QUASI-GORENSTEINESS OF EXTENDED REES ALGEBRAS

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Abstract. Let $R$ be a Noetherian local ring and $I$ a $R$-ideal. It is well-known that if $gr_I(R)$ is Cohen-Macaulay (Gorenstein), then so is $R$, but the converse is not true in general. In this paper we investigate the Cohen-Macaulayness and Gorensteinness of $gr_I(R)$ under the hypothesis of the extended Rees algebra $R[It, t^{-1}]$ is quasi-Gorenstein or the associated graded ring $gr_I(R)$ is a domain.

1. Introduction

A Noetherian ring having a canonical module is called quasi-Gorenstein if it is locally isomorphic to the canonical module. Clearly, a ring is Gorenstein if and only if it is quasi-Gorenstein and Cohen-Macaulay. Murthy [14] showed that a Cohen-Macaulay UFD having a canonical module is Gorenstein. In general, the UFD property implies quasi-Gorensteinness if the ring has a canonical module. There exists a complete UFD’s having a canonical module which are not Cohen-Macaulay, see [5, Theorem 5.8]. This shows that a quasi-Gorenstein ring needs not to be Gorenstein in general. Surprisingly, the quasi-Gorenstein property implies the Gorensteinness for some classes of extended Rees algebras. For definitions of blowup algebras, see Section 2. In this regard, Heinzer, M.-K. Kim, and Ulrich posed the following question.

Question 1.1 ([7, Question 4.11]). Let $(R, m)$ be a local Gorenstein ring and let $I$ be an $m$-primary ideal. Is the extended Rees algebra $R[It, t^{-1}]$ Gorenstein (equivalently Cohen-Macaulay) if it is quasi-Gorenstein?

When the dimension of $R$ is 1, Question 1.1 has an affirmative answer because $R[It, t^{-1}]$ has dimension 2 in this case and quasi-Gorenstein rings satisfy Serre’s condition ($S_2$). The authors showed that Question 1.1 has an affirmative answer when $R$ is a 2-dimensional pseudo-rational ring. The general case still remains open. However, if one removes the condition of $I$ being $m$-primary, then there exists an extended Rees algebra which is a UFD (hence quasi-Gorenstein), but not Gorenstein [12, Example 4.7]. In Section 3 we provide an affirmative answer to Question 1.1 if $I$ is an almost complete intersection, under the additional assumption that the index of nilpotency and the reduction number of $I$ coincide (which is a necessary condition for $R[It, t^{-1}]$ to be Cohen-Macaulay), see Theorem 3.18. We are also able to treat the case when $I$ is a monomial ideal in a polynomial ring in $d$-variables and $I$ has a $d$-generated monomial reduction, see Theorem 3.15. The latter condition, $I$ having such a reduction, is equivalent to the condition that $I$ has only one Rees valuation.

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For monomial ideals in a polynomial ring, the normalization of the extended Rees algebra is Cohen-Macaulay [11]. Therefore, these normalizations are quasi-Gorenstein if and only if they are Gorenstein. In [8] the authors characterized the Gorenstein property of normalized extended Rees algebras of a monomial ideal when the ideal of finite colength has only one Rees valuation. Recall that Rees valuations of monomial ideals correspond to half spaces. We are able to remove the condition of having one Rees valuation, and give a characterization of the Gorenstein property in terms of the half spaces which come from Rees valuations of the ideal.

Since $\text{gr}_I(R) \cong R[It, t^{-1}]/(t^{-1})$ and $t^{-1}$ is a homogeneous non zero-divisor, the associated graded ring $\text{gr}_I(R)$ is Cohen-Macaulay (or Gorenstein) if and only if the extended Rees algebra $R[It, t^{-1}]$ is. Hochster [11, p. 55, Proposition] and Herzog, Simis, Vasconcelos [9, Proposition 1.1] showed that if $R$ is a local Gorenstein ring and the associated graded ring $\text{gr}_I(R)$ is a domain, then $R[It, t^{-1}]$ is quasi-Gorenstein. Hence an affirmative answer to Question 1.1 would imply that $\text{gr}_m(R)$ is Cohen-Macaulay if it is a domain. This version of the question is meaningful even when the ambient ring $R$ is not Gorenstein.

**Question 1.2.** Let $(R, m)$ be a local Cohen-Macaulay ring. Is $\text{gr}_m(R)$ Cohen-Macaulay if it is a domain?

The question has an affirmative answer when $R$ is a complete intersection ring of embedding codimension at most 2, i.e., $\hat{R} \cong S/I$ where $(S, n)$ is a regular local ring and $I$ is a complete intersection ideal of height at most 2. It is natural to ask if the question has an affirmative answer when the ideal $I$ is generated by 3 elements. In Section 5 we prove that this is indeed the case if in addition $I \not\subseteq n^5$ (Theorem 5.7).

We already mentioned that if $\text{gr}_I(R)$ is Cohen-Macaulay, then $R$ is Cohen-Macaulay. Recall that the Cohen-Macaulyness of a ring can be characterized by Serre’s condition $(S_i)$ for every $i$.

**Question 1.3.** Let $R$ be a Noetherian ring and $I$ be an $R$-ideal. If $\text{gr}_I(R)$ satisfies Serre’s condition $(S_i)$ (or $(R_i)$), then does $R[It, t^{-1}]$ satisfy the same condition?

We give a positive answer to Question 1.3 when $R$ is a universally catenary equidimensional local ring and $I$ is not a unit ideal. (Theorem 6.2).

The outline of the paper is as follows: In Section 2 we set up the notation. In Section 3 we start by introducing the graded canonical module of $\mathbb{Z}$-graded rings and study basic properties related to the quasi-Gorensteinness of extended Rees algebras. The two main theorems, Theorem 3.15 and 3.18, are given in this section. In addition, a result on the $a$-invariant of the extended Rees algebra (Theorem 3.25) and the core of powers of the ideal (Theorem 3.21) are presented. In Section 4 we provide a characterization for when the normalized extended Rees algebra of a finite colength monomial ideals in a polynomial ring is Gorenstein. This is a generalization of [8, Theorem 5.6]. In Section 5 we discuss Question 1.2 in detail, and provide a positive answer in the case of almost complete intersections of codimension 2 (Theorem 5.7). In Section 6 we give a positive answer to Question 1.3 when the ring in question is a universally catenary equidimensional local ring and the ideal is not a unit ideal (Theorem 6.2).
2. Preliminaries

All rings are commutative Noetherian with unity. Let \((R, \mathfrak{m})\) be a local ring and \(M\) a finitely generated \(R\)-module. Let \(\mu(M)\) denote the minimal number of generators of \(M\) and let \(\omega_R\) denote a canonical module of \(R\) if it exists. By \(\text{Min}(M)\) and \(\text{Ass}(M)\) we denote the set of minimal primes and associated primes of \(M\), respectively. For \(R\)-ideals \(J \subseteq I\), the ideal \(J\) is called a reduction of \(I\) if there exists an integer \(n\) such that \(JI^n = I^{n+1}\), the smallest such integer is called the reduction number of \(I\) with respect to \(J\), denoted by \(r_J(I)\), and \(r(I) := \min\{r_J(I) : J \text{ a reduction of } I\}\). Let \(\Phi\) be an \(r \times s\) matrix with entries in \(R\). Then \(I_n(\Phi)\) is the ideal generated by \(n \times n\) minors of \(\Phi\). By convention, when \(n \leq 0\), we set \(I_n(\Phi) = R\), and when \(n > \min\{r, s\}\), we set \(I_n(\Phi) = 0\).

For an ideal \(I \subseteq R\), we write
\[
R[It] = \oplus_{i \geq 0} I^i t^i, \quad R[It, t^{-1}] = \oplus_{i \in \mathbb{Z}} I^i t^i, \quad \text{and} \quad gr_t(R) = \oplus_{i \geq 0} I^i / I^{i+1},
\]
and call these rings the Rees algebra, the extended Rees algebra, and the associated graded ring, respectively. Sometimes, they are also called blowup algebras. These are the rings which appear in the construction of blowing up an affine variety along a closed subvariety in algebraic geometry.

A local ring \(R\) having a canonical module \(\omega_R\) is called Gorenstein if \(R\) is Cohen-Macaulay and isomorphic to the canonical module \(\omega_R\). The second property \(R \cong \omega_R\) can be isolated, and it is called the quasi-Gorenstein property. That is, a local ring is quasi-Gorenstein if it is isomorphic to the canonical module \(\omega_R\). Here the canonical module is the dualizing module in the sense of Grothendieck’s local duality theorem. For instance, when \((R, \mathfrak{m})\) is a complete local ring of dimension \(d\), then a canonical module \(\omega_R\) of \(R\) is \(\text{Hom}_R(H^d_{\mathfrak{m}}(R), E_R(R/\mathfrak{m}))\), where \(H^d_{\mathfrak{m}}(-)\) and \(E_R(-)\) denote the \(d\)-th local cohomology with respect to \(\mathfrak{m}\) and the injective envelope, respectively.

Let \(S\) be a Noetherian \(\mathbb{Z}\)-graded ring with unique homogeneous maximal ideal \(\mathfrak{m}\) and assume that \(\mathfrak{m}\) is maximal. The \(a\)-invariant of \(S\), denoted by \(a(S)\), is \(\max\{i \in \mathbb{Z} : \text{soc}(H^d_{\mathfrak{m}}(S)) \neq 0\}\). Here \(\text{soc}(M) = \{0\} :_M \mathfrak{m}\) for any graded \(S\)-module \(M\). If \(S\) has a graded canonical module \(\omega_S\), then by graded local duality \(a(S) = -\min\{i \in \mathbb{Z} : |\omega_S / \mathfrak{m} \omega_S|_i \neq 0\}\). If \(S\) is positively graded, then this number is \(\max\{i \in \mathbb{Z} : |H^d_{\mathfrak{m}}(S)|_i \neq 0\}\).

In the sequel we use the book by Bruns and Herzog [2] as a reference for basic definitions and terminologies.

3. Quasi-Gorensteiness of extended Rees algebras

Graded canonical modules

Let \(R\) be a Noetherian \(\mathbb{Z}\)-graded ring with unique maximal homogeneous ideal \(\mathfrak{m}\). Then the subring \(R_0\) is local. We write \(\mathfrak{m}\) for the maximal ideal of \(R_0\). Let \(E_{R_0}(R_0/\mathfrak{m})\) be the injective envelope of the residue field of \(R_0\). For a homogeneous prime ideal \(p\) of \(R\), we write \(R_{(p)} := S^{-1}R\) where \(S\) is the set of the homogeneous elements of \(R\) which are not in \(p\). Observe that \(R_{(p)}\) is \(\mathbb{Z}\)-graded and has
unique maximal homogeneous ideal \( pR/(p) \). For \( \mathbb{Z} \)-graded \( R \)-modules \( M \) and \( N \), let \(^\ast\)\( \text{Hom}_R(M, N) \) denote the \( R \)-submodule of \( \text{Hom}_R(M, N) \) generated by the homogeneous \( R \)-linear maps of arbitrary degree from \( M \) to \( N \). For a finitely generated \( \mathbb{Z} \)-graded \( R \)-module \( M \), let \( \widehat{M} \) denote the tensor product \( M \otimes_{R_0} \widehat{R}_0 \). We say \( M \) is \(^\ast\)-complete if \( M \cong \widehat{M} \) by the natural isomorphism. In particular, \( \widehat{R} \) is \(^\ast\)-complete and \( R \) is \(^\ast\)-complete if and only if \( R_0 \) is complete.

**Definition 3.1.** Let \( d = \dim R_{\mathfrak{M}} \). A finitely generated graded \( R \)-module \( \omega_R \) is called a graded canonical module of \( R \) if

\[
\omega_R \cong ^\ast\text{Hom}_{R_0}(\mathcal{H}^d_{\mathfrak{M}}(R), E_{R_0}(R_0/\mathfrak{m}))
\]

as graded \( R \)-modules.

**Remark 3.2.** Let \( R \) be a Noetherian \( \mathbb{Z} \)-graded ring with a unique maximal homogeneous ideal. If \( R \) is \(^\ast\)-complete, then \( R \) has a graded canonical module.

