Nonlinear neutral inclusions: Assemblages of spheres
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1. Introduction

A neutral inclusion, when inserted in a matrix containing a uniformly applied electric field, does not disturb the outside field. Mansfield was the first to observe that reinforced holes, "neutral holes", could be cut out of a uniformly stressed plate without disturbing the surrounding stress field in the plate (Mansfield, 1953).

The well known Hashin coated sphere construction (Hashin, 1962) is an example of a neutral coated inclusion for the conductivity problem. In Hashin and Shtrikman (1962,) an exact expression for the effective conductivity of the coated sphere assemblage was found, which coincides with the Maxwell (Maxwell, 1873) approximate formula. Thus the approximate formula is realizable and was shown to be an attainable bound for the effective conductivity of a composite, given the volume fractions of the two materials. This construction was extended to coated confocal ellipsoids in Milton (1981). Ellipsoids are not the only possible shapes for neutral inclusions; indeed in Milton and Serkov (2001) other shapes of neutral inclusions are constructed.

The existence of neutral inclusions was also found in the case of materials with imperfect interfaces, for which the potential (or displacement) field has discontinuities across these interfaces. For these materials neutral inclusions have been studied in Lipton and Vernescu (1996) and Benveniste and Miloh (1999) for the conductivity problem, in Lipton (1997,) for highly conducting interfaces, in Lipton and Vernescu (1995, 1996) and Ru (1998) for the elasticity problem, and for nonlinear materials in Lipton and Talbot (1999).

For other references related to neutral inclusions in composites see also Milton (2002) and Mei and Vernescu (2010) and the references therein.

We consider here nonlinear materials for which the constitutive law relating the current $J$ to the electric field $\nabla u$ is described by a nonlinear constitutive model of the form

$$J = \sigma_1 |\nabla u|^{p-2} \nabla u,$$

Here $u$ is the potential, and $\sigma_1 |\nabla u|^{p-2}$ is a nonlinear conductivity. This constitutive model is used to describe the nonlinear behavior of several materials including nonlinear dielectrics (Bueno et al., 2008; Garroni et al., 2014; Garroni and Kohn, 2003; Levy and Kohn, 1998; Talbot and Willis, 1994a; Talbot and Willis, 1994b), and is also used to model thermo-rheological and electrorheological fluids (Ružička, 1748; Antontsev and Rodrigues, 2006; Berselli et al., 2010), viscous flows in glaciology (Glowinski and Rappaz, 2003), and also in plasticity problems (Atkinson and Champion, 1984; Suquet, 1993; Ponte Castañeda and Suquet, 1997; Ponte Castañeda and Willis, 1985; Idiart, 2008).

In this paper we show that even for nonlinear materials, one can construct neutral inclusions by a suitable coating with a linear material. We consider a particular coated sphere (see Fig. (1), with a nonlinear core of radius $r_c$ and a linear coating with exterior radius $r_e$, $1 < r_c < r_e$. The ratio of the inner sphere radius to the outer sphere radius is related to the volume fractions of the two materials $\theta_1$ and respectively $\theta_2$:

$$\theta_1 = \frac{4/3 \pi r_1^3}{4/3 \pi r_e^3} = \frac{r_1^3}{r_e^3}, \quad \text{and} \quad \theta_2 = 1 - \theta_1. \quad (1.1)$$

The coated sphere is inserted in a matrix formed by a linear material of conductivity $\sigma$. We apply a linear electric field $\mathbf{E} = Ex\mathbf{e}_1$, at infinity, (where for simplicity $\mathbf{E} = \mathbf{e}_1$, with $\mathbf{e}_1 = (1, 0, 0)$ and
\( \mathbf{x} = (x_1, x_2, x_3) \). Thus the problem of finding a neutral inclusion reduces to finding the electric potential \( u \) that solves

\[
\begin{aligned}
\nabla \cdot \left( \sigma_1 |\nabla u|^p \nabla u \right) &= 0 \quad \text{in the core,} \\
\nabla \cdot (\sigma_2 \nabla u) &= 0 \quad \text{in the coating,}
\end{aligned}
\tag{1.2}
\]

where the material conductivities are \( \sigma_1 |\nabla u|^p \) in the core, and \( \sigma_2 \) in the coating, with \( \infty > \sigma_1 > \sigma_2 > 0 \), and satisfies continuity conditions of the electric potential and of the normal component of the current across the interfaces.

In Section 2 we show that we can replace the coated sphere with a sphere composed only of linear material of conductivity \( \sigma_1 \) (see (2.20) and Fig. 2).

Since the equations for conductivity are local equations, one could continue to add similar coated spheres of various sizes without disturbing the prescribed uniform applied field surrounding the inclusions. In fact, one can fill the entire space (aside from a set of measure zero) with assemblages of these coated spheres by adding coated spheres of various sizes ranging to the infinitesimal and it is assumed that they do not overlap the boundary of the unit cell of periodicity (see Fig. (3)). The spheres can be of any size, but the volume fraction of the core and the coating layer is the same for all spheres. While adding the coated spheres, the flux of current and electrical potential at the boundary of the unit cell remains unaltered. Therefore, the effective conductivity does not change.

