Deriving boundary $S$ matrices

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We show how to derive exact boundary $S$ matrices for integrable quantum field theories in 1+1 dimensions using lattice regularization. We do this calculation explicitly for the sine-Gordon model with fixed boundary conditions using the Bethe ansatz for an XXZ-type spin chain in a boundary magnetic field. Our results agree with recent conjectures of Ghoshal and Zamolodchikov, and indicate that the only solutions to the Bethe equations which contribute to the scaling limit are the standard strings.

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1. Introduction

Many interesting statistical-mechanical systems can be described by 1+1-dimensional models with boundaries. These include the Kondo problem and the Anderson model, tunneling in quantum wires, dissipative quantum mechanics, and the Callan-Rubakov effect. At and near a critical point, they can be studied using boundary conformal field theory [1]. Moreover, many of these models are integrable and many properties can be derived exactly even away from critical points [2].

There are several approaches to integrable models, with and without a boundary. One is explicit diagonalization of the (continuum or suitably discretized) Hamiltonian using the Bethe ansatz. Another bypasses this diagonalization by using maximally the constraints resulting from integrability: after making some guesses for the particle content one can find physical quantities like the exact S matrix for particles scattering among themselves and off the boundary. A crucial ingredient in both cases is the Yang-Baxter equation, including the boundary part [3]. In the second approach this equation only allows one to find ratios of S matrix elements and not their overall prefactor. In the bulk such prefactors are controlled by unitarity and crossing symmetry. For the boundary, constraints of the same type were found in [4,5], enabling the determination of S matrix elements up to the usual CDD-type ambiguities. This was done for the Ising model in a boundary magnetic field [4], the sine-Gordon model with a boundary potential [4,6], affine Toda field theories [7,8] and the O(n) sigma models with free or fixed boundary conditions [9].

The purpose of this paper is to use the first approach to confirm the second. We extract boundary S matrices from lattice regularizations of integrable quantum field theories, generalizing a well-known method in the bulk [10,11,12]. Our explicit example is the inhomogeneous XXZ model in a boundary magnetic field, which is expected to provide an integrable regularization of the sine-Gordon model with fixed boundary conditions. Our results confirm results of [4], in particular their cross-unitarity relation, and remove the CDD ambiguity. The interest of the computation lies mainly in the fact that boundary effects are subleading, and a number of subtleties arise in their analysis.

We derive the boundary S matrix elements in the usual manner. We start with the explicit diagonalization of the appropriate lattice model with integrable boundary conditions. This results in the “bare” Bethe ansatz equations, which relate the level densities of the bare particles to the actual filled densities of these particles. Using these equations, we find the ground state of the system, fill the Fermi sea, and identify physical excitations. This results in the “physical” Bethe ansatz equations, which involve the
densities of the actual quasiparticles of the system. On the other hand, physical Bethe-
ansatz equations can also be derived directly from the conjectured $S$ matrix, without
reference to a lattice. We perform such a computation for the $S$ matrix of \[4,6\]. We
compare the two results, and find that they are indeed the same.

The lattice model of interest is the inhomogeneous 6-vertex model \[13,14\] on an open
strip, with integrable boundary conditions as in \[15\]. We take identical boundary conditions
on both sides of the system. In the hamiltonian and homogeneous limit, this model reduces
to the XXZ model with boundary magnetic fields \[16\]

$$H = \epsilon \frac{\gamma}{2\pi \sin \gamma} \left[ \sum_{i=1}^{N-1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \cos \gamma \sigma_i^z \sigma_{i+1}^z \right) + h \sigma_1^z + h \sigma_N^z \right]. \quad (1)$$

In the inhomogeneous 6-vertex model, one gives an alternating imaginary part \(\pm i\Lambda\) to the
spectral parameter on alternating vertices \[13,14\]. The scaling limit is given by taking
\(\Lambda \to \infty, N \to \infty\), and the lattice spacing \(\Delta \to 0\), such that \(L \equiv N\Delta\) remains finite. In
the bulk, this provides a regularization of the sine-Gordon model with Lagrangian

$$L_{SG} = \int_0^L dx \left[ \frac{1}{2} (\partial \phi)^2 + \mu^2 \cos \beta_{SG} \phi \right] \quad (2)$$

where the mass \(\mu \propto \frac{1}{\Delta} \exp(-\text{const}\Lambda)\), and \(\beta_{SG}^2 = 8(\pi - \gamma)\) for the antiferromagnetic case
(\(\epsilon = -1\)), while \(\beta_{SG}^2 = 8\gamma\) for the ferromagnetic one (\(\epsilon = 1\)). The scaling limit of the
gapless hamiltonian \[1\] is given by the ultraviolet limit of \(2\).

