Notes On U(1) Instanton Counting On $A_{l-1}$ ALE Spaces

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Abstract: In this note, we investigate the detailed relationship between the orbifold partition counting and the (l-quotient, l-core) pair counting. We show that the orbifold partition counting is exactly the same as the (l-quotient, l-core) pair counting.

Keywords: $A_{l-1}$ ALE space, U(1) Instantons, Orbifold partitions, Quotient and Core partition, ADHM description, Equivariant cohomology, l-quotient, l-core.
1. Introduction

In recent years, people have made much progress on connecting the four-dimensional supersymmetric gauge theories with the two dimensional conformal theories, e.g. [1, 2, 3]. One of the first examples was found by Nakajima [4]. Further, in [5], Witten and Vafa discussed this in four-dimensional supersymmetric gauge theory based on the ALE space by using the twisting method. Nakajima’s analysis shows that the partition functions of N=4 four-dimensional U(N) gauge theories on the $A_{l-1}$ ALE space is related to the affine character of $\widehat{su}(l)_N$. Since the boundary of $A_{l-1}$ ALE space is the lens space $S^3/\mathbb{Z}_l$, so the U(N) gauge theory can approach some non-trivial flat connection at infinity [6, 7]. Such flat connections are labeled by the N-dimensional representation of $\mathbb{Z}_k$

$$\lambda \in \text{Hom}(\mathbb{Z}, U(N)),$$

which can be decomposed into irreducible representations $R_a$ of $\mathbb{Z}_l$ as

$$\lambda = \sum_a R_a N_a,$$
where $N_a$ are the integers satisfying
\[ \sum_a N_a \dim R_a = N. \]

Thus once we choose boundary condition $\lambda$ at infinity we get a vector-valued partition function whose components are of the form
\[ Z_\lambda(v, \tau) = \chi^{su(l)}_\lambda(v, q). \] (1.1)

Witten and Vafa showed that the partition function of $N=4$ $U(N)$ super-Yang-Mills theory on the ALE spaces is related to the generating function of the Euler number of the instanton moduli spaces [5]. Later, in [8], the authors employed equivariant cohomology techniques to calculate the partition function explicitly. Recently Dijkgraaf and Sulkowski introduce the orbifold partitions and show how to get the affine character of $su(l)_1$ for the $U(1)$ gauge theory [7].

In this note, we investigate the detailed relationship between the orbifold partition counting and the (l-quotient, l-core) pair counting. In section 2, we review the definition of orbifold partitions introduced in [7]. In section 3, we review the basic structures of $U(1)$ instantons on $A_{l-1}$ ALE spaces. In section 4, we investigate the detailed relation between the orbifold partition counting and the (l-quotient, l-core) pair counting and show that the orbifold partition counting is exactly the same as the (l-quotient, l-core) pair counting.

2. Orbifold partitions

In this section we briefly review the first type of orbifold partition counting in [7].

We know that an ideal of functions $\mathcal{I} = \{f(x,y)\} \subset \mathbb{C}[x,y]$ generated by a set of monomials $x^i y^j$ for $i, j \geq 0$ corresponds to an ordinary two-dimensional partition $\lambda$ in such a way that a box $(m, n) \in \lambda$ iff $x^m y^n \notin \mathcal{I}$. For example, the following Young diagram shows the $\mathcal{I}$ which is generated by $y^4, xy^2, x^2 y, x^2$.

\[
\begin{array}{c}
g^4 \\
g^2 \\
y \ y \\
1 \ x
\end{array}
\]

For the $A_{l-1}$ ALE space we will consider ideals of functions having definite transformation properties under the action
\[ (x, y) \rightarrow (\omega x, \bar{\omega} y), \] (2.1)
where $\omega = e^{2\pi i/l}$. All the monomials with the same transformation property form a periodic sub-lattice of $\mathbb{Z}^2$. In particular, the set of invariant monomials is called the invariant sector; all others are called twisted sectors.

