Lipschitz functions with prescribed blowups at many points

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Abstract. In this paper we prove generalizations of Lusin-type theorems for gradients due to Giovanni Alberti, where we replace the Lebesgue measure with any Radon measure $\mu$. We apply this to go beyond the known result on the existence of Lipschitz functions which are non-differentiable at $\mu$-almost every point $x$ in any direction which is not contained in the decomposability bundle $V(\mu, x)$, recently introduced by Alberti and the first author. More precisely, we prove that it is possible to construct a Lipschitz function which attains any prescribed admissible blowup at every point except for a closed set of points of arbitrarily small measure. Here a function is an admissible blowup at a point $x$ if it is null at the origin and it is the sum of a linear function on $V(\mu, x)$ and a Lipschitz function on $V(\mu, x)^\perp$.

Keywords: Lipschitz function, Radon measure, blowup, Lusin type approximation.

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1. Introduction

In [Alb91], Alberti proved a “Lusin type theorem for gradients”: roughly speaking, given any Borel vectorfield $f$ on the Euclidean space $\mathbb{R}^N$ one can find a $C^1$ function $g$ whose gradient coincides with $f$ up to an exceptional open set of arbitrarily small Lebesgue measure. In other words one can prescribe at many points the (unique) blowup of a $C^1$ function in an arbitrary (measurable) way. Rademacher Theorem, which states that Lipschitz functions are differentiable almost everywhere, implies that, even if one weakens the assumptions on $g$, requiring it only to be locally Lipschitz, Alberti’s result is still the best possible: no other blowups than the linear ones can be prescribed on a set of points of positive measure; moreover, one cannot get rid of the small exceptional set without any further assumption on $f$, i.e. in general one cannot find a Lipschitz function $g$ such that the measure of the set $\{Dg \neq f\}$ is zero. However a continuous function $g$ with such property can be found (see [MP08]).

In the present paper, we prove a generalization of Alberti’s result, where the Lebesgue measure is replaced by any Radon measure. Since Rademacher Theorem does not hold in general with respect to a Radon measure and in particular it fails with respect to any singular measure (see Theorem 1.14 of [DR16]), then the following vague question is very natural in our setting. Given a measure $\mu$ on $\mathbb{R}^N$, which blowups is it possible to prescribe for a Lipschitz function, at many points with respect to $\mu$, besides the linear ones?
Let us introduce some basic notations to make the question more precise. We denote by $B(x, r)$ the ball with center $x \in \mathbb{R}^N$ and radius $r > 0$. We simply write $B$ for the unit ball centred at the origin.

**Definition 1.1 (Blowups of a Lipschitz function).** Given a Lipschitz function $g$ defined on an open subset $\Omega \subset \mathbb{R}^N$, and a point $x \in \Omega$, we denote by $\text{Tan}(g, x)$ the set of all the possible limits, with respect to the uniform convergence,

$$
\lim_{j \to \infty} T_{x, r_j} f,
$$

where $r_j \downarrow 0$ and for every $r \leq \text{dist}(x, \Omega^c)$, $T_{x, r} f : B \to \mathbb{R}$ is defined by

$$
T_{x, r} f(y) := r^{-1} (f(x + ry) - f(x)), \quad \text{for every } y \in B.
$$

**Definition 1.2 (Prescribing blowups).** Let $\mu$ be a positive Radon measure on an open set $\Omega \subset \mathbb{R}^N$. Denote by $\text{Lip}(B, 0)$ the space of Lipschitz functions on $B$ which vanish at the origin, endowed with the supremum distance, and let $f : \Omega \subset \mathbb{R}^N \to \text{Lip}(B, 0)$ be a Borel function. We say that $f$ *prescribes the blowups* of a Lipschitz function with respect to $\mu$:

(i) *Weakly*, if there exists a Lipschitz function $g : \Omega \to \mathbb{R}$ such that $f(x) \in \text{Tan}(g, x)$ for $\mu$-a.e. $x \in \Omega$;

(ii) *Weakly in the Lusin sense*, if for every $\varepsilon > 0$ there exists a Lipschitz function $g : \Omega \to \mathbb{R}$ such that

$$
\mu(\{x \in \Omega : f(x) \notin \text{Tan}(g, x)\}) < \varepsilon;
$$

(iii) *Strongly in the Lusin sense*, if for every $\varepsilon > 0$ there exists a Lipschitz function $g : \Omega \to \mathbb{R}$ such that

$$
\mu(\{x \in \Omega : \{f(x)\} \neq \text{Tan}(g, x)\}) < \varepsilon.
$$

In this paper we mainly address the following question.

**Question 1.3.** Given a positive Radon measure $\mu$ on an open set $\Omega \subset \mathbb{R}^N$ with $\mu(\Omega) < \infty$, for which choice of $f$ is it possible to say that $f$ prescribes the blowups of a Lipschitz function wrt $\mu$ weakly/strongly/in the Lusin sense?

As we already observed, when $\mu$ is the Lebesgue measure, Rademacher Theorem is a constraint on the possible choices of a function $f$ for which Question 1.3 may have a positive answer. Namely, in this case, for a.e. point $x$, the corresponding function $f(x)$ must be linear (more precisely, the restriction to $B$ of a linear function). Using the notation that we have introduced above, the content of Alberti’s result, or at least part of it, can be rephrased as follows: if $\Omega \subset \mathbb{R}^N$ is an open set with finite Lebesgue measure, then every Borel function $f : \Omega \to \text{Lip}(B, 0)$ whose values are (restrictions to $B$ of) linear functions almost everywhere, prescribes the blowups of a Lipschitz function wrt the Lebesgue measure strongly in the Lusin sense.

For a general measure $\mu$ Rademacher Theorem does not hold. In particular De Philippis and Rindler in [DR16] completed the proof, also based on other works, that there are Lipschitz functions which are non-differentiable at $\mu_{\text{sing}}$-a.e. point, where $\mu_{\text{sing}}$ is the singular part of $\mu$ wrt Lebesgue. Nevertheless a
suitable weaker version of Rademacher Theorem holds. Indeed, letting $\text{Gr}(\mathbb{R}^N)$ denote the union of the Grassmannians of all vector subspaces of $\mathbb{R}^N$, in [AM16] it is proved that to every Radon measure $\mu$ on $\mathbb{R}^N$ it is possible to associate a Borel function $V(\mu, \cdot) : \mathbb{R}^N \to \text{Gr}(\mathbb{R}^N)$ called the decomposability bundle of $\mu$, with the property that for every Lipschitz function $g$, the restriction of $g$ to the affine subspace $x + V(\mu, x)$ is differentiable at $\mu$-a.e. point $x$ and moreover the bundle is maximal with respect to this property, meaning that there exists a Lipschitz function which is non-differentiable at $\mu$-a.e. point $x$ along any direction which is not in $V(\mu, x)$. In virtue of [AM16, Theorem 1.1 (ii)], the proof that Rademacher Theorem does not hold for singular measures reduces to proving that every singular measure on $\mathbb{R}^N$ has decomposability bundle of dimension at most $N - 1$. This is achieved in [DR16] using a characterization of the decomposability bundle given in [AM16, Theorem 6.4].

Clearly the main result of [AM16] is also a constraint on the possible choices of a function $f$ for which one can expect a positive answer to Question 1.3, indeed one should at least require that $f(x)$ is linear on $V(\mu, x)$ for $\mu$-a.e. point $x$. This observation partially motivates the introduction of the following subset of $\text{Lip}(B, 0)$. Given a vector subspace $V$ of $\mathbb{R}^N$ and a point $y \in \mathbb{R}^N$ we denote respectively $y_V$ and $y_{V^\perp}$ the projections on $V$ and on its orthogonal complement $V^\perp$. Finally we denote the class of admissible blowups by

$$C(\mu, x) := \{ h \in \text{Lip}(B, 0) : h(y) = L(y_V(\mu, x)) + m(y_{V^\perp}(\mu, x)) \}, \quad (1.1)$$

where $L$ is a linear function on $V(\mu, x)$ and $m$ is a Lipschitz function on $V(\mu, x)^\perp$. Now we are ready to state the main results of the paper.

**Theorem 1.4.** Let $\mu$ be a Radon measure on $\mathbb{R}^N$, let $\Omega \subset \mathbb{R}^N$ be an open set with $\mu(\Omega) < \infty$, and let $f$ be as in Definition 1.2. Then the following statements hold:

(I) if $f(x) = L(x)$ at $\mu$-a.e $x$, where $L(x)$ is the restriction to $B$ of a linear function, then $f$ prescribes the blowups of a Lipschitz function with respect to $\mu$ strongly in the Lusin sense;

(II) if $f(x) \in C(\mu, x)$ for $\mu$-a.e. $x$, then $f$ prescribes the blowups of a Lipschitz function with respect to $\mu$ weakly in the Lusin sense.

In Section 1 we exhibit a measure $\mu$ for which one cannot prescribe more blowups that those contained in the class $C(\mu, \cdot)$, proving the sharpness of (II). For $N = 1$ we can prove a stronger statement. In particular we don’t need the restriction that $\Omega$ has finite measure. Firstly we show that the only measures for which one can prescribe strongly some non-linear blowups are the atomic ones. Secondly we prove that any blowup can be prescribed weakly wrt a singular measure $\mu$. More precisely, for the typical 1-Lipschitz function $g$ (in the sense of Baire categories), $\text{Tan}(g, x)$ coincides with the set of all 1-Lipschitz functions in $\text{Lip}(B, 0)$, at $\mu$-a.e. point $x$. 


Given a Borel set $E$, we denote by $\mu \ll E$ the measure defined by

$$\mu \ll E(A) := \mu(A \cap E),$$

for every Borel set $A$.

**Theorem 1.5.** Let $\Omega \subset \mathbb{R}$ be an open set, and let $\mu$ and $f$ be as in Definition 1.2. Then the following statements hold:

(I) $f$ prescribes the blowups of a Lipschitz function with respect to $\mu$ strongly in the Lusin sense, if and only if $f(x)$ is the restriction to $B$ of a positively homogeneous function, for $\mu$-a.e. $x$ and $\mu \ll NL$ is atomic, where

$$NL := \{x \in \Omega : f(x) \text{ is not the restriction to } B \text{ of a linear function}\};$$

(II) if $\mu$ is singular, then $f$ prescribes the blowups of a Lipschitz function with respect to $\mu$ weakly;

**Remark 1.6.**

(i) Statement (I) in Theorem 1.5 is a generalization of Theorem 1 of [Alb91]. In Section 2 we prove a more precise version of this statement, including the possibility to choose the Lipschitz function $g$ in point (iii) of Definition 1.2 of class $C^1$, with arbitrarily small $L^\infty$ norm and with $L^p$ estimates on its gradient for every $p \in [1, \infty]$. A similar result was recently proved by David in [Dav15] in the setting of PI spaces: a class of metric measure spaces which admit a differentiable structure. We point out that statement (I) can also be extended to doubling metric measure spaces, where the differentiable structure is defined using operators called derivations: we will not pursue this issue in the present paper.

(ii) The difference between statement (II) of Theorem 1.5 and statement (II) of Theorem 1.4 is twofold. Firstly, in the 1-dimensional case there is no restriction on the function $f$, due to the fact that the decomposability bundle of a singular measure in $\mathbb{R}$ is always trivial. Secondly, we remark that the blowups in the 1-dimensional case are prescribed weakly, while in the general case are prescribed only weakly in the Lusin sense. More precisely, in dimension $N = 1$ we are able to prove that residually many 1-Lipschitz functions attain, in a set of full measure, every 1-Lipschitz function in Lip$(B, 0)$ as a blowup.

(iii) Statement (I) of Theorem 1.5 is a simple observation, which is already contained in Proposition 4.2 of [Mar17]. The restriction to the family of positively homogeneous functions is clearly necessary in order to prescribe blowups strongly, because if $f \in \text{Tan}(g, x)$ and $h \in \text{Tan}(f, 0)$, then it also holds $h \in \text{Tan}(g, x)$. Presumably, also in dimension larger than 1 the possibility to prescribe strongly some non-linear blowups in the Lusin sense should depend on some property of the measure and intuitively it should fail when the measure is “very diffused”. On the other side, it sounds reasonable that if the measure $\mu$ is supported on a $k$-rectifiable set $E$ in $\mathbb{R}^N$ ($k < N$) then any Borel function $f$ which is $\mu$-a.e. positively homogeneous and linear along the tangent bundle to $E$ prescribes the
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blowups of a Lipschitz function strongly in the Lusin sense. However we
do not pursue this issue in this paper.

On the structure of the paper. The proof of Theorem 1.4 is split in Section
2 for statement (I) and Section 3 for statement (II). In Section 4 we provide an
example of a measure $\mu$ for which every blowup of a Lipschitz function is the
sum of a linear function on $V(\mu, x)$ and a Lipschitz function on its orthogonal,
at $\mu$-a.e. point $x$, in order to justify the choice of the class $C(\mu, \cdot)$ appearing in
statement (II) of Theorem 1.4. Finally, in Section 5 we prove Theorem 1.5.

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2. Proof of Theorem 1.4(I)

In this section we prove statement (I) of Theorem 1.4. As anticipated in Re-
mark 1.6 (i) we will actually prove a stronger statement including some gradient
estimates. In particular $L^\infty$ estimates on the function $g$ of Definition 1.2 and
its gradient are necessary to prove part (II) of Theorem 1.4. The proof of state-
ment (I) is very similar to the one presented in [Alb91]. The new main technical
ingredient is Corollary 2.3.

Theorem 2.1. Let $\mu$ be a Radon measure on $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ open with
$\mu(\Omega) < \infty$. Then for every Borel map $f : \Omega \to \mathbb{R}^N$ and for every $\varepsilon, \zeta > 0$ there
exist a compact set $K$ and a function $g \in C^1_c(\Omega)$ with $\|g\|_\infty \leq \zeta$ such that

$$\mu(\Omega \setminus K) < \varepsilon \mu(\Omega).$$

Moreover, there exists $C = C(N)$ such that

$$\|Dg(x)\|_p \leq C\varepsilon^{1/p-1} \|f\|_p, \text{ for every } p \in [1, \infty],$$

where $\|\cdot\|_p$ denotes the usual norm in $L^p(\Omega, \mu)$.

Let $B_Z$ denote the collection of boxes in $\mathbb{R}^N$ of the form $\prod_{i=1}^N [2n_i - 1, 2n_i + 1]$ for $(n_i)_{i=1}^N \in \mathbb{Z}^N$. For $r > 0$ let $B_Z(r)$ denote the transform of $B_Z$ when the
dilation $x \mapsto rx$ is applied to $\mathbb{R}^N$. For $x \in \mathbb{R}^N$ and $r > 0$ we denote the box with
center $x$ and edge length $2r$ by

$$B_x(x, r) := \left\{ y \in \mathbb{R}^N : \max_{1 \leq i \leq N} |x_i - y_i| \leq r \right\},$$

and for $0 < \varepsilon < 1$ we define the “frame”:

$$Fr(x, r, \varepsilon) := \left\{ y \in B_x(x, r) : |x_i - y_i| \geq (1 - \varepsilon)r \text{ for some } i \in \{1, \cdots, N\} \right\}.$$
Finally we consider the probability measure \( P := \frac{1}{(2r)^N} \mathcal{L}^N \ll B(x, 0, r) \) and for \( \omega \in B(x, 0, r) \) we let \( B_Z(r, \omega) := B_Z(r) + \omega \).

