mid \lambda_s\rangle = \frac{\psi^2}{2} \left( \sum_{i=0}^{r} l^\psi s_i \right) \mid \lambda_s\rangle, \quad f_i \mid \lambda_s\rangle = 0, \quad \text{if } s_i = 0; \quad (7.10)$$

cf. (3.14), where $f_a \equiv E^0_{-\alpha_a}, a = 1, 2, \ldots r$, $f_0 \equiv E^{-1}_{\psi}$, and $s_i$ are the co-prime integers labeling a given integral representation of $\hat{G}$ (2.2), $l^\psi_i$ are the integers in the expansion $\frac{\psi^2}{\psi^2} = \sum_{a=1}^{r} l^\psi a \alpha_a^2$ and $l^\psi_0 = 1$. Such a highest weight state can be realised by

$$\mid \lambda_s\rangle = \bigotimes_{i=0}^{r} \mid \hat{\lambda}_i\rangle^{\pm s_i}, \quad (7.11)$$

where $\mid \hat{\lambda}_i\rangle$ are the highest weight states of the fundamental representations of $\hat{G}$, and $\hat{\lambda}_i$ are the corresponding fundamental weights of $\hat{G}$.

Consider now an integral gradation of $\hat{G}$, with $s'_i = \frac{\psi^2}{\alpha_i^2} s_i, \quad \alpha_0 \equiv -\psi$, and $s_i$ labeling the gradation that defines the model (2.13)–(2.16). Therefore, it follows that the Cartan generators of the special basis introduced in (2.11)–(2.12) provide

$$\bar{H}^0_a \mid \lambda_{s'}\rangle = 0. \quad (7.12)$$

From (7.10) one has that $\mid \lambda_{s'}\rangle$ is annihilated by all negative root step operators with $s'$-zero grade, and, consequently, with $s$-zero grade too. Since it is a highest weight state, $\mid \lambda_{s'}\rangle$ is annihilated by all generators of the subgroup $G_0$. Therefore, taking the expectation value of both sides of (5.16) in such state, one gets (with the gauge choice $h_L(x_-) = h_R(x_+) = 1$)

$$e^{-(\nu+\Omega x_+ x_-)N_s \frac{\psi^2}{\psi^2}} = \langle \lambda_{s'} \mid e^{x_+ E_+} e^{-x_- E_-} \rho e^{x_- E_-} e^{-x_+ E_+} \mid \lambda_{s'}\rangle. \quad (7.13)$$

Now, choosing $\rho$ to be the exponential of an eigenvector of $E_\pm$,

$$[E_\pm, V] = \omega_\pm V, \quad (7.14)$$
we obtain a soliton solution
\[ e^{-(\nu + \Omega + x - x_0) N_s \frac{\psi^2}{l^2}} = \langle \lambda_{s'} | e^{\psi^2 V} | \lambda_s \rangle \] (7.15)

with \( \Gamma = \omega_+ x_+ - \omega_- x_0 \equiv \gamma (x - vt) \).

Suppose \( V \) is an operator in such a representation for which there is a positive integer \( N'_V \), such that
\[ \langle \lambda_{s'} | V^n | \lambda_{s'} \rangle = 0 \quad \text{for} \quad n > N'_V, \] (7.16)

Then the soliton mass is easily obtained from (7.9), where for \( \gamma > 0 \) \( (\gamma < 0) \) only the upper (lower) limit \( x = \infty \) \( (x = -\infty) \) contributes in the integral,
\[ M = \frac{2}{\psi^2} \frac{k N'_V}{l} | \gamma | \sqrt{1 - v^2} = \frac{2}{\psi^2} \frac{k N'_V}{l} \sqrt{\omega_+ \omega_-}. \] (7.17)

Notice that we must have \( \omega_+ \omega_- > 0 \) in order to have the soliton velocity \( v = (\omega_- - \omega_+)/ \sqrt{\omega_- + \omega_+} \), not exceeding the light velocity \( (c = 1) \).

The soliton mass formula (7.17) has some remarkable properties. One of them concerns the relation particle–soliton in the theory, indicating some sort of duality similar to the electromagnetic duality of some four dimensional gauge theories possessing the Bogomolny (monopole) limit \( [30] \). As we have seen, the soliton solutions are created by the eigenvectors \( V \) of \( E^\pm \). From (7.14) one has \([E^+, [E^-, V]] = \omega_+ \omega_- V \). Expanding \( V \) over the eigenvectors of the grading operator \( Q_s \) as \( V = \sum_n V^{(n)} \), one observes that \([E^+, [E^-, V^{(n)}]] = \omega_+ \omega_- V^{(n)} \). Therefore, if some \( V^{(n)} \in \hat{\mathcal{G}}_n \), \( n = 0, \pm 1, \pm 2, \ldots, \pm (l-1) \), does not vanish, it implies that \( V^{(n)} \) must be one of the eigenvectors \( X \) in (7.4). Then we associate a soliton with a fundamental particle. In addition, we have \( \lambda \equiv \omega_+ \omega_- \), and, consequently, from (7.14) and (7.17), the masses of the corresponding soliton and fundamental particle are determined by the same eigenvalue. In fact, we have from (7.4) and (7.17), with \( v_\eta = 1 \), that
\[ M_{\text{sol}} = \frac{2}{\psi^2} \frac{k N'_V}{l} m_{\text{part}}. \] (7.18)

Of course, in the expansion of \( V \), we may have more than one non vanishing \( V^{(n)} \), with \( n = 0, \pm 1, \pm 2, \ldots, \pm (l-1) \). Then we would associate a one–soliton solution to more than one fundamental particle. The counting of one–soliton solutions has to be better analysed in each particular case. We discuss this issue in the section devoted to the examples.

8 Physical properties of the higher grading fields

It is clear from (2.23)-(2.27), that the massive fields associated with non vanishing grade (namely \( F_m^\pm \)), are chiral fields with non vanishing spins, in contrast with the Toda type fields. In fact, we show that the free equations for such fields take the form of the massive Dirac equation, as could be expected from general covariance arguments.

\[ \text{We point out that the soliton mass formula (7.17) could be equally obtained by defining the mass through the momentum formula, instead through the energy like in (7.9), as} \quad \frac{M_{\text{sol}}}{\sqrt{1 - v^2}} \equiv \int dx \Theta^{(0)}_{\text{broken} \, \text{token}}. \quad \text{In this case, we do not have to subtract the vacuum momentum, since it vanishes.} \]
Consider the subspace $\hat{G}_m$ for $0 < m < l$. Let $\hat{G}_m^{(F)}$ be the subspace of $\hat{G}_m$, generated by the eigenvectors of $[E_{-l}, [E_l, \cdot]]$ with non zero eigenvalues, i.e.,

$$\hat{G}_m^{(F)} \equiv \{ T^{(m)} \in \hat{G}_m \mid \lambda^{(m)} \neq 0 \},$$

(8.1)

where $\lambda^{(m)}$ is defined as

$$[E_{-l}, [E_l, T^{(m)}]] = [E_l, [E_{-l}, T^{(m)}]] = \lambda^{(m)} T^{(m)}.$$  

(8.2)

Decompose the subspace $\hat{G}_m$, as a vector space, into the sum

$$\hat{G}_m = \hat{G}_m^{(F)} + \hat{G}_m^{(K)},$$

(8.3)

where $\hat{G}_m^{(K)}$ is the complement of $\hat{G}_m^{(F)}$ in $\hat{G}_m$. Define now

$$T^{(-l+m)} \equiv [E_{-l}, T^{(m)}] \in \hat{G}_{-l+m}.$$  

(8.4)

Since $\lambda^{(m)} \neq 0$, it follows that $T^{(-l+m)}$ is necessarily non vanishing. Therefore, we have

$$[E_l, [E_{-l}, T^{(-l+m)}]] = [E_{-l}, [E_l, T^{(-l+m)}]] = [E_{-l}, [E_l, T^{(m)}]] = \lambda^{(m)} [E_{-l}, T^{(m)}] = \lambda^{(m)} T^{(-l+m)}.$$  

(8.5)

So, whenever we have an eigenvector in $\hat{G}_m$ with eigenvalue $\lambda^{(m)} \neq 0$, we also have a corresponding eigenvector in $\hat{G}_{-l+m}$ with the same eigenvalue. Notice that if $\lambda^{(m)}$ is degenerate, then the corresponding eigenvectors are mapped bijectively. Suppose $T_1^{(m)}$ and $T_2^{(m)}$ have the same eigenvalue, and that they are mapped onto the same element in $\hat{G}_{-l+m}$, i.e.,

$$[E_{-l}, T_1^{(m)}] = [E_{-l}, T_2^{(m)}], \quad \text{and so} \quad [E_{-l}, T_1^{(m)} - T_2^{(m)}] = 0.$$  

(8.6)

However, this relation is in contradiction with the following ones:

$$[E_l, [E_{-l}, T_1^{(m)} - T_2^{(m)}]] = \lambda^{(m)} \left( T_1^{(m)} - T_2^{(m)} \right), \quad \text{and} \quad T_1^{(m)} - T_2^{(m)} \neq 0; \quad \lambda^{(m)} \neq 0.$$  

(8.7)

Analogously, we write

$$\hat{G}_{-l+m} = \hat{G}_{-l+m}^{(F)} + \hat{G}_{-l+m}^{(K)},$$

(8.8)

with

$$\hat{G}_{-l+m}^{(F)} \equiv \{ T^{(-l+m)} \in \hat{G}_{-l+m} \mid \lambda^{(m)} \neq 0 \},$$

(8.9)

and $\hat{G}_{-l+m}^{(K)}$ being the complement of $\hat{G}_{-l+m}^{(F)}$ in $\hat{G}_{-l+m}$. Therefore, from the considerations given above, we conclude that the subspaces $\hat{G}_{-l+m}^{(F)}$ and $\hat{G}_m^{(F)}$ are isomorphic. The mapping is given by the action of $E_{-l}$ on $\hat{G}_m^{(F)}$. Our arguments are applied equally well in the reversed direction, the mapping can also be given by the action of $E_l$ on $\hat{G}_{-l+m}^{(F)}$; and hence

$$[E_{-l}, \hat{G}_m^{(F)}] = \hat{G}_{-l+m}^{(F)}, \quad [E_{-l}, \hat{G}_m^{(F)}] \cap \hat{G}_{-l+m}^{(K)} = \{ 0 \};$$

(8.10)

$$[E_l, \hat{G}_{-l+m}^{(F)}] = \hat{G}_m^{(F)}, \quad [E_l, \hat{G}_{-l+m}^{(F)}] \cap \hat{G}_m^{(K)} = \{ 0 \}.$$  

