TRAVELING WAVES FOR A NONLOCAL DISPERAL SIR MODEL WITH GENERAL NONLINEAR INCIDENCE RATE AND SPATIO-TEMPORAL DELAY

JINLING ZHOU AND YU YANG*

School of Science and Technology
Zhejiang International Studies University, Hangzhou 310012, China

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Abstract. In this paper, we study a nonlocal dispersal SIR model with general nonlinear incidence rate and spatio-temporal delay. By Schauder’s fixed point theorem and Laplace transform, we show that the existence and nonexistence of traveling wave solutions are determined by the basic reproduction number and the minimal wave speed. Some examples are listed to illustrate the theoretical results. Our results generalize some known results.

1. Introduction. As the pioneering work, Schaaf [30] studied the existence of traveling wave solutions in two scalar reaction-diffusion equations with delay. Then, the traveling wave solutions of reaction-diffusion equations with delay have attracted significant attention, see [5, 18, 24, 28, 32, 44].

Since population takes time to move in space and usually is not at the same position in space at previous time, Britton [6, 7] proposed nonlocal delay or spatio-temporal delay. Later, some researchers paid attention to the existence of traveling wave solutions for reaction-diffusion equations with spatio-temporal delays, see [2, 16, 26, 36]. For example, Gourley and Ruan [17] investigated the existence of traveling front solutions for a two-species competition model with nonlocal delays by employing linear chain techniques and geometric singular perturbation theory. Wang et al. [33] established the existence of traveling wave fronts in reaction-diffusion systems with spatio-temporal delays by using monotone iterations. Yu and Yuan [42] considered more general reaction-diffusion systems with distributed and spatio-temporal delays by a new monotone iterations. Recently, Wang et al. [35] incorporated spatio-temporal delay into the classical Kermack-McKendrick model and considered the following nonlocal model

\[
\begin{align*}
\frac{\partial S(x,t)}{\partial t} &= d_1 \Delta S(x,t) - \beta S(x,t) K * I(x,t), \\
\frac{\partial I(x,t)}{\partial t} &= d_2 \Delta I(x,t) + \beta S(x,t) K * I(x,t) - \gamma I(x,t), \\
\frac{\partial R(x,t)}{\partial t} &= d_3 \Delta R(x,t) + \gamma I(x,t),
\end{align*}
\]

where $K * I(x,t)$ is the convolution of $K$ and $I(x,t)$.
where $S(t), I(t)$ and $R(t)$ denote the number of susceptible, infected and removed individuals, respectively. $\beta$ is the transmission coefficient, $\gamma$ is the recovery rate and $d_i > 0 (i = 1, 2, 3)$ are dispersal rates of the corresponding individuals. The operator $(K \ast u)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} K(x - y, t - s) u(y, s) dy ds$, which considers the effect of spatial heterogeneity (geographical movement), nonlocal interaction and time delay such as latent period on the spread of the disease. They proved the existence and nonexistence of traveling wave solutions for system (1) by Schauder’s fixed point theorem and Laplace transform.

The form of incidence rate can have an important role in modeling epidemic dynamics. Liu et al. [27] introduced a nonlinear incidence rate of the form $\beta SI/(1 + \alpha I^q)$, where $p, q$ are positive constants and $\alpha$ is a nonnegative constant. When $p = q = 1$, the incidence rate $\beta SI/(1 + \alpha I)$ was proposed by Capasso and Serio [8] to describe “crowding effect” or “protection measures” in modeling the cholera epidemics in Bari in 1973. When $p = 1, q = 2$, the incidence rate $\beta SI/(1 + \alpha I^2)$ was used by Xiao and Ruan [37] to interpret the “psychological” effect. The most general case $f(S, I)$ was considered by Feng and Thieme [13, 14]. For more details about the nonlinear incidence rates, we refer to see [1, 19, 20, 21, 23, 31, 41].

The diffusion term of system (1) is Laplacian operator that accounts for random motion. However, the movements of individuals which can not be limited to a small area are often free. For example, in ecology, bees often jump from one location to another; in social sciences, it is more suitable to consider nonlocal interactions or movements of agents; it has been suggested that the interaction between neurons is also nonlocal in signal propagation in neural networks [12]. These processes are modeled by a nonlocal dispersal operator (see [15, 22, 43, 69] and the references therein) as follows:

$$J[u](x, t) = \int_{-\infty}^{+\infty} J(x - y)(u(y, t) - u(x, t)) dy,$$

which gives the probability that a particle or agent at location $y$ will jump to location $x$. For more results on traveling wave solutions of nonlocal dispersal problems, see [10, 25, 29, 38, 39]. To our knowledge, there are few works on nonlocal dispersal SIR model with spatio-temporal delay. More recently, Cheng and Yuan [10] investigated the existence and nonexistence of traveling wave solutions for a nonlocal dispersal SIR model with bilinear incidence rate and spatio-temporal delay. Wang et al. [34] considered the existence and nonexistence of traveling wave solutions of a nonlocal dispersal SIR model with standard incidence rate and spatio-temporal delay.

Motivated by the works of [35, 10], in this paper, we investigate the nonlocal dispersal SIR model with more general incidence rate and spatio-temporal delay as follows:

$$\begin{cases}
\frac{\partial S(x, t)}{\partial t} = d_1 (J \ast S(x, t) - S(x, t)) - f(S(x, t))g((K \ast I)(x, t)), \\
\frac{\partial I(x, t)}{\partial t} = d_2 (J \ast I(x, t) - I(x, t)) + f(S(x, t))g((K \ast I)(x, t)) - \gamma I(x, t), \\
\frac{\partial R(x, t)}{\partial t} = d_3 (J \ast R(x, t) - R(x, t)) + \gamma I(x, t),
\end{cases}$$

(2)

where $f(S)$ and $g(I)$ are positive for all $S, I > 0$ and $f(0) = g(0) = 0$. We also need the following conditions:

(H1) $f'(S)$ is continuous and $f'(S) > 0$ for all $S > 0$;
For example, if $K$ is continuous differential, non-increasing for all $I > 0$ and $\lim_{I \to +\infty} g(I)/I = 0$.

It is easy to check that the class of functions $f(S)g(I)$ include some types of incidence rates such as $f(S)g(I) = \beta SI/(1 + \alpha I)$ and $f(S)g(I) = \beta SI/(1 + \alpha I^2)$ for $\beta > 0$ and $\alpha \geq 0$.

The organization of this paper is as follows. In Section 2, we establish the existence of traveling wave solutions as $R_0 > 1$ and $c > c^*$. In Section 3, we prove the nonexistence of traveling wave solutions. In Section 4, some examples are provided to illustrate the main results. The paper ends with a brief conclusion in Section 5.

2. Existence of traveling waves. In this section, we discuss the existence of traveling wave solutions of (2).

Since $R$ does not appear in the first two equations of (2), we let $\xi = x + ct$ and consider the reduced system

$$
\begin{cases}
J^*(\xi) = d_1(J * S(\xi) - S(\xi)) - f(S(\xi))g(K * I(\xi)), \\
J^*(\xi) = d_2(J * I(\xi)) - f(S(\xi))g(K * I(\xi)) - \gamma I(\xi),
\end{cases}
$$

(3)

where

$$
J * S(\xi) = \int_{-\infty}^{+\infty} J(x)S(\xi - x)dx, \quad J * I(\xi) = \int_{-\infty}^{+\infty} J(x)I(\xi - x)dx.
$$

$$
K * I(\xi) = \int_{0}^{+\infty} \int_{-\infty}^{+\infty} K(y,s)I(\xi - y - cs)dyds.
$$

Also, we need the following asymptotic boundary conditions

$$
S(-\infty) = S_0, \quad S(+\infty) < S_0, \quad I(\pm \infty) = 0,
$$

(4)

where $S(\pm \infty) = \lim_{\xi \to \pm \infty} S(\xi)$ and $I(\pm \infty) = \lim_{\xi \to \pm \infty} I(\xi)$.

Suppose that the kernel functions $J(x)$ and $K(x,t)$ satisfy the following conditions.

(A1) $J(x)$ is Lipschitz continuous, $J(x) \in C^1(\mathbb{R})$, $J(-x) = J(x) \geq 0$ and $\int_{-\infty}^{+\infty} J(x)dx = 1$. In addition, there exists $\lambda_J$ such that

$$
\int_{-\infty}^{+\infty} J(x)e^{-\lambda x}dx < +\infty, \quad \text{for} \quad \lambda \in [0, \lambda_J),
$$

and $\int_{-\infty}^{+\infty} J(x)e^{-\lambda x}dx \to +\infty$ as $\lambda \to \lambda_J^+$, where $\lambda_J$ may be $+\infty$. For example, if $J(x) = \frac{1}{2}e^{-|x|}$, then $\lambda_J = 1$; if $J(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$, then $\lambda_J = +\infty$.

(A2) $K(x,t)$ is Lipschitz continuous with $x$, i.e. there exists $L_K$ such that $|K(x_1,t) - K(x_2,t)| \leq L_k|x_1 - x_2|$ for $x_1, x_2, t \in [0,T]$. For any $(x,t) \in \mathbb{R} \times [0,T]$, $K(x,t) = K(-x,t) \geq 0$ and $\int_{0}^{T} \int_{-\infty}^{+\infty} K(x,t)dxdt = 1$. Moreover, for each $c > 0$, there exists $\lambda_{Kc}$ such that

$$
\int_{0}^{T} \int_{-\infty}^{+\infty} K(x,t)e^{-\lambda(x+ct)}dxdt < +\infty \quad \text{for} \quad \lambda \in [0, \lambda_{Kc}).
$$

For example, if $K(x,t) = \delta(t - \tau)\frac{1}{\sqrt{4\pi p}}e^{-\frac{x^2}{4t}}$, then $\lambda_{Kc} = +\infty$. 

