DISTRIBUTION OF TWO-SAMPLE TESTS BASED ON
GEOMETRIC GRAPHS AND APPLICATIONS

BY BHASWAR B. BHATTACHARYA

Stanford University

In this paper central limit theorems are derived for multivariate
two-sample tests based on stabilizing geometric graphs under general
alternatives, in the Poissonized setting. For tests based on stabilizing
graphs, such as the Friedman-Rafsky test [10] and the test based on
the K-nearest neighbor (K-NN) graph [32], the statistic has a limiting
normal distribution, after centering by the conditional mean and
an appropriate scaling. For the K-NN graph and, more generally, for
exponentially stabilizing graphs, this can be strengthened to obtain
the CLT of the statistic when centered by the unconditional mean. As
a consequence, local alternatives for which the K-NN based test have
non-trivial (bounded between 0 and 1) limiting power can be identi-
fied. This validates various applications of these tests and provides a
way to compare their asymptotic performances.

1. Introduction. Let $F$ and $G$ be two continuous distribution func-
tions in $\mathbb{R}^d$ with densities $f$ and $g$ with respect to Lebesgue measure,
respectively. Given independent and identically distributed samples

$$\mathcal{X}_1 = \{X_1, X_2, \ldots, X_{N_1}\} \text{ and } \mathcal{Y}_2 = \{Y_1, Y_2, \ldots, Y_{N_2}\}$$

from two unknown densities $f$ and $g$, respectively, the two-sample problem
is to distinguish the hypotheses

$$H_0 : f = g, \quad \text{versus} \quad H_1 : f \neq g.$$  \hspace{1cm} (1.2)

Asymptotic properties of tests will be derived in the usual limiting regime
where $d$ is fixed, and $N \to \infty$ with

$$\frac{N_1}{N_1 + N_2} \to p \in (0, 1), \quad \frac{N_2}{N_1 + N_2} \to q := 1 - p,$$  \hspace{1cm} (1.3)

and $r := 2pq$.

For univariate data, there are several celebrated nonparametric tests such
as the Kolmogorov-Smirnoff maximum deviation test [33], the Wald-Wolfowitz

MSC 2010 subject classifications: 62F10, 62F12, 60K35, 82B44

Keywords and phrases: Central limit theorem, Hypopaper testing, Geometric graphs,
Stein’s method.

1
runs test [35], and the Mann-Whitney rank test [22] (refer to [11] for more on these tests).

The two-sample problem for multivariate data has been extensively studied, beginning with the work of Weiss [36] and Bickel [6]. Friedman and Rafsky [10] generalized the Wald-Wolfowitz runs test [35] to higher dimensions using the Euclidean minimal spanning tree (MST) of the pooled data. Thereafter many other two-sample tests based on inter-point distances have been proposed. Schilling [32] and Henze [16] used tests based on the $K$-nearest neighbor ($K$-NN) graph of the pooled sample. Later, Rosenbaum [29] and recently Biswas et al. [7] proposed tests based on inter-point distances which are distribution-free under the null. Maa et al. [21] provided certain theoretical motivations for using tests based on inter-point distances.

Another class of multivariate two-sample tests is the Liu-Singh rank sum statistics [20], which generalize the Mann-Whitney rank test using the notion of data-depth. This include tests based on halfspace depth [34], simplicial depth [19], and projection depth [38], among others. For other popular two-sample test refer to [3, 4, 8, 12, 15, 30] and the references therein.

1.1. Graph-Based Two-Sample Tests. Begin by recalling the general framework of graph-based two-sample tests introduced by the author in [5], which includes the tests based on inter-point distances, as well as, those based on data-depth.

A graph functional $G$ in $\mathbb{R}^d$ defines a graph for all finite subsets of $\mathbb{R}^d$, that is, given $S \subset \mathbb{R}^d$ finite, $G(S)$ is a graph with vertex set $S$. A graph functional is said to be undirected/directed if the graph $G(S)$ is an undirected/directed graph with vertex set $S$. We assume that $G(S)$ has no self loops and multiple edges, that is, no edge is repeated more than once in the undirected case, and no edge in the same direction is repeated more than once in the directed case. The set of edges in the graph $G(S)$ will be denoted by $E(G(S))$.

**Definition 1.1 (Bhattacharya [5]).** Let $X_{N_1}$ and $Y_{N_2}$ be i.i.d. samples of size $N_1$ and $N_2$ from densities $f$ and $g$, respectively, as in (1.1). The 2-sample test statistic based on the graph functional $G$ is defined as

$$T(G(X_{N_1} \cup Y_{N_2})) := \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \mathbf{1}\{(X_i, Y_j) \in E(G(X_{N_1} \cup Y_{N_2}))\}. \quad (1.4)$$

If $G$ is an undirected graph functional, then the statistic (1.4) counts the number of edges in the graph $G(X_{N_1} \cup Y_{N_2})$ with one end point in $X_{N_1}$ and the other in $Y_{N_2}$. If $G$ is a directed graph functional, then (1.4) is the number of directed edges with the outward end in $X_{N_1}$ and the inward end
in \( \mathcal{Y}_2 \). This includes the Friedman-Rafsky (FR) test \([10]\) (based on the MST), the test based on the \(K\)-NN graph \([16, 32]\), the cross match test \([29]\) (based on minimum non-bipartite matching), as well as the tests based on the Liu-Singh rank sum statistic \([20]\) (see \([5]\) for details).

Zuo and He \([39]\) proved the asymptotic normality of the Liu-Singh rank sum statistic under general alternatives. However, the limiting distribution of the statistic \((1.4)\) under the alternative, for tests based on geometric graphs, remained open. In this paper, we settle this problem for stabilizing geometric graphs (which include the FR-test and the test based on the \(K\)-NN graph) in the Poissonized setting.

1.2. Poissonization. In the Poissonized setting, instead of taking \(N_1\) samples from the \(F\) and \(N_2\) from \(G\), we take \(\text{Pois}(N_1)\) from \(F\) and \(\text{Pois}(N_2)\) samples from \(G\). The spatial independence of the Poisson process simplifies calculations and yields cleaner formulas for the asymptotic variances.

Suppose \(\{X_i\}_{i \geq 1}\) and \(\{Y_i\}_{i \geq 1}\) be i.i.d. samples from \(f\) and \(g\), respectively, and

\[
\mathcal{X}'_{N_1} = \{X_1, X_2, \ldots, X_{L_{N_1}}\} \quad \text{and} \quad \mathcal{Y}'_{N_2} = \{Y_1, Y_2, \ldots, X_{L_{N_2}}\},
\]

where \(L_{N_1} \sim \text{Pois}(N_1)\) and \(L_{N_2} \sim \text{Pois}(N_2)\) are independent of each other, and of \(\{X_i\}_{i \geq 1}, \{Y_i\}_{i \geq 1}\). Given a graph function \(\mathcal{G}\), the Poissonized two-sample statistic is defined as:

\[
T(\mathcal{G}(\mathcal{X}'_{N_1} \cup \mathcal{Y}'_{N_2})) := \sum_{i=1}^{L_{N_1}} \sum_{j=1}^{L_{N_2}} 1\{(X_i, Y_j) \in E(\mathcal{G}(\mathcal{X}'_{N_1} \cup \mathcal{Y}'_{N_2}))\}.
\]

The distribution of this statistic can be described as follows: Let \(\phi_N(x) := \frac{N_1}{N} f(x) + \frac{N_2}{N} g(x)\) and \(Z_1, Z_2, \ldots\) be independent random variables with common density \(\phi_N(\cdot)\). Let \(L_N\) be an independent Poisson variable with mean \(N_1 + N_2\). Then \(\mathcal{Z}'_N = \{Z_1, Z_2, \ldots, Z_{L_N}\}\) is a non-homogeneous Poisson process in \(\mathbb{R}^d\) with rate function \(N\phi_N = N_1 f + N_2 g\). Label each point of \(z_i \in \mathcal{Z}'_N\) independently with

\[
c_i = \begin{cases} 
1 & \text{with probability } \frac{N_1 f(z_i)}{N_1 f(z_i) + N_2 g(z_i)}, \\
2 & \text{with probability } \frac{N_2 g(z_i)}{N_1 f(z_i) + N_2 g(z_i)}. 
\end{cases}
\]

\(^1\)Note that this is slightly different from the statistic defined in \([5]\), which is normalized by the number of edges in the graph of the pooled sample. In the Poissonized setting, the number of edges in the graph of the pooled sample is random, and for deriving the distribution under the alternative it is convenient to analyze the unnormalized statistic.
Then the sets of points assigned labels 1 and 2 have the same distribution as \( \mathcal{Z}_N^1 \) and \( \mathcal{Z}_N^2 \) (as in (1.5)), respectively. This implies that for a directed graph functional \( \mathcal{G}^2 \), the Poissonized 2-sample test statistic can be re-written as

\[
T(\mathcal{G}(\mathcal{Z}_N')) := \sum_{x,y \in \mathcal{Z}_N'} \psi(c_x, c_y) \mathbf{1}\{(x,y) \in E(\mathcal{G}(\mathcal{Z}_N'))\},
\]

where \( \psi(c_x, c_y) = 1\{c_x = 1, c_y = 2\} \).

Denote by \( \mathbb{E}_0 \) and \( \mathbb{E}_1 \) the expectation under the null and the alternative, respectively. For a directed graph functional \( \mathcal{G} \),

\[
\mathbb{E}_0(T(\mathcal{G}(\mathcal{Z}_N'))) = \frac{N_1N_2}{(N_1 + N_2)^2} \mathbb{E}_0(|E(\mathcal{G}(\mathcal{Z}_N'))|).
\]

Note that for the MST functional, \( \mathbb{E}_0(|E(T(\mathcal{Z}_N'))|) = N - 1 \) and for the K-NN graph functional \( \mathbb{E}_0(|E(N_K(\mathcal{Z}_N'))|) = KN \), respectively. Thus, the level \( \alpha \)-test based on \( \mathcal{G} \) rejects when

\[
N^{-\frac{1}{2}} \left\{ T(\mathcal{G}(\mathcal{Z}_N')) - \mathbb{E}_0(T(\mathcal{G}(\mathcal{Z}_N'))) \right\} \leq \sigma_{\mathcal{G}}^2 z_\alpha,
\]

where \( z_\alpha \) is the standard normal quantile of level \( \alpha \), and \( \sigma_{\mathcal{G}}^2 \) is the limiting null variance of \( N^{-\frac{1}{2}}T(\mathcal{G}(\mathcal{Z}_N')) \), which depends on the graph functional \( \mathcal{G} \), but generally not on the unknown null distribution.

1.3. Stabilizing Graphs. Many geometric graphs such as the MST and the K-NN graph, have local dependence, that is, addition/deletion of a point only effects the edges incident on the neighborhood of that point. This was formalized by Penrose and Yukich [28] using stabilization. To describe the notion of stabilization, few definitions are needed.

Let \( \mathcal{G} \) be a graph functional defined for all locally finite subsets of \( \mathbb{R}^d \).

For \( S \subset \mathbb{R}^d \) nice and \( x \in \mathbb{R}^d \), let \( E(x, \mathcal{G}(S)) \) be the set edges incident on \( x \) in \( \mathcal{G}(S \cup \{x\}) \). Note that \( |E(x, \mathcal{G}(S))| := d(x, \mathcal{G}(S)) \), the (total) degree of the vertex \( x \) in \( \mathcal{G}(S \cup \{x\}) \).

\[\text{Note that every undirected graph functional } \mathcal{G} \text{ can be modified to a directed graph functional } \mathcal{G}_+ \text{ in a natural way: For } S \subset \mathbb{R}^d \text{ finite, } \mathcal{G}_+(S) \text{ is obtained by replacing every edge in } \mathcal{G}(S) \text{ with two directed edges, one in each direction. Thus, without loss of generality, it suffices to consider directed graph functionals.}\]

\[\text{A subset } S \subset \mathbb{R}^d \text{ is locally finite, if } S \cap C \text{ is finite, for all compact subsets } C \subset \mathbb{R}^d.\]

A locally finite set \( S \subset \mathbb{R}^d \) is nice if all the inter-point distances among elements of \( S \) are distinct. If \( S \) is a set of \( N \) i.i.d. points \( W_1, W_2, \ldots, W_N \) from some continuous distribution function \( F \), then the distribution of \( ||W_1 - W_2|| \) does not have any point mass, and \( S \) is nice.
Definition 1.2. Given $S \subset \mathbb{R}^d$ and $y \in \mathbb{R}^d$ and $a \in \mathbb{R}$, denote by $y + S = \{y + z : z \in S\}$ and $aS = \{az : z \in S\}$. A graph functional $G$ is said to be translation invariant if the graphs $G(x + S)$ and $G(S)$ are isomorphic for all points $x \in \mathbb{R}^d$ and all locally finite $S \subset \mathbb{R}^d$. A graph functional $G$ is scale invariant if $G(aS)$ and $G(S)$ are isomorphic for all points $a \in \mathbb{R}$ and all locally finite $S \subset \mathbb{R}^d$.

Let $\mathcal{P}_\lambda$ be the Poisson process of intensity $\lambda \geq 0$ in $\mathbb{R}^d$, and $\mathcal{P}_x^\lambda := \mathcal{P}_\lambda \cup \{x\}$, for $x \in \mathbb{R}^d$. Penrose and Yukich [28] defined stabilization of graph functionals over homogeneous Poisson processes as follows:

Definition 1.3 (Penrose and Yukich [28]). A translation and scale invariant graph functional $G$ stabilizes $\mathcal{P}_\lambda$ if there exists a random but almost surely finite variable $R$ such that

$$E(0, G(\mathcal{P}_\lambda)) = E(0, G(\mathcal{P}_0^\lambda \cap B(0, R) \cup \mathcal{A}))),$$

(1.9)

for all finite $\mathcal{A} \subset \mathbb{R}^d \setminus B(0, R)$, where $B(0, R)$ is the (Euclidean) ball of radius $R$ with center at the point $0 \in \mathbb{R}^d$.

Informally, a graph functional is stabilizing if addition of finitely many points outside a ball of finite radius centered at the origin, does not effect the set of edges incident at the origin. Many popular multivariate two-sample tests are based on stabilizing geometric graphs.

1.3.1. Friedman-Rafsky (FR) Test. Friedman and Rafsky [10] generalized the Wald and Wolfowitz runs test to higher dimensions by using the Euclidean minimal spanning tree of the pooled sample.

Definition 1.4. Given a nice finite set $S \subset \mathbb{R}^d$, a spanning tree of $S$ is a connected graph $T$ with vertex-set $S$ and no cycles. The length $w(T)$ of $T$ is the sum of the Euclidean lengths of the edges of $T$. A minimum spanning tree (MST) of $S$, denoted by $T(S)$, is a spanning tree with the smallest length, that is, $w(T(S)) \leq w(T)$ for all spanning trees $T$ of $S$.

Thus, $T$ defines a graph functional in $\mathbb{R}^d$, and given $\mathcal{N}_1$ and $\mathcal{N}_2$ as in (1.5), the FR-test rejects $H_0$ for small values of

$$T(T(\mathcal{Z}_N')) = \sum_{x, y \in \mathcal{Z}_N'} \psi(c_x, c_y)1\{(x, y) \in E(T(\mathcal{Z}_N'))\}$$

$$= \sum_{(x, y) \in E(T(\mathcal{Z}_N'))} 1\{c_x \neq c_y\},$$
\[ = \sum_{x, y \in Z'} \psi(c_x, c_y) 1\{(x, y) \in E(T(\mathcal{Z}'_N))\}, \quad (1.10) \]

where \( Z'_N = \mathcal{X}'_{N_1} \cup \mathcal{Y}'_{N_2} \) and \( T(\mathcal{Z}'_N) \) is obtained by replacing every (undirected) edge in \( T(\mathcal{Z}'_N) \) with two directed edges, one in each direction.

Friedman and Rafsky [10] calibrated (1.10) as a permutation test, and showed that it has good power in practice for multivariate data, especially against fixed scale alternatives. Later, Henze and Penrose [17] proved that the statistic \( T(T(\mathcal{Z}'_N)) \) is asymptotically normal under \( H_0 \) and is consistent under all fixed alternatives.

Aldous and Steele [1] extended the MST graph functional to locally finite point sets using the Prim’s algorithm. Moreover, by [28, Lemma 2.1] the MST graph functional \( \mathcal{T} \) stabilizes \( \mathcal{P}_1 \).

1.3.2. Test Based on the K-Nearest Neighbor (K-NN) Graph. As in (1.10), a multivariate two-sample test can be constructed using the K-nearest neighbor graph of \( \mathcal{Z}'_N \). This was originally suggested by Friedman and Rafsky [10], and later studied by Schilling [32] and Henze [16].

**Definition 1.5.** Given a nice finite set \( S \subset \mathbb{R}^d \), the (directed) K-nearest neighbor graph (K-NN) is a graph with vertex set \( S \) with a directed edge \((a, b)\), for \( a, b \in S \), if the Euclidean distance between \( a \) and \( b \) is among the \( K \)-th smallest distances from \( a \) to any other point in \( S \). Denote the directed K-NN of \( S \) by \( \mathcal{N}_K(S) \).

Given \( \mathcal{Z}'_N = \mathcal{X}'_{N_1} \cup \mathcal{Y}'_{N_2} \) as in (1.5), the K-NN statistic is

\[ T(\mathcal{N}_K(\mathcal{Z}'_N)) = \sum_{x, y \in \mathcal{Z}'_N} \psi(c_x, c_y) 1\{(x, y) \in E(\mathcal{N}_K(\mathcal{Z}'_N))\}. \quad (1.11) \]

Schilling [35] showed that the test based on the K-nearest neighbor graph is asymptotically normal under \( H_0 \) and consistent under fixed alternatives.\(^4\) The K-NN graph, which can be naturally extended to locally finite points sets, stabilizes \( \mathcal{P}_1 \) [27, Lemma 6.1].

1.4. Summary of Results. In this paper central limit theorems are derived for two-sample tests based on stabilizing geometric graphs under general alternatives, in the Poissonized setting. They provide mathematical justifications for these tests and validates their various applications. In particular, this gives a way to compare the asymptotic performances of tests based

\(^4\)The statistic (1.11) is slightly different from the test used by Schilling [32, Section 2], which can also be re-written as graph-based test (1.4) by allowing multiple edges in the K-NN graph.
on geometric graphs which have zero asymptotic Pitman efficiency, complementing recent results by the author [5]. The results are summarized below:

- It is shown that for tests based on stabilizing geometric graphs, such as the Friedman-Rafsky test (1.10) and the test based on the K-nearest-neighbor (K-NN) graph (1.11), the graph-based two-sample test statistic has a limiting normal distribution, after centering by the conditional mean and scaling by \( N^{-\frac{1}{2}} \) (Theorem 3.1).
- Under the stronger assumption of exponential stabilization [26], the conditional CLT can be strengthened to obtain the central limit theorem (CLT) of the graph-based two-sample statistic, when centered by the unconditional mean (Theorem 3.2). In particular, this implies the CLT for the K-NN test under the alternative, in the Poissonized setting.
- As a consequence, local alternatives for which the K-NN based test have non-trivial (bounded between 0 and 1) limiting power can be identified. In particular, in many situations the test based on the K-NN graph has power going to 1, for alternatives shrinking towards the null slower than \( O(N^{-\frac{1}{4}}) \). This shows that though the K-NN test has zero asymptotic Pitman efficiency, that is, no power against \( O(N^{-\frac{1}{2}}) \) alternatives [5], it is often powerful against alternatives which shrink towards the null more slowly, in particular, at a rate slower than \( O(N^{-\frac{1}{4}}) \). In general, there are two exponents \( \frac{1}{2} < \beta_d \leq \gamma_d < 1 \), depending on the dimension \( d \), such that the K-NN based test has no limiting power for alternatives shrinking towards the null faster than \( O(N^{-\beta_d}) \), and power going to 1 for alternatives shrinking towards the null at a rate slower than \( O(N^{-\gamma_d}) \) (Theorem 4.2).

The rest of the paper is organized as follows: The general consistency result is stated in Section 2. The central limit theorems for the statistic (1.4) are described in Section 3. The application of these results to the K-NN graph functional is given in Section 4. The proofs of the results are given in the appendix.

**2. Consistency.** The consistency of the test based on the K-NN graph was proved by Schilling [32] and more generally by Henze [16]. Later, Henze and Penrose [17] proved the consistency of the FR-test. The arguments here show that the proof in [17] can be easily extended to any stabilizing graph functional, whenever the degree function satisfy the following moment condition:

**Assumption 2.1.** (Degree Moment Condition) A translation and scale
invariant graph functional $\mathcal{G}$ is said to satisfy the $\beta$-degree moment condition if it stabilizes $\mathcal{P}_\lambda$, for all $\lambda \in (0, \infty)$, and

$$\sup_{N \in \mathbb{N}} \sup_{z \in \mathbb{R}^d} \sup_{\mathcal{A} \subseteq \mathbb{R}^d} \mathbb{E} \left\{ d(z, \mathcal{G}(\mathcal{P}_{N\phi_N} \cup \mathcal{A}))^{\beta} \right\} < \infty,$$

where $\mathcal{A}$ ranges over all finite subsets of $\mathbb{R}^d$.

