PIECEWISE POLYNOMIALS ON POLYHEDRAL COMPLEXES

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Abstract. For a $d$-dimensional polyhedral complex $P$, the dimension of the space of piecewise polynomial functions (splines) on $P$ of smoothness $r$ and degree $k$ is given, for $k$ sufficiently large, by a polynomial $f(P, r, k)$ of degree $d$. When $d = 2$ and $P$ is simplicial, in [1] Alfeld and Schumaker give a formula for all three coefficients of $f$. However, in the polyhedral case, no formula is known. Using localization techniques and specialized dual graphs associated to codimension–2 linear spaces, we obtain the first three coefficients of $f(P, r, k)$, giving a complete answer when $d = 2$.

1. Introduction

In [3], Billera used methods of homological and commutative algebra to solve a conjecture of Strang on the dimension of the space of $C^1$ splines on a generic planar simplicial complex. The algebraic approach to the study of splines was further developed in work of Billera and Rose [4], [5], and Schenck and Stillman [16], [17]. For a (not necessarily generic) triangulation $\Delta$ of a simply connected polygonal domain having $f_1^0$ interior edges and $f_0^0$ interior vertices, Alfeld and Schumaker showed that for $k \geq 3r + 1$, the dimension of the vector space $C^r_k(\Delta)$ of splines on $\Delta$ having smoothness $r$ and degree at most $k$ is given by:

Theorem 1.1 ([2]).

$$\dim C^r_k(\Delta) = \binom{k+2}{2} + \binom{k-r+1}{2}f_1^0 - \binom{k+2}{2} - \binom{r+2}{2}f_0^0 + \sigma,$$

where $\sigma = \sum \sigma_i$, $\sigma_i = \sum_j \max\{(r+1+j(1-n(v_i))), 0\}$, and $n(v_i)$ is the number of distinct slopes at an interior vertex $v_i$.

In higher dimensions, a spectral sequence argument [13] shows that the previous theorem continues to hold, in the sense that for $k \gg 0$, the dimension of $C^r_k(\Delta)$ is given by a polynomial in $k$ (the Hilbert polynomial), and the coefficients of the three largest terms of this polynomial are determined by a higher dimensional analog of the formula given by Alfeld and Schumaker in the planar case.

In this note, we study splines on a polyhedral complex $P$. In general, polyhedral complexes do not lend themselves to the full range of techniques available in the simplicial case, and so have been studied less. In [19], Schumaker obtains upper and lower bounds in the planar case, and in [4], Billera and Rose obtained the first two coefficients of the Hilbert polynomial. It is possible to consider all the $C^r_k(P)$ at once by passing to a module over a polynomial ring, and in [20] Yuzvinsky uses
sheaves on posets to obtain results on the freeness of this module. In \[11\], \[12\], Rose studies cycles on the dual graph of \(P\). Our key technical innovation is a refined version of the dual graph, depending on a choice of codimension–2 linear space; used in conjunction with localization techniques.

The main result of this paper is a formula for the third coefficient of the Hilbert polynomial, in the case of a polyhedral complex. In particular, for a planar polyhedral complex, our result gives a formula for the dimension of \(C_k^r(P)\) for all \(k \gg 0\).

**Example 1.2.** Let \(P\) be the polygonal complex depicted below:

![Polygonal complex](image)

The following table gives the dimensions for \(C_k^r(P)\) for small values of \(r\), assuming \(k \gg 0\). For a planar \(P\), Corollary \[3.14\] gives a formula for \(f(P, r, k)\) depending on the combinatorics and geometry of \(P\), "explaining" the values below.

| \(r\) | \(\dim_{\mathbb{R}} C_k^r(P)\) |
|------|----------------------------------|
| 0    | \(2k^2 + 2\)                    |
| 1    | \(2k^2 - 6k + 10\)              |
| 2    | \(2k^2 - 12k + 32\)             |
| 3    | \(2k^2 - 18k + 64\)             |
| 4    | \(2k^2 - 24k + 110\)            |

Throughout the paper, our basic references are de Boor \[7\] (for splines) and Eisenbud \[8\] (for commutative algebra).

2. **Algebraic preliminaries**

**Definition 2.1.** The dual graph \(G_P\) of a \(d\)-dimensional polyhedral complex \(P\) is a graph whose vertices correspond to the \(d\)-faces of \(P\), with an edge connecting two vertices iff the corresponding two \(d\)-faces meet in a \((d-1)\)-face.

