Convergence of Batch Asynchronous Stochastic Approximation
With Applications to Reinforcement Learning

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Abstract

The stochastic approximation (SA) algorithm is a widely used probabilistic method for finding a zero or a fixed point of a vector-valued function, when only noisy measurements of the function are available. In the literature to date, one makes a distinction between “synchronous” updating, whereby every component of the current guess is updated at each time, and “asynchronous” updating, whereby only one component is updated. In this paper, we study an intermediate situation that we call “batch asynchronous stochastic approximation” (BASA), in which, at each time instant, some but not all components of the current estimated solution are updated. BASA allows the user to trade off memory requirements against time complexity. We develop a general methodology for proving that such algorithms converge to the fixed point of the map under study. These convergence proofs make use of weaker hypotheses than existing results. Specifically, existing convergence proofs require that the measurement noise is a zero-mean i.i.d sequence or a martingale difference sequence. In the present paper, we permit biased measurements, that is, measurement noises that have nonzero conditional mean. Also, all convergence results to date assume that the stochastic step sizes satisfy a probabilistic analog of the well-known Robbins-Monro conditions. We replace this assumption by a purely deterministic condition on the irreducibility of the underlying Markov processes.

As specific applications to Reinforcement Learning, we analyze the temporal difference algorithm $TD(\lambda)$ for value iteration, and the $Q$-learning algorithm for finding the optimal action-value function. In both cases, we establish the convergence of these algorithms, under milder conditions than in the existing literature.

1 Introduction and Literature Review

1.1 Literature Review

The stochastic approximation (SA) algorithm is a widely used probabilistic method for finding a solution to an equation of the form $f(\theta) = 0$ where $f : \mathbb{R}^d \to \mathbb{R}^d$, when one has access only to a noisy measurements of the form

$$y_{t+1} = f(\theta_t) + \xi_{t+1},$$

where $\theta_t$ is the current guess and $\xi_{t+1}$ is the measurement noise. In the “standard” SA algorithm, the guess $\theta_t$ is updated according to

$$\theta_{t+1} = \theta_t + \alpha_t y_{t+1} = \theta_t + \alpha_t (f(\theta_t) + \xi_{t+1}),$$

(1)

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where $\alpha_t$ is the step size at time $t$. In the Reinforcement Learning (RL) literature to date, it is more common to study the fixed point formulation of SA, wherein the objective is to find a fixed point of a map $g : \mathbb{R}^d \to \mathbb{R}^d$. This is equivalent to finding a zero of the map $f(\theta) : \theta \mapsto g(\theta) - \theta$. In this case, the updating rule (1) gets modified to

$$
\theta_{t+1} = (1 - \alpha_t) \theta_t + \alpha_t (g(\theta_t) + \xi_{t+1}).
$$

(2)

In this paper, we will study more general formulations of the updating rule (2), where $g(\cdot)$ is replaced by a more general measurement map. The aim remains finding a fixed point of $g(\cdot)$.

The SA method was first introduced in [13] for the case where $d = 1$, with fairly stringent restrictions on the function $f(\cdot)$ and the noise sequence $\{\xi_t\}$. Over the years, the assumptions on both $f(\cdot)$ and on $\{\xi_t\}$ have been relaxed. However, the widely-used assumptions on the step sizes were introduced in [13], and are thus rightly known as the “Robbins-Monro conditions.” Subsequent research has extended the approach to the multi-dimensional case ($d > 1$); see [2] for one of the first such extensions. An excellent survey of the traditional SA algorithm can be found in [9], and a more recent review that encompasses many more variations than [9] can be found in [6].

Since its introduction, the SA algorithm has found applications in many areas, including Reinforcement Learning. One widely used RL algorithm is the so-called $Q$-learning algorithm, introduced in [20], which is used to find the optimal action-value function in Markov Decision Problems (MDPs). While the $Q$-learning algorithm is somewhat reminiscent of the standard SA algorithm, it differs in one important respect. In the current notation, at each instant of time $t$, only one component of the current guess $\theta_t$ is updated, while the rest remain the same. The convergence proofs for the “standard” SA algorithm did not apply to $Q$-learning, so the authors of [20] created a “one-off” method to prove its convergence. In [17], the phrase “asynchronous stochastic approximation” (ASA) was introduced to describe the situation where only one component of $\theta_t$ is updated at any one time $t$. The paper also contains a convergence proof for ASA applied to a class of problems, which includes but is not limited to $Q$-learning. We refer to the updating rule (2) as “synchronous SA” to distinguish it from ASA. Another contemporary paper [8] also provided a convergence proof for $Q$-learning; it also introduced and provided a convergence proof for another algorithm for value iteration known as “Temporal Difference” or $TD(\lambda)$-algorithm; see [8, Eq. (4.7)]. Like $Q$-learning, the version of the $TD(\lambda)$-algorithm introduced in [8] also updates only one component of $\theta_t$ at each time instant.\footnote{These ideas are discussed in detail in Section 3}

When $\theta_t$ is a vector of dimension $d \geq 2$, it is possible to have an intermediate situation where, at each time $t$, some but not all components of $\theta_t$ are updated. We refer to this as batch asynchronous stochastic approximation (BASA). That is the topic of the present paper. The main justification for studying BASA as an alternative to both SA and ASA is this: When the dimension $d$ is large, in SA the memory requirements at each time step $t$ can be very high, due to the need to update all $d$ components at each $t$. At the other extreme, in ASA, the memory requirements are negligible, but the penalty one pays is that the number of iterations becomes very large. To illustrate, if we wish that, on average, each component of $\theta_t$ is updated $N$ times, then $dN$ iterations of ASA would be required. In BASA, it is possible to trade off memory against time. In the above example, if we choose to update, on average $\sqrt{dN}$ components of $\theta_t$ at each time
t, then both the memory requirements as well as the number of iterations would be reasonable. Computational implementations of BASA for function minimization, which illustrate this trade-off, will be published separately. In the present paper, the focus is on a theoretical analysis of BASA.

In the literature to date, the convergence of the asynchronously updated iterates \( \{\theta_t\} \) to a fixed point of the map \( g \) is established under various conditions. In [17], the author assumes that the function \( g \) either satisfies a monotonicity condition, or else is a global contraction with respect to a weighted \( \ell_\infty \)-norm. In that paper, it is also permitted to have delays in the measurements used to do the updating. Note that, aside from the challenge of asynchronous versus synchronous updating, the fact that \( g \) is a contraction with respect to the \( \ell_\infty \)-norm is also a challenge. A classical result in [7] applies to the case where \( g \) is a contraction in the \( \ell_2 \)-norm, but does not apply to the case studied in [17].

In the multidimensional case, a distinction can be made between using a “local” clock versus using a “global” clock to determine the step size at each update. Suppose \( \{\beta_t\} \) is a predetermined (and thus deterministic) sequence of step sizes. If one chooses the step size \( \alpha_t \) at time \( t \) to equal \( \beta_t \), then this would be called using a global clock. In contrast, if the step size \( \alpha_t \) at time \( t \) equals \( \beta_{\nu(i)} \), where the \( i \)-th coordinate of \( \theta_t \) is to be updated at time \( t \), and \( \nu(i) \) is the total number of times that the \( i \)-th coordinate of \( \theta_t \) is updated (including the current occasion), then this would be called a local clock. It appears that this distinction was first introduced in [3]. The updating rule in [17] uses a global clock, as do TD(\( \lambda \)) and Q-learning. This is not surprising because the distinction between local and global clocks was introduced only after these papers were published. One of the contributions of the present paper is to show that, if local clocks are used instead of global clocks, the convergence of both TD(\( \lambda \)) and Q-learning can be established under fewer technical assumptions than with global clocks.

In [3], the asynchronous updating rule uses a local clock, and convergence is established even when the map \( g \) is nonexpansive in the \( \ell_\infty \)-norm; in other words, \( g \) need not be a contraction in the \( \ell_\infty \)-norm. Moreover, it is possible to update multiple components at a time, i.e., batch updating. However, batch updating is treated as a sequence of single-component updatings, which necessarily forces the use of “delayed” measurements even when there are no delays to begin with, thus essentially limiting the theory to the use of local clocks alone. It would be better to set up the theory in such a manner that it can handle batch updating naturally. In order to prove convergence, it is assumed that the measurement error forms a zero-mean i.i.d. sequence; the same assumption is also made in [3]. In contrast, most papers on stochastic approximation require the noise only to be a martingale difference sequence, and not necessarily zero-mean i.i.d. Certainly, the measurement errors are not i.i.d. in temporal difference methods [15] for RL. The i.i.d. assumption does not hold also in Q-learning and in the TD(\( \lambda \)) algorithm. Moreover, in [3], it is assumed that there is a Lyapunov function. In [8], the emphasis throughout is on proving the convergence of the TD(\( \lambda \)) algorithm and the Q-learning algorithm. In contrast, while the motivation in [17] is also to prove the convergence of the Q-learning algorithm, the theory itself is stated in a more general form. In [19], the convergence of the TD(\( \lambda \)) algorithm is proved when function approximation is used to reduce the dimension of the search space. This is but a brief literature survey.

Note that in the RL literature, the adjective “batch” is sometimes used in a sense entirely different from our usage. In particular, it can refer to dividing the time horizon of the learning problem into several shorter blocks of time and then reinitializing the problem at the start of each block. See for instance [19] Section 5.4.5. In this sense, the celebrated AlphaZero algorithm [14] is also a “batch algorithm.” However, that is an entirely different usage of the term, and unrelated to the present usage.

A version of BASA, known as Batch Stochastic Gradient Descent, is widely used in large-scale
optimization problems, both for convex optimization as well as in deep learning where the objective functions are definitely nonconvex. In the language of the present paper, the problem is one of finding a solution to an equation of the form $J'(\theta) = 0$, where $J(\cdot)$ is the objective function. While this problem is similar to the fixed point problem studied here, there are also some important differences. In particular, when BASA is used to determine a fixed point of a map (as in the present paper), it is reasonable to assume that there is a unique fixed point. In contrast, when the objective is to find stationary points of a function $J(\cdot)$, it would be preferable to develop a theory for the case where $J(\cdot)$ has multiple stationary points. This necessitates the use of different techniques. Our results on Batch Stochastic Gradient Descent will be reported elsewhere.

1.2 Contributions of the Paper

In this paper, we present a unified theory for batch asynchronous stochastic approximation (BASA) that contains the above (and other) results as special cases. In particular, we permit batch updating of the current guess, and not just the updating of a single component at a time. In contrast with [3, 4] where the measurement noise sequence is assumed to be a zero-mean i.i.d. sequence, and [17, 8] where the measurement noise sequence is assumed to be a martingale difference sequence, in the present paper we permit the noise sequence to have nonzero conditional mean, i.e., the measurements to be biased. It should be noted that in almost existing theory of SA, the measurement noise is assumed to be a martingale difference sequence. Consequently, the conditional mean of the measurement noise at each time instant is zero. Therefore, the convergence theorem proved here is of independent interest, and can be used to prove the convergence of standard SA iterations with biased measurements. The details of the new approach are given in Section 2.

Then we turn to applications to RL, presented in Section 3. We begin with a technical result on ergodic Markov processes. In the literature to date, the convergence proofs of the $TD(\lambda)$ and the $Q$-learning algorithms are based on stochastic analogs of the Robbins-Monro conditions (3). The technical result allows us to replace these stochastic analogs with purely deterministic conditions on the irreducibility of some state transition matrices, which can be verified algebraically. Next, we study the $TD(\lambda)$-algorithm for value iteration, as enunciated in [8]. This is an ASA (not BASA) algorithm. We study the case where the underlying Markov process is irreducible, and study the use of both global clocks and local clocks. We show that the $TD(\lambda)$ algorithm converges whenever a local clock is used and the conventional deterministic Robbins-Monro conditions hold. The $TD(\lambda)$ algorithm also converges when a global clock is used, provided the step size sequence $\{\beta_t\}$ is nonincreasing. Then we introduce a “batch” version of the $Q$-learning algorithm [20]. Then we show that the batch $Q$-learning algorithm converges when the Markov process corresponding to each fixed action is irreducible. This is an enormous improvement over existing results, which require that the Markov process corresponding to each fixed policy is irreducible. The results of [18] suggest that verifying this assumption could be NP-hard, whereas verifying that the Markov process corresponding to each fixed action is irreducible is very easy. Moreover, as with value iteration, we do away with the stochastic analogs of the Robbins-Monro conditions. Taken together, our results lead to a theory that is very clean and free from unverifiable technical assumptions.

