Light propagation in non-trivial QED vacua

Holger Gies* and Walter Dittrich

Institut für theoretische Physik
Universität Tübingen

Auf der Morgenstelle 14, 72076 Tübingen, Germany

I. INTRODUCTION

The vacuum of classical electrodynamics is defined by the absence of charged matter: \( j^\mu = 0 \). Since the theory is linear, this vacuum is unique and trivial. As an immediate consequence, the light cone condition which characterizes the propagation of light also is uniquely determined and trivial: \( k^2 = 0 \).

Considering the (more) fundamental theory of quantum electrodynamics, we can only demand the absence of external currents \( j^\mu_E = 0 \). But non-charge-like modifications of the vacuum such as external fields, gravitation, temperature or non-trivial topology can influence its ubiquitous quantum fluctuations by acting on the properties of the fluctuating fields (charges, masses, . . . ). In general, the expectation value of the electromagnetic current \( \langle j^\mu \rangle \) will hence not vanish if it is taken with respect to a vacuum that is exposed to a modification \( z \). According to Schwinger [1], we can calculate the induced current to one-loop order with the aid of the formula

\[
\langle j^\mu \rangle_z = -ie \text{tr} \gamma^\mu G(x, x; z),
\]

where \( G(x, x; z) \) denotes the Green’s function (electron propagator) in presence of the modification \( z \) evaluated at the same space-time point \( x \) (closed loop). Since propagating light may couple to this current, the light cone condition alters \( k^2 \neq 0 \) and the vacuum is called non-trivial. Light velocity shifts of this kind were found by, e.g., Adler [2], Brezin and Itzykson [3] for magnetic fields, by Drummond and Hathrell [4] for gravitation, and by Scharnhorst [5] and Barton [6] for a Casimir configuration. Further important examples are studied in refs. [7–10].

In this letter, we investigate a general approach to obtain the light cone condition for a class of non-trivial QED vacua. A first unifying result has been given by Latorre, Pascual and Tarrach [9], who identified the so-called “unified formula” for the velocity shift

\[
\delta v = -\frac{44 \alpha^2}{135 m^4} u,
\]

where \( m \) denotes the electron mass, \( u \) the (renormalized) background energy density and \( \alpha \simeq 1/137 \). Indeed, eq. (2) perfectly describes the polarization and direction averaged findings of refs. [2–10] in the low-energy domain. (For a gravitational background, one \( \alpha \) has to be replaced by the combination \( (G_N m^2) \) involving Newton’s constant). In the case of gravitation, Shore [11] proved a polarization sum rule that represents a generalization of eq. (2). Additionally, he studied the case of weak electromagnetic fields in a consistently covariant manner (cf. the electromagnetic birefringent part of ref. [4]).

In the present work, we address the question of whether a generalization to arbitrary vacua is possible. We especially aim at the inclusion of the high-energy domain where eq. (2) obviously fails, since the energy density is naturally unbounded from above.

II. LIGHT CONE CONDITION

Instead of calculating the current expectation value eq. (1), we make use of the effective action approach. Once the high-energy degrees of freedom of the full quantum theory are integrated out, the effective action by definition describes the non-trivial vacuum as a classical medium. The following assumptions are essential for the formalism and classify the type of vacuum under consideration:

1) The vacuum modification is homogeneous in space and time (or at least slowly varying compared to the wavelength \( \lambda \) of the propagating light \( f^{\mu\nu} \)).
2) The wavelength of the propagating light is large compared to the Compton wavelength (soft photon approximation): \( \omega/m \ll 1 \).
3) Vacuum modifications caused by the propagating light \( f^{\mu\nu} \) itself are negligible.

*E-mail address: holger.gies@uni-tuebingen.de
4) Vacuum modifications (≠ EM fields) behave passively towards EM fields.

Referring to assumption 1) and 2), we can neglect any derivative term of the field strength in the effective Lagrangian. In particular, assumption 1) is equivalent to the geometric optics approximation and assumption 2) excludes dispersive effects from the formalism.

