The universal chiral partition function for exclusion statistics

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Abstract

We demonstrate the equality between the universal chiral partition function, which was first found in the context of conformal field theory and Rogers-Ramanujan identities, and the exclusion statistics introduced by Haldane in the study of the fractional quantum Hall effect. The phenomena of multiple representations of the same conformal field theory by different sets of exclusion statistics is discussed in the context of the $\hat{u}(1)$ theory of a compactified boson of radius $R$.
I. INTRODUCTION

In 1991 Haldane, in the context of the fractional quantum Hall effect, introduced the following “statistical interaction $g_{\alpha\beta}$ through the differential relation

$$\Delta d_{\alpha} = - \sum_{\beta} g_{\alpha\beta} \Delta N_{\beta}$$

(1.1)

where $\{\Delta N_{\beta}\}$ is a set of allowed changes of the particle numbers at fixed size and boundary conditions” and $d_{\alpha}$ is the dimension of the Hilbert space. The key idea embodied in this definition is that the number of states allowed to a particle is linearly dependent on the number of particles in the state. When $g_{\alpha\beta} = 0$ there is no reduction in the number of states and particles are called bosons whereas when $g_{\alpha\beta} = \delta_{\alpha\beta}$ the particles obey the Pauli exclusion principle. The linear exclusion rule (1.1) is “considered a generalization of the Pauli principle” and builds on both previous notions of generalized statistics used in the fractional quantum Hall effect and on solutions to integrable models. Subsequently this somewhat general notion was extended and sharpened by Wu and others and was reapplied to the fractional quantum Hall effect by van Elbergen and Schoutens. In the course of these studies the linear exclusion relation (1.1) has come to be referred to as exclusion statistics.

In 1993, in the context of conformal field theory and the corresponding integrable lattice models one of the authors and his collaborators introduced (for $y_i = 1$) what can be descriptively described as the Universal Chiral Partition Function

$$S_{B}[Q]_{A}(u, y|q) = \sum_{m=0}^{\infty} \left( q^{\frac{1}{2} m B m} - \frac{1}{2} A m \right) \prod_{\alpha=1}^{n} y_{m_{\alpha}}^{m_{\alpha}} \left[ \left( \frac{(1-B)m + \frac{u}{2})}{m_{\alpha}} \right] \right]$$

(1.2)

where for $m$ and $l$ integers the Gaussian polynomials are defined by

$$\left[ \begin{array}{c} l \\ m \end{array} \right] = \begin{cases} \frac{(q)^{m}}{(q)^{m-n}} & \text{for } 0 \leq m \leq l \\
0 & \text{otherwise} \end{cases} \text{ and } (a)_{l} = \prod_{j=0}^{l-1} (1-aq^{j})$$

(1.3)

and we note the limiting property

$$\lim_{l \to \infty} \left[ \begin{array}{c} l \\ m \end{array} \right] = \frac{1}{(q)_{m}}.$$

(1.4)

Here $m, A, Q$ and $u$ are $n$ component vectors, $B$ is an $n \times n$ matrix and the restrictions $Q$ on the sum are such that the arguments of the Gaussian polynomials are integers.

We call (1.2) a chiral partition function because it is indeed a grand partition function for $n$ species of right moving (chiral) particles with fugacities $y_j$

$$S_{B}[Q]_{A}(u, y|q) = \sum_{i} e^{-E_{i}/k_{B}T} y_{j}^{m_{j}}$$

(1.5)

