MARTINGALE INTERPRETATION OF WEAKLY CANCELLING DIFFERENTIAL OPERATORS

D. Stolyarov

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We provide martingale analogs of weakly cancelling differential operators and prove a Sobolev-type embedding theorem for these operators in the martingale setting.

Bibliography: 4 titles.

1. Preliminaries

In [4], Van Schaftingen gives a characterization of linear homogeneous vector-valued elliptic differential operators $A$ of order $k$ in $d > 1$ variables such that the inequality

$$
\| \nabla^{k-1} f \|_{L^\infty (\mathbb{R}^d)} \lesssim \| Af \|_{L^1 (\mathbb{R}^d)}
$$

holds true for any smooth compactly supported function $f$. He calls such operators cancelling. Let $k \geq d$ and let $l \in [1, \ldots, d-1]$. It is also proved in [4] that the operator $A$ is cancelling (assuming the ellipticity) if and only if

$$
\| \nabla^{k-l} f \|_{L^\infty (\mathbb{R}^d)} \lesssim \| Af \|_{L^1}.
$$

However, for the case $l = d$, the cancellation condition is only sufficient. In [3], Raita gives a necessary and sufficient condition on the operator $A$ for the inequality

$$
\| \nabla^{k-d} f \|_{L^\infty} \lesssim \| Af \|_{L^1}
$$

to be true for any $f \in C^\infty_0 (\mathbb{R}^d)$. He calls such operators weakly cancelling operators.

Paper [1] suggests a martingale interpretation of Van Schaftingen’s theorem. It appears that the cancellation condition has a direct analog in a probabilistic model earlier introduced in [2]. The present note provides an analog of Raita’s weak cancellation condition.

We refer the reader to [1] for further historical details and motivation as well as for a more detailed description of notation. See Sec. 4 for comparison of our results with [3] and [4].

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2. Notation and statement

Let $m \geq 2$ be a natural number, let $\mathcal{F} = \{ F_n \}$ be an $m$-uniform filtration on a probability space. By this we mean that each atom of the algebra $\mathcal{F}_n$ is split into $m$ atoms of $\mathcal{F}_{n+1}$ having equal probability. The symbol $\mathcal{A} \mathcal{F}_n$ denotes the set of all atoms in $\mathcal{F}_n$. For each $\omega \in \mathcal{A} \mathcal{F}_n$, we fix a map

$$
J_\omega : [1 \ldots m] \to \{ \omega' \in \mathcal{A} \mathcal{F}_{n+1} \mid \omega' \subset \omega \}.
$$

This fixes the tree structure on the set of all atoms. Each atom in $\mathcal{A} \mathcal{F}_n$ corresponds to a sequence of $n$ integers in the interval $[1, \ldots, m]$, which we call digits. We may go further and consider the set $\mathbb{T}$ consisting of all infinite paths in the tree of atoms. Each path starts from the atom in $\mathcal{F}_0$, then chooses one of its sons in $\mathcal{F}_1$, then one of its sons in $\mathcal{F}_2$, and so on.

*Department of Mathematics and Computer Science, St.Petersburg State University and St.Petersburg Department of Steklov Institute of Mathematics, St.Petersburg, Russia, e-mail: d.m.stolyarov@spbu.ru.

1The notation “$X \lesssim Y$” (as in the inequality above) means there exists a constant $C$ such that $X \leq CY$ uniformly. The parameter with regard to which we apply the term “uniformly” is always clear from the context.

