MINIMAL ALGORITHMIC INFORMATION LOSS METHODS FOR DIMENSION REDUCTION, FEATURE SELECTION AND NETWORK SPARSIFICATION

HECTOR ZENIL, NARSIS A. KIANI, ALYSSA ADAMS, FELIPE S. ABRAHÃO, ANTONIO RUIDA-TOICEN, ALLAN A. ZEA, AND JESPER TEGNER

(Hector Zenil) Machine Learning Group, Department of Chemical Engineering and Biotechnology, University of Cambridge, U.K.. Oxford Immune Algorithmics, Reading, U.K. and Abu Dhabi/Dubai, U.A.E.. Kellogg College, University of Oxford, U.K.. The Alan Turing Institute, British Library, London, U.K.. Algorithmic Dynamics Lab, Unit of Computational Medicine, Karolinska Institutet, Stockholm, Sweden. Algorithmic Nature Group, LABORES for the Natural and Digital Sciences, Paris, France.

(Narsis A. Kiani) Algorithmic Dynamics Lab, Unit of Computational Medicine, and Department of Oncology-Pathology, Center of Molecular Medicine, Karolinska Institutet, Stockholm, Sweden.

(Alyssa Adams) John W. and Jeanne M. Rowe Center for Research in Virology, Morgridge Institute for Research, University of Wisconsin-Madison, Madison, WI USA. Algorithmic Nature Group, LABORES for the Natural and Digital Sciences, Paris, France. Cross Labs, Cross Compass, Japan.

(Felipe S. Abrahão) Centre for Logic, Epistemology and the History of Science, University of Campinas, Brazil. National Laboratory for Scientific Computing (LNCC), 25651-075, Petropolis, RJ, Brazil. Algorithmic Nature Group, LABORES for the Natural and Digital Sciences, Paris, France.

(Antonio Rueda-Toicen) Instituto Nacional de Bioingeniería, Universidad Central de Venezuela, Caracas, Venezuela.

(Allan A. Zea) Department of Mathematics, Universidad Central de Venezuela, Caracas, Venezuela.

(Jesper Tegner) King Abdullah University of Science and Technology (KAUST), Biological and Environmental, Science and Engineering Division, Kingdom of Saudi Arabia. Algorithmic Dynamics Lab, Unit of Computational Medicine, Karolinska Institutet, Stockholm, Sweden.

E-mail addresses: hector.zenil@ki.se, narsis.kiani@ki.se, alyssa.gp.adams@gmail.com, fsa@lncc.br, antonio.rueda.toicen@gmail.com, allan.zea@ciens.ucv.ve, jesper.tegner@ki.se.
Abstract. We introduce a family of unsupervised, domain-free, and asymptotically optimal model-independent algorithms based on the principles of algorithmic probability and information theory designed to minimize the loss of algorithmic information, and thereby avoiding certain deceiving phenomena and distortions known to occur in statistics and entropy-based approaches. Our methods include a lossless-compression-based lossy compression algorithm that can select and coarse-grain data in an algorithmic-complexity fashion (without the use of popular compression algorithms) by collapsing regions that may procedurally be regenerated from a computable candidate model. We show that the method can perform dimension reduction, denoising, feature selection, and network sparsification, while preserving the properties of the objects. As validation case, we demonstrate the methods on image segmentation against popular methods like PCA and random selection, and also demonstrate that the method preserves the graph-theoretic indices measured on a well-known set of synthetic and real-world networks of very different nature, ranging from degree distribution and clustering coefficient to edge betweenness and degree and eigenvector centralities, achieving equal or significantly better results than other data reduction and the leading network sparsification methods (Spectral, Transitive).

Keywords: feature selection; coarse-graining; algorithmic machine learning; renormalisation; lossy compression; data dimension reduction; graph complexity; model-based image segmentation; graph summarisation
1. Introduction

The study of large and complex datasets, or big data, organised as networks, has emerged as one of the central challenges in most areas of science and technology, cellular and molecular networks in biology being one of the prime examples. Hence, a number of techniques for data dimensionality reduction have been developed.

Data reduction consists in the transformation of numerical or alphabetical digital information into a simplified smaller representation preserving certain properties of ‘interest’. The question germane to the area of data reduction concerns how low dimensional structures can be detected in high dimensional data. The main purpose of data dimensionality reduction involves two sides of the same coin. On the one hand there is the minimisation of the loss of information, and on the other, the maximal preservation of the most ‘meaningful’ features characterizing an object (i.e., feature selection). Traditionally, such meaningful features of interest are defined in terms of a user-centric, subjective criterion. For example, linear algebraic (e.g. matrix analysis) and statistically-based dimensionality reduction techniques attempt to minimize statistical information loss under certain algebraic (interpreted as signal and noise) conditions, as a consequence maximizing the statistical mutual information between the desired information and the dimensionally-reduced output.

However, statistical approaches and classical information theory cannot preserve computable features without some statistical signature, no matter how important they may be in characterizing the object (thus making the choice of preserving statistical information arbitrary and fragile) [27, 35, 36]. That is, such techniques (e.g. PCA [23]) will miss any non-linear and algorithmic regularity if it does not show a statistical property. Because the number of algorithmic features outgrows the number of statistical ones (the set of statistical features is a proper subset of the algorithmic), PCA, like all other computable measures for data reduction and clustering techniques, will miss fundamental properties of interest [27] by virtue of being only statistical and not algorithmic—for example, it would miss non-linear embeddings impossible to find with statistical and linear techniques. Improving on some of these limitations, topological data analysis can reduce data by minimizing its size or dimension into a non-linear surface of low algorithmic complexity (aka Kolmogorov complexity, Solomonoff-Kolmogorov-Chaitin complexity, or program-size complexity), e.g. a torus, or an S-shaped function.
The success of both linear and non-linear techniques can thus be simplified by looking for the shortest specification they can achieve. For linear algorithms this is usually approached by traditional statistical techniques, while for non-linear algorithms, some domain-specific subset of algorithms is considered (e.g. the set of all possible geometric shapes). Here, by not confining ourselves to a domain, we take a step forward towards more universal techniques, free of domains and particular implementations. For example, if data points can be embedded in a low-dimensional subspace or topological sub-manifold (such as a torus), an algorithmic loss minimisation process would approximate the shortest description of the generative mechanism of the torus.

The main aim of dimensionality reduction in a network is to approximate a network with a sparse network. There are several methods available in the literature for graph sparsification, which is one kind of graph summarisation [20]. Chew [10] used the shortest-path distance between every pair of vertices as a criterion for sparsifying a network. The concept of cut problems has been utilised for sparsification by Benczur and Karger [6]. In one of the latest methods, spectral similarity of graph Laplacians has been used for sparsification by [26]. For network dimensionality reduction one may choose as a criterion the preservation of graph-theoretic properties such as graph distance, clustering coefficient or degree distribution, or a finite (usually small) combination of these or other indices. But no finitely computable approach can find all possible features of interest in a dataset, for example, all those recursively enumerable features that the set of all Turing machines can characterize, all at the same time [27], which means that the observer is forced to make an arbitrary choice of features of interest (see e.g. [35, 36]).

Here, we introduce a family of semi-computable algorithms that specifically target the preservation of computable properties (hence both statistical and algorithmic). Thus, it can be seen as a generalisation of all dimension reduction procedures. The methods constitute an interesting approach to designing theoretically optimal lossy compression techniques based on principles and estimations to theoretical optimal lossless compression. It is based on successive local perturbations in the form of single edge deletions that produces the minimal loss of algorithmic information in this process. We call our algorithm MILS for Minimal Information Loss Selection.

In this sense, by grounding our theoretical framework on algorithmic information theory (which encompasses algorithmic complexity, algorithmic probability, and algorithmic randomness) [8, 9, 14], we mathematically establish general lower and upper bounds for the algorithmic
complexity oscillations resulting from single (or multiple) edge deletion (or insertion). Then, we prove that some local destructive perturbations in some graphs can in fact play the role of an oracle, so that the sparsified network carries the algorithmic information necessary to compute the value of the original network’s algorithmic complexity (which is a known uncomputable problem in general). Thus, even under the unavoidable uncomputability of the algorithmic complexity, these results indicate that some approximation methods, such as the block decomposition method (BDM) [41], or even brute-force approximation methods in some cases reach optimal values even with limited computational resources. BDM in particular may eventually also surpass the capabilities of other brute-force algorithms in this regard (see Section 3.2).

In this way, using approximations of efficient (polynomial) estimations to algorithmic complexity by recent numerical methods of algorithmic probability [13, 25, 41], we demonstrate how these algorithms can preserve structure properties, performing similarly to, if not always outperforming, state-of-the-art algorithms in e.g. the area of network dimension reduction.

We test our algorithms on non-trivial cases against transitive and non-linear (spectral) methods involving simple graphs where statistical regularities are even easier to conceal and thus may easily fool weaker, linear and computable measures [35].

This approach opens a path towards evaluating the success of all other reduction techniques and for achieving optimal reduction based on the minimisation of algorithmic information loss (thus the non-linear generalisation of all techniques), rather than only preserving statistical or domain-specific algebraic properties. In the present paper, we restrict our attention to edge deletions in graphs in order to describe the MILS algorithm, but MILS can actually be applied to any general object and any type of constructive or destructive perturbation [30] on its constituent elements. For example, MILS can also be easily extended to node deletion/insertion instead of edge deletion/insertion, and to other types of algorithmic perturbations [3] that move the object away from being more incompressible.

While the algorithms introduced are independent of approximating method and can be implemented using Entropy or lossless compression, here we use a method based on [41], which encompasses, and improves

\[^1\]In addition to being asymptotically optimal in general as the computational resources grows indefinitely.
upon, other statistics- and entropic-based methods due to BDM’s intrinsic structural characteristics (see Sections 2.1 and 3.2).

Our results in Section 4 indicate that we either match the results of the best current algorithms or, most of the time, outperform them for both local and global graph properties.

2. Method

2.1. Information difference. As usual, a labeled graph is defined by an ordered pair $G = (V, E)$ comprising a finite set $V(G) = \{1, \ldots, n\}$ of labeled vertices with $n \in \mathbb{N}$ and a set $E(G) \subseteq \mathbb{E}(G)$ of present edges, where $\mathbb{E}(G) := \{(x, y)\mid x, y \in V\} = V \times V$ is the set of all possible edges. Note that in this article we will only deal with labeled graphs. By restricting the set of all possible edges we obtain a undirected graph $G$ without self-loops (or simple graph, for short), where each edge is an unordered pair such that $E(G) \subseteq \mathbb{E}_c(G) := \{\{x, y\}\mid x, y \in V\}$.

All methods employed in this article are built upon the information difference among the elements of an object, or in other words on the information contribution of the elements of a system to the whole, e.g. of nodes or links to a network. This is in turn based on a concept of algorithmic/causal perturbation analysis within the framework of algorithmic information dynamics (AID) [30]. The procedure consists in the perturbation of all elements of a system by the removal of elements whose effects on its algorithmic information content are measured and ranked accordingly. The idea of a dynamic study/calculus of the (possible) changes that can be wrought upon an object to evaluate the contribution of each of its components for different purposes was introduced in [32, 34, 37, 40], and here we extend these ideas to the area of data/network dimension reduction. Formally:

Definition 2.1. Let $G$ be a graph with edges $e_1, \ldots, e_{|E(G)|} \in E(G) \subseteq \mathbb{E}(G)$. Now, let $G \setminus e_i$ denote the graph obtained by removing $e_i$ from $G$, and $I(G, e_i)$ denote the information difference or information value/contribution of $e_i$ to $G$ given by

$$I(G, e_i) := C(G) - C(G \setminus e_i),$$

where $C(G)$ is the algorithmic complexity of graph $G$ (see also Appendix). If one allows multiple edge deletions of a subset $F \subseteq E(G)$, we analogously define

$$I(G, F) := C(G) - C(G \setminus F).$$
Also, for the analogously inverse operation of edge insertion $G \cup e_i$, we denote the information difference by

$$I^{-1}(G, e_i) := C(G) - C(G \cup e_i)$$

and

$$I^{-1}(G, F) := C(G) - C(G \cup F),$$

respectively.

These differences can also be referred to the graph’s nodes, but here we will restrict our attention to the edges. In addition, we focus our results on destructive perturbations, i.e., the edge deletion problem. Nevertheless, in the next Section 3.1, we will also investigate the edge-insertion case (i.e., edge insertion) as an analogous inverse case of edge deletion. This becomes more relevant in the proof of Theorem 3.1 presented in Appendix.

When taking the difference $C(G) - C(G \setminus e_i)$ by itself we will refer to it as the graph (dis)similarity between graph $G$ and $G \setminus e_i$. $I(G, e_i)$ applied to graphs suggests a similarity “distance” between graphs based on algorithmic information content (in \[33\], we show that this similarity measure can classify networks by the family they belong to, differentiating variant synthetic and natural network topologies similar to graph motifs, as shown in \[22\]).

In the description of the algorithm to be presented in Section 2.2, replacing the underlying methods to approximate the algorithmic information content by, e.g., Shannon entropy or lossless compression algorithms represents special cases of the more general algorithm based on algorithmic complexity, and thus it covers all these less powerful cases. This is because BDM calculation subsumes entropy-based approximations to algorithmic information content \[41\]. In addition to algorithmic complexity being an asymptotically optimal measure of irreducible information content, and due to being able to capture algorithmically determined patterns that are not reducible to statistical patterns \[28\] \[35\], Definition 2.1 and our implementation of MILS algorithm offers the advantage of avoiding (both in theory and in practice) certain low-complexity deceiving phenomena that other traditional methods based on entropy and statistics are prone to without compromising (in comparison to these other methods investigated in Section 4) on running time efficiency and data size reduction performance.

2.2. **An algorithm for minimal information loss selection.** MILS is an unsupervised and mostly parameter-free algorithm, i.e., asymptotically independent of model or domain, as it does not need to be
instructed or designed to preserve any particular property, and max-
imises the preservation of all computable elements that contribute to
the algorithmic information content of the data.

Let $G$ be a graph and $k$ denote its number of edges. Ideally, MILS
seeks to reduce $G$ to a graph on $N < k$ edges so that the loss of
information is minimized after each step of the algorithm. It does this
by calculating the information differences between $G$ and $G \setminus S$, where $|S| \geq 1$, and then finding the subset $F \subset E(G)$ such that $I(G, F) \leq I(G, S)$ for all non-empty proper subsets of edges $S \subset E(G)$, repeating
this task with $G = G \setminus F$ until the target size is reached (i.e., when
$|E(G)| = N$). The algorithm’s time complexity is clearly in $O(\exp)$
because of how MILS performs searches and deletions across all subsets
of edges, but significant improvements to this bound are still possible.

A more efficient but suboptimal version of such an exponential-time
MILS iterates over single elements (in the case, edges) or singletons.
Algorithm 1 performs sequential deletions of edges, and then removes
the edge that contributes less to the information content of the graph
before moving to the next step.

