On gravitational interactions for massive higher spins in AdS3

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Abstract
In this paper we investigate gravitational interactions of massive higher spin fields in three dimensional AdS space with arbitrary value of cosmological constant including flat Minkowski space. We use the frame-like gauge description for such massive fields adapted to the three-dimensional case. First, we carefully analyze the procedure of switching on gravitational interactions in the linear approximation in the example of a massive spin-3 field and then proceed with the generalization to the case of an arbitrary integer spin field. As a result we construct a cubic interaction vertex linear in a spin-2 field and quadratic in a higher spin field on the AdS3 background. As in the massless case the vertex does not contain any higher derivative corrections to the Lagrangian and/or gauge transformations. Thus, even after switching on gravitational interactions, one can freely consider any massless or partially massless limits as well as the flat one.

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1. Introduction

Three dimensional higher spin field theories that appear to be much simpler than the higher dimensional ones (for some reviews on higher spin theories see e.g. [1–5]) may provide an arena to gain some useful experience. One of the main reasons is that, in contrast to the situation in \(d \geq 4\) dimensions [6–8], in three dimensions it is not necessary to consider an infinite number of fields to construct a consistent closed theory [9–11]. Also, it is important that in the frame-like formalism many such theories can be considered as Chern–Simons ones corresponding to some non-compact algebras [12, 10, 11].

Until now most works on three dimensional higher spin field theories were devoted to the construction of an interaction for massless fields [13, 14, 11] or parity-odd topologically massive ones [15–17]. However, massless higher spin fields in three dimensions being pure gauge do not have any physical degrees of freedom, while topologically massive ones do.
contain physical degrees of freedom but due to their specific properties can hardly be generalized to higher dimensions. That is why we think that it is important to investigate parity-even interacting theories for massive higher spin fields\(^3\). On the one hand, such theories will certainly have physical interest by themselves, e.g. for the study of various aspects of duality, and on the other hand we may expect that they admit higher dimensional generalizations. It seems natural to begin such investigations with the gravitational interactions that play a fundamental role in any higher spin theory.

In this paper we consider gravitational interactions for massive higher spin fields in AdS\(_3\) space in the linear approximation. This means that we construct a cubic interacting vertex quadratic in a massive higher spin field and linear in a gravitational field on the AdS\(_3\) background. We use the frame-like gauge invariant formalism for massive fields \([21, 22]\) adapted to three dimensions which was elaborated in our recent paper \([23]\). In our opinion, such a formalism is the most convenient one for investigations of massive higher spin interactions. On the one hand, its gauge invariance allows one to extend the constructive approach to the investigation of massless higher spin interactions to the case of an arbitrary combination of massless and/or massive ones with the interaction of massless gravity, with massive higher spins being one important example. On the other hand, such a formalism works nicely in AdS space with an arbitrary value of the cosmological constant including flat Minkowski space, and this allows one to consider all possible massless and partially massless limits that exist for a given field \([24–26, 21]\) as well as the flat limit.

The work is organized as follows. In section 2, as an illustration of the general techniques that we will use for massive fields in subsequent sections, we provide a couple of simple examples of interacting massless theories. Namely, we consider massless \(d = 3\) gravity itself and the model of its interaction with the massless spin-3 field, constructed for the first time in \([11]\). Then in section 3 we carefully analyze the gravitational interactions for the massive spin-3 field. The main result here is that even in the massive case it is possible to switch on the gravitational interaction in the linear approximation, e.g. to construct the cubic vertex linear in a gravitational field and quadratic in a massive higher spin field, without any need to introduce higher derivative corrections to the Lagrangian and/or gauge transformations typical for higher dimensional theories \([27, 28]\). This, in turn, implies that even after switching on gravitational interactions (at least in the linear approximation) one can freely consider any massless or partially massless limits as well as the flat one. At the same time we argue that, in contrast to the massless case, the system gravity plus massive spin-3 is not closed so that construction of the closed consistent theory requires the introduction of other fields and/or interactions. Then in section 4 we generalize these results to the case of an arbitrary integer spin and obtain the corresponding cubic vertex. As in the spin-3 case, this vertex also does not contain any higher derivative corrections and as a result admits non-singular massless, partially massless and flat limits. It was clear from the very beginning that the system under consideration cannot be closed (simply because massless theory is not closed), but let us stress that that the results of the linear approximation do not depend on the presence of any other fields so that the structure of the cubic vertex is model independent.

\section*{Notations and conventions}

We use Greek letters for the world indices and Latin letters for the local ones. We work in \((\Lambda)\text{dS}\) space with an arbitrary value of the cosmological constant \(\Lambda\) and use the notation \(D_\mu\).

\(^3\) The Lagrangian formulation for \(d = 3\) massive higher spin fields was given for the first time in \([18]\). The aspects of massive higher spin fields in AdS\(_3\) are also discussed in \([19]\) and \([20]\).
for the AdS covariant derivative normalized so that

\[ [D_{\mu}, D_{\nu}] \xi^a = \lambda^2 \epsilon_{\mu a} \xi^\nu, \quad \lambda^2 = -\Lambda \]

where \( \epsilon_{\mu a} \) plays the role of a (non-dynamical) frame of the (A)dS background. Also, to write the expression in a totally antisymmetric form on the world indices (which is equivalent to an external product of 1-forms) we will often use the notation

\[ \{ \mu \nu \}_{ab} = e_{\mu a} e_{\nu b} - e_{\mu b} e_{\nu a} \]

and similarly for \( \{ \mu \nu \alpha \}_{abc} \).

2. Massless case

In this section we present a couple of most simple examples of interacting massless theories in AdS3. The reasons for its inclusion are twofold. From one hand they give simple illustration of general techniques that we will use for the massive cases in the subsequent sections. From the other one, it is instructive to compare massless and massive cases because in some aspects they appear to be drastically different.

2.1. Gravity

The free massless spin-2 field in AdS3 space is described by the Lagrangian:

\[ L_0 = \frac{1}{2} \{ \mu \nu \}_{ab} \omega^a \omega^b - \epsilon^{\mu \nu \alpha} \omega^a D_{\alpha} h^a + \frac{\lambda^2}{2} \{ \mu \nu \} h^a h^b, \]

which is invariant under the following gauge transformations:

\[ \delta_0 \omega^a = D_\mu \eta^a + \lambda \epsilon^{\mu \nu} \omega^a D_\nu h^a, \quad \delta_0 h^a = D_\mu \xi^a + \epsilon^{ab} h^b. \]

Now if we introduce two gauge invariant objects (curvature and torsion)

\[ R_{\mu \nu}^a = D_\mu \omega^a + \lambda \omega^a D_\nu h^a, \quad T_{\mu \nu}^a = D_\mu h^a + \epsilon^{ab} \omega^a \omega^b \]

then the Lagrangian can be rewritten as follows

\[ L_0 = -\frac{1}{4} \epsilon^{\mu \nu \alpha} [\omega_{\mu} T_{\nu a}^a + h_{\mu}^a R_{\nu a}^a]. \]

Let us introduce new variables

\[ \hat{\omega}_{\mu}^a = \omega_{\mu}^a + \lambda h_{\mu}^a, \quad \hat{h}_{\mu}^a = \omega_{\mu}^a - \lambda h_{\mu}^a \]

and their corresponding gauge invariant objects

\[ \hat{R}_{\mu \nu}^a = D_\mu \hat{\omega}_{\nu}^a + \lambda \hat{\omega}_{\mu}^a h_{\nu}^a, \quad \hat{T}_{\mu \nu}^a = D_\mu \hat{h}_{\nu}^a - \lambda \epsilon^{ab} \hat{h}_{\nu}^a. \]

Then the Lagrangian takes the form

\[ L_0 = -\frac{1}{8\lambda} \epsilon^{\mu \nu \alpha} [\hat{\omega}_{\mu}^a \hat{R}_{\nu a}^a - \hat{h}_{\mu}^a \hat{T}_{\nu a}^a]. \]

Thus the Lagrangian now consists of two independent parts, each containing only one field, and is invariant under its own gauge transformation

\[ \delta_0 \hat{\omega}_{\mu}^a = D_\mu \hat{\eta}^a + \lambda \epsilon_{\mu ab} \hat{\eta}^b, \quad \delta_0 \hat{h}_{\mu}^a = D_\mu \hat{\xi}^a - \lambda \epsilon_{\mu ab} \hat{\xi}^b, \]

where

\[ \hat{\eta} = \eta^a + \lambda \hat{\xi}^a, \quad \hat{\xi}^a = \eta^a - \lambda \hat{\xi}^a. \]
Now if we suppose that, after switching on the interaction, the possibility to separate the variables also exists, we can greatly simplify all calculations by working with only one field and one gauge transformation. In components the free Lagrangian for the field $\hat{\omega}_\mu^a$ has the form:

$$L_0 = -\frac{1}{4\lambda} \left[ \epsilon^{\mu\nu\alpha\beta} \hat{\omega}_\mu^a D_\nu \hat{\omega}_\alpha^b - \lambda \left\{ \frac{\mu}{ab} \right\} \hat{\omega}_\mu^a \hat{\omega}_\nu^b \right].$$  \tag{10}

To consider possible self interactions we begin with the linear approximation (i.e. cubic terms in the Lagrangian and linear terms in gauge transformations). In this case the only possible cubic vertex and corresponding correction to gauge transformations look like:

$$L_1 = a_0 \left\{ \frac{\mu}{ab} \right\} \hat{\omega}_\mu^a \hat{\omega}_\nu^b \hat{\omega}_\alpha^c, \quad \delta_1 \hat{\omega}_\mu^a = a_0 \epsilon^{abc} \hat{\omega}_\mu^b \hat{\eta}^c. \tag{11}$$

Gauge invariance in the linear approximation requires that

$$\delta_0 L_1 + \delta_1 L_0 = 0$$

and gives us

$$a_0 = -\frac{\alpha_0}{12\lambda}.$$  

Now if we try to go beyond the linear approximation we face the important fact that in the frame-like formalism in $d = 3$ there are not any quartic vertices for all spins $s \geq 2$. Happily, in this particular case it is easy to check that $\delta_1 L_1 = 0$ so that we have a closed consistent theory without any need to introduce other fields. The resulting Lagrangian

$$L = -\frac{1}{4\lambda} \left[ \epsilon^{\mu\nu\alpha\beta} \hat{\omega}_\mu^a D_\nu \hat{\omega}_\alpha^b - \lambda \left\{ \frac{\mu}{ab} \right\} \hat{\omega}_\mu^a \hat{\omega}_\nu^b \right] + \frac{\alpha_0}{3} \left\{ \frac{\mu}{ab} \right\} \hat{\omega}_\mu^a \hat{\omega}_\nu^b \hat{\omega}_\alpha^c \tag{13}$$

is invariant under the following gauge transformations

$$\delta_0 \hat{\omega}_\mu^a = D_\mu \hat{\eta}^a + \lambda \epsilon_{\mu ab} \hat{\eta}^b + \alpha_0 \epsilon^{abc} \hat{\omega}_\mu^b \hat{\eta}^c. \tag{14}$$

In turn, this gauge invariance implies that there exists a deformation for the curvature

$$\hat{\tilde{R}}_{\mu\nu} = \hat{\tilde{R}}_{\mu\nu}^a + \frac{\alpha_0}{2} \epsilon^{abc} \hat{\omega}_\mu^b \hat{\omega}_\nu^c. \tag{15}$$

which transforms covariantly

$$\delta \hat{\tilde{R}}_{\mu\nu}^a = \alpha_0 \epsilon^{abc} \hat{\omega}_\mu^b \hat{\tilde{R}}_{\nu\alpha}^c. \tag{16}$$

