Periodic Solutions to Reversible Second Order Autonomous Systems with Commensurate Delays

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July 21, 2020

Abstract

Existence and spatio-temporal patterns of periodic solutions to second order reversible equivariant autonomous systems with commensurate delays are studied using the Brouwer $O(2) \times \Gamma \times \mathbb{Z}_2$-equivariant degree theory, where $O(2)$ is related to the reversing symmetry, $\Gamma$ reflects the symmetric character of the coupling in the corresponding network and $\mathbb{Z}_2$ is related to the oddness of the right-hand-side. Abstract results are supported by a concrete example with $\Gamma = D_6$ – the dihedral group of order 12.

2010 AMS Mathematics Subject Classification: 34K13, 37J45, 39A23, 37C80, 47H11.

Key Words: Second order delay-differential equations, periodic solutions, commensurate delays, Brouwer equivariant degree, Burnside ring, reversible systems.

1 Introduction

(a) Subject and goal. Existence of periodic solutions to equivariant dynamical systems together with describing their spatio-temporal symmetries constitute an important problem of equivariant dynamics (see, for example, [8, 9] for the equivariant singularity theory based methods and [3, 2, 12] for the equivariant degree treatment). As is well-known, second order systems of ODEs with no friction term exhibit an extra symmetry – the so-called reversing symmetry, i.e. if $x(t)$ is a solution to the system, then so is $x(-t)$. We refer to [10] for a comprehensive exposition of (equivariant) reversible ODEs as well as their applications in natural sciences (see also [1]). It should be stressed that in the context relevant to spatio-temporal symmetries of periodic solutions, reversing symmetry gives rise to extra subgroups of the non-abelian group $O(2)$.

Simple examples show that, in contrast to their ODEs counterparts, second order delay differential equations (in short, DDEs) with no friction term are not reversible, in general. In [6] (see also [13]), we considered space reversible equivariant mixed DDEs of the form

$$\ddot{v}(y) = g(\alpha, v(y)) + a(v(y - \alpha) + v(y + \alpha)), \quad a, \alpha \in \mathbb{R},$$

(1)

This project is supported by the National Natural Science Foundation of China (No. 11871171).

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with equivariant $g : \mathbb{R}^n \to \mathbb{R}^n$ (one can think of equations governing steady-state solutions to PDEs, cf. [16] and references therein). Note that by replacing $g$ by $t$ in (1), one obtains time-reversible DDEs. However, such systems involve using the information from the future by “traveling back in time”, which is difficult to justify from a commonsensical viewpoint.

Time delay systems with commensurate delays play an important role in robust control theory (see, for example, [11] and references therein). A class of such systems exhibiting a reversal symmetry is the main subject of the present paper. To be more specific, given $p > 0$, we are interested in the periodic problem

$$
\begin{align*}
\dot{x}(t) &= f \left( x(t), x \left( t - \frac{p}{m} \right), \ldots, x \left( t - (m-1) \frac{p}{m} \right) \right), \quad t \in \mathbb{R}, \ x(t) \in V = \mathbb{R}^n, \\
x(t) &= x(t+p), \quad \dot{x} = \dot{x}(t+p)
\end{align*}
$$

under the following assumption providing the time reversibility of system (2):

(R) $f(x, y^1, y^2, \ldots, y^{m-2}, y^{m-1}) = f(x, y^{m-1}, y^{m-2}, \ldots, y^2, y^1)$ for all $(x, y^1, \ldots, y^{m-1}) \in V^m$

Assume, in addition, that $\Gamma$ is a finite group and $V$ is an orthogonal $\Gamma$-representation ($\Gamma$ acts on $V = \mathbb{R}^n$ by permuting the vector coordinates in $\mathbb{R}^n$). Put $x := (x,y^1,\ldots,y^{m-1}) \in V^m$ and define on $V^m$ the diagonal $\Gamma$-action by $\gamma x := (\gamma x, \gamma y^1, \ldots, \gamma y^{m-1})$. We make the following assumptions:

(A1) $f$ is $\Gamma$-equivariant, i.e., $f$ is continuous and $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$ and $x \in V^m$;

(A2) $f$ is odd, i.e., $f(-x) = -f(x)$, for all $x \in V^m$;

(A3) $\exists R > 0 \forall x \ |x| > R, |y^j| \leq |x| \Rightarrow x \cdot f(x) > 0$;

(A4) The derivative $A := Df(0) = [A_0, A_1, \ldots, A_{m-1}]$ exists and $A_j A_s = A_s A_j$ for $j, s = 0, 1, \ldots, m-1$.

The goal of the present paper is to study the existence and spatio-temporal properties of solutions to problem (2) under the assumptions (R), (A1), (A3), (A4).

(b) Method. Observe that given an orthogonal $G$-representation $V$ (here $G$ stands for a compact Lie group) and an admissible $G$-pair $(f, \Omega)$ in $V$ (i.e. $\Omega \subset V$ is an open bounded $G$-invariant set and $f : V \to V$ is a $G$-equivariant map without zeros on $\partial \Omega$), the Brouwer degree $d_H := \text{deg}(f^H, \Omega^H)$ is well-defined for any $H \leq G$ (here $\Omega^H := \{ x \in \Omega : hx = x \ \forall h \in H \}$ and $f^H := f|_{\Omega^H}$). If for some $H$, one has $d_H \neq 0$, then the existence of solutions with symmetry at least $H$ to equation $f(x) = 0$ in $\Omega$, can be predicted. Although this approach provides a way to determine the existence of solutions in $\Omega$, and even to distinguish their different orbit types, nevertheless, it comes at a price of elaborate $H$-fixed-point space computations which can be a rather challenging task.

Our method is based on the usage of the Brouwer equivariant degree theory; for the detailed exposition of this theory, we refer to the monographs [3, 13, 12, 15] and survey [2] (see also [4, 5]). In short, the equivariant degree is a topological tool allowing “counting” orbits of solutions to symmetric equations in the same way as the usual Brouwer degree does, but according to their symmetry properties.

To be more explicit, the equivariant degree $G\text{-deg}(f, \Omega)$ is an element of the free $\mathbb{Z}$-module $A(G)$ generated by the conjugacy classes $(H)$ of subgroups $H$ of $G$ with a finite Weyl group $W(H)$:

$$G\text{-deg}(f, \Omega) = \sum_{(H)} n_H \ (H), \ n_H \in \mathbb{Z},$$

(3)
where the coefficients $n_H$ are given by the following Recurrence Formula

$$n_H = \frac{d_H - \sum_{(L) > (H)} n_L n(H, L) |W(L)|}{|W(H)|},$$

and $n(H, L)$ denotes the number of subgroups $L$ in $(L)$ such that $H \leq L$ (see [3]). One can immediately recognize a connection between the two collections: $\{d_H\}$ and $\{n_H\}$, where $H \leq G$ and $W(H)$ is finite. As a matter of fact, $G$-deg($f$, $\Omega$) satisfies the standard properties expected from any topological degree. However, there is one additional functorial property, which plays a crucial role in computations, namely the multiplicativity property. In fact, $A(G)$ has a natural structure of a ring (which is called the Burnside ring of $G$), where the multiplication $\cdot : A(G) \times A(G) \rightarrow A(G)$ is defined on generators by

$$(H) \cdot (K) = \sum_{(L)} m_L (L) \quad (W(L) \text{ is finite}),$$

where the integer $m_L$ represents the number of $(L)$-orbits contained in the space $G/H \times G/K$ equipped with the natural diagonal $G$-action. The multiplicativity property for two admissible $G$-pairs $(f_1, \Omega_1)$ and $(f_2, \Omega_2)$ means the following equality:

$$G\text{-deg}(f_1 \times f_2, \Omega_1 \times \Omega_2) = G\text{-deg}(f_1, \Omega_1) \cdot G\text{-deg}(f_2, \Omega_2).$$

Given a $G$-equivariant linear isomorphism $A : V \rightarrow V$, formula (4) combined with the equivariant spectral decomposition of $A$, reduces the computations of $G\text{-deg}(A, B(V))$ to the computation of the so-called basic degrees $\text{deg}_{\nu_k}$, which can be ‘prefabricated’ in advance for any group $G$ (here $\text{deg}_{\nu_k} := G\text{-deg}(\text{Id}, B(\nu_k))$) with $\nu_k$ being an irreducible $G$-representation and $B(X)$ stands for the unit ball in $X$). In many cases, the equivariant degree based method can be easily assisted by computer (its usage seems to be unavoidable for large symmetry groups).

