MULTI-SCALE MODELING OF PROCESSES IN POROUS MEDIA
- COUPLING REACTION-DIFFUSION PROCESSES IN THE
SOLID AND THE FLUID PHASE AND ON THE
SEPARATING INTERFACES

MARKUS GAHN
Hasselt University, Faculty of Sciences
Campus Diepenbeek, Agoralaan Gebouw D, Diepenbeek 3590, Belgium

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ABSTRACT. The aim of this paper is the derivation of general two-scale compactness results for coupled bulk-surface problems. Such results are needed for example for the homogenization of elliptic and parabolic equations with boundary conditions of second order in periodically perforated domains. We are dealing with Sobolev functions with more regular traces on the oscillating boundary, in the case when the norm of the traces and their surface gradients are of the same order. In this case, the two-scale convergence results for the traces and their gradients have a similar structure as for perforated domains, and we show the relation between the two-scale limits of the bulk-functions and their traces. Additionally, we apply our results to a reaction diffusion problem of elliptic type with a Wentzell-boundary condition in a multi-component domain.

1. Introduction. In this paper, we derive general two-scale compactness results for functions defined on a periodically perforated domain with regular traces on the oscillating boundary, where the norm of the gradients of the traces is of the same order as the norm of the traces themselves. Then, the convergence results for the traces and their gradients have a similar structure as for perforated domains. However, a crucial point is the derivation of the relation between the two-scale limits of the functions and the two-scale limits of their traces. Our results depend on the topological properties of the boundary, i.e., whether it is a connected or a disconnected surface.

We consider a rectangular domain $\Omega \subset \mathbb{R}^n$ that consists of two components $\Omega_1^\epsilon$ and $\Omega_2^\epsilon$ which are separated by the interface $\Gamma^\epsilon$. Both subdomains possess a periodic structure with period $\epsilon$, which is small compared to the size of the whole domain. They are obtained by scaled and shifted reference elements $Y_1$ and $Y_2$ with interface $\Gamma$. The domain $\Omega_1^\epsilon$ is connected, while $\Omega_2^\epsilon$ can be connected or disconnected. We consider functions from the space

$$\mathbb{H}_{j,\epsilon} := \{ u_\epsilon \in H^1(\Omega_j^\epsilon) : u_\epsilon|_{\Gamma_\epsilon} \in H^1(\Gamma_\epsilon) \}.$$

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together with the inner product
\[(u_\epsilon, v_\epsilon)_{H_1^j, \epsilon} := (u_\epsilon, v_\epsilon)_{H_1(\Omega_j^\epsilon)} + \epsilon (u_\epsilon|_{\Gamma_\epsilon}, v_\epsilon|_{\Gamma_\epsilon})_{H_1(\Gamma_\epsilon)}\].

This function space is crucial for the treatment of coupled bulk-surface problems and we show a density result for smooth functions, which is—to the best of our knowledge—not yet known in the literature up to now.

It is well known, see [2, 3, 18], that for a bounded sequence \((u_\epsilon) \subset H^1(\Omega_1^\epsilon)\) with bounded extensions \(\tilde{u}_\epsilon \in H^1(\Omega)\), which exists due to [1], there exist functions \(u_0 \in H^1(\Omega)\) and \(u_1 \in L^2(\Omega, H^1(Y)/\mathbb{R})\), such that up to a subsequence
\[
\tilde{u}_\epsilon \to u_0 \quad \text{in the two-scale sense,}
\]
\[
\nabla \tilde{u}_\epsilon \to \nabla_x u_0(x) + \nabla_y u_1(x,y) \quad \text{in the two-scale sense,}
\]
\[
u_\epsilon|_{\Gamma_\epsilon} \to u_0 \quad \text{in the two-scale sense on } \Gamma_\epsilon.
\]

We show that for a bounded sequence in \(\mathbb{R}^1, \epsilon\) the \(H^1\)-regularity of the traces is inherited to the microscopic variable of the first order term \(u_1\) and it holds the following convergence of the tangential gradient
\[
\nabla_{\Gamma_\epsilon} u_\epsilon \to P_\Gamma(y) \nabla_x u_0(x) + \nabla_{\Gamma_\epsilon} u_1(x,y) \quad \text{in the two-scale sense on } \Gamma_\epsilon,
\]
where \(P_\Gamma(y)\) denotes the orthogonal projection to the tangent space of \(\Gamma\) at \(y \in \Gamma\). However, in general, this results fails for bounded sequences in \(\mathbb{R}^2, \epsilon\) defined on the disconnected domain \(\Omega_2^\epsilon\). For sequences defined only on the oscillating surface \(\Gamma_\epsilon\) the convergence result (1) for the tangential gradient was already stated without proof in [4, 15], however, the case of a disconnected surface when the result fails was not excluded explicitly. In [14], it was proven for functions on a connected surface \(\Gamma_\epsilon\) using the unfolding method together with an extension argument. In [5], the result was shown for the disconnect and connected domain using the unfolding operator under the additional assumption that the two-scale limit is an element of \(H^1(\Omega)\). Here, we prove general two-scale compactness results for the coupled bulk-surface problem, i.e., when the sequence on the surface is given as the trace of functions in the bulk domain, in case of a connected and a disconnected surface \(\Gamma_\epsilon\). Our proof is based on the Helmholtz-decomposition for periodic functions on surfaces and the usual two-scale convergence. Additionally, we give a sufficient condition under which the convergence (1) holds for a sequence \((u_\epsilon)_{\epsilon>0} \subset H^1(\Gamma_\epsilon)\) with \(\sqrt{\epsilon} ||u_\epsilon||_{H^1(\Gamma_\epsilon)}\) bounded. In fact, this convergence holds if the two-scale limit on \(\Gamma_\epsilon\) is an element of \(H^1(\Omega)\). However, we also give an example of a reaction-diffusion problem in a two-component medium, where this regularity for the limit function is not fulfilled.

Our compactness results are crucial for the homogenization of coupled bulk-surface problems in multi-component complex domains. Such type of problems arise for example in heat transfer or reaction-diffusion processes in biological systems. As a test case, we apply our results to an elliptic equation with Wentzell-boundary conditions, i.e., a boundary condition of the form
\[-\nabla u_\epsilon \cdot \nu - \alpha \Delta_{\Gamma_\epsilon} u_\epsilon = g(u_\epsilon) \quad \text{on } \Gamma_\epsilon \text{ for } \alpha > 0,\]
where \(\Delta_{\Gamma_\epsilon}\) denotes the Laplace-Beltrami operator on \(\Gamma_\epsilon\).

In a more general setting the Wentzell-boundary conditions were introduced in [25] as admissible boundary conditions for elliptic operators of second order. For applications involving this boundary condition see e.g., [9, 13, 21]. Wentzell-boundary conditions as in (2) can be obtained by asymptotic analysis, see e.g., [6, 12, 16],
where these boundary conditions were derived via a scale transition for thin layers in a mathematically rigorous way. For further references on the derivation of Wentzell-boundary conditions see e.g., the references in [8].

Our paper is organized as follows. In Section 2, we introduce our geometrical setting and the necessary function spaces. Further, we prove a density result for Sobolev functions with more regular traces, and show a Helmholtz-decomposition for periodic functions on a manifold with boundary. Section 3 is devoted to the two-scale compactness results, and in Section 4, we apply our compactness results to an elliptic problem with a Wentzell-boundary condition.

2. Geometrical setting and function spaces. In this section, we introduce the geometrical setting and suitable functions spaces for our considerations. Especially, we consider some fundamentals from differential geometry as the tangential gradient and the tangential divergence. For \( L^2 \)-functions on a manifold with boundary, we give a Helmholtz-decomposition for homogeneous and periodic boundary conditions.

Throughout this paper, we denote for an arbitrary measurable set \( G \) the inner product on \( L^2(G) \) by \( \langle \cdot, \cdot \rangle_G \). Further, if it is clear from the context, we use the notation \( L^2(G) \) as well for scalar valued and vector valued-functions in \( \mathbb{R}^n \).

2.1. The geometrical setting. We consider the domain \( \Omega = (0, l_1) \times \ldots \times (0, l_n) \subset \mathbb{R}^n \), \( n \geq 2 \) with \( l = (l_1, \ldots, l_n) \in \mathbb{N}^n \) consisting of two components denoted by \( \Omega_1 \) and \( \Omega_2 \). We assume that \( \Omega_1 \) is connected, but \( \Omega_2 \) can be connected (for \( n \geq 3 \)) or disconnected. The two subdomains are constructed in the following way: Let \( Y := (0,1)^n \) and \( Y_2 \subset Y \) be an open and connected subset with Lipschitz boundary. Further, we define \( Y_1 := Y \setminus Y_2 \), \( \Gamma := \partial Y_2 \setminus \partial Y \). For \( k \in \mathbb{Z}^n \), \( j = 1, 2 \), we set \( Y_j := Y_j + k \), \( \Gamma_k := \Gamma + k \). Additionally, we make the assumption that \( Y_1 \) is connected. Further, we define

\[
E_j := \text{int} \bigcup_{k \in \mathbb{Z}} (Y_j + k), \quad \Xi := \partial E_1 = \partial E_2.
\]

For a given sequence \( \epsilon \) with \( \epsilon^{-1} \in \mathbb{N} \), we define the microscopic domains \( \Omega_1 \) and \( \Omega_2 \) in the following way: Let \( K_\epsilon := \{ k \in \mathbb{Z}^n : \epsilon(Y + k) \subset \Omega \} \), and define

\[
\Omega_j := \text{int} \bigcup_{k \in K_\epsilon} \epsilon(Y_j + k), \quad \Omega := \Omega_1 \setminus \Omega_2, \quad \Gamma_\epsilon := \partial \Omega_1 \setminus \partial \Omega, \quad \Gamma := \partial \Omega_2 \setminus \partial \Omega,
\]

i.e., \( \Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_\epsilon \) and the interface \( \Gamma_\epsilon \) separates the two microscopic domains \( \Omega_1 \) and \( \Omega_2 \), see also Figure 1. We note that the set \( \Omega_1 \) is connected and we assume that \( \Gamma_\epsilon \) is of class \( C^2 \). Especially, this implies that sides of \( Y_1 \) and \( Y_2 \) facing each other have to coincide.

Remark 1. For the sake of simplicity, we only consider the case when \( Y_j \) is connected. However, the results can be easily extended to more general cases by considering the connected components.

2.2. Function spaces on manifolds. In this section, we introduce some function spaces on manifolds. We give the definition of the tangential gradient and tangential divergence and summarize some results of Sobolev spaces on manifolds, see [22, Section 2] for an overview about this topic. Further, we introduce the space of \( L^2 \)-vector fields with tangential divergence in a distributional sense. Let us consider a manifold \( M \subset \mathbb{R}^n \) of class \( C^2 \) such that \( M \) is compact. Hence, \( M \) can be a manifold with or without boundary. In the second case, we denote its (manifold) boundary
Figure 1. Microscopic domain for $\varepsilon = \frac{1}{4}$ and $\Omega = (0,1)^2$, and $Y_2$ is strictly included in $Y$.

by $\partial M$. For $M = \mathcal{M}$ it holds that $\partial M = \emptyset$. Later, we consider the cases $M = \Gamma$ or $M = \Gamma_\varepsilon$.