In particular, this means that this model can be considered as a Chern–Simons theory with the algebra $O(2, 1) \cong SL(2)$ \cite{12}. Note however that the interacting Lagrangian cannot be written in the form similar to the free one because

$$\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \hat{\omega}_\mu^a \hat{\tilde{R}}_{\alpha}^a = \epsilon^{\mu\nu\alpha\beta} \hat{\omega}_\mu^a D_\nu \hat{\omega}_\alpha^b - \lambda \left\{ \frac{\mu}{ab} \right\} \hat{\omega}_\mu^a \hat{\omega}_\nu^b + \frac{\alpha_0}{2} \left\{ \frac{\mu}{ab} \right\} \hat{\omega}_\mu^a \hat{\omega}_\nu^b \hat{\omega}_\alpha^c,$$

but the equations of motion following from the Lagrangian are equivalent to

$$\hat{\tilde{R}}_{\mu\nu}^a = 0.$$  

Now we can easily return to the initial variables taking the complete Lagrangian in the form

$$L = L(\hat{\omega}) - L(\hat{h}).$$  \tag{17}

Note that in terms of initial variables each part contains parity-odd higher derivative terms; only when coefficients are equal are these terms canceled and we obtain a parity-even theory with no more than two derivatives. Such a theory is nothing but the usual $d = 3$ gravity in the AdS background with the Lagrangian

$$L = \frac{1}{2} \left\{ \frac{\mu}{ab} \right\} \omega_\mu^a \omega_\mu^b - \epsilon^{\mu\nu\alpha\beta} \omega_\mu^a D_\nu \omega_\alpha^b + \frac{\lambda^2}{2} \left\{ \frac{\mu}{ab} \right\} h_\mu^a h_\nu^b$$

$$- \frac{\alpha_0}{2} \left\{ \frac{\mu}{abc} \right\} \left[ \omega_\mu^a \omega_\nu^b h_\alpha^c + \frac{\lambda^2}{3} h_\mu^a h_\nu^b h_\alpha^c \right].$$  \tag{18}
A free massless spin-3 field in AdS3 space can be described by the following Lagrangian:

\[ \mathcal{L} = -\frac{1}{2} \left( \mu_{ab}^{\mu} \Omega_{v}^{ab} + \varepsilon_{\mu\nu\rho} \Omega_{\mu}^{ab} \mathcal{D}_{\nu} \Phi_{v}^{ab} - \lambda^2 \right) \Omega_{v}^{ab} \Phi_{\mu}^{ac} \Phi_{v}^{bc}, \]

where both \( \Omega_{\mu}^{ab} \) and \( \Phi_{\mu}^{ab} \) are symmetric and traceless on local indices. This Lagrangian is invariant under the following gauge transformations:

\[ \delta \Omega_{\mu}^{ab} = D_{\mu} \eta^{ab} - \lambda^2 \varepsilon_{\mu\nu\rho} c(a) \xi^{bc}, \quad \delta \Phi_{\mu}^{ab} = D_{\mu} \xi^{ab} - \varepsilon_{\mu\nu\rho} c(a) \varepsilon^{bc}, \]

where both \( \eta^{ab} \) and \( \xi^{ab} \) are symmetric and traceless. As in the spin-2 case the Lagrangian can be written in terms of gauge invariant objects (which we will call curvatures) \( G_{\mu\nu}^{ab} \) and \( \mathcal{H}_{\mu\nu}^{ab} \) in the form

\[ \mathcal{L}_0 = \frac{1}{4} \varepsilon_{\mu\nu\rho} [\Omega_{\mu}^{ab} G_{\nu\rho}^{ab} + \Phi_{\mu}^{ab} G_{\nu\rho}^{ab}]. \]

As in the spin-2 case we can introduce new variables

\[ \hat{\Omega}_{\mu}^{ab} = \Omega_{\mu}^{ab} + \lambda \Phi_{\mu}^{ab}, \quad \hat{\Phi}_{\mu}^{ab} = \Omega_{\mu}^{ab} - \lambda \Phi_{\mu}^{ab} \]

and rewrite the free Lagrangian in the form

\[ \mathcal{L}_0 = \frac{1}{8 \lambda} \varepsilon_{\mu\nu\rho} \hat{\Omega}_{\mu}^{ab} \hat{G}_{\nu\rho}^{ab} - \Phi_{\mu}^{ab} \hat{G}_{\nu\rho}^{ab}. \]

Again each half of the Lagrangian contains one field only and is invariant under one gauge transformation

\[ \delta \hat{\Omega}_{\mu}^{ab} = D_{\mu} \hat{\eta}^{ab} - \lambda \varepsilon_{\mu\nu\rho} c(a) \hat{\xi}^{bc}, \quad \delta \hat{\Phi}_{\mu}^{ab} = D_{\mu} \hat{\xi}^{ab} + \lambda \varepsilon_{\mu\nu\rho} c(a) \hat{\eta}^{bc}, \]

where now

\[ \hat{\eta}^{ab} = \eta^{ab} + \lambda \xi^{ab}, \quad \hat{\xi}^{ab} = \eta^{ab} - \lambda \xi^{ab}. \]

Thus, assuming that the possibility to separate variables exists after switching on the interaction, we can work with one field and take care on one gauge transformation. The component form for the \( \hat{\Omega}_{\mu}^{ab} \) field Lagrangian looks as follows

\[ \mathcal{L}_0 = \frac{1}{4 \lambda} \left[ \varepsilon_{\mu\nu\rho} \hat{\Omega}_{\mu}^{ab} \mathcal{D}_{\nu} \hat{\Omega}_{\rho}^{ab} - 2 \lambda \left\{ \mu_{ab}^{\nu} \hat{\Omega}_{\nu}^{ac} \hat{\Omega}_{\mu}^{bc} \right\}. \]
Now let us turn to the gravitational interactions for such particles. The only possible cubic vertex now has the form
\[
\mathcal{L}_1 = \alpha_1 \{ \mu^{\nu a} \} \hat{\Omega}^a_{\mu} \hat{\Omega}^b_{\nu} \hat{\Omega}^c_{\alpha} .
\]  
(30)

while the corresponding corrections to the gauge transformations can be written as follows
\[
\delta \hat{\Omega}^a_{\mu} = \alpha_1 \epsilon^{\mu a b c} \hat{\Omega}^b_{\nu} \hat{\Omega}^c_{\alpha} \hat{\eta}_d , \quad \delta \hat{\Omega}^a_{\mu} = \alpha_1 \epsilon^{\mu a b c} \hat{\Omega}^b_{\nu} \hat{\Omega}^c_{\alpha} \hat{\eta}_d .
\]  
(31)

Now consider gauge invariance in the linear approximation that requires \( \delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 = 0 \). For \( \tilde{\eta}^a \) transformations we obtain
\[
\delta_0 \mathcal{L}_1 = -2 \alpha_1 \{ \mu^{\nu a} \} D_\mu \hat{\Omega}^a_{\nu} \hat{\eta}^b \hat{\eta}^c + 2 \alpha_1 \lambda \epsilon^{\mu a b c} \hat{\Omega}^b_{\nu} \hat{\Omega}^c_{\alpha} \hat{\eta}_d ,
\]
\[
\delta_1 \mathcal{L}_0 = \frac{\alpha_1}{\lambda} \{ \mu^{\nu a} \} D_\mu \hat{\Omega}^a_{\nu} \hat{\eta}^b \hat{\eta}^c - \alpha_1 \lambda \epsilon^{\mu a b c} \hat{\Omega}^b_{\nu} \hat{\Omega}^c_{\alpha} \hat{\eta}_d ,
\]
\[
+ \epsilon^{\mu a b c} \left[(2 \alpha_2 - \frac{\alpha_1}{2}) \hat{\Omega}^a_{\nu} \hat{\Omega}^b_{\mu} \hat{\eta}_d + 2 \alpha_2 \hat{\Omega}^a_{\nu} \hat{\Omega}^b_{\mu} \hat{\eta}_d \hat{\eta}_a \right]
\]

This gives us
\[
\alpha_2 = -\alpha_1 , \quad \alpha_3 = -2 \alpha_1 .
\]

To go beyond the linear approximation let us collect together the self interaction for a graviton and its interaction with spin 3:
\[
\mathcal{L}_1 = -\frac{\alpha_0}{12 \lambda} \{ \mu^{\nu a} \} \hat{\Omega}^a_{\mu} \hat{\eta}_d \hat{\eta}_d + \frac{\alpha_1}{2 \lambda} \{ \mu^{\nu a} \} \hat{\Omega}^a_{\mu} \hat{\Omega}^b_{\nu} \hat{\eta}_d .
\]  
(32)

and all corrections to gauge transformations:
\[
\delta \hat{\Omega}^a_{\mu} = \alpha_0 \epsilon^{a b c} \hat{\Omega}^b_{\nu} \hat{\Omega}^c_{\alpha} \hat{\eta}_d , \quad \delta \hat{\Omega}^a_{\mu} = \alpha_1 \epsilon^{a b c} \hat{\Omega}^b_{\nu} \hat{\Omega}^c_{\alpha} \hat{\eta}_d .
\]  
(33)

Now calculating quadratic variations we obtain
\[
\delta_1 \mathcal{L}_1 = \left(\frac{\alpha_0 - \alpha_1}{\lambda}\right) \epsilon^{a b c} \left[\hat{\Omega}^b_{\mu} \hat{\eta}_d \hat{\eta}_d + \hat{\Omega}^b_{\mu} \hat{\Omega}^c_{\alpha} \hat{\eta}_d \right],
\]
so that for \( \alpha_1 = \alpha_0 \) all variations cancel. As in the pure gravity case the invariance of the Lagrangian implies that there exist deformations of the curvatures
\[
\hat{\tilde{\mathcal{R}}}^a_{\mu \nu} = \hat{\mathcal{R}}^{a b c} \hat{\Omega}^a_{\mu} \hat{\Omega}^b_{\nu} \hat{\eta}_d - \alpha_0 \epsilon^{a b c} \hat{\Omega}^a_{\mu} \hat{\Omega}^b_{\nu} \hat{\eta}_d ,
\]
\[
\hat{\tilde{\mathcal{G}}}^{a b}_{\mu \nu} = \hat{\mathcal{G}}^{a b c} \hat{\Omega}^a_{\mu} \hat{\Omega}^b_{\nu} \hat{\eta}_d + \alpha_0 \epsilon^{a b c} \hat{\Omega}^a_{\mu} \hat{\Omega}^b_{\nu} \hat{\eta}_d ,
\]  
(34)

which transform covariantly
\[
\delta \hat{\tilde{\mathcal{R}}}^{a b}_{\mu \nu} = \alpha_0 \epsilon^{a b c} \hat{\tilde{\mathcal{R}}}^{a b c} - 2 \hat{\mathcal{G}}^{a b c} \hat{\eta}_d \hat{\eta}_d ,
\]
\[
\delta \hat{\tilde{\mathcal{G}}}^{a b}_{\mu \nu} = \alpha_0 \epsilon^{a b c} \hat{\tilde{\mathcal{G}}}^{a b c} - \hat{\mathcal{G}}^{a b c} \hat{\eta}_d \hat{\eta}_d .
\]  
(35)
and such that the equations of motion following from the Lagrangian are equivalent to
\[ \hat{\mathcal{R}}_{\mu\nu}^a = 0, \quad \hat{\mathcal{G}}_{\mu\nu}^{ab} = 0. \]
(36)
In this, such a model can be considered as a Chern–Simons theory with the algebra \( SL(3) \) [11].

