ON THE NUMBER OF FUZZY SUBGROUPS OF A SYMMETRIC GROUP $S_5$

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Abstract

This article computes the number of fuzzy subgroups of symmetric group $S_5$. First, an equivalence relation on the set of all fuzzy subgroups of a group $G$ is defined. Without any equivalence relation on fuzzy subgroups of group $G$, the number of fuzzy subgroups is infinite, even for the trivial group.

The Inclusion-Exclusion principle is used to determine the number of distinct fuzzy subgroups of symmetric group $S_5$. Some inequalities satisfied by this number are also established for $n \geq 5$

Keywords: Fuzzy subgroups, chains of subgroups, maximal chains of subgroups, symmetric groups, recurrence relations.

1 Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965(see[18]). The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in 1971 (see[17]). Since the first paper by Rosenfeld, researchers have sought to characterize the fuzzy subgroups of various groups.

One of the most important problem of fuzzy theory is to classify the fuzzy subgroups of a finite groups. This topic has enjoyed a rapid development in the last few years. In our case the corresponding equivalence classes of fuzzy subgroups are closely connected to the chains of subgroups in $G$. As a...
guiding principle in determining the number of these classes, we first found the number of maximal chains of $G$. Note that an essential role in solving our counting problem is played again by the Inclusion-Exclusion Principle. Sulaiman and Abd Ghafur [10] have counted the number of fuzzy subgroups of symmetric group $S_2, S_3$ and alternating group $A_4$. Sulaiman [9] have constructed the fuzzy subgroups of symmetric group $S_4$ using the Maximal chain method, while Tarnauceanu [16] have also computed the number of fuzzy subgroups of symmetric group $S_4$ by the inclusion -Exclusion Principle. These groups are probably the most important in group theory, because any finite group can be embedded in such a group. They also have remarkable applications in graph theory, in enumerative combinatorics, as well as in many branches of informatics. The paper is organized as follows. In Section 2 we present some preliminary results on fuzzy subgroups. Section 3 deals with the maximal subgroup structure of symmetric group $S_5$. Computing the number of fuzzy subgroups of $S_5$ and inequalities satisfied by this number for the symmetric group $S_n$, $n \geq 5$, are obtained in Section 4. In the final section some conclusions and further research directions are indicated.

2 Preliminaries

Let $G$ be a group with a multiplicative binary operation and identity $e$, and let $\mu : G \to [0, 1]$ be a fuzzy subset of $G$. Then $\mu$ is said to be a fuzzy subgroup of $G$ if (1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, and (2) $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$. The set $\{\mu(x) | x \in G\}$ is called the image of $\mu$ and is denoted by $\mu(G)$. For each $\alpha \in \mu(G)$, the set $\mu_\alpha := \{x \in G | \mu(x) \geq \alpha\}$ is called a level subset of $\mu$. It follows that $\mu$ is a fuzzy subgroup of $G$ if and only if its level subsets are subgroups of $G$. These subsets allow us to characterize the fuzzy subgroups of $G$ (see [3]).

For given two fuzzy subgroups $\mu$ and $\nu$ in $G$, $\mu$ and $\nu$ are equivalent, written as $\mu \sim \nu$, if $\mu(x) \geq \mu(y) \iff \nu(x) \geq \nu(y)$ for all $x, y \in G$. It follows that $\mu \sim \nu$ if and only if $\mu$ and $\nu$ have the same set of level Subgroups and two fuzzy subgroups $\mu, \nu$ of $G$ will be called distinct if $\mu \not\sim \nu$. (see[13]). Hence there exits a one-to-one correspondence between the collection of the equivalence classes of fuzzy subgroups of $G$ and the collection of chains of subgroups of $G$ which end in $G$. So, the problem of counting all distinct fuzzy subgroups of $G$ can be translated into a combinatorial problem on the subgroup lattice $L(G)$ of $G$. This notion of equivalence relation was in [9,10,12,13,15] in order to enumerate fuzzy subgroups of certain families of finite groups. There is another equivalence relation on the set of fuzzy subgroups used by Murali.
and Makamba [5, 6, 7, 8] in order to enumerate fuzzy subgroups of certain families of finite abelian groups. Some other different approaches to classify the fuzzy subgroups can be found in [3] and [4].

