Weyl Symmetry and the Liouville Theory\textsuperscript{*†‡}

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Abstract

Flat-space conformal invariance and curved-space Weyl invariance are simply related in dimensions greater than two. In two dimensions the Liouville theory presents an exceptional situation, which we here examine.

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1 Conformally and Weyl Invariant Scalar Field Dynamics in $d > 2$

Let us begin by recording the $d$-dimensional Lagrange density for a scalar field $\varphi$ with a scale and conformally invariant self interaction.

$$\mathcal{L}_0 = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \lambda \varphi^{\frac{2d}{d-2}}$$

(1.1)

Evidently the expression makes sense only for $d \neq 2$, and we take $d > 2$. The theory is invariant against

$$\delta \varphi = f^\alpha \partial_\alpha \varphi + \frac{d-2}{2d} \partial_\alpha f^\alpha \varphi,$$

(1.2)

where $f^\alpha$ is a (flat-space) conformal Killing vector. The usual, canonical energy momentum tensor

$$\theta_{\mu\nu}^{\text{canonical}} = \partial_\mu \varphi \partial_\nu \varphi - \eta_{\mu\nu} \left( \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - \lambda \varphi^{\frac{2d}{d-2}} \right)$$

(1.3)

is conserved and symmetric, as it should be in a Poincaré invariant theory. But it is not traceless: $\eta_{\mu\nu} \theta_{\mu\nu}^{\text{canonical}} \neq 0$. Nevertheless, because of the conformal invariance (1.2), $\theta_{\mu\nu}^{\text{canonical}}$ can be improved by the addition of a further conserved and symmetric expression, so that the new tensor is traceless [1].

$$\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{canonical}} + \frac{d-2}{4(d-1)} (\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) \varphi^2, \quad \eta^{\mu\nu} \theta_{\mu\nu} = 0$$

(1.4)

A variational derivation of the canonical tensor (1.3) becomes possible after the theory (1.1) is minimally coupled to a metric tensor $g_{\mu\nu}$, and its action integral is varied with respect to $g_{\mu\nu}$. $\theta_{\mu\nu}^{\text{canonical}}$ is regained in the limit $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$. A similar derivation of the improved tensor (1.4) is also possible, provided (1.1), generalized to curved space, is extended by a specific non-minimal coupling [1].

$$\mathcal{L} = \frac{d-2}{8(d-1)} R \varphi^2 + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \lambda \varphi^{\frac{2d}{d-2}}$$

(1.5)

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \delta \mathcal{L} \left( \frac{\delta}{\delta g^{\mu\nu}} \right) = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - \lambda \varphi^{\frac{2d}{d-2}} \right) + \frac{d-2}{4(d-1)} (g_{\mu\nu} D^2 - D_\mu D_\nu + G_{\mu\nu}) \varphi^2$$

(1.6)

Here $G_{\mu\nu}$ is the Einstein tensor, $R$ the Ricci scalar $R = \frac{2}{d-2} g^{\mu\nu} G_{\mu\nu}$, and $D_\mu$ the covariant derivative. In the limit $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ the non-minimal term in $\mathcal{L}$ vanishes, but it survives in the $g^{\mu\nu}$ variation.

$$T_{\mu\nu} \xrightarrow{g_{\mu\nu} \rightarrow \eta_{\mu\nu}} \theta_{\mu\nu}$$

(1.7)

Note that $g^{\mu\nu} T_{\mu\nu} = 0$, with the help of the field equation for $\varphi$.

$$D^2 \varphi + \lambda \frac{2d}{d-1} \varphi^{\frac{d+2}{d-2}} - \frac{d-2}{4(d-1)} R \varphi = 0$$

(1.8)
This ensures the vanishing of $\eta^{\mu\nu} \theta_{\mu\nu}$.

