Braginskii viscosity on an unstructured, moving mesh accelerated with super-time-stepping

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ABSTRACT

We present a method for efficiently modelling Braginskii viscosity on an unstructured, moving mesh. Braginskii viscosity, i.e., anisotropic transport of momentum with respect to the direction of the magnetic field, is thought to be of prime importance for studies of the weakly collisional plasma that comprises the intracluster medium (ICM) of galaxy clusters. Here anisotropic transport of heat and momentum has been shown to have profound consequences for the stability properties of the ICM. Our new method for modelling Braginskii viscosity has been implemented in the moving mesh code Arepo. We present a number of examples that serve to test the implementation and illustrate the modified dynamics found when including Braginskii viscosity in simulations. These include (but are not limited to) damping of fast magneto-sonic waves, interruption of linearly polarized Alfvén waves by the firehose instability and the inhibition of the Kelvin-Helmholtz instability by Braginskii viscosity. An explicit update of Braginskii viscosity is associated with a severe time step constraint that scales with $(\Delta x)^2$ where $\Delta x$ is the grid size. In our implementation, this restrictive time step constraint is alleviated by employing 2nd order accurate Runge-Kutta-Legendre super-time-stepping. We envision including Braginskii viscosity in future large-scale simulations of Kelvin-Helmholtz unstable cold fronts in cluster mergers and AGN-generated bubbles in central cluster regions.

Key words: galaxies: clusters: intracluster medium – conduction – diffusion – plasmas – instabilities – magnetic fields.

1 INTRODUCTION

The present paper concerns itself with an extension of ideal magnetohydrodynamics (MHD) which is known as Braginskii MHD (Braginskii 1965). This extension is appropriate for a fully ionized, weakly collisional and magnetized plasma. The most prominent example of an astrophysical system with these properties is the intracluster medium (ICM) of galaxy clusters.

The ICM has a high temperature ($T \sim 1 - 10$ keV) and low particle density ($n \sim 10^{-2}$ cm$^{-3}$, e.g. Vikhlinin et al. 2006). This leads to a weak collisionality as the Coulomb collision frequency has a $\sim nT^{-3/2}$ dependence on density and temperature (e.g. Hazeltine & Waelbroeck 2004). The ICM also contains a magnetic field with a strength of $\sim 1 \mu$G (Carilli & Taylor 2002). Although this magnetic field is dynamically weak, in the sense that the thermal pressure is much larger than the magnetic pressure, it is strong enough to magnetize the ICM, i.e., both ions and electrons have gyroradii that are much smaller than the characteristic length scale of the plasma. In summary, the temperature, density and magnetic field strength of the ICM lead to the ordering (Schekochihin & Cowley 2006)

$$H \gg \lambda_{\text{mfp}} \gg r_i \gg r_e,$$

where $H$ is a characteristic scale (e.g., the scale height), $\lambda_{\text{mfp}}$ is mean free path of ion Coulomb collisions, $r_i$ is the ion gyroradius and $r_e$ is the electron gyroradius.

As a consequence of this ordering, particles are tied to magnetic field lines and travel long distances along them between each Coulomb collision. This leads to heat and momentum transport which is primarily directed along the local magnetic field direction. A fluid model for this anisotropic transport, which differs from ideal MHD by just a few extra terms, can be constructed. The extra terms describe anisotropic heat conduction and Braginskii viscosity (also known as anisotropic viscosity) and the extension to ideal MHD including both effects is known as Braginskii MHD (Braginskii 1965). Simulations with Braginskii MHD can

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show greatly modified dynamics compared to ideal MHD simulations.

Anisotropic heat conduction fundamentally changes the stability properties of ionized, weakly collisional, magnetized atmospheres (see Balbus & Potter 2016 for a review). The Schwarzschild criterion (Schwarzschild 1958), which states that an atmosphere is stable if the entropy increases with height, assumes a collisional medium and has been found by Balbus (2000, 2001) and Quataert (2008) not to apply to weakly collisional atmospheres. They discovered that stability instead depends on the direction of the temperature gradient and the orientation of the magnetic field. Both temperature gradients anti-parallel and parallel to gravity (increasing and decreasing with height) can be susceptible to instability. The instabilities, known as the magneto-thermal instability (MTI, Balbus 2000, 2001) and the heat-flux-driven buoyancy instability (HBI, Quataert 2008), are both driven by fast heat conduction along magnetic fields. Depending on the magnetic field direction, anisotropic heat conduction can therefore in principle render the outer parts of all clusters MTI-unstable and the core regions of cool-core clusters HBI-unstable (Vikhlinin et al. 2006).

The first studies of the MTI and the HBI considered anisotropic heat conduction but neglected Braginskii viscosity because it acts on a slower time scale. Nevertheless, the inclusion of Braginskii viscosity has subsequently been shown to have important effects. Kunz (2011) showed that the fastest growing mode of the HBI has a vertical wavelength which is longer than the scale height of the plane-parallel atmosphere. This made the local, linear stability analyses utilized thus far inadequate and the theory was consequently extended to a quasi-global setting in Latter & Kunz (2012). Similar considerations and conclusions for the importance of including Braginskii viscosity have been found for a generalization of the HBI in which both the temperature and composition of the plasma is allowed to vary (Pessah & Chakraborty 2013; Berlok & Pessah 2015, 2016b).

One of the key questions which has been addressed by simulations of the HBI is whether it will nonlinearly saturate to wrap the magnetic fields around the cluster core, thus insulating the core from heat transport from the hotter, outer regions (see e.g. Parrish & Quataert 2008; Kunz et al. 2012). If such a re-orientation occurs, it is important that it is accurately modeled by simulations. While simulations without Braginskii viscosity find an efficient insulation, Kunz et al. (2012) found that Braginskii viscosity prevents the HBI from reorienting the magnetic field lines in all but the innermost regions. Besides opposing insulation of the core by the HBI, Braginskii viscosity can also provide a heating mechanism for the ICM via viscous heating (Kunz et al. 2011). Including Braginskii viscosity in cluster simulations thus seems essential for understanding the global energy budget of the ICM.

In general, simulations using Braginskii viscosity of the MTI/HBI, and their composition generalizations mentioned above, show less small-scale mixing and numerical reconnection (Parrish et al. 2012; Kunz et al. 2012; Berlok & Pessah 2016a,b). Simulations of the MTI also lead to a magnetic field structure with long folds and sharp bends, a consequence of Braginskii viscosity suppressing motions that change the magnetic field strength (Kunz et al. 2012). Cavities that appear as depressions in plasma density have been observed in a number of cool-core clusters (e.g. Perseus, see Fabian et al. 2011). The cavities are believed to be AGN-inflated bubbles filled with hot, relativistic plasma buoyantly rising in the ICM. Understanding whether such bubbles can travel long distances into the ICM as well as how they eventually dissipate their energy is an important ingredient in resolving the cooling-flow problem (Fabian 1994). In disagreement with observations, however, bubbles in idealized models are very quickly shedded by disrupting instabilities such as the Rayleigh-Taylor instability and the Kelvin-Helmholtz instability (KHI), see e.g. (Churazov et al. 2001). Several stabilizing agents have been suggested in order to make theory conform with observations, e.g., magnetic fields (Robinson et al. 2004; Ruszkowski et al. 2007), isotropic viscosity (Reynolds et al. 2005) or Braginskii viscosity (Dong & Stone 2009).

Dong & Stone (2009) showed that Braginskii viscosity, like magnetic fields, suppress instabilities in the direction parallel to the magnetic field but that motions in the perpendicular direction are uninhibited. The magnetic field structure in the bubble region, which itself is likely regulated by Braginskii viscosity, is therefore paramount in determining the stability of bubbles (Elwert et al. 2018).

Cold fronts are discontinuities in density and temperature found in galaxy clusters (see Markevitch & Vikhlinin 2007; ZuHone & Roediger 2016 for reviews). In brief, cold fronts are divided into two categories (Tittley & Henriksen 2005): i) remnant core (or merger) cold fronts that arise during a merger when a smaller cluster moves through the ICM of a larger cluster (ZuHone & Roediger 2016) and ii) sloshing cold fronts which arise when low entropy gas from the central part of a large cluster is displaced during a cluster merger (Markevitch & Vikhlinin 2007). Idealized computer models of sloshing cold fronts face an issue similar to the one facing models of AGN-inflated bubbles: the KHI occurs with a vigorousness unmatched by observations of real galaxy clusters. In sloshing cold fronts, the two most promising stabilizing agents are Braginskii viscosity (Zuhone et al. 2015) and a magnetic field which has been shear-amplified to sufficient strength at the location of the cold front (Zuhone et al. 2011). Similarly, simulations have shown that Braginskii viscosity suppresses the KHI in remnant core cold fronts (Suzuki et al. 2013). Whether the KHI is suppressed by viscosity, magnetic field strength or some other agent is still an open question (ZuHone & Roediger 2016).

The magneto-rotational instability (MRI, see Balbus & Hawley 1998 for a review) is a powerful mechanism for transporting angular momentum in accretion discs. Initially discovered within the framework of ideal MHD (Balbus & Hawley 1991), this instability exists also in weakly collisional (Balbus 2004) and collisionless frameworks (Quataert et al. 2002; Sharma et al. 2003; Heinemann & Quataert 2014). Radiatively inefficient accretion flows (RIAFs) are accretion discs found around super-massive black holes and consist of extremely hot and dilute plasma (see Quataert 2003; Yuan & Narayan 2014 for reviews). The characteristic temperatures and densities of RIAFs render them even less collisional than the ICM and a proper treatment requires studying them with methods applicable to collisionless plasmas. Such studies, using particle-in-cell (PIC) simulations, have however found
that micro-scale pitch-angle scattering of particle-waves induces an effective collision frequency which is much higher than the Coulomb collision frequency (Kunz et al. 2014; Hellinger & Trávníček 2015; Sironi & Narayan 2015; Kunz et al. 2016). This has motivated studies of RIAFs using Braginskii viscosity (e.g. Wu et al. 2017; Kempski et al. 2019) and even relativistic variants thereof (Chandra et al. 2017; Foucart et al. 2016, 2017), since such models make it possible to study much larger time and length scales than allowed by costly PIC simulations.

Besides the various applications to specific astrophysical systems, Braginskii viscosity also, quite fundamentally, modifies the MHD wave family and their propagation. It can be shown that the fast and slow magneto-sonic waves are damped while circularly polarized Alfvén waves are unmodified by Braginskii viscosity (see e.g. Parrish et al. 2012 and our derivation of the Braginskii MHD wave dispersion relation in Section 2.2). Only recently, however, was it realized that linearly polarized Alfvén waves can be interrupted by the firehose instability in high-β plasmas (Squire et al. 2016; Squire et al. 2017a). This finding has been shown to have important consequences for the properties of turbulence in weakly collisional plasmas, i.e., Squire et al. (2019) found in the limit of high viscosity, that incompressible turbulent fluid motions are constrained to move in a way in which the magnetic field strength, B, does not change in time. Weakly collisional plasmas with this interesting property are called magneto-immutable in Squire et al. (2019).

The rest of the paper is outlined as follows. In Section 2 we introduce the equations of Braginskii MHD and derive the wave dispersion relation. In Section 3 we describe the spatial discretization on an unstructured, moving mesh and the strategy for updating the equations in time using Runge-Kutta-Legendre second order accurate super-time-stepping. In Section 4 we provide a number of tests including damping of fast magneto-sonic waves (Section 4.3), the interruption of a linearly polarized Alfvén wave by the firehose instability (Section 4.5), and the evolution of the KHI when a magnetic field is oriented along the flow direction and Braginskii viscosity is taken into account (Section 4.6). We conclude in Section 5 where we also provide an outlook to the type of studies that we will be able to perform in the future. Additional details about the Arepo implementation are described in Appendices A and B, while additional details about the analytical and numerical reference solutions (used for testing the implementation) are given in Appendices C and D.