Most recent, the problem of classifying the fuzzy subgroup of finite group $G$ by using a new equivalence relation $\approx$ on the lattice of all fuzzy subgroups of $G$, its definition has a consistent group theoretical foundation, by involving the knowledge of the automorphism group associated to $G$. The approach is motivated by the realization that in a theoretical study of fuzzy groups, fuzzy subgroups are distinguished by their level subgroups and not by their images in $[0, 1]$. Consequently, the study of some equivalence relations between the chains of level subgroups of fuzzy groups is very important. It can also lead to other significant results which are similar with the analogous results in classical group theory (see [14]). In this paper we follow the notion of the equivalence relation used in [15]. This equivalence relation generalizes that used in Murali’s papers [5] - [8]. It is also closely connected to the concept of level subgroup.

One next goal is to describe the method that will be used in counting the chains of subgroups of $G$. Let $M_1, M_2, \ldots, M_k$ be the maximal subgroups of $G$ and denote by $g(G)$ (respectively by $h(G)$) the number of maximal chain of subgroups in $G$ (respectively the number of chains of subgroups of $G$ ending in $G$). The technique developed to obtain $g(G)$ is founded on the following simple remark: every maximal chain in $G$ contains a unique maximal subgroup of $G$. In this way, $g(G)$ and $g(M_i), i = 1, 2, \ldots, k$, are connected by the equality

$$g(G) = \sum_{i=1}^{k} g(M_i)$$

(1)

For finite cyclic groups, this equality leads to the well-known formula

$$g(Z_n) = \left( \begin{array}{c} m_1 + m_2 + \ldots + m_s \\ m_1, m_2, \ldots, m_s \end{array} \right) = \frac{(m_1 + m_2 + \ldots + m_s)!}{m_1!m_2!\ldots m_s!}$$

(2)

In order to compute the number of all distinct fuzzy subgroups of a finite $G$ which is denoted by $h(G)$, we shall apply the inclusion-Exclusion Principle (see[15])

$$h(G) = 2\left( \sum_{i=1}^{k} h(M_i) - \sum_{1 \leq i_1 < i_2 \leq k} h(M_{i_1} \cap M_{i_2}) + \ldots + (-1)^{k-1} h(\bigcap_{i=1}^{k} M_i) \right)$$

(3)
3 Maximal Subgroup Structure Of Symmetric Group $S_5$

One way to investigate the structure of a finite group is to study its maximal subgroups. In nontrivial finite groups, maximal subgroups will always exist because the subgroups form a partially ordered set under inclusion. Since the set of subgroups is finite, this partially ordered set will have a maximal element.

Our problem is very simple for $n = 3$, since the symmetric group $S_3$ is isomorphic to the dihedral group $D_6$. In this way, the number of all maximal chains of subgroups of $S_3$ is

$$g(S_3) = g(D_6) = 4$$

(4)

and the number of all distinct fuzzy subgroups of $S_3$ is

$$h(S_3) = h(D_6) = 10$$

(5)

For the symmetric group $S_4$, it is well-known that any finite group of order $p^m q^n$ ($p$, $q$ primes) is solvable and the index of a maximal subgroup in such a group is a prime power. So, $S_4$ is solvable and given a maximal subgroup $M$ of $S_4$, we have $[S_4 : M] = p^k$ for some prime divisor $p$ of 24 and some positive integer $k$ (see[11]).

Theorem 1.[15] The number $g(S_4)$ of all maximal chains of subgroups of the symmetric group $S_4$ is 44

Theorem 2.[15]. The number $h(S_4)$ of all distinct fuzzy subgroups of the symmetric group $S_4$ is 232

For symmetric groups $S_5$, $S_5$ is the first non-solvable symmetric group. This is an essential part of the proof of the Abel-Ruffini theorem that shows that for every $n > 4$ there are polynomials of degree $n$ which are not solvable by radicals, i.e., the solutions cannot be expressed by performing a finite number of operations of addition, subtraction, multiplication, division and root extraction on the polynomial’s coefficients. It is very difficult to describe all its maximal subgroups of $S_n$ because the general problem of listing all the maximal subgroups of $S_n$ for all degrees $n$ remains intractable (the almost simple case). However we can provide an implicit solution. $S_5$ is a symmetric group of prime power degree. According to the O’Nan-Scott theorem, stated that the symmetric group $(S_n)$ with $n = p^k$ and $p$ prime, there exist maximal subgroup $(M)$, $M = AGL(k, p)$ i.e affine case (see[1]).

