Quantum phase-slip junction under microwave irradiation

A. Di Marco,1 F. W. J. Hekking,1 and G. Rastelli1,2

1LPMMC-CNRS, Université Grenoble Alpes, 25 Avenue des Martyrs B.P. 166, 38042 Grenoble Cedex, France
2Zukunftskolleg, Fachbereich Physik, Universität Konstanz, D-78457, Konstanz, Germany
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We consider the dynamics of a quantum phase-slip junction (QPSJ) – a dual Josephson junction – connected to a microwave source with frequency \( \omega_{\text{mw}} \). With respect to an ordinary Josephson junction, a QPSJ can sustain dual Shapiro steps, consisting of well-defined current plateaus at multiple integers of \( e \omega_{\text{mw}} / \pi \) in the current-voltage (I-V) characteristic. The experimental observation of these plateaus has been elusive up to now. We argue that thermal as well as quantum fluctuations can smear the I-V characteristic considerably. In order to understand these effects, we study a current-biased QPSJ under microwave irradiation and connected to an inductive and resistive environment. We find that the effect of these fluctuations are governed by the resistance of the environment and by the ratio of the phase-slip energy and the inductive energy. Our results are of interest for experiments aimed at the observation of dual Shapiro steps in QPSJ devices for the definition of a new quantum current standard.

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I. INTRODUCTION

The Josephson junction (JJ) is one of the most used superconducting devices in low-temperature condensed matter experiments. A single JJ is the building block of various sensors and electronic components13 and plays an important role in quantum computation and information.4–10 On a more fundamental level, JJs with small capacitance have become the paradigmatic systems for studying decoherence and dissipation of a quantum particle coupled to the external world and for analyzing the transition from quantum to classical states.11–20

Many of the JJ applications are based on the Josephson effect: a Cooper-pair tunneling current \( I_J \) can flow through a JJ in the absence of an applied bias voltage. The amplitude of this supercurrent is a non-linear function of the phase difference \( \Phi \) between the two superconductors of the junction, \( I_J = I_c \sin(\Phi) \). The critical current \( I_c \) is the maximum Cooper pair current that can be carried by the junction. A voltage drop \( V_J = (h/2e)d\Phi/dt \) appears across the junction when \( \Phi \) changes as a function of time. The classical dynamics of \( \Phi \) is ruled by the equations of motion for a fictitious particle moving in a periodic potential. In particular, a phase-locking effect can occur when the JJ is irradiated with microwaves of frequency \( \omega_{\text{mw}} \). Then the so-called Shapiro steps of constant voltage \( V_{J,m} = m(h/2e)\omega_{\text{mw}} \), with \( m \) integer, appear in the current-voltage characteristic in addition to the zero-voltage supercurrent state.21–24 These steps are related only to the fundamental constants of physics (the Planck constant \( \hbar \) and the electron charge \( e \)) and are used in metrology to define the quantum voltage standard.24–27 The necessary metrological accuracy is reached at low temperatures and using junctions with large Josephson energy \( E_J = \Phi_0 L_c / (2\pi) \sim 100 \text{ meV} \) (\( \Phi_0 = \hbar / (2e) \) is the superconducting flux quantum) and small charging energy \( E_C = e^2 / 2C \sim 10 \text{ nV} \), where the capacitance of the junction \( C \) plays the role of the inertial mass in the dynamics of the phase. Moreover, the JJ is typically embedded in a circuit whose resistance \( R \lesssim R_Q \), with \( R_Q = \hbar / (4e^2) = 6.45 \text{ k}\Omega \) the superconducting resistance quantum. Under these conditions, thermal and quantum fluctuations of the phase \( \Phi \) are suppressed efficiently.21–28

The Josephson junction has an exact dual counterpart, the so-called quantum phase-slip junction (QPSJ).29–37 Physical realizations of QPSJs that have been discussed in the literature are a single Josephson junction with a finite capacitance or a linear chain of such Josephson junctions and a narrow superconducting nanowire.32–42 With respect to an ordinary JJ, the role of the phase and the charge in a QPSJ is interchanged. Specifically, Cooper-pair tunneling is replaced by its dual process, i.e., the slippage by \( 2\pi \) of the phase difference between the two superconducting regions of the device. As a consequence, the relations governing the behavior of a QPSJ are exactly dual to the usual Josephson relations. The voltage \( V_J = V_c \sin(\pi q/e) \) across the QPSJ is a non-linear function of the charge variable \( q \), where the critical value \( V_c \) is the maximum voltage that the junction can sustain. The Cooper-pair current \( I_J = dq/dt \) is different from zero only for time-dependent \( q \). As a consequence, under microwave irradiation, a QPSJ should sustain a set of current steps, i.e., the dual Shapiro steps \( I_{J,m} = me \omega_{\text{mw}} / \pi \).

However, experimental evidence for the existence of dual steps has been elusive so far. Indeed, the dual Josephson relations pertain to a QPSJ with a relatively well defined charge \( q \) achieved when phase-slips are produced at an appreciable rate, a condition which is not easily compatible with the existence of a well-defined underlying superconducting state. Actual realizations of a QPSJ are typically operated in a regime where \( V_c \) is not large, so that charge fluctuations are important, and may well mask the dual Shapiro steps.

In this paper, we study the role of both thermal and quantum fluctuations of charge on the properties of the dual Shapiro steps. We present the results of a combined analytical and numerical analysis of a QPSJ irradiated with microwaves and embedded in a resistive (\( R \)) and inductive (\( L \)) electromagnetic environment. We will see, in particular, that an important role is played by the inductance \( L \), the quantity dual to the capacitance \( C \) of a usual Josephson junction. By duality, we expect that the fluctuations of the charge \( q \) are governed by the ratio...
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\omega sults for the be discussed in the following are independent of the specific
tive (in Fig. 1(b) where a QPSJ is connected in parallel to a resis-
tance. The total bias current is the sum of a DC component
and an AC one \(I_{\text{mw}}(t)\). This circuit is related to the voltage-biased one in (a) by the Thévenin-Norton theorem setting
\(I_0 = V_0/R\) and \(|I_{\text{mw}}| = |V_{\text{mw}}|/\sqrt{R^2 + L^2\omega_{\text{mw}}^2}\).

\(U_0/E_L\) of the characteristic phase-slip energy \(U_0 = 2eV_c/(2\pi)\), dual to the Josephson coupling energy \(E_J\), and the inductive energy \(E_L = \Phi_0^2/(2L)\), dual to the charging energy \(E_C\) of a Josephson junction. The larger \(L\), the smaller \(E_L\) and the larger the ratio \(U_0/E_L\), thus favoring a well-defined charge state of the QPSJ. Recent experiments on nanowires and chains of Josephson junctions typically achieve \(U_0/E_L\) ratios that are of the order of \(10^{-2} \text{ to } 10^{-1}\). We will analyze the microwave response of a QPSJ in this regime in detail and study in particular the resolution and accuracy of the dual Shapiro steps.

II. QUALITATIVE DISCUSSION OF THE MAIN RESULTS

The observation of the dual Shapiro steps is expected in the current-voltage characteristic \((I-V)\) of a voltage-biased QPSJ junction in series with an inductance \(L\) and an impedance \(Z(\alpha)\), which hereafter we assume to be frequency independent, i.e., \(Z(\alpha) = R\) [see Fig. 1(a)]. However, in this work, we will focus on the equivalent current-biased circuit shown in Fig. 1(b) where a QPSJ is connected in parallel to a resis-
tive \(R\) and inductive \(L\) electromagnetic environment and is driven by both a DC current source, \(I_0\), and an AC one \(I_{\text{mw}}(t) = I_{\text{mw}}(t) \cos \omega_{\text{mw}}t\) with amplitude \(I_{\text{mw}}\) and microwave fre-
quency \(\omega_{\text{mw}}\). The equivalence between the two circuits in Fig. 1 is provided by the Thévenin-Norton theorem. The results for the \(I_f-V_f\) curve of the QPSJ of Fig. 1(b) that will be discussed in the following are independent of the specific choice of the external bias.