In the present paper, to establish the abstract results on the existence and symmetric properties of periodic solutions, we use the $G$-equivariant Brouwer degree with $G := O(2) \times \Gamma \times \mathbb{Z}_2$, where $O(2)$ is related to the reversing symmetry, $\Gamma$ reflects the symmetric character of the coupling in the corresponding network and $\mathbb{Z}_2$ is related to the oddness of $f$. We also present a concrete illustrating example with $\Gamma := D_6$, where $D_6$ stands for the dihedral group of order 12. Our computations are essentially based on new group-theoretical computational algorithms, which were implemented in the specially created GAP library by Hao-pin Wu (see [17]).

(c) Overview. After the Introduction, the paper is organized as follows. In Section 2 we establish a priori estimates for solutions to problem [3] in the space $C^2(S^1; V)$. In Section 3 we reformulate problem [3] as an $O(2) \times \Gamma \times \mathbb{Z}_2$-equivariant fixed point problem in $C^2(S^1; V)$ and present an abstract equivariant degree based result. This result can be effectively applied to concrete symmetric systems only if a “workable” formula for the degree associated can be elaborated. The latter is a subject of Section 4 where we combine the multiplicativity property of the equivariant degree with appropriate equivariant spectral data of the operators involved. Based on that, in Section 5 we present our main results (see Theorems 5.1 and 5.4) and an illustrating example with the dihedral group $\Gamma = D_6$. We conclude the paper with an Appendix related to the equivariant topology jargon and equivariant degree background.

2 Normalization of Period and A Priori Bounds

We start with standard steps of the degree theory treatment of (autonomous) dynamical systems: normalization of the period and establishing a priori bounds.
2.1 Normalization of period

By substituting \( y(t) = x(\frac{t - \tau}{\pi}) \), system (2) is transformed to

\[
\ddot{y}(t) = \left( \frac{p}{2\pi} \right)^2 \dot{x} \left( \frac{pt}{2\pi} \right)
\]

\[
= \left( \frac{p}{2\pi} \right)^2 f \left( x \left( \frac{pt}{2\pi} \right), x \left( \frac{p}{2\pi} \left( t - \frac{2\pi}{m} \right) \right), \ldots, x \left( \frac{p}{2\pi} \left( t - \frac{2\pi}{m} (m-1) \right) \right) \right)
\]

\[
= \left( \frac{p}{2\pi} \right)^2 f \left( y(t), y \left( t - \frac{2\pi}{m} \right), \ldots, y \left( t - \frac{2\pi}{m} (m-1) \right) \right).
\]

Put \( \tau := \frac{2\pi}{m} \), so system (2) can be written as

\[
\begin{align*}
\dot{y}(t) &= \alpha^2 f(y(t), y(t - \tau), \ldots, y(t - (m-1)\tau)) \\
y(0) &= y(2\pi), \quad \dot{y}(0) = \dot{y}(2\pi),
\end{align*}
\]

where \( \alpha := \frac{p}{2\pi} \). Notice that \( p \)-periodic solutions \( x(t) \) to system (2) are in one-to-one correspondence to \( 2\pi \)-periodic solutions \( y(t) \) to system (7). In order to simplify the notation, replace \( \alpha^2 f \) by \( \dot{f} \) and \( y \) by \( x \), so one can represent system (2) as follows:

\[
\begin{align*}
\dot{x}(t) &= \dot{f}(x_t), \quad t \in \mathbb{R}, \quad x(t) \in V, \\
x(t) &= x(t + 2\pi), \quad \dot{x}(t) = \dot{x}(t + 2\pi),
\end{align*}
\]

where \( x_t := (x(t), x(t - 2\pi/m), \ldots, x(t - (m-1)2\pi/m)) \).

**Remark 2.1.** Notice that \( \dot{f} : V^m \to V \) satisfies conditions (R), \([A_1],[A_4]\) as well.

2.2 A priori bounds

Consider the following modification of system (8):

\[
\begin{align*}
\dot{x}(t) &= \lambda \dot{f}(x_t) - x(t) + x(t), \quad t \in \mathbb{R}, \quad x(t) \in V, \quad \lambda \in [0,1], \\
x(t) &= x(t + 2\pi), \quad \dot{x} = \dot{x}(t + 2\pi),
\end{align*}
\]

where \( \dot{f} : V^m \to V \) is a continuous map. One has the following

**Lemma 2.2.** Assume that \( \dot{f} : V^m \to V \) satisfies \([A_3]\). If \( x : \mathbb{R} \to V \) is a \( C^2 \)-differentiable \( 2\pi \)-periodic function such that \( \max_{t \in \mathbb{R}} |x(t)| > R \), then \( x(t) \) is not a solution to (9) for \( \lambda \in [0,1] \).

**Proof.** Assume for the contradiction that \( x(t) \) is a solution to (9) with \( |x(t_0)| = \max_{t \in \mathbb{R}} |x(t)| > R \), and consider the function \( \phi(t) := \frac{1}{2} |x(t)|^2 \). Then, \( \phi(t_0) = \max_{t \in \mathbb{R}} \phi(t) \), \( \phi'(t_0) = x(t_0) \bullet \dot{x}(t_0) = 0 \) and \( \phi''(t_0) = \dot{x}(t_0) \bullet \dot{x}(t_0) + \ddot{x}(t_0) \bullet x(t_0) \leq 0 \). However, by condition \([A_3]\) one has for \( 1 \geq \lambda > 0 \):

\[
\phi''(t_0) = \dot{x}(t_0) \bullet \dot{x}(t_0) + \ddot{x}(t_0) \bullet x(t_0) \\
\geq \lambda x(t_0) \bullet \dot{f}(x(t_0), x(t_0 - \tau), \ldots, x(t_0 - (m-1)\tau)) + (1 - \lambda) x(t_0) \bullet x(t_0) \\
\geq \lambda x(t_0) \bullet \dot{f}(x(t_0), \ldots, x(t_0 - (m-1)\tau)) \\
> 0,
\]

which is the contradiction with the assumption that \( \phi(t_0) \) is the maximum of \( \phi(t) \), i.e. \( \phi''(t_0) \leq 0 \). In the case \( \lambda = 0 \), the statement is obvious. \( \square \)
Lemma 2.3. Assume that \( f : V^m \to V \) is continuous and satisfies (A). Then, there exists a constant \( M > 0 \) such that for every solution \( x(t) \) to (9) (for some \( \lambda \in [0, 1] \)), one has:
\[
\forall t \in \mathbb{R} \quad |x(t)| < M, \quad |\dot{x}(t)| < M, \quad |\ddot{x}(t)| < M.
\]

Proof. By Lemma 2.2 there exists \( R > 0 \) such that any \( 2\pi \)-periodic solution \( x(t) \) to (9) satisfies \( |x(t)| \leq R \).

Put
\[
K_R := \{(\lambda, x) \in [0, 1] \times V^m : x = (x, y^1, \ldots, y^{m-1}) ; |x| \leq R, |y_j| \leq R, j = 1, \ldots, m-1\}.
\]
Clearly, the set \( K_R \) is compact. Since the map \( \tilde{f}(\lambda, x) := \lambda(f(x) - x), x = (x, y^1, y^2, \ldots, y^{m-1}), \lambda \in [0, 1], \) is continuous, it follows that \( \tilde{f}(K_R) \) is bounded, i.e. there exists \( M_1 \geq 0 \) s.t. \( |\tilde{f}(\lambda, x)| \leq M_1 \) for all \( (\lambda, x) \in K_R \). Therefore, every solution \( x(t) \) to (9) satisfies \( |\dot{x}(t)| \leq M_1 \).