The weak limit of the statistic $\frac{1}{N} T(\mathcal{G}(\mathcal{Z}'_N))$ is given in terms of the Henze-Penrose dissimilarity measure between the two density functions:

**Definition 2.1.** Given $p \in (0, 1)$ and densities $f$ and $g$ in $\mathbb{R}^d$, the Henze-Penrose dissimilarity measure is defined as

$$\delta(f, g, p) = 1 - 2pq \int \frac{f(x)g(x)}{pf(x) + qg(x)} \, dx.$$  \hfill (2.2)

This belongs to a general class of separation measures between distributions [13, 14].

The following proposition gives the weak-limit of $\frac{1}{N} T(\mathcal{G}(\mathcal{Z}'_N))$ for stabilizing graph functionals satisfying the degree moment condition. The proof of the proposition is given in Section A.3.

**Proposition 2.1.** Let $\mathcal{G}$ be a translation and scale invariant directed graph functional which stabilizes $\mathcal{P}_\lambda$ for all $\lambda \in (0, \infty)$. If $\mathcal{G}$ satisfies the $\beta$-degree moment condition for $\beta > 4$, then

$$\frac{1}{N} T(\mathcal{G}(\mathcal{Z}'_N)) \xrightarrow{P} \mathbb{E} \frac{\Delta^+_0}{2} (1 - \delta(f, g, p)), $$ \hfill (2.3)

where $\Delta^+_0 = d^+(0, \mathcal{G}(\mathcal{P}_1))$ is the out-degree of the vertex zero in the graph $\mathcal{G}(\mathcal{P}_1 \cup \{0\})$.

Using the asymptotic normality of $T(\mathcal{G}(\mathcal{Z}'_N))$ under the null and the above result, it follows that the level $\alpha$-test with rejection region (1.8) is consistent for all fixed alternatives (1.2), since $\delta(f, g, p) \geq \delta(f, f, p) = p^2 + q^2$ and the inequality is strict for densities $f$ and $g$ differing on a set of positive measure (see [13, Theorem 1 and Corollary 1]). This generalizes the results in [17, 32] to stabilizing graph functionals.
Remark 2.1. Recently, Arias-Castro and Pelletier [2] showed that Rosenbaum’s cross match test [29] based on non-bipartite matching, has the same limit as in (2.3), thus, it is also consistent for general alternatives. Note that this does not follow from Proposition 2.1, because it is unknown whether the NBM graph functional is stabilizing.

3. Distribution under General Alternatives. This section describes the central limit theorems of the Poissonized two sample statistic $T(\mathcal{G}(Z'_N))$ (recall (1.7)) for stabilizing graph functionals. Let $f$ and $g$ be two densities in $\mathbb{R}^d$, and

$$\phi_N(x) = \frac{N_1}{N} f(x) + \frac{N_2}{N} g(x), \quad \phi(x) = pf(x) + qg(x).$$ (3.1)

Note that $\phi_N \to \phi$ uniformly over the support of $\phi$. Let $Z' = \{Z_1, Z_2, \ldots\}$ be independent random variables with common density $\phi_N$. Then $Z'_N = \{Z_1, Z_2, \ldots, Z_{L_N}\}$, where $L_N \sim \text{Pois}(N)$ is independent of $Z'$. Let $F := \sigma(Z', L_N)$ be the sigma-algebra generated by $Z'$ and the Poisson random variable $L_N$.

In this section, the limiting distribution of $R(\mathcal{G}(Z'_N)) = N^{-\frac{1}{2}} \left\{ T(\mathcal{G}(Z'_N)) - \mathbb{E}_1(T(\mathcal{G}(Z'_N))) \right\}$ (3.2)

is derived for stabilizing graph functionals. Note that $R(\mathcal{G}(Z'_N)) = R_1(\mathcal{G}(Z'_N)) + R_2(\mathcal{G}(Z'_N))$, where

$$R_1(\mathcal{G}(Z'_N)) = N^{-\frac{1}{2}} \left\{ T(\mathcal{G}(Z'_N)) - \mathbb{E}_1(T(\mathcal{G}(Z'_N)) | F) \right\},$$ (3.3)

is the statistic centered by the conditional mean and scaled by $N^{-\frac{1}{2}}$, and

$$R_2(\mathcal{G}(Z'_N)) = N^{-\frac{1}{2}} \left\{ \mathbb{E}_1(T(\mathcal{G}(Z'_N)) | F) - \mathbb{E}_1(T(\mathcal{G}(Z'_N))) \right\}.$$ (3.4)

is the centered conditional mean.

The CLT of (3.2) is derived in two steps: First, conditioning on $F$, the CLT of (3.3) is obtained for stabilizing graph functionals (Theorem 3.1). Then, under the stronger assumption of exponential stabilization (Definition 3.2), the CLT of the conditional mean (3.4) is derived (Theorem 3.2), which combined with the conditional CLT of (3.3) gives the CLT of (3.2) (Theorem 3.3).
3.1. CLT Centered by the Conditional Mean. For a directed graph functional \( G \), \( S \subset \mathbb{R}^d \) finite and a point \( x \in \mathbb{R}^d \), let \( d^\downarrow(x, G(S)) \) be the in-degree, \( d^\uparrow(x, G(S)) \) be the out-degree, and \( d(x, G(S)) = d^\downarrow(x, G(S)) + d^\uparrow(x, G(S)) \) be the total degree of the vertex \( x \) in the graph \( G(S \cup \{x\}) \). Also, let

\[
T_2^\downarrow(x, G(S)) = \left(\frac{d^\downarrow(x, G(S))}{2}\right), \quad T_2^\uparrow(x, G(S)) = \left(\frac{d^\uparrow(x, G(S))}{2}\right)
\]

be the number of outward 2-stars and inward 2-stars incident on \( x \) in \( G(S) \), respectively. Finally, let \( T_2^\downarrow(x, G(S)) \) be the number of 2-stars incident on \( x \) in \( G(S) \) with different directions on the two edges. For notational brevity, denote

\[
\Delta_0^\downarrow = d^\downarrow(0, G(\mathcal{P}_1)), \quad \Delta_0^\uparrow = d^\uparrow(0, G(\mathcal{P}_1)),
\]

and \( \Delta_0^+ := |\{z \in \mathcal{P}_1 : (0, z), (z, 0) \in E(G(\mathcal{P}_{1}))\}| \). Similarly, let

\[
T_2^\downarrow = T_2^\downarrow(0, G(\mathcal{P}_1)), \quad T_2^\uparrow = T_2^\uparrow(0, G(\mathcal{P}_1)),
\]

and \( T_2^+ := T_2^+(0, G(\mathcal{P}_1)) \)

To derive the CLT of (3.3), various assumptions are required on the graph functional \( G \). To begin with recall the degree moment condition (2.1). Another condition that is required is that the graph functional has bounded maximum degree. This is trivially satisfied by bounded degree graph functionals, such as the MST and the K-NN graph. The slightly weaker polynomial upper bound given below includes stabilizing geometric graphs with poly-logarithmic maximum degree, like the Delaunay graph [28].

**Assumption 3.1.** (Maximum Degree Condition) A graph functional \( G \) is said to satisfy the maximum degree condition if

\[
\sup_{z \in \mathcal{P}_N} d(z, G(\mathcal{P}_{N\phi_N})) = o_P(N^\alpha),
\]

for some \( \alpha \leq \frac{1}{40} \).

The following theorem gives the CLT of (3.3) for stabilizing graph functionals satisfying the above conditions.

**Theorem 3.1.** Let \( G \) be a translation and scale invariant directed graph functional which stabilizes \( \mathcal{P}_\lambda \), for all \( \lambda \in (0, \infty) \). If \( G \) satisfies the \( \beta \)-degree moment condition for \( \beta > 4 \) and the maximum degree condition (3.8), then

\[
\mathcal{R}_1(G(Z'_N)) \overset{D}{\to} N(0, \sigma_{\mathcal{G},1}^2),
\]
where

\[
\sigma_{g,1}^2 = \frac{r}{4} \left\{ 2E\Delta_0^\uparrow \int \frac{f(x)g(x)}{\phi(x)} \, dx + 4 \int \frac{f(x)g(x)}{\phi^2(x)} \left( pET_2^\uparrow f(x) + qET_2^\downarrow g(x) \right) \, dx \right.
\]

\[
- r\beta_0 \int \frac{f^2(x)g^2(x)}{\phi^3(x)} \, dx \right\}, \tag{3.9}
\]

and \( \beta_0 := 2ET_2^\uparrow + 2ET_2^\downarrow + 2ET + 2 + 2\Delta_0 + E\Delta_0^\uparrow. \)

The proof of the theorem is given in Section A.4. The limiting conditional variance of \( R_1(G(Z'_N)) \) is derived using properties of stabilizing graphs (Lemma A.4), and the CLT is proved using Stein’s method based on dependency graphs.

**Remark 3.1.** The limiting variance \( \sigma_{g,1}^2 = \sigma_{g,1}(f,g,p) \) in (3.9) depends on the graph functional \( G \), the unknown densities \( f \) and \( g \), and the limiting fraction \( p \) of sample 1. Under the null distribution \( (f = g) \) the expression simplifies to

\[
\sigma_{g,1}^2(f,f,p) = \frac{r}{4} \left\{ 2E\Delta_0^\uparrow + 4 \left( pET_2^\uparrow + qET_2^\downarrow \right) - r\beta_0 \right\}. \tag{3.10}
\]

For example, when \( G = \mathcal{N}_K \) is the \( K \)-NN nearest neighbor graph functional, \( \Delta_0^\uparrow = K, T_2^\uparrow = \frac{K(K-1)}{2}, T_2^\downarrow = \Delta_0^\uparrow - \Delta_0^\downarrow = K\Delta_0^\downarrow - \Delta_0^\downarrow, E\Delta_0 = K \), and (3.10) simplifies to

\[
\sigma_{\mathcal{N}_K,1}^2(f,f,p) = \frac{r}{2} \left\{ Kpq + p^2K^2 + q^2E(\Delta_0^\downarrow)^2 - rK^2 \right\}
\]

\[
= \frac{r}{2} \left\{ Kpq + (p - q)^2K^2 + q^2\text{Var}(\Delta_0^\downarrow) \right\}. \tag{3.11}
\]

For an undirected graph functional \( \mathcal{G} \) (3.10) further simplifies to

\[
\sigma_{\mathcal{G},1}^2(f,f,p) = \frac{r}{4} \left\{ rE\Delta_0 + 2E(\Delta_0^\downarrow)(1 - 2r) \right\}, \tag{3.12}
\]

since \( ET_2^\uparrow = ET_2^\downarrow = E\Delta_0^\downarrow, ET_2^\downarrow = E\Delta_0(\Delta_0 - 1), \) and \( E\Delta_0 = E\Delta_0^\downarrow. \)

For the MST graph functional \( \mathcal{T} \), \( \Delta_0 = 2 \) [1, Lemma 7], and (3.12) becomes

\[
\sigma_{\mathcal{T},1}^2(f,f,p) = \frac{r}{2} \left\{ r + E(\Delta_0^\downarrow)(1 - 2r) \right\}.
\]

In general there seem to be no closed form expressions for \( \text{Var}(\Delta_0^\downarrow) \) for the \( K \)-NN graph, or \( E(\Delta_0^\downarrow) \) for the MST. To compute the rejection regions for these tests, the above quantities need to be estimated through simulations. For the MST, Henze and Penrose [17] computed \( \text{Var}(\Delta_0) \) for small values of \( d \). They also computed \( \lim_{d \to \infty} \text{Var}(\Delta_0) \), using earlier results of Penrose [23], which can be a useful approximation when dimension is large.
3.2. CLT of the Conditional Mean and Applications. Proving the CLT of the conditional mean (3.4) is more difficult, and requires the stronger notion of exponential stabilization [26]. For any locally finite point set \( \mathcal{H} \subset \mathbb{R}^d \) and \( x \in \mathbb{R}^d \), define the out-degree measure of a graph functional \( \mathcal{G} \) as follows:

For all Borel sets \( A \subseteq \mathbb{R}^d \),

\[
d_{out}^A(x, \mathcal{H}, A) = \sum_{y \in \mathcal{H} \cap A} \mathbf{1}\{(x, y) \in E(\mathcal{G}(\mathcal{H}^x))\},
\]

(3.13)

where \( \mathcal{H}^x = \mathcal{H} \cup \{x\} \). In other words, the out-degree measure of a set \( A \), with respect to \( \mathcal{H} \) and \( x \) is the number of edges incident on \( x \) with the other end point in \( \mathcal{H}^x \cap A \) in the graph \( \mathcal{G}(\mathcal{H}^x) \).

**Definition 3.1.** Fix a locally finite point set \( \mathcal{H} \), a point \( x \in \mathbb{R}^d \), and a Borel set \( A \subseteq \mathbb{R}^d \). The radius of stabilization of the degree measure (3.13) at \( x \) with respect to \( \mathcal{H} \) and \( A \) (to be denoted by \( R(x, \mathcal{H}, A) \)) is the smallest \( R \geq 0 \) such that

\[
d_{out}^A(x, x+\{\mathcal{H}\cap B(0,R)\cup Y\}, x+B) = d_{out}^A(x, x+\{\mathcal{H}\cap B(0,R)\}, x+B),
\]

(3.14)

for all finite \( Y \subseteq A \setminus B(0,R) \) and all Borel subsets \( B \subseteq A \), where \( B(0,R) \) is the (Euclidean) ball of radius \( R \) with center at the point \( 0 \in \mathbb{R}^d \). If no such \( R \) exists, then set \( R(x, \mathcal{H}, A) = \infty \).

Let \( S_{f,g} = S_f \cup S_g \), where \( S_f \) and \( S_g \) are the supports of the densities \( f \) and \( g \), respectively. Recalling \( \phi_N \) from (3.1) define:

**Definition 3.2.** Let \( R_N(x) := R(x, \mathcal{P}_{N\phi_N}, S_{f,g}) \) be the radius of stabilization of out-degree measure \( d_{out}^A \) at \( x \) with respect to the Poisson process \( \mathcal{P}_{N\phi_N} \) and the union of the supports \( S_{f,g} \). Define,

\[
\tau(s) := \sup_{N} \sup_{x \in \mathbb{R}^d} \mathbb{P}(R_{N\phi_N}(x) > N^{-1/d}s).
\]

(3.15)

The out-degree measure \( d_{out}^A \) is

- **power law stabilizing of order \( q \)** if \( \sup_{s \geq 1} s^q \tau(s) < \infty \),
- **exponentially stabilizing** if \( \limsup_{s \to \infty} \frac{\log \tau(s)}{s} < 0 \).

The above definition of exponential stabilization from Definition 3.2 is slightly different from [26, Definition 2.4], because the density \( \phi_N \) in the Poisson process \( \mathcal{P}_{N\phi_N} \) depends on \( N \). However, the proof in [26] can be
trivially modified to show the exponential stabilization of the $K$-NN graph functional (see Section A.5.3 for details).

Conditions on the decay of the tail of the radius of stabilization, similar to (3.15) above, is a standard requirement for proving limit theorems of functionals of random geometric graphs [26, 27, 37]. Using this machinery, the following theorem gives the CLT of the conditional mean for exponentially stabilizing random geometric graphs. To this end, assume that

$$\sqrt{N} \left( \frac{N_1}{N_1 + N_2} - p \right) \to 0 \text{ and } \sqrt{N} \left( \frac{N_1}{N_1 + N_2} - q \right) \to 0, \text{ as } N \to \infty.$$ 

THEOREM 3.2. Let $G$ be a translation and scale invariant directed graph functional in $\mathbb{R}^d$ which satisfies the $\beta$-degree moment condition (2.1) for $\beta > 2$. If the out-degree measure $d_G^+\uparrow$ is power law stabilizing of order $q > \frac{\beta}{\beta - 2}$, then

$$\lim_{N \to \infty} \text{Var}(R_2(G(Z_N^'))) = \sigma_{G,2}^2, \quad (3.16)$$

where

$$\sigma_{G,2}^2 = \int \left\{ \left( E[d^+(0, G(P_1^+))d^+(z, G(P_0^0))] - (E\Delta_0^+)^2 \right) dz \right\} + (E\Delta_0^+)^2 \int \frac{f(x)^2 g^2(x)}{\phi^3(x)} dx \quad (3.17)$$

Moreover, if $d_G^+\uparrow$ is exponentially stabilizing then

$$R_2(G(Z_N^')) \overset{D}{\to} N(0, \sigma_{G,2}^2). \quad (3.18)$$

The proof of the theorem is given in Section A.5.1. Combining Theorem 3.1 and Theorem 3.2 the CLT of $R(G(Z_N^'))$ (defined in (3.2)) can be obtained. The proof is in Section A.5.2.

THEOREM 3.3. Let $G$ be a translation and scale invariant directed graph functional which satisfies the $\beta$-degree moment condition for $\beta > 4$ and the maximum out-degree condition (3.8). If the degree measure $d_G^+\uparrow$ is exponentially stabilizing then

$$R(G(Z_N^')) \overset{D}{\to} N(0, \sigma_{G}^2(f, g, p)), \quad (3.19)$$

where $\sigma_{G}^2(f, g, p) = \sigma_{G,1}^2 + \sigma_{G,2}^2$, and $\sigma_{G,1}^2$ and $\sigma_{G,2}^2$ are as defined in (3.9) and (3.17), respectively.
Remark 3.2. If $\mathcal{G} = \mathcal{N}_K$ is the $K$-NN nearest neighbor graph functional, (3.17) reduces to
\[ \sigma^2_{N_K,2}(f,g,p) = \frac{r^2K^2}{4} \int \frac{f^2(x)g^2(x)}{\phi^3(x)} \, dx. \] (3.20)

Under the null ($f = g$), this further reduces to $\sigma^2_{N_K,2}(f,f,p) = \frac{r^2K^2}{4}$. This is expected, because under the null $R^2(\mathcal{N}_K(Z'_{N})) = N_1N_2/N^2$.

4. Test Based on the $K$-NN Graph. Many random geometric graphs, such as the $K$-NN graph and the Delaunay graph are exponentially stabilizing, and Theorem 3.3 can be used to obtain their asymptotic distributions. In particular, this provides mathematical justifications for the two-sample test statistic based on the $K$-NN graph functional, which has been used in various applications. Moreover, using this asymptotic distribution, local alternatives that have non-trivial (bounded between 0 and 1) limiting power can be identified, which provides a way to compare the asymptotic performances of the tests.

4.1. Distribution Under General Alternatives. Assume that the densities $f, g$ are bounded and almost everywhere continuous probability density functions in $\mathbb{R}^d$ with bounded convex support, such that the union of their support $S_{f,g} = S_f \cup S_g$ is convex. In this case, the $K$-NN graph functional $\mathcal{N}_K$ is exponentially stabilizing. This shows the following corollary of Theorem 3.3:

**Corollary 4.1.** For the $K$-NN graph functional $\mathcal{N}_K$ and $f,g$ as above,
\[ R(\mathcal{N}_K(Z'_{N})) \xrightarrow{D} N(0, \sigma^2_K(f,g,p)), \] (4.1)

---

5 Given a locally finite set $S \subset \mathbb{R}^d$ and $x \in S$, the locus of points closer to $x$ than to any other point in $S$ is called the Voronoi cell centered at $x$. The graph with vertex set $S$ in which each pair of adjacent cell centers is connected by an edge is the Delaunay graph of $S$. The Delaunay graph is exponentially stabilizing [26, Section 6.3] (see also [27, 28]).

6 Note that (3.15) is slightly different from [26, Definition 2.4], because the density $\phi_N$ in the Poisson process $\mathcal{P}_{N_{\phi_N}}$ depends on $N$. Nevertheless, the proof of [26, Theorem 6.4] can be easily modified to show the exponential stabilization of the out-degree measure (3.13) for the $K$-NN graph functional $\mathcal{N}_K$, when $f$ and $g$ are as in Corollary 4.1 (see Lemma A.8 for more details).
where \( \sigma^2_K(f, g, p) = \underbrace{\sigma^2_{N_K,1}}_{\text{as defined in (3.9)}} + \frac{2r^2}{4} \int \frac{f(x)g^2(x)}{\phi(x)} \, dx \), and \( \sigma_{N_K,1} \) is as defined in (3.9).

**Proof.** The \( N_K \) graph functional is exponentially stabilizing and satisfies the degree moment condition for \( \beta > 4 \). Therefore, (4.1) holds with \( \sigma^2_K(f, g, p) = \sigma^2_{g,1} + \sigma^2_{g,2} \), where \( \sigma^2_{g,1} \) and \( \sigma^2_{g,2} \) are as defined in (3.9) and (3.17), respectively. The result then follows from (3.20).