**Definition 2.2.** The star of a face \(\sigma\) of \(P\) is the set of all faces \(\tau \in P\) such that there exists a face \(\upsilon \in P\) containing both \(\tau\) and \(\sigma\).

Put another way, the star of \(\sigma\) is the smallest subcomplex of \(P\) containing all faces which contain \(\sigma\). Let \(P\) be a \(d\)-dimensional polyhedral complex embedded in \(\mathbb{R}^d\), such that \(P\) is hereditary; this means for every nonempty face \(\sigma\) of \(P\), the dual graph of the star of \(\sigma\) is connected. Basically, \(P\) should be visualized as a polyhedral subdivision of a manifold with boundary. A \(C^r\)–spline on \(P\) is a piecewise polynomial function (a polynomial is assigned to each \(d\)-dimensional cell of \(P\)), such that two polynomials supported on \(d\)-faces which share a common \((d-1)\)-face \(\tau\) meet with order of smoothness \(r\) along that face. We use \(P_0^i\) to denote the set of interior \(i\) faces of \(P\) (all \(d\)-dimensional faces are considered interior), and...
let \( f_i^0 = |P_i^0| \). The set of splines of degree at most \( k \) (i.e. each individual polynomial is of degree at most \( k \)) is a vector space, which we will denote \( C_k^r(P) \).

2.1. Splines and syzygies. The first important fact is that the smoothness condition is local: for two \( d \)-cells \( \sigma_1 \) and \( \sigma_2 \) sharing a common \((d-1)\)-face \( \tau \), let \( l_\tau \) be a nonzero linear form vanishing on \( \tau \). Billera and Rose ([4], Corollary 1.3) show that a pair of polynomials \( f_i \) supported on \( \sigma_i \), \( i = 1, 2 \) meet with order \( r \) smoothness along \( \tau \) iff

\[
\begin{align*}
l_\tau^{r+1}|f_1 - f_2.
\end{align*}
\]

In [18], Schumaker gave a formula for the dimension of \( C_k^r(P) \) in the planar case, when \( P \) is the star of a vertex, as depicted in Example 2.3 (note that such a \( P \) will always be simplicial). Schumaker observed that a spline is a syzygy on the ideal generated by the \( r + 1 \)st powers of linear forms vanishing on the interior edges. To see this, consider the example below:

**Example 2.3.** A planar \( P \) which is the star of a single interior vertex \( v_0 \) at the origin.

![Diagram of a planar P star vertex](image)

Beginning with the simplex in the first quadrant and moving clockwise, label the polynomials on the triangles \( f_1, \ldots, f_4 \). To obtain a global \( C^r \) function, we require

\[
\begin{align*}
a_1y^{r+1} &= f_1 - f_2
a_2(x - y)^{r+1} &= f_2 - f_3
a_3(x + y)^{r+1} &= f_3 - f_4
a_4x^{r+1} &= f_4 - f_1
\end{align*}
\]

Summing each side above yields the equation \( \sum_{i=1}^{4} a_i l_\tau^{r+1} = 0 \), a syzygy on the ideal

\[
I = (y^{r+1}, (x - y)^{r+1}, (x + y)^{r+1}, x^{r+1}).
\]

It is easy to see that the process can be reversed. The upshot is that if \( P \) is the star of a vertex, then the spline module consists of \( \mathbb{R}[x, y] \oplus \text{syz}(I) \); the summand \( \mathbb{R}[x, y] \) corresponds to the “trivial” splines \( f_1 = f_2 = f_3 = f_4 \). Hence, splines are intimately connected to commutative algebra.

