Hecke algebra trace algorithm and some conjectures on weave knots

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Abstract

In this paper, we perform a case study of a very special family of knots $W(p, m)$ known as weaving knots of type $(p, m)$. This infinite family of knots are hyperbolic and alternating and there are known interesting results as well as conjectures which relate their topological invariants to their geometric invariants. To explore more on the relationship between the topological and geometric invariants of this family of knots, we look upon these knots as closure of $p$ braids written in terms of generators of braid group $B_p$ and consider the image of this braid under representation of $B_p$ into the Hecke algebra $H_p(q)$ over a field $K$. We write an algorithm to explicitly compute the trace of this element in $H_p(q)$ corresponding to parameters $q$ and $z$. We use this algorithm to derive invariants of these knots. We generate data for $W(4, m)$, $W(5, m)$ and $W(6, m)$ families and provide evidence for the conjecture of Dasbach and Lin that higher twist numbers defined by them in [8] provide better volume bound for weaving knots than in [6]. We also look at the asymptotics of the distribution of ranks of Khovanov homology groups of $W(p, m)$ for a fixed $p$ as $m$ grows large.

1 Introduction

Hecke algebras are quotients of group rings of Artin’s braid groups [3] and play important role in the study of many branches of mathematics. One major application of Hecke algebras is in defining link invariants [17]. V. F. R. Jones used representation of braid groups into Hecke algebras which he combined with a special trace function on Hecke algebras and showed the existence of a two variable polynomial as a link invariant. This two variable polynomial turned out to be a universal skein invariant in the sense that all known polynomial invariants could be derived from this only by change of variables. Thus, if one is aiming at computing various polynomial invariants for a given knot or for a given family of knots it is a good idea to compute this two variable polynomial. Assuming that one knows a nice braid representation for the knot under consideration, the most important ingredient in this computation will be finding the trace of the Hecke algebra element which represents the braid. The trace function on Hecke algebras is defined by certain axioms and can be calculated in several steps. Usually what one does

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is express the element in Hecke algebra as a linear combination of its basis elements and find the trace of the basis elements. However, there is no effective way of doing this in general. In [15] the authors had worked out a recursive method of computing the trace for Hecke algebra representation of special 3– braids whose closure gives an infinite family of hyperbolic knots known as weave knots of type \((3, n)\) denoted by \(W(3, n)\). In general \(W(p, n)\), a weave knot of type \((p, n)\), is defined as closure of the \(p\)–braid \((\sigma_1^{e_1} \sigma_2^{e_2} \cdots \sigma_{p-2}^{e_{p-2}} \sigma_{p-1}^{e_{p-1}})^n\) where all \(e_i\) are \(+1\) for \(i\) odd and \(-1\) for \(i\) even. The case for \(W(3, n)\) was simple as the braid group on 3– strands is represented in \(H_3(q)\) which has dimension 3! and hence expressing the corresponding element as linear combination of basis element was easy and could be done by hand. The same technique does not work out for \(W(p, n)\). In this paper, we have given an algorithm to compute the trace of basis elements of \(H_p(q)\). We also write down an algorithm to express the weave braid (whose closure is the weave knot) representation in the Hecke algebra as a linear combination of its basis elements. Combining these two algorithm we obtain the trace of the representations of weave braids in the Hecke algebra. We use this trace computation to calculate the two variable polynomial invariant for all \(W(p, n)\) and hence derive Alexander polynomial, Jones polynomial and HOMELY-PT for all weave links \(W(p, n)\). Since \(W(p, n)\) are alternating knots for \(p\) and \(n\) co-prime, we also utilize results of Lee [4] to compute the Khovanov homology of \(W(p, n)\). We normalize the ranks of Khovanov homology groups and provide evidence that these ranks are normally distributed. In some sense this paper extends the results for \(W(3, n)\) in [15] for the general case \(W(p, n)\) for all \(p\) and \(n\).

On the geometry side after the 1980s William Thurston’s seminal result [1] Corollary 2.5] along with Mostow’s rigidity theorem [11] Theorem 3.1] which ensures that most knot complements have a unique structure of a hyperbolic manifold, there is automatically a strong connection between hyperbolic geometry and knot theory, since knots are determined by their complements. Many mathematicians are naturally interested in finding out if any inference about some of the geometric invariants (such as volume) of a knot can be derived from any of its topological invariants (such as Jones polynomial or colored Jones polynomial). The ultimate curiosity in this direction is the validity of the open conjecture known as volume conjecture [9]. These weaving knots \(W(p, m)\) are good candidates for such exploration. It is conjectured about weaving knots that their complement have maximal volume among all the knots having equal crossing number [1]. Champanerkar, Kofman and Purcell provided asymptotically sharp bounds for the relative volume of these knots (i.e., volume divided by crossing number). They showed that

$$\lim_{p, m \to \infty} \frac{\text{vol}(S^3 \setminus W(p, m))}{c(W(p, m))} = v_{\text{oct}}$$

and hence according to their work in [7] they conclude that weaving knots are geometrically maximal. For us once we have found the Jones polynomial for \(W(p, n)\) we can find ,using a result of Dasbach and Lin [8], a bound in terms of the twist number \(T(K)\) of \(K\). As these knots are alternating \(T(K)\) is nothing but the sum of the modulus of coefficients of \(t^{m+1}\) and \(t^{n-1}\) where \(m\) and \(n\) are the lowest and the highest degrees of the Jones polynomial of \(K\). In the same paper Dasbach and Lin defined invariants such as \(i\)th twist number \(T_i(K)\) of a knot \(K\) to be the sum of the modulus of coefficients of \(t^{m+i}\) and \(t^{n-i}\). They mentioned in their paper that there may be a possibility that \(T_i(K)\) correlate to volume of the knot complement. Since we can write Jones polynomial of all \(W(p, m)\) we can compute the values of \(T_i(W(p, m))\) for \(i\) within the span of the Jones polynomial. We performed experiments on values of \(T_i(W(p, m))\) and came up with 4 set of bounds for the relative volume of \(W(p, m)\) which seem to provide better bounds than given in [6].

The paper is organized as follows. In Section 2 we include the basics on Hecke algebras,
its trace and some relevant results. Section 3 deals with the proof of the lemmas that describe our algorithm. At the end of this section we include the Mathematica program to compute the trace of weave braid representations. We also set up notations that are used in describing the two variable polynomial and all the polynomial invariants derived from that as well as the knot homologies. In Section 4 we explain how to derive two variable polynomial invariants from the trace. We use the appropriate substitutions to write down the Jones polynomial of $W(n, m)$. We also include the additional Mathematica program to compute the Jones polynomial and add a list of examples of Jones polynomial of few weave knots. In Section 5, we discuss the higher twist numbers introduced by Dasbach and Lin, provide important observations about higher twist number of $W(n, m)$ knots and show that the bounds for relative volume of weave knots can be improved from the bounds given in [6]. Section 6 discuss Khovanov homology computation of $W(n, m)$ and the distribution of their normalized ranks. At the end we provide tables displaying the evidence for our conjectures.

## 2 Hecke algebras: some observations

We review briefly the definition of the Hecke algebra $H_{n+1}(q)$ on generators $T_1, T_2, \ldots, T_n$, and we define the representation of the braid group $B_{n+1}$ on $n+1$ strands in $H_{n+1}$. We discuss a suitable basis for $H_{n+1}(q)$ and prove some lemmas that are useful in proving our main results.

**Definition 2.1.** Working over the ground field $K$ containing an element $q \neq 0$, the Hecke algebra $H_{n+1}(q)$ is the associative algebra with 1 on generators $T_1, T_2, \ldots, T_n$ satisfying the following relations:

i) $T_iT_j = T_jT_i$ whenever $|i - j| \geq 2$

ii) $T_iT_{i+1}T_i = T_{i+1}T iT_{i+1}$ for $1 \leq i \leq n-1$,

iii) $T_i^2 = (q-1)T_i + q$ for $i = 1, 2, \ldots, n$.

It is well-known that the dimension of $H_{n+1}$ over $K$ is $(n+1)!$ (for example, see [4]).

Recasting the relation $T_i^2 = (q-1)T_i + q$ in the form $q^{-1}(T_i + (1 - q)) \cdot T_i = 1$ shows that $T_i$ is invertible in $H_{n+1}$ with $T_i^{-1} = q^{-1}(T_i + (1 - q))$.

Consequently, the specification $\rho(\sigma_i) = T_i$, for $i = 1, 2, \ldots, n$, combined with the relations (1) and (2), defines a homomorphism $\rho$ from the group $B_{n+1}$ of braids on $n+1$ strands into the multiplicative monoid of $H_{n+1}$ (see [3] Section 6 in eg).

**Theorem 2.2** ([3]). For $n \geq 1$, there is an unique family of trace functions $Tr : H_{n+1} \to K$ compatible with the inclusions $H_n \to H_{n+1}$ satisfying

i) $Tr(1) = 1$,

ii) $Tr$ is $K$-linear and $Tr(ab) = Tr(ba)$, and

iii) if $a, b \in H_n$, then $Tr(aT_nb) = zTr(ab)$.

Using the properties of $H_{n+1}$, we can observe that trace of an element is a Laurent polynomial in $q$ and $z$. In principle, the above properties of $Tr$ help us compute the trace of any element of $H_{n+1}$. However, the elements of $H_{n+1}$ can be very complicated. So, the first step
Proposition 2.3. The Hecke algebra $H_{n+1}$, for $n \geq 1$, has a basis $B_n$ given as follows:

$$B_n = \{ u_1 u_2 \cdots u_n \in H_{n+1} \mid u_i \in U_i \text{ for } i = 1, 2, \ldots, n \}.$$

This is well known in the theory of Hecke algebra and an elegant proof of this proposition can be found in [5].

For $j \in \mathbb{Z}^+$, let us define $P_j = (-1)^{j-1} \sum_{i=0}^{j-1} (-q)^i$ and $P_{-j} = q^{-j} \sum_{i=0}^{j-1} (-q)^i$ and $P_0 = 0$. For $m \in \mathbb{Z}$, one can check the following:

i) $q P_{m-1} + (q-1) P_m = P_{m+1}$,

ii) for $q = 1$, $P_m = 0$ if $m$ is even and $P_m = 1$ if $m$ is odd,

iii) $P_m = \frac{q^m - (-1)^m}{q+1}$ if $q \neq -1$, and

iv) $P_m = (-1)^{m-1} m$ if $q = -1$.

With these, we have the following:

Lemma 2.4. For $i = 1, 2, \ldots, n$, we have the following:

$$T_i^m = P_m T_i + q P_{m-1} \quad \text{for all } m \in \mathbb{Z}.$$ 

Proof. Note that $T_i^0 = P_0 T_i + q P_{-1}$, since $T_i^0 = 1$, $P_0 = 0$ and $P_{-1} = q^{-1}$. We now prove the lemma for $m \neq 0$. Let us consider the following two cases:

i) **To prove the lemma for $m \in \mathbb{Z}^+$**: We prove the equality $T_i^m = P_m T_i + q P_{m-1}$ by induction on $m$. Note that $T_i^1 = P_0 T_i + q P_0$, since $P_1 = 1$ and $P_0 = 0$. Assume that $T_i^m = P_m T_i + q P_{m-1}$ for $m \geq 1$. We show that $T_i^{m+1} = P_{m+1} T_i + q P_m$. Using the induction hypothesis, we have the following estimate:

$$T_i^{m+1} = T_i^m T_i = (P_m T_i + q P_{m-1}) T_i = P_m T_i^2 + q P_{m-1} T_i = P_m ((q-1) T_i + q) + q P_{m-1} T_i = (q P_{m-1} + (q-1) P_m) T_i + q P_m = P_{m+1} T_i + q P_m,$$

since $T_i^2 = (q-1) T_i + q$ and $q P_m + (q-1) P_m = P_{m+1}$.
ii) To prove the lemma for $m \in \mathbb{Z}^-$: We need to prove that $T_i^{-l} = P_{-l}T_i + qP_{-l-1}$ for $l \in \mathbb{Z}^+$. We prove this equality by induction on $l$. Note that $T_i^{-1} = P_{-1}T_i + qP_{-2}$, since $T_i^{-1} = q^{-1}(T_i + (1-q))$, $P_{-1} = q^{-1}$ and $P_{-2} = q^{-2}(1-q)$. Assume that $T_i^{-l} = P_{-l}T_i + qP_{-l-1}$ for $l \geq 1$. We show that $T_i^{-1} = P_{-l}T_i + qP_{-l-1}$. Using the induction hypothesis, we have the following estimate:

$$T_i^{-l} = (P_{-l}T_i + qP_{-l-1})T_i^{-1} = P_{-l} + qP_{-l-1}q^{-1}(T_i + (1-q))$$

since $T_i^{-1} = q^{-1}(T_i + (1-q))$ and $P_{-l} + (1-q)P_{-l-1} = qP_{-l-2}$.

Let $\mathcal{C}_n$ be the subcollection of $\mathcal{B}_n$ defined as follows:

$$\mathcal{C}_n = \{u_1u_2\cdots u_n \in \mathcal{B}_n \mid u_i \in \{1, T_i\} \text{ for } i = 1, 2, \ldots, n\}.$$  

By convention, let $\mathcal{B}_0 = \mathcal{C}_0 = \{1\}$.

**Lemma 2.5.** For $\gamma_{n-1} \in \mathcal{C}_{n-1}$ and $v_n \in \mathcal{B}_n$, the product $\gamma_{n-1}T_{n+1}v_nT_{n+1}$ can be written as a linear combination of the elements of type $\gamma_n v_{n+1}$, where $\gamma_n \in \mathcal{C}_n$ and $v_{n+1} \in \mathcal{B}_{n+1}$.