Let us first consider the case when the environment is absent, \(R \to \infty\), in the circuit of Fig. 1(b). Then the dual Shapiro steps corresponding to \(m = 0, 1, 2, 3\) for a QPSJ in the absence of environment. The other parameters are: \(\alpha = 1.4\) and \(\hbar \omega_{\text{mw}}/2\pi U_0 = 1\).

\[
V_f^{(m)}(\omega_0) = V_c \sum_{m=-\infty}^{+\infty} J_m(\alpha) \sin (\chi_0 + \omega_0 t + m\omega_{\text{mw}}t), \quad (1)
\]

where \(J_m\) is a Bessel function of the first kind. We defined the parameters \(\alpha = \pi I_{\text{mw}}/(e\omega_{\text{mw}})\) and \(\omega_0 = \pi I_0/e\) as well as the dimensionless charge \(\chi_0 = \pi q_0/e\). From this result we see that whenever \(\omega_0 = m\omega_{\text{mw}}\), the QPSJ will sustain a charge-dependent DC voltage \(V_{I,m} = V_c J_m(\alpha) \sin \chi_0\). In other words, whenever the DC bias current \(I_0\) equals \(m\omega_{\text{mw}}/\pi\), phase-locking occurs, which leads to the appearance of a dual Shapiro step, located at \(I_{J,m} = m\omega_{\text{mw}}/\pi\) in the DC current-voltage characteristic of the QPSJ. The width of this step in voltage \(V_f\) is given by \(2V_c J_m(\alpha)\). Note that the parameter \(\alpha\) acts as the microwave coupling strength: the effect of the microwaves disappears as \(\alpha \to 0\). Figure 2 shows the dual Shapiro steps corresponding to \(m = 0, 1, 2, 3\) for moderate microwave intensity, \(\alpha = 1.4\), and microwave frequency \(\hbar \omega_{\text{mw}}/2\pi U_0 = 1\).

We next turn to the case when the resistance \(R\) of the envi-
ronment is finite. In this case, the total current \(I_f\) will contain two additional components. The first is the current flowing though the resistive-inductive branch of the circuit; it equals \(I_{RL}(t) = \int dt Y(t-t')V_f(t')\), where \(V_f(t) = V_c \sin \pi t\omega_0/\pi\) and \(Y(t)\) is the inverse Fourier transform of the admittance

\[
Y(\omega) = 1/(R - i\omega L), \quad (2)
\]

FIG. 1. (Color online) (a) Circuit of a voltage-biased QPSJ with phase-slip energy \(U_0 = eV_c/\pi\) in series with a resistance \(R\) and an inductance \(L\). The voltage source has a DC component \(V_0\) and an AC one \(V_{\text{mw}}(t)\). (b) Circuit of a current-biased QPSJ embed-
ded in a resistive \((R)\) and inductive \((L)\) electromagnetic environ-
ment.

FIG. 2. (Color on line) Dual Shapiro steps corresponding to \(m = 0, 1, 2, 3\) for a QPSJ in the absence of environment. The other parameters are: \(\alpha = 1.4\) and \(\hbar \omega_{\text{mw}}/2\pi U_0 = 1\).
of the branch. The second component is a fluctuating current $\delta I(t)$ of zero average and the Fourier components of which satisfy the fluctuation-dissipation theorem

$$
\langle \delta I(\omega) \delta I(\omega') \rangle = 2\pi \delta(\omega + \omega') \hbar \omega \Re [Y(\omega)] \coth(\hbar \omega/2k_B T),
$$

(3)

where $T$ is the temperature of the environment. As a result the charge $q$ on the QPSJ satisfies the Langevin equation

$$
dq/dt = I_0 + I_{\text{mw}}(t) - I_{\text{RL}}(t) + \delta I(t).
$$

(4)

In particular, the charge acquires a fluctuating component that will affect the shape of the current-voltage characteristic.

The effect of charge fluctuations as described by Eq. (4) has been analyzed in detail before in the case where microwaves are absent. When $I_{\text{mw}} = 0$, Eq. (4) reduces to the well-known Langevin problem of the quasi-charge dynamics in the overdamped regime. The DC current-voltage characteristic of such a junction has been calculated before in various limits; we briefly recall some of the results here, focussing on the experimentally relevant limit $U_0/E_L < 1$, see also Fig. 3.

Let us first neglect the fluctuating component, $\delta I(t) = 0$. As long as the resistance $R$ is large but finite, so that the dimensionless conductance of the environment, defined as $g = R_0/R$, is still small, the DC current-voltage characteristic of the QPSJ is a so-called Bloch nose. It consists of a zero-current branch at finite voltage up to $V_c$, which bends back to a low-voltage, finite current branch. Setting $\delta I = 0$ and considering the limit $gU_0/E_L \ll 1$, Eq. (4) can be integrated directly to yield the DC voltage

$$
V_j^{(\delta I = 0)} = \frac{R_0 I_0}{g} - \theta \left( \frac{R_0 I_0}{g} - V_c \right) \sqrt{\left( \frac{R_0 I_0}{g} \right)^2 - V_c^2},
$$

(5)

where $\theta(V)$ is the Heaviside step function. The corresponding current-voltage characteristic is shown in the inset of Fig. 3.

Finite charge fluctuations, $\delta I(t) \neq 0$, prevent the formation of a sharp feature in the current-voltage characteristic, even for small $g$. By taking into account these fluctuations, as discussed in Sec. IV – see Eqs. (32) and (33) – the I-V characteristic shows a finite slope at low current and a maximum voltage with a value lower than $V_c$. When the resistance $R$ is reduced further so that $g > 1$, the effect of the environment is stronger. The Bloch nose is smeared into a smooth curve with a maximum voltage at finite current. For very large values of $g$, the current $I_j$ at which the QPSJ sustains the largest voltage approaches the value $\Phi_0/2L$. This phenomenon is dual to the phenomenon of Coulomb blockade found in a Josephson junction in a highly resistive environment, where the voltage at which the Josephson junction sustains the largest current approaches the value $2e/2C$.

We summarize this behavior in the main panel of Fig. 3, where we plotted the QPSJ’s current-voltage characteristic for various values of $g$ at low temperature, $k_B T/U_0 = 0.25$, and for small $U_0/E_L = 0.013$. We stress that the behavior shown in Fig. 3 is essentially nonperturbative in the coupling strength $g$ characterizing the environment. Indeed, it is well-known that perturbation theory in either $g$ or $1/g$ is plagued by divergences and describes at best only parts of the current-voltage characteristic. The complete current-voltage characteristic can only be obtained including the relevant contributions to all orders (see Sec. IV).