Assumption 3) justifies a linearization of the field equations with respect to $f^{\mu \nu}$.

The meaning and necessity of assumption 4) will be explained when we resort to it. Note that we demand neither a low-energy modification of the vacuum nor that the deviation from the Maxwell action be small. Besides, it is understood that we only take into account the real part of the effective action, assuming that the vacuum is sufficiently stable.

First, we consider a vacuum that is purely modified by an electromagnetic background field. Restricted by assumptions 1) and 2), the effective Lagrangian can only depend on the two Lorentz and gauge invariants of the Maxwell field

$$\mathcal{L} = \mathcal{L}(x, y),$$

where we introduced the linearly independent invariants

$$x := \frac{1}{2} F_{\mu \nu} F^{\mu \nu} = \frac{1}{2}(B^2 - E^2)$$

$$y := \frac{1}{2} F_{\mu \nu} \star F^{\mu \nu} = E \cdot B.$$ 

The field strength and its dual are defined as usual:

$$F^{\mu \nu} = \partial \mu A^\nu - \partial \nu A^\mu$$

$$*F^{\mu \nu} = \frac{1}{2} \varepsilon^{\mu \nu \alpha \beta} F_{\alpha \beta}.$$ 

We obtain the field equations from $\mathcal{L}$ by variation

$$0 = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} \right) = \partial_\mu \left( \partial_x \mathcal{L} F^{\mu \nu} + \partial_y \mathcal{L} *F^{\mu \nu} \right) = (\partial_x \mathcal{L}) \partial_\mu F^{\mu \nu} + \left( \frac{1}{2} M^{\mu \nu}_{\alpha \beta} \right) \partial_\mu F^{\alpha \beta},$$

where $\partial_x, \partial_y$ denote the partial derivatives with respect to the field strength invariants $\mathcal{L}$ and $M^{\mu \nu}_{\alpha \beta}$ is given by

$$M^{\mu \nu}_{\alpha \beta} := F^{\mu \nu} F_{\alpha \beta} (\partial_x^2 \mathcal{L}) + *F^{\mu \nu} *F_{\alpha \beta} (\partial_y^2 \mathcal{L}) + \partial_{xy} \mathcal{L} \left( F^{\mu \nu} *F_{\alpha \beta} + *F^{\mu \nu} F_{\alpha \beta} \right).$$

To arrive at the desired light cone condition in the spirit of Shore’s covariant formalism, we make use of the following five pieces of information:

(i) We split up the field strength into its constant (or slowly varying) background part and the propagating part $F^{\mu \nu} \rightarrow F^{\mu \nu} + f^{\mu \nu}$, and linearize with respect to $f^{\mu \nu}$ (assumption 3)).

(ii) In Fourier space, $f^{\mu \nu}$ can be written in terms of the polarization vector $\epsilon^{\mu}$: $f^{\mu \nu} \propto (k^\mu \epsilon^\nu - k^\nu \epsilon^\mu)$. Without loss of generality, we choose the Lorentz gauge $\partial_\mu \epsilon^\mu = 0$.

(iii) The average over polarization states can easily be taken with the aid of the well-known rule $\sum_{\rho \delta} \epsilon^\rho \epsilon^\nu \rightarrow g^{\rho \nu}$, where we use the metric $g = (-, +, +, +)$. Note that the additional terms on the RHS vanish by the antisymmetry properties of $M^{\mu \nu}_{\alpha \beta}$.

