EVENLY STABLE QUADRATIC POLYNOMIALS OVER $\mathbb{Q}$

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Abstract. We study the number of irreducible factors (over $\mathbb{Q}$) of iterates of polynomials of the form $f_r(x) = x^2 + r$ for $r \in \mathbb{Q}$. When the number of such factors is bounded independent of $n$, we call $f_r(x)$ eventually stable (over $\mathbb{Q}$). Previous work of Hamblen, Jones, and Madhu [7] shows that $f_r$ is eventually stable unless $r$ has the form $1/c$ for some non-zero integer $c$, in which case existing methods break down. We study this family, and prove that several conditions on $c$ of various flavors imply that all iterates of $f_{1/c}$ are irreducible. We give an algorithm that checks the eventual stability of $f_{1/c}$ in time $O(\log c)$, and applies to most $c$-values. We also study the two infinite families of $c$-values for which either the first iterate of $f_{1/c}$ is reducible, or the first iterate is irreducible but the second iterate is reducible. We find all $c$-values for which the fourth iterate of $f_{1/c}$ has at least four irreducible factors, and all $c$-values such that $f_{1/c}$ is irreducible but its third iterate has at least three irreducible factors. This last result requires finding all integral points on a genus-2 hyperelliptic curve for which the method of Chabauty and Coleman does not apply; we apply the more recent variant known as elliptic Chabauty. Finally, we use all these results to completely determine the number of irreducible factors of any iterate of $f_{1/c}$, for all $c$ with absolute value at most $10^9$.

1. Introduction

Given a field $K$ with algebraic closure $\overline{K}$, a polynomial $f \in K[x]$, and $\alpha \in K$, denote by $f^n(x)$ the $n$th iterate of $f$, and by $f^n(\alpha)$ the set $\{\beta \in \overline{K} : f^n(\beta) = \alpha\}$. When $f^n(x) - \alpha$ is separable over $K$ for each $n \geq 1$, the set $T_f(\alpha) := \bigcup_{n \geq 0} f^{-n}(\alpha)$ acquires the structure of a rooted tree (with root $\alpha$) if we assign edges according to the action of $f$. A large body of recent work has focused on algebraic properties of properties of $T_f(\alpha)$, particularly the natural action of $\text{Gal}(\overline{K}/K)$ on $T_f(\alpha)$ by tree automorphisms, which yields a homomorphism $\text{Gal}(\overline{K}/K) \to \text{Aut}(T_f(\alpha))$ called the arboreal Galois representation associated to $(f, \alpha)$. A central question is whether the image of this homomorphism must have finite index in $\text{Aut}(T_f(\alpha))$ (see [11] for an overview of work on this and related questions). In the present article we study factorizations of polynomials of the form $f^n(x) - \alpha$, and in particular whether $(f, \alpha)$ is eventually stable over $K$, that is, whether the number of irreducible factors over $K$ of $f^n(x) - \alpha$ is bounded as $n$ grows. Apart from its own interest, eventual stability has proven to be a key link in at least two recent proofs of finite-index results for certain arboreal representations [3, 4]. This is perhaps surprising given that eventual stability is a priori much weaker than finite index of the arboreal representation – the former only implies that the number of Galois orbits on $f^{-n}(\alpha)$ is bounded as $n$ grows, which is an easy consequence of the latter. There are other applications of eventual stability as well; for instance, if $f \in \mathbb{Q}[x]$ is eventually stable over $\mathbb{Q}$, then a finiteness result holds for $S$-integer points in the backwards orbit of 0 under $f$ (see [12] Section 3 and [10]). We refer the reader to [12] for an overview of eventual stability and related ideas. That article defines a notion of eventual stability for rational functions, gives several characterizations of eventual stability, and states some general conjectures on the subject, all of which remain wide open.
In this article, we restrict to the case of polynomial maps, \( K = \mathbb{Q} \), and \( \alpha = 0 \) (the latter restriction could be replaced by another specific choice for \( \alpha \), but taking \( \alpha = 0 \) eases notation). Throughout the article, all statements involving irreducibility are assumed to be over \( \mathbb{Q} \). A special case of [12 Conjecture 1.2] is the following: if \( f \in \mathbb{Q}[x] \) is a polynomial of degree \( d \geq 2 \) such that 0 is not periodic under \( f \), then \( (f, 0) \) is eventually stable over \( \mathbb{Q} \). At present, this conjecture is not known even in the case where \( d = 2 \) and \( f(x) = x^2 + r, r \in \mathbb{Q} \). By generalizing the Eisenstein criterion, Theorem 1.7 of [12] shows that if the \( p \)-adic valuation of \( r \) is positive for some prime \( p \), then \( (x^2 + r, 0) \) is eventually stable. This reduces the problem of proving that \( (x^2 + r, 0) \) is eventually stable for any \( r \in \mathbb{Q} \setminus \{0, -1\} \) to the case where \( r = 1/c \) for \( c \in \mathbb{Z} \setminus \{0, -1\} \) (note that 0 is periodic under \( x^2 + r, r \in \mathbb{Q} \) only when \( r \in \{0, -1\} \)). Our main goal in this article is to give a careful study of eventual stability in this last family. In particular, we offer the following refinement of Conjecture 1.4 of [12], which states that \( (x^2 + \frac{1}{c}, 0) \) is eventually stable for \( c \in \mathbb{Z} \setminus \{0, -1\} \). Denote by \( \mathbb{Z} \setminus \mathbb{Z}^2 \) the set of integers that are not integer squares.

Conjecture 1.1. Let \( f_c(x) = x^2 + r \) with \( r = 1/c \) for \( c \in \mathbb{Z} \setminus \{0, -1\} \). For \( n \geq 1 \), denote by \( k_n \) the number of irreducible factors of \( f_c^n(x) \). Then \( k_n \leq 4 \) for all \( n \geq 1 \). More precisely,

1. If \( c = -m^2 \) with \( m + 1 \in \mathbb{Z} \setminus \mathbb{Z}^2 \) and \( m \neq 4 \), then \( k_n = 2 \) for all \( n \geq 1 \).
2. If \( c = -16 \), then \( k_1 = k_2 = 2 \) and \( k_n = 3 \) for all \( n \geq 3 \).
3. If \( c = -(s^2 - 1)^2 \) for \( s \in \mathbb{Z} \setminus \{3, 5, 56\} \), then \( k_1 = 2 \) and \( k_n = 3 \) for all \( n \geq 2 \).
4. If \( c = -(s^2 - 1)^2 \) for \( s \in \{3, 5, 56\} \), then \( k_1 = 2, k_2 = 3, \) and \( k_n = 4 \) for all \( n \geq 3 \).
5. If \( c = 4m^2(m^2 - 1) \) for \( m \in \mathbb{Z}, m \geq 3 \), then \( k_1 = 1 \) and \( k_n = 2 \) for all \( n \geq 2 \).
6. If \( c = 48 \), then \( k_1 = 1, k_2 = 2, \) and \( k_n = 3 \) for all \( n \geq 3 \).
7. If \( c \) is not in any of the above cases, then \( k_n = 1 \) for all \( n \geq 1 \).

We remark that case (7) of Conjecture 1.1 is precisely the case where \( f_c^2(x) \) is irreducible (see Proposition 2.1), and thus case (7) asserts that if \( f_c^2(x) \) is irreducible, then \( f_c^n(x) \) is irreducible for all \( n \geq 1 \). We state this as its own conjecture:

Conjecture 1.2. Let \( f_c(x) = x^2 + r \) with \( r = 1/c \) for \( c \in \mathbb{Z} \setminus \{0, -1\} \). If \( f_c^2(x) \) is irreducible, then \( f_c^n(x) \) is irreducible for all \( n \geq 1 \).

Observe that Conjecture 1.1 gives a uniform bound for \( k_n \), in contrast to Conjecture 1.4 of [12]. It would be of great interest to have a similar uniform bound for \( f_c(x) \) as \( r \) is allowed to vary over the entire set \( \mathbb{Q} \setminus \{0, -1\} \) (as opposed to just the reciprocals of integers, as in Conjecture 1.1). We pose here a much more general question. Given a field \( K \), call \( f \in K[x] \) normalized (the terminology depressed is also sometimes used, especially for cubics) if \( \deg f = d \geq 2 \) and \( f(x) = a_dx^d + a_{d-2}x^{d-2} + a_{d-3}x^{d-3} + a_1x + a_0 \). Note that every \( f \in K[x] \) is linearly conjugate over \( K \) to a normalized polynomial.

Question 1.3. Let \( K \) be a number field and fix \( d \geq 2 \). Is there a constant \( \kappa \) depending only on \( d \) and \( [K : \mathbb{Q}] \) such that, for all normalized \( f \in K[x] \) of degree \( d \) such that 0 is not periodic under \( f \), and all \( n \geq 1 \), \( f^n(x) \) has at most \( \kappa \) irreducible factors? In the case where \( K = \mathbb{Q} \), \( d = 2 \), and \( f \) is taken to be monic, does the same conclusion hold with \( \kappa = 4 \)?

It is interesting to compare Question 1.3 to [1, Question 19.5], where a similar uniform bound is requested, but under the condition that \( f^{-1}(0) \cap \mathbb{P}^1(K) = \emptyset \).

Our main results give evidence for Conjecture 1.1. We prove the following special cases:

Theorem 1.4. Let notation be as in Conjecture 1.1. Then
(a) We have $k_1 = k_2 = 2$ and $k_3 = 3$ if and only if $c = -16$. In this case $k_n = 3$ for all $n \geq 3$.
(b) We have $k_1 = k_2 = 3$, and $k_3 = 4$ if and only if $c = -(s^2 - 1)^2$ for $s \in \{3, 5, 56\}$. In this case, $k_n = 4$ for all $n \geq 3$.
(c) We have $k_1 = 1$, $k_2 = 2$, and $k_3 = 3$ if and only if $c = 48$. In this case, $k_n = 3$ for all $n \geq 3$.

In order to establish part (c) of Theorem 1.4, we must find all integral points on the hyperelliptic curve

\[ y^2 = 8x^6 - 12x^4 - 4x^3 + 4x^2 + 4x + 1. \]

This curve has genus two, and has Jacobian of rank 2, meaning that the well-known method of Chabauty and Coleman does not apply. On the other hand, we are able to use a variant of the standard method, called elliptic Chabauty, to determine the rational points on the curve in (1.1).

The basic idea of this method, developed in [5, 6], is the following: suppose a curve $C$ admits a suitable map $\phi : C \to E$ to an elliptic curve $E$ defined over some (preferably small) extension $K/\mathbb{Q}$. In particular, if $\pi : E \to \mathbb{P}^1$ is some given non-constant map, then we assume that the image of $C(\mathbb{Q})$ under the composite map $f \circ \phi$ is contained in $\mathbb{P}^1(\mathbb{Q})$. Then, provided that the rank of $E(K)$ is strictly less then the degree of the extension $K/\mathbb{Q}$, we can use the formal group law on $E(K_v)$ for certain completions $K_v/K$ to determine $C(\mathbb{Q})$; see [5] §4.2 and [6] §2. Moreover, under suitable conditions, several components of the elliptic Chabauty method are implemented in MAGMA, and we make use of these implementations here. Our code verifying the calculations in the proof of Theorem 1.5 can be found within the file called Elliptic Chabauty at:

https://sites.google.com/a/alumni.brown.edu/whindes/research

**Theorem 1.5.** The only integral points on the curve (1.1) are those with $x \in \{-2, -1, 0, 1\}$.

To give evidence for the full Conjecture 1.1 we establish several sufficient conditions on $c$ that ensure the conjecture holds. Taken together, they allow us to prove:

**Theorem 1.6.** Conjecture 1.1 holds for all $c$ with $1 \leq c \leq 10^9$.

The proof of Theorem 1.6 is on p. 23. The main engine in the proof is the following verification of Conjecture 1.2 in many special cases.

**Theorem 1.7.** Let $f_s(x) = x^2 + r$ with $r = 1/c$ for $c \in \mathbb{Z} \setminus \{0, 1\}$. Then $f_s^w(x)$ is irreducible for all $n \geq 1$ if $c$ satisfies one of the following conditions:

1. $-c \in \mathbb{Z} \setminus \mathbb{Z}^2$ and $c < 0$;
2. $-c, c + 1 \in \mathbb{Z} \setminus \mathbb{Z}^2$ and $c \equiv -1 \mod p$ for a prime $p \equiv 3 \mod 4$;
3. $-c, c + 1 \in \mathbb{Z} \setminus \mathbb{Z}^2$ and $c$ satisfies one of the congruences in Proposition 3.5 (see Table 1).
4. $-c \in \mathbb{Z} \setminus \mathbb{Z}^2$, $c$ is not of the form $4m^2(m^2 - 1), m \in \mathbb{Z}$, and $c$ is odd;
5. $-c \in \mathbb{Z} \setminus \mathbb{Z}^2$, $c$ is not of the form $4m^2(m^2 - 1), m \in \mathbb{Z}$, and

\[
\frac{\prod_{p|c} p^{\nu_p(c)} \prod_{p|c} \alpha_{p^{\nu_p(c)}}}{\prod_{p|c} p^{\nu_p(c)} \prod_{p|c} \alpha_{p^{\nu_p(c)}}} > 2^{2/15} \approx 1.097,
\]

where $s$ is the largest square divisor of $c$.
6. $c = k^2$ for some $k \geq 2$ and

\[
\frac{\prod_{p|c} p^{\nu_p(c)} \prod_{p|c} \alpha_{p^{\nu_p(c)}}}{\prod_{p|c} p^{\nu_p(c)} \prod_{p|c} \alpha_{p^{\nu_p(c)}}} > 2^{2/15} \approx 1.097,
\]
where \( s' \) is the largest divisor of \( c \) that is a product of (not necessarily distinct) primes equivalent to 1 mod 4.

