Primitive set-theoretic solutions of the Yang-Baxter equation*

F. Cedó  E. Jespers  J. Okniński

Abstract
To every involutive non-degenerate set-theoretic solution \((X, r)\) of the Yang-Baxter equation on a finite set \(X\) there is a naturally associated finite solvable permutation group \(G(X, r)\) acting on \(X\). We prove that every primitive permutation group of this type is of prime order \(p\). Moreover, \((X, r)\) is then a so called permutation solution determined by a cycle of length \(p\). This solves a problem recently asked by A. Ballester-Bolinches. The result opens a new perspective on a possible approach to the classification problem of all involutive non-degenerate set-theoretic solutions.

1 Introduction

A fundamental open problem is to construct all solutions of the Yang-Baxter equation

\[
R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},
\]

where \(V\) is a vector space and \(R: V \otimes V \to V \otimes V\) is a bijective linear transformation. Here \(R_{ij}\) denotes the map \(V \otimes V \otimes V \to V \otimes V \otimes V\) acting as \(R\) on the \((i, j)\) tensor factors and as the identity on the remaining factor. In the context of quantum groups and Hopf algebras such solutions play a fundamental role and are often referred to as R-matrices (see for example [4, 20]). Drinfeld in [12] suggested the study of the set-theoretic solutions of the Yang-Baxter equation, these are the bijective maps \(r: X \times X \to X \times X\), defined for a nonempty set \(X\), such that

\[
r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},
\]

where \(r_{ij}\) denotes the map \(X \times X \times X \to X \times X \times X\) acting as \(r\) on the \((i, j)\) components and as the identity on the remaining component. In their fundamental papers, Gateva-Ivanova and Van den Bergh [14], and Etingof, Schedler and Soloviev [13] introduced a subclass of the set-theoretic solutions, the involutive

---

*The first author was partially supported by the grants MINECO-FEDER MTM2017-83487-P and AGAUR 2017SGR1725 (Spain). The second author is supported in part by Onderzoeksaad of Vrije Universiteit Brussel and Fonds voor Wetenschappelijk Onderzoek (Belgium). The third author is supported by the National Science Centre grant, 2016/23/B/ST1/01045 (Poland). 2010 MSC: Primary 16T25, 20B15, 20F16. Keywords: Yang-Baxter equation, set-theoretic solution, primitive group, brace.
non-degenerate solutions. Recall that a set-theoretic solution \( r : X \times X \to X \times X \) of the Yang-Baxter equation, written in the form \( r(x, y) = (\sigma_x(y), \gamma_y(x)) \), for \( x, y \in X \), is involutive if \( r^2 = \text{id} \), and it is non-degenerate if \( \sigma_x \) and \( \gamma_x \) are bijective maps from \( X \) to \( X \), for all \( x \in X \). Actually, [13] and [14] initiated new perspectives for developing and applying a variety of algebraic methods in this context. In particular, they introduced the associated structure groups and quadratic algebras with very nice structural and arithmetical properties when \( X \) is finite. More recently, a new tool was introduced by Rump in [22], based on algebraic structures called (left) braces. This attracted a lot of attention (including contributions of Bachiller, Brzeziński, Catino, Gateva-Ivanova, Rump, Smoktunowicz, Vendramin, see for example [5, 11, 15, 25] and the references in these papers) and lead to several breakthroughs in the area. In particular it resulted in several new constructions of classes of solutions and it allowed to solve some open problems [1, 3, 9, 10, 25]. Furthermore, many new deep and intriguing connections to a variety of other areas were discovered [14, 29].

Remaining and challenging fundamental problems in the area of involutive non-degenerate set-theoretic solutions of the Yang-Baxter equations are: constructing new classes of solutions and a program to classify all solutions. One of the very fruitful approaches in this context is based on two notions: first, indecomposable solutions and, second, retractable solutions [13]. In both cases, the idea behind these notions is to show that several classes of solutions come from solutions of smaller cardinality. And therefore, the main effort should be to characterize the minimal ‘building blocks’ and to describe the way how they can be used to construct arbitrary solutions.

