Abstract. Oliu-Barton and Ziliotto [8] proved that the constant payoff property holds for discounted stochastic games, as conjectured by Sorin, Venel and Vigeral [11]. That is, the existence of a pair of asymptotically optimal strategies so that the average rewards are constant on any fraction of the game. That a similar property holds for stochastic games with an arbitrary evaluation of the stage rewards is still open. In this paper, we prove that the constant constant payoff property holds for a class of stochastic games which includes the well-known model of absorbing games and stochastic games with two states.

Contents

1 Introduction 2
2 Stochastic games 3
  2.1 Strategies . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
  2.2 The cumulated payoffs . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
  2.3 Optimal and asymptotically optimal strategies . . . . . . . . . . . . . . . . . . . 4
  2.4 The general constant payoff conjecture . . . . . . . . . . . . . . . . . . . . . . . . 5
  2.5 Main result . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
3 Proofs 6
  3.1 Preliminaries . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
  3.2 Absorbing games . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
3.3 The constant payoff in discounted absorbing games . . . . . . . . . . . . . . . . . . 7
3.4 The constant payoff for general absorbing games . . . . . . . . . . . . . . . . . . 9
3.5 Critical stochastic games . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10

1 Introduction

Model. Stochastic games were introduced by Shapley [9] in order to model a repeated interaction between two opponent players in a changing environment. The game proceeds in stages. At each stage $m \in \mathbb{N}$ of the game, players play a zero-sum game that depends on a state variable. Formally, knowing the current state $k_m$, Player 1 chooses an action $i_m$ and Player 2 chooses an action $j_m$. Their choices occur independently and simultaneously and have two consequences: first, they produce a stage payoff $g(k_m, i_m, j_m)$ which is observed by the players and, second, they determine the law $q(k_m, i_m, j_m)$ of the next period’s state $k_{m+1}$. Thus, the sequence of states follows a Markov chain controlled by the actions of both players. To any sequence of nonnegative weights $\theta = (\theta_m)$ and any initial state $k$ correspond the $\theta$-weighted average stochastic game which is one in which Player 1 maximizes the expectation of

$$\sum_{m \geq 1} \theta_m g(k_m, i_m, j_m)$$

given that $k_1 = k$, while Player 2 minimizes this same amount. A crucial aspect in this model is that the current state is commonly observed by the players at every stage. Another one is stationarity: the transition function and stage payoff function do not change over time.

A $\theta$-evaluated stochastic game is thus described by a tuple $(K, I, J, g, q, k, \theta)$ where $K$ is a set of states, $I$ and $J$ are the sets of actions of both players, $g : K \times I \times J \to \mathbb{R}$ is the reward function, $q : K \times I \times J \to \Delta(K)$ is the transition function, $k$ is the initial state and $\theta$ is a normalized sequence of nonnegative weights, i.e. so that $\sum_{m \geq 1} \theta_m = 1$ with no loss of generality. A $\lambda$-discounted stochastic game is one where $\theta_m = \lambda(1 - \lambda)^{m-1}$ for all $m \geq 1$ for some $\lambda \in (0, 1]$. A $T$-stage stochastic game is one where $\theta_m = \frac{1}{T} 1_{\{m \leq T\}}$ for all $m \geq 1$ for some $T \in \mathbb{N}$. Like in Shapley’s seminal paper [9], we assume throughout this paper that $K, I, J$ are finite sets, and identify the set $K$ with $\{1, \ldots, n\}$.

Selected past results. Every stochastic games $(K, I, J, g, q, k, \theta)$ has a value, denoted by $v^k_{\theta}$. Although only stated for the discounted case, this result follows from Shapley [9]. We use the notation $v^k_T$ and $v^k_T$ to refer respectively to the value of a $\lambda$-discounted and a $T$-stage stochastic game. Bewley and Kohlberg [2] proved the convergence of $v^k_T$ as $\lambda$ goes to 0 and the convergence of $v^k_T$ as $T \to +\infty$, to the same limit. Mertens and Neyman [5, 6] proved the existence of the value $v^k$, that is, Player 1 can ensure that the average reward is at least $v^k$ in any $T$-stage stochastic game with $T$ large enough, and similarly Player 2 can ensure that the average reward is at most $v^k$. Neyman and Sorin [7] studied stochastic games with a random number of stages, and proved that the values converge to $v^k$ as the expected number of stages tends to $+\infty$, and the expected number of remaining stages decreases throughout the game. Ziliotto [13] proved that $v^k_{\theta}$ converges
to $v^k$ as $\|\theta\| := \max_{m \geq 1} \theta_m$ goes to 0 provided that $\sum_{m \geq 1} |\theta_{m+1}^p - \theta_m^p|$ converges to zero for some $p > 0$. The value was recently characterized by Attia and Oliu-Bart on [1].

**The constant-payoff property.** A remarkable property, referred to as the constant-payoff property was proved by Sorin, Venel and Vigeral [11] in the framework of single decision-maker problems, and conjectured to hold in any stochastic game. Their conjecture goes as follows.

- **The discounted case.** For any sufficiently small $\lambda$ there exists a pair of optimal strategy so that $\sum_{m=1}^{M} \lambda(1-\lambda)^{m-1} g(k_m, i_m, j_m)$ is approximatively equal to $(\sum_{m=1}^{M} \lambda(1-\lambda)^{m-1}) v^k$ in expectation.

- **The general case.** Because $\sum_{m \geq 1} \lambda(1-\lambda)^{m-1} = 1$, Sorin [10] proposed the interpretation of $\sum_{m=1}^{M} \lambda(1-\lambda)^{m-1}$ as the fraction of the game that has been played at stage $M$. The notion of a fraction of the game extends to any evaluation $\theta$, and the general constant-payoff conjecture is the existence of a pair of strategies so that for any sufficiently small $\|\theta\|$, $\sum_{m=1}^{M} \theta_m g(k_m, i_m, j_m)$ is approximatively equal to $(\sum_{m=1}^{M} \theta_m) v^k$ in expectation.

