In this article, we study the inverse mean value property of solutions to the modified Helmholtz equation. We prove a theorem that characterizes analytically balls in the Euclidean space $\mathbb{R}^m$. For this purpose, we use positive solutions of the modified Helmholtz equation instead of harmonic functions applied in previous results. The obtained Kuran type theorem is based on the volume mean value property of solutions to this equation.

**Abstract**

A theorem characterizing analytically balls in the Euclidean space $\mathbb{R}^m$ is proved. For this purpose, positive solutions of the modified Helmholtz equation are used instead of harmonic functions applied in previous results. The obtained Kuran type theorem is based on the volume mean value property of solutions to this equation.

**1 Introduction and main result**

In 1972 Kuran [5] proved the following inverse of the volume mean value theorem for harmonic functions:

Let $D$ be a domain (= connected open set) of finite (Lebesgue) measure in the Euclidean space $\mathbb{R}^m$ where $m \geq 2$. Suppose that there exists a point $P_0$ in $D$ such that, for every function $h$ harmonic in $D$ and integrable over $D$, the volume mean of $h$ over $D$ equals $h(P_0)$. Then $D$ is an open ball (disk when $m = 2$) centred at $P_0$.

The result was originally obtained by Epstein [2] for a simply connected two-dimensional $D$. Armitage and Goldstein [1] proved this result assuming that the mean value equality holds only for positive harmonic functions which are $L^p$-integrable, $p \in (0, n/(n-2))$. Hansen and Netuka [3] considered some particular class of potentials as the set of test harmonic functions in Kuran’s theorem. A slight modification of his considerations shows that Kuran’s theorem is valid even if $D$ is disconnected; see [8], p. 377.

In the survey article [8], one finds also a discussion of applications of Kuran’s theorem and a possibility of similar results involving some kinds of average over $\partial D$, where $D$ is a bounded domain. One of them (due to Kosmodem’yanuskii [4]) is based on the relation similar to that between the mean values over balls and spheres and reads as follows:

Let $D \subset \mathbb{R}^2$ be a bounded, convex $C^2$-domain. If the equality

$$\frac{1}{|D|} \int_D u(x) \, dx = \frac{1}{|\partial D|} \int_{\partial D} u(x) \, dS_x$$

holds for every function $u \in C^2(D) \cap C^1(\overline{D})$ which is harmonic in $D$, then $D$ is an open disc.
Here and below $|D|$ is the domain’s area (volume if $D \subset \mathbb{R}^m$, $m \geq 3$), whereas $|\partial D|$ is the boundary’s length (area if $D \subset \mathbb{R}^m$, $m \geq 3$), and $|B_r| = \omega_mr^m$ is the volume of a ball $B_r$ of radius $r$; the volume of unit ball is $\omega_m = 2\pi^{m/2}/[m\Gamma(m/2)]$, whereas $\Gamma$ denotes the Gamma function.

In this note, we prove a new analytic characterization of balls. Like Kuran’s theorem, it is based on the $m$-dimensional volume mean value equality, but uses solutions of the modified Helmholtz equation

$$\nabla^2 u - \lambda^2 u = 0, \quad \lambda \in \mathbb{R} \setminus \{0\}$$

instead of harmonic functions; here $\nabla = (\partial_1, \ldots, \partial_m)$ denotes the gradient operator and $\partial_i = \partial/\partial x_i$. Solutions are assumed to be real; indeed, the obtained results can be extended to complex-valued functions by considering the real and imaginary part separately.

Before giving the precise formulation of the main result, let us introduce some notation. By $B_r(x) = \{ y : |y - x| < r \}$ we denote the open ball of radius $r$ centred at $x \in \mathbb{R}^m$; if $D \subset \mathbb{R}^m$ is a bounded domain, then $D_r = D \cup \{ \cup_{x \in \partial D} B_r(x) \}$ is its dilated copy such that the distance from $\partial D_r$ to $D$ is equal to $r$. For a function $f$ integrable over $D$, which has finite Lebesgue measure,