In this direction, a fundamental result of Rump [21] shows that all finite square-free involutive non-degenerate set-theoretic solutions \( (X, r) \), with \( |X| > 1 \), are decomposable (into smaller solutions that are also square-free). However, it turned out that this is no longer true in full generality. The second approach, via the retract relation, allowed to introduce the notion of multipermutation solutions and to define a measure of their complexity - the multipermutation level. Certain positive results in this direction were obtained in [8] and later in [2] it was shown that finite multipermutation solutions are characterized by the condition saying that the structure group of the solution is a poly-Z-group. However, this approach also fails in full generality because there exist solutions that are not retractable. Namely, an example of an indecomposable and irretractable solution was already given in [18] (see Example 8.2.14 in [19]); this was the first example of a solution whose structure group is not a poly-Z group.

Every non-degenerate involutive solution \( (X, r) \) is equipped with a permutation group \( \mathcal{G}(X, r) \) acting on the set \( X \). Recently, Ballester-Bolinches stated (in a talk [5] during a workshop in Oberwolfach [17]) the question of describing all the finite primitive solutions, i.e. those solutions with primitive permutation group.

In this paper we prove that every finite primitive solution \( (X, r) \) is of prime order, i.e. \( |X| \) is a prime number, and it is known that all such solutions admit a very simple description. Namely, they are permutation solutions determined
by a cyclic permutation of length \( p \).

This opens a new perspective on the classification problem of all solutions as the notion of a primitive solution is much better than that of an indecomposable solution, because, roughly speaking, this shows that every solution \((X, r)\) which is not of the form described in Theorem 3.1 is built on an information coming from its imprimitivity blocks, which are sets of smaller cardinality.

The paper is organized as follows. We start in Section 2 with a translation of the problem to a certain brace associated to the given solution \((X, r)\). Then, in Section 3 we prove our main result. The success of our approach is thus another instance of the phenomenon showing that nontrivial connections with the theory of braces allow to attack problems in the area of the Yang-Baxter equation.

## 2 Preliminaries

Let \( X \) be a non-empty set and let \( r : X \times X \rightarrow X \times X \) be a map. For \( x, y \in X \) we put \( r(x, y) = (\sigma_x(y), \gamma_y(x)) \). Recall that \((X, r)\) is an involutive, non-degenerate, set-theoretic solution of the Yang-Baxter equation if \( r^2 = \text{id} \), all the maps \( \sigma_x \) and \( \gamma_y \) are bijective maps from \( X \) to itself and

\[
\begin{align*}
  r_{12}r_{23}r_{12} &= r_{23}r_{12}r_{23},
\end{align*}
\]

where \( r_{12} = r \times \text{id}_X \) and \( r_{23} = \text{id}_X \times r \) are maps from \( X^3 \) to itself. Because \( r^2 = \text{id} \) one easily verifies that \( \gamma_y(x) = \sigma_{\sigma_x(y)}(x) \), for all \( x, y \in X \) (see for example [13, Proposition 1.6]).

**Convention.** Throughout the paper a solution of the YBE will mean an involutive, non-degenerate, set-theoretic solution of the Yang-Baxter equation.

The proof of our result relies on the algebraic structure of the left brace associated to a solution. Hence, we recall some essential background. We refer the reader to [7] for details. A left brace is a set \( B \) with two binary operations, \( + \) and \( \cdot \), such that \((B, +)\) is an abelian group (the additive group of \( B \)), \((B, \cdot)\) is a group (the multiplicative group of \( B \)), and for every \( a, b, c \in B \),

\[
a \cdot (b + c) + a = a \cdot b + a \cdot c. \quad (1)
\]

Note that if we denote by 0 the neutral element of \((B, +)\) and by 1 the neutral element of \((B, \cdot)\), then

\[
1 = 1 \cdot (0 + 0) + 1 = 1 \cdot 0 + 1 \cdot 0 = 0.
\]

In any left brace \( B \) there is an action \( \lambda : (B, \cdot) \rightarrow \text{Aut}(B, +) \), called the lambda map of \( B \), defined by \( \lambda(a) = \lambda_a \) and \( \lambda_a(b) = a \cdot b - a \), for \( a, b \in B \). We shall
write \( a \cdot b = ab \), for all \( a, b \in B \). A trivial brace is a left brace \( B \) such that \( ab = a + b \), for all \( a, b \in B \), i.e. all \( \lambda_a = \text{id} \). The socle of a left brace \( B \) is

\[
\text{Soc}(B) = \{ a \in B \mid ab = a + b, \text{ for all } b \in B \}.
\]

Note that \( \text{Soc}(B) = \text{Ker}(\lambda) \), and thus it is a normal subgroup of the multiplicative group of \( B \). The solution of the YBE associated to a left brace \( B \) is \((B, r_B)\), where \( r_B(a, b) = (\lambda_a(b), \lambda^{-1}_a(b)(a)) \), for all \( a, b \in B \) (see [9, Lemma 2]).