When $e$ is such that $I(G, e) = \text{minLoss}$, we call it a neutral infor-
mation edge [32] because it is the edge that contributes less to the
information content of $G$ (in particular, it minimizes information loss
or the introduction of spurious information into the network according
to the information difference when removed from the original network).
The pseudocode in Algorithm 1 assumes that there is a unique such $e$,
which may not necessarily be the case. Algorithm 2 solves this prob-
lem by performing simultaneous perturbations on all edges with an
information contribution of $\text{minLoss}$. In Algorithm 2 we also introduce
INFORANK, a method that produces a ranking of $e_1, \ldots, e_{|E(G)|}$ from
least informative to most informative edge, i.e., a list of edges sorted in
increasing order by their information contribution to $G$. This ranking
facilitates the search for the most neutral elements of the system (see
Section 3.3), which in turn helps MILS preserve the components that
maximise the information content of the resulting object.

As described in [32], one can easily modify these algorithms in order
to deal with multiple edge deletions at once, as in $I(G, F)$, but possibly
at the expense of much more computational resources. In addition, note
that algorithms 1 and 2 may be applied, mutatis mutandis, to nodes or
to any data element of an object or dataset, e.g., a pixel of an image,
or a row or column in a spreadsheet.

MILS is, by design, optimal in the computability and algorithmic-
information theoretic sense, and only dependent on the method for
Algorithm 1 MILS with sequential perturbation by deletion on edges

1: function Perturb\( (G) \)
2: \( \text{informationLoss} \leftarrow \emptyset \) // for each edge \( e \)
3: \( \text{for } e \in E(G) \text{ do} \)
4: \( \text{store information contribution into } \text{informationLoss} \)
5: \( \text{informationLoss} \leftarrow \text{informationLoss} \cup \{ I(G,e) \} \)
6: \( \text{end for} \) // calculate minimal information loss across all edges
7: \( \text{minLoss} \leftarrow \min(\text{informationLoss}) \)
8: \( \text{for } e \in E(G) \text{ do} \)
9: \( \text{if } I(G,e) = \text{minLoss} \text{ then} \)
10: \( \text{remove } e \)
11: \( \text{return } G \setminus e \)
12: \( \text{end if} \)
13: \( \text{end for} \)
14: \( \text{end function} \)

15: function Sparsify\( (G, N), 1 \leq N \leq |E(G)| \)
16: // until reaching target size
17: \( \text{while } |E(G)| > N \text{ do} \)
18: \( \text{perturb edges in } G \)
19: \( G \leftarrow \text{Perturb}(G) \)
20: \( \text{end while} \)
21: \( \text{return } G \)
22: \( \text{end function} \)

approximating algorithmic complexity in the preservation of any possible feature of interest that contributes to the (algorithmic) information content of a network such as, evidently, its degree distribution and other graph-theoretic, algebraic or topological features, even those not necessarily captured by any graph theoretic measure or classical information approach [35, 36].

In Section 3.3 we will show that Algorithm 2 is deterministic, polynomially time bounded, and describes a criterion to select and remove the most neutral elements of an object. We employ this more efficient version in all our experiments, and even in this limited form the procedure excels at preserving important characteristics of the networks under study. See Theorem 3.3.
Algorithm 2 Minimal Information Loss Selection (MILS)

1: function INFO\textsc{Rank}(G)
2: \quad informationLoss ← \emptyset
\quad // for each edge \( e_i \)
3: \quad for \( e_i \in E(G) \) do
\quad \quad // store information contribution into \( informationLoss \)
4: \quad \quad informationLoss ← informationLoss \cup \{ I(G, e_i) \}
5: \quad end for
6: \quad sort \ informationLoss \ in increasing order
\quad // return information rank
7: \quad return informationLoss
8: end function

9: function MILS(G, N), \( 1 \leq N \leq |E(G)| \)
10: \quad while \( |E(G)| > N \) do
\quad \quad // calculate minimal loss across all edges
11: \quad \quad minLoss ← \min(\text{INFO}\textsc{Rank}(G))
\quad \quad // remove all candidate edges from \( G \)
12: \quad \quad G ← G\{ e_i \in E(G) : I(G, e_i) = minLoss \}
13: \quad end while
14: \quad return G
15: end function

3. Theory

3.1. Upper and lower bounds for the theoretically optimal information difference. As presented in Section 2.1 and formalized in [3, 30, 32, 34], we have that edge deletions (or edge insertion) can be seen as transformations/perturbations that a particular network is being subjected to. This way, it is immediate to ask about the maximum overall impact of these transformations on the algorithmic information content of the network. In other words, one may want to study algorithmic complexity oscillations generated by such perturbations as the network size increases. Indeed, Theorem 3.1 shows that the theoretically optimal information difference (i.e., the information difference calculated by the algorithmic complexity as in Definition 2.1) produced by edge deletions (or insertions) are bounded by terms that depend on the network size \(|V(G)|\), graph sparsity (or edge density when taken in comparison to \( E_c(G) \)), and the number of perturbations \(|F|\). In addition, these lower and upper bounds are always never greater than an quadratic order of the network size (i.e., \(|V(G)|\)), even in the worst case when \(|F| \sim \frac{N^2 - N}{2}\) (see Corollary 3.3).
As expected, since every computational method of approximating $C(G)$ or $C(G \setminus e_i)$ can only reach optimal values for every pair $(G, e_i)$ in an asymptotic limit—remember that the algorithmic complexity is semi-computable—, we show in Theorem 3.1 that there are particular cases in which approximating algorithms to $I(G, e_i)$, such as the one employed within algorithm MILS as in Section 2, calculate values that fall into upper and lower bounds in a logarithmic order of the network size (see also Corollary B.4).

The main idea of the proof of Theorem 3.1 (i.e., Theorem B.1 in the Appendix) is to construct an algorithm for naming the exact edge subset $F \subseteq \mathbb{E}(G)$ that are being deleted (or inserted). Then, we show that, once this enumeration is calculated, this general algorithm employs (or reversibly employs) the perturbation in order to generate $G \setminus F$ from $G$ (or $G$ from $G \setminus F$).

Note that the proof is made for any possible subset $F \subseteq |\mathbb{E}_c(G)|$ of perturbed edges, where $|F| = r \leq |\mathbb{E}_c(G)|$. Therefore, if one takes $|F| = 1$, Theorem 3.1 immediately applies to single edge deletions (or insertions) as in Definition 2.1. In addition, the reader is invited to note that the proof of Theorem 3.1 holds for the classical case of graphs, but it can be extended to directed graphs.

**Theorem 3.1.** Let $G = (V, E)$ be an arbitrary simple graph with $E \subseteq \mathbb{E}_c(G)$ and $|V| = N \in \mathbb{N}$. Let $F \subseteq \mathbb{E}_c(G)$ be an arbitrary subset of edges, where $|F| = r \geq 1$. Then,

$$|I(G, F)| \leq 2 |F| \log_2 (N) + O \left( \log_2 (|F|) \right) + O \left( \log_2 \left( \log_2 (N) \right) \right),$$

if edges in $F \neq \emptyset$, where $F \subseteq E(G)$, are being deleted, or

$$|I^{-1}(G, F)| \leq 2 |F| \log_2 (N) + O \left( \log_2 (|F|) \right) + O \left( \log_2 \left( \log_2 (N) \right) \right),$$

if edges in $F \neq \emptyset$, where $F \subseteq \mathbb{E}_c(G) \setminus E(G)$, are being inserted.

If one wants tighter lower and upper bounds for multiple edge deletions as a function of the edge density (i.e., the size of $|E(G)|$ in comparison to $\frac{N^2-N}{2}$), then one can re-write the statement of Theorem 3.1
and achieve:

\[
(1) \quad -|F| \log_2(|E(G)|) - O(\log_2(|F|)) - \\
O(\log_2(\log_2(|E(G)|))) - O(\log_2(\log_2(|F|))) - O(1) \\
\leq C(G) - C(G \setminus F) \leq \\
|F| \log_2 \left( \frac{N^2 - N}{2} - |E(G)| + |F| \right) + O(\log_2(|F|)) + \\
+ O\left( \log_2 \left( \log_2 \left( \frac{N^2 - N}{2} - |E(G)| + |F| \right) \right) \right) + \\
O(\log_2(\log_2(|F|))) + O(1)
\]

Thus, from Equation (1), we have that these bounds for algorithmic information loss or gain from edge perturbations have a linearly dominant dependence on the number of perturbations $|F|$, except for a multiplicative logarithmic term depending on the edge density (and, if information is lost, also on the number of perturbations). In case the graph is (algorithmically) random, i.e. incompressible, then we get a linearly dominant dependence on the number $|F|$ of edges being perturbed and, unlike in the latter general case of Equation (1), the multiplicative logarithmic term depends only on the network size (see Corollary B.2). Moreover, as one might expected, for single edge perturbations (i.e., $|F| = 1$), these upper and lower bounds become really tight in general (see Corollary B.4).

3.2. Eventual optimal values for graph complexity from edge deletions. The semi-computability of the algorithmic complexity is a well-known phenomenon in algorithmic information theory and theoretical computer science [8, 9, 11, 18]: one can approximate the value of $C(x)$ from above with computable enumerations, but there is no computable function that return the value of $C(x)$ in general for every $x$. Nevertheless, in this section we will explore the possibility of taking advantage of a single perturbation on $G$ with purpose of calculating the value of $C(G)$. In other words, although no algorithm can determine in general the exact edge deletions that produce the desired phenomenon, we will present some particular cases in which an edge deletion works like an oracle that carries the sufficient amount of information for calculating the value of $C(G)$. Thus, approximating algorithms to measures of algorithmic information content, such as in algorithm MILS, can eventually harvest this information in order to achieve these optimal values, should enough computational resources be expended. See Theorem 3.2. These eventually optimal calculations,
although unpredictable in general, highlight the empirical significance of general methods based on algorithmic complexity, such as BDM, when compared with other lossless compression methods that do not (at least in a computable enumeration manner) asymptotically cover the space of partial recursive functions for compressing data.

We will study a case in which the characteristic string (i.e., a bit string that univocally determines the presence or absence of the edges \( \Pi \)) of a compressible and dense enough graph already contains useful information for identifying where the edge deletion occurred, so that the deleted edge’s positioning becomes information itself. So, the aim is to prove a particular case and, thus, show that the phenomenon exists and may occur more often. To achieve Theorem 3.2 with this purpose, the main idea of its proof in Theorem B.7 is to construct some particular cases of simple graphs \( G \) whose characteristic string’s prefix \( x \upharpoonright_k \) (i.e., a \( k \)-bit-length initial segment of \( x \)) carries the information sufficient for computing all the characteristic string \( x \), so that this prefix is logarithmically much smaller than the characteristic string (i.e., \( k = o(\log_2(l(x))) \)), where \( l(x) \) denotes the length of the string \( x \). Moreover, these graphs should be sufficiently dense. Thus, there will be a program that, given the perturbed graph as input, can always decide where a perturbation landed onto the characteristic string, should the edge deletion occur in the characteristic string’s suffix. Hence, one just need to choose the actual index \( i \) of the edge \( e_i \) destroyed (i.e., the bit in the characteristic string in which 1 was replaced with 0) that has

\[
C(i) \leq o(\log_2(|V(G)|))
\]

bits of algorithmic complexity. In addition, the information in the binary encoding of \( i \) carries all the necessary information of the minimum generating program of \( G \), so that

\[
C^*(G^*|i) = O(1)
\]

and

\[
C(i|(G \setminus e_i)) = O(1).
\]

Therefore, we will get that one becomes able to compute \( G^* \) from \( G \setminus e_i \).

Before reaching the desired theorem, we introduce the concept of prefix self-determined graphs. Let \( F_G \) be an infinite family of simple graphs. Let \( p \in \{0, 1\}^* \) only depend on the choice of \( F_G \). By prefix self-determined, we mean that there is a program \( p \) that can return the string \( x \) given only a proper prefix of the characteristic string \( x \) of \( G \) as input. Formally, as in Definition B.1, we say a graph \( G \in F_G \) is prefix self-determined iff there is \( y = x \upharpoonright_n \) with \( n < l(x) \) such that \( U(\langle y, p \rangle) = x \), where \( x \) is the characteristic string of \( G \). In
this article, we are applying the notion of prefix self-determination to
graphs through its characteristic strings, but prefix self-determination
is a property that can be clearly assigned to strings (or any other type
of encodable object) in general. Thus, prefix self-determination is a
stronger notion in comparison to the widely employed self-delimitation
\[8, 9\] from proper prefixes in algorithmic information theory. From
the prefix of a prefix self-determined string, one can not only get the
length of the string, but also promptly compute the very string. In
other words, a prefix self-determined string contains a compression of
itself. It is immediate to see that, for an infinite family of prefix self-
determined strings, if \( y \) is the respective proper prefix of string \( s \), then

\[
C(s) \leq C(y) + O(1) \leq l(y) + O(1) \leq l(s) + O(1).
\]

As a natural extension, one may want to limit a maximum length
for the prefix with respect to the string. This is the case of prefix
self-determined graphs whose initial segment \( y = x \upharpoonright n \) of its charac-
teristic string has length \( n = O(\log_2 (l(x))) \). Let \( F_G \) be an infinite
family of simple graphs. As in Definition \[B.2\] we say a graph \( G \in F_G \)
is logarithmically prefix self-determined iff \( G \) is prefix self-determined
and \( n = O(\log_2 (l(x))) \). Thus, a logarithmically prefix self-determined
graph has a very small subset of edges (which are encoded into the ini-
tial segment of the characteristic string) that algorithmically determine
the whole graph. This property grasps the idea of a small subgraph
carrying all the information sufficient for computably determining the
graph, should the initial segment of the characteristic string be in turn
a characteristic string of a vertex-induced subgraph.

Remember the usual notion of a family of dense graphs being defined
as an infinite family \( F'_G \) of graphs in which every member \( G \in F'_G \) has
\( |E(G)| = \Theta (|V(G)|^2) \).

Now, as demonstrated in Theorem \[B.7\], we obtain one of the par-
ticular cases in which a computer can find nearly optimal values of
algorithmic complexity in polynomial computation time (i.e., running
time) even with brute force:

**Theorem 3.2.** Let \( G \) be a large enough simple graph that belongs to an
infinite family \( F_G \) of logarithmically prefix self-determined dense simple
graphs. Then, there is a polynomially time-bounded exhaustive search
algorithm that, with input \( G \in F_G \), can approximate the integer value
\( C(G) \) with an error bounded by \( o (\log_2 (\log_2 (|V(G)|))) \) bits.