(H2) $g'(0) = \max_{x \in [0, +\infty]} g'(x)$, $g(I)/I$ is continuous differential, non-increasing for all $I > 0$ and $\lim_{I \to +\infty} g(I)/I = 0$. 


Lemma 2.1. Assume that $R_0 = \frac{f(S_0)g'(0)}{\gamma} > 1$. There exists a positive pair of $(\lambda^*, c^*)$ such that

$$\Delta(\lambda^*, c^*) = 0$$

and

$$\frac{\partial \Delta(\lambda^*, c^*)}{\partial \lambda} = 0.$$

Furthermore,

(i) if $0 < c < c^*$, then $\Delta(\lambda, c) > 0$ for all $\lambda \in [0, \lambda_c]$;

(ii) if $c > c^*$, then $\Delta(\lambda, c) = 0$ has two positive roots $\lambda_1 = \lambda_1(c)$ and $\lambda_2 = \lambda_2(c)$ with $0 < \lambda_1 < \lambda^* < \lambda_2 < \lambda_c$ such that $\lambda_1'(c) < 0$, $\lambda_2'(c) > 0$ and

$$\Delta(\lambda, c) \begin{cases} > 0, & \lambda \in (0, \lambda_1) \cup (\lambda_2, \lambda_c), \\ < 0, & \lambda \in (\lambda_1, \lambda_2). \end{cases}$$

Proof. By some calculations, we get

$$\Delta(0, c) = f(S_0)g'(0) - \gamma > 0, \quad \Delta(\lambda, +\infty) = -\infty, \quad \text{for } \lambda \in [0, \lambda_c].$$

Note that

$$\int_{-\infty}^{+\infty} J(x)e^{-\lambda x}dx = \int_{0}^{+\infty} J(x)(e^{-\lambda x} + e^{\lambda x})dx \geq 2 \int_{0}^{+\infty} J(x)dx = 1,$$

and

$$\Re(\lambda, 0) = \int_{0}^{T} \int_{-\infty}^{+\infty} K(x,t)e^{-\lambda x}dxdt$$

$$= \int_{0}^{T} \int_{0}^{+\infty} K(x,t)(e^{-\lambda x} + e^{\lambda x})dxdt$$

$$\geq 2 \int_{0}^{T} \int_{0}^{+\infty} K(x,t)dxdt = 1.$$

Thus,

$$\Delta(\lambda, 0) = d_2 \left( \int_{-\infty}^{+\infty} J(x)e^{-\lambda x}dx - 1 \right) + f(S_0)g'(0)\Re(\lambda, 0) - \gamma > 0.$$

Moreover,

$$\frac{\partial \Delta(0, c)}{\partial \lambda} = -c(1 + f(S_0)g'(0)) \int_{0}^{T} \int_{-\infty}^{+\infty} tK(x,t)dxdt < 0,$$

$$\frac{\partial \Delta(\lambda, c)}{\partial c} = -\lambda(1 + f(S_0)g'(0)) \int_{0}^{T} \int_{-\infty}^{+\infty} tK(x,t)e^{-\lambda x}dxdt < 0,$$

$$\frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2} = d_2 \int_{-\infty}^{+\infty} x^2 J(x)e^{-\lambda x}dx$$

$$+ f(S_0)g'(0) \int_{0}^{T} \int_{-\infty}^{+\infty} (x + ct)^2 K(x,t)e^{-\lambda x}dxdt > 0.$$
By the intermediate value theorem, monotonicity and convexity properties, there exists a point \((\lambda^*, c^*)\) such that

\[
\Delta(\lambda^*, c^*) = 0 \quad \text{and} \quad \frac{\partial \Delta(\lambda^*, c^*)}{\partial \lambda} = 0.
\]  

(5)

We claim that the point \((\lambda^*, c^*)\) is unique. Otherwise, there exists two different points \((\lambda_1, c_1)\) and \((\lambda_2, c_2)\) satisfying (5). Without loss of generality, we assume \(\lambda_1 < \lambda_2\) and \(c_1 < c_2\). Since \(\Delta(\lambda, 0) > 0\), \(\frac{\partial \Delta(\lambda, c)}{\partial c} < 0\) and \(\frac{\partial^2 \Delta(\lambda, c)}{\partial c^2} > 0\), we get

\[
\Delta(\lambda_1, c_1) > \Delta(\lambda_1, c_2) \geq \Delta(\lambda_2, c_2) = 0,
\]

which is a contradiction.

(i) Since \(\Delta(\lambda, 0) > 0\) and \(\frac{\partial \Delta(\lambda, c)}{\partial c} < 0\), we obtain that \(\Delta(\lambda, c) > \Delta(\lambda, c^*)\) for any \(0 < c < c^*\). The convexity \(\Delta(\lambda, c)\) about \(\lambda\) gives

\[
\Delta(\lambda, c^*) > \Delta(\lambda^*, c^*) = 0.
\]

(ii) For any \(c > c^*\), we have

\[
\Delta(\lambda^*, c) < \Delta(\lambda^*, c^*) = 0.
\]

Since \(\Delta(0, c) > 0\), \(\frac{\partial^2 \Delta(\lambda, c)}{\partial c^2} > 0\) and \(\Delta(\lambda, c) \to +\infty\) as \(\lambda \to \lambda^-\), the equation \(\Delta(\lambda, c) = 0\) has two positive roots \(\lambda_1 = \lambda_1(c)\) and \(\lambda_2 = \lambda_2(c)\) with \(0 < \lambda_1 < \lambda^* < \lambda_2 < \lambda_c\) such that \(\lambda_1'(c) < 0, \lambda_2'(c) > 0\) and

\[
\Delta(\lambda, c) \begin{cases} >0, & \lambda \in (0, \lambda_1) \cup (\lambda_2, \lambda_c), \\ <0, & \lambda \in (\lambda_1, \lambda_2). \end{cases}
\]

\[
\Box
\]

In the following, we assume that \(R_0 > 1\) and \(c > c^*\). Define some continuous functions as follows:

\[
\mathcal{S}(\xi) = S_0, \quad \mathcal{I}(\xi) = \begin{cases} e^{\lambda_1 \xi}, & \xi < \xi_1, \\ \frac{e^{\lambda_1 \xi}}{K}, & \xi \geq \xi_1, \end{cases}
\]

\[
\mathcal{S}(\xi) = \begin{cases} S_0 - \sigma e^{\alpha \xi}, & \xi < \xi_2, \\ 0, & \xi \geq \xi_2, \end{cases}, \quad \mathcal{I}(\xi) = \begin{cases} e^{\lambda_1 \xi} (1 - Me^{\eta \xi}), & \xi < \xi_3, \\ 0, & \xi \geq \xi_3, \end{cases}
\]

where \(\sigma, \alpha, \eta, M\) are positive constants to be determined in the following lemmas.

By the conditions (H1) and (H2), we conclude that there exists constant \(K\) such that \(f(S_0)g(K) = \gamma K\).

**Lemma 2.2.** The function \(\mathcal{I}(\xi)\) satisfies

\[
e^{\mathcal{I}(\xi)} \leq d_2(\mathcal{I} \ast \mathcal{I}(\xi)) + f(S_0)g(K \ast \mathcal{I}(\xi)) - \gamma \mathcal{I}(\xi),\]

(6)

for \(\xi \neq \xi_1 := \frac{1}{\lambda_1} \ln K\).

**Proof.** If \(\xi < \xi_1\), then \(\mathcal{I}(\xi) = e^{\lambda_1 \xi}\). Since \(g(I) \leq g'(0)I\), we conclude that (6) holds.

If \(\xi > \xi_1\), then \(\mathcal{I}(\xi) = K\). From the definition of \(K\), we know that (6) holds. \(\Box\)

**Lemma 2.3.** Assume that \(0 < \alpha < \lambda_1\) is sufficiently small and \(\sigma > S_0\) is sufficiently large. Then the function \(\mathcal{S}(\xi)\) satisfies

\[
e^{\mathcal{S}(\xi)} \leq d_1(\mathcal{I} \ast \mathcal{S}(\xi)) - f(S_0)g(K \ast \mathcal{I}(\xi)),
\]

(7)

for \(\xi \neq \xi_2 := \frac{1}{\alpha} \ln \frac{S_0}{\sigma}\).
Proof. Let $\alpha$ be sufficiently small and $\sigma$ be sufficiently large such that $\xi_2 = \frac{1}{\alpha} \ln \frac{S_0}{\sigma} < \xi_1$.

If $\xi > \xi_2$, then $S(\xi) = 0$. This yields that (7) holds.