This configuration of nonlinear materials dissipates energy the same as a linear material with conductivity \( \sigma_1 \) (see (2.20)).

This paper is structured as follows: Section 2 provides the statement of the problem and the main result for an assemblage of coated disks, Section 3 extends the results for an assemblage of coated spheres, Section 4 contains a discussions about the effective conductivity of the assemblage and its relation to \( p \), and Section 5 contains the conclusions and a description of future work.

2. Assemblage of coated spheres: statement of the problem and result

Let \( \mathbf{x} \) be the center of the coated sphere (See Fig. (1)). Inside the sphere, we ask that

\[
\begin{aligned}
\sigma_1 \Delta u &= 0 \quad \text{for } 0 < |\mathbf{x} - \mathbf{x}| < r_c, \\
\sigma_2 \Delta u &= 0 \quad \text{for } r_c < |\mathbf{x} - \mathbf{x}| < r_e,
\end{aligned}
\tag{2.1}
\]

where \( \Delta u = \nabla \cdot (|\nabla u|^p \nabla u) \) represents the \( p \)-Laplacian \((p > 1)\), \( \sigma_1 \) and \( \sigma_2 \) are positive, together with the usual continuity conditions of the electric potential and of the normal component of the current across the interfaces:

\[
\begin{aligned}
u& \text{ continuous across } |\mathbf{x} - \mathbf{x}| = r_c, \\
u &= \mathbf{E} \mathbf{1}, \text{ at } |\mathbf{x} - \mathbf{x}| = r_c, \\
\sigma_1 |\nabla u|^p \nabla u \cdot \mathbf{n} &= \sigma_2 \nabla u \cdot \mathbf{n}, \text{ across } |\mathbf{x} - \mathbf{x}| = r_c, \\
\sigma_2 \nabla u \cdot \mathbf{n} &= \sigma_1 \nabla u \cdot \mathbf{n}, \text{ across } |\mathbf{x} - \mathbf{x}| = r_e.
\end{aligned}
\tag{2.2}
\tag{2.3}
\tag{2.4}
\tag{2.5}
\]

We will next non-dimensionalize the problem (2.1)–(2.5). For this we consider a characteristic length \( L \) and a characteristic conductivity \( \sigma_0 \) and define:

\[
\begin{aligned}
x' &= \frac{x}{L}, & u' &= \frac{u}{\mathbf{E} L}, & \sigma_1' &= \frac{\sigma_1}{\sigma_0}, & \sigma_2' &= \frac{\sigma_2}{\sigma_0}, & \sigma_0 &= \sigma_1, \\
L &= r_L, & r_L &= r_c, & r_e &= r_x, & \nabla' = \frac{\nabla}{L}.
\end{aligned}
\]

We should remark here that in the non-dimensional form, \( \sigma_1' \) is nonlinear with respect to the applied electric field and is a constant only in the case \( p = 2 \).

The problem in non-dimensional form becomes:

\[
\begin{aligned}
\nabla' \cdot \left( \sigma_1' |\nabla u'|^p \nabla u' \right) &= 0, \quad \text{for } 0 < r' < r_c', \\
\nabla' \cdot (\sigma_2' \nabla u') &= 0, \quad \text{for } r_c' < r' < r_e',
\end{aligned}
\tag{2.6}
\]

with the conditions

\[
\begin{aligned}
u' &= \mathbf{1}, \quad \text{on } r' = r_c', \\
u' &= x_1, \quad \text{on } r' = r_e', \\
\sigma_1' \nabla u' \cdot \mathbf{n} &= \sigma_2' \nabla u' \cdot \mathbf{n}, \quad \text{on } r' = r_c', \\
\sigma_2' \nabla u' \cdot \mathbf{n} &= \sigma_1' \nabla u' \cdot \mathbf{n}, \quad \text{on } r' = r_e'.
\end{aligned}
\tag{2.7}
\tag{2.8}
\tag{2.9}
\tag{2.10}
\]

For convenience we will drop the superscript ' and look for a solution \( u \) of the problem (2.6)–(2.10).
Using polar coordinates, we look for a solution \( u \) of (2.6) with radial symmetry, of the form

\[
\begin{align*}
  u &= a_1 r \cos \theta \quad \text{for} \ 0 < r < r_c, \\
  u &= \frac{b_2}{r} \cos \theta + a_2 r \cos \theta \quad \text{for} \ r_c < r < r_e, \\
  u &= r \cos \theta \quad \text{for} \ r = r_e.
\end{align*}
\]  

(2.11)

where \( r = |x - \mathbf{x}| \) and \( \theta \) measures the angle between the unit vector \( \mathbf{v} \) in the direction of the applied field and \( \mathbf{x} \).

We introduce the notation for the unit radial vector \( \mathbf{e}_r = \mathbf{n} = \frac{x - \mathbf{x}}{|x - \mathbf{x}|} \), and denote by \( \mathbf{e}_\theta \) the unit vector perpendicular to \( \mathbf{e}_r \). We have \( \mathbf{v} = \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta \).

