WHICH CLUSTER MORPHISM CATEGORIES ARE CAT(0)

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Abstract. The cluster morphism category of an hereditary algebra was introduced in [5] to show that the picture space of an hereditary algebra of finite representation type is a $K(\pi, 1)$ for the associated picture group, thereby allowing for the computation of the homology of picture groups of finite type as carried out in [7] for the case of $A_n$.

In this paper we show that the cluster morphism category is a CAT(0)-category for hereditary algebras of finite or tame type with only small tubes. As a consequence, we get that the classifying space of the cluster morphism category is a locally CAT(0) space and, as a consequence of that, we get that this classifying space is a $K(\pi, 1)$.

Introduction

We define a CAT(0)-category to be a cubical category (Definition 1.1) whose classifying space is locally CAT(0). To show that the cluster morphism category of an hereditary algebra (Definition 1.3) is CAT(0) we first show it is a cubical category.

Theorem A (Theorem 1.4). The cluster morphism category of any hereditary algebra $\Lambda$ is a cubical category.

In Theorem 2.1 we recall (from [8]) criteria for when a cubical category is a CAT(0)-category. The rest of the paper is devoted to verifying these categorical conditions on the cluster morphism categories for the algebras in our main theorem:

Theorem B (Theorem 3.8). The cluster morphism category of an hereditary algebra of finite or tame representation type is a CAT(0)-category if and only if there are no tubes of rank $\geq 3$ in its Auslander-Reiten quiver.

For hereditary algebras of finite type, we already showed in [5] that the classifying space of the cluster morphism category of $\Lambda$ is a $K(\pi, 1)$ for the picture group of $\Lambda$. We also showed, in [5], that this classifying space is homeomorphic to the “picture space” of $\Lambda$ defined in [7]. The main result of [7] was the calculation of the cohomology of the picture group for $\Lambda$ of type $A_n$, assuming that the picture space is a $K(\pi, 1)$ for the picture group of $\Lambda$. In this paper we show that the classifying space of the cluster morphism category, for $\Lambda$ of finite type or tame with small tubes, is locally CAT(0) by showing that those categories are CAT(0)-categories (Theorem B). This implies that the classifying space is a $K(\pi, 1)$, but being locally CAT(0) is a stronger statement.

The cluster morphism category of an hereditary algebra $\Lambda$ has, as objects, the finitely generated wide subcategories $W$ of $\text{mod}\-\Lambda$. A morphism $[T] : W \to W'$ is given by a partial cluster $T$ in $W$ so that $T^\perp \cap W = W'$ (Definition 1.3). When $\Lambda$ has finite

2020 Mathematics Subject Classification. 16G20; 20F55.
First author supported by Simons Foundation Grant #686616.
type, this is a finite category (having a finite number of objects and morphisms). Thus, its classifying space is a finite cell complex. For any $\Lambda$, this classifying space is finite dimensional since at most $n$ nonidentity morphisms can be composed. This implies, e.g., that the fundamental group of the cluster morphism category, which is the picture group of $\Lambda$, is torsion-free.

In Section 1 we recall the definition of a cubical category (Definition 1.1) and show that the cluster morphism category of any hereditary algebra is cubical (Theorem 1.4). The main property of cubical properties is that every morphism is embedded in a cube of morphisms making its classifying space a union of cubes. In the case of the cluster morphism category, these cubes are given by factoring each cluster morphism of rank $n$ into $n!$ signed exceptional sequences.

In Theorem 2.1 we give conditions under which a cubical category is a $\text{CAT}(0)$-category and we use the rest of this paper to determine when these conditions hold for cluster morphism categories of hereditary algebras of finite or tame type.

1. Cubical categories

We review the definition of a cubical category from [8]. To simplify the discussion, we add the condition that the category has a “rank” filtration, a condition satisfied by the cluster morphism category.

The basic example of a cubical category is the $n$-cube category $I^n$ whose objects are the $2^n$ subsets of the set $[n] = \{1, 2, \cdots, n\}$ and whose morphisms are the inclusion maps. The “rank” of a subset of $[n]$ is defined to be the size of its complement. This is a finite category with at most one morphism between any two objects. Given any morphism $f : X \to Y$ in any category $C$, we denote by $\mathcal{F}ac(f)$ the category of factorizations $X \to Z \to Y$ of $f$. A morphism of factorizations $(X \to Z \to Y) \to (X \to Z' \to Y)$ is a morphism $Z \to Z'$ making the 4 object diagram commute. $\mathcal{F}ac(f)$ has initial object $X \xrightarrow{\sim} X \to Y$ and terminal object $X \to Y \xrightarrow{\sim} Y$. We have a middle object functor $\mathcal{F}ac(f) \to C$ sending $(X \to Z \to Y)$ to $Z$.

Definition 1.1. A cubical category is a small category with graded object set $C = \coprod_{n \in \mathbb{Z}} C_n$ having the following properties where objects $X \in C_n$ are said to have rank $rk X = n$ and a morphism $f : X \to Y$ is said to have rank $rk f = rk X - rk Y$.

1. Every morphism $f : X \to Y$ has $rk f \geq 0$ and $rk f = 0$ only when $X = Y$ and $f = id_X$.
2. For every morphism $f : X \to Y$, the category $\mathcal{F}ac(f)$ of factorizations of $f$ is isomorphic to the $n$-cube category $I^n$ with $n = rk f$ and the “middle object functor”: $\mathcal{F}ac(f) \to C$ is an embedding.
3. A morphism $f : X \to Y$ of rank $n$ is uniquely determined by its $n$ first factors. These are the rank 1 morphisms $t_i : X \to A_i$, $i = 1, \cdots, n$, which can be completed to a factorization $X \to A_i \to Y$ of $f$.
4. A morphism $f : X \to Y$ of rank $n$ is uniquely determined by its $n$ last factors. These are the rank 1 morphisms $x_i : B_i \to Y$, $i = 1, \cdots, n$, which can be completed to a factorization $X \to B_i \to Y$ of $f$. 
Proposition 1.2. Let \( f : X \to Y \) be a morphism in a cubical category \( C \). Then, the functor \( I^n \to C \) given by composing any isomorphism \( I^n \cong \text{Fac}(f) \) with the embedding \( \text{Fac}(f) \to C \), as given in Condition (2) above, preserves ranks of morphisms but increases the rank of each object by \( \text{rk} Y \).