\footnote{In [3] this is handled by a series of one-component updates. This works only when a local clock is used, but not when a global clock is used.}
2 Batch Asynchronous Stochastic Approximation

In this section, we set up the Batch Asynchronous Stochastic Approximation (BASA) algorithm, and establish sufficient conditions for both the almost-sure boundedness and almost-sure convergence of the BASA iterations. Applications to BASA to problems in Reinforcement Learning such as $TD(\lambda)$ and Q-learning are addressed in Section 3.

2.1 Problem Set-Up

Throughout we consider vectors $\theta_t \in \mathbb{R}^d$ where $d$ is fixed. We use $\theta_{t,i}$ to denote the $i$-th component of $\theta$, which belongs to $\mathbb{R}$. The symbol $\|\theta\|_\infty$ denotes the $\ell_\infty$-norm of $\theta$. For $s \leq t$, $\theta_s^t$ denotes $(\theta_s, \cdots, \theta_t)$. In particular, $\theta_0^\infty$ is the sequence $\{\theta_0, \theta_1, \cdots\}$. Note that, if $t$ is finite, then

$$\|\theta_s^t\|_\infty = \max_{s \leq \tau \leq t} \|\theta_{\tau}\|_\infty.$$  

We use $\mathbb{N}$ for the set of natural numbers plus zero, so that $\mathbb{N} = \{0, 1, \cdots\}$. If $\mathcal{F}_t$ is a filtration with $t \in \mathbb{N}$ (that is, an increasing sequence of $\sigma$-algebras), then $\mathcal{M}(\mathcal{F}_t)$ denotes the set of functions that are measurable with respect to $\mathcal{F}_t$. The symbol $[d]$ denotes the set $\{1, \cdots, d\}$.

We begin with a deterministic “step size sequence” $\{\beta_t\}$ where $\beta_t \in (0, 1)$ for all $t$. To capture “asynchronous” updating, we introduce $d$ different $\{0, 1\}$-valued stochastic processes $\{\kappa_{t,i}\}$ for $t \geq 0$ and $i \in [d]$, known as the “update” processes. No assumptions are made regarding the relationship between $\{\kappa_{t,i}\}$ and $\{\kappa_{t,j}\}$ for $i \neq j$. The significance of these processes is that at time $t$, the $i$-th component of $\theta_t$ is updated if and only if $\kappa_{t,i} = 1$. For each $t$, define

$$S(t) = \{i \in [d] : \kappa_{t,i} = 1\}. \tag{4}$$

Thus $S(t)$ denotes the set of components of $\theta_t$ that are updated at time $t$. Let us define the vector $e_{S(t)} \in \{0, 1\}^n$ via

$$(e_{S(t)})_i = \begin{cases} 1, & \text{if } i \in S(t), \\ 0, & \text{if } i \notin S(t). \end{cases} \tag{5}$$

Note that $e_{S(t)} = \sum_{i \in S(t)} e_i$, where $e_i$ is the $i$-th elementary unit vector.

The updating of $\theta_t$ depends on a measurement function $h : \mathbb{N} \times (\mathbb{R}^d)^\mathbb{N} \rightarrow (\mathbb{R}^d)^\mathbb{N}$, which is assumed to be nonanticipative. Thus $h$ satisfies

$$\theta_0^\infty, \phi_0^\infty \in (\mathbb{R}^d)^\mathbb{N}, \theta_0^t = \phi_0^t \implies h(\tau, \theta_0^\infty) = h(\tau, \phi_0^\infty), 0 \leq \tau \leq t. \tag{6}$$

Define

$$\eta_t = h(t, \theta_0^t). \tag{7}$$

Note that the function $\eta_t$ has access to the full measurement vector $\theta_0^t$. In particular, this formulation incorporates the possibility of “delayed information” of the form

$$\eta_{t,i} = g_i(\theta_1(t - \Delta_1(t)), \cdots, \theta_d(t - \Delta_d(t))), \tag{8}$$

where $\Delta_1(t), \cdots, \Delta_d(t)$ are the time-dependent delays. This formulation is analogous to [17, Eq. (2)] and [3, Eq. (1.4)]. Moreover, in the present formulation, the updating function $\eta_t$ is allowed to depend on the entire past history $\theta_0^t$ of the iterations, which makes it more general than both [17, 3].

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3In control and system theory, such a function is also referred to as “causal.”
Using the measurement function, at time \( t \), \( \theta_t \) is updated to \( \theta_{t+1} \) according to
\[
\theta_{t+1} = \theta_t + \alpha_t \circ [\eta_t + \xi_{t+1}],
\]
where \( \alpha_t \) is the \textit{vector of step sizes} belonging to \([0,1)^d\), \( \xi_{t+1} \) is the measurement noise vector belonging to \( \mathbb{R}^d \), and \( \circ \) denotes the Hadamard, or componentwise, product of two vectors of equal size. The use of a step size \textit{vector} instead of a scalar step size is characteristic of BASA. More details about \( \alpha_t \) are now given. Recall the definition of the updating index set \( S(t) \) in (4). Thus
\[
\alpha_{t,i} = 0, \quad \forall i \not\in S(t).
\]
This means that
\[
\theta_{t+1,i} = \theta_{t,i}, \quad \forall i \not\in S(t).
\]
To define \( \alpha_{t,i} \) for \( i \in S(t) \), we distinguish between two cases, namely a global clock versus a local clock. In each case, there is a \textit{deterministic} sequence \( \{\beta_t\} \) where \( \beta_t \in (0,1) \) for all \( t \). If a global clock is used, then
\[
\alpha_{t,i} = \beta_t, \quad \forall i \in S(t).
\]
If a local clock is used, then we first define the local counter
\[
\nu_{t,i} = \sum_{\tau=0}^{t} \kappa_{\tau,i}, i \in [d],
\]
which is the total number of occasions when \( i \in S(\tau), 0 \leq \tau \leq t \). Equivalently, \( \nu_{t,i} \) is the total number of times up to and including time \( t \) when \( \theta_{\tau,i} \) is updated. With this convention, we define
\[
\alpha_{t,i} = \beta_{\nu_{t,i}}, \quad \forall i \in S(t).
\]
The distinction between global clocks and local clocks was apparently introduced in [3]. Traditional RL algorithms such as \( TD(\lambda) \) and \( Q \)-learning (discussed in detail in Section 3) use a global clock. It is shown in Section [4] that the use of local clocks actually simplifies the analysis of these algorithms.

For the above problem formulation, we are interested in studying two questions:

\begin{enumerate}
\item[(Q1.)] Under what conditions is the sequence of iterations \( \{\theta_t\} \) bounded almost surely?
\item[(Q2.)] Define the sequence \( \{\pi_t\} \) by \( \pi_0 = 0 \) and \( \pi_{t+1} = h(t, \pi^*_0) \) for \( t \geq 0 \). Suppose \( \pi_t \to \pi^* \) as \( t \to \infty \). Under what conditions does the sequence of iterations \( \{\theta_t\} \) converge to \( \pi^* \) as \( t \to \infty \)?
\end{enumerate}

\subsection{Assumptions and Theorem Statements}

In this subsection, we first introduce various assumptions, and discuss the implications of these assumptions. As our assumptions are less restrictive than the ones prevailing in the literature, we first state the assumptions and then discuss them. Note that not every assumption is needed in every theorem. After this discussion, we state our main results. Note that we use the generic symbol \( 0 \) to denote a vector with all zero components, whose dimension is determined by the context.

Let \( F_t \) denote the \( \sigma \)-algebra generated by the random variables \( \theta^t_0, \xi^t_i, \kappa^t_{0,i} \) for \( i \in [d] \). (Note that there is no \( \xi^t_0 \).) Then it is clear that \( F_t \subseteq F_{t+1} \), so that \( \{F_t\} \) is a filtration, and that the processes \( \theta^t_0, \xi^t_i, \kappa^t_{0,i} \) are all adapted to this filtration.
(N) The first set of assumptions is on the noise. The noise process \( \{ \xi_t \} \) satisfies the following two assumptions: First, there exists a finite constant \( c'_1 \) and a sequence of constants \( \{ \mu_t \} \) such that
\[
\| E(\xi_{t+1} | F_t) \|_2 \leq c'_1 \mu_t (1 + \| \theta_0^t \|_\infty), \quad \forall t \geq 0.
\]  
(15)

The second assumption is on the conditional variance of \( \xi_{t+1} \). Specifically, there exists a finite constant \( c'_2 \) and a sequence of constants \( \{ \sigma_t \} \) such that
\[
E(\| \xi_{t+1} - E(\xi_{t+1} | F_t) \|_2^2 | F_t) \leq c'_2 \sigma_t^2 (1 + \| \theta_0^t \|_\infty^2), \quad \forall t \geq 0.
\]  
(16)

(S1) The next set of assumptions is on the step size sequence. The random step size sequences \( \{ \alpha_{t,i} \} \) and the sequences \( \{ \mu_t \}, \{ \sigma_t^2 \} \) and satisfy
\[
\sum_{t=0}^{\infty} \alpha_{t,i}^2 < \infty, \quad \sum_{t=0}^{\infty} \sigma_t^2 \alpha_{t,i}^2 < \infty, \quad \sum_{t=0}^{\infty} \mu_t \alpha_{t,i} < \infty, \quad \forall i \in [d].
\]  
(17)

(S2) The random step size sequence \( \{ \alpha_{t,i} \} \) satisfies
\[
\sum_{t=0}^{\infty} \alpha_{t,i} = \infty, \quad \text{a.s., } \forall i \in [d].
\]  
(18)

(F1) The next set of assumptions is on the measurement function. There exist constants \( \rho < 1 \) and \( c'_1 > 0 \) such that
\[
\| h(t, \phi_0^t) \|_\infty \leq \rho \max \{ c'_1, \| \phi_0^t \|_\infty \}, \quad \forall \phi \in (\mathbb{R}^d)^N, t \geq 0.
\]  
(19)

(F2) There exists a \( \pi_0 \in \mathbb{R}^d \) such that the sequence \( \{ \pi_t \} \) defined recursively by \( \pi_{t+1} = h(t, \pi_0^t) \) converges to a limit \( \pi^* \) as \( t \to \infty \).

(F3) There exist an integer \( \Delta \geq 1 \) and a constant \( \gamma \in (0, 1) \) such that
\[
\| h(t, \theta_0^t) - h(t, \pi_0^t) \|_\infty \leq \gamma \| \theta_0^t_{t-\Delta+1} - \pi_0^t_{t-\Delta+1} \|_\infty \quad \forall t \geq \Delta, \quad \forall \theta_0^\infty \in (\mathbb{R}^d)^N.
\]  
(20)

Note that (F2) and (F3) together imply (F1); but we display (F1) separately in the interests of later reference.

(F4) There exist an integer \( \Delta \geq 1 \) and a constant \( \gamma \in (0, 1) \) such that
\[
\| h(t, \theta_0^t) - h(t, \phi_0^t) \|_\infty \leq \gamma \| \theta_0^t_{t-\Delta+1} - \phi_0^t_{t-\Delta+1} \|_\infty \quad \forall t \geq \Delta, \quad \forall \theta_0^\infty, \phi_0^\infty \in (\mathbb{R}^d)^N.
\]  
(21)

Note that (F4) implies (F2) and (F3), which in turn imply (F1). Thus (F4) implies all the rest.

