Collaborative Information Bottleneck
Matias Vera, Student Member, IEEE, Leonardo Rey Vega, Member, IEEE, and Pablo Piantanida, Senior Member, IEEE

Abstract—This paper investigates a multi-terminal source coding problem under a logarithmic loss fidelity which does not necessarily lead to an additive distortion measure. The problem is motivated by an extension of the Information Bottleneck method to a multi-source scenario where several encoders have to build cooperatively rate-limited descriptions of their sources in order to maximize information with respect to other unobserved (hidden) sources. More precisely, we study fundamental information-theoretic limits of the so-called: (i) Two-way Collaborative Information Bottleneck (TW-CIB) and (ii) the Collaborative Distributed Information Bottleneck (CDIB) problems. The TW-CIB problem consists of two distant encoders that separately observe marginal (dependent) components $X_1$ and $X_2$ and can cooperate through multiple exchanges of limited information with the aim of extracting information about hidden variables $(Y_1, Y_2)$, which can be arbitrarily dependent on $(X_1, X_2)$. On the other hand, in CDIB there are two cooperating encoders which separately observe $X_1$ and $X_2$ and a third node which can listen to the exchanges between the two encoders in order to obtain information about a hidden variable $Y$. The relevance (figure-of-merit) is measured in terms of a normalized (per-sample) multi-letter mutual information metric (log-loss fidelity) and an interesting tradeoff arises by constraining the complexity of descriptions, measured in terms of the rates needed for the exchanges between the encoders and decoders involved. Inner and outer bounds to the complexity-relevance region of these problems are derived from which optimality is characterized for several cases of interest. Our resulting theoretical complexity-relevance regions are finally evaluated for binary symmetric and Gaussian statistical models, showing theoretical tradeoffs between the complexity-constrained descriptions and their relevance with respect to the hidden variables.

Index Terms—Multi-terminal source coding; Logarithmic loss; Distributed source coding; Noisy rate-distortion; Side information; Interactive lossy source coding; Information Bottleneck; Shannon theory.

In the last years we have witnessed a monumental proliferation of digital data, leading to new efforts in the understanding of the fundamental principles behind the discovery of relevant information from massive data sets. A good data representation is paramount for performing large-scale data processing and analysis in a computationally efficient (e.g. minimizing communication resources and time of computation) and statistically meaningful manner [1]. In addition to reducing computation time, proper data representations can decrease storage requirements, which translates into reduced inter-node communication allowing to take advantage of different information sources (multi-view analysis) to improve prediction performance.

The challenge of identifying relevant rate-limited information from observed samples, that is the statistical useful information that those observations provide about other hidden variables of interest, is to obtain compressed descriptions that are good enough statistics for inference of these hidden variables. This raises fundamental questions about the information-theoretic principles underlying the process of discovering valuable and relevant knowledge in the form of structured information. In that sense, the standard rate-distortion function of lossy source coding [2] provides an interesting starting point as a means to understand fundamental information-theoretic tradeoffs between relevance (quality of data descriptions) and complexity (size in terms of bits of the descriptions). Relevance can be linked to an appropriate (non-additive) fidelity measure that captures the meaningful characteristics of unobserved data while complexity can be associated to the size of the data descriptions generated from the observed samples.

In this paper, we investigate the fundamental information-theoretic limits of a collaborative and distributed source coding problem with a (not necessarily additive) log-loss fidelity, which is motivated by the Information Bottleneck problem [3]. As opposed to a centralized setting, in our present framework each source observes only a fragment of the total data set to process, where subsets of data tuples (possibly overlapping) are available at different sites. This distributed setup typically imposes a set of constraints on the decoders which are absent in the centralized setup and that could prohibit the transfer of raw data from each of the sites to a central location. We approach this challenging problem from an information-theoretic perspective, studying the exchanges of data descriptions between sites or agents subject to communication (information rates) constraints.
A. Related Work

The idea of obtaining good descriptions of a hidden variable through the compression of an observed dependent one can be formalized through the noisy source-coding problem introduced in [2], where the functions that generate the appropriate descriptions corresponds to the class of rate-limited encoders that compress the observation $X$ with the goal of minimizing a fidelity (distortion) measure with respect to an unobserved variable $Y$. The optimal rate-distortion tradeoff region follows from the function [2]:

$$R(D) = \inf_{P_{U|X} : \mathbb{E}[d(U,Y)] \leq D} I(X;U),$$

where $d : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}_+$ is a per-letter distortion (or loss) measure and $P_{U|X} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{U})$ is conditional distribution that satisfies the Markov chain $U \rightarrow X \rightarrow Y$. Several distortion functions could be of interest in practice such as the Hamming or quadratic loss. In particular, taking the loss $d(u,y) = -\log p_{Y|U}(y|u)$ with $D = H(Y) - \mu$ yields an interesting case of an additive (over the source samples) mutual information as the (single-letter) distortion measure. This measure of relevance was first proposed in [3] giving birth to the Information Bottleneck method. The main idea behind it is finding a compressed description $f(X^n)$ of the data $X^n = (X_1, \ldots, X_n)$ with coding rate $\log |f| \leq nR$ subject to a constraint on the mutual information $I(f(X^n);Y) \geq \mu$, where $Y_i$ depends on $X_i$, and $\mu$ is the minimal level of relevance required and $R$ is the coding rate. As pointed out in [5], this notion of relevance boils down to noisy lossy source coding with logarithmic loss distortion, from which the optimal tradeoff region (rates of complexity $R$ and relevance $\mu$) follows from the rate-relevance function:

$$R(\mu) = \inf_{P_{U|X} : I(U;Y) \geq \mu} I(X;U),$$

where $P_{U|X} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{U})$ forms a Markov chain $U \rightarrow X \rightarrow Y$. The function $\mu \rightarrow R(\mu)$ (or its dual $R \rightarrow \mu(R)$) provides a curve similar to the rate-distortion curve, that provides all tradeoffs between coding rates and levels of information w.r.t. hidden variable $Y$. Interestingly, the same single-letter characterization is also the optimal characterization when the relevance is measured by a multi-letter mutual information $I(f(X^n);Y^n) \geq n\mu$ with $Y^n = (Y_1, \ldots, Y_n)$ which is, in general, a non-additive distortion. This was also observed in [5]. The rate-relevance function given by the classical information bottleneck given by $R(\mu)$ can then be though either, as point-to-point noisy source coding problem with additive single letter distortion given by $d(u,y) = -\log p_{Y|U}(y|u)$ or with multi-letter fidelity criterion given by $I(f(X^n);Y^n)$ as discussed above.

In line with the above mentioned works and modeling the structure of data and its hidden variables by independent and identically distributed samples draw from a known distribution, this paper aims at understanding how proper distributed data descriptions translates into reduced inter-encoder communication when there are several parties involved which observe dependent sources and are interested in extracting useful information about other hidden variables. This clearly should be done by taking advantage of the dependence between the different information sources to recover a good enough statistic that summarizes relevant information about some unobserved hidden variables using cooperation and interaction among all parties involved.

It is worth to further emphasize our motivation behind the use of a multi-letter (non-additive) mutual information as a measure of relevance. Although in principle more difficult to analyze, it appears to be more natural and appealing from a practical perspective, as it allows the possibility of better exploring temporal dependences in the metric of relevance induced by the encoding mapping with respect to the case where an additive metric is considered as in [3]. Despite the fact both additive and non-additive relevances lead asymptotically to the same mathematical problem (the reader may be refer to [17] for further details), the multi-letter form of the relevance is connected to a variety of interesting problems in information theory. More precisely, the multi-letter (non-additive) relevance becomes: the asymptotic exponent corresponding to the second type error probability of distributed testing against independence [8], [9], the asymptotic characterization of images of sets via noisy channels [10] and is also related to the Hypercontraction of the Markov operator [11] and gambling problems [12].

The distributed (non-cooperative) setting of the source coding problem with logarithmic loss distortion, was first investigated in [5], where a complete characterization of the complexity-relevance region was derived, solving completely the Berger-Tung problem [13] under this specific distortion metric. Moreover, the well-known longstanding open CEO problem [14] was also completely solved under this distortion metric. The CEO problem is in fact a well-studied problem which has received a lot of interest in the last years because of its relevance to distributed sensing schemes, specially for the quadratic Gaussian case [15], [16]. A multi-terminal source coding problem —fundamentally different from previous distributed source coding problems— termed information-theoretic biclustering was also investigated in [17]. In this setting, several distributed (non-cooperative) encoders are interested in maximizing, as much as possible, redundant information among their observations. Equally important is the impact that cooperation and interaction can have in distributed source coding scenarios. In this sense, the seminal work by Kaspi [18] has sparked some interest in the recent years, where several papers in the fields of distributed function computation and rate-distortion theory were published [19]–[22].

B. Contributions

In summary, we will consider a multi-point source coding problem where the dependence between the observed and hidden sources can be exploited through cooperation and interaction. As we will be interested in a multi-letter fidelity criterion given by the mutual information between the generated descriptions and the hidden variables we can see that our general setting can be interpreted as a multi-point information bottleneck problem generalizing the above discussed classical point-to-point information bottleneck to a distributed setting.
In more precise terms, in this paper, we first study the so-called Two-way Collaborative Information Bottleneck (TW-CIB) problem, as described in Fig. 1. This scenario consists of two distant encoders that separately observe marginal components $X^n_1$ and $X^n_2$ of a joint memoryless process and wish to cooperate through multiple exchanges of limited (complexity) rate with the goal of extracting relevant information about some hidden variables $(Y^n_1, Y^n_2)$, which can be arbitrarily dependent on $(X^n_1, X^n_2)$. The relevance of the information extracted is measured in terms of the normalized multi-letter mutual information between the generated descriptions and the corresponding hidden variables. We characterize the set of all feasible rates of complexity and relevance, for an arbitrary number of exchange rounds. This result is particularized to some binary symmetric and all possible Gaussian statistical models. In particular, the analysis of the binary symmetric case (even for the simpler half-round case) appears to be rather involved.

Then, we investigate the so-called Collaborative Distributed Information Bottleneck (CDIB) problem, as described in Fig. 2. This differs from the above scenario in that only a single decoder which is not part of the encoders is considered. The decoder wishes to use descriptions from sources $X^n_1$ and $X^n_2$ to maximize the multi-letter mutual information with respect to the hidden (relevant) variable $Y^n$. This scenario can be identified as being the natural extension of the previous works [5], [23]. However, in the present setting, encoders 1 and 2 can interactively cooperate by exchanging pieces of information that should be informative enough about $Y$ but without becoming too complex in order to be transmitted and recovered at the decoder. The central difficult arises in finding the way to explicitly exploit the correlation present between the variables $(X_1, X_2, Y)$ to reduce the cost of communication. We begin by deriving an inner bound to the complexity-relevance tradeoffs. Optimal characterizations are provided in the two specific cases mentioned above. Proofs of the several outer bounds presented in the paper are relegated to Section IV while the inner bounds are developed in the appendices. Gaussian models are investigated in Section V while the binary symmetric model for the TW-CIB problem is studied in Section VI. Finally, in Section VII the conclusions are presented.

**Conventions and Notations**

We use upper-case letters to denote random variables and lower-case letters to denote realizations of random variables. With $x^n$ and $X^n$ we denote vectors and random vectors of $n$ components, respectively. The $i$-th component of vector $x^n$ is denoted interchangeably as $x_i$ or $x[i]$ and with $x[s:t]$ we denote the components with indices ranging from $s$ to $t$ with $s \leq t$. All alphabets are assumed to be finite, except for the Gaussian models discussed in Section V. Entropy is denoted by $H(\cdot)$, differential entropy by $h(\cdot)$, binary entropy by $h_2(\cdot)$ and mutual information by $I(\cdot;\cdot)$. If $X$, $Y$ and $V$ are three random variables on some alphabets their probability distribution is denoted by $p_{X,Y,V}$. When clear from context we will simple denote $p_X(x)$ with $p(x)$. With $\mathcal{P}(\mathcal{X})$ we denote the set of probability distributions over alphabet $\mathcal{X}$. If the probability distribution of random variables $X,Y,V$ satisfies $p(x|yv) = p(x|y)$ for each $x,y,v$, then they form a Markov chain, which is denoted by $X \rightarrow Y \rightarrow V$. When $Z_1$ and $Z_2$ are independent random variables we will denote it as $Z_1 \perp Z_2 $. Conditional variance of $Z_1$ given $Z_2$ is denoted by $\text{Var}[Z_1|Z_2]$. The set of strong typical sequences associated with random variable $X$ is denoted by $T^n_{\epsilon}(X)$, where $\epsilon > 0$. Given $x^n$, the conditional strong typical set given $x^n$ is denoted as $T^n_{\epsilon}[X|x](x^n)$. Typical and conditional typical sets are denoted as $T^n_{\epsilon}$ when clear from the context. The cardinality of set $\mathcal{A}$ is denoted by $|\mathcal{A}|$ and with $2^{|\mathcal{A}|}$ we denote its power set. The complement of $\mathcal{A}$ is denoted by $\mathcal{A}^c$. With $\mathbb{R}_{\geq 0}$ and $\mathbb{Z}_{\geq 0}$ we denote the real and integer numbers greater than 0, respectively. If $a$ and $b$ are real numbers, with $a = b$ we denote $a(1-b) + b(1-a)$. We denote $[a]^+ = \max\{a, 0\}$ when $a \in \mathbb{R}$. All logarithms are taken in base 2.

We finally introduce some convenient notation that will be used through the paper. Let $V_{1,t}$ and $V_{2,t}$ be a sequence of
random variables and let:
\[
W_{1,l} \equiv \{V_{1,k}, V_{2,k}\}_{k=1}^{l-1} \text{ for } l \in [1:K+1], \\
W_{2,l} \equiv \{W_{1,l}, V_{1,l}\} \text{ for } l \in [1:K].
\]

This definition will help to simplify the expressions of the inner and outer bounds of this paper. It will be clear from the following sections that while each \(V_{1,l}, V_{2,l}\) will be used in the generation of the descriptions in encoders 1 and 2 at time \(l\), \(W_{1,l}\) and \(W_{2,l}\) will represent the set of descriptions generated and recovered at both encoders 1 and 2 up to time \(l\).

II. TWO-WAY COLLABORATIVE INFORMATION BOTTLENECK

We begin by introducing the so-called Two-way Collaborative Information Bottleneck (TW-CIB) problem and then state the optimal characterization of the corresponding complexity-relevance region.

A. Problem statement

Consider \((X_1^n, X_2^n, Y_1^n, Y_2^n)\) to be sequences of \(n\) i.i.d. copies of random variables \((X_1, X_2, Y_1, Y_2)\) distributed according to \(p(x_1, x_2, y_1, y_2)\) taking values on \(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}_1 \times \mathcal{Y}_2\), where \(\mathcal{X}_i, \mathcal{Y}_i\) with \(i \in \{1, 2\}\) are finite alphabets. First, encoder 1 generates a (representation) description, based on its observed input sequence \(X_1^n = (X_{11}, \ldots, X_{1n})\) and transmits it to encoder 2. After correctly recovering this description, encoder 2 generates a description based on its observed input sequence \(X_2^n\) and the recovered message from encoder 1 and transmits a description to encoder 2. This process is repeated at both encoders, where each new description is generated based on the observed source realization and the previous description recovered up to that time. The generation of the description at encoder 1 (based on the observed source and previous history) and the recovering at encoder 2 is referred to as a half-round. The addition of the generation of the description at encoder 2 and its recovering at encoder 1 constitutes what we shall call simply a round. After \(K\) rounds have been completed, the information exchange between both encoders concludes. It is expected that the level of relevant information that decoder 1 has gathered about the hidden representation variable \(Y_i^n\) is above a required value \(\mu_1 \geq 0\). Similarly, decoder 2 requires a minimum value of relevant information about sequence \(Y_2^n\) of \(\mu_2 \geq 0\). This problem can be graphically represented in Fig. 1. A mathematical formulation of the described process is given below.

**Definition 1 (K-step code and complexity-relevance region of the TW-CIB problem):** A \(K\)-step \(n\)-length TW-CIB code, for the network model in Fig. 1 is defined by a sequence of encoder mappings:
\[
f_l : \mathcal{X}_1^n \times \mathcal{J}_1 \times \cdots \times \mathcal{J}_{l-1} \rightarrow \mathcal{I}_l, \\
g_l : \mathcal{X}_2^n \times \mathcal{I}_1 \times \cdots \times \mathcal{I}_l \rightarrow \mathcal{J}_l
\]

with \(l \in [1:K]\) and message sets: \(\mathcal{I}_l \equiv \{1, 2, \ldots, |\mathcal{I}_l|\}\) and \(\mathcal{J}_l \equiv \{1, 2, \ldots, |\mathcal{J}_l|\}\). In compact form we denote a \(K\)-step interactive source coding by \((n, \mathcal{F})\) where \(\mathcal{F}\) denote the set of encoders mappings.

An \(4\)-tuple \((R_1, R_2, \mu_1, \mu_2) \in \mathbb{R}_{\geq 0}^2\) is said to be \(K\)-achievable if \(\forall \varepsilon > 0\) exists \(n_0(\varepsilon)\), such that \(\forall n > n_0(\varepsilon)\) exists a \(K\)-step TW-CIB code \((n, \mathcal{F})\) with complexity rates satisfying:
\[
\frac{1}{n} \sum_{l=1}^{K} \log |\mathcal{I}_l| \leq R_1 + \varepsilon, \quad \frac{1}{n} \sum_{l=1}^{K} \log |\mathcal{J}_l| \leq R_2 + \varepsilon, \quad (1)
\]

and normalized multi-letter relevance conditions:
\[
\mu_i - \epsilon \leq \frac{1}{n} I(Y_i^n; \mathcal{I}_K \mathcal{J}_K X_1^n), \quad i \in \{1, 2\}. \quad (2)
\]

The \(K\)-step complexity-relevance region \(\mathcal{R}_{\text{TW-CIB}}(K)\) for the TW-CIB problem is defined as:
\[
\mathcal{R}_{\text{TW-CIB}}(K) \equiv \{(R_1, R_2, \mu_1, \mu_2) : (R_1, R_2, \mu_1, \mu_2) \text{ is } K\text{-achievable}\}.
\]

**Remark 1:** By the memoryless property of \(Y_i^n\), the relevance condition can be equivalently written as:
\[
\frac{1}{n} H(Y_i^n|\mathcal{I}_K \mathcal{J}_K X_1^n) \leq \mu_i + \epsilon, \quad i \in \{1, 2\}
\]

where \(\mu_i \equiv H(Y_i^n) - \mu_i\). In this way, the TW-CIB problem can be recast in the conventional interactive rate-distortion problem [18] using logarithmic-loss distortion [5], where at encoder 1 we put a “soft” decoder whose outputs are probability distributions on \(Y_i^n\) (refer to [17, Lemmas 18, 19] for further details). The descriptions \((\mathcal{I}_K, \mathcal{J}_K)\) can be considered as the indices of the family of probability distributions that the decoder can output. It is also easily shown that, restricting the output probability distributions to products ones should not reduce the optimal complexity-relevance region.

**Remark 2:** \(\mathcal{R}_{\text{TW-CIB}}(K)\) depends on the ordering in the encoding procedure. Above we have defined the encoding functions \(\{f_l, g_l\}_{l=1}^K\) assuming encoder 1 acts first, followed by encoder 2, and the process beginning again at encoder 1. We could consider all possible orderings and take \(\mathcal{R}_{\text{TW-CIB}}(K)\) to be the union of the achievable complexity-relevance pairs over all possible encoding orderings. For sake of clarity and simplicity, we shall not pursue this further.

**Remark 3:** It is straightforward to check that \(\mathcal{R}_{\text{TW-CIB}}(K)\) is convex and closed.

**Remark 4:** We could consider the case in which the number of rounds is arbitrary. In that case we can define the ultimate complexity-relevance region as:
\[
\mathcal{R}_{\text{TW-CIB}} \equiv \bigcup_{K \in \mathbb{Z} > 0} \mathcal{R}_{\text{TW-CIB}}(K)
\]

\[= \{(R_1, R_2, \mu_1, \mu_2) : (R_1, R_2, \mu_1, \mu_2) \text{ is } K\text{-achievable for some } K \in \mathbb{Z} > 0\}. \]

The set limiting operation in the above equation can be easily seen to be well-defined.

B. Characterization of the complexity-relevance region

The next theorem provides the characterization of \(\mathcal{R}_{\text{TW-CIB}}(K)\) in terms of single-letters expressions:

**Theorem 1 (Characterization of the complexity-relevance region for TW-CIB):** Consider an arbitrary pmf \(p(x_1, x_2, y_1, y_2)\).
The corresponding region $\mathcal{R}_{\text{cwb}}(K)$ is the set of tuples $(R_1, R_2, \mu_1, \mu_2) \in \mathbb{R}_{\geq 0}^4$ such that there exists auxiliary random variables $\{V_{1,l}, V_{2,l}\}_{l=1}^K$ satisfying:

$$R_1 \geq I(X_1; W_1, K+1|X_2),$$
$$R_2 \geq I(X_2; W_1, K+1|X_1),$$
$$\mu_1 \leq I(Y_1; W_1, K+1|X_1),$$
$$\mu_2 \leq I(Y_2; W_1, K+1|X_2),$$

taking values in finite discrete alphabets $V_{1,l}$ and $V_{2,l}$ and satisfying Markov chains:

$$V_{1,l} \rightarrow (X_1, W_{1,l}) \rightarrow (X_2, Y_1, Y_2),$$
$$V_{2,l} \rightarrow (X_2, W_{2,l}) \rightarrow (X_1, Y_1, Y_2)$$

for $l \in [1 : K]$. The auxiliary random variables can be restricted to take values in finite alphabets with cardinalities bounds given by:

$$|V_{1,l}| \leq |X_1||W_{1,l}| + 3, \quad \text{for } l = [1 : K]$$

$$|V_{2,l}| \leq |X_2||W_{2,l}| + 3, \quad \text{for } l = [1 : K - 1]$$

$$|V_{2,K}| \leq |X_2||W_{2,K}| + 1,$$

where $|W_{1,l}| = \prod_{i=1}^{l-1} |V_{1,i}|/|V_{2,i}|$ and $|W_{2,l}| = |V_{1,l}|/|V_{2,l}|$ for $l = [1 : K]$.

**Proof:** The proof of the achievability is given in Appendix B while the converse part is relegated to the next section. ■

**Remark 5:** It is immediate to see that the point-to-point classical information bottleneck problem, where observing $X_1$ we are interested in extracting information about $Y_2$ can be seen as an special case of Theorem 1 when $X_2, Y_1 = \emptyset$.

### III. Collaborative Distributed Information Bottleneck

We begin by introducing the so-called Collaborative Distributed Information Bottleneck (CDIB) problem and then provide bounds to the optimal complexity-relevance region. Special cases for which these bounds are tight are also discussed.

#### A. Problem statement

Consider $(X_1^n, X_2^n, Y^n)$ be sequences of $n$ i.i.d. copies of random variables $(X_1, X_2, Y)$ distributed according to $p(x_1, x_2, y)$ taking values on $X_1 \times X_2 \times Y$. We will consider a cooperative setup in which $X_1^n$ and $X_2^n$ are observed at encoders 1 and 2, respectively, and a third party referred as the decoder wishes to “learn” the hidden representation variable $Y^n$. Encoders 1 and 2 cooperatively and interactively generate representations that are perfectly heard by the decoder, through a noiseless but rate-limited broadcast link, as shown in Fig. 2. The cooperation between encoders 1 and 2 permits to save rate during the exchanges and at the same time maintaining an appropriate level of relevance between the generated descriptions and the hidden variable $Y^n$. Encoders 1 and 2 interact as in the TW-CIB problem. After they ceased to exchange their descriptions, the decoder attempts to recover the descriptions generated at encoders 1 and 2, which should have some predefined level of information with respect to $Y^n$.

