ON THE LIMITING DISTRIBUTION OF SOME NUMBERS OF CROSSINGS IN SET PARTITIONS

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Abstract. We study the asymptotic distribution of the two following combinatorial parameters: the number of arc crossings in the linear representation, $c_r(\ell)$, and the number of chord crossings in the circular representation, $c_r(c)$, of a random set partition. We prove that, for $k \leq n/(2 \log n)$ (resp., $k = o(\sqrt{n})$), the distribution of the parameter $c_r(\ell)$ (resp., $c_r(c)$) taken over partitions of $[n] := \{1, 2, \ldots, n\}$ into $k$ blocks is, after standardization, asymptotically Gaussian as $n$ tends to infinity. We give exact and asymptotic formulas for the variance of the distribution of the parameter $c_r(\ell)$ from which we deduce that the distribution of $c_r(\ell)$ and $c_r(c)$ taken over all partitions of $[n]$ is concentrated around its mean. The proof of these results relies on a standard analysis of generating functions associated with the parameter $c_r(\ell)$ obtained in earlier work of Stanton, Zeng and the author. We also determine the maximum values of the parameters $c_r(\ell)$ and $c_r(c)$.

1. Introduction and Main results

1.1. Introduction. Take $n$ points on a circle labeled $1, 2, \ldots, n$ and join them in vertex disjoint simple polygons (the vertices of which are the points $1, 2, \ldots, n$). The resulting configuration is the circular representation of the set partition of $[n] := \{1, 2, \ldots, n\}$ the blocks of which consist of the elements in the same polygons. Alternatively, take $n$ points labeled $1, 2, \ldots, n$ on a line and join them in vertex disjoint directed paths consisting of arcs oriented to the right drawn in the upper half-plane. Arcs are always drawn in such a way that any two arcs cross at most once. The resulting configuration is the linear representation of the set partition of $[n]$ the blocks of which consist of the elements in the same paths. An illustration is given in Figure 1. These representations suggest two natural combinatorial parameters on set partitions: the number of pairs of crossing chords (resp., arcs) in the circular (resp., linear) representation, denoted $c_r(c)$ (resp., $c_r(\ell)$). For instance, if $\pi$ is the set partition represented in Figure 1 then $c_r(c)(\pi) = 9$ and $c_r(\ell)(\pi) = 4$. Throughout this paper, the set of all partitions of $[n]$ will be denoted by $\Pi_n$ and we let $\Pi^k_n$ denote the set of all partitions of $[n]$ into $k$ blocks.
A set partition each block of which has exactly two elements is often called a complete matching and its circular representation is often called a chord diagram. There has been significant interest (e.g., see [14, 11, 3, 5] and the references there) in studying the distribution of the parameter $c^{(c)}$ and $c^{(\ell)}$ in complete matchings (note that $c^{(c)} = c^{(\ell)}$ on the set of matchings). Notably, a remarkable exact counting formula (often called the Touchard-Riordan formula) in terms of the ballot numbers was implicit in the work of Touchard [14] and made explicit later by Riordan [11], and Flajolet and Noy [3] proved that the distribution of the parameter $c^{(c)}$ in a random chord diagram consisting of $n$ chords is asymptotically Gaussian as $n \to \infty$.

The enumeration of (general) set partitions by the parameter $c^{(\ell)}$ has also received considerable interest (e.g., see [1, 5, 8, 7]). This is partly due to the fact that this parameter arises in the combinatorial theory of continued fractions and of a natural $q$-analog of Charlier polynomials [1, 5, 7]. Moreover, there exist combinatorial parameters on set partitions which have the same distribution as the parameter $c^{(\ell)}$ over each $\Pi_n^k$. This is the case for the number of nestings of two arcs [8] and the major index for set partitions introduced in [2].

A classical result is that the number of noncrossing partitions (partitions $\pi$ with $c^{(\ell)}(\pi) = 0$, or equivalently, with $c^{(c)}(\pi) = 0$) of $[n]$ is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. Consider the enumerating polynomial $T_{n,k}(q)$ defined by

$$T_{n,k}(q) = \sum_{\pi \in \Pi_n^k} q^{c^{(\ell)}(\pi)}.$$  

Biane [1] proved the Jacobi type continued fraction expansion

$$\sum_{n \geq k \geq 0} T_{n,k}(q) a^k t^n = \frac{1}{|1 - (a + [0]_q)z|} - \frac{a[1]_q z^2}{|1 - (a + [1]_q)z|} - \frac{a[2]_q z^2}{|1 - (a + [2]_q)z|} - \cdots$$

Stanton, Zeng and the author [7] obtained the expansion (see Equation 28 in [7])

$$\sum_{n \geq k \geq 0} T_{n,k}(q) a^k t^n = \sum_{k=0}^{\infty} \frac{(agt)^k}{\prod_{i=1}^{k} (q^i - q^i[i]_q t + a(1 - q)[i]_q t)},$$

**Figure 1.** Representations of the partition $\pi = 110/2379/4/5612/811$
from which they derived the remarkable formula (see Equation 32 in [7])

\[ T_{n,k}(q) = \sum_{j=1}^{k} (-1)^{k-j} \frac{[j]_q^n}{[j]_q!} q^{-kj} B_{n,k,j}(q), \quad \text{with} \]

\[ B_{n,k,j}(q) = \sum_{i=0}^{k-j} \frac{(1-q)^i}{[k-j-i]_q!} q^{(k-j-i+1)} \left( \binom{n}{i} q^i + \binom{n}{i-1} \right). \]

Here, as usual, \([a]_q! = [1]_q [2]_q \cdots [a]_q\) with \([a]_q = \frac{1-q^a}{1-q}\) = 1 + q + ⋯ + q^{a-1}. Another remarkable formula (see (1.4)) for \(T_{n,k}(q)\) was also recently established by Josuat-Vergès and Rubey [5]:

\[ T_{n,k}(q) = \frac{1}{(1-q)^{n-k}} \sum_{j=0}^{k} \sum_{i=j}^{n-k} A_{i,j}(q), \quad \text{where} \]

\[ A_{i,j}(q) = (-1)^i \left( \binom{n}{k+i} \left( \binom{n}{k-j} - \binom{n}{k+i+1} \binom{n}{k-j-1} \right) \right) [i]_{q}^{(i+1)}. \]

and, as usual, the Gaussian coefficient \([i]_{q}^{(i+1)}\) is given by \([i]_{q}^{(i+1)} = \frac{[i]_q!}{[i]_q [i]_{q^{i+1}}}.\)

Note that if seems extremely hard to get a convenient expression for the distribution of \(c^{r(t)}\) (i.e., for any \(j\), formulas that give the numbers of partitions of \([n]\) satisfying \(c^{r}(\pi) = j\) from the above formulas. Furthermore, except for the formula for the number of noncrossing partitions, it seems that nothing else is known about the “exact” enumeration of set partitions by the parameter \(c^{r(c)}\). In such a situation, it is customary to look for asymptotic approximations (see e.g. [12, Chapter 4] for an accurate study of the blocks of a random set partition or more recently [9] for the study of records). A first step was taken in [6] where the author has obtained exact and asymptotic formulas for the average values of the parameters \(c^{r(t)}\) and \(c^{r(c)}\) in a random set partition. The present paper is mainly devoted to providing further properties of the asymptotic behavior of these parameters. Our approach is straightforward (although sometimes tedious): most of the results presented in this paper are derived from the generating functions (1.2)-(1.3) by using standard methods of analytic combinatorics. We will also determine the maximum values of the parameter \(c^{r(t)}\) and \(c^{r(c)}\) on \(\Pi_n\) and \(\Pi_n^k\).

Before stating our results we need to set up some notation. We let \(X_n\) and \(X_{n,k}\) (resp., \(Y_n\) and \(Y_{n,k}\)) denote the random variables equal to the value of \(c^{r(t)}\) (resp., \(c^{r(c)}\)) taken, respectively, over \(\Pi_n\) and \(\Pi_{n,k}\) endowed with the uniform probability distribution, i.e., for any nonnegative integer \(t\),

\[ Pr(X_n = t) = \frac{\# \{ \pi \in \Pi_n : c^{r(t)}(\pi) = t \}}{B_n}, \]

\[ Pr(X_{n,k} = t) = \frac{\# \{ \pi \in \Pi_n^k : c^{r(t)}(\pi) = t \}}{S_{n,k}}, \]

with similar statements for \(Y_n\) and \(Y_{n,k}\). Here \(B_n\), the \(n\)-th Bell number, is the cardinality of \(\Pi_n\), and \(S_{n,k}\), the \((n,k)\)-th Stirling number of the second kind, is the cardinality of \(\Pi_n^k\). As usual, the mean and the variance of a random variable \(Z\) will be denoted by
E(Z) and Var(Z), and we use \( \rightarrow^d \) (resp., \( \rightarrow^p \)) for convergence in distribution (resp., in probability). We recall that a sequence of random variables \((Z_n)_{n \geq 1}\) is said to be asymptotically Gaussian if \( (Z_n - E(Z_n)) / \sqrt{\text{Var}(Z_n)} \rightarrow^d \mathcal{N}(0,1) \) as \( n \rightarrow \infty \), where \( \mathcal{N}(0,1) \) is the standard normal distribution. In the rest of this paper, we take as granted the elementary asymptotic calculus with usual Landau notations.

1.2. Main results. Let us first recall the closed form expressions for \( E(X_{n,k}) \) and \( E(Y_{n,k}) \) recently obtained by the author.

**Theorem 1.1.** (Kasraoui [6]) For all integers \( n \geq k \geq 1 \), we have

\[
E(X_{n,k}) = \frac{1}{2} n(k-1) - \frac{5}{4} k(k-1) + \frac{3}{2} (n+1-k) \frac{S_{n,k-1}}{S_{n,k}},
\]

\[
E(Y_{n,k}) = \frac{1}{2} n(k-1) - \frac{1}{2} n(4n-5k+1) \frac{S_{n-1,k-1}}{S_{n,k}} - 10 \left( \frac{n}{2} \right) \frac{S_{n-2,k-2}}{S_{n,k}} + \binom{n}{4} \frac{S_{n-4,k-2}}{S_{n,k}}.
\]

Using the well-known summation formula \( S_{n,k} = \frac{k^n}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \), it is easy to see that

\[
S_{n,k} = \frac{k^n}{k!} (1 + o(1)) \quad \text{as} \quad n \rightarrow \infty \quad \text{uniformly for} \quad k \leq n/(2 \log n).
\]

Inserting (1.7) in Theorem 1.1, we immediately obtain the following result.

**Corollary 1.2.** For \( 1 \leq k \leq n/(2 \log n) \) and as \( n \rightarrow \infty \), we have

\[
E(X_{n,k}) = \frac{k-1}{2} n - \frac{5}{4} k(k-1) + o(1),
\]

\[
E(Y_{n,k}) = \frac{k-1}{2} n + o(1).
\]

The above result was explicitly stated for fixed integers \( k \) in [6]. We now present our first non-trivial result which provides a useful expression for the variance of \( X_{n,k} \). Its proof is given in Section 2.