To simplify the comparison with the massive case let us re-express the main formulas in terms of initial variables. The interacting Lagrangian has the form:
\[
\mathcal{L}_1 = \alpha_0 \{ \frac{\mu\nu a}{abc} \} \left[ \Omega_{\mu\nu}^{ad} \Omega_{\gamma\delta}^{bd} A_{\alpha}^{c} + 2\Omega_{\mu\nu}^{ad} \Phi_{\alpha}^{b} \omega_{\alpha}^{c} + 2\lambda^2 \Phi_{\mu}^{ad} \Phi_{\nu}^{bd} \eta_{\alpha}^{c} \right.
\]
\[
\left. - \frac{1}{2} \omega_{\alpha}^{a} \omega_{\beta}^{b} \eta_{\alpha}^{c} - \frac{\lambda^2}{6} G_{\mu\nu}^{ad} A_{\alpha}^{c} \right],
\]
(37)
while corrections to gauge transformations look as follows:
\[
\delta_1 \Omega_{\mu}^{ad} = \alpha_0 \epsilon^{abc} \left[ \omega_{\mu}^{b} \eta^{c} + \lambda^2 G_{\mu}^{bd} \eta^{c} - 2\Omega_{\mu}^{bd} \eta^{cd} - 2\lambda^2 \Phi_{\mu}^{bd} \eta^{cd} \right],
\]
\[
\delta_1 h_{\mu}^{a} = \alpha_0 \epsilon^{abc} \left[ \omega_{\mu}^{b} \eta^{c} + \lambda^2 G_{\mu}^{bd} \eta^{c} - 2\Omega_{\mu}^{bd} \eta^{cd} - 2\lambda^2 \Phi_{\mu}^{bd} \eta^{cd} \right],
\]
\[
\delta_1 \Phi_{\mu}^{ad} = \alpha_0 \epsilon^{abc} \left[ \omega_{\mu}^{b} \eta^{c} + \lambda^2 G_{\mu}^{bd} \eta^{c} - 2\Omega_{\mu}^{bd} \eta^{cd} - 2\lambda^2 \Phi_{\mu}^{bd} \eta^{cd} \right].
\]
(38)
Finally, the deformed curvatures can be written as:
\[
\hat{R}_{\mu\nu}^a = R_{\mu\nu}^a + \frac{\alpha_0}{2} \epsilon^{abc} \left[ \omega_{\mu}^{b} \eta^{c} + \lambda^2 G_{\mu}^{bd} \eta^{c} - 2\Omega_{\mu}^{bd} \eta^{cd} - 2\lambda^2 \Phi_{\mu}^{bd} \eta^{cd} \right],
\]
\[
\hat{G}_{\mu
u}^{ab} = G_{\mu
u}^{ab} + \alpha_0 \epsilon^{abc} \left[ \omega_{\mu}^{b} \eta^{c} + \lambda^2 G_{\mu}^{bd} \eta^{c} - 2\Omega_{\mu}^{bd} \eta^{cd} - 2\lambda^2 \Phi_{\mu}^{bd} \eta^{cd} \right],
\]
\[
\hat{H}_{\mu
u}^{ab} = H_{\mu
u}^{ab} + \alpha_0 \epsilon^{abc} \left[ \omega_{\mu}^{b} \eta^{c} + \lambda^2 G_{\mu}^{bd} \eta^{c} - 2\Omega_{\mu}^{bd} \eta^{cd} - 2\lambda^2 \Phi_{\mu}^{bd} \eta^{cd} \right].
\]
(39)
Thus, the complete model in terms of initial variables is equivalent to a Chern–Simons theory with the algebra \( SL(3) \times SL(3) \) [11]. Let us stress here that in contrast to the higher-dimensional case [27, 28] this model does not contain any higher derivative terms and as a result admits the non-singular flat limit \( \lambda \to 0 \).

2.3. Arbitrary spin-s field

For brevity in this case, from the very beginning we will work in terms of separated variables \( \hat{\Omega}_{\mu}^{a_1...a_{s-1}} \) which are completely symmetric and traceless on local indices. Also, to simplify formulas we will use the compact notations \( \Omega_{\mu}^{a_1...a_{s-1}} = \Omega_{\mu}^{(k)} \) where the index \( k \) denotes just the number of local indices and not the indices themselves. The free Lagrangian for massless spin-s field has the form [23]:
\[
\mathcal{L}_0 = \frac{(-1)^{s+1}}{4\lambda} \left[ \epsilon^{\alpha\beta\gamma} \hat{\Omega}_{\mu}^{(s-1)} D_{\mu} \hat{\Omega}_{\nu}^{(s-1)} - (s-1)\lambda \left\{ \frac{\mu\nu}{abc} \right\} \hat{\Omega}_{\mu}^{a(s-2)} \hat{\Omega}_{\nu}^{b(s-2)} \right].
\]
(40)
It is invariant under the following gauge transformations
\[
\delta_0 \hat{\Omega}_{\mu}^{(s-1)} = D_{\mu} \hat{\eta}^{(s-1)} - \epsilon_{\mu} \left( \hat{\eta}^{(s-2)a} \right),
\]
where the parameter \( \hat{\eta}^{(s-1)} \) is also symmetric and traceless.

Let us consider the interaction of this field with gravity. The only cubic vertex possible looks like:
\[
\mathcal{L}_1 = (-1)^{s+1} a_s \left\{ \frac{\mu\nu a}{abc} \right\} \hat{\Omega}_{\mu}^{a(s-2)} \hat{\Omega}_{\nu}^{b(s-2)} \hat{\Omega}_{\alpha}^{c},
\]
(42)
while the corresponding corrections to gauge transformations can be written as follows:

\[
\delta_1 \tilde{\Omega}_\mu^{(s-1)} = \alpha_1 \epsilon^{abc} \tilde{\Omega}_\mu^{(s-2)a} \tilde{\eta}^b + \alpha_2 \epsilon^{abc} \tilde{\Omega}_\mu^{(s-2)a} \tilde{\omega}_\mu^b \\
\delta_1 \hat{\omega}_\mu^a = \alpha_3 \epsilon^{abc} \tilde{\Omega}_\mu^{(s-2)a} \tilde{\eta}^c .
\]

(43)

Now we have to calculate linear variations. For \( \tilde{\eta}^a \) transformations we obtain:

\[
\delta_0 \mathcal{L}_1 = -2 \alpha_1 \left\{ \mu \nu a \right\} D_\mu \tilde{\Omega}_\nu \tilde{\eta}^a + 2 \alpha_1 \lambda \epsilon^{abc} \tilde{\Omega}_\mu \tilde{\eta}^a \\
\delta_1 \mathcal{L}_0 = \frac{(s-1) \alpha_1}{2 \lambda} \left\{ \mu \nu a \right\} D_\mu \tilde{\Omega}_\nu \tilde{\eta}^a.
\]

Thus we have to put

\[
\alpha_s = \frac{(s-1) \alpha_1}{4 \lambda}.
\]

Similarly, from the variations under the \( \hat{\eta}^a \) transformations we get

\[
\alpha_2 = (-1)^s \alpha_1, \quad \alpha_3 = -(s-1) \alpha_1.
\]

Thus we have achieved gauge invariance in the linear approximation. Combining the self interaction for a graviton with its interaction with the spin \( s \) particle we obtain the cubic Lagrangian

\[
\mathcal{L}_1 = \frac{1}{4 \lambda} \left\{ \mu \nu a \right\} \left[ -\frac{\alpha_0}{3} \hat{\omega}_\mu^a \hat{\omega}_\nu^b \hat{\omega}_\mu^c + (-1)^{s+1} (s-1) \alpha_1 \hat{\eta}^a + (s-1) \hat{\eta}^{b(s-2)} \hat{\eta}^{c(s-2)} \right]
\]

(44)

and the complete set of corrections to the gauge transformations

\[
\delta_1 \hat{\omega}_\mu^a = \alpha_0 \epsilon^{abc} \hat{\omega}_\mu^b \tilde{\eta}^c - (s-1) \alpha_1 \epsilon^{abc} \tilde{\eta}^{b(s-2)} \tilde{\eta}^{c(s-2)} \\
\delta_1 \tilde{\Omega}_\mu^{(s-1)} = \alpha_1 \epsilon^{abc} \tilde{\Omega}_\mu^{(s-2)a} \tilde{\eta}^b + (-1)^s \epsilon^{abc} \tilde{\eta}^{b(s-2)} \tilde{\eta}^c .
\]

(45)

Now let us consider quadratic variations. For the \( \hat{\eta}^a \) transformations we get

\[
\delta_1 \mathcal{L}_1 = \frac{(-1)^{s+1} (s-1) \alpha_1 (\alpha_0 - \alpha_1)}{2 \lambda} \epsilon^{abc} \tilde{\eta}^{b(s-2)} \tilde{\eta}^c
\]

so we have to put \( \alpha_1 = \alpha_0 \). But this still leaves us with the \( \hat{\eta}^{(s-1)} \) variations of the form

\[
\delta_1 \mathcal{L}_1 \sim \tilde{\Omega}_\mu^{(s-1)} \tilde{\eta}^{(s-1)} \tilde{\Omega}_\nu \tilde{\eta}^{(s-1)} ,
\]

which cancel for the \( s = 3 \) case only, so that for any spin \( s \geq 4 \) to obtain a closed consistent theory we have to introduce some other fields and/or interactions. As has been shown in [11], one of the possible solutions is to introduce all intermediate spins \( s = 3, 4, \ldots, (s-1) \) as well as the corresponding additional cubic vertices, with the result being the Chern–Simons theory with the algebra \( SL(s) \).

3. Massive spin-3 field

Self interaction and interacting with gravity for the massive spin-2 field has been considered in the recent paper of one of the present authors [29], and in this section we consider the gravitational interaction for the massive spin-3 one.