**Definition 2 (K-step code and complexity-relevance region of the CDIB problem):** A $K$-step $n$-length CDIB code, for the network model in Fig. 2 is defined by a sequence of encoder mappings:

$$f_l : \mathcal{X}_1^n \times \mathcal{J}_l \times \cdots \times \mathcal{J}_{l-1} \rightarrow \mathcal{I}_l,$$
$$g_l : \mathcal{X}_2^n \times \mathcal{I}_l \times \cdots \times \mathcal{I}_2 \rightarrow \mathcal{J}_l,$$

with $l \in [1 : K]$ and message sets: $\mathcal{I}_l \triangleq \{1, 2, \ldots, |\mathcal{I}_l|\}$ and $\mathcal{J}_l \triangleq \{1, 2, \ldots, |\mathcal{J}_l|\}$. In compact form we denote a $K$-step CDIB code by $(n, \mathcal{F})$ where $\mathcal{F}$ denote the set of encoders mappings.

A 3-tuple $(R_1, R_2, \mu) \in \mathbb{R}^3_{\geq 0}$ is said to be $K$-achievable if $\forall \epsilon > 0$ exists $n_0(\epsilon)$, such that $\forall n > n_0(\epsilon)$ exists a $K$-step source code $(n, \mathcal{F})$ with rates satisfying:

$$\frac{1}{n} \sum_{l=1}^{K} \log |\mathcal{I}_l| \leq R_1 + \epsilon,$$

$$\frac{1}{n} \sum_{l=1}^{K} \log |\mathcal{J}_l| \leq R_2 + \epsilon$$

and normalized multi-letter relevance at the decoder:

$$\mu - \epsilon \leq \frac{1}{n} I(Y^n; I^K J^K).$$

The $K$-step complexity-relevance region $\mathcal{R}_{\text{cib}}(K)$ is defined as:

$$\mathcal{R}_{\text{cib}}(K) \triangleq \{(R_1, R_2, \mu) \text{ is } K\text{-achievable}\}.$$
I have the Markov chain complexity-relevance region descriptions generated in previous rounds. The region \( R^\text{com}_K \) is not present if both encoders – before generating that for the TW-CIB problem, this cooperative binning is not necessary. However, the rate equation in order to recover the last description generated by encoder 1 (encoder 2). The region \( R^\text{com}_K \) is postponed to the next section.

\[
|V_{1,K}| \leq |X_1||W_{1,K}| + 3,
\]
\[
|V_{2,K}| \leq |X_2||W_{2,K}| + 1.
\]

where \( |W_{1,l}| = \prod_{i=1}^{l-1} |V_{1,i}||V_{2,i}| \) and \( |W_{2,l}| = |V_{1,l}||V_{2,l}| \) for \( l = [1 : K] \). Then, \( R^\text{com}_K \subseteq R^\text{com}_K \).

**Proof:** See Appendix [B].

**Remark 6:** As shown in the Appendix this region is achievable using a special cooperative binning between encoders 1 and 2 which was inspired by previous work in [22]. After the information exchange is accomplished, the decoder needs to recover the descriptions generated at encoders 1 and 2. At each round, for example encoder 2, generates its own description after having recovered the ones generated at encoder 1 at the present and previous rounds. So, instead of binning on its last generated description, it can also consider in its binning what he already knows from its past descriptions and the ones from encoder 1 (see Appendix [B]). This allows for an explicit cooperation between encoders 1 and 2 in order to help the decoder to recover both descriptions despite of the fact that it does not have side information and without penalizing the rate constraint (e.g. observe the rate constraint on \( R_1 \) or \( R_2 \)).

Note also that the rate expressions corresponding to \( R_1 \) and \( R_2 \) can be written as:

\[
R_1 \geq \sum_{l=1}^{K} I(X_1; V_{1,l}|W_{1,l}X_2),
\]
\[
R_2 \geq I(X_2; V_{2,K}|W_{2,K}) - I(X_2; V_{2,K}|W_{2,K}X_1) + \sum_{l=1}^{K} I(X_2; V_{2,l}|W_{2,l}X_1),
\]

where the sequential nature of the coding is revealed. We see that for every round / both rate equations present terms \( I(X_1; V_{1,l}|W_{1,l}X_2) \) and \( I(X_2; V_{2,l}|W_{2,l}X_1) \) which correspond to the minimum rates that encoder 2 (encoder 1) needs in order to recover the last description generated by encoder 1 (encoder 2). However, the rate equation \( R_2 \) presents a penalizing term that involves the description generated at encoder 2 in round \( K \). This term appears because the last description generated at encoder 2 will not get benefit from further cooperative binning given that there are not any more rounds. As the decoder has not side information, the encoder 2 has to send an excess rate to compensate for that and help him to recover all generated descriptions. It is clear that for the TW-CIB problem, this cooperative binning is not needed because an external decoder (i.e. different from the encoders) is not present and both encoders – before generating a new description – know (with probability close to 1) the descriptions generated in previous rounds.

The following result gives us an outer bound to the complexity-relevance region \( R^\text{com}_K \) in the special case that \( X_1 \rightarrow Y \rightarrow X_2 \).

**Theorem 3 (Outer bound to \( R^\text{com}_K \)):** Assume that we have the Markov chain \( X_1 \rightarrow Y \rightarrow X_2 \). Let \( R^\text{com}_K \) to be the region of tuples \( (R_1, R_2, \mu) \in \mathbb{R}_+^3 \) such that there exist auxiliary random variables \( \{V_{1,l}, V_{2,l}\}_{l=1}^K \) simultaneously satisfying:

\[
R_1 \geq I(X_1; W_{1,K+1}|X_2), \quad (9)
\]
\[
R_2 \geq I(X_2; W_{1,K+1}|Y) - I(Y; W_{2,K}) + \mu^+, \quad (10)
\]

satisfying the Markov chains (7) and (8) for \( l = [1 : K] \) and \( \mu^+ \) in finite discrete alphabets \( V_{1,l} \) and \( V_{2,l} \) with cardinalities bounded by:

\[
|V_{1,l}| \leq |X_1||V_{1,l}| + 4, \text{ for } l = [1 : K],
\]
\[
|V_{2,l}| \leq |V_{2,l}||V_{2,l}| + 4, \text{ for } l = [1 : K - 1],
\]
\[
|V_{2,K}| \leq |V_{2,K}||V_{2,K}| + 1.
\]

Theorem 4 (Complexity-relevance region when \( X_1 \rightarrow Y \rightarrow X_2 \)): Assume \( K = 1 \): \( X_1 \rightarrow Y \rightarrow X_2 \), then \( R^\text{com}_1 \). Let \( R^\text{com}_1 \) be the region of tuples \( (R_1, R_2, \mu) \in \mathbb{R}_+^3 \) such that there exist auxiliary random variables \( \{V_{1,l}, V_{2,l}\}_{l=1}^K \) simultaneously satisfying:

\[
R_1 \geq I(X_1; V_{1,l}|X_2),
\]
\[
R_2 \geq I(X_2; V_{1,l}|Y) - I(Y; V_{2,l}) + \mu^+, \quad (10)
\]
\[
R_1 + R_2 \geq I(X_1; X_2; W_{1,K+1}|Y) + \mu^+, \quad (10)
\]

where \( |V_{1,l}| = \prod_{i=1}^{l-1} |V_{1,i}||V_{2,i}| \) and \( |V_{2,l}| = |V_{1,l}||V_{2,l}| \) for \( l = [1 : K] \). Then, \( R^\text{com}_K \subseteq R^\text{com}_K \).

**Proof:** The proof is relegated to Section IV.

In general, it appears not possible to show that \( R^\text{com}_K \subseteq R^\text{com}_K \) for every \( K \in \mathbb{Z}_{>0} \) from Theorem 2 and 3. However, in this case when \( K = 1 \) that is, the interaction between encoders 1 and 2 is restricted to only one round.

**Remark 7:** The Markov chain \( X_1 \rightarrow Y \rightarrow X_2 \) turns our problem into the interactive-cooperative CEO problem. This approach has a well-known converse for the sum-rate [24] which has been proved for an additive distortion but can be easily re-adapted. However, the sum-rate constraint provided in this paper is tighter. To check this, we can ignore conditions (9) and (10). Then the corner points of \( R^\text{com}_K \) are:

\[
Q_A = [I(X_1X_2; W_{1,K+1}), (Y; W_{1,K+1})],
\]
\[
Q_B = [I(X_1X_2; W_{1,K+1}Y), 0],
\]

where these components correspond to the sum-rate and relevance, respectively. The resulting corner points meet simultaneously: \( \mu \leq I(Y; U) \) and

\[
R_1 + R_2 \geq I(Y; U) + I(X_1; U|Y) + I(X_2; U|YZ),
\]

where \( Z \) is a random variable independent of \( (X_1, X_2, Y) \), and \( U \) satisfying \( Y \rightarrow (X_1, X_2, Z) \rightarrow U \) and \( X_1 \rightarrow (Y, U, Z) \rightarrow X_2 \). To show this, let us assume that \( Z \equiv z \) almost surely, i.e., \( Z \) is a degenerated random variable, and set \( U \equiv W_{1,K+1} \) and \( Z \equiv z \) or \( U \equiv u \) for the corner points \( Q_A \) and \( Q_B \), respectively.

**C. Characterization of the complexity-relevance region when \( X_1 \rightarrow Y \rightarrow X_2 \) with \( K = 1 \)**

**Theorem 4 (Complexity-relevance region when \( X_1 \rightarrow Y \rightarrow X_2 \) with \( K = 1 \)):** Assume \( K = 1 \): \( X_1 \rightarrow Y \rightarrow X_2 \), then \( R^\text{com}_1 \). Let \( R^\text{com}_1 \) be the region of tuples \( (R_1, R_2, \mu) \in \mathbb{R}_+^3 \) such that there exist auxiliary random variables \( \{V_{1,l}, V_{2,l}\}_{l=1}^K \) simultaneously satisfying:

\[
R_1 \geq I(X_1; V_{1,l}|X_2),
\]
\[
R_2 \geq I(X_2; V_{1,l}|Y) - I(Y; V_{2,l}) + \mu^+, \quad (10)
\]
\[
R_1 + R_2 \geq I(X_1; X_2; W_{1,K+1}|Y) + \mu^+, \quad (10)
\]

where \( |V_{1,l}| = \prod_{i=1}^{l-1} |V_{1,i}||V_{2,i}| \) and \( |V_{2,l}| = |V_{1,l}||V_{2,l}| \) for \( l = [1 : K] \). Then, \( R^\text{com}_K \subseteq R^\text{com}_K \).

**Proof:** The proof of the equality between the regions provided in Theorems 2 and 3 is postponed to the next section.

**Remark 8 (The role of cooperation):** The region \( R^\text{com}_1 \) can be written as:

\[
R_1 \geq I(X_1; V_{1,l}|X_2),
\]
\[
R_2 \geq I(X_2; V_{1,l}|Y) - I(Y; V_{2,l}) + \mu^+, \quad (10)
\]
\[
R_1 + R_2 \geq I(X_1; X_2; W_{1,K+1}|Y) + \mu^+, \quad (10)
\]
\[ R_2 \geq I(X_2; V_2 | V_1), \]
\[ R_1 + R_2 \geq I(X_1 X_2; V_1 V_2), \]
\[ \mu \leq I(Y; V_1 V_2), \]

with \( V_1 \) and \( V_2 \) taking values in finite alphabets \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) and satisfying \( V_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y), V_2 \leftrightarrow (X_1, X_2) \leftrightarrow (X_1, Y) \). It is worth to compare this with the non-cooperative CEO rate-distortion region under logarithmic loss [5] Theorem 3. As it is well known, that region can be expressed in terms of rates \( R_1, R_2 \) and relevance \( \mu \), instead of logarithmic loss distortion level \( \mu' \). In this manner, we can write the following (non-cooperative) complexity-relevance region \( R_{\text{com}} \) as:

\[ R_1 \geq I(X_1; V_1 | V_2), \]
\[ R_2 \geq I(X_2; V_2 | V_1), \]
\[ R_1 + R_2 \geq I(X_1 X_2; V_1 V_2), \]
\[ \mu \leq I(Y; V_1 V_2), \]

where \( V_1 \) and \( V_2 \) take values in finite alphabets \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) satisfying: \( V_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y) \) and \( V_2 \leftrightarrow X_2 \leftrightarrow (X_1, Y) \) form Markov chains. It is clearly seen that \( R_{\text{com}}(1) \geq R_{\text{com1}} \). First, note that:

\[ I(X_1; V_1 | V_2) = I(V_1; X_2 | V_2) + I(X_1; V_1 | X_2) \]
\[ \geq I(X_1; V_1 | X_2). \]

Secondly, the set of probability distributions over which \( R_{\text{com}}(1) \) is constructed is greater than the one corresponding to \( R_{\text{com1}} \). This is seen in the requirement of the auxiliary random variable \( V_2 \), which in the cooperative case can depend on \( V_1 \), reflecting the possibility of cooperation between the encoders.

D. Characterization of the complexity-relevance region when \( X_1 \not\rightarrow X_2 \not\rightarrow Y \) with \( K = 1 \)

**Definition 3:** Let \( \tilde{R}_{\text{com}}(1) \) be the set of tuples \( (R_1, R_2, \mu) \in \mathbb{R}^3_+ \) such that there exists a joint pmf \( p(x_1, x_2, y, v_1, v_2) \) that preserves the joint distribution of the sources \( (X_1, X_2, Y) \) and

\[ R_1 \geq I(X_1; V_1), \]
\[ R_2 \geq I(X_2; V_2 | V_1), \]
\[ \mu \leq I(Y; V_1 V_2), \]

with auxiliary random variables \( V_1, V_2 \) satisfying:

\[ V_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y), V_2 \leftrightarrow (V_1, X_1, X_2) \leftrightarrow Y. \]

Similarly, let \( \tilde{R}_{\text{com}}(1) \) be the set of tuples \( (R_1, R_2, \mu) \in \mathbb{R}^3_+ \) verifying \([11],[13]\) such that there exists a joint pmf \( p(x_1, x_2, y, v_1, v_2) \) that preserves the joint pmf of the sources \( (X_1, X_2, Y) \) while satisfying:

\[ V_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y), V_2 \leftrightarrow (V_1, X_2) \leftrightarrow (X_1, Y). \]

Theorems [5] and [6] will imply the characterization of the corresponding complexity-relevance region. We present first Theorem [5] which gives us inner and outer bounds for \( R_{\text{com}}(1) \) for arbitrary random sources \( X_1, X_2, Y \).

**Theorem 5:** Assume \( K = 1 \) and arbitrary random variables \( X_1, X_2, Y \). Then, we have

\[ \tilde{R}_{\text{com}}(1) \subseteq R_{\text{com}}(1) \subseteq \tilde{R}_{\text{com}}(1). \]

**Proof:** The proof is relegated to the next section. □

The following result implies that \( \tilde{R}_{\text{com}}(1) = \tilde{R}_{\text{com}}(1) \) when \( X_1 \not\rightarrow X_2 \not\rightarrow Y \).

**Theorem 6:** Assume \( K = 1 \) and \( X_1 \not\rightarrow X_2 \not\rightarrow Y \). Then \( \tilde{R}_{\text{com}}(1) \subseteq \tilde{R}_{\text{com}}(1) \).

**Proof:** Assume that \( (R_1, R_2, \mu) \in \tilde{R}_{\text{com}}(1) \). Then, there exists a pmf

\[ p(x_1, x_2, y, v_1, v_2) = p(x_1, x_2, y)p(v_1 | x_1)p(v_2 | x_2, v_1) \]

such that: \( R_1 \geq I(X_1; V_1) \), \( R_2 \geq I(X_2; V_2 | V_1) \) and \( \mu \leq I(Y; V_1 V_2) \). Consider the pmf

\[ \tilde{p}(x_1, x_2, y, v_1, v_2) = p(x_1, x_2, y)p(v_1 | x_1)\tilde{p}(v_2 | x_2, v_1), \]

where

\[ \tilde{p}(v_2 | x_2, v_1) \triangleq \frac{p(x_2, v_2)}{p(x_2, v_1)} = \frac{\sum_{x_1'} p(x_1', x_2) p(v_1 | x_1') p(v_2 | x_1' x_2 v_1)}{\sum_{x_1'} p(x_1', x_2) p(v_1 | x_1')}. \]

By assumption this pmf preserves the sources \( (X_1, X_2, Y) \) while satisfying \([13]\). Moreover, it can be shown without difficulty that \( \tilde{p}(x_1, v_1) = p(x_1, v_1) \) and \( \tilde{p}(x_2, v_1, v_2) = p(x_2, v_1, v_2) \). This implies that \( I(X_1; V_1) \) and \( I(X_2; V_2 | V_1) \) are preserved. If we further assume that \( X_1 \not\leftrightarrow X_2 \not\leftrightarrow Y \), we can write:

\[ \tilde{p}(y, v_1, v_2) = \sum_{x_1, x_2} p(x_1, x_2, y) p(v_1 | x_1) \times \]
\[ \frac{\sum_{x_1'} p(x_1', x_2) p(v_1 | x_1') p(v_2 | x_1' x_2 v_1)}{\sum_{x_1'} p(x_1', x_2) p(v_1 | x_1')} \]
\[ = \sum_{x_1} p(y | x_2) \sum_{x_1} p(x_1, x_2) p(v_1 | x_1) \times \]
\[ \frac{\sum_{x_1'} p(x_1', x_2) p(v_1 | x_1') p(v_2 | x_1' x_2 v_1)}{\sum_{x_1'} p(x_1', x_2) p(v_1 | x_1')} \]
\[ \leq \sum_{x_1} p(y | x_2) p(v_1 | x_1) \]
\[ = p(y, v_1, v_2). \]

As a consequence, the term \( I(Y; V_1 V_2) \) is also preserved and thus \( (R_1, R_2, \mu) \in \tilde{R}_{\text{com}}(1) \).

The next corollary immediately follows.

**Corollary 1:** Provided that \( X_1 \not\rightarrow X_2 \not\rightarrow Y \), we have \( \tilde{R}_{\text{com}}(1) = \tilde{R}_{\text{com}}(1) = R_{\text{com}}(1) \).

It is easily seen that for achieving any \( (R_1, R_2, \mu) \in \tilde{R}_{\text{com}}(1) \) it is not necessary to use binning. First encoder 1 sends its description which can recovered at encoder 2 and the decoder. Then, encoder 2 uses this description –as a coded side information which is also available at the decoder– to generate and sends its own one to the decoder. The previous claim shows this coding scheme is optimal when \( X_1 \not\leftrightarrow X_2 \not\leftrightarrow Y \).

As \( R_{\text{com}}(1) \) is also achievable and \( \tilde{R}_{\text{com}}(1) \subseteq \tilde{R}_{\text{com}}(1) \) (which is trivial to show), we can state an alternative characterization of the complexity-relevance region.

**Corollary 2** (Alternative characterization of \( R_{\text{com}}(1) \) when \( X_1 \not\leftrightarrow X_2 \not\leftrightarrow Y \)): Assume \( K = 1 \) and that \( X_1 \not\leftrightarrow X_2 \not\leftrightarrow Y \).
form a Markov chain, then \( \mathcal{R}_{\text{con}}^{\text{1s}}(1) = \mathcal{R}_{\text{con}}^{\text{max}}(1) = \mathcal{R}_{\text{con}}(1) \).

**Proof:** Follows easily from the above discussion. An alternative proof of this Corollary is presented in Appendix D.

**Remark 9:** From the previous results it should be clear that the coding procedure presented in Theorem 2 is clearly optimal for both cases \( X_1 \Rightarrow Y \Rightarrow X_2 \) and \( X_1 \Rightarrow X_2 \Rightarrow Y \). The first Markov chain corresponds to the typical one considered in the CEO problem [25]. This would be the case where, for example, the hidden variable \( Y \) is related with the observed variables \( X_1 \) and \( X_2 \) through and additive model: \( X_1 = Y + Z_1, X_2 = Y + Z_2 \) where \( Z_1 \) and \( Z_2 \) are independent random variables. For example this situation could appear in a sensor network setting where \( X_1 \) and \( X_2 \) are observed in two geographically separated nodes and in which the fusion center (node 3) desires to obtain a good representation of the hidden variable \( Y \). The case in which \( X_1 \Rightarrow X_2 \Rightarrow Y \) can represent also the case of the distributed sensor network setting, in which the measurements in one of sensors \( (X_2) \) is most informative with respect to the hidden variable \( Y \) that the ones in the other \( (X_1) \). This could represent a situation in which the hidden variable \( Y \) models a physical phenomenon which originates in given point of space and in which the statistical dependence with variables \( X_1 \) and \( X_2 \) at the points of measurements (the sites where nodes 1 and 2 are positioned) depends strongly of their distance to the point of origin. If node 2 is closer than node 1 to the point of origin of \( Y \), \( X_2 \) would have a stronger statistical dependence with \( Y \) and the given Markov chain can be a useful approximate model of this situation.

**Remark 10:** It is worth to mention that the cardinality of the auxiliary variables in this case can be bounded in two different ways. The auxiliary random variables involved in the representation of \( \mathcal{R}_{\text{con}}^{\text{1s}}(1) \) can be restricted to take values in alphabets satisfying:

\[
|V_1| \leq |X_1| + 3, \quad |V_2| \leq |X_2||V_1| + 1.
\]

While the auxiliary random variables involved in the representation of \( \mathcal{R}_{\text{con}}^{\text{1s}}(1) \) can be restricted to take values in alphabets verifying:

\[
|V_1| \leq |X_1| + 3, \quad |V_2| \leq |X_2||V_1| + 1.
\]

**IV. CONVERSES IN THEOREMS [1] [3] [4] AND [5]**

In this section, we provide the proofs to the converses of Theorems [1] [3] and [4]. Together with the inner bounds obtained in Appendix B these results imply the characterization of the corresponding complexity-relevance regions in Theorems [1] [4] and Corollary [3].

**A. Converse result for Theorem [7]**

If a tuple \( (R_1, R_2, \mu_1, \mu_2) \) is achievable, then for all \( \varepsilon > 0 \) there exists \( n_0(\varepsilon) \), such that \( \forall n > n_0(\varepsilon) \) there exists a code \( (n, F) \) with rates and relevance satisfying (1) and (2). For \( t = [1 : n] \), define variables:

\[
V_{1,t} \triangleq I_1, \quad \forall t \in [2 : K]
\]

\[
V_{2,t} \triangleq I_1, \quad \forall t \in [1 : K]
\]

These auxiliary random variables satisfy, for \( t = [1 : n] \) the Markov conditions [3] and [4] and are similar to the choices made in [18]. In that sense, the converse proof follows along similar lines as in [18]. However, for sake of completeness we provide the proof.

**1) Constraint on rate \( R_1 \):** For the first rate, we have

\[
n(R_1 + \varepsilon) \geq H(I^K) \geq \sum_{t=1}^{n} I(Y; J^K; X_t^n|X_t^n) = \sum_{t=1}^{n} I(Y; J^K; X_{t+1:n}^{t+1:n}|X_{t+1:n})
\]

\[
\geq \sum_{t=1}^{n} I(Y; W_1, K+1[t]; X_{t+1:n}|X_{t+1:n}, Q = t) \geq nI(W_1, K+1; X_2|X_1)
\]

where

- step (a) follows from the fact that \( J^K = (J_1, \ldots, J_K) \) is function of \( I^K = (I_1, \ldots, I_K) \) and \( X_t^n \);
- step (b) follows from the use of a time sharing random variable \( Q \) uniformly distributed over the set \([1 : n]\) and independent of the other variables and from the non-negativity of mutual information;
- step (c) follows by defining a new random variable \( \tilde{W}_{1,K+1} \triangleq (W_{1,K+1}[Q], Q) \).