Take \( v \in V \) with \( |v| \leq 1 \) and consider the scalar function \( \psi(t) := x(t) \cdot v \). Since \( x(t) \) is \( 2\pi \)-periodic, there exists \( t_0 \) such that \( \psi(t_0) = 0 \) for some \( t_0 \in \mathbb{R} \), and for \( t_0 + 2\pi \geq t \geq t_0 \) one has:
\[
|\dot{x}(t) \cdot v| = |\psi'(t)| = |\psi'(t_0) + \int_{t_0}^{t} \psi''(s)ds| = |\int_{t_0}^{t} \psi''(s)ds| \\
= |\int_{t_0}^{t} \tilde{x}(s) \cdot vds| \leq \int_{t_0}^{t} |\tilde{x}(s)| |v|ds \leq \int_{t_0}^{t} |\tilde{x}(s)| |v|ds \\
\leq \int_{t_0}^{t} |\tilde{x}(s)| ds \leq \int_{0}^{2\pi} |\tilde{x}(s)| ds \leq 2\pi M_1 =: M_2.
\]
Therefore,
\[
|\dot{x}(t)| = \sup_{|v| \leq 1} |\dot{x}(t) \cdot v| \leq M_2.
\]
Summing up, \( M := \max(R, M_1, M_2) + 1 \) is as required. \( \square \)

3 Operator Reformulation in Function Spaces

3.1 Spaces

Consider the space \( C_{2\pi}(\mathbb{R}; V) \) of continuous \( 2\pi \)-periodic functions equipped with the norm
\[
\|x\|_{\infty} = \sup_{t \in \mathbb{R}} |x(t)|, \quad x \in C_{2\pi}(\mathbb{R}; V).
\]
(10)

Let \( E := C^2_{2\pi}(\mathbb{R}, V) \) denote the space of \( C^2 \)-differentiable \( 2\pi \)-periodic functions from \( \mathbb{R} \) to \( V \) equipped with the norm
\[
\|x\|_{\infty, 2} = \max\{\|x\|_{\infty}, \|\dot{x}\|_{\infty}, \|\ddot{x}\|_{\infty}\}.
\]
(11)

Let \( O(2) \) denote the group of orthogonal \( 2 \times 2 \) matrices. Notice that \( O(2) = SO(2) \cup SO(2)\kappa \), where \( \kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \), and \( SO(2) \) denote the group of rotations \( \begin{bmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{bmatrix} \) which can be identified with \( e^{i\tau} \in S^1 \subset \mathbb{C} \). Notice that \( \kappa e^{i\tau} = e^{-i\tau}\kappa \).

Put \( G := O(2) \times \Gamma \times \mathbb{Z}_2 \) and define the \( G \)-action on \( E \) by
\[
(e^{i\theta}, \gamma, \pm 1)x(t) := \pm \gamma x(t + \theta),
\]
(12)
\[
(e^{i\theta} \kappa, \gamma, \pm 1)x(t) := \pm \gamma x(-t + \theta),
\]
(13)
where \( x \in \mathcal{E} \), \( e^{i\theta}, \kappa \in O(2) \), \( \gamma \in \Gamma \) and \( \pm 1 \in \mathbb{Z}_2 \). Clearly, \( \mathcal{E} \) is an isometric Banach \( G \)-representation. In a standard way, one can identify a \( 2\pi \)-periodic function \( x : \mathbb{R} \to V \) with a function \( \tilde{x} : S^1 \to V \), so one can write \( C^2(S^1, V) \) instead of \( C^2_{2\pi}(\mathbb{R}, V) \). Similar to (12)-(13) formulas define isometric \( G \)-representations on the spaces of periodic functions \( C^2_{2\pi}(\mathbb{R}, V) \) and \( L^2_{2\pi}(\mathbb{R}; V) \) to which appropriate identifications are applied. One can easily describe the \( G \)-isotypic decomposition of \( \mathcal{E} \). Consider, first, \( \mathcal{E} \) as an \( O(2) \)-representation corresponding to its Fourier modes:

\[
\mathcal{E} = \bigoplus_{k=0}^{\infty} V_k, \quad V_k := \{\cos(kt)u + \sin(kt)v : u, v \in V\}, \tag{14}
\]

where each \( V_k \), for \( k \in \mathbb{N} \), is equivalent to the complexification \( V^c := V \oplus iV \) (as a \textit{real} \( O(2) \)-representation) of \( V \), where the rotations \( e^{i\theta} \in SO(2) \) act on vectors \( z \in V^c \) by \( e^{i\theta}(z) := e^{-ik\theta} \cdot z \) (here ‘\( - \)' stands for complex multiplication) and \( \kappa z := z \). Indeed, the linear isomorphism \( \varphi_k : V^c \to V_k \) given by

\[
\varphi_k(x + iy) := \cos(kt)u + \sin(kt)v, \quad u, v \in V,
\]

is \( O(2) \)-equivariant. Clearly, \( V_0 \) can be identified with \( V \) with the trivial \( O(2) \)-action, while \( V_k \), \( k = 1, 2, \ldots \), is modeled on the irreducible \( O(2) \)-representation \( W_k \simeq \mathbb{R}^2 \), where \( SO(2) \) acts by \( k \)-folded rotations and \( \kappa \) acts by complex conjugation.

Next, each \( V_k \), \( k = 0, 1, 2, \ldots \), is also \( \Gamma \times \mathbb{Z}_2 \)-invariant. Let \( V^-_0, V^-_1, V^-_2, \ldots, V^-_r \) be a complete list of all irreducible orthogonal \( \Gamma \times \mathbb{Z}_2 \)-representations on which \( \Gamma \times \mathbb{Z}_2 \)-isotypic components of \( V \simeq V_0 \) are modeled (here “\( \sim \)” stands to indicate the antipodal \( \mathbb{Z}_2 \)-action and \( V^-_0 \) corresponds to the trivial \( \Gamma \)-action). Since \( V^-_{k,l} := W_k \otimes V^-_r \) is an irreducible orthogonal \( G \)-representation, it follows that \( V_0 \) and \( V_k \) (cf. (14)) admit the following \( G \)-isotypic decompositions:

\[
V_0 = V^-_0 \oplus V^-_1 \oplus \cdots \oplus V^-_r \tag{16}
\]

(with the trivial \( O(2) \)-action) and

\[
V_k = V^-_{k,0} \oplus V^-_{k,1} \oplus \cdots \oplus V^-_{k,r}, \tag{17}
\]

where \( V^-_{k,0} \) (resp. \( V^-_{k,1} \)) is modeled on \( V^-_{0,l} \) (resp. \( V^-_{0,l} \) with \( k > 0 \)).

**Remark 3.1.** Clearly, \( x \in C^2(S^1; V) \) is not a constant function if \( G_x \) does not contain \( O(2) \simeq O(2) \times \{1\} \times \{1\} \varsubsetneq G \).

### 3.2 Operators

Define the following operators:

\[
L : \mathcal{E} \to C(S^1, V), \quad Lx := \dot{x} - x
\]

\[
j : \mathcal{E} \to C(S^1, V^m), \quad j(x)(t) := (x(t), x(t - \tau), \ldots, x(t - (m - 1)\tau))
\]

\[
N : C(S^1, V^m) \to C(S^1, V), \quad N(x(t), y^1(t), \ldots, y^{m-1}(t)) = \{x(t), y^1(t), \ldots, y^{m-1}(t)\} - x(t)
\]

which can be illustrated on the diagram following below:
System (9) is equivalent to
\[ Lx = \lambda(N(jx)), \quad x \in \mathcal{E}, \lambda \in [0, 1], \] (18)
which is equivalent to (8) for \( \lambda = 1 \). Since \( L \) is an isomorphism, equation (18) can be reformulated as follows:
\[ \mathcal{F}_\lambda(x) := x - \lambda L^{-1} N(j(x)) = 0, \quad x \in \mathcal{E}, \quad \lambda \in [0, 1]. \] (19)

**Proposition 3.2.** Suppose that \( f \) satisfies conditions \( (R), (A_1), (A_2) \) (cf. Remark 2.1) and the nonlinear operator \( \mathcal{F}_\lambda : \mathcal{E} \to \mathcal{E} \) is given by (19). Then, \( \mathcal{F}_\lambda \) is a \( G \)-equivariant completely continuous field for every \( \lambda \in [0, 1] \).

