ALGEBRAIC THREEEFOLDS OF GENERAL TYPE WITH SMALL VOLUME

YONG HU AND TONG ZHANG

Abstract. It is known that the optimal Noether inequality \( \text{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{20}{3} \) holds for every 3-fold \( X \) of general type with \( p_g(X) \geq 11 \). In this paper, we give a complete classification of 3-folds \( X \) of general type with \( p_g(X) \geq 11 \) satisfying the above equality by giving the explicit structure of a relative canonical model of \( X \). This model coincides with the canonical model of \( X \) when \( p_g(X) \geq 23 \). We also establish the second and third optimal Noether inequalities for 3-folds \( X \) of general type with \( p_g(X) \geq 11 \). These results answer two open questions raised by J. Chen, M. Chen and C. Jiang, and in dimension three an open question raised by J. Chen and C. Lai. A novel phenomenon shows that there is a one-to-one correspondence between the three Noether inequalities and three possible residues of \( p_g(X) \) modulo 3.

1. Introduction

Throughout this paper, we work over the complex number field \( \mathbb{C} \), and all varieties are projective.

1.1. Motivation. The classical Noether inequality, proved by M. Noether [25], asserts that

\[
K_S^2 \geq 2p_g(S) - 4
\]

for every minimal surface \( S \) of general type. This inequality is one of the foundational results in the theory of surfaces. More importantly, it leads to the so-called geography of surfaces of general type which aims to classify surfaces of general type based on the relations of their birational invariants (see [27, §1] for more details). As a typical example, minimal surfaces of general type attaining the equality in (1.1) are said to be on the Noether line, and the explicit classification of such surfaces has been completed by E. Horikawa [14].

It is known that the Noether inequality exists in any dimension: there exist \( a_n, b_n > 0 \) such that \( K_X^n \geq a_n p_g(X) - b_n \) for every minimal \( n \)-fold \( X \) of general type [11]. However, to obtain the optimal \( a_n \) and \( b_n \) for \( n \geq 3 \) is quite challenging (see [20, 2, 5] for \( n = 3 \), as well as [8, 30] for general \( n \)). Recently, J. Chen, M. Chen and C. Jiang [7, 6] proved the following optimal...
Noether inequality that
\begin{equation}
K_X^3 \geq \frac{4}{3} p_g(X) - \frac{10}{3}
\end{equation}
for every minimal 3-fold \( X \) of general type with \( p_g(X) \geq 11 \). This optimal result naturally leads to more challenging problems, just as the classical Noether inequality does for surfaces. In this paper, we are interested in the following two open questions raised by J. Chen, M. Chen and C. Jiang.

**Question A.** [7, Question 1.5] Is there a classification for minimal 3-folds \( X \) of general type satisfying \( K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3} \)? Is there any non-Gorenstein minimal 3-fold \( X \) of general type satisfying \( K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3} \)?

The answer to the first part of Question A can be viewed as an analogue of E. Horikawa’s work [14] in dimension three. Regarding the second part, M. Kobayashi [20] first constructed a series of examples of minimal 3-folds \( X \) of general type satisfying \( K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3} \). Later, Y. Chen and the first named author [12] generalized Kobayashi’s method to obtain more examples. Recently, M. Chen, C. Jiang and B. Li [10] found two new examples via a different method. Note that all these examples are Gorenstein.

**Question B.** [7, Question 1.6] Is there a “second Noether inequality” in dimension three? Namely, is there a real number \( b < \frac{10}{3} \) such that if \( K_X^3 > \frac{4}{3} p_g(X) - \frac{10}{3} \) for a minimal 3-fold \( X \) of general type, then \( K_X^3 \geq \frac{4}{3} p_g(X) - b \)?

For a minimal surface \( S \) of general type, since \( K_S \) is Cartier, \( K_S^3 \) is always an integer. Thus the “second Noether inequality” is naturally that \( K_S^3 \geq 2 p_g(S) - 3 \), and it is actually optimal. However, since the Cartier indices of minimal 3-folds of general type are unbounded, it is not even clear whether \(-\frac{10}{3}\) is an accumulation point of the set
\[ S = \left\{ K_X^3 - \frac{4}{3} p_g(X) \mid X \text{ minimal 3-fold of general type} \right\}. \]

This is the main obstruction to the existence of the “second Noether inequality” in dimension three, not to mention the optimal one.

The main purpose of this paper is to answer the above two questions for 3-folds \( X \) of general type with \( p_g(X) \geq 11 \).

1. We give a complete classification of minimal 3-folds \( X \) of general type with \( p_g(X) \geq 11 \) satisfying \( K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3} \), and we show that they are all Gorenstein.

2. We establish the optimal “second Noether inequality” and even the optimal “third Noether inequality” for minimal 3-folds \( X \) of general type with \( p_g(X) \geq 11 \).

1.2. Threefolds on the Noether line. In this subsection, we introduce our answer to Question A.

**Definition 1.1.** We say a minimal 3-fold \( X \) of general type with \( p_g(X) \geq 11 \) is on the (first) Noether line, if \( K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3} \).
The first main theorem of this paper gives some basic properties of 3-folds on the Noether line.

**Theorem 1.2.** Let $X$ be a minimal 3-fold of general type with $p_g(X) \geq 11$ and on the Noether line. Then the following statements hold:

1. $p_g(X) \equiv 1 \pmod{3}$, and $X$ is Gorenstein;
2. $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$;
3. The canonical image $\Sigma \subseteq \mathbb{P}^{p_g(X)-1}$ of $X$ is a non-degenerate surface of degree $p_g(X) - 2$, and $\Sigma$ is smooth when $p_g(X) \geq 23$;
4. $X$ is simply connected.

Let us compare Theorem 1.2 with the corresponding results in dimension two. Let $S$ be a minimal surface of general type on the Noether line.

1. E. Bombieri proved that $h^1(S, \mathcal{O}_S) = 0$ [1, Lemma 14].
2. E. Horikawa proved that the canonical image $\Sigma \subseteq \mathbb{P}^{p_g(S)-1}$ of $S$ is a non-degenerate surface of degree $p_g(S) - 2$, and $\Sigma$ is smooth when $p_g(S) \geq 7$ [14, §1].
3. E. Horikawa also proved that $S$ is simply connected [14, Theorem 3.4].

Theorem 1.2 (2)–(4) show that similar results hold in dimension three. Note that there exists a minimal 3-fold $X$ on the Noether line with $p_g(X) = 19$ whose canonical image is not smooth [10, Table 10, No. 11]. Thus the lower bound $p_g(X) \geq 23$ in Theorem 1.2 (3) is almost optimal, with the only missing case when $p_g(X) = 22$.

Theorem 1.2 (1) answers the second part of Question A in the negative when $p_g(X) \geq 11$. More interestingly, it reveals a novel phenomenon for 3-folds which does not occur for surfaces. Namely, every minimal 3-fold $X$ of general type with $p_g(X) \geq 11$ and on the Noether line must satisfy $p_g(X) \equiv 1 \pmod{3}$. In contrast, for any $m \geq 3$ there exists a minimal surface $S$ of general type on the Noether line with $p_g(S) = m$. This “congruence modulo 3” phenomenon suggests that there are two missing Noether inequalities for 3-folds with $p_g(X) \equiv 2$ and 0 (mod 3), respectively (which we will establish in Theorem 1.5).

After this paper was finished, J. Chen informed us that assuming the canonical image being a surface, he has got a proof of Theorem 1.2 (1) using a different method.

Now we turn to the first part of Question A, the classification problem. Let $X$ be a minimal 3-fold of general type with $p_g(X) \geq 11$ and on the Noether line. By Theorem 1.2 (1), we may assume that $p_g(X) = 3m - 2$ for an integer $m \geq 5$. A key observation (see Proposition 2.1) is that, there exists a minimal 3-fold $X_1$ birational to $X$ such that $X_1$ admits a fibration $f : X_1 \to \mathbb{P}^1$ with general fiber a $(1, 2)$-surface. Here a $(1, 2)$-surface means a smooth surface $S$ of general type with $\text{vol}(S) = 1$ and $p_g(S) = 2$. Moreover, $f$ has a natural section $\Gamma$ which is the horizontal base locus of the relative canonical map of $X_1$ over $\mathbb{P}^1$ (see §3.1). Let $X_0$ be the relative canonical
model of $X_1$ over $\mathbb{P}^1$ with the induced fibration $f_0 : X_0 \to \mathbb{P}^1$. Then $\Gamma$ descends to a section $\Gamma_0$ of $f_0$.

**Definition 1.3.** We say that $X_0$ is the relative canonical model associated to $X$, and $\Gamma_0$ is the canonical section of $f_0$.

With the above definition, we now introduce the answer to the first part of Question A.

**Theorem 1.4.** Let $X$ be a minimal 3-fold of general type with $p_g(X) = 3m - 2 \geq 11$ and on the Noether line. Let $X_0$ be the relative canonical model associated to $X$ with the fibration $f_0 : X_0 \to \mathbb{P}^1$. Let $X'_0$ be the blow-up of $X_0$ along the canonical section $\Gamma_0$ of $f_0$. Then the induced fibration $f'_0 : X'_0 \to \mathbb{P}^1$ is factorized as

$$f'_0 : X'_0 \xrightarrow{\rho} Y \xrightarrow{q} \mathbb{P}^1$$

with the following properties:

(i) the Hirzebruch surface $\mathbb{F}_e$ is isomorphic to $\mathbb{P}((f_0)_\ast \omega_{X_0})$;

(ii) $q : Y = \mathbb{P}(\mathcal{O}_{\mathbb{F}_e} \oplus \mathcal{O}_{\mathbb{F}_e}(-2s - (m + e)L)) \to \mathbb{F}_e$ is a $\mathbb{P}^1$-bundle, where $s$ is a section on $\mathbb{F}_e$ with $s^2 = -e$ and $L$ is a ruling on $\mathbb{F}_e$;

(iii) $\rho : X'_0 \to Y$ is a flat double cover with the branch locus $B = B_1 + B_2$, where $B_1$ is the relative hyperplane section of $Y$, $B_2 \sim 5B_1 + 5(m + e)q^*L + 10q^*s$ and $B_1 \cap B_2 = \emptyset$.

Moreover, if $p_g(X) \geq 23$, then $X_0$ is exactly the canonical model of $X$.

In one word, $X_0$ is a divisorial contraction of a double cover of a two-tower of $\mathbb{P}^1$-bundles over $\mathbb{P}^1$. Recall that every minimal surface of general type on the Noether line is birationally a double cover of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ [14, §1]. Theorem 1.4 asserts that birationally every minimal 3-fold of general type on the Noether line has a similar structure. A closely related question raised by J. Chen and C. Lai [8, Question 6.4] asks whether a minimal $n$-fold $X$ of general type ($n \geq 3$) with

$$K^n_X = \frac{n+1}{n}p_g(X) - \frac{n^2+1}{n}$$

is, under some extra assumptions, birationally a double cover of a tower of $\mathbb{P}^1$-bundles. Theorem 1.4 gives a general affirmative answer to this question for $n = 3$ when $p_g(X) \geq 11$, and no extra assumptions are needed.

1.3. **Two new Noether inequalities.** In this subsection, we introduce the second and the third Noether inequalities for 3-folds of general type.

**Theorem 1.5.** Let $X$ be a minimal 3-fold of general type with $p_g(X) \geq 11$.

(1) Suppose that $K^3_X > \frac{4}{3}p_g(X) - \frac{4}{3}$. Then we have the optimal inequality

$$(1.3) \quad K^3_X \geq \frac{4}{3}p_g(X) - \frac{19}{6}. $$
If the equality holds, then \( p_g(X) \equiv 2 \pmod{3} \). Moreover, \( X \) has only one non-Gorenstein terminal singularity, and it is of type \( \frac{1}{2}(1,-1,1) \).

(2) Suppose that \( K_X^3 > \frac{4}{3}p_g(X) - \frac{19}{6} \). Then we have the optimal inequality

\[
K_X^3 \geq \frac{4}{3}p_g(X) - 3.
\]

If the equality holds, then \( p_g(X) \equiv 0 \pmod{3} \). Moreover, one of the following two cases occurs:

(i) \( X \) has two non-Gorenstein terminal singularities, and they are of the same type \( \frac{1}{2}(1,-1,1) \);
(ii) \( X \) has only one non-Gorenstein terminal singularity, and it is of type \( cA_1/\mu_2 \).

Moreover, if \( X \) attains the equality in either (1.3) or (1.4), then the statements (2)-(4) in Theorem 1.2 also hold for \( X \).

We remark that the assumption that \( p_g(X) \geq 11 \) in Theorem 1.5 is also optimal, since there exists a minimal 3-fold of general type with \( p_g(X) = 10 \) and \( K_X^3 = \frac{4}{3}p_g(X) - \frac{33}{10} < \frac{4}{3}p_g(X) - \frac{19}{6} \) [10, Table 10, No. 10]. It is clear that Theorem 1.5 (1) answers Question B. More importantly, Theorem 1.5 confirms the aforementioned “congruence modulo 3” phenomenon that has emerged in Theorem 1.2. It is likely that this phenomenon will also show up in other results regarding the explicit geometry of 3-folds of general type.

In a forthcoming paper [17], we will establish the complete classification of 3-folds attaining the equality in (1.3) and (1.4), respectively.

1.4. Idea of the proof. It is clear that to prove the above theorems, we only need to consider the 3-folds close to the Noether line. To illustrate the main idea of the proof, in the following, we assume that \( X \) is a minimal 3-fold of general type with \( p_g(X) \geq 11 \) and \( K_X^3 = \frac{4}{3}p_g(X) - \frac{33}{10} < \frac{4}{3}p_g(X) - \frac{19}{6} \) [10, Table 10, No. 10].

Let \( \phi_{K_X} : X \to \mathbb{P}^{p_g(X)-1} \) be the canonical map of \( X \) with the canonical image \( \Sigma \).

The first key observation is that \( \dim \Sigma = 2 \), i.e., \( \Sigma \) is a surface. In fact, by a result of Kobayashi (Proposition 4.1), we know that \( \dim \Sigma \leq 2 \). The key result we prove here is that \( \dim \Sigma \neq 1 \) (Proposition 4.5). That is to say, there is a gap \( \frac{2}{3} = \frac{10}{6} - \frac{5}{6} \) between 3-folds on the Noether line and 3-folds whose canonical image is a curve. Interestingly, it is within this gap that we discover two new Noether lines.

Second, using the geometry of the surface \( \Sigma \), we prove that up to a birational modification, \( X \) admits a fibration

\[
f : X \to \mathbb{P}^1
\]

with general fiber a minimal \((1,2)\)-surface (Proposition 2.1). This important observation opens the door to studying the explicit geometry of \( X \). With
the help of the fibration $f$, we prove two key estimates in this paper. The first one (Proposition 3.6) is that

$$K_X^3 \geq p_g(X) + \frac{1}{2} \left\lfloor \frac{2(p_g(X) - 1)}{3} \right\rfloor - 3.$$ 

Because of the “round-up” appearing in this inequality, three possible residues of $p_g(X)$ modulo 3 give rise to three different inequalities just as (1.2), (1.3) and (1.4). However, to prove that they are indeed the first three Noether inequalities, we need the second key estimate (Proposition 3.8) that

$$P_2(X) \leq \left\lfloor 2K_X^3 \right\rfloor + \left\lfloor 2K_X^3 - \frac{5(p_g(X) - 1)}{3} \right\rfloor + 7.$$ 

Based on the above two estimates and a careful analysis using Reid’s Riemann-Roch formula, we manage to prove that (1.3) and (1.4) are the desired optimal Noether inequalities. Furthermore, when $X$ attains the equality in any of (1.2), (1.3) and (1.4), we are able to fully describe the non-Gorenstein singularities on $X$. Thus Theorem 1.2 and 1.5 are proved.

To prove Theorem 1.4, suppose that $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$. Let $X_0$ be the relative canonical model of $X$ over $\mathbb{P}^1$. We first show that $\text{Bs}|K_{X_0}| = \Gamma_0$ is the canonical section of $f_0 : X_0 \to \mathbb{P}^1$ (Lemma 5.4). Denote by $X'_0$ the blow-up of $X_0$ along $\Gamma_0$. Then we show that the rational map

$$\varphi : X'_0 \dashrightarrow \mathbb{P}(f_\ast \omega_X)$$

is a flat morphism with all fibers integral curves of genus 2 (Lemma 5.6). Thus $X'_0$ is a double cover of a $\mathbb{P}^1$-bundle over $\mathbb{P}(f_\ast \omega_X)$. From this, we are able to fully determine the structure of $X'_0$. When $p_g(X) \geq 23$, we prove that $K_{X_0}$ is ample (Lemma 5.11). Thus $X_0$ is the canonical model of $X$.