2 EQUATIONS OF BRAGINSKII MHD

We introduce the equations of Braginskii magnetohydrodynamics (MHD) which incorporate anisotropic diffusion of heat and momentum (Braginskii 1965; Kulsrud 1983; Schekochihin et al. 2010; Squire et al. 2017a).

In the following, ρ is the mass density, v is the fluid velocity, B is the magnetic field vector with magnitude B and direction b = B/|B|, T is the temperature and p is thermal pressure given by p = nkB T/µmH where nkB is Boltzmann’s constant, µmH is the proton mass and µ is the mean molecular weight. The ratio of thermal to magnetic pressure is given by β = 2c²/ν² where c = √(p/ρ) is the isothermal sound speed, ν = B/√(µ0ρ) is the Alfvén speed and µ0 is the vacuum permeability. Finally, we define the adiabatic sound speed as c = √(γp/ρ) where γ = 5/3 is the adiabatic index for a fully ionized, non-relativistic gas.

The mass continuity, momentum, induction and entropy equations are then given by

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) ,
\]

\[
\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \nabla \cdot \mathbf{B} \times \mathbf{B} / \mu_0 ,
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{v} \times \mathbf{B} \right) ,
\]

\[
\frac{p}{\gamma - 1} \frac{d}{dt} \left( \frac{\ln(p \rho^{-\gamma})}{\mu} \right) = -\Pi : \nabla \mathbf{v} - \nabla \mathbf{Q} ,
\]

where

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) ,
\]

is the Lagrangian time derivative. In Equations (2)-(5), the extra terms that are included in Braginskii MHD (compared to the equations of ideal MHD) in order to model the weak collisionality are the anisotropic heat flux, \(\mathbf{Q}\), and the anisotropic viscosity tensor, \(\Pi\).

The anisotropic heat flux is given by

\[
\mathbf{Q} = -\chi_{\parallel} (b (b \cdot \nabla T)) ,
\]

where \(\chi_{\parallel}\) is the heat conductivity. This form of the heat flux has a direction along the magnetic field and a magnitude which is proportional to the gradient of temperature projected along the magnetic field. The heat transport is mainly carried by the electrons which can move long distances along the magnetic field between collisions with other electrons. The heat conduction will thus be primarily directed along the magnetic field. This form of anisotropic heat conduction, which drives both the MTI and HBI, has already been implemented in Arepo by Kannan et al. (2016).

The anisotropic viscosity tensor is given by

\[
\Pi = -\Delta p (bb - \frac{1}{3}) ,
\]

where the pressure anisotropy is defined as \(\Delta p = p_{\perp} - p_{\parallel}\) and \(bb\) is a dyadic product. Here \(p_{\parallel}\) and \(p_{\perp}\) are the pressure parallel (perpendicular) to the magnetic field and the total thermal pressure is \(p = p_{\parallel}/3 + 2p_{\perp}/3\).

Plasma motions can easily lead to the production of a pressure anisotropy due to conservation of the first and second adiabatic invariants of plasma particles (Chew et al. 1956). The resulting production of pressure anisotropy is described by the Chew-Goldberger-Low (CGL) equations in a collisionless plasma (Chew et al. 1956; Baumjohann & Treumann 1996). In a collisional plasma such a pressure anisotropy is quickly isotropized by collisions. In a weakly collisional plasma, where the collision frequency is much slower than the gyro-frequency of particles but much faster than the dynamical frequency of the system, the production of pressure anisotropy can be assumed to be balanced by collisions (Schekochihin et al. 2005). This yields an expression for the pressure anisotropy

\[
\Delta p = \rho \gamma \frac{d}{dt} \left( B^2 \rho^{-2} \right) .
\]
where \( \nu \) is the viscosity coefficient (a diffusion coefficient with units \( \text{m}^2/\text{s} \)). We take the viscosity coefficient to be Spitzer (Spitzer 1962; Braginskii 1965; Sarazin 1986) which means it depends on density and temperature as\(^1\) \( \nu \propto T^{5/2}/\rho \). For constant density, we can infer from Equation (9) that regions where the magnetic field strength is increasing in time will have a positive pressure anisotropy while regions where the magnetic field strength is decreasing in time will have a negative pressure anisotropy. Both situations can lead to the excitation of microscale instabilities if the magnitude of the pressure anisotropy exceeds certain thresholds. A dominant perpendicular pressure can lead to the mirror instability (Hasegawa 1969; Southwood & Kivelson 1996; Kivelson & Southwood 1996) while a dominant parallel pressure can excite the firehose instability (Parker 1958; Chandrasekhar et al. 1958; Rosenbluth 1956; Vedenov & Sagdeev 1958; Davidson & Völk 1968). These instabilities are not fully described by Braginskii MHD and require special treatment (see Section 2.1).

Equation (9) can be rewritten by using the equation for mass continuity, Equation (2), and the induction equation, Equation (4), to rewrite the time derivatives of the mass density and magnetic field strength. This yields\(^2\)

\[
\Delta p = \rho \nu (3 bb : \nabla v - \nabla v) ,
\]

(11)

showing that the pressure anisotropy can be evaluated from the instantaneous velocity derivatives, and the density, temperature and local magnetic field direction. Equation (11) is used in Arepo when evaluating the viscosity tensor.

The viscosity tensor enters in the momentum equation, Equation (3), and can be on the order of or larger than the Lorentz force when the magnetic field is weak, i.e., \( \beta > 1 \) (Squire et al. 2017a). The viscosity tensor also enters in the entropy equation as a viscous heating term

\[
-\mathbf{H} : \nabla v = \frac{(\Delta p)^2}{3 \rho \nu} = \frac{\rho \nu}{3} (3 bb : \nabla v - \nabla v)^2 .
\]

(12)

which is always positive. The viscosity tensor, Equation (8), has vanishing bulk viscosity, i.e., the trace of the viscosity tensor is zero.

For our numerical implementation of Braginskii viscosity in Arepo, we will consider the evolution of the total energy density (thermal + kinetic + magnetic)

\[
E = \frac{\rho}{\gamma - 1} + \frac{1}{2} \rho v^2 + \frac{B^2}{2 \mu_0} ,
\]

(13)

instead of solving the entropy equation directly. The required evolution equation for \( E \) can be shown to be given by

\[
\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v} - B(B \cdot \mathbf{v})/\mu_0] = -\nabla \cdot Q - \nabla \cdot (\mathbf{H} : \nabla v) .
\]

(14)

\(^1\) For Spitzer viscosity, the viscosity coefficient is given by (Hazeltine & Waalbroeck 2004)

\[
\nu = \frac{T}{\rho} \sum n_s / v_s ,
\]

(10)

where \( n_s \) is the number density and \( v_s \) is the collision frequency of ion species \( s \).

\(^2\) Here the notation : has the following meaning \( bb : \mathbf{v} = \sum_j \sum_i b_i b_j \partial_i v_j \) which is equivalent to the trace of the matrix product between the matrices \( bb \) and \( \mathbf{v} \).

where \( p_T = p + B^2/2 \mu_0 \) is the total pressure (gas plus magnetic), and the advantage is that this approach conserves total energy.

### 2.1 Firehose and mirror instabilities

We consider two important microscale instabilities for the ions, namely the firehose and the mirror instability. The firehose instability is excited if

\[
\frac{p_\perp}{p_\parallel} - 1 < \frac{2}{\beta} ,
\]

(15)

and the mirror instability is excited if

\[
\frac{p_\perp}{p_\parallel} - 1 > \frac{1}{\beta} ,
\]

(16)

where \( \beta = 2 \mu_0 p_\parallel / B^2 \). These criteria for instability can be rewritten into a criterion for stability (Kunz et al. 2012)

\[
\frac{B^2}{\mu_0} < p_\perp - p_\parallel \left< \frac{B^2}{2 \mu_0} .
\]

(17)

Particle-in-cell simulations of the firehose and mirror instability indicate that these instabilities saturate by driving the pressure anisotropy to marginal stability (Schekochihin et al. 2008; Rosin et al. 2011; Kunz et al. 2014), a prediction that is also supported by observations of the solar wind (Bale et al. 2009; Chen et al. 2016). Motivated by these studies we use an approach that has also previously been used by e.g. Sharma et al. (2006); Kunz et al. (2012); Berlok & Pessah (2016b); Squire et al. (2019), i.e., we add an option to limit the value of the ion pressure anisotropy to the limits set by the firehose and mirror instability thresholds, Equation (17), when evaluating the viscosity tensor.

### 2.2 Dispersion relation for waves

We derive the dispersion relation for MHD waves which are modified when the effect of Braginskii viscosity is taken into account. In order to do so, we linearize Equations (2)-(5). The only nonstandard term in the linearization of these equations is the viscosity term in the momentum equation\(^3\).

We assume that \( \Delta p = 0 \) for the equilibrium (as well as constant \( \rho \) and zero background velocity). We then find that

\[
\delta \Delta p = i \nu (3 bb : k \mathbf{v} - k \cdot \mathbf{v}) ,
\]

(18)

\(^3\) The viscosity term is neglected in the linearized version of the internal energy equation because it is second order in the perturbations.
and the linearized equations are thus
\begin{equation}
-\omega^2 \rho = -i k \cdot \delta \mathbf{v} \quad ,
\end{equation}
(19)
and
\begin{equation}
-\omega \delta \mathbf{v} = -i k \cdot \delta \mathbf{v} - k \cdot \left[ \nabla (\mathbf{b} \cdot \mathbf{v}) \right] ( \mathbf{b} \cdot \delta \mathbf{v} ) ( \mathbf{b} \cdot \mathbf{v}) - 1/3 ]
+ \frac{B^2}{\mu_0 \rho} ( i \mathbf{k} \times \delta \mathbf{b} ) \times \mathbf{b} \quad ,
\end{equation}
(20)
\begin{equation}
-\omega \delta \mathbf{b} = i k \times ( \delta \mathbf{v} \times \mathbf{b} ) \quad ,
\end{equation}
(21)
and
\begin{equation}
\frac{\delta \rho}{\rho} = \frac{\gamma \rho}{\rho} - \frac{\delta P}{P} \quad .
\end{equation}
(22)
We then define the sound speed
\begin{equation}
\tilde{c}^2 \equiv \frac{\gamma \rho}{\rho} \quad ,
\end{equation}
(23)
and the Alfvén speed
\begin{equation}
\tilde{v}_A^2 \equiv \frac{B^2}{\mu_0 \rho} \quad ,
\end{equation}
(24)
and combine the equations above to obtain a single equation for \( \mathbf{\delta v} \), i.e.,

\begin{equation}
\omega^2 \delta \mathbf{v} = k^2 \delta \mathbf{v} - i \omega k \cdot \left[ \nabla (\mathbf{b} \cdot \mathbf{v}) \right] ( \mathbf{b} \cdot \delta \mathbf{v} ) ( \mathbf{b} \cdot \mathbf{v}) - 1/3 ]
+ \frac{B^2}{\mu_0 \rho} ( i \mathbf{k} \times ( \delta \mathbf{b} \times \mathbf{b} ) ) \times \mathbf{b} \quad .
\end{equation}
(25)
Assume now, without loss of generality, that \( \mathbf{b} = \mathbf{e}_z \) and \( k = k_\perp \mathbf{e}_x + k_\parallel \mathbf{e}_z \). Equation (25) can then be written as \( \mathbf{D} \cdot \delta \mathbf{v} = 0 \) where \( \mathbf{D} = \mathbf{M}_{\text{MHD}} + \mathbf{M}_{\text{Brag}} \) and
\begin{equation}
\mathbf{M}_{\text{MHD}} = \begin{pmatrix}
\omega^2 - k_\perp^2 \tilde{c}_\perp^2 & 0 & -k_\parallel \tilde{c}_\perp^2 \\
0 & \omega^2 - k_\parallel^2 \tilde{c}_\parallel^2 & 0 \\
-k_\parallel \tilde{c}_\perp^2 & 0 & \omega^2 - k_\parallel^2 \tilde{c}_\parallel^2
\end{pmatrix} ,
\end{equation}
(26)
takes the standard MHD form (see e.g. Fitzpatrick 2014) and
\begin{equation}
\mathbf{M}_{\text{Brag}} = \omega \frac{\gamma}{3} \begin{pmatrix}
-2 k_\perp^2 & 0 & 2 k_\parallel \\
0 & 0 & 0 \\
0 & 0 & -4 k_\parallel^2
\end{pmatrix} ,
\end{equation}
(27)
contains the Braginskii viscosity physics.