Intermediately, the field equation (8) yields

$$0 = 2(\partial_x \mathcal{L}) k^2 + M^{\mu \nu}_{\alpha \beta} k_\mu k_\nu.$$ 

(iv) Using the fundamental algebraic relations of the field strength tensor and its dual

$$F^{\mu \alpha} F^\alpha_\nu - *F^{\mu \alpha} *F^\alpha_\nu = 2 x g^{\mu \nu},$$

$$F^{\mu \alpha} F^\alpha_\nu = *F^{\mu \alpha} F^\alpha_\nu = y g^{\mu \nu},$$

the Lorentz structure of $M^{\mu \nu}_{\alpha \beta}$ can be decomposed into one term proportional to $\delta_\mu^\alpha$ and another proportional to the Maxwell energy-momentum tensor

$$T^\mu_\nu = F^{\mu \nu} F_{\alpha \beta} - x \delta^\mu_\alpha.$$ 

(v) Only the vacuum expectation value of the energy-momentum tensor is physically meaningful and can be defined by a variation with respect to the metric

$$\langle T^{\mu \nu} \rangle_{xy} = -T^{\mu \nu}(\partial_x \mathcal{L}) + g^{\mu \nu} (\mathcal{L} - x \partial_x \mathcal{L} - y \partial_y \mathcal{L}).$$

Solving eq. (14) for $T^{\mu \nu}$ and inserting into eq. (10), we can present $M^{\mu \nu}_{\alpha \beta}$ in its final shape

$$M^{\mu \nu}_{\alpha \beta} = 2 \left[ -\frac{1}{2} (\partial_x^2 + \partial_y^2) \langle T^\mu_\nu \rangle_{xy} + \delta^\mu_\alpha \left( \frac{1}{2} (\partial_x^2 - \partial_y^2) \mathcal{L} \right) + y \partial_{xy} \mathcal{L} + \frac{1}{2} (\partial_x^2 + \partial_y^2) \langle \mathcal{L} - x \partial_x \mathcal{L} - y \partial_y \mathcal{L} \rangle \right].$$

Substituting $M^{\mu \nu}_{\alpha \beta}$ into eq. (8), we arrive at the desired light cone condition for EM field-modified vacua fulfilling the above-mentioned assumptions.

$$k^2 = Q \langle T^{\mu \nu} \rangle_{xy} k_\mu k_\nu,$$

where

$$Q = \frac{1}{\left( \partial_x \mathcal{L} + (\partial_x \mathcal{L}) \left( \frac{1}{2} (\partial_x^2 - \partial_y^2) + y \partial_{xy} \mathcal{L} + \frac{1}{2} (\partial_x^2 + \partial_y^2) \langle \mathcal{L} - x \partial_x \mathcal{L} - y \partial_y \mathcal{L} \rangle \right) \right)}.\]
vacuum modulations parametrized by the (collective) label \( z \)

\[
k^2 = z \langle 0 | Q (T^{\mu \nu})_{xy} | 0 \rangle z k_{\mu} k_{\nu} = \sum_i z \langle 0 | Q | i \rangle z z \langle i | (T^{\mu \nu})_{xy} | 0 \rangle z k_{\mu} k_{\nu}, \tag{15}\]

where we inserted a complete set of intermediate states in the last line. Falling back on assumption 4), we make use of the fact that the vacuum should behave passively with respect to EM fields. In other words, once in the vacuum state of \( z \), switching on EM fields should not alter the modification \( z \). This prohibition of a backreaction leads to

\[
\langle Q \rangle_z = \langle Q(\mathcal{L}(x, y)) \rangle_z = Q(\mathcal{L}(x, y; z)). \tag{16}\]

Evaluating the expectation value of \( Q \) that is functionally dependent on \( \mathcal{L}(x, y) \), leads back to the definition of \( \mathcal{L} \) via the functional integral over the fluctuating fields. E.g., if the modification \( z \) imposes boundary conditions on the fields, the functional integral has to be taken over the fields which fulfill these boundary conditions. This defines the new effective Lagrangian characterizing the complete non-trivial vacuum

\[
\langle Q \rangle_z = \langle Q(\mathcal{L}(x, y)) \rangle_z = Q(\mathcal{L}(x, y; z)). \tag{17}\]

We finally end up with the light cone condition for a class of non-trivial vacua reconcilable with the above-mentioned assumptions

\[
k^2 = Q(x, y, z) \langle T^{\mu \nu} \rangle_{xyz} k_{\mu} k_{\nu}. \tag{18}\]

As an exact statement in the sense of effective action theories, the validity of eq.(18) is not restricted to perturbation theory.

