REAL ANALYTIC METRICS ON $S^2$ WITH TOTAL ABSENCE OF FINITE BLOCKING

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Abstract. If $(M, g)$ is a Riemannian manifold and $(x, y) \in M \times M$, then a set $P \subset M \setminus \{x, y\}$ is said to be a blocking set for $(x, y)$ if every geodesic from $x$ to $y$ passes through a point of $P$. If no pair $(x, y)$ in $M \times M$ has a finite blocking set, then $(M, g)$ is said to be totally insecure. We prove that there exist real analytic metrics $h$ on $S^2$ such that $(S^2, h)$ is totally insecure.

1. Introduction

Let $(M, g)$ be a $C^\infty$ Riemannian manifold. In this paper, manifolds and surfaces are assumed to be without boundary. We consider a geodesic on $M$ as a mapping $\gamma : I \to M$, where $I$ is an interval of positive length. The trace of $\gamma$, denoted $\text{tr}(\gamma)$, is the image of $\gamma$ in $M$. Unless specified otherwise, we will assume that geodesics are parametrized by arc length. If $I = [a, b]$, $x = \gamma(a)$, and $y = \gamma(b)$, then $\gamma$ is said to be a geodesic from $x$ to $y$. Two geodesics $\gamma_1 : I_1 \to M$, $i = 1, 2$, will be considered the same if and only if $\gamma_1$ is equal to $\gamma_2$ composed with a translation that maps $I_1$ onto $I_2$. A subset $P \subset M$ is called a blocking set for a collection of geodesics $\Gamma$ on $M$, if $\text{tr}(\gamma) \cap P \neq \emptyset$ for every geodesic $\gamma$ in $\Gamma$. If $\Gamma$ consists of all geodesics from $x$ to $y$ and there exists a finite blocking set $P \subset M \setminus \{x, y\}$, then $(x, y)$ is said to be secure; otherwise $(x, y)$ is said to be insecure. The Riemannian manifold $(M, g)$ is defined to be secure if every pair $(x, y) \in M$ is secure. If there exists an insecure pair $(x, y) \in M \times M$, $(M, g)$ is said to be insecure. Moreover, if every pair $(x, y) \in M \times M$ is insecure, then $(M, g)$ is defined to be totally insecure.

E. Gutkin and V. Schroeder [16] showed that flat metrics are secure, and it is conjectured in [6, 19] that these are the only secure metrics. On the other hand, there are many examples of totally insecure metrics [3, 6, 16, 19], and according to [6], it is expected that "most" metrics are totally insecure. V. Bangert and Gutkin [3] proved that any compact Riemannian surface of genus greater than one is totally insecure. In case $M$ is a compact surface of genus one, they showed that there is a $C^2$ open and $C^\infty$ dense collection of metrics on $M$ that are totally insecure. However, in the case of surfaces of genus zero, there were no previously known examples of totally insecure metrics. Gerber and W.-K. Ku [15] showed that on any compact Riemannian manifold $M$ of dimension greater than one there is a dense $G$-delta set of $C^\infty$ metrics, $\mathcal{G}$, such that for each $g \in \mathcal{G}$, there is a dense $G$-delta subset $\mathcal{R} = \mathcal{R}(g)$ of $M \times M$ consisting of insecure pairs $(x, y)$ for $(M, g)$, but this result only provides "generic" insecurity, not total insecurity. The main theorem of our paper is that if $M = S^2$ or $P^2(\mathbb{R})$, then there exist real analytic
metrics $h$ on $M$ such that $(M, h)$ is totally insecure. We present the argument for $M = S^2$, and the case $M = P^2(\mathbb{R})$ follows easily, as indicated in Remark 2.3.

The real analytic metrics $h$ on $S^2$ for which we prove total insecurity are obtained in the same way as the metrics on $S^2$ for which K. Burns and Gerber [8, 9] showed that the geodesic flow is ergodic. Our proof relies on ideas in [3] and techniques in non-uniform hyperbolicity [4, 18]. Burns and Gutkin [6] and, independently, J.-F. Lafont and B. Schmidt [19] showed that compact Riemannian manifolds (of any dimension) with no conjugate points whose geodesic flows have positive topological entropy are totally insecure. In the special case of a compact manifold with negative curvature the methods in [3] provide a different proof of total insecurity. A key step in our paper is to show that there is a closed $h$-geodesic such that for any pair $(x, y) \in S^2 \times S^2$ there is an infinite sequence of $h$-geodesics $(\gamma_n)$ from $x$ to $y$ that stay arbitrarily close to $\rho$ except during a uniformly bounded amount of time at the beginning and at the end of their parameter intervals. (See Proposition 6.9 for a precise statement.) This condition is essentially taken from [3], and it is easy to establish if $(S^2, h)$ is replaced by a manifold of negative curvature and $\rho$ is replaced by any closed geodesic. (For manifolds of negative curvature, the first statement in Proposition 6.6 is also easy to prove, for any closed geodesic $\rho$, but the analog of Proposition 6.9 can be proved directly.)

According to Proposition 6.6, for each $(x, y) \in S^2 \times S^2$, there are $h$-geodesics $\gamma^+ : [0, \infty) \to S^2$ and $\gamma^- : (-\infty, 0] \to S^2$ with $\gamma^+(0) = x$ and $\gamma^-(0) = y$ such that $\gamma^+$ is asymptotic to the closed $h$-geodesic $\rho$ as $t \to \infty$ and $\gamma^-$ is asymptotic to $\rho$ as $t \to -\infty$ (as in Definition 3.3). The sequence of geodesics $(\gamma_n)$ mentioned above is obtained by finding small perturbations $\gamma^+_n$ and $\gamma^-_n$ of $\gamma^+$ and $\gamma^-$, respectively, and large positive $t^+_n$ and $t^-_n$ such that $\gamma^+_n|[0, t^+_n]$ and $\gamma^-_n|[-t^-_n, 0]$ can be smoothly joined at $\gamma^+_n(t^+_n) = \gamma^-_n(-t^-_n)$.

We show (in Proposition 6.11) that no one-element set $\{z\}$ can be a blocking set for any infinite collection of geodesics in $(S^2, h)$ of uniformly bounded length starting at a point $x$. This is clearly true if $(S^2, h)$ is replaced by a manifold with no conjugate points. However, any Riemannian metric on $S^2$ must have conjugate points (Remark 3.4 in Chapter 7 of [11]). In fact, for the metrics from [9], there are conjugate points along any geodesic segment that passes through a cap (as defined in Section 2 below). Our proof of Proposition 6.11 relies on the real analyticity of the metric.

The proof of Proposition 6.6 utilizes the fact that the geodesic flow for $(S^2, h)$ is topologically transitive (which follows from the ergodicity with respect to Liouville measure), but ergodicity is not used in any other way.

2. CONSTRUCTION OF TOTALLY INSECURE REAL ANALYTIC METRICS ON $S^2$

We describe the construction of real analytic metrics on compact surfaces with ergodic geodesic flow, as in [9]. Let $S$ be a compact surface with a real analytic differentiable structure, and let $g$ be a $C^\infty$ metric on $S$ that satisfies the following conditions:

1. There is a finite (non-empty) disjoint collection of monotone curvature caps $C_i$, $i = 1, \ldots, q$, (as defined below) such that $S \setminus (\cup_{i=1}^q C_i)$ has negative curvature with respect to $g$. 
Note that assumption (2) implies that \( g \) is real analytic in a neighborhood of \( \partial C \). Metrics satisfying (1) and (2) exist on every compact real analytic surface \( S \). (See Section 1 of [9].) Such metrics are viewed as having “almost negative curvature,” since many of the properties of geodesic flows on surfaces of negative curvature extend to metrics of this type. They are of interest primarily in the case of surfaces of genus zero or one, which do not support metrics of negative curvature.

A *cap* is defined to be a closed two-dimensional disk with nonnegative curvature such that the boundary circle is the trace of a real analytic geodesic. We say that a cap \( C \) has *monotone curvature* if it is radially symmetric and its curvature is a nondecreasing function of the distance from the boundary of \( C \).

Real analytic metrics \( h \) with ergodic geodesic flow are obtained from the following two theorems in [9]. The proof of Theorem 2.2 is based on Cartan’s theorem B [10].

**Theorem 2.1.** Let \( S \) be a compact surface with a real analytic differentiable structure, and let \( g \) be a \( C^\infty \) Riemannian metric on \( S \) satisfying the above conditions (1) and (2). Choose an open neighborhood \( U_i \) of \( \partial C_i \) for \( i = 1, \ldots, q \), and let \( \mathcal{U} = \bigcup_{i=1}^{q} U_i \). Let \( \mathcal{H}_1 \) be the collection of \( C^3 \) Riemannian metrics on \( S \) that agree with \( g \) to second order on \( \partial C_i \), \( i = 1, \ldots, q \). Then there exist a \( C^2 \) open neighborhood \( \mathcal{H}_2 \) of \( g \) in the set of \( C^3 \) Riemannian metrics on \( S \), and a \( C^3 \) open neighborhood \( \mathcal{H}_3 \) of \( g \) in the set of \( C^3 \) Riemannian metrics on \( \mathcal{U} \), such that the following holds: If \( h \in \mathcal{H}_1 \cap \mathcal{H}_2 \) and \( h|\mathcal{U} \in \mathcal{H}_3 \), then the geodesic flow for \( h \) on \( T^{1,h}S \) is ergodic with respect to Liouville measure.

**Theorem 2.2.** Let \( S \) be a compact surface with a real analytic differentiable structure, and let \( g \) be a \( C^\infty \) Riemannian metric on \( S \). Suppose that \( \Gamma \) is a union of disjoint closed real analytic curves on \( S \) and there exists a neighborhood \( \mathcal{U} \) of \( \Gamma \) on which \( g \) is real analytic. Then for any positive integer \( k \) there exists a real analytic metric \( h \) on \( S \) such that \( g \) and \( h \) agree up to order \( k \) on \( \Gamma \). Moreover, \( h \) can be taken arbitrarily close to \( g \) in the \( C^\infty \) topology.

The main result of this paper is the following:

**Theorem 2.3.** If \( S, g, \mathcal{U}, \mathcal{H}_1 \) are as in Theorem 2.1, then there exist a \( C^2 \) open neighborhood \( \mathcal{H}_2 \) of \( g \) in the set of \( C^3 \) Riemannian metrics on \( S \), and a \( C^3 \) open neighborhood \( \mathcal{H}_3 \) of \( g \mathcal{U} \) in the set of \( C^3 \) Riemannian metrics on \( \mathcal{U} \), such that the following holds: If \( h \in \mathcal{H}_1 \cap \mathcal{H}_2 \) and \( h|\mathcal{U} \in \mathcal{H}_3 \), then \( (S, h) \) is totally insecure.

It follows from Theorem 2.2 that the set of real analytic metrics \( h \) that satisfy the conclusions of Theorems 2.1 and 2.3 is nonempty. In particular, we obtain the following corollary.

**Corollary 2.4.** If \( S \) is a compact surface with a real analytic differentiable structure, then there exist real analytic metrics \( h \) on \( S \) such that \( (S, h) \) is totally insecure.

Theorems 2.3 and Corollary 2.4 are only of interest in the case \( S = S^2 \) or \( P^2(\mathbb{R}) \), since totally insecure metrics for positive genus surfaces were already obtained in [3], as described in our introduction.

**Remark 2.5.** Suppose \( P \) is a non-orientable compact real analytic surface and \( S \) is its orientable double cover, with covering map \( \pi : S \to P \). It follows easily from
the definition of total insecurity that for any Riemannian metric $h_P$ on $P$, $(P, h_P)$ is totally insecure if $(S, \pi^* h_P)$ is totally insecure (see Proposition 1 in [16]). This observation allows us to reduce Theorem 2.3 to the case in which the surface is orientable.

Throughout the rest of this paper we will assume that $S$ is a compact orientable surface with a real analytic differentiable structure and $g$, $\Upsilon$, $\mathbb{H}_1$, $\mathbb{H}_2$, and $\mathbb{H}_3$ are as in Theorem 2.1. As observed above, the reader may as well assume that $S = S^2$, although this does not matter for the proof. Furthermore, we will assume that $h$ is a real analytic metric with $h \in \mathbb{H}_1 \cap \mathbb{H}_2$, and $h|\Upsilon \in \mathbb{H}_3$, where $\mathbb{H}_2 \subset \mathbb{H}_3$ is a $C^2$ open neighborhood of $g$ in the set of $C^3$ Riemannian metrics on $S$, and $\mathbb{H}_3 \subset \mathbb{H}_4$ is a $C^3$ open neighborhood of $g|\Upsilon$ in the set of $C^3$ Riemannian metrics on $\Upsilon$. Additional requirements on $\mathbb{H}_2$ and $\mathbb{H}_3$ will be imposed later in this section and in Section 3.

Since $h$ agrees with $g$ to second order on $\partial C_i$, for $i = 1, \ldots, q$, each $\partial C_i$ is the trace of a closed geodesic for $h$, as it is for $g$. We require $\mathbb{H}_2'$ and $\mathbb{H}_3'$ to be sufficiently small so that for $h \in \mathbb{H}_1 \cap \mathbb{H}_2$, and $h|\Upsilon \in \mathbb{H}_3$, the curvature for $h$ is positive in $\text{int}(\bigcup_{i=1}^{q} C_i)$ and negative in $\mathcal{N}$, which is defined by

$$
(2.1) \quad \mathcal{N} = S \setminus (\bigcup_{i=1}^{q} C_i).
$$

For our argument, we need a closed $h$-geodesic $\rho$ such that for any $(x, y) \in S \times S$, there is a family of $h$-geodesics $(\gamma_n)_{n=1, 2, \ldots}$ which accumulate near $\rho$, as described in Proposition 6.5. Any simple closed geodesic along one of $\partial C_i$, $i \in \{1, \ldots, q\}$, and any closed geodesic whose trace lies in $\mathcal{N}$ could serve as $\rho$. In our argument, we choose to work with a closed geodesic in $\mathcal{N}$, because the estimates required to prove the analog of Proposition 6.6 for a geodesic along one of $\partial C_i$ are more difficult, due to the fact that the curvature of the surface vanishes on $\partial C_i$. (Proposition 6.6 remains true for $\rho$ replaced by a geodesic along one of $\partial C_i$, except in the last line we would either have to replace $\tau_2(\hat{\gamma}_{v+})$ and $\tau_2(\hat{\gamma}_{v-})$ by $|\tau_2(\hat{\gamma}_{v+})|$ and $|\tau_2(\hat{\gamma}_{v-})|$, respectively, or require that the Fermi coordinates along $\partial C_i$ be chosen so that $\tau_2 > 0$ for points near $\partial C_i$ that lie in $\mathcal{N}$. This modification is needed, because no geodesic can be asymptotic to a geodesic along $\partial C_i$ while remaining in $\text{int} C_i$. No additional difficulties in the proof of Theorem 2.3 are caused by this modification.)

We can find a closed $g$-geodesic $\rho_0$ in $\mathcal{N}$ by applying the Birkhoff curve-shortening procedure [5, 13] (for the metric $g$) to a closed curve $\alpha$ in $\mathcal{N}$ that is not homotopic within $\mathcal{N}$ to a point or to any of the boundary components of $\mathcal{N}$. Since each boundary component of $\mathcal{N}$ consists of the trace of a simple closed geodesic, it follows that for any two points $x, y \in \mathcal{N}$ with $\text{dist}_g(x, y) < r$, where $r$ is the injectivity radius of $(S, g)$, the length minimizing $g$-geodesic $\gamma$ from $x$ to $y$ must have $\text{tr}(\gamma) \subset \mathcal{N}$. If we start the curve-shortening procedure by partitioning $\alpha$ into segments of length less than $r$, then all of the curves obtained from $\alpha$ with this procedure, as well as the limiting curve, remain in $\mathcal{N}$. Moreover, all of these curves are homotopic to $\alpha$ within $\mathcal{N}$. The limiting curve is a closed $g$-geodesic $\rho_0$ in $\mathcal{N}$. Since the orbit of the geodesic glow for $g$ along $\rho_0$ is transversally hyperbolic, it follows that if $\mathbb{H}_2'$ is a sufficiently small $C^2$ neighborhood of $g$, there is a closed $h$-geodesic $\rho = h(h)$ that is $C^3$ close to $\rho$ (although we only need it to be $C^0$ close to $\rho$). The neighborhood $\mathbb{H}_2'$ can be chosen so that dist$_h(\text{tr}(\rho), \bigcup_{i=1}^{q} C_i)$ is uniformly bounded away from 0 for all $h \in \mathbb{H}_2$. 


It is possible that there is no simple closed geodesic in \( \mathcal{N} \). For example, if \( S \) is \( S^2 \) with three caps, then any simple closed geodesic in \( \mathcal{N} \) would be homotopic to the boundary of one of the caps, which is impossible by the Gauss-Bonnet Theorem.

There is another type of closed geodesic that may occur, namely one that passes through the interior of one or more of the caps. However, the conjugate points that occur along such a geodesic prevent it from being a suitable choice for \( \rho \) in our argument.

Henceforth, unless otherwise specified, we will assume that geodesics, geodesic flow, distances, lengths of vectors, curvature, etc., for \( S \) are taken with respect to \( h \). For \( x \in S \) and \( v \in T^1_xS \), let \( \gamma_v \) denote the geodesic with \( \gamma_v(0) = x \) and \( \gamma_v'(0) = v \). Let \( \phi' \), \( t \in \mathbb{R} \), be the geodesic flow on \( T^1S \).

3. Stable and Unstable Cone Fields

We will define stable and unstable cone fields at each \( v \in T^1S \). These cone fields are essentially the same as those in [9] except that the definitions are extended to be valid inside the caps. The idea of extending the cone fields into the caps and obtaining continuous stable and unstable line fields (as in our Lemma 3.3) already appeared in V. Donnay’s proof of the existence of \( C^\infty \) metrics on \( S^2 \) with ergodic geodesic flow [14], but the real analytic case is different, because we no longer have invariance of the \( K^+ \) cones (see Definition 3.1) from the time that a geodesic enters a cap until it exits the cap.