Thus, Theorem \[3.2\] shows that there is an algorithm that, with \( G \)
as input, calculates a value \( C'(G) \) that could be considered as a good
approximation of the algorithmic complexity of $G$ in terms of compression rate. Although the actual $C(G)$ and the approximation $C'(G)$ may differ by a number of bits of about a logarithmic term of its own value $C(G)$ (or of the order of a double logarithmic term of the input size), they are both asymptotically comparable with respect to $|V(G)|$. This phenomenon is expected to have an even faster speedup for algorithm-complexity approximating algorithms such as BDM, based on the decomposition into blocks of objects whose complexity is approximated using the Coding Theorem Method (CTM) \cite{13, 24, 25, 39}, which in turn precomputes lookup tables so as to exchange computation time for space/memory \cite{41}. In this sense, BDM can reach optimality for even larger families of graphs beyond that of Theorem 3.2. This is because one can easily extend the latter in order to demonstrate that BDM not only can find the nearly optimal approximations for logarithmically prefix self-determined graphs, but also can find those for graphs that are self-determined by logarithmically size-bounded 2D- or 3D-partitions of most (if not all) of its constituent subgraphs. Additionally, the precomputation of the lookup table enables BDM to take more advantage of larger quantities of constituent blocks in comparison to other brute-force methods of approximating algorithmic complexity from scratch every time. We suggest that this theoretical finding might be one of the reasons MILS is performing better than other size reduction methods. Future research should be pursued with the purpose of revealing other intrinsic structural characteristics underlying MILS and other data size reduction techniques, particularly with respect to modularity \cite{17, 32, 36}.

3.3. Uniqueness and time complexity of the algorithm. For MILS to be well-defined, we need to guarantee that the deletion of an element (e.g. node or edge in a graph) produces a unique object in a deterministic fashion. In the case of edges in networks, the problem is when there are elements $e_i, e_j \in E$ such that $I(G, e_i) = I(G, e_j)$ and the algorithm cannot uniquely decide whether to remove $e_i$ or $e_j$ first, which potentially (and likely) leads to the production of different objects. Algorithm 2 avoids this problem and shows that the algorithm is robust. Note that this algorithm and Theorem 3.3 are stated for graphs, but they can be generalized for multidimensional objects $U$ in general (see Appendix). The proof of Theorem 3.3 can be found in Theorem B.8.

Theorem 3.3. Let $T_{ID}(G)$ be the worst-case computation time of calculating the value $I(G, e_i)$ for any $e_i \in E(G')$ by the chosen algorithm-complexity approximating algorithm, where $G'$ is any edge-induced
subgraph of $G$. Then, MILS (Algorithm 2) is a deterministic algorithm that has a cubic worst-case time complexity with input $E(G)$ and linear worst-case time complexity in $T_{ID}(G)$.

Algorithm 2 also produces a speedup with respect to Algorithm 1 when all elements have the same information value, and thus are to be deleted at the same time: while in Algorithm 1 there would be a time complexity overhead to perform this task, in Algorithm 2, given a previously calculated $\text{INFORank}(G)$, there will be required just one step in the loop and, therefore, just a linear computation time in $O(|\text{informationLoss}|)$, where $\text{informationLoss}$ is the output of $\text{INFORank}(G)$. For example, any attempt to reduce the dimension of the complete graph (either by e.g. single-node or single-edge deletion) will produce an empty graph.

Thus, MILS' Algorithm 2 scales in a tractable manner with respect to the complexity approximating method for very large networks. This is because the worst-case running time is basically dominated by the amount of computational resources expended in the sole approximation to the values of algorithmic complexity as this amount only contributes linearly to the overall time complexity when the network size increases.

In Section 4.1, we present a variation of this algorithm that resorts in heuristically allowing a random selection of elements when they have the same information value. In future research, other variations of Algorithm 2 may be proposed for example in order to improve the sheer cubic dependence on the network size. In this direction, such variations can stem from improvements on the embedded sorting algorithm and its usage within the loop in [Algorithm 2, Step 10].

4. Results

Fig. 1.(A) and Fig. 1.(B) illustrate the application of MILS in the reduction and coarse-graining of two simple cases of the space-time evolution of Elementary Cellular Automata rules 22 and 158. The reduction is by minimisation of algorithmic information loss. MILS effectively extracts the salient elements that characterize each of these systems. The examples in Fig. 1.(A) and Fig. 1.(B) depict the way in which regions with high or low algorithmic content can be ranked, selected or preserved for dimensional reduction purposes. The method performs an unsupervised lossy compression able to preserve the main features (not covered in grey) of both cellular automata, with no intervention and no parameter choice. MILS proceeds by deleting the regions with lowest algorithmic information content contribution and maximizing
the preservation of the features that contribute the most to the algorithmic description of the objects. In general, the extracted features will not be as clear as in these examples as they may pick more complicated patterns even not statistical based on algorithmic probability. Unlike statistical approaches, the algorithm can also approximate (and thus preserve/extract) features that are of an algorithmic nature and which are not statistically apparent as it was in this case (see [35, 41]) and next examples.

**Figure 1.** In (A) and (B) we show a feature selection and image reduction by application of MILS, starting from the original and second step, highlighting the regions that are earmarked to be omitted (in grey) versus the features that are kept along the way, thereby optimally preserving the main properties of these objects, properties whose persistence enables a ranking of such features. Here can be seen how boundaries are favoured as key features. In (C) we see the image reconstruction by preservation of extracted features from the image with the highest algorithmic probability (indicated by red signatures in the first two image processes). The MILS algorithm selects features which are considered most important in the compressed image reconstruction.
Fig. 2 shows the MILS algorithm applied to images, which we call MILR. Fig. 2(A) shows how vertical and horizontal compression preserves features even if distorting the image hence showing a different mechanisms and goals than those from popular image compression algorithms. The purpose of image compression algorithms such as JPEG is to maximise storage compressibility and require a decoder, here the application is directly to the image itself both at storage and visualisation stages. Also, our purpose is not to maximise storage compression but to minimize the loss of algorithmic content in the reduction process. Fig. 2(C) shows how the algorithm preserves the main features of the image leaving almost intact the formulae.

**Figure 2.** Minimal Information Loss Reduction, a MILS-based lossy compression algorithm. A: Row compression preserves features on even highly coarse-grained versions (B starts from A at 0% but sampled to make it a 100 × 100 image from the original 600 × 600) of the same image producing a cartoonish representation. The image in (B) reduced by 23% is 10% the size of the image at 0% in (A). C: Compressed formula by MILS vs random row deletion (D) with resulting images of the same size but MILS preserving text proportion and minimizing information loss while random row deletion distorts the text.
MILS is highly context-sensitive and adapts to different kinds of information such as images. For example, it handles different ways images as seen in Fig. 2. This is because information in Fig. 2(A) is distributed more uniformly than Fig. 2(C) but, according to the algorithm, some features in certain regions are more important than others and thus some distortions are allowed if such features have high algorithmic content according to the underlying computer programs that reconstruct the compressed images. However, in Fig. 2(C), blank space between formulae can be sacrificed first as they do not contain any information. While this behaviour for handling different cases can be replicated in ad-hoc algorithms, MILS handles it naturally and determines when distorting can be allowed or proportion can be preserved, something that image compression algorithms would not do as their goal is to recover the same image dimensions and maximise storage compression and not to preserve algorithmic information content even sacrificing otherwise fundamental properties of the image for those algorithms such as image dimension.

Comparisons to data reduction and sparsification methods are in Figs. 3, 4, 5, 7, and 8 demonstrate how MILS preserves essential local and global properties of synthetic and natural networks of different types and topologies, performing at least as well as but usually better than leading algorithms in graph sparsification. We took a sample of well-known and previously thoroughly studied networks from [22]. These included genetic regulatory networks, protein, power grid and social networks. We applied MILS to each of these networks and compared with two powerful sparsification methods: Transitive reduction [4] and Spectral sparsification [26]. A transitive reduction of a directed graph is a graph with as few edges as possible that has the same reachability relation as the given graph. A good introduction to spectral graph sparsification may be found in [5]. The method was designed to reduce the network dimension based upon spectral similarity of graph Laplacians which guarantees the preservation of important properties of the graph by way of its adjacency matrix Laplacian spectrum.

Fig. 3 and 4 show how MILS preserves the degree distribution and the edge betweenness distribution of a typical synthetically (recursively) generated Erdős-Rényi (ER) random graph (in this example of low edge density) compared with random edge deletion and spectral sparsification.

While MILS is not significantly better at preserving the clustering coefficient of random networks, Fig. 5 show that MILS does significantly better at preserving the clustering coefficient of real-world (biological,
Figure 3. MILS or neutral edge deletion (blue) outperforms random edge deletion (red) at preserving both edge degree distribution (top, showing removed edges) and edge betweenness distribution (bottom) on an Erdős-Rényi random graph of node size 100 and low edge density (~4%) after up to 60 edges were removed (degree distribution comparison) and 150 edges were removed (edge betweenness) out of a total of 200 edges (notice also the scale differences on the x-axis).

social and electric grid) networks taken from [22], outperforming both Transitive and Spectral reduction/sparsification methods.

Figs. 6, 7 and 8 illustrate how MILS outperforms spectral sparsification at preserving edge betweenness, and degree and eigenvector centralities, respectively.

4.1. PyMILS: Implementation in Python. Algorithm 2 was implemented in a Python package called pymils (see Algorithm 3). This package is freely available and can be downloaded through GitHub. To test the validity of pymils, we used several different images that were converted into binary numpy arrays. Given a single input image, the pseudocode for pymils is the following:

A user can tune pymils (Algorithm 3) with the following parameters:

1. size(image)$_{\text{min}}$: Float $0 - 1$. Minimum final image size ratio (cannot be smaller than 4 rows or columns).
2. Sampling: Float $0 - 1$. Percent of rows/columns to randomly sample at each step.

numpy is a Python package that efficiently stores and handles matrices and arrays.
Figure 4. Histograms showing preservation of degree distribution from 20% to 80% edge removal. Green highlights the overlapping and the preserved area of the distributions after random deletion (top), MILS and spectral removal (bottom pairs).

(3) Chunk size $c$: Integer. Size of neighbouring row/column group to remove at each step.

To show the effectiveness of MILS as implemented through pymils, we first test its ability to compress images from the DataSaurus Dozen [21]. Figure 9 shows the 13 images as binary arrays. Each of these images have the same summary statistics, including the x/y mean, the x/y standard deviation, and linear correlation coefficient to two decimal points. This set of different images has been used as a cautionary tale
Figure 5. MILS mean clustering coefficient preservation against two other sophisticated graph sparsification methods based on graph spectral and transitive reduction techniques on biological, electric and social networks taken from [22]. The transitive method does not allow selection of edges to be deleted, and in some cases it either fails to significantly reduce the network size if no cycles are present (such as, generally, in electric and genetic networks) and/or takes the clustering coefficient to 0 (e.g. for protein networks) if cycles are only local. Comparisons with other methods are unnecessary because they destroy local or global properties by design, such as clustering coefficients for the spanning tree algorithm.

against trusting summary statistics alone without looking directly at the underlying dataset.

Figure 10 shows the initial complexity as approximated by BDM and the subsequent change in complexity after pymils was implemented on each image. From Figure 10 it is clear that the different images have different algorithmic complexity values, in contrast with summary statistics. In addition, pymils is able to compress these images such that the resulting image is close to the same complexity as the original image.
Figure 6. Stacked histograms showing edge betweenness preservation of MILS versus spectral sparsification across different families of networks. The similarity in height of each segment is an indication of the preservation of such properties. Blue bars (MILS) approximate yellow (original) bars better than spectral sparsification. On average MILS was 1.5 times the edge betweenness distribution of these representative graphs measured by the area similarity of the respective bars.

We tried a wide variety of other binary array images, including shapes and 1-dimensional elementary cellular automata (ECA) patterns with random initial states. Figure 11 shows an image of a face before and after image compression via pymils. In this example, pymils removed most of the redundancy from the lower half of the original image while preserving complex features in the top half of the original image.
Figure 7. Stacked histograms showing the preservation of degree centrality after application of MILS versus spectral sparsification across different families of networks: bars with height closest to the original graph signify better preservation. Blue bars (MILS) approximate yellow (original) bars compared with spectral sparsification. MILS only slightly outperformed spectral sparsification in this test but never did worse.

To test the ability of pymils to compress an image while preserving algorithmic complexity, we compared it to two other methods: randomly removing rows/columns and PCA. We randomly removed chunks of rows or columns of the same size from images without measuring the resulting complexity values. The PCA analysis was done with two components (for binary values) on each image and then mapping the resulting values onto an image of the same resulting size of
Figure 8. Stacked histograms showing eigenvector centrality preservation of MILS versus spectral sparsification across the different families of networks: bars with height closest to the graph’s original bar signify better preservation. Blue bars (MILS) approximate yellow (original) bars better than spectral sparsification both in distribution shape and individual bar height. On average MILS preserved the eigenvector centrality distribution of these representative networks 1.5 times better.

pymils (50% the original size). Results from this comparison is shown in Figure 12. As a result, pymils performs much better on average at preserving algorithmic complexity ($\Delta BDM$) during image compression. Both random deletion and PCA can introduce additional noise in the image which results in a negative $\Delta BDM$ value.

In addition to testing whole images, the initial and resulting BDM values were tracked for the four image quadrants in each image. Results from Figure 13 suggests pymils performs relatively similar across each image quadrant.
Algorithm 3 Python implementation of the Minimal Information Loss Selection algorithm (PyMILS)

1: function PYMILS(image)
2:     while size(image) > size(image)_{min} do
3:         bdm_{initial} = BDM(image)
4:         Randomly select one of the following: \((rows, columns)\)
5:         For some chunk size \(c\), randomly select clustered groups of size \(c\) of either rows or columns
6:             for each clustered group, do
7:                 remove that group to get \(image_{step}\)
8:                 measure \(bdm_{step} = BDM(image_{step})\)
9:                 replace that group
10:             end for
11:         Remove the clustered group from \(image\) that would result in a \(bdm_{step}\) closest to \(bdm_{initial}\).
12:     end while
13:     return final image
14: end function

Figure 9. The Datasaurus Dozen datasets [21] visualised as binary arrays.

5. Conclusion

We are motivated by the problem of checksum procedures and embedded decompression instructions popular in lossless compression algorithms such as LZW not being sufficiently sensitive to detect minor changes in algorithmic information content, and at the same time we
Figure 10. Scatterplots showing the initial algorithmic complexity as approximated by BDM (top) and the change in complexity after pymils compressed each image to a 50% the original image size.

want to guarantee that the approximations to this content remain robust to changes in the underlying computation model [29, 39]. Furthermore, Shannon entropy has even greater limitations, as it is only constrained to detect trivial statistical regularities when no other updating procedure is available to properly calculate the likelihood and prior of the underlying ensemble [28, 33, 41].

Here we have introduced MILS, studied its time complexity, and demonstrated how they act differently to traditional ones. The algorithm and methods introduced in the present paper are independent of the method used to approximate algorithmic complexity. In particular,
we employ a state-of-the-art method, called BDM, which is based on algorithmic probability as introduced previously in [13, 25, 39, 41].

