Saturated Feedback Stabilizability to Trajectories for the Schlögl Parabolic Equation

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Abstract—In this article, it is shown that there exist a finite number of indicator functions, which allow us to track an arbitrary given trajectory of the Schlögl model, by means of an explicit saturated feedback control input whose magnitude is bounded by a constant independent of the given targeted trajectory. Simulations are presented showing the stabilizing performance of the explicit feedback constrained control. Further, such performance is compared to that of a receding horizon constrained control minimizing the classical energy cost functional.

Index Terms—Control constraints, finite-dimensional control, internal actuators, receding horizon control (RHC), saturated stabilizing feedback, semilinear parabolic equations.

I. INTRODUCTION

We investigate the controlled Schlögl system, for time $t \geq 0$

$$\frac{\partial}{\partial t} y - \nu \Delta y + (y - \zeta_1)(y - \zeta_2)(y - \zeta_3) = h + U_M u$$  \hspace{1cm} (1a)

$$y(0) = y_0 \in W^{1,2}(\Omega), \quad \frac{\partial}{\partial \nu} y |_{\partial \Omega} = 0$$  \hspace{1cm} (1b)

evolving in the Hilbert Sobolev space $W^{1,2}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded rectangular domain, with $d \in \{1, 2, 3\}$ and where $\nu$ is the unit outward normal vector to the boundary $\partial \Omega$ of $\Omega$. Furthermore, $M$ and $M_\sigma$ are positive integers

$$U_M := \{m_{i,M} \mid 1 \leq i \leq M_\sigma \} \subseteq L^2(\Omega)$$  \hspace{1cm} (1c)

is a given family of $M_\sigma$ actuators, which are indicator functions of open subdomains $\omega_i^{M} \subseteq \Omega$ depending on the index $M$, $u = u(t) = (u_1(t), u_2(t), \ldots, u_M(t))$ is a vector of scalar controls (tuning parameters) at our disposal, and

$$U_M u := \sum_{i=1}^{M_\sigma} u_i(t) m_{i,M}, \quad \text{with} \quad \|u(t)\| \leq C_u$$  \hspace{1cm} (1d)

for an a priori given constant $C_u \in [0, +\infty]$ and an a priori given norm $\|\cdot\|$ in $\mathbb{R}^{M_\sigma}$. Finally, $\nu > 0$, $(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$, and $h : (0, +\infty) \to L^2(\Omega)$ is a given external force, with restriction $h |_{(0,T)}$ in the Bochner space $L^2((0, T), L^2(\Omega))$, for each $T > 0$. We are particularly interested in the case where $C_u < +\infty$. Note that the extremal cases $C_u = +\infty$ and $C_u = 0$ correspond, respectively, to the unconstrained case and to the free dynamics case. The usual Euclidean norm in $\mathbb{R}^{M_\sigma}$ shall be denoted by $\| \cdot \|_{\mathbb{R}^{M_\sigma}}$.

Our family of actuators $U_M$ can be chosen so that the total volume covered by the actuators satisfies $\text{vol}(\bigcup_{i=1}^{M_\sigma} \omega_i^{M}) = r \text{vol}(\Omega)$, with an arbitrary a priori given $r \in (0, 1)$. Explicit expressions for $\omega_i^{M}$ shall be given in Section I-A.

System (1) is a model for chemical reactions for nonequilibrium phase transitions; see [22] and [48, Sec. 4]. Also, when coupled with a suitable ordinary differential equation, it gives rise to models in neurology and electrophysiology, namely, to the FitzHugh–Nagumo-like equations [21, 29, 38]. It is also an interesting model from the mathematical point of view. Indeed, the cubic nonlinearity with $\zeta_1 < \zeta_2 < \zeta_3$ and vanishing $(h, u)$, determines the two stable equilibria $\zeta_1$ and $\zeta_3$, and the unstable equilibrium $\zeta_2$. This leads to interesting asymptotic behavior of the solutions and, as we shall see, it lends itself to a nontrivial analysis of the global saturated feedback control mechanism.

The main stabilizability problem under investigation is as follows. We are given a trajectory/solution $\tilde{y}$ of the free dynamics, that is, we assume that $\tilde{y}$ solves

$$\frac{\partial}{\partial t} \tilde{y} - \nu \Delta \tilde{y} + (\tilde{y} - \zeta_1)(\tilde{y} - \zeta_2)(\tilde{y} - \zeta_3) = h$$  \hspace{1cm} (2a)

$$\tilde{y}(0, \cdot) = \tilde{y}_0 \in W^{1,2}(\Omega), \quad \frac{\partial}{\partial \nu} \tilde{y} |_{\partial \Omega} = 0$$  \hspace{1cm} (2b)

and that $\tilde{y}$ has a desired behavior, which we would like to track. Next, we are also given another initial state $\bar{y}_0 \in W^{1,2}(\Omega)$. It turns out that the corresponding solution $y$ of the free dynamics, with $y(0, \cdot) = y_0 \in W^{1,2}(\Omega)$ may present an asymptotic behavior different from the targeted behavior of $\tilde{y}$. For example, in the case $h = 0$, and $\zeta_1 < \zeta_2 < \zeta_3$, we could think...
of the free dynamics equilibrium \( \hat{y}(t, x) = \zeta_2 \), with initial state \( \hat{y}(0, x) = \tilde{y}_0(x) = \zeta_2 \), as our desired targeted behavior. We can see that \( \hat{y}(t, x) = \zeta_2 \) is not a stable equilibrium, and that if \( y_0(x) := c \neq \zeta_2 \) is a constant initial state, then the state \( y(t, x) \), of the free dynamics solution, corresponding to the initial state \( y(0, x) = c \), does not converge to the targeted \( \hat{y}(t, x) \).

Hence, to track a desired trajectory \( \hat{y} \), we (may) need to apply a control. Our goal is to design the control input \( u \) such that the state \( y(t, \cdot) \) of the solution of the system (1) converges exponentially to the targeted state \( \hat{y}(t, \cdot) \) as time increases

\[
\|y(t) - \hat{y}(t)\|_{L^2(\Omega)} \leq e^{-\mu(t-s)} \|y(s) - \hat{y}(s)\|_{L^2(\Omega)}
\]

for all \( t \geq s \geq 0 \), for a suitable constant \( \mu > 0 \). Furthermore, we look for a global stabilizability result, where (3) holds independently of the norm \( \|y_0 - \hat{y}_0\|_{L^2(\Omega)} \).

We shall construct the stabilizing constrained control \( u \) by saturating a suitable unconstrained stabilizing linear feedback control \( K(y - \hat{y}) \), with \( K : W^{1,2}(\Omega) \to \mathbb{R}^{M*} \), through a radial projection as follows:

\[
u = K(y - \hat{y}) := \mathcal{P}_{C_u}(K(y - \hat{y})) \tag{4a}\]

where \( \mathcal{P}_{C_u}(v) := \mathcal{P}_{C_u}(v) \) is given as

\[
\mathcal{P}_{C_u}(v) := \begin{cases} v, & \text{if } \|v\| \leq C_u, \\ \frac{C_u}{\|v\|} v, & \text{if } \|v\| > C_u, \end{cases} \quad v \in \mathbb{R}^{M*}. \tag{4b}\]

Note that we have, for \( v \neq 0 \)

\[
\mathcal{P}_{C_u}(0) = 0 \quad \text{and} \quad \mathcal{P}_{C_u}(v) = \min \left\{ 1, \frac{C_u}{\|v\|} \right\} v \tag{5}\]

and also that, for all \( (v, C_u) \in \mathbb{R}^{M*} \times [0, +\infty) \)

\[
\|\mathcal{P}_{C_u}(v)\| \leq C_u, \quad \mathcal{P}_{0}(v) = 0, \quad \text{and} \quad \mathcal{P}_{+\infty}(v) = v. \]

In particular, the saturated feedback control \( u(t) = K(y(t) - \hat{y}(t)) \) satisfies \( \|u(t)\| \leq C_u \).

The stabilizability of dynamical systems as (1a) is an important problem for applications, even in the case where the “magnitude” \( \|u(t)\| \) of the control is allowed to take arbitrary large values (i.e., the case \( C_u = +\infty \)), as shown by the amount of contributions we can find in the literature (cf. Section I-C).

In applications, we may be faced with physical constraints, for example, with an upper bound for the magnitude of the acceleration/forcing provided by an engine, or with an upper bound for the temperature provided by a heat radiator. For this reason, it is also important to investigate the case of bounded controls (i.e., the case \( C_u < +\infty \)).

Remark 1.1: We consider rectangular spatial domains for the sake of simplicity of exposition. Analogous results can be obtained for more general convex polygonal domains. We shall revisit this point in Remark 2.8.

Remark 1.2: We consider a sequence \( (U_M)_{M \in \mathbb{N}_*} \) of families of indicator functions \( U_M = \{ 1_{\mathcal{C}_M} \mid 1 \leq i \leq M \} \subseteq L^2(\Omega) \) with supports \( \bar{\omega}_M \) depending on the sequence index \( M \). Such dependence on \( M \) is also convenient to be able to consider a sequence of families whose total volume covered by the actuators is fixed a priori, \( \text{vol}(\bigcup_{M \in \mathbb{N}_*} \mathcal{C}_M) = r \text{vol}(\Omega) \), with \( r \in (0, 1) \) independent of \( M \).

Remark 1.3: Considering the difference \( z = y - \hat{y} \), our goal (3) reads \( \|z(t)\|_{L^2(\Omega)} \leq e^{-\mu(t-s)} \|z(s)\|_{L^2(\Omega)} \), for all \( t \geq s \geq 0 \). In this way, we “reduce” the stabilizability to trajectories to the stabilizability to zero. The class of systems globally stabilizable with constrained controls, \( C_u < +\infty \), is strictly smaller than that of systems globally stabilizable with unconstrained controls, \( C_u = +\infty \). This can be illustrated with the following system, where \( r < 0 \) is a constant, \( U_M = \{ 1 \} \), \( z(\cdot) \in \mathbb{R} \), and our control input is \( u(t) = u_1(t) \in \mathbb{R} \).

\[
\frac{d}{dt} z + rz = u_1, \quad |u_1| \leq C_u. \tag{6}\]

We can see that, with \( C_u = +\infty \), the system (6) is globally exponentially stabilizable, and that, with \( \mu > 0 \), \( u_1(t) = (r - \mu)z(t) \) is an admissible stabilizing feedback control. However, for \( C_u < +\infty \) such control is not admissible for large \( z(t) \), as \( \|z(t)\| > \frac{\mu}{r} \). In fact, for finite \( C_u \), the system (6) is not globally stabilizable as a consequence of [52, Sec. 1.6.2, Th. 1.2].

