Radical semistar operations

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ABSTRACT
We introduce and study the set of radical semistar operations of an integral domain $D$. We show that their set is a complete lattice that is the join-completion of the set of spectral semistar operations, and we characterize when every radical operation is spectral (under the hypothesis that $D$ is radical coherent). When $D$ is a Prüfer domain such that every set of minimal prime ideals is scattered, we completely classify stable semistar operations.

ARTICLE HISTORY
Received 16 July 2022
Revised 26 November 2022
Communicated by Alberto Facchini

KEYWORDS
Scattered spaces; semistar operations; stable operations

2020 MATHEMATICS SUBJECT CLASSIFICATION
13A15; 13A18; 13F05; 13F30; 13G05

1. Introduction

Let $D$ be an integral domain. Semistar operations on $D$ are a class of closure operations on the set of $D$-submodules of the quotient field $K$ of $D$, defined by Okabe and Matsuda [14] as a generalization of the concept of star operation, originally introduced by Krull [12] and Gilmer [10, Chapter 32]. Semistar operations enjoy greater flexibility than star operations, making them a good tool to use in order to study several topics relative to the properties of ideals of $D$, as well as the properties of overrings of $D$. There are several subclasses of semistar operations that are particularly of interest, among which we cite finite type operations, eab operations (that are related to the valuation overrings of $D$, cf. [9] and [5, Section 4]) and spectral operations (related to the spectrum of $D$, cf. [2, 3, 8]). See Section 2 for a precise definition.

A semistar operation $\star$ is stable if it distributes over finite intersections, i.e., if $(I \cap J)^\star = I^\star \cap J^\star$ for all $D$-submodules $I, J$; in particular, every spectral semistar operations is stable. Stable operations are naturally connected to localizing systems [8] and singular length functions [18, Theorem 6.5 and subsequent discussion], meaning that there are canonical order-preserving bijections between the sets of stable operations, localizing systems and singular length functions on the same domain $D$ (see Section 7 for details); in particular, any classification of stable semistar operations classifies, as well, localizing systems and singular length functions. However, while spectral semistar operations can be easily classified through subsets of the spectrum of $D$ (see [8, Remark 4.5] and [7, Corollary 4.4]), the same does not hold for stable operations; indeed, a standard representations and a classification of stable operations have only been obtained in very specific situations, like for one-dimensional domains whose maximal space is scattered ([19]; see Section 2.3 below for the definition for scattered space) and for Prüfer domains such that every ideal has only finitely many minimal primes (see [15] and [18]).

In this paper, we study radical semistar operations, i.e., stable semistar operations such that, for every ideal $I$, $1 \in I^\star$ if and only if $1 \in \text{rad}(I)^\star$. While more general that spectral semistar operations, these closures do still retain a very strong link with the spectrum of $D$: indeed, we show that the set $\text{SStar}_{\text{rad}}(D)$ of radical semistar operations is the join-completion of the set $\text{SStar}_{\text{sp}}(D)$ of spectral semistar operations,
and in particular it is a complete lattice (Theorem 4.6). For rad-colon coherent domains (a large class of domain that includes domains with Noetherian spectrum and Prüfer and coherent domains), we also show that the set \( \text{SStar}_{\alpha}(D) \) depends uniquely on the spectrum of \( D \), in the sense that any two such domains with homeomorphic spectra have isomorphic set of radical operations (Theorem 5.3); this notably does not hold for the set \( \text{SStar}_{\alpha}(D) \) of stable semistar operations (see Remark 5.4).

In Section 6, we connect the study of radical operations with the use of the derived set and of scattered topological spaces (following [19–21]) to show that (under the hypothesis that \( D \) is rad-colon coherent) the two sets \( \text{SStar}_{\alpha}(D) \) and \( \text{SStar}_{\alpha}(D) \) coincide if and only if the space \( \text{Min}(I) \) of minimal ideals of \( D \) is scattered for every ideal \( I \). Specializing further to the case of Prüfer domains, we show that this property is enough to obtain a full classification of all stable operations of \( D \) by means of a standard representation (Theorem 6.7), generalizing the results obtained in [15] and [18] for the case where each \( \text{Min}(I) \) is finite; in particular, we show that for these Prüfer domains the set \( \text{SStar}_{\alpha}(D) \) depends only on the spectrum of \( D \) (as a topological space) and on which prime ideals are locally principal (Theorem 6.9). In particular, these results hold when the spectrum of \( D \) is countable.

In Section 7, we define concepts analogous to radical semistar operations in the context of localizing systems and length functions, and show that these natural definitions correspond to radical semistar operations. For almost Dedekind domains, this gives a new way to express a result proved in [21] for the class of SP-scattered domains, a generalization of SP-domains (a domain is an SP-domain if every ideal can be written as a product of radical ideals); if \( D \) is SP-scattered, then every stable semistar operation is radical (Proposition 7.5).

2. Preliminaries

Throughout the paper, \( D \) will denote an integral domain with quotient field \( K \), and \( F(D) \) will denote the set of \( D \)-submodules of \( K \). An overring of \( D \) is a ring between \( D \) and \( K \).

2.1. Semistar operations

A semistar operation on \( D \) is a map \( \star : F(D) \rightarrow F(D), I \mapsto I^\star \), such that, for every \( I, J \in F(D), x \in K \):
- \( I \subseteq I^\star \);
- if \( I \subseteq J \), then \( I^\star \subseteq J^\star \);
- \( (I^\star)^\star = I^\star \);
- \( (xI)^\star = x \cdot I^\star \).

A submodule \( I \) is said to be \( \star \)-closed if \( I = I^\star \). The set of \( \star \)-closed submodules uniquely determines \( \star \).

The set \( \text{SStar}(D) \) of the semistar operations on \( D \) has a natural partial order, where \( \star_1 \leq \star_2 \) if and only if \( I^\star_1 \subseteq I^\star_2 \) for every \( I \in F(D) \), or equivalently if every \( \star_2 \)-closed ideal is \( \star_1 \)-closed. Under this order, \( \text{SStar}(D) \) is a complete lattice: the infimum of a family \( \{ \star_a \}_{a \in A} \) is the map

\[
I \mapsto \bigcap_{a \in A} I^\star_a,
\]

while its supremum is the semistar operation \( \sharp \) such that a submodule \( I \) is \( \sharp \)-closed if and only if it is \( \star_a \)-closed for every \( a \in A \).

An ideal \( I \) of \( D \) is said to be a quasi-\( \star \)-ideal if \( I = I^\star \cap D \); if \( I \) is a prime quasi-\( \star \)-ideal, we say that \( I \) is a quasi-\( \star \)-prime. The set of quasi-\( \star \)-primes is called the quasi-spectrum of \( \star \), and is denoted by \( \text{QSpec}^\star(D) \).

A semistar operation \( \star \) is said to be of finite type if \( \star^\star = \bigcup \{ J^\star \mid J \subseteq I \text{ is finitely generated} \} \), for every \( I \in F(D) \). It is semi-finite (or quasi-spectral) if every quasi-\( \star \)-ideal is contained in a quasi-\( \star \)-prime; every semistar operation of finite type is semi-finite.

A very general way to define semistar operations is through overrings: a family \( \Lambda \) of overrings induces the semistar operation

\[
\star_\Lambda : I \mapsto \bigcap_{T \in \Lambda} IT.
\]
2.3. Derived sets and scattered spaces

When $\Delta$ is a family of localizations of $D$, we say that $\star$ is a spectral semistar operation. A spectral semistar operation can also be defined through a subset of the spectrum $\text{Spec}(D)$ of $D$: given a family $\Delta \subseteq \text{Spec}(D)$, we denote by $s_\Delta$ the semistar operation

$$s_\Delta : I \mapsto \bigcap_{P \in \Delta} ID_P.$$ 

Setting $\Delta^\downarrow := \{Q \in \text{Spec}(D) \mid Q \subseteq P \text{ for some } P \in \Delta\}$, we have that $Q\text{Spec}^{\downarrow}(D) = \Delta^\downarrow$, and that $s_\Delta = s_\Delta^\downarrow$; moreover, $s_\Delta = s_\Lambda$ if and only if $\Delta^\downarrow = \Lambda^\downarrow$ [8, Remark 4.5]. A spectral operation $s_\Delta$ is of finite type if and only if $\Delta$ is compact, with respect to the Zariski topology [7, Corollary 4.4].