The determinant of \( \mathbf{D} \) yields the dispersion relation for waves in Braginskii MHD. We find, as for standard MHD, that the dispersion relation factors into a dispersion relation for the shear Alfvén wave
\begin{equation}
(\omega^2 - k_\parallel^2 \tilde{c}_\parallel^2) = 0 ,
\end{equation}
(28)
and for the fast- and slow magneto-sonic waves
\begin{equation}
\omega^4 + \omega \frac{\gamma}{3} \begin{pmatrix}
4 k_\parallel^2 & k_\perp^2 \\
0 & 0
\end{pmatrix} - \omega^2 k_\parallel^2 \left( \tilde{c}_\parallel^2 + \tilde{c}_\perp^2 \right)
- \omega^2 k_\parallel^2 \left( \tilde{c}_\parallel^2 + \tilde{c}_\perp^2 \right)
+ k_\parallel^2 k_\perp^2 \tilde{c}_\parallel^2 \tilde{c}_\perp^2 = 0 .
\end{equation}
(29)
Equation (28) shows that shear Alfvén waves propagate without damping\(^4\) in Braginskii MHD while Equation (29) differs from the ideal MHD version by terms that contain the viscosity coefficient, \( \eta \). Unlike the ideal MHD result,

\begin{equation}
\rho \frac{\partial v}{\partial t} = -\nabla \cdot \mathbf{v} \quad ,
\end{equation}
(34)
\begin{equation}
\frac{\partial E}{\partial t} = -\nabla \cdot (\mathbf{E} \cdot \mathbf{v}) \quad ,
\end{equation}
(35)
while the density, \( \rho \), and the magnetic field is kept constant during the Braginskii viscosity step. We use the equation for the evolution of the total energy, Equation (14), instead of the equation for the entropy, Equation (5), because this ensures that the algorithm conserves energy.

\(^4\) The linearly polarized Alfvén waves can trigger the firehose instability and damp as a consequence (Squire et al. 2016; Squire et al. 2017a) but this phenomenon is not captured by the linear theory derived here.
3.1 Spatial discretization on a Voronoi mesh

AREPO uses an unstructured, moving Voronoi mesh for solving the equations of motion (Springel 2010). This makes the spatial discretization of Equations (34) and (35) non-trivial. We outline our strategy for evaluating the fluxes in the following three subsections.

3.1.1 A local coordinate system

We define the standard Cartesian coordinate unit vectors as \( e_x, e_y, \) and \( e_z \). We also define a coordinate system \( e_n, e_m, \) and \( e_p \) at each interface between two Voronoi cells. This coordinate system is oriented such that \( e_n \) is the normal to the interface between the two Voronoi cells. The unit vector \( e_m \) is parallel to the component of the magnetic field that lies in the plane of the interface of the Voronoi cells. This is achieved by defining \( e_m \) as

\[
  e_m = \frac{b - (e_n \cdot b)e_n}{|b - (e_n \cdot b)e_n|}.
\]

Consequently, the magnetic field does not have a component along \( e_p \), which is defined by

\[
e_p = e_n \times e_m.
\]

In this new coordinate system, the magnetic field is thus given simply by \( b = b_n e_n + b_m e_m \) where \( b_n \equiv e_n \cdot b \) and \( b_m = e_m \cdot b \). A general vector, such as the velocity, is written as \( \mathbf{v} = v_n e_n + v_m e_m + v_p e_p \) in the local coordinate system.

Finally, we also define the derivative operators \( \partial_n = e_n \cdot \nabla \), \( \partial_m = e_m \cdot \nabla \) and \( \partial_p = e_p \cdot \nabla \), where the gradient operator, \( \nabla \), is given by

\[
  \nabla = e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z},
\]

in the Cartesian coordinate system and by

\[
  \nabla = e_n \partial_n + e_m \partial_m + e_p \partial_p,
\]

in the local coordinate system. Coordinate transformations of vectors and derivative operators are done in the usual way but are detailed in Appendix A for completeness.

3.1.2 Finite volume equations in the local coordinate system

For finite volume codes such as Athena or Arepo (which evolve volume averages of the physical quantities inside grid cells) Equations (34) and (35) are rewritten by taking the volume average over a cell of volume \( V \), i.e.,

\[
  \frac{1}{V} \int_V \frac{\partial \rho}{\partial t} \, dV = - \frac{1}{V} \int_V \nabla \cdot \mathbf{I} \, dV,
\]

\[
  \frac{1}{V} \int_V \frac{\partial E}{\partial t} \, dV = - \frac{1}{V} \int_V \nabla \cdot (\Pi \mathbf{v}) \, dV,
\]

which upon application of the divergence theorem become

\[
  \frac{\partial \rho}{\partial t} = \frac{1}{V} \int_{\partial V} \Pi \cdot \mathbf{e}_n \, dA,
\]

\[
  \frac{\partial E}{\partial t} = \frac{1}{V} \int_{\partial V} (\Pi \mathbf{v}) \cdot \mathbf{e}_n \, dA,
\]

where \( \partial V \) denotes the surface of the volume and \( dA \) is an infinitesimal area. In our Arepo implementation, the surface integrals on the RHS are approximated as a discrete sum of fluxes through the faces of the Voronoi cell. The normal vectors, \( \mathbf{e}_n \), are the ones described in Section 3.1.1.

It is evident that we need to derive expressions for \( \Pi \mathbf{e}_n \) and \( (\Pi \mathbf{v}) \cdot \mathbf{e}_n \). Using the definition of the viscosity tensor, Equation (8), we obtain

\[
  \Pi \mathbf{e}_n = \Delta p (b b - 1/3) \mathbf{e}_n = \Delta p \mathbf{b} - \Delta p \mathbf{e}_n/3.
\]

Since both \( b \cdot e_p = 0 \) and \( e_n \cdot e_p = 0 \) (by construction), Equation (44) shows that the viscous flux in the \( e_p \) direction is zero. This has the advantage that the \( v_p \) component of the velocity does not evolve when working in the local coordinate system.

We detail the conversion of Equations (34) and (35) to the local coordinate system in Appendix B where we find that the momentum and energy equations can be written as

\[
  \rho \frac{\partial \rho}{\partial t} = \frac{1}{V} \int_{\partial V} \Delta \rho (b_n b_n - 1/3) \, dA,
\]

\[
  \rho \frac{\partial v_n}{\partial t} = \frac{1}{V} \int_{\partial V} \Delta \rho b_n b_n \, dA,
\]

\[
  \frac{\partial E}{\partial t} = - \frac{1}{V} \int_{\partial V} \Delta (b_n b_n v_n + b_n b_m v_m - v_n/3) \, dA,
\]

where the pressure anisotropy is given by

\[
  \Delta p = 3 \eta \left[ b_n^2 \partial_n v_n + b_m^2 \partial_m v_m + b_n b_m (\partial_n v_m + \partial_m v_n) \right] - \rho \nu \left[ \partial_n v_n + \partial_m v_m + \partial_p v_p \right].
\]

3.1.3 Estimating derivatives and values at cell interfaces

All quantities and their derivatives are estimated at the corners of Voronoi cells by employing the procedure outlined in Section 2.1 of Pakmor et al. (2016b), i.e. by performing a least-squares fit using values at the three (four) adjacent cell centers in two (three) dimensions.

A scalar quantity, \( \phi \), is then estimated at the interface of a Voronoi cell by performing a weighted mean

\[
  \phi_{\text{face}} = \sum_i w_i \phi_i,
\]

where \( w_i \) is the weight of the corner \( i \).

\[\text{Here the bars on the LHS of the equations denote volume averages. Such bars will not be explicitly shown in the remainder of the paper, as they tend to clutter the notation.}\]

\[\text{The mass density stays constant during a Braginskii viscosity step and can therefore be pulled out from the time derivative in the momentum equations.}\]

\[\text{As described in Pakmor et al. (2016b), the least-squares fit can fail if the mesh contains highly irregular cells. Failed corners are assigned a zero weight and the weights of the remaining corners are adjusted accordingly. We do not calculate the viscous flux}\]
A derivative, for instance $\partial \phi / \partial x$, could in principle also be estimated by using a similar weighting procedure, e.g.,

$$\frac{\partial \phi_{\text{face}}}{\partial x} = \sum_i w_i \frac{\partial \phi_i}{\partial x}. \quad (50)$$

Experience has however shown that more care is needed for anisotropic transport. This was originally realized by Sharma & Hammett (2007) for anisotropic heat conduction where it was shown that heat sometimes flows from cold to hot when a simple averaging procedure is used. For Braginskii viscosity, a similar issue is that a simple average can lead to flow of momentum in the wrong direction (Parrish et al. 2012; Zühne et al. 2015). As this issue arises when not all estimates agree on the sign of the derivative, our procedure for estimating the derivative at interfaces is the following: if all the corners of an interface agree on the sign of the derivative, we estimate the derivative at the interface by calculating a weighted harmonic mean of the values at corners

$$\frac{\partial \phi_{\text{face}}}{\partial x} = \left( \sum_i w_i \frac{\partial \phi_i}{\partial x} \right)^{-1}. \quad (51)$$

If the signs disagree, we set the estimate to zero. In two dimensions (where $w_i = 1/2$), Equation (51) is simply the Van Leer limiter which is also used in the Braginskii viscosity implementation in ATHENA (Stone et al. 2008; Parrish et al. 2012).

### 3.2 Second order accurate super-time-stepping

Stability consideration imply a time step constraint for an explicit update of viscosity,

$$\Delta t \leq C \frac{(\Delta x)^2}{2 \delta v}. \quad (52)$$

where $\Delta x$ is minimum size of the Voronoi cells, $d$ is the dimension of the simulation and $C$ is the Courant number. In comparison, the MHD time step constraint is

$$\Delta t_{\text{MHD}} \leq C \frac{\Delta x}{\delta v_{\text{max}}}, \quad (53)$$

where $\delta v_{\text{max}}$ is the maximum signal speed (flow velocity plus the velocity of the fast magnetosonic wave). Due to the different scalings with $\Delta x$, the MHD time step constraint will thus often be such that $\Delta t_{\text{MHD}} \gg \Delta t$, in particular for high resolution simulations.