**Remark 4.1.** Under the null \((f = g)\), using (3.11) and (3.21), the asymptotic variance (4.1) simplifies to

\[
\sigma^2_K := \sigma^2_K(f, f, p) = \frac{r}{2} \left\{ K(K+1)pq + (p-q)^2 K^2 + q^2 \text{Var}(\Delta^4_0) \right\}. \tag{4.2}
\]

Therefore, the two-sample test based on \( N_K \) rejects when

\[
N^{-\frac{1}{2}} \left\{ T(N_K(Z_N')) - \mathbb{E}_0(T(N_K(Z_N'))) \right\} \leq \sigma_K z_\alpha. \tag{4.3}
\]

This test is asymptotically level \( \alpha \) and distribution free, and consistent against all fixed alternatives. In the following section, we derive the power of this test against local alternatives.

Unfortunately, Theorem 3.3 does not apply to the FR-test, because it is not known whether the MST is exponentially stabilizing. Kesten and Lee [18] and later Penrose [25] used a martingale method to derive central limit theorems for functionals of the MST defined on a homogeneous Poisson process. It might be possible to leverage these techniques with the ideas developed in this section to prove a central limit theorem for the FR-test.

### 4.2. Power Against Local Alternatives.

In [5] it was shown that two-sample tests based on stabilizing graph functionals have zero Pitman efficiency (power against \( O(N^{-\frac{1}{2}}) \) alternatives). This raises the question: Does the \( K \)-NN test have power for alternatives shrinking towards the null at rate slower than \( O(N^{-\frac{1}{2}}) \)? Corollary 4.1 can be used to determine the rate at which the alternatives shrink toward the null, such that the \( K \)-NN test have non-trivial (bounded by 0 and 1) limiting power. This can provide a way to compare between tests which have zero Pitman efficiencies.

To this end, let \( \Theta \subseteq \mathbb{R}^p \) be the parameter space and \( \{ \mathbb{P}_\theta \}_{\theta \in \Theta} \) be a parametric family of distributions in \( \mathbb{R}^d \) with density \( f(\cdot | \theta) \). Let \( \mathcal{X}'_{N_1} \) and \( \mathcal{Y}'_{N_2} \) be samples from \( \mathbb{P}_{\theta_1} \) and \( \mathbb{P}_{\theta_2} \) as in (1.5), respectively, and consider the testing problem

\[
H_0 : \theta_2 - \theta_1 = 0, \quad \text{versus} \quad H_1 : \theta_2 - \theta_1 = \delta_N, \tag{4.4}
\]
for a sequence \( \{ \delta_N \}_{N \geq 1} \) in \( \mathbb{R}^p \), such that \( ||\delta_N|| \to \infty \).

The limiting power of the two-sample test based on \( \mathcal{G} \) is the probability of the rejection region (1.8) under the alternative \( H_1 \) above, as \( N \) goes to infinity.

**Assumption 4.1.** To derive the local power, the following assumptions on the family \( \{ \mathbb{P}_\theta \}_{\theta \in \Theta} \) are needed:

(a) For all \( \theta \in \Theta \), the density \( f(\cdot|\theta) \) has a compact and convex support \( S \subset \mathbb{R}^d \), with a non-empty interior, not depending on \( \theta \).

(b) \( \int_{\partial S} f(z|\theta) \, dz = 0 \), for all \( \theta \in \Theta \), where \( \partial S \) denotes the boundary of \( S \).

(c) For all \( \theta \in \Theta \), the functions \( f(\cdot|\theta) \) and \( \nabla_\theta f(\cdot|\theta) \) are three times continuously differentiable in the interior of \( S \).

(d) For all \( x \in S \), \( f(x|\cdot) \) is three times continuously differentiable in the interior of \( \Theta \).

Define the critical exponents,

\[
\beta_d = \begin{cases} 
\frac{1}{4} & \text{if } d \leq 8 \\
\frac{1}{2} - \frac{2}{d} & \text{if } d \geq 9
\end{cases}, \quad \gamma_d = \begin{cases} 
\frac{1}{4} & \text{if } d \leq 8 \\
\frac{1}{2} & \text{if } d \geq 9
\end{cases}.
\tag{4.5}
\]

The following theorem gives bounds on the rate at which the hypotheses should shrink toward each other to have no power or full power. (Note that for a vector \( x \in \mathbb{R}^p \), \( ||x|| \) denotes the Euclidean norm of \( x \).)

**Theorem 4.2.** Let \( \beta_d \) and \( \gamma_d \) be as defined in (4.5). Consider the two-sample test based on the \( K \)-NN graph functional \( N_k \) with rejection region (4.3) for the testing problem (4.4).

(a) If \( ||N^{\beta_d} \delta_N|| \to 0 \), the limiting power of the test (4.3) is \( \alpha \).

(b) If \( ||N^{\gamma_d} \delta_N|| \to \infty \), the limiting power of the test (4.3) is \( 1 \).

Note that for \( d \leq 8 \), the bounds in the above theorem match, showing that for all hypothesis and all directions \( \delta_N \), the limiting power of the \( K \)-NN test has a sharp phase transition at \( N^{-\frac{1}{4}} \). For \( \delta_N = hN^{-\frac{1}{4}} \), the test has non-trivial limiting power, which can be computed from the proof of Theorem 4.2. This is illustrated for the normal location problem in Section 4.2.1 below. Quite interestingly, the critical exponents \( \beta_d \) and \( \gamma_d \) has a phase-transition at \( d = 8 \). The lower critical exponent \( \beta_d \) decreases with \( d \) to \( \frac{1}{2} \).
(recall the $K$-NN test has no power for $N^{-\frac{1}{2}}$ alternatives [5]), and the upper critical exponent increases with $d$ to 1 (the $K$-NN test always has power against fixed alternatives).

4.2.1. Normal Location Problem. Let $S \subset \mathbb{R}^d$ be a symmetric set\footnote{A set $S \subset \mathbb{R}^d$ is symmetric if $x \in S$ implies $-x \in S$. For example, $S = B(0,1)$, the $d$-dimensional unit ball, is a symmetric set in $\mathbb{R}^d$.} containing the origin $0 \in \mathbb{R}^d$. For $\mu \in \mathbb{R}^d$, let $P^\mu_S$ be the normal distribution with mean $\mu$ and standard deviation 1 truncated to $S$, that is, $dP^\mu_S = f(x|\mu)dx$ and

$$f(x|\mu) = \frac{1}{C_S(\mu)}e^{-\frac{1}{2}(x-\mu)^\top(x-\mu)} \text{ for } x \in S,$$

where $C_S(\mu) = \int_S e^{-\frac{1}{2}(x-\mu)^\top(x-\mu)}dx$. Denote by $E^\mu_S$ the expectation with respect to $P^\mu_S$, and $\mu_S = E^\mu_S(x) = \int_S xe^{-\frac{1}{2}(x-\mu)^\top(x-\mu)}dx$ be the mean of the distribution.

For a fixed symmetric set $S \subset \mathbb{R}^d$ and $\delta_N \in \mathbb{R}^d$, suppose

$$\mathcal{X}_N = \{X_1, X_2, \ldots, X_{L_N}\} \quad \text{and} \quad \mathcal{Y}_N = \{Y_1, Y_2, \ldots, Y_{L_N}\},$$

are independent samples from $P^0_S$ and $P^\delta_N$, respectively, and $L_N \sim \text{Pois}(N_1)$ and $L_{N_2} \sim \text{Pois}(N_2)$ are independent of each other and of $\{X_i\}_{i \geq 1}, \{Y_i\}_{i \geq 1}$.

**Proposition 4.3.** Consider the two-sample test based on the 1-NN graph functional, with rejection region (4.3), for distinguishing the samples in (4.7).

(a) If $||N^{\frac{1}{2}}\delta_N|| \to 0$, the limiting power of the test is $\alpha$.

(b) If $||N^{\frac{1}{2}}\delta_N|| \to \infty$, the limiting power of the test is 1.

(c) If $\delta_N = hN^{-\frac{1}{2}}$, for some $h \in \mathbb{R}^p \setminus \{0\}$, the limiting power of the test is

$$\Phi \left( z_\alpha + \frac{r_2^2}{4\sigma_1} E^0_S(h^\top x)^2 \right),$$

where $V_d = |B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^d$, and $\sigma_1$ is as defined in (4.2).

The proof of the result is given in Section B.4. In fact, Proposition 4.3 easily generalizes to location problems for elliptical distributions. Note that truncated normal distributions are considered, so that Assumption 4.1(a) is satisfied. This is merely a technical condition, and it should be possible to replace this by assumptions of the tails of the distributions.

The following example compares the power of the $K$-NN for the normal location problem in simulations.
Example 4.1 (Normal Location). Consider the parametric family $P_\theta \sim N(\theta, I)$, for $\theta \in \mathbb{R}^d$. To compute the empirical power, consider i.i.d. samples $X_{N_1}$ and $Y_{N_2}$ (as in (1.1)) from $P_0$ and $P_{\delta N}$, where $\delta N$ is a sequence in $\mathbb{R}$ converging to 0.

(a) Figure 1 shows the empirical power (as a function of $h$, where $\delta N = hN^{-\frac{d}{4}}$) of the Hotelling’s $T^2$ test (which is the most powerful test in this case), the FR-test based on the MST, and the tests based on the 1-NN, 2-NN, and 3-NN graphs, repeated over 200 iterations over a grid of 20 values of $h$ in $[0, 3]$ for dimension (a) $d = 3$ and (b) $d = 4$, with sample sizes $N_1 = 1500$ and $N_2 = 1000$. Note that in both dimensions 3 and 4, $\beta_d = \frac{1}{4}$, and the tests based on geometric graphs have non-zero local power $\delta_N = hN^{-\frac{d}{4}}$, as predicted by Theorem 4.2.

(b) For dimension $d = 20$, $\beta_d = 2/5 > 1/4$. Figure 2 shows the empirical power (as a function of $h$) of the Hotelling’s $T^2$ test, the FR-test based on the MST, and the tests based on the 1-NN, 2-NN, and 3-NN graphs, repeated over 200 iterations over a grid of 20 values of $h$ in $[0, 3]$ for dimension $d = 20$, with sample sizes $N_1 = 8000$ and $N_2 = 6000$ for (a) $\delta_N = hN^{-\frac{2}{5}}$, and (b) $\delta_N = hN^{-\frac{1}{4}}$. As expected, the Hotelling’s $T^2$ test has power converging to 1 in both cases. However, as shown in Proposition 4.3, the FR test and the $K$-NN test have little power (as a function of $h$) in Figure 2(a), but have non-trivial power in Figure 2(b).

Acknowledgements. The author is indebted to his advisor Persi Diaconis.
DISTRIBUTION OF GRAPH BASED TWO-SAMPLE TESTS

Fig 2: Empirical power for the normal location problem in Example 4.1 in dimension $d = 20$, for (a) $\delta_N = hN^{-\frac{2}{5}}$, and (b) $\delta_N = hN^{-\frac{1}{4}}$.

for introducing him to graph-based tests and for his constant encouragement and support. The author thanks Riddhipratim Basu, Sourav Chatterjee, Jerry Friedman, Shirshendu Ganguly, and Susan Holmes for illuminating discussions and helpful comments.

References.

[1] D. Aldous, D. and J. M. Steele, Asymptotics for Euclidean minimal spanning trees on random points, *Probab. Theory Related Fields*, Vol. 92, 247–258, 1992.

[2] E. Arias-Castro and B. Pelletier, On the consistency of the crossmatch test, arXiv:1509.05790 [math.ST], 2015.

[3] B. Aslan and G. Zech, New test for the multivariate two-sample problem based on the concept of minimum energy, *Journal of Statistical Computation and Simulation*, Vol. 75, 109–119, 2005.

[4] L. Baringhaus, and C. Franz, On a new multivariate two-sample test, *Journal of Multivariate Analysis*, Vol. 88, 190–206, 2004.

[5] B. B. Bhattacharya, Power of graph-based two-sample tests, arXiv:1508.07530, 2015.

[6] P. J. Bickel, A distribution free version of the Smirnov two sample test in the $p$-variate case, *Annals of Mathematical Statistics*, Vol. 40, 1–23, 1969.

[7] M. Biswas, M. Mukhopadhyay, A. K. Ghosh, A distribution-free two-sample run test applicable to high-dimensional data, *Biometrika*, Vol. 101(4), 913–926, 2014.

[8] H. Chen, and J. H. Friedman, A new graph-based two-sample test for multivariate and object data, arXiv:1307.6294 [stat.ME], 2015.

[9] L. H. Y. Chen and Q.-M. Shao, Normal approximation under local dependence, *Ann. Probab.*, Vol. 32 (3), 1985–2028, 2004.

[10] J. H. Friedman, and L. C. Rafsky, Multivariate generalizations of the Wolfowitz and Smirnov two-sample tests, *Ann. Statist.*, Vol. 7, 697–717, 1979.

[11] J. D. Gibbons and S. Chakraborty, *Nonparametric Statistical Inference*, New York, Marcel Dekker, 2003.
A. Gretton, K. Borgwardt, M. Rasch, B. Scholkopf, and A. Smola, A kernel two-sample test, *Journal of Machine Learning Research*, Vol. 16, 723–773, 2012.

L. Gyorfi, and T. Nemetz, $f$-dissimilarity: A general class of separation measures of several probability measures, *In Topics in Information Theory, Colloq. Math. Soc. Janos Bolany*, Vol. 16, 309–321, 1975.

L. Gyorfi, and T. Nemetz, On the dissimilarity of probability measures, *Problems Control Inform. Theory*, Vol. 6, 263–267, 1977.

P. Hall and N. Tajvidi, Permutation tests for equality of distributions in high-dimensional settings, *Biometrika*, Vol. 89, 359–374, 2002.

N. Henze, A multivariate two-sample test based on the number of nearest neighbor type coincidences, *Ann. Statist.*, Vol. 16, 772–783, 1988.

N. Henze and M. D. Penrose, On the multivariate runs test, *The Annals of Statistics*, Vol. 27 (1), 290–298, 1999.

H. Kesten and H. Lee, The central limit theorem for weighted minimal spanning trees on random points, *Annals of Applied Probability*, Vol. 6, 495–527, 1996.

R. Y. Liu, On a notion of data-depth-based on random simplices, *Ann. Statist.*, Vol. 18, 405–414, 1990.

R. Y. Liu and K. Singh, A quality index based on data-depth and multivariate rank tests, *J. Amer. Statist. Assoc.*, Vol. 88, 252–260, 1993.

J.-F. Maa, D. K. Pearl, and R. Bartoszyński, Reducing multidimensional two-sample data to one-dimensional interpoint comparisons, *The Annals of Statistics*, Vol. 24 (3), 1069–1074, 1996.

H. B. Mann and D. R. Whitney, On a test of whether one of two random variables is stochastically larger than the other, *Annals of Mathematical Statistics*, Vol. 18(1), 50–60, 1947.

M. D. Penrose, The random minimal spanning tree in high dimensions, *Ann. Probab.*, Vol. 24, 1903–1925, 1996.

M. D. Penrose, *Random Geometric Graphs*, Oxford University Press, 2003.

M. D. Penrose, Multivariate spatial central limit theorems with applications to percolation and spatial graphs, *Ann. Probab.*, Vol. 33 (5), 1945–1991, 2005.

M. D. Penrose, Gaussian limits for random geometric measures, *Electronic Journal of Probability*, Vol. 12, 989–1035, 2007.

M. D. Penrose and J. E. Yukich, Central limit theorems for some graphs in computational geometry, *Ann. Appl. Probab.*, Vol. 11, 1005–1041, 2001.

M. D. Penrose and J. E. Yukich, Weak laws of large numbers in geometric probability, *Ann. Appl. Probab.*, Vol. 13, 277–303, 2003.

P. R. Rosenbaum, An exact distribution-free test comparing two multivariate distributions based on adjacency, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, Vol. 67 (4), 515–530, 2005.

V. Rousson, On distribution-free tests for the multivariate two-sample location-scale model, *J. Mult. Anal.*, Vol. 80, 43–57, 2002.

W. Rudin, *Real and Complex Analysis*, 3rd ed. McGraw-Hill, New York, 1987.

M. F. Schilling, Multivariate two-sample tests based on nearest neighbor s, *J. Amer. Statist. Assoc.*, Vol. 81, 799–806, 1986.

N. Smirnoff, On the estimation of the discrepancy between empirical curves of distribution for two independent samples, *Bulletin de Universite de Moscow, Serie internationale (Mathematiques)*, Vol. 2, 3–14, 1939.

J. W. Tukey, Mathematics and picturing data, *In Proc. Intern. Congr. Math. Van- couver 1974*, Vol. 2, 523–531, 1975.

A. Wald and J. Wolfowitz, On a test whether two samples are from the same distri-
APPENDIX A: PROOF OF CLT UNDER GENERAL ALTERNATIVES

In this section the asymptotic distribution of the two-sample test based on stabilizing geometric graphs in the Poissonized setting is derived. The section is organized as follows: Begin by recalling preliminaries about geometric graphs in Section A.1. In Section A.2 a few technical lemmas are proved, which will be required to derive the asymptotic variance of the statistic (3.3). The consistency of these tests under general alternatives (Proposition 2.1) is given in Section A.3. The proof of the conditionally centered CLT of the test statistic (Theorem 3.1) is described in Section 3.2. The CLT of the conditional mean and the proofs of Theorem 3.2 and Theorem 3.3 are given in Section A.5.

A.1. Preliminaries on Stabilizing Graphs. Given a graph functional \( \mathcal{G} \), recall that \( \varphi(z, \mathcal{G}(Z)) \) is a measurable \( \mathbb{R}^+ \) valued function defined for all locally finite set \( Z \subset \mathbb{R}^d \) and \( z \in Z \). For a function \( \varphi \), define

\[
\varphi_N(x, \mathcal{G}(S)) = \varphi(0, \mathcal{G}(N^{\frac{1}{d}}(S - x))).
\]

Note that if both \( \mathcal{G} \) and \( \phi \) are translation and scale invariant, then \( \varphi_N(x, \mathcal{G}(S)) = \varphi(0, \mathcal{G}(S)) \). Moreover, if \( S = \mathcal{P}_\lambda \), is the Poisson process of rate \( \lambda \) in \( \mathbb{R}^d \), then

\[
\varphi_N(x, \mathcal{G}(\mathcal{P}_\lambda)) = \varphi(0, \mathcal{G}(\mathcal{P}_1)), \text{ since } \lambda^{\frac{1}{d}}(\mathcal{P}_\lambda - x) \overset{D}{=} \mathcal{P}_1.
\]

It follows from [28, Lemma 3.2] that, under appropriate moment conditions, \( \varphi_N(z, \mathcal{G}(\mathcal{P}_\kappa)) \) converges to \( \varphi(0, \mathcal{G}(\mathcal{P}_{\kappa(z)})) \), where \( \kappa \) is a density in \( \mathbb{R}^d \). The proof of [28, Lemma 3.2] can be easily modified to show that the same holds for any sequence of densities \( \kappa_N \to \kappa \) uniformly. (Recall the definition of stabilizing functions of graph functionals (Definition 1.3).)

**Lemma A.1.** Let \( \mathcal{G} \) be a translation and scale invariant graph functional in \( \mathbb{R}^d \), and \( \phi_N, \phi \) as in (3.1). Suppose \( \varphi \) is translation invariant and almost surely stabilizing on \( \mathcal{G}(\mathcal{P}_\lambda) \), with limit \( \varphi(0, \mathcal{G}(\mathcal{P}_\lambda)) \) for all \( \lambda \in (0, \infty) \), and for \( \beta > 1 \)

\[
\sup_{N \in \mathbb{N}} \sup_{z \in \mathbb{R}^d} \sup_{A \subset \mathbb{R}^d} \mathbb{E} \left\{ \varphi_N(z, \mathcal{G}(\mathcal{P}_{N\phi_N} \cup A))^\beta \right\} < \infty, \quad (A.1)
\]
where the set $A$ ranges over all finite subsets of $\mathbb{R}^d$.

(a) Then as $N \to \infty$,
\begin{equation}
\varphi_N(z, \mathcal{G}(\mathcal{P}_{N\phi_N})) \to \varphi(0, \mathcal{G}(\mathcal{P}_{\phi(z)})),
\end{equation}
in expectation and in distribution.

(b) For any $y \in \mathbb{R}^d$, as $N \to \infty$,
\begin{equation}
\varphi_N(z + N^{-\frac{1}{d}}y, \mathcal{G}(\mathcal{P}_{N\phi_N})) \to \varphi(0, \mathcal{G}(\mathcal{P}_{\phi(z)})),
\end{equation}
in expectation and in distribution.