$$M(f, D) = \frac{1}{|D|} \int_D f(x) \, dx$$

denotes its volume mean value over $D$. Also, we need the following function

$$a(t) = \Gamma \left(\frac{m}{2} + 1\right) \frac{I_{m/2}(t)}{(t/2)^{m/2}},$$

where $I_\nu$ stands for the modified Bessel function of order $\nu$. The relation

$$[z^{-\nu} I_\nu(z)]' = z^{-\nu} I_{\nu+1}(z) \quad (\text{see [12], p. 79}),$$

where the right-hand side is positive for $z > 0$ and vanishes at $z = 0$, implies that the function $a$ increases monotonically on $[0, \infty)$ from $a(0) = 1$ to infinity; the latter is a consequence of the asymptotic formula valid as $|z| \to \infty$:

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left[ 1 + O(|z|^{-1}) \right], \quad |\arg z| < \pi/2 \quad (\text{see [12], p. 80}).$$

The function $a$ arises in the $m$-dimensional mean value formula for balls

$$a(\lambda r) u(x) = \frac{1}{|B_r|} \int_{B_r(x)} u(y) \, dy, \quad x \in D,$$

which holds, for example, if $u \in C^0(D)$ is a solution of (1) in $D$ and $B_r(x) \subset D$. This equality was obtained by the author recently; see [6], p. 95. Before that only the three-dimensional mean value formula for spheres had been derived by C. Neumann (see his book [10], Chapter 9, Section 3, published in 1896), whereas the $m$-dimensional formula for spheres was given without proof in [11]; its derivation see in the author’s note [7].

Now, we are in a position to formulate the main result.

**Theorem 1.** Let $D \subset \mathbb{R}^m$, $m \geq 2$, be a bounded domain such that its complement is connected, and let $r$ be a positive number such that $|B_r| \leq |D|$. Suppose that there exists a point $x_0 \in D$ such that for some $\lambda > 0$ the mean value equality $u(x_0) a(\lambda r) = M(u, D)$ holds for every positive function $u$ satisfying equation (1) in $D_r$. If also $|D| = |B_r|$ provided $B_r(x_0) \setminus \overline{D} \neq \emptyset$, then $D = B_r(x_0)$. 

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2 Proof of Theorem 1 and discussion

Prior to proving Theorem 1, we introduce the following function

\[ U(x) = \Gamma\left(\frac{m}{2}\right) \frac{I_{(m-2)/2}(\lambda|x|)}{(\lambda|x|/2)^{(m-2)/2}}, \quad x \in \mathbb{R}^m, \tag{5} \]

where the coefficient is chosen so that \( U(0) = 1 \). Let us consider some of its properties. According to (3), this spherically symmetric function monotonically increases as \(|x|\) goes from zero to infinity. Also, it solves equation (1) in \( \mathbb{R}^m \); indeed, the representation

\[ U(x) = \frac{2 \Gamma(m/2)}{\sqrt{\pi} \Gamma((m-1)/2)} \int_0^1 (1 - s^2)^{(m-3)/2} \cosh(\lambda|x|s) \, ds, \tag{6} \]

is easy to differentiate, thus verifying (1). This formula for \( U \) is a consequence of Poisson’s integral (see [9], p. 223):

\[ I_\nu(z) = \left(\frac{z/2}{\nu}\right)^\nu \sqrt{\pi} \Gamma(\nu+1/2) \int_{-1}^1 (1 - s^2)^{\nu-1/2} \cosh zs \, ds. \]

Moreover, (6) takes particularly simple form for \( m = 3 \), namely, \( U(x) = (\lambda|x|)^{-1} \sinh \lambda|x| \). Since formulae (2) and (5) are similar, Poisson’s integral allows us to compare these functions. Indeed, the inequality

\[ [U(x)]|_{|x|=r} > a(\lambda r) \tag{7} \]

immediately follows because \( U(x) \) is spherically symmetric.

Proof of Theorem 1. Without loss of generality, we suppose that the domain \( D \) is located so that \( x_0 \) coincides with the origin. Let us show that the assumption that \( D \neq B_r(0) \) leads to a contradiction.