The precise form of the non-minimal coupling results in the invariance of the curved space action against Weyl transformations, involving an arbitrary function $\sigma$ [2].

\[
\begin{align*}
\eta^{\mu\nu} & \xrightarrow{\text{Weyl}} e^{2\sigma} \eta^{\mu\nu} & (1.9a) \\
\phi & \xrightarrow{\text{Weyl}} e^{\frac{2-d}{2} \sigma} \phi & (1.9b)
\end{align*}
\]

The self coupling is separately invariant against $\mathbf{(1.9)}$. The kinetic term and the non-minimal coupling term are not, but their non-trivial response to the Weyl transformation cancels in their sum. Also it is the Weyl invariance of the action that results in the tracelessness of its $g^{\mu\nu}$-variation i.e. of $T_{\mu\nu}$, just as its diffeomorphism invariance ensures symmetry and covariant conservation of $T_{\mu\nu}$.

Thus we see that Weyl (and diffeomorphism) invariance in curved space is closely linked to conformal invariance in flat space [2]. But can a conformally invariant, flat space theory always be extended to a Weyl and diffeomorphims invariant theory in curved space? Evidently, the answer is “Yes” for the self-interacting scalar theories in $d > 2$, discussed previously [3]. We now examine what happens in $d = 2$.

## 2 Liouville Theory: Conformally Invariant Scalar Field Dynamics in $d = 2$

A 2-dimensional model with non-trivial dynamics that is conformally invariant is the Liouville theory with Lagrange density

\[
L^\text{Liouville}_0 = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - \frac{m^2}{\beta^2} e^{\beta \psi}.
\]  

(2.1)

The conformal symmetry transformations act in an affine manner, so that the exponential interaction is left invariant.

\[
\delta \psi = f^\alpha \partial_\alpha \psi + \frac{1}{\beta} \partial_\alpha f^\alpha
\]

(2.2)

The canonical energy-momentum tensor

\[
\theta^\text{canonical}_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \eta_{\mu\nu} \left( \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi - \frac{m^2}{\beta^2} e^{\beta \psi} \right)
\]

(2.3)

again is not traceless: $\eta_{\mu\nu} \theta^\text{canonical}_{\mu\nu} \neq 0$, but with an improvement it acquires that property.

\[
\theta_{\mu\nu} = \theta^\text{canonical}_{\mu\nu} + \frac{2}{\beta} (\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) \psi, \quad \eta_{\mu\nu} \theta_{\mu\nu} = 0
\]

(2.4)

Again $\theta^\text{canonical}_{\mu\nu}$ arises variationally when the Liouville Lagrange density is minimally extended by an arbitrary metric tensor. Similarly the improved tensor (2.4) is gotten when
a non-minimal interaction is inserted.

\[
\mathcal{L}^{\text{Liouville}} = \frac{1}{\beta} R \psi + \frac{1}{2} g^\mu\nu \partial_\mu \psi \partial_\nu \psi - \frac{m^2}{\beta^2} e^{\beta \psi} \quad (2.5)
\]

\[
T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{|g|} \mathcal{L}^{\text{Liouville}}
\]

\[
= \partial_\mu \psi \partial_\nu \psi - g_{\mu\nu} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi - \frac{m^2}{\beta^2} e^{\beta \psi} \right) + \frac{2}{\beta} (g_{\mu\nu} D^2 - D_\mu D_\nu) \psi \quad (2.6)
\]

However, the curved-space tensor \( T_{\mu\nu} \) is not traceless,

\[
g^{\mu\nu} T_{\mu\nu} = \frac{2}{\beta^2} R \neq 0,
\]

becoming traceless only in the flat-space limit, when \( R \) vanishes. Correspondingly, the action associated with \( 2.5 \) is not invariant against Weyl transformations, which take the following form for the scalar field \( \psi \).

\[
\psi \xrightarrow{\text{Weyl}} \psi - \frac{2}{\beta} \sigma \quad (2.9)
\]

This formula is needed so that the interaction density \( \sqrt{|g|} e^{\beta \psi} \) be invariant. However, the kinetic term together with the non-minimal term are not invariant, so that

\[
I^{\text{Liouville}} = \int \sqrt{|g|} \mathcal{L}^{\text{Liouville}} \quad \xrightarrow{\text{Weyl}} I^{\text{Liouville}} - \frac{2}{\beta^2} \int \sqrt{|g|} (R \sigma + g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma) \quad (2.10)
\]

Note that the change in the action — the last term in \( 2.10 \) — is \( \psi \) independent. So the field equation

\[
D^2 \psi + \frac{m^2}{\beta} e^{\beta \psi} - \frac{1}{\beta} R = 0 \quad (2.11)
\]

enjoys Weyl symmetry, even while the action does not.