Proof. Since \( X, Y \) are initial and terminal in \( \text{Fac}(f) \), they correspond to the initial and terminal objects of \( I^n \), call them \( A, B \). Then every object \( C \) in \( I^n \) lies in a chain of \( n \) rank 1 morphism \( A \to C_{n-1} \to \cdots \to C_2 \to C_1 \to B \) (so that \( \text{rk} C_j = j \). Since \( I^n \cong \text{Fac}(f) \), every object \( Z \) in \( \text{Fac}(f) \) lies in a chain of \( n \) nonidentity morphisms from \( X \) to \( Y \). Since \( \text{rk} X - \text{rk} Y = n \), all of these nonidentity morphisms must have rank 1 and each object \( C_j \) in the chain of morphisms in \( I^n \) corresponds to an object of \( \text{Fac}(f) \) of rank \( \text{rk} Y + j \). Since all objects of \( I^n \) lie in such a chain, the isomorphism \( I^n \cong \text{Fac}(f) \) increases the rank of each object by \( \text{rk} Y \). It follows that corresponding morphisms have the same rank. \( \square \)

We review some basic cluster theory for hereditary algebras which is needed for the definition of a cluster morphism category (Definition 1.3). A module \( E \) is called exceptional if it is a rigid brick where rigid means \( \text{Ext}^1_{\Lambda}(E, E) = 0 \) and a brick is a module whose endomorphism ring is a division algebra. An exceptional module is uniquely determined by its dimension vector \( \dim E \in \mathbb{N}^n \) where \( n \) is the rank of \( \Lambda \), i.e., the number of nonisomorphic simple \( \Lambda \)-modules. A cluster-tilting object for \( \Lambda \) is a rigid object in the cluster category of \( \Lambda \) with \( n \) components \([1]\). This is equivalent to a set of exceptional modules \( E_1, \ldots, E_k \) and shifted indecomposable projective modules \( P_{k+1}[1], \ldots, P_n[1] \) so that \( \text{Ext}^1_{\Lambda}(E, E) = 0 \) and \( \text{Hom}_{\Lambda}(P, E) = 0 \) for \( E = \bigoplus E_i \) and \( P = \bigoplus P_j \). We note that every shifted projective object \( P[1] \) is uniquely determined by its dimension vector \( \dim(P[1]) := -\dim P \in \mathbb{Z}^n \). A partial cluster-tilting set is any subset of the set
\[
\{ E_1, \ldots, E_k, P_{k+1}[1], \ldots, P_n[1] \}.
\]
These are objects of \( D^b(\Lambda) \), the bounded derived category of \( \text{mod-\Lambda} \), which are denoted \( T_1, \ldots, T_n \). Thus \( T = \bigoplus T_i = E \oplus P[1] \in D^b(\Lambda) \). We use the notation \( |T_i| \) for the underlying module of \( T_i \). Thus \( |E_i| = E_i \) and \( |P_j[1]| = P_j \). Also, \( |T| = \bigoplus |T_i| \).

We denote by \( C_\Lambda \) the set of isomorphism classes of exceptional modules \( E \) and shifted indecomposable projective modules \( P[1] \). Thus, every partial cluster-tilting set is a subset of \( C_\Lambda \). Given any partial cluster tilting object \( T \), its (right Hom-Ext) perpendicular category \( T^\perp (= |T|^\perp) \) is the full subcategory of \( \text{mod-\Lambda} \) of all modules \( M \) so that \( \text{Hom}_{\Lambda}(|T|, M) = 0 = \text{Ext}^1_{\Lambda}(|T|, M) \). Then \( T^\perp \) is a finitely generated wide subcategory of \( \text{mod-\Lambda} \). We recall that a full subcategory \( W \) of \( \text{mod-\Lambda} \) is wide if it is an exactly embedded abelian subcategory of \( \text{mod-\Lambda} \) which is closed under extensions. A wide subcategory \( W \) is finitely generated if it contains one object \( P \) so that every other object of \( W \) is a quotient of \( P^k \) for some \( k \). Every finitely generated wide subcategory of \( \text{mod-\Lambda} \) occurs as a perpendicular category \(([11], [4]) \). Also, every finitely generated wide subcategory \( W \) is isomorphic to \( \text{mod-H}_W \) for some hereditary algebra \( H_W \). A cluster-tilting set in \( W \) is defined to be a subset of \( D^b(W) \) corresponding to a cluster-tilting set for \( H_W \). We denote by \( C_W \) the set of isomorphism classes of exceptional objects of \( W \) and shifts \( P[1] \in D^b(W) \) of indecomposable projective objects \( P \in W \). Thus \( C_W \cong C_{H_W} \).
Definition 1.3. The cluster morphism category $\mathcal{X}(\Lambda)$ of an hereditary algebra $\Lambda$ has as objects the finitely generated wide subcategories $\mathcal{W}$ of $\text{mod-}\Lambda$. A morphism $[T] : \mathcal{W} \rightarrow \mathcal{W}'$ in $\mathcal{X}(\Lambda)$ is defined to be a partial cluster tilting set $T$ in $\mathcal{W}$ so that $\mathcal{W}' = T^\perp \cap \mathcal{W}$.

The composition of cluster morphisms

$\mathcal{W} \xrightarrow{[S]} \mathcal{W}'' \xrightarrow{[T]} \mathcal{W}'$

is defined to be the cluster morphism $[R] : \mathcal{W} \rightarrow \mathcal{W}''$ where $R$ is the partial cluster-tilting object in $\mathcal{W}$ given by

$R = S \bigsqcup \sigma_S(t)$

where $\sigma_S : \mathcal{C}_{\mathcal{W}''} \rightarrow \mathcal{C}_\mathcal{W}$ sends $T \in \mathcal{C}_{\mathcal{W}''}$ to the unique element $\sigma_S(T) \in \mathcal{C}_\mathcal{W}$ satisfying the following.

1. $S \bigsqcup \sigma_S(T) = \{S_1, \ldots, S_r, \sigma ST\}$ is a partial cluster tilting set for $\mathcal{W}$.
2. $\dim \sigma_ST - \dim T$ is an integer linear combination of $\dim S_i$.
3. $T^\perp \cap \mathcal{W}'' = (S_1 \oplus \cdots \oplus S_r \oplus \sigma ST)^\perp \cap \mathcal{W}$.