(8.11)
One has
\[
[E_{-l}, \hat{G}_m^{(K)}] \cap \hat{G}^{(F)}_{l+m} = \{0\}, \quad [E_{l}, \hat{G}_m^{(K)}] \cap \hat{G}^{(F)}_m = \{0\},
\] (8.12)
because, otherwise, it would be in a contradiction with the relations
\[
[E_{l}, [E_{-l}, \hat{G}_m^{(K)}]] = 0, \quad [E_{l}, [E_{-l}, \hat{G}^{(K)}_{l+m}]] = 0.
\] (8.13)
Notice that \(\hat{G}_m\) \((\hat{G}_{-l+m})\) does not lie necessarily in the kernel of \(E_{-l} (E_l)\). However, if a given element \(J^{(m)}\) of \(\hat{G}_m^{(K)}\) is in the image of \(E_l\); then it follows from (8.13) that it must be in the kernel of \(E_{-l}\), i.e.,
\[
J^{(m)} = [E_{l}, J^{(-l+m)}] = [E_{-l}, J^{(m)}] = 0, \quad J^{(n)} \in \hat{G}_n^{(K)}.
\] (8.14)
Analogously,
\[
J^{(-l+m)} = [E_{-l}, J^{(m)}] = [E_{l}, J^{(-l+m)}] = 0, \quad J^{(n)} \in \hat{G}_n^{(K)}.
\] (8.15)
In the same way as in section 7, we consider the linear approximation where \(\eta = \eta_0, b = 1, \nu = 0\) (recall equation (2.10)). This gives the free part of equations (2.8) and (2.8),
\[
\partial_- F^+_m = e^{(l-m)\eta_0} [E_{l}, F^-_{l-m}], \quad \partial_+ F^-_{l-m} = -e^{m\eta_0} [E_{-l}, F^+_m].
\] (8.16)
Denote the generators of \(\hat{G}_m^{(F)}\) and \(\hat{G}^{(F)}_{l+m}\), corresponding to the eigenvalue \(\lambda^{(m)} \neq 0\), as \(T^{(m)}_{\lambda^{(m)},i}\) and \(T^{(-l+m)}_{\lambda^{(m)},i}\), respectively, where the index \(i\) stands for a possible degeneracy of \(\lambda^{(m)}\). The basis is chosen in such a way that
\[
[E_{-l}, T^{(m)}_{\lambda^{(m)},i}] = \sqrt{\lambda^{(m)}} T^{(-l+m)}_{\lambda^{(m)},i}, \quad [E_{l}, T^{(-l+m)}_{\lambda^{(m)},i}] = \sqrt{\lambda^{(m)}} T^{(m)}_{\lambda^{(m)},i}.
\] (8.17)
We also use the notations
\[
F^+_m = F^{+(F)}_m + F^{+(K)}_m, \quad F^-_{l-m} = F^{--(F)}_{l-m} + F^{--(K)}_{l-m},
\] (8.18)
with
\[
F^{+(F/K)}_m \in \hat{G}^{(F/K)}_m, \quad F^{--(F/K)}_{l-m} \in \hat{G}^{(F/K)}_{l-m},
\] (8.19)
and
\[
F^{+(F)}_m = \sum_{\lambda^{(m)},i} \psi^{\lambda^{(m)},i}_R T^{(m)}_{\lambda^{(m)},i}, \quad F^{--(F)}_{l-m} = \sum_{\lambda^{(m)},i} \psi^{\lambda^{(m)},i}_L T^{(-l+m)}_{\lambda^{(m)},i}.
\] (8.20)
Since the action of \(E_{\pm l}\) does not mix the subspaces of indices \(F\) and \(K\), we can split equations (8.16) to get
\[
\partial_- \psi^{\lambda^{(m)},i}_R = e^{(l-m)\eta_0} \sqrt{\lambda^{(m)}} \psi^{\lambda^{(m)},i}_L, \quad \partial_+ \psi^{\lambda^{(m)},i}_L = -e^{m\eta_0} \sqrt{\lambda^{(m)}} \psi^{\lambda^{(m)},i}_R.
\] (8.21)
Introduce
\[
\psi^{\lambda^{(m)},i} = \begin{pmatrix} \psi^{\lambda^{(m)},i}_R \\ \psi^{\lambda^{(m)},i}_L \end{pmatrix},
\] (8.22)
and
\[ \gamma_0 = -i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \] (8.23)
satisfying anticommutation relations
\[ \{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu} \mathbb{I}, \] (8.24)
with \( g_{00} = -g_{11} = 1, \ g_{01} = 0. \) Therefore, equations (8.21) can be written as the Dirac type equations
\[ i\gamma^\mu \partial_\mu \psi^{(m),i}_\lambda = m_{\lambda(m)} \left[ \frac{1 + \gamma_5}{2} e^{m\eta_0} + \frac{1 - \gamma_5}{2} e^{(l-m)\eta_0} \right] \psi^{(m),i}_\lambda, \] (8.25)
where, following (7.5), we put
\[ m_{\lambda(m)} \equiv 2\sqrt{\lambda_{(m)}}. \] (8.26)
Hence, the massive degrees of freedom of the model, corresponding to generators of grade \( \pm m \) with \( 0 < m < l \), are, in fact, Dirac fields. However, in general, the mass term involves a \( \gamma_5 \) term, so that parity is violated. A noticeable exception is when \( l = 2, \ m = 1. \)
Notice that \( \psi_{R/L, i}^{\lambda(m)} \) are the projections of \( \psi^{(m),i}_\lambda \) under \( \frac{(1+\gamma_5)}{2} \), with \( \gamma_5 \equiv \gamma_0 \gamma_1. \) Under the gauge transformations (2.30)–(2.31), \( \psi_{R/L, i}^{\lambda(m)} \) are transformed independently under the action of \( h_R(x_+) \) and \( h_L(x_-) \), respectively.
At this point we have seen that the mapping between \( \hat{G}_m \) and \( \hat{G}_{-l+m} \) displayed at the beginning of the section precisely ensures the existence of both chirality components of the Dirac fields. The next question is whether we may write a free action. Suppressing indices for a while, a free field actions would take the form
\[ L = i\bar{\psi}\gamma^\mu \partial_\mu \psi - \bar{\psi}(a + b\gamma_5)\psi. \] (8.27)
Writing
\[ \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \bar{\psi} = (\bar{\psi}_R, \bar{\psi}_L)\gamma_0, \] (8.28)
we obtain
\[ L/i = 2 \left( \bar{\psi}_R \partial_+ \psi_R + \bar{\psi}_L \partial_+ \psi_L \right) - m_R \bar{\psi}_L \psi_R + m_L \bar{\psi}_R \psi_L \] (8.29)
where \( m_R = a + b, \ m_L = a - b. \) Next, equation (8.23) shows that the masses vanish in the limit \( \eta_0 \to \infty. \) Then, we may discuss the conformal properties of the \( \psi \) fields in the usual language, where a field \( A(z_+, z_-) \) is primary with weights \( \Delta_+, \Delta_- \) if \( A(dz_+)^{\Delta_+} (dz_-)^{\Delta_-} \) is invariant by conformal transformations. The kinetic term of equation (8.29) is conformal invariant if we have the following weight assignements.
\[
\begin{array}{cccc}
\Delta_+ & \psi_R & \bar{\psi}_R & \psi_L & \bar{\psi}_L \\
\Delta_- & 0 & 1 - J_R & 0 & 0 \\
\end{array}
\] (8.30)
Consider next the mass term. With the Lorentz transformation \((2.28)\), \(x_{\pm} \to \delta^{\pm 1} x_{\pm}\), \(\bar{\psi}_L \psi_R\) and \(\bar{\psi}_L \psi_R\) will be invariant if \(J_L + J_R = 1\). Returning to our Dirac fields, one finds the identification
\[
\begin{align*}
\psi_R &\Rightarrow \psi^{(m),i}, & \psi^L &\Rightarrow \psi^{(m),i}, \\
\bar{\psi}_R &\Rightarrow \psi^{(l-m),k}, & \bar{\psi}_L &\Rightarrow \psi^{(l-m),\ell},
\end{align*}
\]
(8.31)
for some choice of \(i, j, k, \ell\). Thus the fields of type \(R\) (\(L\)) are to be split between \(\psi\) and \(\bar{\psi}\) type fields. The spectrum of weights is just right to form the needed quadruplets provided the number of \(R\) (\(L\)) fields is even. This is not always true as we shall see. Another difficulty is that in general \(\bar{\psi}\) is not the complex conjugate of \(\psi\). Thus the above free action is not real. This was already the case for the usual affine Toda theories beyond sine–Gordon, and we may expect that this will not be a problem. Another concern is the statistics. At first sight our \(\psi\) fields are c-number fields so that they seem to describe bosons. However, it is well known that in two dimensions the statistics of fields depends upon the coupling constant. Perhaps the latter should be fixed so that the \(\psi\) fields become anticommuting operators. We will have more to say on this below.

9 A special class of models

We now describe a class of models possessing a \(U(1)\) Noether current, which, under some circumstances, is proportional to a topological current. That occurs for those models where the grade \(l\) of the operator \(E_l\), introduced in (2.6), is equal to the integer \(N_s\) defined in (2.7). So, throughout this section we have \(l = N_s\).

The calculations become simpler if we realize the generators of the affine Kac-Moody algebra \(\hat{G}\), in terms of those of the finite simple Lie algebra \(G\) as
\[
H^n_a \equiv z^n H_a, \quad E^n_a \equiv z^n E_a, \quad D \equiv \frac{d}{dz}
\]
(9.1)
with \(z\) being a complex variable. Then, the Lie bracket and trace form on \(\hat{G}\) are given by
\[
[A(z), B(z)] = [A(z), B(z)]_G + C \oint \frac{dz}{2\pi i} \text{Tr} \left( B(z) \frac{d}{dz} A(z) \right)
\]
(9.2)
and
\[
\text{Tr} (A(z) B(z)) = \oint \frac{dz}{2\pi i z} \text{tr} (A(z) B(z))
\]
(9.3)
where \([\cdot, \cdot]_G\) and \(\text{tr} (\cdot \cdot)\) are the Lie bracket and trace form, respectively, on \(G\).

As a first step, we construct two chiral currents associated to the elements \(z^{-1}E_{N_s}\) and \(zE_{-N_s}\) of \(\hat{G}_0\). Multiply eq. (2.8) by \(z^{-1}F^+_{N_s-m}\), sum over \(m\) and take the trace. Then, take the same equation with \(m\) replaced by \(N_s-m\), multiply by \(z^{-1}F^+_m\), sum over \(m\) and take the trace. Add both to get
\[
\sum_{m=1}^{N_s-1} \partial_- \text{Tr} \left( z^{-1}F^+_m F^+_{N_s-m} \right) = -2 \sum_{m=1}^{N_s-1} \text{Tr} \left( z^{-1}E_{N_s} [F^+_m, B^{-1} F^+_m B] \right) + X^+
\]
(9.4)
where
\[
X^+ = \sum_{m=1}^{N_s-2} N_{s-m-1} \sum_{n=1}^{N_s-1-m} \text{Tr} \left( z^{-1} F_{N_s-m}^+ [F_{n+m}^+, B^{-1} F_{n}^- B] \right) \\
+ \sum_{m=2}^{N_s-1} \sum_{n=1}^{N_s-m-1} \text{Tr} \left( z^{-1} F_m^+ [F_{N_s+n-m}^-, B^{-1} F_n^- B] \right) (9.5)
\]

Notice, the first sum in \( m \) ends at \( N_s - 2 \) and the second starts at 2, because the quadratic term in eq. (2.8) does not exist for \( m = N_s - 1 \).