We define the spaces

$$C^k_M(\mathcal{M}) := \{ u \in C^k(\mathcal{M})^n : u \cdot \nu = 0 \text{ on } \mathcal{M} \},$$

$$L^2_M(M) := \{ u \in L^2(M)^n : u \cdot \nu = 0 \text{ a.e. on } M \},$$

for $k \in \{0, 1, 2\}$. On $L^2_M(M)$ we consider the usual inner product from $L^2(M)^n$. It is easy to check, that the space $C^k_M(\mathcal{M})$ is dense in the space $L^2_M(M)$. Now, for every $u \in L^2(M)^n$, we can define an element $P_M u \in L^2_M(M)$, where here $P_M(y)$ for $y \in M$ is the orthogonal projection on the tangent space at $y \in M$, i.e., it holds that

$$P_M(y) u(y) = u(y) - u(y) \cdot \nu(y) \nu(y) \quad \text{for a.e. } y \in M.$$

Let $\phi \in C^1(\mathcal{M})$, due to the regularity of $M$, there exist a tubular neighborhood $U$ of $M$ and an extension $\tilde{\phi} \in C^1(U)$ of $\phi$. We define the tangential gradient $\nabla_M \phi$ of $\phi$ by

$$\nabla_M \phi := P_M \nabla \tilde{\phi} = \nabla \tilde{\phi} - \nabla \tilde{\phi} \cdot \nu \nu \quad \text{on } \mathcal{M}.$$

We emphasize, that this definition is independent of the chosen extension of $\phi$. On the space $C^1(\mathcal{M})$, we introduce the inner product

$$(\phi, \psi)_{H^1(\mathcal{M})} := \int_M \phi \psi + \nabla_M \phi \cdot \nabla_M \psi d\sigma$$

and denote by $\| \cdot \|_{H^1(\mathcal{M})}$ the induced norm. Now, we define by $H^1(M)$ the completion of $C^1(\mathcal{M})$ with respect to the norm $\| \cdot \|_{H^1(\mathcal{M})}$. An equivalent definition of $H^1(M)$ can be given via local coordinates or distributional meaning, see [23, 24, 26]. By definition, for every $\phi \in H^1(M)$ there exists $\nabla_M \phi \in L^2_M(M)$, the tangential gradient in the distributional sense. By $H^1_0(M)$ we denote the closure of $C^1(\mathcal{M})$ in $H^1(M)$. If $M = \mathcal{M}$, we have $H^1_0(M) = H^1(M)$ and for $\partial M \neq \emptyset$, the space $H^1_0(M)$ coincides with the space of functions from $H^1(M)$ with zero traces on $\partial M$. 
Let \( u \in C^1(\overline{M})^n \), then there exists an extension \( \tilde{u} \in C^1(U)^n \) (\( U \) as above a suitable neighborhood of \( M \)) and we define the tangential divergence by
\[
\text{div}_M u := \nabla_M \cdot u := \nabla \cdot \tilde{u} - D\tilde{u} \nu \quad \text{on } \overline{M},
\]
where \( D\tilde{u} \) is the Jacobi-matrix of \( \tilde{u} \). For all \( \phi \in H^1(M) \) and \( u \in C^1(\overline{M}) \), we have the Stokes formula, see [22, Proposition 2.58],
\[
\int_M \nabla_M \phi \cdot u d\sigma = - \int_M \phi \nabla_M \cdot u d\sigma + \int_{\partial M} \phi u \cdot \nu_{\partial M} d\sigma_{\partial M}, \tag{3}
\]
where \( \nu_{\partial M} \) denotes the unit tangent vector to the manifold \( \overline{M} \), normal to the manifold boundary \( \partial M \), pointing outward on \( \partial M \). For an open set \( U \subset \mathbb{R}^n \) we define as usual \( H(\text{div}, U) := \{ u \in L^2(U)^n : \nabla \cdot u \in L^2(U) \} \) and extend this definition in a canonical way to functions in \( L^2_M(M) \): By \( H(\text{div}_M, M) \) we denote the space of tangential vector fields from \( L^2_M(M) \) with distributional tangential divergence in \( L^2(M) \), i.e.,
\[
H(\text{div}_M, M) := \{ u \in L^2_M(M) : \nabla_M \cdot u \in L^2(M) \}.
\]
More precisely, for \( u \in H(\text{div}_M, M) \) it holds for all \( \phi \in C^0_0(M) \)
\[
\int_M \nabla_M \phi \cdot u d\sigma = - \int_M \phi \nabla_M \cdot u d\sigma. \tag{4}
\]
By density of \( C^0_0(M) \) in \( H^1_0(M) \) this identity is also true for all \( \phi \in H^1_0(M) \). With the inner product
\[
(u, v)_{H(\text{div}_M, M)} := (u, v)_{L^2(M)} + (\nabla_M \cdot u, \nabla_M \cdot v)_{L^2(M)}
\]
the space \( H(\text{div}_M, M) \) becomes a Hilbert space. We have the following density result:

**Proposition 1.** The set \( C^1_M(\overline{M}) \) is dense in \( H(\text{div}_M, M) \).

**Proof.** Let \( H \) be the closure of \( C^1_M(\overline{M}) \) in \( H(\text{div}_M, M) \), i.e., we have the orthogonal decomposition \( H(\text{div}_M, M) = H \oplus H^\perp \). Let \( u \in H^\perp \), i.e., we have
\[
(u, v)_{L^2(M)} + (\nabla_M \cdot u, \nabla_M \cdot v)_{L^2(M)} = 0
\]
for all \( v \in C^1_M(\overline{M}) \). We define \( f := \nabla_M \cdot u \in L^2(M) \) and immediately obtain from the equations above, that \( \nabla_M f = u \) in the distributional sense, i.e., \( f \in H^1(M) \). From the Stokes formula (3), we obtain
\[
\int_{\partial M} f v \cdot \nu_{\partial M} d\sigma_{\partial M} = 0
\]
for all \( v \in C^1_M(\overline{M}) \) (the case \( \partial M = \emptyset \) is obvious). For \( y \in \partial M \), the element \( \nu_{\partial M}(y) \) is in the tangent space \( T_y M \). Then, due to the continuity and the surjectivity of the trace operator from \( H^1(M)^n \) to \( H^{1/2}(\partial M)^n \), and the density of \( C^1(\overline{M})^n \) in \( H^1(M)^n \), we can choose \( v = f \nu_{\partial M} \) in the boundary integral equation above and obtain that the trace of \( f \) vanishes on \( \partial M \), and therefore \( f \in H^1_0(M) \). For arbitrary \( v \in H(\text{div}_M, M) \) we can use (4) to obtain
\[
(u, v)_{L^2(M)} + (\nabla_M \cdot u, \nabla_M \cdot v)_{L^2(M)} = (\nabla_M f, v)_{L^2(M)} + (f, \nabla_M \cdot v)_{L^2(M)} = 0,
\]
what implies \( u = 0 \). \( \square \)
2.3. The Helmholtz-decomposition for a manifold with boundary. Let us now assume that \( Y_2 \) touches the boundary \( \partial Y \). In this case, we have to deal with periodic functions on \( \Gamma \). We define for \( k \in \{0,1,2\} \)

\[
C^k_{\text{per}}(\Gamma) := \{ \phi \in C^k(\Xi) : \phi \text{ is } Y\text{-periodic} \},
\]

and the space of periodic Sobolev functions \( H^1_{\text{per}}(\Gamma) \) on \( \Gamma \) as the closure of \( C^1_{\text{per}}(\Gamma) \) in \( H^1(\Gamma) \).

Next, we want to define subspaces of \( H(\text{div}, \Gamma) \) consisting of periodic or vanishing functions on \( \partial \Gamma \) in a generalized sense. First of all, we establish the existence of a generalized trace operator for the normal component of functions from \( H(\text{div}, \Gamma) \).

**Proposition 2.** There exists a surjective, linear, and bounded normal trace operator \( \gamma_\nu : H(\text{div}, \Gamma) \to H^{-\frac{1}{2}}(\partial \Gamma) \), such that for all \( u \in H(\text{div}, \Gamma) \) the generalized Gauss formula holds:

\[
\langle \gamma_\nu u, \phi \rangle_{H^{-\frac{1}{2}}(\partial \Gamma), H^{\frac{1}{2}}(\partial \Gamma)} = \int_{\Gamma} \phi \nabla_\Gamma \cdot u + u \cdot \nabla_\Gamma \phi \, d\sigma,
\]

for all \( \phi \in H^1(\Gamma) \). We usually write \( u \cdot \nu := \gamma_\nu u \).

**Proof.** The case of an open and bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \) can be found in [10, Chapter IX, §1, Theorem 1]. In our case, we have to use the density result from Proposition 1, the continuity of the embedding \( H^1(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\partial \Gamma) \), and the Stokes formula (3). Since the arguments are the same as in the domain-case, we skip the details.

Now, we define the spaces

\[
H_0(\text{div}, \Gamma) := \{ u \in H(\text{div}, \Gamma) : \langle u \cdot \nu, \phi \rangle_{H^{-\frac{1}{2}}(\partial \Gamma), H^{\frac{1}{2}}(\partial \Gamma)} = 0 \forall \phi \in H^1(\Gamma) \},
\]

\[
H_{\text{per}}(\text{div}, \Gamma) := \{ u \in H(\text{div}, \Gamma) : \langle u \cdot \nu, \phi \rangle_{H^{-\frac{1}{2}}(\partial \Gamma), H^{\frac{1}{2}}(\partial \Gamma)} = 0 \forall \phi \in H^1_{\text{per}}(\Gamma) \},
\]

and the subspaces of divergence free functions

\[
H_0(\text{div}_0 \Gamma) := \{ u \in H_0(\text{div}_0 \Gamma) : \nabla_\Gamma \cdot u = 0 \},
\]

\[
H_{\text{per}}(\text{div}_0 \Gamma) := \{ u \in H_{\text{per}}(\text{div}_0 \Gamma) : \nabla_\Gamma \cdot u = 0 \}.
\]

Then, we have the following Helmholtz-decompositions on \( L^2_\Gamma(\Gamma) \):

**Proposition 3.** There are the following orthogonal decompositions of the space \( L^2_\Gamma(\Gamma) \):

\[
L^2_\Gamma(\Gamma) = \nabla_\Gamma H^1_{\text{per}}(\Gamma) \oplus H_{\text{per}}(\text{div}_0, \Gamma),
\]

\[
L^2_\Gamma(\Gamma) = \nabla_\Gamma H^1(\Gamma) \oplus H_0(\text{div}_0, \Gamma),
\]

where \( \nabla_\Gamma H^1_{\text{per}}(\Gamma) \) resp. \( \nabla_\Gamma H^1(\Gamma) \) denotes the space of functions \( u \in L^2_\Gamma(\Gamma) \) such that there exists \( \phi \in H^1_{\text{per}}(\Gamma) \) resp. \( H^1(\Gamma) \) with \( u = \nabla_\Gamma \phi \).

**Proof.** We only give the proof for the first decomposition. The second one follows the same lines. Due to the compactness of the embedding \( H^1(\Gamma) \to L^2(\Gamma) \), we obtain from Peetre’s Lemma, see [20, Lemma 3], that \( H := \nabla_\Gamma H^1_{\text{per}}(\Gamma) \) is closed. It remains to show \( H^\perp = H_{\text{per}}(\text{div}_0, \Gamma) \). For \( u \in H^\perp \) we obtain for all \( \phi \in C^1_0(\Gamma) \)

\[
(\nabla_\Gamma \phi, u)_{L^2(\Gamma)} = 0,
\]
i.e., \( \nabla \cdot u = 0 \) in the distributional sense, especially \( u \in H(\text{div}, \Gamma) \). It remains to establish the periodicity of \( u \). Let \( \phi \in H^1_{\text{per}}(\Gamma) \), then the generalized Gauss formula from Proposition 2 implies
\[
0 = (\nabla \Gamma \phi, u)_{L^2(\Gamma)} = - (\nabla \Gamma \cdot u, \phi)_{L^2(\Gamma)} + \langle u \cdot \nu, \phi \rangle_{H^{\frac{1}{2}}(\partial \Gamma), H^{\frac{1}{2}}(\partial \Gamma)},
\]
i.e., \( u \in H_{\text{per}}(\text{div}, 0, \Gamma) \).