**Proof.** We prove the lemma by considering the following cases:

i) If $v_n = 1$: One can see that $\gamma_{n-1}T_{n+1}v_nT_{n+1} = (q-1)\gamma_{n-1}T_{n+1} + q\gamma_{n-1}$. Since $\gamma_{n-1} \in \mathcal{C}_n$ and $T_{n+1} \in U_{n+1}$, we get $\gamma_{n-1}T_{n+1}v_nT_{n+1} \in \text{Span} \{\gamma_n v_{n+1} \mid \gamma_n \in \mathcal{C}_n \text{ and } v_{n+1} \in U_{n+1}\}$.

ii) If $v_n \neq 1$: We can write $\gamma_{n-1}T_{n+1}v_nT_{n+1} = \gamma_{n-1}T_{n+1}T_nv_{n-1}T_{n+1}$ for some $v_{n-1} \in U_{n-1}$. By relations in the Hecke algebra, $\gamma_{n-1}T_{n+1}T_nv_{n-1}T_{n+1} = \gamma_{n-1}T_{n+1}v_{n-1}T_{n+1} = \gamma_n v_{n+1}$, where $\gamma_n = \gamma_{n-1}T_n$ and $v_{n+1} = T_{n+1}v_{n+1}$. This proves that $\gamma_{n-1}T_{n+1}v_{n+1}T_{n+1} \in \text{Span} \{\gamma_n v_{n+1} \mid \gamma_n \in \mathcal{C}_n \text{ and } v_{n+1} \in U_{n+1}\}$.

**Lemma 2.6.** For $u_n \in \mathcal{B}_n$ and $\gamma_n \in \mathcal{C}_n$, the product $u_n \gamma_n$ can be written as a linear combination of the elements of the type $\gamma_n v_n$, where $\gamma_{n-1} \in \mathcal{C}_{n-1}$ and $v_n \in \mathcal{B}_n$.

**Proof.** We prove this lemma by induction on $n$. Let $u_1 \in U_1$ and $\gamma_1 \in \mathcal{C}_1$. Since $U_1 = \mathcal{C}_1 = \{1, T_1\}$, $u_1\gamma_1 \in \{1, T_1, T_1^2\} \subseteq \text{Span} \{1, T_1\}$. That is $u_1\gamma_1 \in \text{Span} \{\gamma_0 v_1 \mid \gamma_0 \in \mathcal{C}_0 \text{ and } v_1 \in U_1\}$. Thus, the lemma holds for $n = 1$. Assume the lemma for a positive integer $n$. We prove the lemma for the next integer. Let $u_{n+1} \in U_{n+1}$ and $\gamma_{n+1} \in \mathcal{C}_{n+1}$. Consider the following cases:

i) If $u_{n+1} = 1$: We can write $u_{n+1}\gamma_{n+1} = \gamma_{n+1} = \gamma_n v_{n+1}$ for some $\gamma_n \in \mathcal{C}_n$ and $v_{n+1} \in \{1, T_{n+1}\} \subseteq U_{n+1}$.

ii) If $u_{n+1} \neq 1$: One can write $u_{n+1}\gamma_{n+1} = (T_{n+1}u_n)(\gamma_n v_{n+1}) = T_{n+1}(u_n \gamma_n) x_{n+1}$ for some $u_n \in U_n$, $\gamma_n \in \mathcal{C}_n$ and $x_{n+1} \in \{1, T_{n+1}\}$. By induction hypothesis, $u_n \gamma_n \in \text{Span} \{\gamma_{n-1} v_n \mid \gamma_{n-1} \in \mathcal{C}_{n-1} \text{ and } v_n \in U_n\}$, we get $u_{n+1}\gamma_{n+1} \in \text{Span} \{T_{n+1}\gamma_n v_{n+1} x_{n+1} \mid \gamma_n \in \mathcal{C}_n \text{ and } v_n \in U_n\}$. Thus, $u_{n+1}\gamma_{n+1} \in \text{Span} \{\gamma_{n-1} T_{n+1} v_n x_{n+1} \mid \gamma_{n-1} \in \mathcal{C}_{n-1} \text{ and } v_n \in U_n\}$, since by far-commutativity $T_{n+1} \gamma_{n-1} = \gamma_{n-1} T_{n+1}$ for $\gamma_{n-1} \in \mathcal{C}_{n-1}$. Depending on $x_{n+1} = 1$ or $x_{n+1} = T_{n+1}$, we consider the cases as follows:

a) If $x_{n+1} = 1$: Since $\mathcal{C}_n \subseteq \mathcal{C}_n$ and $T_{n+1} v_n \in U_{n+1}$ for $v_n \in U_n$, we get $u_{n+1}\gamma_{n+1} \in \text{Span} \{\gamma_n v_{n+1} \mid \gamma_n \in \mathcal{C}_n \text{ and } v_{n+1} \in U_{n+1}\}$.
b) If \( x_{n+1} = T_{n+1} \): See that \( u_{n+1}\gamma_{n+1} \in \text{Span}\{\gamma_{n-1}T_{n+1}v_nT_{n+1} \mid \gamma_{n-1} \in C_{n-1} \text{ and } v_n \in U_n\} \).

By Lemma 2.5, \( \gamma_{n-1}T_{n+1}v_nT_{n+1} \in \text{Span}\{\gamma_nv_{n+1} \mid \gamma_n \in C_n \text{ and } v_{n+1} \in U_{n+1}\} \) for \( \gamma_{n-1} \in C_{n-1} \text{ and } v_n \in U_n \); thus, \( u_{n+1}\gamma_{n+1} \in \text{Span}\{\gamma_nv_{n+1} \mid \gamma_n \in C_n \text{ and } v_{n+1} \in U_{n+1}\}\).

\[ \Box \]

Lemma 2.7. For \( \beta_n \in B_n \) and \( \gamma_n \in C_n \), there is an algorithm to write the product \( \beta_n\gamma_n \) as a linear combination of the elements in \( B_n \).

Proof. We prove the lemma by induction on \( n \). Let \( \beta_1 \in B_1 \) and \( \gamma_1 \in C_1 \). We have \( \beta_1\gamma_1 \in \{1, T_1, T_1^2\} \), since \( B_1 = \{1, T_1\} \) and \( C_1 = \{1, T_1\} \). Thus, \( \beta_1\gamma_1 \in \text{Span}(B_1) \), since \( T_1^2 = (q-1)T_1 + q \). Hence the lemma holds for \( n = 1 \). Assume the lemma for \( n \geq 1 \). We prove the lemma for the next integer. Let \( \beta_{n+1} \in B_{n+1} \) and \( \gamma_{n+1} \in C_{n+1} \). We can write \( \beta_{n+1}\gamma_{n+1} = \beta_n u_{n+1}\gamma_{n+1} \) for \( \beta_n \in B_n \) and \( u_{n+1} \in U_{n+1} \). By Lemma 2.6, \( u_{n+1}\gamma_{n+1} \in \text{Span}\{\gamma_nv_{n+1} \mid \gamma_n \in C_n \text{ and } v_{n+1} \in U_{n+1}\} \), thus, \( \beta_{n+1}\gamma_{n+1} \in \text{Span}\{\beta_n\gamma_nv_{n+1} \mid \gamma_n \in C_n \text{ and } v_{n+1} \in U_{n+1}\} \). By the induction hypothesis, \( \beta_n\gamma_n \in \text{Span}(B_n) \) for \( \gamma_n \in C_n \); therefore, \( \beta_{n+1}\gamma_{n+1} \in \text{Span}(B_{n+1}) \) for \( \gamma_n \in C_n \) and \( v_{n+1} \in U_{n+1} \). Hence \( \beta_{n+1}\gamma_{n+1} \in \text{Span}(B_{n+1}) \). \[ \Box \]

3 Algorithms to find trace of elements of \( H_{n+1} \)

We write an algorithm to compute the trace of an element in \( H_{n+1} \) corresponding to a weave braid. This requires several steps. In this section, we present step by step algorithms to arrive at final algorithm for computing the trace. We specialize it to weave braid representations to meet our goal.

Lemma 3.1. For \( u_n, v_n \in U_n \), the product \( u_nv_n \) can be written as a linear combination of the elements of the type \( u_nv_n \), where \( u_{n-1} \in U_{n-1} \) and \( w_n \in U_n \).

Proof. We prove this lemma by induction on \( n \). Let \( u_1, v_1 \in U_1 \). See that \( u_1v_1 \in \text{Span}\{u_0w_1 \mid u_0 \in U_0 \text{ and } v_1 \in U_1\} \), since \( U_0 = \{1\} \) and \( U_1 = \{1, T_1\} \) and \( T_1^2 = (q-1)T_1 + q \). Thus, the lemma holds for \( n = 1 \). Assume the lemma for \( n \geq 1 \). We prove the lemma for the next integer.

Let \( u_{n+1}, v_{n+1} \in U_{n+1} \). Consider the following cases:

i) If \( u_{n+1} \neq 1 \) or \( v_{n+1} \neq 1 \): We get \( u_{n+1}v_{n+1} \in \text{Span}\{x_nv_{n+1} \mid x_n \in U_n \text{ and } w_{n+1} \in U_{n+1}\} \).

ii) If \( u_{n+1} \neq 1 \) and \( v_{n+1} \neq 1 \): One can write \( u_{n+1}v_{n+1} = T_{n+1}u_{n+1}T_{n+1}v_{n+1} \) for some \( u_{n-1} \in U_{n-1} \). Using relations in the Hecke algebra, we get \( u_{n+1}v_{n+1} = T_{n+1}u_{n+1}v_{n+1} = T_{n+1}u_{n+1}v_{n+1} = T_{n+1}u_{n+1}v_{n+1} = T_{n+1}u_{n+1}v_{n+1} \). Thus, \( u_{n+1}v_{n+1} \in \text{Span}\{x_nv_{n+1} \mid x_n \in U_n \text{ and } w_{n+1} \in U_{n+1}\} \).

\[ \Box \]
Lemma 3.2. For $\beta_n \in B_n$ and $v_n \in U_n$, there is an algorithm to write the product $\beta_n v_n$ as a linear combination of the elements in $B_n$.

Proof. We prove the lemma by induction on $n$. Let $\beta_1 \in B_1$ and $v_1 \in U_1$. Note that $\beta_1 v_1 \in \text{Span} (B_1)$, since $B_1 = \{1, T_1\}$, $U_1 = \{1, T_1\}$ and $T_1^2 = (q - 1) T_1 + q$. Thus, the lemma holds for $n = 1$. Assume the lemma for $n = k$. We prove the lemma for the next integer. Let $\beta_{k+1} \in B_{k+1}$ and $v_{k+1} \in U_{k+1}$. We can write $\beta_{k+1} v_{k+1} = \beta_n u_{n+1} v_{n+1}$ for $\beta_n \in B_n$ and $u_{n+1} \in U_{n+1}$. By Lemma 3.2, $u_{n+1} v_{n+1} \in \text{Span} \{ v_n w_{n+1} \mid v_n \in U_n \text{ and } w_{n+1} \in U_{n+1} \}$; thus, $\beta_{k+1} v_{k+1} \in \text{Span} \{ \beta_n v_n w_{n+1} \mid v_n \in U_n \text{ and } w_{n+1} \in U_{n+1} \}$. By the induction hypothesis, $\beta_n v_n \in \text{Span} (B_n)$ for $v_n \in U_n$; therefore, $\beta_n v_n w_{n+1} \in \text{Span} (B_{n+1})$ for $v_n \in U_n$ and $w_{n+1} \in U_{n+1}$. Hence $\beta_{k+1} v_{k+1} \in \text{Span} (B_{n+1})$. □

Lemma 3.3. There is an algorithm to compute the trace of elements in $B_n$.

Proof. We prove this lemma by induction on $n$. Note that $B_1 = \{1, T_1\}$. By using the properties of the trace, one can see that $\text{Tr}(1) = 1$ and $\text{Tr}(T_1) = z$. Assume the lemma for $n = k$. We prove the lemma for the next integer. Let $\beta_{k+1} = u_1 u_2 \cdots u_{k+1}$ be element in $B_{k+1}$. Consider the following cases:

i) If $u_{n+1} = 1$: See that $\beta_{n+1} = u_1 u_2 \cdots u_n$, i.e. $\beta_{n+1} \in B_n$. Thus, by induction hypothesis, we can compute $\text{Tr} (\beta_{n+1})$.

ii) If $u_{n+1} \neq 1$: We get $\beta_{n+1} = u_1 u_2 \cdots u_{n+1} T_{n+1} v_n$ for some $v_n \in U_n$. Using the third property for the trace, we can write $\text{Tr} (\beta_{n+1}) = z \text{Tr} (u_1 u_2 \cdots u_{n+1} v_n) = z \text{Tr} (\beta_n v_n)$, where $\beta_n = u_1 u_2 \cdots u_n$. By Lemma 3.2, there is an algorithm to write the product $\beta_n v_n$ as linear combination of the elements of $B_n$. By the linearity of the trace and using the induction hypothesis, one can compute $\text{Tr} (\beta_n v_n)$ and hence $\text{Tr} (\beta_{n+1})$.

