Skyrmionic excitons

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(July 2, 1997)

Abstract

We investigate the properties of a Skyrmionic exciton consisting of a negatively charged Skyrmion bound to a mobile valence hole. A variational wave function is constructed which has the generalized total momentum $P$ as a good quantum number. It is shown that the Skyrmionic exciton can have a larger binding energy than an ordinary magnetoexciton and should therefore dominate the photoluminescence spectrum in high-mobility quantum wells and heterojunctions where the electron-hole separation exceeds a critical value. The dispersion relation for the Skyrmionic exciton is discussed.

78.55.-m, 73.20.Dx
Excitons play a crucial role in our understanding of the optical properties of direct-gap semiconductors. The properties of excitons in systems of reduced dimensionality in a high magnetic field were first investigated by Lerner and Lozovik [1] many years ago, and have been the subject of numerous studies [2] since. Recently, the existence of unusual charged excitations, known as Skyrmions, was predicted theoretically [3], and confirmed experimentally [4–6], in quantum Hall systems with filling factor near \( \nu = 1 \). A Skyrmion is a topological twist in the spin density of the two-dimensional (2D) electron gas. According to the sense of the twist, the Skyrmion carries electrical charge \(-e\) or \(+e\). The size of the Skyrmion is determined by a competition between the Zeeman energy and the electron-electron exchange interaction. The relative strength of these two competing effects is conveniently parameterized by the ratio \( g = \frac{1}{2} |g_e| \mu_B B / (e^2 / \ell) \), where \( g_e \) is the Landé g factor (\( g_e = -0.44 \) in GaAs), \( \ell = \sqrt{\hbar c / eB} \), and \( \epsilon \) is the background dielectric constant. The Zeeman energy favours small Skyrmions, while the exchange interaction prefers large ones [7]. Ordinary spin \( \frac{1}{2} \) quasi-electrons (quasiholes) can be regarded as negatively (positively) charged Skyrmions of zero size.

In this paper we show that, just as an electron and a hole may bind to form an exciton, a negatively charged Skyrmion and a hole may bind to form a Skyrmionic exciton. The Skyrmionic exciton can replace the ordinary magnetoexciton as the lowest energy excitation probed by a photoluminescence (PL) experiment [8,9] at \( \nu = 1 \). Previously [10] we considered the optical recombination of a Skyrmion with a hole localized near a minimum in the disorder potential. The Skyrmionic exciton considered here is different in that the Skyrmion and the mobile hole can move together throughout the plane perpendicular to the magnetic field. The Skyrmionic exciton hence has its own dispersion relation in terms of a quantum number \( P \) which plays the role of the total transverse momentum in a magnetic field. The key element in our approach is the construction of a variational wave function with a definite value of \( P \), which produces an equal probability of finding the Skyrmionic exciton anywhere in the plane perpendicular to the magnetic field. This allows us to compute the energy and the spin of the Skyrmionic exciton as a function of \( P \). We focus in particular on the state with \( P = 0 \) which dominates the optical recombination spectrum at \( \nu = 1 \).

We study a model system consisting of a 2D electron gas lying in the \( x-y \) plane, interacting with a valence hole confined to a plane at a distance \( d \) from the electron gas, in a strong magnetic field along the \( z \) direction. The confinement of the hole along the \( z \) direction occurs naturally in an asymmetrically doped quantum well [8], and may be achieved in a single heterojunction [9] by the application of a bias voltage. The strong magnetic field restricts the electrons and hole to the lowest Landau level. The electrons can have spin up or down, while the hole can have \( m_j = \pm \frac{3}{2} \). The label \( m_j \) plays no role other than to select the polarization of the luminescence [12]. Hence we drop this label and leave it understood that for a right-circularly polarized (RCP) transition the hole must have \( m_j = -\frac{3}{2} \), while for a left-circularly polarized (LCP) transition it must have \( m_j = +\frac{3}{2} \). The Hamiltonian is