**Theorem 1.3.** For all integers \( n \geq k \geq 1 \), the variance of \( X_{n,k} \) satisfies

\[
\text{Var}(X_{n,k}) = \frac{1}{12} (k^2 - 1)n - \frac{1}{72} k(k-1)(2k+5)
\]

\[
+ \frac{1}{12} \left( -15n^2 + (56k - 43)n - 2(k-1)(14k-1) \right) \frac{S_{n,k-1}}{S_{n,k}}
\]

\[
+ \frac{1}{12} (n-k+2)(27n-27k+1) \frac{S_{n,k-2}}{S_{n,k}} - \frac{9}{4} (n+1-k)^2 \left( \frac{S_{n,k-1}}{S_{n,k}} \right)^2.
\]

The above theorem will be essentially derived by means of standard but very tedious computations and manipulations of the generating function expansion (1.2). Combining the previous theorem with (1.7), we arrive at the following result.
Corollary 1.4. For $1 \leq k \leq n/(2 \log n)$ and as $n \to \infty$, we have

$$\text{Var}(X_{n,k}) = \frac{k^2 - 1}{12} n - \frac{1}{72} k(k-1)(2k+5) + o(1).$$  

(1.10)

The two following theorems are probably the most interesting results of the present paper. In Section 3, using expression (1.3) and Curtiss’ theorem for sequences of moment generating functions, we prove a central limit law for the random variable $X_{n,k}$.

Theorem 1.5. For $2 \leq k \leq n/(2 \log n)$, the distribution of $X_{n,k}$ is asymptotically Gaussian as $n \to \infty$.

Let us mention that it seems that the asymptotic distribution of $X_{n,k}$ for other types of interesting ranges for $k$ (for instance, $k \sim an$ with $0 < a < 1$) can be analyzed with a similar method. However, the asymptotic analysis of the moment generating function of $X_{n,k}$ (see Section 3) seems much more difficult for $k \sim an$ than for $k \leq n/(2 \log n)$. This explains why, in this paper, we have limited our study to the latter range.

We now turn our attention to the random variable $Y_{n,k}$. In contrast to the parameter $cr(\ell)$, we have no convenient expression for the distribution of $cr(c)$. Nonetheless, exploiting the combinatorial ‘closeness’ of the parameters $cr(c)$ and $cr(\ell)$ (see Lemma 5.1 for a precise statement) and the results on the distribution of $X_{n,k}$ expounded above, we have managed to obtain interesting (but, necessarily, weaker) results on the asymptotic distribution of $Y_{n,k}$. The following result is demonstrated in Section 5.1.

Theorem 1.6. For $k = o(\sqrt{n})$ and as $n \to \infty$,

(i) the variance of the distribution of $Y_{n,k}$ satisfies

$$\text{Var}(Y_{n,k}) = \frac{k^2 - 1}{12} n + O(k^3 \sqrt{n}),$$  

(1.11)

(ii) the distribution of $Y_{n,k}$ is asymptotically Gaussian.

It seems more difficult to deal with the asymptotic distribution of $X_n$ and $Y_n$. Actually, numerical evidence (see Figure 2) naturally lead to the following conjecture.

Conjecture 1.7. The distributions of $X_n$ and $Y_n$ are asymptotically Gaussian.

We have not been able to prove this conjecture. We can, however, show that the distributions of $X_n$ and $Y_n$ are concentrated around their mean. We first recall the approximation of $E(X_n)$ and $E(Y_n)$ recently obtained by the author.

Theorem 1.8. (Kasraoui [6]) As $n \to \infty$, the means $E(X_n)$ and $E(Y_n)$ are both equal to

$$\frac{n^2}{2 \log n} \left( 1 + \frac{\log \log n}{\log n} (1 + o(1)) \right).$$

In Section 4, we deduce from the expression (1.3) for $E(X_{n,k})$ and the formula of $\text{Var}(X_{n,k})$ in Theorem 1.3 a useful expression for $\text{Var}(X_n)$ from which we will obtain the following fairly good asymptotic approximation of $\text{Var}(X_n)$.
Theorem 1.9. The variance of $X_n$ satisfies, as $n \to \infty$,

$$\text{Var}(X_n) = \frac{n^3}{3(\log n)^2} \left( 1 + 2\frac{\log \log n}{\log n} \left(1 + o(1)\right) \right).$$

(1.12)

Although the proximity of the parameters $cr^{(c)}$ and $cr^{(l)}$ lead us to believe that $\text{Var}(Y_n)$ and $\text{Var}(X_n)$ are asymptotically equivalent, we are currently unable to obtain a satisfactory approximation of $\text{Var}(Y_n)$. In Section 5.2, we derive only the following bound (which seems to be far from sharp but is sufficient for our purpose) by exploiting the closeness of $cr^{(c)}$ to $cr^{(l)}$.

Theorem 1.10. The variance of $Y_n$ satisfies, as $n \to \infty$,

$$\text{Var}(Y_n) = O\left(\frac{n^4}{(\log n)^4}\right).$$

(1.13)

Combining Theorem 1.8 with equations (1.12)–(1.13), we see that $\sqrt{\text{Var}(X_n)/\mathbb{E}(X_n)} \to 0$ and $\sqrt{\text{Var}(Y_n)/\mathbb{E}(Y_n)} \to 0$ as $n \to \infty$. This leads immediately (by Chebyshev’s inequality) to the following result.

Corollary 1.11. As $n \to \infty$, the distributions of $X_n$ and $Y_n$ are concentrated around their mean, i.e., $\frac{X_n}{\mathbb{E}(X_n)} \xrightarrow{p} 1$ and $\frac{Y_n}{\mathbb{E}(Y_n)} \xrightarrow{p} 1$.

Last but not least, in Section 6, we find maximum values of the parameters $cr^{(l)}$ and $cr^{(c)}$ on $\Pi_n$ and $\Pi_n^k$. This answers a question of P. Nadeau [10] which is important.
for the comprehension of the distribution of the parameters \(cr^{(t)}(\pi)\) and \(cr^{(c)}(\pi)\). The answer is, in full generality, far from obvious. Note that it is easy to see that for each \(n \geq k \geq 1\), there is a partition \(\pi\) in \(\Pi_{n,k}\) such that \(cr^{(t)}(\pi) = cr^{(c)}(\pi) = 0\) (there are in fact \(N(n,k)\) such partitions where \(N(n,k)\) is a Narayana number).

**Theorem 1.12.** Let \(M_{n,k}^{(t)}\) (resp., \(M_{n,k}^{(c)}\)) denote the maximum value of \(cr^{(t)}(\pi)\) (resp., \(cr^{(c)}(\pi)\)) taken over all \(\pi \in \Pi_{n,k}^k\). Then, we have

1. (a) if \(1 \leq k \leq \left[\frac{n}{2}\right]\), \(M_{n,k}^{(t)} = (k-1)n - 3\left(\begin{array}{c} k \\ 2 \end{array}\right)\);
   
   (b) if \(\left[\frac{n}{2}\right] \leq k \leq n\), \(M_{n,k}^{(t)} = \left(\frac{n-k}{2}\right)\);

2. (a) if \(k \leq \frac{n}{2}\), \(M_{n,k}^{(c)} = 2\left(\begin{array}{c} k \\ n \end{array}\right) + 2\left(\begin{array}{c} r \\ n \end{array}\right) = (k-1)n - r(k-r)\), where \(r\) is the remainder in the division of \(n\) by \(k\),

   (b) if \(\frac{n}{2} \leq k < \frac{n}{2}\) or \(\frac{n}{2} \leq k \leq n - 6\), \(M_{n,k}^{(c)} = 6\left(\begin{array}{c} \frac{n}{2} \\ 3 \end{array}\right) + 2\left(\begin{array}{c} \frac{n}{2} \\ 2 \end{array}\right)\chi(a \equiv 1 \pmod{2})\) where we set \(a = n - k\),

   (c) if \(k \geq n - 5\) and \(k \geq \frac{n}{2}\), \(M_{n,k}^{(c)} = \left(\frac{n-k}{2}\right)\).

The proof of the above result is far from trivial and relies on a tedious discrete optimization. It is also worth noting that the first part of Theorem 1.12 could be also derived by computing the degree of \(T_{n,k}(q)\) in \(q\) thanks to the expressions (1.3) or (1.4), but this method is inefficient for determining \(M_{n,k}^{(c)}\).

In conjunction with the above result, it is not difficult to determine the global maxima of the functions \(k \mapsto M_{n,k}^{(t)}\) and \(k \mapsto M_{n,k}^{(c)}\) defined on \([n]\). This yields the following result.

**Theorem 1.13.** Let \(M_{n}^{(t)}\) (resp., \(M_{n}^{(c)}\)) denote the maximum value of \(cr^{(t)}(\pi)\) (resp., \(cr^{(c)}(\pi)\)) taken over all \(\pi \in \Pi_n\). Then, we have

1. if \(n \geq 1\), \(M_{n}^{(t)} = \left\lfloor\frac{1}{3}\left(\begin{array}{c} n-1 \\ 2 \end{array}\right)\right\rfloor\),

2. (a) if \(n \geq 5\) and \(n \equiv 0 \pmod{3}\), \(M_{n}^{(c)} = \left\lfloor\frac{2}{3}\left(\begin{array}{c} n-1 \\ 2 \end{array}\right)\right\rfloor\),

   (b) if \(n \geq 5\) and \(n \equiv 1,2 \pmod{3}\), \(M_{n}^{(c)} = \left\lfloor\frac{2}{3}\left(\begin{array}{c} n-2 \\ 2 \end{array}\right)\right\rfloor\).

2. The variance of \(X_{n,k}\): proof of Theorem 1.13

Our proof of Theorem 1.13 essentially relies on the generating function expansion (1.2). Recall that a useful property of the probability generating function \(G(q)\) of a non-negative integer valued random variable \(Z\) is that its \(m\)-th derivative at \(q = 1\) gives the \(m\)-th factorial moment of \(Z\). By definition, the probability generating function of \(X_{n,k}\) is \(p_{n,k}(q) = T_{n,k}(q)/S_{n,k}\), where \(T_{n,k}(q)\) is defined in (1.1). Consequently, we have \(E(X_{n,k}(X_{n,k} - 1)) = T_{n,k}''(1)/S_{n,k}\), whence

\[
(2.1) \quad \text{Var}(X_{n,k}) = \frac{T_{n,k}''(1)}{S_{n,k}} + E(X_{n,k}) - E(X_{n,k})^2.
\]

To find a “convenient” expression for \(\text{Var}(X_{n,k})\), we just have to find a formula for \(T_{n,k}''(1)\) since (1.5) already provides a formula for \(E(X_{n,k})\). We will “extract” a formula for \(T_{n,k}''(1)\) from (1.2) by a routine but unpleasant computation.
Proposition 2.1. For all integers \( n \geq k \geq 1 \), we have
\[
\frac{T''_{n,k}(1)}{S_{n,k}} = \frac{(k-1)^2}{4} n^2 - \frac{k-1}{12} (15k^2 - 16k + 5)n + \frac{k(k-1)}{144} (225k^2 - 229k + 170) + \frac{1}{12} ((18k-33)n^2 - (63k^2 - 137k + 79)n + (k-1)(45k^2 - 73k + 20)) \frac{S_{n,k-1}}{S_{n,k}} + \frac{1}{12} (27n^2 - (54k - 55)n + (k-2)(27k - 1)) \frac{S_{n,k-2}}{S_{n,k}}.
\]

Inserting Proposition 2.1 and (1.5) in (2.1) gives Theorem 1.3. So, to complete the proof of Theorem 1.3, it suffices to verify Proposition 2.1.