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is a natural one-to-one correspondence between points in $T$, i.e., paths, and infinite sequences of digits in $[1, \ldots, m]$. There is also a natural metric on $T$. The distance between the two paths $\gamma_1$ and $\gamma_2$ is defined by the standard formula
\[
\text{dist}(\gamma_1, \gamma_2) = m^{-d}, \quad d = \max\{n \mid \gamma_1(j) = \gamma_2(j) \text{ for all } j < n\}.
\] (2.1)
Define the linear space $V$ by the rule
\[
V = \{x \in \mathbb{R}^m \mid \sum_{j=1}^{m} x_j = 0\}.
\]
Let $\ell$ be an integer. We consider $\mathbb{R}^\ell$-valued martingales adapted to $F$. Let $F = \{F_n\}_n$ be an $\mathbb{R}^\ell$-valued martingale. Define its martingale difference sequence by the rule
\[
f_{n+1} = F_{n+1} - F_n, \quad n \geq 0.
\]
Now, fix an atom $\omega \in \mathcal{A}$. The map $J_\omega$ may be naturally extended to the map that identifies an element of $V \otimes \mathbb{R}^\ell$ with the restriction $f_{n+1}|_\omega$ of a martingale difference to $\omega$. In other words, the map $J_\omega$ identifies $V \otimes \mathbb{R}^\ell$ with the space of $\mathbb{R}^\ell$-valued $\mathcal{F}_{n+1}$-measurable functions on $\omega$ having mean value zero. The said extension will be also denoted by $J_\omega$.

**Definition 2.1.** Let $W \subset V \otimes \mathbb{R}^\ell$ be a linear subspace. Define the martingale Sobolev space by the rule
\[
\mathcal{W} = \{F \text{ is an } L^1 \text{-martingale} \mid \forall n \forall \omega \in \mathcal{A} \ f_{n+1}|_\omega \in J_\omega[W]\}.
\]
The norm in $\mathcal{W}$ is inherited from $L^1$.

We also introduce the martingale analog of the Riesz potential:
\[
I_\alpha[F] = \left\{ \sum_{k=0}^{n} m^{-nk} f_k \right\}_n, \quad \alpha > 0.
\]

**Theorem 2.2** ([1, Theorem 1.9]). If $W$ does not contain nonzero rank-one tensors $v \otimes a$ with $v$ having $m - 1$ equal coordinates, then
\[
\|I_{p-1}[F]\|_{L_p} \lesssim \|F\|_{\mathcal{W}}, \quad p \in (1, \infty].
\] (2.2)

**Remark 2.3.** In fact, a stronger inequality
\[
\sum_{n \geq 0} m^{-\frac{p-1}{p}} \|f_n\|_{L_p} \lesssim \|F\|_{\mathcal{W}}
\] (2.3)
is true if $W$ does not contain rank-one tensors $v \otimes a$ with $v$ having $m - 1$ equal coordinates (see [1, Theorem 1.10]). Moreover, the absence of the said vectors is also necessary for (2.2).

It appears that if we put yet another martingale transform, then the game becomes more interesting, at least for the endpoint case $p = \infty$. Let $\varphi: W \to V$ be a linear operator. When does the inequality
\[
\left\| \sum_{n} m^{-n} \sum_{\omega \in \mathcal{A} \mathcal{F}_n} J_\omega \left[ \varphi(J_\omega^{-1}[f_{n+1}|_\omega]) \right] \right\|_{L_\infty} \lesssim \|F\|_{\mathcal{W}}
\] (2.4)
hold true? By (2.3) and the triangle inequality, it is true provided $W$ does not contain rank-one tensors $v \otimes a$ with $v$ having $m - 1$ equal coordinates. Surprisingly, (2.4) may hold true in other cases. Seemingly, this effect is present for the case $p = \infty$ only.

\[\text{Footnote: There is a small inaccuracy in the notation here. Namely, the image of } J_\omega \text{ is formally defined as a function on } \omega. \text{ In the inequality above, we extend it by zero to the remaining part of the probability space.}\]
Let $D_1, D_2, \ldots, D_m$ be the “nasty” vectors in $V$ that break our inequalities:

$$D_j = (-1, -1, \ldots, -1, m - 1, -1, \ldots, -1).$$

**Theorem 2.4.** The inequality (2.4) holds true if and only if

$$\left( \varphi(D_j \otimes a) \right)_j = 0$$

whenever $D_j \otimes a \in W$.

Formula (2.5) means that the $j$-th coordinate of the vector $\varphi[D_j \otimes a] \in V$ vanishes.

