REMARKS ON SOBOLEV NORMS OF FRACTIONAL ORDERS

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ABSTRACT. When a function belonging to a fractional-order Sobolev space is supported in a proper subset of the Lipschitz domain on which the Sobolev space is defined, how is its Sobolev norm as a function on the smaller set compared to its norm on the whole domain? Do different norms behave differently? This article addresses these issues. We prove some inequalities and disprove some misconceptions by counter-examples.

1. INTRODUCTION

Sobolev spaces of fractional orders are the key function space for the mathematical and numerical analysis of boundary integral equation methods. It is well known that these spaces behave differently with Sobolev spaces of integral orders. For example, if a domain $\mathcal{O}$ in $\mathbb{R}^n$ is partitioned into $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ and if $u \in H^m(\mathcal{O})$ for some non-negative integer $m$, where $H^m(\mathcal{O}) = W^m_{2}(\mathcal{O})$ is the normally-defined Sobolev space with the Lebesgue measure, then

$$\|u\|^2_{H^m(\mathcal{O})} = \|u|_{\mathcal{O}_1}\|^2_{H^m(\mathcal{O}_1)} + \|u|_{\mathcal{O}_2}\|^2_{H^m(\mathcal{O}_2)}. \quad (1.1)$$

It is also obvious that if $u \in H^m(\mathcal{O})$ with $\text{supp}(u) \subset \mathcal{O}_1$, then

$$\|u\|_{H^m(\mathcal{O}_1)} = \|u\|_{H^m(\mathcal{O})}. \quad (1.2)$$

Are the above properties true for fractional-order Sobolev spaces?

Pushing back until the next section the precise definitions of the fractional-order Sobolev spaces $\widetilde{H}^s(\mathcal{O})$ and $\widetilde{W}^s_{2}(\mathcal{O})$, $s > 0$, we mention here that these two spaces coincide when $s - 1/2 \in \mathbb{N}$, the space of non-negative integers; see Subsection 2.3. Their corresponding norms, denoted by $\|\cdot\|_{\widetilde{H}^s(\mathcal{O})}$ and $\|\cdot\|_{\widetilde{W}^s_{2}(\mathcal{O})}$, respectively, are equivalent, i.e., there exist positive constants $C_1$ and $C_2$ satisfying

$$C_1 \|u\|_{\widetilde{H}^s(\mathcal{O})} \leq \|u\|_{\widetilde{W}^s_{2}(\mathcal{O})} \leq C_2 \|u\|_{\widetilde{H}^s(\mathcal{O})} \quad \forall u \in \widetilde{H}^s(\mathcal{O}).$$

We first make a remark (Proposition 2.1) on how the constants $C_1$ and $C_2$ depend on the size of the domain $\mathcal{O}$.

An issue with the Sobolev norm $\|\cdot\|_{\widetilde{H}^s(\mathcal{O})}$ is that instead of (1.1) it is known that

$$\|u\|^2_{\widetilde{H}^s(\mathcal{O})} \leq \|u|_{\mathcal{O}_1}\|^2_{\widetilde{H}^s(\mathcal{O}_1)} + \|u|_{\mathcal{O}_2}\|^2_{\widetilde{H}^s(\mathcal{O}_2)}, \quad (1.3)$$

provided that all norms are well defined. This result is proved in [4, 26]. A counter-example in [2] shows that the opposite inequality is not true. This means there is no constant $C$ independent of $u$ and any other parameter whatsoever such that

$$\|u|_{\mathcal{O}_1}\|^2_{\widetilde{H}^s(\mathcal{O}_1)} + \|u|_{\mathcal{O}_2}\|^2_{\widetilde{H}^s(\mathcal{O}_2)} \leq C \|u\|^2_{\widetilde{H}^s(\mathcal{O})}. \quad (1.4)$$
Inequality (1.3) and the non-existence of (1.4) imply that if \( u \in \tilde{H}^s(\mathcal{O}) \) is supported in \( \overline{\mathcal{O}}_1 \) then, instead of (1.2), we have in general

\[
\|u\|_{\tilde{H}^s(\mathcal{O})} < \|u\|_{\tilde{H}^s(\mathcal{O}_1)}.
\]

We establish in Theorem 3.1 that when \( s = 1/2 \) (a commonly-seen case)

\[
\|u\|_{\tilde{H}^{1/2}(\mathcal{O}_1)} \leq C \|u\|_{\tilde{H}^{1/2}(\mathcal{O})}
\]

but there is no constant \( C' \) satisfying \( \|u\|_{\tilde{H}^{1/2}(\mathcal{O}_1)} \leq C' \|u\|_{\tilde{H}^{1/2}(\mathcal{O})} \) for all \( u \in \tilde{W}^2_2(\mathcal{O}) \) supported in \( \mathcal{O}_1 \).

A third issue arises from the analysis of domain decomposition methods for boundary integral equations of the first kind. Consider for example the hypersingular integral equation; see Section 5 for detail. It is known that the bilinear form \( a(\cdot, \cdot) \) arising from this operator defines a norm equivalent to the \( \tilde{H}^{1/2}(\mathcal{O}) \)-norm. One of the requirements in the analysis is a proof of some inequality of the form

\[
a(u, u) \leq C(a(u|_{\mathcal{O}_1}, u|_{\mathcal{O}_1}) + a(u|_{\mathcal{O}_2}, u|_{\mathcal{O}_2})).
\]

(1.5)

Due to a misconception that the norm \( a(u|_{\mathcal{O}_j}, u|_{\mathcal{O}_j}) \) is equivalent to \( \|u|_{\mathcal{O}_j}\|_{\tilde{H}^{1/2}(\mathcal{O}_j)}^2 \), \( j = 1, 2 \), inequality (1.3) has been used ubiquitously in the literature to obtain the above estimate. Proposition 2.1 and Theorem 3.1 imply that this equivalence is at the cost of the equivalence constants depending on the size of the subdomain \( \mathcal{O}_j \). This may adversely affect the final result; see Section 5 for detail. We prove in Theorem 3.2 that (1.3) can be improved to ensure the following estimate

\[
\|u\|_{\tilde{H}^s(\mathcal{O})} \leq \|u|_{\mathcal{O}_1}\|_{\tilde{H}^s(\mathcal{O}_1)} + \|u|_{\mathcal{O}_2}\|_{\tilde{H}^s(\mathcal{O})},
\]

assuming that \( u|_{\mathcal{O}_j} \) is extended by zero to the exterior of \( \mathcal{O}_j, j = 1, 2 \). This inequality is the right tool to prove (1.5).

The remainder of the article is organised as follows. In Section 2 we present the precise definitions of the Sobolev spaces and norms in consideration. The main results are stated in Section 3, the proofs of which are performed in Section 4. Section 5 discusses applications of these results.

In the sequel, if \( a \leq cb \) where the constant \( c \) is independent of the parameters in concern, for example, the functions and the size of the domain, we will write \( a \lesssim b \). We will also write \( a \simeq b \) if \( a \lesssim b \) and \( b \lesssim a \).

