Pfaffians and the inverse problem for collinear central configurations

D. L. Ferrario

Received: 15 November 2019 / Revised: 17 June 2020 / Accepted: 27 June 2020 / Published online: 13 July 2020
© Springer Nature B.V. 2020

Abstract
We consider, after Albouy–Moeckel, the inverse problem for collinear central configurations: Given a collinear configuration of \( n \) bodies, find positive masses which make it central. We give some new estimates concerning the positivity of Albouy–Moeckel Pfaffians: We show that for any homogeneity \( \alpha \) and \( n \leq 6 \) or \( n \leq 10 \) and \( \alpha = 1 \) (computer assisted) the Pfaffians are positive. Moreover, for the inverse problem with positive masses, we show that for any homogeneity and \( n \geq 4 \) there are explicit regions of the configuration space without solutions of the inverse problem.

Keywords

\( n \)-Body problem · Pfaffian · Central configuration · Inverse problem

1 Introduction
Let \( n \geq 2 \), and \( d \geq 1 \). The configuration space of \( n \) points in the \( d \)-dimensional Euclidean space \( E = \mathbb{R}^d \) is defined as

\[ \mathbb{F}_n(E) = \{ q \in E^n : q_i \neq q_j \}, \]

where \( q = (q_1, q_2, \ldots, q_n) \in E \) and \( \forall j, q_j \in E \). Given a positive parameter \( \alpha > 0 \), and \( n \) positive masses \( m_j > 0 \), the potential function \( U : \mathbb{F}_n(E) \to \mathbb{R} \) is defined as

\[ U(q) = \sum_{1 \leq i < j \leq n} \frac{m_im_j}{\|q_i - q_j\|^\alpha}. \]

A central configuration is a configuration that yields a relative equilibrium solution of the Newton equations of the \( n \)-body problem with potential function \( U \) and can be shown (cf. Moulton 1910; Moeckel 1990; Albouy and Moeckel 2000; Ferrario 2015, 2017a, b) that it is a solution of the following \( n \) equations

\[ \lambda m_jq_j = -\alpha \sum_{k \neq j} m_jm_k \frac{q_j - q_k}{\|q_j - q_k\|^\alpha + 2}. \]
Such configurations have center of mass \( \sum_{j=1}^{n} m_j q_j = 0 \in E \), and the parameter \( \lambda \) turns out to be equal to \( \lambda = -\alpha \frac{U(q)}{\sum_{j=1}^{n} m_j \|q_j\|^2} \). A generic central configuration (with center of mass \( q_0 = \frac{\sum_{j=1}^{n} m_j q_j}{M} \)) not necessarily \( 0 \), where \( M = \sum_{j=1}^{n} m_j \) satisfies the equation

\[
\lambda m_j (q_j - q_0) = -\alpha \sum_{k \neq j} m_j m_k \frac{q_j - q_k}{\|q_j - q_k\|^{\alpha + 2}}.
\]

(1.2)

Now, if for each \( i, j \) denote

\[
Q_{jk} = \frac{q_j - q_k}{\|q_j - q_k\|^{\alpha + 2}}.
\]

Equation (1.2) can be written as

\[
q_j = M^{-1} \sum_{k=1}^{n} m_k q_k - \frac{\alpha}{\lambda} \sum_{k \neq j} m_k Q_{jk}, \quad j = 1, \ldots, n.
\]

(1.3)

The inverse problem, introduced by Moulton (1910) [see also Buchanan (1909)], and considered by Albouy and Moeckel (2000), can be phrased as follows: Given the positions \( q_j \) (or, equivalently, the mutual differences \( q_i - q_j \)) to find the (positive) masses \( m_j \) and \( \lambda < 0 \) such that (1.3) holds. As it is, the equation is not linear in the \((n + 1)\)-tuple \((m_1, \ldots, m_n, \lambda)\), but can be transformed into the following equation

\[
q_j = \hat{c} + \sum_{k \neq j} \hat{m}_k Q_{jk}, \quad j = 1, \ldots, n,
\]

(1.4)

because of the following lemma.

**Lemma 1.5** Given \( q \in \mathbb{R}_+^n \), there exists \((m_1, \ldots, m_n, \lambda)\), with \( m_j > 0 \) satisfying (1.3) if and only if there exists \((\hat{m}_1, \ldots, \hat{m}_n, \hat{c}) \in \mathbb{R}^{n+1}\) such that (1.4) holds and \( \hat{m}_j > 0 \) for each \( j \).

**Proof** If (1.3) holds for \((m_1, \ldots, m_n, \lambda)\) with positive masses, then \( \lambda < 0 \) and simply by setting \( \hat{c} = M^{-1} \sum_{k=1}^{n} m_k q_k \), \( \hat{m}_k = -\frac{\alpha}{\lambda} m_k \) one has that (1.4) holds.

Conversely, assume that \((\hat{m}_1, \ldots, \hat{m}_n, \hat{c})\) satisfies (1.4), with \( \hat{m}_j > 0 \). Then, by putting \( m_k = \hat{m}_k \), \( k = 1, \ldots, n \), \( \lambda = -\alpha \) it follows, multiplying by \( m_j \) (and setting as above \( M = \sum_{j=1}^{n} m_j \)) and summing for \( j = 1, \ldots, n \)

\[
q_j = \hat{c} - \frac{\alpha}{\lambda} \sum_{k \neq j} m_k Q_{jk}, \quad \implies \quad \sum_{j=1}^{n} m_j q_j = M \hat{c} + 0,
\]

and hence (1.3). \( \square \)

**Remark 1.6** Multiplying each equation by \( \hat{m}_j (q_j - \hat{c}) \) and summing for \( j = 1, \ldots, n \), it follows that

\[
\sum_{j=1}^{n} \hat{m}_j \|q_j - \hat{c}\|^2 = \sum_{j=1}^{n} \sum_{k \neq j} \hat{m}_j \hat{m}_k Q_{jk} \cdot (q_j - \hat{c}) = \sum_{1 \leq j < k \leq n} \hat{m}_j \hat{m}_k \|q_j - q_k\|^{-\alpha}.
\]

Hence, whenever (1.3) or (1.4) holds (for positive masses), the corresponding

---

1 In the notation of Albouy and Moeckel (2000), \( q_j = X_j \), \( q_0 = c \), \( A_j = \sum_{k \neq j} m_k Q_{kj} \), so that Eq. (1.2) reads as equation (3) of Albouy and Moeckel (2000) \( \alpha A_j - \lambda (q_j - q_0) = 0 \), \( j = 1, \ldots, n \), for some constant \( \lambda < 0 \).
λ is in any case negative. Moreover, (1.3) holds for \((m_1, \ldots, m_n, \lambda)\) if and only if it holds for \((tm_1, \ldots, tm_n, t\lambda)\) for any \(t > 0\), so that Eqs. (1.3) and (1.4) are equivalent.