The constant-payoff conjecture was established Oliu-Barton and Ziliotto [8] for discounted stochastic games, and the problem remains open for arbitrary evaluations of the rewards.

**Main result.** In this paper we solve the general constant-payoff conjecture for a class of stochastic games (see Theorem 2.4 below) which includes the well-known model of absorbing games introduced by Kohlberg [4], and stochastic games with two states. (It is worth noting that, for absorbing games, the discounted constant-payoff property was proved by Sorin and Vigeral [12] using an independent approach.)

### 2 Stochastic games

In the sequel, let $(K, I, J, g, q, k, \theta)$ denote a $\theta$-evaluated stochastic game. In order to state our results formally, we start by recalling some definitions.

#### 2.1 Strategies

The sequence $(k_1, i_1, j_1, ..., k_m, i_m, j_m, ...)$ generated along the game is called a play. The set of plays is $(K \times I \times J)^\mathbb{N}$.

**Definition 2.1**

- A strategy for a player specifies a mixed action to each possible set of past observations. Formally, a strategy for Player 1 is a collection of maps $\sigma^1 = (\sigma^1_{m \geq 1})$, where $\sigma^1_m : (K \times I \times$
$J)^{m-1} \times K \to \Delta(I)$. Similarly, a strategy for Player 2 is a collection of maps $\sigma^2 = (\sigma^2_m)_{m \geq 1}$, where $\sigma^2_m : (K \times I \times J)^{m-1} \times K \to \Delta(J)$.

- A stationary strategy is one that plays according to the current state only. Formally, a stationary strategy for Player 1 is a mapping $x : K \to \Delta(I)$, and a stationary strategy for Player 2 is a mapping $y : K \to \Delta(J)$.

**Notation.** The sets of strategies for Player 1 and 2 are denoted by $\Sigma$ and $\mathcal{T}$, respectively, and the sets of stationary strategies by $\Delta(I)$ and $\Delta(J)$. For any pair $(\sigma, \tau) \in \Sigma \times \mathcal{T}$ we denote by $P^k_{\sigma,\tau}$ the unique probability measure on the set of plays $(K \times I \times J)^\mathbb{N}$ induced by $(\sigma, \tau)$, $k_1 = k$ and $q$. (Note that the dependence on the transition function $q$ is omitted). This probability is well-defined by the Kolmogorov extension theorem, and the expectation with respect to the probability $P^k_{\sigma,\tau}$ is denoted by $E^k_{\sigma,\tau}$.

### 2.2 The cumulated payoffs

For each normalized sequence of nonnegative weights $\theta$ we introduce the clock function $\varphi(\theta, \cdot) : [0, 1] \to \mathbb{N}$ by setting

$$\varphi(\theta, t) := \inf\{M \geq 1, \sum_{m=1}^M \theta_m \geq t\} \quad \forall t \in [0, 1].$$

The cumulated payoff at time $t$ is defined for any pair of strategies $(\sigma, \tau) \in \Sigma \times \mathcal{T}$ as

$$\gamma^k_\theta(\sigma, \tau; t) := E^k_{\sigma,\tau} \left[ \sum_{m \geq 1} \varphi(\theta, t) \theta_m g(k_m, i_m, j_m) \right].$$

Note that the case $t = 1$ corresponds to the expectation of the $\theta$-evaluation of the stage rewards.

### 2.3 Optimal and asymptotically optimal strategies

**Definition 2.2** An optimal strategy of Player 1 is an element $\sigma \in \Sigma$ so that, for all $\tau \in \mathcal{T}$,

$$\gamma^k_\theta(\sigma, \tau; t) \geq v^k_\theta.$$

An optimal strategies of Player 2 is defined in a similar way. That is, it is an element $\tau \in \Sigma$ so that, for all $\sigma \in \Sigma$ one has $\gamma^k_\theta(\sigma, \tau; t) \leq v^k_\theta$.

**Definition 2.3** Let $\varepsilon \geq 0$. A family of strategies $(\sigma^\varepsilon_\theta)$ indexed by $\theta$ is asymptotically $\varepsilon$-optimal for Player 1 if for any $\tau \in \mathcal{T}$,

$$\liminf_{\lambda \to 0} \gamma^k_\lambda(x^\varepsilon_\lambda, \tau) \geq v^k - \varepsilon.$$

Asymptotically $\varepsilon$-optimal for Player 2 are defined in a symmetric way.
2.4 The general constant payoff conjecture

Sorin, Venel and Vigeral [11] conjectured the existence of a pair of asymptotically 0-optimal strategies \((\sigma(\theta), \tau(\theta))\), indexed by \(\theta\), and so that

\[
\lim_{\|\theta\| \to 0} \gamma^k_\theta(\sigma(\theta), \tau(\theta); t) = tv^k \quad \forall t \in [0,1].
\]

A family of asymptotically 0-optimal strategies. Let \((x_\lambda, y_\lambda)\) a fixed family of optimal stationary strategies so that \(\lambda \mapsto x^k_\lambda(i)\) and \(\lambda \mapsto y^k_\lambda\) admit a Puiseux expansion near 0 for all \((i,j) \in I \times J\). For each \(\theta \in \Delta(N)\) and \(m \geq 1\) set

\[
\lambda_m = \sum_{m' \geq m} \theta_{m'}.
\]

Define then the strategy pair \((\sigma^\theta, \tau^\theta)\) by setting,

\[
(\sigma^\theta_m, \tau^\theta_m) := (x_{\lambda_m}, y_{\lambda_m}) \quad \forall m \geq 1. 
\]

The family \((\sigma^\theta, \tau^\theta)\) is asymptotically 0-optimal by Ziliotto [13], so it is a good candidate for tackling the general constant-payoff conjecture.

2.5 Main result

Consider the following symmetric conditions \((H1)\) and \((H2)\). These conditions preclude going from one state to another and back, and then to a third state, for a fixed pure stationary strategy of the opponent.