The Riemannian metric \( h = \langle \cdot, \cdot \rangle \) on \( S \) induces a Riemannian metric on \( TS \):

\[
\langle \langle \xi, \eta \rangle \rangle = \langle \xi_H, \eta_H \rangle + \langle \xi_V, \eta_V \rangle,
\]

where \( H \) and \( V \) denote the horizontal and vertical components, respectively (see, e.g., Chapter 3, Exercise 2 in [11]). We will identify \( \xi \in TTS \) with \( (\xi_H, \xi_V) \). If \( x \in S \), \( w \in T^1_xS \), and \( \xi \in T_wT^1S \), then \( \langle \xi_V, w \rangle = 0 \). For \( w \in T^1_xS \), we let \( \mathcal{P}(w) \) be the two-dimensional subspace of \( T_wT^1S \) defined by

\[
\mathcal{P}(w) = \{ \xi \in T_wT^1S : \langle \xi_H, w \rangle = 0 \}.
\]

We define \( H, V \) coordinates on \( \mathcal{P}(w) \) by choosing \( N \in T^1_xS \) such that \( \langle N, w \rangle = 0 \) and letting \( \xi = (\xi_H, \xi_V) \in \mathcal{P}(w) \) have coordinates \( (\lambda_1, \lambda_2) \) if \( \xi_H = \lambda_1N \) and \( \xi_V = \lambda_2N \). If \( N \) is replaced by \(-N \), then the coordinates change from \( (\lambda_1, \lambda_2) \) to \((-\lambda_1, -\lambda_2) \), but this does not matter for the cones and the lines through \( 0 \) in \( \mathcal{P}(w) \) that we consider below. The distribution \( w \mapsto \mathcal{P}(w) \), \( w \in T^1S \), is orientable, because we may specify that the ordered pair of vectors given in \( H, V \) coordinates by \((1, 0), (0, 1)\) is positively oriented. This orientation does not depend on the choice of \( N \).

Suppose \( w_0 \in T^1_xS \), \( \xi \in T_{w_0}T^1S \), and \( w(s) \in T^1_{p(s)}S \), \(-s_0 < s < s_0\), is a curve in \( T^1S \) that is tangent to \( \xi \) at \( w_0 = w(0) \) when \( s = 0 \). Then \( J(t) = (d/ds)|_{s=0}^t\gamma_{w(s)}(t) \) is a Jacobi field along \( \gamma_{w_0} \) with \( J(0) = \xi_H \), \( J'(0) = \xi_V \) and \( ((d\phi^j(t))_H, (d\phi^j(t))_V) = (J(t), J'(t)) \). In particular, if \( \xi \in \mathcal{P}(w_0) \), then \( (J(0), \gamma_{w_0}'(0)) = 0 = (J'(0), \gamma_{w_0}'(0)) \), which implies that \( (J(t), \gamma_{w_0}'(t)) \equiv 0 \). Thus \( ((d\phi^j(t))_H, d\phi^j(t)) \equiv 0 \), i.e., the distribution \( w \mapsto \mathcal{P}(w) \) is invariant under the geodesic flow. Moreover, the orbits of the geodesic flow are orthogonal to the distribution \( w \mapsto \mathcal{P}(w) \), since \( \eta_H = w_0 \) and \( \eta_V = 0 \) if \( \eta = (D/dt)|_{t=0}^t\phi^j(w_0) \).

Now assume that the curve \( w(s) \), as above, is a \( C^1 \) regular curve in \( T^1S \) that is everywhere tangent to the distribution \( \mathcal{P} \) (i.e., \( w'(s) \in \mathcal{P}(w(s)) \) for \(-s_0 < s < s_0\)). Then \( w(s) \) is a unit normal field along the curve \( p(s) \) in \( S \). (Throughout this paper,
a regular curve will mean a curve whose derivative is nowhere vanishing. The signed curvature \( k(s) \) of \( p(s) \) with respect to the unit normal field \( w(s) \) is defined by

\[
(3.2) \quad k(s) = -\frac{1}{|p'(s)|^2} \left\langle \frac{D(p'(s))}{ds}, w(s) \right\rangle = \frac{1}{|p'(s)|^2} \left\langle p'(s), \frac{D(w(s))}{ds} \right\rangle.
\]

Our choice of sign is such that unstable [stable] curves (to be defined in Section 3) that lie outside the caps have positive [negative] curvature. The second equality in (3.2) follows from the fact that \( \left( \frac{p'(s)}{ds}, w(s) \right) = 0 \).

If \( J(t) \) is a perpendicular Jacobi field along \( \gamma_{w_0} \), we may write \( J(t) = j(t)N(t) \), where \( N(t) \) is a continuous unit normal field along \( \gamma_{w_0}(t) \), and \( j(t) \) satisfies the scalar Jacobi equation

\[
(3.3) \quad j''(t) + K(\gamma_{w_0}(t))j(t) = 0,
\]

where \( K \) is the Gaussian curvature. If \( w(s), -s_0 < s < s_0, \) with \( w(0) = w_0 \), is a \( C^1 \) regular curve in \( T^1S \) that is everywhere tangent to \( P \) and \( J(t) = (d/\langle s \rangle)|_{s=0}\gamma_{w(s)}(t) = j(t)N(t) \), then

\[
(3.4) \quad j'(t)(/ds)|_{s=0}\gamma_{w(s)}(t) = (d/\langle s \rangle)|_{s=0}(\varphi'/(w(s))),
\]

and \( j'(t)/j(t) = J(t), J'(t)/J(t)^2 \). From the second version of the formula for \( k(s) \) in (3.2), it follows that \( j'(t)/j(t) \) is equal to the signed curvature at \( s = 0 \) of the curve \( s \rightarrow \gamma_{w(s)}(t) \) with respect to the unit normal field \( \varphi'/(w(s)) \). If \( m(t) = j'(t)/j(t) \), the slope of \( d\varphi'/(w(0)) \) in the \( H, V \) coordinate system, then \( m(t) \) satisfies the Riccati equation

\[
(3.5) \quad m^2(t) + m'(t) + K(\gamma_{w_0}(t)) = 0.
\]

The Riccati equation can be transformed by setting \( \theta = \tan^{-1}(m) \), to obtain

\[
(3.6) \quad \theta'(t) + \sin^2(\theta(t)) + K(\gamma_{w_0}(t)) \cos^2(\theta(t)) = 0.
\]

Here \( \theta \in \mathbb{R}/\pi\mathbb{Z} \), which we identify with \( (-\pi/2, \pi/2] \).

**Definition 3.1.** For \( w_0 \in T_{z_0}S \), we define cones \( \mathcal{K}^+_{w_0}, \mathcal{K}^-_{w_0} \subseteq \mathcal{P}(w_0) \) by

\[
\mathcal{K}^+_{w_0} = \{ \xi \in \mathcal{P}(w_0) : \langle \xi_H, \xi_V \rangle \geq 0 \} \quad \text{and} \quad \mathcal{K}^-_{w_0} = \{ \xi \in \mathcal{P}(w_0) : \langle \xi_H, \xi_V \rangle \leq 0 \}.
\]

The \( \mathcal{K}^+_{w_0} \) cones correspond to perpendicular Jacobi fields with \( jj' \geq 0 \). If the curvature is nonpositive along \( \gamma_{w_0}(t) \) for \( t_0 \leq t \leq t_1 \), then \( (j(t)j'(t))' = (j'(t))^2 - K(\gamma_{w_0}(t))j(j(t)) \geq 0 \) for \( t_0 \leq t \leq t_1 \). This implies that

\[
(3.7) \quad d\varphi^{t_1-t_0}\mathcal{K}^+_{w_0}(t_0) \subset \mathcal{K}^+_{w_0}(t_1) \quad \text{and} \quad d\varphi^{t_0-t_1}\mathcal{K}^+_{w_0}(t_1) \subset \mathcal{K}^+_{w_0}(t_0).
\]

If, in addition, the curvature is negative at a point \( \gamma_{w_0}(t) \), for some \( t \in [t_0, t_1] \), then

\[
(3.8) \quad d\varphi^{t_1-t_0}\mathcal{K}^-_{w_0}(t_0) \subset \mathcal{K}^-_{w_0}(t_1) \quad \text{and} \quad d\varphi^{t_0-t_1}\mathcal{K}^-_{w_0}(t_1) \subset \mathcal{K}^-_{w_0}(t_0),
\]

where \( \text{int} \mathcal{K} \), for a cone \( \mathcal{K} \subset \mathcal{P}(w_1) \), means the topological interior of \( \mathcal{K} \) within \( \mathcal{P}(w_1) \) together with \( 0 \in T_{w_1}T^1S \).

For each cap \( C_i, i = 1, \ldots, q \), we choose closed disks \( D_i \) and \( E_i \) in \( S \) that are radially symmetric about the center of \( C_i \) for the \( C^\infty \) metric \( g \) such that \( C_i \subseteq \text{int} D_i \) and \( D_i \subseteq \text{int} E_i \). We require that \( E_i \cap E_j = \emptyset \) if \( i \neq j \), and that the closed \( h \)-geodesic \( \rho \) constructed in Section 2 lie in \( S \setminus (\cup_{i=1}^q E_i) \). Since the curvature for \( g \) is negative in \( E_i \setminus C_i \), \( \text{dist}_g(\gamma(t), \partial C_i) \) is a strictly convex function of \( t \) for any \( g \)-geodesic \( \gamma \) in \( E_i \setminus C_i \). This implies that there exists \( \tilde{\beta}_i > 0 \) such that for any \( g \)-geodesic \( \gamma \) with
Figure 3.1. The tangent vector $\xi = (\xi_H, \xi_V)$ at $w_0$ to the curve $w(s)$ in $T^1 S$ is in $K^+_{w_0}$ for the curve on the left and is in $K^-_{w_0}$ for the curve on the right.

$\gamma(0) \in \partial D_i$ and $\gamma'(0)$ either tangent to $\partial D_i$ or $\gamma'(0)$ pointing strictly out of $D_i$ (i.e., $\gamma'(0)$ and $\partial D_i$ are on opposite sides of the tangent line to $\partial D_i$ at $\gamma(0)$), we have $\gamma((0, \tilde{\beta}_i)) \subset S \setminus D_i$. We will assume that $H'_2$ and $H'_3$ are sufficiently small (i.e., $h$ is sufficiently close to $g$ in the $C^2$ topology) such that the analogous property holds with $g$-geodesics replaced by $h$-geodesics and $\tilde{\beta}_i$ replaced by some $\beta_i > 0$. We will refer to this property (for the $h$-metric) as the strong convexity of $D_i$.

If $x \in \text{int} C_i$ and $v \in T^1_x S$, then there exist $a, b, \hat{a}, \hat{b}$ with $a < \hat{a} < 0 < \hat{b} < b$, such that $\gamma_v(t) \in \text{int} C_i$ for $t \in (\hat{a}, \hat{b})$, $\gamma_v(t) \in \text{int}(D_i \setminus C_i)$ for $t \in (a, \hat{a}) \cup (\hat{b}, b)$, and $\gamma_v(a)$, $\gamma_v(b) \in \partial D_i$. That is, $\gamma_v$ exits $C_i$ in both positive and negative time, and once it exits $C_i$ (in either positive or negative time) it exits $D_i$ without first re-entering $C_i$. This follows from Propositions 2.4 and 4.3 in [9], provided that $g$ and $h$ are sufficiently close in the $C^2$ topology, and $g|\Upsilon_i$ and $h|\Upsilon_i$ are sufficiently close in the $C^3$ topology, where $\Upsilon_i$ is as in Theorem 2.1.

Lemma 3.2. Let $C = C_i$ and $D = D_i$ for some $i \in \{1, \ldots, q\}$ be as above. Let $\Upsilon$ be as in Theorem 2.7. If $\mathbb{H}'_2$ and $\mathbb{H}'_3$ are sufficiently small (i.e., $g$ and $h$ are sufficiently close in the $C^2$ topology, and $g|\Upsilon$ and $h|\Upsilon$ are sufficiently close in the $C^3$ topology), then for any geodesic $\gamma$ in $(D, h)$ with $\gamma(t) \in \text{int} D$ for $t \in (a, b)$ and $\gamma(a), \gamma(b) \in \partial D$, we have

$$d_{\gamma(a)}^{b-a} (K^+_{\gamma(a)}) \subset K^+_{\gamma(b)}.$$
Moreover, if $\gamma(t) \in \text{int} \mathcal{C}$ for $t \in (\hat{a}, \hat{b})$ and $\gamma(\hat{a}), \gamma(\hat{b}) \in \partial \mathcal{C}$, where $a < \hat{a} < \hat{b} < b$, then

\begin{equation}
(3.8) \quad d\varphi^{\hat{a}}_{\gamma(t)}(\mathcal{K}^+_{\gamma(\hat{a})}) \subset \mathcal{K}^+_{\gamma(\hat{b})}.
\end{equation}

**Proof.** If $\text{tr}(\gamma)$ does not intersect $\mathcal{C}$, then (3.7) follows from (3.5), because the curvature is negative in $\mathcal{D} \setminus \mathcal{C}$. The idea for the proof of the $\mathcal{K}^+$ invariance property in (3.7) in the case that $\text{tr}(\gamma)$ intersects $\mathcal{C}$ is that the $h$-geodesic $\gamma$ can be approximated by a $g$-geodesic that also passes through $\mathcal{C}$. From Proposition 2.7 in [9] we know that from the time that the corresponding $g$-geodesic enters $\mathcal{C}$ to the time that it exits $\mathcal{C}$ we have invariance of the $\mathcal{K}^+$ cones under the derivative of the geodesic flow for $g$. This invariance property can be destroyed when the metric $g$ is replaced by the metric $h$. However, following $\gamma$ for additional time before and after it enters $\mathcal{C}$, while it is in the negative curvature region $\mathcal{D} \setminus \mathcal{C}$, allows us to recover the $\mathcal{K}^+$ invariance in (3.7). This is proved in detail in Section 4 of [9]. (See Proposition 4.10 of [9].) Moreover, the estimates in Propositions 4.8, 4.9, and 4.10 in [9] show that it is actually enough to follow $\gamma$ just for additional time before it enters $\mathcal{C}$, which leads to the containment in (3.8). \hfill \Box

**Definition 3.3.** A geodesic $\gamma_1 : [a_1, \infty) \to S$ is said to be **asymptotic as $t \to \infty$** to a closed geodesic $\gamma_2 : [a_2, b] \to S$ if there exists $t_0 \in \mathbb{R}$ such that, after extending the domain of $\gamma_2$ to $(-\infty, \infty)$, we have

\begin{equation}
(3.9) \quad \lim_{t \to \infty} \text{dist}(\gamma_1(t), \gamma_2(t_0 + t)) = 0.
\end{equation}

Similarly, a geodesic $\gamma_1 : (-\infty, a_1] \to S$ is said to be **asymptotic as $t \to -\infty$** to a closed geodesic $\gamma_2$ if there exists $t_0 \in \mathbb{R}$ such that (3.9) holds with “$\lim_{t \to \infty}$” replaced by “$\lim_{t \to -\infty}$”.

From the usual procedure for constructing horocycles in regions of nonpositive curvature (see, e.g., [17]), we know that for each $x \in \mathcal{D}_i \setminus \mathcal{C}_i$ there are exactly two vectors $v_{x,j} \in T^s_2S$, $j = 1, 2$, corresponding to the two possible orientations on $\partial \mathcal{C}_i$, such that $\gamma_{v_{x,j}}(t) \in \text{int}(\mathcal{D}_i \setminus \mathcal{C}_i)$ for all $t < 0$, and $\gamma_{v_{x,j}}$ is asymptotic to a closed geodesic along $\partial \mathcal{C}_i$ as $t \to -\infty$. If $x \in \text{int}(\mathcal{D}_i \setminus \mathcal{C}_i)$ and $w \in T^s_2S$, $w \neq v_{x,j}$, $j = 1, 2$, then $\gamma_w(t)$ exits $\mathcal{D}_i$ in negative time, and one of the following must occur:

1. There exists $a < 0$ such that $\gamma_w(t) \in \text{int}(\mathcal{D}_i \setminus \mathcal{C}_i)$ for $t \in (a, 0]$ and $\gamma_w(a) \in \partial \mathcal{D}_i$; or

2. There exist $a < c < d < 0$ such that $\gamma_w(t) \in \text{int}(\mathcal{D}_i \setminus \mathcal{C}_i)$ for $t \in (a, c) \cup (d, 0]$, $\gamma_w(t) \in \text{int} \mathcal{C}_i$ for $t \in (c, d)$, and $\gamma_w(d) \in \partial \mathcal{D}_i$.

If $x \in \mathcal{C}_i$ and $w \in T^1_2S$ are such that $w$ is not tangent to $\partial \mathcal{C}_i$, then again $\gamma_w(t)$ exits $\mathcal{D}_i$ in negative time and we have:

3. There exist $a < c \leq 0$ such that $\gamma_w(t) \in \text{int} \mathcal{C}_i$ if $c < t < 0$, $\gamma_w(t) \in \text{int}(\mathcal{D}_i \setminus \mathcal{C}_i)$ for $t \in (a, c)$, and $\gamma_w(a) \in \partial \mathcal{D}_i$.

**Definition 3.4.** If $x \in \text{int} \mathcal{D}$ for $\mathcal{D} = \mathcal{D}_i$, $i \in \{1, \ldots, q\}$, and $w \in T^1_2S$ is such that there exists $a < 0$ with $\gamma_w(t) \in \text{int} \mathcal{D}$ for $t \in (a, 0]$ and $\gamma_w(a) \in \partial \mathcal{D}$, we define the **unstable cone** $\mathcal{K}^u_w$ by

$$
\mathcal{K}^u_w = d\varphi^{-a}\mathcal{K}^+_w.
$$

For all other $w \in T^1S$, we define
$\mathcal{K}_w^u = \mathcal{K}_w^\perp$.

Similarly, if $x \in \text{int } D$ for $D = D_i$, $i \in \{1, \ldots, q\}$, and $w \in T^1_2 S$ is such that there exists $b > 0$ with $\gamma_w(t) \in \text{int } D$ for $t \in [0, b)$ and $\gamma_w(b) \in \partial D$, we define the stable cone $\mathcal{K}_w^s$ by

$$\mathcal{K}_w^s = d\varphi^{-b}\mathcal{K}_{\gamma_w(b)}^-.$$  

For all other $w \in T^1 S$, we define

$$\mathcal{K}_w^s = \mathcal{K}_w^-.$$  

With these definitions, it follows from (3.5) and (3.7) that the unstable [stable] cones are invariant for $d\varphi^t$, $t \geq 0$ [$t \leq 0$]. That is,

(3.10)  
$$d\varphi^t\mathcal{K}_w^u \subset \mathcal{K}_{\varphi^t w}^u, \text{ for } t \geq 0,$$

and

(3.11)  
$$d\varphi^t\mathcal{K}_w^s \subset \mathcal{K}_{\varphi^t w}^s, \text{ for } t \leq 0.$$  

Moreover, by (3.6), if the basepoint of $\varphi^t w$, for some $t$ between 0 and $t$, lies in $\mathcal{N}_0$, defined by

(3.12)  
$$\mathcal{N}_0 = S \setminus (\cup_{i=1}^q D_i),$$

then we have

(3.13)  
$$d\varphi^t\mathcal{K}_w^u \subset \text{int } \mathcal{K}_{\varphi^t w}^u, \text{ if } t > 0$$

and

(3.14)  
$$d\varphi^t\mathcal{K}_w^s \subset \text{int } \mathcal{K}_{\varphi^t w}^s, \text{ if } t < 0.$$  

Figure 3.2. Invariance of $\mathcal{K}_w^u$ cones under $d\varphi^t$.
The following lemma will be used to prove a transversality condition that is needed in Lemma 6.7.