For any prime number $P$ dividing the order of $G$, $P$ divides exactly one of the two numbers. The order of $H$ and the index of $H$ in $G$. Hence $S_4$ in $S_5$
The subgroup has order 24 and index 5 in a group of order 120. It is a \(\{2,3\}\) - Hall Subgroups and also a 5 - complement. \(S_5\) is one - headed group since the alternating group \(A_5\) is it’s unique maximal normal subgroup. We distinguish the following four cases.

| MaximalSubgroups | Generatingsets | Order | Number |
|------------------|----------------|-------|--------|
| Direct product of \(S_3\) and \(S_2\) | \(\langle(1,2,3), (1,2), (4,5)\rangle\) | 12    | 10     |
| \(\text{GA (1,5)}\) | \(\langle(1,2,3,4,5), (2,3,5,4)\rangle\) | 20    | 6      |
| \(S_4\) | \(\langle(1,2,3,4), (1,4)\rangle\) | 24    | 5      |
| \(A_5\) | \(\langle(1,2,3,4,5), (1,2,3)\rangle\) | 60    | 1      |

There exist 10 such maximal subgroups, all isomorphic to \(D_{12}\):

1. \(M_2 = \langle(1,2,3), (1,2), (4,5)\rangle\)
2. \(M_3 = \langle(1,2,4), (1,2), (3,5)\rangle\),
3. \(M_4 = \langle(1,2,5), (1,2), (3,4)\rangle\),
4. \(M_5 = \langle(1,3,4), (1,3), (2,5)\rangle\),
5. \(M_6 = \langle(1,3,5), (1,3), (2,4)\rangle\),
6. \(M_7 = \langle(1,4,5), (1,4), (2,3)\rangle\),
7. \(M_8 = \langle(2,3,4), (2,3), (1,5)\rangle\),
8. \(M_9 = \langle(2,3,5), (2,3), (1,4)\rangle\),
9. \(M_{10} = \langle(2,4,5), (2,4), (1,3)\rangle\),
10. \(M_{11} = \langle(3,4,5), (3,4), (1,2)\rangle\),

There exist 6 such maximal subgroups, all isomorphic to \(D_{20}\):

1. \(M_{17} = \langle(2,3,4,5), (2,4)(3,5), (1,2,3,5,4)\rangle\),
2. \(M_{18} = \langle(2,3,5,4), (2,5)(3,4), (1,2,3,4,5)\rangle\),
3. \(M_{19} = \langle(2,4,3,5), (2,3)(4,5), (1,2,4,5,3)\rangle\),
4. \(M_{20} = \langle(2,4,5,3), (2,5)(3,4), (1,2,4,3,5)\rangle\),
5. \(M_{21} = \langle(2,5,3,4), (2,3)(4,5), (1,2,5,4,3)\rangle\),
6. \(M_{22} = \langle(2,5,4,3), (2,4)(3,5), (1,2,5,3,4)\rangle\),


The alternating group $A_5$ possesses 21 maximal subgroups, 6 isomorphic to $D_{10}$

$$g(A_5) = \sum_{i=1}^{21} g(M_i) = 10g(S_3) + 6g(D_{10}) + 5g(A_4)$$

$$g(A_5) = 10(4) + 6(8) + 5(7)$$

$$g(A_5) = 123$$

**Lemma 1.** The number $g(A_5)$ of all the maximal chains of subgroups of the alternating group $A_5$ is 123

Now, we are able to compute $g(S_5)$. By (1), one obtains:

$$g(S_5) = \sum_{i=1}^{22} g(M_i) = g(A_5) + 10g(D_{12}) + 6g(D_{20}) + 5g(S_4)$$

$$g(S_5) = 123 + 10(10) + 6(18) + 5(44)$$

$$g(S_5) = 551$$

**Theorem 3.** The number of $g(S_5)$ of all maximal chains of subgroups of the symmetric group $S_5$ is 551

We conclude that if the maximal subgroup structure of a finite group $G$ is known (i.e., we know the number of maximal subgroups of $G$, their types and their intersections), then from the equalities (1) and (2) some recurrence relation can be inferred which permit us to determine explicitly $g(G)$ and $h(G)$. This fact will be exemplified in the next section for the symmetric group $S_5$.