Since (2.11) satisfies (2.6), it is left to determine the constants so that it satisfies the required conditions (2.7)–(2.10) at the interfaces.

In what follows, we explain how the unknowns \( a_1, a_2, \) and \( b_2 \) and the effective conductivity \( \sigma_e \) are determined from (2.7), (2.8), (2.9) and (2.10).

First, we look at the conditions \( u \) must satisfy when \( r = r_c \). From (2.7), we have

\[
a_1 = a_2 + \frac{b_2}{r_c^2}
\]

(2.12)

and from (2.9), we obtain

\[
\sigma_1 |a_1|^{p-2} a_1 = \sigma_2 \left( a_2 - \frac{2b_2}{r_c^2} \right)
\]

(2.13)

We now look at the conditions that \( u \) must satisfy on the outer interface \( r = r_e \). From (2.8), we have

\[
1 = a_2 + \frac{b_2}{r_e^2}
\]

(2.14)

and from (2.10), we obtain

\[
\sigma_2 \left( a_2 - \frac{2b_2}{r_e^2} \right) = \sigma_c.
\]

(2.15)

Eliminating \( a_2 \) from (2.12) and (2.14), we obtain

\[
a_1 = 1 + \frac{b_2}{r_c^2} \left( \frac{r_c^2}{r_e^2} - 1 \right).
\]

(2.16)

Using (1.1) we denote

\[
A = \frac{r_c^2}{r_e^2} - 1 = \frac{1}{\theta_1} - 1, \quad B = \frac{2r_e^2}{r_c^2} + 1 = \frac{2}{\theta_1} + 1.
\]

(2.17)

Observe that both \( A > 0 \) and \( B > 0 \) are independent of \( r_c \) and \( r_e \), i.e. they are defined only in terms of \( \theta_1 \).

Using (2.13) and (2.14) in (2.16), we obtain the following identity

\[
\sigma_1 \left| 1 + \frac{b_2}{r_c^2} \left( \frac{r_c^2}{r_e^2} - 1 \right) \right|^{p-2} \left[ 1 + \frac{b_2}{r_c^2} \left( \frac{r_c^2}{r_e^2} - 1 \right) \right] = \sigma_2 \left( 1 - \frac{b_2}{r_c^2} - \frac{2b_2}{r_e^2} \right),
\]

which can be rewritten in terms of \( A \) and \( B \) as

\[
\sigma_1 \left| 1 + A \left( \frac{b_2}{r_c^2} \right) \right|^{p-2} \left[ 1 + A \left( \frac{b_2}{r_c^2} \right) \right] - \sigma_2 \left( 1 - B \left( \frac{b_2}{r_c^2} \right) \right) = 0.
\]

(2.18)

At this point, we consider the function

\[
f(x) = \sigma_1 |1 + Ax|^{p-2} (1 + Ax) - \sigma_2 (1 - Bx)
\]

(2.19)

Note that we obtain \( b_2 \) if we can prove that \( f(x) = 0 \) has a (unique) solution. If that is the case, from (2.14) and (2.16), we can obtain \( a_1 \) and \( a_2 \) and from (2.15), we can get an expression for \( \sigma_c \).

Let us study \( f(x) \). If \( 1 + Ax > 0 \), we have

\[
f(x) = \sigma_1 (1 + Ax)^{p-1} - \sigma_2 (1 - Bx).
\]

Taking the derivative of the \( f(x) \), we have

\[
f'(x) = A \sigma_1 (p-1) (1 + Ax)^{p-2} + \sigma_2 B
\]

which is positive for all \( x \), so the function \( f(x) \) is increasing.

If \( 1 + Ax < 0 \), we have

\[
f(x) = -\sigma_1 (-1 - Ax)^{p-1} - \sigma_2 (1 - Bx)
\]

and here

\[
f'(x) = A \sigma_1 (p-1) (-1 - Ax)^{p-2} + \sigma_2 B
\]

is positive for all \( x \) so the function \( f(x) \) is also increasing in this case.

We would like to emphasize that \( f(x) \) approaches \(-\infty \) as \( x \to -\infty \), and as \( x \to \infty \), the function \( f(x) \) approaches \( \infty \). Therefore, we conclude that the equation \( f(x) = 0 \) has a unique solution \( x_0 \). Moreover, observe that the coefficients of \( f(x) \) depend only on \( \sigma_1, \sigma_2, \theta_1, \) and \( p \), thus

\[
x_0 = b_2 \frac{r_c^2}{r_e^2} = K(\sigma_1, \sigma_2, \theta_1, p).
\]

(2.20)

Consequently, from (2.19) we have \( b_2 = x_0 r_c^2 \) which allows us to obtain \( a_2 \) and \( a_1 \) from (2.14) and (2.16), respectively. To obtain an expression for \( \sigma_c \) we use (2.14) in (2.15) as follows

\[
\sigma_c = \sigma_2 \left( 1 - 3 \frac{b_2}{r_c^2} \right)
\]

(2.21)

and from (2.19), we get

\[
\sigma_c = \sigma_2 \left( 1 - 3K(\sigma_1, \sigma_2, \theta_1, p) \right)
\]

(2.22)

Here, we would like to emphasize that (2.20) shows that \( \sigma_c \) does not depend on \( r_c \) or \( r_e \) independently, but it depends only on their ratio. Therefore, we realize that \( \sigma_c \) is independent of scale. The coated sphere thus obtained has an effective conductivity \( \sigma_c \) and thus, when inserted in a matrix of conductivity \( \sigma_c \), it does not perturb the outside field. The rest of the coated spheres inserted into the matrix are chosen identical up to a scale factor, so that when inserted, the outside field is not disturbed. In this way, \( \theta_1 \) is the proportion of nonlinear material in the resulting assemblage of coated spheres and \( \sigma_c \) is its effective conductivity.