2. Sobolev norms

In this section we recall the definitions of Sobolev spaces of fractional orders that we are interested in. Let \( \mathcal{O} \) be a generic bounded and connected domain in \( \mathbb{R}^n \), \( n \geq 1 \), with Lipschitz boundary, and let \( r \) be a positive integer. The spaces \( L^2(\mathcal{O}) \), \( W^2_2(\mathcal{O}) = H^r(\mathcal{O}) \), and \( \tilde{W}^2_2(\mathcal{O}) = H^r_0(\mathcal{O}) \) are defined as usual with norms denoted by \( \|\cdot\|_{L^2(\mathcal{O})} \), \( \|\cdot\|_{W^2_2(\mathcal{O})} \), and \( \|\cdot\|_{\tilde{W}^2_2(\mathcal{O})} \), respectively. Here, the seminorm in \( H^r(\mathcal{O}) \) is used to define the norm in \( H^r_0(\mathcal{O}) \). In the sequel, we define the fractional-order Sobolev spaces of order \( s > 0 \); see e.g., [11, 12].
2.1. Real interpolation spaces $\tilde{H}^s(\mathcal{O})$ for $s > 0$. The general method for constructing real interpolation spaces can be found in [11]. For any $u \in L^2(\mathcal{O})$ and any $t > 0$, the functional $K(t, u)$ is defined by

$$K(t, u) := \inf_{(u_0, u_1) \in \mathcal{X}(u)} \left( \|u_0\|_{0, \mathcal{O}}^2 + t^2 \|u_1\|_{r, \mathcal{O}}^2 \right)^{1/2}$$  \hspace{1cm} (2.1)

where

$$\mathcal{X}(u) = \left\{ (u_0, u_1) \in L^2(\mathcal{O}) \times H_0^s(\mathcal{O}) : u_0 + u_1 = u \right\}.$$  \hspace{1cm} (2.2)

For $s \in (0, r)$, with $\theta = s/r$, we define the interpolation space $[L^2(\mathcal{O}), H_0^s(\mathcal{O})]_\theta$ by

$$[L^2(\mathcal{O}), H_0^s(\mathcal{O})]_\theta := \left\{ u \in L^2(\mathcal{O}) : \|u\|_{[L^2(\mathcal{O}), H_0^s(\mathcal{O})]_\theta} < \infty \right\}$$

where

$$\|u\|_{[L^2(\mathcal{O}), H_0^s(\mathcal{O})]_\theta} := \left( \int_0^\infty \left\| t^{-\theta} K(t, u) \right\|_{L^2(\mathcal{O}), H_0^s(\mathcal{O})} \frac{dt}{t} \right)^{1/2}.$$  \hspace{1cm} (2.3)

We follow [11] to denote this space by $\tilde{H}^s(\mathcal{O})$ and equipped it with the norm

$$\|u\|_{\tilde{H}^s(\mathcal{O})} := \|u\|_{[L^2(\mathcal{O}), H_0^s(\mathcal{O})]_\theta}.$$  \hspace{1cm} (2.4)

If $s - 1/2 \notin \mathbb{N}$ (the set of non-negative integers) then $\tilde{H}^s(\mathcal{O}) = H_0^s(\mathcal{O})$, the closure of $\mathcal{D}(\mathcal{O})$ in $H^s(\mathcal{O})$, where $\mathcal{D}(\mathcal{O})$ is the space of all $C^\infty$ functions with compact support in $\mathcal{O}$. Here, the space $H^s(\mathcal{O}) := [L^2(\mathcal{O}), H_0^s(\mathcal{O})]_\theta$ is defined by interpolation as $\tilde{H}^s(\mathcal{O})$ with the $K$-functional defined with the norm $\|u_1\|_{r, \mathcal{O}}$ instead of the seminorm.

If $s - 1/2 \in \mathbb{N}$, then $\tilde{H}^s(\mathcal{O})$ is a proper subset of $H_0^s(\mathcal{O})$ and is denoted by $H_{00}^s(\mathcal{O})$ in [18]. We follow [11] to use the same notation $\tilde{H}^s(\mathcal{O})$ in both cases.

A special case is when $u \in \tilde{H}^s(\mathcal{O})$ is supported in $\overline{\mathcal{O}}_1$, where $\mathcal{O}_1$ is a bounded Lipschitz domain which is a proper subset of $\mathcal{O}$. The function $u$ also belongs to $\tilde{H}^s(\mathcal{O}_1)$. We want to compare the norm $\|u\|_{\tilde{H}^s(\mathcal{O}_1)}$ with $\|u\|_{\tilde{H}^s(\mathcal{O})}$.

First we clarify how the norm $\|u\|_{\tilde{H}^s(\mathcal{O}_1)}$ is defined. For $r \in \mathbb{N} \setminus \{0\}$, we define two spaces

$$A_0 := \left\{ v \in L^2(\mathcal{O}) : \text{supp}(v) \subseteq \overline{\mathcal{O}}_1 \right\},$$

$$A_1 := \left\{ v \in H_0^s(\mathcal{O}) : \text{supp}(v) \subseteq \overline{\mathcal{O}}_1 \right\},$$

which form a compatible couple $\mathcal{A} = (A_0, A_1)$. By zero extension, we can identify $L^2(\mathcal{O}_1)$ and $H_0^s(\mathcal{O}_1)$ with $A_0$ and $A_1$, respectively. For any $u \in A_0$ we define, similarly to (2.2),

$$\mathcal{X}_1(u) := \left\{ (u_0, u_1) \in A_0 \times A_1 : u_0 + u_1 = u \right\}.$$  \hspace{1cm} (2.5)

We also define two functionals $J : \mathbb{R}^+ \times L^2(\mathcal{O}) \times H_0^s(\mathcal{O}) \to \mathbb{R}$ and $J_1 : \mathbb{R}^+ \times A_0 \times A_1 \to \mathbb{R}$ by

$$J(t, u_0, u_1) := \|u_0\|_{0, \mathcal{O}}^2 + t^2 \|u_1\|_{r, \mathcal{O}}^2, \quad t > 0, \ u_0 \in L^2(\mathcal{O}), \ u_1 \in H_0^s(\mathcal{O}),$$

$$J_1(t, u_0, u_1) := \|u_0\|_{0, \mathcal{O}_1}^2 + t^2 \|u_1\|_{r, \mathcal{O}_1}^2, \quad t > 0, \ u_0 \in A_0, \ u_1 \in A_1.$$  \hspace{1cm} (2.6)

The $K$-functional defined in (2.1) can be written as

$$K(t, u) = \inf \left\{ J(t, u_0, u_1) : (u_0, u_1) \in \mathcal{X}(u) \right\}.$$  \hspace{1cm} (2.7)

Correspondingly, we define

$$K_1(t, u) = \inf \left\{ J_1(t, u_0, u_1) : (u_0, u_1) \in \mathcal{X}_1(u) \right\}.$$  \hspace{1cm} (2.8)
and the corresponding norm

\[ \|u\|_{[A_0, A_1]_\theta} := \left( \int_0^\infty |t^{-\theta}K_1(t, u)|^2 \text{d}t \right)^{1/2}, \quad \theta \in (0, 1). \]  

(2.7)

We note that \( \mathcal{X}_1(u) \) is a proper subset of \( \mathcal{X}(u) \) because in the definition of \( \mathcal{X}(u) \) for \( u \in A_0 \), the two functions \( u_0 \) and \( u_1 \) do not have to be zero in \( O_2 := O \setminus O_1 \), but \( u_0 = -u_1 \) in \( \partial O_2 \). Consequently, for any \( t > 0 \),

\[ K(t, u) \leq K_1(t, u). \]

We will prove later that in general this is indeed a strict inequality.