**Definition 1.7** For each \(q \in \mathbb{F}_n(E)\), let \(\Psi(q)\), \(\tilde{\Psi}(q) \subseteq E^n\) be the subsets

\[
\Psi(q) = \{ q : q_j = \hat{c} + \sum_{k \neq j} \hat{m}_k Q_{jk} : \hat{c} \in E, \hat{m}_j > 0, j = 1, \ldots, n \}
\]

\[
\subseteq \tilde{\Psi}(q) = \{ q : q_j = \hat{c} + \sum_{k \neq j} \hat{m}_k Q_{jk} : \hat{c} \in E, \hat{m}_j \in \mathbb{R}, j = 1, \ldots, n \}.
\]

Hence, given \(q \in \mathbb{F}_n(E)\), there exists a solution of (1.4) if and only if \(q \in \Psi(q)\); furthermore, if \(q \in \Psi(q)\), then \(q \in \tilde{\Psi}(q)\).

We will now deal with the collinear case. First, we will follow Albouy and Moeckel (2000) and consider the inverse problem with real masses; then, we will consider the problem with positive masses and follow Ouyang and Xie (2005) (for \(n = 4\) bodies and \(\alpha = 1\)) and Davis et al. (2018) (for \(n = 5\) bodies and \(\alpha = 1\)), in understanding in which regions the inverse problem has no solutions.

## 2 The case \(d = 1\): collinear configurations and Pfaffians

For \(d = 1\), all configurations are on a line, therefore \(E = \mathbb{R}, c = c\), and \(q \in \tilde{\Psi}(q)\) if and only if there exists \((m_1, \ldots, m_n, c) \in \mathbb{R}^{n+1}\) such that

\[
\begin{bmatrix}
0 & Q_{12} & Q_{13} & \cdots & Q_{1n} \\
-Q_{12} & 0 & Q_{23} & \cdots & Q_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-Q_{1n} - Q_{2n} & \cdots & -Q_{n-1,n} & 0
\end{bmatrix}
\begin{bmatrix}
m_1 \\
m_2 \\
\vdots \\
m_n
\end{bmatrix}
+ c
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
= \begin{bmatrix}
q_1 \\
q_2 \\
\vdots \\
q_n
\end{bmatrix}
\]

(2.1)

where \(Q_{ij} = (q_i - q_j)|q_i - q_j|^{-\alpha-2}\), for \(i, j = 1, \ldots, n\), or

\[
\Longleftrightarrow Qm + cL = q
\]

(2.2)

where \(Q\) is the \(n \times n\) skew-symmetric matrix with entries \(Q_{ij}\), \(m\) the vector of masses, and \(L\) the vector with constant components 1.

Recall that if \(n\) is odd and \(A\) is an antisymmetric \(n \times n\) matrix, \(A^T = -A \implies \det(A) = \det(-A) = (−1)^n \det(A) \implies \det(A) = 0\). If \(n\) is even, then \(\det A = (\mathbf{Pf} A)^2\) [cf., e.g., the combinatorial approach of Godsil (1993), Chap. 7, or the multi-linear algebra approach of Northcott (1984), from page 100]. The (Pfaffian \(\mathbf{Pf} Q\) of a skew-symmetric matrix \(Q\) (for even \(n\)) is defined as follows (in Moulton’s 1910 notation):

\[
\mathbf{Pf} Q = \begin{vmatrix}
Q_{12} & Q_{13} & \cdots & Q_{1n} \\
Q_{23} & \cdots & \cdots & Q_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
Q_{n-1,n} & \cdots & \cdots & Q_{nn}
\end{vmatrix}
= \sum_\sigma (-1)^\sigma Q_{r_1,s_1} Q_{r_2,s_2} \cdots Q_{r_k,s_k},
\]

where \(n = 2k\), and the permutation \(\sigma\) runs over all perfect matchings of \(n = \{1, \ldots, n = 2k\}\): A perfect matching \(\sigma\) is a fixed point free involution of \(n\), which can be represented also as a partition of \(n\) in pairs \([r_1, s_1, r_2, s_2, \ldots, r_k, s_k]\). The sign \((-1)^\sigma\) is the parity of
this permutation. In D. Knuth and Cayley notation (Knuth 1996; Cayley 1849), Pf $A = A[1, 2, \ldots, n]$.

The following recursive identity is the analogue of the Laplace expansion for the determinant:

$$A[1, 2, \ldots, n] = \sum_{j=1}^{n-1} (-1)^{j+1} A_{jn} A[1, \ldots, \hat{j}, \ldots, \hat{n}],$$

(2.3)

where $A[1, \ldots, \hat{j}, \ldots, \hat{n}]$ denotes the Pfaffian of the matrix with the $j$th and $n$th rows and columns canceled out.

An elementary property of Pfaffians is the following: If $A$ is a skew-symmetric matrix, and $B$ the matrix obtained by swapping the $i$th and $j$th columns and the $i$th and $j$th rows, then

$$\text{Pf } A = - \text{Pf } B.$$  

(2.4)

Lemma 2.5 (Halton) Let $A$ be an $n \times n$ skew-symmetric matrix, and $i < j$, with $n$ even. If $A_{ij}$ denotes the matrix $A$ with row $i$ and column $j$ removed, then

$$\det A_{ij} = -A[1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n] \text{ Pf } A.$$  

(2.6)

Remark 2.7 See, for example, lemma 3.2 at page 118 of Godsil (1993), for a proof, where it is used to prove the recursive relation of Pfaffians. See also Stembridge (1990), Dress and Wenzel (1995), and Hamel (2001) for other interesting combinatorial identities for Pfaffians.

Remark 2.8 (Buchanan Albouy–Moeckel Conjecture) Buchanan, in his 1909 article Buchanan (1909), proves a proposition which can be rephrased as follows: for each even $n$, $\alpha = 1$, for each $q \in \mathbb{F}_n(\mathbb{R})$, the Pfaffian is nonzero: Pf $A_n \neq 0$.

As found by Albouy and Moeckel (2000), Buchanan’s proof uses an incorrect argument and cannot be repaired. So, they conjecture it to be true, in the Albouy–Moeckel Conjecture: The Pfaffians are nonzero for all configurations. The partial steps done in the direction of its complete proof are the following: It is true for $n \leq 4$ and $\alpha > 0$, or $\alpha = 1$ and $n \leq 6$, computer assisted (Albouy and Moeckel 2000); it is true for $n \leq 6$ and $\alpha = 1$ (Xie 2014).

The following lemma generalizes Theorem 2.4.(1–2) of Xie (2014); the main conclusion follows from Proposition 5 of Albouy and Moeckel (2000).

Lemma 2.9 If $q_1 > q_2 > q_3 > q_4$ and as above $Q_{ij} = q_{ij}|q_{ij}|^{-\alpha - 2}$, then $Q_{12}Q_{34} > Q_{13}Q_{24}$, and $Q_{23}Q_{14} > Q_{13}Q_{24}$, and hence

$$Q_{12}Q_{34} - Q_{13}Q_{24} + Q_{23}Q_{14} > 0.$$  

Proof

$$Q_{23}Q_{14} > Q_{13}Q_{24} \iff (q_{23}q_{14})^{-\alpha - 1} > (q_{13}q_{24})^{-\alpha - 1} \iff q_{23}q_{14} < q_{13}q_{24} \iff q_{23}(q_{13} + q_{34}) < q_{13}(q_{23} + q_{34}) \iff \frac{q_{13} + q_{34}}{q_{23}} < \frac{q_{23} + q_{34}}{q_{13}} \iff 1 + \frac{q_{34}}{q_{13}} < 1 + \frac{q_{34}}{q_{23}}.$$  

\[ Springer
and the last inequality holds true since \( q_{13} > q_{23} \). Now, this implies

\[
Q_{12} Q_{34} - Q_{13} Q_{24} + Q_{23} Q_{14} > Q_{12} Q_{34} > 0.
\]

\( \square \)

The following lemma generalizes Theorem 2.4.(3) of Xie (2014).

