Stochastically Controlled Stochastic Gradient for the Convex and Non-convex Composition problem

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Abstract

In this paper, we consider the convex and non-convex composition problem with the structure $\frac{1}{n} \sum_{i=1}^{n} F_i(G(x))$, where $G(x) = \frac{1}{n} \sum_{j=1}^{n} G_j(x)$ is the inner function, and $F_i(\cdot)$ is the outer function. We explore the variance reduction based method to solve the composition optimization. Due to the fact that when the number of inner function and outer function are large, it is not reasonable to estimate them directly, thus we apply the stochastically controlled stochastic gradient (SCSG) method to estimate the gradient of the composition function and the value of the inner function. The query complexity of our proposed method for the convex and non-convex problem is equal to or better than the current method for the composition problem. Furthermore, we also present the mini-batch version of the proposed method, which has the improved the query complexity with related to the size of the mini-batch.

1. Introduction

In this paper, we study the problem of the following non-convex composition minimization

$$\min_{x \in \mathbb{R}^N} \left\{ f(x) \overset{\text{def}}{=} F(G(x)) \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} F_i \left( \frac{1}{n} \sum_{j=1}^{n} G_j(x) \right) \right\},$$

(1.1)

where $f: \mathbb{R}^N \to \mathbb{R}$ is a non-convex function, each $F_i: \mathbb{R}^M \to \mathbb{R}$ is a smooth function, each $G_j: \mathbb{R}^N \to \mathbb{R}^M$ is a mapping function, $n$ is the number of $F_i$’s and $G_j$’s. We call $G(x) = \frac{1}{n} \sum_{j=1}^{n} G_j(x)$ the inner function, and $F(w) = \frac{1}{n} \sum_{i=1}^{n} F_i(w)$ the outer function. There are many machine learning applications such as reinforcement learning [1, 2, 3] and nonlinear embedding [4, 5], that can be formed to the composition problem with two finite-sum structure $\frac{1}{n} \sum_{i=1}^{n} F_i \left( \frac{1}{n} \sum_{j=1}^{n} G_j(x) \right)$.

For example,

$$\min_{x} \| E[B]x - E[b] \|^2,$$

where $E[B] = I - \gamma P^\pi$, $\gamma \in (0, 1)$ is a discount factor, $P^\pi$ is the transition probability, $E[b] = r^\pi$, and $r^\pi$ is the expected state transition reward. Another example is the mean-variance in risk-averse learning:

$$\min_{x} \mathbb{E}_{a,b}[h(x; a, b)] + \lambda \text{Var}_{a,b}[h(x; a, b)],$$

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where $h(x; a, b)$ is the loss function with random variables $a$ and $b$. $\lambda > 0$ is a regularization parameter. Stochastic neighbour embedding (SNE) [4] is the non-convex problem that map data from a high dimensional space to a low dimensional space.

$$\min_x \sum_i \sum_t p_{i|t} \log \frac{p_{i|t}}{q_{i|t}},$$

where

$$p_{i|t} = \frac{\exp(-\|z_t - z_i\|^2/2\sigma_t^2)}{\sum_{j: j \neq i} \exp(-\|z_t - z_j\|^2/2\sigma_t^2)}, \quad q_{i|t} = \frac{\exp(-\|x_t - x_i\|^2)}{\sum_{j: j \neq i} \exp(-\|x_t - x_j\|^2)},$$

and $\sigma_t$ is the predefined parameter to control the sensitivity to the distance. $\{z_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^n$ denote the representation of $n$ data points in the high dimensional space and the low dimensional space, respectively.

Recently, there are many stochastic optimization methods solving the composition problem, such as stochastic gradient method [2, 3] and the variance-reduction based method [10, 11, 12]. However, there are two main problems encountered in the composition function: 1) the inner function $G(x)$ is the finite-sum structure. When the number of $G_i(x)$ is large, it will need more computation cost; 2) if the inner function $G(x)$ is estimated, the expectation of the stochastic gradient $f(x)$ with respect to $i_k, j_k \in [n]$ is not equal to the $\nabla f(x)$. That is

$$\mathbb{E}_{i_k, j_k}[(\partial G_{j_k}(x)) \nabla F_{i_k}(\hat{G}(x))] \neq \nabla f(x),$$

where $\hat{G}(x)$ is the estimation of $G(x)$, $\partial G_{j_k}$ is the partial gradient of $G_{j_k}(x)$. Furthermore, we use the query complexity to evaluate the algorithm, that is the number of component function queries used to compute the gradient.

Stochastic gradient method, such as Stochastic composition gradient descent (SCSG) [2] estimates the inner function $G(x)$ by an iterative weighted average of the past values of the $G(x)$, then perform the stochastic quasi-gradient iteration. The advantage of this method is that it does not depend on $n$ but with poor query complexity to the desired point. Variance-reduction method such as Compositional-SVRG [10] estimates the inner function $G(x)$ and the gradient of function $f(x)$ by using the finite-sum structures, which deriving the linear convergence rate with the relationship of $n$. Table 1 present the query complexity result with different algorithms.

Motivated by the recent work [13, 14, 15] that the convergence rate of the finite-sum structure function has the general result under the relationship between $n$ and $\varepsilon$. Here, we use $\varepsilon$ to evaluate the terminal of the convex and non-convex function by $f(x) - f(x^*) \leq \varepsilon$ and $\|\nabla f(x)\|^2 \leq \varepsilon$, respectively, where $x^*$ is the optimal point in the convex function. The core aspect of these kinds of algorithms is similar to the stochastic variance-reduced gradient (SVRG) that using a snapshot vector to compute the “gradient” of the function. The difference lies that the gradient is no longer computed directly but rather using the random subset, called stochastically controlled stochastic gradient (SCSG). We explore the SCSG based method to the composition problem with both convex and non-convex function and analyze the corresponding the convergence and query complexity.

In this paper, we develop a novel stochastic composition optimization through stochastically controlled stochastic gradient (SC-SCSG) method to two finite-sum structure. The main contributions are summarized below:

- We provide the variance reduction based method to estimate the inner function $G(x)$. Similar to the SCSG that estimate the gradient, the function $G(x)$ can also be estimated by a snapshot $\hat{x}_s$, in which $G(\hat{x}_s)$ is not computed directly, but rather based on the random subset from $[n]$. We also analyze the size of the subset such that can lead to the desired precision for both convex and non-convex function.
• After obtaining the estimated inner function, we consider the gradient of the function \( f(x) \). Here, we can also apply the SCSG based method to estimate the gradient. However, there are two situations encountered in the estimate process. 1) the expectation of the gradient is no longer the unbiased estimation. 2) the gradient of \( f(x) \) at the snapshot is formed by two random subsets, which are used for the function \( F_i \) and \( G_j \) respectively. Nevertheless, we also provide the bound of the subset size that we can use the estimated gradient to update the iteration. The details analysis can be referred to Section 4.

• The mini-batch version of the proposed algorithm is also provided for both the convex and non-convex function. The corresponding query complexities are improved based on the size of the mini-batch. More information can be referred to Section 6.

1.1. Results

We give the general query complexity of the composition problem based on SCSG based method. The results present us an intuitive explanation for comparing with other algorithms. Note that the Algorithm 1 can be used to both convex and non-convex problems that deriving the corresponding query complexities. Furthermore, Algorithm 2 present the mini-batch version of the proposed method.

Convex function The query complexity for the convex function is

\[
\mathcal{O}\left(\left(\min \left\{ n, \frac{1}{\mu^2} \right\} + \frac{L^2}{\mu^2} \min \left\{ n, \frac{1}{\mu^2} \right\} \right) \log \left(\frac{1}{\varepsilon}\right)\right),
\]

where \( \mu \) is the constant of strongly convex of \( f(x) \). The result is the same as that of [10] if \( n \leq 1/(\varepsilon \mu^2) \)

Non-convex function The query complexity is \( \mathcal{O}(\min \{1/\varepsilon^{9/5}, n^{4/5}/\varepsilon\}) \), which can be better than that of [3] and comparable to that of [12].

Mini-batch For the mini-batch version, the query complexity can be improved to some extent comparing with above results, that is \( \mathcal{O}(\min \{1/\varepsilon^{9/5}, n^{4/5}/\varepsilon\}/b^{1/5}) \) and

\[
\mathcal{O}\left(\left(\min \left\{ n, \frac{1}{\mu^2} \right\} + \frac{L^2}{\mu^2} \min \left\{ n, \frac{1}{\mu^2} \right\} \right) \log \left(\frac{1}{\varepsilon}\right)\right),
\]

for convex and non-convex function.

1.2. Related work

As the data increase, stochastic optimization has been the popular method in machine learning and deep learning, especially for the finite-sum function. The typical algorithm include (stochastic gradient descent) SGD [16], stochastic variance reduction gradient (SVRG)[17, 18], stochastic dual coordinate ascent (SDCA) [19, 20] and the accelerated method Nesterov’s method [21], accelerated randomized proximal coordinate (APCG) [22, 23] and Katyusha method [24]. As the function is finite-sum structure, the general process for optimization is randomly selected one or a block component function to estimate the gradient. Thus the estimated gradient leads to the large variance of the gradient. Variance reduction method estimates the gradient by using a snapshot in which the gradient of the function is computed at this point, which can appropriately reduce the variance.

The composition function can also be solved by using above algorithms, however, two finite-sum structures prevent implementation directly due to the fact that the computation of the inner function may increase the query complexity. Recently, Wang et al. [2] first proposed the first-order stochastic compositional gradient methods (SCGD) to solve such problems, which used two steps to alternately update the variable and inner function. The SCGD method has the query complexity \( \mathcal{O}(\varepsilon^{-7/2}) \) for the general function and \( \mathcal{O}(\varepsilon^{-5/4}) \) for the strongly convex function. Liu et al. [3] employed Nesterov’s method to accelerate the composition problem with \( \mathcal{O}(\varepsilon^{-5/4}) \) and \( \mathcal{O}(\varepsilon^{-9/4}) \) for strongly convex and non-convex function. However, these methods estimate the inner function by an iterative weighted average of the past function. Such estimation did not take advantage of the finite-sum structure.

Based on the variance reduction technology, Lian et al. [10] first applied the SVRG-based method to estimate the inner function \( G(x) \) and the gradient of the function \( f(x) \) as well. The linear convergence rate is obtained. In the following, Liu et. al [11] apply the duality-free method to the composition problem and derive the linear convergence rate as well. Yu and Huang [25] applied the ADMM-based [26] method and provide an analysis of the convex function without requiring Lipschitz smoothness. Moreover, Liu et. al [12] considered the non-convex function and analyzed the query complexity with different sizes of the inner function and outer function. The details of the query complexity are provided.
There are many recent papers considering the variance reduced method that estimates the gradient using the random subset rather than computing directly. Lei and Jordan [13] proposed an SCG method to the convex finite-sum function, and then applied to the non-convex problem in [14] that using less than a single pass to compute the gradient at the snapshot point. In the following, Allen-Zhu [15] also proposed Natasha1.5 algorithm, in which the gradient for each epoch is based on the random subset. Moreover, the objective function has the regularization term. Liu et al [27] applied the SCG based method to the zeroth-order optimization with the finite-sum function.

The rest of paper is organized as follows: In section 2, we give preliminaries used for analyzing the proposed algorithm. Section 3 presents the SCG-based method for the composition problem. We give the convergence and query complexity for the convex and non-convex function in Section 4 and Section 5, respectively. Section 6 gives the mini-batch version. We conclude our paper in Section 7.

2. Preliminaries

Throughout this paper, we use the Euclidean norm denoted by \( \| \cdot \| \). We use \( i \in [n] \) and \( j \in [m] \) to denote that \( i \) and \( j \) are generated from \([n] = \{1, 2, ..., n\} \) and \([m] = \{1, 2, ..., m\} \). We denote by \((\partial G(x))^{T}\nabla F(G(x))\) the full gradient of the function \( f \), \( \partial G(x) \) the partial gradient of \( G \), and \((\partial G_{jk}(x))^{T}\nabla F_{i}(G(x))\) as the stochastic gradient of the function \( f \), where \( i_{k} \) and \( j_{k} \) are randomly selected from \([n] \) and \([m] \). We use \( E \) to denote the expectation. Note that all the variable such as subset \( A \) and \( B \), element \( i_{k} \) and \( j_{k} \) are independently selected from \([n] \) or \([m] \), in particular, the element in \( A \) and \( B \) are independent. So we use \( E \) in instead of \( E_{i_{k}}, E_{j_{k}}, E_{A}, \) and \( E_{B} \) except particular stated. We use \( A = |A| \) to denote the number of the elements in the set \( D \) and define \( G_{A}(x) = \frac{1}{A} \sum_{1 \leq j \leq A} G_{\partial_{A}j}(x) \). Recall two definitions on Lipschitz function and smooth function.