**2) Constraint on rate \( R_2 \):** The analysis is similar to the case for \( R_1 \) and for that reason is omitted. The final result is:

\[
n(R_2 + \varepsilon) \geq nI(W_1, K+1; X_2|X_1)
\]

**3) Constraint on relevance \( \mu_1 \):** For the first relevance, we have

\[
n(\mu_1 - \varepsilon) \leq \sum_{t=1}^{n} I(Y; I^K J^K; X_t^n|Y_t^{t+1:n}) = \sum_{t=1}^{n} I(Y; I^K J^K X_{t+1:n}^{t+1:n}; Y_{t+1:n}) \leq \sum_{t=1}^{n} I(Y_1; W_{1,K+1[t]} X_{t+1:n}^{t+1:n}; Y_{t+1:n} X_{t+1:n})
\]

\[
\leq \sum_{t=1}^{n} I(Y_1; W_{1,K+1[t]} X_{t+1:n}^{t+1:n}; Y_{t+1:n} X_{t+1:n}) \geq nI(W_1, K+1; X_1)
\]

where

- step (a) follows from the definition of \( W_{1,K+1[t]} \) and non-negativity of mutual information;
\begin{itemize}

- step (b) follows from the Markov chain \( Y_{1:t} \to (W_{1,K+1}[n],X_{1:0}) \) and the use of a time sharing random variable \( Q \) uniformly distributed over the set \([1:n]\) and independent of the other variables;

- step (c) follows by letting a new random variables \( W_{1,K+1} \overset{\text{def}}{=} (W_{1,K+1}[q],Q) \).

4) Relevance \( \mu_2 \): Again, the analysis is similar to the one for \( \mu_1 \). Following similar steps, we obtain:

\[
\begin{align*}
& n(\mu_2 - \varepsilon) \leq nI \left( Y_2; \bar{W}_{1,K+1}X_2 \right).
\end{align*}
\]

B. Converse result for Theorem \([3]\)

If a tuple \((R_1,R_2,\mu)\) is achievable, then for all \( \varepsilon > 0 \) there exists \( n_0(\varepsilon) \), such that \( \forall n > n_0(\varepsilon) \) there exists a code \((n,F)\) with rates and relevance satisfying \((5)\) and \((6)\). For \( t = [1:n] \), define variables:

\[
\begin{align*}
V_{1,1:t} & \overset{\text{def}}{=} (I_1,Y_{[1:t-1]},X_{2([1:n])}) \\
V_{1,t} & \overset{\text{def}}{=} I_t, \quad \forall t \in [2:K] \\
V_{2,t} & \overset{\text{def}}{=} J_t, \quad \forall t \in [1:K].
\end{align*}
\]

These auxiliary random variables satisfy, for \( t = [1:n] \), the Markov conditions \((7)\) and \((8)\).

1) Constraint on rate \( R_1 \): For the first rate, we have

\[
\begin{align*}
n(R_1 + \varepsilon) & \geq H \left( I^K \right) \\
& \overset{(a)}{=} I \left( I^K J^K; X^n_1 | X^n_2 \right) \\
& = \sum_{t=1}^{n} I \left( I^K J^K; X_{1t} | X^n_2 X^n_{1[1:t-1]} \right) \\
& = \sum_{t=1}^{n} I \left( I^K J^K X_{1(t-1)}; X_{2(t-1)},X_{2[1:n+t]}; X_{1t} | X_{2t} \right) \\
& \overset{(b)}{=} \sum_{t=1}^{n} I \left( W_{1,K+1}[t]; Y_{[1:t-1]}; X_{2(t-1)},X_{2[1:n+t]}; X_{1t} | X_{2t} \right) \\
& \overset{(c)}{=} \sum_{t=1}^{n} I \left( W_{1,K+1}[q]; Y_{[1]}; X_{2[1]}; Q = t \right) \\
& \overset{(f)}{=} I \left( \bar{W}_{1,K+1}; X_1 | X_2 \right),
\end{align*}
\]

where

- step (a) follows from the fact that \( J^K \) is function of \( I^K \) and \( X^n_2 \);

- step (b) use the Markov chain \( Y_{[1:t-1]} \to (I^K,J^K,X^n_2 X^n_{1[1:t-1]}) \to X_{1t} \);

- step (c) follows from the fact that a time sharing random variable \( Q \) uniformly distributed over the set \([1:n]\) independent of the other variables;

- step (d) follows by letting a new random variable \( W_{1,K+1} \overset{\text{def}}{=} (W_{1,K+1}[q],Q) \).

2) Constraint on rate \( R_2 \): For the second rate, we have

\[
\begin{align*}
n(R_2 + \varepsilon) & \geq H \left( I^K \right) \\
& \geq H \left( J^{K-1} | X^n_1 Y^n \right) + H \left( J^K J^{K-1} \right) \\
& \overset{(a)}{=} I \left( J^{K-1}; X^n_2 | X^n_1 Y^n \right) + I \left( J^K; X^n_2 | X^n_2 Y^n \right) \\
& \overset{(b)}{=} I \left( J^{K-1}; X^n_2 | X^n_1 Y^n \right) + I \left( J^K; X^n_2 | J^{K-1} Y^n \right) \\
& \overset{(c)}{=} I \left( J^{K-1}; X^n_2 | X^n_1 Y^n \right) + I \left( J^K; X^n_2; J^{K-1} Y^n \right)
\end{align*}
\]

where
• step (a) follows from definition of the code $I^K$ and $J^K$
are functions of $X^1_\nu$ and $X^2_\nu$;
• step (b) follows from \[6\];
• step (c) follows from the use of a time sharing random
variable $Q$ uniformly distributed over the set $[1:n]$
independent of the other variables;
• step (d) follows by letting a new random variables:
$W_{1,K+1} \triangleq (W_{1,K+1}(Q); Q)$.

4. Constraint on the relevance $\mu$: Finally, for the relevance,
we have
$$ n(\mu - \epsilon) \leq \sum_{t=1}^{n} I \left( Y^n; I^K J^K \right) $$
$$ = \sum_{t=1}^{n} I \left( Y_1^n; I^K J^K \left| Y^n_{[1:t-1]} \right. \right) $$
$$ \leq \sum_{t=1}^{n} I \left( Y_1^n; I^K J^K Y^n_{[1:t-1]} X_2^n_{[t+1:n]} \right) $$
$$ = \sum_{t=1}^{n} I \left( Y_1^n; W_1^n, K, X_2^n \right) $$
$$ = \sum_{t=1}^{n} \mu I \left( Y_1^n; W_1^n, K, X_2^n \right) $$

where
• step (a) follows from the use of a time sharing random
variable $Q$ uniformly distributed over the set $[1:n]$
independent of the other variables;
• step (b) follows by letting a new random variables:
$W_{1,K+1} \triangleq (W_{1,K+1}(Q); Q)$.

In this way we conclude the proof that $\mathcal{R}_{\text{CIB}}(K) \supseteq \mathcal{R}_{\text{CIB}}(K)$. The fact that $\mathcal{R}_{\text{CIB}}(K) \subseteq \mathcal{R}_{\text{CIB}}(K)$ is given in Appendix B.

C. Proof of Theorem 4

We now show that $\mathcal{R}_{\text{CIB}}(1) = \mathcal{R}_{\text{CIB}}(1) = \mathcal{R}_{\text{CIB}}(1)$ which implies Theorem 4. When $K = 1$ we have that $\mathcal{R}_{\text{CIB}}(1)$ reads as:

$R_1 \geq I(X_1; V_1 | X_2)$,
$R_2 \geq I(X_2; V_2 | Y_1)$,
$R_1 + R_2 \geq I(X_1; X_2; V_1, V_2)$,
$\mu \leq I(Y; V_1)$.

with $V_1$ and $V_2$ taking values in finite alphabets $V_1$ and $V_2$ and satisfying $V_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y)$, $V_2 \leftrightarrow (V_1, X_2) \leftrightarrow (X_1, Y)$. Similarly $\mathcal{R}_{\text{CIB}}(1)$ can be written as:

$R_1 \geq I(X_1; U_1 | X_2)$,
$R_2 \geq [I(X_2; U_2 | Y_1) - I(Y; U_1) + \mu]^+$,
$R_1 + R_2 \geq I(X_1; X_2; U_1, U_2) Y + \mu$,
$\mu \leq I(Y; U_1)$,

with the auxiliary variables $U_1$ and $U_2$ taking values in finite
alphabets $U_1$ and $U_2$ satisfying: $U_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y)$ and
$U_2 \leftrightarrow (U_1, X_2) \leftrightarrow (X_1, Y)$ (note that $U_1 \leftrightarrow Y \leftrightarrow X_2$ because of $X_1 \leftrightarrow Y \leftrightarrow X_2$). From the previous results it is clear
that $\mathcal{R}_{\text{CIB}}(1) \subseteq \mathcal{R}_{\text{CIB}}(1)$. Similarly to [5], it can be shown
that $\mathcal{R}_{\text{CIB}}(1) \supseteq \mathcal{R}_{\text{CIB}}(1)$. This is accomplished by showing
that, when we fix a distribution on $(U_1, U_2)$ for every point
$(R_1, R_2, \mu) \in \mathcal{R}_{\text{CIB}}(1)$, we can find an appropriate distribution
$(V_1, V_2)$ such that $(R_1, R_2, \mu) \in \mathcal{R}_{\text{CIB}}(1)$. To this end, we
study the extreme points (see Appendix C for a definition) and
directions [26] of the restriction of $\mathcal{R}_{\text{CIB}}(1)$ over the assumed
distribution of $(U_1, U_2)$. The details are given in Appendix C.

D. Converse result for Theorem 5

The proof that $\mathcal{R}_{\text{CIB}}(1) \subseteq \mathcal{R}_{\text{CIB}}(1)$ follows from simple
multiterminal coding arguments and for that reason is omitted.

The relevance level can be obtained using the same ideas that
those in Appendix B. For $\mathcal{R}_{\text{CIB}}(1) \subseteq \mathcal{R}_{\text{CIB}}(1)$ assume that
$(R_1, R_2, \mu) \in \mathcal{R}_{\text{CIB}}(1)$, then for all $\epsilon > 0$ there exists $n_0(\epsilon)$,
such that $\forall n > n_0(\epsilon)$ there exists a code $(n, F)$ with rates
and relevance satisfying (5) and (6). For each $t = [1:n]$, we
define random variables:

$V_{1,t} \triangleq (I_1, Y_{[1:t-1]}, X_{2|[t+1:n]})$, $V_{2,t} \triangleq J_1$. (16)

It is easy to show that these choices verify (14). Using similar
steps as in the previous converse proofs, we can easily obtain the
following bounds:

$R_1 + \epsilon \geq \frac{1}{n} \sum_{t=1}^{n} I \left( I_1 X_{2|[t+1:n]} Y_{[1:t-1]}; X_{1,t} \right)$,
$R_2 + \epsilon \geq \frac{1}{n} \sum_{t=1}^{n} I \left( X_{2,t}; J_1 | I_1 X_{2|[t+1:n]} Y_{[1:t-1]} \right)$,
$\mu - \epsilon \leq \frac{1}{n} \sum_{t=1}^{n} I \left( Y_t; I_1 J_1 X_{2|[t+1:n]} Y_{[1:t-1]} \right)$.

From a time-sharing argument and using (16) we get the rate
conditions corresponding to $\mathcal{R}_{\text{CIB}}(1)$.

V. GAUSSIAN SOURCE MODELS

In this section, we study Gaussian models between source
samples and hidden representations. Although the above
achievability results are strictly valid for random variables
taking values on finite alphabets, the results can be applied
to continuous random variables with sufficiently well
behaved probability density function (e.g. Gaussian random
variables). A simple sequence of coding schemes consisting of
a quantization procedure over the sources and appropriate
test channels (with diminishing quantization steps) followed
by coding schemes as the ones presented in this paper will
suffice (e.g. see [25]).

A. Gaussian TW-CIB model

Let $(X_1, X_2, Y_1, Y_2)$ be Gaussian random variables with
zero-mean. We will assume without loss of generality that we
can write:

$$ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad (17) $$

where $Z_1 \perp (X_1, X_2)$ and $Z_2 \perp (X_1, X_2)$ and matrix $A$
can be obtained from the correlation structure of the random
variables. To this end, define:

$$ a_{12} = \frac{\sigma_{y_1 y_2} \rho_{x_1 y_1} - \rho_{x_1 y_2} \rho_{x_1 x_2}}{1 - \rho_{x_1 x_2}^2} \leq \frac{\sigma_{y_1 y_2}}{\sigma_{x_2}} $$
\[ a_{21} \triangleq \frac{\sigma_{y_2} \rho_{x_1,y_2} - \rho_{y_2,x_2} \rho_{x_1,x_2}}{\sigma_{x_1}}, \]

where \( \sigma^2 \) denotes the variance of a random variable \( B \), and \( \rho_{1,2} \) represents the Pearson product-moment correlation coefficient between random variables \( B_1 \) and \( B_2 \). As \((X_1, X_2, Y_1, Y_2)\) are jointly Gaussian, then \( Z_1 \) and \( Z_2 \) are Gaussian as well. It is easy to check that:

\[ \sigma^2_{z_1} = \frac{\sigma^2_{X_1} \beta}{1 - \rho^2_{x_1,x_2}}, \quad \sigma^2_{z_2} = \frac{\sigma^2_{X_2} \delta}{1 - \rho^2_{x_1,x_2}}, \]

where

\[ \beta \triangleq 1 - \rho^2_{x_1,x_2} - \rho^2_{x_2,y_1} - 2 \rho_{x_1,x_2} \rho_{x_1,y_1} \rho_{x_2,y_1}, \]

\[ \delta \triangleq 1 - \rho^2_{x_1,x_2} - \rho^2_{x_2,y_2} - 2 \rho_{x_1,x_2} \rho_{x_2,x_2} \rho_{x_1,y_2}. \]

We are ready to present our first result.

**Theorem 7 (Complexity-relevance region for the Gaussian TW-CIB model):** When \((X_1, X_2, Y_1, Y_2)\) are jointly Gaussian, for any \( K, R_{\text{tw-cib}}(K) \) is given by:

\[ R_1 \geq \frac{1}{2} \log \left( \frac{1 - \rho^2_{x_1,x_2} (1 - \rho^2_{x_2,y_2}) - \delta}{2 - 2 \mu_2 (1 - \rho^2_{x_1,x_2}) - \delta} \right), \]

\[ 0 \leq \mu_2 < \frac{1}{2} \log \left( \frac{1 - \rho^2_{x_1,x_2}}{\delta} \right), \quad (18) \]

\[ R_2 \geq \frac{1}{2} \log \left( \frac{1 - \rho^2_{x_1,x_2} (1 - \rho^2_{x_2,y_1}) - \beta}{2 - 2 \mu_1 (1 - \rho^2_{x_1,x_2}) - \beta} \right), \]

\[ 0 \leq \mu_1 < \frac{1}{2} \log \left( \frac{1 - \rho^2_{x_1,x_2}}{\beta} \right). \quad (19) \]

**Proof:** We first consider the converse. **Converse:** Assume \((R_1, R_2, \mu_1, \mu_2) \in R_{\text{tw-cib}}(K)\). Consider the relevance level \( \mu_1 \). Using (2):

\[ \mu_1 - \epsilon \leq \frac{1}{n} I \left( Y_1^n; I^K J^K X_1^n \right), \]

\[ = h(Y_1) - \frac{1}{n} h \left( Y_1^n | I^K J^K X_1^n \right), \]

\[ = \frac{1}{2} \log \left( 2 \pi e \sigma^2_{y_1} \right) - \frac{1}{n} h \left( a_{12} X_2^n + Z_1^n | I^K J^K X_1^n \right). \quad (20) \]

From the equation for \( R_2 \) and using the fact that \( J^K \) is a function of \( X_2^n \) and \( I^K \), it is not difficult to obtain:

\[ R_2 + \epsilon \geq \frac{1}{n} I \left( X_2^n, I^K J^K X_1^n \right), \]

\[ = h(X_2 | X_1) - \frac{1}{n} h \left( X_2^n | I^K J^K X_1^n \right), \]

\[ = \frac{1}{2} \log \left( 2 \pi e \text{Var}[X_2 | X_1] \right) - \frac{1}{n} h \left( X_2^n | I^K J^K X_1^n \right). \quad (21) \]

As \( Z_1^n \perp (I^K J^K) \) we can link (a) and (b) using the conditional EPI [27] to write:

\[ 2^{R_2} a_{12}^2 \left( \sigma^2_{x_2} + \sigma^2_{y_1} \right) \geq a_{12}^2 2^{R_1} \left( \sigma^2_{x_2} + \sigma^2_{y_1} \right) + 2 \pi e \sigma^2_{z_1}. \]

From (20) and (21) we can write:

\[ R_2 + \epsilon \geq \frac{1}{2} \log \left( \frac{\text{Var}[X_2 | X_1] \sigma^2_{x_2}}{\sigma^2_{x_2} - 2^{2(\mu_1 - \epsilon)} - \sigma^2_{z_1}} \right). \]

Using the correlation structure implied by (17) we can obtain:

\[ R_2 + \epsilon \geq \frac{1}{2} \log \left( \frac{(1 - \rho^2_{x_1,x_2})(1 - \rho^2_{x_2,y_1}) - \beta}{2 - 2(\mu_1 - \epsilon)(1 - \rho^2_{x_1,x_2}) - \beta} \right). \]

As \( \epsilon > 0 \) is arbitrary we obtain the desired result. The results for \( R_1 \) and \( \mu_2 \) can be obtained similarly.

**Achievability:** We propose the following choices for auxiliary random variables. Let \( V_1^{[2:K]} = V_2^{[2:K]} = 0 \) and \( V_{1,1} = X_1 + P_1 \) and \( V_{2,1} = X_2 + P_2 \), where \( V_{1,1} \) and \( V_{2,1} \) are zero-mean Gaussian random variables with variances:

\[ \mathbb{E}[V_{1,1}^2] = \sigma^2_{x_1} + \sigma^2_{p_1}, \]

\[ \mathbb{E}[V_{2,1}^2] = \sigma^2_{x_2} + \sigma^2_{p_2}, \]

\[ \sigma^2_{p_1} = 2^{-2\mu_1} (1 - \rho^2_{x_1,x_2}) - \delta \]

\[ \sigma^2_{p_2} = 2^{-2\mu_2} (1 - \rho^2_{x_2,x_2}) - \beta \]

and \( P_1, P_2 \) are Gaussian zero-mean random variables such that \( P_1 \perp (X_1, X_2, Y_1, Y_2, P_2) \) and \( P_2 \perp (X_1, X_2, Y_1, Y_2, P_1) \). It is clear these choices satisfy the appropriate Markov chain conditions. Although a bit cumbersome, it is straightforward to calculate the corresponding values of \( I(Y_1; W_{[1:K+1]} X_2), I(Y_2; W_{[1:K+1]} X_2), I(X_1; W_{[1:K+1]} | X_2) \) and \( I(X_2; W_{[1:K+1]} | X_1) \) and conclude the proof.

**Remark 11:** Notice that the maximum values of \( \mu_1 \) and \( \mu_2 \) in (19) and (18) correspond to \( I(X_1 X_2; Y_1) \) and \( I(X_1 X_2; Y_2) \) and are achievable when \( R_2 \to \infty \) and \( R_1 \to \infty \) respectively. Besides, it is clear from the achievability that only one round of interaction suffices to achieve optimality when the sources are jointly Gaussian. In perspective, this is not surprising, and derives from the Wyner-Ziv’s result [28] which states that for Gaussian random variables, the rate-distortion function with side information at the encoder and decoder is not larger than the one with side information only at the decoder. These two cases correspond to two extreme situations: one in which there is no interaction between both encoders and the other in which interaction is not needed because both encoders have access to both observable sources. This conclusion follows easily by noticing that any code for a Gaussian rate-distortion problem, where decoder 1 desires to reconstruct \( Y_1 \) with distortion \( \mu_1 \triangleq \sigma^2_{y_1} 2^{-2\mu_1} \) and decoder 2 desires to recover \( Y_2 \) with distortion \( \mu_2 \triangleq \sigma^2_{y_2} 2^{-2\mu_2} \), is also good for an equivalent CRL problem with desired relevances levels \( \mu_1 \) and \( \mu_2 \).

**B. Gaussian CDIB model:** \( X_1 \leftrightarrow X_2 \leftrightarrow Y \) case

We study the Gaussian case for the region \( R_{\text{cib}}(1) \) investigated in Section III-D when \( X_1 \leftrightarrow X_2 \leftrightarrow Y \). Let \((X_1, X_2, Y)\) be Gaussian random variables with zero-mean. We will assume without loss of generality that we can write:

\[ Y = a X_2 + Z_\alpha, \quad X_2 = b X_1 + Z_\delta, \]

\[ X_2 = b X_1 + Z_\delta, \]

\[ Y = a X_2 + Z_\alpha, \quad X_2 = b X_1 + Z_\delta, \]
where $Z_a \perp (X_1, X_2)$ and $Z_b \perp X_1$ are Gaussian and constants $a$ and $b$ are obtained from the correlation structure of the random variables. That is:

$$a \triangleq \rho_{xy}\frac{\sigma_y}{\sigma_x}, \quad b \triangleq \rho_{x,y}\frac{\sigma_x}{\sigma_x}.$$ 

It is easy to check that

$$\sigma_z^2 = \sigma_y^2(1 - \rho_y^2), \quad \sigma_z^2 = \sigma_x^2(1 - \rho_{x,y}^2).$$

**Theorem 8 (Complexity-relevance region for the Gaussian model when $X_1 \rightarrow X_2 \rightarrow Y$: )** Let $(X_1, X_2, Y_2)$ be jointly Gaussian random variables satisfying $X_1 \rightarrow X_2 \rightarrow Y$. The complexity-relevance region $\mathcal{R}_{\text{comp}}(1)$ is given by (22), with $R_1 \geq 0, R_2 \geq 0$.

**Proof:** We begin with the converse.

**Converse:** Assume $(R_1, R_2, \mu) \in \mathcal{R}_{\text{comp}}(1)$ and consider rate constraint $R_1$. Using the fact that $J_1$ is function of $X_1^n$:

$$R_1 + \epsilon \geq \frac{1}{n} \sum (X_1^n ; I_1) = h(X_1) - \frac{1}{n} h(X_1^n I_1).$$

From rate $R_2$ and using the fact that $J_1$ is function of $X_2^n$ and $I_1$ it is not hard to obtain:

$$R_2 + \epsilon \geq \frac{1}{n} I(J_1;X_2^n I_1) \geq \frac{1}{n} h(bX_1^n + Z_a^n I_1) - \frac{1}{n} h(X_2^n I_1 I_1)$$

$$\geq \frac{1}{2} \log \left( \frac{1}{b^2} \frac{h(X_1^n I_1)}{1 + 2\pi e\sigma_z^2} \right) - \frac{1}{n} h(X_2^n I_1 I_1),$$

$$\geq \frac{1}{2} \log \left( \frac{2\pi e\sigma_y^2 b^2 2^{-2(R_1 + \epsilon)} + 2\pi e\sigma_z^2} {1 - \frac{1}{n} h(X_2^n I_1 I_1)}, \right)$$

where $(a)$ uses the conditional EPI because $Z_b^n \perp I_1$, and $(b)$ use Eq. (23).

