On the abundance of supersymmetric strings in AdS$_3 \times S^3 \times S^3 \times S^1$ describing BPS line operators

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Abstract

We study supersymmetric open strings in type IIB AdS$_3 \times S^3 \times S^3 \times S^1$ with mixed R–R and NS–NS fields. We focus on strings ending along a straight line at the boundary of AdS$_3$, which can be interpreted as line operators in a dual CFT$_2$. We study both classical configurations and quadratic fluctuations around them. We find that strings sitting at a fixed point in $S^3 \times S^3 \times S^1$, i.e. satisfying Dirichlet boundary conditions, are 1/2 BPS. We also show that strings sitting at different points of certain submanifolds of $S^3 \times S^3 \times S^1$ can still share some fraction of the supersymmetry. This allows to define supersymmetric smeared configurations by the superposition of them, which range from 1/2 BPS to 1/8 BPS. In addition to the smeared configurations, there are as well 1/4 BPS and 1/8 BPS strings satisfying Neumann boundary conditions. All these supersymmetric strings are shown to be connected by a network of interpolating BPS boundary conditions. Our study reveals the existence of a rich moduli of supersymmetric open string configurations, for which the appearance of massless fermionic fields in the spectrum of quadratic fluctuations is crucial.

Keywords: AdS/CFT, open string boundary conditions, superconformal line defects

1. Introduction

Line operators, as for example Wilson loops, are perhaps the most basic non-local operators in any QFT, and they are central to the description of gauge field theories. Within the class of line
operators, supersymmetric ones stand out, as their expectation values and correlation functions can be described exactly in some cases [1]. Of special interest are the cases in which the lines also preserve the group of one-dimensional conformal symmetries, as they would constitute superconformal defects. Remarkably, these two features have enabled a wide variety of non-trivial precision tests of the AdS/CFT correspondence, highlighting the importance of BPS line operators within this context.

The literature on supersymmetric line operators, especially for superconformal theories in $d = 4$ and $d = 3$ spacetime dimensions, is vast. As prototypical examples, we could mention Wilson loops in $\mathcal{N} = 4$ super Yang–Mills [2] and $\mathcal{N} = 6$ super Chern–Simons-matter [3–6], whose dual descriptions are in terms of open strings in $\text{AdS}_5 \times S^5$ and $\text{AdS}_4 \times \text{CP}^3$ respectively (see [7] for a review on Wilson loops in Chern–Simons-matter theories). There is however a striking difference between these cases that should be noted. While supersymmetric Wilson loops are pretty much fixed by the contour in $\mathcal{N} = 4$ super Yang–Mills, one discovers a richer family in super Chern–Simons theories, as for a given contour it is possible to define families of supersymmetric Wilson loops containing arbitrary parameters [8, 9]. The reasons why this is the case might not be so obvious in the field theory description. However, from the holographic dual point of view there is a hallmark for this richness in the moduli of supersymmetric line operators: for the dual open strings on $\text{AdS}_4 \times \text{CP}^3$ other boundary conditions than Dirichlet can also be consistent with supersymmetry [10].

The fact that superconformal field theories in $d = 3$ admits a richer moduli of superconformal lines seems to be a general property, irrespective of the aforementioned examples. In the article [11], a classification of superconformal line defects, within superconformal field theories in various spacetime dimensions ($3 \leq d \leq 6$), has been given. This reveals a rich structure of marginal and relevant deformations for lines in $d = 3$ superconformal theories, which are not admitted for $d > 3$. Recently, an extensive study of BPS Wilson loops in three-dimensional supersymmetric gauge theories has been presented in [12–15].

More concretely, we study open strings ending along a straight line at the boundary of $\text{AdS}_3$ in quest of supersymmetric boundary conditions. Our analysis begins with the simplest open string ending along a line while sitting at a point in the compact factors $S^3 \times S^3 \times S^1$. This is an interesting problem, as the holographic dual of the string theory on this background in the general case is more difficult to describe than in the cases in which the factor $S^3 \times S^1$ is replaced either by $T^4$ or $K3$ [16–20], even more if one considers a non-vanishing RR three-form flux. The study of superconformal line defects and the supersymmetries they preserve can set a basis for the description of the CFT$_2$ ambient theory allocating them. Superconformal symmetry constrains the correlation functions on the line, and physical quantities of the CFT$_2$ (such as the central charge or the Bremsstrahlung functions) could be inferred from them [21, 22].

More concretely, we study open strings ending along a straight line at the boundary of $\text{AdS}_3$ in quest of supersymmetric boundary conditions. Our analysis begins with the simplest open string ending along a line while sitting at a point in the compact factors $S^3 \times S^3 \times S^1$, which turns out to be 1/2 BPS. Then, we move to consider open string configurations delocalized within some regions of $S^3 \times S^1 \times S^1$, but still preserving some fraction of the supersymmetry. We do this by smearing open strings, satisfying Dirichlet boundary conditions for different values of the angular positions in $S^3 \times S^3 \times S^1$. These results are presented in section 2.

More general delocalized string configurations, including the ones mentioned in the preceding paragraph, can be accounted in terms of the boundary conditions that are imposed on the quadratic fluctuations around the simplest 1/2 BPS open string. Fluctuations for angular positions in $S^3 \times S^3 \times S^1$ are described in terms of massless scalar fields in $\text{AdS}_3$, whose asymptotic behaviours are of the form
\( \phi^m (\tau, \sigma) = (\alpha^m (\tau) + \ldots) + \sigma (\beta^m (\tau) + \ldots) \),

(1)

where \( \sigma \to 0 \) is the boundary of AdS_2. Since the vanishing mass is inside the Breitenlohner–Freedman window [23, 24], it is possible to set either Dirichlet \( \alpha^m = 0 \) or Neumann \( \beta^m = 0 \) boundary conditions. In conjunction with massless spinor fields, boundary conditions for these scalars more general than Dirichlet can be supersymmetric [10]. For example, the condition \( \dot{\alpha}^m = 0 \). This is like a Dirichlet boundary condition \( \alpha^m = \alpha^m_0 \) without specifying \( \alpha^m_0 \), and should be associated with the smeared configurations mentioned above. A different example of a delocalized string configuration is of course associated with Neumann boundary conditions. This complementary analysis on supersymmetric boundary conditions for the fluctuations is presented in section 3. Many details concerning the derivation of the Killing spinors of the background, the action of quadratic fluctuations and their supersymmetries are relegated to the appendices A and B.

2. Supersymmetric strings on \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \)

We shall look for supersymmetric open strings ending on the boundary of the type IIB supergravity background \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) with metric

\[
\text{d}s^2 = L^2 \left( \text{d}s^2(\text{AdS}_3) + \frac{L^2}{\sin^2 \Omega} \text{d}s^2(S_3^+) + \frac{L^2}{\cos^2 \Omega} \text{d}s^2(S_3^-) + l^2 \text{d}\theta^2 \right).
\]

(2)

The parameter \( \Omega \), that calibrates the radii of the three-spheres, takes values in the range \( 0 < \Omega < \frac{\pi}{2} \). This metric solves the type IIB supergravity equations of motion along with a constant dilaton \( \Phi \) and a three-form flux, which can be the Ramond–Ramond (R–R) one, the Neveu Schwarz–Neveu Schwarz (NS–NS) or a mixture of them. The R–R and NS–NS three-form field strengths are

\[
F_{(3)} = dC_{(2)} = -2 e^{-\Phi} L^2 \cos \lambda \left( \text{vol}(\text{AdS}_3) + \frac{1}{\sin^2 \Omega} \text{vol}(S_3^+) + \frac{1}{\cos^2 \Omega} \text{vol}(S_3^-) \right),
\]

(3)

\[
H_{(3)} = dB_{(2)} = 2L^2 \sin \lambda \left( \text{vol}(\text{AdS}_3) + \frac{1}{\sin^2 \Omega} \text{vol}(S_3^+) + \frac{1}{\cos^2 \Omega} \text{vol}(S_3^-) \right).
\]

(4)

The parameter \( \lambda \), taking values in the range \( 0 \leq \lambda \leq \frac{\pi}{2} \), interpolates between the pure R–R and the pure NS–NS backgrounds.

In the following we will parametrize \( \text{AdS}_3 \) with Poincaré coordinates

\[
\text{d}s^2(\text{AdS}_3) = \frac{1}{z^2} \left( -\text{d}t^2 + \text{d}x^2 + \text{d}z^2 \right) \Rightarrow \text{vol}(\text{AdS}_3) = \frac{1}{z^3} \text{d}t \wedge \text{d}x \wedge \text{d}z,
\]

(5)

and the three-spheres as

\[
\text{d}s^2(S_3^\pm) = d\beta_{\pm}^2 + \cos^2 \beta_{\pm} (d\gamma_{\pm} + \cos^2 \gamma_{\pm} d\varphi_{\pm}^2) \Rightarrow \text{vol}(S_3^\pm) = \cos^2 \beta_{\pm} \cos \gamma_{\pm} d\beta_{\pm} \wedge d\gamma_{\pm} \wedge d\varphi_{\pm}.
\]

(6)

The kind of open strings we shall consider in first place are those ending at a straight line at the boundary of \( \text{AdS}_3 \) and sitting at fixed points in the compact spheres. An ansatz to describe them is the following:

\[
t = \omega \tau, \quad x = x(\sigma), \quad z = \sigma, \quad \beta_{\pm}, \gamma_{\pm}, \varphi_{\pm}, \theta = \text{const}.
\]

(7)
To look for supersymmetric string configurations we need the Killing spinors of the type IIB supergravity solutions we presented above. We give their detailed derivation in the appendix A. Separating into real and imaginary parts

\[ \epsilon = \eta + i \xi \]

we find

\[ \eta = U \epsilon_0 + \frac{\cos \lambda}{\sin \lambda + 1} V \epsilon_1 \]

and

\[ \xi = -\frac{\cos \lambda}{\sin \lambda + 1} U \epsilon_0 + V \epsilon_1, \]

where

\[ U = e^{3M\beta + \gamma_1 \gamma_2} e^{\gamma_2 M\phi} e^{\gamma_1 M_\gamma} e^{\phi M_\beta} e^{2M_\beta e^{-\log z M_t e^{8(M_t + M_\gamma)} e^{(M_t + M_\gamma)}}} \]

and

\[ V = e^{-3M\beta + \gamma_1 \gamma_2} e^{-\gamma_2 M\phi} e^{-\gamma_1 M_\gamma} e^{-\phi M_\beta} e^{-2M_\beta e^{-\log z M_t e^{-8(M_t - M_\gamma)} e^{(M_t - M_\gamma)}}} \]

which are defined in terms of the matrices

\[ M_t = \frac{1}{2} \gamma_1 \gamma_2, \quad M_\phi = \frac{1}{2} \gamma_0 \gamma_1, \quad M_\beta = \frac{1}{2} \gamma_0 \gamma_2, \quad M_\gamma = \frac{1}{2} \gamma_0 \gamma_2. \]

The constant spinors \( \epsilon_0 \) and \( \epsilon_1 \) are Majorana–Weyl spinors with the same chirality

\[ \gamma_{11} \epsilon_0 = -\epsilon_0, \quad \gamma_{11} \epsilon_1 = -\epsilon_1, \]

and are further subject to an additional projection \((110)\)

\[ P^\Sigma \epsilon_0 = 0, \quad P^\Sigma \epsilon_1 = 0. \]

This reduces the number of independent real parameters in the Killing spinors to 16.