First, we have shown that MILS is efficiently time-bounded in both giving the graph and sparsification parameter as inputs and given a chosen algorithmic-complexity approximating algorithm, such as by the BDM. We have shown how MILS outperforms general and leading dimensionality reduction algorithms for networks and, interestingly, that MILS can be generalized to any data, as we have shown on space-time diagrams of discrete dynamical systems (cellular automata) that can also be seen as images. Thus, it makes the algorithms introduced here applicable to challenges of image segmentation. The chief advantage of MILS is that it is optimal whenever an optimal method is used to approximate algorithmic complexity, and it only scales linearly with the worst-time complexity of this chosen method.

To test the algorithm we used a number of well-known networks commonly used in the literature, on which we also applied algorithms that have been reported to outperform previous algorithms. The results provide evidence that our algorithms can outperform some ad-hoc algorithms on most, if not all, indices at preserving features of interest, which we define as all features that are recursively enumerable and therefore possible to characterize using a universal Turing machine,

Figure 11. Example image of a face pasted above blank space before (top) and after (bottom) pymils, with a 50% size reduction.
Figure 12. Boxplots comparing pymils with random row/column deletion and PCA over various test images (top) and ECA images (bottom).

unlike measures that are computable and cannot, even in principle, achieve such a goal.

By presenting theorems, we have investigated the limits of algorithmic-informational impact of perturbations. We have studied upper and lower bounds for the algorithmic information distortions resulting from perturbations. In this way, we have also studied the relationship between local perturbations and the theoretically optimal information content of graphs. Thus, our results demonstrate that numerical approximations to algorithmic information content are sufficiently accurate to outperform other current heuristics—even if computable—techniques for data dimensionality reduction.
Figure 13. Boxplots comparing pymils with random row/column deletion and PCA over various test image quadrants (top) and ECA image quadrants (bottom).

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APPENDIX
Appendix A. Preliminary definitions and background results

A.1. Cellular automata. To illustrate how our algorithm works and can be applied we will use a very well studied dynamical system called a cellular automaton. A cellular automaton is a computer program that applies in parallel a global rule composed of local rules on a tape of cells with symbols (e.g. binary). Broadly speaking, it consists of a set of states, a neighbourhood template and a local rule $f$. Here we will use space-time diagrams of cellular automata to illustrate the way in which the MILS algorithm operates.

Thoroughly studied in [1], Elementary Cellular Automata (or ECA) are defined as one-dimensional cellular automata that take into consideration in their local rules the cells next to the centre cell and the centre cell. In the case of 1-dimensional CA it is common to introduce the radius of the neighbourhood template, which can be written as $\langle -r, -r + 1, \ldots, r - 1, r \rangle$ and has length $2r + 1$ cells. Given a radius $r$, the local rule is a function $f : \mathbb{Z}_{|S|}^{(2r+1)} \rightarrow \mathbb{Z}_{|S|}$, where $S$ is the set of single-cell states.

Elementary Cellular Automata (ECA) have a radius $r = 1$ (closest neighbours), having the neighbourhood template $\langle -1, 0, 1 \rangle$, meaning that the three-cell neighbourhood comprises a central cell. From this it follows that the rule space for ECA contains $2^{2^1} = 256$ rules.

Enumeration of ECA rules: It is common to follow the lexicographic ordering scheme introduced by Wolfram [1]. According to this encoding, the 256 ECA rules can be encoded by 8-bits.

A space-time diagram captures the evolution of a cellular automaton for a given initial condition and is read from the top starting from time step $t = 0$ (the initial condition) and evolving towards the bottom of the diagram (see Figs. 1.(A) and 1.(B) in the main article).

A.2. Classical Information Theory and Shannon Entropy. Central to information theory is the concept of Shannon’s information entropy, which quantifies the average number of bits needed to store or communicate the statistical description of an object.

Definition A.1. Let $X = (R, p)$ be an ensemble, where $R$ is the set of all possible outcomes (the random variable), $n = |R|$ and $p(x_i)$ is the probability of an outcome $x_i \in R$. The Shannon entropy of $X$ is given by

$$H(X) = - \sum_{i=1}^{n} p(x_i) \log_2 p(x_i).$$
It is clear from this definition that in order to calculate \( H(X) \) one has to know (or assume the existence of) a probability mass function of ensemble \( X \).

While entropy can be used as a measure of the information contained in a graph (or another object), we will restrict our analyses to the Block Decomposition Method, which we describe in Section A.6.

A.3. Graph definitions and notation. Graphs (or networks) are mathematical structures that allow a simple representation of systems consisting of a set of agents and a set of interactions between them. Formally, we may define a graph as follows.

**Definition A.2.** A (labeled) graph is an ordered pair \( G = (V, E) \) comprising a finite set \( V(G) = \{1, \ldots, n\} \) of labeled vertices with \( n \in \mathbb{N} \) and a set \( E(G) \subseteq \mathbb{E}(G) = V \times V \) of edges. For simplicity, we will often let \( V = V(G) \) and \( E = E(G) \).

A graph \( H \) is contained in \( G \), and we write \( H \subseteq G \), if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). In this case, we say \( H \) is a subgraph of \( G \). If either \( V(H) \neq V(G) \) or \( E(H) \neq E(G) \), we say \( H \) is a proper subgraph of \( G \), and we normally use \( H \subset G \) to denote this situation. If on the other hand \( V(H) = V(G) \) and \( E(H) = E(G) \), we say the graphs \( H \) and \( G \) are equal.

A directed graph is a graph where the edges have a direction associated with them. In a directed graph (possibly with self-loops), we represent edges by ordered pairs

\[
(v, w) \in \mathbb{E}(G) := \{(x, y) \mid x, y \in V\} = V \times V
\]

instead of unordered pairs as in Definition A.3, such that \( v \) is the source vertex and \( w \) is the target vertex of the edge. We call a traditional directed graph if \( G \) is a directed graph without self-loops, so that

\[
(v, w) \in \mathbb{E}_d(G) := \mathbb{E}(G) \setminus \{(x, x) \mid x \in V\}
\]

**Definition A.3.** A labeled undirected graph \( G = (V, E) \) without self-loops is a labeled graph with a restriction \( \mathbb{E}_e \) in the edge set \( E \) such that each edge is an unordered pair with

\[
E \subseteq \mathbb{E}_e(G) := \{(x, y) \mid x, y \in V\}
\]

We also refer to these graphs as simple labeled (or classical) graphs. In this paper, we will normally refer to simple graphs as just graphs.

\[\text{In this article, we will only deal with labeled graphs.}\]
We say that two vertices $v, w \in V$ are adjacent if there is an edge $e = \{v, w\} \in E$ that connects them, and call $v$ and $w$ the endpoints of such undirected edge. The adjacency matrix $A = (a_{ij})$ of $G$ is a matrix that encodes all the adjacency relationships of the graph. The entries of this matrix are $a_{vw} = 1$ if vertices $v$ and $w$ are adjacent and $a_{vw} = 0$ otherwise. Note that while the adjacency matrix of an undirected graph is always symmetric, this is not necessarily true for a directed graph where adjacency can be asymmetric. The list of eigenvalues of the graph’s adjacency matrix sorted from largest to smallest is known as the spectrum of the graph.

The degree $d(v)$ of $v$ is the number of vertices to which $v$ is connected, that is $d(v) = \sum_{w \in V} a_{vw}$. The degree matrix $D = (d_{ij})$ of $G$ is a diagonal matrix that contains information about the degree of each vertex, and is defined by $d_{ij} = d(i)$ if $i = j$ and $d_{ij} = 0$ otherwise.

We are now in the position to introduce the notion of graph Laplacian, which will be of interest for us in forthcoming sections. Consider the following definition.

**Definition A.4.** Let $G$ be a simple graph on $n$ vertices. The Laplacian matrix (of order $n$) of the graph $G$ is defined as $L = D - A$, where $D$ is the degree matrix and $A$ is the adjacency matrix of $G$.

Throughout the text, and trusting that it does not lead to confusion, we will use the terms nodes and vertices, and links and edges, interchangeably.

**A.4. Spectral sparsification.** Generally speaking, the goal of network sparsification is to approximate a given graph $G$ by a sparse graph $H$ on the same set of vertices. If $H$ is close to $G$ in some appropriate metric, then $H$ can be used as a signature preserving important properties of $G$ for faster computation after reducing the size of $G$ and without introducing too much error. Obvious trivial sparsification methods include edge deletion by some criterion, such as the outermost ones (called the $k$-shell method [21, 37], often used to identify the core and the periphery of the network), but most of these methods (such as this shell one) are rather arbitrary or ad-hoc, rather than general methods aimed at preserving important algebraic, topological or dynamical properties of the original graph, all of which constitute and contribute to the information content of the graph, that is, the information necessary to fully describe a network and reconstruct the network from that description.
A popular sparsification algorithm is the \textit{spanning tree} \cite{16} designed to preserve node distance but clearly destroy all other local node properties, such as the clustering coefficient. Not many non-trivial methods for network sparsification exist today. Some clearly destroy local properties, such as the spanning tree algorithm, which destroys the clustering coefficient. It is acknowledged \cite{5, 26, 57}, however, that spectral graph sparsification is among the most efficient, both at preserving important algebraic and dynamical properties of a network and in terms of fast calculation. In part the dearth of methods is due to a lack of assessment tools to decide whether one method is better than another in general terms, rather than being designed to preserve one or another specific graph theoretic property (e.g. the transitive edge deletion method destroys the clustering coefficient of the original graph \cite{41}). The spectral method considered in this paper is a high-quality algorithm described in \cite{5, 26}.

Transitive reduction was introduced in \cite{4}. A graph $G$ is said to be \textit{transitive} if, for every pair of vertices $u$ and $v$, not necessarily distinct, $(u, v) \in E(G)$ whenever there is a directed path in $G$ from $u$ to $v$. That is, if there is a path from a vertex $x$ to a vertex $y$ in graph $G$, there must also be a path from $x$ to $y$ in the transitive reduction of $G$, and vice versa. If a given graph is a finite directed acyclic graph, its transitive reduction is unique, and is a subgraph of the given graph.

Graph sparsification is the approximation of an arbitrary graph by a sparse graph. We compare MILS against random, simple (e.g. spanning tree) and two powerful graph sparsification and reduction methods (spectral and transitive). Spectral graph sparsification is based on the spectral similarity of graph Laplacians. A spectral sparsifier is a subgraph of the original whose Laplacian quadratic form is approximately the same as that of the original graph on all real vector inputs. Spectral graph sparsification is a stronger notion than cut sparsifiers \cite{57} and is considered one of the most, if not the most, sophisticated sparsification or network reduction method, as it is believed to preserve some of the most important algebraic, topological and potentially dynamical properties of a network.

A.5. \textbf{Graph-theoretic measures.} The global clustering coefficient of $G$ is the fraction of paths of length 2 in $G$ that are closed over all paths of length two in $G$. The mean or average clustering coefficient is the mean over all local clustering coefficients of vertices of $G$.

The betweenness centrality for a vertex $i$ in a connected graph is given by \[
\sum_{s,t \in V \land s \neq i \land t \neq i} \frac{n_{s,t}}{n_{s,t}} \], where $n_{s,t}$ is the number of shortest paths from $s$ to $t$ and $n'_{s,t}$ is the number of shortest paths from $s$ to $t$ passing
through \( i \). The ratio \( \frac{n_{i,t}}{n_{s,t}} \) is taken to be zero when there is no path from \( s \) to \( t \).

Degree centrality is a measure of the centrality of a node in a network and is defined as the number of edges (including self-loops) that lead into or out of the node. The degree centrality of \( G \) is the list of nonnegative integers ("degree centralities") lying between 0 and \( n - 1 \) inclusive, where \( n \) is the number of vertices of \( G \), and identifies nodes in the network by their influence on other nodes in their immediate neighbourhood.

Eigenvector centrality is a list of normalized nonnegative numbers ("eigenvector centralities", also known as Gould indices) that are particular centrality measures of the vertices of a graph. Eigenvector centrality is a measure of the centrality of a node in a network based on the weighted sum of centralities of its neighbours. It therefore identifies nodes in the network that are connected to many other well-connected nodes. For undirected graphs, the vector of eigenvector centralities \( c \) satisfies the equation \( c = \frac{1}{\lambda_1} a . c \), where \( \lambda_1 \) is the largest eigenvalue of the graph’s adjacency matrix \( a \).

A.6. Algorithmic information theory and graph information content. Here, we present some basics concepts regarding algorithmic information theory. For a complete introduction to these concepts and theorems, see [8, 9, 14, 17, 18].

First, for some preliminary notation, let \( \{0,1\}^* \) be the set of all finite binary strings. Let \( |X| \in \mathbb{N} \) denote the number of elements (i.e., the cardinality) in a set, if \( X \) is a finite set, or the number of elements in a data structure, if \( X \) is computational object. In the particular case of strings, let \( l(x) \) denote the length of a finite bit string \( x \in \{0,1\}^* \). Let \( (x)_2 \) denote the binary representation of the number \( x \in \mathbb{N} \). Let \( x \upharpoonright_n \) denote the ordered sequence of the first \( n \) bits of the fractional part in the binary representation of \( x \in \mathbb{R} \). That is, \( x \upharpoonright_n = x_1x_2 \ldots x_n \equiv (x_1, x_2, \ldots, x_n) \), where \( (x)_2 = y.x_1x_2 \ldots x_nx_{n+1} \ldots \) with \( y \in \{0,1\}^* \) and \( x_1, x_2, \ldots, x_n \in \{0,1\} \).

With respect to weak asymptotic dominance of function \( f \) by a function \( g \), we employ the usual \( O(g(x)) \) for the big \( O \) notation, \( \Omega(g(x)) \) for the big \( \Omega \) notation, and \( \Theta(g(x)) \) for the big \( \Theta \) notation; and with respect to strong asymptotic dominance by a function \( g \), we employ the usual \( o(g(x)) \) when \( g \) dominates \( f \), \( \omega(g(x)) \) when \( f \) dominates \( g \), and \( \sim g(x) \) when both \( f \) and \( g \) dominates each other. The reader should pay attention not to confuse the big \( \Omega \) notation (in boldface) \( \Omega(g(x)) \) with the halting probability \( \Omega \) as in Definition [A.10].
A Turing machine \( M \) is a general abstraction of a computer program, similar to cellular automata but sequential, that given an input, may either produce an output (and halt) or never halt. As one of the most fundamental mathematical objects in theoretical computer science, the Turing machine is thus an algorithmically mechanistic causal explanation of a deterministic process’ outcome, and is at the centre of the algorithms introduced here.