A left ideal of a left brace \( B \) is a subgroup \( L \) of the additive group of \( B \) such that \( \lambda_a(b) \in L \), for all \( b \in L \) and all \( a \in B \). An ideal of a left brace \( B \) is a normal subgroup \( I \) of the multiplicative group of \( B \) such that \( \lambda_a(b) \in I \), for all \( b \in I \) and all \( a \in B \). Note that

\[
ab^{-1} = a - \lambda_{ab^{-1}}(b)
\]

for all \( a, b \in B \), and

\[
a - b = a + \lambda_b(b^{-1}) = a\lambda_{a^{-1}}(\lambda_b(b^{-1})) = a\lambda_{a^{-1}b}(b^{-1}),
\]

for all \( a, b \in B \). Hence, every left ideal \( L \) of \( B \) also is a subgroup of the multiplicative group of \( B \), and every ideal \( I \) of a left brace \( B \) also is a subgroup of the additive group of \( B \). It is known that \( \text{Soc}(B) \) is an ideal of the left brace \( B \) (see [22 Proposition 7]).

It also is well-known that the multiplicative group of a finite left brace is solvable (see [2 Theorem 5.2]). Note that if \( B \) is a finite left brace, then, because each \( \lambda_a \in \text{Aut}(B, +) \), the Sylow subgroups \( B_1, \ldots, B_k \) of the additive group of \( B \) are left ideals. Because of (2) and (3) we also get that

\[
B = B_1 + \cdots + B_k = B_1 \cdots B_k \text{ and } B_iB_j = B_i + B_j = B_j + B_i = B_jB_i.
\]

Hence \( \{B_1, \ldots, B_k\} \) is a Sylow system of the solvable group \((B, +)\), i.e. a complete set of multiplicative Sylow subgroups such that any two are permutable.

Recall that if \((X, r)\) is a solution of the YBE, with \( r(x, y) = (\sigma_x(y), y \gamma_y(x)) \), then its structure group \( G(X, r) = \text{gr}(x \in X \mid xy = \sigma_x(y)\gamma_y(x), \text{ for all } x, y \in X) \) has a natural structure of left brace such that \( \lambda_x(y) = \sigma_x(y) \), for all \( x, y \in X \). The additive group of \( G(X, r) \) is the free abelian group with basis \( X \). The permutation group \( G(X, r) = \text{gr}(x \in X) \) of \((X, r)\) is a subgroup of the symmetric group \( \text{Sym}_X \) on \( X \). The map \( x \mapsto \sigma_x \), from \( X \) to \( G(X, r) \) extends to a group homomorphism \( \phi : G(X, r) \rightarrow G(X, r) \) and \( \text{Ker}(\phi) = \text{Soc}(G(X, r)) \). Hence there is a unique structure of left brace on \( G(X, r) \) such that \( \phi \) is a homomorphism of left braces, this is the natural structure of left brace on \( G(X, r) \). In particular \( G(X, r) \) is a solvable group if \( X \) is finite. Note that there is a natural action of \( G(X, r) \) on the additive group of \( G(X, r) \) defined by

\[
a \left( \sum_{i=1}^{n} z_i x_i \right) = \sum_{i=1}^{n} z_i a(x_i),
\]

for all \( a \in G(X, r), x_i \in X \) and \( z_i \in \mathbb{Z} \). It is known that the map \( \varphi : G(X, r) \rightarrow (G(X, r), +) \times G(X, r) \) defined by \( \varphi(g) = (g, \phi(g)) \), for all \( g \in G(X, r) \), is a
monomorphism of groups, and thus $G(X,r)$ is a regular subgroup of the holomorph of $(G(X,r), +)$.