We call \( c \)-values satisfying condition (5) of Theorem 1.7 dominantly odd-powered, and we remark that the condition is satisfied by all squarefree \( c \) but by no square \( c \). We call \( c \)-values satisfying condition (6) of the Theorem dominantly non-residual, and we note that this condition applies only to squares, thus making it orthogonal in some sense to condition (5). We remark that in all the cases of Theorem 1.7, the conditions on \( c \) imply that \( f_r^2 \) is irreducible, though in cases (1), (2), (3), and (6), the conditions on \( c \) are strictly stronger than this.

Theorem 1.7 applies to many values of \( c \). In light of part (1) of the theorem, we restrict our discussion here to \( c > 0 \). Among \( c \) with \( 1 \leq c \leq 10^9 \) and \( f_r^2 \) irreducible, Theorem 1.7 applies to all but 3713 of the \( c \)-values with \( c + 1 \) a non-square (the \( c \) with \( c + 1 \) a square are handled by Corollary 5.5). The reason for this is that the congruence conditions of parts (2)-(4) of the theorem have a much different flavor from the factorization-based conditions of parts (5) and (6). We have further strengthened our capacity to verify many cases of Conjecture 1.2 with an algorithm, which we develop in Section 5 (see Corollary 5.4). Roughly, it reduces the verification of Conjecture 1.2 to a (very fast) finite computation provided that the nearest integer \( \kappa \) to \( \sqrt{c + 1.15} - 1 \) satisfies either \( \kappa \nmid c \) or \( \gcd(\kappa, c/\kappa) > 1 \).

Applying this algorithm to the 3713 numbers mentioned in the previous paragraph, we are left with just a single \( c \)-value, \( c_1 := 33356400 \). Perhaps it is remarkable that \( c_1 \) manages to avoid all the congruences in Table 1, while in addition evading conditions (2) and (5) of Theorem 1.7, as is evident from the factorizations \( c_1 + 1 = 13\cdot 73\cdot 35149 \) and \( c_1 = 2^4\cdot 3^2\cdot 7\cdot 11\cdot 19^2 \). Furthermore, the constant \( \kappa \) from Corollary 5.4 is \( 3\cdot 5^2\cdot 7\cdot 11 \), showing that \( \kappa \mid c \) and \( \gcd(\kappa, c/\kappa) = 1 \). However, \( c_1 \) succumbs to an enlargement of the congruence conditions in Table 1: there are 16 classes modulo 181 that are ruled out, among them 91, and \( c_1 \equiv 91 \mod 181 \).

We now outline our method for proving Theorem 1.7. Our primary tool is the following special case of [10, Theorem 2.2]: for \( n \geq 2 \), \( f_r^n \) is irreducible provided that \( f_r^{n-1} \) is irreducible and \( f_r^n(0) \) is not a square in \( \mathbb{Q} \). The proof of this relies heavily on the fact that \( f_r \) has degree 2, and is essentially an application of the multiplicativity of the norm map. Using ideas from [10, Theorem 2.3 and discussion preceding], one obtains the useful amplification (proven in Section 3) that for \( n \geq 2 \), \( f_r^n \) is irreducible provided that \( f_r^{n-1} \) is irreducible and neither of \( (f_r^{n-1}(0) \pm \sqrt{f_r^n(0)})/2 \) is a square in \( \mathbb{Q} \). When \( r = 1/c \), we have \( f_r(0) = 1/c, f_r^2(0) = (c+1)/c^2, f_r^3(0) = (c^3 + c^2 + 2c + 1)/c^4 \), and so on. The numerator of \( f_r^n(0) \) is obtained by squaring the numerator of \( f_r^{n-1}(0) \), and adding \( c^{2^n-1} \). We thus introduce the family of sequences

\[
(1.2) \quad a_1(c) = 1, \quad a_n(c) = a_{n-1}(c)^2 + c^{2^{n-1}-1} \quad \text{for } n \geq 2.
\]

To ease notation, we often suppress the dependence on \( c \), and write \( a_1, a_2, \) etc. We can then translate the results of the previous paragraph to:

**Lemma 1.8.** Suppose that \( c \in \mathbb{Z} \setminus \{0\} \), \( r = 1/c \), and \( f_r^2 \) is irreducible. Let \( a_n = a_n(c) \) be defined as in (1.2), and set

\[
(1.3) \quad b_n := \frac{a_{n-1} + \sqrt{a_n}}{2} \in \mathbb{Q}.
\]

If for every \( n \geq 3 \), \( b_n \) is not a square in \( \mathbb{Q} \) (which holds in particular if \( a_n \) is not a square in \( \mathbb{Q} \)), then \( f_r^n(x) \) is irreducible for all \( n \geq 1 \).

We make the following conjecture, which by Lemma 1.8 immediately implies Conjecture 1.2:
Conjecture 1.9. Let \( b_n = b_n(c) \) be defined as in (1.3). If \( c \in \mathbb{Z} \setminus \{0, -1\} \), then \( b_n \) is not a square for all \( n \geq 3 \).

Conjecture 1.9 also has strong implications for the density of primes dividing orbits of \( f_r \). We define the orbit of \( t \in \mathbb{Q} \) under \( f_r \) to be the set \( O_{f_r}(t) = \{ t, f_r(t), f_r^2(t), \ldots \} \), and we say that a prime \( p \) divides \( O_{f_r}(t) \) if there is at least one non-zero \( y \in O_{f_r}(t) \) with \( v_p(y) > 0 \). The natural density of a set \( S \) of prime numbers is defined to be

\[
D(S) = \lim_{B \to \infty} \frac{\# \{ p \leq B : p \in S \} }{\# \{ p \leq B \} }.
\]

Note that the elements of \( O_{f_r}(t) \) also form a nonlinear recurrence sequence, where the relation is given by application of \( f_r \). The problem of finding the density of prime divisors in recurrences has an extensive literature in the case of a linear recurrence; see the discussion and brief literature review in [9, Introduction]. The case of non-linear recurrences is much less-studied, though there are some recent results [9, 7, 15]. The following theorem is an application of [7, Theorem 1.1, part (2)]

**Theorem 1.10.** Let \( c \in \mathbb{Z} \), let \( r = 1/c \), suppose that \(-c \) and \( c + 1 \) are non-squares in \( \mathbb{Q} \), and assume that Conjecture 1.9 holds for \( c \). Then

\[
\text{(1.4) for any } t \in \mathbb{Q} \text{ we have } D(\{ p \text{ prime} : p \text{ divides } O_{f_r}(t) \}) = 0.
\]

We remark that in each of the cases of Theorem 1.7 we show that Conjecture 1.9 holds for \( c \). Hence in cases (2), (3), and (6) of Theorem 1.7 and also in cases (1), (4), and (5) with the additional hypothesis that \( c + 1 \) is not a square in \( \mathbb{Q} \), we have that (6.1) holds. We also note that when the hypotheses of Theorem 1.10 are satisfied, we obtain certain information on the action of \( G_b \) on \( T_\infty(0) \); see Section 6

A complete proof of Conjecture 1.9 appears out of reach at present. One natural approach is to prove the stronger statement that \( a_n \) is not a square for each \( n \geq 3 \), or equivalently that the curve

\[
C_n : y^2 = a_n(c)
\]

has no integral points with \( c \notin \{0, -1\} \) for any \( n \geq 3 \). It is easy to see that \( a_n(c) \) is separable as a polynomial in \( c \) (one considers it as a polynomial in \( \mathbb{Z}/2\mathbb{Z}[c] \), where it is relatively prime to its derivative), and because the degree of \( a_n(c) \) is \( 2^n - 1 \), it follows from standard facts about hyperelliptic curves that the genus of \( C_n \) is \( 2^n - 2 - 1 \). Siegel’s theorem then implies that there are only finitely many \( c \) with \( a_n(c) \) a square for given \( n \geq 3 \). However, the size of the genus of \( C_n \) prevents us from explicitly excluding the presence of integer points save in the cases of \( n = 3 \) and \( n = 4 \) (see Proposition 5.2). One idea that has been used to show families of integer non-linear recurrences contain no squares (see e.g. [17, Corollary 1.3] or [9, Lemma 4.3]) is to show that sufficiently large terms of each sequence are sandwiched between squares: they are generated by adding a small number to a large square. In the case of the family \( a_n(c) \), however, the addition of the very large term \( c^{2^n - 1} \) to the square \( a_n^{2^n - 1} \) ruins this approach (see [insert the section where we discuss the growth rate of the sequence \( a_n \)])]. A similar problem is encountered in a family of important two-variable non-linear recurrence sequences first considered in [13] (see [13, Theorem 1.8]). The main idea used in [13] to show the recurrence contains no squares is to rule out certain cases via congruence arguments. This is the essence of our method of proof for cases (2) and (3) of Theorem 1.7. Subsequently Swaminathan [18, Section 4] amplified these congruence arguments and gave new partial results using the idea of sandwiching terms of the sequence between squares.
2. THE CASE WHERE $f_r(x)$ OR $f_r^2(x)$ IS REDUCIBLE

We begin by studying the factorizations of iterates of $f_r(x)$ when either $f_r(x)$ or $f_r^2(x)$ is reducible. The behavior of higher iterates becomes harder to control because of the presence of multiple irreducible factors of the first two iterates, but we are still able to give some results. At the end of this section we prove Theorem 1.4, which gives a complete characterization of certain subcases.

**Proposition 2.1.** Let $f_r(x) = x^2 + r$ with $r = 1/c$ for $c \in \mathbb{Z} \setminus \{0, -1\}$. Then $f_r(x)$ is reducible if and only if $c = -m^2$ for $m \in \mathbb{Z}$. If $f_r(x)$ is irreducible, then $f_r^2(x)$ is reducible if and only if $c = 4m^2(m^2 - 1)$ for $m \in \mathbb{Z}$.

**Proof.** The first statement is clear. Assume now that $f_r(x)$ is irreducible over $\mathbb{Q}$. Let $\alpha$ be a root of $f_r^2(x)$, and observe that $f_r(\alpha)$ is a root of $f_r(x)$, and by the irreducibility of $f_r(x)$, we have $[\mathbb{Q}(f_r(\alpha)) : \mathbb{Q}] = 2$. Now $f_r^2(x)$ is irreducible if and only if $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$, which is equivalent to $[\mathbb{Q}(\alpha) : \mathbb{Q}(f_r(\alpha))] = 2$. But $\alpha$ is a root of $f_r(x) - f_r(\alpha) = x^2 + r - f_r(\alpha)$, and so $[\mathbb{Q}(\alpha) : \mathbb{Q}(f_r(\alpha))] = 2$ is equivalent to $f_r(\alpha) - r$ not being a square in $\mathbb{Q}(f_r(\alpha))$.

Without loss of generality, say $f_r(\alpha) = \sqrt{-r}$. Then $f_r(\alpha) - r$ is a square in $\mathbb{Q}(f_r(\alpha))$ if and only if there are $s_1, s_2 \in \mathbb{Q}$ with

$$-r + \sqrt{-r} = (s_1 + s_2 \sqrt{-r})^2 = s_1^2 - rs_2^2 + 2s_1s_2 \sqrt{-r}.$$

This holds if and only if $2s_1s_2 = 1$ and $s_1^2 - rs_2^2 = -r$. Substituting $s_2 = 1/(2s_1)$ into the second equation and multiplying through by $s_1^2$ gives $s_1^4 + rs_1^2 - r/4 = 0$, which by the quadratic formula holds if and only if

$$s_1^2 = \frac{-r \pm \sqrt{r^2 + 4r}}{2}$$

or equivalently, $2c(-1 \pm \sqrt{1 + c})$ is an integer square (here we have written $1/c$ for $r$ and multiplied both sides of (2.1) by $4c^2$). If $c < -1$, then $\sqrt{1 + c}$ is irrational, so we may assume $c > 0$. We may then discard the $-\,\text{part}$ of the $\pm$, since integer squares are positive. Writing $c = k^2 - 1$ for $k > 0$, we then obtain that $2(k^2 - 1)(-1 + k) = 2(k + 1)(k - 1)^2$ is a square, whence $k + 1 = 2m^2$ for some integer $m$. Thus $c = k^2 - 1 = (2m^2 - 1)^2 - 1 = 4m^4 - 4m^2$, as desired. □

2.1. The case of $f_r$ reducible. When $c = -m^2$ for some $m \geq 1$, we fix the notation

$$(2.2)\quad g_1(x) = x - \frac{1}{m} \quad \text{and} \quad g_2(x) = x + \frac{1}{m},$$

so that $f_r(x) = g_1(x)g_2(x)$. We exclude the case $m = 1$ in what follows, as in that case $f_r(x)$ is not eventually stable.

**Proposition 2.2.** Let $r = 1/c$ and $c = -m^2$ for $m \geq 2$. Let $g_1$ and $g_2$ be as in (2.2). Then the following hold.
Proof. The first item follows from observing that \( g_1(f_r(x)) = x^2 - \frac{m+1}{m^2} \) and \( g_2(f_r(x)) = x^2 + \frac{m-1}{m^2} \). The latter is irreducible because \( m \geq 2 \) implies \((m-1)/m^2 > 0\). The second and third items follow from [3] Proposition 4.2], which implies that for fixed \( j \in \{1,2\} \), we have that \( g_j(f^n_r(x)) \) is irreducible for all \( n \geq 2 \) provided that \( g_j(f_r(x)) \) is irreducible and \( g_j(f^n_r(0)) \) is not a square in \( \mathbb{Q} \) for all \( n \geq 2 \). This immediately proves item (3). To complete the proof of item (2), observe that \( g_1(f^n_r(0)) = f^n_r(0) - \frac{1}{m} \). However, one easily checks that \( x^2 - \frac{1}{m^2} \) maps the interval \((-1/m, 0)\) into itself, and in particular, \( f^n_r(0) < 0 \) for all \( n \geq 1 \). Thus \( g_1(f^n_r(0)) < 0 \) as well, and hence cannot be a square in \( \mathbb{Q} \).