**Lemma 2.10** Assume \( q_1 > q_2 > q_3 > q_4 \), and as above \( Q_{ij} = q_{ij} |y_{ij}|^{-\alpha - 2} \). The function \( f(q_4) = \text{Pf } A_4 = Q_{14} Q_{23} - Q_{24} Q_{13} + Q_{43} Q_{12} \) is monotone increasing in \((-\infty, q_3)\), with \( q_1, q_2, q_3 \) fixed. The function \( g(q_1) = \text{Pf } A_4 \) is monotone decreasing in \((q_2, +\infty)\), with \( q_2, q_3, q_4 \) fixed.

**Proof**

\[
\frac{d(\text{Pf } A_4)}{dq_4} = (\alpha + 1) \left( q_{14}^{-\alpha - 2} Q_{23} - q_{24}^{-\alpha - 2} Q_{13} + q_{34}^{-\alpha - 2} Q_{12} \right)
\]

Since \( Q_{12} Q_{34} > Q_{13} Q_{24} \) and \( Q_{23} Q_{14} > Q_{13} Q_{24} \) by (2.9),

\[
\frac{Q_{12} Q_{34}}{q_{34}} - \frac{Q_{13} Q_{24}}{q_{24}} + \frac{Q_{23} Q_{14}}{q_{14}} > \frac{1}{q_{34}} Q_{13} Q_{24} - \frac{1}{q_{24}} Q_{13} Q_{24} + \frac{1}{q_{14}} Q_{23} Q_{14}
\]

\[= \left( \frac{1}{q_{34}} - \frac{1}{q_{24}} + \frac{1}{q_{14}} \right) Q_{13} Q_{24} > 0.\]

The second part of the statement follows by considering that if \( q_1 > q_2 > q_3 > q_4 \), then one can define \( y_1 = -q_4 > y_2 = -q_3 > y_3 = -q_2 > y_4 = -q_1 \), and the Pfaffian of the corresponding matrix \( Y_{ij} = (y_{ij}) |y_{ij}|^{-\alpha - 2} \), with \( y_{ij} = y_i - y_j \), is equal to

\[
\begin{vmatrix}
Y_{12} & Y_{13} & Y_{14} \\
Y_{23} & Y_{24} & Y_{14} \\
Y_{34} & Y_{23} & Y_{14}
\end{vmatrix}
= \begin{vmatrix}
Q_{12} & Q_{24} & Q_{14} \\
Q_{13} & Q_{23} & Q_{14} \\
Q_{12} & Q_{13} & Q_{14}
\end{vmatrix}
= \begin{vmatrix}
Q_{12} & Q_{13} & Q_{14} \\
Q_{23} & Q_{24} & Q_{14} \\
Q_{12} & Q_{34} & Q_{14}
\end{vmatrix}.
\]

Since \( f(y_4) \) is monotonically increasing in \((-\infty, y_3)\), and \( y_3 = -q_2 \), the function \( g(q_1) = f(-y_4) \) is monotonically decreasing in \((q_2, +\infty)\). \( \square \)

The following lemma is inspired by the proof of Theorem 2.5 of Xie (2014) and in fact generalizes it.

**Lemma 2.11** If \( q \in \mathbb{R}^n \) is a (collinear) configuration with \( q_1 > q_2 > \cdots > q_n \) and \( Q \) denotes the skew-symmetric matrix with entries \( Q_{ij} \), then

\[
\text{Pf } Q = \begin{vmatrix}
Q_{12} & Q_{13} & \cdots & Q_{1n} \\
Q_{23} & Q_{23} & \cdots & Q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{n-1,n} & Q_{n-2,n} & \cdots & Q_{nn}
\end{vmatrix} = \left( \prod_{j=1}^{n-1} Q_{jn} \right) \begin{vmatrix}
\tilde{Q}_{12} & \tilde{Q}_{13} & \cdots & 1 \\
\tilde{Q}_{23} & \tilde{Q}_{23} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{vmatrix},
\]

where for each \( i, j = 1, \ldots, n - 1 \)

\[
\tilde{Q}_{ij} = \left( q_{jn}^{-1} - q_{in}^{-1} \right)^{-\alpha - 1}.
\]
Hence, if the configuration \( \tilde{q} \in \mathbb{R}^{n-1} \) is defined by \( \tilde{q}_j = -q_{jn}^{-1} \) for each \( j = 1, \ldots, n - 1 \), it satisfies

\[
\tilde{q}_1 > \tilde{q}_2 > \cdots > \tilde{q}_{n-1}
\]

and, as for \( Q \), with \( \tilde{q}_{ij} = \tilde{q}_i - \tilde{q}_j \), \( \tilde{Q}_{ij} = \tilde{q}_{ij} |\tilde{q}_{ij}|^{-\alpha/2} \).

**Proof** By multiplying on the left and the right the matrix \( Q \) with the \( n \times n \) matrix with diagonal \( (Q_{11}^{-1}, Q_{2n}^{-1}, \ldots, Q_{n-1,n}^{-1}, 1) \), one obtains a matrix \( \tilde{Q} \) with entries

\[
\tilde{Q}_{ij} = \begin{cases} 
Q_{ij} Q_{in} Q_{jn} & \text{if } 1 \leq i, j \leq n - 1 
\end{cases}
\]

and the proof follows from the fact that if \( i < j \) then

\[
\frac{Q_{ij}}{Q_{in} Q_{jn}} = \left( \frac{q_{ij}}{q_{in} q_{jn}} \right)^{\alpha-1} = \left( \frac{q_{in} - q_{jn}}{q_{in} q_{jn}} \right)^{\alpha-1} = \left( q_{jn}^{-1} - q_{in}^{-1} \right)^{\alpha-1}.
\]

[\( \square \)]

Given an \( n \times n \) skew-symmetric matrix \( Q \), let \( Q^b \) denote the \((n + 1) \times (n + 1)\) skew-symmetric bordered matrix

\[
Q^b = \begin{bmatrix}
0 & Q_{12} & Q_{13} & \cdots & Q_{1n} & 1 \\
-Q_{12} & 0 & Q_{23} & \cdots & Q_{2n} & 1 \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & -Q_{1n} & Q_{2n} & \cdots & -Q_{n-1,n} & 0 & 1 \\
& & & & -1 & \cdots & -1 & 0
\end{bmatrix}.
\]

With this notation, Lemma (2.11) can be written as \( \text{Pf} \ Q = \left( \prod_{j=1}^{n-1} Q_{jn} \right) \text{Pf} \ Q^b \).

**Proposition 2.12** If \( n \) is odd, and for \( q \in \mathbb{F}_n(\mathbb{R}) \) the product of Pfaffians

\[
Q^b[1, \ldots, n, n + 1]Q[1, \ldots, n - 1, \hat{n}] \neq 0
\]

is nonzero, then Eq. (2.2) has solutions.

