EXCEPTIONAL SETS FOR SPINOR REGULAR TERNARY QUADRATIC FORMS

A. G. EARNEST

Abstract. The goal of this note is to provide an analysis of the positive integers that are represented everywhere locally, but not globally, by each of the 29 spinor regular ternary quadratic forms that are not regular.

1. Introduction

It is well-known that there is no local-global principle for the representation of integers by integral quadratic forms. Consequently, it is of interest to identify the exceptional set of integers that are represented everywhere locally, but not globally, by such a form. For ternary forms this set can be either finite or infinite; when it is empty, Dickson called the form regular [5]. That is, the regular forms are those that represent all integers represented by their genus. In terminology later introduced by Benham, Hsia, Hung and the author [2], a ternary form is said to be spinor regular if it represents all the integers represented by its spinor genus.

An integer that is represented by a genus of ternary integral quadratic forms but not by every spinor genus in that genus is referred to as a spinor exceptional integer for the genus. In this note, the general theory of spinor exceptional integers will be used to determine the exceptional sets for each of the spinor regular positive definite ternary primitive integral quadratic forms for which the exceptional set is nonempty. In light of a result of the author and Haensch [7], it is known that there are exactly 29 inequivalent forms with this property. For the 27 of these forms that are alone in their spinor genus, explicit formulas for the numbers of representations of all positive integers by each such form were recently obtained by Aygin, Doyle, Münkel, Pehlivan and Williams [1].

For each of the 29 forms, the exceptional set will be seen to consist of the integers lying in one or more squareclasses whose prime divisors satisfy certain congruence conditions. This phenomenon was observed by Jones and Pall [11] in their study of the genera of regular diagonal ternary forms. For example, they observed that the form

\[ 4x^2 + 9y^2 + 9z^2 + 4xy + 4xz + 2yz \]

represents all those integers represented by its genus except for the odd integers \( m^2 \) for which every prime factor \( p \) of \( m \) satisfies \( p \equiv 1 \pmod{4} \). In all, Jones and Pall listed seven forms with similar properties. A full explanation for this type of behavior later emerged through spinor genus theory. Schulze-Pillot [14] proved that each of these seven forms fails to represent precisely those integers that are spinor exceptional integers for their genus. The present paper can be viewed as an extension of the work of Schulze-Pillot applying the theory of spinor exceptional integers to complete the determination of the exceptional sets for each of the remaining spinor regular ternaries that are not regular.

2010 Mathematics Subject Classification. Primary 11E12; Secondary 11E08 11E20 11E25.

Key words and phrases. Spinor exceptional integers, spinor regular ternary quadratic forms.
We will follow the notations describing the spinor regular ternary forms in [1]. In that paper, the forms are labelled as A1–A13, B1–B12 and C1–C4, where the forms are grouped according to the prime factors of their discriminant. In this paper, the main results for the three groups appear in Propositions 4.1, 5.1 and 6.1, respectively.

The remainder of the paper is organized as follows. Section 2 contains notations and conventions that will be used throughout the paper. Some pertinent facts from the general theory of spinor exceptional integers will be reviewed in Section 3. The statements given there are special cases of general results from [14], stated here only in the generality needed to analyze the genera of the spinor regular ternary forms. On the topic of spinor exceptional integers, the interested reader may also wish to consult the excellent survey [15]. Results for the forms A1–A13, B1–B12 and C1–C4 are given in Sections 4, 5 and 6, respectively. In the final section of the paper, the spinor regular forms that are not alone in their spinor genus, namely B4 and B11, are analyzed in more detail. Since the results on the positive integers not represented by these forms are obtained only for even integers in [1], we give the full statements here as Propositions 7.4 and 7.5.

2. Ternary quadratic forms and lattices

A ternary quadratic form will be described by the sextuple of integers appearing as its coefficients, where \((a, b, c, d, e, f)\) denotes the quadratic form

\[
F = ax^2 + by^2 + cz^2 + dyz + exz + fxy.
\]

This form is referred to as classic if \(d, e\) and \(f\) are even, and non-classic otherwise. The discriminant \(\Delta\) of \(F\) is defined to be \(\frac{1}{2} \det M_F\), where \(M_F\) is the matrix of second partial derivatives of the form.

For referencing the literature on spinor exceptional integers, it will be convenient to freely switch between the language of quadratic forms and lattices. For quadratic lattices, we will follow the terminology and notation of O’Meara’s book [10]. To the ternary form \(F\) given above, we associate the ternary lattice \(L\) having Gram matrix \(\frac{1}{2}M_F\) with respect to some basis. If \(dL\) denotes the discriminant of this lattice, in the sense of [10], then \(\Delta = 4dL\). Let \(\text{gen}L\) and \(\text{spn}L\) denote the genus and spinor genus of \(L\), respectively. For an integer \(n\), we will use the notation \(n \to N\) to indicate that \(n\) is represented by \(N\), where \(N\) can be the lattice \(L\), its \(p\)-adic completion \(L_p\) with respect to some prime \(p\), or its genus \(\text{gen}L\) or spinor genus \(\text{spn}L\). Note that \(n \to L\) if and only if \(n\) is represented by the original form \(F\), and \(n \to \text{gen}L\) if and only if \(n \to L_p\) for all primes \(p\).

3. Determination of spinor exceptional integers

Throughout this section, \(L\) will denote an arbitrary positive definite integral ternary quadratic \(\mathbb{Z}\)-lattice, \(\Delta\) will be the positive integer \(4dL\), and \(n\) will be a positive integer represented by \(\text{gen}L\).

3.1. General criteria for spinor exceptional integers. In light of the results of [14], the spinor exceptional integers for a genus are determined by essentially local information. For a prime \(p\), we denote the \(p\)-adic numbers and \(p\)-adic integers by \(\mathbb{Q}_p\) and \(\mathbb{Z}_p\), respectively, and the group of units of \(\mathbb{Z}_p\) by \(\mathbb{Z}_p^\times\). The order of an element \(\lambda\) of \(\mathbb{Q}_p^\times\) will be denoted by \(\text{ord}_p(\lambda)\); so \(\lambda = p^{\text{ord}_p(\lambda)} \lambda_0\), with \(\lambda_0 \in \mathbb{Z}_p^\times\).