Using the norm defined by (2.7) we can define the interpolation space

\[ [A_0, A_1]_\theta := \left\{ u \in A_0 : \|u\|_{[A_0, A_1]_\theta} < \infty \right\}. \]

This space is the usual space \( \tilde{H}^s(O_1) \) which is equipped with the norm \( \|u\|_{\tilde{H}^s(O_1)} = \|u\|_{[A_0, A_1]_\theta} \), where \( s = \theta r \). Denote

\[ \tilde{H}^s(O_1) := \left\{ u \in A_0 : \|u\|_{\tilde{H}^s(O)} < \infty \right\} \quad \text{and} \quad \|u\|_{\tilde{H}^s(O_1)} := \|u\|_{\tilde{H}^s(O)}. \]

Clearly \( \tilde{H}^s(O_1) \) is a proper subset of \( \tilde{H}^s(O) \). We will prove in Subsection 4.2 that the following subset inclusion is proper

\[ \tilde{H}^s(O_1) \subset \tilde{H}^s(O_1). \]

(2.8)

2.2. Sobolev–Slobodetskii spaces \( W^s_2(O) \) and \( \tilde{W}^s_2(O) \) for \( s > 0 \). Let \( s = m + \sigma \) with \( m \in \mathbb{N} \) and \( \sigma \in (0, 1) \). For every function \( u \) defined in \( O \), we define

\[ \|u\|_{m, O} := \left( \sum_{|\alpha|=0}^m \|\partial^\alpha u\|_{0, O}^2 \right)^{1/2}, \]

\[ |u|_{\sigma, O} := \left( \int_{O \times O} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\sigma}} \text{d}x \text{d}y \right)^{1/2}. \]

\[ \|u\|_{s, O} := \left( \|u\|_{m, O}^2 + \sum_{|\alpha|=m} \|\partial^\alpha u\|_{\sigma, O}^2 \right)^{1/2}. \]

The space \( W^s_2(O) \) is the space of all functions \( u \) defined in \( O \) such that \( \|u\|_{s, O} < \infty \).

The space \( \tilde{W}^s_2(O) \) is defined to be the closure of \( \mathcal{D}(O) \) in \( W^s_2(O) \). Using this space, we define

\[ \tilde{W}^s_2(O) := \left\{ u \in \tilde{W}^s_2(O) : \partial^\alpha u/\rho^\sigma \in L^2(O), \ |\alpha| = m \right\} \]

where \( \rho(x) := \text{dist}(x, \partial O) \) is the distance from \( x \) to the boundary \( \partial O \) of \( O \). This space is equipped with the norm

\[ \|u\|_{s, O} := \left( \int_O \frac{|\partial^\alpha u(x)|^2}{\rho^{2\sigma}(x)} \text{d}x \right)^{1/2}. \]

We note that when \( s = \sigma \in (0, 1) \), i.e., \( m = 0 \), the norm can also be defined by

\[ \|u\|_{s, O} := \left( \int_O \frac{|u(x)|^2}{\rho^{2\sigma}(x)} \text{d}x \right)^{1/2}. \]  

(2.9)
The advantage of this norm is the two terms defining the norm scale similarly when the domain $\mathcal{O}$ is rescaled; see Proposition 2.1.

2.3. Equivalence of norms. If $s = m + 1/2$ for $m \in \mathbb{N}$, then $\tilde{H}^s(\mathcal{O}) = \tilde{W}^s_2(\mathcal{O})$; see [13, Theorem 11.7, page 66]. Moreover,

$$\|u\|_{\tilde{H}^s(\mathcal{O})} \simeq \|u\|_{s,\mathcal{O}}.$$  \hspace{1cm} (2.10)

It is sometimes important to know how the constants depend on the size of $\mathcal{O}$. In the sequel, we assume that all domains are shape regular, i.e., we avoid long and thin shapes. More precisely, the ratio of the diameter of the domain over the diameter of the largest ball inside the domain is less than some constant. In the next proposition, we study the scaling property of the two norms $\|\cdot\|_{\tilde{H}^s(\mathcal{O})}$ and $\|\cdot\|_{s,\mathcal{O}}$, and show how this property depends on the diameter of the domain. A thorough study involving also the diameter of the largest interior ball can be found in [13].

**Proposition 2.1.** Assume that $\Omega$ is a domain in $\mathbb{R}^n$ with Lipschitz boundary satisfying $\tau := \text{diam}(\Omega) < 1$. Then, for $s = m + \sigma$ for $m \in \mathbb{N}$ and $\sigma \in (0, 1)$, show how each norm scales when the domain $\Omega$ is rescaled. Let $\hat{\Omega}$ be a reference set of diameter $1$ satisfying

$$\hat{x} \in \hat{\Omega} \iff \hat{x} = x/\tau, \quad x \in \Omega,$$

and let $\hat{u} : \hat{\Omega} \to \mathbb{R}$ be defined by $\hat{u}(\hat{x}) = u(x)$ for all $\hat{x} \in \hat{\Omega}$ and $x \in \Omega$. Simple calculations reveal

$$\|u\|_{0,\hat{\Omega}}^2 = \tau^m \|\hat{u}\|_{0,\hat{\Omega}}^2 \quad \text{and} \quad \partial_x^\alpha \hat{u}(\hat{x}) = \tau^{\lvert \alpha \rvert} \partial_x^\alpha u(x),$$

Hence, for $m \in \mathbb{N}$,

$$\|u\|_{m,\Omega}^2 = \sum_{\lvert \alpha \rvert = 0}^m \|\partial_x^\alpha u\|_{0,\Omega}^2 = \tau^m \sum_{\lvert \alpha \rvert = 0}^m \|\partial_x^\alpha \hat{u}\|_{0,\hat{\Omega}}^2 = \tau^m \sum_{\lvert \alpha \rvert = 0}^m \tau^{-\lvert \alpha \rvert} \|\partial_x^\alpha \hat{u}\|_{0,\hat{\Omega}}^2.$$

On the other hand, for $\sigma \in (0, 1)$ and $\lvert \alpha \rvert = m$

$$\int_{\Omega \times \Omega} \frac{|\partial_x^\alpha u(x) - \partial_x^\alpha u(y)|^2}{|x - y|^{n+2\sigma}} \, dx \, dy = \int_{\hat{\Omega} \times \hat{\Omega}} \frac{\tau^{-2m} |\partial_x^\alpha \hat{u}(\hat{x}) - \partial_x^\alpha \hat{u}(\hat{y})|^2}{\tau^{n+2\sigma} |\hat{x} - \hat{y}|^{n+2\sigma}} \, \tau^{2\sigma} \, d\hat{x} \, d\hat{y} = \tau^{n-2s} \int_{\hat{\Omega} \times \hat{\Omega}} \frac{\tau^{-2m} |\partial_x^\alpha \hat{u}(\hat{x}) - \partial_x^\alpha \hat{u}(\hat{y})|^2}{\tau^{n+2\sigma} |\hat{x} - \hat{y}|^{n+2\sigma}} \, d\hat{x} \, d\hat{y}$$

and, since $\rho(x) = \text{dist}(x, \partial \Omega) = \tau \text{dist}(\hat{x}, \partial \hat{\Omega}) = \tau \hat{\rho}(\hat{x}),$