3.1. Kinematics of massive spin-3 field

The frame-like gauge invariant formalism for the massive spin-3 field in (A)dS space adapted to three dimensions [23] requires four pairs of fields (\( \Omega^{ab}_\mu, \Phi^{ab}_\mu \)), (\( \Omega^a_\mu, f^{a}_\mu \)), (\( B^i, A_\mu \)) and
components decouple leaving us with the Lagrangian

\[ \mathcal{L}_0 = - \left\{ \frac{\mu^v}{\mu} \right\} \Omega_{\mu}^{\alpha \gamma} \Omega_{\gamma}^{\beta c} + \frac{\epsilon_{\mu \nu \sigma} \Omega_{\mu}^{\alpha \gamma} D_{\nu} \Phi_{\gamma}^{ab} + \frac{1}{2} \left\{ \frac{\mu^v}{\mu} \right\} \Omega_{\mu}^{\alpha \gamma} \Omega_{\nu}^{\beta c} - \frac{\epsilon_{\mu \nu \sigma} \Omega_{\mu}^{\alpha \gamma} D_{\nu} f_{a}^{\mu} a + \frac{1}{2} \left\{ \frac{\mu^v}{\mu} \right\} \Omega_{\mu}^{\alpha \gamma} \Omega_{\nu}^{\beta c} - \frac{\epsilon_{\mu \nu \sigma} \Omega_{\mu}^{\alpha \gamma} D_{\nu} f_{a}^{\mu} a + \frac{1}{2} \right. \]

\[ + \frac{1}{2} B^\nu B^\nu - \frac{\epsilon_{\mu \nu \sigma} B_{\mu} D_{\nu} A_{\sigma} - \frac{1}{2} \pi^a \pi^a + \pi^b \pi_{\mu} \right. \]

\[ + \pi^{\alpha \nu} \pi^{\mu \alpha} f_{a}^{\alpha} + m \Phi_{\mu \nu} \pi_{\alpha}^{\mu \nu} - \bar{m} \Omega_{\mu \nu} \pi_{\alpha}^{\mu \nu} + \bar{m} f_{\mu \nu} B_{\alpha} \right) + M \pi^a A_{\mu} \]

\[ + \left\{ \frac{\mu^v}{\mu} \right\} \left[ - \frac{M^2}{36} \Phi_{\mu}^{\alpha \gamma} \Phi_{\nu}^{\beta c} + \frac{M^2}{8} f_{a}^{\mu} f_{b}^{\nu} \right] + M \bar{m} \pi^a f_{a}^{\mu} \pi + \bar{m}^2 \pi^2 \right) \]

and is invariant under the following set of gauge transformations

\[ \delta_0 \Omega_{\mu}^{ab} = D_\mu \pi^{ab} + \frac{m}{2} \left( e_{\mu} (\pi^{ab} \pi^{bc}) - \frac{2}{3} \delta^{ab} \pi_{\mu} \right) - \frac{M^2}{36} \delta_{\mu} (\pi^{bc} \pi^{ab}) \]

\[ \delta_0 \Phi_{\mu}^{ab} = D_\mu \pi^{ab} - \frac{e_{\mu} (\pi^{b} \pi^{a})}{2} \]

\[ \delta_0 \Omega_{\mu}^{a} = D_\mu \pi^{a} + 3 \pi_{\mu} \pi^{a} + \frac{M^2}{4} \delta_{\mu} \pi^{a} \]

\[ \delta_0 f_{\mu}^{a} = D_\mu \pi^{a} + e_{\mu} (\pi^{b} \pi^{a}) + 3 \pi_{\mu} \pi^{a} + 2 \bar{m} \pi_{\mu} \pi^{a} \]

\[ \delta_0 B^{a} = -2 \bar{m} \pi^{a}, \quad \delta_0 A_{\mu} = D_\mu \pi^{a} + \bar{m} \pi_{\mu} \]

\[ \delta_0 \pi^{a} = M \bar{m} \pi^{a}, \quad \delta_0 \pi = -M \pi \]

where

\[ \bar{m}^2 = 8 \bar{m}^2 + 4 \lambda^2, \quad M^2 = 18 (3 \bar{m}^2 + 2 \lambda^2). \]

Recall that in de Sitter space ($\lambda^2 < 0$) there exist two partially massless limits [24–26, 21]. The first one corresponds to $M \rightarrow 0$, in this scalar the component decouples leaving us with the Lagrangian

\[ \mathcal{L}_0 = - \left\{ \frac{\mu^v}{\mu} \right\} \Omega_{\mu}^{\alpha \gamma} \Omega_{\gamma}^{\beta c} + \frac{\epsilon_{\mu \nu \sigma} \Omega_{\mu}^{\alpha \gamma} D_{\nu} \Phi_{\gamma}^{ab} + \frac{1}{2} \left\{ \frac{\mu^v}{\mu} \right\} \Omega_{\mu}^{\alpha \gamma} \Omega_{\nu}^{\beta c} - \frac{\epsilon_{\mu \nu \sigma} \Omega_{\mu}^{\alpha \gamma} D_{\nu} f_{a}^{\mu} a + \frac{1}{2} \right. \]

\[ + \frac{1}{2} B^\nu B^\nu - \frac{\epsilon_{\mu \nu \sigma} B_{\mu} D_{\nu} A_{\sigma} - \frac{1}{2} \pi^a \pi^a + \pi^b \pi_{\mu} \right) + M \pi^a A_{\mu} \]

\[ + \left\{ \frac{\mu^v}{\mu} \right\} \left[ - \frac{M^2}{36} \Phi_{\mu}^{\alpha \gamma} \Phi_{\nu}^{\beta c} + \frac{M^2}{8} f_{a}^{\mu} f_{b}^{\nu} \right] \]

which is still invariant under the whole set of gauge transformations

\[ \delta_0 \Omega_{\mu}^{ab} = D_\mu \pi^{ab} + \frac{m}{2} \left( e_{\mu} (\pi^{ab} \pi^{bc}) - \frac{2}{3} \delta^{ab} \pi_{\mu} \right) - \frac{M^2}{36} \delta_{\mu} (\pi^{bc} \pi^{ab}) \]

\[ \delta_0 \Phi_{\mu}^{ab} = D_\mu \pi^{ab} - \frac{e_{\mu} (\pi^{b} \pi^{a})}{2} \]

\[ \delta_0 \Omega_{\mu}^{a} = D_\mu \pi^{a} + 3 \pi_{\mu} \pi^{a} + \frac{M^2}{4} \delta_{\mu} \pi^{a} \]

\[ \delta_0 f_{\mu}^{a} = D_\mu \pi^{a} + e_{\mu} (\pi^{b} \pi^{a}) + 3 \pi_{\mu} \pi^{a} + 2 \bar{m} \pi_{\mu} \pi^{a} \]

\[ \delta_0 B^{a} = -2 \bar{m} \pi^{a}, \quad \delta_0 A_{\mu} = D_\mu \pi^{a} + \bar{m} \pi_{\mu} \]

\[ \delta_0 \pi^{a} = M \bar{m} \pi^{a}, \quad \delta_0 \pi = -M \pi \]

In $d = 3$ dimensions such a partially massless particle has one physical degree of freedom only instead of the usual two in the general massive case.

The other partially massless limit corresponds to $\bar{m} \rightarrow 0$, in this scalar and vector components decouple leaving us with the Lagrangian

\[ \mathcal{L}_0 = - \left\{ \frac{\mu^v}{\mu} \right\} \Omega_{\mu}^{\alpha \gamma} \Omega_{\gamma}^{\beta c} + \frac{\epsilon_{\mu \nu \sigma} \Omega_{\mu}^{\alpha \gamma} D_{\nu} \Phi_{\gamma}^{a} + \frac{1}{2} \left\{ \frac{\mu^v}{\mu} \right\} \Omega_{\mu}^{\alpha \gamma} \Omega_{\nu}^{b c} - \frac{\epsilon_{\mu \nu \sigma} \Omega_{\mu}^{\alpha \gamma} D_{\nu} f_{a}^{\mu} a + \frac{1}{2} \right. \]

\[ + \frac{1}{2} B^\nu B^\nu - \frac{\epsilon_{\mu \nu \sigma} B_{\mu} D_{\nu} A_{\sigma} - \frac{1}{2} \pi^a \pi^a + \pi^b \pi_{\mu} \right) + M \pi^a A_{\mu} \]

\[ + \left\{ \frac{\mu^v}{\mu} \right\} \left[ - \frac{M^2}{36} \Phi_{\mu}^{\alpha \gamma} \Phi_{\nu}^{b c} + \frac{M^2}{8} f_{a}^{\mu} f_{b}^{\nu} \right] \]

(50)
which is invariant under the following gauge transformations:

\[
\delta_0 \Omega^{ab}_\mu = D_\mu \eta^{ab} + \frac{m}{2} \left( e_\mu^{(a} \eta^{b)} - \frac{2}{3} g^{ab} \eta_\mu \right) - \frac{M^2}{36} \epsilon^{(a} \Phi^{b)c}.
\]

\[
\delta_0 \Phi^{ab}_\mu = D_\mu \tilde{\xi}^{ab} - \frac{3m}{2} \left( e_\mu^{(a} \eta^{b)} - \frac{2}{3} g^{ab} \tilde{\xi}_\mu \right).
\]

\[
\delta_0 \Omega^{a}_\mu = D_\mu \eta^a + 3m \eta^a - \frac{M^2}{4} \epsilon^{ab} \eta^b.
\]

\[
\delta_0 f^a_\mu = D_\mu \xi^a + \epsilon^{ab} \eta^b + m \tilde{\xi}^a.
\]

In such a limit we obtain a system that does not have any physical degrees of freedom. Finally, recall that in the massless limit in AdS space the massive spin 3 field decomposes into the massless spin-3 field and the massive spin-2 one.

Let us return to the general massive case. Similarly to the massless case, for all fields entering the description of the massive field one can construct the corresponding gauge invariant object (which we will call curvatures, though there are two forms as well as one form among them):

\[
G^{ab}_{\mu \nu} = D_{[\mu} \Omega^{ab}_{\nu]} + \frac{m}{2} \left( e_{[\mu}^{(a} \Omega^{b)}_{\nu]} + \frac{2}{3} g^{ab} \Omega_{[\mu, \nu]} \right) - \frac{M^2}{36} \epsilon_{[\mu}^{\alpha} \Phi^{b)c}_{\nu]}.
\]

\[
\mathcal{H}^{ab}_{\mu \nu} = D_{[\mu} \Phi^{ab}_{\nu]} - \epsilon_{[\mu}^{\alpha \beta \gamma} \Omega^{\alpha}_{\nu]} + 3m \epsilon_{[\mu}^{\alpha \beta \gamma} \Xi^{\alpha}_{\nu]} - \frac{M^2}{4} \epsilon_{[\mu}^{\alpha \beta \gamma} f_{\nu]}.
\]

\[
F^{a}_{\mu \nu} = D_{[\mu} \Omega^{a}_{\nu]} - \frac{2}{3} m \Omega^{a}_{[\mu, \nu]} - \tilde{m} \epsilon^{a}_{[\mu} B^{\nu]} + \frac{2}{3} M \epsilon^{ab} f_{\nu]} - M \tilde{m} \tilde{\xi}^{a}_{\nu]}.
\]

\[
T^{a}_{\mu \nu} = D_{[\mu} f^{a}_{\nu]} + \epsilon_{[\mu}^{\alpha \beta \gamma} B^{a}_{\nu]} - \frac{2}{3} m \tilde{m} \epsilon^{a}_{[\mu} \Xi^{a}_{\nu]} + 2 \tilde{m} \tilde{m} \tilde{\xi}^{a}_{\nu]}.
\]

\[
B^{a}_{\mu} = D_{[\mu} B^{a}_{\nu]} + \frac{M}{3} B^{a}_{[\mu, \nu]} - \frac{2}{3} e^{a b} \pi^{b} - V^{a}_{\mu}.
\]

\[
A^{a}_{\mu} = D_{[\mu} A^{a}_{\nu]} - e^{a} \omega^{b} B^{b} - \tilde{m} \tilde{f}_{[\mu, \nu]}.
\]

\[
\Pi^{a}_{\mu} = D_{[\mu} \pi^{a} - 2 m \tilde{m} \tilde{\xi}^{a}_{\nu]} - 2 m \tilde{m} e^{a} \phi - W^{a}_{\mu}.
\]

\[
\Phi^{a}_{\mu} = D_{[\mu} \phi - \pi^{a} + MA^{a}_{\mu}.
\]

Moreover, using these curvatures the free Lagrangian can be rewritten in the form similar to the Chern–Simons one

\[
\mathcal{L}_0 = \frac{1}{2} e^{\mu v a} \Omega^{a b}_{\mu} \mathcal{H}_{v u}^{a b} + \epsilon^{a b} \Omega^{a b}_{\mu} \mathcal{T}_{v a}^{a b} + \mathcal{H}_{v a}^{a b} \mathcal{G}_{u a}^{a b} - \mathcal{H}_{v a}^{a b} \mathcal{T}_{v u}^{a b} - \mathcal{G}_{v a}^{a b} \mathcal{T}_{v u}^{a b}.
\]

\[
- 2 \lambda_{\mu} B_{c, a} - B_{c a} A_{[c a]} + \frac{1}{2} e^{a} \left[ - \phi \Pi^{a}_{\mu} + \pi^{a} \Phi^{a}_{\mu} \right].
\]