A. Sequence of Families of Actuators

We specify the spatial domain \( \Omega \subset \mathbb{R}^d \) as

\[
\Omega = \Omega^* = \times_{n=1}^d (0, L_n) := (0, L_1) \times \cdots \times (0, L_d)
\]

where \( d \in \{ 1, 2, 3 \} \) and \( \times \) stands for the Cartesian product of subsets \( S_n \subset \mathbb{R} \). Then, the set \( U_M \) of actuators, in (1c), is chosen as in [28, Sec. 4.8] and [30, Sec. 5], with

\[
U_M := \text{span} U_M, \quad \text{dim} U_M = M_\sigma \tag{7a}
\]

where, for a fixed \( M, M_\sigma = M^d \) and

\[
\omega_j = \omega_j^M := \times_{n=1}^d \left( (c_n)_j^M - \frac{rL_n}{2M} (c_n)_j^M + \frac{rL_n}{2M} \right) \tag{7b}
\]

with set of centers \( c = (c_j)^M = ((c_1)_j^M, \ldots, (c_d)_j^M) \)

\[
\left\{ (c_j)_j^M \mid 1 \leq j \leq M_\sigma \right\} = \times_{n=1}^d \left\{ \left( \frac{2k - 1}{2} \right) L_n \middle| 1 \leq k \leq M \right\}. \tag{7c}
\]

See Fig. 1 for an illustration for the case \( d = 2 \). See also [42, Sec. 5.2], [44, Sec. 4], and [45, Sec. 6] where an analogous placement of the actuators/sensors have been used.
B. Main Stabilizability Result and the RHC Framework

For the (ordered) family $U_M$ of linearly independent actuators in (7), let $P_{ts_0}: L^2(\Omega) \rightarrow U_M$ be the orthogonal projection in $L^2(\Omega)$ onto $U_M$. Recall the control operator isomorphism in (1), $U_M: \mathbb{R}^{M_\tau} \rightarrow U_M$, $u \mapsto \sum_{i=1}^{M_\tau} u_i \omega_{M,i}$, and consider the unconstrained explicit feedback control

$$z \mapsto -\lambda P_{ts_0} z, \quad \text{for a given } \lambda \geq 0,$$

where $z = y - \hat{y}$ is the difference to the target $\hat{y}$. Finally, we consider the saturated feedback control

$$K_M(z) := \mathcal{P}_{C_u} (-\lambda (U_M^*)^{-1} P_{ts_0} z).$$

In this way, the difference $z$ will satisfy the system

$$\frac{\partial}{\partial t} z = \nu \Delta z - f(\hat{y}(z) + U_M^* \mathcal{P}_{C_u} (-\lambda (U_M^*)^{-1} P_{ts_0} z)) \quad (9a)$$

$$z(0, \cdot) = z_0, \quad \frac{\partial}{\partial n} z |_{\partial \Omega} = 0, \quad \|u(t)\| \leq C_u \quad (9b)$$

with $z_0 \in W^{1,2}(\Omega)$ and $f(\hat{y}(z) + U_M^* \mathcal{P}_{C_u} (-\lambda (U_M^*)^{-1} P_{ts_0} z))$ for suitable constants $\lambda$, $\xi_1$, $\xi_2$.

Definition 1.4: Given $\nu > 0$ and $\theta \geq 1$, system (9) is globally $(\nu, \theta)$-exponentially stable in the $L^2(\Omega)$-norm if for every initial condition $z_0 \in L^2(\Omega)$, we have $\|z(t)\|_{L^2(\Omega)} \leq c_0 \nu^{-\theta t - s} \|z(0)\|_{L^2(\Omega)}$ for all $t \geq 0$.

Definition 1.4 is inspired from [23, Definition 5.5.1]. Briefly, the main stabilizability result of this article is as follows, whose precise statement shall be given in Theorem 2.1. Hereafter, we write $\mathbb{N}$ for the set of nonnegative integers, and set $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$. Analogously, we set $\mathbb{R}^+ := (0, +\infty)$.

Main Result: For each $\nu > 0$, there exist large enough constants $M \in \mathbb{N}_+, \lambda > 0$, and $C_{u_0} > 0$, such that (9) is globally $(1, \mu)$-exponentially stable in the $L^2(\Omega)$-norm.

Remark 1.5: In general, we may need to take $\theta > 1$ in Definition 1.4. In the particular case that $\theta = 1$, as in the main result, the $L^2(\Omega)$-norm of the state is strictly decreasing for all $t > 0$.

In addition, since in some applications, we may be interested in minimizing the energy spent during the stabilization procedure, we also consider the infinite-horizon constrained optimal control problem

$$\min_{u \in L^2(\mathbb{R}^+; \mathbb{R}^{M_\tau})} \int_0^\infty J_\infty(u; y_0, \hat{y}) \quad \text{subject to } (10a)$$

$$J_\infty(u; y_0, \hat{y}) := \|y - \hat{y}\|_{L^2(\mathbb{R}^+; \mathbb{R}^2)}^2 + \beta \|u\|_{L^2(\mathbb{R}^+; \mathbb{R}^{M_\tau})}^2 \quad (10b)$$

where $\beta > 0$ and the target trajectory $\hat{y}$ is given as the solution to (2) for a pair $(\hat{y}_0, h)$. This problem is an infinite-horizon nonlinear time-varying optimal control problem with control constraints. One efficient approach to deal with (10) is the receding horizon control (RHC). In this approach, the stabilizing control is obtained by concatenating a sequence of finite-horizon open-loop controls. These controls are computed online as the solutions to problems of the following form, for time $t \in T_0 := (t_0, t_0 + T), T > 0$.

$$\frac{\partial}{\partial t} y - \nu \Delta y + (y - \xi_1)(y - \xi_2)(y - \xi_3) = h + U_M^* u \quad (11a)$$

Algorithm 1: RHC$(\delta, T)$.

Require: sampling time $\delta > 0$, prediction horizon $T > \delta$, initial state $y_0 \in W^{1,2}(\Omega)$, targeted trajectory $\hat{y}$ solving (2)

Ensure: Receding horizon control (RHC)

1: Set $t_0 = 0$ and $\hat{y}_0 = y_0$;
2: Find $(y_{\hat{T}}^T (\cdot; t_0, \hat{y}_0), u_{\hat{T}}^T (\cdot; t_0, \hat{y}_0))$ for time in $(t_0, t_0 + T)$ by solving the open-loop problem (12);
3: For all $\tau \in (t_0, t_0 + \delta)$, set $u_{\tau, h}(\tau) = u_{\hat{T}}^T (\tau; t_0, \hat{y}_0);
4: Update: $\hat{y}_0 \leftarrow y_{\hat{T}}^T (t_0 + \delta; t_0, \hat{y}_0);
5: Update: $t_0 \leftarrow t_0 + \delta;
6: Go to step 2;

we consider the problem

$$\min_{u \in L^2(T_0; \mathbb{R}^{M_\tau})} J_T(u; t_0, \hat{y}_0, \hat{y}) \quad \text{subject to } (11)$$

$$J_T(u; t_0, \hat{y}_0, \hat{y}) := \|y - \hat{y}\|_{L^2(T_0; \mathbb{R}^2)}^2 + \beta \|u\|_{L^2(T_0; \mathbb{R}^{M_\tau})}^2 \quad (12b)$$

The receding horizon framework is detailed by the steps of Algorithm 1.

The obtained RHC law is not optimal, as long as $T$ is finite. But, relying on the stabilizability stated in Main Result, we will show (Theorem 3.2) that it is stabilizing and suboptimal. The control constraints are enforced within the finite-horizon open-loop problems.

C. On Previous Related Works in Literature

The literature is rich in results concerning the feedback stabilizability of parabolic like equations under no constraints in the magnitude of the control. For example, we can mention [4], [12], [13], [16], [28], [34], [39], [8, Section 2.2], and references therein. Although we do not address, in the present manuscript, the case of boundary controls, we would like to mention [5], [6], [9], [10], [11], [17], [19], [27], [40], [41], and [43]. When compared to the unconstrained case, there is (it seems) a smaller amount of works in the literature considering an upper bound $C_u$ for the magnitude of the control $u(t)$. For finite-dimensional systems, the literature is still rich, as examples, we refer the reader to [7], [18], [24], [32], [33], [47], [51], [53], [55], and [56]. For infinite-dimensional systems, the amount of works in the literature is more modest, for parabolic equations, we mention [37], and for wave-like equations, we refer the reader to [31] and [49]. See also [50] with an application to the beam equation in [50, Sec. 8.1].

We follow an approach which is common in many works dealing with bounded controls. Namely, we consider the saturation of a given unconstrained stabilizing feedback $u(t) = K(z(t))$, with $z = y - \hat{y}$. This means that, at every instant of time $t \geq 0$, we simply rescale the given unconstrained feedback $u(t)$ if its magnitude violates the constraint. Note that the norm of the (unconstrained) feedback control $u(t) = K(z(t)) \in \mathbb{R}^{M_\tau}$ can
take arbitrary large values (e.g., for linear $K$ and $\gamma > 0$ we have $|K| = |K(z)| = \gamma |K(z)|$). Exponentially stabilizing controls given in linear feedback form $u = Kz$ are often demanded in applications, because such controls are able to respond to small measurement errors; see the numerical simulations in [28] and [45].

We consider a finite-dimensional control input because this is relevant for applications, if we have only a finite number of actuators at our disposal. Note that in (8), we take the saturation of the unconstrained input $-\lambda(U^d)^{-1}P_{d\delta}z(t) \in \mathbb{R}^M$. In the literature, we can also find works where the control input is taken in an infinite-dimensional space. Again control constraints and input saturation can be considered; see [25], [26], [35], and [36].