Proof of Proposition 2.1 By (1.2), the ordinary generating function of the \( T''_{n,k}(1) \)'s satisfies
\[
\sum_{n \geq k \geq 0} T''_{n,k}(1) a^k t^n = \sum_{k \geq 0} (at)^k F_k''(1),
\]
where \( F_k(q) = \prod_{i=1}^k f_i(q) \) with
\[
f_i(q) = f_i(q; a, t) = \frac{q}{q^i - q^i[i]_q t + a(1-q)[i]_q t}.
\]
Using Leibnitz's rule for the derivative of a product, we get
\[
F_k''(1) = F_k(1) \left( \left( \sum_{i=1}^k \frac{f''_i(1)}{f_i(1)} \right)^2 + \sum_{i=1}^k \left( \frac{f''_i(1)}{f_i(1)} - \left( \frac{f'_i(1)}{f_i(1)} \right)^2 \right) \right).
\]
Moreover, using expression (2.3) for \( f_i(q) \), after a routine computation followed by partial fraction decompositions, we arrive at
\[
\frac{f'_i(1)}{f_i(1)} = -\frac{3i}{2} - \frac{1 - 3t + 2at}{2t} + \frac{-1 + t - 2at}{2t(-1 + it)},
\]
\[
\frac{f''_i(1)}{f_i(1)} - \left( \frac{f'_i(1)}{f_i(1)} \right)^2 = -\frac{i^2}{12} + \frac{i(1 + 9t + 12at)}{6t} + \frac{5 + 36at - 17t^2 + 12a^2t^2}{12t^2} + \frac{4 - 3t + 24at - t^2 - 6at^2 + 12a^2t^2}{6t^2(-1 + it)} + \frac{1 - 2t + 4at + t^2 - 4at^2 + 4a^2t^2}{4t^2(-1 + it)^2}.
\]
For \( k \geq 0 \), define power series \( U_k(t) \) and \( V_k(t) \) by
\[
U_k(t) := \sum_{i=1}^k \frac{1}{1 - it} \quad \text{and} \quad V_k(t) := \sum_{i=1}^k \frac{1}{(1-it)^2},
\]
and set \( G_k(t) := t^k F_k(1) \). By specializing (2.3) at \( q = 1 \), we have
\[
G_k(t) = t^k \prod_{i=1}^k f_i(1) = \frac{t^k}{\prod_{i=1}^k (1-it)} = \sum_{n \geq 0} S_{n,k} t^n,
\]
where the last equality is a well-known power series expansion.
Combining (2.4) with (2.5)–(2.8), it is easy (but unpleasant except if we use a computer algebra system) to show that
\[
t^k F''_k(1) = G_k(t) \left( p_k(a, t) + q_k(a, t) U_k(t) + r_k(a, t) (U_k(t))^2 + s_k(a, t) V_k(t) \right),
\]
where \( p_k(a, t) \), \( q_k(a, t) \), \( r_k(a, t) \) and \( s_k(a, t) \) are polynomials in \( \mathbb{C}[a, a^{-1}, t, t^{-1}] \) given by
\[
p_k(a, t) = \frac{k}{144} \left( -98 + 183k - 166k^2 + 81k^3 + 144a^2(1 + k) + 72a(2 - k + 3k^2) \right)
+ \frac{k(1 - 8k + 9k^2 + 12a(3 + k))}{12t} + \frac{k(5 + 3k)}{12t^2},
\]
\[
q_k(a, t) = \frac{1}{12} \left( 2 - 9k + 9k^2 - 24a^2(1 + k) + 6a(2 + 5k - 3k^2) \right)
+ \frac{2 + 5k - 3k^2 - 8a(2 + k)}{4t} + \frac{-4 - 3k}{6t^2},
\]
\[
r_k(a, t) = s_k(a, t) = \frac{1}{4} - a + a^2 + \frac{-1 + 2a}{2t} + \frac{1}{4t^2}.
\]
In conjunction with (2.2), this implies that
\[
T''_{n,k}(1) = [a^k t^n] \sum_{k \geq 0} a^k G_k(t) p_k(a, t) + [a^k t^n] \sum_{k \geq 0} a^k G_k(t) U_k(t) q_k(a, t)
+ [a^k t^n] \sum_{k \geq 0} a^k G_k(t) \left( (U_k(t))^2 + V_k(t) \right) r_k(a, t).
\]
(2.10)

Here, as usual, \([a^i t^j]W(a, t)\) is for the coefficient of \(a^i t^j\) in the power series expansion of \(W(a, t)\).

**Lemma 2.2.** We have the formal power series expansions
\[
C_1(a, t) := \sum_{k \geq 0} a^k G_k(t) = \sum_{n,k \geq 0} S_{n,k} a^k t^n,
\]
\[
C_2(a, t) := \sum_{k \geq 0} a^k G_k(t) U_k(t) = \sum_{n,k \geq 0} nS_{n,k} a^k t^n,
\]
\[
C_3(a, t) := \sum_{k \geq 0} a^k G_k(t) \left( (U_k(t))^2 + V_k(t) \right) = \sum_{n,k \geq 0} n(n + 1)S_{n,k} a^k t^n,
\]
where \(U_k(t), V_k(t)\) and \(G_k(t)\) are defined in (2.7) and (2.8).

**Proof.** The first expansion is immediate from (2.8). If we derive twice each side of (2.8), we obtain the power series expansions
\[
G'_{k}(t) = \frac{1}{t} G_k(t) U_k(t) = \frac{1}{t} \sum_{n \geq 0} nS_{n,k} t^n,
\]
\[
G''_{k}(t) = \frac{1}{t^2} G_k(t) \left( (U_k(t))^2 + V_k(t) - 2U_k(t) \right) = \frac{1}{t^2} \sum_{n \geq 0} n(n - 1)S_{n,k} t^n,
\]
from which it is straightforward to deduce the expansions of \(C_2(a, t)\) and \(C_3(a, t)\). □
It is now a routine matter to derive a formula for $T_{n,k}''(1)$ (which involves only polynomials in $n$ and $k$ and Stirling numbers) from (2.10). Indeed, after routine coefficient extractions in (2.10) based on Lemma 2.2, it is easy to obtain

$$T_{n,k}''(1) = (k - 1)(k - 2) S_{n,k-2} - 2(k - 1) n S_{n,k-2} + n(n + 1) S_{n,k-2}$$

$$+ \frac{1}{2}(k - 1)(3k^2 - 7k + 6) S_{n,k-1} - \frac{1}{2}(k - 3)(3k - 2) n S_{n,k-1}$$

$$- n(n + 1) S_{n,k-1} + \frac{1}{144} k(k - 1)(81k^2 - 85k + 98) S_{n,k}$$

$$+ \frac{1}{12} (3k - 1)(3k - 2) n S_{n,k} + \frac{1}{4} n(n + 1) S_{n,k} + (k + 1)(k + 2) S_{n+1,k-1}$$

$$- 2(k + 1)(n + 1) S_{n+1,k-1} + (n + 1)(n + 2) S_{n+1,k-1}$$

$$+ \frac{1}{12} k(9k^2 - 8k + 1) S_{n+1,k} - \frac{1}{4} (3k + 1)(k - 2)(n + 1) S_{n+1,k}$$

$$- \frac{1}{2}(n + 1)(n + 2) S_{n+1,k} + \frac{1}{12} k(3k + 5) S_{n+2,k}$$

$$- \frac{1}{6} (3k + 4)(n + 2) S_{n+2,k} + \frac{1}{4} (n + 2)(n + 3) S_{n+2,k}.$$

By replacing in the latter equation each occurrence of the left hand sides of the three following identities

$$S_{n+1,k} = S_{n,k-1} + k S_{n,k}, \quad S_{n+1,k-1} = S_{n,k-2} + (k - 1) S_{n,k-1},$$

$$S_{n+2,k} = S_{n,k-2} + (2k - 1) S_{n,k-1} + k^2 S_{n,k},$$

by the corresponding right hand sides, we arrive at Proposition 2.1. \qed

3. LIMITING DISTRIBUTION OF $X_{n,k}$

This section is devoted to proving Theorem 1.5. For simplicity, throughout this section, all asymptotic are meant for $2 \leq k \leq n/(2 \log n)$ and $n \to \infty$ unless otherwise stated and we denote $\mu_{n,k} = E(X_{n,k})$ and $\sigma_{n,k}^2 = \text{Var}(X_{n,k}).$

Let $\tilde{M}_{n,k}(t)$ (resp., $\tilde{P}_{n,k}(q)$) be the moment (resp., probability) generating function of the random variable $\tilde{X}_{n,k} = (X_{n,k} - \mu_{n,k})/\sigma_{n,k}.$ Then, we have $\tilde{M}_{n,k}(t) = \tilde{P}_{n,k}(e^t),$ whence

$$\tilde{M}_{n,k}(t) = \exp \left( -\frac{\mu_{n,k}}{\sigma_{n,k}} \right) \frac{T_{n,k}(e^{t/\sigma_{n,k}})}{\sigma_{n,k}},$$

where $T_{n,k}(q)$ is defined in (1.1). Using expression (1.3) and only elementary asymptotic analysis, we shall prove that $\tilde{M}_{n,k}(t)$ converges pointwise on $\mathbb{R}$ to the function $g(t) := \exp(t^2/2)$. By a celebrated theorem of Curtiss (see e.g. Theorem 2.7 in [12]), this will imply that $\tilde{X}_{n,k} \overset{d}{\to} \mathcal{N}(0,1)$, as stated in Theorem 1.5.