It is clear that either \( B_r(0) \subset D \) or \( B_r(0) \setminus \overline{D} \neq \emptyset \) (the equality \(|B_r| = |D|\) is assumed in the latter case), and we treat these two cases separately. Let us consider the second case first and introduce the bounded open sets \( G_i = D \setminus B_r(0) \) and \( G_e = B_r(0) \setminus \overline{D} \), for which we have \(|G_e| = |G_i| \neq 0\) in view of assumptions about \( D \) and \( r \). The volume mean equality for \( U \) over \( D \) can be written as follows:

\[ |D| a(\lambda r) = \int_D U(y) \, dy; \tag{8} \]

here the condition \( U(0) = 1 \) is taken into account. Since property (1) holds for \( U \) over \( B_r(0) \), we write it in the same way:

\[ |B_r| a(\lambda r) = \int_{B_r(0)} U(y) \, dy. \tag{9} \]

Subtracting (9) from (8), we obtain

\[ 0 = \int_{G_i} U(y) \, dy - \int_{G_e} U(y) \, dy > 0. \]
Indeed, the difference is positive since $U(y)$ (positive and monotonically increasing with $|y|$) is greater than $|U(y)|_{|y|=r}$ in $G_i$ and less than $|U(y)|_{|y|=r}$ in $G_e$, whereas $|G_i| = |G_e|$. This contradiction proves the result in this case.

In the case when $B_r(0) \subset D$, we also have to obtain a contradiction when $B_r(0) \neq D$. Now, after subtraction of (9) from (8) we have

$$((|D| - |B_r|) a(\lambda r) = \int_{G_i} U(y) \, dy > |G_i| \cdot |U(y)|_{|y|=r},$$

where the last inequality is again a consequence of positivity of $U(y)$ and the fact that it increases monotonically with $|y|$. It is clear that $|G_i| = |D| - |B_r| > 0$ because $B_r(0) \subset D$. Therefore, $a(\lambda r) > |U(y)|_{|y|=r}$, which contradicts (7). The proof is complete.

In the limit $\lambda \to 0$, equation (11) turns into Laplace’s, whose solutions are harmonic functions; moreover, the assumption about $r$ becomes superfluous in this case. Thus, letting $\lambda \to 0$ in Theorem 1 leads to an improved formulation of Kuran’s theorem because only positive harmonic functions are involved; see also [1].

The reason why $D$ is supposed to be bounded in Theorem 1 is as follows. It is easy to construct an unbounded domain of finite volume such that $U$ is not integrable over it, and so boundedness of $D$ allows us to avoid formulating rather complicated restrictions on the domain.

In the case of sufficiently smooth $\partial D$, the integral $\int_D u(y) \, dy$ can be replaced by the flux integral $\int_{\partial D} \partial u/\partial n_y \, dS_y$ in the formulation of Theorem 1; here $n$ is the exterior unit normal. Indeed, we have

$$\int_D u(y) \, dy = \lambda^{-2} \int_D \nabla^2 u(y) \, dy = \lambda^{-2} \int_{\partial D} \partial u/\partial n_y \, dS_y.$$

This suggests that the following mean flux equality

$$\frac{\lambda^2 r}{2 \Gamma(\frac{m}{2} + 1)} a(\lambda r) u(x_0) = \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial u}{\partial n_y} \, dS_y$$

(cf. formula (31) in [4]) may also characterize the ball of radius $r$ centred at $x_0 \in D$ provided $D$ has a smooth boundary, the point $x_0$ exists and the equality holds for every solution of equation (11) in $D_r$.

In conclusion we notice that the equality (see [6], Theorem 8)

$$m I_{m/2}(\lambda r) \int_{\partial B_r(x)} u(y) \, dS_y = \lambda r I_{(m-2)/2}(\lambda r) \int_{B_r(x)} u(y) \, dy$$

holds for every point $x$ belonging to a domain $D \subset \mathbb{R}^m$ and all $r$ such that $B_r(x) \subset D$ if and only if $u$ is a solution of equation (11) in $D$. This is analogous to the equality of the mean values over spheres and balls for harmonic functions. In view of Kosmodem’yanskii’s theorem, one might expect that this equality with $B_r(x)$ changed to $D$ characterizes balls in $\mathbb{R}^m$ provided it is valid for every solution of equation (11) in $D$. 

4
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