□

If we can express an element of Hecke algebra $H_{n+1}$ as linear combination of elements of the basis $B_n$, we can effectively compute its trace. In Section 2 we discussed that $\rho : B_{n+1} \rightarrow H_{n+1}$ defined by $\rho (\sigma_i) = T_i$ is a representation of $B_{n+1}$ into the multiplicative monoid of $H_{n+1}$. Thus, if we can express $\rho (\alpha)$ as linear combination of elements of $B_n$ then our trace algorithm can be used to compute the trace of $\rho (\alpha)$ for every $\alpha \in B_{n+1}$ and hence will be useful in computing invariants for links [17].

In this paper our interest is in a special class of braids $\beta_{N,M}$; we call them weave braids of type $(N,M)$. For a pair of positive integers $(N,M)$, $\beta_{N,M} = (\sigma_1 \sigma_2^{-1} \sigma_3 \sigma_4^{-1} \cdots \sigma_{N-1}^{-1})^M \in B_N$. Closure of the braid $\beta_{N,M}$ is $W(N,M)$ known as weave link of type $(N,M)$ having $\text{gcd}(N,M)$ components. When $N$ and $M$ are coprime the weave knots $W(N,M)$ are interesting family of hyperbolic knots that have been studied earlier [6, 7, 15]. A picture of $W(3,4)$ is shown in Figure 1.

We prefer to write $N = n + 1$ and $M = m$, and study $W(n+1,m)$.  

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Theorem 3.4. Let \( b \) denote the braid \( \beta_{n+1,m} = \left( \sigma_1 \sigma_2^{-1} \sigma_3 \sigma_4^{-1} \cdots \sigma_n^{(-1)^{n-1}} \right)^m \) for \( m \geq 1 \). Then there is an algorithm to compute the trace of \( \rho(b) \in H_{n+1} \).

Proof. Let \( r \) be an integer defined as follows:

\[
    r = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}.
\]

Let \( L = \{ (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n \mid l_i \in \{0, 1, \ldots, i\} \} \) for \( i = 1, 2, \ldots, n \). Let \( N \) be set of elements \( (l_1, l_2, \ldots, l_n) \) in \( L \) such that \( l_i = 1 \) if \( i \) is odd and \( l_i \) is either 0 or 1 if \( i \) is even. For \( k = 0, 1, \ldots, r \), let \( L_k \) be the set of elements \( (l_1, l_2, \ldots, l_n) \) in \( N \) such that exactly \( k \) number of \( l_i \)'s are zero. For \( l = (l_1, l_2, \ldots, l_n) \) in \( L \), let \( \beta_l = u_1^{l_1} u_2^{l_2} \cdots u_n^{l_n} \), where for \( i = 1, 2, \ldots, n \),

\[
    u_i^{l_i} = \begin{cases} 1 & \text{if } l_i = 0 \\ T_i T_{i+1} \cdots T_{i-l_i+1} & \text{if } l_i \neq 0. \end{cases}
\]

The basis \( B_n \) for \( H_{n+1} \) can be written as \( B_n = \{ \beta_l \mid l \in L \} \). Note that, for \( k = 0, 1, \ldots, r \) and \( l \in L_k, \beta_l \in C_n \) and it is the word \( T_1 T_2 \cdots T_n \) with exactly \( k \) number of \( T_i \)'s, for \( i \) even, are missing.

Let \( a = \sigma_1 \sigma_2^{-1} \sigma_3 \sigma_4^{-1} \cdots \sigma_n^{(-1)^{n-1}} \). Since \( T_i^{-1} = q^{-1} (T_i + (1-q)) \) for \( i = 1, 2, \ldots, n \), we have the following estimate:

\[
    \rho (a) = T_1 T_2^{-1} T_3 T_4^{-1} \cdots T_n^{(-1)^{n-1}} = \begin{cases} q^{-r} T_1 (T_2 + (1-q)) T_3 (T_4 + (1-q)) \cdots T_n^{-1} (T_n + (1-q)) & \text{if } n \text{ is even} \\ q^{-r} T_1 (T_2 + (1-q)) T_3 (T_4 + (1-q)) \cdots (T_n^{-1} + (1-q)) T_n & \text{if } n \text{ is odd} \end{cases}
\]

\[
= q^{-r} \sum_{k=0}^r (1-q)^k \sum_{l \in L_k} \beta_l 
= q^{-r} \sum_{l \in L} f_l^1(q) \beta_l, \tag{1}
\]

where

\[
f_l^1(q) = \begin{cases} (1-q)^k & \text{if } l \in L_k \text{, for } k = 0, 1, \ldots, r \\ 0 & \text{if } l \notin \bigcup_{k=0}^r L_k \end{cases}.
\]
For an integer \( m \geq 2 \), assume that
\[
\rho (a^{m-1}) = q^{-(m-1)r} \sum_{l \in L} f_l^{m-1}(q) \beta_l ,
\]
for some polynomials \( f_l^{m-1} \) in \( q \). For \( i \in L \) and \( j \in L_k \), since by Lemma 2, there is an algorithm to write \( \beta_i \beta_j \) as linear combination of elements in \( B_n \), we can find the polynomials \( h_l^{ij} \) (for \( l \in L \)) in \( q \) such that \( \beta_i \beta_j = \sum_{l \in L} h_l^{ij}(q) \beta_l \). Consider the following estimate:
\[
\rho (a^m) = \rho (a^{m-1}) \rho (a) = q^{-mr} \left( \sum_{i \in L} f_i^{m-1}(q) \beta_i \right) \left( \sum_{k=0}^{r} (1 - q)^k \sum_{j \in L_k} \beta_j \right)
\]
\[
= q^{-mr} \sum_{k=0}^{r} \sum_{i \in L} \sum_{j \in L_k} (1 - q)^k f_i^{m-1}(q) \beta_i \beta_j
\]
\[
= q^{-mr} \sum_{k=0}^{r} \sum_{i \in L} \sum_{j \in L_k} (1 - q)^k f_i^{m-1}(q) \sum_{l \in L} h_l^{ij}(q) \beta_l
\]
\[
= q^{-mr} \sum_{l \in L} \left( \sum_{k=0}^{r} \sum_{i \in L} \sum_{j \in L_k} (1 - q)^k f_i^{m-1}(q) h_l^{ij}(q) \beta_l \right)
\]
\[
= q^{-mr} \sum_{l \in L} f_l^m(q) \beta_l ,
\]
where
\[
f_l^m(q) = \sum_{k=0}^{r} \sum_{i \in L} \sum_{j \in L_k} (1 - q)^k f_i^{m-1}(q) h_l^{ij}(q) .
\]
Since Tr is linear, by Exp \( 1 \) and Exp \( 2 \), and using Lemma 3, we can compute \( Tr(\rho (a^m)) \) for \( m \geq 1 \). This completes the proof of the theorem. \( \square \)

Note that Eq \( 3 \) is a generalization of recursion relations that were found for \( \beta_3, m \) in [15].

We need to have estimates on the degree of \( Tr(\beta_l) \) (for \( l \in L \)), where \( \beta_l \) is an element in the basis \( B_n \) of \( H_{n+1} \). In this connection, we have the following:

**Lemma 3.5.** Let \( w \) be a word in \( T_1, T_2, \ldots, T_n \). Then the degree of \( Tr(w) \) as a Laurent polynomial in \( q \) and \( z \) is same as the length of the word \( w \).

**Proof.** While computing the trace of a word in \( T_1, T_2, \ldots, T_n \), the relations \( T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i \) for \( i = 1, 2, \ldots, n-1 \) and \( T_iT_j = T_jT_i \) for \( |i - j| \geq 2 \) do not add or subtract any power of \( q \) or \( z \) in the trace thereby do not change the degree. The relation \( T_i^2 = (q - 1)T_i + q \) for \( i = 1, 2, \ldots, n \) together with the properties of the trace contribute a factor \( qz \) in the leading term of the trace and the length of the word decreases by two. In other words, decreased length of the word appears in the form of powers of \( q \) and \( z \) together in the leading term. Now by keeping the properties of the trace in the mind, one can think that the leading term of the trace of any word will be \( q^iz^j \), where \( i + j \) is the length of the word taken and \( j \) is the number of distinct \( T_k \)'s appeared in the word. This completes the proof. \( \square \)
Proposition 3.6. For the weave braid $\beta_{n+1,m} = (\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{n}^{-1})^{m}$, for $m \geq 2$, we have the following:

i) $Tr(\rho(\beta_{n+1,m}))$ as a Laurent polynomial in $q$ and $z$ has the lowest degree equal to $r + d - mr$, where $r = (n - d)/2$ and $d$ is either 0 or 1 depending on $n$ is even or odd respectively.

ii) Degree of $Tr(\rho(\beta_{n+1,m}))$ as a Laurent polynomial in $q$ and $z$ is equal to $mn - mr$.

Proof. i) By the proof of Theorem 3.4, one can write

$$\rho(\beta_{n+1,m}) = q^{-mr} \sum_{k_1=0}^{r} \sum_{k_2=0}^{r} \cdots \sum_{k_m=0}^{r} \sum_{l_1 \in L_{k_1}} \sum_{l_2 \in L_{k_2}} \cdots \sum_{l_m \in L_{k_m}} (1 - q)^{k_1+k_2+\cdots+k_m} \beta_1\beta_2\cdots\beta_{n}.$$ 

Note that $\beta_{l_j}$ (for $l_j \in L_{k_j}$) is obtained by eliminating $k_j$ number of $T_j$‘s, for $j$ even, from the word $T_1T_2\cdots T_n$.

As it is noted in the proof of Lemma 3.3, while computing the trace of a word in $H_{n+1}$, the relations $T_{i+1}T_i + 1 = T_iT_{i+1}$ for $i = 1, 2, \ldots, n - 1$ and $T_nT_1 = T_1T_n$ for $i = 1, 2, \ldots, n - 1$ and $T_1T_2 = T_2T_1$ for $|i - j| \geq 2$ do not add or subtract any power of $q$ or $z$ in the trace. By looking at the relation $T_i = x^{i-1}(y^{1} + y^{2} + \cdots + 1)T_i + x^{i-1}(y^{1} + y^{2} + \cdots + y)$ (see Lemma 2.4), it adds a factor $x^{i-1}z$ or $x^{i-1}y$ in a lowest degree term in the trace of a word $\beta_1\beta_2\cdots\beta_{n}$ (note that this a word in $T_i$’s) and the length of the word (as a word in $T_i$’s) decreases by $j$, where $x = -1$ and $y = -q$. That is, the decreased length appears in the form of powers of $x$, $y$, and $z$. From this observation, we can see that a lowest degree term in the trace of a word $\beta_1\beta_2\cdots\beta_{n}$ is $x^iy^jz^k = (-1)^{i+j}q^iz^k$ for $i + j$ the least integer which is the number of distinct $T_k$’s appeared in the word. Note that $l + i + j$ is the length of the word $\beta_1\beta_2\cdots\beta_{n}$.

One can see that the word $(T_1T_3\cdots T_{2r+c})^m$ is such that it appears the least number of distinct $T_i$’s among all the words $\beta_1\beta_2\cdots\beta_{n}$ appeared in the expression of $\rho(\beta_{n+1,m})$, where $c$ is either $-1$ or 1 depending on $n$ is even or odd respectively. There are $r + d$ number of distinct $T_i$’s appear in the word $(T_1T_3\cdots T_{2r+c})^m$ and the length of this word is $m(r + d) = mn - mr$. A lowest degree term in the trace of the word $(T_1T_3\cdots T_{2r+c})^m$ is $x^iy^jz^k = (-1)^{i+j}q^iz^k$ for $i + j = r + d$ and $l + i + j = mn - mr$. Finally, a lowest degree term in the trace of $\rho(\beta_{n+1,m})$ will be $(-1)^{r+i}q^{-mr}z^k$ for $i + j = r + d$ and $l + i + j = mn - mr$.

ii) In each term in the expression of $\rho(\beta_{n+1,m})$, a product $\beta_1\beta_2\cdots\beta_{n}$ is a word in $T_i$’s of length $mn - (k_1 + k_2 + \cdots + k_m)$, where $l_1, l_2 \in L_{k_1}, l_2 \in L_{k_2}, \ldots, l_m \in L_{k_m}$. By Lemma 3.3, the leading term in the trace of $\beta_1\beta_2\cdots\beta_{n}$ will be $q^iz^k$, where $i + j = mn - (k_1 + k_2 + \cdots + k_m)$ and $j$ is the number of distinct $T_k$’s appeared in $\beta_1\beta_2\cdots\beta_{n}$. Now the leading term in the trace of $(1 - q)^{k_1+k_2+\cdots+k_m} \beta_1\beta_2\cdots\beta_{n}$ becomes $(-1)^{k_1+k_2+\cdots+k_m}q^iz^k$, where $i + j = mn$ and $j$ is the number of distinct $T_k$’s appeared in $\beta_1\beta_2\cdots\beta_{n}$. As it is noted in the first part, the word $(T_1T_3\cdots T_{2r+c})^m$ is such that it appears the least number of distinct $T_i$’s among all the words $\beta_1\beta_2\cdots\beta_{n}$ appeared in the expression of $\rho(\beta_{n+1,m})$. The leading term in the trace of $(1 - q)^{mr}(T_1T_3\cdots T_{2r+c})^m$ will be $(-1)^{mr}q^iz^k$, where $i + j = mn$ and $j = r + d$. Now by looking at the expression for $\rho(\beta_{n+1,m})$, one can see that $(-1)^{mr}q^{-mr}z^k$ is among the leading terms in the trace of $\rho(\beta_{n+1,m})$, where $i + j = mn$ and $j = r + d$. In other words, the degree of the trace of $\rho(\beta_{n+1,m})$ is $mn - mr$.

\[\square\]