With respect to the chosen computational model, let \( L'_U \) denote a binary prefix-free\(^4\) (or self-delimiting) universal programming language for a prefix universal Turing machine \( U \). For example, a program of \( U \) may represent a 2-dimensional (prefix-free) Turing machine (a typical deterministic Turing machine whose single head can move up and down as well as left and right) starting from an empty 2-dimensional grid (instead of the typical 1-dimensional tape Turing machine) as defined in \([40, 41]\).

As in \([14, 17, 18]\):

**Notation A.5.** Let \( \langle \cdot, \cdot \rangle \) denote an arbitrary recursive bijective pairing function. In particular, one can choose a pairing function in which, for every \( x, y \in \{0, 1\}^* \),

\[
l(\langle x, y \rangle) \leq l(x) + l(y) + O\left(\log_2 l(x) + l(y)\right)
\]

holds and the bit string \( \langle x, y \rangle \) is computably recognizable as an ordered pair in polynomial time. For example, with an encoding in the form

\[
\langle x, y \rangle := \underbrace{0 \cdots 0}_{\log_2 l((2)_{\mathbb{N}})} 1 (2)_{\mathbb{N}} \underbrace{0 \cdots 0}_{\log_2 l(x)} 1 (l(x))_{\mathbb{N}} x \\
\underbrace{0 \cdots 0}_{\log_2 l((2)_{\mathbb{N}})} 1 (2)_{\mathbb{N}} \underbrace{0 \cdots 0}_{\log_2 l(y)} 1 (l(y))_{\mathbb{N}} y.
\]

This notation can be recursively extended to \( \langle \cdot, \langle \cdot, \cdot \rangle \rangle \) and, then, to an ordered 3-tuple \( \langle \cdot, \cdot, \cdot \rangle \). Thus, this iteration can be recursively applied with the purpose of defining self-delimited finite ordered \( n \)-tuples \( \langle \cdot, \ldots, \cdot \rangle \).

**Definition A.6.** The (unconditional) prefix\(^5\) algorithmic complexity (also known as K-complexity, prefix Kolmogorov complexity, prefix

---

\(^4\)The group of valid programs constitutes a prefix-free set (no element is a prefix of any other, which is a property necessary to keep \( 0 < m(G) < 1 \)).

\(^5\)In this paper, we will normally refer to prefix algorithmic complexity as just algorithmic complexity.
program-size complexity or prefix Solomonoff-Komogorov-Chaitin complexity) of a finite binary string \( w \), denoted here by \( C(w) \), is the length of the shortest program \( w^* \in L_U' \) such that \( U(w^*) = w \).

In addition, when an input is informed to the Turing machine, we also have that the conditional prefix algorithmic complexity of a binary finite string \( y \) given a binary finite string \( x \), denoted by \( C(y|x) \), is the length of the shortest program \( w \in L_U' \) such that \( U(\langle x, w \rangle) = y \).

Note that \( C(y) = C(y|\epsilon) \), where \( \epsilon \) is the empty string. Moreover, the joint prefix algorithmic complexity of strings \( x \) and \( y \) is denoted by \( C(x,y) := C(\langle x,y \rangle) \).

From algorithmic information theory \([8, 9, 14, 18]\), we know the following results:

**Lemma A.1.** For every \( x,y \in \{0,1\}^* \) and \( n \in \mathbb{N} \),

\[
\begin{align*}
(4) & \quad C(x) \leq l(x) + O(\log_2(l(x))) \\
(5) & \quad C(y|x) \leq C(y) + O(1) \\
(6) & \quad C(y) \leq C(xy) + C(l(x)) + O(1) \\
(7) & \quad C(xy) \leq l(y) + C(l(y)) + C(x) + O(1) \\
(8) & \quad C(x) \leq C(xy) + O(1) \leq C(y) + C(x|y) + O(1) \\
(9) & \quad C(n) = O(\log_2(n))
\end{align*}
\]

**Lemma A.2.** Let \( f_c : \mathbb{N} \to \mathbb{N} \) be a computable function, then

\[
C(f_c(n)) \leq C(n) + O(1)
\]

For graphs (as defined in Section A.3) we use the notation:

**Notation A.7.** Let \( (e_1, \ldots, e_{|E(G)|}) \) be a previously fixed ordering (or indexing) of the set \( E(G) \). For an edge set \( E(G) \), let \( C(E(G)) := C(\langle E(G) \rangle) \), where the edge set string \( \langle E(G) \rangle \) denotes

\[
\langle \langle e_1, z_1 \rangle, \ldots, \langle e_n, z_n \rangle \rangle
\]

such that\(^6\)

\[
z_i = 1 \iff e_i \in E(G)
\]

\(^6\) \( w^* \) denotes the lexicographically first \( p \in L_U' \) such that \( l(p) \) is minimum and \( U(p) = w \).

\(^7\) For the sake of simplicity, we choose to omit the ceiling operator on the logarithm when calculating the length of encoding natural numbers, so that in such cases \( \log_2(x) \) actually stands for \( \lceil \log_2(x) \rceil \) in the present paper. Note that in terms of asymptotic dominance and bounds it is indeed irrelevant.

\(^8\) See also Definition A.8.
where $z_i \in \{0, 1\}$ with $1 \leq i \leq n = |E(G)|$. The same applies analogously to the conditional and joint cases of the algorithmic complexity.

From [9], we remember that the family of all possible simple graphs is always recursively labeled. That is, there is a unified recursive way to encode a sequence of all possible undirected edges given any unordered pair $\{x, y\}$ of natural numbers $x, y \in \mathbb{N}$. Thus, we know that, for any arbitrarily chosen method of recursively encoding a matrix into a string,

$$|C(G) - C(E(G))| = O(1)$$

holds, where $C(G)$ denotes the algorithmic complexity of generating the adjacency matrix of any arbitrarily chosen $G$. Hence, one can interchangeably employ $C(G)$ or $C(E(G))$ to measure the algorithmic information content of a graph when dealing with the theoretical optimal form of the algorithmic complexity as in Definition A.6. Moreover, from [9], we will have that

\begin{align}
10 & \quad C(G | x) = O(1) \\
11 & \quad C(x | G) = O(1) \\
12 & \quad C(x) = C(G) \pm O(1),
\end{align}

where $x \in \{0, 1\}^*$ is the characteristic string of $G$. The characteristic string of a graph is any bit string that determines the presence or absence of edges in $G$.

**Definition A.8.** Let $(e_1, \ldots, e_{|E(G)|})$ be any arbitrarily fixed ordering of all possible edges of a graph $G$ as in Notation A.7. We say that a string $x \in \{0, 1\}^*$ with $l(x) = |E(G)|$ is a characteristic string of a graph $G$ iff, for every $e_j \in E(G)$,

$$e_j \in E(G) \iff \text{the } j\text{-th digit in } x \text{ is } 1,$$

where $1 \leq j \leq l(x)$.

Thus, we have that the algorithmic information content of a graph is equivalently captured (except for a constant that only depends on the chosen universal programming language) by the algorithmic complexity of the adjacency matrix, the edge set string, or of the characteristic string. Note that the same cannot always be said in the case of multilayered or multidimensional networks [1, 2]. See also [9, 10] for more details on characteristic strings and algorithmically characteristic strings.

The concept of algorithmic probability (also known as Levin’s semi-measure) yields a method for approximating algorithmic complexity.

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9And the same definition applies analogously to the simple graph case.
related to the frequency of patterns in the adjacency matrix of a network, including therefore the number of subgraphs in a network. The algorithmic probability \[23, 40, 55\] of a graph \(G\) is a measure that describes the probability that a random computer program \(p\) will produce \(G\) when run on a 2-dimensional tape universal (prefix-free) Turing machine \(U\). That is, \(m(G) = \sum_{p: U(p) = A_G} 1/2^{|p|}\). An example of a popular 2-dimensional tape Turing machine is Langton’s ant [39], commonly referred to as a Turmite.

The probability semi-measure \(m(G)\) is related to algorithmic complexity \(C(G)\) in that \(m(G)\) is at least the maximum term in the summation of programs \(m(G) \geq 2^{-C(G)}\), given that the shortest program carries the greatest weight in the sum. The Algorithmic Coding Theorem [8, 9, 14, 18] further establishes the connection between \(m(G)\) and \(C(G)\) as:

\[
| - \log_2 m(G) - C(G) | < c ,
\]

where \(c\) is some fixed constant, independent of \(G\). This way, the size \(C(G)\) of the algorithmic information content of a graph \(G\) (i.e., its prefix algorithmic complexity) is given by the so-called algorithmic coding theorem [11, 25, 40] by simply rewriting Eq. 13 as:

\[
C(G) = - \log_2 m(G) \pm O(1) .
\]

In [13] a technique was advanced for approximating \(m(G)\) by means of a function that considers all Turing machines of increasing size (by number of states). Indeed, for Turing machines \(M\) with \(n\) states, \(k\) alphabet symbols (usually 2 alphabet symbols only), and empty input \(\lambda\) in which \(n\) is small enough, we have that

\[
D(n, k)(s) = \frac{|\{(M \in (n, k) : M(\lambda) = s)\}|}{|\{(M \in (n, k) : \exists w' (M(\lambda) = w')\}|}
\]

is computable for values of the Busy Beaver problem [52] that are known, providing a means to numerically approximate the algorithmic complexity of small graphs, such as network motifs. In Equation 15 \(M(x) = y\) denotes machine \(M\) returning \(y\) with input \(x\) and, therefore, \(D(n, k)(s)\) gives the frequency of halting Turing machines with \(n\) states and \(k\) alphabet symbols that ended up returning \(s\) as output. The Algorithmic Coding Theorem then establishes that graphs produced with lower frequency by random computer programs have higher algorithmic complexity, and vice versa. In this article, we will use the Block Decomposition Method (BDM) [32, 40, 41] as an estimator of algorithmic complexity \(C(G)\) and, hence, of the information.
difference \( I(G, e_i) \), but the algorithm and methods introduced are independent of the particular method used to approximate algorithmic complexity.

The BDM consists in decomposing the adjacency matrix of a graph into subgraphs of sizes for which complexity values have been estimated, then reconstructing an approximation of the algorithmic complexity of the graph by adding the complexity of the individual pieces according to the rules of information theory, as follows:

\[
K_{\text{log}}(G) = \sum_{(r_u, n_u) \in \text{Adj}(G)_{d \times d}} \log_2(n_u) + K_m(r_u)
\]

where \( \text{Adj}(G)_{d \times d} \) represents the set with elements \((r_u, n_u)\) obtained when decomposing the adjacency matrix of \( G \) into all subgraphs of size \( d \) contained in \( G \) and \( K_m(r_u) \) is an approximation to \( C(r_u) \) using the Coding Theorem Method (CTM) \cite{13, 24, 25, 39} from Equations 14 and 15. In each \((r_u, n_u)\) pair, \( r_u \) is one such submatrix of the adjacency matrix and \( n_u \) its multiplicity (number of occurrences). As can be seen from the formula, repeated subgraphs only contribute to the complexity value with the subgraph BDM complexity value once plus a logarithmic term as a function of the number of occurrences. This is because the information content of subgraphs is only sub-additive, as one would expect from the growth of their description lengths. Applications of \( m(G) \) and \( C(G) \) have been explored in \cite{13, 24, 25, 39}, and include applications to graph theory and complex networks \cite{40} and \cite{39}, where the technique was first introduced.

The only parameters used in the application of BDM are the use of strings up to 12 bits for strings and 4 bits for arrays given the current best CTM approximations \cite{25} and the suggestions in \cite{41} based on an empirical distribution based on all Turing machines with up to 5 states, and with no string/array overlapping in the decomposition for maximum efficiency (as it runs in linear time), and for which the error (due to boundary conditions) has been shown to be bounded \cite{41}.

**Definition A.9.** We say a real number \( x \in [0, 1] \subseteq \mathbb{R} \) is 1-random (\( K \)-incompressible up to a constant, \( K \)-random or prefix algorithmically random) if and only if it satisfies

\[
C(x \mid _n) \geq n - O(1)
\]

where \( n \in \mathbb{N} \) is arbitrary.

In order to avoid ambiguities between plain and prefix algorithmic complexity and ambiguities in relation to randomness deficiencies, we
choose to say that an algorithmically random real number with respect to prefix algorithmic complexity in Definition A.9 is $O(1)$-K-random.

One of the most important results in algorithmic information theory is the investigation and proper formalisation of a mathematical theory for randomness [8, 14, 18]. We will employ one of these important mathematical objects: the halting probability (see Definition A.10). This is a widely known example of infinite binary sequence, or real number, that is algorithmically random with respect to prefix algorithmic complexity.

**Definition A.10.** Let $\Omega \in [0, 1] \subset \mathbb{R}$ denote the *halting probability* (also known as Chaitin’s constant or Omega number). The halting probability is defined by

$$\Omega = \sum_{\exists y (U(p) = y)} \frac{1}{\mathcal{U}(p)}$$

**Theorem A.3.** Let $n \in \mathbb{N}$. Then,

$$C(\Omega \upharpoonright n) \geq n - O(1)$$

That is, $\Omega$ is $O(1)$-K-random.

The same analogous notion of incompressibility can be assigned to classical graphs (see Definition A.3) as in [9, 10]:

**Definition A.11.** We say a simple graph $G$ is $O(1)$-K-random if and only if

$$C(E(G)) \geq \left(\frac{|V(G)|}{2}\right) - O(1)$$

This way, we have from [9] that:

**Corollary A.4.** There is an infinite number of classical graphs $G$ (as in Definition A.3) that are $O(1)$-K-random. In particular, those in which their characteristic strings are long enough initial segments of a $O(1)$-K-random real number.

A.7. **Information difference.** The information difference is based on a concept of algorithmic/causal perturbation analysis as introduced in [32, 34, 40]. The procedure consists in the perturbation of all elements of a system by the removal of elements whose effects on its algorithmic information content are measured and ranked accordingly. Formally:

**Definition A.12** (Definition 2.1 in the main article). Let $G$ be a graph with edges $e_1, \ldots, e_{|E(G)|} \in E(G) \subseteq \mathbb{E}(G)$. Now, let $G \setminus e_i$ denote the
graph obtained by removing $e_i$ from $G$, and $I(G, e_i)$ denote the information difference or information value/contribution of $e_i$ to $G$ given by

$$ I(G, e_i) := C(G) - C(G \setminus e_i) $$

where $C(G)$ is the algorithmic complexity of graph $G$. If one allows multiple edge deletions of a subset $F \subseteq E(G)$, we analogously define

$$ I(G, F) := C(G) - C(G \setminus F) $$

Also, for the analogously inverse operation of edge insertion $G \cup e_i$, we denote the information difference by

$$ I^{-1}(G, e_i) := C(G) - C(G \cup e_i) $$

and

$$ I^{-1}(G, F) := C(G) - C(G \cup F) $$

respectively.