A semistar operation is stable if $(I \cap J)^\star = I^\star \cap J^\star$ for every $I, J \in \mathcal{F}(D)$. Every spectral semistar operation is stable, while every semi-finite stable operation is spectral [1, Theorem 4]. There exist stable operations that are not spectral: for example, if $D$ is a one-dimensional valuation domain with nonprincipal maximal ideal, then the $\nu$-operation $I \mapsto (D : (D : I))$ is stable (since any two ideals are comparable), but not spectral (see for example [11, paragraph after Theorem 2.9]). The localizing system associated to $\star$ is [8, Section 2]

$$F^\star := \{I \text{ ideal of } D \mid I^\star \cap D = D\} = \{I \text{ ideal of } D \mid 1 \in I^\star\};$$

this set uniquely determines $\star$, in the sense that if $\star_1, \star_2$ are stable, then $\star_1 = \star_2$ if and only if $F^{\star_1} = F^{\star_2}$. More precisely, $\star_1 \leq \star_2$ if and only if $F^{\star_1} \subseteq F^{\star_2}$.

The set $\text{SStar}_\star(D)$ of spectral semistar operations of $D$ is closed by infimum, but not by supremum (see [6, Example 4.5] and Example 4.1 below); note, however, that $\text{SStar}_\star(D)$ is a complete lattice (see below the discussion after Corollary 4.3). On the other hand, the set $\text{SStar}_s(D)$ of stable operations is closed by both infima and suprema [16, Proposition 5.3].

2.2. Topologies on the spectrum

Let $\text{Spec}(D)$ denote the spectrum of $D$, i.e., the set of all prime ideals of $D$. We denote by $\mathcal{V}(I)$ and $\mathcal{D}(I)$, respectively, the closed and the open sets of the Zariski topology associated to an ideal $I$; i.e., $\mathcal{V}(I) := \{P \in \text{Spec}(D) \mid I \subseteq P\}$, while $\mathcal{D}(I) := \text{Spec}(D) \setminus \mathcal{V}(I)$.

The spectrum of a ring can also be endowed with two other topologies. The inverse topology is the topology whose subbasic open sets are those in the form $\mathcal{V}(I)$, as $I$ ranges among the finitely generated ideals of $D$; the constructible topology is the topology whose subbasic open sets are the $\mathcal{V}(I)$ and the $\mathcal{D}(I)$, for $I$ ranging among the finitely generated ideals of $D$. In particular, the constructible topology is finer than both the Zariski and the inverse topology, and, furthermore, it is Hausdorff. See [4, Chapter 1] for properties of the inverse and the constructible topologies.

If $I$ is an ideal of $D$ and $\text{Min}(I)$ denotes the set of minimal primes of $D$, the Zariski and the constructible topology agree on $\text{Min}(I)$ (by [4, Corollary 4.4.6(i)], applied to the spectral space $\mathcal{V}(I)$).

2.3. Derived sets and scattered spaces

Let $X$ be a topological space. A point $x \in X$ is isolated if $\{x\}$ is an open set; the set of non-isolated points of $X$ is called the derived set of $X$, and is denoted by $\mathcal{D}(X)$. Given an ordinal $\alpha$, we define the $\alpha$-th derived set as

$$\mathcal{D}^\alpha(X) := \begin{cases} 
\mathcal{D}(\mathcal{D}^\gamma(X)) & \text{if } \alpha = \gamma + 1; \\
\bigcap_{\beta < \alpha} \mathcal{D}^\beta(X) & \text{if } \alpha \text{ is a limit ordinal}.
\end{cases}$$

If $\mathcal{D}^\alpha(X) = \emptyset$ for some $\alpha$, the space $X$ is said to be scattered; equivalently, $X$ is scattered if and only if every nonempty open set has an isolated point. On the other hand, if $\mathcal{D}(X) = X$, then $X$ is said to be perfect.
3. Radical semistar operations

One of the most useful properties of semistar operations of finite type is semifiniteness: every quasi-\(\star\)-ideal is contained in a quasi-\(\star\)-prime, and thus we can obtain useful information on \(\star\) from its quasi-spectrum \(\text{QSpec}^\star(D)\). The following is a weaker form of this property.

**Definition 3.1.** We say that a semistar operation \(\star\) on \(D\) is **quasi-radical** if, whenever \(1/\in I^\star\) for some ideal \(I\) of \(D\), then \(1/\in \text{rad}(I^\star)\).

We collect in the next few propositions the main properties of quasi-radical semistar operations.

**Proposition 3.2.** Let \(D\) be an integral domain and \(\star\) be a semistar operation on \(D\). If \(\star|_{F(D^\star)}\) is quasi-radical as a semistar operation on \(D^\star\), then \(\star\) is quasi-radical.

**Proof.** Let \(I\) be an ideal of \(D\) such that \(1/\in I^\star\). Then, \(1/\in (ID^\star)\), and thus, by hypothesis, \(1/\in \text{rad}(ID^\star)\); hence, \(1/\in \text{rad}(I^\star)\). It follows that \(\star\) is quasi-radical.

**Proposition 3.3.** Let \(D\) be an integral domain and \(\star\) be a semistar operation on \(D\).

(a) If \(\star\) is semi-finite, then it is quasi-radical.

(b) If \(\star\) is of finite type, then it is quasi-radical.

(c) If \(\star\) is induced by overrings, then it is quasi-radical.

(d) If \(\star\) is spectral, then it is quasi-radical.

**Proof.** Suppose \(\star\) is semi-finite, and let \(I\) be an ideal of \(D\) such that \(1/\in I^\star\). Then, \(J := I^\star \cap D\) is a quasi-\(\star\)-ideal such that \(1/\in J^\star\). Since \(\star\) is semi-finite, there is a quasi-\(\star\)-prime ideal \(P\) containing \(J\); thus, \(1/\in P^\star \supseteq \text{rad}(J)^\star \supseteq \text{rad}(I)^\star\). Therefore, \(\star\) is quasi-radical.

The next three points follow from the fact that every semistar operation of finite type is semi-finite, as well as any semistar operation induced by overrings, and that any spectral semistar operation is induced by overrings.

**Proposition 3.4.** Let \(\{\star_\alpha\}_{\alpha \in A}\) be a set of quasi-radical semistar operations on \(D\). Then, \(\inf_{\alpha \in A} \star_\alpha\) is quasi-radical.

**Proof.** Let \(\star := \inf_{\alpha \in A} \star_\alpha\), and let \(I\) be an ideal of \(D\) such that \(1/\in I^\star\). Since \(I^\star = \bigcap_{\alpha \in A} I^\star_\alpha\), it follows that there is a \(\beta \in A\) such that \(1/\in I^\star_\beta\). Since \(\star_\beta\) is quasi-radical, then \(1/\in \text{rad}(I)^\star_\beta\), and thus also \(1/\in \text{rad}(I)^\star\). Hence \(\star\) is quasi-radical.

The previous proposition does not extend to the supremum of a family of quasi-radical operations, as the next example shows.

**Example 3.5.** Let \(D\) be a Prüfer domain of dimension 1 such that \(\text{Max}(D) = \{P, Q_0, Q_1, \ldots, Q_n, \ldots\}\) is countable and with a single non-isolated point, \(P\); suppose also that \(DP\) is not discrete. For every \(n \in \mathbb{N}\), let \(T_n := \bigcap_{i>n} D_{Q_i}\); then, \(T_n\) is a Prüfer domain whose maximal ideals are the extensions of \(Q_i\) (for \(i \geq n\)) and of \(P\); in particular, \(\bigcup_n T_n = DP\).

Recall that a **fractional ideal** of a domain \(T\) is an \(I \in F(T)\) such that \(dI \subseteq T\) for some \(d \in K, d \neq 0\). For every \(n\), let \(\sharp_n\) and \(\star_n\) be the semistar operations defined by

\[
I^\sharp_n := \begin{cases} IT_n & \text{if } IT_n \text{ is a fractional ideal over } T_n \\ K & \text{otherwise,} \end{cases}
\]

and
Proposition 3.4 and stable since every \( \star_n \) is quas-radical. Let \( D \) be an integral domain. Then, the set \( \star_n \) is stable since every \( \star_n \) is quas-radical. Let \( D \) be an integral domain and let \( D \star_n \) to an overring of \( D \). Moreover, if \( t \in D_p \), then \( t \in T_n \) for some \( n \), and thus \( t \in D^{\star_n} \subseteq D^\star \). Hence, \( D^\star = D_p \). For every \( n \), \( PD_p \) is not a fractional ideal over \( T_n \), and thus

\[
(PD_p)^\star = K \cap (PD_p)^{vp} = K \cap D_p = D_p.
\]

Hence,

\[
P^\star = (PD)^\star = (PD^\star)^\star = (PD_p)^\star = D_p.
\]

On the other hand, if \( L \neq P \) is a \( P \)-primary ideal, then \( (LD_p)^{vp} = LD_p \); hence, \( LD_p \) is \( \star \)-closed and thus \( L^\star \subseteq LD_p \cap D \), so that \( 1 \notin L^\star \) while \( 1 \in P^\star = \text{rad}(L)^\star \). Therefore, \( \star \) is not quasi-radical.