$$\int_{\Omega} |\partial_x^\alpha u(x)|^2 \, dx = \int_{\hat{\Omega}} \frac{\tau^{-2m} |\partial_x^\alpha \hat{u}(\hat{x})|^2}{\tau^{2\sigma} \hat{\rho}^{2\sigma}(\hat{x})} \, \tau^n \, d\hat{x} = \tau^{n-2s} \int_{\hat{\Omega}} \frac{\tau^{-2m} |\partial_x^\alpha \hat{u}(\hat{x})|^2}{\tau^{2\sigma} \hat{\rho}^{2\sigma}(\hat{x})} \, d\hat{x}.$$
Consequently,
\[
\|u\|_{\tilde{H}_{r,\Omega}}^2 = \|u\|_{m,\Omega}^2 + \sum_{|\alpha|=m} \int_\Omega \int_{\Omega} \frac{\left|\partial_\alpha^2 u(x) - \partial_\alpha^2 u(y)\right|^2}{|x-y|^{n+2\sigma}} \, dx \, dy + \sum_{|\alpha|=m} \int_\Omega \frac{\left|\partial_\alpha^2 u(x)\right|^2}{\rho^{2\sigma}(x)} \, dx
\]
\[= \tau^n \sum_{|\alpha|=0} \tau^{-2|\alpha|} \|\partial_\alpha^2 \tilde{u}\|_{0,\tilde{\Omega}}^2 + \tau^{-2s} \sum_{|\alpha|=m} \int_\tilde{\Omega} \int_{\tilde{\Omega}} \frac{\left|\partial_\alpha^2 \tilde{u}(\tilde{x}) - \partial_\alpha^2 \tilde{u}(\tilde{y})\right|^2}{|\tilde{x}-\tilde{y}|^{n+2\sigma}} \, d\tilde{x} \, d\tilde{y}
\[+ \tau^{-2s} \sum_{|\alpha|=m} \int_{\tilde{\Omega}} \frac{\left|\partial_\alpha^2 \tilde{u}(\tilde{x})\right|^2}{\rho^{2\sigma}(\tilde{x})} \, d\tilde{x}.
\]
Therefore,
\[
\tau^n \|\tilde{u}\|_{\tilde{H}_{r,\tilde{\Omega}}}^2 \leq \|u\|_{\tilde{H}_{r,\Omega}}^2 \leq \tau^{-2s} \|\tilde{u}\|_{\tilde{H}_{r,\tilde{\Omega}}}^2 \tag{2.13}
\]
When \(m = 0\) if we define the \(\|\cdot\|_{\tilde{H}_{r,\tilde{\Omega}}}\)-norm by (2.11) then
\[
\|u\|_{\tilde{H}_{r,\tilde{\Omega}}}^2 = \tau^{n-1} \|\tilde{u}\|_{\tilde{H}_{r,\tilde{\Omega}}}^2 \tag{2.14}
\]
For the interpolation norm, we have
\[
\|u\|_{L^2(\Omega)}^2 = \tau^n \|\tilde{u}\|_{L^2(\tilde{\Omega})}^2
\]
and
\[
\|u\|_{\tilde{H}_{r}(\Omega)}^2 = \sum_{|\alpha|=r} \|\partial_\alpha u\|_{L^2(\Omega)}^2 = \tau^{-2r} \sum_{|\alpha|=r} \|\partial_\alpha^2 \tilde{u}\|_{L^2(\tilde{\Omega})}^2 = \tau^{-2r} \|\tilde{u}\|_{\tilde{H}_{r}(\tilde{\Omega})}^2 \tag{2.15}
\]
By interpolation
\[
\|u\|_{\tilde{H}_{s}(\Omega)}^2 = \tau^{-2s} \|\tilde{u}\|_{\tilde{H}_{s}(\tilde{\Omega})}^2 \quad 0 \leq s \leq r.
\]
Now consider the case when \(s = m + 1/2\). Using (2.10), (2.13), and (2.15), we deduce
\[
\|u\|_{\tilde{H}_{r,\Omega}}^2 \leq \tau^{-2s} \|\tilde{u}\|_{\tilde{H}_{r,\tilde{\Omega}}}^2 \simeq \tau^{-2s} \|\tilde{u}\|_{\tilde{H}_{r}(\tilde{\Omega})}^2 = \|u\|_{\tilde{H}_{r}(\Omega)}^2
\]
and
\[
\|u\|_{\tilde{H}_{s}(\Omega)}^2 = \tau^{-2s} \|\tilde{u}\|_{\tilde{H}_{s}(\tilde{\Omega})}^2 \simeq \tau^{-2s} \|\tilde{u}\|_{\tilde{H}_{s,\tilde{\Omega}}}^2 \simeq \tau^{-2s} \|u\|_{\tilde{H}_{s,\tilde{\Omega}}}^2,
\]
yielding the first part of the lemma. The constants in the above equivalences \(\simeq\) are the constants in (2.10), which depend on the size of \(\tilde{\Omega}\). Recall that \(\text{diam}(\tilde{\Omega}) = 1\).

In the case when \(s = 1/2\) with norm defined by (2.11), it follows from (2.14) and (2.15) that
\[
\|u\|_{\tilde{H}_{1/2}(\Omega)}^2 = \tau^{-1} \|\tilde{u}\|_{\tilde{H}_{1/2}(\tilde{\Omega})}^2 \simeq \tau^{-1} \|\tilde{u}\|_{\tilde{H}_{1/2,\tilde{\Omega}}}^2 = \|u\|_{\tilde{H}_{1/2,\tilde{\Omega}}}^2,
\]
completing the proof of the lemma.

The following theorem concerning properties of the \(\tilde{H}^s(\Omega)\) norms is proved in [26, Lemma 3.2] and in [1, Theorem 4.1].

**Theorem 2.2.** Let \(\{\Omega_1, \ldots, \Omega_N\}\) be a partition of a bounded Lipschitz domain \(\Omega\) into non-overlapping Lipschitz domains. For \(0 \leq s \leq r\), the following inequalities hold (assuming that all the norms are well defined)
\[
\|u\|_{\tilde{H}^s(\Omega)}^2 \leq \sum_{j=1}^N \|u\|_{\Omega_j}^2 \|\tilde{u}\|_{\tilde{H}^s(\Omega_j)}^2 \tag{2.16}
\]
A consequence of the above theorem is that if $\Omega' \subseteq \Omega$ and $\text{supp}(u) \subset \overline{\Omega'}$ then

$$\|u\|_{H^s(\Omega)}^2 \leq \|u\|_{\tilde{H}^s(\Omega')}^2,$$  \hspace{1cm} (2.17)

provided that all the norms are well defined.

3. The main results

We now state our main results, the proofs of which will be carried out in Section 4. The first theorem confirms that if $u \in \tilde{H}^{1/2}(\Omega)$ is such that $\text{supp}(u) \subset \Omega'$ where $\Omega'$ is a proper subset of $\Omega$, which is itself a Lipschitz domain, then the two norms $\|u\|_{\sim,1/2,\Omega}$ and $\|u\|_{\sim,1/2,\Omega'}$ are not equivalent.

**Theorem 3.1.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 1$. Assume that $u \in \tilde{H}^{1/2}(\Omega)$ satisfies $\text{supp}(u) \subset \Omega'$ where $\Omega'$ is a proper subset of $\Omega$, which is itself a Lipschitz domain, then the two norms $\|u\|_{\sim,1/2,\Omega}$ and $\|u\|_{\sim,1/2,\Omega'}$ are not equivalent.

(i) The following relations between norms of $u$ hold

$$\|u\|_{\sim,1/2,\Omega} \leq C \|u\|_{\sim,1/2,\Omega'},$$

where $C = 4\pi^{n-1} + 1$.

(ii) The opposite inequality is not true, i.e., there is no constant $c$ independent of $u$ and the sizes of $\Omega'$ and $\Omega$ such that $\|u\|_{\sim,1/2,\Omega'} \leq c \|u\|_{\sim,1/2,\Omega}$.

Our next main result improves Theorem 2.2, namely we prove that the norm on the right-hand side of (2.16) can be replaced by $\|u|_{\Omega_j}\|_{\tilde{H}^{1/2}(\Omega)}^2$; cf. (2.17).