We have written a program in Mathematica with the add-on known as Quantum. This program is written using the lemmas and theorems proved in this section. A pdf file of the program is described in the next two pages.
In[1]:= Needs["Quantum`Notation"];

\(n, p = 4, 2\); 

\(n = n - 1; d = (1 - (-1)^n) / 2; r = (n - d) / 2;\)

For[i = 1, i \leq n, i++, T[i] = Symbol["T" <> ToString[i]]];

SetQuantumObject[Table[T[i], (i, n)]]; 

trace[x_ + y_] := trace[x] + trace[y]; trace[a_ \cdot x_] := Expand[a \cdot trace[x]]; /; IntegerQ[a]; 

trace[q^a \cdot x_] := Expand[q^a \cdot trace[x]]; /; IntegerQ[a];

trace[z^a \cdot x_] := Expand[z^a \cdot trace[x]]; /; IntegerQ[a];

trace[q^a] := q^a; /; IntegerQ[a]; trace[q] = q; trace[z] = z;

trace[q^a] := q^a; /; IntegerQ[a];

g[x_ + y_] := g[x] + g[y]; g[a_ \cdot x_] := Expand[a \cdot g[x]]; /; IntegerQ[a];

g[q^a] := Expand[q \cdot g[x]]; g[q^a \cdot x_] := Expand[q^a \cdot g[x]]; /; IntegerQ[a];

h[x_ + y_] := h[x] + h[y]; h[a_ \cdot x_] := Expand[a \cdot h[x]]; /; IntegerQ[a];

h[q^a] := Expand[q \cdot h[x]]; h[q^a \cdot x_] := Expand[q^a \cdot h[x]]; /; IntegerQ[a];

h[a_] := a; /; IntegerQ[a]; h[q] = q; h[q^a] := q^a; /; IntegerQ[a];

g[T1] = T1; g[T1 \cdot T1] = (q - 1) \cdot T1 + q; g[T2] = T2; g[T2 \cdot T1] = T2 \cdot T1;

g[T2 \cdot T2] = (q - 1) \cdot T2 + q; g[T2 \cdot T1 \cdot T2] = T1 \cdot T2 \cdot T1; h[T1] = T1; h[T1 \cdot T1] = (q - 1) \cdot T1 + q;

Lo[0] = Mo[0] = No[0] = \{\}; u[0, 0] = b[\{}\] = bv[\{}\], 0 = c[\{}\] = uc[0, \{}\] = bc[\{}\], \{}\] = 1;

For[i = 1, i \leq n, i++, 
    For[j = 0; Lo[i] = \{}, j \leq i, j++, 
        For[k = 1; Lw[i, j] = Lo[i - 1], k \leq i!, k++, Lw[i, j] = Insert[Lw[i, j], \{k, i\}] ];
        Lo[i] = Union[Lo[i], Lw[i, j]]; 
    For[j = 1; Mw[i, 0] = Mw[i, 1] = Mo[i - 1], j \leq 2^(i - 1), j++, 
        Mw[i, 0] = Insert[Mw[i, 0], \{0, \}\{j, i\}]; Mw[i, 1] = Insert[Mw[i, 1], \{1, \}\{j, i\}];
        Mo[i] = Union[Mw[i, 0], Mw[i, 1]]; 
    For[j = 1; Nw[i, 0] = Nw[i, 1] = No[i - 1], j \leq Length[No[i - 1]], j++, 
        Nw[i, 0] = Insert[Nw[i, 0], \{0, \}\{j, i\}]; Nw[i, 1] = Insert[Nw[i, 1], \{1, \}\{j, i\}];
        No[i] = Union[Nw[i, (1 - (-1)^i) / 2], Nw[i, 1]]; 
    For[j = 1; u[i, 0] = 1, j \leq i, j++, u[i, j] = T[j] \cdot u[i - 1, j - 1]]; Mo[n] = No[n];

For[i = 0, i \leq n, i++, L[i] = \{}; 
    For[j = 1, j \leq 2^n, j++, l = No[n][\{}\]]; 
    If[Sum[l[\{k\}], \{k, n\}] = n - 1, L[i] = Union[L[i], \{\}\{l\}]]; Clear[Lw, Mw, No, Nw]

For[i = 1, i \leq n, i++, 
    For[j = 1, j \leq (i + 1)!, j++, l = Lo[i][\{}\]];
    For[s = 1; b[l] = 1, s \leq i, s++, b[l] = b[l] \cdot u[s, l[\{s\}]]];
In[1] :=
If[lDD > 0, trace[bDD] = Expand[z \cdot trace[bvDelete[l, -1], lDD - 1]]],
trace[bDD] = Expand[trace[bvDelete[l, -1]]];

If[i < n, bv[l, 0] = b[l];
If[lDD > 0,
For[s = 1, s \leq i, s++,
If[s < lDD, bv[l, s] = Expand[bvDelete[l, s] - u[i, lDD]]; bv[l, s] = Expand[(q - 1) \cdot bvDelete[l, s] - u[i, s] + q \cdot bvDelete[l, s] - 1];
For[s = 1, s \leq i, s++,
trace[bDD] = Expand[bv[l, s]]];

For[k = 1, k \leq Length[Mo[i]], k++, m = Mo[i][k];
For[s = 1; c[m] = 1, s \leq i, s++;
uc[s, m] = g[Expand[T[i] \cdot uc[s - 1, Delete[m, -1]]] \cdot u[i, m[i]]]]; uc[0, m] = c[m];

bc[l, m] = h[Expand[bDelete[l, -1]] \cdot u[lDD, m]]]; bc[0, m] = m;

For[i < n - 1, g[T[i + 2] - c[m]] = c[m] \cdot T[i + 2];
g[T[i + 2] \cdot c[m] \cdot T[i + 2]] = (q - 1) \cdot c[m] \cdot T[i + 2] + q \cdot c[m];

For[s = 1, s \leq i + 1, s++,
bc[l, m] = h[Expand[bDelete[l, -1]] \cdot u[lDD, s]]]; bc[0, m] = m;

If[i < n, For[s = 0, s \leq i + 1, s++,
for[h[l, m] \cdot u[i + 1, s] = Expand[bv[l, m] \cdot u[i + 1, s]]]]; Clear[bv, c, g, h, Mo, u, uc]

For[lDD = 1, lDD \leq (n + 1) !, lDD++,
For[k = 0, k < r, k++,
For[i = 1, i \leq (n + 1) !, i++,
For[j = 1, j \leq r \div k !, j++,
For[s = 1, s \leq n, s++,
For[k = 0, k > r, k++, sumi = 0;
For[i = 1, i \leq (n + 1) !, i++,
For[j = 1, j \leq r \div k !, j++,
For[s = 1, s \leq n, s++,
cf = DeleteCases[cf, _?(FreeQ[#, T[s]][i])]; sumj = sumj + cf - 1];
sumi = Expand[sumi + Expand[bv[lDD, i] \cdot sumj]];

TRACE[m] = Expand[f[lDD, 1] \cdot (1 - q) \cdot k \cdot trace[b[lDD]]]]]

For[m = 2, m \leq p, m++,
For[k = 0, k \leq r, k++, sumi = 0;
For[i = 1, i \leq (n + 1) !, i++,
If[Sum[i[s]], {s, n}] > Sum[lDD, {s, n}] - n + k,
sumi = Expand[s
sumi + f[lDD, i] \cdot Sum[Coefficient[bv[lDD, i] \cdot T[s][i]]]], b[lDD], {jj, r ! / k ! / (r - k) !})]]];

f[lDD, 1] = Expand[f[lDD, 1] \cdot (1 - q) \cdot k \cdot sumi];

TRACE[m] = Expand[TRACE[m] + f[lDD, 1] \cdot trace[b[lDD]]]]]; Clear[b, bc, f, L, Lo, trace]

For[m = 1, m \leq p, m++,
Print["Trace of " W[n + 1, m]; Print[q^(-m * r) \cdot TRACE[m]]

Trace of W[4, 1]
z^2 - q z^2 + z^3
q

Trace of W[4, 2]
q^2 - 2 q^3 + q^4 - 2 q z + 8 q^2 z - 8 q^3 z + 2 q^4 z + 2 q^5 z - 7 q z^2 + 3 q^2 z^2 + 7 q^3 z^2 + q^4 z^2 + 5 q^5 z^3 - 5 q z^3 + 5 q^2 z^3 - q^3 z^3
q^2
4 From trace to knot polynomials

We use the construction given in [5, p.288] and work over the function field $K = \mathbb{C}(q, z)$ to obtain expressions for the two-variable HOMFLY-PT polynomial, the one-variable Jones polynomial $V_{W(n,m)}(t)$, and the Alexander polynomial $\Delta_{W(n,m)}(t)$. The expressions are subsequently refined to incorporate information obtained in the previous section.

A polynomial invariant in variables $q$ and $z$ of the knot that is the closure of a braid $\alpha \in B_{n+1}$ is given by the formula

$$V_\alpha(q, z) = \left(\frac{1}{z}\right)^{n+e(\alpha)} \left(\frac{q}{w}\right)^{n-e(\alpha)} Tr(\rho(\alpha)),$$

where $e(\alpha)$ is the exponent sum of the word $\alpha$. The expression defines an element in the quadratic extension $K(\sqrt{q/w})$. For the weaving knot $W(n+1, m)$, viewed as the closure of a braid $\beta_{n+1,m} = (\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1} \cdots \sigma_n\sigma_{n+1}^{-1})^m$, for $m \geq 1$, we have the exponent sum $e = 0$ for $n$ even and $e = m$ for $n$ odd, and

$$\rho((\beta_{n+1,m})) = (T_1 T_2^{-1} T_3 T_4^{-1} \cdots T_{n} T_{n+1}^{-1})^m = q^{-mr} \sum_{l \in L} f_l^m(q) \beta_l$$

where $f_l^m(q)$ are recursively defined as

$$f_l^m(q) = \sum_{k=0}^{r} \sum_{i \in L} \sum_{j \in L_k} (1-q)^k f_{ij}^{m-1}(q) h_{lj}^i(q)$$

with the initial values $f_l^1$ as

$$f_l^1(q) = \begin{cases} 
(1-q)^k & \text{if } l \in L_k \text{, for } k = 0, 1, \ldots, r \\
0 & \text{if } l \notin \bigcup_{k=0}^r L_k .
\end{cases}$$

The polynomial invariant in variables $q$ and $z$ of $W(n+1, m)$ is given by

$$V_{\beta_{n+1,m}}(q, z) = \left(\frac{1}{z}\right)^{n+e} \left(\frac{q}{w}\right)^{n-e} q^{-mr} \sum_{l \in L} f_l^m(q) Tr(\beta_l)$$

From Lemma 3.5, we know that $Tr(\beta_l)$ is a polynomial in variables $q$ and $z$ of degree equal to the length of $\beta_l$.

Following [5], we point out that the universal skein invariant $P_{W(n+1,m)}(\ell, m)$, an element of the Laurent polynomial ring $\mathbb{Z}[\ell, \ell^{-1}, m, m^{-1}]$, is obtained by rewriting $V_{\beta_{n+1,m}}(q, z)$ in terms of $\ell = i(z/w)^{1/2}$ and $m = i(q^{1/2} - q^{-1/2})$, a task easily managed in a computer algebra system by simplifying $V_{\beta_{n+1,m}}(q, z)$ with respect to side relations. Starting from $P_{W(n+1,m)}(\ell, m)$, the Jones polynomial $V_{W(n+1,m)}(t)$ is obtained by setting $\ell = it$ and $m = i(t^{1/2} - t^{-1/2})$, the Alexander polynomial $\Delta_{W(n+1,m)}(t)$ is obtained by setting $\ell = i$ and $m = i(t^{1/2} - t^{-1/2})$, and the HOMFLY-PT polynomial is obtained by setting $\ell = ia$ and $m = iz$.

Proposition 4.1. For a weave braid $\beta_{n+1,m} = (\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1} \cdots \sigma_n\sigma_{n+1}^{-1})^m$, for $m \geq 2$, we have the following:
i) The trailing term in the Jones polynomial of $\beta_{n+1,m}$ is $(-1)^{mn-mr}t^{d/2-3e/2-mr}$, where $r = (n-d)/2$, $e = mn$ and $d$ is either 0 or 1 depending on $n$ is even or odd respectively.

ii) The degree of the Jones polynomial of $\beta_{n+1,m}$ is equal to $d/2 - e/2 + mr$.

**Proof.**

i) One can check that $V_{\beta_{n+1,m}}(t) = (\frac{1}{2}) \frac{n}{2} (\frac{e}{2}) \frac{n}{2} Tr(\beta_{n+1,m}) \big|_{q=t,z=\frac{\sqrt{t}}{1+t,z}}$. This can be simplified as $V_{\beta_{n+1,m}}(t) = (1 + t)^n t^{-\frac{3}{2} - \frac{e}{2}} q^{i-mr} \overline{r}$, by the first part of Proposition 3.6, the trailing term in $V_{\beta_{n+1,m}}(t)$ will be the trailing term in $(-1)^{i+j}(1 + t)^n t^{-\frac{3}{2} - \frac{e}{2}} q^{i-mr} \overline{r}$ for $i, j = r + d$ and $i + j = mn - mr$, and the choice of $i$ and $j$ is such that the degree of $t$ is least. In other words, $i$ must be $r + d$, $j$ must be zero and $l + i$ must be $mn - mr$. That is, the trailing term in $V_{\beta_{n+1,m}}(t)$ will be $(-1)^{mn-mr}t^{d/2-3e/2-2mr}$.

ii) For an alternating knot the span of Jones polynomial is its crossing number. Since the crossing number of $\beta_{n+1,m}$ is $mn$, the proof of this part follows from the first part.