**Lemma 3.5.** For all \( w \in T^1 S \), we have

\[
(3.15) \quad \text{int} K_w^u \cap \text{int} K_w^s = \{0\}.
\]

**Proof.** If \( w \in T^1 S \) for \( x \in S \setminus (\cup_{i=1}^q \text{int} D_i) \), then (3.15) is clear, because \( K_w^u = K_w^+ \) and \( K_w^s = K_w^- \). If \( x \in \cup_{i=1}^q \text{int} D_i \), (3.15) follows from the definition of the stable and unstable cones and (3.15) in Lemma 3.6. \( \Box \)

The unstable cone field is continuous at those \( v \in T_x^1 S \) where one of the following holds:

1. \( x \in (\cup_{i=1}^q \text{int} C_i) \cup (S \setminus (\cup_{i=1}^q \text{int} D_i)) \).
2. For some \( i \in \{1, \ldots, q\} \), \( x \in \partial C_i \) and \( v \) is not tangent to \( \partial C_i \).
3. For some \( i \in \{1, \ldots, q\} \), \( x \in \text{int}(D_i \setminus C_i) \) and \( v \neq v_{x,1}, v_{x,2} \), where \( v_{x,1}, v_{x,2} \) are as defined above.
4. For some \( i \in \{1, \ldots, q\} \), \( x \in \partial D_i \) and \( v \) points strictly into \( D_i \) (that is, \( v \) is not tangent to \( \partial D_i \), and \( v \) and \( D_i \) lie on the same side of the tangent line to \( \partial D_i \) at \( x \)).

Analogous conditions can be given that guarantee that the stable cone field is continuous at certain vectors \( v \in T_x^1 S \).

**Definition 3.6.** If \( v \mapsto K_v \subset \mathcal{P}(v) \) is a cone field defined for \( v \) in a neighborhood of \( v_0 \) in \( T_1 S \) such that each \( K_v \) is closed, then we say that this cone field is *upper semi-continuous* at \( v_0 \) if the following holds: for any sequence of vectors \( (v_n)_{n=1,2,\ldots} \) in this neighborhood of \( v_0 \) and a corresponding sequence \( (\xi_n)_{n=1,2,\ldots} \) with \( \xi_n \in K_{v_n} \) such that \( \lim_{n \to \infty} v_n = v_0 \) and \( \lim_{n \to \infty} \xi_n = \xi_0 \), we must have \( \xi_0 \in K_{v_0} \).

**Lemma 3.7.** The unstable and stable cone fields \( v \mapsto K_v^u \) and \( v \mapsto K_v^s \) given in Definition 3.4 are upper semi-continuous on \( (T^1 S) \setminus (\cup_{i=1}^q \text{int} (\partial C_i)) \).

**Proof.** We will prove this for the unstable cone field. The proof for the stable cone field is similar. Let \( v_0 \in T_x^1 S \) be such that \( v_0 \) is not tangent to any of \( \partial C_i \), \( i = 1, \ldots, q \). The following two cases are not covered by the above cases (1)-(4), at which we have continuity:

(i) For some \( i \in \{1, \ldots, q\} \), \( x \in \partial D_i \) and \( v_0 \) is either tangent to \( \partial D_i \) or points strictly out of \( D_i \).

(ii) For some \( i \in \{1, \ldots, q\} \), \( x \in \text{int}(D_i \setminus C_i) \) and \( v_0 \in \{v_{x,1}, v_{x,2}\} \).

**Case (i).** In this case, \( K_{v_0} = K_{v_0}^+ \). For \( v \) close to \( v_0 \) with basepoint \( y \in \text{int} D_i \), or with basepoint \( y \in \text{int} D_i \) and \( v \in \{v_{y,1}, v_{y,2}\} \), we have \( K_v^u = K_v^+ \). Thus it suffices to consider \( v \) close to \( v_0 \) with basepoint in \( \text{int} D_i \) and \( \gamma_v \) not asymptotic to \( \partial C_i \) as \( t \to -\infty \). Let \( a = a(v), b = b(v), a \in 0 \), be such that \( \gamma_v((a,b)) \subset \text{int} D_i \) and \( \gamma_v((a),b) \) exits \( D_i \). If \( v \) is close to \( v_0 \), then \( b \) is close to \( 0 \). Thus \( K_v^u = d\phi^{(a)}_{\gamma((a))} K_v^+ \), is close to \( K_v^u \), which is contained in \( K_v^+ \) by (3.7) in Lemma 3.2. This establishes upper semi-continuity at \( v_0 \).

**Case (ii).** Again we have \( K_{v_0} = K_{v_0}^+ \). Moreover, \( \gamma_{v_0} \) is asymptotic to a closed geodesic along \( \partial C_i \) as \( t \to -\infty \). For \( v \) close to \( v_0 \) such that \( \gamma_v \) is also asymptotic to \( \partial C_i \) as \( t \to -\infty \), we have \( K_v^u = K_v^+ \). There are two other possibilities: If \( v \) is close to \( v_0 \) and \( \gamma_v \) exits \( D_i \) in negative time without first entering \( C_i \), then \( K_v^u \subset K_v^+ \) by (3.5).
If \( v \) is close to \( v_0 \) and \( \gamma_v \) passes through \( C_i \) in negative time before exiting \( D_i \), let \( a < \tilde{a} < \tilde{b} < 0 \) be such that \( \gamma_v([a,0)) \subset D_i \), \( \gamma_v([\tilde{a}, \tilde{b}]) \subset C_i \), \( \gamma_v(a) \in \partial D_i \), and \( \gamma_v(\tilde{a}) \), \( \gamma_v(\tilde{b}) \in \partial C_i \). Then, by (3.8) in Lemma 3.2 we have \( K_0^{u} = d_{\varphi^{-1}} K^{n}_{\varphi(\alpha)} < K^{n}_{\varphi(\beta)} \). Thus \( K_0^{u} = d_{\varphi^{-1}} K^{n}_{\varphi(\alpha)} \subset d_{\varphi^{-1}} K^{n}_{\varphi(\beta)} \subset K^{n}_{\varphi(\beta)} \), where the last inclusion is by (3.5). Therefore we have upper semi-continuity at \( v_0 \).

\[ \square \]

Definition 3.8 and Remark 3.9 below will be used to obtain upper bounds on the Lyapunov function defined in Section 4, while Definitions 3.10 and 3.12 and Lemma 3.11 will be needed in Propositions 6.9 and 6.9.

**Definition 3.8.** For \( i \in \{1, \ldots, q\} \), let \( Z_i = \{(T^1(\partial C)) \cup \{v \in T^1_S : x \in D_i \text{ and } v \in \{v_{x,1}, v_{x,2}\}\} \).

**Remark 3.9.** The \( K_0^{u} \) cone angle can approach 0 as \( w \) approaches \( \cup_{i=1}^{q} Z_i \), but if \( v \in (T^1 S) \setminus (\cup_{i=1}^{q} Z_i) \), then there exists an open neighborhood \( U \) of \( v \) and an \( \alpha = \alpha(U) > 0 \) such that the cone angle of \( K_0^{u} \) is at least \( \alpha \) for all \( w \in U \). Thus, if \( W \) is a compact subset of \( (T^1 S) \setminus (\cup_{i=1}^{q} Z_i) \), then there is a positive lower bound for the cone angles of \( K_0^{u} \) for \( w \in W \).

**Definition 3.10.** Let \( x \in D = D_i \subset \text{int } E = \text{int } E_i \), for some \( i \in \{1, \ldots, q\} \). If \( x \) is not the center of \( D \) in the radially symmetric g-metric, let \( \gamma \) be the g-geodesic from \( x \) to a point on \( \partial E \) that is of length \( d_{\varphi}(x, \partial E) \). Then the unit vector \( v_0 \) in the \( h \)-metric that is a positive multiple of \( \gamma(0) \) is called a radial vector at \( x \). If \( x \) is the center of \( D \), then any \( v_0 \in T^1_S \) is called a radial vector at \( x \). (As usual, if we do not specify the metric, \( T^1 S \) refers to unit vectors for \( h \).)

**Lemma 3.11.** There exist positive numbers \( \epsilon \) and \( R \) such that if \( x \in \cup_{i=1}^{q} D_i \) and \( v_0 \) is a radial vector at \( x \), then for any \( v \in T^1_S \) with \( d_{\varphi}(v, v_0) < \epsilon \), we have \( v \notin \cup_{i=1}^{q} Z_i \) and \( d_{\varphi}(H = 0) < \text{int } K_0^{u} \), for \( t \geq R \). Here \( H = 0 \) means the line in \( P(v) \) with \( H \) coordinate identically 0.

**Proof.** Let \( R_i \) be the radius of \( E_i \) in the g-metric, and let \( R = \max(R_1, \ldots, R_q) \).

Suppose \( x \in D_i \) and let \( C = C_i \), \( D = D_i \), and \( E = E_i \). Let \( \gamma_0 \) be the (unit speed) g-geodesic with \( \gamma_0(0) = x \) and \( \gamma_0'(0) \) a positive multiple of the radial vector \( v_0 \).

Let \( \gamma_h \) be the h-geodesic with \( \gamma_h(0) = x \) and \( \gamma_h'(0) = v \) for some \( v \in T^1_S \) with \( d_{\varphi}(v, v_0) < \epsilon \), where we describe the choice of \( \epsilon \) later in the argument. We consider the solutions \( \theta_g \) [respectively, \( \theta_h \)] to the transformed Riccati equation (3.3) along \( \gamma_g [\gamma_h] \) with \( \theta = \theta_g [\theta_h] \) and \( K = K_g [K_h] \), the curvature with respect to the g \([h]\) metric. Assume \( \theta_g \) and \( \theta_h \) satisfy the initial condition \( \theta_g(0) = \pi/2 = \theta_h(0) \).

The condition \( \theta_h(0) = \pi/2 \) corresponds to the line \( H = 0 \) in \( P(v) \).

First we consider the case \( x \in C \). Then there exist times \( t_0 = t_0(x), t_1 = t_1(x), t_2 = t_2(x), -2R < t_2 < 0 \leq t_0 < t_1 \leq R \), such that for \( t \geq 0 \), \( \gamma_g(t) \) exits \( E \) at time \( t_1 \), and for \( t < 0 \), \( \gamma_g(t) \) exits \( E \) at time \( t_2 \). It follows from Lemma 2.5 in [1] that \( 0 \leq \theta_g(t_0) \leq \pi/2 \). Since \( K_g(t) \) is negative for \( t_0 < t \leq t_1 \), we obtain \( 0 < \theta_g(t_1) < \pi/2 \). Moreover, by a compactness argument, there is a \( \delta \in (0, \pi/2) \) such that \( \delta < \theta_g(t_1) < (\pi/2) - \delta \), for \( t_1 = t_1(x) \), for all \( x \in C \).

For \( \epsilon \) sufficiently small and \( \epsilon_h \) sufficiently small (i.e., \( h \) sufficiently close to \( g \)), we obtain \( 0 < \theta_h(t_1) < \pi/2 \). In addition, we may assume that \( \gamma_h(t_1) \) is sufficiently close to \( \gamma_g(t_1) \) that \( \gamma_h(t_1) \in S \setminus (\cup_{j=1}^{q} D_j) \), which implies that \( K_0^{u} = K_0^{u} \). Thus \( 0 < \theta_h(t_1) < \pi/2 \) implies that \( d_{\varphi}(H = 0) < \text{int } K_0^{u} \). By the invariance
of the unstable cones it follows that $d\varphi_1^t(H = 0) \subset \text{int} K^u_{\varphi(v)}$, for $t \geq R$. We may also assume that $\theta_d(t_2) \in S \setminus \bigcup_{j=1}^t D_j$, which implies that $v \notin \bigcup_{i=1}^{t} \mathbb{Z}_i$.

Now consider the case $x \in D \setminus \mathbb{C}$. Again let $t_1 = t_2(x)$, $0 < t_1 < R$, be the time at which $\gamma_t$ exits $\mathcal{E}$. Since $K_\varphi(t)$ is negative for $0 \leq t \leq t_1$, we have $0 < \theta_d(t_1) < \pi/2$, and the rest of the argument proceeds as in the case $x \in \mathbb{C}$.

**Definition 3.12.** If $x \in \bigcup_{i=1}^t D_i$, and $v \in T^1_x(S)$ is such that $\text{dist}_h(v, v_0) < \epsilon$, where $v_0$ is a radial vector at $x$ and $\epsilon$ is as in Lemma 3.11, then $v$ is said to be an approximately radial vector at $x$. A $C^1$ regular curve $\sigma(t)$, $a_1 \leq t \leq a_2$, $a_1 < a_2$, is in $(T^1(S) \setminus (\bigcup_{i=1}^t T^1(\partial C_i)))$, is defined to be an approximately stable (approximately unstable) curve if $\sigma'(t)$ is in $K^s_{\sigma(t)}[K^u_{\sigma(t)}]$, for all $t \in [a_1, a_2]$.

**Remark 3.13.** It follows from Lemma 3.11 that if $x \in \bigcup_{i=1}^t D_i$ and $(s(t), a_1 \leq s \leq a_2$, $a_1 < a_2$, is a $C^1$ regular constant basepoint curve in $T^1_x(S)$ such that each $\sigma(s)$ is an approximately radial vector, then $\varphi'(\sigma(s))$, $a_1 \leq s \leq a_2$, is an approximately unstable curve for $t \geq R$. To see this, note that $\sigma'(s) = (\xi_H(s), \xi_V(s))$, where $\xi_H(s) = 0$.

4. **Lyapunov Function and Line Fields**

We now introduce a Lyapunov function $Q = Q_w : P(w) \to \mathbb{R}$ for each $w \in T^1 S$. As we will see in 4.2 below, $Q$ is monotone increasing along the orbits of the geodesic flow. This Lyapunov function will be used to prove that stable and unstable cones intersect down to lines in Lemma 4.1. Similarly constructed line fields in [8, 9] were only shown to exist on some set of full measure, while Lemma 4.1 shows existence everywhere. Moreover, Lemma 4.3 shows continuity of these line fields at all vectors in $T^1 S$ except those that are tangent to the boundary of one of the caps. Our use of Lyapunov functions and the methods in this section are based on ideas in [8, 9].

For each $w \in T^1 S$, let $U_i = U_i(w)$, $i = 1, 2$, be a basis for $P(w)$ such that the unstable cone at $w$ is given by $K^u_w = \{\lambda_1 U_1 + \lambda_2 U_2 : \lambda_1 \lambda_2 \geq 0\}$. We require $U_1, U_2$ to be positively oriented (using the orientation on $P(w)$ given near the beginning of Section 3), $||U_1|| = ||U_2||$, and the parallelogram determined by $U_1$ and $U_2$ to have unit area. This determines $U_1, U_2$ uniquely up to a simultaneous change of sign in both $U_1$ and $U_2$. If $K^u_w = K^u_{w'}$, we let $U_1$ and $U_2$ have $H, V$ coordinates $(1, 0)$ and $(0, 1)$, respectively. For $w \in T^1 S$ and $t \in \mathbb{R}$, let

$$A = A(w, t) = \begin{pmatrix} a(w, t) & b(w, t) \\ c(w, t) & d(w, t) \end{pmatrix}$$

be the matrix for $d\varphi_1^t : P(w) \to P(\varphi^t w)$ with respect to the bases $U_1(w), U_2(w)$ and $U_1(\varphi^t(w)), U_2(\varphi^t(w))$. Since $d\varphi_1^t : P(w) \to P(\varphi^t w)$ is area-preserving and orientation-preserving, $\det A(w, t) = 1$. The inclusion (3.10) implies that if $t \geq 0$, then either all the entries of $A(w, t)$ are nonnegative or all the entries are nonpositive. (Which case occurs may depend on $t$.) Moreover if $t > 0$ and (3.11) holds, then all the entries of $A(w, t)$ are strictly positive or all the entries are strictly negative. In our calculations, it does not matter if $A(w, t)$ is replaced by $-A(w, t)$. Therefore, we may assume that for $t \geq 0$, all of the entries of $A(w, t)$ are nonnegative. We define a Lyapunov function $Q = Q_w : P(w) \to \mathbb{R}$ by $Q(\lambda_1 U_1(w) + \lambda_2 U_2(w)) = \text{sgn}(\lambda_1 \lambda_2) \sqrt{||\lambda_1 \lambda_2||}$. Then $Q_w(\xi) \geq 0$ if and only if
\[ \xi \in \mathcal{K}_w^u. \] Thus, by Lemma 3.5 \( Q_w(\xi) \leq 0 \) if \( \xi \in \mathcal{K}_w^s. \) We define \( F = F_w : \mathcal{P}(w) \to \mathbb{R} \) by \( F(\lambda_1 U_1(w) + \lambda_2 U_2(w)) = \lambda_1 \lambda_2. \)

Let \( \xi = \lambda_1 U_1(w) + \lambda_2 U_2(w) \) and \( a = a(w,t), b = b(w,t), c = c(w,t), d = d(w,t). \) Suppose \( t \geq 0. \) Assume for the moment that \( \lambda_1 \lambda_2 \leq 0. \) Then

\[
\begin{align*}
F(d\varphi^t_w \xi) &= (a\lambda_1 + b\lambda_2)(c\lambda_1 + d\lambda_2) \\
&= ac\lambda^2_1 + bd\lambda^2_2 + (ad + bc)\lambda_1 \lambda_2 \\
&= ac\lambda^2_1 + bd\lambda^2_2 + 2bc\lambda_1 \lambda_2 + \lambda_1 \lambda_2 \\
&\geq ac\lambda^2_1 + bd\lambda^2_2 + 2\sqrt{adbc}\lambda_1 \lambda_2 + \lambda_1 \lambda_2 \\
&= (\sqrt{ac}\lambda_1 + \sqrt{bd}\lambda_2)^2 + \lambda_1 \lambda_2 \\
&\geq \lambda_1 \lambda_2 = F(\xi).
\end{align*}
\]

