Geometric Approaches for Generating
Prolongations for Nonlinear Partial Differential Equations

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Abstract

The prolongation structure of a two-by-two problem is formulated very generally in terms of exterior differential forms on a standard representation of Pauli matrices. The differential system is general without making reference to any specific equation. An integrability condition is provided which gives by construction the equation to be investigated and whose components involve the structure constants of an SU(2) Lie algebra. Along side this, a related, different kind of prolongation, a type of Wahlquist-Estabrook prolongation, over a closed differential ideal is discussed and some applications are given.

Keywords: integrable, prolongation, connection, differential system, fibre bundle, conservation law

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I. INTRODUCTION.

It is a remarkable fact that large classes of partial differential equations including nonlinear equations can be studied from a very general geometric point of view within the confines of the formalism of differential geometry\textsuperscript{1−3}. Not only do the integrable equations possess infinite numbers of conservation law equations as well as Bäcklund transformations, it can also be said in many instances that they can be related to surfaces in, for example, three space and thereby offer a geometric interpretation for many of these associated properties\textsuperscript{4}. Sasaki\textsuperscript{1} made the observation that the equations which are the necessary and sufficient condition for the integrability of a linear problem of the Ablowitz, Kaup, Newell and Segur\textsuperscript{5} (AKNS) type to describe pseudospherical surfaces. Geometrically integrable equations were first considered by Chern and Tenenblat\textsuperscript{6} following in that path. They discovered that, briefly, a differential equation for a real-valued function describes pseudospherical surfaces if it turns out to be the necessary and sufficient condition for the existence of certain smooth functions such that a set of one-forms defined in terms of them satisfy the structure equations of a surface of constant Gaussian curvature.

This is not the only way to regard integrability. There exists a connection between pseudospherical surfaces and integrability of differential equations that goes well beyond the AKNS framework. A differential equation for a real-valued function is kinematically integrable if it is the integrability condition of a one-parameter family of linear problems\textsuperscript{7}. Another approach to integrability is the formal symmetry approach studied by Mikhailov, Shabat and Sokolov\textsuperscript{8}. An equation is formally integrable if it possesses a formal symmetry of infinite rank.

Wahlquist and Estabrook\textsuperscript{9−10} made a significant advance when they found that prolongations can be constructed for nonlinear equations. They were interested at first in studying the Korteweg-de Vries equation and looking at the associated algebra. A kind of prolongation over a fibre bundle was found which corresponds to the Pfaffian system which gives the equation upon projecting to the transversal integral manifold. This approach yields results which can be exploited to develop Lax pairs and to study the Bäcklund properties\textsuperscript{11} of the system, as will be seen here.

The objective here is to find prolongation structures that can be obtained for a large class
of equations given by a two-by-two problem based on an $SU(2)$ Lie algebra and expressed in terms of differential forms. This results in a geometric approach which does not assume the form of any specific equation at the outset. The integrability condition for the Pfaffian system can be expressed as the vanishing of a traceless two-by-two matrix of two forms. This gives by construction the nonlinear equation to be studied. A prolongation structure for a nonlinear equation consists of a system of Pfaffian equations for a set of pseudopotentials, that is functions, which serve as potentials for conservation laws in a generalized sense. It will be shown how these prolongations for the two-by-two system can be derived recursively at first. In the first type of prolongation discussed here, forms are used which satisfy an integrability condition and define a type of connection in terms of pseudopotentials.

A prolongation method of a different but related kind is developed next which is an extension of the Wahlquist and Estabrook approach\textsuperscript{9–10,12–14}. The procedure is somewhat different from their original approach. It relies on finding an underlying Pfaffian system which constitutes a closed differential ideal and reduces to the equation on the transversal integral manifold. This can be regarded as a generalization of the Frobenius Theorem to establish complete integrability\textsuperscript{15}. The method of prolongation introduces over the base manifold a type of fibre bundle which is endowed with a Cartan-Ehresmann connection. The vanishing of the connection form is the necessary and sufficient condition for the existence of this type of prolongation. The theorem developed here for this operation is very suitable for applications and some of these will be mentioned further on\textsuperscript{16–18}. Thus, in this second approach, a differential system which gives the equation on the transversal integral manifold is found, and this differential system is used to solve for the quantities which appear in the connection forms. These two approaches are quite complementary to each other and it might be of interest to put them together here.