We are now in a position to state the main results of this paper, where we study the combined effect of the application of microwaves and the presence of charge fluctuations induced by the resistive-inductive environment. We use an approach that is non-perturbative in both the environmental coupling strength $g$ and the microwave coupling strength $\alpha$. As we will see below, this implies that analytical results can only be obtained in the limit $U_0/E_L < 1$. On the other hand, this corresponds to the relevant experimental situation where QPSJs are studied with relatively low phase-slip rates and not too large inductances. In the limit $U_0/E_L < 1$, we find that, at the first order in $U_0$, the QPSJ’s current-voltage characteristic in the presence of microwaves can be straightforwardly obtained from the DC result without microwaves,

$$
V_j^{(\text{mw})}(\omega_B) = \sum_{m=-\infty}^{\infty} I_m^2(\alpha) V_j^{(\text{DC})}(\omega_B + m\omega_{\text{mw}}),
$$

(6)
in agreement with a general result recently demonstrated in Ref. 50, where $V_j^{(\text{DC})}$ is given by Eqs. (32) and (33). Specifically, this result implies that the current-voltage characteristic of a QPSJ with $U_0/E_L < 1$ under microwave irradiation is obtained by replicating the known DC characteristic of the QPSJ in the absence of microwaves at the positions of the current plateaus $I_{m} = m\alpha_{\text{mw}}/\pi$, which are expected for a QPSJ in the absence of the environment.

We focus on the case $g < 1$, for which dual Shapiro steps clearly appear in the I-V curve. Figure 4 displays a typical current-voltage characteristic obtained in this situation, tak-
ing again \( g = 0.2, \ k_B T / U_0 = 0.25, \) a microwave frequency \( \hbar \omega_{\text{mw}} / 2\pi U_0 = 1 \) and \( \alpha = 1.4. \) We see that the current-voltage characteristics are strongly modified in the simultaneous presence of microwaves and charge fluctuations induced by the environment, combining features of both Fig. 2 and 3.

Rather than being a set of discrete steps, the current-voltage characteristic is a continuous curve, connecting subsequent steps, bending back towards a zero-voltage state in between them. In the presence of microwaves, a replica of the Bloch nose is indeed found for each dual Shapiro step. As expected, in the presence of charge fluctuations, the width of the steps is smaller than the value \( 2V_c I_m (\alpha) \), found for \( g = 0 \); also, the dual steps are no longer strictly horizontal but acquire a small but finite linear slope. Note the role played by the inductance \( L \), which limits the effects of the charge fluctuations. As it is clearly seen in Fig. 4, the larger \( L \), the larger the width of the steps and the smaller their slopes. This can be seen in particular in the inset of Fig. 4, which presents the relative accuracy \( \delta I_m = \pi I_m / me \omega_{\text{mw}} - 1 \) for the first Shapiro step, \( m = 1 \). The inset also shows that the accuracy of the dual step is not only limited by charge fluctuations but also by a systematic shift of the step position, down by about 0.0015 in relative accuracy. This is due to the finite overlap of the various replicas. The shift can be reduced by increasing the microwave frequency so that the replicas are more separated along the \( I_I \)-axis, thus reducing their overlap.

The rest of the paper is structured as follows. In Sec. III, we introduce the model Hamiltonian for a QPSJ connected to a microwave source. We also show the results of the perturbation theory for the dissipative coupling with the external environment and for the coupling with the applied microwaves. In Sec. IV, we develop the non-perturbative approach. In Sec. V, we discuss the results to the leading order in the \( U_0 \) expansion focusing on the accuracy of the dual Shapiro steps and on the Joule heating effects. We draw our conclusions in Sec. VI.

III. CURRENT-BIASED QPSJ

A. QPSJ Hamiltonian

The Hamiltonian of the current-biased QPSJ in the circuit depicted in Fig. 1(b) is given by

\[
\hat{H} = \frac{\pi}{e} \left( \hat{q} + \hat{Q}_{RL} \right) - \frac{\hbar \alpha}{2e} \hat{\phi} + \hat{H}_{\text{env}} \left( \{ \hat{Q}_\lambda \}, \{ \hat{\phi}_\lambda \} \right). \tag{7}
\]

Here the charge and phase operators \( \hat{q} \) and \( \hat{\phi} \) are canonically conjugate, satisfying the commutation relation \( [\hat{q}, \hat{\phi}] = 2ie \). As a consequence, \( \hat{q} \) satisfies the equation of motion \( \ddot{q} + I(t) \) and thus corresponds to the total charge injected into the parallel combination of the QPSJ and the \( R-L \) environment.

The first term in Eq. (7) describes the nonlinear QPSJ with phase-slip energy \( U_0 \), which carries the charge \( \hat{q} + \hat{Q}_{RL} \), where the charge variable \( \hat{Q}_{RL} = \sum_\lambda \hat{Q}_\lambda \) accounts for the charge of the dissipative \( R-L \) environment. We thus model it using an infinite ensemble of harmonic oscillators (Caldeira-Leggett model), which is described by the third term of Hamiltonian (7).

\[
\hat{H}_{\text{env}} \left( \{ \hat{Q}_\lambda \}, \{ \hat{\phi}_\lambda \} \right) = \sum_{\lambda=1}^{\infty} \left( \frac{\hat{Q}_\lambda^2}{2\lambda} + \frac{1}{2\lambda} \left( \frac{\hbar \phi_\lambda}{2e} \right)^2 \right). \tag{8}
\]

The charge \( \hat{Q}_\lambda \) and the phase \( \phi_\lambda \) represent the momentum and position, respectively, of the \( \lambda \)-oscillator with characteristic frequency \( \omega_\lambda = 1 / \sqrt{L_\lambda C_\lambda} \). According to the fluctuation-dissipation theorem, \( \langle [\hat{I}_{\text{RL}}(t), \hat{I}_{\text{RL}}(0)]_+ \rangle = \hbar \omega \text{Re} \Re \left[ Y(\omega) \right] \text{coth} \left( \hbar \omega / 2 \right), \) where \( \hat{I}_{\text{RL}} = \hat{Q}_{RL} \) is the fluctuating current in the \( R-L \) environment and \([\ldots, \ldots]_+\) denotes the anticommutator. This yields the relation

\[
\text{Re} \left[ Y(\omega) \right] = \pi \omega^2 \sum_\lambda \sqrt{\frac{C_\lambda}{L_\lambda}} \delta(\omega^2 - \omega_\lambda^2), \tag{9}
\]

linking the parameters of the Caldeira-Leggett bath with the environmental admittance. Finally, the coupling between the charge operator \( \hat{q} \) and the bias current \( I(t) \) is given by the second term in (7).

Hamiltonian (7) has been used to describe QPSJs based on nanowire Josephson junctions and chains of Josephson junctions. In Appendix A, we show how Hamiltonian (7) can be obtained starting from the well-known Hamiltonian of a current-biased single Josephson junction embedded in an inductive-resistive environment.
B. Current-Voltage Characteristic

The DC current $I_J$ flowing through the QPSJ element is given by the difference between the total DC current $I_0$ and the current flowing through the R-L impedance of the circuit of Fig. 1(b),

$$I_J = I_0 - V_J/R. \quad (10)$$

Here $V_J$ is the DC component of the voltage drop across the QPSJ element. Using the Josephson relation between $\phi$ and $V_J$ and the Heisenberg equation of motion for the operator $\hat{\phi}$ generated by the Hamiltonian $\hat{H}$, this potential reads

$$V_J = \frac{\hbar}{2e} \left\langle \frac{d\phi}{dt} \right\rangle_{DC} = V_c \left\langle \sin \left[ \frac{\pi}{e} (\hat{q} + \hat{Q}_{RL}) \right] \right\rangle_{DC}. \quad (11)$$

The symbol $\langle \ldots \rangle$ denotes the quantum statistical average for the system described by the Hamiltonian $\hat{H}$, Eq. (7).