1.5. Notation and conventions. In this paper, we adopt the following notation and definitions.

1.5.1. Varieties and divisors. Let $V$ be a normal variety of dimension $d$. The geometric genus $p_g(V)$ and the second plurigenus $P_2(V)$ of $V$ are defined as

$$p_g(V) := h^0(V, K_V), \quad P_2(V) := h^0(V, 2K_V).$$

For a Weil divisor $L$ on $V$, the volume $\text{vol}(L)$ of $L$ is defined as

$$\text{vol}(L) := \limsup_{n \to \infty} \frac{h^0(V, nL)}{n^d/d!}.$$ 

The volume $\text{vol}(K_V)$ is called the canonical volume of $V$, and is denoted by $\text{vol}(V)$. We say that $V$ is minimal, if $V$ has at worst $\mathbb{Q}$-factorial terminal singularity and $K_V$ is nef. If $V$ has at worst canonical singularities and $\text{vol}(V) > 0$, we say that $V$ is of general type. If $V$ is of general type and $K_V$ is ample, we say that $V$ is a canonical model. Note that if $V$ is minimal or a canonical model, then $\text{vol}(V) = K_V^d$. 

For a linear system $\Lambda$ on $V$, $\text{Mov}\Lambda$ and $\text{Bs}\Lambda$ denote the movable part and the base locus of $\Lambda$, respectively. The rational map
\[
\phi_{K_V}: V \dashrightarrow \mathbb{P}^{p_g(V)-1}
\]
induced by the canonical linear system $|K_V|$ is called the canonical map of $V$, and $\phi_{K_V}(V)$ is called the canonical image of $V$.

A $\mathbb{Q}$-divisor on $V$ is $\mathbb{Q}$-effective, if it is $\mathbb{Q}$-linear equivalent to an effective $\mathbb{Q}$-divisor.

For an integer $e \geq 0$, $F_e$ denotes the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$.

1.5.2. Birational modifications with respect to divisors. For a normal variety $V$ with $\mathbb{Q}$-factorial singularities and a Weil (thus $\mathbb{Q}$-Cartier) divisor $L$ on $V$ with $h^0(V, L) \geq 2$, we can always find a resolution of singularities of $V$
\[
\alpha: V_0 \rightarrow V
\]
and a successive blow-ups
\[
\beta: V' = V_n \overset{\pi_n}{\rightarrow} V_{n-1} \rightarrow \cdots \rightarrow V_{i+1} \overset{\pi_i}{\rightarrow} V_i \rightarrow \cdots \rightarrow V_1 \overset{\pi_0}{\rightarrow} V_0
\]
with the following properties:

1. $\alpha$ is an isomorphism over the smooth locus of $V$.
2. All $V_i$'s ($i = 0, \ldots, n$) are smooth.
3. Denote $|M_0| = \text{Mov}|(\alpha^*L)|$. Then each $\pi_i$ is a blow-up along a nonsingular center $W_i$ contained in the base locus of $\text{Mov}|(\pi_0 \circ \pi_1 \circ \cdots \circ \pi_{i-1})^*M_0|$.
4. The linear system $\text{Mov}|\beta^*M_0|$ is base point free.

Set
\[
\pi = \alpha \circ \beta: V' \rightarrow V.
\]
This birational modification $\pi$ will be used frequently in this paper.

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2. Fibered minimal model for some threefolds with small volume

Let $X$ be a minimal 3-fold of general type with $p_g(X) \geq 2$. Let
\[
\phi_{K_X}: X \dashrightarrow \mathbb{P}^{p_g(X)-1}
\]
be the canonical map of $X$ with the canonical image $\Sigma$. Denote by $\pi : X' \to X$ the birational modification of $X$ with respect to $K_X$ as in §1.5.2. We have the following commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\psi} & \Sigma' \\
\downarrow{\phi_M} & & \downarrow{\phi_{K_X}} \\
X & \xrightarrow{\phi} & \Sigma
\end{array}
\]

Here $\phi_M$ is the morphism induced by the linear system $|M| = \text{Mov}[|\pi^*K_X|]$, and $X' \xrightarrow{\psi} \Sigma' \xrightarrow{\tau} \Sigma$ is the Stein factorization of $\phi_M$. We may write

$$\pi^*K_X = M + Z,$$

where $Z \geq 0$ is a $\mathbb{Q}$-divisor.

The main result in this section is the following.

**Proposition 2.1.** With the above notation, suppose that $p_g(X) \geq 7$, $\dim \Sigma = 2$ and $K_X^3 < \frac{4}{3}p_g(X) - 2$. Then $\deg \Sigma = p_g(X) - 2$, and $\tau$ is an isomorphism. Moreover, there exists a minimal 3-fold $X_1$ birational to $X$ such that $X_1$ admits a fibration $f : X_1 \to \mathbb{P}^1$ with general fiber $F_1$ a $(1, 2)$-surface.

**Proof.** Since $\Sigma \subseteq \mathbb{P}^{p_g(X)-1}$ is non-degenerate, we have $\deg \Sigma \geq p_g(X) - 2$. In the following, we prove the proposition by steps.

**Step 1.** Denote by $C$ a general fiber of $\psi$. In this step, we show that

$$((\pi^*K_X) \cdot C) \geq 1.$$

Let $S \in |M|$ be a general member. By Bertini’s theorem, $S$ is smooth, and we have

$$M|_S \equiv dC,$$

where $d := (\deg \tau) \cdot (\deg \Sigma) \geq p_g(X) - 2$. Since $p_g(X) \geq 7$, we have

$$K_X^3 < \frac{4}{3}p_g(X) - 2 < 2p_g(X) - 4.$$

By [7, Theorem 4.1], we deduce that $g(C) = 2$. Let $\sigma : S \to S_0$ be the contraction onto the minimal model of $S$, and let $C_0 = \sigma_*C$. Since

$$K_S = (K_{X'} + S)|_S \geq 2M|_S \equiv 2dC \geq 2(p_g(X) - 2)C \geq 10C,$$

we deduce that $K_{S_0} - 10C_0$ is pseudo-effective. In particular,

$$K_{S_0}^2 \geq 10(K_{S_0} \cdot C_0) \geq 10.$$

Since $g(C) = 2$, we have

$$K_{S_0} \cdot C_0 = ((\sigma^*K_{S_0}) \cdot C) \leq (K_S \cdot C) = 2.$$

By the Hodge index theorem, we deduce that $C_0^2 = 0$. This implies that

$$((\sigma^*K_{S_0}) \cdot C) = (K_{S_0} \cdot C_0) = 2.$$
By [7, Corollary 2.3], \(2(\pi^*K_X)|_S - \sigma^*K_{S_0}\) is \(\mathbb{Q}\)-effective. Thus
\[
((\pi^*K_X) \cdot C) \geq \frac{1}{2} ((\sigma^*K_{S_0}) \cdot C) = 1.
\]

**Step 2.** In this step, we prove that \(\deg \Sigma = p_g(X) - 2\) and that \(\tau\) is an isomorphism. Moreover, we construct a relatively minimal fibration from a birational model of \(X\) to \(\mathbb{P}^1\).

By the argument in the proof of [7, Theorem 4.2] and the assumption, we have
\[
\frac{4}{3} p_g(X) - 2 > K_X^3 \geq ((\pi^*K_X)|_S)^2 \geq \frac{2}{3}(2d - 1).
\]
Thus \(d = p_g(X) - 2\), which implies that
\[
\deg \tau = 1, \quad \deg \Sigma = p_g(X) - 2.
\]
By [24, §10], there is a Hirzebruch surface \(F_e\) for some \(e \geq 0\) and a morphism
\[
r : F_e \to \mathbb{P}^{p_g(X)-1}
\]
induced by the linear system \(|s + (e + k)l|\) such that \(\Sigma = r(F_e)\). Here \(l\) is a ruling of the natural fibration \(p : F_e \to \mathbb{P}^1\), \(s\) is a section of \(p\) with \(s^2 = -e\), and \(k \in \mathbb{Z}_{\geq 0}\) such that \(\deg \Sigma = e + 2k\). In particular, \(\Sigma\) is normal. Thus \(\tau\) is an isomorphism.

Replacing \(X'\) by its birational modification, we may assume that there is a surjective morphism \(\varphi : X' \to F_e\) such that \(\psi = r \circ \varphi\). Thus we obtain a fibration
\[
f' := p \circ \varphi : X' \to F_e \to \mathbb{P}^1
\]
with a general fiber \(F' = \varphi^*l\). Let \(\zeta : X' \dashrightarrow X_1\) be the contraction of \(X'\) onto its relative minimal model \(X_1\) over \(\mathbb{P}^1\). Up to a birational modification, we may assume that \(\zeta\) is a morphism. Then we obtain a relatively minimal fibration
\[
f_1 : X_1 \to \mathbb{P}^1
\]
with a general fiber \(F_1\). Here \(\mu := \zeta|_{F'} : F' \to F_1\) is just the contraction onto the minimal model of \(F'\).

**Step 3.** In this step, we prove that \(F_1\) is a \((1, 2)\)-surface.

Since \(\dim \Sigma = 2\), the natural restriction map
\[
H^0(X', K_{X'}) \to H^0(F', K_{F'})
\]
has the image of dimension at least two. In particular, \(p_g(F_1) = p_g(F') \geq 2\). To show that \(F_1\) is a \((1, 2)\)-surface, it suffices to show that \(K_{F_1}^2 = 1\).

From **Step 2** and the assumption that \(p_g(X) \geq 7\), we deduce that \(e + k \geq \frac{1}{2} \deg \Sigma = \frac{1}{2} p_g(X) - 1 \geq \frac{5}{2}\), i.e., \(e + k \geq 3\). Also recall in **Step 2** that \(M = \varphi^*(s + (e + k)l)\). Thus \(\pi^*K_X - (e + k)F' \geq 0\). By [7, Corollary 2.3],
\[
(1 + \frac{1}{e+k})(\pi^*K_X)|_{F'} - \mu^*K_{F_1} \text{ is } \mathbb{Q}\text{-effective. In particular, } \frac{4}{3}(\pi^*K_X)|_{F'} - \mu^*K_{F_1} \text{ is } \mathbb{Q}\text{-effective.}
\]
On the other hand, by the assumption, we have
\[
\frac{4}{3} p_g(X) - 2 > K_X^3 \geq d((\pi^*K_X) \cdot C) = (p_g(X) - 2)((\pi^*K_X) \cdot C).
\]
It follows from Step 1 that

\[
((\mu^* K_{F_1}) \cdot C) \leq \frac{4}{3} (\pi^* K_X|_{F'}) < \frac{4}{3} \frac{4p_g(X) - 2}{p_g(X) - 2} < 2,
\]

where the last inequality follows from the assumption that \( p_g(X) \geq 7 \). Let \( C_1 = \mu^* C \). Then the above inequality means that \( (K_{F_1} \cdot C_1) < 2 \). Thus \( (K_{F_1} \cdot C_1) = 1 \).

**Step 4.** In this step, we show that \( X_1 \) is minimal. By [7, Lemma 3.2], it suffices to show that \( \mu^* K_{F_1} = (\pi^* K_X)|_{F'} \).

By the conclusion in Step 3, we may write

\[
\mu^* K_{F_1} = (\pi^* K_X)|_{F'} + A_1 - A_2,
\]

where \( A_1 \) and \( A_2 \) are both effective \( \mathbb{Q} \)-divisors on \( F' \) supported on the exceptional locus of \( \mu \) with no common irreducible components. Thus we have

\[
0 = ((\mu^* K_{F_1}) \cdot A_2) = (((\pi^* K_X)|_{F'} + A_1 - A_2) \cdot A_2) \geq -A_2^2 \geq 0,
\]

i.e., \( A_2^2 = 0 \). This implies that \( A_2 = 0 \), and thus \( \mu^* K_{F_1} \geq (\pi^* K_X)|_{F'} \). As a result, we have

\[
1 = (\mu^* K_{F_1})^2 \geq (\pi^* K_X)|_{F'}^2 \geq ((\pi^* K_X) \cdot C) \geq 1,
\]

which forces \( (\mu^* K_{F_1})^2 = (\pi^* K_X)|_{F'}^2 \). By the Hodge index theorem again, we deduce that \( \mu^* K_{F_1} = (\pi^* K_X)|_{F'} \). Thus the proof is completed. \( \square \)

3. Geometry of 3-folds fibered by (1, 2)-surfaces

Throughout this section, let \( X \) be a minimal 3-fold of general type with a fibration

\[
f : X \to \mathbb{P}^1
\]

such that the general fiber \( F \) of \( f \) is a (1, 2)-surface. We always assume that the canonical image of \( X \) is a surface. In particular, \( p_g(X) \geq 3 \).

3.1. General Setting. Since \( X \) is minimal, by the adjunction, \( F \) is also minimal. Thus \( |K_F| \) has a unique base point. Consider the relative canonical map \( X \to \mathbb{P}(f_* \omega_X) \) of \( X \) over \( \mathbb{P}^1 \). It is clear that the indeterminacy locus of this rational map contains a section \( \Gamma \) of \( f \) such that \( \Gamma \cap F \) is just the base point of \( |K_F| \).

Following the notation in §2, taking a birational modification \( \pi : X' \to X \) as in §1.5.2 with respect to \( K_X \), we may write

\[
(3.1) \quad \pi^* K_X = M + Z,
\]
where \(|M| = \text{Mov}(|\pi^*K_X|)| is base point free and \(Z \geq 0\) is a \(\mathbb{Q}\)-divisor. We have a similar commutative diagram

\[
\begin{array}{ccc}
    & X' & \\
  f' \downarrow & \downarrow \pi & \downarrow \psi \\
  \mathbb{P}^1 & \rightarrow & \Sigma' \\
  f \downarrow & \downarrow \phi \downarrow & \downarrow \tau \\
  X & \rightarrow & \Sigma \\
  \phi_{K_X} & \end{array}
\]

as in §2, where \(\phi_{K_X}\) is the canonical map of \(X\), \(\phi_M\) is the morphism induced by \(|M|\), \(X' \xrightarrow{\psi} \Sigma' \xrightarrow{\tau} \Sigma\) is the Stein factorization of \(\phi_M\), and \(f' = f \circ \pi\) is the induced fibration. Denote by \(F'\) a general fiber of \(f'\).

Note that \(X\) has at worst terminal singularities. We may write

\[
(3.2) \quad K_{X'} = \pi^*K_X + E_\pi,
\]

where \(E_\pi \geq 0\) is a \(\pi\)-exceptional \(\mathbb{Q}\)-divisor.

Let \(C\) be a general fiber of \(\psi\). Since \(\dim \Sigma = 2\), \(C\) is a curve, and we have

\[
M|_S \equiv dC,
\]

where \(d = (\deg \tau) \cdot (\deg \Sigma) \geq p_g(X) - 2 \geq 1\).

**Remark 3.1.** By Proposition 2.1, if \(p_g(X) \geq 7\) and \(K_X^3 < \frac{4}{3}p_g(X) - 2\), then \(\deg \tau = 1\) and \(\deg \Sigma = p_g(X) - 2\). In particular, \(d = p_g(X) - 2\).

Recall the following two lemmas from [16].

**Lemma 3.2.** [16, Lemma 2.2] We have \(|M|_{F'} = \text{Mov}(|\pi_{F'}|^*K_F|)\), where \(\pi|_{F'} : F' \rightarrow F\) coincides with the blow-up of the unique base point of \(|K_F|\). In particular, \(g(C) = 2\) and there is a rational map \(p : \Sigma' \dashrightarrow \mathbb{P}^1\) such that \(f' = p \circ \psi\).

**Lemma 3.3.** [16, Lemma 2.4] There exists a unique \(\pi\)-exceptional prime divisor \(E_0\) on \(X'\) such that

1. \(\text{coeff}_{E_0}(Z) = \text{coeff}_{E_0}(E_\pi) = 1\);
2. \(\pi(E_0) = \Gamma, \phi_M(E_0) = \Sigma\);
3. \((E_0 \cdot C) = (Z \cdot C) = (E_\pi \cdot C) = ((\pi^*K_X) \cdot C) = 1\).