**Lemma 2.2** (Existence of boxes with negligible frames). Let \( K \) be compact with \( \mu(K) > 0 \) and \( U \supseteq K \) open with \( \mu(U) \leq \frac{3}{4} \mu(K) \). Assume that \( r > 0 \) is such that for each \( B(x, r) \) which intersects \( K \) one has \( B(x, r) \subset U \). For \( \varepsilon > 0 \) and \( \omega \in B(x, 0, r) \) define

\[
B_{g, \omega}^{\text{good}} := \left\{ B(x, r) \in B_Z(r, \omega) : B(x, r) \cap K \neq \emptyset \right\} \quad \text{and} \quad \mu(\text{Fr}(x, r, \varepsilon)) \leq 16N^2 \varepsilon \mu(B(x, r))
\]  

(2.6)

Then for some \( \omega \in B(x, 0, r) \) one has:

\[
\mu(\bigcup B_{g, \omega}^{\text{good}} \cap K) \geq \frac{3}{4} \mu(K).
\]  

(2.7)

**Proof.** We define the \( \varepsilon \)-boundaries of a family of boxes \( G \)

\[
\partial_\varepsilon G := \{ \text{Fr}(x, r, \varepsilon) : B(x, r) \in G \}
\]  

(2.8)

the set of bad boxes

\[
B_{b, \omega}^{\text{bad}} := \left\{ B(x, r) \in B_Z(r, \omega) : B(x, r) \cap K \neq \emptyset \quad \text{and} \quad \mu(\text{Fr}(x, r, \varepsilon)) > 16N^2 \varepsilon \mu(B(x, r)) \right\}
\]  

(2.9)

and the set of bad \( \omega \)'s:

\[
A_{\text{bad}} := \left\{ \omega \in B(x, 0, r) : \mu(\bigcup B_{b, \omega}^{\text{bad}}) \geq \frac{1}{6} \mu(U) \right\}
\]  

(2.10)

The Lemma is proven by showing that \( P(A_{\text{bad}}) < 1 \). Define:

\[
I := \int \chi_{\bigcup \partial_\varepsilon B_{b, \omega}^{\text{bad}}} (x) \chi_{A_{\text{bad}}} (\omega) \, d\mu(x) \, dP(\omega),
\]  

(2.11)

where \( \chi_E \) denotes the characteristic function of the set \( E \), with values 0 and 1. We estimate \( I \) from below integrating first in \( d\mu(x) \):

\[
I = \int \mu(\bigcup \partial_\varepsilon B_{b, \omega}^{\text{bad}}) \chi_{A_{\text{bad}}} (\omega) d\mu(x) dP(\omega)
\]

\[
\geq 16N^2 \varepsilon \int \mu(\bigcup B_{b, \omega}^{\text{bad}}) \chi_{A_{\text{bad}}} (\omega) dP(\omega)
\]  

(2.12)

\[
\geq \frac{16N^2 \varepsilon}{6} \mu(U) P(A_{\text{bad}}).
\]

We estimate \( I \) from above integrating first in \( dP(\omega) \):

\[
I = \int P(\omega \in A_{\text{bad}} : x \in \bigcup \partial_\varepsilon B_{b, \omega}^{\text{bad}}) d\mu(x).
\]  

(2.13)
For fixed $x$ the set $\{\omega : x \in \bigcup \partial \epsilon B_{\epsilon,\omega}^{\text{bad}}\}$ has positive $P$-measure only if $x \in U$ and the one must also have $\omega \in \bigcup_{j=1}^{2N} \text{Fr}(\bar{x}_j, r, \epsilon)$ where the $\{\bar{x}_j\}$ depend only on $x$. As the Lebesgue measure on $\text{Fr}(\bar{x}_j, r, \epsilon)$ is at most $2N(2r)^N \epsilon$ we get

$$I \leq \epsilon (2N)2^N \mu(U) \quad (2.14)$$

and so $P(A_{\text{bad}}) \leq \frac{12}{16} \mu(U) < 1$. Finally for $\omega \in A_{\text{bad}}^c$ we observe:

$$\mu(K \cap \bigcup B_{\epsilon,\omega}^{\text{good}}) \geq \mu(K) - \mu \bigl( \bigcup B_{\epsilon,\omega}^{\text{bad}} \bigr) \geq \frac{3}{4} \mu(K). \quad (2.15)$$

By a standard covering argument, we deduce the following

**Corollary 2.3** (Covering by good boxes). Let $\epsilon > 0$ and $r_0 > 0$; then for every open set $U \subset \mathbb{R}^N$ there is a sequence of disjoint boxes $\{Bx(z_\lambda, r_\lambda)\}_\lambda$ contained in $U$ such that:

$$r_\lambda \leq r_0 \quad (2.16)$$

$$\mu(\text{Fr}(z_\lambda, r_\lambda, \epsilon)) \leq 16N2^N \epsilon \mu(Bx(z_\lambda, r_\lambda)) \quad (2.17)$$

$$\mu(U \setminus \bigcup_\lambda Bx(z_\lambda, r_\lambda)) = 0. \quad (2.18)$$

To prove Theorem 2.1 it is sufficient to perform a straightforward iteration of Lemma 2.4 below. For the proof of the lemma, after we have established Corollary 2.3, we can easily adapt the proof given in [Alb91].

**Lemma 2.4.** Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure $\mu$. Let $f : \Omega \rightarrow \mathbb{R}^N$ be a bounded and continuous function. Then for every $\xi, \eta, \zeta > 0$ there exists a compact set $K \subset \Omega$ and a function $g \in C^1_c(\Omega)$ such that

$$\mu(\Omega \setminus \text{Int}(K)) < \xi \mu(\Omega), \quad (2.19)$$

$$\|g\|_\infty \leq \zeta, \quad (2.20)$$

$$|f(x) - Dg(x)| \leq \eta, \text{ for every } x \in K. \quad (2.21)$$

Moreover there exists $C = C(N)$ such that

$$\|Dg\|_p \leq C \xi^{1/p-1} \|f_{\text{spt}(g)}\|_p, \text{ for every } p \in [1, \infty], \quad (2.22)$$

where $\text{spt}(g)$ is the support of the function $g$.

**Proof.** Suppose $\xi < 1$. Let $K'$ be a compact subset of $\Omega$ such that

$$\mu(\Omega \setminus K') < \mu(\Omega) \xi / 3. \quad (2.23)$$

Let $d' := \text{dist}(K', \mathbb{R}^N \setminus \Omega)$ and $d := \min\{1, d'/2\}$. Denote by $K''$ the compact set

$$K'' := \{x \in \Omega : \text{dist}(x, K') \leq d\}.$$

Since $f$ is uniformly continuous in $K''$, there exists $0 < \delta < d$ such that for all $x \in K''$, $y \in \Omega$ it holds

$$|x - y| < \delta \implies |f(x) - f(y)| < \eta. \quad (2.24)$$
Consider the family of boxes \( \{ B(x_i, r_i) \} \) obtained applying Corollary 2.3 with \( U = \Omega \) and the choice of parameters

\[
r_0 = \min \left\{ \frac{\delta}{2N}, \frac{\zeta}{N\|f\|_{\infty}} \right\} \quad \text{and} \quad \varepsilon = \frac{\xi}{48N^2}. \tag{2.25}\]

Let

\[
\{ B(x_1, r_1), \ldots, B(x_M, r_M) \}
\]

be a finite subfamily such that \( B(x_i, r_i) \cap K' \neq \emptyset \) for \( i = 1, \ldots, M \) and

\[
\mu \left( K' \setminus \bigcup_{i=1}^{M} B(x_i, r_i) \right) < \mu(\Omega) / 3. \tag{2.26}\]

For \( i = 1, \ldots, M \), let \( \phi_i \in C^1(\Omega) \) such that \( 0 \leq \phi_i \leq 1 \), \( \phi_i \equiv 1 \) in the set \( B(x_i, r_i) \setminus \text{Fr}(x_i, r_i, \varepsilon) \), \( \phi_i \equiv 0 \) outside \( B(x_i, r_i) \) and

\[
\|D\phi_i\|_{\infty} \leq \frac{2}{r_i \varepsilon}. \tag{2.27}\]

Denoting

\[
a_i := \int_{B(x_i, r_i)} f \, d\mu \quad \text{and} \quad a_i = \frac{\int_{B(x_i, r_i)} f \, d\mu}{\mu(B(x_i, r_i))},
\]

we set, for all \( x \in \Omega \)

\[
g(x) := \sum_i \phi_i(x)(a_i, x - x_i).
\]

We finally set

\[
K := \bigcup_{i=1}^{M} \text{cl}(B(x_i, r_i) \setminus \text{Fr}(x_i, r_i, \varepsilon)).
\]

It is easy to see that \( g \in C^1(\Omega) \) and \( \|g\|_{\infty} \leq N r_0 \|f\|_{\infty} \), hence property \( (2.20) \) follows from the choice of \( r_0 \) in \( (2.25) \). Property \( (2.19) \) follows from the inequality

\[
\mu(\Omega \setminus \text{Int}(K)) \leq \mu(\Omega \setminus K') + \mu(K' \setminus \left( \bigcup_{i=1}^{M} B(x_i, r_i) \right)) + \mu(\left( \bigcup_{i=1}^{M} B(x_i, r_i) \right) \setminus \text{Int}(K))
\]

by applying \( (2.23) \), \( (2.26) \) and \( (2.17) \), paired with the choice of \( \varepsilon \) in \( (2.25) \). Property \( (2.21) \) follows from \( (2.24) \) by the choice of \( r_0 \) in \( (2.25) \) and that of \( a_i \). To prove \( (2.22) \), in the case \( p \in [1, \infty) \), we compute, using \( (2.27) \) and the definition of \( g \),

\[
\|Dg\|_p^p \leq \sum_i \int_{B(x_i, r_i) \setminus \text{Fr}(x_i, r_i, \varepsilon)} |a_i|^p d\mu + \int_{\text{Fr}(x_i, r_i, \varepsilon)} (2N|a_i|r_i)^p(2/(r_i \varepsilon))^p d\mu.
\]

Combining with \( (2.17) \) we have

\[
\|Dg\|_p^p \leq \sum_i \mu(B(x_i, r_i))(|a_i|^p + 16N^2 \varepsilon^{1-p}(2N|a_i|)^p)
\]

and

\[
\|Dg\|_p^p \leq \sum_i \mu(B(x_i, r_i))(|a_i|^p + 16N^2 \varepsilon^{1-p}(2N|a_i|)^p)
\]
and by the definition of $a_i$, this implies
\[ \|Dg\|_p^p \leq C\varepsilon^{1-p} \sum_i \left( \int_{B(x_i,r_i)} |f| d\mu \right)^p. \]

Finally, by Jensen’s inequality, we get (2.22). The case $p = \infty$ follows immediately from (2.27).

**Proof of Theorem 2.1** Suppose that $\varepsilon < 1$ and $f$ is not $\mu$-almost everywhere 0.

**First case.** $f$ is continuous and bounded. For every $n \geq 1$, set
\[ \eta_n := a\varepsilon^{2-2(n+1)}, \]
where
\[ 0 < a := \inf_{p \in [1,\infty]} \mu(\Omega)^{-1/p} \|f\|_p. \]

We define iteratively a sequence $(\Omega_n, g_n, K_n, f_n)_{n \in \mathbb{N}}$ as follows. Set $\Omega_0 := \Omega, g_0 := 0, K_0 := \emptyset, f_0 := f$. Let $n > 0$ and assume $\Omega_{n-1}, g_{n-1}, K_{n-1}, f_{n-1}$ are given. Apply Lemma 2.4 to obtain compact set $K_n \subset \Omega_{n-1}$ and a function $g_n \in C_1^{1}((\Omega_{n-1})$ such that
\[ \mu(\Omega_{n-1} \setminus \text{Int}(K_n)) < 2^{-n-1}\varepsilon \mu(\Omega_{n-1}), \]
\[ \|g_n\|_\infty \leq 2^{-n} \zeta, \]
\[ \|f_n - Dg_n(x)\| \leq \eta_n, \text{ for every } x \in K_n. \]
\[ \|Dg_n\|_p \leq C(2^{-n-1}\varepsilon)^{1/p-1}\|f_n - \chi_{\text{spt}(g_n)}\|_p, \text{ for every } p \in [1,\infty]. \]

Finally set $\Omega_n := \text{Int}(K_n)$. Define $f_n$ on $\Omega_n$ as $f_n := f_n - Dg_n$. We set $K := \bigcap_{n>0} K_n$ and $g := \sum_{n>0} g_n$. The bound (2.20) is an immediate consequence of (2.29). We prove now that the set $K$ and the function $g$ satisfy (2.1), (2.2) and (2.3). To prove (2.1), notice that by (2.28) and (2.17) it holds
\[ \mu(\Omega \setminus K) = \sum_{n \geq 0} \mu(\Omega_n \setminus \text{Int}(K_{n+1})) \leq \varepsilon \mu(\Omega). \]

Since, for $n \geq 1$, $\text{spt}(g_{n+1}) \subset K_n$, combining (2.30) and (2.31) with $p = \infty$, we get
\[ \|Dg_n\|_\infty \leq C(2^{-(n+1)}\varepsilon)^{-1}\|f_n\|_{C_1^{1}(\text{spt}(g_{n+1}))}\|_\infty \leq C(2^{-(n+1)}\varepsilon)^{-1}\eta_n = C(2^{-(n+1)}\varepsilon)a. \]

This implies that $(\sum_{i=1}^n Dg_i)_{n \in \mathbb{N}}$ (and hence also $(\sum_{i=1}^n g_i)_{n \in \mathbb{N}}$) converges uniformly, therefore $g \in C_1^{1}(\Omega)$. Since for $n \geq 1$ it holds $f_n = f_n - 1 + \sum_{i=0}^{n-1} Dg_i$ on $K_n$, then (2.2) follows immediately from (2.30). To prove (2.3), we compute, using (2.30) and (2.31)
\[ \|Dg\|_p \leq \|Dg\chi_K\|_p + \|Dg\chi_{\Omega_n}\|_p \leq \|f\chi_K\|_p + \sum_{n \geq 0} \|Dg\chi_{\Omega_n \setminus \Omega_{n+1}}\|_p \]
\[ \leq \|f\|_p + \sum_{n \geq 0} \|Dg_{n+1}\chi_{\Omega_n \setminus \Omega_{n+1}}\|_p \leq \|f\|_p + C2^{n+1}\varepsilon^{1/p-1}\sum_{n \geq 0} \|f_n\chi_{\Omega_n}\|_p \]
\[ \leq (1 + 2C\varepsilon^{1/p-1})\|f\|_p + C2^{n+1}\varepsilon^{1/p-1}\sum_{n \geq 1} \|(f_n - Dg_n)\chi_{\Omega_n}\|_p \]
\[ \leq (1 + 2C\varepsilon^{1/p-1})\|f\|_p + C\varepsilon^{1/p-1}\mu(\Omega)^{1/p} \sum_{n \geq 1} \eta_n \leq (1 + 3C\varepsilon^{1/p-1})\|f\|_p \]

\[ \square \]

**Second case.** \(f\) is Borel. Fix \(\varepsilon > 0\). There exists \(r > 0\) such that
\[ \alpha := \mu(\{x \in \Omega : |f(x)| > r\}) \leq \varepsilon/4. \]

By Lusin’s theorem there exists a continuous function \(f_1 : \Omega \to \mathbb{R}^N\) which agrees with \(f\) outside a set of measure \(\mu\) less than \(\alpha\). The function
\[ f_2(x) = \begin{cases} f_1(x) & \text{if } |f_1(x)| \leq r, \\ rf_1(x)/|f_1(x)| & \text{if } |f_1(x)| > r \end{cases} \]
is continuous and bounded and \(\mu(\{x : f(x) \neq f_2(x)\}) \leq \varepsilon/2\). Moreover, for every \(p \in [1, \infty]\), it holds \(\|f_2\|_p \leq 2\|f\|_p\). The theorem follows easily by applying the previous case to the function \(f_2\).

3. **Proof of Theorem 1.4(II)**

The proof of part (II) of Theorem 1.4 is quite involved. The reader might find helpful to read Section 5 before proceeding: although the result in dimension 1 is stronger, the construction presented there requires a considerably smaller amount of technicalities. In the sequel \(L(f)\) denotes the Lipschitz constant of the function \(f\).