1. Dual Shapiro steps in the absence of environment

By setting $\hat{Q}_{RL} = 0$ in Eq. (7), the coupling with the environment vanishes and the system corresponds to an ideal current-biased QPSJ whose Hamiltonian $\hat{H}_0$ contains only the first two terms of $\hat{H}$. Introducing a complete set of discrete phase-states for the QPSJ, $|\phi_n\rangle = 2\pi |n\rangle$ with $n$ integer, we can express $\hat{H}_0$ as

$$\hat{H}_0 = -\frac{U_0}{2} \sum_n (|n\rangle \langle n+1| + \text{h.c}) - \frac{\hbar I(t)}{2e} \sum_n 2\pi n |n\rangle \langle n|, \quad (12)$$

in the phase representation. When $I_{mw} = 0$, Eq. (12) corresponds to the well-known Wannier-Stark ladder problem for a particle moving in a tilted tight-binding lattice, see Fig. 5. The tilt $I_0$ provides an energy difference equal to $\hbar \omega_B$ between two adjacent phase states. The term proportional to $U_0$ induces transitions between adjacent phase-states, i.e., phase-slip events. In the absence of microwaves or a coupling to the environment, we have only coherent Bloch oscillations and the associated energy difference $\hbar \omega_B$ can not be accommodated by the system. Hence no finite DC component is found for the voltage $V_J$ in this case.

Switching on the microwave field, the tilted lattice acquires an additional, oscillatory slope with amplitude $I_{mw} \neq 0$. For this problem, the unitary evolution operator can be evaluated exactly and it reads\(^2\)

$$\hat{U}(t) = e^{i\hat{Q}(t)} e^{-i\frac{\hbar}{2e} \sum_n dt' [\hat{K} \exp(i\omega_B t') + \hat{K}^\dagger \exp(-i\omega_B t')]}, \quad (13)$$

in which we set

$$\mathcal{Q}(t) = \omega_B t + \alpha \sin(\omega_{mw} t). \quad (14)$$

In Eq. (13), we also introduced the number operator $\hat{n} = \sum_n |n\rangle \langle n|$ and the ladder operator $\hat{K} = \sum_n |n\rangle \langle n+1|$. After some algebra, the expectation value of the voltage operator in Eq. (11) on the state $\hat{U}(t) |q_0\rangle$, the time evolved initial quasi-charge state $|q_0\rangle$, is

$$V_J^{(mw)}(t) = V_c \left[ \cos(2\pi \alpha t) - \frac{\hbar I(t)}{2e} t + \frac{\hbar I(t)}{2e} \alpha \sin(\omega_B t) \right] \quad (15)$$

Equation (15) coincides with Eq. (1) and describes the ideal dual Shapiro steps: a non-vanishing DC-voltage now appears each time the bias-current $I_0 = I_J$ satisfies the condition

$$I_J = m \omega_{mw}/\pi, \quad m = 0, 1, \ldots.$$  

This condition is satisfied when the resonant condition $\omega_B = m \omega_{mw} \pi$ is satisfied. For $m = 1$, a photon with energy $\hbar \omega_{mw}$ is exchanged with the microwave source.

2. Perturbation theory

We next analyze the current-voltage characteristic of the QPSJ in terms of perturbation theory in microwave interaction $\hat{\alpha}$ and dissipative coupling $\hat{g}$. We show that this approach systematically leads to divergent behaviour. For simplicity, we assume the bath to be at zero temperature.

Applying the unitary transformation $\hat{U}_{env} = \exp \left[ -i\hat{Q}_{RL} \hat{\phi} / 2e \right]$ to Hamiltonian (7), we obtain the QPSJ Hamiltonian in the form $\hat{H}_s = \hat{H}_0 + \hat{H}_{int}$ in which we consider as the unperturbed Hamiltonian

$$\hat{H}_0 = -U_0 \cos \left( \frac{\pi}{e} q \right) - \frac{\hbar I_0}{2e} \hat{\phi}, \quad (16)$$

FIG. 5. (Color online) Wannier-Stark ladder. The tilt provided by the bias current $I_0$ induces an energy separation $\hbar \omega_B$ between adjacent phase states indicated by red horizontal bars. Phase-locking occurs when the resonant condition $\omega_B = m \omega_{mw}$ is satisfied. For $m = 1$, a photon with energy $\hbar \omega_{mw}$ is exchanged with the microwave source.
and the interaction term
\[ \hat{H}_{\text{int}} = -\frac{\hbar}{2e} \cos(\omega_{\text{mw}} t) \phi + \hat{H}_{\text{env}} \left\{ \{ \hat{Q}_h \}, \{ \phi_h + \phi \} \right\}. \]

In this canonical form, the voltage operator is given by
\[ V_J = V_c \left( \sin \left( \frac{\pi}{e} q \right) \right)_{\text{DC}}. \]

Using the interaction picture, we expand the unitary time evolution operator in terms of \( \hat{H}_{\text{int}} \), Eq. (17), to calculate \( V_J \), Eq. (18). After some algebra, for vanishing microwave strength \( \alpha = 0 \), we obtain for the DC component of the voltage
\[ V_J^{(\text{DC})} = g V_c^2 / (2 R_Q I_0). \]

This result is indeed linear in \( g \) and corresponds to the first order expansion of the classical solution. Its validity requires \( V_J / V_c \ll 1 \), hence \( I_0 \gg g V_c / R_Q \). We conclude that perturbation theory breaks down in the limit of vanishing DC current bias.

In the presence of microwaves, \( \alpha \neq 0 \), the result generalizes to
\[ V_J^{(\text{mw})} = g V_c^2 / (2 R_Q) \sum_{n=0}^{+\infty} \frac{f_m^2(\alpha)}{n} \int dt \cos \omega_{\text{mw}} n t, \]
which shows that the divergent behavior found for \( I_0 \to 0 \) is repeated at the positions \( I_0 \to n \omega_{\text{mw}} / \pi \) at which the dual Shapiro steps are expected.

Although the perturbative approach is divergent and is inappropriate to describe the dual Shapiro steps in the presence of dissipation, it is useful for giving a simple picture of the QPSJ’s dynamics: The incoherent tunneling of the localized states results in a cascade of Shapiro steps. The incoherent tunneling of the localized states results in a cascade of Shapiro steps. The incoherent tunneling of the localized states results in a cascade of Shapiro steps. The incoherent tunneling of the localized states results in a cascade of Shapiro steps. The incoherent tunneling of the localized states results in a cascade of Shapiro steps. The incoherent tunneling of the localized states results in a cascade of Shapiro steps. The incoherent tunneling of the localized states results in a cascade of Shapiro steps. The incoherent tunneling of the localized states results in a cascade of Shapiro steps. The incoherent tunneling of the localized states results in a cascade of Shapiro steps. The incoherent tunneling of the localized states results in a cascade of Shapiro steps.