Conversely, for \( u \in H_{\text{per}}(\text{div}, 0, \Gamma) \), we immediately obtain using again the Gauss formula for all \( \phi \in H^1_{\text{per}}(\Gamma) \)
\[
(u, \nabla \phi)_{L^2(\Gamma)} = - (\text{div} \phi, u)_{L^2(\Gamma)} + \langle u \cdot \nu, \phi \rangle_{H^{\frac{1}{2}}(\partial \Gamma), H^{\frac{1}{2}}(\partial \Gamma)} = 0.
\]

Let us define for \( k \in 0, 1, 2 \) the spaces
\[
C^k_{\Gamma, 0}(\Gamma) := \{ u \in C^k_{\text{per}}(\Gamma) : u \text{ has compact support in } \Gamma \},
\]
\[
C^k_{\Gamma, \text{per}}(\Gamma) := C^k_{\text{per}}(\Gamma)^n \cap C^k(\bar{\Gamma}).
\]

Additionally, we have the following density results:

**Proposition 4.** (i) The space of functions \( C^2_{\Gamma, 0}(\Gamma) \) is dense in \( H_0(\text{div}, \Gamma) \).

(ii) The space of functions \( C^2_{\Gamma, \text{per}}(\Gamma) \) is dense in \( H_{\text{per}}(\text{div}, \Gamma) \).

**Proof.** The proofs are very similar to the proof of Proposition 1 and we skip the details.

**Corollary 1.** For all \( \theta \in H_{\text{per}}(\text{div}, \Gamma) \) and \( \phi \in C^1(\bar{\Omega}) \) it holds that \( \Phi_\varepsilon := \phi \theta_\varepsilon := \phi \theta (\frac{\cdot}{\varepsilon}) \in H(\text{div}, \Gamma_\varepsilon) \) with
\[
\nabla \Gamma_{\varepsilon} \cdot \theta_\varepsilon = \nabla \phi \cdot \theta (\frac{\cdot}{\varepsilon}) + \frac{1}{\varepsilon} \phi \nabla \Gamma \cdot \theta (\frac{\cdot}{\varepsilon}).
\]

**Proof.** Due to Proposition 4, it is enough to show the result for smooth functions. Then, this formula is obtained immediately from the definition of the tangential divergence.

### 2.4. Sobolev spaces for functions with regular traces

In this section, we introduce some function spaces which are needed to deal with coupled bulk-surface problems including Wentzell-boundary conditions of second order. In fact, such kind of problems lead to solution spaces of Sobolev-functions of first order with more regular traces, see Section 4. We define for \( j = 1, 2 \)
\[
\mathbb{H}_{j, \varepsilon} := \{ u_\varepsilon \in H^1(\Omega_j^\varepsilon) : u_\varepsilon|_{\Gamma_j} \in H^1(\Gamma_j) \},
\]
with the inner product
\[
(u_\varepsilon, v_\varepsilon)_{\mathbb{H}_{j, \varepsilon}} := (u_\varepsilon, v_\varepsilon)_{H^1(\Omega_j^\varepsilon)} + \varepsilon (u_\varepsilon|_{\Gamma_j}, v_\varepsilon|_{\Gamma_j})_{H^1(\Gamma_j)}.
\]

This space is isometric isomorph to the space
\[
\tilde{\mathbb{H}}_{j, \varepsilon} := \{ (u_\varepsilon, u_\varepsilon|_{\Gamma_j}^\varepsilon) \in H^1(\Omega_j^\varepsilon) \times H^1(\Gamma_j) : u_\varepsilon|_{\Gamma_j} = u_\varepsilon|_{\Gamma_j} \},
\]
together with the inner product
\[
((u_\varepsilon, u_\varepsilon|_{\Gamma_j}^\varepsilon), (v_\varepsilon, v_\varepsilon|_{\Gamma_j}^\varepsilon))_{\tilde{\mathbb{H}}_{j, \varepsilon}} := (u_\varepsilon, v_\varepsilon|_{\Gamma_j}^\varepsilon)_{H^1(\Omega_j^\varepsilon)} + \varepsilon (u_\varepsilon|_{\Gamma_j}, v_\varepsilon|_{\Gamma_j})_{H^1(\Gamma_j)}.
\]

Hence, we identify a function \( u_\varepsilon \in \mathbb{H}_{j, \varepsilon} \) with the function \( (u_\varepsilon, u_\varepsilon|_{\Gamma_j}) \in \tilde{\mathbb{H}}_{j, \varepsilon}, \) i.e., we also consider \( u_\varepsilon \) as a function in the product space \( H^1(\Omega_j^\varepsilon) \times H^1(\Gamma_j) \). The
product space $H^1(\Omega_j^\varepsilon) \times H^1(\Gamma_\varepsilon)$ is separable with respect to the $\bar{H}_{j,\varepsilon}$-norm. Since $H_{j,\varepsilon} \cong \bar{H}_{j,\varepsilon} \subset H^1(\Omega_j^\varepsilon) \times H^1(\Gamma_\varepsilon)$, the space $H_{j,\varepsilon}$ is also separable, i.e., it is a separable and reflexive space. Further, we define the the space $\mathbb{L}_{\varepsilon,j}$ as the space of functions from $L^2(\Omega_j^\varepsilon) \times L^2(\Gamma_\varepsilon)$ with inner product $((u_\varepsilon, v_\varepsilon), (u_\varepsilon, v_\varepsilon))_{\mathbb{L}_{\varepsilon,j}} := (u_\varepsilon, \eta_\varepsilon(\cdot, -\varepsilon))_{\Omega_j^\varepsilon} + (u_\varepsilon^{\Gamma_\varepsilon}, v_\varepsilon^{\Gamma_\varepsilon})_{\Gamma_\varepsilon}$. Thus, we obtain the Gelfand-triple

$$ H_{j,\varepsilon} \subset \mathbb{L}_{\varepsilon,j} \subset \bar{H}_{j,\varepsilon}. $$

In the same way, for $j = 1, 2$, we define the spaces on the reference element:

$$ \mathbb{H}_j := \{ u \in H^1(Y_j) : u|_\Gamma \in H^1(\Gamma) \}, $$

with the inner product $((u, v)_{\mathbb{H}_j} := (u, v)_{H^1(Y_j)} + (u|_\Gamma, v|_\Gamma)_{H^1(\Gamma)}$. Obviously, $\mathbb{H}_j$ is a Hilbert space. If $Y_2$ touches the outer boundary, we denote by $\mathbb{H}_{j,\text{per}}$ the space of functions from $\mathbb{H}_j$ which are $Y$-periodic. Further, we denote by $H^2_{\text{per}}(Y_j)$ the space of functions from $H^1_{\text{per}}(Y_j) \cap H^3(\Gamma_1)$, and together with the inner product on $H^2_{\text{per}}(Y_j)$ this space becomes a Hilbert space.

In the following remark, we introduce a periodic covering of the domains $E_j$ for $j = 1, 2$.

**Remark 2.** In the following, we suppress the index $j$, since both cases $j = 1$ and $j = 2$ can be treated in the same way. We consider $U_{\text{per}} := \{ U^k_i \}_{i=0,\ldots,m; \ k \in \mathbb{Z}^n}$ for a suitable $m \in \mathbb{N}$, such that $U_{\text{per}}$ is a covering of $E_j$ which fulfills the following properties:

(C1) $U^k_i$ is open and bounded for all $i \in \{0, 1, \ldots, m\}$, $k \in \mathbb{Z}^n$,

(C2) $U^k_i = U^k_0 + k$ for all $i \in \{0, 1, \ldots, m\}$, $k \in \mathbb{Z}^n$,

(C3) $U^k_0 \cap \partial E_j = \emptyset$ for $i = 1, \ldots, m$,

(C4) $\overline{U^k_0} \cap \partial E_j = \emptyset$ and $U^k_0 \cap Y_j \neq \emptyset$,

(C5) There exist $C^2$-diffeomorphisms $\Phi^k_i : U^k_i \to (-1, 1)^n$ for all $i \in \{1, \ldots, m\}$, $k \in \mathbb{Z}^n$, such that $\Phi^k_i(U^k_0 \cap E_j) = (-1, 1)^{n-1} \times (0, 1)$ and $\Phi^k_i(U^k_0 \cap \partial E_j) = (-1, 1)^{n-1} \times \{0\}$, see e.g., [17, 26] for more details. Further, the $\Phi^k_i$ fulfill

$$ \Phi^k_i = \Phi^k_0(\cdot - k). $$

We mention, that $\{ U^k_0 \}_{i=1}^m$ is in general not a covering of $Y_j$.

Further, let $\{ \eta^k_i \}_{i=1}^m$ be a subordinated partition of unity of $U_{\text{per}}$ with $\eta^k_0 = \eta^k_0(\cdot - k)$.

The following Lemma provides some technical results which are needed to derive density results in the case, when $Y_2$ touches the boundary $\partial Y$. We denote by $C^\infty_{\per}(Y_j)$ for $j = 1, 2$ the space of functions from $C^\infty(E_j)$ which are $Y$-periodic.

**Lemma 2.1.** For $j = 1, 2$, we have the following results:

(i) The space $C^\infty_{\per}(Y_j)$ is dense in $H^3_{\per}(Y_j)$.

(ii) There exists a continuous and linear trace operator $\gamma_{\per} : H^2_{\per}(Y_j) \to H^1_{\per}(\Gamma)$, such that $\gamma_{\per}(\phi) = \phi|_\Gamma$ for all $\phi \in C^\infty_{\per}(Y_j)$.

(iii) There exists a continuous and linear operator $Z_{\per} : H^1_{\per}(\Gamma) \to H^3_{\per}(Y_j)$, such that $\gamma_{\per} \circ Z_{\per} = 1_{H^1_{\per}(\Gamma)}$, i.e., the operator $\gamma_{\per}$ is surjective.

(iv) The set $C^\infty_{\per,0}(Y_j) := C^\infty_{\per}(Y_j) \cap C^\infty_0(E_j)$ is dense in $H^1_{\per,0}(Y_j) := \{ u \in H^1_{\per}(Y_j) : u|_\Gamma = 0 \} \subset H^1_{\per}(Y_j)$. 
Remark 3.  (i) The case $j = 2$ is standard, if $\overline{Y}_2 \subset Y$. Then, we can easily ignore the index “per”.

(ii) For $u \in H^2_{\text{per}}(Y_j)$ we have $\gamma_{\text{per}}(u) = \gamma_0(u)$ on $\Gamma$, where $\gamma_0$ denotes the usual trace operator from $H^1(Y_j)$ into $H^{1/2}(\partial Y_j)$. This follows from the continuity of the operator $\gamma_{\text{per}}$, the density of $C^\infty(\overline{Y}_j)$ in $H^2_{\text{per}}(Y_j)$, and the fact that $\gamma_{\text{per}}(\phi) = \phi|_{\Gamma}$ for $\phi \in C^\infty_{\text{per}}(Y_j)$.

Proof. We use the notations and properties from Remark 2 and $j \in \{1, 2\}$. More details on the local construction of trace operators and extension operators can be found in [17, 26].