From relevance condition we use the same idea:

$$\mu - \epsilon \leq \frac{1}{n} I(Y^n;I_1 J_1)$$

$$= h(Y) - \frac{1}{n} h(aX_2^n + Z_a^n | I_1 J_1)$$

$$\leq \frac{1}{2} \log \left( 2\pi e\sigma_y^2 \right) - \frac{1}{2} \log \left( a^2 \frac{h(X_2^n I_1 J_1)}{1 + 2\pi e\sigma_z^2} \right),$$

where $(c)$ use the conditional EPI because $Z_a^n \perp (I_1 J_1)$. Then, (23) is proved using Eq. (24) and the fact that $\epsilon > 0$ is arbitrary.

**Achievability:** We propose the following choices for auxiliary random variables. Let $V_1 = X_1 + P_1$ and $V_2 = X_2 + P_2$, where $V_1$ and $V_2$ are zero-mean Gaussian random variables with variances:

$$E[V_1^2] = \sigma_x^2 + \sigma_p^2, \quad E[V_2^2] = \sigma_x^2 + \sigma_p^2,$$

$$\sigma_{p_1}^2 = \sigma_x^2 \frac{2 - 2R_1}{1 - 2 - 2R_1},$$

$$\sigma_{p_2}^2 = \sigma_x^2 \frac{2 - 2R_2}{1 - 2 - 2R_2} (1 - \rho_{x,y}^2 + \rho_{x,y}^2 2^{-2R_1} \rho_{x,y}^2 2^{-2R_2} \rho_{x,y}^2 2^{-2R_1}),$$

and $P_1, P_2$ are Gaussian zero-mean random variables such that $P_1 \perp (X_1, X_2, Y, P_2)$ and $P_2 \perp (X_1, X_2, X_1, Y, P_1)$. It is clear these choices satisfy the appropriate Markov chain conditions. Although a bit cumbersome, it is straightforward to calculate the corresponding values of $I(X_1;V_1), I(X_2;V_2|V_1)$ and $I(Y;V_1,V_2)$. This concludes the proof.

This region can also be written as:

$$R_1 \geq 0,$$

$$R_2 \geq \frac{1}{2} \log \left( \frac{\rho_{x,y}^2 \rho_{x,y}^2 2^{-2R_1} + \rho_{x,y}^2 2^{-2R_1} (1 - \rho_{x,y}^2)} {2^{-2\mu} - (1 - \rho_{x,y}^2)} \right).$$

In Fig. we plot this alternative parametrization for different values of $\mu$. Taking into account that $\mu_{\max} = I(Y;X_2)$ it is seen how when increasing $R_1$ the value of $R_2$ tends to saturate. If the value of $\mu$ required is small enough, after increasing sufficiently $R_1$, the information about $Y$ provided by the second encoder would be not useful. In fact, it can be proved that the critical value of $R_1$ (if exists) for which $R_2 = 0$ satisfy:

$$R_1 = \frac{1}{2} \log \left( \frac{\rho_{x,y}^2 \rho_{x,y}^2 2^{-2R_1} + \rho_{x,y}^2 2^{-2R_1} (1 - \rho_{x,y}^2)} {2^{-2\mu} - (1 - \rho_{x,y}^2)} \right).$$

Moreover, it can be proved that there will be a critical value for $R_1$ if and only if the required level of relevance satisfy $\mu \leq I(Y;X_1)$. If the value for $\mu$ is greater than this quantity, it is not possible to have $R_2 = 0$ independently of the value of $R_1$. This not a surprise because it means, that for the level of relevance required, encoding of only $X_1$ is sufficient. If $\mu > I(Y;X_1)$ s required node 3 will require information from $X_2$ (remember that $X_1 \rightarrow X_2 \rightarrow Y$) which leads to $R_2 > 0$.

**C. Gaussian CDIB model: $X_1 \rightarrow Y \rightarrow X_2$ case**

We will consider the Gaussian case for the region $\mathcal{R}_{\text{comp}}(1)$ when $X_1 \rightarrow Y \rightarrow X_2$, studied in section [IV-C] Let $(X_1, X_2, Y)$ be Gaussian random variables with zero-mean. We will assume without loss of generality, that we can write:

$$Y = a_1 X_1 + a_2 X_2 + Z,$$

where $Z \perp (X_1, X_2)$ is Gaussian and constants $a_1$ and $a_2$ can be obtained from the correlation structure of the random variables, using the Markov chain $X_1 \rightarrow Y \rightarrow X_2$. This is:

$$a_1 \triangleq \frac{\sigma_y}{\sigma_x} \frac{\rho_{x,y}(1 - \rho_{x,y}^2)} {1 - \rho_{x,y}^2 \rho_{x,y}^2},$$

$$a_2 \triangleq \frac{\sigma_y}{\sigma_x} \frac{\rho_{x,y}(1 - \rho_{x,y}^2)} {1 - \rho_{x,y}^2 \rho_{x,y}^2}.$$

$$\mu \leq \frac{1}{2} \log \left( \frac{1 - \rho_{x,y}^2 + \rho_{x,y}^2 2^{-2R_1} - \rho_{x,y}^2 \rho_{x,y}^2 2^{-2R_2} + \rho_{x,y}^2 \rho_{x,y}^2 2^{-2R_1} \rho_{x,y}^2 2^{-2R_2} \rho_{x,y}^2 2^{-2R_1}} {1 - \rho_{x,y}^2 \rho_{x,y}^2 2^{-2R_1} + \rho_{x,y}^2 2^{-2R_1} \rho_{x,y}^2 2^{-2R_2} \rho_{x,y}^2 2^{-2R_1} \rho_{x,y}^2 2^{-2R_2} \rho_{x,y}^2 2^{-2R_1}} \right).$$

(22)
For $R_2 - \mu$, doing a similar analysis:

\[
R_2 - \mu + 2\epsilon \geq \frac{1}{n}H (J_1) - \frac{1}{n}I (Y^n; I_1 J_1),
\]

\[
geq \frac{1}{n} I (X^n_1 X^n_2; I_1 | J_1),
\]

\[
= r_1 - \frac{1}{2} \log \left( \frac{\var{Y}^{1/2}}{\var{X}^{1/2}} \right),
\]

\[
= r_1 - \frac{1}{2} \log \left( \frac{1 - \rho_{xy}^2}{1 - \rho_{xz}^2} \right).
\]

For $R_2 - \mu$, we have:

\[
R_2 - \mu = \frac{1}{n} I (X^n; I_1 J_1 | Y^n),
\]

\[
= \frac{1}{n} I (X^n_1; I_1 J_1 | Y^n) + \frac{1}{n} I (X^n_2; I_1 J_1 | X^n_1 Y^n),
\]

\[
= r_1 + r_2.
\]

Finally, this term is bounded by:

\[
2 \frac{\hat{h}(Y^n | I_1)}{\sigma^2} \geq 2 \frac{\hat{h} (a_1 X^n_1 + a_2 X^n_2 | I_1)}{\sigma^2} + 2\pi \epsilon \sigma^2,
\]

\[
\geq \frac{\left[ a_1^2 \frac{\hat{h}(X^n_1 | I_1)}{\var{X^n_1}} + a_2^2 \frac{\hat{h}(X^n_2 | I_1)}{\var{X^n_2}} \right]}{\sigma^2} + 2\pi \epsilon \sigma^2,
\]

\[
= \frac{\left[ a_1^2 \frac{\hat{h}(X^n_1 | Y^n)}{\var{X^n_1}^{1/2}} + a_2^2 \frac{\hat{h}(X^n_2 | Y^n)}{\var{X^n_2}^{1/2}} \right]}{\sigma^2} + 2\pi \epsilon \sigma^2.
\]

Finally, this term is bounded by:

\[
2 \frac{\hat{h}(Y^n | I_1)}{\sigma^2} \geq 2 \frac{\hat{h} (a_1 X^n_1 + a_2 X^n_2 | I_1)}{\sigma^2} + 2\pi \epsilon \sigma^2,
\]

Then, the bound of $R_2 - \mu$ can be written as:

\[
R_2 - \mu + 2\epsilon \geq r_2
\]

\[
- \frac{1}{2} \log \left( \frac{\var{Y}^{1/2}}{\var{X}^{1/2}} \right),
\]

\[
= \frac{1}{n} H (I_1 J_1 | Y^n),
\]

\[
= \frac{1}{n} H (I_1 J_1 | Y^n),
\]

\[
= \frac{1}{n} I (X^n_1 X^n_2; I_1 J_1 | Y^n),
\]

\[
= \frac{1}{n} I (X^n_1; I_1 J_1 | Y^n) + \frac{1}{n} I (X^n_2; I_1 J_1 | X^n_1 Y^n),
\]

\[
= r_1 + r_2.
\]

For $R_2 - \mu$, doing a similar analysis:

\[
R_2 - \mu + 2\epsilon \geq \frac{1}{n} H (I_1) - \frac{1}{n} I (Y^n; I_1 J_1),
\]

\[
geq \frac{1}{n} I (X^n_1 X^n_2; I_1 | J_1),
\]

\[
= r_1 - \frac{1}{2} \log \left( \frac{\var{Y}^{1/2}}{\var{X}^{1/2}} \right),
\]

\[
= r_1 - \frac{1}{2} \log \left( \frac{1 - \rho_{xy}^2}{1 - \rho_{xz}^2} \right).
\]

For $R_1 + R_2 - \mu$, using Markov chain $X_1 \rightarrow Y \rightarrow X_2$, we have:

\[
R_1 + R_2 - \mu + 3\epsilon \geq \frac{1}{n} H (I_1 J_1) - \frac{1}{n} I (Y^n; I_1 J_1),
\]

\[
= r_1 + r_2,
\]

\[
= r_1 + r_2.
\]

Finally, this term is bounded by:

\[
2 \frac{\hat{h}(Y^n | I_1)}{\sigma^2} \geq 2 \frac{\hat{h} (a_1 X^n_1 + a_2 X^n_2 | I_1)}{\sigma^2} + 2\pi \epsilon \sigma^2,
\]

\[
\geq \frac{\left[ a_1^2 \frac{\hat{h}(X^n_1 | I_1)}{\var{X^n_1}} + a_2^2 \frac{\hat{h}(X^n_2 | I_1)}{\var{X^n_2}} \right]}{\sigma^2} + 2\pi \epsilon \sigma^2,
\]

\[
= \frac{\left[ a_1^2 \frac{\hat{h}(X^n_1 | Y^n)}{\var{X^n_1}^{1/2}} + a_2^2 \frac{\hat{h}(X^n_2 | Y^n)}{\var{X^n_2}^{1/2}} \right]}{\sigma^2} + 2\pi \epsilon \sigma^2.
\]

Finally, this term is bounded by:

\[
2 \frac{\hat{h}(Y^n | I_1)}{\sigma^2} \geq 2 \frac{\hat{h} (a_1 X^n_1 + a_2 X^n_2 | I_1)}{\sigma^2} + 2\pi \epsilon \sigma^2,
\]

\[
\geq \frac{\left[ a_1^2 \frac{\hat{h}(X^n_1 | Y^n)}{\var{X^n_1}^{1/2}} + a_2^2 \frac{\hat{h}(X^n_2 | Y^n)}{\var{X^n_2}^{1/2}} \right]}{\sigma^2} + 2\pi \epsilon \sigma^2.
\]

Finally, the relevance condition can be bounded as:

\[
\mu - \epsilon \leq \frac{1}{n} I (Y^n; I_1 J_1),
\]

\[
= h (Y) - \frac{1}{n} h (Y^n | I_1 J_1),
\]
\[
R_2 \geq \frac{1}{2} \log \left( \frac{\sigma_2^2}{\sigma_z^2} \left[ \rho_z^2 - (a_z^2 \text{Var}[X_1|Y])^{2-2r_1} + a_2^2 \text{Var}[X_2|Y]^{2-2r_2} \right] \right) + \mu,
\]

As \( \epsilon > 0 \) is arbitrary, we obtain (26).

An inner bound for \( R_{\text{con}}(1) \) can be obtained defining \( V_1 = X_1 + P_1 \) and \( V_2 = X_2 + V_1 + P_2 \), where \( P_1 \) and \( P_2 \) are Gaussian variables with \( P_1 \perp (X_1, X_2, Y, P_2) \) and \( P_2 \perp (X_1, X_2, Y, P_1) \) and variances \( \sigma_1^2 \) and \( \sigma_2^2 \). Numerically choosing \( \sigma_1^2 \) and \( \sigma_2^2 \) to satisfy the relevance condition, we can plot the resulting inner bound and compare with the obtained outer bound. The results are shown in Fig. [3]. We observe that there is small gap between both regions (the parameters \( \rho_{x_1y} \) and \( \rho_{x_2y} \) were chosen to maximize the observed difference).

Although we were unable to prove it, we suspect that in this special Gaussian case there is no gain from cooperation and that the observed gap is indeed not achievable. This suspicion is motivated by the fact that the non-cooperative Gaussian CEO region for this problem, which can be easily obtained from the corresponding CEO result with Gaussian inputs and quadratic distortion in [15], was also numerically shown to be equal to the upper bound for the cooperative case. It is interesting to observe that, if true, the conclusion that cooperation is not helpful should hold for the cooperative Gaussian CEO problem with quadratic distortion as well. In the case the gap were achievable, this would be due to possible gains in the individuals rates \( R_1 \) and \( R_2 \). The sum-rate and relevance do not increase when cooperation is allowed. This is rooted in the well-known result that when cooperation is allowed, the relevance does not increase when cooperation is allowed. This result is rooted in the well-known result that cooperation is not helpful.

In this section, we will consider a binary example for the region obtained related to the TW-CIB problem. The study of the binary examples with multiple rounds proves to be rather challenging for which closed forms remain elusive to obtain. Our approach to the problem will be the following. We will consider the problem in which decoder 1 is intended to learn a hidden variable \( Y_1 \) while decoder 2 desires to learn \( X_2 \). Exchanges between encoder 1 and the decoder 2, and between encoder 2 and decoder 1, are through a two decoupled half-rounds as will be explained below. First we will consider the problem where both encoders know \( X_1 \) and \( X_2 \). From the perspective of each encoder-decoder pair this is reminiscent of a noisy rate-distortion problem with side information at both the encoder and the decoder where the metric of interest is given by the relevance \( \frac{1}{n} I(Y^n_1; J_1 X^n_1) \) and \( \frac{1}{n} I(Y^n_2; I X^n_2) \), respectively. Let us refer this region to as \( R_{\text{con}}^{\text{D}}(1/2) \). Secondly, we will consider the more interesting problem in which \( X_2 \) is not known at encoder 1 and \( X_1 \) is not known at encoder 2. This is reminiscent of a noisy rate-distortion problem with side information only at the decoder. We refer to this region to as \( R_{\text{con}}^{\text{D}}(1/2) \). Notice that in these two regions there is not interaction between the encoders. In the first case, interaction is not needed because each node has full knowledge of the side information of the other node. In the second case, neglect any interaction. Encoder 1 sends its description to decoder 2 who uses its side information \( X_2 \) for decoding. Similar, and without consider the previous description received from node 1, encoder 2 sends its own description to decoder 1 who recover it with its side information \( X_1 \). It is clear that we have the following:

\[
R_{\text{con}}^{\text{D}}(1/2) \subseteq R_{\text{con}}^{\text{D}} \subseteq R_{\text{con}}^{\text{ED}}(1/2).
\]

As a consequence, the existent gap between \( R_{\text{con}}^{\text{D}}(1/2) \) and \( R_{\text{con}}^{\text{D}}(1/2) \) can be thought to be an upper bound to the potential gain to be obtained from multiple interactions. In more specific terms, each of the above regions can be characterized by two relevance-rate functions (one for each encoder-decoder pair). For instance, for the encoder 1-decoder 2 pair, we have:

\[
\mu_{\text{con}}^{\text{ED}}(R_1) = \sup \left\{ \mu_2 : (R_1, \mu_2) \in R^{\text{ED}}_{\text{con}}(1/2) \right\},
\]

\[
\mu_{\text{con}}^{\text{D}}(R_1) = \sup \left\{ \mu_2 : (R_1, \mu_2) \in R^{\text{D}}_{\text{con}}(1/2) \right\}.
\]

Similar definitions are valid for the relevance-rate functions \( \mu_{\text{con}}^{\text{ED}}(R_2) \) and \( \mu_{\text{con}}^{\text{D}}(R_2) \) corresponding to the encoder 2-decoder 1 pair. It is also clear that as the encoding and decoding of the encoders and decoders are decoupled, a full characterization of these functions for the encoder 1-decoder 2 pair also leads to the full characterization of the functions for the other pair. These functions which are concave (see Appendix [[E]]) are to be computed when \( (X_1, X_2, Y_1, Y_2) \) satisfy \( (X_1, X_2, Y_1, Y_2) \sim \text{Bern}(1/2) \) and subject to \( Y_1 \rightarrow X_2 \rightarrow X_1 \not\rightarrow Y_2 \). This implies that \( X_1 = X_2 \oplus Z \) with \( Z \sim \text{Bern}(q) \), \( q \in (0, 1/2) \), \( Z \perp X_2, Y_2 = X_1 \oplus W_1 \) and \( Y_1 = X_2 \oplus W_2 \) with \( W_1 \sim \text{Bern}(p_1) \), \( p_1 \in (0, 1/2) \), \( W_1 \perp (X_1, X_2) \) for \( i = 1, 2 \).

In the above, we will assume that \( p_1 = p_2 \). In this way the above relevance-rate functions for both pairs of encoders and decoders are the same and we can work with only one encoder-decoder pair satisfying \( X_2 \rightarrow X_1 \rightarrow Y \), where the decoder has access to \( X_2 \) and wishes to learn \( Y \). With this in mind, we begin with the characterization of \( \mu_{\text{con}}^{\text{D}}(R) \). We have the following result.

\[
R_2 \geq \frac{1}{2} \log \left( \frac{1 - \rho_{x_1y}^2 \rho_{x_2y}^2 - \rho_{x_1y}^2 (1 - \rho_{x_2y}^2)^{2-2r_1} - \rho_{x_2y}^2 (1 - \rho_{x_1y}^2)^{2-2r_2}}{(1 - \rho_{x_1y}^2)(1 - \rho_{x_2y}^2)} \right) + \mu,
\]

\[
\mu \leq \frac{1}{2} \log \left( \frac{1 - \rho_{x_1y}^2 \rho_{x_2y}^2 - \rho_{x_1y}^2 (1 - \rho_{x_2y}^2)^{2-2r_1} - \rho_{x_2y}^2 (1 - \rho_{x_1y}^2)^{2-2r_2}}{(1 - \rho_{x_1y}^2)(1 - \rho_{x_2y}^2)} \right).
\]
Theorem 10 (Relevance-rate function for binary sources with side information to the encoder and the decoder): Consider random binary sources \((X_1, X_2, Y) \sim \text{Bern}(1/2)\) with \(X_2 \rightarrow X_1 \rightarrow Y\) such that \(X_1 = X_2 + Z\) with \(Z \sim \text{Bern}(q)\), \(q \in (0, 1/2)\), \(Z \perp X_2\) and \(Y = X_1 + W\) with \(W \sim \text{Bern}(p)\), \(p \in (0, 1/2)\), \(W \perp (X_1, X_2)\). The relevance-rate function \(\mu_{\text{TW-CIB}}(R)\) can be put as:

\[
\mu_{\text{TW-CIB}}(R) = 1 - h_2 \left( h_2(q) - R \right) p.
\]

Proof: For the converse, we can without loss of generality begin from a single letter description. If \((R, \mu)\) is achievable, it is clear that there exists \(U\) such that \(U \rightarrow (X_1, X_2) \rightarrow Y\) and

\[
R \geq I(X_1; U|X_2), \quad \mu \leq I(Y; U|X_2).
\]

is straightforward to obtain:

\[
H(X_1|X_2U) \geq h_2(q) - R, \quad \mu \leq 1 - H(Y|X_2U).
\]

As \(Y = X_1 + W\) with \(W \sim \text{Bern}(p)\) and \(W \perp (X_1, X_2)\) it is clear that \(W \perp (U, X_1)\). This allows us to use Mrs. Gerber lemma [20] to obtain:

\[
H(Y|X_2U) \geq h_2 \left( h_2^{-1} \left( h_2(q) - R \right) + p \right),
\]

which implies

\[
\mu_{\text{TW-CIB}}(R) \leq 1 - h_2 \left( h_2^{-1} \left( h_2(q) - R \right) + p \right).
\]

The achievability is straightforward choosing \(U = U_0 \mathbb{I}\{X_2 = 0\} + U_1 \mathbb{I}\{X_2 = 1\}\), where \(U_i, i = 0, 1\) are binary random variables which are schematized in Fig. 5 and the value of \(s\) is given by \(s = h_2^{-1} \left( h_2(q) - R \right)\).

Now consider the problem of obtaining \(\mu_{\text{TW-CIB}}^0(R)\). Unfortunately in this case, as \(U\) should depend only on \(X_1\) (and not on \(X_2\)) we cannot apply Mrs. Gerber Lemma to obtain a tight upper bound to \(\mu_{\text{TW-CIB}}^0(R)\). The converse and achievability in this case are more involved requiring the use of convex analysis. The following theorem provides the characterization of \(\mu_{\text{TW-CIB}}^0(R)\) and its proof is deferred to Appendix 1

Theorem 11 (Relevance-rate function for binary sources with side information only to the decoder): Consider random Binary sources \((X_1, X_2, Y) \sim \text{Bern}(1/2)\) with \(X_2 \rightarrow X_1 \rightarrow Y\) such that \(X_1 = X_2 + Z\) with \(Z \sim \text{Bern}(q)\), \(q \in (0, 1/2)\), \(Z \perp X_2\) and \(Y = X_1 + W\) with \(W \sim \text{Bern}(p)\), \(p \in (0, 1/2)\), \(W \perp (X_1, X_2)\). The relevance-rate function \(\mu_{\text{TW-CIB}}^0(R)\) can be put as:

\[
\mu_{\text{TW-CIB}}^0(R) = \begin{cases} 
1 - h_2(p \ast q) + f \left( g^{-1}(R) \right) R & 0 \leq R \leq R_c, \\
1 - h_2(p \ast q) + f \left( g^{-1}(R) \right) , & R_c < R \leq h_2(q), \\
1 - h_2(p) & R > h_2(q),
\end{cases}
\]

where \(R_c\) is given by:

\[
\frac{f'(g^{-1}(R_c))}{g'(g^{-1}(R_c))} = \frac{f \left( g^{-1}(R_c) \right)}{R_c},
\]

and \(g(\cdot)\) and \(f(\cdot)\) are defined in (44) and (45).

It is important to emphasize, as it is discussed in Appendix 1 that this region is achieved by time-sharing. This is similar to the Wyner-Ziv problem for binary sources [28].

Remark 12: The proof in Appendix 1 can be generalized to the cases in which \(X_1, X_2\) and \(Y\) are Bernoulli random variables with other parameters than 1/2. Moreover, a similar (but even more cumbersome) analysis can be carried over for arbitrary discrete random sources that satisfy the above Markov chain.