In [7] the authors defined two type orbifold partitions. We only review the first type:
Definition 2.0.1 (Orbifold partition of the first type) It is an ordinary two dimensional partition, with some subset of its boxes distinguished; these distinguished boxes, as points in $\mathbb{Z}^2$ lattice, correspond to monomials with a definite transformation property under the action $(x, y) \rightarrow (\omega x, \bar{\omega} y)$, where $\omega = e^{2\pi i / l}$; we define a weight of such a partition as the number of these distinguished boxes.

In [7] the authors also pointed out that this kind of orbifold partitions are related to states of a Fermi sea.

We define the generating functions of the generalized partitions of the first type for ALE spaces of $A_{l-1}$ type by

$$Z_{r,orbifold}^l = \sum_{\text{first type orbifold partitions}} q^{\#(\text{black boxes})},$$

(2.2)

where $r = 0, \cdots, l - 1$ and $r$ specifies the power of $\omega$ in the action (2.1).

The orbifold partitions of the first type can be identified with a blended partition [7].

Let us review the definition of the blended partition [9, 10]. Consider a colored partition $\vec{R} = \{ R_0, \cdots, R_{l-1} \}$ with charges $p_i, i = 0, \cdots, l - 1$. The blended partition $K = (K_i)_{i \in \mathbb{N}}$ is defined by the set of integers

$$\{ p + K_m - m | m \in \mathbb{N} \} = \{ k(R_{i,m} - m + p_i) + i | i = 0, \cdots, l - 1, m \in \mathbb{N} \}. \tag{2.3}$$

The total number of boxes of $K$ is

$$|K| = \sum_{i=1}^{l} \left( l|R_i| + \frac{l}{2}p_i^2 + ip_i \right) - \frac{(l+1)p^2}{2} - \frac{p^2}{2}, \tag{2.4}$$

where $p = \sum_{i=0}^{l-1} p_i$. In Appendix C, we’ll show that $\vec{R}$ is just the l-core of K, where K is placed at point $(p, 0)$.

In [7], after claiming that the orbifold $\mathbb{Z}_l$—partitions of the first type are in one-to-one correspondence with the blended partition $K$ obtained from k-colored partition $\vec{R}$, such that

- an orbifold $\mathbb{Z}_l$—partition has the same shape as the corresponding blended partition,
- a weight of an orbifold partition (as given by the number of distinguished boxes it contains) is specified by the total weight of a state of k fermions related to $\vec{R}$,

the authors get the following formula about the number of distinguished boxes

$$|K| - \sum_{i=0}^{l-1} i p_i + \frac{(l+1)p}{2} + \frac{(r+1)r}{2} = \sum_{i=0}^{l-1} (l|R_i|) + \frac{l}{2}p_i^2 - \frac{(l+1)p}{2} - \frac{p^2}{2} + \frac{(l+1)p}{2} + \frac{(r+1)r}{2}, \tag{2.5}$$

\[\text{The definition of blended partition can be found in Appendix A. The relationship between the blended partition and free fermions can be found in [7].}\]
where the total charge is \( p = nl + r, 0 \leq r \leq l - 1 \). Then the partition function \((2.2)\) becomes

\[
Z_{r, orbifold}^l = \sum_{\{R_0, \ldots, R_{l-1}\}} q^{\sum_i |R_i|} \sum_{\{p_i | \sum_{i=0}^{l-1} p_i = pln + r\}} q^{\frac{l^2}{24} \sum (\frac{1}{2} + \frac{1}{3} l_n + \frac{1}{4} (r+1)r)}
\]

\[
= \frac{q^{l/24}}{\eta(q)} \sum_{n_1, \ldots, n_{l-1}} q^{\sum_i (n_i^2 - n_i n_{i+1}) + \frac{r}{2} + n_1 r - \frac{r}{2}}
\]

\[
= \frac{q^{l/24} \eta^2 q^{-\frac{r}{2}}}{\eta(q)} \chi_{\hat{su}(l)}(0),
\]

where \( \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \) is the Dedekind eta function and \( n_1, \ldots, n_{l-1} \in \mathbb{N}^{l-1} \) are defined by

\[
(p_0, \ldots, p_{l-1}) = \left(\begin{array}{c}
n + n_1 + r \\
n - n_1 + n_1 \\
n - n_2 + n_3 \\
\vdots \\
n - n_{l-2} + n_{l-1} \\
n - n_{l-1}
\end{array}\right) \in \mathbb{Z}^l.
\]

3. Instantons on \( A_{l-1} \) ALE space

In this section we’ll review the ADHM construction of instantons on the \( \mathbb{C}^2 \) and \( A_{k-1} \) ALE space [11, 8, 12, 13].