(i) Let $u \in H^2_{\text{per}}(Y_j)$ periodically extended to $E_j$. The function $u^k : E_j \cap U^k \to \mathbb{R}$ defined by $u^k := \eta^k u$ for $k \in \mathbb{Z}^n$ and $i \in \{1, \ldots, m\}$ can be extended to a function $\tilde{u}^k \in H^2_0(U^k)$, due to the regularity of $\partial E_j$. Further, we define $\tilde{u}^k_0 := u^k : \eta^k u$. Due to the properties of $U_{\text{per}}$ and the partition of unity $\{\eta^k\}$, as well as the periodicity of $u$, we obtain $\tilde{u}^k_i(x + l) = \tilde{u}^k_i(x)$ for all $k, l \in \mathbb{Z}^n$ and $i \in \{0, 1, \ldots, m\}$. Hence, the function $\tilde{u} := \sum_{i=0}^m \sum_{k \in \mathbb{Z}^n} \tilde{u}^k_i$ is an element of $H^2_{\text{per}}(\mathbb{R}^n) \cap H^1_{\text{per}}(Y)$. Then, the convolution $\tilde{u}\delta := \phi_\delta * \tilde{u}$ with a standard mollifier $\phi_\delta$, $\delta > 0$, is an element of $C^\infty_c(Y)$ with $\tilde{u}\delta \to \tilde{u}$ in $H^2_{\text{per}}(\mathbb{R}^n)$, what gives us the desired result.

(ii) Let $u \in H^2_{\text{per}}(Y_j)$, extended periodically to $E_j$. We use the same notations as above. Using standard techniques, we can define a local trace operator $\gamma^l_{\text{per}}(u^k_i) \in H^0_0(U^k_i \cap \partial E_j)$ for $i \in \{1, \ldots, m\}$ (again, we used the regularity of $\partial E_j$), and extend this function by zero to $\partial E_j$. Now, we define $\gamma_{\text{per}}(u) := \sum_{i=1}^m \sum_{k \in \mathbb{Z}^n} \gamma^l_{\text{per}}(u^k_i)|_{\Gamma}$. By construction (using the properties of $U_{\text{per}}$) the function $\gamma_{\text{per}}(u)$ is $Y$-periodic. The continuity and the fact $\gamma_{\text{per}}(\phi) = \phi|_{\Gamma}$ for all $\phi \in C^\infty_{\text{per}}(Y)$, follows from the properties of the local trace operators.

(iii) The proof follows the same ideas as in part (ii) above, where here we have to construct local extension operators. Therefore, we skip the details. The property $\gamma_{\text{per}} \circ Z_{\text{per}} = Id_{H^2_{\text{per}}(\Gamma)}$ follows from the corresponding property of the local operators.

(iv) Let $u \in H^1_{\text{per}, 0}(Y_j)$, extended periodically to $E_j$. Again with the same notations as above, we have $u^k_i \in H^0_0(U^k_i)$ for $i \in \{0, 1, \ldots, m\}$. Hence, there exist sequences $\{u^k_i, l \in \mathbb{N} \subset C^\infty_c(U^k_i)\}$ with $u^k_i \to u^k_i$ for $l \to \infty$ in $H^1(U^k_i)$. We define $u_l := \sum_{i=0}^m \sum_{k \in \mathbb{Z}^n} u^k_{i,l} \in C^\infty_{\text{per}}(Y_j) \cap C^\infty_{\text{loc}}(E_j)$ (by construction) and we have $u_l \to u$ in $H^1(Y_j)$.

\[\square\]

Proposition 5.  (i) If $Y_2$ is strictly included in $Y$, then the space $C^\infty(\overline{Y}_j)$ is dense in $H_{\text{per}}$ and $C^\infty_{\text{per}}(Y_1)$ is dense in $H^1_{\text{per}}$.

(ii) If $Y_2$ touches the boundary $\partial Y$, then the space $C^\infty_{\text{per}}(Y_j)$ is dense in $H_{\text{per}}$.

(iii) In both cases, we have $C^\infty(\overline{Y}_j)$ is dense in $H^1_{\text{per}}(Y)$.

Proof. In the following, we denote by $\langle \cdot, \cdot \rangle_U$ the duality pairing between $H^{-1/2}(U)$ and $H^{1/2}(U)$ for a suitable set $U$.

(i) We only consider the case $j = 1$, since the other case can be treated in a similar way, where we can ignore the outer boundary $\partial Y$. 
Let $H$ be the closure of $C^\infty(Y_1)$ with respect to the inner product on $H^1$, i.e., we have $H^1 = H \oplus H^\perp$. Let $u \in H^\perp$, i.e., it holds that
\[ 0 = (u, \phi)_{H^1(Y_1)} = (u, \phi)_{H^1(Y_1)} + (u, \phi)_{H^1(\Gamma)} \quad \text{for all } \phi \in C^\infty(Y_1). \]
Especially, for all $\phi \in C^\infty_0(Y_1)$, we have
\[ 0 = (u, \phi)_{L^2(Y_1)} + (\nabla u, \nabla \phi)_{L^2(Y_1)}, \]
i.e., \( \Delta u = u \) and therefore we have \( \nabla u \in H(\text{div}, Y_1) \), and the normal-trace \( \nabla u \cdot \nu \) is an element of \( H^{-\frac{1}{2}}(\partial Y_1) \).

Now, from the generalized Green's formula, we obtain for all $\phi \in C^\infty(Y_1)$:
\[ 0 = (u, \phi)_{H^1(Y_1)} + (u, \phi)_{H^1(\Gamma)} = \langle \nabla u \cdot \nu, \phi \rangle_{\partial Y_1} + (u, \phi)_{H^1(\Gamma)}. \]

More precisely, we showed that
\[ 0 = \langle \nabla u \cdot \nu, \psi \rangle_{\partial Y_1} + (u, \psi)_{H^1(\Gamma)} \quad \text{(5)} \]
for all $\psi : \partial Y_1 \to \mathbb{R}$, such that $\psi$ is the trace of a function from $C^\infty(Y_1)$. Due to the regularity of $\Gamma$ and dist($\Gamma, \partial Y$) > 0, the trace operator maps $H^{\frac{3}{2}}(Y_1)$ continuously and surjective to $\{ v \in H^{\frac{3}{2}}(\partial Y_1) : v|_\Gamma \in H^1(\Gamma) \}$ (with respect to the norm $\| \cdot \|_{H^{\frac{3}{2}}(\partial Y_1)} + \| \cdot \|_{H^1(\Gamma)}$). Hence, due to the density of $C^\infty(Y_1)$ in $H^{\frac{3}{2}}(Y_1)$, we can choose $\psi = u|_{\partial Y_1}$ and obtain using again the Green formula
\[ 0 = \langle \nabla u \cdot \nu, u \rangle_{\partial Y_1} + \| u \|_{H^1(Y_1)} = \| u \|_{H^1(Y_1)} + \| u \|_{H^1(\Gamma)}, \]
i.e., $u = 0$ and therefore $H^\perp = \{ 0 \}$, which gives us the density of $C^\infty(Y_j)$ in $H_j$. The second statement follows by similar ideas as in the proof of (ii) below.

(ii) In this case, the set $\Gamma$ intersects the outer boundary $\partial Y$, i.e., dist($\Gamma, \partial Y$) = 0, and we have to argue in a slightly different way. We use the same notations as in part (i). Let $H$ be the closure of $C^\infty_{\text{per}}(Y_j)$ with respect to the inner product on $H_j$, i.e., we have $H_{\text{per}} = H \oplus H^\perp$. Let $u \in H^\perp$. With the same methods as above, we obtain $\nabla u \in H(\text{div}, Y_j)$ with $\Delta u = u$, and
\[ \langle \nabla u \cdot \nu, u \rangle_{\partial Y_j} + (u, \phi)_{H^1(\Gamma)} = 0 \quad \text{for all } \phi \in C^\infty_{\text{per}}(Y_j), \quad \text{(6)} \]
\[ \langle \nabla u \cdot \nu, \phi \rangle_{\partial Y_j} = 0 \quad \text{for all } \phi \in C^\infty_{\text{per}}(Y_j) \cap C^\infty_0(E_j). \quad \text{(7)} \]
The surjectivity of $\gamma_{\text{per}}$ implies the existence of a function $\tilde{u} \in H^{\frac{3}{2}}_{\text{per}}(Y_j)$, such that $\gamma_{\text{per}}(\tilde{u}) = u|_\Gamma$ (in the following we use the same expression for a function and its trace). From Lemma 2.1(i) follows the existence of a sequence $\{ \tilde{u}_k \} \subset C^\infty_{\text{per}}(Y_j)$ with $\tilde{u}_k \to \tilde{u}$ in $H^{\frac{3}{2}}(Y_j)$, and the continuity of $\gamma_{\text{per}}$ implies $\tilde{u}_k \to \tilde{u}$ in $H^{\frac{3}{2}}(\partial Y_j)$. Hence, we can choose $\phi = \tilde{u}$ in (6), i.e., we have
\[ \langle \nabla u \cdot \nu, \tilde{u} \rangle_{\partial Y_j} + (u, \tilde{u})_{H^1(\Gamma)} = 0. \]
Due to Lemma 2.1(iv), the equation (7) is also true for all $\phi \in H^1_{\text{per,0}}(Y_j)$. Since $\tilde{u} - u \in H^1_{\text{per,0}}(Y_j)$, we obtain
\[ \langle \nabla u \cdot \nu, \tilde{u} \rangle_{\partial Y_j} = \langle \nabla u \cdot \nu, \tilde{u} \rangle_{\partial Y_j}, \]
and therefore
\[ 0 = \langle \nabla u \cdot \nu, u \rangle_{\partial Y_j} + (u, u)_{H^1(\Gamma)} = \| u \|_{H^2(Y_j)}^2 + \| u \|_{H^1(\Gamma)}^2. \]
(iii) For $Y_2$ strictly included in $Y$, we can conclude in the same way as in (i). In
the other case, we define $H$ as the closure of $C^\infty(\overline{\Omega}_j)$ in $H^1_{\mathrm{per}}(\Omega)$, and obtain as
in (ii) for $u \in H^1$:
\begin{align*}
(\nabla u \cdot \nu, \phi)_{\partial \Omega_j^*} + (u, \phi)_{H^1(\Gamma_\epsilon)} &= 0 \quad \text{for all } \phi \in C^\infty(\overline{\Omega}_j), \\
(\nabla u \cdot \nu, \phi)_{\partial \Omega_j^*} &= 0 \quad \text{for all } \phi \in C^\infty_{\Gamma_\epsilon,0}(\overline{\Omega}_j),
\end{align*}
where $C^\infty_{\Gamma_\epsilon,0}(\overline{\Omega}_j)$ denotes the space of all functions from $C^\infty(\overline{\Omega}_j)$ vanishing in
a neighborhood of $\Gamma_\epsilon$. Since $C^\infty_{\Gamma_\epsilon,0}(\overline{\Omega}_j)$ is dense in $\{u \in H^1(\Omega_j^*): u|_{\Gamma_\epsilon} = 0\}$,
see [7, Theorem 3.1], the desired result is obtained by using similar arguments
as in (ii).

\rightline{$\square$}

3. Two-scale compactness results including regular traces. In this section, we
derive new two-scale compactness results for functions with more regular traces.
First of all, we repeat the definition of two-scale convergence for periodic domains
and surfaces. The theory for periodic domains was first introduced in [19] and
further developed in [2]. Later, it was extended to periodic surfaces in [3, 18].

Definition 3.1. (i) A sequence of functions $u_\epsilon \in L^2(\Omega)$ is said to converge in
the two-scale sense to the limit function $u_0 \in L^2(\Omega \times Y)$, if for every $\phi \in
C^0(\overline{\Omega}, C^0_{\mathrm{per}}(Y))$ the following relation holds
\[
\lim_{\epsilon \to 0} \int_{\Omega} u_\epsilon(x) \phi \left( x, \frac{x}{\epsilon} \right) dx = \int_{\Omega} u_0(x,y,\phi(x,y)dydx.
\]
(ii) A sequence of functions $u_\epsilon \in L^2(\Gamma_\epsilon)$ is said to converge in the two-scale sense
on the surface $\Gamma_\epsilon$ to a limit $u_0 \in L^2(\Omega \times \Gamma)$, if for every $\phi \in C^0(\overline{\Omega}, C^0_{\mathrm{per}}(\Gamma))$
it holds that
\[
\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} u_\epsilon(x) \phi \left( x, \frac{x}{\epsilon} \right) d\sigma = \int_{\Gamma} u_0(x,y)\phi(x,y)d\sigma_ydx.
\]
Both definitions carry over in a natural way to the vector-valued case $\xi_\epsilon \in L^2(\Omega)^n$
resp. $\xi_\epsilon \in L^2(\Gamma_\epsilon)^n$.