In this work, we also continue the investigation on the receding horizon framework initiated in [1] for the stabilization (to zero) of linear nonautonomous (time-varying) systems. In this framework, no terminal cost or constraints is needed, and the stability is obtained by an appropriate concatenation scheme on a sequence of overlapping temporal intervals. Recall that, in theory, stabilizing system (1) to a given time-dependent trajectory $\tilde{y} = \tilde{y}(t)$ is equivalent to stabilizing the nonautonomous error dynamics (9) to zero. We adapt the analysis given in [1] for (10), with the differences that here, first, the dynamics is nonlinear, second, control constraints are imposed, and finally, numerically, instead of stabilizing (9) to zero, we stabilize the original system (1) to the trajectory $\tilde{y}$.

D. Contents and Notation

This article is organized as follows. Section II is dedicated to the proof of main result. In Section III, we discuss the receding horizon algorithm including the existence of optimal controls for the finite-horizon subproblems. The results of numerical simulations showing the stabilizing performance of both the explicit saturated feedback and the RHC are discussed in Section IV, and finally, Section V concludes this article.

Concerning notation, given Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if the inclusion $X \subseteq Y$ is continuous. The space of continuous linear mappings from $X$ into $Y$ is denoted by $\mathcal{L}(X,Y)$. We write $\mathcal{L}(X) := \mathcal{L}(X,X)$. The discontinuous dual of $X$ is denoted $X' := \mathcal{L}(X,\mathbb{R})$. The adjoint of an operator $L \in \mathcal{L}(X,Y)$ will be denoted $L^* \in \mathcal{L}(Y',X')$. The space of continuous functions from $X$ into $Y$ is denoted by $C(X,Y)$.

The orthogonal complement to a given subset $B \subset H$ of a Hilbert space $H$, with scalar product $(\cdot, \cdot)_H$, is denoted $B^\perp := \{h \in H \mid (h,s)_H = 0 \text{ for all } s \in B\}$.

Given two closed subspaces $F \subseteq H$ and $G \subseteq H$ of the Hilbert space $H = F + G$, with $F \cap G = \{0\}$, we denote by $P_F \in \mathcal{L}(H,F)$ the oblique projection in $H$ onto $F$ along $G$. That is, writing $h \in H$ as $h = h_F + h_G$ with $(h_F, h_G) \in F \times G$, we have $P_F h := h_F$. The orthogonal projection on $H$ is denoted by $P = P_H$. Notice that $P_F^* = P_F$. By $C[a_1, \ldots, a_n]$, we denote a nonnegative function that increases in each of its nonnegative arguments $a_i$, $1 \leq i \leq n$.

Finally, $C, C_i$, $i = 0, 1, \ldots$, stand for unessential positive constants.

II. EXPONENTIAL STABILIZABILITY

We fix the data in the Schlägl system (1a) as

$$\nu > 0, \quad h \in L_1^2(\mathbb{R}_+,L^2(\Omega)), \quad (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3. \quad (13)$$

We recall the free dynamics

$$\frac{\partial}{\partial t}y - \nu \Delta y + (y - \zeta_1)(y - \zeta_2)(y - \zeta_3) = h \quad (1a)$$

$$y(0) = y_0 \in W^{1,2}(\Omega), \quad \frac{\partial}{\partial \nu}y |_{\partial\Omega} = 0 \quad (1b)$$

whose solution, with initial state $y(0) = y_0$, will be denoted by $S(y_0; t) := y(t)$. We show here that a saturated control allows us to track arbitrary solutions of the free dynamics. Let us assume that the trajectory $\tilde{y}(t) = S(\tilde{y}_0; t)$ has a desired behavior. Our goal is to construct a control that stabilizes the system to this trajectory. We consider the system

$$\frac{\partial}{\partial t}y - \nu \Delta y + (y - \zeta_1)(y - \zeta_2)(y - \zeta_3) = h + U^d_M \tilde{y}(y - \tilde{y}) \quad (1a)$$

$$y(0) = y_0 \in W^{1,2}(\Omega), \quad \frac{\partial}{\partial \nu}y |_{\partial\Omega} = 0 \quad (1b)$$

with the saturated feedback control

$$\tilde{K}_M(y - \tilde{y}) = \Psi_{C_a}(-\lambda(U^d)^{-1}P_{d\delta}(y - \tilde{y})). \quad (1c)$$

Let us denote the solution of (15) by $y(t) := S^\nu_{\text{feed}}(y_0; t)$.

Theorem 2.1: For arbitrary $\mu > 0$, there exists $M_\mu \subset \mathbb{N}$ such that, for every $M \geq M_\mu$, there exists $\lambda_* > 0$ such that, for every $\lambda > \lambda_*$, there exists $C_{C^*} \subset \mathbb{R}_+$ such that, for all $C_a > C_{C^*}$, it holds that: for each $(\tilde{y}_0, y_0) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, the solutions $\tilde{y}(t) := S(\tilde{y}_0; t)$ of (14) and $y(t) := S^\nu_{\text{feed}}(y_0; t)$ of (15) satisfy for all $t \geq s \geq 0$

$$|y(t) - \tilde{y}(t)|_{L^2(\Omega)} \leq e^{-\mu(t-s)}|y(s) - \tilde{y}(s)|_{L^2(\Omega)}. \quad (16)$$

Furthermore, $M_\mu \leq \bar{C}[\mu, [\zeta|_{\mathbb{R}}]),$ where $|\zeta| := \max_{1 \leq i \leq 3} |\zeta_i|_{\mathbb{R}}$.

The proof of Theorem 2.1 is given in Section II-E. Observe that Theorem 2.1 states that every trajectory $\tilde{y}$ of the free-dynamics system (14) can be tracked exponentially fast. Note also that $M \geq M_\mu$, does not depend on the pair $(\tilde{y}_0, y_0)$ of initial states, which means that the number $M_\mu$ of actuators can be chosen independently of both the targeted trajectory $\tilde{y}$ and the initial error $y_0 - \tilde{y}_0$.

For simplicity, we shall often denote $L^2 := L^2(\Omega)$ endowed with the usual scalar product, and we shall denote $V := W^{1,2}(\Omega)$ endowed with the scalar product

$$(w, z)_V := \nu(\nabla w, \nabla z)_{L^2(\Omega)} + (w, z)_{L^2} \quad (17)$$

We also write, for more general Lebesgue and Sobolev spaces,

$L^p := L^p(\Omega), \quad W^{s,p} := W^{s,p}(\Omega), \quad p \geq 1, \quad s \geq 0.$

A. On the Well-Posedness of Strong Solutions

The free dynamics (14) is the particular case of (15) when we take $\lambda = 0$. When referring to “the solution of system (15),” we
mean the strong solution \( y \in W_{\text{loc}}(\mathbb{R}^+, W^{2,2}, L^2) \), with \( W_{\text{loc}}(\mathbb{R}^+, X, Y) := \{ y \mid y \in W((0, T), X, Y), \ T > 0 \} \)
\[ W((0, T), X, Y) := \{ y \in L^2((0, T), X) \mid \dot{y} \in L^2((0, T), Y) \} \]
whose existence and uniqueness can be derived as a weak limit of the solutions of finite-dimensional Galerkin approximations and suitable \textit{a priori} “energy” estimates; see, for example, [54, ch. 3, Sec. 3.2 and 3.7] and [46, Sec. 4.3]. We skip the details here, but refer to Remark 2.2 and provide the essential \textit{a priori} energy estimates next. For simplicity, we denote \( A \gamma := -\nu A y + y \) and
\[ \tilde{h}_\gamma(t) := U_M^\circ \mathcal{M} M (y(t) - \tilde{y}(t)) \]
\[ = U_M^\circ \mathcal{M} M (\lambda (U_M^\circ)^{-1} P_M (y(t) - \tilde{y}(t))), \quad \lambda \geq 0 \]
from which we find
\[ \| \tilde{h}_\gamma(t) \|_{L^2} \leq \lambda \| P_M (y(t) - \tilde{y}(t)) \|_{L^2} \leq \lambda \| y(t) - \tilde{y}(t) \|_{L^2}. \]
Multiplying the dynamics by \( 2\lambda y \), we obtain
\[ \frac{d}{dt} \| y \|_{L^2}^2 \leq -2 |A y|_{L^2}^2 + 2(y, A y)_{L^2} - 2(y - \zeta_1)(y - \zeta_2)(y - \zeta_3) + h \| \tilde{h}_\gamma \|_{L^2}^2 \leq -4 h \| \tilde{h}_\gamma \|_{L^2}^2 + 3|y|_{L^2}^2 + 3|\zeta|_{L^2}^2 |y - \tilde{y}|_{L^2}^2 + 2(-y |+| \xi_2 |y - \zeta_2| + (\xi_1 + 1) |y + \zeta_0|, A y)_{L^2} \]
where
\[ \begin{align*}
(\xi_2, \xi_1, \xi_0) := (\zeta_1 + \zeta_2 + \zeta_3 - \zeta_1 \zeta_2 - \zeta_1 \zeta_3 - \zeta_2 \zeta_3) \end{align*} \]
which gives us
\[ \frac{d}{dt} \| y \|_{L^2}^2 \leq -|A y|_{L^2}^2 + 3|\zeta|_{L^2}^3 + 3|\tilde{h}_\gamma|_{L^2}^2 + 3|y - \tilde{y}|_{L^2}^2 + 2(-y |+| 3|y - \zeta_2| + (\xi_1 + 1) |y + \zeta_0|, A y)_{L^2} \]
Next, we claim that the solutions are in \( W_{\text{loc}}((0, T), W^{2,2}, L^2) \) for arbitrary \( T > 0 \), for both the free and the controlled dynamics.