**Definition 1.** A function \( p \) is called a Lipschitz function on \( \mathcal{X} \) if there is a constant \( B_{p} \) such that \( \|p(x) - p(y)\| \leq B_{p}\|x - y\| \), \( \forall x, y \in \mathcal{X} \).

**Definition 2.** A function \( p \) is called a \( L_{p} \)-smooth function on \( \mathcal{X} \) if there is a constant \( L_{p} \) such that \( \|\nabla p(x) - \nabla p(y)\| \leq L_{p}\|x - y\| \), and equal to \( p(x) \leq p(x) + \langle \nabla p(x), y - x \rangle + L_{p}/2\|y - x\|^{2} \), \( \forall x, y \in \mathcal{X} \).

We make the following assumptions used for the discussion of the convergence rate and complexity analysis.

**Assumption 1.** For function \( f: \mathbb{R}^{M} \rightarrow \mathbb{R} \), all \( i \in [n] \),

- \( f \) is \( \mu \)-strongly convex satisfying \( f(y) \geq f(x) + \langle f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|^{2} \).
- \( f \) has the optimal point \( x^{*} \), then \( \langle f(x_{k}), x^{*} - x_{k} \rangle \leq -\mu\|x_{k} - x^{*}\|^{2} \).

**Assumption 2.** For function \( G_{j}: \mathbb{R}^{N} \rightarrow \mathbb{R}^{M} \), all \( j \in [m] \),

- \( G_{j} \) has the bounded Jacobian with a constant \( B_{G_{j}} \), that is \( \|\partial G_{j}(x)\| \leq B_{G_{j}}, \forall x \in \mathbb{R}^{N} \), then \( G_{j}(x) \) is also a Lipschitz function that satisfying \( \|G_{j}(x) - G_{j}(y)\| \leq B_{G}\|x - y\| \), \( \forall x, y \in \mathbb{R}^{N} \).
- \( G_{j} \) is \( L_{G} \)-smooth satisfying \( \|\partial G_{j}(x) - \partial G_{j}(y)\| \leq L_{G}\|x - y\| \), \( \forall x, y \in \mathbb{R}^{N} \).

**Assumption 3.** For function \( F_{i}: \mathbb{R}^{M} \rightarrow \mathbb{R} \), all \( i \in [n] \),

- \( F_{i} \) has the bounded gradient with a constant \( B_{F_{i}} \), that is \( \|\nabla F_{i}(y)\| \leq B_{F_{i}}, \forall y \in \mathbb{R}^{M} \).
- \( F_{i} \) is \( L_{F} \)-smooth satisfying \( \|\nabla F_{i}(x) - \nabla F_{i}(y)\| \leq L_{F}\|x - y\| \), \( \forall x, y \in \mathbb{R}^{M} \).

**Assumption 4.** For function \( F_{i}(G(x)): \mathbb{R}^{N} \rightarrow \mathbb{R} \), all \( i \in [n] \), there exist a constant \( L_{F} \) satisfying

\[
\|\langle \partial G_{j}(x) \rangle^{T}\nabla F_{i}(G(x)) - \langle \partial G_{j}(y) \rangle^{T}\nabla F_{i}(G(y))\| \leq L_{F}\|x - y\|, \forall j \in [m], \forall x, y \in \mathbb{R}^{N}.
\]

**Assumption 5.** We assume that \( i_{k} \) and \( j_{k} \) are independently and randomly selected from \([n] \) and \([m] \), \( z \in \mathbb{R}^{M}, x \in \mathbb{R}^{N} \),

\[
E[\langle \partial G_{j_{k}}(x) \rangle^{T}\nabla F_{i_{k}}(z)] = \langle \partial G(x) \rangle^{T}\nabla F(z),
\]
3. Stochastic Composition via SCSG for the composition problem

In this section, we present the variance-reduction based method for the composition problem, which can be used for both the convex and non-convex function. Before describing the proposed algorithm, we recall the original SVRG [17]. The general process of the SVRG works as follows. The update process is divided into $S$ epochs, each of the epoch consists of $K$ iterations. At the beginning of each epoch, SVRG define a snapshot vector $\tilde{x}_s$, and then compute the full gradient $\nabla f(\tilde{x}_s)$. In the inner iteration of the current epoch, SVRG defines the estimated gradient by randomly selecting $i_k$ from $[n]$ at the $k$-th iteration,

$$
(\partial G(x_k))^T \nabla F_i (G_k) - (\partial G(\tilde{x}_s))^T \nabla F_i (G(\tilde{x}_s)) + \nabla f(\tilde{x}_s).
$$

However, for the composition problem, there are also variance-reduction based methods in [10], [11] and [12]. The difference with SVRG is that there is another estimated function for $G(x)$, as $G(x)$ is also the finite-sum structure. There methods defined the estimate function as

$$
\tilde{G}_k = G_A(x_k) - G_A(\tilde{x}_s) + G(\tilde{x}_s),
$$

where $A$ is the mini-batch formed by randomly sampling from $[n]$. Whereas, as the number of the inner function $G_j$ and the outer function $F_i$ increase, it is not reasonable to compute the full gradient of $f(x)$ and the full function $G(x)$ directly for each epoch.

Extended from the SCSG [14][13] and Natasha1.5 [15], we present a new algorithm for the composition problem as shown in Algorithm 1. First of all, we introduce the two subset $D_1$ and $D_2$, which are independent with each other and formed by

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**Algorithm 1 SC-SCSG for the composition problem**

**Require:** $K$, $S$, $\eta$ (learning rate), $\tilde{x}_0$ and $D = [D_1, D_2]$

for $s = 0, 1, 2, \ldots, S-1$ do

Sample from $[n]$ for $D$ times to form mini-batch $D_1$
Sample from $[n]$ for $D$ times to form mini-batch $D_2$

\[ \nabla f_D(\tilde{x}_s) = (\partial G_{D_1}(\tilde{x}_s))^T \nabla F_{D_2}(G_{D_1}(\tilde{x}_s)) \] $	riangleright$ D Queries

$x_0 = \tilde{x}_s$

for $k = 0, 1, 2, \ldots, K-1$ do

Sample from $[n]$ to form mini-batch $A$

$G_k = G_A(x_k) - G_A(\tilde{x}_s) + G_{D_1}(\tilde{x}_s)$

$\triangleright$ A Queries

Uniformly and randomly pick $i_k$ and $j_k$ from $[n]$

Compute the estimated gradient $\nabla f_k$ from (3.4)

$x_{k+1} = x_k - \eta \nabla f_k$

\[ x_{s+1} = x_K \]

end for

end for

**Output:** $\tilde{x}_s$ is uniformly and randomly chosen from $s \in \{0, \ldots, S-1\}$ and $k \in \{0, \ldots, K-1\}$.

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**Assumption 6.** We assume that $H_1$ and $H_2$ are the upper bounds on the variance of the functions $G(x)$ and $(\partial G(x))^T \nabla F(y)$, respectively, that is,

\[
\frac{1}{n} \sum_{i=1}^{n} \| G(x) - G_i(x) \|^2 \leq H_1.
\]

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\| (\partial G(x))^T \nabla F_i(y) - (\partial G_j(x))^T \nabla F_i(y) \right\|^2 \leq H_2.
\]

In the paper, we denote by $x^*_k$ the $k$-th inner iteration at $s$-th epoch. But in each epoch analysis, we drop the superscript $s$ and denote by $x_k$ for $x^*_k$. We let $x^*$ be the optimal solution of $f(x)$. Throughout the convergence analysis, we use $O(\cdot)$ notation to avoid many constants, such as $B_F$, $B_G$, $L_F$, $L_G$ and $L_f$, ... that are irrelevant with the convergence rate and provide insights to analyze the iteration and query complexity.
randomly selecting from \([n]\), respectively. We define \(D = [D_1, D_2]\) for a new variable. \(D_1\) is used for estimating the inner function. Based on the variance reduction technology, the estimated inner function at \(k\)-th iteration of \(s\)-th epoch is
\[
\hat{G}_k = G_A(x_k) - G_A(\bar{x}_s) + G_{D_1}(\bar{x}_s),
\]
where the subset of \(A\) is the same as in (3.2). Note that \(A\) and \(D\) are independent with each other. The difference with (3.2) is computing the third term that is under the subset \(D_1\) rather than \([n]\) as in (3.2). Throughout the paper, we assume that \(|A| \leq |D|\). \(D_2\) is used to estimate the outer function \(F\). The key distinguish with \([14, 13, 15]\) is the biased full gradient of \(f(\bar{x}_s)\). We define this estimated full gradient of \(f(\bar{x}_s)\) for each epoch as \(\nabla \tilde{f}_D(\bar{x}_s) = (\partial G_{D_1}(\bar{x}_s))^T \nabla F_{D_2}(G_{D_1}(\bar{x}_s))\). However, \(\mathbb{E}_{A,D}[\nabla \tilde{f}_D(\bar{x}_s)] \neq \nabla f(\bar{x}_s)\). Then, we estimate the gradient of the \(f(x_k)\) by
\[
\nabla \tilde{f}_k = (\partial G_{j_k}(x_k))^T \nabla F_{i_k}(\hat{G}_k) - (\partial G_{j_k}(\bar{x}_s))^T \nabla F_{i_k}(G_{D_1}(\bar{x}_s)) + \nabla \tilde{f}_D(\bar{x}_s),
\]
where \(i_k\) and \(j_k\) are randomly selected from \([n]\) at the \(k\)-th iteration for function \(F\) and \(G\), respectively. Furthermore, \(\mathbb{E}_{i_k,j_k,A,D}[\nabla \tilde{f}_k] \neq \nabla f(x_k)\) as well. This gives us more discussion about the upper bound with respect to the estimated function and the new random subset \(D\).

### 3.1. Technical Tool

For the subset \(A \subseteq [n]\), we present the following lemma that the variance of a random variable decreases by a factor \(|A|\) if we choose \(|A|\) independent element from \([n]\) and average them. The proof process is trivial and can be referred to Appendix. However, it presents an important tool for analyzing the querying complexity under the different size of the subset.

**Lemma 1.** If \(v_1, ..., v_m \in \mathbb{R}^d\) satisfy \(\sum_{i=1}^m v_i = 0\), and \(A\) is a non-empty, uniform random subset of \([n]\), \(A = |A|\), then
\[
\mathbb{E}_A \left\| \frac{1}{A} \sum_{b \in A} v_b \right\|^2 \leq \frac{1}{m} \sum_{i=1}^m v_i^2.
\]
Furthermore, if the elements in \(A\) are independent, then
\[
\mathbb{E}_A \left\| \frac{1}{A} \sum_{b \in A} v_b \right\|^2 = \frac{1}{m} \sum_{i=1}^n v_i^2.
\]

Based on Lemma 1, we can obtain the inequality with two-variables \(D_1\) and \(D_2\), which are used for the gradient of \(f(x)\) with the partial gradient \(\partial G(x)\).

