The resolution and representation of time series in Banach space

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Abstract

We describe a new systematic procedure for resolution and representation of unit root processes on Banach space. Each unit root process is a marginally stable time series with a uniquely defined resolvent operator which is singular at unity. The proposed method uses infinite-length Jordan chains to find the key spectral projections which enable separation and solution of the fundamental equations for the Laurent series coefficients. We use the coefficients to find two distinct forms of the Granger-Johansen representation for the time series—a natural form, expressed as the sum of a series of negative powers and a series of nonnegative powers of backward differences, in complementary subspaces defined by the key projections and justified by additional assumptions on the resolvent—and an extended form requiring no additional assumptions, which avoids the potentially divergent series of nonnegative powers but preserves the natural spectral separation. The representations remain valid when the resolvent has an isolated essential singularity at unity.

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1. Introduction

A time series is a sequence of observed values indexed according to time in order of occurrence. The observed data is often stochastic in nature but is nevertheless used to make predictions about future values of the observed variables. Common examples of time series include regular observations of climate variables such as sunshine intensity and rainfall and regularly updated values of various social science metrics such as stock market indices. While the most basic time series are often sequences of real numbers it is now commonplace to consider vector-valued variables where several real numbers are observed at each time. More complex observations are also possible. Functional data analysis [37] treats observations as values of some continuous function. In such cases the corresponding observation at each time could be an infinite sequence of coefficients—such as a Fourier series—taking values in some abstract vector space. While the theoretical structures used to analyse time series are well developed for vectors in finite-dimensional Euclidean space the development of a theoretical basis for the resolution and representation of time series in infinite-dimensional space is an active area of research [9, 10, 16, 17, 21, 23, 41, 45].

An autoregressive moving average $ARMA(p, q)$ process of order $(p, q)$ is a time series where a linear combination of the current value $x(t)$ and the $p$ previous values $\{x(s)\}_{s=t-p}$ is modelled as a linear combination of the current value $n(t)$ and the $q$ previous values $\{n(s)\}_{s=t-q}$ of a white noise process. The order of an $ARMA$ process can be determined from the relevant autocorrelation functions. For further discussion see Box et al. [13] and Brockwell and Davis [14]. Brockwell and Davis recommend using the Akaike information criterion [1]. Alternatively one could use the closely related Bayesian information criterion [42]. Estimation of the $ARMA$ coefficients in Banach space can be accomplished using a linear minimum norm regression. Model fitting is not considered in this paper. Our main contribution is to propose and demonstrate a new method for resolution and representation of an $ARMA(p, q)$ unit root process on Banach space.

2. Contribution

We propose a systematic procedure for resolution and representation of an $ARMA(p, q)$ unit root process taking values in Banach space.

1. We apply a Maclaurin transform to an $ARMA(1, 1)$ unit root process in order to obtain a linear operator equation for the transformed time series. The operator is defined by the characteristic linear pencil for the time series and the corresponding resolvent operator is found by solving a doubly infinite system of fundamental equations to find the Laurent series coefficients. The resolvent operator is used to solve the linear operator equation and find a resolution of the transformed time series.

2. We use an inverse Maclaurin transform on the resolved transformed $ARMA(1, 1)$ time series to recover the desired representation and hence derive two different forms of a generalized Granger–Johansen representation [19, 26, 29, 35, 36].
We note that an ARMA$(p, q)$ process with bounded linear operator coefficients that maps one Banach space onto another can be reduced to an equivalent ARMA$(1, 1)$ process with augmented bounded linear operator coefficients that map one suitably defined Cartesian product Banach space onto another. We outline this reduction in a special case later in the paper.

A time series is said to be stable if the resolvent $R(z)$ of the characteristic polynomial is analytic on and inside the unit circle in the complex plane. Our particular interest is the solution of marginally stable unit root processes where $R(z)$ is analytic on a set $V_\epsilon = U_{1+\epsilon}(0) \setminus \{1\} = \{z \in \mathbb{C} \mid |z| < 1 + \epsilon \text{ with } z \neq 1\}$ for some $\epsilon \in \mathbb{R}$ with $\epsilon > 0$. We will use the notation $U_r(a) = \{z \in \mathbb{C} \mid |z-a| < r \leq \infty\}$ and $U_{s,r}(a) = \{z \in \mathbb{C} \mid 0 \leq s < |z-a| < r \leq \infty\}$ for $a \in \mathbb{C}$ and $r, s \in [0, \infty)$ throughout the paper. We assume that the resolvent has an isolated singularity at $z = 1$ which may be a finite-order pole or an essential singularity. Our proposed techniques are new and will include, extend and unify the current methods used in the literature.

We will use recently established theory to construct the resolvent operator $\mathcal{M}$ and so a key aim of our paper will be to explain the proposed method of analysis, and to present a collection of suitable examples that show how the new method can be applied to more general time series representation problems on Banach space. Many of the current methods are restricted to Euclidean space or Hilbert space for pencils where $R(z)$ has a finite-order pole at $z = 1$ or where calculation of $R(z)$ depends on the existence of various orthogonal decompositions. Other methods can only be applied in Banach space if certain direct sum decompositions are assumed. We will show that many of these restrictions can be removed by the systematic use of the natural non-orthogonal spectral separation projections for the resolvent.

### 3. Structure of the paper

In Section 4 we review the relevant literature. The main results are described briefly in Section 5. Section 6 is the main section. The Maclaurin transform and the transformed equation are explained in Section 6.1 and the resolvent operator for the characteristic linear pencil and the resolution of the transformed equation are discussed in Section 6.2. The backward difference operator is discussed in Section 6.3. A natural representation for the time series is established in Section 6.4 by assuming that the resolvent is analytic on a set $\mathcal{V}_\epsilon \cup \mathcal{W}_\theta$ for some $\epsilon > 0$ and some $\theta > 0$ where

$$\mathcal{W}_\theta = U_{0,1+\theta}(1) = \{z \in \mathbb{C} \mid 0 < |z - 1| < 1 + \theta\}.$$
The natural representation is defined in two distinct parts by the spectral separation projections using negative and nonnegative powers of the backward difference operator and can be interpreted as either a non-standard or standard form of the Granger–Johansen representation. In Section 6.5 we show that the natural representation obtained here for $I(1)$ and $I(2)$ processes is closely aligned to the representations obtained by Beare and Seo [10]. The natural representation is replaced in Section 6.6 by an extended form of the representation that requires only the minimal assumption that the resolvent is analytic on $\mathcal{V}_\epsilon$ for some $\epsilon > 0$. The extended representation no longer uses nonnegative powers of the backward difference operator but nevertheless can still be interpreted as a generalized form of the Granger–Johansen representation. In particular we show that the spectral separation of the two distinct parts is preserved. In Section 7 we present an informal discussion of the methodology used by Albrecht et al. [2, 4, 5] to justify the representation of the resolvent operator. The main results are illustrated for three specific applications in Section 8. In Section 9 we discuss augmented linear pencils and reduction of order for ARMA processes. In Section 10 we draw some brief conclusions. The necessary background material is outlined in Appendices A, B and C.

4. Literature review

The calculation of resolvent operators for linear pencils has a long history. In particular we cite key papers by Stummel [46], Langenhop [32, 33], Bart and Lay [8], Howlett et al. [28] and Albrecht et al. [2, 5]. See also [6, 18, 20, 22, 27, 31, 40, 43, 38, 47]. For a more extensive review of the literature on inversion of linear pencils see [2, 3, 4, 5]. Additional information can be found in the books by Gohberg et al. [24], Kato [30], Yosida [49], Campbell and Meyer [15] and Avrachenkov et al. [7].

We will limit our detailed review of the time series literature to the most relevant recent papers. For a full review of the literature on unit root processes we refer to Beare and Seo [9, 10], Franchi and Paruolo [21, 23] and Seo [41] and references therein. For a more general view and some applications see Chang et al. [16, 17]. The paper by Spangenberg [45] does not consider unit root processes but does discuss stationarity for ARMA processes with values in Banach space and contains some interesting examples. See also the text by Bosq [12] for the background theory of linear stochastic processes. Our main interest is the Granger–Johansen representation—a multivariate extension of the earlier Beveridge–Nelson representation [11, 36, 44]. The papers by Hansen [25, 26] provide a nice introduction to the Granger–Johansen representation. We quote from [25].

The [Granger–Johansen] representation theorem states that a co-integrated vector autoregressive process can be decomposed into four components: a random walk, a stationary process, a deterministic part, and a term that depends on the initial conditions.

The paper by Beare and Seo [10] is the main focus of our review. Let $(\Omega, \Sigma, \mu)$ be a probability space and let $H$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Beare
and Seo consider an autoregressive AR($p$) process where $p \in \mathbb{N}$ and \{x(\omega, t)\}_{t \in \mathbb{N}} \in H^N$ is a random time series defined by an equation

$$x(t) = \sum_{j=1}^{p} \Phi_j x(t - j) + \epsilon(t)$$

for $t \in \mathbb{N}$ and the initial values $x(-p + 1), \ldots, x(0)$. The coefficients $\Phi_j \in \mathcal{B}(H)$ are compact bounded linear operators for each $j = 1, \ldots, p$ and \{\epsilon(\omega, t)\}_{t \in \mathbb{Z}} \in H^Z$ is a time series of independent identically distributed (i.i.d.) random variables with finite variance

$$\sigma^2 = \int_{\Omega} ||\epsilon(\omega, t)||^2 d\mu(\omega) < \infty$$

and zero mean

$$\mu = \int_{\Omega} \epsilon(\omega, t) d\mu(\omega) = 0$$

and with positive definite covariance $\Sigma \in \mathcal{B}(H)$ defined by

$$\Sigma(u) = \int_{\Omega} \langle u, \epsilon(\omega, t) \rangle \epsilon(\omega, t) d\mu(\omega)$$

for each fixed $u \in H$ and each $t \in \mathbb{Z}$. The characteristic function for the process is a polynomial of degree $p$ defined by

$$\Phi(z) = I - \sum_{j=1}^{p} \Phi_j z^j$$

for all $z \in \mathbb{C}$. Beare and Seo assume that the operator polynomial $\Phi(z)$ is a nonsingular mapping on the closed unit disk centred at $z = 0$ except at $z = 1$. It follows that the resolvent operator $\Phi(z)^{-1}$ is analytic on the closed unit disk except for an isolated singularity at $z = 1$. The analytic Fredholm theorem \cite[pp 203–204]{24} shows that for some $\epsilon > 0$ the resolvent operator can be written in the form

$$\Phi(z)^{-1} = \sum_{k \in \mathbb{N} - 1 - d} \Upsilon_k (z - 1)^k$$

for all $z \in U_{0, \epsilon}(1)$ and some $d \in \mathbb{N}$, where $\Upsilon_k \in \mathcal{B}(H)$ is a finite rank operator for each $k = -d, \ldots, -1$ and $\Upsilon_d \neq 0$.

If $d = 1$ let $H_1 = \Phi(1)(H)$ denote the range space and let $K_1 = [\Phi(1)^{-1}]^{-1}([0])$ denote the null space of $\Phi(1) \in \mathcal{B}(H)$. Note that $\dim(H_1^1) = \dim(K_1) < \infty$. Beare and Seo show \cite[Theorem 3.1]{10} that $\Upsilon_{-1}(H) = K_1$ and that

$$x(t) = z_0 - \Upsilon_{-1} \left( \sum_{s=1}^{t} \epsilon(s) \right) + \sum_{\ell \in \mathbb{N} - 1} \Upsilon_\ell (\epsilon(t - \ell)) / \ell!$$

for all $t \in \mathbb{N}$ where $z_0 = z_0(\omega) \in H$ is a random variable with mean zero and square integrable norm. If $z_0 \in K_1$ they conclude that for fixed $u \in H$ the time series \{\langle u, x(t) \rangle\}_{t \in \mathbb{N}}$ is $I(0)$ if $u \in K_1^1$ and is $I(1)$ if $u \notin K_1^1$.

If $d = 2$ let $H_2 = H_1 + [\Phi'(1)(K_1)]$ and $K_2 = [K_1^1 + \Phi'(1)(H_1^1)]^1$. Beare and Seo show \cite[Theorem 4.1]{10} that $\Upsilon_{-2}(H) = K_2$ has finite dimension and that

$$x(t) = z_0 + z_1 t + \Upsilon_{-2} \left( \sum_{s=1}^{t} \sum_{r=1}^{s} \epsilon(r) \right) - \Upsilon_{-1} \left( \sum_{s=1}^{t} \epsilon(s) \right) + \sum_{\ell \in \mathbb{N} - 1} \Upsilon_\ell (\epsilon(t - \ell)) / \ell!$$
for all $t \in \mathbb{N}$ where $z_\ell = z_\ell(\omega) \in H$ are random variables with mean zero and square integrable norm for each $\ell = 1, 2$. If $z_\ell \in K_\perp$ for each $\ell = 1, 2$ they conclude that for fixed $u \in H$ the time series $\{(u, x(t))\}_{t \in \mathbb{N}}$ is either $I(0)$ or $I(1)$ if $u \in K_\perp^1$, and that the time series is $I(2)$ if $u \notin K_\perp^1$.

Beare and Seo define necessary and sufficient conditions for $d = 1$, in which case $\Phi(z)^{-1}$ has a first-order pole at $z = 1$, and for $d = 2$, in which case $\Phi(z)^{-1}$ has a second-order pole at $z = 1$. These conditions are algebraically straightforward when $d = 1$ and tolerably so when $d = 2$. Franchi and Paruolo [23] present a comprehensive analysis for unit root processes on Hilbert space when $\Phi(z)^{-1}$ has a pole of order $d$ at $z = 1$ for all values $d \in \mathbb{N}$. The necessary and sufficient conditions given in [23] for a pole of order $d$ are more complicated when $d > 2$. A similar orthogonal decomposition on Hilbert space was proposed by Howlett et al. [28] and used to calculate the resolvent operator near a finite-order pole. This calculation is also described in the book by Avrachenkov et al. [7, pp 270–285]. These methods were later extended, using Zorn’s lemma, by Albrecht et al. [3] to find the required Laurent series expansion on Hilbert space when the resolvent has an isolated essential singularity.

A recent paper by Seo [41] extends the results in [10] for $I(1)$ and $I(2)$ processes to Banach space under less restrictive assumptions but nevertheless assumes that certain key subspaces can be complemented—an assumption that cannot be guaranteed and would therefore require a posteriori validation in Banach space.

The main advantages of our proposed method are (i) the method is valid for all Banach spaces, (ii) the required subspaces are defined by the spectral separation projections and hence are always complemented, (iii) both the key subspaces and their complementary subspaces may be infinite dimensional, (iv) the necessary and sufficient conditions that define the resolvent are simply the elementary conditions that guarantee convergence of the Laurent series, (v) a general method, described in Appendix A, has been proposed for solution of the fundamental equations in separable Banach spaces using infinite-length singular and regular Jordan chains, and (vi) the method can be applied when the resolvent has an isolated essential singularity at $z = 1$.

5. The main results

The main purpose of this paper is to establish a version of the Granger–Johansen representation for marginally stable $ARMA(1,1)$ autoregressive moving average unit root processes on Banach space. A unit root process is uniquely defined by a characteristic resolvent operator $R(z)$ which is necessarily analytic on a set of the form

$$[U_1(0)^a \setminus \{1\}] \cup U_{0,\delta}(1) = \{z \in \mathbb{C} \mid |z| \leq 1\} \setminus \{1\} \cup \{z \in \mathbb{C} \mid 0 < |z - 1| < \delta\}$$

for some $\delta > 0$. In the above expression we have followed Yosida [49] and written $U_1(0)^a$ to denote the closure of $U_1(0)$. In Proposition 1 and Theorem 1 we establish natural preliminary forms of the Granger–Johansen representation using sums defined by the
negative and nonnegative powers of the backward difference operator under the stronger assumption that $R(z)$ is analytic on $\mathcal{V}_\epsilon \cup \mathcal{W}_\theta = [\mathcal{U}_{1+\epsilon}(0) \setminus \{1\}] \cup \mathcal{U}_{0,1+\theta}(1)$ for some $\epsilon > 0$ and $\theta > 0$. The negative and nonnegative powers of the backward difference operator generate terms in different complementary subspaces defined by the spectral separation projection operators at $z = 1$. In Proposition 2 and Theorem 2 we establish extended forms of the Granger–Johanssen representation using only the minimal assumption that $R(z)$ is analytic on a set $\mathcal{V}_\epsilon = [\mathcal{U}_{1+\epsilon}(0) \setminus \{1\}]$ for some $\epsilon > 0$. In this regard we will show that $R(z)$ is analytic on a set $\mathcal{V}_\epsilon$ for some $\epsilon > 0$ if and only if $R(z)$ is analytic on a set $[\mathcal{U}_1(0)^\alpha \setminus \{1\}] \cup \mathcal{U}_{0,\delta}(1)$ for some $\delta > 0$. The extended forms do not make direct use of the nonnegative powers of the backward difference operator because the relevant sums are no longer guaranteed to converge. The convenient bipartite spectral separation obtained in the natural forms is nevertheless preserved.

For matrix operators we know that $R(z) = A(z)^{-1}$ is analytic on $\mathcal{V}_\epsilon$ for some $\epsilon > 0$ with a finite order pole at $z = 1$ if and only if $\det[A(z)] \neq 0$ for $z \in \mathcal{V}_\epsilon$ and $\det[A(1)] = 0$. For general linear operators the relationship between $A(z) = C_0 + C_1(z-1)$ and $R(z) = A(z)^{-1}$ is more complex and a more direct approach is required. Albrecht et al. [2] showed that the required Laurent series expansion $R(z) = \sum_{j \in \mathbb{Z}} T_j (z-1)^j$ is well-defined if and only if there exists a solution $\{T_j\}_{j \in \mathbb{Z}} \subseteq \mathcal{B}(Y, X)$ to a doubly-infinite set of fundamental equations defined by the coefficients $C_0, C_1 \in \mathcal{B}(X, Y)$. This is essentially an implicit condition on $A(z)$. In a subsequent paper Albrecht et al. [5] proposed a general solution procedure for these equations when $X, Y$ are separable Banach spaces. The procedure is discussed informally in Section 8. The relevant theory is outlined in Appendix A.

6. Formulation and general resolution

Let $X, Y$ be Banach spaces. We assume that $\{x(t)\}_{t \in \mathbb{N}-1} : \Omega \to X^{N-1}$ is a random series whose value $\{x(t)\}_{t \in \mathbb{N}-1}(\omega) = \{x(\omega, t)\}_{t \in \mathbb{N}-1} \in X^{N-1}$ depends on the outcome $\omega \in \Omega$ of some random process in a probability space $(\Omega, \Sigma, \mu)$. Thus for each outcome of the process the value of the series is simply defined as the corresponding series of values. We include the possibility that $\omega = \{\omega(t)\}_{t \in \mathbb{Z}} \in \mathbb{C}^\mathbb{Z}$ where $x(\omega, t) = x(\omega(t), t)$ for each $t \in \mathbb{N}-1$. If $A \in \mathcal{B}(X, Y)$ is a bounded linear operator then we define $[A\{x(t)\}_{t \in \mathbb{N}-1}](\omega) = \{Ax(\omega, t)\}_{t \in \mathbb{N}-1} \in Y^{N-1}$ for each $\omega \in \Omega$. We will outline the solution procedure for an ARMA(1,1) autoregressive moving average process satisfying an equation in the form

$$A_0 x(t) + A_1 x(t-1) = F_0 n(t) + F_1 n(t-1)$$

for each $t \in \mathbb{N}$, where a linear combination of the current value $x(t)$ and the most recently observed value $x(t-1)$ of the time series is modelled as a linear combination of the current value $n(t)$ and the most recently observed value $n(t-1)$ of a strong white noise process. The noise $\{n(t)\}_{t \in \mathbb{Z}} \in X^\mathbb{Z}$ is defined by a sequence of i.i.d. random vectors $n(t) : \Omega \to X$ with value $n(\omega, t) \in X$ for each $\omega \in \Omega$ and all $t \in \mathbb{Z}$. Thus we assume that $n(t)$ is strictly stationary with

$$\sigma^2 = \mathbb{E}[\|n\|^2] = \int_\Omega \|n(\omega, t)\|^2 d\mu(\omega) < \infty$$
and

$$\mu = \mathbb{E}[n] = \int_{\Omega} n(\omega, t) d\mu(\omega) = 0$$

for each $t \in \mathbb{Z}$. Thus we say that the noise process $n(t)$ is $\mu$-square integrable with finite variance and zero mean. Note that

$$\mathbb{E}[\|n(t)\|^2] = (\int_{\Omega} \|n(\omega, t)\| d\mu(\omega))^2 \leq \int_{\Omega} \|n(\omega, t)\|^2 d\mu(\omega) \cdot \int_{\Omega} d\mu(\omega) = \mathbb{E}[\|n(t)\|^2] < \infty$$

implies that the noise process is also $\mu$-integrable. The observations begin at $t = 0$ and so it is necessary to impose an initial condition in the form $x(-1) = c \in X$. The operators $A_0, A_1, F_0, F_1 \in \mathcal{B}(X, Y)$ are bounded linear operators. It is convenient to define a related time series $\{g(t)\}_{t \in \mathbb{Z}} = \{g(\omega, t)\}_{t \in \mathbb{Z}} \in Y^\mathbb{Z}$ by setting

$$g(t) = F_0 n(t) + F_1 n(t - 1)$$

for all $t \in \mathbb{Z}$. The series $\{g(t)\}_{t \in \mathbb{Z}} \in Y^\mathbb{Z}$ defined in (3) is a moving average MA(1) process defined by a sequence of identically distributed random vectors with finite variance and zero mean.