**Lemma 3.3** (cf. [2, Corollary 3.6.14]). Let \( R \) be a \( \mathbb{Z} \)-graded Cohen-Macaulay ring with a unique maximal homogeneous ideal. Assume that \( R \) has a graded canonical module \( \omega_R \). If \( \underline{x} = x_1, \ldots, x_n \) form a homogeneous regular sequence on \( R \), then \( \underline{x} \) form a regular sequence on \( \omega_R \) and we have

\[
\omega_{R/(\underline{x})} \cong (\omega_R/\underline{x}\omega_R)(\sum_{i=1}^n \deg(x_i)).
\]

**Proof.** It suffices to show the statement when \( n = 1 \). Let \( \mathfrak{N} \) be the unique maximal homogeneous ideal, \( d = \dim R_{\mathfrak{N}} \), and \( \mathfrak{m} \) the maximal ideal of \( R_0 \). Since \( R \) and \( R/(x) \) are Cohen-Macaulay, the exact sequence

\[
0 \to R(-\deg(x)) \xrightarrow{x} R \to R/(x) \to 0
\]

induces an exact sequence

\[
0 \to \mathcal{H}^{d-1}_{\mathfrak{N}}(R/(x)) \xrightarrow{x} \mathcal{H}^d_{\mathfrak{N}}(R(-\deg(x)) \xrightarrow{x} \mathcal{H}^d_{\mathfrak{N}}(R) \to 0.
\]

Taking \(^\ast\)\( \text{Hom}_{R_0}(-, E_{R_0}(R_0/\mathfrak{m})) \) we obtain the exact sequence

\[
0 \to \omega_R \xrightarrow{x} \omega_R(\deg(x)) \to ^\ast\text{Hom}_{R_0}(\mathcal{H}^{d-1}_{\mathfrak{N}}(R/(x)), E_{R_0}(R_0/\mathfrak{m})) \to 0.
\]

This shows that \( x \) is a non zerodivisor of \( \omega_R \). Hence we are done once we have shown that \( \omega_{R/(x)} \cong ^\ast\text{Hom}_{R_0}(\mathcal{H}^{d-1}_{\mathfrak{N}}(R/(x)), E_{R_0}(R_0/\mathfrak{m})) \). From the ring homomorphism \( R \to R/(x) \), we obtain a surjective ring homomorphism \( R_0 \to [R/(x)]_0 \) of local rings. We write \( (\overline{R}_0, \overline{\mathfrak{m}}) \) for the local ring \( [R/(x)]_0 \). By [13, Exercise 13] we have \( E_{\overline{R}_0}(\overline{R}_0/\overline{\mathfrak{m}})) \cong \text{Hom}_{\overline{R}_0}(\overline{R}_0, E_{R_0}(R_0/\mathfrak{m})) \). Therefore, by the hom-tensor adjointness we have

\[
^\ast\text{Hom}_{\overline{R}_0}(-, E_{\overline{R}_0}(\overline{R}_0/\overline{\mathfrak{m}})) \cong ^\ast\text{Hom}_{\overline{R}_0}(-, ^\ast\text{Hom}_{R_0}(\overline{R}_0, E_{R_0}(R_0/\mathfrak{m})))
\]

\[
\cong ^\ast\text{Hom}_{R_0}(- \otimes_{\overline{R}_0} \overline{R}_0, E_{R_0}(R_0/\mathfrak{m}))
\]

\[
\cong ^\ast\text{Hom}_{R_0}(-, E_{R_0}(R_0/\mathfrak{m})
\]

for any \( R/(x) \)-module in the first variable. Since \( \mathcal{H}^{d-1}_{\mathfrak{N}}(R/(x)) \cong \mathcal{H}^{d-1}_{\mathfrak{N}}(R/(x)) \), this shows the statement.

**Remark 3.4** ([4, Exercise 7.5]). Let \( M, N \) be a finitely generated \( \mathbb{Z} \)-graded \( R \)-module. If \( \widehat{M} \cong \widehat{N} \), then \( M \cong N \).
The following lemma is a partial converse of Lemma 3.3.

**Lemma 3.5.** Let $R$ be a $\mathbb{Z}$-graded Cohen-Macaulay ring with a unique maximal homogeneous ideal. Assume that $\underline{x} = x_1, \ldots, x_n$ form a homogeneous regular sequence, and $R/(x_1, \ldots, x_n)$ has a graded canonical module $\omega_{R/(x_1, \ldots, x_n)}$. Write $\rho = \sum_{i=1}^n \deg(x_i)$. If $\omega_R/(x_1, \ldots, x_n)(a)$ for some $a \in \mathbb{Z}$, then $\omega_R \cong R(a - \rho)$. In particular, $R$ has a graded canonical module.

**Proof.** It suffices to show the statement when $n = 1$. By Remark 3.2 $\omega_R$ exists, and by Remark 3.4 it suffices to show that $R(a - \rho) \cong \omega_R$. Hence we may assume that $R$ is $^*$-complete. By Lemma 3.3 we have $\omega_R/x\omega_R(\deg(x)) \cong \omega_R/(x) \cong R/(x)(a)$. By Nakayama’s lemma we see that $\omega_R$ is a cyclic $R$-module. Consider the exact sequence

$$0 \to K \to R(a - \deg(x)) \to R \to 0.$$  

We tensor Equation (2) with $R/(x)$. By Lemma 3.3 $x$ is a non zero divisor on $\omega_R$. This implies that $\text{Tor}_R^d(\omega_R, R/(x)) = 0$, i.e., Equation (2) remains exact after applying $- \otimes_R R/(x)$. Since $R/(x)(a - \deg(x)) \cong \omega_R/x\omega_R$, one has $K/xK = 0$. By Nakayama’s lemma we obtain $K = 0$. Indeed this implies $R(a - \deg(x)) \cong \omega_R$. □

**Theorem 3.6** (cf. [15, Theorem 5.12]). Let $R$ and $S$ be Noetherian $\mathbb{Z}$-graded rings with unique maximal homogeneous ideals $\mathfrak{M}$ and $\mathfrak{N}$, respectively. Let $\phi : S \to R$ be a graded ring homomorphism. Assume that $R$ is a finitely generated $S$-module, $\phi(S_0) = R_0$, and $S$ is Cohen-Macaulay. Write $\dim S_0 = n$ and $\dim R_0 = d$. If $S$ has a graded canonical module $\omega_S$, then one has

$$\omega_R \cong ^*\text{Ext}_S^{n-d}(R, \omega_S).$$

**Proof.** By Remark 3.4 it suffices to show the isomorphism after $^*$-completions. Since $\phi(S_0) = R_0$, by $^*$-completing both $R$ and $S$ as $S$-modules we may assume that $R$ and $S$ are $^*$-complete. Let $m_{S_0}$ and $m_{R_0}$ denote the maximal ideals of $S_0$ and $R_0$, respectively. Write $E': = E_{S_0}(S_0/m_{S_0})$ and $E := E_{R_0}(R_0/m_{R_0})$. By [13, Exercise 13] we have $E \cong \text{Hom}_{S_0}(R_0, E')$. By hom-tensor adjointness and the graded version of the local duality theorem [2, Theorem 3.6.19(b)] we have

$$\omega_R \cong ^*\text{Hom}_{R_0}(H_{\mathfrak{M}}^d(R), E)$$

$$\cong ^*\text{Hom}_{R_0}(H_{\mathfrak{N}}^d(R), \text{Hom}_{S_0}(R_0, E'))$$

$$\cong ^*\text{Hom}_{S_0}(H_{\mathfrak{N}}^d(R_0 \otimes_{R_0} R_0, E'))$$

$$\cong ^*\text{Hom}_{S_0}(H_{\mathfrak{N}}^d(R, E'))$$

$$\cong ^*\text{Ext}_S^{n-d}(R, \omega_S).$$

□

**Corollary 3.7.** Let $R$ be a Noetherian $\mathbb{Z}$-graded ring with unique maximal homogeneous ideal $\mathfrak{M}$. Let $S = A[X_1, \ldots, X_n]$ be a $\mathbb{Z}$-graded polynomial ring over a Gorenstein local ring $A$. Assume that there exists a surjective graded ring homomorphism $\phi : S \to R$ with $\phi(A) = R_0$ and $\mathfrak{M}$ is maximal. Let $\mathfrak{N} := \phi^{-1}(\mathfrak{M})$, $g = \text{ht} \ker(\phi)$, and $\rho = \sum_{i=1}^m \deg(X_i)$. Then one has

$$\omega_R \cong ^*\text{Ext}_{S_{(\mathfrak{M})}}^g(R, S_{(\mathfrak{M})})(-\rho)$$

$$\cong ^*\text{Ext}_S^g(R, S)(-\rho).$$

Furthermore, $\omega_{S_{(\mathfrak{M})}} \cong S_{(\mathfrak{M})}(-\rho)$.
Proof. Since \( \operatorname{Ext}^g_R(R, S) \) is a graded \( R \)-module and \( R = R_{(\mathfrak{m})} \), we have
\[
\operatorname{Ext}^g_R(R, S) \cong \operatorname{Ext}^g_{R_{(\mathfrak{m})}}(R, S) \\
\cong \operatorname{Ext}^g_{R_{(\mathfrak{m})}}(R_{(\mathfrak{m})}, S_{(\mathfrak{m})}) \\
\cong \operatorname{Ext}^g_{R_{(\mathfrak{m})}}(R_{(\mathfrak{m})}, S(\mathfrak{m})) \\
\cong \operatorname{Ext}^g_{R_{(\mathfrak{m})}}(R, S(\mathfrak{m})).
\]

Therefore, it suffices to show the first isomorphism.

First we show that we can reduce to the case where all \( X_i \) are in \( \mathfrak{m} \). If \( \deg(X_i) \neq 0 \), then \( \phi(X_i) \) is a homogeneous element of degree not equal to zero in \( R \). Since \( \mathfrak{m} \) is maximal, there is no homogeneous unit of degree not equal to zero. Hence \( \phi(X_i) \in \mathfrak{m} \), i.e., \( X_i \in \mathfrak{m} \). Therefore, if \( X_i \notin \mathfrak{m} \), then \( \deg(X_i) = 0 \). Suppose \( X_i \notin \mathfrak{m} \). Since \( \phi(X_i) \in R_0 = \phi(A) \), there exist \( z_i \in A \subseteq S_0 \) such that \( z_i = \phi(X_i) \). Since \( X_i - z_i \) is in \( \ker(\phi) \), \( X_i - z_i \in \mathfrak{m} \). Replacing the variable \( X_i \) by \( X_i - z_i \), we may assume that \( X_i \in \mathfrak{m} \).

Since \( X_i \in \mathfrak{m} \) for all \( i \), we have \( \mathfrak{m} = (m, X_1, \ldots, X_n) \) where \( m \) is the maximal ideal of \( A \). Let \( S' = S_{(\mathfrak{m})} \). Since \( \mathfrak{m} = \phi^{-1}(\mathfrak{m}) \), \( \phi \) factors through \( S' \). Write \( \phi' : S' \to R \) for the ring homomorphism induced by \( \phi \). Since \( S' \) and \( R \) have unique maximal homogeneous ideals, \( \phi' \) surjective, and \( \phi'(A) = R_0 \), by Theorem 3.6 we are done once we have shown that \( \omega_{S'} \cong S'(-\rho) \). By Remark 3.4 it suffices to show the isomorphism after \( \ast \)-completion. Hence we may assume that \( S' \) is \( \ast \)-complete. We have shown that \( \omega_{S'} \cong S'(-\rho) \).

A Noetherian \( \mathbb{Z} \)-graded ring \( R \) with a unique maximal homogeneous ideal is called quasi-Gorenstein if \( \omega_R \cong R(a) \) for some \( a \in \mathbb{Z} \). If the unique maximal ideal is maximal, the number \( a \) is well-defined, and it is called the \( a \)-invariant of \( R \). A Gorenstein \( \mathbb{Z} \)-graded ring is a quasi-Gorenstein ring which is Cohen-Macaulay. The ring \( S_{(\mathfrak{m})} \) in the remark above is a Gorenstein ring. Notice that this definition agrees with [2, Theroem 3.6.19] when \( R \) is Cohen-Macaulay. However, we do no require a ring to be Cohen-Macaulay.

**Graded canonical modules of extended Rees algebras**

**Proposition 3.8.** Let \( (R, m) \) be a Cohen-Macaulay local ring having a canonical module \( \omega_R \). Let \( I \subseteq m \) be an \( R \)-ideal. If \( R[I^*t^{-1}] \) is quasi-Gorenstein, then \( R \) is Gorenstein.

**Proof.** The canonical module of \( R[I^*t^{-1}] \) is isomorphic to \( \omega_R[I^*t^{-1}] \) [1, Proposition 4.1]. Since \( R[I^*t^{-1}] = R[I^*t^{-1}]_{(I^*t^{-1})} \) and the canonical module localizes, \( \omega_R[I^*t^{-1}] \cong (\omega_R[I^*t^{-1}]_{(I^*t^{-1})})_{(I^*t^{-1})} \cong (R[I^*t^{-1}]_{(I^*t^{-1})})_{(I^*t^{-1})} = R[I^*t^{-1}]_{(I^*t^{-1})} \) where \( a \) is the \( a \)-invariant of \( R[I^*t^{-1}] \). Hence we have \( R \cong \omega_R \), i.e., \( R \) is Gorenstein. \( \square \)
Lemma 3.9. Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring having a canonical module \(\omega_R\) and \(I \subseteq \mathfrak{m}\) an ideal. If \(R[It, t^{-1}]\) is quasi-Gorenstein, then the canonical module \(\omega_{gr_I(R)}\) of \(gr_I(R)\) is of rank 1, and there exists an inclusion \(gr_I(R)(\rho) \subseteq \omega_{gr_I(R)}\) of graded \(gr_I(R)\)-modules where \(\rho = a(R[It, t^{-1}]) - 1\).

Proof. Write \(T = R[It, t^{-1}]\) and \(G = gr_I(R)\). We need to show that \((\omega_G)_p \cong G_p\) for all associated primes of \(\omega_G\). Since \(G\) is unmixed, \(\text{Ass}(\omega_G) = \text{Min}(G)\). Let \(\pi : T \xrightarrow{\text{nat}} G\) and \(p \in \text{Min}(G)\). Write \(P = \pi^{-1}(p)\). Since \(T\) satisfies Serre’s condition \((S_2)\) and \(P \in \text{Min}(T/(t^{-1}))\), \(P\) is of height 1. Hence \(T_P\) is Gorenstein. Since \(G_P \cong T_P/(t^{-1})_P\), \(G_P\) is Gorenstein. Since a canonical module localizes, indeed this shows that \((\omega_G)_p \cong G_p\).

