Towards OPE based local quark-hadron duality: Light-quark channels

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Abstract

Various light-quark channel current-current correlators are subjected to
the concept of a non-perturbative component of coarse graining in opera-
tor product expansions introduced in a parallel work. This procedure allows
for low-energy structure of the OPE-derived spectral function. With naive
vacuum saturation for 4 quark operators and using lattice data for the gauge
invariant scalar quark correlator the results are far off the experimentally mea-
sured behavior. However, using the correlation length of the gauge invariant
vector quark correlator, which is about 10 times smaller than the scalar one,
the qualitative results are rather realistic. Namely, the input of information
on the mass of the lowest resonance in one channel yields the correspond-
ing masses within acceptable errors in other channels. Still, the shapes of
the calculated spectral functions are considerably deformed as compared to
experiment. This may be a consequence of vacuum saturation and the trunca-
tion at a mass dimension which is below the critical dimension from which on
the asymptotic expansion does not approximate anymore. To improve on this
high-resolution lattice information on gauge invariant $n > 2$ point correlators
would be needed. Motivated by the small effective correlation length in the 4
quark contributions the relevance of the approach for heavy quark physics, in
particular in the calculation of nonleptonic, inclusive $\Delta\Gamma$, is discussed.
1 Introduction

The basic assumption made in QCD sum rules (QSR) in particular [1] and in applications of the operator product expansion (OPE) to hadronic physics in general [2] is quark-hadron duality, namely the possibility to express the low-energy dispersive part of a correlator of QCD currents in terms of hadronic cross sections. For external momentum \( q \) with \( q^2 = -Q^2 < 0 \) correlators are believed to be (asymptotically) approximated by the OPE which is a power series in \( Q^{-2} \). A first estimate for the critical dimension \( d_c \sim 12 \) from which on the entire coefficients, being products of Wilson coefficients and averages over local, gauge invariant operators, are dominated by short distance effects was performed in ref. [1] for pure gluodynamics. Thereby, the vacuum was described by a dilute instanton gas. Powers with \( d > d_c \) have no potential to reflect the hadronic properties of the respective channel which relate to the operator averages via dispersion relations in QSR (global duality) and via the optical theorem and analytical continuation in applications such as calculations of nonleptonic decay widths of heavy mesons (local duality).

Apart from some puzzles\(^1\) the tremendous success of the sum rule method seems to support the global version of quark-hadron duality.

The observation that an analytical continuation of the OPE to time-like, external momenta does not yield the phenomenological resonance structure in the imaginary part has lead to the notion of local duality violation [2, 4, 5]. The up-to-date claim is that the asymptotic nature of the expansion does not allow for additive, exponential-like terms to appear. These terms are to be responsible for a “wiggling” of the low-energy part of the spectral function [2]. This, however, seems to contradict the QSR philosophy [1]: If only the first few terms of the OPE (up to the critical dimension \( d_c \)) are needed to describe the lowest resonances in a given channel via a dispersion relation then operators of dimension \( d \geq d_c \) should also be forgotten in an OPE-based construction of the spectral function.

Therefore, we propose an alternative in [6]. The observation is that the non-perturbative behavior of the QCD vacuum, when probed with low euclidean momenta, is not only characterized by nonvanishing vacuum expectation values (VEV’s) of local, gauge invariant operators but also by finite correlation lengths of the corresponding gauge invariant correlators [7, 8]. As a consequence the fundamental field operators must loose their relevance with decreasing resolution (see next section).

From the knowledge of gauge invariant correlation functions of fundamental operators at high resolution \( Q_0 \) evolution equations for the VEV’s \( A(Q) \) of the relevant local, effective operators can be derived for \( Q < Q_0 \). For \( d = 4 \), where 2-point functions are needed, we have \( A(Q) \sim \exp[4/5(\lambda Q)^{-1}] \) with \( \lambda \) being the correlation length. Compared with the conventional power correction the effect becomes noti-

\(^1\)Large channel-to-channel variation of spectral continuum thresholds, see ref. [3], large error in vacuum averages of local operators, unclarified status of such approximations as vacuum saturation in vacuum averages over 4-quark operators, very large non-perturbative effects in the scalar and pseudoscalar channels
cable if $Q \sim \lambda^{-1}$. For gluon and quark condensates the lattice implies $A(Q_0)$’s at $Q_0 \sim 2$ GeV which are compatible with the phenomenological values obtained from conventional QSR’s at $\mu \sim 1$ GeV. Since the inverse correlation lengths are below the mass of the $\rho$-resonance (assuming naive vacuum saturation at dimension 6) light-quark channel QCS sum rules are hardly touched by the exponential decrease of the operator VEV’s $[3]$. However, calculating the spectral function from the truncated OPE these large correlation lengths do not reflect the resonance physics in the respective channels. A much smaller, effective correlation length is needed at dimension 6 to yield more realistic spectra. To investigate the properties of OPE-based spectral functions in various light-quark channels is the main purpose of this paper.

