Application of energy and angular momentum balance to gravitational radiation reaction for binary systems with spin-orbit coupling

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Abstract We study gravitational radiation reaction in the equations of motion for binary systems with spin-orbit coupling, at order \((v/c)^7\) beyond Newtonian gravity, or \(O(v/c)^2\) beyond the leading radiation reaction effects for non-spinning bodies. We use expressions for the energy and angular momentum flux at infinity that include spin-orbit corrections, together with an assumption of energy and angular momentum balance, to derive equations of motion that are valid for general orbits and for a class of coordinate gauges. We show that the equations of motion are compatible with those derived earlier by a direct calculation.

Keywords Gravitational Radiation · Binary Systems · Spinning Bodies

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1 Introduction and Summary

The backreaction of the emission of gravitational radiation on the system emitting the radiation is a problem both of formal interest within general relativity and of practical interest for gravitational-wave detection. A leading candidate source for laser-interferometric gravitational-wave observatories, both on the ground and in space, is the radiation-reaction induced inspiral of a binary system of two compact objects (black holes or neutron stars). In order to develop accurate theoretical predictions for the gravitational waveforms emitted by such systems, one must know their evolution under the dissipative effects of gravitational-wave emission to high accuracy. In addition, particularly for systems containing black holes, the effects of spin may be important. Spin-orbit and spin-spin couplings can result in precessions of the bodies’ spins and of the orbital angular momentum, leading to modulations in the gravitational waveform [1], and can affect the rate of decay of the orbit [2,3].
As a result, substantial effort has gone into determining the effects of spin in binary systems. Except for the final few orbits, much of the inspiral of such systems can be described by the post-Newtonian approximation, which is an expansion of Einstein’s equations in powers of \( \epsilon \sim (v/c)^2 \sim Gm/rc^2 \), where \( v \), \( m \) and \( r \) represent typical velocities, masses and separations in the system, and \( G \) and \( c \) are the gravitational constant and speed of light. Each power of \( \epsilon \) represents one “post-Newtonian” (PN) order in the series (\( \epsilon^{1/2} \) represents one-half, or 0.5PN orders).

Formally, spin effects first enter the equations of motion at the 1PN level, and have been derived by numerous authors from a variety of points of view, ranging from formal developments of the GR equations of motion in multipole expansions \([4, 5]\), to post-Newtonian calculations \([6]\), to treatments of linearized GR as a spin-two quantum theory \([7, 8]\). For a review of these various approaches, see \([9]\).

Spin also affects gravitational radiation reaction, and radiation reaction can affect spin; it is straightforward to show that such effects first occur at 3.5PN order. In earlier work, we derived, from first principles, the radiation-reaction effects of spin-orbit and spin-spin coupling, by integrating the post-Newtonian hydrodynamic equations of motion, including 1PN, 2.5PN and 3.5PN terms, over bodies consisting of rotating fluid \([10, 11]\). As a check, we found that the loss of energy and angular momentum (including spin) induced by the radiation reaction terms, matched precisely the expressions for energy and angular momentum flux derived by Kidder \(\text{et al.}\, [2, 3]\).

An alternative approach to obtaining equations of motion with radiation reaction at higher PN orders was studied by Iyer and Will \([12, 13]\). There, we wrote down the most general form that the 2.5PN and 3.5PN radiation-reaction terms could take in the equations of motion for a binary system of spinless bodies, in terms of arbitrary coefficients. We then used the assumption of energy and angular momentum balance, combined with energy and angular-momentum flux expressions accurate to PN order beyond the quadrupole approximation to impose constraints on the arbitrary coefficients used in the equations of motion. After taking into account a fundamental ambiguity in the definitions of energy and angular momentum at 2.5PN and 3.5PN orders, we were left with equations of motion with coefficients that are fixed up to two arbitrary coefficients at 2.5PN order and 6 arbitrary coefficients at 3.5PN order. It was then straightforward to show that these eight degrees of freedom correspond precisely to the effects, mapped onto the two-body equations of motion, of coordinate transformations at the relevant PN orders. At 2.5PN order, for example, one choice of the two arbitrary coefficients gives the equations in the so-called Burke-Thorne gauge \([14]\), in which the radiation reaction terms are obtained from a gradient of the potential \((G/5c^2)x^i x^j \partial^5 I^{\leq\leq}/\partial x^5\), where \(I^{\leq\leq}\) is the trace-free moment of inertia tensor of the system, while another choice gives the Damour-Deruelle gauge, which is more directly tied to harmonic gauge \([15, 16]\). For spinless systems, this approach was extended to determine the 4.5PN terms in the equations of motion using flux expressions accurate to 2PN order beyond quadrupole \([17]\).

It is the purpose of this paper to extend this approach to include spin-orbit radiation reaction effects at 3.5PN order. We assume that the equation of motion for the relative vector \( \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \) in a binary system may be written in a PN expansion in the form

\[
a = -\frac{\mathbf{n}}{r^2} n + a_{\text{PN-SO}} + \ldots + a_{2.5\text{PN}} + \ldots + a_{3.5\text{PN-SO}},
\]

where \( r \equiv |\mathbf{x}| \), \( \mathbf{n} \equiv \mathbf{x}/r \) and \( m \equiv m_1 + m_2 \); \( a_{\text{PN-SO}} \) is the post-Newtonian spin-orbit contribution, \( a_{2.5\text{PN}} \) is the leading radiation-reaction contribution, and \( a_{3.5\text{PN-SO}} \) is
the 3.5PN spin-orbit contribution. Here and for the rest of this paper, we use units in which $G = c = 1$. We have not displayed the point mass 1PN, 2PN, 3PN, and 3.5PN terms, as they will play no role in our analysis. We also will not use the bookkeeping parameter $\epsilon$ explicitly to keep track of PN orders, since we will be considering only specific orders. It is sufficient to recall that, since spin scales as $m v r$, then $S/r^2 \sim v(m/r) \sim \epsilon^{3/2}$. This, plus explicit labelling of terms throughout, should make clear the PN order of terms being discussed.