The main problem of the previous example is that the restriction of a quasi-radical operation on \( D \) to an overring of \( D \) is not quasi-radical (as it happens for \( \star \mid_{F(D_p)} \)); this in turn is due to the fact that the property of being quasi-radical depends only on the ideals of \( D \), rather than on all \( D \)-submodules of \( K \). For this reason, we are only interested in the following subclass of semistar operations.

Definition 3.6. We say that a semistar operation \( \star \) on \( D \) is radical if it is quasi-radical and stable.

Lemma 3.7. Let \( \star \) be a radical semistar operation, and suppose that \( T \) is an overring of \( D \). Then \( \star \mid_{F(T)} \) is radical.

Proof. Let \( I \) be a \( T \)-ideal such that \( 1 \notin I^\star \). Then, \( 1 \notin (I \cap D)^\star \), and since \( \star \) is radical we have \( 1 \notin (\text{rad}(I \cap D))^\star \). However, \( \text{rad}(I \cap D) = \text{rad}(I) \cap D \); hence

\[
1 \notin \text{rad}(I \cap D) = (\text{rad}(I) \cap D)^\star = (\text{rad}(I))^\star \cap D^\star.
\]

Thus \( 1 \notin (\text{rad}(I))^\star \) and so \( \star \mid_{F(T)} \) is radical, as claimed. \( \square \)

Proposition 3.8. Let \( D \) be an integral domain and let \( \star \) be a radical semistar operation on \( D \) such that \( D = D^\star \). Let \( J \) be an ideal of \( D \) such that \( J = J^\star \). Then, \( \text{rad}(J)^\star = \text{rad}(J) \).

Proof. Let \( s \in \text{rad}(J)^\star \), and let \( t \in s^{-1} \text{rad}(J) \cap D \). Then, \( st \in \text{rad}(J) \), and thus there is an \( n \) such that \( s^n t^n \in J \), i.e., \( t^n \in s^{-n}J \cap D \). Hence \( t \in \text{rad}(s^{-n}J \cap D) \) and so \( s^{-1} \text{rad}(J) \cap D \subseteq \text{rad}(s^{-n}J \cap D) \).

Since \( s \in \text{rad}(J)^\star \), we have \( 1 \notin s^{-1} \text{rad}(J)^\star \); hence also \( 1 \in \text{rad}(s^{-n}J \cap D)^\star \). Since \( \star \) is radical, it follows that \( 1 \in (s^{-n}J \cap D)^\star \); thus \( 1 \in s^{-n}J^\star \) and \( s^n \in J^\star = J \). Therefore, \( s \in \text{rad}(J) \), and \( \text{rad}(J)^\star = \text{rad}(J) \). \( \square \)

Theorem 3.9. Let \( D \) be an integral domain. Then, the set \( S\Star_{\text{rad}}(D) \) of radical semistar operations is a complete sublattice of \( S\Star(D) \).

Proof. Let \( \{\star_\alpha\}_{\alpha \in A} \) be a family of radical semistar operations. Then, its infimum is quasi-radical by Proposition 3.4 and stable since every \( \star_\alpha \) is stable, and thus \( S\Star_{\text{rad}}(D) \) is closed by infima. Let \( \star \) be the supremum of \( \{\star_\alpha\}_{\alpha \in A} \).

Let \( T := D^\star \); then, \( T \) is \( \star_\alpha \)-closed for every \( \alpha \). By Proposition 3.2, it suffices to show that \( \star \mid_{F(T)} \) is radical; furthermore, by Lemma 3.7, each \( \star_\alpha \mid_{F(T)} \) is radical. Therefore, without loss of generality, we can actually suppose that \( T = D \), i.e., that \( D \) is \( \star_\alpha \)-closed for every \( \alpha \).

Let \( J \) be an ideal of \( D \) such that \( 1 \notin J^\star \). Let \( L := J^\star \); then, \( L \) is an ideal of \( D \) that is \( \star_\alpha \)-closed for every \( \alpha \), and thus by Proposition 3.8 also \( \text{rad}(L) \) is \( \star_\alpha \)-closed for every \( \alpha \); thus, \( \text{rad}(L) = \text{rad}(L)^\star \). In particular, \( 1 \notin \text{rad}(L)^\star \); the claim now follows from the fact that \( \text{rad}(J) \subseteq \text{rad}(L) \). \( \square \)
4. Radical operations as a completion

By Proposition 3.3, each spectral semistar operation \( s_\Delta \) is radical; in this section, we explore the link between these two classes of semistar operations. Following [6, Example 4.5], we first give an example of a radical operation that is not spectral.

**Example 4.1.** Let \( \bar{\mathbb{A}} \) be the ring of all algebraic integers, i.e., the integral closure of \( \mathbb{Z} \) in \( \overline{\mathbb{Q}} \). Then, \( \bar{\mathbb{A}} \) is a Bézout domain (every finitely generated ideal is principal) and, for every maximal ideal \( P \), we have that \( \bar{\mathbb{A}} = \bigcap \{ \mathbb{A}Q \mid Q \in \text{Max}(\bar{\mathbb{A}}) \setminus \{P\} \} \). Hence, for each \( P \) the spectral operation \( \sharp(P) := s_{\text{Max}(\bar{\mathbb{A}}) \setminus \{P\}} \) closes \( \bar{\mathbb{A}} \), and thus the supremum \( \star \) of all the \( \sharp(P) \) closes \( \bar{\mathbb{A}} \) too, and thus it closes every principal ideal (since \( (x\bar{\mathbb{A}})^* = x \cdot \bar{\mathbb{A}} = x \cdot \bar{\mathbb{A}} \).

As the supremum of a family of radical operations, \( \star \) is itself radical. However, for every \( P \)-primary ideal \( Q \), we have \( Q^{\sharp(P)} = \mathbb{A} \); therefore, \( \text{QSpec}^\star(D) \) contains only the zero ideal. In particular, every \( \star \)-spectral, it should be equal to \( s_{(0)} \), and in particular \( 1 \) would belong to ideal \( I \) for every nonzero ideal \( \star \), contradicting the fact that principal ideals are closed. Hence \( \star \) is radical, but not spectral.

The following proposition characterizes which radical operations are spectral.

**Proposition 4.2.** Let \( \star \) be a radical semistar operation on \( D \). Then, \( \star \) is spectral if and only if, for every radical ideal \( I \),

\[
I^\star \cap D = \bigcap \{ P \mid P \in \mathcal{V}(I) \cap \text{QSpec}^\star(D) \}.
\]

**Proof.** Suppose first that \( \star \) is spectral, say \( \star = s_\Delta \) with \( \Delta = \Delta^\downarrow \). For every \( P \in \Delta \), the ideal \( ID_P \) is radical, and its minimal primes are the minimal primes of \( I \) contained in \( P \); all of them belong to \( \Delta \), and thus they are all in \( \mathcal{V}(I) \cap \text{QSpec}^\star(D) \). Hence,

\[
I^\star \cap D = \bigcap_{P \in \Delta} \{ QD_P \cap D \mid Q \in \text{Min}(I), Q \subseteq P \} = \bigcap_{Q \in \text{Min}(I) \cap \Delta} Q.
\]

The claim follows.

Conversely, suppose that the equality holds, and let \( \Delta := \text{QSpec}^\star(D) \). For every \( P \in \Delta \), \( PD_P \) is \( \star \)-closed, and thus \( \star \) is the identity on \( F(D_P) \); it follows that \( I^\star \subseteq ID_P \) for every \( P \in \Delta \), and thus \( \star \subseteq s_\Delta \).

Suppose that \( \star < s_\Delta \); then, there is an ideal \( I \) of \( D \) such that \( I^\star \subseteq I^{s_\Delta} \). Let \( x \in I^{s_\Delta} \setminus I^\star \) and let \( J := (I : ID x) \). Since \( \star \) is stable, we have \( 1 \in I^{s_\Delta} \) while \( 1 \notin J^\star \); since both \( s_\Delta \) and \( \star \) are radical, it follows that \( 1 \in \text{rad}(J)^{s_\Delta} \) while \( 1 \notin \text{rad}(J)^\star \). However, by the hypothesis and the first part of the proof, \( \text{rad}(J)^{\Delta} \cap D = \text{rad}(J)^\star \cap D \); this is a contradiction, and thus \( \star \) must be equal to \( s_\Delta \). In particular, \( \star \) is spectral, as claimed.

**Corollary 4.3.** Let \( D \) be an integral domain such that every ideal has only finitely many minimal primes. Then, every radical semistar operation is spectral.