But if \( \lambda_1 \lambda_2 \geq 0, \) then (4.1) still holds, and this shows that \( F(d\varphi^t_w \xi) \geq F(\xi). \) Thus we obtain \( F(d\varphi^t_w \xi) \geq F(\xi), \) for all \( t \geq 0 \) and all \( \xi \in \mathcal{P}(w). \) This implies that

\[
(4.2) \quad Q(d\varphi^t_w \xi) \geq Q(\xi), \quad \text{for all } t \geq 0 \text{ and all } \xi \in \mathcal{P}(w).
\]

If we let

\[
\tau(w, t) = \left( 2b(t,w)c(t,w) + 1 \right)^{1/2},
\]

then it follows from (4.1) that

\[
(4.3) \quad Q(d\varphi^t_w \xi) \geq \tau(w, t)Q(\xi) \geq 0, \quad \text{for } \xi \in \mathcal{K}_w^u \text{ and } t \geq 0.
\]

If \( t > 0 \) and (3.13) holds, then \( \tau(w, t) > 1. \) Similarly,

\[
(4.4) \quad Q(\xi) \leq \tau(w, t)Q(d\varphi^t_w \xi) \leq 0, \quad \text{for } \xi \in d\varphi^{-t}\mathcal{K}_w^{s, u} \text{ and } t \geq 0.
\]

If \( \delta > 0, \) then there exists \( C = C(\delta) > 0 \) such that if \( \mathcal{K}_w^u = \mathcal{K}_w^+ \) and \( \mathcal{K}_w^+ \subset \mathcal{K}_w^u \) is a cone such that the slopes of the boundary lines of \( \mathcal{K}_w \) are \( \delta \) and \( 1/\delta \) (or if \( \mathcal{K}_w^u = \mathcal{K}_w^+ \) and \( \mathcal{K}_w^+ \subset \mathcal{K}_w^u \) is a cone such that the slopes of the boundary lines of \( \mathcal{K}_w \) are \( -\delta \) and \( -1/\delta \), then

\[
(4.5) \quad C||\xi|| \leq |Q(\xi)|, \quad \text{for all } \xi \in \mathcal{K}_w^u.
\]

Also note that for \( w \) such that \( \mathcal{K}_w^u = \mathcal{K}_w^+, \) in particular for \( w \in T^1 \mathcal{N}_0, \) where \( \mathcal{N}_0 = S \setminus (\cup_{i=1}^d D_i), \) we have

\[
(4.6) \quad |Q(\xi)| \leq ||\xi||, \quad \text{for all } \xi \in \mathcal{P}(w).
\]

If \( \mathcal{W} \) is a compact subset of \( (T^1S) \setminus (\cup_{i=1}^d Z_i), \) then it follows from Remark 3.9 that there is a constant \( \bar{C} = \bar{C}(\mathcal{W}) > 0 \) such that

\[
(4.7) \quad |Q(\xi)| \leq \bar{C}||\xi||, \quad \text{for all } \xi \in \mathcal{P}(w), \quad \text{for all } w \in \mathcal{W}.
\]

**Lemma 1.** For all \( w \in T^1S, \) if we let \( E^w_u \) and \( E^s_w \) be defined by

\[
E^w_u := \bigcap_{t \geq 0} d\varphi^t(\mathcal{K}_w^{s, u}) \text{ and } E^s_w := \bigcap_{t \geq 0} d\varphi^{-t}(\mathcal{K}_w^{s, u}),
\]

then \( E^w_u \) and \( E^s_w \) are lines in \( T_w(T^1S). \) Moreover, if \( w \in (T^1S) \setminus (\cup_{i=1}^d T^1(\partial\mathcal{C}_i)), \) then \( E^w_u \subset \text{int} \mathcal{K}_w^u \) and \( E^s_w \subset \text{int} \mathcal{K}_w^s. \)
The intersection of the unit disk in $P_d\phi$ lines of other hand, (4.10) implies that this image contains the intersection of the disk of $0$ for all $t \leq 0$ and it follows from the cone invariance (3.10) that may assume that $t$ in $T^1\mathbb{N}_0$ before exiting (if it exists at all). Thus there is a sequence $(t_n)_{n=1,2,\ldots}$ such that for $n = 1, 2, \ldots$, we have $-t_{n+1} < -t_n - \beta < -t_n < 0$, and $\varphi^t w \in T^1\mathbb{N}_0$ for all $t \in [-t_n - \beta, -t_n]$. It follows from (3.13), (4.3) and a compactness argument on $T^1\mathbb{N}_0$ that there exists $\tau_0 > 1$ such that for all $v \in T^1\mathbb{S}$ with $\varphi^t v \in T^1\mathbb{N}_0$ for all $t \in [-\beta, 0]$ we have

$$Q_v(d\varphi^t_{\varphi^t v, \xi}) \geq \tau_0 Q_{\varphi^t v}(\xi), \quad \text{for all } \xi \in K_{\varphi^t v}^u.$$

It also follows from (4.13) and a compactness argument on $T^1\mathbb{N}_0$ that there exists $\delta > 0$ such that for all $v \in T^1\mathbb{S}$ with $\varphi^t v \in T^1\mathbb{N}_0$ for all $t \in [-\beta, 0]$ we have

$$d\varphi^t K_{\varphi^t v}^u \subset \tilde{K}_v,$$

where $\tilde{K}_v \subset P(v)$ is a cone whose boundary lines have slopes $\delta$ and $1/\delta$ in the $H, V$ coordinates. Let $C = C(\delta)$ be as in (4.5).

Define $C_\beta := \sup\{||d\varphi^t_{v, \xi}|| : v \in T^1\mathbb{S}\}$. Let $\xi \in K_{\varphi^{-t_{n+1}} v}^u$ and suppose $||\xi|| = 1$. Then it follows from (4.5) and (4.9) that

$$Q_{\varphi^{-t_{n+1}} v}(d\varphi^t_{v, \xi}) \geq C \cdot ||d\varphi^t_{v, \xi}|| \geq C C_\beta^{-1}.$$

Moreover, by (4.2), (4.10), and (4.13), we have

$$||d\varphi^{t_{n+1} - t_1}_{v, \xi}|| \geq Q_{\varphi^{-t_{n+1}} v}(d\varphi^{t_{n+1} - t_1} \cdot d\varphi^t_{v, \xi}) \geq \tau_0^n Q_{\varphi^{-t_{n+1}} v}(d\varphi^t_{v, \xi}) \geq \tau_0^n C C_\beta^{-1}.$$

The intersection of the unit disk in $P(\varphi^{-t_{n+1} - \beta} w)$ and $K_{\varphi^{-t_{n+1} - \beta} w}^u$ has area $\pi/2$, and the image of this intersection under $d\varphi^{t_{n+1} - t_1 + \beta}$ also has area $\pi/2$. On the other hand, (4.10) implies that this image contains the intersection of the disk of radius $\tau_0^n C C_\beta^{-1}$ with $d\varphi^{t_{n+1} - t_1 + \beta} K_{\varphi^{-t_{n+1} - \beta} w}^u$. Since $\lim_{n \to \infty} \tau_0^n C C_\beta^{-1} = \infty$, the cone angle of $d\varphi^{t_{n+1} - t_1 + \beta} K_{\varphi^{-t_{n+1} - \beta} w}^u$ goes to zero as $n \to \infty$. Thus $E_{\varphi^{-t_{n+1}} w}^u$ is a line, and it follows from the cone invariance (3.13) that $E_{\varphi^{-t_{n+1}} w}^u$ is a line.

Case 2. Either $w$ is tangent to $\partial \mathbb{C}$ or $\gamma_w(t)$ is asymptotic to $\partial \mathbb{C}$ as $t \to -\infty$. Then there exists $t_0 \leq 0$ such that $K_{\varphi^{-t_0} w}^u(t) = K_{\varphi^{-t_0} w}^+$ and the curvature $K(\gamma_w(t)) \leq 0$ for all $t \leq t_0$. As in case 1, if $E_{\varphi^{-t_0} w}^u$ is a line, then so is $E_{\varphi^{-t_0} w}^u$. Therefore we may assume that $t_0 = 0$.

Let $\epsilon > 0$ and let $T = T(\epsilon) \leq 0$ be such that $-\epsilon \leq K(\gamma_w(t)) \leq 0$ for all $t \leq T$. For $B > 0$, let $m_{1,B}(t)$ and $m_{2,B}(t)$ be solutions to the Riccati equation along $\gamma_w(t)$ (i.e., (3.3) with $w$ replaced by $w$) such that $m_{1,B}(T-B) = 0$ and $\lim_{t \to (T-B)} + m_{2,B}(t) = \infty$. Since the boundary lines of $K_{\varphi^{-t_0} w}^u$ have slopes 0 and $\infty$, $m_{1,B}(t)$ and $m_{2,B}(t)$ represent the slopes of the boundary lines of $d\varphi^{t-(T-B)}(K_{\varphi^{-t_0} w}^u)$ for $t > T - B$. If $K(\gamma_w(t))$ were replaced by the constant $-\epsilon$ in (3.3), then the solution $\bar{m}_{2,B}$ with $\lim_{t \to (T-B)^+} \bar{m}_{2,B}(t) = \infty$ would...
Therefore, by a comparison lemma (see, e.g., [2]), we obtain
\[ \lim \sup \] will be called the stable and unstable line fields.

If \( \epsilon \) was arbitrary, \( E_w^u \) is contained in a cone within \( K_w^+ \) that is bounded by lines whose slopes differ by at most \( \sqrt{\epsilon} \). Since \( \epsilon \) was arbitrary, \( E_w^u \) is a line.

In both cases, if \( w \) is not tangent to the boundary of a cap, then there exist \( r_1 > r_2 > 0 \) such that \( d\varphi^1 - r^1, K_w^u \cap \subset \int K_w^u - r_w^u \). Thus \( E_w^u \subset \varphi^2(\int K_w^u - r_w^u) \subset \int K_w^u \). A similar argument shows that if \( w \) is not tangent to the boundary of a cap, then \( E_w^s \subset \int K_w^u \).

**Definition 4.2.** The line fields \( v \to E_v^s \) and \( v \to E_v^u \) on \( T^1S \) obtained in Lemma 4.1 will be called the stable and unstable line fields, respectively. Note that \( d\varphi^E_v = E_v^u \) and \( d\varphi^E_v = E_v^s \), for \( v \in T^1S \) and \( t \in \mathbb{R} \). A \( C^1 \) regular curve \( \sigma(t), a_1 \leq t \leq a_2, a_1 < a_2 \), in \( (T^1S) \setminus (\bigcup_{i=1}^n T^1(\partial C_i)) \) is defined to be a stable [unstable] curve if \( \sigma(t) \in E_v^u, E_v^s \) for all \( t \in [a_1, a_2] \).

The following lemma allows us to integrate the stable and unstable line fields to obtain stable and unstable curves.

**Lemma 4.3.** The stable and unstable line fields given in Lemma 4.1 are continuous on \( (T^1S) \setminus (\bigcup_{i=1}^n T^1(\partial C_i)) \).

**Proof.** We will give the proof for the unstable line field. The proof for the stable line field is similar.

In order to compare a cone in \( \mathcal{P}(w_1) \) with a cone in \( \mathcal{P}(w_2) \), or a line through the origin in \( \mathcal{P}(w_1) \) with a line through the origin in \( \mathcal{P}(w_2) \), we will use \( H, V \) coordinates on both \( \mathcal{P}(w_1) \) and \( \mathcal{P}(w_2) \) to identify \( \mathcal{P}(w_1) \) with \( \mathcal{P}(w_2) \).

Suppose \( v \in T^1S \) and \( v \) is not tangent to the boundary of a cap. Let \( \epsilon > 0 \) and let \( K_v, v \) be the closed cone in \( \mathcal{P}(v) \) centered at \( E_v^u \) and of cone angle \( \epsilon \) in the \( H, V \) coordinate system. By Lemma 4.1 and 3.10, there exists \( T > 0 \) such that \( \partial \mathcal{P}(v, T) \subset \ int K_v, v \). By the continuity of \( \varphi^T \), there exists a closed cone \( \mathcal{K}_{v, r} \subset \mathcal{P}(\varphi^{-T}) \) such that \( K_v, v \subset \ int \mathcal{K}_{v, r} \) and \( \varphi^T \mathcal{K}_{v, r} \subset \ int K_v, v \). From the continuity of \( \varphi^{-T} \) at \( v \) and the upper semi-continuity of the unstable cone field at \( \varphi^{-T}v \), we know that for \( w \in T^1S \) sufficiently close to \( v \), we have \( K_w^u \subset \mathcal{K}_{v, r} \subset \mathcal{K}_{v, r} \). Where \( \mathcal{K}_{v, r} \subset \mathcal{P}(\varphi^{-T}w) \) is a copy of \( \mathcal{K}_{v, r} \) (using the \( H, V \) coordinates as described in the preceding paragraph). Moreover, by the continuity of \( \varphi^{-T} \) and \( \varphi^{-T} \), for \( w \) sufficiently close to \( v \), we have \( d\varphi^{-T}(\mathcal{K}_{v, r} \subset \int K_w, v \), where \( K_w, v \subset \mathcal{P}(w) \) is a copy of \( K_v, v \). Therefore

\[ E_w^u \subset d\varphi^{-T}(K_v^u, v) \subset d\varphi^{-T}(K_v, v) \subset \ int K_v, v \]

which implies that \( E_w^u \) makes angle less than \( \epsilon \) with \( E_v^u \).
5. Tubular Neighborhoods of $\text{tr}(\rho)$ and Their Lifts

For the rest of this paper, we let $\mathcal{N} = S \setminus \bigcup_{i=1}^{g} C_i$, $\mathcal{N}_0 = S \setminus \bigcup_{i=1}^{g} D_i$, and we let $\mathcal{N}_1$ and $\mathcal{N}_2$ be open subsets of $S$ such that $\overline{\mathcal{N}_2} \subset \mathcal{N}_1$, $\overline{\mathcal{N}_1} \subset \mathcal{N}_0$, and the closed geodesic $\rho : [0, L] \to \mathcal{N}$ described in Section 2 has $\text{tr}(\rho) \subset \mathcal{N}_2$. We let $(\tau_1, \tau_2)$ be Fermi coordinates along $\rho$, where $\tau_1 \in \mathbb{R}/L\mathbb{Z}$ is the coordinate along $\rho$ and $\tau_2 \in [-\epsilon_0, \epsilon_0]$ is the coordinate along geodesics perpendicular to $\rho$. Here $\epsilon_0 > 0$ is chosen sufficiently small so that all points with Fermi coordinates in $(\mathbb{R}/L\mathbb{Z}) \times [-\epsilon_0, \epsilon_0]$ are contained in $\mathcal{N}_2$. For $0 < \epsilon \leq \epsilon_0$, let

\begin{align*}
F(\epsilon) = \{ p \in S : \text{dist}(p, \text{tr}(\rho) \leq \epsilon \}.
\end{align*}

Each point in $F(\epsilon)$ has Fermi coordinates $(\tau_1, \tau_2)$ in $(\mathbb{R}/L\mathbb{Z}) \times [-\epsilon_0, \epsilon_0]$, but if $\rho$ is not simple, then some of the points in $F(\epsilon)$ will have more than one such representation in Fermi coordinates. In order to handle the case in which $\rho$ is not simple, we let

\begin{align*}
\tilde{F}(\epsilon) = (\mathbb{R}/L\mathbb{Z}) \times [-\epsilon, \epsilon],
\end{align*}

and let $\tilde{\pi} : \tilde{F}(\epsilon) \to F(\epsilon)$ be the projection that takes $(\tau_1, \tau_2)$ to the point in $S$ with Fermi coordinates $(\tau_1, \tau_2)$. Define $\tilde{h} = \tilde{\pi}^* h$ to be the covering metric. Let $\hat{\rho} : [0, L] \to \tilde{F}(\epsilon)$ be the simple closed geodesic in $(\tilde{F}(\epsilon), \hat{h})$ that is the lift of $\rho$. We also define

\begin{align*}
\tilde{F}(\epsilon) = \mathbb{R} \times [-\epsilon, \epsilon],
\end{align*}

which is the universal covering space of $\tilde{F}(\epsilon)$. Let $\tilde{\pi} : \tilde{F}(\epsilon) \to \tilde{F}(\epsilon)$ be the covering map, and let $\tilde{h} = \tilde{\pi}^* h$ be the covering metric. We let $\tilde{\rho} : \mathbb{R} \to \tilde{F}(\epsilon)$ be the geodesic in $(\tilde{F}(\epsilon), \hat{h})$ that is the lift of $\hat{\rho}$.

If $(\tau_1, \tau_2)$ are Fermi coordinates along $\hat{\rho}$ or $\tilde{\rho}$ and $\gamma : I \to \tilde{F}(\epsilon)$ or $\gamma : I \to \tilde{F}(\epsilon)$ is a geodesic whose trace is contained in the region in which $\tau_2 \neq 0$, then the negative curvature of $(\tilde{F}(\epsilon), \hat{h})$ and $(\tilde{F}(\epsilon), \tilde{h})$ implies that $t \mapsto |\tau_2(\gamma(t))|$ is a strictly convex function.

It follows from the simple connectivity of $\tilde{F}(\epsilon)$ and the negative curvature of $(\tilde{F}(\epsilon), \tilde{h})$ that for each $\tilde{\rho} \in \tilde{F}(\epsilon)$, there is a unique vector $\tilde{Z}(\tilde{\rho}) \in T^1_{\tilde{\rho}}(\tilde{F}(\epsilon))$ such that $\gamma_{\tilde{Z}(\tilde{\rho})}$ is asymptotic to $\tilde{\rho}$ as $t \to \infty$. We let $\tilde{Z}$ be the (unique) vector field on $\tilde{F}(\epsilon)$ obtained by applying $d\tilde{\pi}$ to $\tilde{Z}$. Then for each $\tilde{\rho} \in \tilde{F}(\epsilon)$, $\gamma_{\tilde{Z}(\tilde{\rho})}$ is asymptotic to $\hat{\rho}$ as $t \to \infty$.