**Proof.** Let $\tilde{\mathcal{P}}_1$ be a homogeneous Poisson process of rate 1 on $\mathbb{R}^d \times [0, \infty)$. Define coupled point processes $\tilde{\mathcal{P}}(N)$ and $H^x_N$ as follows: Let $\tilde{\mathcal{P}}(N)$ be the image of the restriction of $\tilde{\mathcal{P}}_1$ to the set \{(x, t) \in \mathbb{R}^d \times [0, \infty) : t \leq N\phi_N(x)\} under the projection $(x, t) \to x$. Then $\tilde{\mathcal{P}}(N)$ is a Poisson process in $\mathbb{R}^d$ with intensity function $N\phi_N(\cdot)$, consisting of $C_N$ random points with common density $\phi_N$. Let $H^x_N$ be the restriction of $\tilde{\mathcal{P}}_1$ to the set \{(x, t) : t \leq Nf(z)\} under the mapping $(x, t) \to N^{-\frac{1}{d}}(x - z)$. Note that $H^x_N$ is a homogeneous Poisson process on $\mathbb{R}^d$ of intensity $\phi(z)$.

By [28, Lemma 3.1]), given $K > 0$ and $z$ a Lebesgue point of $\phi$,
\begin{equation}
\lim_{N \to \infty} \mathbb{P}(N^{-\frac{1}{d}}(\tilde{\mathcal{P}}(N) - z) \cap B(0, K) = H^x_N \cap B(0, K)) = 1.
\end{equation}
Hence, there exists $K$ such that for $N$ large enough,
\begin{align}
&\mathbb{P}(|\varphi_N(z, \mathcal{G}(\mathcal{P}_{N\phi_N})) - \varphi(0, \mathcal{G}(\mathcal{P}_{\phi(z)}))| > \varepsilon) \\
&\leq \mathbb{P}(N^{-\frac{1}{d}}(\tilde{\mathcal{P}}(N) - z) \cap B(0, K) \neq H^x_N \cap B(0, K)) \\
&\quad + \mathbb{P}(|\mathcal{G}(H^x_N), K - \varphi(\mathcal{G}(H^x_N), K| > \varepsilon) \\
&\leq \varepsilon,
\end{align}
(A.4)
since $\varphi$ stabilizes $\mathcal{P}_{\phi(z)}$. This implies that $\varphi_N(z, \mathcal{G}(\tilde{\mathcal{P}}(N))) \overset{D}{\to} \varphi(0, \mathcal{G}(\mathcal{P}_{\phi(z)})$.

By assumption (A.1), $\varphi_N(z, \mathcal{G}(\tilde{\mathcal{P}}(N)))$ is uniformly integrable. Therefore,
\begin{equation}
\lim_{N \to \infty} \mathbb{E}\varphi_N(z, \mathcal{G}(\tilde{\mathcal{P}}(s))) = \mathbb{E}\varphi(0, \mathcal{G}(\mathcal{P}_{\phi(z)})).
\end{equation}
This completes the proof of (A.2).
To prove (A.3) consider a slight modification of the same coupling: Let $\tilde{P}_1$ be a homogeneous Poisson process of rate 1 on $\mathbb{R}^d \times [0, \infty)$. Define coupled point processes $\tilde{P}(N)$ and $H_N^Z$: Let $\tilde{P}(N)$ be the image of the restriction of $\tilde{P}_1$ to the set 
\[ \{(x, t) \in \mathbb{R}^d \times [0, \infty) : t \leq N\phi_N(x)\} \]
under the projection $(x, t) \to x$. Also, let $H_N^Z$ be the restriction of $\tilde{P}_1$ to the set 
\[ \{(x, t) : t \leq N\phi(z)\} \]
under the mapping $(x, t) \to N^{1/2} (x - \{z + N^{-1/2} y\})$. $H_N^Z$ is a homogeneous Poisson process on $\mathbb{R}^d$ of intensity $\phi(z)$.

Now, by [26, Lemma 3.1]
\[
\lim_{N \to \infty} P(N^{1/2}(\tilde{P}(s) - \{z + N^{-1/2} y\}) \cap B(0, K) = H_N^Z \cap B(0, K)) = 1.
\]
By stabilization, as in (A.4), it follows that
\[ \varphi_N(z + N^{-1/2} y, \mathcal{G}(\mathcal{P}_{N\phi_N})) \overset{D}{\to} \varphi(0, \mathcal{G}(\mathcal{P}_{\phi(z)})). \]
The convergence in expectation follows from uniform integrability by assumption (A.1).

**A.2. Technical Lemmas.** In this section a few technical lemmas required for deriving the limit of the conditional variance in Theorem 3.1 are proved. Begin with a few definitions: For $A \subset \mathbb{R}^d$, denote by $|A|$ the Lebesgue measure of the set $A$. A point $x \in \mathbb{R}^d$ is a Lebesgue point of $\phi$ if
\[
\lim_{\varepsilon \to 0} \frac{1}{|B(x, \varepsilon)|} \int_{B(x, \varepsilon)} |\phi(y) - \phi(x)| dy = 0,
\]
where $B(x, \varepsilon)$ is the Euclidean ball in $\mathbb{R}^d$ with center at $x$ and radius $\varepsilon$. Almost every point $x \in \mathbb{R}^d$ is a Lebesgue point of $\phi$ [31, Theorem 7.7].

Let $\phi_N, \phi$ as in (3.1), and $h : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ a symmetric and jointly measurable function, such that for almost every $x \in \mathbb{R}^d$, $h(x, \cdot)$ is measurable and $x$ a Lebesgue point of the function $\phi(\cdot)h(x, \cdot)$. Define
\[
\kappa_N(z) = \sum_{w \in \mathcal{P}_{N\phi_N}} h(z, w) 1\{(z, w) \in E(\mathcal{G}(\mathcal{P}_{N\phi_N}))\}. \tag{A.5}
\]

**Lemma A.2.** Let $\mathcal{G}$ be a translation and scale invariant graph functional in $\mathbb{R}^d$ which satisfies the $\beta$-degree moment condition (2.1) for $\beta > 2$. Then for $h : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ as above
\[
\lim_{N \to \infty} E\kappa_N(z) = h(z, \cdot) E\Delta^\dagger_0, \tag{A.6}
\]

- Note that the notation $|S|$ is also used to denote the cardinality of a finite set $S$, depending on the context.
if \( z \) is a Lebesgue point of \( \phi \) and \( h(z, \cdot)\phi \). Moreover, as \( N \to \infty \),

\[
\frac{1}{N} \sum_{z \in \mathcal{P}_N \phi_N} \kappa_N(z) \mathbb{E} \Delta_0^\top \int_{\mathbb{R}^d} h(z, z) \phi(z) dz. \tag{A.7}
\]

The lemma is proved below in Section A.2.1. The same proof shows that

\[
\frac{1}{N} \sum_{z \in \mathcal{P}_N \phi_N} \kappa_N^+(z) \mathbb{E} \Delta_0^\top \int_{\mathbb{R}^d} h(z, z) \phi(z) dz, \tag{A.8}
\]

where \( \kappa^+_N(z) = \sum_{w \in \mathcal{P}_N \phi_N} h(z, w) \mathbf{1}\{(z, w), (z, w) \in E(\mathcal{G}(\mathcal{P}_N \phi_N))\} \) and \( \Delta_0^+ \) is as defined in (3.6).

Next, define \( \omega^\uparrow, \omega^\downarrow: \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, 1] \) as follows:

\[
\omega^\uparrow(x, y, z) = \frac{p^2 g(x) f(y) f(z)}{(p f(x) + q g(x))(p f(y) + q g(y))(p f(z) + q g(z))}. \tag{A.9}
\]

and

\[
\omega^\downarrow(x, y, z) = \frac{pq^2 f(x) g(y) f(z)}{(p f(x) + q g(x))(p f(y) + q g(y))(p f(z) + q g(z))}. \tag{A.10}
\]

Let

\[
\tau^\uparrow_N(z) := \sum_{w_1 \neq w_2 \in \mathcal{P}_N \phi_N} \omega^\uparrow(z, w_1, w_2) \mathbf{1}\{(z, w_1), (z, w_2) \in E(\mathcal{G}(\mathcal{P}_N \phi_N))\},
\]

and

\[
\tau^\downarrow_N(z) := \sum_{w_1 \neq w_2 \in \mathcal{P}_N \phi_N} \omega^\downarrow(z, w_1, w_2) \mathbf{1}\{(w_1, z), (w_2, z) \in E(\mathcal{G}(\mathcal{P}_N \phi_N))\}.
\]

where \( \phi_N \) is defined in (3.1).

**Lemma A.3.** Let \( \mathcal{G} \) be an translation and scale invariant graph functional in \( \mathbb{R}^d \) which satisfies the \( \beta \)-degree moment condition (2.1) for \( \beta > 4 \). Then

\[
\frac{1}{N} \sum_{z \in \mathcal{P}_N \phi_N} \tau^\uparrow_N(h, z) \xrightarrow{L^2} \mathbb{E} T^\uparrow_2 \int_{\mathbb{R}^d} \omega^\uparrow(z, z, z) \phi(z) dz, \tag{A.11}
\]

for \( \omega^\uparrow \) as in (A.9). The same result holds for \( \tau^\downarrow_N(h, z) \), with \( \mathbb{E} T^\downarrow_2 \) replaced by \( \mathbb{E} T^\uparrow_2 \), and \( \omega^\uparrow \) replaced by \( \omega^\downarrow \).

The proof of Lemma A.2 is given in Section A.2.1. The proof of Lemma A.3 is described in Section A.2.2.
A.2.1. Proof of Lemma A.2. The proof of Lemma A.2 is organized as follows: Begin with the proof of (A.6) below. This together with uniform integrability, which follows from the degree moment condition (2.1), implies the convergence in expectation in (A.7). Following this the convergence of (A.7) in $L^2$ is shown, by computing the limit of the second moment.

**Proof of (A.6) and Convergence in Expectation:** Fix $K > 0$. By the Palm theory of Poisson processes [24, Theorem 1.6],

$$
\mathbb{E} \sum_{w \in \mathcal{P}_{N\phi_N}} |h(z, w) - h(z, z)|1\{w \in B(z, KN^{\frac{1}{2}})\} = N \int_{B(z, KN^{\frac{1}{2}})} |h(z, w) - h(z, z)|\phi_N(w)dw,
$$

(A.12)

which tends to zero as $N \to \infty$, if $z$ is a Lebesgue point of both $\phi$ and $h(z, \cdot)\phi(\cdot)$. Since $h$ has range $[0, 1]$, this implies that

$$
\lim sup_{N \to \infty} \mathbb{E} \sum_{w \in \mathcal{P}_{N\phi_N}} |h(z, w) - h(z, z)|1\{(z, w) \in \mathcal{I}(\mathcal{P}_{N\phi_N})\}
\leq \lim sup_{N \to \infty} \mathbb{E}d_K^\uparrow(z, \mathcal{I}(\mathcal{P}_{N\phi_N})),
$$

(A.13)

where $d_K^\uparrow(z, \mathcal{I}(\mathcal{P}_{N\phi_N}))$ is the number of edges $(z, w) \in E(\mathcal{I}(\mathcal{P}_{N\phi_N}))$ incident on $z$ such that $w \notin B(z, KN^{\frac{1}{2}})$.

Since $\mathcal{I}$ stabilizes $\mathcal{P}_\lambda$ for all $\lambda \in (0, \infty)$, the functions $d^\uparrow$ and $d_K^\uparrow$ defined on $\mathcal{I}$ stabilizes $\mathcal{P}_\lambda$, for all $\lambda \in (0, \infty)$ (as in Definition 1.3). Therefore, by Lemma A.1,

$$
\lim sup_{K \to \infty} \lim sup_{N \to \infty} \mathbb{E}d_K^\uparrow(z, \mathcal{I}(\mathcal{P}_{N\phi_N})) = \lim sup_{K \to \infty} \mathbb{E}d_K^\uparrow(0, \mathcal{I}(\mathcal{P}_{\phi(z)})) = 0.
$$

(A.14)

Now, recall the definition of $\kappa_N(\cdot)$ from (A.5). From (A.13) and (A.14) and Lemma A.1, it follows that for every $z$ which is a Lebesgue point of $\phi$ and $h(z, \cdot)\phi$,

$$
\lim_{N \to \infty} \mathbb{E}\kappa_N(z) = h(z, z) \lim_{N \to \infty} \mathbb{E}d_K^\uparrow(z, \mathcal{I}(\mathcal{P}_{N\phi_N})) = h(z, z)\mathbb{E}d_K^\uparrow(0, \mathcal{I}(\mathcal{P}_{\phi(z)})) = h(z, z)\mathbb{E}d_K^\uparrow(0, \mathcal{I}(\mathcal{P}_1)),
$$

(A.15)

where the last equality uses $\mathcal{I}$ is scale invariant. Therefore, as $N \to \infty$

$$
\mathbb{E}\left(\frac{1}{N} \sum_{z \in \mathcal{P}_{N\phi_N}} \kappa_N(z)\right) = \int \phi_N(z)\mathbb{E}\kappa_N(z) \to \mathbb{E}\Delta_0^\uparrow \int \phi(z)h(z, z)dz
$$
by (A.15) and the Dominated Convergence Theorem.

**Proof of (A.7) (Convergence in $L^2$):** By the Palm theory of Poisson processes [24, Theorem 1.6],

$$
\mathbb{E}\left( \frac{1}{N} \sum_{z \in \mathcal{P}_N} \kappa_N(z) \right)^2 = \frac{1}{N} \int \phi_N(z) \mathbb{E} \kappa_N^2(z) dz + Q_2, \quad (A.17)
$$

where

$$
Q_2 = \int \phi_N(z_1) \phi_N(z_2) \mathbb{E} \kappa_N(z_1) \kappa_N(z_2) dz_1 dz_2.
$$

Since $h$ is bounded in $[0, 1]$, $\mathbb{E} \kappa_N(z)^2 \leq \mathbb{E} (d'(z, \mathcal{P}_N \phi_N))^2 \to \mathbb{E} (d'(z, \mathcal{P}_\phi(z)))^2$, as (2.1) holds for $\beta > 2$. Thus, $\int \phi_N(z) \mathbb{E} \kappa_N^2(z) dz = O(1)$, and the first term in (A.17) goes to 0 as $N \to \infty$.

Therefore, it suffices to consider $Q_2$. Fix $K > 0$ and let $z_1$ and $z_2$ be Lebesgue points of $\phi$. Define $A(z_1, z_2) := B(z_1, KN^{-\frac{1}{2}}) \times B(z_2, KN^{-\frac{1}{2}})$. Then by triangle inequality, for almost all $z_1, z_2$

$$
N^2 \int_{A(z_1, z_2)} |\phi(w_1)\phi(w_2) - \phi(z_1)\phi(z_2)| dw_1 dw_2 \to 0, \quad (A.18)
$$

as $N \to \infty$.

Similarly, if $z_1, z_2$ are Lebesgue points of $h(z_1, \cdot)\phi(\cdot)$ and $h(z_2, \cdot)\phi(\cdot)$, respectively, then as $N \to \infty$

$$
N^2 \int_{A(z_1, z_2)} |h(z_1, w_1)h(z_2, w_2)\phi(w_1)\phi(w_2) - h(z_1, z_1)h(z_2, z_2)\phi(z_1)\phi(z_2)| dw_1 dw_2 \\
\leq N^2 \int_{A(z_1, z_2)} \phi(w_2)h(z_2, w_2)|h(z_1, w_1)\phi(w_1) - h(z_1, z_1)\phi(z_1)| dw_1 dw_2 \\
+ N^2 \int_{A(z_1, z_2)} h(z_1, z_1)\phi(z_1)|h(z_2, w_2)\phi(w_2) - h(z_2, z_2)\phi(z_2)| dw_1 dw_2 \\
\to 0. \quad (A.19)
$$

Let $S_{w_1, w_2} = \{w_1, w_2 \in \mathcal{P}_N : (z_1, w_1), (z_2, w_2) \in E(\mathcal{G}(\mathcal{P}_N \phi_N))\}$. Then, since $h$ has range $[0, 1]$, (A.18) and (A.19) gives

$$
\limsup_{N \to \infty} \mathbb{E} \sum_{S_{w_1, w_2}} |h(z_1, w_1)h(z_2, w_2) - h(z_1, z_1)h(z_2, z_2)|
$$
DISTRIBUTION OF GRAPH BASED TWO-SAMPLE TESTS

\[ \limsup_{N \to \infty} T_{N,K}(z_1) + \limsup_{N \to \infty} T_{N,K}(z_2). \]  
(A.20)

where

\[ \limsup_{N \to \infty} T_{N,K}(z_1) = \limsup_{N \to \infty} E \frac{d^2}{K}(z_1, G(P_{N\phi_N})) d^2(z_2, G(P_{N\phi_N})) \leq \left( E\{d^2(0, G(P_{\phi(z_2)}))\}^2 E\{d^2(0, G(P_{\phi(z_1)}))\}^2 \right)^{\frac{1}{2}}, \]  
(A.21)

by Lemma A.1, since (2.1) holds for \( \beta > 2 \).

Similarly,

\[ \limsup_{N \to \infty} T_{N,K}(z_2) = \limsup_{N \to \infty} E \frac{d^2}{K}(z_1, G(P_{N\phi_N})) d^2(z_2, G(P_{N\phi_N})) \leq \left( E\{d^2(0, G(P_{\phi(z_2)}))\}^2 E\{d^2(0, G(P_{\phi(z_1)}))\}^2 \right)^{\frac{1}{2}}. \]  
(A.22)

Combining (A.21) and (A.22) and taking \( K \to \infty \) it follows that the LHS of (A.20) goes to zero. Therefore,

\[ \lim_{N \to \infty} E\kappa_N(z_1) \kappa_N(z_2) \]

\[ = \lim_{N \to \infty} \sum_{w_1, w_2 \in P_{N\phi_N}} h(z_1, w_1) h(z_2, w_2) 1\{(z_1, w_1), (z_2, w_2) \in E(G(P_{N\phi_N})) \}
\]

\[ = h(z_1, z_1) h(z_2, z_2) \lim_{N \to \infty} E \frac{d^2}{K}(z_1, G(P_{N\phi_N})) d^2(z_2, G(P_{N\phi_N})) \]  
(A.23)

for \( z_1, z_2 \) Lebesgue points of \( \phi \) and \( h(z_1, \cdot) \phi \) and \( h(z_2, \cdot) \phi \), respectively. Now, by a modification of the coupling argument used in Lemma A.1, similar to the proof of [28, Lemma 3.1], it can be shown that

\[ \lim_{N \to \infty} E \frac{d^2}{K}(z_1, G(P_{N\phi_N})) d^2(z_2, G(P_{N\phi_N})) = E \frac{d^2}{K}(0, G(P_{\phi(z_1)})) E \frac{d^2}{K}(0, G(P_{\phi(z_2)})) \]

\[ = \left( E \frac{d^2}{K}(0, G(P_1)) \right)^2, \]  
(A.24)

where the last step uses \( G \) is scale invariant. Combining (A.24) with (A.23) gives

\[ \lim_{N \to \infty} E\kappa_N(z_1) \kappa_N(z_2) = \left( E \frac{d^2}{K}(0, G(P_1)) \right)^2 h(z_1, z_1) h(z_2, z_2). \]

Thus, taking limit as \( N \to \infty \) in (A.17) gives

\[ \lim_{N \to \infty} \left( \frac{1}{N} \sum_{z \in P_{N\phi_N}} \kappa_N(z) \right)^2 = \mu(G, h)^2, \]  
(A.25)
where $\mu(\mathcal{G}, h)$ is defined in (A.16). Combining (A.16) and (A.25) gives the $L^2$ convergence in (A.7).

**Proof of** (A.8): It follows from (A.12), (A.13), and (A.14) that if $z$ is a Lebesgue point of both $\phi$ and $h(z, \cdot)\phi(\cdot)$, then

$$\mathbb{E} \sum_{w \in \mathcal{P}N\phi_N} |h(z, w) - h(z, z)|1\{(z, w, (w, z) \in \mathcal{G}(\mathcal{P}N\phi_N)\} \to 0.$$  

Lemma A.1 then implies that $\mathbb{E}\kappa_N^+(z) \to \mathbb{E}\Delta_0^+ h(z, z)$, where $\Delta_0^+$ is as defined in (3.6). This shows convergence in expectation. The $L^2$ convergence is similar to the proof of (A.7).

A.2.2. **Proof of Lemma A.3.** The proof is very similar to Lemma A.2. Without loss of generality consider the function $\omega^\uparrow$ defined in (A.9) (the proof for $\omega^\downarrow$ is identical).

Fix $K > 0$. Define $B_2(z) := B(z, KN^{-\frac{1}{2}}) \times B(z, KN^{-\frac{1}{2}})$, then by Palm theory,

$$\mathbb{E} \sum_{w_1 \neq w_2 \in \mathcal{P}N\phi_N} |\omega^\uparrow(z, w_1, w_2) - \omega^\uparrow(z, z, z)|1\{w_1, w_2 \in B(z, KN^{-\frac{1}{2}})\} = N^2 \int_{B_2(z)} |\omega^\uparrow(z, w_1, w_2) - \omega^\uparrow(z, z, z)| \phi(w_1)\phi(w_2)d\mu_1d\mu_2 + o(1),$$

(A.26)

since $\phi_N \to \phi$ uniformly.