II. PROLONGATION STRUCTURE FOR A TWO-BY-TWO PROBLEM.

Consider the Pfaffian system which is given by

$$\xi_i = 0, \quad \xi_i = dy_i - \Omega_{ij}y_j, \quad i, j = 1, 2. \quad (1)$$
In (1), $\Omega$ is a traceless two-by-two matrix which consists of a set of one forms. They can be thought of as quite general, but may be taken to constitute a one-parameter family of forms which, projected onto the solution manifold, depend on the independent variables, the dependent variables and their derivatives. The form of the matrix of one-forms $\Omega$ is given explicitly as

$$\Omega = (\Omega_{ij}) = \omega_l \sigma_l = \begin{pmatrix} \omega_3 & \omega_1 - i\omega_2 \\ \omega_1 + i\omega_2 & -\omega_3 \end{pmatrix},$$

where $\sigma_l$, $l = 1, 2, 3$ are the Pauli matrices. Using (2) for $\Omega$, the exterior differential system in (1) takes the form,

$$\xi_1 = dy_1 - y_1 \omega_3 - y_2 (\omega_1 - i\omega_2), \quad \xi_2 = dy_2 - y_1 (\omega_1 + i\omega_2) + y_2 \omega_3.$$  

(3)

The integrability conditions for (1) are expressed as the vanishing of a traceless two-by-two matrix of two-forms $\Theta$,

$$\Theta = 0,$$

$$\Theta = d\Omega - \Omega \wedge \Omega.$$  

(4)

This gives by construction the nonlinear equation which is of interest. The components of $\Theta$ can be expressed in the form,

$$\Theta = (\Theta_{ij}) = \vartheta_l \sigma_l, \quad \vartheta_l = d\omega_l - i\epsilon_{lmn} \omega_m \wedge \omega_n.$$  

(5)

In (5), the $\epsilon_{lmn}$ represents the totally antisymmetric constants of an $SU(2)$ Lie algebra, which is the case considered now. The nonlinear system to be considered is specified then by

$$\Theta = 0, \quad \vartheta_l = 0, \quad l = 1, 2, 3.$$  

(6)

By exterior differentiation of $\Theta$ in (5), it is found that

$$d\Theta = \Omega \wedge \Theta - \Theta \wedge \Omega.$$  

(7)

This establishes that the exterior derivatives of the two-forms $\{\vartheta_l\}$ are contained in the ring generated by the set $\{\vartheta_l\}$.

It is important to realize that (1) and integrability condition (4) are both invariant under the following type of gauge transformation

$$y \to y' = Qy, \quad \Omega \to \Omega' = Q\Omega Q^{-1} + dQQ^{-1}, \quad \Theta \to \Theta' = Q\Theta Q^{-1}.$$  

(8)
In (8), Q is an arbitrary space-time dependent two-by-two matrix with determinant one. In other words, the gauge transformation of Ω does not change the solution manifold of the nonlinear equation. The matrix of one-forms Ω has the interpretation of being a connection on a gauge field, the two-form is Θ, a curvature or gauge field strength, and the closure property (7), a Bianchi identity.

**Theorem 2.1.** The exterior derivatives of the forms ξ₁ and ξ₂ have the form,

\[ dξ₁ = -y_2(∂₁ - i∂₂) - y_1∂_3 + ω_3 ∧ ξ₁ + (ω_1 - iω_2) ∧ ξ₂, \]  
\[ dξ₂ = -y_1(∂₁ + i∂₂) + y_2∂_3 + (ω_1 + iω_2) ∧ ξ₁ - ω_3 ∧ ξ₂. \]

Consequently, the derivatives of ξ₁ and ξ₂ are contained in the ring of forms spanned by \{∂₁\} and \{ξ₁\}.