In order to compare these two extreme cases, where there is no interaction with an example where there is some coupling between the two pairs of encoder-decoder, we study the full interactive case with one round for random binary sources that satisfy \(Y_1 \rightarrow X_2 \rightarrow X_1 \rightarrow Y_2\) with \(p_1 = p_2\). Assume that in the first half round, encoder 1 transmits a description to decoder 2, who wants to learn hidden variable \(Y_2\). After that, encoder 2 sends a description to decoder 1. In this case, and according to Theorem 1, encoder 2 should transmit with rates and relevances satisfying:

\[
R_1 \geq I(X_1; V_1|X_2),
\]

\[
R_2 \geq I(X_2; V_2|X_1),
\]

\[
\mu_1 \leq I(Y_1; V_2; X_1) = I(Y_1; V_2 X_1)
\]

\[
\mu_2 \leq I(Y_2; V_1 X_2),
\]
Figure 6: $\mu_{TW-CIB}^B(R_1), \mu_{TW-CIB}^D(R_1), \mu_{TW-CIB}^{ED}(R_1), \mu_{TW-CIB}^{ED}(R_2)$ and $\mu_{TW-CIB}^{INT}(R_2)$ as functions of $R_1$ and $R_2$ respectively and when $p_1 = p_2 = 0.1$ and $q = 0.1$.

where $V_1 \rightarrow X_1 \leftarrow (X_2, Y_1, Y_2)$ and $V_2 \rightarrow (V_1, X_2) \leftarrow (X_1, Y_1, Y_2)$. It is clear that the tradeoff between $R_1$ and $\mu_2$ is given by the function $\mu_{TW-CIB}^B(R_1)$, with the optimal choice of $V_1$ given in Appendix E. Regarding the choice of $V_2$, we should consider the following problem (with $V_1$ fixed with the mentioned optimal choice):

$$\max_{p(v_2|x_2v_1)} I(Y_1; V_2 X_1) \text{ s.t. } R_2 \geq I(X_2; V_2|X_1 V_1). \quad (27)$$

This problem is similar to the one considered for $\mu_{TW-CIB}^B(R_1)$. It is, however, a little more subtle and difficult to solve. It can be seen that it corresponds to a source coding problem where both encoder and decoder have side information ($V_1$ and $X_1$ respectively), but the side information of the encoder is degraded with respect to that of the decoder. We simply evaluated the resulting rate region by numerically generating random probability distributions of $V_2$ with cardinalities no lower than 7, as indicated in Theorem 1 for each value of $R_2$. Taking the maximum of the generated value of $\mu_2$ for each value of $R_2$ and considering the concave envelope of the resulting curve (allowing for time-sharing between different points in the curve), we obtained the function $\mu_{TW-CIB}^{INT}(R_2)$ which is clearly achievable. In Fig. 6 we plot this function along with $\mu_{TW-CIB}^B(R_1)$, $\mu_{TW-CIB}^D(R_1)$ (plotted as only one curve, as these are equivalent) and $\mu_{TW-CIB}^{ED}(R_1)$, $\mu_{TW-CIB}^{ED}(R_2)$ (again, plotted together because they are equivalent). It is seen that, in contrast with the corresponding Gaussian TW-CIB case analyzed in Section VII, where interaction does not help and both encoder-decoder pairs operate in a complete decoupled manner, interaction clearly helps in this binary setting. Actually, during the second half round, the first description sent by encoder 1 is useful for encoder 2 and decoder 1 in the task of learning $Y_1$.

VII. SUMMARY AND DISCUSSION

We investigated a multi-terminal collaborative source coding problem with a non-additive logarithmic distortion. This work intended to characterize tradeoffs between rates of complexity and relevance to the source-coding problem of cooperatively extracting information about hidden variables from some observed and physically distributed ones. Two different scenarios are distinguished: the so-called Two-way Collaborative Information Bottleneck (TW-CIB) and the Collaborative Distributed Information Bottleneck (CDIB). These problems differ from each other in the fact that the decoder may or may not be one of the encoders, necessitating fundamentally different approaches. Inner and outer bounds to the complexity-relevance region of these problems are derived and optimality is characterized for several cases of interest.

Specific applications of our results to binary symmetric and Gaussian statistical models were also considered and optimality is characterized for most of the cases. These results show that cooperation does not improve the rates of relevance in presence of Gaussian statistical models in most cases. This can be explained from the well-known result by Wyner-Ziv [28] which implies that side information at the encoder does not improve the quadratic distortion in presence of Gaussian sources. In contrast, we have shown that cooperation clearly helps in the TW-CIB scenario. In particular, the converse to the complexity-relevance region of the binary model appears to be rather involved and required the use of tools of convex analysis. It will be the purpose of future work to study the binary source model within the CDIB framework for which we conjecture that cooperation also helps.

APPENDIX A

STRONGLY TYPICAL SEQUENCES AND RELATED RESULTS

In this appendix we introduce standard notions in information theory but suited for the mathematical developments and proof needed in this work. The results presented can be easily derived from the standard formulations provided in [25] and [31]. Be $X$ and $Y$ finite alphabets and $(x^n, y^n) \in X^n \times Y^n$. With $\mathcal{P}(X \times Y)$ we denote the set of all probability distributions on $X \times Y$. We define the strongly $\delta$-typical sets as:

**Definition 4 (Strongly typical set):** Consider $p \in \mathcal{P}(X)$ and $\delta > 0$. We say that $x^n \in X^n$ is $p\delta$-strongly typical if $x^n \in \mathcal{T}^n_{p\delta}$ with:

$$\mathcal{T}^n_{p\delta} = \left\{ x^n \in X^n : \left| \frac{N(a|x^n)}{n} - p(a) \right| \leq \frac{\delta}{|X|} \right\} \quad \forall a \in X \text{ such that } p(a) \neq 0,$$

where $N(a|x^n)$ denotes de number of occurrences of $a \in X$ in $x^n$ and $p \in \mathcal{P}(X)$. When $X \sim p_X$ we can denote the corresponding set of strongly typical sequences as $\mathcal{T}^n_{X\delta}$. Similarly, given $p_{XY} \in \mathcal{P}(X \times Y)$ we can construct the set of $\delta$-jointly typical sequences as:

$$\mathcal{T}^n_{X|Y\delta} = \{(x^n, y^n) \in X^n \times Y^n : \left| \frac{N(a,b|x^n,y^n)}{n} - p_{X,Y}(a,b) \right| \leq \frac{\delta}{|X|} \right\} \quad \forall a, b \in X \text{ such that } p_{X,Y}(a,b) \neq 0.$$
when (28) is satisfied.

Then:

\[ \frac{|N(a,b|x^n,y^n)|}{n} - p_{XY}(a,b) \leq \frac{\delta}{|X||Y|}, \]

\[ \forall (a,b) \in X \times Y \text{ such that } p_{XY}(a,b) \neq 0. \]

We also define the \textit{conditional} typical sequences. In precise terms, given \( x^n \in X^n \) we consider the set:

\[ T_{|X|}^n(x^n) = \{ y^n \in Y^n : \frac{|N(a,b|x^n,y^n)|}{n} - p_{XY}(a,b) \leq \frac{\delta}{|X||Y|}, \]

\[ \forall (a,b) \in X \times Y \text{ such that } p_{XY}(a,b) \neq 0 \}. \]

Notice that we the following is an alternative writing of this set:

\[ T_{|X|}^n(x^n) = \{ y^n \in Y^n : (x^n,y^n) \in T_{|X|}^n(x^n) \}. \]

We have several useful and standard lemmas, which will be presented without proof:

\textbf{Lemma 1 (Conditional typicality lemma [27])}: Consider de product measure \( \prod_{i=1}^N p_{XY}(x_i,y_i) \). Using that measure, we have the following

\[ \Pr \left\{ T_{|X|}^n(x^n) \right\} \geq 1 - O \left( c_1^{-n f(\epsilon)} \right), \quad c_1 > 1 \]

where \( f(\epsilon) \rightarrow 0 \) when \( \epsilon \rightarrow 0 \). In addition, for every \( x^n \in T_{|X|}^n(x^n) \) with \( \epsilon' < \frac{\epsilon}{|X|} \) we have:

\[ \Pr \left\{ T_{|X|}^n(x^n) \right\} \geq 1 - O \left( c_2^{-n g(\epsilon',\epsilon')} \right), \quad c_2 > 1 \]

where \( g(\epsilon,\epsilon') \rightarrow 0 \) when \( \epsilon, \epsilon' \rightarrow 0 \).

\textbf{Lemma 2 (Covering Lemma [25])}: Be \((U,V,X) \sim p_{UVX}\) and \((x^n,u^n) \in T_{|X|}^n(x^n)\), \( \epsilon' < \frac{\epsilon}{|X|} \) and \( \epsilon < \epsilon''. \) Consider also \( V^n(m) \) \( m=1 \) random vectors which are independently generated according to \( \frac{1}{|T_{|X|}^n(x^n)|} \{ u^n \in T_{|X|}^n(x^n) \}. \). Then:

\[ \Pr \left\{ V^n(m) \notin T_{|V|}^n(x^n) \right\} \rightarrow 0 \]

uniformly for every \( (x^n,u^n) \in T_{|X|}^n(x^n) \) if:

\[ R > I(V;X|U) + \delta(\epsilon,\epsilon',\epsilon'',n) \]

where \( \delta(\epsilon,\epsilon',\epsilon'',n) \rightarrow 0 \) when \( \epsilon,\epsilon',\epsilon'' \rightarrow 0 \) and \( n \rightarrow \infty \).

\textbf{Corollary 3}: Assume the conditions in Lemma 2 and also:

\[ \Pr \left\{ (X^n,U^n) \in T_{|X|}^n(x^n) \right\} \rightarrow 1 \]

Then:

\[ \Pr \left\{ (U^n,X^n,V^n(m)) \notin T_{|U|}^n(x^n) \right\} \rightarrow 0 \]

when (28) is satisfied.

\textbf{Lemma 3 (Packing Lemma [23])}: Be \((U_1,U_2,W_1,W_2) \sim p_{U_1U_2W_1W_2X}\), \((x^n,u^n,v^n_1,v^n_2) \in T_{|X|}^n(x^n)\) and \( \epsilon' < \frac{\epsilon}{|X|^2} \) and \( \epsilon < \min \{ \epsilon_1,\epsilon_2 \}. \) Consider random vectors \( \{U_1(m_1)\}_{m_1=1}^{A_1} \) and \( \{U_2(m_2)\}_{m_2=1}^{A_2} \) which are independently generated according to

\[ \frac{1}{T_{|U|}^n(x^n)} \left\{ u^n \in T_{|U|}^n(x^n) \right\}, \quad i = 1, 2 \]

and \( A_1, A_2 \) are positive random variables independent of everything else. Then

\[ \Pr \left\{ (U_1^n(m_1),U_2^n(m_2)) \in T_{|U|}^n(x^n,u^n,v^n_1,v^n_2) \right\} \rightarrow 0 \]

uniformly for every \((x^n,u^n,v^n_1,v^n_2) \in T_{|X|}^n(x^n)\).

Consider \((x^n,y^n) \in T_{|X|}^n(x^n)\), and random vectors \( U^n \) generated according to:

\[ \Pr \left\{ U^n \sim u^n \right\} \rightarrow 1 \]

For sufficiently small \( \epsilon,\epsilon',\epsilon'' \) the following holds uniformly for every \((x^n,y^n) \in T_{|X|}^n(x^n)\):

\[ \Pr \left\{ U^n \notin T_{|U|}^n(x^n) \right\} \rightarrow 0 \]

where \( \epsilon > 1 \).

\textbf{Corollary 5}: Assume the conditions in Lemma 4 and also:

\[ \Pr \left\{ (X^n,Y^n) \in T_{|X|}^n(x^n) \right\} \rightarrow 1 \]

and that uniformly for every \((x^n,y^n) \in T_{|X|}^n(x^n)\):

\[ \Pr \left\{ U^n \notin T_{|U|}^n(x^n,y^n) \right\} \rightarrow 0 \]

we obtain:

\[ \Pr \left\{ (U^n,X^n,Y^n) \in T_{|U|}^n(x^n,y^n) \right\} \rightarrow 1 \]

We next present a result which will be very useful to us. In order to use the Markov lemma we need to show that the descriptions induced by the encoding procedure in each encoder satisfies (30). A proof of this result can be found in [22].
Lemma 5 (Encoding induced distribution): Consider a pmf \( p_{U|X|W} \) belonging to \( P(U \times X \times W) \) and \( \epsilon' \geq \epsilon \). Be \( \{U^m(m)\}_{m=1}^S \) random vectors independently generated according to
\[
\Pr \left\{ U^m(M) = u^n | x^n, u^n, U^m(M) \in T^n_{U|X|W}[x^n, w^n] \right\} = \frac{1}{|T^n_{U|X|W}[x^n, w^n]|} \quad \text{and where} \quad (w^n, X^n) \text{ are generated with an arbitrary distribution. Once these vectors are generated, and given } x^n \text{ and } w^n, \text{ we choose one of them if:}
\[
(u^n(m), w^n, x^n) \in T^n_{U|W|X}[m], \text{ for some } m \in \{1 : S\}.
\]
If there are various vectors \( u^n \) that satisfies this we choose the one with smallest index. If there are none we choose an arbitrary one. Let \( M \) denote the index chosen. Then we have that:
\[
\Pr \{ U^n(M) = u^n | x^n, w^n, U^n(M) \in T^n_{U|X|W}[x^n, w^n] \} = \frac{1}{|T^n_{U|X|W}[x^n, w^n]|}
\]

APPENDIX B
ACHIEVABILITY PROOFS

We will begin with the proof of Theorem 1. The proof of Theorem 2 can be seen as a simple extension with some minor differences to be discussed next.

A. Proof of Theorem 2

Let us describe the coding generation, encoding and decoding procedures. We will consider the following notation. With \( m_{i,l} \) we will generically denote the indices used for the descriptions \( V^n_{i,l} \) generated at encoder \( i \) at round \( l \). With \( M_{i,l} \) we will denote the actual index corresponding to the actual description \( V^n_{i,l} \) generated at encoder \( i \) at round \( l \). With \( M_{W_{i,l}} \) we denote the indices used for the sets of descriptions generated just before encoder \( i \) generated its own description at round \( l \) and with \( M_{W_{i,l}} \) the actual corresponding indices. Similarly, \( p_{i,l} \) will denote the bin indices used at encoder \( i \) at round \( l \) and \( P_{i,l} \) will denote the actual bin index generated at encoder \( i \) at round \( l \). With \( M_{i,l}(j) \) where \( i \neq j \) we denote the estimation at encoder \( j \) of the actual index generated at encoder \( i \) at round \( l \), where \( i \in \{1, 2\} \) and \( j \in \{1, 2, 3\} \). We will fix codeword length \( n \) and a distribution which satisfies the Markov chains (7) and (8). We will describe the coding procedure to be used.

Coding generation: Consider the round \( l \in \{1 : K\} \). For each \( M_{W_{i,l}} \), we generate \( 2^{nR_{1,l}} \) independent and identically distributed \( n \)-length codewords \( V^n_{i,l}(m_{i,l}, M_{W_{i,l}}) \) according to:
\[
\Pr \{ V^n_{i,l}(m_{i,l}, M_{W_{i,l}}) = v^n_{i,l} \} = \frac{1}{\sum_{v^n_{i,l} \in T^n_{V_{i,l}|W_{i,l}}[v^n_{1,l}]}}, \epsilon(l, 1) > 0
\]
where \( m_{i,l} \in \{1 : 2^{nR_{1,l}}\} \) and let \( M_{W_{i,l}} \) denote the indices of the descriptions \( W^n_{i,l} \) generated at encoders 1 and 2 in the past rounds \( t \in \{1 : l - 1\} \) as explained above. For example, \( M_{W_{1,l}} = \{m_{1,t}, m_{2,t}, l^{-1}_{t=1}\} \). With \( w^n_{i,l} \) we denote the set of \( n \)-length codewords (which are realizations of \( W^n_{i,l} \)) from previous rounds corresponding to the indices \( M_{W_{i,l}} \). Constant \( \epsilon(1, l) \) is chosen to be sufficiently small. It is clear that there exists \( 2^{nR_{1,l} + \sum_{k=1}^{R_{1,l}} R_{1,l} + 2R_{2,k}} \) \( n \)-length codewords \( V^n_{i,l}(m_{i,l}, M_{W_{i,l}}) \). These codewords are distributed independently and uniformly over \( 2^{nR_{1,l}} \) bins which are denoted as \( B_{i,l}(p_{i,l}) \) with \( p_{i,l} \in \{1 : 2^{nR_{1,l}}\} \). Similarly, for encoder 2 and each \( M_{W_{2,l}} \), we generate \( 2^{nR_{2,l}} \) independent and identically distributed \( n \)-length codewords \( V^n_{i,l}(m_{i,l}, M_{W_{i,l}}) \) according to:
\[
\Pr \{ V^n_{i,l}(m_{i,l}, M_{W_{i,l}}) = v^n_{i,l} \} = \frac{1}{\sum_{v^n_{i,l} \in T^n_{V_{i,l}|W_{i,l}}[v^n_{1,l}]}}, \epsilon(2, l) > 0
\]
These \( 2^{nR_{1,l} + \sum_{k=1}^{R_{1,l}} R_{1,l} + 2R_{2,k}} \) \( n \)-length codewords are distributed independently and uniformly over \( 2^{nR_{2,l}} \) bins which are denoted as \( B_{i,l}(p_{i,l}) \) with \( p_{i,l} \in \{1 : 2^{nR_{1,l}}\} \). It is clear that we should impose that
\[
\begin{align*}
R_{1,l} & < \tilde{R}_{1,l} + \sum_{k=1}^{l-1} \tilde{R}_{1,k} + \tilde{R}_{2,k}, \\
R_{2,l} & < \tilde{R}_{2,l} + \tilde{R}_{1,l} + \sum_{k=1}^{l-1} \tilde{R}_{1,k} + \tilde{R}_{2,k},
\end{align*}
\]
for each \( l \in \{1 : K\} \).

This procedure for the codebooks generation is done sequentially beginning at encoder 1 and round 1 and terminated at encoder 2 and round \( K \). After that, the codebooks are revealed to all parties.

Encoding: Consider encoder 1 at round \( l \in \{1 : K\} \). Upon observing \( x^n \) and given all of its encoding and decoding history up to round \( l \), encoder 1 first looks for a codeword \( v^n_{1,l}(m_{1,l}, \tilde{M}_{W_{1,l}}(1)) \) such that
\[
\begin{align*}
(x^n, w^n_{1,l}(\tilde{M}_{W_{1,l}}(1)), v^n_{1,l}(m_{1,l}, \tilde{M}_{W_{1,l}}(1))) \in T^n_{V_{1,l}|X_{1,l}W_{1,l}}[\epsilon(1, l)], \quad \text{where } \epsilon(1, l) > 0.
\end{align*}
\]
Notice that some components in \( \tilde{M}_{W_{1,l}}(1) \) are generated at encoder 1 and are perfectly known. If more than one codeword satisfies this condition, then we choose the one with the smallest index. Otherwise, if no such codeword exists, we choose an arbitrary index and declare an error. With the chosen index \( m_{1,l} \) and with \( \tilde{M}_{W_{1,l}}(1) \), we determine the index \( p_{1,l} \) of the bin \( B_{i,l}(p_{1,l}) \) to which \( v^n_{i,l}(m_{1,l}, \tilde{M}_{W_{1,l}}(1)) \) belongs. Then, the index \( p_{1,l} \) is transmitted to encoder 2 and 3. Similarly, encoder 2 looks for a codeword \( v^n_{2,l}(m_{2,l}, \tilde{M}_{W_{2,l}}(2)) \) such that
\[
\begin{align*}
(x^n, w^n_{2,l}(\tilde{M}_{W_{2,l}}(2)), v^n_{2,l}(m_{2,l}, \tilde{M}_{W_{2,l}}(2))) \in T^n_{V_{2,l}|X_{2,l}W_{2,l}}[\epsilon(2, l)], \quad \text{where } \epsilon(2, l) > 0.
\end{align*}
\]
If more than one codeword satisfies this condition, then we choose the one with the smallest index. Otherwise, if no such codeword exists, we choose an arbitrary index and declare an error. With the chosen index \( m_{2,l} \) and with \( \tilde{M}_{W_{2,l}}(2) \), we determine the index \( p_{2,l} \) of the bin \( B_{i,l}(p_{2,l}) \) to which \( v^n_{i,l}(m_{2,l}, \tilde{M}_{W_{2,l}}(2)) \) belongs. Then, the index \( p_{2,l} \) is transmitted to encoder 1 and the decoder.
Decoding: At round \( l \in [1 : K] \) encoder 1, after receiving \( p_{1,l-1} \) looks for \( m_{2,l-1} \) such that

\[
\left( x_{1}^{n}, w_{1}^{n} | m_{2,l-1}(1) \right) \in T_{v_{1}}(x_{1}^{n}, w_{1}^{n} | m_{2,l-1}(1) \in B(p_{2,l-1}) \text{ if there are more than one pair of codewords, or}
\]

\( \left( (m_{2,l-1}, m_{W_{1},l-1}(1)) \right) \in B(p_{2,l-1}) \text{. If there are more than one pair of codewords, or}
\]

\( \text{node 3 could recover that index.} \]

The joint-typicality decoding at node 2 would be successful with high probability if

\[
\hat{R}_{1} > I(V_{1}; X_{1})
\]

The index \( M_{1} \) of \( V_{1}^{n}(M_{1}) \) with \( M_{1} \in [1 : 2^{n\hat{R}_{1}}] \) such that

\[
(X_{1}^{n}, V_{1}^{n}(M_{1})) \text{ are typical. This would the case with high probability if}
\]

\[
\hat{R}_{1} > I(V_{1}; X_{1})
\]

The occurrence of event \( \hat{R}_{1} > I(V_{1}; X_{2}) \)

Of course, it is not guaranteed that node 3 could recover that index because it does not have side information. In this way, the information sent by node 2 should provide something to be used by node 3 to recover not only the index generated at node 2 but also the index generated at node 1. First, node 2 choose \( V_{1}^{n}(m_{1}, m_{2}) \) with \( m_{2} \in [1 : 2^{n\hat{R}_{2}}] \) such that

\[
(X_{2}^{n}, V_{1}^{n}(M_{1}), V_{2}^{n}(M_{2}, m_{1})) \text{ is typical, where } M_{1} \text{ is}
\]

\( \text{estimation of } M_{1} \text{ at node 2 (which with probability close to}
\]

\( \text{one will be equal to the true } M_{1}). \) In order to achieve this with high probability:

\[
\hat{R}_{2} > I(V_{2}; X_{2} | V_{1})
\]

After that, node 2 look for the bin index where \( \text{both} \ (M_{1}(2), M_{2}) \) live \( (P_{2}) \) and send it to node 3. Notice that as explained before, at node 2 the bins contain all possible pairs \( (m_{1}, m_{2}) \) (distributed in uniform fashion). This is the key fact. Node 2 bins both indices: the one recovered from node 1 and the one it generates. In this way an explicit cooperation is achieved between node 1 and 2 through binning in order to help the decoder in node 3 to recover both \( M_{1} \) and \( M_{2}. \)

Clearly, the number of bins in node 3 should satisfy:

\[
R_{2} < \hat{R}_{2} + \hat{R}_{1}.
\]

Finally, node 3 should recover \( M_{1} \) and \( M_{2} \) from the bin indices \( P_{1} \) and \( P_{2}. \) This is simply done by looking for \( (m_{1}, m_{2}) \) such that \( m_{1} \in B(P_{1}) \) and \( (m_{1}, m_{2}) \in B(P_{2}) \) are jointly typical. As the bins formations in node 1 and 2 are done with uniform distributions over the indices sets, the probability of failure of this procedure is shown to go to zero exponentially fast if:

\[
\hat{R}_{1} < R_{1} + R_{2}, \ 
\hat{R}_{2} < R_{2}, \ 
\hat{R}_{1} + \hat{R}_{2} < R_{1} + R_{2}.
\]