In order to alleviate this constraint, we have implemented a second order accurate super-time-stepping (STS) version of Braginskii viscosity. This implementation is based on Meyer et al. (2012); Meyer et al. (2014) which describe a second-order accurate Runge-Kutta Legendre method (RK12). The RK12 method has been implemented in the Pluto code (Vaidya et al. 2017) for anisotropic heat conduction but has not previously been used for anisotropic viscosity. We briefly outline the merits of STS and how we have applied it to Braginskii viscosity

across an interface when it has less than two good corners. Mesh regularization in Arepo ensures that such failed interfaces rarely occur. We test the implementation on an irregular, random grid in Section 4.3.

Super-time-stepping introduces several new stages for each super time step. The main advantage is that the maximum time step, $\tau$, is given by

$$\tau = \frac{\Delta t}{4} \left( s^2 + s - 2 \right), \quad (54)$$

where $\Delta t$ is the maximum time step for an explicit update and $s$ is the number of stages in the super-time-stepping. For large $s$, $\tau \propto s^2$ and a speedup by a factor $s$ is therefore expected compared to subcycling of an explicit update.

In Braginskii viscosity, the momentum equation is a diffusion equation which has a diffusion coefficient that depends on temperature. The viscous heating in the energy equation can change the temperature and thus the viscosity coefficient. The equations are thus coupled. In equation form, we can write

$$\frac{\partial n}{\partial t} = L(T, \vartheta), \quad (55)$$

$$\frac{\partial E}{\partial t} = \nabla \cdot F_E(T, \vartheta), \quad (56)$$

where $L(T, \vartheta)$ is the parabolic operator on the RHS of Equation (34) and $F_E(T, \vartheta)$ is the energy flux on the RHS of Equation (35). Both equations depend on both velocity and temperature. In order to solve the momentum equation using super-time-stepping, we decouple the momentum and energy equations by assuming that the viscosity coefficient simply takes the value at the beginning of the time step. This allows us to write the velocity update from time $t^n$ to time $t^{n+1} = t^n + \tau$ as (Meyer et al. 2012)

$$Y_0 = \vartheta^n, \quad (57)$$

$$Y_1 = Y_0 + \mu_1 \tau L(T^n, Y_0), \quad (58)$$

$$Y_j = \mu_j Y_{j-1} + v_j Y_{j-2} + (1 - \mu_j - v_j) Y_0 + \tau \tau L(T^n, Y_{j-1}) + \hat{\gamma}_j \tau L(T^n, Y_0) \quad \text{for } 2 \leq j \leq s \quad (59)$$

$$\vartheta^{n+1} = Y_s. \quad (60)$$

Here the coefficients $\mu_1$, $\mu_j$, $v_j$, $\hat{\gamma}_j$ are the super-time-stepping parameters\(^9\) given in equations 16-17 in Meyer et al. (2012).

In order to also update the total energy, we use the trapezoidal rule, i.e.,\(^{10}\)

$$E^{n+1} = E^n + \frac{\tau}{2} \left[ \nabla \cdot F_E(T^n, \vartheta^n) + \nabla \cdot F_E(T^n, \vartheta^{n+1}) \right]. \quad (62)$$

The number of super-time-stepping stages needed for each time step is likely to change as a function of time. In the current implementation we determine the number of necessary stages by first computing the minimum values of

\(^9\) We retain their notation but stress that these parameters should not be confused with physical constants such as viscosity, adiabatic index etc.

\(^{10}\) An alternative to this approach would be to update the total energy by performing an Euler step for each stage of the super-time-step, i.e.,

$$E_{j+1} = E_j + \delta t_j \nabla \cdot F_E(T^j, \vartheta^j) \quad \text{for } 0 \leq j < s, \quad (61)$$

where the time increment for each stage is given by $\delta t_0 = \Delta t / 3$, $\delta t_j = 2 \Delta t / 3$, $\delta t_j = (j + 1) \Delta t / 2$ for $j > 1$ and fulfill $\sum_j \delta t_j = \tau$. This would allow for the viscosity coefficient to vary during the super-time-step and could potentially provide an increased accuracy when using many stages.
Δt and Δc_{\text{MHD}} on the grid. We then compute s such that Δt(s^2 + s - 2) ≥ Δc_{\text{MHD}} with 3 ≤ s ≤ s_{\text{max}} where s_{\text{max}} is a pre-defined maximum number of stages. We only use odd values for s as recommended by Meyer et al. (2012).

4 TESTS AND EXAMPLES

4.1 Decay of a velocity profile I

We test the numerical method for solving Equations (34) and (35) by using a simple initial condition for which we can also derive the analytical solution. The MHD solver is turned off for this test and the magnetic field is static. We consider an initial condition with velocity

\[ \mathbf{v}(r) = A \sin(k \cdot r) \frac{1}{k} \],

(63)

a uniform magnetic field, \( \mathbf{B} \), and an initially constant internal energy, \( \epsilon_0 \). Here \( A \) is the amplitude of the perturbation, \( r = x_e + y_e + z_e \) is a position vector and \( k \) is a wavevector with magnitude \( k \). If the magnetic field, \( \mathbf{B} \), wave vector, \( \mathbf{k} \), and velocity, \( \mathbf{v} \), are aligned, the time evolution of velocity and internal energy can be shown to be given by

\[ \mathbf{v}(t, r) = A \sin(k \cdot r) e^{-\gamma t} \frac{1}{k} \],

(64)

\[ \mathbf{v}(t, r) = \epsilon_0 + \frac{\rho A^2}{2} \cos^2(k \cdot r) \left( 1 - e^{-2\gamma t} \right) \],

(65)

where

\[ \gamma = \frac{4v_\parallel}{3} k^2 \].

(66)

is the decay rate.

We perform a simulation in a three-dimensional (3D) periodic, cubic box of size \( L \) on a static Voronoi mesh with 2N^3 mesh-generating points where \( N = 32 \). This mesh is created by interlacing two Cartesian meshes, i.e., with the first mesh displaced to have its cell centers at the cell corners of the second mesh. The magnetic field, wave vector and velocity field are all oriented along \( e_\parallel \), but nothing is aligned with the Voronoi grid. This means that the simulation is automatically testing all the terms in the implementation. We use STS with \( s = 31 \) stages to accelerate the simulation, yielding a speedup by a factor \( ~8 \). The result is shown in Fig. 1 and shows excellent agreement with the analytical solution. Physically, the viscous heating is maximal at the nodes of the velocity profile because the slope of the velocity profile is maximal there.

4.2 Decay of a velocity profile II

We consider a simulation inspired by the test problem in (Hopkins 2017). We take \( \mathbf{B} = B(e_\parallel + e_\perp) / \sqrt{2} \) with an initial velocity profile given by \( \mathbf{v} = c q(x) e_\parallel \) where

\[ q(x) = \frac{3}{2} \left( 1 - \frac{1}{2} \left( \text{erf} \left( \frac{x - x_0}{a} \right) - \text{erf} \left( \frac{x + x_0}{a} \right) \right) \right) \],

(67)

with \( x_0 = 1/4L \) and \( a = 0.05L \). Unlike our first example in Section 4.1, Equations (34) and (35) cannot be reduced to a simple one-dimensional diffusion equation for this setup and we instead have a coupled set of equations. This set of equations can however still be solved analytically and we present a derivation of the analytical solution in Appendix C.3. Our derivation uses the fact that \( q(x) \) can be represented by the Fourier series

\[ q(x) = \sum_{n=0}^{\infty} a_n \cos(k_n x) \],

(68)

where \( k_n = 2\pi n / L \) and the Fourier coefficients are given by

\[ a_n = \left\{ \begin{array}{ll} \frac{2}{2 \sin (\pi n / 2)} e^{-n^2 x_0^2 / 400} & \text{for } n = 0 \ , \\ -2 \sin (\pi n / 2) e^{-n^2 x_0^2 / 400} & \text{for } n > 0 \ . \end{array} \right. \]

(69)

We find that the evolution of \( v_x \) and \( v_y \) can be described by

\[ v_x(t, x) = -c \sum_{n=0}^{\infty} \frac{3a_n}{10} \cos(k_n x) (1 - e^{-\gamma_n t}) \ ,

(70)

and

\[ v_y(t, x) = c \sum_{n=0}^{\infty} \frac{a_n}{10} \cos(k_n x) (1 + 9e^{-\gamma_n t}) \ ,

(71)

where \( \gamma_n = \frac{5v_\parallel k_n^2}{6} \).

We conclude from the time dependence of Equations (70) and (71) that \( v_x = 0 \) at \( t = 0 \) but that a non-zero component will grow as a function of time. This happens at the expense of \( v_y \) which decreases in time. The mixing of \( v_x \) and \( v_y \) happens because the magnetic field is misaligned with respect to the initial velocity profile. Interestingly, the asymptotic solution with \( \gamma_n t \gg 1 \) has both \( v_x \) and \( v_y \) vary as a function of the \( x \)-coordinate. For isotropic viscosity both velocity components would be constant in space after a long duration of time. For Braginskii viscosity, things are not so simple. Instead, the solution at \( \gamma_n t \gg 1 \) is such that a particular linear combination of \( v_x \) and \( v_y \) is constant in space, i.e., one can see from Equations (70) and (71) that \( v_x + 3v_y = 6c \) at late times. This follows from the geometry of the problem and is outlined in more detail in Appendix C. We conclude that velocity profiles can change in complicated ways in the presence of anisotropic Braginskii viscosity.

The velocity profiles yield a pressure anisotropy given by

\[ \Delta p(x, t) = -\frac{3 \rho c y_\parallel}{2} \sum_{n=1}^{\infty} k_n a_n \sin(k_n x) e^{-\gamma_n t} \ ,

(73)

and the effect of Braginskii viscosity is to drive this pressure anisotropy towards zero, as seen by considering the \( \gamma_n t \gg 1 \) limit of Equation (73).

The viscous damping has an associated heating rate that will increase the internal energy in regions where \( \Delta p \neq 0 \). We obtain an expression for the evolution of the internal energy given by

\[ \epsilon(t) = \epsilon_0 + \frac{9c y_\parallel}{10} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_n a_m \sqrt{n m} \min \left( \frac{\gamma_n}{\gamma_m} \right) \sin(k_n x) \sin(k_m y) \left( 1 - e^{-(\gamma_n + \gamma_m) t} \right) \ .

(74)

where \( \epsilon_0 \) is the initially constant internal energy. As we might have been able to conclude already from the evolution of the pressure anisotropy, we observe that the internal
Figure 1. Viscous damping of a velocity profile (left panel) and the associated increase in internal energy due to viscous heating (right panel). The data from the 3D simulation (solid lines) agrees with the analytical solution (black dashed lines).

Figure 2. Anisotropic diffusion in a magnetic field $B = B(e_x + e_y)/\sqrt{2}$ creates a non-zero $v_x$ (upper left panel) from a velocity field which is initially strictly in the y-direction (upper right panel). The pressure anisotropy (lower left panel) causes viscous heating which modifies the internal energy (lower right panel). We compare data from Arepo (blue solid lines) and analytical theory (orange dashed lines) at $ct/L = 25$. The initial condition ($ct/L = 0$) and the asymptotic solution ($\gamma nt \gg 1$) are shown with black dashed and black solid lines, respectively.
energy stops increasing and becomes constant in time as $\gamma_{nl} > 1$.