Further useful representations of eq.(18) are obtained by choosing a certain reference frame and introducing

\[
\tilde{k}^\mu = \frac{k^\mu}{|k|} = \left( \frac{k^0}{|k|}, \frac{k}{|k|} \right) =: (v, \hat{k}). \tag{19}\]

Here we defined the phase velocity by \( v := k^0/|k| \). For eq.(18), we obtain

\[
v^2 = 1 - Q \langle T^{00} \rangle \hat{k}_\mu \hat{k}_\nu. \tag{20}\]

Averaging over propagation directions, the light cone condition yields

\[
v^2 = 1 - \frac{4}{3} Q (T^{00}) = 1 - \frac{4}{3} Q u, \tag{21}\]

where we assumed that \( Q(T^{00}) \ll 1 \) and \( u \) denotes the (renormalized) energy density of the modified vacuum. These representations indicate that the light cone condition is a generalization of the “unified formula” of Latorre, Pascual and Tarrach [3]. However, the Q-factor generally depends on all the variables and parameters of \( \mathcal{L} \) and we will have to verify that its low-energy value approaches the constant pre-factor of eq.(3).

### III. APPLICATIONS TO THE LIGHT CONE CONDITION

Up to now, our formalism has only worked in a simple manner at the expense of dealing with effective Lagrangians. Of course, the calculation of the latter generally lacks simplicity. Indeed, we will only deal with effective Lagrangians of first-order perturbation theory, but the implementation of higher-loop corrections is directly controlled by the effective action approach. Due to the perturbative properties of QED, first-order effective Lagrangians are appropriately characterized by

\[
\mathcal{L} = \mathcal{L}_M + \mathcal{L}_c; \quad \frac{\mathcal{L}_c}{\mathcal{L}_M} \ll 1, \tag{22}\]

where \( \mathcal{L}_M = -x \) denotes the Maxwell Lagrangian and \( \mathcal{L}_c \) contains the correction terms. For this class of Lagrangians, the denominator of the Q-factor in eq.(14) simplifies to

\[
\text{denom.}(Q) = 1 + \mathcal{O}(\mathcal{L}_c), \tag{23}\]

and the approximation \( Q = \frac{1}{2} (\partial^2_x + \partial^2_y) \mathcal{L} \) is justified.

#### A. Weak EM Fields

The weak field limit of the one-loop effective Lagrangians of QED, i.e., the Heisenberg-Euler Lagrangian, is given by

\[
\mathcal{L} = -x + c_1 x^2 + c_2 y^2, \tag{24}\]

where

\[
c_1 = \frac{8\alpha^2}{45m^4}, \quad c_2 = \frac{14\alpha^2}{45m^4}. \tag{25}\]

Employing eq.(24), we immediately find the propagation and direction averaged velocity

\[
Q = c_1 + c_2, \quad \Rightarrow \quad v = 1 - \frac{4\alpha^2}{135m^4} \left[ \frac{1}{2} (E^2 + B^2) \right]. \tag{26}\]

Here, we have rediscovered the “universal constant” of the “unified formula”, eq.(3). Since all of the low-energy calculations are indeed based on the Heisenberg-Euler Lagrangian, the seeming universality of this factor is not astonishing. As will be pointed out in the following, this combination of constants only represents a natural low-energy limit of the respective modifications. Of course, higher-loop corrections to the Heisenberg-Euler Lagrangian will also modify this factor.
B. Strong Magnetic Fields

The vacuum modified by EM fields of arbitrary strength consistent with the one-loop approximation is described by Schwinger’s famous Lagrangian \[ \mathcal{L}_0 = -\frac{1}{8\pi^2} \int_0^{s_0} ds e^{-m^2 s(1+|y|\cot(\sqrt{x^2+y^2+x}^2))} \times \cot(\sqrt{x^2+y^2-x}^2) - \frac{2}{3}(m^2)^2 x - 1. \]