Free boundary conditions in the lattice model correspond to fixed ones in the sine-
Gordon model. This can easily be seen by recalling that \(\phi\) is dual to the arrows of the vertex
model; since in the approach of \[13,14\] the time direction is across the diagonal of a vertex,
the field at the boundary is necessarily constant. Thus we have \(\phi(0) = \phi(L) = \phi_0\), where
\(\phi_0\) depends on the boundary magnetic field \(h\) (at \(h = 0\) we have \(\phi_0 = 0\)). These boundary
conditions preserve the topological $U(1)$ symmetry giving conservation of soliton number,
but they break the $Z_2$ charge conjugation symmetry relating soliton and antisoliton (unless
\(h = 0\)).

Because we are mainly interested in checking the physical $S$ matrices and paving
the way towards a thermodynamic analysis, we shall discuss the lattice model and the
underlying inverse scattering problem in the presence of boundaries as little as possible.
We refer the reader to \[13,17\] for details. We shall simply use the resulting bare Bethe
ansatz equations without deriving them.
2. The bare Bethe ansatz equations

The wave function of the inhomogeneous six-vertex model can be expressed in terms of a set of “roots” $\alpha_j$, where $j = 1 \ldots n$. They must be solutions of the set of equations

$$
N \left[ f(\alpha_j + \Lambda, \gamma) + f(\alpha_j - \Lambda, \gamma) \right] + 2f(\alpha_j, \gamma H) = 2\pi l_j + \sum_{m=1, m \neq j}^n \left[ f(\alpha_j - \alpha_m, 2\gamma) + f(\alpha_j + \alpha_m, 2\gamma) \right],
$$

where $l_j$ is an integer. The function $f$ is defined as

$$
f(a, b) = -i \ln \left( \frac{\sinh(ib-a)/2}{\sinh(ib+a)/2} \right)
$$

and

$$
H \equiv \frac{1}{\gamma} f(i\gamma, -i \ln(h + \cos \gamma)).
$$

For $h = 0$, $\gamma H = \pi - \gamma$. By construction of the Bethe-ansatz wave function, $\alpha_j > 0$. Even though there is a solution of (3) with one vanishing root for any $N$ and $n$, we emphasize that $\alpha_j = 0$ is not allowed because the wave function vanishes identically in this case.

The solutions of these equations are quite intricate for arbitrary $\gamma$. For simplicity, we restrict to the case $\gamma = \pi/t$ where $t$ is an integer, and consider both choices $\epsilon = \pm 1$. In the sine-Gordon model, these fall in the attractive and repulsive regimes respectively. We make the standard assumption that all the solutions of interest are collections of “$k$-strings” for $k = 1, 2 \ldots t-1$ and antistrings $a$. A $k$-string is a group of $\alpha_j$ in the pattern $\alpha^{(k)} - i\pi (k-1), \alpha^{(k)} - i\pi (k-3), \ldots, \alpha^{(k)} + i\pi (k-1)$ where $\alpha^{(k)}$ is real. The antistring has $\alpha_j = \alpha^{(a)} + i\pi$, where $\alpha^{(a)}$ is real. We shall comment on the validity of this assumption (which is more crucial when one considers subleading effects) in the conclusion.

The thermodynamic limit is obtained by sending $N \to \infty$. In this case, we can define densities of the different kinds of solutions. The number of allowed solutions of (3) of type $k$ in the interval $(\alpha, \alpha + d\alpha)$ is $(\rho_k(\alpha) + \rho^h_k(\alpha)) 2N d\alpha$, where $\rho_k$ is the density of “filled” solutions (those which appear in the sum in the right-hand-side of (3) ) and $\rho^h_k$ is the density of “holes” (unfilled solutions). The densities $\rho_a$ and $\rho^h_a$ are defined likewise for the antistring. The “bare” Bethe ansatz equations follow from taking the derivative of (3).

For $\gamma = \pi/t$ they can be written in the form:

$$
2\pi(\rho_k + \rho^h_k) = \frac{1}{2} [a_k(\alpha + \Lambda) + a_k(\alpha - \Lambda)] - \phi_{k,t-1} * \rho_a + \sum_{l=1}^{t-1} \phi_{kl} * \rho_l + \frac{1}{2N} u_k
$$

$$
2\pi(\rho_a + \rho^h_a) = 2\pi(\rho_{t-1} + \rho^h_{t-1}) + \frac{1}{2N}(u_a - u_{t-1})
$$

(5)
where * denotes convolution:

\[ f * g(\alpha) \equiv \int_{-\infty}^{\infty} d\alpha' f(\alpha - \alpha') g(\alpha'). \]