We then write down the most general 3.5PN spin-orbit expression that (a) contains terms each involving a single spin (either $S_1$ or $S_2$), (b) is a vector, and (c) is antisymmetric under the interchange $1 \leftrightarrow 2$. This turns out to involve 30 arbitrary coefficients.

Because the bodies have intrinsic spin, we must make an assumption about their spin evolution. At 1PN order, we assume that they obey the standard spin-orbit precession equations (see Eq. (2) below). These 1PN equations produce only precession; the magnitudes of the spins do not change. At 3.5PN order, we likewise assume that gravitational radiation reaction produces only precessions of the spin. This is a reasonable assumption, because, for a rotating axisymmetric body, it is impossible to see how gravitational radiation can cause it to spin up or down, to the 3.5PN order being considered. Such effects must involve specific couplings of radiation to deformations of the bodies, either due to rotational flattening or due to tidal couplings, which are beyond the scope of our assumption of almost point-like bodies with spin. In this case, we can then show that the most general 3.5PN expression for the evolution of each spin that (a) is a pseudovector; (b) depends only on the spin itself and on orbital variables; and (c) is orthogonal to the spin, can in fact be written as a total time derivative of spin and orbital variables, which can then be absorbed into a meaningless 3.5PN-order correction to the definition of the spin.

Consequently, we can calculate the loss of energy and angular momentum using only the parametrized equations of motion and the 1PN spin precession equations. There is no contribution to the evolution of the spins at 3.5PN order. However, we must incorporate the freedom to add arbitrary terms of 3.5PN spin-orbit order into the definitions of total energy and total angular momentum, just as in the spinless case. There are 6 such terms in $E$ and 26 in $J$. Thus there is a total of 62 arbitrary coefficients to be determined. We then equate the time derivative of these expressions for $E$ and $J$ with the corresponding expressions obtained from the far-zone gravitational-wave flux, including spin-orbit terms, as calculated by Kidder et al. [2, 3], and compare them term by term. This leads to 54 constraints on the coefficients; however 4 of these constraints are not linearly independent of others, and thus we have 50 constraints on 62 coefficients, leaving 12 undetermined coefficients. Finally we show that these 12 free coefficients in the equation of motion correspond precisely to the effects of 3.5PN order coordinate transformations, mapped onto the two-body equations of motion with spin-orbit coupling.

The remainder of this paper provides details. In Sec. 2 we review the known equations of motion and spin evolution through 2.5PN order. Section 3 applies energy and angular momentum balance to determine the 3.5PN spin-orbit terms in the two-body equations of motion, while Sec. 4 shows that the remaining undetermined coefficients are directly related to gauge freedom. Section 5 presents concluding remarks, while certain detailed formulae are relegated to Appendices.
Two-body equations of motion with spin-orbit coupling

The PN-SO and 2.5PN terms in Eq. (1) are given by conventional expressions

\[ a_{\text{PN-SO}} = \frac{1}{r^3} \left\{ \frac{3}{2} \mathbf{n} \cdot (4\mathbf{S} + 3\xi) - \mathbf{v} \times (4\mathbf{S} + 3\xi) + \frac{3}{2} \mathbf{r} \mathbf{n} \times (4\mathbf{S} + 3\xi) \right\}, \tag{2} \]

\[ a_{2.5\text{PN}} = \frac{8\mu m}{5r^4} \left\{ \left[ 3(1 + \beta)v^2 + \frac{1}{3}(23 + 6\alpha - 9\beta) \frac{m}{r} - 5\beta r^2 \right] \mathbf{r} \mathbf{n} \right. \]
\[ - \left. \left[ (2 + \alpha)v^2 + (2 - \alpha) \frac{m}{r} - 3(1 + \alpha)r^2 \right] \mathbf{v} \right\}, \tag{3} \]

where \( v = dx/dt \) is the relative velocity, \( \mu \equiv m_1 m_2 / m \) is the reduced mass, \( \mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2 \) is the total spin, \( \xi = (m_2/m_1)\mathbf{S}_1 + (m_1/m_2)\mathbf{S}_2 \) is a second spin parameter, \( \mathbf{L}_N = \mathbf{x} \times \mathbf{v} \) is the orbital angular momentum per unit reduced mass, and \( \mathbf{r} = \mathbf{v} \cdot \mathbf{n} \).

The coefficients \( \alpha \) and \( \beta \) in \( a_{2.5\text{PN}} \) reflect the possibility of different gauges for expressing radiation reaction at 2.5PN order \[12, 13\]. The choice \( \alpha = 4, \beta = 5 \) corresponds to Burke-Thorne gauge \[14\], while the choice \( \alpha = -1, \beta = 0 \) leads to the Damour-Deruelle radiation-reaction formula \[15, 16\]. Any choice of \( \alpha \) and \( \beta \) leads to the same loss of energy and angular momentum at 2.5PN order, corresponding to quadrupole approximation energy and angular momentum flux.