Note that assumption (N) is about the noise, (S1) and (S2) are about the step size sequence, and (F1) through F4) are about the behavior of the measurement function \( h(\cdot, \cdot) \).

Now we discuss the assumptions further.

1. The assumptions \( \{15\} \) and \( \{16\} \) represent significant relaxation of currently used assumptions. In most of the literature to date, it is assumed that \( \{ \xi_t \} \) is a martingale difference sequence with respect to the filtration \( \{ F_t \} \). In contrast, \( \{15\} \) is more general. Similarly, in the current literature, the number \( \sigma_t^2 \) is assumed to be uniformly bounded with respect to \( t \). There is no such restriction here. Instead, the restrictions are that \( \{17\} \) holds, or that \( \{18\} \) holds as well.
2. To illustrate, suppose $\alpha_{t,i} = \beta_t$ for all $i \in S(t)$ (global clock), and choose

$$
\mu_t = \frac{1}{\log(t + 2)} \sigma_t^2 = \frac{1}{(t + 2) \log(t + 2)}.
$$

Then all three equations in (17) hold. Note that if we also impose (18), then $\mu_t$ must approach zero at $t \to \infty$. Note that (15) implies that

$$
|E(\xi_{t+1,i}|F_t)| \leq c_1 \mu_t (1 + \|\theta_0^t\|_{\infty}), \forall t \geq 0, \forall i \in [d]. \quad (22)
$$

3. Now we come to the assumptions on the measurement function $h(\cdot)$. Suppose Assumption (F4) holds. Then a ready consequence of (21) is that

$$
\|h(t, \theta_0^t) - h(\phi_0^t)\|_{\infty} \leq \gamma |t/\Delta| \|\theta_0^{\Delta-1} - \phi_0^{\Delta-1}\|_{\infty}, \forall t \geq \Delta, \forall \theta_0^\infty, \phi_0^\infty \in (\mathbb{R}^d)^N. \quad (23)
$$

Therefore, for every sequence $\phi_0^\infty$, the iterations $h(t, \phi_0^t)$ converge to a unique fixed point $\pi^*$. Moreover, Assumptions (F1) through (F3) follow readily. Hence (F4) is an easily stated and easily verified assumption. However, the proofs use only (F1) through (F3).

4. We further explore the implications of Assumption (F4). In many applications, the measurement map $h$ is “memoryless,” that is,

$$
h(t, \theta_0^t) = g(\theta_t), \quad (24)
$$

where $g(\cdot)$ is a contraction with respect to $\| \cdot \|_{\infty}$. Let $\gamma$ be a contraction constant of $g$. In [17], a more general measurement is considered, namely

$$
h(t, \theta_0^t) = g(\theta_1(t - \Delta_1(t)), \ldots, \theta_d(t - \Delta_d(t))), \quad (25)
$$

where $\Delta_i(t)$ are time-varying delays. We wish to develop a theory that is general enough to handle the above and yet more general measurement maps.

First, suppose $h$ is given by (24). Then it is obvious that

$$
\|h(t, \theta_0^t) - h(t, \pi_0^t)\|_{\infty} = \|g(\theta_t) - g(\pi_t)\|_{\infty} \leq \gamma \|\theta_t - \pi_t\|_{\infty}.
$$

Hence (25) holds with $\Delta = 1$. Next, suppose $h$ is given by (25). Then, for each index $i \in [d]$, we have that

$$
|\theta(t - \Delta_i(t)) - \pi(t - \Delta_i(t))| \leq \|\theta_{t-\Delta+1}^t - \pi_{t-\Delta+1}^t\|_{\infty},
$$

provided we define

$$
\Delta := \max_i \max_{t} \Delta_i(t) + 1.
$$

In turn this implies (20). Another possibility is that $h(t, \theta_0^t)$ can consist of an average of several delayed terms of the form (25).

Having discussed the implications of various assumptions, we now state the two main results on the behavior of BASA.

**Theorem 1.** Suppose that Assumptions (N) about the noise sequence, (S1) about the step size sequence, and (F1) about the function $h$ hold. Then $\sup_t \|\theta_t\|_{\infty} < \infty$ almost surely.

**Theorem 2.** Suppose that Assumptions (N) about the noise sequence, (S1) and (S2) about the step size sequence, and ((F2) plus (F3) about the function $h$ hold. Then $\theta_t$ converges almost surely to $\pi^*$ as $t \to \infty$. 
2.3 A New Convergence Theorem for Stochastic Processes

Before we begin with the proofs of Theorems 1 and 2, we state and prove a preliminary result on the convergence of stochastic processes. Usually such results assume that the additive noise term has zero conditional mean, so that it forms a martingale difference sequence. We do away with this requirement. Because the result is of independent interest in the convergence theory of stochastic processes, we state it as a theorem.

**Theorem 3.** Suppose $U_0$ is a constant, and define \( U_t, t \geq 1 \) by

\[
U_{t+1} = (1 - \alpha_t)U_t + \alpha_t \zeta_{t+1}, \tau \geq 0
\]

where \( \{\alpha_t\}, \{\zeta_t\} \) are stochastic processes adapted to some filtration \( \{F_t\} \), and in addition, \( \alpha_t \in [0, 1) \).

Suppose there exist sequences of constants \( \{\mu_t\}, \{\sigma_t\} \) such that

\[
|E(\zeta_{t+1}|F_t)| \leq \mu_t \text{ a.s. } \forall t \geq 0,
\]

\[
E\left((\zeta_{t+1} - E(\zeta_{t+1}|F_t))^2|F_t\right) \leq \sigma_t^2(1 + U_t^2) \text{ a.s. } \forall t \geq 0
\]

Under these conditions, if almost surely

\[
\sum_{t=0}^{\infty} \alpha_t^2 < \infty, \sum_{t=0}^{\infty} \mu_t \alpha_t < \infty, \sum_{t=0}^{\infty} \sigma_t^2 \alpha_t^2 < \infty.
\]

Then \( \{U_t\} \) is bounded almost surely, and converges to a random variable. If in addition

\[
\sum_{t=0}^{\infty} \alpha_t = \infty,
\]

then \( U_t \to 0 \) almost surely.

The proof is based on the “almost supermartingale theorem” of Robbins & Siegmund [12, Theorem 1]. This paper is a little difficult to locate. However, the result is also proved as [11, Lemma 2, Section 5.2]. Also see a recent survey paper as [6, Lemma 4.1]. The Robbins-Siegmund theorem states the following:

**Lemma 1.** Suppose \( \{z_t\}, \{\beta_t\}, \{\gamma_t\}, \{\zeta_t\} \) are stochastic processes taking values in \([0, \infty)\), adapted to some filtration \( \{F_t\} \), satisfying

\[
E(z_{t+1}|F_t) \leq (1 + \beta_t)z_t + \gamma_t - \theta_t \text{ a.s. } \forall t.
\]

Then, on the set

\[
\Omega_0 := \{\omega : \sum_{t=0}^{\infty} \beta_t(\omega) < \infty \} \cap \{\omega : \sum_{t=0}^{\infty} \gamma_t(\omega) < \infty \},
\]

we have that \( \lim_{t \to \infty} z_t(\omega) \) exists, and in addition, \( \sum_{t=0}^{\infty} \theta_t(\omega) < \infty \).

**Proof.** (Of Theorem 3) Define \( \eta_t := E(\zeta_{t+1}|F_t), \psi_{t+1} := \zeta_{t+1} - \eta_t \). Then it follows from (27) and (28) that the relationships below hold almost surely for all \( t \geq 0 \).

\[
|\eta_t| \leq \mu_t, E(\psi_{t+1}|F_t) = 0, E(\psi_{t+1}^2|F_t) \leq \sigma_t^2.
\]
Now let us estimate $E(U_{t+1}^2 | F_t)$ using the above facts. This gives

$$E(U_{t+1}^2 | F_t) = (1 - \alpha_t)^2 U_t^2 + \alpha_t^2 \eta_t^2 + \alpha_t^2 E(u_{t+1}^2 | F_t) + 2(1 - \alpha_t)U_t \alpha_t \eta_t.$$ 

Now observe that $2U_t \leq 1 + U_t^2$, and $1 - \alpha_t \leq 1$, and invoke (27) and (28). This gives

$$E(U_{t+1}^2 | F_t) \leq (1 + \alpha_t^2) U_t^2 + \alpha_t^2 \mu_t^2 + \alpha_t^2 \sigma_t^2 (1 + U_t^2) + (1 + U_t^2) \alpha_t \mu_t - 2 \alpha_t U_t^2$$

(32)

$$= U_t^2 (1 + \alpha_t^2 + \alpha_t^2 \sigma_t^2 + \alpha_t \mu_t) + \alpha_t^2 \mu_t^2 + \alpha_t \mu_t - 2 \alpha_t U_t^2.$$ 

(33)

Now apply Lemma 1 with

$$z_t = U_t^2, \beta_t = \alpha_t^2 + \alpha_t^2 \sigma_t^2 + \alpha_t \mu_t, \gamma_t = \alpha_t^2 \mu_t^2 + \alpha_t \mu_t, \theta_t = 2 \alpha_t U_t^2.$$ 

If (29) holds, then

$$\sum_{t=0}^{\infty} \alpha_t \mu_t < \infty \implies \sum_{t=0}^{\infty} \alpha_t^2 \mu_t^2 < \infty.$$ 

Therefore the sequences $\{\beta_t\}$ and $\{\gamma_t\}$ are summable almost everywhere, and $P(\Omega_0) = 1$. It therefore follows that $\{U_t^2\}$ is bounded almost surely, and converges almost surely to some random variable $\chi$. Moreover

$$\sum_{t=0}^{\infty} \alpha_t(\omega) U_t^2(\omega) < \infty \ a.s.$$

(34)

Now let us add (30). Suppose there exists an $\omega \in \Omega_0$ such that $\sum_{t=0}^{\infty} \alpha_t(\omega) = \infty$, and $\chi(\omega) > 0$, say $\chi(\omega) \geq 2\epsilon$. Pick a $T$ such that $U_t^2(\omega) \geq \epsilon$ for all $t \geq T$. Then

$$\sum_{t=T}^{\infty} \alpha_t(\omega) U_t^2(\omega) \geq \epsilon \sum_{t=T}^{\infty} \alpha_t(\omega) = \infty,$$

which contradicts (34). Hence the set of $\omega \in \Omega_0$ such that $\chi(\omega) > 0$ is a set of measure zero. We conclude that $U(t) \to 0$ almost surely.

2.4 Proof of Theorem 1

Throughout we are dealing with stochastic processes. So in principle we should be writing, for instance, $\theta_t(\omega)$, where $\omega$ is the element of the probability space that captures the randomness. We do not do this in the interests of brevity, but the presence of the argument $\omega$ is implicit throughout, and what we present are “sample path-wise” arguments. Where necessary, $\omega$ is displayed.

To aid in proving the results, we introduce a sequence of “renormalizing constants.” For $t \geq 0$, define

$$\Lambda_t := \max\{||\theta_t||_{\infty}, c_1^t\},$$

(35)

where $c_1^t$ is defined in (19). With this definition, it follows from (19) that $\eta_t = h(t, \theta_0^t)$ satisfies

$$||\eta_t||_{\infty} \leq \rho \Lambda_t, \forall t.$$ 

(36)

Define $\xi_{t+1} = \Lambda_t^{-1} \xi_{t+1}$ for all $t \geq 0$. Now observe that $\Lambda_t^{-1} \leq c_1^{-1}$, and $\Lambda_t^{-1} \leq (||\theta_0^t||_{\infty})^{-1}$. Hence

$$||E(\xi_{t+1} | F_t)||_{\infty} \leq c_1 \mu_t (c_1^{-1} + 1) =: c_2 \mu_t,$$ 

(37)
we have then $P(D(39))$. Therefore with $D$ in Theorem 1 are satisfied, and moreover, there is no term analogous to Lemma 2.