4 Counting the number of fuzzy subgroups of symmetric group $S_5$

The problem of counting all distinct fuzzy subgroups of $G$ can be translated into combinatorial problem on the subgroup lattice $L(G)$ of $G$: finding the number of all chain of subgroups of $G$ that terminates in $G$. Clearly, we obtain that in any group with at least two elements there are more distinct fuzzy subgroup than subgroups

Since the maximal subgroups $M_i$ of $S_5$ have precisely determined with the
help of the computational group theory system GAP, we can describe all their intersection by a direct calculation. We have:

\[ M_1 \cap M_2 = \{(1, 2, 3), (2, 3)(4, 5)\} \cong S_3, M_2 \cap M_{12} = \{(2, 3), (1, 2, 3)\} \cong S_3, \]
\[ M_6 \cap M_{14} \cap M_{16} = \{(1, 3), (3, 5)\} \cong S_3, M_{11} \cap M_{13} \cap M_{16} = \{(4, 5), (3, 4)\} \cong S_3, \]
\[ M_1 \cap M_{17} = \{(2, 4)(3, 5), (1, 2, 3, 5, 4)\} \cong D_{10}, M_1 \cap M_{22} = \{(2, 4)(3, 5), (1, 2)(4, 5)\} \cong D_{10}, \]
\[ M_1 \cap M_{12} = \{(2, 3, 4), (1, 3, 2)\} \cong A_4, M_1 \cap M_{16} = \{(1, 2, 3), (1, 5, 2)\} \cong A_4, \]
\[ M_{12} \cap M_{17} = \{(1, 4, 2, 3)\} \cong C_4, M_{13} \cap M_{17} = \{(2, 3, 4, 5), (2, 4)(3, 5)\} \cong C_4, \]
\[ M_{16} \cap M_{19} \cap M_{22} = \{(1, 2, 3, 5), (1, 3)(2, 5)\} \cong C_4, \]
\[ M_2 \cap M_7 = \{(4, 5), (2, 3)\} \cong C_2 \times C_2, M_2 \cap M_{10} \cap M_{14} = \{(4, 5), (1, 3)\} \cong C_2 \times C_2, \]
\[ M_1 \cap M_2 \cap M_{12} = \{(1, 3, 2)\} \cong C_3, M_1 \cap M_{14} \cap M_{15} = \{(1, 4, 5)\} \cong C_3, \]
\[ M_2 \cap M_3 = \{(1, 2)\} \cong C_2, M_4 \cap M_{19} = \{(1, 2)(3, 4)\} \cong C_2, M_8 \cap M_{15} \cap M_{19} = \{(1, 5)(2, 4)\} \cong C_2, \]
\[ M_2 \cap M_{11} \cap M_{12} \cap M_{15} \]

This has been used in [12,15] to obtain explicit formulas of \( h(D_{2n}) \) for some classes of positive integers \( n \). Recall here only that

\[ h(D_{2n}) = \frac{2^{m_1}}{p_1 - 1} \left( P_1^{m_1+1} + p_1 - 2 \right) i f n = p_1^{m_1} \]

and, in particular , \( h(D_4) = 8, h(D_6) = 10 \), \( h(D_8) = 32 \), \( h(D_{10}) = 68 \) and \( h(D_{20}) = 100 \)

\[ h(C_2 \times C_2) \cong h(C_4) \cong h(D_4) = 8 \]
\[ h(A_3) \cong h(C_3) = 2 \]

It follows that:

\( M_{i_1} \cap M_{i_2} \cap \ldots \cap M_{i_r} = \{e\} \), for all \( r \geq 8 \) and all \( 0 \leq i_1 < i_2 < \ldots < i_r \leq 22 \)