2.2. Graded modules, Hilbert polynomial and series. As first observed by Billera and Rose, one way to study the dimension of the vector space \( C_k^r(P) \) is to embed \( P \) in the hyperplane \( x_{d+1} = 1 \subseteq \mathbb{R}^{d+1} \), and form the cone \( \hat{P} \) over \( P \), with vertex at the origin. If \( C_k^r(\hat{P}) \) is the set of splines on \( \hat{P} \) such that each polynomial is homogeneous of degree \( k \), then Billera and Rose show ([4], Theorem 2.6) there is a vector space isomorphism between \( C_k^r(\hat{P}) \) and \( C_k^r(P) \). It is easy to see that the set of splines of all degrees

\[
C^r(\hat{P}) = \bigcup_{k \geq 0} C_k^r(\hat{P})
\]
is a graded module over the polynomial ring $R$ in $d + 1$ variables. It follows from basic commutative algebra (see [14]) that the dimension of all the graded pieces of $C^r(\hat{P})$ is encoded by the Hilbert series:

$$HS(C^r(\hat{P}), t) = \sum_k \dim C^r_\tau(\hat{P}) t^k = \frac{g(t)}{(1-t)^{d+1}};$$

for some $g(t) \in \mathbb{Z}[t]$, and that for $k \gg 0$, the dimension of $C^r_\tau(P)$ is given by the Hilbert polynomial $HP(C^r(\hat{P}), k)$, which is an element of $\mathbb{Q}[k]$ of degree $d$. An easy induction shows that

$$HS(R, t) = \frac{1}{(1-t)^{d+1}}$$

and

$$HP(R, k) = \binom{d+k}{d}.$$

We need one last bit of notation: $R(-i)$ will denote the polynomial ring $R$, considered as a graded module over $R$, but with generator in degree $i$:

$$HS(R(-i), t) = \frac{t^i}{(1-t)^{d+1}}$$

and

$$HP(R(-i), k) = \binom{d+k-i}{d}.$$

**Lemma 2.4.** [Billera-Rose, [4]] Let $P$ be a $d$–dimensional polyhedral complex. Then there is a graded exact sequence:

$$0 \rightarrow C^r(\hat{P}) \rightarrow R^{f_d} \oplus R^{f_{d-1}}(-\tau - 1) \xrightarrow{\phi} R^{f_{d-1}} \rightarrow N \rightarrow 0$$

where $\phi = \begin{bmatrix} \partial_d & \tau_1^{r_1+1} & \cdots & \tau_m^{r_m+1} \end{bmatrix}$

Write $[\partial_d \mid D]$ for $\phi$. To describe $\partial_d$, note that the rows of $\partial_d$ are indexed by $\tau \in P^0_{d-1}$. If $\sigma_1, \sigma_2$ denote the $d$–faces adjacent to $\tau$, then in the row corresponding to $\tau$ the smoothness condition means that the only nonzero entries occur in the columns corresponding to $\sigma_1, \sigma_2$, and are $\pm(1, -1)$. When $P$ is simplicial, $\partial_d$ is the top boundary map in the (relative) chain complex.

A main result of [4] is that $N$ is supported on primes of codimension at least two. Combining this with the exact sequence of Lemma 2.4 allows Billera and Rose to determine the first two coefficients of the Hilbert polynomial (though the result is phrased in terms of the Hilbert series).

### 3. Main Theorem

We begin by sketching our strategy. It follows from additivity of the Hilbert polynomial on exact sequences and Lemma 2.4 that obtaining the coefficient of $k^{d-2}$ in the Hilbert polynomial of $C^r(\hat{P})$ is equivalent to obtaining the coefficient of $k^{d-2}$ in the Hilbert polynomial of $N$. Since

$$N \simeq \bigoplus_{\tau \in P^0_{d-1}} R/\tau^{r+1} \partial_d,$$

every element of $N$ is torsion. Using localization, we first show that the codimension–2 associated primes of $N$ must be linear, then give a precise description of which codimension–2 linear primes actually occur. This leads to an explicit description of the submodule of $N$ supported in codimension–2. Elements of this submodule are the only elements of $N$ which contribute to the $k^{d-2}$ coefficient of the Hilbert polynomial, and the formula follows.
3.1. The codimension two associated primes of $N$.

Lemma 3.1. Any codimension–2 prime ideal $J$ associated to $N$ contains a linear form $l_{\tau}$, for some $\tau \in P_{d-1}^0$.

Proof. From the description

$$N \simeq \left( \bigoplus_{\tau \in P_{d-1}^0} R/l_{\tau}^{r+1} \right)/\partial_d,$$

it follows that if no $l_{\tau}$ is in $J$, then all the $l_{\tau}$ are invertible in $R_J$, so that $N_J$ vanishes. □

Lemma 3.2. Let $\xi$ be a codimension–2 linear space. If $\sigma \in P_d$ has at most one facet whose linear span contains $\xi$, then every generator of $N$ corresponding to a facet of $\sigma$ is mapped to zero in the localization $N_{I(\xi)}$.