Now, we show the inclusion \(gr_I(R)(\rho) \subseteq \omega_{gr_I(R)}\). Let \(S := R[X_1, \ldots, X_n]\) be a \(\mathbb{Z}\)-graded polynomial ring over \(R\) such that \(\phi : S \rightarrow R\) is a surjective homogeneous ring homomorphism. Let \(H = \ker(\phi)\) and \(g = \lambda t h\). Observe that \(G = S/(H, h)\) where \(\phi(h) = t^{-1}\). From the exact sequence of graded \(T\)-modules \(0 \rightarrow T(1) \xrightarrow{t^{-1}} T \rightarrow G \rightarrow 0\), we have an exact sequence of graded Ext modules

\[
0 \rightarrow \text{Ext}^2_S(T, S)(a) \xrightarrow{t^{-1}} \text{Ext}^2_S(T(1), S)(a) \rightarrow \text{Ext}^3_S(G, S)(a)
\]

where \(a\) is the \(a\)-invariant of \(T\). By Corollary 3.7 this shows that

\[
0 \rightarrow \omega_T \xrightarrow{t^{-1}} \omega_T(-1) \rightarrow \omega_G.
\]

The result follows since \(\text{Ext}^3_S(G, S)(a) \cong \omega_G\) and \(\omega_T \cong T(a)\).

Theorem 3.10. Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring having a canonical module \(\omega_R\) and \(I \subseteq \mathfrak{m}\) an ideal. Assume that a graded canonical module \(\omega_{R[It, t^{-1}]}\) of \(R[It, t^{-1}]\) satisfies Serre’s condition \((S_3)\). We have

(a) If \(R[It, t^{-1}]\) is quasi-Gorenstein, then \((\omega_{R[It, t^{-1}]/t^{-1}\omega_{R[It, t^{-1}]}})(1)\) is a graded canonical module of \(gr_I(R)\).

(b) \(R[It, t^{-1}]\) is quasi-Gorenstein if and only if \(gr_I(R)\) is quasi-Gorenstein. Furthermore, if the conditions hold, then \(R[It, t^{-1}]\) satisfies Serre’s condition \((S_3)\).

Proof. Let \(G = gr_I(R)\) and \(T = R[It, t^{-1}]\). Let \(\omega_G\) be a graded canonical module of \(G\).

(a): We have

\[
0 \rightarrow (\omega_T/t^{-1}\omega_T)(-1) \rightarrow \omega_G \rightarrow L \rightarrow 0
\]

where \(L\) is the cokernel of the natural map in (3). We show that \(L = 0\). It is equivalent to showing that \(L_p = 0\) for all \(p \in \text{Ass}(L)\). Since \(\omega_T\) satisfies Serre’s condition \((S_3)\), \(\omega_T/t^{-1}\omega_T\) satisfies Serre’s condition \((S_2)\) as a \(G\)-module. Let \(P \in \text{Ass}(L)\). Since \(T_P\) is Gorenstein for all prime ideals of height at most 2, \(G_P\) is Gorenstein for all prime ideals of height at most 1. Hence if \(P \in \text{Ass}(L)\), then \(ht p \geq 2\). Since \(p\) is an associated prime of \(L\), \(\text{depth}(L_p) = 0\). However, \(\text{depth}(\omega_T/t^{-1}\omega_T)_p \geq 2\) and \(\text{depth}(\omega_G)_p \geq 2\). This implies that \(\text{depth}(L_p) \geq 1\), and this is a contradiction.

(b): The forward direction follows immediately from part (a). For the other direction, we only need to show that \(\omega_T\) is a cyclic faithful \(T\)-module. We have \(G(a) \cong (T/t^{-1}T)(a)\) where \(a\) is the \(a\)-invariant of \(G\), and \(\mu_T(\omega_T) = \lambda_T(\omega_T/m_T\omega_T) = \mu_G(\omega_T/t^{-1}\omega_T) = 1\) where \(\mathfrak{m}_T\) is the maximal homogeneous ideal of \(T\) and \(\lambda(-)\)
denotes the length of a module. Hence $\omega_T$ is cyclic. Also $\omega_T$ is faithful since $T$ is unmixed cf. [1, (1.8)(a)(c)]. Therefore $\omega_T \cong T(a(G) + 1)$.

Now, we show the second part of (b). Since $G$ satisfies Serre’s condition $(S_2)$, $T$ satisfies Serre’s condition $(S_3)$ for all ideals which contain $t^{-1}$. Let $P \in \text{Spec}(T)$ and $t^{-1} \notin P$. Since $t^{-1} = t$, $T_P = (R[It, t^{-1}]_{t^{-1}})_P = R[t, t^{-1}]_P$. The ring $R[t, t^{-1}]$ is Cohen-Macaulay since $R$ is. Hence $R[t, t^{-1}]_P$ is Cohen-Macaulay.

\[\Box\]

**A QUESTION AND THE MAIN THEOREMS**

It is interesting to see under which conditions a quasi-Gorenstein extended Rees algebra is Gorenstein. To this end, Heinzer, M.-K. Kim, and Ulrich posed the next question. In this section we present two cases which give an affirmative answer to this question.

**Question 3.11** ([7, Question 4.11]). Let $(R, \mathfrak{m})$ be a local Gorenstein ring. Let $I$ be an $\mathfrak{m}$-primary ideal. If the extended Rees algebra $R[It, t^{-1}]$ is quasi-Gorenstein, then is it Gorenstein?

In the same paper, the authors characterized the quasi-Gorenstein property of the extended Rees algebra in terms of colon ideals.

**Proposition 3.12** ([7, Theorem 4.1]). Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $d$, and let $I$ be an $\mathfrak{m}$-primary ideal. Assume that $J \subseteq I$ is a reduction of $I$ with $\mu(J) = d$. Let $r := r_J(I)$ be the reduction number of $I$ with respect to $J$ and $k \geq r$. Then the graded canonical module $\omega_{R[It, t^{-1}]}$ of $R[It, t^{-1}]$ has the form

\[\omega_{R[It, t^{-1}]} \cong \bigoplus_{i \in \mathbb{Z}} (J^{i+k} :_R I^k)^{i+d-1} \cdot \]

In particular, for $a \in \mathbb{Z}$, the following are equivalent:

(a) $R[It, t^{-1}]$ is quasi-Gorenstein with $\mathbf{a}$-invariant $a$.
(b) $J^i :_R I^j = I^{i+e-(r-d+1)}$ for every $i \in \mathbb{Z}$.

**Definition 3.13**. Let $R$ be a Noetherian local ring. Let $I$ be an ideal and $J$ a minimal reduction of $I$. Then the index of nilpotency, denoted by $s_J(I)$, is

\[\min \{i \mid I^{i+1} \subseteq J \}, \quad \text{and} \quad s(I) = \max \{s_J(I) \mid J \text{ is a minimal reduction of } I \}; \]

**Lemma 3.14** ([7, Remark 4.4]). We use the setting of Proposition 3.12. In addition assume that $R[It, t^{-1}]$ is quasi-Gorenstein.

(a) One has $s_J(I) - d + 1 \leq a(R[It, t^{-1}]) \leq r_J(I) - d + 1$.
(b) Write $r = r_J(I)$. One has $\max \{n \mid I^n \subseteq J^n \} = r - d + 1 - a(R[It, t^{-1}])$.

Now, we are ready to provide a positive answer to Question 3.11 for a class of monomial ideals in a polynomial ring.

**Theorem 3.15**. Let $R$ be a polynomial ring in $d$-variables over a field, and $I$ a monomial ideal of height $d$. Assume that $I$ has a $d$-generated monomial reduction. If $R[It, t^{-1}]$ is quasi-Gorenstein, then $R[It, t^{-1}]$ is Gorenstein.

**Proof.** We show that $\text{gr}_J(R)$ is Cohen-Macaulay. Let $J = (g_1, \ldots, g_d)$ be the monomial reduction of $I$, and write $r := r_J(I)$. Let $u = r - d + 1 - a(R[It, t^{-1}])$ where $a(R[It, t^{-1}])$ is the $\mathbf{a}$-invariant of $R[It, t^{-1}]$. Since $R[It, t^{-1}]$ is quasi-Gorenstein, we have $J^i : I^j = I^{i-u}$ for all $i$ by Proposition 3.12(b). The Cohen-Macaulayness
of \( \text{gr}_J(R) \) follows by the Valabrega-Valla criterion \cite[Theorem 1.1]{21} once we have shown that \( J \cap I^i \subseteq JI^i \) for \( 0 \leq i \leq r \). Recall that all ideals in question are monomial ideals. Let \( a \) be an arbitrary monomial in \( J \cap I^i \). Since \( a \in J \) and \( J \) is a monomial ideal generated by the \( g_i \)'s, we can write \( a = a'g \) where \( a' \in R \) and \( g = g_j \) for some \( j \). We want to prove that \( a' \in I^{i-1} \). For an arbitrary element \( z \in I^r \), we have \( az \in I^{i+r} = JI^r \subseteq JI^j \), where the last inclusion follows from Lemma 3.14(b). Then \( az = a'zg \in JI^{i+u} \), and this implies \( a'z \in JI^{i+u-1} \). Since \( g + J^2 \) is a non-zero-divisor on \( \text{gr}_J(R) \). Since \( z \in I^i \) is arbitrary, we conclude that indeed \( a' \in JI^{i+u-1} : I^r = I^{i-1} \). \( \square \)

In the rest of this section we use the setting of Proposition 3.12 and study quasi-Gorenstein extended Rees algebras under the condition that \( s_J(I) = r_J(I) \) for some \( d \)-generated minimal reduction \( J \) of \( I \). This is a necessary condition if the associated graded ring \( \text{gr}_J(R) \) is Cohen-Macaulay by \cite[Theorem 1.1]{21}.

Remark 3.16. Let \((R, m)\) be a \( d \)-dimensional Gorenstein local ring and \( I \) an \( m \)-primary ideal. Assume that \( s_J(I) = r_J(I) \) for some \( d \)-generated minimal reduction \( J \) of \( I \). Then \( R[I, t^{-1}] \) is quasi-Gorenstein if and only if \( J^i : I^r = I^i \) for all \( i \in \mathbb{Z} \).

Definition 3.17 \((\cite{19})\). Let \((R, m)\) be a Noetherian local ring and \( I \) an \( m \)-primary ideal. The ideal \( I \) called \( n \)-standard if \( J \cap I^i = JI^{i-1} \) for all \( i \leq n \).

Recall that when \( R \) is a local Cohen-Macaulay ring having a canonical module and \( I \) an \( R \)-ideal, if \( R[I, t^{-1}] \) is quasi-Gorenstein, then \( R \) is Gorenstein.

Theorem 3.18. Let \((R, m)\) be a \( d \)-dimensional Cohen-Macaulay local ring having a canonical module and \( I \) an \( m \)-primary ideal. Assume that \( R[I, t^{-1}] \) is quasi-Gorenstein and \( s_J(I) = r_J(I) \) for some \( d \)-generated minimal reduction \( J \) of \( I \). Then \( I \) is \( 2 \)-standard.

Proof. Since \( R[I, t^{-1}] \) is quasi-Gorenstein and \( s_J(I) = r_J(I) \), we have \( J^i : I^r = I^i \) for all \( i \in \mathbb{Z} \) by Remark 3.16. We show that \( J \cap I^2 \subseteq JI \). Let \( J = (x_1, \ldots, x_d) \). Let \( a \in J \cap I^2 \). Then we may write \( a = \sum a_ix_i \) for some \( a_i \) in \( R \). For any \( z \in I^r \), \( az \in I^{i+2} = I^{i+2} \subseteq JI^{i+2} \subseteq J^2I^r \subseteq J^2 \), that is \( z \sum a_ix_i = \sum za_ix_i \) in \( J^2 \). Since \( J/J^2 \) is a free \( R/J \)-module with basis \( x_1 + J^2, \ldots, x_d + J^2 \), we obtain \( za_i \in J \). This implies that indeed \( a_i \in J : I^r = I \). \( \square \)

Lemma 3.19. Let \((R, m, k)\) be a \( d \)-dimensional Cohen-Macaulay local ring having a canonical module with infinite residue field \( k \). Let \( I \) be an \( m \)-primary ideal. Assume that \( R[I, t^{-1}] \) is quasi-Gorenstein and \( s_J(I) = r_J(I) \) for some minimal reduction \( J \) of \( I \).

(a) One has \( r_J(I) = r(I) \) and \( s_J(I) = s(I) \); in particular, \( s(I) = r(I) \).
(b) \( a(\text{gr}_J(R)) = a(R[I, t^{-1}]) - 1 = d - r_J(I) \) where \( a(\text{gr}_J(R)) \) and \( a(R[I, t^{-1}]) \) are the \( a \)-invariants of \( \text{gr}_J(R) \) and \( R[I, t^{-1}] \), respectively.