In defining spin, we must specify the center of mass of each body using a procedure commonly known as the “spin supplementary condition (SSC); the definition used in this paper corresponds to the value \( k_{\text{SSC}} = 1/2 \) (see \[18, 3, 10\] for further discussion).

The equations of evolution for the spins may be written in the form

\[ \dot{\mathbf{S}}_1 = (\dot{\mathbf{S}})_\text{PN} + \ldots + (\dot{\mathbf{S}})_\text{3.5PN-SO}, \tag{4} \]

where

\[ (\dot{\mathbf{S}})_\text{PN} = \frac{\mu}{r^3} \mathbf{L}_N \times \mathbf{S}_1 \left( 2 + \frac{3}{2} \frac{m_2}{m_1} \right), \tag{5} \]

and where the equations for spin 2 can be obtained by the interchange \( 1 \leftrightarrow 2 \). We have not included conservative 2PN and 3PN contributions, and can show that the leading radiation reaction contributions come at 3.5PN order \[10\].

Up to 2.5PN order, the motion is conservative and the energy and angular momentum are constant. Including only the Newtonian and spin-orbit terms, they are given by

\[ E = E_N = \mu \left( \frac{1}{2} v^2 - \frac{m}{r} \right), \tag{6} \]
\[ J = \mu \mathbf{L}_N + \mathbf{S} + \frac{\mu}{2r} \mathbf{n} \times \left[ \mathbf{n} \times (4\mathbf{S} + 3\xi) \right]. \tag{7} \]

In our chosen spin supplementary condition, there is no spin-orbit contribution to the conserved energy, while \( J \) contains the orbital angular momentum, the total spin, and a PN spin-orbit contribution. These conserved quantities can be derived from the equations of motion by constructing \( \frac{1}{2} dv^2 / dt \equiv \mathbf{v} \cdot \mathbf{a}, \) and \( d(x \times \mathbf{v}) / dt \equiv \mathbf{x} \times \mathbf{a}, \) and showing that, after substituting the equations of motion and spin-precession equations carried to the appropriate order, everything can be expressed as total time derivatives.
3 Spin-orbit radiation reaction via \(E\) and \(J\) balance

We now write down the most general 3.5PN spin-orbit terms as

\[
a_{3.5PN-SO} = -\frac{\mu}{5!r^4} \left[ A_S \frac{\dot{\mathbf{n}}}{r} (\mathbf{L}_N \cdot \mathbf{S}) + B_S \frac{\mathbf{v}}{r} (\mathbf{L}_N \cdot \mathbf{S}) + C_S \dot{\mathbf{v}} \times \mathbf{S} + D_S \mathbf{n} \times \mathbf{S} \right] + A_\xi \frac{\dot{\mathbf{n}}}{r} (\mathbf{L}_N \cdot \xi) + B_\xi \frac{\mathbf{v}}{r} (\mathbf{L}_N \cdot \xi) + C_\xi \dot{\mathbf{v}} \times \xi + D_\xi \mathbf{n} \times \xi \right].
\]

The form of Eq. (8) is dictated by the fact that it must be a correction to the Newtonian acceleration, (i.e. be proportional to a mass /\(r^2\)); must vanish in the test body limit when gravitational radiation vanishes, (i.e. be proportional to \(\mu\)); must be dissipative, or odd in velocities; must be linear in the spins; must be a vector, not a pseudovector; and must change sign under the interchange \(1 \leftrightarrow 2\). Note that other possible terms, such as \(\mathbf{L}_N (\mathbf{n} \cdot \mathbf{S})\) can be seen to be linear combinations of the terms above using standard vector identities. The prefactor 1/5 is chosen for convenience. To make the terms of \(O(\epsilon^7/2)\) beyond Newtonian order, \(A_S, B_S, C_S, A_\xi, B_\xi\) and \(C_\xi\) must be of \(O(\epsilon)\), and \(D_S\) and \(D_\xi\) must be of \(O(\epsilon^2)\). The only orbital variables available to construct expressions of the relevant order are \(v^2, m/r\) and \(\dot{r}^2\). Thus \(A_S, B_S, C_S\) and \(D_S\) can be written in terms of 15 arbitrary coefficients, in the form

\[
A_S = a_1 v^2 + a_2 \frac{m}{r} + a_3 \dot{r}^2,
B_S = a_4 v^2 + a_5 \frac{m}{r} + a_6 \dot{r}^2,
C_S = a_7 v^2 + a_8 \frac{m}{r} + a_9 \dot{r}^2,
D_S = a_{10} v^4 + a_{11} v^2 \dot{r}^2 + a_{12} \dot{r}^4 + a_{13} v^2 \frac{m}{r} + a_{14} \dot{r}^2 \frac{m}{r} + a_{15} \frac{m^2}{r^2}.
\]

In a parallel manner, we can write \(A_\xi, B_\xi, C_\xi\) and \(D_\xi\) in terms of its own set of 15 coefficients. Because all expressions involving spin-orbit terms divide naturally into those involving the total spin \(\mathbf{S}\) and those involving the spin parameter \(\xi\), we can solve for each set using identical methods; we will focus on the \(S\)-terms. Our goal is to evaluate these thirty coefficients by imposing energy and angular momentum balance.