**Proof:** Expressions for the \(dy_i\) follow from (3), and from (5) \(dω_i\) can be obtained by writing

\[ dω₁ = ∂₁ + 2iω₂ ∧ ω₃, \quad dω₂ = ∂₂ - 2iω₁ ∧ ω₃, \quad dω₃ = ∂₃ + 2iω₁ ∧ ω₂. \]

The exterior derivative of \(ξ₁\) from (3) is given by

\[-y_1∂_3 - 2iy₁ω₁ ∧ ω₂ + ω₃ ∧ (ξ₁ + y₁ω₃ + y₂(ω₁ - iω₂)) \]

\[-y_2(∂₁ + 2iω₂ ∧ ω₃ - i∂₂ - 2ω₁ ∧ ω₃) + (ω₁ - iω₂) ∧ (ξ₂ + y₁(ω₁ + iω₂) - y₂ω₃) \]

\[-y_2∂₁ + iy₂∂₂ - y₁∂_3 + ω₃ ∧ ξ₁ + (ω₁ - iω₂) ∧ ξ₂ \]

\[-2iy₁ω₁ ∧ ω₂ + y₂ω₃ ∧ (ω₁ - iω₂) - 2iy₂ω₂ ∧ ω₃ + 2y₂ω₁ ∧ ω₃ + 2iy₁ω₁ ∧ ω₂ - y₂(ω₁ - iω₂) ∧ ω₃. \]

The second line in the final result vanishes and we are left with (9). The proof of (10) proceeds in the same way.

**Corollary 2.1.** The exterior derivatives (9) and (10) can be expressed in terms of the matrix elements of Ω and Θ in (2) and (5) for \(i = 1, 2\) as follows,

\[ dξ_i = -Θ_{ij}y_j - Ω_{ij} ∧ ξ_j. \]

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The one-forms $\xi_1$ and $\xi_2$ can be used to generate an ideal which assumes a standard Riccati form by taking particular linear combinations of them. The new one-forms which result are called $\xi_3$ and $\xi_4$, and are defined by calculating in the following way; first,

$$y_1^2\xi_3 = y_1\xi_2 - y_2\xi_1 = y_1dy_2 - y_2dy_1 - y_1^2(\omega_1 + i\omega_2) + 2y_1y_2\omega_3 + y_2^2(\omega_1 - i\omega_2).$$

Therefore, $\xi_3$ is given by

$$\xi_3 = d\left(\frac{y_2}{y_1}\right) - (\omega_1 + i\omega_2) + 2\left(\frac{y_2}{y_1}\right)\omega_3 + \left(\frac{y_2}{y_1}\right)^2(\omega_1 - i\omega_2). \quad (12)$$

In a similar fashion,

$$y_2^2\xi_4 = y_2\xi_1 - y_1\xi_2 = y_2dy_1 - y_1dy_2 - y_2^2(\omega_1 - i\omega_2) - 2y_1y_2\omega_3 + y_1^2(\omega_1 + i\omega_2),$$

and so $\xi_4$ is given by

$$\xi_4 = d\left(\frac{y_1}{y_2}\right) - (\omega_1 - i\omega_2) - 2\left(\frac{y_1}{y_2}\right)\omega_3 + \left(\frac{y_1}{y_2}\right)^2(\omega_1 + i\omega_2). \quad (13)$$

Introducing the new projective variables $\xi_3$ and $\xi_4$ defined to be

$$y_3 = \frac{y_2}{y_1}, \quad y_4 = \frac{y_1}{y_2} \quad (14)$$

into the expressions for $\xi_3$ and $\xi_4$, Pfaffian system (11) takes the Riccati form,

Theorem 2.2. The exterior derivatives of the forms $\xi_3$ and $\xi_4$ in (13)-(14) are given by

$$d\xi_3 = -d(\partial_1 + i\partial_2) + y_3^2(\partial_1 - i\partial_2) + 2y_3\partial_3 - 2(\omega_3 + y_3(\omega_1 - i\omega_2)) \wedge \xi_3, \quad (15)$$

$$d\xi_4 = -d(\partial_1 - i\partial_2) + y_4^2(\partial_1 + i\partial_2) - 2y_4\partial_4 + 2(\omega_3 - y_4(\omega_1 + i\omega_2)) \wedge \xi_4. \quad (16)$$

Thus, the derivatives of $\xi_3$ and $\xi_4$ given in (13)-(14) are contained in the ring of forms spanned by $\{\partial_i\}$ and $\{\xi_i\}$.