\[
H = \varepsilon_h \sum_m \hat{h}_m^\dagger \hat{h}_m + \frac{1}{2} g_e \mu_B B \sum_m (\hat{e}_{m \uparrow}^\dagger \hat{e}_{m \uparrow} - \hat{e}_{m \downarrow}^\dagger \hat{e}_{m \downarrow}) + \frac{1}{2} \sum_{\sigma \sigma'} \sum_{mm'nn'} V_{ee}^{\sigma \sigma'} \hat{e}_{m \sigma}^\dagger \hat{e}_{m' \sigma'}^\dagger \hat{e}_{n' \sigma'} \hat{e}_{n \sigma}
\]
Here $e_m^{\uparrow} \sigma$ creates an electron with spin $\sigma$ ($\sigma = \uparrow$ or $\downarrow$) in the state $\phi_m(r) = (2^{m+1}\pi m!)^{-1/2} r^m e^{-im\sigma} e^{-r^2/4}$ ($\ell = 1$), and $h_m^\dagger$ creates a hole in the state $\phi_m^*(r)$. The energy of the hole ($\varepsilon_h$) is measured relative to the position of the Fermi level at $\nu = 1$. A uniform neutralizing background is added to the Hamiltonian in the usual way.

The ground state of $H$ at $\nu = 1$ is the fully polarized state $|0\rangle = \prod_{m=0}^\infty e_m^{\uparrow} |\text{vac}\rangle$, where $|\text{vac}\rangle$ is the empty conduction band. The positive background ensures that the state $|0\rangle$ is electrically neutral. The initial states probed by a PL experiment at $\nu = 1$ lie in the sector of the Hilbert space with $n_h = 1$ and $n_\downarrow - n_\uparrow = 1$. Here $n_h$ is the number of holes in the valence band, and $n_\downarrow$ and $n_\uparrow$ are the number of spin down electrons and spin up holes in the conduction band, respectively. The initial states have one hole in the valence band, and a negatively charged excitation in the conduction band. For a magnetoexciton the negatively charged excitation is a spin down electron. For a Skyrmionic exciton the negatively charged excitation is a Skyrmion.

A magnetoexciton has the generalized total momentum $P$ as a good quantum number [13]. In general, conserved quantum numbers arise from the invariance of the system under a group of symmetry operations. The symmetry group associated with $P$ is the magnetic translation group [14]. Magnetic translations differ from ordinary translations by a gauge transformation, which compensates for the shift in the argument of the vector potential due to the translation. Unlike ordinary translations, magnetic translations do not commute when acting on charged excitations such as $e_m^{\uparrow} \sigma$ and $h_m^\dagger$. Hence $P$ is not a good quantum number for a charged particle in a magnetic field. However, magnetic translations do commute when acting on neutral excitations such as $e_m^{\uparrow} h_m^\dagger$ and $e_m^{\uparrow} e_m^{\uparrow}$. The initial states can therefore be chosen as eigenstates of $P$.

In the Hartree-Fock approximation, the wave function for a Skyrmion and a hole localized at the origin is given by [7]

$$ a^\dagger|0\rangle = \prod_{m=0}^{M-1} (-u_m e_{m+1\downarrow} e_m^{\uparrow} + v_m) e_0^{\uparrow} h_0^\dagger |0\rangle. \quad (2) $$

A variational wave function for a Skyrmionic exciton of generalized momentum $P$ is constructed according to

$$ a^\dagger(P)|0\rangle = \int d^2 R e^{iP \cdot R} a^\dagger(R)|0\rangle, \quad (3) $$

with

$$ a^\dagger(R) = M_R a^\dagger M_R^\dagger, \quad (4) $$

where $M_R$ is the magnetic translation operator. This construction is analogous to the tight-binding method for constructing extended Bloch states from localized atomic orbitals. Since $M_R M_{R'} = M_{R+R'}$ when acting on neutral excitations,