Lemma 3.1. Let $t \in \mathbb{R}$ and $u := \exp(t/\sigma_{n,k}).$ Then, for $2 \leq k \leq n/(2 \log n)$ and as $n \to \infty$, we have

$$T_{n,k}(u) = u^{-k^2} \left[ k \right]_{\mu}^n \left( 1 + o(1) \right).$$
Inserting Lemma 3.1 and the approximation (1.7) in (3.1), we arrive at

\[ \tilde{M}_{n,k}(t) = u^{-k^2 - \mu_{n,k}} \left( \frac{[k]_u}{k} \right)^n \frac{k!}{[k]_u!} (1 + o(1)) . \]

Lemma 3.2. Let \( t \in \mathbb{R} \) and \( u := \exp(t/\sigma_{n,k}) \). Then, for \( 2 \leq k \leq n/(2 \log n) \) and as \( n \to \infty \), we have

(a) \( \log \left( \frac{[k]_u}{k} \right) = \frac{k - 1}{2 \sigma_{n,k}} t + \frac{k^2}{24 \sigma_{n,k}^2} t^2 + o \left( \frac{k^2}{\sigma_{n,k}^2} \right) , \)

(b) \( \log \left( \frac{[k]_u!}{k!} \right) = \frac{k(k - 1)}{4 \sigma_{n,k}} t + o(1) . \)

Inserting Lemma 3.2 in (3.3), and then using the approximations (1.8) and (1.10) for \( \mu_{n,k} \) and \( \sigma_{n,k}^2 \), after a routine computation, we obtain

\[ \log \tilde{M}_{n,k}(t) = -\frac{k^2 - \mu_{n,k}}{\sigma_{n,k}} t + n \log \left( \frac{[k]_u}{k} \right) - \log \left( \frac{[k]_u!}{k!} \right) + o(1) \]

\[ = \left( -k^2 - \mu_{n,k} + n \frac{k - 1}{2} - \frac{k(k - 1)}{4} \right) \frac{t}{\sigma_{n,k}} + \frac{n k^2}{24 \sigma_{n,k}^2} t^2 + o(1) \]

\[ = o(k) \cdot \frac{t}{\sigma_{n,k}} + \frac{t^2}{2} + o(1) = \frac{t^2}{2} + o(1) , \]

as desired. So, to complete the proof of Theorem 1.5 it suffices to prove Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.2 (a) By definition, \( [k]_u = (e^{tk/\sigma_{n,k}} - 1) (e^{t/\sigma_{n,k}} - 1)^{-1} \). Using the asymptotic expansion \( e^x = 1 + x + x^2/2 + x^3/6 + o(x^2) \) (valid as \( x \to 0 \)), it is easy to check that, for any (fixed) real \( t \), if \( w/v \to 0 \) as \( (v, w) \to (0, 0) \), then we have, as \( (v, w) \to (0, 0) \),

\[ (e^{vt} - 1) (e^{wt} - 1)^{-1} = v w^{-1} \left( 1 + (v - w) t/2 + v^2 t^2/6 + o(v^2) \right) . \]

Noting that \( k/\sigma_{n,k} \) is \( O(n^{-1/2}) \) by (1.10) and specializing the above formula at \( v = k/\sigma_{n,k} \) and \( w = 1/\sigma_{n,k} \), we arrive at

\[ [k]_u = k \left( 1 + \frac{k - 1}{2 \sigma_{n,k}} t + \frac{k^2}{6 \sigma_{n,k}^2} t^2 \right) . \]

This implies, by the expansion \( \log(1 + x) = x - x^2/2 + o(x^2) \) as \( x \to 0 \), that

\[ \log \left( \frac{[k]_u}{k} \right) = \frac{k - 1}{2 \sigma_{n,k}} t + \frac{k^2}{6 \sigma_{n,k}^2} t^2 - \frac{1}{2} \left( \frac{k - 1}{2 \sigma_{n,k}} t + \frac{k^2}{6 \sigma_{n,k}^2} t^2 \right)^2 + o \left( \frac{k^2}{\sigma_{n,k}^2} \right) , \]

which leads, after straightforward simplifications, to the desired result.
(b) Let $I_k(q) := \frac{|k|}{\ell!}$. Sachkov (see e.g. [12, Section 1.3.1, p.29]) proved that

$$
(3.5) \quad \log I_k(e^x) = \frac{k(k-1)}{4} x + \sum_{\ell=1}^{\infty} b_{2\ell} \frac{x^{2\ell}}{2\ell (2\ell)!} \sum_{j=1}^{k} (j^{2\ell} - 1) \quad (|x| < 2\pi),
$$

where $b_{2\ell}$ are the Bernoulli numbers. Since for all $\ell \geq 1$

$$
\sigma_{n,k}^{2\ell} \sum_{j=1}^{k} (j^{2\ell} - 1) = \sigma_{n,k}^{2\ell} \cdot O \left( k^{2\ell+1} \right) = O \left( \frac{k}{n^\ell} \right) \quad \text{as } k \to \infty,
$$

we have, in view of the expansion (3.5),

$$
\log I_k(\exp(t/\sigma_{n,k})) - \frac{k(k-1)}{4} t = \sum_{\ell=1}^{\infty} b_{2\ell} \frac{t^{2\ell}}{2\ell (2\ell)!} \sigma_{n,k}^{2\ell} \sum_{j=1}^{k} (j^{2\ell} - 1) \to 0.
$$

This is exactly the formula (b) in Lemma 3.2. □

**Proof of Lemma 3.1**

**Step 1.** Consider the $B_{n,k,j}(q)$’s defined in (1.3). We claim that, for any positive real number $q$, we have

$$
(3.6) \quad \left| T_{n,k}(q) - B_{n,k,j}(q) \right| \leq \left( 1 + 2q^k \right) \exp \left( \frac{n|1-q|}{q} \right) \sum_{j=1}^{k-1} \frac{[j]_q^n}{[j]_q} q^{-kj} \quad (1 \leq j \leq k-1, \ q > 0).
$$

To prove (3.7), first observe that, for any real $q > 0$, we have

$$
\frac{q^{(\ell+1)}_i}{[\ell]_q} = q^{\ell} \prod_{i=1}^{\ell} \frac{q^{-i} \cdots q^{-1}}{[i]_q} \leq q^\ell \quad (\ell \text{ integer } \geq 0).
$$

This, combined with (1.3) and the relation $\left( \frac{n}{i} \right) \leq \left( \frac{n}{i} \right)$ which is valid for $i \leq k \leq n/(2 \log n)$ if $n$ is enough large, implies that we have, for $j = 1, \ldots, k-1$,

$$
\left| B_{n,k,j}(q) \right| \leq \sum_{i=0}^{k-j} \frac{|1-q|^i}{[k-j-i]_q} q^{-j-i+1} \left( \binom{n}{i} q^i + \binom{n}{i-1} \right)
$$

$$
\leq q^{k-j}(1 + q^j) \sum_{i=0}^{k-j} \left( \frac{n|1-q|}{q} \right)^i \frac{1}{i!} \leq q^{k-j}(1 + q^j) \exp \left( \frac{n|1-q|}{q} \right).
$$

Equation (3.7) is an immediate consequence of the last inequality and the relation $q^{k-j}(1 + q^j) \leq (1 + 2q^k)$ valid for $q > 0$.

**Step 2.** Let $u = \exp(t/\sigma_{n,k})$. We claim that, as $k \leq n/(2 \log n)$ and $n \to \infty$, we have

$$
(3.8) \quad \sum_{j=1}^{k-1} \frac{[j]_u^n}{[j]_u} u^{-kj} \leq k \frac{[k-1]_u^n}{[k-1]_u} u^{-k(k-1)}.
$$
Set $r_{n,k,j}(u) := \frac{[n]_u}{[k]_u} u^{-kj}$. Clearly, in order to prove (3.8), it suffices to show that $r_{n,k,j+1}(u) \geq r_{n,k,j}(u)$ for $j = 1, \ldots, k - 1$. We have, for $1 \leq j \leq k - 1$,

$$
(3.9) \quad \frac{r_{n,k,j+1}(u)}{r_{n,k,j}(u)} = \frac{u^{-k}}{[j+1]_u} \left(1 + \frac{u^j}{[j]_u}\right)^n \geq \frac{u^{-k}}{[k]_u} \left(1 + \frac{u^k}{[k]_u}\right)^n,
$$

where the inequality follows from the relation $q^{-j}[j]_q \leq q^{-j-1}[j+1]_q$ ($q > 0$). Using the asymptotic approximations (3.4) and $u^k = 1 + tk/\sigma_{n,k} (1 + o(1))$ (by definition, $u^k = e^{tk/\sigma_{n,k}}$), it is easy to check that we have

$$
(3.10) \quad \frac{u^{-k}}{[k]_u} \left(1 + \frac{u^k}{[k]_u}\right)^n = \frac{1}{k} \exp \left(\frac{n}{k}(1 + o(1))\right) \to \infty.
$$

This, combined with (3.9), implies that $r_{n,k,j+1}(u) \geq r_{n,k,j}(u)$ for $1 \leq j \leq k$ if $n$ is enough large.

**Conclusion.** By (1.3), $B_{n,k,k}(q) = q^{-k^2+k}[k]_q^n/[k]_q!$ for all $n \geq k \geq 1$. Combining (3.6) and (3.8), we arrive at

$$
(3.11) \quad \left| \frac{T_{n,k}(u)}{B_{n,k,k}(u)} - 1 \right| \leq k \left(1 + 2u^k\right) \exp \left(\frac{n|1-u|}{u}\right) [k]_u \left(\frac{[k-1]_u}{[k]_u}\right)^n.
$$

Let $R_{n,k}$ be the right-hand member of (3.11). Using (3.4), one can check that

$$
|1-u|/u = n \left|\exp(-t/\sigma_{n,k}) - 1\right| = n |t|/\sigma_{n,k}(1 + o(1)) = o(n/k),
$$

$$
\left(\frac{[k-1]_u}{[k]_u}\right)^n = \left(1 - \frac{1}{k}(1 + o(1))\right)^n = \exp \left(-\frac{n}{k}(1 + o(1))\right),
$$

from which it is easy to deduce that

$$
R_{n,k} = 3k^2 \exp \left(-\frac{n}{k}(1 + o(1))\right) = o(1).
$$

Combining the latter approximation with (3.11), we immediately obtain

$$
T_{n,k}(u) = B_{n,k,k}(u) (1 + o(1)) = u^{-k^2+k} \frac{[k]_q^n}{[k]_q!} (1 + o(1)),
$$

which is obviously equivalent to Lemma 3.1 since $u^k = e^{tk/\sigma_{n,k}} = 1 + o(1)$. \qed

### 4. The Variance of $X_n$: Proof of Theorem 1.9

This section is dedicated to proving Theorem 1.9. Let us first recall the closed form expression for $E(X_n)$ recently obtained by the author in [6]

$$
(4.1) \quad E(X_n) = -\frac{5B_{n+2}}{4B_n} + \left(\frac{n}{2} + \frac{9}{4}\right) \frac{B_{n+1}}{B_n} + \frac{n}{2} + \frac{1}{4}.
$$