**Proof** Observe that Eq. (2.2) has solutions if the rank of the \( n \times (n + 1) \) matrix

\[
\begin{bmatrix}
0 & Q_{12} & Q_{13} & \cdots & Q_{1n} & 1 \\
-Q_{12} & 0 & Q_{23} & \cdots & Q_{2n} & 1 \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & -Q_{1n} & Q_{2n} & \cdots & -Q_{n-1,n} & 0 & 1 \\
& & & & -1 & \cdots & -1 & 0
\end{bmatrix}
\]

is equal to \( n \), which happens if for some \( j \in \{1, \ldots, n\} \) the \( n \times n \) square matrix obtained by removing the \( j \)th column is non-singular. Now, this is the same as the matrix obtained by removing the \((n + 1)\)th row and the \( j \)th column of the bordered matrix \( Q^b \). By (2.6) (on transposed matrices), its determinant is equal to

\[
Q[1, \ldots, \hat{j}, \ldots, n] \text{Pf} \ Q^b.
\]

By taking \( j = n \), the conclusion follows. [\( \square \)]
Note that that statement holds with $j$ chosen as any index from 1 to $n$, instead of $n$; moreover, because of (2.3), there exists $j$ such that $Q^b[1, \ldots, n, 1]Q[1, \ldots, \hat{j}, \ldots, n] \neq 0$ if and only if $Q^b[1, \ldots, n, n + 1] \neq 0$. See also Theorem 1 of Albouy and Moeckel (2000), where shorter proofs or more general results are presented, using exterior algebra as a computational device.

Let $n$ be odd and $q$ a configuration. Then, the corresponding $Q_n$ is a $n \times n$ singular matrix. The two matrices in (2.12) are the $(n - 1) \times (n - 1)$ skew-symmetric matrix $\tilde{Q}_{n-1}$ corresponding to the configuration with the $n$th body removed, and the $(n + 1) \times (n + 1)$ matrix $Q_n^b$. Because of (2.11), the Pfaffian $\text{Pf} Q_{n-1}$ is nonzero if and only if the Pfaffian of the corresponding $\tilde{Q}_{n-1}$ is nonzero. But $\tilde{Q}_{n-1}$ is an $(n - 2) \times (n - 2)$ matrix. So, for odd $n$ the existence of solutions to (2.2) follows from the calculation of Pfaffians of the even-dimensional matrices $Q_n^b$ and $\tilde{Q}_{n-1}^b$. (The existence of solutions for $n = 5$ was proven in Theorem 2.6 of Xie (2014) in a different way.)

On the other hand, let $n$ be even, and $q$ a configuration and $Q_n$ as above. By (2.11), the existence of solutions to (2.2) follows from the calculation of the Pfaffian of $\tilde{Q}_n^b$, where $\tilde{Q}_n$ is a matrix with odd size.

**Theorem 2.13** For all $\alpha > 0$, and any $n \leq 6$, the Pfaffian of $Q$ (for even $n$) or of $Q^b$ (for odd $n$) is nonzero; hence, for each configuration $q$ Eq. (2.2) has solutions with real masses $m_j$.

**Proof** By Lemma (2.11), as explained before, the Pfaffian of the matrix corresponding to a collinear configuration $q \in \mathbb{F}_n(\mathbb{R})$ with $n$ even is nonzero, if it is nonzero the Pfaffian of the bordered matrix $Q^b$ corresponding to collinear $n - 1$ bodies. For $n = 5$, one can apply (2.3) and obtain, given that $Q_{j6}^b = 1$ for $j = 1, \ldots, 5$,

$$\text{Pf} Q^b = Q[\hat{1}, 2, 3, 4, 5] - Q[1, \hat{2}, 3, 4, 5]$$

$$+ Q[1, 2, \hat{3}, 4, 5] - Q[1, 2, 3, \hat{4}, 5] + Q[1, 2, 3, 4, \hat{5}].$$

Without loss of generality, one can assume $q_1 > q_2 \cdots > q_5$: since by Lemma (2.10) the Pfaffian $Q[2, 3, 4, 5]$ is decreasing in $q_2$, and $q_1 > q_2$, one has $Q[1, 3, 4, 5] < Q[2, 3, 4, 5]$; since $Q[1, 2, 3, 4]$ is increasing in $q_4$, and $q_4 > q_5$, $Q[1, 2, 3, 4] > Q[1, 2, 3, 5]$. Therefore $\text{Pf} Q^b > Q[1, 2, 3, 4, 5]$, which is strictly positive by (2.9).

**Remark 2.14** Such a nice argument, introduced already by Xie (2014), unfortunately does not work as it is for $n > 6$: When $n \geq 8$ in the (symmetric) sum of seven terms, only the two consecutive terms at both endpoints can be estimated by monotonicity. It is very interesting that, at least for $\alpha = 1$ when the Pfaffian is a rational function of the mutual distances, it is possible to prove its positivity by checking that all the coefficients of the polynomials are positive. This was found by Albouy and Moeckel (2000): In the following, we show how we computed the polynomial for $n = 8$ and 10, finding that it has all positive coefficients.

It is maybe worth noting that in the notation of Albouy and Moeckel (2000), the following equalities hold: If $n = 2k$, then $K_n = k! \text{Pf} Q$, while if $n = 2k + 1$, then $K_n^L = k! \text{Pf} Q^b$.

**Lemma 2.15** Let $\alpha = 1$, $q \in \mathbb{F}_n(\mathbb{R})$ an ordered collinear configuration (with $q_1 > q_2 > \cdots > q_n$, and as above $q_{ij} = q_i - q_j$), and $n$ even. Let $P$ be the skew-symmetric matrix defined for each $i < j$ by $P_{ij} = \text{the product of all } q_{ab} \text{ such that } a \in \{i, j\} \text{ or } b \in \{i, j\} \text{ and } a < b$:

$$P_{ij} = \prod_{\substack{1 \leq a < b \leq n \text{ or } (a, b) \neq (i, j) \text{ or } (a, b) \neq (i, j)}} q_{ab}.$$
Its Pfaffian and the Pfaffian of the antisymmetric matrix with terms \( Q_{ij} = q_{ij}^2 \) for \( i < j \) satisfy the identity

\[
Pf \ P = \left( \prod_{1 \leq i < j \leq n} q_{ij}^2 \right) Pf \ Q.
\]

**Proof** Let \( P' \) denote the matrix obtained by multiplying the \( j \)th row and column of \( Q \) by the factor \((-1)^{j-1} \prod_{1 \leq i \leq n, i \neq j} q_{ij} \), for \( j = 1, \ldots, n \). It follows that

\[
Pf \ P' = \left( \prod_{1 \leq j \leq n} (-1)^{j-1} \prod_{1 \leq i \leq n, i \neq j} q_{ij} \right) Pf \ Q = \left( \prod_{1 \leq i < j \leq n} q_{ij}^2 \right) Pf \ Q
\]
since

\[
\prod_{1 \leq i \leq n, i \neq j} q_{ij} = \left( \prod_{1 \leq i < j} q_{ij} \right) \left( \prod_{j < i \leq n} q_{ij} \right) = (-1)^{n-j-1} \left( \prod_{1 \leq i < j} q_{ij} \right) \left( \prod_{j < i \leq n} q_{ji} \right).
\]