Similarly, multiply (2.9) by \( zF_{N_s-m}^- \), sum over \( m \), and take the trace. Then, take the same equation with \( m \) replaced by \( N_s - m \), multiply by \( zF_m^- \), sum over \( m \) and take the trace. Add them to get
\[
\sum_{m=1}^{N_s-1} \partial_+ \text{Tr} \left( zF_m^- F_{N_s-m}^- \right) = 2 \sum_{m=1}^{N_s-1} \text{Tr} \left( zE_{-N_s} [F_m^-, BF_m^+ B^{-1}] \right) - X^- (9.6)
\]

where
\[
X^- = \sum_{m=1}^{N_s-2} N_{s-m-1} \sum_{n=1}^{N_s-1-m} \text{Tr} \left( zF_{N_s-m}^- [F_{n+m}^-, B F_n^+ B^{-1}] \right) \\
+ \sum_{m=2}^{N_s-1} \sum_{n=1}^{N_s-m-1} \text{Tr} \left( zF_m^- [F_{N_s+n-m}^-, B F_n^+ B^{-1}] \right) (9.7)
\]

Using the fact that
\[
\sum_{m=1}^{N_s-2} N_{s-m-1} \sum_{n=1}^{N_s-1-m} = \sum_{m'=2}^{N_s-1} \sum_{n=1}^{m'-1} , \quad \text{where } m' = m + n (9.8)
\]

one gets\( ^9 \)
\[
X^+ = X^- = 0 (9.9)
\]

Projecting (2.7) onto \( z^{-1} E_{N_s} \) and then onto \( zE_{-N_s} \), and comparing with (9.4) and (9.6), one gets\( ^{10} \)
\[
\partial_- J = 0 , \quad \partial_+ \bar{J} = 0 (9.10)
\]

where
\[
J(x_+) = \text{Tr} \left( z^{-1} E_{N_s} B^{-1} \partial_+ B \right) - \frac{1}{2} \sum_{m=1}^{N_s-1} \text{Tr} \left( z^{-1} F_m^+ F_{N_s-m}^+ \right) \\
\bar{J}(x_-) = \text{Tr} \left( zE_{-N_s} \partial_- B B^{-1} \right) - \frac{1}{2} \sum_{m=1}^{N_s-1} \text{Tr} \left( zF_m^- F_{N_s-m}^- \right) (9.11)
\]

\(^9\)In fact, each double sum in \( X^\pm \) vanishes separately. Use that \( \sum_{m=2}^{N_s-1} \sum_{n=1}^{m-1} = \sum_{n=1}^{N_s-2} \sum_{m=n+1}^{N_s-1} \), and notice that the terms for (fixed \( n \)) \( m = n + k \) and \( m = N_s - k \) cancel.

\(^{10}\)In the projection onto \( z^{-1} E_{N_s} \), it is easier to use the equivalent form of (2.7), \( \partial_- (B^{-1} \partial_+ B) = [E_{N_s}, B^{-1} E_{-N_s} B] + \sum_{n=1}^{N_s-1} [F_n^+, B^{-1} F_n^- B] \)
We now consider those models where \( z^{-1}E_{N_a} \) and \( zE_{-N_a} \) are parallel, and lie in the center of \( \hat{G}_0 \), i.e.
\[
zE_{-N_a} = \mu z^{-1}E_{N_a} \equiv E_0 \in \text{center of } \hat{G}_0 \tag{9.12}
\]
where \( \mu \) is some constant independent of \( z \). Using such condition one observes, from (9.4) and (9.6), that the current
\[
\tilde{J}_+ = -\frac{\mu}{2} \sum_{m=1}^{N_s-1} \text{Tr} \left( z^{-1}F_m^+ F_{N_a-m}^+ \right), \quad \tilde{J}_- = \frac{1}{2} \sum_{m=1}^{N_s-1} \text{Tr} \left( zF_m^- F_{N_a-m}^- \right) \tag{9.13}
\]
is conserved
\[
\partial_\mu \tilde{J}^\mu = 0 \tag{9.14}
\]
In addition, the condition (9.12) implies that the current
\[
\tilde{j}_+ = - \text{Tr} \left( E_0 B^{-1} \partial_+ B \right), \quad \tilde{j}_- = \text{Tr} \left( E_0 \partial_- BB^{-1} \right) \tag{9.15}
\]
is a topological current, i.e. it is conserved independently of the equations of motion
\[
\partial_\mu \tilde{j}^\mu = 0 \tag{9.16}
\]
We now come to a very interesting property of these models. Under the conformal transformations (2.22)-(2.27), the chiral currents (9.11) transform as
\[
\mathcal{J}(x_+) \rightarrow (f'(x_+))^{-1} \left( \mathcal{J}(x_+) - \frac{1}{\mu N_s} \text{Tr} \left( E_0 Q_s \right) (\ln f'(x_+))' \right)
\]
\[
\mathcal{J}(x_-) \rightarrow (g'(x_-))^{-1} \left( \mathcal{J}(x_-) - \frac{1}{N_s} \text{Tr} \left( E_0 Q_s \right) (\ln g'(x_-))' \right) \tag{9.17}
\]
Therefore, if
\[
\text{Tr} \left( E_0 Q_s \right) \neq 0 \tag{9.18}
\]
one concludes that, given a solution of the model, one can always map it, under a conformal transformation, into a solution where\(^\text{13}\)
\[
\mathcal{J}(x_+) = \mathcal{J}(x_-) = 0 \tag{9.19}
\]
Such a procedure amounts to a gauge fixing of the conformal symmetry. We are choosing one solution in each orbit of the conformal group. Another way of saying it, is that (9.19) are constraints and we are performing a Hamiltonian reduction. The degree of freedom eliminated corresponds to the field \( \eta \). So, in the submodel defined by (9.19) one observes from (9.11), that the Noether and topological currents (9.13) and (9.15) are equal
\[
\tilde{J}_\mu = \tilde{j}_\mu \tag{9.20}
\]
\(^{11}\)Obviously, \( E_0 \) can not have any component in the direction of the central term \( C \) or the derivation \( D \), since it is the projection of elements belonging to \( \hat{G}_{\pm N_a} \).
\(^{12}\)We use symbols with a tilde since the currents we are defining right now are not yet properly normalised (see below).
\(^{13}\)The exceptions occur when \( \mathcal{J}(x_+) \) and/or \( \mathcal{J}(x_-) \) present singularities.
as a result of the field equations. As we will see on explicit examples, the true Noether and topological currents differ from the one just defined by overall constants:

\[ J_\mu = c_{\text{Noether}} \tilde{J}_\mu, \quad j_\mu = c_{\text{topol}} \tilde{J}_\mu \]  

(9.21)

Thus the topological and Noether currents are proportional

\[ J_\mu = c_{\text{Noether}} c_{\text{topol}} j_\mu. \]  

(9.22)

This is a very important property of such models, which can lead, in some cases as we will discuss below, to the confinement of the matter fields. In general, that is if \( J \neq 0 \), or \( \bar{J} \neq 0 \), one gets from eqs. (9.11), (9.13) and (9.15),

\[ \mu J = \tilde{J}_+ - \tilde{j}_+ \quad \bar{J} = -\tilde{J}_- + \tilde{j}_- \]  

(9.23)

Therefore the Noether and topological charges

\[ q_{\text{Noether}} \equiv \int_{-\infty}^{\infty} dx J_t \quad q_{\text{topol}} \equiv \int_{-\infty}^{\infty} dx j_t \]  

(9.24)

satisfy

\[ \frac{q_{\text{Noether}}}{c_{\text{Noether}}} - \frac{q_{\text{topol}}}{c_{\text{topol}}} = \frac{1}{2} (q_R - q_L) \]  

(9.25)

where we have denoted

\[ q_R \equiv \mu \int_{-\infty}^{\infty} dx J(x_+); \quad q_L \equiv \int_{-\infty}^{\infty} dx \bar{J}(x_-) \]  

(9.26)

We now comment on the relationship between the masses of particles and solitons of the theory and symmetries associated to \( E_0 \). From (9.12) one gets that \( E_0 \) commutes with \( E_{\pm N_s} \) and therefore it generates a diagonal \( U(1) \) global gauge symmetry of the type described in (2.33), namely

\[ B \rightarrow e^{i\theta E_0} B e^{-i\theta E_0} = B, \quad F_m^\pm \rightarrow e^{i\theta E_0} F_m^\pm e^{-i\theta E_0} \]  

(9.27)

with \( \theta = \text{constant} \). The charges of the fields associated to such \( U(1) \) symmetry are, of course, given by the eigenvalues of \( E_0 \) in the subspace \( \hat{G}_{\text{fields}} \equiv \bigoplus_{n=-N_s+1}^{N_s-1} \hat{G}_n \), i.e.

\[ [E_0, T] = q T, \quad T \in \hat{G}_{\text{fields}} \]  

(9.28)

The masses of the fields are determined by the eigenvalue of \( [E_{N_s}, [E_{-N_s}, \cdot]] \) (see (7.3)). So, using (9.12)

\[ [E_{N_s}, [E_{-N_s}, T]] = \frac{1}{\mu} [E_0, [E_0, T]] = \frac{q^2}{\mu} T \]  

(9.29)

and so

\[ m_T^2 = \frac{4}{\mu} q^2 \]  

(9.30)
The solitons of such theory are created by the operators

\[ V_T(\zeta) = \sum_{n=-\infty}^{\infty} \zeta^{-n} T^n \quad (9.31) \]

since they are eigenvectors of \( E_{\pm N_s} \), (see (5.16)-(5.19), \( T^n \equiv z^n T \))

\[
[E_{N_s}, V_T(\zeta)] = \frac{\zeta}{\mu} q V_T(\zeta) \quad (9.32)
\]

\[
[E_{-N_s}, V_T(\zeta)] = \zeta^{-1} q V_T(\zeta) \quad (9.33)
\]

Therefore, if the expansion of the exponential of \( V_T(\zeta) \) truncates, we get, from (7.17), that the one-soliton masses are

\[
M_{\text{sol}} \sim \frac{q}{\sqrt{\mu}} \quad (9.34)
\]

Therefore, the masses of the fundamental particles and solitons of such theory are proportional to the \( U(1) \) charge. We have here a situation similar to four dimensional gauge theories with Higgs in the adjoint and in the BPS limit, where the masses of particles and monopoles (dyons) are given by \( \text{mass} \sim \sqrt{q_{\text{elect.}}^2 + q_{\text{mag.}}^2} \). That point has to be further investigated, because we believe it indicates these systems possess some sort of duality involving particles and solitons [30].