Let \(\theta(O^+(L_p))\) denote the group of spinor norms of rotations on \(L_p\), \(\theta(L_p, n)\) the relative spinor norm group defined in [14] Definition 1, p. 531], and \(N_p(\sqrt{-n\Delta})\) the group of local norms at \(p\) of \(\mathbb{Q}(\sqrt{-\text{sqf}(n\Delta)})\), where \(\text{sqf}(\gamma)\) denotes the squarefree...
part of the positive integer $\gamma$. Concretely,

$$N_p(-n\Delta) = \{ \gamma \in \mathbb{Q}_p : (\gamma, -n\Delta)_p = +1 \},$$

where $(\cdot, \cdot)_p$ denotes the Hilbert symbol at $p$. Note that $N_p(-n\Delta)$ is unchanged when the argument is changed by a square factor; in particular, $N_p(-n\Delta) = N_p(-\text{sqf}(n\Delta))$.

**Lemma 3.1.** If $-n\Delta \in \mathbb{Q}_p^2$, then $\theta(L_p, n) = N_p(-n\Delta)$.

**Proof.** It follows from the definition of $\theta(L_p, n)$ that

$$N_p(-n\Delta) \subseteq \theta(L_p, n) \subseteq \mathbb{Q}_p^2.$$

The result is immediate from this. \Box

**Lemma 3.2.** If $Z_p^\times \mathbb{Q}_p^2 \subseteq N_p(-n\Delta)$, then $\text{ord}_p(n\Delta)$ is even. Moreover, the reverse implication is true when $p$ is odd.

**Proof.** Suppose first that $\text{ord}_p(-n\Delta)$ is odd. Then $(\varepsilon_p, -n\Delta)_p = (\varepsilon_p, p)_p = -1$ by [10] 63:11a, where $\varepsilon_p$ denotes a unit of $\mathbb{Z}_p$ of quadratic defect $4Z_p$. This establishes the first implication. If $p$ is odd and $\text{ord}_p(n\Delta)$ is even, then $-n\Delta \in Z_p^\times \mathbb{Q}_p^2$. Hence $(\gamma, -n\Delta)_p = +1$ for any $\gamma \in \mathbb{Q}_p^\times \mathbb{Q}_p^2$ by [10] 63:12. \Box

The following theorem of Schulze-Pillot [13] Satz 2] gives complete criteria for $n$ to be a spinor exceptional integer for gen$L$.

**Theorem 3.3.** The integer $n$ is a spinor exceptional integer for gen$L$ if and only if

$$\theta(O^+(L_p)) \subseteq N_p(-n\Delta) \quad \text{and} \quad \theta(L_p, n) = N_p(-n\Delta)$$

for all primes $p$.

The local conditions appearing in the statement of this theorem will be discussed in more detail in the following two subsections.

### 3.2. Primes not dividing $2\Delta$.

Throughout this subsection, $p$ will denote a prime not dividing $2\Delta$. Consequently, $p$ is odd, $\text{ord}_p(\Delta) = 0$, and $L_p$ is unimodular. So $\theta(O^+(L_p)) = Z_p^\times \mathbb{Q}_p^2$ by [10] 92:5.

**Lemma 3.4.** $\theta(O^+(L_p)) \subseteq N_p(-n\Delta) \iff \text{ord}_p(n\Delta)$ is even.

**Proof.** This follows from Lemma 3.2 since $\text{ord}_p(n\Delta) = \text{ord}_p(n)$. \Box

**Lemma 3.5.** Assume that $\text{ord}_p(n)$ is even.

(i) If $\text{ord}_p(n) = 0$, then $\theta(L_p, n) = N_p(-n\Delta)$.

(ii) If $\text{ord}_p(n) > 0$, then

$$\theta(L_p, n) = N_p(-n\Delta) \iff -n\Delta \in \mathbb{Q}_p^2.$$

**Proof.** If $-n\Delta \in \mathbb{Q}_p^2$, then $\theta(L_p, n) = N_p(-n\Delta)$ by Lemma 3.1. So suppose that $-n\Delta \notin \mathbb{Q}_p^2$. Here $\text{ord}_p(n)$ even implies that $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$ by Lemma 3.4. Since $p \nmid \Delta$ and $\text{ord}_p(n)$ is even, $p$ is unramified in $\mathbb{Q}(\sqrt{-\text{sqf}(n\Delta)})$. So [13] Satz 3(a) applies with $r = s = 0$. Hence, $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$ holds if and only if $\text{ord}_p(n) = 0$. \Box

For a positive integer $t$, let $M_t$ denote the multiplicative semigroup generated by 1 and the set of all primes $p$ such that $-t \in \mathbb{Q}_p^2$. For any $s \in \mathbb{Q}$, note that $M_t = M_s^t$; in particular, $M_1 = M_{\text{sqf}(t)}$. Also, $M^2$ will denote $\{w^2 : w \in M_1\}$.

**Corollary 3.6.** Assume that $\text{ord}_p(n)$ is even for all $p \nmid 2\Delta$. Write $n = kw^2$, where all prime divisors of $k$ divide $2\Delta$ and g.c.d.$(w, 2\Delta) = 1$. Then

$$\theta(L_p, n) = N_p(-n\Delta) \iff w \in M_{n\Delta}.$$
3.3. Primes dividing $2\Delta$. Throughout this subsection, $p$ will denote a prime divisor of $2\Delta$.

**Lemma 3.7.** Let $p$ be an odd prime such that $-n\Delta \not\in \mathbb{Q}_p^2$ and $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$.

(i) If $L_p \cong (1, p, p^2)$ or $L_p \cong (1, p^2, p^3)$, then 
\[ \theta(L_p, n) = N_p(-n\Delta) \iff \text{ord}_p(n) \leq 1. \]

(ii) If $L_p \cong (1, p, p^2)$, then 
\[ \theta(L_p, n) = N_p(-n\Delta) \iff \text{ord}_p(n) \leq 2. \]

**Proof.** By the assumptions, we are in the situation of Satz 3(b) of [14]. For $L_p \cong (1, p, p^2)$ or $L_p \cong (1, p^2, p^3)$, the stated result follows from subcase (ii) of Satz 3(b) with $r = 1$, $s = 2$ or $3$. For $L_p \cong (1, p^2, p^3)$, it follows from subcase (i) with $r = 2, s = 3$. \[ \square \]

For the prime 2, there are numerous possibilities for the nature of a Jordan splitting of $L_2$. So in this case, we will state only the general form of the needed result, and provide specific references to the relevant subcases of the original result in [14, Satz 4] as they arise.

**Lemma 3.8.** Assume that $-n\Delta \not\in \mathbb{Q}_2^2$ and $\theta(O^+(L_2)) \subseteq N_2(-n\Delta)$. Then there exists a nonnegative integer $\lambda$ such that 
\[ \theta(L_2, n) = N_2(-n\Delta) \iff \text{ord}_2(n) \leq \lambda. \]

**Corollary 3.9.** Let $p$ be a prime dividing $2\Delta$ and assume that $\lambda \geq 1$ if $p = 2$. If $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$ and $\text{ord}_p(n) = 0$, then $\theta(L_p, n) = N_p(-n\Delta)$.