Note that

$$\omega^\uparrow(z, w_1, w_2)\phi(w_1)\phi(w_2) = \frac{pq^2f(z)g(w_1)g(w_2)}{pf(z) + gg(z)}.$$  

Therefore, if $z$ is a Lebesgue point of both $f$ and $g$, then, as $N \to \infty$,

$$N^2 \int_{B_2(z)} |\omega^\uparrow(z, w_1, w_2)\phi(w_1)\phi(w_2) - \omega^\uparrow(z, z, z)\phi^2(z)|d\mu_1d\mu_2 \to 0. \quad (A.27)$$

Moreover,

$$\phi(z) \lim_{N \to \infty} N^2 \int_{B_2(z)} |\phi(w_2) - \phi(z)|d\mu_1d\mu_2 = 0. \quad (A.28)$$

and by the Lebesgue differentiation theorem

$$\lim_{N \to \infty} N^2 \int_{B(z, KN^{-\frac{1}{2}})} |\phi(w_1) - \phi(z)|dy_1 \int_{B(z, KN^{-\frac{1}{2}})} \phi(w_2)d\mu_2 = 0. \quad (A.29)$$
Combining (A.28) and (A.29) gives
\[
\lim_{N \to \infty} N^2 \int_{B^2(z)} \omega^\dagger(z, z, z) \phi_N(w_1) \phi(w_2) - \phi^2(z) \, dw_1 dw_2 \to 0. \quad (A.30)
\]
The triangle inequality combined with (A.27) and (A.30) implies that the RHS of (A.26) goes to 0 as \( N \to \infty \). This implies that
\[
E \sum_{w_1 \neq w_2 \in P_{N\phi_N}} \left| \omega^\dagger(z, w_1, w_2) - \omega^\dagger(z, z, z) \right| 1\{(z, w_1), (z, w_1) \in E(\mathcal{G}(P_{N\phi_N}))\}
\leq ET_{2,K}^\dagger(z, \mathcal{G}(P_{N\phi_N})) + o(1), \quad (A.31)
\]
where
\[
T_{2,K}^\dagger(z, \mathcal{G}(P_{N\phi_N})) = \sum_{w \in P_{N\phi_N}} 1\{(z, w_1), (z, w_2) \in E(\mathcal{G}(P_{N\phi_N}))\} 1\{w_1 \text{ or } w_2 \notin B(z, KN^{-\frac{1}{2}})\}.
\]
Now, since \( \mathcal{G} \) stabilizes \( P_\lambda \) for all \( \lambda \in (0, \infty) \), the function \( T_{2,K}^\dagger \) and hence \( T_{2,K}^\dagger \) stabilizes \( P_\lambda \), for all \( \lambda \in (0, \infty) \). Moreover, \( T_{2,K}^\dagger(z, \mathcal{G}(P_{N\phi_N})) \) satisfies the bounded moment condition (A.1) for \( \beta > 1 \), since \( d^\dagger(z, \mathcal{G}(P_{N\phi_N})) \) satisfies (2.1) for some \( \beta > 2 \). Therefore, by Lemma A.1,
\[
\lim sup_{K \to \infty} \lim sup_{N \to \infty} ET_{2,K}^\dagger(z, \mathcal{G}(P_{N\phi_N})) = 0. \quad (A.32)
\]
From (A.31) and (A.32) it follows that for every \( z \) which is a Lebesgue point of both \( f \) and \( g \),
\[
\lim_{N \to \infty} E \tau_N^\dagger(z) = \omega^\dagger(z, z, z) \lim_{N \to \infty} ET_{2}^\dagger(z, \mathcal{G}(P_{N\phi_N})) = \omega^\dagger(z, z, z)ET_{2}^\dagger(0, \mathcal{G}(P_{\phi(z)})) \quad (A.33)
\]
where (A.33) uses Lemma A.1 and (A.34) uses \( \mathcal{G} \) is scale invariant. Therefore, by the Dominated Convergence Theorem,
\[
\lim_{N \to \infty} E \left( \frac{1}{N} \sum_{z \in P_{N\phi_N}} \tau_N^\dagger(z) \right) = \lim_{N \to \infty} \int \phi_N(z) \mathbb{E} \tau_N^\dagger(z) \phi(z) \omega^\dagger(z, z, z) \, dz. \quad (A.35)
\]
Now, as in Lemma A.2, it can be shown that the second moment of the quantity \( \frac{1}{N} \sum_{z \in P_{N\phi_N}} \tau_N^\dagger(z) \) converges to \( \left( ET_{2}^\dagger \int \phi(z) \omega^\dagger(z, z, z) \, dz \right)^2 \). This proves the \( L^2 \) convergence of (A.11).
A.3. Proof of Proposition 2.1. Recall from (1.7) the definition of $T(\mathcal{G}(Z'_N))$. Define
\[ h_N(x,y) = \frac{N_1 N_2 f(x) g(y)}{(N_1 f(x) + N_2 g(x))(N_1 f(y) + N_2 g(y))}. \]
Note that $h_N(x,y) \to h(x,y) = \frac{pq f(x) g(y)}{(pf(x) + qg(x))\{pf(y) + qg(y))}$, uniformly in $x, y \in \mathbb{R}^d$ as $N \to \infty$. Then, with $\phi_N$ and $\phi$ as in (3.1), using Lemma A.7
\[
\frac{1}{N} \mathbb{E}(T(\mathcal{G}(Z'_N))|\mathcal{F}) = \frac{1}{N} \sum_{1 \leq i \neq j \leq L_N} h_N(Z_i, Z_j) 1\{(Z_i, Z_j) \in E(\mathcal{G}(Z'_N))\}
\]
\[
= \frac{1}{N} \sum_{1 \leq i \neq j \leq L_N} h(Z_i, Z_j) 1\{(Z_i, Z_j) \in E(\mathcal{G}(Z'_N))\} + o(1)
\]
\[
= \frac{1}{N} \sum_{z,w \in P_{N\phi_N}} h(z,w) 1\{(z,w) \in E(\mathcal{G}(Z'_N))\} + o(1)
\]
\[
\frac{L^2}{\mathbb{E}\Delta_0^\uparrow} \int h(z,z) \phi(z) dz = \frac{\mathbb{E}\Delta_0^\uparrow}{2}(1 - \delta(f,g,p)), \quad (A.36)
\]
where $\Delta_0^\uparrow = d(0, \mathcal{G}(\mathcal{P}_1))$ and $\delta(f,g,p)$ is defined in (2.2).

The limit in (A.36) shows convergence in expectation. To show convergence in probability, compute
\[
\text{Var}(T(\mathcal{G}(Z'_N))) = \mathbb{E}(\text{Var}(T(\mathcal{G}(Z'_N))|\mathcal{F})) + \text{Var}(\mathbb{E}(T(\mathcal{G}(Z'_N))|\mathcal{F})).
\]
From (A.36) it follows that $\text{Var}(\frac{1}{N} \mathbb{E}(T(\mathcal{G}(Z'_N))|\mathcal{F})) \to 0$. Moreover, by Lemma A.4 below,
\[
\frac{1}{N} \mathbb{E}(\text{Var}(T(\mathcal{G}(Z'_N))|\mathcal{F})) = \mathbb{E}(\text{Var}(\mathcal{R}_1(\mathcal{G}(Z'_N))|\mathcal{F})) = \sigma_{\mathcal{G},1}^2.
\]
This implies $\frac{1}{N^2} \text{Var}(T(\mathcal{G}(Z'_N))) \to 0$, completing the proof of (2.3).

A.4. Proof of Theorem 3.1. This section is organized as follows: Section A.4.1 below derives the limit of the conditional variance, using results proved in Section A.2. The proof of the CLT is given in Section A.4.2.

A.4.1. Limiting Conditional Variance. The limit of the conditional variance $\text{Var}(\mathcal{R}_1(\mathcal{G}(Z'_N))|\mathcal{F})$ can be computed as a function of the graph functional $\mathcal{G}$ and the unknown densities $f, g$. 


Lemma A.4. Let $\mathcal{G}$ as in Theorem 3.1 and $\mathcal{R}_1(\mathcal{G}(Z'_N))$ as in (3.3). Then,
\[
\text{Var}(\mathcal{R}_1(\mathcal{G}(Z'_N)) | \mathcal{F}) \overset{L^2}{\to} \sigma^2_{\mathcal{G},1},
\] (A.37)
where $\sigma_{\mathcal{G},1}$ is as defined in Theorem 3.1.

Proof. Define the function $h_N : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ as
\[
h_N(x, y) = \frac{N_1N_2f(x)g(y)}{(N_1f(x) + N_2g(x))(N_1f(y) + N_2g(y))}.
\] (A.38)

By construction $Z'_N$ is a Poisson process in $\mathbb{R}^d$ with intensity function $N\phi_N$, where $\phi_N \to \phi$ uniformly as $N \to \infty$ (as defined in (3.1)). Therefore, for $x, y \in Z'_N$, $P(\psi(c_x, c_y) | \mathcal{F}) = h_N(x, y)$. Moreover,
\[
h_N(x, y) \to h(x, y) = \frac{pqf(x)g(y)}{(pf(x) + qg(x))(pf(y) + qg(y))},
\]
uniformly in $x, y \in \mathbb{R}^d$ as $N \to \infty$.

Now, let
\[
V_{x,y} := (\psi(c_x, c_y) - h_N(x, y))1\{(x, y) \in E(\mathcal{G}(Z'_N))\}.
\] (A.39)

Then by (3.3), $\mathcal{R}_1(\mathcal{G}(Z'_N)) := \frac{1}{\sqrt{N}} \sum_{x,y \in Z'_N} V_{x,y}$. Therefore,
\[
\text{Var}(\mathcal{R}_1(\mathcal{G}(Z'_N)) | \mathcal{F}) = \frac{1}{N} \sum_{x,y \in Z'_N} \text{Var}(V_{x,y} | \mathcal{F}) + 2 \sum_{x,y,z \in Z'_N} \text{Cov}(V_{x,y}, V_{x,z} | \mathcal{F})
\] (A.40)

By Lemma A.2, as $N \to \infty$,
\[
\frac{1}{N} \sum_{x,y \in Z'_N} \text{Var}(V_{x,y} | \mathcal{F}) \overset{P}{\to} \Delta_0^\phi \int \left\{ h(x,x)(1 - h(x,x)) \right\} \phi(x) dx.
\]

It remains to compute the limit of the covariance term in (A.40). To this end, observe that
\[
\text{Cov}(V_{x,y}, V_{x,z} | \mathcal{F})
\]
\[
\{\omega_N(x, y, z) - h_N(x, y)h_N(x, z)\}1\{(x, y), (x, z) \in E(G(Z'_N))\},
\]
where
\[
\omega^\uparrow_N(x, y, z) := \frac{N_1^2N_2f(x)f(y)g(z)}{(N_1f(x) + N_2g(x))(N_1f(y) + N_2g(y))(N_1f(z) + N_2g(z))} \rightarrow \omega^\uparrow(x, y, z),
\]
where \(\omega^\uparrow(\cdot, \cdot, \cdot)\) is defined in (A.9). The convergence above is uniformly in \(x, y, z \in \mathbb{R}^d\), as \(N \to \infty\). Therefore, by Lemma A.3,
\[
\frac{1}{N} \sum_{x,y,z \in Z'_N} \text{Cov}(V_{x,y}, V_{x,z} | F) \overset{P}{\rightarrow} \mathbb{E} T^\uparrow 2 \hat{\phi}(x) dx,
\]
(A.41)
where \(T^\uparrow 2 = T^\uparrow 2(0, \mathcal{G}(P_1))\) is as defined in (3.7). Similarly, define
\[
\omega^\downarrow_N(x, y, z) := \frac{N_1N_2^2f(x)g(y)g(z)}{(N_1f(x) + N_2g(x))(N_1f(y) + N_2g(y))(N_1f(z) + N_2g(z))} \rightarrow \omega^\downarrow(x, y, z),
\]
where \(\omega^\downarrow(\cdot, \cdot, \cdot)\) is defined in (A.10). Then, by Lemma A.3,
\[
\frac{1}{N} \sum_{x,y,z \in Z'_N} \text{Cov}(V_{y,x}, V_{z,x} | F) \overset{P}{\rightarrow} \mathbb{E} T^\downarrow 2 \hat{h}^2(x, x) \phi(x) dx,
\]
(A.42)
where \(T^\downarrow 2 = T^\downarrow 2(0, \mathcal{G}(P_1))\) is as defined in (3.7).
Similarly, it can be shown that
\[
\frac{1}{N} \sum_{x,y,z \in Z'_N} \text{Cov}(V_{x,y}, V_{x,z} | F) \overset{P}{\rightarrow} - \mathbb{E} T^\uparrow 2 \hat{h}^2(x, x) \phi(x) dx,
\]
(A.43)
where \(T^\uparrow 2\) is as defined in (3.7).
Also, from (A.8)
\[
\text{Cov}(V_{x,y}, V_{y,x} | F) = - \frac{1}{N} \sum_{x,y \in Z'_N} h_N(x, y)h_N(y, x)1\{(x, y), (y, z) \in E(G(Z'_N))\}
\]
\[
\overset{P}{\rightarrow} - \mathbb{E} \Delta^+_0 \int h(x, x)^2 \phi(x) dx,
\]
(A.44)
where $\Delta_0^+$ is as defined in (3.6).

Finally, observe $h(x, x) = \frac{pqf(x)g(x)}{\varphi^2(x)}$ and $\omega^\uparrow(x, x, x) = \frac{pq^2f(x)g(x)^2}{\varphi^3(x)}$. Using this, and combining together (A.41), (A.42), (A.43), (A.44) with (A.40), the limit in (A.37) follows. \hfill \square

A.4.2. Completing the Proof of Theorem 3.1. The CLT of $R_1(G(Z_N'))$ will be proved using Stein's method based on dependency graphs given below:

**Theorem A.1 (Chen and Shao [9]).** Let $\{W_i, i \in V\}$ be random variables indexed by the vertices of a dependency graph $H = (V, E)$ with maximum degree $D$. If $W = \sum_{i \in V} W_i$ with $E(W_i) = 0$, $E W^2 = 1$ and $E |W_i|^3 \leq \theta^3$ for all $i \in V$ and for some $\theta > 0$, then

$$\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \lesssim D^{10} |V|^3 \theta^3. \quad (A.45)$$

Using this, the proposition below shows that the statistic $T(G(Z_N'))$, centered by the conditional mean and scaled by the conditional variance converges to $N(0, 1)$. This, along with Lemma A.4, completes the proof of Theorem 3.1.

**Proposition A.2.** Let $T(G(Z_N'))$ be as defined in (3.3) and $\Phi$ the standard normal distribution function. Then for all $x \in \mathbb{R}$,

$$\lim_{N \to \infty} \left| P\left( \frac{T(G(Z_N')) - E(T(G(Z_N'))|F)}{\sqrt{\text{Var}(T(G(Z_N'))|F)}} \leq x \mid F \right) - \Phi(x) \right| = 0, \quad (A.46)$$

**Proof.** Let $W := \frac{T(G(Z_N')) - E(T(G(Z_N'))|F)}{\sqrt{\text{Var}(T(G(Z_N'))|F)}}$. Recall the definition of $V_{x,y}$ from (A.39) and let

$$U_{x,y} = \frac{V_{x,y}}{\sqrt{\text{Var}(T(G(Z_N'))|F)}}.$$

Note that $W = \sum_{x,y \in Z_N} U_{x,y}$, and $E(U_{x,y}|F) = 0$ and $EW^2 = 1$. Let $D$ be the maximum degree of dependency graph of the random variables $\{U_{x,y}, (x,y) \in E(G(Z_N'))\}$. It is easy to see that $D \leq 2\Delta(G(Z_N'))^2 = o_P(N^{\frac{1}{2}})$, by Assumption 3.1. Moreover, for $(x,y) \in E(G(Z_N'))$

$$|U_{x,y}|^3 \leq \frac{1}{(\text{Var}(T(G(Z_N'))|F))^2}.$$
Therefore, by Theorem A.1 above, conditional on \( F \),
\[
\mathbb{P}(W \leq x| F) - \Phi(x) \lesssim D^{10} \frac{|Z_N'|}{(\text{Var}(T(\mathcal{G}(Z_N'))|F))^2} = \frac{D^{10} |Z_N'|}{\sqrt{N}(\text{Var}(R_1(\mathcal{G}(Z_N'))|F))^{3/2}}. \tag{A.47}
\]

Note that \( \frac{|Z_N'|}{N} \overset{p}{\to} 1 \), as \( |Z_N'| = L_N \) is a Poisson random variable with mean \( N \), and by Lemma A.4, \( \text{Var}(R_1(\mathcal{G}(Z_N'))|F) \overset{p}{\to} \sigma^2 \). Therefore, (A.47) goes to zero in probability, since \( D^{10} = o(\sqrt{N}) \), by Assumption 3.1. This completes the proof of (A.46).

A.5. Proofs of Theorem 3.2 and Theorem 3.3. The proof of Theorem 3.2 is given in Section A.5.1 below. Theorems 3.1 and 3.2 can be easily combined to complete the proof of Theorem 3.3 (see Section A.5.2).

A.5.1. Proof of Theorem 3.2. Recall the definition of \( h_N(x,y) \) from (A.38). Note that \( h_N(x,y) \to h(x,y) = \frac{pqf(x)g(y)}{(pf(x) + qg(y))}, \) uniformly in \( x,y \in \mathbb{R}^d \) as \( N \to \infty \). It is easy to see that
\[
\lim_{N \to \infty} \sup_{x,y \in \mathbb{R}^d} \sqrt{N} \left| \frac{h_N(x,y)}{h(x,y)} - 1 \right| = 0,
\]

since \( \sqrt{N} \left( \frac{N_1}{N_1 + N_2} - p \right) \to 0 \) and \( \sqrt{N} \left( \frac{N_1}{N_1 + N_2} - q \right) \to 0 \). Therefore, for any \( \varepsilon > 0 \)
\[
N^{-\frac{1}{2}} \sum_{x,y \in Z_N'} |h_N(x,y) - h(x,y)| 1\{ (x,y) \in E(\mathcal{G}(Z_N')) \} \leq \frac{\varepsilon}{N} \sum_{x,y \in Z_N'} |h(x,y)| 1\{ (x,y) \in E(\mathcal{G}(Z_N')) \} \leq \frac{\varepsilon |E(\mathcal{G}(Z_N'))|}{N}
\]

for \( N \) large enough. By stabilization, \( \frac{|E(\mathcal{G}(Z_N'))|}{N} \overset{p}{\to} \mathbb{E} \Delta_0^+ \), as \( N \to \infty \). Therefore, the RHS above is arbitrarily small as \( N \to \infty \), and
\[
R_2(\mathcal{G}(Z_N')) = N^{-\frac{1}{2}} \sum_{x,y \in Z_N'} (J_{x,y} - \mathbb{E}(J_{x,y})) + o(1), \tag{A.48}
\]

where \( J_{x,y} = h(x,y) 1\{ (x,y) \in E(\mathcal{G}(Z_N')) \} \).
Let \( w(x) = \frac{pf(x)}{pf(x) + qg(x)} \). For \( x \in \mathbb{R}^d \), a locally finite point set \( H \) and any Borel set \( A \subseteq \mathbb{R}^d \), define
\[
\zeta_{H}^x(A) := \zeta(x, H, A) := w(x) \sum_{y \in H \cap A} 1\{ (x, y) \in E(\mathcal{G}(H^x)) \},
\]
where \( d^1_g(\cdot, \cdot, \cdot) \) is the out-degree measure defined in (3.13).