Similarly the same as [17, Lemma 2].

In the other direction, (40) gives a closed-form solution for the recursion (41).

Note that for $s, k_0$ be given. Let us note that, by virtue of (41), $D_i(s, k_1)$ converges to zero almost surely. This holds for each $i \in [d]$. Therefore, if we define

$$
\Omega_1 = \{ \omega \in \Omega_1 : D_i(0, k_1)(\omega) \to 0 \text{ as } t \to \infty \forall i \in [d] \},
$$

then $P(\Omega_1) = 1$. We can see that for $\omega \in \Omega_1$ we can choose $r_i^*(\omega, \epsilon)$ such that $\forall k \geq r_i^*(\omega, \epsilon), i \in [d]$ we have

$$|D_i(0, k_1)(\omega)| \leq \frac{1}{2}\epsilon.
$$

Note that for $s \leq k$

$$
D_i(0, k_1)(\omega) = \left[ \prod_{t=s}^{k} (1 - \alpha_{t,i}(\omega)) \right] D_i(0, s)(\omega) + D_i(s, k_1)(\omega).
$$

As a consequence

$$
|D_i(s, k_1)(\omega)| \leq |D_i(0, k_1)(\omega)| + |D_i(s, k_1)(\omega)|.
$$

Thus (42) holds for the choice of $r_i^*(\omega, \epsilon)$ made above.

**Lemma 3.** There exists $\Omega_2 \subset \Omega$ with $P(\Omega_2) = 1$ and $r_i^*: \Omega_1 \times \mathbb{N} \times (0, 1) \to \mathbb{N}$ such that

$$
\prod_{t=j}^{k} (1 - \alpha_{t,i}(\omega)) \leq \epsilon, \forall k \geq r_i^*(\omega, j, \epsilon), i \in [d], \omega \in \Omega_2.
$$

where $c_2 = c'_1(c_1^{-1} + 1)$. In particular, the above implies that

$$
|E(\zeta_{t+1,i} | \mathcal{F}_t)| \leq c_2 \mu_t, \forall t \geq 0.
$$

Similarly

$$E(||\zeta_{t+1} - E(\zeta_{t+1,i} | \mathcal{F}_t)||^2_2 | \mathcal{F}_t) \leq c_3 \sigma_i^2, \forall t \geq 0,
$$

for some constant $c_3$. for some constant $c_3$. We also introduce, for $i \in [d]$ and $0 \leq s \leq k < \infty$, the doubly-indexed stochastic process

$$
D_i(s, k + 1) = \sum_{t=s}^{k} \left[ \prod_{r=t+1}^{k} (1 - \alpha_{r,i}) \right] \alpha_{t,i} \zeta_{t+1,i},
$$

where an empty product is taken as 1. Then it is easy to verify that $\{D_i(s, k)\}$ satisfies the recursion

$$
D_i(s, k + 1) = (1 - \alpha_{k,i}) D_i(s, k) + \alpha_{k,i} \zeta_{k+1,i}, D_i(s, s) = 0.
$$

In the other direction, (40) gives a closed-form solution for the recursion (41).

Recall that $\mathbb{N}$ denotes the set of non-negative integers $\{0, 1, 2, \ldots, \}$. The next lemma is basically the same as [17, Lemma 2].

**Lemma 2.** There exists $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$ and $r_i^*: \Omega_1 \times (0, 1) \to \mathbb{N}$ such that

$$
|D_i(s, k_1)(\omega)| \leq \epsilon, \forall s \leq k, i \in [d], k \geq r_i^*(\omega, \epsilon), \omega \in \Omega_1.
$$

**Proof.** Let $\epsilon > 0$ be given. Let us note that, by virtue of (41), $D_i$ satisfies the recursion

$$
D_i(0, t + 1) = (1 - \alpha_{t,i}) D_i(0, t) + \alpha_{t,i} \zeta_{t+1,i},
$$

with $D_i(0, 0) = 0$. Let us fix $i \in [d]$. The assumed conditions on $\zeta_{t+1,i}$ and $\alpha_{t,i}$ yield that conditions in Theorem 1 are satisfied, and moreover, there is no term analogous to $U_2^i$ on the right side of (39). Therefore $D_i(0, k_1)$ converges to zero almost surely. This holds for each $i \in [d]$. Therefore, if we define

$$
\Omega_1 = \{ \omega \in \Omega_1 : D_i(0, k_1)(\omega) \to 0 \text{ as } t \to \infty \forall i \in [d] \},
$$

then $P(\Omega_1) = 1$. We can see that for $\omega \in \Omega_1$ we can choose $r_i^*(\omega, \epsilon)$ such that $\forall k \geq r_i^*(\omega, \epsilon), i \in [d]$ we have

$$|D_i(0, k_1)(\omega)| \leq \frac{1}{2}\epsilon.
$$

Note that for $s \leq k$

$$
D_i(0, k_1)(\omega) = \left[ \prod_{t=s}^{k} (1 - \alpha_{t,i}(\omega)) \right] D_i(0, s)(\omega) + D_i(s, k_1)(\omega).
$$

As a consequence

$$
|D_i(s, k_1)(\omega)| \leq |D_i(0, k_1)(\omega)| + |D_i(s, k_1)(\omega)|.
$$

Thus (42) holds for the choice of $r_i^*(\omega, \epsilon)$ made above. 

**Lemma 3.** There exists $\Omega_2 \subset \Omega$ with $P(\Omega_2) = 1$ and $r_i^*: \Omega_1 \times \mathbb{N} \times (0, 1) \to \mathbb{N}$ such that

$$
\prod_{t=j}^{k} (1 - \alpha_{t,i}(\omega)) \leq \epsilon, \forall k \geq r_i^*(\omega, j, \epsilon), i \in [d], \omega \in \Omega_2.
$$
Proof. In view of the assumption (S), if we define

\[ \Omega_2 = \left\{ \omega \in \Omega : \sum_{t=j}^{\infty} \alpha_{t,i}(\omega) = \infty \forall i \in [d] \right\}, \]

then \( P(\Omega_2) = 1 \). For all \( \omega \in \Omega_2 \), we have

\[ \sum_{t=j}^{\infty} \alpha_{t,i}(\omega) = \infty. \]

Using the elementary inequality \((1 - x) \leq \exp\{-x\}\) for all \( x \in [0, \infty) \), it follows that

\[ \prod_{t=j}^{k}(1 - \alpha_{t,i}(\omega)) \leq \exp\left\{- \sum_{t=j}^{k} \alpha_{t,i}(\omega) \right\}. \]

Hence for \( \omega \in \Omega_2 \), \( \prod_{t=j}^{k}(1 - \alpha_{t,i}(\omega)) \) converges to zero. Thus we can choose \( r_2^*(\omega, j, \epsilon) \) with the required property.

In the rest of this section, we will fix \( \omega \in \Omega_1 \cap \Omega_2 \), the functions \( r_1^* \), \( r_2^* \) obtained in Lemma 2 and Lemma 3 respectively and prove that if (F1) holds, then \( \|\theta_t(\omega)\|_\infty \) is bounded, which proves Theorem 1.

Let us rewrite the updating rule (9) as

\[ \theta_{t+1,i} = (1 - \alpha_{t,i})\theta_{t,i} + \alpha_{t,i}(\eta_{t,i} + \Lambda_t \zeta_{t+1,i}), \quad i \in [d], \quad t \geq 0, \quad (44) \]

By recursively invoking (44) for \( t \in [0, k] \), we get

\[ \theta_{k+1,i} = A_{k+1,i} + B_{k+1,i} + C_{k+1,i}, \quad (45) \]

where

\[ A_{k+1,i} = \prod_{t=0}^{k}(1 - \alpha_{t,i})\theta_{t,i}, \quad (46) \]

\[ B_{k+1,i} = \sum_{t=0}^{k} \left( \prod_{r=t+1}^{k} (1 - \alpha_{r,i}) \right) \alpha_{t,i} \eta_{t,i}, \quad (47) \]

\[ C_{k+1,i} = \sum_{t=0}^{k} \left( \prod_{r=t+1}^{k} (1 - \alpha_{r,i}) \right) \alpha_{t,i} \Lambda_t \zeta_{t+1,i}. \quad (48) \]

Lemma 4. For \( i \in [d] \),

\[ |C_{k+1,i}| \leq \Lambda_k \sup_{0 \leq r \leq k} |D_i(r, k + 1)|. \quad (49) \]

Proof. We begin by establishing an alternate expression for \( C_{k,i} \), namely

\[ C_{k+1,i} = \Lambda_0 D_i(0, k + 1) + \sum_{t=1}^{k} (\Lambda_t - \Lambda_{t-1}) D_i(t, k + 1). \quad (50) \]
For this purpose, observe that $C_{k+1,i}$ satisfies

$$C_{k+1,i} = \Lambda_k \alpha_{k,i} \zeta_{k+1,i} + (1 - \alpha_{k,i}) C_{k,i} = \Lambda_k D_i(k, k + 1) + (1 - \alpha_{k,i}) C_{k,i},$$  \hspace{1cm} (51)$$

because $\alpha_{k,i} \zeta_{k+1,i} = D_i(k, k + 1)$ due to (41) with $s = k$. The proof of (50) is by induction. It is evident from (48) that

$$C_{1,i} = \Lambda_0 \alpha_{0,1} \zeta_{1,i} = \Lambda_0 D_i(0, 1).$$

Thus (50) holds when $k = 0$. Now suppose by way of induction that

$$C_{k,i} = \Lambda_0 D_i(0, k) + \sum_{t=1}^{k-1} (\Lambda_t - \Lambda_{t-1}) D_i(t, k).$$  \hspace{1cm} (52)$$

Using this assumption, and the recursion (51), we establish (50).

Substituting from (52) into (51) gives

$$C_{k+1,i} = \Lambda_k D_i(k, k + 1) + \sum_{t=1}^{k-1} (\Lambda_t - \Lambda_{t-1}) (1 - \alpha_{k,i}) D_i(t, k),$$  \hspace{1cm} (53)$$

Now (41) implies that

$$(1 - \alpha_{k,i}) D_i(t, k) = D_i(t, k + 1) - \alpha_{k,i} \zeta_{k+1,i} = D_i(t, k + 1) - D_i(k, k + 1).$$

Therefore the summation in (53) becomes

$$\sum_{t=1}^{k-1} (\Lambda_t - \Lambda_{t-1}) (1 - \alpha_{k,i}) D_i(t, k) = \sum_{t=1}^{k-1} (\Lambda_t - \Lambda_{t-1}) D_i(t, k)
- D_i(k, k + 1) \sum_{t=1}^{k-1} (\Lambda_t - \Lambda_{t-1}) = S_1 + S_2 \text{ say.}$$

Then $S_2$ is just a telescoping sum and equals

$$S_2 = -\Lambda_{k-1} D_i(k, k + 1) + \Lambda_0 D_i(k, k + 1).$$

The second term in (53) equals

$$\Lambda_0 (1 - \alpha_{k,i}) D_i(0, k) = \Lambda_0 [D_i(0, k + 1) - \alpha_{k,i} \zeta_{k+1,i}] = \Lambda_0 D_i(0, k + 1) - \Lambda_0 D_i(k, k + 1).$$

Putting everything together and observing that the term $\Lambda_0 D_i(k, k + 1)$ cancels out gives

$$C_{k+1,i} = \Lambda_0 D_i(0, k + 1) + (\Lambda_k - \Lambda_{k-1}) D_i(k, k + 1) + \sum_{t=1}^{k-1} (\Lambda_t - \Lambda_{t-1}) D_i(t, k).$$

This is the same as (53). This completes the induction step and thus (49) holds. Using the fact that $\Lambda_t \geq \Lambda_{t-1}$, the desired bound (49) follows readily.
**Proof.** (Of Theorem 1) As per the statement of the theorem, we assume that (F1) holds. We need to prove that

\[
\sup_{t \geq 0} \Lambda_t < \infty.
\]