Remark 4. For a two-scale convergent sequence $(\xi_\epsilon) \subset L^2(\Gamma_\epsilon)$ with two-scale
limit $\xi_0 \in L^2(\Omega \times \Gamma)^n$, we immediately obtain $\xi_0 \in L^2(\Omega, L^2(\Gamma))$.

In the following, we denote the zero extension of a function $u_\epsilon$ defined on $\Omega_j^*$
to the whole domain $\Omega$ by $\tilde{u}_\epsilon$.

Lemma 3.2. Let $u_\epsilon \in H^1(\Omega_j^*)$ be a sequence with $\|u_\epsilon\|_{H^1(\Omega_j^*)} \leq C$. Then, there
exist $u_0 \in L^2(\Omega)$ and $U_1 \in L^2(\Omega, H^1(\Omega_j^*)/\mathbb{R})$, such that up to a subsequence
$\tilde{u}_\epsilon \to \chi_{Y_j}u_0$ in the two-scale sense,
\[
\nabla u_\epsilon \to \chi_{Y_j} \nabla x U_1 \quad \text{in the two-scale sense}.
\]
Additionally, for every $\psi \in D(\Omega) \otimes L^2_{\mathrm{per}}(\Omega_j^*)/\mathbb{R}$, it holds that
\[
\int_{\Omega_j} \frac{u_\epsilon(x)}{\epsilon} \psi \left( x, \frac{x}{\epsilon} \right) dx 
\rightarrow \int_{\Omega_j} \int_{\Omega_j} U_1(x,y)\psi(x,y)dydx + \int_{\Omega_j} \int_{\Omega_j} u_0(x)\nabla x \cdot (y\psi(x,y))dydx.
\]
Proof. The convergence of $\bar{u}_\varepsilon$ is well known, see [2], so we have to check the convergence of the gradients. There exists a two-scale limit $\xi_0 \in L^2(\Omega \times Y)^n$ for a subsequence of the zero-extension of the gradients and $\xi_0 = 0$ almost everywhere in $\Omega \times Y_j \setminus Y_j$. Now, for all $\Phi \in C^0_j(\Omega \times Y_j)^n$ with $\nabla_y \cdot \Phi = 0$, it holds that
\[
\int_{\Omega} \int_{Y_j} \xi_0(x, y) \Phi(x, y) dy dx = \lim_{\varepsilon \to 0} \int_{Y_j} \nabla u_\varepsilon(x) \cdot \Phi \left( x, \frac{x}{\varepsilon} \right) dx
\]
\[
= -\lim_{\varepsilon \to 0} \int_{Y_j} u_\varepsilon(x) \nabla_x \cdot \Phi \left( x, \frac{x}{\varepsilon} \right) dx
\]
\[
= -\int_{\Omega} \int_{Y_j} u_0(x) \nabla_x \cdot \Phi(x, y) dy dx = 0,
\]
since
\[
\int_{Y_j} \Phi_1(x, y) dy = \int_{Y_j} \Phi(x, y) \cdot \nabla_y y_i dy = \int_{\partial Y_j} \Phi(x, y) \cdot \nu_y d\sigma_y = 0.
\]
The Helmholtz-decomposition, see e.g., [10, Chapter IX, §1, Proposition 1] implies the existence of a function $U_1 \in L^2(\Omega, H^1(Y_j)/\mathbb{R})$ with $\xi_0 = \chi_{Y_j} \nabla_y U_1$.

Now we prove the second part of the lemma. It is enough to consider functions of the form $\psi(x, y) = \phi(x) \theta(y)$ with $\phi \in D(\Omega)$ and $\theta \in L^2_{\text{per}}(Y_j)/\mathbb{R}$. Let $\theta \in H^1(Y_j)/\mathbb{R}$ be the unique weak solution of
\[
-\Delta_y \theta = \theta \quad \text{in } Y_j,
\]
\[-\nabla_y \theta \cdot \nu = 0 \quad \text{on } \partial Y_j.
\]
We define $u_\varepsilon^k(y) := u_\varepsilon(\varepsilon(y + k))$ and $\phi_\varepsilon^k(y) := \phi(\varepsilon(y + k))$ for $y \in Y_j$ and $k \in K_\varepsilon$, and obtain
\[
\int_{Y_j} \frac{u_\varepsilon(x)}{\varepsilon} \phi(x) \theta \left( \frac{x}{\varepsilon} \right) dx = \frac{1}{\varepsilon} \sum_{k \in K_\varepsilon} \int_{Y_j} u_\varepsilon(x) \phi(x) \theta \left( \frac{x}{\varepsilon} \right) dx
\]
\[
= \varepsilon^{n-1} \sum_{k \in K_\varepsilon} \int_{Y_j} \nabla_y (u_\varepsilon^k(y) \phi_\varepsilon^k(y)) \cdot \nabla_y \theta(y) dy
\]
\[
= \varepsilon^n \sum_{k \in K_\varepsilon} \int_{Y_j} \left( \nabla u_\varepsilon(\varepsilon(k + y)) \phi_\varepsilon^k(y) + u_\varepsilon^k(y) \nabla \phi(\varepsilon(k + y)) \right) \cdot \nabla_y \theta(y) dy
\]
\[
= \int_{\Omega} \phi(x) \nabla u_\varepsilon(x) \cdot \nabla_y \theta \left( \frac{x}{\varepsilon} \right) + \int_{\Omega} \int_{Y_j} u_\varepsilon(x) \nabla \phi(x) \cdot \nabla_y \theta(y) dy dx
\]
\[
\to 0 \quad \int_{\Omega} \int_{Y_j} \phi(x) \nabla U_1(x, y) \cdot \nabla_y \theta(y) dy dx + \int_{\Omega} \int_{Y_j} u_0(x) \nabla \phi(x) \cdot \nabla_y \theta(y) dy dx
\]
\[
= \int_{\Omega} \int_{Y_j} U_1(x, y) \phi(x) \theta(y) dy dx + \int_{\Omega} \int_{Y_j} u_0(x) \nabla \phi(x) \cdot \nabla_y \theta(y) dy dx.
\]

Remark 5. It is well known, that for $u_\varepsilon \in H^1(\Omega_1)$ bounded, there exists an extension $\hat{u}_\varepsilon \in H^1(\Omega)$, which is also bounded. Further, there exist $\hat{u}_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega, H^1_{\text{per}}(Y_j)/\mathbb{R})$, such that up to subsequence it holds that
\[
\hat{u}_\varepsilon \to \hat{u}_0 \quad \text{in the two-scale sense},
\]
\[
\nabla \hat{u}_\varepsilon \to \nabla_x \hat{u}_0 + \nabla_y u_1 \quad \text{in the two-scale sense}.
\]
With the notations from Lemma 3.2, we have
\[
\tilde{u}_0(x)\chi_Y(y) = u_0(x)\chi_Y(y) \quad \text{in } \Omega \times Y,
\]
\[
U_1(x,y) = \nabla_x \tilde{u}_0(x) \cdot y + u_1(x,y) \quad \text{in } \Omega \times Y_j.
\]
Especially, Lemma 3.2 yields for all \( \psi \in \mathcal{D}(\Omega) \otimes L^2_{per}(Y_j)/\mathbb{R} \)
\[
\lim_{\epsilon \to 0} \int_{\Omega} \tilde{u}_\epsilon(x) \psi \left( x, \frac{x}{\epsilon} \right) dx \to \int_{\Omega} \int_{Y} u_1(x,y)\psi(x,y)dydx.
\]
This result was already obtained in [11, Theorem 2.3].

In the next Lemma, we prove for sequences defined on \( \Gamma_\epsilon \) results of the same type as those in Lemma 3.2.

**Lemma 3.3.** Let \( u_\epsilon \in H^1(\Gamma_\epsilon) \) be a sequence with \( \sqrt{\epsilon}\|u_\epsilon\|_{H^1(\Gamma_\epsilon)} \leq C \). Then there exist \( u_0^\Gamma \in L^2(\Omega) \) and \( U_1^\Gamma \in L^2(\Omega, H^1(\Gamma)/\mathbb{R}) \), such that up to a subsequence
\[
u_\epsilon \to u_0^\Gamma \quad \text{in the two-scale sense on } \Gamma_\epsilon,
\]
\[
\nabla_\Gamma u_\epsilon \to \nabla_\Gamma U_1^\Gamma \quad \text{in the two-scale sense on } \Gamma_\epsilon.
\]
Additionally, for all \( \Phi \in \mathcal{D}(\Omega) \otimes L^2_{per}(\Gamma)/\mathbb{R} \) it holds that
\[
\epsilon \int_{\Gamma_\epsilon} \frac{u_\epsilon(x)}{\epsilon} \Phi \left( x, \frac{x}{\epsilon} \right) d\sigma 
\]
\[
\quad \to \int_{\Omega} \int_{\Gamma} U_1^\Gamma(x,y)\Phi(x,y)d\sigma_ydx + \int_{\Omega} \int_{\Gamma} u_0^\Gamma(x)\nabla_x \cdot (y\Phi(x,y))d\sigma_ydx.
\]