1) \textbf{Free Dynamics:} From (19) we obtain, for \( \lambda = 0 \),
\[ \frac{d}{dt} \| y \|_{L^2}^2 \leq -|A y|_{L^2}^2 + 3|\zeta|_{L^2}^3 + 3|\tilde{h}_\gamma|_{L^2}^2 + 2C |y|_{L^2}^2. \]
Then time integration gives us, for an arbitrary \( T > 0 \),
\[ \| y(T) \|_{L^2}^2 + |A y|_{L^2(0,T)}^2 \leq |y(0)|_{L^2}^2 + C_1(|\zeta|_{0(0,T),L^2}^2 + |\tilde{h}|_{0(0,T),L^2}^2 + |y|_{0(0,T),V}^2), \]
thus \( y \in L^\infty((0, T), V) \cap L^2((0, T), D(A)) \). It turns out that the (graph) norm of the domain \( D(A) \) of \( A \) is equivalent to the usual norm in \( W^{2,2}(\Omega) \). Thus, to show that \( y \in W((0, T), W^{2,2}, L^2) \), it remains to show that \( y \in L^2((0, T), L^2) \). For this purpose, we observe that the nonlinearity \( f(y) := (y - \zeta_1)(y - \zeta_2)(y - \zeta_3) \) satisfies
\[ |f(y)|_{L^2}^2 = |y^3 - 3y \xi_2 \zeta_2 - (y - \zeta_1 + 1) y - \zeta_0|_{L^2}^2 \leq C_1 |y|_{L^6}^6 + 1 \]
for a suitable constant \( C_1 > 0 \). From \( V \mapsto L^6 \), we can derive that \( f(y) \in L^\infty((0, T), L^2) \). Then, from the dynamics equation in (14), it follows that \( \dot{y} \in L^2((0, T), L^2) \).

2) \textbf{Controlled Dynamics:} By assumption the targeted state \( \tilde{y} \) satisfies the free dynamics. In particular \( \tilde{y} \in L^2((0, T), L^2) \). Next, if \( \lambda > 0 \) and if \( y \) satisfies the corresponding controlled dynamics with targeted trajectory \( \tilde{y} \), by (19) we find that
\[ \frac{d}{dt} \| y \|_{L^2}^2 \leq -|A y|_{L^2}^2 + 3|\zeta|_{L^2}^3 + 3|\tilde{h}_\gamma|_{L^2}^2 + 2C |y|_{L^2}^2 + 6 \lambda^2 |y|_{L^2}^2 \]
and, since \( V \mapsto L^2 \), it follows that
\[ \frac{d}{dt} \| y \|_{L^2}^2 \leq -|A y|_{L^2}^2 + 3|\zeta|_{L^2}^3 + 3|\tilde{h}_\gamma|_{L^2}^2 + 2C + 6 \lambda^2 |y|_{L^2}^2 \]
for a suitable constant \( C_1 > 0 \). We can argue as in the free dynamics case to conclude that \( y \in W((0, T), W^{2,2}, L^2) \).

B. \textbf{Dynamics of the Error}

Given a solution \( \tilde{y}(t) \) of the free dynamics (2), we can reduce the stabilizability toward the targeted trajectory \( \tilde{y} \) to the stabilizability of the error \( z := y - \tilde{y} \) toward zero, where \( y(t) \) is the solution of the controlled dynamics (15). For the dynamics of \( z \), we find
\[ \frac{\partial}{\partial t} z - \nu \Delta z + f(y) - f(\tilde{y}) = U_M^\circ \mathcal{M} M (z) \]
\[ z(0) = z_0 := y_0 - \tilde{y}_0, \quad \frac{\partial}{\partial n} z |_{\partial N} = 0 \]
where \( \nu \) and \( \zeta \) are as in (13) and
\[ f(w) := (w - \zeta_1)(w - \zeta_2)(w - \zeta_3). \]
Here, we derive auxiliary results concerning this dynamics, which we shall use later on.
We start by observing that \( f(w) = w^3 + \xi_2 w^2 + \xi_1 w + \xi_0 \) with the \( \xi_j \)'s as in (18), which leads us to
\[
f(z + \tilde{y}) = z^3 + 3\bar{y}z^2 + 3\bar{y}^2z + \tilde{y}^3 + \xi_2(z^2 + 2\bar{y}z + \bar{y}^2) + \xi_1(z + \tilde{y}) + \xi_0
\]
\[
= z^3 + (3\bar{y} + \xi_2)z^2 + (3\bar{y}^2 + 2\xi_2 \bar{y} + \xi_1)z + f(\bar{y})
\]
and thus,
\[
\frac{\partial}{\partial t} z = \nu \Delta z - z^3 - \tilde{f}(z) + U_M^\circ \overline{K}_M(z)
\]
\[
\tilde{f}(z) := (3\bar{y} + \xi_2)z^2 + (3\bar{y}^2 + 2\xi_2 \bar{y} + \xi_1)z.
\]
By multiplying the difference dynamics by \( 2z \), we find
\[
\frac{d}{dt} |z|^2 _{L^2} = -2 |z|^2 _{V^2} + 2(z - z^3, z)_{L^2} - 2(\tilde{f}(z), z)_{L^2}
\]
\[
= 2(U_M^\circ \overline{K}_M(z), z)_{L^2}
\]
Observe that
\[
2(z^3 + \tilde{f}(z), z)_{L^2}
\]
and by writing
\[
z^2 + \bar{y}z + \bar{y}^2 = \left( \frac{15}{16} \right)^2 z^2 + \frac{(16)}{10} \bar{y}^2 + \left( \frac{15}{16} \right)^2 \bar{y}^2 + \frac{3}{10} \bar{y}^2
\]
we arrive at
\[
\frac{d}{dt} |z|^2 _{L^2} = -2 |z|^2 _{V^2} + 2 \left( \left( \frac{15}{16} \right)^2 z + \frac{16}{10} \bar{y} \right)^2 \bar{y}^2,
\]
\[
\leq -2 \left( \frac{31}{256} z^2 + \frac{44}{10} \bar{y}^2 + \xi_2 z + 2\xi_2 \bar{y} + \xi_1, z^2 \right)_{L^2}
\]
\[
= \frac{31}{128} |z|^4 _{L^4} - \left( \frac{22}{25} \bar{y}^2 + 4\xi_2 \bar{y} + 2\xi_1, z^2 \right)_{L^2} - 2(\xi_2 z, z^2)_{L^2}.
\]

The Cauchy–Schwarz and Young inequalities give us
\[
-2 (\xi_2 z, z^2)_{L^2} \leq 2 |\xi_2|_{\mathbb{R}} |z|_{L^2} |z^2|_{L^4}
\]
\[
\leq \gamma^{-1} |\xi_2|_{\mathbb{R}} |z|^2 _{L^2} + \gamma |z|^4 _{L^4} \quad \text{for all } \gamma > 0.
\]
Furthermore, observe that
\[
\left( -\frac{22}{25} \bar{y}^2 - 4\xi_2 \bar{y} - 2\xi_1, z^2 \right)_{L^2} \leq \tilde{C} |z|^2 _{L^2}
\]
where \( \tilde{C} \leq \overline{C}_{||\xi||_2} \), namely,
\[
\tilde{C} := \frac{50}{11} \xi_2^2 - 2\xi_1 = \max_{s \in \mathbb{R}} \left( -\frac{22}{25} \bar{y}^2 - 4\xi_2 \bar{y} - 2\xi_1 \right)
\]
and
\[
\frac{d}{dt} |z|^2 _{L^2} \leq -\frac{8}{\nu} |z|^4 _{V^2} + \frac{128}{15} |\xi_2|_{\mathbb{R}} |z|^2 _{L^2} + \tilde{C} |z|^2 _{L^2}.
\]
Then, together with (21), we arrive at the inequality
\[
\frac{d}{dt} |z|^2 _{L^2} \leq -\frac{8}{\nu} |z|^4 _{V^2} + \left( \frac{128}{15} |\xi_2|_{\mathbb{R}} |z|^2 _{L^2} + \tilde{C} + 2 \right) |z|^2 _{L^2} + 2(U_M^\circ \overline{K}_M(z), z)_{L^2}
\]
that we shall use later, together with the following lemmas.

**Lemma 2.3:** Let \( (\beta_0, \beta_1, \beta_2, \gamma, P) \in \mathbb{R}_+^5 \). Then, we have
\[
-\beta_2 \bar{x}^2 + \beta_0 \bar{x} \leq -\beta_1 \bar{x} \frac{\bar{x} + 1}{\bar{x}^2 + \beta_2 \bar{x} + \beta_1}.
\]

**Proof:** We have \( \bar{x} \geq \frac{\gamma}{\beta_1} \), we can rewrite \(-\beta_2 \bar{x}^2 + \beta_0 \bar{x} \leq -\beta_1 \frac{\bar{x} + 1}{\bar{x}^2 + \beta_2 \bar{x} + \beta_1} \).

**Lemma 4.2:** The constrained feedback operator satisfies
\[
\tilde{K}(t) := (U_M^\circ \overline{K}_M(z)(t), z(t))_{L^2}
\]
\[
= \left\{ \begin{array}{ll}
-\lambda \min \left\{ 1, \frac{C_u}{\|v(t)\|} \right\} |P_{dt} z(t)|^2 _{L^2} & \text{if } P_{dt} z(t) \neq 0 \\
0, & \text{if } P_{dt} z(t) = 0
\end{array} \right.
\]
where \( v(t) := -\lambda (U_M^\circ)^{-1} P_{dt} z(t) \).

**Proof:** Recalling (5) and the feedback in (15), with \( v(t) = -\lambda (U_M^\circ)^{-1} P_{dt} z(t) \), we find
\[
\tilde{K}(t) := (U_M^\circ \mathcal{Q}_\gamma (-\lambda (U_M^\circ)^{-1} P_{dt} z(t)), z(t))_{L^2}
\]
\[
= \min \left\{ 1, \frac{C_u}{\|v(t)\|} \right\} (U_M^\circ v(t), z(t))_{L^2} \quad \text{if } v(t) \neq 0
\]
from which we can conclude the proof.

**C. Norm Decrease for Large Error**

We consider again system (15). We show that if the error norm is large, at a given instant of time, then such norm is decreasing at that instant of time. Here, the nonlinear term plays a crucial role. Also, recall that \( \tilde{y}_0 = \tilde{y}(0) \) is the initial state of the targeted trajectory \( \tilde{y} \) solving (14).