Below is the program written in Mathematica in continuation of the trace program discussed in the previous section to compute the Jones polynomial:

```mathematica
q = t; z = t^2 / (1 + t); For[m = 2, m <= p, m++, Print["Jones Polynomial of " W[n+1,m]]; e = d * m; V = Expand[Simplify[t^n (t^-3 * e / 2 - m * r) * (1 + t)^n * Trace[m]]; print[V]; Clear[q, z]
```

Using this program, we can compute the Jones polynomial of any weave knot. Below are is some examples:

- $V_{W(3,9)}(t) = -t^{-9} + 9t^{-8} - 36t^{-7} + 93t^{-6} - 189t^{-5} + 324t^{-4} - 480t^{-3} + 630t^{-2} - 737t^{-1} + 778 - 737t + 630t^{-2} - 480t^{-3} + 324t^{-4} - 189t^{-5} + 93t^{-6} - 36t^{-7} + 9t^{-8} - t^{-9}$
- $V_{W(4,2)}(t) = t^{-9/2} - t^{-7/2} + 3t^{-5/2} - 2t^{-3/2} + 2t^{-1/2} - 2t^{1/2} + 2t^{3/2}$
- $V_{W(4,3)}(t) = t^{-7} - 4t^{-6} + 8t^{-5} - 11t^{-4} + 13t^{-3} - 13t^{-2} + 11t^{-1} - 8 + 5t - t^{2}$
- $V_{W(4,7)}(t) = t^{-17} - 8t^{-16} + 36t^{-15} - 113t^{-14} + 281t^{-13} - 589t^{-12} + 1084t^{-11} - 1787t^{-10} + 3528t^{-9} - 4242t^{-8} + 4606t^{-7} - 4523t^{-6} - 4099t^{-5} + 3187t^{-4} - 239t^{-3} - 1358t^{-2} - 686 - 273t - 77t^{2} + 13t^{3} - t^{4}$
- $V_{W(5,2)}(t) = t^{-4} - 2t^{-3} + 4t^{-2} - 5t^{-1} - 5t + 5 + 4t^{2} - 2t^{3} + t^{4}$
- $V_{W(5,6)}(t) = t^{-12} - 12t^{-11} + 72t^{-10} - 292t^{-9} + 960t^{-8} - 2296t^{-7} + 4935t^{-6} - 9175t^{-5} + 14934t^{-4} - 21518t^{-3} + 27709t^{-2} - 32138t^{-1} + 33749 - 32138t + 27709t^{2} - 21518t^{3} + 14934t^{4} - 9175t^{5} + 4935t^{6} - 966t^{7} - 292t^{8} + 72t^{10} - 12t^{11} + t^{12}$
- $V_{W(6,5)}(t) = t^{-17} + 11t^{-16} - 66t^{-15} + 276t^{-14} - 898t^{-13} + 2413t^{-12} - 5540t^{-11} + 11095t^{-10} - 19613t^{-9} + 30823t^{-8} - 43327t^{-7} + 54797t^{-6} - 62699t^{-5} + 65029t^{-4} - 61266t^{-3} + 52274t^{-2} - 40151t^{-1} + 27496 - 16577t + 8666t^{2} - 3855t^{3} + 1428t^{4} - 425t^{5} + 95t^{6} - 14t^{7} + t^{8}$
- $V_{W(6,7)}(t) = -3549707 - t^{-24} + 15t^{-23} - 120t^{-22} + 66t^{-21} - 256t^{-20} + 10144t^{-19} - 30777t^{-18} + 82162t^{-17} - 19665t^{-16} + 427553t^{-15} - 851900t^{-14} + 1564255t^{-13} - 2655194t^{-12} + 4172378t^{-11} - 6072825t^{-10} + 8188029t^{-9} - 10228204t^{-8} + 11839044t^{-7} - 12697719t^{-6} + 12613052t^{-5} - 11588995t^{-4} + 982624t^{-3} - 7661335t^{-2} + 546938t^{-1} + 2083392t^{-1} - 1097273t^{2} + 514481t^{3} - 212798t^{4} + 767395^{4} - 237236^{4} + 6125t^{7} - 12678 + 196^{9} - 20^{10} + t^{11}$
We note that $W(3,9)$ and $W(4,2)$ are links with 3 and 2 components having their Jones polynomial is Laurent polynomial in $t$ and $t^{1/2}$ respectively. To compute the Alexander polynomial and the HOMFLY-PT, all we need is to make different substitution as discussed in the beginning of this section.

5 Higher twist numbers versus volume

In [8], Dasbach and Lin introduced higher twist numbers of a knot in terms of the coefficients of its Jones polynomial $V_K(t)$ with the idea that these invariants also correlate with the hyperbolic volume of the knot complement. The twist numbers are defined as follows:

**Definition 5.1.** Let

$$V_K(t) = \lambda_m t^m + \lambda_{m+1} t^{m+1} + \cdots + \lambda_{n-1} t^{n-1} + \lambda_n t^n,$$

then the $j$th twist number of $K$ is $T_j(K) = |\lambda_{m+j}| + |\lambda_{n-j}|$.

Note that twist numbers $T_j(K)$ are only defined for $j$ within the span of the Jones polynomial. In the case of weaving knots $W(n+1,m)$, the relevant twist numbers are defined for $1 \leq j \leq \mu - 1$, where $\mu = d/2 + 3e/2 - mr$ is the lowest degree in the Jones polynomial of $W(n+1,m)$. Here $d$ is 0 or 1 depending on $n$ being even or odd, and $r = \frac{n-d}{2}$.

According to [6] the volume of weaving knots $W(n+1,m)$ for $n \geq 2$ and $m \geq 7$ is estimated as

$$v_{\text{oct}}(n-1)m \left( 1 - \frac{(2\pi)^2}{m^2} \right)^{3/2} \leq \text{vol}(W(n+1,m)) < (v_{\text{oct}}(n-2) + 4 v_{\text{tet}}) m. \tag{5}$$

Here $v_{\text{oct}}$ and $v_{\text{tet}}$ denote the volumes of the ideal octahedron and ideal tetrahedron respectively. Champanerkar, Kofman, and Purcell call these bounds asymptotically sharp because their ratio approaches 1, as $n$ and $m$ tend to infinity. Since the crossing number of $W(n+1,m)$ is known to be $n \cdot m$, the volume bounds in (5) imply

$$v_{\text{oct}} n \cdot (n-1) \left( 1 - \frac{(2\pi)^2}{m^2} \right)^{3/2} \leq \frac{\text{vol}(W(n+1,m))}{nm} < \frac{v_{\text{oct}}(n-2) + 4 v_{\text{tet}}}{n} \tag{6}$$

We notice that for a fixed $n$, in the family of knots $W(n+1,m)$ the upper bound for the ratio $\frac{\text{Volume}}{\text{Crossing number}}$ is constant for all $m$ but for a fixed value of $m$ there is a gap between the upper and lower bounds. Let us denote $\frac{v_{\text{oct}}(n-2)+4 v_{\text{tet}}}{n}$ by $U_n$ and $\frac{v_{\text{oct}} n \cdot (n-1)}{n}$ by $L_n$. In these notations, according to [6] we have

$$L_n \left( 1 - \frac{(2\pi)^2}{m^2} \right)^{3/2} \leq \frac{\text{vol}(W(n+1,m))}{nm} < U_n \tag{7}$$

By taking limit as $m \to \infty$ have the following:

$$L_n \leq \lim_{m \to \infty} \frac{\text{vol}(W(n+1,m))}{nm} \leq U_n \tag{8}$$

We can ask whether or not better bounds on the relative volume of weaving knots $W(n+1,m)$ can be observed in terms of the higher twist numbers of these knots. In [15] the authors
took the $k$th root of $T_k(W(3,m))$ and then divided by the crossing number $2m$ to obtain an expression whose limit as $m$ tends to infinity is finite. Multiplying by a normalization constant $C_k$ so that

$$\lim_{m \to \infty} C_k \frac{\sqrt[k]{T_k(W(3,m))}}{2m} = 2v_{tet}. $$

they plotted the upper bound, the lower bound, the values $\text{vol}(W(3,m))/2n$ according to SnapPy and $\tau_k = C_k \cdot \frac{\sqrt[k]{T_k(W(3,m))}}{2m}$ for $k = 2, 3$, and $4$ and showed that all three of these $\tau_k$ curves provide better lower bounds on the relative volume than the lower bound given in [6].

To explore on this conjecture we use our program for Jones polynomial and extend it to compute various twist numbers. We run the program for $W(4,m)$ and $W(5,m)$ for various values of $m$ and we used the program SnapPy [10] to compute estimates of the volume. The first experiment that we perform and observe that $T_k(m) \approx \frac{(r^k+(r+d)^k)m^k}{k!}$ as $m$ grows large. This suggests that $T_k(m) \approx \frac{(r^k+(r+d)^k)m^k}{k!}$. Let us denote $\frac{T_k(m)}{(r^k+(r+d)^k)m^k}$ by $f_k(m)$ and $\frac{T_k(m)k!}{(r^k+(r+d)^k)m^k}$ by $g_k(m)$ and our data suggests that both $f_k(m)$ and $g_k(m)$ converge to 1 as $m$ grows large for every $k$. Table 6 to Table 9 at the end are showing the values of $f_k(m)$ and $g_k(m)$ for various values of $m$. Let $v_n^\tau(m)$ denote the relative volume of the knot $W(n+1,m)$. Now, we consider the following functions:

$$L_k^1(m) = L_n \cdot (1 + |1 - f_k(m)|)$$
$$L_k^2(m) = \frac{L_n}{1 - |1 - f_k(m)|}$$
$$L_k^3(m) = L_n \cdot (1 + |1 - g_k(m)|)$$
$$L_k^4(m) = \frac{L_n}{1 - |1 - g_k(m)|}$$

Note that from our observations made earlier regarding $f_k(m)$ and $g_k(m)$ each $L_k^1(m) \geq L_n$ and from our experiment we conjecture that when $m$ grows large $v_n^\tau(m) \geq L_k^1(m)$. Thus, each $L_k^1(m)$ seem to provide a better lower bound for the relative volume than given in [6]. For a family of knots $W(4,m)$ and $W(5,m)$, Figure 2 and Figure 3 below display the graphs of $L_k^1$ for various values of $k$ along with the graphs of bounds on relative volume as in [6] and the value of relative volume computed from SnapPy.
Figure 2: Comparing bounds of $W(4, m)$ with $L^3_k$

Figure 3: Comparing bounds of $W(5, m)$ with $L^3_k$

Here the horizontal axis is values of $m$, top horizontal line and the lower most curve represents upper bound and the lower bound for the relative volume given in [6], the dotted curve in red is of the values of relative volume from SnapPy and the curve in blue, orange, pink, green
and purple represent the graphs of $L^3_k$ for $k = 2, 3, 4, 5$ and 6 respectively. It is evident from the graphs that as $k$ value is higher, the lower bound for the relative volume gets better. In the limiting case when $m \to \infty$ these $L^k_k(m)$ and the lower bound given in [6] converge to the same limit.

Similarly we consider the functions $U^i_k(m) \leq U_n$ for $i = 1, 2, 3, 4$ as:

\begin{align*}
U^1_k(m) &= L_n \cdot (1 - |1 - f_k(m)|) \\
U^2_k(m) &= \frac{L_n}{1 + |1 - f_k(m)|} \\
U^3_k(m) &= L_n \cdot (1 - |1 - g_k(m)|) \\
U^4_k(m) &= \frac{L_n}{1 + |1 - g_k(m)|}
\end{align*}

such that each $U^i_k(m) \leq U_n$ and our data seems to provide evidence for $v^i_n(m) \leq U^i_k(m)$, making us believe that $U^i_k(m)$ provide better upper bound for $v^i_n(m)$. As a sample for $W(4, m)$ and $W(5, m)$ families, Figure 4 and Figure 5 below display the graphs of $U^3_k$ for various values of $k$ along with the graphs of bounds on relative volume as in [6] and the value of relative volume computed from SnapPy.
Figure 5: Comparing bounds of $W(5, m)$ with $U^3_k$

Here the horizontal axis is values of $m$, top horizontal line and the lower most curve represents upper bound and the lower bound for the relative volume given in [6], the dotted curve in red is of the values of relative volume from SnapPy and the curve in blue, orange, pink, green and purple represent the graphs of $U^3_k$ for $k = 2, 3, 4, 5$ and 6 respectively. It is evident from the graphs that as $k$ value is higher, the upper bound for the relative volume gets better. In the limiting case when $m \to \infty$ these $U^3_k(m)$ and the upper bound given in [6] converge to the same limit justifying the results in [7] that $W(n + 1, m)$ are geometrically maximal.

**Remark 5.2.** We have discussed above that $L_n \leq \lim_{m \to \infty} \frac{\text{vol}(W(n+1,m))}{nm} \leq U_n$. The functions $L_k^1(m)$ will certainly provide a lower bound for the relative volume of $W(n + 1, m)$ which is better that the lower bound in [6]. However, the situation on the other side of the inequality may be tricky. If $\lim_{m \to \infty} \frac{\text{vol}(W(n+1,m))}{nm} = U_n$ then the functions $U_k^1(m)$ may not remain upper bound for the relative volume. This case arises for $W(3, m)$ family.