These densities are originally defined for \( \alpha > 0 \), but the equations allow us to define \( \rho_k(-\alpha) \equiv \rho_k(\alpha) \) in order to rewrite the integrals to go from \(-\infty\) to \(\infty\). The kernels in these equations are defined most easily in terms of their Fourier transforms:

\[ \tilde{\phi}_{kl}(x) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} e^{i\alpha tx/\pi} \phi_{kl}(\alpha) = \delta_{ab} - 2\frac{\cosh x \sinh(t-k)x \sinh lx}{\sinh x \sinh tx}, \tag{6} \]

for \( k \geq l \) with \( \phi_{lk} = \phi_{kl} \), and

\[ \tilde{a}_k = \frac{\sinh(t-k)x}{\sinh tx}, \]

\[ \tilde{u}_k = 2\frac{\sinh(t-H)x \sinh kx}{\sinh x \sinh tx} + \frac{\sinh(t-2k)x/2}{\sinh tx/2} - 1, \]

\[ \tilde{u}_a = 2\frac{\sinh Hx}{\sinh tx} - \frac{\sinh(t-2)x/2}{\sinh tx/2} - 1. \tag{7} \]

The boundary manifests itself in the first term in \( u_k \); notice that even for \( h = 0 \) it still modifies the equations. A few technicalities account for the other terms (these are relevant here because we are interested in subleading boundary effects). The second term in \( u_k \) arises from the fact that the sum in (3) does not include the term \( m = j \); the integration over densities includes such a contribution and so it must be subtracted off by hand. The third term in \( u_k \) arises because \( \rho \) and \( \rho^h \) are defined for allowed solutions, while as already explained, \( \alpha = 0 \) is not allowed because it does not give a valid wavefunction. Since it is a valid solution of (3) but is not included in the densities, we must subtract an explicit \( \frac{2\pi}{2N} \delta(\alpha) \) (corresponding to \( 1/2N \) in Fourier space). One expects all other corrections to be suppressed by additional powers of \( 1/N \).

3. The physical Bethe ansatz equations

We now derive the physical Bethe ansatz equations by extending the computation done in [12] for periodic boundary conditions. The two cases \( \varepsilon = \pm 1 \) are of course very different.

Case i: \( \varepsilon = 1 \)

In this case the ground state is filled with anti-strings so that \( \rho_k = 0 \) and \( \rho^h_a = 0 \). We can use (3) to find the densities in the ground state, but we will not need these explicitly
here. Excited states are given by including string solutions ($\rho_k \neq 0$) and/or holes in the antistring distribution ($\rho_h^A \neq 0$). The physical equations are obtained by rewriting the bare equations (5) so that physical densities (i.e. $\rho_k$ and $\rho_h^A$) appear on the right-hand side. This is done simply in Fourier space by solving for the Fourier transform $\tilde{\rho}_a$ in the last equation in (5) and substituting it into the others. The result is

$$2\pi(\rho_k + \rho_h^A) = \frac{1}{2} [A_k(\alpha + \Lambda) + A_k(\alpha - \Lambda)] + \Phi_{k,t-1} * \rho_h^A + \sum_{l=1}^{t-1} \Phi_{kl} * \rho_l + \frac{1}{2N}U_k$$

$$2\pi(\rho_a + \rho_h^A) = 2\pi(\rho_{t-1} + \rho_{t-1}^h) + \frac{1}{2N}(U_a - U_{t-1})$$

(8)

where the kernels this time are

$$\tilde{\Phi}_{kl} = \delta_{kl} - 2\frac{\cosh x \cosh(t-1-k)x \sinh tx}{\cosh(t-1)x \sinh x} \quad k, l \neq t-1; \quad k \geq l$$

$$\tilde{\Phi}_{t-1,k} = -\frac{\cosh x \sinh kx}{\cosh(t-1)x \sinh x} \quad k \neq t-1$$

$$\tilde{\Phi}_{t-1,t-1} = -\frac{\sinh(t-2)x}{2 \cosh(t-1)x \sinh x}$$

(9)

where $\Phi_{lk} = \Phi_{kl}$, and.