**Proof.** First of all, from the results in [18], see also Remark 4, it follows the existence of \( u_0^\Gamma \in L^2(\Omega \times \Gamma) \) and \( \xi \in L^2(\Omega, L^2(\Gamma)) \), such that up to a subsequence \( u_\epsilon \) and \( \nabla_\Gamma u_\epsilon \) converge in the two-scale sense to \( u_0^\Gamma \) and \( \xi \), respectively. Now, choose \( \Phi(x,y) = \phi(x)\theta(y) \) with \( \phi \in \mathcal{D}(\Omega) \) and \( \theta \in C^2_{\Gamma,0}(\Gamma) \). By integration by parts, we obtain with Corollary 1
\[
0 \leftarrow \epsilon^2 \int_{\Gamma_\epsilon} \nabla_\Gamma u_\epsilon(x) \cdot \Phi \left( x, \frac{x}{\epsilon} \right) d\sigma
\]
\[
- = -\epsilon^2 \int_{\Gamma_\epsilon} u_\epsilon(x) \left( \nabla \phi(x) \cdot \theta \left( \frac{x}{\epsilon} \right) + \frac{1}{\epsilon} \phi(x) \nabla_\Gamma \cdot \theta \left( \frac{x}{\epsilon} \right) \right) d\sigma
\]
\[
\quad \to - \int_{\Omega} \int_{\Gamma} u_0^\Gamma(x,y)\phi(x)\nabla_\Gamma \cdot \theta(y)d\sigma_ydx.
\]
Hence, the function \( u_0^\Gamma \) is independent of \( y \), i.e., \( u_0^\Gamma(x,y) = u_0^\Gamma(x) \). Now we show that \( \xi \) is the gradient of a function. Let \( \Phi(x,y) = \phi(x)\theta(y) \) with \( \phi \in \mathcal{D}(\Omega) \) and \( \theta \in H_0(\text{div}_\Gamma 0, \Gamma) \). We get, using again Corollary 1,
\[
\int_{\Omega} \int_{\Gamma} \xi(x,y) \cdot \Phi(x,y)d\sigma_ydx = \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \nabla_\Gamma u_\epsilon(x) \cdot \Phi \left( x, \frac{x}{\epsilon} \right) d\sigma
\]
\[
\quad = \lim_{\epsilon \to 0} -\epsilon \int_{\Gamma_\epsilon} u_\epsilon(x)\nabla \phi(x) \cdot \theta \left( \frac{x}{\epsilon} \right) d\sigma
\]
\[
\quad = - \int_{\Omega} u_0^\Gamma(x)\nabla \phi(x) \cdot \int_{\Gamma} \theta(y)d\sigma_ydx = 0,
\]
since
\[
\int_{\Gamma} \theta_\epsilon(y)d\sigma_y = \int_{\Gamma} \nabla_\Gamma y_i \cdot \theta d\sigma_y = 0.
\]
As in the proof of Lemma 3.2, we define $u_{\varepsilon}$, i.e., for all $v$ with $\varepsilon > 0$ and $v \in H^1(\Gamma)$, we have
$$-\Delta_{\Gamma} \vartheta = \psi \quad \text{on } \Gamma,$$
$$-\nabla_{\Gamma} \vartheta \cdot \nu = 0 \quad \text{on } \partial \Gamma,$$
i.e., for all $v \in H^1(\Gamma)$ it holds that
$$\int_{\Gamma} \nabla_{\Gamma} \vartheta \cdot \nabla_{\Gamma} v \, d\sigma = \int_{\Gamma} \psi v \, d\sigma.$$As in the proof of Lemma 3.2, we define $u_{\varepsilon}^k(y) := u_{\varepsilon}(\varepsilon(y + k))$ and $\phi_{\varepsilon}^k(y) := \phi(\varepsilon(y + k))$ for $y \in \Gamma$ and $k \in K_{\varepsilon}$, and with similar arguments we get:
$$\int_{\Gamma_{\varepsilon}} u_{\varepsilon}(x) \Phi\left(\frac{x}{\varepsilon}\right) \, d\sigma = \sum_{k \in K_{\varepsilon}} \varepsilon^{n-1} \int_{\Gamma} u_{\varepsilon}^k(y) \phi_{\varepsilon}^k(y) \theta(y) \, d\sigma$$
$$= \sum_{k \in K_{\varepsilon}} \varepsilon^{n-1} \int_{\Gamma} \nabla_{\Gamma}(u_{\varepsilon}^k \phi_{\varepsilon}^k)(y) \cdot \nabla_{\Gamma} \vartheta(y) \, d\sigma$$
$$= \sum_{k \in K_{\varepsilon}} \varepsilon^{n-1} \int_{\Gamma} \phi_{\varepsilon}^k(y) \nabla_{\Gamma} u_{\varepsilon}^k(y) \cdot \nabla_{\Gamma} \vartheta(y) + u_{\varepsilon}^k(y) \nabla_{\Gamma} \phi_{\varepsilon}^k(y) \cdot \nabla_{\Gamma} \vartheta(y) \, d\sigma$$
$$= \varepsilon \int_{\Gamma_{\varepsilon}} \phi(x) \nabla_{\Gamma} u_{\varepsilon}(x) \cdot \nabla_{\Gamma} \vartheta\left(\frac{x}{\varepsilon}\right) + u_{\varepsilon}(x) \nabla_{\Gamma} \phi\left(\frac{x}{\varepsilon}\right) \, d\sigma.$$Passing to the limit $\varepsilon \to 0$ gives us
$$\lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} \frac{u_{\varepsilon}(x) \Phi\left(\frac{x}{\varepsilon}\right)}{\varepsilon} \, d\sigma$$
$$= \int_{\Omega} \int_{\Gamma} \phi(x) \nabla_{\Gamma} U_1^\Gamma(x, y) \cdot \nabla_{\Gamma} \vartheta(y) \, d\sigma \, dx + \int_{\Omega} \int_{\Gamma} \int_{\Gamma} u_0^\Gamma(y) P_\Gamma(y) \nabla \phi(x) \cdot \nabla_{\Gamma} \vartheta(y) \, d\sigma \, dx$$
$$= \int_{\Omega} \int_{\Gamma} \phi(x) U_0^\Gamma(x, y) \psi(y) \, d\sigma \, dx + \int_{\Omega} \int_{\Gamma} u_0^\Gamma(x) \nabla \phi(x) \cdot y \psi(y) \, d\sigma \, dx,$$where at the end we used $P_\Gamma(y) \nabla \phi(x) = \nabla_{\Gamma}(\nabla \phi(x) \cdot y)$. This gives us the desired result. \(\square\)

Analogously to Remark 5, we can get more information about the limit $U_1^\Gamma$, if additional assumptions are made on the domain or on the regularity of the two-scale limit $u_0^\Gamma$, see Remark 6 and Corollary 2.

**Remark 6.** If in Lemma 3.3 it additionally holds that $u_0^\Gamma \in H^1(\Omega)$, then from the proof of Theorem 3.4 below, see also [14] for the case when $\Gamma_{\varepsilon}$ is connected, we get
$$u_{\varepsilon} \to u_0^\Gamma \quad \text{in the two-scale sense on } \Gamma_{\varepsilon},$$
$$\nabla_{\Gamma_{\varepsilon}} u_{\varepsilon} \to P_\Gamma(y) \nabla u_0^\Gamma(x) \quad \text{in the two-scale sense on } \Gamma_{\varepsilon},$$
with $u_1^\Gamma \in L^2(\Omega, H^1_{per}(\Gamma)/\mathbb{R})$. With the notations from Lemma 3.3 we get
$$U_1^\Gamma(x, y) = \nabla u_0^\Gamma(x) \cdot y + u_1^\Gamma(x, y) \quad \text{in } \Omega \times \Gamma,$$
and for all $\Phi \in D(\Omega) \otimes L^2_{\text{per}}(\Gamma)/\mathbb{R}$ it holds that
\[
\epsilon \int_{\Gamma_\epsilon} \frac{u_\epsilon(x)}{\epsilon} \Phi \left( \frac{x}{\epsilon} \right) d\sigma \xrightarrow{\epsilon \to 0} \int_{\Omega} \int_{\Gamma} u^\Gamma_{\epsilon}(x,y) \Phi(x,y) d\sigma_y dx.
\]

The following theorem gives a compactness result for bounded sequences in $H^1_{1,\epsilon}$, i.e., for functions defined on the connected domain $\Omega_\epsilon$ with more regular traces in $H^1(\Gamma_\epsilon)$.

**Theorem 3.4.** Let $u_\epsilon$ be a bounded sequence in $H^1_{1,\epsilon}$, i.e., it holds that
\[
\|u_\epsilon\|_{H^1(\Omega_\epsilon)} + \sqrt{\epsilon} \|u_\epsilon\|_{H^1(\Gamma_\epsilon)} \leq C.
\]

Then there exist an extension $\tilde{u}_\epsilon \in H^1(\Omega)$ with $\|\tilde{u}_\epsilon\|_{H^1(\Omega)} \leq C\|u_\epsilon\|_{H^1(\Omega_\epsilon)}$, and functions $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega, H^1_{\text{per}}(\Gamma))/\mathbb{R}$ such that $\int_{\Gamma} u_1(x,y) d\sigma_y = 0$ for almost every $x \in \Omega$, such that up to a subsequence it holds that
\[
\begin{align*}
\tilde{u}_\epsilon &\to u_0 \quad \text{ (strongly) in } L^2(\Omega), \\
\nabla \tilde{u}_\epsilon &\to \nabla_x u_0(x) + \nabla_y u_1(x,y) \quad \text{in the two-scale sense}, \\
u_\epsilon &\to u_0 \quad \text{ in the two-scale sense on } \Gamma_\epsilon, \\
\nabla_{\Gamma_\epsilon} u_\epsilon &\to P_{\Gamma}(y) \nabla_x u_0(x) + \nabla_{\Gamma} u_1(x,y) \quad \text{in the two-scale sense on } \Gamma_\epsilon.
\end{align*}
\]

**Proof.** First of all, the existence of an extension $\tilde{u}_\epsilon \in H^1(\Omega)$ with
\[
\|\tilde{u}_\epsilon\|_{H^1(\Omega)} \leq C\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C,
\]
follows, since the domain $\Omega_\epsilon$ is connected, see [1]. Further, there exist functions $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega, H^1_{\text{per}}(\Gamma))/\mathbb{R}$ with $\int_{\Gamma} u_1(x) d\sigma_y = 0$, $u_0^\Gamma \in L^2(\Omega, L^2_{\text{per}}(\Gamma))$, such that up to a subsequence
\[
\begin{align*}
\tilde{u}_\epsilon &\to u_0 \quad \text{ in } L^2(\Omega), \\
\nabla \tilde{u}_\epsilon &\to \nabla_x u_0(x) + \nabla_y u_1(x,y) \quad \text{in the two-scale sense}, \\
u_\epsilon &\to u_0^\Gamma \quad \text{ in the two-scale sense on } \Gamma_\epsilon, \\
\nabla_{\Gamma_\epsilon} u_\epsilon &\to \xi_0^\Gamma \quad \text{ in the two-scale sense on } \Gamma_\epsilon.
\end{align*}
\]

It is well known, see [3, Proposition 2.6], that $u_0^\Gamma = u_0$, and we only have to check $\xi_0^\Gamma(x,y) = P_{\Gamma}(y) \nabla_x u_0(x) + \nabla_{\Gamma} u_1(x,y)$. We choose $\Psi(x,y) := \phi(x)\theta(y)$ with $\phi \in D(\Omega)$ and $\theta \in H^1_{\text{per}}(\text{div}_1\Gamma, \Gamma)$. We obtain from Corollary 1
\[
\int_{\Omega} \int_{\Gamma} \xi_0^\Gamma(x,y) \cdot \theta(y)\phi(x) d\sigma_y dx = \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \phi(x) \nabla_{\Gamma_\epsilon} u_\epsilon(x) \cdot \theta \left( \frac{x}{\epsilon} \right) d\sigma
\]
\[
= - \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} u_\epsilon(x) \nabla \phi(x) \cdot \theta \left( \frac{x}{\epsilon} \right) d\sigma
\]
\[
= - \int_{\Omega} \int_{\Gamma} u_0(x) \nabla \phi(x) \cdot \theta(y) d\sigma_y dx
\]
\[
= \int_{\Omega} \int_{\Gamma} \phi(x) \nabla u_0(x) \cdot \theta(y) d\sigma_y dx
\]
\[
= \int_{\Omega} \int_{\Gamma} \phi(x) P_{\Gamma}(y) \nabla u_0(x) \cdot \theta(y) d\sigma_y dx.
\]

Now, the Helmholtz-decomposition from Proposition 3 implies the existence of a function $u_1^\Gamma \in L^2(\Omega, H^1_{\text{per}}(\Gamma)/\mathbb{R})$ with $\int_{\Gamma} u_1^\Gamma(x,y) d\sigma = 0$, such that
\[
\xi_0^\Gamma(x,y) = P_{\Gamma}(y) \nabla_x u_0(x) + \nabla_{\Gamma} u_1^\Gamma(x,y).
\]
It remains to show that $u_1^\epsilon = u_1$ on $\Gamma$. We choose $\Phi(x, y) = \phi(x)\theta(y)$ with $\phi \in \mathcal{D}(\Omega)$ and $\theta \in C^\infty_{\text{per}}(Y_1)^n$, such that $\int_{Y_1} \nabla \cdot \theta(y) dy = 0$. Due to the divergence theorem, we obtain

$$0 = \int_{Y_1} \nabla \cdot \theta(y) dy = \int_{\Gamma} \theta(y) \cdot \nu d\sigma$$