**Lemma 2.** If \(w_1, ..., w_n \in \mathbb{R}^{M \times N}\) and \(v_1, ..., v_n \in \mathbb{R}^M\) satisfy \((\frac{1}{n} \sum_{i \in [n]} w_i)^T (\frac{1}{n} \sum_{j \in [n]} v_j) = w^T \tilde{v},\) and \(D = [D_1, D_2]\) is a non-empty, uniform random subset consist of \(D_1\) and \(D_2\), which are independently and uniformly selected from \([n]\), \(D = |D_1| = |D_2|\), then
\[
\mathbb{E}_D \left\| \frac{1}{|D_1||D_2|} \left( \sum_{d_1 \in D_1} w_{d_1} \right)^T \left( \sum_{d_2 \in D_2} v_{d_2} \right) - \tilde{w} \tilde{v} \right\|^2 = \mathbb{E}_D \left\| \frac{1}{D^2} \left( \sum_{|d_1, d_2| \in D} \left((w_{d_1})^T v_{d_2} - \tilde{w}^T \tilde{v}) \right) \right\|^2 \leq \frac{1}{D^2} \frac{1}{n^2} \sum_{i,j=1}^n \left((w_{i})^T v_{j} - \tilde{w}^T \tilde{v}) \right)^2.
\]

### 3.2. Bounds analysis of the estimated function and the gradient

Here, we mainly give different kinds of bounds for the proposed algorithm, such as \(\mathbb{E}_{A,D_1} \| \hat{G}_k - G(x_k) \|^2, \mathbb{E}_{A,D} \| E_{i_k,j_k} \nabla \tilde{f}_k - \nabla f(x_k) \|^2.\) These bounds will be used to analyze the convergence rate and query complexity. We assume that these bounds are all base on Assumption 2-6. Parameters such as \(B_G, B_F, L_G, L_F\) and \(L_f\) in the bound are all from those Assumptions. We do not define the exact value of parameters such as \(h, A\) and \(D\), which have great influence on the convergence and will be clearly defined in the query analysis. Our proposed bound are similar to that of \([10, 11]\) and \([12]\), but, the difference lies that there is an extra subset \(D\), which shows an interesting phenomenon. That is when the subset \(D\) is equal to the \([n]\), the corresponding bounds are the same as in \([10, 11]\) and \([12]\). However, it is the independent subset \(D\) that gives more general query complexity result for the problem (1.1). The following bounds are all used for the composition problem for both convex and non-convex problem based on the Lemma 1 and Lemma 2. The more details of the proof can be referred to Appendix. For simplicity, we drop the superscript \(i_k, j_k, A\) and \(D\) for the expectation with \(E\) in the proof.
Lemma 3. Suppose Assumption 2 and 6 holds, for $\hat{G}_k$ defined in (3.3) with $D = |D_1|$ and $A = |A|$, we have

$$\mathbb{E}_{A,D_1}||\hat{G}_k - G(x_k)||^2 \leq 4 \left( \frac{\mathbb{I}(A < n)}{A} + \frac{\mathbb{I}(D < n)}{D} \right) B_G^2 \mathbb{E}||x_k - \tilde{x}_s||^2 + 2 \frac{\mathbb{I}(D < n)}{D} H_1.$$  

Lemma 4. Suppose Assumption 2, 3, 5 and 6 holds, for $\hat{G}_k$ defined in (3.3) and $\nabla \hat{f}_k$ defined in (3.4) with $D = |D_1, D_2|$ and $D = |D_1| = |D_2|$, we have

$$\mathbb{E}_{A,D}||\nabla \hat{f}_k - \nabla f(x_k)||^2 \leq 4 B_G^2 L_P^2 \left( \frac{\mathbb{I}(A < n)}{A} + \frac{\mathbb{I}(D < n)}{D} \right) \mathbb{E}||x_k - \tilde{x}_s||^2 + 16 B_G^2 L_P^2 \frac{\mathbb{I}(D < n)}{D} H_1 + 4 \frac{\mathbb{I}(D^2 < n^2)}{D^2} H_2.$$  

Lemma 5. Suppose Assumption 2-6 holds, for $\hat{G}_k$ defined in (3.3) and $\nabla \hat{f}_k$ defined in (3.4) with $D = |D_1, D_2|$ and $D = |D_1| = |D_2|$, we have

$$\mathbb{E}_{i_k,j_k,A,D}||\nabla \hat{f}_k - \nabla f(x_k)||^2 \leq 5 B_G^2 L_P \frac{L_2}{B_G^2 L_P} + 4 \frac{\mathbb{I}(A < n)}{A} + 4 \frac{\mathbb{I}(D < n)}{D} \mathbb{E}||x_k - \tilde{x}_s||^2 + 20 B_G^2 L_P^2 \frac{\mathbb{I}(D < n)}{D} H_1 + 5 \frac{\mathbb{I}(D^2 < n^2)}{D^2} H_2.$$  

As can be seen from the above results directly, when $A$ and $D$ increase, the upper bounds are more approximating the related bounds as in [10, 11, 12]. Though there are extra terms with respect to $A$ and $D$, it gives us another direction for analyzing the convergence rate and query complexity. As the convergence rate not only depends on the convergence sequence, but also the terms including the event function $\mathbb{I}$. Thus, we can obtain the lower bound range of $A$ and $D$ that is related to $\varepsilon$. The result in Lemma 4 and 5 are similar except the extra term $L_2^2 \mathbb{E}||x_k - \tilde{x}_s||^2$. This is due to the fact that the order of the expectation is different. This difference derives from the proof process by using the smoothness of the function $f(x)$ and the update of $x$ in Algorithm 1. Furthermore, these two lemmas can be both applied to analyze the convergence rate and query complexity of the convex and non-convex composition problem.

4. Stochastic Composition via SCSG method for the Convex Composition problem

In this section, we analyze the proposed method for the convex composition problem. We first present the convergence of the proposed algorithm, and then give the query complexity. Thought the proof is similar to that of [10] and [28], we present a more clear and simple process as there is an extra step deriving from the subset $D$. In order to ensure the convergence of the proposed algorithm, we obtain the desired parameters’ setting, such as $A, D, K, \eta$ and $h$. Based on the setting, we can obtain the corresponding query complexity, which is better than or equal to the SVRG-based method in [10] and [11]. This is in fact that the event function $\mathbb{I}$ has the influence on the size of $A$ and $D$.

4.1. Convergence analysis for the convex problem

Based on the strong convex and smoothness of the function of $f(x)$, we provide the convergence sequence, in which the parameters are not defined. But the sequences motivate us to consider the parameters’ setting such that lead to the desired convergence rate.

Theorem 1. Suppose Assumption 1-6 holds, in Algorithm 1, let $h > 0, \eta > 0, A = |A|, D = |D_1| = |D_2|$, $K$ is the number of the inner iteration, $x^*$ is the optimal point, we have

$$\rho_1 \mathbb{E}||x_{s+1} - x^*||^2 \leq \left( \frac{1}{K} + \rho_2 \right) \mathbb{E}||\tilde{x}_s - x^*||^2 + \rho_3,$$

where

$$V = B_G^2 L_P^2 \left( \frac{\mathbb{I}(A < n)}{A} + \frac{\mathbb{I}(D < n)}{D} \right),$$

$$V_1 = 20 B_G^2 L_P^2 \frac{\mathbb{I}(D < n)}{D} H_1 + \frac{\mathbb{I}(D^2 < n^2)}{D^2} H_2,$$
\[
\rho_1 = \left(2\mu - h - 4V \frac{1}{h} - (12L_f^2 + 10V) \eta \right) \eta, \\
\rho_2 = 2 \left(2V \frac{1}{h} + 5 \left(L_f^2 + V \right) \eta \right) \eta, \\
\rho_3 = \frac{1}{\eta} \frac{4}{5} V_1 + 2\eta^2 V_1.
\] (4.3) (4.4) (4.5)

We do not give the convergence form for the update of iteration as we do not sure whether \(\rho_1\) is positive or not. Based on above equality in Lemma 1, we assume that \(\rho_1 > 0\) in (4.3), then we can obtain
\[
\mathbb{E}[\|\tilde{x}_S - x^*\|^2] \leq \rho S \mathbb{E}[\|\tilde{x}_0 - x^*\|^2] + \sum_{s=0}^{S} \rho^s \\
\leq \rho S \mathbb{E}[\|\tilde{x}_0 - x^*\|^2] + \frac{\rho_3 (1 - \rho^S)}{\rho_1 (1 - \rho)}.
\] (4.6)

where \(\rho = \left(\frac{1}{\mu} + \rho_2\right)/\rho_1\), \(\rho_2\) and \(\rho_3\) defined in (4.4) and (4.5), the last inequality is based on the formula of geometric progression. Thus, if the \(\tilde{x}_S\) converge to the optimal point \(x^*\), we need to require that \(\rho < 1\) and the second term \(\rho_3(1 - \rho^S)/(\rho_1(1 - \rho))\) is less than \(\varepsilon/2\). Actually, if \(D = n\), the second term is equal to zero satisfying the requirement directly, which is similar to the convergence results in [10] and [11].

### 4.2. Query complexity analysis for the convex problem

Based on the above result in (4.6), we analyze the query complexity. Furthermore, we also present the parameters’ setting, and then derived the query complexity, in which the details information can be referred to the Appendix.

**Corollary 1.** Suppose Assumption 1-6 holds, in Algorithm 1, let \(h = \mu\), the step size is \(\eta \leq \mu/(135L_f^2)\), the subset size of \(A\) is \(A = \min\{n, 128B_C^2L_f^2/\mu^2\}\), the subset size of \(D_1\) and \(D_2\) are both \(D = \min\{n, 5(16C^4L_f^2H_1 + 4H_2)/(4\mu^2)\}\), the number of the inner iteration is \(K \geq 540L_f^2/\mu^2\), the number of outer iteration is \(S \geq 1/(\log(1/\rho))\log(2E[\|\tilde{x}_0 - x^*\|^2]/\varepsilon)\).

The query complexity is
\[
(D + KA)S = \mathcal{O} \left( \left( \min\left\{ n, \frac{1}{\varepsilon \mu^2} \right\}, \frac{L_f^2}{\mu^2} \min\left\{ n, \frac{1}{\mu^2} \right\} \right) \log(1/\varepsilon) \right).
\]

As can be seen from the above result, Corollary 1 present the general query complexity under different parameters. Comparing \(n\) with corresponding parameters, we analyze the query complexity separately. We remove the parameters such as \(B_C^2, L_f^2, H_1, H_2\), and analyze the size with the order of \(1/\mu\). Though the comparison is not exactly correct, we present the results to illustrate the corresponding different algorithms. We can directly obtain that \(1/\mu^2 < 1/(\varepsilon \mu^2)\). We consider three situations comparing with \(n\), that is to present the value of the \textit{min} function,

- \(\frac{1}{\mu^2} \leq \frac{1}{\varepsilon \mu^2} \leq n\). When \(n\) is large enough such that we can obtain the query complexity is \(\mathcal{O}((1/(\varepsilon \mu^2) + L_f^2/\mu^4) \log(1/\varepsilon))\).
  
  This result avoids the situation that computing the full gradient of \(f(x)\) and the full function \(G(x)\) for the large-scale number of \(n\). What’s more, this result is better than Compositional-SVRG [10] and [11].

- \(\frac{1}{\mu^2} \leq n \leq \frac{1}{\varepsilon \mu^2}\). When \(n\) is smaller than \(1/(\varepsilon \mu^2)\), the query complexity becomes \(\mathcal{O}((n + L_f^2/\mu^4) \log(1/\varepsilon))\), which is the same as Compositional-SVRG [10]. That is we need to compute the full gradient of \(\nabla f(\tilde{x}_s)\) as in (3.1). The estimation of inner function \(G(x)\) is the same as in [10].

- \(n \leq \frac{1}{\mu^2} \leq \frac{1}{\varepsilon \mu^2}\). When \(n\) is small, the query complexity becomes \(\mathcal{O}((n + L_f^2n/\mu^2) \log(1/\varepsilon))\). The result has the similar form to SVRG [17]. This also gives us the intuition that the inner function should be computed directly rather than estimated.

### 5. Stochastic Composition via SCSG method for the Non-convex composition problem

In this section, we give the analysis of the convergence analysis and the query complexity under the proposed algorithm for the non-convex composition. We first present the new reformed sequence with respect to \(E[f(x_k)] + c_k E\|x_k - \tilde{x}_s\|^2\), in which the parameters are not well defined. Then, we sum-up this sequence based on the SVRG-based on the framework, in which there is a snapshot point \(\tilde{x}_s\) in each epoch. The last not least, we present the query complexity analysis and derive the optimal parameters’ setting such that improve the query complexity.
5.1. Convergence analysis for the non-convex problem

We first present the new form sequence under the Lyapunov function based on the smoothness of \(f(x)\) and the update of \(x\). The new parameters such as \(c_k\), \(u_k\) and \(J_k\) will be used to form sequence such that we can obtain the convergence sequence.