### 3.2. Non-canonical vertices

In our investigations of gravitational vertices, we will call a vertex canonical if it corresponds to switching on standard minimal interactions, i.e. to the replacement of the background frame $e_{\mu}^{a}$ by the dynamical one $e_{\mu}^{a} - h_{\mu}^{a}$, and the background Lorentz derivative $D_{\mu}$ by the fully covariant one $D_{\mu} - a_{\mu}$. For the massive spin-3 case, due to the presence of Stueckelberg fields, we have two possible non-canonical vertices. The first one is of the type $2 - 2 - 0$ where one of the spin 2 is the graviton, while the other is a Stueckelberg field. Such a vertex has been
considered in the work of one of the present authors [29] devoted to the massive spin-2 case, where it was shown that such a vertex can be completely removed by field redefinitions. The second possibility is the vertex of type $3 - 2 - 1$ where spin-2 is again the graviton, while spin-1 is a Stueckelberg field. The most general ansatz for such a vertex can be written as:

$$\mathcal{L}_1 = \left\{ \mu^{ab} \right\} [a_1 \Omega_{\mu}^{ab} h_{\nu}^{\alpha} B^{\alpha} + a_2 \Omega_{\mu}^{ac} h_{\nu b} B^{\alpha} + a_3 \Phi_{\mu}^{ab} \omega_{\nu}^{b} B^{\alpha} + a_4 \Phi_{\mu}^{ac} \omega_{\nu}^{b} B^{b}] + \epsilon^{ab} [a_5 h_{\mu}^{ab} D_{\alpha} \Phi_{\mu}^{ab} B^{\alpha} + a_6 D_{\mu} h_{\nu}^{ab} \Phi_{\mu}^{ab} B^{b}].$$

(54)

Recall that all vertices which have the same or greater number of derivatives as free Lagrangian are always defined up to possible field redefinitions. For the case at hand we have the following possibilities:

$$\Omega_{\mu}^{ab} \Rightarrow \Omega_{\mu}^{ab} + \kappa_1 h_{\mu}^{ab} B^{b} - \text{Tr}, \quad \omega_{\mu}^{a} \Rightarrow \omega_{\mu}^{a} + \kappa_2 \Phi_{\mu}^{ab} B^{b}.$$

(55)

Let us consider variations under gravitational $\xi^a$ transformations:

$$\delta_0 \mathcal{L}_1 = \left\{ \mu^{ab} \right\} [a_1 D_{\mu} \Omega_{\nu}^{ab} B^{\alpha} \xi^{c} + a_2 D_{\nu} \Omega_{\nu}^{ab} B^{\alpha} \xi^{b} - a_1 \Omega_{\mu}^{ac} \xi^{c} D_{\nu} B^{b} - a_2 \Omega_{\mu}^{ab} \xi^{b} D_{\nu} B] - a_3 \epsilon^{ab} \mu_{\nu}^{\alpha} D_{\nu} \Phi_{\mu}^{ab} (\xi^{b} D_{\mu} B^{b} - \xi^{b} D_{\mu} B^{b}).$$

To compensate for these variations we introduce the following corrections to the gauge transformations:

$$\delta_1 \Phi_{\mu}^{ab} = \alpha_1 \xi^{c} (\alpha B^{a}) B^{b} + \alpha_2 \xi^{c} (\beta B^{a}) B^{b} - \text{Tr}$$

$$\delta_1 \Omega_{\mu}^{ab} = \alpha_3 \xi^{c} (\alpha D_{\mu} B^{b} - \xi^{c} D_{\mu} B).$$

They produce the following variations:

$$\delta_1 \mathcal{L}_0 = 2 \left\{ \mu^{ab} \right\} [D_{\mu} \Omega_{\nu}^{ab} (\alpha B^{a} B^{b} + \alpha_2 \xi^{c} B^{b}) - \alpha_1 \Omega_{\mu}^{ac} (\xi^{b} D_{\nu} B^{b} - \xi^{b} D_{\nu} B^{b})] + 2 \alpha_3 \epsilon^{ab} \mu_{\nu}^{\alpha} D_{\nu} \Phi_{\mu}^{ab} \xi^{c} D_{\mu} B^{b}.$$

Thus we obtain

$$a_1 = a_2 = -2a_3, \quad a_5 = 2a_3 \implies a_1 = a_2 = -a_5$$

but such relations on $a_{1,2,5}$ mean that these terms can be removed by a field redefinition with the parameter $\kappa_1$. This leaves us with

$$\mathcal{L}_1 = \left\{ \mu^{ab} \right\} [a_3 \Phi_{\mu}^{ab} \omega_{\nu}^{c} B^{b} + a_4 \Phi_{\mu}^{ac} \omega_{\nu}^{b} B^{\alpha}] + a_6 \epsilon^{ab} \mu_{\nu}^{\alpha} h_{\nu}^{ab} \Phi_{\mu}^{ab} B^{b}.$$

Now let us consider the variations under $\xi^{ab}$ transformations

$$\delta_0 \mathcal{L}_1 = \xi^{ab} \left\{ \mu^{ab} \right\} [a_3 D_{\mu} \omega_{\nu}^{c} B^{b} - a_4 D_{\nu} \omega_{\nu}^{b} B^{b} + a_3 \omega_{\nu}^{c} D_{\nu} B^{b} + a_4 \omega_{\nu}^{b} D_{\nu} B^{b}] - \epsilon^{ab} \mu_{\nu}^{\alpha} h_{\nu}^{ab} \Phi_{\mu}^{ab} D_{\nu} B^{b}.$$

Then we introduce the corresponding corrections to gauge transformations

$$\delta \omega_{\mu}^{a} = \beta_1 \xi^{ab} \eta_{\mu}^{b} B^{b}, \quad \delta h_{\mu}^{ab} = \beta_2 \xi^{ab} \eta_{\mu}^{b} B^{b},$$

which produce

$$\delta_1 \mathcal{L}_0 = \beta_1 \left\{ \mu^{ab} \right\} \omega_{\mu}^{a} \xi^{bc} \partial_{\nu} B^{b} - \beta_1 \epsilon^{ab} \mu_{\nu}^{\alpha} \partial_{\nu} h_{\mu}^{ab} B^{b} + \beta_2 \xi^{ab} \mu_{\nu}^{\alpha} \omega_{\nu}^{b} B^{b} - \beta_2 \epsilon^{ab} \mu_{\nu}^{\alpha} \omega_{\nu}^{b} \xi^{bc} B^{b}.$$

Thus we obtain

$$a_3 = 0, \quad a_4 = \beta_1 = -a_6$$

and it means that all the remaining terms can be removed by a field redefinition with the parameter $\kappa_2$. 
3.3. Canonical vertices

In this subsection we consider canonical vertices, i.e. those that correspond to the standard minimal interaction with the replacement of the background frame $e^a_{\mu}$ by the dynamical one $e^a_{\mu} - h^a_{\mu}$, and the AdS covariant derivatives $D_\mu$ by the total Lorentz covariant one $D_\mu - \omega_\mu$.

To be sure that the results obtained are the most general ones, initially we take all such terms with arbitrary coefficients. Thus the ansatz for cubic Lagrangian will look as follows:

$$\mathcal{L}_1 = \{ \mu_{abc} \} \left[ c_1 h^a_{\mu} \Omega_{\mu}^b \Omega_{\mu}^c + c_2 \omega^a_{\mu} \Phi_{a \mu} \phi_{\mu} + c_3 h^a_{\mu} \Omega_{\mu}^b \Omega_{\mu}^c + c_4 f_\mu^a \omega_{\mu} \Omega_{\mu}^a \right]$$

$$+ c_5 h_{\mu}^a B^a_{\mu} + c_6 v_{\mu}^a B^a_{\mu} D_\mu A_\mu + c_7 \pi_{\mu}^a \pi_{\mu}^a + c_8 \{ \mu_{abc} \} h_{\mu}^a \pi^b D_\pi^c$$

$$+ \epsilon^{\mu_{abc}} [d_1 h^a_{\mu} \Omega_{\mu}^b f_\sigma + d_2 h^a_{\mu} \Phi_{a \mu} f_\sigma + d_3 h^a_{\mu} \Omega_{\mu}^a f_\sigma] + d_4 h^a_{\mu} f_\mu^a f_\nu \omega_{\mu} f_\nu \pi^a$$

$$+ \{ \mu_{abc} \} \left[ e_1 h^a_{\mu} \Phi_{a \mu} \phi_{\mu} + e_2 h^a_{\mu} f_\mu^a f_\nu \epsilon_\nu \right] + e_3 \{ \mu_{abc} \} h_{\mu}^a f_\nu f_\pi \epsilon_\nu + c_{45} \pi_{\mu}^a \pi_{\mu}^a.$$

Let us begin with gravitational Lorentz transformations. Calculating variations under the $\eta^a$ transformations we obtain

$$\delta_0 \mathcal{L}_1 = \{ \mu_{abc} \} \left[ -c_2 D_\mu \Omega_{\mu}^a f_\sigma \phi_{\sigma} + c_2 D_\mu \Phi_{a \mu} f_\sigma \phi_{\sigma} - c_4 D_\mu \Omega_{\mu}^a f_\sigma \pi_{\sigma} + c_4 D_\mu f_\mu^a \Omega_{\mu}^a \pi_{\sigma} \right]$$

$$+ \epsilon^{\mu_{abc}} \left[ 2 \epsilon_{\mu} \Omega_{\mu}^a f_\sigma \pi_{\sigma} + 2 \epsilon_{\mu} \Phi_{a \mu} f_\sigma \pi_{\sigma} + c_6 \pi_{\mu}^a \pi_{\mu}^a \phi_{\sigma} \right] + c_8 \{ \mu_{abc} \} \pi_{\mu}^a \pi_{\mu}^a.$$ 

By analyzing the terms with explicit derivatives, it is not hard to determine the corresponding corrections to the gauge transformations:

$$\delta_1 \Omega_{\mu}^a = \frac{c_2}{2} e^{\mu \nu \rho} \Omega_{\mu}^\nu h^\rho_{\nu}, \quad \delta_1 \Phi_{\mu}^a = \frac{c_2}{2} e^{\mu \nu \rho} \Phi_{\mu}^\nu h^\rho_{\nu},$$

$$\delta_1 B^a_{\mu} = \frac{c_2}{2} e^{\mu \nu \rho} B^a_{\mu} h^\rho_{\nu}, \quad \delta_1 \pi_{\mu}^a = \frac{c_2}{2} e^{\mu \nu \rho} \pi_{\mu}^\nu h^\rho_{\nu}.$$

They produce

$$\delta_1 \mathcal{L}_0 = \{ \mu_{abc} \} \left[ c_2 D_\mu \Omega_{\mu}^a f_\sigma \phi_{\sigma} + c_2 D_\mu \Phi_{a \mu} f_\sigma \phi_{\sigma} + c_4 D_\mu \Omega_{\mu}^a f_\sigma \pi_{\sigma} + c_4 D_\mu f_\mu^a \Omega_{\mu}^a \pi_{\sigma} \right]$$

$$+ \epsilon^{\mu_{abc}} \left[ -2 \epsilon_{\mu} \Omega_{\mu}^a f_\sigma \pi_{\sigma} + 2 \epsilon_{\mu} \Phi_{a \mu} f_\sigma \pi_{\sigma} + c_6 \pi_{\mu}^a \pi_{\mu}^a \phi_{\sigma} \right] + c_8 \{ \mu_{abc} \} \pi_{\mu}^a \pi_{\mu}^a.$$ 