**Lemma 2.1** Let $(X,r)$ be a solution of the YBE. Then $\lambda_g(\sigma_x) = \sigma_g(x)$, for all $g \in G(X,r)$ and all $x \in X$.

**Proof.** Let $g \in G(X,r)$ and let $x \in X$. There exist a non-negative integer $s$, and $x_1, \ldots, x_s \in X$ and $\varepsilon_1, \ldots, \varepsilon_s \in \{ -1,1 \}$, such that $g = \sigma_{x_1}^{\varepsilon_1} \cdots \sigma_{x_s}^{\varepsilon_s}$. Let $\phi: G(X,r) \rightarrow G(X,r)$ be the homomorphism of left braces such that $\phi(x) = g$. Now we have

\[
\lambda_g(\sigma_x) = \sigma_{x_1}^{\varepsilon_1} \cdots \sigma_{x_s}^{\varepsilon_s} \sigma_x - \sigma_{x_1}^{\varepsilon_1} \cdots \sigma_{x_s}^{\varepsilon_s} = \phi(x_1^{\varepsilon_1} \cdots x_s^{\varepsilon_s} x - x_1^{\varepsilon_1} \cdots x_s^{\varepsilon_s}) = \lambda_{\phi(x)}(\sigma_x).
\]

The following definition is due to Ballester-Bolinches [3].

**Definition 2.2** A finite solution $(X,r)$ of the YBE is said to be primitive if its permutation group $G(X,r)$ acts primitively on $X$.

**Remark 2.3** Note that if $(X,r)$ is a finite primitive solution of cardinality $|X| > 1$, then it is indecomposable because $G(X,r)$ acts transitively on $X$. By a well-known result, that Huppert attributes to Galois [4] II, 3.2 Satz], since $G(X,r)$ is solvable and because $G(X,r)$ acts primitively on $X$, there exist a prime $p$ and a positive integer $m$ such that $|X| = p^m$, and $G(X,r)$ has a unique non-trivial normal abelian subgroup $N$. Furthermore, $N \cong (\mathbb{Z}/p)^m$ is the unique minimal normal subgroup of $G(X,r)$, $N$ acts transitively on $X$, $G(X,r) = N \rtimes A$, where $A$ is the stabilizer of some $x \in X$ and $N$ is self-centralizing in $G(X,r)$.

By [13] Theorem 2.13, for each prime $p$, there is a unique indecomposable solution $(X,r)$ of the YBE of cardinality $p$. In this case $G(X,r) \cong \mathbb{Z}/(p)$, and $(X,r)$ is primitive.

Recall that a solution $(X,r)$ is said to be irretractable if $\sigma_x \neq \sigma_y$ for all distinct elements $x, y \in X$, otherwise the solution $(X,r)$ is retractable.

The following is a result of Ballester-Bolinches [3].

**Lemma 2.4** Let $(X,r)$ be a primitive solution of cardinality $p^m$, for some prime $p$ and some integer $m > 1$. Then $(X,r)$ is irretractable.

**Proof.** Suppose that $(X,r)$ is retractable. Let $x, y \in X$ be distinct elements such that $\sigma_x = \sigma_y$. Let $[x] = \{ y \in X | \sigma_x = \sigma_y \}$. Let $g \in G(X,r)$. We shall prove that $g[x] = [g(x)]$, i.e. $\sigma_{g(x)} = \sigma_{g(t)}$, for all $t \in [x]$. By Lemma 2.1

\[
\sigma_{g(x)} = \lambda_g(\sigma_x) = \lambda_g(\sigma_t) = \sigma_{g(t)}.
\]

Therefore, $g[x] = [g(x)]$. Thus $[x]$ is a subset of imprimitivity. Since $(X,r)$ is a primitive solution and $|[x]| > 1$, we have that $[x] = X$. But then $G(X,r)$
is cyclic of order $p^m$, in contradiction with the fact that the group $G(X, r)$ is primitive.  

Let $S_1$, $S_2$ be two non-empty sets. Let $G_i$ be a group of permutations of $S_i$, for $i = 1, 2$. Recall that $G_1$ and $G_2$ are permutationally isomorphic if there exist a bijection $f_1 : S_1 \to S_2$ and an isomorphism $f_2 : G_1 \to G_2$ such that $f_1(g(s)) = f_2(g)(f_1(s))$, for all $s \in S_1$ and all $g \in G_1$. In this case $(f_1, f_2)$ is called a permutational isomorphism from $G_1$ to $G_2$ (see [24 Definition 2.1.2]).