**Proof.** For odd $p$, this follows either from Lemma 3.1 or from Satz 3(b) of [14]. For $p = 2$, it follows either from Lemma 3.1 or Lemma 3.8. \[ \square \]

4. Discriminants divisible only by 2

The spinor regular ternaries with discriminants divisible only by 2 are labelled as A1–A13 in [1]. Table [2] contains a list of representatives for all of the classes of forms in the genus of each of these forms, along with their separation into spinor genera and their discriminant. These representatives and their separation into spinor genera were obtained by the use of Magma [4]. In each case, the spinor regular form is alone in its spinor genus, which is labelled Spinor Genus I, and the remaining classes constitute a second spinor genus labelled Spinor Genus II. These spinor genera will subsequently be referred to simply as SGI and SGII.

For each of the forms A1–A13, the data needed to determine the candidates for spinor exceptional integers for their genus is given in Table [2]. The second column of the table gives the splitting of $L_2$ for the ternary lattice $L$ corresponding to the form. These splittings can be obtained by routine calculation using one of the representative forms in the genus. In all cases, the 2-adic splitting is of the type 
\[ \langle b_1, b_2 2^r, b_3 2^s \rangle, \]
where $b_i \in \mathbb{Z}_2^*$ and $r \leq s$ are nonnegative integers. The third column contains the local spinor norm group $\theta(O^+(L_2))$ for each lattice $L$. For the lattices for which the splitting of $L_2$ has $0 < r < s$ (that is, A4, A6, A7, A8, A9, A11 and A12), the corresponding spinor norm groups $\theta(O^+(L_2))$ can be determined by Propositions 1.4, 1.6, 1.7 and 1.8 and Theorem 2.7 of [5]. For the remainder of the cases, $L_2$ is split by a multiple of (1, 1). Here $\theta(O^+(1, 1)) = \{\gamma \in \mathbb{Q}_2^2 : (\gamma, -1)_2 = +1\} = \{1, 2, 5, 10\}\mathbb{Q}_2^2$ by Proposition B of [9]. Then $\theta(O^+(L_2)) = \{1, 2, 5, 10\}\mathbb{Q}_2^2$ follows

1Throughout the paper, $\{a_1, \ldots, a_k\}\mathbb{F}^2$ will be used to denote $a_1\mathbb{F}^2 \cup \cdots \cup a_k\mathbb{F}^2$. 


Table 1. Genera containing spinor regular ternaries with 2-power discriminant

| #  | Spinor Genus I  | Spinor Genus II  | $\Delta$ |
|----|-----------------|------------------|---------|
| A1 | (2,2,5,2,2,0)   | (1,1,16,0,0,0)   | $2^6$   |
| A2 | (1,4,9,4,0,0)   | (2,2,9,-2,2,0)   | $2^7$   |
| A3 | (2,5,8,4,0,2)   | (2,2,17,2,-2,0)  | $2^8$   |
| A4 | (4,4,5,0,4,0)   | (1,4,16,0,0,0)   | $2^8$   |
| A5 | (4,9,9,2,4,4)   | (1,16,16,0,0,0)  | $2^{10}$ |
| A6 | (4,5,13,2,0,0)  | (1,16,20,-16,0,0)| $2^{10}$ |
| A7 | (2,2,9,-2,2,0)  | (1,4,17,-4,0,0)  |         |
| A8 | (4,4,17,0,4,0)  | (1,8,64,0,0,0)   |         |
| A9 | (5,8,8,0,4,4)   | (1,4,64,0,0,0)   | $2^{10}$ |
| A10| (5,13,16,0,0,2) | (4,16,21,16,0,0) | $2^{14}$ |
| A11| (9,17,32,0,0,4) | (1,64,64,0,0,0)  | $2^{14}$ |
| A12| (9,16,36,16,4,8)| (4,33,33,2,4,4)  |         |
| A13| (9,16,36,16,4,8)| (4,17,64,0,0,-4) |         |

Table 2. Data for A1–A13

from Theorem 3.14(iv) of [8]. The value of $\lambda$ in Lemma 3.8 can be determined by one of the subcases of [14, Satz 4]. For each form, the specific subcase that applies is identified in the fourth column of the table, and the value of $\lambda$ appears in the last column.

Table 2. Data for A1–A13

| #  | $\mathcal{L}_2$       | $\#(\mathcal{O}^+(\mathcal{L}_2))$ | subcase | $\lambda$ |
|----|-----------------------|-------------------------------------|---------|---------|
| A1 | (1, 1, 2)             | (1, 2, 5, 10) $Q_2^2$               | (b)(iii)| 1      |
| A2 | (1, 1, 2)             | (1, 2, 5, 10) $Q_2^2$               | (b)(iii)| 2      |
| A3 | (1, 1, 2)             | (1, 2, 5, 10) $Q_2^2$               | (b)(iii)| 3      |
| A4 | (1, 2, 2, 2)          | (1, 5) $Q_2^4$                      | (b)(iii)| 1      |
| A5 | (1, 2, 2, 2)          | (1, 2, 5, 10) $Q_2^2$               | (b)(i)  | 1      |
| A6 | (5, 2, 2, 5, 2)       | (1, 5) $Q_2^2$                      | (b)(ii) | 1      |
| A7 | (1, 2, 2, 5)          | (1, 5) $Q_2^2$                      | (b)(iii)| 3      |
| A8 | (1, 2, 2, 5)          | (1, 2, 3, 6) $Q_2^2$                | (c)(iii)| 1      |
| A9 | (1, 2, 2, 5)          | (1, 5) $Q_2^2$                      | (b)(iii)| 3      |
| A10| (1, 2, 2, 5)          | (1, 2, 5, 10) $Q_2^2$               | (b)(iv) | 1      |
| A11| (5, 2, 2, 5)          | (1, 5) $Q_2^2$                      | (b)(ii) | 3      |
| A12| (1, 2, 2, 5)          | (1, 5) $Q_2^2$                      | (b)(iii)| 5      |
| A13| (1, 2, 2, 5)          | (1, 2, 5, 10) $Q_2^2$               | (b)(i)  | 3      |
Proposition 4.1. Let $f$ be one of the forms $A1$–$A13$, and let $n$ be a positive integer. Then $n$ is represented everywhere locally, but not globally, by $f$ if and only if $n$ lies in:

(i) $M_1^2$ for $A1$, $A4$, $A5$, $A6$, $A10$;
(ii) $2M_1^2$ for $A2$;
(iii) $M_1^2$, $4M_1^2$ for $A3$, $A7$, $A9$, $A13$;
(iv) $M_2^2$ for $A8$;
(v) $4M_2^2$ for $A11$;
(vi) $M_2^2$, $4M_2^2$, $16M_2^2$ for $A12$.