The proof of (4.1) in [6] mainly relies on combinatorial arguments. Note that (4.1) can also be extracted from the generating function expansion (1.2). We can go even further
and derive from (1.2) (or more directly by relying on Proposition 2.1) the expression

\[ E(X_n^2 - X_n) = \frac{25}{16} \frac{B_{n+4}}{B_n} - \left( \frac{5}{4} n + \frac{407}{72} \right) \frac{B_{n+3}}{B_n} + \left( \frac{1}{4} n^2 + \frac{13}{12} n + \frac{223}{48} \right) \frac{B_{n+2}}{B_n} \]

\[ + \left( \frac{1}{2} n^2 - \frac{73}{18} \right) \frac{B_{n+1}}{B_n} + \left( \frac{1}{4} n^2 - \frac{1}{3} n - \frac{59}{144} \right). \]

A proof is given at the end of this section. Combining the above formula with (4.1) quickly gives a compact expression for \( \text{Var}(X_n) \) that we don’t explicitly state here due to lack of space. To analyze asymptotically this formula, we shall use the approximations

\[ \frac{B_{n+s+t}}{B_{n+s}} = \left( \frac{n}{\log n} \right)^t \left( 1 + \frac{\log \log n}{\log n} (1 + o(1)) \right), \]

\[ \frac{B_{n+s+t}}{B_{n+s}} - \frac{B_{n+t}}{B_n} = \sum_{r=1}^{t} \left( \frac{n^{t-r}}{\log n} \right)^r \left( 1 + \frac{\log \log n}{\log n} (1 + o(1)) \right), \]

which are valid as \( n \to \infty \) for any (fixed) integers \( s \) and \( t \). These approximations can be derived from earlier work of Salvy and Schakell [13]. For a proof of (4.3), we refer the reader to [6, Equation (3.19)] and its proof there. Proof details of (4.4) are given at the end of this section.

It is now a routine matter to prove Theorem 1.9. Inserting expressions (4.2) and (4.1) in the relation \( \text{Var}(X_n) = E(X_n^2 - X_n) + E(X_n) - E(X_n)^2 \), then using approximation (4.3), little rearrangement, gives (details are left to the reader)

\[ \text{Var}(X_n) = \frac{n^2}{4} \frac{B_{n+1}}{B_n} \left( \frac{B_{n+2}}{B_{n+1}} - \frac{B_{n+1}}{B_n} \right) - \frac{5}{4} n \frac{B_{n+2}}{B_n} \left( \frac{B_{n+3}}{B_{n+2}} - \frac{B_{n+2}}{B_n} \right) \]

\[ + \frac{25}{16} \frac{B_{n+2}}{B_n} \left( \frac{B_{n+4}}{B_{n+3}} - \frac{B_{n+3}}{B_n} \right) + 9 n \left( \frac{B_{n+2}}{B_n} - \left( \frac{B_{n+1}}{B_n} \right)^2 \right) \]

\[ + \frac{28}{12} n \frac{B_{n+2}}{B_n} + O \left( \left( \frac{n}{\log n} \right)^3 \right). \]

This, combined with (4.3)–(4.4), immediately leads to Theorem 1.9. Before closing this section, we give some proof details of (4.2) and (4.4).

**Proof of (4.2).** By the law of total expectation, we have

\[ E(X_n(X_n - 1)) = \frac{1}{B_n} \sum_{k=1}^{n} S_{n,k} E(X_{n,k}(X_{n,k} - 1)). \]

The following result will enable us to “simplify” sums of the form \( \sum_{k=1}^{n} P(k) S_{n,k} \) for any polynomial \( P \).

**Lemma 4.1.** For all integers \( n, r \geq 0 \), set \( B_n^{(r)} := \sum_{k=1}^{n} k^r S_{n,k} \). Then we have: \( B_n^{(r)} = \sum_{i=0}^{r} a_i^{(r)} B_{n+i} \), where the family \( \left( a_i^{(r)} \right)_{0 \leq i \leq r} \) is defined recursively by \( a_0^{(0)} = 1 \) and \( a_i^{(r+1)} = a_i^{(r)} - \sum_{\ell=1}^{r} \binom{r}{\ell} a_{i-\ell}^{(r)} \), with, by convention, \( a_{-1}^{(r)} = 0 \) for \( r \geq 0 \).
Clearly, Lemma 4.1 is equivalent to the relation $B^{(r+1)}_n = B^{(r)}_{n+1} - \sum_{\ell=0}^{r} \binom{r}{\ell} B^{(\ell)}_n$, which can be derived as follows:

$$B^{(r+1)}_n = \sum_{k=1}^{n} k^r kS_{n,k} = B^{(r)}_{n+1} - \sum_{k=1}^{n} (k+1)^r S_{n,k} = B^{(r)}_{n+1} - \sum_{\ell=0}^{r} \sum_{k=1}^{n} \binom{r}{\ell} k^\ell S_{n,k},$$

where the second equality results from the identity $kS_{n,k} = S_{n+1,k} - S_{n,k-1}$. The first values of the $B^{(r)}_n := \sum_{k=1}^{n} k^r S_{n,k}$ read

\begin{align*}
B^{(0)}_n &= B_n, & B^{(1)}_n &= B_{n+1} - B_n, & B^{(2)}_n &= B_{n+2} - 2B_{n+1} \\
B^{(3)}_n &= B_{n+3} - 3B_{n+2} + B_n, & B^{(4)}_n &= B_{n+4} - 4B_{n+3} + 4B_{n+1} + B_n.
\end{align*}

If we insert the expression of $E(\pi_{n,k}(\pi_{n,k} - 1))$ given in Proposition 2.1 (recall that $E(\pi_{n,k}(\pi_{n,k} - 1)) = T_{n,k}(1)/S_{n,k}$) in (4.6), and then use (4.7) to simplify the resulting sum, we painlessly arrive at (4.2).

**Proof of (4.4).** All asymptotic in what follows are meant for $n \to \infty$. Our demonstration relies on (4.3) and the approximation found by Salvy and Schakell (see Section 3.3 in [13])

$$\frac{B_{n+u+2}}{B_{n+u}} - \left(\frac{B_{n+u+1}}{B_{n+u}}\right)^2 = \frac{n}{(\log n)^2} \left(1 + 2\frac{\log \log n}{\log n}(1 + o(1))\right),$$

valid for any fixed integer $u$. Note that if we multiply both sides of Equation (4.8) by $B_{n+u}/B_{n+u+1}$, then use the specialization of (4.3) at $t = -1$ in the right-hand side of the resulting equation, we get the approximation

$$\frac{B_{n+u+2}}{B_{n+u+1}} - \frac{B_{n+u+1}}{B_{n+u}} = \frac{1}{\log n} \left(1 + \frac{\log \log n}{\log n}(1 + o(1))\right).$$

Let $s$ and $t$ be two nonnegative integers. It is easily checked that

$$\frac{B_{n+s+t}}{B_{n+s}} - \frac{B_{n+t}}{B_n} = \sum_{i=0}^{s-1} \sum_{\ell=0}^{t-1} \frac{B_{n+i+\ell+2}}{B_{n+i+\ell+1}} \left(\frac{B_{n+i+\ell+1}}{B_{n+i+\ell}} - \frac{B_{n+i+\ell}}{B_{n+i}}\right).$$

Combining (4.3) and (4.9) shows that each of the $st$ summands in (4.10) is asymptotically equal to $\frac{n^{t-1}}{(\log n)^t} \left(1 + t\frac{\log \log n}{\log n}(1 + o(1))\right)$, whence (4.4).

### 5. Limiting distribution of $Y_{n,k}$

The purpose of this section is to demonstrate Theorems 1.6 and 1.10. As we mentioned in the introduction, our proof mainly relies on results about the distribution of $X_{n,k}$ expounded in the introduction and the following result which quantifies the combinatorial “closeness” of the parameters $cr^{(c)}$ and $cr^{(t)}$.

**Lemma 5.1.** For any set partition $\pi \in \Pi_n^k$, we have

$$cr^{(t)}(\pi) \leq cr^{(c)}(\pi) \leq cr^{(t)}(\pi) + 2k(k - 1).$$
Proof. Let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a partition in $\Pi_n^k$. Clearly, if two arcs in the linear representation of $\pi$ cross, then the corresponding chords in the circular representation of $\pi$ cross. This proves that $cr^c(\pi) \geq cr^f(\pi)$.

For $i = 1, 2, \ldots, k$, let $e_i$ be the chord in the circular representation of $\pi$ that joins $\min(B_i)$ and $\max(B_i)$ and denote by $c(e_i)$ the number of chords in the circular representation of $\pi$ which cross with $e_i$. Then, it is easy to check (see e.g. Figure 1) that we have

$$cr^c(\pi) \leq cr^f(\pi) + \sum_{i=1}^k c(e_i).$$

To conclude the proof, it suffices to show that $c(e_i) \leq 2(k - 1)$ for $i = 1, \ldots, k$. This is due to the fact that the chord $e_i$ can cross with at most two chords coming from the block $B_j$ for any integer $j \neq i$ (see e.g. Figure 1).

\[\square\]

5.1. Limiting distribution of $Y_{n,k}$. In this section, all asymptotic are meant for $k = o(\sqrt{n})$ and $n \to \infty$ unless otherwise stated. Note that Lemma 5.1 asserts that

$$0 \leq Y_{n,k} - X_{n,k} \leq 2k(k - 1), \quad (n \geq k \geq 1) \quad (5.1)$$

whence $\text{Var}(Y_{n,k} - X_{n,k}) = O(k^4)$. This, combined with the approximation (1.10) of $\text{Var}(X_{n,k})$ and the well-known Cauchy-Schwartz inequality involving the covariance of two random variables $U$ and $V$

$$|\text{Cov}(U, V)| \leq \sqrt{\text{Var}(U)\text{Var}(V)},$$

leads, after a routine computation, to

$$\text{Var}(Y_{n,k}) = \text{Var}(X_{n,k}) + \text{Var}(Y_{n,k} - X_{n,k}) + 2\text{Cov}(X_{n,k}, Y_{n,k} - X_{n,k})$$

$$= \text{Var}(X_{n,k}) + O(k^3 \sqrt{n}) = \frac{k^2 - 1}{12} n + O(k^3 \sqrt{n}),$$

as stated in the first part of Theorem 1.6. We now turn our attention to the second part of Theorem 1.6. First note that

$$\frac{Y_{n,k} - \text{E}(Y_{n,k})}{\sqrt{\text{Var}(Y_{n,k})}} = B_{n,k} \frac{X_{n,k} - \text{E}(X_{n,k})}{\sqrt{\text{Var}(X_{n,k})}} + C_{n,k},$$

where $B_{n,k}$ and $C_{n,k}$ are the random variables defined on $\Pi_{n,k}$ by

$$B_{n,k} = \sqrt{\frac{\text{Var}(X_{n,k})}{\text{Var}(Y_{n,k})}} \quad \text{and} \quad C_{n,k} = \frac{(Y_{n,k} - X_{n,k}) - \text{E}(Y_{n,k} - X_{n,k})}{\sqrt{\text{Var}(Y_{n,k})}}.$$ 