The mapping $\sigma_S(T)$ is a bijection from $\mathcal{C}_{\mathcal{W}''}$ to the set of all objects of $\mathcal{C}_{\mathcal{W}}$ which are compatible with $S$ [5, Prop. 1.8].

Theorem 1.4 (Theorem A). The cluster morphism category of any hereditary algebra $\Lambda$ is a cubical category.

Proof. Let $\mathcal{C}$ denote the cluster morphism category of $\Lambda$. The objects of $\mathcal{C}$ are finitely generated wide subcategories $\mathcal{W}$ of $\text{mod-}\Lambda$. Each of these is equivalent to a module category of an hereditary algebra $H_{\mathcal{W}}$ of finite rank which we define to be the rank of $\mathcal{W}$. Also, $\mathcal{W}$ is the perpendicular category of some partial cluster tilting set $T = \{T_1, \ldots, T_i\}$, i.e., $\mathcal{W} = T^\perp$ and $rk \mathcal{W} = n - t$.

A cluster morphism $[R] : \mathcal{W} \rightarrow \mathcal{W}'$ is, by definition, a partial cluster-tilting set in $\mathcal{W}$:

$R = \{R_1, \ldots, R_r\} \subset \mathcal{C}_\mathcal{W}$

so that $\mathcal{W}' = \mathcal{W} \cap R^\perp$. It follows that $r = rk \mathcal{W} - rk \mathcal{W}'$. When $r = 0$, $\mathcal{W}' = \mathcal{W}$ and $[\emptyset] : \mathcal{W} \rightarrow \mathcal{W}$ is the identity map. Thus, objects and morphisms of $\mathcal{C}$ have ranks so that Condition (1) in Definition 1.3 is satisfied.

Every factorization of a cluster morphism $[R] : \mathcal{W} \rightarrow \mathcal{W}'$ of rank $r$ is given by

$\mathcal{W} \xrightarrow{[S]} \mathcal{W}'' \xrightarrow{[T]} \mathcal{W}'$

where $S$ is a subset of $R$ of size, say $s$ and $T \subset \mathcal{C}_{\mathcal{W}''}$ is a partial cluster tilting set in $\mathcal{W}'' = S^\perp \cap \mathcal{W}$ so that

$R = S \bigsqcup \sigma_S(T)$

where $\sigma_S : \mathcal{C}_{\mathcal{W}''} \rightarrow \mathcal{C}_\mathcal{W}$ is described in Definition 1.3. The objects $\mathcal{W}'' = S^\perp \cap \mathcal{W}$ are distinct for different $S$ since, otherwise we would have $(S')^\perp = S^\perp = (S \cup S')^\perp$ which is impossible since $S \cup S'$ is larger than $S$. (The size of $S$ is $rk \mathcal{W}'' - rk \mathcal{W}'$.). This gives an isomorphism between the category of factorizations of the cluster morphism $[R] : \mathcal{W} \rightarrow \mathcal{W}'$ with the cube category $\mathcal{T}$. So, Condition (2) is satisfied.

The possible first factors of the cluster morphism $[R]$ are the elements $[R_i]$ of $R$. So, $[R]$ is determined by its first factors and Condition (3) is satisfied.
To verify the last property (4), let \([L_i : W_i \to W']\) be the last factors of \([R]\). Then we have the following factorizations of the morphism \([R]\).

\[
W \xrightarrow{[S]} W_i \xrightarrow{[L]} W'.
\]

where \([S] = [R_1, \cdots, \tilde{R}_i, \cdots, R_r]\). Since \(W_i = W \cap (S)^\perp\), we have the following linear equation

\[
\langle \dim R_i, \dim L_j \rangle = 0
\]

for \(j \neq i\) where \(\langle \cdot, \cdot \rangle\) is the Euler-Ringel form uniquely determined by the formula

\[
\langle \dim A, \dim B \rangle = \dim K \text{Hom}_\Lambda(A, B) - \dim K \text{Ext}^1_\Lambda(A, B)
\]

and \(\dim M \in \mathbb{N}^n\) is the dimension vector of any \(\Lambda\)-module \(M\). If we choose a cluster tilting set \(T = \bigoplus T_k\) for \(W' = S^\perp \cap W\), we also have

\[
\langle \dim R_i, \dim T_k \rangle = 0
\]

for all \(i, k\). Since \(\sigma_{S_i}(T_i) = R_i\), the characterizing properties of the mapping \(\sigma_{S_i} : C_{W'} \to C_W\) also give one more linear equation:

\[
\langle \dim R_i, \dim L_i \rangle = \langle \dim L_i, \dim L_i \rangle.
\]

This is the dimension over \(K\) of the division algebra \(\text{End}_\Lambda(L_i)\). The wide subcategory \(W\) is determined by the last factors \(L_i : W_i \to W'\) since \(W\) is generated by the union of the \(W_i\). The rank of \(W\) is equal to \(r\) plus the rank of \(W'\). Since the Euler-Ringel form, restricted to \(W\) is nonsingular, the linear equations above have a unique solution for \(\dim R_i\) given the last factors \(L_j\). Each \(R_i \in W \bigsqcup W[1]\) is determined by its dimension vector \(\dim R_i \in \mathbb{Z}^n\). Therefore, the last factors \(L_i\) determine the first factors \(R_i\) of the cluster morphism \([R]\). And the first factors determine \([R]\). Thus, Condition (4) is satisfied. So, \(C\), the cluster morphism category of \(\Lambda\) is a cubical category.

2. Pairwise Compatibility

Let \(C\) be a cubical category (Definition 1.1). We say that \(C\) is a \(\text{CAT}(0)\)-category if its classifying space is locally \(\text{CAT}(0)\). The following Theorem 2.1 from [8] converts this into a sufficient algebraic condition. This is the theorem that we use in order to determine when a cluster morphism category is \(\text{CAT}(0)\).

We recall that the classifying space of a cubical category is a locally cubical space in the sense that its universal covering is cubical. Gromov [2] gave necessary and sufficient conditions for a simply connected cubical space to be \(\text{CAT}(0)\). In [8] we used Gromov’s theorem to prove Theorem 2.1 below.