$$ M_R a^\dagger(P) M_R^\dagger = e^{-iP \cdot R} a^\dagger(P). \quad (5) $$
This shows $a^\dagger(P)|0\rangle$ is a state with a definite value of $P$. The quantum number $P$ labels the irreducible representations of the magnetic translation group within the sector of the Hilbert space containing the initial states. Because $M_R$ commutes with $H$, the states $a^\dagger(P)|0\rangle$ are orthogonal and uncoupled by $H$. The energy of the Skyrmionic exciton may therefore be obtained by minimizing the expectation value of $H$ in the state $a^\dagger(P)|0\rangle$. The values of $u_m$ and $v_m$ that minimize the energy will depend on $P$. For $u_m = 0$ and $v_m = 1$ we recover the wave function for a magnetoexciton in the lowest Landau level [1].

Using a variational wave function with $M = 14$ parameters, we have minimized the energy of the state $a^\dagger(P = 0)|0\rangle$, for various values of the separation $d$ between the electron and hole planes. Our results are shown in Fig. 1. Figure 1(a) shows the difference in energy between the Skyrmionic exciton and the ordinary magnetoexciton. Figure 1(b) shows the corresponding difference in spin. The size of the Skyrmionic exciton (i.e. the number of spin flips) and its energy are controlled by three competing effects: the electron-electron exchange interaction, the Zeeman energy, and the electron-hole interaction. Just as for a localized Skyrmion [7], the exchange interaction favours a large number of spin flips whilst the Zeeman energy opposes this. The hole interacts with the charge distribution of the Skyrmion, which becomes more diffuse as the Skyrmion size increases. The electron-hole interaction favours a strongly peaked charge distribution and as such moving the hole towards the electron plane should decrease the number of spin flips.

The three-way competition is manifest in Figure 1. As the Zeeman energy is lowered, the Skyrmionic exciton becomes larger in size and increasingly lower in energy than the magnetoexciton. When the hole is moved towards the electron plane (decreasing $d$) the size of the Skyrmionic exciton drops and its energy moves closer to that of the magnetoexciton. A similar reduction in the size of the Skyrmion due to its interaction with a charged particle has been found for a Skyrmion bound to a localized valence hole [10] and for a Skyrmion bound to a charged impurity [11]. For any given separation $d$ there is a threshold to Skyrmionic exciton formation. When the hole is far removed from the electron plane ($d = \infty$) the Skyrmionic exciton forms below $g = 0.025$. As the hole moves towards the electron plane the threshold drops to lower $g$. For the case of $d < \ell$ there is no Skyrmionic exciton formation. This means that magnetoexciton states are the appropriate initial states for systems in which the hole is close the electron plane [12]. However for systems in which $d > \ell$ the $P = 0$ magnetoexciton is not the lowest energy state, this now being the $P = 0$ Skyrmionic exciton. As an example consider a high-mobility GaAs/Ga$_{1-x}$Al$_x$As one-side doped quantum well, with an electron density of $n_s = 10^{11}$ cm$^{-2}$ and a well width of 400 Å. For this density, the magnetic length at $\nu = 1$ is $\ell = 126$ Å. Based on our calculations we expect that Skyrmionic excitons have a lower energy than magnetoexcitons in such quantum wells.

The qualitative behaviour of the $P = 0$ Skyrmionic exciton is unaffected by the number of variational parameters, $M$; more parameters simply scale the results to larger numbers of spin flips and a correspondingly larger energy difference. The maximum number of parameters we can use for the $P = 0$ state is $M = 14$ to yield a tractable computing problem. We believe that this yields all the qualitative physical features of the Skyrmionic exciton. For a given number of variational parameters (fixed $M$) we find that the energy and spin differences for the $P = 0$ state are always larger than for the localized state studied in Ref. [10]. By comparison with our earlier calculations for the localized state using $M = 60$ parameters we therefore expect Skyrmionic excitons with at least 3 spin flips in a 400 Å wide quantum...
well with \( n_s = 10^{11} \text{ cm}^{-2} \).