Using the two-scale convergence of $\nabla u_\epsilon$, we get

$$\int_{\Omega^1} \nabla u_\epsilon(x) \cdot \Phi \left( x, \frac{x}{\epsilon} \right) dx \rightharpoonup_{\text{two-scale}} \int_{\Omega} \int_{Y_1} \left( \nabla_x u_0(x) + \nabla_y u_1(x, y) \right) \cdot \Phi(x, y) dy dx,$$

and by integration by parts, Remark 5 and 6, we obtain

$$\int_{\Omega^1} \nabla u_\epsilon(x) \cdot \Phi \left( x, \frac{x}{\epsilon} \right) dx = \epsilon \int_{\Gamma^1} \frac{u_\epsilon(x)}{\epsilon} \Phi \left( x, \frac{x}{\epsilon} \right) \cdot \nu d\sigma$$

$$- \int_{\Omega^1} u_\epsilon(x) \left( \nabla_x \phi(x) \cdot \theta \left( \frac{x}{\epsilon} \right) + \frac{1}{\epsilon} \phi(x) \nabla_y \theta \left( \frac{x}{\epsilon} \right) \right) dx$$

$$\rightharpoonup_{\text{two-scale}} \int_{\Omega} \int_{\Gamma} u_1^\epsilon(x, y) \Phi(x, y) \cdot \nu d\sigma y dx$$

$$- \int_{\Omega} \int_{Y_1} u_0(x) \nabla_x \phi(x) \cdot \theta(y) + u_1(x, y) \phi(x) \nabla_y \theta(y) dy dx$$

$$= \int_{\Omega} \int_{\Gamma} u_1^\epsilon(x, y) \Phi(x, y) \cdot \nu d\sigma y dx + \int_{\Omega} \int_{Y_1} \left( \nabla_x u_0(x) + \nabla_y u_1(x, y) \right) \cdot \Phi(x, y) dy dx$$

$$- \int_{\Omega} \int_{\Gamma} u_1(x, y) \Phi(x, y) \cdot \nu d\sigma y dx$$

Altogether, we obtain

$$\int_{\Gamma} u_1^\epsilon(x, y) \theta(y) \cdot \nu d\sigma y = \int_{\Gamma} u_1(x, y) \theta(y) \cdot \nu d\sigma y.$$

for almost every $x \in \Omega$ and all $\theta \in C^\infty_{\text{per}}(Y_1)^n$ with $\int_{Y_1} \nabla \cdot \theta(y) dy = 0$, and therefore $u_1 = u_1^\epsilon$ on $\Gamma$, what finishes the proof.  

A crucial point in the proof of Theorem 3.4 is the $H^1$-regularity of the two-scale limit $u_1^\epsilon$ of the traces, which is directly obtained from the coupling to the bulk-domain $\Omega^1$. However, we can immediately show that convergence result for the surface gradients from Theorem 3.4 carries over to functions defined only on the surface $\Gamma_\epsilon$, if we assume $H^1$-regularity of the two scale limit:

**Corollary 2.** Let $u_\epsilon \in H^1(\Gamma_\epsilon)$ with

$$\sqrt{\epsilon} \| u_\epsilon \|_{H^1(\Gamma_\epsilon)} \leq C,$$

and $u_\epsilon$ converges in the two-scale sense on $\Gamma_\epsilon$ to $u_0 \in H^1(\Omega)$. Then there exists $u_1^\epsilon \in L^2(\Omega, H^1_{\text{per}}(\Gamma)/\mathbb{R})$, such that

$$\nabla_{\Gamma_\epsilon} u_\epsilon \rightharpoonup P_\Gamma(y) \nabla u_0(x) + \nabla_{\Gamma_\epsilon} u_1^\epsilon(x, y) \quad \text{in the two-scale sense on } \Gamma_\epsilon.$$

The assumptions of Corollary 2 are fulfilled, for example, if there exists an extension $\tilde{u}_\epsilon \in H^1(\Omega)$ of $u_\epsilon$, such that the sequence $(\tilde{u}_\epsilon)$ is bounded in $H^1(\Omega)$. This is always the case when the surface $\Gamma_\epsilon$ is connected, see [14]. However, in Section 4 we show, that the $H^1$-regularity of the two-scale limit is in general not guaranteed if $\Gamma_\epsilon$ is disconnected, i.e., for suitable $u_\epsilon \in H^1(\Gamma_\epsilon)$ with $\sqrt{\epsilon} \| u_\epsilon \|_{H^1(\Gamma_\epsilon)} \leq C$ we show the
existence of a two-scale limit \( u_0 \in L^2(\Omega) \setminus H^1(\Omega) \). If the domain \( \Omega_2 \) is disconnected, we obtain the following two-scale compactness result:

**Theorem 3.5.** Let \( \Omega_2 \) be strictly included in \( \Omega \), i.e., the microscopic domain \( \Omega_2 \) is disconnected. Let \( u_\epsilon \in \mathbb{H}_{2,\epsilon} \) be a bounded sequence, i.e.,

\[
\|u_\epsilon\|_{H^1(\Omega_2)} + \sqrt{\epsilon}\|u_\epsilon\|_{H^1(\Gamma_\epsilon)} \leq C.
\]

Then there exist \( u_0 \in L^2(\Omega) \) and \( U_1 \in L^2(\Omega, \mathbb{H}_2/\mathbb{R}) \) with \( \int_\Gamma U_1(x,y)d\sigma = 0 \), such that up to a subsequence it holds that

- \( \bar{u}_\epsilon \to \chi_{\Omega_2}u_0 \) in the two-scale sense,

- \( \nabla u_\epsilon \to \chi_{\Omega_2} \nabla_y U_1 \) in the two-scale sense,

- \( u_\epsilon \to u_0^\Gamma \) in the two-scale sense on \( \Gamma_\epsilon \),

- \( \nabla_\Gamma u_\epsilon \to \nabla_\Gamma U_1^\Gamma \) in the two-scale sense on \( \Gamma_\epsilon \).

**Proof.** From Lemma 3.2 and 3.3 follows the existence of functions \( u_0, u_1^\Gamma \in L^2(\Omega) \), \( u_1 \in L^2(\Omega, H^1(\Omega_2)) \), and \( u_1^\Gamma \in L^2(\Omega, H^1(\Gamma)) \), such that \( \int_\Gamma u_1(x,y)d\sigma = 0 = \int_\Gamma u_1^\Gamma(x,y)d\sigma \) for almost every \( x \in \Omega \), and

- \( \bar{u}_\epsilon \to \chi_{\Omega_2}u_0 \) in the two-scale sense,

- \( \nabla u_\epsilon \to \chi_{\Omega_2} \nabla_y U_1 \) in the two-scale sense,

- \( u_\epsilon \to u_0^\Gamma \) in the two-scale sense on \( \Gamma_\epsilon \),

- \( \nabla_\Gamma u_\epsilon \to \nabla_\Gamma U_1^\Gamma \) in the two-scale sense on \( \Gamma_\epsilon \).

Again, the equality \( u_0 = u_0^\Gamma \) is standard. It remains to prove \( U_1 = U_1^\Gamma \). We choose \( \phi \in \mathcal{D}(\Omega) \) and \( \theta \in C^\infty(\Omega_2)^n \) with \( 0 = \int_{\Omega_2} \nabla \cdot \theta(y)d\sigma = \int_{\Gamma} \theta(y) \cdot \nu d\sigma_d \), and extend the function \( \theta \) periodically, what is possible since \( \Omega_2 \) does not touch the boundary \( \partial Y \). Then, by integration by parts and Lemma 3.2 and 3.3, we get

\[
\int_{\Omega_2} \nabla u_\epsilon(x) \cdot \phi(x) \theta \left( \frac{x}{\epsilon} \right) dx = \epsilon \int_{\Gamma_\epsilon} u_\epsilon(x) \phi(x) \theta \left( \frac{x}{\epsilon} \right) \cdot \nu d\sigma \\
- \int_{\Omega_2} u_\epsilon(x) \left( \nabla \phi(x) \cdot \theta \left( \frac{x}{\epsilon} \right) + \frac{\phi(x)}{\epsilon} \nabla_y \cdot \theta \left( \frac{x}{\epsilon} \right) \right) dx \\
\xrightarrow{\epsilon \to 0} \int_{\Gamma} \int_{\Omega_2} u_0(x) \nabla \phi(x) \cdot \theta(y) + U_1^\Gamma(x,y) \phi(x) \theta(y) \cdot \nu d\sigma_d dx \\
- \int_{\Omega} \int_{\Omega_2} u_0(x) \left( \nabla \phi(x) \cdot \theta(y) + \nabla \phi(x) \cdot \nabla_y \theta(y) \right) + U_1(x,y) \phi(x) \nabla_y \theta(y) dy dx \\
= \int_{\Gamma} \int_{\Omega} U_1^\Gamma(x,y) \phi(x) \theta(y) \cdot \nu d\sigma_d dx + \int_{\Omega} \int_{\Omega_2} \nabla_y U_1(x,y) \cdot \phi(x) \theta(y) dy dx \\
- \int_{\Omega} \int_{\Gamma} U_1(x,y) \phi(x) \theta(y) \cdot \nu d\sigma_d dx.
\]

Further, it holds that

\[
\lim_{\epsilon \to 0} \int_{\Omega_2} \nabla u_\epsilon(x) \cdot \phi(x) \theta \left( \frac{x}{\epsilon} \right) dx = \int_{\Omega} \int_{\Omega_2} \nabla_y U_1(x,y) \cdot \phi(x) \theta(y) dy dx,
\]

and altogether we obtain

\[
\int_{\Omega} \int_{\Gamma} (U_1(x,y) - U_1^\Gamma(x,y)) \phi(x) \theta(y) \cdot \nu d\sigma_d dx = 0.
\]
for all \( \phi \in \mathcal{D}(\Omega) \) and \( \theta \in C^\infty(\overline{\Omega}^n) \) with \( 0 = \int_\Gamma \theta(y) \cdot \nu d\sigma_y \). This implies the equality of \( U_1 \) and \( U_1^\Gamma \) up to a constant, and due to the mean-value property of both functions, even equality.

\[ \boxed{\square} \]

4. Application to coupled bulk-surface problem with Wentzell-boundary conditions. We assume that \( Y_2 \) is strictly included in \( Y \), i.e., the microscopic domain \( \Omega^2 \) is disconnected, as well as its surface \( \Gamma^e \). We consider the following problem:

\[
\begin{align*}
-\Delta u_{1e} + u_{1e} &= f_1 & \text{in } \Omega^1, \\
-\Delta u_{2e} + u_{2e} &= f_2 & \text{in } \Omega^2, \\
-\nabla u_{1e} \cdot \nu_{1e} &= 0 & \text{on } \partial \Omega, \\
-\nabla u_{1e} \cdot \nu_{1e} &= \epsilon (\Delta_{\Gamma^e} u_{1e} + (u_{1e} - u_{2e})) & \text{on } \Gamma^e, \\
-\nabla u_{2e} \cdot \nu_{2e} &= \epsilon (\Delta_{\Gamma^e} u_{2e} - (u_{1e} - u_{2e})) & \text{on } \Gamma^e,
\end{align*}
\]

where \( \Delta_{\Gamma^e} \) denotes the Laplace-Beltrami operator on \( \Gamma^e \), \( f_j \in L^2(\Omega) \), and \( u_{je} \) for \( j = 1, 2 \) is the outer unit normal vector on \( \partial \Omega^j \) with respect to \( \Omega^j \). This model describes stationary reaction-diffusion processes for two different phases, e.g., a solid and a liquid phase, with additional reaction-diffusion processes on the interface \( \Gamma^e \).