**Theorem 2.1.** [8] The following are sufficient conditions for a cubical category \(C\) to be a \(\text{CAT}(0)\)-category.

1. \(n\) morphisms \(X \to A_i\) form the first factors of a morphism \(X \to Y\) of rank \(n\) if and only if every pair of them form the first factors of a morphism of rank \(2\).
2. \(n\) morphisms \(B_i \to Y\) form the last factors of a morphism \(X \to Y\) of rank \(n\) if and only if every pair of them form the last factors of a morphism of rank \(2\).
3. There exists a faithful functor from \(C\) to a group \(G\) considered as a category with one object.
Furthermore, the first two conditions are necessary.

It is immediate that the cluster morphism category satisfies the first condition since the objects $T_i \in \mathcal{C}(W)$ in a morphism $[T] : W \to W'$ are given by the pairwise compatibility condition that they do not extend each other. In this section we will verify the second condition for $\Lambda$ of finite type and of tame type with only small tubes. For this we use the following result of Speyer and Thomas.

**Theorem 2.2.** [12] The dimension vectors $\dim X_i \in \mathbb{Z}^n$ of a collection of $n$ exceptional modules and shifted exceptional modules $X_i$ (with underlying modules $M_i$) form the $c$-vectors of a cluster tilting object for an hereditary algebra $\Lambda$ if and only if the following two conditions hold.

1. Objects $X_i, X_j$ of the same sign are Hom-orthogonal.
2. The modules $M_i$ form an exceptional sequence with the negative modules to the right of the positive modules.

The second conditions is equivalent to the following three conditions.

1. The positive modules $X_i$ can be arranged in an exceptional sequence.
2. The underlying modules $M_j$ of the shifted objects $X_j = M_j[1]$ can be arranged in an exceptional sequence.
3. For every $X_i = M_i$ and $X_j = M_j[1]$, $(M_i, M_j)$ forms an exceptional pair.

**Corollary 2.3.** $c$-vectors for $\Lambda$ are given by a pairwise compatibility condition if and only if the following condition is satisfied:

(*) Every collection of Hom-orthogonal exceptional modules having the property that any pair is an exceptional pair can be arranged in an exceptional sequence.

**Corollary 2.4.** The cluster morphism category of $\Lambda$ satisfies the second condition in Theorem 2.1 if and only if it satisfies Condition (*) in Corollary 2.3 holds.

**Proof.** The linear equations relating first factors $R_i$ and last factors $L_j$ of a cluster morphism $R = [R_1, \ldots, R_k]$ were given in the proof of Theorem 1.4:

$$\langle \dim R_i, \dim L_j \rangle = \begin{cases} \dim_K \text{End}(L_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

By [4, Theorem 4.3.1] these equations imply that the last factors of any cluster morphism $[R] : \text{mod-}\Lambda \to 0$ have, as dimension vectors, the negatives of the $c$-vectors of the cluster tilting object $R$ of $\Lambda$. By the previous Corollary 2.3 Condition (*) is necessary for these last factors to be given by a pairwise compatibility condition as required by (2) in Theorem 2.1.

To show sufficiency, suppose that (*) holds and $[L_i] : \mathcal{W}_i \to \mathcal{W}$ is a collection of rank 1 cluster morphisms which are pairwise compatible in the sense that any pair $[L_i], [L_j]$ form the last factors of a rank 2 cluster morphism $[L_{ij}] : \mathcal{W}_{ij} \to \mathcal{W}$. Then each object $L_i$ is left perpendicular to $\mathcal{W}$, i.e., lies in the wide subcategory $\perp \mathcal{W}$ and the same objects $L_i$ give rank 1 morphisms $[L_i] : \mathcal{W}_i \cap \perp \mathcal{W} \to 0$ which are pairwise compatible. Thus, by (*) they form the last factors of some morphism $[R] : \mathcal{W}' \to 0$ where $\mathcal{W}' \subset \perp \mathcal{W}$. Equivalently,

$$\langle \dim R_i, \dim L_j \rangle = \delta_{ij} \langle \dim L_j, \dim L_j \rangle$$
which implies that \([R]\) gives a morphism \(W'' \to W\) with last factors \([L_i] : W_i \to W\).

Thus condition (2) in Theorem 2.1 holds. □

It remains to determine when Condition (*) holds.

**Theorem 2.5.** Let \(\Lambda\) be an hereditary algebra which is either of finite or tame representation type. Then Condition (*) in Corollary 2.3 holds if and only if the Auslander-Reiten quiver of \(\Lambda\) has no tubes of rank \(\geq 3\).

**Proof.** If \(\Lambda\) has a tube of rank \(\geq 3\) then (*) fails since the quasi-simple objects of the mouth of this tube are Hom orthogonal and any pair of them forms an exceptional sequence, but, taken together, they do not form an exceptional sequence since they extend each other in a cycle.

Conversely suppose that \(\Lambda\) is either of finite type or tame with only small tubes (of rank \(\leq 2\)). Let \(M_i\) be a collection of Hom-orthogonal exceptional \(\Lambda\) modules so that any two form an exceptional pair. Then, each exceptional tube can have at most one of the objects \(M_i\). We can then arrange these module by taking first the preinjective modules, arranged in order of distance from the injective modules, i.e., from right-to-left, in the preinjective component, then taking the regular modules and finally the preprojective modules in decreasing distance from the projective modules, i.e., right-to-left in the preprojective component. In finite type, we simply arrange the modules right to left in the AR-quiver. This produces an exceptional sequence since extensions go right-to-left in the AR-quiver (or around the tubes). □

### 3. Faithful Group Functor

In this section we complete the proof that the cluster morphism category is \(CAT(0)\) by constructing a faithful group functor to the picture group \(G(\Lambda)\) for all hereditary algebras \(\Lambda\) of finite or tame type. We view \(G(\Lambda)\) as a groupoid with one object and we construct a faithful contravariant functor

\[ F : X(\Lambda) \to G(\Lambda) \]

This is equivalent to the faithful covariant functor \(F'([X]) = F([X])^{-1}\) since inversion makes every groupoid isomorphic to its opposite category.