The presentation is set up as follows: In the next section we briefly review the idea of non-perturbative coarse graining of local operator VEV’s as it is developed in $[6]$. Thereby, the focus is on correlation functions which are represented by so-called connected diagrams. Section 3 investigates the vacuum saturation hypothesis for 4-quark operators $[1]$ in the light of non-perturbative coarse graining. The low-energy parts of the spectral functions in the $\rho$, $a_1$, $\pi$, and $\phi$ channels are calculated in section 4 by analytical continuation of the OPE to time-like external momenta. Implications for the calculation of the difference of nonleptonic inclusive decay widths in neutral $B$-meson systems are discussed in section 5. The last section summarizes the results.

2 Euclidean exponentials and Minkowskian oscillations

In this section we briefly review the work of $[3]$ on non-perturbative coarse graining for VEV’s of local, gauge invariant operators.

At a large euclidean momentum $Q \equiv \sqrt{Q^2}$, where we expect the description of the dynamics in terms of the continuum action and local operators made of fundamental fields to be sufficiently accurate, we start by expanding the current-current correlator into a conventional OPE. The evolution to lower momenta is obtained by running the Wilson coefficients perturbatively via the running coupling $\alpha_s(Q^2)$ and the anomalous operator dimensions $[1]$. According to $[3]$ the non-perturbative running of an operator average is governed by the non-perturbative part of the corresponding gauge invariant correlator in euclidean position space. This correspondence can be expressed as

$$\langle F(0) \rangle_Q^{np} \equiv \langle F_1(0) \cdots F_n(0) \rangle_Q^{np} \rightarrow 1/N \sum \langle F_1(0) \cdots F_n(x_n) \rangle_Q^{np} .$$

In eq. (1) parallel transporters

$$S(0, x) \equiv \mathcal{P} \exp \left[ ig \int_0^x dz_\mu A_\mu \right]$$

(2)
are appropriately contained in the non-local expression to define gauge invariant correlations. The sum runs over all relevant \[2\] piecewise straight \[7\] trajectories of parallel transport, and the symbol \(P\) demands path ordering. With the normalization factor \(1/\mathcal{N}\), which depends on \(n\) and the numbers of fields transforming under the fundamental and adjoint representation, the correlation function reduces to the "condensate" in the limit \(x_1, \ldots, x_n \to 0\). Making the convention that an arrow pointing from \(x_i\) towards \(x_j\) stands for the parallel transport \(S(x_i, x_j)\), we have a way to depict correlators. Note that points with fields transforming under the fundamental (adjoint) representation are connected to one (two) lines of parallel transport. There are disconnected and connected diagrams. In this section we only consider the latter.

To coarse grain from resolution \(Q\) to resolution \(Q - dQ\) we average the non-perturbative part of the correlator corresponding to a connected diagrams over a (euclidean) ball of radius \(dR_Q\) with

\[
dR_Q = \frac{1}{Q - dQ} - \frac{1}{Q} = \frac{1}{Q} \left( \frac{1}{1 - \frac{dQ}{Q}} - 1 \right) \sim \frac{dQ}{Q^2}. \tag{3}\]

Using the short-hand notation of eq. (1), this is written as

\[
\langle F(0) \rangle_{Q - dQ}^{np} = \frac{1}{(V(dR_Q))^{(n-1)}} \int_{|x_1|, \ldots, |x_n| \leq dR_Q} d^4x_1 \cdots d^4x_n \langle F_1(0) \cdots F_n(x_n) \rangle^Q_{np}, \tag{4}\]

where

\[
V(dR_Q) = \frac{1}{2} \pi^2(dR_Q)^4 = \frac{1}{2} \pi^2 \left( \frac{dQ}{Q^2} \right)^4. \tag{5}\]

In Fig. 1 there are 3 examples for connected diagrams.

Let us now focus on 2-point functions as they are relevant for dimension 3 and 4 quark and gluon operators, respectively. Only the gauge invariant bilocal quark correlator \[8\]

\[
\langle \text{tr} \bar{q}(x) S(x, 0) q(0) \rangle \tag{6}\]

and the gluonic field strength correlator \[7, 8\]

\[
\langle \text{tr} F_{\mu \nu}(x) S(x, 0) F_{\kappa \lambda} S^\dagger(x, 0) \rangle \tag{7}\]

have been measured on the lattice. The results imply that there exists an additive decomposition into a perturbative, power-like in \(|x|\), and a non-perturbative, exponential in \(|x|\) piece \[8\]. We are interested in the non-perturbative part, which in short-hand reads

\[
\langle F_1(0)F_2(x) \rangle_{Q}^{np} = A(Q) \exp(-|x|/\lambda) \tag{8}\]

As explained in \[8\] \(\lambda\) is expected to have direct phenomological meaning, and so it can not depend on the resolution \(Q\). Hence, coarse graining may only affect the

\[2\] This will be specified later.
pre-exponential factor \(A(Q)\). An evolution equation for \(A(Q)\) can be derived if we combine eqs. (4) and (8):

\[
\left( V(dQ/Q^2) \right)^{-1} \int_{|x| \leq dQ} d^4x A(Q)e^{-|x|/\lambda} = A(Q - dQ)e^{-0/\lambda} = A(Q - dQ). \tag{9}
\]

Note that for \(\lambda \to \infty\) \(A\) is invariant under coarse graining. Therefore, the conventional treatment of non-perturbative corrections in the framework of the OPE is recovered, and condensates do not depend on the resolution in this limit. In particular, they could be determined at a large resolution, where fundamental fields make sense. In the real world, however, correlation lengths are finite. We will see later how this is reflected in the existence of hadronic resonances.