This implies also that the \( ij \)-entry of \( P' \) is equal to

\[
P_{ij}' = q_{ij}^{-2} \left( \prod_{1 \leq a < b \leq n, i \leq a} q_{ab} \right) \left( \prod_{1 \leq a < b \leq n, j \leq b} q_{ab} \right) = P_{ij}.
\]

\[\square\]

**Remark 2.16** For even \( n \), if the matrix \( P \) of (2.15) is computed starting from the matrix \( \tilde{Q}^b \) of (2.11) instead of \( Q \), it can be renamed \( \tilde{P} \): Its Pfaffian is a polynomial in the \( n - 2 \) variables \( \tilde{x}_j = \tilde{q}_j - \tilde{q}_{j+1} = \tilde{q}_{j,j+1} \) for \( j = 1, \ldots, n - 2 \), where \( \tilde{q}_j = -\tilde{q}_{jn}^{-1} \), and for each \( 1 \leq i < j < n \) the equality \( \tilde{q}_{ij} = \frac{q_{ij}}{q_{i\min} q_{j\min}} \) holds, and \( \tilde{q}_{in} = 1 \). Note that \( \tilde{q}_n \) is not defined, and \( \tilde{q}_{in} \) is not \( \tilde{q}_i - \tilde{q}_n \); hence, \( \tilde{q}_{in} = 1 \), for \( i = 1, \ldots, n - 1 \), does not imply \( \tilde{q}_1 = \cdots = \tilde{q}_{n-1} \).

**Theorem 2.17** The Pfaffian of the matrix \( P \), defined in (2.15), is a polynomial with nonnegative integer coefficients, for each even \( n \leq 8 \), with respect to the variables \( x_1, \ldots, x_{n-1} \), defined as \( x_j = q_j - q_{j+1} = q_{j,j+1} \) for \( j = 1, \ldots, n - 1 \).

The Pfaffian of the matrix \( \tilde{P} \), defined in (2.16), is a polynomial with nonnegative integer coefficients, for each even \( n \leq 10 \), with respect to the variables \( \tilde{x}_1, \ldots, \tilde{x}_{n-2} \), defined as \( \tilde{x}_j = \tilde{q}_j - \tilde{q}_{j+1} = \tilde{q}_{j,j+1} \) for \( j = 1, \ldots, n - 2 \), where \( \tilde{q}_j = -\tilde{q}_{jn}^{-1} \).

As a consequence, for each even \( n \leq 10 \) the Pfaffian of \( Q \) is positive.

**Proof (Computer assisted)** The proof is just a computer computation, performed on some computer algebra systems. The output numbers for the first cases are as follows.

For \( P \):

- \( n = 4 \): minimum of coefficients = 1, maximum of coefficients = 19. Total of 25 nonzero coefficients in the \( n - 1 \) variables \( x_1, x_2, x_3 \).
\( n = 6 \): minimum of coefficients = 1, maximum of coefficients = 6217712. Polynomial of degree 24 in 5 variables with 7993 nonzero coefficients.

\( n = 8 \): minimum of coefficients = 1, maximum of coefficients = 1974986029814430328. Polynomial of degree 48 in 7 variables with 8863399 nonzero coefficients.

For \( \tilde{P} \):

\( n = 4 \): minimum of coefficients = 1, maximum of coefficients = 2. Total of 5 nonzero coefficients in the \( n - 2 \) variables \( \tilde{x}_1, \tilde{x}_2 \). The Pfaff is the polynomial of degree \((n - 2)^2 = 4\)

\[ \tilde{x}_1^4 + 2\tilde{x}_1^3\tilde{x}_2 + \tilde{x}_1^2\tilde{x}_2^2 + 2\tilde{x}_1\tilde{x}_2^3 + \tilde{x}_2^4. \]

\( n = 6 \): minimum of coefficients = 1, maximum of coefficients = 3018. Total of 519 nonzero coefficients in the 4 variables of degree \((n - 2)^2 = 16\).

\( n = 8 \) minimum of coefficients = 1, maximum of coefficients = 922577565632. Total of 306016 nonzero coefficients in \( n - 2 \). Degree = \((n - 2)^2 = 36\).

If \( n = 10 \), then the number of perfect matchings is \( \frac{10!}{2^5(5)!} = 945 \): For each one, a polynomial of degree 64 in 8 variables is added. So, in theory computations even in dense multivariate polynomials with integer coefficients could fit into the memory of a normal computer. The minimum of the coefficients is 1; the maximum is 8181820494918819340996488. There are a total of 488783941 nonzero coefficients. (The runtime was approximately 10 days.) \( \Box \)

For \( n = 12 \), an empirical estimate of the time needed to perform the calculation with this algorithm would be of the order of 4–5 years on the same computer.

3 Positive masses

Consider now the inverse problem with real and positive masses: Let \( X_0 \subset E^n = \mathbb{R}^n \) be the subset \( X_0 = \{ q \in E^n : \sum_{j=1}^{n} q_j = 0 \} \), which is the orthogonal complement of \( \mathbf{L} \) in \( E^n \). The \( n \) columns of the antisymmetric matrix \( Q \) (which can be denoted as \( Q_1, \ldots, Q_n \)) generate a subspace of dimension \( n \) (for even \( n \)) or \( n - 1 \) (for odd \( n \)) in \( E^n \). Let \( \Pi \) denote the orthogonal projection of \( E^n \) onto \( X_0 \); then, if \( x \in X_0 \), Eq. (2.2) is equivalent to

\[ Q(x)m + c\mathbf{L} = x \iff x = \Pi Q(x)m. \] (3.1)

In fact, if \( x = Q(x)m + cL \), then by projecting one obtains \( \Pi x = x = \Pi Q(x)m \) since \( \Pi \mathbf{L} = 0 \). Conversely, if \( x = \Pi Q(x)m \), then \( \Pi Q(x)m = Q(x)m \in \ker \Pi = \text{Span}(\mathbf{L}) \), since \( \Pi^2 = \Pi \), and hence, there exists \( c \in \mathbb{R} \) such that \( \Pi Q(x)m = Q(x)m + cL \), that is, \( x = Q(x)m + cL \). For a different set of variables, see Ouyang and Xie (2005) (for \( n = 4 \) bodies and \( \alpha = 1 \)) and Davis et al. (2018) (for \( n = 5 \) bodies and \( \alpha = 1 \)); for the general problem with positive masses, see again Albouy and Moeckel (2000).