A string configuration is supersymmetric if there are Killing spinors satisfying the \( \kappa \)-symmetry projection

\[ \Gamma \epsilon = \epsilon, \]

where

\[ \Gamma = -\frac{\partial X^\mu \partial X^\nu \Gamma_{\mu \nu}}{\sqrt{-h}} K. \]

In this projector, \( K \) stands for the complex conjugation operation and \( h \) is the determinant of the induced metric. For the ansatz (7) we obtain

\[ \Gamma = -\frac{2}{\sqrt{1 + (x')^2}} (M_t - x'M_\gamma) K : \hat{\Gamma} K. \]
Separating into real and imaginary parts, the \( \kappa \)-symmetry projection (17) implies

\[
(1 - \tilde{\Gamma}) U \epsilon_0 = -\frac{\cos \lambda}{\sin \lambda + 1} (1 - \tilde{\Gamma}) V \epsilon_1,
\]

(20)

\[
(1 + \tilde{\Gamma}) U \epsilon_0 = \frac{\sin \lambda + 1}{\cos \lambda} (1 + \tilde{\Gamma}) V \epsilon_1,
\]

(21)

and from their combination we obtain

\[
\epsilon_0 = \tan \lambda U^{-1} V \epsilon_1 + \frac{1}{\cos \lambda} U^{-1} \tilde{\Gamma} V \epsilon_1.
\]

(22)

Evaluated on the ansatz (7), the matrices defining the Killing spinors are

\[
U = U_1 e^{\log \sigma M_t \in \text{e}^{(x(\sigma) + \omega \tau)(M_t + M_z)}},
\]

(23)

\[
V = V_7 e^{-\log \sigma M_t \in \text{e}^{(x(\sigma) - \omega \tau)(M_t - M_z)}},
\]

(24)

where \( U_7 \) and \( V_7 \) are defined in the appendix. Both \( U_7 \) and \( V_7 \) commute with \( M_t, M_x \) and \( M_z \). Then, we get

\[
U^{-1} V = \frac{U_7^{-1} V_7}{2 \sigma} \left[ 1 - \omega^2 \tau^2 + x(\sigma)^2 + \sigma^2 - 4 \omega \tau M_t - 4 \tau x(\sigma) M_x 
+ 2(1 + \omega^2 \tau^2 - x(\sigma)^2 - \sigma^2) M_z \right],
\]

(25)

\[
U^{-1} \tilde{\Gamma} V = \frac{U_7^{-1} V_7}{2 \sigma \sqrt{1 + (x')^2}} \left[ x'(1 - \omega^2 \tau^2 + x(\sigma)^2 - \sigma^2) + 2 \tau x(\sigma) - 4 \tau x(\sigma) M_t 
- 4(x' x(\sigma) + \sigma) M_x + \left(2 x'(1 + \omega^2 \tau^2 - x(\sigma)^2 + \sigma^2) - 4 \sigma x(\sigma) \right) M_z \right].
\]

(26)

Implementing these expressions, equation (22) becomes

\[
\epsilon_0 = \frac{U_7^{-1} V_7}{2 \sigma \cos \lambda} \left[ 2 \sigma x(\sigma) + x' \left(1 + x(\sigma)^2 - \sigma^2\right) \sqrt{1 + (x')^2} + \left(1 + x(\sigma)^2 + \sigma^2\right) \sin \lambda \right]
- 2 \left(\frac{2 \sigma x(\sigma) - x' \left(1 - x(\sigma)^2 + \sigma^2\right)}{\sqrt{1 + (x')^2}} \right) \left(1 - x(\sigma)^2 - \sigma^2\right) \sin \lambda M_t
- 4 \left(\frac{x' x(\sigma) + \sigma}{\sqrt{1 + (x')^2}} + x(\sigma) \sin \lambda\right) M_x \right] \epsilon_1
- \frac{U_7^{-1} V_7}{2 \sigma \cos \lambda} \left[ -x' \left(x(\sigma) + \frac{1}{\sqrt{1 + (x')^2}} + x(\sigma) \sin \lambda\right) \right] \left(4 \omega \tau M_t + \left(1 - 2 M_z \right) \omega^2 \tau^2 \right) \epsilon_1.
\]

(27)

Since \( \epsilon_0 \) and \( \epsilon_1 \) are constant spinors, the string configuration will be supersymmetric if \( x(\sigma) \) can be chosen so that all the dependence on \( \sigma \) and \( \tau \) goes away from the rhs of (27). Only the second term is \( \tau \)-dependent, which is cancelled provided that

\[
x'(\sigma) = -\tan \lambda, \quad \Leftrightarrow \quad x(\sigma) = x_0 - \tan \lambda \sigma.
\]

(28)

If this is the case, (27) becomes

\[
\epsilon_0 = U_7^{-1} V_7 (x_0 - 2 x_0 M_z - 2 M_z) \epsilon_1.
\]

(29)
Therefore, the string configuration
\[ t = \omega \tau, \quad x = x_0 - \tan \lambda \sigma, \quad z = \sigma, \quad \beta_\pm, \gamma_\pm, \varphi_\pm, \theta = \text{const}, \] (30)
is 1/2 BPS. The worldsheet ends along the line \( x = x_0 \) at the boundary and it is tilt by an angle \(-\lambda\). The induced geometry turns out to be that of an AdS_2 space. In what follows we will choose
\[ \omega = \frac{1}{\cos \lambda}, \] (31)
so that the induced metric becomes
\[ ds^2 = \frac{L^2}{\sigma^2 \cos^2 \lambda} \left( -d\tau^2 + d\sigma^2 \right), \] (32)
and the radius of the AdS_2 space is \( R := L/\cos \lambda \). As one might have expected, the configuration (30) solves the equations of motion for an open string that couples to the NS–NS B-field (see appendix B). It is precisely this coupling the reason for the tilt. The configuration (30) has been previously found as a classical solution in [25].

Finally, though in the rest of this paper we will focus on the case \( 0 \leq \lambda < \frac{\pi}{2} \), it is interesting to see what happens in the pure NS–NS limit. When \( \lambda \to \frac{\pi}{2} \) the worldsheet becomes parallel to the boundary, so the limit of (30) is just one particular case of a more general solution of the type
\[ t = \tau, \quad x = -\sigma, \quad z = z_0, \quad \beta_\pm, \gamma_\pm, \varphi_\pm, \theta = \text{const}. \] (33)
In this case we get
\[ \tilde{\Gamma} = -2M_z, \] (34)
and the \( \kappa \)-symmetry projection (155) translates into
\[ 2U^{-1}M_t U \epsilon_0 = -\epsilon_0, \] (35)
\[ 2V^{-1}M_t V \epsilon_1 = \epsilon_1. \] (36)
Using (10) and (11) we get
\[ 2U^{-1}M_t U = 2M_t [1 + 2(M_t + M_\sigma)(\tau - \sigma)], \] (37)
\[ 2V^{-1}M_t V = 2M_t [1 - 2(M_t - M_\sigma)(\tau + \sigma)]. \] (38)
Thus, in order to remove the \( \tau \)-dependence in (37) and (38) and get a BPS string, it suffices to impose
\[ (M_t + M_\sigma)\epsilon_0 = 0, \quad \Leftrightarrow \quad 2M_z \epsilon_0 = -\epsilon_0, \] (39)
\[ (M_t - M_\sigma)\epsilon_1 = 0, \quad \Leftrightarrow \quad 2M_z \epsilon_1 = \epsilon_1. \] (40)
The strings satisfying (33) preserve 8 independent Killing spinors for all values of \( z_0 \), and we get therefore a one-parameter family of 1/2 BPS strings which extend parallel to the boundary of AdS_3. It is immediate to see that the induced metric for these strings is flat.
2.1. Supersymmetric smeared strings

Generically, and because $U_7$ and $V_7$ depend on the values $\beta_\pm, \gamma_\pm, \varphi_\pm$, different string configurations (30) at different points of the internal space $S^3 \times S^3$ preserve different sets of supersymmetries. To conclude this section, we shall observe the existence of subspaces $\mathcal{M} \subset S^3 \times S^3 \times S^1$ such that all the strings sitting in $\mathcal{M}$ share some fraction of supersymmetry, which enables to have supersymmetric smeared string configurations. By smeared configurations we mean that the corresponding open string partition function should account for the superposition of strings with Dirichlet boundary conditions sitting at different points of a given $\mathcal{M}$.

One obvious possibility is to take $\mathcal{M}_1 = S^1$, as the Killing spinors are independent of the angle $\theta$. A configuration of strings smeared along that circle should therefore be 1/2 BPS as well. This is in stark contrast with the analogue cases in AdS$_5 \times S^5$ or AdS$_4 \times$ CP$^3$. Smearing open strings along compact subspaces in those cases breaks the supersymmetry either fully or partially.

Less obvious examples involve smearing in the $S^3 \times S^3$ factor. The angular dependence of the projection (29) is given by the matrix

$$U_7^{-1} V_7 = e^{-\varphi_+ M_{\varphi_+} - \varphi_- M_{\varphi_-}} e^{-\gamma_+ M_{\gamma_+} - \gamma_- M_{\gamma_-}} e^{-2(\beta_+ M_{\beta_+} + \beta_- M_{\beta_-})} \times e^{-\gamma_+ M_{\gamma_+} - \gamma_- M_{\gamma_-}} e^{-\varphi_+ M_{\varphi_+} - \varphi_- M_{\varphi_-}}. \quad (41)$$

As a concrete example, let us consider strings sitting in two maximal circles of the three-spheres, by setting $\beta_\pm = \gamma_\pm = 0$. Then

$$U_7^{-1} V_7 \big|_{\beta_\pm = \gamma_\pm = 0} = e^{-2(\varphi_+ M_{\varphi_+} + \varphi_- M_{\varphi_-})} e^{-\varphi_+ M_{\varphi_+} + \varphi_- M_{\varphi_-}} \times e^{-\varphi_+ M_{\varphi_+} - \varphi_- M_{\varphi_-}}, \quad (42)$$

which implies that, for spinors also satisfying

$$(M_{\varphi_+} + M_{\varphi_-}) \epsilon_1 = 0, \quad (43)$$

the matrix (42) acts trivially on $\epsilon_1$ if the strings additionally satisfy that $\varphi_+ - \varphi_- = 0$. Thus, a second example is obtained by additionally smearing along this diagonal circle. Strings sitting at the submanifold $\mathcal{M}_2 = S^1_0 \times S^1$

$$\beta_\pm = \gamma_\pm = 0, \quad \varphi_+ - \varphi_- = 0, \quad (44)$$

will be invariant under common supersymmetry transformations with 4 real parameters, because the additional projection (43) halves the number of supersymmetries. More specifically,

$$(M_{\varphi_+} + M_{\varphi_-}) \epsilon_1 = 0, \quad (45)$$
$$\epsilon_0 = (x_0 - 2x_0 M_t - 2M_t) \epsilon_1. \quad (46)$$

In analogy with the case of type IIA strings on AdS$_4 \times$ CP$^3$ analyzed in [3], we conclude that a string smeared over the region $S^1_0 \times S^1$ defined by (44) is 1/4 BPS. For different choices of the maximal circles one would encounter different projections but the smeared configurations will continue to be 1/4 BPS.
It is also possible to smear over larger submanifolds. A next example arises when we consider strings sitting in two maximal two-spheres, setting in this case $\beta_\pm = 0$. Smearing over $\mathcal{M}_3 = S^3_0 \times S^1 \supset \mathcal{M}_2$, where the diagonal two-sphere is defined by

$$\beta_\pm = 0, \quad \gamma_+ - \gamma_- = 0, \quad \varphi_+ - \varphi_- = 0,$$

the configuration is supersymmetric if, in addition to (45) and (46), one also imposes that

$$(M_{\gamma_+} + M_{\gamma_-})\epsilon_1 = 0.$$\hspace{1cm}(47)

With this additional projection the Killing spinors contain 2 real parameters, which means this smearing is 1/8 BPS.

As a final example, we can consider a string smeared over $\mathcal{M}_4 = S^3_0 \times S^1 \supset \mathcal{M}_3$ defined by

$$\beta_+ - \beta_- = 0, \quad \gamma_+ - \gamma_- = 0, \quad \varphi_+ - \varphi_- = 0.$$\hspace{1cm}(49)

To have common supersymmetries in this larger submanifold it should also be required that

$$(M_{\beta_+} + M_{\beta_-})\epsilon_1 = 0.$$\hspace{1cm}(50)

Notice, however, that

$$4M_{\beta_+}M_{\beta_-} = (4M_{\gamma_+}M_{\gamma_-})(4M_{\varphi_+}M_{\varphi_-}),$$

(51)

which implies that (50) is not an independent constraint with respect to (45) and (48). Then, a string smeared over the larger $\mathcal{M}_4$ will also be 1/8 BPS.