We can use the analytical solution given by Equations (70), (71), (73), and (74) to test our implementation of Braginskii viscosity. In order to do so, we construct a 2D grid with resolution $N_x = 128$ and $N_y = 4$ and size $L_x = L_y = N_x/N_y$. The Voronoi cell centers are set to be a regular Cartesian, uniform grid where every second row has been displaced by 0.45\text{a}_x$ thus creating a hexagonal mesh (Pakmor et al. 2016b). We set $\nu_{\parallel}/(Lc) = 10^{-3}$ and evolve the simulation to $ct/L = 25$ with the MHD solver turned off. The resulting profiles for $\nu_{\parallel}$ who reached the same conclusion for Parrish et al. We find that with Arepo data as blue solid lines and the theory as orange dashed lines. The initial condition and the $\gamma_{nl} > 1$ analytical solution are also shown with black dashed lines and black solid lines, respectively. As expected from the analytical solution, we observe that the $\nu_{\parallel}$-component grows (upper left panel) while the initial $\nu_{\perp}$-component decays (upper right panel). The pressure anisotropy (lower left panel), which initially has two sharp peaks of opposed sign decreases as a function of time. At $ct/L = 25$ most of the high wavenumber modes have already decayed and the pressure anisotropy looks almost like the fundamental sine wave (which decays at the slowest rate). The internal energy has developed two prominent peaks at the locations where the pressure anisotropy was initially peaked ($x/L = \pm 1/4$) and the internal energy has almost achieved its final value.

### 4.3 Fast magnetosonic modes

We consider a 2D domain in the $xy$-plane of size $[0, L] \times [0, L]$ with a background magnetic field perpendicular to this plane, i.e., $B = B_0 \mathbf{e}_z$ where $B_0$ is such that $B = 25$. We initialize a standing, fast magnetosonic mode with initial velocity, density and magnetic field given by the linear wave solution which we derived in Section 2.2, i.e.,

$$
\begin{align*}
\mathbf{v}(r, t) &= -A \sin(k \cdot r) [\omega_0 \cos(\omega_0 t)] - \gamma \sin(\omega_0 t) e^{-\gamma t} \frac{k}{k^2}, \\
\frac{\delta \mathbf{p}}{P_0} &= \frac{\delta B_z}{B_0} = A \cos(k \cdot r) \sin(\omega_0 t) e^{-\gamma t},
\end{align*}
$$

where $k = k_x \mathbf{e}_x + k_y \mathbf{e}_y$, with $k_x = k_y = k_{L}/\sqrt{2} = 2\pi/L$ and $A = 10^{-3}$ is the perturbation amplitude. Here $\omega_0 = \text{Re}(\omega)$ and $\gamma = -\text{Im}(\omega)$ are the real and imaginary parts of the complex frequency, $\omega$, which is given by

$$
\omega = \pm k_L \sqrt{\frac{2}{\gamma^2} + \varepsilon^2} - \frac{k_{\perp}^2}{6} - \frac{\gamma^2}{12} k_{\perp}^2.
$$

In ideal MHD, fast magnetosonic waves propagate freely ($\gamma = 0$) and are non-dispersive ($\omega_0/k_L = \sqrt{\varepsilon^2 + \gamma^2}$ is independent of $k_L$) but Equation (77) shows that Braginskii viscosity both damps and makes the waves dispersive. This also means that the evolution of the velocity field is not perfectly out of phase with the density and magnetic field evolution, i.e., Equation (75) has a $\sin(\omega_0 t)$ contribution when $\gamma \neq 0$.

Using the full expressions for the eigenmode, given by Equations (75) and (76), is therefore important for obtaining good agreement when comparing simulations and linear theory.

As an illustration of fast waves with Braginskii viscosity, we show the evolution of the amplitude of the density perturbation for a simulation with $\nu_{\parallel}/(Lc) = 0.05$ in the first panel of Fig. 3. We can estimate the decay rate, $\gamma$, by fitting an exponential decay function to the peaks in amplitude and the real frequency, $\omega_0$, by measuring the distance between them. The agreement with the theoretical prediction for the decay rate, which is given by $\gamma = \nu_{\parallel}^2 k_{L}^2/6$ as long as $k_{L} \nu_{\parallel} < 6 \sqrt{\varepsilon^2 + \gamma^2}$, is excellent.

More quantitatively, we perform two parameter studies of damping of fast magnetosonic waves. Firstly, we vary the value of the viscosity coefficient, $\nu_{\parallel}$, at a fixed numerical resolution of $32 \times 32$. We calculate the damping rate and frequency for each simulation and show the result in the middle panel of Fig. 3. The agreement is excellent except for the data point with $\nu_{\parallel}/(Lc) = 10^{-3}$ where the damping rate is higher in the simulation than it should be according to the linear theory. This is due to numerical dissipation at the rather low numerical resolution employed (see also Parrish et al. 2012 who reached the same conclusion for ATHENA simulations).

The theoretical curves in the middle panel of Fig. 3 are computed from Equation (77) which remains valid even in the regime of strong damping and dispersion of fast waves (which occurs when $k_{L} \nu_{\parallel}$ is high, i.e., when $k_{L} \nu_{\parallel}$ is not much smaller than $\sqrt{\varepsilon^2 + \gamma^2}$). We are therefore able to extend the results of Parrish et al. (2012) and test AREPO even in this regime. We find that AREPO is able to well describe the change in real frequency and the purely exponential decay which occurs to the right of the dashed vertical line in the middle panel of Fig. 3. These simulations were performed with the RKL2 super-time-stepping method in order to expedite the process. This was particularly useful for the two simulations with $\nu_{\parallel}/(Lc) = 5$ where we took $\tau_{\text{max}} = 17$.

Secondly, we vary the numerical resolution at a fixed value of the viscosity coefficient, $\nu_{\parallel}/(Lc) = 10^{-2}$. We then

---

12 Hopkins (2017) a similar test was performed with the MHD solver turned on, with the justification that MHD dynamics would be sub-dominant with a magnetic field strength of $B_0 = 10^{-6}$ in code units. While this argument is valid for ideal MHD, Braginskii MHD is susceptible to microscale instabilities when Equation (17) is violated. These appear to have modified the solution presented in Hopkins (2017) which exhibits an asymmetry around $x = 0$. As apparent in Fig. 2, the initial condition has a significant pressure anisotropy such that the mirror (firehose) instability threshold is violated at $x/L = -1/4$ ($x/L = 1/4$). We therefore turn off the MHD solver in AREPO which prevents microscale instabilities from modifying the evolution of the simulation. We have also performed the test with ATHENA (not shown here) with the MHD solver turned off and find agreement between ATHENA, AREPO and the analytical solution.

13 Parrish et al. (2012) estimated the damping rate as

$$
\gamma = \frac{\nu_{\parallel} k_{L}^2}{6} \left[ \left( \frac{e - \dot{\theta}}{\dot{\theta}} \right)^2 - 3b \cdot \hat{k} \cdot (\hat{b} - \hat{\theta}) \right]^2,
$$

where $\dot{\theta} = \dot{k}/k$ and $\hat{b} = b/n$ but found that the damping rates in their simulations were lower than this estimate at high $\nu_{\parallel}$. A solution of our Equation (20) could in principle be used to understand their simulations which considered a general $k$ and both slow and fast waves.
measure the relative error in the measured damping rate, \((\gamma_s - \gamma_t)/\gamma_t\), where \(\gamma_s\) is the damping rate in the simulation and \(\gamma_t\) is the theoretical damping rate. This error is shown as a function of resolution \(N = N_x = N_y\) in Fig. 3. We find third order convergence, in agreement with Parrish et al. (2012).

As a final test, we consider fast magnetosonic waves on an irregular, random mesh in 3D, with a significant amplitude \((A = 0.1)\) and \(\nu_\parallel/(Lc) = 0.05\). We orient \(k = k_\parallel\) and construct a regular 3D grid by interlacing two standard Cartesian 3D grids. This gives a grid with \(2N^3\) points where \(N = 16\). The noisy grid is then constructed by displacing...
every mesh-generating point in all three directions by a random offset drawn from the interval $[-0.2, 0.2]L/N$.

We perform four simulations, i.e., MHD and Braginskii MHD on both the regular (for reference) and the irregular grid. The resulting velocity profile at $ct/L = 1$ is shown in Fig. 4. The random mesh leads to noise in the simulation results but overall, the solutions obtained with the random mesh closely follows the results obtained with the regular mesh.

We also note that the significant initial amplitude of the wave ($A = 0.1$) leads to the development of small scale variation in the velocity profile in the ideal MHD simulation only. In the Braginskii MHD such variations are damped on an even shorter time scale than the damping of the large scale motion (because the damping rate is inversely proportional to the length scale of the variation, squared).

4.4 Circularly polarized Alfvén waves

Circularly polarized Alfvén waves are not modified by Braginskii viscosity because the associated pressure anisotropy is exactly zero. The magnetic field strength and density is constant in time for such waves, which directly follows from Equation (9). This property should be respected by the numerical implementation of Braginskii viscosity in Arepo. In order to ensure this, we perform simulations of circularly polarized Alfvén waves. Any damping of their amplitude will be due to inherent numerical viscosity in the main MHD solver of Arepo or due to interpolation errors in the Braginskii viscosity implementation.

We initialize a traveling, circularly polarized Alfvén wave by setting a background magnetic field $B = B(e_x + e_y)/(\sqrt{2})$ and a perturbation of the form

\[
\frac{\delta B}{B} = A \left( \cos(kx - \omega t)e_y - \sin(kx - \omega t)e_x \right),
\]

and

\[
\frac{\partial v}{\partial t} = -\frac{\omega}{k} \frac{\delta B}{B}.
\]

Here the wavevector is oriented along the background magnetic field, $k = k_0 \mathbf{b}$ with $k_0 = 2\pi / (2L)$ and the frequency of the Alfvén wave is $\omega = k_0 v_A$.

The simulation domain is a cubic box of size $L$ with $2N^3$ mesh-generating points where $N = 16$ and we let the wave propagate for two wave periods, $\omega t = 4\pi$. We compare a simulation with a large viscosity coefficient, $\nu_{\parallel} / (Lc) = 1$ (using RKL2 and $s_{\text{max}} = 17$), with a reference simulation where Braginskii viscosity is not included. Due to the low grid resolution, the amplitude of the wave decays during its evolution in both simulations, see Fig. 5. For the ideal MHD reference simulation, the decay is purely due to numerical viscosity. By measuring the exponential decay rate, we find that $\gamma_{\text{num}} / (Lc) \approx 0.043$. We can estimate the corresponding numerical viscosity coefficient, by assuming that the numerical viscosity can be described as an isotropic viscosity of the form $-\nu_{\text{num}} \nabla^2 \mathbf{v}$ such that $\nu_{\text{num}} = k^2 \gamma_{\text{num}}$. This estimate gives $\nu_{\text{num}} / (Lc) \approx 10^{-3}$. The Braginskii viscosity simulation with $\nu_{\parallel} / (Lc) = 1$ does not show increased decay and this means that the extra numerical viscosity added by the Braginskii viscosity is less than $\nu_{\text{num}}$.

Only at an explicit Braginskii viscosity coefficient of $\nu_{\parallel} / (Lc) = 10$ do we see a significant deviation from the reference simulation, see Fig. 5. Note that this value of $\nu_{\parallel}$ is extremely large and would lead to a decay rate of $\gamma_{\text{iso}} = k^2 \nu_{\text{iso}} \approx 400$ if it was added as isotropic viscosity. The explicit decay rate would thus be $\sim 10^4$ times larger than the numerical decay rate instead of just a few times larger as in Fig. 5. From the simulations we can conclude that the extra numerical viscosity added by the Braginskii viscosity module is less than $10^{-3} \nu_{\parallel}$. Similar encouraging findings have previously been found for the Athena MHD code (see the Appendix in Parrish et al. 2012).