**Theorem 3.2.** Let $\{\Omega_1, \ldots, \Omega_N\}$ be a partition of a bounded Lipschitz domain $\Omega$ into non-overlapping Lipschitz domains. For $0 \leq s \leq r$, let $u \in \tilde{H}^s(\Omega)$ be such that $u_j \in \tilde{H}^s(\Omega_j)$, where $u_j$ is the zero extension of $u|_{\Omega_j}$ onto $\Omega \setminus \overline{\Omega_j}$, $j = 1, \ldots, N$. Then the following inequalities hold

$$\|u\|_{\tilde{H}^s(\Omega)}^2 \leq \sum_{j=1}^N \|u_j\|_{\tilde{H}^s(\Omega)}^2.$$  \hspace{1cm} (3.1)

A direct consequence of Theorem 3.2 is the following corollary which generalises Theorem 2.2 and has applications discussed in Section 5.

**Corollary 3.3.** Under the assumption of Theorem 3.2 we have

$$\|u\|_{\tilde{H}^{1/2}(\Omega)}^2 \leq \sum_{j=1}^N \|u_j\|_{\tilde{H}^{1/2}(\Omega)}^2$$  \hspace{1cm} (3.2)

where $u_j$ is the zero extension of $u|_{\Omega_j}$ onto $\Omega \setminus \overline{\Omega_j}$.

**Proof.** The result is a direct consequence of Theorem 3.2, noting that $\tilde{H}^{1/2}(\Omega_j) \subset \tilde{H}^{1/2}(\Omega_j)$.

4. Proofs of the main results

4.1. Proof of Theorem 3.1.
Proof. We first prove part (i). Recall the definition of $\|v\|_{\sim,1/2,\Omega}$:

$$\|u\|_{\sim,1/2,\Omega}^2 = \|u\|_{1/2,\Omega}^2 + \int_\Omega \frac{|u(x)|^2}{\dist(x, \partial\Omega)} \, dx.$$ 

Clearly,

$$\int_\Omega \frac{|u(x)|^2}{\dist(x, \partial\Omega)} \, dx = \int_{\Omega'} \frac{|u(x)|^2}{\dist(x, \partial\Omega')} \, dx \leq \int_{\Omega'} \frac{|u(x)|^2}{\dist(x, \partial\Omega')} \, dx. \quad (4.1)$$

On the other hand, since $\supp(u) \subset \Omega'$

$$|u|_{1/2,\Omega}^2 = \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+1}} \, dx \, dy + 2 \int_{\Omega'} \left( \int_{\Omega \setminus \Omega'} \frac{dy}{|x-y|^{n+1}} \right) |u(x)|^2 \, dx.$$

It will be proved in Lemma 4.1 below that

$$\int_{\Omega \setminus \Omega'} \frac{dy}{|x-y|^{n+1}} \leq \frac{2\pi^{n-1}}{\dist(x, \partial\Omega')} \quad \forall x \in \Omega'.$$

Hence

$$|u|_{1/2,\Omega}^2 \leq |u|_{1/2,\Omega}^2 + 4\pi^{n-1} \int_{\Omega'} \frac{|u(x)|^2}{\dist(x, \partial\Omega')} \, dx \leq 4\pi^{n-1} \|u\|_{\sim,1/2,\Omega'}^2$$

so that, with the help of (4.1),

$$\|u\|_{\sim,1/2,\Omega}^2 \leq 4\pi^{n-1} \|u\|_{1/2,\Omega}^2 + \int_{\Omega'} \frac{|u(x)|^2}{\dist(x, \partial\Omega')} \, dx \leq (4\pi^{n-1} + 1) \|u\|_{\sim,1/2,\Omega'}^2.$$

This proves part (i).

Part (ii) is proved by the following counter-example, which is a modification of the counter-example in the appendix of [2]. Consider $D$ to be the upper half of the unit disk in the $(x,y)$-plane, i.e.,

$$D = \{(r, \theta) : r \in [0,1], \ \theta \in [0,\pi]\}$$

where $(r,\theta)$ denote polar coordinates. Then define

$$\Omega = \{(r, \theta) : r \in [0,1], \ \theta = 0 \ or \ \theta = \pi\} = [-1,1] \times \{0\}$$

$$\Omega' = \{(r, \theta) : r \in [0,3/4], \ \theta = 0\} = [0,3/4] \times \{0\}.$$

For $\epsilon \in (0,1/2)$, define $U_\epsilon : D \to \mathbb{R}$ and $U : D \to \mathbb{R}$ by

$$U_\epsilon(r, \theta) = \begin{cases} 0, & 0 \leq r < \epsilon, \\ (-\log r)^{-1/2} - (-\log \epsilon)^{-1/2}, & \epsilon \leq r < 1/2, \\ (3 - 4r)(\log 2)^{-1/2} - (-\log \epsilon)^{-1/2}, & 1/2 \leq r < 3/4, \\ 0, & 3/4 \leq r \leq 1, \end{cases}$$

and

$$U(r, \theta) = \begin{cases} 0, & r = 0, \\ (-\log r)^{-1/2}, & 0 < r < 1/2, \\ (3 - 4r)(\log 2)^{-1/2}, & 1/2 \leq r < 3/4, \\ 0, & 3/4 \leq r \leq 1, \end{cases}$$

These two functions are first studied in [2]. We now define $V_\epsilon : D \to \mathbb{R}$ by