Proof. (a): By Lemma 3.14(a) we have \( s_J(I) - d + 1 = a := a(R[I, t^{-1}]) = r_J(I) - d + 1 \). Let \( L, K \subseteq I \) be minimal reductions with \( r_L(I) = r(I) \) and \( s_K(I) = s(I) \), respectively. Apply Lemma 3.14(a) to see that \( s(I) - d + 1 \leq a \leq r(I) - d + 1 \). This shows that \( s(I) \leq s_J(I) \) and \( r(I) \geq r_J(I) \). The other direction of the inequalities follows from the definition of \( s(I) \) and \( r(I) \).
(b): Let $T := R[It, t^{-1}]$ and $G := \text{gr}_I(R)$. From the exact sequence (3), we have $a(G) \geq a(T) - 1$. We apply [20, Proposition 3.2] to get the other inequality, $a(G) \leq r(I) - d$. 

\textbf{Theorem 3.20.} Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring having a canonical module with infinite residue field. Let $I$ be an $\mathfrak{m}$-primary ideal. Assume that $I$ is an almost complete intersection ideal, i.e., $\mu(I) \leq \text{ht} I + 1$. The following are equivalent:

(a) $R[It, t^{-1}]$ is Gorenstein

(b) $R[It, t^{-1}]$ is quasi-Gorenstein and $s_J(I) = r_J(I)$ for some minimal reduction $J$ of $I$.

\textit{Proof.} Recall that if $I$ is a complete intersection, i.e., $\mu(I) = \text{ht} I$, then $R[It, t^{-1}]$ is a Cohen-Macaulay ring, and the equivalence follows immediately. Suppose that $\mu(I) = \text{ht} I + 1$. The implication $(a) \implies (b)$ is obvious. Let $d = \dim R$ and $J$ be a minimal reduction such that $r = r_J(I) = s_J(I)$. Choose a generating set $x_1, \ldots, x_d$ of $J$. Since $J$ is a minimal reduction of $I$, a generating set of $J$ can be extended to that of $I$. Hence we may write $I = J + (x)$ for some $x$ in $R$. By [21, Theorem 1.1] it suffices to show that $J \cap I^i \subseteq JI^{i-1}$ for $1 \leq i \leq r$. We claim that $J \cap (x)^i \subseteq I^{i+1}$. Since $I = J + (x)$,

\[ J \cap I^i \subseteq JI^{i-1} \iff J \cap (J + (x))^i \subseteq JI^{i-1} \]

\[ \iff J \cap (J^i + xJ^{i-1} + \cdots + (x)^i) \subseteq JI^{i-1} \]

\[ \iff J^i + xJ^{i-1} + \cdots + x^{i-1}J + J \cap (x)^i \subseteq JI^{i-1} \]

\[ \hspace{1cm} \text{(since } J \supseteq J^i, x^iJ) \]

\[ \iff J \cap (x)^i \subseteq JI^{i-1}. \]

Observe that $J \cap (x)^i = J \cap (x^i) = x^i(J :_R I^i) = x^i(\mathfrak{m} : I^i)$. One has $I^i x^j(J : I^i) \subseteq I^{i+1} (J : I^i) \subseteq I^{i+j} (J : I^i) = J^i I^{i+j} (J : I^i) = J^i I^{i+j} I^i (J : I^i) \subseteq J^i I^{i+j} I^i \subseteq J^{i+j}$. Therefore $x^j(J : I^i) \subseteq J^{i+j+1} = I^{i+1}$. This implies that $J \cap (x)^i \subseteq J \cap I^{i+1}$.

We apply decreasing induction on $i$. When $i = r$, $J \cap (x)^r \subseteq I^{r+1} = JI^{r+1}$. For $i < r$, $J \cap I^{i+1} = JI^i$ by the induction hypothesis. Hence we have $J \cap (x)^i \subseteq J \cap I^{i+1} = JI^i \subseteq JI^{i-1}$.

\section*{Quasi-Gorenstein extended Rees algebra and the core}

Let $R$ be a local ring and $I$ an $R$-ideal. Since the extended Rees algebra $R[It, t^{-1}]$ depends not only on the ring $R$, but also on the ideal $I$, it is interesting to see which properties of the ideal $I$ can be deduced if $R[It, t^{-1}]$ is quasi-Gorenstein. In this section, we compute the core of powers of an ideal $I$ when $R[It, t^{-1}]$ is quasi-Gorenstein.

\textbf{Theorem 3.21.} Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring having a canonical module and $k = R/\mathfrak{m}$. Let $I$ be an $\mathfrak{m}$-primary ideal. Assume that $R[It, t^{-1}]$ is quasi-Gorenstein and either characteristic of $k$ is zero or greater than $r(I)$. Let $a := a(R[It, t^{-1}])$ be the $a$-invariant of $R[It, t^{-1}]$. Then $\text{core}(I^u) = I^{du+a}$ for all $u \in \mathbb{Z}$.

\textit{Proof.} We may assume that the residue field is infinite. Let $J$ be a minimal reduction of $I$ with $r := r_J(I) = r(I)$. Fix a minimal generating set $x_1, \ldots, x_d$ of $J$ where $d = \dim R$. Let $J' = J^{[u]} := (x_1^u, \ldots, x_d^u)$ and $I' = I^u$. Then $J'$ is a minimal
reduction of \( I' \). By [16, Theorem 4.5] \( \text{core}(I) = J^{n+1} : I^n \) for \( n \geq r(I) \). We compute the core of \( I' \). We use [17, Lemma 2.2] which shows \( J^{n+1} : I^n = J[n] : I^{dn} \) for \( n \gg 0 \) in our setting. For \( n \gg 0 \) one has
\[
\text{core}(I') = J^{n+1} : I^n = J[n+1] : I^{dn} = (J[n])^{n+1} : (I^u)^{dn} = J[nu+u] : I^{udn} = J[nu+u] : I^{udn-r} I^r
\]
\[
= J[nu+u] : J^{udn-r} I^r = (J[nu+u] : J^{udn-r}) : I^r = J^{d(nu+u-1)+1-(udn-r)} : I^r = J^{du-d+1+r} : I^r = J^{du+a}
\]
where the last equality follows from Proposition 3.12(b). 

**More on the a-invariant**

In this subsection we study the \( a \)-invariant of the extended Rees algebras when it is quasi-Gorenstein and its relative integral closure is Cohen-Macaulay.

**Definition 3.22.** Let \( S \) be a \( \mathbb{Z} \)-graded ring. Let \( M \) be a graded \( S \)-module. The initial degree of \( M \), denoted by \( \text{indeg}_S(M) \), is the inf \( \{ i \in \mathbb{Z} \mid [M]_i \neq 0 \} \).

**Remark 3.23.** The number \( \text{indeg}_S(M) \) can be \( -\infty \) in general. However, if \( S \) is Noetherian and has unique maximal homogeneous ideal \( \mathfrak{M} \) which is maximal and \( M \) is finitely generated, then \( \text{indeg}_S(M/\mathfrak{M}M) \) is a finite number. In this case, \( \text{indeg}_S(M/\mathfrak{M}M) \) is the minimum degree among of a minimal homogeneous generating set of \( M \).

**Lemma 3.24.** Let \( (R, \mathfrak{m}) \) be an analytically unramified Cohen-Macaulay local ring having a canonical module. Let \( I \) be an \( \mathfrak{m} \)-primary ideal. Write \( T := R[It, t^{-1}] \), and let \( \mathfrak{M}_T \) be maximal homogenous ideal of \( T \). Let \( \mathcal{T} \) be the integral closure of \( T \) in \( R[t, t^{-1}] \). Let \( C \) denote the conductor ideal, i.e., \( C = T :_T \mathcal{T} \). If \( T \) is quasi-Gorenstein, then \( a(T) = a(T) + q \) where \( q = -\text{indeg}(C/\mathfrak{M}_T C) \). Furthermore, we have \( q \geq 0 \), and if \( T \) is not integrally closed, then \( q > 0 \).

**Proof.** If \( T \) is integrally closed, then there is nothing to prove. We assume that \( T \) is not integrally closed. We may assume that residue field of \( R \) is infinite. Let \( d = \dim R \) and \( A := R[It, t^{-1}] \) where \( J \) is a minimal reduction of \( I \). Then \( A \) is a Cohen-Macaulay ring with \( A_0 = [T]_0 = R \), and since \( R \) is analytically unramified, \( A \subseteq \mathcal{T} \) is module-finite extension. Hence the graded canonical modules \( \omega_T \) and \( \omega_\mathcal{T} \) of \( T \) and \( \mathcal{T} \) are \( \text{Hom}_A(T, \omega_A) \) and \( \text{Hom}_A(\mathcal{T}, \omega_A) \), respectively where \( \omega_A \) is the graded canonical module of \( A \). We claim that \( \omega_\mathcal{T} \cong \text{Hom}_T(T, \omega_T) \). Observe that \( \text{Hom}_T(\mathcal{T}, \omega_T) \cong \text{Hom}_T(\mathcal{T}, \text{Hom}_A(T, \omega_A)) \cong \text{Hom}_A(\mathcal{T} \otimes_T T, \omega_A) \cong \text{Hom}_A(\mathcal{T}, \omega_A) \), where the last module is a graded canonical module of \( \mathcal{T} \) see Theorem 3.6. Since \( \mathcal{T} \) has a unique maximal homogeneous ideal which is maximal, it has well-defined
a-invariant – \( \text{indeg}(\omega_T / \mathcal{M}_T \omega_T) \).

Since \( T \subset \overline{T} \) is birational, i.e., they have the same total quotient ring, we have \( \text{Hom}_T(\overline{T}, T) \cong (T :_T \overline{T}) \) where the last module is the conductor ideal \( \mathcal{C} \). Recall that since \( T \) is quasi-Gorenstein, \( \omega_T \cong T(a(T)) \). Therefore \( \omega_T \cong C(a(T)) \). This implies that \( a(T) = - (\text{indeg}(C(a(T))/\mathcal{M}_T C(a(T)))) = - (\text{indeg}(C/\mathcal{M}_T C) - a(T)) \).

It remains to show that \( \text{indeg}(C/\mathcal{M}_T C) = \text{indeg}(C/\mathcal{M}_T C) \). Since \( R \) is analytically unramified, there exists a positive integer \( q \) such that \( \overline{T} = I^i \cdot q \overline{T} \) for \( i \geq q \geq 0 \) [18, Theorem 1.4]. This shows that \( t^{-q} \in \mathcal{C} \). We choose the smallest positive integer \( q \) with this property. Now, we have \( \text{indeg}_T(C/\mathcal{M}_T C) = -q \). We have \( [T],_i = [\overline{T}],_i \) for all \( i \leq 0 \) and \( [\mathcal{M}_T],_i = [\mathcal{M}_T],_i \) for \( i < 0 \). This shows that \( \text{indeg}(C/\mathcal{M}_T C) = \text{indeg}(C/\mathcal{M}_T C) \). \( \square \)

**Theorem 3.25.** Let \((R, \mathfrak{m})\) be a \( d \)-dimensional analytically unramified Cohen-Macaulay local ring having a canonical module with infinite residue field. Let \( I \) be an \( \mathfrak{m} \)-primary ideal. Let \( \{F_i\}_{i \in \mathbb{Z}} \) where \( F_i = \overline{T} \) be the integral closure filtration where \( F_1 = R \) when \( i \leq 0 \). Assume that \( \overline{T} = \bigoplus_{i \in \mathbb{Z}} F_it^i \) is Cohen-Macaulay. If \( R[It, t^{-1}] \) is quasi-Gorenstein, then the index of nilpotency does not depend on a minimal reduction of \( I \). Furthermore, the a-invariant of \( R[It, t^{-1}] \) is \( s(I) - d + 1 \).

**Proof.** Let \( T = R[It, t^{-1}] \). Let \( J \) be a minimal reduction of \( I \). Since \( \overline{T} \) is Cohen-Macaulay, \( a(T) = s_J(\overline{T}) - d + 1 \) where \( s_J(\overline{T}) := \min\{i | F_{i+1} \subseteq J\} \). We show that \( s_J(I) = s_J(\overline{T}) - q \). By Lemma 3.24 \( a(T) + q = a(T) \). Therefore we have \( a(T) = s_J(\overline{T}) - q - d + 1 \). By Lemma 3.14(a) we have \( s_J(I) - d + 1 \leq a(T) \), and this shows that \( s_J(I) + q \leq s_J(\overline{T}) \). Since \( t^{-q} \in \mathcal{C} \) is Cohen-Macaulay, we have \( \overline{T}t^{-q} \subseteq \overline{T} \). Now, we have \( J : \overline{T}t^{-q} \subseteq J : t^n \), and this implies the other inequality \( s_J(I) + q \geq s_J(\overline{T}) \). Since \( a(T) = s_J(I) - d + 1 \) for any minimal reduction \( J \), we have \( s_J(I) = s(I) \). \( \square \)

**Remark 3.26.** The ring \( \bigoplus_{i \in \mathbb{Z}} F_it^i \) is not Cohen-Macaulay in general. However, Hochster [10] showed that it is Cohen-Macaulay when \( R \) is a polynomial ring (localised at the origin) over a field and \( F_1 \) is a monomial ideal.