If $\xi < \xi_2$, then $S(\xi) = S_0 - \sigma e^{\alpha \xi}$. Since $g(I) \leq g'(0)I$ and $\xi < \xi_2 = \frac{1}{\alpha} \ln \frac{S_0}{\sigma}$, we have

$$e^{S'(\xi)} - d_1(J * S(\xi) - F(\xi)) + f(S(\xi))g(K * T(\xi))$$

$$\leq e^{S'(\xi)} - d_1(J * S(\xi) - F(\xi)) + f(S_0)g'(0)K * T(\xi)$$

$$= -\sigma \alpha + d_1 \sigma \left( \int_{-\infty}^{+\infty} J(x)e^{-\alpha x} dx - 1 \right) + f(S_0)g'(0)e^{(\lambda_1 - \alpha)\xi}\Re(\lambda_1, c)$$

$$\leq -\sigma \alpha + d_1 \sigma \left( \int_{-\infty}^{+\infty} J(x)e^{-\alpha x} dx - 1 \right) + f(S_0)g'(0) \left( \frac{S_0}{\sigma} \right)^{\lambda_1 - \alpha} \Re(\lambda_1, c).$$

Let $\sigma = 1$, $\sigma > S_0$ be sufficiently large and $0 < \alpha < \lambda_1$ be sufficiently small. It is clear that (7) holds. \hfill \Box

Lemma 2.4. Assume that $0 < \eta < \min\{\lambda_2 - \lambda_1, \lambda_1, \frac{\alpha}{\sigma}\}$. Then there exists $M$ such that the function $I(I)$ satisfies

$$cI'(\xi) \leq d_2(J * I(\xi) - I(\xi)) + f(S(\xi))g(K * I(\xi)) - \gamma I(\xi), \quad (8)$$

for $\xi \notin \xi_3 := \frac{1}{\eta} \ln \frac{1}{M}.$

Proof. Let $M$ be sufficiently large such that $\xi_3 < \xi_2 < \xi_1$.

If $\xi > \xi_3$, then $I(I) = 0$ and (8) holds.

If $\xi < \xi_3,$ then $S(\xi) = S_0 - \sigma e^{\alpha \xi}$. Define $h(I) = g(I)/I$ for $I > 0$ and $h(0) = g'(0)$. Using the condition (H2), we conclude that $h$ is non-decreasing and there exists $L_h$ such that $|h'(I)| \leq L_h$ for $I \geq 0$. Denote $L_f = \max_{S \in [0, S_0]} f'(S)$. Since

$$f(S_0) - \sigma e^{\alpha \xi} = f(S_0) + f'(\theta_1(\xi))(-\sigma e^{\alpha \xi}) \geq f(S_0) - L_f \sigma e^{\alpha \xi},$$

$$g'(0) - h(K * I(\xi)) = h(0) - h(K * I(\xi)) = -h'(\theta_2(\xi))K * I(\xi) \leq L_h K * I(\xi),$$

we get

$$cI'(\xi) - d_2(J * I(\xi) - I(\xi)) - f(S(\xi))g(K * I(\xi)) + \gamma I(\xi)$$

$$= cI'(\xi) - d_2(J * I(\xi) - I(\xi)) - f(S(\xi))h(K * I(\xi))K * I(\xi) + \gamma I(\xi)$$

$$= c^\lambda \xi f(S_0)g'(0)\Re(\lambda_1, c) + Me^{(\lambda_1 + \eta)\xi}\Delta(\lambda_1 + \eta, c) - f(S_0)g'(0)\Re(\lambda_1 + \eta, c)$$

$$- f(S(\xi))h(K * I(\xi))\Re(\lambda_1, c) - Me^{(\lambda_1 + \eta)\xi}\Re(\lambda_1 + \eta, c),$$

$$= c^\lambda \xi \Re(\lambda_1, c)|f(S_0)g'(0) - f(S(\xi))h(K * I)|$$

$$- Me^{(\lambda_1 + \eta)\xi}\Re(\lambda_1 + \eta, c)|f(S_0)g'(0) - f(S(\xi))h(K * I)|$$

$$\leq c^\lambda \xi \Re(\lambda_1, c)|f(S_0)L_r \Re(\lambda_1, c)e^{\lambda_1 \xi} + L_f h(0)\sigma e^{\alpha \xi}|$$

$$+ Me^{(\lambda_1 + \eta)\xi}\Delta(\lambda_1 + \eta, c)$$

$$= e^{-(\lambda_1 + \eta)\xi} \{ f(S_0)L_r \Re(\lambda_1, c)e^{(\lambda_1 - \eta)\xi} + L_f h(0)\sigma \Re(\lambda_1, c)e^{(\alpha - \eta)\xi} + M\Delta(\lambda_1 + \eta, c) \}.$$
Let $X > \max\{\xi_1, \xi_2, \xi_3\}$ and define

$$
\Gamma_X = \left\{ (\phi(\cdot), \varphi(\cdot)) \in C([-X, X], \mathbb{R}^2) \mid \begin{array}{l}
\phi(-X) = S(-X), \\
\varphi(-X) = I(-X), \\
\forall \xi \in [-X, X] \end{array} \right\}.
$$

Clearly, $\Gamma_X$ is a closed, bounded and convex subset of $C([-X, X], \mathbb{R}^2)$. For any $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_X$, define

$$
\hat{\phi}(\xi) = \begin{cases}
S(\xi), & \xi \leq -X, \\
\phi(\xi), & -X < \xi < X, \\
\varphi(\xi), & \xi \geq X.
\end{cases}
$$

Consider the initial value problems

\begin{align}
\alpha & \cdot S'(\xi) = d_1 J * \hat{\phi}(\xi) + \alpha_1 \phi(\xi) - f(\phi(\xi))g(K * \hat{\varphi}(\xi)) - (d_1 + \alpha_1)S(\xi), \quad (9) \\
\alpha & \cdot I'(\xi) = d_2 J * \hat{\varphi}(\xi) + f(\phi(\xi))g(K * \hat{\varphi}(\xi)) - (d_2 + \gamma)I(\xi), \quad (10)
\end{align}

with

\begin{align}
S(-X) = S(-X), \quad I(-X) = I(-X),
\end{align}

where $\alpha_1 > L_{fg'(0)}K$. Applying the ODE theory, we obtain that $[9], [11]$ admit a unique solution $S_X(\cdot), I_X(\cdot) \in C^1([-X, X])$. Then, we define an operator $F = (F_1, F_2) : \Gamma_X \rightarrow \Gamma_X$ such that

$$
F_1[\phi, \varphi](\xi) = S_X(\xi), \quad F_2[\phi, \varphi](\xi) = I_X(\xi), \quad \text{for} \ \xi \in [-X, X].
$$

**Lemma 2.5.** The operator $F$ maps $\Gamma_X$ to $\Gamma_X$.

**Proof.** For any given $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_X$, we first show

$$
S(\xi) \leq F_1[\phi, \varphi](\xi) \leq S_0.
$$

It is easy to check that $S_0$ is the upper solution of $[9]$. Thus, $S(\xi) \leq S_0$ for $\xi \in [-X, X]$. Since $\alpha_1 > L_{fg'(0)}K$, we obtain that $\alpha_1 S - f(S)g(I)$ is increasing for $S \geq 0$. By Lemma 2.3 we get

\begin{align}
cS'(\xi) & \leq d_1 J * \hat{\phi}(\xi) - \alpha_1 \phi(\xi) + f(\phi(\xi))g(K * \hat{\varphi}(\xi)) + (d_1 + \alpha_1)S(\xi) \\
& \leq cS'(\xi) - d_1 J * S(\xi) - \alpha_1 S(\xi) + f(S(\xi))g(K * \hat{\varphi}(\xi)) + (d_1 + \alpha_1)S(\xi) \\
& \leq 0,
\end{align}

for $\xi \neq \xi_2$. Therefore, $S_X(\xi) \geq S(\xi)$ for $\xi \in [-X, X]$.

Following, we consider $F_2[\phi, \varphi](\xi)$. It follows from Lemma 2.2 that

\begin{align}
cI'(\xi) & \leq d_2 J * \hat{\varphi}(\xi) - f(\phi(\xi))g(K * \hat{\varphi}(\xi)) + (d_2 + \gamma)I(\xi) \\
& \geq cI'(\xi) - d_2 J * I(\xi) - f(I(\xi))g(K * \hat{\varphi}(\xi)) + (d_2 + \gamma)I(\xi) \\
& \geq 0,
\end{align}

for $\xi \neq \xi_1$. Similarly, by Lemma 2.4 we have

\begin{align}
cI'(\xi) & \leq d_2 J * \hat{\varphi}(\xi) - f(\phi(\xi))g(K * \hat{\varphi}(\xi)) + (d_2 + \gamma)I(\xi) \\
& \leq cI'(\xi) - d_2 J * I(\xi) - f(I(\xi))g(K * I(\xi)) + (d_2 + \gamma)I(\xi) \\
& \leq 0,
\end{align}

for $\xi \neq \xi_3$. Using the Sturm Comparison Theorem, we conclude that $I(\xi) \leq I_X(\xi) \leq I(\xi)$ for $\xi \in [-X, X]$.

Lemma 2.6. The operator $F : \Gamma_X \to \Gamma_X$ is completely continuous.