**Proof.** Let \( I \) be a radical ideal and \( P_1, \ldots, P_n \) be its minimal primes. Then, \( I = P_1 \cap \cdots \cap P_n \), and thus \( I^\star = P_1^\star \cap \cdots \cap P_n^\star \). Since \( \star \) is stable, for each \( i \) the ideal \( P_i^\star \cap D \) is either equal to \( P_i \) or to \( D \) [15, Lemma 3.1] hence, \( I^\star \cap D \) is equal to the intersection of the minimal primes that are quasi-\( \star \)-ideals. By Proposition 4.2, \( \star \) is spectral.

The following result is a variant of [15, Lemma 3.1].

**Proposition 4.4.** Let \( \star \) be a stable semistar operation, and let \( I \) be a radical ideal of \( D \). Then, \( I^\star \cap D \) is either \( D \) or a radical ideal.
Proof. Suppose \( J^* \cap D \neq D \). Let \( s \in D \) be such that \( s^n \in J^* \) for some integer \( n \). Let \( L := s^{-1}J \cap D: \) since \( \ast \) is stable, \( 1 \in L^* \). We claim that \( s^{-1}J \cap D = \text{rad}(L) \). Indeed, if \( x \in s^{-1}J \cap D \) then \( x \in J \) and thus also \( s^nx \in J \), i.e., \( x \in s^{-n}J \cap D = L \subseteq \text{rad}(L) \). On the other hand, if \( x \in \text{rad}(L) \), then \( x^k \in s^{-n}J \) for some \( k \), and thus \( x^k s^n \in J \). Since \( x, s \in D \), we have \( x^k s^n \in J \), where \( N := \max(n, k) \); since \( J \) is radical, it follows that \( x^k \in J \), that is, \( x \in s^{-1}J \cap D \). Thus \( s^{-1}J \cap D = L = \text{rad}(L) \).

Since \( 1 \in L^* \), it follows that \( 1 \in (s^{-1}J)^* \), that is, \( s \in J^* \). Hence \( J^* \) is radical, as claimed. \( \square \)

Proposition 4.5. Let \( D \) be a domain, let \( I \) be a radical ideal of \( D \) and \( \Delta = \Delta_\downarrow \subseteq \text{Spec}(D) \). Then, the following are equivalent:

(i) \( I = I^{\Delta \downarrow} \cap D \);
(ii) \( V(I) \cap \Delta \) is dense in \( V(I) \);
(iii) \( \text{Min}(I) \cap \Delta \) is dense in \( \text{Min}(I) \).

Proof. By Proposition 4.4, the ideal \( I := I^{\Delta \downarrow} \cap D \) is a radical ideal of \( D \) containing \( I \); therefore, \( V(I) \subseteq V(I) \) is a closed set, and \( V(J) \) contains \( V(I) \cap \Delta \) since, if \( P \in V(I) \cap \Delta \), then \( J = I^{\Delta \downarrow} \cap D \subseteq P^{\Delta \downarrow} \cap D = P \). In particular, if \( V(I) \cap \Delta \) is dense then it must be \( I = J \). On the other hand, if \( V(I) \cap \Delta \) is not dense, then there is an ideal \( L \) such that \( \Delta \cap V(L) \subseteq V(I) \); thus, \( ID_P = LDP \) for every \( P \in \Delta \), and \( J \supseteq L \), so that \( I \neq J \). Thus, the first two conditions are equivalent.

If \( \text{Min}(I) \cap \Delta \) is dense in \( \text{Min}(I) \), then \( \text{Min}(I) \) is contained in the closure of \( V(I) \cap \Delta \); then, \( V(I) \cap \Delta \) is dense since \( \text{Min}(I) \) is dense in \( V(I) \). Conversely, suppose \( V(I) \cap \Delta \) is dense in \( V(I) \) and take \( P \in \text{Min}(I) \). For every open set \( \Omega \) meeting \( V(I) \), \( \Omega \cap \Delta \cap V(I) \) is nonempty; if \( Q \) belongs to the intersection, then \( \Omega \cap \Delta \cap \text{Min}(I) \) contains the minimal primes of \( I \) contained in \( Q \). Hence, \( \Delta \cap \text{Min}(I) \) is dense in \( \text{Min}(I) \). Thus also the last two conditions are equivalent. \( \square \)

Let \( D \) be an integral domain. The space \( \text{SS}^*_{\text{rad}}(D) \) of spectral semistar operation on \( D \) is a complete lattice: indeed, let \( X : = \{ s_{\Delta_\alpha} \mid \alpha \in A \} \) be a subset of \( \text{SS}^*_{\text{rad}}(D) \) with \( \Delta_\alpha = \Delta_{\alpha \uparrow} \). Then, setting \( \Delta_{\uparrow} = \bigcup_\alpha \Delta_\alpha \) and \( \Delta_{\cap} = \bigcap_\alpha \Delta_\alpha \), it is easy to see that the infimum of \( X \) in \( \text{SS}^*_{\text{rad}}(D) \) is \( s_{\Delta_{\uparrow}} \) and that its supremum is \( s_{\Delta_{\cap}} \).

However, while \( s_{\Delta_{\uparrow}} \) is also the infimum of \( X \) as a subset of \( \text{SS}(D) \), the same does not hold for \( s_{\Delta_{\cap}} \) (see Example 4.1). We now want to prove that the set \( \text{SS}^*_{\text{rad}}(D) \) of radical semistar operations is the join-completion of \( \text{SS}^*_{\text{rad}}(D) \) in \( \text{SS}(D) \). In particular, the construction of Example 4.1 is the only way to obtain non-spectral radical semistar operations.

Theorem 4.6. Let \( D \) be an integral domain. Then:

(a) \( \text{SS}^*_{\text{rad}}(D) \) is join-dense in \( \text{SS}^*_{\text{rad}}(D) \);
(b) \( \text{SS}^*_{\text{rad}}(D) \) is the completion of \( \text{SS}^*_{\text{rad}}(D) \) in \( \text{SS}(D) \).

Proof. Since \( \text{SS}^*_{\text{rad}}(D) \) is a complete sublattice of \( \text{SS}(D) \) (Theorem 3.9), we only need to prove that every radical semistar operation is the supremum of a family of spectral operations.

Fix thus \( \ast \in \text{SS}^*_{\text{rad}}(D) \). Let \( \Delta \subseteq \text{Spec}(D) \) be such that \( \Delta \cap V(I) \) is dense in \( V(I) \) for every radical ideal \( I \) such that \( I = I^\ast \cap D \). Then, \( s_{\Delta} \leq \ast \); indeed, if \( J \) is an ideal such that \( 1 \in J^{\Delta \ast} \) and \( 1 \neq J^\ast \), then \( \Delta \cap V(J^{\ast \downarrow}) \) would be dense in \( V(J^{\ast \downarrow}) \), and thus by Proposition 4.5 \( J^{\ast \downarrow} \cap D \) would be quasi-\( s_{\Delta} \)-closed, against the fact that \( 1 \in J^{\Delta \ast} \). Hence, \( s_{\Delta} \leq \ast \). Let \( \sharp \) be the supremum of all such \( s_{\Delta} \); by construction, \( \sharp \leq \ast \).

We claim that \( \ast = \sharp \). Let \( J \) be a proper radical ideal: if \( 1 \in J^\ast \), then \( 1 \in J^\ast \) since \( \ast \leq \sharp \). Suppose that \( 1 \in J^\ast \). We claim that \( D(J) \cap V(I) \) is dense in \( V(I) \) for every radical ideal \( I \) such that \( I = I^\ast \cap D \). If not, there is a \( P \in V(I) \) that is not in the closure of \( D(L) \cap V(I) \); hence, there is a radical ideal \( L \) such that \( P \in D(L) \) and \( D(L) \cap D(J) \cap V(I) = \emptyset \). Since \( D(L \cap D(J) = D(L \cap J) \), it follows that \( D(L \cap J) \cap V(I) = \emptyset \), and thus \( V(I) \subseteq V(L \cap J) \). Thus, \( L \cap J \subseteq I \), and \( L^\ast \cap J^\ast = (L \cap J)^\ast \subseteq I^\ast \). Hence

\[
(J^\ast \cap D) \cap (L^\ast \cap D) \subseteq I^\ast \cap D = I.
\]
By hypothesis, $J'$ contains 1; hence, $J' \cap D = D$ and $L' \cap D \subseteq I$, so that $L \subseteq I$. In particular, $\mathcal{D}(L) \subseteq \mathcal{D}(I)$; it follows that $\mathcal{D}(L) \cap \mathcal{V}(I) = \emptyset$, against the hypothesis that $P \in \mathcal{D}(L) \cap \mathcal{V}(I)$. Therefore, $\mathcal{D}(J) \cap \mathcal{V}(I)$ is dense in $\mathcal{V}(I)$ for all radical ideal $I$ such that $I = I' \cap D$; thus, $s_{\mathcal{D}(I)}$ is one of the spectral operations used to define $\sharp$; hence, $s_{\mathcal{D}(I)} \leq \sharp$. It follows that $1 \in J^{s_{\mathcal{D}(I)}} \subseteq J^\sharp$. Therefore, $1 \in J^\sharp$ if and only if $1 \in J^\sharp$; since $\star$ and $\sharp$ are stable, it follows that $\star = \sharp$, as claimed, and $\star$ is in the completion of $\text{SSStar}_{\text{sp}}(D)$.

