Photon Damping Caused by Electron-Positron Pair Production in a Strong Magnetic Field

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Abstract

Damping of an electromagnetic wave in a strong magnetic field is analyzed in the kinematic region near the threshold of electron-positron pair production. Damping of the electromagnetic field is shown to be noticeably nonexponential in this region. The resulting width of the photon $\gamma \to e^+ e^-$ decay is considerably smaller than previously known results.

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The problem of propagation of electromagnetic fields through an active medium is inherent in a variety of physical phenomena. The birth and evolution of supernova and neutron stars, where the matter density can be on the order of nuclear density $\rho \approx 10^{14} - 10^{15} \text{g/cm}^3$ and the temperature can achieve several tens of MeVs, are the largest scale and the most interesting such phenomena. In addition to dense and hot matter, a strong magnetic field, which can be several orders of magnitude as high as so-called critical, or Schwinger, value $B_c = m_e^2/e \approx 4.41 \cdot 10^{13} \text{G}$ can be generated in the above-mentioned objects [1, 2]. This strong magnetic field can induce new phenomena which can considerably affect the evolution of these astrophysical objects. Electromagnetic-field damping caused by electron-positron pair production in an external magnetic field is one of these phenomena. Recall that the $\gamma \rightarrow e^+e^-$ process is kinematically forbidden in vacuum. The magnetic field changes the kinematics of charged particles, electrons and positrons, allowing the production of an electron-positron pair in the kinematic region $q_0^2 = q_0^2 - q_3^2 \geq 4m_e^2$, where $q_0$ is the photon energy and $q_3$ is the momentum component along the magnetic field.

In 1954, Klepikov [3] examined the production of an electron-positron pair by a photon in a magnetic field and obtained the amplitude and width of the $\gamma \rightarrow e^+e^-$ decay in the semiclassical approximation. Later, the authors of [4]–[9] considered this process in the context of its astrophysical applications. It was pointed out in [7, 8] that the use of the expression derived in [3] for the width considerably overestimates the result in the strong magnetic field limit. In this case, one should use an exact expression for the width of one-photon production of a pair when electrons and positrons occupy only the ground Landau level. However, it was found that the expression for the decay width in the limit of strong magnetic field has a root singularity at the point $q_3^2 = 4m_e^2$. Shabad [9] emphasized that this behavior indicates that the decay width calculated in the perturbation theory cannot be treated as a damping coefficient. In this case, the damping coefficient is primarily determined from the time evolution of the photon wave function in the presence of a magnetic field. Shabad [9] suggested that this dependence be obtained by solving the dispersion equation with account taken of the vacuum polarization in a magnetic field with complex values of photon energy. In our opinion, this method has several disadvantages. First, it is well known but rarely mentioned that the dispersion equations with complex energies have no solutions in the physical sheet. Solutions are in the nonphysical Riemannian sheets (analyticity region of

1We use the system of units where $\hbar = c = 1$.

2Hereafter, we consider the magnetic field directed along the third axis.
the polarization operator), which are generally infinite in number. As a result, the dispersion equation has the infinite number of solutions with both positive and negative imaginary parts of energy. The physical status of these solutions requires a separate investigation.

Shabad [9] used the asymptotic form of the polarization operator near the pair production threshold and erroneously treated it as a two-sheet complex function. This circumstance led to the existence of two complex conjugate solutions, one of which is physically meaningless because it has a positive imaginary part and, therefore, provides exponentially increasing amplitude of electromagnetic wave. Therefore, to obtain physically meaningful result, one should artificially discard the redundant solutions.

Second, this approach cannot correctly describe the substantially nonexponential damping in the near-threshold region in a strong field.

Thus, damped electromagnetic waves in a magnetic field cannot be completely described by solving the dispersion equation.

In this work, we use a method that is applied in the field theory at finite temperatures and in plasma physics (see, e.g., [10]). It consists of the determination of a retarded solution to the electromagnetic field equation that includes an external source and takes into account the vacuum polarization in a magnetic field. Time damping of the electromagnetic wave is analyzed in a uniform external magnetic field, whose intensity is the largest parameter of the problem, $B_e \gg q^2, m_e^2$. To describe the time evolution of electromagnetic wave $A_\alpha(x)$ [$x_\mu = (t, \mathbf{x})$], we consider a linear response of the system ($A_\alpha(x)$ and a vacuum polarized in magnetic field) to an external source, which is adiabatically turned on at $t = -\infty$ and turned off at $t = 0$. At $t > 0$, the electromagnetic wave evolves independently. Thus, the source is necessary for creating an initial state. For simplicity, we consider the evolution of a monochromatic wave. In this case, the source function should be taken in the form

$$J_\alpha(x) = j_\alpha e^{i\mathbf{kx}} e^{\epsilon t} \theta(-t), \quad \epsilon \rightarrow 0^+.$$  