These differences can also be referred to the graph’s nodes, but here we will restrict our attention to the edges. In addition, we focus our results on destructive perturbations, i.e., the edge deletion problem. Nevertheless, we will investigate in the next Section B.1 the edge-insertion case (i.e., edge insertion) as an analogous inverse case of edge deletion. This will become most relevant in the proof of Theorem B.1.

When taking the difference $C(G) - C(G \setminus e_i)$ by itself we will refer to it as the graph (dis)similarity between graph $G$ and $G \setminus e_i$. $I(G, e_i)$ applied to graphs suggests a similarity “distance” between graphs based on algorithmic information content (in [33], we show that this similarity measure can classify networks by the family they belong to, differentiating variant synthetic and natural network topologies similar to graph motifs, as shown in [22]).

In the description of the algorithm to be presented in Section A.8, replacing the underlying methods to approximate the algorithmic information content (which is the theoretically optimal measure of information content) by, e.g., Shannon entropy or lossless compression algorithms represents special cases of the more general algorithm based on algorithmic complexity, and thus it covers all these less powerful cases. The idea of a dynamic study/calculus of the (possible) changes that can be wrought upon an object to evaluate the contribution of each of its components for different purposes was introduced in [32, 37], and here we extend these ideas to the area of data/network dimension reduction.

Note that MILS can actually be applied to any general object and only to graphs or networks. Let $U$ be an object (possibly multidimensional, like those in [1, 2]) and $I(U, u_i)$ be the information difference of
an element \( u_i \in U \) to \( U \), defined by \( I(U, u_i) = C(U) - C(U \setminus u_i) \) [30]. If \( u_i \) is a neutral element, then by definition it does not contribute to the algorithmic information content of \( U \), and so \( I(U, u_i) = \log |U| \), which indicates that \( u_i \) is part of the dynamical causal path of \( U \) and \( U \) can regenerate \( u_i \) by its normal dynamical course. In general, however, if there is no element \( u_i \) such that \( I(U, u_i) = \log |U| \), there is always the most neutral element (possibly not unique) in \( U \) to be removed in the application of MILS. Moreover, for the most neutral element \( u_i \), we will have that

\[
I(U, u_i) = \min\{|C(U) - C(U \setminus u_i)| \sim \log |U|\}
\]

that is, the element with an information contribution closest to \( \log |U| \). This result can be directly obtained by an analogous argument to those in Section B.1.

A.8. An algorithm for minimal information loss selection (MILS).

MILS is an unsupervised and mostly parameter-free algorithm, i.e., asymptotically independent of model or domain, as it does not need to be instructed or designed to preserve any particular property, and maximises the preservation of all computable elements that contribute to the algorithmic information content of the data.

Let \( G \) be a graph and \( k \) denote its number of edges. Ideally, MILS seeks to reduce \( G \) to a graph on \( N < k \) edges so that the loss of information is minimized after each step of the algorithm. It does this by calculating the information differences between \( G \) and \( G \setminus S \), where \( |S| \geq 1 \), and then finding the subset \( F \subset E(G) \) such that \( I(G, F) \leq I(G, S) \) for all non-empty proper subsets of edges \( S \subset E(G) \), repeating this task with \( G = G \setminus F \) until the target size is reached (i.e., when \( |E(G)| = N \)). The algorithm’s time complexity is clearly in \( O(\exp) \) because of how MILS performs searches and deletions across all subsets of edges, but significant improvements to this bound are still possible.

A more efficient but suboptimal version of such an exponential-time MILS iterates over single elements (in the case, edges) or singletons. Algorithm 4 performs sequential deletions of edges, and then removes the edge that contributes less to the information content of the graph before moving to the next step.

When \( e \) is such that \( I(G, e) = \text{minLoss} \), we call it a neutral information edge [32] because it is the edge that contributes less to the information content of \( G \) (in particular, it minimizes information loss or the introduction of spurious information into the network according to the information difference when removed from the original network). The pseudocode in Algorithm 4 assumes that there is a unique such \( e \),
Algorithm 4 MILS with sequential perturbation by deletion on edges

1: function Perturb\( G \)
2: \[
\text{informationLoss} \leftarrow \emptyset
\] // for each edge \( e \)
3: \textbf{for} \( e \in E(G) \) \textbf{do}
4: \[
\text{informationLoss} \leftarrow \text{informationLoss} \cup \{I(G,e)\}
\]
5: \textbf{end for}
6: // calculate minimal information loss across all edges
7: \( \text{minLoss} \leftarrow \min(\text{informationLoss}) \)
8: \textbf{for} \( e \in E(G) \) \textbf{do}
9: \[
\text{if } I(G,e) = \text{minLoss} \text{ then}
\]
10: \[
\text{return } G \setminus e
\]
11: \textbf{end if}
12: \textbf{end for}
13: \textbf{end function}

14: function Sparsify\( (G, N), 1 \leq N \leq |E(G)| \)
15: // until reaching target size
16: \textbf{while} \( |E(G)| > N \) \textbf{do}
17: \[
G \leftarrow \text{Perturb}(G)
\]
18: \textbf{end while}
19: \textbf{return} \( G \)
20: \textbf{end function}

which may not necessarily be the case. Algorithm 5 solves this problem by performing simultaneous perturbations on all edges with an information contribution of \( \text{minLoss} \). In Algorithm 5, we also introduce InfoRank, a method that produces a ranking of \( e_1, \ldots, e_{|E(G)|} \) from least informative to most informative edge, i.e., a list of edges sorted in increasing order by their information contribution to \( G \). This ranking facilitates the search for the most neutral elements of the system (see Section B.8), which in turn helps MILS preserve the components that maximise the information content of the resulting object.

As described in [32], one can easily modify these algorithms in order to deal with multiple edge deletions at once, as in \( I(G,F) \), but possibly at the expense of much more computational resources. In addition, note that algorithms 4 and 5 may be applied, mutatis mutandis, to nodes or...
Algorithm 5 Minimal Information Loss Selection (MILS)

1: function InfoRank($G$)
2:    $\text{informationLoss} \leftarrow \emptyset$
3:     // for each edge $e_i$
4:     for $e_i \in E(G)$ do
5:         // store information contribution into $\text{informationLoss}$
6:         $\text{informationLoss} \leftarrow \text{informationLoss} \cup \{I(G,e_i)\}$
7:     end for
8:     sort $\text{informationLoss}$ in increasing order
9:     // return information rank
10:    return $\text{informationLoss}$
11: end function

12: function MILS($G,N$), $1 \leq N \leq |E(G)|$
13:     while $|E(G)| > N$ do
14:         // calculate minimal loss across all edges
15:         $\text{minLoss} \leftarrow \min(\text{InfoRank}(G))$
16:         // remove all candidate edges from $G$
17:         $G \leftarrow G \{e_i \in E(G) : I(G,e_i) = \text{minLoss}\}$
18:     end while
19:     return $G$
20: end function

to any data element of an object or dataset, e.g. a pixel of an image, or a row or column in a spreadsheet.

MILS is, by design, optimal in the computability and algorithmic-information theoretic sense, and only dependent on the method for approximating algorithmic complexity in the preservation of any possible feature of interest that contributes to the (algorithmic) information content of a network such as, evidently, its degree distribution and other graph-theoretic, algebraic or topological features, even those not necessarily captured by any graph theoretic measure or classical information approach \[35, 36\].

In Section B.8 we will show that Algorithm 5 is deterministic, polynomially time bounded, and describes a criterion to select and remove the most neutral elements of an object. We employ this more efficient version in all our experiments, and even in this limited form the procedure excels at preserving important characteristics of the networks under study. See Theorem B.8.

A.9. MILS and other dimensionality reduction methods. Other dimension reduction methods such as UMAP, AE, VAE, t-SNE, PHATE,
and others (for surveys see [7, 12, 15]) are, in a deep fundamental sense, special cases of MILS. This is because MILS implements a universal algorithm that is ultimately a generalisation of all other computable methods based on probability distributions, algebra or topology. For example, t-SNE and UMAP, are heavily based on building probability distributions thus they will suffer a similar fate to those reported in [35]. In any event, the ergodicity of the algorithmic complexity approximation implemented by the coding theorem method (CTM) over the software space implies that the BDM will eventually encompass all them if enough computational resources are expended. In other words, even the most sophisticated algorithms implementing algebraic or topological feature selection beyond those implementing classical probability approaches will be eventually instantiated by CTM (if the algebraic or topological features are recursively enumerable) and therefore MILS can be seen as an asymptotic generalisation of all other algorithms. The question is simply which one may extract certain features of interest faster and which one will be less biased and more universal in the sense that features are not pre-selected (e.g. algebraic or topological) like when picking a method that will maximise but also bias towards some properties of the data in an embedding. MILS, in that sense, is the most universal and agnostic but will not necessarily overtake all others in all applications at the same rate. Only with unbounded computational resources it will overtake all.
Appendix B. Theoretical results

B.1. Algorithmic information oscillations from local perturbations.

Theorem B.1 (Theorem 3.1 in the main article). Let $G = (V, E)$ be an arbitrary simple graph with $E \subseteq \mathbb{E}_c(G)$ and $|V| = N \in \mathbb{N}$. Let $F \subseteq \mathbb{E}_c(G)$ be an arbitrary subset of edges, where $|F| = r \geq 1$. Then,

$$|I(G, F)| \leq 2 |F| \log_2 (N) + O(\log_2 (|F|)) + O(\log_2 (\log_2 (N))) \ ,$$

if edges in $F \neq \emptyset$, where $F \subseteq E(G)$, are being deleted, or

$$|I^{-1}(G, F)| \leq 2 |F| \log_2 (N) + O(\log_2 (|F|)) + O(\log_2 (\log_2 (N))) \ ,$$

if edges in $F \neq \emptyset$, where $F \subseteq \mathbb{E}_c(G) \setminus E(G)$, are being inserted.

Proof. We will split the proof in two cases. First, we will prove

\begin{equation}
C(G) \leq C(G \setminus F) + 2 |F| \log_2 (N) + O(\log_2 (|F|)) + O(\log_2 (\log_2 (N))) \nonumber \end{equation}

and then

\begin{equation}
C(G \setminus F) \leq C(G) + 2 |F| \log_2 (N) + O(\log_2 (|F|)) + O(\log_2 (\log_2 (N))) \nonumber \end{equation}

First, let $F \subseteq \mathbb{E}_c(G)$ be an arbitrary subset of edges. Let $p_1 \in \{0, 1\}^*$ be a string that represents the algorithm running on a prefix universal Turing machine $U$ such that $p_1$ receives $a \in L_U$, $b \in \{0, 1\}$, and $k, r \in \mathbb{N}$ as inputs. Then, $p_1$ calculates $U(a)$ and check whether or not $U(a) = A_{G'}$ for some graph $G'$, according to an arbitrarily chosen recursively bijective encoding of an arbitrary adjacency matrix into a bit string. Then, $p_1$ execute the following procedure:

- If $U(a) = A_{G'}$, then:
  - if $b = 0$ (i.e., edges were deleted), then:
    $p_1$ enumerates the subsets of size $r$ in the power set $\mathcal{P}(\overline{E}(G'))$,
    which comprises all the subsets of the set
    $$\overline{E}(G') = \mathbb{E}_c(G') \setminus E(G')$$
    of absent undirected edges (i.e., the symmetric zeros) in $A_{G'}$. Then, $p_1$ replaces in $A_{G'}$ every 0 with 1 that corresponds to the elements in the $k$-th subset $X_k \subseteq \overline{E}(G')$ in
this enumeration of subsets, where $|X_k| = r$. Finally, $p_1$ returns this resulting adjacency matrix of graph $G' \cup X_k$.

- if $b = 1$ (i.e., edges were inserted), then:
  $p_1$ enumerates the subsets of size $r$ in the power set $\mathcal{P}(E(G'))$, which comprises all the subsets of the set $E(G') \subseteq E_{c}(G')$ of present undirected edges (i.e., the symmetric 1’s) in $A_{G'}$. Then, $p_1$ replaces in $A_{G'}$ every 1 with 0 that corresponds to the elements in the $k$-th subset $X_k \subseteq E(G')$ in this enumeration of subsets. Finally, $p_1$ returns this resulting adjacency matrix of graph $G' \setminus X_k$.

- Otherwise, $p_1$ starts any arbitrarily chosen non-halting computation.

Note that, since $(G \setminus F) \cup F = G$ and $(G \cup F) \setminus F = G$ and $C(p_1)$ only varies by $O(1)$ on choosing between deletions and insertions, we can study only the edge-deletion case without loss of generality.

Also note that one can always define a recursively bijective encoding of an arbitrary adjacency matrix into a bit string. Therefore, if $a = (A_{G'})^*$ is the shortest program that generates the adjacency matrix $A_{G'}$ of an arbitrary graph $G'$, then program $p_1$ always halts in the cases:

- $b = 0$, $k \leq \left( \left\lceil \frac{|E(G')|}{r} \right\rceil \right)$, and $1 \leq r \leq |E(G')|$ or;
- $b = 1$, $k \leq \left( \left\lceil \frac{|E(G')|}{r} \right\rceil \right)$, and $1 \leq r \leq |E(G')|$.