Because the equations of motion at 2.5PN order and 3.5PN order have dissipative terms, the energy and angular momentum are no longer conserved explicitly. Furthermore, they are ambiguous because one has the freedom to add arbitrary terms to \(E\) and \(J\) at 2.5PN order and 3.5PN order to redefine them without affecting their conservation through 2PN order. Similarly, the spins are strictly defined only up to the order at which radiation reaction begins, and so one has the freedom to add a 3.5PN term to each spin, without changing its behavior at “conservative” orders.

Adding to \(E\) and \(J\) the appropriate 2.5PN terms to account for the coefficients \(\alpha\) and \(\beta\) in Eq. (3), and adding the most general 3.5PN spin-orbit terms, with arbitrary coefficients, we can define new quantities \(E^*\), and \(J^*\) according to

\[
E^* = E_N + \delta E_{2.5PN} + \delta E_{3.5PN-SO},
J^* = \mu \mathbf{L}_N + S + \frac{\mu}{2r} \mathbf{n} \times (\mathbf{n} \times (4 \mathbf{S} + 3 \xi)) + \delta J_{2.5PN} + \delta J_{3.5PN-SO}.
\]
where, from our earlier work [12, 13], we can write

\[ \delta E_{2.5\text{PN}} = \frac{8}{5} \frac{\mu m^2}{r^2} \left[ (2 + \alpha) v^2 - \beta r^2 \right], \]

\[ \delta J_{2.5\text{PN}} = \frac{8}{5} \frac{\mu m^2}{r^2} \vec{r} \cdot \vec{L}_\text{N}, \]  

and for the 3.5PN-SO expressions we write the general parametrized form

\[ \delta E_{3.5\text{PN-SO}} = - \frac{1}{5} \frac{\mu^2}{r^2} \left( \vec{L}_\text{N} \cdot \vec{S} \right) \left( \alpha_1 v^2 + \alpha_2 r^2 + \alpha_3 \frac{m}{r} \right) + (S \to \xi), \]

\[ \delta J_{3.5\text{PN-SO}} = - \frac{1}{5} \frac{\mu^2}{r^2} \left( \vec{L}_\text{N} \cdot \vec{S} \right) \left( \gamma_1 v^2 + \gamma_2 r^2 + \gamma_3 \frac{m}{r} \right) + \vec{v} \cdot \left( \vec{r} + \frac{\vec{v} \times \vec{S}}{\vec{r}} \right), \]

\[ \delta J_{3.5\text{PN-SO}} = - \frac{1}{5} \frac{\mu^2}{r^2} \left( \vec{L}_\text{N} \cdot \vec{S} \right) \left( \gamma_1 v^2 + \gamma_2 r^2 + \gamma_3 \frac{m}{r} \right) + \vec{v} \cdot \left( \vec{r} + \frac{\vec{v} \times \vec{S}}{\vec{r}} \right), \]

\[ + \frac{1}{5} \frac{\mu^2}{r^2} \left( \vec{L}_\text{N} \cdot \vec{S} \right) \left( \gamma_1 v^2 + \gamma_2 r^2 + \gamma_3 \frac{m}{r} \right) + \vec{v} \cdot \left( \vec{r} + \frac{\vec{v} \times \vec{S}}{\vec{r}} \right), \]

where the notation \( S \to \xi \) means repeat the preceding terms replacing \( S \) with \( \xi \), with an appropriate set of arbitrary coefficients. This gives a total of 32 arbitrary coefficients.

We now take time derivatives of \( E^* \) and \( J^* \) in Eqs. (10), substituting the Newtonian and PN spin-orbit accelerations explicitly, to obtain

\[ \dot{E}^* = \mu v \cdot \left( a_{2.5\text{PN}} + a_{3.5\text{PN-SO}} \right) + \frac{d}{dt} \delta E_{2.5\text{PN}} + \frac{d}{dt} \delta E_{3.5\text{PN-SO}}; \]

\[ \dot{J}^* = \dot{S}_{3.5\text{PN-SO}} + \mu \times \left( a_{2.5\text{PN}} + a_{3.5\text{PN-SO}} \right) + \frac{d}{dt} \delta J_{2.5\text{PN}} + \frac{d}{dt} \delta J_{3.5\text{PN-SO}} \]

In fact, we will show in Appendix [15] that, if we assume that \((\dot{S}_1)_{3.5\text{PN-SO}}\) is orthogonal to \( S_1 \) (and similarly for spin 2), then the most general 3.5PN expression for \((\dot{S}_1)_{3.5\text{PN-SO}}\) turns out to be a total time derivative, which can be absorbed into a meaningless 3.5PN correction to the definition of \( S_1 \). Hence we can assume henceforth that \( \dot{S}_{3.5\text{PN-SO}} = (\dot{S}_1)_{3.5\text{PN-SO}} + (\dot{S}_2)_{3.5\text{PN-SO}} = 0 \).