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i.e., the charge-charge correlation function

$$J(t) = \sum_{\lambda} \langle \dot{Q}_\lambda (t) \dot{Q}_\lambda (0) - \dot{Q}_\lambda^2 (0) \rangle ,$$

(25)

which quantifies the fluctuations of the tunneling phase due to the thermal bath. In particular, \( J(t) \) gives the coupling strength between the QPSJ and the environment. For the current-biased configuration of Fig. 1(b), we have

$$J(t) = 2RQ \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \Re e[Y(\omega)] \left( e^{i\omega t} - 1 \right) \left( 1 - e^{-\omega T_B} \right),$$

(26)

where \( Y(\omega) \) is the admittance of the circuit and \( T_B = 1/k_B T \) the inverse temperature. An exact calculation yields

$$A(t) = \pi g \left( 1 - e^{-\omega_{\text{RL}} |t|} \right),$$

(27)

$$M(t) = 2g \frac{\pi |t|}{\hbar \beta} - A(t) \cot \left( \frac{\hbar \omega_{\text{RL}} \beta}{2} \right) + 2g \sum_{n=m}^{\infty} \frac{1 - e^{-\nu_n |t|}}{n - (\nu_n / \omega_{\text{RL}})^2}.$$

(28)

Here \( \nu_n = 2\pi n / \hbar \beta \) is the \( n \)-th Matsubara frequency, and \( \omega_{\text{RL}} = R / L \) is the frequency scale characterizing the environment fluctuations at vanishing temperature.

The Jacobi-Anger expansion \( \exp[i\alpha \sin(x)] = \sum_{n=0}^{\infty} J_n(\alpha) \exp[i nx] \) allows to cast \( F_q \) in terms of the Bessel functions of the first kind \( J_n(\alpha) \),

$$F_q = \sum_{m_0=-\infty}^{\infty} J_{m_0} (\alpha) \sum_{m_2=-\infty}^{\infty} J_{m_2+1} (\alpha) \exp \left[ i \sum_{k=0}^{2n+1} \left( \omega_B + \omega_{\text{MW}} m_k \right) \eta_k \right].$$

(29)

Performing the change of variables \( \tau_k = t_k - t_{k-1} \), each time \( t_k \) can be expressed as \( t_k = t_0 - \sum_{k=1}^{k} \tau_k \) with \( k \geq 1 \). Then Eq. (29) becomes

$$F_q = \sum_{\{ m_k \}} \left( \prod_{m_k} J_{m_k} \right) \exp \left[ i \omega_{\text{MW}} t_0 \sum_{k=0}^{2n+1} \eta_k m_k \right] \times \exp \left[ -i \sum_{k=0}^{2n+1} \left( \omega_B + \omega_{\text{MW}} m_k \right) \eta_k \sum_{k=1}^{k} \tau_k \right].$$

(30)

where we used the sum rule \( \sum_k \eta_k = 0 \). Unlike the functions \( M(t_{k'} - t_k) \) and \( A(t_{k'} - t_k) \) in Eq. (22), which depend only on the time difference \( t_{k'} - t_k = \sum_{k=1}^{k} \tau_k - \sum_{k'=1}^{k'} \tau_{k'} \), Eq. (30) is a function of the time \( t_0 \) at which we calculate the voltage across the QPSJ. From Eq. (30) we observe that the frequency spectrum of Eq. (21) at the time \( t_0 \) involves integer components of the single fundamental frequency \( \omega_{\text{MW}} \) applied to the dual junction. This frequency mixing is due to the QPSJ which operates as a non-linear capacitance, i.e., it is related to the cosine dependence of the QPSJ energy as a function of the charge \( q \). Thus, in the steady state regime, we can extract the DC component by considering the time average of the general signal as \( \bar{f}(t) = \frac{1}{T_{\text{MW}}} \int_{t_i}^{t_i+T_{\text{MW}}} dt f(t) \) over a microwave period \( T_{\text{MW}} = 2\pi / \omega_{\text{MW}} \) where \( t_i \) is an arbitrary initial time. Then, the DC voltage reads

$$V_{J_c} = \frac{V(t_0)}{V_c} = \ldots F_q(0) = \ldots \frac{1}{T_{\text{MW}}} \int_{t_i}^{t_i+T_{\text{MW}}} e^{i\omega_{\text{MW}} t_0} \sum_{k=0}^{2n+1} \eta_k m_k .$$

(31)

The latter quantity is different from zero only if the sum rule \( \sum_{k=0}^{2n+1} \eta_k m_k = 0 \) is satisfied for each arbitrary configuration of the variables \( \{ \eta_k \} \) at given set of the integers \( \{ m_k \} \) associated to the expansion of the Bessel functions.

V. LOWEST ORDER RESULTS

A general analysis of the \( U_0 \)-expansion Eq. (31) is only possible in limiting cases. We focus here on the experimentally most relevant limit of relatively small QPSJ energy \( U_0 \), typically encountered in Josephson junction-based QPSJs. Then Eq. (31) can be approximated with its first term. We discuss the range of validity of this approximation below. Considering \( n = 0 \) only, the non-zero dichotomic variables are \( \eta_0 = \pm \) and \( \eta_1 = \pm \). Since they have to satisfy the constraint \( \sum_k \eta_k = \eta_0 + \eta_1 = 0 \), it follows that the allowed configurations \( \{ \eta_k \} \) are \((0, 0)\) and \((\pm, \mp)\), i.e., \( \eta_0 \) and \( \eta_1 \) have opposite sign. This means that the time-average given by Eq. (31) is different from zero if the indices \( m_0 \) and \( m_1 \) of the two possible sums of Bessel functions in Eq. (29) are equal.

A. DC-current-biased QPSJ

Let us first consider the case without microwave irradiation. Setting \( \alpha = 0 \) in Eq. (31), and retaining the term \( n = 0 \) only, the voltage drop on the QPSJ as a function of \( \omega_B \) reads

$$V_{J_c}^{\text{(DC)}}(\omega_B) \approx \frac{\pi}{2} U_0 \left[ P(h \omega_B) - P(-h \omega_B) \right],$$

(32)

FIG. 7. (Color online) Environment-assisted transitions between adjacent states in the Wannier-Stark ladder lead to the appearance of a finite voltage across the QPSJ element.
where we defined the function
\[ P(\Delta E) \equiv \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} d\tau \, e^{i\tau} e^{\frac{i}{\hbar} \Delta E \tau} . \] (33)

The function \( P(\Delta E) \) represents the probability density that the QPSJ absorbs \( (\Delta E > 0) \) or emits \( (\Delta E < 0) \) an amount of energy \( |\Delta E| \) from or to the external environment respectively during a phase-slip event. It is dual to the well-known function \( P(E) \) used to describe charge tunneling in the presence of an environment.\(^{36,56}\) We see that an incoherent phase slippage by \( \Delta \theta = 2\pi \) in the Wannier-Stark Ladder takes place only if the system exchanges the energy \( \bar{\Delta E} = \hbar \omega_B = (\Delta \phi) \hbar / (2e) \) with the environment, see Fig. [7] As the energy spectrum of the bath is continuous, the QPSJ has a dissipative behavior for any value of the applied DC-current \( I_0 \).

The validity of Eq. (32) is given by the condition \( V_{j}^{(DC)}/V_c \ll 1 \), yielding \( U_0 |P(h\omega_B)| \ll 1 \).\(^{32} \) The current-voltage characteristics displayed in Fig. [3] have been obtained from Eqs. (32) and (33) by direct numerical integration, using the correlation function Eq. (26). However, analytical results are available, for instance, in the limit of low temperature and small conductance so that \( \beta E_L / 2\pi^2 g \gg 1 \) and \( \beta E_B / 2\pi^2 g \gg 1 \). Then

\[ \frac{V_j^{(DC)}}{V_c}(\omega_B) \approx u \left[ \frac{\Gamma(g + i\beta h\omega_B/2\pi)^2}{\Gamma(2g)} \right] \sinh(\beta h\omega_B/2) , \] (34)

where \( u = (\beta U_0/4\pi)(\beta E_B / 2\pi^2 g)^{1/2} - 2\pi \gamma \) with \( \gamma = 0.577 \ldots \) the Euler constant. Hence we find a linear conductance \( G_0 \) at vanishing current \( I_j \) and voltage \( V_j \) to \( h\omega_B/2\pi \ll 1 \), given by

\[ G_0 R_Q \approx 4(U_0/\beta)^{2(\gamma-1)} \left( \frac{E_B}{U_0} \right) ^{2g} \left( \frac{1}{2\pi^2 g} \right) ^{2g} \frac{\Gamma(2g)}{\Gamma^2(g)} e^\gamma - g . \] (35)

We note that \( G_0 \sim T^{2-2\gamma} \) and thus decreases with decreasing temperature; similarly \( G_0 \sim E_B^{2-2\gamma} \sim 1/T^{2\gamma} \) and thus decreases with increasing inductance. Moreover, \( G_0 \) decreases with decreasing \( g \).