**Proof.** Combining (14) and (15) with the definition of \( L \) yields:
\[ L|_{V_0} = -(k^2 + 1) \text{Id} : V^c \to V^c \quad \text{and} \quad L|_{V_0} = -\text{Id} \quad (k > 0). \] (20)
In particular, \( L \) (and, therefore, \( L^{-1} \)) is \( G \)-equivariant. Since \( j \) is the embedding, it is \( G \)-equivariant as well. Since \( L \) and \( N \) are continuous and \( j \) is a compact operator, it follows that \( \mathcal{F}_\lambda \) is a completely continuous field. Also, by assumption \( (A_1) \) (resp. \( (A_2) \), the operator \( N \) is \( \Gamma \)-equivariant (resp. \( \mathbb{Z}_2 \)-equivariant). Since system (2) is autonomous, it follows that \( N \circ j \) is \( SO(2) \)-equivariant. To complete the proof, one only needs to show that \( N \circ j \) commutes with the \( \kappa \)-action. In fact, combining the definitions of the actions of \( N, j \) and \( \kappa \) with 2\( \pi \)-periodicity of \( x \in \mathcal{E} \) and condition \( (R) \), one obtains for all \( t \):
\[
N(j(\kappa x))(t) = N(j(x))(-t) = N(x(-t), x(-t - \tau), \ldots, x(-t - (m - 1)\tau))
\]
\[
= f(x(-t), x(-t - \tau), \ldots, x(-t - (m - 1)\tau)) - x(-t)
\]
\[
= f(x(-t), x(-t - (m - 1)\tau), \ldots, x(-t - \tau)) - x(-t)
\]
\[
= f(x(-t), x(-t + \tau - 2\pi), \ldots, x(-t + (m - 1)\tau - 2\pi)) - x(-t)
\]
\[
= f(x(-t), x(-t + \tau), \ldots, x(-t + (m - 1)\tau)) - x(-t)
\]
\[
= \kappa(f(x(t), x(t - \tau), \ldots, x(t - (m - 1)\tau)) - x(t))
\]
\[
= \kappa(N(jx))(t)
\]
Lemma 3.3. Under the assumptions (R), \( (\sigma) \) is satisfied, put

\[
\mathcal{A} := D\mathcal{F}_1(0) : \mathcal{E} \longrightarrow \mathcal{E}.
\]  

Then,

\[
\mathcal{A} = \text{Id} - L^{-1}(DN(0)) \circ j : \mathcal{E} \longrightarrow \mathcal{E}.
\]  

One can easily check that \( \mathcal{A} \) is a Fredholm operator of index zero. Therefore, \( \mathcal{A} \) is an isomorphism if and only if \( 0 \notin \sigma(\mathcal{A}) \). Also, since \( G \) does not move the origin, it follows that \( \mathcal{A} \) is \( G \)-equivariant.

**Lemma 3.3.** Under the assumptions (R), \( (A_2) \) and \( (A_4) \), suppose, in addition, that \( 0 \notin \sigma(\mathcal{A}) \) (here \( \sigma(\mathcal{A}) \) stands for the spectrum of \( \mathcal{A} \)). Then, for a sufficiently small \( \epsilon > 0 \), the map \( \mathcal{F} := \mathcal{F}_1 \) (cf. (19)) is \( \Omega_\varepsilon \)-admissibly \( G \)-equivariant homotopy to \( \mathcal{A} \) given by (21) (here \( \Omega_\varepsilon := \{ x \in \mathcal{E} : \|x\| < \epsilon \} \).

**Proof.** Put \( \mathcal{H}_t(x) := (1 - t)\mathcal{A}(x) + t\mathcal{F}(x), x \in \mathcal{E}, t \in [0, 1] \), and show that there exists a sufficiently small \( \epsilon > 0 \) such that \( \mathcal{H}_t(\cdot) \) is an \( \Omega_\varepsilon \)-admissible homotopy. Indeed, assume for contradiction, that there exist sequences \( \{x_n\} \subset \mathcal{E} \) and \( \{t_n\} \subset [0, 1] \) such that \( x_n \to 0, t_n \to t_0 \) and

\[
\mathcal{H}_{t_n}(x_n) = \mathcal{A}(x_n) - t_n(\mathcal{A}(x_n) - \mathcal{F}(x_n)) = 0 \quad \text{for all} \quad n \in \mathbb{N}.
\]

Then, by linearity and differentiability, one has:

\[
\frac{\mathcal{A}(x_n)}{\|x_n\|} = \mathcal{F} \left( \frac{x_n}{\|x_n\|} \right) = t_n \frac{\mathcal{A}(x_n) - \mathcal{F}(x_n)}{\|x_n\|} \to 0 \quad \text{as} \quad n \to \infty.
\]  

Put \( v_n := \frac{x_n}{\|x_n\|} \). Combining (23) with (22) yields:

\[
\mathcal{A}(v_n) = v_n - L^{-1}(DN(0))(j(v_n))) \to 0 \quad \text{as} \quad n \to \infty.
\]  

Since \( j \) is a compact operator, there exist \( y_0 \) and a subsequence \( \{v_{n_k}\} \) such that \( L^{-1}(DN(0))(j(v_{n_k}))) \to y_0 \). Hence, by continuity of \( \mathcal{A} \) combined with (24), one has \( v_{n_k} \to y_0 \) and \( \|y_0\| = 1 \). Therefore, \( \mathcal{A}(y_0) = 0 \) which is impossible since \( \mathcal{A} \) is an isomorphism.

**3.3 Abstract equivariant degree based result**

To formulate an equivariant degree based result related to problem (8), we need additional concepts.

**Definition 3.4.** (a) An orbit type \( (H) \) in the space \( \mathcal{E} \) is said to be of **maximal kind** if there exists \( k \geq 0 \) and \( u \neq 0, u \in \mathbb{V}_k \), such that \( H = G_u \) and \( (H) \) is a maximal orbit type in \( \Phi(G, \mathbb{V}_k \setminus \{0\}) \).

(b) Take \( x \in \mathcal{E} \) and assume that there exists \( p \in \mathbb{N} \) such that \( \phi_p(G_x) = (H) \), where \( (H) \) is of maximal kind and the homomorphism \( \phi_p : O(2) \times \Gamma \times \mathbb{Z}_2 \to O(2) \times \Gamma \times \mathbb{Z}_2 \) is given by

\[
\phi_p(g, h, \pm 1) = (\mu_p(g), h, \pm 1), \quad g \in O(2), \quad h \in \Gamma
\]

(here \( \mu_p : O(2) \to O(2)/\mathbb{Z}_p \cong O(2) \) is the natural \( p \)-folding homomorphism of \( O(2) \) into itself). Then, \( x \) is said to have an **extended orbit type** \( (H) \).

**Remark 3.5.** The above concepts have a very transparent meaning. Without extra assumptions, typical equivariant degree based results provide **minimal spacious symmetries** of the corresponding periodic solutions (cf. Definition 3.4(a)) and do not provide an information on its **minimal period** (Definition 3.4(b)).
Under the assumptions (R), \((A_1)\), \((A_2)\) and \((A_4)\), the \(G\)-equivariant degree \(G\)-deg\((\mathcal{A}, B(E))\) is correctly defined provided that 0 \(\not\in\) \(\sigma(\mathcal{A})\) (here \(B(E)\) denotes the unit ball in \(E\)). Put
\[
\omega := (G) - G\text{-deg}(\mathcal{A}, B(E)).
\] (25)

We are now in a position to formulate the abstract result.

**Proposition 3.6.** Assume that \(f : V^m \to V\) satisfies conditions (R), \((A_1)\)–\((A_4)\). Assume, in addition, that 0 \(\not\in\) \(\sigma(\mathcal{A})\) (cf. (19), (21), (22)). Assume, finally,
\[
\omega = n_1(H_1) + n_2(H_2) + \cdots + n_m(H_m), \quad n_j \neq 0, (H_j) \in \Phi_0(G)
\] (26)
(cf. (25)). Then:

(a) for every \(j = 1, 2, \ldots, m\), there exists a \(G\)-orbit of \(2\pi\)-periodic solutions \(x \in E \setminus \{0\}\) to \(\mathcal{S}\) such that \((G_x) \geq (H_j)\);

(b) if \(H_j\) is finite, then the solution \(x\) is non-constant;

(c) if \((H_j)\) is of maximal kind, then the solution \(x\) has the extended orbit type \((H_j)\).

**Proof.** (a) Take \(\Omega\) provided by Lemma 3.3. Then, \(\mathcal{F}\) is \(\Omega\)-admissible and, by equivariant homotopy invariance of the equivariant degree,
\[
G\text{-deg}(\mathcal{F}, \Omega_\epsilon) = G\text{-deg}(\mathcal{A}, B(E)).
\] (27)

Similarly, take \(M\) provided by Lemma 2.2 and put \(\Omega_M := \{x \in E : \|x\| < M\}\). Then, \(\mathcal{F} = \mathcal{F}_1\) is \(\Omega_M\)-admissible and equivariantly homotopic to \(\mathcal{F}_0 = Id\). Hence,
\[
G\text{-deg}(\mathcal{F}, \Omega_M) = (G).
\] (28)

Combining (25), (26), (27), (28) with the existence property of the equivariant degree yields part (a).