### 9.1 The example of the principal gradation

Some special examples of models which possesses the structures described above can be constructed as follows. Let \( \hat{G} \) be any untwisted affine Kac-Moody algebra furnished with the principal gradation, where \( s = (1, 1, \ldots, 1) \), and corresponding grading operator, \( Q_{\text{ppal}} \), given by (2.2) with \( N_s = h = \text{Coxeter number} \). Therefore

\[
\hat{G}_0 = \{ H_a^0, a = 1, 2, \ldots, r; C; Q_{\text{ppal}} \}
\]

\[
\hat{G}_m = \{ E_{\alpha(m)}^0, E_{-\alpha(h-m)}^1 \}
\]

\[
\hat{G}_{-m} = \{ E_{-\alpha(m)}^0, E_{\alpha(h-m)}^1 \}
\]

(9.35)

where \( 0 < m < h \), and \( \alpha^{(m)} \) are positive roots of height \( m \), i.e. they contain \( m \) simple roots in their expansion. Following (2.10) we parametrize \( B \) as

\[
B = e^{\varphi \cdot \tilde{H}^0} e^{\nu \cdot C} e^{\eta Q_{\text{ppal}}} = e^{\varphi \cdot \tilde{H}^0} e^{\tilde{\nu} \cdot C} e^{\eta Q_{\text{ppal}}}
\]

(9.36)

where \( \tilde{H}^0 \) is defined in (2.11), and \( \tilde{\nu} = \nu - \frac{2}{h} \delta \cdot \varphi \), with \( \delta = \sum_{a=1}^{r} \frac{\lambda_a}{\alpha_a^2} \), and \( \lambda_a \) being the fundamental weights of \( G \). We then choose \( E_{\pm h} \) to be parallel

\[
E_{\pm h} = m \cdot H^{\pm 1}
\]

(9.37)

where we shall choose \( m \) to be a vector inside the Fundamental Weyl chamber (FWC), and so \( m \cdot \alpha > 0 \) for \( \alpha > 0 \). Consequently

\[
E_0 \equiv m \cdot H^0
\]

(9.38)
satisfies (9.12) (with \( \mu = 1 \)), since \([E_0, \hat{G}_0] = 0\). Since the masses are determined by the eigenvalues of \([E_h, [E_{-h}, \cdot]]\) (see (7.3)) we conclude that the \(\varphi\)'s fields are massless, and the fields in the direction of the step operators have masses \(m_\alpha^2 = 4(\mathbf{m} \cdot \alpha)^2\). In addition, since \(\mathbf{m}\) lies inside the FWC, one has \(m_\alpha \neq 0\) for any \(\alpha\). Therefore, according to the discussion in section (8), all the fields in the direction of the step operators are Dirac fields. Then, following (8.17) and (8.20), we write

\[
F^+_m = \sum_{\alpha(m)} \sqrt{im_{\psi^\alpha(m)}} \psi^\alpha_R E_{\alpha(m)}^0 + \sum_{\alpha(h-m)} \sqrt{im_{\psi^\alpha(h-m)}} \psi^\alpha_R E_{-\alpha(h-m)}^1
\]

\[
F^-_{h-m} = \sum_{\alpha(m)} \sqrt{im_{\psi^\alpha(m)}} \psi^\alpha_L E^1_{\alpha(m)} - \sum_{\alpha(h-m)} \sqrt{im_{\psi^\alpha(h-m)}} \psi^\alpha_R E^0_{-\alpha(h-m)}
\]

(9.39)

with \(0 < m < h\), and where we have denoted

\[
m_{\psi^\alpha(m)} = m_{\varphi^\alpha(m)} = 2m \cdot \alpha(m)
\]

(9.40)

Consequently, associated to each positive root \(\alpha(m)\), we have two Dirac fields

\[
\psi^\alpha(m) \equiv \left( \begin{array}{c} \psi^\alpha_R(m) \\ \psi^\alpha_L(m) \end{array} \right); \quad \varphi^\alpha(m) \equiv \left( \begin{array}{c} \varphi^\alpha_R(m) \\ \varphi^\alpha_L(m) \end{array} \right)
\]

(9.41)

Notice such notation is in accordance with (8.31), since \(\psi^\alpha_R(m)\) and \(\varphi^\alpha_R(m)\) are in the direction of \(E^0_{\alpha(m)} \in \hat{G}_m\) and \(E^1_{-\alpha(m)} \in \hat{G}_{h-m}\) respectively. Similarly, \(\psi^\alpha_L(m)\) and \(\varphi^\alpha_L(m)\) are in the direction of \(E^1_{\alpha(m)} \in \hat{G}_{h+m}\) and \(E^0_{-\alpha(m)} \in \hat{G}_{-m}\) respectively.

These systems possess a \((U(1)_L)^r \otimes (U(1)_R)^r\) gauge symmetry of the type (2.29)-(2.31), with

\[
h_L(x_+) = e^{i\theta L(x_+)} H^0; \quad h_R(x_+) = e^{i\theta R(x_+)} H^0
\]

(9.42)

since these \(h_L/R\) satisfy (2.32). They also possess a global gauge symmetry of the type (2.33), namely

\[
\varphi \rightarrow \varphi; \quad \nu \rightarrow \nu; \quad \eta \rightarrow \eta; \quad F^\pm_m \rightarrow e^{i\theta H^0} F^\pm_m e^{-i\theta H^0}
\]

(9.43)

with \(\theta = \text{constant}\). Notice that the charges of the fields with respect to \(U(1)\) global symmetry (9.27) are (see (9.28))

\[
q_\varphi = q_\eta = q_\varphi = 0; \quad q_{\psi^\alpha(m)} = -q_{\varphi^\alpha(m)} = m \cdot \alpha(m)
\]

(9.44)

The equations of motion for these systems, obtained from (2.13)-(2.16), are

\[
\partial^2 \varphi = -4 \sum_{m=1}^{h-1} \frac{2\alpha(m)}{\alpha(m)^2} m_{\psi^\alpha(m)} \psi^\alpha(m) e^{\gamma_5 (\alpha(m) \cdot \varphi + mn)} \left( \frac{1 + \gamma_5}{2} - e^{h_\eta (1 - \gamma_5)} \right) \psi^\alpha(m)
\]

(9.45)

\[
\partial^2 \tilde{\nu} = -2 \sum_{m=1}^{h-1} \frac{2}{\alpha(m)^2} m_{\psi^\alpha(m)} e^{h_\eta \gamma_5 \psi^\alpha(m)} e^{\gamma_5 (\alpha(m) \cdot \varphi + mn)} (1 - \gamma_5) \psi^\alpha(m) - 4m^2 e^{h_\eta}
\]

(9.46)
\[ i \gamma^\mu \partial_\mu \psi^{(m)} = m_{\psi^{(m)}} e^{\frac{\gamma_5}{2}} (e^{\gamma_5} + e^{-\gamma_5}) \psi^{(m)} + U^{(m)} \] (9.47)  
\[ i \gamma^\mu \partial_\mu \tilde{\psi}^{(m)} = m_{\psi^{(m)}} e^{-\gamma_5} (e^{\gamma_5} - e^{-\gamma_5}) \tilde{\psi}^{(m)} + \tilde{U}^{(m)} \] (9.48)  
\[ \partial^2 \eta = 0 \] (9.49)

where the gamma matrices are defined in (8.23), \( \tilde{\psi}^{(m)} \equiv (\tilde{\psi}^{(m)})^T \gamma_0 \), and \( \psi^{(m)} \).

\[ U^{(m)} = \begin{pmatrix} U_R^{(m)} \\ U_L^{(m)} \end{pmatrix} \quad \tilde{U}^{(m)} = \begin{pmatrix} \tilde{U}_R^{(m)} \\ \tilde{U}_L^{(m)} \end{pmatrix} \] (9.50)

with

\[ U_R^{(m)} = -\frac{\alpha^{(m)} \gamma_5}{\sqrt{m_{\psi^{(m)}}}} \sum_{n=1}^{m-1} e^{\gamma_5} \text{Tr} \left( E_{-\alpha^{(m)}} F_{h-m+n}^{-} e^{\gamma_5} H_0^0 F_n^+ e^{\gamma_5} H_0^0 \right) \]  
\[ U_L^{(m)} = \frac{\alpha^{(m)} \gamma_5}{m_{\psi^{(m)}}} \sum_{n=1}^{h-m-1} e^{\gamma_5} \text{Tr} \left( E_{-\alpha^{(m)}} F_{m+n}^{+} e^{\gamma_5} H_0^0 F_n^+ e^{\gamma_5} H_0^0 \right) \]  
\[ \tilde{U}_R^{(m)} = \frac{\alpha^{(m)} \gamma_5}{\sqrt{m_{\psi^{(m)}}}} \sum_{n=1}^{h-m-1} e^{\gamma_5} \text{Tr} \left( E_{\alpha^{(m)}} F_{m+n}^{+} e^{\gamma_5} H_0^0 F_n^+ e^{\gamma_5} H_0^0 \right) \]  
\[ \tilde{U}_L^{(m)} = \frac{\alpha^{(m)} \gamma_5}{\sqrt{m_{\psi^{(m)}}}} \sum_{n=1}^{h-m-1} e^{\gamma_5} \text{Tr} \left( E_{\alpha^{(m)}} F_{h-m+n}^{+} e^{\gamma_5} H_0^0 F_n^+ e^{\gamma_5} H_0^0 \right) \] (9.51)

Since these systems satisfy (9.12), they possess the conserved Noether and topological currents (9.13) and (9.13), respectively. In order to correctly define these currents, we have to be more precise about normalizations. First, concerning the topological current, the potential terms in Eqs. (9.45)–(9.49) involve \( \varphi \) only through \( \exp(\pm \alpha^{(m)} \cdot \varphi) \). We see that the potential is invariant under

\[ \varphi \to \varphi + 2\pi i \mu^v \] (9.52)

where \( \mu^v \) is a co-weight of \( G \), i.e. \( \mu^v = \sum_n a_n \frac{2\pi}{\alpha_a} \), with \( \lambda_a \) being the fundamental weights of \( G \), satisfying \( \frac{2\lambda_a m_a}{\alpha_a} = \delta_{ab} \), and \( \alpha_a \) being the simple roots of \( G \). Therefore, \( \mu^v \cdot \alpha \) is an integer for any root \( \alpha \) of \( G \). We shall choose \( m = \mu_0 \sum_a \alpha_a m_a \) with \( m_a \in \mathbb{Z} \), and where \( \mu_0 \) is an overall scale parameter with the dimension of a mass.\(^14\) Under Eq. (9.52) we have \( m \cdot \varphi / \mu_0 \to m \cdot \varphi / \mu_0 + 2\pi i \mathbb{Z} \). The appropriate definition of the topological current is

\[ j^\mu = \frac{1}{2\pi i \mu_0} e^{i \mu v} \partial_\nu (m \cdot \varphi) . \] (9.53)