3.1. **Preliminary results.**

**Definition 3.1 (Local behaviour of a Lipschitz function).** Let \(S\) be a set and let \(\alpha \geq 0\) and \(r_0 > 0\). A real-valued Lipschitz function \(f\) whose domain contains \(S\) is said to be \(\alpha\)-Lipschitz on \(S\) below scale \(r_0\) if whenever \(x, y \in X\) are such that \(\text{dist}(x, S), \text{dist}(y, S) \leq r_0\) and \(d(x, y) \leq r_0\) one has:
\[ |f(x) - f(y)| \leq \alpha d(x, y). \]

(3.1)

The Lipschitz function \(f\) is said to be asymptotically flat on \(S\) if for each \(\varepsilon > 0\) there is an \(r_\varepsilon > 0\) such that \(f\) is \(\varepsilon\)-Lipschitz on \(S\) below scale \(r_\varepsilon\).

Before moving on we need to recall something about the general differentiability theory for real-valued Lipschitz functions in the metric setting developed in [Sch16a] and the differentiability theory, wrt singular Radon measures, for real-valued Lipschitz functions defined on Euclidean spaces studied in [AM16]. We do not want to dispirit the reader: both theories can essentially be treated as black-boxes to understand the results here, as they only intervene through the Localized Approximation Scheme, Theorem 3.4.

**Definition 3.2 (Alberti representations).** Let \(\mu\) be a Radon measure on a metric space \(X\) and let \(\text{Frag}(X)\) denote the set of 1-Lipschitz maps \(\gamma : \text{dom} \gamma \to X\) where \(\text{dom} \gamma\) is a compact subset of \(\mathbb{R}\). We topologize \(\text{Frag}(X)\) with the Hausdorff distance between graphs. An Alberti representation of \(\mu\) is a pair \((Q, w)\) where \(Q\)
is a Radon measure on Frag(X) and $w$ is a locally bounded Borel map $w : X \to [0, \infty)$ such that:

$$\mu = \int_{\text{Frag}(X)} w_{\gamma} (L^1, \text{dom} \gamma) \, dQ(\gamma), \tag{3.2}$$

where $\gamma_{\#}(L^1, \text{dom} \gamma)$ denotes the push-forward, using $\gamma$, of the 1-dimensional Lebesgue measure on $\text{dom} \gamma$. More precisely, (3.2) should be understood as follows: for each $g : X \to \mathbb{R}$ continuous and in $L^1(\mu)$ one has:

$$\int_X g \, d\mu = \int_{\text{Frag}(X)} dQ(\gamma) \int_{\text{dom} \gamma} w \circ \gamma(t) g \circ \gamma(t) \, dt. \tag{3.3}$$

**Definition 3.3** (The norm of the Weaver differential). Let $\mu$ be a Radon measure on $X$ and $f : X \to \mathbb{R}$ Lipschitz. We denote by $|df|_{\mathcal{E}(\mu)}$ the local norm of $df$ [Sch16a Defn. 2.101 & 2.123], which is an $L^\infty(\mu)$-function which is $\geq 0$ $\mu$-a.e. For this paper we do not need the explicit definition of $|df|_{\mathcal{E}(\mu)}$ but the following characterization [Sch16a Sec. 3.3]: $|df|_{\mathcal{E}(\mu)} \geq \alpha$ on a Borel set $S \subset X$ if and only if for each $\varepsilon > 0$ the measure $\mu_{\downarrow} S$ has an Alberti representation $(Q_\varepsilon, w_\varepsilon)$ such that for $Q_\varepsilon$-a.e. $\gamma$ for $L^1$-a.e. $t \in \text{dom} \gamma$ one has $(f \circ \gamma)'(t) \geq \alpha - \varepsilon$. From the definition of the decomposability bundle in [AM16] in terms of the Alberti representations we see that if $f : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz for $\mu$-a.e. $x$ one has:

$$|df|_{\mathcal{E}(\mu)}(x) = \|dV(x, \mu)f\|_2, \tag{3.4}$$

where $dV(x, \mu)f$ is the derivative of $f$ at $x$ in the direction of $V(x, \mu)$.

**Theorem 3.4** (Localized Approximation Scheme). Let $(X, \mu)$ be a locally compact metric measure space ($\mu$ being Radon). Let $f$ be a real-valued Lipschitz function defined on $X$ and $K \subset X$ a compact subset on which $|df|_{\mathcal{E}(\mu)} \leq \alpha$ for some $\alpha > 0$. Then for each $\varepsilon > 0$ there are an $r_\varepsilon > 0$, a compact $K_\varepsilon \subset K$ and a real-valued Lipschitz function $f_\varepsilon$ defined on $X$ such that:

**(Apx1):** For any open set $U \subset X$ containing $K$ the Lipschitz constant $L(f_\varepsilon|U)$ of the restriction $f_\varepsilon|U$ is at most the Lipschitz constant $L(f|U)$ of the restriction $f|U$. In particular, taking $U = X$, $L(f_\varepsilon) \leq L(f)$.

**(Apx2):** $\|f_\varepsilon - f\|_\infty \leq \varepsilon$ and $f$ is $\alpha$-Lipschitz on $K_\varepsilon$ below scale $r_\varepsilon$.

**(Apx3):** $\mu(K \setminus K_\varepsilon) \leq \varepsilon$.

**Proof.** The proof of this result is rather technical and corresponds to Theorem 3.66 of [Sch16a], proved in Section 5.1 of [Sch16a], in the special case where $g = 1$ (i.e. without discussing cones). However, here we need two slight modifications of that result: that $X$ is locally compact and (Apx1). We will refer to the notation and proof in Section 5.1. of [Sch16a].

That in Theorem 3.66 of [Sch16a] one can take $X$ locally compact is not surprising because in the argument only the compactness of $K$ is directly used.

On the other hand, to obtain (Apx1) we must inspect the construction more carefully. We have first constructed a convex metric space $Z$ (i.e. any pair of points is joined by a geodesic) and obtained an isometric embedding $i : X \hookrightarrow Z$. Without loss of generality we have assumed $L(f) = 1$, considered the cylinder
Cyl = Z × ℝ (here we use ℝ instead of a finite interval because X is only known to be locally compact) with metric:
\[ d_{Cyl}((z_1, t_1), (z_2, t_2)) := \max(|t_1 - t_2|, d_Z(z_1, z_2)). \] (3.5)

We now identify X with a subset of Cyl via \( x \mapsto (i(x), f(x)) \). Note that the projection
\[ \tau : Cyl \to ℝ \]
\[(z, t) \mapsto t, \] (3.6)
extends \( f \) as \( \tau | X = f \). The goal has then become to approximate \( \tau \) and this has been accomplished by covering \( \mu \)-a.e. point of \( K \) (thus in (Apx3) we pass to a subset \( K_x \)) by strips whose union is \( T_ε \) (see the definition of \( T_ε \) above equation (5.41) in [Sch16a]). As \( K \) is compact \( T_ε \) lies in \( Z × [a, b] \) for some \( a, b \) and the approximation \( \tau_ε \) is obtained by setting \( \tau_ε = \tau \) on \( Z × [−∞, a) \) and
\[ \tau_ε(z, t) := a + \int_a^t \chi_{T_ε}(z, s) \, ds \quad \text{elsewhere.} \] (3.7)

In (5.48) of [Sch16a] we have proved that \( \tau_ε \) is 1-Lipschitz with respect to the distance:
\[ D_α((z_1, t_1), (z_2, t_2)) := \max(|t_1 - t_2|, αd_Z(z_1, z_2)). \] (3.8)

In particular, as \( K \subset U, \alpha \leq L(f|X) \). Now pick \((z_i, t_i) \in U \) for \( i \in \{1, 2\} \) such that \( z_i = i(x_i) \) and \( t_i = f(x_i) \); then:
\[ D_α((z_1, t_1), (z_2, t_2)) \leq L(f|U)d_X(x_1, x_2), \] (3.9)
which proves the theorem.

\[ \square \]

Lemma 3.5 (Step 1 of Construction). Let \( K \subset ℝ^N \) be a compact set and assume that the decomposability bundle of \( \mu \ll K \) has constant dimension \( N_0 \) and let \( π_x \) denote its fibre at \( x \) and \( π_x^⊥ \) its orthogonal complement. Assume that for some \( N_0 \)-dimensional hyperplane \( π \) one has \( \|π_x − π\|_{∞} \leq ε_0 \) and let \( h : π^⊥ \to ℝ \) be \( 1 \)-Lipschitz. Then there is a constant \( C = C(N, N_0) \) (indep. of \( h \)) such that for each choice of parameters \((ε_s, ε_m, σ, r_0) ∈ (0, 1/2)^4 \) there are a \( √3 \)-Lipschitz function \( g : ℝ^N \to ℝ \), compact subsets \( J^\text{good} \subset J \subset K \) and a scale \( r > 0 \) such that:

(a) \( μ(K \setminus J) ≤ ε_m μ(K), \|g\|_{∞} ≤ ε_s \) and \( g \) is \( Cε_0 \)-Lipschitz on \( J \) below scale \( r \).
(b) \( μ(J^\text{good}) ≥ C^{-1}σ^{N−N_0}μ(J) \).
(c) One can decompose \( J^\text{good} \) as a finite disjoint union \( \bigcup_{a=1}^MC_a \) such that for each \( a \in \{1, \cdots, M\} \) there is an \( 0 < r_a ≤ r_0 \) such that whenever \( x \in C_a \) one has:
\[ \|T_{x, r_a}g − h\|_{∞, \delta} ≤ C(ε_s + σ), \] (3.10)

where in (3.10) \( T_{x, r_a} \) is the map introduced in Definition 1.1 and we have implicitly extended \( h \) as a map \( h : ℝ^N = π ⊕ π^⊥ \to ℝ \) by letting \( h(y, \tilde{y}) = h(\tilde{y}) \).

Lemma 3.5 is proven using the following intermediate results.
Lemma 3.6 (A good rectangle). Let \( \mu \) be a Radon measure on \( \mathbb{R}^N \) and \( r_0 > 0 \). Fix parameters \( (L, \sigma) \in [8, \infty) \times (0, 1/2) \) and define for \( 0 < r \leq r_0 \) the following sets:

\[
E(x, r) := x + \left[ -\frac{L^2r}{2}, \frac{L^2r}{2} \right]^N \times [-2r, 2r]^{N-N_0},
\]

(3.11)

\[
S(x, r) := x + \left[ -\frac{L^2r}{2} + \frac{Lr}{2}, \frac{L^2r}{2} - \frac{Lr}{2} \right]^N \times [-2\sigma r, 2\sigma r]^{N-N_0}.
\]

(3.12)

Given a compact set \( K \subset \mathbb{R}^N \) with \( \mu(K) > 0 \), define the bad set:

\[
\text{Bad}(K, r) := \left\{ x \in K : 0 < \mu(S(x, r)) \leq c \mu(E(x, r)) \right\},
\]

(3.13)

where \( c := \sigma^{N-N_0}2^{-2N-N_0-1}(1 + 1/6)^{-1} \). Then there exists \( r < r_0 \) (possibly depending on \( K \)) such that \( \mu(\text{Bad}(K, r)) \leq \frac{1}{4}\mu(K) \).

Heuristically, \( E(x, r) \) is a rectangle at \( x \) at scale \( r \) which is \( L^2 \)-times bigger in the direction of the first \( N_0 \) coordinates, while \( S(x, r) \) is a core of \( E(x, r) \) which is much smaller (generally \( \sigma \ll 1 \)) in the transverse direction of the last \( N - N_0 \) coordinates. Even though \( \mu \) is not the Lebesgue measure, Lemma 3.6 says that the ratio \( \mu(S(x, r))/\mu(E(x, r)) \) is up to a constant at least the same ratio that one would have for Lebesgue measure. Besicovitch Covering Theorem implies that:

Corollary 3.7 (Covering by good rectangles). Let \( \mu \) be a Radon measure on \( \mathbb{R}^N \) and \( K \) compact with \( \mu(K) > 0 \) and \( r_0 > 0 \). Let \( L, \sigma \), be as above. Then for any \( \varepsilon > 0 \) there are finitely many pairwise disjoint \( \{E(x_i, r_i)\}_i \) such that:

\[
0 < r_i \leq r_0,
\]

(3.14)

\[
\mu(S(x_i, r_i) \cap K) \geq \sigma^{N-N_0}2^{-2N-N_0-1}(1 + 1/6)^{-1} \mu(E(x_i, r_i) \cap K) > 0,
\]

(3.15)

\[
\mu(K \setminus \bigcup_i E(x_i, r_i)) \leq \varepsilon \mu(K).
\]

(3.16)

Proof of Lemma 3.6. Choose an open set \( U \supset K \) such that

\[
\mu(U) \leq \left(1 + \frac{1}{6}\right) \mu(K).
\]

(3.17)

Then choose \( r \leq r_0 \) such that \( E(x, 4r) \cap K \neq \emptyset \) implies that \( E(x, 4r) \subset U \). Then let \( I \) denote the integral:

\[
I := \int \chi_{\text{Bad}(K, r)}(x_1)\chi_{S(x_1, r/2)}(x_2) \, d\mu(x_1) \, d\mathcal{L}^N(x_2);
\]

(3.18)

if we integrate first in \( x_2 \) we get:

\[
I = r^N \sigma^{N-N_0}(L^2 - L)^{N_0}2^{N-2N_0}\mu(\text{Bad}(K, r)).
\]

(3.19)
If we integrate first in $x_1$ we get:

$$I = \int \mu \left( (\text{Bad}(K, r) \cap \mathcal{S}(x_2, r/2)) \right) d\mathcal{L}^N(x_2). \quad (3.20)$$

Choose $x_2$ such that Bad$(K, r) \cap \mathcal{S}(x_2, r/2) \neq \emptyset$ and let $x_1 \in K$ denote a point in this non-empty intersection. Then for $i \in \{1, \cdots, N_0\}$ we get:

$$|x_1^i - x_2^i| \leq \left( \frac{L_i^2}{2} - \frac{L}{2} \right) r; \quad (3.21)$$

for $i > N_0$ one has $|x_1^i - x_2^i| \leq \sigma r$. In particular, Bad$(K, r) \cap \mathcal{S}(x_2, r/2) \subset \mathcal{S}(x_1, r)$; but as $x_1 \in \text{Bad}(K, r)$ one has $\mu(\mathcal{S}(x_1, r)) \leq c \mu(E(x_1, r))$. Using the triangle inequality we observe $E(x_1, r) \subset E(x_2, 2r)$ from which we conclude $E(x_2, 2r) \subset E(x_1, 4r) \subset U$. We thus obtain the upper bound:

$$I \leq c \int \mu(U \cap E(x_2, 2r)) d\mathcal{L}^N(x_2)$$

$$= c \int \chi_U(x_1)\chi_{E(x_2, 2r)}(x_1) d\mathcal{L}^N(x_2) d\mu(x_1)$$

$$= c \int \chi_U(x_1)\chi_{E(x_2, 2r)}(x_2) d\mathcal{L}^N(x_2) d\mu(x_1)$$

$$= cr^N(L^2)^{N_0} 2^{3N-2N_0} \mu(U) \leq cr^N(L^2)^{N_0} 2^{3N-2N_0} \left( 1 + \frac{1}{6} \right) \mu(K). \quad (3.22)$$

The proof is completed combining (3.22) with (3.19) and the choice of $c$ which gives $\mu(\text{Bad}(K, r)) \leq \frac{1}{2} \mu(K)$.

**Proof of Lemma 3.5** Step 1: Construction of auxiliary functions.

Without loss of generality we will assume that $\pi_0$ is the plane $\mathbb{R}^{N_0} \times \{0\}$.