**Remark 2.1.** If \( p = 2 \), the function

\[
f(x) = \sigma_1 (1 + Ax) - \sigma_2 (1 - Bx)
\]

has a unique root \( x_0 = \frac{\sigma_2 - \sigma_1}{A \sigma_1 - B \sigma_2} \), and in this case, \( \sigma_c \) is given by

\[
\sigma_c = \sigma_2 \left( 1 - \frac{3(\sigma_2 - \sigma_1)}{A \sigma_1 - B \sigma_2} \right) = \sigma_2 + \frac{3 \beta_3 \sigma_2 (\sigma_1 - \sigma_2)}{3 \beta_3 + \beta_2 (\sigma_1 - \sigma_2)}
\]

(2.23)

which is the Hashin–Shtrikman formula.

### 3. Assemblage of coated disks: statement of the problem and result

Following the same method, we obtain similar results in the two-dimensional case of assemblages of coated disks. Consider a particular coated disk of center \( \mathbf{x} \) with core radius \( r_c \) and exterior radius \( r_e \), subject to linear boundary conditions at the exterior boundary of the disk. The problem in non-dimensional form is to find a solution \( u \) that solves (2.6) together with the usual compatibility conditions at the interfaces. The ratio of the inner disk radius to the outer disk radius is fixed, or equivalently the area fractions of both materials \( \theta_1 \) and \( \theta_2 \) are fixed:

\[
\frac{r_1^2}{r_e^2} = \frac{r_2^2}{r_e^2} = \frac{r_1^2}{r_2^2} \quad \text{and} \quad \bar{\theta}_1 = 1 - \bar{\theta}_2.
\]

(3.1)
Accordingly, using polar coordinates, we look for a solution \( u \) of (2.6) satisfying (2.7)-(2.10), with radial symmetry, of the form

\[
\begin{align*}
  u &= \tilde{a}_1 r \cos \theta & & \text{for } 0 < r < r_c, \\
  u &= \frac{b_2}{r_c^2} \cos \theta + \tilde{a}_2 r \cos \theta & & \text{for } r_c < r < r_e,
\end{align*}
\]

with \( r, \theta, e, \) and \( e_i \) described in a similar way as in the previous section.

The corresponding equations to (2.12)-(2.15) are

\[
\tilde{a}_1 = \tilde{a}_2 + \frac{b_2}{r_c^2},
\]

\[
\sigma_1 |\tilde{a}_1|^{p-2} \tilde{a}_1 = \sigma_2 \left( \frac{\tilde{a}_2 - \frac{b_2}{r_c^2}}{r_c^2} \right).
\]

where \( p \) is the Hashin–Shtrikman formula.

Consequently, from (3.3) we have

\[
\frac{\sigma_2 \left( \tilde{a}_2 - \frac{b_2}{r_c^2} \right)}{r_c^2} = \sigma_2.
\]

Eliminating \( \tilde{a}_2 \) from (3.3) and (3.5), we obtain

\[
\tilde{a}_1 = 1 + \frac{b_2}{r_c^2} \left( \frac{r_c^2}{r_c^2} - 1 \right).
\]

Let us denote

\[
C = \frac{r_c^2}{r_c^2} - 1 = \frac{1}{\theta_1} - 1 \quad \text{and} \quad D = \frac{r_c^2}{r_c^2} + 1 = \frac{1}{\theta_1} + 1.
\]

Observe that \( C > 0, \ D > 0 \) and both are independent of \( r_c \) and \( r_e \) (they are defined only in \( \theta_1 \)).

Using (3.4) and (3.5) in (3.7), we obtain the following identity

\[
\sigma_1 \left[ 1 + C \left( \frac{b_2}{r_c^2} \right) \right]^{p-2} \left[ 1 + D \left( \frac{b_2}{r_c^2} \right) \right] - \sigma_2 \left( 1 - D \left( \frac{b_2}{r_c^2} \right) \right) = 0
\]

At this point, we consider the function

\[
g(x) = \sigma_1 |1+\theta x|^{p-2}(1+\theta x) - \sigma_2 (1-Dx).
\]

We conclude that \( g(x) \) has a unique solution \( \theta_0 \). Moreover, we observe that the coefficients of \( g(x) \) depend only on \( \sigma_1, \sigma_2, p, \) and \( \theta_1 \), thus

\[
\theta_0 = \frac{b_2}{r_c^2} = \bar{K}(\sigma_1, \sigma_2, p, \theta_1).
\]