Increasing \( \omega_B \) till \( \omega_B^{\text{max}} \approx 2\pi g / (\hbar \beta) \), we reach the back-bending point corresponding to the maximum value

\[ \left( \frac{V_j^{(DC)}}{V_c} \right)_{\text{max}} \approx \pi u \left( \frac{1}{4} (\beta U_0)^{1-2\gamma} \frac{U_0}{E_B} \right) ^{2g} (2\pi^2 g)^{2g} e^{-2\gamma} , \] (36)

for \( g < 1 \). We see that the lower the temperature \( T \), the larger is the inductance \( L \) and the smaller the conductance \( g \), the closer \( V_j,\text{max} \) is to the maximum value \( V_c \). Beyond the back-bending point, the system enters into the Bloch oscillation branch where the bias energy \( h\omega_B \) becomes dominant with respect to both quantum and thermal fluctuations and the DC-voltage \( V_j \) decreases exponentially to zero.

Another interesting limit is the high-conductance regime \( g \gg 1 \). In this limit, the QPSJ is strongly coupled to the external electromagnetic dissipative environment and

\[ P(\Delta E) \approx \frac{1}{\sqrt{4\pi \hbar k_B T}} \exp \left\{ - (\Delta E - E_L)^2 / (4E_L k_B T) \right\} . \] (37)

As a result, the Bloch nose broadens into a Gaussian with a width \( \sqrt{E_L k_B T} \) and peaked at the inductive energy \( E_L \), as can be seen in Fig. [3] Lowering the temperature such that \( \beta E_L \gg 1 \), \( P(\Delta E) \to \delta(\Delta E - E_L) \). As a result, phase-slip events in a current-biased QPSJ can only occur if the energy \( h\pi I_0 / e \) exchanged with the inductive environment equals \( E_L \). This is the phenomenon dual to the Coulomb blockade of Cooper pair tunneling in a voltage-biased Josephson junction embedded in a highly resistive environment, where the transfer of Cooper pairs is possible only if the energy \( 2eV \) exchanged with the environment equals to the charging energy \( E_C \).

**B. Microwave irradiated QPSJ**

In the presence of the microwave source, the \( n = 0 \) term in the time-averaged expansion Eq. (31) can be written as

\[ V_j^{(\text{mw})}(\omega_B) = \sum_{m=\pm1} J_m^2(\alpha) V_j^{(DC)}(\omega_B + m\omega_{\text{mw}}) . \] (38)

Comparing Eq. (38) with Eq. (39), we see that, under the effect of the microwave radiation, the first-order voltage across the QPSJ junction is the superposition of an infinite number of zero-microwave potentials shifted by an integer multiple \( m \) of \( \omega_{\text{nw}} \). Unlike Eq. (15), the weight of the \( m \)-th term in Eq. (38) is determined by the squared first-kind Bessel function of the \( m \)-th order, \( \tilde{J}_m^2(\alpha) \). This result is in agreement with the general theorem proved in Ref. [51]. Since the sum rule \( \sum_j \tilde{J}_m^2(\alpha) = 1 \), holds, the larger is \( \alpha \) the smaller is the amplitude of the voltage corresponding to \( m = 0 \) and consequently the more important is the contribution of the higher-order terms. In other words, changing the amplitude \( \alpha \), the constant total weight \( \sum_j \tilde{J}_m^2(\alpha) \) re-distributes among the infinite terms of Eq. (38).

Using Eq. (10) in combination with Eq. (38), we find that the \( I_j-V_j \) characteristic of the QPSJ consists of \( (m\omega_{\text{nw}}) \)-shifted and rescaled copies of the QPSJ’s characteristic in the absence of microwaves, Eq. (32), obtained for \( I_{\text{nw}} = 0 \). These features occurring at \( I_{m_j} = m\omega_{\text{nw}} / \pi \) represent the dual or current Shapiro steps smeared by quantum and thermal fluctuations induced by the thermal bath. These results are shown in Fig. [4] obtained by direct numerical evaluation of Eq. (32) in combination with Eq. (38) for \( g < 1 \). The plotted smeared \( I_j-V_j \) curves result from the competition and interference between the environment-assisted phase slippage and the pure photon-assisted tunneling of the phase induced by the microwave field. In order for these features to be resolved, the microwave frequency \( \omega_{\text{nw}} \) has to be much larger than \( \omega_B^{\text{max}} \approx 2\pi g / (\hbar \beta) \), the bias current corresponding to the back-bending point \( (V_j^{(DC)}/V_c)_{\text{max}} \), see Eq. (36).

When \( g > 1 \), the current-voltage characteristics of the microwave-irradiated QPSJ typically look like the ones plotted in Fig. [3] We find that they consist of replicas of the smeared current-voltage characteristics for \( g > 1 \) and \( I_{\text{nw}} = 0 \), see Fig. [3] centered around the positions of the ideal Shapiro steps shown in Fig. [2]. Since the \( I_j-V_j \) characteristics for \( g > 1 \) are more smeared than the ones found in the low-conductive case, a higher microwave frequency \( h\omega_{\text{nw}} / 2\pi U_0 = 20 \) has
been used to resolve the various replicas and obtain Fig. 8. When increasing the inductance \( L \) for \( g > 1 \), the smearing effects are reduced. The inset of Fig. 8 shows the relative accuracy \( \delta I_m = \pi J_1 / me\omega_{\text{mw}} - 1 \) of the structure found at \( m = 1 \) with respect to a perfect first Shapiro step.

C. Accuracy of the current Shapiro steps

The reduction of quantum and thermal fluctuations affecting the dual Shapiro steps is crucial for their experimental observation as well as their potential applications, such as in metrology. In this respect, it is important to analyze the accuracy of the dual steps. We focus on the relevant regime of low conductance, \( g < 1 \), where actual well-defined dual Shapiro steps are found and examine the smearing of the \( m \)-th step by considering the relative deviation \( \delta I_m = \pi J_1 / me\omega_{\text{mw}} - 1 \). Based on the asymptotic results of Eqs. (35) and (36), we expect a minimal smearing when \( T \) and \( g \) are chosen as small as possible and \( L \) large.

The behavior of \( \delta I_m \) as a function of some of the relevant system parameters is studied numerically in Fig. 9 and Fig. 10 for the first dual Shapiro step, \( m = 1 \). In these figures, the solid, dashed and dotted lines correspond to three different microwave strengths \( \alpha = 1.4, 2.2, \) and 3.2. Also shown (dashed-dotted line) is the behaviour of the unperturbed dual Shapiro step for \( \alpha = 2.2 \) (see text).
finite overlap of the $m = 1$ replica of the Bloch nose with all the other replicas $m \neq 1$. This suggests that increasing the microwave frequency should yield a better accuracy of the step position as it separates the replicas more, thereby reducing their overlap and, at the same time, improving their individual resolution. The result of an increasing of $\omega_{\text{mw}}$ on the step position can be seen by comparing Fig. 9(b) with Fig. 10. We notice, for instance, that when $\alpha = 2.2$ the relative offset reduced from about 0.02 in the former to about 0.0004 in the latter by increasing $\omega_{\text{mw}}$ by a factor of 10.