Define

\[
\delta = \min\left\{\frac{1 - \rho}{2\rho}, \frac{1}{2}\right\}.
\]

Choose \(r^*_1 = r^*_1(\delta)\) as in Lemma 2 such that

\[
|D_i(s, k + 1)| \leq \delta \quad \forall k \geq s \geq r^*_1, \quad \forall i \in [d].
\]

It is now shown that

\[
\Lambda_t \leq (1 + 2\delta)\Lambda_{r^*_1} \quad \forall t, \quad \forall i \in [d].
\]

By the monotonicity of \(\{\Lambda_t\}\), it is already known that \(\Lambda_t \leq \Lambda_{r^*_1}\) for \(t \leq r^*_1\). Hence, once (54) is established, it will follow that

\[
\sup_{0 \leq t < \infty} \Lambda_t \leq (1 + 2\delta)\Lambda_{r^*_1}.
\]

The proof of (54) is by induction on \(t\). Accordingly, suppose (54) holds for \(t \leq k\). Using (49), we have

\[
|C_{k+1,i}| \leq \delta \Lambda_k \leq \Lambda_{r^*_1}(1 + 2\delta).
\]

It is easy to see from its definition that

\[
|A_{k+1,i}| \leq \Lambda_{r^*_1} \left[\prod_{t=0}^{k} (1 - \alpha_{t,i})\right]
\]

Using the induction hypothesis that \(\Lambda_t \leq (1 + 2\delta)\Lambda_{r^*_1}\) for \(t \leq k\), we have

\[
|B_{k+1,i}| \leq \sum_{t=0}^{k} \left[\prod_{r=t+1}^{k} (1 - \alpha_{r,i})\right] \alpha_{t,i} \eta_{t,i}
\]

\[
\leq \sum_{t=0}^{k} \left[\prod_{r=t+1}^{k} (1 - \alpha_{r,i})\right] \alpha_{t,i} \rho \Lambda_t
\]

\[
\leq \rho(1 + 2\delta)\Lambda_{r^*_1} \sum_{t=0}^{k} \left[\prod_{r=t+1}^{k} (1 - \alpha_{r,i})\right] \alpha_{t,i}.
\]

Now note that \(\rho(1 + 2\delta) \leq 1\). Also, the following identity is easy to prove by induction.

\[
\left[\prod_{t=0}^{T} (1 - \alpha_{t,i})\right] + \sum_{t=0}^{T} \left[\prod_{r=t+1}^{T} (1 - \alpha_{r,i})\right] \alpha_{t,i} = 1 \quad \forall T < \infty
\]

Combining these bounds gives

\[
|A_{k+1,i}| + |B_{k+1,i}| \leq \Lambda_{r^*_1}.
\]

Combining this with (45) and (55) leads to

\[
\theta_{k+1,i} \leq \Lambda_{r^*_1}(1 + \delta(1 + 2\delta)) \leq \Lambda_{r^*_1}(1 + 2\delta).
\]

Therefore \(\|\theta_{k+1}\|_\infty \leq \Lambda_{r^*_1}(1 + 2\delta)\), and

\[
\Lambda_{k+1} = \max\{\|\theta_{k+1}\|_\infty, \Lambda_k\} \leq \Lambda_{r^*_1}(1 + 2\delta).
\]

This proves the induction hypothesis completes the proof of Theorem 1. \(\square\)
2.5 Proof of Theorem 2

Proof. (Of Theorem 2) We now come to the proof of the convergence of \( \{\theta_t\} \). Accordingly, it is assumed that (S1, (S2)), (N), and (F1) through (F3) hold. Theorem 1 applies to this case. As a result, \( \bar{\Lambda} := \lim_{t \to \infty} \Lambda_t \) is well-defined and finite. Also, since the sequence \( \{\pi_t\} \) is convergent, the constant \( c_\pi := \sup_t \|\pi_t\|_\infty \) is well-defined and finite. Also, Assumption (S2) implies that

\[
\prod_{t=0}^{k} (1 - \alpha_{t,i}) \to 0 \quad \text{as} \quad t \to \infty, \quad \text{a.s.} \quad (57)
\]

The objective here is to prove that \( \theta_{k+1,i} - \pi_{k+1,i} \to 0 \) as \( k \to \infty \), for each index \( i \in [d] \). Let us recall from (45) and (56) that

\[
\theta_{k+1,i} = A_{k+1,i} + B_{k+1,i} + C_{k+1,i},
\]

while it follows from (56) that

\[
\pi_{k+1,i} = \left[ \prod_{t=0}^{k} (1 - \alpha_{t,i}) \right] \pi_{k+1,i} + \sum_{t=0}^{k} \left[ \prod_{r=t+1}^{k} (1 - \alpha_{r,i}) \right] \alpha_{t,i} \pi_{k+1,i}
\]

Now let us recall the definitions of \( A_{k+1,i} \), \( B_{k+1,i} \) and \( C_{k+1,i} \) from (46), (47) and (48), and regroup terms. This gives

\[
\theta_{k+1,i} - \pi_{k+1,i} = \bar{A}_{k+1,i} + \bar{B}_{k+1,i} + \bar{C}_{k+1,i} + \bar{F}_{k+1,i} - \bar{G}_{k+1,i},
\]

where \( \bar{C}_{k+1,i} \) is as in (48) as before, and

\[
\bar{A}_{k+1,i} = \prod_{t=0}^{k} (1 - \alpha_{t,i}) \left( \theta_{t,i} - \pi_{k+1,i} \right),
\]

\[
\bar{B}_{k+1,i} = \sum_{t=0}^{k} \left[ \prod_{r=t+1}^{k} (1 - \alpha_{r,i}) \right] \alpha_{t,i} \left( \eta_{t,i} - \pi_{t+1,i} \right)
\]

\[
\bar{F}_{k+1,i} = \sum_{t=0}^{k} \left[ \prod_{r=t+1}^{k} (1 - \alpha_{r,i}) \right] \alpha_{t,i} \left( \pi_{t+1,i} - \pi_i^* \right)
\]

\[
\bar{G}_{k+1,i} = \sum_{t=0}^{k} \left[ \prod_{r=t+1}^{k} (1 - \alpha_{r,i}) \right] \alpha_{t,i} \left( \pi_{k+1,i} - \pi_i^* \right)
\]

We now show in succession that each of these five terms approaches zero.

We begin with \( \bar{A}_{k+1,i} \). Observe that \( |\theta_{t,i} - \pi_{k+1,i}| \leq \bar{\Lambda} + c_\pi \). Therefore

\[
|\bar{A}_{k+1,i}| \leq (\bar{\Lambda} + c_\pi) \left[ \prod_{t=0}^{k} (1 - \alpha_{t,i}) \right] \to 0 \quad \text{as} \quad k \to \infty,
\]

from (57).

Next, we study \( \bar{B}_{k+1,i} \). Note that \( \eta_{t,i} - \pi_{t+1,i} = h(t, \theta_0^i) - h(t, \pi_0^i) \). Assumption (F3) implies that

\[
\|h(t, \theta_0^i) - h(t, \pi_0^i)\|_\infty \leq \gamma \|\theta_{t-\Delta+1}^i + \pi_{t-\Delta+1}^i\|_\infty.
\]
Repeated application of the above leads to
\[ \|\theta_{t+1} - \pi_{t+1}\|_\infty \leq \gamma^{[t/\Delta]} \|\theta_0^{\Delta-1} - \pi_0^{\Delta-1}\|_\infty = M\gamma^{[t/\Delta]}, \]
after defining \( M := \|\theta_0^{\Delta-1} - \pi_0^{\Delta-1}\|_\infty \). Therefore
\[ |\tilde{B}_{k+1,i}| \leq \sum_{t=0}^{k} \left[ \prod_{r=t+1}^{k} (1 - \alpha_{r,i}) \right] \alpha_{t,i} M \gamma^{[t/\Delta]}. \]
Let \( W_{k+1,i} \) denote the quantity on the right side, and observe that \( W \) satisfies the recursion
\[ W_{k+1,i} = (1 - \alpha_{k,i}) W_{k,i} + \alpha_{k,i} M \gamma^{[k/\Delta]}. \]
Now we apply Theorem 3 to the above. Note that the role of the “measurement noise” \( \xi_{k+1,i} \) is played here by the deterministic sequence \( \{M \gamma^{[k/\Delta]}\} \). Observe that \( \alpha_{k,i} \in [0, 1] \), while
\[ \sum_{t=0}^{\infty} \gamma^{[t/\Delta]} < \infty. \] (63)
Hence both (15) and (16) hold. As a result, \( W_{k,i} \to 0 \) as \( k \to \infty \).

Next, it has already been shown that \( C_{k+1,i} \to 0 \) as \( t \to \infty \).
Next we study \( F_{k+1,i} \). It is evident from (61) that \( F_{k,i} \) satisfies the recursion
\[ F_{k+1,i} = (1 - \alpha_{k,i}) F_{k,i} + \alpha_{k,i} (\pi_{k+1,i} - \pi^*_i), \]
Here the “noise term” is the deterministic sequence \( \{ \pi_{k+1,i} - \pi^*_i \} \), which is \( O(\gamma^{[k/\Delta]}) \). So we invoke (63) to conclude that \( F_{k+1,i} \to 0 \) as \( k \to \infty \). Finally, we come to \( G_{k+1,i} \). Write
\[ G_{k+1,i} = H_{k+1,i} (\pi_{k+1,i} - \pi^*_i), \]
where
\[ H_{k+1,i} = \sum_{t=0}^{k} \left[ \prod_{r=t+1}^{k} (1 - \alpha_{r,i}) \right] \alpha_{t,i}. \]
Therefore \( H_{k+1,i} \) satisfies the recursion
\[ H_{k+1,i} = (1 - \alpha_{k,i}) H_{k,i} + \alpha_{k,i}. \]
In this equation, the “noise” term is always equal to 1. Applying Theorem 3 to the above recursion shows that \( H_{k+1,i} \to 0 \) as \( k \to \infty \). Because \( \{ \pi_{k,i} \} \) converges to \( \pi^*_i \), the difference \( |\pi_{k+1,i} - \pi^*_i| \to 0 \) as \( k \to \infty \). So the product \( G_{k+1,i} \to 0 \) as \( k \to \infty \).

3 Applications to Reinforcement Learning

In this section, we apply the results proved in Section 2 to the problems of value iteration and Q-learning in Markov Decision Problems (MDPs), which are among the major problems in RL. The following are our contributions:

1. We state and prove a new result on ergodic Markov chains, which allows us to replace the probabilistic assumption (18) by easily verifiable algebraic conditions. See Theorem 4 below.
2. The version of the $TD(\lambda)$-algorithm for value iteration, as enunciated in [8], is an ASA (not BASA) algorithm. Moreover, the convergence result for $TD(\lambda)$ requires a probabilistic condition analogous to [18]; see [8, Theorem 3]. We study the case where the underlying Markov process is irreducible, and study the use of both global clocks and local clocks. By invoking Theorem 4, we show that the $TD(\lambda)$ algorithm converges whenever a local clock is used and the conventional deterministic Robbins-Monro conditions hold. The $TD(\lambda)$ algorithm also converges when a global clock is used, provided the step size sequence $\{\beta_t\}$ is nonincreasing. Thus we eliminate the need for probabilistic assumptions like (18).

3. The traditional convergence proofs for Q-learning, such as those in [20, 17, 8] require an assumption that every state-action pair is visited infinitely often. One way to ensure this is to assume that every policy leads to an irreducible process. Verifying this assumption is likely to be NP-hard, because a closely related problem is NP-hard [18]. Instead, we introduce the concept of “batch” Q-learning, which is BASA. In this approach, it is enough to assume that every action leads to an irreducible process, which is easy to verify. We again analyze the use of both global as well as local clocks, and prove convergence theorems. So far as we are aware, there are no similar results in the literature.