Consequently, from (3.9) we have \( \frac{\tilde{a}_2}{r_c^2} = \frac{b_2}{r_c^2} \) which allows us to obtain \( \tilde{a}_1 \) and \( \tilde{a}_2 \) from (3.5) and (3.7), respectively, and

\[
\sigma_2 = \sigma_2 \left( 1 - 2\bar{K}(\sigma_1, \sigma_2, p, \theta_1) \right)
\]

Remark 3.1. If \( p = 2 \),

\[
g(x) = \sigma_1 (1 + \theta x) - \sigma_2 (1-Dx)
\]

has a unique root \( \theta_0 = \frac{\sigma_2 - \theta_1}{\sigma_2 - \sigma_1} \) and in this case \( \sigma \) becomes

\[
\sigma = \sigma_2 \left( 1 - 2\bar{K}(\sigma_1, \sigma_2, p, \theta_1) \right) / (\sigma_1 + D\sigma_2).
\]

which is the Hashin–Shtrikman formula.

Remark 3.2. When \( p = 2 \), the effective conductivity \( \sigma \) of any isotropic composite of phases 1 and 2 (both linear in this case) satisfies the Hashin–Shtrikman bounds (Hashin and Shtrikman, 1962; Milton, 2002)

\[
\sigma_1 + \frac{d\sigma_1}{d\theta_1}(\sigma_2 - \sigma_1) \geq \sigma_2 + \frac{d\theta_1\sigma_2}{d\theta_2}(\sigma_2 - \sigma_1) \geq \sigma_2 + \frac{d\sigma_2}{d\theta_2}(\sigma_2 - \sigma_1)
\]

where \( d = 2, 3 \) is the dimensionality of the composite and it is assumed that the phases have been labeled so that \( \sigma_1 > \sigma_2 \). Thus, for \( p = 2, \) the coated sphere (2.21) and coated disk assemblages (3.11) with phase 1 as core and phase 2 as coating attain the lower bound and with the phases inverted, attain the upper bound. They are examples of isotropic materials that, for fixed volume fractions \( \theta_1 \) and \( \theta_2 = 1 - \theta_1 \), attain the minimum or maximum possible effective conductivity.

4. Discussion on \( \sigma \) and \( x_0 \)

In this section, we discuss and analyze \( \sigma \) and \( x_0 \) and also their behavior with respect to \( p \).

First, observe that if we evaluate the function \( f \) in (2.18) at \( x = -\frac{1}{A} \), we obtain

\[
f \left( -\frac{1}{A} \right) = -\sigma_2 \left( 1 + \frac{B}{A} \right) < 0;
\]

and if we evaluate it at \( x = \frac{1}{A} \), we obtain

\[
f \left( \frac{1}{B} \right) = \sigma_1 \left( 1 + \frac{A}{B} \right)^{p-1} > 0.
\]

Therefore, we can conclude that \( x_0 \) in (2.19) satisfies

\[
-\frac{1}{A} < x_0 < \frac{1}{B},
\]

which implies \( 1 + Ax_0 > 0 \) and \( 1 - Bx_0 > 0 \); and the effective conductivity \( \sigma \) (2.20) satisfies

\[
\sigma_2 \left( 1 - \frac{3}{B} \right) \leq \sigma_1 \leq \sigma_2 \left( 1 + \frac{3}{A} \right)
\]

or equivalently

\[
\sigma_2 - 3\sigma_2 \theta_1 \leq \sigma_1 \leq \sigma_2 + 3\sigma_2 \theta_1
\]

where \( A = \frac{1}{A} - 1 > 0 \) and \( B = \frac{1}{B} + 1 > 0 \). If \( \theta_1 = 1 \) (the sphere is made only of nonlinear material), we have \( A = 0, B = 3, \) and \( f(x) = \sigma_1 - \sigma_2 (1-3x) \); from where \( x_0 = \frac{\sigma_2 - \sigma_1}{3\sigma_2} \) and \( \sigma_1 = \sigma_1 \). If \( \theta_1 = 0, \) from above we obtain \( \sigma_1 = \sigma_2 \).

Let us also observe that \( f(0) = \sigma_1 - \sigma_2 > 0, \) therefore we can improve (4.1) and we obtain

\[
-\frac{1}{A} < x_0 < 0,
\]

which implies \( 0 < 1 + Ax_0 < 1 \). Note that, in this case,

\[
\frac{\partial \sigma_0}{\partial p} - \frac{\sigma_1 (1 + Ax_0)^{p-1} \ln(1 + Ax_0)}{\sigma_1 (p-1)A(1 + Ax_0)^{p-2} + \sigma_2 B} > 0
\]

and it is equal to 0 if \( A = 0 \) which corresponds to the case when \( \theta_1 = 1. \) So \( x_0 \) is an strictly increasing function of \( p \), except when \( \theta_0 = 1 \) (all nonlinear material), in which case it is a constant function \( x_0 = \frac{\sigma_2 - \sigma_1}{3\sigma_2} \).