3.1 ADHM on \( \mathbb{C}^2 \)

The moduli space of U(1) instantons with instanton number k on \( \mathbb{C}^2 \) has a very beautiful description—the ADHM description. Basically the moduli space is a \( U(k) \) quotient of a hypersurface on \( \mathbb{C}^{2k^2 + 2k} \) defined by the ADHM constraints

\[
[B_1, B_2] + IJ = 0
\]

\[
[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \xi 1_{k \times k},
\]

where \( B_\alpha, \alpha = 1, 2 \) are linear transformations from a k-dimensional complex vector space \( V \) to itself, \( I \) is a linear map from \( V \) to a 1-dimensional complex vector space \( W \) and \( J \) is a linear map from \( W \) to \( V \). So \( B_\alpha, \alpha = 1, 2 \) are \( k \times k \) matrices, \( I \) is a \( k \times 1 \) matrix and \( J \) is a \( 1 \times k \) matrix. The U(k) action is defined by

\[
B_\alpha \mapsto UB_\alpha U^\dagger, \quad I \mapsto UI, \quad J \mapsto JU^\dagger,
\]

for \( U \in U(k) \). There is a \( U(1)^2 \) action on \( \mathbb{C}^2 \). It also induces a \( U(1)^2 \) on the instanton moduli space. According to [14], the fixed points of \( U(1)^2 \) action on the instanton moduli

\[ -4 - \]
space is in one-to-one correspondence with the Young diagram. The tangent space of the
instanton moduli space is
\[ TM_k = V^* \otimes V \otimes (Q - \wedge^2 Q - 1) + W^* \otimes V + V^* \otimes W \otimes \wedge^2 Q, \]  
(3.4)
where \( Q \) is a two-dimensional \( U(1)^2 \) module.

### 3.2 ADHM on \( A_{l-1} \) ALE space

The \( A_{l-1} \) type ALE space is defined by the blowup of the quotient \( \mathbb{C}^2/\Gamma \) where \( \Gamma \) is the \( \mathbb{Z}_l \) action:
\[
\Gamma : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{2\pi i/l} & 0 \\ 0 & e^{-2\pi i/l} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.
\]  
(3.5)

In fact the \( U(1) \) instantons on \( \mathbb{C}^2/\Gamma \) can be described by the set \( \{(Y,r)\} \) where \( Y \) is a Young diagram with \( l \) types of boxes and \( r \) is an integer mod \( p \). The integer \( r \) specifies the \( \mathbb{Z}_l \) representation \( R_r \) under which the first box in \( Y \) transforms. We know that there exists a \( U(1)^2 \) action on the instantons on \( \mathbb{C}^2 \). The specified \( r \) gives rise to an embedding of \( \Gamma \) into \( U(1)^2 \):
\[
\Gamma : R_a \mapsto e^{2\pi ia/l} R_a, \quad T_1 \mapsto e^{2\pi ia/l} T_1, \quad T_2 \mapsto e^{-2\pi ia/l} T_2.
\]  
(3.6)

Under this action \( V \) and \( W \) have the following decomposition:
\[
V = \sum_{a=0}^{l-1} V_a \otimes R_a, \quad \dim V_a = k_a,  
\]  
(3.7)
\[
W = \sum_{a=0}^{l-1} W_a \otimes R_a, \quad \dim W_a = N_a.  
\]  
(3.8)

Since we are dealing with the \( U(1) \) instantons, only one of \( N_a \) is non-vanishing, and it depends on the \( a \) specified. Thus the tangent space of the instanton moduli space is given by the \( \Gamma \)-invariant component of the \( \mathbb{C}^2 \) result \[8, 12\]
\[ TM_Y = \left( V^* \otimes V \otimes (Q - \wedge^2 Q - 1) + W^* \otimes V + V^* \otimes W \otimes \wedge^2 Q \right)^\Gamma. \]  
(3.9)

The dimension of the instanton moduli space is
\[
\dim_{\mathbb{C}} M_Y = (k_a k_{a+1} + k_a k_{a-1} - 2k_a^2 + sk_a N_a)  
= -\hat{C}_{ab} k_a k_b + 2k_a N_a,  
\]  
(3.10)
\[ (3.11) \]
where \( \hat{C}_{ab} = 2\delta_{ab} - \delta_{a,b+1} - \delta_{a,b-1} \) is the extended \( A_{l-1} \) Cartan matrix.