where the sum is over all states $i$ whose energy $E_{i}$ is given in terms of single particle energies of particles with a linear dispersion relation as...
Here the single particle momenta are chosen from the set
\[ P_j^\alpha \in \{ P_{\text{min}}(m), P_{\text{min}}(m) + \frac{2\pi}{M}, \ldots, P_{\text{max}}(m) \}, \] (1.7)
where the Fermi exclusion rule (Pauli principle) holds
\[ P_j^\alpha \neq P_k^\alpha \text{ for } j \neq k \text{ and all } \alpha, \] (1.8)
the minimum and maximum momenta are
\[ P_{\text{min}}(m) = \frac{\pi}{M} [(B - 1)m - A + 1], \quad P_{\text{max}}(m) = -P_{\text{min}}(m) + \frac{2\pi}{M} (\frac{u}{2} - A) \] (1.9)
and \( q = e^{-\frac{2\pi}{m} M \beta} \). The expression (1.2) is obtained from (1.5) by use of the identity
\[ \sum_{N=0}^{\infty} Q_m(N, N') q^N = q^{m(m-1)} \left[ \frac{N'}{m} + 1 \right] \] (1.10)
where \( Q_m(N, N') \) is the number of additive partitions of \( N \geq 0 \) into \( m \) distinct non-negative integers each less than or equal to \( N' \). We refer to (1.7)-(1.9) as fermionic counting rules.

We refer to (1.2) as universal because in a long series of papers (see refs. 18-35 and references contained therein) it has been seen that the characters of conformal field theories and branching functions of affine Lie algebras may be universally written in this form (in the conformal limit \( T \to 0, M \to \infty \) with \( q \) fixed.)

The connection of (1.2) with bosons and fermions is easily seen by using elementary identities in \( q \) series that date back to Euler and the \( q \)-analogue of the binomial theorem.

For the connection with a free fermion we set \( n = B = A = 1, u = \infty \), and consider the unrestricted sum (\( Q = 0 \)) and use 1.3.16 on page 9 of ref. 36 to find
\[ S_1[0\;1] (\infty, y|q) = \sum_{n=0}^{\infty} q^{\frac{1}{2} m(m-1)} y^m = (-y)_\infty = \prod_{j=0}^{\infty} (1 + yq^j). \] (1.11)
The righthand side is manifestly the partition function for a free (chiral) fermion with a linear dispersion relation. From (1.9) we see that \( P_{\text{min}}(m) = 0 \) is independent of \( m \) as should be the case for a free fermion.

For the connection with a free boson we set \( n = 1, u = \infty, A = 0 \) and consider the unrestricted sum as before, but now we set \( B = 0 \) and use (18) of page xiv of ref. 36 to find
\[ S_0[0\;0] (\infty, y|q) = \sum_{n=0}^{\infty} \frac{y^m}{(q)_m} = \frac{1}{(y)_\infty} = \frac{1}{\prod_{j=0}^{\infty} (1 - yq^j)}. \] (1.12)
The right hand side is manifestly the partition function for a free (chiral) boson with a linear dispersion relation. From (1.9) we see that \( P_{\text{min}}(m) = \frac{\pi}{M} (1 - m) \). Thus we see that a particle with a Pauli exclusion principle can indeed have a bosonic partition function. This
cannot not be considered as strange since the identity (1.12) has been known for well over 200 years. The extension to $n$ free bosons with $g_{\alpha\beta} = 0$ or free fermions with $g_{\alpha\beta} = \delta_{\alpha\beta}$ is obvious.

We can now easily compare the universal chiral partition function (1.2) with $u = \infty$ with the exclusion statistics given by (1.1). The rule (1.1) gives a linear exclusion of states governed by the matrix $g_{\alpha\beta}$. The universal chiral partition function (1.2) comes from the state counting formula (1.7)-(1.9) with a linear exclusion of states governed by the matrix $B_{\alpha\beta}$. We have just seen that the case $g_{\alpha\beta} = B_{\alpha\beta} = 0$ gives free bosons and $g_{\alpha\beta} = B_{\alpha\beta} = \delta_{\alpha\beta}$ gives free fermions. Therefore the identification is almost obvious that if we set

$$B_{\alpha\beta} = g_{\alpha\beta} \quad (1.13)$$

then the exclusion statistics (1.1) of Haldane will lead to the universal chiral partition function (1.2) with $u = \infty$. The only difference in the two formulations is that the rule (1.1) is excluding states from a bosonic Fock space while the counting rules (1.7)-(1.9) are excluding or adding states to a fermionic Fock space. The virtue of the fermionic formulation is that the state counting (1.7)-(1.9) is very explicit while for the bosonic construction no such simple explicit formula is known.