**Remark 1.** In our formulation the ARMA process has no deterministic driver. Thus our representation is reduced to three characteristic components: a random walk, a stationary process and a component that depends on the initial conditions.

**Remark 2.** A range of different assumptions is possible for the noise process. We make the simplest possible assumption—that $\{n(\omega, t)\}_{t \in \mathbb{Z}}$ is a strong white noise process. In other words $\{n(\omega, t)\}_{t \in \mathbb{Z}}$ is a sequence of i.i.d. random vectors with finite variance and zero mean. This assumption is sufficient to establish almost sure convergence of the strictly stationary term in the standard form of the Granger–Johansen representation. We do not use the square integrability as such and could simply assume that the noise is i.i.d. and integrable with zero mean. Other weaker assumptions are possible. Our representation remains valid if $\{n(\omega, t)\}_{t \in \mathbb{Z}}$ is a conventional weak white noise process. The condition that $\|n(\omega, t)\|$ be integrable for all $t \in \mathbb{Z}$ could potentially be replaced by the weaker condition that $\log^+ \|(A_1 A_0^{-1} F_0 + F_1) n(\omega, t)\|$ be integrable for all $t \in \mathbb{Z}$. See [43, Proposition 2.1] for discussion of the relevant issues.

### 6.1. The Maclaurin transform and the transformed equation

The ARMA(1, 1) model can be formally solved using a Maclaurin transform—which we shall refer to as an $\mathcal{M}$-transform. The $\mathcal{M}$-transform $X(z) = \mathcal{M}[\{x(t)\}_{t \in \mathbb{N} - 1}](z)$ of the time series $\{x(t)\}_{t \in \mathbb{N} - 1} \in X^{\mathbb{N} - 1}$ is a random power series $X(z) : \Omega \to X$ in the variable $z \in \mathbb{C}$ with value

$$X(\omega, z) = \sum_{t \in \mathbb{N} - 1} x(\omega, t) z^t$$

for each outcome $\omega \in \Omega$ of the random process. For a completely arbitrary time series we cannot decide a priori if $X(z)$ will converge for any particular value of $z \in \mathbb{C}$ but we will
show that if the time series \( \{x(t)\}_{t \in \mathbb{N}} \) can be modelled as an ARMA(1, 1) process then the \( \mathcal{M} \)-transform can be used to find a Granger–Johansen representation for the series.

We can then recover a retrospective answer to the convergence question for \( X(z) \) and the corresponding representation for \( \{x(t)\}_{t \in \mathbb{N}} \). If we apply an \( \mathcal{M} \)-transform to (2) we get

\[
A(z)X(z) + A_1 c = G(z) \iff A(z)X(z) = G(z) - A_1 c
\]

where \( c = x(-1) \), the linear pencil \( A(z) = A_0 + A_1 z \) is the characteristic function, and we have defined \( X(z) = \sum_{t \in \mathbb{N}} x(t)z^t \) and used (3) to define \( G(z) = \sum_{t \in \mathbb{N}} g(t)z^t \).

Solution of (5) depends on the existence of a resolvent operator \( R(z) = A(z)^{-1} \) on some suitable region \( z \in \mathcal{U} \) of the complex plane.

6.2. The resolvent operator and the resolution

We are particularly interested in those marginally stable time series known as unit root processes where the resolvent operator \( R(z) = A(z)^{-1} \) is singular at \( z = 1 \) but is otherwise analytic on and inside the unit circle. To this end we will assume that \( R(z) \) is analytic on the set \( \mathcal{V}_\epsilon \) defined in (11) for some \( \epsilon > 0 \) and has a singularity at \( z = 1 \) which may be a pole of finite order or an isolated essential singularity. Hence \( R(z) \) is analytic on the region \( \mathcal{U}_{0, \epsilon}(1) \). We must necessarily analyze the singularity at \( z = 1 \) so we begin by writing

\[
A(z) = C_0 + C_1(z - 1)
\]

where \( C_0 = A_0 + A_1 \in \mathcal{B}(X, Y) \) and \( C_1 = A_1 \in \mathcal{B}(X, Y) \). Albrecht et al. [2] have shown that the resolvent operator can be expressed as a Laurent series

\[
R(z) = \sum_{j \in \mathbb{Z}} T_j(z - 1)^j
\]

for all \( z \in \mathcal{U}_{0, \epsilon}(1) \) if and only if the coefficients \( \{T_j\}_{j \in \mathbb{Z}} \) satisfy a system of left and right fundamental equations at \( z = 1 \) given respectively by

\[
T_{j-1}C_1 + T_jC_0 = \begin{cases} 
I_X & \text{if } j = 0 \\
0_X & \text{if } j \in \mathbb{Z}, j \neq 0
\end{cases}
\]

and

\[
C_1T_{j-1} + C_0T_j = \begin{cases} 
I_Y & \text{if } j = 0 \\
0_Y & \text{if } j \in \mathbb{Z}, j \neq 0
\end{cases}
\]

and the coefficients \( \{T_j\}_{j \in \mathbb{Z}} \) also satisfy the magnitude constraints

\[
\lim_{k \to \infty} \|T_{-k}\|^{1/k} = 0 \quad \text{and} \quad \lim_{\ell \to \infty} \|T_\ell\|^{1/\ell} \leq 1/\epsilon.
\]

If (10) is satisfied then for each \( \eta > 0 \) we can find a real constant \( c_\eta > 0 \) with \( \|T_{-k}\| \leq c_\eta \eta^{-k} \) for all \( k \in \mathbb{N} \) and a real constant \( d_\epsilon > 0 \) with \( \|T_\ell\| \leq d_\epsilon / \epsilon^\ell \) for all \( \ell \in \mathbb{N} - 1 \).

If (8), (9) and (10) are all satisfied then

\[
T_{-k} = (-1)^{k-1}(T_{-1}C_0)^{k-1}T_{-1} \in \mathcal{B}(Y, X)
\]
for all $k \in \mathbb{N}$ and
\[ T_\ell = (-1)^\ell (T_0 C_1)^\ell T_0 \in \mathcal{B}(Y, X) \tag{12} \]
for all $\ell \in \mathbb{N} - 1$. Thus the solution to [8] and [9] on $\mathcal{U}_{0, \epsilon}(1)$ is completely determined by the basic solution $\{T_{-1}, T_0\}$ on $\mathcal{U}_{0, \epsilon}(1)$. The operators $P = T_{-1} C_1 \in \mathcal{B}(X)$ and $P^c = I_X - P = T_0 C_0 \in \mathcal{B}(X)$ are complementary key projections on $X$ and the operators $Q = C_1 T_{-1} \in \mathcal{B}(Y)$ and $Q^c = I_Y - Q = C_0 T_0 \in \mathcal{B}(Y)$ are corresponding complementary key projections on $Y$. We also note that $T_{-1} C_i T_0 - T_0 C_i T_{-1} = 0_{Y, X}$ for each $i = 0, 1$. We will write
\[ R(z) = R_{\sin}(z) + R_{\text{reg}}(z) \tag{13} \]
for all $z \in \mathcal{U}_{0, \epsilon}(1)$ where the singular part of the Laurent series
\[ R_{\sin}(z) = [I_X (z - 1) + T_{-1} C_0]^{-1} T_{-1} = \sum_{k \in \mathbb{N}} T_{-k}(z - 1)^{-k} \tag{14} \]
converges for all $z \in \mathcal{U}_{0, \infty}(1)$ and the regular part of the Laurent series
\[ R_{\text{reg}}(z) = [I_X + T_0 C_1 (z - 1)]^{-1} T_0 = \sum_{\ell \in \mathbb{N} - 1} T_{\ell}(z - 1)^\ell \tag{15} \]
converges for all $z \in \mathcal{U}_\epsilon(1)$. See Appendix A for a formal statement of the key results in Albrecht et al. [2]. Since the resolvent operator satisfies the equation $R(z) A(z) = I_X$ for all $z \in \mathcal{U}_{0, \epsilon}(1)$ we obtain a formal resolution of the transformed equation [5] given by
\[ X(z) = R(z)[G(z) - A_1 c] = R_{\sin}(z)[G(z) - A_1 c] + R_{\text{reg}}(z)[G(z) - A_1 c] \tag{16} \]
for all $z \in \mathcal{U}_{0, \epsilon}(1)$. We will show that the formal resolution in [10] leads naturally to an intuitive time series representation using sums of negative and nonnegative powers of the backward difference operator acting on a white noise process.

6.3. The backward difference operator

The backward difference operator\(^2\) for a time series $\{u(t)\}_{t \in \mathbb{Z}} \in \mathcal{X}$ is defined by
\[ \nabla \{u(t)\}_{t \in \mathbb{Z}} = \{\nabla u(t)\}_{t \in \mathbb{Z}} = \{u(t) - u(t - 1)\}_{t \in \mathbb{Z}}. \]

The backward difference operator is closely related to the lag operator
\[ L\{u(t)\}_{t \in \mathbb{Z}} = \{Lu(t)\}_{t \in \mathbb{Z}} = \{u(t - 1)\}_{t \in \mathbb{Z}}. \]

Indeed we have $\nabla = I_X - L$ where $I_X \in \mathcal{B}(X)$ is the identity operator. We can define integer powers of the backward difference operator by the formulæ
\[ \nabla^{-k} u(t) = (I_X - L)^{-k} u(t) = \sum_{s \in \mathbb{N} - 1} (-1)^{k-s} \binom{s+k-1}{s} L^s u(t) = \sum_{s \in \mathbb{N} - 1} \binom{s+k-1}{s} u(t - s) \tag{17} \]

\(^2\)In the time series literature the backward difference operator is often denoted by $\Delta$. In this paper we use the more conventional mathematical notation and denote the backward difference operator by $\nabla$. 10
for all $k \in \mathbb{N}$, and
\[
\nabla^\ell u(t) = (I_x - L)^\ell u(t) = \sum_{s=0}^\ell {\binom{\ell}{s}} (-1)^s L^s u(t) = \sum_{s=0}^\ell {\binom{\ell}{s}} (-1)^s u(t - s) \tag{18}
\]
for all $\ell \in \mathbb{N} - 1$. It is a matter of elementary algebra to show that
\[
\mathcal{M}[\{\nabla^j u_+(t)\}_{t \in \mathbb{Z}}](z) = (1 - z)^j U(z)
\]
for each $j \in \mathbb{Z}$ where we have defined
\[
u_+(t) = \begin{cases} u(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{19}
\]
Since $\nu_+(t) = 0$ for $t < 0$ it follows from (17) and (18) that
\[
\nabla^{-k} u_+(t) = \sum_{s \in \mathbb{N} - 1} {\binom{s+k-1}{s}} u_+(t - s) = \sum_{s=0}^{\min(\ell,t)} {\binom{\ell}{s}} (-1)^s u(t - s) \tag{20}
\]
and
\[
\nabla^\ell u_+(t) = \sum_{s=0}^\ell {\binom{\ell}{s}} (-1)^s u_+(t - s) = \sum_{s=0}^{\min(\ell,t)} {\binom{\ell}{s}} (-1)^s u(t - s). \tag{21}
\]

6.4. A natural representation of the time series

We assume that $R(z)$ is analytic on the set $\mathcal{V}_\epsilon \cup \mathcal{W}_\theta$ for some $\epsilon > 0$ and some $\theta > 0$.

Consider the term $X_{\sin}(z) = R_{\sin}(z)[G(z) - C_1 c]$. We have
\[
X_{\sin}(z) = \sum_{k \in \mathbb{N}} T_{-k}(z - 1)^{-k} [G(z) - C_1 c] \\
= (-1) \sum_{k \in \mathbb{N}} (T_{-1} C_0)^{k-1} T_{-1} (1 - z)^{-k} [G(z) - C_1 c] \\
= (-1) \sum_{k \in \mathbb{N}} (T_{-1} C_0)^{k-1} T_{-1} \mathcal{M} \left[ \{\nabla^{-k}(g - C_1 c \delta)_+(t)\}_{t \in \mathbb{Z}} \right] (z) \tag{22}
\]
where the time series $\{\delta(t)\}_{t \in \mathbb{Z}}$ is defined by $\delta(t) = 0$ if $t \neq 0$ and $\delta(0) = 1$. Now we can apply (20) to deduce that
\[
X_{\sin}(z) = (-1) \sum_{k \in \mathbb{N}} (T_{-1} C_0)^{k-1} T_{-1} \mathcal{M} \left[ \{\sum_{t=0}^{\ell} {\binom{s+k-1}{s}} (g - C_1 c \delta)(t - s)\}_{t \in \mathbb{Z}} \right] (z) \\
= \mathcal{M} \left[ \{(-1) \sum_{k \in \mathbb{N}} (T_{-1} C_0)^{k-1} T_{-1} \sum_{s=0}^{\ell} {\binom{s+k-1}{s}} (g - C_1 c \delta)(t - s)\}_{t \in \mathbb{Z}} \right] (z) \\
= \mathcal{M} \left[ \{\sum_{k \in \mathbb{N}} (-1)^k T_{-k} \nabla^{-k} g_+(t)\}_{t \in \mathbb{Z}} \right] (z) \\
+ \mathcal{M} \left[ \{\sum_{k \in \mathbb{N}} {\binom{\ell}{s}} (T_{-1} C_0)^{k-1} T_{-1} C_1 c\}_{t \in \mathbb{Z}} \right] (z). \tag{23}
\]
We justify convergence of (23) as follows. We have assumed that $R(z)$ is analytic on $\mathcal{V}_\epsilon$ for some $\epsilon > 0$ and so $R_{\sin}(z) = \sum_{k \in \mathbb{N}} T_{-k}(z - 1)^{-k}$ is analytic for $z \in \mathcal{U}_{0,\infty}(1)$. Hence, for each $\eta > 0$, we can find $c_\eta > 0$ such that
\[
\|T_{-k}\| = \|(T_{-1} C_0)^{k-1} T_{-1}\| \leq c_\eta \eta^k
\]
Now consider the term \(X\) for all \(t\) for all \(z \neq 1\) is also well defined. The function \(R_{\sin}(z) = \sin((z - 1) + T_{-1}C_0)^{-1}T_{-1}\) is analytic for all \(z \neq 1\) and hence is analytic at \(z = 0\) with
\[
R_{\sin}(z) = \sum_{s \in \mathbb{N} - 1} U_s z^s \tag{24}
\]
for all \(z \in \mathcal{U}_1(0)\) where the coefficients \(\{U_s\}_{s \in \mathbb{N} - 1}\) are given by the formula
\[
U_s = (1/2)^s \sum_{t=0}^{s} \binom{s}{t} (z - 1)^t (1 - z)^{s-t} \tag{25}
\]
for each \(s \in \mathbb{N} - 1\). If we define
\[
\alpha_{\sin,+}(t) = \sum_{k \in \mathbb{N}} (-1)^k T_{-k} \nabla^{-k} g_+(t) + (I_X - T_{-1}C_0)^{-1}T_{-1}C_1 c
\]
\[
= \sum_{k \in \mathbb{N}} (-1)^k T_{-k} \nabla^{-k} g_+(t) - U_t C_1 c \tag{26}
\]
for \(t \in \mathbb{N} - 1\) then we have shown that \(M [\{\alpha_{\sin,+}(t)\}_{t \in \mathbb{Z}}] (z) = X_{\sin}(z)\).

Now consider the term \(X_{\text{reg}}(z) = R_{\text{reg}}(z)[G(z) - C_1c]\). We have
\[
X_{\text{reg}}(z) = \sum_{t \in \mathbb{N} - 1} T_t(z - 1)^t [G(z) - C_1c]
\]
\[
= \sum_{t \in \mathbb{N} - 1} (T_0C_1)^t T_0 (1 - z)^t [G(z) - C_1c]
\]
\[
= \sum_{t \in \mathbb{N} - 1} (T_0C_1)^t T_0 M [\{\nabla^t(g - C_1c \delta + t)\}_{t \in \mathbb{Z}}] (z) \tag{27}
\]
By applying (21) we can see that
\[
X_{\text{reg}}(z) = \sum_{t \in \mathbb{N} - 1} (T_0C_1)^t T_0 M [\{\nabla^t(g - C_1c \delta + t)\}_{t \in \mathbb{Z}}] (z)
\]
\[
= M [\{\sum_{s \in \mathbb{N} - 1} (T_0C_1)^t T_0 \sum_{s=0}^{\min(t,\ell)} \binom{t}{s} (-1)^s (g - C_1c \delta)(t - s)\}_{t \in \mathbb{Z}}] (z)
\]
\[
= M [\{\sum_{t \in \mathbb{N} - 1} (-1)^t T_t \nabla^t g_+(t)\}_{t \in \mathbb{Z}}] (z)
\]
\[
- M [\{\sum_{s \in \mathbb{N} - 1} \delta(t - s)\}_{t \in \mathbb{Z}}] (z)
\]
\[
= M [\{(-1)^t \sum_{t \in \mathbb{N} - 1+t} (T_0C_1)^t c\}_{t \in \mathbb{Z}}] (z) \tag{28}
\]
We can justify convergence of (28) in the following way. We have assumed that $R(z)$ is analytic on $\mathcal{W}_\theta$ for some $\theta > 0$ and so $R_{\text{reg}}(z) = \sum_{\ell \in \mathbb{N}-1} T_\ell (z - 1)^\ell$ is analytic for all $z \in U_{1+\theta}(1)$. Hence we can find $d_\theta > 0$ such that

$$
\|T_\ell\| = \|(T_0 C_1)^\ell T_0\| \leq d_\theta/(1+\theta)^\ell
$$

for all $\ell \in \mathbb{N} - 1$. It follows that

$$
\mathbb{E} \left[ \|\sum_{\ell \in \mathbb{N}-1} (1)^\ell T_\ell \nabla^\ell g_+(t)\| \right]
= \mathbb{E} \left[ \|\sum_{\ell \in \mathbb{N}-1} (T_0 C_1)^\ell T_0 \sum_{s=0}^{\min\{\ell, t\}} \binom{\ell}{s} (-1)^s g(t-s)\| \right]
\leq d_\theta \mathbb{E} \left[ \|g\| \right] \sum_{s=0}^t \sum_{\ell \in \mathbb{N}-1+s} \binom{\ell}{s} \left[(1+\theta)^t\right]^{\ell-s}
= d_\theta \mathbb{E} \left[ \|g\| \right] (1+\theta) \sum_{s=0}^t \theta^{s+1}
$$

is well defined and finite for all $t \in \mathbb{N} - 1$. A similar argument shows that

$$
\sum_{\ell \in \mathbb{N}-1+t} (1)^\ell \epsilon T_\ell C_1 \epsilon^{t+1} C_1 = (I_X - T_0 C_1)^{-t-1} (T_0 C_1)^{t+1} C_1
$$

is also well defined. The function $R_{\text{reg}}(z) = [I_x + T_0 C_1 (z - 1)]^{-1} T_0$ is analytic for $z \in U_{1+\theta}(1)$ and hence is analytic at $z = 0$ with

$$
R_{\text{reg}}(z) = \sum_{s \in \mathbb{N} - 1} V_s z^s
$$

for all $z \in U_\theta(0)$ where the coefficients $\{V_s\}_{s \in \mathbb{N} - 1}$ are given by the formula

$$
V_s = (1/s!) \left[d^s R_{\text{reg}}(z)/dz^s\right]_{z=0}
= (1/s!) \left[d^s [I_x + T_0 C_1]^{-1} T_0/dz^s\right]_{z=0}
= (-1)^s (I_X - T_0 C_1)^{-s-1} (T_0 C_1)^s T_0
$$

for each $s \in \mathbb{N} - 1$. If we set

$$
x_{\text{reg},+}(t) = \sum_{\ell \in \mathbb{N}-1} (1)^\ell T_\ell \nabla^\ell g_+(t) + (-1)^{t+1} (I_X - T_0 C_1)^{-t-1} (T_0 C_1)^{t+1} C_1
= \sum_{\ell \in \mathbb{N}-1} (1)^\ell T_\ell \nabla^\ell g_+(t) - V_t C_1 C_1
$$

for all $t \in \mathbb{N} - 1$ then $\mathcal{M} \{x_{\text{reg},+}(t)\}_{t \in \mathbb{Z}}(z) = X_{\text{reg}}(z)$. We can now combine (26) and (31) to state a non-standard natural form of the Granger–Johansen representation.