(28)

The convergence is implicitly insured by the prescription \( m^2 \rightarrow m^2 - \delta \). For purely magnetic fields, i.e., on the positive \( x \)-axis in field space, we find the \( Q \)-factor by differentiation

\[ Q(h) = -\frac{1}{2\alpha^2} \int_0^{s_0} \frac{dz}{z} e^{-2h(z)} \left[ \frac{z \coth z - 1}{\sinh^2 z} - \frac{1}{3} \frac{z \coth z}{z} \right], \]

(29)

where we introduced the convenient dimensionless parameter \( h := \frac{-2e\alpha}{m^2} \). With some effort, the evaluation of the integral is analytically achievable. Details will be given in a forthcoming paper [12]. Our findings are

\[ Q(h) = \frac{1}{2B^2} \left[ 2h^2 - \frac{h}{3} \right] \Psi(1+h) - h - 2h - 4h \ln \Gamma(h) + 2h \ln 2\pi + \frac{1}{3} \left[ 4\zeta'(1+h) + \frac{1}{6h} \right]. \]

(30)

where \( \Psi \) denotes the logarithmic derivative of the \( \Gamma \)-function and \( \zeta' \) is the first derivative of the Hurwitz Zeta function with respect to the first argument.

The zero-field limit coincides with eq. (20). For strong fields, the last term of eq. (30) \( \propto \frac{1}{t^2} \propto B \) dominates the expression in the square brackets. Hence, the \( Q \)-factor decreases with

\[ Q(B) \approx \frac{1}{6\pi^2} \frac{\alpha}{B} \frac{1}{B}, \text{ for } B \rightarrow \infty. \]

(31)

Since the loop-corrections to \( \langle T^{\mu \nu} \rangle \) are of higher order and can be neglected, the velocity of propagating light will be modified according to

\[ v^2 = 1 - \frac{\alpha \sin^2 \theta}{\pi^2} \left[ 2h^2 - \frac{h}{3} \right] \Psi(1+h) - 4h \ln \Gamma(h) - 3h^2 - h + 2h \ln 2\pi + \frac{1}{3} + 4\zeta'(1+h) + \frac{1}{6h}. \]

(32)

One can show [12] that eq. (32) coincides with the findings of Tsai and Erber [10]. Although the velocity shift increases proportional to the magnetic field for large \( B \), (last term \( \propto \frac{1}{t^2} \) in eq. (22)), its total amount remains comparably small

\[ \delta v \approx 9.58 \cdot 10^{-5} \text{ at } B = B_{cr} = \frac{m^2}{e}. \]

(33)

for strong \( B \)-fields consistent with the one-loop approximation, i.e., \( \frac{B}{B_{cr}} < \frac{\alpha}{\pi} \approx 430 \). In order to let \( Q(T_{\mu \nu}) \bar{k}_\mu \bar{k}_\nu \) be bounded, we expect that higher loop corrections promote the decrease of \( Q(B) \).

C. Finite Temperature

An effective Lagrangian describing a finite temperature vacuum state can always be decomposed into its temperature-independent \( (T = 0) \) part and its thermal correction

\[ \mathcal{L}(T) = \mathcal{L}(T = 0) + \Delta \mathcal{L}(T), \]

(34)

whereby \( \mathcal{L}(T = 0) \) denotes the usual zero-temperature Lagrangian, e.g., eq. (28). In a similar manner, we can decompose the \( Q \)-factor

\[ Q(T) = Q(T = 0) + \Delta Q(T) \]

\[ = \frac{1}{2} \left[ (\partial_x^2 + \partial_y^2) \mathcal{L}(T = 0) + (\partial_x^2 + \partial_y^2) \Delta \mathcal{L}(T) \right]. \]

(35)

Including a magnetic background field, \( \Delta \mathcal{L}(B, T) \) was calculated by Dittrich [13]

\[ \Delta \mathcal{L}(B, T) = -\frac{\sqrt{\pi}}{4\pi^2} \int_0^{s_0} \frac{ds}{s^2} e^{-m^2 s \cot xB} \cot yB \times T \left[ \Theta_2(0, 4\pi i sT^2) - \frac{1}{2T \sqrt{\pi s}} \right], \]

where \( \Theta_2 \) denotes the second Jacobi \( \Theta \)-function [14].