Then requiring that $\delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 = 0$ we obtain

$$c_2 = 2c_1, \quad 2c_3 = c_4 = -c_1,$$

$$d_1 = 3m c_1, \quad d_2 = m c_1, \quad d_3 = -2m c_1, \quad d_4 = m c_1, \quad d_5 = -m c_6, \quad d_6 = M c_8,$$

$$e_1 = \frac{M c_1}{36}, \quad e_2 = \frac{-M c_1}{8}, \quad e_3 = -M c_1.$$
Now let us consider variations under the $\tilde{\xi}^a$ transformations
\[
\delta_0 L_1 = \left\{ \begin{array}{l}
\frac{\mu}{\nu} \left\{ \begin{array}{l}
-2c_1 D_\alpha \tilde{\xi}^a D_\beta \Omega^a_{\alpha} \tilde{\xi}^c + 2c_3 D_\alpha \tilde{\xi}^a D_\beta \Omega^a_{\beta} \\
-2c_3 D_\alpha \tilde{\xi}^a D_\beta \phi^a - c_b D_\alpha \tilde{\xi}^a D_\beta \Omega^a_{\alpha} D_\alpha D_\alpha D_\beta A_\alpha - 2c_3 \tilde{\xi}^a \phi^a D_\alpha A_\alpha - c_b \left\{ \begin{array}{l}
\frac{\mu}{\nu} D_\alpha \phi^a D_\beta D_\phi \\
+ \epsilon_{\mu \nu} D_\alpha \phi^a D_\mu D_\phi + d_1 \tilde{\xi}^a D_\alpha \phi^a D_\beta D_\phi - d_1 \tilde{\xi}^a D_\phi \phi^a D_\mu D_\phi + d_1 \tilde{\xi}^a \phi^a D_\mu D_\phi
\right. \\
+ \epsilon_{\mu \nu} D_\alpha \phi^a D_\beta D_\phi + d_1 \tilde{\xi}^a D_\phi \phi^a D_\mu D_\phi + d_1 \tilde{\xi}^a \phi^a D_\mu D_\phi
\end{array} \right\}
\right. \\
- \frac{d_2}{2} \left( D_\alpha \phi^a D_\beta D_\phi - 2 \tilde{\xi}^a \phi^a D_\mu D_\phi + \epsilon_{\mu \nu} \phi^a D_\mu D_\phi + 2 \tilde{\xi}^a \phi^a D_\mu D_\phi \right)
\end{array} \right\} \\
- \left\{ \begin{array}{l}
2c_2 D_\alpha \phi^a D_\beta D_\phi + \phi^a D_\beta D_\phi + \epsilon_{\mu \nu} \phi^a D_\mu D_\phi
\end{array} \right\}
\right\}
\]
In this case we have a lot of terms with explicit derivatives and the corresponding corrections to the gauge transformations look as follows:
\[
\delta \Omega^a_{\mu} = \frac{d_3}{2} \left( \Omega^a_{\mu} \tilde{\xi}^a - \frac{2}{3} \phi^a D_\alpha \phi^a D_\mu \tilde{\xi}^a \right) + e_1 \epsilon_{\mu \nu} \phi^a D_\nu \phi^a \tilde{\xi}^a
\]
\[
\delta \Phi^a_{\mu} = c_1 \epsilon_{\mu \nu} \phi^a D_\nu \phi^a \tilde{\xi}^a + d_4 \left( \phi^a \tilde{\xi}^a - \frac{2}{3} \phi^a \phi^a \tilde{\xi}^a \right)
\]
\[
\delta \Omega^a_{\mu} = \frac{d_3}{2} \left( \Omega^a_{\mu} \tilde{\xi}^a - \frac{2}{3} \phi^a D_\alpha \phi^a D_\mu \tilde{\xi}^a \right) + e_1 \epsilon_{\mu \nu} \phi^a D_\nu \phi^a \tilde{\xi}^a
\]
\[
\delta \Phi^a_{\mu} = c_1 \epsilon_{\mu \nu} \phi^a D_\nu \phi^a \tilde{\xi}^a + d_4 \left( \phi^a \tilde{\xi}^a - \frac{2}{3} \phi^a \phi^a \tilde{\xi}^a \right)
\]
\[
\delta \Omega^a_{\alpha} = \frac{d_3}{2} \left( \Omega^a_{\alpha} \tilde{\xi}^a - \frac{2}{3} \phi^a D_\alpha \phi^a D_\alpha \tilde{\xi}^a \right) + e_1 \epsilon_{\alpha \beta} \phi^a D_\beta \phi^a \tilde{\xi}^a
\]
\[
\delta \Phi^a_{\alpha} = c_1 \epsilon_{\alpha \beta} \phi^a D_\beta \phi^a \tilde{\xi}^a + d_4 \left( \phi^a \tilde{\xi}^a - \frac{2}{3} \phi^a \phi^a \tilde{\xi}^a \right)
\]
\[
\delta \pi^a = c_5 \left( \tilde{\xi}^a D_\alpha \pi^a - \tilde{\xi}^a \phi^a (D \pi) \right) + e_3 \left( f \tilde{\xi}^a - f \tilde{\phi}^a \right) + 2e_4 \phi \tilde{\xi}^a
\]
\[
\delta \phi = c_5 \left( \tilde{\xi}^a \phi^a \pi^a \right)
\]
By straightforward but rather long calculations we can see that all variations can be canceled provided that the following relations hold:
\[
-2c_3 = c_6 = 2c_7 = -c_8 = c_1, \quad e_4 = -3m^2 c_1.
\]
Thus the invariance under the gravitational gauge transformations already completely fixes all the coefficients in the Lagrangian and in the corresponding corrections to the gauge transformations up to one arbitrary coupling constant. Choosing the same coupling constant $\alpha_1$ as in the massless spin 3 case we obtain the following final form of the cubic vertex (compare with the free Lagrangian (46)):
and corrections to the gravitational gauge transformations:

\[ \delta_1 \Omega_{\mu}^{ab} = \alpha_1 \left[ \varepsilon^{cd(a} \Omega_{\mu c}^{b)} \eta^d + \frac{m}{2} \left( \Omega_{\mu}^{(a} \eta^{b)} - \frac{2}{3} g^{ab} \Omega_{\mu}^{c} \xi^c \right) + \frac{M^2}{36} \varepsilon^{cd(a} \Phi_{\mu c}^{b)} \xi^d \right] \]

\[ \delta_1 \Phi_{\mu}^{ab} = \alpha_1 \left[ \varepsilon^{cd(a} \Phi_{\mu c}^{b)} \eta^d + \varepsilon^{cd(a} \Omega_{\mu c}^{b)} \xi^d + \frac{3m}{2} \left( f_{\mu}^{(a} \eta^{b)} - \frac{2}{3} g^{ab} f_{\mu}^{c} \xi^c \right) \right] \]

\[ \delta_1 \Omega_{\mu}^a = \alpha_1 \left[ \varepsilon^{abc} \Omega_{\mu}^b \xi^c + 3m \Omega_{\mu}^{ab} \xi^b - \tilde{m} B_{\mu} \xi^a + \tilde{m} e_{\mu}^a B^b \xi^b + \frac{M^2}{4} \varepsilon^{abc} f_{\mu}^b \xi^c - \tilde{M} \varepsilon_{\mu}^a \xi^b \phi^b \right] \]

\[ \delta_1 B_{\mu} = \alpha_1 [\varepsilon_{\mu}^a B^b \xi^b - \tilde{\xi}^a D_{\mu} B^a - \tilde{m} \tilde{\Omega}_{\mu}^a \xi^a - \tilde{M} \varepsilon_{\mu}^a \phi^b \xi^b] \]

\[ \delta_1 \pi^a = \alpha_1 [\varepsilon_{\mu}^a \pi^b \xi^c - (\xi^a D_{\pi} \pi^a - \tilde{\xi}^a D_{\pi}) - \tilde{M} \tilde{\Omega}^a \xi^b - \tilde{M} \varepsilon_{\mu}^a \phi^b \xi^b] \]

\[ \delta_1 \phi = -\alpha_1 (\pi \tilde{\xi}) \tag{58} \]

But we still have to take care on the whole set of massive spin-3 gauge transformations \( \eta^{ab}, \xi^a, \eta^a, \xi, \xi \). Note that there is a striking difference between \( d = 3 \) and \( d \geq 4 \) cases here. In \( d \geq 4 \) dimensions variations of the Lagrangian under the higher spin gauge transformations produce terms proportional to gravitational curvature and to compensate for these terms one has to introduce a lot of higher derivative corrections to the Lagrangian and gauge transformations [27, 28]. Moreover, the higher the spin one will try to consider, the more derivatives one will have to introduce. In this, the most important terms are those containing the Weyl tensor because all the terms containing the Ricci tensor can be compensated for by corresponding corrections to the graviton transformations and/or removed by field redefinitions. The same is true for the massive higher spins as well. A complete analysis for the massive spin-3 field gravitational interactions in \( d \geq 4 \) dimensions is still absent but the results of [30] clearly show that in this case the four derivative terms containing the full Riemann tensor play a crucial role. However, in three dimensions the Weyl tensor is identically zero and as a result, we will show in the example of the massive spin-3 field now, it is possible to achieve gauge invariance in the linear approximation without introduction of any higher derivative corrections. Thus, after switching on gravitational interactions we can still consider any massless, partially massless or flat limits. Moreover, as will be seen from the next section, this holds true for the arbitrary spins as well.

So, our cubic vertex is completely fixed and we now have to find corrections to the massive spin-3 gauge transformations to achieve invariance in the linear approximation. The procedure is exactly the same as was shown above: we calculate variations of the cubic vertex \( \delta_0 \mathcal{L}_1 \), analyze the terms containing explicit derivatives, find the form of the necessary corrections to the gauge transformations and adjust the coefficients to achieve \( \delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 = 0 \). All the calculations are straightforward but rather long, so we will not give the details of them here but simply give the final results. For massive spin-3 fields the corrections to the gauge transformations appear to be:

\[ \delta_1 \Omega_{\mu}^{ab} = \alpha_1 \left[ -\varepsilon^{cd(a} \Omega_{\mu c}^{b)} \omega_{\mu d} - \frac{m}{2} \left( h_{\mu}^{(a} \eta^{b)} - \frac{2}{3} g^{ab} h_{\mu}^{c} \xi^c \right) - \frac{M^2}{36} \varepsilon^{cd(a} \Phi_{\mu c}^{b)} h_{\mu}^{d} \right] \]

\[ \delta_1 \Phi_{\mu}^{ab} = \alpha_1 \left[ -\varepsilon^{cd(a} \Phi_{\mu c}^{b)} h_{\mu}^{d} - \varepsilon^{cd(a} \Omega_{\mu c}^{b)} \omega_{\mu d} - \frac{3m}{2} \left( h_{\mu}^{(a} \eta^{b)} - \frac{2}{3} g^{ab} h_{\mu}^{c} \xi^c \right) \right] \]

\[ \delta_1 \Omega_{\mu}^a = \alpha_1 \left[ \varepsilon^{abc} \omega_{\mu}^b \eta^c - 3m \eta^{ab} h_{\mu}^{b} + \frac{M^2}{4} \varepsilon^{abc} h_{\mu}^{b} \xi^c \right] \]
\[ \delta_1 f_{\mu} a = \alpha_1 \left[ e^{abc} h_{\mu} b f_{\mu} c + e^{abc} \omega_{\mu} b \xi c - m_0 e^{ab} h_{\mu} b - 2 \bar{m} h_{\mu} a \right] \]

\[ \delta_1 A_{\mu} = -\alpha_1 \bar{m} h_{\mu} a \xi. \]  