**Lemma 3.4.** The sheaf \(f_*\omega_X\) is locally free of rank two over \(\mathbb{P}^1\), and it is nef. Moreover, \(f_*\omega_X\) is not ample if and only if \(\Sigma\) is a cone over a smooth rational curve of degree \(p_g(X) - 2\).

**Proof.** It is clear that \(f_*\omega_X\) is torsion free over \(\mathbb{P}^1\) of rank \(p_g(F) = 2\). Thus it is locally free. We may write

\[
f_*\omega_X = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)
\]

with \(a, b \in \mathbb{Z}\) and \(a \geq b\). Since the canonical image \(\Sigma\) is a surface, we deduce that \(b \geq 0\). Otherwise, \(\Sigma \simeq \mathbb{P}^1\). Therefore, \(a \geq b \geq 0\), and \(f_*\omega_X\) is nef.

Consider the relative canonical map

\[
\phi : X' \dashrightarrow V := \mathbb{P}(f'_*\omega_{X'})
\]
of $X'$ over $\mathbb{P}^1$. It is known that $\phi$ is induced by the linear system $|K_{X'} + tF'|$ for a sufficiently large integer $t$. Since $f'_s$ is left exact, we know that $f'_s\mathcal{O}_{X'}(M)$ is a subsheaf of $f'_s\omega_{X'}$. Note that $f'_s\mathcal{O}_{X'}(M)$ is also a direct sum of two line bundles over $\mathbb{P}^1$ and $h^0(X', M) = h^0(X', K_{X'})$. It follows that $f'_s\mathcal{O}_{X'}(M) = f'_s\omega_{X'}$. By the projection formula, we have $f'_s\mathcal{O}_{X'}(M + tF') = f'_s\mathcal{O}_{X'}(K_{X'} + tF')$. We deduce that $h^0(X', M + tF') = h^0(X', K_{X'} + tF')$. In particular, $\text{Mov}|K_{X'} + tF'| = |M + tF'|$ is base point free. Thus $\phi$ is a morphism. Since $\phi$ is induced by the natural morphism $f''_s f'_s\omega_{X'} \to \omega_{X'}$. We have $K_{X'} \geq \phi^*H$, where $H$ is a relative hyperplane section of $V$. As $f_s\omega_X$ is nef over $\mathbb{P}^1$, $f_s\omega_X$ is generated by global sections. Since $X$ is terminal, we have $f'_s\omega_{X'} = f_s\omega_X$. It follows that $|H|$ is base point free and $h^0(V, H) = h^0(\mathbb{P}^1, f_s\omega_X) = p_g(X) = h^0(X', M)$. In particular, we have $$|M| = \phi^*|H|.$$ As a result, $\Sigma$ is just the image of $V$ under the morphism induced by $|H|$. Since $V$ is a Hirzebruch surface, we know that $\Sigma$ is a cone if and only if $H$ is not ample, which is equivalent to the non-amenability of $f_s\omega_X$. Thus the proof is completed. \hfill \Box

**Lemma 3.5.** We have $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$. In particular, $\chi(\omega_X) = p_g(X) - 1$.

**Proof.** By [21, Theorem 2.1], $R^1f_s\omega_X$ and $R^2f_s\omega_X$ are both torsion free sheaves. Now $q(F) = 0$. We deduce that $R^1f_s\omega_X = 0$. Moreover, $R^2f_s\omega_X = \omega_{\mathbb{P}^1}$. Thus

$$h^1(X, \mathcal{O}_X) = h^2(X, \omega_X) = h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}) + h^1(\mathbb{P}^1, R^1f_s\omega_X) = 0.$$ By Lemma 3.4, $h^1(\mathbb{P}^1, f_s\omega_X) = 0$. It follows that

$$h^2(X, \mathcal{O}_X) = h^1(X, \omega_X) = h^1(\mathbb{P}^1, f_s\omega_X) + h^0(\mathbb{P}^1, R^1f_s\omega_X) = 0.$$ Finally, we conclude that

$$\chi(\omega_X) = p_g(X) + h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) - 1 = p_g(X) - 1.$$ The proof is completed. \hfill \Box

3.2. **Two key estimates.** In this subsection, we prove two important estimates.

**Proposition 3.6.** The following inequalities hold:

1. $(K_X \cdot \Gamma) \geq \frac{1}{2} \left[ \frac{2(d - 2)}{3} \right]$;
2. $K_X^3 \geq d + \frac{1}{2} \left[ \frac{2(d - 2)}{3} \right]$.

Moreover, if the equality in (2) holds, so does the equality in (1).

**Proof.** Let $E_0$ be the unique $\pi$-exceptional prime divisor as in Lemma 3.3. Take a general member $S \in |M|$. By Bertini’s theorem and Lemma 3.3 (2), $S$ is smooth and $S|_{E_0}$ is irreducible. By Lemma 3.2, $S \cap F' = C$. Now a
Denote $\Gamma_S = E_0|_S$. By Lemma 3.3 (2) and (3), $\Gamma_S$ is a section of $\psi|_S$. Therefore, by Lemma 3.3 (1), we may write
\begin{equation}
E_\pi|_S = \Gamma_S + E_V, \quad Z|_S = \Gamma_S + Z_V.
\end{equation}
Here $E_V \geq 0$ and $Z_V \geq 0$ are $S$-divisors on $S$. By Lemma 3.3 (3),
\begin{equation}
(E_V \cdot C) = ((E_\pi - E_0) \cdot C) = 0, \quad (Z_V \cdot C) = ((Z - E_0) \cdot C) = 0.
\end{equation}
We deduce that both $E_V$ and $Z_V$ are vertical with respect to $\psi|_S$.

By the adjunction formula, (3.1), (3.2) and (3.3),
\begin{equation}
K_S = (K_X + S)|_S \equiv 2dC + 2\Gamma_S + E_V + Z_V.
\end{equation}

Denote by $\sigma : S \to S_0$ the contraction onto its minimal model $S_0$. By [7, Corollary 2.3], we have
\begin{equation}
\pi^* K_X|_S \sim_{\mathbb{Q}} \frac{1}{2} \sigma^* K_{S_0} + H_S,
\end{equation}
where $H_S \geq 0$ is a $\mathbb{Q}$-divisor on $S$. Therefore, by Lemma 3.3 (3),\begin{equation}
((\pi^* K_X) \cdot C) \leq 2((\pi^* K_X) \cdot C) = 2. \quad \text{Let } C_0 = \sigma_\ast C \text{ and } \Gamma_{S_0} = \sigma_\ast \Gamma_S. \text{ Then we have}
(K_{S_0} \cdot C_0) \leq 2. \quad \text{On the other hand, by (3.4), we have}
\end{equation}
\begin{equation}
K_{S_0} \equiv 2dC_0 + 2\Gamma_{S_0} + \sigma_\ast (E_V + Z_V).
\end{equation}

We deduce that $(K_{S_0} \cdot C_0) \geq 2dC_0^2 \geq 2C_0^2$. By parity, it follows that $(K_{S_0} \cdot C_0) = 2$ and $C_0^2 = 0$. In particular, the fibration $\psi|_S$ descends to a fibration $S_0 \to \mathbb{P}^1$ whose general fiber is $C_0$ with $g(C_0) = 2$. Moreover,
\begin{equation}
(H_S \cdot C) = ((\pi^* K_X) \cdot C) - \frac{1}{2} (K_{S_0} \cdot C_0) = 0.
\end{equation}
This implies that $H_S$ is vertical with respect to $\psi|_S$.

Since $\Gamma_{S_0}$ is a section of the fibration $S_0 \to \mathbb{P}^1$, we have $g(\Gamma_{S_0}) = 0$. By the adjunction formula on $S_0$ and (3.6), we have
\begin{equation}
-2 = (K_{S_0} \cdot \Gamma_{S_0}) + \Gamma_{S_0}^2
\end{equation}
\begin{equation}
= (K_{S_0} \cdot \Gamma_{S_0}) + \frac{1}{2} ((K_{S_0} - 2dC_0 - \sigma_\ast (E_V + Z_V)) \cdot \Gamma_{S_0})
\end{equation}
\begin{equation}
= \frac{3}{2} (K_{S_0} \cdot \Gamma_{S_0}) - d - \frac{1}{2} (\sigma_\ast (E_V + Z_V) \cdot \Gamma_{S_0}).
\end{equation}

We deduce that
\begin{equation}
(K_{S_0} \cdot \Gamma_{S_0}) = \frac{2}{3} d - \frac{4}{3} + \frac{1}{3} (\sigma_\ast (E_V + Z_V) \cdot \Gamma_{S_0}) \geq \left\lfloor \frac{2(d - 2)}{3} \right\rfloor,
\end{equation}
where the last inequality holds because $\sigma_\ast (E_V + Z_V)$ is vertical with respect to the fibration $S_0 \to \mathbb{P}^1$ and $(K_{S_0} \cdot \Gamma_{S_0})$ is an integer. Together with (3.5) and the fact that $(H_S \cdot \Gamma_S) \geq 0$, we deduce that
\begin{equation}
(K_X \cdot \Gamma) = ((\pi^* K_X)|_S \cdot \Gamma_S) \geq \frac{1}{2} ((\pi^* K_{S_0}) \cdot \Gamma_S) \geq \frac{1}{2} \left\lfloor \frac{2(d - 2)}{3} \right\rfloor.
\end{equation}
Thus the inequality (1) is proved.
For (2), note that
\[ K_X^3 \geq (\pi^*K_X \cdot M^2) + ((\pi^*K_X)|_S \cdot Z|_S). \]

By Lemma 3.3 (3),
\[ (\pi^*K_X \cdot M^2) = d((\pi^*K_X) \cdot C) = d. \]

By (1), we have
\[ ((\pi^*K_X)|_S \cdot Z|_S) \geq (\pi^*K_X)|_S \cdot \Gamma_S = (K_X \cdot \Gamma) \geq \frac{1}{2} \left\lceil \frac{2(d - 2)}{3} \right\rceil. \]

Combine the above inequalities together, and we deduce that
\[ K_X^3 \geq d + \frac{1}{2} \left\lceil \frac{2(d - 2)}{3} \right\rceil. \]

This proves (2).

**Proposition 3.7.** Keep the same notation as in the proof of Proposition 3.6. Suppose that the equality in Proposition 3.6 (2) holds. Then we have the following equalities:

1. \[ K_{S_0}^2 = 4d + 2 \left\lceil \frac{2(d - 2)}{3} \right\rceil; \]
2. \[ (K_{S_0} \cdot \sigma_*(E_V + Z_V)) = 0; \]
3. \[ \sigma^*K_{S_0} \sim_\mathbb{Q} 2(\pi^*K_X)|_S. \]

Moreover, if \( d \equiv 2 \pmod{3} \), then \( \sigma_*(E_V + Z_V) = 0 \) and \( K_{S_0} = 2dC_0 + 2\Gamma_{S_0} \).

**Proof.** By (3.6) and (3.8), we have
\[
K_{S_0}^2 = 2d(K_{S_0} \cdot C_0) + 2(K_{S_0} \cdot \Gamma_{S_0}) + (K_{S_0} \cdot \sigma_*(E_V + Z_V)) \\
greater 4d + 2 \left\lceil \frac{2(d - 2)}{3} \right\rceil + (K_{S_0} \cdot \sigma_*(E_V + Z_V)).
\]

Together with (3.5), we deduce that
\[
K_X^3 \geq ((\pi^*K_X)|_S)^2 \geq \frac{1}{4} K_{S_0}^2 \geq d + \frac{1}{2} \left\lceil \frac{2(d - 2)}{3} \right\rceil + \frac{1}{4} (K_{S_0} \cdot \sigma_*(E_V + Z_V)).
\]

Now by our assumption, \( K_X^3 = d + \frac{1}{2} \left\lceil \frac{2(d - 2)}{3} \right\rceil. \) We conclude that all the above inequalities become equalities. In particular, we have \( K_{S_0}^2 = 4K_X^3 \) and \( (K_{S_0} \cdot \sigma_*(E_V + Z_V)) = 0. \) Thus the equalities (1) and (2) hold. Moreover, we have \( 4((\pi^*K_X)|_S)^2 = 2((\pi^*K_X)|_S \cdot \sigma^*K_{S_0}) = K_{S_0}^2. \)

By the Hodge index theorem, (3.5) and the above equalities, we deduce that \( H_S \equiv 0. \) Since \( H_S \) is effective, we have \( H_S = 0. \) Thus
\[
\sigma^*K_{S_0} \sim_\mathbb{Q} 2(\pi^*K_X)|_S.
\]

Suppose that \( d \equiv 2 \pmod{3} \). Then \( \frac{2d}{3} - \frac{4}{3} = \left\lceil \frac{2(d - 2)}{3} \right\rceil. \) Note that (3.8) now becomes an equality. Thus
\[
(\sigma_*(E_V + Z_V) \cdot \Gamma_{S_0}) = 0.
\]
Moreover, by (3.6) and the equality (2), we conclude that
\[
(\sigma_*(E_V + Z_V))^2 = - (\sigma_*(E_V + Z_V) \cdot (2\Gamma + 2dC_0)) = 0.
\]
Thus \(\sigma_*(E_V + Z_V) = 0\). By (3.6), we have \(K_{S_0} \equiv 2dC_0 + 2\Gamma S_0\). The whole proof is completed.

**Proposition 3.8.** One of the following inequalities holds:

1. \(P_2(X) \leq \left|2K_X^3\right| + \left|2K_X^3 - \frac{5(p_g(X) - 1)}{3}\right| + 7;\)
2. \(K_X^3 \geq \frac{4}{3} p_g(X) - \frac{17}{6}.\)

**Proof.** Set \(|M_0| = \text{Mov}2K_{X'}, |M_1| = \text{Mov}2K_{X'} - M\) and \(|M_2| = \text{Mov}2K_{X'} - 2M\). Replacing \(X'\) by a further blow-up, we may assume that \(|M_0|, |M_1|\) and \(|M_2|\) are all base point free. By Bertini’s theorem, we may take a smooth general member \(S \in |M|\). It is easy to see that

\[
P_2(X) = u_0 + u_1 + h^0(X', 2K_{X'} - 2M),
\]

where

\[
u_i = \dim \text{Im} \left( H^0(X', M_i) \to H^0(S, M_i|S) \right) \quad (i = 0, 1).
\]

Note that we always have

\[
K_X^3 \geq \left((\pi^* K_X)|S\right)^2.
\]

**Step 1.** In this step, we prove that

\[
u_0 \leq \left|2K_X^3\right| + 2.
\]

Consider the complete linear system \(|M_0|_S\). By our assumption, it induces a morphism \(\phi_0 : S \to \mathbb{P}^{h^0(S, M_0|S) - 1}\). If \(\dim \phi_0(S) = 2\), by [26, Lemma 1.8],

\[
4 \left((\pi^* K_X)|S\right)^2 \geq (M_0|S)^2 \geq 2h^0(S, M_0|S) - 4.
\]

If \(\dim \phi_0(S) = 1\), since \(M_0|S \geq M|S\), the general fiber of \(\phi_0\) is identical to \(C\). Now \(M_0|S \equiv bC\), where \(b \geq h^0(S, M_0|S) - 1\). Thus by Lemma 3.3 (3),

\[
2 \left((\pi^* K_X)|S\right)^2 \geq b \left((\pi^* K_X)|C\right) \geq h^0(S, M_0|S) - 1.
\]

Thus in both cases, we always have

\[
2 \left((\pi^* K_X)|S\right)^2 \geq h^0(S, M_0|S) - 2.
\]

Together with (3.12), it follows that

\[
u_0 \leq h^0(S, M_0|S) \leq 2 \left((\pi^* K_X)|S\right)^2 + 2 \leq 2K_X^3 + 2.
\]

Since \(u_0\) is an integer, we deduce that \(u_0 \leq \left|2K_X^3\right| + 2\).