Now let us consider the tautological bundle \[11\]. We know the \( A_{l-1} \) singularity is resolved by replacing the singularity with \( l - 1 \) intersecting \( \mathbb{P}^1 \). This leads to new self-dual connections with non-trivial fluxes along the exceptional divisors. The \( U(1) \) bundles \( \Upsilon^a, a = 0, \ldots, l-1 \) carry the unit of flux though the exceptional divisors. \( \Upsilon^0 \) is the trivial bundle. These \( \Upsilon^a, a = 0, \ldots, l-1 \) have the following property:
\[
\int C_1(\Upsilon^a) \wedge c_1(\Upsilon^b) = -C_{ab}, \quad \int_{C_a} c_1(\Upsilon^b) = \delta_a^b,  
\]  
(3.12)
where $C^{ab}$ is the inverse of the $A_{l-1}$ Cartan matrix and $C_a$ is the $a^{th}$ exceptional divisor.

The gauge bundle $F_Y$ is given by [11, 12]

$$F_Y = (V^* \otimes \Upsilon \otimes (Q - \lambda^2 Q - 1) + W^* \otimes \Upsilon)^\Gamma,$$

(3.13)

where $\Upsilon = \sum_{l=0}^{l-1} \Upsilon^a R_a$ is the tautological bundle.

The Chern characters are given by [12]

$$ch_1(F_Y) = \sum_a u_a ch_1(\Upsilon^a),$$

(3.14)

$$ch_2(F_Y) = \sum_a u_a ch_2(\Upsilon^a) - \frac{K}{l} \Omega,$$

(3.15)

where $K = \sum_a k_a$, $\Omega$ is the normalized volume form of the manifold and

$$u_a := N_a + k_{a+1} + k_{a-1} - 2k_a = N_a - \hat{C}_{ab} k_b.$$

(3.16)

Further, the instanton number $k \in \mathbb{Z}$ is defined by

$$k = -\int_M ch_2(F_Y) = \frac{1}{2} \sum_a C^{aa} u_a + \frac{K}{k} = k_0 + \frac{1}{2} \sum_a C^{aa} N_a,$$

(3.17)

where $C^{aa} = \frac{1}{l}(l - a)a$.

According to [5, 12], the partition function is given by

$$Z(q, z_a) = \sum_{k, u_a} \chi(M_{k,u_a}) q^k e^{-z^a u_a} 2,$$

(3.18)

where $\chi(M_{k,u_a})$ is the Euler number of the instanton moduli space with first and second Chern characters $u_a$ and $k$ respectively.

### 4. Regular and fractional instantons

According to [8, 12, 13], the partition function of instantons on $A_{l-1}$-ALE space can be factorized into a product of contributions of regular and fractional instantons. We define the $r$ sector of the partition function $Z(q, z_a = 0)$ by

$$Z^l_r = Z_{reg} Z^{l}_{r,frac},$$

(4.1)

where the detailed definitions of $Z_{reg}$ and $Z^{l}_{r,frac}$ can be found in the subsection 4.1 and subsection 4.2. We’ll show that

$$Z^l_r \equiv Z^{l}_{r,orbifold}.$$
4.1 Regular instantons

The regular instantons are instantons in the regular representation of \( \Gamma = \mathbb{Z}_l \). They are free to move on \( \mathbb{C}^2/\Gamma \). The moduli space is

\[
\mathcal{M}^\text{reg}_{kl} = (\mathbb{C}^2/\mathbb{Z}_l)^k/S_k,
\]