$$V_\epsilon(r, \theta) = \begin{cases} U_\epsilon(r, \theta) \cos \theta, & 0 \leq \theta < \pi, \\ 0, & \pi \leq \theta \leq \pi. \end{cases}$$
Let \( u_\epsilon \) be the trace of \( V_\epsilon \) on the boundary of \( D \). Then
\[
\text{supp}(u_\epsilon) = [\epsilon, 3/4] \times \{0\} \subset \Omega'.
\]
For \((r, \theta) \in D\),
\[
|V_\epsilon(r, \theta)| \leq |U_\epsilon(r, \theta)| \leq |U(r, \theta)|
\]
\[
\left| \frac{\partial V_\epsilon}{\partial r}(r, \theta) \right| \leq \left| \frac{\partial U_\epsilon}{\partial r}(r, \theta) \right| \leq \left| \frac{\partial U}{\partial r}(r, \theta) \right|
\]
\[
\left| \frac{\partial V_\epsilon}{\partial \theta}(r, \theta) \right| \leq |U_\epsilon(r, \theta)| \leq |U(r, \theta)|
\]
(4.3)
We note that
\[
\frac{\partial U}{\partial r}(r, \theta) = \begin{cases} \frac{1}{2}r^{-1}(- \log r)^{-3/2}, & 0 < r < 1/2, \\ -4(\log 2)^{-1/2}, & 1/2 < r < 3/4, \\ 0, & 3/4 < r < 1,
\end{cases}
\]
so that \( U \in H^1(D) \). Indeed,
\[
\int_{D} |U(r, \theta)|^2 \, dx \, dy = \pi \int_{0}^{1/2} \frac{r}{- \log r} \, dr + \pi \int_{1/2}^{3/4} \frac{1}{\log 2} (3 - 4r)^2 r \, dr < \infty
\]
and
\[
\int_{D} \left| \frac{\partial U}{\partial r}(r, \theta) \right|^2 \, dx \, dy = \pi \int_{0}^{1/2} \frac{dr}{r(- \log r)^3} + \pi \int_{1/2}^{3/4} \frac{16}{\log 2} r \, dr < \infty.
\]
It follows from (4.3) that \( V_\epsilon, U_\epsilon \in H^1(D) \) and
\[
\|V_\epsilon\|_{1,D} \lesssim \|U_\epsilon\|_{1,D} \lesssim \|U\|_{1,D}, \quad 0 < \epsilon < 1.
\]
Consequently, by the definition of the Slobodetski norm and the trace theorem
\[
\|u_\epsilon\|_{1/2,\Omega} \leq \|u_\epsilon\|_{1/2,\partial D} \lesssim \|V_\epsilon\|_{1,D} \lesssim 1.
\]
Since \( \text{supp}(u_\epsilon) = [\epsilon, 3/4] \times \{0\} \), we have
\[
\|u_\epsilon\|_{\infty,1/2,\Omega}^2 = \|u_\epsilon\|_{1/2,\Omega}^2 + \int_{-1}^{1} \frac{|u_\epsilon(r,0)|^2}{\min\{1-r,1+r\}} \, dr
\]
\[
= \|u_\epsilon\|_{1/2,\Omega}^2 + \int_{\epsilon}^{3/4} \frac{|u_\epsilon(r,0)|^2}{\min\{1-r,1+r\}} \, dr.
\]
Due to
\[
\|u_\epsilon\|_{L^2(\Omega)}^2 = \|u_\epsilon\|_{L^2(\Omega')}^2 \lesssim \int_{\epsilon}^{3/4} \frac{|u_\epsilon(r,0)|^2}{\min\{1-r,1+r\}} \, dr,
\]
we deduce
\[
\|u_\epsilon\|_{\infty,1/2,\Omega'}^2 \lesssim \|u_\epsilon\|_{1/2,\Omega}^2 + \|u_\epsilon\|_{L^2(\Omega')}^2 = \|u_\epsilon\|_{1/2,\Omega}^2,
\]
so that \( \|u_\epsilon\|_{\infty,1/2,\Omega'}^2 \lesssim 1 \). On the other hand, a simple calculation reveals that
\[
\|u_\epsilon\|_{\infty,1/2,\Omega'}^2 \geq \int_{\Omega'} \frac{|u_\epsilon(r,0)|^2}{\text{dist}(r,\partial \Omega')} \, dr \geq \int_{\epsilon}^{1/2} \frac{|u_\epsilon(r,0)|^2}{r} \, dr
\]
\[
= \log |\log \epsilon| + \frac{4(\log 2)^{1/2}}{(\log(1/\epsilon))^{1/2}} - \frac{\log 2}{\log(1/\epsilon)} - \log |\log 2| - 3.
\]
Hence, \( \|u_\epsilon\|_{\infty,1/2,\Omega'} \to \infty \) as \( \epsilon \to 0^+ \), while \( \|u_\epsilon\|_{\infty,1/2,\Omega} \) is bounded. This proves part (iii) completing the proof of the theorem. \( \Box \)
We now prove the claim (4.2).

**Lemma 4.1.** Let $\Omega$ and $\Omega'$ be two open bounded domains in $\mathbb{R}^n$, $n = 1, 2, 3$, satisfying $\Omega' \subset \Omega$, and let $x \in \Omega'$.

(i) The following inequality holds
\[
\int_{\Omega \setminus \Omega'} \frac{dy}{|x - y|^{n+1}} \leq \frac{2\pi^{n-1}}{\text{dist}(x, \partial \Omega')}.
\]

(ii) The opposite inequality is not true, i.e., there is no constant $C$ independent of $x$ such that
\[
\frac{1}{\text{dist}(x, \partial \Omega')} \leq C \int_{\Omega \setminus \Omega'} \frac{dy}{|x - y|^{n+1}}.
\]

**Proof.** To prove part (i), consider first the case when $x = 0$, $\Omega = B_R = B_R(0)$, and $\Omega' = B_{R'} = B_{R'}(0)$, $R' < R$. Here $B_r(z)$ is the ball centred at $z$ and having radius $r$.

The required statement becomes
\[
\int_{B_R \setminus B_{R'}} \frac{dy}{|y|^{n+1}} \leq \frac{2\pi^{n-1}}{R'}.
\] (4.4)

The result for $n = 1$ is easily seen. We prove the result for $n = 2$. The case when $n = 3$ can be proved similarly. By using polar coordinates
\[
\int_{B_R \setminus B_{R'}} \frac{dy}{|y|^3} = 2\pi \int_{R'}^R \frac{dr}{r^2} = 2\pi \left( \frac{1}{R'} - \frac{1}{R} \right) \leq \frac{2\pi}{R'}.
\]

In the general case, let

\[
R' = \text{dist}(x, \partial \Omega'), \quad R = \max_{z \in \partial \Omega} |x - z|
\]

so that $R' < R$ and that

\[
B_R(x) \subset \Omega' \subset \Omega \subset B_R(x).
\]

Consequently,
\[
\int_{\Omega \setminus \Omega'} \frac{dy}{|x - y|^3} \leq \int_{B_{R_+}(x) \setminus B_{R'}(x)} \frac{dy}{|x - y|^3}.
\]

Since
\[
\int_{B_{R_+}(x) \setminus B_{R'}(x)} \frac{dy}{|x - y|^3} = \int_{B_{R_+} \setminus B_{R'}} \frac{dy}{|y|^3}
\]

it follows from (4.4) that
\[
\int_{\Omega \setminus \Omega'} \frac{dy}{|x - y|^3} \leq \frac{2\pi}{R'} = \frac{2\pi}{\text{dist}(x, \partial \Omega')}.
\]

To prove part (ii), we revisit the proof of part (i) of Theorem 3.1. If the opposite of (4.2) holds, then
\[
\int_{\Omega \setminus \Omega'} \frac{dy}{|x - y|^{n+1}} \leq \frac{1}{\text{dist}(x, \partial \Omega')}.
\]

It can be derived from that proof that
\[
\|v\|_{\sim, 1/2, \Omega} \simeq \|v\|_{\sim, 1/2, \Omega'}^2,
\]

which cannot be true as shown by the counter-example in part (ii) of Theorem 3.1. $\square$
4.2. **Proof of claim (2.5)**. To prove the proper inclusion $\tilde{H}^s(O_1) \subsetneq \tilde{H}^s_2(O_1)$, it suffices to prove the following lemma with $r = 1$.

**Lemma 4.2.** Assume that $O$ has a smooth boundary. Let $v \in A_0$ be such that $v \in C^4(O)$ and $v > 0$ in $O_1$. Then, for any $t > 0$,

$$K(t, v) < K_1(t, v)$$

where $K$ and $K_1$ are defined by (2.5) and (2.6), respectively.

**Proof.** Fix $t > 0$. Let $(v_0^*, v_1^*) \in X(v)$, see (2.2), be such that

$$\{v_0^*, v_1^*\} = \text{argmin} \{J(t, v_0, v_1) : (v_0, v_1) \in X(v)\} \quad (4.5)$$

where $J(t, v_0, v_1)$ is defined in (2.4). It suffices to show that $(v_0^*, v_1^*) \notin X_1(v)$; see (2.3). The problem (4.5) is an optimisation problem with constraint, the constraint being $g(v_0, v_1) = 0$ where

$$g : L^2(O) \times H^1_0(O) \rightarrow L^2(O), \quad g(v_0, v_1) := v_0 + v_1 - v.$$ 

For each $t > 0$, the Lagrangian functional $L : L^2(O) \times H^1_0(O) \times L^2(O) \rightarrow \mathbb{R}$ is defined by

$$L(v_0, v_1, p) := J(t, v_0, v_1) + G(v_0, v_1, p)$$

where $G(v_0, v_1, p) := \langle p, g(v_0, v_1) \rangle_{L^2(O)} = \langle p, v_0 + v_1 - v \rangle_{L^2(O)}$. Here $p$ is the Lagrangian multiplier. It is well known that

$$\min_{(v_0, v_1) \in X(v)} J(t, v_0, v_1) = \inf_{v_0 \in L^2(O)} \sup_{v_1 \in H^1_0(O)} \sup_{p \in L^2(O)} L(v_0, v_1, p). \quad (4.6)$$