4.5 Linearly polarized Alfvén waves

In contrast to circularly polarized Alfvén waves, linearly polarized Alfvén waves have an associated pressure anisotropy. Quite recently, it was realized that this can prevent large amplitude linearly polarized Alfvén waves from propagating in a weakly collisional, high-$\beta$ plasma because the oscillation is interrupted by the firehose instability (Squire et al. 2016).

Extensive modeling of this phenomenon, using both a collisionless Landau fluid closure and weakly-collisional Braginskii MHD, has been presented in Squire et al. (2017a). The consequences for the magneto-rotational instability (MRI) have also been investigated (Squire et al. 2018) and the interruption was studied using hybrid kinetic simulations in Squire et al. (2017b).

Here we compare an AREPO simulation with two reference solutions. These are found by using the MHD code ATHENA (Stone et al. 2008) as well as a solution obtained by numerically integrating a simplified set of 1D equations. Details of the latter calculation, which is performed with spectral accuracy in space and a fourth order accurate Runge-Kutta-Legendre update in time, can be found in Appendix D.

We initialize the simulations at $t = 0$ with a magnetic field $B = B_0 \mathbf{e}_x$ and a perturbation of the form

\[
\frac{\delta B}{B_0} = -A \cos(kx)e_y,
\]

where $k = 2\pi / L$ and the perturbation amplitude is $A = 1/2$. This initial profile is shown with a dashed black line in the upper left panel of Fig. 6. The initial magnetic field strength is such that $B_0 = 10^4$ and the initial velocity is zero (shown in the upper right panel of Fig. 6).

In the absence of Braginskii viscosity, the resulting wave motion is a linearly polarized Alfvén wave with

\[
\frac{\delta B}{B_0} = -A \cos(kx) \cos(\omega t)e_y,
\]

\[
\frac{\mathbf{v}}{v_A} = A \sin(kx) \sin(\omega t)e_y,
\]

where $\omega = ke_B$ is the frequency of the wave.

When Braginskii viscosity is included in the simulation, this motion is severely modified (Squire et al. 2016; Squire et al. 2017a). This is evident in Fig. 6 where both $\delta B_x$ and $v_y$ have acquired highly non-sinusoidal shapes. The reason for this disruption of the wave is a self-generated pressure anisotropy which can trigger the firehose instability if the circumstances are right (see below, Squire et al. 2016).

Unlike the circularly polarized Alfvén wave, which has
a constant in time magnetic field strength, the linearly polarized Alfvén wave given by Equation (82) and (83) has a magnetic field strength that varies as

$$B(t) = B_0 \sqrt{1 + A^2 \cos^2(kx) \cos^2(\omega t)}.$$  \hspace{1cm} (84)

This decaying magnetic field strength leads to a pressure anisotropy given by

$$\Delta p(x,t) = 3 \nu \frac{d \ln B}{dt} = \frac{3 \nu \chi A^2 \omega \cos^2(kx) \sin(2\omega t)}{2(1 + A^2 \cos^2(kx) \cos^2(\omega t))},$$  \hspace{1cm} (85)

which eventually triggers the firehose instability if there are regions where Equation (15) is fulfilled, i.e., if the following inequality is satisfied

$$\Delta p \mu_0 B_0 \mathbf{B}^2 < -1.$$  \hspace{1cm} (86)

From Equation (86) we observe that the pressure anisotropy will attain its extrema at $x/L = 0$ and $x/L = 1/2$ while the pressure anisotropy at $x/L = 1/4$ and $x/L = 3/4$ remains zero.

As in Squire et al. (2017a), we can estimate whether the firehose instability will be triggered at the extrema of the pressure anisotropy by setting $\cos^2(kx) = 1$ in Equation (86). If $A$ is reasonably small, we can furthermore ignore the squared parenthesis in the denominator. We then find that

$$\mu_0 \Delta p B_0 \mathbf{B}^2 \sim \frac{3 \nu \chi A^2 \omega \sin(2\omega t)}{2e^2} < -1.$$  \hspace{1cm} (87)

Since the maximum of $\sin(2\omega t)$ is 1, the criterion for interruption of the linearly polarized Alfvén wave becomes

$$A \gtrsim \sqrt{\frac{2e^2}{3\nu \chi \omega}}.$$  \hspace{1cm} (88)

As in Squire et al. (2016), we choose the viscosity coefficient such that the RHS of Equation (88) is $1/5$, i.e.,

$$\nu = \frac{50e^2}{3\chi \omega},$$  \hspace{1cm} (89)

which has the numerical value $\nu/(Lc) = 3.75 \times 10^{-2}$. With a perturbation amplitude $A = 1/2$, we thus expect the wave to be interrupted by the firehose instability.

This prediction is fulfilled by the simulations, where interruption of the wave leads to large regions in which the pressure anisotropy divided by the magnetic field strength squared, i.e., $\mu_0 \Delta p/B^2$, is constant. Its value is either zero or pinned to the firehose instability threshold, $\mu_0 \Delta p/B^2 = -1$ (see the lower left panel of Fig. 6). In both types of regions, the wave evolves more slowly than it would have in the absence of Braginskii viscosity. This happens because the restoring force of the wave is the sum of the Lorentz force and the pressure anisotropy. The regions where $\mu_0 \Delta p/B^2$ is zero also have zero magnetic field fluctuation, as can be seen by comparing the lower and upper left panels of Fig. 6. There is thus no restoring force for the wave. In the regions where the magnetic field fluctuation is non-zero, the restoring force is canceled by the pressure anisotropy, $\Delta p + B^2/\mu_0 = 0$. In real space, the wave evolution leads to a zigzag pattern in
the magnetic field structure, with segments of straight field lines that are connected with curved field segments in which the firehose instability is at the instability threshold (see the lower right panel of Fig. 6).

A more detailed account of the test problem described in this section as well as the behavior found for traveling waves can be found in Squire et al. (2016) and Squire et al. (2017a).

4.6 The viscous Kelvin-Helmholtz instability

We study the KHI in a weakly collisional, magnetized plasma by including Braginskii viscosity in our analysis. This test is of particular relevance for cold fronts in galaxy clusters (see the discussion in Section 1). We consider a 2D periodic domain where the \(x\)-component of the velocity has a smooth profile given by \(v_0(x) = c(x)\delta_y\) with

\[
v(x, z) = v_0 \left[ \tanh \left( \frac{z - z_1}{a} \right) - \tanh \left( \frac{z - z_2}{a} \right) \right],
\]

where \(z_1 = 0.5L\), \(z_2 = 1.5L\), the smoothing parameter is \(a = 0.05L\) and \(2L\) is the extent of the domain in the \(z\)-direction. Here the flow speed is \(v_0/c = 1\) where \(c\) is the constant, isothermal sound speed of the background. Finally, we include a dynamically weak background magnetic field, \(B = B_0 x\), with \(B\) such that \(\beta = 10^3\).

Using a smooth profile such as Equation (90), instead of a discontinuous velocity profile, is essential for obtaining converged results for the KHI (McNally et al. 2012; Lecoanet et al. 2016). The linear theory, however, becomes analytically intractable due to the \(z\)-dependence of the background velocity shear (Berlok & Pfrommer 2019). It is nevertheless possible to numerically calculate the growth rate of the KHI for smooth shear profiles by linearizing Equations (2)-(5) and using a pseudo-spectral method to generate a matrix eigenvalue problem. This procedure is described in detail in Berlok & Pfrommer (2019) where it is applied to the KHI in various regimes, e.g., in sub- and supersonic flows, with and without background density variations, magnetic field and Braginskii viscosity. The linear calculations are partially automated by using Pseca15 and the linear solutions can be used to compare with nonlinear computer simulations. Such a verification test was developed in Berlok & Pfrommer (2019) for \(v_0/(Lc) = 0.01\) (referred to as Mv in Table 2 therein) and will here be used to test Braginskii viscosity in ARePo.

The growth rates obtained with Pseca15 for the KHI with a Braginskii viscosity coefficient \(v_0/(Lc) = 0.01\) are shown in the left panel of Fig. 7 along with the inviscid solution for comparison. As expected, the solution including Braginskii viscosity has a slightly lower growth rate than the inviscid solution. Furthermore, the growth rate maximum is moved to longer wavelengths (lower wavenumbers) because viscosity acts most efficiently on short spatial scales. The maximum growth rate is \(\sigma_{\text{max}}/Lc = 1.7087545\) and occurs at \(k_{\text{max}}L = 4.5470431\) (Berlok & Pfrommer 2019).

We perform two simulations of the KHI using ARePo with Braginskii viscosity enabled with \(v_0/(Lc) = 0.01\). One simulation in which the pressure anisotropy is limited to lie within the threshold for stability of microscale instabilities (see Section 2.1 for details) and one in which the pressure anisotropy is allowed to freely evolve. We set the extent of the domain in the \(x\)-direction, \(L_x\), to be such that the fastest growing mode fits exactly inside the domain, \(L_x = 2\pi/k_{\text{max}}\), and the instability is seeded by initializing all quantities with the linear solution for the perturbations (see Berlok & Pfrommer (2019) for details). We use a moving mesh with 354 \times 512 mesh-generating points and the RKL2 method with \(k_{\text{max}} = 11\) in order to accelerate the simulation. The initial amplitude of the perturbations is \(10^{-2}\) but as the system evolves the perturbations grow exponentially until they nonlinearly saturate (see right panel of Fig. 7). The exponential growth closely matches the theoretical value (the fit to \(\delta p/\rho\) yields \(\sigma L/c = 1.7084\) in the simulation without \(\Delta p\)-limiters).

We show the evolution of the KHI in Fig. 8 with black solid lines indicating the initially straight magnetic field lines. A passive scalar is shown in order to track the movement of the gas with red (blue) indicating initially right-moving (left-moving) fluid. The nonlinear evolution leads to the classical KHI rolls, and associated winding up of the magnetic field lines, see Fig. 8. We show the simulation with \(\Delta p\)-limiters (upper row of panels) and the simulation without \(\Delta p\)-limiters (lower row of panels) where microscale are allowed to grow without bounds. Microscale instabilities make the magnetic field become more jagged which makes the simulation without \(\Delta p\)-limiters (bottom row) appear less viscous than the one with \(\Delta p\)-limiters (upper row) where the solution remains smooth. However, the simulation without \(\Delta p\)-limiters is actually more viscous because large pressure anisotropies lead to increased damping of the velocity-field. This can be seen from the less evolved stage of the KHI rolls in the lower row of Fig. 8 in comparison to the upper row.

As mentioned above, the linear solution is seeded exactly with a single mode in the \(x\)-direction, where, e.g., the \(\delta v_z\)-component of the solution can be written as

\[
\delta v_z(x, z, t) = c[C_k(z) \cos(kx) + S_k(z) \sin(kx)].
\]

Here \(C_k(z)\) and \(S_k(z)\) are \(z\)-dependent Fourier-amplitudes that grow exponentially in time. In order to assess how well ARePo retains the correct shape of the linear solution, we calculate \(C_k(z)\) and \(S_k(z)\) from simulation snapshots at four different times and compare with the linear theory in Fig. 9. We find an exact match at \(t = 0\) (simply due to the seeding), and a very good match after amplification by a factor of \(\sim 5\) at \(ct/L = 1\). Nonlinear interactions start to play a significant role at \(ct/L = 2\) and cause the linear solution to over-predict the amplitudes by a factor of two at \(ct/L = 3\).