**Step 2.** In this step, we prove that

\[
u_1 \leq \left[2K_X^3 - \frac{5(p_g(X) - 1)}{3}\right] + 4.
\]
Similarly as in Step 1, the complete linear system $|M_1|_S$ induces a morphism $\phi_1 : S \to \mathbb{P}^{h^0(S, M_1|_S) - 1}$. If $\dim \phi_1(S) = 2$, by [26, Lemma 1.8], $(M_1|_S)^2 \geq 2h^0(S, M_1|_S) - 4$. It follows that

$$4((\pi^*K_X)|_S)^2 \geq (M_1|_S + M|_S)^2 \geq 2h^0(S, M_1|_S) - 4 + 2(M_1|_S \cdot M|_S).$$

Since $\phi_1$ does not contract $C$, the linear system $|M_1|_S||C$ induces a finite morphism on $C$. Note that $g(C) = 2$ by Lemma 3.2. We deduce that $(M_1|_S \cdot C) \geq 2$. Recall that we have $M|_S \equiv dC$, where $d \geq p_g(X) - 2$. Thus

$$(M_1|_S \cdot M|_S) \geq d(M_1|_S \cdot C) \geq 2d \geq 2p_g(X) - 4.$$

Combine the above inequalities with (3.12) together. We deduce that

$$h^0(S, M_1|_S) \leq 2(1 + (\pi^*K_X)|_S)^2 + 2 - (2p_g(X) - 4) \leq 2K_X^3 + 6 - 2p_g(X).$$

Since $p_g(X) \geq 3$, we deduce that

$$u_1 \leq h^0(S, M_1|_S) \leq 2K_X^3 + \frac{17}{3} - \frac{5}{3}p_g(X) = 2K_X^3 - \frac{5p_g(X) - 1}{3} + 4.$$

Since $u_1$ is an integer, we know that (3.14) holds in this case.

If $\dim \phi_1(S) = 1$, since $M_1|_S \geq M|_S$, the general fiber of $\phi_1$ is just $C$. Thus $M_1|_S \equiv bC$, where $b \geq h^0(S, M_1|_S) - 1$. Note that $M|_S \equiv dC$ as before. Therefore, the divisor $2(\pi^*K_X)|_S = (b + d)C$ on $S$ is pseudo-effective. Let $\sigma : S \to S_0$ be the contraction morphism onto the minimal model of $S$. As has been proved in Proposition 3.6, the fibration $\psi|_S : S \to \mathbb{P}^1$ descends to a fibration $S_0 \to \mathbb{P}^1$ with a general fiber $C_0 = \sigma_*C$. In the meantime, by (3.6), we may write

$$K_{S_0} = 2dC_0 + 2\Gamma_{S_0} + \Delta_0,$$

where $\Gamma_{S_0}$ is a section of the fibration $S_0 \to \mathbb{P}^1$ and $\Delta_0$ is an effective divisor which is vertical with respect to the fibration $S_0 \to \mathbb{P}^1$. By the adjunction formula, we have

$$-2 = (K_{S_0} \cdot \Gamma_{S_0}) + \Gamma_{S_0}^2 = 2d + 3\Gamma_{S_0}^2 + (\Delta_0 \cdot \Gamma_{S_0}).$$

We deduce that

$$\Gamma_{S_0}^2 = -\frac{2}{3}d - \frac{2}{3} - \frac{(\Gamma_{S_0} \cdot \Delta_0)}{3}.$$

This implies that the divisor

$$K_{S_0} - \frac{2(d - 2)}{3}C_0 \equiv \frac{2}{3} \left( (d + 2)C_0 + 3\Gamma_{S_0} + \frac{3}{2}\Delta_0 \right)$$

on $S_0$ is nef. Therefore, it follows that

$$\left( (\sigma^*(K_{S_0} - \frac{2(d - 2)}{3}C_0) \right) \cdot (2(\pi^*K_X)|_S = (b + d)C) \geq 0,$$

i.e.,

$$2((\pi^*K_X)|_S \cdot (\sigma^*K_{S_0}) \geq \frac{4(d - 2)}{3}((\pi^*K_X) \cdot C) + (b + d)((\sigma^*K_{S_0}) \cdot C).$$
Note that \((K_{S_0} \cdot C_0) = 2\) and \(((\pi^*K_X) \cdot C) = 1\) by Proposition 3.6 and Lemma 3.3. Thus the above inequality becomes
\[
((\pi^*K_X)|_S \cdot (\sigma^*K_{S_0})) \geq \frac{2(d - 2)}{3} + (b + d) = b + \frac{5}{3}d - \frac{4}{3}.
\]
On the other hand, by (3.5) and (3.12), we have
\[
((\pi^*K_X)|_S \cdot (\sigma^*K_{S_0})) \leq 2((\pi^*K_X)|_S)^2 \leq 2K_X^3.
\]
The above two inequalities imply that
\[
b \leq 2K_X^3 + \frac{4}{3} - \frac{5}{3}d \leq 2K_X^3 - \frac{5(p_g(X) - 1)}{3} + 3,
\]
where the last inequality holds since \(d \geq p_g(X) - 2\). It follows that
\[
u_1 \leq h^0(S, M_1|_S) \leq b + 1 \leq 2K_X^3 - \frac{5(p_g(X) - 1)}{3} + 4.
\]
Again, since \(u_1\) is an integer, (3.14) holds also in this case.

**Step 3.** In this step, we prove that if \(h^0(X', M_2) \geq 2\), then
\[
h^0(X', M_2 - F') > 0.
\]
Suppose that \(h^0(X', M_2) \geq 2\). Then we have \(2\pi^*K_X \geq 2M + M_2\). Thus
\[
2(\pi^*K_X)|_{F'} \geq 2M|_{F'} + M_2|_{F'}.
\]
By Lemma 3.2, \(M|_{F'} \sim C\) and \((\pi^*K_X)|_{F'} = (\pi|_{F'})^*K_F\). We deduce that
\[
(((\pi|_{F'})^*K_F) \cdot M_2|_{F'}) = 0.
\]
Since \(M_2|_{F'}\) is base point free, we conclude \(M_2|_{F'} = 0\) by the Hodge index theorem. Note that we have the exact sequence
\[
0 \rightarrow H^0(X', M_2 - F') \rightarrow H^0(X', M_2) \rightarrow H^0(F', M_2|_{F'})
\]
Thus the result follows.

**Step 4.** In this step, we finish the whole proof of the proposition.

By (3.11), (3.13) and (3.14), we have
\[
P_2(X) \leq \left[2K_X^3\right] + \left[2K_X^3 - \frac{5(p_g(X) - 1)}{3}\right] + 6 + h^0(X', 2K_{X'} - 2M).
\]
If \(h^0(X', 2K_{X'} - 2M) = 1\), then the above inequality is identical to the inequality (1) in the proposition.

Suppose that \(h^0(X', 2K_{X'} - 2M) \geq 2\). By **Step 3**, \(h^0(X', M_2 - F') > 0\). Thus we have
\[
2\pi^*K_X \geq 2M + M_2 \geq 2M + F'.
\]
Note that by (3.9), (3.10) and the fact that \(d \geq p_g(X) - 2\), we have
\[
((\pi^*K_X)^2 \cdot M) = ((\pi^*K_X) \cdot M^2) + ((\pi^*K_X)|_S \cdot Z|_S) \geq p_g(X) + \frac{1}{2} \left[\frac{2(p_g(X) - 4)}{3}\right] - 2.
\]
The above two inequalities imply that
\[ K_X^3 \geq (\pi^*K_X)^2 \cdot M + \frac{1}{2} \left( \left( \pi^*K_X \right)^2 \cdot F' \right) \]
\[ \geq p_g(X) + \frac{1}{2} \left( \frac{2(p_g(X) - 4)}{3} \right) - \frac{3}{2} \]
\[ \geq \frac{4}{3}p_g(X) - \frac{17}{6}. \]
This is the inequality (2) in the proposition. The proof is completed. □

3.3. A special case. In this subsection, we consider the case when
\[ f_\ast \omega_X = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1} \]
with \( a = p_g(X) - 2 \). Since \( p_g(X) \geq 3 \), we have \( a \geq 1 \).

By Lemma 3.4, \( \Sigma \) is a cone, and is birational to \( \mathbb{F}_a \). Moreover, \( \Sigma \) is normal. Thus \( \tau \) is an isomorphism. Now we have the following commutative diagram:

\[ X' \xrightarrow{\psi_0} \mathbb{F}_a \xrightarrow{p} \mathbb{P}^1 \]

where \( \mathbb{F}_a \to \Sigma \) is the blow-up of the cone singularity \( v \) of \( \Sigma \), \( \psi_0 \) is induced by this blow-up. Let \( s \) be the section on \( \mathbb{F}_a \) with \( s^2 = -a \) and \( l \) be a ruling on \( \mathbb{F}_a \).

**Lemma 3.9.** The rational map \( \psi_0 \) is a morphism. Moreover, \( p \circ \psi_0 = f' \) and
(3.15) \[ |M| = \psi_0^\ast s + al. \]

**Proof.** This is actually implied by the proof of Lemma 3.4. □

Recall that by Lemma 3.3, there is a unique \( \pi \)-exceptional prime divisor \( E_0 \) satisfying the condition therein.

**Lemma 3.10.** There exists a unique prime divisor \( D_0 \) such that

1. \( \text{coeff}_{D_0}(\psi_0^\ast s) = 1 \);
2. \( (D_0 \cdot E_0 \cdot F') = 1 \) and \( (\pi^*K_X) \cdot D_0 \cdot F' = 1 \).

**Proof.** By the abuse of notation, we still denote by \( C \) the general fiber of \( \psi_0 \). By Lemma 3.2 and 3.9, we have
\[ C \equiv M|_{F'} \sim (\psi_0^\ast (s + al))|_{F'} \equiv (\psi_0^\ast s)|_{F'} . \]

Thus by Lemma 3.3 (3), \( (\psi_0^\ast s) \cdot E_0 \cdot F' = (E_0 \cdot C) = 1 \). Since \( \psi_0^\ast s \) is Cartier and \( E_0 \notin \text{Supp}(\psi_0^\ast s) \), for any prime divisor \( D \) with \( \text{coeff}_{D}(\psi_0^\ast s) > 0 \), we have \( \text{coeff}_{D}(\psi_0^\ast s) \geq 1 \) and \( (D \cdot E_0 \cdot F') \) is a non-negative integer. Thus there exists a unique prime divisor \( D_0 \) with \( \text{coeff}_{D_0}(\psi_0^\ast s) = 1 \) such that
\[ (D_0 \cdot E_0 \cdot F') = 1 . \]
Note that $\left(\pi^*K_X\right)|_{F'} = (\pi|_{F'})^*K_F \equiv C + E_0|_{F'}$. Therefore,

$$((\pi^*K_X) \cdot D_0 \cdot F') = (E_0 \cdot D_0 \cdot F') = 1.$$  

The proof is completed. \hfill \Box

**Lemma 3.11.** Let $A$ be an ample Cartier divisor on $\mathbb{F}_a$. For any integer $m > 0$, there exists an integer $c > 0$ and an effective divisor $H_m \sim cm(K_{X'|\mathbb{F}_a} + E_0) + c\psi_0^*A$ such that $E_0 \not\subseteq \operatorname{Supp}(H_m)$, where $K_{X'|\mathbb{F}_a} = K_{X'} - \psi_0^*K_{\mathbb{F}_a}$ and $E_0$ is the $\pi$-exceptional divisor as in Lemma 3.3.

**Proof.** The proof of [7, Claim 4.9] works verbatim in our setting, and we only need to replace $W$, $g$, $\mathbb{F}_a$ and $E_0$ loc. cit. by $X'$, $\psi_0$, $\mathbb{F}_a$ and $E_0$ in our context. \hfill \Box

**Lemma 3.12.** For any nef $\mathbb{Q}$-divisor $L$ on $X'$, we have

$$((3\pi^*K_X - (a - 2)F') \cdot D_0 \cdot L) \geq 0,$$

where $D_0$ is the divisor as in Lemma 3.10.

**Proof.** The proof is just a slight modification of that of [7, Claim 4.10]. By (3.1), (3.2) and Lemma 3.9, we have

$$K_{X'|\mathbb{F}_a} + E_0 = \left(\pi^*K_X + E_\pi\right) + \psi_0^*(2s + (a + 2)t) + E_0$$

(3.16)

$$= \pi^*K_X + 2M - (a - 2)\psi_0^*l + E_\pi + E_0$$

$$= 3\pi^*K_X - (a - 2)\psi_0^*l + E_\pi + E_0 - 2Z.$$

Write $E_\pi + E_0 - 2Z = N_+ - N_-$, where $N_+$ and $N_-$ are both effective $\mathbb{Q}$-divisors with no common irreducible components. By Lemma 3.3 (1), we deduce that $E_0 \not\subseteq \operatorname{Supp}(N_+)$ and $E_0 \not\subseteq \operatorname{Supp}(N_-)$.

Choose an ample divisor $A = at_1l + t_2s$ on $\mathbb{F}_a$, where $t_1 > t_2$ are two positive integers such that $t_2K_X$ is Cartier. Let $m$ be a positive integer such that $mK_X$ is Cartier. By Lemma 3.11, there exists an integer $c > 0$ and an effective divisor $H_m \sim cm(K_{X'|\mathbb{F}_a} + E_0) + c\psi_0^*A$ such that $E_0 \not\subseteq \operatorname{Supp}(H_m)$. Thus we have

$$H_m + cmN_- + ct_2Z$$

$$\sim cm(K_{X'|\mathbb{F}_a} + E_0) + c\psi_0^*A + cmN_- + ct_2Z$$

$$= cm(3\pi^*K_X - (a - 2)\psi_0^*l + E_\pi + E_0 - 2Z + N_-) + c\psi_0^*A + ct_2Z$$

$$= cm(3\pi^*K_X - (a - 2)\psi_0^*l + cmN_+ + c(at_1\psi_0^*l + t_2(\psi_0^*s + Z))$$

$$= cm(3\pi^*K_X - (a - 2)\psi_0^*l + cmN_+ + c(a(t_1 - t_2)\psi_0^*l + t_2(M + Z))$$

$$= cm(3\pi^*K_X - (a - 2)\psi_0^*l + cmN_+ + c(a(t_1 - t_2)\psi_0^*l + t_2\pi^*K_X).$$

Here the first equality is by (3.16), and the last two equalities are by (3.15) and (3.1), respectively. By Lemma 3.9, $\phi_0^*l = F'$. This implies

$$H_m + cmN_- + ct_2Z - cmN_+$$

$$\sim cm\pi^*\left(3K_X - (a - 2)F\right) + c\pi^*\left(a(t_1 - t_2)F + t_2K_X\right).$$
Note that $N_+$ is $\pi$-exceptional. We deduce that $cmN_+$ is contained in the fixed part of $[H_m + cmN_- + ct_2Z]$. In particular, $H_m + cmN_- + ct_2Z - cmN_+$ is effective.

Let $G_m = \frac{1}{cm}(H_m + cmN_- + ct_2Z - cmN_+)$. Since $E_0 \notin \text{Supp}(H_m) \cup \text{Supp}(N_+) \cup \text{Supp}(N_-)$, by Lemma 3.3 (1), $\text{coeff}_{E_0}(G_m) = \frac{t_2}{m}\text{coeff}_{E_0}(Z) = \frac{t_2}{m}$. By Lemma 3.10 (2),

$$\left(\left(G_m - \frac{t_2}{m}E_0\right) \cdot E_0 \cdot F'\right) \geq \mu_m(D_0 \cdot E_0 \cdot F') \geq \mu_m,$$

where $\mu_m = \text{coeff}_{D_0}(G_m)$. Since $E_0$ is $\pi$-exceptional and both $G_m$ and $F'$ are $\pi$-trivial, we have $(G_m \cdot E_0 \cdot F') = 0$. It follows that

$$- \frac{t_2}{m}(E_0^2 \cdot F') \geq \mu_m \geq 0.$$

In particular, $\lim_{m \to \infty} \mu_m = 0$. Thus for any nef $\mathbb{Q}$-divisor $L$ on $X'$, we have

$$\lim_{m \to \infty} (G_m \cdot D_0 \cdot L) = \lim_{m \to \infty} ((G_m - \mu_m D_0) \cdot D_0 \cdot L) \geq 0.$$

By the definition of $G_m$, the above inequality just implies that

$$((3\pi^*K_X - (a - 2)F') \cdot D_0 \cdot L) = \lim_{m \to \infty} (G_m \cdot D_0 \cdot L) \geq 0.$$

The proof is completed. □

**Proposition 3.13.** Under the above assumption, if $K_X - bF$ is nef, then we have

$$K_X^3 \geq \frac{4}{3}a + b - \frac{2}{3}.$$

**Proof.** By (3.1) and (3.15), $\pi^*K_X = aF' + \psi_0^*s + Z$. Thus

$$K_X^3 \geq a (\left(\pi^*K_X\right)^2 \cdot F') + \left(\left(\pi^*K_X\right)^2 \cdot (\psi_0^*s)\right) \geq a + \left(\left(\pi^*K_X\right)^2 \cdot D_0\right),$$

where $D_0$ is the unique divisor as in Lemma 3.10. By Lemma 3.10 (2) and Lemma 3.12, we have

$$0 \leq \left(\left(3\pi^*K_X - (a - 2)F'\right) \cdot D_0 \cdot (\pi^*K_X - bF')\right)$$

$$= 3 \left(\left(\pi^*K_X\right)^2 \cdot D_0\right) - (a + 3b - 2) \left(\pi^*K_X \cdot D_0 \cdot F'\right)$$

$$= 3 \left(\pi^*K_X\right)^2 \cdot D_0 - (a + 3b - 2).$$

That is, $(\pi^*K_X)^2 \cdot D_0 \geq \frac{a + 3b - 2}{3}$. Thus it follows that

$$K_X^3 \geq a + \frac{a + 3b - 2}{3} = \frac{4}{3}a + b - \frac{2}{3}.$$

The proof is completed. □

**Lemma 3.14.** If $a \geq 21$, then $f_*\mathcal{O}_X(2K_X)$ is an ample vector bundle over $\mathbb{P}^1$ of rank 4. In particular, $f_*\mathcal{O}_X(2K_X) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is nef.