Theorem 2.2 is proved along the same lines as the previous theorem where $dy_3$ and $dy_4$ are obtained from (13) and (14) respectively.

The results of Theorem 2.2 can be cast in the general form,

$$d\xi_3 = \beta_1 \wedge \xi_3 + \sum_{j=1}^{3} c_{3j}\partial_j, \quad d\xi_4 = \beta_2 \wedge \xi_4 + \sum_{j=1}^{3} c_{4j}\partial_j, \quad (17)$$
where the coefficients $c_{ij}$ are function valued quantities and the $\beta_i$ are one-forms defined to be

$$\beta_1 = -2\omega_3 - 2y_3(\omega_1 - i\omega_2), \quad \beta_2 = 2\omega_3 - 2y_4(\omega_1 + i\omega_2).$$

(18)

It is interesting to note that the exterior derivatives of the forms $\beta_i$ in (18) have the same generic form as that expressed on the right side of (17). Based on these results, there are a series of prolongation results which can be stated and proved along lines similar to the ones given. These results will be collected together in Theorem 2.3.

**Theorem 2.3.** (i) Define one-forms $\xi_5$ and $\xi_6$ to have the form,

$$\xi_5 = dy_5 + 2\omega_3 + 2y_3(\omega_1 - i\omega_2), \quad \xi_6 = dy_6 - 2\omega_3 + 2y_4(\omega_1 + i\omega_2).$$

(19)

The exterior derivatives of $\xi_5$ and $\xi_6$ can be expressed in the form,

$$d\xi_5 = 2y_3(\vartheta_1 - i\vartheta_2) + 2\vartheta_3 + 2\xi_3 \wedge (\omega_1 - i\omega_2), \quad d\xi_6 = 2y_4(\vartheta_1 + i\vartheta_2) - 2\vartheta_3 + 2\xi_4 \wedge (\omega_1 + i\omega_2).$$

(20)

(ii) Define the pair of one-forms

$$\xi_7 = dy_7 - e^{y_5}(\omega_1 - i\omega_2), \quad \xi_8 = dy_8 - e^{y_6}(\omega_1 + i\omega_2).$$

(21)

The exterior derivatives of $\xi_7$ and $\xi_8$ are given by

$$d\xi_7 = -e^{y_5}(\vartheta_1 - i\vartheta_2 + \xi_5 \wedge (\omega_1 - i\omega_2)), \quad d\xi_8 = -e^{y_6}(\vartheta_1 + i\vartheta_2 + \xi_6 \wedge (\omega_1 + i\omega_2)).$$

(22)

Theorem 2.3 is shown along lines identical to that used for Theorem 2.1 by evaluating exterior derivatives of the relevant forms, substituting known derivatives and then simplifying the resulting expression.

These results are quite significant in that they provide numerous ways in which to generate conservation laws explicitly. In fact, an infinite number of them will be seen to emerge.

**A. CONSERVATION LAWS.**

It is worth illustrating how conservation laws can be generated from the forms which have been produced thus far. For example, the equations which express $d\beta_1$ and $d\beta_2$ are conservation laws in $1 + 1$ dimensions. Under a solution of the original nonlinear equation $\Theta = 0$ with $y_3, y_4$
solutions of $\xi_3 = 0$ and $\xi_4 = 0$, we can equate to zero $d\beta_i = 0$, $i = 1, 2$. Expressing the one-form $\beta = \beta_i$ in terms of the basis set $dx$, $dt$ so that $\beta = I\,dx + J\,dt$, then $d\beta = 0$ implies

$$
\frac{\partial I}{\partial t} - \frac{\partial J}{\partial x} = 0.
$$

(23)

This signifies that $I$ is a conserved density and $J$ is a conserved current. In fact $d\beta_i = 0$ gives a one-parameter family of conservation laws as do most other results.