**Lemma 2.5** Let $(X, r)$ be a finite irretractable solution of the YBE. Consider the permutation group $G = G(X, r)$ of $(X, r)$ with its natural structure of left brace. Then, the map $\varphi : X \to G$ defined by $\varphi(x) = \sigma_x$ for all $x \in X$ is an injective homomorphism of solutions of the YBE from $(X, r)$ to the solution $(G, r_G)$ associated to the left brace $G$. Furthermore, $\text{Soc}(G) = \{0\}$. Let $\bar{\varphi} : G \to G(G, r_G)$ be the map defined by $\bar{\varphi}(g) = \lambda_g$, for all $g \in G$. Let $\varphi' : X \to \varphi(X)$ be the bijection defined by $\varphi'(x) = \sigma_x$, for all $x \in X$. Then $(\varphi', \bar{\varphi})$ is a permutational isomorphism from $G$ to $G(G, r_G)$.

**Proof.** Since $(X, r)$ is irretractable, it is clear that $\varphi$ is injective. Since $r(x, y) = (\sigma_x(y), \sigma_{\sigma_x(y)}(x))$, and by Lemma 2.1

$$r_G(\sigma_x, \sigma_y) = (\lambda_{\sigma_x}(\sigma_y), \lambda_{\lambda_{\sigma_x}(\sigma_y)}(\sigma_x)) = (\sigma_x(y), \lambda_{\lambda_{\sigma_x}(\sigma_y)}(\sigma_x)) = (\sigma_x(y), \sigma_{\lambda_{\sigma_x}(\sigma_y)}(\sigma_x)),$$

we have that $\varphi$ is a homomorphism of solutions of the YBE. Recall that the lambda map of the left brace $G$ is a homomorphism of groups from the multiplicative group of $G$ to the group $\text{Aut}(G, +)$. Hence $\bar{\varphi}$ is a homomorphism of groups and clearly it is surjective. Furthermore, $\text{Ker}(\bar{\varphi}) = \text{Soc}(G)$. Let $g \in \text{Soc}(G)$. We have, by Lemma 2.1 $\sigma_x = \lambda_g(\sigma_x) = \sigma_{g(x)}$, for all $x \in X$. Since $(X, r)$ is irretractable, $g(x) = x$, for all $x \in X$. Hence $g = \text{id}$, and thus $\text{Soc}(G) = \{0\}$. Therefore $\bar{\varphi}$ is an isomorphism of groups.

By Lemma 2.1 we have that

$$\varphi'(g(x)) = \sigma_{g(x)} = \lambda_g(\sigma_x) = \bar{\varphi}(g)(\varphi'(x)),$$

for all $x \in X$ and all $g \in G$. Therefore $(\varphi', \bar{\varphi})$ is a permutational isomorphism from $G$ to $G(G, r_G)$, and the result is proven.

The following result is well-known.

**Lemma 2.6** Let $B$ be a finite left brace. Let $p_1, \ldots, p_k$ be the prime factors of $|B|$. Let $B_i$ be the Sylow $p_i$-subgroup of the additive group of $B$ and let $b_i \in B_i$, for $i = 1, \ldots, k$. Since $B_iB_j = B_jB_i$, for $i \neq j$, there exist unique $a_{i,j} \in B_j$ and $c_{i,j} \in B_i$ such that $b_ib_j = a_{i,j}c_{i,j}$. Then $\lambda_{b_i}(b_j) = a_{i,j}$.

**Proof.** Let $i \neq j$. By [21 Lemma 2(i)], $b_ib_j = \lambda_{b_i}(b_j)\lambda_{b_i}(b_j)^{-1}(b_i)$. Since $B_i$ is a left ideal of $B$, for all $l = 1, \ldots, k$, we have that $\lambda_{b_i}(b_j) \in B_j$ and $\lambda_{b_i}(b_j)^{-1}(b_i) \in B_i$. Since $B_i \cap B_j = \{0\}$, we get that $a_{i,j} = \lambda_{b_i}(b_j)$.  