**Proof.** Since $P_2(F) = 4$, we deduce that $f_*\mathcal{O}_X(2K_X)$ is locally free of rank 4. Taking a birational modification of $X'$, we may assume that $\psi_0^*s + Z + E_\pi$
is a simple normal crossing $\mathbb{Q}$-divisor. Let $\mu := \pi|_{F'} : F' \to F$. By (3.1) and Lemma 3.9, we have

$$\mu^* K_F \sim (\pi^* K_X)|_{F'} = M|_{F'} + Z|_{F'} = (\psi^*_0 B + Z)|_{F'}.$$ 

Thus we may write $(\psi^*_0 B + Z)|_{F'} = \mu^* B$ for a divisor $B \sim_Q K_F$. By a result of J. Kollár [7, Theorem A.1], $\text{glct}(F) \geq \frac{1}{10}$. Thus the pair $(F, \frac{2}{a} \mu^* B)$ is klt, and $\mu_* \mathcal{O}_{F'}(K_{F'} - \frac{2}{a} \mu^* B) = \mathcal{O}_F$. It follows that

$$h^0 \left( F', K_{F'} + \left[ \mu^* K_F - \frac{2}{a} \mu^* B \right] \right) = h^0 \left( F', 2 \mu^* K_F + K_{F'} - \frac{2}{a} \mu^* B \right)$$

$$= h^0(F, 2K_F) = 4.$$

Therefore, $f'_* \mathcal{O}_{F'}(K_{X'} + [\pi^* K_X - \frac{2}{a}(\psi^*_0 B + Z)])$ is a vector bundle of rank 4 over $\mathbb{P}^1$. Let $F'_1$ and $F'_2$ be two different smooth fibers of $f'$. Consider the divisor

$$D := \pi^* K_X - \frac{2}{a}(\psi^*_0 B + Z) - F'_1 - F'_2.$$

Now the fractional part of $D$ is simple normal crossing. By Lemma 3.9, $D \sim_Q (1 - \frac{2}{a}) \pi^* K_X$ is nef and big. Thus by the Kawamata-Viehweg vanishing theorem, we have

$$h^1(X', K_{X'} + [D]) = 0.$$

By the Leray spectral sequence, the above vanishing implies that

$$h^1 \left( \mathbb{P}^1, f'_* \mathcal{O}_{F'} \left( K_{X'} + \left[ \pi^* K_X - \frac{2}{a}(\psi^*_0 B + Z) \right] \right) \right) \otimes \mathcal{O}_{\mathbb{P}^1}(-2) = 0.$$

We conclude that $f'_* \mathcal{O}_{X'}(K_{X'} + [\pi^* K_X - \frac{2}{a}(\psi^*_0 B + Z)])$ is ample. Since $f'_*$ is left exact, we obtain an inclusion

$$f'_* \mathcal{O}_{X'} \left( K_{X'} + \left[ \pi^* K_X - \frac{2}{a}(\psi^*_0 B + Z) \right] \right) \hookrightarrow f'_* \omega_{X'}^{\otimes 2} = f_* \mathcal{O}_X(2K_X)$$

between two vector bundles over $\mathbb{P}^1$ of the same rank. Therefore, $f_* \mathcal{O}_X(2K_X)$ is also ample. Thus $f_* \mathcal{O}_X(2K_X) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is nef.

**Proposition 3.15.** If $p_g(X) \geq 23$, then $2K_X - F$ is nef. In particular,

$$K_X^3 \geq \frac{4}{3} p_g(X) - \frac{17}{6}.$$

**Proof.** Let $\phi : X \dashrightarrow \mathbb{P}(f_* \mathcal{O}_X(2K_X))$ be the relative bicanonical map of $X$ with respect to $f$. Then $\phi$ is induced by the linear system $|2K_X + tF|$ for a sufficiently large integer $t$. Since $F'$ is a minimal $(1,2)$-surface, $|2K_{F'}|$ is base point free [15, Lemma 2.1]. We deduce that the indeterminacy locus of $\phi$ is vertical with respect to $f$. Let $\pi_1 : X_1 \to X$ be the blow-up of the indeterminacy locus of $\phi$, and let $\phi_1 : X_1 \to \mathbb{P}(f_* \mathcal{O}_X(2K_X))$ be the induced morphism by $\pi_1$. Then we have $\phi^*_1 H \sim 2\pi_1^* K_X - E_1$, where $H$ is a relative hyperplane section of $\mathbb{P}(f_* \mathcal{O}_X(2K_X))$, and $E_1 \geq 0$ is a vertical $\mathbb{Q}$-divisor with respect to the fibration $f_1 : X_1 \to \mathbb{P}^1$. Since $a = p_g(X) - 2 \geq 21$, by Lemma 3.14, $f_* \mathcal{O}_X(2K_X) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is nef. Thus the divisor $\pi_1^*(2K_X - F) - E_1$ is also nef.
Suppose that $A$ is an integral curve on $X$. If $A$ is horizontal with respect to $f_1$, then
\[(\pi_1^*(2K_X - F) \cdot A) \geq (E_1 \cdot A) \geq 0.\]
If $A$ is vertical with respect to $f_1$, then
\[(\pi_1^*(2K_X - F) \cdot A) = 2(\pi_1^*K_X \cdot A) \geq 0.\]
As a result, $2K_X - F$ is nef. By Proposition 3.13, we deduce that
\[K_X^3 \geq \frac{4}{3}p_g(X) - \frac{17}{6}.\]
The whole proof is completed. \qed

4. NOETHER INEQUALITIES FOR 3-FOLDS OF GENERAL TYPE

In this section, we establish three Noether inequalities for 3-folds of general type. Within this section, let $X$ be a minimal 3-fold of general type with $p_g(X) \geq 3$. Let
\[\phi_{K_X} : X \rightarrow \Sigma \subseteq \mathbb{P}^{p_g(X) - 1}\]
be the canonical map of $X$ with the image $\Sigma$.

4.1. The case when $\dim \Sigma = 3$. We start from the easiest case.

**Proposition 4.1.** Suppose that $\dim \Sigma = 3$. Then
\[K_X^3 \geq 2p_g(X) - 6.\]

**Proof.** This is just [20, Theorem 2.4]. \qed

4.2. The case when $\dim \Sigma = 2$. Given a 3-fold $X$ with at worst canonical singularities, there is an associated basket $B_X$ according to Reid [28]. By [18], $B_X$ is uniquely determined by $X$. Recall the Riemann-Roch formula in [28, Corollary 10.3] for $P_2(X)$:
\[P_2(X) = \frac{1}{2}K_X^3 + 3\chi(\omega_X) + l_2(X).\]
Here the correction term
\[l_2(X) = \sum_{Q} \frac{b_Q(r_Q - b_Q)}{2r_Q},\]
where the sum $\sum_Q$ runs over all singularities $Q \in B_X$ with the type $\frac{1}{r_Q}(1,-1,b_Q)$ ($b_Q$ and $r_Q$ are coprime, and $0 < b_Q \leq \frac{1}{2}r_Q$). In particular, $l_2(X) \geq 0$. Moreover, we have the following facts:

1. $l_2(X) = 0$ if and only if $X$ is Gorenstein. Otherwise, $l_2(X) \geq \frac{1}{4}$.
2. $l_2(X) = \frac{1}{4}$ if and only if $X$ has only one non-Gorenstein terminal singularity, and it is of type $\frac{1}{2}(1,-1,1)$.
3. $l_2(X) = \frac{1}{2}$ if and only if one of the following two cases occurs:
   - $X$ has two non-Gorenstein terminal singularities, and they are of type $\frac{1}{2}(1,-1,1)$;
– $X$ has only one non-Gorenstein terminal singularity, and it is of type $cA_1/\mu_2$.

We refer the reader to [28] for more details regarding the above formula.

**Proposition 4.2.** Suppose that $\dim \Sigma = 2$, $p_g(X) \geq 7$ and $p_g(X) \equiv 1 \pmod{3}$. Then

$$(4.3) \quad K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}.$$  

If the equality holds, then $X$ is Gorenstein, and it follows that $X$ is factorial. If the equality does not hold, then

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{17}{6}.$$  

**Proof.** To prove the proposition, we may assume that $K_X^3 < \frac{4}{3}p_g(X) - 2$. By Proposition 2.1, we may further assume that $X$ admits a fibration $f : X \to \mathbb{P}^1$ such that the general fiber $F$ is a minimal $(1,2)$-surface. Thus all results in §3 apply here.

Since $p_g(X) \equiv 1 \pmod{3}$, by Proposition 3.6 and Remark 3.1, we deduce that

$$K_X^3 \geq p_g(X) - 2 + \frac{1}{2} \left\lfloor \frac{2(p_g(X) - 4)}{3} \right\rfloor = \frac{4}{3}p_g(X) - \frac{10}{3}.$$  

Thus the inequality (4.3) holds.

Now suppose that $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$. Since $p_g(X) \equiv 1 \pmod{3}$, $K_X^3$ is an even integer. By Proposition 3.8 (1), we deduce that

$$P_2(X) \leq 4K_X^3 - \frac{5(p_g(X) - 1)}{3} + 7 = \frac{11}{3}p_g(X) - \frac{14}{3}.$$  

On the other hand, by Lemma 3.5, we have $\chi(\omega_X) = p_g(X) - 1$. Thus the Riemann-Roch formula (4.1) for $X$ is equivalent to

$$P_2(X) = \frac{1}{2}K_X^3 + 3(p_g(X) - 1) + l_2(X) = \frac{11}{3}p_g(X) - \frac{14}{3} + l_2(X).$$  

As a result, we have $l_2(X) = 0$. Thus $X$ is Gorenstein. By [19, Lemma 5.1], $X$ is factorial.

In the following, we assume that $K_X^3 > \frac{4}{3}p_g(X) - \frac{10}{3}$. Suppose on the contrary that $K_X^3 < \frac{4}{3}p_g(X) - \frac{17}{6}$. Since $p_g(X) \equiv 1 \pmod{3}$, we deduce that

$$\left\lfloor 2K_X^3 \right\rfloor = \frac{8}{3}p_g(X) - \frac{20}{3}.$$  

By Proposition 3.8 (1) again, we have

$$P_2(X) \leq 2\left\lfloor 2K_X^3 \right\rfloor - \frac{5(p_g(X) - 1)}{3} + 7 = \frac{11}{3}p_g(X) - \frac{14}{3}.$$  

On the other hand, by the Riemann-Roch formula (4.1) and Lemma 3.5, we deduce that

$$P_2(X) = \frac{1}{2}K_X^3 + 3(p_g(X) - 1) + l_2(X) > \frac{11}{3}p_g(X) - \frac{14}{3}.$$
This is a contradiction. Therefore, we have
\[ K_X^3 \geq \frac{4}{3} p_g(X) - \frac{17}{6}. \]
The proof is completed.

\[ \Box \]

**Proposition 4.3.** Suppose that \( \dim \Sigma = 2 \), \( p_g(X) \geq 7 \) and \( p_g(X) \equiv 2 \pmod{3} \). Then
\[ (4.4) \quad K_X^3 \geq \frac{4}{3} p_g(X) - \frac{19}{6}. \]
If the equality holds, then \( X \) has only one non-Gorenstein terminal singularity, and it is of type \( \frac{1}{2}(1, -1, 1) \). If the equality does not hold, then
\[ K_X^3 \geq \frac{4}{3} p_g(X) - \frac{17}{6}. \]

**Proof.** As in the proof of Proposition 4.2, we may assume that \( K_X^3 < \frac{4}{3} p_g(X) - 2 \). Thus all results in §3 apply.

Since \( p_g(X) \equiv 2 \pmod{3} \), we have \( \lceil \frac{2(p_g(X) - 4)}{3} \rceil = \frac{2p_g(X) - 7}{3} \). By Proposition 3.6 and Remark 3.1, we deduce that
\[ K_X^3 \geq p_g(X) - 2 + \frac{2p_g(X) - 7}{6} = \frac{4}{3} p_g(X) - \frac{19}{6}. \]
Thus the inequality (4.4) holds.

Now suppose that \( K_X^3 = \frac{4}{3} p_g(X) - \frac{19}{6} \). Since \( p_g(X) \equiv 2 \pmod{3} \), we know that \( 2K_X^3 = \frac{8}{3}(p_g(X) - 2) - 1 \) is an odd integer. Thus \( X \) is not Gorenstein. By the Riemann-Roch formula (4.1) and Lemma 3.5, we have
\[ P_2(X) = \frac{1}{2} K_X^3 + 3(p_g(X) - 1) + l_2(X) = \frac{11}{3}(p_g(X) - 2) + \frac{11}{4} + l_2(X). \]
On the other hand, by Proposition 3.8 (1), we deduce that
\[ P_2(X) \leq 4K_X^3 - \left\lfloor \frac{5(p_g(X) - 1)}{3} \right\rfloor + 7 = \frac{11}{3}(p_g(X) - 2) + 3. \]
Therefore, we have \( l_2(X) = \frac{1}{4} \). By (4.2), \( X \) has only one non-Gorenstein singularity, and it is of type \( \frac{1}{2}(1, -1, 1) \).

In the following, we assume that \( K_X^3 > \frac{4}{3} p_g(X) - \frac{19}{6} \). Suppose on the contrary that \( K_X^3 < \frac{4}{3} p_g(X) - \frac{19}{6} \). Then \( \frac{8}{3} p_g(X) - \frac{19}{6} < 2K_X^3 < \frac{8}{3} p_g(X) - \frac{17}{3} \).
Since \( p_g(X) \equiv 2 \pmod{3} \), we deduce that
\[ \left\lfloor 2K_X^3 \right\rfloor = \frac{8}{3} p_g(X) - \frac{19}{3} \]
and
\[ \left\lfloor 2K_X^3 - \frac{5(p_g(X) - 1)}{3} \right\rfloor = p_g(X) - 5. \]
Thus by Proposition 3.8 (1), we have
\[ P_2(X) \leq \frac{11}{3} p_g(X) - \frac{13}{3} = \frac{11}{3}(p_g(X) - 2) + 3. \]
On the other hand, now $K_X^3$ is not an integer. Thus $X$ is non-Gorenstein. By (4.2), $l_2(X) \geq \frac{1}{4}$. By the Riemann-Roch formula (4.1) and Lemma 3.5, we have

$$P_2(X) = \frac{1}{2}K_X^3 + 3(p_g(X) - 1) + l_2(X) > \frac{11}{3}(p_g(X) - 2) + 3.$$ 

This is a contradiction. Therefore, we have

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{17}{6}.$$ 

The proof is completed. □

**Proposition 4.4.** Suppose that $\dim \Sigma = 2$, $p_g(X) \geq 7$ and $p_g(X) \equiv 0 \pmod{3}$. Then

$$K_X^3 \geq \frac{4}{3}p_g(X) - 3. \tag{4.5}$$

If the equality holds, then one of the following two cases occurs:

- $X$ has two non-Gorenstein terminal singularities, and they are of the same type $\frac{1}{2}(1, -1, 1)$;
- $X$ has only one non-Gorenstein terminal singularity, and it is of type $cA_1/\mu_2$.