**Proposition 1.** If $R(z)$ is analytic on the set $V_\epsilon \cup \mathcal{W}_\theta$ for some $\epsilon > 0$ and $\theta > 0$ and if $x(-1) = c$ then we can write

$$
x(t) = \sum_{k \in \mathbb{N}} (1)^k T_{-k} \nabla^{-k} g_+(t) - U_t C_1 c + \sum_{\ell \in \mathbb{N}-1} (1)^\ell T_\ell \nabla^\ell g_+(t) - V_t C_1 c
$$

for all $t \in \mathbb{N} - 1$. □

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Proposition 1 shows that \( x(t) \) may be written as the sum of a *stochastic trend* depending only on negative powers of the backward difference operator, deterministic components \((-1)^tU_tC_1c\) and \((-1)^tV_tC_1c\) which do not depend on \(g_+\), and the term

\[
h_t(t) = \sum_{t \in \mathbb{N} - 1} (-1)^{-t}T_t \nabla^t g_+(t) = \sum_{s=0}^t \sum_{t \in \mathbb{N} - 1 + s} \binom{t}{s} (-1)^{t+s}T_t g(t - s) \tag{33}
\]

The expression (32) is non-standard because \( h_t(t) \) is only asymptotically stationary rather than strictly stationary. However one advantage is that \( h_t(t) \) is *a posteriori* observable because \( g(t) = A_0x(t) + A_1x(t - 1) \) can be recovered from the observed values \( x(t) \) for \( t \in \mathbb{N} - 1 \) and the initial value \( x(-1) \) if the coefficients \( \{A_0, A_1\} \) of the model are known. This assumes, in essence, that the model fitting was successful [1, 13, 14, 42]. We will now show that (32) can be converted into standard form. Define

\[
h_\infty(t) = \sum_{t \in \mathbb{N} - 1} (-1)^t T_t \nabla^t g(t) = \sum_{s \in \mathbb{N} - 1} \sum_{t \in \mathbb{N} - 1 + s} \binom{t}{s} (-1)^{t+s}T_t g(t - s) \tag{34}
\]

and write \( h_t(t) = h_\infty(t) - \sum_{t \in \mathbb{N} - 1} (-1)^t T_t \nabla^t [g - g_+](t) \) for all \( t \in \mathbb{N} - 1 \). Now

\[
\begin{align*}
\sum_{t \in \mathbb{N} - 1} (-1)^t T_t \nabla^t [g - g_+](t) & = \sum_{s \in \mathbb{N} - 1} \sum_{t \in \mathbb{N} - 1 + s} \binom{t}{s} (-1)^s (T_0C_1)^t T_0 [g - g_+](t - s) \\
& = \sum_{s \in \mathbb{N} + t} \sum_{t \in \mathbb{N} - 1 + s} \binom{t}{s} (-1)^s (T_0C_1)^t T_0 g(t - s) \\
& = \sum_{t \in \mathbb{N} - 1} \sum_{m \in \mathbb{N} - 1} \sum_{r \in \mathbb{N}} \left( \binom{m + r + t}{r} \right) (-1)^m (T_0C_1)^{m + r + t} T_0 g(r) \\
& = \sum_{t \in \mathbb{N} - 1} \left( \binom{t}{r} \right) (T_0C_1)^{-t} T_0 g(-r) \\
& = (-1)^t (I_X - T_0C_1)^{-t - 1} T_0 g(-r) \\
& = V_tC_1 k
\end{align*}
\]

for all \( t \in \mathbb{N} - 1 \) where we have defined

\[
k = \sum_{t \in \mathbb{N}} (-1)^r (I_X - T_0C_1)^{-r} (T_0C_1)^{r-1} T_0 g(-r) \in \mathcal{P}^c(X). \tag{35}
\]

The significance of the expressions in (34) for \( h_\infty(t) \) and (35) for \( k \) is the dependence on the function \( g(t) \) which is strictly stationary. Thus, in each case, the entire expression is strictly stationary. By comparison the expression in (33) for \( h_t(t) \) depends on the truncated function \( g_+(t) \) which is not strictly stationary. We can now state a natural form of the standard Granger–Johansen representation.

**Theorem 1.** If \( R(z) \) is analytic on the set \( \mathcal{V}_\epsilon \cup \mathcal{W}_\theta \) for some \( \epsilon > 0 \) and \( \theta > 0 \) and if \( x(-1) = c \) then we can write

\[
x(t) = \sum_{k \in \mathbb{N}} (-1)^k T_{-k} \nabla^{-k} g_+(t) - U_tC_1c + \sum_{t \in \mathbb{N} - 1} (-1)^t T_t \nabla^t g(t) - V_tC_1c - k(t) \tag{36}
\]

where we have defined \( k(t) = V_tC_1 k \) for all \( t \in \mathbb{N} - 1 \). \( \square \)

In the natural form (36) of the standard Granger–Johansen representation the term \( h_\infty(t) \) defined in (34) is strictly stationary but non-observable because \( g(t - s) \) is only observed for
s \leq t$. In practice one would need to check the observed properties of $h_t(t)$ to determine whether the unobservable $h_{\infty}(t)$ is truly strictly stationary. The natural form of the Granger–Johansen representation shows that the key terms derived from the negative and nonnegative powers of the backward difference operator lie in complementary subspaces defined by the spectral separation projection $P = T_{-1}C_1$.

**Corollary 1.** Let $P = T_{-1}C_1 \in \mathcal{B}(X)$ and $P^c = I_X - P = T_0C_0 \in \mathcal{B}(X)$ denote the spectral separation projections centred at $z = 1$ on $X$. If $R(z)$ is analytic on the set $\mathcal{V}_\epsilon \cup \mathcal{W}_\theta$ for some $\epsilon > 0$ and $\theta > 0$ and if $x(-1) = c$ then we have

\[
x_{\text{sin}}(t) = \sum_{k \in \mathbb{N}} (-1)^k T_{-k} \nabla^{-k} P g_+(t) - U_tT_{-1}C_1c \in P(X) \quad (37)
\]

and

\[
x_{\text{reg}}(t) = \sum_{\ell \in \mathbb{N} - 1} (-1)\ell T_\ell \nabla^\ell P^c g(t) - V_tC_1c - k(t) \in P^c(X) \quad (38)
\]

for all $t \in \mathbb{N} - 1$. □

**Proof.** We have $T_{-k}g(t) = (-1)(T_{-1}C_0)^{k-1}T_{-1}(C_1T_{-1}g(t)) = T_{-k}Pg(t)$ for each $k \in \mathbb{N}$ and $T_\ell g(t) = (-1)\ell(T_0C_1)^\ell T_0(C_0T_0g(t)) = T_\ell P^c g(t)$ for each $\ell \in \mathbb{N} - 1$. □

We know that $\|(T_{-1}C_0)^k\|^{1/k} \to 0$ and that $\|(C_0T_{-1})^k\|^{1/k} \to 0$ as $k \to \infty$. We have the following related result.

**Lemma 1.** The operators $T_{-1}C_0 \in \mathcal{B}(X)$ and $C_0T_{-1} \in \mathcal{B}(Y)$ are either both nilpotent of index $d + 1$ for some finite $d \in \mathbb{N}$ or both quasi-nilpotent. □

**Proof.** If $(T_{-1}C_0)^k \neq 0_X$ then $(T_{-1}C_0)^k(T_{-1}C_1 + T_0C_0) \neq 0_X$. Since $T_{-1}C_0T_0 = 0_{Y,X}$ we know that $T_{-1}(C_0T_{-1})^kC_1 \neq 0_X$. Therefore $(C_0T_{-1})^k \neq 0_Y$. A similar argument shows that the reverse implication is also true. In the first case $T_{-k} = 0_{Y,X}$ for all $k \geq d + 1$ and the resolvent $R(z)$ has a pole of order $d$ at $z = 1$. In the second case $T_{-k} \neq 0_{Y,X}$ for all $k \in \mathbb{N}$ and the resolvent $R(z)$ has an isolated essential singularity at $z = 1$. □

**Corollary 2.** Suppose $R(z)$ is analytic on the set $\mathcal{V}_\epsilon \cup \mathcal{W}_\theta$ for some $\epsilon > 0$ and $\theta > 0$ and that $x(-1) = c$. If $R(z)$ has a pole of order $d$ at $z = 1$ then $T_{-k-1} = 0_{Y,X}$ for $k \geq d + 1$ and $T_{-d} \neq 0_{X,Y}$ and we have

\[
x(t) = \sum_{k=1}^{d} (-1)^k T_{-k} \nabla^{-k} g_+(t) - U_tC_1c + \sum_{\ell \in \mathbb{N} - 1} (-1)\ell T_\ell \nabla^\ell g(t) - V_tC_1c - k(t) \quad (39)
\]

for all $t \in \mathbb{N} - 1$. We say that the time series is integrated of order $d$ and we write \{x_+(t)\}_{t \in \mathbb{Z}} is $I(d)$. □
6.5. Special cases

Beare and Seo [10] consider specific results for \( I(d) \) processes when \( d = 1, 2 \). We have the following remarks. We suppose \( R(z) \) is analytic on the set \( V_\epsilon \cup W_\theta \) for some \( \epsilon > 0 \) and \( \theta > 0 \) and that \( \mathbf{x}(-1) = \mathbf{c} \). From a purely algebraic viewpoint it is more convenient to refer the following arguments to the non-standard representation in Proposition [11]. When \( d = 1 \) we know that \( T_{-k} = 0 \) for \( k > 1 \) and so \( T_{-2} = T_{-1}C_0T_{-1} = 0 \). Therefore the standard series expansion gives

\[
U_t = (-1)[I_X - T_{-1}C_0]^{-1}T_{-1} = (-1)[T_{-1} + (T_{-1}C_0)T_{-1} + \cdots] = -T_{-1}.
\]

Hence (32) becomes

\[
\mathbf{x}(t) = (-1)T_{-1}\nabla^{-1}\mathbf{g}_+(t) + T_{-1}C_1\mathbf{c} + \sum_{t \in \mathbb{N}-1}(-1)^t T_{t}\nabla^{t}\mathbf{g}_+(t) - V_tC_1\mathbf{c} \tag{40}
\]

for all \( t \in \mathbb{N} - 1 \). In the above expression \( \nabla^{-1}\mathbf{g}_+(t) = \sum_{s=0}^t \mathbf{g}(s) \) for all \( t \in \mathbb{N} - 1 \).

**Remark 3.** Let \( f \in X^* \) be a bounded linear functional on \( X \). We follow Luenberger [34], Sections 5.6, 5.7, pp 115–118 and use the scalar product notation \( \langle \mathbf{x}, f \rangle = f(\mathbf{x}) \) to denote the value of \( f \) at the point \( \mathbf{x} \in X \). For each \( S \subseteq X \) the orthogonal complement is

\[
S^\perp = \{ f \in X^* \mid \langle \mathbf{x}, f \rangle = 0 \text{ for all } \mathbf{x} \in S \} \subseteq X^*.
\]

We know from Theorem 4 in Appendix A that

\[
Y = C_1T_{-1}(Y) \oplus C_0T_0(Y) = Q(Y) \oplus Q^c(Y) = Y_{\sin} \oplus Y_{\reg}.
\]

Since \( T_{-1}C_1(X_{\sin}) = P[P(X)] = P(X) = X_{\sin} \) and \( T_{-1}C_0(X_{\reg}) = T_{-1}C_0[T_0C_0(X)] = \{0\} \) it follows that

\[
T_{-1}(Y) = T_{-1}(Y_{\sin}) \oplus T_{-1}(Y_{\reg}) = T_{-1}C_1(X_{\sin}) + T_{-1}C_0(X_{\reg}) = X_{\sin}.
\]

Now consider the time series \( \{\langle \mathbf{x}(t), f \rangle\}_{t \in \mathbb{N}-1} \) formed by the action of \( f \) on \( \{\mathbf{x}(t)\}_{t \in \mathbb{N}-1} \). If \( \mathbf{x}(-1) = \mathbf{c} \in T^{-1}(Y) \) then \( T_0C_1\mathbf{c} = T_0C_1T_{-1}\mathbf{d} \) for some \( \mathbf{d} \in Y \) and so \( T_0C_1\mathbf{c} = 0 \). Therefore \( V_tC_1\mathbf{c} = 0 \) for all \( t \in \mathbb{N} - 1 \). We also have

\[
\sum_{t \in \mathbb{N}-1}(-1)^t T_{t}\nabla^{t}\mathbf{g}_+(t) = \sum_{t \in \mathbb{N}-1}(-1)^t T_{t} \left[ \sum_{s=0}^{\min\{t,t\}} \binom{s}{t}\binom{t}{t-s}(-1)^{t-s}\mathbf{g}(t-s) \right]
\]

\[
= \sum_{s=0}^t \left[ \sum_{t \in \mathbb{N}-1}(-1)^{t-s}T_{t-s}\binom{s}{t}\right] \mathbf{g}(t-s)
\]

\[
= \sum_{s=0}^t (-1)^s \left[ \sum_{m=N-1}^{s-m}(s+m)(T_0C_1)^m\right] (T_0C_1)^sT_0\mathbf{g}(t-s)
\]

\[
= \sum_{s=0}^t (-1)^s (I_X - T_0C_1)^{-s-1}(T_0C_1)^sT_0\mathbf{g}(t-s)
\]

\[
= \sum_{s=0}^t V_s\mathbf{g}(t-s)
\]

and so (40) reduces to

\[
\mathbf{x}(t) = T_{-1} \left[ C_1\mathbf{c} - \sum_{s=0}^t \mathbf{g}(t-s) \right] + \sum_{s=0}^t V_s\mathbf{g}(t-s) \tag{41}
\]
for all \( t \in \mathbb{N} - 1 \). Now when \( f \in X^\perp_{\text{sin}} = T_{-1}(Y)^\perp \subseteq X^* \) we can see that

\[
\langle x(t), f \rangle = \sum_{s=0}^{t} V_s \langle g(t-s), f \rangle
\]

(42)

for all \( t \in \mathbb{N} - 1 \) and so \( \{\langle x(t), f \rangle \}_{t \in \mathbb{N} - 1} \) is I(0). Hence \( x(t) \) is co-integrated. In this regard we recall that \( R_{\text{reg}}(z) = \sum_{s \in \mathbb{N} - 1} V_s z^s \) for all \( z \in \mathcal{U}_\theta(0) \) and that \( R_{\text{reg}}(z) = \sum_{s \in \mathbb{N} - 1} T_{\ell}(z-1)^s \)

for all \( z \in \mathcal{U}_{1+\theta}(1) \). Thus we know that \( R_{\text{reg}}(1) = T_0 \) is nonsingular on the subspace \( P^c(X) \). Therefore \( R_{\text{reg}}(1) = T_0 \neq 0 \) provided \( P^c \neq 0 \). Thus (41) and (42) provide a natural extension of the results obtained by Beare and Seo. \( \square \)

When \( d = 2 \) we know that \( T_{-k} = (-1)^{k-1}(T_{-1}C_0)^{k-1}T_{-1} = 0 \) for \( k > 2 \) and so the standard series expansion gives

\[
U_tC_1c = (-1)(I_X - T_{-1}C_0)\sum_{s=0}^{t-1}T_{-1}C_1c = (-1)[T_{-1}C_1c + (t+1)T_{-1}C_0T_{-1}C_1c]
\]

for all \( t \in \mathbb{N} - 1 \). Therefore (42) becomes

\[
x(t) = T_{-2}\nabla^{-2}g_+(t) - T_{-1}\nabla^{-1}g_+(t) + T_{-1}C_1c - (t+1)T_{-2}C_1c + \sum_{s=0}^{t} V_s g(t-s) - V_t C_1c
\]

(43)

for all \( t \in \mathbb{N} - 1 \). In the above expression \( \nabla^{-1}g_+(t) = \sum_{s=0}^{t} g(t-s) \) and \( \nabla^{-2}g_+(t) = \sum_{s=0}^{t} (s+1)g(t-s) \).

**Remark 4.** If \( x(-1) = c \in T_{-2}(Y) \) then \( c = T_{-2}d \) for some \( d \in Y \). Since \( T_{0}C_1T_{-2} = 0 \) it follows that \( T_{0}C_1c = 0 \). We also have \( T_{-1}C_1c = (T_{-1}C_1 + T_{0}C_0)T_{-2}d = T_{-2}d \). Therefore (43) reduces to

\[
x(t) = T_{-2}\nabla^{-2}g_+(t) - T_{-1}\nabla^{-1}g_+(t) + T_{-2}d + (t+1)T_{-2}C_0T_{-2}d + \sum_{s=0}^{t} V_s g(t-s).
\]

(44)

If we also have \( f \in T_{-2}(Y)^\perp \) then (44) implies

\[
\langle x(t), f \rangle = -T_{-1}\sum_{s=0}^{t} \langle g(t-s), f \rangle + \sum_{s=0}^{t} V_s \langle g(t-s), f \rangle
\]

for all \( t \in \mathbb{N} - 1 \). Thus \( \{\langle x(t), f \rangle \}_{t \in \mathbb{N} - 1} \) is I(1) if \( f \notin T_{-1}(Y)^\perp \) and is I(0) if \( f \in T_{-1}(Y)^\perp \). Therefore \( x(t) \) is co-integrated. Once again we note that \( R_{\text{reg}}(1) = T_0 \neq 0 \) provided \( P^c \neq 0 \). Note that \( T_{-2}(Y) = T_{-1}C_0T_{-1}(Y) \subseteq T_{-1}(Y) \) implies \( T_{-1}(Y)^\perp \subseteq T_{-2}(Y)^\perp \). If \( f \notin T_{-2}(Y)^\perp \) then \( \{\langle x(t), f \rangle \}_{t \in \mathbb{N} - 1} \) is I(2). \( \square \)

### 6.6. The extended representation

Now suppose only that \( \{x(t)\}_{t \in \mathbb{Z}} \) is a well-defined unit root process. Therefore \( R(z) \) is analytic for \( z \in [\mathcal{U}_1(0)^a \setminus \{1\}] \cup \mathcal{U}_{0,\delta}(1) \) for some \( \delta > 0 \). We have the following result.

**Lemma 2.** The resolvent operator \( R(z) \) is analytic on a set \( [\mathcal{U}_1(0)^a \setminus \{1\}] \cup \mathcal{U}_{0,\delta}(1) \) for some \( \delta > 0 \) if and only if \( R(z) \) is analytic on a set \( \mathcal{V}_\epsilon = \mathcal{U}_{1+\epsilon}(0) \setminus \{1\} \) for some \( \epsilon > 0 \). \( \square \)
Proof. Suppose \( R(z) \) is analytic on \( V_\epsilon \) for some \( \epsilon > 0 \). Since
\[
[ U_1(0)^a \setminus \{1\} ] \cup U_{0,\epsilon}(1) \subseteq V_\epsilon
\]
it follows that \( R(z) \) is analytic on \([ U_1(0)^a \setminus \{1\} ] \cup U_{0,\epsilon}(1) \).