5. Isomorphic sets of radical operations

Let $D_1, D_2$ be two integral domains. If $\phi : \text{Spec}(D_1) \longrightarrow \text{Spec}(D_2)$ is an order isomorphism, then $\phi$ induces an order isomorphism $\Phi : \text{SSStar}_{\text{sp}}(D_1) \longrightarrow \text{SSStar}_{\text{sp}}(D_2)$ by setting $\Phi(s_\Delta) = s_{\phi(\Delta)}$ for every $\Delta \subseteq \text{Spec}(D_1)$. However, $\Phi$ does not, in general, extend to a similar isomorphism between the set of radical semistar operations, for example because it may be $\text{SSStar}_{\text{sp}}(D_1) = \text{SSStar}_{\text{rad}}(D_1)$ while $\text{SSStar}_{\text{sp}}(D_2) \neq \text{SSStar}_{\text{rad}}(D_2)$ (take for example $D_1 := K[X]$ and $D_2 := \mathbb{A}$, where $K$ is a field of the same cardinality of $\text{Max}(\mathbb{A})$).

In this section, we extend this result to radical operations by using the Zariski topology. We work in a particular class of domains: we say that a domain is rad-colon coherent if, for every $x \in K$, the radical of the ideal $(D :_D x)$ is the radical of a finitely generated ideal. This property is linked with the relationship between the Zariski, inverse and constructible topology of $\text{Spec}(D)$ and the Zariski, inverse and constructible topology of $\text{Over}(D)$. Every Noetherian domain (or, more generally, every domain with Noetherian spectrum) is rad-colon coherent; likewise, every Prüfer domain and every coherent domain are rad-colon coherent, as well as every polynomial ring in finitely many variables over a Prüfer domain. See [17] for applications of this property and for an example of a domain that is not rad-colon coherent.

In our context, the reason why we use this notion is essentially the following lemma.

**Lemma 5.1.** Let $D$ be a rad-colon coherent domain and let $I$ be a radical ideal. Define $T := \bigcap \{P_D | P \in \text{Min}(I)\}$. If $\star$ is a radical semistar operation such that $I = I^\star \cap D$, then $T^\star = T$ and $(IT)^\star = IT$.

**Proof.** Suppose first that $\star = s_\Delta$ is spectral, with $\Delta = \Delta^\downarrow$. Then, by Proposition 4.5, $\Delta \cap \mathcal{V}(I)$ is dense in $\mathcal{V}(I)$ and $\Delta \cap \text{Min}(I)$ is dense in $\text{Min}(I)$, with respect to the Zariski topology. By [4, Corollary 4.4.6(iii)], the Zariski and the constructible topology agree on $\text{Min}(I)$; hence, $\Delta \cap \text{Min}(I)$ is dense in $\text{Min}(I)$ also with respect to the constructible topology.

Let $x \in T^\star$, and let $J := (D :_D x) = x^{-1}D \cap D$. We claim that $\mathcal{V}(J) \cap \text{Min}(I) \cap \Delta = \emptyset$. Indeed, let $P \in \text{Min}(I) \cap \Delta$. Since $x \in T^\star \subseteq D_P^\star$, we have $1 \in (x^{-1}D_P)^\star$, and thus $1 \in (JD_P)^\star$; however, if $P \in \mathcal{V}(J)$ then $(JD_P)^\star \subseteq (PD_P)^\star = PD_P$ since $P \in \Delta$. Therefore, $\mathcal{V}(J) \cap \text{Min}(I) \cap \Delta = \emptyset$, and thus $\text{Min}(I) \cap \Delta \subseteq \mathcal{D}(J)$. Since $D$ is rad-colon coherent, $\text{rad}(J)$ is the radical of a finitely generated ideal, and thus $\mathcal{D}(J)$ is a closed subset, with respect to the constructible topology; thus $\mathcal{D}(J) \cap \text{Min}(I)$ is closed in $\text{Min}(I)$. Since $\text{Min}(I) \cap \Delta$ is dense in $\text{Min}(I)$, it follows that $\mathcal{D}(J) \cap \text{Min}(I)$ must be equal to the whole of $\text{Min}(I)$, that is, $\mathcal{V}(J) \cap \text{Min}(I) = \emptyset$. Thus, $JD_P = DP$ for every $P \in \text{Min}(I)$, and $x \in T$. Hence, $T^\star = T$.

This also implies that $(IT)^\star$ is a radical ideal of $T$ contained in $PT$ for every $P \in \text{Min}(I)$. Hence $(IT)^\star = IT$, as claimed.

Suppose now that $\star$ is any radical operation. By Theorem 4.6, $\star$ is the supremum of a family $Y$ of spectral semistar operation. For each $\sharp \in Y$, we have $\sharp \leq \star$, and thus $I = I^\sharp \cap D$; by the previous part of the proof, $T^\sharp = T$ and $(IT)^\sharp = IT$. Hence, also $T^\star = T$ and $(IT)^\star = IT$, as claimed.

**Proposition 5.2.** Let $D$ be a rad-colon coherent domain and let $I$ be a radical ideal. Let $Y$ be a family of radical semistar operations and let $\sup : Y$. If $I = I^\star \cap D$ for every $\star \in Y$, then $I = I^\sharp \cap D$.

**Proof.** Let $T := \bigcap \{P_D | P \in \text{Min}(I)\}$. By Lemma 5.1, $(IT)^\star$ is closed by every $\star \in Y$, and thus also $(IT)^\sharp$ is closed. Then, $I^\sharp \cap D \subseteq (IT)^\sharp \cap D = I$, and thus $I = I^\sharp \cap D$. 


Theorem 5.3. Let $D_1, D_2$ be rad-colon coherent integral domains, and suppose that there is a homeomorphism $\phi : \text{Spec}(D_1) \rightarrow \text{Spec}(D_2)$. Then, there is an order isomorphism

$$\Phi : \text{Star}_{\text{rad}}(D_1) \rightarrow \text{Star}_{\text{rad}}(D_2)$$

such that $\Phi(s_\Delta) = s_{\phi(\Delta)}$ for every $\Delta \subseteq \text{Spec}(D_1)$.

Proof. Let $X_i := \text{Star}_{\text{sp}}(D_i)$ and $Y_i := \text{Star}_{\text{rad}}(D_i)$ for $i = 1, 2$.

By Theorem 3.9, $Y_1$ is a join-completion of $X_1$; hence, we can consider $Y_1$ as a sublattice of the set $\mathcal{L}(X_1)$ of lower sets of $X_1$ by the map $\epsilon_1$, defined by $\epsilon_1(y) = \{x \in X_1 \mid x \leq y\}$ for every $y \in Y_1$. In particular, $\epsilon_1(x) = \{x\}^\downarrow$ for every $x \in X_1$. Likewise, we can consider $Y_2$ as a sublattice of $\mathcal{L}(X_2)$ through a map $\epsilon_2$ defined analogously.

The map

$$\Phi : X_1 \rightarrow X_2,$$

$$s_\Delta \mapsto s_{\phi(\Delta)}$$

is an order isomorphism; thus, it can be extended to a map $\widetilde{\Phi}$ between $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$, that remains an order isomorphism. We claim that $\widetilde{\Phi}(\epsilon_1(Y_1)) = \epsilon_2(Y_2)$, and to do so it is enough to prove that, if $A \subseteq X_1$, then the supremum $\sup_{\epsilon_1} A$ in $Y_1$ (that is, the supremum of $A$ as a semistar operation) is spectral if and only if $\sup_{\epsilon_2} \Phi(A)$ is spectral.

Suppose first that $\star := \sup_{\epsilon_1} A$ is not spectral, and let $\sharp = s_\Delta$ be the supremum of $A$ in $X_1$. Let $\ast$ and $\sharp'$ be, respectively, the supremum of $\Phi(A)$ in $Y_2$ and $X_2$. By construction, $\ast < \sharp'$, and thus there is a radical ideal $I$ such that $I = \ast \cap D_1$ while $I^\sharp = D_2^\sharp$. Let now $J$ be the radical ideal such that $\mathcal{V}(J) = \phi(\mathcal{V}(I))$; we claim that $J = J^\ast \cap D$ while $J^\sharp = D_2^\sharp$.