- \((H1)\) There does not exist a triplet of different states \((\ell, \ell', \ell)\) and a tuple of actions \((i, j, i', j')\), \(i \neq i'\) so that

  \[
  q(\ell' | \ell, i, j)q(\ell' | i', j')q(\ell | i, j) > 0.
  \]

- \((H2)\) There does not exist a triplet of different states \((\ell, \ell', \ell)\) and a tuple of and a tuple of actions \((i, j, i', j')\), \(j \neq j'\) so that

  \[
  q(\ell' | \ell, i, j)q(\ell' | i', j')q(\ell | i, j) > 0.
  \]

**Theorem 2.4** Any stochastic game \((K, I, J, g, q, k, \theta)\) satisfying \((H1)\) and \((H2)\) satisfies general constant-payoff property. More precisely, the family of asymptotically 0-optimal strategies defined in \((2.1)\) satisfies

\[
\lim_{\|\theta\| \to 0} \gamma^k_\theta(\sigma^\theta, \tau^\theta; t) = tv^k \quad \forall t \in [0,1].
\]

**Definition 2.5** A stochastic game \((K, I, J, g, q, k, \theta)\) is absorbing if for every state \(\ell \neq k\) is absorbing, i.e. \(q(\ell | \ell, i, j) = 1\) for all \((i,j) \in I \times J\).
The following result is a direct consequence of Theorem 2.4.

**Corollary 2.6** Suppose that \((K, I, J, g, q, k, \theta)\) is an absorbing game. Then,
\[
\lim_{\|\theta\| \to 0} \gamma_{\theta}^k(\sigma^\theta, \tau^\theta; t) = tv^k \quad \forall t \in [0, 1].
\]

3 Proofs

Theorem 2.4 relies on the two following properties, which hold for all \(t \in [0, 1]\):

- \(\lim_{\lambda \to 0} \gamma_{\lambda}^k(x_{\lambda}, y_{\lambda}; t) = tv^k\).
- \(\lim_{\|\theta\| \to 0} \gamma_{\theta}^k(\sigma^\theta, \tau^\theta; t)\) exists and does not depend on the family of vanishing evaluations.

We start by recalling some useful results from [3, 8] on discounted stochastic games. Then, for the sake of simplicity, we will establish Corollary 2.6 first. Finally, we will extend this result to the class of stochastic games of Theorem 2.4.

3.1 Preliminaries

We recall the following results from [3, 8] for discounted stochastic games.

**Theorem 3.1** For every \(t \in [0, 1]\) the limit \(\Pi_t := \lim_{\lambda \to 0} \sum_{m \geq 1} \lambda(1 - \lambda)^{m-1} Q_{\lambda}^{m-1} \in \mathbb{R}^{n \times n}\) exists. Furthermore, there exist \(p \in \mathbb{N}\), which stands for the number of payoff-relevant cycles, \(\Phi \in \mathbb{R}^{n \times p}\), \(A \in \mathbb{R}^{p \times p}\) and \(M \in \mathbb{R}^{p \times n}\) so that
\[
\Pi_t = \int_0^t \Phi e^{-\ln(1-s)A} M ds.
\]

**Corollary 3.2** The map \(t \mapsto \Pi_t\) is twice differentiable on \((0, 1)\) and
\[
\frac{\partial^2}{\partial t^2} \Pi_t = \frac{1}{1-t} \Phi e^{-\ln(1-t)A} AM \quad \forall t \in (0, 1).
\]

**Lemma 3.3** Suppose that \(AMg_0 = 0\). Then, \(\lim_{\lambda \to 0} \gamma_{\lambda}(x_{\lambda}, y_{\lambda}; t) = tv\).

Indeed, in this case one has \(\frac{\partial^2}{\partial t^2} \Pi_t g_0 = 0\). Therefore, there exists vectors \(\alpha, \beta \in \mathbb{R}^n\) so that \(\Pi_t g_0 = \alpha t + \beta\). The boundedness of \(g_0\) implies that
\[
\lim_{\lambda \to 0} \gamma_{\lambda}(x_{\lambda}, y_{\lambda}; t) = \Pi_t g_0 \quad \forall t \in [0, 1].
\]

In particular one has \(\Pi_0 g_0 = 0\) because \(\Pi_0 = 0\) by definition, and \(\Pi_1 g_0 = v\) by the optimality of \((x_{\lambda}, y_{\lambda})\). Consequently, \(\alpha = v\) and \(\beta = 0\).
3.2 Absorbing games

Throughout this section, \((K, I, J, g, q, k, \theta)\) is an absorbing game, where all states except \(k\) are absorbing. For any initial state \(\ell \neq k\), the stochastic game \((K, I, J, g, q, \ell, \theta)\) is equivalent to the matrix game \(g^\ell \in \mathbb{R}^{I \times J}\) so that \(v_0^\ell = v^\ell = \text{val} g^\ell\). For this reason, we restrict our attention to the game with initial state \(k\).

3.3 The constant payoff in discounted absorbing games

Proposition 3.1 Let \((x_\lambda, y_\lambda)\) be a pair of optimal strategies that admit an expansion in Puiseux series near 0. Then, for all \(t \in [0, 1]\), \(\lim_{\lambda \to 0} \gamma_k^t(x_\lambda, y_\lambda; t) = tv^k\).

Let \(p \in \mathbb{N}\), \(A \in \mathbb{R}^{p \times p}\) and \(M \in \mathbb{R}^{p \times n}\) be as in Theorem 3.1. By Lemma 3.3 it is enough to prove that \(AM g_0 = 0\). First of all, note that any absorbing state \(\ell \neq k\) is payoff-relevant cycle, and that \(A^{\ell, \ell} = 0\) because \(A^{\ell, \ell}\) is the the normalized exit rate from \(\ell\). We distinguish two cases, depending on whether \(k\) is a payoff-relevant cycle or not.

Case 1: \(k\) is not a payoff-relevant cycle. In this case \(p = n - 1\) and \(A = 0\), so that \(AM g_0 = 0\).