Proof. First consider the lattice $L$ corresponding to the form $A1$. In this case, we seek to show that

$$n \not\rightarrow L \iff n \in M_1^2.$$ 

We first prove the forward implication. From $n \not\rightarrow L$ it follows that $n \not\leftrightarrow \text{spn}L$, since $L$ is spinor regular. So $n$ is a spinor exceptional integer for gen $L$. So by Theorem 3.3 and Lemma 3.4 we see that ord$p$(n) is even for all odd $p$; thus,

$$n = 2^t w^2,$$

with $w$ odd. As $\Delta = 2^6$, then we have $sqf(n\Delta) = 1$ or 2, depending upon whether $t$ is even or odd, respectively. Since $\theta(O^+(L_2)) \subseteq N_2(-n\Delta)$ and $5 \in \theta(O^+(L_2)) \setminus N_2(-2)$, it must be that $t$ is even and $M_{n\Delta} = M_1$. By Lemma 3.5 it follows that $w \in M_1$. Since $-1 \notin \mathbb{Q}_2^*$, it follows from Lemma 3.8 that

$$t = \text{ord}_2(n) \leq \lambda = 1.$$ 

Since $t$ is even, this implies that $t = 0$. Hence, $n \in M_1^2$ as claimed.

For the reverse implication, assume that $n \in M_1^2$. So $n = w^2$, $w \in M_1$ and $sqf(n\Delta) = 1$. Note that $w$ is odd since $2 \notin M_1$, as $-1 \notin \mathbb{Q}_2^*$. For $p$ odd, ord$p$(n) even implies that $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$, by Lemma 3.4 and $\theta(L_p, n) = N_p(-n\Delta)$ by Lemma 3.6. Also $\theta(O^+(L_2)) \subseteq N_2(-1)$. Since ord$2$(n) = 0, it follows from Lemma 3.8 that $\theta(L_2, n) = N_2(-n\Delta)$. By Theorem 3.3 $n$ is a spinor exceptional integer for gen $L$. The form $(1, 1, 16)$ lies in $\text{SGII}$ for the genus of $A1$. So $1 \rightarrow \text{SGII}$ and hence $n \not\rightarrow \text{SGII}$ since $n$ is a square. But then it must be that $n \not\rightarrow \text{SGII}$ and so $n \not\rightarrow L$. This completes the proof for $A1$.

The proofs for the forms $A4$, $A5$, $A6$ and $A10$ are identical. For $A3$, $A7$, $A9$, $A11$ and $A13$, the proof remains the same except that $\lambda = 3$ for these cases. Thus the conditions for $\theta(L_2, n) = N_2(-n\Delta)$ in Lemma 3.8 hold also with $t = 2$, and integers of the type $4M_1^2$ are also spinor exceptional integers for these genera. Note that for $A11$, $1 \not\rightarrow \text{gen}L$, so the integers in $M_1^2$ need not be listed among the spinor exceptional integers. In this one instance, $4 \rightarrow \text{SGII}$, and so all integers of the type $4M_1^2$ are represented by $\text{SGII}$, and hence not by $L$. For the form $A12$, the proof is again unchanged except that in this case $\lambda = 5$; thus the conditions for $\theta(L_2, n) = N_2(-n\Delta)$ in Lemma 3.8 hold also with $t = 4$, and the integers of the type $M^2_1$, $4M_1^2$ and $16M_2^2$ are spinor exceptional integers for the genus.

This leaves two remaining cases, $A2$ and $A8$. The only difference for the proof of $A2$ is that ord$_2$(\(\Delta\)) is odd. Since the other aspects of the proof remain the same, including that sqf\((n\Delta) = 1\), we see in this case that $n = 2^t w^2$, where $t$ and $w$ are odd integers. As $\lambda = 1$, it must be that $t = 1$ and so $n = 2w^2$. Here $2 \rightarrow \text{SGII}$, so we conclude that no elements of the type $2M_1^2$ are represented by $L$.

Finally, in the case of $A8$, we have $\theta(O^+(L_2)) = \{1, 2, 3, 6\} \mathbb{Q}_2^*$. So $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$ holds for all $p$ if and only if sqf\((n\Delta) = 2\). Hence sqf\((n) = 1\) for any spinor exceptional integer $n$ for this genus, since and ord$_2$(\(\Delta\)) is odd. As $\lambda = 1$, the spinor exceptional integers for the genus must be of the form $w^2$, for some odd integers.
integer \( w \). In this case, the prime divisors \( p \) of \( w \) must satisfy the condition that \(-2 \in \mathbb{Q}_p^2\). Hence, \( w \in M_2 \), as asserted. \( \square \)

**Remark.** The set \( M_1 \) appearing in the statement of the proposition is generated by 1 and the primes congruent to 1 modulo 4, and the set \( M_2 \) is generated by 1 and the primes congruent to 1 or 3 modulo 8.

5. **Discriminants divisible by 2 and 3**

The spinor regular ternaries with discriminants divisible by both 2 and 3 are labelled as B1–B12 in [1]. Table 3 contains a list of representatives for all of the classes of forms in the genus of each of these forms, along with their separation into spinor genera and their discriminant. In each case, as for the previous grouping, the genus splits into two spinor genera, which are labelled as before. The spinor regular forms B4 and B11 are the first forms listed in SGI for their genus.

| #  | Spinning Genus I | Spinning Genus II | \( \Delta \) |
|----|------------------|-------------------|-----------|
| B1 | (3,3,4,0,0,3)    | (1,1,36,0,0,1)    | 2^3 3^4   |
| B2 | (3,4,4,3,3)      | (1,3,10,-3,1,0)  | 2^3 3^4   |
| B3 | (1,7,12,0,0,1)   | (3,3,13,-3,3,-3) | 2^3 3^4   |
|    |                  | (1,108,0,0,1)    |           |
| B4 | (3,7,5,3,3)      | (1,144,0,0,1)    | 2^3 3^5   |
|    | (3,3,16,0,0,-3)  | (1,3,37,1,0)     |           |
| B5 | (4,4,9,0,0,4)    | (1,12,12,12,0)   | 2^3 3^2   |
| B6 | (3,4,9,0,0,0)    | (1,3,36,0,0,0)   | 2^3 3^4   |
| B7 | (4,9,12,0,0,0)   | (1,12,36,0,0,0)  | 2^3 3^4   |
| B8 | (4,9,28,0,4,0)   | (1,3,36,-36,0,0) | 2^3 3^3   |
|    |                  | (9,13,13,-10,-6,-6) |       |
| B9 | (9,16,16,16,0,0) | (1,4,8,48,48,0)  | 2^3 3^4   |
| B10| (13,13,16,-8,8,10)| (4,13,37,-24,-4) | 2^3 3^4   |
| B11| (9,16,48,0,0,0)  | (1,4,144,144,0,0)| 2^4 3^3   |
|    | (16,25,25,-14,16,-16)| (4,49,49,-46,44) |           |
| B12| (9,16,112,16,0,0)| (1,144,144,144,0,0)| 2^3 3^7 | (9,49,49,-46,66) | (4,144,144,144,0,0) |