\(^14\) Notice that we are not assuming \( \tilde{\psi}^{(m)} = \psi^{(m)} \ast \).
\(^15\) We use the following normalization for the trace form, \( \text{Tr} (x \cdot H^a y \cdot H^{-a}) = x \cdot y \), and \( \text{Tr} (E^a E^{-a}) = \frac{2}{\pi \varphi} \).
\(^16\) Notice that \( \mu_0 \) and all \( m_a \)’s have to have the same sign in order for \( m \) to lie in the Fundamental Weyl Chamber (see (9.37))
Indeed, the vacua are infinitely degenerate and for a soliton solution the asymptotic values of \( \varphi \), at \( x = \pm \infty \), have to differ by values appearing on the right hand side of Eq.(9.52). Thus it follows from the argument just given that \( Q_{\text{topol}} = \int dx \partial \varphi \in \mathbb{Z} \). Concerning the Noether current, we have to take account of the fact that, with our field normalization, the free part of the Lagrangian has an overall factor \( k \), where \( k \) is the coupling constant. The correct definition of the Noether current is

\[
J_\mu = \frac{k}{\mu_0} \sum_{m=1}^{h-1} \sum_{\alpha(m)} \frac{2}{\alpha(m)^2} m_{\alpha(m)} \bar{\psi}_{\alpha(m)} \gamma_\mu \psi_{\alpha(m)} \equiv \frac{k}{\mu_0} \sum_{\alpha > 0} \frac{2}{\alpha^2} m_\alpha \bar{\psi}_\alpha \gamma_\mu \psi_\alpha
\]

where \( \alpha \) stands for any positive root of \( G \). Using the special form of \( m \) just introduced, we obtain

\[
J_\mu = \sum_{\alpha > 0} \sum_a \frac{2}{\alpha^2} \alpha_\alpha \, k \bar{\psi}_\alpha \gamma_\mu \psi_\alpha.
\] (9.55)

For each term \( N_\alpha \equiv \int \bar{\psi}_\alpha \gamma_0 \psi_\alpha \) is such that

\[
i \{ \psi_\alpha, N_\beta \}_{\text{P.B.}} = \delta_\alpha,\beta \psi_\alpha.
\]

Moreover, as is well known, \( \frac{2}{\alpha^2} \alpha_\alpha \in \mathbb{Z} \). Thus, the above definition of the Noether current is such that the Noether charge \( Q_{\text{Noether}} = \int dx J_0 \) has integer eigenvalues, as it should.

Next let us compare the two currents so defined. Clearly \( \text{Tr} (Q_{\text{ppal}} \mathcal{E}_0) = 2 \hat{\delta} \cdot \mathbf{m} \neq 0 \) (\( \hat{\delta} = \sum_{a=1}^{\lambda} \frac{\lambda_a}{\alpha_a} \)), since \( \mathbf{m} \) lies inside the FWC. Therefore (9.18) is satisfied, and one can always find one solution in each orbit of the conformal group such that (9.19) is true. Consequently, for such solutions we have

\[
i \sum_{\alpha > 0} \frac{2}{\alpha^2} m_\alpha \bar{\psi}_\alpha \gamma_\mu \psi_\alpha \equiv \epsilon^{\mu
u} \partial_\nu \left( \mathbf{m} \cdot \varphi \right)
\] (9.56)

Consequently the Noether and topological currents are proportional:

\[
j^\mu = \frac{1}{k\pi} J^\mu, \quad Q_{\text{topol}} = \frac{1}{k\pi} Q_{\text{Noether}}
\] (9.57)

This equation has an important consequence. It will certainly hold at the quantum level, after a suitable redefinition. Since the eigenvalues of the Noether charge will be integers, and, since the topological charge is also an integer, we obtain that \( k \pi \) should be rational. One may expect that this will lead to the correct statistics for the \( \psi \) fields, following the argument given at the end of the previous section. A quantification of \( k \) is of course expected, since our actions are related with the one of WZNW. On the other hand, one may regard eqs.(9.57) as classical versions of a formulae of the Atiyah Patodi Singer type (see e.g. ref.[31] for a review). In practice, Eqs.(9.57) mean that the Noether density is non zero only where \( \partial \varphi \neq 0 \), that is inside the solitons. Thus the \( \psi \) fields are confined inside the solitons. We shall come back to this on the simplest example below.

The soliton solutions are obtained as follows. The operators diagonalizing the adjoint action of \( E_{\pm h} \), given in (9.37), are

\[
V_\alpha (\zeta) \equiv \sum_{n \in \mathbb{Z}} \zeta^{-n} E_\alpha^n
\] (9.58)
for any root $\alpha$ (positive or not) of $G$. Indeed

$$[E_{\pm h}, V_{\alpha}(\zeta)] = \zeta^{\pm 1} (m \cdot \alpha) V_{\alpha}(\zeta)$$

(9.59)

Notice that $V_{\alpha}(\zeta)$ and $V_{-\alpha}(-\zeta)$ have the same eigenvalues, and it turns out that the one-soliton solutions are obtained from (5.16), by choosing the constant group element $\rho$ to be an exponentiation of (see discussion below (5.16))

$$V_{\alpha}\left(a^{\alpha}_\pm, \zeta \right) \equiv a^{\alpha}_\pm V_{\alpha}(\zeta) + a^{\alpha}_\mp V_{-\alpha}(-\zeta)$$

(9.60)

We then have a one-soliton solution (with two parameters $a^{\alpha}_\pm$) for each pair of Dirac fields $\psi_{\alpha}$ and $\tilde{\psi}_{\alpha}$. The masses of the solitons and Dirac particle are proportional to $m \cdot \alpha$.

The construction of the soliton solutions, as well as the physical properties of such models, are discussed in detail in section 10 for the case of $sl(2)$.

## 10 Example of the principal gradation for $sl(2)^{(1)}$

In this section we discuss two examples, for $l = 2$ and $l = 3$, associated with the principal gradation of the untwisted affine Kac-Moody algebra $sl(2)^{(1)}$. Let us denote by $H^n$, $E_{\pm}^n$, $D$ and $C$ the Chevalley basis generators of the $sl(2)^{(1)}$. The commutation relations are

$$[H^n, H^m] = 2mC \delta_{m+n,0},$$
$$[H^n, E_{\pm}^m] = \pm 2E_{\pm}^{m+n},$$
$$[E_+^m, E_-^n] = H^{m+n} + mC \delta_{m+n,0},$$
$$[D, T^m] = mT^m, \quad T^m \equiv H^m, E_{\pm}^m;$$

all other commutation relations are trivial. The grading operator for the principal gradation ($s = (1, 1)$) is

$$Q \equiv \frac{1}{2}H^0 + 2D.$$  

(10.5)

Then the eigensubspaces are

$$\hat{G}_0 = \{H^0, C, Q\};$$
$$\hat{G}_{2n+1} = \{E_+, E_{n+1}^-\} \quad n \in \mathbb{Z};$$
$$\hat{G}_{2n} = \{H^n\}, \quad n \in \{\mathbb{Z} - 0\}.$$  

(10.6)

The mapping $B$ is parametrised as

$$B = e^{\varphi H^0} e^{\tilde{\nu} C} e^{\eta Q} = e^{\varphi \tilde{H}^0} e^{\nu C} e^{\eta Q},$$

(10.7)

where $\tilde{H}^0 = H^0 - \frac{1}{2} C$ is the Cartan generator in the special basis introduced in (2.11), and so $\tilde{\nu} = \nu - \frac{1}{2} \varphi$.  

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10.1 Case $l = 2$

Consider the case $l = 2$, and choose

$$E_2 \equiv m H^1, \quad E_{-2} \equiv m H^{-1},$$

(10.8)

where $m$ is a constant. We then have

$$[E_2, [E_{-2}, E_{\pm}]] = 4m^2 E_{\pm}.$$  

(10.9)

Therefore, each of the subspaces $\hat{G}_{\pm 1}$ has two generators with the same eigenvalue $4m^2$.

Following (8.17) and (8.20) we write

$$F_1^+ = 2\sqrt{i}m \left( \psi_R E_+^0 + \bar{\psi}_R E_+^1 \right), \quad F_1^- = 2\sqrt{i}m \left( \psi_L E_-^1 - \bar{\psi}_L E_-^0 \right),$$

(10.10)

and introduce the Dirac fields

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}; \quad \bar{\psi} = \begin{pmatrix} \bar{\psi}_R \\ \bar{\psi}_L \end{pmatrix}$$

(10.11)

From (7.5) we obtain the masses of the particles,

$$m_\phi = m_\tilde{\nu} = m_\eta = 0; \quad m_\psi = 4m;$$

(10.12)

The equations of motion derived from (2.13)–(2.16), are

$$\partial^2 \varphi = -4m_\psi \bar{\psi} \gamma_5 e^{\eta + 2\varphi \gamma_5} \psi,$$

(10.13)

$$\partial^2 \tilde{\nu} = -2m_\psi \bar{\psi} (1 - \gamma_5) e^{\eta + 2\varphi \gamma_5} \psi - \frac{1}{2} m_\psi^2 e^{2\eta},$$

(10.14)

$$\partial^2 \eta = 0,$$

(10.15)

$$i \gamma^\mu \partial_\mu \psi = m_\psi e^{\eta + 2\varphi \gamma_5} \psi,$$

(10.16)

$$i \gamma^\mu \partial_\mu \bar{\psi} = m_\psi e^{-\eta - 2\varphi \gamma_5} \bar{\psi},$$

(10.17)

where the gamma matrices are defined in (8.23), and $\gamma_5 = \gamma_0 \gamma_1$, and $\bar{\psi} \equiv \psi^T \gamma_0$. Recall that $\partial^2 = \partial_t^2 - \partial_x^2$, $x_\pm = t \pm x$. The corresponding Lagrangian has the form

$$\frac{1}{k} \mathcal{L} = \frac{1}{4} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{4} \partial_\mu \varphi \partial^\mu \eta + \frac{1}{2} \partial_\mu \tilde{\nu} \partial^\mu \eta - \frac{1}{8} m_\psi^2 e^{2\eta}$$

$$+ i \bar{\psi} \gamma^\mu \partial_\mu \psi - m_\psi \bar{\psi} e^{\eta + 2\varphi \gamma_5} \psi.$$  

(10.18)

It is real (for $\eta$ = real constant) if $\bar{\psi}$ is the complex conjugate of $\psi$, and if $\varphi$ is pure imaginary. This will be true for the soliton solution as we shall see below.