**Lemma 2.5:** For every \( \mu > 0 \), there is a constant \( D \geq 1 \) such that for all \( (z_0, \tilde{y}_0) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega) \), the solution of system (20) satisfies
\[
\frac{d}{dt} |z(t)|^2 _{L^2} \leq -\mu |z(t)|^2 _{L^2} \quad \text{if } |z(t)|_{L^2(\Omega)} \geq D,
\]
and
\[
|z(t)|_{L^2(\Omega)} \leq D \quad \text{for all } t \geq (\mu^2 D^{-2})^{-1}.\]
Moreover, if for some \( s \geq 0 \), we have that \( |z(s)|_{L^2(\Omega)} \leq D \), then \( |z(t)|_{L^2(\Omega)} \leq D \) for all \( t \geq s \). Furthermore, \( D \leq C_{[\mu, \xi]} \) is independent of \((\oz, \oy, C_u)\).

**Proof:** By (23) and Lemma 2.4, we find
\[
\frac{d}{dt} |z|^2_{L^2} \leq -\frac{1}{8} |z|^4_{L^4} + \left( \frac{128}{15} |\xi_2|^2 + C + 2 \right) |z|^2_{L^2}
\]
which, together with
\[
|z|^2_{L^2} \leq \|\Omega\| \frac{1}{\|\Omega\|} \int_{\Omega} 1 \, dx
\]
gives us
\[
\frac{d}{dt} |z|^2_{L^2} \leq -\frac{1}{8} \|\Omega\|^{-1} |z|^4_{L^2} + C_1 |z|^2_{L^2}
\]
where \( 0 \leq C_1 = \frac{128}{15} |\xi_2|^2 + C + 2 \leq C_{[\xi, \xi]} \); with \( \xi_2 \) and \( C \) defined in (18) and (22), respectively.

By taking \((\beta_0, \beta_1, \beta_2, \alpha, \rho) = (C_1, 2\mu, \frac{1}{8} \|\Omega\|^{-1}, |z|^2_{L^2}, 2)\) in Lemma 2.3, we find
\[
\frac{d}{dt} |z|^2_{L^2} \leq -2\mu (|z|^2_{L^2})^2
\]
while \( |z|_{L^2} \geq \tilde{D} : = \frac{2\mu + \sqrt{4\mu^2 + \frac{1}{2} \|\Omega\|^{-1} C_1}}{\frac{1}{2} \|\Omega\|^{-1}} \). (24)

In particular
\[
\frac{d}{dt} |z|^2_{L^2} \leq -2\mu (|z|^2_{L^2})^2 \quad \text{and} \quad \frac{d}{dt} |z|^2_{L^2} \leq -2\mu |z|^2_{L^2} \quad \text{(25a)}
\]
while \( |z|_{L^2} \geq D : = \max\{1, \tilde{D}\} \). (25b)

Next, observe that for \( r > 1 \), the solution \( \varphi(t) \in \mathbb{R} \) of
\[
\varphi = -2\mu \varphi^r, \quad \varphi(0) = \varphi_0 \geq 0, \quad t \geq 0
\]
is given by
\[
\varphi(t) = \frac{\varphi_0}{(1 + 2\mu (r - 1) \varphi_0^{-1} t)^{1/r}}.
\]

Hence, from (25), we conclude that if \( |\oz|_{L^2} > D \), then
\[
|z(t)|_{L^2} \leq |\oz|^2_{L^2} \left( 1 + \mu |\oz|^2_{L^2} t \right)^{-1/2}, \quad \text{while} \quad |z(t)|_{L^2} \geq D.
\]

In particular
\[
|z(t)|_{L^2} \leq (\mu t)^{-2}, \quad \text{while} \quad |z(t)|_{L^2} \geq D, \quad \text{if} \quad |\oz|_{L^2} > D.
\]

Observe also that if \( |\oz|_{L^2} > D \), then
\[
(\mu t)^{-2} = D \quad \iff \quad t = \tau := (\mu^2 D)^{-\frac{1}{2}}.
\]

Therefore, if \( |\oz|_{L^2} > D \), then \( |z(t)|_{L^2} = D \) for some \( t \leq \tau \). By (25), it also follows that, if \( |z(t_0)|_{L^2} \leq D \) for some \( t_0 \geq 0 \), then \( |z(t)|_{L^2} \leq D \) for all \( t \geq t_0 \). In particular, for every initial error \( \oz \in \dot{W}^{1,2}(\Omega) \), it holds that \( |z(t)|_{L^2} \leq D \), for all \( t \geq t \). Furthermore, from the second inequality in (25), we can conclude that if \( |\oz|_{L^2} > D \), then
\[
|z(t)|_{L^2} \leq e^{-2\mu (t-s)} |z(s)|_{L^2} \quad \text{while} \quad |z(t)|_{L^2} \geq D.
\]

Finally, the constants \( \tilde{D} \leq C_{[\mu, \xi]} \) and \( D \leq C_{[\mu, \xi]} \), defined in (24) and (25), are independent of \((\oz, \oy, C_u)\). \( \square \)

**D. Property of the Sequence of Families of Actuators**

We present an auxiliary result concerning the sequence \((\mu_M)_{M \in \mathbb{N}^+}\) in Section 1-A. Recall that \( \mu_M \) stands for the orthogonal projection in \( L^2(\Omega) \) onto \( U_M \). Further, for simplicity, we denote by \( S^r := \dot{S}_r \) (the orthogonal complement in \( L^2(\Omega) \) of a subset \( S \subseteq L^2(\Omega) \)).

**Lemma 2.6:** For every \( \omega > 0 \), there exists \( M_\omega \in \mathbb{N}^+ \) such that for all \( M \geq M_\omega \), we can find \( \lambda_\omega = \lambda_\omega(M) = C_{[w]} > 0 \) such that
\[
|w|_{L^2}^2 + 2\lambda_\omega |\mu_M w|_{H}^2 \geq \omega |w|_{L^2}^2 \quad \text{for all} \quad w \in V = \dot{W}^{1,2}(\Omega).
\]

**Proof:** In fact Lemma 2.6 follows from the result in [30, Corollary 3.1] for general diffusion-like operators, which include the shifted Laplacian \( \lambda = -\Delta + 1 \), as mentioned in [30, Sec. 5]. From the proof of [30, Corollary 3.1], we find
\[
|w|_{L^2}^2 + 2\lambda_\omega |\mu_M w|_{H}^2 \geq |w|_{V}^2 + 2\lambda_\omega |\mu_M P_{H} w|_{L^2(H)}^2 |\mu_M P_{H} w|_{H}^2
\]
where \( U_M \) is an auxiliary finite-dimensional space, satisfying \( L^2(\Omega) = U_M + \dot{U}_M^\perp \) and \( U_M \cap \dot{U}_M^\perp = \{0\} \), and \( P_{H} \) is the orthogonal projection in \( L^2(\Omega) \) onto \( U_M \), along \( \dot{U}_M^\perp \). From [45, Sec. 6], we know that by choosing the auxiliary space as the span of “regularized” actuators as in [45, eq. (6.8)], then the norm of the oblique projection is independent of \( M \). Hence,
\[
|w|_{L^2}^2 + 2\lambda_\omega |\mu_M w|_{H}^2 \geq |w|_{V}^2 + 2\lambda_\omega \|P_{H} w\|_{L^2(H)}^2 \quad \text{with} \quad \Xi \text{ independent of} \quad M.
\]
By the proof of [30, Lemma 3.5], the desired result follows if
\[
\beta_{M_\omega} \geq 4 \omega \quad \text{and} \quad \lambda_\omega \geq (2\omega |1|^2_{L^2(V, H)} + 1) \frac{\beta_{M_\omega}^2}{2\omega}
\]
where (see [30, eqs. (2.2) and (3.1)])
\[
\beta_{M_\omega} := \inf_{\theta \in \Theta \cap \dot{U}_M (0)} \frac{|\theta|_V}{|\theta|_H} \quad \text{and} \quad \beta_M := \sup_{\theta \in \Theta_M (0)} \frac{|\theta|_V}{|\theta|_H}.
\]
From [44, Sec. 5], it follows that \( \beta_{M_\omega} = C_1 M^2 + 1 \). Therefore, we can choose \( M_\omega = \min\{M \in \mathbb{N}^+ \mid \beta_{M_\omega} \geq 4 \omega \} \) and \( \lambda_\omega = (2\omega |1|^2_{L^2(V, H)} + 1) \frac{\beta_{M_\omega}^2}{2\omega} \).

**Remark 2.7:** From [44, Sec. 5] (see also [45, Th. 6.1]), it follows that \( \beta_M \geq C_2 M^2 + 1 \). Therefore, for large \( M \), we may need to take large \( \lambda_\omega \).

**Remark 2.8:** The divergence \( \beta_M \to +\infty \) plays a crucial role in the derivation of Lemma 2.6. The proof of such divergence, shown in [44, Sec. 5] [45, Th. 6.1] for rectangular domains (boxes), can be adapted for general polygonal domains, which are the union of a finite number of triangles (simplexes). The proof in [44] and [45] is based on the fact that a rectangle can be partitioned into rescaled copies of itself. Note that a triangle can also be partitioned into smaller triangles. Indeed, for planar triangles, \( d = 2 \), we obtain four similar congruent triangles by connecting the middle points of the edges, and iterating the procedure, we obtain finer partitions into congruent triangles. For the case \( d = 3 \), it may be possible to partition a triangle \( T \) (tetrahedron) into smaller triangles all congruent to \( T \), however, the partition is possible into triangles where the
number of congruent classes does not exceed 3; see [20, Sec. 3 and Fig. 5] and [15, Th. 4.1 and Fig. 5]. This fact allows us to repeat/adapt the arguments in [44] and [45].

On the other hand, the satisfiability of the divergence $\beta_M \to +\infty$ for smooth domains is an open nontrivial question (cf., [46, Conj. 4.6] and discussion thereafter).