### 3 Obtaining the \( d = 2 \) Liouville theory from the \( d > 2 \) Weyl invariant theories

We see that the 2-dimensional situation is markedly different from what is found for \( d > 2 \): for the latter theories there exists a Weyl-invariant precursor, with a traceless energy-momentum tensor in curved space, which leads to a traceless energy-momentum tensor in flat space. For \( d = 2 \) the precursor is not Weyl invariant and the energy-momentum tensor becomes traceless only in the flat-space limit.

To get a better understanding of the 2-dimensional behavior, we now construct a limiting procedure that takes the Weyl invariant models at \( d > 2 \), \( 1.5 \), to two dimensions. Thereby we expose the steps at which Weyl invariance is lost.
In order to derive the $d = 2$ Liouville theory (2.5) from the $d > 2$, Weyl invariant models with polynomial interaction (1.5), we set
\[
\varphi = \frac{2d}{\beta(d-2)} \, e^{\frac{\beta}{2(d-2)} \varphi} \quad (d > 2),
\]
and take the limit $d \to 2$, from above. We examine each of the three terms in (1.5) separately.

For the self interaction, we have
\[
\lambda \varphi^{2(d-2)} = \lambda \left( \frac{2d}{\beta(d-2)} \right)^{\frac{2d}{d-2}} e^{\frac{\beta}{2(d-2)} \varphi} \rightarrow \frac{m^2}{\beta^2} e^{\frac{\beta}{2(d-2)} \varphi}. \quad (3.2)
\]
In the last step, to absorb the singular factor we renormalize the constant $\lambda$ by defining $\frac{m^2}{\beta^2}$.

For the kinetic term, the limit is immediate.
\[
\frac{1}{2} g_{\mu
u} \partial_{\mu} \varphi \partial_{\nu} \varphi \rightarrow \frac{1}{2} g_{\mu
u} \partial_{\mu} \psi \partial_{\nu} \psi \quad (3.3)
\]
But the non-minimal term has no limit, so we expand the exponential.
\[
\frac{d-2}{8(d-1)} R \varphi^2 = \frac{d^2}{2\beta^2(d-1)(d-2)} R e^{\frac{\beta}{2(d-2)} \varphi} = \frac{d^2}{2\beta^2(d-1)(d-2)} R + \frac{d}{2\beta(d-1)} R \psi + \cdots \quad (3.4)
\]
In the $d = 2$ limit, (3.2) and (3.3) and the last term in (3.4) lead to the curved space Liouville Lagrange density (2.5). The first term in (3.4) gives a indeterminate result in the action.
\[
\int \sqrt{|g|} \ L_{d > 2} \ d \downarrow \frac{1}{2} \int \sqrt{|g|} \ L_{\text{Liouville}} + \frac{2}{\beta^2} \frac{\int \sqrt{|g|} R}{d-2} \quad (3.5)
\]
The indeterminacy arises from the fact that in two dimensions $\sqrt{|g|} R$ is the Euler density and its integral is just a surface term – effectively vanishing as far as bulk properties are concerned. So the last term in (3.5) gives $0/0$ at $d = 2$. Evidently, the Liouville model is regained when $0/0$ is interpreted as 0, but this leads to a loss of Weyl invariance. To maintain Weyl invariance on the limit $d \downarrow 2$, we must carefully evaluate the $\psi$-independent $\int \sqrt{|g|} R/(d-2)$ quantity – we need a kind of L’Hospital’s rule for dimensional reduction.

It turns out that a precise evaluation of $\int \sqrt{|g|} R/(d-2)$ in the limit $d \downarrow 2$ can be found, by reference to Weyl’s original ideas.