We now substitute the appropriate terms from the equations of motion [4] and [8], and calculate explicitly the time derivatives of the 2.5PN and 3.5PN-SO contributions to \( E^* \) and \( J^* \). These time-derivative terms may be calculated using the identities shown in Appendix [15] which are derived using the Newtonian equations of motion and the 1PN spin-orbit terms. When evaluating \( \frac{d}{dt} \delta E_{2.5\text{PN}} \) and \( \frac{d}{dt} \delta J_{2.5\text{PN}} \), in order to obtain all terms that contribute at 3.5PN-SO order, we must include the 1PN spin-orbit terms present in the expressions in Appendix [15]. The result is

\[ \dot{E}^* = - \frac{8 \mu^2 m^2}{15 r^4} \left( 12 v^2 - 11 r^2 \right) - \frac{8 \mu^2 m^2}{10 r^6} \vec{L}_\text{N} \cdot \left( 4 \vec{S} + 3 \xi \right) \left[ (2 + \alpha) v^2 + 3 \beta r^2 \right] \]

\[ - \frac{2}{5 r^4} \left( \vec{L}_\text{N} \cdot \vec{S} \right) \left( \mathcal{P}_1 v^2 + \mathcal{P}_2 v^2 r^2 + \mathcal{P}_3 r^4 + \mathcal{P}_4 v^2 m^2 r + \mathcal{P}_5 r^2 m^2 r + \mathcal{P}_6 m^2 r^2 \right) \]

\[ + (S \to \xi) \]

\[ \dot{J}^* = - \frac{8 \mu^2 m^2}{5 r^4} \vec{L}_\text{N} \left( 2 v^2 - 3 r^2 + 2 \frac{m}{r} \right) \]
\[-\frac{8\mu^2 m\alpha}{5r^4} \left\{ -\frac{1}{2r^2} \mathbf{L}_N \cdot (4\mathbf{S} + 3\mathbf{\xi}) + \dot{r} \mathbf{n} \times \left[ (\mathbf{v} - \frac{3}{r} \mathbf{r}) \times (4\mathbf{S} + 3\mathbf{\xi}) \right] \right\} \]

\[-\frac{\mu^2}{3r^4} \left\{ \mathbf{S} \left( R_1 v^4 + R_{2i} r^2 + R_{4i} r^4 + R_4 v^2 \frac{m}{r} + R_5 r^2 \frac{m}{r} + R_6 m^2 \right) \right\} \]

\[+ \mathbf{n}(\mathbf{n} \cdot \mathbf{S}) \left( R_7 v^4 + R_{8i} r^2 + R_9 r^4 + R_{10} v^2 \frac{m}{r} + R_{11} r^2 \frac{m}{r} + R_{12} m^2 \right) \]

\[+ \dot{r} \mathbf{n}(\mathbf{n} \cdot \mathbf{S}) \left( R_{13} v^2 + R_{14} r^2 + R_{15} \frac{m}{r} \right) + \dot{r} \mathbf{v}(\mathbf{n} \cdot \mathbf{S}) \left( R_{16} v^2 + R_{17} r^2 + R_{18} \frac{m}{r} \right) \]

\[+ \mathbf{v}(\mathbf{v} \cdot \mathbf{S}) \left( R_{19} v^2 + R_{20} r^2 + R_{21} \frac{m}{r} \right) + (\mathbf{S} \rightarrow \mathbf{\xi}) \right\} \]. \hspace{1cm} (15)

The first term in each of Eqs. (14) and (15) is the 2.5PN quadrupole, or Newtonian loss term, while the second term in each case comes from the spin-orbit correction terms in Appendix A applied to \(d\delta E_{2.5\text{PN}}/dt\) and \(d\delta J_{2.5\text{PN}}/dt\). In the third set of terms in each case, the 27 coefficients \(P_n, n = 1 \ldots 6\) and \(R_n, n = 1 \ldots 21\) in the \(S\)-dependent terms are functions of the 15 coefficients \(a_n\) from the equations of motion \([8]\) and \([9]\), and of the 16 coefficients \(a_n\) and \(\gamma_n\) from the 3.5PN ambiguity terms in \(E^*\) and \(J^*\). A parallel set of 27 coefficients appear in the \(\xi\)-dependent terms, with identical dependences on the corresponding 15 + 16 arbitrary coefficients.

We now use the assumption of energy and angular momentum balance to equate the rate of energy and angular momentum loss to the corresponding far-zone fluxes \([2]\) \([3]\). The lowest-order Newtonian and the 1PN spin-orbit contributions are given by

\[\dot{E}_{\text{far zone}} = \frac{8\mu^2 m^2}{15r^4} (12v^2 - 11\dot{r}) \]

\[-\frac{8\mu^2 m}{15r^3} \left\{ \mathbf{L}_N \cdot \mathbf{S} \left( 27v^2 - 37v^2 - 12 \frac{m}{r} \right) \right\} \]

\[\mathbf{J}_{\text{far zone}} = \frac{8\mu^2 m}{5r^3} \mathbf{L}_N (2v^2 - 3v^2 + 2 \frac{m}{r}) \]

\[-\frac{4\mu^2}{5r^3} \left\{ \mathbf{S} \left( 6v^2 r^2 - 6v^4 - \frac{50}{3} v^2 \frac{m}{r} + \frac{50}{3} v^2 \frac{m}{r} - \frac{2}{m^2} \right) \right\} \]

\[+ \mathbf{n}(\mathbf{n} \cdot \mathbf{S}) \left( 18v^2 - 30v^2 r^2 + 25v^2 \frac{m}{r} + 6v^2 \frac{m}{r} + 3 \frac{m^2}{r^2} \right) \]

\[+ \dot{r} \mathbf{n}(\mathbf{v} \cdot \mathbf{S}) \left( 6v^2 - 21 \frac{m}{r} \right) - \dot{r} \mathbf{v}(\mathbf{n} \cdot \mathbf{S}) \left( 18v^2 - 30v^2 + 33 \frac{m}{r} \right) \]