### E. Proof of Theorem 2.1

For an arbitrary given $\mu > 0$, by (23), we have that
\[
\frac{d}{dt} |z(t)|^2_{L^2} \leq - |z(t)|^2_V + \left(\frac{128}{15} |z(t)|^2 + \bar{C} + 2 \right) |z(t)|^2_{L^2} + 2(U_M, \overline{K}_M(z), z)_{L^2} \leq - |z(t)|^2_V + 2(U_M, \overline{K}_M(z), z)_{L^2} + \varpi |z(t)|^2_{L^2} - 2\mu |z(t)|^2_{L^2}
\]
with
\[
\varpi := 2\mu + \frac{128}{15} |z(t)|^2 + \bar{C} + 2
\]
and $\varpi$ and $\bar{C}$ as in (18) and (22). With $M_\varpi = C_{\varpi} |z(t)| \leq C_{\varpi} |z(t)|$ and $\lambda_\varpi > 0$ be given by Lemma 2.6, we arrive at
\[
|z(t)|^2_{L^2} \leq 2(U_M, \overline{K}_M(z), z)_{L^2} + 2\lambda |P_{t_{\varpi}} z| |z(t)|^2_{L^2} - 2\mu |z(t)|^2_{L^2}.
\]
Observe that, by Lemma 2.4, we have
\[
(U_M, \overline{K}_M(z(t)), z(t))_{L^2} = -\lambda |P_{t_{\varpi}} z(t)|^2_{L^2} \leq C_u
\]
where $|(U_M, \overline{K}_M(z(t)))_{L^2}| := \max_{t \in t_{\varpi}(\cdot), 0} \frac{|(U_M, \overline{K}_M(z(t)))_{L^2}|}{|z(t)|^2_{L^2}}$.

From Lemma 2.5, there is a constant $D \geq 1$ such that for
\[
t_a := \min\{t \geq 0 \mid |z(t)|^2_{L^2} \leq D\} \leq (\mu^2 D)^{-\frac{1}{2}}.
\]
we have that
\[
|z(t)|^2_{L^2} \leq e^{-\mu(t-s)} |z(s)|^2_{L^2} \quad \text{for all } 0 \leq s \leq t \leq t_a \quad (29a)
\]
\[
|z(t)|^2_{L^2} \leq D \quad \text{for all } t \geq t_a \quad (29b)
\]

Now, motivated by (28), we set
\[
C_u := \lambda \left\| (U_M, \overline{K}_M(z(t)))_{L^2} \right\| D.
\]

We can conclude that
\[
\left\| -\lambda |P_{t_{\varpi}} z(t)|^2_{L^2} \right\| \leq C_u, \text{ if } C_u \geq C_u \text{ and } t \geq t_a.
\]

By (28), we have $U_M, \overline{K}_M(z(t))_{L^2} = -\lambda |P_{t_{\varpi}} z(t)|^2_{L^2}$ if $C_u \geq C_u$ and $t \geq t_a$. Therefore, by (27),
\[
\frac{d}{dt} |z(t)|^2_{L^2} \leq -2\mu |z(t)|^2_{L^2} \quad \text{for all } t \geq t_a, \text{ if } C_u \geq C_u
\]
which implies that
\[
|z(t)|^2_{L^2} \leq e^{-\mu(t-s)} |z(s)|^2_{L^2} \quad \text{if } t \geq s \geq t_a, \text{ if } C_u \geq C_u.
\]

Combining (29) and (31), we find
\[
|z(t)|^2_{L^2} \leq e^{-\mu(t-t_a)} e^{-\mu(t-t_a-s)} |z(s)|^2_{L^2} = e^{-\mu(t-s)} |z(s)|^2_{L^2}
\]
for all $t \geq t_a \geq s \geq 0$, if $C_u \geq C_u$. (32)

By (29), (31), and (32), we have that (16) holds true.

**Remark 2.9:** Note that $C_u$ as in (30) depends on $\lambda = \lambda(M)$. Due to Remark 2.7, we can expect $\lambda(M)$ to increase with $M$. From the proof of Theorem 2.1, it also follows that the triple $(M_{\lambda}, \lambda, C_u)$ can be chosen independent of $\delta$, however, the same triple depends on $\nu$ through the used Lemma 2.6, because by (17), the V-norm depends on $\nu$.

### III. Receding Horizon Control (RHC)

We investigate the performance and stability of Algorithm 1 applied to the infinite-horizon problem (10). To present our results, we introduce the finite- and infinite-horizon value functions.

**Definition 3.1:** Let $\tilde{y} \in W_{loc}(R_{+}, W^{1,2}, L^2)$ be a solution to (2) for a given pair $(\tilde{y}, h)$. The infinite-horizon value function $V_{\infty} : W^{1,2} \to R_{+}$ and the finite-horizon value function $V_T : R_{+} \times W^{1,2,1} \to R_{+}$, with $T > 0$, are as follows: see (10) and (12). With $I_t^y := (t_0, t_0 + T)$
\[
V_{\infty}(y_0) := \inf_{u \in L^2(R_{+}, R^{M \|})} \{ J_{\infty}(u; y_0, \tilde{y}) \text{ subject to } (1) \}
\]
\[
V_T(t_0, y_0) := \inf_{u \in L^2(I_{T_{\infty}}^y, R^{M \|})} \{ J_T(u; t_0, y_0, \tilde{y}) \text{ subject to } (11) \}.
\]

**Theorem 3.2:** Let $\tilde{y} \in W^{1,2}$ and suppose that for a chosen $(\mu, \lambda, C_u)$, the system (15) is globally $(1, \mu)$ exponentially stable around the reference trajectory $\tilde{y}$ of (2), that is, (16) holds for every $y_0 \in W^{1,2}$. Then, for every given sampling time $\delta > 0$, there are numbers $T^* > \delta$, and $\alpha \in (0, 1)$, such that for every prediction horizon $T > T^*$, and every $y_0 \in W^{1,2}$, the RHC $u_{rhi}$ given by Algorithm 1 satisfies the suboptimality inequality
\[
\alpha V_{\infty}(y_0) \leq \alpha V_{\infty}(u_{rhi}; t_0, y_0, \tilde{y}) \leq V_{\infty}(y_0)
\]
and provides the asymptotic behavior $|y_{rhi}(t) - \tilde{y}(t)|_{L^2(0,T)} \to 0$ as $t \to +\infty$.

**Proof:** We consider the dynamics of the error $z := y - \tilde{y}$, see (9), with a general control $u$ in the time interval $I_{T_{\infty}}^y = (t_0, t_0 + T)$, with $t_0 \geq 0$ and $T \in (0, +\infty)$
\[
\frac{\partial}{\partial t} z - \nu \Delta z + \theta(z) = U_M u, \quad \frac{\partial}{\partial t} z \mid_{\partial} = 0, \quad z(t_0) = \tilde{z}_0
\]
where $\theta(z) = z^3 + (3\tilde{y} + \xi_3)^2 + (3\tilde{y}^2 + 2\xi_2 + \xi_1) z$.

In this way, for the solution $z = z(t_0, z, u)$ of (33), $\beta > 0$, we define the following performance index:
\[
J_T^y(u; t_0, \tilde{z}_0) := |z(t)|_{L^2(I_{T_{\infty}}^y, L^2)}^2 + \beta |u|^2_{L^2(I_{T_{\infty}}^y, R^{M \|})}.
\]
Algorithm 2: RHC2(δ, T).

Require: sampling time δ > 0, prediction horizon T > δ, initial state g0 ∈ W^{1,2}, targeted trajectory ĝ solving (2)

Ensure: Receding horizon control (RHC)

1: u_{rh} ∈ L^2([0,T], M^k).
2: Set t₀ = 0 and z₀ := y₀ − ĝ₀;
3: Find (z₉(t₀), z₀) for time in (t₀, t₀ + T) by solving the open-loop problem (35);
4: Solve: t₀ ← t₉(t₀); update: z₀ ← z₉(t₀);
5: Update: t₀ ← t₀ + δ;
6: Go to step 2;

Then, we can also define the following value functions:

\[ V_∞(z₀) := \inf_{u ∈ L^2([0,T], M^k)} \{ J^P_∞(u; 0, z₀) subject to (33) \} \]

\[ V_T(t₀, z₀) := \inf_{u ∈ L^2([t₀,T], M^k)} \{ J^P_T(u; t₀, z₀) subject to (33) \} \]

(for given initial pairs (0, z₀) and (t₀, z₀), respectively). Next, for a given triple (t₀, T, z₀), we define the open-loop problem

\[ \min_{u ∈ L^2([t₀,T], M^k)} J^P_T(u; t₀, z₀) subject to (33) \] (35)

and write Algorithm 2. By induction, it can be easily shown that both of Algorithms 1 and 2 deliver the same RHC u_{rh}.

Further, for z₀ = y₀ − ĝ₀, it can be seen that \( V_∞(y₀) = V_∞(z₀) \), \( V_T(0, y₀) = V_T(0, z₀) \), and \( J^P_T(u_{rh}; 0, z₀) = J_∞(u_{rh}; 0, y₀) \). Thus, we can restrict ourselves to show that the following suboptimality inequalities and asymptotic behavior hold for Algorithm 2

\[ α V_∞(z₀) ≤ α J^P_∞(u_{rh}; 0, z₀) ≤ V_∞(z₀) \] (36a)

\[ |z_{rh}(t)|_{L₂} → 0 as t → +∞. \] (36b)

Verification of (36a): Since the stabilizability result for the time-varying system (33) holds globally, the proof follows with similar arguments given in [1, Th. 2.6]. Therefore, we omit the proof here and restrict ourselves to the verification of Properties P1 and P2 in [1, Th. 2.6] as follows.

P1: There is a continuous, nondecreasing, and bounded function γ₂ : \( \mathbb{R}_+ \to \mathbb{R}_+ \) such that, for all (t₀, z₀) ∈ \( \mathbb{R}_+ \times W^{1,2} \):

\[ V_T(t₀, z₀) ≤ γ₂(T)|z₀|_{L₂}. \] (37)

First, we show that for every given (t₀, z₀), there exist a control \( \hat{u} ∈ L^2([t₀, T+\infty), M^k) \), satisfying (33b), for which the solution of (33a) satisfies

\[ |z(t)|_{L₂} ≤ e^{-\mu(t-t₀)}|g₀(t)|_{L₂} \] for t ≥ t₀.