#### 3.1. Formula for the Group Functor

Recall first that the picture group \(G(W)\) of any finitely generated wide subcategory \(W\) of \(mod-\Lambda\) is naturally embedded as a subgroup of the picture group \(G(\Lambda)\) since the generators of \(G(W)\) are \(x(\beta)\) for all dimension vectors \(\beta\) of exceptional modules \(M_\beta \in W\) and these are some of the generators of \(G(\Lambda)\). Furthermore, any relations among these \(x(\beta)\) are relations which hold in \(G(\Lambda)\) since \(W\) is extension closed in \(mod-\Lambda\). Thus we have a canonical homomorphism

\[ \varphi_W : G(W) \to G(\Lambda). \]

Similarly, for any \(W' \subset W\), we have a homomorphism \(\varphi_{W'}^W : G(W') \to G(W)\) so that

\[ \varphi_{W'} = \varphi_W \circ \varphi_{W'}^W : G(W') \to G(\Lambda). \]

With this notation we will prove the following.
Theorem 3.1. Let $\Lambda$ be an hereditary algebra of finite or tame type. Then, to every morphism $[X] : W \to W'$ in the cluster morphism category of $\Lambda$, there is $\gamma_W[X] \in G(W)$ with the property that, for all morphisms $[Y] : W' \to W''$, we have

$$\gamma_W([Y] \circ [X]) = \gamma_W[X] \varphi_W^W(\gamma_W[Y]).$$

Furthermore, for any two distinct morphisms $[X] \neq [X'] : W \to W'$, the corresponding group elements are distinct: $\varphi_W(\gamma_W[X]) \neq \varphi_W(\gamma_W[X']) \in G(\Lambda)$.

In other words, $F([X]) = \varphi_W(\gamma_W[X])^{-1}$ is a $G(\Lambda)$ valued faithful group functor on the cluster morphism category of $\Lambda$.

This theorem, assigning a picture group element to every partial cluster tilting object of $W$, will follow from the special case of (complete) cluster tilting objects, the case which is already known by [7, 5, 6] and [3].

Proposition 3.2. For any hereditary algebra $\Lambda$ of finite or tame representation type there is a function $\overline{\varphi}$ which assigns to every cluster tilting object $T = T_1 \oplus \cdots \oplus T_n$ an element $\overline{\varphi}(T)$ in the picture group $G(\Lambda)$ satisfying the following properties.

1. For the cluster tilting object $P[1]$ consisting of the $n$ shifted projective modules, $\overline{\varphi}(P[1]) = e \in G(\Lambda)$, the identity element.

2. If $T' = T \setminus T_k \oplus T'_k$ is the mutation of $T$ in direction $k$, $\overline{\varphi}(T') = \overline{\varphi}(T)[x(\beta)]^\epsilon$ when $D(\beta)$ is the wall separating the two clusters, i.e., $M_\beta$ is the unique indecomposable object in $(T \setminus T_k)^\perp$. The sign $\epsilon$ is positive when $T'_k$ is on the positive side of $D(\beta)$, i.e., when $\text{Hom}_\Lambda(T'_k, M_\beta) \neq 0$, and $\epsilon = -1$ otherwise.

3. If $T \not\cong T'$ then $\overline{\varphi}(T) \neq \overline{\varphi}(T') \in G(\Lambda)$.

Proof. Statement (1) is by definition and (2) is proved in [5, 7, 6] for finite type and follows from [3] in the tame case since, by the polygonal deformation lemma, there is a finite sequence of mutations from any cluster to the initial cluster (of shifted projective objects) which is well-defined up to polygonal deformation and thus has well-defined picture group element associated to it by multiplying the generators $x(\beta)$ for the walls that are crossed.

To prove (3) we need to review these arguments. We fix the clusters $T, T'$ and choose finite mutation paths from the initial cluster to $T, T'$. We keep track of all the walls of all of the clusters along those two paths. Then, we can enumerate the dimension vectors $\beta_i$ of the exceptional modules in order of length up to and including the lengths of those $\beta$ for which $D(\beta)$ occurs as a wall in one of those two mutation paths. Thus $\beta_1, \ldots, \beta_n$ are the simple roots. Let $S_k = \{\beta_1, \ldots, \beta_k\}$ and let $G(S_k)$ be the subgroup of the picture group $G(\Lambda)$ generated by $x(\beta_i)$ where $i \leq k$. Then, the canonical homomorphism $G(S_k) \to G(S_{k+1})$ has a retraction $\rho : G(S_{k+1}) \to G(S_k)$ given by sending $x(\beta_{k+1})$ to 1.

We recall [6] Theorem 1.18: The walls $D(\beta_j)$ for $j \leq k$ separate $\mathbb{R}^n$ into convex regions called compartments and denote $U_{\epsilon}$ in [6]. To each such region we can associate a picture group element $g_k(U_{\epsilon}) \in G(S_k)$ so that, if two regions $U_{\epsilon_1}, U_{\epsilon_2}$ are adjacent and separated by a wall $D(\beta)$, $\beta \in S_k$ with $U_{\epsilon_2}$ on the positive side of $D(\beta)$ then

$$g_k(U_{\epsilon_2}) = g_k(U_{\epsilon_1}) x(\beta).$$
(This is statement (2).) Furthermore, these group elements are uniquely determined if we stipulate that the component \( U_- \) on the negative side of all the walls \( D(\beta_j) \) is assigned the identity element of the group: \( g_k(U_-) = e \in G(S_k) \).

For example, when \( k = 1 \), there is only one wall \( D(\beta_1) \) which is a hyperplane. This divides \( \mathbb{R}^n \) into two regions \( U_- \) and \( U_+ \) with \( g_1(U_-) = e \) and \( g_1(U_+) = x(\beta_1) \).

When we go from \( S_k \) to \( S_{k+1} \) we add the wall \( D(\beta_{k+1}) \). This being part of a hyperplane will either cut the compartment \( U_i \) into two pieces \( U_{i+} \), on the positive side of \( D(\beta_{k+1}) \), and \( U_{i-} \) on the negative side of \( D(\beta_{k+1}) \) or \( U_i \) will be disjoint from \( D(\beta_{k+1}) \) in which case we re-label the compartment as \( U_0 \). In either case, the retraction \( \rho : G(S_{k+1}) \rightarrow G(S_k) \) will send the new group element \( g_{k+1}(U_*) \) to \( g_k(U_*) \) since \( g_{k+1}(U_*) \) is the product of \( x(\beta) \) for the walls \( D(\beta) \) which are crossed to get from \( U_- \) to the compartment \( U_+ \) and, when the walls \( D(\beta_{k+1}) \) are deleted, this get the group element \( \rho(g_{k+1}(U_*)) = g_k(U_*) \).