**Theorem 3.** The number \( h(A_5) \) of all distinct fuzzy subgroups of the alternating group \( A_5 \) is 402

\[ c_r = (-1)^{r-1} \sum_{0 \leq i_1 < i_2 < \ldots < i_r \leq 22} h(M_{i_1} \cap M_{i_2} \cap \ldots \cap M_r) \]

We have

\[ c_1 = (h(A_5)) + 5h(S_4) + 6h(D_{20}) + h(D_{12}) = 2842 \]
\[ c_2 = -(5h(A_4) + 10h(S_3) + 5h(D_{10}) + 45h(C_2 \times C_2) + 45h(C_4) + 90h(C_2)) = -1504 \]
\[ c_3 = \binom{22}{3} - 630 + 10h(S_3) + 15h(C_4) + 15h(C_2 \times C_2) + 30h(C_3) + 560h(C_2) = 2430 \]
\[ c_4 = -\binom{22}{4} - 586 + 576h(C_2) + 10h(C_3) = -7901 \]
\[
c_5 = \binom{22}{5} - 300 + 300h(C_2) = 26634 \quad c_6 = -\binom{22}{6} - 85 + 85h(C_2) = -74698
\]
\[
c_7 = \binom{22}{7} - 10 + 10h(C_2) = 170555 \quad c_8 = -\binom{22}{8} = -319770
\]
\[
c_9 = \binom{22}{9} = 497420 \quad c_{10} = -\binom{22}{10} = -646646
\]
\[
c_{11} = \binom{22}{11} = 705432 \quad c_{12} = -\binom{22}{12} = -646646
\]
\[
c_{13} = \binom{22}{13} = 497420 \quad c_{14} = -\binom{22}{14} = -319770
\]
\[
c_{15} = \binom{22}{15} = 170544 \quad c_{16} = -\binom{22}{16} = -74613
\]
\[
c_{17} = \binom{22}{17} = 26334 \quad c_{18} = -\binom{22}{18} = -7315
\]
\[
c_{19} = \binom{22}{19} = 1540 \quad c_{20} = -\binom{22}{20} = -231
\]
\[
c_{21} = \binom{22}{21} = 21 \quad c_{22} = -\binom{22}{22} = -1
\]

\[
\therefore h(S_5) = 2 \sum_{r=1}^{22} c_r = 4154
\]

Hence the theorem holds.

**Theorem 4.** The number \( h(S_5) \) of all distinct fuzzy subgroups of the symmetric group \( S_5 \) is 4154.

**Theorem 4.2[15].** For \( n \geq 5 \), the number \( h(S_n) \) of all distinct fuzzy subgroups of the symmetric group \( S_n \) satisfies the following inequality:

\[
h(S_n) \geq 2 \left( \sum_{r=0}^{n} (-1)^r \binom{n}{r} h(A_{n-r}) + \sum_{r=0}^{n-1} \binom{n}{r+1} h(S_{n-r-1}) \right)
\]  \hspace{1cm} (6)

From Theorem 4.2 such a bound can be also inferred for the number of all distinct fuzzy subgroups of \( (S_n) \). For \( n = 5 \), we can easily see that \((6)\) becomes

\[
h(S_5) \geq 1940 + 2h(A_5)
\]

It implies that

\[
h(S_5) > 1942
\]  \hspace{1cm} (7)

It is obvious that \((7)\) is the lower bound for \( h(S_5) \)
5 Conclusion and Further Research

The study concerning the classification of the fuzzy subgroups of (finite) groups is a significant aspect of fuzzy group theory. The problem of counting the number of distinct fuzzy subgroups relative to the notion of the equivalence relation. Without any equivalence relation on fuzzy subgroups of finite group, the number of fuzzy subgroups is infinite, even for the trivial group. These equivalence relations provide settings for classifying the fuzzy subgroups of finite groups. Classifying the fuzzy subgroups can be made with respect to some natural equivalence relations on the fuzzy group lattices, as the Murali’s equivalence relation used in [5] - [8], used in [12,13,15], or used in [14]. The number of fuzzy subgroups of finite symmetric group $S_n$ depends on the notion of equivalence relation on its fuzzy subgroups. The group structures can be classified by assigning equivalence classes to its fuzzy subgroups. This will surely constitute the subject of further research on the classification of the fuzzy subgroups of finite symmetric groups.

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