For example, by taking \( \sigma_1 = 10, \sigma_2 = 1, \) and \( \theta_1 = 0, 0.0, 0.2, 0.4, 0.5, 0.6, 0.8, 1, \) the behavior of \( x_0 \) for different values of \( p \) can be observed in Fig. 4 and the values of \( x_0 \) for \( p = 1, 1.1, 1.3, 1.6, 2, 2.7, 4, 10 \) can be found in Table 1. In this case, observe that when \( \theta_1 = 1, \ x_0 = \frac{1-10}{1-10} = -3. \)
Since \( \sigma_1 = \sigma_2(1 - 3x_0) \), we have

\[
\frac{\partial \sigma_1}{\partial p} = -3\sigma_2 \frac{\partial x_0}{\partial p} \leq 0,
\]

and it is equal to 0 if \( A = 0 \) which corresponds to the case when \( \theta_1 = 1 \). So, \( \sigma_1 \) is a strictly decreasing function of \( p \), except when \( \theta_1 = 1 \) (all nonlinear material), in which case it is a constant function \( \sigma_1 = \sigma_1 (=10 \text{ with the values given previously}) \). The behavior of \( \sigma_1 \) for different values of \( p \) can be observed in Fig. 5 and the values of \( \sigma_1 \) for \( p = 1.1, 1.3, 1.6, 2, 2.7, 4, 10 \) can be found in Table 2.

Also, as a function of \( \theta_1 \), the effective conductivity \( \sigma \) satisfies

\[
\frac{\partial \sigma}{\partial \theta_1} = -3\sigma_2 \frac{x_0}{\theta_1} \left( \frac{\sigma_1(p - 1)(1 + A \theta_0)^p}{\sigma_1(p - 1)A(1 + A \theta_0)^{p - 2} + B \sigma_2} \right) > 0.
\]

Therefore \( \sigma \) is an increasing function with respect to \( \theta_1 \). The plot of \( \sigma \) varying with respect to the volume fraction \( \theta_1 \) in the case when \( \sigma_1 = 10 \text{ and } \sigma_2 = 1 \) can be observed in Fig. 6. The dashed plot corresponds to the case when \( p = 2 \) (Hashin–Shtrikman lower bound for different values of \( \theta_1 \)). All the curves have fixed values \( \sigma = \sigma_2 \) for \( \theta_1 = 0 \) and \( \sigma = \sigma_1 \) for \( \theta_1 = 1 \). In the case when \( \sigma_1 = 10 \), and \( \sigma_2 = 1 \), then \( \sigma = 1 \) for \( \theta_1 = 0 \) and \( \sigma = 10 \) for \( \theta_1 = 1 \) (see Fig. 6).

As it was observed in the previous section, \( \sigma_1^2 \) depends nonlinearly on the applied field \( E \). To better understand the effect of the non-linearity in what follows we will analyze the behavior of \( \sigma_1 \) and \( x_0 \) in their dimensional form with respect to \( p \) and \( E \). Thus we consider the cases \( 0 < E < 1 \text{ and } E > 1 \). For these cases, we have that

\[
\frac{\partial x_0}{\partial p} = \frac{-\sigma_1(1 + A \theta_0)^{p - 1} \ln(1 + A \theta_0)}{\sigma_1(p - 1)A(1 + A \theta_0)^{p - 2} + B \sigma_2^2}.
\] (4.3)

Also, since \( \sigma_1 = \frac{E}{x_0} (E - 3x_0) \), we have

\[
\frac{\partial \sigma_1}{\partial p} = \frac{\sigma_2 \frac{\partial x_0}{\partial p}}{E}.
\] (4.4)
With respect to \( \theta_1 \), we obtain

\[
\frac{\partial x_0}{\partial \theta_1} = x_0 \left[ \frac{\sigma_1(p-1)(E + Ax_0)^{p-2} + 2\sigma_2}{\sigma_1(p-1)A(E + Ax_0)^{p-2} + \sigma_2 B} \right]
\]

(4.5)

and the change of the effective conductivity \( \sigma_e \) with respect to the volume fraction \( \theta_1 \) is given by

\[
\frac{\partial \sigma_e}{\partial \theta_1} = \frac{-3\sigma_1 x_0}{E \theta_1^2} \left[ \frac{\sigma_1(p-1)(E + Ax_0)^{p-2} + 2\sigma_2}{\sigma_1(p-1)A(E + Ax_0)^{p-2} + \sigma_2 B} \right]
\]

(4.6)

We start with \( 0 < \theta_1 < 1 \). In this case, since \( 0 < E + Ax_0 < 1 \), we have from (4.3) that \( \frac{\partial x_0}{\partial \theta_1} > 0 \), and we conclude \( x_0 \) is an strictly increasing function of \( p \). If \( E = 0.7 \), \( \sigma_1 = 10 \), \( \sigma_2 = 1 \), and \( \theta_1 = 0.00 \), \( 0.1 \), \( 0.2 \), \( 0.5 \), \( 0.6 \), \( 0.8 \), \( 1 \), the behavior of \( x_0 \) for different values of \( p \) can be observed in Fig. 7; and the values of \( x_0 \) for \( p = 1, 1.1, 1.3, 1.6, 2, 2.7, 4, 10 \) can be found in Table 3.