Expanding the l.h.s. and r.h.s. of eq. (9) in \(\frac{dQ}{Q^2}\) and comparing coefficients of the linear terms, we obtain the following equation:

\[
\frac{d}{dQ} A(Q) = \frac{4}{5\lambda} Q^{-2} A(Q), \tag{10}
\]

with solution

\[
A(Q) = A(Q_0) \exp \left[ -\frac{4}{5\lambda} \left( \frac{1}{Q} - \frac{1}{Q_0} \right) \right]. \tag{11}
\]

At dimension 4 there are no anomalous operator dimensions and being content with a determination of Wilson coefficients at the lowest possible order in \(\alpha_s\) the generic form of a non-perturbative correction is

\[
\frac{A_4(Q_0)}{(Q_0)^2} \exp \left[ -\frac{4}{5\lambda_4} \left( \frac{1}{Q} - \frac{1}{Q_0} \right) \right]. \tag{12}
\]

Eq. (11) implies that compared to conventional dimension 4 power corrections there is a noticeable suppression if \(Q\) is of the order of \(\lambda^{-1}\) or less.
Lattice measurements with $N_F = 4$ staggered fermions of mass $m_q = 0.01$ and lattice resolution $Q_0 = a^{-1} \sim 2$ GeV suggest that scalar fermionic and gluonic correlation lengths are $\lambda_s \sim 3.2$ GeV$^{-1}$ and $\lambda_g \sim 1.7$ GeV$^{-1}$, respectively [8]. For the vector fermionic correlation length $\lambda_v \sim 1/10 \lambda_s^{-1}$ was found [8]. So for dimension 4 and, assuming naive vacuum saturation, also for dimension 6 this does not seem to pose a problem for the sum rule analysis of light quark correlators ($Q_0 = a^{-1} \sim 2$ GeV $A_q(Q_0)$ as well as $A_g(Q_0)$ are compatible with their QSR determined values at $\mu \sim 1$ GeV [8]). However, as we will see later, naive vacuum saturation gives light-quark channel spectra which are completely off the experimentally measured behavior. Qualitatively more realistic spectral functions can be obtained using the much smaller $\lambda_v^{-1}$.

The contribution of dimension 4 to the spectral function $\rho(s)$ is (up to a normalization!) obtained by analytically continuing to $Q^2 = -(s + i\epsilon)$ or $Q = -i\sqrt{s}$, $(s > 0)$, and taking the imaginary part. A term like the one in eq. (12) corresponds to a term

$$\frac{A_4(Q_0)}{s^2} \exp \left[ \frac{4}{5\lambda_4 Q_0} \right] \sin \left[ -\frac{4}{5\lambda_4 \sqrt{s}} \right]$$

(13)

in $\rho(s)$. The oscillatory behavior manifests itself in a quite different way than it was suggested in refs. [8]: $\sin[\sqrt{s}^{-1}]$ instead of $\sin[\sqrt{s}]$ or $\sin[s]$. There, oscillations are present everywhere though power suppressed at high $s$. Here, oscillations only start if $\sqrt{s}^{-1}$ is larger than the correlation length of the corresponding 2-point function.

3 Vacuum saturation in the context of gauge invariant correlations

After the treatment of connected diagrams in the previous section we turn to disconnected diagrams in this section. A sufficient condition for factorized coarse graining is the factorization of the non-perturbative part in the corresponding correlation function. It is unlikely that such a factorization occurs for connected diagrams.

Let us focus on the case of 4 quark operators since generalizations to higher dimensions are straightforward. Disconnected diagrams are associated with 4 quark operators $\bar{q}(0)\Gamma q(0)\bar{q}(0)\Gamma q(0)$ composed of color singlet currents (Fig. 2). Thereby, $q$ is a single flavor quark field, and $\Gamma$ denotes one of $1, \gamma_5, \gamma_\mu, \cdots$ or a product of them. Diagrams with permuted arguments are irrelevant because of the integrations in eq. (4) and the fact that we may assume translational, $P$, and $T$ invariance [9]. In practical applications one encounters color octet currents in 4 quark operators. These structures lead to special forms of disconnected diagrams which are due to the fact that gauge invariant correlation functions can only be defined if 2 of the 4 arguments coincide, hence yielding 3 point functions. For example, an operator $\bar{q}(0)\Gamma t^a q(0)\bar{q}(0)\Gamma t^a q(0)$ demands non-local contributions of the form

$$\langle \bar{q}(0)\Gamma S(0, x_1) t^a q(x_1) \bar{q}(x_2) S(x_2, 0) \Gamma t^a q(0) \rangle ;$$

5
Thereby, the color matrices are normalized as $\text{tr} t^a t^b = 2 \delta^{ab}$. The corresponding diagrams are depicted in Fig 3. Lines do not meet at the points 0 (for (i)) and $x_1$ (for (ii)) because they connect to different fields.