Case 2: \(k\) is a payoff-relevant cycle. In this case \(p = n\) and \(M = \text{Id}\) so it is enough to prove \(A g_0 = 0\). If \(A^{k, k} = 0\), then again \(A = 0\) so that \(A g_0 = 0\). So suppose that \(|A^{k, k}| > 0\). This is the interesting case, as it corresponds to a situation where the absorption occurs after a positive, but random fraction of the game. We now prove this case.

Proof. For all \((i, j) \in I \times J\), let \((c(i), e(i), c'(j), e'(j)) \in \mathbb{R}^4_+\) so that \(x_k^j(i) = c(i) \lambda^{e(i)} + o(\lambda^{e(i)})\) and \(y_k^j(j) = c'(j) \lambda^{e'(j)} + o(\lambda^{e'(j)})\), and so that \(c = 0\) implies \(e = 0\). Introduce the sets of actions:

\[
\begin{align*}
I_0 &= \{i \in I \mid e(i) = 0\} & J_0 &= \{j \in J \mid e'(j) = 0\} \\
I_\bullet &= \{i \in I \mid e(i) \in (0, 1)\} & J_\bullet &= \{j \in J \mid e'(j) \in (0, 1)\} \\
I_1 &= \{i \in I \mid e(i) = 1\} & J_1 &= \{j \in J \mid e'(j) = 1\} \\
I_+ &= \{i \in I \mid e(i) > 1\} & J_+ &= \{j \in J \mid e'(j) > 1\}.
\end{align*}
\]

The sets \(I_0, I_\bullet, I_1, I_+\) partition \(I\) while \(J_0, J_\bullet, J_1, J_+\) partition \(J\). Because \(k\) is a payoff-relevant cycle and \(|A^{k, k}| > 0\), the normalized exit rate from \(k\) to any other state \(\ell \neq k\) is given by \(A^{k, \ell}\), which is the sum of the three normalized exit rates corresponding to \(I_0 \times J_1\), the pairs \((i, j) \in I_\bullet \times J\) so that \(e(i) + e'(j) = 1\), and \(I_1 \times J_0\), denoted respectively by \(A^{k, \ell}_{10}, A^{k, \ell}_{\bullet}, A^{k, \ell}_{01}\). Thus, \(A = A_{10} + A_\bullet + A_{01}\), where these are four \(n \times n\) Markov matrices. The situation can be visualized as follows.
The actions in $I_0 \times J_0$ determine the payoff $g_0^k$, while the shaded areas correspond to the pairs of actions which determine the transitions from $k$ to the set of absorbing states. The actions in $I_+ \times J_+$ are irrelevant as they do not affect neither $g_0^k$ nor $A_{k,\ell}$ for all $\ell \neq k$.

**Case 2a:** Suppose by contradiction that $g_0^k > v^k$ and $|A_{01}^{k,k}| > 0$. In this case, Player 2 can deviate from $y_\lambda$ to a strategy $\tilde{y}_\lambda$ which changes the probabilities of playing actions in $J_1$ to $c'(j)\lambda^{1-\varepsilon}$ for a sufficiently small $\varepsilon$ (say, smaller than all nonzero $e(i)$ and $e'(j)$). By doing so, the probability that the state $k$ is left before stage $t/\lambda$ goes to 1 for any $t > 0$. Consequently,

$$
\lim_{\lambda \to 0} \gamma^k_\lambda(x_\lambda, \tilde{y}_\lambda) = \sum_{\ell \neq k} \frac{A_{k,\ell}^k v^\ell}{|A_{k,k}|}.
$$

(3.1)

On the other hand, if Player 1 deviates from $x_\lambda$ to a strategy $\tilde{x}_\lambda$ which plays actions outside $I_0$ to 0, then the transition from $k$ to the set of absorbing states depends only on $A_{01}$, and one has

$$
\lim_{\lambda \to 0} \gamma^k_\lambda(\tilde{x}_\lambda, y_\lambda) = \frac{g_0^k + \sum_{\ell \neq k} A_{01}^{k,\ell} v^\ell}{1 + |A_{01}^{k,k}|}.
$$

(3.2)

Yet, the optimality of $(x_\lambda)$ and $(y_\lambda)$ implies that

$$
\lim_{\lambda \to 0} \gamma^k_\lambda(\tilde{x}_\lambda, y_\lambda) \leq v \leq \lim_{\lambda \to 0} \gamma^k_\lambda(x_\lambda, \tilde{y}_\lambda).
$$

(3.3)

The relations (3.1), (3.2) and (3.3) are not compatible with $g_0^k > v^k$, a contradiction. **Case 2b:** Suppose by contradiction that $g_0^k > v^k$ and $A_{01}^{k,k} = 0$. In this case, for the strategy $(\tilde{x}_\lambda)$ described in the previous case, one has

$$
\lim_{\lambda \to 0} \gamma^k_\lambda(\tilde{x}_\lambda, y_\lambda) = g_0^k.
$$

This contradicts the optimality of $(y_\lambda)$.

We thus conclude that $g_0^k \leq v^k$. Similarly, reversing the roles of the players one obtains $g_0^k \geq v^k$ so that $g_0^k = v^k$. Now one the one hand, $g^\ell = v^\ell$ for all $\ell \neq k$ and on the other $A_{\ell,\ell'}^\ell = 0$ for all $\ell \neq k$ and $\ell'$. Therefore, $Ag = Av = 0$ as soon as $A_{\ell,\ell'}^\ell v^\ell + \sum_{\ell \neq k} A_{\ell,\ell'}^k v^\ell = 0$, which follows from

$$
v^k = \lim_{\lambda \to 0} \gamma^k_\lambda(x_\lambda, y_\lambda) = \frac{g_0^k + \sum_{\ell \neq k} A_{k,\ell}^k v^\ell}{1 + |A_{k,k}|}.
$$

(3.4)
3.4 The constant payoff for general absorbing games

Proposition 3.2 Let \((\sigma^\theta, \tau^\theta)\) be the family of asymptotically 0-optimal strategies defined in \((2.1)\). Then, for all \(t \in [0, 1]\), \(\lim_{\|\theta\| \to 0} \gamma^\theta_t(\sigma^\theta, \tau^\theta; t)\) exists and does not depend on the vanishing evaluations.