**Proposition 2.3.** Let \( r = 1/c \) and \( c = -m^2 \) for \( m \geq 2 \), and let \( g_1 \) and \( g_2 \) be as in (2.2). Then \( g_2(f^n_r(0)) \) is a square in \( \mathbb{Q} \) if and only if \( m = 4 \). Moreover, \( g_2(f^n_r(x)) \) is reducible if and only if \( m = 4 \).

**Proof.** Observe that

\[
g_2(f^n_r(0)) = \frac{m^3 - m^2 + 1}{m^4},
\]

and hence \( g_2(f^n_r(0)) \) is a square in \( \mathbb{Q} \) if and only if the elliptic curve \( y^2 = x^3 - x^2 + 1 \) has an integral point with \( x = m \). This is curve 184.a1 in the LMFDB [14], and has only the integral points \((0, \pm 1), (1, \pm 1), (4, \pm 7)\). Because \( m \geq 2 \), the only \( m \)-value for which \( g_2(f^n_r(0)) \) is a square is \( m = 4 \).

To prove the second assertion, note that if \( m \neq 4 \), then [3] Proposition 4.2[ shows that \( g_2(f^n_r(0)) \) being a non-square in \( \mathbb{Q} \) implies that \( g_2(f^n_r(x)) \) is irreducible. On the other hand, if \( m = 4 \), then

\[
g_2(f^n_r(x)) = (x^2 - x + 7/16)(x^2 + x + 7/16)
\]

showing that \( g_2(f^n_r(x)) \) is reducible. We return to the analysis of the case \( m = 4 \) in Proposition 2.9.

**Definition 2.4.** A sequence \((s_n)_{n \geq 1}\) is a rigid divisibility sequence if for all primes \( p \) we have the following:

1. If \( v_p(s_n) = e \) then \( v_p(s_{mn}) = e \) for all \( m \geq 1 \), and
2. If \( v_p(s_n) > 0 \) and \( v_p(s_j) > 0 \), then \( v_p(s_{\gcd(n,j)}) > 0 \).

**Remark 2.5.** If \((s_n)_{n \geq 1}\) is a rigid divisibility sequence and \( s_1 = 1 \), then from (2) it follows that if \( p \mid \gcd(s_n, s_{n-1}) \) then \( p \mid s_1 = 1 \), which is impossible. Hence \( \gcd(s_n, s_{n-1}) = 1 \) for all \( n \geq 2 \). A similar argument shows that for \( q \) prime we have \( \gcd(s_q, s_i) = 1 \) for all \( 1 \leq i < q \).
Proposition 2.6. Let $r = 1/c$ and $c = -m^2$ for $m \geq 2$, and let $g_2$ be as in (2.2). Then $g_2(f_r^n(x))$ is irreducible for all $n \geq 2$ provided that $m \neq 4$ and at least one of the following holds:

- $m \equiv 3 \pmod{4}$
- $m \equiv 2, 5, 6 \pmod{7}$
- $m \equiv 8, 10 \pmod{13}$
- $m \equiv 3, 5, 11 \pmod{19}$
- $m \equiv 3, 19, 26 \pmod{29}$
- $m \equiv 6, 20 \pmod{37}$
- $m \equiv 15, 30 \pmod{43}$

If in addition $m - 1$ is not a square in $\mathbb{Q}$, then the following congruences also suffice:

- $m \equiv 2 \pmod{3}$
- $m \equiv 18 \pmod{19}$
- $m \equiv 8, 10, 14 \pmod{29}$
- $m \equiv 13, 31 \pmod{37}$
- $m \equiv 36, 39, 42 \pmod{43}$

Proof. By part (3) of Proposition 2.2 it suffices to show that $g_2(f_r^n(0))$ is not a square in $\mathbb{Q}$ for all $n \geq 2$. Note that for each $n \geq 1$, $g_2(f_r^{n-1}(0))$ is a positive rational number with denominator $m^{2n}$, and numerator prime to $m$. We take $b_n$ to be the numerator of $g_2(f_r^{n-1}(0))$. We first observe that the proof of [9, Proposition 5.4] shows that the sequence $(b_n)_{n \geq 1}$ is a rigid divisibility sequence. In particular, if $b_2$ is not a square in $\mathbb{Q}$, then because $b_2 > 0$ we must have some prime $p$ dividing $b_2$ to odd multiplicity, and the rigid divisibility condition implies that $b_{2j}$ is not a square for all $j \geq 2$. A similar argument shows that if $b_2$ is not a square in $\mathbb{Q}$, then neither is $b_{2j}$ for all $j \geq 1$.

By Proposition 2.3 and our assumption that $m \neq 4$, we have that $g_2(f_r^0(0))$ is not a square in $\mathbb{Q}$. It follows that $b_{2j}$ is a non-square for all $j \geq 1$.

Now for a given modulus $k$ and $m \equiv 0 \pmod{k}$, the sequence $(g_2(f_r^n(0)) \pmod{k})_{n \geq 1}$ eventually lands in a repeating cycle, and we search for values of $k$ and congruence classes of $m$ modulo $k$ such that $g_2(f_r^n(0)) \pmod{k}$ fails to be a square for all $n \geq 2$. Note that this method works even when $g_2(f_r^{3j-1}(0)) \pmod{k}$ is a square for all $j \geq 1$, since we have shown in the previous paragraph that $b_{3j}$ is a non-square for all $j \geq 1$. A computer search yields the congruences given in the first part of the proposition. If in addition $m - 1$ is a non-square in $\mathbb{Q}$, then we have $b_{2j}$ not a square in $\mathbb{Q}$ for all $j \geq 1$, and the congruences in the second part of the proposition show that $b_{2j+1} = g_2(f_r^{2j}(0)) \pmod{k}$ is a non-square for all $j \geq 1$.

Proposition 2.7. Let $r = 1/c$ and $c = -m^2$ for $m \geq 2$, and let $g_2$ be as in (2.2). If $m \equiv -1 \pmod{p}$ for a prime $p \equiv 7 \pmod{8}$, then $g_2(f_r^n(x))$ is irreducible for all $n \geq 2$. The same conclusion holds if $m - 1$ is not a square in $\mathbb{Q}$ and $m \equiv -1 \pmod{p}$ for a prime $p \equiv 3 \pmod{8}$.

Proof. By part (3) of Proposition 2.2 it suffices to show that $g_2(f_r^n(0))$ is not a square in $\mathbb{Q}$ for all $n \geq 2$. We have $c = -m^2 \equiv -1 \pmod{p}$, and so $(f_r^n(0) \pmod{p})_{n \geq 0}$ is the sequence $0, -1, 0, -1, \ldots$. Thus $(g_2(f_r^n(0)) \pmod{p})_{n \geq 0}$ is the sequence $-1, -2, -1, -2, -1, \ldots$. If $p \equiv 7 \pmod{8}$, then both $-1$ and $-2$ are non-squares modulo $p$, and the proof is complete. If $p \equiv 3 \pmod{8}$, then $-1$ is
a non-square modulo $p$ but $-2$ is a square, meaning we can only conclude that $g_2(f_r^{2j}(0))$ is a non-square in $\mathbb{Q}$ for $j \geq 1$. However, as in the proof of Proposition 2.6, this implies that $b_{2j+1}$ is a non-square for all $j \geq 1$. If in addition $m - 1$ is not a square, then $b_{2j}$ is not a square for all $j \geq 1$, completing the proof. □

Propositions 2.6 and 2.7 allow us to prove a case of Theorem 1.6.

**Corollary 2.8.** Let $r = 1/c$ and $c = -m^2$ for $m \geq 2$, and let $g_2$ be as in (2.2). Suppose that $m \neq 4$ and $m^2 \leq 10^9$. Then $g_2(f_r^n(x))$ is irreducible for all $n \geq 1$. If in addition $m + 1$ is not a square in $\mathbb{Q}$, then $f_r^n(x)$ is a product of two irreducible factors for all $n \geq 1$.

**Proof.** By part (3) of Proposition 2.2 it suffices to show that $g_2(f_r^n(0))$ is not a square in $\mathbb{Q}$ for all $n \geq 2$. Because $m \neq 4$, we may apply both Propositions 2.6 and 2.7. The first group of congruences in Proposition 2.6 applies to all $m$ with $2 \leq m \leq 10^{9/2}$ except for a set of 1642 $m$-values. After applying the first part of Proposition 2.7 that number decreases to 1258. Of these, 14 have the property that $m - 1$ is a square. We apply the second group of congruences in Proposition 2.6 and the second part of Proposition 2.7 to the remaining 1244 values, and only 242 survive. This leaves 256 values of $m$ that we must handle via other methods.

To do this, we employ a new method to search for primes $p$ such that $g_2(f_r^n(0))$ is a non-square modulo $p$ for all but finitely many $n$. We search for $p$ such that:

\[(2.3)\] the sequence $(g_2(f_r^n(0)) \mod p)_{n \geq 0}$ eventually assumes a non-square constant value

or eventually cycles between two distinct values, both of which are non-squares modulo $p$.

If we find such a $p$, it implies that all but finitely many terms of the sequence $(g_2(f_r^n(0)))_{n \geq 2}$ are non-squares in $\mathbb{Q}$. We then reduce modulo other primes to show that the remaining terms are non-squares, in the same manner as the last paragraph of Section 5.

The method proves quite effective. Of the 256 $m$-values left over from the first paragraph of this proof, all have a prime $p < 500$ that satisfies (2.3). For each such $m$ and $p$, we take the finitely many terms of the sequence $(g_2(f_r^n(0)))_{n \geq 2}$ that have still not been proven non-square by (2.3), and reduce modulo small primes until all have been proven non-square. The $m$-value producing the largest number of such terms is $m = 4284$, where we must check that each of $g_2(f_r(0)), g_2(f_{21}^n(0)), \ldots, g_2(f_{242}^n(0))$ is a non-square. In all cases the desired result is achieved by reducing modulo primes less than 100. □

We now consider the case $m = 4$. As shown in Proposition 2.3 it is the only one with $m \geq 2$ for which $g_2(f_r^2(x))$ is reducible; indeed, we have

\[(2.4)\] $g_2(f_r^2(x)) = (x^2 - x + 7/16)(x^2 + x + 7/16) := g_{21}(x)g_{22}(x),$

and we note that both $g_{21}(x)$ and $g_{22}(x)$ are irreducible.

**Proposition 2.9.** Let $r = -1/16$ and let $g_{21}$ and $g_{22}$ be as in (2.4). For all $n \geq 1$, both $g_{21}(f_r^n(x))$ and $g_{22}(f_r^n(x))$ are irreducible for all $n \geq 1$. In particular, $f_r^n(x)$ has precisely three irreducible factors for all $n \geq 3$.

**Proof.** Because $m + 1$ is not a square, Proposition 2.2 shows that $g_1(f_r^n(x))$ is irreducible for all $n \geq 1$. The proof of [9, Proposition 4.2] shows that it suffices to prove that neither $g_{21}(f_r^n(0))$ nor $g_{22}(f_r^n(0))$ is a square in $\mathbb{Q}$ for all $n \geq 1$. (This conclusion holds for $n \geq 1$ rather than for $n \geq 2$, as in other invocations of [9, Proposition 4.2] so far, because $g_{21}$ and $g_{22}$ have even degree.) Observe that $f_r^n(0) \equiv 5 \mod 11$ for $n \geq 3$, and $g_{21}(5) \equiv 6 \mod 11$. Because 6 is a non-square modulo
11, we must only verify that neither of \(g_{21}(f_2(0))\) or \(g_{21}(f_2^6(0))\) is a square in \(\mathbb{Q}\). The former is 129/256 and the latter is \((19 \cdot 1723)/2^{16}\), neither of which is a square in \(\mathbb{Q}\). For \(g_{22}(f_n(0))\) we have a simpler argument using \(p = 5\): observe that \(g_{22}(0) \equiv g_{22}(-1) \equiv 2 \mod 5\) and \(f_n(0) \equiv 0 \) or \(-1\) for all \(n \geq 1\).

We now consider the case where \(m + 1\) is a square. Say \(m + 1 = s^2\) with \(s \geq 2\), so that \(f_r(x) = x^2 - 1/m^2 = x^2 - 1/(s^2 - 1)^2\). We have

\[
g_1(f_r(x)) = x^2 - \frac{s + 1}{m^2} = \left(x - \frac{s}{s^2 - 1}\right)\left(x + \frac{s}{s^2 - 1}\right) := h_1(x)h_2(x).
\]

Now \(h_1(f_r(x)) = x^2 - \frac{s^2 - s + 1}{(s^2 - 1)^2}\). Thus \(h_1(f_r(x))\) is irreducible unless \(s\) is the \(x\)-coordinate of an integral point on the elliptic curve \(y^2 = x^3 - x + 1\). This is curve 92.a1 in LMFDB, and has an unusually large number of non-trivial integral points: \((0, \pm 1)\), \((1, \pm 1)\), \((-1, \pm 1)\)\((3, \pm 5)\), \((5, \pm 11)\), \((56, \pm 419)\).

We assume for a moment that \(s \not\in \{3, 5, 56\}\), so that \(h_1(f_r(x))\) is irreducible. Observe that \(x^2 - \frac{s}{s^2 - 1}\) maps the interval \((-1/m, 0)\) into itself, and in particular, \(f_r(0) = 0\) for all \(n \geq 1\). Thus \(h_1(f_r(0)) < 0\) as well, and hence cannot be a square in \(\mathbb{Q}\). Then [9] Proposition 4.2] proves that \(h_1(f_r^n(x))\) is irreducible for all \(n \geq 1\).

**Corollary 2.10.** Let \(r = 1/c\) and \(c = -(s^2 - 1)^2\) for \(s \geq 2\), and let \(g_2\) be as in (2.2) and \(h_1, h_2\) as in (2.5). Suppose that \((s^2 - 1)^2 \leq 10^9\). Then for all \(n \geq 1\) we have \(g_2(f_r^n(x))\) and \(h_2(f_r^n(x))\) irreducible. If in addition \(s \not\in \{3, 5, 56\}\) then for all \(n \geq 1\) we have \(h_1(f_r^n(x))\) irreducible. In particular if \((s^2 - 1)^2 \leq 10^9\) and \(s \not\in \{3, 5, 56\}\), then \(f_r^n(x)\) is a product of three irreducible factors for all \(n \geq 2\).