Now, define the following coefficients, for \( i = 1, \ldots, n \) and \( j = 0, \ldots, n - 1 \):

\[ \beta_{ij} = \begin{cases} 1 & \text{if } j = 0; \\ 1 - \frac{j}{n} & \text{if } i \leq j; \\ -\frac{j}{n} & \text{if } i > j. \end{cases} \] (3.2)

Consider the \( n \) variables \( x_0, x_1, \ldots, x_{n-1} \), where as above \( x_j = q_j - q_{j-1} \) for \( j = 1, \ldots, n - 1 \), and \( x_0 = \frac{1}{n} (q_1 + \cdots + q_n) \). Note that for each \( i = 1, \ldots, n \) and \( j = 2, \ldots, n - 1 \) one has
\begin{align*}
\beta_{ij} - \beta_{i,j-1} &= \begin{cases}
-\frac{i}{n} + \frac{i-1}{n} = -\frac{1}{n} & \text{if } j < i \\
\frac{n-1}{n} & \text{if } j = i \\
\frac{-1}{n} & \text{if } j > i
\end{cases},
\end{align*}

and therefore, since \( \beta_{il} = 1 \), for each \( i = 1 \ldots n \) the following identities hold

\[ q_i = \sum_{j=0 \ldots n-1} \beta_{ij} x_j \quad \text{&} \quad x_0 = \frac{1}{n} \sum_{i=1 \ldots n} q_i, j > 0 \implies x_j = q_j - q_{j+1}. \tag{3.3} \]

Equation (3.3) can be written in matrix form as follows:

**Lemma 3.4** Let \( B \) be the matrix with coefficients \( b_{ij} = \beta_{i,j-1} \) defined above, \( x \) the column vector with components \( x_0, \ldots, x_{n-1} \) and \( q \) the column vector with components \( q_1, \ldots, q_n \). Then, \( B \) is an invertible matrix such that \( q = Bx \).

Given Eq. (3.1), and the permutation symmetries of the potential, we can restrict the problem to the cone

\[ X_0^+ = \{ q \in X_0 : q_1 > q_2 > \cdots > q_n \}, \]

which in coordinates \( x \) can be written as

\[ X_0^+ = \{ x : x_0 = 0, x_i > 0, i = 1, \ldots, n-1 \}. \]

In such coordinates, Eq. (3.1) is transformed in

\[ x_i = (B^{-1} Qm)_i, \quad i = 1, \ldots, n-1, \tag{3.5} \]

with suitable substitutions in the expressions of \( Q \). For example, if \( n = 3 \), one has to consider only the second and third rows of the following equation

\[
\begin{bmatrix}
x_0 \\
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
1/3 & 1/3 & 1/3 \\
1 & -1 & 0 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
0 & Q_{12} & Q_{13} \\
-Q_{12} & 0 & Q_{23} \\
-Q_{13} & -Q_{23} & 0
\end{bmatrix}
\begin{bmatrix}
m_1 \\
m_2 \\
m_3
\end{bmatrix},
\]

which turns out to be

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
Q_{12} & Q_{13} - Q_{23} \\
-Q_{12} + Q_{13} & Q_{23}
\end{bmatrix}
\begin{bmatrix}
m_1 \\
m_2
\end{bmatrix}.
\]

As above, \( Q_{ij} = q_{ij}^{\alpha-1} \), for \( i < j \), and hence, the last equation can be written as

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
x_1^{\alpha-1} - x_1^{\alpha-1} \frac{x_1^{\alpha-1} - x_2^{\alpha-1}}{x_1^{\alpha-1} - x_2^{\alpha-1}} (x_1 + x_2)^{\alpha-1} - x_2^{\alpha-1} \\
-x_1^{\alpha-1} + (x_1 + x_2)^{\alpha-1}
\end{bmatrix}
\begin{bmatrix}
m_1 \\
m_2
\end{bmatrix}.
\]

Another way of writing Eq. (3.5) is as follows: If now \( x \) denotes the \((n-1)\)-dimensional vector of positive coordinates \( x_1, \ldots, x_{n-1} > 0 \),

\[ x = \sum_{k=1}^{n} m_k Y_k \text{ with } m_k > 0, \tag{3.6} \]

where \( Y_k \) is the \((n-1)\)-dimensional vector with components \( Y_{ik} = Q_{i,k} - Q_{i+1,k} \) for \( i = 1, \ldots, n-1 \) and \( k = 1, \ldots, n \). Given that for each \( k \)

\[ \sum_{i=i \ldots n-1} Y_{ik} = \sum_{i=1 \ldots n-1} (Q_{i,k} - Q_{i+1,k}) = Q_{1,k} + Q_{k,n} > 0, \]

\( \triangledown \) Springer
x and all \( Y_k \) belong to the half-space \( x_1 + x_2 + \cdots + x_{n-1} > 0 \) and can be centrally projected on the hyperplane \( x_1 + x_2 + \cdots + x_{n-1} = 1 \). Let \( \Delta^{n-2} \) denote the standard Euclidean simplex in coordinates \( x_i \), and \( X_1 \) the affine subspace \( X_1 = \{ x \in X_0 : x_1 + x_2 \cdots + x_{n-1} = 1 \} \). Let \( p \) denote central projection \( p(x) = \frac{x}{\sum_{i=1}^{n-1} x_i} \), partially defined \( p : X_0 \to X_1 \).

**Lemma 3.7** The vector \( x \) is a solution of (3.6) if and only if its projection \( p(x) \) is a solution of

\[
x = \sum_{k=1}^{n} m_k^p(Y_k) \text{ with } m_k^p > 0,
\]

with \( \sum_k m_k = 1 \) and \( x \in X_1 \).

**Proof** As we have seen, \( p \) is well defined on \( x \) (since all \( x_j \) are positive) and on all \( Y_k \). If \( x = \sum_{k=1}^{n} m_k Y_k(x) \), then by homogeneity if we let \( \lambda_0 = x_1 + \cdots + x_{n-1} \) and \( \lambda_k = Q_{1,k} + Q_{k,n} > 0 \) for each \( k \),

\[
\sum_{k=1}^{n} m_k^p(Y_k(\lambda_0^{-1} x)) = \sum_{k=1}^{n} m_k^p(Y_k(x)) = \sum_{k=1}^{n} m_k^p \lambda_k^{-1} Y_k(x)
\]

\[
\implies \sum_{k=1}^{n} m_k^p(Y_k(\lambda_0^{-1} x)) = p(x) = \lambda_0^{-1} x \iff m_k^p \lambda_k^{-1} \lambda_0 = m_k.
\]

Now, if \( x \) and all \( Y_k \) belong to \( X_1 \),

\[
1 = \sum_{j=1}^{n-1} x_j = \sum_{j=1}^{n} \sum_{k=1}^{n} m_k Y_{jk} = \sum_{k=1}^{n} m_k \sum_{j=1}^{n-1} Y_{jk} = \sum_{k=1}^{n} m_k.
\]

\( \square \)

We can summarize the above facts in the following theorem.

**Theorem 3.9** Let \( f : \Delta^{n-2} \to X_1 \) the multi-valued map defined as follows: \( f(x) = \text{CH}[Y_1(x), \ldots, Y_n(x)] \) is the convex hull of the \( n \) points \( Y_1, \ldots, Y_n \) in \( X_1 \). Then, \( x \in f(x) \) if and only if any corresponding configuration \( q \) solves the inverse central configuration problem.