**Proof of the upper bound**[17]: We have that there are only

$$
|E(G \setminus F)| = |E_c(G \setminus F)| - |E(G \setminus F)| = |E_c(G)| - |E(G \setminus F)| =
$$

$$
= \frac{N^2 - N}{2} - |E(G)| + |F| =
$$

$$
\leq 2 \left( \frac{N^2 - N}{2} \right) = O(N^2)
$$

possible absent edges in $G \setminus F$. We know that the exact collection $F$ of edges that were deleted in $G$ can be univocally encoded in less than or equal to

$$
\log_2 \left( \left\lceil \frac{|E(G \setminus F)|}{r} \right\rceil \right) +
$$

$$
O \left( \log_2 \left( \log_2 \left( \left\lceil \frac{|E(G \setminus F)|}{r} \right\rceil \right) \right) \right)
$$

(20)
bits. Note that
\begin{equation}
\binom{\binom{E(G \setminus F)}{r}}{r} = \frac{\binom{E(G \setminus F)}{r}!}{r!(\binom{E(G \setminus F)}{r} - r)!} < 2|E(G \setminus F)|.
\end{equation}

In addition, we have from Equation 19 that, if \( r = |F| \), then
\begin{equation}
\frac{\binom{E(G \setminus F)}{r}!}{r!(\binom{E(G \setminus F)}{r} - r)!} = \frac{(|E_c(G \setminus F)| - |E(G)| + |F|)!}{|F|!(|E_c(G \setminus F)| - |E(G)| + |F|)!} = \frac{|F|!(|E_c(G \setminus F)| - |E(G)|)!}{(|E_c(G \setminus F)| - |E(G)|)!} = \frac{(|E_c(G \setminus F)| - |E(G)| + 1) \cdots (|E_c(G \setminus F)| - |E(G)| + |F|)}{|F| \cdots 1} \leq (|E_c(G)| - |E(G)| + |F|)^{|F|} \leq \left(2 \left(\frac{N^2 - N}{2}\right)^{|F|}\right) \leq (N^2)^{|F|}.
\end{equation}

Now, let \( X_k \) corresponds to the exact subset \( F \) of edges, \( a = (A_{G \setminus F})^* \), \( b = 0 \), and \( r = |F| \). Note that, by supposition and construction, \( |F| \leq |E(G \setminus F)| \) and \( |F| \geq 1 \). Therefore, by the minimality of the prefix algorithmic complexity \[9, 14, 18\], the construction of program \( p_1 \), and Equations 20, 21, and 22 we will have that
\begin{equation}
C(G) \leq l((a, b, k, r, p_1)) \leq \frac{C(G \setminus F) + \log_2 \left\lceil \binom{|E(G \setminus F)|}{r} \right\rceil}{r!(\binom{|E(G \setminus F)|}{r} - r)!} + O\left(\log_2 \left(\log_2 \left(\frac{|E(G \setminus F)|}{r}\right)\right)\right) + O(1) \leq C(G \setminus F) + 2 |F| \log_2 (N) + O\left(\log_2 (2 |F| \log_2 (N)) \right) + \log_2 (|F|) + O\left(\log_2 (\log_2 (|F|))\right) + O(1) \leq C(G \setminus F) + 2 |F| \log_2 (N) + O\left(\log_2 (|F|)\right) + O\left(\log_2 (\log_2 (N))\right)
\end{equation}

Proof of the lower bound \[18\]: We have that there are only
\begin{equation}
|E(G)| \leq \frac{N^2 - N}{2} = O\left(N^2\right)
\end{equation}
present edges in $G$. We know that the exact collection $F$ of edges that were deleted in $G$ (or inserted in $G \setminus F$) can be univocally encoded in

$$\log_2 \left( \left( \frac{|E(G)|}{r} \right) \right) + O \left( \log_2 \left( \log_2 \left( \left( \frac{|E(G)|}{r} \right) \right) \right) \right)$$

bits. From Equation 27, we also analogously have as in Equation 19 that, if $r = |F|$, then

$$\left( \frac{|E(G)|}{r} \right) = \frac{|E(G)|!}{|F|!(|E(G)| - |F|)!} \leq \frac{|E(G)|}{|F| \cdots 1} \leq |E(G)|^{|F|} \leq \left( \frac{N^2 - N}{2} \right)^{|F|} \leq (N^2)^{|F|}.$$

Now, let $X_k$ corresponds to the exact subset $F$ of edges, $a = (A_G)^*$, $b = 1$, and $r = |F|$. Note that, by supposition, $|F| \leq |E(G)|$ and $|F| \geq 1$. Therefore, by the minimality of the prefix algorithmic complexity [9, 14, 18], the construction of program $p_1$, and Equations 28 and 29, we will have that

$$C(G \setminus F) \leq l((a, b, k, r, p_1)) \leq$$

$$\leq C(G) + \log_2 \left( \left( \frac{|E(G)|}{r} \right) \right) + O \left( \log_2 \left( \log_2 \left( \left( \frac{|E(G)|}{r} \right) \right) \right) \right) + O(1) \leq$$

$$\leq C(G) + 2|F| \log_2 (N) + O(\log_2 (2|F| \log_2 (N))) + \log_2 (|F|) + O(\log_2 (\log_2 (|F|))) + O(1) \leq$$

$$\leq C(G) + 2|F| \log_2 (N) + O(\log_2 (|F|)) + O(\log_2 (\log_2 (N))) \leq$$

$$\square$$

If one wants tighter lower and upper bounds for multiple edge deletions as a function of the edge density (i.e., the size of $|E(G)|$ in comparison to $\frac{N^2 - N}{2}$), then one can re-write the statement of Theorem B.1.
and achieve:

\[
(32) \quad -|F| \log_2 (|E(G)|) - O(\log_2 |F|) - O(\log_2 (|E(G)|)) - O(\log_2 (\log_2 |F|)) - O(1) \\
\leq C(G) - C(G \setminus F) \leq \\
|F| \log_2 \left( \frac{N^2 - N}{2} - |E(G)| + |F| \right) + O(\log_2 |F|) + \\
O \left( \log_2 \left( \log_2 \left( \frac{N^2 - N}{2} - |E(G)| + |F| \right) \right) \right) + \\
O(\log_2 (\log_2 |F|)) + O(1)
\]

As one might intuitively expect, when the frequency of edge occurrences is nearly equidistributed (in particular, when the graph is incompressible), we can also retrieve from algorithmic information theory [8, 9, 14, 18] some of the particular cases in which the loss and gain of algorithmic information are both dominated by a slightly more-than-linear factor of the number of destructive perturbations:

**Corollary B.2.** Let \( G = (V, E) \) be an arbitrary \( O(1) \)-K-random simple graph (as in Definition A.11) with \( E \subseteq E_c(G) \) and \( |V| = N \), where \( N \in \mathbb{N} \) is large enough such that

\[
|E(G)| - \frac{N^2 - N}{4} \leq O \left( \sqrt{\frac{N^2 - N}{2} \log_2 \left( \frac{N^2 - N}{2} \right)} \right)
\]

holds. Let \( F \subseteq E_c(G) \) be an arbitrary subset of edges, where \( |F| = r \geq 1 \). Then,

\[
-|F| O(\log_2 (N)) - O(\log_2 |F|) - O(\log_2 (\log_2 (N))) \leq \\
C(G) - C(G \setminus F) \leq \\
|F| O(\log_2 (N)) + O(\log_2 |F|) + O(\log_2 (\log_2 (N)))
\]

if edges in \( F \neq \emptyset \), where \( F \subseteq E(G) \), are being deleted.

**Proof.** From [9], as stated in Section A.6, we have that

\[
C(E(G)) = C(G) = C(x) \pm O(1)
\]

where \( x \) is the characteristic string of \( G \). Therefore, in accordance with Definitions A.8 and A.11, we will have that

\[
C(x) \geq l(x) - O(1)
\]

In this case, from the Borel normality of the string \( x \) [8, 18], we know there is a large enough characteristic string of length \( N_0 - N_0 \) such that the condition in Equation A.33 is always satisfied for every \( N \geq N_0 \in \mathbb{N} \).
Now, remember Equation 19. Therefore, to achieve the desired proof, just replace $|E(G)|$ with
\[
\frac{N^2 - N}{4} \pm O \left( \sqrt{\frac{N^2 - N}{2} \log_2 \left( \frac{N^2 - N}{2} \right)} \right)
\]
into Equation 32. □

Note that, if the number of perturbations are about the same order of the present edges in $G$, then Equation 32 would tell us that the algorithmic information loss is allowed to be logarithmically stronger than the algorithmic information gain.

One can modify Theorem B.1 (or Equation 32) and find an alternative form that does not depend on $|F|$ and edge density:

**Corollary B.3.** Let $G = (V, E)$ be an arbitrary simple graph (as in Definition A.3) with $E \subseteq \mathbb{E}_e(G)$ and $|V| = N \in \mathbb{N}$. Let $F \subseteq \mathbb{E}_e(G)$ be an arbitrary subset of edges, where $|F| = r \geq 1$. Then,
\[
|I(G, F)| = O(N^2),
\]
if edges in $F \neq \emptyset$, where $F \subseteq E(G)$, are being deleted, or
\[
|I^{-1}(G, F)| = O(N^2),
\]
if edges in $F \neq \emptyset$, where $F \subseteq \mathbb{E}_e(G) \setminus E(G)$, are being inserted.

**Proof.** First, remember that
\[
\binom{n}{k} \leq 2^n
\]
for very $n \in \mathbb{N}$ and $0 \leq k \leq n$. As described in the beginning of the proof of Theorem B.1, we can focus on the edge deletion case without loss of generality, so that the edge insertion case derives in an analogous inverse manner. This way, we have by definition that
\[
|F| \leq |E(G)| \leq \frac{N^2 - N}{2}
\]
and
\[
|\bar{E}(G \setminus F)| \leq \frac{N^2 - N}{2}.
\]
Therefore, the proof follows analogously from Theorem B.1 except for subsequentially replacing Equations 20 and 28 with
\[
|\bar{E}(G \setminus F)| + O \left( \log_2 \left( |\bar{E}(G \setminus F)| \right) \right)
\]
and
\[
|E(G)| + O \left( \log_2 \left( |E(G)| \right) \right),
\]
respectively in the proof of Theorem B.1.

Moreover, for single edge perturbations (i.e., $|F| = 1$), these upper and lower bounds become really tight in general, as expected:

**Corollary B.4.** Let $G = (V, E)$ be an arbitrary simple graph with $E \subseteq \mathbb{E}_c(G)$ and $|V| = N \in \mathbb{N}$. Then,

$$|I(G, e_i)| = |C(G) - C(G \setminus e_i)| = O(\log_2(N)),$$

if an edge is being deleted, or

$$|I^{-1}(G, e_i)| = |C(G) - C(G \cup e_i)| = O(\log_2(N)),$$

if an edge is being inserted.

**Proof.** First, note that in this particular case we have $r = |F| = 1$. Therefore, the proof follows directly from Theorem B.1.

---

**B.2. Eventual optimal values for graph complexity from edge deletions.**

**Definition B.1.** Let $F_G$ be an infinite family of simple graphs. Let $p \in \{0, 1\}^*$ only depend on the choice of $F_G$. We say a graph $G \in F_G$ is **prefix self-determined** iff there is $y = x |_n$ with $n < l(x)$ such that $U(\langle y, p \rangle) = x$, where $x$ is the characteristic string of $G$.

**Definition B.2.** Let $F_G$ be an infinite family of simple graphs. We say a graph $G \in F_G$ is **logarithmically prefix self-determined** iff $G$ is prefix self-determined and $n = O(\log_2(l(x)))$.

In order to prove that Definition B.2 and, as a direct consequence, Definition B.1, are non-empty (i.e., satisfiable), we can choose a particular type of self-delimiting program that returns itself concatenated with repetitions of itself and that is nearly incompressible:

**Lemma B.5.** There is an infinite family $F_G$ of logarithmically prefix self-determined dense graphs whose respective proper prefixes $v \in \{0, 1\}^*$ follows

$$|C(v) - l(v)| = O(\log_2(l(v)))$$

and

$$l(v) = o(\log_2(l(x))),$$

where $x$ is the characteristic string of $G$. 

Proof. This proof is based on constructing characteristic strings of the form \( w r \upharpoonright_k w' \), where \( r \in [0, 1] \subseteq \mathbb{R} \) is an arbitrary \( O(1) \)-K-random (i.e., prefix algorithmically random as in Definition A.9) real number and \( w, r \upharpoonright_k w' \in \{0, 1\}^* \). Moreover, we want that

\[
 l(w) + k = o \left( \log_2 \left( l(w r \upharpoonright_k w') \right) \right) = O \left( \log_2 \left( l(w r \upharpoonright_k w') \right) \right),
\]

and

\[
|E(G)| = \Theta \left( |V(G)|^2 \right),
\]

To this end, first let \( p_2 \in \{0, 1\}^* \) be a string that represents the algorithm running on a prefix universal Turing machine \( U \) that receives \( x \in \{0, 1\}^* \) as input and:

1. first, decides whether or not there are \( w \in L' \) and \( y \in \{0, 1\}^* \) such that \( x = wy \), where \( l(w) < l(x) \);
2. then,
   - (a) if \( x = wy \):
     - (i) build the string
       \[
       z = y y y \cdots (y \upharpoonright_h)
       \]
     - by concatenating the string \( y \) until finding \( h \in \mathbb{N} \), where \( h \leq l(y) \), for which
       \[
       l(wz) = \frac{n^2 - n}{2},
       \]
     - where
       \[
       n = \min \left\{ t \left| \frac{t^2}{2} \geq 2^{l(y)^2} + l(w) \right. \right\};
       \]
     - (ii) return \( wz \).
   - (b) Otherwise, return “error”.

Note that \( p_2 \) always halts for every input and we immediately have that either \( U(\langle x, p_2 \rangle) = wz \) or \( U(\langle x, p_2 \rangle) = “error” \). Since \( w \) and \( y \) were arbitrary in the construction of \( p_2 \), let \( w \) encode the length of \( y \).

\footnote{For example, one may choose \( r \) to be the halting probability \( \Omega \) without loss of generality in this proof.}
in language $L'_U$. Hence, we will have that

\begin{align}
  l(\text{wy}) &= l(w) + l(y) \\
  &\leq O\left(\log_2(l(y))\right) + l(y) \\
  &\leq o(l(y)) + l(y) = o(l(y)^2) \\
  &= o\left(\log_2\left(l(w) + 2^{l(y)^2}\right)\right) = o\left(\log_2(l(wz))\right).
\end{align}

(41)

From [9], we know the family of all possible simple graphs can be recursively labeled with a unique way of ordering the sequence of all possible edges. Thus, since by construction there is $n \in \mathbb{N}$ such that $l(wz) = n^2 - n^2$, let $G$ be a simple graph whose characteristic string is $wz = U(\langle wy, p_2 \rangle)$. This way, we have that

\begin{align}
  |V(G)|^2 - |V(G)| &= l(wz) \geq 2^{l(y)^2} + l(w) \\
  \text{and, by the construction of } p_2 \text{ and the minimality of } |V(G)|,
\end{align}

(42)

\begin{align}
  O\left(2^{l(y)^2} + l(w)\right) + 2^{l(y)^2} + l(w) &\geq \frac{|V(G)|^2 - |V(G)|}{2} = l(wz)
\end{align}

(43)

Until here, we proved the logarithmically prefix self-determination part and proved that

\[ l(\text{wy}) = o\left(\log_2(l(wz))\right). \]

Now, let $r \in [0, 1] \subseteq \mathbb{R}$ be an arbitrary $O(1)$-K-random real number. Note that, for any large enough length $l(wz)$, the string $wz$ cannot ever be $O(1)$-K-random, since it is based on multiple repetitions of the same string. From the Borel normality of $r$ (as in the proof of Corollary B.2), we also know there is a large enough $k \in \mathbb{N}$ such that the number $\#_1(r \upharpoonright k)$ of occurrences of 1’s in $r \upharpoonright k$ follows

\begin{align}
  \left|\#_1(r \upharpoonright k) - \frac{k}{2}\right| &= O\left(\sqrt{k \log_2(k)}\right).
\end{align}

(44)