### 3. Fluctuations around classical solutions

In this section, the fact that delocalized string configurations can be 1/2 BPS, 1/4 BPS or 1/8 BPS will be derived in an alternative way. More precisely, we will analyze the fluctuations around the classical solution (30) up to quadratic order and study boundary conditions, other than Dirichlet, that can be consistent with supersymmetry. The mass spectrum for quantum corrections was calculated in [25] for the mixed-flux type IIB theory in $AdS_3 \times S^3 \times T^4$. We should extend the analysis to the $AdS_3 \times S^3 \times S^3 \times S^1$ case. We do this in detail in the appendix B. As shown there, a quadratic expansion of the action around (30) yields 7 massless scalars corresponding to fluctuations in the directions of $S^3 \times S^3 \times S^1$ and 1 massive scalar $\phi$, with mass $m_\phi^2 = 2/R_2^2$, for the fluctuation transverse to the worldsheet in $AdS_3$. The classical solution is sitting at a point in each of the three-spheres, which breaks their isometries $SO(4)$ down to $SO(3)$. The fluctuations must therefore accommodate into representations of these residual symmetries. Three of the scalar fluctuations $\phi_{a}^m$ form a $\mathbf{\Box \Box}_+$ of the $SO(3)_+$,\footnote{\textit{a, b} takes values 1 or 2 and $\phi_{a}^m$ is traceless.} another three $\phi_{a}^m$ form a $\mathbf{\Box \Box}_-$ of the $SO(3)_-$, and the remaining $\phi^0$ and $\phi^9$ are in the trivial representation of both $SO(3)$.

Concerning the fermionic fluctuations, after reducing them to $AdS_2$ spinors, we see that 4 of them are massless $\psi_{a}^m$, while the remaining 4 $\chi^{a\dagger}$ have masses $m_F = 1/R$. These two sets transform in the $\mathbf{\Box \Box}_+ \otimes \mathbf{\Box \Box}_-$ of $SO(3)_+ \times SO(3)_-$. 

The action for the quadratic fluctuations can be then written as
\[ S = -\frac{1}{2} \int d^2 \sigma \sqrt{-h} \left( \partial_a \phi^b \partial^a \phi^b + \frac{2}{R^2} (\phi^b)^2 + \frac{1}{2} \partial_a \phi^a \partial^b \phi^b + \frac{1}{2} \partial_a \phi^a \partial^b \phi^b + \partial_a \phi^a \partial^b \phi^b \right) \]

\[ - \frac{i}{2} \int d^2 \sigma \sqrt{-h} \left( \tilde{\psi}_{ia} \{ D^{(2)} \tilde{\psi}^{iai} + \tilde{\chi}_{ia} \left( \psi^{(2)} - \frac{1}{R} \right) \chi^{iai} \right), \]  

(52)

where \( h_{a,b} \) is the induced metric on the worldsheet, evaluated on the classical solution (30), and \( D^{(2)} \) is the covariant derivative in AdS2. In the following, the two-dimensional Dirac matrices will be denoted as \( \gamma^0 \) and \( \gamma^1 \), with \( \tau^3 := \gamma^0 \gamma^1 \) the chirality matrix. The supersymmetry transformations of the action (52), derived in the appendix B.2, are
\[ \delta \chi^{iai} = \frac{1}{2} \left( \tilde{\psi}_{ia} \gamma^0 \chi^{iai} + \frac{i}{2} \left[ \sin \Omega \ (\tilde{\psi} \gamma^0 \frac{\chi^{iai}}{R} \right) \kappa^{iai} + \cos \Omega \ (\tilde{\psi} \gamma^0 \frac{\chi^{iai}}{R} \right) \kappa^{iai} \right), \]  

(53)

\[ \delta \psi^{iai} = \frac{1}{2} \tilde{\psi}_{ia} \chi^{iai} + \frac{i}{2} \left( \cos \Omega \ (\tilde{\psi} \gamma^0 \frac{\chi^{iai}}{R} \right) - \sin \Omega \ (\tilde{\psi} \gamma^0 \frac{\chi^{iai}}{R} \right), \]  

(54)

\[ \delta \phi^0 = -\frac{1}{2} \tilde{\psi}_{ia} \chi^{iai}, \]  

(55)

\[ \delta \phi^b = -\frac{1}{2} \tilde{\psi}_{ia} \chi^{iai}, \]  

(56)

\[ \delta \phi^b = -\frac{i}{2} \left[ \cos \Omega \ (2 \tilde{\psi}_{ia} \kappa^{iai} - \delta^a_p \tilde{\psi}_{ia} \kappa^{iai} \right) - \sin \Omega \ (2 \tilde{\psi}_{ia} \kappa^{iai} - \delta^a_p \tilde{\psi}_{ia} \kappa^{iai} \right), \]  

(57)

\[ \delta \phi^b = \frac{i}{2} \left[ \sin \Omega \ (2 \tilde{\psi}_{ia} \kappa^{iai} - \delta^a_p \tilde{\psi}_{ia} \kappa^{iai} \right) + \cos \Omega \ (2 \tilde{\psi}_{ia} \kappa^{iai} - \delta^a_p \tilde{\psi}_{ia} \kappa^{iai} \right), \]  

(58)

where \( \kappa^{iai} \) are AdS2 Killing spinors, which satisfy
\[ \kappa^{iai} = a^{-1/2} e^{iai} + a^{1/2} \gamma^0 \tilde{e}^{iai}, \]  

(59)

with \( e(\tau) \) a spinor such that
\[ \tau^1 e^{iai} = -\tilde{e}^{iai}, \quad \tilde{e}^{iai} = 0. \]  

(60)

These constraints, in addition to the property (59) imply that \( e^{iai} \) parametrizes 8 real degrees of freedom, as expected.

3.1 Supersymmetric boundary conditions

In order to search for supersymmetric boundary conditions appropriate to describe BPS strings we shall now derive the asymptotic expansion of the suppressed transformations (53) to (58). Taking into account the usual \( \sigma \to 0 \) expansion for AdS2 fields we get
\[ \chi^{iai}(\tau, \sigma) = a^{-1/2} (\alpha^{iai}(\tau) - i a^{1/2} \gamma^0 \tilde{e}^{iai}(\tau) + \ldots) + a^{3/2} \left( \tilde{\psi}_{ia} \chi^{iai}(\tau) + \frac{1}{3} \sigma \tau^1 \tilde{\chi}^{iai}(\tau) + \ldots \right), \]  

(61)
\[
\psi^{\alpha i}(\tau, \sigma) = \sigma^{1/2} \left( \alpha^{\alpha i}_\chi(\tau) + \sigma \tau_3 \dot{\alpha}^{\alpha i}_\psi(\tau) + \ldots \right) + \sigma^{1/2} \left( \beta^{\alpha i}_\psi(\tau) + \sigma \tau_3 \dot{\beta}^{\alpha i}_\psi(\tau) + \ldots \right), \tag{62}
\]

\[
\phi^\alpha_j(\tau, \sigma) = (\alpha^\alpha_j(\tau) + \ldots) + \sigma (\beta^\alpha_j(\tau) + \ldots), \tag{63}
\]

\[
\phi^\alpha_\chi(\tau, \sigma) = \sigma^{-1} (\alpha^\alpha_\chi(\tau) + \ldots) + \sigma^2 (\beta^\alpha_\chi(\tau) + \ldots), \tag{64}
\]

\[
\phi^\alpha_\psi(\tau, \sigma) = \left( \alpha^\alpha_\psi(\tau) + \ldots \right) + \sigma \left( \beta^\alpha_\psi(\tau) + \ldots \right), \tag{65}
\]

\[
\phi^\alpha_\psi(\tau, \sigma) = \left( \alpha^\alpha_\psi(\tau) + \ldots \right) + \sigma \left( \beta^\alpha_\psi(\tau) + \ldots \right), \tag{66}
\]

for the fermionic and bosonic fluctuations, where

\[
\tau^4 \alpha^{\alpha i}_\chi = -\alpha^{\alpha i}_\chi, \quad \tau^4 \beta^{\alpha i}_\chi = \beta^{\alpha i}_\chi, \tag{67}
\]

\[
\tau^4 \alpha^{\alpha i}_\psi = -\alpha^{\alpha i}_\psi, \quad \tau^4 \beta^{\alpha i}_\psi = \beta^{\alpha i}_\psi. \tag{68}
\]

Due to the Breitenlohner–Freedman bound [23, 24] we must always take

\[
\alpha^{\alpha i}_\chi = 0 \quad \forall \alpha, \dot{a}; \quad \alpha_{\dot{a}} = 0, \tag{69}
\]

as boundary conditions for the \( \chi^{\alpha i} \) and \( \phi^\alpha \) fields. However, we have some freedom to impose more general boundary conditions on the other fluctuations. In the following, conditions such as \( \alpha^{\alpha i}_\chi = 0 \quad \forall \alpha, \dot{a} \) will be written simply as \( \alpha^{\alpha i}_\chi = 0 \), without explicitly specifying that indices without contraction take all possible values.

Then, taking (69) into consideration and inserting (61)–(66) into the transformations (53)–(58), the transformations relevant to the analysis of supersymmetric boundary conditions become

\[
\delta \alpha^{\alpha i}_\chi = \frac{i}{2R} \left( \sin \Omega \alpha^{\alpha i}_{\dot{a} \dot{b}} + \cos \Omega \alpha^{\alpha i}_{\dot{a} e} \right) \tag{70}
\]

\[
\delta \alpha^{\alpha i}_\psi = \frac{-1}{2R} \beta^{\alpha i}_{\dot{a} \dot{b}} - \frac{i}{2R} \left( \cos \Omega \beta^{\alpha i}_{\dot{a} e} - \sin \Omega \beta^{\alpha i}_{\dot{a} e} \right) \tag{71}
\]

\[
\delta \beta^{\alpha i}_\chi = \frac{-1}{2R} \alpha^{\alpha i}_{\dot{a} \dot{b}} - \frac{i}{2R} \tau^3 \left( \cos \Omega \alpha^{\alpha i}_{\dot{a} e} - \sin \Omega \alpha^{\alpha i}_{\dot{a} e} \right) \tag{72}
\]

\[
\delta \alpha_{\dot{a}} = \frac{1}{2} \beta^{\alpha i}_{\dot{a} \dot{b}} \epsilon^{\alpha i}_{\dot{a} \dot{b}} \tag{73}
\]

\[
\delta \beta_{\dot{a}} = \frac{1}{2} \left( \alpha^{\alpha i}_{\dot{a} \dot{b}} \tau^3 \epsilon^{\alpha i}_{\dot{a} \dot{b}} + \alpha^{\alpha i}_{\dot{a} e} \tau^3 \epsilon^{\alpha i}_{\dot{a} e} \right) \tag{74}
\]

\[
\delta \alpha_{\dot{a}} = 0 \tag{75}
\]

\[
\delta \alpha^{\alpha i}_\psi = -\frac{i}{2} \cos \Omega \left( 2 \beta^{\alpha i}_{\dot{a} \dot{b}} \epsilon^{\alpha i}_{\dot{a} \dot{b}} - \delta^{\alpha i}_\psi \beta^{\alpha i}_{\dot{a} \dot{b}} \epsilon^{\alpha i}_{\dot{a} \dot{b}} \right) \tag{76}
\]

\[
\delta \beta^{\alpha i}_\psi = -\frac{i}{2} \cos \Omega \left[ -2 \left( \alpha^{\alpha i}_{\dot{a} \dot{b}} \tau^3 \epsilon^{\alpha i}_{\dot{a} \dot{b}} + \alpha^{\alpha i}_{\dot{a} e} \tau^3 \epsilon^{\alpha i}_{\dot{a} e} \right) + \delta^{\alpha i}_\psi \left( \alpha^{\alpha i}_{\dot{a} \dot{b}} \tau^3 \epsilon^{\alpha i}_{\dot{a} \dot{b}} + \alpha^{\alpha i}_{\dot{a} e} \tau^3 \epsilon^{\alpha i}_{\dot{a} e} \right) \right] + \frac{i}{2} \sin \Omega \left( 2 \beta^{\alpha i}_{\dot{a} \dot{b}} \epsilon^{\alpha i}_{\dot{a} \dot{b}} - \delta^{\alpha i}_\psi \beta^{\alpha i}_{\dot{a} \dot{b}} \epsilon^{\alpha i}_{\dot{a} \dot{b}} \right) \tag{77}
\]
\[ \delta \alpha^a_b = i \frac{\sin \Omega}{2} \left( 2 \gamma^a_{\psi \epsilon} \epsilon^\epsilon \eta^\eta - \delta^a_b \gamma^a_{\psi \epsilon} \epsilon^\epsilon \right) \]

\[ \delta \beta^a_b = i \frac{\sin \Omega}{2} \left[ -2 \left( \gamma^a_{\psi \epsilon} \epsilon^\epsilon \eta^\eta \right) + \delta^a_b \gamma^a_{\psi \epsilon} \epsilon^\epsilon \right] + i \frac{\cos \Omega}{2} \left( 2 \gamma^a_{\psi \epsilon} \epsilon^\epsilon - \delta^a_b \gamma^a_{\psi \epsilon} \epsilon^\epsilon \right). \]