(4.3)

which is related to the Hilbert scheme of \( k \)-points on \( \mathbb{C}^2/\Gamma \) via the Hilbert-Chow morphism. The first Chern class of the regular instanton moduli space is vanishing. Further, according to [8, 12, 13], the regular instantons correspond to the \( l \)-quotients \( \vec{R} \) of Young diagrams whose definition can be found in Appendix B. Thus the instanton number \( k \) has following relationship with the number of boxes of the \( l \)-quotients \( \vec{R} = (R_0, R_1, \cdots, R_{l-1}) \) of Young diagram \( K \) [12]:

\[
k = \sum_{i=0}^{l-1} |R_i|.
\]

(4.4)

Hence it is not hard to find that the partition function of the regular instantons is [13]

\[
Z_{\text{reg}} = \sum_k q^k \chi(\mathcal{M}^\text{reg}_{kl}) = \frac{q^{\frac{l}{2}^3}}{\eta(q)^3},
\]

(4.5)

where \( \chi(\mathcal{M}^\text{reg}_{kl}) \) is the Euler number of \( \mathcal{M}^\text{reg}_{kl} \). Further, the formula (4.5) is consistent with the formula (2.6) and the Young diagram \( K \) is just the blended partition of \( \vec{R} \). In Appendix C, we show that the \( \vec{R} \) is just the \( l \)-quotient of the blended partition \( K \) of \( \vec{R} \).

4.2 Fractional instantons

According to [8, 12, 13], the fractional instantons correspond to the Young diagrams which do not have any boxes whose hook length \( \ell(s) \) satisfies \( \ell(s) = 0 \mod l \). In fact, the fractional instantons correspond to the \( l \)-cores of Young diagrams whose definition can be found in Appendix B. Following [13], we define the partition function \( Z_{r, \text{frac}}^l \) by

\[
Z_{r, \text{frac}}^l := \sum_{Y \in \mathcal{C}(l)} q^{N_{kl-r}},
\]

(4.6)

where \( \mathcal{C}(l) \) is the subset of all Young diagrams consisting of \( l \)-cores whose definition can be found in Appendix B. According to [13], we have the following formula:

\[
Z_{r, \text{frac}}^l = \sum_{\{p_i|\sum_i p_i=0\}} q^{\frac{1}{2} \sum_{i=0}^{l-1} p_i^2 + \sum_{i=-r}^{l-1} p_i} \chi(\mathcal{M}^\text{reg}_{kl}) = \sum_{\{\tilde{p}_i|\sum_i \tilde{p}_i=p=ln+r\}} \frac{q^{\frac{1}{2} \sum_{i=0}^{l-1} \tilde{p}_i^2 + \sum_{i=0}^{l-1} \tilde{p}_i} p^{\binom{l+1}{2} - \binom{ln}{2} - \binom{ln}{2} - \binom{r+1}{2} + \binom{r+1}{2}}}{l},
\]

(4.7)

where the definition of \( \tilde{p}_i \) and the detailed derivation can be found in the equations (B.4, B.8, B.9) in Appendix B. According to the Proposition 2.28 in [13], the blended partition

\[\text{Comparing the } q \text{ here with notations in theorem 4.6 in [13], our } q \text{ is equal to } t_0 \text{ and } q = t_i = 1, \text{ for } i \neq 0 \mod l.\]
$K$ is uniquely determined by the pair $(l$-quotient, $l$-core). Hence counting the blended partitions is equivalent to counting the pair of $(l$-quotient, $l$-core). Furthermore, according to [7], the shape of blended partition is the same as the orbifold partitions. So we have

$$Z_{r, orbifold} = Z_{r} = Z_{reg} Z_{r, frac}. \quad (4.9)$$

5. Conclusion

In this note, we have investigated the detailed relationship between the orbifold partitions of first type introduced in [7] and the pair $(l$-quotient, $l$-core) introduced in [13]. We find that orbifold partition counting presented in [7] is exactly the same as counting the $(l$-quotient, $l$-core) pair. According to [12, 13] the U(N) partition function factorizes into

$$Z_{U(N)} = Z_{U(1)}^N = (Z_{U(1), reg} Z_{U(1), frac})^N. \quad (5.1)$$

It would be interesting to see how to generalize the orbifold partition counting to the U(N) case.