Table 3. Genera containing spinor regular ternaries with discriminant divisible by 2 and 3

The data necessary to determine the spinor exceptional integers for these genera is summarized in Table 4. We first record the local splittings at the primes 2 and 3 for the corresponding lattices. The splittings for \( L_3 \) are given in the second column of Table 4. In every case, it follows from [13, Satz 3] that

\[
\theta(O^+(L_3)) = \{1, 3\} \mathbb{Q}_p^2.
\]

The 2-adic splittings appear in the third column. Some additional explanation is in order here. This grouping contains both classic and non-classic forms. For the non-classic forms B1–B4, the splitting given in the table is for the lattice \( L'_2 \) obtained by scaling \( L_2 \) by 2. For the remaining forms B5–B12, which are classic, the splitting of \( L'_2 = L_2 \) is given. Since the spinor norm group \( \theta(O^+(L_2)) \) is unaffected by scaling, this makes it possible to directly apply the results of [8] to determine \( \theta(O^+(L_2)) \) from \( L'_2 \) in all cases. We use the notations \( \mathbb{H} \) and \( \mathbb{A} \) to denote the binary \( \mathbb{Z}_2 \)-lattices with Gram matrices \((-1 0)

We then use the notations \( \mathbb{H} \) and \( \mathbb{A} \) to denote the binary \( \mathbb{Z}_2 \)-lattices with Gram matrices \((-1 0)

\[\begin{pmatrix}-1 & 0 \\
0 & 1\end{pmatrix}\) and \[\begin{pmatrix}1 & 1 \\
1 & 1\end{pmatrix}\], respectively. If \( M \) is one of these matrices and \( \alpha \in \mathbb{Z} \), \( \alpha M \) will denote the matrix resulting from multiplying each entry of \( M \) by

\[\begin{pmatrix}1 & 1 \\
1 & 1\end{pmatrix}\].
α. In all cases occurring in Table 4, $L_2$ has a binary Jordan component either of odd order or of even order (see [5] Definition 3.1, p. 79) for an explanation of this terminology), and the complementary component has the same order. It follows from Theorem 3.14(i) of [8] that in all cases

$$\theta(O^+(L_2)) = \mathbb{Z}_2^2 \mathbb{Q}_2^2.$$  

The value of λ in Lemma 3.8 can be determined by one of the subcases of [14, Satz 4(a)]. For each form, the specific subcase that applies is identified in the fourth column of the table, and the value of λ appears in the last column.

| #  | $L_3$ | $L_2$ | subcase | λ  |
|----|-------|-------|---------|----|
| B1 | (1, 3, 3^2) | $A \perp (2^3)$ | (ii)(β) | 1 |
| B2 | (1, 3, 3^2) | $A \perp (5 \cdot 2^6)$ | (ii)(α) | 1 |
| B3 | (1, 3, 3^2) | $A \perp (3 \cdot 2^3)$ | (ii)(β) | 1 |
| B4 | (1, 3, 3^2) | $A \perp (2^3)$ | (ii)(β) | 3 |
| B5 | (1, 3, 3^2) | (1) $\perp 2A$ | (ii)(γ) | 1 |
| B6 | (1, 3, 3^2) | (1, 3, 2^3) | (i)(β) | 1 |
| B7 | (1, 3, 3^2) | (1, 2^2, 3 \cdot 2^3) | (i)(α) | 1 |
| B8 | (1, 3^2, 3^4) | (1) $\perp 2A$ | (ii)(γ) | 1 |
| B9 | (1, 3^2, 3^4) | (1) $\perp 2^3A$ | (ii)(γ) | 3 |
| B10 | (1, 3^2, 3^4) | (5) $\perp 2^6H$ | (ii)(γ) | 3 |
| B11 | (1, 3^2, 3^4) | (1, 2^4, 3 \cdot 2^3) | (i)(α) | 3 |
| B12 | (1, 3^2, 3^4) | (1) $\perp 2^4A$ | (ii)(γ) | 3 |

Table 4. Data for B1–B12

**Proposition 5.1.** Let $f$ be one of the forms B1–B12, and let $n$ be a positive integer. Then $n$ is represented everywhere locally, but not globally, by $f$ if and only if $n$ lies in:

(i) $M_3^2$ for B1, B2, B5, B6, B7, B8;
(ii) $3M_2^2$ for B3;
(iii) $M_2^3$, $4M_3^2$ for B4, B9, B11, B12;
(iv) $4M_2^3$ for B10.

**Proof.** First consider the lattice $L$ corresponding to the form B1. Assume first that $n \rightarrow \text{gen}L$ but $n \not\rightarrow L$. Then $n \not\rightarrow \text{spin}L$, since $L$ is spinor regular. So $n$ is a spinor exceptional integer for gen$L$. From Theorem 3.3, Lemma 3.4, and Lemma 3.8, it follows that $\text{ord}_p(n)$ is even for all $p \neq 3$. So $\text{sqf}(n\Delta) = 1$ or 3. Since $\theta(O^+(L_3)) \not\subseteq N_3(-1)$, it must be that $\text{sqf}(n\Delta) = 3$. So $n = 2^t3^2w^2$ with $s, t$ even and $w \in M_3$, by Lemma 3.4. By Lemma 3.5, $\text{ord}_p(n) \leq 1$ for $p = 2, 3$. Hence, $s = t = 0$ and the conclusion follows.