### 3.1. Dirichlet boundary conditions.

We will say that a set of boundary conditions is supersymmetric when the variation of them under (70)–(79) is vanishing for some non-trivial choice of parameters \( e^{a\alpha} \). It is straightforward to see from these expressions that Dirichlet boundary conditions

\[ \alpha^{a\alpha} = 0, \quad \beta^{a\alpha} = 0, \quad \gamma = 0, \quad \delta^a_b = 0, \quad \eta = 0, \quad \zeta = 0, \quad \tau = 0, \quad \nu = 0, \]

preserve the full set of supersymmetries of the action (52), and thus matches the fact that the string given by (30) is 1/2 BPS².

### 3.1.2. Smeared boundary conditions.

Let us now discuss the possibility of having delocalized supersymmetric string configurations. We presented various instances of supersymmetric smeared strings in section 2. It should also be possible to describe all of them in terms boundary conditions for the fluctuations.

Let us first focus on the case of strings delocalized over the submanifold \( \mathcal{M}_1 = S^\perp \). As discussed above, the partition function of a string smeared over a submanifold \( \mathcal{M} \) can be obtained as an integral over the partition functions of Dirichlet strings sitting on \( \mathcal{M} \). In terms of the fluctuations, a uniform smearing over \( \mathcal{M}_1 \) should correspond to imposing \( \alpha = \text{const.} \) and integrating over the constant, which is equivalent to imposing \( \alpha = 0 \). Therefore, our proposal is that the string uniformly smeared over \( \mathcal{M}_1 \) should be described by the following boundary conditions on the fluctuations:

\[ \alpha^{a\alpha} = 0, \quad \beta^{a\alpha} = 0, \quad \gamma = 0, \quad \delta^a_b = 0, \quad \eta = 0, \quad \zeta = 0, \quad \tau = 0. \]

By looking at the transformations (70)–(79) we see that these boundary conditions are invariant under the action of the 8 supercharges parametrized by the \( e^{a\alpha} \) spinors, in agreement with the results obtained in section 2 when analyzing the classical limit of this delocalized string.

On the other hand, for the string smeared over the submanifold \( \mathcal{M}_2 = S^\perp \times S^\perp \) given at (44) we must impose now

\[ \zeta^{\beta \pm} = 0, \quad \zeta^{\gamma \pm} = 0, \quad \zeta^{\gamma^+ - \gamma^-} = 0, \quad \zeta^\delta = 0, \]

on the \( \zeta^{\mu} \) fluctuations defined in appendix B. Using the \( \phi^a_{\mu} \) and \( \phi^a_{\nu} \) fields (see (194) and (195) for a definition in terms of \( \phi^a_{\mu} = e^{a\mu} \zeta^{\nu} \)) these boundary conditions can be expressed as

\[ \alpha_1^2 = \alpha_2^2 = 0, \quad \alpha_1^1 = \alpha_2^1 = 0, \quad \sin \Omega \alpha_1^1 - \cos \Omega \alpha_1^1 = 0, \quad \gamma = 0, \]

where the \( \sin \Omega \) and \( \cos \Omega \) factors come from the vielbein that relates the \( \zeta^{\mu} \) fields with the scalar fluctuations (we have assumed the classical string to be sitting at the origin). Conditions

²All the supersymmetries of the action for quadratic fluctuations are one half of the supersymmetries of the background.
(83) have to be supplemented with a condition that accounts for the smearing along the diagonal $S_0^2$. This condition is simply written as $\cos \Omega \dot{\alpha}_1^1 + \sin \Omega \dot{\alpha}_1^2 = 0$.

Looking again at the transformations (70)–(79), we see that

$$
\alpha_{x}^{\alpha} = 0, \quad \beta_{x}^{\alpha} = 0, \quad \alpha_{r} = 0, \quad \dot{\alpha}_{0} = 0,
\alpha_{1}^{1} = \alpha_{1}^{2} = 0, \quad \alpha_{2}^{1} = \alpha_{2}^{2} = 0, \quad \sin \Omega \alpha_{1}^{1} - \cos \Omega \alpha_{1}^{2} = 0, \quad \cos \Omega \dot{\alpha}_{1}^{1} + \sin \Omega \dot{\alpha}_{1}^{2} = 0,
$$

(84)
define a set of boundary conditions, which are invariant only under the supersymmetries parametrized by the spinors $\varepsilon^{12}$ and $\varepsilon^{21}$. The latter becomes clear when demanding that $\delta \alpha_{x}^{\alpha}$ must be 0. Thus, the boundary conditions (84) preserve 4 real supercharges, in agreement with the results presented in section 2 for the 1/4 BPS classical string smeared over $M_2$.

As for the string smeared over the submanifold $M_3 = S_0^3 \times S^1$ given at (47), in this case we must impose

$$
\zeta^{\beta 
= 0, \quad \zeta^{+} - \zeta^{-} = 0, \quad \zeta^{\varphi^{+}} - \zeta^{\varphi^{-}} = 0, \quad \zeta^{\theta} = 0,
$$

(85)
or, equivalently,

$$
\alpha_{1}^{1} + \alpha_{2}^{2} = 0, \quad \alpha_{1}^{2} + \alpha_{2}^{1} = 0, \quad \sin \Omega \alpha_{1}^{1} - \cos \Omega \alpha_{1}^{2} = 0, \quad \sin \Omega \alpha_{1}^{2} - \cos \Omega \alpha_{1}^{1} = 0, \quad \dot{\alpha}_{0} = 0.
$$

(86)
The conditions associated to the smearing are in this case $\cos \Omega \dot{\alpha}_{1}^{1} + \sin \Omega \dot{\alpha}_{1}^{2} = 0$ and $\cos \Omega \dot{\alpha}_{1}^{2} + \sin \Omega \dot{\alpha}_{1}^{1} = 0$. Then, the complete set of boundary conditions in this case is

$$
\alpha_{x}^{\alpha} = 0, \quad \beta_{x}^{\alpha} = 0, \quad \alpha_{r} = 0, \quad \dot{\alpha}_{0} = 0,
\alpha_{1}^{1} + \alpha_{2}^{2} = 0, \quad \sin \Omega \alpha_{1}^{1} - \cos \Omega \alpha_{1}^{2} = 0, \quad \cos \Omega \dot{\alpha}_{1}^{1} + \sin \Omega \dot{\alpha}_{1}^{2} = 0,
\alpha_{1}^{2} + \alpha_{2}^{1} = 0, \quad \sin \Omega \alpha_{1}^{2} - \cos \Omega \alpha_{1}^{1} = 0, \quad \cos \Omega \dot{\alpha}_{1}^{2} + \sin \Omega \dot{\alpha}_{1}^{1} = 0,
$$

(87)
and they are invariant under (70)–(79) for transformations which satisfy

$$
\varepsilon^{11} = \varepsilon^{22} = 0, \quad \varepsilon^{12} = -\varepsilon^{21}.
$$

(88)
These constraints are solved by 2 real supercharges, and therefore the conditions (87) match the results obtained previously for the classical string smeared over $M_3$.

The boundary conditions (87) can be slightly modified in order to describe the string smeared over the submanifold $M_4 = S_0^3 \times S^1$. In this case it suffices to impose

$$
\alpha_{x}^{\alpha} = 0, \quad \beta_{x}^{\alpha} = 0, \quad \alpha_{r} = 0, \quad \dot{\alpha}_{0} = 0,
\sin \Omega \alpha_{1}^{1} - \cos \Omega \alpha_{1}^{2} = 0, \quad \cos \Omega \dot{\alpha}_{1}^{1} + \sin \Omega \dot{\alpha}_{1}^{2} = 0,
\sin \Omega \alpha_{1}^{2} - \cos \Omega \alpha_{1}^{1} = 0, \quad \cos \Omega \dot{\alpha}_{1}^{2} + \sin \Omega \dot{\alpha}_{1}^{1} = 0,
$$

(89)
and they are invariant under (70)–(79) for transformations which satisfy

$$
\varepsilon^{11} = \varepsilon^{22} = 0, \quad \varepsilon^{12} = -\varepsilon^{21}.
$$

These conditions are again preserved only by the supersymmetries which solve (88), in agreement with the results expected for the 1/8 BPS string smeared over $M_4$. 

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3.1.3. Neumann boundary conditions. The analysis of fluctuations not only appears to be consistent with the results obtained when studying classical strings, but also enables us to characterize some other delocalized supersymmetric strings not accounted in section 2.1. Instead of smearing over regions $M \subset S^3 \times S^3 \times S^3$ we can also impose Neumann boundary conditions on the coordinates spanning them and ask in what cases they can be supersymmetric.

We can start by considering the imposition of a Neumann boundary condition for the fluctuation along $M_1$ and Dirichlet for the rest

$$\beta_0 = 0, \quad \alpha_0^\alpha = 0, \quad \alpha_0^\delta = 0, \quad \alpha_{tr} = 0. \quad (90)$$

A quick inspection of the transformations (70) to (79) reveals that (90) cannot preserve all the supersymmetries. In order to have $\delta \beta_0 = 0$ with the most general parameters the vanishing of all $\alpha_{vi}^{ai}$ would be needed. The variation of the latter depend on $\beta_0^a$ and $\beta_0^b$, which are not necessarily vanishing when imposing Dirichlet boundary conditions on $\phi_0^a$ and $\phi_0^b$.

Nonetheless, (90) can preserve some fraction of the supersymmetry if for the fermionic fluctuations we demand that

$$\alpha_{vi}^{ai} = 0, \quad \alpha_{vi}^{12} - \alpha_{vi}^{21} = 0, \quad \beta_{vi}^{12} + \beta_{vi}^{21} = 0, \quad \beta_{vi}^{11} = \beta_{vi}^{22} = 0. \quad (91)$$

In particular, (90) and (91) are invariant under transformations with $\epsilon_{11}^{11} = \epsilon_{22}^{22} = 0$ and $\epsilon_{12}^{12} = \epsilon_{21}^{21}$. This example represents a notorious difference between the AdS$_3 \times S^3 \times S^3 \times S^3$ and the AdS$_4 \times$ CP$^1$ cases. In this case, imposing $\dot{\alpha} = 0$ or $\beta = 0$ on the fluctuations of a given submanifold does not preserve the same amount of supersymmetry. This reinforces our proposal that the delocalized configurations presented in section 2.1 are accounted in terms of boundary conditions that fix $\dot{\alpha}$ for the corresponding scalar fluctuations.