The mathematical details of the proof of this fact can be found in appendix B in [22] (setting \( X_{3} = V_{1} = V_{2} = \phi \)). Eliminating \( \hat{R}_{1} \) and \( \hat{R}_{2} \) through a Fourier-Motzkin elimination procedure we obtain:

\[
R_{1} \geq I(X_{1}; V_{1} | X_{2}), \ 
R_{2} \geq I(X_{2}; V_{2} | V_{1}), \ 
R_{1} + R_{2} \geq I(X_{1}; X_{2} | V_{1} V_{2}),
\]

In the following, we will provide the detailed mathematical proof for case with arbitrary \( K. \) In order to maintain expressions simple, in the following when we denote a description without the corresponding index, i.e. \( V_{i}^{n} \) or \( W_{i}^{n} \) for \( i \in \{1,2\} \), we will assume that the corresponding index is the true one generated in the corresponding encoders through the detailed encoding procedure. Consider round \( l \) and the event

\[
D_{l} = G_{l} \cap F_{l}, \text{ where for } \epsilon_{l} > 0,
\]

\[
G_{l} = \left\{ (X_{1}^{n}, X_{2}^{n}, Y^{n}, W_{l}^{n}) \in T_{X_{1}X_{2}YW_{l,1}^{n}}(1) \right\},
\]

\( \text{for all } l \in [1 : K + 1] \) and

\[
F_{l} = \left\{ M_{1,l}(2) = M_{1,t}, \ M_{2,l}(1) = M_{2,t}, \ t \in [1 : l - 1] \right\},
\]

for all \( l \in [1 : K] \). We also define

\[
F_{K+1} = \left\{ M_{1,t}(3) = M_{1,t}, \ M_{2,t}(3) = M_{2,t}, \ t \in [1 : K] \right\}.
\]

Sets \( G_{l} \) tell us that all the descriptions generated up to round \( l \) are jointly typical with the sources \( X_{1}, X_{2}, Y. \) This is an event that clearly depend on the encoding procedure at encoders 1 and 2. Sets \( F_{l} \) indicate that encoders are able to recover without error the indices generated in the other encoders. Clearly, this event depends on the decoding procedure employed. The occurrence of event \( D_{l} \) guarantees that encoders 1 and 2 share a common path of descriptions \( W_{l}^{n} \) which are typical with \( (X_{1}^{n}, X_{2}^{n}, Y^{n}). \) Similarly, \( D_{K+1} = F_{K+1} \cap G_{K+1} \) guarantees that all the generated descriptions are typical with \( (X_{1}^{n}, X_{2}^{n}, Y^{n}) \) and are perfectly recovered at the decoder. Let
us also define the event $E_l = \{ \text{there exists at least an error at the encoding or decoding in a encoder during round } l \}$

$$E_l = \mathcal{E}_{\text{enc}}(1,l) \cup \mathcal{E}_{\text{dec}}(2,l) \cup \mathcal{E}_{\text{enc}}(2,l) \cup \mathcal{E}_{\text{dec}}(1,l),$$

where $\mathcal{E}_{\text{dec}}(i,l)$ considers the event that at encoder $i$ during round $l$ there is a failure at recovering an index generated previously in the other encoder and $\mathcal{E}_{\text{enc}}(i,l)$ contains the errors at the encoding in encoder $i$ during round $l$. In precise terms:

$$\mathcal{E}_{\text{enc}}(1,l) = \left\{ (X^n_1, W^n_{2,1}(\hat{M}_{1,1}(1)), V^n_{1,1}(m_{1,l}, \hat{M}_{1,1}(1)) \notin T^n_{W_1,1} v_{1,1}(1,l) \forall m_{1,l} \in [1 : 2^{nR_l}(1)] \right\},$$

$$\mathcal{E}_{\text{enc}}(2,l) = \left\{ (X^n_2, W^n_{2,2}(\hat{M}_{2,2}(2)), V^n_{2,2}(m_{2,l}, \hat{M}_{2,2}(2)) \notin T^n_{W_2,2} v_{2,2}(2,l) \forall m_{2,l} \in [1 : 2^{nR_l}(2)] \right\},$$

$$\mathcal{E}_{\text{dec}}(1,l) = \left\{ \hat{M}_{2,l}(1) \neq M_{2,l} \right\},$$

$$\mathcal{E}_{\text{dec}}(2,l) = \left\{ \hat{M}_{1,l}(2) \neq M_{1,l} \right\},$$

for all $l \in [1 : K]$. Defining the fictitious round $K + 1$, where there are not descriptions generation and exchanges but only a decoding procedure at encoder 3, we can write:

$$\mathcal{E}_{K+1} = \mathcal{F}_{K+1}^{c} = \left\{ \hat{M}_{1,l}(3) \neq M_{1,l}, \hat{M}_{2,l}(3) \neq M_{2,l}, \right.$$ for some $l \in [1 : K] \right\}.$

The main goal is to prove the occurrence of $\mathcal{D}_{K+1}$ (with high probability) which guarantees that the descriptions generated at each encoder are jointly typical with the sources and are perfectly recovered at the decoder 3. We can write:

$$\Pr\{D_{K+1}^{c}\} = \Pr\{\mathcal{D}_{K+1}^{c} \cap D_{K}\} + \Pr\{\mathcal{D}_{K+1}^{c} \cap D_{K}^{c}\}$$

$$\leq \Pr\{\mathcal{D}_{K+1}^{c} \cap D_{K}\} + \Pr\{\mathcal{D}_{K}^{c}\}$$

$$\leq \Pr\{\mathcal{D}_{K}^{c}\} + \Pr\{\mathcal{D}_{K+1}^{c} \cap (D_{K} \cap \mathcal{E}_{K})\}$$

$$\leq \Pr\{\mathcal{D}_{K}^{c}\} + \sum_{l=1}^{K} \Pr\{\mathcal{D}_{l} \cap \mathcal{E}_{l}\}$$

$$+ \sum_{l=1}^{K} \Pr\{\mathcal{D}_{l}^{c} \cap (D_{K} \cap \mathcal{E}_{l})\}.$$  

Notice that

$$D_{l} = \left\{ (X^n_1, X^n_2, Y^n) \in T^n_{X_1, X_2, Y} v_{l}(\epsilon_{l}), \epsilon_{l} > 0 \right\}.$$  

From Lemma 3 we see that for every $\epsilon_{l} > 0$, $\Pr\{D_{l}\} \xrightarrow{n \to \infty} 0.0$. Then, it is easy to see that $\Pr\{D_{K+1}^{c}\} \xrightarrow{n \to \infty} 1$ will hold if the coding generation, the encoding and decoding procedures described above allow us to have the following:

1) If $\Pr\{D_{l}\} \xrightarrow{n \to \infty} 1$ then $\Pr\{D_{l+1}\} \xrightarrow{n \to \infty} 1 \forall l \in [1 : K - 1];$

2) $\Pr\{D_{l} \cap \mathcal{E}_{l}\} \xrightarrow{n \to \infty} 0 \forall l \in [1 : K];$

3) $\Pr\{\mathcal{D}_{l+1}^{c} \cap (D_{K} \cap \mathcal{E}_{l})\} \xrightarrow{n \to \infty} 0.$

In the following we will prove these facts. Observe that, at round $l$ the encoders act sequentially:

Encoding at encoder 1 $\rightarrow$ Decoding at decoder 2 $\rightarrow$ Encoding at encoder 2 $\rightarrow$ Decoding at decoder 1. Then, using (32) we can write condition 2 as:

$$\Pr\{D_{l} \cap \mathcal{E}_{l}\} = \Pr\{D_{l} \cap \mathcal{E}_{\text{enc}}(1,l)\}$$

$$+ \Pr\{\mathcal{D}_{l} \cap \mathcal{E}_{\text{dec}}(2,l) \cap \mathcal{E}_{\text{enc}}(1,l)\}$$

$$+ \Pr\{D_{l} \cap \mathcal{E}_{\text{dec}}(2,l) \cap \mathcal{E}_{\text{enc}}(1,l) \cap \mathcal{E}_{\text{dec}}(2,l)\}.$$  

Assume then that at the beginning of round $l$ we have $\Pr\{D_{l}\} \xrightarrow{n \to \infty} 1$. This implies that $\Pr\{G_{l}\}, \Pr\{F_{l}\} \xrightarrow{n \to \infty} 1.$ Clearly, we have:

$$\Pr\{\left( X^n_1, W^n_{1,l} \right) \in T^n_{X_1, W_1} v_{l}(\epsilon_{l}) \} \xrightarrow{n \to \infty} 1.$$  

We can clearly write:

$$\Pr\{\left( X^n_1, W^n_{1,l}, V^n_{1,l}(m_{1,l}, M_{W_1,l}) \right) \notin T^n_{X_1, W_1 V_1,l} v_{l}(\epsilon_{l}) \forall m_{1,l} \in [1 : 2^{nR_l}(l)] \}.$$  

We can use lemma 2 to show that:

$$\Pr\{\left( X^n_1, W^n_{1,l}, V^n_{1,l}(m_{1,l}, M_{W_1,l}) \right) \notin T^n_{X_1, W_1, V_1,l} v_{l}(\epsilon_{l}) \forall m_{1,l} \in [1 : 2^{nR_l}(l)] \} \xrightarrow{n \to \infty} 0,$$

if $R_{1,l} > I(V_1;l; X_1(W_1,l)) + \delta_{c,1}$, (33)

where $\delta_{c,1}$ can be made arbitrarily small. In this situation we clearly guarantee that:

$$\Pr\{\left( X^n_1, W^n_{2,l} \right) \in T^n_{X_2, W_2} v_{l}(\epsilon_{l}) \} \xrightarrow{n \to \infty} 1.$$  

The conditions in Lemma 5 are also satisfied implying:

$$\Pr\{V_{1,l}(m_{1,l}) = v_{1,l}(a^n_1, a^n_2, b^n, w^n_{1,l}, V_{1,l}(m_{1,l})) \in T^n_{V_1,l, X_1, W_1, \lambda_{l}}(x^n_1, w^n_{1,l})\}$$

$$= \frac{1}{|T^n_{V_1,l, X_1, W_1, \lambda_{l}}(x^n_1, w^n_{1,l})|}.$$  

Given that we imposed the Markov chain $V_{1,l} \leftrightarrow (X_1, W_1,l) \leftrightarrow (X_2, Y)$ we can use lemma 4 and its corresponding corollary to obtain:

$$\Pr\{\left( X^n_1, X^n_2, Y^n, W^n_{2,l} \right) \in T^n_{X_1, X_2, Y, W_2} v_{l}(\epsilon_{l}) \} \xrightarrow{n \to \infty} 1,$$

with $\epsilon_{l}$ sufficiently small. At this point we have that all descriptions generated up to round $l$, including the one generated at encoder 1 at round $l$ are jointly typical with the sources $X^n_1, X^n_2, Y^n$ with probability arbitrarily close to 1. Next, we should analyze the decoding at encoder 2. We can write:

$$\Pr\{D_{l} \cap \mathcal{E}_{\text{dec}}(2,l) \cap \mathcal{E}_{\text{enc}}(1,l)\} \leq \Pr\{D_{l} \cap \mathcal{E}_{\text{dec}}(2,l)$$

$$\cap \mathcal{E}_{\text{enc}}(1,l) \cap \left\{ (X^n_1, X^n_2, Y^n, W^n_{2,l}) \notin T^n_{X_1, X_2, Y, W_2} v_{l}(\epsilon_{l}) \right\}\}$$

$$+ \Pr\{\left( X^n_1, X^n_2, Y^n, W^n_{2,l} \right) \notin T^n_{X_1, X_2, Y, W_2} v_{l}(\epsilon_{l}) \}$$

$$\leq \Pr\{\left( X^n_2, W^n_{2,l} \right) \in T^n_{X_2, W_2} v_{l}(\epsilon_{l}) \cap F_{l} \cap \mathcal{E}_{\text{dec}}(2,l)\}.$$
Clearly, the second term in the RHS of (34) goes to zero when \( n \to \infty \). The first term is bounded by:

\[
\Pr \left\{ (X^n_1, X^n_2, Y^n, W^n_{2,l}) \notin T^n_{\{X_1, X_2, Y W_{l+1}\}} \right\}.
\]

From lemma 3 we can easily obtain that:

\[
\Pr \{ \mathcal{K}_{2,l} \} \xrightarrow{n \to \infty} 0,
\]

if

\[
\frac{1}{n} \log \mathbb{E} \left[ \left| \tilde{m}_{l,1} : (\tilde{m}_{l,1}, M_{W_{l,1}}) \in B(P_{l,1}) \right| \right] \leq I(X_2; V_{l,1}|W_{l,1}) - \delta',
\]

where \( \delta' \) can be made arbitrarily small. It is very easy to show that:

\[
\mathbb{E} \left[ \left| \tilde{m}_{l,1} : (\tilde{m}_{l,1}, M_{W_{l,1}}) \in B(P_{l,1}) \right| \right] = 2^{n(R_{l,1} - R_{l,1})},
\]

which implies that:

\[
R_{l,1} - R_{l,1} < I(X_2; V_{l,1}|W_{l,1}) - \delta'.
\]

At this point, we should analyze the encoding at encoder 2. This is done along the same lines of thought used for the encoding at encoder 1. The same can be said of the decoding at encoder 1. In summary we obtain the following rate equations:

\[
\tilde{R}_{l,1} - R_{l,1} < I(X_2; V_{l,1}|W_{l,1}) - \delta'.
\]

It is straightforward to see that \( \Pr \{ \mathcal{D}_l \cap \mathcal{E}_l \} \xrightarrow{n \to \infty} 0 \) \( \forall l \in [1 : K] \). The analysis for the joint typicality of all descriptions generated up to round \( l \), including the one generated at encoder 1 at round \( l \) are jointly typical with the sources \( X^n_1, X^n_2, Y^n \) with probability arbitrarily close to 1, can be repeated at encoder 2 obtaining:

\[
\Pr \left\{ (X^n_1, X^n_2, Y^n, W^n_{1,l+1}) \notin T^n_{\{X_1, X_2, Y W_{l+1}\}} \right\} \xrightarrow{n \to \infty} 1,
\]

which is a restatement of \( \Pr \{ \mathcal{G}_{l+1} \} \xrightarrow{n \to \infty} 1 \). In conjunction with the fact the above rate conditions guarantee that there are not errors at the encoding and decoding at encoder 1 and 2 during round \( l \) we have that \( \Pr \{ \mathcal{D}_{l+1} \} \xrightarrow{n \to \infty} 1 \). In this manner we can conclude that \( \Pr \{ \mathcal{D}_l \} \xrightarrow{n \to \infty} 1 \) implies that:

\[
\Pr \{ \mathcal{D}_{l+1} \} \xrightarrow{n \to \infty} 1 \quad \text{for } l \in [1 : K - 1].
\]

In order to complete the error probability analysis we need to prove that:

\[
\Pr \{ \mathcal{D}^c_{K+1} \cap (\mathcal{D}_K \cap \mathcal{E}^c_K) \} \xrightarrow{n \to \infty} 0.
\]

In order to do this we need to analyze the decoding at encoder 3. It is easy to show that:

\[
\Pr \{ \mathcal{D}^c_{K+1} \cap (\mathcal{D}_K \cap \mathcal{E}^c_K) \} \leq \Pr \{ \mathcal{G}_{K+1} \} + \Pr \{ \mathcal{G}^c_{K+1} \},
\]

where

\[
\mathcal{G}_{K+1} = \left\{ (X^n_1, X^n_2, Y^n, W^n_{1,K+1}) \in T^n_{\{X_1, X_2, Y W_{K+1}\}} \right\}.
\]

From the previous analysis it is easy to see that \( \Pr \{ \mathcal{G}^c_{K+1} \} \xrightarrow{n \to \infty} 0 \). The first term in the RHS of (38) can be bounded as:

\[
\Pr \{ \mathcal{G}_K \cap \mathcal{F}^c_{K+1} \} \leq \Pr \left\{ \left\{ W^n_{1,K+1} \in T^n_{\mathcal{F}_{K+1}} \right\} \cap \mathcal{F}^c_{K+1} \right\} \leq \Pr \{ \mathcal{K}_3 \},
\]

where

\[
\mathcal{K}_3 = \left\{ \exists \left\{ \tilde{m}_{l,1}, \tilde{m}_{l,2} \right\}_{l=1}^K : \left( \tilde{m}_{l,1}, \tilde{m}_{l,2} \right) \notin \left( \mathcal{M}_{l,1}, \mathcal{M}_{l,2} \right)_{l=1}^K \right\}
\]

We can write:

\[
\Pr \{ \mathcal{K}_3 \} = \mathbb{E} \left[ \Pr \left\{ \mathcal{K}_3 : \left( \mathcal{M}_{l,1}, \mathcal{M}_{l,2} \right)_{l=1}^K = \left( m_{l,1}, m_{l,2} \right)_{l=1}^K, \left( p_{l,1}, p_{l,2} \right)_{l=1}^K \right\} \right]
\]

where

\[
\mathcal{A} = \left\{ (m_{l,1}, m_{l,2})_{l=1}^K \right\},
\]

considering that \( \{ p_{l,1}, p_{l,2} \}_{l=1}^K \) are the functions of \( \{ m_{l,1}, m_{l,2} \}_{l=1}^K \) generated by the described encoding
procedure. In order to compute \( \mathbb{E} \left[ A \left( \{m_{1,l}, m_{2,l}\}^{K}_{l=1} \right) \right] \) we will consider a relabelling of the indices of the exchanged descriptions. We define for every \( s \in [1 : 2K] \):

\[
m_s = \begin{cases} 
\frac{i + 1}{2} & \text{s odd} \\
\frac{p_s + 1}{2} & \text{s even}
\end{cases}, \quad p_s = \begin{cases} 
\frac{i + 1}{2} & \text{s odd} \\
\frac{p_s + 1}{2} & \text{s even}
\end{cases}.
\]

Clearly, we can write:

\[
A \left( \{m_{1,l}, m_{2,l}\}^{K}_{l=1} \right) = \{\{m_s\}^{2K}_{s=1} \neq \{m_s\}^{2K}_{s=1} : \{m_s\}^{2K}_{s=1} \in \mathcal{B}(p_s), s \in [1 : 2K] \}.
\]

Consider \( \mathcal{M} = [1 : 2K] \) and its power set \( 2^\mathcal{M} \). It is straightforward to obtain:

\[
\mathbb{E} \left[ A \left( \{m_{1,l}, m_{2,l}\}^{K}_{l=1} \right) \right] = \sum_{\mathcal{H} \in 2^\mathcal{M}} \mathbb{E} \left[ \{\{m_s\} : s \in \mathcal{M} : m_s \neq M_s \forall s \in \mathcal{H}, \{m_l\}^{2K}_{l=1} \in \mathcal{B}(p_s), s \in \mathcal{M} \} \right].
\]

Let us analyze each term in the above sum. Consider \( \mathcal{H} \in 2^\mathcal{M} \) and \( s_{\min}(\mathcal{H}) = \min \{s : s \in \mathcal{H} \} \). We are interested in computing:

\[
\mathbb{E} \left[ \{\{m_s\} : s \in \mathcal{M} : m_s \neq M_s \forall s \in \mathcal{H}, \{m_l\}^{2K}_{l=1} \in \mathcal{B}(p_s), s \in \mathcal{M} \} \right].
\]

It is clear that the number of indices such that \( m_s \neq M_s \forall s \in \mathcal{H} \) is given by:

\[
\prod_{s \in \mathcal{H}} (q^n \tilde{R}_s - 1) \leq 2^n \sum_{s \in \mathcal{H}} \tilde{R}_s.
\]

As all indices of the generated codewords are independently and uniformly distributed in each of the bins used in the encoders 1 and 2, the probability that each of the above indices \( \{m_s\} \subseteq \mathcal{M} \) belongs to the bins \( \{B(p_s)\} \subseteq \mathcal{M} \) is given by \( 2^{-n} \sum_{s=s_{\min}(\mathcal{H})}^{2K} \tilde{R}_s \) for any sequence \( \{p_s\} \subseteq \mathcal{M} \). Then we can write:

\[
\mathbb{E} \left[ A \left( \{m_{1,l}, m_{2,l}\}^{K}_{l=1} \right) \right] \leq \sum_{\mathcal{H} \in 2^\mathcal{M}} 2^n \sum_{s=s_{\min}(\mathcal{H})}^{2K} \tilde{R}_s - \sum_{s=s_{\min}(\mathcal{H})}^{2K} R_s \to 0 \text{ if } \mathcal{H} \in 2^\mathcal{M}:
\]

\[
\sum_{s \in \mathcal{H}} \tilde{R}_s - \sum_{s=s_{\min}(\mathcal{H})}^{2K} R_s < 0.
\]

Clearly, \( \mathbb{E} \left[ A \left( \{m_{1,l}, m_{2,l}\}^{K}_{l=1} \right) \right] \to 0 \) if each \( \mathcal{H} \in 2^\mathcal{M} \):

\[
\sum_{s \in \mathcal{H}} \tilde{R}_s - \sum_{s=s_{\min}(\mathcal{H})}^{2K} R_s < 0.
\]

Consider this equation for every \( \mathcal{H} \in 2^\mathcal{M} \). Clearly \( s_{\min}(\mathcal{H}) \in [1 : 2K] \) when \( \mathcal{H} \) ranges over \( 2^\mathcal{M} \). Consider the sets \( \mathcal{H} \in 2^\mathcal{M} \) such that \( s_{\min}(\mathcal{H}) = r \). It is clear that over these sets, the one which put the more stringent condition in (40) is \([r : 2K]\). In this way, the \( 2^{2K} \) equations in (40) can be replaced by only \( 2K \) equations given by:

\[
\sum_{s=r}^{2K} (\tilde{R}_s - R_s) < 0, \quad r \in [1 : 2K].
\]

Using the relabelling in (39) it is easy to see that these equations can be put in the following manner in terms of \( \tilde{R}_{l,l} \) and \( \tilde{R}_{l,l} \) with \( l \in [1 : K] \) and \( i \in \{1, 2\} \):

\[
\tilde{R}_{l,i} + \tilde{R}_{l,i} + \sum_{k=l+1}^{K} (\tilde{R}_{l,k} + \tilde{R}_{2,k}) < R_{l,1} + R_{2,1} + \sum_{k=l+1}^{K} (R_{1,k} + R_{2,k})
\]

\[
\tilde{R}_{2,l} + \sum_{k=l+1}^{K} (\tilde{R}_{1,k} + \tilde{R}_{2,k}) < R_{2,1} + \sum_{k=l+1}^{K} (R_{1,k} + R_{2,k}).
\]

At this point we can use equations (31), (33), (35), (36), (37) jointly with the fact that the total rates at encoders 1 and 2 can be written as:

\[
\tilde{R}_{l,1} = \sum_{l=1}^{K} R_{l,1}, \quad \tilde{R}_{2,l} = \sum_{l=1}^{K} R_{2,l},
\]

in a Fourier-Motzkin elimination procedure to obtain:

\[
R_1 > I(X_1; W_{1,K+1} | X_2), \quad R_2 > I(X_2; V_{2,K} | W_{2,K}) + I(X_2; W_{2,K} | X_1), \quad R_1 + R_2 > I(X_1; X_2; W_{1,K+1}).
\]