This follows by applying the control law (15c), with \( (\mu, \lambda, C_u) \) given by Theorem 2.1, to the following time-shifted system with \( \tilde{z}(s) := z(s + t₀), \tilde{g}(s) := g₀(t₀ + s) \), for s ≥ 0,

\[ \frac{∂}{∂s} \tilde{z} - νt\tilde{z} + f̂(\tilde{z}) = Û_M \hat{u}, \quad \frac{∂}{∂n} \tilde{z} |_{g₀} = 0, \quad \tilde{z}(0,) = z₀ \]

with \( ĝ \) the solution of the free dynamics (2) with forcing function \( \tilde{h} := h(· + t₀) \) and initial function \( g₀(t₀) \) in place of \( h \) and \( \tilde{y}_₀ \), respectively. Since \( h ∈ L^∞(\mathbb{R}_+, L^2(Ω)) \) and \( \tilde{y}_₀ ∈ W^{1,2} \), it follows that \( h ∈ L^∞(\mathbb{R}_+, L^2(Ω)) \) and \( g₀(t₀) \) in \( W^{1,2} \), and thus, the saturated control \( Û_M \hat{u} = \sum_{i=1}^{M} u_i L^i := Û_M \hat{C}(\cdot)(Û_M \hat{C}(·)^{-1} P_{h(t)} z) \) is exponentially stabilizing. Therefore, (34) holds for \( \hat{u}(t) := \hat{u}(t-t₀) \) for t ≥ t₀, where \( \lambda, C_u \), and \( M \) are independent of t₀ and \( \hat{u} \). Evaluating (34) at \( t = \hat{u} \), we obtain

\[ \nabla T(t₀, z₀) ≤ J^P_T(\hat{u}; t₀, z₀) ≤ \frac{1}{2\mu}(1 - e^{-2\mu T})|z₀|_{L₂}^2 \]

where \( \hat{C} \) depends on \( K, C_u \), and \( Û_M \), see (4).

P2: For every \( (t₀, z₀) ∈ \mathbb{R}_+ × W^{1,2} \), every finite horizon optimal control problem of the form (35) admits a solution: We use a standard argument from calculus of variations. Since the set of admissible controls

\[ U_{ad} := \{ u ∈ L^2(I_{T₀}, M^k) : \| u(t) \| ≤ C_u \} \]

is nonempty and \( J^P_T \) is nonnegative, we can select an admissible minimizing sequence \( \{ u^n, z^n \}_n ∈ L^∞(I_{T₀}, M^k) × W^{1,2}(T₀, T), W^{2,2}, L^2 \) satisfying \( J^P_T(u^n; t₀, z₀) \to V_T(t₀, z₀) \). Due to the fact that \( U_{ad} \) is a closed and bounded and convex subset of \( L^2(I_{T₀}, M^k) \) and using the energy estimate

\[ |z|_{W(I_{T₀}, W^{2,2}, L^2)} ≤ C_z \| z₀ \|_{W^{1,2}} + |u|_{L^2(I_{T₀}, M^k)} \]

with \( C_z > 0 \) for (33a), which is justified by similar arguments given in Section II-A, we can infer that there exists a weakly convergent subsequence, still denoted by \( \{ u^n, z^n \}_n \) so that

\[ z^n \to \hat{z} \quad \text{in} \quad W(I_{T₀}, W^{2,2}, L^2) \quad \text{and} \quad u^n \to \hat{u} \quad \text{in} \quad L^2(I_{T₀}, M^k). \]

We verify that \( \hat{z} \) is the strong solution of (33a) corresponding to \( \hat{u} \). To show this, we need to pass the limit in the variational formulation for (33a). From (41), we find that

\[ \left( \frac{∂}{∂t} z^n, Δ z^n, Û_M \hat{u} \right)_{L^2(I_{T₀}, L^2)^3} \left( \frac{∂}{∂t} z^n, Δ z^n, Û_M \hat{u} \right)_. \]

It remains to show that \( f̂(\hat{z}) - \hat{f}(\hat{z}) \), writing \( \delta z^n := z^n - \hat{z} \) and \( \delta \hat{z} := \hat{f}(\hat{z}) - \hat{f}(\hat{z}) \), we find

\[ \| \delta \hat{z} \|_{L^2} ≤ \| z^n \|^3 - (z^n)^3 \|_{L^2} + \| \hat{f}(\hat{z}) - \hat{f}(\hat{z}) \|_{L^2} \]

\[ ≤ \| z^n \|^3 - (z^n)^3 \|_{L^2} + \| \hat{f}(\hat{z}) - \hat{f}(\hat{z}) \|_{L^2} \]

and, estimating separately the terms on the right-hand side,

\[ \| z^n \|^3 - (z^n)^3 \|_{L^2} ≤ \| z^n \|^2 + (z^n)^2 \|_{L^2} \| z^n \|_{L^6} \]

\[ ≤ \| \hat{f}(\hat{z}) - \hat{f}(\hat{z}) \|_{L^2} \]

(42a)

and, estimating separately the terms on the right-hand side,

\[ \| \hat{f}(\hat{z}) - \hat{f}(\hat{z}) \|_{L^2} ≤ \| \hat{z} \|^2 + \| z^n \|^2 \|_{L^2} \]

(42b)
with $\Psi_2 := |3\hat{y} + \xi_2|_{L^6} (|z^m|_{L^6} + |z^*|_{L^6})$ \hspace{1cm} (42c)

$$|(3\hat{y}^2 + 2\xi_2\hat{y} + \xi_1)^2|_{L^2} \leq |3\hat{y}^2 + 2\xi_2\hat{y} + \xi_1|_{L^3} |\delta^n|_{L^6}$$

$$\leq \Psi_3 |\delta^n|_{L^6}, \text{ with } \Psi_3 := \left(3|\hat{y}|_{L^6}^2 + 2|\xi_2\hat{y} + \xi_1|_{L^3}\right). \hspace{1cm} (42d)$$

Now, since the terms $|z^m|_{W(\Gamma_0^T, W^{2,2}, L^2)}$, $|\hat{y}|_{W(\Gamma_0^T, W^{2,2}, L^2)}$, and $\{|z^n|_{W(\Gamma_0^T, W^{2,2}, L^2)}\}_n$ are bounded and since $W(\Gamma_0^T, W^{2,2}, L^2) \rightarrow L^\infty(\Gamma_0^T, W^{1,2})$, we have

$$\Psi_i(t) \leq C_{\phi} \quad \text{for all } t \in I_0^T \text{ and } i \in \{1, 2, 3\}.$$ 

Therefore, from (42), we can conclude

$$|\delta^n|^2_{L^2(I_0^T, L^2)} \leq 9C_{\phi}^2 |\xi|^2_{L^2(W^{1,2}, L^6)} |\delta^n|^2_{L^2(I_0^T, W^{1,2})}. \hspace{1cm} (43)$$

Using (41), (43), and the fact that the embedding $W(\Gamma_0^T, W^{2,2}, L^2) \rightarrow L^2(I_0^T, W^{1,2})$ is compact, we conclude that $f\hat{y}(z^n) \xrightarrow{L^2(I_0^T, L^2)} f\hat{y}(z^*)$, and as a consequence, $z^*$ is the strong solution associated to $u^*$. Finally, since $z^n \xrightarrow{L^2(I_0^T, L^2)} z^*$ and $J_P^P$ is a convex and continuous mapping in $u$, it is weakly lower semicontinuous, and thus, we obtain that:

$$J_P^P(u^*; t_0, \bar{z}_0) \leq \liminf_{n \rightarrow \infty} J_P^P(u^n; t_0, \bar{z}_0) = \nabla T(t_0, \bar{z}_0).$$

Hence, $(z^*, u^*)$ is optimal. The rest of proof follows as in the proof of [1, Th. 2.6] by using a dissipation inequality for the finite-horizon value function $\nabla T$. It is derived by applying (37) for all initial pairs $(t_i, z_{rh}(t_i))$ with $i > 1$, where $t_i = \delta + t_{i-1}$ and $z_{rh}(t_i) = z^n(t_i; t_{i-1}, z_{rh}(t_{i-1}))$. Therefore, it is essential that $z_{rh}(t_i) \in W^{1,2}$ for all $i \geq 1$. This fact can be justified using induction and estimate (40) repeatedly.

**Verification of (36b):** Using (36a) and (39), for $t_0 = 0$ and $T \rightarrow \infty$, we obtain

$$|z_{rh}|^2_{L^2(\mathbb{R}^+, L^2)} + \beta |u_{rh}|^2_{L^2(\mathbb{R}^+, \mathbb{R}^{n_M})} \leq \nabla T_{\alpha}(z_0) \leq \frac{(1+\beta C)}{2\alpha} |z_0|^2_{L^2}.$$ 

Therefore, there exists a constant $C_J$ such that

$$\max \left\{|z_{rh}|^2_{L^2(\mathbb{R}^+, L^2)}, |u_{rh}|^2_{L^2(\mathbb{R}^+, \mathbb{R}^{n_M})}\right\} \leq C_J |z_0|^2_{L^2}. \hspace{1cm} (44)$$

Furthermore, similarly to the estimate (26), we can find

$$\frac{d}{dt}|z_{rh}|^2_{L^2} \leq -|z_{rh}|^2_{L^2} + D_0|z_{rh}|^2_{L^2} + 2(z_{rh}, U^e u_{rh})_{L^2}$$

where $D_0 := \frac{128}{15} |\xi|^2_{\mathbb{R}^3} + \tilde{C} + 2$ was defined for (26). After time integration over the interval $(s, t)$, with $t > s$, we obtain

$$\Xi(t, s) := |z_{rh}(t)|^2_{L^2} - |z_{rh}(s)|^2_{L^2} \hspace{1cm} (45)$$

$$\leq D_0 \int_s^t |z_{rh}|^2_{L^2} dt + 2C_B \int_s^t |z_{rh}|_{L^2} |u_{rh}|_{\mathbb{R}^{n_M}} dt$$

where $C_B > 0$ depends only on $U_M$. Using (45) with $s = 0$, (44), and Young’s inequality, we obtain

$$|z_{rh}|_{L^\infty(\mathbb{R}^+, L^2)} \leq C_{inf} |z_0|_{L^2} \hspace{1cm} (46)$$

for some $C_{inf} > 0$. From (45), with $Y := L^2((s, t), L^2)$

$$\Xi(t, s) \leq D_0 |z_{rh}|^2_Y + 2C_B |z_{rh}|_Y |u_{rh}|_{L^2((s, t), \mathbb{R}^{n_M})}$$

$$\leq C_B(t - s)^{\frac{1}{2}} |z_0|^2_{L^2} \hspace{1cm} (47)$$

for every $t \geq s$, where $C_B := C_{inf}(D_0 + 2C_B)C_J$, and (44) and (46) were used. The rest of proof follows the same lines as in the proof of [1, Th. 6.4] based on (44) and (47).