Since $B_{n,k} \overset{p}{\to} 1$ and $C_{n,k} \overset{p}{\to} 0$ (by (5.3) and (5.1)) and $X_{n,k}$ is asymptotically Gaussian (by Theorem 1.5), the second part of Theorem 1.6 is an immediate consequence of (5.4) and the following basic result in probability theory:

If $(A_n)_{n \geq 1}$, $(B_n)_{n \geq 1}$ and $(C_n)_{n \geq 1}$ are sequences of random variables such that $A_n \overset{d}{\to} A$, $B_n \overset{p}{\to} b$ and $B_n \overset{p}{\to} c$, where $b$ and $c$ are constant, then $A_nB_n + C_n \overset{d}{\to} bA + c.$
5.2. An upper bound for the variance of $Y_n$: proof of Theorem 1.10. The same reasoning as in the proof of (5.3) shows that

$$\text{Var}(Y_n) \leq \text{Var}(X_n) + \text{Var}(Y_n - X_n) + 2\sqrt{\text{Var}(X_n)\text{Var}(Y_n - X_n)}. \tag{5.5}$$

Combining the above inequality with Theorem 1.9, we see that Theorem 1.10 is an immediate corollary of the following property:

$$\text{Var}(Y_n - X_n) = O\left(\frac{n^4}{(\log n)^4}\right) \quad \text{as } n \to \infty. \tag{5.6}$$

To prove (5.6), first observe that, by the law of total expectation, we have

$$\text{Var}(Y_n - X_n) \leq E( (Y_n - X_n)^2) = \frac{1}{B_n} \sum_{k=1}^{n} S_{n,k} E((Y_{n,k} - X_{n,k})^2) \tag{5.7}$$

where the last inequality is a consequence of (5.1). Using expressions (1.5) and (1.6) for $E(X_{n,k})$ and $E(Y_{n,k})$, after a routine computation, we get

$$E(Y_{n,k} - X_{n,k}) = \frac{5}{4} k(k-1) - \frac{3}{2} (n+1-k) \frac{S_{n,k-1}}{S_{n,k}} - \frac{1}{2} n(4n-5k+1) \frac{S_{n-1,k-1}}{S_{n,k}}$$

$$- 10 \left(\frac{n}{2}\right) \frac{S_{n-2,k-2}}{S_{n,k}} + \left(\frac{n}{4}\right) \frac{S_{n-4,k-2}}{S_{n,k}} \leq \frac{5}{4} k(k-1) + \frac{5}{2} n k \frac{S_{n-1,k-1}}{S_{n,k}} + \left(\frac{n}{4}\right) \frac{S_{n-4,k-2}}{S_{n,k}}.$$  

Inserting the latter relation in (5.7) gives, after some manipulations,

$$\text{Var}(Y_n - X_n) \leq \frac{1}{B_n} \left(\frac{5}{2} B_n^{(4)} + 5n B_n^{(3)} + 2 \left(\frac{n}{4}\right) B_n^{(2)}\right), \tag{5.8}$$

where, as in Lemma 4.1, $B_n^{(r)} = \sum_{k=1}^{r} k^r S_{n,k}$. Combining the above inequality with the relations in (4.7) and the approximation (1.3), we finally arrive at

$$\text{Var}(Y_n - X_n) = O\left(\frac{B_{n+4}}{B_n}\right) = O\left(\frac{n^4}{(\log n)^4}\right),$$

as stated in (5.6). This concludes the proof of Theorem 1.10.

6. Maximum values of the parameters $c_r^{(f)}$ and $c_r^{(c)}$

This section contains the proof of Theorems 1.12 and 1.13. It seems difficult (in general) to determine $M_{n,k}^{(f)}$ and $M_{n,k}^{(c)}$ directly from their combinatorial definition. The key idea is to convert our original problem (find global maxima of functions defined on set partitions) to a maximization problem of functions defined on integer partitions (these partitions are often easier to handle than set partitions).

Recall that a partition of a positive integer $n$ is a finite nonincreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The $\lambda_i$ are called
the parts of the partition. We often write $\lambda = (1^{m_1}2^{m_2}3^{m_3} \ldots)$ where exactly $m_i$ of the $\lambda_j$ are equal to $i$. It is usual to associate with a set partition $\pi = B_1/B_2/\ldots/B_k$ its block-size vector $\text{bl}(\pi)$, which is the integer partition whose parts are $|B_1|, |B_2|, \ldots, |B_k|$. For instance, if $\pi = 17/23/8/4/56$, we have $\text{bl}(\pi) = (3,2,2,1) = (1^22^23)$. In the sequel, we let $\mathcal{P}(n,k)$ denote the set of all (integer) partitions of $n$ into exactly $k$ parts.

For an integer partition $\lambda$ of $n$, set
\[
(6.1) \quad M^{(\ell)}(\lambda) = \max_{\pi \in \Pi_n: \text{bl}(\pi) = \lambda} cr^{(\ell)}(\pi) \quad \text{and} \quad M^{(c)}(\lambda) = \max_{\pi \in \Pi_n: \text{bl}(\pi) = \lambda} cr^{(c)}(\pi).
\]

In Section 6.1 we prove the following result.

**Theorem 6.1.** If $\lambda = (1^{m_1}2^{m_2}3^{m_3} \ldots r^{m_r})$, we have
\[
(6.2) \quad M^{(\ell)}(\lambda) = \sum_{s=2}^{r} (2s-3) \binom{m_s}{2} + \sum_{1 \leq s < t \leq r} 2(s-1)m_sm_t,
\]
\[
(6.3) \quad M^{(c)}(\lambda) = \binom{m_2}{2} + \sum_{s=3}^{r} 2s \binom{m_s}{2} + 2m_2 \sum_{t=3}^{r} m_t + \sum_{3 \leq s < t \leq r} 2sm_sm_t.
\]

For every integers $n \geq k \geq 1$ and $u \in \{\ell, c\}$, let $\mathcal{M}^{(u)}_{n,k}$ be the set of the maximas of $M^{(u)}(\lambda)$, where $\lambda$ runs over all partitions of $n$ into $k$ blocks, i.e.
\[
(6.4) \quad \mathcal{M}^{(u)}_{n,k} = \{\lambda \in \mathcal{P}(n,k): M^{(u)}(\lambda) = \max(M^{(u)}(\tau) : \tau \in \mathcal{P}(n,k))\}.
\]

Note that, by (6.1) and (6.4), we have
\[
(6.5) \quad M^{(u)}_{n,k} = M^{(u)}_{n,k}(\lambda) \quad \text{for } u \in \{c, \ell\} \text{ and every } \lambda \in \mathcal{M}^{(u)}_{n,k}.
\]

In the sequel, we let $\lambda^{*}_{n,k}$ denote the (unique) partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$ such that $\left\lfloor \frac{n}{r} \right\rfloor \leq \lambda_i \leq \left\lceil \frac{n}{r} \right\rceil$ for $i = 1, \ldots, k$. For instance, we have $\lambda^{*}_{3,3} = (2^23^1) = (3,2,2)$. The following result is proved in Section 6.2.

**Theorem 6.2.** Suppose $n \geq k \geq 1$. Then,

1. $\mathcal{M}^{(\ell)}_{n,k} = \{\lambda^{*}_{n,k}\}$.
2. (a) $\mathcal{M}^{(c)}_{n,k} = \{\lambda^{*}_{n,k}\}$ if $n \geq 3k$,
   (b) $\mathcal{M}^{(c)}_{3k-j,k} = \{(1^s2^{j-2s}3^{k-j-s}) : s = \left\lceil \frac{j}{2} \right\rceil\}$ if $0 \leq j < k$ or $k \leq j \leq 2k - 6$,
   (c) $\mathcal{M}^{(c)}_{k+j,k} = \{(1^k2^j), (1^k2^j2^j3^2)\}$ if $4 \leq j \leq 5$ and $k \geq j$,
   (d) $\mathcal{M}^{(c)}_{k+j,k} = \{(1^j2^j)\}$ if $0 \leq j \leq 3$ and $k \geq j$.

Note that Equation (6.5), in conjunction with the two above theorems, easily leads to Theorem 6.12. For instance, if $n = qk + r$ with $q = \left\lceil \frac{n}{k} \right\rceil$, we have $\lambda^{*}_{n,k} = (q^r(rq + 1)^r)$. If $q \geq 2$ (i.e., $n \geq 2k$), by (6.5) and Theorems 6.1 and 6.2, this implies that
\[
M^{(\ell)}_{n,k} = M^{(\ell)}_{n,k}(\lambda^{*}_{n,k}) = (2q-3)\binom{k-r}{2} + (2q-1)\binom{r}{2} + 2(q-1)r(k-r).
\]
Using the relation \( \binom{k-r}{2} + \binom{r}{2} + r(k - r) = \binom{k}{2} \) and replacing \( q \) by \( (n - r)/k \) in the above equality, we arrive at Theorem 1.12(1a). The proof of the other assertions in Theorem 1.12 are so similar that we leave the details to the reader.

In Section 6.3, we deduce Theorem 1.13 from Theorem 1.12. We conclude this section with remarks on the maximas of the parameter \( cr^{(\ell)} \) and \( cr^{(c)} \) in Section 6.4.

### 6.1. Proof of Theorem 6.1
Using only (6.1) and the combinatorial definition of \( cr^{(\ell)} \) and \( cr^{(c)} \), it is easy to compute \( M^{(\ell)}(\lambda) \) and \( M^{(c)}(\lambda) \) for integer partitions \( \lambda \) into two parts. A look at Fig. 3 and a little moment’s thought (we refer the reader to Sections 5 and 6 in [6] for more details) will convince the reader that

\[
M^{(\ell)}(a, 1) = M^{(c)}(a, 1) = 0 \quad \text{if} \quad a \geq 1, \\
M^{(\ell)}(a, 2) = M^{(c)}(a, 2) = 2 \quad \text{if} \quad a \geq 3, \\
M^{(\ell)}(2, 2) = M^{(c)}(2, 2) = 1, \\
M^{(\ell)}(a, b) = 2(b - 1) - \chi(a = b) \quad \text{and} \quad M^{(c)}(a, b) = 2b \quad \text{if} \quad a \geq b \geq 3.
\]

(6.6)

To go further, we will rely on the obvious fact that \( cr^{(\ell)} \) and \( cr^{(c)} \) are \( Z \)-parameters (see [6]), i.e., for any set partition \( \pi = B_1/B_2/\cdots/B_k \), we have

\[
\begin{align*}
cr^{(\ell)}(\pi) &= \sum cr^{(\ell)}(st(B_i/B_j)) \\
&= \sum cr^{(c)}(st(B_i/B_j)),
\end{align*}
\]

where the summations are over all pairs \( (i, j) \) with \( 1 \leq i < j \leq k \) and \( st \) is the standardization map defined as follows. Recall that, given a subset \( S \subseteq \mathbb{P} \) with cardinality \( |S| = n \), the standardization map \( st \) is the (unique) order-preserving bijection \( st : S \to [n] \). We let \( st \) act element-wise on objects built using \( S \) as label. For instance, the set partition \( \pi = 29/410/5/711/8 \) of \( S = \{2, 4, 5, 7, 8, 9, 10, 11\} \) is sent after standardization to the set partition \( st(\pi) = 16/27/3/48/5 \).