**Lemma 6.** Suppose Assumption 2-6 holds, in Algorithm 1, we can obtain the following new sequence with respect to \(f(x_k)\) and \(||x_k - \tilde{x}_s||^2\), let \(h > 0, \eta > 0, A = |A|\) and \(D = |D_1| = |D_2|\), we have

\[
\mathbb{E}[f(x_{k+1})] + c_{k+1} \mathbb{E}[\|x_{k+1} - \tilde{x}_s\|^2] \leq \mathbb{E}[f(x_k)] + c_k \mathbb{E}[\|x_k - \tilde{x}_s\|^2] - u_k \|\nabla f(x_k)\|^2 + J_k,
\]

where

\[
W = B_0^2 L_F^2 \left( 4 \frac{1(A < n)}{A} + 4 \frac{1(D < n)}{D} \right),
\]

\[
c_k = c_{k+1} \left( 1 + \left( \frac{2}{h} + 4hW \right) \eta + 10 \left( L_f^2 + W \right) \eta^2 \right) + 2W \eta + 5(L_f^2 + W) L_f \eta^2,
\]

\[
u_k = \left( \frac{1}{2} - hc_{k+1} \right) \eta - (L_f + 2c_{k+1}) \eta^2,
\]

\[
W_1 = 20B_0^2 L_F^2 \frac{1(D < n)}{D} H_1 + 5 \frac{1(D^2 < n^2)}{D^2} H_2,
\]

\[
J_k = \left( \frac{1}{2} + hc_{k+1} \right) \frac{4}{5} W_1 \eta + (L_f + 2c_{k+1}) W_1 \eta^2.
\]

Based on the above inequality with respect to the sequence \(\mathbb{E}[f(x_k)] + c_k \mathbb{E}[\|x_k - \tilde{x}_s\|^2\) and Algorithm 1, we can obtain the convergence form in which the parameters are not clearly defined.

**Theorem 2.** Suppose Assumption 2-6 holds, in Algorithm 1, we can obtain the following new sequence with respect to \(f(x_k)\) and \(||x_k - \tilde{x}_s||^2\). \(K\) is the number of inner iterations, \(S\) is the number of inner iterations, we have

\[
u_0 \mathbb{E}[\|\nabla f(\hat{x}_k^\ast)\|^2] \leq \frac{f(x_0) - f(x^*\text{)}\}}{KS} + J_0,
\]

where \(\hat{x}_k\) is the output point, \(J_0\) and \(u_0\) are defined in (5.5) and (5.3).

5.2. Query complexity analysis for the non-convex problem

Consider the convergence form above, we actually can’t obtain the convergence rate if the parameter \(u_k\) in (5.3) is negative. Furthermore, there is extra term \(J_0\) derived from the subset \(D\). We need to consider the size of the subset \(D\) such that we can keep the \(J_0\) under our desired degree of accuracy \(\epsilon\). What’s more, the parameter \(c_k\) in (5.2) is not a constant, which has a relationship with \(K\) and \(\eta\). Based on these influence element, we consider the parameters’ setting and give the query complexity.

**Corollary 2.** Suppose Assumption 2-6 holds, in Algorithm 1, for the step \(\eta > 0\), by setting \(h = \sqrt{1/\eta}\), the set-size of the subset \(D_1\) and \(D_2\) are \(D = \min \{n, O(1/\epsilon)\}\), the set-size of \(A\) is \(A = \min \{n, O(1/\eta)\}\), the number of inner iteration is \(K \leq O \left( 1/\eta^{3/2} \right)\), the total number of iteration is \(T = O \left( 1/ (\epsilon \eta) \right)\), then we can obtain \(\mathbb{E}[\|\nabla f(\hat{x}_k^\ast)\|^2] \leq \epsilon\).

The above corollary gives the parameters’ setting except the step \(\eta\), note that the outer number of iteration \(S\) has the relationship with \(T\) and \(K\), that is \(T = SK\). Here, we present the optimal step \(\eta\) such that we reach the improved the query complexity.

**Corollary 3.** Suppose Assumption 2-6 holds, in Algorithm 1, the step size is \(\eta = \min \{1/n^{2/5}, \epsilon^{2/5}\}\), then the query complexity is

\[
O \left( \min \left\{ \frac{1}{\epsilon \eta}, \frac{n^{4/5}}{\epsilon} \right\} \right),
\]

**Proof.** Based on the parameters’ setting, that is \(D = \min \{n, O(1/\epsilon)\}\), \(A = \min \{n, O(1/\eta)\}\), \(K \leq O \left( 1/\eta^{3/2} \right)\), and \(T = O \left( 1/ (\epsilon \eta) \right)\), we have,

\[
O \left( \frac{T}{K} (D + KA) \right) = O \left( \frac{1}{\epsilon \eta} \left( \frac{D}{K} + A \right) \right)
\]
Algorithm 2 Mini-batch version of SC-SCSG for the composition problem

Require: \( K, S, \eta \) (learning rate), \( \hat{x}_0 \) and \( D = |D_1, D_2| \)
for \( s = 0, 1, 2, \ldots, S - 1 \) do
    Sample from \([n]\) for \( D \) times to form mini-batch \( D_1 \)
    Sample from \([n]\) for \( D \) times to form mini-batch \( D_2 \)
    \( \nabla f_D(\hat{x}_s) = (\partial G_{D_1}(\hat{x}_s))^T \nabla F_{D_2}(G_{D_1}(\hat{x}_s)) \)  \( \triangleright \) \( D \) Queries
    for \( k = 0, 1, 2, \ldots, K - 1 \) do
        Sample from \([m]\) to form mini-batch \( A \)
        \( \tilde{G}_k = G_A(x_k) - G_A(\hat{x}_s) + G_{D_1}(\hat{x}_s) \)
        \( \Lambda_0 = 0 \)
        for \( t = 1, \ldots, b \) do
            Uniformly and randomly pick \( i_k \) and \( j_k \) from \([n]\)
            Compute the estimated gradient \( \nabla f_k \) from (3.4)  \( \triangleright \) \( 4 \) Queries
            \( \Lambda_{t+1} = \Lambda_t + \nabla f_k \)
        end for
        \( \Lambda = \Lambda_{b}/b \)
        \( x_{k+1} = x_k - \eta \Lambda \)
    end for
    Update \( \hat{x}_{s+1} = x_K \)
end for
Output: \( \hat{x}_k \) is uniformly and randomly chosen from \( s \in \{0, \ldots, S - 1\} \) and \( k \in \{0, \ldots, K - 1\} \).

\[
= \mathcal{O} \left( \frac{1}{\varepsilon \eta} \left( \min \left \{ \frac{n}{\varepsilon}, \frac{1}{\varepsilon} \right \} \eta^{3/2} + \frac{1}{\eta} \right) \right) \\
= \mathcal{O} \left( \frac{1}{\varepsilon} \left( \min \left \{ \frac{n}{\varepsilon}, \frac{1}{\varepsilon} \right \} \eta^{1/2} + \frac{1}{\eta^2} \right) \right) \\
\geq \mathcal{O} \left( \min \left \{ \frac{1}{\varepsilon^{9/5} \eta^{4/5}}, \frac{1}{\varepsilon^{2/5}} \right \} \right),
\]

where the optimal \( \eta = \min \{1/n^{2/5}, \varepsilon^{2/5}\} \).

As can be seen from the above result, we can see that when \( n \) is large enough the query complexity become \( \mathcal{O}(1/\varepsilon^{9/5}) \), that is the gradient and the inner function are estimated rather than computed the full value directly. The corresponding is better than the accelerated method in [3], in which the query complexity does not depend on \( n \). Furthermore, when \( n \leq 1/\varepsilon \), the query complexity is \( \mathcal{O}(n^{4/5}/\varepsilon) \), which is the same as in [12] for the case of the problem in (1.1).

6. Mini-batch version of SC-SCSG for the composition problem

In this section, we present the mini-batch version of the proposed method in Algorithm 2. The difference with Algorithm 1 is the computation of the gradient of the \( f(x) \). Furthermore, the convergence proof with the related upper bounds are almost the same except the following lemma. By using the Lemma 1, we derive the similar bound but the first term is reduced by a factor of \( b \), where \( b \) is the number of the mini-batch. Note that here the element in the mini-batch are independent, we can obtain the result directly. The details can be referred to Appendix.

Lemma 7. Suppose Assumption 2-6 holds, for \( \tilde{G}_k \) defined in (3.3) and \( \Lambda \) defined in Algorithm 2 with \( D = |D_1, D_2| \) and \( D = |D_1| = |D_2| \), we have

\[
\mathbb{E}_{i_k, j_k, A, D} \| \Lambda - \nabla f(x_k) \|^2 \\
\leq 5D_1^4 L_F^2 \left( \frac{L_G^2}{bB_G^2 L_F^2} + 4 \mathbb{I}(A \leq n) \frac{A}{A} + 4 \mathbb{I}(D < n) \frac{D}{D} \right) \mathbb{E} \| x_k - \hat{x}_s \|^2 + 20B_G^2 L_F^2 \mathbb{I}(D < n) \frac{D}{D} H_1 + 5 \mathbb{I}(D < n^2) \frac{D^2}{D^2} H_2,
\]
Based on the above lemma, we can obtain the query complexity for both convex and non-convex problem. As the process of the proof are similar to that of Corollary 1 and Corollary 2, we give the following result directly. The difference of the parameters' setting are $K$, and $\eta$ due to the fact of the mini-batch.

**Corollary 4.** Suppose Assumption 1-6 holds, in Algorithm 2, for convex problem, let $h = \mu$, the step size is $\eta \leq b\mu / (135L_F^2)$, the subset size of $A$ is $A = \min\{n, 128B_1L_F^2L^2/m^2\}$, the subset size of $D_1$ and $D_2$ are both $D = \min\{n, 5(16B_1L_F^2H_1 + 4H_2)/(4c\mu^2)\}$, the number of the inner iteration is $K \geq 540L_F^2/(b\mu^2)$, the number of outer iteration is $S \geq 1/(\log(1/\rho)) \log(2E\|\tilde{x}_0 - x^*\|^2/\varepsilon)$. The query complexity is

$$
(D + KA) S = O\left(\left(\min\left\{n, \frac{1}{\eta^2}\right\} + \frac{L^2}{\eta^2} \min\left\{n, \frac{1}{\mu^2}\right\}\right) \log\left(1/\varepsilon\right)\right).
$$

**Corollary 5.** Suppose Assumption 2-6 holds, in Algorithm 2, Let $h = \sqrt{b/\eta}$, the step size is $\eta = b^{3/5} \min\{1/\varepsilon^{2/5}, \varepsilon^{2/5}\}$, the set-size of $A$ is $A = \min\{n, \mathcal{O}(b/\eta)\}$, the set-size of the subset $D_1$ and $D_2$ are $D = \min\{n, \mathcal{O}(1/\varepsilon)\}$, the number of inner iteration is $K \leq \mathcal{O}\left(b^{1/2}/(\eta^{3/2})\right)$, the total number of iteration is $T = \mathcal{O}\left(1/(\varepsilon\eta)\right)$, in order to obtain $E[\|\nabla f(\tilde{x}_k)\|^2] \leq \varepsilon$. The query complexity is

$$
\frac{1}{b^{1/5}} \mathcal{O}\left(\min\left\{1/\varepsilon^{1/5}, \frac{n^{4/5}}{\varepsilon}\right\}\right)
$$

From the above result of the query complexity of the convex and non-convex problem, we can see that both of their step size $\eta$ and the number of inner iteration $K$ increase. These two key parameters lead to the improved the query complexity of both convex and non-convex problem.

**7. Conclusion**

In this paper, we propose the variance reduction based method for the convex and non-convex composition problem. We apply the stochastically controlled stochastic gradient to estimate inner function $G(x)$ and the gradient of $f(x)$. The query complexity of our proposed algorithm is better than or equal to the current methods on both convex and non-convex function. Furthermore, we also present the corresponding mini-batch version of the proposed method, in which the query complexities are improved as well. In the future, we can consider the non-smooth function of the composition problem with the method of the stochastically controlled stochastic gradient.