We start by two technical lemmas.

Lemma 3.4 Let \(0 \leq t < t + h < 1\). Then, for all \(e \geq 0\),

\[
\lim_{\|\theta\| \to 0} \sum_{m=\varphi(t, \theta)} (\lambda^\theta_m)^e = \begin{cases} +\infty & \text{if } e < 1 \\ \ln \left(1 - \frac{h}{1 - t}\right) & \text{if } e = 1 \\ 0 & \text{if } e > 1. \end{cases}
\]

Proof. For each \(m \geq 1\) set \(t^\theta_m := \sum_{m=1}^{m-1} \theta_m\) so that \(\lambda^\theta_m = \frac{\theta_{m+1}}{1 - t^\theta_m}\). Then, for any \(m\) between \(\varphi(t, \theta)\) and \(\varphi(t + h, \theta)\) one has \(t \leq t^\theta_m \leq t + h\), so that

\[
\left(\frac{\theta_{m+1}}{1 - t}\right)^e \leq \left(\frac{\theta_{m+1}}{1 - t^\theta_m}\right)^e \leq \left(\frac{\theta_{m+1}}{1 - t - h}\right)^e \tag{3.5}
\]

We distinguish three cases, depending on whether \(e = 1\), \(e > 1\) or \(e < 1\).

Case \(e = 1\). It is enough to note that \(\lim_{\|\theta\| \to 0} \sum_{m=\varphi(t, \theta)} \varphi(t + h, \theta) \lambda^\theta_m = h\), add the inequalities of \((3.5)\) for all \(\varphi(\theta, t) \leq m \leq \varphi(\theta, t + h)\) and then take \(\|\theta\|\) to 0, to obtain the desired result, i.e.

\[
\frac{h}{1 - t} \leq \lim_{\|\theta\| \to 0} \sum_{m=\varphi(t, \theta)} \varphi(t + h, \theta) \lambda^\theta_m = \frac{h}{1 - t - h}.
\]

It follows that, as \(h\) tends to 0, \(\lim_{\|\theta\| \to 0} \sum_{m=\varphi(t, \theta)} \varphi(t + h, \theta) \lambda^\theta_m = h\), and hence the result.

Case \(e < 1\). In this case \(\theta_{m+1}^e \geq \|\theta\|^{e-1} \theta_{m+1}\). From \((3.5)\) one derives

\[
\sum_{m=\varphi(t, \theta)} (\lambda^\theta_m)^e \geq \|\theta\|^{e-1} \sum_{m=\varphi(t, \theta)} \varphi(t + h, \theta) \lambda^\theta_m. \tag{3.6}
\]

The result follows, since \(\lim_{\|\theta\| \to 0} \|\theta\|^{e-1} = +\infty\) and \(\lim_{\|\theta\| \to 0} \sum_{m=\varphi(t, \theta)} \varphi(t + h, \theta) \lambda^\theta_m = h\).

Case \(e > 1\). In this case \(\theta_{m+1}^e \leq \|\theta\|^{e-1} \theta_{m+1}\), and the result follows from \((3.5)\) like the previous case, since \(\lim_{\|\theta\| \to 0} \|\theta\|^{e-1} = 0\) in this case.

Lemma 3.5 Let \((a^\theta_m)_{m \geq 1}\) be a sequence in \([0, 1]\) and let \(f(\cdot, \theta) : [0, 1] \to \mathbb{N}\) be a nondecreasing mao so that \(\lim_{\|\theta\| \to 0} \sum_{m=1}^{f(t, \theta)} \theta_m = t\). Suppose that \(\lim_{\|\theta\| \to 0} |a^\theta_{f(t, \theta)} - a(t)| = 0\) for all \(t \in [0, 1]\) and some measurable function \(a : [0, 1] \to [0, 1]\). Then,

\[
\lim_{\|\theta\| \to 0} \sum_{m=1}^{f(t, \theta)} \theta_m a^\theta_m = \int_0^t a(s)ds.
\]
The proof of this result is omitted.

**Proof of Proposition 3.2.** For each \( m \geq 1 \), let \( Q^\theta_m \in \mathbb{R}^{n \times n} \) be the transition matrix induced by \((\sigma^\theta, \tau^\theta)\) at stage \( m \). By the definition of these strategies, there exist coefficients \( c_\ell \geq 0 \) and exponents \( e_\ell \geq 0 \) for all \( 1 \leq \ell \leq n \) so that

\[
(Q^\theta_m)^{k,\ell} = \begin{cases} 
  c_\ell (\lambda_m^\theta)^{e_\ell} + o((\lambda_m^\theta)^{e_\ell}) & \text{if } \ell \neq k \\
  1 - c_k (\lambda_m^\theta)^{e_k} + o((\lambda_m^\theta)^{e_k}) & \text{if } \ell = k.
\end{cases}
\]

Moreover one can assume without loss of generality that \( e_\ell = 0 \) whenever \( c_\ell = 0 \) for all \( \ell \neq k \), so that \( e_k \geq e_\ell \) for all \( \ell \neq k \). Let \( p(t, \theta) \) be the probability of being at \( k \) after \( \varphi(t, \theta) \) stages. Then

\[
p(t, \theta) = \prod_{m=1}^{\varphi(t, \theta)} (Q^\theta_m)^{k,k} = \prod_{m=1}^{\varphi(t, \theta)} \left( 1 - c_k (\lambda_m^\theta)^{e_k} + o((\lambda_m^\theta)^{e_k}) \right).
\]

Taking \( ||\theta|| \) to 0 and setting \( p(t) := \lim_{||\theta|| \to 0} p(t, \theta) \) one has, by Lemma 3.4,

\[
p_t = \lim_{||\theta|| \to 0} \exp \left( -c_k \sum_{m=1}^{\varphi(t, \theta)} (\lambda_m^\theta)^{e_k} \right) = \begin{cases}
  0 & \text{if } e_k < 1 \\
  (1 - t)^c & \text{if } e_k = 1 \\
  1 & \text{if } e_k > 1.
\end{cases}
\]