Since gauge invariant $n > 2$ point functions have yet not been measured on the lattice we do have to think of approximations involving only 2-point functions. The (quite intuitive) hypothesis of vacuum saturation can be implemented in two ways: 1) vacuum saturation on the level of local operators according to the formula of ref. [1] with separate coarse graining for each factor and 2) vacuum saturation on the level of the correlation function, that is, first delocalization of the operator and then vacuum insertion. There is one obvious reason why the two prescriptions lead to different results. In case 1) we essentially square the result for the quark condensate, which implies correlations between points of maximal separation $dQ/Q^2$, whereas in case 2) one factor may contain correlations between points of maximal separation $2dQ/Q^2$ (Fig. 2 or case (ii) in Fig. 3). Another, not-so-obvious reason becomes
apparent if we insert the vacuum state into the correlator for singlet currents
\[ \langle \bar{q}(0) \Gamma \Sigma(0, x_1)q(x_2)\bar{q}(x_1)\Gamma \Sigma(x_2, x_3)q(x_3) \rangle . \] (15)

Then information of the 2-point function
\[ \langle \bar{q}(0) \Gamma \Sigma(0, x)q(x) \rangle \] (16)
is needed. From ref. [9] we know that on a lattice the correlation length \( \lambda_v \) of the vector correlator (\( \Gamma = \gamma_\mu \)) is about 10 times smaller than the correlation length \( \lambda_s \) of the scalar correlator (\( \Gamma = 1 \)). This result has been obtained for a quark mass \( m \) with \( a \cdot m = 0.01 \) and \( a^{-1} \sim 2 \text{ GeV} \) in the \( N_F = 4 \) theory with staggered fermions. In the octet case vacuum insertion in between the currents factorizes the correlator into gauge variant 2 point functions. The gauge invariant product of them has not yet been measured on the lattice so we can not compare the corresponding correlation length with the singlet case.

Let us keep all these points in mind in the next section, where for simplicity we apply prescription 1. With the necessary lattice information available we hope to investigate case 2 in a separate publication.

4 Light-quark channels

After writing down the coarse grained OPE’s in the \( \rho, a_1, \pi, \) and \( \phi \) channels, in this section we calculate the respective spectral functions by analytical continuation from euclidean to time-like momenta. We restrict ourselves to light-quark channels because here the two needed correlation functions for corrections up to dimension 6 (assuming vacuum saturation in the sense of 1) have been measured on the lattice. We investigate with two sets of parameters. Set (A) takes the \( N_F = 4 \) results of refs. [8] for the lowest quark mass \( (a \cdot m = 0.01) \) literally, that is
\[
(A) : \\
\lambda_q = 3.1 \text{ GeV}^{-1}, \quad A_q(a^{-1} = Q_0 \sim 2 \text{ GeV}) = (0.212 \text{ GeV})^3 ; \\
\lambda_g = 1.7 \text{ GeV}^{-1}, \quad A_g(a^{-1} = Q_0 \sim 2 \text{ GeV}) = 0.015 \text{ (GeV)}^4 .
\] (17)

Due to the uncertainty in the way dimension 6 contributions are treated and motivated by the 10 times smaller vector correlation length \( \lambda_v \) we use \( \lambda_q = \lambda_v \) in (B)
\[
(B) : \\
\lambda_q = 0.3 \text{ GeV}^{-1}, \quad A_q(a^{-1} = Q_0 \sim 2 \text{ GeV}) = (0.212 \text{ GeV})^3 ; \\
\lambda_g = 1.7 \text{ GeV}^{-1}, \quad A_g(a^{-1} = Q_0 \sim 2 \text{ GeV}) = 0.015 \text{ (GeV)}^4 .
\] (18)
4.1 $\rho$-correlator

Here, we investigate the correlator of currents $j_\mu^\rho = 1/2(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)$ which at $Q \sim Q_0$ and up to dimension 6 has the following conventional OPE 

\[
\int d^4 e^{ix} \left\langle T\{j_\mu^\rho(x) j_\mu^\rho(0)\}\right\rangle = (g_\mu q_\nu - g_\mu q^2) \times
\left\{ \frac{-1}{8\pi^2} \left( 1 + \frac{\alpha_s(Q^2)}{\pi} \right) \log \frac{Q^2}{Q_0^2} + \frac{1}{Q^4} \left[ \frac{1}{2} \langle m_u \bar{u}u + m_d \bar{d}d \rangle_{Q_0} + \frac{1}{24} \left( \frac{\alpha_s}{\pi} F^a_{\mu \nu} F^a_{\mu \nu} \right)_{Q_0} \right] - \frac{\pi \alpha_s(Q^2)}{Q^6} \left[ \frac{1}{2} \langle (\bar{u}\gamma_\mu \gamma_5 t^a u - \bar{d}\gamma_\mu \gamma_5 t^a d)^2 \rangle_{Q_0} + \frac{1}{9} \langle (\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d) \sum_{q=u,d,s} \tilde{q} \gamma_\mu t^a q \rangle_{Q_0} \right] \right\}.
\]