II. THE EQUATIONS OF WU

The argument just given is very general and is valid for any number of quasi particles. For the case of one quasi particle, however, a much more detailed treatment of exclusion statistics was made by in 1994 by Wu who showed that the energy of systems with exclusion statistics in the thermodynamic limit where $M \to \infty$ with $T$ fixed is

$$E_{wu}(g) = \rho_0 \int_0^\infty d\epsilon \epsilon n_g(\epsilon) \quad (2.1)$$

where

$$n_g(\epsilon) = \frac{1}{w(\epsilon) + g} \quad \text{and} \quad w(\epsilon)^g[1 + w(\epsilon)]^{1-g} = y^{-1} e^{\epsilon/k_B T}. \quad (2.2)$$

or, equivalently setting $z = \epsilon/k_B T$

$$E_{wu}(g) = \rho_0 (k_B T)^2 \int_0^\infty dz \ z \ \bar{n}_g(z) \quad (2.3)$$

with

$$\bar{n}_g(z) = \frac{1}{\bar{w}(z) + g} \quad \text{and} \quad \bar{w}(z)^g[1 + \bar{w}(z)]^{1-g} = y^{-1} e^{z}. \quad (2.4)$$

In this section we will show the equivalence of these results with the corresponding results obtained from the universal chiral partition function.

For $n = 1$ the universal chiral partition function of (1.2) with $u \to \infty$, $A = 0$ and no restrictions $Q$ is
\[ S_B \left[ 0 \right] (\infty, y|q) = \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}Bm^2} y^m}{(q)_m}. \]  

(2.5)

From the definition of \( q \) we see that to study the thermodynamic limit \( M \to \infty \) with \( T \) fixed we need to study the behavior of (2.5) as \( q \to 1 \). This limit has been studied extensively in ref.19-20 by means of the method of steepest descents in the context of the computation of the central charge of conformal field theory. In the present case we note that as \( q \to 1 \) the sum in (2.5) is dominated by terms where \( m \sim 1/\ln q^{-1} \) and thus we may write

\[ S_B \left[ 0 \right] (\infty, y|q) \sim \sum_{m=0}^{\infty} \exp\{-\frac{B}{2} m^2 \ln q^{-1} + m\ln y - \frac{1}{\ln q^{-1}} \int_0^{m\ln q^{-1}} dt \ln(1 - e^{-t})\}. \]  

(2.6)

The sum is dominated by its value at the steepest descents point determined from

\[ 0 = \frac{d}{dm} \left\{-\frac{B}{2} m^2 \ln q^{-1} + m\ln y - \frac{1}{\ln q^{-1}} \int_0^{m\ln q^{-1}} dt \ln(1 - e^{-t})\right\} \]

\[ = -mB\ln q^{-1} + \ln y - \ln(1 - e^{-m\ln q^{-1}}). \]  

(2.7)

Thus, setting \( x = m\ln q^{-1} \) we write (2.7) as

\[ ye^{-Bx} = (1 - e^{-x}) \]  

(2.8)

and using this value of \( x \) in (2.6) and recalling the definition of \( q \) we find that as \( M \to \infty \)

\[ \ln S_B \left[ 0 \right] (\infty, y|q) \sim \frac{k_B T M}{2\pi v} \left\{-\frac{B}{2} x^2 + x\ln y - \int_0^{x} dt \ln(1 - e^{-t})\right\}. \]  

(2.9)

The free energy per site \( f \) is defined as

\[ f = -k_B T \lim_{M \to \infty} \frac{1}{M} \ln S_B \left[ 0 \right](\infty, y|q) \]  