Notice that such model is invariant under the transformations

$$x_+ \leftrightarrow x_-; \quad \psi_R \leftrightarrow \epsilon \bar{\psi}_L; \quad \bar{\psi}_R \leftrightarrow -\epsilon \psi_L; \quad \varphi \leftrightarrow \varphi; \quad \eta \leftrightarrow \eta; \quad \nu \leftrightarrow \nu$$

(10.19)

where $\epsilon = \pm 1$. It should be interpreted as the product CP of charge conjugation times parity. Parity alone is clearly violated.
The generator \(H^0 \in \hat{G}_0\) commutes with \(E_{\pm 2}\), and, therefore, the gauge symmetry (2.29)–(2.31) of the model is \(U(1)_L \otimes U(1)_R\),

\[
h_L(x_-) = e^{\xi^-(x_-)H^0}, \quad h_R(x_+) = e^{\xi^+(x_+)H^0}.
\]

Since the generators of \(U(1)_L\) and \(U(1)_R\) are the same, we have the global gauge transformations (2.33) generated by \(h_D \equiv h_L h_R^{-1} \equiv e^{i\theta H^0/2} (\theta = \text{const.})\). The fields are transformed as

\[
\psi \to e^{i\theta} \psi, \quad \bar{\psi} \to e^{-i\theta} \bar{\psi}, \quad \varphi \to \varphi, \quad \bar{\nu} \to \bar{\nu}, \quad \eta \to \eta;
\]

and the corresponding Noether current is

\[
J^\mu = \bar{\psi} \gamma^\mu \psi, \quad \partial_\mu J^\mu = 0.
\]

The fields \(\psi\) and \(\bar{\psi}\) have charges 1 and \(-1\), respectively; and \(\varphi, \bar{\nu}\) and \(\eta\) have charge zero.

Let us next see how the general arguments given above concerning Noether and topological charges apply here. The topological current and charges are

\[
j^\mu = \frac{1}{2\pi i} \epsilon^{\mu \nu} \partial_\nu \varphi, \quad Q_{\text{topol.}} \equiv \int dx j^0,
\]

Indeed, the Lagrangian (10.18) has infinitely many degenerate vacua due to the invariance under \(\varphi \to \varphi + i\pi\). Making use of the field equations, one easily verifies that

\[
\partial_- J = 0, \quad \partial_+ \bar{J} = 0.
\]

Combining this equation with the conservation of the vector current \(\bar{\psi} \gamma^\mu \psi\), one deduces that there exist two charges defined by

\[
\mathcal{J} = -i\bar{\psi}_R \gamma^\mu \psi_R + \frac{1}{2} \partial_+ \varphi, \quad \bar{\mathcal{J}} = i\bar{\psi}_L \gamma^\mu \psi_L + \frac{1}{2} \partial_- \varphi
\]

which satisfy \(\partial_- \mathcal{J} = 0, \partial_+ \bar{\mathcal{J}} = 0\). Applying the general argument of the previous section, we conclude that we may choose the solution so that Eqs.9.19 hold, so that \(\mathcal{J} = \bar{\mathcal{J}} = 0\). This gives, altogether,

\[
\frac{1}{2\pi i} \epsilon^{\mu \nu} \partial_\nu \varphi = \frac{1}{\pi} \bar{\psi} \gamma^\mu \psi,
\]

so that the topological and Noether currents are proportional. As discussed in section 9, that is a consequence of the fact that \(E_{\pm 2}\) satisfies (9.12).

Let us turn to the Noether charge which here is simply the fermion number. It should be defined such that it satisfies the Poisson bracket relation

\[
i \{ \psi, Q_{\text{Noether}} \}_{\text{P.B.}} = \psi
\]

Since the coupling constant \(k\) was taken as an overall factor, this is satisfied by

\[
Q_{\text{Noether}} = k \int dx \bar{\psi} \gamma^0 \psi
\]
so that

\[ Q_{\text{topol.}} = \frac{1}{k\pi} Q_{\text{Noether}} \] (10.28)

As argued in general, this means that \( k \) should only take discrete values as expected, since our actions are related with the one of WZNW.

Let us now construct the soliton solutions. The operators \( E_{\pm 2} \) given in (10.8), lie in the homogeneous Heisenberg subalgebra generated by \( H^n \), with the commutation relations (10.1). Such a subalgebra has no generators of grade \( \pm 1 \) for the principal gradation. Therefore, the model under consideration has no vacuum solutions of type (2.20). Then, from (4.1), we get

\[ E_{\pm} = E_{\pm 2} = m H^{\pm 1}. \] (10.29)

We perform the dressing transformation starting from the vacuum solution (2.18), namely

\[ \varphi = \eta = \psi = \tilde{\psi} = 0, \quad \tilde{\nu} = -\frac{1}{8} m^2 \dot{x}_+ x_- \equiv \nu_0. \] (10.30)

Now, let \( | \hat{\lambda}_0 \rangle \) and \( | \hat{\lambda}_1 \rangle \) be the highest weight states of two fundamental representations of the affine Kac–Moody algebra \( sl(2)^{(1)} \), respectively the scalar and spinor ones. Then, from (5.16) with \( \eta = 0 \), we obtain the solutions on the orbit of the vacuum (10.30),

\[ e^{-\varphi} = \frac{\langle \hat{\lambda}_1 | G | \hat{\lambda}_1 \rangle}{\langle \hat{\lambda}_0 | G | \hat{\lambda}_0 \rangle}, \quad e^{-(\tilde{\nu} - \nu_0)} = \langle \hat{\lambda}_0 | G | \hat{\lambda}_0 \rangle, \]

\[ \psi_R = \sqrt{\frac{m}{i}} \frac{\langle \hat{\lambda}_0 | E^1 G | \hat{\lambda}_0 \rangle}{\langle \hat{\lambda}_0 | G | \hat{\lambda}_0 \rangle}, \quad \tilde{\psi}_R = -\sqrt{\frac{m}{i}} \frac{\langle \hat{\lambda}_1 | E^0 G | \hat{\lambda}_1 \rangle}{\langle \hat{\lambda}_1 | G | \hat{\lambda}_1 \rangle} \]

\[ \psi_L = -\sqrt{\frac{m}{i}} \frac{\langle \hat{\lambda}_1 | G E^0 | \hat{\lambda}_1 \rangle}{\langle \hat{\lambda}_1 | G | \hat{\lambda}_1 \rangle}, \quad \tilde{\psi}_L = -\sqrt{\frac{m}{i}} \frac{\langle \hat{\lambda}_0 | G E^{-1} | \hat{\lambda}_0 \rangle}{\langle \hat{\lambda}_0 | G | \hat{\lambda}_0 \rangle}, \] (10.31)

where

\[ G \equiv e^{x_+ \varepsilon_+} e^{-x_- \varepsilon_-} \rho e^{x_- \varepsilon_-} e^{-x_+ \varepsilon_+}. \] (10.32)

In order to get the soliton solutions, we choose the fixed mapping \( \rho \) to be an exponentiation of an eigenvector of \( E_{\pm} \) (solitonic specialization); namely, \( \rho = e^V \), with \( [E_{\pm}, V] = \omega_{\pm} V \). Therefore,

\[ G = \exp (e^{\Gamma} V) \quad \text{with} \quad \Gamma = \omega_+ x_+ - \omega_- x_- \equiv \gamma (x - vt). \] (10.33)

In this case the eigenvectors of \( E_{\pm} \) are

\[ V_{\pm}(z) = \sum_{n \in \mathbb{Z}} z^{-n} E_{\pm}^n. \] (10.34)

Indeed,

\[ [E_+, V_{\pm}(z)] = \pm 2m z V_{\pm}(z) \equiv \omega_{\pm} V_{\pm}(z), \] (10.35)

\[ [E_-, V_{\pm}(z)] = \pm \frac{2m}{z} V_{\pm}(z) \equiv \omega_{\pm} V_{\pm}(z). \] (10.36)
The solution, associated with $V_+(z)$, is
\begin{equation}
\nu = \nu_0, \quad \varphi = \bar{\psi} = 0, \quad \psi = \sqrt{m_1} e^{\Gamma} \left( z \right); \tag{10.37}
\end{equation}
while those, associated with $V_-(z)$, is given by
\begin{equation}
\nu = \nu_0, \quad \varphi = \psi = 0, \quad \bar{\psi} = -\sqrt{m_1} e^{-\Gamma} \left( \frac{1}{1/z} \right), \tag{10.38}
\end{equation}
where
\begin{equation}
\Gamma = 2m(x_+ - \frac{1}{z}x_-) \equiv \gamma(x - vt). \tag{10.39}
\end{equation}
The masses of these solutions are obtained from (7.17). Here the relevant state $| \lambda' \rangle$ in (7.16) is
\begin{equation}
| \lambda' \rangle = | \hat{\lambda}_0 \rangle \otimes | \hat{\lambda}_1 \rangle. \tag{10.40}
\end{equation}
Using level one vertex operators, one can verify that
\begin{equation}
\langle \hat{\lambda}_i | (V_\pm(z))^n | \hat{\lambda}_i \rangle = 0, \quad \text{for } n \geq 1 \text{ and } i = 0, 1. \tag{10.41}
\end{equation}
Therefore, $N'_\nu = 0$ in (7.16), and from (7.17) one gets that the masses of the solutions (10.37) and (10.38) vanish. Such solutions correspond to the objects which travel with velocities $v = \pm (1 - z^2)/(1 + z^2)$; and keeping $z^2 > 0$, one has $| v | < 1$. Therefore, these solutions cannot be interpreted as solitons (particles), since they would correspond to massless particles traveling with velocity smaller than light velocity. We should interpret them as vacuum configurations, since they have the same energy as vacuum (10.30).