For any functional $F : (v, p) \mapsto F(v, p)$, we denote by $\partial_v F(v, p)(\varphi)$ the $v$-Fréchet derivative of $F$ at $(v, p)$, acting on $\varphi$. Similarly, $\partial_p F(v, p)(q)$ denotes the $p$-Fréchet derivative of $F$ at $(v, p)$, acting on $q$. The minimiser $(v_0^*, v_1^*)$ and the solution $(v_0^*, v_1^*, p^*)$ to the minimax problem (4.6) solve the following equations

$$\partial_{v_0} L(v_0, v_1, p) = 0, \quad \partial_{v_1} L(v_0, v_1, p) = 0, \quad \partial_p L(v_0, v_1, p) = 0.$$ 

Since (see e.g. [1])

$$\partial_{v_0} J(t, v_0, v_1)(\varphi) = 2 \langle v_0, \varphi \rangle_{L^2(O)} \quad \forall \varphi \in L^2(O),$$

$$\partial_{v_1} J(t, v_0, v_1)(\psi) = 2t^2 \langle \nabla v_1, \nabla \psi \rangle_{L^2(O)} \quad \forall \psi \in H^1(O),$$

$$\partial_{v_0} G(v_0, v_1, p)(\varphi) = \langle p, \varphi \rangle_{L^2(O)} \quad \forall \varphi \in L^2(O),$$

$$\partial_{v_1} G(v_0, v_1, p)(\psi) = \langle p, \psi \rangle_{L^2(O)} \quad \forall \psi \in H^1_0(O),$$

$$\partial_p G(v_0, v_1, p)(q) = \langle q, v_0 + v_1 - v \rangle_{L^2(O)} \quad \forall q \in L^2(O),$$

we have

$$\partial_{v_0} L(v_0, v_1, p)(\varphi) = 2 \langle v_0, \varphi \rangle_{L^2(O)} + \langle p, \varphi \rangle_{L^2(O)} \quad \forall \varphi \in L^2(O),$$

$$\partial_{v_1} L(v_0, v_1, p)(\psi) = 2t^2 \langle \nabla v_1, \nabla \psi \rangle_{L^2(O)} + \langle p, \psi \rangle_{L^2(O)} \quad \forall \psi \in H^1_0(O),$$

$$\partial_p L(v_0, v_1, p)(q) = \langle q, v_0 + v_1 - v \rangle_{L^2(O)} \quad \forall q \in L^2(O).$$

Hence, $(v_0^*, v_1^*, p^*) \in L^2(O) \times H^1_0(O) \times L^2(O)$ satisfies

$$2 \langle v_0^*, \varphi \rangle_{L^2(O)} + \langle p^*, \varphi \rangle_{L^2(O)} = 0 \quad \forall \varphi \in L^2(O), \quad (4.7)$$

$$2t^2 \langle \nabla v_1^*, \nabla \psi \rangle_{L^2(O)} + \langle p^*, \psi \rangle_{L^2(O)} = 0 \quad \forall \psi \in H^1_0(O), \quad (4.8)$$

$$\langle q, v_0^* + v_1^* - v \rangle_{L^2(O)} = 0 \quad \forall q \in L^2(O). \quad (4.9)$$
It follows from (4.7) and (4.8) that
\[ t^2 \langle \nabla v^*_1, \nabla \psi \rangle_{L^2(O)} - \langle v^*_0, \psi \rangle_{L^2(O)} = 0 \quad \forall \psi \in H^1_0(O). \]
This and (4.9) give
\[ t^2 \langle \nabla v^*_1, \nabla \psi \rangle_{L^2(O)} + \langle v^*_1, \psi \rangle_{L^2(O)} = \langle v, \psi \rangle_{L^2(O)} \quad \forall \psi \in H^1_0(O). \]
This is a weak formulation of the following boundary value problem
\[ -t^2 \Delta v^*_1 + v^*_1 = v \quad \text{in} \quad O, \]
\[ v^*_1 = 0 \quad \text{on} \quad \partial O. \]  (4.10)
Since \( O \) has smooth boundary and \( v \in C^4(O) \), we deduce that \( v^*_1 \in C(O) \). Moreover, since \( v \geq 0 \) in \( O \), due to the strong maximum principle, see e.g. [6, Corollary 9.37], either \( v^*_1 > 0 \) in \( O \) or \( v^*_1 \equiv 0 \) in \( O \). If \( v^*_1 \equiv 0 \) then \( v^*_0 \equiv 0 \) on \( O \) due to (4.7). This contradicts the assumption on \( v \). Hence \( v^*_1 > 0 \) on \( O \), which implies \( (v^*_0, v^*_1) \notin X_1(v) \).

**Remark 4.3.** The above result is consistent with the well-known fact that the values of the solution \( v^*_1 \) of (4.10) in a subdomain \( O_2 \subset O \) depends on the values of \( v \) not only in \( O_2 \) but in all of \( O \); see e.g. [6, page 307].

**4.3. Proof of Theorem 3.2.** The proof follows along the lines of the proof of [4, Theorem 4.1].

**Proof.** Introduce the product space
\[ \tilde{\Pi}^s := \prod_{j=1}^N \tilde{H}^s(\Omega_j), \quad 0 \leq s \leq r, \]
with a norm defined from the interpolation norms by
\[ \|u\|_{\tilde{\Pi}^s}^2 := \sum_{j=1}^N \|u_j\|_{\tilde{H}^s(\Omega_j)}^2 = \sum_{j=1}^N \|u_j\|_{\tilde{H}^r(\Omega)}^2, \]
where \( u = (u_1, \ldots, u_N) \). If \( s = \theta r \) for some \( \theta \in (0, 1) \), then
\[ \tilde{\Pi}^s = [\tilde{\Pi}^0, \tilde{\Pi}^r]_{\theta}. \]
On the product set \( \tilde{\Pi}^s \), consider the sum operator \( S : \tilde{\Pi}^s \to \tilde{H}^s(\Omega) \) defined by
\[ Su := \sum_{j=1}^N u_j, \quad u_j \in \tilde{H}^s(\Omega_j). \]
Recalling that \( \|\|_{\tilde{H}^s(\Omega_j)} = \|\|_{L^2(\Omega)} \) and \( \|\|_{\tilde{H}^r(\Omega)} = |\|_{H^r(\Omega)} \), we deduce
\[ \|Su\|_{\tilde{H}^s(\Omega)}^2 = \sum_{j=1}^N \|u_j\|_{\tilde{H}^s(\Omega_j)}^2 = \|u\|_{\tilde{\Pi}^s}^2, \quad s = 0 \text{ or } s = r. \]
By interpolation
\[ \|Su\|_{\tilde{H}^s(\Omega)} \leq \|u\|_{\tilde{\Pi}^s} \quad \text{for} \quad 0 \leq s \leq r. \]
Now for any function \( u \in \tilde{H}^s(\Omega) \) such that \( u_j \in \tilde{H}^s(\Omega_j) \), \( j = 1, \ldots, N \), where \( u_j \) is the zero extension of \( u|_{\Omega_j} \) onto \( \Omega \setminus \Omega_j \), we define \( u = (u_1, \ldots, u_N) \) Then \( u = Su \) because \( \{\Omega_1, \ldots, \Omega_N\} \) is a partition of \( \Omega \). Consequently

\[
\|u\|_{\tilde{H}^s(\Omega)}^2 = \|Su\|_{\tilde{H}^s(\Omega)}^2 \leq \sum_{j=1}^{N} \|u_j\|_{\tilde{H}^s(\Omega)}^2,
\]

proving (3.1).