Since $w$ and $y$ were arbitrary in the construction of $p_2$, let $y := r \upharpoonright k$. Therefore, since $k > k_0$ can be arbitrarily large given a large enough $k_0$, we will obtain an infinite family $F_G$ of distinct graphs with characteristic strings of the form $wz$. Remember that, by construction, $w$ encodes the integer value $k$ and that $k > k_0$ can be arbitrarily large given that large enough $k_0$. Therefore, from Equations 42, 43, and 44.
by construction of \( p_2 \) and since \( l(w) = \mathcal{O}(\log_2(k)) \), we will have that

\[
\frac{|V(G)|^2 - |V(G)|}{2} \geq |E(G)| \geq \frac{2^{k^2} - k}{2} - \left( \frac{2^{k^2} - k}{k} \right) \mathcal{O}\left( \sqrt{k \log_2(k)} \right) = \Omega\left( \frac{|V(G)|^2 - |V(G)|}{2} \right).
\]

This proves \( |E(G)| = \Theta\left( |V(G)|^2 \right) \). Now, the fact that \( C(r |_k) = C(y) \geq k - \mathcal{O}(1) \) holds follows directly from Definition A.9. In addition, from Lemma A.1 we have that

\[
C(wy) \leq l(r |_k) + \mathcal{O}(\log_2(k)) + \mathcal{O}(\log_2(\log_2(k))) \leq k + \mathcal{O}(\log_2(k))
\]

and

\[
C(y) \leq C(wy) + \mathcal{O}(\log_2(\log_2(k)))
\]

Therefore, by making \( v := wy \), where \( l(y) = k \) and \( l(w) = \mathcal{O}(\log_2(k)) \), and achieving

\[
k - \mathcal{O}(\log_2(\log_2(k))) \leq C(v) \leq k + \mathcal{O}(\log_2(k))
\]

from the incompressibility of \( r \) and Equations 46 and 47, we will have that

\[
l(v) - \mathcal{O}(\log_2(l(v))) - \mathcal{O}(\log_2(\log_2(l(v)))) \leq C(v) \leq l(v) + \mathcal{O}(\log_2(l(v)))
\]

Now we show in Theorem B.6 that the compression of the characteristic string and its density are so strong that enables one to pick and delete the right edge in a way such that this very deletion encodes the optimal compression of \( G \):

Theorem B.6. There is an infinite family \( F_G \) of logarithmically prefix self-determined dense simple graphs (in particular, any family satisfying Lemma B.5) such that, for every large enough member \( G \in F_G \), there is at least one edge deletion so that

\[
C(G^* | (G \setminus e_i)) = \mathcal{O}(1).
\]
Proof. Let $F\_G$ be an arbitrary infinite family of graphs satisfying Lemma B.5. Thus, there is $v = x \upharpoonright n$ with $n < l(x)$ such that $U(\langle v, p_2 \rangle) = x$, $n = o(\log_2(l(x)))$, and $|C(v) - l(v)| = O(\log_2(l(v)))$, where $x$ is the characteristic string of $G$. Let $p_3 \in \{0, 1\}^*$ be a string that represents the algorithm running on a prefix universal Turing machine $U$ that receives the adjacency matrix $A\_G'$ and $p_2$ as inputs and:

1. applies any previous arbitrarily chosen method of recursively encoding a matrix into a string and check whether or not $A\_G'$ is a matrix;
2. if $A\_G'$ is a matrix, then:
   a. reversively applies the previous arbitrarily chosen method of recursively encoding a matrix into a string to produce the correspondent characteristic string $x'$;
   b. decides whether or not there are $w' \in L'\_U$ and $z' \in \{0, 1\}^*$ such that $x' = w'z'$, where $l(w') < l(x')$ and $w'$ encodes the binary representation of a natural number $h \in \mathbb{N}$ in language $L'\_U$;
   c. if $x' = w'z'$ holds in this way, then:
      i. makes $v := w'y'$, where $y'$ is the string of the next $U(w')$ bits in $x'$ right after the end of $w'$;
      ii. makes $x := U(\langle v, p_2 \rangle)$;
      iii. decides whether or not $x' = x$;
      iv. if $x' < x$ and $l(x) = l(x')$, then:
         A) enumerates all the 1’s in $x$;
         B) search for the first bit of $x$ that displays value 1 while the same bit of $x'$ displays value 0;
         C) calculates de index $i \in \mathbb{N}$, where the $i$-th 1 in $x$ corresponds to this first bit in which the values differ from each other;
         D) returns the binary representation $(i)\_2 \in \{0, 1\}^*$ of the integer $i \in \mathbb{N}$.
      v. otherwise, returns “error”.
   d. otherwise, returns “error”.
3. otherwise, returns “error”.

Note that, if $U$ halts on $\langle v, p_2 \rangle$, then $p_3$ always halts. We will have by hypothesis, the minimality of algorithmic complexity, and Lemmas A.1 and A.2 that

$$C(x) \leq C(v) + O(\log_2(l(v))) + O(1) \leq$$

(50)

$$l(v) + O(\log_2(l(v))) + O(1) =$$

$$= o(\log_2(l(x))) ,$$
because $U(⟨v, p_2⟩) = x$ and $l(v) = o(\log_2(l(x)))$ hold. Therefore, once a method of recursively encoding a matrix into a string to produce the correspondent string $x'$ is fixed, we will have that, from Lemmas A.1 and A.2 and the fact that $x$ is the characteristic string of $G$,

$$(51) \quad l(G^*) = C(G) \leq o\left(\log_2\left(\frac{|V(G)|^2 - |V(G)|}{2}\right)\right) + O(1)$$

holds. Therefore, since there are only up to

$$(52) \quad 2^{o\left(\log_2\left(\frac{|V(G)|^2 - |V(G)|}{2}\right)\right)} + O(1)$$

possible bit strings of length $\leq l(G^*)$ and $G$ is dense, then there is at least one present edge that is the $i$-th bit of $x$, where $i > l(v)$, and $i \in \mathbb{N}$ corresponds to the program $G^* \in L'_U$. Then, $U(⟨A_{G^*}, p_2, p_3⟩) = G^*$ and, since $p_2$ and $p_3$ are fixed, $C(G^*|G\setminus e_i) = O(1)$. □

Identifying complex systems that displays such a phenomenon, as the graphs in Theorem B.6, may reveal those systems that are capable of carrying the sufficient information about where the perturbation occur and, hence, it enables future investigation on how this correlates with robustness or self-repair.

Moreover, one can take advantage of Lemma B.5 in order to obtain one of the particular cases in which a computer can already find nearly optimal values of algorithmic complexity in polynomial computation time (i.e., running time) even with brute force:

**Theorem B.7** (Theorem 3.2 in the main article). Let $G$ be a large enough simple graph that belongs to an infinite family $F_G$ of logarithmically prefix self-determined dense simple graphs. Then, there is a polynomially time-bounded exhaustive search algorithm that, with input $G \in F_G$, can approximate the integer value $C(G)$ with an error bounded by $o(\log_2(\log_2(|V(G)|)))$ bits.

**Proof.** This proof is based on the fact that, although program $p_2$ demands an exponential computation time with input $wy$, the computation time of emulating program $⟨wy, p_2⟩$ is still polynomial with respect to $|E(G)|$, since $G$ is dense from Lemma B.5. This way, from the “almost” incompressibility of the prefix $wy$ also demonstrated in Lemma B.5, we will achieve the desired upper bound for the error with respect to the optimal value $C(G)$. To such an end, first we choose without loss of generality a self-delimiting universal programming language $L'_U$ that is recognizable in polynomial time. Let $P$ be the usual time complexity class of polynomially time-bounded deterministic Turing machines [17, 50]. As a direct consequence, then we will have that
Steps (I) or (II)b in the proof of Lemma 3.5 can be solved in $\mathbf{P}$. Note that Step (II)(a)ii is basically the concatenation of the string $w$ obtained from Step (I) with the string resulting from Step (II)(a)i. Thus, since we know the concatenation operation and Step (I) are in $\mathbf{P}$, we can just focus on analyzing the time complexity of Step (II)(a)i, which is indeed where the computation is most demanding. In addition, from the minimality of $n$ and construction of $p_2$, we have that at most

$$O\left(2^{\ell(y)^2 + \ell(w)} \right)$$

concatenations will be needed to find the minimum $n$ with

$$l(wz) = \frac{n^2 - n}{2} \geq 2^{\ell(y)^2} + \ell(w).$$

Now, given $k \geq 4$ tapes with one tape filled with the binary representation of this maximum number of concatenations, one tape filled with $w$, and another tape filled with $y$, we will have that there is a $k$-tape Turing machine that returns $wz$ in at most

$$O\left(\left(\frac{2^{\ell(y)^2 + \ell(w)}}{\ell(y)}\right) \left(\max \{l(y), \ell(w)\}\right)^C\right)$$

computation steps. Therefore, since the worst-case time complexity of simulating any $k$-tape Turing machine is also in $\mathbf{P}$ [17, 50], then the worst-case time complexity of $p_2$ is indeed exponential in $l(wy)$ (i.e., $p_2$ represents an algorithm in $\text{EXPTIME}$). In fact, since $wz$ is the output of $p_2$, we will necessarily also have a best-case time complexity (for one-way single-tape Turing machines) of

$$2^{\ell(y)^2 + \ell(w)} - \ell(wy)$$

by just comparing the lengths of the output and input. Thus, since $l(wy) = o(\log_2(l(wz)))$, $p_2$ is also exponential in the best case scenario. On the other hand, we have by hypothesis that $G$ is dense, i.e., $|E(G)| = \Theta(|V(G)|^2)$. Then, we will have that $|E(G)| = \Theta(l(wz))$. Now, note that there are only $2^{l(\langle wy, p_2 \rangle)}$ possible bit strings with length at most $l(\langle wy, p_2 \rangle)$. Since the worst-case time complexity of simulating any Turing machine in a universal Turing machine is polynomially bounded with respect to the original computation time of the machine being simulated [17, 50], then we have that exhaustively searching all possible outputs of polynomially time-bounded programs with length $\leq l(\langle wy, p_2 \rangle)$ has a worst-case computation time bounded by

$$O\left(\left(\left(\frac{l(\langle wy, p_2 \rangle)}{2^{l(\langle wy, p_2 \rangle)}}\right)\right)^C\right).$$
for some large enough constant $C \in \mathbb{N}$. Remember our choice of the pairing function in Notation $\mathtt{A.5}$ so that, since $p_2$ is fixed, we will have that

$$l(\langle wy, p_2 \rangle) \leq l(wy) + l(p_2) + O(l(wy) + l(p_2)) = O(l(wy))$$

Therefore, since $l(wy) = o(\log_2 (wz))$ by hypothesis and, as we have shown, $p_2$ represents an algorithm in $\textsc{EXPTIME}$ given $wy$ as input, we will have that there is an exhaustive search algorithm with $G$ as input that will eventually test $\langle wy, p_2 \rangle$ as the generating program of $G$ in worst-case computation time bounded by

$$O\left(|E(G)|^{c''}\right),$$

for some large enough constants $c', c'' \in \mathbb{N}$. Therefore, such an algorithm returns a value $C'(G)$ in polynomial computation time with input $G$. Finally, in order to show that

$$|C'(G) - C(G)| = o\left(\log_2 \left(\log_2 \left(|V(G)|\right)\right)\right),$$

just note that $|C(wy) - l(wy)| = O\left(\log_2 \left(l(wy)\right)\right)$, $G$ is dense, and $l(wy) = o\left(\log_2 \left(l(wz)\right)\right)$ hold from Lemma $\mathtt{B.5}$. □

B.3. Minimal Information Loss Selection (MILS).

Theorem B.8 (Theorem 3.3 in the main article). Let $T_{ID}(G)$ be the worst-case computation time of calculating the value $I(G, e_i)$ for any $e_i \in E(G')$ by the chosen algorithmic-complexity approximating algorithm, where $G'$ is any edge-induced subgraph of $G$.\footnote{For example, by choosing the BDM, $T_{ID}(G)$ is the worst-case computation time of calculating the value of $K_{\log}(G') - K_{\log}(G' \setminus e_i)$ for every $e_i \in E(G')$, where $G'$ is any edge-induced subgraph of $G$. See Definition $\mathtt{A.12}$ and Equation $\mathtt{16}$.} Then, MILS (Algorithm 5) is a deterministic algorithm that has a cubic worst-case time complexity with input $E(G)$ and linear worst-case time complexity in $T_{ID}(G)$.

Proof. Let $e_i$ and $e_j$ be any two edges of a graph $G$ (whose size is to be reduced by the application of MILS) with $I(G, e_i) = I(G, e_j)$, then MILS will remove $e_i$ and $e_j$ from $G$ simultaneously. This element deletion condition helps MILS to always produce the same output for a given input, which makes it a deterministic algorithm. To prove this claim, note that $I(G, e_i)$ calculated by any algorithmic-complexity approximating algorithm (e.g., by the BDM) always produce an integer,
ordering any set of integers is decidable and always produce unique outputs for the same set of integers. Therefore, for every iteration of the loop in [Algorithm 5, Step 10], there will be a unique size-reduced graph $G_{n-i-1}$ from a previous graph $G_{n-i}$ by deleting the edges in \( \{ e_i \in E(G_{n-i}) : I(G_{n-i}, e_i) = \text{minLoss} \} \). In fact, MILS can only vary the output for the same input if more computation time is expended on the CTM’s precomputation before the BDM. In order to study the time complexity of MILS, note that the worst-case of function MILS\((G, N)\) is given when $N = 1$ and $\text{InfoRank}(G)$ outputs a list of $|E(G)|$ distinct entries. This way, since one has $|E(G)|$ iterations in the loop in [Algorithm 5, Step 3] by insertion (thus, an operation in $O(1)$) of the result of the approximation of $I(G, e_i)$ in the array $\text{informationLoss}$, which has size $|E(G)|$, and sorting algorithms have quadratic worst-case time complexity, we will have a worst-case time complexity of function $\text{InfoRank}(G)$ in

\[
O(|E(G)| (T_{ID}(G)) + O(|E(G)|^2). \tag{56}
\]

In addition, since one has up to $|E(G)|$ iterations in the loop, finding minimums in the first dimension of arrays takes a linear worst-case computation time, and deleting elements in an array also takes a linear worst-case computation time, we will have a worst-case time complexity of function MILS\((G, N)\) in

\[
O (|E(G)| (T_{IR}(G) + |E(G)|)), \tag{57}
\]

where $T_{IR}(G)$ is the worst-case computation time of function $\text{InfoRank}(G)$ for any $G$ (which was already studied in Equation 56 so that

\[
T_{IR}(G) = O (|E(G)| (T_{ID}(G)) + |E(G)|^2) \tag{58}
\]

holds). Now, suppose without loss of generality that the inputs of Algorithm 5 are $E(G)$ and $N$. Therefore, since composing functions is closed under polynomial computation time as in [Algorithm 5, Step 11] and function $\text{InfoRank}(G)$ is called recursively, we will have that the worst-case time complexity of Algorithm 5 is

\[
O (|E(G)| (|E(G)| (T_{ID}(G)) + |E(G)|^2 + |E(G)|^3)) = O (|E(G)|^3 + |E(G)|^2 (T_{ID}(G))) \tag{59}
\]

□
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