3. **Proof of Theorem 2.4**

3.1. **Necessity.** Assume that $j \in [1, \ldots, m]$ and a vector $a \in \mathbb{R}^\ell \setminus \{0\}$ such that $D_j \otimes a \in W$ and

$$\left( \varphi(D_j \otimes a) \right)_j = \theta \neq 0.$$  

Consider the martingale $F$ defined as follows:

$$F_n = a \cdot m^n \chi_{\omega_n}, \quad \text{where } \omega_n \in \mathcal{A}_F, \quad n \geq 0,$$

corresponds to the sequence $\{j, j, j, \ldots, j\}$. Then

$$f_{n+1} = J_\omega[D_j \otimes a] \cdot m^n \chi_{\omega_n}.$$

Let us stop our martingale at step $N$ and plug the stopped martingale into (2.4). Then the sum on the left hand-side of (2.4) is equal to $N\theta$ on the atom $\omega_N$. Hence, the left hand-side tends to infinity as $N \to \infty$, whereas the right hand-side is identically equal to one. Thus, if $\theta \neq 0$, the inequality (2.4) cannot be true.

3.2. ** Sufficiency**

**Lemma 3.1.** Let $G$ be a real finite dimensional linear space, let $E$ and $F$ be its subspaces. Let $\psi$ be a linear functional on $E$, which vanishes on $E \cap F$. There exists a linear functional $\Psi$ on $G$ such that $\Psi$ is an extension of $\psi$ and it vanishes on $F$.

**Proof.** Consider the diagram

The arrow (1) exists because $\psi|_{E \cap F} = 0$. The arrow (2) is constructed from (1) with the help of the Hahn–Banach theorem. The map $\Psi$ is then restored by commutativity of the diagram. □
We want to extend \( \varphi \) to the whole space \( V \otimes \mathbb{R}^\ell \) preserving the condition (2.5). To do that, we consider coordinate functionals \( \varphi_j : W \rightarrow \mathbb{R} \), who are simply \( j \)-th coordinates of \( \varphi \), and we try to extend them. Consider the spaces \( \mathcal{D}_j \) defined as
\[
\mathcal{D}_j = \left\{ D_j \otimes a \mid a \in \mathbb{R}^\ell \right\}.
\]
Formula (2.5) means that \( \varphi_j|_{W \cap \mathcal{D}_j} = 0 \) exactly. We apply Lemma 3.1 with \( G := V \otimes \mathbb{R}^\ell \), \( E := W \), \( F := \mathcal{D}_j \), and \( \varphi_j \) in the role of \( \psi \), and we obtain a functional \( \Phi_j := \Psi \) on \( V \otimes \mathbb{R}^\ell \), which vanishes on \( \mathcal{D}_j \) and extends \( \varphi_j \). Compose a linear operator \( \Phi : V \otimes \mathbb{R}^\ell \rightarrow \mathbb{R}^m \) from the functionals \( \Phi_j \):
\[
\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_m).
\]
This operator extends \( \varphi \) and satisfies the condition
\[
\forall j \in [1, \ldots, m] \quad \forall a \in \mathbb{R}^\ell \quad (\Phi[D_j \otimes a])_j = 0. \tag{3.1}
\]
It suffices to prove an \textit{a priori} stronger version of (2.4):
\[
\left\| \sum_n m^{-n} \sum_{\omega \in AF_n} J_\omega \left[ \Phi(J_{\omega - 1}[f_{n+1}|_\omega]) \right] \right\|_{L_\infty} \lesssim \|F\|_{L_1} \tag{3.2}
\]
for any \( L_1 \)-martingale \( F \).\(^3\) We use the fact that any \( L_1 \)-martingale adapted to \( F \) has the limit \( \mathbb{R}^\ell \)-valued measure \( \mu \) of bounded variation on \( T \) (the measure is defined on the Borel \( \sigma \)-algebra on \( T \) defined by metric (2.1)) related to \( F \) by the formula
\[
F_n = \sum_{\omega \in AF_n} \mu(\omega) \cdot m^n \chi_\omega. \tag{3.3}
\]
Thus, inequality (3.2) is an estimate of a linear operator on the space of measures. It suffices to verify it for the case where \( \mu \) is a delta measure.