It is interesting to investigate why the curve for $\alpha = 2.2$ is less affected by the offset than the one for $\alpha = 1.4$, although the step size is the same for both curves. Indeed, the value of the squared Bessel functions $J^2_0(\alpha)$ determining the $m = 1$ step width is almost equal for the two curves. However, the value $J^2_0(\alpha)$ is very different: $J^2_0(2.2) \approx 0.01$ whereas $J^2_0(1.4) \approx 0.32$. In other words, the $m = 0$ Shapiro step will strongly influence the step $m = 1$ for $\alpha = 1.4$, leading to a large offset, whereas it influences the $m = 1$ step much less for $\alpha = 2.2$. The step corresponding to $\alpha = 3.2$ is more or less structureless, as its weight is very small, $I^2_0(3.2) \approx 0.07$.

As far as the smearing is concerned around the actual plateau position, a comparison between Fig. 9(a) and Fig. 9(b) shows the effect of the inductance. Increasing the inductance by a factor of 4 reduces the relative width of the step from about 0.1 in Fig. 9(a) to about 0.05 in Fig. 9(b).

**D. The effect of Joule heating**

In this Section, we discuss an important aspect related to the experiment aimed to detect dual Shapiro steps, namely the effect of Joule heating in the I-V characteristic of the QPSJ.

As we have seen above, we expect to approach the ideal dual Shapiro steps of Fig. 4 under the condition $g \ll 1$. This means that the QPSJ is ideally embedded in a highly-dissipative environment. Such an environment is expected to produce also unwanted Joule heating which in turn would enhance the smearing of the steps. Indeed, in the low-

FIG. 10. (Color online) Relative deviation $\delta I_m$ for the first Shapiro step, $m = 1$, for $k_B T/U_0 = 0.1$, $\hbar \omega_{\text{mw}}/2\pi U_0 = 2$ and $U_0/E_L = 0.013$. The plotted I-V characteristics have been obtained using three values of the microwave strength $\alpha$: 1.4 (red dashed), 2.2 (blue solid and dot-dashed) and 3.2 (green dotted). The (blue) dot-dashed line correspond to the unperturbed Shapiro step for $\alpha = 2.2$ (see text).

FIG. 11. (Color online) Effect of Joule heating on the dual Shapiro steps obtained from the numerical evaluation of Eq. (38) in the low-conductive regime, $g = 0.2$. In both panels, for the (red) dashed $I_1$-V$_I$ curves the temperature is fixed to $k_B T/U_0 = 0.25$. The (blue) solid curves in (a) and (b) have been determined using the effective temperatures $T_{\text{eff}}$ which are the solutions of Eq. (39) and Eq. (41) respectively, with $k_B T_{\text{ph}}/U_0 = 0.25$, and $U_0 = 4$ GHz. The electron-phonon coupling constant and the volume of there-sisance $R$ are $\Sigma = 10^9$ W m$^{-3}$ K$^{-5}$ and $\Omega = 10^{-19}$ m$^3$ respectively. All the I-V characteristics in (a) and (b) are determined setting $U_0/E_L = 0.141$, $\hbar \omega_{\text{mw}}/2\pi U_0 = 1$, and $\alpha = \pi \omega_{\text{mw}}/(\epsilon \omega_{\text{mw}} \sqrt{R^2 + L^2 \omega_{\text{mw}}^2}) = 1.4$, as for the green dotted line in Fig. 4. The insets show the rescaled effective temperature $T_{\text{eff}}/T$ as a function of the current through the QPSJ.
conductance limit, $R \gg R_Q$, quantum effects due to the external bath become small, whereas thermal ones induced by heating may become dominant. In this context, the effective electronic temperature $T_{\text{eff}}$ of the $R$-$L$ series can be much larger than the phonon temperature $T_{\text{ph}}$. For the circuit of Fig. 11(b), the current flowing through the $R$-$L$ branch is $V_J/R$, then the power dissipated by the resistance is $P_I = V_J^2/R$, where $V_J$ is a function of the temperature [see Eq. (38)]. It follows that the effective temperature $T_{\text{eff}}$ can be estimated by the self-consistent equation

$$ T_{\text{eff}}^5 = T_{\text{ph}}^5 + V_J^2(T_{\text{eff}}, \omega_B)/(R \Sigma \Omega). $$

In this last relation, $\Sigma$ is the material-dependent electron-phonon coupling constant, and $\Omega$ the volume of $R$. Figure 11(a) shows the I-V curve of a QPSJ embedded in an environment with $g \ll 1$ and fixed temperature, $k_B T/U_0 = 0.25$, where the Joule heating is not taken into account, together with the dual Shapiro steps smeared by the voltage-dependent effective temperature Eq. (39) which accounts for the exchange of energy between the electrons and the phonons in the resistance $R$. We notice a reduction of the width of the steps, as one expects. From the inset of Fig. 11(a), we see that $T_{\text{eff}}$ follows the oscillating trend of $V_J$. In particular, it coincides with $T_{\text{ph}}$ whenever $V_J = 0$ and reaches its relative maxima for the values of $V_J$ around the maximum amplitude of the steps: the wider are the steps in the absence of Joule heating the larger is their effective thermal smearing.

On the other hand, Joule heating affects differently the dual Shapiro steps appearing in the I-V characteristic of a voltage-biased QPSJ as in the circuit of Fig. 11(a). In this configuration, the power $P_I = I_J^2 R$, which is dissipated by the resistance $R$, is determined by the current flowing through both $R$ and the QPSJ, i.e.,

$$ I_J = (V_0 - V_J)/R. $$

Here $V_J = V_J(T, V_0)$ is obtained from Eq. (38) replacing $I_0$ with $V_0/R$ and $|V_{\text{mw}}|$ with $|V_{\text{mw}}|/\sqrt{R^2 + L^2 \omega_{\text{mw}}^2}$. As a result, the effective temperature of the environment can be written as

$$ T_{\text{eff}}^5 = T_{\text{ph}}^5 + I_J^2(T_{\text{eff}}, V_0)/R/(\Sigma \Omega). $$

Inserting into Eq. (40) the temperatures $T_{\text{eff}}$ obtained by solving self-consistently Eq. (41) for different values of the DC voltage bias $V_0$, we obtain the (blue) solid QPSJ’s I-V characteristic shown in Fig. 11(b). Notice that this curve is more smeared than the one found in the current-biased case and plotted in Fig. 11(a) using the same set of parameters. As shown in the inset of Fig. 11(b), $T_{\text{eff}}$ increases with $|I_J|$ and is equal to $T_{\text{ph}}$ only when $I_J = 0$. In particular, the effective temperature given by Eq. (41) is much larger than $T_{\text{ph}}$ when $I_J$ is close to $e\varphi_{\text{mw}}/\pi$. Consequently, the Joule heating affects the steps for $m \neq 0$ more than the one occurring for $m = 0$, as one can see from Fig. 11(b), thereby compromising their experimental observation. The reduction of this effect is possible, for instance, with the decreasing of the microwave frequency $\varphi_{\text{mw}}$. However, the use of smaller $\varphi_{\text{mw}}$ leads also to the increasing of the offset of the steps which we discussed in Sec. V C.