\[+ \mathbf{v}(\mathbf{v} \cdot \mathbf{S}) \left( 6v^2 - 12v^2 + 23 \frac{m}{r} \right) \]

\[+ \xi \left( 5v^2 - 2v^2 r^2 - \frac{10}{3} v^2 + \frac{22}{3} v^2 \frac{m}{r} + \frac{23}{3} v^2 \frac{m}{r} - \frac{4m^2}{3r^2} \right) \]

\[+ \mathbf{n}(\mathbf{n} \cdot \mathbf{\xi}) \left( 13v^2 - 20v^2 r^2 + \frac{41}{3} v^2 \frac{m}{r} + 6v^2 \frac{m}{r} + 4m^2 \right) \]

\[+ \dot{r} \mathbf{n}(\mathbf{v} \cdot \mathbf{\xi}) \left( 7v^2 - 5v^2 r^2 - \frac{34m}{3r} \right) - \dot{r} \mathbf{v}(\mathbf{n} \cdot \mathbf{\xi}) \left( 13v^2 - 20v^2 + 64m \right) \]
After rewriting some of the terms in Eq. (15) using standard vector identities, we compare Eqs. (14) and (15) to Eqs. (16) and (17) term by term to obtain 54 constraints on the 62 coefficients. It turns out, however, that 4 of these constraints are not linearly independent of others, so there are 50 non-trivial constraints, leaving 12 undetermined degrees of freedom. The specific choice of the free coefficients is somewhat arbitrary; one choice gives the following values for the coefficients (9) in the equations of motion (5):

\begin{align*}
a_1 &= 2820 [2160] + 15 \gamma_4 + 45 \gamma_7 + 45 \gamma_9 + 15 \gamma_{11} + 15 \gamma_{12} - 3 \alpha_2, \\
a_2 &= -1728 [-1348] - 13 \gamma_4 - 39 \gamma_7 - 42 \gamma_9 - 11 \gamma_{11} - 42 \gamma_{12} + 3 \alpha_2, \\
   &\quad + 48 [36] (\alpha - \beta), \\
a_3 &= -6020 [-4620] - 35 \gamma_4 - 105 \gamma_7 - 105 \gamma_9 - 35 \gamma_{11} - 105 \gamma_{12} + 7 \alpha_2, \\
a_4 &= -220 [-164] - \gamma_4 - 3 \gamma_7 - 3 \gamma_9 - 2 \gamma_{11} - 3 \gamma_{12}, \\
a_5 &= \frac{68}{3} [36] + \gamma_4 + 3 \gamma_7 + 3 \gamma_9 + 2 \gamma_{11} + 3 \gamma_{12} + 16 [12] \alpha, \\
a_6 &= 860 [640] + 5 \gamma_4 + 15 \gamma_7 + 15 \gamma_9 + 10 \gamma_{11} + 15 \gamma_{12}, \\
a_7 &= -788 [-608] - 4 \gamma_4 - 6 \gamma_7 - 15 \gamma_9 - 4 \gamma_{11} - 15 \gamma_{12}, \\
a_8 &= \frac{3152}{3} [808] + 4 \gamma_4 + 16 \gamma_7 + 24 \gamma_9 + 4 \gamma_{11} + 16 \gamma_{12} - 32 [24] \alpha, \\
a_9 &= 2460 [1900] + 10 \gamma_4 + 30 \gamma_7 + 45 \gamma_9 + 10 \gamma_{11} + 45 \gamma_{12}, \\
a_{10} &= -148 [-112] - 2 \gamma_4 - 3 \gamma_7 - 3 \gamma_9 - 2 \gamma_{11} - 3 \gamma_{12}, \\
a_{11} &= 3320 [2540] + 25 \gamma_4 + 60 \gamma_7 + 60 \gamma_9 + 25 \gamma_{11} + 60 \gamma_{12}, \\
a_{12} &= -6020 [-4620] - 35 \gamma_4 - 105 \gamma_7 - 105 \gamma_9 - 35 \gamma_{11} - 105 \gamma_{12}, \\
a_{13} &= \frac{1276 [968]}{3} + 4 \gamma_4 + 11 \gamma_7 + 9 \gamma_9 + 4 \gamma_{11} + 5 \gamma_{12}, \\
a_{14} &= -4392 [-3372] - 23 \gamma_4 - 87 \gamma_7 - 78 \gamma_9 - 23 \gamma_{11} - 54 \gamma_{12} + 48 [36] \alpha, \\
a_{15} &= -376 \left[ -\frac{872}{3} \right] - 2 \gamma_4 - 8 \gamma_7 - 6 \gamma_9 - 2 \gamma_{11} - 2 \gamma_{12},
\end{align*}

where the numbers in square brackets represent the values to be used, along with the corresponding set of six free coefficients, for the terms in Eq. (5) involving $\xi$.