By taking a particular structure for the $\omega_i$, expressions for the conservation laws can be determined recursively. Suppose that

$$
\omega_1 + i\omega_2 = r\,dx + C\,dt, \quad \omega_1 - i\omega_2 = q\,dx + B\,dt, \quad \omega_3 = \eta\,dx + A\,dt,
$$

(24)

where $\eta$ is a parameter and $A$, $B$, $C$ are a one-parameter $\eta$ family of functions of $q$ and $r$. The one-form $\xi_3$ can be expressed as

$$
\xi_3 = (y_3)_x\,dx + (y_3)_t - (r\,dx + C\,dt) + 2y_3(\eta\,dx + A\,dt) + y_3^2(q\,dx + B\,dt).
$$

The coefficients of the basis forms $dx$ and $dt$ yield the equations

$$
(y_3)_x - r + 2\eta y_3 + q y_3^2 = 0, \quad (y_3)_t - C + 2A y_3 + B y_3^2 = 0.
$$

In order to solve this system, substitute $y_3 = \sum_{n=1}^{\infty} \eta^{-n}W_n$ into the first of these equations with $W_0 = 0$ to obtain

$$
-r + W_1 + \sum_{n=1}^{\infty} \eta^{-n}(W_{n,x} + 2W_{n+1} + q \sum_{k=1}^{n-1} W_{n-k}W_k) = 0.
$$

Equating the coefficients of $\eta^{-n}$ equal to zero, the following recursion results,

$$
W_1 = r, \quad W_{n,x} + 2W_{n+1} + q \sum_{k=1}^{n-1} W_{n-k}W_k = 0.
$$

(25)

This implies $W_{n+1}$ is a polynomial in $q$, $r$ and their $x$-derivatives. Moreover, the consistency of solving $\xi_3 = 0$ by using only its $x$ dependent part is guaranteed by complete integrability. Moreover, the one-form $\beta_1 = -2(\omega_3 + (\omega_1 - i\omega_2)y_3)$ can be expressed in terms of $W_n$ as follows

$$
-\frac{1}{2}\beta_1 = (\eta + q \sum_{n=1}^{\infty} \eta^{-n}W_n)\,dx + (A + B \sum_{n=1}^{\infty} \eta^{-n}W_n)\,dt.
$$
Consequently, the $n$-th conserved density $I_n$ is given by

$$I_n = qW_n.$$ 

This result would be common to all systems described by (24). Clearly, the expression of the conserved current depends on the particular equation considered. As a remark, notice that by substituting (24) into $\xi_1$ and $\xi_2$, these forms have the structure $\xi_1 = dy_1 - (\eta y_1 + qy_2)\,dx - (Ay_1 + By_2)\,dt$ and $\xi_2 = dy_2 - (ry_1 + \eta y_2)\,dx - (Cy_1 - Ay_2)\,dt$. This type of structure in a prolongation will be exploited in a different setting next.

III. A VERSION OF WAHLQUIST-ESTABROOK PROLONGATION.

A prolongation structure which is different from those discussed thus far will be developed. This is of the kind used by Wahlquist and Estabrook. Although the pseudopotentials which will be introduced serve as potentials for conservation laws in a generalized sense, the kind of results obtained have other applications and consequences. In the results thus far, the Pfaffian systems had the property that their exterior derivatives were contained in a certain ring of forms. Similarly, the Pfaffian equations here will reduce to

$$d\tilde{\xi}_i = \sum_j A_{ij} \wedge \tilde{\xi}_j + \sum_l F_{il} \vartheta_l.$$  

(26)

The structure of the result in (26) can be regarded as a generalization of the Frobenius condition for complete integrability of the Pfaffian equation.

Following Wahlquist and Estabrook, consider the manifold $M = \mathbb{R}^m$ which has coordinates $(u_1, u_2, \ldots, u_m)$, with projection map $\pi : M \to \mathbb{R}^2$ defined by $\pi(u_1, u_2, \ldots, u_m) = (u_1, u_2)$. Let there be defined on $M$ the exterior differential system $\{\xi_i\}$ and denote by $\mathcal{I}(\xi_i)$ the differential ideal of forms on $M$ generated by $\{\xi_i\}$. From the point of view of integral manifolds, the exterior differential system is completely determined by the associated ideal $\mathcal{I}(\xi_i)$. It will be the case that the $\{\xi_i\}$ are chosen so that $d\mathcal{I} \subset \mathcal{I}$. As a consequence, the Frobenius Theorem indicates that the Pfaff system $\{\xi_i\}$ is completely integrable. In the applications which involve nonlinear equations, the variables $u_1$ and $u_2$ will be identified with the independent variables $x$ and $t$ in
the equation, and $u_3$ the dependent variable. The system $\{\xi_i\}$ is constructed in such a way that solutions $u = u(x,t)$ of an evolution equation correspond to the two-dimensional transversal integral manifolds.