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A. Technical Tool

**Lemma.** 1. If \( v_1, \ldots, v_n \in \mathbb{R}^M \) satisfy \( \sum_{i=1}^n v_i = 0 \), and \( A \) is a non-empty, uniform random subset of \( [m] \), then

\[
\mathbb{E}_{A} \left\| \frac{1}{A} \sum_{b \in A} v_b \right\|^2 \leq \frac{\|A < n\|}{A} \frac{1}{n} \sum_{i=1}^n v_i^2.
\]

Furthermore, if the elements in \( A \) are independent, then

\[
\mathbb{E}_{A} \left\| \frac{1}{A} \sum_{b \in A} v_b \right\|^2 = \frac{1}{An} \sum_{i=1}^n v_i^2.
\]

**Proof.** Based on the \( \sum_{i=1}^n v_i = 0 \), and permutation and combination, For the case that \( A \) is a non-empty, uniformly random subset of \( [m] \), we have

\[
\mathbb{E}_{A} \left\| \sum_{b \in A} v_b \right\|^2 = \mathbb{E}_{A} \left[ \sum_{b \in A} \| v_b \|^2 \right] + \frac{1}{C_n^A} \sum_{i \in [n]} \left\langle v_i, \frac{C_n^{A-1}}{n-1} \sum_{i \neq j} v_j \right\rangle = A \frac{1}{n} \sum_{i=1}^n v_i^2 + \frac{A(A-1)}{n(n-1)} \sum_{i \in [n]} \left\langle v_i, \sum_{i \neq j} v_j \right\rangle.
\]
\[ = A \frac{1}{n} \sum_{i=1}^{n} v_i^2 + \frac{A(A-1)}{n(n-1)} \sum_{i \in [n]} \langle v_i, -v_i \rangle \]
\[ = A \frac{(n-A)}{(n-1)} \frac{1}{n} \sum_{i=1}^{n} v_i^2 \]
\[ \leq A \mathbb{P}(A < n) \frac{1}{n} \sum_{i=1}^{n} v_i^2. \]

Thus, we have
\[ \mathbb{E}_A \left[ \frac{1}{4} \sum_{b \in A} \|v_b\|^2 \right]^2 = \frac{1}{A^2} \mathbb{E}_A \left[ \sum_{b \in A} \|v_b\|^2 \right] \leq \frac{\mathbb{P}(A < n)}{A} \frac{1}{n} \sum_{i=1}^{n} v_i^2. \]

For the case that the element in \( A \) is randomly and independently selected from \([m]\), we have
\[ \mathbb{E}_A \left[ \sum_{b \in A} \|v_b\|^2 \right] = \mathbb{E}_A \left[ \sum_{b \in A} \|v_b\|^2 \right] + 2 \mathbb{E}_A \left[ \sum_{1 \leq b < A} \langle v_b, \sum_{b < k \leq A} v_k \rangle \right] \]
\[ = A \frac{1}{n} \sum_{i=1}^{n} \|v_i\|^2 + 2 \mathbb{E}_A \left[ \sum_{1 \leq b < A} \mathbb{E} [v], \sum_{b < k \leq A} v_k \right] \]
\[ = A \frac{1}{n} \sum_{i=1}^{n} \|v_i\|^2 + A (A - 1) \mathbb{E} [\|v\|^2] \]
\[ = A \frac{1}{n} \sum_{i=1}^{n} \|v_i\|^2. \]

\[ \square \]

**Lemma 8.** For the sequences that satisfy \( c_{k-1} = c_k Y + U \), where \( Y > 1, U > 0, k \geq 1 \) and \( c_0 > 0 \), we can get the geometric progression
\[ c_k + \frac{U}{Y-1} = \left( \frac{Y}{Y-1} \right) \left( c_{k-1} + \frac{U}{Y-1} \right), \]
then \( c_k \) can be represented as decrease sequences,
\[ c_k = \left( \frac{Y}{Y-1} \right)^k \left( c_0 + \frac{U}{Y-1} \right) - \frac{U}{Y-1}. \]

**B. Bound analysis of SC-SCSG for the composition problem**

**Lemma.** Suppose Assumption 2 and 6 holds, for \( \hat{G}_k \) defined in (3.3) with \( D = |D_1| \) and \( A = |A| \), we have
\[ \mathbb{E}_{A,D_1} \| \hat{G}_k - G(x_k) \|^2 \leq 4 \left( \frac{\mathbb{I}(A < n)}{A} + \frac{\mathbb{I}(D < n)}{D} \right) B^2 \mathbb{E} \|x_k - \bar{x}_s\|^2 + 2 \frac{\mathbb{I}(D < n)}{D} H_1. \]  
\[ \text{(B.1)} \]

**Proof.** By the definition of \( \hat{G}_k \) in (3.3), we have
\[ \mathbb{E} \| \hat{G}_k - G(x_k) \|^2 = \mathbb{E} \| G_k - G_{D_1}(x_k) + G_{D_1}(x_k) - G(x_k) \|^2 \]
\[ \overset{\circ}{\leq} 2 \mathbb{E} \| G_k - G_{D_1}(x_k) \|^2 + 2 \mathbb{E} \| G_{D_1}(x_k) - G(x_k) \|^2 \]
\[ \overset{\circ}{\leq} 4 \left( \frac{\mathbb{I}(A < n)}{A} + \frac{\mathbb{I}(D < n)}{D} \right) B^2 \mathbb{E} \|x_k - \bar{x}_s\|^2 + 2 \frac{\mathbb{I}(D < n)}{D} H_1, \]

where \( \overset{\circ}{\leq} \) follows from \( ||a_1 + a_2||^2 \leq 2a_1^2 + 2a_2^2 \); \( \overset{\circ}{=} \) is based on Assumption 6 and the following inequality: Through adding and subtracting the term \( G(x_k) - G(\bar{x}_s) \), we have
\[ \mathbb{E} \| G_k - G_{D_1}(x_k) \|^2 \]
\[ = \mathbb{E} \| G_A(x_k) - G_A(\bar{x}_s) + G_{D_1}(x_k) - G_{D_1}(x_k) \|^2 \]
\[ = \mathbb{E} \| G_A(x_k) - G_A(\bar{x}_s) - (G(x_k) - G(\bar{x}_s)) + (G(x_k) - G(\bar{x}_s)) + G_{D_1}(x_k) - G_{D_1}(x_k) \|^2 \]
\[ \overset{\circ}{\leq} 2 \mathbb{E} \| G_A(x_k) - G_A(\bar{x}_s) - (G(x_k) - G(\bar{x}_s)) \|^2 + 2 \mathbb{E} \| G_{D_1}(x_k) - G_{D_1}(x_k) - (G(\bar{x}_s) - G(x_k)) \|^2 \]
Lemma. 4 Suppose Assumption 2, 3, 5 and 6 holds, for \( \hat{G}_k \) defined in (3.3) and \( \nabla \hat{f}_k \) defined in (3.4) with \( D = |D_1| \leq |D_2| \), and

\[
E_{A,D} \left[ \nabla \hat{f}_k - \nabla f(x_k) \right]^2 
\leq 4B_G^2 L_F^2 + \frac{4}{A} \left( \frac{I(A < n)}{D} + \frac{I(D < n)}{D} \right) \| \nabla \hat{f}_k - \nabla f(x_k) \| \| \nabla \hat{f}_k - \nabla f(x_k) \| + \frac{4I(D < n)}{D} H_1 + \frac{4I(D < n)^2}{D^2} H_2,
\]

where \( \odot \) follows from \( \|a + b\|^2 \leq 2a^2 + 2b^2 \); \( \ominus \) is based on Assumption 6; \( \odot \) follows from the bounded function of \( G \) in Assumption 2.

Proof. Through adding and subtracting the terms of \( (\partial G(x_k))^T \nabla F(G(x_k)) \), \( (\partial G_{D_1}(\hat{x}_s))^T \nabla F_{D_1}(G(\hat{x}_s)) \), \( (\partial G(\hat{x}_s))^T \nabla F(G(\hat{x}_s)) \), we have

\[
E_{A,D} \left[ \nabla \hat{f}_k - \nabla f(x_k) \right]^2 
= \mathbb{E} \left[ (\partial G(x_k))^T \nabla F(G(x_k)) - (\partial G(\hat{x}_s))^T \nabla F(\hat{x}_s) + \nabla f_{D}(\hat{x}_s) - \nabla f(x_k) \right]^2
\]

\[
\leq 4E \left[ (\partial G(x_k))^T \nabla F(G(x_k)) - (\partial G(\hat{x}_s))^T \nabla F(G(\hat{x}_s)) \right]^2
+ 4E \left[ (\partial G(\hat{x}_s))^T \nabla F(\hat{x}_s) - (\partial G_{D_1}(\hat{x}_s))^T \nabla F_{D_1}(\hat{x}_s) \right]^2
+ 4E \left[ \nabla f_{D}(\hat{x}_s) - (\partial G_{D_1}(\hat{x}_s))^T \nabla F_{D_1}(\hat{x}_s) \right]^2
+ 4E \left[ (\partial G_{D_1}(\hat{x}_s))^T \nabla F_{D_1}(\hat{x}_s) - (\partial G(\hat{x}_s))^T \nabla F(G(\hat{x}_s)) \right]^2
\]

\[
\leq 4B_G^2 L_F^2 E \left[ \hat{G}_k - G(x_k) \right]^2 + 4B_G^2 L_F^2 E \left[ \hat{G}(\hat{x}_s) - G_{D_1}(\hat{x}_s) \right] + 4B_G^2 L_F^2 E \left[ \hat{G}(\hat{x}_s) - G_{D_1}(\hat{x}_s) \right]^2 + 4I(D < n) \left[ H_1 + \frac{I(D < n)^2}{D^2} H_2 \right]
\]

Lemma. 5 Suppose Assumption 2-6 holds, for \( \hat{G}_k \) defined in (3.3) and \( \nabla \hat{f}_k \) defined in (3.4) with \( D = |D_1| \leq |D_2| \), and

\[
E_{A,D} \left[ \nabla \hat{f}_k - \nabla f(x_k) \right]^2 
\leq 5B_G L_F^2 \left( \frac{L_F^2}{B_G^2 L_F^2} + \frac{4}{A} \left( \frac{I(A < n)}{D} + \frac{I(D < n)}{D} \right) \right) \| \nabla \hat{f}_k - \nabla f(x_k) \| \| \nabla \hat{f}_k - \nabla f(x_k) \| + \frac{5I(D < n)}{D} H_1 + \frac{5I(D < n)^2}{D^2} H_2,
\]

where \( \odot \) follows from \( \|a + b\|^2 \leq 2a^2 + 2b^2 \); \( \ominus \) is based on the bounded Jacobian of \( G \) and the smoothness of \( F \) in Assumption 2 and 3, and the upper bound of variance in Assumption 6 and Lemma 2. \( \odot \) is based on Lemma 3 and Assumption 6.

Proof. Through adding and subtracting the term of \( (\partial G_j(x_k))^T \nabla F_j(G(x_k)) \), \( (\partial G_j(\hat{x}_s))^T \nabla F_j(\hat{x}_s) \), \( (\partial G(\hat{x}_s))^T \nabla F(G(\hat{x}_s)) \), \( (\partial G_{D_1}(\hat{x}_s))^T \nabla F_{D_1}(G(\hat{x}_s)) \), we have

\[
E \left[ (\partial G_j(x_k))^T \nabla F_j(G(x_k)) - (\partial G_j(\hat{x}_s))^T \nabla F_j(\hat{x}_s) + \nabla f_{D}(\hat{x}_s) - \nabla f(x_k) \right]^2
\]

\[
= \mathbb{E} \left[ (\partial G_j(x_k))^T \nabla F_j(G(x_k)) - (\partial G_j(\hat{x}_s))^T \nabla F_j(\hat{x}_s) + \nabla f_{D}(\hat{x}_s) - \nabla f(x_k) \right]^2
\]
Lemma 9. For $f(x)$ is $\mu$-strongly convex, by setting $\eta = O\left(\frac{\mu}{L_f^2}\right), k = O\left(\frac{1}{\eta}\right) = O\left(\frac{L_f^2}{\mu}\right)$, we have the geometric convergence in expectation:

$$E\|\tilde{x}_{k+1} - x^*\|^2 \leq \rho^k E\|\tilde{x}_0 - x^*\|^2$$

where $\rho = \frac{1}{2(\mu - 2L_f^2\eta)\eta K} + \frac{L_f^2\eta}{(\mu - 2L_f^2\eta)} < 1$. The gradient complexity is

$$O\left((n + K) \log \left(\frac{1}{\epsilon}\right) \right) = O\left(n + \frac{L_f^2}{\mu} \right) \log \left(\frac{1}{\epsilon}\right) \quad (C.1)$$

Proof.