In other words, \( p_t \) is well-defined and does not depend on the family of vanishing evaluations. Similarly, conditional on reaching an absorbing at stage \( m + 1 \), the probability that \( k_{m+1} = \ell \) is given by

\[
\mathbb{P}_{k_{m+1} = \ell | k_m = k} = \frac{(Q^\theta_m)^{k,\ell}}{\sum_{\ell' \neq k} (Q^\theta_m)^{k,\ell'}} = \frac{c_\ell (\lambda_m^\theta)^{e_\ell} + o((\lambda_m^\theta)^{e_\ell})}{c_k (\lambda_m^\theta)^{e_k} + o((\lambda_m^\theta)^{e_k})}.
\]

Thus, the limits as \( ||\theta|| \) goes to 0 exist for all \( \ell \neq k \), and do not depend on the family of vanishing evaluations. Let \( a_\ell := \prod_{m=0}^{\infty} \mathbb{I}_{\{e_m = e_\ell\}} \) denote this limit. Together, these two results imply that for all \( t \in [0, 1] \) the following limit exists and does not depend on the family of vanishing evaluations:

\[
P_{t^{k,\ell}} := \lim_{||\theta|| \to 0} \mathbb{P}_{k_{\varphi(\theta, t)} = \ell | k_m = k} = p_t \mathbb{I}_{\{e_k = e_\ell\}} + (1 - p_t) a_\ell \mathbb{I}_{\{\ell \neq k\}} \quad \forall 1 \leq \ell \leq n.
\]

Hence, the same is true for the cumulated times \( \Pi_{t^{k,\ell}} := \int_0^t P_{s^{k,\ell}} ds \). Finally, \( \lim_{||\theta|| \to 0} g(\sigma^\theta, \tau^\theta) = g_0 \) exists and does not depend on the family of vanishing evaluations either. The result follows then from Lemma 3.3, since it implies in particular that \( \Pi_t = \lim_{||\theta|| \to 0} \sum_{m=1}^{\varphi(t, \theta)} \theta_m \prod_{m'=1}^{m} Q_{m'}^\theta, \) so that

\[
\lim_{\lambda \to 0} \gamma_\theta(x, y; t) = \Pi_t g_0.
\]

3.5 Critical stochastic games

In this section we extend the constant-payoff property to any critical stochastic game, a class that includes stochastic games satisfying \((H1)\) and \((H2)\).
A critical stochastic game is one with the following property. For each \( \varepsilon > 0 \) there exists a family of strategies \((x_\lambda, y_\lambda)\) in the discounted stochastic game which satisfies

- \((x_\lambda)\) is asymptotically \(\varepsilon\)-optimal for Player 1 and \((y_\lambda)\) is asymptotically \(\varepsilon\)-optimal strategy for Player 2, that is for any \((\sigma, \tau) \in \Sigma \times T\),
  \[
  \liminf_{\lambda \to 0} \gamma_\lambda^K(x_\lambda, \tau) \geq v^K - \varepsilon \quad \text{and} \quad \limsup_{\lambda \to 0} \gamma_\lambda^K(\sigma, y_\lambda) \leq v^K + \varepsilon .
  \]

- The stochastic matrix \((Q^K_\lambda)\) on the state space induced by \((x_\lambda, y_\lambda)\) is critical, that is, there exists a transition matrix \(M \in \mathbb{R}^{n \times n}\) (of a continuous-time Markov chain, so that for all \(1 \leq \ell \leq n\), \(M^{\ell, \ell'} \geq 0\) for all \(\ell' \neq \ell\), and \(M^{\ell, \ell'} + \sum_{\ell' \neq \ell} M^{\ell, \ell'} = 0\)) so that
  \[
  Q^K_\lambda = \text{Id} + M\lambda + o(\lambda) .
  \]

**Proposition 3.3** Every stochastic games satisfying the assumptions \((H1)\) and \((H2)\) is critical.

**Proof.** Let \((x_\lambda)\) be a family of optimal stationary strategies so that \(\lambda \mapsto x_\lambda^K(i)\) admit a Puiseux expansion near 0 for all \((k, i) \in K \times I\) and let \(c(k, i)\) and \(e(k, i)\) so that \(x_\lambda^K(i) = c(k, i)\lambda^{e(k, i)} + o(\lambda^{e(k, i)})\). For any \(T \in \mathbb{N}\) and \(\lambda\) consider the strategy \(x_\lambda^T\) of Player 1 defined by \((x_\lambda^T)^k(i) := 0\) if \(e(k, i) > 0\) and otherwise

\[
(x_\lambda^T)^k(i) := \frac{c(k, i)T^{1-e(k, i)}\lambda}{\sum_{i' \in I} c(k, i')T^{1-e(k, i')}\lambda^{e(k, i') \leq 1}} .
\]

One defines a strategy \(y_\lambda^T\) of Player 2 in a symmetric way. The family of Markov chains \((Q^K_\lambda)\) is critical by construction, so it is enough to prove that the strategies \((x_\lambda^T)\) and \((y_\lambda^T)\) are asymptotically \(\varepsilon\)-optimal for \(T\) large enough or, equivalently, that for any \(j \in J^n\),

\[
\lim_{T \to +\infty} \lim_{\lambda \to 0} \gamma_\lambda(x_\lambda^T, j) = \lim_{\lambda \to 0} \gamma_\lambda(x_\lambda, j) = v \in \mathbb{R}^n .
\]

This result follows from the fact that \((H1)\) implies that the \(\lim_{\lambda \to 0} \gamma_\lambda(x_\lambda, j)\) depends only on the exist terms of highest order for each state, so that the loss of \((x_\lambda^T)\) is bounded by the difference between the exit distributions, which tends to 0. Similarly, \((H2)\) implies an analogue property for \((y_\lambda^T)\).