**Proof.** Just as before, we may assume that $K_X^3 < \frac{4}{3}p_g(X) - 2$ so that all results in §3 apply.

Since $p_g(X) \equiv 0 \pmod{3}$, we have $\left\lceil \frac{2(p_g(X) - 4)}{3} \right\rceil = \frac{2p_g(X) - 6}{3}$. By Proposition 3.6 and Remark 3.1, we deduce that

$$K_X^3 \geq p_g(X) - 2 + \frac{p_g(X) - 3}{3} = \frac{4}{3}p_g(X) - 3.$$ 

Thus the inequality (4.5) holds.

Suppose that $K_X^3 = \frac{4}{3}p_g(X) - 3$. Then $K_X^3$ is an odd integer. Thus $X$ is non-Gorenstein by [4, §2.2]. By the Riemann-Roch formula (4.1) and Lemma 3.5,

$$P_2(X) = \frac{1}{2}K_X^3 + 3(p_g(X) - 1) + l_2(X) = \frac{11}{3}p_g(X) - \frac{9}{2} + l_2(X).$$

By Proposition 3.8, we deduce that

$$P_2(X) \leq 4K_X^3 - \left\lceil \frac{5(p_g(X) - 1)}{3} \right\rceil + 7 = \frac{11}{3}p_g(X) - 4.$$ 

As a result, $l_2(X) = \frac{1}{4}$, and the description of the non-Gorenstein singularities on $X$ simply follows from (4.2). □

4.3. **The case when** $\dim \Sigma = 1$. We have the following proposition.

**Proposition 4.5.** Suppose that $\dim \Sigma = 1$ and $p_g(X) \geq 11$. Then

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{8}{3}.$$
Proof. Suppose that the connected component of a general fiber of \( \phi_KX \) is not a \((1,2)\)-surface. By [7, Theorem 4.4 and Theorem 4.5], we have
\[
K_X^3 > 2p_g(X) - 6 > \frac{4}{3}p_g(X) - \frac{8}{3}.
\]

In the following, we assume that the general fiber of \( \phi_KX \) is a \((1,2)\)-surface. Since \( p_g(X) \geq 11 \), by [6, Corollary 3], after replacing \( X \) by another minimal model, we may assume that there is a fibration \( f : X \to B \) over a smooth curve \( B \) with general fiber \( F \) a minimal \((1,2)\)-surface. Moreover, \( \phi_KX \) factors through \( f \). If \( g(B) > 0 \), by [9, Lemma 4.5 (i)], \( g(B) = h^1(X, \mathcal{O}_X) = 1 \) and \( h^2(X, \mathcal{O}_X) = 0 \). Thus by [16, Theorem 1.1], we have
\[
K_X^3 \geq \frac{4}{3} \chi(\omega_X) = \frac{4}{3}p_g(X).
\]

From now on, we assume that \( B \simeq \mathbb{P}^1 \). Since \( f_*\omega_X/\mathbb{P}^1 \) is nef, we may write
\[
f_*\omega_X = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b),
\]
where \( a = p_g(X) - 1 \) and \( 1 \leq b \leq 2 \). Let \( \nu : \tilde{B} \to B \) be a double cover branched along \( 2b \) general points on \( B \). Let \( \tilde{f} : \tilde{X} \to \tilde{B} \) be the base change of \( f \) via \( \nu \). Then it is clear that \( \tilde{X} \) is minimal, and the general fiber \( \tilde{F} \) of \( \tilde{f} \) is a minimal \((1,2)\)-surface. Moreover, we have
\[
\tilde{f}_*\omega_{\tilde{X}} = \mathcal{L} \oplus \mathcal{O}_{\tilde{B}},
\]
where \( \mathcal{L} \) is a line bundle on \( \tilde{B} \) with degree \( 2a + 2b \). Thus the canonical image of \( \tilde{X} \) has dimension two. Denote by \( \mu : \tilde{X} \to X \) the induced double cover. Then we have
\[
K_{\tilde{X}}^3 = 2(K_X + bF)^3 = 2K_X^3 + 6b.
\]
If \( b = 1 \), by the Hurwitz formula, we know that \( \tilde{B} \simeq \mathbb{P}^1 \). Note that \( K_{\tilde{X}} - 2\tilde{F} = \mu^*K_X \) is nef. By Proposition 3.13 and (4.6), we have
\[
K_{\tilde{X}}^3 = \frac{1}{2}K_X^3 - 3 \geq \frac{1}{2} \left( \frac{4}{3}(2a + 2) + 2 - \frac{2}{3} \right) - 3 = \frac{4}{3}p_g(X) - \frac{7}{3}.
\]
If \( b = 2 \), by the Hurwitz formula, we know that \( g(\tilde{B}) = 1 \). Now \( K_{\tilde{X}} - 4\tilde{F} = \mu^*K_X \) is nef. By [16, Remark 2.12] and (4.6), we have
\[
K_{\tilde{X}}^3 = \frac{1}{2}K_X^3 - 6 \geq \frac{1}{2} \left( \frac{4}{3}(2a + 4) + 4 \right) - 6 = \frac{4}{3}p_g(X) - \frac{8}{3}.
\]
Thus the proof is completed. \( \square \)

4.4. Three Noether inequalities. Now we are ready to state the main theorem in this section.

**Theorem 4.6.** Let \( X \) be a minimal 3-fold of general type with \( p_g(X) \geq 11 \).

1. (First Noether inequality) We have the optimal inequality
\[
K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}.
\]
If the equality holds, then \( p_g(X) \equiv 1 \pmod{3} \) and \( X \) is Gorenstein. Moreover, it follows that \( X \) is factorial.
(2) (Second Noether inequality) If \( K_X^3 > \frac{4}{3}p_g(X) - \frac{10}{3} \), then we have the optimal inequality

\[
K_X^3 \geq \frac{4}{3}p_g(X) - \frac{19}{6}.
\]

If the equality holds, then \( p_g(X) \equiv 2 \pmod{3} \). Moreover, \( X \) has only one non-Gorenstein terminal singularity, and it is of type \( \frac{1}{2}(1,-1,1) \).

(3) (Third Noether inequality) If \( K_X^3 > \frac{4}{3}p_g(X) - \frac{10}{6} \), then we have the optimal inequality

\[
K_X^3 \geq \frac{4}{3}p_g(X) - 3.
\]

If the equality holds, then \( p_g(X) \equiv 0 \pmod{3} \). Moreover, one of the following two cases occurs:

- \( X \) has two non-Gorenstein terminal singularities, and they are of type \( \frac{1}{2}(1,-1,1) \);
- \( X \) has only one non-Gorenstein terminal singularity, and it is of type \( cA_1/\mu_2 \).

**Proof.** This is a combination of Proposition 4.1, 4.2, 4.3, 4.4 and 4.5. By the examples in Proposition 5.18 in §5.2, the above three inequalities are all optimal. Thus they are indeed the first three Noether lines. \( \square \)

**Theorem 4.7.** Let \( X \) be a minimal 3-fold of general type with \( p_g(X) \geq 11 \) and \( K_X^3 < \frac{4}{3}p_g(X) - \frac{8}{3} \). Then the following statements hold:

1. \( h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0 \);
2. The canonical image \( \Sigma \subseteq \mathbb{P}^{p_g(X)-1} \) of \( X \) is a non-degenerate surface of degree \( p_g(X) - 2 \), and \( \Sigma \) is smooth when \( p_g(X) \geq 23 \);
3. \( X \) is simply connected.

**Proof.** Since \( p_g(X) \geq 11 \) and \( K_X^3 < \frac{4}{3}p_g(X) - \frac{8}{3} \), by Proposition 4.1 and 4.5, we know that the canonical image of \( X \) is a surface. By Proposition 2.1, we may assume that \( X \) admits a fibration \( f : X \to \mathbb{P}^1 \) with general fiber \( F \) a \((1,2)\)-surface. Then the statement (1) just follows from Lemma 3.5.

By Proposition 2.1, the canonical image \( \Sigma \subseteq \mathbb{P}^{p_g(X)-1} \) of \( X \) is a non-degenerate surface of degree \( p_g(X) - 2 \). Now we have

\[
f_*\omega_X = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)
\]

with \( a \geq b \geq 0 \) and \( a + b = p_g(X) - 2 \). If \( p_g(X) \geq 23 \), by Proposition 3.15, we have \( b > 0 \). By Lemma 3.4, \( \Sigma \) is smooth. Thus the statement (2) holds.

For (3), let \( \alpha : X_1 \to X \) be a resolution of singularities of \( X \) such that \( \alpha \) is an isomorphism over the smooth locus of \( X \), and let \( f_1 : X_1 \to \mathbb{P}^1 \) be the induced fibration with a general fiber \( F_1 \). Recall that \( f \) has a natural section \( \Gamma \). Since \( X \) has only terminal singularities, we conclude that \( \Gamma_1 := \alpha_*^{-1}\Gamma \) is also a section of \( f_1 \). In particular, \( f_1 \) has no multiple fibers. Now we may use the same argument as in the proof of [16, Theorem 4.7] to deduce that

\[
\pi_1(X_1) \simeq \pi_1(\mathbb{P}^1) \simeq \{1\},
\]
i.e., \( X_1 \) is simply connected. Thus by a result of S. Takayama [29, Theorem 1.1], \( X \) is simply connected.

**Remark 4.8.** Note that if two smooth complex projective varieties are birational to each other, then their fundamental groups are isomorphic. Thus the above proof actually shows that any smooth model of \( X \) is simply connected.

5. Threefolds on the Noether lines

In this section, we first classify minimal 3-folds of general type with \( p_g(X) \geq 11 \) which attain the Noether equality in Theorem 4.6 (1). Then we construct examples of minimal 3-folds of general type which attain the three Noether equalities in Theorem 4.6, respectively.

5.1. Classification of 3-folds on the first Noether line. Throughout this subsection, let \( X \) be a minimal 3-fold of general type with \( p_g(X) \geq 11 \) satisfying \( K_X^3 = \frac{4}{3} p_g(X) - \frac{16}{3} \). By Theorem 4.6 (1), \( K_X \) is Cartier and \( p_g(X) = 3m - 2 \) for an integer \( m \geq 5 \). By Theorem 4.7, the canonical image of \( X \) is a surface of degree \( p_g(X) - 2 \). Thus by Proposition 2.1, replacing \( X \) by another minimal model if necessary, we may assume that \( X \) admits a fibration \( f : X \to \mathbb{P}^1 \) with a general fiber \( F \) a \((1,2)\)-surface.

Just as in §3.1, we have the following commutative diagram:

\[
\begin{array}{ccc}
X' & \xrightarrow{\psi} & \Sigma \\
\downarrow{\pi} & & \downarrow{\phi_{K_X}} \\
\mathbb{P}^1 & \xrightarrow{f} & X
\end{array}
\]

Here \( \phi_{K_X} \) is the canonical map of \( X \) with the canonical image \( \Sigma \), \( \pi \) is a birational modification with respect to \( K_X \) as in §1.5.2, \( \psi \) is the morphism induced by \( |M| = \operatorname{Mov}|\pi^*K_X| \), and \( f' = f \circ \pi \). Note that by Proposition 2.1, \( \psi \) has connected fibers.

Let

\[
Z = \pi^*K_X - M, \quad E_{\pi} = K_{X'} - \pi^*K_X.
\]

Since \( X \) is Gorenstein and terminal, both \( Z \) and \( E_{\pi} \) are effective Cartier divisors. Denote by \( C \) a general fiber of \( \psi \) and by \( F' \) a general fiber of \( f' \). Recall that \( f \) admits a section \( \Gamma \) such that \( \Gamma \cap F \) is the unique base point of \( |K_F| \). Note that \( \Gamma \simeq \mathbb{P}^1 \) is a smooth curve.

5.1.1. More properties of \( X \). We recall some results in the proof of Proposition 3.6. Let \( S \in |M| \) be a general member. By Bertini’s theorem, \( S \) is smooth. Let \( \sigma : S \to S_0 \) be the contraction to the minimal model of \( S \). As in the proof of Proposition 3.6, the natural fibration \( \psi|_S : S \to \mathbb{P}^1 \) descends to a fibration \( S_0 \to \mathbb{P}^1 \) with a general fiber \( C_0 = \sigma_*C \).

Let \( E \) be the unique \( \pi \)-exceptional prime divisor as in Lemma 3.3. Denote \( \Gamma_S = E|_S \). By Lemma 3.3, \( \Gamma_S \) is a section of \( \psi|_S \), and we may write

\[
E_{\pi}|_S = \Gamma_S + EV, \quad Z|_S = \Gamma_S + ZV.
\]
Here both $E_V$ and $Z_V$ are effective divisors on $S$, and $(E_V \cdot C) = (Z_V \cdot C) = 0$.

**Lemma 5.1.** The following statements hold:

1. $(K_X \cdot \Gamma) = \frac{p_g(X) - 4}{3}$;
2. $\sigma_p(E_V + Z_V) = 0$;
3. $K_{S_0} \equiv 2(p_g(X) - 2)C_0 + 2\Gamma_{S_0}$, where $\Gamma_{S_0} := \sigma_p \Gamma_S$;
4. $\sigma^* K_{S_0} \sim 2(\pi^* K_X)|_S$;
5. $E_V - Z_V$ is an effective divisor on $S$.

**Proof.** Since $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$, $X$ attains the equality in Proposition 3.6 (2) in which $d = p_g(X) - 2$. Thus by Proposition 3.6, we have

$$(K_X \cdot \Gamma) = K_X^3 - d = \frac{p_g(X) - 4}{3}.$$

The statements (2)–(4) follow directly from Proposition 3.7. For (5), note that

$$E_V - Z_V = E_\pi|_S - Z|_S = (K_X + S)|_S - 2(\pi^* K_X)|_S = K_S - 2(\pi^* K_X)|_S.$$

Since $K_S \geq \sigma^* K_{S_0}$, by (4), there exist an integer $n > 0$ and an effective divisor $D$ on $S$ such that $nE_V \sim D + nZ_V$. By (2), we have $h^0(S, nE_V) = h^0(S_0, O_{S_0}) = 1$. Thus

$$nE_V = D + nZ_V.$$

It follows that $E_V - Z_V = \frac{1}{n}D$ is effective. \qed

Denote $S_X = \pi_* S$. Then we have the following lemma.

**Lemma 5.2.** The linear system $|K_X|$ has no fixed part. Moreover, the surface $S_X \in |K_X|$ is normal with at worst canonical singularities. Moreover, $\pi|_S : S \to S_X$ factors through $\sigma : S \to S_0$.

**Proof.** The proof is just identical to that of [16, Lemma 4.4 and 4.5]. Thus we omit the proof here and leave it to the interested reader. \qed

**Proposition 5.3.** The following statements hold:

1. $\text{Bs}|K_X| = \Gamma$, and $\Gamma$ lies in the smooth locus of $X$;
2. $S_X = S_0$ is smooth;
3. Let $\pi_\Gamma : X_\Gamma \to X$ be the blow-up of $X$ along $\Gamma$. Then $|\pi_\Gamma^* K_X - E_\Gamma|$ is base point free, where $E_\Gamma$ is the $\pi_\Gamma$-exceptional divisor.

**Proof.** By Lemma 5.2, $|K_X|$ has no fixed part. Thus $\text{Bs}|K_X| = \pi(Z \cap S)$. Recall that $Z|_S = \Gamma_S + Z_V$. By Lemma 3.3 (2), $\pi(\Gamma_S) = \Gamma$. Thus to prove (1), we only need to show that $\pi(Z_V) \subseteq \Gamma$.

By Lemma 5.1 (2), $\sigma(Z_V)$ consists of finitely many points on $S_0$. By Lemma 5.2, $S_0$ is the minimal resolution of $S_X$. We deduce that $\pi(Z_V)$ also consists of finitely many points on $S_X$. Suppose that there is a point $p \in \pi(Z_V)$ with $p \notin \Gamma$. Then we may write

$$Z_V = Z_1 + Z_2,$$
where $Z_1$ and $Z_2$ are effective divisors on $S$, $\pi(Z_1) = p$ and $p \notin \pi(Z_2)$. Now by Lemma 5.1 (3) and (4), we know that

$$\langle \pi^* K_X \rangle |_S = M|_S + Z|_S \equiv (p_g(X) - 2)C + \Gamma_S + Z_1 + Z_2.$$  

However, since $(Z_1 \cdot C) = 0$ and $Z_1$ does not intersect $Z_2$ or $\Gamma_S$, we deduce that

$$(Z_1 \cdot ((p_g(X) - 2)C + \Gamma_S + Z_2)) = 0.$$

This is a contradiction, because $(\pi^* K_X)|_S$ is nef and big, thus 1-connected. As a result, $\pi(Z_V) \subset \Gamma$ and $Bs[K_X] = \Gamma$.