3.1 A Result About Ergodic Markov Processes

Suppose that $\{N_t\}$ is an integer-valued Markov process assuming values in the set $[d] = \{1, \ldots, d\}$, with the state transition matrix $A \in [0,1]^{d \times d}$. Thus
\[
\Pr\{N_{t+1} = j | N_t = i\} = a_{ij}, \quad \forall i, j \in [d].
\]

The following result is a corollary of a far more general result proved in [5].

**Lemma 5.** Suppose $\{N(t)\}$ is a Markov process on $[d]$ with a state transition matrix $A$, and suppose further that $A$ is irreducible. Let $\nu$ denote the unique stationary distribution. Define the function $f_i : [d] \to \{0, 1\}$ by $f_i(j) = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta. Then
\[
\frac{1}{T} \sum_{t=0}^{T-1} f_i(N_t) \to \nu_i \text{ as } T \to \infty, \text{ a.s.}, \forall i \in [d].
\] (64)

Using this result, we can replace (18) by an algebraic condition. Note that in reality $N_t = N_t(\omega)$ where $\omega \in \Omega$, the probability space that defines the Markov process. Since the set $[d]$ is finite, it follows that the set
\[
\Omega_0 := \left\{ \omega \in \Omega : \frac{1}{T} \sum_{t=0}^{T-1} f_i(N_t(\omega)) \to \nu_i \text{ as } T \to \infty, \forall i \in [d] \right\}
\]
has measure one. Therefore we can interchange the order of the quantifiers, and rewrite (64) as
\[
\frac{1}{T} \sum_{t=0}^{T-1} f_i(N_t(\omega)) \to \nu_i \text{ as } T \to \infty, \forall i \in [d], \forall \omega \in \Omega_0.
\] (65)

**Lemma 6.** Fix an integer $i \in [d]$, and suppose $\omega \in \Omega_0$. Then there exists an integer $M_i(\omega)$ such that, whenever $2^k > M_i(\omega)$, we have
\[
\frac{1}{2^k} \left( \sum_{t=2^k+1}^{2^{k+1}} f_i(N(t)(\omega)) \right) \geq \frac{\nu_i}{2}.
\] (66)
Proof. Recall that \( \nu_i > 0 \), and suppose that \( \omega \in \Omega_0 \). Choose \( \epsilon = \nu_i / 2 \), and choose an integer \( M_i(\omega) \) such that

\[
\left| \frac{1}{T} \sum_{t=0}^{T-1} f_i(N_t(\omega)) - \nu_i \right| = \left| \frac{1}{T} \sum_{t=0}^{T-1} (f_i(N_t(\omega)) - \nu_i) \right| < \frac{\epsilon}{3}, \forall T \geq M_i(\omega).
\]

Thus, if \( 2^k > M_i(\omega) \), we have that

\[
\left| \frac{1}{2^k} \sum_{t=2^k+1}^{2^k+1} (f_i(T(t)(\omega)) - \nu_i) \right| \leq \left| \frac{1}{2^k} \sum_{t=1}^{2^k+1} (f_i(T(t)(\omega)) - \nu_i) \right| + \left| \frac{1}{2^k} \sum_{t=1}^{2^k} (f_i(T(t)(\omega)) - \nu_i) \right| < \frac{2}{3} \epsilon + \frac{1}{3} \epsilon = \epsilon.
\]

This implies (66).

\[ \square \]

**Theorem 4.** Suppose \( \{N(t)\} \) is a Markov process on \([d]\) with a state transition matrix \( A \) that is irreducible. Suppose \( \{\beta_t\}_{t \geq 0} \) is a sequence of real numbers in \((0, 1)\) such that \( \beta_{t+1} \leq \beta_t \) for all \( t \), and

\[
\sum_{t=0}^{\infty} \beta_t = \infty. \tag{67}
\]

Then

\[
\sum_{t=0}^{\infty} \beta_t I_{\{N(t) = i\}}(\omega) = \sum_{t=0}^{\infty} \beta_t f_i(N(t)(\omega)) = \infty, \forall i \in [d], \forall \omega \in \Omega_0, \tag{68}
\]

where \( I \) denotes the indicator function.

Proof. Since the set \([d]\) is finite, it is sufficient to prove (68) for each index \( i \in [d] \). So fix an index \( i \), and suppose \( \omega \in \Omega_0 \). Then

\[
\sum_{t=0}^{\infty} \beta_t f_i(N(t)(\omega)) \geq \sum_{k=1}^{\infty} \left( \sum_{t=2^k+1}^{2^k+1} \beta_t f_i(N(t)(\omega)) \right) \geq \sum_{k=1}^{\infty} \left( \sum_{t=2^k+1}^{2^k+1} \beta_{2^k+1} f_i(N(t)(\omega)) \right) = \sum_{k=1}^{\infty} \beta_{2^k+1} \left( \sum_{t=2^k+1}^{2^k+1} f_i(N(t)(\omega)) \right) \geq \sum_{k=1}^{\infty} \beta_{2^k+1} \frac{2^k \nu_i}{2} = \frac{\nu_i}{4} \sum_{k=1}^{\infty} \beta_{2^k+1} 2^{k+1} = \frac{\nu_i}{4} \sum_{k=1}^{\infty} \beta_{2^k+1} 2^{k+1} = \frac{\nu_i}{4} \sum_{k=1}^{\infty} \beta_t = \infty.
\]

This completes the proof.

\[ \square \]

### 3.2 Value Iteration Using Temporal Differences

Consider the following standard problem in reinforcement learning: Suppose \( \{X_t\}_{t \geq 0} \) is a stationary Markov process taking values in a finite set \( \mathcal{X} = \{x_1, \ldots, x_n\} \). We order the elements of \( \mathcal{X} \) in

\[ \text{While applying the contents of Section 2 to value iteration, we should take } d = n, \text{ the cardinality of the state space. When studying } Q \text{-learning, one should take } d = nm, \text{ the product of the cardinalities of the state space and the action space.} \]
some fashion, and use the same ordering throughout. Let \( R : \mathcal{X} \rightarrow \mathbb{R} \) be a “reward” function, and define the reward vector \( \mathbf{r} := [ R(x_1) \ldots R(x_n) ] \).

Let \( \gamma \in (0, 1) \) denote the “discount factor.” For each \( x_i \in \mathcal{X} \), define the expected discounted future reward, or “value,” of the state \( x_i \) as

\[
V^*(x_i) := \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t R(X_t) | X_0 = x_i \right].
\]

(69)

Finally, define the value vector \( \mathbf{v}^* \) as

\[
\mathbf{v}^* := [ V^*(x_1) \ldots V^*(x_n) ].
\]

The objective is to determine the vector \( \mathbf{v}^* \).

Let \( A \in [0, 1]^{n \times n} \) denote the state transition matrix of the Markov process, with elements

\[
a_{ij} := \text{Pr}\{X_{t+1} = x_j | X_t = x_i\}.
\]

Then it is easy to show that \( \mathbf{v}^* \) satisfies the equation

\[
\mathbf{v}^* = \mathbf{r} + \gamma A \mathbf{v}^*.
\]

(70)

Note that the \( \ell_\infty \)-induced norm of the matrix \( A \) is exactly one. Therefore the map \( g : \mathbf{v} \mapsto \mathbf{r} + \gamma A \mathbf{v} \) is a contraction with respect to the \( \ell_\infty \)-norm, with contraction constant \( \gamma \). Therefore \( \mathbf{v}^* \) is the unique fixed point of the map \( g \). In principle one could just write \( \mathbf{v}^* = (I_n - \gamma A)^{-1} \mathbf{r} \). However, if the state transition matrix \( A \) is unknown, then it is not possible to carry out the contraction iterations. In such a case, one seeks alternative approaches.

One such approach is the so-called “temporal difference” family of algorithms, first introduced in [15]. In this approach, it is assumed that the learner has available a sample path \( \{(X_t, R_t)\} \) of the Markov process under study, together with the associated reward at each time. In principle, by observing the sample path for a sufficiently long duration, it is possible to make a reliable estimate of \( A \). However, a key feature of the temporal difference algorithm is that it is a “direct” method, which works directly with the sample path, without attempting to infer the underlying Markov process. With the sample path \( \{X_t\} \) of the Markov process, one can associate a corresponding “index process” \( \{N_t\} \) taking values in \([n]\), as follows:

\[
N_t = i \text{ if } X_t = x_i \in \mathcal{X}.
\]

It is obvious that the index process has the same transition matrix \( A \) as the process \( \{X_t\} \). The idea is to start with an initial estimate \( \hat{\mathbf{v}}_0 \), and update it at each time \( t \) based on the sample path \( \{(X_t, R_t)\} \).

Now we introduce the \( TD(\lambda) \) algorithm studied in this paper. Note that the version introduced in [15] and analyzed further in [19] is slightly different, and the difference is discussed later. The version of the \( TD(\lambda) \) algorithm studied in this paper comes from [8 Eq. (4.7)], and is as follows: At time \( t \), let \( \hat{\mathbf{v}}_t \in \mathbb{R}^n \) denote the current estimate of \( \mathbf{v}^* \). Let \( \{N_t\} \) be the index process defined above. Define the “temporal difference”

\[
\delta_{t+1} := R_{N_t} + \gamma \hat{V}_{t,N_{t+1}} - \hat{V}_{t,N_t}, \forall t \geq 0,
\]

For simplicity we take the reward function to be deterministic. The extension of the arguments below to the case of random rewards is straightforward and left to the reader.
where \( \hat{V}_{t,N_t} \) denotes the \( N_t \)-th component of the vector \( \hat{v}_t \). Equivalently, if the state at time \( t \) is \( x_i \in X \) and the state at the next time \( t+1 \) is \( x_j \), then

\[
\delta_{t+1} = R_i + \gamma \hat{V}_{t,j} - \hat{V}_{t,i}. \tag{72}
\]

Next, choose a number \( \lambda \in [0,1) \). Define the “eligibility vector”

\[
\mathbf{z}_t = \sum_{\tau=0}^{t} (\gamma \lambda)^\tau I_{\{N_{t-\tau} = N_t\}} \mathbf{e}_{N_t - \tau}, \tag{73}
\]

where, as before \( \mathbf{e}_{N_s} \) is a unit vector with a 1 in location \( N_s \) and zeros elsewhere. Since the indicator function in the above summation picks up only those occurrences where \( N_{t-\tau} = N_t \), the vector \( \mathbf{z}_t \) can also be expressed as

\[
\mathbf{z}_t = \mathbf{z}_t \mathbf{e}_{N_t}, \quad \mathbf{z}_t = \sum_{\tau=0}^{t} (\gamma \lambda)^\tau I_{\{N_{t-\tau} = N_t\}}. \tag{74}
\]

Thus the support of the vector \( \mathbf{z}_t \) consists of the singleton \( \{N_t\} \). Finally, update the estimate \( \hat{v}_t \) as

\[
\hat{v}_{t+1} = \hat{v}_t + \delta_{t+1} \alpha_t \mathbf{z}_t, \tag{75}
\]

where \( \alpha_t \) is the step size discussed in Section 2. Note that, at time \( t \), only the \( N_t \)-th component of \( \hat{v}_t \) is updated, and the rest remain the same. Therefore \( TD(\lambda) \) is an ASA, not a BASA, algorithm.

The choice \( \lambda = 0 \) results in \( \mathbf{z}_t = \mathbf{e}_{N_t} \), and results in the update rule

\[
\hat{v}_{t+1} = \hat{v}_t + \delta_{t+1} \alpha_t \mathbf{e}_{N_t},
\]

which is known (naturally enough) as the \( TD(0) \) algorithm. \( TD(0) \) is also an ASA, not a BASA, algorithm.