By induction on \( k \) we have the following.

**Statement for \( S_k \):** Distinct compartments have different elements of the picture group assigned to them, i.e., \( g_k(U_i) \neq g_k(U_j) \) when \( U_i \neq U_j \).

To show that this holds for \( S_{k+1} \), suppose not. Then there are two compartments \( U, U' \) in the complement of the walls \( D(\beta_j) \) for \( j \leq k + 1 \) which have the same group element \( g_{k+1}(U) = g_{k+1}(U') \in G(S_{k+1}) \). Then \( \rho(g_{k+1}(U)) = \rho(g_{k+1}(U')) \) in \( G(S_k) \). Therefore, by induction on \( k \), \( U, U' \) lie in the same compartment \( U_\epsilon \) for \( S_k \). Since \( U \neq U' \), by symmetry we must have \( U = U_+, U' = U_- \). But then,

\[
g_{k+1}(U) = g_{k+1}(U_+) = g_{k+1}(U_-) x(\beta_k) = g_{k+1}(U') x(\beta_k+1).
\]

So, \( g_{k+1}(U) \neq g_{k+1}(U') \). So, the statement holds for all \( S_k \). Taking \( \gamma \) to be \( g_k \) for \( k \) maximal we obtain statement (3). So, the proposition holds. \( \square \)

By Proposition 3.2 above, the function \( \gamma \) taking a cluster tilting object \( T \) to \( \gamma[T] \in G(\Lambda) \) is given by

\[
\gamma[T] = x(\beta_1)^{\varepsilon_1} x(\beta_2)^{\varepsilon_2} \cdots x(\beta_k)^{\varepsilon_k}
\]

where \( \beta_1, \ldots, \beta_k \) are the \( c \)-vectors of any mutation sequence from the shifted projected cluster \( P_{1}[1], \ldots, P_{n}[1] \) to \( T \) and the sign \( \varepsilon_i \) is positive/negative depending on whether the mutation is green/red. (For more explanations, see [9], [10], [4], [7], [5].)

Recall that the picture group of \( \Lambda \) is the group with generators \( x(\beta) \) for all real Schur roots \( \beta \) which are also the dimension vectors of all exceptional \( \Lambda \)-modules. The relations are given as follows where \( [a, b] := b^{-1}aba^{-1} \).

\[
[x_{\beta_i}, x_{\beta_j}] = x_{\gamma_1} \cdots x_{\gamma_r}
\]

for \( \beta_i, \beta_j \) any pair of hom orthogonal roots (roots whose corresponding modules are Hom-orthogonal) so that \( \text{Ext}(\beta_i, \beta_j) = 0 \) and \( \gamma_k \) are the dimension vectors of indecomposable extensions \( M_{\gamma_k} \) of \( M^a_{\beta_i} \) by \( M^b_{\beta_j} \) in increasing order of the fraction \( a_k/b_k \). For example, in type \( B_2 \) we have:

\[
x_\alpha x_\beta = x_\beta x_\alpha + x_\alpha + x_\beta x_\alpha.
\]

3.2. **Proof of Theorem 3.1** We first prove the special case when \( [X] : \mathcal{W} \rightarrow \mathcal{W}' \) has rank 1, i.e., \( X \) is either an exceptional module in \( \mathcal{W} \) or a shifted indecomposable projective object of \( \mathcal{W} \).
Lemma 3.3. Suppose that \([X] : \mathcal{W} \rightarrow \mathcal{W}'\) has rank 1 and \([Y]\) is a cluster tilting set in \(\mathcal{W}'\). Then

\[
\gamma_{\mathcal{W}}([Y] \circ [X]) = \gamma_{\mathcal{W}}[X] \varphi_{\mathcal{W}}[\gamma_{\mathcal{W}}[Y]]
\]

Proof. Since finitely generated wide subcategories are equivalent to module categories, we can take \([X]\) to be a rank 1 morphism \(\text{mod-}\Lambda \rightarrow \mathcal{W}\) and \([Y]\) a cluster-tilting set in \(\mathcal{W}\). Since \(\Lambda\) is tame, \(\mathcal{W}\) has finite type. Let \([Z] = \sigma_X[Y]\). Then \([X, Z]\) is a cluster-tilting set for \(\text{mod-}\Lambda\). Given that \(X\) is an indecomposable exceptional module or shifted projective module, it gives a vertex in the picture of \(\Lambda\). The cluster \([X, Z]\) gives a chamber \(\tilde{U}'\) in the picture of \(\Lambda\) with vertices \(X\) and the elements of \([Z]\). This chamber has an associated picture group element

\[
\gamma(\tilde{U}') = [X, Z] \in G(\Lambda).
\]

Let \([P[1]]\) be the set of shifted projective objects of \(\mathcal{W}\) and let \([Q] = \sigma_X[P[1]]\). Then \([X, Q]\) is another cluster-tilting set for \(\Lambda\) corresponding to another chamber \(\tilde{U}\) in the picture of \(\Lambda\) and, by definition of \(\gamma([X])\), we have

\[
\gamma(\tilde{U}) = [X, Q] = \gamma[X] \in G(\Lambda).
\]

The statement of Lemma 3.3 is

\[
\gamma(\tilde{U}') = \gamma(\tilde{U}) \gamma_{\mathcal{W}}[Y]
\]

where \(\gamma_{\mathcal{W}}[Y] \in G(\mathcal{W})\) is the picture group element assigned to the chamber \(\tilde{U}'\) in the picture for \(\mathcal{W}\) with vertices given by \([Y]\). By definition, this is the product of all labels \(x(\beta_i)\) corresponding to the walls \(D(\beta_i)\) that need to be crossed to get from the chamber \(\tilde{U}\) given by \([P[1]]\) to the chamber \(\tilde{U}'\) given by \([Y]\). However, the picture for \(\mathcal{W}\) is isomorphic to the local picture for \(\Lambda\) at the point \(X\) and the corresponding walls have the same labels. The vertices of the picture for \(\mathcal{W}\) are objects of \(\mathcal{W}\) (or shifted projective objects) and the corresponding vertices of the picture for \(\Lambda\) are given by applying \(\sigma_X\) to these labels. Since \(\tilde{U}, \tilde{U}'\) correspond to \(\tilde{U}, \tilde{U}'\) by construction and the walls separating them have identical labels, \(\gamma_{\mathcal{W}}[Y]\) is equal to the product of the labels of the walls in the picture for \(\Lambda\) separating \(\tilde{U}\) from \(\tilde{U}'\). Therefore,

\[
\gamma_{\mathcal{W}}[Y] = \gamma(\tilde{U})^{-1} \gamma(\tilde{U}').
\]

This completes the proof of Lemma 3.3.