$$E_{i,j}\|x_{k+1} - x^*\|^2$$

$$=E\|x_k - x^*\|^2 - 2\eta E\langle \nabla f_k, x_k - x^* \rangle + \eta^2 E\|\nabla f_k\|^2$$

$$=E\|x_k - x^*\|^2 - 2\eta E\langle f(x_k), x_k - x^* \rangle + \eta^2 E\|\nabla f_k\|^2$$

$$\leq E\|x_k - x^*\|^2 - 2\eta \mu E\|x_k - x^*\|^2 + 2\eta^2 E\|\nabla f(x_k)\|^2 + 2\eta^2 E\|\nabla f_k - \nabla f(x_k)\|^2$$

$$=E\|x_k - x^*\|^2 - 2\eta \mu E\|x_k - x^*\|^2 + 2\eta^2 \left(2E\|\nabla f(x_k)\|^2 + 2L_f^2E\|x_k - \tilde{x}_s\|^2\right)$$

$$\leq E\|x_k - x^*\|^2 - 2\eta \mu E\|x_k - x^*\|^2 + 2\eta^2 \left(2L_f^2E\|x_k - x^*\|^2 + 2L_f^2E\|x_k - x^*\|^2 + 2L_f^2E\|\tilde{x}_s - x^*\|^2\right)$$

$$=E\|x_k - x^*\|^2 - 2\left(\mu - 2L_f^2\eta\right)\eta E\|x_k - x^*\|^2 + 2L_f^2\eta^2 E\|\tilde{x}_s - x^*\|^2$$

Summing up from $k = 0$ to $k = K - 1$, we have

$$E\|x_K - x^*\|^2 \leq E\|x_0 - x^*\|^2 - 2\left(\mu - 2L_f^2\eta\right)\eta K E\|\tilde{x}_{s+1} - x^*\|^2 + 2L_f^2\eta^2 K E\|\tilde{x}_s - x^*\|^2$$

For $x_0 = \tilde{x}_s$, we have

$$E\|\tilde{x}_{s+1} - x^*\|^2 \leq \frac{1 + 2L_f^2\eta^2 K}{2\left(\mu - 2L_f^2\eta\right)\eta K} E\|\tilde{x}_s - x^*\|^2 - \frac{1}{2\left(\mu - 2L_f^2\eta\right)\eta K} E\|x_K - x^*\|^2$$
\[
\begin{align*}
\leq & \frac{1 + 2L_j^2 \eta^2 K}{2 \left( \mu - 2L_j^2 \eta \right)} E \|\tilde{x}_s - x^*\|^2 \\
= & \left( \frac{1}{2 \left( \mu - 2L_j^2 \eta \right) \eta K} + \frac{L_j^2 \eta}{\mu - 2L_j^2 \eta} \right) E \|\tilde{x}_s - x^*\|^2
\end{align*}
\]

By setting \( \eta = O \left( \frac{\mu}{L_j^2} \right) \), \( K = O \left( \frac{1}{\eta} \right) = O \left( \frac{L_j^2}{\mu} \right) \), we have the geometric convergence in expectation:
\[
E \|\tilde{x}_{s+1} - x^*\|^2 \leq \rho^s E \|\tilde{x}_0 - x^*\|^2
\]
where \( \rho = \frac{1}{2(\mu - 2L_j^2 \eta) \eta K} + \frac{L_j^2 \eta}{\mu - 2L_j^2 \eta} < 1 \)

Proof of Theorem 1

Proof. By the update of \( x_k \) in Algorithm 1, we have
\[
\begin{align*}
\mathbb{E}_{i,j} & \|x_{k+1} - x^*\|^2 \\
= & \mathbb{E}\|x_k - x^*\|^2 - 2\eta \mathbb{E}\langle \nabla \tilde{f}_k, x_k - x^* \rangle + \eta^2 \mathbb{E}\|\nabla \tilde{f}_k\|^2 \\
= & \mathbb{E}\|x_k - x^*\|^2 - 2\eta \mathbb{E}\langle \nabla f(x_k), x_k - x^* \rangle + \eta^2 \mathbb{E}\|\nabla \tilde{f}_k\|^2 \\
= & \mathbb{E}\|x_k - x^*\|^2 - 2\eta \mathbb{E}\langle \nabla f(x_k), x_k - x^* \rangle + 2\eta \mathbb{E}\|\nabla f(x_k) - \nabla f(x_k)\|^2 \\
\leq & \mathbb{E}\|x_k - x^*\|^2 - 2\eta\mu \mathbb{E}\|x_k - x^*\|^2 + 2\eta \mathbb{E}\|\nabla f(x_k) - \nabla f(x_k)\|^2 \\
& + 2\eta^2 \left( \mathbb{E}\|\nabla f(x_k)\|^2 + \mathbb{E}\|\nabla \tilde{f}_k - \nabla f(x_k)\|^2 \right) \\
\leq & \mathbb{E}\|x_k - x^*\|^2 - 2\eta\mu \mathbb{E}\|x_k - x^*\|^2 + 2\eta \mathbb{E}\|\nabla f(x_k) - \nabla f(x_k)\|^2 \\
& + 2\eta^2 \left( L_j^2 \mathbb{E}\|x_k - x^*\|^2 + 5 \left( L_j^2 + V \right) \|x_k - \tilde{x}_s\|^2 + V_1 \right) \\
\leq & \mathbb{E}\|x_k - x^*\|^2 - \left( 2\mu - 4V \frac{1}{h} - (12L_j^2 + 10V) \eta \right) \mathbb{E}\|x_k - x^*\|^2 \\
& + 2 \left( 2V \frac{1}{h} + 5 \left( L_j^2 + V \right) \eta \right) \mathbb{E}\|\tilde{x}_s - x^*\|^2 + \frac{1}{K} V_2 + 2\eta^2 V_3,
\end{align*}
\]
where
\[
\begin{align*}
V &= B_G^1 L_P^2 \left( 4 \frac{\|A \prec n\|}{A} + 4 \frac{\|D \prec n\|}{D} \right), \\
V_1 &= 20 B_G^1 L_P^2 \frac{\|D \prec n\|}{D} H_1 + 5 \frac{\|D \prec n^2\|}{D^2} H_2, \\
V_2 &= 16 B_G^2 L_P^2 \frac{\|D \prec n\|}{D} H_1 + 4 \frac{\|D \prec n^2\|}{D^2} H_2 = \frac{4}{5} V_1.
\end{align*}
\]

\( \dagger \) is based on \( \|a_1 + a_2\|^2 \leq 2a_1^2 + 2a_2^2 \) and \( \langle a_1, a_2 \rangle \leq h \|a_1\|^2 + \frac{h}{n} \|a_2\|^2, h > 0 \); \( \ddagger \) is based on strongly-convex of \( f \) in Assumption 1; \( \dagger \) following from Lemma 4 and Lemma 5.
Summing up from $k = 0$ to $k = K - 1$, we have
\[
\mathbb{E}\|x_K - x^*\|^2 \leq \mathbb{E}\|x_0 - x^*\|^2 - \rho_1 K \mathbb{E}\|	ilde{x}_{s+1} - x^*\|^2 + \rho_2 K \mathbb{E}\|	ilde{x}_s - x^*\|^2 + \rho_3 K,
\]
where
\[
\begin{aligned}
\rho_1 &= \left(2\mu - h - 4V \frac{1}{h} - (12L_f^2 + 10V) \eta\right) \eta = (\mu - 2L_f^2 \eta) \eta - \rho_2, \\
\rho_2 &= 2 \left(2V \frac{1}{h} + 5(L_f^2 + V) \eta\right) \eta, \\
\rho_3 &= \frac{1}{h} \eta V_2 + 2\eta^2 V_1.
\end{aligned}
\]

For $x_0 = \tilde{x}_s$, by arrange, we have
\[
\rho_1 \mathbb{E}\|	ilde{x}_{s+1} - x^*\|^2 \leq \frac{1}{K} \mathbb{E}\|x_0 - x^*\|^2 + \rho_2 \mathbb{E}\|	ilde{x}_s - x^*\|^2 + \rho_3 - \frac{1}{K} \mathbb{E}\|x_K - x^*\|^2 \\
\leq \left(\frac{1}{K} + \rho_2\right) \mathbb{E}\|	ilde{x}_s - x^*\|^2 + \rho_3.
\]

\[
\square
\]

**Proof of Corollary 1**

**Proof.** In order to keep the proposed algorithm converge, we consider the parameters’ setting, we first ensure that $\rho_1 > 0$ in (4.3), and then define
\[
\rho = \left(\frac{1}{K} + \rho_2\right) / \rho_1,
\]
that require $\rho < 1$, where $\rho_2$ defined in (4.4). Thus, the convergence sequence is
\[
\mathbb{E}\|	ilde{x}_s - x^*\|^2 \leq \rho^s \mathbb{E}\|	ilde{x}_0 - x^*\|^2 + \frac{\rho}{\rho_1} \sum_{s=0}^8 \rho^s \leq \rho^S \mathbb{E}\|	ilde{x}_0 - x^*\|^2 + \frac{\rho}{\rho_1} \frac{1}{1 - \rho}.
\]

We ensure $\frac{\rho}{\rho_1} \frac{1}{1 - \rho} \leq \frac{1}{4} \varepsilon$, where $\rho_3$ defined in (4.5), that we can derive the size of the $D$. In the following we analyze the parameters’ setting such that satisfying above requirement.

1. In order to ensure $\rho_1 > 0$ in (4.3), we consider the parameter $h$, $\eta$ and $A$,
   (a) $h = \mu$, consider $\rho_1$ in (4.3), we should require that $h \leq \mu$, however, $V$ in (4.1) has the relationship with $A$ and $D$. In order to keep $A$ small enough, we set the upper bound of $h$. Thus, we set $h = \mu$.
   (b) $A = \min \left\{ n, 128B_G^4L_F^2 \frac{1}{\mu^2} \right\}$, based on the setting of $h$, we require that $V/h < \frac{\mu}{16}$. Thus, we have
   \[
   V = B_G^4L_F^2 \left(4 \frac{I(A \leq n)}{A} + 4 \frac{I(D < n)}{D} \right) \leq 8B_G^4L_F^2 \frac{I(A \leq n)}{A} \leq \frac{1}{16} \mu^2.
   \]
   For $V$ defined in (4.1), if $A < n$, we have
   \[
   A \geq 128B_G^4L_F^2 \frac{1}{\mu^2},
   \]
   otherwise, $A = n$ satisfy the requirement. Thus, we have $A = \min \left\{ n, 128B_G^4L_F^2 \frac{1}{\mu^2} \right\}$.
   (c) $\eta \leq \frac{3\mu}{8L_f}$, back to the target of $\rho_1 > 0$, we require that $\eta \leq \frac{3\mu}{8L_f} \leq \frac{\mu}{12L_f^2 + 10V} \leq \frac{\mu}{12L_f^2 + 10V} = \frac{2\mu - h - \frac{1}{2}V}{2L_f^2 + 10(L_f^2 + V)}$, note that $\mu \leq L_f$ by the definition in preliminaries.
2. In order to ensure $\rho < 1$ in (C.3), we first consider $\rho_1$ and $\rho_2$ in (4.3) and (4.4). By the setting of $h = \mu$ and $V < \mu^2/16$, we have,

$$
\rho_1 \geq \left( \mu - 2L_f^2\eta - \frac{1}{4}\mu + 10 \left( L_f^2 + \frac{1}{16}\mu^2 \right) \eta \right) \eta \geq \left( \frac{3}{4}\mu - \frac{101}{8} L_f^2 \eta \right) \eta, \tag{C.4}
$$

$$
\rho_2 \leq 4\frac{1}{\mu} \frac{1}{16}\mu^2 + 10 \left( L_f^3 + \frac{1}{16}\mu^2 \right) \eta^2 \leq \left( \frac{1}{4}\mu + 10 \left( L_f^3 + \frac{1}{16}\mu^2 \right) \eta \right) \eta \geq \left( \frac{1}{4}\mu + \frac{85}{8} L_f^3 \eta \right) \eta. \tag{C.5}
$$

We require that $\rho = \frac{1}{K \rho_1} + \frac{\rho_2}{\rho_1} < 1$, and analyze the two term separately,

(a) In order to $\frac{\rho_2}{\rho_1} < \frac{1}{2}$, that is

$$
\frac{\rho_2}{\rho_1} < \frac{\left( \frac{1}{4}\mu + \frac{85}{8} L_f^3 \eta \right) \eta}{\left( \frac{3}{4}\mu - \frac{101}{8} L_f^2 \eta \right) \eta} < \frac{1}{2}.
$$

We get $\eta \leq \frac{\mu}{135L_f^2}$.

(b) In order to $\frac{1}{K \rho_1} < \frac{1}{2}$, that is

$$
\frac{1}{K \rho_1} < \frac{1}{2} \frac{1}{K \rho_2} \leq \frac{1}{2K} \left( \frac{1}{4}\mu + 10 \left( L_f^2 + \frac{1}{16}\mu^2 \right) \eta \right) \eta
\leq \frac{1}{2K} \left( \frac{1}{4}\mu + \frac{85}{8} L_f^3 \eta \right) \eta
\leq \frac{1}{2}.
$$

Thus, we have $K \geq 540L_f^2$.