(1)

The time dependence of $A_\alpha(x)$ is determined by the equation

$$(g_{\alpha\beta} \partial_\mu - \partial_\alpha \partial_\beta) A_\beta(x) + \int d^4x' P_{\alpha\beta}(x - x') A_\beta(x') = J_\alpha(x),$$  

(2)

where $P_{\alpha\beta}(x - x')$ is the photon polarization operator in a magnetic field. We note that, for the source on the right-hand side of Eq. (2) to be conserved, $\partial_\alpha J_\alpha = 0$, the current $j_\alpha$ must have the form $j_\alpha = (0, \mathbf{j})$, $\mathbf{j} \cdot \mathbf{k} = 0$. The evolution of $A_\alpha(x)$ is
described by the retarded solution
\[ A_R^\alpha(x) = \int d^4x' G_{\alpha\beta}^R(x-x') J_\beta(x'), \] (3)
where \( G_{\alpha\beta}^R(x-x') \) is the retarded Green’s function, which is defined through the commutator of the Heisenberg operators of electromagnetic field as (see, e.g., \[ ]) \( G_{\alpha\beta}^R(x-x') = -i\langle 0| [\hat{A}_\alpha(x), \hat{A}_\beta(x')] |0 \rangle \theta(t-t') \), (4)

It is instructive to express this function in terms of the causal Green’s function \( G_{\alpha\beta}^C(x-x') = -i\langle 0| T \hat{A}_\alpha(x) \hat{A}_\beta(x') |0 \rangle \), (5)

by using the relationship \( G_{\alpha\beta}^R(x-x') = 2 \text{Re} G_{\alpha\beta}^C(x-x') \theta(t-t') \). (6)

In the presence of a magnetic field, it is convenient to decompose Green’s function (5) in the eigenvectors of polarization operator \[ ]:
\[ G_{\alpha\beta}^C(x-x') = \int \frac{d^4q}{(2\pi)^4} G_{\alpha\beta}^C(q) e^{-iqx} \] (7)
\[ G_{\alpha\beta}^C(q) = \sum_{\lambda=1}^3 \frac{b_\alpha^{(\lambda)} b_\beta^{(\lambda)}}{(b^{(\lambda)})^2} \frac{1}{q^2 - \mathcal{P}^{(\lambda)}(q)}, \] (8)
where \( \mathcal{P}^{(\lambda)}(q) \) are the eigenvalues of polarization operator. The eigenvectors
\[ b_\alpha^{(1)} = (q\varphi)_\alpha, \]
\[ b_\alpha^{(2)} = (q\tilde{\varphi})_\alpha, \]
\[ b_\alpha^{(3)} = q^2(q\varphi\varphi)_\alpha - (q\varphi\varphi q)q_\alpha, \] (9)
(10)
together with the 4-vector \( q_\alpha \) form a complete orthogonal basis in the Minkowski 4-space. In Eqs. (10), \( \varphi_\alpha = F_\alpha / B, \tilde{\varphi}_\alpha = \frac{1}{2}\varepsilon_\alpha\beta\mu\nu\varphi_{\mu\nu} \) are dimensionless magnetic-field tensor and dual tensor, respectively, \( (q\varphi)_\alpha = q_\alpha \varphi_\sigma, (q\tilde{\varphi}q) = q_\alpha \varphi_\beta \varphi_\beta q_\sigma \).

Substituting Eqs. (1) and (3) into Eq. (3) and using Eqs. (7) and (8), we obtain after simple integration the following result at \( t > 0 \):
\[ A_R^\alpha(x) = \sum_{\lambda=1}^3 V_\alpha^{(\lambda)}(x) = 2 e^{ikx} \text{Re} \sum_{\lambda=1}^3 \int \frac{dq_0}{2\pi i} \frac{b_\alpha^{(\lambda)} (b^{(\lambda)} j) / (b^{(\lambda)})^2}{(q_0 - i\varepsilon) (q_0^2 - k^2 - \mathcal{P}^{(\lambda)}(q))}, \] (11)
where $q_\alpha = (q_0, k)$. Note that the definition of the integral in Eq. (11) should be completed because the integrand can include singularities, which are due, on the one hand, to zeros of its denominator and, on the other, to the domain of its definition. To analyze these singularities, it is necessary to know the explicit form and analytical properties of the eigenvalues $P^{(\lambda)}(q)$ of the polarization operator, which was examined in detail in a number of works. In the limit of strong magnetic field, the functions $P^{(\lambda)}(q)$, which we are interested in, can be borrowed, e.g., from [9, 12, 13] and represented as [with the $O(1/B)$ accuracy]