Finally, we show the spatial profiles for all the components of the system at \(ct/L = 1\) in Fig. 10. This figure shows an excellent (but at this numerical resolution not perfect) match between linear theory and ARePo simulation. The very sharp variations with \(z\) in the profiles (in particular for \(\delta b_z = \delta pB_y/\rho B\)) explain why a high grid resolution is required to accurately simulate the linear evolution of the KHI with magnetic fields and Braginskii viscosity.

\[15\] Pseca15 (Pseudo-Spectral Eigenvalue Calculator with an Automated Solver) is freely available online (Berlok & Pfrommer 2019).

\[16\] The simulation with \(\Delta p\)-limiters shows a slight \(-2\%\) deviation in the growth rate. A deviation is expected since the \(\Delta p\)-limiters were not included in the theory of Berlok & Pfrommer (2019).
Braginskii viscosity on a moving mesh

Figure 7. Left: growth rates as a function of wavenumber, $kL$, for $\nu_{\parallel} = 0$ and $\nu_{\parallel}/(Lc) = 0.01$ obtained with Psecas (solid lines) and the growth rate found in the Arepo simulation without pressure anisotropy limiters (red cross). Braginskii viscosity inhibits growth of the KHI and moves the maximally unstable mode to longer wavelengths. Right: exponential growth of deviations from the background equilibrium as measured in the Arepo simulation.

Figure 8. Evolution of passive scalar field (false color image) and magnetic field lines (black solid lines) as a function of time. The simulation in the upper panels has the pressure anisotropy limited to lie inside the microscale instability thresholds (Equation 17) while the pressure anisotropy is allowed to freely evolve in the simulation shown in the lower panels.

4.7 Verification of RKL2 super-time-stepping method

The RKL2 method has been employed in several of the tests presented in the previous subsections. In this section we briefly verify that the method is second order accurate in time and that it gives the expected reduction in computational cost.

We consider the anisotropic diffusion test presented in Section 4.2 but increase the value of $\nu_{\parallel}$ by a factor of 10. We perform six simulations with increasing spatial resolu-
Figure 9. Growth of the perturbation to $v_z$ in the simulation (solid lines) compared to the exponential growth predicted by theory (dashed black lines). Blue and orange lines show the $z$-dependent amplitude of the sine perturbation, $S_k(z)$, and the cosine perturbation, $C_k(z)$. Nonlinear effects cause saturation of the instability and the linear theory therefore over-predicts the amplitude at $ct/L = 3$. Note the change in scale on the vertical axis between panels.

Figure 10. Eigenmode structure of the KHI with Braginskii viscosity at $ct/L = 1$. At this point in time the perturbations have grown by a factor of 5 and we still see agreement between simulation (blue and orange solid lines) and the linear theory (dashed black lines).

Following the procedure used in Vaidya et al. (2017), we set the ratio of super time step to grid spacing, $\tau/\Delta x$, to be constant. The resulting number of STS stages then increase in increments of 2 from $s = 3$ at the lowest resolution to $s = 13$ at the highest. We calculate the root-mean-square (RMS) differences between analytic theory and internal energy, $\epsilon$, in the simulations (at $ct/L = 1$) and find second order convergence, see the upper panel in Fig. 11. We show the corresponding computational cost, calculated as the total number of computational steps required to reach $ct/L = 1$ (i.e. $N_{\text{tot}} = t/\tau s$), in the lower panel of Fig. 11. We find, in agreement with expectations (Vaidya et al. 2017), that the number of total steps scales as $N_{\epsilon}^{3/2}$. This is a factor $\sqrt{N_{\epsilon}}$.
less than the scaling for the number of steps needed for a standard explicit update.

In general, the different scalings of the explicit Braginskii time step, $\Delta t \propto \Delta x^2$, and the MHD time step, $\Delta t_{\text{MHD}} \propto \Delta x$, makes the number of required stages scale as $s \propto 1/\sqrt{\Delta x}$ (see Equations 52, 53 and 54). The total number of computational steps then scales as $N_{\text{tot}} \propto s/\Delta t_{\text{MHD}} \propto \Delta x^{-3/2}$ when using RKL2 and as $N_{\text{tot}} \propto \Delta x^{-2}$ when using an explicit update.\footnote{In practice, there is a maximum number of stages, $s_{\text{max}}$, above which operator splitting of diffusion and MHD and/or the accuracy of RKL2 breaks down. The best value for $s_{\text{max}}$ is likely to be problem dependent and we are not aware of a systematic approach to determining it. One potential idea, see the Appendix in Hopkins (2017), is to define a signal speed for the diffusive flux, $v_{\text{diff}}$, and use that to calculate the maximum super-time-step (and thereby $s_{\text{max}}$) as $s = C\Delta x/v_{\text{diff}}$ at the beginning of each super-time-step.} We conclude that the RKL2 method can accelerate the Braginskii viscosity update while retaining second order accuracy.

Figure 11. Verification of RKL2 super-time-stepping. Upper: The RMS difference between simulation and analytic theory at $ct/L = 1$ as a function of grid resolution. Lower: The total number of computational steps needed to reach $ct/L = 1$ as a function of grid resolution.

5 CONCLUSION AND FUTURE PROSPECTS

We have presented a new scheme for modeling Braginskii viscosity on an unstructured, moving mesh. This scheme has been implemented in the MHD code Arepo (Springel 2010; Pakmor et al. 2011) which already had the capability of modeling anisotropic heat conduction (Kannan et al. 2016). With methods for describing both anisotropic heat conduction and viscosity in place, Arepo is now capable of describing weakly collisional plasmas within the Braginskii MHD framework.

The Braginskii viscosity implementation has been thoroughly tested against analytical solutions (some newly derived) and independent numerical solutions. The examples show that the inclusion of Braginskii viscosity can change, sometimes fundamentally, the system evolution. The first example considered the decay of a velocity profile aligned with the magnetic field. Secondly, we rotated the magnetic field as an illustration of how Braginskii viscosity can mix velocity components. We were able to provide additional insight into this example, originally presented by Hopkins (2017), by deriving and comparing with the analytical solution and also analyzing the associated viscous heating.

Thirdly, we tested the ability of our code to propagate the MHD wave family with the correct modified behavior due to Braginskii viscosity. We found, in agreement with linear theory, that fast magnetosonic waves are damped by dissipation of their self-generated pressure anisotropy. We also verified that circularly polarized Alfvén waves propagate without damping\footnote{Even circularly polarized waves are in fact damped by numerical dissipation. What we have verified is that the damping rate is not increased by including explicit Braginskii viscosity.}, as they should because they have a constant magnetic field strength and density and, consequently, zero pressure anisotropy (see Equation 9). Linearly polarized Alfvén waves, on the other hand, have a time-varying magnetic field strength with an associated pressure-anisotropy that can lead to non-trivial evolution of the wave structure. This was realized only recently by Squire et al. (2016) who found that triggering of the firehose instability happens in a high-$\beta$ plasma whenever the wave-amplitude exceeds a certain limit (given in Equation 88). The Alfvén wave plays a special role in magnetized turbulence theory (Goldreich & Sridhar 1995), and Alfvén wave interruption therefore modifies turbulence properties in weakly collisional plasmas (Squire et al. 2017a; Squire et al. 2019). It is therefore promising, that our implementation in Arepo is able to reproduce Alfvén wave interruption and that we find agreement with both the well-tested MHD code Athena and an independent numerical solution obtained with spectral methods.

We concluded the tests by performing a simulation of the KHI in a weakly collisional, magnetized plasma. In this test, we initialized the simulation with a highly pre-varying magnetic field strength that can lead to non-trivial evolution of the wave structure. We find that Braginskii viscosity mixes and leads to saturation. We find that Braginskii viscosity inhibits the growth rate of the KHI in agreement with theory (Suzuki et al. 2013; Berlok & Pfrommer 2019) and previous numerical studies of cold fronts in galaxy clusters (Zuhone et al. 2015).

Our implementation employs 2nd order accurate (in time) Runge-Kutta-Legendre super-time-stepping (RKL2, Meyer et al. 2014) which is used to alleviate the severe time step constraint associated with diffusion operators in simula-
tions. This method has previously been used for anisotropic heat conduction on a Cartesian static grid but it is, to our knowledge, the first use of the method for Braginskii viscosity and the first use of this method on an unstructured, moving grid. The use of RKL2 will allow us to perform Braginskii MHD simulations of AGN-inflated bubbles and sloshing cold fronts in galaxy clusters without the computational cost becoming prohibitively expensive.

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APPENDIX A: COORDINATE TRANSFORMATIONS

Any vector, for instance the velocity vector $\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z$, can be expressed in terms of the new coordinate vectors $\mathbf{e}_n$, $\mathbf{e}_m$, and $\mathbf{e}_p$ as

$$\mathbf{v} = (e_n \mathbf{v}) \mathbf{e}_n + (e_m \mathbf{v}) \mathbf{e}_m + (e_p \mathbf{v}) \mathbf{e}_p = v_n \mathbf{e}_n + v_m \mathbf{e}_m + v_p \mathbf{e}_p \ .$$  \hfill (A1)

Similarly the vector

$$\mathbf{v} = v_n \mathbf{e}_n + v_m \mathbf{e}_m + v_p \mathbf{e}_p \ ,$$  \hfill (A2)

in the local coordinate system, is given by

$$\mathbf{v} = (e_x \mathbf{v}) \mathbf{e}_x + (e_y \mathbf{v}) \mathbf{e}_y + (e_z \mathbf{v}) \mathbf{e}_z = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z \ ,$$  \hfill (A3)

in the Cartesian coordinate system. We can write out the transformation rules explicitly by defining the components of the unit vectors $\mathbf{e}_n$, $\mathbf{e}_m$, and $\mathbf{e}_p$ in the Cartesian basis. These are given by $\mathbf{e}_n = n_x \mathbf{e}_x + n_y \mathbf{e}_y + n_z \mathbf{e}_z$, $\mathbf{e}_m = m_x \mathbf{e}_x + m_y \mathbf{e}_y + m_z \mathbf{e}_z$, and $\mathbf{e}_p = p_x \mathbf{e}_x + p_y \mathbf{e}_y + p_z \mathbf{e}_z$.

APPENDIX B: EQUATIONS IN THE LOCAL COORDINATE SYSTEM

We convert the momentum equations to the local coordinate system and find using Equation (8) that

$$\Pi \mathbf{e}_n = \Delta p (b - 1/3) \mathbf{e}_n = \Delta p b \mathbf{n} - \Delta p \mathbf{e}_n / 3 \ .$$  \hfill (B1)

The terms needed in the momentum equations are then

$$\Pi \mathbf{e}_n \mathbf{e}_n = \Delta p b_n b_n - \Delta p / 3 \ ,$$  \hfill (B2)

$$\Pi \mathbf{e}_m \mathbf{e}_m = \Delta p b_m b_m \ ,$$  \hfill (B3)

$$\Pi \mathbf{e}_p \mathbf{e}_p = 0 \ ,$$  \hfill (B4)

and we observe that the last component of $\mathbf{v}$ does not evolve when working in the local coordinate system.