$$\tilde{A}_k = \frac{\cosh(t-1-k)x}{\cosh(t-1)x} \quad k \neq t-1$$

$$\tilde{A}_{t-1} = \frac{1}{2 \cosh(t-1)x}$$

$$\tilde{U}_k = 2\frac{\cosh(t-1-H)x \sinh kx}{\cosh(t-1)x \sinh x} - \frac{\cosh x \sinh kx}{\cosh(t-1)x \sinh x} \left( \frac{\sinh(t-2)x/2}{\sinh tx/2} + 1 \right)$$

$$+ \frac{\sinh(t-2k)x/2}{\sinh tx/2} - 1 \quad k \neq t-1$$

$$\tilde{U}_{t-1} = \frac{\sinh(2t-2-H)x}{\cosh(t-1)x \sinh x} - \frac{\sinh(t-2)x/2 \cosh 1/2 x}{\cosh(t-1)x \sinh x} - \frac{\sinh(t-2)x}{2 \cosh(t-1)x \sinh x} - 1$$

$$\tilde{U}_a = \frac{\sinh Hx}{\cosh(t-1)x \sinh x} - \frac{\sinh(t-2)x/2 \cosh 1/2 x}{\cosh(t-1)x \sinh x} - \frac{\sinh(t-2)x}{2 \cosh(t-1)x \sinh x} - 1.$$  

(10)

(We did not simplify some of the trigonometric sums to allow further identification of the various terms.) These are the physical Bethe ansatz equations, governing how the actual quasiparticle excitations interact with each other. Notice that the kernels are now symmetric, as opposed to (5). Each density corresponds to a quasiparticle; it is easy to read off the bulk $S$ matrix elements. In the next section, we will discuss how to do this and how to find the boundary $S$ matrix.
Case ii: $\epsilon = -1$

In this case the ground state is made of real solutions. The physical densities are therefore $\rho^h_1$, $\rho_k$ and $\rho_a$. Eliminating $\rho_1$ from the right hand side of the bare equations gives

\[
2\pi(\rho_1 + \rho^h_1) = \frac{1}{2}[s(\alpha - \Lambda) + s(\alpha + \Lambda)] + s \cdot a^{(t-1)}_1 \cdot \rho^h_1
\]
\[
- \sum_{k=2}^{t-1} \phi_k^{(t-1)} \cdot \rho_k + a^{(t-1)}_{t-2} \cdot \rho_a + \frac{1}{2N} \rho_0
\]

\[
(11)
\]

\[
2\pi(\rho_k + \rho^h_k) = a^{(t-1)}_{k-1} \cdot \rho^h_1 + \sum_{l=2}^{t-1} \phi^{(t-1)}_{k-1,l-1} \cdot \rho_l - \phi^{(t-1)}_{k-1,t-1} \cdot \rho_a + \frac{1}{2N} u^{(t-1,H-1)}_{k-1}
\]

\[
2\pi(\rho_a + \rho^h_a) = 2\pi(\rho_{t-1} + \rho^h_{t-1}) + \frac{1}{2N} \left[u^{(t-1,H-1)}_a - u^{(t-1,H-1)}_{t-2}\right],
\]

where the quantities $a^{(t-1)}$, $\phi^{(t-1)}$ and $u^{(t-1,H-1)}_{k-1}$ are the same as in the bare equation (3)-(7) with $t, k, H$ there replaced by $t-1, k-1, H-1$ respectively, and

\[
\tilde{u}_0 = \frac{\sinh(t-H)x}{\cosh x \sinh(t-1)x} + \frac{\cosh tx/2}{\cosh x \sinh(t-1)x} + \frac{\sinh(t-2)x}{2 \cosh x \sinh(t-1)x} - 1
\]

\[
\tilde{s} = \frac{1}{2 \cosh x}.
\]

It is more difficult to read off the quasiparticle spectrum and the bulk and boundary $S$ matrices here because the bulk scattering is non-diagonal and most densities correspond in fact to pseudoparticles. We will discuss this in the next section.

4. The $S$ Matrix

In this section we derive the physical Bethe ansatz equations from the bulk and boundary $S$ matrices by quantizing the momenta of a set of relativistic quasiparticles on a line. We identify these with the physical equations derived from the lattice model in the cases $\epsilon = \pm 1$.

In a relativistic quantum field theory the energy and momentum of a particle of mass $m$ can be parametrized in terms of the rapidity $\theta$: $(E, P) = (m \cosh \theta, m \sinh \theta)$. Lorentz invariance requires that $S$ matrix elements depend only on rapidity differences.

First let us discuss the case when the $S$ matrix is diagonal. When there are $p$ different species of particle, we have the two-particle $S$-matrix elements $S_{bc}(\theta_1 - \theta_2)$, where $b$ and $c$ run from 1 to $p$. These $S$-matrix elements give the phase shift in the wavefunction...
when two particles are exchanged. We also have the boundary $S$ matrix elements $R_b(\theta)$, which gives the phase shift when a particle of species $b$ bounces off a wall and changes its rapidity from $\theta$ to $-\theta$. (We do not consider the case where a particle changes species when bouncing off the boundary.) We have a gas of $N$ particles on a line of length $L$, with the $i$th particle of species $b_i$. For the fixed boundary conditions we consider, we demand that the wavefunction vanish at both ends of the line. This requires making a stationary wave of states with opposite momenta, with the momenta subject to the constraints