(59)

Comparing these expressions with the initial gauge transformations (47), one can see that they indeed correspond to standard substitution rules. At the same time the gravitational fields also have non-trivial transformations:

\[ \delta_1 \omega_{\mu} a = \alpha_1 \left[ -2 e^{abc} \Omega_{\mu} b \eta_{\mu} c + e^{abc} \Omega_{\mu} b \eta c + 3 m \eta_{\mu} f_{\mu} b + m e^{ab} \Omega_{\mu} b - m \Phi_{\mu} a \eta b - 2 \bar{m} A_{\mu} \eta a \right. \]

\[ - \frac{M^2}{18} e^{abc} \Phi_{\mu} b d e c + \frac{M^2}{4} e^{abc} f_{\mu} b e c - M \bar{m} e_{\mu} a \xi \eta b \]

\[ \delta_1 h_{\mu} a = \alpha_1 e^{abc} \left[ -2 \Phi_{\mu} b \eta_{\mu} c - 2 \Omega_{\mu} b e c + f_{\mu} b \eta c + \Omega_{\mu} b \xi c \right]. \]

(60)

Having at our disposal linear corrections to all gauge transformations, we can calculate their commutators in the lowest order and extract composition laws for gauge parameters showing us the structure of the gauge algebra behind these transformations. For the massive spin-3 gauge parameters from (58) and (59), we obtain (up to the common multiplier \( \alpha_1^2 \)):

\[ \hat{\eta}_{ab} = e^{cd a} \left( \eta^{bc} + \frac{M^2}{36} \xi^{bc} \right) + \frac{m}{2} (\xi^{(a} \xi^{b)}) - \text{Tr} \]

\[ \hat{\xi}_{ab} = e^{cd a} \left( \eta^{bc} \eta^{c} + \frac{3m}{2} (\xi^{(a} \xi^{b)}) - \text{Tr} \right) \]

\[ \hat{\eta}^a = e^{abc} \left( \eta^{bc} + \frac{M^2}{4} \xi^{bc} \right) + 3 \eta^{ab} \xi^{bc} \]

\[ \hat{\xi}^a = e^{abc} \left( \eta^{bc} \eta^{c} + \frac{3m}{2} \eta^{ab} \eta^{bc} + 2 \bar{m} \eta^{ab} \right) \]

and for the graviton gauge parameters, correspondingly:

\[ \hat{z}_{ab} = e^{abc} \left( -2 \eta^{bd} \xi^{cd} - \frac{M^2}{18} \xi^{bd} \xi^{cd} + \eta^{bc} \xi^{c} + \frac{M^2}{4} \xi^{bd} \xi^{c} \right) + 3 \eta^{ab} \xi^{bc} + m \xi^{ab} \eta^{bc} + 2 \bar{m} \eta^{ab} \xi^{c} \]

\[ \hat{z}^a = e^{abc} \left( -2 \eta^{bd} \xi^{cd} + \eta^{bc} \xi^{c} \right). \]  

(61)

We see that the structure of the algebra drastically differs from that of the pure massless spin-3 case interacting with gravity. The main reason is that the Stueckelberg spin-2 and spin-1 fields are gauge fields themselves having their own gauge transformations so that the algebra must necessarily be extended with the corresponding generators. We have already mentioned that in the massless limit massive spin-3 particles decompose into massless spin-3 and massive spin-2 ones. Due to the universality of gravitational interactions, even in the massless limit \( m = 0 \) (in this \( M^2 = \lambda^2 \) this massive spin-2 field is still present and we obtain different algebra.

As in the massless case, the invariance of the Lagrangian implies that there exist deformations for all gauge invariant curvatures such that under corrected gauge transformations they transform covariantly, and this provides highly non-trivial check for all our calculations. All additional terms for the massive spin-3 curvatures contain one forms \( \omega_{\mu} a \) or \( h_{\mu} a \) and are completely determined by the structure of gauge transformations. The results:

\[ \Delta G_{\mu \nu}^{ab} = \alpha_1 \left[ e^{cd a} \Omega_{\mu} b c \omega_{\nu} d + \frac{m}{2} \Omega_{\mu} (a h_{\nu}) c - \frac{2}{3} \eta^{ab} \Omega_{\mu} c h_{\nu} c \right] + \frac{M^2}{36} e^{cd a} \Phi_{\mu} b c h_{\nu} d \]

\[ \Delta H_{\mu \nu}^{ab} = \alpha_1 \left[ e^{cd a} (\Phi_{\mu} b c \omega_{\nu} d + \Omega_{\mu} b c h_{\nu} d) + \frac{3m}{2} \left( f_{\mu} (a h_{\nu}) c - \frac{2}{3} \eta^{ab} f_{\mu} c h_{\nu} c \right) \right] \]
\[ \Delta \mathcal{F}_{\mu \nu}^a = \alpha_l \left[ \epsilon^{abc} \Omega_{[\mu}^a \omega_{\nu]} c + 3m \Omega_{[\mu}^a h_{\nu]}^b - \bar{m} B_{[\mu}^a h_{\nu]}^b + \bar{m} \epsilon_{[\mu}^a h_{\nu]}^b B^b - \frac{M^2}{4} \epsilon^{abc} f_{[\mu}^a h_{\nu]}^b c - M \bar{m} \epsilon_{[\mu}^a h_{\nu]}^b \phi \right] \]

\[ \Delta T_{\mu \nu}^a = \alpha_l [\epsilon^{abc} (f_{[\mu}^a h_{\nu]}^c + \Omega_{[\mu}^c h_{\nu]}^a) + m \Phi_{[\mu}^a h_{\nu]}^b + 2 \bar{m} A_{[\mu}^a h_{\nu]}^b] \]

\[ \Delta B_{\mu}^a = \alpha_1 \left[ \epsilon^{abc} \omega_{\mu}^b B^c + \frac{M}{2} \epsilon^{abc} \pi_{\mu}^b c + V_{\mu}^b \right] \]

\[ \Delta A_{\mu \nu} = \alpha [\epsilon^{abc} \omega_{[\mu}^a b_{\nu]}^c + \bar{m} f_{[\mu}^a h_{\nu]}^b] \]

\[ \Delta \Pi_{\mu}^a = \alpha [\epsilon^{abc} \omega_{\mu}^a \pi^c + 2 \bar{m} \pi_{\mu}^a + W_{\mu}^a] \]

\[ \Delta \Phi_{\mu} = \alpha_1 h_{\mu}^a \pi^a \] (63)

clearly show these deformations (as well as the corrections to the gauge transformations) exactly correspond to standard substitution rules. As for the deformations for gravitational curvature and torsion, all terms containing one forms are also determined by the very structure of the corrections to the gauge transformations, while the terms quadratic in zero forms are determined by the requirement that deformed curvatures transform covariantly. By straightforward but rather lengthy calculations we obtain

\[ \Delta R_{\mu \nu}^a = \alpha_1 \left[ - \epsilon^{abc} \Omega_{[\mu}^a \omega_{\nu]} c + \frac{1}{2} \Omega_{[\mu}^a \Omega_{\nu]}^c - 3m \Omega_{[\mu}^a f_{\nu]}^b - m \Phi_{[\mu}^a \Omega_{\nu]}^b + 2 \bar{m} \Omega_{[\mu}^a A_{\nu]}^b - \bar{m} f_{[\mu}^a B_{\nu]}^c \right] \]

\[ - \frac{M^2}{36} \epsilon^{abc} \Phi_{[\mu}^a \Phi_{\nu]}^c + \frac{M^2}{8} \epsilon^{abc} f_{[\mu}^a f_{\nu]}^c - M \bar{m} \epsilon_{[\mu}^a f_{\nu]}^b \phi \]

\[ - \epsilon^{abc} b_{\mu}^a B^c + \frac{1}{2} \epsilon^{abc} b_{\mu}^a B^c - \epsilon_{\mu}^a \pi^a + \frac{1}{2} \epsilon_{\mu}^a \pi^a + \frac{1}{2} \epsilon_{\mu}^a \phi^2 + 3 \bar{m}^2 \epsilon_{\mu}^a \phi^2 ] \] (64)

We have already mentioned that the system gravity plus massless spin-3 provides a closed consistent theory without any need to introduce other fields. But if we consider quadratic variations to check if \( \delta L_1 = 0 \) we immediately recover that, in contrast to the massless case, the system gravity plus massive spin-3 is not closed. The crucial point is that the variations under the main spin-3 transformations \( \eta^{ab} \) and \( \xi^{ab} \) contain one forms \( \Omega_{[\mu}^a \Omega_{\nu]}^b, \Phi_{[\mu}^a \Phi_{\nu]}^b, \Omega_{[\mu}^a A_{\nu]}^b \) and \( f_{[\mu}^a h_{\nu]}^b \). In three dimensions there are no quartic vertices constructed out of one forms, while zero forms \( B^a, \pi^a \) and \( \phi \) do not have non-trivial transformations and can not be of any help. Thus to construct a consistent theory (if it exists at all) we must introduce other fields into the system\(^4\).

Let us stress however that the existence and the very structure of the cubic vertex does not depend on the presence or absence of any other fields so that the results obtained here are universal and model independent.

4. Massive arbitrary spin-\( s \) field

In this section we consider the cubic gravitational vertex for the massive field with arbitrary integer spin \( s \). We already know that the system gravity plus massless spin \( s \) is not closed, so there is no chance that this can be the case for the massive field. But as we have already mentioned, the existence and structure of the cubic vertex do not depend on the presence of

\(^4\) In the case under consideration, the most natural candidate for the role of the new field is the massive spin-4 field which will generate the new interaction terms. In turn the new massive spin-4 field will almost certainly require the introduction of even higher spins and so on up to infinity. Therefore one can suggest that the nonlinear theory of massive higher-spin fields does not have a truncated version, unlike the massless case where we can use a finite set of fields.
any other fields. Moreover the mere existence of such a cubic vertex is crucial for any further investigations.