Let \( \mu_N := \sum_{x \in \mathcal{P}_{N\phi_N}} \zeta_{\mathcal{P}_{\phi_N}}^x \) and \( v(y) = qg(y) + \frac{pf(y)}{pf(y) + qg(y)} \). Therefore,
\[
\langle v, \zeta_{\mathcal{P}_{\phi_N}}^x \rangle = \int v(y) \zeta_{\mathcal{P}_{\phi_N}}^x (dy) = w(x) \sum_{y \in \mathcal{P}_{N\phi_N}} v(y) 1\{ (x, y) \in E(\mathcal{G}(\mathcal{P}_{N\phi_N})) \}
\]
and
\[
\langle v, \mu_N \rangle = \sum_{x \neq y \in \mathcal{P}_{N\phi_N}} h(x, y) 1\{ (x, y) \in E(\mathcal{G}(\mathcal{P}_{N\phi_N})) \}.
\]

Thus, from (A.48)
\[
\mathcal{R}_2(\mathcal{G}(Z'_N)) = N^{-\frac{1}{2}} (\langle v, \mu_N \rangle - \mathbb{E}(\langle v, \mu_N \rangle)) + o(1). \tag{A.50}
\]

**Proof of (3.16):** By [26, Lemma 4.1]
\[
\frac{1}{N} \text{Var} \mathbb{E}(\mathcal{R}(\mathcal{G}(Z'_N))|\mathcal{F}) = \frac{1}{N} \text{Var}(\langle v, \mu_N \rangle) := a_N + b_N, \tag{A.51}
\]
where \( a_N = \int \mathbb{E}\{ \langle v, \zeta_{\mathcal{P}_{\phi_N}}^x \rangle^2 \} \phi_N(x) dx \), and
\[
b_N = \int_{\mathbb{R}^2} \left( \mathbb{E}\{ \langle v, \zeta_{\mathcal{P}_{\phi_N}}^{x_N(z)} \rangle \langle v, \zeta_{\mathcal{P}_{\phi_N}}^{x_N(z)} \rangle \} - \mathbb{E}\{ \langle v, \zeta_{\mathcal{P}_{\phi_N}}^{x_N(z)} \rangle \} \mathbb{E}\{ \langle v, \zeta_{\mathcal{P}_{\phi_N}}^{x_N(z)} \rangle \} \right) \phi_N(x) \phi_N(x_N(z)) dz dx, \tag{A.52}
\]
where \( x_N(z) := x + N^{-1/d} z \).

From Lemma A.7 and the assumption that \( \mathcal{G} \) is \( \beta \)-degree bounded for \( \beta > 4 \), we get
\[
a_N \rightarrow \mathbb{E}(\Delta_0^2) \int h^2(x, x) \phi(x) dx. \tag{A.53}
\]

Therefore, to compute of (A.51), it suffices to derive the limit of the \( b_N \). To this end, we have the following lemma, which is proved similarly to Lemma A.7.
Lemma A.5. Let \( \mathcal{G} \) be as in Theorem 3.2. For \( x \) a Lebesgue point of \( \phi \) and \( v(\cdot)\phi(\cdot) \),

\[
\lim_{N \to \infty} \mathbb{E}\{v(x, x^N_\phi)\} = h(x, x)\mathbb{E}\Delta_0^\dagger. \tag{A.54}
\]

Moreover, any \( z \in \mathbb{R}^d \) and \( x_N(z) := x + N^{-1/d}z \),

\[
\lim_{N \to \infty} \mathbb{E}\{v(x, x^N_\phi)\} = h(x, x)\mathbb{E}\Delta_0^\dagger. \tag{A.55}
\]

Proof. The limit in (A.54) follows from (A.6) in Lemma A.7, since \( h(x, y) = w(x)\phi(y) \).

It remains to show (A.55). By the Palm theory of Poisson processes [24, Theorem 1.6],

\[
\mathbb{E}\sum_{w \in \mathcal{P}_N\phi_N} |v(w) - v(x)| \mathbf{1}\{w \in B(x_N(z), KN^{-\frac{1}{2}})\} = N \int_{B(x_N(z), KN^{-\frac{1}{2}})} |v(w) - v(x)| \phi_N(w) \, dw \leq N \int_{B(x_N(z), KN^{-\frac{1}{2}}}} |v(w)\phi_N(w) - v(x)\phi_N(x)| + v(x)|\phi_N(w) - \phi_N(x)| \, dw,
\]

which tends to zero as \( N \to \infty \), if \( z \) is a Lebesgue point of both \( \phi \) and \( v(\cdot)\phi(\cdot) \).

Since \( v \) has range \([0, 1]\), this implies that

\[
\limsup_{N \to \infty} \mathbb{E}\sum_{w \in \mathcal{P}_N\phi_N} |v(w) - v(x)| \mathbf{1}\{(x_N(z), w) \in \mathcal{G}(\mathcal{P}_N\phi_N)\} \leq \limsup_{N \to \infty} \mathbb{E}d_K^\dagger(x_N(z), \mathcal{G}(\mathcal{P}_N\phi_N)) \tag{A.56}
\]

where

\[
d_K^\dagger(x_N(z), \mathcal{G}(\mathcal{P}_N\phi_N)) = \sum_{w \in \mathcal{P}_N\phi_N} \mathbf{1}\{(x_N(z), w) \in E(\mathcal{G}(\mathcal{P}_N\phi_N))\} \mathbf{1}\{w \notin B(x_N(z), KN^{-\frac{1}{2}})\}.
\]

Therefore, by Lemma A.1(b),

\[
\limsup_{K \to \infty} \limsup_{N \to \infty} \mathbb{E}d_K^\dagger(x_N(z), \mathcal{G}(\mathcal{P}_N\phi_N)) = \limsup_{K \to \infty} \mathbb{E}d_K^\dagger(0, \mathcal{G}(\mathcal{P}_\phi(z))) = 0. \tag{A.57}
\]
From (A.56) and (A.57), for every \( x \) which is a Lebesgue point of \( \phi \) and \( v(\cdot)\phi(\cdot) \),
\[
\lim_{N \to \infty} E\{\langle v, \zeta^{xN}(z) \rangle\} = \lim_{N \to \infty} w(xN(z))v(x)Ed^\dagger(xN(z), \mathcal{G}(P_{N\phi_N}))
\]
\[
= E\Delta^\dagger h(z, z), \quad (A.58)
\]
completing the proof of (A.55).

By [26, Lemma 3.6] it follows that
\[
\langle v, \zeta^{xN}(z) \rangle \frac{D}{D} h^2(x, x)E\{d^\dagger(0, \mathcal{G}(P^z_1))d^\dagger(z, \mathcal{G}(P^0_1))\}.
\]
Moreover, since \( \mathcal{G} \) is \( \beta \)-degree moment bounded for \( \beta > 4 \) by assumption, the LHS above is uniformly integrable, which implies the convergence in expectation. This is summarized in the following lemma:

**Lemma A.6** (Penrose [26, Lemma 3.6]). Let \( \mathcal{G} \) be as in Theorem 3.2. For every Lebesgue point \( x \) of \( \phi \) and any \( z \in \mathbb{R}^d \),
\[
\lim_{N \to \infty} E\{\langle v, \zeta^{xN}(z) \rangle\} \langle v, \zeta^{xN}(z) \rangle = h^2(x, x)E\{d^\dagger(0, \mathcal{G}(P^z_1))d^\dagger(z, \mathcal{G}(P^0_1))\}. \quad (A.59)
\]

Using this the limit of \( b_N \) (defined in (A.52)) can be derived. The limit in (3.16) then follows from (A.53) and the following lemma.

**Lemma A.7.** Let \( \mathcal{G} \) be as in Theorem 3.2 and \( b_N \) as in (A.52). Then as \( N \to \infty \)
\[
b_N \to r^2 \int_{\mathbb{R}^2} \left( E\{d^\dagger(0, \mathcal{G}(P^z_1))d^\dagger(z, \mathcal{G}(P^0_1))\} - (E\Delta^\dagger_0)^2 \right) dz \int \frac{f(y)^2g^2(y)}{\phi^3(y)}dy.
\]

**Proof.** Let \( u(x) = \frac{f(x)^2g^2(x)}{\phi^2(x)} \). Since \( \mathcal{G} \) is power-law stabilizing (recall Definition 3.2), by [26, Theorem 2.1] and Lemma A.6 it follows that
\[
b_N \to r^2 \int_{\mathbb{R}^2} \left( E\{d^\dagger(0, \mathcal{G}(P^z_\phi(x)))d^\dagger(z, \mathcal{G}(P^0_\phi(x)))\} - (E\Delta^\dagger_0)^2 \right) u(x)dxdz. \quad (A.60)
\]
Note that
\[
E\{d^\dagger(0, \mathcal{G}(P^z_\phi(x)))d^\dagger(z, \mathcal{G}(P^0_\phi(x)))\} = E\{d^\dagger(0, \mathcal{G}(P^\phi(x)^{-\frac{1}{2}}z))d^\dagger(\phi(x)^{-\frac{1}{2}}z, \mathcal{G}(P^0_1))\},
\]
by the scale invariance of the degree function. Now, substituting \( y = \phi(x)^{-\frac{1}{2}}z \) in (A.60), the result follows.
Proof of (3.18) The proof of the central limit theorem follows from [26, Theorem 2.2] (the slight modification required to adapt the result in our setup is straightforward and the details are omitted).

A.5.2. Proof of Theorem 3.3. Recall from (3.3) and (3.4), $R(\mathcal{G}(Z'_N)) = R_1(\mathcal{G}(Z'_N)) + R_2(\mathcal{G}(Z'_N))$. Let $\sigma_{y,1}$ and $\sigma_{y,2}$ be as defined in Theorems 3.1 and 3.2, and $V \sim N(0, \sigma_{y,2}^2)$. Then, for $t \in \mathbb{R}$, from Proposition A.2 and Theorem 3.1,

$$\mathbb{E}(e^{itR(\mathcal{G}(Z'_N))} \mid \mathcal{F}) = e^{itR_2(\mathcal{G}(Z'_N))} \mathbb{E}(e^{itR_1(\mathcal{G}(Z'_N))} \mid \mathcal{F}) \xrightarrow{D} \mathcal{N}(e^{\frac{1}{2} \sigma_{y,1}^2}t, t^2).$$

Now, by the Dominated Convergence Theorem

$$\mathbb{E}(e^{itR(\mathcal{G}(Z'_N))}) = \mathbb{E}\mathbb{E}(e^{itR(\mathcal{G}(Z'_N))} \mid \mathcal{F}) \rightarrow e^{\frac{1}{2} \sigma_{y,1}^2} + \sigma_{y,2}^2 t^2,$$

and the proof is complete.

A.5.3. Exponential Stabilization of the K-NN Graph. Recall the definition of exponential stabilization from Definition 3.2. Note that this is slightly different from [26, Definition 2.4], because the density $\phi_N$ in the Poisson process $\mathcal{P}_{N\phi_N}$ depends on $N$. Nevertheless, the proof of [26, Theorem 6.4] can be trivially modified to show the exponential stabilization of the out-degree measure (3.13) for the K-NN graph functional $N_K$, as shown below:

**Lemma A.8.** For $f$ and $g$ be as in Corollary 4.1, the out-degree measure $d^+_N$ is exponentially stabilizing for the K-nearest neighbor graph functional $N_K$.

**Proof.** Suppose $f$ and $g$ are both supported on some bounded convex set $\Gamma \in \mathbb{R}^d$. Let $\{C_i\}_{1 \leq i \leq \gamma_d}$ be a finite collection of infinite open cones in $\mathbb{R}^d$ with angular radius $\pi/12$ and apex at 0 in $\mathbb{R}^d$. For $x \in \Gamma$ and $i \in [\gamma_d]$ denote by $C_i(x)$ the translate if $C_i$ with apex at $x$, and $C_i^+(x)$ the open cone concentric to $C_i(x)$ with apex at $x$ and angular radius $\pi/6$. For $x \in \mathcal{P}_{N\phi_N}$ and $i \in [\gamma_d]$, let $R_i(x)$ be the distance from $x$ to $K$-th nearest neighbor of $x$ in $\mathcal{P}_{N\phi_N} \cap C_i^+(x)$, if such a point exists and is less than the diameter of $C_i(x) \cap \Gamma$. Otherwise, set $R_i(x)$ to be the diameter of $C_i(x) \cap \Gamma$. Define $R_+(x) = \max_{i \in [\gamma_d]} R_i(x)$. It follows from Penrose [26, Section 6.3] that for every $x \in \Gamma \cap \mathcal{P}_{N\phi_N}$, $R_N(x) \leq \lceil R_+(x) \rceil$.

To show exponential stabilization, we begin by bounding $\mathbb{P}(R_i(x) > N^{-1/d}s)$. This probability is zero unless $\text{diam}(C_i(x) \cap \Gamma) \geq N^{-1/d}s$. In this case, choose $y \in C_i(x) \cap \Gamma$ such that $|y - x| = \lambda^{-1/d}s$. By the convexity of $C_i(x) \cap \Gamma$, $\frac{x+y}{2} \in C_i(x) \cap \Gamma$. Note that $B_{\gamma\lambda^{-1/d}s} \left(\frac{x+y}{2}\right) \subseteq C_i^+(x)$. Hence, if the
Therefore, there exists a constant $\delta$ (independent of $i, x, s$) such that

\[
\int_{B_{\eta\lambda^{-1/d}s} \left( \frac{x+d}{2} \right)} f(x)dx \geq \delta N^{-1}s^d, \quad \text{and} \quad \int_{B_{\eta\lambda^{-1/d}s} \left( \frac{x+d}{2} \right)} g(x)dx \geq \delta N^{-1}s^d,
\]

which implies that $\int_{B_{\eta\lambda^{-1/d}s} \left( \frac{x+d}{2} \right)} \phi_N(x)dx \geq \delta N^{-1}s^d$. This implies that

\[
\mathbb{P}(R^+(x) \geq N^{-1/d}s) \leq \gamma de^{-\delta s^d} \sum_{r=0}^{K-1} \frac{s^d}{r!} = O_\delta(K)s^d e^{-\delta s^d},
\]

and the exponential stabilization of the degree function for the $K$-NN graph functional follows. \hfill \square

APPENDIX B: POWER AGAINST LOCAL ALTERNATIVES

In this section the proofs of Theorem 4.2 and Proposition 4.3 are presented. To simplify calculations, the theorem is proved for $K = 1$. The proof for general $K$ can be done similarly.

Let $\mathcal{X}'_{N_1}$ and $\mathcal{Y}'_{N_2}$ be samples from $\mathbb{P}_{\theta_1}$ and $\mathbb{P}_{\theta_2}$ as in (4.4), respectively. Let $\phi_{N_1}^{\theta_1, \theta_2}(x) = \frac{N_1}{N} f(x|\theta_1) + \frac{N_2}{N} f(x|\theta_2)$, and suppose

\[
h_{N}^{\theta_1, \theta_2}(x, y) = \frac{N_1 N_2 f(x|\theta_1) f(y|\theta_2)}{(N_1 f(x|\theta_1) + N_2 f(x|\theta_2))(N_1 f(y|\theta_1) + N_2 f(y|\theta_2))}.
\]

Moreover, denote $\rho_{N}^{\theta_1, \theta_2}(x, y) = \mathbb{P}((x, y) \in E(N_1(P_N^y \phi_{N})))$. Then, as in (A.36),

\[
\mathbb{E}(T(\mathcal{G}(Z_N))) = N^2 \int_{S \times S} h_{N}^{\theta_1, \theta_2}(x, y) \mathbb{P}((x, y) \in E(N_1(P_N^y \phi_{N})) \phi_{N}^{\theta_1, \theta_2}(x) \phi_{N}^{\theta_1, \theta_2}(y)dx dy
\]

\[
= N_1 N_2 \int_{S \times S} f(x|\theta_1) f(y|\theta_2) \rho_{N}^{\theta_1, \theta_2}(x, y)dx dy
\]

\[
:= \frac{N_1 N_2}{N^2} \beta_N(\theta_1, \theta_2), \quad (B.1)
\]

where $\beta_N(\theta_1, \theta_2) = N^2 \int_{S \times S} f(x|\theta_1) f(y|\theta_2) \rho_{N}^{\theta_1, \theta_2}(x, y)dx dy$.

Under the null, $\mathbb{E}_0(T(\mathcal{G}(Z_N))) = \frac{N_1 N_2}{N^2} \beta_N(\theta_1, \theta_1) = \frac{N_1 N_2}{N^2} \mathbb{E}(|E(N_1(P_N^y \phi_{N})))| = \frac{N_{1} N_{2}}{N}$. Thus, to derive power against local alternatives (4.4), the distribution of

\[
W := N^{-\frac{1}{2}} \left\{ T(\mathcal{G}(Z_N')) - \frac{N_1 N_2}{N^2} \beta_N(\theta_1, \theta_1) \right\}
\]
has to be derived, when $\theta_2 - \theta_1 = \delta_N \to 0$. Note that

$$W = W_1 + pqN^{-\frac{1}{2}} \{\beta_N(\theta_1, \theta_2) - \beta_N(\theta_1, \theta_1)\} + o(1),$$

where $W_1 = N^{-\frac{1}{2}} \{T(\mathcal{G}(\mathcal{Z}'_N)) - \frac{N_1N_2}{N^2} \beta_N(\theta_2, \theta_1)\} \overset{D}{\to} N(0, \sigma_1^2)$, where, as in (4.1), $\sigma_1 = \sigma_1(f(\cdot|\theta_1), f(\cdot|\theta_1), p)$. Therefore, it suffices to derive the limit of

$$D := N^{-\frac{1}{2}} \{\beta_N(\theta_1, \theta_2) - \beta_N(\theta_1, \theta_1)\}$$

$$= \frac{\delta_N^T \nabla_{\theta_2} \beta_N(\theta_1, \theta_1)}{\sqrt{N}} + \frac{1}{2} \frac{\delta_N^T H_{\theta_2} \beta_N(\theta_1, \theta_1) \delta_N}{\sqrt{N}} + E_N, \quad \text{(B.2)}$$

where

$$\nabla_{\theta_2} \beta_N(\theta_1, \theta_1) := \left(\frac{\partial}{\partial \theta_{2i}} \beta_N(\theta_1, \theta_2) \bigg|_{\theta_2 = \theta_1}\right)_{1 \leq i \leq p},$$

is the gradient vector, and

$$H_{\theta_2} \beta_N(\theta_1, \theta_1) := \left(\frac{\partial^2}{\partial \theta_{2i} \partial \theta_{2j}} \beta_N(\theta_1, \theta_2) \bigg|_{\theta_2 = \theta_1}\right)_{1 \leq i, j \leq p},$$

is the Hessian matrix of $\beta_N(\theta_1, \theta_2)$, at $\theta_2 = \theta_1$, and the error term

$$E_N = \frac{1}{6\sqrt{N}} \sum_{i,j,k} \delta_{N_i} \delta_{N_j} \delta_{N_k} \left. \frac{\partial^3 \beta_N(\theta_1, \theta_2)}{\partial \theta_{2i} \partial \theta_{2j} \partial \theta_{2k}} \bigg|_{\theta_2 = \zeta(\theta_1, \theta_2)} \right), \quad \text{(B.3)}$$

for some $\zeta(\theta_1, \theta_2) \in B(\theta_1, ||\theta_1 - \theta_2||)$.

The rest of the section is organized as follows: The limits of the gradient term and the Hessian term are derived in Section B.1 and Section B.2, respectively. The proof of Theorem 4.2 is then completed in Section B.3, by analyzing the error term.

**B.1. Limit of the Gradient Term.** Recall that $\mathcal{N}_1(\mathcal{P}_0^d)$ is the 1-NN graph on the Poisson process of rate 1 in $\mathbb{R}^d$, with the point 0 added to it. For $s \geq 1$, define $C_s$ to be the expected length of the $s$-th power of the outward edge incident on 0 in the graph $\mathcal{N}_1(\mathcal{P}_0^d)$. Note that

$$C_d = \int_{\mathbb{R}^d} ||x||^d e^{-||x||^d V_d} dx = S_d \int_0^\infty t^{d-1} e^{-t^d V_d} dt = \frac{1}{V_d} \int_0^\infty t e^{-t} dt = \frac{1}{V_d}, \quad \text{(B.4)}$$
where \( V_d = |B(0, 1)| \) and \( S_d = |\partial B(0, 1)| \) is the volume and the surface area of the unit ball in \( \mathbb{R}^d \), respectively. Similarly,
\[
C_{2d} = \int_{\mathbb{R}^d} ||x||^{2d} e^{-||x||^2} dV_d = S_d \int_0^\infty r^{3d-1} e^{-r^2} V_d \]
\[
= \frac{1}{V_d^2} \int_0^\infty t^{2}e^{-t} = \frac{2}{V_d^2}. \tag{B.5}
\]

The following proposition shows that the gradient term has a finite limit when \( \delta_N = \frac{h}{N^{\frac{1}{q}}} \).