Suppose $N \subset \mathbb{R}^2$ is coordinatized by the variables $(x,t)$ and $\pi : M \to N$, with $s : N \to M$ a cross section of $\pi : M \to N$. The integral manifolds can then be written as sections $S$ in $M$ specified by

$$s(x,t) = (x, t, u_3(x,t), \ldots, u_m(x,t)).$$

A bundle can now be constructed based on $M$ so that $\tilde{M} = M \times \mathbb{R}^n$. We write $B = (\tilde{M}, \tilde{\pi}, M)$ and $\mathbb{R}^n$ is coordinatized with coordinates $y = (y_1, \ldots, y_n)$ and whose number at first can be left undetermined. The $y$ will be referred to as prolongation variables. Everything done so far can be lifted up to $\tilde{M}$. Thus, consider the exterior differential system in $\tilde{M}$ specified by

$$\tilde{\xi}_i = \tilde{\pi}^*\xi_i = 0, \quad i = 1, \ldots, l, \quad \tilde{\omega}_j = 0, \quad j = 1, \ldots, n.$$  

(28)

The forms $\{\tilde{\omega}_j\}$ have been included in order to specify a Cartan-Ehresmann connection on $B$. System (28) is called a Cartan prolongation if it is closed and whenever $S$ is a transversal solution of $\{\xi_i = 0\}$ there should also exist a transverse local solution $\tilde{S}$ of (28) with $\tilde{\pi}(\tilde{S}) = S$. It remains to discuss more carefully the details of the connection.

The definition of connection can be stated in various ways. A Cartan-Ehresmann connection on $B$ can be regarded as a field $H$ of horizontal contact elements on $\tilde{M}$ which is supplementary to the field $V$ of the $\tilde{\pi}$-vertical contact elements. Also $H$ is assumed complete, so every complete vector field $X$ on $M$ has a complete horizontal lift $\tilde{X}$ on $\tilde{M}$. Alternatively, a Cartan-Ehresmann connection $H$ introduced on $B$ is a system of one-forms $\{\tilde{\omega}_i\}$, $i = 1, \ldots, n$ with the property that the mapping $\tilde{\pi}_*$ from $H_{\tilde{m}} = \{X \in T_{\tilde{q}}| \tilde{\omega}_i(\tilde{x}) = 0, i = 1, \ldots, n\}$ onto the tangent space $T_{\tilde{q}}$ is a bijection for all $\tilde{q} \in \tilde{M}$. The ideal $\tilde{\mathcal{I}}$ of differential forms on $\tilde{M}$, which is generated by $\tilde{\pi}^*\mathcal{I} \cup H^*$ determines on $\tilde{M}$ the exterior differential system, which we continue to write as $\{\xi_i = 0\}$. Here $H^*$ is the set of one-forms on $\tilde{M}$ which vanish on the field $H$.

It remains to specify an explicit expression for the connection that is considered here. In terms
of the coordinates of $B$, the connection is designated to have the general form

$$\tilde{\omega}^k = dy^k - F^k(u_1, \cdots, u_m, y) dt - G^k(u_1, \cdots, u_m, y) dx \equiv dy^k - \eta^k, \quad k = 1, \cdots, n. \quad (29)$$

The intention then is to enlarge the differential ideal of forms by combining the collection $\tilde{\omega}^k$ with the original set of forms, as noted in (28).

The integrability condition requires that the prolonged differential ideal \{\xi_i, \tilde{\omega}^k\} remain closed. This implies that the differentials of $\tilde{\omega}^k$ can be expressed in the form (26),

$$d\tilde{\omega}^k = \sum_{j=1}^l f^{kj} \xi_j + \sum_{j=1}^n \eta^{kj} \wedge \tilde{\omega}_j. \quad (30)$$

The $f^{kj}$ in (30) represent dependent functions of the bundle coordinates and $\eta^{ki}$ represent a matrix of one-forms.