The unique choice of the twelve coefficients

\begin{align*}
a_2 &= \frac{45}{2} \gamma_4 = \frac{287}{2}, \quad \gamma_7 = -\frac{89}{6}, \quad \gamma_9 = -\frac{140}{3}, \quad \gamma_{11} = -\frac{263}{2}, \quad \gamma_{12} = -1,
\end{align*}

for the $S$ terms, and

\begin{align*}
a_2 &= -\frac{105}{2} \gamma_4 = \frac{181}{2}, \quad \gamma_7 = -\frac{155}{6}, \quad \gamma_9 = -34, \quad \gamma_{11} = -\frac{105}{2}, \quad \gamma_{12} = 2,
\end{align*}

for the $\xi$ terms, along with the values $\alpha = -1$, and $\beta = 0$ for the harmonic Damour-Deruelle gauge at 2.5PN order, gives precisely the 3.5PN spin-orbit radiation reaction terms derived in [10].
4 Gauge Freedom and Arbitrary Coefficients in the Equation of Motion

The formulas for energy and angular momentum flux in the far zone are gauge invariant, while the equations of motion are gauge, or coordinate dependent. Any coordinate transformation \( x^\mu \rightarrow x^\mu + \zeta^\mu \), where \( \zeta^\mu \) is, in a suitable sense, of 2.5PN and 3.5PN order relative to \( x^\mu \), will induce changes in the variables of a binary system, such as the relative vector \( x \) and the spin vectors. Notice that a transformation of coordinate time simply induces a velocity-dependent change in \( x \) via \( x(t + \delta t) = x(t) + v \delta t \).

As for the spin, any change induced by a gauge transformation at 2.5PN or 3.5PN order can always be reabsorbed into a new definition of spin, since it is ambiguous at radiation-reaction orders. Therefore we will only consider coordinate transformation induced changes in the relative vector \( x \) at 2.5PN and 3.5PN-SO orders, according to

\[
x' = x + \delta x_{2.5PN} + \delta x_{3.5PN-SO}.
\]

The 2.5PN order coordinate change that corresponds to the arbitrary coefficients \( \alpha \) and \( \beta \) in Eq. (3) was calculated in [12, 13], and is given by

\[
\delta x_{2.5PN} = \frac{8 \mu m}{15r} \left[ \beta \dot{r} n + (2\beta - 3\alpha)v \right].
\]

(22)

We can derive directly

\[
\begin{align*}
v' &= v + \delta x_{2.5PN} + \delta x_{3.5PN-SO}, \\
dv' &= \frac{dv}{dt} + \delta \dot{x}_{2.5PN} + \delta \dot{x}_{3.5PN-SO}, \\
m \frac{dx'}{r^3} &= m \frac{dx}{r^3} + m \frac{\dot{x}}{r^3} \left( \delta x_{2.5PN} - 3n \cdot \delta x_{2.5PN} \right) \\
&\quad+ m \frac{\dot{x}}{r^3} \left( \delta x_{3.5PN-SO} - 3nn \cdot \delta x_{3.5PN-SO} \right).
\end{align*}
\]

(23)

The 2.5PN terms in these equations must also be used to determine the induced change in the 1PN spin-orbit acceleration terms in Eq. (2). In evaluating \( \delta \dot{x}_{2.5PN} \) explicitly using Eq. (22), the 1PN spin-orbit equations must be employed wherever an acceleration occurs. The result is that the equation of motion \( \delta \dot{x} \) changes between the original and the new coordinates by a quantity \( Q \) given by

\[
Q = \left\{ \frac{8 \mu m}{5r^3} \left[ \left( 3\beta v^2 + (2\alpha - 3\beta) \frac{m}{r} - 5\beta \dot{r}^2 \right) \dot{r} n - \left( v^2 - \frac{m}{r} - 3\dot{r}^2 \right) \alpha v \right] \right\}
\]

\[
- \delta x_{3.5PN-SO} - m \frac{\dot{x}}{r^3} \left( \delta x_{3.5PN-SO} - 3nn \cdot \delta x_{3.5PN-SO} \right)
\]

\[
- \frac{8 \mu m}{5r^3} \left[ \frac{1}{2r} \mathbf{L}_N \cdot (4S + 3\xi) (3(\alpha - \beta) \dot{r} n + \alpha v) - \dot{r} v \times (4S + 3\xi) \left( \alpha + \frac{1}{3} \beta \right) \right]
\]

\[
- \frac{1}{6} n \times (4S + 3\xi) \left( \beta v^2 - \beta \frac{m}{r} - (9\alpha + 6\beta) \dot{r}^2 \right). \]

(24)

Note that the 2.5PN terms in Eq. (24) match exactly the arbitrary terms in Eq. (8). We now want to find a form for \( \delta x_{3.5PN-SO} \) so that the 3.5PN-SO terms in Eq. (24) match the terms in (8) generated by the arbitrary coefficients in Eq. (9). This can be done either by direct integration to find \( \delta x_{3.5PN-SO} \), or by assuming a suitable
form for $\delta x_{3.5\text{PN-SO}}$ and seeing if one can solve for a set of coefficients. Remarkably, a solution can be found, and is given by

$$
\delta x_{3.5\text{PN-SO}} = -\frac{\mu}{5r^2} \left[ \frac{\tilde{r}_N \cdot S}{r} \left( \gamma_4 + 3\gamma_7 + 3\gamma_9 + \gamma_{11} + 3\gamma_{12} - \frac{1}{5} \alpha_2 \right) \right] \\
+ \frac{v}{3r} \left( \tilde{L}_N \cdot S \right) \left( \gamma_4 + 3\gamma_7 + 3\gamma_9 + 3\gamma_{12} - \frac{1}{5} \alpha_2 \right) \\
+ \frac{1}{4} n \times S \left[ \gamma_7 \gamma_9 + \gamma_{12} \right] + \frac{1}{4} n \times S \left( \gamma_7 + \gamma_9 + \gamma_{12} \right), \quad (25)
$$

The 12 ($6 + 6$) coefficients correspond precisely to the 12 degrees of freedom in Eqs. (13).