Before describing this, let us remark that the conformal and Weyl transformation rules for $\varphi$, (2.2) and (2.9), are correctly obtained by substituting (3.1) into the corresponding rules for $\psi$, (1.2) and (1.9b), and passing to limit $d \downarrow 2$. The same connection exists between the equations of motion (2.11) and (1.8). However, the reduction of the $\varphi$ energy-momentum tensor (1.6) produces the $\psi$ tensor (2.6) plus the term $\frac{1}{\beta^2} G_{\mu\nu}/(d-2)$, which is indeterminate at $d = 2$, since both the numerator and denominator vanish. Notice that taking the trace of this quantity, before passing to $d \downarrow 2$, leaves $\frac{1}{\beta^2} (1-d/2)R/(d-2) = -\frac{2}{\beta^2} R$, which cancels the non-vanishing trace of Liouville energy-momentum tensor. This again identifies the indeterminacy as the source of Weyl non-invariance.
4 Weyl's Weyl Invariance

To obtain a definite value for the behavior of \( \int \sqrt{|g|} \frac{R}{(d - 2)} \) in the limit of \( d \downarrow 2 \), we examine once again the Weyl transformation properties of the kinetic term for a scalar field theory in \( d \) dimensions. (The self-interaction is Weyl invariant, and needs no further discussion.) As already remarked, the kinetic term is not Weyl invariant, and this is compensated by the non-minimal interaction, to produce the Weyl invariant kinetic action.

\[
I_{\text{kinetic}} = \int \sqrt{|g|} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{d - 2}{8(d - 1)} R \varphi^2 \right)
\]  

(4.1)

However, Weyl proposed a different mechanism for the construction of a Weyl invariant kinetic term: Rather than using a non-minimal interaction, he introduced a “gauge potential” \( W_\mu \) to absorb the non-variance. One verifies that

\[
I_{\text{Weyl}} = \int \sqrt{|g|} \left( \frac{1}{2} g^{\mu\nu} \left[ \partial_\mu \varphi + (d - 2) W_\mu \varphi \right] \left[ \partial_\nu \varphi + (d - 2) W_\nu \varphi \right] \right)
\]

(4.2)

is invariant against (1.9), provided \( W_\mu \) transforms as

\[
W_\mu \xrightarrow{\text{Weyl}} W_\mu - \frac{1}{2} \partial_\mu \sigma.
\]

(4.3)

We now demand that \( I_{\text{kinetic}} \) in (4.1) coincides with \( I_{\text{Weyl}} \) in (4.2). This is achieved when the following holds.

\[
\frac{R}{4(d - 1)} = D^\mu W_\mu + (d - 2) g^{\mu\nu} W_\mu W_\nu
\]

(4.4)

this curious Riccati-type equation is familiar in \( d = 2 \), where it states that \( \sqrt{|g|} R \) is a total derivative; a condition that is generalized by (4.4) to arbitrary \( d > 2 \).

With the help of (4.4) we evaluate, before passing to \( d \downarrow 2 \), the ambiguous contribution to the action – the last term in (3.5). We have from (4.4)

\[
\frac{\int \sqrt{|g|} R}{4(d - 1)(d - 2)} = \frac{1}{d - 2} \int \partial_\mu (\sqrt{|g|} W^\mu) + \int \sqrt{|g|} g^{\mu\nu} W_\mu W_\nu.
\]

(4.5)

The first term does not contribute, even when \( d \neq 2 \), because the integrand is a total derivative for all \( d \), while the remainder leaves

\[
\lim_{d \downarrow 2} \frac{\int \sqrt{|g|} R}{d - 2} = 4 \int \sqrt{|g|} g^{\mu\nu} w_\mu w_\nu
\]

(4.6)

where \( w_\mu \equiv W_\mu|_{d=2} \) satisfies, according to (3.4),

\[
4D^\mu w_\mu = R \quad \text{at } d = 2.
\]

(4.7)

[Note that (4.3) and (4.7) are consistent with the Weyl transformation formula for \( R \) at \( d = 2 \): \( R_{\text{Weyl}} \propto e^{-2\sigma} (R - 2D^2 \sigma) \).]
Thus to achieve Weyl invariance, the action should be supplemented by the metric-dependent, but $\psi$-independent term.

$$\triangle I = \frac{8}{\beta^2} \int \sqrt{|g|} \ g^{\mu\nu} w_\mu w_\nu$$  (4.8)

According to (4.3) and (4.7), the Weyl variation of $\triangle I$ is

$$\triangle I \rightarrow \triangle I + \frac{2}{\beta^2} \int \sqrt{|g|} \ (R \sigma + g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma).$$  (4.9)

This cancels the Weyl non-invariant response of $I_{\text{Liouville}}$; see (2.10).