This proof is illustrated in Figure 1 in the case \(X = P_2\), \([Y] = [S_3, S_1[1]]\) and \([P[1]] = [S_1[1], S_3[1]]\). Then \([Z] = [P_3, S_2]\) and \([Q] = [S_2, P_3[1]]\). The chamber labels \(\mathcal{U}, \mathcal{U}', \tilde{U}, \tilde{U}'\) agree with the labels in the proof. \(\square\)

Theorem 3.1 will follow from the following Key Lemma.

Lemma 3.4. Let \(\mathcal{W}\) be a finitely generated wide subcategory of \(\text{mod-}\Lambda\) of rank \(k\) and let \([P_\mathcal{W} = [P_1, \ldots, P_k]\) be the set of projective objects of \(\mathcal{W}\). Let \([P_\mathcal{W}[1]]\) be the set of shifted projective objects \(P_i[1]\) in the bounded derived category of \(\mathcal{W}\). Then, for any pair of morphisms \([X] : \text{mod-}\Lambda \rightarrow \mathcal{W}\), \([Y] : \mathcal{W} \rightarrow 0\) in the cluster morphism category we have the following equation in the picture group of \(\Lambda\).

\[
\gamma([X] \bigsqcup \sigma_X[Y]) = \gamma([X] \bigsqcup \sigma_X[P_\mathcal{W}[1]]) \varphi_{\mathcal{W}}(\gamma_{\mathcal{W}}([Y])).
\]
Proof. This will follow from Lemma 3.3. Thus, suppose we have:

\[ \text{mod-}\Lambda \xrightarrow{[X]} \mathcal{W} \xrightarrow{[Y]} 0 \]

and \([P[1]] : \mathcal{W} \to 0\) is the cluster morphism given by the shifted projective objects in \(\mathcal{W}\). We need to show that

\[ (3.1) \quad \gamma([Y] \circ [X]) = \gamma([X]) \coprod [P[1]] \varphi_{\mathcal{W}}(\gamma_{\mathcal{W}}([Y])). \]

When \([X]\) has rank 1, this holds by Lemma 3.3. So, suppose \([X]\) has rank \(\geq 2\). Then \([X]\) factors as a composition

\[ [X] = [B] \circ [A] : \text{mod-}\Lambda \xrightarrow{[A]} \mathcal{W}^* \xrightarrow{[B]} \mathcal{W}. \]

Since \([A], [B]\) have smaller rank than \([X]\) we have

\[ \gamma([Y] \circ [X]) = \gamma([Y] \circ [B] \circ [A]) = \gamma([A])\gamma_{\mathcal{W}^*}([Y] \circ [B]) = \gamma([A])\gamma_{\mathcal{W}^*}([B] \gamma_{\mathcal{W}}([Y]). \]

In the special case \(Y = P[1]\) we get:

\[ \gamma([X]) = \gamma([P[1]] \circ [X]) = \gamma([A])\gamma_{\mathcal{W}^*}([P[1]] \circ [B]) = \gamma([A])\gamma_{\mathcal{W}^*}([B]). \]

So,

\[ \gamma([Y] \circ [X]) = \gamma([A])\gamma_{\mathcal{W}^*}([B] \gamma_{\mathcal{W}}([Y]) = \gamma([X])\gamma_{\mathcal{W}}([Y]). \]

This proves the Key Lemma 3.4. \(\Box\)

We need one more lemma to obtain a faithful group functor for the cluster morphism category of \(\Lambda\) or, equivalently, for any finitely generated wide subcategory \(\mathcal{W}\) of finite or tame type in \(\text{mod-}\mathcal{H}\) for any hereditary algebra \(\mathcal{H}\). Indeed, for any such wide subcategory \(\mathcal{W}\) and any partial cluster tilting object \(X\) in \(\mathcal{W}\) we define \(\gamma_{\mathcal{W}}([X]) \in G(\mathcal{W})\) to be

\[ (3.2) \quad \gamma_{\mathcal{W}}([X]) := \gamma_{\mathcal{W}}([X]) \coprod \sigma_X[P[1]] \]

where \(P[1]\) is the set of all shifted projective objects in the wide subcategory \(\mathcal{W}' = X^\perp \cap \mathcal{W}\) of \(\mathcal{W}\). Then, as a special case of Lemma 3.4 we have

\[ \gamma_{\mathcal{W}^*}([Y] \circ [X]) = \gamma_{\mathcal{W}^*}([X]) \coprod \sigma_X([Y]) = \gamma_{\mathcal{W}^*}([X]) \varphi_{\mathcal{W}^*}(\gamma_{\mathcal{W}^*}([Y])) \]

for any cluster tilting set \(Y\) in \(\mathcal{W}^*\).