As it is easily seen (we omit the details), Theorem 6.1 is immediate from (6.6) and the following lemma.

**Lemma 6.3.** For every integer partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), we have

\[
\begin{align*}
M^{(\ell)}(\lambda) &= \sum_{1 \leq i < j \leq k} M^{(\ell)}(\lambda_i, \lambda_j) \\
M^{(c)}(\lambda) &= \sum_{1 \leq i < j \leq k} M^{(c)}(\lambda_i, \lambda_j).
\end{align*}
\]

(6.8)

The proof of above result relies on (6.7) and properties of certain set partitions that we describe below. Note that it is immediate from (6.7) that \( M^{(u)}(\lambda) \leq \sum M^{(u)}(\lambda_i, \lambda_j) \) for \( u \in \{c, \ell\} \).

The set partitions \( \pi(\lambda) \). Suppose \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathcal{P}(n, k) \) and consider the Ferrers diagram \( F \) of \( \lambda \) (which is an array of square cells having left-justified rows with
row \( i \) containing \( \lambda_i \) cells. Then, we put the integers 1, 2, \ldots, \( n \) in increasing order in the cells of \( F \) "from top to bottom and left to right", i.e., starting with the leftmost column, filling its cells with the integers 1, 2, \ldots, \( k \) from top to bottom, then filling the next column with the integers \( k + 1, k + 2, \ldots, k + a_2 \) where \( a_2 \) is the number of cells in the second column of \( F \) and working up to the right. Let \( \pi(\lambda) \) be the set partition of \([n]\) the blocks of which consist of the elements in the same row of the filling of \( F \). As an example, the Ferrers diagram of \((4,2,1)\) and its corresponding filling are

\[
\begin{array}{cccc}
\text{\begin{tabular}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
5 & 6 & 7 & 8 \\
\hline
\end{tabular}}
\end{array}
\]

whence \( \pi(4,2,1) = 1467/253 \).

Suppose \( \pi(\lambda) = B_1/B_2/\ldots/B_k \). Clearly, by definition of \( \pi(\lambda) \), we have \( |B_i| = \lambda_i \) for \( i = 1, \ldots, k \), and \( \text{st}(B_i/B_j) = \pi(|B_i|, |B_j|) = \pi(\lambda_i, \lambda_j) \) for \( 1 \leq i < j \leq k \). Moreover, by construction of \( \pi(a,b) \), for all integers \( a \geq b \geq 1 \), we have

\[
\pi(b,b) = \{1, 3, \ldots, 2b - 1\}, \{2, 4, \ldots, 2b\},
\]

\[
\pi(a,b) = \{1, 3, \ldots, 2b - 1, 2b + 1, 2b + 2, \ldots, a + b\}, \{2, 4, \ldots, 2b\} \quad \text{if} \ a > b.
\]

For instance, \( \pi(2,2) = 13/24 \) and \( \pi(4,2) = 1356/24 \). Using the above expressions for \( \pi(a,b) \), Fig. 4 and (6.6), it is easy to compute \( cr^{(o)}(\pi(a,b)) \) and \( cr^{(c)}(\pi(a,b)) \) and check that \( cr^{(u)}(\pi(a,b)) = M^{(u)}(a,b) \) for \( u \in \{c, \ell\} \) and all integers \( a \geq b \geq 1 \) (details are left to the reader). To resume, we have seen that \( \text{bl}(\pi(\lambda)) = \lambda \) and, \( cr^{(u)}(\text{st}(B_i/B_j)) = M^{(u)}(\lambda_i, \lambda_j) \) for \( u \in \{c, \ell\} \) and \( 1 \leq i < j \leq k \). By (6.7), this implies that \( M^{(u)}(\lambda) \geq \sum M^{(u)}(\lambda_i, \lambda_j) \) for \( u \in \{c, \ell\} \). This ends the proof of Lemma 6.3 and thus completes the proof of Theorem 6.1.

6.2. Proof of Theorem 6.2. For simplicity, we introduce auxiliary functions \( R, T^{(\ell)} \) and \( T^{(c)} \) defined for a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) = (1^{m_1}2^{m_2}3^{m_3} \cdots r^{m_r}) \) by

\[
R(\lambda) := \sum_{s=1}^{r} 2s \binom{m_s}{2} + \sum_{1 \leq s < t \leq r} 2sm_sm_t = \sum_{i=1}^{k} 2(i-1)\lambda_i,
\]

\[
T^{(\ell)}(\lambda) := \sum_{s=2}^{r} \binom{m_s}{2} + 2\binom{\ell(\lambda)}{2},
\]

\[
T^{(c)}(\lambda) := 2(m_1 + m_2)\ell(\lambda) - 2\binom{m_1 + m_2 + 1}{2} + \binom{m_2}{2},
\]

\[\text{Diagram Figure 4. Sketch of circular and linear representation of } \pi(a,b)\text{.}\]
where \( \ell(\lambda) \) is the number of parts of \( \lambda \). So, by Theorem 6.1, we have
\[
M^{(\ell)}(\lambda) = R(\lambda) - T^{(\ell)}(\lambda) \quad \text{and} \quad M^{(c)}(\lambda) = R(\lambda) - T^{(c)}(\lambda).
\]
The following result is the key ingredient in the proof of Theorem 6.2.

**Lemma 6.4.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) = (1^{m_1}2^{m_2} \cdots r^{m_r}) \) be a partition with two nonconsecutive parts \( u, v \) with \( u < v \) (i.e., \( v - u \geq 2 \) and \( m_u m_v > 0 \)). Let \( \mathcal{I}^+_{u,v} \) be the partition obtained from \( \lambda \) by decreasing the rightmost part of \( \lambda \) equal to \( v \) by 1 and increasing the leftmost part of \( \lambda \) equal to \( u \) by 1. Then, if we set \( \tilde{\lambda} = \mathcal{I}^+_{u,v} \), we have
\[
R(\tilde{\lambda}) - R(\lambda) = 2 \left( 1 + \sum_{u < t < v} m_t \right),
\]
\[
T^{(\ell)}(\tilde{\lambda}) - T^{(\ell)}(\lambda) = 2 - m_u + m_{u+1} + m_{v-1} - m_v + \chi(u + 1 = v - 1) + (m_u - 1)\chi(u = 1),
\]
\[
T^{(c)}(\tilde{\lambda}) - T^{(c)}(\lambda) = \begin{cases} 2(\ell(\lambda) - m_1) - 1, & \text{if } u = 1, v = 3; \\ m_2, & \text{if } u = 1, v \geq 4; \\ -2(\ell(\lambda) - m_1) + m_2 + 1, & \text{if } u = 2; \\ 0, & \text{if } u \geq 3. \end{cases}
\]

**Proof.** Set \( d = \sum_{i=1}^{r} m_i \) and \( s = 1 + \sum_{i=u+1}^{r} m_i \). By definition of \( \tilde{\lambda} \), we have
\begin{enumerate}
\item \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_d) \) with \( \tilde{\lambda}_d = \lambda_d - 1 \), \( \tilde{\lambda}_s = \lambda_s + 1 \) and \( \tilde{\lambda}_i = \lambda_i \) for \( i \neq d, s \);
\item \( \tilde{\lambda} = (1^{m_1}2^{m_2} \cdots r^{m_r}) \) with \( m_j = m_j - 1 \) for \( j \in \{u, v\} \), \( m_j = m_j + 1 + \chi(v = u + 2) \) for \( j \in \{u + 1, v - 1\} \) and \( m_j = m_j \) for \( j \notin \{u, u + 1, v - 1\} \).
\end{enumerate}
Using expression (a) for \( \tilde{\lambda} \) and (6.9), we arrive at \( R(\tilde{\lambda}) - R(\lambda) = \sum_{i=1}^{k} 2(i - 1)(\tilde{\lambda}_i - \lambda_i) = 2(s - d) \). This proves the first assertion. Similarly, the other assertions can be obtained by using expression (b) for \( \tilde{\lambda} \) and (6.9). We omit the details.

A very useful consequence of Lemma 6.4 is the following result.

**Proposition 6.5.** Suppose we are given integers \( n, k \geq 1 \) and let \( \lambda \) be a partition in \( \mathcal{P}(n, k) \) which contains two nonconsecutive parts \( u, v \) with \( u < v \). Then, (1) \( \lambda \) is not a maxima of \( M^{(\ell)} \) on \( \mathcal{P}(n, k) \); (2) If in addition \( (u, v) \neq (1, 3) \), then \( \lambda \) is not a maxima of \( M^{(c)} \) on \( \mathcal{P}(n, k) \).

**Proof.** Set \( \tilde{\lambda} = \mathcal{I}^+_{u,v} \), with \( \mathcal{I}^+_{u,v} \) defined as in Lemma 6.4. Observe that \( \tilde{\lambda} \in \mathcal{P}(n, k) \). By (6.11), we have
\[
R(\tilde{\lambda}) - R(\lambda) \geq 2 + m_{u+1} + m_{v-1}.
\]
This inequality, combined with (6.16) and (6.12), implies that
\[
M^{(\ell)}(\tilde{\lambda}) - M^{(\ell)}(\lambda) \geq m_u + m_v - \chi(v = u + 2) - (m_u - 1)\chi(u = 1) \geq 1
\]
since \( m_u \) and \( m_v \) are positive integers. This proves the first assertion.
Suppose now \((u, v) \neq (1, 3)\) (and thus \(v \geq 4\)). Then, it is immediate from (6.13) that \(T^{(c)}(\lambda) - T^{(c)}(\lambda) \leq m_2 \chi(u = 1)\). If we combined this inequality with (6.14) and (6.10), we arrive at

\[
M^{(c)}(\lambda) - M^{(c)}(\lambda) \geq 2 + m_{u+1} + m_{v-1} - \chi(v = u + 2) - m_2 \chi(u = 1) \geq 2.
\]

This proves the second assertion.

We now have enough tools to verify Theorem 6.2.

**Proof of Theorem 6.2.** It is easily checked that any partition \(\lambda\) in \(\mathcal{P}(n, k)\) which is distinct from \(\lambda^*_{n,k}\) contains two nonconsecutive parts \(u, v\) such that (i) \(u < \left\lfloor \frac{n}{k} \right\rfloor < v\) or (ii) \(u = \left\lfloor \frac{n}{k} \right\rfloor\) and \(v > \left\lfloor \frac{n}{k} \right\rfloor\). By Proposition 6.5, this leads to assertions (1) and (2a).