Now we are set to prove analyze the average level or relevance. Let us denote with \( C \) the random realization of one codebook and be \( C = c \) one of its realizations. The average level of relevance over all random codebooks can be written as:

\[
\mathbb{E}_C \left[ \frac{1}{n} I (Y^n; M_{W_{1,K+1}} | C = c) \right] = \frac{1}{n} I (Y^n; M_{W_{1,K+1}} | C) = \frac{1}{n} I (Y^n; M_{W_{1,K+1}} | C)
\]

\[
\geq \frac{1}{n} I (Y^n; M_{W_{1,K+1}} | C),
\]

using the independence of the random generated codes with \( Y^n \). The following decomposition can be obtained introducing the indices recovered at encoder 3, which will denote as \( M_{W_{1,K+1}} \):

\[
\frac{1}{n} I (Y^n; M_{W_{1,K+1}}) = \frac{1}{n} H (Y^n) - \frac{1}{n} H (\hat{Y} | M_{W_{1,K+1}}, \hat{M}_{W_{1,K+1}})
\]

\[
- \frac{1}{n} H (Y^n | M_{W_{1,K+1}}, \hat{M}_{W_{1,K+1}}) - \frac{1}{n} H (\hat{M}_{W_{1,K+1}} | M_{W_{1,K+1}}).
\]

Last term in the above expression can be negligible when \( n \to \infty \) by a simple application of Fano inequality. To bound the other conditional entropy term we consider the following random variable:

\[
\hat{Y}_{\delta} = \left\{ \begin{array}{ll}
Y^n & \text{if } M_{W_{1,K+1}} = \hat{M}_{W_{1,K+1}} \\
\delta & \text{else}
\end{array} \right\}
\]

where, \( W_{1,K+1} \delta \hat{M}_{W_{1,K+1}} (\hat{M}_{W_{1,K+1}}) \). This auxiliary variable allows us to bound the conditional entropy as follows:

\[
H (Y^n | M_{W_{1,K+1}}, \hat{M}_{W_{1,K+1}})
\]
the relevance condition can be bounded as
\[
H(Y^n, Y^n | M_{W_{1,K+1}}, \hat{M}_{W_{1,K+1}}),
\]

\[
\leq H(\hat{Y}^n | M_{W_{1,K+1}}, \hat{M}_{W_{1,K+1}}) + H(Y^n | \hat{Y}^n),
\]

(a) \leq \log \left( \frac{nI(Y^n | W_{1,K+1}) + 1}{n} + \delta + H(Y^n | \hat{Y}^n) \right),

(b) \leq n (H(Y | W_{1,K+1}) + \epsilon) + \delta + H(Y^n | \hat{Y}^n),

\]

(c) \leq n [H(Y | W_{1,K+1}) + \epsilon] + \delta + 1

+ n \Pr (Y^n \neq \hat{Y}^n) \log |\mathcal{Y}|,

where (a) follows from the fact that the uniform distribution maximize entropy, (b) stems from standard properties of a conditional typical set and (c) is consequence of Fano inequality. We define the error probability \( P_e = \Pr \{ Y^n \neq \hat{Y}^n \} \). Then, the relevance condition can be bounded as

\[
\frac{1}{n} I(Y^n; M_{W_{1,K+1}})
\]

\[
\geq H(Y) - H(Y | W_{1,K+1}) - P_e \log |\mathcal{Y}| - \kappa_n
\]

\[
\geq I(Y; W_{1,K+1}) + \kappa_n
\]

where \( \kappa_n \) goes to zero with \( n \) large enough. In this way,

\[
E_c \left[ \frac{1}{n} I(Y^n; M_{W_{1,K+1}} | C = c) \right] \geq I(Y; W_{1,K+1}) - \epsilon_n,
\]

with \( \epsilon_n \rightarrow 0 \) when \( n \rightarrow \infty \). This show that every relevance level \( \mu \leq I(Y; W_{1,K+1}) \) is achievable in an average sense over all random codebook. For that reason, there must exists at least one good codebook.

**B. Achievability in Theorem 1**

The coding scheme is basically the same as the previous one. In this case encoder 1 and 2 operate sequentially in the same manner as above until the last round. As there is no encoder 3, only the first part of error probability analysis given is relevant for this case. The calculation of the relevance levels at encoder 1 and 2 follows also the same lines and for that reason is also omitted. We should mention that, at a given round, the bins generated, for example, at encoder 1 needs to contain only the index of latest generated description. It is not needed to generate larger bins in order to contain also all previous generated descriptions at encoder 1 and 2. In this way, instead of (31), the following are to be satisfied in the bins generation:

\[
R_{1,l} < R_{1,l}, \quad R_{2,l} < R_{2,l}, \quad l \in [1, K],
\]

which simplifies the analysis and the needed Fourier-Motzkin elimination procedure. The reason for this difference is given by the absence of the decoder in node 3.

**APPENDIX C**

**CORNER POINTS FOR \( \mathcal{R}^{\text{max}}_{\text{CODB}}(1) \)**

Let any fix distribution of \( U_1 \) and \( U_2 \) according to the corresponding Markov chains. This distribution induces 4 different corner points in \( \mathcal{R}^{\text{max}}_{\text{CODB}}(1) \), namely:

\[
Q_1 = \{ I(X_1; U_1 | X_2), I(U_1 U_2; X_2), I(Y; U_1 U_2) \},
\]

\[
Q_2 = \{ I(X_1; U_1), I(U_2; X_2 | U_1), I(Y; U_1 U_2) \},
\]

\[
Q_3 = \{ I(U_1; X_1), I(Y; U_1) - I(U_2; X_2 | U_1 Y) \},
\]

\[
Q_4 = \{ I(U_1; X_1 | X_2), 0, I(X_1; U_1 X_2) - I(U_1 U_2; X_1 X_2 | Y) \}
\]

The involved directions are given by the vectors (0, 1) and (1, 0) and do not enter in the analysis. The inclusion of \( Q_1 \) and \( Q_2 \) in \( \mathcal{R}^{\text{max}}_{\text{CODB}}(1) \) is easily proved by simply choosing \( V_1 = U_1 \) and \( V_2 = U_2 \). For \( Q_3 \) simply choose \( V_1 = U_1 \) and \( V_2 = v_2 \) with \( v_2 \in \mathcal{V}_2 \). The analysis of \( Q_4 \) is slightly more sophisticated. We need to use time sharing. We define a random variable \( Z \sim \text{Bern}(\lambda) \), with \( \lambda \in (0,1) \), and independent of everything else. We select

\[
V_1 = U_1 \mathbb{I} \{ Z = 1 \} + v_1 \mathbb{I} \{ Z = 0 \}, \quad v_1 \in \mathcal{V}_1,
\]

\[
V_2 = v_2, \quad v_2 \in \mathcal{V}_2.
\]

Thanks to an appropriate choice of the time-sharing parameter \( \lambda \), we will show that the point \( Q_4 \) is in \( \mathcal{R}^{\text{max}}_{\text{CODB}}(1) \). That choice is given by

\[
\lambda = \frac{\lambda(Y) - I(X_1; U_1 | X_2) + I(U_1 U_2; X_1 X_2 | Y)}{I(X_1; U_1)} = 1 - \frac{I(X_2; U_1)}{I(X_1; U_1)}.
\]

It is easy to see that the following conditions are met:

\[
R_1 \geq \lambda I(U_1; X_1 | X_2),
\]

\[
R_2 \geq 0,
\]

\[
R_1 + R_2 \geq \lambda I(U_1; X_1),
\]

\[
\mu \leq \lambda I(Y; U_1).
\]

With this specific choice it is easy to show that we meet the rate conditions in \( \mathcal{R}^{\text{max}}_{\text{CODB}}(1) \). It remains to analyze the relevance condition \( \mu Q_4 \leq \lambda (Y; U_1) \). To this end, let us consider: \( A \triangleq \lambda I(Y; U_1) - I(X_1; U_1 | X_2) + I(U_1 U_2; X_1 X_2 | Y) \). We can easily check that:

\[
A = \lambda - \lambda I(X_1; U_1) + I(U_1 U_2; X_1 X_2 | Y)
\]

\[
= - \lambda I(X_1; U_1 | Y) + I(U_1 U_2; X_1 X_2 | Y)
\]

\[
= -(1 - \lambda) I(X_1; U_1 | Y) + I(X_1 X_2 U_1 U_2 | Y).
\]

We have clearly that \( A \geq 0 \) which implies the relevance condition. Then, \( Q_4 \in \mathcal{R}^{\text{max}}_{\text{CODB}}(1) \). For every choice of the distributions of \( U_1 \) and \( U_2 \) (with the appropriate Markov chains), the extreme points of the outer bound are contained in \( \mathcal{R}^{\text{max}}_{\text{CODB}}(1) \), which implies that \( \mathcal{R}^{\text{CODB}}_{\text{CODB}}(1) \subseteq \mathcal{R}^{\text{max}}_{\text{CODB}}(1) \) from which the desired conclusion is obtained.
Wyner \[6\], \[33\] provide a good summary of convex analysis.

We can obtain the following conditions on random variable \(W_1\) and \(W_2\) we see that \(Q_1 \in \mathcal{R}_{\text{con}}(1)\). Let us analyze \(Q_2\). Consider the random variables \(V_1 = (V_1', Z)\) and \(V_2 = (V_2', Z)\) where

\[
V_1' = V_1 I \{ Z = 1 \} + v_1 I \{ Z = 0 \}, \quad v_1 \in \mathcal{V}_1
\]

\[
V_2' = V_2 I \{ Z = 1 \} + W I \{ Z = 0 \},
\]

where \(Z \sim \text{Bern}(\lambda)\) is independent of everything else with \(\lambda \in (0, 1)\) and \(W\) is random variable that satisfies \(W \perp X_2 - X_1 Y\). From these definitions we see that the following are satisfied:

\[
\tilde{V}_1 \sim X_1 \sim (X_2, Y), \quad \tilde{V}_2 \sim \tilde{V}_1 X_2 \sim (X_1, Y). \tag{41}
\]

Consider \(\lambda = \frac{I(X_1; V_1|X_2)}{I(X_1; V_1|X_2)} = 1 - \frac{I(X_2; V_1)}{I(X_1; V_1|X_2)}\). The following relations are easy to obtain:

\[
I(X_1; V_1) = I(X_1; V_1|X_2),
\]

\[
I(X_2; V_2|\tilde{V}_1) = \lambda I(X_1; V_1|X_2) + (1 - \lambda)I(X_2; W),
\]

\[
I(Y; \tilde{V}_2) = \lambda I(Y; V_2) + (1 - \lambda)I(Y; W).
\]

From these equations, and in order to show that \(Q_2 \in \mathcal{R}_{\text{con}}(1)\), we can obtain the following conditions on random variable \(W\):

\[
I(X_2; W) \leq I(X_2; \tilde{V}_2) + I(X_1; \tilde{V}_1|X_2), \tag{42}
\]

\[
I(Y; \tilde{V}_1, \tilde{V}_2) \leq I(Y; W). \tag{43}
\]

Consider the distribution \(p_{V_1, V_2|X_2}\) given by:

\[
p_{V_1, V_2|X_2}(v_1, v_2|x_2) = \sum_{x_1} p(x_1|x_2)p(v_1|x_1)p(v_2|x_1 x_2).
\]

We choose random variable \(W\) such that \(p_{W|X_2} \sim p_{V_1, V_2|X_2}\). With this choice we obtain \(I(X_2; W) = I(V_1; V_2)\) which clearly satisfies condition \((42)\). Up to this point we have not used the condition \(X_1 \perp X_2 \perp Y\). Using this condition we can obtain \(p_{W|Y} \sim p_{V_1, V_2|Y}\), which implies that \(I(Y; W) = I(Y; \tilde{V}_2)\), satisfying condition in \((43)\). So, we were able to find \((\tilde{V}_1, \tilde{V}_2)\) that satisfies \((41)\) and

\[
I(X_1; \tilde{V}_1) = I(X_1; V_1|X_2),
\]

\[
I(X_2; \tilde{V}_2) \leq I(X_2; V_2),
\]

\[
I(Y; \tilde{V}_1 \tilde{V}_2) = I(Y; W).
\]

This shows definitively that \(Q_2 \in \mathcal{R}_{\text{con}}(1)\). As for any pair \((V_1, V_2)\) we have that \((Q_1, Q_2) \in \mathcal{R}_{\text{con}}(1)\), then \(\mathcal{R}_{\text{con}}(1) \subseteq \mathcal{R}_{\text{con}}(1)\) and \(\mathcal{R}_{\text{con}}(1) = \mathcal{R}_{\text{con}}(1)\).

**APPENDIX E**

**PROOF OF THEOREM** \[11\]

As the proof relies heavily on convex analysis notions, we begin recalling basic facts of convex analysis that will be used during the proof. These results are presented without proofs which can be consulted in several well-known references on convex analysis as \[26\]. The works by Witsenhausen and Wyner \[6\], \[33\] provide a good summary of convex analysis for information-theoretic problems. Consider a compact and connected set \(A \subseteq \mathbb{R}^n\). We define \(\mathcal{C} = \text{co}(A)\) to be the convex hull of \(A\). Let \(m \leq n\) be the dimension of \(C\) (that is, the dimension of its affine hull). We say that \(x \in C\) is an extreme point of \(C\) if there not exist \(\lambda \in (0, 1)\) and \(x_1, x_2 \in C\) such that \(x = \lambda x_1 + (1 - \lambda)x_2\). We say that \(f : \mathbb{R}^n \to \mathbb{R}\) is convex if its effective domain (the set where \(f(x) < \infty\)) is convex and:

\[
f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2),
\]

with \(\lambda \in [0, 1]\) and \(x_1, x_2 \in \mathbb{R}^n\). When the inequality is strict for every \(\lambda \in (0, 1)\), \(x_1, x_2 \in \mathbb{R}^n\) we say that \(f(x)\) is strictly convex. When \(-f(x)\) is convex (strictly convex), we say that \(f(x)\) is concave (strictly concave). Some useful results are presented without proof:

(i) \(\mathcal{C}\) is compact;

(ii) Every extreme point of \(\mathcal{C}\) belongs to \(A\) and it is on the boundary of \(\mathcal{C}\);

(iii) Fenchel-Eggleston’s theorem \[34\]: If \(A\) has \(m\) or less connected components, every point of \(\mathcal{C}\) is the convex combination of no more that \(m\) points \(A\);

(iv) Dubin’s theorem \[35\]: Every point of the intersection of \(\mathcal{C}\) with \(k\) hyperplanes is the convex combination of no more that \(k + 1\) extreme points of \(\mathcal{C}\);

(v) Krein-Milman’s theorem \[30\]: \(\mathcal{C}\) is the convex hull of its extreme points;

(vi) Supporting hyperplanes \[25\]: On every point of the boundary (relative boundary if \(m < n\)) of \(\mathcal{C}\) there exists a supporting hyperplane of dimension \(m - 1\) such that \(\mathcal{C}\) is contained in one of the half-spaces determined by that hyperplane. Indeed, \(\mathcal{C}\) is the intersection of all half-spaces that contain \(\mathcal{C}\);

(vii) Consider the functions \(f : \mathbb{R}^{n-1} \to \mathbb{R}\) and \(g : \mathbb{R}^{n-1} \to \mathbb{R}\) defined as:

\[
f(y) \triangleq \inf \{ x : (x, y) \in \mathcal{C}\},
\]

\[
g(y) \triangleq \sup \{ x : (x, y) \in \mathcal{C}\}.
\]

Then \(f(y)\) is convex and \(g(y)\) is concave. Moreover, the points in the graphs of \(f(y)\) and \(g(y)\) are extreme points\(^2\) of \(\mathcal{C}\);

(viii) Be \(\{f_\alpha(y)\}\) and \(\{g_\beta(y)\}\) families of convex and concave functions respectively. Then \(\sup_\alpha f_\alpha(y)\) and \(\inf_\beta g_\beta(y)\) are convex and concave functions;

(ix) Let \(f : \mathbb{R}^{n-1} \to \mathbb{R}\) be an arbitrary upper semi-continuous function that nowhere has the value \(-\infty\). We define the convex envelope \(\text{cvx}(f)(y)\) of \(f(y)\) as the point-wise supremum of all affine functions that are smaller than \(f(y)\). Similarly, if \(f : \mathbb{R}^{n-1} \to \mathbb{R}\) is an arbitrary upper semi-continuous function that nowhere has the value \(+\infty\)

\(^2\)For the points \(y \in \mathbb{R}^{n-1}\) where \(f(y)\) and \(g(y)\) are strictly convex and concave respectively it is immediate to show that \((y, f(y))\) and \((y, g(y))\) are extreme points of \(\mathcal{C}\). If they are simply convex and concave, it means that they could be affine functions over some closed set of their effective domains. In such a case, that part of the graph of \(f(y)\) and \(g(y)\) constitutes a non-zero dimensional face \[25\] of the set \(\mathcal{C}\) which can thought as the set of points of \(\mathcal{C}\) where a certain linear functional achieves its maximum over \(\mathcal{C}\). But any linear functional achieves its maximum over a compact and convex set \(\mathcal{C}\) at an extreme point of \(\mathcal{C}\).
we define the concave envelope $\text{conc}(f)(y)$ of $f(y)$ as the point-wise infimum of all affine functions that are
greater than $f(y)$.

Let us consider the set $\mathcal{P}(\mathcal{U})$ where $\mathcal{U}$ is an arbitrary finite alphabet (cardinality equal to 3 suffices). From Theorem [1] it is clear that we can write:

$$\mathcal{R}^0_{\text{tw-cm}}(1/2) = \{(R, \mu) : R \geq I(X_1; U|X_2), \mu \leq I(Y; UX_2), U \sim p(u) \in \mathcal{P}(\mathcal{U}), U \rightsquigarrow X_1 \rightsquigarrow X_2Y \}.$$ 

The desired function $\mu^0_{\text{tw-cm}}(R)$ can be obtained from

$$\mu^0_{\text{tw-cm}}(R) = \sup \{\mu : (R, \mu) \in \mathcal{R}^0_{\text{tw-cm}}(1/2)\}.$$ 

We define the following functions:

$$g(r) \triangleq h_2(r * q) - h_2(r)$$
$$f(r) \triangleq h_2(p * q) - (1 - q * r)h_2\left(\frac{q}{1 - q * r}\right)$$

(44)

$$= h_2(r * q) - (1 - q * r)h_2\left(\frac{1 - q * r}{q * r}\right)$$

$$= h_2(p * q) - (1 - p * q)h_2\left(\frac{r * p}{1 - p * q}\right)$$

(45)

where $p, q \in (0, 1/2)$ and $r \in [0, 1]$. It can be easily shown that these functions are strictly convex, continuous and twice continuously differentiable as functions of $r$. In addition, they are symmetric with respect to $r = \frac{1}{2}$ and $0 \leq f(r) \leq g(r)$ for all $r \in [0, 1]$. In fact, it is not difficult to check that:

$$g(r) = I(X_1; U|X_2), \quad f(r) = I(Y; U|X_2),$$

when $(X_1, X_2, Y) \sim \text{Bern}(1/2)$, $U \rightsquigarrow X_1 \rightsquigarrow X_2Y$ and $X_1 = U \oplus Y$ with $Y \sim \text{Bern}(r)$, $U \sim \text{Bern}(1/2)$ and $U \perp V$.

The following lemma can be easily proved:

**Lemma 6 (Alternative characterization of $\mathcal{R}^0_{\text{tw-cm}}(1/2)$ for Binary sources):** Consider Binary sources $(X_1, X_2, Y) \sim \text{Bern}(1/2)$ with $X_2 \rightsquigarrow X_1 \rightsquigarrow Y$ such that $X_1 = X_2 \oplus Z$ with $Z \sim \text{Bern}(q)$, $q \in (0, 1/2)$, $Z \perp X_2$ and $Y = X_1 \oplus W$ with $W \sim \text{Bern}(p)$, $p \in (0, 1/2)$, $W \perp (X_1, X_2)$.

Region $\mathcal{R}^0_{\text{tw-cm}}(1/2)$ is equivalent to:

$$\mathcal{R}^0_{\text{tw-cm}}(1/2) = \left\{(R, \mu) : R \geq \sum_{u \in \mathcal{U}} p(u)g(r(u)), \mu \leq 1 - h(p * q) + \sum_{u \in \mathcal{U}} p(u)f(r(u)), \frac{1}{2} = \sum_{u \in \mathcal{U}} p(u)r(u), \ r(u) \in [0, 1] \ \forall u \in \mathcal{U} \right\}$$

**Proof:** Consider $(R, \mu) \in \mathcal{R}^0_{\text{tw-cm}}(1/2)$. Then, it should exist $U \rightsquigarrow X_1 \rightsquigarrow (X_2, Y)$ with $p(u) \in \mathcal{P}(\mathcal{U})$ such that $R \geq I(X_1; U|X_2)$ and $\mu \leq I(Y; UX_2)$. In a first place, we consider $I(X_1; U|X_2)$:

$$I(X_1; U|X_2) = H(X_1|X_2) - H(X_1|UX_2)$$
$$= h(q) - H(X_1|UX_2)$$
$$= h_2(q) - \sum_{(x_2, u) \in X_2 \times \mathcal{U}} p(x_2, u)H(X_1|U = u, X_2 = x_2)$$

Using the fact that $U \rightsquigarrow X_1 \rightsquigarrow (X_2, Y)$, it is not difficult to check that:

$$p(X_1 = 1|U = u, X_2 = 0) = \frac{q^2}{1 - q * r(u)},$$

$$p(X_1 = 1|U = u, X_2 = 1) = \frac{(1 - q)q}{1 - q * r(u)},$$

$$p(X_2 = 0, U = u) = (1 - q * r(u))p(u),$$

$$p(X_2 = 1, U = u) = (q * r(u))p(u),$$

where $r(u) = p(X_1 = 1|U = u)$. Using these equations, and from the fact that $X_1$ conditioned on $X_2$ and $U$ is a binary random variable we have that:

$$H(X_1|U = u, X_2 = x_2) = h_2\left(p(X_1 = 1|U = u, X_2 = x_2)\right),$$

from which $I(X_1; U|X_2) = \sum_{u \in \mathcal{U}} p(u)g(r(u))$ is easily obtained.

For $I(Y; U|X_2)$ we have:

$$I(Y; U|X_2) = I(Y; X_2) + I(Y; U|X_2)$$
$$= 1 - h_2(p * q) + I(Y; U|X_2).$$

The analysis of $I(Y; U|X_2)$ is similar to that of $I(X_1; U|X_2)$, obtaining:

$$I(Y; U|X_2) = \sum_{u \in \mathcal{U}} p(u)f(r(u)).$$

The requirement that $\sum_{u \in \mathcal{U}} p(u)r(u) = \frac{1}{2}$ follows from the fact that $p(X_1 = 1) = \frac{1}{2}$.