**Proposition B.1.** Let \( \delta_N = \frac{h}{N^{\frac{1}{q}}} \), for some \( h \in \mathbb{R}^p \setminus \{0\} \). Then, under the assumptions in Theorem 4.2,
\[
\frac{1}{\sqrt{N}} \delta_N \nabla \theta_2 \beta_N(\theta_1, \theta_1) = O(1). \]

**B.1.1. Proof of Proposition B.1.** Recall the definition of \( \beta_N(\theta_1, \theta_2) \) from (B.1).

Note that for \( x, y \) in the interior of \( S \),
\[
\rho_1^{\theta_1, \theta_2}(x, y) = \mathbb{P}( (x, y) \in E(\mathcal{N}_1(\mathcal{P}_{x,y}^{\theta_1, \theta_2}))) = e^{-\int_{B_S(\|x-y\|)} N_1 f(z|\theta_1) + N_2 f(z|\theta_2) dz},
\]
where \( B_S(\|x-y\|) = B(x, ||x-y||) \cap S \). Then differentiating with respect to \( \theta_2 \) under the integral sign gives,
\[
\nabla \theta_2 \beta_N(\theta_1, \theta_2) = N^2 \int_{S \times S} f(x|\theta_1) \nabla \theta_2 f(y|\theta_2) \rho_N^{\theta_1, \theta_2}(x, y) \]
\[
- N^2 \int_{S \times S} f(x|\theta_1) f(y|\theta_2) \left( \int_{B_S(\|x-y\|)} \nabla \theta_2 f(z|\theta_2) dz \right) \rho_N^{\theta_1, \theta_2}(x, y).
\]

Therefore,
\[
\delta_N \nabla \theta_2 \beta_N(\theta_1, \theta_1) = T_1 - N_2 T_2, \quad \text{where} \]
\[
T_1 := N^2 \int_{S \times S} f(x|\theta_1) \delta_N \nabla \theta_1 f(y|\theta_1) \rho_N^{\theta_1, \theta_1}(x, y) dx dy
\]
\[
T_2 := N^2 \int_{S \times S} f(x|\theta_1) f(y|\theta_1) \left( \int_{B_S(\|x-y\|)} \delta_N \nabla \theta_1 f(z|\theta_1) dz \right) \rho_N^{\theta_1, \theta_1}(x, y) dx dy. \tag{B.6}
\]

The orders of \( T_1 \) and \( T_2 \) are derived in the following lemma. Proposition B.1 follows from this lemma by using (B.6): \( N^{-\frac{1}{2}} \delta_N \nabla \theta_2 \beta_N(\theta_1, \theta_1) = N^{-\frac{3}{2}} \{ T_1 - N_2 T_2 \} \).
Lemma B.1. Let $\delta_N = \frac{h}{\sqrt{N}}$, for some $h \in \mathbb{R}^p \setminus \{0\}$. Then the following holds:

(a) $\frac{T_1}{\sqrt{N}} \to O(1)$.
(b) $\frac{N^2 T_2}{\sqrt{N}} = O(1)$.

To prove the above lemma, begin with the following result about the derivatives of integrals of functions defined on small balls.

Proposition B.2. Let $h : \mathbb{R}^d \to \mathbb{R}$ be a three times continuously differentiable function. For $x \in \mathbb{R}^d$ in the interior of $S$ and $r > 0$, define

$$w_x(r) = \int_{B_S(x,r)} h(z)dz,$$

and $w_x^{(s)}(r) = \frac{\partial^s}{\partial r^s} w_x(r)$. Then

(a) $w_x^{(s)}(0) = 0$, for $s \in [d - 1]$.
(b) $w_x^{(d+1)}(0) = 0$,
(c) $w_x^{(d+2)}(0) = \frac{1}{2}(d + 1)! V_d \text{tr}(H_x h(x)).$

Proof. Note that

$$w_x(r) = \int_0^r \int_{\partial B_S(x,t)} h(z)dz dt = \int_0^r H_x(t)dt$$

and $H_x(t) = \int_{\partial B_S(x,t)} h(z)dz$.

Choose $r \geq t > 0$ small enough so that $B(x,r) \subseteq B(x,t) \subseteq S$. Note that, for all $x \in \mathbb{R}^d$, $\int_{\partial B(x,d)} (x - z, \nabla_x h(x))dz = 0$, by symmetry. Thus, by the Taylor series expansion of $h(z)$ around $x$,

$$H_x(t) = h(x)t^{d-1}S_d + \frac{1}{2} \int_{\partial B(x,t)} (x - z)^\top H_x h(x)(x - z)dz + O(t^{d+2}),$$

(B.7)

where $S_d = |\partial B(0,1)|$ is the surface area of the unit ball in $\mathbb{R}^d$. Consider the spectral decomposition $H_x h(x) = P_x \Lambda(x) P_x$, where

$$\Lambda(x) = \text{diag}(\lambda_1(x), \lambda_2(x), \ldots, \lambda_d)$$

is the diagonal matrix of eigenvalues of the Hessian matrix $H_x h(x)$. Under orthogonal the transformation $z \mapsto P_x(z - x)$,
\[
\int_{\partial B(x,t)} (x - z)^\top H_x h(x)(x - z) \, dz = \int_{\partial B(0,t)} \sum_{i=1}^d \lambda_i(x) z_i^2 \, dz
\]

\[
= \frac{S_d}{d} \text{tr}(H_x h(x)) t^{d+1}
\]

\[
= V_d \text{tr}(H_x h(x)) t^{d+1}, \quad (B.8)
\]

since \( V_d = \frac{S_d}{d} \).

Combining (B.7) and (B.8), \( H_x(t) = \frac{1}{d} V_d \text{tr}(H_x h(x)) t^{d+1} + O(t^{d+2}) \), and \( w_x(r) = h(x) r^d V_d + \frac{1}{2(d+2)} V_d \text{tr}(H_x h(x)) r^{d+2} + O(r^{d+3}) \). Then it is easy to see that \( w_x(0) = 0 \) and \( w_x^{(s)}(0) = 0 \), for all \( s \in [d - 1] \).

Moreover, \( w_x^{(d)}(0) = d V_d h(x) \) and \( w_x^{(d+1)}(0) = 0 \). Moreover, \( w_x^{(d+2)}(0) = \frac{1}{2} (d + 1)! V_d \text{tr}(H_x h(x)) \), as required.

The following simple observation will be useful:

**Observation B.1.** Let \( A \subseteq \mathbb{R}^d \) be compact and convex with non-empty interior, and \( z \in S^{d-1} \) be a fixed unit vector. Then for any positive integer \( r > 0 \), and \( N \) large enough,

\[
\int_{\mathbb{R}^d \setminus N^{\frac{1}{r}} A} ||t||^r |\langle t, z \rangle| e^{-||t||^d} \, dt \lesssim d e^{-\frac{1}{2} N^{\frac{1}{2} + \frac{r}{2}}}.
\]

**Proof.** Choose \( N \) large enough, such that \( B(0, \frac{1}{2}) \subset N^{\frac{1}{r}} A \). Then \( B(0, \frac{N^{\frac{1}{r}}}{2}) \subset N^{\frac{1}{r}} A \), and by the Cauchy-Schwarz inequality

\[
\int_{\mathbb{R}^d \setminus N^{\frac{1}{r}} A} ||t||^r |\langle t, z \rangle| e^{-||t||^d} \, dt \leq \int_{\mathbb{R}^d \setminus B(0, \frac{N^{\frac{1}{r}}}{2})} ||t||^r e^{-||t||^d} \, dt
\]

\[
\lesssim d \int_{N^{\frac{1}{r}} A} e^{r+d} e^{-y^d} \, dz
\]

\[
\lesssim d \int_{N^{\frac{1}{r}}} y^\frac{r+1}{d} e^{-y} \, dy
\]

\[
\lesssim d \int_{N^{\frac{1}{r}}} e^{-y^d} \, dy \lesssim d e^{-\frac{1}{2} N^{\frac{1}{2} + \frac{r}{2}}},
\]

using \( y \leq e^{\frac{d}{r} y^d} \) for \( y \) large enough. \( \square \)

The next two propositions bound the rate of convergence of certain functions of the 1-NN graph functional.
PROPOSITION B.3. Let \( w : S \to \mathbb{R} \) be a continuously differentiable function such that \( \int_S w(x) f(x|\theta_1)^{1-\frac{d}{4}} dx = 0 \), for some integer \( r \geq 0 \). Then,

\[
\mathbb{E} \sum_{x,y \in \mathcal{P}_N f(\cdot|\theta_1)} w(x)||x-y||^r \cdot \mathbf{1}\{(x,y) \in E(N_1(\mathcal{P}_N f(\cdot|\theta_1)))\} = O(N^{1-\frac{r+2}{4}}).
\]

PROOF. Recall

\[
\rho_{\theta_1,\theta_1}^N(x,y) = \mathbb{P}((x,y) \in E(N_1(\mathcal{P}_N f(\cdot|\theta_1))) = e^{-N \int_{\mathcal{B}_S(x,||x-y||)} f(z|\theta_1) dz}.
\]

Define, for \( x \in S \),

\[
V_x := N \int_S ||x-y||^r f(y|\theta_1) e^{-N \int_{\mathcal{B}_S(x,||x-y||)} f(z|\theta_1) dz} dy
:= V_x^{(1)} + V_x^{(2)} + V_x^{(3)},
\]

where

\[
\begin{align*}
V_x^{(1)} &= N f(x|\theta_1) \int_S ||x-y||^r \rho_{\theta_1,\theta_1}^N(x,y) dy, \\
V_x^{(2)} &= N \int_S ||x-y||^r \langle y-x, \nabla f(x|\theta_1) \rangle \rho_{\theta_1,\theta_1}^N(x,y) dy, \\
V_x^{(3)} &= \frac{N}{2} \int_S ||x-y||^r \langle y-x, H_x f(\zeta_{x,y}|\theta_1) \rangle (y-x) \rho_{\theta_1,\theta_1}^N(x,y) dy,
\end{align*}
\]

for some \( \zeta_{x,y} \in B(x,||x-y||) \).

To begin with, bound the \( V_x^{(2)} \) term. Observe that for \( x \in S \),

\[
\begin{align*}
N^2 \int_S ||x-y||^r \langle y-x, \nabla f(x|\theta_1) \rangle e^{-N||x-y||^r f(x|\theta_1) V_4} dy &= \frac{N^{1-\frac{d}{4}}}{f(x|\theta_1)^{\frac{r+d+1}{4}}} \left| \int_{(Nf(x|\theta_1))^{\frac{1}{2}}(S-x)} ||t||^r \langle t, \nabla f(x|\theta_1) \rangle e^{-||t||^r V_4} dt \right| \\
&= \frac{N^{1-\frac{d}{4}}}{f(x|\theta_1)^{\frac{r+d+1}{4}}} \left| \int_{\mathbb{R}^d \setminus (Nf(x|\theta_1))^{\frac{1}{2}}(S-x)} ||t||^r \langle t, \nabla f(x|\theta_1) \rangle e^{-||t||^r V_4} dt \right| \\
&\lesssim d^{-\frac{1}{2}} N^{\frac{1}{2}} e^{-\frac{1}{2} N^{\frac{1}{2}}} = O(e^{-\frac{1}{2} N^{\frac{1}{2}}}).
\end{align*}
\]

The first inequality follows by the variable change \( t = (Nf(x|\theta_1))^{\frac{1}{2}}(y-x) \), the second equality uses symmetry, and the last inequality uses Observation B.1.
Now, as in Proposition B.2,

\[
\rho_N^{\theta_1, \theta_1}(x, y) = \int_{B_S(x, ||x-y||)} f(z|\theta_1)dz = ||x - y||^d f(x|\theta_1)V_d + R_{x,y},
\]

where

\[
|R_{x,y}| = \left| \frac{1}{2} \int_{B_S(x, ||x-y||)} (z - x)^\top H_x f(\zeta_{x,z}|\theta_1)(z - x)dz \right| \lesssim \int_{B_S(x, ||x-y||)} ||z - x||^2dz \lesssim_d ||x - y||^{d+2},
\]

by the uniform boundedness of the spectral norm (Assumption 4.1(c)). Then using (B.10),

\[
|NV_x^{(2)}| \leq N\left| \int_S ||x - y||^r(y - x, \nabla_x f(x|\theta_1))e^{-N||x-y||^d f(x|\theta_1)V_d} (1 - e^{-NR_{x,y}}) dy \right| + O(e^{-\frac{1}{4}N^\frac{1}{2}})
\]

\[
\lesssim N^3 \int_S ||x - y||^{r+d+3}e^{-N||x-y||^d f(x|\theta_1)V_d}dy + O(e^{-\frac{1}{4}N^\frac{1}{2}})
\]

\[
\leq \frac{N^2}{(N f(x|\theta_1))^{\frac{r+d+3}{d}}} f(x|\theta_1) \int_S ||z||^{r+d+3}e^{-||z||^d V_d}dz + O(e^{-\frac{1}{4}N^\frac{1}{2}})
\]

\[
= O(N^{1-\frac{r+3}{d}}). \quad (B.11)
\]

The second inequality uses $1 - e^{-x} \leq x$, the Cauchy-Schwarz inequality, and the fact $\sup_x ||\nabla_x f(x|\theta_1)|| < \infty$ (Assumption 4.1(c)). This implies that

\[
N|\int w(x)f(x|\theta_1)V^{(2)}_2(x)dx| = O(N^{1-\frac{r+3}{d}}).
\]

Next, we bound the $V^{(3)}_x$ term. As in (B.10),

\[
N^2 \int_S ||x - y||^{r+2}e^{-NV_d||x-y||^d f(x|\theta_1)}dy \leq \frac{N^2}{(N f(x|\theta_1))^{\frac{r+d+2}{d}}} f(x|\theta_1) \int_{\mathbb{R}^d} ||t||^{r+2}e^{-V_d||t||^d}dt = O(N^{1-\frac{r+4}{d}}). \quad (B.12)
\]

By Assumption 4.1(c), the maximum eigenvalue of $H_x f(z|\theta_1)$ is uniformly bounded, that is, $K_1 := \sup_z ||H_x f(z|\theta_1)||_\infty < \infty$. Then by (B.12) and the expansion of $\rho_N^{\theta_1, \theta_1}(x, y)$ as above,

\[
|NV^{(3)}_x| \lesssim N^2 \int_S ||x - y||^{r+2}\rho_N^{\theta_1, \theta_1}(x, y)dy = O(N^{1-\frac{r+4}{d}}), \quad (B.13)
\]
that is \( N \int_S w(x)f(x|\theta_1)V_3(x) \, dx = O(N^{1-\frac{r+2}{d}}) \), since \( K_2 := \sup_x |w(x)| < \infty \).

It remains to bound \( V_x^{(1)} \). First, by the transformation \( z = (Nf(x|\theta_1))^{\frac{1}{2}}(x-y) \),

\[
N V_x^{(1)} = N^2 f(x|\theta_1) \int_S ||x-y||^r e^{-N||x-y||^d} f(x|\theta_1) V_d \, dy \\
= \frac{1}{f(x|\theta_1)} \int (Nf(x|\theta_1))^{\frac{1}{2}}(x-y) \ ||z||^{d} e^{-V_d ||z||^d} \, dz \\
\rightarrow \frac{C_d}{f(x|\theta_1)^{\frac{r}{2}}}.
\]

By Observation B.1, the convergence above is exponentially fast. Therefore, ignoring this exponentially small error term, and following arguments as (B.10) and (B.11), (using \( 1 - e^{-x} \leq x \) and \( |R_{x,y}| \lesssim ||x - y||^{d+2} \),

\[
\left| N V_x^{(1)} - \frac{C_d}{f(x|\theta_1)^{\frac{r}{2}}} \right| \\
\lesssim N^2 f(x|\theta_1) \int_S ||x-y||^r e^{-N||x-y||^d} f(x|\theta_1) V_d \left( 1 - e^{-N R_{x,y}} \right) \, dy \\
\lesssim N^3 f(x|\theta_1) \int_S ||x-y||^{r+d+2} e^{-N||x-y||^d} f(x|\theta_1) V_d \, dy \\
= \frac{N^2}{(Nf(x|\theta_1))^{\frac{d+r+2}{d}}} \int_S ||z||^{2d+2} e^{-||z||^d} V_d \, dz = O(N^{1-\frac{r+2}{d}}). \tag{B.14}
\]

Then by (B.9), (B.13), (B.11), and (B.14),

\[
E \sum_{x,y \in P_{Nf(x|\theta_1)}} w(x)||x-y||^d 1\{ (x,y) \in E(N_1(P_{Nf(x|\theta_1)}) \}
= N \int S \, w(x)f(x|\theta_1)V_2 \, dx \\
= N \int S \, w(x)f(x|\theta_1)(V_x^{(1)} + V_x^{(2)} + V_x^{(3)}) \, dx \\
= O(N^{1-\frac{r+2}{d}})
\]

and the result follows. \(\square\)

**Proposition B.4.** Let \( w : S \rightarrow \mathbb{R} \) be a twice continuously differentiable function such that \( \int_S w(y)f(y|\theta_1) \, dy = 0 \). Then

\[
E \sum_{x,y \in P_{Nf(x|\theta_1)}} w(y) 1\{ (x,y) \in E(N_1(P_{Nf(x|\theta_1)}) \} = O(N^{1-\frac{r}{d}}).
\]
Proof. By Assumption 4.1(b) all the points in $\mathcal{P}_{Nf(\cdot|\theta_1)}$ are in the interior of the support $S$ of $f(\cdot|\theta_1)$ with probability 1. The Taylor-series expansion of $w$ around $x$ gives,

$$\mathbb{E} \sum_{x,y \in \mathcal{P}_{Nf(\cdot|\theta_1)}} w(y)1\{(x, y) \in E(\mathcal{N}_1(\mathcal{P}_{Nf(\cdot|\theta_1)}))\} := J_1 + J_2 + J_3,$$

where

$$J_1 := \mathbb{E} \sum_{x \in \mathcal{P}_{Nf(\cdot|\theta_1)}} w(x)d^\top(x, \mathcal{N}_1(\mathcal{P}_{f(x|\theta_1)})),$$

$$J_2 := \mathbb{E} \sum_{x, y \in \mathcal{P}_{Nf(\cdot|\theta_1)}} (y - x)^\top \nabla_x w(x)1\{(x, y) \in E(\mathcal{N}_1(\mathcal{P}_{Nf(\cdot|\theta_1)}))\},$$

$$J_3 := \mathbb{E} \sum_{x, y \in \mathcal{P}_{Nf(\cdot|\theta_1)}} (y - x)^\top \mathbf{H}_x w(\zeta_{x,y})(y - x)1\{(x, y) \in E(\mathcal{N}_1(\mathcal{P}_{Nf(\cdot|\theta_1)}))\},$$

for some $\zeta_{x,y} \in B(x, ||x - y||).$

To begin with, observe that $d^\top(x, \mathcal{N}_1(\mathcal{P}_{Nf(x|\theta_1)})) = 1\{|\mathcal{P}_{Nf(x|\theta_1)}| \geq 1\}$. Therefore, $\mathbb{E}d^\top(x, \mathcal{N}_1(\mathcal{P}_{Nf(x|\theta_1)})) = 1 - e^{-N}$, and

$$J_1 = \int_S w(x)\mathbb{E}d^\top(x, \mathcal{N}_1(\mathcal{P}_{f(x|\theta_1)}))f(x|\theta_1)dx = (1 - e^{-N})\int_S w(x)f(x|\theta_1)dx = 0. \quad (B.15)$$

Next, using [26, Lemma 3.5], for $x \in S$

$$\frac{1}{N^{1 - \frac{d}{2}}}N\int_S (y - x)^\top \nabla_x w(x)f(y|\theta_1)e^{-N\int_{B_S(x, ||x - y||)} f(z|\theta_1)dz}dy \rightarrow \mathbb{E} \sum_{y \in x + \mathcal{P}_{f(x|\theta_1)}} (y - x)^\top \nabla_x w(x)1\{(x, y) \in E(\mathcal{N}_1(\mathcal{P}_{f(x|\theta_1)}))\}$$

$$= \mathbb{E} \sum_{z \in \mathcal{P}_0^y} z^\top \nabla_x w(x)1\{(0, z) \in E(\mathcal{N}_1(\mathcal{P}_0^y))\}f^{-\frac{1}{2}}(x|\theta_1)$$

$$= \left(\int_{\mathbb{R}^d} z^\top \nabla_x w(x)e^{-\frac{1}{4}d||z||^2}dz\right)f^{-\frac{1}{2}}(x|\theta_1) = 0.$$

Therefore, by the Dominated Convergence Theorem, $\frac{J_2}{N^{1 - \frac{d}{2}}} \rightarrow 0$. In fact, by arguments similar to (B.14), it can be shown that $J_2 = O(N^{1 - \frac{2}{d}})$.

Finally, as $w$ is twice continuously differentiable over a compact set $S$, the spectral norm of the Hessian of $w$ is bounded, and

$$J_3 \leq \mathbb{E} \sum_{x, y \in \mathcal{P}_{Nf(\cdot|\theta_1)}} ||y - x||^21\{(x, y) \in E(\mathcal{N}_1(\mathcal{P}_{Nf(\cdot|\theta_1)}))\} = O(N^{1 - \frac{2}{d}}),$$
Proof of Lemma B.1(a): Recall from (B.1) that
\[
\frac{T_1}{\sqrt N} = \frac{1}{N^{1-\frac{d}{2}}} \mathbb{E} \sum_{x,y \in \mathcal{P}_{NF(\cdot|\theta_1)}} \frac{h^T \nabla_{\theta_1} f(y|\theta_1)}{f(y|\theta_1)} \mathbb{1}\{ (x,y) \in E(\mathcal{N}_1(\mathcal{P}_{NF(\cdot|\theta_1)})) \}.
\]
Then by Proposition B.4 (with \( w(y) = \frac{h^T \nabla_{\theta_1} f(y|\theta_1)}{f(y|\theta_1)} \)), \( \frac{T_1}{\sqrt N} = O(1) \).