3 Primitive solutions

We are ready to prove our main result.

**Theorem 3.1** Let \((X, r)\) be a finite primitive solution of the YBE with \(|X| > 1\). Then \(|X|\) is prime. Furthermore, \(\sigma_x = \sigma_y\), for all \(x, y \in X\), and \(\sigma_x\) is a cycle of length \(|X|\).

**Proof.** Let \((X, r)\) be a finite primitive solution of the YBE. Suppose that \(|X| > 1\) and that \(|X|\) is not prime. Let \(\mathcal{G} = \mathcal{G}(X, r)\). Then, by Remark 2.3, \((X, r)\) is indecomposable and \(|X| = p^m\), for some prime \(p\) and integer \(m > 1\). Furthermore, \(\mathcal{G}\) is solvable and \(\mathcal{G} = N \rtimes A\), where \(N \cong \mathbb{Z}/(p)^m\) is the unique minimal normal subgroup of \(\mathcal{G}\) and \(A\) is the stabilizer of some \(x \in X\) and a maximal subgroup of \(\mathcal{G}\). Moreover, \(N\) is self-centralizing in \(\mathcal{G}(X, r)\).

Recall that \(\mathcal{G}\) has a natural structure of left brace. By Lemma 2.5, the multiplicative group of the left brace \(\mathcal{G}\) acts primitively on \(\hat{X} = \{\sigma_x \mid x \in X\}\), by the lambda map. Furthermore \(A = \text{Stab}_{\mathcal{G}}(\sigma_x)\), for some \(x \in X\).

As mentioned earlier, \(\mathcal{G} = B_1 \oplus \cdots \oplus B_k\), where \(B_i\) is the Sylow \(p_i\)-subgroup of \((\mathcal{G}, +)\), and each \(B_i\) is a left ideal of the left brace \(\mathcal{G}\) (here \(p_1, \ldots, p_k\) are the prime factors of \(|\mathcal{G}|\)). Furthermore, \(\mathcal{G} = B_1 \cdots B_k\) and \(B_iB_j = B_jB_i\), for all \(i, j\).

Without loss of generality, we may assume \(p = p_1\). Since the multiplicative group of \(\mathcal{G}\) is solvable, and \(B_2 \cdots B_k = B_2 \oplus \cdots \oplus B_k\) is a Hall \(p'_1\)-subgroup of the multiplicative group of \(\mathcal{G}\), there exists \(g \in \mathcal{G}\) such that \(B_2 \cdots B_k \subseteq gAg^{-1}\).

Since \(gAg^{-1}\) is the stabilizer of \(\sigma_{g(x)}\), we may assume that \(B_2 \oplus \cdots \oplus B_k \subseteq A\). Since \(N < \mathcal{G}\), \(N \subseteq B_1\).

Let \(2 \leq i \leq k\) and \(b \in B_i\). Since \(N\) is a normal subgroup of \((\mathcal{G}, \cdot)\), for every \(n \in N\), there exists \(n' \in N\) such that \(n \cdot b = b \cdot n'\). Hence, because \(N \subseteq B_1\), by Lemma 2.6, we have that \(\lambda_n(b) = b\). So \((\lambda_n)_{|B_i} = \text{id}\), for every \(i = 2, \ldots, k\) and every \(n \in N\). Since \(\lambda_n \in \text{Aut}(\mathcal{G}, +)\), it follows that

\[
(\lambda_n)_{|B_2 \oplus \cdots \oplus B_k} = \text{id}. \tag{4}
\]

Write \(\sigma_x = x_1 + \cdots + x_k\) with each \(x_j \in B_j\). As \(\lambda_n(\sigma_x) = \sigma_x\), for all \(a \in A\) and because each \(B_i\) is a left ideal, we get that \(\lambda_n(x_i) = x_i\) for any \(a \in A\). Hence,

\[
\lambda_g(x_i) = x_i, \text{ for any } i \geq 2 \text{ and any } g \in \mathcal{G}. \tag{5}
\]

Note that, because of (4), for every \(n \in N\),

\[
\lambda_n(\sigma_x) = \lambda_n(x_1) + \cdots + \lambda_n(x_k) = \lambda_n(x_1) + x_2 + \cdots + x_k
\]