Lemma 3.5. Let \([X] : \mathcal{W} \to \mathcal{W}'\), \([Y] : \mathcal{W}' \to \mathcal{W}''\) be morphisms in the cluster morphism category of \(\Lambda\). Then

\[ \gamma_{\mathcal{W}}([X] \coprod \sigma_X[Y]) = \gamma_{\mathcal{W}''}([X]) \varphi_{\mathcal{W}''}^{\mathcal{W}}(\gamma_{\mathcal{W}''}([Y])). \]

Proof. Let \(Z\) be any cluster tilting set in \(\mathcal{W}''\) and let \([A] = [Y] \circ [X], [B] = [Z] \circ [Y], [C] = [Z] \circ [A] = [B] \circ [X]\). Since \([B] = [Z] \circ [Y]\) is a cluster tilting set for \(\mathcal{W}'\) we have

\[ \gamma_{\mathcal{W}^*}(\gamma_{\mathcal{W}''}([B])) = \gamma_{\mathcal{W}''}([Y]) \gamma_{\mathcal{W}''}([Z]) \]

Since \([C] = [B] \circ [X] = [Z] \circ [A]\) is a cluster tilting set for \(\mathcal{W}\) we also have

\[ \gamma_{\mathcal{W}}([C]) = \gamma_{\mathcal{W}}([B] \circ [X]) = \gamma_{\mathcal{W}}([X]) \gamma_{\mathcal{W}}([B]) = \gamma_{\mathcal{W}}([X]) \gamma_{\mathcal{W}''}([Y]) \gamma_{\mathcal{W}''}([Z]) \]

\[ \gamma_{\mathcal{W}}([C]) = \gamma_{\mathcal{W}}([Z] \circ [A]) = \gamma_{\mathcal{W}''}([A]) \gamma_{\mathcal{W}''}([Z]) \]

Multiplying by \(\gamma_{\mathcal{W}''}([Z])^{-1}\) we obtain

\[ \gamma_{\mathcal{W}}([X]) \gamma_{\mathcal{W}''}([Y]) = \gamma_{\mathcal{W}''}([A]) = \gamma_{\mathcal{W}''}([X] \coprod \sigma_X[Y]) \]
as claimed. □

**Example 3.6.** Consider the standard example: \( \Lambda = KA_3 \), for the linearly oriented quiver \( A_3 \):

\[
1 \leftarrow 2 \leftarrow 3.
\]

Consider the cluster morphisms

\[
\text{mod}\nobreak-\Lambda \xrightarrow{[X]} \mathcal{W} \xrightarrow{[Y]} \mathcal{W}'
\]

where \( X = P_2 \), with \( \mathcal{W} = P_2^\perp \) having indecomposable objects \( S_1 \) and \( S_3 = Y \). Then \( \sigma_{P_2} : \mathcal{C}_W \to \mathcal{C}_\Lambda \) sends the objects \( S_1, S_3, S_1[1], S_3[1] \) to \( S_1, P_3, S_2, P_3[1] \) resp. So, \( \sigma_X(Y) = \sigma_{P_2}(S_3) = P_3 \) which makes \( \mathcal{W}' = (P_2 \oplus P_3)^\perp \) with one indecomposable object \( S_1 \).

By (3.2), the picture group element corresponding to \( X = P_2 \) is

\[
\gamma(P_2) = \gamma(P_2 \bigsqcup \sigma_{P_2}(S_1[1], S_3[1])) = \gamma(P_2, S_2, P_3[1]) = x(S_2)x(P_2) \in G(\Lambda)
\]

since, to get to the chamber \( \tilde{U} \) in Figure 1 with corners \( P_2, S_2, P_3[1] \) from the unbounded chamber, we need to pass through the walls \( D(S_2) \) and \( D(P_2) \).

Since \( \mathcal{W} \) is semi-simple, \( G(\mathcal{W}) = \mathbb{Z}^2 \) is the free abelian group generated by \( x(S_1), x(S_3) \). The picture group element for \( Y = S_3 \) is

\[
\gamma_{\mathcal{W}}(S_3) = \gamma_{\mathcal{W}}(S_3 \bigsqcup \sigma_{S_3}(S_1[1])) = \gamma_{\mathcal{W}}(S_3, S_1[1]) = x(S_3).
\]

Since \( \sigma_{P_2,P_3}(S_1[1]) = \sigma_{P_2}(S_1[1]) = S_2 \), the product of these is

\[
\gamma(P_2, P_3) = \gamma(P_2 \bigsqcup \sigma_{P_2,P_3}(S_1[1])) = \gamma(P_2, P_3, S_2) = x(S_2)x(P_2)x(S_3) = \gamma(P_2)\gamma_{\mathcal{W}}(S_3)
\]
Indeed, one must pass through the walls \( D(S_2), D(P_2), D(S_3) \) to get to the chamber \( \tilde{U}' \) in Figure 1 with corners \( P_2, P_3, S_2 \).

**Theorem 3.7.** Equation (3.2) gives a faithful (contravariant) group functor from the cluster morphism category of \( \Lambda \) to its picture group \( G(\Lambda) \) for \( \Lambda \) a hereditary algebra of finite or tame representation type.

*Proof.* Lemma 3.5 shows that \( F([X]) = \gamma_W([X])^{-1} \) is a group functor on the cluster morphism category of \( \Lambda \). It remains to show that it is faithful. To show this, suppose not. Then there are cluster morphisms \([X],[Y] : W \rightarrow W'\) with the same group value \( \gamma_W[X] = \gamma_W[Y] \). This would imply that the cluster tilting objects \([X] \oplus P[1]\) and \([Y] \oplus P[1]\) in \( W \) have the same group value in \( G(W) \) which is not possible by Proposition 3.2. \( \square \)

We now prove our main theorem.

**Theorem 3.8** (B). *The cluster morphism category of an hereditary algebra of finite or tame representation type is a \( CAT(0) \)-category if and only if there are no tubes of rank \( \geq 3 \) in its Auslander-Reiten quiver.*

*Proof.* Recall that a \( CAT(0) \)-category is a cubical category whose classifying space is locally \( CAT(0) \) and thus a \( K(\pi,1) \) by [2]. In Theorem 1.4 we showed that the cluster morphism category \( X(\Lambda) \) of any finite dimensional hereditary algebra satisfies the definition of a cubical category (Definition 1.1).

Theorem 2.1 gives criteria for when a cubical category is \( CAT(0) \). The first two conditions are necessary and the three conditions together are sufficient. The first condition is immediate since it requires that cluster-tilting objects be given by a pairwise compatibility condition which is true by definition for any hereditary algebra. Theorem 2.5 shows that the second condition (that the \( c \)-vectors of a cluster-tilting object are given by a pairwise compatibility condition) fails for tame algebras having an exceptional tube of rank \( \geq 3 \) and holds for all other \( \Lambda \) of finite or tame representation type. Finally, Theorem 3.7 proves the last condition, that there exists a faithful group functor from the cluster morphism category to some group when \( \Lambda \) has finite or tame representation type.

This concludes the proof of our main theorem, Theorem B. \( \square \)

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