Recall that the 1-Lipschitz retraction of $\mathbb{R}^{N_0}$ onto $B \subset \mathbb{R}^{N_0}$ is given by:

$$J(x) := \begin{cases} x & \text{if } |x| \leq 1 \\ \frac{r}{|r|} & \text{otherwise.} \end{cases} \quad (3.23)$$

Fix the parameter $L \gg 1$; we define a $\frac{1}{L}$-Lipschitz cut-off function on $\mathbb{R}$:

$$\varphi(r) := \begin{cases} 1 & \text{if } |r| \in [0, L^2/2 - L/4], \\ 1 - \frac{1}{L}(|r| - L^2/2 + L/4) & \text{if } |r| \in (L^2/2 - L/4, L^2/2], \\ 0 & \text{otherwise.} \end{cases} \quad (3.24)$$

We also define the 1-Lipschitz cut-off function on $\mathbb{R}$:

$$\psi(r) := \begin{cases} 1 & \text{if } |r| \leq 1, \\ 2 - |r| & \text{if } |r| \in (1, 2], \\ 0 & \text{otherwise.} \end{cases} \quad (3.25)$$

We now replace $h$ by $h \circ J$ so that we can assume $\|h\|_\infty \leq 1$ and $\partial_i h = 0$ on $\mathcal{B}^c$. 
We define the building block of our construction:

$$F(y, \tilde{y}) := \varphi(|y|)\psi(|\tilde{y}|)h(\tilde{y}).$$

(3.26)

We now collect some properties of $F$:

(F1): $F$ is $(4/L + \sqrt{2})$-Lipschitz (note that on $\mathcal{B}^c$ $\nabla \psi$ and $\nabla h$ give orthogonal contributions to the derivative of $F$).

(F2): $F = 0$ outside of $B = \left[-\frac{L^2}{2}, \frac{L^2}{2}\right]^{N_0} \times [-2, 2]^{N-N_0}$.

(F3): $F = h$ on the core $S = \left[-\frac{L^2}{2} + \frac{L^2}{4}, \frac{L^2}{2} - \frac{L^2}{4}\right]^{N_0} \times [-1, 1]^{N-N_0}$.

(F4): $\|F\|_\infty \leq \|h\|_\infty \leq 1$.

(F5): We can assume $|df|_{\varepsilon(\mu)} \leq C\varepsilon_0$, where $C$ depends possibly only on $N$ and $N_0$.

(F6): $F$ decays linearly to 0 when approaching the boundary of $B$: $|F(y, \tilde{y})| \leq d((y, \tilde{y}), \partial B)$.

Only (F5) and (F6) require justification. For (F5) observe that by the Leibniz rule $df = \psi dh \varphi + \varphi dh \psi + \varphi \psi dh$; observe also that $d\psi$ and $dh$ give an $O(\varepsilon_0)$ contribution to the Weaver differential (we take $\mu$ as the reference measure) as $\|\pi_x - \pi_0\|_\infty \leq \varepsilon_0$. Thus, $|df|_{\varepsilon(\mu)} \leq \frac{1}{L} + C\varepsilon_0$, and, choosing $L$ large enough and inflating $C$, we get (F5). For (F6) we have three cases.

The first: $|y| \geq L^2/2 - L/4$ so that:

$$|F(y, \tilde{y})| \leq |\varphi(y)| \leq 1 - \frac{4}{L} \left(|y| - \frac{L^2}{2} + \frac{L}{4}\right)$$

$$= \frac{4}{L} \left(\frac{L^2}{2} - |y|\right)$$

$$\leq \frac{4}{L} d((y, \tilde{y}), \partial B),$$

(3.27)

and (F6) holds as long as $L \geq 4$. The second: $|y| \leq L^2/2 - L/4$ and $|\tilde{y}| \geq 1$:

$$|F(y, \tilde{y})| \leq \psi(\tilde{y}) = 2 - |\tilde{y}| \leq d((y, \tilde{y}), \partial B).$$

(3.28)

The third: $|y| \leq L^2/2 - L/4$ and $|\tilde{y}| \leq 1$: $d((y, \tilde{y}), \partial B) \geq 1$ and so we conclude by (F4).

Step2: Covering $K$ by good boxes.

We apply Corollary 3.7 and find finitely many pairwise disjoint $\{E(x_i, r_i)\}_{i=1}^M$ such that:

$$0 < r_i \leq \min(r_0, \varepsilon_0/2),$$

(3.29)

$$\mu(S(x_i, r_i) \cap K) \geq C^{-1}o^{N-N_0} \mu(E(x_i, r_i) \cap K) > 0,$$

(3.30)

$$\mu(K \setminus \bigcup_i E(x_i, r_i)) \leq \frac{\varepsilon_m}{2} \mu(K).$$

(3.31)

We let $\tilde{J} := K \cap \bigcup_i E(x_i, r_i)$ and $\tilde{J}^{\text{good}} := K \cap \bigcup_i E(x_i, r_i)$ which give (b) and the first inequality in (a) if we replace $J$ with $\tilde{J}$: the set $J$ will be chosen later to be a subset of $\tilde{J}$ and $\tilde{J}^{\text{good}}$ will be set to be $J \cap \tilde{J}^{\text{good}}$. 
Write \( x_i = (y_i, \tilde{y}_i) \) and define the \((A/L + \sqrt{2})\)-Lipschitz function \( F_i \) supported on \( E(x_i, r_i) \):

\[
F_i(y, \tilde{y}) = \sigma r_i F \left( \frac{y - y_i}{r_i}, \frac{\tilde{y} - \tilde{y}_i}{\sigma r_i} \right). \tag{3.32}
\]

Because of \((F6)\) the function \( F_i \) can be glued together to get a \((A/L + \sqrt{2})\)-Lipschitz function \( f \) as in \cite[Thm. 4.8]{Sch16b}. Note that the choice of the \( r_i \)'s implies \( \|f\|_\infty \leq \varepsilon_s/2 \).

If \( x \in S(x_i, r_i) \) lies on the core center, i.e. \( x \) is of the form \( x = (y, \tilde{y}_i) \), then:

\[
\|T_{x, \sigma r_i} f - h\|_{\infty, B} = 0. \tag{3.33}
\]

Thus, as \( f \) is \((A/L + \sqrt{2})\)-Lipschitz, for all \( x \in S(x_i, r_i) \) we have:

\[
\|T_{x, \sigma r_i} f - h\|_{\infty, B} \leq C\sigma. \tag{3.34}
\]

**Step 3:** Applying the approximation scheme.

Note that \((F5)\) implies that \( |df|_{L(\mu)} \leq C\varepsilon_0 \) and applying Theorem 3.4 we can find a \((A/L + \sqrt{2})\)-Lipschitz function \( g \), a compact set \( J \subset \tilde{J} \) and \( r > 0 \) such that:

\[
\|g - f\|_\infty \leq \frac{\varepsilon_s}{2} \min \left( \frac{1}{2}, \min_{1 \leq i \leq M} (\sigma r_i) \right), \tag{3.35}
\]

\[
\mu(J \setminus J) \leq \frac{\varepsilon_m}{2} \mu(K), \tag{3.36}
\]

\[
\mu(J \cap J^{\text{good}}) \leq \frac{C^{-1}\varepsilon_m}{16} \sigma^N N_0 \mu(\tilde{J}), \tag{3.37}
\]

and \( g \) is \((C\varepsilon_0)\)-Lipschitz on \( J \) below scale \( r \). Thus, if we let \( J^{\text{good}} := J^{\text{good}} \cap J \), then (a) and (b) follow. For (c) we just combine \([3.34]\) with:

\[
\|f - g\|_\infty \leq \frac{\varepsilon_s}{2} \tag{3.38}
\]

We finally choose \( L \) large enough so that \((F6)\) holds and \( \frac{1}{L} \leq \varepsilon_s \). \( \square \)

**Lemma 3.8** (Step 2 of Construction). Let \( K \subset \mathbb{R}^N \) be a compact subset and \( \alpha > 0 \) be such that \( |df|_{L(\mu)} \leq \alpha \) on \( K \). Then there is a constant \( C = C(N) \) (indep. of \( f \)) such that for any choice of parameters \( (\varepsilon_s, \varepsilon_m) \in (0, 1/2)^2 \) there are a Lipschitz function \( \tilde{f} \) and a compact set \( \tilde{K} \) such that:

(a) \( \|\tilde{f} - f\|_\infty \leq \varepsilon_s \) and \( \mu(K \setminus \tilde{K}) \leq \varepsilon_m \mu(K) \).

(b) \( \tilde{f} \) is asymptotically flat on \( \tilde{K} \).

(c) \( \tilde{f} \) is \((L(f) + C \frac{\alpha}{\varepsilon_m})\)-Lipschitz.

**Proof.** **Step 1:** Killing the gradient of \( f \).

We apply Theorem 2.1 with parameters \( \varepsilon := \varepsilon_m^{(1)} > 0, \zeta := \varepsilon_m^{(1)} > 0 \) to find a \( \frac{C\alpha}{\varepsilon_m^{(1)}} \)-Lipschitz function \( f_1 \) and a compact \( K_1 \subset K \) such that \( \|f_1\|_\infty \leq \varepsilon_s^{(1)} \), \( df_1 = df \) on \( K_1 \) and

\[
\mu(K \setminus K_1) \leq \varepsilon_m^{(1)} \mu(K). \tag{3.39}
\]
We let \( g_1 := f - f_1 \) and observe that \( f_1 \) is \((L(f) + \frac{C_\alpha}{\varepsilon_m^{(1)}})\)-Lipschitz with \( dg_1 = 0 \) on \( K_1 \). We fix the parameters \((\alpha_1, \varepsilon_m^{(1)}, \eta_m^{(1)}) \in (0, 1)^3 \) and use Theorem 3.4 to find an \((L(f) + \frac{C_\alpha}{\varepsilon_m^{(1)}})\)-Lipschitz function \( \hat{g}_1 \) and a compact \( H_1 \subset K_1 \) and a scale \( r_1 > 0 \) such that:

\[
\|\hat{g}_1 - g_1\|_\infty \leq \eta_m^{(1)} \tag{3.40}
\]

\[
\mu(K_1 \setminus H_1) \leq \eta_m^{(1)} \mu(K_1), \tag{3.41}
\]

and \( \hat{g}_1 \) is \( \alpha_1 \)-Lipschitz on \( H_1 \) below scale \( r_1 \).

**Step 2: The general iteration.**

We apply Theorem 2.1 with parameters \( \varepsilon := \varepsilon_m^{(j+1)} > 0, \zeta := \varepsilon_s^{(j+1)} > 0 \) to find a \( \frac{C_\alpha}{\varepsilon_m^{(j+1)}} \)-Lipschitz function \( f_{j+1} \) and a compact \( K_{j+1} \subset H_j \) such that:

\[
\|f_{j+1}\|_\infty \leq \varepsilon_s^{(j+1)} \tag{3.42}
\]

\[
df_{j+1} = df_j \quad \text{on } K_{j+1} \tag{3.43}
\]

\[
\mu(H_j \setminus K_{j+1}) \leq \varepsilon_m^{(j+1)} \mu(H_j). \tag{3.44}
\]

We let \( g_{j+1} := \hat{g}_j - f_{j+1} \) and observe that \( g_{j+1} \) satisfies:

\[
L(g_{j+1}) \leq L(f) + \frac{C_\alpha}{\varepsilon_m^{(j+1)}} + \sum_{l \leq j} \frac{C_\alpha_l}{\varepsilon_m^{(l+1)}}, \tag{3.45}
\]

and satisfies \( dg_{j+1} \) on \( K_{j+1} \). Moreover, by the inductive step and because of \((\text{Apx1})\) in Theorem 3.4 we can assume that for \( l \leq j \) the function \( g_{j+1} \) is \((\alpha_l + \sum_{k=l}^j \frac{C_\alpha_k}{\varepsilon_m^{(k+1)}})\)-Lipschitz on \( H_l \) below scale \( r_l \).

We fix the parameters \((\alpha_{j+1}, \varepsilon_s^{(j+1)}, \eta_m^{(j+1)}) \in (0, 1)^3 \) and use Theorem 3.4 to find an \((L(f) + \frac{C_\alpha}{\varepsilon_m^{(j+1)}} + \sum_{l \leq j} \frac{C_\alpha_l}{\varepsilon_m^{(l+1)}})\)-Lipschitz function \( \hat{g}_{j+1} \) and a compact \( H_{j+1} \subset K_{j+1} \) and a scale \( r_{j+1} \in (0, r_j) \) such that:

\[
\|\hat{g}_{j+1} - g_{j+1}\|_\infty \leq \eta_s^{(j+1)} \tag{3.46}
\]

\[
\mu(K_{j+1} \setminus H_{j+1}) \leq \eta_m^{(j+1)} \mu(K_{j+1}), \tag{3.47}
\]

and \( \hat{g}_{j+1} \) is \( \alpha_{j+1} \)-Lipschitz on \( H_{j+1} \) below scale \( r_{j+1} \). Also by \((\text{Apx1})\) in Theorem 3.4 we see that \( \hat{g}_{j+1} \) is \((\alpha_l + \sum_{k=l}^j \frac{C_\alpha_k}{\varepsilon_m^{(k+1)}})\)-Lipschitz on \( H_l \) below scale \( r_l \).

**Step 3: Choice of parameters.**

The parameters \( \alpha_j, \varepsilon_s^{(j)}, \varepsilon_s^{(j)} \) and \( \eta_m^{(j)} \) can be chosen arbitrarily small at each stage, while with the parameters \( \varepsilon_m^{(j)} \) we must be careful otherwise the Lipschitz constants of the functions involved at each stage will diverge to \( \infty \). For \( j \geq 1 \) we thus let:

\[
\varepsilon_m^{(j+1)} := \sqrt{\alpha_j}. \tag{3.48}
\]
We have the bound:

\[
\mu(K \setminus H_j) \leq \mu(K \setminus K_1) + \sum_{l \leq j} \mu(K_l \setminus H_l) + \sum_{k=1}^{j-1} \mu(H_l \setminus K_{l+1})
\]

\[
\leq \varepsilon_m \left( \frac{1}{m} + \sum_{l \leq j} \eta_m \mu(K_l) + \sum_{k=1}^{j-1} \varepsilon_{m(k+1)} \right) \mu(K). \tag{3.49}
\]

We let \(H_\infty = \bigcap_j H_j\) and want this set to contain a significant part of the measure of \(K\). Thus, if we choose \(\varepsilon_m := \varepsilon_m / 3\) and the \(\eta_m, \alpha_l\) so that:

\[
\sum_l \eta_m \leq \frac{\varepsilon_m}{3}, \tag{3.50}
\]

\[
\sum_l \sqrt{\alpha_l} \leq \frac{\varepsilon_m}{3}, \tag{3.51}
\]

we get \(\mu(K \setminus H_\infty) \leq \varepsilon_m \mu(K)\) and can finally let \(\hat{K} := H_\infty\).