Here, we encounter a subtlety of the formalism. The \( Q \)-factor is evaluated by deriving \( \mathcal{L} \) with respect to \( x \) and \( y \). However, the finite-temperature formalism demands that the electric field vanishes in order to fulfill the principle of thermal equilibrium. Hence, we first have to preserve the (unphysical) \( x \) and \( y \) dependence of the thermal Lagrangian, then perform the differentiation and, subsequently, set \( E = 0 \). In addition, the vanishing of the electric field is required for the vacuum in order to remain passive according to assumption 4. The appropriate expression is simply obtained by replacing

\[ \mathcal{L}(T = 0) + \Delta \mathcal{L}(T) \]

by the gauge and Lorentz invariant terms

\[ (ces)^2 |y| \cot(ces (\sqrt{x^2+y^2}), ) \cot(ces (\sqrt{x^2+y^2} - x)^2) \]

(37a)

in analogy to eq. (28). Executing the differentiation and setting \( E = 0 \), the thermal \( Q \)-factor reads
\[
\Delta Q(B,T) = -\frac{\alpha}{\pi B^2} \int_0^{\infty} \frac{ds}{s} e^{-s} \left[ \frac{esB \coth esB - 1}{\sinh^2 esB} \frac{esB}{3 \coth esB} \right] \\
= -\frac{\alpha}{\pi B^2} \int_0^{\infty} \frac{ds}{s} e^{-s} \sum_{n=1}^{\infty} e^{-\alpha n} e^{-\frac{esB^2}{s}}.
\]

An analytic evaluation of eq.(38) is only possible for certain limiting cases. First, we consider a purely temperature-modified vacuum with vanishing field strength. The proper-time integration then yields

\[
\Delta Q(B=0,T) = \frac{22\alpha^2}{45 m^4} \sum_{n=1}^{\infty} (-1)^n \left( \frac{m}{T} n \right)^2 K_2(\sqrt{T} n),
\]

where \( K_2 \) denotes a modified Bessel function and reflects the \( S^1 \times \mathbb{R}^3 \) topology of the finite-temperature coordinate space.

The low-temperature limit is easily obtained, since the Bessel function is exponentially damped for large values of its argument. Thus, only the first term of the sum in eq.(38) is important and we find

\[
\Delta Q(B=0,T \to 0) \simeq -\frac{22\alpha^2}{45 m^4} \sqrt{\frac{\pi}{2}} \left( \frac{m}{T} \right)^{\frac{3}{2}} e^{-\frac{\pi}{2}} \to 0. 
\]

Here, we find the substantiation of why the \( Q \)-factor of the low-temperature velocity shift is simply given by its constant value at the origin in field space \( Q(B=0,T=0) = c_1 + c_2 \) eq.(29). Finite-temperature corrections vanish exponentially in the low-temperature limit.

The same conclusion holds for the parallel-plate Casimir configuration, the so-called Scharnhorst effect, due to the similarities of both effects concerning periodic boundary conditions: the corrections to the “universal” constant of eq.(2) regarding this Casimir vacuum vanish exponentially with the plate separation \( a \). In this sense, the derivation of the Scharnhorst velocity shift is a trivial application of our light cone condition.