It remains to determine $w_\mu$ by solving (4.7). We are of course interested in a local solution, so that the Weyl-invariant Liouville action be local. Such a solution has been found [4]. It is

$$w^\mu = \frac{\varepsilon^{\mu\nu}}{4\sqrt{|g|}} \left( \frac{\varepsilon^{\alpha\beta}}{\sqrt{|g|}} \partial_\alpha g_{\beta\nu} + (\cosh \omega - 1) \partial_\nu \gamma \right).$$  (4.10)

The second term in the parenthesis is the canonical SL (2, R) 1-form, with

$$\cosh \omega = \frac{g^{++} - g^{--}}{\sqrt{|g|}} \quad \text{and} \quad e^\gamma = \sqrt{\frac{g^{++}}{g^{--}}}. \quad (4.11)$$

$[(+,-) \text{ refer to light-cone components } \frac{1}{\sqrt{2}}(x^0 \pm x^1)].$ This portion of $w^\mu$ is Weyl invariant, while the rest verifies the transformation law (4.3). The solution (4.10) is not unique. One may add to (4.10) any Weyl-invariant term of the form $\frac{\varepsilon^{\mu\nu}}{\sqrt{|g|}} \partial_\nu X$, since this will not contribute to (4.7).

Remarkably $w^\mu$ in (4.10) is not a contravariant vector, even though $D_\mu w^\mu$ is the scalar $R/4$. Consequently our Weyl invariant Liouville action $I_{\text{Liouville}} + \triangle I$ is not diffeomorphism invariant. Its $g^{\mu\nu}$ variation defines a traceless energy-momentum tensor, which however is not (covariantly) conserved.

We do not know what to make of this. Perhaps the above mentioned ambiguity can be used to remedy the diffeomorphism non-invariance, but we have not been able to do so. It would seem therefore that a local, curved-space Liouville action can be either diffeomorphism invariant or Weyl invariant, but not both.

If this conjecture is true, we are facing an “anomalous” situation in a classical field theory, which has previously been seen only in a quantized field theory. It is know that in two dimensions, the diffeomorphism invariant Lagrange density $\frac{1}{2\sqrt{|g|}} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$ is also invariant against Weyl transformations that transform the metric tensor, but not the scalar field $\varphi$ \[ i.e. Eq. (1.9) at $d = 2 \]. However, the effective quantum action that is obtained by performing the functional integral over $\varphi$, yields a metric expression which is either diffeomorphism invariant or Weyl invariant but not both [5].

If locality is abandoned, one may readily construct a covariant solution for $w_\mu$ in the form $\partial_\mu w$,

$$\omega_\mu = \partial_\mu \omega,$$  (4.12)

with $w$ transforming under a Weyl transformation as [compare (1.8)]

$$w \rightarrow w - \frac{\sigma}{2}. \quad (4.13)$$
Evidently

\[ D^2 w = R/4, \]

\[ w(x) = \frac{1}{4} \int d^2 y \sqrt{|g(y)|} \frac{1}{D^2(x, y)} R(y), \]  
(4.14)

where the Green’s function is defined by

\[ D_x^2 \frac{1}{D^2(x, y)} = \frac{1}{\sqrt{|g|}} \delta^2(x - y). \]  
(4.15)

Eq. (4.13) is verified by (4.14), and the addition to the Liouville action is just the Polyakov action [5].

\[ \Delta I = \frac{1}{2\beta^2} \int \partial^2 x d^2 y \sqrt{|g(x)|} R(x) \frac{1}{D^2(x, y)} \sqrt{|g(y)|} R(y) \]  
(4.16)

This then provides a diffeomorphism and Weyl invariant action for the Liouville theory, which however is non-local. Whether locality can be also achieved remains an open question.

References

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