We will call $\hat{Z}$ and $\tilde{Z}$ asymptotic vector fields for $\hat{\rho}$ and $\tilde{\rho}$, respectively. These vector fields can be obtained by applying the geodesic flow to a stable horocycle through a tangent vector to $\hat{\rho}$ or $\tilde{\rho}$. Such horocycles are known to be $C^\infty$ [1], because $(\tilde{F}(\epsilon), \tilde{h})$ can be extended to a closed surface of negative curvature. It follows that $\hat{Z}$ and $\tilde{Z}$ are $C^\infty$ vector fields. In this paper, we will only use the fact that they are continuous.

6. Proof of Total Insecurity of $(S, h)$

Let $\mathcal{P}(w), w \in T^1 S$, be the two-dimensional distribution on $T^1 S$ defined in (5.1), and let $Q = Q_w : \mathcal{P}(w) \to \mathbb{R}$ be the Lyapunov function defined at the beginning of Section 4. Let $\mathcal{N}, \mathcal{N}_0, \mathcal{N}_1$, and $\mathcal{N}_2$ be as in Section 5.

**Definition 6.1.** If a $C^1$ regular curve $\sigma(s), a_1 \leq s \leq a_2, a_1 < a_2$, in $T^1 S$ is everywhere tangent to the distribution $\mathcal{P}$, then we define the Lyapunov length of $\sigma$
as $L_Q(\sigma) = \int_{a_1}^{a_2} |Q(\sigma'(s))| \, ds$. We let $L(\sigma) = \int_{a_1}^{a_2} ||\sigma'(s)|| \, ds$ be the usual length of $\sigma$.

**Remark 6.2.** The Lyapunov length, like the usual length, is independent of the parametrization of $\sigma$, because $Q$ is homogeneous of degree 1. If $\sigma$ is everywhere tangent to $\mathcal{P}$, and for each tangent vector $w = \sigma'(s)$ of $\sigma$ we have $K_w^+ = K_w^-$ (in particular if $tr(\sigma) \subset \mathcal{N}_0$), then by (4.6), we know that $L_Q(\sigma) \leq L(\sigma)$. If $\sigma$ is as in Definition 6.1 and $\sigma(s) \in (T^1 S) \setminus (\cup_{i=1}^{q} Z_i)$ for all $s \in [a_1, a_2]$, then it follows from (4.7) that $L_Q(\sigma) < \infty$.

Lemma 6.3 shows that the length of certain stable or approximately stable curves goes to zero under the application of $d\varphi^t$ as $t$ goes to infinity along certain sequences. The uniformity of the contraction in part (1) of Lemma 6.3 allows us to obtain expansion of approximately unstable curves in Corollary 6.4.

**Lemma 6.3.** Suppose $\sigma(s), s \in [0, a], \ a > 0$, is a $C^1$ regular curve in $(T^1 S) \setminus (\cup_{i=1}^{q} T^1(\mathcal{D}_i))$ that is everywhere tangent to the distribution $\mathcal{P}$. Let $v_0 = \sigma(s_0)$, for some $s_0 \in [0, a]$. The two statements below are true for $\sigma$.

1. For every $\epsilon > 0$, there exists $M = M(\epsilon) > 1$, independent of the choice of $\sigma$, such that the following holds: If $m \geq M$ is an integer and there exists a finite sequence $(t_n)_{n=1, \ldots, m+1}$ such that $t_1 \geq 0$, $t_{n+1} \geq 1 + t_n$ for $n = 1, 2, \ldots, m$, $tr(\varphi^{t_n}(\sigma)) \subset T^1 N_1$, for $n = 1, 2, \ldots, m$, and $(\varphi^{t_n+1}(\sigma))$ is an approximately stable curve, then $L(\varphi^{t_n}(\sigma)) < \epsilon L(\varphi^{t_1}(\sigma))$.

2. If $\sigma$ is a stable curve with $L_Q(\sigma) < \infty$, and there exists a sequence $(t_n)_{n=1, \ldots}$ such that $t_1 \geq 0$, $t_{n+1} \geq 1 + t_n$ for $n = 1, 2, \ldots$, and $\varphi^{t_n}(v_0) \in T^1 N_2$ for $n = 1, 2, \ldots$, then $\lim_{n \to \infty} L(\varphi^{t_n}(\sigma)) = 0$.

**Proof.** Since $\mathcal{N}_1 \subset \mathcal{N}_0$, there exists $\eta \in (0, 1)$ such that for every $v \in \overline{T^1 N_1}$, $\gamma_v([0, \eta])$ lies in $\mathcal{N}_0$. By (3.14) and a compactness argument, there exists $\delta \in (0, 1)$ such that for $v \in \overline{T^1 N_1}$, $d\varphi^{-\eta}(K_{\varphi^{t_n} v}) \subset \tilde{K}_v \subset K_v^-$, where $\tilde{K}_v$ is bounded by lines of slopes $-\delta$ and $-1/\delta$ in the $H, V$ coordinate system on $\mathcal{P}(v)$. If $\beta \geq 1 > \eta$, then $d\varphi^{-\beta}(K_{\varphi^{t_n} v}) \subset d\varphi^{-\eta}(K_{\varphi^{t_n} v})$. Thus, by (4.3) and (4.4) there exists a constant $C > 0$ such that if $\xi \in d\varphi^{-\beta}(K_{\varphi^{t_n} v})$ for some $\beta \geq 1$ and some $v \in \overline{T^1 N_1}$, then

$$C||\xi|| \leq |Q(\xi)| \leq ||\xi||.$$  

By (3.13) and (4.4), there exists $\kappa \in (0, 1)$ such that if $\xi \in d\varphi^{-\eta}(K_{\varphi^{t_n} v})$ for some $v \in \overline{T^1 N_1}$, then

$$|Q(d\varphi^{t_n}\xi)| \leq \kappa|Q(\xi)|.$$  

If $\beta \geq 1$ and $\xi \in d\varphi^{-\beta}(K_{\varphi^{t_n} v})$ for some $v \in \overline{T^1 N_1}$, then $|Q(d\varphi^{t_n}\xi)| \leq |Q(d\varphi^{t_n}\xi)|$ and we obtain

$$|Q(d\varphi^{t_n}\xi)| \leq \kappa|Q(\xi)|.$$  

Thus, if $\bar{\sigma}$ is a stable curve in $\overline{T^1 N_1}$, or more generally, if $\bar{\sigma}$ is in $\overline{T^1 N_1}$ and $\varphi^{t_n}\bar{\sigma}$ is an approximately stable curve for some $\beta \geq 1$, then

$$C L(\bar{\sigma}) \leq L_Q(\bar{\sigma}) \leq L(\bar{\sigma})$$  

and

$$L_Q(\varphi^{t_n}(\bar{\sigma})) \leq \kappa L_Q(\bar{\sigma}).$$

Let $\epsilon > 0$ and take $M > 1$ sufficiently large so that $\kappa^{M-1} < C\epsilon$. Suppose $m \geq M$ is an integer and $(t_n)_{n=1, \ldots, m+1}$ and $\sigma$ are as in the hypothesis of part (1).
Since $\varphi^{m+1}(\sigma)$ is approximately stable and $\text{tr}(\varphi^m(\sigma)) \subset T^1N_1$, it follows from (6.2) that $CL(\varphi^m(\sigma)) \leq Q(\varphi^m(\sigma))$. Note that (3.11) implies that $\varphi^{m-1}(\sigma)$ is approximately stable for $t \geq 0$. Applying (6.3) to $\tilde{\varphi} = \varphi^1(\sigma), \cdots, \varphi^{m-1}(\sigma)$, and $\beta = t_2 - t_1, \ldots, t_m - t_{m-1}$, respectively, we obtain $LQ(\varphi^m(\sigma)) \leq k^{m-1}LQ(\varphi^1(\sigma))$. Since $\varphi^1(\sigma)$ is approximately stable, (6.2) implies that $LQ(\varphi^1(\sigma)) \leq C\ell L(\varphi^1(\sigma))$. To summarize, we have

$$
CL(\varphi^m(\sigma)) \leq Q(\varphi^m(\sigma)) \leq k^{m-1}LQ(\varphi^1(\sigma)) \leq k^{m-1}L(\varphi^1(\sigma)) < C\ell L(\varphi^1(\sigma)),
$$

and this completes the proof of part (1).

It follows from (6.2) that there exists $\tilde{c} > 0$ such that any stable curve $\tilde{\sigma}$ that contains a vector in $T^1N_2$ and has $LQ(\tilde{\sigma}) \leq \tilde{c}$ must be contained in $T^1N_1$. Suppose now that $\sigma$ and the sequence $(t_n)_{n=1,2,\ldots}$ are as in the hypothesis of part (2). If $LQ(\varphi^m(\sigma)) \leq \tilde{c}$ for some $n \geq 1$, then $\varphi^m(\sigma) \subset T^1N_1$ and $LQ(\varphi^m(\sigma)) \leq (1/C)LQ(\varphi^1(\sigma))$. Therefore to prove part (2), it suffices to show that $\lim_{n \to \infty} LQ(\varphi^m(\sigma)) = 0$.

We may assume that $v_0$ is an endpoint of $\sigma$, say $v_0 = \sigma(0)$. Let $0 = s_0 < s_1 < \cdots < s_k = a$ be chosen so that $LQ(\varphi^i(\sigma|[s_\ell, s_{\ell+1}])) < \tilde{c}/2$ for $\ell = 0, \ldots, k - 1$. Let $\sigma_\ell = \sigma|[0, s_\ell]$ for $\ell = 1, \ldots, k$. We will use induction on $\ell$ to show that $\lim_{n \to \infty} LQ(\varphi^n(\sigma_\ell)) = 0$ for $l = 1, \ldots, k$. Since $\sigma_k = \sigma$, the conclusion of part (2) will follow.

Since $LQ(\varphi^i(\tilde{\sigma}))$ is a nonincreasing function of $t$ for any stable curve $\tilde{\sigma}$, we have $LQ(\varphi^i(\sigma_{t_1})) \leq LQ(\varphi^i(\sigma_{t})) < \tilde{c}/2$ for all $n \geq 1$. Thus $\text{tr}(\varphi^i(\sigma_{t})) \subset T^1N_1$ for all $n \geq 1$. Therefore part (1) implies that $\lim_{n \to \infty} LQ(\varphi^i(\sigma_{t})) = 0$. Since $LQ(\varphi^i(\sigma_{t})) \leq L(\varphi^i(\sigma))$, we have $\lim_{n \to \infty} LQ(\varphi^i(\sigma)) = 0$.

Now suppose that $\ell \in \{1, \ldots, k - 1\}$ and assume that $\lim_{n \to \infty} LQ(\varphi^n(\sigma_\ell)) = 0$. Let $M_\ell$ be sufficiently large so that $LQ(\varphi^n(\sigma_{t_\ell})) < \tilde{c}/2$ for $n \geq M_\ell$. Since $LQ(\varphi^n(\sigma|[s_\ell, s_{\ell+1}])) \leq LQ(\varphi^n(\sigma|[s_\ell, s_{\ell+1}])) < \tilde{c}/2$ for all $n \geq 1$, it follows that $LQ(\varphi^n(\sigma_{t_{\ell+1}})) < \tilde{c}$ for $n \geq M_\ell$. Then $\text{tr}(\varphi^n(\sigma_{t_{\ell+1}})) \subset T^1N_1$ for $n \geq M_\ell$. By applying part (1), we obtain $\lim_{n \to \infty} LQ(\varphi^n(\sigma_{t_{\ell+1}})) = 0$, which completes the inductive proof.

**Corollary 6.4.** There exists $r_0 > 0$ such that if $\sigma(\cdot), s \in [0, a], a > 0$, is an approximately unstable curve and there exist $v_0 = \sigma(s_0)$ for some $s_0 \in [0, a]$ and a sequence $(t_n)_{n=1,2,\ldots}$ with $t_1 \geq 1, t_{n+1} \geq 1 + t_n$ for $n = 1, 2, \ldots$, and $\varphi^n(v_0) \in T^1N_2$ for $n = 1, 2, \ldots$, then $\lim_{n \to \infty} LQ(\varphi^n(\sigma)) > r_0$.

**Proof.** Let $b = \text{dist}(T^1N_2, \partial(T^1N_1)) > 0$ and let $r_0 = \min(1/2, b/2)$. Let $\sigma(\cdot), s \in [0, a], a > 0$, be an approximately unstable curve and let $v_0 = \sigma(s_0)$ and $(t_n)_{n=1,2,\ldots}$ be as in the hypothesis of the corollary. If $\text{tr}(\varphi^n(\sigma))$ is not contained in $T^1N_1$, then $LQ(\varphi^n(\sigma)) \geq b > r_0$. Thus, we may assume that $\text{tr}(\varphi^n(\sigma)) \subset T^1N_1$ for $n = 1, 2, \ldots$.

If $p(s)$ is a curve in $S$ and $\alpha(s)$ is a curve in $T^1S$ such that $\alpha(s) \in T^1\varphi^0(S)$, then we let $\alpha_{\text{rev}}$ denote the curve in $T^1\varphi^1(S)$ with the same basepoints as $\alpha$, but the vectors $\alpha(s) \in T^1\varphi^0(S)$ are replaced by $-\alpha(s) \in T^1\varphi^0(S)$. Since $\mathcal{K}_v^0 = \mathcal{K}_v^+$ and $\mathcal{K}_v^0 = \mathcal{K}_v^-$ for $v \in T^1N_1$, a $C^1$ curve $\alpha$ in $T^1N_1$ is approximately stable [stable] if and only if $\alpha_{\text{rev}}$ is approximately unstable [stable]. Note that $\varphi^i((\varphi^i(\alpha))_{\text{rev}}) = \alpha_{\text{rev}}$ for all $t \in \mathbb{R}$.

Since $\sigma$ is approximately unstable, so is $\varphi^n(\sigma)$, for $n = 1, 2, \ldots$. Let $\epsilon = L(\varphi^1(\sigma)) > 0$, and let $M = M(\epsilon)$ be as in part (1) of Lemma 6.3. Suppose there exists $m \geq M$ such that $LQ(\varphi^m(\sigma)) < 2r_0 \leq 1$. Let $\tilde{\sigma} = (\varphi^m(\sigma))_{\text{rev}}$ and let $t_0 = 0$.
Then for $0 \leq k \leq m$, we have $(\varphi^{t_k}(\sigma))_{\text{rev}} = \varphi^{t_m-t_k}(\tilde{\sigma})$. Applying part (1) of Lemma 6.3 to $\tilde{\sigma}$ (instead of $\sigma$), using the sequence $0, t_m - t_{m-1}, t_m - t_{m-2}, \ldots, t_m - t_1, t_m$ (instead of $t_1, t_2, \ldots, t_{m+1}$), and noting that $\varphi^{t_m}(\tilde{\sigma}) = \sigma_{\text{rev}}$ is approximately stable, we obtain $L((\varphi^{t_k}(\sigma))_{\text{rev}}) = L(\varphi^{t_m-t_k}(\tilde{\sigma})) < \epsilon L(\tilde{\sigma})) < \epsilon$, which is a contradiction, since $L((\varphi^{t_k}(\sigma))_{\text{rev}}) = L(\varphi^{t_k}(\sigma)) = \epsilon$. Thus $L(\varphi^{t_k}(\sigma)) \geq 2\sigma_0$ for all $m \geq M$. \qed

Lemma 6.5 below provides a comparison between the Euclidean curvature and the curvature within a manifold $(\tilde{M}, \tilde{\rho})$ for a given curve $\alpha$ in a neighborhood of a geodesic $\gamma$. In our application of this lemma, $\gamma$ will be replaced by $\tilde{\rho}$, and $\tilde{h}$ will be replaced by $h$. If $\alpha(t)$ is a regular $C^2$ curve in the Euclidean plane and $N(t)$ is a unit normal field along $\alpha(t)$, then the signed curvature of $\alpha$ with respect to $N$ is defined by

$$\kappa(t) = \frac{-\alpha''(t) \cdot N(t)}{|\alpha'(t)|^2}$$

The choice of sign is consistent with that in equation (6.2).

Suppose $a < b$ and $\gamma : [a, b] \to M$ is a geodesic in a complete Riemannian surface $(\tilde{M}, \tilde{\rho})$ such that $\gamma$ is either one-to-one on $[a, b]$ or $\gamma$ is a closed geodesic that is one-to-one on $[a, b)$. In the former case, let $I = [a, b]$, and in the latter case, let $I = [a, b]/\sim$, where $a \sim b$. Let $(\tau_1, \tau_2)$ be Fermi coordinates along $\gamma$, where $\tau_1$ is the parameter along $\gamma$ and $\tau_2$ is the parameter along geodesics perpendicular to $\gamma$. Let $\epsilon_1 > 0$ be sufficiently small so that points that are within distance at most $\epsilon_1$ from $\text{tr}(\gamma)$ have a unique representation in Fermi coordinates $(\tau_1, \tau_2) \in I \times [-\epsilon_1, \epsilon_1]$. For $\epsilon$ with $0 < \epsilon \leq \epsilon_1$, define

$$\overline{F}(\epsilon) = \{ (\tau_1, \tau_2) : \tau_1 \in I, -\epsilon \leq \tau_2 \leq \epsilon \}.$$

On $\overline{F}(\epsilon)$ we have the Riemannian metric induced by $\tilde{\rho}$ through the identification of points within distance at most $\epsilon$ of $\text{tr}(\gamma)$ in $(\tilde{M}, \tilde{\rho})$ and their Fermi coordinates $(\tau_1, \tau_2)$. We also have the Euclidean metric $\langle \partial/\partial \tau_i, \partial/\partial \tau_j \rangle = \delta_{ij}$, for $1 \leq i, j \leq 2$ on $\overline{F}(\epsilon)$. For a regular $C^2$ curve $\alpha(s)$, $-\ell \leq s \leq \ell$, in $\overline{F}(\epsilon)$ we may compare the signed curvature of $\alpha$ in these two metrics. We will use the subscripts $\tilde{h}$ and $\epsilon$ to distinguish inner products with respect to the metric $\tilde{h}$ and the Euclidean metric.

**Lemma 6.5.** Suppose $\gamma : [a, b] \to M$ is a geodesic on a complete Riemannian surface $(\tilde{M}, \tilde{\rho})$ such that $\gamma$ is either one-to-one on $[a, b]$ or $\gamma$ is a closed geodesic that is one-to-one on $[a, b)$. Suppose $I$, $\epsilon_1$, Fermi coordinates $(\tau_1, \tau_2)$ along $\gamma$, and $\overline{F}(\epsilon)$ for $0 < \epsilon \leq \epsilon_1$, are as defined above. Let $\kappa_0 > 0$ and let $0 < \zeta < 1$. Then there exists $\epsilon > 0$, depending only on $\gamma$, $\tilde{\rho}, \epsilon_1, \kappa_0$, and $\zeta$, such that the following properties hold for any regular $C^2$ curve, $\alpha(s) = (\tau_1(s), \tau_2(s))$, $-\ell \leq s \leq \ell$, in $F(\epsilon)$.