We now compare the optical recombination spectrum of the Skyrmionic exciton with the recombination spectrum of a magnetoexciton against a filled \( \nu = 1 \) background. The selection rules require the conservation of \( P \), and a change in the \( z \) component of the total angular momentum by \(+1 (-1)\) unit for a RCP (LCP) transition. At temperature \( T = 0 \) the initial state prior to recombination is the state with the lowest energy in the sector of initial states, i.e. the \( P = 0 \) Skyrmionic exciton. The selection rules give a final state containing \( |S_z = \pm \frac{1}{2}\rangle \) spin waves with generalized total momentum \( P \). For a magnetoexciton \( (S_z = -\frac{1}{2}) \), the final state contains no spin waves in the case of RCP, and one spin wave in the case of LCP. By Larmor’s theorem, the energy of a \( P = 0 \) spin wave is \( |g_e|\mu_B B \). Hence the recombination spectrum of a magnetoexciton against a \( \nu = 1 \) background consists, at \( T = 0 \) and in the absence of disorder, of a sharp line in either polarization. For a Skyrmionic exciton \( (S_z < -\frac{1}{2}) \) the final state contains multiple spin waves. For \( |S_z = \pm \frac{1}{2}\rangle \) there is a continuum of final states, with energies between \( |S_z = \pm \frac{1}{2}\rangle |g_e|\mu_B B \) (corresponding to all spin waves having \( P = 0 \)), and \( |S_z = \pm \frac{1}{2}\rangle (|g_e|\mu_B B + \sqrt{\pi/2 e^2/\ell}) \) (corresponding to widely separated spin down electrons and spin up holes \([13]\)). Hence the recombination spectrum of a Skyrmionic exciton has an intrinsic width of order \( |S_z = \pm \frac{1}{2}\rangle \sqrt{\pi/2 e^2/\ell} \) in RCP, and of order \( |S_z = -\frac{1}{2}\rangle \sqrt{\pi/2 e^2/\ell} \) in LCP. Since one more spin wave is left after LCP recombination, the intrinsic width of the LCP line exceeds that of the RCP line. A similar broadening of the line shape due to the formation of a spin texture has been predicted in the electromagnetic absorption spectrum of a Skyrmion bound to a charged impurity \([11]\). Whereas an observation of the broadening of the absorption spectrum would require the growth of specialized structures, the broadening of the luminescence lines should be observable in standard high-mobility quantum wells and heterojunctions. An additional extrinsic broadening of the luminescence lines is due to finite temperature and the presence of disorder. The extrinsic broadening of the recombination spectrum of a magnetoexciton also yields an LCP line whose width exceeds that of the RCP line \([12]\), which can account for the difference in line width observed in narrow quantum wells \([16]\). In wide quantum wells both the extrinsic and intrinsic mechanisms will contribute to the broadening of the luminescence lines. The difference in line width between the LCP and RCP lines should persist to low temperatures and high mobilities in wide wells, where the intrinsic mechanism dominates, but not in narrow wells, where only the extrinsic mechanism operates.

The detailed PL spectrum is difficult to calculate owing to the complicated final state after recombination of a Skyrmionic exciton. However, the average PL energy can be found directly from the initial state \( |i\rangle \) using \([17]\)

\[
\langle \omega \rangle = \frac{\langle i|L^\dagger[L,H]|i\rangle}{\langle i|L^\dagger L|i\rangle},
\]