Indeed, if $s_\Delta \in \Phi(A)$, then $\phi^{-1}(\Delta) \cap \mathcal{V}(I)$ is dense in $\mathcal{V}(I)$, and thus $\Delta \cap \mathcal{V}(J)$ is dense in $\mathcal{V}(J)$; since $D_2$ is rad-colon coherent, by Proposition 5.2 $J = J^\sup \Phi(A) \cap D_2$, i.e., $J = J^\ast \cap D_2$. On the other hand, $\sharp = s_\Delta$ for some $\Delta$ such that $\Delta \cap \mathcal{V}(I) = \emptyset$; hence, $\sharp' = \phi(\sharp) = \phi(s_\Delta) = s_{\phi(\Delta)}$, where $\phi(\Delta) \cap \mathcal{V}(J)$ is empty. Hence, $J^\sharp = D_2^\sharp$. Thus, $\ast \neq \sharp'$, and $\sup_{\epsilon_2} \Phi(A)$ is not spectral.

The opposite implication follows by applying the same reasoning to the homeomorphism $\phi^{-1}$ (which induces the map $\Phi$ on the sets of spectral semistar operations).

Therefore, $\Phi$ restricts to an isomorphism between $\epsilon_1(Y_1)$ and $\epsilon_2(Y_2)$; since $Y_i \simeq \epsilon_i(I_i)$ for $i = 1, 2$, it follows that $Y_1 = \text{Star}_{\text{rad}}(D_1)$ and $Y_2 = \text{Star}_{\text{rad}}(D_2)$ are isomorphic, as claimed.

Remark 5.4. Theorem 5.3 does not hold for the set of all stable operations; indeed, if $D_1$ is a DVR and $D_2$ is a one-dimensional non-discrete valuation ring, then $\text{Spec}(D_1) \simeq \text{Spec}(D_2)$ (since they both contain two prime ideals) but $\text{Star}_{\ast}(D_1)$ contains only two semistar operations (the identity and $s_{(0)}$), while $\text{Star}_{\ast}(D_2)$ has three elements (the two of the previous case and the $v$-operation). We shall see in the next section (Theorem 6.9) how to generalize Theorem 5.3 to (some) Prüfer domains.

6. When every radical operation is spectral

We have seen that, in general, not every radical semistar operation is spectral, although the two sets are equal when every ideal has only finitely many minimal primes (Corollary 4.3). In this section, we characterize when the two sets are equal for rad-colon coherent domains; specializing to Prüfer domain, we also show that under some hypothesis we can obtain a standard representation of all stable operations.

We start with two topological lemmas.

Lemma 6.1. Let $D$ be an integral domain and $I$ a radical ideal that is not prime. Then, $\text{Min}(I)$ is not perfect if and only if there are a prime ideal $Q$ and a radical ideal $J \neq I$ such that $I = Q \cap J$. 


\textbf{Proposition 6.4.} Let $D$ be a domain such that $\text{Spec}(D)$ is countable. Then, $D$ is min-scattered.

\textbf{Proof.} The space $\text{Spec}(D)$, endowed with the constructible topology, is Hausdorff, compact and countable, and thus scattered [13]. Therefore, for every ideal $I$, $\text{Min}(I)$ is scattered, with respect to the constructible topology; however, on each $\text{Min}(I)$ the constructible and the Zariski topology coincide. Hence $D$ is min-scattered.

\textbf{Theorem 6.5.} Let $D$ be a rad-colon coherent domain. Then, the following are equivalent:

(i) $D$ is min-scattered;
(ii) every radical semistar operation is spectral.

\textbf{Proof.} (ii) $\implies$ (i) Suppose that there is a radical operation $\ast$ that is not spectral. By Proposition 4.2, there is an ideal $I$ such that $I \ast \subseteq \bigcap\{P \mid P \in \text{V}(I) \cap \text{QSpec}^\ast(D)\}$; without loss of generality we can suppose that $I = I^\ast$. Let $J$ be equal to the intersection, and let $\Gamma := \text{Min}(I) \setminus \text{V}(I)$. By construction, $\Gamma$ is nonempty.

Indeed, since $\text{Min}(I)$ is perfect each $\text{Min}(I) \setminus \{P\}$ is dense, and thus by Proposition 4.2, $I = I^\ast \cap D$ for every $P$; since $D$ is rad-colon coherent, by Proposition 5.2 we have $I = I^\ast \cap D$. However, $P^\ast \subseteq 1$ for every $P \in \text{Min}(I)$; hence, $1 \in P^\ast$ for every $P \in \text{V}(I)$. By Proposition 4.2, $\ast$ cannot be spectral.

(i) $\implies$ (ii) Suppose that there is a radical operation $\ast$ that is not spectral. By Proposition 4.2, there is an ideal $I$ such that $I^\ast \cap D \subseteq \bigcap\{P \mid P \in \text{V}(I) \cap \text{QSpec}^\ast(D)\}$; without loss of generality we can suppose that $I = I^\ast$. Let $J$ be equal to the intersection, and let $\Gamma := \text{Min}(I) \setminus \text{V}(I)$. By construction, $\Gamma$ is nonempty.

The set $\Gamma$ does not contain isolated points of $\text{Min}(I)$: if $Q \in \Gamma$ is isolated, then $I = Q \cap I_0$ for some $I_0 \ subseteq I$, and thus $I^\ast \cap D = (Q \cap I_0)^\ast \cap D = Q^\ast \cap I_0 \cap D$ can only be equal to $I$ if $Q = Q^\ast \cap D$, i.e., $Q \in \text{QSpec}^\ast(D)$ and $J \subseteq Q$.

For every $P \in \Gamma$, let $\gamma(P)$ be the minimal ordinal number such that $P \notin \mathcal{D}^\gamma(\Gamma)$. Note that $\gamma(P)$ exists since $\text{Min}(I)$ is scattered and no element of $\Gamma$ is isolated; furthermore, $\gamma(P)$ is a successor ordinal. Let $\gamma$ be the minimal element of the set of all $\gamma(P)$, and let $Q \in \text{Min}(I)$ be such that $\gamma(Q) = \gamma$. Let also $\beta$ be such that $\gamma = \beta + 1$. 

\textbf{Lemma 6.2.} Let $D$ be an integral domain. Then, the following are equivalent:

(i) $\text{Min}(I)$ is scattered for every ideal $I$;
(ii) $\text{Min}(I)$ is not perfect for every ideal $I$.

\textbf{Proof.} (i) $\implies$ (ii) is obvious. To show (ii) $\implies$ (i), let $I$ be a radical ideal and let $X := \bigcap_\alpha D^\alpha(\text{Min}(I))$: then, $X$ is perfect. Let $J := \bigcap\{Q \mid Q \in X\}$; then, $I \subseteq J \subseteq P$ for all $P \in X$, and thus $X \subseteq \text{Min}(I)$. We claim that $\text{Min}(J)$ is perfect. Indeed, suppose not: then, it has an isolated point $P$, and $P$ cannot belong to $X$, since $X$ is perfect. Since $P$ is isolated, there is a finitely generated ideal $L$ such that $D(L) \cap \text{Min}(J) = \{P\}$; therefore, $L \not\subseteq P$ while

\[ L \subseteq \bigcap_{Q \in \text{Min}(J) \setminus \{P\}} Q = J, \]

a contradiction. Thus $\text{Min}(J)$ is perfect, as claimed.

\textbf{Definition 6.3.} We say that $D$ is min-scattered if $\text{Min}(I)$ is a scattered space for every ideal $I$.

\textbf{Proposition 6.4.} Let $D$ be a domain such that $\text{Spec}(D)$ is countable. Then, $D$ is min-scattered.

\textbf{Proof.} If $I$ is not perfect, there is an isolated point $Q$ of $\text{Min}(I)$, and $\text{Min}(I) \setminus \{Q\} = \text{Min}(I) \cap \text{V}(I)$ for some radical ideal $J$. By construction, $J \subseteq I$ and $\text{V}(I) \cup \text{V}(Q) = \text{V}(I)$, so that $I = Q \cap J$. Conversely, if $I = Q \cap J$, then $\text{V}(I) = \text{V}(Q) \cup \text{V}(J)$. Since $I \neq J$, $\text{V}(J)$ cannot contain all minimal primes of $I$; therefore, $Q$ must be contained in $\text{Min}(I)$. Hence, $\{Q\} = \text{Min}(I) \setminus \text{V}(I)$ is open in $\text{Min}(I)$ and $Q$ is isolated; thus $\text{Min}(I)$ is not perfect. 

\[ \square \]
By construction, $Q$ is a limit point of $\text{Min}(I)$, while $Q$ is isolated in $\mathcal{D}^\beta(\text{Min}(I))$. Hence, $Q$ is a limit point of $\text{Min}(I) \setminus \mathcal{D}^\beta(\text{Min}(I))$. The latter set is contained in $\mathcal{V}(J)$, by definition of $Q$; since $\mathcal{V}(J)$ is closed, it follows that also $Q \in \mathcal{V}(J)$. This is a contradiction, and thus $\mathcal{I}^*$ must be empty, i.e., there cannot be a radical non-spectral semistar operation. The claim is proved.