Since \(\mathcal{G}\) acts transitively on \(\hat{X}\), \(\mathcal{G} = NA\) and \(A\) stabilizes \(\sigma_x\),

\[
\hat{X} = \{\lambda_g(\sigma_x) \mid g \in \mathcal{G}\} = \{\lambda_n(\sigma_x) \mid n \in N\}
\]

\[
= \{\lambda_n(x_1) + x_2 + \cdots + x_k \mid n \in N\}.
\]

For every subset \(S\) of \(\mathcal{G}\), we denote by \(\langle S \rangle\) the subgroup of \((\mathcal{G}, \cdot)\) generated by \(S\), and by \(\langle S \rangle_+\) will denote the subgroup of \((\mathcal{G}, +)\) generated by \(S\). By (2) and (3), we have that

\[
\mathcal{G} = \langle \hat{X} \rangle = \langle \hat{X} \rangle_+ = \langle X_1 \rangle_+ + \cdots + \langle X_k \rangle_+,
\]
where \( X_1 = \{ \lambda_n(x_1) \mid n \in \mathbb{N} \} \), and \( X_i = \{ x_i \} \), for \( i = 2, \ldots, k \). Thus, \( B_j = \langle X_j \rangle_+ \), for every \( j \), and
\[
G = \langle X_1 \rangle_+ \oplus \langle x_2 \rangle_+ \oplus \cdots \oplus \langle x_k \rangle_+ = B_1 \oplus B_2 \cdots \oplus B_k. \tag{6}
\]

Furthermore, from [5] and [6] we get that
\[
(\lambda g)_{B_2 \oplus \cdots \oplus B_k} = \text{id} \quad \text{for any } g \in G. \tag{7}
\]

Using also Lemma [2.6] we obtain, for \( b_1 \in B_1 \) and \( b \in B_i \) with \( i \geq 2 \), that there exists \( b' \in B_1 \) such that
\[
b^{-1}b_1b = b^{-1}\lambda_{b_1}(b)b' = b^{-1}bb' = b' \in B_1.
\]

Hence, we have shown that
\[
(B_1, \cdot) \lhd (G, \cdot).
\]

Let \( Z(B_1) \) be the center of \( (B_1, \cdot) \). As \( B_1 \supseteq N \) is a nontrivial \( p \)-group, we have that \( Z(B_1) \) is a normal non-trivial subgroup of \( (G, \cdot) \). Since \( N \) is the unique minimal normal subgroup of \( (G, \cdot) \), we have that \( N \subseteq Z(B_1) \). Since \( N \) is self-centralizing in \( (G, \cdot) \), we have that \( N = Z(B_1) = B_1 \).

So, \( N = B_1 \) is an ideal of the left brace \( G \), \( A = B_2 \oplus \cdots \oplus B_k \) is a left ideal. Note that by (7), \( A = B_2 \oplus \cdots \oplus B_k \) is a trivial brace. Thus, using (7) again, we have
\[
\sigma_x = x_1 + \cdots + x_k = x_1 + \lambda_{x_1}(x_2 + \cdots + x_k) = x_1(x_2 + \cdots + x_k) = x_1x_2 \cdots x_k,
\]
and \( x_1 \in N \) and \( x_2 \cdots x_k \in A \). Since \( A = \text{Stab}_G(\sigma_x) \), for every \( a \in A \), we have
\[
\sigma_x = \lambda_{a}(\sigma_x) = a\sigma_x - a = ax_1x_2 \cdots x_k - a
\]
\[
= ax_1a^{-1}ax_2 \cdots x_k - a = ax_1a^{-1} + \lambda_{ax_1a^{-1}}(ax_2 \cdots x_k) - a
\]
\[
= ax_1a^{-1} + ax_2 \cdots x_k - a \quad \text{(by (7))}
\]
\[
= ax_1a^{-1} + x_2 \cdots x_k \quad \text{(since } A \text{ is a trivial brace)}
\]
\[
= ax_1a^{-1} + \lambda_{ax_1a^{-1}}(x_2 \cdots x_k) \quad \text{(by (7))}
\]
\[
= ax_1a^{-1}x_2 \cdots x_n.
\]

Hence \( ax_1a^{-1} = x_1 \), for all \( a \in A \). Since \( x_1 \in N \), we get that \( \langle x_1 \rangle \) is a normal subgroup of \( G \), and thus \( N = \langle x_1 \rangle \). But \( N \) is not cyclic, a contradiction. Therefore \( |X| \) is prime.