(b) Follows from Remark 3.1

(c) Follows from Definition 3.4. \(\square\)

### 4 Computation of \(G\)-deg\((\mathcal{A}, B(E))\)

Proposition 3.6 reduces the study of problem (8) to computing \(G\)-deg\((\mathcal{A}, B(E))\). In this section, we will develop a “workable” formula for \(G\)-deg\((\mathcal{A}, B(E))\) and analyze the non-triviality of some of its coefficients.

#### 4.1 Spectrum of \(\mathcal{A}\)

To begin with, we collect the equivariant spectral data related to \(\mathcal{A}\). Since \(\mathcal{A}\) is \(G\)-equivariant, it respects isotypic decomposition (14). Put \(\gamma := e^{i2\pi} \) and \(\mathcal{A}_k := \mathcal{A}|_{V_k}\). Keeping in mind the commensurateness of delays in problem (8) and taking into account (20), one easily obtains:
\[
\mathcal{A}_k = \text{Id} + \frac{1}{k^2 + 1} \left( \sum_{j=0}^{m-1} \gamma^{jk} A_j - \text{Id} \right), \quad k = 0, 1, 2 \ldots
\] (29)
where $A_j$ stands for the derivative of $f$ with respect to $j$-th variable (to simplify notations, we keep for derivatives of $f$ the same symbols as for the ones of $f$; cf. assumption (A1)). By assumption (R), $A_j = A_{m-j}$ for $j = 1, \ldots, m-1$, hence (29) can be simplified as follows:

$$
\mathcal{A}_k = \text{Id} + \frac{1}{k^2+1} \left( A_0 + \sum_{j=1}^r 2 \cos \frac{2\pi j k}{m} A_j - \varepsilon_m A_r - \text{Id} \right), \quad k = 0, 1, \ldots, r = \left\lfloor \frac{m-1}{2} \right\rfloor, \quad (30)
$$

where

$$
\varepsilon_m = \begin{cases} 
1 & \text{if } m \text{ is even;} \\
0 & \text{otherwise.}
\end{cases} \quad (31)
$$

Since the matrices $A_j$ are $\Gamma$-equivariant, one has $\mathcal{A}_k(V_{k,l}^-) \subset V_{k,l}^-$ ($k = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots, r$). In particular, $A_j(V_l^-) \subset V_l^-$, so put

$$
A_{j,l} := A_j|_{V_l^-}, \quad l = 0, 1, 2, \ldots, r.
$$

To simplify the computations, we will assume that instead of (A4) the following condition is satisfied:

$(\mathcal{A}_4')$ $A_{j,l} = \mu_l^j \text{Id}$ for $l = 0, 1, 2, \ldots, r$ and $j = 0, 1, \ldots, m-1$.

Clearly, under condition $(\mathcal{A}_4')$, the matrices $A_j$ commute with each other, therefore, condition (A4) follows. In particular, their corresponding eigenspaces coincide: $E(\mu_l^j) = E(\mu_l^j)$. This way, one obtains the following description of the spectrum of $\mathcal{A}$:

$$
\sigma(\mathcal{A}) = \bigcup_{k=0}^{\infty} \sigma(\mathcal{A}_k),
$$

(32)

where

$$
\sigma(\mathcal{A}_k) = \left\{ 1 + \frac{1}{1+k^2} \left( \mu_0^l + \sum_{j=1}^r 2 \cos \frac{2\pi j k}{m} \mu_j^l - \varepsilon_m \mu_r^l - 1 \right) : l = 0, 1, \ldots, r, \quad r = \left\lfloor \frac{m-1}{2} \right\rfloor \right\}. \quad (33)
$$

4.2 Computation of $G$-deg($\mathcal{A}, B(\mathcal{E})$): reduction to basic $G$-degrees

Observe that $\xi_{k,l} \in \sigma(\mathcal{A}_k)$ contributes $G$-deg($\mathcal{A}, B(\mathcal{E})$) only if $\xi_{k,l} < 0$. Clearly (cf. (33)),

$$
\xi_{k,l} := 1 + \frac{1}{1+k^2} \left( \mu_0^l + \sum_{j=1}^r 2 \cos \frac{2\pi j k}{m} \mu_j^l - \varepsilon_m \mu_r^l - 1 \right), \quad l = 0, 1, \ldots, r, \quad r = \left\lfloor \frac{m-1}{2} \right\rfloor, \quad k = 0, 1, \ldots, (34)
$$

is negative (i.e. $\xi_{k,l} \in \sigma_{-}(\mathcal{A})$) if and only if

$$
k^2 < -\mu_0^l - \sum_{j=1}^r 2 \cos \frac{2\pi j k}{m} \mu_j^l + \varepsilon_m \mu_r^l, \quad l = 0, 1, \ldots, r, \quad r = \left\lfloor \frac{m-1}{2} \right\rfloor, \quad k = 0, 1, \ldots
$$

(35)

By condition $(\mathcal{A}_4')$, the $V_l^-$-isotypic multiplicity of $\mu_j^l$ is independent of $j$ and is equal to

$$
m^l := \dim E(\mu_j^l)/\dim V_l^- = \dim V_l^-/\dim V_l^-.
$$

(36)
Put (cf. (35)-(36))
\[ m_{k,l} := \begin{cases} m^l & \text{if } k^2 < -\mu_0^l - \sum_{j=1}^r 2 \cos \frac{2\pi j k}{m} \mu_j^l + \varepsilon_m \mu_r^l \\ 0 & \text{otherwise.} \end{cases} \] (37)

Then,
\[ G\text{-deg}(\mathcal{A}, B(\mathcal{B})) = \prod_k \prod_{l=0}^c (\text{deg} V_{k,l})^{m_{k,l}} \] (38)

Notice that in the product (38), one has \( m_{k,l} \neq 0 \) for finitely many values of \( k \) and \( l \) (cf. (37)). Hence, for almost all the factors in (38), one has \( \text{deg} V_{k,l} \neq 0 \), which is the unit element in \( A(G) \). Thus, formula (38) is well-defined.

**Remark 4.1.** Using the relation \( \text{deg} V_{k,l}^2 = (G) \), one can further simplify formula (38). Clearly, only the exponents \( m_{k,l} \neq 0 \) which are odd will contribute to the value of (38).

### 4.3 Maximal orbit types in products of basic \( G \)-degrees

In order to effectively apply Proposition 3.6(c), one should answer the following question: which orbit types of maximal kind (see Definition 3.4) appearing in the right-hand side of formula (38) will “survive” in the resulting product?

To begin with, take \( \text{deg} V_{k,l} \) appearing in (38) and let \((H_o)\) be a maximal orbit type in \( V^-_{k,l} \). Then,
\[ \text{deg} V_{k,l} = (G) - x_o(H_o) + a, \quad x_o := \frac{(-1)^{\dim V_{k,l}^H_o} - 1}{|W(H_o)|}, \] (39)
where \( a \in A(G) \) has a zero coefficient corresponding to \((H_o)\). Then, by (39), one has
\[ x_o = \begin{cases} 0 & \text{if } \dim V_{k,l}^H_o \text{ is even} \\ 1 & \text{if } \dim V_{k,l}^H_o \text{ is odd and } |W(H_o)| = 2 \\ 2 & \text{if } \dim V_{k,l}^H_o \text{ is odd and } |W(H_o)| = 1. \end{cases} \] (40)

Notice that for two different irreducible \( G \)-representations \( V^-_{k,l} \) and \( V^-_{k',l'} \), it is possible that \( \text{deg} V_{k,l} = \text{deg} V_{k',l'} \) (see, for example, [3], p. 183, where irreducible \( \Gamma = A_5 \)-representations are discussed). In this case, one has \( \text{deg} V_{k,l} \cdot \text{deg} V_{k',l'} = (G) \) (cf. Remark 4.1).

**Lemma 4.2.** Suppose that \((H_o)\) is a maximal orbit type in \( V^-_{k,l} \setminus \{0\} \) and \( V^-_{k',l'} \setminus \{0\} \) and both \( V^-_{k,l}^H_o \) and \( V^-_{k',l'}^H_o \) are of odd dimension. Then:

(i) \( \text{deg} V_{k,l} \) and \( \text{deg} V_{k',l'} \) share the same coefficient \( x_o \) standing by \((H_o)\);

(ii) the coefficient standing by \((H_o)\) in \( \text{deg} V_{k,l} \cdot \text{deg} V_{k',l'} \) is zero.