For the reverse implication, assume that $n = w^2$ with $w \in M_3$. The lattice corresponding to (1, 1, 36, 0, 0, 1) lies in SGI and represents 1, and so $n \rightarrow SGI$; in particular, $n \rightarrow \text{gen}L$. Since $\Delta = 2^t3^3$, we have $\text{sqf}(n\Delta)$ is 3; thus, $N_p(-n\Delta) = N_p(-3)$ and $M_{p\Delta} = M_3$. Since $\text{ord}_p(n\Delta)$ is even for $p \neq 2, 3$, it follows from Lemma 3.4 that $\theta(O^+(L_3)) \subseteq N_p(-n\Delta)$, and, by Corollary 3.6, $\theta(L, n) = N_p(-n\Delta)$ for all $p \neq 2, 3$. For the primes 2 and 3, the direct computations show that $\theta(O^+(L_2)) = \mathbb{Z}_2^2 \mathbb{Q}_2^2 = N_2(-3) = N_2(-n\Delta)$ and $\theta(O^+(L_3)) = \{1, 3\} \mathbb{Q}_3^2 = N_3(-3) = N_3(-n\Delta)$. It then follows by Corollary 3.9 that $\theta(L, n) = N_p(-n\Delta)$. So the criteria of Theorem 3.8 are met and $n$ is a spinor exceptional integer for gen$L$. So $n \not\rightarrow SGI$ and, in particular, $n \not\rightarrow L$. This completes the proof for B1.
The proofs for the forms B2, B5, B6, B7 and B8 proceed in exactly the same way. For B4, B9, B11 and B12, the proof is analogous except that \( \lambda = 3 \) for these cases. Thus the conditions for \( \theta(L_2, n) = N_2(-n\Delta) \) in Lemma 3.3 hold also with \( t = 2 \), and integers of the type \( 4M_3^2 \) are also spinor exceptional integers for these genera.

This leaves two remaining cases, B3 and B10. The only difference for the proof of B3 is that \( \text{ord}_3(\Delta) \) is even. Since the other aspects of the proof remain the same, including that \( \text{sqf}(n\Delta) = 1 \), we see in this case that if \( n \) is a spinor exceptional integer for \( \text{gen} L \), then \( n = 2^t3^sw^2 \), where \( t \) is even, \( s \) is odd, and \( w \in M_3 \). As \( \lambda = 1 \), it must be that \( t = 0 \), and it follows from Lemma 3.7 that \( s = 1 \); so \( n = 3w^2, w \in M_3 \). In this case, \( 3 \rightarrow SGII \), so we conclude that no elements of the type \( 3M_3^2 \) are represented by \( L \).

For the case B10, the proof of the forward implication proceeds as before up to the point where we conclude that \( n = 2^tw^2 \) with \( t \) even and \( w \in M_3 \). In this case, \( L_2 \sim = \langle 5 \rangle \perp \langle 2^3 \rangle A \). So the assumption that \( n \rightarrow L_2 \) implies that \( t \neq 0 \). Hence, \( n \in 4M_3^2 \), as claimed. For the reverse implication, observe that \( 4 \rightarrow SGII \), and so \( 4M_3^2 \rightarrow SGII \). The remainder of the argument then proceeds as before. □

Remark. The set \( M_3 \) appearing in the statement of the proposition is generated by 1 and the primes congruent to 1 modulo 3.

6. Discriminants divisible by 2 and 7

The spinor regular ternaries with discriminants divisible by 2 and 7 are labelled as C1–C4 in [1]. Table 5 contains a list of representatives for all of the classes of forms in the genus of each of these forms, along with their separation into spinor genera and their discriminant. In each case, as for the previous two groupings, the genus splits into two spinor genera, which are labelled as before.

| # | Spinous Genus I | Spinous Genus II | \( \Delta \) | \( L_2' \) |
|---|---|---|---|---|
| C1 | \( 2,7,8,7,1,0 \) | \( 1,7,14,7,0,0 \) | \( 2^47^4 \) | \( H \perp \langle 2 \rangle \) |
| C2 | \( 7,8,9,6,7,0 \) | \( 4,7,15,7,10,0 \) | \( 2^47^4 \) | \( A \perp \langle 5 \cdot 2^3 \rangle \) |
| C3 | \( 8,9,25,2,4,8 \) | \( 1,28,56,7,-28,0 \) | \( 2^57^4 \) | \( \langle 1 \rangle \perp 2H \) |
| C4 | \( 29,32,36,32,12,24 \) | \( 4,29,197,-4,24 \) | \( 2^47^4 \) | \( \langle 5 \rangle \perp 2^2A \) |

Table 5. Genera containing spinor regular ternaries with discriminant divisible by 2 and 7

For each lattice \( L \) corresponding to one of the forms C1–C4, we have \( L_7 \cong \langle 1,7,7^2 \rangle \), and it follows from [13 Satz 3] that

\[
\theta(O^+(L_7)) = \{ 1,7 \} \mathbb{Q}_7^2 = N_7(-7).
\]

The 2-adic splittings for these lattices are given in the last column of Table 5. The forms C1 and C2 are non-classic and the splittings listed for them are for the lattice \( L_2' \) obtained from \( L_2 \) by scaling by 2; for C3 and C4, which are classic, the splittings listed are for \( L_2' = L_2 \). In all cases, it follows from [8 Theorem 3.14(i)] that

\[
\theta(O^+(L_2)) = \mathbb{Z}_2^4 \mathbb{Q}_2^2.
\]
Proposition 6.1. Let \( f \) be one of the forms \( C1-C4 \), and let \( n \) be a positive integer. Then \( n \) is represented everywhere locally, but not globally, by \( f \) if and only if \( n \) lies in:

(i) \( M^2_2 \) for \( C1, C2, C3 \);
(ii) \( 4M^2_2 \) for \( C4 \).

Proof. Let \( L \) be a lattice corresponding to one of the forms \( C1-C3 \). Assume first that \( n \to \text{gen}L \) but \( n \not\to L \). As before, \( n \) is a spinor exceptional integer for \( \text{gen}L \). From Theorem 3.3, Lemma 3.4, and the computation of \( \theta(O^+(L_2)) \), it follows that \( \text{ord}_p(n\Delta) \) is even for all \( p \neq 7 \). So \( \text{sqf}(n\Delta) = 1 \) or 7. Since \( \theta(O^+(L_7)) \nsubseteq N_7(-1) \), it must be that \( \text{sqf}(n\Delta) = 7 \). So \( n = 2^s7^t w^2 \) with \( s, t \) even and \( w \in M_7 \), by Lemma 5.6. By Lemma 3.7, \( \text{ord}_7(n) \leq 1 \) and so \( t = 0 \). Since \( 2 \in M_7 \), it follows that \( n \in M^2_2 \) as claimed.