$$e^{iLm_{b_i}\sinh \theta_i} \prod_{j=1, j \neq i}^N S_{b_ib_j}(\theta_i - \theta_j)R_{b_i}(\theta_i)$$

$$= e^{-iLm_{b_i}\sinh \theta_i} \prod_{j=1, j \neq i}^N S_{b_ib_j}(-\theta_i - \theta_j)R_{b_i}(-\theta_i)$$

for all $i$. When the scattering is trivial, this reduces to the familiar $P = n\pi/L$, where $n$ is a positive integer. Unitarity requires that $S^{-1}(\theta) = S(-\theta)$ and $R^{-1}(\theta) = R(-\theta)$, so this can be rewritten as

$$e^{2im_{b_i}\sinh \theta_i L} \prod_{j=1, j \neq i}^N S_{b_ib_j}(\theta_i - \theta_j)S_{b_ib_j}(\theta_i + \theta_j)R_{b_i}^2(\theta_i) = 1.$$  

Alternatively we can deduce these equations from systematic application of the Zamolodchikov-Faddeev algebra together with its boundary counterpart.

As in the lattice model, we define densities of solutions of (13) so that $(\rho_b(\theta) + \rho_b^h(\theta))2Ld\theta$ gives the number of allowed rapidities between $\theta$ and $\theta + d\theta$ for species $b$, and $\rho_b$ gives the density of filled states. Taking the logarithm of (13) gives an equation of the form (3); taking the derivative of this gives an equation for the densities. We again define the densities at negative rapidity by $\rho_b(-\theta) = \rho_b(\theta)$. We have one equation for every type of particle:

$$2\pi(\rho_b + \rho_b^h) = m_b \cosh \theta + \sum_{c=1}^p \varphi_{bc} \ast \rho_c + \frac{1}{2L} \Theta_b,$$

where

$$\varphi_{bc}(\theta) = -i \frac{d}{d\theta} \ln S_{bc}(\theta)$$

$$\Theta_b(\theta) = -i \frac{d}{d\theta} 2 \ln R_b(\theta) + i \frac{d}{d\theta} \ln S_{bb}(2\theta) - 2\pi \delta(\theta)$$

$$= 1.$$  

$$e^{2im_{b_i}\sinh \theta_i L} \prod_{j=1, j \neq i}^N S_{b_ib_j}(\theta_i - \theta_j)S_{b_ib_j}(\theta_i + \theta_j)R_{b_i}^2(\theta_i) = 1.$$  

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$$2\pi(\rho_b + \rho_b^h) = m_b \cosh \theta + \sum_{c=1}^p \varphi_{bc} \ast \rho_c + \frac{1}{2L} \Theta_b,$$

where

$$\varphi_{bc}(\theta) = -i \frac{d}{d\theta} \ln S_{bc}(\theta)$$

$$\Theta_b(\theta) = -i \frac{d}{d\theta} 2 \ln R_b(\theta) + i \frac{d}{d\theta} \ln S_{bb}(2\theta) - 2\pi \delta(\theta)$$

$$= 1.$$  

Alternatively we can deduce these equations from systematic application of the Zamolodchikov-Faddeev algebra together with its boundary counterpart.
The additional terms in $\Theta_b$ result from the same considerations as in the lattice model: the facts that the product in (13) has no term $i = j$ and that the solution with $\theta = 0$ should not be included. As before, we expect additional corrections to be suppressed by powers of $1/L$.

The equations (14) are the physical Bethe ansatz equations derived directly from the $S$ matrix description. They enable us to compare a diagonal $S$ matrix to lattice results. We do this in case $i$ below. If the scattering is not diagonal, the analysis is more complicated: one must “diagonalize” the multiparticle state, including the boundary $S$ matrices. Basically, one must find the eigenvalues $\Lambda(\theta_i|\{\theta_j\})$ of the object

$$T(\theta_i|\{\theta_j\})_{bc}R_c(\theta)T^{-1}(-\theta_i|\{\theta_j\})_{cb}R_b^{-1}(-\theta_i)$$

where $T(\theta_i|\{\theta_j\})_{bc}$ is the matrix which describes scattering a particle of species $b$ and rapidity $\theta_i$ through all the other particles and ending up with that particle being of species $c$. The generalization of (13) then is

$$e^{2im_b\sinh\theta_i\Lambda(\theta_i|\{\theta_j\})} = 1.$$  

To diagonalize (16), one uses the Bethe ansatz and the analysis proceeds as before, but the calculation is a little delicate. It is however simplified by using results of [15]; we discuss this in case $ii$ below.