Take any fiber $T$ of $f$. Then $(\Gamma \cdot T) = 1$. Since every irreducible component of $T$ is Cartier by Proposition 4.2, there is exactly one irreducible component $T_0$ of $T$ such that $\Gamma \cap T_0 \neq \emptyset$. Moreover, $(\Gamma \cdot T_0) = 1$ and $\text{coeff}_{T_0}(T) = 1$. It implies that $T$ is smooth at the point $\Gamma \cap T_0$, so is $X$. Thus (1) is proved.

To prove (2), suppose that $S_X$ is singular. By Bertini’s theorem and (1), the singular locus of $S_X$ is contained in $\Gamma$. Let $p \in \Gamma$ be a singularity on $S_X$. By Lemma 5.2, we have the minimal resolution $\sigma_0 : S_0 \to S_X$ such that $K_{S_0} = \sigma_0^* K_{S_X}$. Let $E_p$ be the exceptional divisor on $S_0$ lying over $p$, and let $\Gamma_{S_0} = (\sigma_0)^{-1}_* \Gamma$ be the strict transform of $\Gamma$. Then we have

$$(K_{S_0} \cdot E_p) = 0, \quad (\Gamma_{S_0} \cdot E_p) > 0.$$  

On the other hand, by Lemma 5.1 (3), we have

$$(K_{S_0} \cdot E_p) = 2(p_g(X) - 2)(C_0 \cdot E_p) + 2(\Gamma_{S_0} \cdot E_p) \geq 2(\Gamma_{S_0} \cdot E_p).$$

This is a contradiction. As a result, $S_X = S_0$ is smooth, and (2) is proved.

For (3), let $\pi_\Gamma : X_\Gamma \to X$ be the blow-up of $X$ along $\Gamma$ with $E_\Gamma$ the $\pi_\Gamma$-exceptional divisor. By (1) and (2), we have

$$K_{X_\Gamma} = \pi_\Gamma^* K_X + E_\Gamma, \quad \pi_\Gamma^* K_X = M_\Gamma + E_\Gamma,$$

where $M_\Gamma = \pi_\Gamma^* K_X - E_\Gamma$. Let $S_\Gamma \in |M_\Gamma|$ be a general member. It suffices to show that the restricted linear system $|M_\Gamma||_{S_\Gamma}$ on $S_\Gamma$ is base point free. Suppose on the contrary that $Bs(|M_\Gamma||_{S_\Gamma}) \neq \emptyset$. Since $S_\Gamma$ is the blow-up of $S_X$ along the divisor $\Gamma$, we know that $\pi_\Gamma|_{S_\Gamma} : S_\Gamma \to S_X = S_0$ is an isomorphism. In particular, $S_\Gamma$ admits a fibration $S_\Gamma \to \mathbb{P}^1$ and $(\pi_\Gamma|_{S_\Gamma})^* \Gamma = E_\Gamma|_{S_\Gamma}$. Now

$$M_\Gamma|_{S_\Gamma} = (\pi_\Gamma^* K_X)|_{S_\Gamma} - E_\Gamma|_{S_\Gamma} = (\pi_\Gamma|_{S_\Gamma})^* (K_X|_{S_X} - \Gamma).$$

By the adjunction formula and Lemma 5.1 (3), we also have

$$K_X|_{S_X} = \Gamma = \frac{1}{2} K_{S_0} - \Gamma \equiv (p_g(X) - 2) C_0.$$  

Thus $Bs(|M_\Gamma||_{S_\Gamma})$ is a vertical divisor with respect to the fibration $S_\Gamma \to \mathbb{P}^1$. On the other hand, by (1), $Bs(|M_\Gamma||_{S_\Gamma}) = (Bs|M_\Gamma|) \cap S_\Gamma \subseteq E_\Gamma|_{S_\Gamma}$. This is a contradiction. As a result, $|M_\Gamma||_{S_\Gamma}$ is base point free, and (3) is proved. □
5.1.2. The relative canonical model $X_0$. Let

$$\epsilon : X \to X_0$$

be the contraction from $X$ onto its relative canonical model $X_0$ over $\mathbb{P}^1$. Let $f_0 : X_0 \to \mathbb{P}^1$ be the induced fibration. Let $\Gamma_0 = \epsilon_* \Gamma$. Then $\Gamma_0$ is a section of $f_0$. Let

$$\pi_0 : X'_0 \to X_0$$

be the blow-up along $\Gamma_0$ with the exceptional divisor $E_0$.

Lemma 5.4. We have $\text{Bs}|K_{X_0}| = \Gamma_0$, and $\Gamma_0$ lies in the smooth locus of $X_0$. Moreover, $|\pi^*_0 K_{X_0} - E_0|$ is base point free.

Proof. Since $\epsilon^* K_{X_0} = K_X$, by Proposition 5.3 (1), $\epsilon^{-1}(\text{Bs}|K_{X_0}|) = \text{Bs}|K_X| = \Gamma$. Thus $\text{Bs}|K_{X_0}| = \Gamma_0$. Moreover, $\epsilon^{-1}(\Gamma_0) = \Gamma$. Now $\epsilon|\Gamma : \Gamma \to \Gamma_0$ is an isomorphism. By Zariski’s main theorem and the fact that $\epsilon$ is birational, we deduce that $\epsilon$ is an isomorphism over a neighbourhood of $\Gamma$. Thus by Proposition 5.3 (1) again, $\Gamma_0$ lies in the smooth locus of $X_0$. The rest part of the lemma simply follows from Proposition 5.3 (3). \qed

Let $M_0 = \pi^*_0 K_{X_0} - E_0$. By Lemma 5.4, we obtain a morphism

$$\phi : X'_0 \to \Sigma$$

induced by the linear system $|M_0|$, and $\phi$ has connected fibers. By the abuse of notation, we still denote by $C$ a general fiber of $\phi$. Then by Lemma 3.2, $g(C) = 2$. By Lemma 3.3 (3), we have $((\pi^*_0 K_{X_0}) \cdot C) = 1$. Thus $(E_0 \cdot C) = ((K_{X'_0} - \pi^*_0 K_{X_0}) \cdot C) = 1$. In particular, $\phi|_{E_0} : E_0 \to \Sigma$ is birational. Let $f'_0 := f_0 \circ \pi_0 : X'_0 \to \mathbb{P}^1$ be the induced fibration with a general fiber $F'_0$. Then there is a natural $\mathbb{P}^1$-bundle structure $f'_0|_{E_0} : E_0 \to \mathbb{P}^1$ on $E_0$. Thus we may assume that $E_0$ is isomorphic to the Hirzebruch surface $\mathbb{F}_e$ for some $e \geq 0$. Since $\deg \Sigma = p_g(X) - 2$, by [24, Theorem 7], either $\Sigma \simeq \mathbb{F}_e$ or $\Sigma$ is a cone over a smooth rational curve of degree $p_g(X) - 2$. Let

$$r : \mathbb{F}_e \to \Sigma$$

be the blow-up of the cone singularity of $\Sigma$ if $\Sigma$ is a cone, or be the identity morphism if $\Sigma$ is smooth. Let $p : \mathbb{F}_e \to \mathbb{P}^1$ be the natural projection.

Lemma 5.5. The rational map

$$\varphi := r^{-1} \circ \phi : X'_0 \dashrightarrow \mathbb{F}_e$$

is a morphism. In particular, $f'_0 = p \circ \varphi$.

Proof. The proof is the same as that for Lemma 3.9. Thus we leave it to the interested reader. \qed

Lemma 5.6. The morphism $\varphi$ is flat with all fibers integral.
Lemma 5.7. Let $E = \varphi_* O_{X_0'}(2E_0)$. Then $E$ is a locally free sheaf of rank two on $F_e$.

Proof. Take any fiber $C$ of $\varphi$. Since a general fiber of $\varphi$ is of genus 2, by Lemma 5.6, it follows that $C$ is an integral curve of arithmetic genus 2. We deduce that $h^1(C, O_C) = 2$. By Theorem 4.6 (1) and Lemma 5.4, $X_0'$ is Cohen-Macaulay and the dualizing sheaf $\omega_{X_0'}$ is invertible. Since $C$ is a fiber of $\varphi$ and $\Sigma$ is smooth, $C$ is Cohen-Macaulay and the dualizing sheaf $\omega_C = \omega_{X_0'}|C$ is invertible. We deduce that

$$h^0(C, K_C) = h^0(C, \omega_C) = h^1(C, O_C) = 2,$$

where the last equality follows from the Serre duality. On the other hand, note that $K_{X_0'} = \pi_0^* K_{X_0} + E_0 = M_0 + 2E_0$. Thus $h^0(C, 2E_0|C) = h^0(C, K_{X_0'}|C) = h^0(C, K_C) = 2$. By Grauert’s theorem [13, III, Corollary 12.9], the result follows. □

5.1.3. Explicit description of $X_0'$. Let $Y = \mathbb{P}(E)$ be the $\mathbb{P}^1$-bundle over $F_e$, and denote by $\varphi : Y \to F_e$ the natural projection. Since all fibers of $\varphi$ are integral, by [3, Theorem 3.3], $|K_W| = |2E_0|_W$ is base point free for any fiber $W$ of $\varphi$. Thus we obtain a morphism

$$\rho : X_0' \to Y$$

which is just the relative canonical map of $X_0'$ over $F_e$. Since a general fiber $C$ of $\varphi$ is of genus 2, we deduce that $\rho$ is a finite morphism of degree two. Let $E_Y = \rho(E_0)$. Then $E_Y$ is a section of $q$. Thus we have the following
commutative diagram

$\begin{array}{ccccccccccc}
X' & \xrightarrow{\rho} & Y \\
\downarrow{\pi_0} & \downarrow{\varphi} & \downarrow{j} \\
X_0 & \xrightarrow{\varphi} & \mathbb{F}_e \\
\downarrow{f_0} & \downarrow{p} & \downarrow{\mathbb{P}^1} \\
\end{array}$

where $j : \mathbb{F}_e \to Y$ corresponds to the section $E_Y$. Since $\rho^*E_Y = 2E_0$, we have $E = \varphi^*(\rho^*O_Y(E_Y)) = q_*O_Y(E_Y)$. Thus $E_Y$ is just the relative hyperplane section of $Y$.

From now on, by the abuse of notation, we identify $E_0$ and $E_Y$ with $\mathbb{F}_e$ under the isomorphism $\varphi|_{E_0}$ and $q|_{E_Y}$, respectively. Denote by $l$ a ruling on $\mathbb{F}_e$ and by $s$ the section on $\mathbb{F}_e$ with $s^2 = -e$. Recall that $r : \mathbb{F}_e \to \Sigma$ is induced by the linear system $|s + (e + k)l|$ on $\mathbb{F}_e$ for some $k \geq 0$ with $e + 2k = p_g(X) - 2$. Thus we have

$$(5.1) \quad M_0 = \varphi^* (s + (e + k)l),$$

In particular,

$$(5.2) \quad M_0|_{E_0} = s + (e + k)l.$$

By Lemma 5.1 (1), we have

$$(5.3) \quad (K_{X_0} \cdot \Gamma_0) = (K_X \cdot \Gamma) = \frac{p_g(X) - 4}{3} = e + 2k - \frac{2}{3}.$$

Thus $(\pi_0^* K_{X_0})|_{E_0} \sim \frac{e + 2k - 2}{3} l$. It follows that

$$(5.4) \quad E_0|_{E_0} = (\pi_0^* K_{X_0} - M_0)|_{E_0} \sim -s - \frac{2e + k + 4}{3} l.$$

For any fiber $C$ of $\varphi$, $\varphi|_C : C \to \varphi(C) \cong \mathbb{P}^1$ is just the canonical map of $C$. Note that $2E_0|_C \in |K_C|$. Thus we have $\rho^*E_Y = 2E_0$ and $\rho|_{E_0} : E_0 \to E_Y$ is an isomorphism. We deduce that

$$(5.5) \quad E_Y|_{E_Y} \sim -2s - \frac{4e + 2k + 4}{3} l.$$

Push forward by $q$ on the following short exact sequence

$$0 \to O_Y \to O_Y(E_Y) \to O_{E_Y}(E_Y) \to 0.$$ 

Since $R^1 q_* O_Y = 0$, we obtain

$$(5.6) \quad 0 \to O_{\mathbb{F}_e} \to \mathcal{E} \to j^*O_{E_Y}(E_Y) \to 0.$$ 

Since $j : \mathbb{F}_e \to E_Y$ is an isomorphism, by (5.5), we know that

$$(5.7) \quad \det \mathcal{E} = j^*O_{E_Y}(E_Y) = O_{\mathbb{F}_e} \left( -2s - \frac{4e + 2k + 4}{3} l \right).$$

Furthermore, since $K_{\mathbb{F}_e} = -2s - (e + 2)l$, we deduce that

$$(5.8) \quad K_Y \sim -2E_Y - q^* \left( 4s + \frac{7e + 2k + 10}{3} l \right).$$
Let $B$ be the branch locus of $\rho$. Since $X'_0$ has at worst canonical singularities, $B$ is a reduced divisor on $Y$. Note that $E_Y$ is contained in $B$. Thus we may write

$$B = E_Y + B',$$

where $E_Y \subseteq B'$. Since $\rho$ is a double cover, there is a divisor $L$ on $Y$ such that $B \sim 2L$.

**Lemma 5.8.** We have

1. $L \sim 3E_Y + 5q^*(s + \frac{2e + k + 2}{3}l)$;
2. $B' \sim 5E_Y + 10q^*(s + \frac{2e + k + 2}{3}l)$;
3. $B' \cap E_Y = \emptyset$.

**Proof.** By (5.1), we have $K_{X'_0} = \varphi^*(s + (e + k)l) + 2E_0$. Note that $\rho^*E_Y = 2E_0$. From (5.8), we deduce that $\rho^*L \sim K_{X'_0} - \rho^*K_Y \sim \rho^* \left(3E_Y + 5q^*(s + \frac{2e + k + 2}{3}l)\right)$. Since Pic($Y$) is torsion-free, we obtain the linear equivalence in (1). Since $B' \sim 2L - E_Y$, it is clear that (2) just follows from (1). Finally, by (2) and (5.5), we deduce that $B'|_{E_Y} \sim 0$. Since $E_Y \not\subseteq B'$, (3) follows.

**Lemma 5.9.** The short exact sequence (5.6) splits. As a result,

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}_e} \oplus \mathcal{O}_{\mathbb{P}_e} \left(-2s - \frac{4e + 2k + 4}{3}l\right).$$

**Proof.** The proof is identical to that of [16, Lemma 5.8]. Thus we leave it to the interested reader.

**Lemma 5.10.** We have

$$f_*\omega_X = (f_0)_*\omega_{X_0} = \mathcal{O}_{\mathbb{P}_1}(k) \oplus \mathcal{O}_{\mathbb{P}_1}(e + k).$$

In particular, $\varphi$ coincides with the relative canonical map of $X_0$ over $\mathbb{P}_1$.

**Proof.** By (5.8) and Lemma 5.8, we have

$$\varphi_*\omega_{X'_0} = \omega_Y \oplus \omega_Y(L) = \omega_Y \oplus \mathcal{O}_Y (E_Y + q^*(s + (e + k)l)),$$

Since $q_*\omega_Y = 0$ and $\mathcal{E} = q_*\mathcal{O}_Y(E_Y)$, by Lemma 5.9 and the projection formula, we deduce that

$$\varphi_*\omega_{X'_0} = \mathcal{O}_{\mathbb{P}_e} (s + (e + k)l) \oplus \mathcal{O}_{\mathbb{P}_e} \left(-s - \frac{e - k + 4}{3}l\right).$$

Thus it follows that

$$(f_0)_*\omega_{X_0} = (f'_0)_*\omega_{X'_0} = p_* \mathcal{O}_{\mathbb{P}_e} (s + (e + k)l) = \mathcal{O}_{\mathbb{P}_1}(k) \oplus \mathcal{O}_{\mathbb{P}_1}(e + k).$$

Note that $\mathbb{P}_e = \mathbb{P}((f_0)_*\omega_{X_0})$. Thus $\varphi$ coincides with the relative canonical map of $X_0$ over $\mathbb{P}_1$.