Let \( j = \{j_n\}_n \) be a sequence of digits, i.e., a point in \( T \), let \( a \in \mathbb{R}^\ell \). Consider the martingale \( F \) that represents \( a \cdot \delta_j \) via formula (3.3). In this case,
\[
f_{n+1} = J_\omega \left[ D_{j_{n+1}} \otimes a \right] \cdot m^n, \quad \text{where } \omega = \{j_1, j_2, \ldots, j_n\}.
\]
Condition (3.1) makes the summands in the inner sum in (3.2) have disjoint supports. Indeed,
\[
J_\omega \left[ \Phi(J_{\omega - 1}[f_{n+1}|_\omega]) \right] = J_{\omega n} \left[ \Phi(D_{j_{n+1}} \otimes a) \right].
\]
By (3.1), this function is zero on the atom \( \{j_1, j_2, \ldots, j_{n}, j_{n+1}\} \), where all the functions \( f_k \) with \( k > n + 1 \) are supported.

Therefore, (3.2) follows from the trivial estimate \( \|f_{n+1}\|_{L_\infty} \lesssim m^n \).

4. COMPARISON WITH THE REAL-VARIABLE CASE

Assume now that \([1, \ldots, m]\) is equipped with the structure of an abelian group \( G \). Let \( \Gamma \) be the dual group of \( G \). We may think of \( V \) and \( W \) as of spaces of functions on \( G \) having zero means.\(^4\) Assume further that \( W \) is translation invariant with respect to the action of \( G \). In this case, there exist spaces \( W_\gamma \subset \mathbb{R}^\ell \), \( \gamma \in \Gamma \), such that
\[
W = \left\{ w \in V \otimes \mathbb{R}^\ell \mid \forall \gamma \in \Gamma \setminus \{0\} \quad \hat{w}(\gamma) \in W_\gamma \right\}.
\]
\(^3\)Note that \( L_\infty \) is formally defined as a map on \( V \), and now we apply it to an element of \( \mathbb{R}^m \); this does not cause any problem though.

\(^4\)Since we will be working with the Fourier transform, one might wish to switch to complex scalars here. This does not lead to any problems.
As it is proved in [1], the condition that $W$ does not contain rank-one tensors $v \otimes a$ with $v$ having $m - 1$ equal coordinates may be reformulated as

$$\bigcap_{\gamma \in \Gamma \setminus \{0\}} W_\gamma = \{0\}.$$  

This perfectly matches Van Schaftingen’s cancelling condition in [4].

Let also the operator $\varphi$ be translation invariant. This means there exist functionals $\varphi_\gamma$ on the spaces $W_\gamma$, $\gamma \neq 0$, such that

$$\widehat{\varphi[w]}(\gamma) = \varphi_\gamma(\widehat{w}(\gamma)), \quad \gamma \in \Gamma \setminus \{0\}, \quad w \in W.$$  

Let us express (2.5) in Fourier terms using the Plancherel theorem (by translation invariance, it suffices to consider the case $j = 0$ only):

$$\varphi[D_0 \otimes a](0) = \langle \varphi[D_0 \otimes a], \delta_0 \rangle = \sum_{\gamma} \varphi[D_0 \otimes a](\gamma) = \sum_{\gamma \in \Gamma \setminus \{0\}} \varphi_\gamma[a], \quad D_0 \otimes a \in W.$$  

Therefore, condition (2.5) is equivalent to

$$\sum_{\gamma \in \Gamma \setminus \{0\}} \varphi_\gamma[a] = 0, \quad \forall a \in \bigcap_{\gamma \in \Gamma \setminus \{0\}} W_\gamma,$$  

which perfectly matches Raita’s weak cancelling condition in [3].

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REFERENCES

1. R. Ayuosh, D. Stolyarov, and M. Wojciechowski, “Martingale approach to Sobolev embedding theorems,” https://arxiv.org/abs/1811.08137.

2. S. Janson, “Characterizations of $H^1$ by singular integral transforms on martingales and $R^n$,“ Math. Scand., 41, 140–152 (1977).

3. B. Raita, “Critical differentiability of $BV^A$-maps and cancelling operators,” https://arxiv.org/pdf/1712.01251v2, to appear in Trans. Amer. Math. Soc.

4. J. Van Schaftingen, “Limiting Sobolev inequalities for vector fields and canceling linear differential operators,” J. Eur. Math. Soc., 15, No. 3, 877–921 (2013).