Figure 6 and Figure 7 represent the comparison of bounds of $W(3, m)$ family with functions $L_k^3$ and $U_k^3$ respectively.
Figure 6: Comparing bounds of $W(3, m)$ with $L_k^3$

Figure 7: Comparing bounds of $W(3, m)$ with $U_k^3$
6 Khovanov homology

Weave knots \( W(n + 1, m) \) are alternating knots therefore its Khovanov homology groups are supported on two lines and can be determined by Jones polynomial and its signature. We quote the following:

**Theorem 6.1** (Theorem 1.2 in [4]). For any alternating knot \( L \) the Khovanov invariants \( \mathcal{H}^{i,j}(L) \) are supported only in two lines

\[
 j = 2i - \sigma(L) \pm 1.
\]

In [15], the authors proved the following result regarding the signature of weave knots:

**Proposition 6.2.** For a weaving knot \( (W(p, m), \) the signature \( \sigma(W(p, m)) \) is 0 when \( p \) is odd and it is \( 1 - m \) when \( p \) is even.

Therefore direct application of results in [4] will provide the following

**Theorem 6.3.** For a weaving knot \( W(2k + 1, m) \) the non-vanishing Khovanov homology \( \mathcal{H}^{i,j}(W(2k + 1, m)) \) lies on the lines

\[
 j = 2i \pm 1.
\]

For a weaving knot \( W(2k, m) \) the non-vanishing Khovanov homology \( \mathcal{H}^{i,j}(W(2k, m)) \) lies on the lines

\[
 j = 2i + m - 1 \pm 1
\]

The range for \( i, j \) and the rank of non-zero \( \mathcal{H}^{i,j}(W(n + 1, m)) \) is obtained from the Jones polynomial using the results of Lee in [4]. The authors described the procedure in detail for \( W(3, m) \) in [15] and we follow the same methodology that follows a straightforward algorithm once one has the information about the Jones polynomial. In the continuation of our Mathematica program in the earlier sections, we add the following program.

```mathematica
q = -t*Q^2; z = t^2*Q^4; H1 - t*Q^2L;
For@m = 2, m£ p, m++,
If@GCD@n + 1, mD Š 1, Print@"KhovanovHomologyof"W@n + 1, mDD; e = d* ... 1, Qd+2D = ";
Print@MatrixForm@Table@dimH@i, Qd-2* jD, 8j,-2, Qd-2<, 8i,tld-2, td<DDDD;
Clear@dimH, q, zD
```

Using this program we can compute the ranks of the Khovanov homology groups of any weave knot. Here are some examples.
Table 1: Khovanov homology of $W(4, 3)$

\[
\begin{pmatrix}
\begin{array}{cccccccccc}
i & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 \\
j & 5 & 4 & 5 & 6 & 7 & 5 & 6 & 7 & 5 & 6 \\
\end{array}
\end{pmatrix}
\]

Table 2: Khovanov homology of $W(4, 5)$

\[
\begin{pmatrix}
\begin{array}{cccccccccccc}
i & -14 & -13 & -12 & -11 & -10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 \\
j & 8 & 115 & 133 & 133 & 138 & 138 & 122 & 98 & 64 & 35 & 16 & 5 & 1 & 5 & 1 \\
\end{array}
\end{pmatrix}
\]

Table 3: Khovanov homology of $W(5, 2)$

\[
\begin{pmatrix}
\begin{array}{cccccccc}
i & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
j & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\end{pmatrix}
\]
Table 4: Khovanov homology of $W(5, 3)$

\[
\begin{array}{cccccccccc}
\hline
i & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
j & 13 & 11 & 9 & 7 & 5 & 3 & 1 & -1 & -3 & -5 & -7 & -9 & -11 & -13 \\
\hline
\end{array}
\]

The authors in [15] have already pointed out that once we know the ranks of Khovanov homology groups of alternating knots, we can have the complete information on their integral Khovanov homology using recent results of Shumakovich [11].

Returning to the ranks of the rational Khovanov homology, we take advantage of the “knight move” periodicity and simplify by recording the Betti numbers for $W(n, m)$ from only along the line $j - 2i = 1$ for $n$ odd and along the line $j - 2i = m$ for $n$ even. In order to study the asymptotic behavior of Khovanov homology we have to normalize the data. This is done by computing the total rank of the Khovanov homology along the line and dividing each Betti number by the total rank. We obtain normalized Betti numbers that sum to one.

This raises the possibility of approximating the distribution of normalized Betti numbers by a probability distribution. For our baseline experiments we choose to use the normal $N(\mu, \sigma^2)$
probability density function
\[ f_{\mu,\sigma^2}(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(x-\mu)^2}{2\sigma^2} \right) \]

Here we compute the value of \( \mu \) and \( \sigma \) from our data. For a fixed \( n \), we regard \( W(n,m) = X_m \) as a countable family of random variables for \( m = 1, 2, \ldots \). For a particular \( i \) let \( p_i \) denote the normalized rank of \( H^{i,j}(W(n,m)) \) along the line \( j - 2i = 1 \) for \( n \) odd and \( j - 2i = m \) for \( n \) even. Now, from this data we find the mean \( \mu_m \) as first moment using the formula
\[ \mu_m = E(X_m) = \Sigma_i i \cdot p_i. \]
After knowing the value of \( \mu \) we find the standard deviation \( \sigma \) using the formula
\[ \sigma^2 = E(X_m) = \Sigma_i i^2 \cdot p_i. \]
We use these values of \( \mu_m \) and \( \sigma_m \) and consider the density function
\[ f_{\mu_m,\sigma^2}(x) = \frac{1}{\sigma_m \sqrt{2\pi}} \exp \left( -\frac{(x-\mu_m)^2}{2\sigma_m^2} \right) \]

We plot the graph of these density functions along with the normalized ranks of Khovanov homology groups of \( W(n,m) \). We see that for a fixed \( n \), the normalized ranks converge to the normal distribution as \( m \to \infty \). For \( n \) odd this is very clear but for \( n \) even, there seems to be some deviation. We have presented in Table 10 to Table 13 the data showing a comparison of the actual ranks with the proposed density function which provides sufficient evidence that as \( m \) grows large the ranks in a family \( W(n,m) \) are normally distributed. Below are few graphs that demonstrate this.
Figure 9: Comparing normalized homology of $W(3, 59)$ with density function

Figure 10: Comparing normalized homology of $W(4, 7)$ with density function

Figure 11: Comparing normalized homology of $W(4, 49)$ with density function
Figure 12: Comparing normalized homology of $W(5, 6)$ with density function

Figure 13: Comparing normalized homology of $W(5, 46)$ with density function

Figure 14: Comparing normalized homology of $W(6, 5)$ with density function
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Table 6: Values of $f_k(m)$ for $W(3,m)$

| m   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 10  | 1.0 | 0.9 | 0.78| 0.696| 0.6504| 0.63 | 0.63 | 0.64512|
| 20  | 1.05| 0.95| 0.87| 0.78075| 0.6969| 0.625388| 0.65412|
| 30  | 1.96| 0.983333| 0.822222| 0.759881| 0.684711| 0.615571| 0.554454|
| 40  | 1.975| 0.93| 0.871031| 0.804103| 0.744231| 0.665975| 0.601467|
| 50  | 1.98| 0.94| 0.835867| 0.835867| 0.772256| 0.707836| 0.64463|
| 60  | 1.983333| 0.99| 0.904726| 0.904726| 0.822222| 0.759881| 0.684711|
| 70  | 1.985| 0.95| 0.871031| 0.804103| 0.744231| 0.665975| 0.601467|
| 80  | 1.9875| 0.96375| 0.893667| 0.817094| 0.763157| 0.707836| 0.64463|
| 90  | 1.99| 0.975| 0.871031| 0.804103| 0.744231| 0.665975| 0.601467|
| 100 | 1.995| 0.98| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 110 | 1.9975| 0.99| 0.904726| 0.904726| 0.822222| 0.759881| 0.684711|
| 120 | 1.999| 0.995| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 130 | 1.9995| 0.995| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 140 | 1.99975| 0.9975| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 150 | 1.9999| 0.999| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 160 | 1.99995| 0.9995| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 170 | 1.999975| 0.99975| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 180 | 1.99999| 0.9999| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 190 | 1.999995| 0.99995| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 200 | 1.9999975| 0.999975| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 210 | 1.999999| 0.99999| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 220 | 1.9999995| 0.999995| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 230 | 1.99999975| 0.9999975| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 240 | 1.9999999| 0.999999| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 250 | 1.99999995| 0.9999995| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 260 | 1.999999975| 0.999999975| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 270 | 1.99999999| 0.99999999| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 280 | 1.999999995| 0.999999995| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| 290 | 1.9999999975| 0.9999999975| 0.880769| 0.890769| 0.822222| 0.759881| 0.684711|
| m | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 |
|---|---|---|---|---|---|---|---|---|
| k | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 0.938272 | 0.791038 | 0.61411 | 0.453799 | 0.327863 | 0.23531 | 0.169542 |
| 2 | 0.967901 | 0.88283 | 0.759031 | 0.620184 | 0.485789 | 0.367512 | 0.27017 |
| 3 | 0.978326 | 0.919101 | 0.82925 | 0.71807 | 0.597512 | 0.479751 | 0.36427 |
| 4 | 0.983642 | 0.938272 | 0.866551 | 0.777021 | 0.678501 | 0.578229 | 0.481671 |
| 5 | 0.986864 | 0.950131 | 0.88283 | 0.759031 | 0.620184 | 0.485789 | 0.367512 |
| 6 | 0.989026 | 0.958166 | 0.919101 | 0.827925 | 0.718087 | 0.602362 | 0.490418 |
| 7 | 0.990577 | 0.963973 | 0.938286 | 0.866551 | 0.777021 | 0.678501 | 0.578229 |
| 8 | 0.991744 | 0.968366 | 0.950131 | 0.88283 | 0.759031 | 0.620184 | 0.485789 |
| 9 | 0.992653 | 0.971804 | 0.958166 | 0.919101 | 0.827925 | 0.718087 | 0.602362 |
| 10 | 0.993583 | 0.974569 | 0.963973 | 0.938286 | 0.866551 | 0.777021 | 0.678501 |
| 11 | 0.994479 | 0.978739 | 0.968366 | 0.950131 | 0.88283 | 0.759031 | 0.620184 |
| 12 | 0.995263 | 0.98035 | 0.971804 | 0.958166 | 0.919101 | 0.827925 | 0.718087 |
| 13 | 0.995983 | 0.981734 | 0.974569 | 0.963973 | 0.938286 | 0.866551 | 0.777021 |
| 14 | 0.996679 | 0.98383 | 0.978739 | 0.968366 | 0.950131 | 0.88283 | 0.759031 |
| 15 | 0.997231 | 0.985046 | 0.98035 | 0.971804 | 0.958166 | 0.919101 | 0.827925 |
| 16 | 0.997763 | 0.986492 | 0.98383 | 0.974569 | 0.963973 | 0.938286 | 0.866551 |
| 17 | 0.998206 | 0.987876 | 0.985046 | 0.978739 | 0.968366 | 0.950131 | 0.88283 |
| 18 | 0.998578 | 0.989283 | 0.987876 | 0.98035 | 0.971804 | 0.958166 | 0.919101 |
| 19 | 0.998911 | 0.990466 | 0.989283 | 0.98383 | 0.974569 | 0.963973 | 0.938286 |
| 20 | 0.999231 | 0.991101 | 0.990466 | 0.985046 | 0.978739 | 0.968366 | 0.950131 |
| 21 | 0.999543 | 0.991946 | 0.991101 | 0.990466 | 0.985046 | 0.978739 | 0.968366 |
| 22 | 0.999843 | 0.992417 | 0.991946 | 0.991101 | 0.990466 | 0.985046 | 0.978739 |
| 23 | 0.999899 | 0.992707 | 0.992417 | 0.991946 | 0.991101 | 0.990466 | 0.985046 |
| 24 | 0.999983 | 0.992936 | 0.992707 | 0.992417 | 0.991946 | 0.991101 | 0.990466 |
| 25 | 0.999983 | 0.992936 | 0.992707 | 0.992417 | 0.991946 | 0.991101 | 0.990466 |
| 26 | 0.999983 | 0.992936 | 0.992707 | 0.992417 | 0.991946 | 0.991101 | 0.990466 |
| 27 | 0.999983 | 0.992936 | 0.992707 | 0.992417 | 0.991946 | 0.991101 | 0.990466 |
| 28 | 0.999983 | 0.992936 | 0.992707 | 0.992417 | 0.991946 | 0.991101 | 0.990466 |
| 29 | 0.999983 | 0.992936 | 0.992707 | 0.992417 | 0.991946 | 0.991101 | 0.990466 |
| 30 | 0.999983 | 0.992936 | 0.992707 | 0.992417 | 0.991946 | 0.991101 | 0.990466 |
Table 8: Values of $f_k(m)$ for $W(5, m)$