We now want to guarantee convergence of the sequence \(\{\hat{g}_j\}\). First note:

\[
\hat{g}_j - f = \sum_{l \leq j} (\hat{g}_l - g_l) - \sum_{l \leq j} f_l, \tag{3.52}
\]

from which we deduce:

\[
\|\hat{g}_j - f\| \leq \sum_{l \leq j} (\eta_m + \varepsilon_m); \tag{3.53}
\]

we will choose the parameters \(\eta_m, \varepsilon_m\) so that:

\[
\sum_l (\eta_m + \varepsilon_m) \leq \varepsilon_m. \tag{3.54}
\]

Now \(\hat{g}_{j+1} - \hat{g}_j = \hat{g}_{j+1} - g_{j+1} - f_j\) and thus:

\[
\|\hat{g}_{j+1} - \hat{g}_j\| \leq \eta_{m(j+1)} + \varepsilon_{m(j)}. \tag{3.55}
\]

Therefore for \(j \to \infty\) we have \(\hat{g}_j \to f\) uniformly, \(f\) being a continuous function; but:

\[
L(\hat{g}_j) \leq L(f) + \frac{3C}{\varepsilon_m} + C \sum_{l \leq j} \sqrt{\alpha_l}, \tag{3.56}
\]

and if we choose the \(\alpha_j\) to satisfy:

\[
\sum_j \sqrt{\alpha_j} \leq \frac{\alpha}{\varepsilon_m}, \tag{3.57}
\]
and then inflate \( C \) we conclude that \( \hat{f} \) is \((L(f) + \frac{C a}{\varepsilon_m})\)-Lipschitz. Finally for \( l \leq j \) the function \( \hat{g}_j \) is \((\alpha_l + C \sum_{l \leq k \leq j} \sqrt{\alpha_k})\)-Lipschitz on \( H_l \cup H_\infty = \hat{K} \) below scale \( r_l \), and thus \( \hat{f} \) is asymptotically flat on \( \hat{K} \). \( \square \)

**Lemma 3.9** (Step 3 of Construction). Let \( K \subset \mathbb{R}^N \) be compact and assume that the decomposability bundle of \( \mu \leq K \) has constant dimension \( N_0 \). Assume also that for some \( N_0 \)-dimensional hyperplane \( \pi \) one has \( \| \pi_x - \pi \|_\infty \leq \varepsilon_0 \) for each \( x \in K \) where \( \varepsilon_0 > 0 \). Let \( h : \pi \to \mathbb{R}^N \) be 1-Lipschitz with \( h(0) = 0 \) and extend it to \( \mathbb{R}^N \) as in Lemma 3.5. Then there are constants \( C_0, C_1 \), which depend only on \( N \) and \( N_0 \), such that the following holds: for each choice of parameters \( (\varepsilon_s, \varepsilon_m, r_0) \in (0, 1/2)^3 \) there are a \((\sqrt{3} + C_0 \frac{\varepsilon_0}{\varepsilon_m})\)-Lipschitz function \( g : \mathbb{R}^N \to \mathbb{R} \) and a compact \( J \subset K \) such that:

(a) \( g \) is asymptotically flat on \( J \), \( \| g \|_\infty \leq C_1 \varepsilon_s \) and \( \mu(K \setminus J) \leq C_1 \varepsilon_m \mu(K) \).

(b) One can write \( J \) as a finite disjoint union \( J = \bigcup_{a=1}^M C_a \) and for each \( a \in \{1, \ldots, M\} \) there is an \( 0 < r_a \leq r_0 \) such that if \( x \in C_a \) one has:

\[
\| T_{x,r_a} g - h \|_{\infty,B} \leq C_1 (\varepsilon_0 + \varepsilon_1^{\frac{1}{2}}(N-N_0)).
\]

(3.58)

**Proof. Step 1:** Applying Lemma 3.5.

We let \( C \) denote the maximum of the constants \( C, C_0 \) and \( C_1 \) from Lemmas 3.5 and 3.8. We fix parameters \( (\varepsilon_s^{(1)}, \varepsilon_m^{(1)}, \sigma, r^{(1)}) \in (0, 1)^4 \) to be chosen later. For the moment we just remark we will need \( r^{(1)} \leq r_0 \) and \( \varepsilon_1^{\frac{1}{2}}(N-N_0) > \varepsilon_0^{(1)} \). We now apply Lemma 3.5 to obtain an \( \sqrt{3} \)-Lipschitz function \( g_1 \) and compact subsets \( J_1^{\text{good}} \subset J_1 \subset K \) and a scale \( 0 < \rho_1 \leq r^{(1)} \) such that:

(\( g_1: a \)) \( : \mu(K \setminus J_1) \leq \varepsilon_s^{(1)} \mu(K) \), \( \| g_1 \|_\infty \leq \varepsilon_s^{(1)} \) and \( g_1 \) is \( C \varepsilon_0 \)-Lipschitz on \( J_1 \) below scale \( \rho_1 \).

(\( g_1: b \)) \( : \mu(J_1^{\text{good}}) \geq C_1 \varepsilon_m^{(1)} \mu(J_1) \).

(\( g_1: c \)) \( : \) One can decompose \( J_1^{\text{good}} \) as a finite disjoint union \( \bigcup_{a=1}^M C_a^{(1)} \) such that for each \( a \in \{1, \ldots, M \} \) there is an \( 0 < r^{(1)}_a \leq r^{(1)} \) such that whenever \( x \in C_a^{(1)} \) one has:

\[
\| T_{x,r^{(1)}_a} g_1 - h \|_{\infty,B} \leq C (\varepsilon_s^{(1)} + \sigma).
\]

(3.59)

**Step 2:** Applying Lemma 3.8.

We fix parameters \( (\varepsilon_s^{(1)}, \varepsilon_m^{(1)}) \in (0, 1)^2 \) to be chosen later. We apply Lemma 3.8 (to the function \( g_1 \) and the compact set \( J_1 \)) to find a Lipschitz function \( \hat{g}_1 \) and a compact set \( \hat{J}_1 \subset J_1 \) such that:

(\( \hat{g}_1: a \)) \( : \| \hat{g}_1 - g_1 \|_\infty \leq \varepsilon_s^{(1)} \) and \( \mu(J_1 \setminus \hat{J}_1) \leq \varepsilon_m^{(1)} \mu(J_1) \).

(\( \hat{g}_1: b \)) \( : \) \( \hat{g}_1 \) is asymptotically flat on \( \hat{J}_1 \).

(\( \hat{g}_1: c \)) \( : \) \( \hat{g}_1 \) is \((\sqrt{3} + C^2 \varepsilon_0^{\frac{1}{2}})\)-Lipschitz.
As in (g1:c) pick \( x \in C_a^{(1)} \) and combine [3.59] with (g1:a) to obtain:

\[
\left\| T_{x,r_a^{(1)}} \hat{g}_1 - h \right\|_{\infty,B} \leq \left\| T_{x,r_a^{(1)}} \hat{g}_1 - T_{x,r_a^{(1)}} g_1 \right\|_{\infty,B} + C(\varepsilon_s^{(1)} + \sigma) \\
\leq \frac{2\varepsilon_s^{(1)}}{r_a^{(1)}} + C(\varepsilon_s^{(1)} + \sigma),
\]

(3.60)

and observe that \( \varepsilon_s^{(1)} \) will be chosen later to be insignificant next to \( \min_{1 \leq a \leq M_1} r_a^{(1)} \).

**Step3:** The construction of \( g_2, \hat{g}_2 \) and \( G_2 \).

We now want to follow the first two steps, but first need an intermediate construction. Fix parameters \((\alpha_1, \varepsilon_p^{(1)}) \in (0,1)^2 \) and choose \( R_1 \leq \alpha_1 \min_{1 \leq a \leq M_1} r_a^{(1)} \) such that \( \hat{g}_1 \) is \( \alpha_1 \)-Lipschitz on \( \hat{J}_1 \) below scale \( R_1 \). Find a compact \( J_1^{1/2} \subset \hat{J}_1 \) good such that:

\[
\mu \left( (\hat{J}_1 \setminus J_1^{1/2}) \setminus J_1^{1/2} \right) \leq \varepsilon_p^{(1)} \mu(\hat{J}_1 \setminus J_1^{1/2}),
\]

(3.61)

and find a \( \tau_1 > 0 \) such that the \( \tau_1 \)-neighbourhoods of \( J_1^{1/2} \) and \( J_1^{1/2} \) are disjoint.

We now fix parameters \((\varepsilon_2^{(1)}, \varepsilon_m^{(1)}, r^{(2)}) \in (0,1)^3 \) (for the moment imposing the constraint that \( 2r^{(2)} < \tau_1 \)) and apply Lemma 3.5 using the parameters \((\varepsilon_2^{(1)}, \varepsilon_m^{(1)}, \alpha, r^{(2)}) \) and the compact set \( J_1^{1/2} \) to find an \( \sqrt{3} \)-Lipschitz function \( g_2 \) and compact subsets \( J_2^{1/2} \subset J_2 \subset J_1^{1/2} \) and a scale \( 0 < \rho_2 \leq r^{(2)} \) such that:

**(g2:a):** \( \mu(J_1^{1/2} \setminus J_2) \leq \varepsilon_m^{(2)} \mu(J_1^{1/2}), \| g_2 \|_{\infty} \leq \varepsilon_s^{(2)} \) and \( g_2 \) is \( C\varepsilon_0 \)-Lipschitz on \( J_2 \) below scale \( \rho_2 \).

**(g2:b):** \( \mu(J_2^{1/2}) \geq C^{-1} \sigma N - N_0 \mu(J_2) \).

**(g2:c):** One can decompose \( J_2^{1/2} \) as a finite disjoint union \( J_2^{1/2} = \bigcup_{a=1}^{M_2} C_a^{(2)} \) such that for each \( a \in \{1, \cdots, M_2 \} \) there is an \( 0 < r_a^{(2)} \leq r^{(2)} \) such that whenever \( x \in C_a^{(2)} \) one has:

\[
\left\| T_{x,r_a^{(2)}} g_2 - h \right\|_{\infty,B} \leq C(\varepsilon_s^{(2)} + \sigma).
\]

(3.62)

We fix parameters \((\varepsilon_2^{(2)}, \varepsilon_m^{(2)}) \in (0,1)^2 \) to be chosen later. We apply Lemma 3.8 (to the function \( g_2 \) and the compact set \( J_2 \)) to find a Lipschitz function \( \hat{g}_2 \) and a compact subset \( \hat{J}_2 \subset J_2 \) such that:

**(g2:a):** \( \| \hat{g}_2 - g_2 \|_{\infty} \leq \varepsilon_s^{(2)} \) and \( \mu(J_2 \setminus \hat{J}_2) \leq \varepsilon_m^{(2)} \mu(J_2) \).

**(g2:b):** \( \hat{g}_2 \) is asymptotically flat on \( \hat{J}_2 \).

**(g2:c):** \( \hat{g}_2 \) is \( (\sqrt{3} + C^2 \frac{\alpha}{\varepsilon_m^{(2)}}) \)-Lipschitz.

We now modify \( \hat{g}_2 \) so that it vanishes on the \((\tau_1)\)-neighbourhood of \( J_1^{1/2} \) and stays the same on the \((\tau_1/2)\)-neighbourhood of \( J_1^{1/2} \). This is accomplished by replacing \( \hat{g}_2 \) with

\[
\hat{g}_2 \max \left( 0, 1 - \frac{1}{\tau_1} \text{dist}(\cdot, (J_1^{1/2})_{\tau_1/2}) \right),
\]

(3.63)
where \((J_1^0)_{	au_1/2}\) denotes the \((\tau_1/2)\)-neighbourhood of \(J_1^0\). As \((J_1^{\text{good}})_{\tau_1}\) and \((J_1^0)_{\tau_1}\) are disjoint, \(\hat{g}_2\) now vanishes on \((J_1^{\text{good}})_{\tau_1}\). One also obtains an upper bound on the Lipschitz constant of \(\hat{g}_2\):

\[
L(\hat{g}_2) \leq \sqrt{3} + C^2 \frac{\varepsilon_0}{\varepsilon_{m(1)}^2} + 2 \frac{\varepsilon_s^2 + \varepsilon_a^2}{\tau_1}.
\] (3.64)

To get the last term in (3.64) to be \(\leq \alpha_1\) we impose the restriction \(2(\varepsilon_s^2 + \varepsilon_a^2) \leq \alpha_1\).

We now let \(G_2 = \hat{g}_1 + \hat{g}_2\) and to get a good upper bound on \(L(G_2)\) impose the restriction \(2(\varepsilon_s^2 + \varepsilon_a^2) \leq \alpha_1 R_1\). In fact, we now verify that

\[
L(G_2) \leq \sqrt{3} + C^2 \frac{\varepsilon_0}{\min(\varepsilon_{m(1)}^2, \varepsilon_{m(2)}^2)} + 2 \alpha_1;
\] (3.65)

the first case is when \(d(x, y) \geq R_1\) in which we have:

\[
|G_2(x) - G_2(y)| \leq |\hat{g}_1(x) - \hat{g}_1(y)| + 2 \|\hat{g}_2\|_\infty \leq \left(\sqrt{3} + C^2 \frac{\varepsilon_0}{\varepsilon_{m(1)}^2} + \alpha_1\right) d(x, y),
\] (3.66)

and the second case is when \(d(x, y) \leq R_1\) in which case we have:

\[
|G_2(x) - G_2(y)| \leq |\hat{g}_2(x) - \hat{g}_2(y)| + |\hat{g}_1(x) - \hat{g}_1(y)| \leq \left(\sqrt{3} + C^2 \frac{\varepsilon_0}{\varepsilon_{m(2)}^2} + 2 \alpha_1\right) d(x, y).
\] (3.67)

We now show that \(G_2\) is asymptotically flat on \(\hat{J}_2 \cup (J_1^{\text{good}} \cap \hat{J}_1)\). In fact, \(\hat{g}_1\) is asymptotically flat on \(\hat{J}_1\) and hence on \(\hat{J}_2 \cup (J_1^{\text{good}} \cap \hat{J}_1)\), and \(\hat{g}_2\) is asymptotically flat on \(\hat{J}_2\) and vanishes on \((J_1^{\text{good}})_{\tau_1}\). We finally establish analogues of (3.60) and (3.62). Pick \(x \in C_a^{(1)}\); then:

\[
\left\|T_{x, r_a^{(1)}} G_2 - h\right\|_\infty, B \leq \left\|T_{x, r_a^{(1)}} \hat{g}_1 - h\right\|_{\infty, B} + \frac{2 \|\hat{g}_2\|_\infty}{r_a^{(1)}} \leq \frac{2 \varepsilon_s^{(1)}}{r_a^{(1)}} + C(\varepsilon_a^{(1)} + \sigma) + \alpha_1.
\] (3.68)

Pick \(x \in C_a^{(2)}\); then:

\[
\left\|T_{x, r_a^{(2)}} G_2 - h\right\|_\infty, B \leq \frac{\|\hat{g}_1 - \hat{g}_1(x)\|_{\infty, B(x, r_a^{(2)})}}{r_a^{(2)}} + \left\|T_{x, r_a^{(2)}} \hat{g}_2 - h\right\|_\infty, B \leq \alpha_1 + C(\varepsilon_a^{(2)} + \sigma) + 2 \frac{\varepsilon_s^{(2)}}{r_a^{(2)}}.
\] (3.69)

In connection with (3.69) we observe that \(\varepsilon_s^{(2)}\) will be chosen later to be insignificant next to \(\min_{1 \leq a \leq M_2} r_a^{(2)}\).
Step4: The construction of \( G_{j+1} \), for \( j \geq 2 \).

We fix parameters \( (\alpha_j, \varepsilon_{(j)}^0) \in (0, 1)^2 \) and choose
\[
R_j \leq \alpha_j^2 \min \{ r_a^{(l)} : 1 \leq l \leq j, 1 \leq a \leq M_l \} \tag{3.70}
\]
such that \( G_j \) is \( \alpha_j \)-Lipschitz on \( \hat{J}_j \) below scale \( R_j \). We then find a compact set \( J_{j,2}^j \subset \hat{J}_j \setminus J_j^{\text{good}} \) such that:
\[
\mu \left( (\hat{J}_j \setminus J_j^{\text{good}}) \setminus J_{j,2}^j \right) \leq \varepsilon_{(j)}^0 \mu(\hat{J}_j \setminus J_{j}^{\text{good}}), \tag{3.71}
\]
and find a \( \tau_j > 0 \) such that \((J_{j,2}^j)_{\tau_j} \cap (J_{j}^{\text{good}})_{\tau_j} = \emptyset \).