$$P^{(1)}(q) \simeq -\frac{\alpha}{3\pi} q_2^2 - q^2 \Lambda(B, q^2),$$

$$P^{(3)}(q) \simeq -q^2 \Lambda(B, q^2),$$

$$P^{(2)}(q) \simeq -\frac{2\alpha eB}{\pi} \left( \frac{1}{\sqrt{z(1-z)}} \frac{\arctan \sqrt{\frac{z}{1-z}}}{1-z} \right) - q^2 \Lambda(B, q^2),$$

where

$$\Lambda(B, q^2) = \frac{\alpha}{3\pi} \left[ 1.792 - \ln(B/B_e) \right] + \pi(q^2),$$

$$z = q_\parallel^2 / 4m_e^2, \quad q_\perp^2 = q_1^2 + q_2^2,$$

and $q^2 \pi(q^2)$ is the photon polarization operator in the absence of a magnetic field. Note that the contribution from the pole $q_\parallel^2 = 0$ that results from the normalization of the basis vectors $b^{(2)}_\alpha$ and $b^{(3)}_\alpha$ is nonphysical and, taking into account explicit form (12)–(14) of the polarization operator, can be removed by gauge transformation after summation over polarizations. Thus, the contribution to the solution can be made only by the poles corresponding to the dispersion equation

$$q^2 - P^{(\lambda)}(q) = 0.$$  

Using solution (11), one can demonstrate on the basis of Eqs. (12)–(14) that only two modes, $\lambda = 1$ and $\lambda = 2$ with the polarization vectors

$$\xi^{(1)}_\alpha = \frac{b_\alpha^{(1)}}{\sqrt{(b^{(1)})^2}} = \frac{(q\varphi)_\alpha}{\sqrt{q^2}}, \quad \xi^{(2)}_\alpha = \frac{b_\alpha^{(2)}}{\sqrt{(b^{(2)})^2}} = \frac{(q\bar{\varphi})_\alpha}{\sqrt{q^2}},$$

are physically meaningful for real photons.

Modes with the polarizations $\xi^{(1)}_\alpha$ and $\xi^{(2)}_\alpha$ correspond to so-called parallel ($\parallel$) and perpendicular ($\perp$) modes, respectively, in the Adler notation.  

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3 Modes with the polarizations $\xi^{(1)}_\alpha$ and $\xi^{(2)}_\alpha$ correspond to so-called parallel ($\parallel$) and perpendicular ($\perp$) modes, respectively, in the Adler notation.
A photon of the third, $\lambda = 3$, mode is nonphysical. Indeed, substitution of the expression for $P^{(3)}(q)$ into Eq. (13) gives the equation that has the only solution $q^2 = 0$. Therefore, the contribution of the third mode to solution (11) is proportional to the total divergence and can be eliminated by the corresponding gauge transformation.

In the limit of strong magnetic field, only the mode with the polarization vector $\varepsilon^{(2)}_\mu$ can decay into an electron-positron pair, because only the eigenvalue of the polarization operator $P^{(2)}(q)$ (14) has the imaginary part at $q^\parallel \geq 4m_e^2$. Therefore, to analyze time damping of the electromagnetic field, it is sufficient to consider only the term with $V^{(2)}_\alpha(x)$ in Eq. (11).

The further calculations can be simplified by going over to the reference frame, where $k = (k_1, k_2, 0)$, which can always be done without disturbing the properties of the external magnetic field. In this frame, $q^2 = q_0^2$ and the polarization vector of the second mode takes the form $\varepsilon^{(2)}_\alpha = (0, 0, 0, -1)$. As a result, $V^{(2)}_\alpha(x)$ is expressed as

$$V^{(2)}_\alpha(x) = V^{(2)}_\alpha(0, x) \frac{\text{Re}F(t)}{\text{Re}F(0)},$$

(17)

where

$$F(t) = \int_C \frac{dq_0}{2\pi i} \frac{e^{-i q_0 t}}{(q_0 - i\varepsilon)(q_0^2 - k^2 - P^{(2)}(q))},$$

(18)

$$V^{(2)}_\alpha(0, x) = 2 \varepsilon^{(2)}_\alpha j_3 e^{i kx} \text{Re}F(0).$$

The path of integration $C$ in Eq. (18) is determined by the analytical properties of $P^{(2)}(q)$ and is shown in Fig. 1. The function $P^{(2)}(q)$ is analytical in the complex plane $q_0$ with cuts along the real axis (see Fig. 1). In the kinematic region $|q_0| < 2m_e$, the eigenvalue $P^{(2)}(q)$ is real and Eq. (15) has real solutions which govern the photon dispersion in this region.