By (4.4), we can also conclude that \( \sigma_e \) is an strictly decreasing function of \( p \). The behavior of \( \sigma_e \) for different values of \( p \), in the case when \( E = 0.7 \), \( \sigma_1 = 10 \), and \( \sigma_2 = 1 \), can be observed in Fig. 8 and the values of \( \sigma_e \) for \( p = 1, 1.1, 1.3, 1.6, 2, 2.7, 4, 10 \) can be found in Table 4.

The rate of change of the effective conductivity \( \sigma_e \) with respect to the volume fraction \( \theta_1 \) (4.6) is determined by the sign of \( x_0 \). If \( x_0 < 0 \), then \( \sigma_e \) is an strictly increasing function of \( \theta_1 \), and if \( x_0 > 0 \) it is decreasing. We obtain have that \( \sigma_e \) is a constant function of \( \theta_1 \) if \( \frac{\partial x_0}{\partial \theta_1} = 0 \). A plot of \( \sigma_e \) as a function of \( \theta_1 \) can be observed in Fig. 9 for the values \( E = 0.7 \), \( \sigma_1 = 10 \), and \( \sigma_2 = 1 \). The dashed plot corresponds to the case when \( p = 2 \) (Hashin–Shtrikman lower bound for different values of \( \theta_1 \)). The curves have values \( \sigma_e = \sigma_2 = 1 \) for \( \theta_1 = 0 \) and \( \sigma_e = \sigma_1 E^{p-2} = 10(0.7)^{1.73} \) for \( \theta_1 = 1 \). Observe for example, for \( p = 1.1 \) and \( \theta_1 = 1 \), \( \sigma_e = 10(0.7)^{1.73} \approx 13.8 \) and for \( p = 1 \). \( \sigma_e = 10(0.7)^{1.73} \approx 58.1 \). Observe in Fig. 9 that for \( p = 1 \), the effective conductivity \( \sigma_e \) is decreasing with respect to \( \theta_1 \), this is obtained from the fact that \( x_0 > 0 \) for \( p = 1 \). \( E = 0.7 \), \( \sigma_1 = 10 \), and \( \sigma_2 = 1 \).

When \( E > 1 \), we have \( \frac{\partial x_0}{\partial \theta_1} > 0 \) if \( \sigma_1 \geq \sigma_2 \left( \frac{E - 2 \theta_1}{1 - \theta_1} \right) \), otherwise \( \frac{\partial x_0}{\partial \theta_1} < 0 \). For example, if \( E = 2, \sigma_1 = 10, \sigma_2 = 1 \), and \( \theta_1 = 0.0 \), \( 0.2 \), \( 0.4 \), \( 0.5 \), \( 0.6 \), \( 0.8 \), \( 1 \), the values of \( x_0 \) for \( p = 1, 1.3, 1.6, 2, 2.7, 4, 10 \) can be found in Table 5. Observe that, for instance, for \( \theta_1 = 0.4 \), \( x_0 \) is increasing as \( p \) increases, but for \( \theta_1 = 0.8 \), \( x_0 \) is decreasing as \( p \) increases.

The values of \( \sigma_e \), for \( p = 1, 1.1, 1.3, 1.6, 2, 2.7, 4, 10 \) can be found in Table 6. By (4.4), we have \( \sigma_e \) it is increasing iff \( \sigma_1 > \sigma_2 \left( \frac{E - 2 \theta_1}{1 - \theta_1} \right) \), otherwise \( \frac{\partial x_0}{\partial \theta_1} < 0 \). For example, for \( \theta_1 = 0.4 \), the value of \( \sigma_e \) decreases as \( p \) increases but for \( \theta_1 = 0.8 \), \( \sigma_e \) increases as \( p \) increases.

Observe that if \( f(0) = \sigma_1 E^{p-2} - \sigma_2 E > 0 \) if \( E > 1 \), therefore \( x_0 < 0 \) and, by (4.6), the effective conductivity \( \sigma_e \) is an strictly increasing function of \( \theta_1 \). The effective conductivity \( \sigma_e \) as a function of the volume fraction \( \theta_1 \) for the case when \( E = 2, \sigma_1 = 10, \) and \( \sigma_2 = 1 \) can be observed in Fig. 10. The dashed plot corresponds to the case when \( p = 2 \) (Hashin–Shtrikman lower bound for different values of \( \theta_1 \)).

5. Final remarks

We considered the problem of constructing neutral inclusions from nonlinear materials. In particular, we studied the case of a

Table 3

| \( \theta_1 \) | \( p \) | 1.1 | 1.3 | 1.6 | 2 | 2.7 | 4 | 10 |
|----------|-----|-----|-----|-----|---|-----|---|-----|
| 0.0      | 0.00| 0.00| 0.00| 0.00| 0.00| 0.00| 0.00| 0.00|
| 0.1      | -0.17| -0.17| -0.15| -0.12| -0.09| -0.05| 0.01|
| 0.2      | -0.47| -0.45| -0.38| -0.30| -0.21| 0.12| 0.02|
| 0.3      | -0.70| -0.65| -0.54| -0.42| -0.29| 0.17| 0.03|
| 0.4      | -0.94| -0.92| -0.74| -0.37| -0.40| 0.22| 0.04|
| 0.5      | -1.21| -1.73| -1.35| -1.05| -0.73| -0.41| 0.06|
| 0.6      | -1.99| -2.75| -2.45| -2.10| -1.58| -0.91| 0.10|