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A. Colored partitions , charged partitions and blended colored partitions

In this Appendix, we shall review the definitions of colored partitions, charged partitions and blended partitions in [9].

**Definition A.0.1 (Colored partition)** The colored partition $\vec{R}$ is the $l$-tuple of partitions:

$$\vec{R} = (R_0, \cdots, R_{l-1}), \quad (A.1)$$

where

$$R_k = (R_{k,1} \geq R_{k,2} \cdots \geq R_{k,n_k} > R_{k,n_k+1} = 0 = \cdots). \quad (A.2)$$

$$|\vec{R}| := \sum_{k,i} R_{k,i} \quad (A.3)$$

**Definition A.0.2 (Charged partition)** The charged partition $(p, \vec{R})$ is the set of non-increasing integers $\vec{R}_i = R_i + p$, where

$$\vec{R} = (R_0 \geq R_1 \geq \cdots), \quad (A.4)$$

is a partition, and $p \in \mathbb{Z}$. The limit $\vec{R}_\infty \equiv p$ is called the charge.

**Definition A.0.3 (Blending of colored partition)** Given a vector $\vec{p} = (p_0, \cdots, p_{l-1})$, with

$$\sum_{i=0}^{l-1} p_i = p \quad (A.5)$$

and an $N$-tuple of partition $\vec{R}$, we define the blended partition $K$, as follows:

$$\{p + K_m - m | m \in \mathbb{N}\} = \{k(R_{i,m} - m + p_i) + i | i = 0, \cdots, l - 1, m \in \mathbb{N}\}. \quad (A.6)$$
B. Quotients and cores for Young diagrams

In this appendix, we shall review the definitions of quotients and cores for Young diagrams [13, 12, 15].

**Definition B.0.4 (Maya diagram)** A Maya diagram is a sequence \( \{\mu(k)\}_{k \in \mathbb{Z}} \) which consists of 0 or 1 and satisfies the following property: there exist \( N, M \in \mathbb{Z} \) such that for all \( k > N \) (resp. \( k < N \)), \( \mu(k) = 1 \) (resp. \( \mu(k) = 0 \)).

There is a one-to-one correspondence between the set of Maya diagrams and the set of Young diagrams. We can place the Young diagram \( Y \) on the \((x, y)\)-plane by the following way: the bottom-left corner of the Young diagram is at the origin \((0,0)\) and a box in the Young diagram is an unit square. We call an upper-right borderline of \( \{x - \text{axis}\} \cup \{y - \text{axis}\} \cup Y \) the extended borderline of \( Y \) and denote it by \( \partial Y \). The line defined by \( y = x \) is called the medium. The Maya diagram \( \{\mu(k)\}_{k \in \mathbb{Z}} \) corresponds to a Young diagram \( Y \) as follows.

1. Give the direction to the extended borderline \( \partial Y \) of \( Y \), which goes from \((0, +\infty)\) to \((+\infty, 0)\) (see Figure 1). Then each edge of the extended borderline is numbered by \( k \in \mathbb{Z} \), if we set the edge which is located at the next to the medium to be 0. Next we encode a edge \( \downarrow \) (resp. \( \uparrow \)) to 0 (resp. 1). By this way, we have a 0/1 sequence \( \{\mu(k)\}_{k \in \mathbb{Z}} \), where each \( \mu_Y(k) \) corresponds to a edge of \( \partial Y \).

2. **Definition B.0.5 (Quotients)** For each \( 0 \leq i \leq l - 1 \), we define
   
   \[ \mu_{Y_i}(k) := \mu_Y(lk + i), \quad k \in \mathbb{Z}. \]  

   Then we have an \( p \)-tuple \( \left( \{\mu_{Y_0}(k)\}_{k \in \mathbb{Z}}, \ldots, \{\mu_{Y_{l-1}}(k)\}_{k \in \mathbb{Z}} \right) \) of subsequences \( \mu_Y(k)_{k \in \mathbb{Z}} \) and each \( \{\mu_{Y_i}(k)\}_{k \in \mathbb{Z}} \) is also a Maya diagram. The \( p \)-quotient for \( Y \) is the \( l \)-tuple of Young diagrams \( \vec{Y} := (Y_0^*, \ldots, Y_{l-1}^*) \) corresponding to the Maya diagrams \( \left( \{\mu_{Y_0}(k)\}_{k \in \mathbb{Z}}, \ldots, \{\mu_{Y_{l-1}}(k)\}_{k \in \mathbb{Z}} \right) \).
Remark B.0.6 Here sometime there does not exist a Young diagram corresponding to the Maya diagram $\{\mu_{Y_i^*}(k)_{k \in \mathbb{Z}}\}$, if so, we denote this Young diagram as the empty set $\emptyset$.