Now suppose \( R(z) \) is analytic on \([ U_1(0)^a \setminus \{1\} ] \cup U_{0,\delta}(1) \) for some \( \delta > 0 \). Define the
compact set \( C = \{ z \in \mathbb{C} \mid |z| = 1 \text{ and } z \notin U_{\delta/2}(1) \} \). The resolvent \( R(z) \) is analytic on
\( C \) and so for each \( w \in C \) we can find \( \epsilon(w) > 0 \) such that \( R(z) \) is analytic on \( U_{\epsilon(w)}(w) \).

Therefore
\[
C \subseteq \bigcup_{w \in C} U_{\epsilon(w)}(w).
\]
Since \( C \) is compact there exists a finite sub-covering. Thus we have a finite collection of
neighbourhoods \( \{ U_{\epsilon_i}(w_i) \}_{i=1}^n \) with \( w_i \in C \) and \( \epsilon_i > 0 \) for each \( i = 1, \ldots, n \) such that
\[
C \subseteq \bigcup_{i=1}^n U_{\epsilon_i}(w_i).
\]
Thus there is some \( \epsilon \in (0, \delta/2) \) such that
\[
N(C, \epsilon) = \{ z \in \mathbb{C} \mid |z - w| < \epsilon \text{ for some } w \in C \} \subseteq \bigcup_{i=1}^n U_{\epsilon_i}(w_i).
\]

Therefore \( R(z) \) is analytic for \( z \in N(C, \epsilon) \). Hence \( R(z) \) is analytic for
\[
z \in V_\epsilon \subseteq N(C, \epsilon) \cup [ U_1(0)^a \setminus \{1\} ] \cup U_{0,\delta}(1).
\]
This completes the proof. \( \square \).

Lemma 2 shows that if \( \{ x(t) \} \) is a well-defined unit root process then there is some
\( \epsilon > 0 \) such that \( R(z) \) is analytic for \( z \in V_\epsilon \). It is still true that the function \( R_{\sin}(z) \) is
analytic for all \( z \neq 1 \) and hence is analytic at \( z = 0 \) with
\[
R_{\sin}(z) = \sum_{s \in \mathbb{N} - 1} U_s z^s
\]
for all \( z \in U_1(0) \) where the coefficients \( \{ U_s \}_{s \in \mathbb{N} - 1} \) are given by the formula
\[
U_s = (-1)(I_X - T_{-1}C_0)^{-s-1}T_{-1}
\]
for each \( s \in \mathbb{N} - 1 \). The resolvent \( R(z) = (A_0 + A_1 z)^{-1} \) is also analytic for \( z \in U_1(0) \) and
so \( A_0^{-1} \) is well defined. Hence we can use a Neumann expansion to write
\[
R(z) = \sum_{s \in \mathbb{N} - 1} R_s z^s
\]
for all \( z \in U_1(0) \) where
\[
R_s = (-1)^s(A_0^{-1}A_1)^s A_0^{-1}
\]
for all \( s \in \mathbb{N} - 1 \). Since \( R(z) \) is singular at \( z = 1 \) it follows that \( \lim_{s \to \infty} \| R_s \|^{1/s} = \lim_{s \to \infty} \| (A_0^{-1}A_1)^s \|^{1/s} = 1 \). We also know that
\[
R_{\text{reg}}(z) = R(z) - R_{\sin}(z)
\]
for $z \in U_{\epsilon}(1)$. Since $R(z)$ and $R_{\sin}(z)$ are analytic for $z \in \mathcal{V}_\epsilon$ and since $R_{\reg}(z)$ is analytic at $z = 1$ we can use (46) to extend $R_{\reg}(z)$ to an analytic function on $U_{1+\epsilon}(0)$. Therefore

$$R_{\reg}(z) = \sum_{s \in N-1} Q_s z^s$$

(47)

for $z \in U_{1+\epsilon}(0)$ where $Q_s \in \mathcal{B}(Y, X)$ and $\lim_{s \to \infty} \|Q_s\|^{1/s} \leq 1/(1 + \epsilon)$. Now (46) gives

$$Q_s = R_s - U_s = (-1)^s (A_0^{-1} A_1)^s A_0^{-1} + (I_X - T_0 C_0)^{-s-1} T_0$$

(48)

for each $s \in N - 1$. We have $C_0 = A_0 + A_1$ and so we can calculate $\{Q_s\}_{s \in N-1}$ from a knowledge of the coefficients $A_0$, $A_1$ and $T_0$.

The inequality $\lim_{s \to \infty} \|Q_s\|^{1/s} \leq 1/(1 + \epsilon)$ means that we can find a real constant $c > 0$ such that $\|Q_s\| \leq c/(1 + \epsilon)^s$ for all $s \in N - 1$. If we write $\zeta = z - 1$ we have

$$R_{\reg}(z) = \sum_{s \in N-1} Q_s (1 + \zeta)^s$$

$$= \sum_{s \in N-1} Q_s \left[ \sum_{\ell=0}^{s} \binom{s}{\ell} \zeta^\ell \right]$$

$$= \sum_{\ell \in N-1} \left[ \sum_{s \in N-1} \binom{s}{\ell} Q_s \right] \zeta^\ell$$

$$= \sum_{\ell \in N-1} \left[ \sum_{r \in N-1} \binom{\ell + r}{\ell} Q_{\ell + r} \right] (z - 1)^\ell$$

(49)

for all $z \in U_{\epsilon}(1)$. It follows from (49) that as well as the formula $T_\ell = (-1)^\ell (T_0 C_1)^\ell T_0$ for each $\ell \in N - 1$ we can also calculate $\{T_\ell\}_{\ell \in N-1}$ indirectly using the formula

$$T_\ell = \sum_{r \in N-1} \binom{\ell + r}{\ell} Q_{\ell + r}$$

(50)

for each $\ell \in N - 1$. Since $\|Q_{\ell + r}\| \leq c/(1 + \epsilon)^{\ell + r}$ it follows that (50) converges absolutely with $\|T_\ell\| \leq c(1 + \epsilon)/\epsilon^{\ell + 1}$. Before continuing we will illustrate our considerations with an elementary matrix example.

**Example 1.** Let $\epsilon \in (0, 1) \subseteq \mathbb{R}$ and define

$$A_0 = \begin{bmatrix} 1 & -\epsilon \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$ 

Therefore

$$A(z) = \begin{bmatrix} 1 - z & -\epsilon \\ 1 - z & 1 - z \end{bmatrix}$$

for all $z \in \mathbb{C}$ and

$$R(z) = (1 + \epsilon - z)^{-1} \begin{bmatrix} 1 & \epsilon(1 - z)^{-1} \\ -1 & 1 \end{bmatrix} = (1 + \epsilon - z)^{-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + (1 - z)^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

for all $z \in \mathbb{C}$ with $z \notin \{1, 1 + \epsilon\}$. If we wish to find expansions centred at $z = 1$ then we have $A(z) = C_0 + C_1 (z - 1)$ where

$$C_0 = \begin{bmatrix} 0 & -\epsilon \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$
and \( R(z) = T_{-1}(z - 1)^{-1} + \sum_{\ell \in \mathbb{N} - 1} T_{\ell}(z - 1)^{\ell} \) where

\[
T_{-1} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad T_{\ell} = \epsilon^{-(\ell+1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \epsilon^{-\ell} \begin{bmatrix} \epsilon^{-1} & -\epsilon^{-1} \\ -\epsilon^{-1} & \epsilon^{-1} \end{bmatrix} = \epsilon^{-\ell} T_0
\]

for each \( \ell \in \mathbb{N} - 1 \). Since \( \|T_{\ell}\|_2 = 2\epsilon^{-(\ell+1)} \) it follows that the Laurent series for \( R(z) \) centred at \( z = 1 \) converges for \( z \in \mathcal{U}_{0,\epsilon}(1) \). The spectral separation projections are

\[
P = T_{-1}C_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = C_1T_{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.
\]

We note that \( PT_{-1} = T_{-1} \Rightarrow T_{-1} \in P(X) \) and \( P^c T_{\ell} = (I - P)T_{\ell} = T_{\ell} \Rightarrow T_{\ell} \in P^c(X) \) for each \( \ell \in \mathbb{N} - 1 \). Although the singular part of the resolvent can be written directly in the form

\[
R_{\text{sin}}(z) = T_{-1}(z - 1)^{-1} = \sum_{s \in \mathbb{N} - 1} (-1)^s T_{-1} z^s \in P(X)
\]
as a legitimate Maclaurin transform the same direct approach for the regular part gives

\[
R_{\text{reg}}(z) = \sum_{\ell \in \mathbb{N} - 1} \epsilon^{-\ell} T_0 (z - 1)^{\ell} = \sum_{\ell \in \mathbb{N} - 1} (-1)^\ell \epsilon^{-\ell} T_0 \sum_{r=0}^{\ell} \binom{\ell}{r} z^r = \sum_{r \in \mathbb{N} - 1} \left[ \sum_{\ell \in \mathbb{N} + r - 1} (-1)^\ell \epsilon^{-\ell} T_0 z^r \right] = \sum_{r \in \mathbb{N} - 1} \left[ \sum_{m \in \mathbb{N} - 1} \binom{m+r}{r} (-1)^m \epsilon^{-m} T_0 (z/\epsilon)^r \right]
\]

which is not a legitimate Maclaurin transform because each coefficient of \( z^r \) involves the sum of a divergent series. Thus we must take a different approach. We can write

\[
R_{\text{reg}}(z) = R(z) - R_{\text{sin}}(z) = (1 + \epsilon - z)^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - (1 - z)^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

for all \( z \in \mathcal{U}_{1+\epsilon}(0) \).

It is important to note that in Example 1 we have \( Q_s \in P^c(X) \) for all \( s \in \mathbb{N} - 1 \). We will now argue that this is true in general.

**Lemma 3.** Let

\[
Q(z) = \begin{cases} 
R(z) - R_{\text{sin}}(z) & \text{for } z \in \mathcal{U}_{1+\epsilon}(0) \setminus \{1\} \\
R_{\text{reg}}(z) & \text{for } z = 1
\end{cases}
\]

and suppose that the Maclaurin series representation is given by \( Q(z) = \sum_{s \in \mathbb{N} - 1} Q_s z^s \) for all \( z \in \mathcal{U}_{1+\epsilon}(0) \). Then \( Q_s \in B(P^c(X)) \) for all \( s \in \mathbb{N} - 1 \). \( \square \)
Proof. Let \( P = T_{-1}C_1 \) and let \( PQ(z) = \sum_{s \in \mathbb{N} - 1} PQ_s z^s \) for \( z \in U_{1+\epsilon}(0) \) denote the projection of \( Q(z) \) onto the subspace \( P(X) \). We have \( PT_\ell = 0 \) for all \( \ell \in \mathbb{N} - 1 \) and so
\[
PQ(z) = P[R(z) - R_{\sin}(z)] = PR_{\text{reg}}(z) = \sum_{\ell \in \mathbb{N} - 1} PT_\ell(z - 1)^\ell = 0
\]
for all \( z \in U_\epsilon(1) \). Since \( PQ(z) \equiv 0 \) on a non-trivial open set it follows by analytic continuation that \( PQ(z) \equiv 0 \) for all \( z \in U_{1+\epsilon}(0) \) and hence that \( PQ_s = 0 \) for all \( s \in \mathbb{N} - 1 \). Thus we deduce that \( Q_s = P^c Q_s \in \mathcal{B}(P^c(X)) \) for all \( s \in \mathbb{N} - 1 \). \( \square \)

We can now find a more general representation for \( \{x_{\text{reg},+}(t)\}_{t \in \mathbb{Z}} \). We have
\[
X_{\text{reg}}(z) = R_{\text{reg}}(z)[G(z) - C_1 c]
= \sum_{s \in \mathbb{N} - 1} Q_s z^s \sum_{r \in \mathbb{N} - 1} (g - C_1 c \delta_+(r)) z^r
= \sum_{t \in \mathbb{N} - 1} \left[ \sum_{s \in \mathbb{N} - 1} Q_s (g - C_1 c \delta_+(t - s)) \right] z^t
= \sum_{t \in \mathbb{N} - 1} \left[ \sum_{s=0}^{t} Q_s (g(t - s)) - Q_t C_1 c \right] z^t
= \mathcal{M} \left[ \{ \sum_{s=0}^{t} Q_s g(t - s) \}_{t \in \mathbb{N} - 1} - \{ Q_t C_1 c \}_{t \in \mathbb{N} - 1} \right] (z)
\]
for all \( z \in U_{1+\epsilon}(0) \). Therefore
\[
x_{\text{reg}}(t) = \sum_{s=0}^{t} Q_s (g(t - s)) - Q_t C_1 c
\]
for all \( t \in \mathbb{N} - 1 \). It is now possible to state a more general form of the non-standard Granger–Johansen representation.

Proposition 2. If \( R(z) \) is analytic on the set \( \mathcal{V}_\epsilon \) for some \( \epsilon > 0 \) then we can write
\[
x(t) = \sum_{k \in \mathbb{N}} (-1)^k T_{-k} \nabla^{-k} g_+(t) - U_tC_1 c + \sum_{s=0}^{t} Q_s g(t - s) - Q_t C_1 c
\]
for all \( t \in \mathbb{N} - 1 \). \( \square \)

The term
\[
\hat{h}_t(t) = \sum_{s=0}^{t} Q_s g(t - s)
\]
is observable but is only asymptotically stationary. If we define
\[
\hat{h}_\infty(t) = \sum_{s \in \mathbb{N} - 1} Q_s g(t - s)
\]
for all \( t \in \mathbb{N} - 1 \) then we have
\[
\hat{h}_t(t) = \hat{h}_\infty(t) - \sum_{s \in \mathbb{N} + t} Q_s g(t - s) = \hat{h}_\infty(t) - \sum_{r \in \mathbb{N}} Q_{r+t} g(-r)
\]
for all \( t \in \mathbb{N} - 1 \). The extended term \( \hat{h}_\infty(t) \) is strictly stationary but is no longer \textit{a posteriori} observable.

The original relationships (2) and (3) show that
\[
A_0 x(t) + A_1 x(t - 1) = g(t)
\]
for all \( t \in \mathbb{N} \). Therefore
\[
x(t) = A_0^{-1}g(t) - A_0^{-1}A_1x(t-1)
\]
for all \( t \in \mathbb{N} \). If we extend this relationship to all \( t \in \mathbb{Z} \) then repeated application gives
\[
x(-1) = \sum_{r=1}^{n}(-1)^{r-1}(A_0^{-1}A_1)^{-1}A_1g(-r) + (-1)^{n}(A_0^{-1}A_1)^{n}x(-n-1)
\]
for each \( n \in \mathbb{N} \). We can also write the original relationship in the form
\[
(C_0 - C_1)x(t) + C_1x(t-1) = g(t)
\]
for all \( t \in \mathbb{N} \). Thus we have
\[
T_{-1}C_0x(t) - Px(t) + Px(t-1) = T_{-1}g(t)
\]
where we have written \( P = T_{-1}C_1 \). Since \( T_{-1}C_0T_0 = 0 \) and \( T_{-1}C_1 + T_0C_0 = I_X \) it follows that \( P x(t) - T_{-1}C_0x(t) = (I_X - T_{-1}C_0)Px(t) \). Some elementary algebra now shows that
\[
P x(t) = (-1)(I_X - T_{-1}C_0)^{-1}T_{-1}g(t) + (I_X - T_{-1}C_0)^{-1}Px(t-1)
\]
for all \( t \in \mathbb{N} \). If we extend this relationship to all \( t \in \mathbb{Z} \) then repeated application gives
\[
P x(-1) = \sum_{r=1}^{n}(-1)(I_X - T_{-1}C_0)^{-r}T_{-1}g(-r) + (I_X - T_{-1}C_0)^{-n}Px(-n-1)
\]
for each \( n \in \mathbb{N} \). By applying (55) and (56) and using (25) and (45) we deduce that
\[
(-1)^{t+1}R_tC_1x(-1) - U_tC_1x(-1) = (-1)^{t+1}(A_0^{-1}A_1)^{t+1}A_1x(-1) - (I_X - T_{-1}C_0)^{-t-1}Px(-1)
\]
\[
= \sum_{r=1}^{n}(-1)^{r+t}(A_0^{-1}A_1)^{r+t}A_1g(-r) + (-1)^{n+t+1}(A_0^{-1}A_1)^{n+t}A_1x(-n-1)
\]
\[
+ \sum_{r=1}^{n}(I_X - T_{-1}C_0)^{-r-t-1}T_{-1}g(-r) - (I_X - T_{-1}C_0)^{-n-t-1}Px(-n-1)
\]
\[
= \sum_{r=1}^{n}Q_{t+r}g(-r) - R_{n+t}C_1x(-n-1) + U_{n+t}C_1x(-n-1)
\]
for all \( t \in \mathbb{N} - 1 \) and all \( n \in \mathbb{N} \). Rearranging and writing \( x(-1) = c \) gives
\[
\sum_{r=1}^{n}Q_{t+r}g(-r) = [R_{n+t}C_1x(-n-1) - U_{n+t}C_1x(-n-1)] - [R_tC_1c - U_tC_1c]
\]
\[
= Q_{n+t}C_1x(-n-1) - Q_tC_1c
\]
for all \( t \in \mathbb{N} - 1 \). We have
\[
\mathbb{E}[\|\sum_{r=1}^{n}Q_{t+r}g(-r)\|] \leq c\mathbb{E}[\|g\|]\sum_{r=1}^{n}1/(1 + \epsilon)^{t+r} = c\mathbb{E}[\|g\|]/[\epsilon(1 + \epsilon)^t] \rightarrow 0
\]
as \( t \to \infty \). Hence (58) shows that we can define a \( \mu \)-integrable time series \( \{k(t)\}_{t \in \mathbb{N} - 1} \) by setting
\[
k(t) = \lim_{n \to \infty} [Q_{n+t}C_1x(-n-1) - Q_tC_1c]
\]
for each \( t \in \mathbb{N} - 1 \). Thus we can now write
\[
\sum_{r \in \mathbb{N}} Q_{t+r}g(-r) = k(t)
\]
for all \( t \in \mathbb{N} - 1 \). Therefore
\[
h_{t}(t) = \hat{h}(t) - k(t)
\]
for all \( t \in \mathbb{N} - 1 \). This relationship allows us to state a general form of the standard Granger–Johansen representation. This is our main result.
Theorem 2. If $R(z)$ is analytic on the set $\mathcal{V}_e$ for some $\epsilon > 0$ and if $x(-1) = c$ then we can write
\[
x(t) = \sum_{k \in \mathbb{N}} (-1)^k T_{-k} V^{-k} g_+(t) - U_t C_1 c + \sum_{s \in \mathbb{N} - 1} Q_s g(t - s) - Q_t C_1 c - k(t)
\] for all $t \in \mathbb{N} - 1$.

The extended representation in Theorem 2 is more robust than the natural representation in Theorem 1 and while we can no longer use the coefficients $\{T_\ell\}_{\ell \in \mathbb{N} - 1}$ in the extended representation the spectral separation of the singular and regular parts is nevertheless preserved. In each case we have $X_{\sin}(z) \in P(X)$ and $X_{\reg}(z) \in P^c(X)$.