Table 4

| \( \theta_1 \) | \( p \) | 1.1 | 1.3 | 1.6 | 2 | 2.7 | 4 | 10 |
|----------|-----|-----|-----|-----|---|-----|---|-----|
| 0.0      | 1.00| 1.00| 1.00| 1.00| 1.00| 1.00| 1.00|
| 0.2      | 1.75| 1.74| 1.66| 1.53| 1.38| 1.22| 0.96|
| 0.4      | 3.00| 2.93| 2.62| 2.29| 1.90| 1.52| 0.92|
| 0.5      | 4.00| 3.80| 3.30| 2.80| 2.26| 1.72| 0.89|
| 0.6      | 5.47| 4.96| 4.15| 3.46| 2.70| 1.96| 0.85|
| 0.8      | 10.3| 8.42| 6.79| 5.51| 4.13| 2.76| 0.74|
| 1        | 13.8| 12.8| 11.5| 7.78| 4.90| 4.90| 0.58|
coated sphere in which the core was nonlinear and the coating was a linear material. We showed that the coated sphere is equivalent to a sphere composed only of linear material of conductivity \( \sigma_1 \). One could continue to add coated spheres of various sizes but with fixed volume fraction \( \phi \), without disturbing the prescribed uniform applied electric field surrounding the inclusions, fill the entire space with an assemblage of these coated spheres and this configuration of nonlinear materials dissipates energy the same as a linear material with thermal conductivity \( \sigma_1 \). We then studied the two-dimensional case of a coated sphere in which the core was nonlinear and the coating was a linear material.

**Table 5**

Values of \( \omega_0 \), when \( E = 2 \), \( \sigma_1 = 10 \), and \( \sigma_2 = 1 \).

| \( \phi_1/p \) | 1.1 | 1.3 | 1.6 | 2 | 2.7 | 4 | 10 |
|-------------|-----|-----|-----|---|-----|---|----|
| 0.0         | -0.0 | 0.0 | -0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.2         | -0.49 | -0.43 | -0.39 | -0.35 | -0.32 | -0.30 | -0.27 |
| 0.4         | -1.14 | -1.01 | -0.92 | -0.86 | -0.80 | -0.76 | -0.70 |
| 0.5         | -1.48 | -1.35 | -1.26 | -1.20 | -1.14 | -1.09 | -1.04 |
| 0.6         | -1.79 | -1.73 | -1.67 | -1.64 | -1.59 | -1.56 | -1.52 |
| 0.8         | -2.38 | -2.56 | -2.78 | -3.0 | -3.25 | -3.5 | -3.81 |
| 1           | -2.90 | -3.43 | -4.40 | -6.0 | -10.1 | -26.0 | -1710 |

**Table 6**

Values of \( \omega_0 \), when \( E = 2 \), \( \sigma_1 = 10 \), and \( \sigma_2 = 1 \).

| \( \phi_1/p \) | 1.1 | 1.3 | 1.6 | 2 | 2.7 | 4 | 10 |
|-------------|-----|-----|-----|---|-----|---|----|
| 0.0         | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.2         | 1.73 | 1.65 | 1.58 | 1.53 | 1.48 | 1.44 | 1.40 |
| 0.4         | 2.71 | 2.52 | 2.38 | 2.28 | 2.20 | 2.13 | 2.05 |
| 0.5         | 3.22 | 3.02 | 2.89 | 2.80 | 2.71 | 2.64 | 2.56 |
| 0.6         | 3.68 | 3.60 | 3.50 | 3.46 | 3.38 | 3.34 | 3.28 |
| 0.8         | 4.57 | 4.84 | 5.15 | 5.50 | 5.90 | 6.25 | 6.71 |
| 1           | 5.35 | 6.15 | 7.60 | 10.0 | 16.2 | 40.0 | 2560 |

**Fig. 9.** Plot of \( \sigma_r \) with respect to \( \theta_2 \) for \( E = 0.7 \). \( \sigma_1 = 10 \), \( \sigma_2 = 1 \), \( p = 1.1, 1.3, 1.6, 2, 2.7, 4, 10 \). The dashed plot corresponds to \( p = 2 \), above this curve are the plots for \( p < 2 \) and below for \( p > 2 \) in decreasing and increasing order respectively.

**Fig. 10.** Plot of \( \sigma_r \) with respect to \( \theta_2 \) for \( E = 2 \). \( \sigma_1 = 10 \), \( \sigma_2 = 1 \), \( p = 1.1, 1.3, 1.6, 2, 2.7, 4, 10 \). The dashed plot corresponds to \( p = 2 \), above this curve are the plots for \( p < 2 \) and below for \( p > 2 \) in decreasing and increasing order respectively.

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**Acknowledgment**

The authors thank Robert P. Lipton and Burt S. Tilley for fruitful discussions and helpful suggestions.

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