We define $p_i(Y) \in \mathbb{Z}(i = 0, \cdots, l - 1)$ by the following condition:

\[ \sharp \{ \mu_{Y_i^*}(k) = 1 | n < p_i(Y) \} = \sharp \{ \mu_{Y_i^*}(k) = 0 | n \geq p_i(Y) \}. \] (B.2)

Remark B.0.7

\[ \sum_{i=0}^{l-1} p_i(Y) = 0. \] (B.3)

Remark B.0.8 Notice that here we place the vertex of Young diagram to the origin. If we place the vertex at $(p, 0)$, then we have

\[ \tilde{p}_i = \begin{cases} p_0 + n, \\ p_1 + n, \\ \vdots \\ p_{l-m-1} + n, \\ p_{l-m} + n + 1, \\ p_{l-m+1} + n + 1, \\ \vdots \\ p_{l-1} + n + 1, \end{cases} \] (B.4)

where $p = nl + m$ and $\tilde{p}_i$ is the new one satisfying (B.2) in the new Young diagram whose vertex is placing at $(p, 0)$. Hence we have

\[ \sum_{i=0}^{l-1} \tilde{p}_i = p. \] (B.5)

Definition B.0.9 (l-core) A Young diagram is called l-core if its l-quotient $\vec{Y}$ is empty. Denote $\mathcal{C}^{(l)}$ as the subset of all Young diagrams consists of l-cores.

Definition B.0.10 Let $Y^{(l)}$ be the Young diagram obtained by removing as many hooks of length $l$ as possible from $Y$. It is called the l-core of $Y$.

Proposition B.0.11 For any $l \geq 2$, a Young diagram $Y$ is uniquely determined by its l-core $Y^{(l)}$ and l-quotient $\vec{Y}$. Then we have

\[ |Y| = |Y^{(l)}| + l|\vec{Y}|. \] (B.6)

Proposition B.0.12 For $Y \in \mathcal{C}^{(l)}$ which determined by $(p_0, p_1, \cdots, p_{l-1})$ and $\sum_i p_i = 0$, then the number of boxes on $y = x - nl + j \ N_{nl-j}(Y)$ is

\[ \sum_{k \in \mathbb{Z}} N_{kt-j}(Y) = \frac{1}{2} \sum_{i=0}^{l-1} p_i(Y)^2 + \sum_{i=l-j}^{l-1} p_i(Y). \] (B.7)
If we place the Young diagram at \((p,0)\), where \(p = ln + r (0 \leq r \leq l - 1)\), then using (B.4) to replace \(p_i\) by \(\tilde{p}_i\) in the formula (B.7), then we have the following formula

\[
\sum_{k \in \mathbb{Z}} N_{kl-j}(Y) = \frac{1}{2} \left\{ \sum_{i=0}^{l-1} (\tilde{p}_i - n)^2 + \sum_{i=l-j}^{l-1} (\tilde{p}_i - n - 1)^2 \right\} + \sum_{i=l-j}^{l-1} (\tilde{p}_i - n - 1)
\]

\[
= \frac{1}{2} \left\{ \sum_{i=0}^{l-1} (\tilde{p}_i^2 - 2n\tilde{p}_i + n^2) + \sum_{i=l-j}^{l-1} [\tilde{p}_i^2 - 2(n + 1)\tilde{p}_i + (n + 1)^2] \right\} + \sum_{i=l-j}^{l-1} (\tilde{p}_i - n - 1)
\]

\[
= \frac{1}{2} \sum_{i=0}^{l-1} \tilde{p}_i^2 - n \sum_{i=0}^{l-1} \tilde{p}_i + \frac{n^2l}{2} + \frac{(2n + 1)j}{2} - (n + 1)j
\]

\[
= \frac{1}{2} \sum_{i=0}^{l-1} \tilde{p}_i^2 - nr - \frac{n^2l}{2} - \frac{j}{2}.
\]  