7. The resolvent operator for a linear pencil on Banach space

The methods described in this paper depend on a systematic methodology for calculation of the resolvent operator for a linear pencil on Banach space. The basic methodology was introduced earlier in Section 6.2. In this section we pick up where that earlier discussion left off. Albrecht et al. [2, 4, 5] show that the resolvent operator $R(z) = (C_0 + C_1 (z - 1))^{-1}$ exists and is defined by the Laurent series $R(z) = \sum_{j \in \mathbb{Z}} T_j (z - 1)^j$ if and only if the coefficients $\{T_j\}_{j \in \mathbb{Z}}$ satisfy the left and right fundamental equations (8), (2) and the magnitude constraint (10) in which case the coefficients are defined by (11) and (12). It follows from (10) and (11) that
\[
\lim_{k \to \infty} ||T_{-k}||^{1/k} = 0 \iff \lim_{k \to \infty} ||(T_{-1} C_0)^k||^{1/k} = 0 \iff \lim_{k \to \infty} ||(C_0 T_{-1})^k||^{1/k} = 0.
\]

We will show that the operators $T_{-1} C_0 \in B(X)$ and $C_0 T_{-1} \in B(Y)$ are either both nilpotent of index $d$ for some finite $d \in \mathbb{N}$ or both quasi-nilpotent. We argue as follows. If $(T_{-1} C_0)^k \neq 0_X$ then $(T_{-1} C_0)^k (T_{-1} C_1 + T_0 C_0) \neq 0_X$. Since $T_{-1} C_0 T_0 = 0_{Y,X}$ we know that $T_{-1} C_0 (T_{-1} C_0)^k C_1 \neq 0_X$. Therefore $(C_0 T_{-1})^k \neq 0_Y$. A similar argument shows that the reverse implication is also true. In the first case $T_{-k} = 0_{Y,X}$ for all $k \geq d + 1$ and the resolvent $R(z)$ has a pole of order $d$ at $z = 1$. In the second case $T_{-k} \neq 0_{Y,X}$ for all $k \in \mathbb{N}$ and the resolvent $R(z)$ has an isolated essential singularity at $z = 1$.

We can find the resolvent $R(z)$ by finding a basic solution to the fundamental equations. The task is much easier if we begin by finding the key spectral separation projections $P = T_{-1} C_1 \in B(X)$ and $Q = C_1 T_{-1} \in B(Y)$.

7.1. Calculation of the key spectral separation projections

Albrecht et al. [3] have used infinite-length Jordan chains to find the projections $P \in B(X)$ and $Q \in B(Y)$. The set $J_{\sin} \subseteq X^N$ of all infinite-length singular Jordan chains for the pencil $A(z)$ on $\mathcal{U}_s(1)$ is the set of sequences $\{x_{-n}\}_{n \in \mathbb{N}} \in X^N$ such that $C_0 x_{-n} + C_1 x_{-n-1} = 0_Y$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} ||x_{-n}||^{1/n} \leq s$. The subspace $X_{\sin} = \{x_{-1} \mid \{x_{-n}\}_{n \in \mathbb{N}} \in J_{\sin}\}$
The key projections. The set \( J_{\text{reg}} \subseteq X^N \) of all infinite-length regular Jordan chains for the pencil \( A(z) \) on \( U_{0,r}(1) \) is the set of sequences \( \{x_n\}_{n \in \mathbb{N}} \in X^N \) such that \( C_0 x_{n+1} + C_1 x_n = 0 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \|x_n\|^{1/n} \leq 1/r \). The subspace
\[
X_{\text{reg}} = \{ x_1 \mid \{ x_n \}_{n \in \mathbb{N}} \in J_{\text{reg}} \}
\]
is the generating set for \( J_{\text{reg}} \). Albrecht et al. [5] show that \( X_{\text{sin}} = P(X) \) and \( X_{\text{reg}} = P^c(X) = (I_X - P)(X) \) and that \( X = X_{\text{sin}} \oplus X_{\text{reg}} = X_{\text{sin}} \times X_{\text{reg}}. \) They also show that \( Y_{\text{sin}} = Q(Y) = C_0(X_{\text{sin}}) \) and \( Y_{\text{reg}} = Q^c(Y) = (I_Y - Q)(Y) = C_1(X_{\text{reg}}) \) and that \( Y = Y_{\text{sin}} \oplus Y_{\text{reg}} = Y_{\text{sin}} \times Y_{\text{reg}}. \)

The spectral separation projections \( P \in B(X) \) and \( Q \in B(Y) \) can be found by calculating general solutions to the defining equations for the Jordan chains subject to the limit requirements on the chains determined by the annular region \( U_{s,r}(1) \). This allows us to find the spaces \( X_{\text{sin}} \) and \( X_{\text{reg}} \) in the domain space and the corresponding spaces \( Y_{\text{sin}} \) and \( Y_{\text{reg}} \) in the range space and hence to find the key projections.

For matrix pencils the full generality of the Jordan chain methodology is not necessary. This is also true for some operator pencils on infinite-dimensional spaces. We will consider a set of specific examples in Section 5 to illustrate various options.

### 7.2. Separation of the fundamental equations

The key projections \( P = T_{-1}C_1 \in B(X) \) and \( Q = C_1T_{-1} \in B(Y) \) allow us to separate each of \( (\mathfrak{S}) \) and \( (\mathfrak{Q}) \) into two singly infinite sets of equations—one for the singular part of the resolvent and the other for the regular part. We prefer to write \( M = P(X) = X_{\text{sin}} \) and \( M^c = P^c(X) = X_{\text{reg}} \) so that \( X = M \times M^c \) and \( N = Q(Y) = Y_{\text{sin}} \) and \( N^c = Q^c(Y) = Y_{\text{reg}} \) so that \( Y = N \times N^c \). Now we define restricted operators \( \mathfrak{C}_i \in B(M,N) \), \( \mathfrak{C}^c_i \in B(M^c,N^c) \) for each \( i = 0, 1, \mathfrak{T}_{-k} \in B(N, M) \) for each \( k \in \mathbb{N} \) and \( \mathfrak{T}_\ell \in B(N^c, M^c) \) for each \( \ell \in \mathbb{N} - 1 \) by setting
\[
\mathfrak{C}_i \cong \begin{bmatrix} Q & P \\ Q^c & P^c \end{bmatrix} C_i \begin{bmatrix} P & P^c \\ P^cC_i & P^cC_i^c \end{bmatrix} = \begin{bmatrix} \mathfrak{C}_i & 0_{M^c,N} \\ 0_{M,N^c} & \mathfrak{C}^c_i \end{bmatrix},
\]
\[
T_{-k} \cong \begin{bmatrix} P & P^c \\ P^cT_{-k} & P^cT_{-k}^c \end{bmatrix} = \begin{bmatrix} \mathfrak{T}_{-k} & 0_{N^c,M} \\ 0_{N,M^c} & \mathfrak{T}^c_{-k} \end{bmatrix}
\]
and
\[
T_\ell \cong \begin{bmatrix} P & P^c \\ P^cT_\ell & P^cT_\ell^c \end{bmatrix} = \begin{bmatrix} 0_{N,M} & 0_{N^c,M^c} \\ 0_{N,M^c} & \mathfrak{T}^c_\ell \end{bmatrix}.
\]

If we restrict our attention to \( M \) the left fundamental equations \( (\mathfrak{S}) \) become a singly infinite set of left fundamental equations for the singular part of the resolvent given by
\[
\mathfrak{T}_{-1}\mathfrak{C}_1 = \mathfrak{T}_{-k-1}\mathfrak{C}_1 + \mathfrak{T}_{-k}\mathfrak{C}_0 = 0_M \quad \text{for } k \in \mathbb{N}
\]

(63)
where \( \mathcal{I}_M \in \mathcal{B}(M) \) denotes the identity operator on \( M \). If we restrict our attention to \( M^c \) the left fundamental equations (8) become a singly infinite set of left fundamental equations for the regular part of the resolvent given by

\[
\mathcal{T}_0^c \mathcal{T}_0^c = \mathcal{I}_{M^c} \quad \mathcal{T}_{\ell-1}^c \mathcal{C}_0^c + \mathcal{T}_{\ell}^c \mathcal{C}_0^c = 0_{M^c} \quad \text{for } \ell \in \mathbb{N} \tag{64}
\]

where \( \mathcal{I}_{M^c} \in \mathcal{B}(M^c) \) denotes the identity operator on \( M^c \). The systems (63) and (64) are completely separate. If we restrict our attention to \( N \) the right fundamental equations (9) become a singly infinite set of right fundamental equations for the singular part of the resolvent given by

\[
\mathcal{C}_1 \mathcal{T}_{-1} = \mathcal{I}_N \quad \mathcal{C}_1 \mathcal{T}_{-k-1} + \mathcal{C}_0 \mathcal{T}_{-k} = 0_N \quad \text{for } k \in \mathbb{N} \tag{65}
\]

where \( \mathcal{I}_N \in \mathcal{B}(N) \) denotes the identity operator on \( N \). If we restrict our attention to \( N^c \) the right fundamental equations (9) become a singly infinite set of right fundamental equations for the regular part of the resolvent given by

\[
\mathcal{C}_0^c \mathcal{T}_{\ell-1}^c + \mathcal{C}_0^c \mathcal{T}_{\ell}^c = 0_{N^c} \quad \text{for } \ell \in \mathbb{N} \tag{66}
\]

where \( \mathcal{I}_{N^c} \in \mathcal{B}(N^c) \) denotes the identity operator on \( N^c \). The systems (65) and (66) are completely separate. In infinite-dimensional space the analysis depends on both the left and right sets of fundamental equations. We need both \( \mathcal{T}_{-1} \mathcal{C}_1 = \mathcal{I}_M \) and \( \mathcal{C}_1 \mathcal{T}_{-1} = \mathcal{I}_N \) to deduce that \( \mathcal{T}_{-1} = \mathcal{C}_1^{-1} \in \mathcal{B}(N, M) \) is the uniquely-defined inverse of \( \mathcal{C}_1 \). Similarly we need both \( \mathcal{T}_0^c \mathcal{C}_0^c = \mathcal{I}_{M^c} \) and \( \mathcal{C}_0^c \mathcal{T}_0^c = \mathcal{I}_{N^c} \) to deduce that \( \mathcal{T}_0^c = [\mathcal{C}_0^c]^{-1} \in \mathcal{B}(N^c, M^c) \) is the uniquely-defined inverse of \( \mathcal{C}_0^c \).

The existence of \( \mathcal{C}_1^{-1} \) and \( [\mathcal{C}_0^c]^{-1} \) means that \( M \) is isomorphic to \( N \) and that \( M^c \) is isomorphic to \( N^c \). Thus \( X = M \times M^c \) and \( Y = N \times N^c \) are isomorphic. Nevertheless there are situations where we may wish to regard these isomorphic spaces as different. See [2, 28] and [2] pp 282–285] for some specific instances.

### 7.3. Solution of the fundamental equations

The systems (63) and (65) have a unique solution \( \{ \mathcal{T}_{-k} \}_{k \in \mathbb{N}} \) where

\[
\mathcal{T}_{-k} = (-1)^{k-1} (\mathcal{C}_1^{-1} \mathcal{C}_0)^{k-1} \mathcal{C}_1^{-1} \quad \text{for each } k \in \mathbb{N} \tag{67}
\]

for each \( k \in \mathbb{N} \) and the systems (64) and (66) have a unique solution \( \{ \mathcal{T}_{\ell} \}_{\ell \in \mathbb{N}-1} \) where

\[
\mathcal{T}_{\ell} = (-1)^{\ell} ([\mathcal{C}_0^c]^{-1} \mathcal{C}_1^c)^{\ell} [\mathcal{C}_0^c]^{-1} \quad \text{for each } \ell \in \mathbb{N}-1 \tag{68}
\]

for each \( \ell \in \mathbb{N}-1 \). The formulæ (67) and (68) validate (11) and (12). It follows that if we can find the key spectral separation projections \( P \in \mathcal{B}(X) \) and \( Q \in \mathcal{B}(Y) \) on \( \mathcal{U}_{s,r}(1) \) then it is a simple matter to define the resolvent operator by the Laurent series

\[
R(z) = \sum_{j \in \mathbb{Z}} T_j (z - 1)^j \text{ for all } z \in \mathcal{U}_{s,r}(1).
\]
8. Some specific applications

We consider three specific examples to demonstrate our proposed method for calculation of the resolvent operator. Example 2 is taken from Seo [41, Example 4.1, pp 13–14 and Example 4.3, p 15]. Example 3 concerns a Volterra integral operator and amplifies the discussions in Seo [41, version 4, Example 4.2, p 14] and Spanenberg [45, Example 3.4, p 135]. Example 4 is a discrete time analogue with added noise of a model proposed by Albrecht et al. [4, Section 6, pp 170–173]. In each case \( X, Y \) are separable Banach spaces and \( A(z) \in B(X, Y) \) is a linear operator pencil. Our primary task is to find the resolvent operator \( R(z) = A(z)^{-1} \in B(Y, X) \) on an annular region \( U_{0, r}(1) \) for some \( r > 0 \). In Examples 2 and 4 we make explicit use of Jordan chains to find the key projection operators and hence solve the fundamental equations. In Example 3, where \( P^c = 0_X \iff Q^c = 0_Y \), we show that overt use of Jordan chains can be avoided by using a Neumann expansion instead. In this case the regular part of the Laurent series is absent. Our aim is to demonstrate that the existing Hilbert space methods—which invariably depend on a knowledge of advanced functional analysis—can be replaced by systematic procedures that are more intuitive and more general.

Example 2. Let \( X = Y = c_0 \) be the Banach space of all sequences \( x = \{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}^\mathbb{N} \) with \( \lim_{n \to \infty} x_n = 0 \) and norm defined by \( \|x\|_\infty = \max_{n \in \mathbb{N}} |x_n| \). The space \( c_0 \) is a closed separable subspace of \( \ell^\infty \) with the standard basis \( \{e_j\}_{j \in \mathbb{N}} \). Let \( \lambda \in \mathbb{R} \) with \( \lambda \in (0, 1) \) and define \( A(z) = A_0 + A_1z \in B(X, Y) \) where \( A_0 = I \) is the identity mapping and

\[
A_1 = (-1) \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \lambda^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.
\]

Near \( z = 1 \) we can write \( A(z) = A_0 + A_1z = C_0 + C_1(z - 1) \) where

\[
C_0 = I + A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 - \lambda & 0 & \cdots \\ 0 & 0 & 0 & 1 - \lambda^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\]

and \( C_1 = A_1 \). We wish to find a Laurent expansion for the resolvent operator \( R(z) = A(z)^{-1} \in B(Y, X) \) in the form of a Laurent series \( R(z) = \sum_{j \in \mathbb{Z}} T_j (z - 1)^j \) which converges on some region \( U_{0, \rho}(1) = \{z \in \mathbb{C} \mid 0 < |z - 1| < \rho\} \). Thus we need to find \( \{T_j\}_{j \in \mathbb{Z}} \in B(Y, X) \) satisfying the left fundamental equations (8) and the right fundamental equations (9). It is sufficient to find the basic solution \( \{T_{-1}, T_0\} \in B(Y, X) \) on some region \( U_{0, \rho}(1) \).
We begin by finding a general form for the infinite-length singular Jordan chains \( \{x_n\}_{n \in \mathbb{N}} \) satisfying
\[
C_0 x_n + C_1 x_{n-1} = 0
\]
for all \( n \in \mathbb{N} \) with \( \|x_n\|_{1/n}^\infty \to 0 \) as \( n \to \infty \). We have
\[
-x_{n,2} - x_{n-1,1} - x_{n-1,2} = 0
\]
\[
-x_{n-1,2} = 0
\]
\[
(1 - \lambda)x_{n,3} - \lambda x_{n-1,3} = 0
\]
\[
(1 - \lambda^2)x_{n,4} - \lambda^2 x_{n-1,4} = 0
\]
\[
\vdots \quad \vdots
\]
from which we deduce
\[
x_{n-1,1} = -x_{n,2}
\]
\[
x_{n-1,2} = 0
\]
\[
x_{n-1,3} = (1/\lambda - 1)x_{n,3}
\]
\[
x_{n-1,4} = (1/\lambda^2 - 1)x_{n,4}
\]
\[
\vdots \quad \vdots
\]
and so on. If we let \( x_{-1} = t = \{t_j\}_{j \in \mathbb{N}} \in c_0 \) then we have \( x_{-n,2} = 0 \) for \( n \geq 2 \) and \( x_{-n,1} = -x_{-n+1,2} = 0 \) for \( n \geq 3 \). In general we have \( x_{-n,j} = (1/\lambda^j - 1)^{n-1}t_j \) for all \( n \in \mathbb{N} \) and all \( j \geq 3 \). If \( t_j \neq 0 \) then
\[
\|x_{-n}\|_{1/n}^\infty \geq |1/\lambda^j - 1|^{1-1/n}|t_j|^{1/n} \to |1/\lambda^j - 1|
\]
as \( n \to \infty \). Since \( \lambda \neq 1 \) it is not possible to have \( \lim_{n \to \infty} \|x_{-n}\|_{1/n}^\infty = 0 \). Consequently we must choose \( t_j = 0 \) for \( j \geq 3 \). Therefore the subspace \( X_{\sin} \subseteq c_0 \) generated by the infinite-length singular Jordan chains is the closed subspace \( S(\{e_1, e_2\}) \) generated by \( e_1 \) and \( e_2 \).

Our next step is to find a general form for the infinite-length regular Jordan chains \( \{x_n\}_{n \in \mathbb{N}} \) satisfying
\[
C_0 x_{n+1} + C_1 x_n = 0
\]
for all \( n \in \mathbb{N} \) with \( \|x_n\|_{1/n}^\infty \to \rho \) for some \( \rho > 0 \). We have
\[
-x_{n+1,2} - x_{n,1} - x_{n,2} = 0
\]
\[
-x_{n,2} = 0
\]
\[
(1 - \lambda)x_{n+1,3} - \lambda x_{n,3} = 0
\]
\[
(1 - \lambda^2)x_{n+1,4} - \lambda^2 x_{n,4} = 0
\]
\[
\vdots \quad \vdots
\]
from which we deduce

\[
\begin{align*}
\begin{bmatrix}
  x_{n+1,2} & = & -x_{n,1} \\
  x_{n,2} & = & 0 \\
  x_{n+1,3} & = & [1/(1 - \lambda) - 1]x_{n,3} \\
  x_{n+1,4} & = & [1/(1 - \lambda^2) - 1]x_{n,4} \\
  & \vdots & \vdots
\end{bmatrix}
\end{align*}
\]

and so on. If we set \( x_1 = s = \{s_j\}_{j \in \mathbb{N}} \) then \( x_{1,2} = s_2 = 0 \). We also have \( x_{2,2} = 0 \) and \( x_{2,1} = -s_1 \). Therefore \( s_1 = 0 \) as well. The remaining equations show that \( x_{n,j} = [1/(1 - \lambda^j) - 1]^{n-1}s_j \) for all \( n \in \mathbb{N} \) and all \( j \geq 3 \). If we choose \( x_1 = e_k \) for some \( k \geq 3 \) then \( x_n = [1/(1 - \lambda^{k-2}) - 1]^{n-1}e_k \) and hence

\[
\|x_n\|^{1/n}_\infty = [1/(1 - \lambda^{k-2}) - 1]^{1-1/n} \to [1/(1 - \lambda^{k-2}) - 1] = \lambda^{k-2}/(1 - \lambda^{k-2}) \leq \lambda/(1 - \lambda)
\]

as \( n \to \infty \). It follows that if we define the subspace \( X_{\text{reg}} \subset c_0 \) as the closed subspace \( S(\{e_3, e_4, \ldots\}) \) generated by \( \{e_n\}_{n \in \mathbb{N} + 2} \) then \( X = X_{\text{sin}} \oplus X_{\text{reg}} \cong X_{\text{sin}} \times X_{\text{reg}} \) is the desired spectral decomposition of the domain space for the region \( U_{0, \rho} \) where \( \rho = 1/\lambda - 1 \). The corresponding decomposition for \( Y = c_0 \) is defined by \( Y_{\text{sin}} = C_1(X_{\text{sin}}) \) and \( Y_{\text{reg}} = C_0(X_{\text{reg}}) \). We have

\[
C_1 e_1 = -e_1 \quad \text{and} \quad C_1 e_2 = -e_1 - e_2
\]

which means that \( Y_{\text{sin}} \subset c_0 \) is the subspace \( S(\{e_1, e_2\}) \) and

\[
C_0 e_j = (1 - \lambda^j)e_j
\]

for each \( j \geq 3 \). Therefore \( Y_{\text{reg}} \subset c_0 \) is the subspace \( S(\{e_3, e_4, \ldots\}) \) and \( Y = Y_{\text{sin}} \oplus Y_{\text{reg}} \cong Y_{\text{sin}} \times Y_{\text{reg}} \) is the corresponding spectral decomposition of the range space on \( U_{0, \rho}(1) \). The corresponding key projections are clearly

\[
P = Q = \begin{bmatrix}
  1 & 0 & 0 & 0 & \cdots \\
  0 & 1 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