For a connection such as (29) the prolongation condition can be expressed equivalently using the summation over repeated indices in what follows as,

$$-d\eta^i = \frac{\partial \eta^i}{\partial y} \wedge (dy^i - \eta^i), \quad \text{mod } \tilde{\pi}^*(\mathcal{I}). \quad (31)$$

This result can be rewritten using the identity

$$d\eta^i = d_M \eta^i - \left(\frac{\partial \eta^i}{\partial y^j}\right) \wedge dy^j.$$  

Here $d_M$ is understood as differentiation with respect to the variables of the base manifold. The prolongation condition then becomes,

$$d_M \eta^i - \left(\frac{\partial \eta^i}{\partial y^j}\right) \wedge \eta^j = 0, \quad \text{mod } \tilde{\pi}^*(\mathcal{I}).$$

Introduce the vertical valued one-form as well as the definitions

$$\eta = \eta^i \frac{\partial}{\partial y^i}, \quad d\eta = (d_M \eta^i) \frac{\partial}{\partial y^i}, \quad [\eta, \omega] = (\eta^i \wedge \frac{\partial \omega^j}{\partial y^i} + \omega^j \wedge \frac{\partial \eta^i}{\partial y^i}) \frac{\partial}{\partial y^j}.$$  

The prolongation condition then takes the concise form

$$d\eta + \frac{1}{2} [\eta, \eta] = 0, \quad \text{mod } \tilde{\pi}^*(\mathcal{I}). \quad (32)$$
A particular version of connection form (29) of use in generating Lax pairs is,

\[ \tilde{\Omega}^k = dy^k - \eta^k = dy^k - \sum_{i=1}^{n} F^{ki}(u)y^i dt - \sum_{i=1}^{n} G^{ki}(u)y^i dx. \]  

(33)

The commutator in (32) can be worked out using (33). It is given explicitly by

\[ [\eta, \eta] = (G^{ji} F^{\nu j} y^i dx \wedge dt + F^{ji} G^{\nu j} y^i dt \wedge dx + F^{ji} G^{\nu j} y^i dt \wedge dx + G^{ji} F^{\nu j} y^i dx \wedge dt) \frac{\partial}{\partial y^\nu} \]

\[ = 2[F, G]^{\nu i} y^i \frac{\partial}{\partial y^\nu} dx \wedge dt. \]  

(34)

The prolongation condition takes the form

\[ (\frac{\partial F^{vi}}{\partial u_j} du_j \wedge dt + \frac{\partial G^{vi}}{\partial u_j} du_j \wedge dx) y^i \frac{\partial}{\partial y^\nu} + [F, G]^{vi} y^i \frac{\partial}{\partial y^\nu} dx \wedge dt = 0, \quad \text{mod } \pi^*(I). \]  

(35)

If the ideal of forms is specified by the system of two forms \{\xi_i\} closed over \( I \), then (35) takes the equivalent form

\[ (\frac{\partial F^{vi}}{\partial u_j} du_j \wedge dt + \frac{\partial G^{vi}}{\partial u_j} du_j \wedge dx) + [F, G]^{vi} dx \wedge dt \equiv \lambda_j^{vi} \xi_j. \]  

(36)

The objective in any given case is to produce the differential ideal \( I \) relevant to the equation and then solve (36) for the components of the connection \( F^{vi} \) and \( G^{vi} \). In effect, the following theorem has been established\(^7\,13\).

**Theorem 3.1.** Each prolongation of Pfaffian system \( \{\xi_i = 0\} \) which corresponds to a nonlinear equation on the integral manifold by a Cartan-Ehresmann connection determines a geometrical realization of a Wahlquist-Estabrook partial Lie algebra \( L \) by solving (36). Conversely, every geometrical realization of \( L \) corresponds to such a prolongation by constructing (29). Moreover, on a two-dimensional solution submanifold of the differential ideal, the one-forms are annihilated and there exists the differential Lax pair

\[ y_x = -Fy, \quad y_t = -Gy. \]  