In the sequel, let \((x_\lambda, y_\lambda)\) a fixed family of optimal stationary strategies so that \(\lambda \mapsto x_\lambda^K(i)\) and \(\lambda \mapsto y_\lambda^K\) admit a Puiseux expansion near 0 for all \((i, j) \in I \times J\), and let \((Q_\lambda)\) be the corresponding family of stochastic matrices. We assume that \((Q_\lambda)\) is critical. Let \((\sigma^\theta, \tau^\theta)\) be the pair of strategies indexed by \(\theta\) defined in Section 2.5 and for each \(m \geq 1\) let \(Q_\lambda^m \in \mathbb{R}^{n \times n}\) be the transition matrix induced by \((\sigma^\theta, \tau^\theta)\) at stage \(m\). By the choice of \((x_\lambda, y_\lambda)\) for each \(1 \leq \ell, \ell' \leq n\) and \(m \geq 1\), there exist \(c(\ell, \ell') \geq 0\) and \(e(\ell, \ell') \geq 0\) so that

\[
Q^K_\lambda(\ell, \ell') = c(\ell, \ell')(\lambda^K_\lambda)^{e(\ell, \ell')} .
\]
A family of continuous-time processes indexed by $\theta$. First of all, the family $(Q_\lambda)$ being critical, every state is a payoff-relevant cycle. Fix an initial state $1 \leq k \leq n$. Let $(Y^{k,\theta}_m)_{m \geq 1}$ be the random process of states $(k_m)$ under the law $P^k_{\sigma_\theta, \tau_\theta}$, which is an inhomogeneous Markov chain with transition matrices $(Q^\theta_m)$. For any $t, h \geq 0$ so that $0 \leq t \leq t + h \leq 1$, let $J^\theta_{[t, t+h]}$ be the number of jumps (i.e. changes of state) of the process $(Y^{k,\theta}_m)_{m \geq 1}$ in the interval $[\varphi(t, \theta), \varphi(t + h, \theta)]$. Finally, let $(X^{k,\theta}_t)$ be the time-changed process defined on $[0, 1]$ by

$$X^{k,\theta}_t := Y^{k,\theta}_{\varphi(t, \theta)} \quad \forall t \in [0, 1].$$

**Notation.** In the sequel, we will use the following notation.

- For any $t, h \geq 0$ so that $0 \leq t \leq t + h \leq 1$ define

$$P^\theta_{t, t+h} := \prod_{m = \varphi(t, \theta)} Q^\theta_m.$$

- For all $t \in [0, 1]$ and $1 \leq \ell \leq n$, let $P^\ell_t$ denote the conditional probability on $\{X^{k,\theta}_t = \ell\}$, so that for all $1 \leq \ell, \ell' \leq n$ and $0 \leq t \leq t + h \leq 1$,

$$P^\ell_t(X^{k,\theta}_{t+h} = \ell') := P(X^{k,\theta}_{t+h} = \ell' \mid X^{k,\theta}_t = \ell) = (P^\theta_{t, t+h})^\ell, \ell'.$$

**Proposition 3.4** Let $(Q_\lambda)_{\lambda}$ be so that $Q_\lambda = \text{Id} + A\lambda + o(\lambda)$ for a transition matrix $A \in \mathbb{R}^{n \times n}$. Then, for all $t \in [0, 1)$,

(i) $\lim_{\|\theta\| \to 0} P^\ell_t(X^{k,\theta}_{t+h} = \ell) = 1 + \frac{A^\ell}{1-t} h + o(h)$.

(ii) $\lim_{\|\theta\| \to 0} P^\ell_t(X^{k,\theta}_{t+h} = \ell') = \frac{A^\ell, \ell'}{1-t} h + o(h)$.

**Proof.** (i) Conditional to $\{X^{k,\theta}_t = \ell\}$, the event $\{X^{k,\theta}_{t+h} = \ell\}$ is the disjoint union of $\{J^\theta_{[t, t+h]} = 0\}$ and $\{X^{k,\theta}_{t+h} = \ell\} \cap \{J^\theta_{[t, t+h]} \geq 2\}$. For the former, one has

$$\lim_{\|\theta\| \to 0} P^\ell_t(J^\theta_{[t, t+h]} = 0) = 1 + \frac{A^\ell}{1-t} h + o(h).$$

(3.7)
Indeed, 
\[
\lim_{\|\theta\| \to 0} \mathbb{P}_t(J_{t,t+h}^\theta = 0) = \lim_{\|\theta\| \to 0} \prod_{m=\varphi(t,\theta)}^{\varphi(t+h,\theta)} \mathbb{P}_t(X_{m+1}^{k,\theta} = X_{m}^{k,\theta}),
\]
\[
= \lim_{\|\theta\| \to 0} \prod_{m=\varphi(t,\theta)}^{\varphi(t+h,\theta)} \left(1 - \sum_{s' \neq \ell} (Q_m^{\theta})_{\ell,s'}\right),
\]
\[
= \lim_{\|\theta\| \to 0} \prod_{m=\varphi(t,\theta)}^{\varphi(t+h,\theta)} \left(1 - \lambda_m^\theta |A_{\ell,\ell'}| + o(\lambda_m^\theta)\right),
\]
\[
= \lim_{\|\theta\| \to 0} \exp\left(-|A_{\ell,\ell'}| \sum_{m=\varphi(t,\theta)}^{\varphi(t+h,\theta)} \lambda_m^\theta\right),
\]
and the result follows from Lemma 3.4. For the latter, namely \(\{X_{t+h}^{k,\theta} = \ell\} \cap \{J_{[t,t+h]}^\theta \geq 2\}\), one has
\[
\mathbb{P}_t(J_{[t,t+h]}^\theta \geq 2) \leq \max_{1 \leq \ell' \leq m} \mathbb{P}_t^{\ell'}(J_{[0,h]}^\theta \geq 1)^2 = \max_{1 \leq \ell' \leq m} \left(1 - \mathbb{P}_t^{\ell'}(J_{[t,t+h]}^\theta = 0)\right)^2. \tag{3.8}
\]
Therefore, \(\lim_{\|\theta\| \to 0} \mathbb{P}_t(J_{[t,t+h]}^\theta \geq 2) = o(h)\), which together with (3.7) proves the desired result.