Proof. In $R_{I(\xi)}$, any $l_{\tau}$ such that $\xi \not\subseteq V(l_{\tau})$ becomes invertible. As $N$ is the cokernel of

$$\phi = \begin{bmatrix} \partial_d & | & l_{r_1}^{r+1} & \cdots & l_{r_m}^{r+1} \end{bmatrix},$$

in the right hand submatrix $D$ of $\phi$, all the forms (or all save one) $l_{r_i}^{r+1}$ such that $\tau$ is a facet of $\sigma$ become units. As the column of the left hand (\partial_d) matrix corresponding to $\sigma$ has nonzero entries only in rows corresponding to facets of $\sigma$, this means that every generator corresponding to a facet of $\sigma$ has zero image in the localization, and the result follows. □

Theorem 3.3. Any codimension–2 prime ideal $J$ associated to $N$ is of the form $\langle l_{\tau_1}, l_{\tau_2} \rangle$ for $\tau_i \in P_{d-1}^0$ such that $V(l_{\tau_1}, l_{\tau_2})$ has codimension–2.

Proof. If there do not exist two $l_i$ as above, then by Lemma 3.1 $V(J)$ is contained in exactly one hyperplane which is the linear span of $\tau \in P_{d-1}^0$. Thus, in $R_J$, all but one of the $l_i$ become units, and the proof of Lemma 3.2 shows that $N_J$ vanishes. □

Theorem 3.3 gives an explicit set of candidates for the codimension two primes of $N$. As noted earlier, in [4], Billera and Rose showed that all associated primes of $N$ have codimension at least two (this also follows from the arguments above), so the theorem identifies all candidates for the associated primes of minimal codimension.

Proposition 3.4. Let $N$ be a finitely generated, graded $R = \mathbb{K}[x_1, \ldots, x_d]$ module of dimension $m < d$. Let $\mathcal{P}$ be the set of minimal (codimension $d-m$) associated primes of $N$, and for $Q \in \mathcal{P}$, define

$$N(Q) = \{ n \in N \mid \text{ann}(n) \subseteq Q \}.$$

If $HP(N, k) = a_m k^m + O(k^{m-1})$, then

$$HP\left( \bigoplus_{Q \in \mathcal{P}} N(Q), k \right) = a_m k^m + O(k^{m-1}).$$

Proof. First, note that $N(Q) \hookrightarrow N$. Then the direct sum of the $N(Q)$, taken over the finite set of $Q \in \mathcal{P}$, maps to $N$, and so there is an exact sequence

$$0 \to K \to \bigoplus_{Q \in \mathcal{P}} N(Q) \to N \to C \to 0.$$
By construction, $C$ is supported in codimension greater than $d - m$, so has Hilbert polynomial of degree at most $m - 1$. Next we show that $N(P)_Q = 0$ for any $P, Q \in \mathcal{P}$ such that $P \neq Q$. From the definition of $N(P)$, for any $n \in N(P)$, $\sqrt{\text{ann}(n)} = P$. As $Q \neq P$, there exists $p \in P, p \notin Q$. Since $p^m \cdot n = 0$ for some $m$, and since $p$ is a unit in $R_Q$, we see that $N(P)_Q = 0$. Localization is an exact functor, and so localizing the exact sequence above at $Q \in \mathcal{P}$ yields an exact sequence:

$$0 \rightarrow K_Q \rightarrow N(Q)_Q \rightarrow N_Q \rightarrow C_Q \rightarrow 0.$$  

Localizing the exact sequence $0 \rightarrow N(Q) \rightarrow N$ yields an exact sequence

$$0 \rightarrow N(Q)_Q \rightarrow N_Q,$$

which implies that $K_Q$ must vanish. Thus, $K$ is also supported in codimension greater than $d - m$, so has Hilbert polynomial of degree at most $m - 1$. The result follows by additivity of Hilbert polynomials in an exact sequence.  