(2.10)

and the internal energy per site is

\[ E(B) = \frac{\partial}{\partial B} f \bigg|_y \]  

(2.11)

and thus we have

\[ E(B) = \frac{(k_B T)^2}{2\pi v} \left\{-\frac{B}{2} x^2 + x\ln y - \int_0^{x} dt \ln(1 - e^{-t})\right\}. \]  

(2.12)

This may also be written in terms of the Rogers dilogarithm function

\[ L(w) = -\frac{1}{2} \int_0^w dv \left[\frac{\ln v}{1 - v} + \frac{\ln(1 - v)}{v}\right] \]

\[ = -\int_0^w dv \ln(1 - v) + \frac{1}{2} \ln \ln(1 - w) = -\int_0^w dv \ln v - \frac{1}{2} \ln \ln(1 - w) \]  

(2.13)
by use of the relation
\[ L(w) + L(1 - w) = L(1) = \frac{\pi^2}{6} \] (2.14)
as
\[ E(B) = \frac{(k_B T)^2}{2 \pi v} \left\{ \frac{1}{2} x \ln y + L(1 - e^{-x}) \right\} = \frac{(k_B T)^2}{2 \pi v} \left\{ \frac{B}{2} x^2 + \frac{1}{2} x \ln (1 - e^{-x}) + L(1 - e^{-x}) \right\}. \] (2.15)

We will show that if we identify \( B \) in (2.15) with \( g \) in (2.3) then
\[ E(g) = E_{wu}(g). \] (2.16)
We do this by first noting that from (2.4)
\[ z = g \ln \bar{w} + (1 - g) \ln (1 + \bar{w}) + \ln y \quad \text{and} \quad \frac{d\bar{w}}{dz} = \frac{\bar{w}(1 - \bar{w})}{\bar{w} + g}. \] (2.17)
Moreover, we see from (2.17) that if \( z = \infty \) then \( w = \infty \) and by comparing with (2.8) (with \( B = g \)) we see that if \( z = 0 \) then \( w = 1/(e^x - 1) \). Thus we rewrite (2.3) using \( \bar{w} \) as the independent variable instead of \( z \) and obtain
\[ E_{wu}(g) = \rho_0 (k_B T)^2 \int_{1/e-1}^{\infty} \frac{d\bar{w}}{\bar{w}(\bar{w} + 1)} \{ g \ln \bar{w} + (1 - g) \ln (1 + \bar{w}) + \ln y \} \]
\[ = \rho_0 (k_B T)^2 \{ g x^2 + x \ln (1 - e^{-x}) + \int_{1/e-1}^{\infty} \frac{d\bar{w}}{\bar{w}(1 + \bar{w})} \{ g \ln \bar{w} + (1 - g) \ln (1 + \bar{w}) \} \} \] (2.18)
where in the last line \( y \) has been eliminated in favor of \( x \) by use of (2.8) (with \( B = g \)). In the remaining integral let
\[ v = \frac{1}{1 + \bar{w}} \quad \text{with} \quad d\bar{w} = -\frac{dv}{v^2} \] (2.19)
to find
\[ E_{wu}(g) = \rho_0 (k_B T)^2 \{ \frac{g}{2} x^2 + x \ln (1 - e^{-x}) - \int_{0}^{1/e-1} dv \frac{\ln v}{1 - v} \} \]
\[ = \rho_0 (k_B T)^2 \{ \frac{g}{2} x^2 + \frac{1}{2} x \ln (1 - e^{-x}) + L(1 - e^{-x}) \} \] (2.20)
where in the last line we have used the definition of \( L(w) \) of the last line of (2.13). Upon comparing with (2.13) we see that (2.16) does indeed hold. Thus we have shown that the energy given by the universal chiral partition function (1.2) with \( n = 1 \) and \( u = \infty \) is identical with the energy computed by Wu[4] for the exclusion statistics of Haldane[6].
III. MULTIPLE REPRESENTATIONS FOR $\hat{U}(1)$

Perhaps the most interesting and least understood property of the universal chiral partition function is that there are often identities between forms (1.2) with different $B$ matrices which may even have different dimensions. Thus, for the minimal model $M(3,4)$ there is an identity between the universal chiral partition function with a one dimensional and an eight dimensional $B$ matrix. This multiplicity of representations is often identified with the various integrable perturbation of the conformal field theory.