Nevertheless, \( Q \) ceases to remain constant if we move noticeably away from the origin in field/parameter space. The high-temperature limit of the thermal correction eq.(29) to the \( Q \)-factor serves as an example. In this limit, the complete infinite sum has to be taken into account. With some effort, one obtains

\[
\Delta Q(T \gg m) = -\frac{22\alpha^2}{45 m^4} \left[ 1 - \frac{k_1 m^4}{4 T^4} + \mathcal{O}\left( \frac{m^6}{T^6} \right) \right],
\]

where \( k_1 = 0.123749077470 \ldots \) = const.. We immediately find the complete \( Q \)-factor

\[
Q(T \gg m) = Q(T = 0) + \Delta Q(T \gg m) = \frac{11}{90} k_1 \frac{\alpha^2}{T^4} + \mathcal{O}\left( \frac{m^2}{T^2} \right),
\]

which exhibits a rapid decrease \( \propto 1/T^4 \). Numerical analyses show that eq.(12) is already valid below the \( e^+e^- \) threshold where \( T/m < 1 \).

With the aid of the finite-temperature VEV of the energy-momentum tensor

\[
\langle T_{\mu\nu} \rangle_T = \frac{\pi^2}{90} (N_B + \frac{\pi}{3} N_F) T^4 \text{diag}(3,1,1,1),
\]

where the integer variables \( N_B \) and \( N_F \) denote the number of bosonic and fermionic degrees of freedom at a given temperature, we recover the well-known low-temperature result \[1\]

\[
v = 1 - \frac{44\pi^2}{2025} a^2 \frac{T^4}{m^2},
\]

where we used \( N_B = 2, N_F = 0 \), and eqs.(21,41,43). In the high-temperature limit, the velocity of soft photons moving in a photon and ultra-relativistic \( e^+e^- \) gas, i.e., \( N_B = 2 \) and \( N_F = 4 \), is found by employing eq.(12)

\[
v = 1 - \frac{121}{8100} k_2 \pi^2 a^2 + \mathcal{O}\left( \frac{m^2}{T^2} \right)
\]

\[
= 1 - 9.72 \ldots \times 10^{-7} + \mathcal{O}\left( \frac{m^2}{T^2} \right) = \text{const.} + \mathcal{O}\left( \frac{m^2}{T^2} \right).
\]

Starting with an increase proportional to \( T^4 \) for low temperature, the velocity shift approaches a constant value in the high-temperature limit. The magnitude of this constant indicates its subdominant importance even in extremely hot surroundings (early universe).

**D. Finite Temperature and Magnetic Fields**

A non-trivial interplay of the various effects is found in the domain of strong fields in hot surroundings. In any other limiting case, thermal phenomena decouple from the electromagnetic ones. Therefore, we evaluate the \( Q \)-factor eq.(28) in the limit \( B \gg B_{cr} \)

\[
\Delta Q(T,B \gg B_{cr}) = -\frac{\alpha}{3\pi B^2} \frac{B}{B_{cr}} \sum_{n=1}^{\infty} (-1)^n \frac{m}{T} n K_1(\sqrt{T} n).
\]

Subsequently, we take the limit \( T/m \gg 1 \) and find

\[
\Delta Q(T,B) = -\frac{\alpha}{6\pi B^2} \frac{B}{B_{cr}} + \frac{\alpha}{6\pi} k_2 e + \mathcal{O}\left( \frac{m^2}{B T^2} \right)
\]

where \( k_2 = 0.213139199408 \ldots \) = const.. Again, the first term in this expansion exactly cancels the zero-temperature strong-field contribution to \( Q \) found in eq.(31) and we obtain for the complete \( Q \)-factor
where we have introduced the convenient dimensionless variables $\tilde{B} = \frac{B}{m^2}$ and $\tilde{T} = \frac{T}{m}$ which satisfy $\tilde{B}, \tilde{T} \gg 1$. Note that the energy density $\langle T^{00} \rangle$ consists of three contributions: a thermal (eq.\((43)\)), a magnetic (\(\frac{1}{2}B^2\)) and a mixed part that can be obtained from the high-temperature limit of eq.\((48)\). Finally, we arrive at the polarization and propagation direction averaged light velocity for a strong $B$-field and high-temperature modified vacuum

\[
v = 1 - \frac{11\pi^2}{135} k_2 \alpha^2 \frac{\tilde{T}^2}{\tilde{B}} - \frac{k_2}{18\pi} \frac{\tilde{B}}{\tilde{T}^2} \frac{\alpha}{2\tilde{r}} \alpha^2. \tag{49}
\]

One finds a minimal velocity shift at $\tilde{T}^2/\tilde{B} \simeq 1.74$ where $|\delta v| \simeq 3.20 \cdot 10^{-5}$. This magnitude corresponds to the velocity shifts we typically found in the present work for strong fields consistent with the one-loop approximation, e.g., in eq.\((33)\).