**Proof.** Observe that \((s^2 - 1)^2 \leq 10^9\) if and only if \(s \leq 177\). We have shown in Corollary 2.8 that \(g_2(f_r^n(x))\) is irreducible for all \(s \geq 2\) and \(s \leq 177\). In the paragraph preceding the present corollary, we showed that \(s \not\in \{3, 5, 56\}\) implies that \(h_1(f_r^n(x))\) is irreducible for all \(n \geq 1\). To show that \(h_2(f_r^n(x))\) is irreducible for \(n \geq 1\), it suffices by [9] Proposition 4.2] to show that \(h_2(f_r(x))\) is irreducible and that \(h_2(f_r^n(0))\) is a non-square in \(\mathbb{Q}\) for all \(n \geq 1\). Note that \(h_2(f_r(x)) = x^2 + \frac{s^2 - s + 1}{(s^2 - 1)^2}\), and we have \(s^3 - s - 1 > 0\) for \(s \geq 2\). Hence \(h_2(f_r(x))\) is irreducible. To verify that \(h_2(f_r^n(0))\) is a non-square in \(\mathbb{Q}\) for all \(n \geq 2\), we search for primes \(p\) satisfying the condition (2.3), with \(h_2\) replacing \(g_2\). We find that there exists a prime \(p \leq 500\) with the desired property for all \(s \geq 2\) and \(s \leq 177\) except for \(s = 153\). For that \(s\)-value, the prime \(p = 1051\) suffices.

For each such \(s\) and \(p\), we take the finitely many terms of the sequence \((h_2(f_r^n(0)))_{n\geq2}\) that have still not been proven non-square, and reduce modulo small primes until all have been proven non-square. Unsurprisingly, the \(s\)-value producing the largest number of such terms is \(s = 153\), where we must check that each of \(h_2(f_r(0)), h_2(f_r^2(0)), \ldots, h_2(f_r^{107}(0))\) is a non-square. In all cases the desired result is achieved by reducing modulo primes less than 100.

Finally, we handle the case of \(s \in \{3, 5, 56\}\). These are precisely the \(s\)-values for which \(s^3 - s + 1\) is a square. In this case, \(h_1(f(x))\) is no longer irreducible; indeed, we have

\[
h_1(f(x)) = \left(x - \frac{s^3 - s + 1}{s^2 - 1}\right)\left(x + \frac{s^3 - s + 1}{s^2 - 1}\right) := h_{11}(x)h_{12}(x).
\]

**Proposition 2.11.** Let \(r = 1/c\) and \(c = -(s^2 - 1)^2\) for \(s \in \{3, 5, 56\}\). Let \(g_2\) be as in (2.2), \(h_2\) as in (2.5), and \(h_{11}\) and \(h_{12}\) as in (2.6). Then for all \(n \geq 1\) we have \(g_2(f_r^n(x)), h_2(f_r^n(x)), h_{11}(f_r^n(x)), h_{12}(f_r^n(x))\) irreducible; in particular, \(f_r^n(x)\) is a product of four irreducible factors for all \(n \geq 3\).
Proof. Corollary 2.10 shows that for \( s \in \{3, 5, 56\} \), we have \( g_2(f_r^n(x)) \) and \( h_2(f_r^n(x)) \) irreducible for all \( n \geq 1 \). To show that \( h_{11}(f_r^n(x)) \) and \( h_{12}(f_r^n(x)) \) are irreducible for \( n \geq 1 \), it suffices by \([9] \) Proposition 4.2 to show that \( h_{11}(f_r(x)) \) and \( h_{12}(f_r(x)) \) are irreducible and \( h_{11}(f_r^n(0)) \) and \( h_{11}(f_r^n(0)) \) are non-squares in \( \mathbb{Q} \) for all \( n \geq 2 \). Note that \( h_{11}(f_r(x)) = x^2 - v \), where \( v = ((s^2 - 1)(\sqrt{s^2 - s + 1}) + 1)/(s^2 - 1)^2 \). For \( s = 3, 5, 56 \) respectively, the prime factorization of the numerator of \( v \) is \( 41, 5 \cdot 53, 2 \cdot 656783 \). Hence in all three cases \( h_{11}(f_r(x)) \) is irreducible. Note that \( h_{12}(f_r(x)) = x^2 + u \) where \( u > 0 \), and hence is irreducible.

One readily sees that \( h_{11}(f_r^n(0)) < 0 \) for all \( n \geq 2 \), showing that \( h_{11}(f_r(0)) \) is a non-square in \( \mathbb{Q} \) for \( n \geq 2 \). For \( s = 3 \), we reduce the sequence \( (h_{12}(f_r^n(0)))_{n \geq 2} \) modulo \( 29 \) and find that it cycles among the four values \( 17, 15, 26, 21 \), none of which is a square modulo \( 29 \). For \( s = 5 \) we reduce modulo \( 23 \) and find that the sequence in question cycles between \( 10 \) and \( 11 \), which are both non-squareds modulo \( 23 \). For \( s = 56 \) we reduce modulo \( 31 \) and find that the sequence takes only the value \( 6 \), i.e. \( h_{12}(f_r^n(0)) \equiv 6 \pmod{31} \) for all \( n \geq 2 \). But \( 6 \) is non-square modulo \( 31 \). \( \square \)

2.2. The case of \( f_r \) irreducible, \( f_r^2 \) reducible. Assume now that \( c = 4m^2(m^2 - 1) \) for some \( m \geq 2 \), in which case we have

\[
(2.7) \quad f_r^2(x) = \left( x^2 - \frac{1}{m} x + \frac{2m^2 - 1}{4m^2(m^2 - 1)} \right) \left( x^2 + \frac{1}{m} x + \frac{2m^2 - 1}{4m^2(m^2 - 1)} \right) =: q_1(x)q_2(x).
\]

We note that \( q_1 \) and \( q_2 \) both have discriminant \(-1/(m^2 - 1)\), and so are irreducible.

Observe that for \( m = 2 \) we have the factorization

\[
(2.8) \quad q_2(f_r(x)) = (x^2 - (1/2)x + 19/48)(x^2 + (1/2)x + 19/48).
\]

However, this is the only \( m \)-value for which such a factorization occurs, as the next two results show.

Proposition 2.12. Let \( r = 1/c \) and \( c = 4m^2(m^2 - 1) \) for \( m \geq 2 \). If \( f_3(x) \) has strictly more than two irreducible factors, then either

\[
8m^6 - 12m^4 + 4m^3 + 4m^2 - 4m + 1 \quad \text{or} \quad 8m^6 - 12m^4 - 4m^3 + 4m^2 + 4m + 1
\]

is a square in \( \mathbb{Q} \).

Proof. Observe that \( f_r^3(x) \) has strictly more than two irreducible factors if and only if \( q_i(f_r(x)) \) is reducible for at least one \( i \in \{1, 2\} \). Assume that \( q_i(f_r(x)) \) is irreducible, let \( \alpha \) be a root of \( q_i(f_r(x)) \), and observe that \( f_r(\alpha) := \beta \) is a root of \( q_i(x) \). By the irreducibility of \( q_i(x) \), we have \( [\mathbb{Q}(\beta) : \mathbb{Q}] = 2 \). Because \( q_i(f_r(x)) \) is irreducible, we have \( [\mathbb{Q}(\alpha) : \mathbb{Q}] < 4 \), which implies \( [\mathbb{Q}(\alpha) : \mathbb{Q}(\beta)] = 1 \), and thus \( \alpha \in \mathbb{Q}(\beta) \). But \( \alpha \) is a root of \( f_r(x) - \beta = x^2 + r - \beta \), and so \( \alpha \in \mathbb{Q}(\beta) \) is equivalent to \( \beta - r \) being a square in \( \mathbb{Q}(\beta) \). Letting \( \beta' \) be the other root of \( q_i(x) \), we have

\[
N_{\mathbb{Q}(\beta)/\mathbb{Q}}(\beta - r) = (\beta - r)(\beta' - r) = q_i(r) = \frac{8m^6 - 12m^4 \pm 4m^3 + 4m^2 + 4m + 1}{(4m^3 - 4m^2)^2}.
\]

The multiplicativity of the norm map implies that the rightmost expression is a square in \( \mathbb{Q} \). \( \square \)

We now prove Theorem 1.5 which we restate here.

Theorem 2.13. The only integral points on the curve \( y^2 = 8x^6 - 12x^4 - 4x^3 + 4x^2 + 4x + 1 \) are those with \( x \in \{-2, -1, 0, 1\} \).
Proof. We note first that the polynomial $F(x) = 8x^6 - 12x^4 + 4x^3 + 4x^2 - 4x + 1$ factors over a small extension of $\mathbb{Q}$. Namely, if $\beta$ is an algebraic number satisfying $\beta^3 - 64\beta + 512 = 0$, then

$$F(x) = \left(x^2 - \frac{1}{8}\beta x + \frac{1}{128}(\beta^2 - 64\right)\left(x^4 + \frac{1}{8}\beta x^3 + \frac{1}{128}(\beta^2 - 128)x^2 + \frac{1}{16}(-\beta - 8)x - \frac{1}{32}\beta\right).$$

In particular, if $(x, y)$ is a rational point on the hyperelliptic curve $y^2 = F(x)$, then there exists $y_1, y_2$, and $\alpha \in \mathbb{Q}(\beta)$ such that

$$\alpha y_1^2 = F_1(x) = x^2 - \frac{1}{8}\beta x + \frac{1}{128}(\beta^2 - 64)$$

$$\alpha y_2^2 = F_2(x) = x^4 + \frac{1}{8}\beta x^3 + \frac{1}{128}(\beta^2 - 128)x^2 + \frac{1}{16}(-\beta - 8)x - \frac{1}{32}\beta;$$

simultaneously; this follows from the fact that $F_1(x)$ and $F_2(x)$ lie in the same square-class in $K = \mathbb{Q}(\beta)$. In particular, if $x \in \mathbb{Z}$ and $p$ is a prime in $K$ such that $p \nmid 2$ and $p|\alpha$, then $F_1$ and $F_2$ have a common root modulo $p$. In particular, $p$ must divide the resultant $r = 1/128(b^2 - b - 40)$ of $F_1$ and $F_2$. On the other hand, $K$ has class number 1, and since we may assume (without loss of generality) that $\alpha$ is square-free, we see that

$$\alpha = (-1)^{e_0} \cdot (b/8)^{e_2} \cdot (1/64b^2 + 3/8b - 1)^{e_3}$$

for some $e_i \in \{0, 1\}$ and $0 \leq i \leq 3$; here we use Sage to factor the fractional ideal generated by $r$ and find generators $-1$ and $\beta/8$ of the unit group of $K$. In particular, we have deduced that if $(x, y) \in \mathbb{Q}^2$ is a rational point (integral $x$-coordinate) on the hyperelliptic curve $y^2 = F(x)$, then $(x, y_2)$ is a $K$-point on

$$V_\alpha : \alpha y_2^2 = F_2(x),$$

for some $y_2 \in K$ and some $\alpha$ in (2.10). In particular, for such $\alpha$ it must be the case that $V_\alpha(K_v)$ is non-empty for every completion $K_v/K$. However, we check with MAGMA that only the curves $V_\alpha$ corresponding to $\alpha = 1$ and $\alpha = -\beta/8$ have points everywhere locally. On the other hand, $V_\alpha(K)$ is non-empty for both $\alpha = 1$ and $\alpha = -\beta/8$. Therefore, there exist computable elliptic curves $E_1$ and $E_2$ (in Weierstrass form) together with birational maps $\phi_1 : E_1 \to V_1$ and $\phi_2 : E_2 \to V_{-\beta/8}$ all defined over $K$. In particular, it suffices to compute the sets

$$T_i = \{P \in E_i(K) : x(\phi_i(P)) \in \mathbb{P}^1(\mathbb{Q})\}$$

for $i \in \{1, 2\}$, to classify the integral points on $y^2 = F(x)$. However, $E_1(K)$ and $E_2(K)$ both have rank 2. In particular, $\text{rank}(E_1(K))$ and $\text{rank}(E_2(K))$ are both strictly less than $[K : \mathbb{Q}] = 3$. Therefore, $T_1$ and $T_2$ are finite sets, and we may use the elliptic Chabauty method do describe them [5] §4.2]. Moreover, since both $E_1$ and $E_2$ are in Weierstrass form and we succeed in finding explicit generators for their Mordell-Weil groups, we may use an implementation of the elliptic Chabauty method in MAGMA to describe $T_1$ and $T_2$; see the file named Elliptic Chabauty at the website above for the relevant code. In particular, we deduce that if $(x, y)$ is a rational point on $y^2 = F(x)$ such that $x \in \mathbb{Z}$, then $x \in \{0, \pm 1, -2\}$ as claimed. \hfill \Box

**Corollary 2.14.** Let $r = 1/c$ and $c = 4m^2(m^2 - 1)$ for $m \geq 2$. Then $f_r^3(x)$ has more than two irreducible factors if and only if $m = 2$.

**Proof.** The sufficiency is clear from (2.8). To see that $m = 2$ is also necessary, assume that $f_r^3(x)$ has more than two irreducible factors. From Proposition 2.12 we have that $m$ or $-m$ is the $x$-coordinate of an integral point on the curve $y^2 = 8x^6 - 12x^4 - 4x^3 + 4x^2 + 4x + 1$. It then follows from Theorem 2.13 that $\pm m \in \{-2, -1, 0, 1\}$. Since $m \geq 2$, the only possibility is $m = 2$. \hfill \Box
We have now assembled enough ingredients to prove Theorem 1.4.