3. Consider the term $\rho^3 E\|\hat{x}_0 - x^*\|^2 + \rho_1 \frac{1}{1-p}$, we analyze them separately,

(a) In order to ensure $\frac{\rho_3}{\rho_1} \frac{1}{1-p} \leq \frac{1}{2} \varepsilon$, that is

$$
\frac{\rho_3}{\rho_1} \frac{1}{1-p} \leq \frac{\rho_3}{\rho_1} \frac{1}{1-p} \leq \frac{\rho_3}{\rho_1} \frac{1}{1-p} \leq \frac{2\rho_3}{\rho_1} \frac{1}{1-p} \leq \frac{1}{2} \varepsilon.
$$

Based on the bound of $\rho_1$ in (C.4), the definition of $V_1$ in (4.2) and the step size $\eta$ mentioned above, we have

i. For $V$

$$
V_1 \leq \frac{4}{5}\varepsilon \mu^2 \leq \left( \frac{4}{5} - \frac{101}{8} \frac{1}{135} \right) \varepsilon \mu^2 \leq \frac{4}{5} \mu + \frac{101}{8} \frac{2}{135} \mu \varepsilon \leq \frac{3}{4} \mu - \frac{101}{8} L_f^2 \eta \varepsilon \leq \frac{3}{4} \mu - \frac{101}{8} L_f^2 \eta \varepsilon
$$

ii. If $D < n$, we can obtain $D \geq \frac{n}{\varepsilon^{2p^2}} (20B_1^2L_f^2H_1 + 5H_2)$, otherwise $D = 0$, the above inequality is correct.

Thus, we obtain $D = \min \left\{ n, (16B_1^2L_f^2H_1 + 4H_2) \right\}$.

(b) In order to ensure $\rho^3 E\|\hat{x}_0 - x^*\|^2 \leq \frac{1}{2} \varepsilon$, we need the number of the outer iterations

$$
S \geq \frac{1}{\log (1/\rho)} \log \frac{2E\|\hat{x}_0 - x^*\|^2}{\varepsilon}.
$$
All in all, we consider the query complexity based on above parameters’ setting. For each outer iteration, there will be \((D + KA)\) queries. Thus, the query complexity is

\[
(D + KA) S = O \left( \left( \min \left\{ \frac{n}{\varepsilon \mu^2}, \frac{L^2}{\mu^2} \min \left\{ \frac{n}{\mu^2} \right\} \right\} \log \left( \frac{1}{\varepsilon} \right) \right) \right).
\]

D. Proof of SC-SCSG method for Non-convex composition problem

**Proof of Lemma 6**

**Proof.** Consider the upper bound of \(f(x_{k+1})\) and \(||x_{k+1} - x_s||^2\), respectively.

- Base on the smoothness of \(f\) in Assumption 4 and take expectation with respect to \(i_k, j_k\), we have

\[
\begin{align*}
E_{i,j}[f(x_{k+1})] & \leq E[f(x_k)] - \eta E(\nabla f(x_k), \nabla \tilde{f}_k) + \frac{L_f \eta^2}{2} E \left\| \nabla \tilde{f}_k \right\|^2 \\
& = E[f(x_k)] - \eta E(\nabla f(x_k), \nabla \tilde{f}_k - \nabla f(x_k) + \nabla f(x_k)) + \frac{L_f \eta^2}{2} E \left\| \nabla \tilde{f}_k \right\|^2 \\
& = E[f(x_k)] - \eta E(\nabla f(x_k), \nabla f(x_k)) - \eta(\nabla f(x_k), E[\nabla \tilde{f}_k] - \nabla f(x_k)) + \frac{L_f \eta^2}{2} E \left\| \nabla \tilde{f}_k - \nabla f(x_k) + \nabla f(x_k) \right\|^2 \\
& \leq E[f(x_k)] - \eta E\|\nabla f(x_k)\|^2 + \frac{1}{2} \eta E\|\nabla f(x_k)\|^2 + \frac{1}{2} \eta E E_{i,j} \left[ \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2 \\
& \quad + \frac{L_f \eta^2}{2} \left( 2E\|\nabla f(x_k)\|^2 + 2E \left\| \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2 \right) \\
& = E[f(x_k)] - \frac{1}{2} \eta E\|\nabla f(x_k)\|^2 + \frac{1}{2} \eta E E_{i,j} \left[ \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2 + L_f \eta^2 \left( E\|\nabla f(x_k)\|^2 + E \left\| \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2 \right) \\
& = E[f(x_k)] - \left( \frac{1}{2} \eta - L_f \eta^2 \right) E\|\nabla f(x_k)\|^2 + \frac{1}{2} \eta E E_{i,j} \left[ \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2 + L_f \eta^2 E \left\| \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2,
\end{align*}
\]

where the last inequality is based on \(||a_1 + a_2||^2 \leq 2a_1^2 + 2a_2^2||\).

- Base on the update of \(x_k\) in Algorithm 1 and take expectation with respect to \(i_k, j_k\), we have,

\[
\begin{align*}
E_{i,j}[x_{k+1} - x_s]^2 & = E\|x_k - x_s\|^2 - 2\eta E(\nabla \tilde{f}_k, x_k - x_s) + \eta^2 E \left\| \nabla \tilde{f}_k \right\|^2 \\
& = E\|x_k - x_s\|^2 - 2\eta E(\nabla \tilde{f}_k - \nabla f(x_k) + \nabla f(x_k), x_k - x_s) + \eta^2 E \left\| \nabla \tilde{f}_k \right\|^2 \\
& = E\|x_k - x_s\|^2 - 2\eta E(\nabla f(x_k), x_k - x_s) - 2\eta E \left[ \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2 + \eta^2 E \left\| \nabla \tilde{f}_k - \nabla f(x_k) + \nabla f(x_k) \right\|^2 \\
& \leq E\|x_k - x_s\|^2 + h\eta E\|\nabla f(x_k)\|^2 + h\eta E \left[ \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2 + \frac{2}{h} \eta^2 E\|x_k - x_s\|^2 \\
& \quad + \eta^2 \left( 2E\|\nabla f(x_k)\|^2 + 2E \left\| \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2 \right) \\
& = \left( 1 + \frac{2}{h} \eta \right) E\|x_k - x_s\|^2 + (h\eta + 2\eta^2) E\|\nabla f(x_k)\|^2 + h\eta E \left[ \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2 + 2\eta^2 E \left\| \nabla \tilde{f}_k - \nabla f(x_k) \right\|^2,
\end{align*}
\]

where the inequality is based on \(2\langle a_1, b_2 \rangle \leq 1/h\langle a_1 ||^2 + h\langle a_2 ||^2, \forall h > 0, \text{ and } ||a_1 + a_2||^2 \leq 2a_1^2 + 2a_2^2.\)
Combine above equalities and Lemma 4, 5, we form a Lyapunov function,

\[
\begin{align*}
E[f(x_{k+1})] + c_{k+1} E\|x_{k+1} - \bar{x}_s\|^2 \\
= & E[f(x_k)] - \left(\frac{1}{2} \left( h + 2c_{k+1}\right) - \frac{1}{2} \left( h + 2c_{k+1}\right)\right) E\|x_k - \bar{x}_s\|^2 \\
& + \frac{1}{2} \left( h + 2c_{k+1}\right) E\|x_k - \bar{x}_s\|^2 + \frac{1}{2} \left( h + 2c_{k+1}\right) E\|x_k - \bar{x}_s\|^2 \\
& + \frac{1}{2} \left( h + 2c_{k+1}\right) E\|x_k - \bar{x}_s\|^2 + \frac{1}{2} \left( h + 2c_{k+1}\right) E\|x_k - \bar{x}_s\|^2 \\
& + \left( h + 2c_{k+1}\right) E\|x_k - \bar{x}_s\|^2 + \frac{1}{2} \left( h + 2c_{k+1}\right) E\|x_k - \bar{x}_s\|^2 \\
& \leq E[f(x_k)] + c_k E\|x_k - \bar{x}_s\|^2 - u_k \|\nabla f(x_k)\|^2 + J_k,
\end{align*}
\]

where

\[
\begin{align*}
u_k &= \left(\frac{1}{2} - \frac{h c_{k+1}}{2}\right) \left( h + 2c_{k+1}\right) \\
W_1 &= 20 B_2^2 L_f^2 \|D \leq n\| D_1 + 5 \|D^2 \leq n^2\| D_2 \\
W_2 &= \frac{4}{5} W_1 \\
J_k &= \left(\frac{1}{2} + \frac{h c_{k+1}}{2}\right) W_2 \|D \leq n\| D_1 + \left(\frac{1}{2} + \frac{h c_{k+1}}{2}\right) W_2 \|D \leq n\| D_1 \\
W &= B_2^2 L_f^2 \left(\frac{4 |A| \leq n\| A_1 + 4 |D| \leq n\| D_1\| D_2\right) \\
c_k &= c_{k+1} \left(1 + \left(\frac{2}{2} + 4hW\right) \eta + 10 \left( L_f^2 + W\right) \eta^2\right) + 2W \eta + 5(L_f^2 + W) L_f \eta^2.
\end{align*}
\]

Based on the above inequality with respect to the sequence \(E[f(x_k)] + c_k E\|x_k - \bar{x}_s\|^2\) and Algorithm 1, we can obtain the convergence form in which the parameters are not clear defined.

**Proof of Theorem 2**

**Proof.** Based on the update for \(c_k\) in (5.2), we can see that \(c_k < c_{k+1}\). As \(c_k\) is a decreasing sequence, we have \(u_0 < u_k\) and \(J_k < J_0\). Then, we get

\[
u_0 E[\|\nabla f(x_k)\|^2] \leq E[f(x_k)] + c_k E[\|x_k - \bar{x}_s\|^2] - (E[f(x_{k+1})] + c_{k+1} E[\|x_{k+1} - \bar{x}_s\|^2]) + J_0.
\]

Sum from \(k = 0\) to \(k = K - 1\), we can get

\[
\frac{1}{K} \sum_{k=0}^{K-1} u_0 E[\|\nabla f(x_k)\|^2] \leq \frac{E[f(x_0)] - (E[f(x_K)] + c_K E[\|x_K - \bar{x}_s\|^2])}{K} + J_0
\]

\[
\leq \frac{E[f(x_0)] - E[f(x_K)]}{K} + J_0.
\]

Since \(x_0 = \bar{x}_s\), let \(\bar{x}_{s+1} = x_K\), we obtain,

\[
\frac{1}{K} \sum_{k=0}^{K-1} u_0 E[\|\nabla f(x_k)\|^2] \leq \frac{E[f(\bar{x}_s)] - E[f(\bar{x}_{s+1})]}{K} + J_0.
\]

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Summing the outer iteration from $s = 0$ to $S - 1$, we have
\[ u_0 \mathbb{E} \| \nabla f(\tilde{x}_k^s) \|^2 = \frac{1}{K} \sum_{k=0}^{K-1} \sum_{s=0}^{S-1} u_0 \mathbb{E} \| \nabla f(\tilde{x}_k^s) \|^2 + J_0 \leq \frac{\mathbb{E}[f(\tilde{x}_0)] - \mathbb{E}[f(\tilde{x}_S)]}{KS} + J_0 \leq \frac{f(x_0) - f(x^*)}{KS} + J_0, \]
where $x_k^s$ indicates the $s$-th outer iteration at $k$-th inner iteration, and $\tilde{x}_k^s$ is uniformly and randomly chosen from $s = \{0, \ldots, S-1\}$ and $k=\{0, \ldots, K-1\}$.