We then construct \( g_{j+1} \) and \( \hat{g}_{j+1} \) as we did for \( g_2 \) and \( \hat{g}_2 \) in Step3: one needs only to adjust the indexes. We will refer to the variants of properties \((g_{2:a})-(g_{2:c})\) and \((\hat{g}_{2:a})-(\hat{g}_{2:c})\) by \((g_{j+1:a})-(g_{j+1:c})\) and \((\hat{g}_{j+1:a})-(\hat{g}_{j+1:c})\).

We then have to modify \( \hat{g}_{j+1} \) so that it vanishes on \( J_{j}^{\text{good}} \) and stays the same on \( (J_{j,2}^j)_{\tau_j/2} \). This is accomplished by replacing \( \hat{g}_{j+1} \) with:
\[
\hat{g}_{j+1} \max \left( 0, 1 - \frac{2}{\tau_j} \text{dist}(\cdot, (J_{j,2}^j)_{\tau_j/2}) \right); \tag{3.72}
\]
we also record the upper bound
\[
L(\hat{g}_{j+1}) \leq \sqrt{3} + C^2 \frac{\varepsilon_0}{\varepsilon_{(j+1)}^0} + 2 \frac{\varepsilon_{(j+1)}^0 + \varepsilon_{(j+1)}^1}{\tau_j}, \tag{3.73}
\]
and impose the restriction \( 2(\varepsilon_{(j+1)}^0 + \varepsilon_{(j+1)}^1) \leq \alpha_j \tau_j \) to get the last term in (3.73) to be \( \leq \alpha_j \).

We now let \( G_{j+1} = G_j + \hat{g}_{j+1} \) and, akin to (3.65), we obtain the upper bound:
\[
L(G_{j+1}) \leq \sqrt{3} + C^2 \frac{\varepsilon_0}{\min_{1 \leq j \leq 1} \varepsilon_{(j)}^{(l)}} + 2\alpha_j, \tag{3.74}
\]
after imposing the restriction \( 2(\varepsilon_{(j)}^1 + \varepsilon_{(j+1)}^1) \leq \alpha_j R_j \).

Compared to Step3, the analogues of (3.68), (3.69) require some modifications because the errors cumulate additively; keep also in mind that for us an empty sum like \( \sum_{j<1} c_j \) defaults to 0. For \( x \in C_{a}^{(l)} \) where \( l \leq j \) we obtain: then:
\[
\left\| T_{x,a}^{(l)} G_{j+1} - h \right\|_{\infty,B} \leq \left\| G_{l-1} - G_{l-1}(x) \right\|_{\infty,B(x,r_a^{(l)})} + \left\| T_{x,a}^{(l)} \hat{g}_l - h \right\|_{\infty,B} + \sum_{1 < k \leq j+1} \left\| \hat{g}_k - \hat{g}_k(x) \right\|_{\infty,B(x,r_a^{(l)})}. \tag{3.75}
\]

The first term in (3.75) is bounded observing that \( G_{l-1} \) is \( \alpha_{l-1} \) Lipschitz on \( B(x, r_a^{(l)}) \) for \( k < l \). The second term is bounded using \((g_l:c)\) and \((\hat{g}_l:a)\). Finally
the third term is bounded using $\|\hat{g}_k - \hat{g}_k(x)\|_{\infty,B(x,r_a^{(l)})} \leq 2\|\hat{g}_k\|_{\infty}$ and mind-
ing that we imposed the restriction $2(\varepsilon_s^{(k+1)} + \varepsilon_s^{(k+1)}) \leq \alpha_k R_k$. We thus get from (3.75):

$$
\|T_{x,r_a^{(l)}}G_{j+1} - h\|_{\infty,B} \leq \alpha_{l-1} + C(\varepsilon_s^{(l)} + \sigma) + 2\varepsilon_s^{(l)}r_a^{(l)} + \sum_{l<k<j+1} \alpha_k = \sum_{l-1 \leq k < j+1} \alpha_k + C(\varepsilon_s^{(l)} + \sigma) + 2\varepsilon_s^{(l)}r_a^{(l)}. \quad (3.76)
$$

Pick $x \in C_{a^{(j+1)}}$; then:

$$
\|T_{x,r_a^{(j+1)}}G_{j+1} - h\|_{\infty,B} \leq \|G_j - G_j(x)\|_{\infty,B(x,r_a^{(j+1)})} + \|T_{x,r_a^{(j+1)}}\hat{g}_{j+1} - h\|_{\infty,B}. \quad (3.77)
$$

Using that $G_j$ is $\alpha_j$-Lipschitz on $B(x,r_a^{(j+1)})$ and $(g_{j+1};c)$, $(\hat{g}_{j+1};a)$ we finally get:

$$
\|T_{x,r_a^{(j+1)}}G_{j+1} - h\|_{\infty,B} \leq \|G_j - G_j(x)\|_{\infty,B(x,r_a^{(j+1)})} + \|T_{x,r_a^{(j+1)}}\hat{g}_{j+1} - h\|_{\infty,B}. \quad (3.78)
$$

**Step 5: Choice of the parameters.**

There are two kinds of parameters:

- Parameters that can be chosen arbitrarily small: $\varepsilon_{s}^{(l)}$, $\varepsilon_{s}^{(l)}$, $\varepsilon_{m}^{(l)}$, $\varepsilon_{p}^{(l)}$ and $\alpha_l$.
- Parameters that can’t be chosen arbitrarily small: $\varepsilon_{s}^{(l)}$ and $\sigma$. In particular, as $\varepsilon_{m}^{(l)} \to 0$ the Lipschitz constant of $G_l$ blows-up.

We thus choose:

$$
\varepsilon_{m}^{(l)} = \begin{cases} \min\left(\frac{\varepsilon_{m}^{(l)} \cdot 1}{180\varepsilon_0}\right) & \text{for } l \geq 2, \\ \left(\frac{\varepsilon_{m}^{(2)}}{2}\right)^2 & \text{for } l = 1. \end{cases} \quad (3.79)
$$

We now estimate:

$$
\mu(J_1 \setminus J_1^{\text{good}}) \leq \mu(J_1 \setminus \hat{J}_1) + \mu(\hat{J}_1 \setminus J_1^{\text{good}}) \leq (\varepsilon_{s}^{(1)} + 1 - C^{-1}\sigma^{N-N_0})\mu(J_1); \quad (3.80)
$$

thus, if we choose $\sigma = (1.5C\varepsilon_{m}^{(2)})^{1/(N-N_0)}$ we get:

$$
\mu(J_1 \setminus J_1^{\text{good}}) \leq \left(1 - \varepsilon_{s}^{(2)}\right) \mu(J_1). \quad (3.81)
$$

For $l \geq 2$ the same argument for (3.81) yields:

$$
\mu(J_1 \setminus J_1^{\text{good}}) \leq \left(1 - \varepsilon_{s}^{(2)}\right) \mu(J_1); \quad (3.82)
$$
as $J_l \subset J_{l-1} \setminus J_{l-1}^\text{good}$ induction grants:

$$
\mu(J_l) \leq \left( 1 - \frac{\hat{\varepsilon}_m^{(2)}}{2} \right)^{l-1} \mu(K) \quad (3.83)
$$

$$
\mu(J_l \setminus J_l^\text{good}) \leq \left( 1 - \frac{\hat{\varepsilon}_m^{(2)}}{2} \right)^l \mu(K). \quad (3.84)
$$

The construction in Step 3 will be iterated finitely many times and we just need an upper bound for the smallest number of iterations which will give the desired approximation in measure (third inequality in (a)). We get:

$$
\mu \left( K \setminus \bigcup_{k=1}^l (\hat{J}_k \cap J_k^\text{good}) \right) \leq \mu(K \setminus J_1) + \sum_{k=1}^{l-1} \mu(J_{k+1}^\text{good} \setminus J_{k+1}^\text{good}) + \mu(J_l \setminus \hat{J}_l)
$$

$$
+ \sum_{k=1}^{l-1} \mu \left( (\hat{J}_k \setminus J_k^\text{good}) \setminus J_k^\text{good} \right) + \mu(J_l \setminus J_l^\text{good})
$$

$$
\leq \left( \varepsilon_m^{(1)} + \sum_{k=1}^{l-1} \varepsilon_m^{(k+1)} + \varepsilon_m^{(1)} \sum_{k=1}^l (1 - \frac{\varepsilon_m^{(2)}}{2})^{k-1} + \sum_{k=1}^{l-1} \varepsilon_p^{(k)} + \left( 1 - \frac{\varepsilon_m^{(2)}}{2} \right)^l \right) \mu(K).
$$

(3.85)

Recall that at each stage we can choose $\varepsilon_m^{(k)}$ and $\varepsilon_p^{(k)}$ arbitrarily small and observe that:

$$
\varepsilon_m^{(1)} \sum_{k=1}^l (1 - \frac{\varepsilon_m^{(2)}}{2})^{k-1} \leq 2 \frac{\varepsilon_m^{(1)}}{\varepsilon_m^{(2)}} = 2 \varepsilon_m^{(2)} \leq \varepsilon_m.
$$

(3.86)

Thus there is a universal constant $C_1$ such that if $l := \lceil \log_{1 - \varepsilon_m^{(2)}}{\varepsilon_m} \rceil$ one has:

$$
\mu \left( K \setminus \bigcup_{k=1}^l (\hat{J}_k \setminus J_k^\text{good}) \right) \leq C_1 \varepsilon_m \mu(K). \quad (3.87)
$$

Thus we will let $J := \bigcup_{k=1}^l (\hat{J}_k \setminus J_k^\text{good})$. We then have:

$$
\|G_l\|_\infty \leq 2 \sum_{k=1}^l (\varepsilon_s^{(k)} + \varepsilon_s^{(k)})
$$

(3.88)

which can be made $\leq C_1 \varepsilon_s$ as the parameters $\varepsilon_s^{(k)}$, $\varepsilon_s^{(k)}$ can be chosen arbitrarily small at each stage. Also the parameters $\alpha_k$ can be chosen arbitrarily small; thus, as $\varepsilon_m^{(2)} \simeq \varepsilon_m$ we obtain a universal constant $C_0$ such that:

$$
L(G_l) \leq \left( \sqrt{3} + C_0 \frac{\varepsilon_0}{\varepsilon_m} \right).
$$

(3.89)
We thus let \( g := G_t \) and (3.58) now follows from (3.76), (3.78) if we choose at each step the parameters \( \alpha_k, \varepsilon^{(k)}_s \) and \( \hat{\varepsilon}^{(k)}_s \) sufficiently small. \( \square \)

3.2. Proof of Theorem 1.4(II). The proof will be achieved by an iteration of the Lemma 3.10 below which is a simple consequence of Lemma 3.9. More precisely, the function \( g \) required in Definition 1.2 is obtained as a sum of functions \( g_i \) given by Lemma 3.10. One of the subtle points in the process is that in principle it could be \( L(g_i) \geq \sqrt{3} \) for every \( i \), hence the sum could fail to be Lipschitz in general. However, the fact that the \( g_i \)'s can be chosen asymptotically flat and with arbitrarily small norm, allows one to control the Lipschitz constant of the sum.

**Lemma 3.10.** Let \( f : \Omega \to \text{Lip}(B,0) \) be a Borel map such that, for \( \mu \)-a.e. \( x \in \Omega \), \( f(x) \in C(\mu,x) \) and moreover the corresponding function \( L \) in (1.1) vanishes. Let \( K \subset \Omega \) be a compact set such that \( L(f(x)) \leq 1 \) for every \( x \in K \) and let \( \varepsilon > 0 \) be fixed. There are constants \( C_0 \) and \( C_1 \), depending only on \( N \), \( \alpha \), \( (C_0 + \sqrt{3}) \)-Lipschitz function \( g : \Omega \to \mathbb{R} \) and a compact \( J \subset K \) such that:

(a) \( g \) is asymptotically flat on \( J \), \( \|g\|_{\infty} \leq \varepsilon \) and \( \mu(K \setminus J) \leq 3C_1\varepsilon\mu(K) \).

(b) There are \( 0 < r_1 \leq r_0 \leq \varepsilon \) and for every \( x \in J \) there are \( r_1 \leq r(x) \leq r_0 \) such that:

\[
\|T_{x,r(x)}g - f(x)\|_{\infty,B} \leq 3C_1\varepsilon. \tag{3.90}
\]

(c) \( g \) is supported on the tubular neighborhood of \( K \) with radius \( \varepsilon \).

**Proof.** Let \( C_0 \) and \( C_1 \) be the constants in Lemma 3.9. Since \( f(x) \in C(\mu,x) \) for \( \mu \)-a.e. \( x \), by the Lusin’s theorem we can find at most \( N+1 \) disjoint compact sets \( K_j \subset K \) \( (j = 0, \ldots, N) \) of positive measure, such that

\[
\mu(K \setminus \bigcup_{j=0}^{N} K_j) < C_1\varepsilon\mu(K). \tag{3.91}
\]

and \( f(x) \in C(\mu,x) \) for every \( x \in K_j \), for every \( j \). Moreover, on each \( K_j \) both \( V(\mu,x) \) and \( f(x) \) vary continuously in \( x \) and \( V(\mu,\cdot) \) has constant dimension \( j \).

Since the Grassmannian of \( j \)-planes in \( \mathbb{R}^N \) is totally bounded, and since \( V(\mu,x) \) and \( f(x) \) vary continuously in \( x \) on each \( K_j \), we can find finitely many disjoint non-empty compact subsets \( K_j^\ell \subset K_j \) \( (\ell = 1, \ldots, k_j) \) such that

\[
\|V(\mu,x) - V(\mu,y)\|_{\infty} \leq \varepsilon^{2N} \tag{3.92}
\]

and

\[
\|f(x) - f(y)\|_{\infty} \leq C_1\varepsilon \tag{3.93}
\]

for every pair \( (x,y) \) of points in \( K_j^\ell \) and moreover

\[
\mu(K_j \setminus \bigcup_{\ell} K_j^\ell) \leq \frac{C_1\varepsilon}{(N+1)}\mu(K). \tag{3.94}
\]
Choose for each $j, l$ a point $x_{j}^{l}$ and let $f_{j}^{l} := f(x_{j}^{l}) \in \text{Lip}$. Notice that, by assumption, $L(f_{j}^{l}) \leq 1$. Let
\[
d := \min\{1, \text{dist}(K, (\mathbb{R}^{N} \setminus \Omega)), \min\{\text{dist}(K_{j}^{l}, K_{k}^{m}) : l \neq m\}\}.
\]
For every $(j, \ell)$, apply Lemma 3.9 with $K := K_{j}^{l}$, $\pi := V(\mu, x_{j}^{l})$, $h := f_{j}^{l} \cdot \pi^{\perp}$ and $\varepsilon_{0} := 2^{N}, r_{0} := \frac{d}{4}, \varepsilon_{m} := \varepsilon^{N}, \varepsilon_{a} := C_{4}^{-1}\varepsilon \min\{1, \frac{d}{4}\}$. By the lemma, for every $(j, \ell)$ there exist a $(\sqrt{3} + C_{0})$-Lipschitz function $g_{j}^{l} : \mathbb{R}^{N} \to \mathbb{R}$ and a compact set $J_{j}^{l} \subset K_{j}^{l}$ such that:

(a) $g_{j}^{l}$ is asymptotically flat on $J_{j}^{l}$, $\|g_{j}^{l}\|_{\infty} \leq \varepsilon \min\{1, \frac{d}{4}\}$ and $\mu(K_{j}^{l} \setminus J_{j}^{l}) \leq C_{4}\varepsilon^{N}(K_{j}^{l})$.