5 Concluding remarks

We have used energy and angular momentum balance to deduce the general form of the 3.5PN spin-orbit radiation reaction terms in the two-body equations of motion, and showed that the remaining undetermined degrees of freedom correspond to the freedom to change gauges or coordinates at the corresponding post-Newtonian order. A specific choice of the free coefficients yields 3.5PN spin-orbit terms in the equations of motion identical with those derived from first principles. The results were subject to the physically reasonable assumption that gravitational radiation reaction has no effect on the magnitude of the individual spins, to 3.5PN order.

A natural extension of this work is to determine the contribution of spin-spin interactions in radiation reaction using balance arguments and to compare the results with those calculated from first principles by Wang and Will [11]. This work is in progress.

A Extracting total time derivatives

Using the Newtonian equations of motion plus the 1PN spin-orbit terms, it is straightforward to establish a number of identities, which may be used to relate collections of terms to total time derivatives of other expressions. For any non-negative integers $s$, $p$ and $q$, we obtain

$$
\frac{d}{dt} \left( \frac{v^{2s+p}}{r^q} \right) = \frac{v^{2s-2}\dot{p}^{p-1}}{r^{q+1}} \left[ \frac{1}{2} (p+q)v^2 r^2 - 2sp^2 \frac{m}{r} - s \frac{m}{r} \right] \\
+ \frac{p^2 v^2}{2} \tilde{L}_N \cdot (4S + 3\xi),
$$

$$
\frac{d}{dt} \left( \frac{v^{2s+p}}{r^q} \tilde{L}_N \right) = \tilde{L}_N \frac{d}{dt} \left( \frac{v^{2s+p}}{r^q} \right) \\
- \left( \frac{v^{2s+p}}{r^{q+2}} \right) n \times \left( \frac{3}{2} \dot{v} n \right) \times (4S + 3\xi), \quad (26)
$$
Another set of identities, to be used only in 3.5PN terms, require only the Newtonian equations of motion:

\[
\frac{d}{dt} \left( \frac{v^2 s^p}{r^q} x^j \right) = \frac{v^{2s-2} s^p - 1}{r^{q+1}} \left\{ \left[ p^4 - (p + q)v^2j^2 - 2s^2 m/r \right] x^j + 2v^2 \hat{r} x^{(i) j} \right\},
\]

\[
\frac{d}{dt} \left( \frac{v^2 s^p}{r^q} v^i v^j \right) = \frac{v^{2s-2} s^p - 1}{r^{q+1}} \left\{ \left[ p^4 - (p + q)v^2j^2 - 2s^2 m/r \right] v^i v^j - 2m^2 \hat{r} v^i (v^{(i) j}) \right\},
\]

\[
\frac{d}{dt} \left( \frac{v^2 s^p}{r^q} v^i \right) = \frac{v^{2s-2} s^p - 1}{r^{q+1}} \left\{ \left[ p^4 - (p + q)v^2j^2 - 2s^2 m/r \right] v^i + v^2 \hat{r} \left( v^i v^j - \frac{m}{r} n^i n^j \right) \right\}
\]

(27)

**B Evolution of spins at 3.5PN order**

In this appendix we justify our assumption that the individual spins are unaffected by 3.5PN spin-orbit effects, i.e. that \((d\hat{S}_1/dt)_{3.5PN-SO} = 0\), and similarly for body 2. First we write down the general form that 3.5PN spin-orbit terms could take, consistent with the assumptions used in earlier sections, namely

\[
(\hat{S}_1)_{3.5PN-SO} = \frac{\mu^2}{r^4} \left\{ N_1 \hat{S}_1 + N_2 \hat{m} \cdot \hat{n} \cdot \hat{S}_1 + N_3 \hat{n} \cdot \hat{v} \cdot \hat{v} \cdot \hat{S}_1 + N_4 \hat{v} \cdot \hat{v} \cdot \hat{S}_1 + N_5 \hat{v} \cdot \hat{v} \cdot \hat{S}_1 \right\},
\]

(28)

where \(N_1\) and \(N_2\) are each linear combinations of \(v^4, v^2j^2\) and \(v^2 m/r\) etc. at \(O(e^2)\) (containing 6 terms each), and \(N_3\), \(N_4\) and \(N_5\) are each linear combinations of \(v^2, j^2\) and \(m/r\). Note that other possible terms, such as \(\hat{L}_N \times \hat{S}_1\), or \(\hat{L}_N \cdot \hat{S}_1\) can be rewritten as linear combinations of the terms above.

We now impose the physically reasonable constraint \(\hat{S}_1 \cdot (\hat{S}_1)_{3.5PN-SO} = 0\), which implies that radiation reaction does not change the magnitude of the body’s spin, only its orientation. That constraint implies that only the third and fourth terms in Eq. (28) survive, and then only in an antisymmetric combination that leaves \((\hat{S}_1)_{3.5PN-SO}\) in the general form

\[
(\hat{S}_1)_{3.5PN-SO} = \frac{\mu^2}{r^4} \hat{L}_N \times \hat{S}_1 \left( c_1 v^2 + c_2 r^2 + c_3 \frac{m}{r} \right).
\]

(29)

However, it is straightforward to show, using the identities in Appendix A that the right-hand side of Eq. (29) can be written as a total time derivative and therefore can be absorbed into \(\hat{S}_1\), independently of the values of \(c_1\), \(c_2\) and \(c_3\). This is in accord with the result derived from first principles in [10].

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