Imposing Neumann boundary conditions for the fluctuation along $M_2$ can be supersymmetric if some fermionic boundary conditions are accordingly changed. More precisely, from (70) to (79) we find that the boundary conditions

$$\alpha_{tr} = 0, \quad \beta_0 = 0, \quad \alpha_{vi}^{ai} = 0, \quad \beta_{vi}^{11} = \beta_{vi}^{22} = \alpha_{vi}^{12} = \alpha_{vi}^{21} = 0,$$

$$\alpha_{2}^{1} = \alpha_{1}^{2} = 0, \quad \alpha_{1}^{1} = \alpha_{1}^{1} = 0, \quad \sin \Omega \alpha_{1}^{1} - \cos \Omega \alpha_{1}^{1} = 0, \quad \cos \Omega \beta_{1}^{1} + \sin \Omega \beta_{1}^{1} = 0,$$

(92)

are invariant under the supersymmetries parametrized by $\epsilon_{12}^{12}$ and $\epsilon_{21}^{21}$, and thus describe a 1/4 BPS string. In contrast to the previous example, in this case the supersymmetries preserved are exactly the same as the ones preserved in the smearing over $M_3$.

When turning to Neumann boundary conditions in $M_3$, we can impose

$$\alpha_{vi}^{ai} = 0, \quad \alpha_{vi}^{12} = \alpha_{vi}^{21} = 0,$$

$$\alpha_{vi}^{11} + \alpha_{vi}^{22} = 0, \quad \beta_{vi}^{11} + \beta_{vi}^{22} = 0,$$

$$\alpha_{1}^{1} + \alpha_{1}^{2} = \alpha_{2}^{1} + \alpha_{2}^{1} = 0, \quad \alpha_{tr} = 0, \quad \beta_0 = 0. \quad (93)$$

$$\sin \Omega \left( \alpha_{2}^{1} - \alpha_{2}^{2} \right) - \cos \Omega \left( \alpha_{2}^{1} - \alpha_{2}^{2} \right) = 0, \quad \sin \Omega \alpha_{1}^{1} - \cos \Omega \alpha_{1}^{1} = 0,$$

$$\cos \Omega \left( \beta_{2}^{1} - \beta_{2}^{2} \right) + \sin \Omega \left( \beta_{2}^{1} - \beta_{2}^{2} \right) = 0, \quad \cos \Omega \beta_{1}^{1} + \sin \Omega \beta_{1}^{1} = 0.$$

These conditions preserve the supersymmetries which satisfy (88) and therefore describe a 1/8 BPS string, as in the case (89) for the smearing over $M_3$. 

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Our final example is the case of Neumann boundary conditions for the fluctuations along \( M_4 \). In that case, the conditions
\[
\alpha^a_{\chi} = 0, \quad \alpha^a_{\psi} = 0, \quad \alpha_{\tau} = 0, \quad \beta_9 = 0,
\]
\[
\sin \Omega \alpha_2 - \cos \Omega \alpha_1 = 0, \quad \sin \Omega \alpha_1 - \cos \Omega \alpha_2 = 0,
\]
\[
\cos \Omega \beta_2 + \sin \Omega \beta_1 = 0, \quad \cos \Omega \beta_1 + \sin \Omega \beta_2 = 0,
\]

preserve only the supersymmetries which satisfy (88), and thus correspond to a 1/8 BPS string.

3.1.4. Interpolations. In the previous sections we have found diverse sets of 1/2 BPS, 1/4 BPS and 1/8 BPS strings which are described either by Dirichlet, smeared or Neumann boundary conditions. As we shall see in the following, there is a network of supersymmetric boundary conditions that allows us to interpolate between all those BPS strings.

Let us begin our analysis by studying the interpolation between the Dirichlet and smeared strings discussed in the previous sections. For this it is useful to consider first the case of the string smeared over \( M_1 = S^1 \), and to note that a Dirichlet string can be thought as a limiting case of such delocalized configuration. For the uniformly smeared string the partition function can be obtained by taking Dirichlet strings with boundary condition \( \phi^9 = \text{const} \) and then integrating over the values of such constant. We could in principle perform this integration with an arbitrary weight function and the configuration would continue to be supersymmetric. We can then think of the Dirichlet string as the limit in which the region of support of the weight function collapses to a point. Thus, by smoothly deforming the domain of support we can interpolate between the Dirichlet string and the string uniformly smeared over \( M_1 \). This idea can be generalized to interpolate between any of the strings considered in section 2. These configurations preserve the same amount of supersymmetry as the least supersymmetric endpoint of the interpolation and describe a marginal deformation in the dual dCFT.

We can also interpolate between the smeared strings of section 3.1.2 and the Neumann strings analyzed at section 3.1.3. This is similar to the mixed boundary conditions presented in [10] for the AdS_4 × CP^3 case. For example, we can interpolate between the smeared and Neumann boundary conditions over \( M_1 \) if we impose
\[
\alpha^a_{\chi} = 0, \quad \alpha^a_{\psi} = 0, \quad \alpha_{\tau} = 0, \quad \beta_9 = 0, \\
\beta_{12} = \beta_{21} = 0, \quad \left( \Lambda \alpha_{\psi}^{12} + \tau_3 \beta_{\psi}^{12} \right) - \left( \Lambda \alpha_{\psi}^{21} + \tau_3 \beta_{\psi}^{21} \right) = 0,
\]

\[
\alpha^a_b = \alpha^a_h = 0, \quad \alpha_{\tau} = 0, \quad \Lambda \beta_9 + \epsilon_9 = 0,
\]

(95)

where \( \Lambda \in \mathbb{R} \). As expected, this set of conditions is invariant under 2 of the supersymmetries of the action (52) for \( \Lambda > 0 \), and in the limit \( \Lambda = 0 \) preserves all the supersymmetries. Moreover, \( \Lambda \) is a dimensionless parameter and, from the perspective of the 1d dual dCFT, this interpolating boundary condition describes a superconformal marginal deformation.

In a similar way we can interpolate between all the smeared and Neumann boundary conditions presented in the previous sections. All those interpolating conditions preserve the same amount of supersymmetry as the least supersymmetric end of the flow, and describe marginal deformations in the dual dCFT (figure 1).
Figure 1. Examples of BPS delocalized string configurations. Arrows indicate BPS interpolations between them. The submanifolds are $\mathcal{M}_1 = S^1$, $\mathcal{M}_2 = S^1 \times S^1$, $\mathcal{M}_3 = S^2 \times S^1$ and $\mathcal{M}_4 = S^3 \times S^1$.

4. Conclusions

As we have argued in this paper, there must exist a rich family of line operators in the two-dimensional CFTs dual to string theories on $\text{AdS}_3 \times S^3 \times S^3 \times S^1$. The knowledge of their symmetries and supersymmetries can be useful to account for correlators in the superconformal lines and, from them, to infer physical information about the CFTs themselves.

The richness of BPS lines is associated with the diversity of supersymmetric open string boundary conditions that can be implemented in the holographic formulation: as shown, either Dirichlet or a wide variety of delocalized boundary conditions can be imposed in the compact space $S^3 \times S^3 \times S^1$ while still preserving supersymmetry. We have found that Dirichlet conditions account for 1/2 BPS strings, whereas delocalized strings range from 1/2 BPS to 1/8 BPS. In the case of delocalized strings, we have found that both smeared (i.e. fixing $\alpha$) for the scalar fields) or Neumann (i.e. fixing $\beta$) boundary conditions can be imposed preserving some amount of supersymmetry. In contrast to what was previously found in other backgrounds such as $\text{AdS}_5 \times S^5$ or $\text{AdS}_4 \times \text{CP}^3$, we have found cases in which smeared and Neumann boundary conditions along a same direction do not preserve the same amount of supersymmetry, and cases in which strings with smeared boundary conditions preserve the same number of supersymmetries as in the Dirichlet string. Finally, we have found that all the supersymmetric strings described are connected between each other by a network of interpolating BPS boundary conditions.

The fact that worldsheets ending along lines at the boundary can be supersymmetric for boundary conditions other than Dirichlet is due to the existence of massless fermionic excitations [10]. This is somehow reminiscent to the richness of the protected operators in spectrum of integrable AdS/CFT_2 backgrounds [26–28], whose origin is also attributed to the existence of massless fermionic excitations in the integrable description [29]. It would be interesting to further investigate the relation between these two properties.

A further characterization of the line operators described in the above sections could be achieved by computing correlations between the components of the displacement multiplet, which are in correspondence with the fluctuations around classical worldsheet configurations. This could be done by extending the analysis of [30, 31, 37] to the open strings presented in this paper for the $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ background. The displacement multiplet constitutes a particular representation of the supergroup of symmetries of the line defect. Correspondingly, excitations on the worldsheet should fit into representations of the symmetry group of the classical string. The symmetry group of the type IIB theory in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ is the direct product $D(2, 1; \sin^2 \Omega) \otimes D(2, 1; \sin^2 \Omega)$, where $D(2, 1; \sin^2 \Omega)$ is an exceptional supergroup whose bosonic subalgebra is $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ [26, 32]. This suggests that the classical string described in this paper is invariant under a single $D(2, 1; \sin^2 \Omega)$, and the
fluctuations in the case of Dirichlet boundary conditions should form a representation of this supergroup. Similarly, fluctuations in the case of smeared or Neumann boundary conditions should transform in representations of smaller subgroups of $D(2, 1; \sin^2 \Omega)$. The existence of a rich moduli of BPS line operators in the dual CFTs allows for many future non-trivial comparisons between different computations of correlators of displacement multiplet components by employing superconformal bootstrap techniques in the CFT side or by using Witten diagram on the worldsheet, as in [36, 37].

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. $\text{AdS}_3 \times S^1 \times S^3 \times S^1$ Killing spinors

For the coordinates (5) and (6) and the choice of vielbeins

$$e^0 = \frac{L}{z} \, dt, \quad e^1 = \frac{L}{z} \, dx, \quad e^2 = \frac{L}{z} \, dz,$$

$$e^3 = \frac{L}{\sin \Omega} \, d\beta_+, \quad e^4 = \frac{L}{\sin \Omega} \cos \beta_+ \, d\gamma_+, \quad e^5 = \frac{L}{\sin \Omega} \cos \beta_+ \cos \gamma_+ \, d\varphi_+,$$

$$e^6 = \frac{L}{\cos \Omega} \, d\beta_- , \quad e^7 = \frac{L}{\cos \Omega} \cos \beta_- \, d\gamma_- , \quad e^8 = \frac{L}{\cos \Omega} \cos \beta_- \cos \gamma_- \, d\varphi_- ,$$

$$e^9 = l \, d\theta ,$$

the non-vanishing components of the spin connections turn out to be

$$\omega^{20} = \frac{1}{z} \, dt, \quad \omega^{21} = \frac{1}{z} \, dx,$$

$$\omega^{34} = \sin \beta_+ \, d\gamma_+ , \quad \omega^{35} = \sin \beta_+ \cos \gamma_+ \, d\varphi_+, \quad \omega^{45} = \sin \gamma_+ \, d\varphi_+ ,$$

$$\omega^{67} = \sin \beta_- \, d\gamma_-, \quad \omega^{68} = \sin \beta_- \cos \gamma_- \, d\varphi_-, \quad \omega^{78} = \sin \gamma_- \, d\varphi_-.$$
\[ G_{(3)} = -e^{-\Phi/2}H_{(3)} - i e^{\Phi/2}F_{(3)} \]
\[ = 2iL^2 e^{-\Phi/2+\lambda} \left( \text{vol(AdS}_3) + \frac{1}{\sin^2 \Omega} \text{vol(S}^3_+) + \frac{1}{\cos^2 \Omega} \text{vol(S}^3_-) \right). \quad (103) \]