The identification of the universal chiral partition function (1.2) with the exclusion statistics (1.1) means that this same phenomena of a multiplicity of representations must occur for exclusion statistics also. Here we illustrate this multiplicity phenomena for the $\hat{u}(1)$ affine Lie algebra.

The conformal field theory of affine Lie algebra $\hat{u}(1)$ appears in several contexts. In one context it is the Gaussian model which describes a boson compactified with a radius $R$ with conformal dimensions

$$h_{n,m} = \frac{1}{2}(n/R + mR/2)^2. \quad (3.1)$$

In the context of the XXZ spin chain

$$H_{XXZ} = \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cos \mu \sigma_j^x \sigma_{j+1}^x) \quad (3.2)$$

the radius $R$ (using the normalization of ref. 38) is

$$R = \left( \frac{2}{1 - \mu/\pi} \right)^{1/2}, \quad 0 \leq \mu \leq \pi. \quad (3.3)$$

When $R^2 = 2p'/p$ is rational the extended characters are given for $p' > p$ relatively prime as

$$\chi_l^{(pp')}(q, y) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} q^{pp'(j + \frac{m_1 - m_2}{pp'})} y^{j}, \quad 0 \leq l \leq 2pp' \quad (3.4)$$

It was shown in ref. 33 that if we write $l = p'Q + Q'$ with $Q' = 0, 1, \ldots, p' - 1$ and $Q \in Z_{2p}$ then $\chi(q, y)$ may be written in the form (1.2) with $n = 2$

$$B = \left( \begin{array}{cc} \frac{R^2}{4} & \frac{R^2}{4} \\ 1 - \frac{R^2}{4} & \frac{R^2}{4} \end{array} \right), \quad A = \frac{Q'}{p}(1, -1), \quad m_1 - m_2 \equiv Q(\text{mod} 2p') \quad (3.5)$$

and $y_1 = y_2^{-1} = y^{1/2}$. In this $2 \times 2$ matrix $B$ the two particles appear in a symmetric fashion.

The $\hat{u}(1)$ conformal field theory also appears in the context of the Cologero-Sutherland model

$$H_{CS} = \sum_{i} \frac{p_i^2}{2} + \left( \frac{\pi}{L} \right)^2 \sum_{i<j} \frac{\beta^2 \sin^2(x_i - x_j)\pi/L}{\sin^2(x_i - x_j)\pi/L} \quad (3.6)$$
where now the radius of compactification is $R^2 = \beta$. Here it has been shown that there is again a two-particle basis, (now called anyons and superfermions in and g-ons in) but now these two particles do not appear symmetrically. In this context there is an isomorphism with the fractional quantum Hall effect where the basis particles are called electrons and quasi-particles. In the Cologero-Sutherland model (and because of the isomorphism of models also in the fractional quantum Hall effect) the truncated partition sum for g-ons (quasi-electrons for the filling fraction $\nu = 1/g = r/s$ with $r$ and $s$ relatively prime) satisfies the recursion relation for integer $L + \frac{s}{2}$

$$X_L(y, q, r, s) = X_{L-r}(y, q, r, s) + yq^\frac{L}{r}X_{L-s}(y, q, r, s).$$  \hspace{1cm} (3.7)

and in it was shown for $r = 1$ that the universal chiral partition function $F_L(y, q, r, s, a)$ with

$$n = 1, \quad B = \frac{s}{r}, \quad \frac{u}{2} = \frac{L}{r} + \frac{s}{2r} - a \hspace{1cm} (3.8)$$