(b) One can write every $J_{j}^{l}$ as a finite disjoint union $J_{j}^{l} = \bigcup_{a=1}^{M} J_{j,a}^{l}$ and for each $a \in 1, \ldots, M$ ($M$ may depend on $j$ and $\ell$) there is an $0 < r_{a} := r_{a}(\ell, j) \leq r_{0}$ such that if $x \in J_{j,a}^{l} \setminus J_{j}^{l}$ one has:
\[
\left\|T_{x,r_{a}}g_{j}^{l} - f\right\|_{\infty,B} \leq 2C_{4}\varepsilon.
\]

Via a simple cut-off, we can modify each $g_{j}^{l}$ to a $(\sqrt{3} + C_{0})$-Lipschitz function $\tilde{g}_{j}^{l}$ supported on $\Omega$ such that
\[
\tilde{g}_{j}^{l} = g_{j}^{l} \quad \text{on} \quad \left\{x : \text{dist}(x, K_{j}^{l}) < \frac{d}{4}\varepsilon\right\}
\]
and
\[
\tilde{g}_{j}^{l} = 0 \quad \text{on} \quad \left\{x : \text{dist}(x, K_{j}^{l}) > \frac{d}{2}\varepsilon\right\}
\]
Finally we define the function $g : \Omega \to \mathbb{R}$ by
\[
g := \sum_{j, \ell} \tilde{g}_{j}^{l}
\]
and we observe that, by (a), $\|g\|_{\infty} \leq \varepsilon$. Denoting $J := \bigcup_{j, \ell} J_{j}^{l}$, by (3.91), (3.94) and (a), we get
\[
\mu(K \setminus J) \leq 3C_{4}\varepsilon\mu(K).
\]
Moreover, denoting $r_{1} := \min_{j, \ell, a}\{r_{a}(\ell, j)\}$ and setting for every $x \in J_{j,a}^{l}$ $r(x) := r_{a}(\ell, j)$, we get by (3.93) and (b) that it holds $0 < r_{1} \leq r(x) \leq r_{0} \leq \varepsilon$ and
\[
\left\|T_{x,r(x)}g - f(x)\right\|_{\infty,B} \leq 3C_{4}\varepsilon,
\]
where we observe that by the choice of $r_{0}$, it holds $T_{x,r(x)}g = (g_{j}^{l})_{x,r(x)}$ for every $x \in J_{j}^{l}$. \hfill $\square$

**proof of Theorem 1.4(II).** Step 1: Prescribing the linear part. Fix $\varepsilon > 0$. It is not restrictive to assume that $\mu(\Omega) = 1$. Let $K$ be a compact set such that $\mu(K) < \varepsilon/4$, $f(x) \in C(\mu, x)$, and $L(f(x)) \leq D$, for every $x \in K$ and for some $D > 0$. Without loss of generality we can assume that $D = 1$. 
Firstly, for every \( x \in K \) we extend the linear function \( L \) which \( f(x) \) defines on \( V(\mu, x) \) (see [1.1]) to a linear function \( \tilde{L} \) defined on \( \mathbb{R}^N = (V(\mu, x), V(\mu, x)^\perp) \) as

\[
\tilde{L}(x, y) := L(x).
\]

Then we take any Borel measurable extension \( \bar{L} \) of \( \tilde{L} \) defined on the set \( \Omega \) and preserving the bound \( L(\bar{L}) \leq 1 \). By Theorem 2.1 applied to \( f = \bar{L} \) and \( \zeta = 1 \), we can find a compact set \( K^0 \subset K \) and a function \( g_0 \in C^1_c(\Omega) \) with \( \|g_0\|_\infty \leq C \) such that

\[
\mu(\Omega \setminus K^0) < \varepsilon/2
\]

\[
Dg_0(x) = \bar{L}(x) = \tilde{L}, \quad \text{for every } x \in K^0
\]

and \( L(g_0) \leq C \).

**Step 2: Prescribing the non-linear part.** Consider the function \( f_0: \Omega \to \text{Lip} \) such that \( f_0(x) \equiv 0 \) for every \( x \in \Omega \setminus K \) and \( f_0(x) = f(x) - \tilde{L}(x) \) for every \( x \in K \). We will apply Lemma 3.10 to a sequence of sets \( (K_i)_{i \in \mathbb{N}} \) (with \( K_0 := K \)), the map \( f_0 \) and a sequence of parameters \( (\varepsilon_i)_{i \in \mathbb{N}} \) with \( \varepsilon_i \to 0 \) and we will obtain respectively functions \( g^i \), compact sets \( J_i := K_{i+1} \), and for every \( x \in J_i \) a radius \( r^i(x) \leq r^i(x) \leq r^0 \leq \varepsilon_i \).

Since we can choose the \( \varepsilon_i \) inductively, we can assume that for every \( i \) it holds

\[
\varepsilon_i r^i(x) \geq \sum_{j>i} \varepsilon_j
\]

so that, for every \( i \) it holds, for every \( x \in J_i \)

\[
\|T_{x,r^i(x)}(g^i + \sum_{j>i} g^j) - f_0(x)\|_{\infty,B} \leq \|T_{x,r^i(x)} g^i \|_{\infty,B} + \|T_{x,r^i(x)} \sum_{j>i} g^j\|_{\infty,B} \leq 3C_1 \varepsilon_i + \sum_{j>i} \varepsilon_j \frac{r^j(x)}{r^i(x)} \leq (3C_1 + 1) \varepsilon_i.
\]

Moreover, since the \( g^j \)'s are asymptotically flat on the \( J_j \)'s, we can add the further restriction on the inductive choice of the \( \varepsilon_i \)'s that, for every \( j \) and for every \( i > j \), \( g_j \) is \((\varepsilon_j)^j\)-Lipschitz on a tubular neighbourhood of \( J_j \) of radius \( \varepsilon_i \), below scale \( \varepsilon_i \).
Hence, since \( r^i(x) \leq r^i_0 \leq \varepsilon_i \), we have, for every \( x \in J_i \)

\[
\left\| T_{x,r^i(x)}(g^i + \sum_{j<i} g^j) - f_0(x) \right\|_{\infty,B} \\
\leq \left\| T_{x,r^i(x)}g^i - f_0(x) \right\|_{\infty,B} + \left\| T_{x,r^i(x)} \sum_{j<i} g^j \right\|_{\infty,B} \\
\leq \left\| T_{x,r^i(x)}g^i - f_0(x) \right\|_{\infty,B} + \sum_{j<i} \left\| g^j \right\|_B \sum_{j<K_i}(x) \\
\leq 3C_1\varepsilon_i + \sum_{j<i} (\varepsilon_j)^i.
\] (3.98)

Denote \( g_1 := \sum_{i \in \mathbb{N}} g^i \) and \( J := \cap_i K_i \). Combining (3.97) and (3.98) we have that, provided \((\varepsilon_i)_{i \in \mathbb{N}}\) respects the choices made above and provided \( \sum_j (\varepsilon_j)^i \to 0 \) as \( i \to \infty \), and \( \sum_j (\varepsilon_j) \leq \varepsilon/2 \) it holds that \( \mu(\Omega \setminus J) \leq \varepsilon/2 \) and moreover, for every \( x \in J \),

\[
\left\| T_{x,r^i(x)}g_1 - f_0(x) \right\|_{\infty,B} \to 0 \quad \text{as } i \to \infty.
\]

**Step 3: Lipschitz estimates.** It remains to show that \( g_1 \) is Lipschitz. Let \( x, y \in \Omega \) with \( x \neq y \). Firstly observe that if \( |x - y| \geq \frac{1}{2} \sum_i \varepsilon_i \), then the estimate is very simple, indeed

\[
|g_1(y) - g_1(x)| \leq 2 \sum_{i=1}^{\infty} \|g^i\|_\infty \leq 2 \sum_i \varepsilon_i \leq 4|x - y|.
\] (3.99)

Otherwise let us consider different cases. Here we make the following assumption on the sequence \((\varepsilon_j)_{j \in \mathbb{N}}\): for every \( j \) it holds \( \varepsilon_j \geq 2\varepsilon_k \) for every \( k > j \), hence in particular \( \varepsilon_j \geq \sum_{k>j} \varepsilon_k \).

- if either \( x \in J \) or \( y \in J \). Let \( j_0 \) be the first index \( j \) such that \( |y - x| \geq \varepsilon_j \). In particular \( x \) and \( y \) are in \( B_{\varepsilon_{j_0-1}}(J^{j_0-1}) \). Since we know that for every \( j < j_0 - 1 \), the function \( g^j \) is \((\varepsilon_j)^{j_0-1}\)-Lipschitz on the tubular neighbourhood of \( B_{\varepsilon_{j_0-1}}(J^j) \), below scale \( \varepsilon_{j_0-1} \), this implies that \( \sum_{j=1}^{j_0-2} g^j \) is \((\varepsilon_j)^{j_0-1}\)-Lipschitz on \( B_{\varepsilon_{j_0-1}}(J^{j_0-1}) \) below scale \( \varepsilon_{j_0-1} \) and in particular \( |(\sum_{j=1}^{j_0-2} g^j)(y) - (\sum_{j=1}^{j_0-2} g^j)(x)| \leq |y - x| \). Moreover, since
\( \varepsilon_{j_0} \geq \sum_{k > j_0} \varepsilon_k \), it holds \( |y - x| \geq \frac{1}{2} \sum_{j = j_0}^{\infty} \varepsilon_j \). Hence we can write

\[
|g_1(y) - g_1(x)| \leq |\left( \sum_{j=1}^{j_0-2} g^j(y) \right) - \left( \sum_{j=1}^{j_0-2} g^j(x) \right)| + |g^{j_0-1}(y) - g^{j_0-1}(x)| + |\left( \sum_{j=j_0}^{\infty} g^j(y) \right) - \left( \sum_{j=j_0}^{\infty} g^j(x) \right)|
\]

\[
\leq (1 + \sqrt{3} + C_0)|y - x| + 2 \sum_{j=j_0}^{\infty} \varepsilon_j \leq (5 + \sqrt{3} + C_0)|y - x|.
\]

(3.100)

- If \( x \notin J \) and \( y \notin J \), let \( i_0 \) (respectively \( j_0 \)) be the first index \( i \) such that \( x \notin B_{\varepsilon_i}(J^i) \) (respectively \( y \notin B_{\varepsilon_j}(J^j) \)). We can assume, without loss of generality, that \( i_0 \leq j_0 \).

If \( |x - y| < \varepsilon_{i_0+1} \), then necessarily \( j_0 - i_0 \leq 1 \). In this case (recalling (c) in Lemma 3.10) \( g^j(x) = g^j(y) = 0 \) for every \( j \geq i_0 + 1 \). Moreover for every \( i < i_0 - 1 \), the function \( g^i \) is \((\varepsilon_i)^{i_0-1}\)-Lipschitz on the tubular neighbourhood of \( B_{\varepsilon_{i_0-1}}(J^i) \), below scale \( \varepsilon_{i_0-1} \), hence, as in the previous case, we can estimate

\[
|g_1(y) - g_1(x)| \leq |\left( \sum_{i=1}^{i_0-2} g^i(y) \right) - \left( \sum_{i=1}^{i_0-2} g^i(x) \right)| + |g^{i_0}(y) - g^{i_0}(x)| + \sum_{i=i_0-1}^{i_0} (\varepsilon_i)^{i_0}|y - x| + 2(\sqrt{3} + C_0)|y - x| \leq (1 + 2\sqrt{3} + 2C_0)|y - x|.
\]

(3.101)

The last case to analyse is when \( |x - y| \geq \varepsilon_{i_0+1} \). In this case let \( k_0 \leq i_0 + 1 \) be the first index \( k \) such that \( |x - y| \geq \varepsilon_k \). Note that if \( k_0 = 1 \), then we fall in the first case considered, because we have required in particular that \( \varepsilon_1 \geq \sum_{j > 1} \varepsilon_j \), and hence (3.99) provides the Lipschitz estimate. Therefore we can consider only the case \( k_0 \geq 2 \).

Since \( |x - y| \leq \varepsilon_{k_0-1} \), then for every \( i < k_0 - 1 \), the function \( g^i \) is \((\varepsilon_i)^{k_0-1}\)-Lipschitz on the tubular neighbourhood of \( B_{\varepsilon_{k_0-1}}(J^i) \), below
scale $\varepsilon_{k_0-1}$, hence

$$|g_1(y) - g_1(x)| \leq \left( \sum_{j=1}^{k_0-2} |g^j(y)| - \sum_{j=1}^{k_0-2} |g^j(x)| \right) + |g^{k_0-1}(y) - g^{k_0-1}(x)|$$

$$+ |g^{k_0}(y) - g^{k_0}(x)| + 2 \sum_{j=k_0+1}^{\infty} \varepsilon_j$$

$$\leq \left( \sum_{j=1}^{k_0-2} (\varepsilon_j^{k_0-1} + 2\sqrt{3} + 2C_0) \right) |y - x| + 2\varepsilon_{k_0}$$

$$\leq (4 + 2\sqrt{3} + 2C_0) |y - x|.$$

**Step4:** Conclusion of the proof. Consider the Lipschitz function $g := g_0 + g_1$. It is easy to see that it holds

$$\mu(\{x \in \Omega : f(x) \not\subset \Tan(g, x)\}) \leq \mu(\Omega \setminus (K^0 \cap J)) < \varepsilon.$$

Indeed $d_{V^0(\mu, x)}g_1 = 0$ on $J$ and $d_{V^\perp(\mu, x)}g_0 = 0$ on $K^0$. Hence, since $\varepsilon$ can be chosen arbitrarily small, $f$ prescribes the blowups of a Lipschitz function weakly in the Lusin sense.

$$\square$$

**Remark 3.11.** As we will show in the proof of Theorem 1.5 (II), once it is possible to prescribe weakly (in the Lusin sense) a blowup in a closed class of admissible functions, it is also possible to prescribe all the admissible blowups “simultaneously”. By this we mean that it is possible to find a Lipschitz function attaining all admissible blowups on an arbitrarily large set of points. In the previous proof it is sufficient to select (in a measurable way) a countable dense set of admissible blowups $\{g_i(x)\}$ for every point $x$, and, selecting in a suitable way different blowups at different scales, one can build a function $f$ attaining at many points all the $g_i$’s as blowups. To conclude, it is sufficient to observe that the set of all blowups at one point is closed.

**4. Optmality of the class $C(\mu, \cdot)$**

We now give an example of a measure for which one cannot prescribe more blowups than those contained in $C(\mu, \cdot)$. In general it seems a hard problem to characterize the largest set of blowups one can prescribe in terms of structural properties of the Radon measure.

Given a Radon measure $\mu$ on $\mathbb{R}^N$, $r > 0$ and a point $x$ we define the measure $T_{x, r}\mu$ on $B$ by

$$T_{x, r}\mu(A) := \mu(x + rA),$$

for every Borel set $A \subset B$. 

We denote by $\text{Tan}(\mu, x)$ the set of the blowups of $\mu$ at $x$, i.e. all the possible limits of the form
\[
\lim_{r_i \searrow 0} \kappa_i
\]
where
\[
\kappa_i := \frac{T_{x,r_i}(\mu)}{\mu(B(x,r_i))}.
\]

Fix $k \in \{1, \ldots, N-1\}$ and let $\nu_\mathbb{R}$ be a (doubling) Radon measure on $\mathbb{R}$ such that $\nu_\mathbb{R}$ is singular with respect to the Lebesgue measure, its support is $\mathbb{R}$ and for $\nu_\mathbb{R}$-a.e $x \in \mathbb{R}$ the set $\text{Tan}(\nu_\mathbb{R}, x)$ contains only the Lebesgue measure. Examples of such measures are discussed in [Pre87] or can be obtained modifying the example of [GKS10].