Proof. From [9]-[11], the unique solution $S_X(\cdot), I_X(\cdot) \in C^1([-X, X])$ and

$$S_X(\xi) = \mathcal{S}(-X)e^{-\frac{d_1+\alpha_1}{c}(\xi+X)} + \frac{1}{c} \int_{-X}^{\xi} e^{-\frac{d_1+\alpha_1}{c}(\xi-\eta)} f_1(\eta) d\eta,$$

$$I_X(\xi) = I(-X)e^{-\frac{d_2}{c}(\xi+X)} + \frac{1}{c} \int_{-X}^{\xi} e^{-\frac{d_2}{c}(\xi-\eta)} f_2(\eta) d\eta,$$

where

$$f_1(\eta) = d_1 J * \hat{\phi}(\eta) + \alpha_1 \phi(\eta) - f(\phi(\eta)) g(K * \hat{\varphi}(\eta)),$$

and

$$f_2(\eta) = d_2 J * \hat{\varphi}(\eta) + f(\phi(\eta)) g(K * \hat{\varphi}(\eta)).$$

Let $(\phi_i(\cdot), \varphi_i(\cdot)) \in \Gamma_X, \ i = 1, 2$. Then, we get

$$|J * \hat{\phi}_1(\eta) - J * \hat{\phi}_2(\eta)| \leq \left| \int_{-\infty}^{+\infty} J(\eta-y)(\hat{\phi}_1(y) - \hat{\phi}_2(y)) dy \right|$$

$$\leq \left| \int_{-X}^{X} J(\eta-y)(\phi_1(y) - \phi_2(y)) dy \right|$$

$$+ \left| \int_{X}^{+\infty} J(\eta-y)(\phi_1(X) - \phi_2(X)) dy \right|$$

$$\leq 2 \max_{\eta \in [-X,X]} |\phi_1(\eta) - \phi_2(\eta)|.$$

Similarly,

$$|J * \hat{\varphi}_1(\eta) - J * \hat{\varphi}_2(\eta)| \leq 2 \max_{\eta \in [-X,X]} |\varphi_1(\eta) - \varphi_2(\eta)|,$$

$$|K * \hat{\phi}_1(\eta) - K * \hat{\phi}_2(\eta)|$$

$$\leq \int_0^T \int_{-\infty}^{+\infty} K(\eta-y-cs,s)|\hat{\phi}_1(\eta) - \hat{\phi}_2(\eta)| dy ds$$

$$\leq \int_0^T \int_{-X}^{X} K(\eta-y-cs,s)|\phi_1(\eta) - \phi_2(\eta)| dy ds$$

$$+ \int_0^T \int_{X}^{+\infty} K(\eta-y-cs,s)|\phi_1(X) - \phi_2(X)| dy ds$$

$$\leq 2 \max_{\eta \in [-X,X]} |\phi_1(\eta) - \phi_2(\eta)| \int_0^T \int_{-X}^{+\infty} K(\eta-y-cs,s) dy ds.$$

By the definition of operator $F$, [12] and [13], we get that $F$ is continuous. Since $S_X(\xi) \leq S_0$ and $I_X(\xi) \leq K$, it follows from [9] and [10] that

$$|cs_X(\xi)| \leq d_1(|J * \hat{\phi}(\xi)| + |S_X(\xi)|) + 2\alpha_1 S_0 + |f(S_0)g'(0)K * \hat{\varphi}(\xi)|$$

$$\leq 2d_1 S_0 + 2\alpha_1 S_0 + f(S_0)g'(0)K,$$

and

$$|cI_X(\xi)| \leq d_2(|J * \hat{\phi}(\xi)| + |I_X(\xi)|) + |f(S_0)g'(0)K * \hat{\varphi}(\xi)| + \gamma |I_X(\xi)|$$

$$\leq d_2 (S_0 + K) + f(S_0)g'(0)K + \gamma K.$$
Theorem 2.7. For the operator \( F : \Gamma_X \rightarrow \Gamma_X \), there exists a fixed point in \( \Gamma_X \).

Proof. It is obvious that \( \Gamma_X \) is closed, bounded and convex. By Schauder’s fixed point theorem, Lemma 2.5 and 2.6 there exists \((S_X(\cdot), I_X(\cdot)) \in \Gamma_X\) such that
\[
(S_X(\xi), I_X(\xi)) = F[S_X, I_X](\xi), \quad \text{for} \quad \xi \in [-X, X].
\]

Define
\[
C^{1,1}([-X, X]) = \left\{ \phi(\xi) \in C([-X, X]) \mid |\phi(\xi) - \phi(\eta)| \leq C|\xi - \eta|, \quad |\phi'(\xi) - \phi'(\eta)| \leq C|\xi - \eta|, \quad \forall \xi, \eta \in [-X, X] \right\}.
\]
where \( C \) is a constant. Then we have the following result.

Lemma 2.8. There exists a constant \( C > 0 \) such that
\[
\|S_X(\xi)\|_{C^{1,1}([-X, X])} \leq C, \quad \|I_X(\xi)\|_{C^{1,1}([-X, X])} \leq C,
\]
for any \( X > \max\{\xi_1, \xi_2, \xi_3\} \).

Proof. Clearly, \((S_X(\xi), I_X(\xi))\) satisfies
\[
cS'_X(\xi) = d_1(\ast \tilde{S}_X(\xi) - S_X(\xi)) - f(S_X(\xi))g(K \ast \tilde{I}_X(\xi)),
\]
\[
cI'_X(\xi) = d_2(\ast \tilde{I}_X(\xi) - I_X(\xi)) + f(S_X(\xi))g(K \ast \tilde{I}_X(\xi)) - \gamma I_X(\xi),
\]
for \( \xi \in (-X, X) \).

From \(14\) and \(15\), we know that there exists a constant \( C_1 \) such that
\[
|S_X(\xi) - S_X(\eta)| \leq C_1 |\xi - \eta| \quad \text{and} \quad |I_X(\xi) - I_X(\eta)| \leq C_1 |\xi - \eta|,
\]
for any \( \xi, \eta \in (-X, X) \).

By \(16\), we have
\[
c |S'_X(\xi) - S'_X(\eta)| \leq d_1 \left| \int_{-\infty}^{+\infty} [J(\xi - y) - J(\eta - y)]\tilde{S}_X(y)dy \right| + d_1 |S_X(\xi) - S_X(\eta)|
\]
\[
+ |f(S_X(\xi))g(K \ast \tilde{I}_X(\xi)) - f(S_X(\eta))g(K \ast \tilde{I}_X(\eta))| := d_1S_1 + d_1S_2 + S_3.
\]

Using the assumption (A1), we obtain
\[
S_1 = \left| \int_{-\infty}^{+\infty} [J(\xi - y) - J(\eta - y)]\tilde{S}_X(y)dy \right|
\]
\[
\leq \left| \int_{-\infty}^{-X} [J(\xi - y) - J(\eta - y)]\tilde{S}(y)dy \right| + \left| \int_{-X}^{X} [J(\xi - y) - J(\eta - y)]S_X(y)dy \right|
\]
\[
+ \left| \int_{-\infty}^{+\infty} [J(\xi - y) - J(\eta - y)]S_X(y)dy \right|
\]
\[
\leq S_0 \left| \int_{\xi + X}^{\eta + X} J(z)dz \right| + \sigma \int_{-\infty}^{-X} |J(\xi - y) - J(\eta - y)| e^{\alpha y}dy
\]
where $L_j$ is the Lipschitz constant of kernel $J$ and $\| J \|_{L^\infty(\mathbb{R})} = \sup_{x \in \mathbb{R}} |J(x)|$.

From (16), we get

$$S_3 = \int_{-\infty}^{\infty} | J(x) | dx = \max_{S \in \mathcal{S}_0} \int_{-\infty}^{\infty} | J(x) | dx$$

By calculation, we have

$$S_3 = \int_{-\infty}^{\infty} | J(x) | dx = \max_{S \in \mathcal{S}_0} \int_{-\infty}^{\infty} | J(x) | dx$$

where $L_f = \max_{x \in \mathcal{S}_0} |J(x)|$.

By calculation, we have

$$S_3 = \int_{-\infty}^{\infty} | J(x) | dx = \max_{S \in \mathcal{S}_0} \int_{-\infty}^{\infty} | J(x) | dx$$

where $L_f = \max_{x \in \mathcal{S}_0} |J(x)|$.

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where $L_f = \max_{x \in \mathcal{S}_0} |J(x)|$.

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where $L_f = \max_{x \in \mathcal{S}_0} |J(x)|$.

By calculation, we have

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where $L_f = \max_{x \in \mathcal{S}_0} |J(x)|$.

By calculation, we have

$$S_3 = \int_{-\infty}^{\infty} | J(x) | dx = \max_{S \in \mathcal{S}_0} \int_{-\infty}^{\infty} | J(x) | dx$$

where $L_f = \max_{x \in \mathcal{S}_0} |J(x)|$.

By calculation, we have

$$S_3 = \int_{-\infty}^{\infty} | J(x) | dx = \max_{S \in \mathcal{S}_0} \int_{-\infty}^{\infty} | J(x) | dx$$

where $L_f = \max_{x \in \mathcal{S}_0} |J(x)|$.