The true soliton solutions of the system are constructed as follows. Notice that $V_+(z)$ and $V_-(z)$ have the same eigenvalues. Therefore, any linear combination of them leads to solutions traveling with a constant velocity without dispersion. So, we let
\begin{equation}
V(a_{\pm}, z) \equiv \sqrt{i} (a_+ V_+(z) + a_- V_-(z)); \tag{10.42}
\end{equation}
\begin{equation}
[\mathcal{E}_+, V(a_{\pm}, z)] = 2mz V(a_{\pm}, z), \quad [\mathcal{E}_-, V(a_{\pm}, z)] = \frac{2m}{z} V(a_{\pm}, z), \tag{10.43}
\end{equation}
and so $\omega_+ = 2mz$ and $\omega_- = \frac{2m}{z}$. The particular factor $\sqrt{i}$ is chosen such that the reality condition will be obeyed with $a_- = a_+^*$. Again, using level one vertex operators, one can verify that
\begin{equation}
\langle \lambda_{\nu'} | V(a_{\pm}, z)^n | \lambda_{\nu'} \rangle = 0 \quad \text{for } n > 4. \tag{10.44}
\end{equation}
Therefore, $N'_\nu = 4$ in (7.16), and from (7.17) with $\psi^2 = 2$, and $\psi$ being the highest root of $sl(2)$, one gets that the mass of such solutions is
\begin{equation}
M = 8k m = 2k m \psi, \tag{10.45}
\end{equation}
\footnote{Notice that the truncation occurs for powers greater than 4, and not 2, because $| \lambda_{\nu'} \rangle$ lies in the tensor product representation, see (10.40).}
where \( k \) is the coupling constant appearing in the Lagrangian (10.18), see (6.12). The solutions generated by (10.42), have two parameters, namely \( a_\pm \). One parameter is always present, because one can scale an eigenvector of \( E_\pm \) without changing the width \( \gamma \) and velocity \( v \) of the soliton, obtained from the eigenvectors \( \omega_\pm \); see (5.19). However, in this case, the second parameter comes from a symmetry. As we have pointed out in (5.21), associated to the fixed element \( \rho = e^{V(a_\pm,z)} \), we have an orbit of equivalent solutions due to the global transformations (10.21),

\[
V(a_\pm, z) \to \pm \sqrt{i} \left( a_+ e^{i\theta} V_+(z) + a_- e^{-i\theta} V_-(z) \right). 
\] (10.46)

The explicit form of the solutions generated by (10.42), is obtained using (10.31),

\[
\varphi = \log \left( \frac{1 + i\sigma e^{2\Gamma}}{1 - i\sigma e^{2\Gamma}} \right), 
\] (10.47)

\[
\tilde{\nu} = -\log \left( 1 + i\sigma e^{2\Gamma} \right) - \frac{1}{8} m_\psi^2 x_+ x_-, 
\] (10.48)

\[
\eta = 0; 
\] (10.49)

and

\[
\psi = a_+ \sqrt{m} e^{\Gamma} \left( \frac{\frac{z}{1+i\sigma e^{2\Gamma}}}{1-i\sigma e^{2\Gamma}} \right), \quad \bar{\psi} = a_- \sqrt{m} e^{\Gamma} \left( \frac{\frac{z}{1-i\sigma e^{2\Gamma}}}{1+i\sigma e^{2\Gamma}} \right); 
\] (10.50)

where \( \Gamma \) is given in (10.39), and \( \sigma = a_+ a_- z / 4 \). Keeping \( m \) and \( z \) real, we have the mass \( M \) of the soliton, from (10.45), real and positive, and also the parameters \( \gamma \) and \( v \) (10.39) are real. The reality condition is obeyed if \( a_- = a^*_+ \), as anticipated. At this point, it is useful to re-express the expressions just given in terms of the physical parameters of the soliton. Using equations (8.23), and (10.12), one deduces that

\[
\gamma = m_\psi / \sqrt{1 - v^2}, \quad z = \sqrt{(1 - v)/(1 + v)}. 
\] (10.51)

Moreover, since \( a_\pm \) are complex conjugate, we may write

\[
a_\pm = e^{\pm i\theta} 2 \sqrt{\frac{\sigma}{z}}. 
\] (10.52)

The dependence upon space-time appears only through \( \sqrt{\sigma} \exp(\Gamma) \). We will write \( \sqrt{\sigma} \exp(\Gamma) \) where \( x_0 \) is the position of the soliton at time zero. Then we have

\[
\varphi = 2i \arctan \left( \exp \left( 2m_\psi (x - x_0 - vt) / \sqrt{1 - v^2} \right) \right), 
\] (10.54)

which is the sine–Gordon soliton. The Dirac fields are given by

\[
\psi = e^{i\theta} \sqrt{m_\psi} e^{m_\psi (x - x_0 - vt) / \sqrt{1 - v^2}} \left( \begin{array}{c} \frac{1-v}{1+v} \frac{1}{1+ie^{2m_\psi(x-x_0-\nu t)/\sqrt{1-v^2}}} \\ \frac{1+v}{1-v} \frac{1}{1-ie^{2m_\psi(x-x_0-\nu t)/\sqrt{1-v^2}}} \end{array} \right). 
\] (10.55)

\(^{18}\)by convention, we choose \( \sigma \) to be positive
and $\bar{\psi}$ is the complex conjugate of $\psi$. Thus the only parameters are the soliton mass and velocity, together with the angle $\theta$ which reflects the global invariance (10.21). Notice that the sign of the topological charge can be reversed by reversing the sign of $z$. Therefore, the solutions (10.54)–(10.55) contain the sine–Gordon soliton and anti–soliton.

Finally, we come to the very important feature of the present model already mentioned above in general, namely it is clear from the explicit expressions Eqs.10.55 that $\psi$ vanishes exponentially when $x - x_0 \to \pm \infty$, so that the Dirac field is confined inside the soliton. That this must be true is of course a general consequence of Eq.10.25 which may be verified directly on the explicit solution. This phenomenon has been much studied for electron phonon systems. Models of a similar type describe the electron self-localization in quasi-one-dimensional dielectrics (for recent reviews see [32], [33]). At low temperature these systems go over to dielectric states characterized by charge density waves which can be constructed on the basis of the Peierls model. The continuous limits are described by Lagrangians similar to Eq.10.18. Discussing this important issue is beyond the scope of the present article, so we will not dwell upon it here. Let us simply recall that the typical example of the polyacteline molecule was much discussed in connection with fermion number fractionization [34, 35]. Clearly, on the other hand one may regard our soliton solution a sort of one dimensional bag model for QCD. In this connection let us note that, if we introduce the two-by-two matrix $U = \exp(\eta + 2\varphi\gamma_5)$, we may rewrite the Lagrangian Eq.10.18 as

$$L = \frac{1}{16} \left\{ \text{tr} \left[ U^{-1} \partial_\mu U \frac{1}{2} \gamma_5 U^{-1} \partial^\mu U \right] - \frac{1}{2} \text{tr} \left[ U^{-1} \partial_\mu U \right] \text{tr} \left[ U^{-1} \partial^\mu U \right] \right\}$$

$$+ i \bar{\psi} \gamma_\mu \partial_\mu \psi - \bar{\psi} U \psi - \frac{m^2}{8} \det(U),$$

(10.56)

which is similar to a two-dimensional version of the low energy effective action for QCD (see e.g. [36]).

10.2 Case $l = 3$

In this case we choose

$$E_3 = q_+ E_+^1 + q_- E_-^1, \quad E_{-3} = q_+ E_-^2 + q_- E_-^1;$$

(10.57)

and so

$$[E_3, E_{-3}] = 3q_+ q_- C \equiv \beta C.$$

(10.58)

Introduce also the notations

$$E_1 = q_+ E_+^0 + q_- E_-^1, \quad E_{-1} = q_+ E_-^1 + q_- E_0^+, \quad f_1 = q_+ E_+^0 - q_- E_-^1, \quad f_{-1} = q_+ E_-^1 - q_- E_0^-, \quad f_2 = -\sqrt{q_+ q_-} H^1, \quad f_{-2} = -\sqrt{q_+ q_-} H^{-1}. $$

(10.59)

The fields of the model are those introduced in (10.7), and the matter fields $\psi^i_{R/L}, i = 1, 2,$ and $\chi_\pm$ defined as

$$F_1^+ = \psi_R^2 f_1 + \chi_+ E_1, \quad F_2^+ = \psi_R^1 f_2;$$

$$F_1^- = \psi_L^1 f_{-1} + \chi_- E_{-1}, \quad F_2^- = \psi_L^2 f_{-2}. $$

(10.60)
According to (8.21), $\psi^i_{R/L}$, $i = 1, 2$, are the components of two Dirac fields which we shall denote by $\psi^i$.

One can easily verify that

\[
[E_3, [E_{-3}, E_{\pm 1}]] = 0 \\
[E_3, [E_{-3}, H^0]] = 4q_+ q_+ \tilde{H}^0 \\
[E_3, [E_{-3}, f_i]] = 4q_+ q_- f_i, \ i = \pm 1, \pm 2. \tag{10.61}
\]

Therefore, from (7.3), the masses of the particles are

\[
m_\nu = m_\eta = m_\chi = 0, \quad m_\varphi = m_{\psi^i} = 4\sqrt{q_+ q_-}, \ i = 1, 2 \tag{10.62}
\]
and so we have to choose $q_\pm$, such that $q_+ q_- > 0$.

Then, from (2.13)-(2.16), the equations of motion are

\[
\partial_+ \partial_- \varphi = -q_+ q_- \left( (e^{2\varphi} - e^{-2\varphi}) (e^{3\eta} - e^\eta (\psi^1_R \psi^2_R - \chi_+ \chi_-)) - e^\eta (e^{2\varphi} + e^{-2\varphi}) (\psi^1_L \chi_+ - \psi^2_L \chi_-) \right), \tag{10.63}
\]

\[
\partial_+ \partial_- \nu = -q_+ q_- \left( (e^{2\varphi} + e^{-2\varphi}) \left( \frac{3}{2} e^{3\eta} - \frac{1}{2} e^\eta (\psi^1_L \psi^2_R - \chi_+ \chi_-) \right) - \frac{1}{2} e^\eta (e^{2\varphi} - e^{-2\varphi}) \left( \psi^1_L \chi_+ - \psi^2_L \chi_- \right) + 2 e^{2\eta} \psi^1_L \psi^1_R \right), \tag{10.64}
\]

\[
\partial_- \psi^1_R = \sqrt{q_+ q_-} e^\eta \left( \psi^1_L (e^{2\varphi} + e^{-2\varphi}) - \chi_- (e^{2\varphi} - e^{-2\varphi}) \right), \tag{10.65}
\]

\[
\partial_+ \psi^1_L = 2 \sqrt{q_+ q_-} \left( -e^{2\eta} \psi^1_R + 2 \frac{1}{2} e^\eta \psi^2_L \left( \psi^1_R (e^{2\varphi} - e^{-2\varphi}) + \chi_+ (e^{2\varphi} + e^{-2\varphi}) \right) \right), \tag{10.66}
\]

\[
\partial_- \psi^2_R = 2 \sqrt{q_+ q_-} \left( e^{2\eta} \psi^2_L + \frac{1}{2} e^\eta \psi^1_R \left( \psi^1_L (e^{2\varphi} - e^{-2\varphi}) - \chi_- (e^{2\varphi} + e^{-2\varphi}) \right) \right), \tag{10.67}
\]

\[
\partial_+ \psi^2_L = \sqrt{q_+ q_-} e^\eta \left( -\psi^2_R (e^{2\varphi} + e^{-2\varphi}) - \chi_+ (e^{2\varphi} - e^{-2\varphi}) \right), \tag{10.68}
\]

\[
\partial_- \chi_+ = \sqrt{q_+ q_-} e^\eta \psi^1_R \left( \chi_- (e^{2\varphi} - e^{-2\varphi}) - \psi^2_R (e^{2\varphi} + e^{-2\varphi}) \right), \tag{10.69}
\]

\[
\partial_+ \chi_- = \sqrt{q_+ q_-} e^\eta \psi^2_R \left( \chi_+ (e^{2\varphi} - e^{-2\varphi}) + \psi^1_R (e^{2\varphi} + e^{-2\varphi}) \right), \tag{10.70}
\]

\[
\partial_+ \partial_- \eta = 0. \tag{10.71}
\]