4.1. Kinematics

We will use the frame-like gauge invariant description for the massive spin $s$ field elaborated in our previous work [23]. Such a description uses a collection of one forms $\Omega_{\mu}^{(k)}$, $\Phi_{\mu}^{(k)}$, $1 \leq k \leq s - 1$ (in the same compact notations) and a pair of zero forms $B^a$, $\pi^a$. The free Lagrangian has the form:

$$L_0 = \sum_{k=1}^{s-1} (-1)^{k+1} \left[ \frac{k}{2} \right]_{ab} \left( \Omega_{\mu}^{a(k-1)} \Omega_{\nu}^{b(k-1)} + \beta_k^2 \Phi_{\mu}^{a(k-1)} \Phi_{\nu}^{b(k-1)} \right) - \epsilon^{\mu\nu\rho} \Omega_{\mu}^{(k)} D_{\nu} \Phi_{\rho}^{(k)}$$

and is invariant under the following gauge transformations:

$$\delta_0 \Omega_{\mu}^{(k)} = D_\mu \eta^{(k)} + \alpha_k \left[ \left( \eta^{(k-1)} - \frac{2}{2k - 1} \eta^{(k-2)} \right) + \frac{\beta_k^2 \eta^{(k-3)} b^{(k-1)} \eta^{(k-1)}}{2k - 1} \right]$$

$$= \delta_0 \Phi_{\mu}^{(k)}$$

$$\delta_0 B^a = \gamma_0 \eta^a, \quad \delta_0 \pi^a = -\frac{\gamma_0}{2} \pi^a$$

where

$$\alpha_k^2 = \frac{(k+1)(s-k)(s+k)}{s^2 - \lambda^2} \left[ \alpha_{s-k-1}^2 + (s-k-1)(s+k-1) \lambda^2 \right], \quad k = 2, 3, \ldots, (s-1)$$

$$\beta_k = \frac{s(s+1)}{s^2 - \lambda^2} \beta_{k-1}, \quad k = 1, 2, \ldots, (s-1)$$

$$\gamma_k = \frac{(k+1)(k+2)}{k} \alpha_{k+1}, \quad k = 1, 2, \ldots, (s-2)$$

Recall that in de Sitter space ($\Lambda = -\lambda^2 > 0$) there exists a number of so called partially massless limits [24–26, 21]. This happens each time when one of the parameters $\alpha_k$ goes to zero. In this, the whole system decomposes into two independent subsystems, one of them describing the partially massless spin $s$ particle, while the other subsystem describes the massive spin $k$ one.

4.2. Cubic vertex

Let us turn to the gravitational interactions for such a massive spin $s$ field. As we have already mentioned, in three dimensions even for arbitrary spin it is possible to achieve gauge invariance in the linear approximation without the introduction of any higher derivative corrections to the Lagrangian and/or gauge transformations. This time, from the very beginning we take the cubic vertex corresponding to standard substitution rules, i.e. the replacement of the AdS derivative
by the totally covariant one \( D_\mu = \omega_\mu \), and the background frame \( e_\mu^a = h_\mu^a \), though we have explicitly checked that it is indeed the most general solution. Such a vertex is determined up to one arbitrary coupling constant, which for simplicity we set to be 1, and looks as follows:

\[ L_1 = \sum_{k=1}^{s-1} (-1)^k \frac{\mu \nu \alpha \beta c}{2} \left[ \frac{1}{2} h_\mu^a \varepsilon_\nu^{ab(k-1)} \Omega_\alpha^a c^{(k-1)} + \alpha_\mu^a \Omega_\nu^{b(k-1)} \Phi_a c^{(k-1)} \right] \]

\[ - \frac{1}{2} B^\mu B^\mu + \omega_\mu^a B_\nu \pi^b + \varepsilon^a \omega_\mu^b \Phi_a + B_\nu D_\mu \pi^a \]

\[ + \sum_{k=2}^{s-1} (-1)^k \frac{\mu \nu \alpha \beta c}{2} \left[ \frac{k}{k-1} h_\mu^a \Omega_\nu^{b(k-1)} \Phi_a c^{(k-1)} - h_\mu^a \Omega_\nu^{b(k-1)} \Phi_a c^{(k-1)} \right] + \gamma_0 \varepsilon^a \omega_\mu^b \Phi_a c^{(k-1)} - \frac{1}{2} B^\mu B^\mu \pi^a. \]

Note that as in the spin-3 case the structure of the vertex is completely determined (up to possible field redefinitions) by the invariance under the gravitational \( \tilde{\eta}^a \) and \( \tilde{\xi}^a \) transformations. In this, appropriate corrections to the gauge transformations are found to be:

\[ \delta_1 \Omega_{\mu}^{(k)} = \varepsilon^{abc}(\Omega_{\mu}^{(k-1)a} \tilde{\eta}^b + \beta_2 \Phi_{\mu}^{(k-1)a} \tilde{\xi}^b) + \frac{(k+1)(k+2)}{k} \alpha_{k+1} \Omega_{\mu}^{a(k)} \tilde{\xi}^a \]

\[ + \alpha_k \left[ \Omega_{\mu}^{(k-1)} \tilde{\xi}^a - \frac{2}{2k-1} \varepsilon^{abc} \Omega_{\mu}^{a(k-2)} \tilde{\xi}^a \right] \]

\[ \delta_1 \Phi_{\mu}^{(k)} = \varepsilon^{abc}(\Phi_{\mu}^{(k-1)a} \tilde{\eta}^b + \Omega_{\mu}^{(k-1)a} \tilde{\xi}^b) + (k+1) \alpha_{k+1} \Phi_{\mu}^{a(k)} \tilde{\xi}^a \]

\[ + \frac{k}{k-1} \alpha_k \left[ \Phi_{\mu}^{(k-1)} \tilde{\xi}^a - \frac{2}{2k-1} \varepsilon^{abc} \Phi_{\mu}^{a(k-2)} \tilde{\xi}^a \right] \]

\[ \delta_1 \Omega_{\mu}^{a} = \varepsilon^{abc} \Omega_{\mu}^{b} \tilde{\xi}^c + 6 \varepsilon^{abc} \Omega_{\mu}^{b} \tilde{\xi}^c - \frac{\gamma_0}{2} e_\mu^a \Phi_{\nu}^{b} \tilde{\xi}^b \]

\[ \delta_1 \Phi_{\mu}^{a} = \varepsilon^{abc} \Phi_{\mu}^{b} \tilde{\xi}^c + 2 \varepsilon^{abc} \Phi_{\mu}^{b} \tilde{\xi}^c + \gamma_0 e_\mu^a \Phi_{\nu}^{b} \tilde{\xi}^b \]

\[ \delta_1 B_{\mu} = \varepsilon_{\mu}^{abc} B^{(b} \tilde{\xi}^{c)} + \tilde{\xi}^{a} D_{\mu} B^{\nu} + \gamma_0 \Omega_{\mu}^{a} \tilde{\xi}^c - 4 \beta_1^2 \varepsilon_{\mu}^{abc} \tilde{\xi}^c \]

\[ \delta_1 \pi_{\mu} = \varepsilon_{\mu}^{abc} \pi_{(b} \tilde{\xi}_{c)} + \tilde{\xi}^{a} D_{\mu} \pi^{\nu} - \varepsilon_{\mu}^{abc} B_{(b} \tilde{\xi}^{c)} + \gamma_0 \Phi_{\mu}^{a} \tilde{\xi}^c. \]

Now we have to take care on all massive spin gauge transformations and check if we indeed have invariance in the linear approximation. As for the higher spin fields, their corrections to the gauge transformations again exactly correspond to standard substitution rules:

\[ \delta_1 \Omega_{\mu}^{(k)} = -\varepsilon^{abc}(k^{(k-1)}a \omega_{\mu}^b + \beta_2 k^{(k-1)}a h_{\mu}^b - \frac{(k+1)(k+2)}{k} \alpha_{k+1} h_{\mu}^a \eta_{a(k)} \]

\[ - \alpha_k \left[ h_{\mu}^{(1)} \eta^{(k-1)} - \frac{1}{2k-1} \varepsilon^{abc} k^{(k-2)}a h_{\mu}^a \right] \]

\[ \delta_1 \Phi_{\mu}^{(k)} = -\varepsilon^{abc}(k^{(k-1)}a h_{\mu}^b + \tilde{\xi}^{(k-1)}a \omega_{\mu}^b - \frac{(k+1)(k+2)}{k} \alpha_{k+1} h_{\mu}^a \xi_{a(k)} \]

\[ - \alpha_k \left[ h_{\mu}^{(1)} \xi^{(k-1)} - \frac{1}{2k-1} \varepsilon^{abc} k^{(k-2)}a h_{\mu}^a \right] \]

as can be easily seen by comparing these expressions with the free ones (66). But to find higher spin transformations for a graviton requires more work. By straightforward but lengthy calculations we obtain:
\[
\delta_1 a_\mu^a \equiv \sum_{k=1}^{s-1} (-1)^{k+1} k e^{abc} \left[ \Omega^{b(k-1)}_\mu \eta^{c(k-1)} + \beta_k^2 \Phi^{b(k-1)}_\mu \xi^{c(k-1)} \right] \\
+ \sum_{k=2}^{s-1} (-1)^{k} k \alpha_k \left[ k + 1 \frac{k - 1}{k} \Phi^{a(k-1)}_\mu \eta^{(k-1)} - \Phi^{a(k-1)}_\mu \eta^{(k-1)} \right] \\
+ \Omega^{a(k-1)}_\mu \eta^{(k-1)} - \frac{k + 1}{k - 1} \Omega^{a(k-1)}_\mu \xi^{(k-1)} - \gamma_0 (\eta_\mu \pi^a - \pi_\mu \eta^a) 
\]

\[
\delta_1 h_\mu^a = \sum_{k=1}^{s-1} (-1)^{k+1} k e^{abc} \left[ \Phi^{b(k-1)}_\mu \eta^{c(k-1)} + \Omega^{b(k-1)}_\mu \xi^{c(k-1)} \right]. 
\]

Thus we have constructed the cubic vertex and all corresponding corrections to gauge transformations such that the resulting theory is invariant in the linear approximation. Unlike the higher dimensional case [27, 28], the cubic vertex under consideration does not require any higher derivative terms. Note also that, as well as in the spin-3 case, for the arbitrary \(s\)-case after switching on the gravitational interaction we still have the possibility to take any of the massless, partially massless or flat limits. In particular, this allows one to consider interactions for such partially massless fields that can possess the specific properties and have a simpler structure in comparison with generic massive fields. As we have already mentioned, to go beyond the linear approximation one has to introduce other fields into the system, and the spectrum of massive fields that one has to consider to obtain a consistent theory is an open question that certainly deserves further study.

5. Conclusion

In this paper we have studied gravitational interactions for massive higher spin fields in three dimensions using a frame-like gauge invariant description. Firstly, after providing a couple of simple interacting massless models illustrating our general techniques, we constructed gravitational interactions for the massive spin-3 field in the linear approximation, e.g. found the cubic vertex linear in the gravitational field and quadratic in the spin-3 field. Recall that the linear approximation for any field does not depend on the presence of any other fields in the system so that the very existence of such a cubic vertex is crucial for any further investigations. In contrast to the massless case—and this may be the main lesson from our work—we argue that the system gravity plus the massive spin-3 field is not closed. The main reason is that the Stueckelberg spin-2 and spin-1 fields are gauge fields themselves, and have their own gauge symmetries, so that the algebra must necessarily be extended with the corresponding generators. In this, even in the massless limit, we obtain a system containing the graviton, massless spin-3 and massive spin-2 fields. This means that to construct a consistent theory we have to introduce some other field, the most natural candidate being the spin-4 one. But this almost certainly will require the introduction of even higher spins and so on. This simple example shows that the structure, spectrum of fields and gauge algebra for massive and massless theories may be drastically different. To proceed in this direction, one first of all has to know the structure of the general cubic vertices for massive higher spins in three dimensions, which are still absent. Recall also, that for the massive spin-3 case we have shown, that after switching on the interactions these gauge invariant curvatures admit nonlinear deformations so that they transform covariantly under corrected gauge transformations. This, in turn, suggests that the most natural route to deal with such theories is some extension of the Fradkin–Vasiliev formalism adapted to three dimensions.
We then managed to generalize these results to the case of the massive field with arbitrary integer spin. It is important that it turns out to be possible without introduction of any higher derivative corrections to the Lagrangian and/or gauge transformations, and this is one of the reasons why three dimensional theories appear to be much simpler than higher dimensional ones. Note also that even after switching on the gravitational interaction (at least in the linear approximation) we still have the possibility to take any massless, partially massless or flat limits.

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