Now we digress to discuss briefly the difference between the \( TD(\lambda) \) algorithm posed in [8] (which is studied here) and that posed in [15] and analyzed further in [19]. In the latter, the approximation \( \hat{v} \) does not belong to \( \mathbb{R}^n \), but belongs a lower dimensional space. Thus \( \hat{v} \) is a function of some parameter \( \mathbf{p} \in \mathbb{R}^r \), where \( r \ll n \), the number of states. The most widely used representation is

\[
\hat{v}_t = \Psi \mathbf{p}_t,
\]

where \( \Psi \in \mathbb{R}^{n \times r} \) is a basis matrix. The eligibility vector \( \mathbf{z}_t \) is now a summation of all past gradients \( (\psi^{N_t})^\top \), where \( \psi^k \) denotes the \( k \)-th row of \( \Psi \). Thus \( \mathbf{z}_t \in \mathbb{R}^p \). At each time instant, the parameter vector \( \mathbf{p}_t \) is updated as

\[
\mathbf{p}_{t+1} = \mathbf{p}_t + \delta_{t+1} \alpha_t \mathbf{z}_t,
\]

This can be seen most clearly on [19, p. 676], in the right column. This version of \( TD(\lambda) \) is a fully synchronous algorithm, in that every component of \( \mathbf{p}_t \) is updated at each time instant. Hence it is not discussed further.

Now we return to the problem of analyzing the convergence of the \( TD(\lambda) \)-algorithm as described above. To complete the problem specification, we need to specify how the step size \( \alpha_t \) is chosen in (75). The two possibilities studied here are: global clocks and local clocks. The meaning of these

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6The case \( \lambda = 1 \) involves more technicalities, so we do not study it.
phrases is already discussed in (12) and (14), but repeated here for the convenience of the reader. If a global clock is used, then $\alpha_t = \beta_t$, whereas if a local clock is used, then $\alpha_t = \beta_{\nu_t,i}$, where

$$\nu_{t,i} = \sum_{\tau=0}^{t} I_{\{z_{\tau,i} \neq 0\}}.$$  

Note that in the traditional implementation of the $TD(\lambda)$ algorithm suggested in [15, 19, 8], a global clock is used. Perhaps this is not surprising, because the distinction between a global clock and a local clock was introduced only later, in [3]. As we shall see below, the convergence proofs when local clocks are used involve slightly fewer assumptions than when global clocks are used. Moreover, neither involves probabilistic conditions such as (18). This is in sharp contrast to existing theorems in this domain, including [17, 8].

Next we present two theorems regarding the convergence of the $TD(0)$ algorithm. As the hypotheses are slightly different, they are presented separately. But the proofs are quite similar.

**Theorem 5.** Consider the $TD(\lambda)$ algorithm using a local clock to determine the step size. Suppose that the state transition matrix $A$ is irreducible, and that the deterministic step size sequence $\{\beta_t\}$ satisfies the Robbins-Monro conditions (S2), namely

$$\sum_{t=0}^{\infty} \beta_t = \infty, \sum_{t=0}^{\infty} \beta_t^2 < \infty. \quad (76)$$

Then $v_t \rightarrow v^*$ almost surely as $t \rightarrow \infty$.

**Proof.** Define the measurement error

$$\xi_{t+1,i} = \delta_{t+1} - [R_i + \gamma (A\hat{v}_t)_i - \hat{V}_i] = \delta_{t+1} - ([g(\hat{v}_t)]_i - \hat{V}_{t,i}).$$

With this definition, we have that

$$\delta_{t+1} = [g(\hat{v}_t)]_i - \hat{V}_{t,i} + \xi_{t+1,i}.$$  

Therefore the $TD(\lambda)$ algorithm is just an asynchronous update of the map $\hat{v} \mapsto g(\hat{v}) - \hat{v}$ using noisy measurements, and the step size

$$\bar{\alpha}_{t,i} = \alpha_t z_t I_{\{N_t = i\}}, \quad (77)$$

where the constant $z_t$ is defined in (74).

The first step is to show that the noise sequence $\{\xi_t\}$ satisfies Assumption (N), c., that (15) and (16) hold.

Since the index process $\{N_t\}$ has the same dynamics as the Markov process $\{X_t\}$, we have that

$$\Pr\{N_{t+1} = j | N_t \} = a_{ij},$$

where $i, j$ denote $N_t, N_{t+1}$ respectively. Therefore

$$E[\delta_{t+1} | N_t = i] = R_{N_t} + \gamma \sum_{j=1}^{n} a_{ij} V_{t,j} - V_{t,i} = g_i(v_t) - V_{t,i}.$$  

Since the above holds for every $i \in [n]$, it follows that $\delta_{t+1}$ is an unbiased estimator of the $N_t$-th component of $r + \gamma A\hat{v}_t - \hat{V}_t$. Thus (15) holds with $\mu_t = 0$ for all $t$.
Next let us estimate the conditional variance. For this purpose, let \( a^i \) denote the \( i \)-th row of the matrix \( A \), and observe that

\[
g_i(\hat{v}_t) = R_i + \gamma a^i \hat{v}_t.
\]

Hence, if \( N_t = i \) and \( N_{t+1} = j \), then the measurement error is

\[
\xi_{t+1,i} = R_i + \hat{V}_{t,j} - \left[ R_i + \gamma a^i \hat{v}_t \right] \text{ w.p. } a_{ij},
\]

\[
E[\xi_{t+1,i}^2 | N_t = i] = \sum_{j=1}^{n} a_{ij} \left[ \hat{V}_{t,j} - a^i \hat{v}_t \right]^2.
\]

It is not necessary to simplify this expression further. All that is needed is that the right side is quadratic in \( \|\hat{v}_t\|_2 \). Hence (16) holds with \( \sigma_t^2 \) equal to some fixed constant for all \( t \). Therefore the noise assumptions (N) are satisfied.

The rest of the proof is based on Theorem 2. The fact that \( A \) is irreducible implies, from Theorem 4, that for each index \( i \), the state process \( \{X_t\} \) equals \( x_i \) infinitely often, or equivalently, \( \nu_{t,i} \to \infty \) as \( t \to \infty \) for each index \( i \in [d] \). It is obvious from (74) that

\[
1 \leq z_t \leq \frac{1}{1 - \gamma \lambda}.
\]

Therefore, when a local clock is used, for each index \( i \in [n] \), we have

\[
\sum_{t=0}^{\infty} \alpha_{t,i}^2 \leq \frac{1}{1 - \gamma \lambda} \sum_{t=0}^{\infty} \beta_t^2 < \infty.
\]

Moreover, since \( \mu_t = 0 \) and \( \sigma_t^2 \) equals some constant for all \( t \), we see that (18) is satisfied. In the other direction, we have

\[
\sum_{t=0}^{\infty} \alpha_{t,i} = \sum_{t=0}^{\infty} \beta_{\nu_{t,i}, z_t} \geq \sum_{t=0}^{\infty} \beta_{\nu_{t,i}} = \infty,
\]

because \( \nu_{t,i} \) ranges over all the integers due to the local clock being used. Hence (18) is also satisfied. Therefore the desired conclusion follows from Theorem 2.

\[\text{Theorem 6.} \] Consider the TD(\( \lambda \)) algorithm using a global clock to determine the step size. Suppose that the state transition matrix \( A \) is irreducible, and that the deterministic step size sequence is nonincreasing (i.e., \( \beta_{t+1} \leq \beta_t \) for all \( t \)), and satisfies the Robbins-Monro conditions (76). Then \( \nu_t \to \nu^* \) almost surely as \( t \to \infty \).

The proof is essentially the same as for Theorem 5. Because \( A \) is irreducible, and \( \sum_{t=0}^{\infty} \beta_t = \infty \), it follows from Theorem 4 that

\[
\sum_{t=0}^{\infty} \beta_t I\{N_t = i\} = \infty \text{ a.s.}
\]

for every index \( i \in [n] \). The other details are the same as above.
3.3 Batch Q-Learning

In this subsection we study the convergence of the Q-learning algorithm, first introduced in [20]. This algorithm aims to find the optimal policy for a Markov decision process (MDP), *without* knowing the dynamics of the underlying MDP. Instead, the learner is able to observe the sample path of the MDP, one time instant at a time, for the current state-action pair. The original proof of convergence given in [20] used some techniques that were specifically tailored to the problem at hand. Subsequently, more general convergence proofs, based on asynchronous stochastic approximation, were given in [17] and [8]. In this subsection, we first formulate the problem of finding an optimal policy in an MDP, and then the Q-learning algorithm. As with the TD(λ) algorithm, the convergence proofs given in [17, 8] require conditions analogous to (17) and (18). Verifying these assumptions could be NP-hard. We show that, by introducing “batch” Q-learning, it is possible to eliminate such hypotheses, and to prove a theorem without such assumptions. Also, until now no one seems to have considered using local clocks in the updating step. We show how this can be done within the context of Q-learning, and the advantages thereof.

We begin by introducing the optimal action-value function, and the corresponding Q-iteration. Though this material is standard, the discussion serves to fix notation. Let \( \mathcal{X} = \{x_1, \ldots, x_n\} \) be a finite state space, and \( \mathcal{U} = \{u_1, \ldots, u_m\} \) be a finite “action space.” Whereas in a Markov process there is just one state transition matrix, in a MDP there are \( m \) different state transition matrices \( A_{uk}, k \in [m] \). Thus

\[
q_{ij}^{uk} = \Pr\{X_{t+1} = x_j | X_t = x_i, U_t = u_k\}.
\]

The reward function now depends on both the current state and current action. It is denoted by \( R: \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R} \). For convenience the reward is taken to be deterministic. A policy \( \pi \) is any map, deterministic or probabilistic, from \( \mathcal{X} \) into \( \mathcal{U} \). If \( \pi \) is deterministic, then \( \pi: \mathcal{X} \rightarrow \mathcal{U} \) assigns exactly one action to every state. If \( \pi \) is a probabilistic policy, then \( \pi: \mathcal{X} \rightarrow \mathbb{S}(\mathcal{U}) \), where \( \mathbb{S}(\mathcal{U}) \) denotes the set of probability distributions on \( \mathcal{U} \). Thus, if \( \pi \) is a probabilistic policy and

\[
\phi_{ik} = \Pr\{U_t = u_k | X_t = x_i\},
\]

then

\[
\pi(x_i) = [ \phi_{i1} \cdots \phi_{im} ],
\]

(78)

With each policy \( \pi \), deterministic or probabilistic, one can associate a corresponding reward function \( R_\pi: \mathcal{X} \rightarrow \mathbb{R} \), and a Markov process with state transition matrix \( A_\pi \in [0,1]^{n \times n} \), as follows: If \( \pi \) is deterministic, then

\[
R_\pi(x) = R(x, \pi(x_i)), (A_\pi)_{ij} = a_{ij}^{\pi(x_i)}.
\]

If \( \pi \) is probabilistic as in (78), then

\[
R_\pi(x_i) = \sum_{k=1}^{m} \phi_{ik} R(x_i, u_k), (A_\pi)_{ij} = \sum_{k=1}^{m} \phi_{ik} a_{ij}^{uk}.
\]

Either way, this defines a Markov process that depends on the policy chosen.

With each policy \( \pi \), we can associate a corresponding discounted future reward as in (69), with \( V(x_i) \) replaced by \( V_\pi(x_i) \). Now a key problem in MDPs is to identify an “optimal” value function and a corresponding optimal “policy.” Let \( \Pi_d, \Pi_p \) denote respectively the set of deterministic and of probabilistic policies. For each \( x_i \in \mathcal{X} \), define the corresponding *optimal value*

\[
V^*(x_i) = \max_{\pi \in \Pi_p} V_\pi(x_i).
\]

(79)
Then there are a number of facts that are not immediately obvious. For details, see [11]. First, there is an optimal policy \( \pi^* \in \Pi_d \) such that

\[
V_{\pi^*}(x_i) = V^*(x_i), \quad \forall x_i \in \mathcal{X}.
\]

In other words, there is one common policy that is optimal for every state \( x_i \in \mathcal{X} \). Second, this optimal policy can be chosen from \( \Pi_d \). In other words, allowing probabilistic policies does not result in larger optimal values. Third, the optimal value function \( V^* \) satisfies the Bellman optimality equation

\[
V^*(x_i) = \max_{u_k \in \mathcal{U}} \left[ R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} V^*(x_j) \right].
\]

Finally, any policy \( \pi \) such that \( V_{\pi}(x_i) = V^*(x_i) \) is an optimal policy.