**Lemma 5.11.** If $p_\delta(X) \geq 23$, then $K_{X_0}$ is ample.
Proof. By Theorem 4.7 (2), the canonical image $\Sigma \simeq \mathbb{F}_e$ is smooth. Thus by Lemma 3.4 and 5.10, $k \geq 1$. Let $F_0$ be a general fiber of $f_0$. We have

$$\pi_0^*(K_{X_0} - F_0) = M_0 + E_0 - \pi_0^*F_0 = \varphi^*(s + (e + k - 1)t) + E_0.$$ 

By (5.3), we have

$$(\pi_0^*(K_{X_0} - F_0)|_{E_0} = \frac{e + 2k - 5}{3}l.$$ 

Since $e + 2k = p_g(X) - 2 \geq 21$, we deduce that $(\pi_0^*(K_{X_0} - F_0)|_{E_0}$ is nef. Note that $\varphi^*(s + (e + k - 1)t)$ is nef and $E_0$ is irreducible. We conclude that $K_{X_0} - F_0$ is nef. On the other hand, since $K_{X_0}$ is $f_0$-ample, $K_{X_0} + tF_0$ is ample for a sufficiently large integer $t$. Thus $K_{X_0} = \frac{1}{t+1}(K_{X_0} - F_0) + \frac{1}{t+1}(K_{X_0} + tF_0)$ is ample. The proof is completed. \qed

Now we are ready to state the classification theorem for 3-folds on the (first) Noether line.

**Theorem 5.12.** Let $X$ be a minimal 3-fold of general type on the Noether line with $p_g(X) = 3m - 2 \geq 11$ and with a fibration $f : X \to \mathbb{P}^1$ such that a general fiber of $f$ is a $(1,2)$-surface. Let $X_0$ be the relative canonical model of $X$ over $\mathbb{P}^1$ with the fibration $f_0 : X_0 \to \mathbb{P}^1$. Then $B_s|_{K_{X_0}} = \Gamma_0$ is a section of $f_0$. Let $\pi_0 : X_0' \to X_0$ be the blow-up along $\Gamma_0$. Then the induced fibration $f_0' : X_0' \to \mathbb{P}^1$ is factorized as

$$f_0' : X_0' \xrightarrow{p} Y \xrightarrow{q} \mathbb{F}_e \xrightarrow{p} \mathbb{P}^1$$

with the following properties:

(i) the Hirzebruch surface $\mathbb{F}_e$ is isomorphic to $\mathbb{P}(f_0^*\omega_X)$;

(ii) $q : Y = \mathbb{P}(\mathcal{O}_{\mathbb{F}_e} \oplus \mathcal{O}_{\mathbb{F}_e}(-2s - (m + e)t)) \to \mathbb{F}_e$ is a $\mathbb{P}^1$-bundle, where $s$ is a section on $\mathbb{F}_e$ with $s^2 = -e$ and $t$ is a ruling on $\mathbb{F}_e$;

(iii) $\rho : X_0' \to Y$ is a flat double cover with the branch locus $B = B_1 + B_2$, where $B_1$ is the relative hyperplane section of $Y$, $B_2 \sim 5B_1 + 5(m + e)q^*t + 10q^*s$ and $B_1 \cap B_2 = \emptyset$.

Moreover, if $p_g(X) \geq 23$, then $X_0$ is the canonical model of $X$.

**Proof.** Note that $\frac{4e + 2k + 1}{3} = \frac{p_g(X) + 2}{3} + e = m + e$. Thus the classification is a combination of Lemma 5.4, 5.8, 5.9 and 5.10. If $p_g(X) \geq 23$, by Lemma 5.11, $K_{X_0}$ is ample. Thus $X_0$ is the canonical model of $X$. \qed

### 5.2. Examples of threefolds on the three Noether lines

In this subsection, we construct 3-folds of general type that satisfy each Noether equality in Theorem 4.6.

Let $\mathbb{F}_e$ be the Hirzebruch surface for some $e \geq 3$. Denote by $t$ a ruling on $\mathbb{F}_e$ and by $s$ the section of $\mathbb{F}_e$ over $\mathbb{P}^1$ with $s^2 = -e$. Fix an integer $a \geq 2e$. Let $D = 2s + at$, and let

$$p : Y := \mathbb{P}((\mathcal{O}_{\mathbb{F}_e} \oplus \mathcal{O}_{\mathbb{F}_e}(-D)) \to \mathbb{F}_e$$
be the $\mathbb{P}^1$-bundle over $\mathbb{F}_e$. Denote by $V$ the effective relative hyperplane section of $Y$. Then we have

\begin{equation}
K_Y = -2V + p^*(K_{\mathbb{F}_e} - D) = -2V - p^*(4s + (a + e + 2)l).
\end{equation}

**Lemma 5.13.** The linear system $|V + p^*D|$ is base point free. Moreover, a general member in $|V + p^*D|$ does not intersect $V$.

**Proof.** Since $a \geq 2e$, it is clear that $|D|$ is base point free. We only need to prove that the restricted linear system $|V + p^*D||_V$ on $V$ is base point free. By the definition of $V$, we have $\mathcal{O}_V(V) = \mathcal{O}_V(-p^*D)$. Moreover, using the spectral sequence twice, we deduce that

\[ h^1(Y, p^*D) = h^1(\mathbb{F}_e, D) = h^1(\mathbb{F}_e, O_{\mathbb{P}^1}(a - 2e) \oplus O_{\mathbb{P}^1}(a - e) \oplus O_{\mathbb{P}^1}(a)) = 0. \]

Therefore, the restriction map

\[ H^0(Y, V + p^*D) \to H^0(V, V + p^*D) = H^0(V, \mathcal{O}_V) \]

is surjective, which implies that $|V + p^*D||_V$ is base point free. Note that $\mathcal{O}_V(V + p^*D) = \mathcal{O}_V$. Thus a general member in $|V + p^*D|$ will not intersect $V$. \qed

Fix another integer $b$ with $2b \geq 5a$. Choose a general member $H \in |5V + p^*(10s + 2bl)| = |5V + 5p^*D + (2b - 5a)p^*l|$. By Bertini’s theorem and Lemma 5.13, $H$ is smooth, and the divisor $B = H + V$ is simple normal crossing. Let

\[ \rho : X' \to Y \]

be the double cover branched along $B$. Set $\psi := p \circ \rho : X' \to \mathbb{F}_e$. By [22, Corollary 2.31 (3) and Proposition 5.20 (3)], we deduce that $X'$ has at worst canonical singularities. Moreover, if $2b = 5a$, by Lemma 5.13, $H \cap V = \emptyset$, and $X'$ is smooth.

Denote $L = 3V + p^*(5s + bl)$. Then $2L \sim B$. By (5.9), we have

\[ K_{X'} \sim p^*(K_Y + L) = p^*V + p^*(s + (b - a - e - 2)l). \]

For simplicity, we denote $N = s + (b - a - e - 2)l$. Since $2b \geq 5a$ and $a \geq 2e \geq 6$, we have

\[ b - a - e - 2 \geq \frac{3}{2}a - e - 2 \geq 2e - 2. \]

Thus $N$ is ample. Recall that $V$ is contained in the branch locus of $\rho$. We may write $p^*V = 2E$, where $E \simeq \mathbb{F}_e$ is a section of $\psi$. Let

\[ A = K_{X'} - E \sim E + \rho^*N. \]

**Lemma 5.14.** The divisor $A$ is nef and big. Moreover,

\[ A^3 = 3b - \frac{7}{2}a - 4e - 6. \]
Proof. Note that $\rho|_E : E \to V$ is an isomorphism, both of which are isomorphic to $\mathbb{P}_e$. Since $\mathcal{O}_E(2E) = (\rho|_E)^*\mathcal{O}_V(V)$, under the above isomorphism, we have
\begin{equation}
E|_E \equiv \frac{1}{2} V|_V \equiv -s - \frac{1}{2}a t.
\end{equation}
Thus
\begin{equation}
A|_E \equiv E|_E + (\rho^*N)|_E \equiv \left(b - \frac{3}{2}a - e - 2\right) l.
\end{equation}
Since $b - \frac{3}{2}a - e - 2 \geq 0$, we deduce that $A|_E$ is nef.

Let $\Gamma$ be an integral curve on $X'$. If $\Gamma \subseteq E$, by the nefness of $A|_E$, we have $(A\cdot\Gamma) \geq 0$. If $\Gamma \not\subseteq E$, by the ampleness of $N$, we have $(A\cdot\Gamma) \geq ((\rho^*N)\cdot\Gamma) \geq 0$. Thus $A$ is nef.

By (5.11), $(A^2\cdot E) = (A|_E)^2 = 0$. Thus
\begin{equation}
A^3 = (A^2\cdot (\rho^*N)) = (E^2\cdot (\rho^*N)) + 2 (E\cdot (\rho^*N)^2).
\end{equation}
By (5.10), we have
\begin{equation}
(E^2\cdot (\rho^*N)) = \left(-s - \frac{1}{2}a t\right) (s + (b - a - e - 2)t) = -b + \frac{3}{2}a + 2e + 2,
\end{equation}
and
\begin{equation}
(E\cdot (\rho^*N)^2) = N^2 = (s + (b - a - e - 2)t)^2 = 2b - 2a - 3e - 4.
\end{equation}
Put the above equalities together, and it follows that
\begin{equation}
A^3 = 3b - \frac{7}{2}a - 4e - 6.
\end{equation}
Since $3b - \frac{7}{2}a - 4e - 6 \geq 0$, we conclude that $A$ is big. \qed

Lemma 5.15. Let $\Gamma$ be an integral curve on $X'$. Then $(A\cdot \Gamma) = 0$ if and only if $\Gamma$ is a ruling on $E$. Moreover, if $\Gamma$ is a ruling on $E$, then $(K_{X'}\cdot \Gamma) = (E\cdot \Gamma) = -1$.

Proof. Suppose that $\Gamma$ is a ruling on $E$. By (5.11), $(A\cdot \Gamma) = 0$. By (5.10), $(E\cdot \Gamma) = (E|_E \cdot \Gamma) = -1$. Thus $(K_{X'}\cdot \Gamma) = (A\cdot \Gamma) + (E\cdot \Gamma) = -1$.

Now suppose that $(A\cdot \Gamma) = 0$. If $\Gamma$ is vertical with respect to $\psi$, then $(A\cdot \Gamma) = (E\cdot \Gamma) > 0$. Thus $\Gamma$ is horizontal with respect to $\psi$. Note that $N$ is ample, which implies that $(E\cdot \Gamma) = (A\cdot \Gamma) - ((\rho^*N)\cdot \Gamma) < 0$. We conclude that $\Gamma \subseteq E$. By (5.11) again, we deduce that $\Gamma$ is a ruling on $E$. \qed

Lemma 5.16. We have
\begin{equation}
p_g(X') = 2b - 2a - 3e - 2.
\end{equation}
Proof. Recall that $K_{X'} \sim \rho^*(V + p^*N)$. By the projection formula, we have
\begin{equation}
\rho_*\omega_X = \mathcal{O}_Y(V + p^*N) \oplus \mathcal{O}_Y(V + p^*N - L).
\end{equation}
Note that \( h^0(Y, V + p^*N - L) = 0 \). Thus \( p_g(X') = h^0(Y, V + p^*N) \). By the projection formula again, we have
\[
p_* \mathcal{O}_Y(Y, V + p^*N) = \mathcal{O}_{\mathbb{P}_e}(N) \oplus \mathcal{O}_{\mathbb{P}_e}(N - D).
\]
Since \( h^0(\mathbb{P}_e, N - D) = 0 \), we deduce that \( p_g(X') = h^0(\mathbb{P}_e, N) \). Finally, by the projection formula once more, we deduce that
\[
h^0(\mathbb{P}_e, N) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b - a - e - 2)) + h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b - a - 2e - 2)) = 2b - 2a - 3e - 2.
\]
Thus the result follows. \( \square \)

By Lemma 5.15, \( K_{X'} \) is not nef. Let \( \pi : X' \to X \) be the contraction of a \( K_{X'} \)-negative extremal ray \( R \).

**Lemma 5.17.** The morphism \( \pi \) is just the contraction of the rulings on \( E \). Moreover, \( X \) has at worst canonical singularities, and \( K_X \) is nef and big. In particular, we have
\[
(5.12) \quad K_X^3 = 3b - \frac{7}{2}a - 4e - 6,
\]
and
\[
(5.13) \quad p_g(X) = 2b - 2a - 3e - 2.
\]

**Proof.** Let \( \Gamma \) be an integral curve which generates the \( K_{X'} \)-negative extremal ray \( R \), i.e., \( R = \mathbb{R}_{\geq 0}[\Gamma] \), where \([\Gamma]\) is the numerical class of \( \Gamma \). Since \( K_{X'} = A + E \) and \( A \) is nef by Lemma 5.14, we have \( (E \cdot \Gamma) < 0 \). Thus \( \Gamma \subseteq E \).

Since \( E \cong \mathbb{P}_e \), we may write \( \Gamma \equiv ms + nl \), where \( m, n \geq 0 \). Thus \([ms] + [nl] = [\Gamma] \in R \). Since \( R \) is extremal, we have \([ms] \in R \) and \([nl] \in R \).

By (5.10) and (5.11),
\[
(K_{X'} \cdot s) = (A|_E \cdot s) + (E|_E \cdot s) = b - 2a - 2 \geq \frac{1}{2}a - 2 \geq e - 2 > 0.
\]
Thus \( m = 0 \), and \( R \) is generated by \( l \). As a result, \( \pi \) is just the contraction of the rulings on \( E \).

Since \( \pi \) is a divisorial contraction, by [22, Corollary 3.43 (3)], \( X \) has at worst canonical singularities. Thus we may write
\[
K_{X'} = \pi^* K_X + \lambda E,
\]
where \( \lambda \geq 0 \). By Lemma 5.15, \( -\lambda = \lambda(E \cdot l) = (K_{X'} \cdot l) = -1 \). Thus \( \lambda = 1 \).

It implies that \( \pi^* K_X = K_{X'} - E = A \). By Lemma 5.14, we conclude that \( K_X \) is nef and big. The equalities (5.12) and (5.13) follow immediately from Lemma 5.14 and 5.16. \( \square \)

**Proposition 5.18.** Let \( e \geq 3 \) and \( k \geq 0 \) be two integers.

1. There exists a minimal 3-fold \( X_{e,k} \) of general type with \( p_g(X_{e,k}) = 3e + 6k - 2 \) and \( K_{X_{e,k}}^3 = 4e + 8k - 6 \). In particular,
\[
K_{X_{e,k}}^3 = \frac{4}{3} p_g(X_{e,k}) - \frac{10}{3}.
\]
(2) There exists a minimal 3-fold \( X_{e,k} \) of general type with \( p_g(X_{e,k}) = 3e + 6k + 2 \) and \( K^3_{X_{e,k}} = 4e + 8k - \frac{1}{2} \). In particular,
\[
K^3_{X_{e,k}} = \frac{4}{3} p_g(X_{e,k}) - \frac{19}{6}.
\]

(3) There exists a minimal 3-fold \( X_{e,k} \) of general type with \( p_g(X_{e,k}) = 3e + 6k \) and \( K^3_{X_{e,k}} = 4e + 8k - 3 \). In particular,
\[
K^3_{X_{e,k}} = \frac{4}{3} p_g(X_{e,k}) - 3.
\]

Proof. For the 3-fold in (1), let \( a = 2(e + k) \) and \( b = 5(e + k) \). By Lemma 5.17, the above construction for \( a \) and \( b \) gives rise to a minimal 3-fold \( X \) with \( K^3_X = 4e + 8k - 6 \) and \( p_g(X) = 3e + 6k - 2 \). For the 3-fold in (2) (resp. (3)), we may take \( a = 2(e + k) + 1 \) and \( b = 5(e + k) + 3 \) (resp. \( a = 2(e + k) \) and \( b = 5(e + k) + 1 \)). Thus the proof is completed. \( \square \)

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(Y.H.) **School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, People’s Republic of China**

*Email address*: yonghu@sjtu.edu.cn

(T.Z.) **School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, People’s Republic of China**

*Email address*: tzhang@math.ecnu.edu.cn, mathtzhang@gmail.com