**IV. NUMERICAL SIMULATIONS**

We present the results of numerical simulations showing the stabilizing performance of the proposed saturated feedback. We compute the targeted trajectory $\hat{y}$, solving (14) as follows:

$$\frac{\partial}{\partial t} \hat{y} - \nu \Delta \hat{y} + f(\hat{y}) = h, \quad \frac{\partial}{\partial n} \hat{y} \mid_{\partial \Omega} = 0, \quad \hat{y}(0) = y_0$$

where $f(w) := (w - \zeta_1)(w - \zeta_2)(w - \zeta_3)$, and the controlled trajectory $y$ solving (15) as follows:

$$\frac{\partial}{\partial t} y - \nu \Delta y + f(y) = h + U^e y \mid_{\mathbb{R}^{n_M}} \frac{\partial}{\partial n} y \mid_{\partial \Omega} = 0$$

with the saturated feedback control

$$u_{rh} = \mathcal{K}_M(y - \hat{y}).$$

We also report on numerical experiments associated with Algorithm 1 and draw a comparison between the saturated controls and the RHC laws. We have chosen the parameters

$$\nu = 0.1 \quad \text{and} \quad (\zeta_1, \zeta_2, \zeta_3) = (-1, 0, 2).$$

The spatial domain is the unit square $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$. As spatial discretization, we have taken a standard piecewise linear finite-element approximation, for the triangulations given in Fig. 2. Most of the simulations correspond to the case $M^\sigma = 9$ with npts $\Omega = 3328$ mesh points (degrees of freedom). For
the temporal discretization, we have taken a standard Crank–Nicolson/Adams–Bashford scheme with stepsize $k = 10^{-3}$. For solving the open-loop problems within Algorithm 1, we employed a projected gradient method to the associated reduced problems, namely, we used the iteration

$$u_{\text{opt}}^{t+1} = P_{\text{opt}}(u_{\text{opt}}^t - \alpha J_{\text{opt}}(u_{\text{opt}}^t)),$$

where $J_{\text{opt}}$ stands for the gradient of the reduced problem and the stepsize $\alpha$ is computed by a nonmonotone line search algorithm, which uses the Barzilai–Borwein step-sizes $[2, 14]$ corresponding to $F$ as the initial trial step size; see [3] and the references therein for more details. For every problem, we used the saturated control with $\lambda = 175$ as the initial iterate. Furthermore, the optimization algorithm was terminated when the norm of the difference corresponding to two successive iterations was less than $10^{-4}$.

We will consider the cases of 1, 4, 9, and 16 actuators, whose locations and supports are illustrated in Fig. 2.

**Example 4.1:** We take as initial states the constant functions

$$\hat{y}_0(x) = \zeta_2 \quad \text{and} \quad y_0(x) = \zeta_3$$

and we take the vanishing external force

$$h(t, x) = 0.$$ 

Note that $\hat{y}_0(x)$ is an unstable equilibrium and $y_0(x)$ is a stable equilibrium, for the free dynamics. Thus, our goal is to leave a stable state and to approach and stabilize an unstable one. In Fig. 3, we can see that the saturated feedback control is able to stabilize the error dynamics in case $C_u \geq e^{1.5}$.

On the other hand, for magnitudes $C_u \leq e^1$, in Fig. 4, we can see that the saturated control is not able to stabilize the error dynamics.

**Example 4.2:** We compare the performance of the control generated by Algorithm 1 (RHC) with that of the explicitly given saturated control. For Algorithm 1, we set $T = 1.25$ and $\delta = 0.5$ and run the algorithm until the final computational time $T_\infty$ for two control cost parameters $\beta \in \{10^{-3}, 10^{-5}\}$. As initial states, we take the constant functions

$$\hat{y}_0(x) = \zeta_2 \quad \text{and} \quad y_0(x) = \zeta_3$$

and as external force, we take the time-periodic function

$$h(t, x) = \frac{1}{2} \alpha \left\{ \sin(6 \pi \|x\|) \Re(t) \right\} 1(x \in \Omega) \left( 1(x \in \Omega) \right).$$

The initial states are stable equilibria of the free dynamics in the case of a vanishing external force, $h = 0$. Here, we are taking a nonzero time-periodic external forcing, which induces the asymptotic periodic-like behavior shown in Fig. 5 for the norm of the targeted trajectory $\hat{y}$. From Figs. 6–10, we can see that the saturated feedback control is able to stabilize the error dynamics for $C_u \geq e^{1.5}$, whereas RHC is stabilizing for $C_u \geq e^1$. This can be seen from Fig. 7, where the saturated control fails to track $\hat{y}$, while RHC succeeds. For the case $C_u = e^{1.5}$, it can be seen in Fig. 6 that, none of the control laws is able to track $\hat{y}$. As illustrated in Figs. 6–10, the saturated control is active for a longer interval compared to RHC. Furthermore, observing the plots related to $|y - \hat{y}|_{L^2(0, T_\infty), L^2}$, we can see that the influence of $\beta$ is only recognizable for $C_u \geq e^2$.

The value of the performance index function

$$J_{T_\infty}(u; y_0, \hat{y}) = |y - \hat{y}|_{L^2(0, T_\infty), L^2}^2 + \beta |u|_{L^2(0, T_\infty), \mathcal{M}_u}^2$$

Fig. 3. Norms of error and control. Large control constraint (Ex. 4.3).

Fig. 4. Norms of error and control. Small control constraint (Ex. 4.3).

Fig. 5. Norm of targeted trajectory (Ex. 4.2).

Fig. 6. Norms of error and control. $(C_u, T_\infty) = (e^{0.5}, 25)$ (Ex. 4.2).
for different control laws is reported in Table I. As we would expect, in all the cases, RHC delivers better results than the saturated controls concerning the value of $J_{\infty}$. We can also see that as $C_u$ is getting larger the performance of the saturated control and RHC are getting closer to each other (except the case $C_u = e^{\infty}$).

Note that for large time, the logarithm of the norm of the error $y - \hat{y}$ stays close to a small value, approximately around $-35$. This can be explained due to the accuracy/precision used in the numerical computations, in fact that value is relatively close to the standard MATLAB precision $\varepsilon \approx 10^{-16}$ we have used; indeed $e^{-35} \approx 6 \times 10^{-16}$.

Recall that we are also computing the targeted trajectory $\hat{y}$, whose values are then used (in the control) to compute the controlled trajectory $y$. We cannot expect that the computational errors associated with the solution of the two systems will cancel each other. Hence, the aforementioned behavior for the norm of $y - \hat{y}$ is consistent.

Example 4.3: Now we take as initial conditions

$$\hat{y}_0(x) = 10 - 20x_1x_2; \quad y_0(x) = -10x_1 + x_2.$$ Their norms as well as the norm of the initial error $z_0 = y_0 - \hat{y}_0$ are large when compared to those in Examples 4.1 and 4.2. The external forcing is taken as (48). In Fig. 11, we can see that the norm of the targeted trajectory $\hat{y}$ decreases fast as long as it is large (cf., Section II-C). Asymptotically it exhibits again a periodic-like behavior near 2.
Comparing Figs. 12 and 13 we see again that we achieve the stability of the controlled error dynamics if, and only if, the magnitude $C_u$ of the control constraint is large enough.

**Example 4.4:** We take the initial states and external force again as in Example IV.3. But, instead of investigating the role played by the control constraint $C_u$, we focus on parameters $M$ and $\lambda$ determining the feedback law. In Fig. 12, we see that by increasing the control constraint $C_u$, we approach the behavior of the unconstrained limit case $C_u = +\infty$. To verify that an arbitrary exponential decrease rate $\mu$ can be achieved with a sufficiently large $C_u = C_u(\mu)$, it is enough to show that this rate can be achieved with the unconstrained feedback. Indeed, in Fig. 14, we confirm that by increasing $M$ and $\lambda$, we reach larger exponential decrease rates. This is consistent with our theoretical result that says that we can achieve an arbitrary large exponential decrease rate $\mu$. We recall the idea of the proof: for small time, the exponential decrease rate is guaranteed for large initial errors, with norm larger than a suitable constant $D = D(\mu)$; for such $D$, we choose $M$ and $\lambda$ large enough to achieve such exponential decrease rate with the unconstrained control; finally, we choose $C_u = C_u(D(\mu))$ large enough so that the constraint is inactive for an error norm smaller than $D$.

**Remark 4.5:** In Fig. 14, we see that with a single actuator, $M_x = 1$, we are able to achieve the exponential stability of the error dynamics, for a suitable rate $\mu_1$. This does not follow from, nor contradict, our theoretical results, from which we have that exponential stability with an a priori given rate $\mu$ holds for large enough $M_x$. This leads us to the following question: (when, if possible) can we guarantee/achieve exponential stability with a single actuator? This could be an interesting problem for future research.

Previously, we have taken $\lambda \geq 100$, which allow us to obtain large exponential decrease rates for the error norm. Fig. 15 shows that, for $M_x = 16$ actuators, we can achieve exponential stability of the error dynamics with $\lambda \geq 5$. However, the figure also shows that $\lambda$ cannot be taken arbitrarily small, since the error dynamics is not exponentially stable with $\lambda \leq 1$.

Fig. 15. Norm of error for small $\lambda$. $C_u = +\infty$ (Ex. 4.5).

**V. CONCLUDING REMARKS**

In this article, we have shown the global stabilizability to trajectories for the Schlögl model for chemical reactions, with a finite number of internal actuators and under control magnitude constraints. The number of actuators and the magnitude of the controls depend on the diffusion coefficient and on the nonlinearity, but they are independent of the external forcing and the targeted trajectory. The stabilizing controls can be taken as the saturation of an explicit feedback operator. Its performance was compared to an RHC approach.

An interesting topic for future work could be the investigation of systems coupling the Schlögl parabolic equation with an ordinary differential equation. These are systems of FitzHugh–Nagumo type modeling phenomena in neurology and electrophysiology.

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