### IV. CONCLUSIONS

In this letter, we investigated light propagation in non-trivial QED vacua in the geometric optics approximation. For a certain class of vacua fulfilling moderate assumptions, we derived the light cone condition averaged over polarization states which represents a generalization of the "unified formula" found by Latorre, Pascual and Tarrach [1]. The latter was identified as the low-energy limit of our light cone condition. Especially, the "universal constant" of eq.\((33)\) turned out to be a derivative combination of the Lagrangian evaluated at the origin in field/parameter space.

Within this conceptual framework, we calculated the velocity shifts induced by various vacuum modifications. In the low-energy limit, we recovered the well-known results that were already perfectly described by the unified formula. As a high-energy example, we reproduced the findings of Tsai and Erber [10] for intense magnetic fields using our comparably simple formalism.

In the high-temperature limit, we observed that the velocity shift approaches a constant but comparably small value in contrast to the rapid increase $\propto T^4$ for low temperature. Of course, this result is of minor importance from an experimental viewpoint. However, on the one hand, it at least confirms the consistency of the formalism, and on the other hand, as a non-trivial prediction of QED, it also indicates that the theory perfectly controls the velocity shifts. In the latter sense, this result confirms the consistency of QED.

The only formally unbounded velocity shifts were discovered in the limit of strong fields at zero as well as high temperature. With regard to the validity of the loop expansion, we expect higher-order or even non-perturbative Lagrangians to cure this problem. Then the light cone condition might serve as an indicator of consistency.

Since the calculated velocities are phase and group velocities of soft photons, i.e., in the zero-frequency limit, the whole formalism is not immediately appropriate for discussing questions of causality that arise from the possibility of positive velocity shifts, ($v > 1$ for negative energy densities, e.g., Casimir vacua); the maximal signal velocity, namely, is evaluated in the infinite frequency limit. For indirect predictions, the reader is referred to refs. [11]. The question of causality is extensively discussed in ref. [1]. In concordance with these authors, let us just say that we find no grounds for a violation of the causal structure of space time by quantum vacuum effects.

---

[1] J. Schwinger, Phys. Rev. 82, 664 (1951).
[2] S.L. Adler, Ann. Phys. (N.Y.) 67, 599 (1971).
[3] E. Brezin and C. Itzykson, Phys. Rev. D 3, 618 (1971).
[4] I.T. Drummond and S.J. Hathrell, Phys. Rev. D 22, 343 (1980).
[5] K. Scharnhorst, Phys. Lett. B 236, 354 (1990).
[6] G. Barton, Phys. Lett. B 237, 559 1990.
[7] R.D. Daniels and G.M. Shore, Nucl. Phys. B 425, 634 (1994).
[8] R.D. Daniels and G.M. Shore, Phys. Lett. B 367, 75 (1996).
[9] J.L. Latorre, P. Pascual and R. Tarrach, Nucl. Phys. B 437, 60 (1995).
[10] Wu-yang Tsai and T. Erber, Phys. Rev. D 12, 1132 (1975).
[11] G.M. Shore, Nucl. Phys. B 460, 379 (1996).
[12] W. Dittrich and H. Gies, in preparation
[13] W. Dittrich, Phys. Rev. D 19, 2385 (1979).
[14] I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series and Products, Academic Press (1965).
[15] K. Scharnhorst and G. Barton, J. Phys A 26, 2037 (1993).