**Proof.**

(i) Follows immediately from (40).

(ii) Consider the following product
\[ \text{deg} V_{k,l} \cdot \text{deg} V_{k',l'} = ((G) - x_o(H_o) + a) \cdot ((G) - x_o(H_o) + a') \]
\[ = (G) - 2x_o(H_o) + x_o^2(H_o) \cdot (H_o) + b, \]

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where \( a, a', b \in A(G) \) are elements with zero coefficient standing by \((H_o)\). Then, by using the Recurrence Formula for multiplication, one obtains:

\[
(H_o) \cdot (H_o) = y_o(H_o) + d,
\]

\[
y_o := \frac{n(H_o, H_o)^2|W(H_o)^2|}{|W(H_o)|} = |(W(H_o))|.
\]

Hence,

\[
-2x_o(H_o) + x_o^2(H_o) \cdot (H_o) = \begin{cases} 
-2 + 2 & \text{if } x_o = 1, \quad |W(H_o)| = 2 \cdot (H_o) = 0, \\
-4 + 4 & \text{if } x_o = 2, \quad |W(H_o)| = 1 \cdot (H_o) = 0,
\end{cases}
\]

and the statement follows.

Observe that the set \( \bigcup_{k>0} \Phi_0(G, V_k) \setminus \{(G)\} \) has no maximal elements. Indeed, for an orbit type \((H)\) in \( \Phi_0(G, V_k) \setminus \{(G)\} \), any orbit type \((H') \in \Phi_0(G, V_k) \setminus \{(G)\} \) such that \( \phi_p(H') = (H) \), satisfies \((H') > (H)\) (see Definition \[34\]). Notice that if \((H_o)\) is maximal in \( \Phi_0(G, V_k) \setminus \{(G)\} \), then \((H'_o) > (H'_o)\) in \( \Phi_0(G, V_k) \setminus \{(G)\} \) if and only if \((H'_o)\) is maximal in \( \Phi_0(G, V_k) \setminus \{(G)\}\) for some \( p > 1 \) and \( \phi_p(H'_o) = (H_o) \). To be more precise, in this a case, if \((H_o)\) is a maximal orbit type in \( V_{k,l}^- \setminus \{0\}\), then \((H'_o)\) is a maximal orbit type in \( V_{p,k,l}^- \setminus \{0\}\). Moreover, the statement following below shows that if a maximal type is presented in \( \deg V_{k,l}^- \) then it “survives” after the multiplication by a “folded” basic degree.

**Lemma 4.3.** Suppose that \((H_o)\) is a maximal orbit type in \( V_{k,l}^- \setminus \{0\}\) and let \((H'_o)\) be the maximal type in \( V_{p,k,l}^- \setminus \{0\}\) such that \( \phi_p(H'_o) = (H_o) \) (see Definition \[34\]). Then:

(i) \( \deg V_{k,l}^- \) and \( \deg V_{p,k,l}' \) share the same coefficient standing by \((H_o)\) and \((H'_o)\), respectively;

(ii) if, in addition, the dimension of \( V_{k,l}^- \) is odd (cf. Lemma \[4.2\]), and \( x_o \) is the coefficient standing by \((H_o)\) in \( \deg V_{k,l}^- \), then the coefficient standing by \((H_o)\) in \( \deg V_{p,k,l}' \cdot \deg V_{p,k,l}' \) is equal to \(-x_o\) (in particular, different from zero).

**Proof.** (i) Take

\[
\deg V_{k,l}^- = (G) - x_o(H_o) + a \quad \text{and} \quad \deg V_{p,k,l}' = (G) - x'_o(H'_o) + a',
\]

where \( a, a' \in A(G) \) have zero coefficients standing by \((H_o)\) and \((H'_o)\), respectively. Since \( |W(H_o)| = |W(H'_o)| \), the statement follows from \[41\].

(ii) Under the notations of (i), one has

\[
\deg V_{k,l}^- \cdot \deg V_{p,k,l}' = (G) - x_o(H_o) - x'_o(H'_o) + x_o^2(H_o) \cdot (H'_o) + b,
\]

where \( b \in A(G) \) is the element with zero coefficients standing by \((H_o)\) and \((H'_o)\). Then, again by Recurrence Formula, one obtains:

\[
(H'_o) \cdot (H_o) = y_o(H_o) + d,
\]

\[
y_o = \frac{n(H'_o, H_o)n(H_o, H_o)|W(H'_o)||W(H_o)|}{|W(H_o)|} = |(W(H_o))|,
\]

where \( d \in A(G) \) is an element with zero coefficients standing by \((H_o)\) and \((H'_o)\). Hence,

\[
-2x_o(H_o) + x_o^2(H_o) \cdot (H'_o) = \begin{cases} 
-1 + 2 & \text{if } x_o = 1, \quad |W(H_o)| = 2 \cdot (H_o) = 0, \\
-2 + 4 & \text{if } x_o = 2, \quad |W(H_o)| = 1 \cdot (H_o) = 0,
\end{cases}
\]

Consequently,

\[
\deg V_{k,l}^- \cdot \deg V_{p,k,l}' = (G) - x_o(H'_o) + x_o(H_o) + b.
\]
We summarize the above discussion in the statement following below. To this end, we need additional notations.

**Definition 4.4.** (i) For any \( (H_o) \in \Phi_0(G) \), define the function \( \text{coeff}^{H_o} : A(G) \to \mathbb{Z} \) assigning to any \( a = \sum_{(H)} n_H(H) \in A(G) \) the coefficient \( n_{H_o} \) standing by \( (H_o) \).

(ii) Given an orbit type \( (H_o) \in \Phi_0(G, \mathcal{E}) \) of maximal kind (see Definition 3.4(a)) and \( k = 0, 1, 2, \ldots, \) define the integer

\[
 n_{H_o}^k := \sum_{l=0}^{r} k_{H_o}^{l} \cdot m_{k,l},
\]

(42)

where \( m_{k,l} \) is given by (37) and

\[
 k_{H_o}^{l} := \begin{cases} 
 1 & \text{if } \dim V_{H_o}^{l} \text{ is odd} \\
 0 & \text{otherwise}
\end{cases}
\]

(43)

(cf. formulas (38)–(40)).

The following statement is an immediate consequence of Remark 4.1 and Lemmas 4.2 and 4.3.

**Lemma 4.5.** Let \( (H_o) \in \Phi_0(G, \mathcal{E}) \) be an orbit type of maximal kind (see Definition 3.4(a)) and assume that for some \( k \geq 0 \), the number \( n_{H_o}^k \) is odd (see Definition 4.4). Then,

\[
 \text{coeff}^{H_o}(G\text{-deg}((\mathcal{A}, B(\mathcal{E}))) = \pm x_o,
\]

(44)

where \( x_o \) is given by (40).

5 Main Results and Example

5.1 Main result: non-degenerate version

In this section, we will present our main results and describe a class of illustrating examples with \( G = O(2) \times D_6 \times Z_2 \). The “non-degenerate” version of the main result is:

**Theorem 5.1.** Assume that \( f : V^m \to V \) satisfies conditions (R), (A_1) and (A_3). Assume, in addition, that \( 0 \notin \sigma(\mathcal{A}) \), where \( \sigma(\mathcal{A}) \) is given by (32)–(33) (see also (31)). Assume, finally, that there exist \( k \in \mathbb{N} \) and an orbit type \( (H_o) \) in \( \Phi_0(G, \mathcal{E}) \) of maximal kind such that \( n_{H_o}^k \) is odd (see Definitions 3.4(a) and 4.4).

Then, system (8) admits a non-constant \( 2\pi \)-periodic solution with the extended orbit type \( (H_o) \).