**Example 3.10** The case \( n = 3 \) as expected is rather simple: Given that \( x_1 + x_2 = 1 \), the matrix \( Y \) turns out to be

\[
\begin{bmatrix}
x_1^{-1} & x_1^{-1} & 1 - x_2^{-1} \\
1 - x_1^{-1} & x_2^{-1} & x_2^{-1}
\end{bmatrix},
\]

and the projections on \( p(Y_k) \) on \( X_1 \) are the columns of the following matrix

\[
\begin{bmatrix}
x_1^{-1} & x_1^{-1} & 1 - x_2^{-1} \\
1 - x_1^{-1} & x_2^{-1} & x_2^{-1}
\end{bmatrix}
\]

Given that for each \( x_1 \in (0, 1) \)

\[
1 - x_2^{-1} < 0 < x_1 < 1 < x_1^{-1},
\]
for each \( x = (x_1, x_2) \in \Delta^1 \) one has \( x \in \text{CH}[Y_1, Y_3] \subset f(x) \), and hence, there are positive masses solving the inverse central configuration problem.

**Example 3.11** Consider the case \( n = 4 \), and \( \alpha > 0 \). The matrix \( Y \), given that \( x_1 + x_2 + x_3 = 1 \),

\[
Y = \begin{bmatrix}
Q_{11} - Q_{21} & Q_{12} - Q_{22} & Q_{13} - Q_{23} & Q_{14} - Q_{24} \\
Q_{21} - Q_{31} & Q_{22} - Q_{32} & Q_{23} - Q_{33} & Q_{24} - Q_{34} \\
Q_{31} - Q_{41} & Q_{32} - Q_{42} & Q_{33} - Q_{43} & Q_{34} - Q_{44}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
x_1^{\alpha - 1} & x_1^{\alpha - 1} & (x_1 + x_2)^{\alpha - 1} - x_2^{\alpha - 1} & 1 - (x_1 + x_2)^{\alpha - 1} \\
x_2^{\alpha - 1} & x_2^{\alpha - 1} & x_2^{\alpha - 1} - x_1^{\alpha - 1} & (x_2 + x_3)^{\alpha - 1} - x_3^{\alpha - 1} \\
1 - (x_1 + x_2)^{\alpha - 1} & -x_2^{\alpha - 1} + (x_2 + x_3)^{\alpha - 1} & x_3^{\alpha - 1} & x_3^{\alpha - 1}
\end{bmatrix}
\]

The projections on \( X_1 \) are

\[
p(Y_1) = Y_1, \quad p(Y_4) = Y_4
\]

and

\[
p(Y_2) = \frac{Y_2}{x_1^{\alpha - 1} + (x_2 + x_3)^{\alpha - 1}}, \quad p(Y_3) = \frac{Y_3}{x_3^{\alpha - 1} + (x_1 + x_2)^{\alpha - 1}}.
\]

Note that the second components of \( p(Y_1) \) and \( p(Y_4) \) are negative:

\[-x_1^{\alpha - 1} + (x_1 + x_2)^{\alpha - 1} < 0, \quad (x_2 + x_3)^{\alpha - 1} - x_3^{\alpha - 1} < 0.\]

The second components of \( p(Y_2) \) and \( p(Y_3) \) are

\[
x_2^{\alpha - 1} \quad \text{and} \quad x_3^{\alpha - 1}.
\]

If \( x_2 > \frac{1}{2} \), then \( x_2^{\alpha - 1} < 2^{\alpha + 1} \); since \( x_1 + x_2 + x_3 = 1 \), and by convexity

\[
x_1^{\alpha - 1} + (x_2 + x_3)^{\alpha - 1} = x_1^{\alpha - 1} + (1 - x_1)^{\alpha - 1} > 2^{\alpha + 2},
\]

\[
x_3^{\alpha - 1} + (x_1 + x_2)^{\alpha - 1} = x_3^{\alpha - 1} + (1 - x_3)^{\alpha - 1} > 2^{\alpha + 2}.
\]

Hence, if \( x_2 > 1/2 \), the second components of \( p(Y_2) \) and \( p(Y_3) \) satisfy the inequalities

\[
x_2^{\alpha - 1} < \frac{2^{\alpha + 1}}{2^{\alpha + 2}} = 2^{-1}
\]

\[
x_3^{\alpha - 1} < \frac{2^{\alpha + 1}}{2^{\alpha + 2}} = 2^{-1}.
\]

But this means that for any \( x \) with \( x_2 > 1/2 \), the second components of \( p(Y_k) \) are smaller than \( 1/2 \) for each \( k \), and hence, \( x \notin \text{CH}[Y_1, Y_2, Y_3, Y_4] \): The inverse problem does not have solutions in this region. For \( \alpha = 1 \), a plot of the region where the inverse problem has solutions is represented in Fig. 1. The four simplices are represented in Fig. 2. The plane \( x_1 + x_2 + x_3 = 1 \) is projected to the \( x_1 x_2 \)-plane. The symmetry \( (x_1, x_2, x_3) \mapsto (x_3, x_2, x_1) \), which comes from the symmetry \( (q_1, \ldots, q_n) \mapsto (-q_n, \ldots, -q_1) \), is projected to the affine reflection \( (x_1, x_2) \mapsto (1 - x_1 - x_2, x_2) \).
Note that if $x_1 > \frac{1}{2}$, then $(x_2 + x_3 < 1/2 \implies (x_2 + x_3)^{-\alpha - 1} > 2^{\alpha + 1}$) the following inequalities hold true:

\begin{align}
(x_1 + x_2)^{-\alpha - 1} &< x_1^{-\alpha} \\
(x_2 + x_3)^{-\alpha - 1} &< x_3^{-\alpha} \\
(x_2 + x_3)^{-\alpha - 1} &> 1 \\
x_1^{-\alpha - 1} - (x_2 + x_3)^{-\alpha - 1} &= x_1^{-\alpha - 1} - (1 - x_1)^{-\alpha - 1} < 0.
\end{align}

(3.12)

Now write the projections $p(Y_1), p(Y_2), p(Y_4)$ in barycentric coordinates with respect to the affine frame $P_1' = (1, 0, 0), P_2' = (1/2, 1/2, 0), P_3' = (1/2, 0, 1/2)$ in $X_1$: 
\[ p(Y_1) = Y_1 = (2x_1^{-\alpha-1} - 1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2((x_1 + x_2)^{-\alpha-1} - x_1^{-\alpha-1}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2(1 - (x_2 + x_3)^{-\alpha-1}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \]

\[ p(Y_4) = Y_4 = (1 - 2(x_2 + x_3)^{-\alpha-1}) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2((x_2 + x_3)^{-\alpha-1} - x_3^{-\alpha-1}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2(x_3^{-\alpha-1}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \]

\[ \lambda p(Y_2) = Y_2 = (x_1^{-\alpha-1} - (x_2 + x_3)^{-\alpha-1}) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2(x_2^{-\alpha-1}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2(-x_1^{-\alpha-1} + (x_2 + x_3)^{-\alpha-1}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \]

where \( \lambda = x_2^{-\alpha-1} + (x_2 + x_3)^{-\alpha-1} > 0. \)

Now, by inequalities (3.12), the signs of the barycentric coordinates are \( Y_1 \mapsto (+, -, -), \) \( Y_2 \mapsto (-, +, -), \) \( Y_4 \mapsto (-, -, +), \) and hence, the 2-simplex \( \sigma \) with vertices \( P_1', P_2' \) and \( P_3' \) is contained in \( \text{CH}[Y_1, Y_2, Y_4] \) for each \( x \in \sigma, \) which means that the inverse problem has solutions.