1. If $N_{\overline{h}}(s)$ and $N_\epsilon(s)$ are continuous unit normal fields along $\alpha(s)$ for the metric $\overline{h}$ and the Euclidean metric, respectively, then $N_{\overline{h}}(s)$ and $N_\epsilon(s)$ lie on the same side of the tangent line to $\alpha$ at $s$ when $s = 0$ (and therefore, by continuity, for all $s \in [-\ell, \ell]$) if and only if the same choice of sign replaces $\pm$ in (6.12) as in (6.13) below.

2. Suppose $N_{\overline{h}}(s)$ and $N_\epsilon(s)$ are as in part (1), lying on the same side of the tangent lines to $\alpha$. Let the signed curvatures $k(s)$ and $\kappa(s)$, denote, respectively, the signed curvature of $\alpha(s)$ with respect to $N_{\overline{h}}(s)$ on $(\overline{F}(\epsilon), \overline{h})$ (as defined in (6.2) ) and the signed curvature of $\alpha(s)$ with respect $N_\epsilon(s)$.
in the Euclidean metric on $\overline{F}(\epsilon)$ (as defined in (6.3)). If $|k(s)| \geq k_0$ for all $s \in [-\ell, \ell]$, then
\[
1 - \zeta < \frac{\kappa(s)}{k(s)} < 1 + \zeta, \text{ for all } s \in [-\ell, \ell].
\]

Proof. Since the curvatures (in either metric) and the unit normal fields along $\alpha$ are invariant under reparametrization, we may assume that $\alpha(s)$ is a unit speed curve in $(\overline{F}(\epsilon), \overline{h})$. Note that in Fermi coordinates $(\tau_1, \tau_2)$, if $\overline{h}_{ij} = \langle \partial/\partial \tau_i, \partial/\partial \tau_j \rangle$, and $\Gamma^m_{ij}$ are the Christoffel symbols for $\overline{h}$, then
\[
\overline{h}_{11}(\tau_1, 0) = 1, \quad \overline{h}_{12}(\tau_1, \tau_2) = 0, \quad \overline{h}_{22}(\tau_1, \tau_2) = 1,
\]
\[
\frac{\partial \overline{h}_{11}}{\partial \tau_2}(\tau_1, 0) = 0, \quad \frac{\partial \overline{h}_{12}}{\partial \tau_2}(\tau_1, \tau_2) = 0, \quad \frac{\partial \overline{h}_{22}}{\partial \tau_2}(\tau_1, \tau_2) = 0,
\]
\[
(6.7)
\overline{h}_{ij}(\tau_1, \tau_2) = \delta_{ij} + o(|\tau_2|),
\]
and
\[
(6.8)
\Gamma^m_{ij}(\tau_1, \tau_2) = O(|\tau_2|).
\]
for $i, j, m \in \{1, 2\}$. Here, and throughout this proof, $O(|\tau_2|)$ and $o(|\tau_2|)$ denote functions whose absolute values are bounded, respectively, by constant multiples of $|\tau_2|$ and $\tau_2^2$, where the constants can be chosen independently of $\alpha$ and $\epsilon$.

Since
\[
(6.9)
\alpha'(s) = \tau'_1(t) \frac{\partial}{\partial \tau_1} + \tau'_2(t) \frac{\partial}{\partial \tau_2}
\]
and $< \alpha'(s), \alpha'(s) >_\overline{h} = 1$, we have
\[
(6.10)
\overline{h}_{11}(\tau'_1)^2 + (\tau'_2)^2 = 1.
\]
It follows that $|\tau'_1| \leq (\overline{h}_{11})^{-1/2}$ and $|\tau'_2| \leq 1$. From (6.7) and (6.10), we obtain
\[
(6.11)
(\tau'_1)^2 + (\tau'_2)^2 = 1 + o(|\tau_2|).
\]
Note that
\[
(6.12)
N_{\overline{h}}(s) = \pm \left[ -\frac{-\tau'_2(s)}{\sqrt{\overline{h}_{11}}} \frac{\partial}{\partial \tau_1} + \tau'_1(s) \sqrt{\overline{h}_{11}} \frac{\partial}{\partial \tau_2} \right],
\]
where the same choice of $\pm$ is made for all $s$. For the Euclidean normal vector, we have
\[
(6.13)
N_\epsilon(s) = \pm \left[ -\frac{-\tau'_2(s)}{\sqrt{(\tau'_1(s))^2 + (\tau'_2(s))^2}} \frac{\partial}{\partial \tau_1} + \tau'_1(s) \sqrt{(\tau'_1(s))^2 + (\tau'_2(s))^2} \frac{\partial}{\partial \tau_2} \right],
\]
also with the same choice of $\pm$ for all $s$. If we make the same choice of $\pm$ in (6.12) as in (6.13), then it follows from (6.10) and (6.7) for $(i, j) = (1, 1)$ that
\[
(6.14)
\angle(N_{\overline{h}}(s), N_\epsilon(s)) = o(|\tau_2|),
\]
where $\angle$ denotes the angle measured in the Euclidean coordinate system. In this case, if $\epsilon$ is sufficiently small and $|\tau_2| < \epsilon$, then $N_{\overline{h}}(s)$ and $N_\epsilon(s)$ lie on the same side of the tangent line to $\alpha$ at $s$. Conversely, if we had made the opposite choice of signs in (6.12) from that in (6.13), then $N_{\overline{h}}(s)$ and $N_\epsilon(s)$ would lie on opposite sides of the tangent line to $\alpha$ at $s$. The assertion in part (1) follows.
In the proof of part (2), we may assume that $\pm$ in both (6.12) and (6.13) is taken to be $+$. Taking the covariant derivative $D/ds$, with respect to the metric $\tilde{h}$, of (6.9), we obtain
\[
\frac{D}{ds}\alpha'(s) = \sum_{m=1}^{2} \left( r''_m(s) + \sum_{1 \leq i, j \leq 2} \Gamma_1^{p}_{i, j}(s) r'_i(s) r'_j(s) \right) \frac{\partial}{\partial r_m}.
\] (6.15)

Then the signed curvature of $\alpha(s)$ with respect to $N_h$ in the metric $\tilde{h}$ is
\[
k(s) = -\left( \frac{D}{ds} \alpha'(s), N_h(s) \right) \frac{\partial}{\partial r_m}.
\] (6.16)

Next we find the signed curvature $\kappa(s)$ of $\alpha(s)$ with respect to $N_e$ in the Euclidean metric. By (6.4) and (6.13), we have
\[
\kappa(s) = \frac{-(\alpha''(s), N_e(s))}{||\alpha'(s)||^2_e} = \frac{r''_1 r'_2 - r'_1 r''_2}{((r'_1)^2 + (r'_2)^2)^{3/2}}.
\] (6.17)

If $|\kappa(s)| \geq k_0$ and $\epsilon$ is sufficiently small, then (6.6) follows from (6.11), (6.16), and (6.17). □

In Proposition 6.6 below, we will consider the closure of an $\epsilon_0$-tubular neighborhood of $\tilde{\rho}$, $(\tilde{F}(\epsilon_0), \tilde{h})$, as described in Section 5. Either component of $\tilde{F}(\epsilon_0) \setminus \text{tr}(\tilde{\rho})$ could serve as the region in which the second of the Fermi coordinates $(\tau_1, \tau_2)$ is positive, depending on how the Fermi coordinates are chosen. For any geodesic $\gamma : [a, b] \rightarrow S$, we let $-\gamma$ denote the geodesic $\gamma$ transversed in the opposite direction: $(-\gamma)(t) = \gamma(a + b - t)$, for $a \leq t \leq b$.

**Proposition 6.6.** If $\rho : [0, L] \rightarrow S$ is the closed geodesic constructed in Section 2 and $x \in S$, then there exist $v_+, v_- \in T^1_x S$ such that $\gamma_{v_+}(t)$ and $\gamma_{v_-}(t)$ are asymptotic to $\rho$ and $-\rho$, respectively, as $t \rightarrow \infty$. If $x \in \cup_{i=1}^N \mathcal{D}_i$, then $v_+$ and $v_-$ can be chosen to be approximately in the radial direction (see Definition 5.12). In addition, if we are given a choice of Fermi coordinates $(\tau_1, \tau_2)$ on $(\tilde{F}(\epsilon_0), \tilde{h})$, then we can choose $v_+$ and $v_-$ so that for every $\epsilon > 0$ there exists $T = T(\epsilon) > 0$ such that $\gamma_{v_+}||T, \infty)$ and $\gamma_{v_-}||T, \infty)$ have lifts $\tilde{\gamma}_{v_+}||T, \infty)$ and $\tilde{\gamma}_{v_-}||T, \infty)$, respectively, to $(\tilde{F}(\epsilon_0), \tilde{h})$ such that
\[
0 < \tau_2(\tilde{\gamma}_{v_+}(t)) < \epsilon \text{ and } 0 < \tau_2(\tilde{\gamma}_{v_-}(t)) < \epsilon, \text{ for } t \geq T.
\]

**Proof.** Let $x \in S$ and let $\sigma : [-a_0, a_0] \rightarrow T^1_x S$, $a_0 > 0$, be a regular $C^1$ curve (with constant basepoint $x$). If $x \in \cup_{i=1}^N \mathcal{D}_i$, then $\sigma$ is chosen so that $\sigma(s)$ is approximately in the radial direction for all $s \in [-a_0, a_0]$. We will show that for some $s_2 \in (-a_0, a_0)$, the vector $\sigma(s_2)$ satisfies the conditions required of $v_+$. Let $R$ be as in Lemma 6.11 and let $t_0 > R$ be such that $x$ is not conjugate to $\gamma_{\sigma(t_0)}(t_0)$ along $\gamma_{\sigma(t_0)}[0, t_0]$. Let $0 < a < a_0/2$ and $0 < b < (t_0 - R)/2$, and assume $a, b$ are...
sufficiently small so that \(\exp_x : \{t \sigma(s) \in T_x S : -2a < s < 2a, t_0 - 2b < t < t_0 + 2b\} \to S\) is a diffeomorphism onto its image. Define
\[
\mathcal{A}_0 = \mathcal{A}_0(a, b, t_0) = \{\varphi^t(\sigma(s)) : -a < s < a, t_0 - b < t < t_0 + b\}
\]
and
\[
\mathcal{A}_1 = \mathcal{A}_1(a, b, t_0) = \{\varphi^t(\sigma(s)) : -a/2 \leq s \leq a/2, t_0 - (b/2) \leq t \leq t_0 + (b/2)\}.
\]
Note that \(\varphi^t(\sigma(s)) \subset (T^1 S) \setminus (\cup_{i=1}^q Z_i)\), for \(s \in [-a_0, a_0]\) and \(t \geq 0\) (where \(Z_i\) is in Definition 3.8), because \(\varphi(s) \notin (\cup_{i=1}^q Z_i)\) and \(\varphi^t(\sigma(s)) \subset (T^1 S) \setminus (\cup_{i=1}^q Z_i)\) for all \(t \geq 0\). We require \(a, b\) to be sufficiently small so that there exist an open subset \(W\) of \(T^1 S\) with \(\overline{\mathcal{A}_0} \subset W \subset (T^1 S) \setminus (\cup_{i=1}^q Z_i)\) and a coordinate chart \(\Psi : W \to \mathbb{R}^3\) such that \(\Psi(\mathcal{A}_0)\) is an open subset of \(\mathbb{R}^2\). Here, and in the proof of Lemma 6.7, we identify \(\mathbb{R}^2\) with \(\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3\). For \(\eta > 0\), let \(A = A(\eta)\) be an \(\eta\)-neighborhood of \(\mathcal{A}_1\).

We will continue with the proof of Proposition 6.6 after Lemma 6.7 and Corollary 6.8.

**Lemma 6.7.** If \(\eta = \eta(a, b, W, \Psi)\) is sufficiently small, then every vector \(w \in A(\eta)\) can be joined to a vector in \(A_0\) by a stable curve with finite Lyapunov length.

**Proof.** By Lemma 3.11 the curves \(s \mapsto \varphi^s(\sigma(s)), -a_0 \leq s \leq a_0\), are approximately unstable for \(t \geq R\). Thus each tangent plane to \(\mathcal{A}_0\) is spanned by a vector tangent to an orbit of the geodesic flow and a vector in an unstable cone. By Lemma 3.11 the \(E^s\) line field is contained in the interiors of the stable cones at vectors in \(\mathcal{A}_0\). Hence Lemma 3.5 implies that \(E^s\) is transversal to the tangent plane to \(\mathcal{A}_0\) at each vector in \(\mathcal{A}_0\). Since \(E^s\) is continuous on \(W\), it follows that there exists \(c > 0\) such that \(\Psi(\mathcal{A}_0) \times (-c, c) \subset \Psi(W)\) and the line field \(d\Psi(E^s)\) is uniformly transversal to horizontal planes in \(\Psi(\mathcal{A}_0) \times (-c, c)\). Let \(\zeta > 0\) be such that within \(\Psi(\mathcal{A}_0) \times (-c, c)\) any unit vector along \(d\Psi(E^s)\) has a component of absolute value greater than \(\zeta\) in the vertical direction. Let \(h_0 = \text{dist}(\Psi(A_1), \Psi(\partial \mathcal{A}_0))\). We choose \(\eta\) sufficiently small so that \(A(\eta) \subset W \setminus \text{dist}(\Psi(A_1), \Psi(\partial \mathcal{A}_0)) > h_0/2\), and for any \((p_1, p_2, p_3) \in \Psi(A(\eta))\), we have \(|p_3| < \min(c, h_0\zeta/2)\). Let \(w \in A(\eta) \setminus \mathcal{A}_0\). Then \(\Psi(w) = (p_1, p_2, p_3)\), where \(p_3 \neq 0\). We suppose \(p_3 > 0\). Since \(d\Psi(E_s)\) is a continuous line field on \(d\Psi(W)\), there exists a unit speed curve \(\beta\) that is everywhere tangent to \(d\Psi(E^s)\) and starts at \(\beta(0) = p_3\) with \(\beta'(0)\) having a negative component in the vertical direction. Then \(\beta\) must reach \(\mathbb{R}^2 \times \{0\}\) in time less than \(p_3/\zeta\), unless it first exits the region \(\Psi(\mathcal{A}_0) \times \mathbb{R}\). But it cannot exit this region in time less than \(h_0/2\), and by our choice of \(\eta\), we have \(p_3/\zeta < h_0/2\). Thus, for some \(t_1 \in (0, p_3/\zeta)\), the curve \(\beta(t), 0 \leq t \leq t_1\), connects \((p_1, p_2, p_3)\) to a point in \(\Psi(\mathcal{A}_0)\). Then \(\beta(t) = \Psi^{-1}(\beta(t))\), 0 \leq t \leq t_1\), is a stable curve that joins \(w\) to a vector in \(\mathcal{A}_0\). Since \(\text{tr}(\beta)\) is a compact subset of \((T^1 S) \setminus (\cup_{i=1}^q Z_i)\), it follows from Remark 6.2 that \(\beta\) has finite Lyapunov length. The case \(p_3 < 0\) can be handled similarly. \(\square\)

**Corollary 6.8.** If \(U_0\) is a nonempty open subset of \(T^1 N_2\), where \(N_2\) is as in Lemma 6.3 then there exists \(s_0 \in (-a, a)\) and a sequence \((t_n)_{n=1,2,...}\) with \(t_1 \geq 0\) and \(t_{n+1} > 1 + t_n\) for \(n = 1, 2,\ldots\), such that \(\varphi^{s_n}(\sigma(s_0)) \in U_0\), for \(n = 1, 2,\ldots\).

**Proof.** Let \(U_0\) be a nonempty open subset of \(U_0\) such that \(U_1 \subset U_0\), and let \(\eta > 0\) be such that Lemma 6.7 holds for \(A = A(\eta)\). Then \(U_1\) and \(A\) each have positive Liouville measure. By the ergodicity of the geodesic flow \(\varphi^t\), we know that there exists \(w \in A\) and a sequence \((\tilde{t}_n)_{n=1,2,...}\) such that \(\tilde{t}_1 \geq 0\) and \(\tilde{t}_{n+1} > 1 + \tilde{t}_n\) for
n = 1, 2, . . . , and φ^i_n(w) ∈ U_0 for n = 1, 2, . . . . If w ∉ A_0 then by Lemma 6.7 there is a stable curve β from w to a vector v in A_0. The vector v can be written as v = φ^i_0(σ(s_0)) for some s_0 ∈ (−a, a) and t_0 − b < t_0 < t_0 + b. By part (2) of Lemma 6.3, lim_{n→∞} L(φ^i_n(β)) = 0. Thus there exists N such that t_{N+1} > t_0, and for n > N, φ^i_n(v) ∈ U_0. Then the Corollary holds with t_n = t_{N+n} + t_0. If w ∈ A_0, then we already have w = φ^i_0(σ(s_0)) for some s_0 ∈ (−a, a) and t_0 − b < t_0 < t_0 + b and the Corollary holds with t_n = t_{n+1} + t_0.