We now restrict to the case of Prüfer domains, extending results proved in [15] and mostly following the general method of that paper. Given a semistar operation $\star$ on the Prüfer domain $D$, we define the pseudo-spectrum $\text{PsSpec}^\star(D)$ as the set of those prime ideals $Q$ such that $1 \in Q^\star$, but there is a $Q$-primary ideal $L$ such that $L = L^* \cap D$. Using the quasi-spectrum and the pseudo-spectrum, we can define from $\star$ a new semistar operation $\hat{\star}$, called the normalized stable version of $\star$, as

$$
\hat{\star} : I \mapsto \bigcap_{P \in \text{PsSpec}^\star(D)} ID_P \cap \bigcap_{Q \in \text{PsSpec}^\star(D)} (ID_Q)^v_Q,
$$

where $v_Q$ is the $v$-operation on the valuation domain $D_Q$. Note that $v_P$ is different from the identity on $D_P$ if and only if $P$ is idempotent.

**Lemma 6.6.** Let $\star$ be a radical semistar operation. Then, $\star = \hat{\star}$ if and only if $\star$ is spectral.

**Proof.** If $\star$ is spectral, then $\star = s_{\text{PsSpec}^\star(D)} = \hat{\star}$. Conversely, if $\star = \hat{\star}$ but $\star$ is not spectral, there is a $Q \in \text{PsSpec}^\star(D)$. By definition, $1 \in Q^\star$, while $1 \notin L^*$ for some $Q$-primary ideal $L$; since $\text{rad}(L) = Q$, this contradicts the fact that $\star$ is radical. Hence $\star$ must be spectral.

**Theorem 6.7.** Let $D$ be a Prüfer domain. Then, the following are equivalent:

(i) $D$ is min-scattered;
(ii) every radical semistar operation is spectral;
(iii) $\star = \hat{\star}$ for every stable semistar operation $\star$.

**Proof.** (i) $\iff$ (ii) follows from Theorem 6.5, since a Prüfer domain is rad-colon coherent, while (iii) $\implies$ (ii) follows from Lemma 6.6.

To prove (i) $\implies$ (iii), fix a stable semistar operation $\star$. By [15, Theorem 3.9], $\star \leq \hat{\star}$, and thus if $1 \in I^*$ then also $1 \in \hat{I}^*$. Suppose that $1 \in \hat{I}^*$ while $1 \notin I^*$. Then, $I := I^* \cap D$ is a proper ideal of $D$ that is quasi-$\star$-closed. Changing notation from $I$ to $J$, we can suppose without loss of generality that $I = \hat{I}^* \cap D$.

Since $\text{Min}(I)$ is not perfect, there is an isolated point $Q$. Since $\text{Min}(I) \setminus \{Q\}$ is closed, it is equal to $\mathcal{V}(I_1) \cap \text{Min}(I_1)$ for some radical ideal $I_1$. Let $T := \bigcap\{DP : P \in \mathcal{V}(Q)\}$ and $S := \bigcap\{DP : P \in \mathcal{V}(I_1)\}$; Then, $I = IS \cap IT$, and in particular

$$
I^* \cap D = (IS)^* \cap D \cap (IT)^* \cap D.
$$

The radical of $IT \cap D$ is $Q$, which is a prime ideal. By the proof of [15, Theorem 4.5], since $1 \in (IT \cap D)^*$, we also have $1 \in (IT \cap D)^\star$, and thus $(IT)^\star \cap D = D$. On the other hand, $IS \cap D$ is not contained in $Q$; hence, neither does $(IS)^\star \cap D$. By construction, $I = I^* \cap D$; this is a contradiction, and thus $\star$ and $\hat{\star}$ must be equal, as claimed.

**Corollary 6.8.** Let $D$ be a domain such that $\text{Spec}(D)$ is countable. Then, every radical semistar operation is spectral. If $D$ is Prüfer, moreover, $\star = \hat{\star}$ for every stable semistar operation $\star$.

**Proof.** If $\text{Spec}(D)$ is countable, then $D$ is min-scattered by Proposition 6.4. The claims now follow from Theorems 6.5 and 6.7.

The following is a version of Theorem 5.3 for stable operations on a Prüfer domain; it can also be seen as a variant of [18, Theorem 5.12] (in view of [18, Section 6]).
Theorem 6.9. Let $D_1, D_2$ be Prüfer domains. Suppose that there is a homeomorphism $\phi : \text{Spec}(D_1) \rightarrow \text{Spec}(D_2)$ such that a prime ideal $P$ is idempotent if and only if $\phi(P)$ is idempotent. If $D_1$ is min-scattered, then there is an isomorphism $\Phi : \text{SStar}_{\text{st}}(D_1) \rightarrow \text{SStar}_{\text{st}}(D_2)$.

Proof. We first note that if $J$ is an ideal of $D_2$, then $\text{Min}(J)$ is the set of minimal elements of the closed set $V(J)$; then, $\phi^{-1}(V(J))$ is a closed set of $\text{Spec}(D_1)$, and thus it is equal to $V(J')$ for some ideal $J'$ of $D_1$. By hypothesis, $\text{Min}(J')$ is scattered, and thus also $\phi(\text{Min}(J')) = \text{Min}(J)$ is scattered. Hence also $D_2$ is min-scattered.

Given a stable semistar operation $\ast$ on $D_1$, we define $\Phi(\ast)$ as the map

$$
\Phi(\ast) : I \mapsto \bigcap_{P \in \phi(Q\text{Spec}^*(D_1))} (I(D_2)_P \cap \bigcap_{Q \in \phi(P\text{Spec}^*(D_1))} (I(D_2)_Q)^{\psi Q}.
$$

We claim that $Q\text{Spec}^{\Phi(\ast)}(D_2) = \phi(Q\text{Spec}^*(D_1))$ and $P\text{Spec}^{\Phi(\ast)}(D_2) = \phi(P\text{Spec}^*(D_1))$.

Indeed, let $P \in \text{Spec}(D_1)$ and let $Q := \phi(P)$. If $P \in Q\text{Spec}^*(D_1)$, then

$$
Q^{\Phi(\ast)} \cap D \subseteq Q(D_2)_Q \cap D_2 = Q,
$$

and thus $Q \in Q\text{Spec}^{\Phi(\ast)}(D_2)$; conversely, if $Q \in Q\text{Spec}^{\Phi(\ast)}(D_2)$, then either $Q(D_2)_A \neq (D_2)_A$ for some $A \in \phi(Q\text{Spec}^*(D_1))$ or $(Q(D_2)_B)^{\psi \neq} Q(D_2)_B$ for some $B \in \phi(P\text{Spec}^*(D_1))$. In the former case, $P \subseteq A$; since the quasi-spectrum is closed by generizations [15, Proposition 3.4(a)], $P \in Q\text{Spec}^*(D_1)$ and thus $Q \in \phi(Q\text{Spec}^*(D_1))$. In the latter case it must be $Q \subseteq B$, and thus $Q \in \phi(Q\text{Spec}^*(D_1))$ since every $B' \subseteq B$ is in the quasi-spectrum [15, Proposition 3.4(b)]. Therefore, $Q\text{Spec}^{\Phi(\ast)}(D_2) = \phi(Q\text{Spec}^*(D_1))$.

Suppose now that $P \in P\text{Spec}^*(D_1)$. By the previous paragraph, $Q \notin Q\text{Spec}^{\Phi(\ast)}(D_2)$. There is a $P$-primary ideal $L \subseteq P$ such that $L = L' \cap D_1$ in particular, $P$ is not branched in the valuation domain $(D_1)_P$. The map $\phi$ induced a homeomorphism between $\text{Spec}((D_1)_P)$ and $\text{Spec}((D_2)_Q)$; therefore, neither $Q$ is branched, and thus there exist a $Q$-primary ideal $L' \subseteq Q$. By definition,

$$
(L')^{\Phi(\ast)} \cap D_2 \subseteq L(D_2)_Q \cap D_2 = L',
$$

and thus $Q \in P\text{Spec}^{\Phi(\ast)}(D_2)$. Conversely, if $Q \in P\text{Spec}^{\Phi(\ast)}(D_2)$, then there is a $Q$-primary ideal $L \subseteq Q$ such that $L^{\Phi(\ast)} \cap D_2 = L$. By the previous part of the proof, $P \notin Q\text{Spec}^*(D_1)$; if $P$ is not even in $P\text{Spec}^*(D_1)$, then $L^{\Phi(\ast)}$ would just be equal to $D^{\Phi(\ast)}$, a contradiction. Hence, $Q = \phi(P) \in \phi(P\text{Spec}^*(D_1))$. Therefore, $P\text{Spec}^{\Phi(\ast)}(D_2) = \phi(P\text{Spec}^*(D_1))$.