(ii) Similarly, conditional on \(\{X_{t+h}^{k,\theta} = \ell\}\),
\[
\{X_{t+h}^{k,\theta} = \ell'\} = \{X_{t+h}^s = \ell'\} \cap \left(\{J_{[t,t+h]}^\theta = 1\} \cup \{J_{[t,t+h]}^\theta \geq 2\}\right).
\]
Together with (3.8) this equality yields
\[
\lim_{\|\theta\| \to 0} \mathbb{P}_t(X_{t+h}^{k,\theta} = \ell') = \lim_{\|\theta\| \to 0} \mathbb{P}_t(J_{[t,t+h]}^\theta = 1, X_{t+h}^{k,\theta} = \ell') + o(h).
\]
Conditional on leaving the state \(\ell\) at stage \(m\), the probability of going to \(\ell' \neq \ell\) is given by
\[
\mathbb{P}(X_{m+1}^{k,\theta} = \ell' \mid X_m^{k,\theta} = \ell, X_{m+1}^{k,\theta} \neq \ell) = \frac{(Q_m^{\theta})_{\ell,\ell'}}{\sum_{\ell'' \neq \ell} (Q_m^{\theta})_{\ell,\ell''}}.
\]
By assumption, this converges to \(\frac{A_{\ell,\ell'}}{|A_{\ell,\ell'}|}\) as \(\|\theta\|\) goes to 0. On the other hand, (3.8) implies
\[
\lim_{\|\theta\| \to 0} \mathbb{P}_t(J_{[t,t+h]}^\theta = 1) = \lim_{\|\theta\| \to 0} 1 - \mathbb{P}_t(J_{[t,t+h]}^\theta = 0) + o(h).
\]
Consequently, one has
\[
\lim_{\|\theta\| \to 0} \mathbb{P}_t(J_{[t,t+h]}^\theta = 1, X_{t+h}^{k,\theta} = \ell') = \lim_{\|\theta\| \to 0} \frac{(Q_m^{\theta})_{\ell,\ell'}}{\sum_{\ell'' \neq \ell} (Q_m^{\theta})_{\ell,\ell''}} \left(1 - \mathbb{P}_t(J_{[t,t+h]}^\theta = 0) + o(h)\right)
\]
\[
= \frac{A_{\ell,\ell'}}{|A_{\ell,\ell'}|} \left(\frac{|A_{\ell,\ell'}|}{1 - t} h + o(h)\right)
\]
\[
= \frac{A_{\ell,\ell'}}{1 - t} h + o(h),
\]
where we used (3.7) to deduce the second equality. \(\blacksquare\)
Corollary 3.6  The processes \((X^{k,\theta}_t)_{t \in [0,1]}\) converge, as \(\theta\) tends to 0, to a inhomogeneous Markov process with generators \(\left(\frac{1}{1-\theta^2} A\right)_{t \in [0,1]}\).

**Proof.** The limit is identified by Proposition 3.4. The tightness is a consequence of (ii). Indeed, it implies that for any \(T > 0\), uniformly in \(\theta\):
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( \exists t_1, t_2 \in [0, T] \mid t_1 < t_2 < t_1 + \varepsilon, \ X^{k,\theta}_{t_1} \neq X^{k,\theta}_{t_1}, \ X^{k,\theta}_{t_2} \neq X^{k,\theta}_{t_2} \right) = 0,
\]
which is precisely the tightness criterion for càdlàg process with discrete values.

Corollary 3.7  For all \(t \in [0,1]\) the following limit exist:
\[
\Pi_t := \lim_{\|\theta\| \to 0} \sum_{m=1}^{\varphi(\theta,t)} \theta_m \prod_{m'=1}^{m} Q_{m'}^\theta = \int_0^t e^{-\ln(1-s)A} ds.
\]
Corollary 3.7 follows from Corollary 3.6 together with Lemma 3.5.

**References**

[1] L. Attia and M. Oliu-Barton, *A formula for the value of a stochastic game*, Proceedings of the National Academy of Sciences 116 (2019), no. 52, 26435–26443.

[2] T. Bewley and E. Kohlberg, *The asymptotic theory of stochastic games*, Mathematics of Operations Research 1 (1976), 197–208.

[3] B. Jaffuel and M. Oliu-Barton, *Occupation times in stochastic games*, Preprint (2013).

[4] E. Kohlberg, *Repeated games with absorbing states*, Annals of Statistics 2 (1974), 724–738.

[5] J.-F. Mertens and A. Neyman, *Stochastic games*, International Journal of Game Theory 10 (1981), 53–66.

[6] ______, *Stochastic games*, Proceedings of the National Academy of Sciences of the United States of America 79 (1982), 2145–2146.

[7] A. Neyman and S. Sorin, *Repeated games with public uncertain duration process*, International Journal of Game Theory 39 (2010), 29–52.

[8] M. Oliu-Barton and B. Ziliotto, *Constant payoff in zero-sum stochastic games*, ArXiv:1811.04518, 2018.

[9] L.S. Shapley, *Stochastic games*, Proceedings of the National Academy of Sciences of the United States of America 39 (1953), 1095–1100.
[10] S. Sorin, *A First Course on Zero-Sum Repeated Games*, vol. 37, Springer Science & Business Media, 2002.

[11] S. Sorin, X. Venel, and G. Vigeral, *Asymptotic properties of optimal trajectories in dynamic programming*, Sankhya A 72 (2010), 237–245.

[12] S. Sorin and G. Vigeral, *Limit optimal trajectories in zero-sum stochastic games*, hal-01959326v2f, 2019.

[13] B. Ziliotto, *A Tauberian theorem for nonexpansive operators and applications to zero-sum stochastic games*, Mathematics of Operations Research 41 (2016), 1522–1534.