3.2. The graph associated to $P$ and a codimension two subspace. In this section, we analyze the codimension–2 associated primes of $N$ in greater depth. In the simplicial, planar case, the Alfeld-Schumaker formula shows that there is a contribution to the constant term of the Hilbert polynomial from the local behavior at interior vertices. As noted in Example 2.3, a syzygy on the linear forms adjacent to a single fixed vertex corresponds to a loop around that vertex.

Example 3.5. For the planar polyhedral complex $P$ appearing in Example 1.2, let $v_i$ be the vertex associated to the face labelled $f_i$. The dual graph $G(P)$ is:

In [12], Rose shows that if the dual graph $G_P$ of $P$ has a basis of disjoint cycles, then the projective dimension and Hilbert series of $C^r(\hat{P})$ are determined by the case when $G_P$ has a single cycle, which Rose analyzed in [11]. We now define a refined version of dual graph, which depends on $P$ and the choice of a codimension–2 linear subspace.

Definition 3.6. Let $P$ be a $d$–dimensional polyhedral complex embedded in $\mathbb{R}^d$, and $\xi$ a codimension–2 linear subspace. $G_\xi(P)$ is a graph whose vertices correspond to those $\sigma \in P_d$ such there exists a $(d - 1)$–face of $\sigma$ whose linear span contains $\xi$. Two vertices of $G_\xi(P)$ are joined iff the corresponding $d$–faces share a common $(d - 1)$–face whose linear span contains $\xi$. 
Example 3.7. We continue to analyze the complex $P$ of Example 1.2. For each interior vertex $v$ of $P$, $G_v(P)$ consists of a triangle. However, there is another codimension–2 space to consider: if $\xi$ is the point at which (the linear hulls of) the three edges connecting interior vertices to boundary vertices meet, then $G_\xi(P)$ is

Let $P'$ be obtained by moving the top vertex of $P$ a bit to the right. Then $P$ and $P'$ are combinatorially equivalent, but in $P'$ there are no sets of $\geq 3$ concurrent $V(l_\tau), \tau \in P_{d-1}^0$, except at the interior vertices. In particular, $G_\xi(P')$ is acyclic.

Lemma 3.8. For any $\sigma \in P_d$, there are at most two facets of $\sigma$ whose linear spans contain a given codimension–2 linear space $\xi$.

Proof. Suppose the linear spans of three facets $\tau_1, \tau_2, \tau_3$ of $\sigma$ meet in a codimension–2 linear space $\xi$. For each $V(l_\tau), \sigma$ lies on one side of the hyperplane; so $\sigma$ lies between $V(l_{\tau_1})$ and $V(l_{\tau_2})$. Since $V(l_{\tau_3})$ contains $\xi = V(l_{\tau_1}) \cap V(l_{\tau_2})$, this means $V(l_{\tau_3})$ would split $\sigma$, a contradiction. □

Corollary 3.9. $G_\xi(P)$ is homotopic to a disjoint union of circles and segments.

Proof. By Lemma 3.8, the valence of any vertex $v \in G_\xi(P)$ is at most two. □

Theorem 3.10. For a polyhedral complex $P$ and codimension–2 linear prime $\xi$,

$$N_I(\xi) \cong \bigoplus_{\psi \in H_1(G_\xi(P))} (R/I_\psi) / I_\psi$$

where $\psi \in H_1(G_\xi(P))$ means $\psi$ is a component of $G_\xi(P)$ homotopic to $S^1$, and $I_\psi = \langle l_\tau^{r+1} \mid \tau \in P_{d-1}^0 \text{ corresponds to an edge of } \psi \rangle$.

Proof. By Corollary 3.9 $G_\xi(P)$ consists of a disjoint union of cycles and segments. By Lemma 3.8 all generators of $N$ which lie in a segment are mapped to zero in the localization $N_I(\xi)$. For each $d$–face $\sigma$ corresponding to a vertex in a cycle, note that there are two $(d-1)$–faces $\tau_1, \tau_2$ of $\sigma$ such that $l_{\tau_1}, l_{\tau_2}$ are not units in $R_I(\xi)$; every other linear form defining a facet of $\sigma$ becomes a unit. Reducing the column of $\partial_d$ corresponding to $\sigma$ by the columns of $D_I(\xi)$ having a unit entry gives a column with nonzero entries only in rows corresponding to $\tau_1$ and $\tau_2$. Repeating the process shows that the cycle corresponds to a principal submodule of $N_I(\xi)$, with the generator quotiented by the $(r+1)^{st}$ powers of the forms corresponding to the edges of the cycle. □

Corollary 3.11. Let $Q$ be a codimension–2 associated prime of $N$ with $V(Q) = \xi$, and let $N(Q) \subseteq N$ be as in Proposition 3.4. Then

$$N(Q) \cong \bigoplus_{\psi \in H_1(G_\xi(P))} R/I_\psi,$$
where the direct sum on the right hand side is as in Theorem 3.10.