(19)

After application of the vacuum saturation hypothesis, going to the SU(2) chiral limit $m_u = m_d = 0$, and implementing the non-perturbative coarse graining of operator averages\[\] the restriction $Q \sim Q_0$ can be dropped, and the scalar part of the correlator (curly brackets on the r.h.s. of eq. (19)) reduces to

\[
\Pi^\rho(Q^2) = \frac{-1}{8\pi^2} \left( 1 + \frac{\alpha_s(Q^2)}{\pi} \right) \log \frac{Q^2}{Q_0^2} + \frac{1}{24 Q^4} \exp\left[ -\frac{4}{5\lambda_q} \left( \frac{1}{Q} - \frac{1}{Q_0} \right) \right] \left\langle \frac{\alpha_s}{\pi} F^a_{\mu \nu} F^a_{\mu \nu} \right\rangle_{Q_0} - \frac{\pi \alpha_s(Q^2)}{Q^6} \left[ \frac{112}{81} \exp\left[ -\frac{8}{5\lambda_q} \left( \frac{1}{Q} - \frac{1}{Q_0} \right) \right] \langle \bar{q}q \rangle_{Q_0} \right]^2.
\]

(20)

4.2 $\alpha_1$-correlator

The vacuum averaged OPE for the correlator of the current $j_\mu^{a_1} = 1/2(\bar{u}\gamma_\mu \gamma_5 u - \bar{d}\gamma_\mu \gamma_5 d)$ at $Q \sim Q_0$ and up to dimension 6 reads in the SU(2) chiral limit \[\]

\[
\int d^4 e^{ix} \left\langle T\{j_\mu^{a_1}(x) j_\mu^{a_1}(0)\}\right\rangle = (g_\mu q_\nu - g_\mu q^2) \times \left\{ \frac{-1}{8\pi^2} \left( 1 + \frac{\alpha_s(Q^2)}{\pi} \right) \log \frac{Q^2}{Q_0^2} + \frac{1}{24 Q^4} \left( \frac{\alpha_s(Q^2)}{\pi} \right) \log \frac{Q^2}{Q_0^2} + \frac{1}{24 Q^4} \left\langle \frac{\alpha_s}{\pi} F^a_{\mu \nu} F^a_{\mu \nu} \right\rangle_{Q_0} - \frac{\pi \alpha_s(Q^2)}{Q^6} \left[ \frac{1}{2} \langle (\bar{u}\gamma_\mu t^a u)^2 - (\bar{d}\gamma_\mu t^a d)^2 \rangle_{Q_0} + \langle u \leftrightarrow d \rangle_{Q_0} \right] \right\}.
\]

(21)

\[\]

\[\] At dimension 6 there is logarithmic running of the Wilson coefficients due to the nonvanishing anomalous dimensions of the corresponding operators. Together with the running coupling $\alpha_s(Q^2)$ these dependences almost cancel, and hence we will omit them throughout what follows. Operators of dimension 4 are perturbatively invariant under the renormalization group. We take the value $\alpha_s(Q_0^2) = (2 \text{ GeV})^2 \sim 0.2$. This is motivated by ref. where a non-perturbative evolution of $\alpha_s$ has been obtained in two flavor lattice QCD using the Schrödinger functional scheme. With the conversion $\Lambda_{\overline{\text{MS}}} = 2.382A$ and taking $\Lambda_{\overline{\text{MS}}} = 0.4 \text{ GeV}$, one obtains $\alpha_s(2 \text{ GeV})^2 \sim 0.19$. We have varied the coupling up to $\alpha_s = 0.4$ but no drastic changes occur.
After vacuum saturation and implementation of non-perturbative operator running the restriction $Q \sim Q_0$ can be forgotten, and we have

$$
\Pi^\alpha(Q^2) = -\frac{1}{8\pi^2} \left(1 + \frac{\alpha_s(Q^2)}{\pi}\right) \log \frac{Q^2}{Q_0^2} + \frac{1}{24Q^4} \exp\left[-\frac{4}{5\lambda_g} \left(\frac{1}{Q} - \frac{1}{Q_0}\right)\right] \left\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^a F_{\mu\nu}^a \right\rangle Q_0
$$

$$
\frac{\pi \alpha_s(Q_0^2)}{Q^6} \frac{176}{81} \exp\left[-\frac{8}{5\lambda_g} \left(\frac{1}{Q} - \frac{1}{Q_0}\right)\right] \left\langle \bar{q}q \right\rangle Q_0^2.
$$

(22)

### 4.3 $\pi$-correlator

The pion current is given as $j_\pi^n = 1/2i(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)$, and in the SU(2) chiral limit its correlator can be expanded as [1]

$$
i \int d^4e^{iqx} \left\langle T\{j_\pi^n(x)j_\pi^n(0)\} \right\rangle = -3Q^2 \left\{ -\frac{1}{16\pi^2} \left(1 + \frac{\alpha_s(Q^2)}{\pi}\right) \log \frac{Q^2}{Q_0^2} - \frac{1}{48Q^4} \left\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^a F_{\mu\nu}^a \right\rangle Q_0 - \frac{\pi \alpha_s(Q_0^2)}{Q^6} \left[ \frac{1}{12} \left\langle (\bar{u}\sigma_{\mu\nu}\gamma_5 t^a u - \bar{d}\sigma_{\mu\nu}\gamma_5 t^a d)^2 \right\rangle_q Q_0 + \frac{1}{18} \left\langle (\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d) \sum_{q=u,d,s} \bar{q}q \gamma_\mu t^a q \right\rangle Q_0 \right] \right\}.
$$