**Case i: $\beta^2 = 8\pi/t$**

As discussed in the introduction, the scaling limit of the inhomogeneous XXZ model with $\epsilon = 1$ is the sine-Gordon model at $\beta^2 = 8\pi/t$. The particles and their bulk $S$ matrix in this case are well known [13]. There are $t-2$ breather states with mass $m_a = 2M\sinh\frac{\pi a}{2(t-1)}$, along with a soliton and an antisoliton of mass $M$. All of these states scatter diagonally with each other. The boundary $S$ matrices for the soliton and antisoliton are conjectured in [4], and those for the breathers in [6].

Using these $S$ matrix elements, we can now compare the physical Bethe ansatz equations (14) with the lattice ones (8). First we have to extract the scaling limit of the lattice model. This is easily done by taking the limit $\Lambda \to \infty$ at finite bare rapidity $\alpha$. The source term in equations (8) reproduces the expected spectrum with

$$M \propto \frac{1}{\Delta} e^{-t\Lambda/2(t-1)}$$
and the physical rapidity

\[ \theta = \frac{t}{2(t-1)} \alpha, \]

provided we identify the breathers with the string solutions \( k = 1 \ldots t-2 \), the soliton with the string solution \( t-1 \), and the antisoliton with the antistring \( a \).

It is already known [12] that the kernels \( \Phi_{jk} \) in (9) are the same as the \( \varphi_{bc} \) that result from the known \( S_{bc} \) scattering matrix. It is straightforward but rather tedious to check that the kernels \( U_k \) in (10) too are the same as \( \Theta_b \) resulting from (15) and the \( S_{bc}(\theta) \) and \( R_b(\theta) \) of [19,4,6], completing the boundary identification. Let us discuss the case of soliton and antisoliton only for simplicity, denoting them + and − respectively. Using the integral representation for the log of a gamma function, the result of [4] can be written

\[ -i \ln R_{\pm}(\theta) = \int \frac{dx}{x} \sin \frac{2}{\pi} (t-1)x \theta \]

\[ \times \left[ \frac{\sinh(t-1 \pm 2\xi/\pi)x}{2 \cosh(t-1)x \sinh x} - \frac{\sinh \frac{t}{2}(t-1)x \sinh \frac{t}{2}-1)x}{\sinh x/2 \sinh 2(t-1)x} \right], \]

where \( \xi \) depends on \( \phi_0 \). Comparing this with (10) allows us to identify

\[ \xi = \frac{\pi}{2} (t-1-H). \]

To identify the remaining terms we recall that

\[ -i \ln S_{++}(\theta) = - \int \frac{dx}{x} \frac{\sinh(t-2)x}{2 \sinh x \cosh(t-1)x} \sin \frac{2}{\pi} (t-1)x \theta. \]

The identity

\[ \frac{\sinh 3(t-1)x/2 \sinh(t-2)x/2}{\sinh x/2 \sinh 2(t-1)x} = \frac{\sinh(t-2)x/2 \cosh tx/2}{2 \cosh(t-1)x \sinh x} = \frac{\sinh(t-2)x/2}{4 \sinh x/2 \cosh(t-1)x/2} + \frac{\sinh(t-2)x}{4 \sinh x \cosh(t-1)x} \]

completes the desired identification.

This confirms the \( S \) matrices conjectured in [4,6] at these values of \( \beta \). We emphasize that it is absolutely crucial to have included the correction terms to \( u_k \) in (7) and \( \Theta_b \) in (15) to get this result.

Case ii: \( \beta^2 = 8\pi(t-1)/t \)

The continuum limit of the lattice model at \( \gamma = \pi/t \) is the sine-Gordon model at \( \beta^2 = 8\pi(t-1)/t \). The only states in the spectrum are the soliton and antisoliton, and they
scatter with a nondiagonal $S$ matrix [19]. With our fixed boundary conditions, soliton number is conserved, so the scattering off the wall is diagonal but only if $h = 0$ ($H = t - 1$) is it equal for soliton and antisoliton [4].