The second part of the result follows by [13, Theorem 2.13].

References

[1] D. Bachiller, Counterexample to a conjecture about braces, J. Algebra 453 (2016), 160–176.
[2] D. Bachiller, F. Cedó and L. Vendramin, A characterization of finite multipermutation solutions of the Yang-Baxter equation, Publ. Mat. 62 (2018), 641–649.

[3] A. Ballester-Bolinches, Finite groups versus finite left braces, talk in Mini-Workshop: Algebraic Tools for Solving the Yang-Baxter Equation, Oberwolfach, November 2019.

[4] K. A. Brown and K. R. Goodearl, Lectures on Algebraic Quantum Groups, Birkhäuser Verlag, Basel, 2002.

[5] T. Brzeziński, Trusses: Between braces and rings, Trans. Amer. Math. Soc. 372 (2019), 4149–4176.

[6] F. Catino, I. Colazzo and P. Stefanelli, Regular subgroups of the affine group and asymmetric product of braces, J. Algebra 455 (2016), 164–182.

[7] F. Cedó, Left braces: solutions of the Yang–Baxter equation, Adv. Group Theory Appl. 5 (2018), 33–90.

[8] F. Cedó, E. Jespers and J. Okniński, Retractability of the set theoretic solutions of the Yang-Baxter equation, Adv. Math. 224 (2010), 2472–2484.

[9] F. Cedó, E. Jespers and J. Okniński, Braces and the Yang–Baxter equation, Commun. Math. Phys. 327 (2014), 101–116.

[10] F. Cedó, E. Jespers and J. Okniński, An abundance of simple left braces with abelian multiplicative Sylow subgroups, Rev. Mat. Iberoam. (online first) DOI 10.4171/RMI/1168.

[11] A. Doikou and A. Smoktunowicz, From braces to Hecke algebras and quantum groups, arXiv:1912.03091v2 [math-ph].

[12] V. G. Drinfeld, On some unsolved problems in quantum group theory. Quantum Groups, Lecture Notes Math. 1510, Springer-Verlag, Berlin, 1992, 1–8.

[13] P. Etingof, T. Schedler and A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, Duke Math. J. 100 (1999), 169–209.

[14] T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I-type, J. Algebra 206 (1998), 97–112.

[15] T. Gateva-Ivanova, Set-theoretic solutions of the Yang-Baxter equation, braces and symmetric groups, Adv. Math. 338 (2018), 649–701.

[16] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin 1967.

[17] E. Jespers, V. Lebed, W. Rump and L. Vendramin, Mini-Workshop: Algebraic Tools for Solving the Yang-Baxter Equation, Report No. 51/2019 DOI: 10.4171/OWR/2019/51.

[18] E. Jespers and J. Okniński, Monoids and groups of I-type, Algebr. Represent. Theory 8 (2005), 709–729.

[19] E. Jespers and J. Okniński, Noetherian Semigroup Algebras, Springer, Dordrecht 2007.

[20] C. Kassel, Quantum Groups, Springer-Verlag, 1995.

[21] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, Adv. Math. 193 (2005), 40–55.
[22] W. Rump, Braces, radical rings, and the quantum Yang-Baxter equation, J. Algebra 307 (2007), 153–170.
[23] W. Rump, The brace of a classical group, Note Mat. 34 (2014), no. 1, 115–144.
[24] M. W. Short, The primitive soluble permutation groups of degree less than 256, Lecture Notes in Math. 1519, Springer, Berlin 1992.
[25] A. Smoktunowicz, On Engel groups, nilpotent groups, rings, braces and the Yang-Baxter equation, Trans. Amer. Math. Soc. 370 (2018), 6535–6564.

F. Cedó
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona), Spain
cedo@mat.uab.cat

E. Jespers
Department of Mathematics
Vrije Universiteit Brussel
Pleinlaan 2, 1050 Brussel, Belgium
Eric.Jespers@vub.be

J. Okniński
Institute of Mathematics
Warsaw University
Banacha 2, 02-097 Warsaw, Poland
okninski@mimuw.edu.pl