Consider the continuous mapping $L : [0, 1] \rightarrow [0, 1] \times [0, 1/2] \times [0, 1] - h_2(p * q) + f(r(u))$. Consider the image of this mapping to be $\mathcal{A}$. As $[0, 1]$ is a compact and connected subset of $\mathbb{R}$ and $L(r)$ is continuous, $\mathcal{A}$ is compact and connected. Let us consider $\mathcal{C} = \text{co}(\mathcal{A})$. This set, thanks to Fenchel-Eggleston theorem, we have:

$$\mathcal{C} = \left\{(r, \xi, \eta) : (\lambda_i, r_i)\frac{3}{i=1} \in [0, 1], \sum_{i=1}^{3} \lambda_i = 1, \right.\left. r = \sum_{i=1}^{3} \lambda_i r_i, \ \xi = \sum_{i=1}^{3} \lambda_i g(r_i), \right.\left. \eta = 1 - h_2(p * q) + \sum_{i=1}^{3} \lambda_i f(r_i) \right\}.$$

We also define the convex set

$$\mathcal{C}_{1/2} = \mathcal{C} \cap \left\{(r, \xi, \eta) : r = \frac{1}{2} \right\}|_{(\xi, \eta)},$$

that is the projection of $\mathcal{C} \cap \{(r, \xi, \eta) : r = \frac{1}{2}\}$ onto the plane $(\xi, \eta)$. Define the concave function $\tilde{\mu}(R)$ as:

$$\tilde{\mu}(R) = \sup \left\{ \eta : \left(\frac{1}{2}, R, \eta \right) \in \mathcal{C} \right\} = \sup \left\{ \eta : (R, \eta) \in \mathcal{C}_{1/2} \right\}.$$
As $C$ is compact, $C_{1/2}$ is also compact. Moreover, it is easy to see that it is not empty for $R \in [0, h(q)]$. This means that:

$$ \hat{\mu}(R) = \max \{ \eta : (R, \eta) \in C_{1/2} \}. $$

As the graph of $\hat{\mu}(R)$ is the upper boundary of the convex set $C_{1/2}$, by (vii), each point $(R, \hat{\mu}(R))$ is a extreme point of $C_{1/2}$ or a convex combination of extreme points of $C_{1/2}$. From Dubin’s theorem, as $C_{1/2}$ is the intersection of $C$ with one hyperplane, every extreme point of $C_{1/2}$ is a convex combination of no more that 2 extreme points of $C$ which also belong to $A$. This means that there exist $\lambda^* \in [0, 1]$ and $r_1, r_2 \in [0, 1]$ such that:

$$ \hat{\mu}(R) = 1 - h_2(p \ast q) + \lambda^* f(r_1^*) + (1 - \lambda) f(r_2^*) $$

with $R = \lambda^* g(r_1^*) + (1 - \lambda^*) g(r_2^*)$. Notice that is not necessarily true that $\frac{1}{2} = \lambda^* r_1^* + (1 - \lambda^*) r_2^*$. However, using the symmetry of functions $f(r)$ and $g(r)$ it is not difficult to show that:

$$ \hat{\mu}(R) = \max \left\{ 1 - h_2(p \ast q) + \lambda f(r_1) + (1 - \lambda) f(r_2) : \lambda g(r_1) + (1 - \lambda) g(r_2) = R, \lambda, r_1, r_2 \in [0, 1], \frac{1}{2} = \lambda r_1 + (1 - \lambda) r_2 \right\}, $$

obtaining an alternative characterization for $\hat{\mu}(R)$, from which it is easy to show that is an upper semi-continuous function.

From the Lemma and the definition of $C$ it is clear that we can write:

$$ R^u_{\text{tw-cub}}(1/2) = \{ (\xi, \eta) : \exists (\xi', \eta') \in C_{1/2}, \xi \geq \xi', \eta \leq \eta' \}. $$

This clearly implies that $\hat{\mu}(R) \leq R^u_{\text{tw-cub}}(R)$. It is easy to show that if $R \mapsto \hat{\mu}(R)$ is not decreasing then $\hat{\mu}(R) \geq R^u_{\text{tw-cub}}(R)$, which implies that $\hat{\mu}(R) = R^u_{\text{tw-cub}}(R)$. The following lemma establish the non-decreasing property of $\hat{\mu}(R)$.

**Lemma 7:** Consider random binary sources $(X_1, X_2, Y) \sim$ Bern(1/2) with $X_2 \rightarrow X_1 \rightarrow Y$ such that $X_1 = X_2 \oplus Z$ with $Z \sim$ Bern(q), $q \in (0, 1/2)$, $Z \perp X_2$ and $Y = X_1 \oplus W$ with $W \sim$ Bern(p), $p \in (0, 1/2)$, $W \perp (X_1, X_2)$. Then, for all $R \in [0, h_2(q)]$:

$$ 1 - h_2(p \ast q) + \frac{h_2(p \ast q) - h_2(p)}{h_2(q)} \leq R \leq \hat{\mu}(R) \leq 1 - h_2(p \ast q) + R $$

and $\hat{\mu}(R)$ is not decreasing in $R$.

**Proof:** From the assumptions, Lemma 6 definitions of $C$ and $\hat{\mu}(R)$, we have:

$$ \hat{\mu}(R) = \max \{ I(V; U X_2) : I(X_1; U | X_2) = R \}, $$

and $U \sim p(u) \in \mathcal{P}(U), U \rightarrow X_1 \rightarrow (X_2, Y).$

**From data processing inequality** it is easy to see that for all variables $U$ such that $U \rightarrow X_1 \rightarrow (X_2, Y)$, $I(V; U X_2) \leq I(V; X_1 X_2) \leq 1 - h_2(p)$ and $I(X_1; U | X_2) \leq H(X_1 | X_2) = h_2(q)$. This implies that $\hat{\mu}(R) \leq 1 - h_2(p)$ for all $R \in [0, h_2(q)]$. Consider $U = X_1$. In this case $R = h_2(q)$ and $I(V; U X_2) = 1 - h_2(p)$, allowing us to conclude that $\hat{\mu}(h_2(q)) = 1 - h_2(p)$. When $U$ is constant, we obtain $I(X_1; U | X_2) = 0$ and $I(Y; U X_2) = 1 - h(p \ast q)$. In fact, it is not hard to check that $\hat{\mu}(0) = 1 - h(p \ast q)$. As $\hat{\mu}(R)$ is concave, the lower bound on $\hat{\mu}(R)$ follows immediately. The

proposition of the upper bound is straightforward and for that reason is omitted. To prove the non-decreasing property consider any $R \in [0, h_2(q)]$ and $R_1 \leq R$. Then, exists $\lambda \in [0, 1]$ such that $R = \lambda R_1 + (1 - \lambda) h_2(q)$. As $R \rightarrow \hat{\mu}(R)$ is concave, we have:

$$ \hat{\mu}(R) \geq \lambda \hat{\mu}(R_1) + (1 - \lambda) \hat{\mu}(h_2(q)) \geq \hat{\mu}(R_1), $$

from which the result follows.

From the previous results we can conclude that:

$$ \mu^0_{\text{tw-cub}}(R) = \max_{(\lambda, r_1, r_2) \in [0, 1]} 1 - h_2(p \ast q) + \lambda f(r_1) + (1 - \lambda) f(r_2) \quad \text{s.t.} \quad \lambda g(r_1) + (1 - \lambda) g(r_2) = R. $$

This problem can be solved numerically to obtain, for each $R$, the exact value of $\mu^0_{\text{tw-cub}}(R)$. However, more can be said of $\mu^0_{\text{tw-cub}}(R)$. As the graph of $\mu^0_{\text{tw-cub}}(R)$ is an upper boundary of $C_{1/2}$, which is convex and compact, on each point of this boundary exists a supporting hyperplane. Consider point $(R_0, \mu^0_{\text{tw-cub}}(R_0))$. The supporting hyperplane for this point is defined by the pair $(\alpha, \psi(\alpha))$, such that $\mu^0_{\text{tw-cub}}(R_0) = \alpha R_0 + \psi(\alpha)$ and $\mu^0_{\text{tw-cub}}(R) \leq \alpha R + \psi(\alpha)$ for other $R \in [0, h_2(q)]$.

This implies that:

$$ \psi(\alpha) = \max \{ \mu^0_{\text{tw-cub}}(R) - \alpha R : R \in [0, h_2(q)] \} = \max \{ \eta - \alpha \xi : (\xi, \eta) \in C_{1/2} \}, $$

$$ \psi(\alpha) = \max \{ \eta - \alpha \xi : (\xi, \eta) \in C \}. $$

From (viii) above is immediate to see that $\psi(\alpha)$ is a convex function of $\alpha$. From its concavity and upper semi-continuity we know that $\mu^0_{\text{tw-cub}}(R)$ can be expressed alternatively as the point-wise infimum of affine functions that are greater that $\mu^0_{\text{tw-cub}}(R)$. In fact, it is not difficult to show that:

$$ \mu^0_{\text{tw-cub}}(R) = \min \{ \psi(\alpha) + \alpha R : \alpha \in \mathbb{R} \}. $$

From the results of Lemma 7, it is not difficult to see that in (47), it suffices to restrict $\alpha$ to the interval $[0, 1]$. Consider now a fixed value of $\alpha \in [0, 1]$ and define $\tilde{\nu}(r, \alpha) = 1 - h_2(p \ast q) + \nu(r, \alpha)$ where $\nu(r, \alpha) = \psi(r, \alpha) = \psi(\alpha) + \alpha r$ for $r \in [0, 1]$. Define $A^\alpha$ to be the graph of $\tilde{\nu}(r, \alpha)$ and $C^\alpha = \text{co}(A^\alpha)$. It is not hard to see that:

$$ C^\alpha = \{ (r, \eta - \alpha \xi) : (r, \xi, \eta) \in C \}, $$

and that the upper-boundary of $C^\alpha$ (which is compact) is the graph of the concave envelope of $\nu(r, \alpha)$. In fact, if we define $\psi(r, \alpha)$ as:

$$ \psi(r, \alpha) = \max \{ \omega : (r, \omega) \in C^\alpha \} = \max \{ \eta - \alpha \xi - (r, \xi, \eta) \in C \}, $$

we have that $\text{conc}(\tilde{\nu}(r, \alpha)) = \psi(r, \alpha)$. It is clear that $\psi(1/2, \alpha)$ is equal to $\psi(\alpha)$ defined in (46). That is:

$$ \psi(\alpha) = \text{conc}(\tilde{\nu}(r, \alpha)) \bigg|_{r=1/2} = 1 - h_2(p \ast q) + \nu(1/2, \alpha) \bigg|_{r=1/2}. $$

Note that $\nu(r, \alpha)$ is symmetric with respect to $r = 1/2$ for every $\alpha$ and that $\nu(1/2, \alpha) = 0$. This symmetry implies that:

$$ \text{conc}(\nu(r, \alpha)) \bigg|_{r=1/2} = \max \{ \nu(\alpha, r) \}. $$
Again as \( r \mapsto \nu(r, 0) \) is strictly convex and \( \alpha \mapsto \frac{\partial^2 \nu(r, \alpha)}{\partial r^2} \) is continuous, it must exists a maximal value \( \alpha^{**} < \alpha^* \) such that for all \( \alpha \in [0, \alpha^{**}] \)

\[
\frac{\partial^2 \nu(r, \alpha)}{\partial r^2} \geq 0, \quad \forall r \in [0, 1/2],
\]

which implies the convexity of \( \nu(\alpha, r) \) for all \( r \in [0, 1/2] \) for \( \alpha \in [0, \alpha^{**}] \). For every \( \alpha < \alpha^* \) we must have that there exists \( r \in (0, 1/2) \) such that \( \nu(r, \alpha) > 0 \). This, jointly with the continuity of \( (r, \alpha) \mapsto \nu(r, \alpha) \), implies that for \( \alpha^* \) there must exist \( r_{\alpha^*} \in (0, 1/2) \) with \( \nu(r_{\alpha^*}, \alpha^*) = 0 \). Similarly, it can be argued that \( \frac{\partial \nu(r_{\alpha^*})}{\partial r} \bigg|_{r_{\alpha^*}} = 0 \). This means that for \( \alpha \in [\alpha^*, 1] \), \( \max_{r \in [0,1/2]} \nu(\alpha, r) = 0 \) and \( \psi(\alpha) = 1 - h_2(p*q) \). When \( \alpha^{**} < \alpha < \alpha^* \), \( \max_{r \in [0,1/2]} \nu(\alpha, r) > 0 \) and \( \psi(\alpha) < 1 - h_2(p*q) \). Consider \( r_{\alpha} \in (0, 1/2) \) to be the point at which the maximum is achieved. At this point \( \frac{\partial \nu(r_{\alpha})}{\partial r} \bigg|_{r_{\alpha}} = 0 \). We have, by the implicit function theorem, that:

\[
\alpha = \frac{f'(r_{\alpha})}{g'(r_{\alpha})}, \quad \psi(\alpha) = 1 - h_2(p*q) + f(r_{\alpha}) - \alpha g(r_{\alpha}), \quad \psi'(\alpha) = -g(r_{\alpha}) < 0.
\]

At point \( \alpha^* \), the derivative of \( \psi(\alpha) \) could not exist, but the limit from the left exists and satisfies:

\[
\lim_{\alpha \to \alpha^*} \psi'(\alpha) = -g(r_{\alpha^*}) < 0.
\]

Finally, when \( \alpha \in [0, \alpha^{**}] \), as \( \nu(r, \alpha) \) is convex, its maximum value has to be achieved at a boundary point of \([0, 1/2]\). It is clear that this point should be \( r = 0 \). In this manner \( \max_{r \in [0,1/2]} \nu(\alpha, r) = f(0) - \alpha g(0) = h(p*q) - h_2(p) - \alpha h_2(q) \) and \( \psi(\alpha) = 1 - h_2(p) - \alpha h_2(q) \), which is an affine function in \( \alpha \) with slope \( h_2(q) \). With these results, we see that \( \psi(\alpha) \) must have the shape shown in Fig. 7. From (47) and (48), the obtained properties of \( \psi(\alpha) \) and the fact that beyond \( R > h_2(q) \), \( \mu_{\text{wcm}}(R) \) takes the value of \( 1 - h_2(p) \), it is easy show that:

\[
\mu_{\text{wcm}}(R) = \begin{cases} 
1 - h_2(p*q) + \alpha^* R & 0 \leq R \leq g(r_{\alpha^*}), \\
1 - h_2(p*q) + f \left( g^{-1}(R) \right) & g(r_{\alpha^*}) < R \leq h_2(q), \\
1 - h_2(p) & R > h_2(q).
\end{cases}
\]

Let us define \( R_c \equiv g(r_{\alpha^*}) \). As \( \mu_{\text{wcm}}(R_c) \) is concave it is not difficult to see that \( R_c \) and \( \alpha^* \) should satisfy:

\[
f' \left( g^{-1}(R_c) \right) = f' \left( g^{-1}(R_c) \right) \frac{R_c}{R_c}, \quad \alpha^* = f \left( g^{-1}(R_c) \right) \frac{R_c}{R_c}.
\]

From the final expression in (??), it is pretty clear how should be the scheme to be used to achieve \( \mu_{\text{wcm}}(R) \). When \( R > R_c \), auxiliary random variable \( U \) should be chosen such that: \( U = X_1 \oplus V \), where \( V \sim \text{Bern} \left( g^{-1}(R_c) \right) \). When \( R \leq R_c \), a time-sharing scheme should be used. It is not difficult to show that \( U \) should be chosen as: \( U \equiv \mathbb{1} \{ T = 0 \} + V \mathbb{1} \{ T = 1 \} \), where \( V \sim \text{Bern} \left( g^{-1}(R_c) \right) \) and \( T \sim \text{Bern} \left( \frac{R_c}{R} \right) \). When \( R > h_2(q) \), \( U \equiv X_1 \).
ACKNOWLEDGEMENT
The authors wish to thank the Associate Editor and the anonymous reviewers for the detailed suggestions and comments which significantly improved the manuscript.

REFERENCES
[1] V. Chandrasekaran and M. I. Jordan, “Computational and statistical tradeoffs via convex relaxation,” Proc. of the Nat. Academy of Sciences, vol. 110, no. 13, E1181–E1190, 2013.
[2] C. E. Shannon, “Coding theorems for a discrete source with a fidelity criterion,” in Claude Elwood Shannon: collected papers, N. J. A. Sloane and A. D. Wyner, Eds. IEEE Press, 1993, pp. 325–350.
[3] N. Tishby, F. C. Pereira, and W. Bialek. “The information bottleneck method,” in Proceedings of the Annual Allerton Conference on Communication, Control and Computing, 1999, pp. 368–377.
[4] R. Dobrushin and B. Tsybakov, “Information transmission with additional noise,” IEEE Transactions on Information Theory, vol. 8, no. 5, pp. 293–304, September 1962.
[5] T. Courtade and T. Weissman, “Multiterminal source coding under logarithmic loss,” Information Theory, IEEE Trans. on, vol. 60, no. 1, pp. 740–761, 2014.
[6] H. Witsenhausen and A. Wyner, “A conditional entropy bound for a pair of discrete random variables,” Information Theory, IEEE Transactions on, vol. 21, no. 5, pp. 493–501, 1975.
[7] G. Pichler, P. Piantanida, and G. Matz, “Distributed information-theoretic biclustering of two memoryless sources,” in Proceedings of the 53rd Annual Allerton Conference on Communication, Control and Computing, 2015.
[8] R. Ahlswede and I. Csiszar, “Hypothesis testing with communication constraints,” Information Theory, IEEE Transactions on, vol. 32, no. 4, pp. 533–542, Jul 1986.
[9] G. Katz, P. Piantanida, and M. Debbah, “Distributed binary detection with lossy data compression,” Information Theory, IEEE Transactions on, October 2015, (submitted). [Online]. Available: http://arxiv.org/abs/1601.01152
[10] J. Korner and K. Marton, “Images of a set via two channels and their role in multi-user communication,” Information Theory, IEEE Transactions on, vol. 23, no. 6, pp. 751–761, Nov 1977.
[11] R. Ahlswede and P. Gacs, “Spreading of sets in product spaces and hypercontraction of the markov operator,” Ann. Probab., vol. 4, no. 6, pp. 925–939, 12 1976. [Online]. Available: http://dx.doi.org/10.1214/aop/1176995937
[12] E. Erkip and T. M. Cover, “The efficiency of investment information,” vol. 44, no. 3, pp. 1026–1040, May 1998.
[13] S. Y. Tung, “Multiterminal source coding,” Ph.D. Dissertation, Electrical Engineering, Cornell University, Ithaca, NY, May 1978.
[14] T. Berger, Z. Zhang, and H. Viswanathan, “The CEO problem,” Information Theory, IEEE Trans. on, vol. 42, no. 3, pp. 887–902, 1996.
[15] V. Prabhakaran, D. Tse, and K. Ramchandran, “Rate region of the quadratic gaussian CEO problem,” in IEEE International Symposium on Information Theory, ISIT 2004, 2004, p. 119.
[16] Y. Oohama, “Rate-distortion function for Gaussian multiterminal source coding systems with several side informations at the decoder,” IEEE Transactions on Information Theory, vol. 51, no. 7, pp. 2577–2593, Jul. 2005.
[17] G. Pichler, P. Piantanida, and G. Matz, “Distributed information-theoretic biclustering,” CoRR, vol. abs/1404.6055, 2016. [Online]. Available: http://arxiv.org/abs/1404.6055
[18] A. Kossieris, “Two-way source coding with a fidelity criterion,” IEEE Transactions on Information Theory, vol. 31, no. 6, pp. 735 – 740, Nov 1985.
[19] N. Ma and P. Ishwar, “Some results on distributed source coding for interactive function computation,” IEEE Transactions on Information Theory, vol. 57, no. 9, pp. 6180–6195, 2011.
[20] H. H. Permuter, Y. Steinberg, and T. Weissman, “Two-way source coding with a helper,” IEEE Transactions on Information Theory, vol. 56, no. 6, pp. 2905–2919, June 2010.
[21] Y. K. Chia, H. H. Permuter, and T. Weissman, “Cascade, triangular, and two-way source coding with degraded side information at the second user,” IEEE Transactions on Information Theory, vol. 58, no. 1, pp. 189–206, Jan 2012.
[22] L. R. Vega, P. Piantanida, and A. O. Hero, “The three-terminal interactive lossy source coding problem,” IEEE Transactions on Information Theory, vol. 63, no. 1, pp. 532–562, Jan 2017.
[23] M. Vera, L. R. Vega, and P. Piantanida, “The two-way cooperative information bottleneck,” in IEEE International Symp. on Information Theory, ISIT 2015, 2015, pp. 2131–2135.
[24] V. Prabhakaran, K. Ramchandran, and D. Tse, “On the role of interaction between sensors in the CEO problem,” in Proceedings of the Annual Allerton Conference on Communication, Control and Computing, 2004.
[25] A. E. Gamal and Y.-H. Kim, Network Information Theory. New York, NY, USA: Cambridge University Press, 2012.
[26] R. T. Rockafellar, Convex Analysis. Princeton University Press, Jun. 1970.
[27] O. Rioul, “Information theoretic proofs of entropy power inequalities,” IEEE Transactions on Information Theory, vol. 57, no. 1, pp. 33–55, Jan. 2011.
[28] A. D. Wyner and J. Ziv, “The rate-distortion function for source coding with side information at the decoder,” IEEE Trans. Inform. Theory, vol. 22, pp. 1–10, 1976.
[29] A. B. Wagner, S. Tavildar, and P. Viswanathan, “Rate region of the quadratic gaussian two-encoder source-coding problem,” IEEE Trans. on Information Theory, vol. 54, no. 5, pp. 1938–1961, May 2008.
[30] A. D. Wyner and J. Ziv, “A theorem on the entropy of certain binary sequences and applications: Part I,” IEEE Transactions on Information Theory, vol. 19, no. 6, pp. 769–772, 1973.
[31] I. Csiszar and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems. New York: Academic, 1981.
[32] P. Piantanida, L. Vega, and A. O. Hero, “A proof of the generalized Markov lemma with countable infinite sources,” in Information Theory (ISIT), 2014 IEEE International Symposium on, June 2014, pp. 591–595.
[33] H. Witsenhausen, “Some aspects of convexity useful in information theory,” IEEE Transactions on Information Theory, vol. 26, no. 3, pp. 265–271, May 1980.
[34] H. Eggleston, Convexity. New York:Cambridge University, 1963.
[35] L. Dubin, “On extreme points of convex sets,” Journal of Mathematical Analysis and Applications, vol. 5, pp. 237–244, 1962.
[36] M. Krein and D. Milman, “On extreme points of regular convex sets,” Studia Mathematica, vol. 9, pp. 133–138, 1940.

Matias Vera (S’16) received the B.Sc. and M.Sc. degrees in electrical engineering from the University of Buenos Aires, Buenos Aires, Argentina, in 2014. He is currently a PhD student at the Department of Electronics, School of Engineering, Universidad de Buenos Aires, where he also works as a Teaching Assistant. His current research interests include machine learning, information theory and speaker recognition.

Leonardo Rey Vega (M’11) received the M.Sc (with honors) and PhD (summa cum laude) degrees in Electrical Engineering from the University of Buenos Aires (Argentina) in 2004 and 2010, respectively. In 2007 and 2008 he was invited at the INRS-EMT in Montreal, Canada and in 2012 he was a visitor at the Department of Telecommunications at SUPELEC, France. He is currently an Associate Professor at the University of Buenos Aires and member of the National Scientific and Technical Research Council in Argentina. Dr. Rey Vega’s research interests include statistical signal processing, information theory, representation learning and wireless sensor networks.

Pablo Piantanida (SM’16) received both B.Sc. in Electrical Engineering and the M.Sc. (with honors) from the University of Buenos Aires (Argentina) in 2003, and the Ph.D. from Université Paris-Sud (Orsay, France) in 2007. Since October 2007 he has joined the Laboratoire des Signaux et Systèmes (L2S), at CentraleSupélec together with CNRS (UMR 8506) and Université Paris-Sud, as an Associate Professor of Network Information Theory. He is currently associated with Montreal Institute for Learning Algorithms (MILA) at Université de Montréal. He is an IEEE Senior Member, and General Co-Chair of the 2019 IEEE International Symposium on Information Theory (ISIT). His research interests lie broadly in information theory and its interactions with other fields, including multi-terminal information theory, Shannon theory, machine learning and representation learning, statistical inference, cooperative communications, communication mechanisms for security and privacy.