We now complete the proof of Proposition 6.6. Let ε_0 ∈ (0, 1), F(ε_0), and \( \hat{F}(\epsilon_0) \) be as described in Section 5 with Fermi coordinates (\( \tau_1, \tau_2 \)) along \( \hat{\rho} \) in the metric \( \hat{h} \) defined in \( [\mathbb{R}/L\mathbb{Z}] \times [-\epsilon_0, \epsilon_0] \). Let \( \hat{Z} \) be the unit vector field on \( \hat{F}(\epsilon_0) \) that is asymptotic to \( \hat{\rho} \), as in Section 5

Since F(ε_0) ⊂ S\((\cup_{i=1}^q D_i)\), it follows from (5.13) that if v ∈ T^1 S has its basepoint in F(ε_0) and dφ^(-1)v ∈ K^u_{\hat{F}(\epsilon_0)}, then dφ^(-1)(K^u_{\hat{F}(\epsilon_0)}) ⊂ int K^u_\epsilon. Moreover, by a compactness argument, there exist k_0, k_1, 0 < k_0 < k_1 < ∞, such that for all such v, dφ^(-1)(K^u_{\hat{F}(\epsilon_0)}) is contained in a cone in K^u_\epsilon that is bounded by lines of slopes k_0 and k_1 in the H, V coordinates. Therefore, if t ≥ R + 1 and σ_0 is the restriction of σ to a subinterval of [−a_0, a_0] such that \( \varphi^t\sigma_0 \) has all of its basepoints in F(ε_0), then the curvature of the curve of basepoints of \( \varphi^t\sigma_0 \) (with respect to the metric \( h \) and the normal field given by \( \varphi^t\sigma_0 \)) is in the interval [k_0, k_1], because \( \varphi^t\sigma_0 = \varphi^t(\varphi^(-1)\sigma_0) \) and \( \varphi^(-1)\sigma_0 \) is approximately unstable (by Remark 3.3). Likewise, the curve of basepoints of any lift of such a \( \varphi^t\sigma_0 \) to \( T^1(\hat{F}(\epsilon_0)) \) has curvature (with respect to the metric \( \hat{h} \)) in [k_0, k_1].

We now apply Lemma 6.5 with γ = \( \hat{\rho} \), \( \epsilon_1 = \epsilon_0 \), \( \bar{F}(\epsilon_0) = \hat{F}(\epsilon_0) \), and \( \bar{h} = \hat{h} \). Let \( \epsilon \in (0, \epsilon_0) \) be such that the conclusion of Lemma 6.5 holds for \( \zeta = 1/2 \) and \( k_0 \) as above. Let \( \kappa_0 = k_0/2 \) and \( \kappa_1 = (3/2)k_1 \), and let \( \hat{F}(\epsilon) \subset \hat{F}(\epsilon_0) \) be defined as in (5.2). By (6.14), there is a constant \( \hat{C}_0 > 0 \) such that along any regular \( C^2 \) curve \( \hat{a}(s) = (\tau_1(s), \tau_2(s)) \), 0 ≤ s ≤ \( \hat{\ell} \), in \( \hat{F}(\epsilon) \), the unit normal vectors \( \hat{N}_\epsilon \) and \( \hat{N}_\epsilon \) to \( \hat{a} \) (in the \( \hat{h} \) metric and the Euclidean metric, respectively) satisfy

\[
\angle(\hat{N}_\epsilon, N_\epsilon) ≤ \hat{C}_0 \tau_2^2,
\]

where \( \angle \) denotes the angle measured in the Euclidean coordinate system. Since, by (6.7), the ratio of the Euclidean length to the length in the \( \hat{h} \) metric is close to 1 for \( \tau_2 \) close to 0, we may also assume that \( \epsilon \) is sufficiently small so that

\[
||v||_{\hat{h}} ≤ 2||v||_\epsilon,
\]

for all \( v ∈ TS \) with basepoint in \( \hat{F}(\epsilon) \). After choosing \( \epsilon \), we choose \( \delta \) so that

\[
0 < \delta < \frac{1}{\hat{C}_0 + 2} \min \left( \epsilon, \frac{\epsilon \kappa_0}{2}, \frac{\kappa_0}{2k_1}, \frac{\tau_0 \kappa_0}{2 \sqrt{1 + k_1^2}} \right),
\]

where \( \tau_0 \) is as in Corollary 6.4.

Let \( \hat{U}_0 = \{ \hat{w} ∈ T^1_p(\hat{F}(\epsilon)) : p = (\tau_1, \tau_2), 0 < \tau_2 < \delta, \angle(\hat{Z}(p), \partial/\partial \tau_1) < \delta, \angle(\hat{w}, \hat{Z}(p)) < \delta, \angle(\hat{w}, \partial/\partial \tau_2) < \pi/2, \text{ and } \angle(\hat{w}, \partial/\partial \tau_1) > \angle(\hat{Z}(p), \partial/\partial \tau_1) \} \). For the rest of the proof of this proposition, the signed Euclidean angle from one vector to another at the same basepoint in \( \hat{F}(\epsilon) \) will be taken to be in \((-π, π]\) and the counterclockwise direction will be the positive direction. In particular, for \( \hat{w} ∈ \hat{U}_0 \cap T^1_p(\hat{F}(\epsilon)) \) the signed Euclidean angle from \( \hat{Z}(p) \) to \( \hat{w} \) is negative. Let \( \hat{U}_0 \)
be the image of $\hat{U}_0$ under the projection $\hat{\pi}$ from $T^1(\hat{F}(\epsilon))$ to $T^1(F(\epsilon))$. Let $s_0 \in (-a, a)$ and the sequence $(t_n)_{n=1,2,...}$ be as in Corollary 6.8 applied to this choice of $\hat{U}_0$. If $n$ is sufficiently large, then by Corollary 6.4 $\mathcal{L}(\varphi^{t_n}(\sigma|[s_0,a])) > r_0$ and $\mathcal{L}(\varphi^{t_n}(\sigma|(-a,s_0])) > r_0$. Now fix a choice of such an $n$, where we also require that $t_n \geq R + 1$.

Let $w = \varphi^{t_n}(\sigma(s_0))$ and let $\hat{w} \in \hat{U}_0$ be such that $\hat{w}$ projects to $w$. Let $\hat{\sigma}$ be the lift of $\varphi^{t_n}(\sigma(s))$, $s_0 \leq s \leq a_1$, to $T^1(\hat{F}(\epsilon))$, where $\varphi^{t_n}(\sigma(s_0))$ lifts to $\hat{w}$ and the curve $\hat{\sigma}$ is truncated, if necessary, at $s = a_1$, where $\hat{\sigma}$ exits $T^1(\hat{F}(\epsilon))$. (If it doesn’t exit $T^1(\hat{F}(\epsilon))$, we take $a_1 = a$.) Let $\hat{\alpha}(s)$, $s_0 \leq s \leq a_1$, be the curve of basepoints of $\hat{\sigma}$. Then $\hat{\alpha}'(s_0)$ is orthogonal to $\hat{w}$ in the $\hat{h}$ metric. Let $N_{\hat{h}}$ and $N_\epsilon$ be unit normal fields along $\hat{\alpha}(s)$ in the $\hat{h}$-metric and the Euclidean metric, chosen so that $N_{\hat{h}}(0) = \hat{w}$ and $N_\epsilon(s)$ lie on the same side of the tangent line to $\hat{\alpha}$ at all $s \in [0,a_1]$. Since $\mathcal{L}(\hat{w},\partial/\partial\tau_1) < 2\delta < \pi/2$ and $\mathcal{L}(\hat{w},-\partial/\partial\tau_2) < \pi/2$, $N_{\hat{h}}(0)$ has a positive component in the $\partial/\partial\tau_1$ direction and a negative component in the $\partial/\partial\tau_2$ direction, which, according to equations (6.12) and (6.13), implies that the same is true of $N_\epsilon(0)$. In particular, we see that $\hat{\alpha}'(s_0)$ must have a nonzero component in the $\partial/\partial\tau_2$ direction. We will assume that this component is positive. (If not, we would replace $\varphi^{t_n}(\sigma|[s_0,a])$ by $\varphi^{t_n}(\sigma|(-a,s_0])$ in our argument.) We may assume that the parametrization of $\sigma$ is such that $s_0 = 0$ and $\hat{\alpha}(s)$ is parametrized by Euclidean arc length.

By the choice of $\epsilon$ and by (6.13), we know that $\mathcal{L}(N_\epsilon(0),N_\epsilon(0)) < C_\delta \delta^2 < C_\epsilon$. Thus $\mathcal{L}(N_\epsilon(0),\partial/\partial\tau_1) < (C_\epsilon + 2)\delta$. According to Lemma 6.5, the Euclidean curvature of $\hat{\alpha}(s)$ is between $\kappa_0$ and $\kappa_1$ for $0 \leq s \leq a_1$. Thus $N_\epsilon(s)$ rotates in the counterclockwise direction at a rate between $\kappa_0$ and $\kappa_1$ radians per unit time. For $s \in [0,\min(a_1, (C_\epsilon + 2)\delta/\kappa_0)]$, the signed Euclidean angle from $\partial/\partial\tau_1$ to $N_\epsilon(s)$, is strictly between $-(C_\epsilon + 2)\delta + \kappa_0 s$ and $\kappa_1 s$, which implies it is strictly between $-(C_\epsilon + 2)\delta$ and $\pi/2$. Thus $\hat{\alpha}'(s)$ has a non-zero component in the $\partial/\partial\tau_2$ direction for $s \in [0,\min(a_1, (C_\epsilon + 2)\delta/\kappa_0)]$. In fact, this component must be positive, because $\hat{\alpha}'(0)$ has a positive component in the $\partial/\partial\tau_2$ direction. Since the component of $\hat{\alpha}'(s)$ in the $\partial/\partial\tau_2$ direction is at most 1, and $0 < \tau_2(\hat{\alpha}(0)) < \delta < \epsilon/2$, we obtain $0 < \tau_2(\hat{\alpha}(s)) < \epsilon/2 + (C_\epsilon + 2)\delta/\kappa_0 < \epsilon$ for $s \in [0,\min(a_1, (C_\epsilon + 2)\delta/\kappa_0)]$.

![Figure 6.1. Rotation of $N_{\hat{h}}$.](image-url)
The length of $\hat{\sigma}(s)$, $0 \leq s \leq \min(a_1, (C_0 + 2)\delta/\kappa_0)$, in $T^1S$ with the metric induced by $\hat{h}$, is at most $(1 + k_1^2)1/2$ times the $\hat{h}$-length of $\hat{\alpha}(s)$, $0 \leq s \leq \min(a_1, (C_0 + 2)\delta/\kappa_0)$. By (6.19), the $\hat{h}$-length of $\hat{\alpha}(s)$, $0 \leq s \leq \min(a_1, (C_0 + 2)\delta/\kappa_0)$, is at most $2(C_0 + 2)\delta/\kappa_0$. Thus the length of $\hat{\sigma}(s)$, $0 \leq s \leq \min(a_1, (C_0 + 2)\delta/\kappa_0)$, is at most $2(1 + k_1^2)/(C_0 + 2)\delta/\kappa_0$, which is less than $r_0$. Thus $\hat{\sigma}(s)$ can neither exit $\tilde{T}^1_1(F(\varepsilon))$ nor reach length $r_0$ by time $s = \min(a_1, (C_0 + 2)\delta/\kappa_0)$. Hence $a_1 \geq (C_0 + 2)\delta/\kappa_0$.

Since $N_\varepsilon(s)$ rotates counter-clockwise at a rate of at least $\kappa_0$ radians per unit time, there is an $s_1 \in (0, (C_0 + 2)\delta/\kappa_0)$ at which $N_\varepsilon(s)$ has a positive component in the $\partial/\partial \tau_2$ direction, and by (6.12), $N_\varepsilon(s_1)$ also has a positive component in the $\partial/\partial \tau_2$ direction. Moreover, $s_1$ may be chosen so that $N_\varepsilon(s)$ and $N_\rho(s)$ have positive components in the $\partial/\partial \tau_1$ direction for all $s \in [0, s_1]$. Each vector in $\tilde{Z}$ with basepoint in the $\tau_2 > 0$ region has a positive component in the $\partial/\partial \tau_1$ direction and a negative component in the $\partial/\partial \tau_2$ direction. Thus the signed Euclidean angle from $\tilde{Z}_\alpha(s)$ to $N_\varepsilon(s)$ changes from being negative at $s = 0$ to being positive at $s = s_1$. Both $N_\rho(s)$ and $\tilde{Z}_\alpha(s)$ are continuous unit vector fields along $\hat{\alpha}(s)$, $0 \leq s \leq s_1$. By the intermediate value theorem, there exists $s_2 \in (0, s_1)$ such that $N_\rho(s_2) = \tilde{Z}_\alpha(s_2)$.

Since $N_\varepsilon(s_2)$ is a lift to $T^1_1(F(\varepsilon))$ of $\varphi^{t_n}(\sigma(s_2))$, it follows that $v_+ = \sigma(s_2)$ has the required properties. The existence of $v_-$ also follows, since we may replace $\rho$ by $-\rho$ in the above proof. 

**Proposition 6.9.** Let $\rho$ be the closed geodesic constructed in Section 6. For each $(x, y) \in S \times S$ there exists an infinite family of distinct geodesics $\gamma_n : [0, L_n] \to S$, $n = 1, 2, \ldots$, from $x$ to $y$ with $\lim_{n \to \infty} L_n = \infty$ satisfying the following: for every $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that for $n \geq N$ and $t \in [T, L_n - T]$, we have $\operatorname{dist}(\gamma_n(t), \operatorname{tr}(\rho)) < \varepsilon$. Suppose $\tilde{\rho}$ and $(\hat{F}(\varepsilon), \hat{h})$ are as in Section 5 and we are given a choice of Fermi coordinates $(\tau_1, \tau_2)$ on $(\tilde{F}(\varepsilon), \tilde{h})$ such that $\tau_1$ is the coordinate along $\tilde{\rho}$ and $\tau_2$ is the coordinate along geodesics perpendicular to $\tilde{\rho}$. Then the geodesics $\gamma_n$ can be chosen so that if $0 < \varepsilon < \varepsilon_0$ and $n > N$, there exists a lift $\tilde{\gamma}_n|\tau_1, L_n - T]$ of $\gamma_n|\tau_1, L_n - T]$ to $\tilde{F}(\varepsilon_0)$ such that $0 < \tau_2(\tilde{\gamma}_n(t)) < \varepsilon$, for $t \in [T, L_n - T]$.

**Proof.** Let $x, y \in S$, and for $0 < \varepsilon \leq \varepsilon_0$, let $F(\varepsilon)$, $\tilde{F}(\varepsilon)$ be the closures of the $\varepsilon$-tubular neighborhoods of $\rho$, $\tilde{\rho}$, $\hat{\rho}$, respectively, as defined in Section 5. Suppose we are given a choice of Fermi coordinates $(\tau_1, \tau_2)$ along $\tilde{\rho}$ on $\tilde{F}(\varepsilon_0)$. Let $(\varepsilon_n)_{n=1,2,\ldots}$ be any sequence with $\varepsilon_n \downarrow 0$ and $\varepsilon_1 < \varepsilon_0$. We will construct a sequence of geodesics $\gamma_n : [0, L_n] \to S$, $n = 1, 2, \ldots$, from $x$ to $y$ and sequences $(T_n)_{n=1,2,\ldots}$ with $0 < T_n < L_n/2$, $T_n \uparrow \infty$, such that

$$\operatorname{dist}(\gamma_n(t), \operatorname{tr}(\rho)) < \varepsilon_m, \text{ for } t \in [T_m, L_n - T_m]\text{ and } n \geq m \geq 1.$$

Moreover, we will show that there is a lift $\tilde{\gamma}_n$ of $\gamma_n|\tau_1, L_n - T_1]$ to $\tilde{F}(\varepsilon_0)$ such that

$$0 < \tau_2(\tilde{\gamma}_n(t)) < \varepsilon_m, \text{ for } t \in [T_m, L_n - T_m] \text{ and } n \geq m \geq 1.$$

This will imply the conclusion of the proposition, because for any given $\varepsilon > 0$, we may choose $m$ such that $\varepsilon_m < \varepsilon$, and let $N(\varepsilon) = m$ and $T(\varepsilon) = T_m$.

By Proposition 6.6 there exist vectors $v_x \in T^1_xS$, $v_y \in T^1_yS$ such that $\gamma_{v_x}$ and $\gamma_{v_y}$ are asymptotic to $\rho$ and $-\rho$, respectively, and for $T$ sufficiently large, the second Fermi coordinate of the lifts of $\gamma_{v_x}|\tau_1, \infty)$ and $\gamma_{v_y}|\tau_1, \infty)$ to $\tilde{F}(\varepsilon_0)$ is always positive. Let $\sigma_x : [0, a_0] \to T^1_xS$ and $\sigma_y : [0, b_0] \to T^1_yS$ for some $a_0, b_0 > 0$, be one-to-one
regular $C^1$ curves with $\sigma_x(0) = v_x$, $\sigma_y(0) = v_y$, such that for all sufficiently large $t$, the derivative with respect to $s$ at $s = 0$ of the curve of basepoints of $\varphi^t(\sigma_x(s))$ has a positive component in the $\partial/\partial s_2$ direction, and similarly for $\varphi^t(\sigma_y(s))$. If $x \in \bigcup_{i=1}^b D_i$, then $v_x$ and the curve $\sigma_x$ can be chosen so that $\sigma_x(s)$ is approximately in the radial direction for all $s \in [0, a_i]$, and similarly if $y \in \bigcup_{i=1}^b D_i$. Thus the curves $\varphi^t\sigma_x$ and $\varphi^t\sigma_y$ are approximately unstable for $t \geq R$, where $R$ is as in Lemma 3.11.

Let $T_0 = R$, and let $a_0$ and $b_0$ be as above. We will choose $T_n$, $\bar{t}_{x,n}$, $\bar{t}_{y,n}$, $a_n$, and $b_n$ inductively so that for $n = 1, 2, \ldots$, we have the following: $T_n > T_{n-1}$, $T_n > \max(\bar{t}_{x,n}, \bar{t}_{y,n})$, $0 < a_n < a_{n-1}$, and $0 < b_n < b_{n-1}$. Further conditions on these parameters will be imposed below. We require $T_n$ to be sufficiently large so that $\gamma_{x,v}[T_n, \infty)$, $\gamma_{y,v}[T_n, \infty)$ have lifts $\tilde{\gamma}_{x,v}, \tilde{\gamma}_{y,v}$, respectively, in $F(\epsilon_0)$ so that $\text{dist}(\tilde{\gamma}_{x,v}(t), \text{tr}(\bar{\rho})) < \epsilon_n/2$ and $\text{dist}(\tilde{\gamma}_{y,v}(t), \text{tr}(\bar{\rho})) < \epsilon_n/2$ for $t \geq T_n$. We let $\bar{\sigma}_{x,v}$, $\bar{\sigma}_{y,v}$ be lifts to $F(\epsilon_0)$ of $\varphi^{T_n}(\sigma_x[0, a_n])$, $\varphi^{T_n}(\sigma_y[0, b_n])$, respectively, where $a_n$, $b_n$ are chosen so that the curve of basepoints of $\bar{\sigma}_{x,v}(s)$, for $0 \leq s \leq a_n$, and the curve of basepoints of $\bar{\sigma}_{y,v}(s)$, for $0 \leq s \leq b_n$, have length less than $\epsilon_n/2$. Later we will impose an additional condition relating the choices of the lift $\bar{\sigma}_{x,v}$ and the lift $\bar{\sigma}_{y,v}$. We let $\tilde{\gamma}_{x,v}$ and $\tilde{\gamma}_{y,v}$ be lifts of $\gamma_{x,v}[T_n, \infty)$ and $\gamma_{y,v}[T_n, \infty)$ chosen so that $\tilde{\gamma}_{x,v}'(T_n) = \bar{\sigma}_{x,v}(0)$ and $\tilde{\gamma}_{y,v}'(T_n) = \bar{\sigma}_{y,v}(0)$. We will show that there exist $\tilde{t}_{x,n}, \tilde{t}_{y,n} > T_n$ and $\bar{a}_n \in (0, a_n)$, $\bar{b}_n \in (0, b_n)$ such that $\gamma_{x,v}(\bar{a}_n)[0, \tilde{t}_{x,n}]$ joins smoothly to $-\langle \gamma_{x,v}(\bar{a}_n)[0, \tilde{t}_{x,n}] \rangle$ at $\gamma_{x,v}(\bar{a}_n)(\tilde{t}_{x,n}) = \gamma_{x,v}(\bar{a}_n)(\bar{t}_{x,n})$, to form a geodesic $\gamma_n$ from $x$ to $y$ of length $L_n > 2T_n$. Our construction will be such that dist($\gamma_n(t), \text{tr}(\bar{\rho})) < \epsilon_m$ for $t \in [T_m, L_n - T_m]$, for $1 \leq m \leq n$.