is an exact solution of (3.7) for all $a$. Here we demonstrate this solution by writing the specialization of (3.8) of (1.2) as

$$F_L(y, q, r, s, a) = \sum_{L+s/2-am \equiv ra \pmod{r}}^{\infty} q^{\frac{L}{r}m^2 + am} y^m \left[ \frac{L}{r} - \frac{(s/r - 1)m + \frac{s}{2r} - a}{m} \right] \hspace{1cm} (3.9)$$

and use the recursion relation for Gaussian polynomials

$$\left[ \begin{array}{c} l \\ m \end{array} \right] = \left[ \begin{array}{c} l - 1 \\ m \end{array} \right] + q^{l-m} \left[ \begin{array}{c} l - 1 \\ m - 1 \end{array} \right] \hspace{1cm} (3.10)$$

to obtain

$$F_L(y, q, r, s, a) = \sum_{L+s/2-am \equiv ra \pmod{r}}^{\infty} q^{\frac{L}{r}m^2 + am} y^m \left[ \frac{L}{r} - \frac{(s/r - 1)m + \frac{s}{2r} - a}{m} \right]$$

$$+ q^{L+s/2-(s/r - 1)m-a-m} \left[ \frac{L}{r} - \frac{(s/r - 1)m + \frac{s}{2r} - a}{m - 1} \right]. \hspace{1cm} (3.11)$$

The first term is recognized as $F_{L-r}(y, q, r, s, a)$ and in the second term we set $m - 1 = j$ and thus we obtain

$$F_L(y, q, r, s, a) = F_{L-r}(y, q, r, s, a)$$

$$+ \sum_{L-s+s/2-sj \equiv ra \pmod{r}}^{\infty} q^{\frac{L}{r}(j+1)^2 + a(j+1) + \frac{L}{r} + \frac{s}{2r} - \frac{(s/r - 1)(j+1) - a}{j}} y^{j+1} \left[ \frac{L}{r} - 1 + \frac{s}{2r} - \frac{(s/r - 1)(j+1) - a}{j} \right]. \hspace{1cm} (3.12)$$

Finally after simplifying the exponent and comparing with the definition (3.5) the second term is recognized as $yq^{L/r}F_{L-s}(y, q, r, s, a)$ and thus we obtain the final result

$$F_L(y, q, r, s, a) = F_{L-r}(y, q, r, s, a) + yq^{L/r}F_{L-s}(y, q, r, s, a) \hspace{1cm} (3.13)$$

which is precisely the recursion relation (3.7).
This derivation makes clear that the restrictions in the sum of the universal chiral partition function arise whenever \( B \) is fractional. These restrictions in essence reduce the partition function to \( 1/r \) of the unrestricted partition function and thus will not effect the exponential behavior of the \( M \to \infty \) thermodynamic limit. Consequently these restrictions have no effect on the computation of the energy done in the preceding section.

The partition function for the fractional quantum Hall effect representation of \( \hat{u}(1) \) is obtained from the function \( F_{L}(y, q, r, s, a) \) for the electrons and \( F_{L}(y, q, s, r, a) \) for the quasi particle for finite \( L \) by means of the following elegant polynomial identity for \( s \geq r \)

\[
\sum_{j=L(a \mod s), s=a \mod r}^{\infty} q^\frac{j^2}{2rs} y^{-\frac{j}{rs}} \left[ \frac{(\frac{1}{r} + \frac{1}{s})L + (\frac{1}{r} - \frac{1}{s})j}{L} \right] = \sum_{a=1-r}^{s-r} q^\frac{a^2}{2rs} y^\frac{a}{rs} F_{L-s/2}(y^{\frac{1}{r}}, q, r, s, a) F_{L-r/2}(y^{-\frac{1}{s}}, q, s, r, \theta(a > 0) - a/s) \tag{3.14}
\]

and \( \theta(a > 0) = 1 \) if \( a > 0 \) and zero otherwise. We note that there are in general \( rs \) different limits as \( L \to \infty \) depending on the congruences \( L \equiv l(\text{mod } rs) \) with \( l = 0, 1, \ldots, rs - 1 \). We also note that (3.14) is in general not invariant under \( y \to y^{-1} \). When \( r = s = 1 \) the identity (3.14) is the polynomial form of the Jacobi triple product identity (see page 49 of ref. 48).