By calculation, we have

$$S_3 = \int_{-\infty}^{\infty} | J(x) | dx = \max_{S \in \mathcal{S}_0} \int_{-\infty}^{\infty} | J(x) | dx$$

where $L_f = \max_{x \in \mathcal{S}_0} |J(x)|$.
Theorem 2.9. According to the above discussion, we show that there exists a nontrivial bounded positive pair of \((S(\xi), I(\xi))\) satisfying (3) and (4). Furthermore, we have
\[
\int_{-\infty}^{+\infty} I(\xi) d\xi = \frac{c(S_0 - S(+\infty))}{\gamma}.
\]

Proof. According to the above discussion, we show that there exists a nontrivial bounded positive pair of \((S(\xi), I(\xi))\) satisfying
\[
c S'\xi = d_1 (J * S(\xi) - S(\xi)) - f(S(\xi)) g(K * I(\xi)),
\]
\[
c I'\xi = d_2 (J * I(\xi) - I(\xi)) + f(S(\xi)) g(K * I(\xi)) - \gamma I(\xi).
\]

Next, we only need to prove that \((S(\xi), I(\xi))\) satisfies (4). By the definition of \(S(\xi), I(\xi)\), it is easy to see that \(S(-\infty) = S_0\) and \(I(-\infty) = 0\). It remains to show that \(S(+\infty) < S_0\) and \(I(+\infty) = 0\).
Integrating (17) from $-Y$ to $Y$, we have
\[
\left| \int_{-Y}^{Y} f(S(\xi))g(K * I(\xi))d\xi \right| \leq d_1 \left| \int_{-Y}^{Y} [J * S(\xi) - S(\xi)]d\xi \right| + c |S(Y) - S(-Y)|
\]
\[
\leq d_1 \left| \int_{-Y}^{Y} [J * S(\xi) - S(\xi)]d\xi \right| + cS_0.
\]

Note that
\[
\left| \int_{-Y}^{Y} [J * S(\xi) - S(\xi)]d\xi \right| = \left| \int_{-Y}^{Y} \int_{-\infty}^{\infty} J(y) [S(\xi - y) - S(\xi)]dyd\xi \right|
\]
\[
= \left| \int_{-\infty}^{\infty} J(y) \int_{-Y}^{Y} [S(\xi - y) - S(\xi)]d\xi dy \right|
\]
\[
= \left| \int_{-\infty}^{\infty} J(y) \left[ \int_{-\infty}^{-Y-y} S(\eta)d\eta - \int_{Y-y}^{\infty} S(\eta)d\eta \right] dy \right|
\]
\[
\leq 4S_0 \int_{0}^{+\infty} |yJ(y)| dy.
\]

Thus, for any constant $Y$, we get
\[
\left| \int_{-Y}^{Y} f(S(\xi))g(K * I(\xi))d\xi \right| \leq \left( 4d_1 \int_{0}^{+\infty} |yJ(y)| dy + c \right) S_0.
\]

Integrating (18) from $-Y$ to $Y$, we obtain
\[
\left| \gamma \int_{-Y}^{Y} I(\xi)d\xi \right| \leq d_2 \left| \int_{-Y}^{Y} [J * I(\xi) - I(\xi)]d\xi \right| + \left| \int_{-Y}^{Y} f(S(\xi))g(K * I(\xi))d\xi \right| + cK
\]
\[
\leq \left( 4d_2 \int_{0}^{+\infty} |yJ(y)| dy + c \right) K + \left( 4d_1 \int_{0}^{+\infty} |yJ(y)| dy + c \right) S_0.
\]

The arbitrariness of $Y$ implies that
\[
\left| \gamma \int_{-Y}^{Y} I(\xi)d\xi \right| < +\infty.
\]

It follows from the proof of Lemma 2.6 that $I(\xi)$ and $I'(\xi)$ are uniformly bounded. Thus, we conclude that $I(+\infty) = 0$.

Since $S(\xi) \leq S_0$, we have $\limsup_{n \to +\infty} S(\xi) \leq S_0$. We claim that $\liminf_{n \to +\infty} S(\xi) < S_0$. Otherwise, $S(+\infty) = S_0$. Integrating (17) from $-Y$ to $Y$, we get
\[
c(S(Y) - S(-Y)) = d_1 \int_{-Y}^{Y} [J * S(\xi) - S(\xi)]d\xi - \int_{-Y}^{Y} f(S(\xi))g(K * I(\xi))d\xi
\]
\[
= d_1 \int_{-\infty}^{+\infty} J(y) \int_{0}^{1} [S(-Y-ty) - S(Y-ty)]dtdy
\]
\[
- \int_{-Y}^{Y} f(S(\xi))g(K * I(\xi))d\xi.
\]

Let $Y \to +\infty$, we have
\[
0 = 0 - \int_{-\infty}^{+\infty} f(S(\xi))g(K * I(\xi))d\xi,
\]
which contradicts $\int_{-\infty}^{+\infty} f(S(\xi))g(K * I(\xi))d\xi > 0$. 


We now prove the existence of \( \lim_{n \to +\infty} S(\xi) \). Otherwise,

\[
\liminf_{n \to +\infty} S(\xi) = m_1 < \limsup_{n \to +\infty} S(\xi) = m_2.
\]

Thus, we can find two point sequences \( \{ \xi_n \} \) and \( \{ \eta_n \} \) such that

\[
\begin{align*}
\lim_{n \to +\infty} S(\xi_n) &= \liminf_{n \to +\infty} S(\xi_n) = m_1, & S'(\xi_n) &= 0, \\
\lim_{n \to +\infty} S(\eta_n) &= \limsup_{n \to +\infty} S(\eta_n) = m_2, & S'(\eta_n) &= 0.
\end{align*}
\]

Therefore, \( \{ \xi_n \} \) yields that

\[
0 = cS'(\xi_n) = d_1 [ J * S(\xi_n) - S(\xi_n) ] - f(S(\xi_n))g(K * I(\xi_n)).
\]

Letting \( n \to +\infty \), we have \( \lim J * S(\xi_n) = \lim S(\xi_n) = m_1 \). We prove that \( S(\xi_n + z) \to m_1 \) as \( n \to +\infty \) for any \( z \in [-L, L] \) with any \( L > 0 \). Choose a sufficiently small \( \varepsilon > 0 \), let \( \bar{S}_n(z) = S(\xi_n + z) \) and \( I_\varepsilon = [-L, L] \cap \{ z : \liminf_{n \to +\infty} \bar{S}_n(z) > m_1 + \varepsilon \} \). Hence, we have

\[
m_1 = \lim_{n \to +\infty} \lim_{L \to +\infty} J * S(\xi_n) = \lim_{n \to +\infty} \lim_{L \to +\infty} \int_{-L}^{L} J(z) \bar{S}_n(z) dz
\]

\[
= \lim_{L \to +\infty} \lim_{n \to +\infty} \int_{-L}^{L} J(z) \bar{S}_n(z) dz
\]

\[
\geq \lim_{L \to +\infty} \liminf_{n \to +\infty} \int_{[-L, L] \setminus I_\varepsilon} J(z) \bar{S}_n(z) dz + \lim_{L \to +\infty} \liminf_{n \to +\infty} \int_{I_\varepsilon} J(z) \bar{S}_n(z) dz
\]

\[
\geq m_1 \lim_{L \to +\infty} \int_{[-L, L] \setminus I_\varepsilon} J(z) dz + (m_1 + \varepsilon) \lim_{L \to +\infty} \int_{I_\varepsilon} J(z) dz
\]

\[
= m_1 \lim_{L \to +\infty} \int_{[-L, L]} J(z) dz + \varepsilon \lim_{L \to +\infty} \int_{I_\varepsilon} J(z) dz
\]

\[
= m_1 + \varepsilon \int_{I_\varepsilon} J(z) dz,
\]

where \( \mu \) denotes the measure. This yields that \( \mu(I_\varepsilon) = 0 \). Thus, we have \( S(\xi_n + z) \to m_1 \) for almost everywhere in \([-L, L]\) as \( n \to +\infty \). Since \( \{ \bar{S}_n(z) \} \) is equicontinuous, the convergence is everywhere in \([-L, L]\). Using the similar arguments, we conclude that \( S(\eta_n + z) \to m_2 \) as \( n \to +\infty \) for any \( z \in [-L, L] \).

Integrating \( \{ \xi_n \} \) from \( \xi_n \) to \( \eta_n \), we get

\[
c(S(\eta_n) - S(\xi_n)) = d_1 \int_{\xi_n}^{\eta_n} \left[ J * S(\xi) - S(\xi) \right] d\xi - \int_{\xi_n}^{\eta_n} f(S(\xi))g(K * I(\xi)) d\xi.
\]

Letting \( n \to +\infty \), we have

\[
0 < c(m_2 - m_1) \leq d_1 \lim_{n \to +\infty} \lim_{L \to +\infty} \int_{-L}^{L} \int_{0}^{1} J(y) [S(\xi_n - ty) - S(\eta_n - ty)] dy dy
\]

\[
= (m_1 - m_2) \lim_{L \to +\infty} \int_{-L}^{L} y J(y) dy = 0.
\]

This a contradiction. Thus, \( \liminf_{\xi \to +\infty} S(\xi) = \limsup_{\xi \to +\infty} S(\xi) \), which implies that

\[
\lim_{\xi \to +\infty} S(\xi) = \lim_{\xi \to +\infty} S(\xi) = S(+) < S_0.
\]
Integrating (17) from $-\infty$ to $+\infty$, we obtain
\[ c(S(+) - S_0) = d_1 \int_{-\infty}^{+\infty} [J * S(\xi) - S(\xi)]d\xi - \int_{-\infty}^{+\infty} f(S(\xi))g(K * I(\xi))d\xi. \]

Using Fubini’s theorem, we have
\[
\int_{-\infty}^{+\infty} [J * S(\xi) - S(\xi)]d\xi = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [J(y)S(\xi - y) - S(\xi)]dyd\xi
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J(y)y \int_{0}^{1} S'(\xi - ty)dt dyd\xi
= \int_{-\infty}^{+\infty} J(y)y \int_{0}^{1} [S(+) - S_0]dy dy = 0.
\]

Therefore, we get
\[ c(S_0 - S(+) = \int_{-\infty}^{+\infty} f(S(\xi))g(K * I(\xi))d\xi. \]

Integrating (18) from $-\infty$ to $+\infty$, we obtain
\[ 0 = \int_{-\infty}^{+\infty} f(S(\xi))g(K * I(\xi))d\xi - \int_{-\infty}^{+\infty} \gamma I(\xi)d\xi. \]

This yields that
\[ \int_{-\infty}^{+\infty} I(\xi)d\xi = \frac{c(S_0 - S(+)\gamma}{\gamma}. \]

3. **Nonexistence of traveling waves.** In this section, we study the nonexistence of traveling wave solutions of (3).