To derive the Bethe ansatz equations for this nondiagonal $S$ matrix, we need the eigenvalues of the matrix (16). We now discuss some technical aspects of this computation; the uninterested reader can jump directly to the next paragraph. The equivalent problem for periodic boundary conditions, where one diagonalizes $T_{bb}$, is well understood. In this case, a simplifying feature is that $S(0)$ is simply the permutation operator. Hence, instead of $T_{bb}$ one can consider the matrix describing the scattering of one particle through all the others including itself. The unwanted contribution of a particle scattering with itself is trivial anyway and does not have to be discarded. The advantage is that the new diagonalization problem fits perfectly in the inverse scattering framework and is easy to solve. In the presence of a boundary the situation is different. If one tries the same approach there are now two unwanted contributions, since the particle at rapidity $\theta$ can “scatter with itself” at rapidity $\pm \theta$. The process with $S(\theta - \theta) = S(2\theta)$ leads to non-trivial terms. When the scattering was diagonal this was easy to discard by hand, but the situation is more complicated here. On the other hand, the inverse scattering formalism in the presence of boundaries developed e.g. in [15,17] is treating a slightly different problem: $R_{b}^{-1}(-\theta)$ is replaced there with $R_{b}^{-1}(-\theta - i\pi)$. Fortunately these two complications essentially cancel each other; changing the rapidity $-\theta$ to $-\theta - i\pi$ acts as a projector which cancels out the unwanted processes resulting from the self-scattering term. The final expression for the eigenvalue $\Lambda(\theta_{i}|\{\theta_{j}\})$ turns out to be the same as that of [15], up to some missing prefactors that produce a phase, as required. The complete proof is rather tedious and will be provided upon request. The interested reader can also check the final result by direct calculation in the two-particle case.

The result is that the auxiliary problem determining the eigenvalue $\Lambda$ is similar to the bare problem, up to a redefinition of some parameters. Let us define $\sigma(\theta)$, which is the density of particles (soliton or antisoliton). In this case we have

$$-i \ln R_{\pm}(\theta) = \int \frac{dx}{x} \left[ \frac{\sinh(1 \pm 2\xi(t - 1)/\pi)x}{2 \cosh x \sinh(t - 1)x} + \frac{\sinh^{3/2}x \sinh(t - 2)x/2 \sinh 2x}{\sinh(t - 1)x/2 \sinh 2x} \right] \sin \frac{2}{\pi} x \theta$$

together with

$$-i \ln S_{++}(\theta) = -\int \frac{dx}{x} \frac{\sinh(t - 2)x}{2 \cosh x \sinh(t - 1)x} \sin \frac{2}{\pi} x \theta.$$
The first part of the quantization equation (the derivative of the log of (13)) reads

\[ 2\pi (\sigma + \sigma^h) = m \cosh \theta - \frac{i}{2L} \frac{d}{d\theta} \ln \Lambda, \]

with

\[- \frac{i}{2L} \frac{d}{d\theta} \ln \Lambda = \left( -\frac{d}{d\theta} \ln S_{++}(\theta) \right) \ast \sigma - \sum_{b=1}^{t-2} a_{b}^{(t-1)} \ast \sigma_b + a_{t-2}^{(t-1)} \ast \sigma_a + \frac{1}{2L} U, \]

and

\[ U = -i \frac{d}{d\theta} 2 \ln R_+(\theta) + i \frac{d}{d\theta} \ln S_{++}(2\theta) - 2\pi \delta(\theta). \]

For the other densities \( \sigma_b \) and \( \sigma_b^h \) \((b = 1 \ldots t-2)\), we have the bare equations (5) with three changes. First, \( t \) is replaced by \( t-1 \) because there are \( t-1 \) solutions to the new Bethe equations. There is also a new source term \( a_{b}^{(t-1)} \ast \sigma \) because the "vertices" in the problem of diagonalizing (16) correspond to the real particles, and have fluctuating numbers and rapidities. These two changes are identical to the case with periodic boundary conditions. In addition, we have a boundary term which from [15] is the same as the one in the bare equations with \( H \) replaced by \( H - 1 \). This shift is of course related to the shift of \( t \). For instance, if in the original bare equations \( h = 0 \) so that \( H = t - 1 \), then in the auxiliary problem we again expect soliton-antisoliton symmetry yielding \( H = (t - 1) - 1 \).

We can thus identify (19) and the equations for \( \sigma_b \) with (11) provided we set

\[ \xi = \frac{\pi}{2} \left( 1 - \frac{H}{t-1} \right) \]

by using again the identity (18) with \( x \) replaced by \((t-1)x\) and \( t-1 \) by \( 1/(t-1) \). This confirms results of [4,6] at values of \( \beta \) where the scattering is non-diagonal.

5. Conclusion

We have verified in both attractive and repulsive regimes that the boundary \( S \) matrices conjectured in [4,6] are in fact correct. We think that the main interest of our computation lies in the delicate analysis of all the boundary terms, which are crucial to make the computation consistent. In particular, observe from the example of \( \alpha = 0 \) that at order \( 1/N \), the modification of even a single root in the Bethe equations affects the results. This puts the string hypothesis to a much stronger test than the bulk computations done so far. If we believe that the picture of [4] is consistent and that the inhomogeneous XXZ model provides also a regularization of the sine-Gordon model in the presence of boundaries, we
are forced to conclude that strings are the **only** solutions of the Bethe equations that contribute to the scaling limit, i.e. to the low-energy excitations. This does not contradict the known examples of non-string solutions, which either exist in finite numbers \[10, 20\] (so we show that the finite number is zero in the scaling limit) or are at very high energy \[21\].