The basic solution is found by solving the equations \( T_{-1}C_0 = P \) and \( C_0 T_{-1} = Q \) to find \( T_{-1} \) and the equations \( T_0C_0 = P^c = I - P \) and \( C_0 T_0 = Q^c = I - Q \) to find \( T_0 \). Some elementary algebra gives gives

\[
T_{-1} = \begin{bmatrix}
  -1 & 1 & 0 & 0 & \cdots \\
  0 & -1 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\quad \text{and} \quad
T_0 = \begin{bmatrix}
  0 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & \cdots \\
  0 & 1/(1 - \lambda) & 0 & \cdots \\
  0 & 0 & 1/(1 - \lambda^2) & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

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Although the algebra is simply Gaussian elimination we note that calculation of $T_0$ requires consideration of both $T_0C_0 = P^c$ and $C_0T_0 = Q^c$. Now we can calculate the entire Laurent series with

$$T_{-k} = (-1)^{k-1}(T_{-1}C_0)^{k-1}T_{-1}$$

for all $k \in \mathbb{N}$ which gives

$$T_{-2} = (-1)T_{-1}C_0T_{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and $T_{-k} = 0$ for $k \geq 3$. Therefore $R(z)$ has a pole of order 2 at $z = 1$. We can also calculate

$$T_\ell = (-1)^\ell(T_0C_1)^\ell T_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/(1 - \lambda)^\ell & 0 & \cdots \\ 0 & 0 & 0 & 1/(1 - \lambda^2)^\ell & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

for each $\ell \in \mathbb{N} - 1$. The singular part of the resolvent is

$$R_{\sin}(z) = \begin{bmatrix} -1/(z - 1) & -1/(z - 1)^2 + 1/(z - 1) & 0 & 0 & \cdots \\ 0 & -1/(z - 1) & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and the regular part is

$$R_{\text{reg}}(z) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & (1 - \lambda)/(1 - \lambda - (z - 1)) & 0 & \cdots \\ 0 & 0 & (1 - \lambda^2)/(1 - \lambda^2 - (z - 1)) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and the Laurent series converges for $z \in U_{0,\rho}(1)$ where $\rho = 1/(1 - \lambda)$. Our series solution relies only on intuitive use of elementary algebra and is identical to the solution found by Seo [41] on the same example. \qed

**Example 3.** Let $X = Y = L^2([0, 1])$ be the Hilbert space of all complex-valued square-integrable functions $x : [0, 1] \to \mathbb{C}$. Let $V \in \mathcal{B}(X,Y)$ be the Volterra integral operator defined by

$$[Vx](t) = \int_{[0,t]}x(s)ds$$
for all $t \in [0, 1]$. We consider the operator pencil $A(z) = I - z(I - V) = V - (z - 1)(I - V)$. We will show that the resolvent $R(z) = A(z)^{-1}$ has an isolated essential singularity at $z = 1$. We will prove that $V$ is quasi-nilpotent but not nilpotent. That is we will show that $\|V^n\|^{1/n} \to 0$ as $n \to \infty$ but that $V^n \neq 0$ for all $n \in \mathbb{N}$. Clearly $V^n 1 = \frac{t^n}{n!}$ where we have defined $1(t) = 1$ and $t^n(t) = t^n$ for all $t \in [0, 1]$ and all $n \in \mathbb{N}$. Hence $V^n \neq 0$ for all $n \in \mathbb{N}$. The Cauchy–Schwartz inequality shows that

$$|V x(t)| \leq \left[\int_{[0, t]} |x(s)|^2ds\right]^{1/2} \left[\int_{[0, t]} 1^2ds\right]^{1/2} \leq \|x\| \cdot t^{1/2} \leq \|x\|$$

for all $t \in [0, 1]$. Now it follows $|V^n x(t)| \leq \|x\| t^{n-1}/(n - 1)! \leq \|x\|/(n - 1)!$ for all $t \in [0, 1]$ and all $n \in \mathbb{N}$. Consequently $\|V^n\| \leq 1/(n - 1)!$ for all $n \in \mathbb{N}$. Choose $p \in \mathbb{N}$. An elementary argument shows that $(n - 1)! > p^n$ for $n \in \mathbb{N}$ sufficiently large. Therefore we can find $m = m(p) \in \mathbb{N}$ such that $\|V^n\|^{1/n} < 1/p$ for all $n \geq m$. Since $p \in \mathbb{N}$ was chosen arbitrarily it follows that $\lim_{n \to \infty} \|V^n\|^{1/n} = 0$.

The adjoint operator $V^* \in \mathcal{B}(Y, X)$ is defined by

$$\langle V^* y, x \rangle = \langle y, V x \rangle = \int_{[0, 1]} y(t) \left[\int_{[0, t]} \overline{f}(s)ds\right] dt = \int_{[0, 1]} \left[\int_{[s, 1]} y(t)dt\right] \overline{f}(s)ds$$

for all $x \in X$ and $y \in Y$. Therefore

$$V^* y(s) = \int_{[s, 1]} y(t)dt$$

for all $s \in [0, 1]$ and so

$$V^* V x(s) = \int_{[s, 1]} \left[\int_{[0, t]} x(r)dr\right] ds$$

for all $x \in X$. Since $V^* V$ is self-adjoint and compact it has a discrete spectrum on the non-negative real axis [49, pp 282–286]. Suppose $x \in X$ is an eigenvector corresponding to a real eigenvalue $\lambda \geq 0$. Then we must have

$$\int_{[s, 1]} \left[\int_{[0, t]} x(r)dr\right] ds = \lambda x(s)$$

and differentiation shows that

$$(-1) x(s) = \lambda x''(s)$$

for all $s \in [0, 1]$. Therefore

$$x(s) = c \cos(s/\sqrt{\lambda}) + d \sin(s/\sqrt{\lambda}).$$

Direct calculation gives

$$\int_{[s, 1]} \left[\int_{[0, t]} x(r)dr\right] dt = \lambda [c \cos(s/\sqrt{\lambda}) + d \sin(s/\sqrt{\lambda})]$$

$$- [c \cos(1/\sqrt{\lambda}) + d \sin(1/\sqrt{\lambda})] + d\sqrt{\lambda}[1 - s]$$
and so we must have $d = 0$ and $\cos(1/\sqrt{\lambda}) = 0$. Thus the eigenvalues are

$$\lambda_n = \frac{4}{(2n + 1)^2 \pi^2}$$

and the corresponding eigenvectors $x_n \in X$ are defined by

$$x_n(s) = \cos(s/\sqrt{\lambda_n})$$

for all $s \in [0, 1]$ and each $n \in \mathbb{N} - 1$. Therefore $\|V^*V\| = \max_{n \in \mathbb{N} - 1} \lambda_n = 4/\pi^2$ which implies $\|V\| = 2/\pi < 1$. Hence we can apply the Neumann expansion to see that

$$(I - V)^{-1} = I + V + V^2 + \cdots$$

is well defined with $\|(I - V)^{-1}\| \leq \sum_{k \in \mathbb{N} - 1} (2/\pi)^k = \pi/(\pi - 2)$. By once more applying the Neumann expansion we have

$$R(z) = [V - (z - 1)(I - V)]^{-1}$$

$$= -(I - V)^{-1} [I - V(I - V)^{-1}/(z - 1)]^{-1}/(z - 1)$$

$$= -(I - V)^{-1} [I + V(I - V)^{-1}/(z - 1) + V^2(I - V)^{-2}/(z - 1)^2 + \cdots] / (z - 1)$$

$$= - [(I - V)^{-1}/(z - 1) + V(I - V)^{-2}/(z - 1)^2 + V^2(I - V)^{-3}/(z - 1)^3 + \cdots]$$

for all $z \neq 1$ because $\|V^n(I - V)^{-n}\|^{1/n} \leq \|V^n\|^{1/n} \|(I - V)^{-n}\|^{1/n} = \|V^n\|^{1/n} \cdot \pi/(\pi - 2) \to 0$ as $n \to \infty$. Thus $V(I - V)^{-1}$ is quasi-nilpotent.

In this example explicit solution of the fundamental equations can be suppressed because $A(z) = C_0 + C_1(z - 1)$ where $C_0 = V$ and $C_1 = (-1)(I - V)$ is invertible. It follows that $T_{-1} = C_1^{-1}$ and that the key spectral separation projection is $P = T_{-1}C_1 = I$ with $P^c = 0$. Thus the Laurent series for $R(z)$ at $z = 1$ contains only a singular part and no regular part. The solution to the fundamental equations is given by $T_{-k} = (-1)^{k-1}T_{-1}(C_0T_{-1})^{k-1} = (-1)(I - V)^{-1}[V(I - V)^{-1}]^{k-1}$ for all $k \in \mathbb{N}$ with $T_{\ell} = 0$ for all $\ell \in \mathbb{N} - 1$. Thus $R(z) = \sum_{k \in \mathbb{N}} T_{-k} / (z - 1)^k$ for all $z \in \mathcal{U}_{0, \infty}(1)$. \hfill \square

**Example 4.** Let $\lambda_n \in \mathbb{R}$ with $0 < \lambda_n < 1$ for $n \in \mathbb{N}$ and $\lambda_n^{1/n} \downarrow 0$ as $n \uparrow \infty$ and let $\sigma = \sum_{n \in \mathbb{N}} \lambda_n < 1$. Let $f_n : \mathbb{Z} \to \mathbb{R}$ for $n \in \mathbb{N}$ and $g : \mathbb{Z} \to \mathbb{R}$ be real-valued functions with $f_n(t) = 0$ for $t \leq 0$ and with $g(0) = 1$ and $g(t) = 0$ for $t < 0$ and suppose that

$$f_n(t + 1) - f_n(t) - \lambda_n[f_{n+1}(t) + g(t)] = 0$$

for each $n \in \mathbb{N}$ and

$$g(t + 1) - (1 - \sigma)g(t) = \epsilon w(t)$$

for all $t \in \mathbb{N} - 1$ where $\{w(t)\}_{t \in \mathbb{Z}}$ is a random process with $\mathbb{P}[w(t) = 0] = p$ and $\mathbb{P}[w(t) = 1] = 1 - p$ for each $t \in \mathbb{Z}$ and some $p \in (0, 1)$. We could regard $g(t)$ as an investment fund with an initial endowment that is progressively distributed over time to feed an infinite hierarchy of subsidiary assets $\{f_n(t)\}_{n \in \mathbb{N}}$, each of which contribute, in turn,
the incremental growth of the next lowest asset in the hierarchy. The total return for the investor is the lowest ranked asset \( f_1(t) \) which is the only one that can be realized. The noise term means that the investment fund receives a regular random supplement.

An \( \mathcal{M} \)-transform gives

\[
(1 - z)F_n(z) - \lambda_n z[F_{n+1}(z) + G(z)] = 0 \quad (69)
\]

\[
[1 - (1 - \sigma)z]G(z) = \varepsilon zW(z) + 1 \quad (70)
\]

for each \( n \in \mathbb{N} \) where we have used the notation

\[
F(z) = \mathcal{M}[\{f(t)\}_{t \in \mathbb{N}-1}](z) = \sum_{t \in \mathbb{N}-1} f(t)z^t
\]

to denote the \( \mathcal{M} \)-transform of a function \( \{f(t)\}_{t \in \mathbb{N}-1} \). In this case it is convenient to define \( \zeta = 1/z \) and write \( F_n(z) = H_n(\zeta) \), \( G(z) = K(\zeta) \) and \( W(z) = Y(\zeta) \). Now (69) and (70) become

\[
(\zeta - 1)H_n(\zeta) - \lambda_n[H_{n+1}(\zeta) + K(\zeta)] = 0 \quad (71)
\]

\[
[\zeta - 1 + \sigma]K(\zeta) = \varepsilon Y(\zeta) + \zeta \quad (72)
\]

for each \( n \in \mathbb{N} \). If we assume we can find \( \{m_n\}_{n \in \mathbb{N}} > 0 \) with \( \sum_{n \in \mathbb{N}} m_n < \infty \) and \( r > 0 \) such that \( |H_n(\zeta)| \leq m_n \) for \( 0 < |\zeta - 1| < r \) then we can define vectors

\[
\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_3 \end{bmatrix} \in \ell^1 \quad \text{and} \quad H(\zeta) = \begin{bmatrix} H_1(\zeta) \\ H_2(\zeta) \\ \vdots \end{bmatrix} \in \ell^1
\]

for \( 0 < |\zeta - 1| < r \). Now we can write the equations (71) and (72) as a vector equation

\[
\begin{bmatrix} (\zeta - 1)I - U \\ 0^* \end{bmatrix} \begin{bmatrix} -\lambda \\ \zeta - 1 + \sigma \end{bmatrix} \begin{bmatrix} H(\zeta) \\ K(\zeta) \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon Y(\zeta) - 1 \end{bmatrix} \quad (73)
\]

or equivalently \( A(\zeta)H(\zeta) = L(\zeta) \) where \( A(\zeta) = C_0 + C_1(\zeta - 1) \in \mathcal{B}(\ell^1 \times \mathbb{C}) \),

\[
C_0 = \begin{bmatrix} -U \\ 0^* \end{bmatrix} \begin{bmatrix} -\lambda \\ \sigma \end{bmatrix}, \quad C_1 = \begin{bmatrix} I \\ 0^* \end{bmatrix} \in \mathcal{B}(\ell^1 \times \mathbb{C}),
\]

\[
L(\zeta) = \begin{bmatrix} 0 \\ (1 - \sigma)[Y(\zeta) - 1] \end{bmatrix} \in \ell^1 \times \mathbb{C}
\]

and where

\[
U = \begin{bmatrix} 0 & \lambda_1 e_1 & \lambda_2 e_2 & \lambda_3 e_3 & \cdots \end{bmatrix} \in \mathcal{B}(\ell^1).
\]

The set \( X_{\text{sin}} = \{ x_n \in X \mid \text{there exists} \ \{ x_n \}_{n \in \mathbb{N}} \in \mathbb{N} \ \text{with} \ \{ C_0 x_n + C_1 x_{n-1} = 0 \ \text{for all} \ n \in \mathbb{N} \ \text{and} \ \lim_{n \to \infty} \| x_n \|^{1/n} = 0 \} \) is the generating set for the infinite-length singular Jordan chains on \( \mathcal{U}_{0,\infty}(1) \) for the linear pencil \( A(z) \). Albrecht et al. [3] have shown that \( X_{\text{sin}} = \)
The set of all regular Jordan chains on some set \( U \) for each \( n \in \mathbb{N} \). Some elementary algebra shows that
\[
C_n = \left[ \begin{array}{c} \sum_{k=0}^{n-1} \sigma^k (-1)^{n-k} U^{n-1-k} \lambda \end{array} \right].
\]
Now we have
\[
U^2 = \left[ \begin{array}{cccc} 0 & 0 & \lambda_1 \lambda_2 e_1 & \lambda_2 \lambda_3 e_2 \\
0 & 0 & \lambda_1 \lambda_3 e_1 & \lambda_2 \lambda_3 \lambda_4 e_2 \\
& & & \lambda_3 \lambda_4 \lambda_5 e_3 
\end{array} \right],
\]
and so on. It is clear that the first \( n \) columns of \( U^n \) are zero columns. Consequently \( U^n e_j = 0 \) for \( n > j \). It follows that
\[
x_{-1} = \begin{bmatrix} e_j \\ 0 \end{bmatrix} \implies x_{-n} = (-1)^{n-1} C_{n-1}^{-1} x_{-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
for all \( n > j + 1 \). Therefore \( \|x_{-n}\|_1^{1/n} = 0 \) for \( n > j + 1 \) and hence \( \ell^1 = S(\{e_1, e_2, \ldots\}) \subseteq X_{\sin} \). We can also see that
\[
x_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies x_{-n} = \begin{bmatrix} \sum_{k=0}^{n-1} \sigma^k (-1)^{k+1} U^{n-1-k} \lambda \\ (-1)^n \sigma \end{bmatrix}
\]
from which it follows that \( \|x_{-n}\|_1 \geq \sigma^{n-1} \Rightarrow \|x_{-n}\|_1^{1/n} = \sigma^{-1/n} \rightarrow \sigma > 0 \) as \( n \rightarrow \infty \). Therefore
\[
\begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin X_{\sin}.
\]
Hence we have shown that \( X_{\sin} = \ell^1 \subseteq \ell^1 \times \mathbb{C} \).
The set \( X_{\reg} = \{ x_1 \in X \mid \text{there exists } \{x_n\}_{n \in \mathbb{N}} \text{ with } C_0 x_{n+1} + C_1 x_n = 0 \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n \rightarrow \infty} \|x_n\|_1^{1/n} \leq \rho \} \) for some \( \rho > 0 \) is the generating set for the infinite-length regular Jordan chains on some set \( U_\rho(1) \). Albrecht et al. have shown that \( X_{\reg} = P^c(X) \) where \( P^c = I_X - P = T_0 C_0 \) is the complementary key spectral projection on \( U_\rho(1) \). To find \( X_{\reg} \) we consider the equation
\[
C_0 x_{n+1} + C_1 x_n = 0 \implies (-1)^{n-1} C_0^{n-1} x_n = x_1
\]
for all \( n \in \mathbb{N} \). Therefore we must have
\[
\lim_{n \rightarrow \infty} (-1)^{n-1} C_0^{n-1} x_n = x_1.
\]
We have already seen that
\[
(-1)^{n-1} C_0^{n-1} \begin{bmatrix} e_j \\ 0 \end{bmatrix} = 0
\]
for \( n > j + 1 \) and so we might expect to find

\[
x_1 = \lim_{n \to \infty} (-1)^{n-1} C_0^{n-1} \begin{bmatrix} 0 \\ \alpha_n \end{bmatrix} = \lim_{n \to \infty} \alpha_n (-1)^{n-1} C_0^{n-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

for some suitably chosen sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \in \mathbb{R} \) with \( \alpha_n \neq 0 \) for all \( n \in \mathbb{N} \). Our earlier calculations showed us that

\[
(-1)^n C_0^n / \sigma^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lim_{n \to \infty} \alpha_n (-1)^{n-1} C_0^{n-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma I + U \end{bmatrix}^{-1} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = v
\]

as \( n \to \infty \). Therefore \((C_0 / \sigma)v = v \iff C_0 v = \sigma v\). If we choose \( x_1 = v \) then \( x_n = (-1)^{n-1} C_0^{n-1} v = (-1)^{n-1} \sigma^{n-1} v \) for all \( n \in \mathbb{N} \). Thus \( \|x_n\|_1 = \sigma^{n-1} \|v\|_1 \) and hence \( \|x_n\|_{1/n} = \sigma^{1-1/n} \|v\|_{1/n} \to \sigma \) as \( n \to \infty \). It follows that \( v \in X_{\text{reg}} \). Since this is the only possible solution we have shown that \( X_{\text{reg}} = \{x \in X \mid x = \alpha v \text{ for some } \alpha \in \mathbb{C}\} \). We have also shown that \( \|x_n\|_{1/n} \to \sigma \) for every infinite-length regular Jordan chain. Hence \( \rho = \sigma \).