Notice that these equations are invariant under the CP transformation

\[
x_+ \leftrightarrow x_-, \ \psi^1_R \leftrightarrow \psi^2_L, \ \psi^2_R \leftrightarrow -\psi^1_L, \ \chi_+ \leftrightarrow \chi_-, \ \varphi \leftrightarrow \varphi, \ \nu \leftrightarrow \nu, \ \eta \leftrightarrow \eta. \tag{10.72}
\]

Since, there are no generators of $\mathcal{G}_0$, given in \([10.6]\), that commute with $E_{\pm 3}$ in \([10.57]\), we do not have any gauge symmetry of the type \([2.29]-[2.31]\). In the linear approximation for the $\psi$ and $\chi$ fields with $\varphi = 0$ and $\eta = \text{constant}$, one verifies that the $\psi$ field equations may be deduced from a Lagrangian of the type Eq. \([8.27]\), with $\psi^1 \leftrightarrow \psi$, and $\psi^2 \leftrightarrow \tilde{\psi}$. Such is not the case for the $\chi$ fields although they are massless in the linear approximation. Indeed, $\chi_{\pm}$ has weights \((\frac{3}{2}, 0)\) and \((0, \frac{3}{2})\) respectively, so that we cannot introduce a conjugate field to write down a covariant kinetic term. As a matter of fact, we have been unable to derive the above field equations from a local action.

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The sine–Gordon (or sinh–Gordon) model is a submodel of the system (10.63)-(10.71). Indeed, \( \psi^i = \chi_\pm = \eta = 0 \) is a solution of the equations of motion, and then \( i \varphi (\varphi) \) has to satisfy the sine–Gordon (sinh–Gordon) equation.

The operators \( E_{\pm 3} \) belong to the principal Heisenberg subalgebra of \( \mathfrak{sl}(2) \)

\[
[E_{2m+1}, E_{2n+1}] = q_+q_- (2m + 1) C \delta_{n+m+1,0}
\]

where

\[
E_{2m+1} \equiv q_+ E_+^m + q_- E_-^{m+1}, \quad m \in \mathbb{Z}
\]

They are eigenvectors of the grading operator (10.5)

\[
[Q, E_{2m+1}] = (2m + 1) E_{2m+1}
\]

The adjoint action of \( E_{2m+1} \) is diagonalized by the operators

\[
V(z) \equiv -\sqrt{q_+q_-} \sum_{n=-\infty}^{\infty} z^{-2n} H^n + \sum_{n=-\infty}^{\infty} z^{-2n-1} \left( q_+ E_+^n - q_- E_-^{n+1} \right) - \frac{1}{2} \sqrt{q_+q_-} C
\]

where

\[
[Q, V_n] = n V_n
\]

Indeed, one gets

\[
[E_{2m+1}, V(z)] = 2\sqrt{q_+q_-} z^{2m+1} V(z)
\]

Therefore, \( V(z) \) are eigenvectors of \( E_{\pm 3} \) with eigenvalues

\[
\omega_{\pm} = 2\sqrt{q_+q_-} z^{\pm 3}
\]

Notice that, if one shifts \( z \to \omega z \), with \( \omega^3 = 1 \), the eigenvalues do not change. Therefore

\[
V(a_j, z) \equiv a_0 V(z) + a_1 V(\omega z) + a_2 V(\omega^2 z)
\]

are also eigenvectors of \( E_{\pm 3} \). Therefore, if one performs the dressing transformations from the vacuum (2.18), namely

\[
\varphi = \eta = \chi_\pm = \psi^1 = \psi^2 = 0, \quad \nu = -3q_+q_- x_+ x_-
\]

one obtains solutions traveling with constant velocity, without dispersion, by taking the constant group element \( \rho \) in (5.16) as the exponentiation of (10.80) (see (5.19)).

The operator \( V(z) \) introduced in (10.76) can be realized through the principal vertex operator construction [37, 13]. Using such contruction, and taking into account that the highest weight state (10.40) lies in the tensor product representation, one gets

\[
\langle \lambda_{s'} | (V(a_j, z))^n | \lambda_{s'} \rangle = 0; \quad \text{for } n > 2, \text{ with just one non vanishing } a_j \text{'s (10.82)}
\]

\[
\langle \lambda_{s'} | (V(a_j, z))^n | \lambda_{s'} \rangle = 0; \quad \text{for } n > 4, \text{ with two non vanishing } a_j \text{'s (10.83)}
\]

\[
\langle \lambda_{s'} | (V(a_j, z))^n | \lambda_{s'} \rangle = 0; \quad \text{for } n > 6, \text{ with all three } a_j \text{'s, non vanishing (10.84)}
\]
Using the mass formula (7.17), one gets that the masses of the solitons created by the operator $V(a_j, z)$ in (10.80), from the vacuum (10.81) are

$$M_1 = \frac{8}{3} k \sqrt{q_+ q_-} = 2 k m_\varphi; \quad \text{with just one non vanishing } a_j \text{’s}$$

(10.85)

$$M_2 = \frac{16}{3} k \sqrt{q_+ q_-} = 4 k m_\varphi; \quad \text{with two non vanishing } a_j \text{’s}$$

(10.86)

$$M_3 = 8 k \sqrt{q_+ q_-} = 6 k m_\varphi; \quad \text{with all three } a_j \text{’s non vanishing}$$

(10.87)

Now, if one performs the dressing transformations from the vacuum (2.20), namely

$$\varphi = \eta = \psi^1 = \psi^2 = 0, \quad \chi_\pm = c_1^\pm, \quad \nu = -3q_+ q_- x_+ x_-$$

(10.88)

with $c_1^\pm$ = const., then the soliton type solutions are created by eigenvectors of

$$E_\pm \equiv E_{\pm 3} + c_1^\pm E_{\pm 1}$$

(10.89)

Again, $V(z)$ are eigenvectors of $E_\pm$ with eigenvalues

$$\omega_\pm = 2 \sqrt{q_+ q_-} \left( z^{\pm 3} + c_1^\pm z^{\pm 1} \right)$$

(10.90)

Therefore from (7.17) one gets that the mass of the corresponding soliton is ($N_V = 2$ from (10.82))

$$M' = \frac{8}{3} k \sqrt{q_+ q_-} (1 + c_1^+ c_1^- + c_1^+ z^2 + c_1^- / z^2)$$

(10.91)

The spectrum of the soliton solutions of such model is rather complicated. There remains to perform a more detailed analysis of those solutions, in order to understand, among other things, the physical consequences of the existence of (at least) two classical vacuum configurations.

11 Outlook

The models we introduced in this paper constitute a generalization of the affine Toda systems, in the sense that they contain matter fields coupled to the usual (gauge) Toda fields. We believe such models open the study of a variety of massive integrable theories with very interesting physical properties. As we have already mentioned, from the point of view of non perturbative aspects of quantum field theories, we hope these models will be useful, as a laboratory, in the understanding of the quantum theory of solitons, some confinement mechanisms and also to obtain exact results on the strong coupling regime. On the other hand, these systems can be used to describe very interesting phenomena in condensed matter physics and statistical mechanics like electron–phonon systems, electron self–localization in quasi-one-dimensional dielectrics and polyacetylene molecules.

The next step in achieving such program is to consider the quantum theory of these models, trying to obtain exact results by exploring some of their special physical properties. In this sense the most promising class of models is that described in section 9, which simplest example corresponds to the model associated to $sl(2)$, described in subsection 10.1.
That class of models possesses a $U(1)$ Noether current depending only on the matter fields which, under a special gauge fixing of the conformal symmetry, is equivalent to a topological current depending only on the (gauge) Toda fields. We believe the full consequences of such equivalence have not been understood yet and, the use of it, may shed light on several non perturbative aspects of the theory. We have already pointed out that such equivalence leads, at the classical level, to the localization of the matter fields inside the soliton. It is very plausible that, at the quantum level, this gives a confinement mechanism, which can be regarded as a one dimensional bag model for QCD. Another special aspect of these models concerns the mass formula. The masses of solitons and particles are both determined by the eigenvalues of the operators $E_{\pm l}$ appearing in the flat connection defining the models. In addition, the soliton masses have a topological character due to the spontaneous breakdown of the conformal symmetry. The models presenting the equivalence between Noether and topological currents have an additional feature; the masses of particles and solitons are proportional to the $U(1)$ Noether charge. This situation resembles very much the one of four dimensional gauge theories with Higgs in the adjoint representation and in the BPS limit. These facts may indicate the existence of a sort of duality in these models involving solitons and particles [30]. These points certainly deserve further study and we hope to explore them in a future work. They may have a direct connection with supersymmetric Yang-Mills theories along the line of ref.[1].

In addition to those, there are still many interesting practical problems to be solved following the lines of the present paper and references herein. We mention first the study the $W$–symmetries of the system under consideration, in particular, generalizations of the $W$–algebras of the standard Toda systems, $W$–geometries associated with them in the spirit of [38], [39], [23], including differential and algebraic geometry setting of the Toda systems coupled to the matter fields. For this questions, the formulation of the corresponding problem in terms of Lax operators of a Generalized mkdV hierarchy [40] should be useful. Second, it is very believable that the general solution for the matter fields can be presented in a compact form; in other words, it seems to be possible to resolve the recurrent formula (3.13). Third, it would be useful to obtain, in some suitable parametrisation for the mappings $B$ and $F^{\pm}$, an explicit expression for the Lagrange function or the effective action corresponding to our system (2.7)–(2.9). Having such an expression, one can directly get the formula for the energy–momentum tensor, and hence calculate in a more simple way the masses of solitons generated by the spontaneously breakdown of the conformal symmetry. In cases where this is not possible, one can work with the WZNW fields, in terms of which the matter fields can be written locally.

The construction of the multisoliton solutions can be made in a straightforward way using the methods presented in this paper. That would be important in the study of the classical scattering of the solitons, which could give us valuable information to construct the corresponding S–matrix, using a bootstrap programme [11], [42]. Here, the question of massless particles is of special importance [44].

Another point to be explored is the construction of the local conserved charges of these systems using for instance, the methods of refs. [15], [16], where the flat connection is gauge transformed into an abelian (Heisenberg) subalgebra of the affine Kac-Moody algebra $\hat{G}$. 

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