where \( L \) is the luminescence operator. Previously \([10]\) we calculated the average PL energy for a disordered system using the initial state \( |i\rangle = a^\dagger|0\rangle \). We have repeated the calculation for a disorderless system using the initial state \( |i\rangle = a^\dagger(P = 0)|0\rangle \). As before, the \( g \) dependence of the red shift of the LCP line at \( \nu = 1 \) provides means to distinguish between the optical recombination of a Skyrmionic exciton and the recombination of a magnetoexciton against a filled \( \nu = 1 \) background. For a Skyrmionic exciton the red shift increases with \( g \), while for a magnetoexciton the red shift is independent of \( g \). Using a wave function with
\(M = 14\) parameters, we find the red shift increases by \(0.03\) \(e^2/\ell\) when \(g\) varies from 0.025 to 0.02, with \(d = 3\ell\). In practice, the \(g\) factor may be varied by tilting the magnetic field \([4]\) or by applying hydrostatic pressure \([5]\). In a tilted-field experiment, the deformation of the hole \(z\) wave function by the parallel magnetic field \((B_\parallel)\) leads to an additional angle dependence of the luminescence energy. The red shift remains unaffected by \(B_\parallel\), provided the hole confinement potential is the same on both sides of \(\nu = 1\). This is the case for an asymmetrically doped quantum well \([6]\). In a single heterojunction, the hole is confined near the electron plane on the low-\(B\) side of \(\nu = 1\), but unconfined on the high-\(B\) side. The disparity in the confinement potential causes a decrease of the red shift with tilting angle opposing the Skyrmion signature. This problem may be overcome by applying a gate voltage to the heterojunction, which confines the hole along \(z\) on both sides of \(\nu = 1\).

Thus far our discussion has focused on the \(P = 0\) state, which is the state that determines the PL spectrum at \(T = 0\) in the absence of disorder. States with \(P \neq 0\) can also be probed, for example using PL with a grating on the sample, or by means of resonant light scattering. In the remainder of the paper we derive two exact properties of the \(P \neq 0\) Skyrmionic exciton: its electric dipole moment and the asymptotic behaviour of the dispersion in the high \(|P|\) limit. We also describe features of the dispersion relation obtained from our variational state \(a^\dagger(P)|0\rangle\).

The electric dipole moment of the Skyrmionic exciton is given by

\[
d = \frac{\epsilon}{B^2} \mathbf{P} \times \mathbf{B}. \tag{7}
\]

Accordingly the electric dipole moment of the Skyrmionic exciton is independent of its spin, and equal to the electric dipole moment of an ordinary magnetoexciton \([1]\). This can be shown explicitly for the state given by Eqs. \((2)-(4)\), or more generally as follows. The Gor’kov momentum for a system of \(N\) charges in a magnetic field is given by

\[
\mathbf{P} = \sum_{i=1}^{N} \mathbf{P}_i + \frac{1}{c} \mathbf{B} \times \mathbf{d}, \tag{8}
\]

where \(\mathbf{P}_i = -i \nabla_i - \frac{\hbar}{e} \mathbf{A}(\mathbf{r}_i)\), and \(\mathbf{d} = \sum_{i=1}^{N} q_i \mathbf{r}_i\) is the electric dipole moment. We now take the expectation value of Eq. \((8)\) in an eigenstate of \(\mathbf{P}\). For a state in the lowest Landau level the expectation value of \(\sum_{i=1}^{N} \mathbf{P}_i\) is equal to zero. Hence \(\mathbf{P} = \frac{1}{c} \mathbf{B} \times \mathbf{d}\), where \(\mathbf{P}\) is the eigenvalue of \(\mathbf{P}\). Taking the cross product with \(\mathbf{B}\) yields Eq. \((7)\). Equation \((7)\) is a general property of a Skyrmionic exciton in the lowest Landau level, which does not rely on the specific form of the wave function in Eqs. \((2)-(4)\).