(37)
The following is an example of a differential system which gives rise to two important equations. Let $M = \mathbb{R}^5$ and let $\{u_1, \cdots, u_5\} = \{x, t, u, p, q\}$. Define the following system of two forms to be

$$
\begin{align*}
\xi_1 &= du \wedge dt - p dx \wedge dt, \\
\xi_2 &= dp \wedge dt - q dx \wedge dt, \\
\xi_3 &= -du \wedge dx + dq \wedge dx + u du \wedge dt - u dq \wedge dt + \beta(u - q) du \wedge dt,
\end{align*}
$$

where $\beta$ in (38) is a real, non-zero constant. Exterior differentiation of the system of $\xi_i$ in (38) yields,

$$
\begin{align*}
d\xi_1 &= dx \wedge \xi_2, \\
d\xi_2 &= \frac{1}{u} dx \wedge (-\xi_3 + u((1 + \beta)u - q)\xi_1), \\
d\xi_3 &= (1 - \beta)[(dq - p dx) \wedge \xi_1 + p dt \wedge \xi_3].
\end{align*}
$$

Clearly, all of the $d\xi_i$ vanish modulo the set of $\xi_i$ in (38), therefore $d\mathcal{I} \subset \mathcal{I}$ as required. On the transversal integral manifold, it is determined that

$$
\begin{align*}
0 &= \xi_1|_S = s^*\xi_1 = (u_x - p) dx \wedge dt, \\
0 &= \xi_2|_S = s^*\xi_2 = (p_x - q) dx \wedge dt, \\
0 &= \xi_3|_S = s^*\xi_3 = (u_t - q_t + u(u_x - q_x) + \beta(u - q)u_x) dx \wedge dt.
\end{align*}
$$

Thus, sectioning of the differential system (38) generates the following system of equations

$$
\begin{align*}
p &= u_x, \\
q &= p_x, \\
(u - q)_t + u(u - q)_x + \beta(u - q)u_x &= 0.
\end{align*}
$$

The first two equations in (41) imply that $q = u_{xx}$ and putting this in the third equation, it is found that

$$
(u - u_{xx})_t + u(u - u_{xx})_x + \beta(u - u_{xx})u_x = 0.
$$

For $\beta = 2$, (42) becomes the Camassa-Holm equation, and for $\beta = 3$ it takes the form of the Degasperis-Procesi equation. Solutions of (36) for the components of the connection based on systems such as (38) have been given\textsuperscript{16–17}.

**IV. SUMMARY.**
It has been seen that a prolongation structure can be obtained for (1) with associated integrability condition (4). This leads to a geometric formulation of conservation laws, which can be written down. Moreover, the two approaches complement each other to a certain degree. This has also been explored for \( SL(2, \mathbb{R}) \), \( O(3) \) and \( SU(3) \), but not reported here. The whole prolongation structure is gauge covariant, reflecting the gauge invariance of (4). In fact, although not addressed here, there is a geometrical application\(^3\) that can be outlined now. Start with a general two-dimensional Riemannian manifold \( M \). Take an orthonormal basis \( \{e_i\} \) on the tangent plane \( T_p \) at each point \( p \). Then the structure equations for \( M \) read
\[
dp = \alpha^1 e_1 + \alpha^2 e_2, \quad de_1 = \omega e_2, \quad de_2 = -\omega e_1, \tag{43}
\]
where \( \alpha^1, \alpha^2 \) are one-forms dual to \( \{e_i\} \) and \( \omega \) is the connection one-form. The integrability conditions are given by
\[
d\alpha^1 = \omega \wedge \alpha^2, \quad d\alpha^2 = -\omega \wedge \alpha^1, \quad d\omega = -K \alpha^1 \wedge \alpha^2, \tag{44}
\]
with \( K \) the Gaussian curvature of \( M \). In this instance, a surface could arise if we take (2) and identify
\[
\alpha^1 = \omega_2 + \omega_3, \quad \alpha^2 = -2\omega_1, \quad \omega = \omega_2 - \omega_3. \tag{45}
\]
The set of one-forms \( (\alpha^1, \alpha^2, \omega) \) satisfying these equations describes a surface through these structure equations. It is remarkable that there is a unification of such diverse fields under this kind of approach.

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