| m | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|---|----|----|----|----|----|----|----|----|
| 2 | 1  | 0.5| 0.46875 | NA | NA | NA | NA | NA |
| 4 | 1  | 1  | 1.00781 | 1.03711 | 1.06934 | 1.11786 | 1.18 | NA |
| 6 | 1  | 1  | 1.01389 | 1.04861 | 1.10725 | 1.18996 | 1.29053 | 1.40038 |
| 8 | 1  | 1  | 1.00781 | 1.02832 | 1.06476 | 1.11923 | 1.19288 | 1.28594 |
| 10 | 1 | 1 | 1.005 | 1.0185 | 1.04303 | 1.08045 | 1.13219 | 1.19331 |
| 12 | 1 | 1 | 1.00347 | 1.01302 | 1.03064 | 1.05788 | 1.09604 | 1.14621 |
| 14 | 1 | 1 | 1.00195 | 1.00745 | 1.01778 | 1.03405 | 1.05719 | 1.08805 |
| 16 | 1 | 1 | 1.00154 | 1.00592 | 1.0142 | 1.02731 | 1.04606 | 1.07118 |
| 18 | 1 | 1 | 1.00125 | 1.00481 | 1.0116 | 1.02239 | 1.03788 | 1.05873 |
| 20 | 1 | 1 | 1.00103 | 1.00399 | 1.00965 | 1.01868 | 1.03171 | 1.04928 |
| 22 | 1 | 1 | 1.00087 | 1.00336 | 1.00815 | 1.01583 | 1.02693 | 1.04194 |
| 24 | 1 | 1 | 1.00074 | 1.00287 | 1.00698 | 1.01358 | 1.02315 | 1.03613 |
| 26 | 1 | 1 | 1.00064 | 1.00248 | 1.00604 | 1.01178 | 1.02012 | 1.03144 |
| 28 | 1 | 1 | 1.00056 | 1.00217 | 1.00528 | 1.01032 | 1.01764 | 1.02761 |
| 30 | 1 | 1 | 1.00049 | 1.00191 | 1.00466 | 1.00911 | 1.0156 | 1.02445 |
| 32 | 1 | 1 | 1.00043 | 1.00169 | 1.00414 | 1.0081 | 1.01389 | 1.02179 |
| 34 | 1 | 1 | 1.00039 | 1.00151 | 1.0037 | 1.00725 | 1.01245 | 1.01955 |
| 36 | 1 | 1 | 1.00035 | 1.00136 | 1.00333 | 1.00653 | 1.01122 | 1.01763 |
| 38 | 1 | 1 | 1.00031 | 1.00123 | 1.00301 | 1.00591 | 1.01016 | 1.01599 |
| 40 | 1 | 1 | 1.00028 | 1.00111 | 1.00273 | 1.00538 | 1.00925 | 1.01456 |
| 42 | 1 | 1 | 1.00026 | 1.00102 | 1.0025 | 1.00491 | 1.00846 | 1.01332 |
| 44 | 1 | 1 | 1.00024 | 1.00093 | 1.00229 | 1.0045 | 1.00776 | 1.01223 |
| 46 | 1 | 1 | 1.00022 | 1.00085 | 1.0021 | 1.00414 | 1.00714 | 1.01127 |
| 48 | 1 | 1 | 1.0002 | 1.00079 | 1.00194 | 1.00383 | 1.0066 | 1.01041 |
| 50 | 1 | 1 | 1.00018 | 1.00073 | 1.0018 | 1.00354 | 1.00612 | 1.00965 |
| 52 | 1 | 1 | 1.00017 | 1.00068 | 1.00167 | 1.00329 | 1.00568 | 1.00898 |
| 54 | 1 | 1 | 1.00016 | 1.00063 | 1.00155 | 1.00306 | 1.00529 | 1.00837 |
| 56 | 1 | 1 | 1.00015 | 1.00059 | 1.00145 | 1.00286 | 1.00494 | 1.00782 |
| 58 | 1 | 1 | 1.00014 | 1.00055 | 1.00135 | 1.00268 | 1.00463 | 1.00732 |
| 60 | 1 | 1 | 1.00013 | 1.00051 | 1.00127 | 1.00251 | 1.00434 | 1.00687 |
| 62 | 1 | 1 | 1.00012 | 1.00048 | 1.00119 | 1.00236 | 1.00408 | 1.00646 |
| 64 | 1 | 1 | 1.00011 | 1.00045 | 1.00112 | 1.00222 | 1.00384 | 1.00608 |
| 66 | 1 | 1 | 1.00011 | 1.00043 | 1.00106 | 1.00209 | 1.00362 | 1.00574 |
| 68 | 1 | 1 | 1.0001 | 1.0004 | 1.001 | 1.00198 | 1.00342 | 1.00542 |
| 70 | 1 | 1 | 1.0001 | 1.00038 | 1.00094 | 1.00187 | 1.00324 | 1.00513 |
| 72 | 1 | 1 | 1.00009 | 1.00036 | 1.00089 | 1.00177 | 1.00307 | 1.00487 |
| 74 | 1 | 1 | 1.00009 | 1.00034 | 1.00085 | 1.00168 | 1.00291 | 1.00462 |
| 76 | 1 | 1 | 1.00008 | 1.00033 | 1.00081 | 1.0016 | 1.00277 | 1.00439 |
| 78 | 1 | 1 | 1.00008 | 1.00031 | 1.00077 | 1.00152 | 1.00263 | 1.00418 |
| 80 | 1 | 1 | 1.00007 | 1.00029 | 1.00073 | 1.00145 | 1.00251 | 1.00398 |
| 82 | 1 | 1 | 1.00007 | 1.00028 | 1.0007 | 1.00138 | 1.00239 | 1.0038 |
| 84 | 1 | 1 | 1.00007 | 1.00027 | 1.00066 | 1.00132 | 1.00229 | 1.00363 |
Table 9: Values of $f_k(m)$ for $W(6, m)$

| m | k | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|---|---|-----|-----|-----|-----|-----|-----|-----|-----|
| 2 | 0.5| 0.461538 | 0.364286 | 0.340206 | NA   | NA   | NA   | NA   |
| 3 | 1  | 0.905983 | 0.793651 | 0.702558 | 0.619529 | 0.56046 | 0.505702 | NA   |
| 4 | 1  | 0.990385 | 0.948214 | 0.894974 | 0.831392 | 0.764305 | 0.694829 | 0.627057 |
| 5 | 1  | 0.990769 | 0.961371 | 0.920808 | 0.875241 | 0.825431 | 0.771134 | 0.713298 |
| 6 | 1  | 0.991453 | 0.963492 | 0.923062 | 0.876936 | 0.828974 | 0.779769 | 0.728834 |
| 7 | 1  | 0.992151 | 0.966097 | 0.926315 | 0.879296 | 0.829744 | 0.779752 | 0.729614 |
| 8 | 1  | 0.992788 | 0.968638 | 0.930433 | 0.883953 | 0.834092 | 0.783585 | 0.733358 |
| 9 | 1  | 0.993352 | 0.970958 | 0.934632 | 0.889473 | 0.840281 | 0.790126 | 0.740344 |
| 10| 1   | 0.993846 | 0.973029 | 0.938623 | 0.895117 | 0.847111 | 0.79782  | 0.748871 |
| 11| 1   | 0.994278 | 0.974863 | 0.942305 | 0.900567 | 0.853999 | 0.805847 | 0.757929 |
| 12| 1   | 0.994658 | 0.976488 | 0.945662 | 0.905692 | 0.860667 | 0.813796 | 0.767009 |
| 13| 1   | 0.994993 | 0.977931 | 0.948706 | 0.910451 | 0.866992 | 0.821463 | 0.775852 |
| 14| 1   | 0.99529  | 0.979217 | 0.951466 | 0.914841 | 0.872928 | 0.828755 | 0.784332 |
| 15| 1   | 0.995556 | 0.980368 | 0.953969 | 0.918883 | 0.878465 | 0.835635 | 0.792392 |
| 16| 1   | 0.995793 | 0.981403 | 0.956245 | 0.922601 | 0.883618 | 0.842097 | 0.800013 |
| 17| 1   | 0.996007 | 0.982338 | 0.958319 | 0.926024 | 0.888408 | 0.848154 | 0.807199 |
| 18| 1   | 0.996201 | 0.983186 | 0.960215 | 0.92918  | 0.89286  | 0.853825 | 0.813966 |
| 19| 1   | 0.996378 | 0.983959 | 0.961953 | 0.932094 | 0.897002 | 0.859135 | 0.820334 |
| 20| 1   | 0.996538 | 0.984664 | 0.963551 | 0.934791 | 0.90086  | 0.864109 | 0.826328 |
| 21| 1   | 0.996686 | 0.985312 | 0.965024 | 0.937292 | 0.904458 | 0.868773 | 0.831972 |
| 22| 1   | 0.996821 | 0.985907 | 0.966386 | 0.939616 | 0.907818 | 0.873149 | 0.83729  |
| 23| 1   | 0.996946 | 0.986458 | 0.967648 | 0.941781 | 0.910961 | 0.87726  | 0.842305 |
| 24| 1   | 0.997062 | 0.986967 | 0.968821 | 0.9438  | 0.913907 | 0.88128  | 0.84704  |
| 25| 1   | 0.997169 | 0.98744 | 0.969913 | 0.945688 | 0.916671 | 0.884771 | 0.851514 |
| 26| 1   | 0.997269 | 0.98788 | 0.970933 | 0.947456 | 0.919269 | 0.888207 | 0.855747 |
| 27| 1   | 0.997362 | 0.98829 | 0.971887 | 0.949116 | 0.921716 | 0.891453 | 0.859757 |
| 28| 1   | 0.997449 | 0.988674 | 0.972782 | 0.950676 | 0.924022 | 0.894521 | 0.863559 |
| 29| 1   | 0.99753 | 0.989034 | 0.973622 | 0.952145 | 0.926201 | 0.897427 | 0.867167 |
| 30| 1   | 0.997607 | 0.989371 | 0.974412 | 0.953531 | 0.92826  | 0.900182 | 0.870597 |
| $m$ | Total dimension | $\mu$ | $\sigma$ | $L^1$-Comparison | $L^2$-Comparison |
|-----|----------------|------|---------|----------------|----------------|
|  5  |     61.        | 0.491803 | 2.06154 | 0.0927555      | 0.0388521      |
| 10  |     7563.      | 0.499934 | 2.94869 | 0.0401467      | 0.0110943      |
| 20  | $1.14413 \times 10^8$ | 0.5 | 4.19983 | 0.0184547      | 0.00430564     |
| 25  | $1.40719 \times 10^7$ | 0.5 | 4.7022  | 0.014319       | 0.00320955     |
| 35  | $2.12865 \times 10^4$ | 0.5 | 5.5727  | 0.0101674      | 0.00207271     |
| 40  | $2.61807 \times 10^16$ | 0.5 | 5.96046 | 0.00877244     | 0.00174506     |
| 50  | $3.96035 \times 10^20$ | 0.5 | 6.66868 | 0.00700549     | 0.00131088     |
| 55  | $4.87091 \times 10^22$ | 0.5 | 6.99596 | 0.00634725     | 0.00116064     |
| 65  | $7.36823 \times 10^26$ | 0.5 | 7.6084  | 0.00533794     | 0.000938172    |
| 70  | $9.06233 \times 10^28$ | 0.5 | 7.89683 | 0.00495697     | 0.000853839    |
| 80  | $1.37086 \times 10^33$ | 0.5 | 8.44418 | 0.00432092     | 0.000720759    |
| 85  | $1.68604 \times 10^35$ | 0.5 | 8.70496 | 0.00404986     | 0.000667466    |
| 95  | $2.55048 \times 10^39$ | 0.5 | 9.20438 | 0.00362903     | 0.000579817    |
| 100 | $3.13688 \times 10^41$ | 0.5 | 9.44419 | 0.00344996     | 0.000543405    |
| 110 | $4.74516 \times 10^45$ | 0.5 | 9.90641 | 0.00312469     | 0.000481757    |
| 115 | $5.83616 \times 10^47$ | 0.5 | 10.1296 | 0.00298159     | 0.000455466    |
| 125 | $8.82836 \times 10^51$ | 0.5 | 10.5619 | 0.00274385     | 0.000409984    |
| 130 | $1.08582 \times 10^54$ | 0.5 | 10.7715 | 0.00264327     | 0.000390201    |
| 140 | $1.64251 \times 10^58$ | 0.5 | 11.179  | 0.00244937     | 0.000355405    |
| 145 | $2.02016 \times 10^60$ | 0.5 | 11.3773 | 0.00235921     | 0.00034035     |
| 155 | $3.0559 \times 10^64$ | 0.5 | 11.7638 | 0.00220931     | 0.000312643    |
| 160 | $3.7585 \times 10^66$ | 0.5 | 11.9523 | 0.0021421      | 0.000300394    |
| 170 | $5.68549 \times 10^70$ | 0.5 | 12.3208 | 0.00201439     | 0.000278328    |
| 175 | $6.99269 \times 10^72$ | 0.5 | 12.501  | 0.00195534     | 0.00026836     |
| 185 | $1.05778 \times 10^77$ | 0.5 | 12.8538 | 0.00184678     | 0.000250244    |
| 190 | $1.30099 \times 10^79$ | 0.5 | 13.0266 | 0.00179875     | 0.000241991    |
| 200 | $1.968 \times 10^83$ | 0.5 | 13.3655 | 0.00171018     | 0.000226879    |
| 205 | $2.42049 \times 10^85$ | 0.5 | 13.5317 | 0.00166766     | 0.000219946    |
| 215 | $3.66147 \times 10^89$ | 0.5 | 13.8583 | 0.00158861     | 0.000207168    |
| 220 | $4.50331 \times 10^91$ | 0.5 | 14.0187 | 0.00155132     | 0.00020127     |
| 230 | $6.81215 \times 10^95$ | 0.5 | 14.3342 | 0.0014836      | 0.000190339    |
| 235 | $8.3784 \times 10^97$ | 0.5 | 14.4893 | 0.00145309     | 0.000185266    |
| 245 | $1.2674 \times 10^{102}$ | 0.5 | 14.7948 | 0.00139347     | 0.00017582     |
| 250 | $1.5588 \times 10^{104}$ | 0.5 | 14.9451 | 0.00136587     | 0.000171416    |
| 260 | $2.358 \times 10^{108}$ | 0.5 | 15.2414 | 0.00131178     | 0.00016318     |
| 265 | $2.90014 \times 10^{110}$ | 0.5 | 15.3874 | 0.00128662     | 0.000159324    |
| 275 | $4.38705 \times 10^{114}$ | 0.5 | 15.6754 | 0.00123966     | 0.000152087    |
| 280 | $5.39571 \times 10^{116}$ | 0.5 | 15.8174 | 0.00121825     | 0.000148686    |
| 290 | $8.16209 \times 10^{120}$ | 0.5 | 16.0976 | 0.00117612     | 0.000142281    |
| 295 | $1.00387 \times 10^{123}$ | 0.5 | 16.2359 | 0.0011564      | 0.000139261    |
| 300 | $1.51856 \times 10^{127}$ | 0.5 | 16.5091 | 0.0011175      | 0.000133557    |
| 310 | $1.8677 \times 10^{129}$ | 0.5 | 16.644  | 0.00109934     | 0.00013086     |
| 320 | $2.82527 \times 10^{133}$ | 0.5 | 16.9105 | 0.00106411     | 0.000125751    |
Table 11: Density function and comparisons for $W(4, m)$