For the reverse implication, assume that \( n = w^2 \) with \( w \in M_7 \). It can be seen from the representatives of \( \text{SGII} \) that \( 1 \to \text{SGII} \), and so \( n \to \text{SGII} \); in particular, \( n \to \text{gen}L \). Since \( \text{ord}_2(\Delta) \) is even and \( \text{ord}_7(\Delta) \) is odd, we have \( \text{sqf}(n\Delta) = 7 \); thus, \( \text{ord}_p(n\Delta) = N_p(-7) \) for all \( p \), and \( M_n\Delta = M_7 \). Since \( \text{ord}_p(n\Delta) \) is even for \( p \neq 2,7 \), it follows from Lemma 3.3 that \( \theta(O^+(L_2)) \subseteq N_p(-n\Delta) \), and from Corollary 3.6 that \( \theta(L_2, n) = N_p(-n\Delta) \) for all \( p \neq 2, 7 \). Also, \( \theta(O^+(L_7)) = N_7(-7) = N_7(-n\Delta) \) and \( \text{ord}_7(n) = 0 \) gives \( \theta(L_7, n) = N_7(-n\Delta) \), by Corollary 3.9. Since \( -7 \in \mathbb{Q}_2^2 \), we have \( \theta(O^+(L_2)) = \mathbb{Z}_2^2 \mathbb{Q}_2^2 \subseteq N_2(-7) = N_2(-n\Delta) \), and, by Lemma 3.1, \( \theta(L_7, n) = N_2(-n\Delta) \). So the criteria of Theorem 3.3 are met and \( n \) is a spinor exceptional integer for \( \text{gen}L \). So \( n \not\to \text{SGI} \) and, in particular, \( n \not\to L \). This completes the proof for \( C1-C3 \).

For the case \( C4 \), the proof of the forward implication proceeds as above to the point where we conclude that \( n = 2^s w^2 \) with \( s \) even and \( w \in M_7 \). In this case, \( L_2 \cong (5) \perp 2^s A \). So the assumption that \( n \to L_2 \) implies that \( s \neq 0 \). Hence, \( n \not\in 4M^2_2 \), as claimed. For the reverse implication, observe that \( 4 \to \text{SGII} \), and so \( 4M^2_2 \to \text{SGII} \). The remainder of the argument then proceeds as before. \( \square \)

Remark. The set \( M_2 \) appearing in the statement of the proposition is generated by 1 and the primes congruent to 1, 2 or 4 modulo 7.

7. Forms not alone in their spinor genus

For each of the 27 spinor regular ternaries that are alone in their spinor genus, Aygin et al [11] summarize in Table A.17 the list of all positive integers not represented by the form. As detailed in that paper, the results for several of these forms appeared earlier in work of Lomadze [12] and Berkovich [3]. However, for the remaining two forms which are not alone in their spinor genus, the method of [11] yields only the even integers that fail to be represented. For completeness, we will state here the full results for these two forms, which are B4 and B11 in the list. Before stating the results, we will establish three lemmas needed to analyze the local obstructions to representation by these forms.

Lemma 7.1. Let \( L \) be a ternary quadratic \( \mathbb{Z} \)-lattice such that \( L_2 \cong M \perp (16) \), where \( M \) is a lattice corresponding to \( x^2 + xy + y^2 \), and let \( n \) be a positive integer. Then \( n \not\to L_2 \) if and only if \( n = 2 + 4\ell \) or \( 8 + 16\ell \), for some nonnegative integer \( \ell \).

Proof. Since \( \mathbb{Z}_2^* \to M \), it follows that \( \mathbb{Z}_2^*, 4\mathbb{Z}_2^* \to L_2 \). Also, for \( x, y \in \mathbb{Z}_2^* \), \( x^2 + xy + y^2 \) lies in either \( \mathbb{Z}_2^* \) or \( 4\mathbb{Z}_2^* \). So all odd integers and integers of the type \( 4n_0 \) with \( n_0 \) odd are represented by \( L_2 \), and no integer of the type \( 2 + 4\ell \) can be represented by \( L_2 \).

Consider \( n = 4n_0 \) with \( n_0 \) even. If there exist \( x, y \in \mathbb{Z}_2^* \) such that

\[
(7.1) \quad n = x^2 + xy + y^2 + 16z^2,
\]
then $x, y \in 2\mathbb{Z}_2$ and the right-hand side of Equation (7.1) is in $4\mathbb{Z}_2$. Hence, no element of the type $8 + 16\ell$ is represented by $L_2$.

Finally, let $n = 16n_0, n_0 \in \mathbb{Z}$. Since $\mathbb{Z}_2^e \to M$, either $n_0$ or $n_0 - 1$ is represented by $M$. So there exist $x_0, y_0, z \in \mathbb{Z}_2$ such that
\[
    n_0 = x_0^2 + x_0y_0 + y_0^2 + z^2.
\]
It follows that Equation (7.1) is satisfied with $x = 4x_0, y = 4y_0$. This completes the proof.

**Lemma 7.2.** Let $L$ be a ternary quadratic $\mathbb{Z}$-lattice such that $L_2 \cong (1, 16, 48)$, and let $n$ be a positive integer. Then $n \nmid L_2$ if and only if $n = 5 + 8\ell, 2 + 4\ell, 3 + 4\ell, 8 + 16\ell$, or $12 + 16\ell$, for some nonnegative integer $\ell$.

**Proof.** Here $n \to L_2$ if and only if there exist $x, y, z \in \mathbb{Z}_2$ such that
\[
    n = x^2 + 16y^2 + 48z^2.
\]
If $2 \nmid n$, then Equation (7.3) is solvable if and only if $n \equiv 1 \pmod{8}$, thus ruling out integers of the type $5 + 8\ell$ and $3 + 4\ell$. If $2 \mid n$, then $x = 2x_0$ for some $x_0 \in \mathbb{Z}_2$, so
\[
    n = 4x_0^2 + 16y^2 + 48z^2.
\]
So it must be that $4 \mid n$, ruling out integers of the type $2 + 4\ell$. Write $n = 4n_0$, with $n_0 \in \mathbb{Z}$. Dividing Equation (7.3) through by 4 then gives
\[
    n_0 = x_0^2 + 4y^2 + 12z^2.
\]
If $2 \mid n_0$, then Equation (7.5) is solvable if and only if $n_0 \equiv 1 \pmod{4}$. This rules out integers of the type $4(3 + 4\ell) = 12 + 16\ell$. If $2 \nmid n_0$ then $x_0 \in 2\mathbb{Z}_2$, and so the right-hand side of Equation (7.5) is in $4\mathbb{Z}_2$. This rules out integers of the type $8 + 16\ell$. Finally, the lattice $(1, 1, 3)$ is isotropic over $\mathbb{Z}_2$ and is therefore $\mathbb{Z}_2$-universal by [6, Proposition 4.1]. Hence, all positive integers divisible by 16 are represented by $L_2$. This completes the proof.