**Theorem 3.1.** Assume that $R_0 \leq 1$. For any $c > 0$, system (3) has no nontrivial bounded positive solution satisfying (4).

**Proof.** Suppose that $(S(\xi), I(\xi))$ is a traveling wave solution of (3) satisfying (4).

Integrating (18) from $-\infty$ to $+\infty$, we have
\[ 0 = d_2 \int_{-\infty}^{+\infty} [J * I(\xi) - I(\xi)]d\xi + \int_{-\infty}^{+\infty} f(S(\xi))g(K * I(\xi))d\xi - \gamma \int_{-\infty}^{+\infty} I(\xi)d\xi. \]

Since
\[ \int_{-\infty}^{+\infty} [J * I(\xi) - I(\xi)]d\xi = 0 \]

and
\[ f(S(\xi)) \leq f(S_0), \quad g(I) \leq g'(0)I, \]

we get
\[
\gamma \int_{-\infty}^{+\infty} I(\xi)d\xi = \int_{-\infty}^{+\infty} f(S(\xi))g(K * I(\xi))d\xi
< f(S_0)g'(0) \int_{-\infty}^{+\infty} K * I(\xi)d\xi
= f(S_0)g'(0) \int_{-\infty}^{+\infty} I(\xi)d\xi.
\]
It follows from (18) that
\[ h > \gamma \int_{-\infty}^{+\infty} I(\xi) d\xi. \]

It is a contradiction. \( \square \)

**Theorem 3.2.** If \( R_0 > 1 \) and \( 0 < c < c^* \), then system [3] has no nontrivial bounded positive solution satisfying [4].

**Proof.** Assume that \( (S(\xi), I(\xi)) \) is a solution of [3] satisfying [4]. By [4] and the definition of \( h(I) \), we have \( f(S(\xi))h(K * I(\xi)) \to f(S_0)g'(0) \) as \( \xi \to -\infty \). Using the continuity on \( \xi \), there exists \( \xi_* \) such that
\[ f(S(\xi))g(K * I(\xi)) > \frac{f(S_0)g'(0) + \gamma}{2} K * I(\xi), \text{ for } \xi < \xi_. \]

It follows from (18) that
\[ cI'(\xi) > d_2(J * I(\xi) - I(\xi)) + \frac{f(S_0)g'(0) + \gamma}{2} (K * I(\xi) - I(\xi)) \]
\[ + \frac{f(S_0)g'(0) - \gamma}{2} I(\xi). \tag{19} \]

Integrating (19) from \( -\infty \) to \( \xi < \xi_* \), we get
\[ c(I(\xi) - I(-\infty)) > d_2 \int_{-\infty}^{\xi} [J * I(\eta) - I(\eta)] d\eta + \frac{f(S_0)g'(0) - \gamma}{2} \int_{-\infty}^{\xi} I(\eta) d\eta \]
\[ + \frac{f(S_0)g'(0) + \gamma}{2} \int_{-\infty}^{\xi} [K * I(\eta) - I(\eta)] d\eta. \tag{20} \]

Let \( H(\xi) = \int_{-\infty}^{\xi} I(\eta) d\eta, \xi \in \mathbb{R} \). Hence, \( H(\xi) \) is non-decreasing, \( H(-\infty) = 0 \) and \( H(\xi) \leq \frac{c(S_0 - S(\xi^+) - \gamma)}{\gamma} \).

By computation, we obtain
\[ \int_{-\infty}^{\xi} J * I(\eta) d\eta = \int_{-\infty}^{\xi} \int_{-\infty}^{+\infty} J(y) I(\eta - y) dy d\eta \]
\[ = \int_{-\infty}^{+\infty} J(y) \int_{-\infty}^{\xi} I(\eta - y) dy d\eta \]
\[ = \int_{-\infty}^{+\infty} J(y) H(\xi - y) dy = J * H(\xi). \]

Similarly,
\[ \int_{-\infty}^{\xi} K * I(\eta) d\eta = \int_{-\infty}^{\xi} \int_{0}^{+T} K(y, s) I(\eta - y - cs) dy ds d\eta \]
\[ = \int_{0}^{+T} \int_{-\infty}^{+\infty} K(y, s) \int_{-\infty}^{\xi} I(\eta - y - cs) dy ds d\eta \]
\[ = \int_{0}^{+T} \int_{-\infty}^{+\infty} K(y, s) H(\xi - y - cs) dy ds = K * H(\xi). \]