Let \(\mathcal{P}(n, k; \leq 3)\) denote the set of all partitions in \(\mathcal{P}(n, k)\) which have no part greater than 3. If \(n \leq 3k\), any partition in \(\mathcal{P}(n, k) \setminus \mathcal{P}(n, k; \leq 3)\) contains two parts \(u, v\) with \(u < 3 < v\). By Proposition 6.5(2), this implies that \(M^{(c)}_{n,k} \subseteq \mathcal{P}(n, k; \leq 3)\). Moreover, by (6.3), we have \(M^{(c)}(1^{a}2b3^{d}) = (\frac{a}{2}) + 6(\frac{d}{2}) + 2bd\), from which it is easily seen that \(M^{(c)}(1^{a}2b+23^{d}) < M^{(c)}(1^{a+1}2b3^{d+1})\) and \(M^{(c)}(1^{a}2b+6) < M^{(c)}(1^{a+3}2b3^{3})\) for all \(a, b \geq 0\) and \(d \geq 1\). Altogether, this shows that

\[
k \geq \frac{n}{3} \quad \text{and} \quad \lambda \in M^{(c)}_{n,k} \Rightarrow \lambda = (1^{a}2b3^{d}) \quad \text{with} \quad b \leq 1 \quad \text{or} \quad (c = 0 \quad \text{and} \quad b \leq 5)\.
\]

On the other hand, (solving the system \(\{a + b + c = k, a + 2b + 3c = 3k - j\}\)) we see that, for \(0 \leq j \leq 2k\),

\[
\mathcal{P}(3k - j, k; \leq 3) = \left\{(1^{s}2^{j-2s}3^{k-j+s}) : \max(0, j - k) \leq s \leq \frac{j}{2}\right\}.
\]

Combining (6.15) with (6.16) gives assertion (2b). This also shows that, if \(j \in \{4, 5\}\), \(M^{(c)}_{k+j, k} \subseteq \{1^{k-j}2^{j} \setminus \{1^{k-j+2}2^{j-4}3^{2}\}\}\), while, by (6.3), we have (if \(j \in \{4, 5\}\) \(M^{(c)}(1^{k-j}2^{j}) = M^{(c)}(1^{k-j+2}2^{j-4}3^{2})\). This gives assertion (2c). Assertion (2d) can be obtained in a similar way.

**6.3. Proof of Theorem 1.13.** It is a simple matter to derive Theorem 1.13 from Theorem 1.12. By definition, we have \(M^{(w)}_{n} = \max \left(M^{(w)}_{n,k} : 1 \leq k \leq n\right)\) for \(u \in \{\ell, c\}\).

We first show that \(M^{(\ell)}_{n} = \left\lfloor \frac{1}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor\). For \(1 \leq n \leq 3\), \(M^{(\ell)}_{n} = 0\) and this assertion is true. Suppose \(n \geq 4\). Then, use of the expressions given in Theorem 1.12 for \(M^{(\ell)}_{n,k}\) and a basic study of the function \(k \mapsto M^{(\ell)}_{n,k}\) (which is a piecewise function each part of which is a quadratic function in \(k\)) the details of which are left to the reader shows that

- if \(n \equiv 0 \pmod{3}\), the function \(k \mapsto M^{(\ell)}_{n,k}\) has global maxima at exactly two points \(k = \frac{n}{3}\) and \(k = \frac{n}{3} + 1\), and the maximum is \(M^{(\ell)}_{n,\frac{n}{3}} = M^{(\ell)}_{n,\frac{n}{3}+1} = \frac{n(n-3)}{6} = \left\lfloor \frac{1}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor\);

- if \(n \equiv 1, 2 \pmod{3}\), the function \(k \mapsto M^{(\ell)}_{n,k}\) has a unique global maximum at \(k = \left\lfloor \frac{n}{3} \right\rfloor\), and the maximum is \(M^{(\ell)}_{n,\left\lfloor \frac{n}{3} \right\rfloor} = \frac{1}{3} \left\lfloor \frac{n-1}{2} \right\rfloor\).

This concludes the proof of Theorem 1.13(1). The proof of the second part relies on the following result.
Lemma 6.6. For \( n \geq k \geq 1 \), let \( g_n(k) = (k-1)n - r_k(k-r_k) \), where \( r_k \) is the remainder of the division of \( n \) by \( k \). For \( 1 \leq k \leq n-1 \), we have \( g_n(k) < g_n(k+1) \).

Proof. Let \( q_{k+1} := \lfloor \frac{n}{k+1} \rfloor \). Then, noticing that \( n = (k+1)q_{k+1} + r_{k+1} \) and using the definition of \( g_n \), we obtain

\[
(6.17) \quad g_n(k+1) - g_n(k) = (k+1)q_{k+1} + r_k(k-r_k) - r_{k+1}(k-r_{k+1}).
\]

If \( q_{k+1} \geq k \), this yields \( g_n(k) < g_n(k+1) \) (since \( x(a-x) \leq a^2/4 \) for \( a, x \in \mathbb{R} \)).

Suppose \( q_{k+1} < k \). Then, \( q_{k+1} + r_{k+1} < 2k \), and since \( n \equiv q_{k+1} + r_{k+1} \equiv r_k \pmod{k} \), we have \( r_{k+1} = ks + r_k - q_{k+1} \) with \( s \in \{0,1\} \). Inserting this identity in (6.17) gives, after a routine computation,

\[
g_n(k+1) - g_n(k) = \begin{cases} 
q_{k+1}(1 + q_{k+1}) + 2r_k(k - q_{k+1}), & \text{if } s = 0; \\
q_{k+1}(1 + q_{k+1}) + 2r_{k+1}(k - r_k), & \text{if } s = 1,
\end{cases}
\]

from which it immediately results that \( g_n(k+1) - g_n(k) > q_{k+1}(1 + q_{k+1}) > 0 \). \( \square \)

Theorem 1.12(2a), in conjunction with the above result, implies that \( M_{n,[\frac{n}{3}]}^{(c)} > M_{n,k}^{(c)} \) for \( k < \lfloor \frac{n}{3} \rfloor \). On the other hand, using the formulas (2b) and (2c) in Theorem 1.12, a simple study of the sign of \( M_{n,k+1}^{(c)} - M_{n,k}^{(c)} \) for \( k > \lfloor \frac{n}{3} \rfloor \), shows that \( M_{n,[\frac{n}{3}]}^{(c)} > M_{n,k}^{(c)} \) for \( k > \lfloor \frac{n}{3} \rfloor \). Finally, by comparing \( M_{n,[\frac{n}{3}]}^{(c)} \) and \( M_{n,[\frac{n}{2}]}^{(c)} \), we arrive at the following result:

- if \( n \equiv 0 \pmod{3} \), the function \( k \mapsto M_{n,k}^{(c)} \) has a unique global maximum at \( k = n/3 \) and the maximum is \( M_{n,[\frac{n}{3}]}^{(c)} = 6\left(\frac{2}{3}\right) = \left\lfloor \frac{2}{3} \left( \frac{n-1}{2} \right) \right\rfloor \);
- if \( n \equiv 1 \pmod{3} \), the function \( k \mapsto M_{n,k}^{(c)} \) has global maxima at exactly two points \( k = \lfloor \frac{n}{3} \rfloor \) and \( k = \left\lfloor \frac{2n}{3} \right\rfloor \), and the maximum is \( M_{n,[\frac{n}{3}]}^{(c)} = M_{n,[\frac{2n}{3}]}^{(c)} = 6\left(\frac{2}{3}\right) = \left\lfloor \frac{2}{3} \left( \frac{n-2}{2} \right) \right\rfloor \);
- if \( n \equiv 2 \pmod{3} \), the function \( k \mapsto M_{n,k}^{(c)} \) has a unique global maximum at \( k = \left\lfloor \frac{2n}{3} \right\rfloor \), and the maximum of \( f_n \) is \( M_{n,[\frac{n}{3}]}^{(c)} = 6\left(\frac{n-1}{3}\right) + 2\left[\frac{n}{3}\right] = \frac{2}{3} \left( \frac{n-2}{2} \right) \).

This completes the proof of Theorem 1.13.

6.4. Remarks on the maxims of the parameter \( cr^{(l)} \). This section presents results concerning the set partitions which maximize the parameter \( cr^{(l)} \). We don’t give proofs due to lack of space but the results announced can be painlessly deduced from Theorem 6.2 and properties of the set partitions \( \pi(\lambda) \) described in Section 6.2.

Let \( a_{n,k}^{(l)} \) (resp., \( a_n^{(l)} \)) denote the number of partitions \( \pi \in \Pi_n^k \) (resp., \( \pi \in \Pi_n \)) satisfying \( cr^{(l)}(\pi) = M_{n,k}^{(l)} \) (resp., \( cr^{(l)}(\pi) = M_n^{(l)} \)). Then, we have

\[
a_{n,k}^{(l)} = \begin{cases} 
1, & \text{if } 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor; \\
\left(\begin{array}{c}
\binom{n}{2n-2k} \end{array}\right), & \text{if } \left\lfloor \frac{n}{2} \right\rfloor \leq k \leq n;
\end{cases}
\]

\[
a_n^{(l)} = \begin{cases} 
2, & \text{if } n \equiv 0 \pmod{3}, \\
1, & \text{if } n \equiv 1, 2 \pmod{3};
\end{cases}
\]
If $n \geq 4$ and $\left\lfloor \frac{n}{2} \right\rfloor \leq k \leq n$, the partitions $\pi \in \Pi^k_n$ satisfying $cr^{(t)}(\pi) = M^{(t)}_{n,k}$ are the partitions of $[n]$ which consist of exactly $n - k$ mutually disjoint and crossing arcs. If $n \geq 4$ and $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, the (unique) partition $\pi \in \Pi^k_n$ satisfying $cr^{(t)}(\pi) = M^{(t)}_{n,k}$ is the partition $\pi(\lambda^*_{n,k})$ as defined in Section 6.2. Furthermore,

- if $n \equiv 0 \pmod{3}$, the two partitions $\pi \in \Pi_n$ satisfying $cr^{(t)}(\pi) = M^{(t)}_{n,k}$ are $\pi(\lambda^*_{n,\frac{n}{3}})$ and $\pi(\lambda^*_{n,\frac{n}{3}+1})$;
- if $n \equiv 1, 2 \pmod{3}$, the unique partition $\pi \in \Pi_n$ satisfying $cr^{(t)}(\pi) = M^{(t)}_{n,k}$ is $\pi(\lambda^*_{n,\left\lfloor \frac{n}{3} \right\rfloor})$.

Finally, note that similar results for the parameter $cr^{(c)}$ should exist but seem much more complicated (except if $n \equiv 0 \pmod{3}$).

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