Proof of Theorem 1.4. Part (a) is proven in Propositions 2.3 and 2.9. Part (b) follows from Proposition 2.11 and the remarks after (2.5).

The first assertion of part (c) is proven in Corollary 2.14. To prove the second assertion, let \( m = 2 \), let \( q_1 \) and \( q_2 \) be as in (2.7), and set \( v_1(x) = x^2 - (1/2)x + 19/48 \) and \( v_2(x) = x^2 + (1/2)x + 19/48 \), so that \( q_2(f_r(x)) = v_1(x)v_2(x) \). We must show that \( q_1(f_r^n(x)) \) and \( v_j(f_r^n(x)) \) (\( j \in \{1, 2\} \)) are irreducible for all \( n \geq 1 \). Because \( q_1, v_1, \) and \( v_2 \) have even degree, we may use the proof of [9, Proposition 4.2] (or a straightforward adaptation of the proof of Proposition 2.12) to show that it suffices to prove \( q_1(f_r^n(0)) \) and \( v_j(f_r^n(0)) \) are not a squares in \( \mathbb{Q} \) for all \( n \geq 1 \).

We now search for primes \( p \) satisfying the condition (2.3), with \( q_1 \) and \( v_j \) replacing \( g_2 \). We reduce the sequence \( q_1(f_r^n(0)) \) modulo 239, and find that it only takes the non-square value 13 for \( n \geq 7 \). For \( n \) with \( 1 \leq n \leq 6 \), one verifies directly that \( q_1(f_r^n(0)) \) is not a square. We reduce the sequence \( v_1(f_r^n(0)) \) modulo 239, and find that it only takes the non-square value 73 for \( n \geq 7 \). For \( n \) with \( 1 \leq n \leq 6 \), one verifies directly that \( v_1(f_r^n(0)) \) is not a square. We reduce the sequence \( v_2(f_r^n(0)) \) modulo 41, and find that it only takes the non-square value 24 for \( n \geq 7 \). For \( n \) with \( 1 \leq n \leq 6 \), one verifies directly that \( v_2(f_r^n(0)) \) is not a square.

We close this section with a proof of one case of Theorem 1.6.

Proposition 2.15. Let \( r = 1/c \) and \( c = 4m^2(m^2 - 1) \) for \( m \geq 3 \), and let \( q_1 \) and \( q_2 \) be as in (2.7). Suppose that \( 4m^2(m^2 - 1) \leq 10^9 \). Then for all \( n \geq 1 \) we have \( q_1(f_r^n(x)) \) and \( q_2(f_r^n(x)) \) irreducible. Hence \( f_r^n(x) \) is a product of two irreducible factors for all \( n \geq 2 \).

Proof. Observe that \( 4m^2(m^2 - 1) \leq 10^9 \) if and only if \( m \leq 125 \). Because \( q_1 \) and \( q_2 \) have even degree, we may use the proof of [9, Proposition 4.2] (or a straightforward adaptation of the proof of Proposition 2.12) to show that it suffices to prove \( q_1(f_r^n(0)) \) and \( q_2(f_r^n(0)) \) are non-squares in \( \mathbb{Q} \) for all \( n \geq 1 \). We search for primes \( p \) satisfying the condition (2.3), with \( q_1 \) and \( q_2 \) replacing \( g_2 \).

For \( q_1(f_r^n(0)) \), we find that there exists a prime \( p \leq 500 \) (indeed, \( p \leq 337 \)) with the desired property for all \( m \) with \( 3 \leq m \leq 125 \). For \( q_2(f_r^n(0)) \), we also find that there exists a prime \( p \leq 500 \) with the desired property for all \( m \) with \( 3 \leq m \leq 125 \).

For each such \( m \) and \( p \), we take the finitely many terms of the sequence \( (q_1(f_r^n(0)))_{n \geq 2} \) (resp. \( (q_2(f_r^n(0)))_{n \geq 2} \)) that have still not been proven non-square, and reduce modulo small primes until all have been proven non-square.

3. The proof of cases (1)-(4) of Theorem 1.7

In the last section, we saw the primary importance of whether or not \( p(f_r^n(0)) \) is a square, for various polynomials \( p(x) \). For the remainder of this article, we use similar ideas to study the irreducibility of \( f_r(x) \) in the case where \( f_r^2(x) \) is irreducible. However, we use a refinement of [9, Proposition 4.2], similar to [10, Theorem 2.3], that is more powerful; see Lemma 1.8 (restated as Lemma 3.1 below).

Recall from the introduction that \( r = 1/c \), and that \( f_r^n(0) \) is a rational number with denominator \( c^{2n-1} \). We define \( a_n(c) \) to be the numerator of \( f_r^n(0) \). Hence \( a_n(c) \) is described by the recurrence

\[
(3.1) \quad a_1(c) = 1, \quad a_n(c) = a_{n-1}(c)^2 + c^{2n-1} \quad \text{for } n \geq 2.
\]
To ease notation, we often suppress the dependence on $c$, and write $a_1, a_2, \ldots$. Recall also that we define

$$
(3.2) \quad b_n := \frac{a_n - 1 + \sqrt{a_n}}{2} \in \mathbb{Q}.
$$

We now prove Lemma 1.8, which we restate here.

**Lemma 3.1.** Suppose that $c \in \mathbb{Z} \setminus \{0\}$, $r = 1/c$, and $f_r^n$ is irreducible. Let $a_n = a_n(c)$ and $b_n$ be defined as in (3.1) and (3.2), respectively. If for every $n \geq 3$, $b_n$ is not a square in $\mathbb{Q}$ (which holds in particular if $a_n$ is not a square in $\mathbb{Q}$), then $f_r^n(x)$ is irreducible for all $n \geq 1$.

**Proof.** This proof is essentially the same as the proof of [10, Theorem 2.3], but for completeness we give the argument here. By hypothesis $f_r^n(x)$ is irreducible; assume inductively that $f_r^n(x)$ is irreducible for some $n \geq 2$. Let $s$ be a root of $f_r^{n+1}(x)$, and observe that $f_r(s) := \beta$ is a root of $f_r^n(x)$. By our inductive assumption, we have $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2^n$. Now $f_r^{n+1}(x)$ is irreducible if and only if $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^{n+1}$, which is equivalent to $[\mathbb{Q}(\alpha) : \mathbb{Q}(\beta)] = 2$. This holds if and only if $f_r(x) - \beta$ is irreducible over $\mathbb{Q}(\beta)$, i.e. $\beta - r$ is a square in $\mathbb{Q}(\beta)$. Now factor $f_r^n(x)$ over $K_1 := \mathbb{Q}(\sqrt{-r})$. We have $f_r^n(x) = (f_r^n(x) - \sqrt{-r})(f_r^n(x) + \sqrt{-r})$, and because $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2^n$, we must have $[\mathbb{Q}(\beta) : \mathbb{Q}(\beta^1)] = 2^{n+1}$, which implies that the minimal polynomial of $\beta$ over $K_1$ is one of $f_r^{n-1}(x) = \pm \sqrt{-r}$. It follows that $N_{\mathbb{Q}(\beta)/K_1}(\beta - r)$ is the product of $(\beta' - r)$, where $\beta'$ varies over all roots of $f_r^{n-1}(x) = \pm \sqrt{-r}$; this product is just $f_r^{n-1}(r) = \pm \sqrt{-r}$ (here we use that $n \geq 2$, so the degree of $f_r^{n-1}(x)$ is even and we may replace the product of $(\beta' - r)$ with the product of $(r - \beta')$).

To summarize, we have

$$
N_{\mathbb{Q}(\beta)/K_1}(\beta - r) = f_r^{n-1}(r) = f_r^n(0) = \sqrt{-r}.
$$

Suppose now that $f_r^{n+1}(x)$ is reducible, and hence $\beta - r$ is a square in $\mathbb{Q}(\beta)$. Because the norm map is multiplicative, this implies $N_{\mathbb{Q}(\beta)/K_1}(\beta - r)$ is a square in $K_1$, i.e. there exist $s_1, s_2 \in \mathbb{Q}$ with $(s_1 + s_2\sqrt{-r})^2 = f_r^n(0) = \sqrt{-r}$. Elementary calculations show this last equality implies $s_2 = \frac{1}{2} s_1$ and $s_1 - rs_2 = f_r^n(0)$, whence

$$
s_1^2 = \frac{f_r^n(0) + \sqrt{f_r^{n+1}(0)}}{2} = \frac{a_n + \sqrt{a_{n+1}}}{2c^{n-1}}.
$$

Now $n \geq 2$, and hence we have that one of $(a_n + \sqrt{a_{n+1}})/2$ is a square in $\mathbb{Q}$. As $a_{n+1} = a_n^2 + c^{n-1} > a_n^2 > 0$, we have $(a_{n+1} - \sqrt{a_{n+1}})/2 < 0$, implying that $(a_n + \sqrt{a_{n+1}})/2$ is a square in $\mathbb{Q}$. But this is contrary to the hypotheses of the Lemma, and we thus conclude that $f_r^{n+1}(x)$ is irreducible. \(\square\)

**Proposition 3.2.** Let $c \in \mathbb{Z} \setminus \{0, -1\}$. Then neither $a_3$ nor $a_4$ is a square in $\mathbb{Q}$.

**Proof.** We have $a_3(c) = c^3 + c^2 + 2c + 1$, and so if $a_3(c) = y_0^2$ for $y_0 \in \mathbb{Q}$, then necessarily $y_0 \in \mathbb{Z}$, and $(c, y_0)$ is an integer point on the elliptic curve $y^2 = x^3 + x^2 + 2x + 1$. This curve has conductor 92, and is curve 92.b2 in the LMFDB [14]. Besides the point at infinity, it has only the rational points $(0, \pm 1)$, but $c = 0$ is excluded by hypothesis.

We now address $a_4(c)$. As in the previous paragraph, if $a_4(c) = y_0^2$ for $y_0 \in \mathbb{Q}$, then $(c, y_0)$ is an integer point on the hyperelliptic curve

$$
C : y^2 = x^7 + x^6 + 2x^5 + 5x^4 + 6x^3 + 6x^2 + 4x + 1.
$$

One easily checks that $x^7 + x^6 + 2x^5 + 5x^4 + 6x^3 + 6x^2 + 4x + 1$ has no repeated roots, and hence $C$ has genus 3. Denote by $J$ the Jacobian of $C$. A two-descent using MAGMA [2] shows that $J(\mathbb{Q})$
has rank zero, and hence consists only of torsion. We now use standard reduction techniques to determine all torsion in $J(\mathbb{Q})$ [8, Theorem C.1.4 and Section C.2]. We have a commutative diagram

\begin{equation}
\begin{array}{ccc}
C(\mathbb{Q}) & \longrightarrow & J(\mathbb{Q}) \\
\downarrow & & \downarrow \\
C(\mathbb{F}_3) & \longrightarrow & J(\mathbb{F}_3)
\end{array}
\end{equation}

where the vertical maps are reduction modulo 3 and the horizontal maps are the Abel-Jacobi maps taking $P$ to the divisor class of $(P - \infty)$. The latter are known to be injective [8, Corollary A.6.3.3]. The discriminant of $C$ is $2^{12} \cdot 23 \cdot 2551$, and it follows that $C$, and hence hence $J$ [8, p. 164], has good reduction at all primes $p \not\in \{2, 23, 2551\}$. Because $J(\mathbb{Q})$ is torsion, it follows that for any such prime $p$, the reduction map $J(\mathbb{Q}) \rightarrow J(\mathbb{F}_p)$ is injective on prime-to-$p$ torsion [8, Theorem C.1.4]. Turning again to MAGMA, we find $\#J(\mathbb{F}_{11}) = 1372$, and because $3 \nmid \#J(\mathbb{F}_{11})$, we conclude that $J(\mathbb{Q})$ has trivial 3-torsion. Thus the right vertical map in (3.3) is injective, and it follows that the left vertical map is injective as well. But one verifies that $\#C(\mathbb{F}_3) = 4$, and hence $C(\mathbb{Q}) = \{\infty, (0, \pm 1), (-1, 0)\}$. Because we have excluded $c = 0, -1$, we arrive at the desired contradiction.

One may attempt the same argument with $a_5(c)$, but a 2-descent on the Jacobian $J$ of the associated genus-7 hyperelliptic curve shows only that the rank of $J(\mathbb{Q})$ is at most 2.

**Proposition 3.3.** The sequence $(a_n)_{n \geq 1}$ is a rigid divisibility sequence. (See Definition 2.4).

**Proof.** This is a straightforward application of [7, Lemma 2.5]. □

**Proposition 3.4.** If $c < 0$, then $a_n$ is not a square in $\mathbb{Q}$ for all $n \geq 2$.

**Proof.** Let $r = 1/c$ and $f_r(x) = x^2 + r$, and consider the image of the interval $I = (-\sqrt{-r}, 0)$ under $f_r : \mathbb{R} \rightarrow \mathbb{R}$. We have $f_r(-\sqrt{-r}) = 0$ and $f_r(0) = r \in I$, so as $f_r$ is a continuous function with no critical points in $I$, it follows that $f_r(I) \subset I$. As $f_r(0) = r \in I$, inductively, $f_r^n(0) \in I$ for all $n \geq 1$. Hence $0 > f^n(r)(0) = a_n/c^{2n}$, and hence $a_n < 0$ for $n \geq 1$, proving that $a_n$ is not a square in $\mathbb{Q}$.

**Proposition 3.5.** Suppose that $c + 1$ is not a square in $\mathbb{Z}$. If $c$ satisfies any of the congruences in Table 1, then $a_n$ is not a square in $\mathbb{Q}$ for all $n \geq 2$.