5. Applications

Inequality (3.2) is needed in the analysis of domain decomposition methods for boundary integral equations. Consider for example the exterior Neumann boundary value problem

\[
-\Delta U = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega},
\]
\[
\frac{\partial U}{\partial n_i} = g_i \quad \text{on} \quad \Omega_i, \quad i = 1, 2,
\]
\[
\frac{\partial U}{\partial r} = o(1/r) \quad \text{as} \quad r = |x| \to \infty,
\]

(5.1)

where \( \Omega \) is a screen in \( \mathbb{R}^3 \) and \( \Omega_i, \ i = 1, 2, \) are two sides of \( \Omega \) determined by two opposite normal vectors \( n_i \). It is well known that [19, 20] if \( \phi := [U]_{\Omega} \) denotes the jump of \( U \) across the screen \( \Omega \), then (5.1) is equivalent to the boundary integral equation

\[
\mathcal{M}\phi(x) = -g(x), \quad x \in \Omega,
\]

(5.2)

where \( \mathcal{M} \) is the hypersingular integral operator defined by

\[
\mathcal{M}\phi(x) := -\frac{1}{2\pi} \frac{\partial}{\partial n_x} \int_{\Omega} \frac{\partial}{\partial n_y} \left( \frac{1}{|x-y|} \right) \phi(y) \, ds_y.
\]

It is also well known that [3] [19] [20] that \( \mathcal{M} : \tilde{H}^{1/2}(\Omega) \to H^{-1/2}(\Omega) \) is bijective, where \( H^{-1/2}(\Omega) \) is the dual of \( \tilde{H}^{1/2}(\Omega) \) with respect to the \( L^2 \)-dual pairing. A weak formulation for equation (5.2) is

\[
a(\phi, \psi) = -\langle g, \psi \rangle \quad \forall \psi \in \tilde{H}^{1/2}(\Omega)
\]

where \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \)-inner product and the bilinear form \( a(\cdot, \cdot) \) is defined by \( a(\varphi, \psi) = \langle \mathcal{M}\varphi, \psi \rangle \) for all \( \varphi, \psi \in \tilde{H}^{1/2}(\Omega) \). It is known that this bilinear form defines a norm which is equivalent to the \( \tilde{H}^{1/2}(\Omega) \)-norm, i.e.,

\[
a(\psi, \psi) \simeq \|\psi\|_{\tilde{H}^{1/2}(\Omega)}^2 \quad \forall \psi \in \tilde{H}^{1/2}(\Omega).
\]

Together with (2.12) this implies

\[
a(\psi, \psi) \simeq \|\psi\|_{\sim, 1/2, \Omega}^2 \quad \forall \psi \in \tilde{H}^{1/2}(\Omega).
\]

(5.3)

(5.4)

The boundary element method applied to this equation results in the following equation which computes an approximate solution \( \varphi_h \in \mathcal{V}_h \)

\[
a(\varphi_h, \psi_h) = -\langle g, \psi_h \rangle \quad \forall \psi_h \in \mathcal{V}_h
\]

(5.5)

where \( \mathcal{V}_h \) is a finite-dimensional subspace of \( \tilde{H}^{1/2}(\Omega) \). This equation yields a symmetric and dense matrix system

\[
Ax = b.
\]

(5.6)
Let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ be the maximum and minimum eigenvalues of $A$, respectively. The condition number $\kappa(A)$ is defined by $\kappa(A) = \lambda_{\max}(A)/\lambda_{\min}(A)$. The matrix $A$ is ill-conditioned, namely $\kappa(A)$ increases significantly with the size of $A$. Therefore, when the size of $A$ is large, a direct solver to solve (5.6) performed on a computer results in inaccurate solutions due to machine errors. If an iterative method like the conjugate gradient method is used to solve (5.6), it requires a large number of iterations to produce satisfactory solutions. To solve this ill-conditioned system efficiently, a preconditioner $C$ is required. Instead of solving (5.6), one solves

$$C^{-1}Ax = C^{-1}b,$$

with the preconditioner $C$ designed such that $C^{-1} \approx A^{-1}$ so that $\kappa(C^{-1}A) \approx \kappa(I) = 1$. Here $I$ is the identity matrix of the same size as $A$.

Preconditioners by domain decomposition have been studied for (5.6); see e.g. [2, 3, 9, 10, 12, 13, 14, 16, 17, 21, 22, 23, 24, 25]. The method can be briefly described as follows. Partition the domain $\Omega$ into subdomains $\Omega_1, \ldots, \Omega_N$. On each subdomain we define $V_j = V_h \cap H^{1/2}(\Omega_j)$ and decompose $V_h$ by

$$V_h = V_1 + \cdots + V_N. \quad (5.7)$$

A preconditioner $C$ is defined using this subspace decomposition. To estimate the condition number $\kappa(C^{-1}A)$, one needs to show, among other things, the following two statements:

(i) For any $v \in V_h$, there exists a decomposition $v = v_1 + \cdots + v_N$ with $v_j \in V_j$, $j = 1, \ldots, N$, such that

$$C_1 \sum_{j=1}^N a(v_j, v_j) \leq a(v, v). \quad (5.8)$$

(ii) For any $v \in V_h$ and any decomposition $v = v_1 + \cdots + v_N$ with $v_j \in V_j$, $j = 1, \ldots, N$, the following inequality holds

$$a(v, v) \leq C_2 \sum_{j=1}^N a(v_j, v_j). \quad (5.9)$$

The positive constants $C_1$ and $C_2$ are independent of $v \in V_h$. Ideally, they are independent of the parameter $h$ defining the problem (5.5), or depend at most logarithmically on $h$. Statement (i) yields $C_1 \leq \lambda_{\min}(C^{-1}A)$ and is called the stability of the decomposition (5.7), while Statement (ii) yields $\lambda_{\max}(C^{-1}A) \leq C_2$ and is called the coercivity of the decomposition. The condition number $\kappa(C^{-1}A)$ is then bounded by $C_2/C_1$.

Due to (5.3) and (5.4), either $\|\cdot\|_{H^{1/2}(\Omega)}$ or $\|\cdot\|_{H^{-1/2,\Omega}}$ can be used to prove (5.8) and (5.9). It turns out that the norm $\|\cdot\|_{H^{-1/2,\Omega}}$ is more suitable to prove (5.8) while $\|\cdot\|_{H^{1/2}(\Omega)}$ is good for proving (5.9). There has been a belief that

$$a(v_j, v_j) \approx \|v_j\|_{H^{1/2}(\Omega_j)}^2 \approx \|v_j\|_{H^{-1/2,\Omega_j}}^2, \quad (5.10)$$

and thus, ubiquitously in the literature, a common practice has been to use Theorem 2.2 to prove

$$\|v\|_{H^{1/2}(\Omega)}^2 \leq C_2 \sum_{j=1}^N \|v_j\|_{H^{1/2}(\Omega_j)}^2$$
to derive (5.9). Theorem 3.1 and Proposition 2.1 imply that the equivalences (5.10) hold with constants depending on the diameter of $\Omega_j$, which is proportional to $h$. This may result in more than logarithmic dependence on $h$ of the constant $C_2$. To avoid this unsatisfactory result, one has to prove

$$\|v\|_{H^{1/2}(\Omega)}^2 \leq C_2 \sum_{j=1}^{N} \|v_j\|_{H^{1/2}(\Omega)}^2.$$

This inequality can be obtained by invoking Corollary 3.3.

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