The next step is to study the system at finite temperature and to derive the thermodynamic Bethe ansatz equations for the free energy. This should yield the “$g$-function”, which measures the number of degrees of freedom on the boundary \[22\]. At a fixed point, this can be calculated from conformal field theory \[23, 22\], and was calculated for the free boson (the critical limit of the sine-Gordon model) in \[24\]. It is simple to get the bulk part of the free energy from the physical Bethe ansatz equations, but additional subtleties appear because the $g$-function is a subleading term. For example, the usual thermodynamic Bethe ansatz free energy is the log of the partition function only at leading order in $L$; there are system-dependent corrections at lower order. However, when one looks at massless $S$ matrices to describe a flow between ultraviolet and infrared fixed points, the ratio $g_{\text{UV}}/g_{\text{IR}}$ should be independent of these corrections. It is indeed simple to calculate this ratio from the $S$ matrix, for example, in the Kondo problem \[2, 25\] or in the flows between the minimal models. We hope to discuss these more physical aspects in the near future.

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References

[1] For a review, see I. Affleck, “Conformal Field Theory Approach to Quantum Impurity Problems”, UBCTP-93-25, cond-mat/9311054.
[2] N. Andrei, K. Furuya, and J. Lowenstein, Rev. Mod. Phys. 55 (1983) 331; A.M. Tsvelick and P.B. Wiegmann, Adv. Phys. 32 (1983) 453.
[3] I. Cherednik, Theor. Math. Phys. 61 (1984) 977.
[4] S. Ghoshal and A.B. Zamolodchikov, “Boundary State and Boundary S Matrix in Two-Dimensional Integrable Field Theory”, RU-93-20, hep-th/9306002.
[5] A. Fring and R. Köberle, “Factorized Scattering in the Presence of Reflecting Boundaries”, USP-IFQSC/TH/93-06, hep-th/9304141.
[6] S. Ghoshal, “Bound State Boundary S Matrix of the Sine-Gordon Model”, RU-93-51, hep-th/9310188.
[7] A. Fring and R. Köberle, “Affine Toda Field Theory in the Presence of Reflecting Boundaries”, USP-IFQSC/TH/93-12, hep-th/9309142.
[8] R. Sasaki, “Reflection Bootstrap Equations for Toda Field Theory”, hep-th/9311027.
[9] S. Ghoshal, “Boundary S Matrix of the O(n) Symmetric Nonlinear Sigma Model” RU-94-02, hep-th/9401005.
[10] C. Destri and J.H. Lowenstein, Nucl. Phys. B205 (1982) 369.
[11] G.I. Dzapardize, A.A. Nersesyan and P.B. Wiegmann, Phys. Scr. 27 (1983) 5.
[12] A.N. Kirillov and N. Reshetikhin, J. Phys. A20 (1987) 1565, 1587.
[13] C. Destri and H. de Vega, J. Phys. A22 (1989) 1329.
[14] N. Reshetikhin and H. Saleur, “Lattice Regularization of Massive and Massless Field Theories”, USC-93-020, hep-th/9309135.
[15] E.K. Sklyanin, J. Phys. A21 (1988) 2375.
[16] F. Alcaraz, M. Barber, M. Batchelor, R. Baxter and G. Quispel, J. Phys. A20 (1987) 6397.
[17] L. Mezincescu and R.I. Nepomechie, Int. J. Mod. Phys. A6 (1991) 5231; Int. J. Mod. Phys. A7 (1992) 565; H.J. de Vega and A. Gonzalez-Ruiz, preprint LPTHE-93/38.
[18] M. Takahashi and M. Suzuki, Prog. Th. Phys. 48 (1972) 2187.
[19] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1980) 253.
[20] F. Woynarovich, J. Phys. A15 (1982) 2985; O. Babelon, H.J. de Vega and C.M. Viallet, Nucl. Phys. B220 (1983) 13.
[21] F. Essler, V. Korepin and K. Schoutens, J. Phys A25 (1992) 4115.
[22] I. Affleck and A. Ludwig, Phys. Rev. Lett. 67 (1991) 161.
[23] J. Cardy, Nucl. Phys. B324 (1989) 581.
[24] S. Eggert and I. Affleck, Phys. Rev. B46 (1992) 10866.
[25] P. Fendley, Phys. Rev. Lett. 71 (1993) 2485, cond-mat/9304031.