**Proof.** By Proposition 3.4 it suffices to consider $c > 0$. Because $a_2 = c + 1 > 0$ is nonsquare by assumption, there is a prime $q$ with $v_q(c + 1)$ odd. Proposition 3.3 then implies that $a_{2m}$ is nonsquare for all $m \geq 1$, so we need only check that $a_n$ is nonsquare for odd $n \geq 2$. To do this, we let $p$ be a fixed prime with $p < 100$ and $p \nmid c$. Let $c_0 \in \{1, \ldots, p - 1\}$ satisfy $1/c \equiv c_0 \mod p$ and put $f = x^2 + c_0 \in \mathbb{F}_p[x]$. Now $a_n = c^{2n-1}f^n(0)$, and it follows that if $f^n(0)$ is not a square in $\mathbb{F}_p$, then $a_n$ is not a square in $\mathbb{Q}$. The sequence $(f^n(0) \mod p)_{n \geq 1}$ eventually lands in a repeating cycle. When this sequence is such that $\hat{f}^{2n+1}(0)$ is a non-square in $\mathbb{F}_p$ for all $n \geq 2$, then $a_{2n+1}$ is a non-square in $\mathbb{Z}$ for all $n \geq 1$ (the $n = 1$ case is by Proposition 3.2). Most of the pairs of $p, c$ listed in Table 1 yield such a result. For instance, when $p = 3$ and $c \equiv 1 \mod p$, we have $f^n(0) = 2$ for all $n \geq 2$. When $p = 5$ and $c \equiv 3 \mod p$, the sequence $f^n(0)$ is $2, 1, 3, 1, 3, \ldots$, and hence $\hat{f}^{2n+1}(0)$ is a non-square for all $n \geq 1$. The remaining pairs $p, c$ in Table 1 satisfy the condition that both $\hat{f}^{3(n+1)+1}(0)$ and $\hat{f}^{3n+2}(0)$ are non-squares for $n \geq 1$ (the $n + 1$ comes from the fact that $a_4$ is automatically a non-square by Proposition 3.2). Thus $a_{3n+1}$ and $a_{3n+2}$ are
\[ c \equiv 1, 2 \pmod{3} \]
\[ c \equiv 2, 3 \pmod{5} \]
\[ c \equiv 1, 2, 5, 6 \pmod{7} \]
\[ c \equiv 3, 5, 7, 10 \pmod{11} \]
\[ c \equiv 3, 4, 5, 6, 8, 11 \pmod{13} \]
\[ c \equiv 6, 10, 14, 15 \pmod{17} \]
\[ c \equiv 1, 4, 9, 11, 12, 13, 15, 16, 18 \pmod{19} \]
\[ c \equiv 10, 12, 18, 20, 22 \pmod{23} \]
\[ c \equiv 2, 12, 14, 17, 18, 27 \pmod{29} \]
\[ c \equiv 1, 13, 16, 22, 30 \pmod{31} \]
\[ c \equiv 6, 18, 23, 31, 32, 35 \pmod{37} \]
\[ c \equiv 7, 8, 11, 19, 25, 28, 35, 36 \pmod{41} \]
\[ c \equiv 1, 4, 5, 9, 14, 15, 21, 27, 33, 37, 42 \pmod{43} \]
\[ c \equiv 6, 7, 9, 10, 24, 25, 28, 33, 46 \pmod{47} \]
\[ c \equiv 5, 18, 21, 23, 26, 30, 37, 40, 45, 46, 47 \pmod{53} \]
\[ c \equiv 10, 16, 29, 37, 55, 57, 58 \pmod{59} \]
\[ c \equiv 2, 3, 11, 15, 27, 30, 32, 40, 45, 50 \pmod{61} \]
\[ c \equiv 10, 15, 20, 33, 38, 41, 49, 53, 66 \pmod{67} \]
\[ c \equiv 4, 10, 49, 51, 53, 61, 70 \pmod{71} \]
\[ c \equiv 1, 3, 43, 44, 50, 51, 71 \pmod{73} \]
\[ c \equiv 3, 25, 58, 78 \pmod{79} \]
\[ c \equiv 15, 19, 23, 25, 29, 31, 37, 41, 56, 59, 68, 82 \pmod{83} \]
\[ c \equiv 13, 25, 49, 63 \pmod{89} \]
\[ c \equiv 3, 21, 59, 79, 89 \pmod{97} \]

Table 1. Congruences that ensure \( a_n \) is not a square for \( n \geq 2 \), provided that \( c + 1 \) is not a square.

non-squares in \( \mathbb{Z} \) for all \( n \geq 1 \). But by Proposition 3.2 we have that \( a_3 \) is not a square in \( \mathbb{Z} \), and it follows from Proposition 3.3 that \( a_{3n} \) is a non-square in \( \mathbb{Z} \) for all \( n \geq 1 \). An example is when \( p = 7 \) and \( c \equiv 5 \pmod{p} \), for which the sequence \( \bar{f}^n(0) \) is \( 3, 5, 0, 3, 5, 0, \ldots \). \( \square \)

We now prove cases (1)-(4) of Theorem 1.7, which we restate here.

**Theorem 3.6.** Let \( f_r(x) = x^2 + r \) with \( r = 1/c \) for \( c \in \mathbb{Z} \setminus \{0, -1\} \), and let \( a_n \) and \( b_n \) be as in (3.1) and (3.2). Assume that \( c \) satisfies one of the following conditions:

1. \( -c \in \mathbb{Z} \setminus \mathbb{Z}^2 \) and \( c < 0 \);
2. \( -c, c + 1 \in \mathbb{Z} \setminus \mathbb{Z}^2 \) and \( c \equiv -1 \pmod{p} \) for a prime \( p \equiv 3 \pmod{4} \);
(3) \(-c, c + 1 \in \mathbb{Z} \setminus \mathbb{Z}^2\) and \(c\) satisfies one of the congruences in Proposition 3.2 (see Table 1);

(4) \(-c \in \mathbb{Z} \setminus \mathbb{Z}^2\), \(c\) is not of the form \(4m^2(m^2 - 1), m \in \mathbb{Z}\), and \(c\) is odd;

In cases (1)-(3), \(a_n\) is not a square in \(\mathbb{Q}\) for any \(n \geq 2\), while in case (4), \(b_n\) is not a square for any \(n \geq 2\). In all cases, \(f^2(x)\) is irreducible for all \(n \geq 1\).

Proof. Observe that conditions (1)-(4) each imply that \(f^2(x)\) is irreducible, by Proposition 2.1 (note that \(c = 4m^2(m^2 - 1)\) implies that \(c + 1 = (2m - 1)^2\)). We now argue that in cases (1)-(3) \(a_n\) is not a square in \(\mathbb{Q}\) for any \(n \geq 2\) and in case (4), \(b_n\) is not a square for any \(n \geq 2\). In all these cases, Lemma 1.8 proves that \(f^2(x)\) is irreducible for all \(n \geq 1\).

If we are in case (1), then the desired conclusion holds by Proposition 3.4.

Assume we are in case (2). By Proposition 3.4 it suffices to consider \(c > 0\). Because \(1/c \equiv -1 \text{ mod } p\), we see that modulo \(p\), the orbit of \(0\) under \(f_r\) is \(0 \mapsto -1 \mapsto 0 \mapsto \cdots\). Moreover, \(-1\) is not a square modulo \(p\) by assumption, and so \(a_{2n+1}\) is not a square for all \(n \geq 3\). Because \(a_2 = c + 1 \geq 2\) is assumed non-square, it must be divisible by some prime to odd multiplicity.

From Proposition 3.3 it then follows that \(a_{2n}\) is not a square in \(\mathbb{Q}\) for all \(n \geq 1\).

In case (3) the desired conclusion holds by Proposition 3.5.

In case (4), if \(a_n\) is not a square in \(\mathbb{Q}\) then \(b_n\) cannot be a square in \(\mathbb{Q}\), and so we are done. If \(a_n\) is square in \(\mathbb{Q}\), then from the recursion in (3.1) and the fact that any integer equals its square

Thus modulo 2, we have \(a_{n} - 1 + \sqrt{a_n} \equiv 2a_n - 1 \equiv 1\), whence \(v_2 \left( \frac{a_n + \sqrt{a_n}}{2} \right) = -1\), proving that \(b_n = \frac{a_n + \sqrt{a_n}}{2}\) is not a square in \(\mathbb{Q}\).

\[
\sqrt{a_n} \equiv a_n \equiv a_{n-1}^2 + c^{2n-1-1} \equiv a_{n-1}^2 + 1 \equiv a_{n-1} + 1 \pmod{2}.
\]

4. Proof of cases (5) and (6) of Theorem 1.7

Here we prove cases (5) and (6) of Theorem 1.7 as specific instances of a more general method.

To this end, we consider the consequences of \(b_n = \frac{a_{n-1} + \sqrt{a_n}}{2}\) being a square in \(\mathbb{Q}\) in greater detail. Throughout this section, we denote the set of positive integers by \(\mathbb{Z}^+\).

Lemma 4.1. Let \(c \in \mathbb{Z} \setminus \{0\}\). For all primes \(p\) dividing \(c\), we have \(a_n \equiv 1 \text{ mod } p\) for all \(n \geq 1\).

Proof. We have \(a_1 \equiv 1 \text{ mod } p\) and \(p \mid c^{2n-1-1}\) for all \(n > 1\). The definition of \(a_n\) and a simple induction give the desired result.

The following lemma is the main tool in the arguments of this section.

Lemma 4.2. Let \(c \in \mathbb{Z}^+\) be even, and let \(d^2\) be the greatest square integer divisor of \(c\), so that \(c = \delta d^2\) for a squarefree integer \(\delta\). Suppose \(\frac{a_{n-1} + \sqrt{a_n}}{2}\) is a square in \(\mathbb{Q}\), where \(n \geq 2\). Then there exist coprime integers \(x\) and \(y\) such that

\[
\begin{align*}
(a_{n-1} = x^2 - \frac{y^2}{\delta^{2n-1-1}}, & \quad d^{2n-1-1} = \frac{2xy}{\delta^{2n-1-1}}, \quad \sqrt{a_n} = x^2 + \frac{y^2}{\delta^{2n-1-1}}. \\
\end{align*}
\]

Proof. Observe that if \(\frac{a_{n-1} + \sqrt{a_n}}{2}\) is a square in \(\mathbb{Q}\), then certainly \(a_n\) is a square in \(\mathbb{Q}\), and thus

\[
(a_{n-1}, d^{2n-1-1}, \sqrt{a_n})
\]

is an integer solution to the quadratic Diophantine equation \(X^2 + kY^2 = Z^2\), where \(k = \delta^{2n-1-1}\). From Proposition 3.3 we have \(\gcd(a_{n-1}, \sqrt{a_n}) = 1\), and from Lemma 4.1 we have \(\gcd(a_i, c) = 1\) for all \(i \geq 1\), implying that the entries of (4.2) are pairwise relatively prime.
Now \( X^2 + kY^2 = Z^2 \) is equivalent to \( \frac{Z}{k}X = \frac{Y}{x} \). Setting this equal to \( \frac{z}{y} \) for relatively prime positive integers \( x, y \), we may then set up linear equations for \( \frac{z}{y} \) and \( \frac{x}{y} \), which we solve to get \( \frac{z}{y} = \frac{kx^2 - y^2}{2xy} \) and \( \frac{x}{y} = \frac{kx^2 + y^2}{2xy} \). After dividing the fractions on the right-hand sides through by \( \gcd(kx^2 - y^2, 2xy) \) (which coincides with \( \gcd(kx^2 + y^2, 2xy) \)), we may equate numerators and denominators. Applying this to the solution in (3.2) gives

\[
(4.3) \quad a_{n-1} = \frac{\delta^{2n-1-1}x^2 - y^2}{w}, \quad d^{2n-1-1} = \frac{2xy}{w}, \quad \sqrt{a_n} = \frac{\delta^{2n-1-1}x^2 + y^2}{w},
\]

where \( w = \gcd(\delta^{2n-1-1}x^2 - y^2, 2xy) \). It remains to demonstrate that \( w = \delta^{2n-1-1} \), or equivalently that \( v_p(w) = v_p(\delta^{2n-1-1}) \) for all primes \( p \).

Observe that because \( \delta \) is square-free, \( v_p(\delta^{2n-1-1}) = 2n-1 - 1 \) or 0 for all primes \( p \). But \( n \geq 2 \), so \( v_p(\delta^{2n-1-1}) \) is either 0 or odd. Note that \( (a_{n-1} + \sqrt{a_n})/2 = \delta^{2n-1-1}x^2/w \), and hence \( \delta^{2n-1-1}/w \) is a square in \( \mathbb{Q} \). Therefore if \( v_p(w) = 0 \), then \( v_p(\delta^{2n-1-1}) \) is even, whence \( v_p(\delta^{2n-1-1}) = 0 \).

Now consider the case \( v_p(w) > 0 \). Because \( d^{2n-1-1} \) and \( a_{n-1} \) are integers, the first two equations of (4.3) give

\[
(4.4) \quad v_p(2xy) \geq v_p(w) > 0 \quad \text{and} \quad v_p(\delta^{2n-1-1}x^2 - y^2) \geq v_p(w) > 0.
\]

If \( v_p(x) > 0 \), then the co-primality of \( x \) and \( y \) gives \( v_p(y) = 0 \), yielding the contradiction \( v_p(\delta^{2n-1-1}x^2 - y^2) = 0 \). Thus \( v_p(x) = 0 \), and from the first equation of (4.4), we have \( v_p(2y) > 0 \). If \( v_p(y) = 0 \), then \( p = 2 \) and \( v_p(2xy) = 1 \). But \( v_p(w) > 0 \), and from the middle equation of (4.3) we have \( v_p(w) = 1 \), and hence \( d \) is odd. But \( c \) is even by hypothesis, whence \( \delta \) is even, giving the contradiction \( v_p(\delta^{2n-1-1}x^2 - y^2) = 0 \). Therefore \( v_p(y) > 0 \). In order to get \( v_p(\delta^{2n-1-1}x^2 - y^2) > 0 \) we must have \( v_p(\delta^{2n-1-1}) > 0 \), implying \( v_p(\delta^{2n-1-1}) = 2n-1 - 1 \). We have shown \( v_p(\delta^{2n-1-1}) = 2n-1 - 1 \), and thus the proof of the Lemma will be complete if we show \( v_p(w) = 2n-1 - 1 \).