| $m$ | Total dimension | $\mu$   | $\sigma$ | $L^1$-Comparison | $L^2$-Comparison |
|-----|-----------------|---------|----------|------------------|------------------|
| 3   | 38              | -2.92105| 1.96525  | 0.149409         | 0.0579667       |
| 9   | 316013          | -10.8938| 3.32161  | 0.04861          | 0.01299         |
| 15  | $1.42313 \times 10^9$ | -18.4827| 4.46909  | 0.0682443        | 0.0159307       |
| 21  | $5.38341 \times 10^{12}$ | -26.0208| 5.52066  | 0.0795165        | 0.0168891       |
| 27  | $1.8702 \times 10^{16}$ | -33.542 | 6.51972  | 0.0863526        | 0.0170625       |
| 33  | $6.17622 \times 10^{19}$ | -41.0554| 7.48648  | 0.0913297        | 0.0169847       |
| 39  | $1.97224 \times 10^{23}$ | -48.5647| 8.43176  | 0.0951959        | 0.0168515       |
| 45  | $6.14883 \times 10^{26}$ | -56.0716| 9.36192  | 0.0987784        | 0.0167358       |
| 51  | $1.88293 \times 10^{30}$ | -63.5768| 10.281   | 0.1024           | 0.016661        |
| 57  | $5.68624 \times 10^{33}$ | -71.0809| 11.1917  | 0.106094         | 0.0166295       |
| 63  | $1.69815 \times 10^{37}$ | -78.5843| 12.0959  | 0.109813         | 0.0166357       |
| 69  | $5.92539 \times 10^{40}$ | -86.087 | 12.9949  | 0.113637         | 0.016671        |
| 75  | $1.47594 \times 10^{44}$ | -93.5894| 13.8898  | 0.117603         | 0.0167272       |
| 81  | $4.30701 \times 10^{47}$ | -101.091| 14.7813  | 0.121735         | 0.0167968       |
| 87  | $1.24996 \times 10^{51}$ | -108.593| 15.6699  | 0.125943         | 0.016874        |
| 93  | $3.61031 \times 10^{54}$ | -116.095| 16.5561  | 0.130083         | 0.0169541       |
| 99  | $1.03844 \times 10^{58}$ | -123.596| 17.4403  | 0.134139         | 0.0170336       |
| 105 | $2.97592 \times 10^{61}$ | -131.097| 18.3228  | 0.138126         | 0.0171102       |
| 111 | $8.50043 \times 10^{64}$ | -138.598| 19.2038  | 0.142253         | 0.0171821       |
| 117 | $2.42097 \times 10^{68}$ | -146.099| 20.0835  | 0.146262         | 0.0172481       |
| 123 | $6.87691 \times 10^{71}$ | -153.6  | 20.962   | 0.150135         | 0.0173077       |
| 129 | $1.94878 \times 10^{75}$ | -161.101| 21.8396  | 0.153875         | 0.0173604       |
| 135 | $5.51052 \times 10^{78}$ | -168.601| 22.7163  | 0.157501         | 0.0174063       |
| 141 | $1.55512 \times 10^{82}$ | -176.102| 23.5922  | 0.161119         | 0.0174453       |
| 147 | $4.38073 \times 10^{85}$ | -183.602| 24.4673  | 0.1646           | 0.0174776       |
| 153 | $1.23199 \times 10^{89}$ | -191.103| 25.3419  | 0.167959         | 0.0175036       |
| 159 | $3.45937 \times 10^{92}$ | -198.603| 26.2159  | 0.171226         | 0.0175244       |
| 165 | $9.69995 \times 10^{95}$ | -206.104| 27.0894  | 0.174393         | 0.0175376       |
| 171 | $2.71623 \times 10^{99}$ | -213.604| 27.9624  | 0.177517         | 0.0175464       |
| 177 | $7.59677 \times 10^{102}$| -221.105| 28.835  | 0.18052          | 0.0175502       |
| 183 | $2.12232 \times 10^{106}$| -228.605| 29.7072  | 0.183406         | 0.0175494       |
| 189 | $5.92227 \times 10^{109}$| -236.105| 30.5791  | 0.186182         | 0.0175443       |
| 195 | $1.651 \times 10^{113}$  | -243.606| 31.4506  | 0.188878         | 0.0175352       |
| 201 | $4.59826 \times 10^{116}$| -251.106| 32.3219  | 0.191529         | 0.0175224       |
| 207 | $1.27954 \times 10^{120}$| -258.606| 33.1929  | 0.194079         | 0.0175063       |
| 213 | $3.55752 \times 10^{123}$| -266.107| 34.0636  | 0.19654         | 0.0174872       |
| 219 | $9.88319 \times 10^{126}$| -273.607| 34.9341  | 0.198945         | 0.0174652       |
| 225 | $2.7436 \times 10^{130}$  | -281.107| 35.8043  | 0.201291         | 0.0174406       |
| 231 | $7.61089 \times 10^{133}$| -288.607| 36.6744  | 0.203594         | 0.0174137       |
| 237 | $2.10988 \times 10^{137}$| -296.108| 37.5443  | 0.205812         | 0.0173846       |
| 243 | $5.84521 \times 10^{140}$| -303.608| 38.414   | 0.20795         | 0.0173536       |
| 249 | $1.61837 \times 10^{144}$| -311.108| 39.2836  | 0.210015         | 0.0173208       |
| 255 | $4.47821 \times 10^{147}$| -318.608| 40.153   | 0.212031         | 0.0172864       |
Table 12: Density function and comparisons for $W(5, m)$

| $m$ | Total dimension | $\mu$   | $\sigma$    | $L^1$-Comparison | $L^2$-Comparison |
|-----|----------------|---------|-------------|-----------------|-----------------|
| 2   | 15.            | 0.46667 | 1.85712     | 0.283683        | 0.110559        |
| 3   | 181.           | 0.497238| 2.22647     | 0.0865806       | 0.0270373       |
| 6   | 130863.        | 0.499996| 2.96808     | 0.0265589       | 0.00749205      |
| 8   | $8.4444\times 10^6$ | 0.5 | 3.40303     | 0.0151096       | 0.00407916      |
| 9   | $7.18681\times 10^7$ | 0.5 | 3.60626     | 0.0125854       | 0.00324384      |
| 11  | $4.70654\times 10^9$ | 0.5 | 3.98674     | 0.00953972      | 0.0022939       |
| 12  | $3.80183\times 10^{10}$ | 0.5 | 4.1653      | 0.00853743      | 0.00200708      |
| 14  | $2.47746\times 10^{12}$ | 0.5 | 4.50234     | 0.0070473       | 0.00161378      |
| 16  | $1.61335\times 10^{14}$ | 0.5 | 4.81649     | 0.00611831      | 0.0013525       |
| 17  | $1.30183\times 10^{15}$ | 0.5 | 4.96621     | 0.00576828      | 0.00125091      |
| 22  | $4.45229\times 10^{19}$ | 0.5 | 5.65592     | 0.00444872      | 0.00090282      |
| 23  | $3.5924\times 10^{20}$ | 0.5 | 5.78402     | 0.00423478      | 0.00085379      |
| 26  | $1.88707\times 10^{23}$ | 0.5 | 6.15232     | 0.00373578      | 0.000732032     |
| 28  | $1.22854\times 10^{25}$ | 0.5 | 6.38607     | 0.00348866      | 0.000667054     |
| 29  | $9.91263\times 10^{25}$ | 0.5 | 6.4998      | 0.00337018      | 0.000638335     |
| 31  | $6.45343\times 10^{27}$ | 0.5 | 6.72147     | 0.00315237      | 0.000587124     |
| 32  | $5.20704\times 10^{28}$ | 0.5 | 6.82961     | 0.00305441      | 0.000564211     |
| 34  | $3.38994\times 10^{30}$ | 0.5 | 7.04091     | 0.00286292      | 0.000529217     |
| 36  | $2.20696\times 10^{32}$ | 0.5 | 7.24006     | 0.00270004      | 0.000486759     |
| 37  | $1.78071\times 10^{33}$ | 0.5 | 7.34648     | 0.00262126      | 0.000470325     |
| 42  | $6.08972\times 10^{37}$ | 0.5 | 7.8293      | 0.00231933      | 0.000401242     |
| 43  | $4.91358\times 10^{38}$ | 0.5 | 7.92233     | 0.00226405      | 0.000389583     |
| 46  | $2.58107\times 10^{41}$ | 0.5 | 8.1951      | 0.00211847      | 0.000358014     |
| 48  | $1.68036\times 10^{43}$ | 0.5 | 8.372       | 0.0020257       | 0.000339423     |
| 49  | $1.35582\times 10^{44}$ | 0.5 | 8.45907     | 0.00198427      | 0.000330767     |
| 51  | $8.8268\times 10^{45}$ | 0.5 | 8.63057     | 0.00190276      | 0.000314598     |
| 52  | $7.12203\times 10^{46}$ | 0.5 | 8.71505     | 0.00186274      | 0.000307038     |
| 54  | $4.63666\times 10^{48}$ | 0.5 | 8.88161     | 0.00179739      | 0.00029286      |
| 56  | $3.01861\times 10^{50}$ | 0.5 | 9.0451      | 0.00173609      | 0.000279819     |
| 57  | $2.43561\times 10^{51}$ | 0.5 | 9.12575     | 0.00170566      | 0.000273684     |
| 62  | $8.32934\times 10^{55}$ | 0.5 | 9.51874     | 0.00156901      | 0.000246328     |
| 63  | $6.72065\times 10^{56}$ | 0.5 | 9.59541     | 0.0015428       | 0.000241441     |
| 66  | $3.53031\times 10^{59}$ | 0.5 | 9.82182     | 0.0014718       | 0.000227778     |
| 68  | $2.29834\times 10^{61}$ | 0.5 | 9.9699      | 0.00142608      | 0.00021942      |
| 69  | $1.85445\times 10^{62}$ | 0.5 | 10.0431     | 0.00140353      | 0.000215446     |
| 71  | $1.2073\times 10^{64}$ | 0.5 | 10.188      | 0.00136579      | 0.000207874     |
| 72  | $9.74131\times 10^{64}$ | 0.5 | 10.2597     | 0.00134785      | 0.000204266     |
| 74  | $6.34189\times 10^{66}$ | 0.5 | 10.4015     | 0.00131214      | 0.000197377     |
| 76  | $4.12876\times 10^{68}$ | 0.5 | 10.5415     | 0.00127808      | 0.000190896     |
| 77  | $3.33135\times 10^{69}$ | 0.5 | 10.6107     | 0.00126212      | 0.000187797     |
| 82  | $1.13926\times 10^{74}$ | 0.5 | 10.9506     | 0.00118425      | 0.000173573     |
| 83  | $9.1923\times 10^{74}$  | 0.5 | 11.0173     | 0.00117013      | 0.00017096      |
| 86  | $4.82866\times 10^{77}$ | 0.5 | 11.215      | 0.00112823      | 0.000163527     |
Table 13: Density function and comparisons for $W(6, m)$

| $m$ | Total dimension | $\mu$     | $\sigma$ | $L^1$-Comparison | $L^2$-Comparison |
|-----|-----------------|-----------|----------|------------------|------------------|
| 5   | 254403          | -5.66801  | 2.99802  | 0.0264335        | 0.00759289       |
| 7   | $5.68714 \times 10^7$ | -8.2106  | 3.47257  | 0.0220519        | 0.00579802       |
| 11  | $2.2178 \times 10^{12}$ | -13.1145 | 4.32385  | 0.0303306        | 0.00702358       |
| 13  | $4.12427 \times 10^{14}$ | -15.5304 | 4.71055  | 0.0337645        | 0.00747458       |
| 17  | $1.33526 \times 10^{19}$ | -20.3325 | 5.4275   | 0.0385126        | 0.00800467       |
| 19  | $2.34815 \times 10^{21}$ | -22.725  | 5.76377  | 0.0403142        | 0.00816211       |
| 23  | $7.03737 \times 10^{25}$ | -27.5002 | 6.40244  | 0.043491         | 0.00836267       |
| 25  | $1.20359 \times 10^{28}$ | -29.8846 | 6.70782  | 0.0446933        | 0.00842603       |
| 29  | $3.45657 \times 10^{32}$ | -34.6489 | 7.29618  | 0.0470285        | 0.00850699       |