In fact, consider the \( 3 \times 3 \) matrix whose columns are the coordinates of \( pY_1, pY_2, pY_4. \) It is of type

\[ A = \begin{bmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{bmatrix}, \]

where the sum of the columns is 1. Hence, if \( D \) is the matrix with diagonal \( (a_{11}, a_{22}, a_{33}), \) \( A = (I - \hat{A})D, \) where \( \hat{A} \) is

\[ \hat{A} = \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix}, \]

with all \( b_{ij} > 0 \) and the sum of the columns is < 1. Therefore, \( A^{-1} = D^{-1} \sum_{k=0}^{\infty} \hat{A}^k \) as convergent \( \| \hat{A} \|_1 < 1, \) with all entries positive. This implies that for each \( x \) in the vertices of the triangle \( x_1 > 1/2 \) are in the interior of the 2-simplex \( \text{CH}[pY_1, pY_2, pY_4] \) (because their barycentric coordinates are proportional to the columns of \( A^{-1} \)).

**Remark 3.13** Because of the homogeneity, one can use the following procedure to check if \( x \in \text{CH}[pY_1, \ldots, pY_n]: \) For each \( j = 1 \ldots n, \) compute the inverse \( C_j^{-1} \) of the square matrix \( C_j \) of order \( n - 1 \) obtained by removing the first row and the \( j \)th column of the matrix \( B^{-1}Q \) (written in terms of coordinates \( x_1 \).) Then, \( x \in X_1 \) satisfy \( x \in \text{CH}[pY_1, \ldots, \hat{p}Y_k, \ldots, pY_n] \) (with the \( k \)th entry removed) if and only if the vector \( C_j^{-1}X \) has all \( n - 1 \) positive components.
which correspond to multiples of barycentric coordinates of \( x \) with respect to the vertices in \( \text{CH}[pY_1, \ldots, \widehat{pY}_k, \ldots, pY_n] \).

**Theorem 3.14** Let \( q \in F_n(\mathbb{R}) \) be a collinear configuration such that \( q_1 > q_2 > \cdots > q_n \). If for an index \( j \) with \( 2 \leq j \leq n-2 \) the inequality \( 2(q_j - q_{j+1}) > q_1 - q_n \) holds true, then the inverse problem does not have solutions for this configuration. Any positive masses \( m_j \) exist such that \( q \) is a central configuration with respect to the masses \( m_j \).

**Proof** The assertion follows if we prove that if for some \( i \) such that \( 2 \leq i \leq n-2 \) the inequality \( x_i > 1/2 \) holds for the point \( x \in X_1 \) defined with coordinates \( x_i = \frac{q_i - q_n}{q_1 - q_n} \), then \( x \) does not belong to \( \text{CH}[pY_1, \ldots, pY_n] \). In fact, consider the matrix \( \tilde{Y} \) with columns the vectors \( pY_k \): Its coefficients are, for \( j = 1, \ldots, n-1 \) and \( k = 1, \ldots, n \),
\[
\tilde{Y}_{jk} = \frac{Q_{j,k} - Q_{j+1,k}}{Q_{1k} + Q_{kn}}
\]
If \( x_i > \frac{1}{2} \), for some \( 2 \leq i \leq n-2 \), then consider the terms \( Y_{ik} \): If \( k \in \{i, i + 1\} \), then
\[
Q_{1k} + Q_{kn} = (x_1 + \cdots + x_{k-1})^{-\alpha-1} + (x_k + \cdots + x_n)^{-\alpha-1} > 2^{\alpha+1}
\]
by convexity, and
\[
Q_{i,i+1} = x_i^{-\alpha-1} < 2^{\alpha+1}
\]
by monotonicity; hence, the following inequalities hold
\[
Y_{ik} = \frac{Q_{ik} - Q_{i+1,k}}{Q_{1k} + Q_{kn}} = \begin{cases} 
-\frac{Q_{ki} + Q_{k,i+1}}{Q_{1k} + Q_{kn}} < 0 < \frac{1}{2} & \text{if } k < i \\
\frac{Q_{1k} + Q_{kn}}{Q_{i,i+1}} < \frac{1}{2} & \text{if } k = i \\
\frac{Q_{1i} + Q_{1n}}{Q_{i,i+1}} < \frac{1}{2} & \text{if } k = i + 1 \\
\frac{Q_{ik} - Q_{i+1,k}}{Q_{1k} + Q_{kn}} < 0 < \frac{1}{2} & \text{if } k > i + 1.
\end{cases}
\]
Since all the \( i \)th coordinates of the points \( pY_k \) are less than \( \frac{1}{2} \), while \( x_i > \frac{1}{2} \), the point \( x \) does not belong to \( \text{CH}[pY_1, \ldots, pY_n] \). \( \square \)

**References**

Albouy, A., Moeckel, R.: The inverse problem for collinear central configurations. Celest. Mech. Dyn. Astron. 77(2), 77–91 (2000)

Buchanan, H.E.: On certain determinants connected with a problem in celestial mechanics. Bull. Am. Math. Soc. 15(5), 227–232 (1909)

Cayley, A.: Sur les déterminants gauches. J. Reine Angew. Math. 38(1), 239–251 (1871)

Davis, C., Geyer, S., Johnson, W., Xie, Z.: Inverse problem of central configurations in the collinear 5-body problem. J. Math. Phys. 59(5), 052902 (2018)

Dress, A., Wenzel, W.: A simple proof of an identity concerning Pfaffians of skew symmetric matrices. Adv. Math. 112(1), 120–134 (1995)

Ferrario, D.L.: Fixed point indices of central configurations. J. Fixed Point Theory Appl. 17(1), 239–251 (2015)

Ferrario, D.L.: Central configurations and mutual differences. Symmetry Integr. Geom. Methods Appl. 13, Paper No. 021 (2017a)

Ferrario, D.L.: Central configurations, Morse and fixed point indices. Bull. Belg. Math. Soc. Simon Stevin 24(4), 631–640 (2017b)

Godsil, C.D.: Algebraic Combinatorics. Chapman and Hall Mathematics Series. Chapman & Hall, New York (1993)

Hamel, A.M.: Pfaffian identities: a combinatorial approach. J. Combin. Theory Ser. A 94(2), 205–217 (2001)

Knuth, D.E.: Overlapping Pfaffians. Electron. J. Combin. 3, 2 (1996). Research Paper 5, approx. 13
Moeckel, R.: On central configurations. Math. Z. 205(1), 499–517 (1990)
Moulton, F.R.: The straight line solutions of the problem of $n$ bodies. Ann. Math. Second Ser. 12(1), 1–17 (1910)
Northcott, D.G.: Multilinear Algebra. Cambridge University Press, Cambridge (1984)
Ouyang, T., Xie, Z.: Collinear central configuration in four-body problem. Celest. Mech. Dyn. Astron. 93(1), 147–166 (2005)
Stembridge, J.R.: Nonintersecting paths, Pfaffians, and plane partitions. Adv. Math. 83(1), 96–131 (1990)
Xie, Z.: An analytical proof on certain determinants connected with the collinear central configurations in the $n$-body problem. Celest. Mech. Dyn. Astron. 118(1), 89–97 (2014)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.