For further analysis, it is convenient to transform the path of integration to the path shown in Fig. 2. In this case, the integral in Eq. (18) is represented as

$$F(t) = F_{\text{pole}}(t) + F_{\text{cut}}(t),$$

(19)

where the first term is determined by the residue at the point $q_0 = \omega$, which is the solution to dispersion Eq. (14). This term corresponds to the undamped solution in the region $\omega < 2m_e$. The second term determines the time dependence of the
electromagnetic field above the threshold of electron-positron pair production and has the form of the Fourier integral

\[ F_{\text{cut}}(t) = \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} F_{\text{cut}}(q_0) e^{-iq_0t}, \]

\[ F_{\text{cut}}(q_0) = \frac{2\theta(q_0 - 2m_e) I}{q_0 (|q_0^2 - k^2 - R|^2 + I^2)}, \]

where

\[ R \equiv \text{Re} \mathcal{P}^{(2)}(q_0) = \frac{\alpha}{\pi} eB \left( \frac{1}{\sqrt{z(z-1)}} \ln \frac{\sqrt{z} + \sqrt{z-1}}{\sqrt{z} - \sqrt{z-1}} + 2 \right), \]

\[ I \equiv -\text{Im} \mathcal{P}^{(2)}(q_0 + i\varepsilon) = \frac{\alpha eB}{\sqrt{z(z-1)}}, \quad z = \frac{q_0^2}{4m_e^2}. \]

Expressions (20)–(23) together with Eq. (17) determine the time evolution of the photon wave function above the pair production threshold in a strong magnetic field.

Strictly speaking, because of the threshold behavior of the Fourier transform \( F_{\text{cut}}(q_0) \), time damping of the function \( F_{\text{cut}}(t) \) and, therefore, of the wave function \( A_\alpha(t) \) is nonexponential. However, in some characteristic time interval (the inverse effective width of the \( \gamma \rightarrow e^+e^- \) decay can naturally be chosen as such an interval), the time dependence of the wave function can approximately be represented as exponentially damping harmonic oscillations:

\[ A_\mu(t) \sim e^{-\gamma_{\text{eff}} t/2} \cos(\omega_{\text{eff}} t + \phi_0). \]

Here, \( \omega_{\text{eff}} \) and \( \gamma_{\text{eff}} \) are, respectively, the effective frequency and width of photon decay, which should be found by using Eqs. (20)–(23) for each value of momentum \( k \) to determine the effective photon dispersion law above the threshold of electron-positron pair production. The quantity \( \gamma_{\text{eff}} \), which governs the intensity of photon absorption due to \( e^+e^- \) pair production in a magnetic field, is important for astrophysical applications. The absorption coefficient obtained from the \( \gamma \rightarrow e^+e^- \) decay probability and containing a root singularity is usually employed in astrophysics (see, e.g., [15]). Shabad [9] pointed out that this leads to the overestimation of the intensity of \( e^+e^- \)-pair production. Our analysis demonstrates that the calculation of the absorption coefficient (decay width) by using the complex solution in the second Riemannian sheet [9] also leads to a considerable overestimation in the near-threshold region, as is seen from Figs. 3 and 4.
Nonexponential damping in the near-threshold region is known for the processes in vacuum and matter [16, 17]. However, as far as we know, it has not been considered in an external field so far. In contrast to vacuum or medium, the near-threshold effect in the magnetic field is kinematically enhanced due to the singular behavior of the polarization operator in this field. Therefore, this phenomenon is not only topical for astrophysical application but is of conceptual interest.

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Figure 1: The path of integration $C$ in the complex $q_0$ plane. The crosses are the poles corresponding to the real solutions of dispersion Eq. (15). The shaded parts of the real axis are cuts.
Figure 2: The path of integration $C$ after the transformation allowing one to separate the pole $F_{pole}(t)$ and cut $F_{cut}(t)$ contributions.
Figure 3: The frequency dependence of the $\gamma \to e^+ e^-$ decay width in the near-threshold region for the magnetic field $B = 200B_e$. Line 1 is the tree approximation including the root singularity; line 2 is obtained from the complex solution of the dispersion equation in the second Riemannian sheet [9]; and line 2 is $\gamma_{\text{eff}}$ from approximation (24).
Figure 4: The decay width vs. the magnetic field for the frequency $\omega = 2.5m_e$. The meaning of lines is the same as in Fig. 3.