**Proof.** The proof follows immediately from Lemma 4.5 and Proposition 3.6(c). \( \square \)

5.2 Example

(a) Growth at infinity. We start with describing a class of maps satisfying condition (A_3). Take \( V := \mathbb{R}^n \) equipped with the norm \( \max \), and consider a map \( F : V \times V^{m-1} \to V \) given by

\[
 F(x, y) = (p_1(x, y), p_2(x, y), ..., p_n(x, y))^T \in V \quad (x \in V, \ y \in V^{m-1}),
\]

(45)

where

\[
 p_s(x, y) = x_s^T Q_s(x, y) + q_s(x, y) \quad (s = 1, \ldots, n),
\]

(46)
$r \geq 5$ is an odd integer, $Q_s(x,y)$ is a polynomial which is positive for all $x$ and $y$, and $q_s(x,y)$ is a homogeneous polynomial of degree $3 \leq t < r$ (we assume that $t$ is also odd). Choose (for a moment, arbitrary) matrices $A_j : V \rightarrow V$, $j = 0, 1, \ldots, m - 1$, and define

$$f(x, y) := A_0x + \sum_{j=1}^{m-1} A_jy^j + F(x, y) \quad (x \in V, \ y \in V^{m-1}).$$

(47)

Put $q_o = \min\{Q_s(x,y) : s = 1, \ldots, n\}$. Since $Q_s$ is a polynomial, $q_o > 0$, thus, for $|y| \leq |x|$, one has:

$$x \cdot f(x, y) \geq x \cdot A_0x + x \cdot \sum_{j=1}^{m-1} A_jy^j + \sum_{s=1}^{n} (x_s^{r+1}Q_s(x, y)) + x_sq_s(x, y)$$

$$\geq \sum_{s=1}^{n} x_s^{r+1}q_o - |A_0||x|^2 - \sum_{j=1}^{m-1} |A_j||y^j||x| - \sum_{s=1}^{n} |x_sq_s(x, y)|$$

$$\geq q_o|x|^{r+1} - \left(|A_0| + \sum_{j=1}^{m-1} |A_j|\right)|x|^2 - C|x|^{t+1} - D$$

Since $r \geq 5$ and $t < r$, it follows that there exists a constant $R > 0$ such that if $|x| > R$, then

$$x \cdot f(x, y) \geq q_o|x|^{r+1} - \left(|A_0| + \sum_{j=1}^{m-1} |A_j|\right)|x|^2 - C|x|^{t+1} - D > 0,$$

which implies that the assumption $(A_3)$ is satisfied.

(b) Symmetries. Assume, further, that $V = \mathbb{R}^n$ is an orthogonal $\Gamma$-representation, where $\Gamma \leq S_n$ acts on the vectors in $\mathbb{R}^n$ by permuting their coordinates. Then, one can easily identify the $\Gamma$-symmetric interactions between the coordinates in $V$. Following these interactions, one can easily chose monomials for $Q_s$ and $q_s$, $s = 1, \ldots, n$, in such a way that the map $f$ given by (47) satisfies condition $(A_1)$.

Assume, in addition, that the $\Gamma$-isotypic decomposition of $V$ is given by

$$V = V_0 \oplus \cdots \oplus V_r,$$

where $V_l$, $l = 0, 1, 2, \ldots, r$, is equivalent to an irreducible $\Gamma$-representation $V_l$ of a real type. Then clearly, for each $j = 0, 1, 2, \ldots, m - 1$ and $l = 0, 1, 2, \ldots, r$, there exists $\mu^j_l \in \mathbb{R}$ such that

$$A_j|_{V_l} = \mu^j_l \text{Id}_{V_l}$$

(just consider any $A_j$ in $V$ in an “isotypic” basis). Then, since $t > 2$, it follows that the map $f$ also satisfies $(A^*_4)$.

Observe that condition $(A_2)$ is nicely compatible with other assumptions on $f$ listed in item (a) – it is enough to assume, in addition, that $Q_s$ is an even polynomial for all $s$. Also, condition $(R)$ can be easily satisfied.

(c) $O(2) \times D_6 \times \mathbb{Z}_2$-equivariant problems. Let $\Gamma := D_6$ be the dihedral group given by

$$D_6 := \{1, \gamma, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \kappa, \gamma\kappa, \gamma^2\kappa, \gamma^3\kappa, \gamma^4\kappa, \gamma^5\kappa\},$$
with \( \gamma := e^{\frac{2\pi}{3}} \) and \( \kappa z = z \), \( z \in \mathbb{C} \). Let \( V := \mathbb{R}^6 \) and assume that \( \Gamma \) acts on \( V \) by permuting the coordinates of the vectors \( x = (x_1, x_2, \ldots, x_6) \) the same way as it permutes vertices of a regular hexagon on the plane. Take \( V^6 = V \times \cdots \times V \) and assume that \( \Gamma \) acts diagonally on it. The list of irreducible \( D_6 \)-representation is given by the table of their characters:

| \( \chi \) | (1) | (\( \kappa \)) | (\( \gamma \)) | (\( \gamma^2 \)) | (\( \kappa \gamma \)) | (\( \gamma^3 \)) |
|---|---|---|---|---|---|---|
| \( \chi_1 \) | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \chi_2 \) | 1 | -1 | -1 | 1 | 1 | -1 |
| \( \chi_3 \) | 1 | -1 | 1 | 1 | -1 | 1 |
| \( \chi_4 \) | 1 | 1 | -1 | 1 | -1 | -1 |
| \( \chi_5 \) | 2 | 0 | 1 | -1 | 0 | 2 |
| \( \chi_6 \) | 2 | 0 | -1 | -1 | 0 | 2 |

Table 1: Character Table of \( D_6 \)

Notice that the character \( \chi \) of the representation \( V \) is:

| \( \chi \) | (1) | (\( \kappa \)) | (\( \gamma \)) | (\( \gamma^2 \)) | (\( \kappa \gamma \)) | (\( \gamma^3 \)) |
|---|---|---|---|---|---|---|
| \( \chi_6 \) | 2 | 0 | -1 | -1 | 0 | 2 |

which implies the following \( D_6 \times \mathbb{Z}_2 \)-isotypic decomposition of \( V \):

\[ V := V^+ \oplus V^- \oplus V^+_\chi \oplus V^-_\chi. \tag{48} \]

Let \( f : V^6 \to V \) be given by (47) and satisfy conditions \((A_1)\)–\((A_3)\) (i.e. in this case \( m = 6 \) and \( n = 6 \)). Assume that

\[ Df(0) = [\hat{d}A, a\hat{A}, b\hat{A}, c\hat{A}, b\hat{A}, a\hat{A}], \tag{49} \]

where

\[ \hat{A} = \begin{bmatrix} -1 & 1/10 & 0 & 0 & 0 & 1/10 \\ 1/10 & -1 & 1/10 & 0 & 0 & 0 \\ 0 & 1/10 & -1 & 1/10 & 0 & 0 \\ 0 & 0 & 1/10 & -1 & 1/10 & 0 \\ 0 & 0 & 0 & 1/10 & -1 & 1/10 \\ 1/10 & 0 & 0 & 0 & 1/10 & -1 \end{bmatrix}, \tag{50} \]

and the numbers \( a, b, c \) and \( d \) will be specified later. One can easily verify that \( \hat{A} \) is \( \Gamma \)-equivariant and

\[ \sigma(\hat{A}) = \{ 1\mu = -8/10, 2\mu = -9/10, 3\mu = -11/10, 4\mu = -12/10 \}. \tag{51} \]

Since each isotypic component in (48) is irreducible, it follows that condition \((A'_4)\) is automatically
satisfied, and one can verify that:

\[ \mu_j^1 := \begin{cases} 
  d \cdot 1 \mu & \text{if } j = 0 \\
  a \cdot 1 \mu & \text{if } j = 1, 5 \\
  b \cdot 1 \mu & \text{if } j = 2, 4 \\
  c \cdot 1 \mu & \text{if } j = 3 
\end{cases} \]

\[ \mu_j^2 := \begin{cases} 
  a \cdot 3 \mu & \text{if } j = 1, 5 \\
  b \cdot 3 \mu & \text{if } j = 2, 4 \\
  c \cdot 3 \mu & \text{if } j = 3 
\end{cases} \]

\[ \mu_j^3 := \begin{cases} 
  d \cdot 2 \mu & \text{if } j = 0 \\
  a \cdot 2 \mu & \text{if } j = 1, 5 \\
  b \cdot 2 \mu & \text{if } j = 2, 4 \\
  c \cdot 2 \mu & \text{if } j = 3 
\end{cases} \]

\[ \mu_j^4 := \begin{cases} 
  d \cdot 4 \mu & \text{if } j = 0 \\
  a \cdot 4 \mu & \text{if } j = 1, 5 \\
  b \cdot 4 \mu & \text{if } j = 2, 4 \\
  c \cdot 4 \mu & \text{if } j = 3 
\end{cases} \]