Consider now the map $\Psi : \text{SStar}_{\text{st}}(D_2) \rightarrow \text{SStar}_{\text{st}}(D_1)$ defined by

$$
\Psi(\ast) : I \mapsto \bigcap_{P \in \phi^{-1}(Q\text{Spec}^2(D_2))} (I(D_1)_P \cap \bigcap_{Q \in \phi^{-1}(P\text{Spec}^2(D_2))} (I(D_1)_Q)^{\psi Q}
$$

for every ideal $I$ of $D_1$ and every $\ast \in \text{SStar}_{\text{st}}(D_2)$. Then, $\Psi$ is the map associated to the homeomorphism $\phi^{-1}$ by the previous construction; hence,

$$
\Psi \circ \Phi(\ast) : I \mapsto \bigcap_{P \in \phi^{-1}(Q\text{Spec}^{\Phi(\ast)}(D_2))} (I(D_1)_P \cap \bigcap_{Q \in \phi^{-1}(P\text{Spec}^{\Phi(\ast)}(D_2))} (I(D_1)_Q)^{\psi Q} = \bigcap_{P \in \text{Spec}^*(D_1)} I(D_1)_P \cap \bigcap_{Q \in \text{Spec}^*(D_1)} (I(D_1)_Q)^{\psi Q} = I^\ast
$$

since $\phi$ is a homeomorphism. By Theorem 6.7, $\ast = \ast$, and thus $\Psi \circ \Phi(\ast) = \ast$, i.e., $\Psi \circ \Phi$ is the identity on $\text{SStar}_{\text{st}}(D_1)$. By symmetry, also $\Phi \circ \Psi$ is the identity on $\text{SStar}_{\text{st}}(D_2)$; hence, $\Phi$ and $\Psi$ are inverses one of each other. It is straightforward to see that they are also order-preserving; thus, they establish a isomorphism between $\text{SStar}_{\text{st}}(D_1)$ and $\text{SStar}_{\text{st}}(D_2)$, as claimed.

Corollary 6.10. Let $D_1, D_2$ be Prüfer domains with countable spectrum. Suppose that there is a homeomorphism $\phi : \text{Spec}(D_1) \rightarrow \text{Spec}(D_2)$ such that a prime ideal $P$ is idempotent if and only if $\phi(P)$ is idempotent. Then, there is an isomorphism $\Phi : \text{SStar}_{\text{st}}(D_1) \rightarrow \text{SStar}_{\text{st}}(D_2)$.
Proof. If Spec($D_1$), Spec($D_2$) are countable, then $D_1$, $D_2$ are min-scattered by Proposition 6.4. The claim now follows from Theorem 6.9.

7. Other versions

Stable semistar operations are linked to two other structures on a ring: localizing systems and length functions.

A localizing system on a domain $D$ is a set of ideals $\mathcal{F}$ such that:
- if $I \in \mathcal{F}$ and $I \subseteq J$, then $J \in \mathcal{F}$;
- if $I \in \mathcal{F}$ and $(I : p) D \in \mathcal{F}$ for all $p \in I$, then $J \in \mathcal{F}$.

The map $\star \mapsto \mathcal{F}^\star := \{ I \mid 1 \in I^\star \}$ establishes a bijective correspondence between the set of stable semistar operations and the set of all localizing systems, whose inverse is given by the map associating to $\mathcal{F}$ the semistar operation [20, Section 2]

$$\star_\mathcal{F} : I \mapsto \bigcup_{J \in \mathcal{F}} (I : D J).$$

We say that a localizing system is radical if, for every ideal $I$ such that $\text{rad}(I) \in \mathcal{F}$, we have $I \in \mathcal{F}$. This notion corresponds exactly to radical semistar operations.

**Proposition 7.1.** Let $\star$ be a stable semistar operation. Then, $\star$ is radical if and only if $\mathcal{F}^\star$ is a radical localizing system.

**Proof.** If $\star$ is radical and $\text{rad}(I) \in \mathcal{F}^\star$, then $1 \in \text{rad}(I)^\star$ and thus $1 \in I^\star$ by definition, so that $I \in \mathcal{F}$ and $\mathcal{F}^\star$ is radical. Conversely, if $\star$ is not radical there is an ideal $I$ such that $1 \notin I^\star$ while $1 \in \text{rad}(I)^\star$: then, $\text{rad}(I) \in \mathcal{F}^\star$ while $I \notin \mathcal{F}^\star$, so that $\mathcal{F}^\star$ is not radical. 

Therefore, the bijection between stable operations and localizing systems restricts to a bijection between $\text{SStar}_{\text{rad}}(D)$ and the set $L_{\text{rad}}(D)$ of radical localizing systems; it follows that Theorems 5.3 and 6.5 can be expressed also in the terminology of localizing systems.

A singular length function on $D$ is a map $\ell$ from the set of all $D$-modules to $\{0, \infty\}$ such that:
- $\ell$ is additive on exact sequences;
- for each module $M$, $\ell(M)$ is the supremum of $\ell(N)$, as $N$ ranges among the finitely generated submodules of $M$.

A singular length function is uniquely determined by its ideal colength $\tau$, where $\tau$ is the map that associates to each ideal $I$ the length $\ell(D/I)$ [22, Proposition 3.3]. We denote by $L_{\text{sing}}(D)$ the set of singular length functions on $D$.

There is a bijective correspondence between localizing systems and singular length functions, where we associate to a localizing system $\mathcal{F}$ the colength [4, Section 6]

$$\tau_\mathcal{F}(I) = \begin{cases} 0 & \text{if } I \in \mathcal{F}, \\ \infty & \text{if } I \notin \mathcal{F}. \end{cases}$$

We say that a length function $\ell$ with associated colength $\tau$ is radical if $\tau(I) = \tau(\text{rad}(I))$ for every ideal $I$. This definition corresponds to radical semistar operations and radical localizing systems.

**Proposition 7.2.** Let $\mathcal{F}$ be a localizing system. Then, $\mathcal{F}$ is radical if and only if the associated length function $\ell_\mathcal{F}$ is radical.

**Proof.** If $I$ is an ideal, by definition $\tau(I) = 0$ if and only if $I \in \mathcal{F}$. Therefore, $\tau(I) = \tau(\text{rad}(I)) = 0$ if and only if $I, \text{rad}(I) \in \mathcal{F}$, while $\tau(I) = \tau(\text{rad}(I)) = \infty$ if and only if $I, \text{rad}(I) \notin \mathcal{F}$. The claim follows.
Let $D$ be a Prüfer domain. To every singular length function $\ell$ with associated colength $\tau$ we can associate the space

$$\Sigma(\ell) := \{P \in \Spec(D) \mid \tau(Q) > 0 \text{ for some } P\text{-primary ideal } Q\},$$

and the length function

$$\ell^{\#} := \sum_{P \in \Sigma(\ell)} \ell \otimes D_P$$

(where $(\ell \otimes D_P)(M) := \ell(M \otimes D_P)$). Then, we get an analogue of Theorem 6.7.

**Proposition 7.3.** Let $D$ be a Prüfer domain. The following are equivalent:

(i) $D$ is min-scattered;
(ii) $\ell = \ell^{\#}$ for every singular length function $\ell$.

**Proof.** Let $\Phi$ be the isomorphism between $\SStar_D(D)$ and $\Lsing_D(D)$ obtained composing the bijections of the two with the set of localizing systems. The claim follows from Theorem 6.7 and the fact that $\Phi(\hat{\ast}) = \Phi(\ast)^{\#}$ [18, Proposition 6.8]. □

As a consequence, we also obtain a version of Theorem 6.9 (compare with [18, Theorem 5.12]).

**Theorem 7.4.** Let $D_1, D_2$ be Prüfer domain. Suppose that there is a homeomorphism $\phi : \Spec(D_1) \rightarrow \Spec(D_2)$ such that a prime ideal $P$ is idempotent if and only if $\phi(P)$ is idempotent. If $D_1$ is min-scattered, then there is an isomorphism $\Phi : \Lsing(D_1) \rightarrow \Lsing(D_2)$.

To conclude, we express [21, Corollary 7.5] in the terminology of this paper; see [21] for the definition of SP-scattered domain.

**Proposition 7.5.** Let $D$ be an SP-scattered domain. Then, every stable semistar operation on $D$ is radical.

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