4. The Gorenstiness of the integral closure of extended Rees algebras

Let \( I \) be a monomial ideal in a polynomial ring \( R = k[x_1, \ldots, x_d] \) over a field \( k \). The integral closure of the extended Rees algebra \( A = \overline{R[It, t^{-1}]} \) is Cohen-Macaulay by [10]. In [8, Theorem 5.6] the authors characterized the Gorenstiness of \( A \) when \( I \) has a minimal reduction \( J \) which is generated by powers of variables, i.e., \( J = (x_1^{a_1}, \ldots, x_d^{a_d}) \) for some \( a_i \in \mathbb{N} \). This condition having such a minimal reduction is equivalent to the assumption that \( I \) has only one Rees valuation. In this section we generalize this result by removing the condition of the number of Rees valuations. We are able to interpret the reduction number that appears in [8, Proposition 5.4] in terms of the a-invariant of \( R[It, t^{-1}] \), and this leads to a lower bound on the reduction number. We follow the notation of Chapters 5 and 6 of [2].

**Setting 4.1.** Let \( R = k[x_1, \ldots, x_d] \) be a polynomial ring in \( d \)-variables over a field \( k \) and \( \mathfrak{m} = (x_1, \ldots, x_d)R \). We assign a \( \mathbb{Z}^{d+1} \)-grading to the Laurent polynomial ring \( R[t, t^{-1}] \) by setting \( \text{deg}(x_1^{a_1} \cdots x_d^{a_d} t^{a_{d+1}}) = (a_1, \ldots, a_d, a_{d+1}) \). This determines
the grading, since \( \{ \deg(x_1), \ldots, \deg(x_d), \deg(t) \} \) forms a \( \mathbb{Z} \)-basis for \( \mathbb{Z}^{d+1} \). Let \( A \) be a \( \mathbb{Z}^{d+1} \)-graded subring of \( R[t, t^{-1}] \). Then the affine semigroup

\[
\{ \deg(m) \mid m \text{ homogeneous element in } A \} \subseteq \mathbb{Z}^{d+1}
\]

is called the \textit{affine semigroup of} \( A \) and denoted by \( C_A \). For an affine semigroup \( C \), \( \text{relint}(C) \) denotes the \textit{relative interior} of \( C \). Finally, for \( n \in \mathbb{N} \), let \( \{ e_i \}_{i=1}^n \) denote the standard basis of \( \mathbb{Z}^n \), i.e., \( e_i = (1, 0, \ldots, 0) \).

**Lemma 4.2.** With Setting 4.1, let \( C \) be the affine semigroup of \( R[\mathfrak{m} t, t^{-1}] \). Let \( W = \mathbb{Z}^{d+1} \) and \( \{ e_i \} \) be the standard basis of \( W \). Let \( \phi \in \text{Aut}_{\mathbb{Z}}(W) \) be an automorphism of \( W \) defined as follows: \( \phi(e_i) = e_i + e_{d+1} \) for \( i = 0, \ldots, d \) and \( \phi(e_{d+1}) = -e_{d+1} \). Then \( \phi|_C \) is an embedding of \( C \) into \( \mathbb{Z}_{\geq 0}^{d+1} \).

**Proof.** One can easily check that \( \phi \) is an automorphism on \( W \). We show that \( \phi|_C \) is an embedding. Since \( \mathbb{C} \) does not have any inverse (in the sense of affine semigroups), \( \ker(\phi|_C) = 0 \). It remains to show that \( \phi(C) \subseteq \mathbb{Z}^{d+1} \). Since \( \phi(\deg(x_i)) = (0, \ldots, 0, 1, 0, \ldots, 1) \) and \( \phi(\deg(t^{-1})) = (0, \ldots, 0, 1) \), it suffices to show that

\[
\phi(\deg(x_1^{a_1} \cdots x_d^{a_d} t^b)) \in \mathbb{Z}_{\geq 0}^{d+1}
\]

where \( a_i \in \mathbb{Z}_{\geq 0} \) and \( \sum_{i=0}^d a_i \geq b \). Indeed we have

\[
\phi(\deg(x_1^{a_1} \cdots x_d^{a_d} t^b)) = \sum_{i=0}^d a_i \phi(\deg x_i) + b\phi(\deg t)
\]

\[
= \sum_{i=0}^d a_i \phi(\deg x_i) - b\phi(\deg(t^{-1}))
\]

\[
= (a_1, \ldots, a_d, \sum_{i=0}^d a_i) - b(0, \ldots, 0, 1)
\]

\[
= (a_1, \ldots, a_d, \sum_{i=0}^d a_i - b) \in \mathbb{Z}_{\geq 0}^{d+1}.
\]

**Corollary 4.3.** With Setting 4.1, let \( \mathcal{F} = \{ \mathcal{F}_i \}_{i \in \mathbb{Z}} \) be a filtration where \( \mathcal{F}_i = R \) when \( i \leq 0 \) and \( \mathcal{F}_i \) are monomial ideals contained in \( \mathfrak{m}^i \). Then the affine semigroup of \( \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i t^i \) can be embedded into \( \mathbb{Z}_{\geq 0}^{d+1} \). In particular, the affine semigroup of \( R[\mathfrak{I} t, t^{-1}] \) can be embedded into \( \mathbb{Z}_{\geq 0}^{d+1} \) when \( I \) is a monomial ideal.

For a monomial ideal \( I \) in \( R \), its integral closure \( \overline{I} \) can be determined by a Newton polyhedron. We would like to describe the ring \( R[\mathfrak{I} t, t^{-1}] \) using the half-spaces in \( \mathbb{Z}^{d+1} \) that corresponds to the ones that determine \( \overline{I} \) in \( \mathbb{Z}^d \). Let \( \langle \ , \ , \rangle \) denote the inner product in \( \mathbb{R}^n \).

**Lemma 4.4.** With Setting 4.1, let \( I \) be a monomial ideal in \( R \). Let \( H_i^+ = \{ v \in \mathbb{Z}^d \mid \langle (a_{i,1}, \ldots, a_{i,d}), v \rangle \geq h_i \} \) where \( h_i \in \mathbb{Z}_{\geq 0} \) be the half-spaces in \( \mathbb{Z}^d \) that determine the Newton polyhedron of \( I \). Define the half-spaces \( \overline{H}_i^+ \) that correspond to each \( H_i^+ \) in \( \mathbb{Z}^{d+1} \) as \( \overline{H}_i^+ := \{ v \in \mathbb{Z}^{d+1} \mid \langle (a_{i,1} - h_i, \ldots, a_{i,d} - h_i, h_i), v \rangle \geq 0 \} \). Let \( C \) be the affine semigroup of \( R[\mathfrak{I} t, t^{-1}] \). Then \( \cap \overline{H}_i^+ \) is the affine semigroup \( \phi(C) \) where \( \phi \) is the embedding in Lemma 4.2.
Proof. Let $H := H^+_i$ for some $i$ and write $H = \{v \in \mathbb{Z}^d \mid \langle (a_1, \ldots, a_d), v \rangle \geq h \}$. An exponent vector $(z_1, \ldots, z_{d+1})$ in $\phi(C)$ is the image $\phi((z_1, \ldots, z_d, -z_{d+1} + (z_1 + \cdots + z_d)))$. One has $(z_1, \ldots, z_d, -z_{d+1} + (z_1 + \cdots + z_d)) \in C$ if and only if the monomial $x^{z_1} \cdots x^{z_d}$ is in $\prod_{i=1}^{d+1} (x_i^{z_i} + \cdots + x_d^{z_d})$. In terms of half-spaces this corresponds to the condition $\langle (a_1, \ldots, a_d), (z_1, \ldots, z_d) \rangle \geq (-z_{d+1} + (z_1 + \cdots + z_d)h$ for all $H$. Let $\tilde{H} = \{v \in \mathbb{Z}^{d+1} \mid \langle (a_i - h, \ldots, a_d - h, +h), v \rangle \geq 0 \}$. Therefore, we obtain $\cap H^+_i$ is $\phi(C)$. □

Example 4.5. Let $R = \mathbb{C}[x_1, \ldots, x_d]$ and $I = (x_1, \ldots, x_d)$. Let $\{e_i\}_{i=1}^{d+1}$ be the standard base of $\mathbb{Z}^{d+1}$. Then one can easily see that $\phi(C) = \{(z_1, \ldots, z_{d+1}) \in \mathbb{Z}^{d+1} \mid z_i \geq 0 \text{ for all } i \}$ is part of a minimal generating set for the canonical ideal for $R[It, t^{-1}]$.

Lemma 4.6. With Setting 4.1 assume that $I$ is an $m$-primary monomial ideal. Let $C$ be the affine semigroup of $\overline{R[It, t^{-1}]}$ and $\phi$ the embedding in Lemma 4.2. Then there exists an exponent vector of the form $(1, \ldots, 1, q)$ in $\phi(C)$ for some integer $q$ with $1 \leq q \leq d + 1$ which is part of a minimal generating set for the canonical ideal for $R[It, t^{-1}]$.

Proof. Since $I$ is $m$-primary, the half-spaces of the form $\{v \in \mathbb{Z}^d \mid (0, \ldots, 1, \ldots, 0) \cdot v \geq 0 \}$ are part of the boundary of the Newton polyhedron of $I$. By Lemma 4.4 these will be part of boundary half-spaces in $\phi(C)$. For instance, the half-space $\{v \in \mathbb{Z}^d \mid (1, \ldots, 0) \cdot v \geq 0 \}$ corresponds to the half-space $\{v \in \mathbb{Z}^{d+1} \mid (1, \ldots, 0) \cdot v \geq 0 \}$. Hence if $(z_1, \ldots, z_d, z_{d+1}) \in \text{relint}(C)$, then $z_i \geq 1$ for $i = 0, \ldots, d$.

By [2, Theorem 6.3.5(b)] it suffices to show that if $(1, \ldots, 1, q)$ is in $\text{relint}(\phi(C))$, then $(1, \ldots, 1, q - 1)$ is not in $\phi(C)$. Since $x_1 \cdots x_d \in R \subseteq \overline{R[It, t^{-1}]}$, $\phi((1, \ldots, 1, 0) = (1, \ldots, 1, d)$ is in $\phi(C)$. If $(1, \ldots, 1, d)$ is on the boundary, then $(1, \ldots, 1, d + 1)$ in $\text{relint}(\phi(C))$. In this case we set $q = d + 1$. If $(1, \ldots, 1, d)$ is not on the boundary, we can choose $q \leq d$ be the minimal in the last component since we have $\phi(t^{-1}) = (0, \ldots, 0, 1)$ in $\phi(C)$. □

Theorem 4.7. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring in $d$-variables over a field $k$ and $m = (x_1, \ldots, x_d)$. Let $I$ be an $m$-primary monomial ideal and $H_i$ the half-spaces that determine the Newton Polyhedron of $I$ where $H_i = \{v \in \mathbb{Z}^d \mid \langle (a_1, \ldots, a_d), v \rangle \geq h_i \}$ for $h_i \in \mathbb{Z}_{\geq 0}$. Let $q$ as in Lemma 4.6 and $w_i := \langle (a_{i-1} - h_i, \ldots, a_i - h_i, 1, \ldots, 1, q) \rangle$. Define $N_i^+ := \{v \in \mathbb{Z}^{d+1} \mid \langle (a_{i-1} - h_i, \ldots, a_i - h_i, 1, \ldots, 1, q) \rangle \geq w_i \}$. Let $C$ be the affine semigroup of $R[It, t^{-1}]$ and $\phi$ the embedding in Lemma 4.2. Then $R[It, t^{-1}]$ is Gorenstein if and only if the relative interior of $\phi(C)$ is contained in $\cap N_i^+$, equivalently $\text{relint}(\phi(C)) = \cap N_i^+$. 

Proof. For $v \in N_i^+$ for any $i$, since

\[\langle (a_{i-1} - h_i, \ldots, a_i - h_i, 1, \ldots, 1, q) \rangle \geq w_i = \langle (a_{i-1} - h_i, \ldots, a_i - h_i, 1, \ldots, 1, q) \rangle,\]

we have

\[\langle (a_{i-1} - h_i, \ldots, a_i - h_i, 1, \ldots, 1, q) \rangle \geq 0.\]

Hence $v - (1, \ldots, 1, q)$ is in $\phi(C)$ by Lemma 4.4. In other words, $\cap N_i^+ = (1, \ldots, 1, q) + \phi(C)$. Since $(1, \ldots, 1, q) \in \text{relint}(C)$, we have $\cap N_i^+ \subseteq \text{relint}(C)$. 

Suppose $\overline{R[H, t^{-1}]}$ is Gorenstein. Then the canonical ideal is principal. By Lemma 4.6 the canonical ideal is generated by the element that corresponds to the exponent vector $(1, \ldots, 1, q) \in \phi(C)$. This shows that $\text{relint}(\phi(C)) = C^N$. Conversely, suppose $\text{relint}(\phi(C)) = C^N$. Then $\text{relint}(C)$ is generated by the exponent vector $(1, \ldots, 1, q)$. Hence the canonical ideal is principal (c.f. [2, Theorem 6.3.5.(b)]), and this shows that $\overline{R[H, t^{-1}]}$ is Gorenstein. □

The following example illustrates the above theorem when there is only one half-space which determines the integral closure of $I$. This will help one to understand and prove Corollary 4.10 which is Theorem 5.6 in [8].