From the \( \gcd \) definition of \( w \),

\[
(4.5) \quad v_p(w) = \min\{v_p(\delta^{2n-1-1}x^2 - y^2), v_p(2y)\}.
\]

Now, \( v_p(\delta^{2n-1-1}x^2) = 2n-1 - 1 \equiv 1 \) mod 2, while \( v_p(y^2) = 2v_p(y) \equiv 0 \) mod 2; hence the two valuations are distinct, and so

\[
(4.6) \quad v_p(\delta^{2n-1-1}x^2 - y^2) = \min\{2n-1 - 1, 2v_p(y)\} \leq 2n-1 - 1.
\]

It follows from (4.5) that \( v_p(w) \leq 2n-1 - 1 \). Suppose for contradiction that \( v_p(w) < 2n-1 - 1 \). Then

\[
v_p(a_{n-1} + \sqrt{a_n}) = v_p(2\delta^{2n-1-1}x^2/w) = v_p(2) + 2n-1 - 1 - v_p(w) > 0.
\]

From (1.2), \( e^{2n-1-1} = (a_{n-1} + \sqrt{a_n})(\sqrt{a_n} - a_{n-1}) \), implying that \( v_p(c) > 0 \). It follows that \( v_p(a_{n-1}) = 0 \); otherwise (1.2) would imply \( v_p(a_{n-1}) \) and \( v_p(a_{n-1}) \) are both positive, contradicting Proposition 3.3. Hence the first equation of (1.3) gives \( v_p(w) = v_p(\delta^{2n-1-1}x^2 - y^2) \). From (4.6) and our assumption that \( v_p(w) < 2n-1 - 1 \), we obtain \( v_p(w) = 2v_p(y) \). Thus \( v_p(\delta^{2n-1-1}/w) = 2n-1 - 1 - 2v_p(y) \equiv 1 \) mod 2, contradicting the assumption that \( \delta^{2n-1-1}x^2/w \) is a square in \( \mathbb{Q} \). □
Definition 4.3. Suppose that $\varphi(p, l)$ is a logical proposition defined for any prime $p$ and nonnegative integer $l$. Then let $\Psi_{\varphi} : \mathbb{Z}^+ \rightarrow \mathbb{Q}$ be defined by
\[
\Psi_{\varphi}(k) = \frac{\prod_{p : \varphi(p, v_p(k))} p^{v_p(k)}}{\prod_{p : \neg \varphi(p, v_p(k))} p^{v_p(k)}}.
\]

Lemma 4.4. Let $c \in \mathbb{Z}^+$ be even. Suppose that $\varphi$ is a logical proposition with the property that whenever $\frac{a_{n-1} + \sqrt{a_n}}{2}$ is a square in $\mathbb{Q}$ and $\varphi(p, v_p(c))$ is true for some prime $p$, then $p | y$, where $y$ is as in the parameterization (4.1). If $\frac{a_{n-1} + \sqrt{a_n}}{2}$ is a square in $\mathbb{Q}$ for some $n \geq 2$, then
\[
\Psi_{\varphi}(c) < 2^{\frac{n}{2} - 1}.
\]

Proof. Assume that $\frac{a_{n-1} + \sqrt{a_n}}{2}$ is a square in $\mathbb{Q}$, and let $x$ and $y$ be as in (4.1). We achieve the bound in the Lemma by factoring $c^{2^n - 1}$ into coprime factors derived from $x$ and $y$. Recall from Lemma 4.2 that $x$ and $y$ are coprime and $c = \delta^2$. The middle equation of (4.1) yields
\[
c^{2^n - 1} = \delta^{2^n - 1} \left( d^{2^n - 1} \right)^2 = \frac{4x^2y^2}{\delta^{2^n - 1}}.
\]
The first equation of (4.1) shows that $\frac{\frac{x^2}{\delta^{2^n - 1}}}$ is an integer and $x^2 > \frac{4y^2}{\delta^{2^n - 1}}$. Moreover, $x^2$ and $\frac{4y^2}{\delta^{2^n - 1}}$ are coprime because $x$ and $y$ are coprime. The additional factor of 4 on the right-hand side of (4.8) can be finessed by defining
\[
\epsilon = \begin{cases} 
1 & \text{if } y \text{ is even} \\
-1 & \text{if } y \text{ is odd}
\end{cases}
\]
Then $2^{1-\epsilon}x^2$ and $2^{1+\epsilon}y^2/\delta^{2^n - 1}$ are coprime integers whose product is $c^{2^n - 1}$.

If $\varphi(p, v_p(c))$ holds, then by hypothesis $v_p(y) > 0$, and so $v_p(2^{1-\epsilon}x^2) = 0$. Therefore, $\varphi(p, v_p(c))$ implies $v_p \left( 2^{1+\epsilon}y^2/\delta^{2^n - 1} \right) = v_p \left( c^{2^n - 1} \right)$. This gives
\[
\prod_{p : \varphi(p, v_p(c))} p^{v_p(c^{2^n - 1})} = \prod_{p : \neg \varphi(p, v_p(c))} p^{v_p(2^{1+\epsilon}y^2/\delta^{2^n - 1})} \leq \prod_p p^{v_p(2^{1+\epsilon}y^2/\delta^{2^n - 1})} = \frac{2^{1+\epsilon}y^2}{\delta^{2^n - 1}}.
\]
Dividing both sides by $c^{2^n - 1}$ and noting $v_p(c^{2^n - 1}) = (2^n - 1)v_p(c)$, we obtain
\[
\prod_{p : \neg \varphi(p, v_p(c))} p^{(2^n - 1)v_p(c)} \leq \frac{1}{2^{1-\epsilon}x^2}.
\]

Multiplying the inequalities in (4.10) and (4.11) yields
\[
\Psi_{\varphi}(c)^{2^n - 1} \leq \frac{2^{1+\epsilon}y^2}{2^{1-\epsilon} \delta^{2^n - 1} x^2} < 2^{2\epsilon},
\]
where the last inequality follows from $x^2 > \frac{4y^2}{\delta^{2^n - 1}}$. Hence $\Psi_{\varphi}(c) < 2^{\frac{2n}{2} - 1} \leq 2^{\frac{n}{2} - 1}$. □

Corollary 4.5. Let $f_r(x) = x^2 + r$ with $r = 1/c$ for $c$ an even positive integer, and let $\varphi$ be a logical proposition with the property described in Lemma 4.4. Suppose that $f_r^r(x)$ is irreducible. If $\Psi_{\varphi}(c) > 2^{\frac{2n}{2}}$, then $f_r^n(x)$ is irreducible for all $n \geq 1$. 

4.2. The result is that, under a relatively mild hypothesis on growth of an integer satisfying mild hypotheses, the algorithm operates by combining analytic bounds on the ϕ of case (5) give Ψ if a is an integer, this implies p that assume that

Proof of cases (5) and (6) of Theorem 1.7. Thanks to parts (1) and (4) of Theorem 1.7 we may assume that c > 0 and c is even.

Assume we are in case (5), and let φ(p, l) be the proposition that l ≡ 1 mod 2. The conditions of case (5) give Ψφ(c) > 2^{2/15} and fr^2 irreducible. Hence by Corollary 4.5 it suffices to show that if \( \frac{a_{n-1} + \sqrt{a_n}}{2} \) is a square in \( \mathbb{Q} \) and vp(c) ≡ 1 mod 2 for some prime p, then p \| y, where y is as in (4.1). However, this is straightforward to show: if vp(c) ≡ 1 mod 2, then p \| δ. But (4.1) implies \( y^2/δ^{2n-1-1} \) is an integer, showing that vp(y) > 0, as desired.

Assume now that we are in case (6). Let φ′(p, l) be the proposition that p ≠ 1 mod 4 and vp(c) > 0. Because c = k^2 for k > 2, Proposition 2.1 gives that \( f_r^2(x) \) is irreducible. Suppose that \( \frac{a_{n-1} + \sqrt{a_n}}{2} \) is a square in \( \mathbb{Q} \), and note that by Proposition 3.2 we may assume n > 4. Let x, y, d, and δ be as in Lemma 4.2. The conditions of case (6) give δ = 1 and Ψφ′(c) > 2^{2/15}. If φ′(p, vp(c)) holds, then p ≠ 1 mod 4, and so −1 is not a quadratic residue modulo p. Because δ = 1, the first equation of (4.1) gives \( x^2 = y^2 + a_{n-1} \), and so by Lemma 4.1 we have \( x^2 \equiv y^2 + 1 \mod p \). Hence \( x^2 \not\equiv 0 \mod p \), so p \| x. If also p \| y, then vp(d^{2n-1-1}) = vp(2xy) ≤ 1. Because n > 4 and vp(d) is an integer, this implies vp(d) = 0, which contradicts the assumption vp(c) > 0; thus we conclude that p \| y. The desired result follows from Corollary 4.5.

5. An algorithm

In this section, we give an algorithm that tests the stability of \( f_r(x) \) for \( r = 1/c \) and c a positive integer satisfying mild hypotheses. The algorithm operates by combining analytic bounds on the growth of \( a_n \) with the restrictions on the value of \( a_n \) imposed by the parameterization of Lemma 4.2. The result is that, under a relatively mild hypothesis on c, we can conclude that \( \frac{a_{n-1} + \sqrt{a_n}}{2} \) is non-square for all n except those less than roughly \((\log_2 c)/2\) (see Lemma 5.2). The remaining values of \( \frac{a_{n-1} + \sqrt{a_n}}{2} \) can be checked for squares quite rapidly (see Remark 5.6). If appropriately implemented, given a particular c the resulting stability test either demonstrates the stability of \( f_r \) or reports an inconclusive result in time that is logarithmic in c.

We first derive an analytic upper bound on \( a_n \). In combination with this upper bound, the trivial lower bound \( a_n > 0 \) proves sufficient.

Lemma 5.1. Let c ∈ \( \mathbb{Z} \), c > 16. Then for all n,

\[
1 \leq \frac{a_n}{c^{2n-1-1}} < \frac{c}{2} \left( 1 - \sqrt{\frac{c-4}{c}} \right) < 1 + \frac{\mu}{c},
\]

where \( \mu = 16(7 - 4\sqrt{3}) \) is a constant independent of c and n satisfying 1.14 ≤ μ ≤ 1.15.

Proof. Observe that the fixed points of \( f_r \) are \( k_\pm := \frac{1}{2} \left( 1 \pm \sqrt{\frac{c-4}{c}} \right) \), the roots of \( f_r(x) - x = x^2 - x + 1/c \). We note that \( f_r(x) \geq x \) and \( f_r^2(x) \geq 0 \) on the interval \([0, k_-]\), and so for all \( x \in [0, k_-] \) we have

\[
0 \leq x \leq f_r(x) \leq f_r(k_-) = k_-.\]
Hence $f_r([0, k_-]) \subseteq [0, k_-]$, and thus for any $s \in [0, k_-)$, $\{f^n(s)\}_{n \geq 0}$ is a strictly increasing sequence on the interval $[0, k_-)$. In particular, the sequence $\{f^n(0)\}_{n \geq 1} = \{\frac{a_n}{c^{l_n-1}}\}_{n \geq 1}$ is confined to this interval, and so for all $n \in \mathbb{N}$, we have

$$\frac{a_n}{c^{2n-1}} < ck_- = \frac{c}{2} \left( 1 - \sqrt{\frac{c-4}{c}} \right).$$

Because the sequence $\{\frac{a_n}{c^{2n-1}}\}_{n \geq 1}$ is strictly increasing, $a_1/c^{2n-1} = a_1 = 1$ bounds it from below.

Finally, note that $(\mu - 1)c^2 - 2\mu c - \mu^2$ has $c = 16$ as a root, and is positive for all $c > 16$. Thus for all $c > 16$, we have $(\mu - 1) - \frac{2\mu}{c} - \frac{\mu^2}{c^2} > 0$, implying

$$\frac{c^2}{4} - c > \frac{c^2}{4} - c - (\mu - 1) + \frac{2\mu}{c} + \frac{\mu^2}{c^2} = \left( \frac{c}{2} - \frac{\mu}{c} \right)^2,$$

implying $1 + \frac{\mu}{c} > \frac{c}{2} - \sqrt{\frac{c^2}{4} - c} = \frac{c}{2} \left( 1 - \sqrt{\frac{c-4}{c}} \right)$, as desired. \qed

**Lemma 5.2.** Let $c \in \mathbb{Z}$ such that $c > 32$, and let $\mu$ be as in Lemma 5.1. Let $\kappa \in \mathbb{Z}$ be such that $\kappa + 1$ is the nearest integer to $\sqrt{c + \frac{\mu}{2}}$. If either $\kappa \not| c$ or $\gcd(c, \kappa/c) \neq 1$, then $\frac{a_n}{c^{2n-1}}$ is non-square for all $n$ with $n \geq C_1 + (1/2) \log_2 c$, where $C_1 = 8 + \log_2 (2^{1/31} - 1)$ satisfies $2.53 \leq C_1 \leq 2.54$.

**Proof.** Prove by considering the contrapositive: suppose that $\frac{a_n}{c^{2n-1}}$ is a square in $\mathbb{Q}$ for some $c > 32$ and $n \geq C_1 + (1/2) \log_2 c$. Then it suffices to show that $\kappa$ and $c/\kappa$ are coprime integers.