Since $\sigma_x$ and $\sigma_y$ and their images under $\varphi^t$, for $t \geq R$, are approximately unstable, the curvatures of $\tilde{\sigma}_{x,v}$ and $\tilde{\sigma}_{y,v}$ are positive. For $t \geq T_n$, let $E_{x,v}(t)$ be a unit normal field along $\tilde{\gamma}_{x,v}(t)$ chosen so that $E_{x,v}(T_n)$ is in the same direction as the derivative at $s = 0$ of the curve of basepoints of $\tilde{\sigma}_{x,v}(s)$. Let $J_{x,v}(t) = J_{x,v}(t)E_{x,v}(t)$ be the Jacobi field $J_{x,v}(t) = (d/ds)|_{s=0}\gamma_{x,v}(s)(t-T_n)$, for $t \geq T_n$. Since $J_{x,v}(T_n)/J_{x,v}(T_n)$ is equal to the curvature at $s = 0$ of the curve $s \mapsto \gamma_{x,v}(s)(0)$ (as explained in Section 3) and $J_{x,v}(T_n) > 0$, we have $J_{x,v}(T_n) > 0$. Therefore, if we extend the geodesic $\tilde{\gamma}_{x,v}[T_n, \infty)$ for $t < T_n$ up to the point where $\tilde{\gamma}_{x,v}$ exits $\tilde{F}(\epsilon_0)$ and let $\text{tr}(\tilde{\gamma}_{x,v})$ denote the trace of this extended geodesic, then for $a_n$ sufficiently small
and \( s \in (0, a_n] \), the distance between \( \gamma_{\sigma_{x,n}(a_n)}(t) \) and \( \text{tr}(\gamma_{\sigma_{x,n}(0)}) \) is an increasing function of \( t \) near \( t = 0 \). By the convexity of this function, it follows that \( \gamma_{\sigma_{x,n}(s)}(t) \) must leave \( \tilde{F}(\epsilon_n) \) at some \( t_{x,n,s} > 0 \).

The geodesic \( \tilde{\gamma}_{x,n}(t), \ t \geq T_n \), and the geodesics \( \gamma_{\sigma_{x,n}(s)}(t), \ 0 \leq t \leq t_{x,n,s}, \ 0 < s \leq a_n \), form a foliation of a region \( \mathcal{R}_{x,n} \) in \( \tilde{F}(\epsilon_n) \) bounded on four sides by \( \tilde{\gamma}_{x,n}(t) \), \( t \geq T_n \); the curve of basepoints of \( \sigma_{x,n}(s), \ 0 \leq s \leq a_n ; \gamma_{\sigma_{x,n}(a_n)}(t), \ 0 \leq t \leq t_{x,n,a_n} \); and \{ \( \tau_1, \tau_2 \) \in \( \tilde{F}(\epsilon_n) : \tau_1 \geq \tau_{x,n}, \tau_2 = \epsilon_n \} \), where \( \tau_{x,n} \) is the \( \tau_1 \)-coordinate of \( \gamma_{\sigma_{x,n}(a_n)}(t) \) at the time it leaves \( \tilde{F}(\epsilon_n) \). The geodesics in this foliation cannot intersect each other, because the curvature is negative in \( \tilde{F}(\epsilon_n) \), which implies that there are no focal points in \( \tilde{F}(\epsilon_n) \). Let \( X_n \) be the unit vector field on \( \mathcal{R}_{x,n} \) consisting of the geodesics forming this foliation. There exists \( \mathcal{F}_{x,n} \) such that the region \( \mathcal{R}_{x,n} \) contains all points in \( \tilde{F}(\epsilon_n) \) with \( \tau_1 \geq \mathcal{F}_{x,n} \) that lie above \( \text{tr}(\tilde{\gamma}_{x,n}) \). Similarly, we may construct a vector field \( Y_n \) on a region \( \mathcal{R}_{y,n} \) in \( \tilde{F}(\epsilon_n) \) foliated by the geodesic \( \tilde{\gamma}_{y,n}(t), \ t \geq T_n \), and the geodesics \( \gamma_{\sigma_{y,n}(s)}(t), \ 0 \leq t < t_{y,n,s}, \ 0 < s \leq b_n \), where \( t_{y,n,s} \) is the time at which \( \gamma_{\sigma_{y,n}(s)}(t) \) exits \( \tilde{F}(\epsilon_n) \). There exists \( \mathcal{F}_{y,n} \) such that the region \( \mathcal{R}_{y,n} \) contains all points in \( \tilde{F}(\epsilon_n) \) with \( \tau_1 \leq \mathcal{F}_{y,n} \) that lie above \( \text{tr}(\tilde{\gamma}_{y,n}) \). Let \( T_{x,n} > T_n \) and \( T_{y,n} > T_n \) be times at which \( \tau_1(\tilde{\gamma}_{x,n}(T_{x,n})) \geq \mathcal{F}_{x,n} \) and \( \tau_1(\tilde{\gamma}_{y,n}(T_{y,n})) \geq \mathcal{F}_{y,n} \).

Consider the projections \( \tilde{\gamma}_{x,n}, \tilde{\gamma}_{y,n} \) of \( \gamma_{x,n}, \gamma_{y,n} \), respectively, to \( \tilde{F}(\epsilon_n) \). Since \( \tilde{\gamma}_{x,n} \) and \( \tilde{\gamma}_{y,n} \) approach \( \tilde{\rho} \) from the same side \( (\tau_2 > 0) \), but with opposite orientations, they intersect infinitely often. Thus there exist \( t_{x,n} > T_{x,n} \) and \( t_{y,n} > T_{y,n} \) such that \( \tilde{\gamma}_{x,n}(t_{x,n}) = \tilde{\gamma}_{y,n}(t_{y,n}) \). We now impose the additional condition that the lifts \( \tilde{\sigma}_{x,n} \) and \( \tilde{\sigma}_{y,n} \) be chosen so that \( \tilde{\gamma}_{x,n}(t_{x,n}) = \tilde{\gamma}_{y,n}(t_{y,n}) \).

Let \( (\tau_1, \tau_2) \) be the \( (\tau_1, \tau_2) \)-coordinates of \( \tilde{\gamma}_{x,n}(t_{x,n}) \). At \( (\tau_1, \tau_2) \) both \( X_n \) and \( Y_n \) have negative components in the \( \partial/\partial\tau_2 \) direction, and \( X_n \) has a positive component in the \( \partial/\partial\tau_1 \) direction, while \( Y_n \) has a negative component in the \( \partial/\partial\tau_1 \) direction.
direction. At $(τ_{1,n}, ε_n)$ both $X_n$ and $Y_n$ have positive components in the $∂/∂τ_2$ direction, and $X_n$ still has a positive component in the $∂/∂τ_1$ direction, while $Y_n$ still has a negative component in the $∂/∂τ_1$ direction. Therefore the Euclidean angle from $X_n$ to $Y_n$ measured in the counterclockwise direction changes from being greater than $π$ to being less than $π$ along the vertical segment $τ_1 = τ_{1,n}$, $τ_{2,n} ≤ τ_2 ≤ ε_n$. By the intermediate value theorem, there is a point $p_n$ along this segment such that the angle from $X_n$ to $Y_n$ is $π$. We let $a_n ∈ (0, a_n)$, $b_n ∈ (0, b_n)$, and $l_x, l_y, n > T_n$ be such that $γ_{σ_x(a_n)}(l_x, n) = p_n = γ_{σ_y(b_n)}(l_y, n)$ and $γ'_{σ_x(a_n)}(l_x, n) = −(γ'_{σ_y(b_n)}(l_y, n))$. We join $γ_{σ(a_n)}$ and $−γ_{σ(b_n)}$ at $p_n$ to form a geodesic $β_n$ from the basepoint of $σ(a_n)$ to the basepoint of $σ(b_n)$ in $F(ε_n)$, and we let $β_n$ be the image of $β_n$ under the projection from $F(ε_n)$ to $F(ε_n)$. The geodesic $γ_n : [0, L_n] → S$ from $x$ to $y$ is defined to be the concatenation of $γ_{σ_x(a_n)}(0, T_n)$, $β_n$, and $−γ_{σ_y(b_n)}(0, T_n)$. It follows from the construction that the second Fermi coordinate of $β_n$ is everywhere positive and less than $ε_n$, as required. Moreover, if $1 ≤ m < n$, then $β_n$ can be extended by joining it to lifts to $F(ε_n)$ of $γ_{σ_x(a_n)}(0, T_n)$ and $−γ_{σ_y(b_n)}(0, T_n)$. For this extension of $β_n$, the second Fermi coordinate is everywhere positive and less than $ε_m$. This implies $[0,T]$.

The following theorem is a special case of a theorem of S. Łojasiewicz [20]. (See also Theorem 4.4 in the expository article [7].)

**Theorem 6.10.** Suppose $M$ is a connected real analytic surface, $K$ is a compact subset of $M$, and $f : M → ℝ$ is a real analytic function. Assume that $f$ does not vanish identically on $M$. Then there exists a finite set of points $P ⊂ M$ and a set $A$ consisting of the union of finitely many real analytic curves on $M$, where $P$ and/or $A$ may be empty, such that

$$\{ y ∈ K : f(y) = 0 \} = K \cap (P \cup A).$$

**Proposition 6.11.** If $T > 0$, and $(x, z) ∈ S × S$, then there are at most finitely many unit speed geodesics from $x$ to $z$ of length less than or equal to $T$.

**Proof.** Suppose the lemma were false. Then there exists an infinite sequence $γ_n : [0, L_n] → S$ of unit speed geodesics with $γ_n(0) = x$ and $γ_n(t_n) = z$ for some $t_n ∈ [0, L_n]$. By passing to a subsequence of $(γ_n)_{n=1, 2, ...}$ and reindexing, we may assume that $\lim_{n→∞} t_n = t_0 ∈ (0, T]$ and $\lim_{n→∞} γ'_n(0) = v_0 ∈ T_xM$. Let $f : T_xS → ℝ$ be defined by $f(v) = (\text{dist}(\exp_x v, z))^2$. Since $y → (\text{dist}(y, z))^2$ is a real analytic function in a neighborhood of $z$, there exists an open disk $M$ about $t_0v_0$ in $T_xS$ such that $f$ restricted to $M$ is real analytic. Let $K$ be a closed disk about $t_0v_0$ that is contained in $M$. Since $f$ vanishes on an infinite subset of $K$, Theorem 5.11 implies that there exists a non-trivial real analytic curve $α(s)$, $−δ < s < δ$, in $T_xM$ such that $f(α(s)) = 0$ for all $s ∈ (−δ, δ)$. Consider the variation by (not necessarily unit speed) geodesics, $s → \exp(tα(s))$, where $0 ≤ t ≤ 1$. These geodesics pass through $x$ when $t = 0$, and they pass through $z$ when $t = 1$. By the first variation formula for arc length (see, e.g., [12]), $(d/ds)(|α(s)|) ≡ 0$. Thus there exists $L > 0$ such that $|α(s)| = L$ for all $s ∈ (−δ, δ)$. This implies that $f$ vanishes along an arc of the circle $|v| = L$. Therefore $f$ vanishes identically on this circle. Thus every unit speed geodesic starting at $x$ passes through $z$ at time $L$. We can repeat this argument at $z$ to conclude that every unit speed geodesic starting at $z$ passes through $x$ at time $L$. Hence every unit speed geodesic starting at $x$ is at a point conjugate to
there is a geodesic starting at $x$ along that geodesic at times $t = L, 2L, 3L, \ldots$. However, by Proposition 6.6 there is a geodesic starting at $x$ that eventually remains in the negative curvature region, which implies there are at most finitely many points conjugate to $x$ along this geodesic. This is a contradiction. \qed

**Lemma 6.12.** Let $(x, y) \in S \times S$ and suppose $\gamma_n : [0, L_n] \to S$, $n = 1, 2, \ldots$, is an infinite family of distinct geodesics from $x$ to $y$ as in Proposition 6.9. If $\Omega$ is a finite subset of $S$, then there exists an infinite subsequence $(\gamma_{n_k})_{k=1,2,\ldots}$ of $(\gamma_n)_{n=1,2,\ldots}$ such that $\gamma_{n_k}((0, L_{n_k})) \cap \Omega = \emptyset$ for $k = 1, 2, \ldots$.

**Proof.** Let $\Sigma \subset S$ be the set of self-intersection points of $\rho$. If $\Sigma \neq \emptyset$, then there exist $\varepsilon_1 > 0$ and $C > 1$, depending on the angles made by $\rho$ at points in $\Sigma$, such that if $z$ is in the $\varepsilon_1$-neighborhood of $\operatorname{tr}(\rho)$, and $z$ has at least one representation in Fermi coordinates $(\tau_1, \tau_2)$ along $\rho$ with $\tau_2 \neq 0$, then $z \notin \Sigma$ and

$$0 < \operatorname{dist}(z, \Sigma) < C|\tau_2|.$$  

Let $\alpha_1 = \min\{\operatorname{dist}(z, \operatorname{tr}(\rho)) : z \in \Omega \setminus \operatorname{tr}(\rho)\}$ and $\alpha_2 = \min\{\operatorname{dist}(z, \Sigma) : z \in (\Omega \cap \operatorname{tr}(\rho)) \setminus \Sigma\}$. (We define $\min(\emptyset) = \infty$.) Let $0 < \varepsilon < \min(\varepsilon_1, \alpha_1, \alpha_2/C)$, and let $T = T(\varepsilon)$ and $N = N(\varepsilon)$ be as in Proposition 6.9. For $n > N$, there exists a smooth choice of the coordinate $\tau_2$ along $\gamma_n([T, L_n - T])$ such that

$$0 < |\tau_2(\gamma_n(t))| < \varepsilon,$$

for $t \in [T, L_n - T]$. If a point $\gamma_n(t)$, for some $n > N$ and some $t \in [T, L_n - T]$, is in $\operatorname{tr}(\rho)$, then it is not in $\Sigma$, and it is closer to $\Sigma$ than any point in $(\Omega \cap \operatorname{tr}(\rho)) \setminus \Sigma$. If it is not in $\operatorname{tr}(\rho)$, then it is closer to $\operatorname{tr}(\rho)$ than any point in $\Omega \setminus \operatorname{tr}(\rho)$. Therefore $\gamma_n([T, L_n - T]) \cap \Omega = \emptyset$, for $n > N$. By applying Proposition 6.11 to points of the form $(x, z)$ or $(y, z)$, where $z \in \Omega$, we see that there exist infinitely many $n > N$ such that $\gamma_n((0, T) \cup (L_n - T, L_n)) \cap \Omega = \emptyset$. \qed

Below is the proof of our main result.

**Proof of Theorem 2.3.** Let $(x, y) \in S \times S$, and let $\gamma_n : [0, L_n] \to S$, $n = 1, 2, \ldots$, be an infinite family of distinct geodesics from $x$ to $y$ as in Proposition 6.9. By applying Lemma 6.12 to $\Omega = \{x, y\}$ and passing to a subsequence and reindexing, we may assume that the geodesics $\gamma_n$ pass through $x$ and $y$ only at the endpoints.

We will prove inductively that there exists a strictly increasing sequence of positive integers $n_1, n_2, \ldots$ such that

$$\text{(6.22)} \quad \text{no three of } \gamma_{n_1}, \gamma_{n_2}, \ldots, \gamma_{n_k} \text{ are concurrent except at } x \text{ and at } y.\text{ We may take } n_1 = 1 \text{ and } n_2 = 2. \text{ Then } (6.22) \text{ is clearly satisfied for } k = 1, 2. \text{ Now assume } k \geq 2 \text{ and we have found } n_1 < n_2 < \cdots < n_k \text{ such that } (6.22) \text{ holds. We will show that we can choose } n_{k+1} > n_k \text{ such that } (6.22) \text{ holds with } k \text{ replaced by } k + 1. \text{ We now apply Lemma 6.12 with }$$

$$\Omega = \bigcup_{1 \leq i < j \leq k} (\gamma_{n_i}((0, L_{n_i})) \cap \gamma_{n_j}((0, L_{n_j}))).$$

Since the geodesics $\gamma_n$ pass through $x$ and $y$ only at the endpoints, $\Omega$ is a finite set. Thus Lemma 6.12 implies that there exists $n_{k+1} > N$ such that $\gamma_{n_{k+1}}((0, L_{n_{k+1}})) \cap \Omega = \emptyset$, and (6.22) holds with $k$ replaced by $k + 1$. Therefore there exists an infinite sequence of positive integers $n_1, n_2, \ldots$ such that no three of $\gamma_{n_1}, \gamma_{n_2}, \ldots$ are concurrent except at $x$ and at $y$. Since any point can be in at most two of
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$\gamma_n(\{0, L_{n_1}\}), \gamma_n(\{0, L_{n_2}\}), \ldots$, it follows that there does not exist a finite blocking set for $(x, y)$.

7. Acknowledgments

We thank Eugene Gutkin for his encouragement and helpful correspondence, and we thank Ji-Ping Sha for making a correction to an earlier version of our proof of Proposition 6.11.

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