(B.8)

If \(j = r\), then we have

\[
\sum_{k \in \mathbb{Z}} N_{kl-r}(Y) = \sum_{i=0}^{l-1} \tilde{p}_i^2 - nr - \frac{n^2l}{2} - \frac{r}{2}
\]

\[
= \sum_{i=0}^{l-1} \left( \frac{l \tilde{p}_i^2}{2} - \frac{(l+1)p}{2} - \frac{(r+1)r}{2} \right),
\]  

(B.9)

where \(p = ln + r\).

C. Blended partition, l-quotient and l-core

Suppose we have a blended partition \(K\) as follows:

\[
\{p + K_m - m | m \in \mathbb{N}\} = \{(R_i, m - m + p_i) + i | i = 0, \ldots, l - 1, m \in \mathbb{N}\}. \tag{C.1}
\]

According to the l-quotient’s definition (B.0.5), it is easy to find that the right hand side of the formula (C.1) tells us the l-quotient of K is \(\tilde{R} = (R_0, \ldots, R_l)\) and the positions of 0 are \((R_i, m - m + p_i, 0)\) in Maya diagram corresponding to the Young diagram \(R_i\). Thus it is not hard to find that \(R_i\) is placed at \((p_i, 0)\) position. Further, the left hand side tells us that the positions of 0 of the Maya diagram of \(K\) are \((p + K_m - m, 0)\). It implies that the Young diagram K is placed at \((p,0)\).

Moreover, since the decomposition of \(K\) into a pair (l-quotient, l-core) is unique, so it is not hard to imagine that counting the orbifold partition K is equivalent to counting pairs (l-quotient, l-core).

Now let us see an example. Figure 2 is obtained from [7]. It shows the blended partition coming from \(k=3\) fermions. Notice the total charge \(p = p_0 + p_1 + p_2 = 0\). Table 1 shows the Maya diagrams corresponding to this blended partition. In Table 1, the central vertical line is the position of the origin. It is not hard to see that the diagrams \(R_0, R_1, R_2\) are
Figure 2: The blended partition copied from [7].

Table 1: Maya diagrams with vanishing total charge $p=0$

|   | $K$ | $R_0$ | $R_1$ | $R_2$ |
|---|-----|-------|-------|-------|
|   | 0 1 0 1 0 1 0 1 | 0 0 1 0 1 1 1 0 | 0 0 1 0 1 1 1 0 | 0 0 1 0 1 1 1 0 |
| $p_0$ | 1 | 1 | 1 | 1 |
| $p_1$ | 1 | 1 | 1 | 1 |
| $p_2$ | 1 | 1 | 1 | 1 |

Table 2: Maya diagrams with $p=1$

|   | $K$ | $R_0$ | $R_1$ | $R_2$ |
|---|-----|-------|-------|-------|
|   | 0 1 0 1 0 1 0 1 | 0 0 1 0 1 1 1 0 | 0 0 1 0 1 1 1 0 | 0 0 1 0 1 1 1 0 |
| $p_0$ | 1 | 1 | 1 | 1 |
| $p_1$ | 1 | 1 | 1 | 1 |
| $p_2$ | 1 | 1 | 1 | 1 |

exactly same as the left Young diagrams in Figure 2. It also shows the correct position of the vertices of $R_0, R_1, R_2$. Now let us examine the situation of $p \neq 0$. For simplicity, let us assume $p = 1$, so that the Maya diagram is changed to one shown in Table 2. It shows that $p_2 = -1$ and $p_0, p_1$ does not change; which means the $R_2$ is placed at $(-1, 0)$. Now if we blend $R_0, R_1, R_2$ to a single Young diagram by using the formula (C.1), it is easily to see that this blended Young diagram is exactly same as the $K$ here.

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The 3-core of the Young diagram $K$ in figure 2 is as follows:

\[
\begin{array}{c}
  b \\
  b \\
  b
\end{array}
\]

where $b$ stands for black color.

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