(23)

After vacuum saturation and with non-perturbative operator running (no restriction $Q \sim Q_0$ anymore) the piece in curly brackets becomes

$$
\Pi_\pi(Q^2) = -\frac{1}{16\pi^2} \left(1 + \frac{\alpha_s(Q^2)}{\pi}\right) \log \frac{Q^2}{Q_0^2} - \frac{1}{48Q^4} \exp\left[-\frac{4}{5\lambda_g} \left(\frac{1}{Q} - \frac{1}{Q_0}\right)\right] \left\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^a F_{\mu\nu}^a \right\rangle Q_0
$$

$$
\frac{\pi \alpha_s(Q_0^2)}{Q^6} \frac{56}{81} \exp\left[-\frac{8}{5\lambda_g} \left(\frac{1}{Q} - \frac{1}{Q_0}\right)\right] \left\langle \bar{q}q \right\rangle Q_0^2.
$$

(24)

### 4.4 $\phi$-correlator

The $\phi$-meson current is defined as $j_\phi^\mu = -1/3\bar{s}\gamma_\mu s$. If we keep the $s$-quark mass finite, let $m_u = m_d = 0$, assume that the $s$-quark condensate at $Q = Q_0$ does not deviate from the ones of $u$ or $d$ quarks, then at $Q \sim Q_0$ the OPE of the $\phi$-correlator becomes [1]

$$
i \int d^4e^{iqx} \left\langle T\{j_\phi^\mu(x)j_\phi^\nu(0)\} \right\rangle = (q_\mu q_\nu - g_{\mu\nu}q^2) \times \left\{ \frac{2}{9} \left( -\frac{1}{8\pi^2} \left(1 + \frac{\alpha_s(Q^2)}{\pi}\right) \log \frac{Q^2}{Q_0^2} + \frac{1}{Q^4} \left[ \left\langle m_\phi \bar{q}q \right\rangle Q_0 + \frac{1}{24} \left\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^a F_{\mu\nu}^a \right\rangle Q_0 \right] - \frac{\pi \alpha_s(Q_0^2)}{Q^6} \left[ \left\langle (\bar{s}\gamma_\mu \gamma_5 t^a s)^2 \right\rangle Q_0 + \frac{2}{9} \left\langle \bar{s}\gamma_\mu t^a s \sum_{q=u,d,s} \bar{q}q \gamma_\mu t^a q \right\rangle Q_0 \right] \right\} \right\}
$$

(25)
Figure 4: The spectral functions $\text{Im}\Pi(Q^2 - s - i\varepsilon), (s > 0)$ for sets (A) and (B), where (i), (ii), (iii), and (iv) correspond to the $\rho$, $a_1$, $\pi$, and $\phi$ channels, respectively.

Building in vacuum saturation and non-perturbative running of the operator VEV’s, the restriction $Q \sim Q_0$ does not apply anymore, and the scalar piece in curly brackets becomes

$$
\Pi^\pi(Q^2) = \frac{2}{9} \left( -\frac{1}{8\pi^2} \left( 1 + \frac{\alpha_s(Q^2)}{\pi} \right) \log \frac{Q^2}{Q_0^2} + \right.
$$

$$
\frac{1}{Q^4} \left[ \exp\left[ \frac{4}{5\lambda_q} \left( \frac{1}{Q} - \frac{1}{Q_0} \right) \right] \langle m_s\bar{q}q \rangle_{Q_0} + \frac{1}{24} \exp\left[ -\frac{4}{5\lambda_g} \left( \frac{1}{Q} - \frac{1}{Q_0} \right) \right] \left\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^a F_{\mu\nu}^a \right\rangle_{Q_0} \right] -
$$

$$
\frac{\pi\alpha_s(Q_0^2) 112}{Q^6} \frac{81}{81} \exp\left[ -\frac{8}{5\lambda_q} \left( \frac{1}{Q} - \frac{1}{Q_0} \right) \right] \langle \bar{q}q \rangle_{Q_0}^2 \right) .
$$

4.5 Spectral functions

Here, we calculate the spectral functions of the respective channels from the OPE by taking the imaginary part of $\Pi(Q^2)$ at analytical continued $Q^2 = -s - i\varepsilon$ or $Q = \sqrt{Q^2} = -i\sqrt{s}$ with real $s$ and $s > 0$. As in the conventional sum rule approach we expect to obtain information on the lowest resonances (but now without the apriori assumption of quark-hadron duality). Fig. 4 shows the results of the calculations for $\alpha_s(Q_0^2 = (2\text{GeV})^2) = 0.2$ we use $m_s = 120$ MeV. Exact vacuum saturation in the sense of 1) is assumed with the commonly introduced “correction” factor $k \geq 1$ set equal to unity. Due to the truncation of the OPE at dimension 6 and other errors (large quark mass and $N_F = 4$ in the lattice calculation, vacuum saturation, chiral $\text{SU}(2)$ limit, and a relatively low “fundamentality scale” $Q_0$) we have to cut
off the spectra at some lower bounds. Since spectral functions ought to be positive definite natural candidates $\sqrt{s_0}$ for these cutoffs are the first zeros encountered when decreasing $\sqrt{s}$.