Using Eq. \((7)\) we can establish the exact asymptotic form of the dispersion relation of the Skyrmionic exciton in the high \(P\) limit. Suppose we were given the exact ground state \(s_0^\dagger|0\rangle\) and energy \(\varepsilon_{SK}\) for a Skyrmion localized at the origin. Exact Skyrmion eigenstates have been found for a hard-core model Hamiltonian \([14]\). We now use the state \(a^\dagger|0\rangle = s_0^\dagger h_0^\dagger|0\rangle\) to construct the Skyrmionic exciton, instead of Eq. \((4)\). The resulting state \(a^\dagger(P)|0\rangle\) is an exact eigenstate of \(H\) for \(d = \infty\) with energy \(\varepsilon_{h} + \varepsilon_{SK}\). If \(s_0^\dagger|0\rangle\) is within the lowest Landau level, the dipole moment of \(a^\dagger(P)|0\rangle\) is given by Eq. \((8)\). Because the Skyrmion has no \(internal\) dipole moment in its ground state, the separation between the Skyrmion and the hole in the state \(a^\dagger(P)|0\rangle\) is \(r_0 = d/e\). The electron-hole interaction breaks the degeneracy of the states \(a^\dagger(P)|0\rangle\) with respect to \(P\). For large \(|P|\) the energy shift is \(-\epsilon^2/(\epsilon|r_0|)\). The
exact asymptotic behaviour of the dispersion law in the limit $|\mathbf{P}| \to \infty$ is therefore given by
\[ \varepsilon_h + \varepsilon_{SK} - e^2/(|\mathbf{P}|\ell^2). \]

We have obtained a dispersion law for the Skyrmionic exciton by minimizing the energy of the state $a^\dagger(\mathbf{P})|0\rangle$ with $M = 10$ parameters as a function of $|\mathbf{P}|$. The dispersion is parabolic near $\mathbf{P} = 0$ with an effective mass that differs by less than 7% from the effective mass of a magnetoexciton. The energy and spin differences between the Skyrmionic exciton and the magnetoexciton become smaller as $|\mathbf{P}|$ increases. While the variational state $a^\dagger(\mathbf{P})|0\rangle$ gives reliable results for the dispersion at small ($< 1/\ell$) values of $|\mathbf{P}|$, this state does not adequately describe the dispersion at large ($> 1/\ell$) values of $|\mathbf{P}|$. Above a certain value of $|\mathbf{P}|$ the magnetoexciton becomes the lowest energy state within the variational space, yielding a dispersion that approaches $\varepsilon_h + \frac{1}{2}|g_e|\mu_B B$ in the limit $|\mathbf{P}| \to \infty$. However, from our previous discussion we know that the exact dispersion must approach $\varepsilon_h + \varepsilon_{SK}$. The incorrect behaviour of the variational state $a^\dagger(\mathbf{P})|0\rangle$ in the high-$|\mathbf{P}|$ limit occurs because this state forces the Skyrmion to develop an internal dipole moment as $|\mathbf{P}|$ increases. The internal dipole moment—which must be absent in the exact state $a^\dagger(\mathbf{P})|0\rangle$—prevents the formation of a spin texture at high $|\mathbf{P}|$. Increasing the number of parameters cannot suppress the internal dipole moment and thus does not remedy the failure of our variational state at high $|\mathbf{P}|$. The construction of a variational state that does reproduce the correct asymptotic behaviour of the dispersion in the high-$|\mathbf{P}|$ limit is left for future research.

We thank Andrew Turberfield, Darren Leonard, Stuart Trugman and Igor Lerner for helpful discussions. This work was supported by EPSRC Grant No. GR/K 15619.
FIGURES

FIG. 1. Difference in (a) energy and (b) spin between a $P = 0$ Skyrmionic exciton and a $P = 0$ magnetoeexciton. The energy difference $\Delta E$ and spin difference $|\Delta S_z|$ are plotted vs the reduced Zeeman energy $g = \frac{1}{2} g_{\text{e}} \mu_B B/(e^2/\ell)$, for various values of the separation $d$ between the electron and hole planes. The number of variational parameters is $M = 14$. 
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\[ \Delta E \left[ \frac{e^2}{\epsilon l} \right] \]

\[ d = \infty \]
\[ d = 4l \]
\[ d = 3l \]
\[ d = 2l \]
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\[ \Delta S_z = d = 4l, \quad d = 3l, \quad d = 2l \]