Therefore

\[
P^c = \begin{bmatrix} 0 \\ \sigma I + U \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \lambda \end{bmatrix}
\]

and

\[
P = I - P^c = \begin{bmatrix} I \\ \sigma I + U \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \lambda \end{bmatrix}
\]

are the key spectral separation projections on \( U_{0,\sigma}(1) = U_{\sigma}(1) \cap U_{0,\infty}(1) \). For each

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in X
\]

we can write \( x = Px + P^c x \) to find the unique decomposition

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha + \beta (\sigma I + U)^{-1} \lambda \\ 0 \end{bmatrix} + \begin{bmatrix} -\beta (\sigma I + U)^{-1} \lambda \\ \beta \end{bmatrix} \in X_{\text{sin}} \oplus X_{\text{reg}}.
\]

Since \( C_1 = I \) and \( T_{-1} C_1 = P \) we deduce that \( T_{-1} = P \). Thus we have

\[
T_{-1} = \begin{bmatrix} I \\ \sigma \end{bmatrix}^{-1} \begin{bmatrix} \sigma I + U \end{bmatrix}^{-1} \begin{bmatrix} \lambda \\ 1 \end{bmatrix}.
\]

We also have

\[
T_0 C_0 = P^c \iff \begin{bmatrix} T_{0,11} \\ s^* \gamma \\ 0^* \end{bmatrix} \begin{bmatrix} -U \\ -\lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0^* \\ -(\sigma I + U)^{-1} \lambda \\ 1 \end{bmatrix}
\]

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which implies $T_{0,11}U = 0$, $-T_{0,11}\lambda + \sigma t = -(\sigma I + U)^{-1}\lambda$, $s^*U = 0^*$, and $-s^*\lambda + \gamma\sigma = 1$. Since $U = [0, \lambda_1e_1, \lambda_2e_2, \ldots]$ it follows that $T_{0,11}U = 0$ and $s^*U = 0^*$ implies $T_{0,11} = 0$ and $s = 0$. Therefore

$$T_0 = \begin{bmatrix} 0 & -(\sigma I + U)^{-1}\lambda/\sigma \\ 0^* & 1/\sigma \end{bmatrix}.$$ 

We can now calculate

$$T_{-k} = (-1)^{k-1}(T_{-1}C_0)^{k-1}T_{-1} = \begin{bmatrix} U^{k-1} & U^{k-1}(\sigma I + U)^{-1}\lambda \\ 0^* & 0 \end{bmatrix}$$

for each $k \in \mathbb{N}$ and

$$T_\ell = (-1)^\ell(T_0C_1)^\ellT_0 = (-1)^\ell \begin{bmatrix} 0 & -(\sigma I + U)^{-1}\lambda/\sigma^{\ell+1} \\ 0^* & 1/\sigma^{\ell+1} \end{bmatrix}$$

for each $\ell \in \mathbb{N} - 1$. An elementary series argument shows that the singular part of the resolvent is

$$R_{\text{sing}}(\zeta) = \sum_{k \in \mathbb{N}} T_{-k}(\zeta - 1)^{-k} = \begin{bmatrix} [(\zeta - 1)I - U]^{-1} & [(\zeta - 1)I - U]^{-1}(\sigma I + U)^{-1}\lambda \\ 0^* & 0 \end{bmatrix}$$

for all $\zeta \neq 1$ and that the regular part is

$$R_{\text{reg}}(\zeta) = \sum_{\ell \in \mathbb{N} - 1} T_\ell(\zeta - 1)^\ell = \begin{bmatrix} 0 & -(\zeta - 1 + \sigma)^{-1}(\sigma I + U)^{-1}\lambda \\ 0^* & (\zeta - 1 + \sigma)^{-1} \end{bmatrix}$$

for $|\zeta - 1| < \sigma$. Thus the resolvent is

$$R(\zeta) = \begin{bmatrix} [(\zeta - 1)I - U]^{-1} & (\zeta - 1 + \sigma)^{-1}[(\zeta - 1)I - U]^{-1}\lambda \\ 0^* & (\zeta - 1 + \sigma)^{-1} \end{bmatrix} \in \mathcal{B}(\ell^1 \times \mathbb{C}) \quad (74)$$

for $\zeta \in \mathcal{U}_{0,\sigma}(1)$ where we have used the rather devious identity

$$\left\{ (\zeta - 1 + \sigma)[(\zeta - 1)I - U]^{-1} - I \right\} (\sigma I + U)^{-1} = [(\zeta - 1)I - U]^{-1}.$$ 

We can use (74) to resolve (73) and obtain

$$\begin{bmatrix} H(\zeta) \\ K(\zeta) \end{bmatrix} = \begin{bmatrix} (\zeta - 1 + \sigma)^{-1}[(\zeta - 1)I - U]^{-1}\lambda[eY(\zeta) + \zeta] \\ (1 - \sigma)(\zeta - 1 + \sigma)^{-1}[eY(\zeta) + \zeta] \end{bmatrix}. \quad (75)$$

Some elementary algebra shows that

$$F_m(z) = \sum_{n \in \mathbb{N} - 1}(-1)^n \left[ \prod_{j=m}^{m+n}\lambda_j \right] z^{n+1}[1 - (1 - \sigma)z]^{-1}(1 - z)^{-n-1}[\epsilon zW(z) + 1].$$

and

$$G(z) = \frac{1}{[1 - (1 - \sigma)z]}[\epsilon zW(z) + 1].$$
An inverse Maclaurin transform can now be applied to find the solutions. The inversion is straightforward but algebraically complicated. The details will not be presented but we note that the partial fraction expansion
\[
\frac{1}{[1 - (1 - \sigma)z](1 - z)^{n+1}} = \frac{(1/\sigma - 1)^{n+1}}{[1 - (1 - \sigma)z]} + \frac{(1/\sigma - 1)^n/\sigma}{(1 - z)} + \ldots
\]

\[
\ldots + \frac{(1/\sigma - 1)^2/\sigma}{(1 - z)^{n-1}} + \frac{(1/\sigma - 1)/\sigma}{(1 - z)^n} + \frac{1/\sigma}{(1 - z)^{n+1}}
\]

for each \(n \in \mathbb{N} - 1\) can be used to assist the inversion process. The solution is given by
\[
\{f_m(t)\}_{t \in \mathbb{N} - 1} = \sum_{n \in \mathbb{N} - 1} (-1)^n \left[ \prod_{j=m}^{m+n} A_j \right] (1/\sigma - 1)^{n+1} \{(1 - \sigma)^t\}_{t \in \mathbb{N} - 1} \ast \{v_{n+1}(t)\}_{t \in \mathbb{N} - 1}
\]

\[
+ \sum_{n \in \mathbb{N} - 1} (-1)^n \left[ \prod_{j=m}^{m+n} A_j \right] \sum_{k=1}^{n+1} [(1/\sigma - 1)^{k-1}/\sigma] \setminus \nabla^{-n+k-2} v_{n+1}(t) \}_{t \in \mathbb{N} - 1}.
\]

and
\[
\{g(t)\}_{t \in \mathbb{N} - 1} = \{(1 - \sigma)^t\}_{t \in \mathbb{N} - 1} \ast \{v(t)\}_{t \in \mathbb{N} - 1}
\]

where we have defined \(v(t) = \epsilon w(t - 1) + \delta(t)\) and where \(\delta(0) = 1\) and \(\delta(t) = 0\) for \(t \neq 0\). We have written \(v_{n+1}(t) = v(t - n - 1)\) for convenience. The convolution operator is defined by \([p \ast q](t) = \sum_{s=0}^{t} p(t-s)q(s)\). \(\Box\)

9. The augmented linear pencil

We have restricted our attention to linear pencils because we can use augmented operators to show that a polynomial pencil inversion can always be replaced by an equivalent linear pencil inversion. We use an elementary augmentation rather than the companion operator construction that is often used in the literature [45, Section 2.2, p 131].

Suppose \(X, Y\) are complex Banach spaces and consider the polynomial pencil \(A(z) = \sum_{i=0}^{p} C_i(z - 1)^i \in \mathcal{B}(X, Y)\) for all \(z \in \mathbb{C}\) defined by the bounded linear operators \(C_i \in \mathcal{B}(X, Y)\) for each \(i = 0, 1, \ldots, p\). If we suppose the existence of a resolvent operator \(R(z) = \sum_{j \in \mathbb{Z}} T_j(z - 1)^j\) for all \(z \in \mathcal{U}_{s,r}(1)\) where \(0 \leq s < r \leq \infty\) and where \(T_j \in \mathcal{B}(Y, X)\) for each \(j \in \mathbb{Z}\) then we can equate coefficients for the various powers of \(z\) and the identities \(R(z) A(z) = I_X\) and \(A(z) R(z) = I_Y\) to show that the left fundamental equations for the polynomial pencil take the form
\[
\sum_{i=0}^{p} T_{j-p+i} C_{p-i} = \begin{cases} I_X & \text{if } j = 0 \\ 0_X & \text{if } j \neq 0 \end{cases}
\]

and the right fundamental equations take the analogous form
\[
\sum_{i=0}^{p} C_{p-i} T_{j-p+i} = \begin{cases} I_Y & \text{if } j = 0 \\ 0_Y & \text{if } j \neq 0. \end{cases}
\]

(76)

(77)
Define augmented operator matrices \( C_0, C_1 \in \mathcal{B}(X^p, Y^p) \) and an associated augmented linear pencil \( A(z) = C_0 + C_1(z - 1) \in \mathcal{B}(X, Y) \) for all \( z \in \mathbb{C} \) by setting

\[
C_0 = \begin{bmatrix}
C_0 & 0 & 0 & \cdots & 0 \\
C_1 & C_0 & 0 & \cdots & 0 \\
C_2 & C_1 & C_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_{p-1} & C_{p-2} & C_{p-3} & \cdots & C_0
\end{bmatrix}
\quad \text{and} \quad
C_1 = \begin{bmatrix}
C_p & C_{p-1} & C_{p-2} & \cdots & C_1 \\
0 & C_p & C_{p-1} & \cdots & C_2 \\
0 & 0 & C_p & \cdots & C_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & C_p
\end{bmatrix}
\]

where \( 0 = 0_{X,Y} \in \mathcal{B}(X,Y) \) and define corresponding augmented operator matrices \( T_j \in \mathcal{B}(Y^p, X^p) \) for each \( j \in \mathbb{Z} \) and an associated augmented resolvent operator \( R(z) = \sum_{j \in \mathbb{Z}} T_j(z - 1)^j \) by setting

\[
T_j = \begin{bmatrix}
T_{jp} & T_{jp-1} & T_{jp-2} & \cdots & T_{jp-p+1} \\
T_{jp+1} & T_{jp} & T_{jp-1} & \cdots & T_{jp-p+2} \\
T_{jp+2} & T_{jp+1} & T_{jp} & \cdots & T_{jp-p+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
T_{jp+p-1} & T_{jp+p-2} & T_{jp+p-3} & \cdots & T_{jp}
\end{bmatrix}
\]

It follows from the theory of linear pencils that the resolvent operator \( R(z) \) is analytic for \( z \in \mathcal{U}_{s,r} \) if and only if

\[
\lim_{k \to \infty} \|T_{-k}\|^{1/k} \leq s \quad \text{and} \quad \lim_{\ell \to \infty} \|T_{\ell}\|^{1/\ell} \leq 1/r
\]

and the coefficients \( \{T_j\}_{j \in \mathbb{Z}} \) satisfy the augmented left fundamental equations

\[
T_iC_0 + T_{i-1}C_1 = \begin{cases} \mathcal{I}_X & \text{if } i = 0 \\
\vartheta_X & \text{if } i \in \mathbb{Z}, i \neq 0 \end{cases}
\]

and the augmented right fundamental equations

\[
C_0T_i + C_{1i-1} = \begin{cases} \mathcal{I}_Y & \text{if } i = 0 \\
\vartheta_Y & \text{if } i \in \mathbb{Z}, i \neq 0 \end{cases}
\]

where \( \mathcal{I}_X \in \mathcal{B}(X^p) \) and \( \mathcal{I}_Y \in \mathcal{B}(Y^p) \) are the respective identity mappings and \( \vartheta_X \in \mathcal{B}(X^p) \) and \( \vartheta_Y \in \mathcal{B}(Y^p) \) are the respective zero mappings. It follows that the solution to (78) and (79) is completely determined by the basic solution \( \{T_{-1}, T_0\} \) on \( \mathcal{U}_{s,r}(1) \) and is given by \( T_{-k} = (-1)^{k-1}(T_{-1}C_0)^{k-1}T_0 \) for all \( k \in \mathbb{N} \) and \( T_{\ell} = (-1)^{\ell}(T_0C_1)^{\ell}T_0 \) for all \( \ell \in \mathbb{N} - 1 \). By considering the element by element identities in (78) and (79) it can be seen that these identities are precisely the identities (76) and (77). Thus we deduce that the resolvent \( R(z) \) for the polynomial pencil \( A(z) \) exists and is analytic for \( z \in \mathcal{U}_{s,r} \) if and only if the resolvent \( R(z) \) for the augmented linear pencil \( A(z) \) exists and is analytic for \( z \in \mathcal{U}_{s,r} \). The basic solution to (76) and (77) is completely determined by \( \{T_{-1}, T_0\} \equiv \{T_j\}_{j=-2p+1}^{p-1} \).
9.1. Order reduction for an ARMA process

We can use augmented operators to show that an ARMA\((p, q)\) process on Banach spaces \(X, Y\) can be reduced to an ARMA\((1, 1)\) process on Banach spaces \(X^r, Y^r\) where \(r = \max\{p, q\}\). Rather than describe the general reduction which requires separate considerations for \(p \geq q\) and \(p < q\) we illustrate the procedure for a particular case.

Example 5. Consider the augmentation procedure for an ARMA\((2, 3)\) process. Suppose

\[
\sum_{i=0}^{2} A_i x(t - i) = \sum_{j=0}^{3} F_j w(t - r)
\]

where \(\{x(t)\}_{t \in \mathbb{N}} \in X^{N-1}\) is a time series, \(\{w(t)\}_{t \in \mathbb{Z}} \in X^{\mathbb{Z}}\) is an i.i.d., zero mean noise process where \(A_i \in \mathcal{B}(X, Y)\) for each \(i = 0, 1, 2\) and \(F_j \in \mathcal{B}(X, Y)\) for each \(j = 0, \ldots, 3\) are bounded linear operators. Let

\[
y(t) = \begin{bmatrix} x(3t) \\ x(3t + 1) \\ x(3t + 2) \end{bmatrix} \in X^3, \quad \text{and} \quad n(t) = \begin{bmatrix} w(3t) \\ w(3t + 1) \\ w(3t + 2) \end{bmatrix} \in X^3
\]

for each \(t \in \mathbb{Z}\) define an augmented process \(\{y(t)\}_{t \in \mathbb{Z}} \in (X^3)^{\mathbb{Z}}\) and an augmented i.i.d., zero mean noise process \(\{n(t)\}_{t \in \mathbb{Z}} \in (X^3)^{\mathbb{Z}}\) respectively. Suppose that

\[
\mathcal{A}_0 = \begin{bmatrix} A_0 & 0 & 0 \\ A_1 & A_0 & 0 \\ A_2 & A_1 & A_0 \end{bmatrix} \in \mathcal{B}(X^3, Y^3) \quad \text{and} \quad \mathcal{A}_1 = \begin{bmatrix} 0 & A_2 & A_1 \\ 0 & 0 & A_2 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{B}(X^3, Y^3)
\]

and

\[
\mathcal{F}_0 = \begin{bmatrix} F_0 & 0 & 0 \\ F_1 & F_0 & 0 \\ F_2 & F_1 & F_0 \end{bmatrix} \in \mathcal{B}(X^3, Y^3) \quad \text{and} \quad \mathcal{F}_1 = \begin{bmatrix} F_3 & F_2 & F_1 \\ 0 & F_3 & F_2 \\ 0 & 0 & F_3 \end{bmatrix} \in \mathcal{B}(X^3, Y^3)
\]

are corresponding augmented bounded linear operators. Now the original ARMA\((2, 3)\) equation for \(\{x(t)\}_{t \in \mathbb{Z}}\) can be replaced by an equivalent ARMA\((1, 1)\) equation

\[
\mathcal{A}_0 y(t) + \mathcal{A}_1 y(t - 1) = \mathcal{F}_0 n(t) + \mathcal{F}_1 n(t - 1) \quad \text{(80)}
\]

for \(\{y(t)\}_{t \in \mathbb{Z}}\). The characteristic polynomial for the augmented series is the linear pencil \(\mathcal{A}(z) = (\mathcal{A}_0 + \mathcal{A}_1 z)\). Resolution of (80) requires calculation of \(\mathcal{R}(z) = \mathcal{A}(z)^{-1}\). \(\Box\)

10. Conclusions

We have used recent results relating to the systematic calculation of resolvent operators for linear pencils on Banach space to establish a Granger–Johansen representation for an ARMA\((1, 1)\) unit root process. Our results extend the existing published results for unit root processes with a finite-order pole from Hilbert space to Banach space and establish new results for unit root processes with an isolated essential singularity. We have also demonstrated effective algorithms for calculation of the key spectral separation projections and the Laurent series coefficients. Our future research will look more specifically at applications of these new results.
11. Acknowledgements

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Appendix A. The resolvent operator for a linear pencil

The following results are taken from the papers by Albrecht et al. [2, 5]. We consider a linear pencil $A(z) = A_0 + A_1z \in \mathcal{B}(X, Y)$ where $X, Y$ are complex Banach spaces and $A_0, A_1 \in \mathcal{B}(X, Y)$. We will apply these results to analysis of the resolvent operator $R(z) = A(z)^{-1}$ on an annular region $\mathcal{U}_{s,r}(1) = \{ z \in \mathbb{C} | s < |z - 1| < r \}$ where $0 \leq s < r \leq \infty$. Thus we rewrite $A(z) = C_0 + C_1(z - 1)$ where $C_0 = A_0 + A_1$ and $C_1 = A_1$. We wish to write $R(z) = \sum_{j \in \mathbb{Z}} T_j(z - 1)^j$ for all $z \in \mathcal{U}_{s,r}(1)$. Albrecht et al. [2] proved the following theorem.

**Theorem 3.** The coefficients $\{T_j\}_{j \in \mathbb{Z}} \in \mathcal{B}(Y, X)$ satisfy (8), (9) and (10) if and only if the following are all satisfied: (i) $P = T_{-1}C_1 \in \mathcal{B}(X)$ and $P^c = I_X - P = T_0C_0 \in \mathcal{B}(X)$ are complementary projections on $X$; and $Q = C_1T_{-1} \in \mathcal{B}(Y)$ and $Q^c = I_Y - Q = C_0T_0 \in \mathcal{B}(Y)$ are corresponding complementary projections on $Y$; (ii) $C_i = QC_iP + Q^cC_iP^c$ for $i = 0, 1$; (iii) $T_{-k} = PT_{-k}Q$ for $k \in \mathbb{N}$ and $T_\ell = P\ell T_0 Q^c$ for $\ell \in \mathbb{N} - 1$; (iv) $T_{-k} = (-1)^{k-1}(T_{-1}C_0)^{k-1}T_{-1}$ for $k \in \mathbb{N}$ and $T_\ell = (-1)^\ell(T_0C_1)^\ell T_0$ for $\ell \in \mathbb{N} - 1$; and (v) $\lim_{k \to \infty} \|(T_{-1}C_0)^k\|^{1/k} \leq s$ and $\lim_{\ell \to \infty} \|(T_0C_1)^\ell\|^{1/\ell} \leq 1/r$. \hfill $\square$

This result suggests that the coefficients $\{T_j\}_{j \in \mathbb{Z}}$ can be found directly by solving the fundamental equations without prior knowledge of $R(z)$. In this regard Albrecht et al. [2] showed that it is sufficient to find a basic solution $\{T_{-1}, T_0\}$.

**Corollary 3.** The resolvent $R : \mathcal{U}_{s,r}(1) \to \mathcal{B}(Y, X)$ is analytic if and only if there exist operators $T_{-1}, T_0 \in \mathcal{B}(Y, X)$ such that (i) $T_{-1}C_1 + T_0C_0 = I_X$ and $C_1T_{-1} + C_0T_0 = I_Y$; (ii) $T_{-1}C_0T_0 = 0$ and $T_0C_1T_{-1} = 0$ for each $i = 0, 1$; and (iii) $\lim_{k \to \infty} \|(T_{-1}C_0)^k\|^{1/k} \leq s$ and $\lim_{\ell \to \infty} \|(T_0C_1)^\ell\|^{1/\ell} \leq 1/r$. If these conditions are satisfied the basic solution $\{T_{-1}, T_0\}$ on $\mathcal{U}_{s,r}(1)$ is uniquely defined and

$$
R(z) = R_{\text{sin}}(z) + R_{\text{reg}}(z) \\
= PR(z)Q + P^cR(z)Q^c \\
= T_{-1}[I_Y(z - 1) + C_0T_{-1}]^{-1} + T_0[I_Y + V_1T_0(z - 1)]^{-1} \\
= [I_X(z - 1) + T_{-1}C_0]^{-1}T_{-1} + [I_X + T_0C_1(z - 1)]^{-1}T_0 \\
= (A.1)
$$

for $z \in \mathcal{U}_{s,r}(1)$ where $R_{\text{sin}}(z)$ is the singular part and $R_{\text{reg}}(z)$ is the regular part. \hfill $\square$

The expression (A.1) provides a closed form for the resolvent. The fundamental equations can be solved easily for matrix operators [6, 7, 27, 33] but are much more difficult to
solve for bounded linear operators on infinite-dimensional space \([7, 8, 28, 2, 4, 5]\). When \(X, Y\) are separable Banach spaces the key spectral separation projections \(P \in \mathcal{B}(X)\) and \(Q \in \mathcal{B}(Y)\) can be found using infinite-length singular and regular Jordan chains. The key projections can be used to separate each set of fundamental equations into two separate solvable systems—one for the singular part of the resolvent and one for the regular part—and enables implementation of a systematic solution procedure. Albrecht et al. [5] proposed the following definitions for the Jordan chains and established the associated critical properties.