Looking at the spectra belonging to set (A) above these cutoffs, there is no resemblance of the measured behavior. In contrast, set (B) seems to give a more realistic behavior: There is higher spectral strength at lower energy in the $\rho$ than there are in the $a_1$ or $\phi$ channels. In the latter we have varied the strange quark mass: For $m_s < 50$ MeV the $\phi$ channel behaves $\rho$-like, that is, there is a large peak at the lower bound of the spectrum. For $m_s > 50$ MeV the $\phi$-spectrum is $a_1$-like, that is, the spectral strength is steadily decreasing from its perturbative value down to its first zero with decreasing energy. Similarly, the $\pi$ channel exhibits a large concentration of spectral strength at low energies.

Let us process the spectral information contained in Fig. 4 to more quantitative statements. One may ask where the “center-of-mass”

$$M = \frac{\int_{\sqrt{s_0}}^{\sqrt{s_1}} (d\sqrt{s}) \sqrt{s} \Im(s)}{\int_{\sqrt{s_0}}^{\sqrt{s_1}} (d\sqrt{s}) \Im(s)} \quad (27)$$

of a given spectrum within a low-energy domain $\sqrt{s_0} \leq \sqrt{s} \leq \sqrt{s_1}$ is. This is motivated by the experimental fact that the lowest resonance dominates the spectrum within a large range of energies. Note that using a narrow resonance model, the residue of this single resonance cancels in the definition of $M$ (ratio of moments). We may now define the universal upper bound $\sqrt{s_1}$ of this low-energy domain by demanding $M_\rho$ to coincide with the mass $m_\rho = 770$ MeV of the lowest $\rho$-resonance. With $\sqrt{s_0} = 420$ MeV eq. (27) then yields $\sqrt{s_1} = 1.2$ GeV. Using this, we calculate $M$ for the other channels with $\sqrt{s_0} = \sqrt{s'_0}$ always being the first zero of the respective spectral function. The result can be compared with the measured mass of the lowest resonance. We have

$$\begin{align*}
\sqrt{s_0^{a_1}} &= 540 \text{ MeV} , & M_{a_1} &= 855 \text{ MeV} , & \frac{M_{a_1}}{m_{a_1}} &= \frac{855}{1260} \sim 0.68 ; \\
\sqrt{s_0^\pi} &= 420 \text{ MeV} , & M_\pi &= 680 \text{ MeV} , & \frac{M_\pi}{1/2[m_\pi + m_\pi(1300)]} &= \frac{680}{720} \sim 0.94 ; \\
\sqrt{s_0^\phi} &= 680 \text{ MeV} , & M_\phi &= 960 \text{ MeV} , & \frac{M_\phi}{m_\phi} &= \frac{960}{1020} \sim 0.94 . \quad (28)
\end{align*}$$

Due to the distinguished role of the $\pi$-meson as a Goldstone-boson we have taken along the next pion resonance ($\pi(1300)$) assuming their residues to be equal. The differences between the measured (still using the narrow resonance model) and the computed ratios of moments is about 30% for the $a_1$ channel and less than 10% level for $\pi$ and $\phi$ channels. We do emphasize at this point that conventional OPE’s would have given the same value for $M$ in all channels (apart from small perturbative corrections to the Wilson coefficients).
Why do the shapes deviate considerably from the experimental ones (as a general feature, they seem to be shifted to lower energies)? Although the use of the small correlation length in set \((B)\) was motivated by ambiguities concerning vacuum saturation we do believe that the truncation of the OPE at dimension 6 is the major source of deviation. It is quite plausible that higher operator dimensions introduce shorter and shorter effective correlation lengths and therefore higher mass scales to govern the operator VEV’s at low resolution \([6]\). Viewed in this context, the use of a small correlation length at dimension 6 may be an effective way to simulate higher mass dimensions. Apart from this there are, of course, the unresolved problems linked to vacuum saturation as such and the way it is being implemented as they were discussed in the previous section.

5 Implications for the calculation of \(\Delta \Gamma_B\)

Motivated by the occurrence of a large (effective) mass scale in the previous section we discuss in this section how the notion of non-perturbative coarse graining of local operator averages may influence the calculation of nonleptonic, inclusive width differences in neutral \(B\)-meson systems.