More precisely, for a real representation of Dirac matrices \( \Gamma_\mu = e^{\eta_\mu} \gamma_m \), a Killing spinor \( \epsilon \) satisfies
\[ \Gamma^{\mu\nu\rho} G_{\mu\nu\rho} \epsilon = 0, \quad (104) \]
\[ D_\mu \epsilon + \frac{e^{\Phi/2}}{96} \left( 9 \Gamma^{\mu\nu} G_{\mu\nu\rho} - \Gamma^{\mu\nu\rho} G_{\mu\nu\rho} \right) \epsilon^* = 0, \quad (105) \]
where \( D_\mu \) stands for the covariant derivative
\[ D_\mu = \partial_\mu + \frac{1}{4} \omega^m_{\mu \nu} \gamma_m. \quad (106) \]

Additionally, the Killing spinors are Weyl spinors obeying
\[ \gamma_{11} \epsilon = -\epsilon, \quad (107) \]
where \( \gamma_{11} = \gamma_0\gamma_1 \ldots \gamma_9 \). Defining
\[ \gamma_+ = \gamma_0 \gamma_1 \gamma_2, \quad \gamma_+^* = i \gamma_3 \gamma_4 \gamma_5, \quad \gamma_- = i \gamma_6 \gamma_7 \gamma_8, \quad (108) \]
and
\[ P_\Sigma^\pm := \frac{1}{2} (1 \pm \Sigma) \quad \text{for} \ \Sigma := i \left( \sin \Omega \gamma_+ \gamma_- + \cos \Omega \gamma_+ \gamma_- \right) \quad (109) \]
the condition \( (104) \) becomes
\[ P_\Sigma^\epsilon \epsilon = 0. \quad (110) \]

Since the condition \( (105) \) relates \( \epsilon \) with \( \epsilon^* \) it is convenient to separate the Killing spinor into
\[ \epsilon = \eta + i \xi. \quad (111) \]

For the different values of \( \mu \), and using the condition \( (110) \), the equation \( (105) \) becomes
\[ \partial_t \eta - \frac{1}{z} M_\eta \eta - \frac{1}{z} \sin \lambda M_\eta - \frac{1}{z} \cos \lambda M_\xi \eta = 0, \quad (112) \]
\[ \partial_t \xi - \frac{1}{z} M_\xi \eta + \frac{1}{z} \sin \lambda M_\xi + \frac{1}{z} \cos \lambda M_\eta \xi = 0, \]
\[ \partial_x \eta - \frac{1}{z} M_\eta \xi - \frac{1}{z} \sin \lambda M_\xi \eta - \frac{1}{z} \cos \lambda M_\xi = 0, \quad (113) \]
\[ \partial_x \xi - \frac{1}{z} M_\xi \xi + \frac{1}{z} \sin \lambda M_\xi \xi - \frac{1}{z} \cos \lambda M_\xi \xi = 0. \]
Replacing in (116) we find a relation between disentangled by taking an additional derivative with respect to $\beta$ the Killing spinors are independent of expression for the Killing spinors. We can start with equation (115), which simply state that $M$

where the matrices $M$ are the ones defined in (12)–(14).

By solving the Killing equations (112)–(120) one by one, we will construct a factorized expression for the Killing spinors. We can start with equation (115), which simply state that the Killing spinors are independent of $\theta$. Next, we consider the equation (116). They can be disentangled by taking an additional derivative with respect to $\beta_\pm$, which gives

$$\partial^2_{\beta_\pm} \eta = M^2_{\beta_\pm} \eta, \quad \partial^2_{\beta_\pm} \xi = M^2_{\beta_\pm} \xi.$$  \hspace{1cm} (121)

Let us first solve for the $\beta_+$ dependence. In order to fulfill (121) we need

$$\eta = e^{\beta_+ M_{\beta_+}} a + e^{-\beta_+ M_{\beta_+}} b,$$  \hspace{1cm} (122)

$$\xi = e^{\beta_+ M_{\beta_+}} c + e^{-\beta_+ M_{\beta_+}} d.$$  \hspace{1cm} (123)

Replacing in (116) we find a relation between $a$ and $c$ and between $b$ and $d$. Defining $U_2 = e^{\beta_+ M_{\beta_+}}$ and $V_2 = e^{-\beta_+ M_{\beta_+}}$:

$$\eta = U_2 \epsilon_0^{(2)} + \frac{\cos \lambda}{\sin \lambda + 1} V_2 \epsilon_1^{(2)},$$  \hspace{1cm} (124)

$$\xi = -\frac{\cos \lambda}{\sin \lambda + 1} U_2 \epsilon_0^{(2)} + V_2 \epsilon_1^{(2)},$$  \hspace{1cm} (125)

where $\epsilon_0^{(2)}$ and $\epsilon_1^{(2)}$ are not constant but independent of $\theta$ and $\beta_\pm$. In the same way we can deal with the $\beta_-$ dependence. Now we define $U_3 = e^{\beta_- M_{\beta_-}} e^{\beta_- M_{\beta_-}}$ and $V_3 = e^{-\beta_- M_{\beta_-}} e^{-\beta_- M_{\beta_-}}$ and then
\[ \eta = U_3 \epsilon_0^{(3)} + \frac{\cos \lambda}{\sin \lambda + 1} V_3 \epsilon_1^{(3)}, \quad (126) \]

\[ \xi = -\frac{\cos \lambda}{\sin \lambda + 1} U_3 \epsilon_0^{(3)} + V_3 \epsilon_1^{(3)}. \quad (127) \]

It is straightforward to extend this to account completely for the dependence on the compact space coordinates

\[ \eta = U_7 \epsilon_0^{(7)}(t, x, z) + \frac{\cos \lambda}{\sin \lambda + 1} V_7 \epsilon_1^{(7)}(t, x, z), \quad (128) \]

\[ \xi = -\frac{\cos \lambda}{\sin \lambda + 1} U_7 \epsilon_0^{(7)}(t, x, z) + V_7 \epsilon_1^{(7)}(t, x, z), \quad (129) \]

where

\[ U_7 = e^{\beta + \gamma + \delta + \epsilon + \zeta} \cos \lambda \sin \lambda + 1 \]

\[ V_7 = e^{-\beta - \gamma - \delta - \epsilon - \zeta}. \quad (130) \]

Finally, one can solve for the remaining equations (112)–(114) to obtain

\[ \epsilon_0^{(7)}(t, x, z) = e^{\log \sqrt{z} + (x + y)(M_t + M_x)} \epsilon_0 \]

\[ = \frac{1}{2} \left[ \frac{1}{\sqrt{z}} (1 - 2M_z) (1 + 2(x + t)M_t + \sqrt{z}(1 + 2M_z)) \epsilon_0, \quad (132) \right. \]

\[ \epsilon_1^{(7)}(t, x, z) = e^{-\log \sqrt{z} + (x - y)(M_t - M_x)} \epsilon_1 \]

\[ = \frac{1}{2} \left[ \frac{1}{\sqrt{z}} (1 + 2M_z) (1 + 2(x - t)M_t + \sqrt{z}(1 - 2M_z)) \epsilon_1. \quad (133) \right. \]

Appendix B. Quadratic fluctuations

Before presenting the mass spectrum for the quadratic fluctuations around (30), let us see that it also emerges as a solution to the string equations of motion. The action for the bosonic coordinates of the string coupled to the NS–NS B-field is

\[ S_B = -\int d^2 \sigma \sqrt{-h} + \frac{1}{2} \int d^2 \sigma \epsilon^{\alpha \beta} B_{\alpha \beta}. \quad (134) \]

Here \( h_{\alpha \beta} \) and \( B_{\alpha \beta} \) are the induced metric and the pullback of the NS–NS B-field on the worldsheet, respectively. For the B-field we will use the gauge

\[ B = 2L^2 \sin \lambda \left( \frac{1}{\sqrt{z}} dx \wedge dz + \frac{\varphi_+}{\sin^2 \Omega} \cos^2 \beta_+ \cos \gamma_+ d\beta_+ \wedge d\gamma_+ \right. \]

\[ + \frac{\varphi_-}{\cos^2 \Omega} \cos^2 \beta_- \cos \gamma_- d\beta_- \wedge d\gamma_-. \quad (135) \]
Making the ansatz
\[ t = \omega \tau, \quad x = x(\sigma), \quad z = \sigma, \quad \beta_\pm, \gamma_\pm, \varphi_\pm, \theta = \text{const.} \quad (136)\]
the equations of motions reduce to
\[ 2 \left( 1 + (x')^2 \right) \left( x' + \sin \lambda \sqrt{1 + (x')^2} \right) - \sigma x'' = 0, \quad (137)\]
which is straightforwardly solved by \( x' = -\tan \lambda \).

Let us now turn to the mass spectrum of the bosonic fluctuations. The standard approach would be to perform a series expansion of the action \((134)\) in powers of \( \delta X^\mu := X^\mu - X^\mu_{\text{cl}} \), with \( X^\mu_{\text{cl}} \) the classical solution. However, and because the coefficients of such expansion will not be manifestly covariant \([33, 34]\), it is more convenient to make the expansion in terms of Riemann normal coordinates. Let \( X^\mu(q) \) be a geodesic such that \( X^\mu(0) = X^\mu_{\text{cl}} \). Using the geodesic equation we get
\[ X^\mu(q) = X^\mu_{\text{cl}} + q \zeta^\mu - \frac{q^2}{2} \Gamma^\mu_{\nu\sigma}(X^\mu_{\text{cl}}) \zeta^\nu \zeta^\sigma + O(q^3), \quad (138)\]
where we have defined
\[ \zeta^\mu := \frac{dX^\mu}{dq} \bigg|_{q=0}. \quad (139)\]
If we now look for a geodesic with \( X^\mu(1) = \bar{X}^\mu \), we then have
\[ \bar{X}^\mu = X^\mu_{\text{cl}} + \zeta^\mu - \frac{1}{2} \Gamma^\mu_{\nu\sigma}(X^\mu_{\text{cl}}) \zeta^\nu \zeta^\sigma + O(\zeta^3), \quad (140)\]
which serves as an expansion of \( \bar{X}^\mu \) around \( X^\mu_{\text{cl}} \) in powers of a vector \( \zeta^\mu \). Moreover, in order to get an expansion in terms of scalar fields, it is useful to define
\[ \phi^m = \zeta^\mu \epsilon^\mu_m, \quad (141)\]
where \( \epsilon^\mu_m \) are the inverse of the background-space vielbeins given in \((96)–(99)\).