Let $\mu$ be the product measure
\[
\mu := \mathcal{L}^k \otimes \nu_\mathbb{R} \otimes \cdots \otimes \nu_\mathbb{R}
\]
and consider a Lipschitz function $f$ on $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$. The decomposability bundle $V(\mu, \cdot)$ coincides with $\mathbb{R}^k$ as $\mu$, by the properties of $\nu_\mathbb{R}$, is concentrated on a set which intersects each $C^1$-curve $\gamma$ whose tangent vector does not lie in $\mathbb{R}^k$ in a set of zero 1-dimensional Hausdorff measure, and thus we have a well defined derivative $d_{\mathbb{R}^k}f$ in the direction of $\mathbb{R}^k$. Observe that $\mu$ is also doubling and fix a point $P$ of approximate continuity for $d_{\mathbb{R}^k}f$. We can assume that at $P$ all blowups of $\mu$ are positive multiples of $\mathcal{L}^N$.

Let $g$ be a blowup of $f$ at $P$ and let $(r_i)_{i \in \mathbb{N}}$ be a sequence of radii for which $g = \lim_{i \to \infty} T_{P,r_i}f$. Let $\tilde{\mu}$ be the limit of any converging subsequence of $\kappa_i$, defined in (4.1) with $x = P$. Since the support of $\tilde{\mu}$ is the whole ball $B$ and since $d_{\mathbb{R}^k}f$ is approximately continuous at $P$, for every $q \in \mathbb{R}^N$ and $v \in \mathbb{R}^k$ such that $q + v \in B$ we get
\[
g(q + v) - g(q) = \langle d_{\mathbb{R}^k}f(P), v \rangle.
\]
Let $m$ be the (Lipschitz) restriction of $g$ to $\mathbb{R}^{N-k} \cap B$. Then for every $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{N-k}$ such that $x + y \in B$ it holds
\[
g(x, y) = m(y) + \langle d_{\mathbb{R}^k}f(P), x \rangle,
\]
hence $g \in C(\mu, P)$.

5. PROOF OF THEOREM 1.5

As we already observed in point (iii) of Remark 1.6 statement (I) of Theorem 1.5 is contained in Proposition 4.2 of [Mar17]. Regarding statement (II), we will prove a stronger (perhaps surprising) statement: namely we will prove that if $\mu$ is singular, then the generic 1- Lipschitz function (in the sense of Baire categories) attains every 1-Lipschitz function as blowup at $\mu$-almost every point.

In this section we denote by $X$ the complete metric space of 1-Lipschitz functions on $\mathbb{R}$ endowed with the supremum norm. By $\mu$ we denote a singular probability measure on $\mathbb{R}$. We begin with the following lemma.
Lemma 5.1 (Covering by intervals with non-negligible centres). Let $U \subset \mathbb{R}$ be an open set. For every $r_0 > 0$ and $n \in \mathbb{N}$ there is a sequence of closed intervals $\{[x_j - r_j, x_j + r_j]\}_j$ contained in $U$ with disjoint interiors such that:

$$r_j \leq r_0$$

$$\mu(\bigcup_j [x_j - (8n)^{-1}r_j, x_j + (8n)^{-1}r_j]) \geq \frac{1}{16} n^{-1} \mu(\bigcup_j [x_j - r_j, x_j + r_j])$$

$$\mu(U \setminus \bigcup_j [x_j - r_j, x_j + r_j]) = 0.$$ 

**Proof.** Given any measure $\mu$ on $\mathbb{R}$ and $0 < \lambda < 1$, it is known that for $\mu$-a.e. $x$ there holds

$$\limsup_{r \to 0} \frac{\mu(B(x, \lambda r))}{\mu(B(x, r))} \geq \lambda.$$ 

The lemma then follows from Besicovitch Covering Theorem. We thank the referee for suggesting this short proof. We also show a constructive proof, in order to help the reader to understand the ideas behind Corollary 3.7.

Through the proof the closed interval $[x - r, x + r]$ will be denoted by $I(x, r)$.

Firstly we apply Corollary 2.3 with $\varepsilon = 2^{-6}$ obtaining a sequence of disjoint intervals $\{I(z_\lambda, r_\lambda)\}_\lambda$. By the choice of $\varepsilon$, for every $\lambda$ it holds that

$$\mu(I(z_\lambda, (1 - 2^{-6})r_\lambda)) \geq \frac{1}{2} \mu(I(z_\lambda, r_\lambda)).$$

Now we “split” each interval $I(z_\lambda, (1 - 2^{-6})r_\lambda)$, into $2^7 - 2$ sub-intervals

$$\{I_i^\lambda := I(z_\lambda^i, 2^{-7}r_\lambda)\}_{i=1}^{2^7-2}.$$ 

with disjoint interiors and length $2^{-6}r_\lambda$. Denote by $\bar{I}_\lambda^i$ the “central part” of $I_i^\lambda$, i.e.

$$\bar{I}_\lambda^i := I(z_\lambda^i, 2^{-10}n^{-1}r_\lambda).$$

Observe that, for every $\lambda$, the family

$$\left\{ \bigcup_{i=1}^{2^7-2} \bar{I}_\lambda^i + j2^{-9}n^{-1}r_\lambda \right\}_{j=0,\ldots,8n-1}$$

covers the set $I(z_\lambda, (1 - 2^{-6})r_\lambda)$. Hence for at least one index $j_0$, the set

$$\bigcup_{i=1}^{2^7-2} \bar{I}_\lambda^i + j_02^{-9}n^{-1}r_\lambda$$

satisfies

$$\mu\left( \bigcup_{i=1}^{2^7-2} \bar{I}_\lambda^i + j_02^{-9}n^{-1}r_\lambda \right) \geq \frac{1}{8} n^{-1} \mu(I(z_\lambda, (1 - 2^{-6})r_\lambda)) \geq \frac{1}{16} n^{-1} \mu(I(z_\lambda, r_\lambda)).$$
Moreover, for every $j = 0, \ldots, 8n - 1$ and every $i = 1, \ldots, 2^7 - 2$ the interval $I_i^j + j2^{-9}r^{-1}r$ is contained in the interior of $I(z, r)$ The result follows by adding to these intervals the two intervals $[z - r, a]$ and $[b, z + r]$, where $a$ and $b$ are respectively the minimum and the maximum of the set in (5.5). Of course the procedure above should be also repeated for every $\lambda$. □

**Proposition 5.2.** Let $\mu$ be a singular probability measure on $\mathbb{R}$. Let $f : [-1, 1] \to \mathbb{R}$ be a 1-Lipschitz function with $f(0) = 0$. Then the set

$$X_f := \{ g \in X : f \in \tan(g, x) \text{ for } \mu - \text{almost every } x \}$$

is residual in $X$ (i.e. it contains the intersection of countably many open and dense sets).

**Proof.** For $n \in \mathbb{N}$ and $g \in X$, consider the set

$$E^n_g := \{ x \in \mathbb{R} : \exists \rho < n^{-1} \text{ s.t. } |f - T_{x, \rho}g| < n^{-1} \}.$$

First of all we notice that $E^n_g$ is open. Indeed if $x \in E^n_g$, $\rho \leq n^{-1}$ satisfies $|f - T_{x, \rho}g| < n^{-1}$ and $y \in \mathbb{R}$ is so that

$$2|y - x| < \rho(n^{-1} - |f - T_{x, \rho}g|),$$

then, using that $g$ is 1-Lipschitz, we deduce

$$|f - T_{y, \rho}g| \leq |f - T_{x, \rho}g| + |T_{x, \rho}g - T_{y, \rho}g| \leq |f - T_{x, \rho}g| + 2\rho^{-1}|y - x| < n^{-1},$$

hence $y \in E^n_g$. Now we define

$$A_n := \{ g \in X : \mu(E^n_g) > 1 - n^{-1} \}.$$

**Step1:** $A_n$ is open. Fix $g \in A_n$ and consider the multifunction $\varrho : E^n_g \to 2^{(0, n^{-1})}$ defined by

$$x \mapsto \{ \rho \in (0, n^{-1}) \text{ s.t. } |f - T_{x, \rho}g| < n^{-1} \}.$$

Notice that the values of $\varrho$ are non-empty open sets because the function $(x, \rho) \mapsto |f - T_{x, \rho}g|$ is continuous in the variable $\rho$. Moreover, since the function is continuous also in the variable $x$, for $\delta > 0$ the sets

$$U_\delta := \{ x \in E^n_g : \rho_0(x) := \sup \{ \varrho(x) \} > \delta \text{ and there exists } \rho(x) \text{ s.t. } \delta < \rho(x) \in \varrho(x) \text{ and } |f - T_{x, \rho(x)}g| < n^{-1} - \delta \}$$

are open and $\bigcup_{\delta > 0} U_\delta = E^n_g$. Moreover $\mu(E^n_g) > 1 - n^{-1}$, since $g \in A_n$. Then there exists $\delta > 0$, with $\mu(U_\delta) > 1 - n^{-1}$.
If we consider now \( h \in X \) such that \( 2 |g - h| < \delta^2 \), we deduce that for every \( x \in U_\delta \) it holds

\[
|f - T_{x, \rho(x)} h| \leq |f - T_{x, \rho(x)} g| + |T_{x, \rho(x)} g - T_{x, \rho(x)} h| \\
\leq |f - T_{x, \rho(x)} g| + 2|g - h| \delta^{-1} < (n^{-1} - \delta) + \delta.
\]

The last inequality guarantees that \( A_n \) is open.

**Step 2:** \( A_n \) is dense. Let \( g \in X \) and fix \( \varepsilon > 0 \). We want to show that there exists \( h \in A_n \) such that \( |h - g| \leq \varepsilon \).

Consider inductively a sequence of functions \( h_i \) defined as follows. Let \( h_0 := g, M_0 := \mathbb{R}, \alpha_0 := \varepsilon \) and for \( i = 1, 2, \ldots \) let \( U_i \subset M_{i-1} \) be an open set such that

\[
\mathcal{L}^1(U_i) \leq \frac{\alpha_{i-1}}{16n},
\]

and

\[
\mu(M_{i-1} \setminus U_i) < \frac{1}{n2^{i+2}}.
\]

Moreover by Corollary 5.1 we can select \( I_{1}^i, \ldots, I_{m(i)}^i \subset U_i \) closed intervals with center \( x_j^i \), length \( 4\ell_j^i \) and disjoint interiors such that firstly

\[
\mu(U_i \setminus \bigcup_{j=1}^{m(i)} I_j^i) < \frac{1}{n2^{i+2}}
\]

and secondly, denoting \( I_j^i \) the closed interval with center \( x_j^i \) and length \( (2n)^{-1}\ell_j^i \),

\[
\mu(\bigcup_{j=1}^{m(i)} I_j^i) \geq (16n)^{-1} \mu(\bigcup_{j=1}^{m(i)} I_j^i), \quad \text{for every } i.
\]

Or first aim is to perturb \( h_{i-1} \) obtaining a new function \( h_i \) such that all the points in \( \bigcup_{j=1}^{m(i)} I_j^i \) belong to \( E_{h_i}^n \). Let \( f_i \) be the 1-Lipschitz function

\[
f_i(x) := h_{i-1}(x - |(-\infty, x) \cap \bigcup_{j=1}^{m(i)} I_j^i|).
\]

Observe that \( f_i \) is differentiable with \( f'_i \equiv 0 \) on the set \( \bigcup_{j=1}^{m(i)} I_j^i \). Denote by \( k_i \) the 1-Lipschitz function

\[
k_i(x) := f_i(x) + \sum_{j=1}^{m(i)} f_j^i(x),
\]

where \( f_j^i : \mathbb{R} \to \mathbb{R} \) is any 1-Lipschitz function such that \( f_j^i \equiv 0 \) on \( \mathbb{R} \setminus I_j^i \) and

\[
T_{x_j^i, \ell_j^i} f_j^i = f
\]
(observe that such a function exists because \( f \) is 1-Lipschitz, \( g(0) = 0 \) and the length of the interval \( I_j^i \) is \( 4\ell_j^i \)). Eventually we define the 1-Lipschitz function \( h_i := f_i + k_i \).

Denote, for every \( i \)
\[
\alpha_i := \min_{j=1}^{m(i)} \{ \ell_j^i \}; \quad M_i := (\bigcup_{j=1}^{m(i)} I_j^i) \setminus (\bigcup_{j=1}^{m(i)} \bar{I}_j^i).
\]

Note that the following properties hold, for every \( i,j \)

(i) \(|h_i - h_{i-1}| \leq 2L^1(U^i) \leq \frac{\alpha_{i-1}}{8n}\),

(ii) \(|T_{x,\ell_j^i}h_i - f| < (2n)^{-1}, \) for every \( x \in \bar{I}_j^i \).

Comparing (ii) with the definition of \( E_n^m \), it is evident not only that every \( x \in \bar{I}_j^i \) belongs to \( E_{h_n}^m \), but also that there is still “room” for some additional perturbation. Namely for every function \( \tilde{h} \) with \(|h_i - \tilde{h}| < (4n)^{-1}\alpha_i \) it holds that every \( x \in \bar{I}_j^i \) belongs to \( E_{\tilde{h}}^m \), indeed

\[
|T_{x,\ell_j^i}\tilde{h} - f| \leq |T_{x,\ell_j^i}h_i - f| + |T_{x,\ell_j^i}h_i - \tilde{h}|
\]
\[
\leq 2(\ell_j^i)^{-1}|h_i - \tilde{h}| + (2n)^{-1} < (2n)^{-1}(\ell_j^i)^{-1}\alpha_i + (2n)^{-1}
\]
\[
\leq n^{-1}.
\]

In particular, for every \( i,j \), for every \( x \in \bar{I}_j^i \) and for every \( m > i \) it holds \( x \in E_{h_m}^n \), since

\[
|h_i - h_m| \leq \sum_{j=i}^{m-1} \frac{\alpha_j}{8n} \leq \frac{1}{8n} \sum_{j=0}^{m-i-1} (16)^{-i} < (4n)^{-1}\alpha_i.
\]

Moreover, by (i) and the choice of \( \alpha_0 \) it follows that \(|g - h_m| < \varepsilon\), for every \( m \).

Combining (5.7), (5.8) and (5.9) we deduce that for \( i_0 \) large enough we have

\[
\mu(\bigcup_{i \leq i_0} (\bigcup_{j=1}^{m(i)} \bar{I}_j^i)) > 1 - n^{-1},
\]

hence denoting \( h := h_{i_0} \), we have that \( h \in A_n \).

**Step3: Conclusion of the proof.** Clearly every function which belongs to the intersection of the \( A_n \)'s is also in \( X_f \), hence \( X_f \) is a residual set and in particular, by the Baire theorem, it is dense in \( X \).

**Proof of theorem 1.5(II).** Without loss of generality we can assume that the Lipschitz constant of \( f(x) \) is bounded by 1 for \( \mu \)-a.e. \( x \) and that \( \Omega = \mathbb{R} \), because Proposition 5.2 also holds when \( \mathbb{R} \) is replaced by an open subset \( \Omega \) (clearly in this case the space \( X \) will be replaced by the space \( X^\Omega \) of 1-Lipschitz functions on \( \Omega \)).
First case. \( \mu \) is a finite measure. Consider the metric space \( Z \) made by the 1-Lipschitz functions on \([-1, 1]\) with value 0 at the origin, endowed with the supremum distance. Let \((f_i)_{i \in \mathbb{N}}\) be dense in \( Z \). Up to rescaling, we may assume that \( \mu \) is a probability measure. By Proposition 5.2 each set \( X_{f_i} \) is residual in \( X \), and so it is \( Y := \bigcap_i X_{f_i} \). This means that for all \( g \in Y \) and for \( \mu \)-a.e. \( x \), every \( f_i \) belongs to \( \text{Tan}(g, x) \). The theorem is then a consequence of the simple observation that \( \text{Tan}(g, x) \) is always a closed subset of \( Z \).

Second case. \( \mu \) is any Radon measure. Write \( \mathbb{R} \) as a countable union of sets \( E_i, i = 1, 2, \ldots \) with finite measure. Consider for every \( i \) the space \( Y_i \) defined above relatively to the measure \( \mu \mid_{E_i} \). Since each \( Y_i \) is residual in \( X \), so it is the set \( Y_\infty := \cap_i Y_i \).

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