To be specific, we take the example of the \(B_s\bar{B}_s\) system as it was treated in refs. [12]. On the level of an effective weak Hamiltonian, which is obtained by integrating out the heavy bosons \(Z^0, W^\pm\) of the fundamental theory (also including pQCD corrections) and which, omitting Cabibbo suppressed contributions, is of the form

\[
\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{cb} V_{cs} \left( \sum_{i=1}^{6} c_i O_i + c_8 O_8 \right),
\]

the width difference \(\Delta \Gamma_{B_s}\) between the mass eigenstates \(\vert B_{H/L} \rangle\) can be calculated by virtue of the optical theorem as (relativistic normalization of states)

\[
\Delta \Gamma_{B_s} = \frac{1}{2M_{B_s}} \left\langle \bar{B}_s \left| \text{Im} \int d^4x \mathcal{T} \mathcal{H}_{eff}(x) \mathcal{H}_{eff} \right| B_s \right\rangle.
\]

The first six \(\Delta B = 1\) operators of eq. (29) are of the four quark type, for example

\[O_1 = (\bar{b}_\alpha c_\beta)_{V-A} (c_\beta s_\alpha)_{V-A},\]

and \(O_8 = g/(8\pi^2) m_b \bar{b}_\alpha \sigma^{\mu\nu}(1 - \gamma_5) F_\mu^n \epsilon_{\alpha\beta} s_\beta\).

To be able to handle eq. (30) one expands in position space the T-product into an OPE which contains \(\Delta B = 2\) operators, for example

\[Q = (\bar{b}_\alpha s_\alpha)_{V-A} (\bar{b}_\beta s_\beta)_{V-A}.\]

In view of the subsequent \(x\) integration the OPE only makes sense if the momentum \(q\) of the external states is in the euclidean (or space-like) domain. So for a meaningful calculation formally an analytical continuation of the time-like momentum squared \(q^2 \sim m^2_B\) of the heavy mesons at rest to \(q^2 = -Q^2 < 0\) must be performed. After a determination of the Wilson coefficients in the euclidean the result is analytically
continued back to $q^2 \sim m_q^2$. Thereby, one has to deal with two renormalization scales $\mu_{1,2}$. The former is due to the expansion of the fundamental, electroweak (together with QCD corrections) interaction into the effective Hamiltonian, whereas the latter comes in via the separate scale dependence of the $\Delta B = 2$ operators and Wilson coefficients. It is claimed in refs. [12] that in the $\Delta B = 2$ OPE the dependence on $\mu_1$ is almost cancelled on the level of the Wilson coefficients at the same order in $\alpha_s$. Since the $\overline{\text{MS}}$-scheme was used in [12] the $\overline{\text{MS}}$ matrix elements had to be matched to the matrix elements obtained in lattice renormalization at the low lattice renormalization point $2\sim \text{GeV}$. Subsequently, they were run up to $\sim m_b$.

Keeping the scale of the matrix element fixed in a purely perturbative renormalization group evolution, the transcendental dependences on the external momentum scale are powers of logarithms due to their resummation. Violating local duality, we have seen in the previous sections that these logarithmic dependences do not introduce resonance structure in the spectral functions of light-quark channel vacuum correlators. If the scale of the process is comparable to the inverse (effective) correlation length then there is a much stronger dependence of the product of Wilson coefficient and matrix element of a local operator than the logarithmic one. Along the lines of section 3 it would therefore be important to measure the 4 quark correlators corresponding to the $\Delta B = 2$ matrix elements of eq. (wi) in order to decide whether mass scales (inverse correlation lengths) occur which are dangerous for the heavy quark expansion. This should also be done for contributions of higher dimensional operators formally corresponding to higher powers in $1/m_b$. After all, the inverse, effective correlation length, used in the OPE’s of vacuum correlators in light-quark channels (set (B) in section 4), is not too small compared to $m_b$ ($10/3 \text{ GeV} \ll 4.5 \text{ GeV}$).

6 Summary

In this paper we investigated the consequences of non-perturbative coarse graining of operator VEV’s as it was proposed in ref. [6]. The focus was on light-quark correlators.

After a brief review of OPE coarse graining, based on the knowledge of non-perturbatively calculated, gauge invariant $n$-point functions, we addressed the issue of vacuum saturation in the case of so-called disconnected diagrams. In particular, it was pointed out that there are ambiguities in the way approximations to the general prescription are implemented for coarse graining VEV’s of local 4 quark operators. Using the simplest option for vacuum saturation, the machinery was applied to the light-quark correlators in the $\rho$, $a_1$, $\pi$, and $\phi$ channels. The spectral functions of the respective channels were calculated from the OPE by analytical continuation to time-like external momenta. Using lattice data on the gauge invariant field strength and scalar quark correlators, the spectra were found to be far off their experimentally measured behavior. With a 10 times smaller correlation length at dimension
6 the basic phenomenological features turned out to be contained in the spectral functions. However, in general the spectra are shifted to lower energies and resonance information is vague in the $a_1$ and $\phi$ channels. This may be a consequence of excluded higher mass dimensions and the treatment of the VEV’s of 4 quark operators. Despite the obvious shortcomings a calculation of the ratios of the first and zeroth moments of the spectral distributions in the $a_1$, $\pi$, and $\phi$ channels gave quite realistic results after fixing this ratio in the $\rho$ channel. Finally, we discussed the potential impact non-perturbative coarse graining may have on the theoretical determination of the inclusive, nonleptonic $\Delta \Gamma$ in $B_s$-meson decays.

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