When \( L \to \infty \) the case \( r = 1, s = 2 \) was first proven by Ramanujan in the famous “lost notebook” (see eqn. 2.3.2 of ref. 43) and the general case for the limits \( L \to \infty \) is a consequence of the partition counting theorems of Andrews and of Kadell on representations of \( 1/(q)^{\infty} \) by means of \( r \times s \) Durfee rectangles. The proof of the most general case with \( L \) finite is obtained by adapting the \( L \to \infty \) proof to include an upper bound on the number and size of the parts of the partitions. Indeed a more general result than (3.14) can be proven which generalizes 2.3.1 of ref. 43. The details of these proofs will be published elsewhere. The special case of \( L \to \infty \) with all values of \( l \) summed together and \( y = 1 \) was conjectured in ref. 43.

To compare the fermi representations of the \( \hat{u}(1) \) characters (3.4) given by (3.14) and the representation of ref. 43 with the \( B \) matrix (3.3) we first note that in (3.4) the case \( p = p' = 1 \) \( R^2 = 2 \) is the self dual point there the characters are the same as the two characters of \( \tilde{su}(2)_1 \). We note that the same \( \tilde{su}(2)_1 \) characters are obtained from (3.14) with \( r = 1, s = 2 \) in the \( L \to \infty \) limit since \( j \equiv L(\text{mod } s) \). More generally we relate the two fermionic forms by using \( R^2 = 2p'/p = s/r \).

But the two different representations of \( \chi_L^{(pp')}(q, y) \) given by (3.3) and by (3.14) do not exhaust the representations of the \( \hat{u}(1) \) characters (3.4) in terms of the universal chiral partition function (1.2). For example the case \( s = 3, r = 1 \) has an \( N = 2 \) supersymmetry, and for this case the three quasi particle representation of the \( c = 1 \) \( N = 2 \) supersymmetric characters give a three quasi particle representation of the characters. For \( R^2 = 2n \) and \( n > 3 \) there is a representation with \( n \) quasi particles and \( B = 2C_{D_n}^{-1} \) with \( C_{D_n} \) the Cartan matrix of the Lie algebra \( D_n \). The points \( n = N^2 \) with \( N \geq 2 \) are the points shown in ref. 43 to have an extra \( \tilde{su}(N) \) symmetry.

In general for \( R^2 \) rational there is a representation which originates in the Bethe’s Ansatz solution to the XXZ model of Takahashi and Suzuki. This representation is analogous to the representation of the characters of the \( M(p, p') \) minimal model proven in detail in ref. 43.
These representations differ from the cases discussed above in that in general most of the parameters $u$ in (1.2) are finite.

We thus see that there are at least three ways to describe the $\hat{u}(1)$ compactified boson of rational radius in terms of exclusion statistics. This multiplicity of representations at times leads to a confusion of language. For example it is said in ref. 22 that an anyon representation of the characters for $r^2 = 2$ was given in ref. 24. However the anyons of ref. 43 lead to the case $r = 1, s = 2$ of (3.14) whereas the two representations of ref. 24 are the representation (3.5) and a form with an infinite number of quasi particles first proposed by Melzer 57. It could be argued that the term “anyon” should be reserved for the excitations in (3.14) and the term “spinon” should be reserved for (3.5). But there is no uniformity in the usage of these terms and the term spinon is sometimes used to denote the Takahashi-Suzuki representations as well. Indeed since for general rational $R^2$ there are more that two representation further new names are needed.