Proof of Lemma B.1(b): Recall from (B.6) that
\[
T_2 = \mathbb{E} \sum_{x,y \in \mathcal{P}_{NF(\cdot|\theta_1)}} \int_{B_S(x,||x-y||)} \delta_N^T \nabla_{\theta_1} f(z|\theta_1) dz \mathbb{1}\{ (x,y) \in E(\mathcal{N}_1(\mathcal{P}_{NF(\cdot|\theta_1)})) \}
\]
\[
= \mathbb{E} \sum_{x,y \in \mathcal{P}_{NF(\cdot|\theta_1)}} w_{x,\theta_1}(||x-y||, \delta_N) \mathbb{1}\{ (x,y) \in E(\mathcal{N}_1(\mathcal{P}_{NF(\cdot|\theta_1)})) \}. \tag{B.16}
\]
where
\[
w_{x,\theta_1}(r,b) = \int_{B_S(x,r)} b^T \nabla_{\theta_1} f(z|\theta_1) dz, \tag{B.17}
\]
for \( b \in \mathbb{R}^p, \theta_1 \in \Theta \) and \( r \geq 0 \).

Let \( w_{x,\theta_1}^{(s)}(r,b) = \frac{\partial^s}{\partial x^s} w_{x,\theta_1}(r,b) \). Then it is easy to see that \( w_{x,\theta_1}(0,b) = 0 \) and \( w_{x,\theta_1}^{(s)}(0,b) = 0 \), for all \( s \in [d-1] \). Moreover, by Proposition B.2, \( w_{x,\theta_1}^{(d+1)}(0,b) = 0 \), and by the Taylor series expansion of \( w_{x,\theta_1}(||x-y||, \delta_N) \) around \( 0 \), \( T_2 = T_{21} + T_{22} \) (recall the definition of \( T_2 \) from (B.16)), where
\[
T_{21} = \frac{1}{d!} \mathbb{E} \sum_{(x,y) \in E(\mathcal{N}_1(\mathcal{P}_{NF(\cdot|\theta_1)}))} w_{x,\theta_1}^{(d)}(0, \delta_N) ||x-y||^d
\]
\[
T_{22} = \frac{1}{(d+2)!} \mathbb{E} \sum_{(x,y) \in E(\mathcal{N}_1(\mathcal{P}_{NF(\cdot|\theta_1)}))} w_{x,\theta_1}^{(d+2)}(z, \delta_N) \cdot ||x-y||^{d+2},
\]
for some \( z \in (0,||x-y||) \). By Proposition B.2, \( w_{x,\theta_1}^{(d)}(0, \delta_N) = d! V_d \delta_N^T \nabla_{\theta_1} f(x|\theta_1) \) and
\[
\sqrt N T_{21} = V_d N^{-\frac{d}{2}} \mathbb{E} \sum_{(x,y) \in E(\mathcal{N}_1(\mathcal{P}_{NF(\cdot|\theta_1)}))} h^T \nabla_{\theta_1} f(x|\theta_1) ||x-y||^d = O(1), \tag{B.18}
\]
by Proposition B.3 (with \( r = d \) and \( w(x) = h^T \nabla_{\theta_1} f(x|\theta_1) \)).
Finally, by Assumption 4.1, sup_{z \in S} |w_{x, \theta_1}^{(d+2)}(z, \delta_N)| < \infty, and using [28, Theorem 2.5], it is easy to verify that,

$$\sqrt{N}T_{22} \lesssim N^{\frac{3}{2}} \mathbb{E} \sum_{(x, y) \in E(N_1(\mathcal{P}_N(\theta_1)))} ||x - y||^{d+2} = O(N^{-\frac{3}{2}}). \quad (B.19)$$

Lemma B.1(b) now follows from (B.18) and (B.19) above, and by recalling $T_2 = T_{21} + T_{22}$.

### B.2. Limit of the Hessian Term.

The following proposition gives the limit of the Hessian term, when $\delta_N = hN^{-\frac{1}{2}}$.

**Proposition B.5.** Let $\delta_N = hN^{-\frac{1}{2}}$, for some $h \in \mathbb{R}^p \setminus \{0\}$. Then, under the assumptions in Theorem 4.2,

$$\frac{\delta_N^T H_{\theta_2} \beta_N(\theta_1, \theta_1) \delta_N}{2\sqrt{N}} \to -pq\mathbb{E} \left( h^T \nabla_{\theta_1} \log f(x|\theta_1) \right)^2. \quad (B.20)$$

**Proof of Proposition B.5:** Recall the definition of $\beta_N(\theta_1, \theta_2)$ from (B.1). Differentiating with respect to $\theta_2$ twice under the integral signs gives,

$$H_{\theta_2, \beta_N(\theta_1, \theta_2)} = N^2 \int_{S \times S} f(x|\theta_1)H_{\theta_2}(y|\theta_2)\rho_{N, \theta_1, \theta_2}(x, y)dxdy$$

$$- 2N^2 N_2 \int_{S \times S} f(x|\theta_1)f(y|\theta_2)\nabla_{\theta_2} f(y|\theta_2)(W_{x, \theta_2}(r))^T \rho_{N, \theta_1, \theta_2}(x, y)dxdy$$

$$- N^2 N_2 \int_{S \times S} f(x|\theta_1)f(y|\theta_2) \left( \int_{B_S(x, ||x-y||)} H_{\theta_2}(z|\theta_2)dz \right) \rho_{N, \theta_1, \theta_2}(x, y)dxdy$$

$$+ N^2 N_2^2 \int_{S \times S} f(x|\theta_1)f(y|\theta_2)W_{x, \theta_2}(r)(W_{x, \theta_2}(r))^T \rho_{N, \theta_1, \theta_2}(x, y)dxdy,$$

where $W_{x, \theta_2}(r) = \int_{B_S(x, r)} \nabla_{\theta_2} f(z|\theta_2)dz$. Therefore, $\delta_N^T H_{\theta_2} \beta_N(\theta_1, \theta_1) \delta_N = T_{21} - T_{22} - T_{23} + T_{24}$, where

$$T_{21} = N^2 \int_{S \times S} f(x|\theta_1)\delta_N^T H_{x, \theta_1} f(y|\theta_1)\delta_N \rho_{N, \theta_1, \theta_1}(x, y)dxdy$$

$$T_{22} = 2N^2 N_2 \int_{S \times S} f(x|\theta_1)\delta_N^T \nabla_{\theta_1} f(y|\theta_1)w_{x, \theta_1}(||x-y||) \delta_N \rho_{N, \theta_1, \theta_1}(x, y)dxdy$$

$$T_{23} = N^2 N_2 \int_{S \times S} f(x|\theta_1)f(y|\theta_1) \left( \int_{B_S(x, ||x-y||)} \delta_N^T H_{\theta_1}(z|\theta_1)\delta_N dz \right) \rho_{N, \theta_1, \theta_1}(x, y)dxdy$$

$$T_{24} = N^2 N_2 \int_{S \times S} f(x|\theta_1)f(y|\theta_1) W_{x, \theta_1}(r)(W_{x, \theta_1}(r))^T \rho_{N, \theta_1, \theta_1}(x, y)dxdy.$$
\[ T_{24} = N_2^2 N^2 \int_{S \times S} f(x|\theta_1) f(y|\theta_1) w_{x,\theta_1}^2(||x - y||, \delta_N) \rho_N^{\theta_1, \theta_1} (x, y) dx dy, \]

(B.22)

where \( w_{x,\theta_1}(r, b) \) is defined in (B.17). The limit of \( T_{21}, T_{22}, T_{23}, \) and \( T_{24} \) are given in the following lemma. Using this Proposition B.5 follows.

**Lemma B.2.** Let \( \delta_N = h N^{-\frac{1}{4}} \), for some \( h \in \mathbb{R}^p \setminus \{0\} \). Then the following holds:

(a) \( \frac{T_{21}}{\sqrt{N}} \to 0 \).

(b) \( \frac{T_{22}}{\sqrt{N}} \to 2q E \left( h^\top \nabla_{\theta_1} \log f(x|\theta_1) \right)^2 \).

(c) \( \frac{T_{23}}{\sqrt{N}} \to 0 \).

(c) \( \frac{T_{24}}{\sqrt{N}} \to 2q^2 E \left( h^\top \nabla_{\theta_1} \log f(x|\theta_1) \right)^2 \).

**Proof.** By Lemma A.2,

\[
\frac{T_{21}}{\sqrt{N}} = \frac{1}{N} E \sum_{x,y \in \mathcal{P}_{Nf(|\theta_1|)}} \frac{h^\top H_{\theta_1} f(y|\theta_1) h}{f(y|\theta_1)} 1\{ (x, y) \in E(\mathcal{N}_1(\mathcal{P}_{Nf(|\theta_1|)})) \}
\]

\[
\to \int_{S} h^\top H_{\theta_1} f(x|\theta_1) h dx = 0,
\]

since

\[
\int_{S} h^\top H_{\theta_2} f(x|\theta_2) h dx = \int_{S} \sum_{1 \leq i, j \leq p} \frac{\partial^2}{\partial \theta_1 i \partial \theta_1 j} f(x|\theta_1) h_i h_j
\]

\[
= \sum_{1 \leq i, j \leq p} \frac{\partial^2}{\partial \theta_1 i \partial \theta_1 j} \int_{S} f(x|\theta_1) h_i h_j = 0,
\]

for \( \theta_2 \in \Theta \). This completes the proof of (a).

To show (b), observe that

\[
N^2 \int_{S \times S} \frac{f(x|\theta_1) f(y|\theta_1) h^\top \nabla_{\theta_1} f(y|\theta_1) h^\top \nabla_{\theta_1} f(x|\theta_1)}{f(y|\theta_1)} ||x - y||^d \rho_N^{\theta_1, \theta_1} (x, y) dx dy
\]

\[
= E \sum_{(x,y) \in \mathcal{P}_{Nf(|\theta_1|)}} \frac{h^\top \nabla_{\theta_1} f(y|\theta_1) h^\top \nabla_{\theta_1} f(x|\theta_1)}{f(y|\theta_1)} ||x - y||^d 1\{ (x, y) \in E(\mathcal{N}_1(\mathcal{P}_{Nf(|\theta_1|)})) \}
\]

\[
\to C_d \int_{S \times S} \frac{(h^\top \nabla_{\theta_1} f(x|\theta_1))^2}{f(x|\theta_1)} dx dy
\]

\[
= \frac{1}{V_d} E \left( \log h^\top \nabla_{\theta_1} f(x|\theta_1) \right)^2,
\]
since \( C_d = \frac{1}{V_d} \) by (B.4). Now, using

\[
T_{22} = \frac{1}{\sqrt{N}} \int_{B_S(x,r)} b^T \nabla \theta_1 f(z) |z| dz = V_d r^d b^T \nabla \theta_1 f(x) + O(r^{d+1}),
\]

we get

\[
\frac{T_{22}}{\sqrt{N}} \rightarrow 2q\mathbb{E} \left( \log h^T \nabla \theta_1 f(x) \right)^2,
\]

proving (b).

The proof of (c) can be done similarly. To prove (d) recall that

\[
w_{x_1,\theta_1}(r,b) = \frac{1}{N} \sum_{x,y \in P} f(x) \log f(x) / f(y)
\]

since \( C_d = \frac{1}{V_d} \) by (B.5).

**B.3. Completing the Proof of Theorem 4.2.** Recall the definition of the error term (B.3). First, it is shown that \( E_N \rightarrow 0 \), for \( N^{\beta_d} \delta_N \rightarrow 0 \), where \( \beta_d \) is as defined in (4.5). To this end, define

\[
T_N(\theta_1, \theta_2) = \sum_{i,j,k} \delta_N, \delta_N, \delta_N \frac{\partial^3 \beta_N(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_1 \partial \theta_1 k}.
\]

Note that the error term \( E_N = \frac{1}{6\sqrt{N}} T_N(\theta_1, \zeta_N(\theta_1, \theta_2)) \), where \( \zeta_N(\theta_1, \theta_2) \in B(\theta_1, ||\theta_1 - \theta_2||) \).

Control of the error term, needs the third derivatives of \( \beta_N(\theta_1, \theta_2) \) with respect to \( \theta_2 \). This involves several terms which can computed by taking another derivative with respect to \( \theta_2 \) of the Hessian of \( \beta_N(\theta_1, \theta_2) \) in (B.21). Each of the resulting terms can be bounded in a similar manner. In the
following, this is illustrated for one such term $T_0$, obtained by taking the derivative of $\rho_N^{\theta_1, \theta_2}$ with respect to $\theta_2$ of the fourth term in (B.21).

Let $\alpha_d$ be a positive constant depending on $d$. Let $K_h = \sup_z |h^\top \nabla \theta_2 f(z|\theta_2)| < \infty$, by Assumption (4.1). Then for $N$ large,

\[
T_0 := \frac{N^2 N_2^3}{\sqrt{N}} \int f(x|\theta_1) f(y|\theta_2) \left( \int_{B_3(x, \|x-y\|)} \delta_N^\top \nabla \theta_2 f(z|\theta_2) dz \right)^3 \rho_N^{\theta_1, \theta_2}(x, y) dxdy
\]

\[
\leq K_h \frac{N^2 N_2^3}{N^{\frac{3}{2}+3\alpha_d}} \int_{S \times S} f(x|\theta_1) f(y|\theta_2) \|x - y\|^3 \rho_N^{\theta_1, \theta_2}(x, y) dxdy
\]

\[
\leq 2 K_h N_2^{\frac{3}{2} - 3\alpha_d} \mathbb{E} \sum_{x, y} \|x - y\|^3 \mathbb{1}\{(x, y) \in \mathcal{E}(\mathcal{P}_{N_1 f|\theta_1} + N_2 f|\theta_2))\}
\]

\[
\leq O(N^{\frac{1}{2} - 3\alpha_d}), \tag{B.23}
\]

using [37, Corollary 8.4]. The other terms in the third derivative can be also be bounded similarly. This implies that $|E_N| = O(N^{\frac{1}{2} - 3\alpha_d})$.

Recall $\beta_d$ defined in (4.5). Now, let $N^{\beta_d} \delta_N = h$ such that $\|h\| \to 0$. Taking $\alpha_d = \beta_d$ in (4.5) it follows that $E_N \to 0$. Also, by Proposition B.1 and Proposition B.5, $\frac{\delta_N^\top \nabla \theta_2 \beta_N(\theta_1, \theta_1)}{\sqrt{N}} \to 0$ and $\frac{1}{2} \frac{\delta_N^\top \nabla \theta_2 \beta_N(\theta_1, \theta_1) \delta_N}{\sqrt{N}} \to 0$, respectively. Then by (B.2), $D \to 0$ and limiting power of the test (1.8) is $\alpha$.

Next, suppose $d \leq 8$ and $N^{\frac{1}{2} \delta_N} = h$ is such that $\|h\| \to \infty$. By Proposition B.5, $\frac{1}{2} \frac{\delta_N^\top \nabla \theta_2 \beta_N(\theta_1, \theta_1) \delta_N}{\sqrt{N}} = \Theta(\|h\|^2)$. Moreover, from (B.23) above (taking $\beta_d = 1/4$), $|E_N| = O(\|h\|^3 N^{-\frac{1}{4}})$, and $\|h\|^2 \gg N^{-\frac{1}{4}} \|h\|^3$, whenever $\|\delta_N\| \to \infty$. Also, by Proposition B.1, $\frac{\delta_N^\top \nabla \theta_2 \beta_N(\theta_1, \theta_1)}{\sqrt{N}} = O(\|h\|) \ll \Theta(\|h\|^2)$, which shows that $D \to \infty$, and the limiting power of the test (1.8) is 1.

Finally, suppose $d \geq 9$ and $N^{\frac{1}{2} \delta_N} = h$ is such that $\|h\| \to \infty$. Then by Proposition B.1, $\frac{\delta_N^\top \nabla \theta_2 \beta_N(\theta_1, \theta_1)}{\sqrt{N}} = \Theta(N^{\frac{1}{2} - \frac{3}{4}} \|h\|)$, and by Proposition B.5

$\frac{1}{2} \frac{\delta_N^\top \nabla \theta_2 \beta_N(\theta_1, \theta_1) \delta_N}{\sqrt{N}} = \Theta(N^{\frac{1}{2} - \frac{5}{4}} \|h\|)^2$. Moreover, from (B.23) above, $|E_N| = O(\|h\|^3 N^{\frac{3}{2} - \frac{5}{4}})$, and the Hessian term $N^{\frac{1}{2} - \frac{3}{4}} \|h\|^2 \gg \|h\|^3 N^{\frac{3}{2} - \frac{5}{4}}$, whenever $\|\delta_N\| \to \infty$. Therefore, the Hessian term dominates and $D \to \infty$, completing the proof of the result.

**B.4. Proof of Proposition 4.3.** Recall the normal density $f(\cdot | \mu)$ truncated at the set $S$, as in (4.6). Note that the gradient of the normalizing constant is $\nabla_{\mu} \mathcal{C}_S(\mu) = \int_S (x - \mu)e^{-\frac{1}{2}(x-\mu)\top(x-\mu)}dx = \mu_S - \mu$. Therefore,

\[
\nabla_{\mu} f(x|\mu) = \left\{ (x - \mu) - \frac{\nabla_{\mu} \mathcal{C}_S(\mu)}{\mathcal{C}_S(\mu)} \right\} f(x|\mu). \tag{B.24}
\]
Recall that the gradient term $\delta_N^T \nabla_{\mu} \beta_N(\mathbf{0}, \mathbf{0}) = T_1 - N_2T_2$, where $T_1$ and $T_2$ are defined in (B.6). By symmetry of $S$, when $\mu = 0$, then $\mu_S = 0$. Then by (B.24), $\nabla_{\mu} f(x|\mathbf{0}) = xf(x|\mathbf{0})$. Then, 

$$T_1 = N^2 \int_S \int_S f(x|\mathbf{0}) \delta_N^T \nabla_{\mu} f(y|\mathbf{0}) e^{-f_{BS(x,|x-y|)} N f(z|\mathbf{0})} dz \, dx \, dy$$

$$= N \int_S \delta_N^T y \mathbb{E}(d^y(y, N_K(\mathcal{P}^y_{N f(\mathbf{0})}))) f(y|\mathbf{0}) dy = 0,$$  \hspace{1cm} (B.25)

since $S = -S$ and $\mathbb{E}(d^y(y, N_K(\mathcal{P}^y_{N f(\mathbf{0})}))) = \mathbb{E}(d^y(-y, N_K(\mathcal{P}^{-y}_{N f(\mathbf{0})})))$, by symmetry.

Next, compute $T_2$. Recall (B.17). Then by symmetry,

$$w_{x,0}(r,b) := \int_{BS(x,r)} b^T \nabla f(z|\mathbf{0}) dz = \int_{BS(x,r)} b^T z f(z|\mathbf{0}) dz = -w_{-x,0}(r),$$

which implies,

$$T_2 = N^2 \int_S \int_S f(x|\mathbf{0}) f(y|\mathbf{0}) w_{x,0}(||x - y||, \delta_N) e^{-f_{BS(x,|x-y|)} N f(z|\mathbf{0})} dz \, dx \, dy = 0.$$  \hspace{1cm} (B.26)

From (B.25) and (B.26), it follows that $\delta_N^T \nabla_{\mu} \beta_N(\mathbf{0}, \mathbf{0}) = T_1 - N_2T_2 = 0$.

Next, by Proposition B.5, for $\delta_N = hN^{-\frac{1}{4}}$, 

$$\frac{\delta_N^T \nabla_{\mu} \beta_N(\mathbf{0}, \mathbf{0}) \delta_N}{2\sqrt{N}} \quad \rightarrow \quad -pq\mathbb{E}_S^0 \left(h^T \nabla_{\mu} \log f(x|\mathbf{0})\right)^2 = -pq\mathbb{E}_S^0 \left(h^T x\right)^2.$$ 

This implies that $\mathcal{D} \rightarrow -pq\mathbb{E}_S^0 \left(h^T x\right)^2$, which implies (c).

The above argument also shows that for $||N^{\frac{1}{2}} \delta_N|| \rightarrow 0$, $\frac{\delta_N^T \nabla_{\mu} \beta_N(\mathbf{0}, \mathbf{0}) \delta_N}{2\sqrt{N}} \rightarrow 0$ and so, $\mathcal{D} \rightarrow 0$. This implies that the power of the test goes to 0, proving (a).

Similarly, if $||N^{\frac{1}{2}} \delta_N|| \rightarrow \infty$, $\frac{\delta_N^T \nabla_{\mu} \beta_N(\mathbf{0}, \mathbf{0}) \delta_N}{2\sqrt{N}} \rightarrow \infty$, and $\mathcal{D} \rightarrow \infty$, which implies that the power of the test goes to 1, proving (b).