In principle, Joule effect can be reduced by increasing the inductance $L$ of the environment rather than the resistance $R$. $L$ plays the same role of $R$ in the reduction of the fluctuations, as shown previously. As the dual Shapiro steps are replicas of the I-V characteristic at low current, we can estimate the leading dependence for the smearing by considering Eq. (35). We obtain the slope

$$ G_0 R_Q \approx 2g \left( \frac{k_B T}{U_0} \right)^2 \left( \frac{V_J}{U_0} \right)^2, $$

for $g \ll 1$. We observe that the smearing due to the temperature can partially be compensated by increasing the inductance of the environment.

VI. CONCLUSIONS

In this paper, we discussed the microwave response of a QPSJ embedded in an inductive-resistive environment. We focused on the regime of relatively small ratio of phase-slip energy $U_0$ over inductive energy $E_L$. The response consists of a series of well-defined current Shapiro steps, located at multiples of $e\varphi_{\text{mw}}/\pi$, if the environmental resistance is sufficiently large, such that the dimensionless conductance $g < 1$. These steps are in fact replicas of the QPSJ’s Bloch nose, observed in the absence of microwave. Charge fluctuations induced by the environment smear the steps. The smearing can be reduced by decreasing the dimensionless environmental conductance $g$, decreasing the dimensionless temperature $k_B T/U_0$ and increasing the ratio $U_0/E_L$, which can be achieved by increasing environmental inductance $L$. Finally, we showed that the conductance $g$ can not be increased indefinitely, as heating effects may develop in the environment.

The results presented in this paper are relevant for recent experiments on Josephson junction chains and nanowires. In these works, typical phase-slip energies $U_0$ are in the range of $1 \div 10$ GHz, whereas the environmental inductances $L$ are $50 \div 500$ nH. This motivated the parameter choices used in this paper: $U_0/E_L$ ranges from $0.001 \div 0.1$; at typical cryostat temperatures $k_B T/U_0 \sim 0.1 \div 0.2$. We found that, although dual Shapiro-like features could be visible experimentally for these parameters, their relative accuracy remains limited to about 0.001 by fluctuation effects.

To date, a systematic evidence for the existence of dual Shapiro steps is still lacking. The reason for this might well be that fluctuation effects have so far masked the steps for QPSJs with intermediate ratios of the parameter $U_0/E_L$ and not too small conductance $g$. Work on nanowire-based QPSJs with larger values of the ratio $U_0/E_L$ and lower conductances $g$ seems promising at the same time these systems suffer from substantial heating effects. We conclude that further work is necessary, both on nanowires and on Josephson junction chains.
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Appendix A: QPSJ Hamiltonian for an underdamped Josephson junction

In this Appendix, we will study the Hamiltonian of the circuit of Fig. 12, which is formed by a Josephson junction (JJ), biased by a time-dependent current \( I(t) = I_0 + I_{mw} \cos \omega_{0w} t \), in parallel with a capacitance \( C \) and an external electromagnetic environment composed by a resistance \( R \) and an inductance \( L \) in series. In particular, we will show that this Hamiltonian reduces to the QPSJ Hamiltonian used in the main text under suitable conditions.

Neglecting the contribution of the quasi-particle excitations, the Hamiltonian corresponding to the circuit of Fig. 12 is given by the sum of the charging energy of the capacitance \( C \), the non-linear Josephson energy and the energy of the environment,

\[
\hat{H}_s = \frac{1}{2C} \left[ \int_{-\infty}^{t} dt' \dot{I}(t') + \hat{Q}_{RL} + \hat{\hat{Q}} \right]^2 - E_J \cos(\hat{\phi}) + \hat{H}_{env}.
\]

(A1)

The phase operator \( \hat{\phi} \) is the phase-difference between the two superconductors forming the junction and \( \hat{\hat{Q}} \) is its conjugate charge operator \( [\hat{\phi}, \hat{\hat{Q}}] = 2e \hat{I} \), i.e., the charge tunneling through the junction. In Eq. (A1), we also introduced \( \hat{Q}_{RL} = \sum \lambda \hat{Q}_{\lambda} \), which accounts for the charge noise produced by the \( R-L \) environment, as discussed in the main text.

The equivalence between Hamiltonian (A1) and QPSJ Hamiltonian (7) can be demonstrated through the following steps. First, we apply the gauge and the unitary transformations \( \hat{U}_{\hat{g}}(t) = \exp \left[-i\hat{\phi} \int_{-\infty}^{t} dt' \dot{I}(t') / 2e\right] \) and \( \hat{U}_{env} = \exp \left[-i\hat{\phi} \int_{-\infty}^{t} dt' \dot{I}(t') / 2e\right] \) respectively to Eq. (A1) and we get

\[
\hat{H}_s' = \frac{\hat{\hat{Q}}^2}{2C} - E_J \cos(\hat{\phi}) - \frac{\hbar I(t)}{2e} \hat{\phi} + \hat{H}_{env} \{ [\hat{Q}_\lambda], \{ \hat{\phi}_\lambda + \hat{\phi} \} \}.
\]

(A2)

Here the first and second term correspond to the standard Hamiltonian \( \hat{H}_J \) of an isolated JJ. In the tight-binding regime, \( E_J \gg E_C \), \( \hat{H}_s' \) becomes

\[
\hat{H}_s'' = -U_0 \cos \left( \frac{\pi}{e} \hat{q} \right) - \frac{\hbar I(t)}{2e} \hat{\phi} + \hat{H}_{env} \{ [\hat{Q}_\lambda], \{ \hat{\phi}_\lambda + \hat{\phi} \} \}.
\]

(A3)

where \( \hat{q} \) is the quasi-charge operator and \( U_0 = 8 \sqrt{E_J \hbar \omega_p / \pi \exp(-\sqrt{8E_J / E_C})} \) \( \approx \nu_c / \pi \) the half-bandwidth of the first Bloch band of \( \hat{H}_J \). Within this limit, an energy gap of the order of the plasma frequency \( \hbar \omega_p = \sqrt{8E_J E_C} \) separates the first from the second Bloch band. We neglect the possibility of inter-band Landau-Zener transitions assuming the low temperature and bias current limit \( (k_B T, \hbar \omega_p / 2e, \hbar I_{mw} / 2e) \ll \hbar \omega_p \) as well as considering an off-resonance microwave field, \( \hbar I_{mw} \ll \omega_p \).

Finally, we apply the inverse unitary transformation \( \hat{U}_{env}^{-1} \) to Eq. (A3) and we obtain the effective low-energy Hamiltonian

\[
\hat{H} = -U_0 \cos \left( \frac{\pi}{e} \left( \hat{q} + \hat{Q}_{RL} \right) \right) - \frac{\hbar I(t)}{2e} \hat{\phi} + \hat{H}_{env} \{ [\hat{Q}_\lambda], \{ \hat{\phi}_\lambda \} \}.
\]

(A4)

This is the energy operator (2) of the main text describing a current-biased quantum phase-slip junction coupled to an external \( R-L \) electromagnetic environment, as depicted in Fig. 1(b).

\[\text{FIG. 12.} \text{ Current-biased Josephson junction with Josephson energy } E_J \text{ in parallel with a capacitance } C \text{ and embedded in a resistive } (R) \text{ and inductive } (L) \text{ electromagnetic environment. The circuit is biased with a time-dependent current } I(t). \]

\[\text{FIG. 1(b).} \text{ Extensive calculations and detailed discussions of }

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