Proof. Let $K(\xi)$ be the submodule of $N$ generated by $\tau \in P^0_{d-1}$ such that $\xi$ is not contained in the linear span of $\tau$. These generators correspond exactly to the columns of $D$ in which $l_\tau$ is a unit in the localization at $Q$. Localize the exact sequence

$$0 \rightarrow K(\xi) \rightarrow N \rightarrow N/K(\xi) \rightarrow 0.$$ 

at $Q$. By construction,

$$(N/K(\xi))_Q \cong N(Q)_Q \cong (\bigoplus_{P \in \mathcal{P}} N(P))_Q.$$ 

So we have that $N(Q) \cong N/K(\xi)$. To finish, we need to show that $N/K(\xi)$ decomposes as the direct sum (over cycles in $G_\xi(P)$) of the $R/I_\psi$. In $N/K(\xi)$, we may use the columns in $D$ which contain a unit to reduce each column of $\partial_d$. After reducing, a column corresponding to $\sigma \in P_d$ has at most two nonzero entries. If $\sigma$ is part of a segment in $G_\xi(P)$, then in $N/K(\xi)$ the generators corresponding to $\tau \subseteq \sigma$ reduce to zero. Otherwise, $\sigma$ corresponds to a vertex in a cycle of $G_\xi(P)$, and has exactly two (modulo $K(\xi)$) nonzero entries. So in each cycle, all the generators are equivalent in $N/K(\xi)$, and are quotiented by the appropriate $l_\tau^{r+1}$.

Theorem 3.12. Let $M$ be a graded module, and let $a_{d-2}(M)$ denote the coefficient of $k^{d-2}$ in $HP(M,k)$. As in Proposition 3.4, let $\mathcal{P}$ be the set of codimension-2 primes associated to $N$, and for $Q_i \in \mathcal{P}$ let $\xi_i = V(Q_i)$. Then $a_{d-2}(C^r(P)) = a_{d-2}(R^{f_d^{d-1}_{d-1}} + a_{d-2}(R(-r-1)^{f_0^{d-1}_{d-1}}) + \sum_{Q_i \in \mathcal{P}} \sum_{\psi_j \in H_1(G_{\xi_i}(P))} a_{d-2}(R/I_{\psi_j}).$

Proof. By Lemma 2.4 and the additivity of Hilbert polynomials on exact sequences,

$$a_{d-2}(C^r(P)) = a_{d-2}(R^{f_d^{d-1}_{d-1}} + a_{d-2}(R(-r-1)^{f_0^{d-1}_{d-1}}) + a_{d-2}(N).$$

It follows from Proposition 3.4 that

$$a_{d-2}(N) = \sum_{Q_i \in \mathcal{P}} a_{d-2}(N(Q_i)).$$

By Corollary 3.11

$$N(Q_i) \cong \bigoplus_{\psi_j \in H_1(G_{\xi_i}(P))} R/I_{\psi_j},$$

and the result follows.

The value of $a_{d-2}(R(-m))$ is a Stirling number of the first kind, and can be written out explicitly. In order for Theorem 3.12 to be useful, we need to know $a_{d-2}(R/I_\psi)$, which is provided by the following lemma:

Lemma 3.13. Let $I_\psi = \langle t_1^{r+1}, \ldots, t_n^{r+1} \rangle \subseteq K[x_0, \ldots, x_d]$ be a codimension-2 ideal, minimally generated by the $n$ given elements. Define

$$\alpha(\psi) = \binom{n+1}{r+1}, \quad s_1(\psi) = (n-1)\alpha(\psi) + n - r - 2, \quad s_2(\psi) = r + 1 - (n-1)\alpha(\psi).$$