Example 4.8. Let $R = \mathbb{C}[x, y, z]$ and $I = (x^2, y^2, z^2)$. Then the integral closure of $I$ is determined by the half spaces $\{v \in \mathbb{Z} \mid \langle(2, 2, 1), v \rangle \geq 4\}$ and $\{v \in \mathbb{Z} \mid \langle(e_i, v) \geq 0\}$ for $i = 1, 2, 3$ where $\{e_i\}$ denote the standard bases of $\mathbb{C}^3$. Let $\nu$ be the valuation corresponding to the half space $\{v \in \mathbb{Z} \mid \langle(2, 2, 1), v \rangle \geq 4\}$; then one has $\nu(x) = 2, \nu(y) = 2$, and $\nu(z) = 1$. Let $\overline{R[H, t^{-1}]}$ and $C$ be the corresponding affine semigroup. Then $\phi(C)$ is determined by the following half spaces represented as a matrix

$$M = \begin{bmatrix}
-2 & -2 & -3 & 4 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 
\end{bmatrix}$$

in the sense that an exponent vector $(z_1, \ldots, z_4) \in \phi(C)$ if and only if all the entries of $M(z_1, \ldots, z_4)^{tr} \geq 0$. Here $M \geq c$ where $c \in \mathbb{Z}$ if all the entries of $M$ are greater than or equal to the number $c$. One can easily check that $(1, 1, 1, 2) \in \phi(C)$, but $(1, 1, 1, 1) \notin \phi(C)$. Hence $q = 2$. Furthermore,

$$M \cdot (1, 1, 1, 2)^{tr} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

If $w \in \text{relint} \phi(C)$, then $M \cdot w \geq 0$, i.e., $M \cdot w \geq 1$. Since $M \cdot w \geq 1$ and $M \cdot (1, 1, 1, 2)^{tr} = 1$, we have $\frac{M \cdot w}{M \cdot (1, 1, 1, 2)^{tr}} = 1$, i.e., $M \cdot (w - (1, 1, 1, 2)^{tr}) \geq 0$. Hence $w - (1, 1, 1, 2)^{tr} \in \phi(C)$. This implies the exponent vector $(1, 1, 1, 2)$ generates $\text{relint} \phi(C)$. Therefore, $\overline{R[H, t^{-1}]}$ is Gorenstein by Theorem 4.7. Furthermore, by [2, Corollary 6.3.6] the ideal corresponding the relint $\phi(C)$ is the graded canonical ideal of $\overline{R[H, t^{-1}]}$, and it is generated by $xyz$ which corresponds to $(1, 1, 1, 2)$. This shows that $a = a(\overline{R[H, t^{-1}]} = -1 = 2 - (1 + 1 + 1) = q - d$ where $d = \text{dim} R = 3$.

Since $\overline{R[H, t^{-1}]}$ is Cohen-Macaulay, $a = r - d + 1$ where $r = r(F)$ the reduction number of the filtration $F = \{I_t\}_{t \in \mathbb{Z}}$. Therefore, $r = -1 + 3 - 1 = 1$.

Corollary 4.9. With the setting of Theorem 4.7, one has $a(\overline{R[H, t^{-1}]} \geq q - d$ and $r(F) \geq q - 1$. Furthermore, if $\overline{R[H, t^{-1}]}$ is Gorenstein, then $a(\overline{R[H, t^{-1}]} = q - d$ and $r(F) = q - 1$.

Corollary 4.10 ([8, Theorem 5.6]). With the setting of Theorem 4.7, assume that $I = (x_1^{a_1}, \ldots, x_d^{a_d})$. Let $L = \text{lcm}(a_1, \ldots, a_d)$. Write $L/a_1 + \cdots + L/a_d = jL + p$, where $j \geq 0$ and $1 \leq p \leq L$. Then $\overline{R[H, t^{-1}]}$ is Gorenstein if and only if $p = 1$.

Proof. Let $\{e_i\}$ be the standard basis of $\mathbb{Z}^d$. Let $H_i^+ = \{v \in \mathbb{Z}^d \mid \langle e_i, v \rangle \geq 0\}$ for $i = 1, \ldots, d$. Observe that the half-spaces which determine the integral closure of
Let \( \rho \) indeed shows that \( \rho \eta = 1 \). Theorem I corresponds to the Rees valuation of \( I \). Now, we proceed as in Example 4.8. The affine semigroup \( \phi(C) \) is determined by the following half spaces represented as a matrix

\[
M = \begin{bmatrix}
L/a_1 - L & L/a_2 - L & \cdots & L/a_d - L & L \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\( \phi(C) = \{(x_1, \ldots, x_{d+1}) \in \mathbb{Z}^{d+1} \mid M \cdot (x_1, \ldots, x_{d+1}) \geq 0\} \). Therefore one has

\[
\text{relint} \phi(C) = \{(x_1, \ldots, x_{d+1}) \in \mathbb{Z}^{d+1} \mid M \cdot (x_1, \ldots, x_{d+1}) > 0\}.
\]

Let \( q \) be as in Lemma 4.6. One has

\[
\begin{bmatrix} 1 \\ 1 \\ \vdots \\ q \end{bmatrix} = \begin{bmatrix} L/a_1 + \cdots + L/a_d + L(q-d) \\ 1 \\ \vdots \\ 1 \end{bmatrix},
\]

and \( \cap N_i^+ = \{(x_1, \ldots, x_{d+1}) \in \mathbb{Z}^{d+1} \mid M \cdot (x_1, \ldots, x_{d+1}) \geq M \cdot (1, \ldots, 1, q)\} \). By Theorem 4.7 \( R[It, t^{-1}] \) is Gorenstein if and only if \( \text{relint} \phi(C) \subseteq \cap N_i^+ \), i.e.,

\[
\{(x_1, \ldots, x_{d+1}) \in \mathbb{Z}^{d+1} \mid M \cdot (x_1, \ldots, x_{d+1}) \geq 0\} \subseteq \{(x_1, \ldots, x_{d+1}) \in \mathbb{Z}^{d+1} \mid M \cdot (x_1, \ldots, x_{d+1}) > 0\}.
\]

Write \( \rho := \min\{(x_1, \ldots, x_{d+1}) \in \text{relint}(C) \mid \langle (L/a_1 - L, L/a_2 - L, \cdots, L/a_d - L, L), (x_1, \ldots, x_{d+1}) \rangle \geq 0\} \) and \( \eta := M \cdot (1, \ldots, 1, q) = L/a_1 + \cdots + L/a_d + L(q-d) \). Observe that \( \text{relint} \phi(C) \subseteq \cap N_i^+ \) if and only if \( \rho \geq \eta \). Since \( (1, \ldots, 1, q) \) is in \( \text{relint}(\phi(C)) \), one has \( \eta \geq 0 \), i.e., \( \eta \geq 1 \). Assume that \( \rho = 1 \). Then \( \text{relint} \phi(C) \subseteq \cap N_i^+ \) if and only if \( \eta = 1 \), and this is equivalent to the condition of \( p = 1 \). Hence it suffices to show that \( \rho = 1 \). First, we claim that \( \gcd(L/a_1 - L, L/a_2 - L, \cdots, L/a_d - L, L) = 1 \). Here we take the gcd to be positive. Recall that \( L = \text{lcm}(a_1, \ldots, a_d) \). Since \( \delta := \gcd(L/a_1 - L, L/a_2 - L, \cdots, L/a_d - L, L) = \gcd(L/a_1, \cdots, L/a_d, L) = \gcd(L/a_1, \cdots, L/a_d) \), we have \( \delta | (L/a_i) \) for all \( i \). This implies that \( a_i | (L/\delta) \) for all \( i \) since \( \delta \) divides \( L \). We have \( L/\delta \geq \text{lcm}(1, \ldots, a_d) = L \); hence \( \delta = 1 \). We claim that there exists \( (b_1, \ldots, b_d) \) where \( b_i > 0 \) for \( i = 1, \ldots, d \) such that \( \langle (L/a_1 - L, L/a_2 - L, \cdots, L/a_d - L, L), (b_1, \ldots, b_{d+1}) \rangle = 1 \). Since \( \gcd(L/a_1 - L, L/a_2 - L, \cdots, L/a_d - L, L) = 1 \), there exist such \( b_i \)’s in \( \mathbb{Z} \). We show that one can modify the \( b_i \)’s so that \( b_i > 0 \) for all \( i = 1, \ldots, d \). Suppose that \( i \leq d \) is the least index where \( b_i \leq 0 \). Let \( n \) be an integer such that \( b_i + nL > 0 \). By replacing \( b_i \) by \( b_i + nL \) and \( b_{d+1} \) by \( b_{d+1} - n(L/a_i - L) \). Therefore, we may assume that \( b_i > 0 \). This is a contradiction. This shows that the first row of \( M \cdot (b_1, \ldots, b_{d+1}) \) is 1 and all the other rows are positive. This indeed shows that \( \rho = 1 \).

\( \square \)

**Remark 4.11.** One may ask if the numbers \( w_i \) in Theorem 4.7 are the minimum in \( \{(a_1 - h_i, \ldots, a_d - h_i, b_i) \mid v \in \mathbb{Z}^{d+1}\} \) for each \( i \). The following example shows that it can happen that \( w_i \) are not the minimum for each halfspace, but it is the
minimum in the intersection, i.e., it is the minimum in the relative interior.

Let \( R = \mathbb{C}[x, y] \). Let \( I = (x^3, xy, y^4) \). Let \( T = \mathbb{R}[t, t^{-1}] \). Then \( T \) is Gorenstein, and \( \phi(C_T) \) is determined by the half-spaces

\[
\begin{bmatrix}
-1 & -2 & 3 \\
-2 & -1 & 3 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

One can easily see that \( q = 2 \). But \( \langle (-1, -2, 3), (3, 1, 2) \rangle = 1 \) whereas \(\langle -1, -2, 3 \rangle, (1, 1, 2) \rangle = 3 \).

5. The Cohen-Macaulayness of the associated graded ring

Let \((R, \mathfrak{m})\) be a regular local ring and \(I\) an ideal. Let \(J\) be a proper ideal. We define a function \(\text{ord}_J : R \rightarrow \mathbb{N}_{\geq 0} \cup \{\infty\}\) as follows; \(\text{ord}_J(x) := \sup \{i \mid x \in J^i\}\) for \(x \in R\) and \(\text{ord}_J(I) := \inf \{\text{ord}_J(x) \mid x \in I\}\). By the Krull intersection theorem, this number is finite if \(x \neq 0\). In general, we have \(\text{ord}_m(xy) \geq \text{ord}_m(x) + \text{ord}_m(y)\). Since \(\text{gr}_m(R)\) is a domain, \(\text{ord}_m\) is a valuation, i.e., \(\text{ord}_m(xy) = \text{ord}_m(x) + \text{ord}_m(y)\) for \(x, y \in R\). For \(x, y \in R\), let \(x^* := x + m^{\text{ord}_m(x)+1} \in m^{\text{ord}_m(x)} / m^{\text{ord}_m(x)+1}\) denote its image in \(\text{gr}_f(R)\). We call \(x^* \) the leading form of \(x\) and \(I^* := (x^* | x \in I)\) the leading ideal of \(I\), respectively.

Remark 5.1. Since \((R, \mathfrak{m})\) is a regular local ring, so is \((\hat{R}, \hat{\mathfrak{m}})\) where \(\hat{\cdot}\) denotes \(m\)-adic completion. Because \(\text{gr}_m(R) = \text{gr}_m(\hat{R})\), we have \(\text{ord}_m(x) = \text{ord}_{\hat{m}}(x)\) and \(\text{ord}_m(I) = \text{ord}_{\hat{m}}(I)\). In particular, we have \(I^* = (\hat{I})^*\) in \(\text{gr}_m(R) = \text{gr}_m(\hat{R})\).

Since \(R\) is complete, we may write \(x = \sum_{i \geq 0} [x]_i = \sum_{i \geq \text{ord}_m(x)} [x]_i\) where \([x]_i \in \hat{m}^i \setminus \hat{m}^{i+1} \cup \{0\}\) and \(x^* = ([x]_{\text{ord}_m(x)})^*\).

Lemma 5.2. Let \((R, \mathfrak{m})\) be a regular local ring with maximal ideal \(\mathfrak{m}\). Let \(I\) be an ideal which is minimally generated by the 2 by 2 minors of the 2 by 3 matrix \(M\) with entries in \(\mathfrak{m}\) where

\[
M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}.
\]

Write \(G := \text{gr}_m(R)\). Let \(L = (b^* w^* - c^* v^*, -(a^* w^* - c^* u^*), a^* v^* - b^* u^*)\) and \(C_*\) be the complex

\[
0 \rightarrow G^2 \xrightarrow{\begin{bmatrix} a^* & u^* \\ b^* & v^* \\ c^* & w^* \end{bmatrix}} G^3 \xrightarrow{\begin{bmatrix} b^* w^* - c^* v^* & -(a^* w^* - c^* u^*) & a^* v^* - b^* u^* \end{bmatrix}} G.
\]

If \(\text{ht} L = 2\), then \(C_*\) is acyclic. Furthermore, we have \(I^* = ((bw - cv)^*, (aw - cu)^*, (av - bu)^*)\).