**Definition 1.** Let \(0 \leq s < r \leq \infty\) and suppose \(R(z)\) is analytic for all \(z \in \mathcal{U}_{s, r}(1)\). If \(\{x_{-n}\}_{n \in \mathbb{N}} \subseteq X\) with \(C_0 x_{-n} + C_1 x_{-n-1} = 0_Y\) for all \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} \|x_{-n}\|^{1/n} = s\) then we say that \(\{x_{-n}\}_{n \in \mathbb{N}}\) is an infinite-length singular Jordan chain for the pencil \(A(z) = C_0 + C_1 (z - 1)\) on the annular region \(\mathcal{U}_{s, \infty}(1)\). Note that for all \(k \in \mathbb{N}\) and \(s > 0\) this definition excludes all singular Jordan chains \(\{x_{-n}\}_{n \in \mathbb{N}}\) of length \(k\) with \(x_{-n} = 0_X\) for all \(n > k\). When \(s = 0\) the finite-length singular Jordan chains are included.

**Definition 2.** Let \(0 \leq s < r \leq \infty\) and suppose \(R(z)\) is analytic for all \(z \in \mathcal{U}_{s, r}(1)\). Let \(X_{\text{sin}} \subseteq X\) be the subspace \(X_{\text{sin}} = \{x_{-1} \in X \mid \text{there exists } \{x_{-n}\}_{n \in \mathbb{N}} \text{ with } C_0 x_{-n} + C_1 x_{-n-1} = 0_Y \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n \to \infty} \|x_{-n}\|^{1/n} = s\}\). The space \(X_{\text{sin}}\) is the generating set for all infinite-length singular Jordan chains on the region \(\mathcal{U}_{s, \infty}(1)\).

**Lemma 4.** Let \(0 \leq s < r \leq \infty\) and suppose \(R(z)\) is analytic on \(\mathcal{U}_{s, r}(1)\). Then \(X_{\text{sin}} = P(X)\) where \(P = T_1 C_1\) is the key spectral separation projection.

**Definition 3.** Let \(0 \leq s < r \leq \infty\) and suppose \(R(z)\) is analytic for all \(z \in \mathcal{U}_{s, r}(1)\). If \(\{x_n\}_{n \in \mathbb{N}} \subseteq X\) with \(C_1 x_n + C_0 x_{n+1} = 0_Y\) for all \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} \|x_n\|^{1/n} = 1/r\) then we say that \(\{x_n\}_{n \in \mathbb{N}}\) is an infinite-length regular Jordan chain for the pencil \(A(z) = C_0 + C_1 (z - 1)\) on the annular region \(\mathcal{U}_{0, r}(1)\). Note that for all \(k \in \mathbb{N}\) and \(r < \infty\) this definition excludes all regular Jordan chains \(\{x_n\}_{n \in \mathbb{N}}\) of length \(k\) with \(x_n = 0_X\) for all \(n > k\). When \(r = \infty\) the finite-length regular Jordan chains are included.

**Definition 4.** Let \(0 \leq s < r \leq \infty\) and suppose \(R(z)\) is analytic for all \(z \in \mathcal{U}_{s, r}(1)\). Let \(X_{\text{reg}} \subseteq X\) be the subspace \(X_{\text{reg}} = \{x_1 \in X \mid \text{there exists } \{x_n\}_{n \in \mathbb{N}} \text{ with } C_1 x_n + C_0 x_{n+1} = 0_Y \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n \to \infty} \|x_n\|^{1/n} = 1/r\}\). The space \(X_{\text{reg}}\) is the generating set for all infinite-length regular Jordan chains on the region \(\mathcal{U}_{0, r}\).

**Lemma 5.** Let \(0 \leq s < r \leq \infty\) and suppose \(R(z)\) is analytic on \(\mathcal{U}_{s, r}(1)\). Then \(X_{\text{reg}} = P^c(X)\) where \(P^c = T_0 C_0\) is the complementary spectral separation projection.

The above results lead to the following theorem.
Theorem 4. Suppose $R(z)$ is analytic for $z \in \mathcal{U}_{s,r}(1)$. Let $P = T_{-1}C_1 \in \mathcal{B}(X)$ and $P^c = I_X - P = T_0C_0 \in \mathcal{B}(X)$ be the complementary key projections on the domain space that separate the bounded and unbounded parts of the spectral set for $A(z)$ relative to the annular region $\mathcal{U}_{s,r}(1)$ and let $Q = C_1T_{-1} \in \mathcal{B}(Y)$ and $Q^c = I_Y - Q = C_0T_0 \in \mathcal{B}(Y)$ be the corresponding complementary key projections on the range space. Now let $X_{\sin} \subseteq X$ and $X_{\text{reg}} \subseteq X$ be the respective generating subspaces for the infinite-length singular and regular generalized Jordan chains for $X$ and $X_{\text{reg}}$ also have regular generalized Jordan chains for $X$. Let $X$ be a Banach space over the field $\mathbb{C}$ of complex numbers with norm $\| \cdot \| : X \to [0, \infty)$. We say that a function $f : \Omega \to X$ is Bochner $\mu$-integrable if and only if the function $\| f \| : \Omega \to [0, \infty)$ defined by $\| f \|(\omega) = \| f(\omega) \|$ for all $\omega \in \Omega$ is $\mu$-integrable in which case
\[
\| \int_E f(\omega)\mu(d\omega) \| \leq \int_E \| f(\omega) \|\mu(d\omega)
\]
for each $E \in \Sigma$. \hfill $\Box$

Corollary 4. Let $X$ and $Y$ be Banach spaces and suppose that $A \in \mathcal{B}(X,Y)$. If the function $f : \Omega \to X$ is Bochner $\mu$-integrable then the function $g = Af : \Omega \to Y$ defined by $g(\omega) = Af(\omega)$ for $\mu$-almost all $\omega \in \Omega$ is Bochner $\mu$-integrable with
\[
\int_E g(\omega)\mu(d\omega) = A\int_E f(\omega)\mu(d\omega)
\]
for each $E \in \Sigma$. \hfill $\Box$

Let $f : \Omega \to X$ be a Bochner $\mu$-integrable random function taking values in the Banach space $X$. The expected value of $f$ is defined by
\[
\mathbb{E}[f] = \int_\Omega f(\omega)\mu(d\omega)
\]
and we note from Theorem 5 that $\| \mathbb{E}[f] \| \leq \mathbb{E}[\| f \|]$. When $A \in \mathcal{B}(X,Y)$ is a bounded linear map from the Banach space $X$ to the Banach space $Y$, it follows from Corollary 4 that $\mathbb{E}[Af] = A\mathbb{E}[f]$. 

Appendix B. The Bochner integral—outline of the standard theory

Let $X$ be a Banach space over the field $\mathbb{C}$ of complex numbers with norm $\| \cdot \| : X \to [0, \infty)$. We say that a function $f : \Omega \to X$ is a vector-valued random function or simply a random function. The necessary definitions and basic theory for Bochner integrals of vector-valued random functions can be found in the text by Yosida [49, pp 130–134]. We mention only the most critical results.

Theorem 5. A strongly $\Sigma$-measurable function $f : \Omega \to X$ is Bochner $\mu$-integrable if and only if the function $\| f \| : \Omega \to [0, \infty)$ defined by $\| f \|(\omega) = \| f(\omega) \|$ for all $\omega \in \Omega$ is $\mu$-integrable in which case
\[
\| \int_E f(\omega)\mu(d\omega) \| \leq \int_E \| f(\omega) \|\mu(d\omega)
\]
for each $E \in \Sigma$. \hfill $\Box$

Theorem 4. Suppose $R(z)$ is analytic for $z \in \mathcal{U}_{s,r}(1)$. Let $P = T_{-1}C_1 \in \mathcal{B}(X)$ and $P^c = I_X - P = T_0C_0 \in \mathcal{B}(X)$ be the complementary key projections on the domain space that separate the bounded and unbounded parts of the spectral set for $A(z)$ relative to the annular region $\mathcal{U}_{s,r}(1)$ and let $Q = C_1T_{-1} \in \mathcal{B}(Y)$ and $Q^c = I_Y - Q = C_0T_0 \in \mathcal{B}(Y)$ be the corresponding complementary key projections on the range space. Now let $X_{\sin} \subseteq X$ and $X_{\text{reg}} \subseteq X$ be the respective generating subspaces for the infinite-length singular and regular generalized Jordan chains for $A(z)$ on $\mathcal{U}_{s,\infty}(1)$ and $\mathcal{U}_{0,r}(1)$. Then we have (i) $X_{\sin} = P(X)$ and $X_{\text{reg}} = P^c(X)$ with $X = X_{\sin} \oplus X_{\text{reg}} \cong X_{\sin} \times X_{\text{reg}}$ and (ii) $Y_{\sin} = Q(Y) = C_1(X_{\sin})$ and $Y_{\text{reg}} = Q^c(Y) = C_0(X_{\text{reg}})$ with $Y = Y_{\sin} \oplus Y_{\text{reg}} \cong Y_{\sin} \times Y_{\text{reg}}$. We also have $X_{\sin} = T_{-1}(Y_{\sin})$ and $X_{\text{reg}} = T_0(Y_{\text{reg}})$. \hfill $\Box$
Finally we note that if $\|f\|^2$ is integrable then the standard theory of integration for real-valued functions gives us the inequality
\[
\mathbb{E}[\|f\|^2] = \left[ \int_{\Omega} \|f(\omega)\| \cdot 1 \cdot \mu(d\omega) \right]^2 \leq \left[ \int_{\Omega} \|f(\omega)\|^2 \mu(d\omega) \right] \cdot \left[ \int_{\Omega} 1^2 \mu(d\omega) \right] = \mathbb{E}[\|f\|^2].
\]

It follows that if $f : \Omega \to H$ is strongly $\Sigma$-measurable and $\|f\|^2 : \Omega \to [0, \infty)$ is $\mu$-integrable, then $\mathbb{E}[\|f\|^2] \leq \mathbb{E}[\|f\|^2] \leq \mathbb{E}[\|f\|^2]$.

Appendix C. Probability Theory

We require a basic theory of convergence for series with random coefficients. The book by Williams [48] is an important reference for this topic. We outline the key arguments.

Appendix C.1. The tail $\sigma$-algebra

Consider an infinite sequence of independent random variables. A tail event is one which does not depend on any finite subsequence of these random variables. The Kolmogorov 0-1 law tells us that tail events almost certainly occur or almost certainly do not occur. This concept is critical in understanding the convergence of random sequences.

**Definition 5.** Let $X$ be a Banach space, $(\Omega, \Sigma, \mu)$ a probability space and $\{s_k\}_{k \in \mathbb{N}} \in \Sigma$ a sequence of random variables, $s_k : \Omega \to X$ for each $k \in \mathbb{N}$. Let
\[
\Sigma_n = \sigma(s_k)_{k \in \mathbb{N}, k \geq n} = \sigma(x_n, x_{n+1}, \ldots)
\]
be the $\sigma$-algebra generated by $\{s_k\}_{k \in \mathbb{N}, k \geq n}$. The tail $\sigma$-algebra $\Sigma_\infty = \sigma(s_k)_{k \in \mathbb{N}}$ is the $\sigma$-algebra defined by $\Sigma_\infty = \bigcap_{n \in \mathbb{N}} \Sigma_n$. $\square$

**Theorem 6 (Kolmogorov 0-1 law).** Let $X$ be a Banach space, $(\Omega, \Sigma, \mu)$ a probability space and $\{s_k\}_{k \in \mathbb{N}}$ a sequence of independent random variables $s_k : \Omega \to X$ for each $k \in \mathbb{N}$ with tail $\sigma$-algebra
\[
\Sigma_\infty = \bigcap_{n \in \mathbb{N}} \Sigma_n = \bigcap_{n \in \mathbb{N}} \sigma(s_k)_{k \in \mathbb{N}, k \geq n}.
\]
Then (i) $E \in \Sigma_\infty \Rightarrow \mu(E) \in \{0, 1\}$ and (ii) if $X$ is separable and $x : \Omega \to X$ is a $\Sigma_\infty$-measurable random variable then there exists a vector $a \in X$ such that $\mu(x^{-1}(a)) = \mu(\{\omega \mid x(\omega) = a\}) = 1$. $\square$

Appendix C.2. Random series

Let $(\Omega, \Sigma, \mu)$ be a probability space and suppose, for a moment, that we restrict our attention to a sequence $\{s_k\}_{k \in \mathbb{N} - 1}$ of independent extended real-valued random variables $s_k : \Omega \to \mathbb{R} \cup \{\pm \infty\}$ for each $k \in \mathbb{N} - 1$. Because lower and upper limits of measurable functions are also measurable the respective lower and upper limits
\[
s_\infty = \liminf_{k \to \infty} s_k \in \Sigma_n \quad \text{and} \quad s^\infty = \limsup_{k \to \infty} s_k \in \Sigma_n
\]
(C.1)
are also random variable. In \( C_{\Omega} \) we have defined \( s_\infty(\omega) = \liminf_{k \to \infty} s_k(\omega) \) and \( s^\infty(\omega) = \limsup_{k \to \infty} s_k(\omega) \) for each \( \omega \in \Omega \) and used the notation \( \Sigma_n = \sigma(\{s_k\}_{k \in \mathbb{N}, k \geq n}) \) to denote the \( \sigma \)-algebra generated by \( \{s_k\}_{k \in \mathbb{N}, k \geq n} \). The upper and lower limit are not changed if we exclude any finite subsequence and so we also have \( s_\infty \in \Sigma_\infty \) and \( s^\infty \in \Sigma_\infty \) where \( \Sigma_\infty \) is the corresponding tail \( \sigma \)-algebra. We conclude, from the Kolmogorov 0-1 Law, that there are two extended real numbers \( c \) and \( d \) with \(-\infty \leq c \leq d \leq \infty \) such that \( s_\infty(\omega) = c \) and \( s^\infty(\omega) = d \) for \( \mu \)-almost all \( \omega \in \Omega \).

Now let \( X \) be a Banach space and let \( \{s_k\}_{k \in \mathbb{N} - 1} \) be a sequence of independent random variables \( s_k : \Omega \to X \). Let \( \{s_k\}_{k \in \mathbb{N} - 1} \) be a corresponding sequence of independent extended real-valued random variables \( s_k : \Omega \to \mathbb{R} \cup \{\pm \infty\} \) defined by

\[
s_k(\omega) = ||s_k(\omega)||
\]

for each \( \omega \in \Omega \) and each \( k \in \mathbb{N} - 1 \). Let us suppose that \( \limsup_{k \to \infty} s_k = s^\infty \). It follows that there is some extended real number \( c \in [0, \infty) \) with \( s^\infty(\omega) = c \) for \( \mu \)-almost all \( \omega \in \Omega \). Therefore we have

\[
\mathbb{E}[s^\infty] = \int_\Omega s^\infty(\omega)d\mu(\omega) = c.
\]

If we define the random variable \( m_n : \Omega \to [0, \infty] \) by setting \( m_n(\omega) = \sup_{k \geq n} s_k(\omega) \) for each \( \omega \in \Omega \) then we have \( m_n(\omega) \downarrow s^\infty(\omega) \) for all \( \omega \in \Omega \). Let us suppose there is some random variable \( m : \Omega \to [0, \infty) \) with \( \mathbb{E}[m] < \infty \) such that \( m_n \leq m \) for all \( n \in \mathbb{N} \). By dominated convergence it follows that

\[
\mathbb{E}[m_n] = \int_\Omega m_n(\omega)d\mu(\omega) \downarrow \int_\Omega s^\infty(\omega)d\mu(\omega) = \mathbb{E}[s^\infty] = c.
\]

Since \( \mathbb{E}[s_k] \leq \mathbb{E}[m] < \infty \) for all \( n \in \mathbb{N} \) there is some constant \( K \) such that \( \mathbb{E}[s_k] \leq K \) for all \( k \in \mathbb{N} \). Therefore \( c \in [0, K] \). It is also true that \( \mathbb{E}[m_k] \leq K \) for all \( k \in \mathbb{N} \).

Let us now suppose that we define another sequence \( \{u_k\}_{k \in \mathbb{N} - 1} \) of independent random variables by setting \( u_k(\omega) = (M/a^k)s_k(\omega) \) for each \( k \in \mathbb{N} - 1 \) where \( M, a \in \mathbb{R} \) with \( M, a > 0 \). We claim that the random power series

\[
U(\omega, \zeta) = \sum_{n \in \mathbb{N} - 1} u_k(\omega)\zeta^k \iff U(z) = \sum_{n \in \mathbb{N} - 1} u_k\zeta^k
\]

is \( \mu \)-almost surely absolutely convergent on the region \( |\zeta| < a \). To justify this claim we argue as follows. If we take expected values then we have

\[
\mathbb{E}[||U(\zeta)||] = \int_\Omega ||U(\omega, \zeta)||d\mu(\omega)
\]

\[
\leq \int_\Omega \sum_{k \in \mathbb{N} - 1} ||u_k(\omega)|||\zeta|^k d\mu(\omega)
\]

\[
= M\sum_{k \in \mathbb{N} - 1} (\int_\Omega s_k(\omega)d\mu(\omega)) (|\zeta|/a)^k
\]

\[
= M\sum_{k \in \mathbb{N} - 1} \mathbb{E}[s_k](|\zeta|/a)^k
\]

\[
\leq MK/(1 - |\zeta|/a).
\]
Therefore \( \mathbb{E}[\|U(\zeta)\|] < \infty \) for \(|\zeta| < a\) and hence \( U(\omega, \zeta) \) is finite \( \mu \)-almost everywhere when \(|\zeta| < a\). It follows that \( U(\omega, \zeta) \) converges almost surely for \(|\zeta| < a\) as claimed. We may be able to say more. Suppose \(|\zeta| = a + \delta\) for some \( \delta > 0 \) and suppose too that \( c > 0 \). Therefore \( U(\omega, \zeta) = \sum_{k \in \mathbb{N}} u_k(\omega) \zeta^k \) where

\[
\limsup_{k \to \infty} \|u_k(\omega)\zeta^k\| = \limsup_{k \to \infty} M_{sk}(\omega)(1 + \delta/a)^k = \infty
\]

for almost all \( \omega \in \Omega \). We conclude that the series diverges almost everywhere for \(|\zeta| > a\). Thus we say that the random power series has radius of convergence \( r = a \). Roters \[39,\] Theorem 2.1, p 122 provides a precise statement about the radius of convergence for a random power series.

Similar arguments can be applied to more general series. If we suppose there is some sequence \( \{a_k\}_{k \in \mathbb{N}-1} \subset \mathbb{C} \) with \( \sum_{k \in \mathbb{N}-1} |a_k| = L < \infty \) and we define a sequence \( \{s_k\}_{k \in \mathbb{N}-1} \) by setting \( s_k(\omega) = a_k s_k(\omega) \) for each \( k \in \mathbb{N}-1 \) and each \( \omega \in \Omega \) then the random series

\[
g(\omega) = \sum_{k \in \mathbb{N}-1} a_k s_k(\omega) \iff g = \sum_{k \in \mathbb{N}-1} a_k s_k
\]

is \( \mu \)-almost surely absolutely convergent. We have \( \mathbb{E}[\|g\|] = \int_{\Omega} \|g(\omega)\|d\mu(\omega) \leq KL \) which means that \( \|g(\omega)\| < \infty \) for \( \mu \)-almost all \( \omega \in \Omega \).

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