Plugging \((140)\) into \((134)\) we get, up to some boundary terms which can be discarded by the addition of counterterms,
\[ S_B = S_{\text{cl}} - \frac{1}{2} \int d^2 \sigma \sqrt{-h_{\text{cl}}} \left[ \partial_\alpha \phi^\mu \partial^\alpha \phi^\mu + \frac{2}{R^2} (\phi^\mu)^2 + \sum_{j=3}^9 \left( \partial_\alpha \phi^j \partial^\alpha \phi^j \right) \right], \quad (142)\]
where \( S_{\text{cl}} \) and \((h_{\alpha\beta})_{\text{cl}}\) stand respectively for the action and the induced metric evaluated in the classical solution (in what follows we will refer to \((h_{\alpha\beta})_{\text{cl}}\) simply as \( h_{\alpha\beta}\)), and \( R = L / \cos \lambda \) is the radius of the induced \( \text{AdS}_2 \). \( \phi^\mu \) stands for the transverse fluctuation in \( \text{AdS}_3 \), defined as
\[ \phi^\mu(\tau, \sigma) := \cos \lambda \phi^1(\tau, \sigma) + \sin \lambda \phi^2(\tau, \sigma). \quad (143)\]
Then, we see from \((142)\) that the spectrum of bosonic fluctuations consists in 1 massive (with mass \( m_\mu^2 = \frac{2}{R^2} \)) and 7 massless scalar fields.
Regarding the fermionic fluctuations $\Theta^1$ and $\Theta^2$, and in order to simplify the calculations, we will fix the $\kappa$-symmetry gauge condition to be $\Theta^1 = \Theta^2 := \Theta$, as customary for open strings in type IIB backgrounds [33]. Then, up to quadratic order in the fluctuations the action reads [35]

$$S_F = S_{F1} + S_{F2} + S_{F3},$$

with

$$S_{F1} := -2i \int d^2 \sigma \sqrt{-h} \, h^{\alpha \beta} \bar{\Theta} \Gamma_{\alpha} D_{\beta} \Theta,$$

$$S_{F2} := -\frac{ie^\phi}{24} \int d^2 \sigma \sqrt{-h} \, h^{\alpha \beta} \bar{\Theta} \Gamma_{\alpha} \Gamma^{\lambda \rho} F_{\lambda \rho} \Gamma_{\beta} \Theta = \frac{i}{R} \int d^2 \sigma \sqrt{-h} \, \Theta \gamma_\gamma P \Sigma \Theta,$$

$$S_{F3} := -\frac{i}{2} \int d^2 \sigma \, \epsilon^{\alpha \beta} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu \bar{\Theta} \Gamma^\mu \Gamma^\rho H_{\rho \nu} \Theta = 0,$$

where

$$\Gamma_{\alpha} := \partial_{\alpha} X^\mu \Gamma_{\mu}, \quad D_{\alpha} := \partial_{\alpha} + \frac{1}{4} \tilde{\omega}_{ab} \gamma_{ab},$$

with \( \tilde{\gamma}^a := \tilde{e}_a^a \Gamma^a \), for \( \tilde{e}_a^a \) and \( \tilde{\omega}_{ab} \) the vielbeins and spin connections of AdS$_2$. Further splitting $\Theta$ as

$$\Theta = \Theta_0 + \Theta_1, \quad \text{for } P^\Sigma_+ \Theta = \Theta_0, \quad P^\Sigma_0 \Theta = \Theta_1,$$

the total action for the fermionic fluctuations becomes

$$S_F = -2i \int d^2 \sigma \sqrt{-h} \, \Theta_0 D_\alpha \Theta_0 + \bar{\Theta}_1 \left( \Gamma^\alpha D_\alpha - \frac{1}{R} \right) \Theta_1.$$

### B.1. Supersymmetry of the fluctuations

The Green–Schwarz action is invariant under supersymmetry transformations

$$\delta_{\text{susy}} X^\mu = \bar{\eta} \Gamma^\mu \Theta^1 + \bar{\xi} \Gamma^\mu \Theta^2, \quad \delta_{\text{susy}} \Theta^1 = \eta, \quad \delta_{\text{susy}} \Theta^2 = \xi,$$

for $\eta$ and $\xi$ the real and imaginary parts of the Killing spinors (111). It is also invariant under the $\kappa$-symmetry transformations

$$\delta_{\kappa} X^\mu = \bar{\Theta}^1 \Gamma^\mu P_+^\kappa \Theta^1 + \bar{\Theta}^2 \Gamma^\mu P_-^\kappa \Theta^2, \quad \delta_{\kappa} \Theta^1 = P_+^\kappa \Theta^1, \quad \delta_{\kappa} \Theta^2 = P_-^\kappa \Theta^2,$$

where the projectors

$$P_{\pm}^\kappa = \frac{1}{2} \left( 1 \pm \Gamma \right), \quad \Gamma = \frac{\epsilon^{\alpha \beta \mu} \Pi_{\alpha}^\mu \Pi_{\beta}^\mu}{2 \sqrt{-H}},$$

are defined in terms of

$$\Pi_{\mu}^\kappa = \partial_{\alpha} X^\mu - \bar{\Theta}^1 \Gamma^\mu \partial_{\alpha} \Theta^1 - \bar{\Theta}^2 \Gamma^\mu \partial_{\alpha} \Theta^2, \quad H_{\alpha \beta} = g_{\mu \nu} \Pi_{\alpha}^\mu \Pi_{\beta}^\nu.$$
Whenever a combined transformation \( \delta := \delta_{\text{susy}} + \delta_\kappa \) leaves a configuration invariant, we say it is supersymmetric. A classical configuration with vanishing fermions is then supersymmetric for \( \kappa_1 = -\eta \) and \( \kappa_2 = -\xi \) if
\[
P^c_+ \eta = 0, \quad P^c_+ \xi = 0, \quad \Leftrightarrow \quad \tilde{\Gamma} K \epsilon = \epsilon,
\]
where \( K \) indicates complex conjugation.

As we have shown, the string configuration (30) is supersymmetric. In what follows we shall derive the supersymmetry transformations for the fluctuations around it. The preservation of the \( \kappa \)-symmetry gauge condition requires \( \delta \Theta_1 = \delta \Theta_2 \) and from this (labelling with \((0),(1), \ldots \) the successive orders of the expansion in powers of the fluctuations), we get
\[
\eta_{(1)} + P^c_{+ (0)} \eta_{(1)} - P^c_{+ (1)} \eta_{(0)} = \xi_{(1)} + P^c_{- (0)} \xi_{(1)} - P^c_{- (1)} \xi_{(0)}.
\]

For the variation of the bosonic transverse fluctuations we obtain
\[
\delta \phi^tr = -2 \bar{\Theta} \gamma^r (\eta_{(0)} + \xi_{(0)}), \quad \delta \phi^a = -2 \bar{\Theta} \gamma^a (\eta_{(0)} + \xi_{(0)}), \quad a = 3, \ldots, 9
\]

while for the fermionic fluctuations we get
\[
\delta \Theta = \frac{1}{2} (\eta_{(1)} + \xi_{(1)}) - \frac{1}{2} \tilde{\Gamma}_{(0)} (\eta_{(0)} + \xi_{(0)}),
\]
where
\[
\tilde{\Gamma}_{(0)} = -2 \sin \lambda M_z - 2 \cos \lambda M_x := -2 M_{tr},
\]
\[
\tilde{\Gamma}_{(1)} = -2 \left[ \frac{\sigma}{R} \partial_r \phi^r M_r - \frac{1}{R} (\phi^r + \sigma \partial_\sigma \phi^r) M_{lg} + \frac{2}{2R} \sum_{a=3}^{9} \gamma^a \left( \partial_r \phi^a \gamma^r + \partial_\sigma \phi^a \gamma^0 \right) \right],
\]
with
\[
M_{lg} = \cos \lambda M_z - \sin \lambda M_x.
\]

For getting (159) we have used that
\[
\eta_{(0)} - \xi_{(0)} = \tilde{\Gamma}_{(0)} (\eta_{(0)} + \xi_{(0)}).
\]

Using this relation, it is also possible to express all the dependence of \( \delta \Theta \) on the Killing spinors through \( \eta_{(0)} + \xi_{(0)} \), since

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3The total transformation must preserve the static gauge choice, which requires \( \delta \chi^0 = \delta \chi^9 = 0 \). In order to achieve this, a diffeomorphism should be added to the total transformation \( \delta \). As the transformation should be first order in the fluctuations, \( \delta \ln X^\mu = -\partial_\mu \chi^0 \delta \sigma^0 \). Then, this additional transformation does not affect the transverse fluctuations and \( \delta \sigma^a \) can be chosen so that \( \delta \chi^0 = \delta \chi^9 = 0 \). Similarly, \( \delta \eta^a = -\partial_\mu \eta^0 \delta \sigma^0 = 0 \) and so the diffeomorphism does not modifies the transformation of the fermionic fluctuations either.
\[ \eta(1) + \xi(1) = -\frac{1}{L} \left[ \left( \phi^\mu M_\mu + \sin \Omega \sum_{a=3}^{5} \phi^a M_a + \cos \Omega \sum_{a=6}^{8} \phi^a M_a \right) \right. \\
\times \left( \cos \lambda - \sin \lambda \tilde{\Gamma}(0) \right) \right] (\eta(0) + \xi(0)), \]

\[ \eta(1) - \xi(1) = \frac{1}{L} \left[ \left( \phi^\mu M_\mu + \sin \Omega \sum_{a=3}^{5} \phi^a M_a + \cos \Omega \sum_{a=6}^{8} \phi^a M_a \right) \right. \\
\times \left( \sin \lambda + \cos \lambda \tilde{\Gamma}(0) \right) \right] (\eta(0) + \xi(0)). \]

Thus, splitting into the variation of massless and massive fermions, taking into account that

\[ \Theta_0 = P^\Sigma(\Theta), \Theta_1 = P^\Sigma_\perp(\Theta) \]

and that \( P^\Sigma(\eta(0) + \xi(0)) = 0 \), we have

\[ \delta \Theta_0 = \left( - \cos \Omega \sum_{a=3}^{5} \Gamma^a \partial_\alpha \phi^a M_a + \sin \Omega \sum_{a=6}^{8} \Gamma^a \partial_\alpha \phi^a M_a + \frac{1}{2} \Gamma^a \partial_\alpha \phi^a \right) \gamma^9(\eta(0) + \xi(0)), \]

\[ \delta \Theta_1 = \left( \frac{1}{2} \Gamma^a \partial_\alpha \phi^a \gamma^9 - \frac{1}{R} \phi^\mu M_\mu + \sin \Omega \sum_{a=3}^{5} \left( \Gamma^a \partial_\alpha \phi^a + \frac{1}{R} \phi^a \right) M_a \right) \gamma^9(\eta(0) + \xi(0)). \]

B.2. Fluctuations as AdS$_2$ fields

In the previous subsection, fermionic fluctuations and Killing fields are 32-component spinors. We can alternatively describe the quadratic fluctuations and their supersymmetry transformations in terms of two-components AdS$_2$ spinors. In order to do that, it is necessary to identify longitudinal and transverse directions (to the worldsheet) and decompose the Dirac matrices into products as \( SO(1,1) \) and \( SO(8) \) matrices:

\[ \gamma^0 = \tau^0 \otimes 1_{16}, \]

\[ \gamma^9 = - \sin \lambda \gamma^1 + \cos \lambda \gamma^2 = \tau^1 \otimes 1_{16}, \]

\[ \gamma^\mu = \cos \lambda \gamma^1 + \sin \lambda \gamma^2 = \tau_3 \otimes \rho^8, \]

\[ \gamma^m = \tau_3 \otimes \rho^{m-2}, \text{ for } m \geq 3, \]

where \( \tau^a \) and \( \rho^4 \) are \( SO(1,1) \) and \( SO(8) \) Dirac matrices. Then, the 32-component spinors can be written as \( \Theta = \psi(\tau, \sigma) \otimes \varphi \), where \( \psi(\tau, \sigma) \) are two-component worldsheet spinors and \( \varphi \) are constant 16-component spinors. All these spinors can be accommodated into representations of the \( SU(2) \) rotations that leave invariant the \( \Sigma \) projections (149). Thus,
\[ \Theta_0 = \psi_{\mu}^{\dot{a}} \otimes \varphi_{\mu a}^{+++} + \psi_{-}^{\dot{a}} \otimes \varphi_{\mu a}^{-} \quad \Theta_1 = \chi_{+}^{\dot{a}} \otimes \varphi_{\mu a}^{++} + \chi_{-}^{\dot{a}} \otimes \varphi_{\mu a}^{--}, \]

where \( a \) and \( \dot{a} \) are fundamental SU(2) indices and

\[ \tau_3 \psi_{\pm} = \pm \psi_{\pm}, \quad \rho \varphi^{\pm \pm} = \pm \varphi^{\pm \pm}, \quad \sigma \varphi^{\pm \pm} = \pm \varphi^{\pm \pm}, \]

for \( \tau_3 = \tau^0 \tau^1, \quad \rho = \tau^1 \ldots \tau^8, \quad \sigma = \sin \Omega \rho^0 \rho^1 \rho^2 \rho^3 + \cos \Omega \rho^4 \rho^5 \rho^6. \]

It is easy and convenient to use a basis in which the constant spinors satisfy

\[ (\varphi_{\pm}^{\dot{a}})_{\mp \pm} = \frac{1}{4} \delta_{\pm}^{\dot{a}} \delta_{\mp \pm}^{a}, \quad (\varphi_{\pm}^{\dot{a}})_{\pm \pm} = \frac{1}{4} \delta_{\pm}^{\dot{a}} \delta_{\pm \pm}^{a}, \]

and 0 otherwise.