Then, (20) becomes
\[ \frac{f(S_0)g'(0) - \gamma}{2} H(\xi) \leq cI(\xi) - d_2[J * H(\xi) - H(\xi)] \]
\[ - \frac{f(S_0)g'(0) + \gamma}{2} [K * H(\xi) - H(\xi)]. \tag{21} \]
Integrating (21) from $-\infty$ to $\xi < \xi_*$, we have
\[
\frac{f(S_0)g'(0) - \gamma}{2} \int_{-\infty}^{\xi} H(\eta) d\eta \leq c \int_{-\infty}^{\xi} I(\eta) d\eta - d_2 \int_{-\infty}^{\xi} [J \ast H(\eta) - H(\eta)] d\eta
\]
\[
- \frac{f(S_0)g'(0) + \gamma}{2} \int_{-\infty}^{\xi} [K \ast H(\eta) - H(\eta)] d\eta.
\]
By calculation, we get
\[
\int_{-\infty}^{\xi} [J \ast H(\eta) - H(\eta)] d\eta = \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} J(y)[H(\eta - y) - H(\eta)] dy d\eta
\]
\[
= \int_{-\infty}^{\xi} J(y) \int_{-\infty}^{\xi} [H(\eta - y) - H(\eta)] dy d\eta
\]
\[
= - \int_{-\infty}^{\xi} \int_{0}^{1} yJ(y)H(\xi - \theta y) d\theta dy.
\]
Similarly,
\[
\int_{-\infty}^{\xi} [K \ast H(\eta) - H(\eta)] d\eta = - \int_{0}^{T} \int_{-\infty}^{\xi} \int_{0}^{1} (y + cs)K(y, s)H(\xi - \theta(y + cs)) d\theta dy ds.
\]
Thus, (22) is equivalent to
\[
\frac{f(S_0)g'(0) - \gamma}{2} \int_{-\infty}^{\xi} H(\eta) d\eta
\]
\[
\leq cH(\xi) + d_2 \int_{-\infty}^{\xi} \int_{0}^{1} yJ(y)H(\xi - \theta y) d\theta dy
\]
\[
+ \frac{f(S_0)g'(0) + \gamma}{2} \int_{0}^{T} \int_{-\infty}^{\xi} \int_{0}^{1} (y + cs)K(y, s)H(\xi - \theta(y + cs)) d\theta dy ds.
\]
Since $yH(\xi - \theta y)$ is monotone decreasing with respect to $\theta$ for $\theta \in [0,1]$, we have $yH(\xi - \theta y) \leq yH(\xi)$ and $(y + cs)H(\xi - \theta(y + cs)) \leq (y + cs)H(\xi)$. Then, we obtain
\[
\frac{f(S_0)g'(0) - \gamma}{2} \int_{-\infty}^{\xi} H(\eta) d\eta \leq cH(\xi) + d_2 \int_{-\infty}^{\xi} yJ(y)H(\xi) dy
\]
\[
+ \frac{f(S_0)g'(0) + \gamma}{2} \int_{0}^{T} \int_{-\infty}^{\xi} (y + cs)K(y, s)H(\xi) dy ds
\]
\[
= \left[ c + \frac{f(S_0)g'(0) + \gamma}{2} \int_{0}^{T} \int_{-\infty}^{\xi} csK(y, s) dy ds \right] H(\xi)
\]
\[
:= LH(\xi).
\]
It follows from $H(\xi)$ is nondecreasing that
\[
\int_{-\infty}^{\xi} H(\eta) d\eta = \int_{0}^{+\infty} H(\xi - \eta) d\eta \geq \int_{0}^{\omega} H(\xi - \eta) d\eta \geq \omega H(\xi - \omega), \quad \text{for } \omega > 0.
\]
Therefore, we have $\frac{f(S_0)g'(0) - \gamma}{2} \omega H(\xi - \omega) \leq LH(\xi)$ for $\xi < \xi_*$. Thus, there exists some $\omega$ such that $H(\xi - \omega) \leq H(\xi)$. Define $P(\xi) = H(\xi) e^{-\mu_0 \xi}$ for $0 < \mu_0 < \lambda_c$. Since $P(\xi - \omega) = H(\xi - \omega) e^{-\mu_0 (\xi - \omega)} \leq \frac{H(\xi)}{2} e^{-\mu_0 (\xi - \omega)} = \frac{e^{\mu_0 \omega}}{2} P(\xi)$, we choose $\mu_0 < \min\{\lambda_c, \frac{\ln 2}{\omega}, \alpha\}$ such that $P(\xi - \omega) < P(\xi)$. The boundness of $H(\xi)$ implies that $P(\xi)$ is bounded as $\xi \to -\infty$. There
exists a constant $H_0$ such that $P(\xi) = H(\xi) e^{-\mu_0 \xi} \leq H_0$. This yields that $H(\xi) \leq H_0 e^{\mu_0 \xi}$ for $\xi \in \mathbb{R}$. By the definition of $H(\xi)$, we have
\[
\sup_{\xi \in \mathbb{R}} \{I(\xi) e^{-\mu_0 \xi}\} < +\infty.
\]
Hence,
\[
J * I(\xi) e^{-\mu_0 \xi} = e^{-\mu_0 \xi} \int_{-\infty}^{+\infty} J(y) e^{\mu_0 (\xi - y)} I(\xi - y) e^{-\mu_0 (\xi - y)} dy \\
\leq \sup_{\xi \in \mathbb{R}} \{I(\xi) e^{-\mu_0 \xi}\} \int_{-\infty}^{+\infty} J(y) e^{-\mu_0 y} dy < +\infty,
\]
and
\[
K * I(\xi) e^{-\mu_0 \xi} = e^{-\mu_0 \xi} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} K(y, s) e^{\mu_0 (\xi - y - cs)} I(\xi - y - cs) e^{-\mu_0 (\xi - y - cs)} dy ds \\
\leq \sup_{\xi \in \mathbb{R}} \{I(\xi) e^{-\mu_0 \xi}\} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} K(y, s) e^{\mu_0 (y + cs)} dy ds < +\infty.
\]
For $\lambda \in \mathbb{C}$ with $0 < \text{Re}\lambda < \mu_0$, we can define a two-sided Laplace transform of $I(\xi)$ by
\[
L(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda \xi} I(\xi) d\xi.
\]
The second equation of \footnotesize{(3)} is equivalent to
\[
c' I(\xi) = d_2 (J * I(\xi) - I(\xi)) + f(S_0) g'(0) K * I(\xi) - \gamma I(\xi) \\
+ [f(S(\xi)) g(K * I(\xi)) - f(S_0) g'(0) K * I(\xi)]. \tag{25}
\]
Define
\[
F(\xi) = f(S_0) g'(0) K * I(\xi) - f(S(\xi)) g(K * I(\xi)).
\]
Note that $h(I) = \frac{d_1}{K}$ for $I > 0$ and $h(0) = g'(0)$. The condition (H2) means that $h$ is non-increasing and there exists $L_h$ such that $|h'(I)| \leq L_h$ for $I \geq 0$. Denote $L_f = \max_{S \in [0, S_0]} f'(S)$. Thus, we obtain
\[
F(\xi) = [f(S_0) h(0) - f(S(\xi)) h(K * I(\xi))] K * I(\xi) \\
= [f(S_0) h(0) - h(K * I(\xi))] + h(K * I(\xi))(f(S_0) - f(S(\xi))) K * I(\xi) \\
\leq [f(S_0) L_h K * I(\xi) + h(0) L_f (S_0 - S(\xi))] K * I(\xi). \tag{26}
\]
Next, we prove that there exists $\nu > 0$ such that $\sup_{\xi \in \mathbb{R}} \{(S_0 - S(\xi)) e^{-\nu \xi}\} < +\infty$. Let $U(\xi) = S_0 - S(\xi)$, we have $0 \leq U(\xi) \leq S_0$ for any $\xi \in \mathbb{R}$ and $\lim_{\xi \to -\infty} U(\xi) = 0$. According to the first equation of \footnotesize{(3)}, we get
\[
c U'(\xi) = d_1 (J * U(\xi) - U(\xi)) - f(S(\xi)) g(K * I(\xi)). \tag{27}
\]
Let $\chi \in C^\infty(\mathbb{R}, [0, 1])$ be a nonnegative nondecreasing function satisfying $\chi = 0$ for $\xi \leq -2$ and $\chi = 1$ for $\xi \geq -1$. For $N \in \mathbb{N}$, set $\chi_N(\xi) = \chi(\frac{\xi}{N})$. Then, taking $0 < \nu < \mu_0$ and multiplying \footnotesize{(27)} by $e^{-\nu \xi} \chi_N(\xi)$ and integrating over $\mathbb{R}$, we have
\[
c \int_{-\infty}^{+\infty} U'(\xi) e^{-\nu \xi} \chi_N d\xi = d_1 \int_{-\infty}^{+\infty} (J * U(\xi) - U(\xi)) e^{-\nu \xi} \chi_N d\xi \tag{28}
\]
In addition, we have
\[
\int_{-\infty}^{\infty} f(S(\xi)) g(K * I(\xi)) e^{-\nu \xi} \chi_N d\xi.
\]
Notice that for some constant \( r \), we obtain
\[
\int_{-\infty}^{\infty} J * U(\xi) e^{-\nu \xi} \chi_N d\xi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(\xi - y) U(y) e^{-\nu \xi} \chi_N dy d\xi
\]
\[
= \int_{-\infty}^{\infty} U(y) e^{-\nu y} \int_{-\infty}^{\infty} J(z) e^{-\nu z} \chi_N(z + y) dz dy
\]
\[
= \int_{-\infty}^{\infty} U(y) e^{-\nu y} \left\{ \int_{-\infty}^{r} J(z) e^{-\nu z} \chi_N(z + y) dz + \int_{r}^{+\infty} J(z) e^{-\nu z} \chi_N(z + y) dz \right\}. 
\]
Choosing some constant \( 1 < k_1 < 1 + \frac{a_0}{d_1} \), it is obvious that \( \int_{r}^{+\infty} J(z) dz < k_1 \). By the continuity, there exists some \( \nu_0 > 0 \) such that
\[
\int_{r}^{+\infty} J(z) e^{-\nu z} dz < k_1 e^{\nu r}, \quad \text{for any } 0 < \nu < \nu_0.
\]
Let \( k_2 = \int_{-\infty}^{r} J(z) e^{-\nu z} dz \), we get
\[
\int_{-\infty}^{\infty} (J * U(\xi) - U(\xi)) e^{-\nu \xi} \chi_N d\xi \leq \int_{-\infty}^{\infty} U(y) e^{-\nu y} (k_1 e^{\nu r} + k_2 - \chi_N(y)) dy. \tag{29}
\]
In addition, we have
\[
\int_{-\infty}^{\infty} U'(\xi) e^{-\nu \xi} \chi_N d\xi = \nu \int_{-\infty}^{\infty} U(\xi) e^{-\nu \xi} \chi_N d\xi - \int_{-\infty}^{\infty} U(\xi) e^{-\nu \xi} \chi'_N d\xi, \tag{30}
\]
and for some constant \( M_1 > 0 \) independent of \( N \)
\[
\int_{-\infty}^{\infty} f(S(\xi)) g(K * I(\xi)) e^{-\nu \xi} \chi_N d\xi \leq M_1.
\]
Using (28), (29) and (30), we obtain
\[
M_1 \geq (c\nu + d_1) \int_{-\infty}^{\infty} U(y) e^{-\nu y} \chi_N dy - d_1 (k_1 e^{\nu r} + k_2) \int_{-\infty}^{\infty} U(y) e^{-\nu y} dy 
\]
\[
- c \int_{-\infty}^{\infty} U(\xi) e^{-\nu \xi} \chi'_N d\xi.
\]
Then, we can choose \( \nu < \min\{\mu_0, \nu_0\} \) and \( r \) small enough such that
\[
b = c\nu + d_1 - d_1 (k_1 e^{\nu r} + k_2) > 0.
\]
Due to \( \chi'_N \to 0 \) and \( \chi_N \to 1 \) as \( N \to \infty \), we have
\[
\int_{-\infty}^{\infty} U(y) e^{-\nu y} dy \leq M_1 b^{-1}.
\]
This implies that \( S_0 - S(\xi) \leq M_1 b^{-1} e^{\nu \xi} \). It follows from (26) that
\[
F(\xi) \leq \left[ f(S_0) L_b K * I(\xi) + h(0) L_f M_1 b^{-1} e^{\nu \xi} \right] K * I(\xi).
\]
Denote $Q(\xi) = f(S_0) L_h K \ast I(\xi) + h(0) L_f M b^{-1} e^{\nu \xi}$ for $\xi \leq \xi_1$. In view of (23) and $\mu_0 < \nu$, we obtain

$$
\sup_{\xi \leq \xi_1} \{ Q(\xi) e^{-\mu_0 \xi} \} \leq f(S_0) L_h \sup_{\xi \leq \xi_1} \{ K \ast I(\xi) e^{-\mu_0 \xi} \} + h(0) L_f M b^{-1} \sup_{\xi \leq \xi_1} \{ e^{(\nu - \mu_0) \xi} \} < +\infty. 
$$

(31)