This multiplicity of fermionic representations (or equivalently the multiplicity of possible exclusion statistics description) is a very important physical effect. In the case of the models $M(p, p + 1)$ the different fermionic descriptions correspond to different massive integrable perturbation (such as $\phi_{1,2}$, $\phi_{1,3}$ and $\phi_{2,1}$). But these different representations also correspond to different ways of constructing the massless theory itself and these constructions may be thought of as involving different ultraviolet regularizing procedures needed to define the theory. In this interpretation we see that the different regularizing procedures can lead to different particle descriptions which are in general not local with respect to one another. This phenomena is most important in the applications of $\hat{u}(1)$ to the fractional quantum Hall effect.

IV. CONFORMAL FIELD THEORY AND ROGERS-RAMANUJAN IDENTITIES

The fundamental principle behind the identification of the exclusion statistics of Haldane with the fermionic counting rules (1.7)–(1.9) is the equivalence of the bosonic and fermionic description of the underlying Hilbert space. This Bose/Fermi equivalence is the principle behind all Rogers-Ramanujan identities and is the reason why there is an equivalence between the fermionic description of conformal field theory arising from the thermodynamic Bethe’s Ansatz and the corresponding bosonizations of Kac–Moody algebras. In lattice statistical mechanics and in conformal field theory this equivalence is well known19–34. However in the study of the exclusion statistics on the fractional quantum Hall effect this equivalence does not seem to be widely and explicitly recognized22. We will thus conclude with a few suggestions as to why this identification has not previously been made.

There are perhaps two obvious obstacles to the identification of the the exclusion statistics of Haldane with the fermionic counting rules (1.7)–(1.9) and the universal chiral partition function (1.2). The first is that in most applications of the universal chiral partition function (1.2) to conformal field theory only the special case $y_j = 1$ occurs because in the CFT applications there was no conservation law imposed on the number of excitations. In particular while Fermi/Bose (Rogers–Ramanujan identities) are known for all minimal models $M(p, p')$ the bosonic form for the partition function with a fugacity $y \neq 1$ is only known for the special cases $M(2, 2n + 1)$ (see chapter 7 of the book of Andrews23).
The second obstacle to identification is that in the definition of exclusion statistics of Haldane\textsuperscript{1} all values of $g_{\alpha\beta}$ are allowed whereas in the applications of (1.2) to conformal field theory only very specific values of the matrices $B$ are allowed. For example only three cases of the scalar case $n = 1$ are related to conformal field theories: $b = 2$ is $M(2,5)$, $b = 1$ is $M(3,4)$ and $b = 1/2$ is $M(3,5)$. We note that the case $b = 1/2$ is sometimes referred to as “semionic” and there is an equality of the characters of $SU(2)_1$ and $U(1)_2$. For these three values of $b$ there are two special additional properties of (1.2) with $u = \infty$ and $y = 1$. First of all the dilogarithms in (2.15) are all rational multiples of $L(1)$ and secondly (1.2) transforms under a representation of the modular group. This second property is of great importance in the theory of Kac–Moody algebras\textsuperscript{2} and conformal field theory\textsuperscript{3}. In a similar fashion these three special cases seem to be the only ones which are related to Bethe’s Ansatz models\textsuperscript{4}. From this point of view we are here proposing that the word “universal” in the universal chiral partition function is to be used in a much wider sense than the conformal field theory context where it first appeared.

V. CONCLUSION

We conclude with the statement that all of the identifications made here of exclusion statistics of ref.\textsuperscript{1} with the universal chiral partition function (1.2) and the fermionic counting rules (1.7)-(1.9) in section 2 and 3 for the scalar case can be extended to the matrix case as well. Thus, since the fermionic counting rules (1.7)-(1.9) are more general than the exclusion statistics (1.1) and they include (1.1) as the special case $u = \infty$, we propose that in one dimension (1.7)-(1.9) is a more natural and general definition of “exclusion statistics” and in the future should be taken as the definition of the term instead of (1.1).

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