Proof of Corollary 2

Proof. In order to have $\mathbb{E} [\| \nabla f(\tilde{x}_k^s) \|^2] \leq \varepsilon$, that is
\[ \mathbb{E} [\| \nabla f(\tilde{x}_k^s) \|^2] \leq \frac{L_f(f(x_0) - f(x^*))}{u_0 KS} + J_0 / u_0 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon, \]
we consider the corresponding parameters’ setting:

1. For the first term, consider $c_k$ defined in (5.2) define $c_k = c_{k+1} Y + U$, based on Lemma 8, for $k = K$, we have
\[ c_K = \left( \frac{1}{Y} \right)^K \left( c_0 + \frac{U}{Y - 1} \right) - \frac{U}{Y - 1}, \]
where
\[ Y = 1 + \left( 2 \frac{h}{h} + 4hW \right) \eta + 10 \left( B_0^2 L_F^2 + W \right) \eta^2, \]
\[ U = 2W \eta + 5(L_f^2 + W) L_f \eta^2 > 0. \]
By setting $c_K \to 0$, we obtain
\[ c_0 = \frac{UY^K}{Y - 1} - \frac{U}{Y - 1} = \frac{U (Y^K - 1)}{Y - 1}. \]
Then, putting the $Y$ and $U$ into the above equation. We have
\[ c_0 = \frac{2W \eta + 5(L_f^2 + W) L_f \eta^2}{(2 \frac{h}{h} + 4hW) \eta + 10 \left( L_f^2 + W \right) \eta^2} C = \frac{2W + 5(L_f^2 + W) L_f \eta}{(2 \frac{h}{h} + 4hW) + 10 \left( L_f^2 + W \right) \eta} C, \tag{D.1} \]
where $C = Y^K - 1$. Because $c_0$ has the influence on the parameters such as $K$, $C$ and $u_0$, we analyze them separately,

(a) For $K$ and $C$, based on the character of function $\left( 1 + \frac{1}{t_2} \right)^{t_1} \to e^1$ as $t_1, t_2 \to +\infty$ and $t_1 t_2 < 1$, and the function is also the increasing function with an upper bound of $e$, we require
\[ K < \frac{1}{\left( \frac{2}{\frac{h}{h} + 4hW} \right) \eta + 10 \left( L_f^2 + W \right) \eta^2}, \tag{D.2} \]
thus, we have $C < e - 1$.

(b) For $u_0$ defined in (5.3), in order to keep $u_k > 0$, we need to keep $c_0 h < 1/4$. If $c_0 h < 1/4$, there exits a constant $\bar{u}$ such that $u_0 = \bar{u} \eta$. In order to satisfy $c_0 h < 1/4$, combine with (D.1) and $C < e - 1$, that is
\[ c_0 h \leq \frac{2W + 5(L_f^2 + W) L_f \eta}{(2 \frac{h}{h} + 4hW) + 10 \left( L_f^2 + W \right) \eta} h (e - 1) \leq \frac{1}{4}, \]

\[ \text{Here the 'e' is the Euler number, approximate to 2.718.} \]
i. By setting \( h = \frac{1}{\sqrt[5]{L_f}} \), there exist \( \hat{w} > 0 \), based on above inequality, we have

\[
W \leq \frac{16L_f^3\eta + 50L_f^3\sqrt[5]{\eta}}{9.6 + 34L_f^3\eta - 50L_f^3\eta} < \hat{w}L_f^3\eta
\]

Thus, combine with the definition of \( W \) in (5.1), we require that

\[
W = B_G^4L_P^4 \left( 4 \frac{\| A < n \|}{A} + 4 \frac{\| D < n \|}{D} \right) \\
\leq 8B_G^4L_P^4 \frac{\| A < n \|}{A} \leq \hat{w}L_f^3\eta = O \left( L_f^3\eta \right).
\]

If \( A < n \), we require \( A \geq O \left( B_G^4L_P^2/(L_f^3\eta) \right) \). Thus, we have \( A = \min \{ n, O(1/\eta) \} \).

ii. Based on the setting of \( h \) and \( W \), combing with (D.2), we have

\[
K < \frac{1}{\left( 10\sqrt{L_f^3\eta} + \frac{4}{5\sqrt{L_f^3\eta}} \hat{w}L_f^3\eta \right)^2 + 10 \left( L_f^2 + \hat{w}L_f^3\eta \right) \eta^2} = \frac{1}{\left( 10\sqrt{L_f^3\eta} + \frac{4}{5\sqrt{L_f^3\eta}} \hat{w}L_f^3\eta \right)^2 + 10 \left( L_f^2 + \eta \right) \eta^2} = O \left( \frac{1}{(L_f\eta)^3/2} \right).
\]

2. For the second term about \( J_0 \), as \( u_0 = w_1\eta \), we require

\[
\frac{J_0}{\hat{w}\eta} = \frac{1}{u} \left( \frac{1}{2} + hc_0 \right) W_2 + (L_f + 2c_0) W_1 \eta \\
\leq \frac{1}{u} \left( \frac{3}{5} + L_f \eta + \frac{1}{2} \eta \sqrt{\eta} \right) W_1 \eta \\
\leq \frac{1}{u} \left( 20B_G^2L_P^2H_1 + 5H_2 \right) \left( \frac{3}{5} + L_f \eta + \eta \sqrt{\eta} \right) \frac{\| D < n \|}{D} \leq \frac{1}{2} \varepsilon.
\]

Then, if \( D < n \), we require that

\[
D \geq \frac{\hat{w}}{\varepsilon} \left( 20B_G^2L_P^2H_1 + 5H_2 \right) \left( \frac{3}{5} + \frac{1}{2}L_f \eta + c_0 \eta \sqrt{\eta} \right) = O \left( \frac{1}{\varepsilon} \right).
\]

Thus, we set \( D = \min \{ n, O(1/\varepsilon) \} \).

3. Based on the first term \( \frac{L_f(f(x_n) - f(x^*))}{n\eta} \leq \frac{1}{2} \varepsilon \), the total number of iteration is \( T = SK = \frac{2L_f(f(x_n) - f(x^*))}{\eta^2} \).

Thus, based on the above parameters’ setting, we can ensure that \( E[\| \nabla f(\hat{x}_k) \|^2] \leq \varepsilon \).

\[ \square \]

E. Proof for the Mini-batch of the SC-SGSG to the composition problem

Lemma 7 Suppose Assumption 2-6 holds, for \( \hat{G}_k \) defined in (3.3) and \( \Lambda \) defined in Algorithm 2 with \( D = [D_1, D_2] \) and \( D = |D_1| = |D_2| \), we have

\[
\mathbb{E}_{(k, j, k, A, D)} \| \Lambda - \nabla f(x_k) \|^2 \\
\leq 5B_G^4L_P^4 \left( \frac{L_f^2}{bB_G^4L_P^2} + 4 \frac{\| A < n \|}{A} + 4 \frac{\| D < n \|}{D} \right) \mathbb{E}\| x_k - \bar{x}_s \|^2 + 20B_G^4L_P^4 \frac{\| D < n \|}{D} H_1 + 5 \frac{\| D_2 < n^2 \|}{D^2} H_2.
\]

Proof. Through adding and subtracting the term of \( \frac{1}{f} \sum_{(i,j) \in I_0} \langle \partial G_j(x_k) \rangle^T \nabla F_i(G(x_k)) \), \( \frac{1}{f} \sum_{(i,j) \in I_0} \langle \partial G_i(\bar{x}_s) \rangle^T \nabla F_i(G(\bar{x}_s)) \), and \( \langle \partial G(\bar{x}_s) \rangle^T \nabla F(G(\bar{x}_s)), \langle \partial G_{D_1}(\bar{x}_s) \rangle^T \nabla F_{D_1}(G(\bar{x}_s)) \), we have

\[
\mathbb{E}\| \Lambda - \nabla f(x_k) \|^2
\]
\[ \leq 5E \left\| \frac{1}{b} \sum_{(i,j) \in E_k} (\partial G_j(x_k))^T \nabla F_i(G(x_k)) - (\partial G_j(\bar{x}_s))^T \nabla F_i(G(\bar{x}_s)) \right\|^2 \\
+ 5E \left\| \frac{1}{b} \sum_{(i,j) \in E_k} (\partial G_j(x_k))^T \nabla F_i(\bar{G}_k) - (\partial G_j(x_k))^T \nabla F_i(G(x_k)) \right\|^2 \\
+ 5E \left\| (\partial G_k(\bar{x}_s))^T \nabla F_i(G(\bar{x}_s)) - (\partial G_k(\bar{x}_s))^T \nabla F_i(G(D_k(\bar{x}_s))) \right\|^2 \\
+ 5E \left\| (\partial G_{D_1}(\bar{x}_s))^T \nabla F_{D_2}(G(\bar{x}_s)) - (\partial G(\bar{x}_s))^T \nabla F(G(\bar{x}_s)) \right\|^2 \\
\leq \frac{5}{b} L_f^2 E\|x_k - \bar{x}_s\|^2 + 5B_G^2 L_F^3 E\|\bar{G}_k - G(x_k)\|^2 + 5B_G^2 L_F^3 E\|G(\bar{x}_s) - G(D_1(\bar{x}_s))\|^2 \\
+ 5B_G^2 L_F^3 E\|G(\bar{x}_s) - G(D_1(\bar{x}_s))\|^2 + 5\left\| \frac{(D^2 < n)}{D^2} \right\| H_2 \\
\leq 5B_G^4 L_F^3 \left( \frac{L_f^2}{B_G^3 L_F^2} + 4\frac{\|A < n\|}{A} + 4\frac{\|D < n\|}{D} \right) E\|x_k - \bar{x}_s\|^2 + 20B_G^2 L_F^3 \frac{\|(D < n)\|}{D} H_1 + 5\left\| \frac{(D^2 < n^2)}{D^2} \right\| H_2, \]

where (\(1\)) follows from \(|a_1 + a_2 + a_3 + a_4 + a_5|\leq 5a_1^2 + 5a_2^2 + 5a_3^2 + 5a_4^2 + 5a_5^2\), and Lemma 1. (2) is based on \(E[\|X - E[X]\|^2] = \text{Var}[X] \leq \text{Var}[X^2]\), the smoothness of \(F_i\) in Assumption 4, the bounded Jacobian of \(G(x)\) and the smoothness of \(F\) in Assumption 2 and 3, and the upper bound of variance in Assumption 6 and Lemma 2. (3) is based on Lemma 3 and Assumption 6.

**Corollary.** 5 Suppose Assumption 2-6 holds, in Algorithm 2, let \(h = \sqrt{b/\eta}\), the step size is \(\eta = b^{1/5} \min\left\{1/n^{2/5}, \varepsilon^{2/5}\right\}\), the set-size of \(A\) is \(A = \min\left\{n, \mathcal{O}(b/\eta)\right\}\), the set-size of the subset \(D_1\) and \(D_2\) are \(D = \min\left\{n, \mathcal{O}(1/\varepsilon)\right\}\), the number of inner iteration is \(K \leq \mathcal{O}\left(\frac{b^{1/2}}{\eta^{3/2}}\right)\), the total number of iteration is \(T = \mathcal{O}\left(1/(\varepsilon\eta)\right)\), in order to obtain obtain \(E[\|\nabla f(\bar{x}_k)\|^2] \leq \varepsilon\).The query complexity is

\[ \frac{1}{b^{1/5}} \mathcal{O}\left( \min\left\{ \frac{1}{\varepsilon^{1/5}}, \frac{n^{4/5}}{\varepsilon} \right\} \right) \]

**Proof.** Based on the parameters’ setting, that is \(D = \min\left\{n, \mathcal{O}(1/\varepsilon)\right\}, A = \min\left\{n, \mathcal{O}(b/\eta)\right\}, K \leq \mathcal{O}\left(\frac{b^{1/2}}{\eta^{3/2}}\right), \) and \(T = \mathcal{O}\left(1/(\varepsilon\eta)\right)\), we have,

\[ \mathcal{O}\left( \frac{T}{K} (D + KA) \right) = \mathcal{O}\left( \frac{\eta^{3/2}}{\varepsilon b^{1/2} \eta} \left( \min\left\{n, \frac{1}{\varepsilon}\right\} + \frac{b^{1/2} b}{\eta^{3/2} \eta} \right) \right) = \mathcal{O}\left( \frac{\eta^{1/2}}{\varepsilon b^{1/2}} \left( \min\left\{n, \frac{1}{\varepsilon}\right\} + \frac{b^{1/2}}{\eta^{3/2}} \right) \right) \]

\[ = \frac{1}{\varepsilon b^{1/2}} \mathcal{O}\left( \min\left\{n, \frac{1}{\varepsilon}\right\} \eta^{1/2} + \frac{b^{1/2}}{\eta^{2}} \right) \]

\[ \geq \frac{1}{b^{1/5}} \mathcal{O}\left( \min\left\{ \frac{n^{4/5}}{\varepsilon}, \frac{1}{\varepsilon^{9/5}} \right\} \right) \]

where the optimal \(\eta = b^{3/5} \min\left\{1/n^{2/5}, \varepsilon^{2/5}\right\}\).