Proof. The acyclicity of \(C_*\) follows by Buchbaum-Eisenbud acyclicity criterion [3, Corollary 1]. Since \(C_*\) is acyclic and all the entries of the maps are in the maximal homogenous ideal of \(\text{gr}_m(R)\), the ideal \((b^* w^* - c^* v^*, -(a^* w^* - c^* u^*), a^* v^* - b^* u^*)\) is minimally generated by these three elements. In particular, these elements are not zero. This implies that \((bw - cv)^* = b^* w^* - c^* v^*, (aw - cu)^* = a^* w^* - c^* u^*, (av - bu)^* = a^* v^* - b^* u^*\).
Let $I = (f_1, f_2, f_3)$ where $f_1 = bw - cv$, $f_2 = -(aw - cu)$, $f_3 = av - bu$. By Remark 5.1 we may assume that $R$ is complete. By definition, $(f_1^*, f_2^*, f_3^*) \subseteq I^*$. Suppose $(f_1^*, f_2^*, f_3^*) \neq I^*$. Then there exists $x \in I$ such that $x^* \in I^* \setminus (f_1^*, f_2^*, f_3^*)$. Since $x \in I$, we can write $x = g_1 f_1 + g_2 f_3 + g_3 f_3$ for some $g_i \in R$.

Since we are in a complete local ring, we can write $x = \sum_{i \geq 0} [x]_i$ as in Remark 5.1. Since $\text{ord}_m(x) = \text{ord}_m(g_1 f_1 + g_2 f_3 + g_3 f_3) \geq \min\{\text{ord}_m(g_i f_i)\}_{i=1,2,3}$, the set $\Gamma = \{\text{ord}_m(g_i f_i)\}_{i=1,2,3}$ for a triple $g_i$ such that $x = \sum g_i f_i$ is finite. We choose $g_i$’s such that the number $\min\{\text{ord}_m(g_i f_i)\}_{i=1,2,3}$ is the maximum in $\Gamma$. We are going to construct $g_i$’s such that $x = g'_1 f_1 + g'_2 f_2 + g'_3 f_3$ and $\min\{\text{ord}_m(g'_i f_i)\} \geq \min\{\text{ord}_m(g_i f_i)\}$. This will contradict the maximality.

Let $n := \min\{\text{ord}_m(g_i f_i)\}$. If $\text{ord}_m(x) = n$, then $x^* \in (f_1^*, f_2^*, f_3^*)$. Therefore we may assume that $n < \text{ord}_m(x)$. Let $\delta_i := \text{ord}_m(f_i)$ for $i = 1,2,3$. We have

$$(g_1 f_1 + g_2 f_2 + g_3 f_3)^* = ([g_1]_{n-\delta_1}^*)^*(f_1^*)^* + ([g_2]_{n-\delta_2}^*)^*(f_2^*)^* + ([g_3]_{n-\delta_3}^*)^*(f_3^*)^* = 0.$$

Since the complex $C_\bullet$ is acyclic, there exists $s,t$ in $R$ such that

$$(g_1^{n-\delta_1})^* = s^* a^* + t^* u^*, (g_2^{n-\delta_2})^* = s^* b^* + t^* v^*, (g_3^{n-\delta_3})^* = s^* e^* + t^* w^*.$$

Since the complex

$$0 \to R^2 \xrightarrow{\begin{pmatrix} a & u \\ b & v \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} f_1 & -f_2 & f_3 \end{pmatrix}} I \to 0$$

is a complex in $R$, we have $(sa + tw)f_1 - (sb + tv)f_2 + (sc + tw)f_3 = sa f_1 - bf_2 + cf_3 + tu f_1 - tv f_2 + w f_3 = 0$. Let $g'_1 = g_1 - (sa + tw), g'_2 = g_2 + (sb + tv)$, and $g'_3 = g_3 - (sc + tw)$. Then $x = g'_1 f_1 + g'_2 f_2 + g'_3 f_3$ and $\min\{\text{ord}_m(g'_i f_i)\} \leq \min\{\text{ord}_m(g_i f_i)\}$. This is a contradiction.

We first state a useful remark when proving next two lemmas.

Remark 5.3. Let $(R, m)$ be a Noetherian local ring. Let $x,y,u,v$ be elements in $m$ such that $xy - uv \neq 0$. If $(xy - uv)^*$ is a prime element in $\gr_m(R)$, then $\text{ord}_m(xy) = \text{ord}_m(uv)$.

**Proof.** Assume to the contrary $\text{ord}_m(xy) \neq \text{ord}_m(uv)$. Without loss of generality, we may assume that $\text{ord}_m(xy) < \text{ord}_m(uv)$. Then $(xy - uv)^* = (xy)^* = x^* y^*$. Since $x^*$, $y^*$ are in the maximal homogeneous ideal of $\gr_m(R)$, neither of them is a unit. Therefore the product $x^* y^*$ is not a prime element. This is a contradiction.

Remark 5.4. Let $(R, m)$ be a Noetherian local ring. Let $x,y$ be elements in $m$. If $\text{ord}_m(x) = \text{ord}_m(y)$ and $x^* + y^* \neq 0$, then $(x + y)^* = x^* + y^*$. In particular, $\text{ord}_m(x + y) = \text{ord}_m(x)$.

**Proof.** Let $G = \gr_m(R)$. Recall that $\text{ord}_m(x + y) = \text{ord}_{G_+}((x + y)^*)$ where $G_+ = \oplus_{i > 0} G_i$. We have $\text{ord}_{G_+}(x^* + y^*) \geq \min\{\text{ord}_{G_+}(x^*), \text{ord}_{G_+}(y^*)\}$. If this is a strict inequality, then $(x + y)^* = x^* + y^*$, and this is a contradiction. Also if $\text{ord}_{G_+}(x^*) \neq \text{ord}_{G_+}(y^*)$, then $(x + y)^* = x^* + y^*$, and this is a contradiction.

Lemma 5.5. Let $(R, m)$ be a regular local ring with maximal ideal $m$. Let $I$ be an ideal of height 2 which is minimally generated by the $2 \times 2$ minors of the $2 \times 3$ matrix $M$ with entries in $m$. Assume that the leading forms of any two minors form part of a minimal generating set of $I^*$. If $I^*$ is a prime ideal and $I_1(M) \not\subseteq m^2$, then we can find a matrix

$$\tilde{M} = \begin{pmatrix} a' & b' & c' \\ u' & v' & w' \end{pmatrix}$$
such that \( I = I_2(\tilde{M}) \) and \( I^* = ((b'w' - c'v')*, (a'w' - c'u')*, (a'v' - b'u')*) \) is perfect of height 2.

**Proof.** Write

\[
M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}.
\]

By switching rows and columns, which does not change minors, we may assume that \( a \in \mathfrak{m} \setminus \mathfrak{m}^2 \). Henceforth we will only use the assumption that \((av - bu)^*, (aw - cv)^*\) form part of a minimal generating set of \( I^* \). We are going to modify \( M \) applying row and column operations to obtain \( \tilde{M} \) with the desired properties. Observe that adding a multiple of a row to the other does not change 2 \times 2 minors, but adding a multiple of a column to another changes one minor, but does not change the other two in general. In both cases the ideal \( I_2(-) \) does not change. In the proof the only column operations we perform are adding a multiple of the first column to the second or the third column. This does not change the minors \( av - bu, aw - cu, \) but the minor \( bw - cv \) will be changed to \( bw - cw + f(aw - bu) + g(aw - cu) \) for some \( f, g \) in \( R \).

We claim that after performing row and column operations on \( M \), we may assume that \((av - bu)^* = a^*v^* - b^*u^*\). Assume to the contrary that \((av - bu)^* \neq a^*v^* - b^*u^*\).

Since \((av - bu)^*\) is part of a minimal generating set of a prime ideal \( I^* \), \((av - bu)^*\) is a prime element. By Remark 5.3 we have \( \text{ord}_m(av) = \text{ord}_m(bu) \). Therefore, by Remark 5.4 one has \( a^*v^* - b^*u^* = 0 \) equivalently \( a^*v^* = b^*u^* \). Since \( \text{gr}_m(R) \) is a UFD and \( a^* \) is of degree 1, it is a prime element. Hence we have \( a^*|b^* \) or \( a^*|u^* \). Since \( a^*|b^* \), then there exists \( \delta \) in \( R \) such that \( a^*\delta = b^* \), and this implies \( \text{ord}_m(b) \leq \text{ord}_m(b - \delta a) \). We subtract the first column multiplied by \( \delta \) from the second column to obtain

\[
M' = \begin{pmatrix} a & b - \delta a & c \\ u & v - \delta u & w \end{pmatrix}.
\]

Since \( \text{ord}_m(b) \leq \text{ord}_m(b - \delta a) \), we have \( \text{ord}_m(ub) \leq \text{ord}_m(u(b - \delta a)) \). We replace \( M \) by \( M' \). The column operation changes the minors \( av - bu, aw - cu, bw - cv \) to \( av - bu, aw - cu, bw - cv + \delta(aw - cu) \). If \( a^*|u^* \), then we perform a row operation to obtain new \( u, v, w \). We note that the row operation does not change the minors.

We claim that this process terminates. Each time we replace \( M \) by \( M' \), either \( \text{ord}_m(b) \) or \( \text{ord}_m(u) \) strictly increases whereas \( \text{ord}_m(av - bu) \) is fixed. By Remark 5.3, we have \( \text{ord}_m(av) = \text{ord}_m(bu) \), and this implies \( \text{ord}_m(av - bu) \geq \min\{\text{ord}_m(\text{ord}_m(av)), \text{ord}_m(bu)\} = \text{ord}_m(bu) \). The number \( \text{ord}_m(av - bu) \) is fixed whereas \( \text{ord}_m(bu) \) is strictly increasing after each process. Therefore this will terminate, and we obtain \((av - bu)^* = a^*v^* - b^*u^*\).

We are going to show that by subtracting a multiple of the first column from the third column, we can obtain a matrix \( M \) such that \((av - bu)^* = a^*v^* - b^*u^*\), \((aw - cu)^* = a^*w^* - c^*u^*\). In particular, this will not change the entries \( a, b, u, v \) of the matrix \( M \); hence we preserve the property \((av - bu)^* = a^*v^* - b^*u^* \). Suppose \((aw - cu)^* \neq a^*w^* - c^*u^* \). Since \((aw - cu)^*\) is part of a minimal generating set of \( I^* \), it is a prime element. By Remark 5.3 we have \( \text{ord}_m(aw) = \text{ord}_m(cu) \) and \( a^*w^* = c^*u^* \). If \( a^*|u^* \), then the prime element \((av - bu)^* = a^*v^* - b^*u^* \) is divisible by \( a^* \). This is a contradiction. Therefore we have \( a^*|c^* \). Then there exists \( \delta \) in \( R \).
such that \( a^* \delta^* = c^* \). We subtract the first column multiplied by \( \delta \) from the third column. In particular, this does not change the entries \( a, b, u, v \) of \( M \). This process terminates, and we have \( (av - bu)^* = a^*v^* - b^*u^* \) and \( (av - ub)^* = a^*v^* - u^*b^* \).

By to Lemma 5.2 it suffices to show that \( \text{ht}(a^*w^* - c^*u^*, a^*v^* - b^*u^*) = 2 \). The images \( (av - bu)^*, (aw - cu)^* \) form a part of a minimal generating set of \( I^* \). This implies that \( ((av - bu)^*) \) is a prime ideal in \( \text{gr}_m(R) \) and \( (aw - cu)^* \notin ((av - bu)^*) \). Hence height of the ideal \( \text{ht}((av - bu)^*, (aw - cu)^*) \) is 2. Indeed our new \( M \) is \( M \) in the statement.

**Lemma 5.6.** Let \((R, m)\) be a regular local ring with maximal ideal \( m \). Let \( I \) be an ideal of height 2 which is minimally generated by the 2 by 2 minors of the 2 by 3 matrix \( M \) with entries in \( m \) where

\[
M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}.
\]

Assume that \((av - bu)^*, (aw - cu)^* \) is part of a minimal generating set of \( I^* \). If \( I^* \) is a prime ideal and \((av - bu)^* = a^*v^* - b^*u^* \), then we have \( I^* = ((bv - cu + f(aw - bu))^*, (aw - cu)^*, (av - bu)^*) \) for some \( f \) in \( R \), and this ideal is perfect of height 2.

**Proof.** Suppose \((aw - cu)^* \neq a^*w^* - c^*u^* \). Since \((av - bu)^* = a^*v^* - b^*u^* \), \((aw - cu)^* \) form part of a minimal generating set of a prime ideal, they are prime elements. By Remark 5.3 we have \( \text{ord}_m(aw) = \text{ord}_m(cu) \). Therefore we have \( a^*w^* - c^*u^* = 0 \). Since \( a^*v^* - b^*u^* \) is a prime element and \( \text{gr}_m(R) \) is a UFD, \( \text{gcd}(a^*, u^*) \sim 1 \). This implies that \( a^*|e^* \) and \( u^*|w^* \). Hence we may write \( c^* = \delta^*a^* \) and \( w^* = \delta^*u^* \) for some \( \delta \) in \( R \). Let \( M' \) be a matrix modified by subtracting the first column of \( M \) multiplied by \( \delta \) from the third column. This column operation does not change the entries \( a, b, u, v \) of the matrix \( M \). Notice that \( \text{ord}_m(c - \delta a) \geq \text{ord}_m(c) \) and \( \text{ord}_m(w - \delta u) \geq \text{ord}_m(w) \). As in the proof of Lemma 5.5, this process will terminate. Hence \((aw - cu)^* = a^*w^* - c^*u^* \). Notice that the 2 \times 2 minors of \( M' \) are \( av - bu, aw - cu, bw - cv + f(aw - bu) \) for some \( f \) in \( R \).

Since \((av - ub)^*, (aw - cu)^* \) form a part of a minimal generating set of \( I^* \) and \((av - ub)^* \) is a prime element, we have \( \text{ht}((av - ub)^*, (aw - cu)^*) = 2 \). The result follows by applying Lemma 5.2 to the matrix \( M' \).