We are looking for weak solutions \( u_j = (u_{1e}, u_{2e}) \in \mathbb{H}_{1,e} \times \mathbb{H}_{2,e} \), which fulfill for all \( (\phi_1, \phi_2) \in \mathbb{H}_{1,e} \times \mathbb{H}_{2,e} \) and \( j = 1, 2 \) the following variational equation:

\[
\int_{\Omega^j} u_{je} \phi_j + \nabla u_{je} \cdot \nabla \phi_j dx \\
+ \epsilon \int_{\Gamma^e} \nabla_{\Gamma^e} u_{je} \cdot \nabla \phi_j + (-1)^j (u_{2e} - u_{1e}) \phi_j d\sigma = \int_{\Omega^j} f_j \phi_j dx.
\]

(9)

Due to the Lax-Milgram Lemma, there exists a unique weak solution. Choosing \( (\phi_1, \phi_2) = (u_{1e}, u_{2e}) \) and adding up the both equalities from (9), we obtain

\[
\sum_{j=1}^2 \left[ \|u_{je}\|_{H^1(\Omega^j)} + \sqrt{\epsilon} \|\nabla_{\Gamma^e} u_{je}\|_{L^2(\Gamma^e)} \right] + \sqrt{\epsilon} \|u_{1e} - u_{2e}\|_{L^2(\Gamma^e)} \leq C.
\]

Using the trace estimate

\[
\sqrt{\epsilon} \|u_{je}\|_{L^2(\Gamma^e)} \leq C \left( \frac{1}{\sqrt{\epsilon}} \|u_{je}\|_{L^2(\Omega^j)} + \sqrt{\epsilon} \|\nabla_{\Gamma^e} u_{je}\|_{L^2(\Omega^j)} \right),
\]

we obtain

\[
\|u_{1e}\|_{\mathbb{H}_{1,e}} + \|u_{2e}\|_{\mathbb{H}_{2,e}} \leq C.
\]

Therefore, we can apply Theorem 3.4 and 3.5 from Section 3 and obtain:

**Proposition 6.** Let \( \tilde{u}_{1e} \in H^1(\Omega) \) be an extension of \( u_{1e} \) such that \( \|\tilde{u}_{1e}\|_{H^1(\Omega)} \leq C\|u_{1e}\|_{H^1(\Omega^1)} \). There exist \( u_{1,0} \in H^1(\Omega) \) and \( u_{1,1} \in L^2(\Omega, \mathbb{H}_{1,per}/\mathbb{R}) \), such that up to a subsequence

\[
\begin{align*}
\tilde{u}_{1e} &\to u_{1,0} \quad \text{(strongly) in } L^2(\Omega), \\
\nabla \tilde{u}_{1e} &\to \nabla_x u_{1,0}(x) + \nabla_y u_{1,1}(x,y) \quad \text{in the two-scale sense,} \\
u_{1e} &\to u_{1,0} \quad \text{(strongly) in } \mathbb{H}_{1,1}, \\
\nabla_{\Gamma^e} u_{1e} &\to P_\Gamma(y) \nabla_x u_{1,0}(x) + \nabla_y u_{1,1}(x,y) \quad \text{in the two-scale sense on } \Gamma^e,
\end{align*}
\]

in the two-scale sense on \( \Gamma^e \).
Further, there exist \( u_{2,0} \in L^2(\Omega) \), such that up to a subsequence
\[
\tilde{u}_{2\varepsilon} \to \chi_{Y_2} u_{2,0} \quad \text{in the two-scale sense,}
\]
\[
\nabla u_{2\varepsilon} \to 0 \quad \text{in the two-scale sense,}
\]
\[
u_{2\varepsilon} \to u_{2,0} \quad \text{in the two-scale sense on } \Gamma_\varepsilon,
\]
\[
\nabla_{\Gamma, u_{2\varepsilon}} \to 0 \quad \text{in the two-scale sense on } \Gamma_\varepsilon.
\]

**Proof.** The first part is exactly Theorem 3.4. For the second part, we apply Theorem 3.5 to obtain functions \( u_{2,0} \in L^2(\Omega) \) and \( U_{2,1} \in L^2(\Omega, \mathbb{H}_2/\mathbb{R}) \), such that
\[
\tilde{u}_{2\varepsilon} \to \chi_{Y_2} u_{2,0} \quad \text{in the two-scale sense,}
\]
\[
\nabla u_{2\varepsilon} \to \chi_{Y_2} \nabla_y U_{2,1} \quad \text{in the two-scale sense,}
\]
\[
u_{2\varepsilon} \to u_{2,0} \quad \text{in the two-scale sense on } \Gamma_\varepsilon,
\]
\[
\nabla_{\Gamma, u_{2\varepsilon}} \to \nabla_{\Gamma, U_{2,1}} \quad \text{in the two-scale sense on } \Gamma_\varepsilon.
\]

It remains to prove \( U_{2,1} = 0 \). We test equation (9) for \( j = 2 \) with \( \phi_2 = \epsilon \Phi \left( x, \frac{x}{\epsilon} \right) := \epsilon \phi(x) \theta \left( \frac{x}{\epsilon} \right) \), where \( \phi \in D(\Omega) \) and \( \theta \in C^\infty(\mathbb{Y}_2) \):
\[
\epsilon \int_{\Omega_2} u_{2\varepsilon}(x) \phi(x) \theta \left( \frac{x}{\epsilon} \right) + \nabla u_{2\varepsilon}(x) \cdot \left[ \nabla \phi(x) \theta \left( \frac{x}{\epsilon} \right) + \frac{1}{\epsilon} \phi(x) \nabla_y \theta \left( \frac{x}{\epsilon} \right) \right] dx
\]
\[
+ \epsilon^2 \int_{\Gamma_\varepsilon} \nabla_{\Gamma, u_{2\varepsilon}}(x) \cdot \left[ P_{\Gamma} \left( \frac{x}{\epsilon} \right) \nabla \phi(x) \theta \left( \frac{x}{\epsilon} \right) + \frac{1}{\epsilon} \phi(x) \nabla_{\Gamma} \theta \left( \frac{x}{\epsilon} \right) \right]
\]
\[
- \left[ u_{1\varepsilon} - u_{2\varepsilon} \right] \phi(x) \theta \left( \frac{x}{\epsilon} \right) d\sigma_x = \epsilon \int_{\Omega_2} f_2(x) \phi(x) \theta \left( \frac{x}{\epsilon} \right) dx
\]
Passing to the limit \( \epsilon \to 0 \) gives us
\[
0 = \int_{\Omega} \int_{\mathbb{Y}_2} \nabla_y U_{2,1}(x, y) \cdot \nabla_y \Phi(x, y) dy dx + \int_{\Omega} \int_{\Gamma} \nabla_{\Gamma} U_{2,1}(x, y) \cdot \nabla_{\Gamma, \Phi}(x, y) d\sigma_y dx.
\]
Because of the density result from Proposition 5(i), we can choose \( \Phi = U_{2,1} \), and therefore \( U_{2,1} = 0 \). \( \square \)

In the next step, we derive the macroscopic model for the limit functions \( u_{1,0} \) and \( u_{2,0} \).

**Theorem 4.1.** The functions \( u_{1,0} \) and \( u_{2,0} \) are the unique weak solutions of the following problem:
\[
-\nabla \cdot (D\nabla u_{1,0}) + Au_{1,0} = F \quad \text{in } \Omega,
\]
\[
-D\nabla u_{1,0} \cdot \nu = 0 \quad \text{on } \partial \Omega,
\]
\[
u_{2,0} = \frac{|Y_2| f_2 + |\Gamma| u_{1,0}}{|Y_2| + |\Gamma|},
\]
with the effective coefficients
\[
A = \frac{|\Gamma| |Y_1| + |\Gamma| |Y_2| + |Y_1||Y_2|}{|\Gamma| + |Y_2|},
\]
\[
F = |Y_1| f_1 + \frac{|Y_2||\Gamma|}{|Y_2| + |\Gamma|} f_2,
\]
\[
D_{ij} = \int_{Y_1} (e_i + \nabla_y w_i) \cdot (e_j + \nabla_y w_j) dy + \int_{\Gamma} (P_{\Gamma} e_i + \nabla_{\Gamma} w_i) \cdot (P_{\Gamma} e_j + \nabla_{\Gamma} w_j) d\sigma_y,
\]
where $w_i \in \mathbb{H}_{1, \text{per}}/\mathbb{R}$ is the unique solution of the cell problem

\begin{align*}
-\nabla_y (\epsilon_i + \nabla_y w_i) &= 0 & \text{ in } Y_1,
- (\epsilon_i + \nabla_y w_i) \cdot \nu = -\nabla_y \cdot (P_\Gamma e_i + \nabla_y w_i) & \text{ on } \Gamma,
\end{align*}

(11)

$$w_i \text{ is } Y\text{-periodic, } \int_\Gamma w_i d\sigma_y = 0.$$  

**Proof.** We test the variational equation (9) for $j = 1$ with $\phi_j(x) = \epsilon \phi(x) \theta_{1/\epsilon}(\frac{x}{\epsilon})$, where $\phi \in \mathcal{D}(\Omega)$ and $\theta \in C^\infty_\text{per}(\Omega)$. Passing to the limit $\epsilon \to 0$, we obtain from Proposition 6

$$0 = \int_\Omega \int_{Y_1} (\nabla u_{1,0}(x) + \nabla_y u_{1,1}(x, y)) \cdot \nabla_y \theta(y) \phi(x) dy \ dx$$

$$+ \int_\Omega \int_{\Gamma} (P_\Gamma(y) \nabla u_{1,0}(x) + \nabla_\Gamma u_{1,1}(x, y)) \cdot \nabla_\Gamma \theta(y) \phi(x) d\sigma_y d\tau.$$

By density, see Proposition 5, this result holds for all $\theta \in \mathbb{H}_{1, \text{per}}$. Hence, we have

$$u_{1,1}(x, y) = \sum_{i=1}^n \partial_x u_{1,0}(x) w_i(y),$$

where $w_i$ is the unique solution of the cell problem (11). Now, we test equation (9) for $j = 2$ with $\phi \in \mathcal{D}(\Omega)$, and with the results from Proposition 6 we pass to the limit $\epsilon \to 0$:  

$$\int_\Omega |Y_1| f_1 \phi \ dx = \int_\Omega |Y_1| f_2 \phi \ dx = \int_\Omega |Y_2| f_2 \phi \ dx,$$

what gives us the identity for $u_{2,0}$. We proceed in the same way for equation (9) with $j = 1$ and $\phi \in C^\infty(\Omega)$, and obtain

$$|Y_1| \int_\Omega f_1 \phi \ dx = \int_\Omega |Y_1| u_{1,0} \phi + \int_\Omega \nabla u_{1,0} + \nabla_y u_{1,1} \cdot \nabla \phi \ dy \ dx$$

$$+ \int_\Omega |\Gamma| u_{1,0} \phi \ dx + \int_\Omega (P_\Gamma \nabla u_{1,0} + \nabla_\Gamma u_{1,1}) \cdot P_\Gamma \nabla \phi d\sigma_y d\tau.$$

Plugging in the representation for $u_{1,1}$ and $u_{2,0}$, we get

$$\int_\Omega F \phi \ dx = \int_\Omega A u_{1,0} \phi + D \nabla u_{1,0} \cdot \nabla \phi \ dx.$$

Hence, $(u_{1,0}, u_{2,0})$ solves (10), and due to the structure of $D$ and since $A > 0$, problem (10) is elliptic and therefore the solution is unique.

**Remark 7.** From the identity (10) in Theorem 4.1, we immediately obtain $u_{2,0} \in L^2(\Omega) \setminus H^1(\Omega)$ for $f_2 \in L^2(\Omega) \setminus H^1(\Omega)$. Hence, we found a sequence $u_{2k}[\Gamma_\ast] \in H^1(\Gamma_\ast)$ with $\sqrt{c\|u_{2k}\|_{H^1(\Gamma_\ast)}} \leq C$, such that its two-scale limit $u_{2,0}$ is not an element of $H^1(\Omega)$.

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E-mail address: markus.gahn@uhasselt.be