**Lemma 7.3.** Let $L$ be a ternary quadratic $\mathbb{Z}$-lattice such that $L_3 \cong (1, 3, 9)$, and let $n$ be a positive integer. Then $n \nmid L_3$ if and only if $n = 2 + 3\ell$ or $n = 9^k(6 + 9\ell)$, for some nonnegative integers $k, \ell$.

**Proof.** Let $V$ denote the underlying quadratic space. By a computation of Hasse symbols, it follows that $V_3$ is anisotropic; hence $\alpha \not\sim V_3$ for any $\alpha \in -dV = -3Q_3^9$ (see, e.g., [6, Lemma 2.2]). As $9^k(6 + 9\ell) = 3^k \cdot 3(2 + 3\ell) \in -3Q_3^9$, no such integer can be represented by $L_3$. Integers of the type $n = 2 + 3\ell$ are ruled out for representation by $L_3$ by the Local Square Theorem [10, 63:1].

It remains to show that all other integers are represented by $L_3$. Write $n = 3^a n_0$, with $t$ a nonnegative integer and $n_0 \equiv 1, 2 \pmod{3}$. If $n_0 \equiv 1 \pmod{3}$, then there exists $\lambda \in \mathbb{Z}_3^\times$ such that $n_0 = \lambda^2$. So if $t = 0$ and $n_0 \equiv 1 \pmod{3}$ then $n = \lambda^2 \to L_3$.

If $t = 2k + 1$ is odd and $n_0 \equiv 1 \pmod{3}$, then $n = 3(3^k\lambda)^2 \to L_3$. By [10, 92:1b], $\mathbb{Z}_3^\times \to (1, 1)$. So $3^2Z_3^\times \to (1, 3^2)$. It then follows that $n \to L_3$ whenever $t$ is even and $t \geq 2$. This exhausts all cases and completes the proof.

**Proposition 7.4.** The form $3x^2 + 7y^2 + 7z^2 + 3xy + 3xz + 5yz$ represents a positive integer $n$ if and only if $n$ is not of the type
\[
    2 + 3\ell, 9^k(6 + 9\ell), 4^a(2 + 4\ell) or 4^a M_3^\ell,
\]
where $k, \ell$ are nonnegative integers and $a \in \{0, 1\}$.

**Proof.** This is the form B4. Let $L$ be a lattice corresponding to this form and let $n$ be a positive integer. By [10, 92:1b], $n \to L_p$ for all primes $p \neq 2, 3$. For $p = 2$ and $p = 3$, the conditions for representation by $L_p$ are given in Lemmas 7.1 and 7.3 respectively. Consequently, $n \to \text{gen}L$ if and only if $n$ is not of one of the types
2 + 3\ell, 9^k(6 + 9\ell) or 4^a(2 + 4\ell) for some nonnegative integers \(k, \ell\) and \(a \in \{0, 1\}\). If \(n\) is not of one of these types, then \(n \to L\) if and only if \(n\) does not lie in \(M_2^2\) or \(4M_2^2\), by Proposition 5.1(iii).

**Proposition 7.5.** The form \(9x^2 + 16y^2 + 48z^2\) represents a positive integer \(n\) if and only if \(n\) is not of the type

\[5 + 8\ell, 4^a(2 + 4\ell), 4^a(3 + 4\ell)\] or \(4^a M_2^2\),

where \(\ell\) is a nonnegative integer and \(a \in \{0, 1\}\).

**Proof.** This is the form B11. Let \(L\) be a lattice corresponding to this form and let \(n\) be a positive integer. By [10] 92.1b, \(n \to L_p\) for all primes \(p \neq 2, 3\). For \(p = 2\) and \(p = 3\), the conditions for representation by \(L_p\) are given in Lemmas [7,2] and [7,3] respectively. Consequently, \(n \to \text{gen}L\) if and only if \(n\) is not of one of the types \(5 + 8\ell, 4^a(2 + 4\ell)\) or \(4^a(3 + 4\ell)\) for some nonnegative integers \(\ell\) and \(a \in \{0, 1\}\). If \(n\) is not of one of these types, then \(n \to L\) if and only if \(n\) does not lie in \(M_2^2\) or \(4M_2^2\), by Proposition 5.1(iii).

**References**

[1] Z.S. Aygin, G. Doyle, F. Münkel, L. Pehlivan and K.S. Williams, Representation numbers of spinor regular ternary quadratic forms, *Integers* 21 (2021), #A99 (110 pages).

[2] J.W. Benham, A.G. Earnest, J.S. Hsia and D.C. Hung, Spinor regular ternary quadratic forms, *J. London Math. Soc.* 42 (1990), 1–10.

[3] A. Berkovich, On the Gauss EΥPHKA theorem and some allied inequalities, *J. Number Theory* 148 (2015), 1–18.

[4] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* 24 (1997), 235–265.

[5] L.E. Dickson, Ternary quadratic forms and congruences, *Ann. of Math.* 28 (1926/27), 333–341.

[6] A.G. Earnest and G.L.K. Gunawardana, Local criteria for universal and primitively universal quadratic forms, *J. Number Theory* 225 (2021), 260–280.

[7] A.G. Earnest and A.N. Haensch, Completeness of the list of spinor regular ternary quadratic forms, *Mathematika* 65 (2019), 213–235.

[8] A.G. Earnest and J.S. Hsia, Spinor norms of local integral rotations II, *Pacific J. Math.* 61 (1975), 71–86.

[9] J.S. Hsia, Spinor norms of local integral rotations I, *Pacific J. Math.* 57 (1975), 199–206.

[10] O.T. O’Meara, *Introduction to Quadratic Forms*, Springer-Verlag, Berlin, 1973.

[11] B.W. Jones and G. Pall, Regular and semi-regular positive ternary quadratic forms, *Acta Math.* 70 (1940), 165–191.

[12] G.A. Lomadze, Formulas for the number of representations of numbers by certain regular and semiregular ternary quadratic forms belonging to two-class genera (Russian), *Acta Arith.* 34 (1977/78), 131–162.

[13] M. Kneser, Klassenzahlen indefiniter quadratischer Formen in drei oder mehr Veränderlichen, *Arch. Math. (Basel)* 7 (1956), 323–332.

[14] R. Schulze-Pillot, Darstellung durch Spinorgeschlechter ternärer quadratischer Formen, *J. Number Theory* 12 (1980), 529–540.

[15] R. Schulze-Pillot, Exceptional integers for genera of integral ternary positive definite quadratic forms, *Duke Math. J.* 102 (2000), 351–357.

Department of Mathematics, Southern Illinois University, Carbondale, IL, 62901, U.S.A.

Email address: aeearnest@siu.edu