**Theorem 5.7.** Let \( S \cong R/I \), where \((R, m)\) is a regular local ring and \( I \) is a height 2 perfect ideal. Assume that \( \text{gr}_n(S) \) is an integral domain where \( n = m/I \). If \( \mu(I) \leq 2 \) or \( \mu(I) = 3 \) and \( I \notin m^3 \), then \( \text{gr}_n(S) \) is Cohen-Macaulay.

**Proof.** Since \( \text{gr}_n(S) \cong \text{gr}_m(R)/I^* \) \([4, Exercise 5.3]\), it suffices to show that \( I^* \) is a Cohen-Macaulay ideal. There exists a minimal generating set of \( I \) such that its leading forms are part of a minimal generating set of \( I^* \). We fix a generating set of \( I \) with this property.

Case 1: When \( \mu(I) = 2 \): Let \( I = \langle f, g \rangle \) for some \( f \) and \( g \) in \( S \). We claim that \( I^* = \langle f^*, g^* \rangle \). Since \( I^* \) is a prime ideal and \( f^*, g^* \) form a part of minimal generating set of \( I^* \), \( f^* \) is a prime element. Since \( f^*, g^* \) are part of a minimal generating set of \( I^* \), \( g^* \notin \langle f^* \rangle \). Since \( G/\langle f^* \rangle \) is a domain, the image of \( g^* \) in this ring is a non-zerodivisor. By \([4, Exercise 5.2]\), the image of \( I^* \) is generated by the image of
$g^*$. Hence $I^* = (f^*, g^*)$.

Case 2: When $\mu(I) = 3$: Since $I$ is a height 2 perfect ideal, by the Hilbert-Burch theorem [2], $I$ can be generated by the 2 by 2 minors of a 2 by 3 matrix $M$ where the minors are the chosen generators. Write

$$M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}.$$ 

If $\text{ord}_m(I) \leq 3$, $I_2(M) = I \nsubseteq m^3$, hence $I_1(M) \nsubseteq m^2$. Now, the result follows by applying Lemma 5.5 to the matrix $M$.

Suppose $\text{ord}_m(I) = 4$. If there exists an entry of $M$ which has order 1, then we may apply Lemma 5.5. We may assume that no entry of $M$ has order 1. Since $\text{ord}_m(I) = 4$, at least one of the $2 \times 2$ minors of $M$ has order 4. Without loss of generality, we may assume that $\text{ord}_m(av - bu) = 4$. By the assumption on orders, $\text{ord}_m(a), \text{ord}_m(b), \text{ord}_m(u)$, and $\text{ord}_m(v)$ are greater than or equal to 2. Since $4 = \text{ord}_m(av - bu) \geq \min\{\text{ord}_m(av), \text{ord}_m(bu)\} \geq 4$, we have $\text{ord}_m(a) = \text{ord}_m(b) = \text{ord}_m(u) = \text{ord}_m(v) = 2$. This implies $(av - bu)^* = a^*v^* - b^*u^*$. Now, we are done by Lemma 5.6.

**Remark 5.8.** One can not relax the condition of $\text{gr}_I(R)$ a domain. Let $R = \mathbb{C}[a, b, c, d, e]$ and $I$ be the 2 by 2 minors of the matrix $M$,

$$\begin{pmatrix} a^2 + c^3 & 0 & ad + c^3 \\ ab + c^3 & ae + a^3 & 0 \end{pmatrix}.$$ 

Then $I^*$ is not prime, and $G/I^*$ is not Cohen-Macaulay.

One can see from Lemma 5.5 and Lemma 5.6, that once we can find a minor which commutes with taking $^*$, then we can find a matrix $M$ where the images of the minors generate the leading ideal. The following theorem analyzes the case when none of the minors commute with taking $^*$.

**Theorem 5.9.** Let $(R, \mathfrak{m})$ be a regular local ring with maximal ideal $\mathfrak{m}$. Let $I$ be an ideal which is minimally generated by the 2 by 2 minors of a 2 by 3 matrix $M$ with entries in $\mathfrak{m}$ where

$$M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}.$$ 

Let

$$M^* = \begin{pmatrix} a^* & b^* & c^* \\ u^* & v^* & w^* \end{pmatrix}.$$ 

Suppose that $(av - bu)^*, (aw - cu)^*$ form part of a minimal generating set of $I^*$ and $I^*$ is a prime ideal. If $(av - bu)^* \neq a^*v^* - b^*u^*$ and $(aw - cu)^* \neq a^*w^* - c^*u^*$, then either $\text{ht} I_1(M^*) \leq 2$ or one of the rows of $M^*$ divides the other.

**Proof.** Since the images of the minors form part of a minimal generating set of a prime ideal $I^*$, we have $a^*v^* = b^*u^* = c^*u^*$. Let $p^* = \gcd(a^*, u^*)$ where $p$ in $R$. Write $a^* = p^*(a')^*$, $u^* = p^*(u')^*$ where $a', u'$ in $R$. Then $(a')^*v^* = v^*(u')^*$ where $\gcd((a')^*, (u')^*) \sim 1$. Therefore $(a')^*|b^*$ and $(u')^*|v^*$. Let $q$ in $R$ such that $(a')^*q^* = b^*$ and $(u')^*q^* = v^*$. From $a^*w^* = c^*u^*$, we have $(a')^*w^* = c^*(u')^*$.
Therefore there exists \( r \) in \( R \) such that \((a')^*r^* = c^* \) and \((u')^*r^* = w^* \). Now, we have

\[
M^* = \left( (a')^*p^* (a')^*q^* (a')^*r^* \right) (u')^*p^* (u')^*q^* (u')^*r^*.
\]

Therefore \( I_1(M^*) \subseteq ((a')^*,(u')^*) \), and this implies \( \text{ht}(I_1(M^*)) \leq 2 \) if \((a')^*,(u')^* \) is not a unit ideal. If \((a')^*,(u')^* \) is a unit ideal, then either \((a')^* \) or \((u')^* \) is a unit. Without loss of generality, assume that \((a')^* \) is a unit. Then one can easily see that indeed the first row divides the second row. \( \square \)

### 6. Serre’s conditions

In this section we show that when a ring \( R \) is local, equidimensional, and universally catenary, if \( \text{gr}_I(R) \) satisfies Serre’s condition \((S_1)\) (or \((R_i)\)), then \( \text{R}[It,t^{-1}] \) satisfies Serre’s condition \((S_1)\) (or \((R_i)\)), and \( R \) satisfies Serre’s condition \((S_1)\).

When \( R \) is a \( \mathbb{Z} \)-graded ring and \( p \) is a prime ideal, let \( p^* \) denote the ideal ideal generated by all homogeneous elements in \( p \). This is a homogeneous prime ideal which has height exactly one less than that of \( p \) if \( p \) is not a homogenous ideal. Recall that a Noetherian ring \( R \) satisfies Serre’s condition \((S_i)\) if for every prime ideal \( p \) of \( R \), \( \text{depth } R_p \geq \min\{i, \text{dim } R_p\} \). A Noetherian ring \( R \) satisfies Serre’s condition \((R_i)\) if for every prime ideal \( p \) of \( R \) with \( \text{dim } R_p \leq i \), the ring \( R_p \) is regular.

**Lemma 6.1** ([2, Theorem 1.5.9 and Exercise 2.1.27, 2.2.24]). Let \( R \) be a Noetherian \( \mathbb{Z} \)-graded ring.

(a) For \( p \in \text{Spec}(R) \) the localization \( R_p \) is regular (Cohen-Macaulay) if and only if \( R_p^* \) is.

(b) Let \( p \in \text{Spec}(R) \). If \( p \) is not homogeneous, then \( \text{depth } R_p = \text{depth } R_{p^*} + 1 \).

**Theorem 6.2.** Let \( (R, m) \) be a local equidimensional universally catenary ring. Let \( I \subseteq m \) be an \( R \)-ideal. Consider the following conditions:

(a) The ring \( \text{gr}_I(R) \) satisfies Serre’s condition \((S_1)\) (or \((R_1)\)).

(b) The ring \( \text{R}[It,t^{-1}] \) satisfies Serre’s condition \((S_1)\) (or \((R_1)\)).

(c) The ring \( R \) satisfies Serre’s condition \((S_1)\).

We have (a) \( \implies \) (b) \( \implies \) (c).

**Proof.** (a) \( \implies \) (b): Let \( \pi : \text{R}[It,t^{-1}] \twoheadrightarrow \text{gr}_I(R) \) be the natural surjective ring homomorphism. By Lemma 6.1(a)(b) it suffices to show Serre’s condition \((S_1)\) (or \((R_1)\)) for homogeneous prime ideals. Let \( P \subseteq \text{R}[It,t^{-1}] \) a homogeneous prime ideal. Since \( R \) is universally catenary and equidimensional, so is \( \text{R}[It,t^{-1}] \). This implies that \( \text{ht}(P + (t^{-1})) \leq \text{ht } P + 1 \). We can choose a minimal prime \( Q \) of \( P + (t^{-1}) \) of height \( \text{ht}(P + (t^{-1})) \). We first show the statement for Serre’s condition \((S_1)\). Since \( \text{gr}_I(R) \) satisfies Serre’s condition \((S_1)\), depth \( \text{gr}_I(R)_{\pi(Q)} \geq \min\{i, \text{dim } \text{gr}_I(R)_{\pi(Q)}\} \). Since depth \( \text{gr}_I(R)_{\pi(Q)} = \text{depth } \text{R}[It,t^{-1}]_{Q-1} \) and \( \text{dim } \text{gr}_I(R)_{\pi(Q)} = \text{dim } \text{R}[It,t^{-1}]_{Q-1} \), we have

\[
\text{depth } \text{R}[It,t^{-1}]_{Q} \geq \min\{i + 1, \text{dim } \text{R}[It,t^{-1}]_{Q}\}.
\]

If \( P = Q \), i.e., \( t^{-1} \in P \), then we are done. Suppose \( P \subseteq Q \). We need to show that \( \text{depth } \text{R}[It,t^{-1}]_{P} \geq \min\{i, \text{dim } \text{R}[It,t^{-1}]_{P}\} \). Since \( \text{dim } \text{R}[It,t^{-1}]_{P} =
\[ \dim R[I, t^{-1}]_p - 1, \] by Equation (4) it suffices to show that \( \operatorname{depth} R[I, t^{-1}]_p \geq \operatorname{depth} R[I, t^{-1}]_Q + 1 \). This follows immediately once we have shown that
\[ \operatorname{Ext}^j_{R[I, t^{-1}]}((R[I, t^{-1}]/P)_Q, R[I, t^{-1}]_Q) = 0 \] for \( j < \operatorname{depth} R[I, t^{-1}]_Q - 1 \).
Since \( \dim (R[I, t^{-1}]/P)_Q = 1 \), this follows from Lemma 6.3.

Now, suppose that \( \operatorname{gr}_I(R) \) satisfies Serre’s condition \( (R_i) \). When \( \operatorname{ht} P \leq i \), since \( \operatorname{ht} \pi(Q) \leq \operatorname{ht} P \leq i \), \( \operatorname{gr}_I(R_{\pi(Q)}) \) is regular. Since \( R[I, t^{-1}]/(t^{-1}) \cong \operatorname{gr}_I(R) \) and \( t^{-1} \) is a regular element, \( R[I, t^{-1}]_Q \) is regular. Since \( P \subseteq Q \), \( R[I, t^{-1}]_P \) is regular.

(b) \( \implies \) (c): Let \( p \) be a prime ideal of \( R \) of height \( c \). Recall that \( R[I, t^{-1}]/(t^{-1} - 1) \cong R \). Let \( P \) be the pre-image of \( p \) in \( R[I, t^{-1}] \). Since \( R[I, t^{-1}] \) is equidimensional and universally catenary, \( \operatorname{ht} P = c + 1 \). Since \( P \) contains \( t^{-1} - 1 \), it is a non homogeneous prime ideal of \( R[I, t^{-1}] \). Therefore, \( P^* \) is a homogeneous prime ideal of height \( c \). Since \( R[I, t^{-1}] \) satisfies Serre’s condition \( (S_i) \), we have \( \dim R[I, t^{-1}]_P^* \geq \min \{ i, \dim R[I, t^{-1}]_P \} \). Recall that \( t^{-1} - 1 \) is a regular element in \( R[I, t^{-1}] \) and \( R \cong R[I, t^{-1}]/(t^{-1} - 1) \). We have depth \( R[I, t^{-1}]_P^* = \dim R[I, t^{-1}]_P - 1 = \dim R_p + 1 - 1 = \dim R_p \) and \( \dim R[I, t^{-1}]_P^* = \dim R[I, t^{-1}]_P - 1 = \dim R_p + 1 - 1 = \dim R_p \) where the first equality follows from Lemma 6.1(b). Therefore, depth \( R_p \geq \min \{ i, \dim R_p \} \).

**Lemma 6.3** (Ischebeck [6, (15.E) Lemma 2]). Let \( (R, \mathfrak{m}) \) be a Noetherian local ring and \( M, N \) be finitely generated \( R \)-modules which are not zero. Then
\[ \operatorname{Ext}^i_R(N, M) = 0 \]
for \( i < \operatorname{depth} M - \dim N \).

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