Then the minimal free resolution of $R/I_\psi$ is:

$$0 \rightarrow R(-r-1 - \alpha(\psi))^{s_1(\psi)} \oplus R(-r-2 - \alpha(\psi))^{s_2(\psi)} \rightarrow R(-r-1)^n \rightarrow R \rightarrow R/I_\psi \rightarrow 0.$$
Proof. See Theorem 3.1 of [16]; the key step involves showing that a certain matrix has full rank, which was established by Schumaker in [18]. □

It follows from Lemma 3.13 that the Hilbert polynomial of $R/I_\psi$ is given by:

$$
\binom{k+d}{d}-n\binom{k+d-r-1}{d}+s_1(\psi)\binom{k+d-r-1-\alpha(\psi)}{d}+s_2(\psi)\binom{k+d-r-2-\alpha(\psi)}{d}.
$$

Corollary 3.14. If $P$ is a hereditary planar polyhedral complex, then

$$
HP(C^r(\hat{P}), k) = \frac{f_2}{2}k^2 + \frac{3f_2-2(r+1)f^0_1}{2}k + f_2 + \left(\binom{r}{2} - 1\right)f_1^0 + \sum_{\psi_j \in H_1(G_\psi(P))} c_j,
$$

where

$$
c_j = 1 - n(\psi_j)\binom{r}{2} + s_1(\psi_j)\left(\binom{r + \alpha(\psi_j)}{2}\right) + s_2(\psi_j)\left(\binom{r + \alpha(\psi_j) + 1}{2}\right)
$$

$$
= \left(\frac{r + 2}{2}\right) + \frac{\alpha(\psi_j)}{2}\left(2r + 3 + \alpha(\psi_j) - n(1 + \alpha(\psi_j))\right).
$$

4. Examples and connection to simplicial case

We close with some examples, and a discussion of the relation to the simplicial case. We begin by applying Corollary 3.14 to Example 1.2. As we saw in Example 3.7, there are four $\xi$ at which $H_1(G_\xi(P)) \neq 0$, and each $I_\psi$ has three generators. Hence the $c_j$ are all the same, and equal to

$$
\left(\frac{r + 2}{2}\right) + \frac{\alpha(\psi_j)}{2}\left(2r + 3 + \alpha(\psi_j) - 3(1 + \alpha(\psi_j))\right),
$$

which simplifies to

$$
\left(\frac{r + 2}{2}\right) + \left\lfloor \frac{r + 1}{2}\right\rfloor\left(r - \left\lfloor \frac{r + 1}{2}\right\rfloor\right).
$$

Applying Corollary 3.14 yields:

| $r$ | \text{dim} \mathcal{C}_k^r(P) | \frac{k^2}{2} + \frac{3f_2-2(r+1)f^0_1}{2}k | f_2 + \left(\frac{r}{2}\right)f_1^0 | 4\left(\frac{r^2}{2} + \alpha(r - \alpha)\right) |
|-----|---------------------|----------------------|----------------------|----------------------|
| 0   | $2k^2 + 2$          | $2k^2$               | $-2$                | 4                    |
| 1   | $2k^2 - 6k + 10$    | $2k^2 - 6k$          | $-2$                | 12                   |
| 2   | $2k^2 - 12k + 32$   | $2k^2 - 12k$         | 4                    | 28                   |
| 3   | $2k^2 - 18k + 64$   | $2k^2 - 18k$         | 16                   | 48                   |
| 4   | $2k^2 - 24k + 110$  | $2k^2 - 24k$         | 34                   | 76                   |

As in Example 3.7, consider the configuration $P'$ obtained by perturbing a vertex in Example 1.2 so the three edges defining $\xi$ no longer meet. Then there are only three nontrivial $c_j$, and we have
Example 4.1. In this example, we show that $G_\xi(P)$ can have several disjoint cycles. Let $P$ be as below:

![Graph Diagram]

If $\xi$ corresponds to the central vertex, then $G_\xi(P)$ is:

![Graph Diagram]