We derive an equation for the energy flux $\Pi \mathbf{e}_n$ in the local coordinate system. Using Equation (8) we obtain

$$\Pi \mathbf{e} = \Delta p (b - 1/3) \mathbf{e} = \Delta p (b_n \mathbf{n} + b_m \mathbf{m} - \mathbf{e} / 3) \ ,$$  \hfill (B5)

and the expression needed in the energy equation is thus

$$\Pi \mathbf{e} \mathbf{e}_n = \Delta p (b_n \mathbf{n} + b_m \mathbf{m} - \mathbf{e} / 3) \ .$$  \hfill (B6)

The pressure anisotropy given by Equation (9) can be expressed in the local coordinate system by combining

$$b_\mathbf{b} : \nabla p = \hat{b}_n^2 \partial_n \partial_n v_n + \hat{b}_m^2 \partial_m \partial_m v_n + b_n b_m (\partial_n v_n + \partial_n v_m) \ .$$  \hfill (B7)
APPENDIX C: ANALYTICAL SOLUTIONS TO THE DIFFUSION EQUATIONS

We consider a plasma with a constant density and a uniform magnetic field. We also assume that the viscosity coefficient is constant. This assumption makes the velocity equation independent from the internal energy evolution. In 2D we work in a Cartesian coordinate system aligned with the magnetic field by defining unit vectors parallel, $\parallel$, and perpendicular, $\perp$, to the magnetic field.

In 3D we consider polar coordinates $(r, \phi, \phi)$ corresponding to the parallel, perpendicular and azimuthal direction. We assume for simplicity that all fields are independent of the $\phi$ coordinate in 3D. The merit of this coordinate system is that the equations simplify considerably. We have that

$$\mathbf{b} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

(C1)

$$\nabla \cdot \mathbf{v} = \partial_t \rho_n + \partial_m \mathbf{v}_m + \partial_p \mathbf{p}_p,$$

to give Equation (48).

\section{C1 1D solution used as test}

Consider an initial condition with velocity

$$\mathbf{v}(r) = A \sin(k \cdot r) \mathbf{b},$$

(C9)

where $A$ is the amplitude of the perturbation and $r = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z$ is a position vector. If the magnetic field, $\mathbf{B}$, and wave vector, $k$, are aligned, $k = k \mathbf{b}$, then the initial velocity is directed along the magnetic field and only varies in magnitude along this direction. The solution is therefore found by integrating Equation (C6). We find

$$v_\parallel(x, t) = A \sin(k \cdot r) e^{-\gamma t} \mathbf{b},$$

(C10)

with the decay rate

$$\gamma = \frac{4\nu}{3} k^2.$$

(C11)

The heating rate given by Equation (12) leads to a time evolution given by

$$\varepsilon(r, t) = \varepsilon_0 + \frac{\rho A^2}{2} \cos^2(k \cdot r) \left(1 - e^{-2\gamma t}\right),$$

(C12)

found by simple integration. Here we have assumed that the internal energy is initially constant in space with $\varepsilon(t = 0) = \varepsilon_0$.

\section{C2 General solution}

If there is variation in both parallel and perpendicular directions we can proceed by Fourier transforming Equations (C4) and (C5) by assuming that the domain is periodic. We obtain

$$\frac{\partial \hat{v}_\parallel}{\partial t} = -\frac{2}{3} \left(2 \hat{\theta}_\parallel \hat{v}_\parallel - \hat{\phi} \hat{v}_\perp - \hat{\phi} \hat{v}_\parallel \right),$$

(C4)

$$\frac{\partial \hat{v}_\perp}{\partial t} = -\frac{1}{3} \left(2 \hat{\phi}_\parallel \hat{v}_\parallel - \hat{\phi} \hat{v}_\perp - \hat{\phi} \hat{v}_\parallel \right),$$

(C5)

which are coupled equations. In the special case where there is only variation in the parallel direction Equation (C4) however reduces to

$$\frac{\partial \hat{v}_\parallel}{\partial t} = \frac{4}{3} \frac{\partial^2 \hat{v}_\parallel}{\partial \tau^2},$$

(C6)

and, conversely, Equation (C5) reduces to

$$\frac{\partial \hat{v}_\perp}{\partial t} = \frac{1}{3} \frac{\partial^2 \hat{v}_\perp}{\partial \tau^2},$$

(C7)

if there is only variation in the perpendicular direction. In these limiting cases the problem thus reduces to solving a 1D diffusion equation. This is useful for testing purposes as the 1D diffusion equation has a known analytical solution.

The entropy equation gives the following evolution equation

$$\frac{\partial \xi}{\partial t} = -\mathbf{v} \cdot \hat{\nabla} \hat{\xi} = \frac{(\Delta \rho)^2}{3 \nu \partial_\parallel}$$

(C8)

for the internal energy $\xi = p/(\gamma - 1)$, assuming that the initial internal energy is uniform.

\footnote{This transformation assumes that both $k_\parallel$ and $k_\perp$ are not zero. If one of them is zero, a solution can be found directly without the need for a transformation, i.e., as outlined above.}
The Cartesian solution is then \( \vec{b} = \hat{t}_n \vec{b} + \hat{\perp} n \) where \( n \) is a unit vector normal to \( \vec{b} \).

### C 2D solution used as test

The test presented in Section 4.2 has \( \vec{b} = (\vec{e}_x + \vec{e}_y)/\sqrt{2} \) with variation in \( x \) only. We thus have the special case \( k_{\parallel} = k_{\perp} = k/\sqrt{2} \) of the previous section for each mode \( n \). The solution given by Equations (C19) and (C20) was for a single mode. For a sum of modes we have

\[
v_{\parallel}(t) = \frac{2}{5} \sum_n \left[ \xi_n(0) - \xi_n(0) e^{-\gamma_n t} \right], \tag{C21}\]

\[
v_{\perp}(t) = \frac{4}{5} \sum_n \left[ \xi_n(0) + \xi_n(0) e^{-\gamma_n t} \right], \tag{C22}\]

where

\[
\gamma_n = \frac{5 v_{\perp}}{3} k_n^2. \tag{C23}\]

The variables \( \xi_n \) and \( \zeta_n \) are related to \( v_{\parallel,n} \) and \( v_{\perp,n} \) by \( \xi_n = v_{\parallel,n}/2 + v_{\perp,n} \) and \( \zeta_n = -2 v_{\parallel,n} + v_{\perp,n} \). Since \( v_{\parallel} = v_{\perp}/\sqrt{2} \) and \( v_{\perp} = -v_{\perp}/2 \) at \( t = 0 \) it follows that

\[
\xi_n(0) = -\frac{a_n}{2\sqrt{2}} \cos(k_n x), \tag{C24}\]

\[
\zeta_n(0) = -\frac{3a_n}{2\sqrt{2}} \cos(k_n x), \tag{C25}\]

where \( a_n \) are the Fourier coefficients for the initial \( y \)-velocity given in Equation (60). We thus find the solution for parallel and perpendicular velocity profiles

\[
v_{\parallel}(t) = -\sum_n \frac{1 - 6 e^{-\gamma_n t}}{5\sqrt{2}} a_n \cos(k_n x), \tag{C26}\]

\[
v_{\perp}(t) = -\sum_n \frac{2 + 3 e^{-\gamma_n t}}{5\sqrt{2}} a_n \cos(k_n x), \tag{C27}\]

which enables us to obtain \( v_{t}(t) = (v_{\parallel}(t) + v_{\perp}(t))/\sqrt{2} \) and \( v_{t}(t) = (v_{\parallel}(t) - v_{\perp}(t))/\sqrt{2} \) as given in Equations (70) and (71), respectively.

The pressure anisotropy for this solution can be found using Equation (C3) and the expressions for \( v_{\parallel} \) and \( v_{\perp} \). We find that the pressure anisotropy is given by Equation (73), i.e.,

\[
\Delta p(x, t) = \frac{3\rho v_{\parallel}^3}{2} \sum_{n=1}^{\infty} n_k a_n \sin(k_n x) \sinh(\gamma_n t). \tag{C28}\]

The associated heating rate is evaluated using Equation (C8) which yields

\[
\frac{\partial \varepsilon}{\partial t} = \frac{3\rho v_{\parallel}^3}{4} \left( \sum_n a_n k_n \sin(k_n x) \sinh(\gamma_n t) \right)^2 = \frac{9\rho v_{\parallel}^3}{16} \sum_n \sum_m a_n b_m \sqrt{\gamma_n \gamma_m} \sin(k_n x) \sin(k_m x) \sinh(\gamma_n \gamma_m t) \tag{C29}\]

The \( k_n k_m = \sqrt{\gamma_n \gamma_m b}/(5v_{\parallel}) \) was used to simplify the double sum in the last step. Integration of Equation (C29) with respect to time then gives \( \varepsilon(x, t) \) as written in Equation (74).

### APPENDIX D: INTERRUPTION BY THE FIREHOSE INSTABILITY

We derive a set of two coupled wave equations which model the Alfvén wave interruption test that we present in Section 4.5. We use the same geometry as in Squire et al. (2017a), i.e., we take the mean field to be in the \( z \)-direction, the perpendicular perturbation to be in the \( x \)-direction and variation to be in the \( z \)-direction only.

We have \( \vec{B} = B_{0} \vec{e}_z + \delta \vec{B} \) with \( \delta \vec{b} = \delta \vec{B}/B_0 = \delta \vec{e}_x \) and the velocity \( \vec{v} = \vec{v}_x \). The induction equation, Equation (4), then yields

\[
\frac{\partial \delta \vec{b}}{\partial t} = \frac{\partial \vec{v}}{\partial z}. \tag{D1}\]

The momentum equation becomes

\[
\frac{\partial \vec{v}}{\partial t} = \frac{\vec{v}}{\rho} \nabla \vec{p} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\Delta \rho}{1 + \delta b^2} \delta \vec{b} \right), \tag{D2}\]

where, since \( \rho \) is constant, Equation (9) yields

\[
\Delta p(z, t) = 3\rho v_{\parallel}^3 \frac{d\ln B}{dt}. \tag{D3}\]

As

\[
B(t) = B_0 \sqrt{1 + \delta b^2}, \tag{D4}\]

we find that

\[
\Delta p(z, t) = \frac{3\rho v_{\parallel}^3}{1 + \delta b^2} \frac{\partial \delta b}{\partial t} = \frac{3\rho v_{\parallel}^3}{1 + \delta b^2} \frac{\partial}{\partial z} \left( \frac{3\rho v_{\parallel}^3}{1 + \delta b^2} \frac{\partial}{\partial z} \delta b \right), \tag{D5}\]

where the induction equation was used in the last step. Combining the expression for \( \Delta p \) given by Equation (D5) and Equation (D2) gives us two coupled equations

\[
\frac{\partial \delta b}{\partial t} = \frac{\partial \vec{v}}{\partial z}, \tag{D6}\]

\[
\frac{\partial \vec{v}}{\partial t} = \frac{\vec{v}}{\rho} \nabla \vec{p} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{3\rho v_{\parallel}^3}{1 + \delta b^2} \frac{\partial}{\partial z} \delta b \right), \tag{D7}\]

for \( \vec{v} \) and \( \delta \vec{b} \). We solve these coupled equations with the same method as Squire et al. (2016); Squire et al. (2017a), i.e., spectral derivatives with the Fast-Fourier transform (FFT), hyper-viscous damping of grid scale noise with a \( k^b \) diffusion operator and a fourth order Runge-Kutta update in time.

Our coupled equations differ, albeit only slightly, from the ones in Squire et al. (2017a) because we include the \( 1/(1+\delta b^2) \) nonlinearity in Equation (D5) for \( \Delta p \). Squire et al. (2017a), who considered small perturbations deliberately only kept it in the momentum equation (Squire, private communication). The inclusion of the nonlinearity in the expression for \( \Delta p \) turns out to be necessary for a match between the solutions obtained here and with Athena and Arepo (shown in Fig. 6).

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