Note that by assumption $n \geq 2.53 + (1/2) \log_2(32) > 5$, implying that $n \geq 6$. Because $\frac{a_n}{c^{2n-1}}$ is a square in $\mathbb{Q}$, we may apply Lemma 4.2 to obtain the parameterization given in (4.1). In particular, recall from the proof of Lemma 4.3 that if $\epsilon \in \{-1, +1\}$ is defined as in (4.9) and $x, y, \delta$ are as in Lemma 4.2, then $2^{1-\epsilon} \alpha^2$ and $2^{1+\epsilon} \gamma^2/\delta^{2n-1}$ are coprime integers whose product is $\alpha^{2n-1}$. Therefore, the $(2n-1 - 1)^{\text{th}}$ roots of these two factors are coprime integers whose product is $c$. More precisely, taking

$$l = (2y^2/\delta^{2n-1})^{1/(2n-1)},$$

then $2^{l}/(2^{n-1})l$ and $c/(2^{l}/(2^{n-1})l)$ are coprime integers. We will demonstrate that $\kappa$ and $c/\kappa$ are coprime by proving that $\kappa = 2^{l}/(2^{n-1})l$.

Taking the difference of the first and third equalities in (4.1), we have $\sqrt{a_n} - a_{n-1} = 2y^2/\delta^{2n-1} = l^{2n-1}$. Two applications of Lemma 5.1 yield

$$\sqrt{c}(1 - \left(1 + \frac{\mu}{c}\right)) < \sqrt{c} \left(\frac{\sqrt{a_n}}{c^{2n-2-1/2}}\right) - \frac{a_{n-1}}{c^{2n-2-1}} < \sqrt{c} \left(1 + \frac{\mu}{c}\right) - 1,$$

or more concisely, $\sqrt{c} - \mu/c - 1 < l^{2n-1}/c^{2n-2} < \sqrt{c} + \mu - 1$. Assume that $l \leq \sqrt{c} - \mu/c - 1$, and note that then

$$l^{2n-1} < (\sqrt{c} - \mu/c - 1)(\sqrt{c} - \mu/c - 1)^{2n-2} < (\sqrt{c} - \mu/c - 1)c^{2n-2},$$

contradicting the lower bound on $l$. A similar argument gives an upper bound on $l$, leaving us with

$$\sqrt{c} - \mu/c - 1 < l < \sqrt{c} + \mu - 1.$$

Observe that $\sqrt{c} + \mu - \sqrt{c} + \mu/c$ is a decreasing function of $c$; thus $(\sqrt{c} + \mu - 1) - (\sqrt{c} - \mu/c - 1) < \sqrt{32} + \mu - \sqrt{32} + \mu/32 < 1/4$. Clearly, this implies $|((\sqrt{c} + \mu - 1) - l)| < 1/4.$
We claim that \(|2^r/(2^{n-1})l - l| \leq 1/4\). Assuming this for a moment, the triangle inequality yields \(|(\sqrt{c+\mu} - 1) - 2^r/(2^{n-1})l| < 1/2\). Because \(2^r/(2^{n-1})l\) is an integer, it thus must be the nearest integer to \(\sqrt{c+\mu} - 1\), completing the proof.

To prove \(|2^r/(2^{n-1})l - l| \leq 1/4\), begin by observing that for \(c > 32\) we have \(\sqrt{c+\mu} - 1 < \sqrt{c}\), and so from (5.1) we have \(l < \sqrt{c}\). To momentarily ease notation, let \(\sigma = 2^{1/(2^{n-1})}\). Then \(\sigma - 1 = \sigma(1 - \sigma^{-1})\), and the fact that \(|\sigma| > 1\) implies \(|\sigma - 1| > |\sigma^{-1} - 1|\), whence \(|\sigma^{-1} - 1| \leq |\sigma - 1| = \sigma - 1\). We have shown

\[
|2^r/(2^{n-1})l - l| \leq \sqrt{c}|2^r/(2^{n-1}) - 1| \leq \sqrt{c} \left(2^{1/(2^{n-1})} - 1\right).
\]

From (5.2), it suffices to show \(2^{1/(2^{n-1})} - 1 \leq 1/(4\sqrt{c})\), which is equivalent to

\[
\log_2(2^{1/(2^{n-1})} - 1) + 2 \leq -(1/2) \log_2 c.
\]

By hypothesis, \(C_1 - n \leq -(1/2) \log_2 c\), and so the proof will be complete if we show that

\[
(5.3) \quad \log_2(2^{1/(2^{n-1})} - 1) + n + 2 \leq C_1
\]

for all \(n \geq 6\). From Lemma [5.3] we have that the left-hand side is a decreasing function of \(n\) for \(n \in [6, \infty)\). But when \(n = 6\) the left-hand side gives \(C_1\), proving that (5.3) holds for \(n \geq 6\). \(\square\)

**Lemma 5.3.** The function

\[
f(x) = \log_2(2^{1/(2^{x-1})} - 1) + x + 2
\]

is decreasing on the interval \((1, \infty)\).

**Proof.** Begin by writing \(f(x) = \log_2(g(x) - 1) + x + 2\), with \(g(x) = 2^{1/(2^{x-1})}\). Then

\[
f'(x) = \frac{g'(x)}{(\ln 2)(g(x) - 1)} + 1,
\]

and because \(g(x) > 1\) on \((1, \infty)\), it follows that \(f'(x) < 0\) provided that \(1/\ln 2 g'(x) + g(x) < 1\).

Now let \(g(x) = 2^h(x)\), with \(h(x) = 1/(2^{x-1}) - 1\). Then \(g'(x) = (\ln 2)h'(x)2^h(x)\), and hence \(1/\ln 2 g'(x) + g(x) < 1\) is equivalent to \(2^h(x)h'(x) + 1 < 1\). Let \(r(x) = 2^h(x)h'(x) + 1\). We argue that \(\lim_{x \to 1} r(x) = -\infty\), \(\lim_{x \to \infty} r(x) = 1\), and \(r'(x)\) does not vanish on \((1, \infty)\); together, these imply \(r(x) < 1\) on \((1, \infty)\), as desired. The first two assertions are readily verified. For the third, note that

\[
r'(x) = 2^h(x)h''(x) + (\ln 2)h'(x)(1 + h'(x)),
\]

and so it suffices to show that \(h''(x) + (\ln 2)h'(x)(1 + h'(x))\) does not vanish on \((1, \infty)\). Let \(s = s(x) = 2^{x-1}\), so that

\[
h(x) = 1/(s - 1), \quad h'(x) = -(\ln 2)s/(s - 1)^2, \quad h''(x) = (\ln 2)^2s(s^2 - 1)/(s - 1)^4.
\]

Algebra then yields

\[
h''(x) + (\ln 2)h'(x)(1 + h'(x)) = \frac{-2 + s(2 + \ln 2)}{(s - 1)^4},
\]

and the latter expression does not vanish for \(s > 1\). \(\square\)

**Corollary 5.4** (Stability Test). Let \(r = 1/c\), and let \(c \in \mathbb{Z}\) such that \(c > 32\), let \(\mu = 16(7 - 4\sqrt{3})\) as in Lemma [5.7], and let \(\kappa\) be the nearest integer to \(\sqrt{c + \mu} - 1\). Then \(f_\kappa''(x)\) is irreducible for all \(n \geq 1\) provided the following hold:
Theorem 6.1. Let $c$ and the discussion immediately following the statement of Theorem 1.7 on p. 3. If of Theorem 1.6 follow from Theorem 1.4, Corollaries 2.8 and 2.10, and Proposition 2.15.

Corollary 5.5 to handle the remaining cases. If assume that Conjecture 1.9 holds for $c$ bound (say less than 100). One simply recursively computes the sequence $(\ldots)$.

Proof of Theorem 1.6. Write $p$ a fixed small prime. We have now at last assembled all the ingredients required to prove Theorem 1.6. Proof of Theorem 1.6. If $f_r^n(x)$ is irreducible for all $n$ with $1 \leq n < 2.54 + (1/2)\log_2(c)$. If $c + 1$ is a square in $\mathbb{Z}$, then $f_r^n(x)$ is irreducible for all $n \geq 1$.

Remark 5.6. For given $c$ it is very quick to check that $a_n$ is not a square for all $n$ up to some small bound (say less than 100). One simply recursively computes the sequence $(a_n \bmod p)_{1 \leq n \leq 100}$ for a fixed small prime $p$, and records the non-squares in each position of the sequence. As long as one uses enough primes $p$, each term of the sequence will be a non-square modulo one of these primes.

We have now at least assembled all the ingredients required to prove Theorem 1.6. Proof of Theorem 1.6. If $f_r^n(x)$ is irreducible and $c + 1$ is not a square, then Theorem 1.6 holds by the discussion immediately following the statement of Theorem 1.7 on p. 3. If $f_r^n(x)$ is irreducible and $c + 1$ is a square, then we may apply parts (1) and (4)-(6) of Theorem 1.7, and then use Corollary 5.5 to handle the remaining cases. If $f_r(x)$ or $f_r^2(x)$ is reducible, then the relevant cases of Theorem 1.6 follow from Theorem 1.4, Corollaries 2.8 and 2.10, and Proposition 2.15. □

6. Applications to the density of primes dividing orbits

In this section, we prove Theorem 1.10, which we restate here for the reader’s convenience.

Theorem 6.1. Let $c \in \mathbb{Z}$, let $r = 1/c$, suppose that $-c$ and $c + 1$ are non-squares in $\mathbb{Q}$, and assume that Conjecture 1.8 holds for $c$, i.e. that $\frac{a_n - 1 + \sqrt{n}}{2}$ is not a square in $\mathbb{Q}$ for all $n \geq 2$. Then

(6.1) for any $t \in \mathbb{Q}$ we have $D(\{p \text{ prime : } p \text{ divides } O_{f_r}(t)\}) = 0$.

Remark 6.2. Observe that the hypothesis that $\frac{a_n - 1 + \sqrt{n}}{2}$ not be a square for $n \geq 2$ is strictly weaker than $a_n$ not being a square for $n \geq 2$; in the latter case the conclusion of Theorem 6.1 follows immediately from part (2) of [7, Theorem 1.1]. To prove Theorem 6.1 we must apply [7, Theorem 1.1] in a non-trivial way.

Remark 6.3. When the hypotheses of Theorem 6.1 are satisfied, we also obtain certain information on the action of $G_\mathbb{Q}$ on $T_\infty(0)$. The index-two subgroup $G_\mathbb{Q}(\sqrt{-r})$ acts on both $T_\infty(\sqrt{-r})$ and $T_\infty(-\sqrt{-r})$. Both of these actions are transitive on each level of the tree, i.e., on $f_r^n(\sqrt{-r})$ (resp. $f_r^n(-\sqrt{-r})$), and the images of the maps $G_\mathbb{Q}(\sqrt{-r}) \to \text{Sym}(f_r^n(±\sqrt{-r})) \cong S_n$ cannot lie in the alternating subgroup.
Proof. Let $K = \mathbb{Q}(\sqrt{-r})$, so that $f_r = (x + \sqrt{-r})(x - \sqrt{-r})$ over $K$. Let $g_1 = (x + \sqrt{-r})$ and $g_2 = (x - \sqrt{-r})$. To apply part (2) of [7, Theorem 1.1], we must show that for $i = 1, 2$, $g_i(f_r^{n-1}(0))$ is a non-square in $K$ for all $n \geq 2$. But $g_i(f_r^{n-1}(0)) = f_r^{n-1}(0) \pm \sqrt{-r}$. As in the final part of the proof of Lemma 3.1, $f_r^{n-1}(0) \pm \sqrt{-r}$ is a square in $K$ if and only if $(f_r^{n-1}(0) \pm \sqrt{f_r^{n}(0)})/2$ is a square in $\mathbb{Q}$, which in turn is equivalent to $(a_{n-1} + \sqrt{\alpha_n})/2$ being a square in $\mathbb{Q}$. But by assumption $(a_{n-1} + \sqrt{\alpha_n})/2$ is not a square in $\mathbb{Q}$, and so we may apply part (2) of [7, Theorem 1.1] twice to show

\begin{equation}
0 = \lim_{B \to \infty} \frac{\# \{ p \in S : N(p) \leq B \}}{\# \{ p : N(p) \leq B \}},
\end{equation}

where $N(p)$ is the norm of the ideal $p$ and $S$ is the set of primes $p$ in the ring of integers $\mathcal{O}_K$ of $K$ that divide $g_i(f_r^{n-1}(t))$ for at least one value of $i \in \{1, 2\}$ and $n \geq 2$.

If we exclude the finite set of ramified primes, then the primes $p$ in $\mathcal{O}_K$ come in two flavors: those with norm $p$, where necessarily $p$ splits in $\mathcal{O}_K$; and those with norm $p^2$, where necessarily $p$ is inert in $\mathcal{O}_K$. Note that the set \{ $p^2 : p \leq B$ \} has asymptotic density zero in the set \{ $p : p \leq B$ \}, and so (6.2) is equivalent to

\begin{equation}
0 = \lim_{B \to \infty} \frac{\# \{ p \in S : N(p) = p \leq B \} }{\# \{ p : N(p) = p \leq B \} }.
\end{equation}

Suppose $p$ in $S$, and say $p | g_i(f_r^{n-1}(t))$ for $n \geq 2$. Then $N(p) | N_{K/\mathbb{Q}}(g_i(f_r^{n-1}(t))) = f_r^{n}(t)$, where $N_{K/\mathbb{Q}}$ is the usual field norm. Let $p = \mathbb{Z} \cap \mathcal{O}_K$ be the prime lying below $p$. Note that $N(p) = p$ if $p$ splits in $\mathcal{O}_K$, i.e. if $-r$ is a quadratic residue modulo $p$, and $N(p) = p^2$ otherwise. But $0 \equiv f_r(f_r^{n-1}(t)) \equiv (f_r^{n-1}(t))^2 + r \pmod{p}$ and hence $-r$ must be a quadratic residue modulo $p$. Thus $N(p) = p$. It follows that the numerator of (6.3) is $2\# \{ p : p \leq B$ and $p$ divides $O_f(t) \}$. Clearly the denominator is $2\# \{ p : p \leq B$ and $-r$ is a quadratic residue modulo $p \}$. But by quadratic reciprocity and Dirichlet’s theorem on primes in arithmetic progressions, the latter is asymptotic to $\# \{ p : p \leq B \}$. It follows that $D(\{ p : p$ divides $O_f(t) \}) = 0$, as desired. \qed

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