For these two cycles, $I_\psi$ has only two generators, and the $c_j$ value for such an ideal is always $(r + 1)^2$. There are four cycles for which $I_\psi$ has three generators, so by Corollary 3.14 we have (this table omits column for $\frac{f_2}{2}k^2 + \frac{3f_2 - 2(r + 1)f_0}{2}k$):

| $r$ | $\dim_k C_\psi^r(P)$ | $f_2 + \left(\binom{r}{2} - 1\right)f_0^0$ | $4\left(r + \frac{1}{2}\right) + \alpha(r - \alpha)$ | $2(r + 1)^2$ |
|-----|----------------------|--------------------------------|--------------------------------|----------------|
| 0   | $4k^2 + 2$           | $4$                          | $4$                          | $2$            |
| 1   | $4k^2 - 12k + 16$    | $-4$                         | $12$                         | $8$            |
| 2   | $4k^2 - 24k + 54$    | $8$                          | $28$                         | $18$           |
| 3   | $4k^2 - 36k + 112$   | $32$                         | $48$                         | $32$           |
| 4   | $4k^2 - 48k + 194$   | $68$                         | $76$                         | $50$           |
Example 4.2. In our final example, we look at a honeycomb configuration, that is, the polyhedral complex consisting of seven hexagons, pictured below:

For each interior vertex $v$, $G_v(P)$ will be a triangle and a line segment; while if $\xi$ is the point at the center of the diagram, $G_\xi(P)$ is a hexagon. However, each of the $I_\psi$ is generated by three elements, and so Corollary 3.14 yields:

$$r \dim R C^r_k(P) = f_2 + \left( \binom{k}{2} - 1 \right) f_1^0 + 7 \left( \binom{r+2}{2} + \alpha(r-\alpha) \right)$$

| $r$ | $\dim R C^r_k(P)$ | $\frac{3}{2} f_2 - 2(r+1) f_1^0$ | $f_2 + \left( \binom{k}{2} - 1 \right) f_1^0$ | $7 \left( \binom{r+2}{2} + \alpha(r-\alpha) \right)$ |
|-----|---------------------|---------------------------------|---------------------------------|---------------------------------|
| 0   | $\frac{3}{2} k^2 - \frac{3}{2} k + 2$ | $\frac{3}{2} k^2 - \frac{3}{2} k$ | $-5$ | $7$ |
| 1   | $\frac{3}{2} k^2 - \frac{3}{2} k + 16$ | $\frac{3}{2} k^2 - \frac{3}{2} k$ | $-5$ | $21$ |
| 2   | $\frac{3}{2} k^2 - \frac{3}{2} k + 56$ | $\frac{3}{2} k^2 - \frac{3}{2} k$ | $7$ | $49$ |
| 3   | $\frac{3}{2} k^2 - \frac{3}{2} k + 115$ | $\frac{3}{2} k^2 - \frac{3}{2} k$ | $31$ | $84$ |
| 4   | $\frac{3}{2} k^2 - \frac{3}{2} k + 200$ | $\frac{3}{2} k^2 - \frac{3}{2} k$ | $67$ | $133$ |

4.1. Connection to the simplicial case. In the case where $P$ is actually a simplicial complex, the first point to notice is that when two facets $\tau_1, \tau_2$ of $\sigma \in P_d$ meet, then they actually meet in a face of $\sigma$, necessarily of codimension–2. Thus, in the simplicial case, the only $\xi$ such that $H_1(G_\xi(P)) \neq 0$ are $\xi \in P_d^{0}$. For a planar $P$ which is simplicial, there are $f_0^0$ such faces. The formula for the value $c_j$ which appears in Corollary 3.14 can be rewritten as:

$$\left( \begin{array}{c} r + 2 \\ 2 \end{array} \right) + \sigma_i,$$

where $\sigma_i$ is as in the Alfeld-Schumaker formula. So in the planar, simplicial case, the formula of Corollary 3.14 is the polynomial appearing in Theorem 1.1.

Concluding Remarks and Questions:

1. To obtain the next coefficient of the Hilbert polynomial will be difficult; indeed, even in the simplicial case no general formula is known.
2. As noted in [15], the value for which the dimension of $C^r_k(P)$ becomes polynomial is the Castelnuovo-Mumford regularity the sheaf associated to $C^r(P)$; it would be interesting to determine the regularity of $C^r(P)$.
3. It is possible to generalize these results to the case of mixed smoothness, which was considered for planar simplicial complexes in [9]; we leave this to the interested reader.
4. In the simplicial, planar case, [6] shows that $C^r(\hat{P})$ free $\rightarrow C^{r-1}(\hat{P})$ free. Is this true for $P$ polyhedral?

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