Therefore,

$$
\sup_{\xi \leq \xi_1} \{ F(\xi) e^{-(\mu_0 + \mu_0^*) \xi} \} \leq \sup_{\xi \leq \xi_1} \{ R(\xi) e^{-\mu_0 \xi} \} \sup_{\xi \leq \xi_1} \{ K \ast I(\xi) e^{-\mu_0 \xi} \} < +\infty.
$$

Since $F(\xi)$ is bounded, we get

$$
\sup_{\xi \in \mathbb{R}} \{ F(\xi) e^{-(\mu_0 + \mu_0^*) \xi} \} < +\infty. 
$$

(32)

Now, we take two-sided Laplace transform on (25) and get

$$
\Delta(\lambda, c) L(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda \xi} \left[ f(S_0) g'(0) K \ast I(\xi) - f(S(\xi)) g(K \ast I(\xi)) \right] d\xi,\tag{33}
$$

where $L(\lambda)$ is defined by (24) for $\lambda \in \mathbb{C}$ with $0 < \Re \lambda < \mu_0$. It follows from (32) that the right-hand side of (33) is defined for $\lambda \in \mathbb{C}$ with $0 < \Re \lambda < w_0 + \mu_0^*$. By the definition of $\Delta(\lambda, c)$, we conclude that $\Delta(\lambda, c) > 0$ for $0 < c < c^*$ and $0 < \lambda < \lambda_c$. Hence, $L(\lambda)$ is well-defined for all $0 < \Re \lambda < \min\{\mu_0 + \mu_0^*, \lambda_c\}$.

Next, we show that $L(\lambda)$ is well-defined for all $0 < \Re \lambda < \lambda_c$. Applying the property of Laplace transforms, we know that either there exists a real number $\eta$ such that $L(\lambda)$ is analytic for $\lambda \in \mathbb{C}$ with $0 < \Re \lambda < \eta$ and $\lambda = \zeta$ is a singular point of $L(\lambda)$, or $L(\lambda)$ is well defined for $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$. If $\xi = \infty$, then $L(\lambda)$ is well-defined for all $0 < \Re \lambda < \lambda_c$. If $\mu_0 + \mu_0^* < \zeta < \lambda_c$, then for any $0 < k < 1$ and $\zeta_1 = \mu_0 + \mu_0^* + k(\zeta - \mu_0 - \mu_0^*) < \zeta$, we obtain

$$
\int_{-\infty}^{+\infty} e^{-\xi_1} I(\xi) d\xi < +\infty.
$$

This means that

$$
\sup_{\xi \in \mathbb{R}} \{ I(\xi) e^{-\xi_1} \} < +\infty.
$$

Similar to the argument of (23), we have

$$
\sup_{\xi \in \mathbb{R}} \{ K \ast I(\xi) e^{-\xi_1} \} < +\infty.
$$

Repeating the process of (31) to (32), we get

$$
\sup_{\xi \in \mathbb{R}} \{ F(\xi) e^{-(\xi_1 + \mu_0^*)} \} < +\infty.
$$

Hence, the right-hand side of (33) is defined for $\lambda \in \mathbb{C}$ with $0 < \Re \lambda < \zeta_1 + \mu_0^*$. It gives that $L(\lambda)$ is defined for $0 < \Re \lambda < \min\{\zeta_1 + \mu_0^*, \lambda_c\}$. When $1 > k > \max\{0, \frac{c^* - 2\mu_0^* - \mu_0^*}{\zeta_1 + \mu_0^* - \mu_0^*}\}$, we have $\zeta_1 + \mu_0^* > \zeta$. So, we know that $\min\{\zeta_1 + \mu_0^*, \lambda_c\} > \zeta$, which contradicts to the singularity of $\zeta$. Thus, we conclude that $L(\lambda)$ is well-defined for all $0 < \Re \lambda < \lambda_c$.

Also, (33) can be re-written as

$$
\int_{-\infty}^{+\infty} e^{-\lambda \xi} \left[ \Delta(\lambda, c) I(\xi) - f(S_0) g'(0) K \ast I(\xi) + f(S(\xi)) g(K \ast I(\xi)) \right] d\xi = 0,
$$

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where $0 < \lambda < \lambda_c$. By the definition of $\Delta(\lambda, c) > 0$ and conditions (A1) and (A2), we obtain $\Delta(\lambda, c) \to +\infty$ as $\lambda \to \lambda_c^-$. This yields that the integral function is positive for sufficiently large $\lambda$. Hence, it is impossible that the integral is zero. □

4. Examples. In this section, we present some examples to illustrate the theoretical results.

**Example 1.** Consider $f(S) = \beta S$ and $g(I) = \frac{I}{1 + \alpha I}$, system (2) reduces to

\[
\begin{align*}
\frac{\partial S(x, t)}{\partial t} &= d_1(J * S(x, t) - S(x, t)) - \frac{\beta S(x, t)(K * I)(x, t)}{1 + (K * I)(x, t)^2}, \\
\frac{\partial I(x, t)}{\partial t} &= d_2(J * I(x, t) - I(x, t)) + \frac{\beta S(x, t)(K * I)(x, t)}{1 + (K * I)(x, t)^2} - \gamma I(x, t), \\
\frac{\partial R(x, t)}{\partial t} &= d_3(J * R(x, t) - R(x, t)) + \gamma I(x, t).
\end{align*}
\]

Applying Theorems 2.9, 3.1 and 3.2 we have the following theorem.

**Theorem 4.1.** There exists a constant $c^* > 0$ such that

(i) if $R_0 = \frac{\beta S_0}{\gamma} > 1$ and $c > c^*$, then system (34) has a traveling wave solution satisfying $S(-\infty) = S_0, S(+\infty) < S_0, I(\pm \infty) = 0$;

(ii) if $R_0 \leq 1$ or $R_0 > 1$ and $0 < c < c^*$, then system (34) has no nontrivial positive traveling wave solution satisfying $S(-\infty) = S_0, S(+\infty) < S_0, I(\pm \infty) = 0$.

**Remark 1.** For the case $K(x, t) = \delta(x)\delta(t - \tau)$, Zhang and Wu [3] established the traveling wave solutions of system (2) with Laplacian diffusion. When $f(S)g(K * I(x, t)) = \beta S \int_0^h k(\tau)g(I(x, t - \tau))d\tau$, Bai and Zhang [4] considered the traveling wave solutions of system (2) with Laplacian diffusion.

**Example 2.** Consider $f(S) = \beta S$ and $g(I) = \frac{I}{1 + \alpha I^2}$, system (2) becomes to

\[
\begin{align*}
\frac{\partial S(x, t)}{\partial t} &= d_1(J * S(x, t) - S(x, t)) - \frac{\beta S(x, t)(K * I)(x, t)}{1 + \alpha((K * I)(x, t))^2}, \\
\frac{\partial I(x, t)}{\partial t} &= d_2(J * I(x, t) - I(x, t)) + \frac{\beta S(x, t)(K * I)(x, t)}{1 + \alpha((K * I)(x, t))^2} - \gamma I(x, t), \\
\frac{\partial R(x, t)}{\partial t} &= d_3(J * R(x, t) - R(x, t)) + \gamma I(x, t).
\end{align*}
\]

By Theorems 2.9, 3.1 and 3.2 we obtain the following results.

**Theorem 4.2.** There exists a constant $c^* > 0$ such that

(i) if $R_0 = \frac{\beta S_0}{\gamma} > 1$ and $c > c^*$, then system (35) has a traveling wave solution satisfying $S(-\infty) = S_0, S(+\infty) < S_0, I(\pm \infty) = 0$;

(ii) if $R_0 \leq 1$ or $R_0 > 1$ and $0 < c < c^*$, then system (35) has no nontrivial positive traveling wave solution satisfying $S(-\infty) = S_0, S(+\infty) < S_0, I(\pm \infty) = 0$.

5. Conclusion. In this paper, we have investigated the existence and nonexistence of traveling wave solutions for a nonlocal dispersal SIR model with general nonlinear incidence rate and spatio-temporal delay. From Theorems 2.9, 3.1 and 3.2 we conclude that whether the disease can spread or not depends on $R_0$ and $c^*$. 
Here the minimal wave speed $c^*$ is determined by the following equations

$$\Delta(\lambda, c) = d_2 \left( \int_{-\infty}^{+\infty} J(x)e^{-\lambda x} dx - 1 \right) - c\lambda + f(S_0)g'(0)\mathcal{R}(\lambda, c) - \gamma = 0,$$

$$\frac{\partial \Delta(\lambda, c)}{\partial \lambda} = -d_2 \int_{-\infty}^{+\infty} xJ(x)e^{-\lambda x} dx - c(1 + f(S_0)g'(0) \int_0^T \int_{-\infty}^{+\infty} tK(x,t)dxdt) = 0,$$

where

$$\mathcal{R}(\lambda, c) = \int_0^T \int_{-\infty}^{+\infty} K(x,t)e^{-\lambda(x+ct)}dxdt.$$ 

For the case $f(S) = \beta S$ and $g(I) = I$, system (1) is similar to that considered by Cheng and Yuan [10]. The model could also be improved by further generalizing the incidence rate, for example, according to Korobeinikov [23], Huang and Takeuchi [21]. Furthermore, we can study the asymptotic speed of propagation, the uniqueness and stability of traveling wave solutions. We leave this for future work.

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E-mail address: jlzhou@amss.ac.cn
E-mail address: yangyyj@126.com