The challenge thus arises as to how an optimal policy can be determined. This task is greatly simplified by the introduction of the action-value function in [20].

**Theorem 7.** Define \( Q^* : \mathcal{X} \times \mathcal{U} \to \mathbb{R} \) by

\[
Q^*(x_i, u_k) = R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} V^*(x_j).
\]

Then \( Q^*(\cdot, \cdot) \) satisfies the following relationships:

\[
Q^*(x_i, u_k) = R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} \max_{w_l \in \mathcal{U}} Q^*(x_j, w_l).
\]

Moreover, every policy \( \pi \in \Pi_d \) such that

\[
\pi^*(x_i) = \arg \max_{u_k \in \mathcal{U}} Q^*(x_i, u_k)
\]

is optimal.

The difficulty with the above theorem is that the characterization (82) is implicit, as the unknown quantity \( Q^* \) occurs on both sides. An alternate characterization of \( Q^* \) as the fixed point of a contraction provides a way of computing \( Q^* \) iteratively. Define \( G : \mathbb{R}^{n \times m} \to \mathbb{R}^{n \times m} \) by

\[
(G(Q))(x_i, u_k) := R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} \max_{w_l \in \mathcal{U}} Q(x_j, w_l).
\]

Then it is known that the map \( G \) is monotone and is a contraction with respect to the “vector \( \ell_\infty \)-norm”. Specifically, if we view \( Q \) in (82) as an \( n \times m \) matrix, and \( G \) defined in (83) as a map from \( \mathbb{R}^{n \times m} \) into itself, then

\[
\max_{i,j} |(GQ)_{ij} - (GW)_{ij}| \leq \gamma \max_{i,j} |Q_{ij} - W_{ij}|, \quad \forall Q, W \in \mathbb{R}^{n \times m},
\]

and of course \( \gamma < 1 \). Therefore for all \( Q_0 : \mathcal{X} \times \mathcal{U} \to \mathbb{R} \), the sequence of iterations \( \{G^t(Q_0)\} \) converges to \( Q^* \) as \( t \to \infty \). However, at each iteration, all \( nm \) components of \( Q \) are updated. Moreover, carrying out the iteration requires knowledge of the parameters of the MDP.

The \( Q \)-learning algorithm proposed in [20] has the characterization (82) of \( Q^* \) as its starting point, and is an asynchronous updating procedure, as follows:
1. Choose an arbitrary initial guess $Q_0 : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and an initial state $X_0 \in \mathcal{X}$.

2. At time $t$, with current state $X_t = x_i$, choose a current action $U_t = u_k \in \mathcal{U}$, and let the Markov process run for one time step. Observe the resulting next state $X_{t+1} = x_j$. Then update the function $Q_t$ as follows:

$$
Q_{t+1}(x_i, u_k) = Q_t(x_i, u_k) + \beta_t [R(x_i, u_k) + \gamma V_t(x_j) - Q_t(x_i, u_k)],
$$

(86)

$$
Q_{t+1}(x_s, w_l) = Q_t(x_s, w_l), \forall (x_s, w_l) \neq (x_i, u_k).
$$

where

$$
V_t(x_j) = \max_{w_l \in \mathcal{U}} Q_t(x_j, w_l),
$$

(87)

and $\{\beta_t\}$ is a deterministic sequence of step sizes.

3. Repeat.

In earlier work such as [17, 8], it is shown that the Q-learning algorithm converges to the optimal action-value function $Q^*$ provided

$$
\sum_{t=0}^{\infty} \beta_t I(X_t, U_t) = (x_i, u_k) = \infty, \forall (x_i, u_k) \in \mathcal{X} \times \mathcal{U},
$$

(88)

$$
\sum_{t=0}^{\infty} \beta_t^2 I(X_t, U_t) = (x_i, u_k) < \infty, \forall (x_i, u_k) \in \mathcal{X} \times \mathcal{U}.
$$

(89)

Similar hypotheses are present in all existing results in asynchronous SA. Note that in the Q-learning algorithm, there is no guidance on how to choose the next action $U_t$. Presumably $U_t$ is chosen so as to ensure that (88) and (89) are satisfied. However, we now demonstrate a way to avoid such conditions, by using Theorem 4. We also introduce batch updating and show that it is possible to use a local clock instead of a global clock.

The batch Q-learning algorithm introduced here is as follows:

1. Choose an arbitrary initial guess $Q_0 : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, and $m$ initial states $X_0^k \in \mathcal{X}, k \in [m]$, in some fashion (deterministic or random). Note that the $m$ initial states need not be distinct.

2. At time $t$, for each action index $k \in [m]$, with current state $X_t^k = x_i^k$, choose the current action as $U_t = u_k \in \mathcal{U}$, and let the Markov process run for one time step. Observe the resulting next state $X_{t+1}^k = x_j^k$. Then update function $Q_t$ as follows, once for each $k \in [m]$:

$$
Q_{t+1}(x_i^k, u_k) = \begin{cases} 
Q_t(x_i^k, u_k) + \alpha_{t,i,k}[R(x_i, u_k) + \gamma V_t(x_j^k) - Q_t(x_i^k, u_k)], & \text{if } x_s = x_i^k, \\
Q_t(x_s^k, u_k), & \text{if } x_s^k \neq x_i^k.
\end{cases}
$$

(90)

where

$$
V_t(x_j^k) = \max_{w_l \in \mathcal{U}} Q_t(x_j^k, w_l).
$$

(91)

Here $\alpha_{t,i,k}$ equals $\beta_t$ for all $i, k$ if a global clock is used, and equals

$$
\alpha_{t,i,k} = \sum_{\tau=0}^{t} I(X^k_{\tau}=x_i)
$$

(92)

if a local clock is used.
3. Repeat.

**Remark:** Note that \( m \) different simulations are being run in parallel, and that in the \( k \)-th simulation, the next action \( U_t \) is always chosen as \( u_k \). Hence, at each instant of time \( t \), exactly \( m \) components of \( Q(\cdot, \cdot) \) (viewed as an \( n \times m \) matrix) are updated, namely the \( (X^k_t, u_k) \) component, for each \( k \in [m] \). In typical MDPs, the size of the action space \( m \) is much smaller than the size of the state space \( n \). For example, in the Blackjack problem discussed in [16] Chapter 4, \( n \approx 2^{100} \) while \( m = 2! \). Therefore the proposed batch \( Q \)-learning algorithm is quite efficient in practice.

Now, by fitting this algorithm into the framework of Theorems 1 and 2 and using Theorem 4, we can prove the following general result.

**Theorem 8.** Suppose that each matrix \( A^{u_k} \) is irreducible, and that the step size sequence \( \{\beta_t\} \) satisfies the Robbins-Monro conditions (76). With this assumption, we have the following:

1. If a local clock is used as in (90), then \( Q_t \) converges almost surely to \( Q^* \).

2. If a global clock is used (i.e., \( \alpha_{t,i,k} = \beta_t \) for all \( t, i, k \)), and \( \{\beta_t\} \) is nonincreasing, then \( Q_t \) converges almost surely to \( Q^* \).

**Remark:** Note that, in the statement of the theorem, it is not assumed that every policy \( \pi \) leads to an irreducible Markov process – only that every action leads to an irreducible Markov process. In other words, the assumption is that the \( m \) different matrices \( A^{u_k}, k \in [m] \) correspond to irreducible Markov processes. This is a substantial improvement. It is shown in [18] that the transition matrix being irreducible, plus possibly some transient states. Our problem is slightly different, because we don’t permit any transient states. Nevertheless, this problem is also likely to be very difficult. By not requiring any condition of this sort, and also by dispensing with conditions analogous to (88) and (89), the above theorem statement is more useful.

**Proof.** The quantity being multiplied by \( \alpha_{t,i,k} \) in (90) is \( R(x_i, u_k) + \gamma V_t(x_j) - Q_t(x_i, u_k) \), where \( x_j \) is a randomly generated next state corresponding to the action \( u_k \). The key observation, which is apparently made for the first time in [17], is this:

\[
E \left[ \max_{w_l \in \mathcal{U}} Q_t(X_{t+1}, w_l) \mid (X_t, U_t) = (x_i, u_k) \right] = \sum_{j=1}^n a_{ij}^{u_k} \max_{w_l \in \mathcal{U}} Q_t(x_j, w_l).
\]

Hence the quantity

\[
V_t(x_j) = \max_{w_l \in \mathcal{U}} Q_t(x_j, w_l)
\]

is an unbiased estimate of

\[
\max_{w_l \in \mathcal{U}} Q_t(X_{t+1}, w_l) \mid (X_t, U_t).
\]

As a result, the quantity multiplying \( \alpha_{t,i,k} \) in (90) is an unbiased estimate of \( [G(Q)](x_i, u_k) - Q_t(x_i, u_k) \). If we define the measurement error as

\[
\xi_{t,i,k} = R(x_i, u_k) + \gamma \sum_{j=1}^n a_{ij} V_t(x_j) - Q_t(x_i, u_k) - \{R(x_i, u_k) + \gamma V_t(x_j) - Q_t(x_i, u_k)\},
\]

then \( \xi_{t,i,k} \) has zero conditional mean. It also has conditional variance that grows quadratically as a function of the Euclidean norm-squared of \( Q(\cdot, \cdot) \), using by now familiar arguments. Finally, it is already known that the map \( G \) is a contraction. Therefore, the desired conclusions follows from reasoning entirely analogous to that used in the proofs of Theorems 5 and 6. \( \square \)
4 Conclusions and Future Work

In this paper, we have proposed a unified formulation of batch asynchronous stochastic approximation (BASA) algorithms, and developed a general methodology for proving that such algorithms converge. As a part of our approach, we have stated and proved a result on the convergence of SA-like algorithms wherein the measurement noise is allowed to have a nonzero conditional mean and/or unbounded conditional variance. This result is of independent interest, and its applications will also be explored elsewhere. In addition to the general convergence framework for BASA algorithms, we have replaced previously prevalent probabilistic assumptions on the divergence of step sizes by a purely deterministic condition when the underlying Markov process is irreducible. As specific applications, we have (i) studied the $TD(\lambda)$-algorithm for value determination of a Markov Reward Process, and (ii) we have introduced a “batch” version of the $Q$-learning algorithm. In both cases, we have introduced alternate versions of the algorithms that use local clocks instead of a global clock. Then we have established the convergence of these algorithms under fewer technical conditions than in earlier papers.

A version of BASA, known as Batch Stochastic Gradient Descent, is widely used in large-scale optimization problems, both for convex optimization as well as in deep learning where the objective functions are definitely nonconvex. In the language of the present paper, the problem is one of finding a solution to an equation of the form $J'(\theta) = 0$, where $J(\cdot)$ is the objective function. While this problem is similar to the fixed point problem studied here, there are also some important differences. In particular, when BASA is used to determine a fixed point of a map (as in the present paper), it is reasonable to assume that there is a unique fixed point. In contrast, when the objective is to find stationary points of a function $J(\cdot)$, it would be preferable to develop a theory for the case where $J(\cdot)$ has multiple stationary points. This necessitates the use of different techniques. Our results on Batch Stochastic Gradient Descent will be reported elsewhere.

In principle, the use of local clocks should lead to faster convergence than the use of global clocks. This has not been established here. We surmise that the new and rapidly evolving field of “finite-time stochastic approximation” can be fruitfully applied to carry out such analysis.

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