Substituting in (150), the action for the fermionic fluctuations becomes

\[ S_F = -\frac{i}{2} \int d^2 \sigma \sqrt{-h} \left( \bar{\psi}_{\mu a}^{\dot{a}} \mathcal{D}^{(2)} \psi_{\mu a}^{\dot{a}} + \bar{\chi}_{\mu}^{\dot{a}} \left( \mathcal{D}^{(2)} - \frac{1}{R} \right) \chi_{\mu}^{\dot{a}} \right), \]

where

\[ \psi_{\mu a}^{\dot{a}} = \psi_{\mu a}^{\dot{a}} + \psi_{\mu a}^{\dot{a}}, \quad \chi_{\mu}^{\dot{a}} = \chi_{\mu}^{\dot{a}} + \chi_{\mu}^{\dot{a}}, \]

and (2) indicates that Dirac matrices and covariant derivatives are now those of AdS_2. This is the action of four massless and four massive (with mass \( \frac{1}{2} \)) worldsheet fermions.

We would like to also have expressions for the supersymmetry transformations in terms of AdS_2 spinors. For this we would need the decomposition of \( \eta_0 + \xi_0 \) as well. In terms of the matrices \( U \) and \( V \) (8) and (9) we can write

\[ \eta_0(0) + \xi_0(0) = (1 - p(\lambda)) U(0) \xi_0 + (1 + p(\lambda)) V(0) \xi_1, \quad \text{for} \ p(\lambda) = \frac{\cos \lambda}{\sin \lambda + 1}. \]

4 In the need of an explicit basis we shall use:

\[ \tau^0 = i \sigma^3, \quad \tau^1 = \sigma^3, \]

and

\[ \rho^1 = \sigma^2 \otimes \sigma^2 \otimes \sigma^1 \otimes I_1, \quad \rho^2 = \sigma^2 \otimes \sigma^2 \otimes \sigma^1 \otimes I_2, \quad \rho^3 = \sigma^2 \otimes \sigma^3 \otimes I_1 \otimes \sigma^2, \quad \rho^4 = \sigma^2 \otimes \sigma^3 \otimes I_2 \otimes \sigma^2, \]

\[ \rho^{(2)} = \sigma^2 \otimes I_1 \otimes \sigma^1 \otimes \sigma^1, \quad \rho^{(3)} = \sigma^2 \otimes I_2 \otimes \sigma^2 \otimes \sigma^3, \quad \rho^{(4)} = \sigma^3 \otimes I_1 \otimes \sigma^1 \otimes 1_2, \quad \rho^{(5)} = - \sigma^3 \otimes I_2 \otimes I_1 \otimes I_2, \]
Evaluating (132) and (133) on the classical solutions, using the relation (29) and defining
\[ \varepsilon(\tau) = \frac{1}{2(\sin \lambda + 1)} \left[ 1 + 2M_{\|} \right] (1 + 2M_\sigma) (1 + 2M_z) \left[ 1 + 2(\chi_0 - \tau \sec \lambda)M_t \right] \varepsilon_1, \]

we get
\[ \eta(0) + \xi(0) = \sigma^{-1/2} \varepsilon + \sigma^{1/2} 2M_t \varepsilon. \]  
(177)

It is useful to note that
\[ 2M_{\|} \varepsilon = \varepsilon \quad \text{and} \quad \ddot{\varepsilon} = 0. \]  
(179)

Then, writing \( \eta(0) + \xi(0) \) and \( \varepsilon \) as
\[ \eta(0) + \xi(0) = \kappa^{a\dot{a}} \otimes \varphi^{a\dot{a}}_{+} + \kappa^{a\dot{a}} \otimes \varphi^{a\dot{a}}_{-} \quad \text{and} \quad \varepsilon = \dot{\varepsilon}^{a\dot{a}} \otimes \varphi^{a\dot{a}}_{+} + \dot{\varepsilon}^{a\dot{a}} \otimes \varphi^{a\dot{a}}_{-}, \]  

we obtain
\[ \kappa^{a\dot{a}} = \sigma^{-1/2} \dot{\varepsilon}^{a\dot{a}} + \sigma^{1/2} \tau^{0} \dot{\varepsilon}^{a\dot{a}}, \]  
(181)

where
\[ \kappa^{a\dot{a}} = \kappa^{a\dot{a}}_{+} + \kappa^{a\dot{a}}_{-} \quad \text{and} \quad \dot{\varepsilon}^{a\dot{a}} = \dot{\varepsilon}^{a\dot{a}}_{+} + \dot{\varepsilon}^{a\dot{a}}_{-}. \]  
(182)

The properties (179) can now be expressed as
\[ \tau^{1} \dot{\varepsilon}^{a\dot{a}} = -\dot{\varepsilon}^{a\dot{a}} \quad \text{and} \quad \ddot{\varepsilon}^{a\dot{a}} = 0. \]  
(183)

Choosing a representation of \( \tau \) and \( \rho \) matrices such that the \( \varphi \) spinors satisfy
\[ (\varphi^{1})_{a\dot{a}} \cdot \rho^3 \cdot \rho^5 \cdot \varphi^{3\dot{a}}_{a\dot{b}} = -\frac{i}{4} 6^a_{b} (\sigma^3)^a_{b}, \]
\[ (\varphi^{1})_{a\dot{a}} \cdot \rho^3 \cdot \rho^5 \cdot \varphi^{3\dot{a}}_{bb} = \frac{i}{4} 6^a_{b} (\sigma^1)^a_{b}, \]  
(184)

\[ (\varphi^{1})_{a\dot{a}} \cdot \rho^3 \cdot \rho^5 \cdot \varphi^{3\dot{a}}_{a\dot{b}} = \frac{i}{4} 6^a_{b} (\sigma^2)^a_{b}, \]
\[ (\varphi^{1})_{a\dot{a}} \cdot \rho^3 \cdot \rho^5 \cdot \varphi^{3\dot{a}}_{bb} = \frac{i}{4} 6^a_{b} (\sigma^2)^a_{b}, \]  
(185)

\[ (\varphi^{1})_{a\dot{a}} \cdot \rho^3 \cdot \rho^4 \cdot \varphi^{3\dot{a}}_{a\dot{b}} = \frac{i}{4} 6^a_{b} (\sigma^3)^a_{b}, \]
\[ (\varphi^{1})_{a\dot{a}} \cdot \rho^3 \cdot \rho^4 \cdot \varphi^{3\dot{a}}_{bb} = \frac{i}{4} 6^a_{b} (\sigma^3)^a_{b}, \]  
(186)

\[ (\varphi^{1})_{a\dot{a}} \cdot \rho^7 \cdot \rho^8 \cdot \varphi^{3\dot{a}}_{a\dot{b}} = \frac{i}{4} 6^a_{b} (\sigma^1)^a_{b}, \]
\[ (\varphi^{1})_{a\dot{a}} \cdot \rho^7 \cdot \rho^8 \cdot \varphi^{3\dot{a}}_{bb} = -\frac{i}{4} 6^a_{b} (\sigma^1)^a_{b}, \]  
(187)
\[(\varphi^\dagger)_{\pm \pm} \cdot \rho^9 \cdot \rho^8 \cdot \varphi^\dagger_{bb} = \frac{i}{4} \delta^9_b (\sigma^2)_b, \quad \varphi^\dagger_{\pm \pm} \cdot \rho^9 \cdot \rho^8 \cdot \varphi^\dagger_{bb} = \frac{i}{4} \delta^9_b (\sigma^2)_b, \quad (188)\]

\[(\varphi^\dagger)_{\pm \pm} \cdot \rho^9 \cdot \rho^7 \cdot \varphi^\dagger_{bb} = \frac{i}{4} \delta^9_b (\sigma^3)_b, \quad \varphi^\dagger_{\pm \pm} \cdot \rho^9 \cdot \rho^7 \cdot \varphi^\dagger_{bb} = \frac{i}{4} \delta^9_b (\sigma^3)_b, \quad (189)\]

\[(\varphi^\dagger)_{\pm \pm} \cdot \rho^9 \cdot \rho^4 \cdot \rho^5 \cdot \varphi^\dagger_{bb} = \pm \frac{i}{4} \delta^9_b (\sigma^2)_b, \quad \varphi^\dagger_{\pm \pm} \cdot \rho^9 \cdot \rho^4 \cdot \rho^5 \cdot \varphi^\dagger_{bb} = \pm \frac{i}{4} \delta^9_b (\sigma^2)_b, \quad (190)\]

\[(\varphi^\dagger)_{\pm \pm} \cdot \rho^9 \cdot \rho^3 \cdot \varphi^\dagger_{bb} = \pm \frac{i}{4} \delta^9_b (\sigma^3)_b, \quad \varphi^\dagger_{\pm \pm} \cdot \rho^9 \cdot \rho^3 \cdot \varphi^\dagger_{bb} = \pm \frac{i}{4} \delta^9_b (\sigma^3)_b, \quad (191)\]

\[(\varphi^\dagger)_{\pm \pm} \cdot \rho^9 \cdot \varphi^\dagger_{bb} = \frac{1}{4} \delta^9_b \delta^9_b, \quad (193)\]

and 0 otherwise. Defining

\[\phi^g_b := \phi^g + (\sigma^1)_b, \quad \phi^g + (\sigma^2)_b - \phi^g + (\sigma^3)_b \quad (194)\]

\[\phi^{\dagger}_b := \phi^{\dagger}_b + (\sigma^1)_b, \quad \phi^{\dagger}_b + (\sigma^2)_b - \phi^{\dagger}_b + (\sigma^3)_b \quad (195)\]

we get that the supersymmetry transformations can be expressed as

\[
\delta \chi^{\dagger}_{\mu} = \frac{1}{2} \left( \partial \phi^{tr} + \phi^{tr}_R \right) \nabla^3 \kappa^{\mu},
\]

\[
- \frac{1}{2} \left[ \sin \Omega \left( \partial \phi^g_b \cdot \phi^g_R \nabla^b + \cos \Omega \left( \partial \phi^g_b \cdot \phi^g_R \nabla^b \right) \kappa^{ab} \right), \quad (196)\]

\[
\delta \psi^{\dagger}_{\mu} = \frac{1}{2} \partial \phi^g_b \kappa^{\mu} + \frac{i}{2} \left( \cos \Omega \partial \phi^g_b \kappa^{\mu} - \sin \Omega \partial \phi^g_b \kappa^{\mu} \right), \quad (197)\]

\[
\delta \phi^g = - \frac{1}{2} \psi^{\dagger}_{\mu} \kappa^{\mu}, \quad (198)\]

\[
\delta \phi^r = - \frac{1}{2} \nabla^3 \kappa^{\mu}, \quad (199)\]

\[
\delta \phi^g_b = - \frac{i}{2} \left[ \cos \Omega \left( \partial \bar{\psi}_{\mu} \kappa^{\mu} - \delta^g_b \bar{\psi} \kappa^{\mu} \right) \right] - \sin \Omega \left( \partial \psi^{\dagger}_{\mu} \kappa^{\mu} - \delta^g_b \psi^{\dagger} \kappa^{\mu} \right), \quad (200)\]

\[
\delta \phi^g_b = \frac{i}{2} \left[ \sin \Omega \left( \partial \bar{\psi} \kappa^{\mu} - \delta^g_b \bar{\psi} \kappa^{\mu} \right) + \cos \Omega \left( \partial \psi^{\dagger} \kappa^{\mu} - \delta^g_b \psi^{\dagger} \kappa^{\mu} \right) \right]. \quad (201)\]
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