(cf. (51)). Also (cf. (36)), \( m^l = 1 \) for all \( l = 1, 4, 5, 6 \). In addition (cf. (29)-(33)), one has:

\[ d + a\gamma^k + b\gamma^{2k} + c\gamma^{3k} + b\gamma^{4k} + a\gamma^{5k} = \begin{cases} 
  d + 2a + 2b + c & k = 0 \text{ mod } 6 \\
  d + a - b - c & k = 1 \text{ mod } 6 \\
  d - a - b - c & k = 2 \text{ mod } 6 \\
  d - 2a + 2b - c & k = 3 \text{ mod } 6 \\
  d - a - b + c & k = 4 \text{ mod } 6 \\
  d + a - b - c & k = 5 \text{ mod } 6 
\end{cases} \]

Take

\[ a := 4, \quad b := 1, \quad c := 3, \quad d := 6.9. \] (52)

Then (cf. (51)-(55)), one has the following table:

| \( k \backslash \ell \) | 1 | 4 | 5 | 6 |
|------------------------|---|---|---|---|
| 0                      | − | − | − | − |
| 1                      | − | − | − | − |
| 2                      | + | − | − | − |
| 3                      | + | + | + | + |

Table 2: Eigenvalues of \( \mathcal{A} \)

Notice that there is no non-negative eigenvalue for \( k > 2 \). Also, \( 0 \notin \mathcal{A} \). Therefore (cf. (38)),

\[ G\text{-deg}(\mathcal{A}, B(\mathcal{A})) = \deg_{V_{0,1}} \cdot \deg_{V_{0,4}} \cdot \deg_{V_{0,5}} \cdot \deg_{V_{0,6}} \cdot \deg_{V_{1,1}} \cdot \deg_{V_{1,4}} \cdot \deg_{V_{1,5}} \cdot \deg_{V_{1,6}} \cdot \deg_{V_{2,4}} \cdot \deg_{V_{2,5}} \cdot \deg_{V_{2,6}}. \] (53)

The orbit types of maximal kind occurring in \( V_{k,l}^- \) with \( k > 0 \) and \( l = 1, 4, 5, 6 \) related to (53) are:
Using the argument similar to the one utilized in the proof of Theorem 5.1, one can establish the following "degenerate" version of the main result.

### Remark 5.2.

(i) For any subgroup $S \leq D_6$, the symbol $S^p$ stands for $S \times \mathbb{Z}_2$.

(ii) Given two subgroups $H \leq O(2)$ and $K \leq D_6^p$, we refer to Appendix, item (a), for the "amalgamated notation" $H^Z \times K^R$.

(iii) We refer to \cite{3} for the explicit description of the (sub)groups $\tilde{D}_b$, $D^d_k$, $D^d_k$, and $\mathbb{Z}_2$.

We summarize our considerations in the statement following below.

### Proposition 5.3.

Let $V$ be a $D_6 \times \mathbb{Z}_2$-representation admitting isotypic decomposition \cite{48} and let $V^g$ be equipped with the diagonal $D_6 \times \mathbb{Z}_2$-action. Let $f : V^u \to V$ be given by \eqref{49} with $Q_s$ being an even polynomial ($s = 1, \ldots, n$), and suppose that conditions (R) and \eqref{51} are satisfied. Assume, further, that $Df(0)$ is given by \eqref{50} and \eqref{52}. Let $(\mathcal{H})$ be one of the orbit types listed in \cite{51}. Then, system \eqref{53} admits a non-constant $2\pi$-periodic solution with the extended orbit type $(\mathcal{H})$.

### 5.3 Main result: degenerate version

Using the argument similar to the one utilized in the proof of Theorem 5.1, one can establish the following "degenerate" version of the main result.

### Theorem 5.4.

Assume that $f : V^m \to V$ satisfies conditions (R), \eqref{51} (A$_3$) and (A$'_4$). Put

$$\mathcal{C} := \left\{ k \in \mathbb{N} \cup \{0\} : k^2 = -\mu_0 - \sum_{j=1}^{r} 2 \cos \frac{2\pi j k}{m} \mu_j + \varepsilon_m \mu^l \right\}, \quad l = 0, 1, 2, \ldots, r, \quad r := \left\lfloor \frac{m-1}{2} \right\rfloor$$

and choose $s \in \mathbb{N}$ such that

$$\mathcal{C} \cap \{(2k-1)s : k \in \mathbb{N}\} = \emptyset.$$  \quad (55)

Assume that there exist $k \in \mathbb{N}$ and an orbit type $(H_o)$ in $\Phi_0(G, \mathcal{C})$ of maximal kind such that $n^{(2k-1)}$ is odd (see Definitions \cite{4}, (a) and \cite{4}). Then, system \eqref{53} admits a non-constant $2\pi$-periodic solution with the extended orbit type $(H_o)$.

### Remark 5.5.

It is easy to extend the setting considered in Proposition 5.3 to the one supporting Theorem 5.4. We leave this task to the reader.
A Equivariant Brouwer Degree Background

(a) Amalgamated notation. Given two groups \( G_1 \) and \( G_2 \), the well-known result of É. Goursat (see [7, 10]) provides the following description of a subgroup \( \mathcal{H} \leq G_1 \times G_2 \): there exist subgroups \( H \leq G_1 \) and \( K \leq G_2 \), a group \( L \), and two epimorphisms \( \varphi : H \rightarrow L \) and \( \psi : K \rightarrow L \) such that

\[
\mathcal{H} = \{(h, k) \in H \times K : \varphi(h) = \psi(k)\}.
\]

The widely used notation for \( \mathcal{H} \) is

\[
\mathcal{H} := H \varphi \times^\psi_L K, \tag{56}
\]

in which case \( H \varphi \times^\psi_L K \) is called an amalgamated subgroup of \( G_1 \times G_2 \).

In this paper, we are interested in describing conjugacy classes of \( \mathcal{H} \)-pairs and let \( \mathbb{A} \)-pairs. The symbol \( (\mathcal{H}, \mathcal{K}) \) stands for the Burnside ring of \( \mathcal{H} \). The following notation is called the \( \mathcal{H} \)-isotypic decomposition of \( V \) if \( \mathcal{H} \)-invariant subsets of \( V \) are \( \mathcal{H} \)-isotypic.

Let \( \mathcal{H} \) be a finite-dimensional \( \mathcal{G} \)-space, then a continuous map \( f : X \rightarrow Y \) is called equivariant if \( f(gx) = gf(x) \) for each \( x \in X \) and \( g \in \mathcal{G} \). Let \( V \) be a finite-dimensional \( \mathcal{G} \)-representation (without loss of generality, orthogonal). Then, \( V \) decomposes into a direct sum

\[
V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \tag{58}
\]

where each component \( V_i \) is modeled on the irreducible \( \mathcal{G} \)-representation \( \mathcal{V}_i \), \( i = 0, 1, 2, \ldots, r \), that is, \( V_i \) contains all the irreducible subrepresentations of \( V \) equivalent to \( \mathcal{V}_i \). Decomposition \( \mathcal{M}^2 \) is called \( \mathcal{G} \)-isotypic decomposition of \( V \).

(b) Axioms of Equivariant Brouwer Degree. Denote by \( \mathcal{M}^2 \) the set of all admissible \( \mathcal{G} \)-pairs and let \( A(\mathcal{G}) \) stand for the Burnside ring of \( \mathcal{G} \) (see Introduction, item (b)). The following result (cf. [3]) can be considered as an axiomatic definition of the \( \mathcal{G} \)-equivariant Brouwer degree.

Theorem A.1. There exists a unique map \( \mathcal{G} \)-deg : \( \mathcal{M}^2 \rightarrow A(\mathcal{G}) \), which assigns to every admissible \( \mathcal{G} \)-pair \( (f, \Omega) \) an element \( \mathcal{G} \)-deg\( (f, \Omega) \) \( \in A(\mathcal{G}) \)

\[
\mathcal{G} \text{-deg}(f, \Omega) = \sum_{(H)} n_H(H) = n_{H_1}(H_1) + \cdots + n_{H_m}(H_m), \tag{59}
\]

satisfying the following properties:

(Existence) If \( \mathcal{G} \)-deg\( (f, \Omega) \neq 0 \), i.e., \( n_{H_i} \neq 0 \) for some \( i \) in \( \Omega \), then there exists \( x \in \Omega \) such that \( f(x) = 0 \) and \( (G_x) \geq (H_i) \).

(Additivity) Let \( \Omega_1 \) and \( \Omega_2 \) be two disjoint open \( \mathcal{G} \)-invariant subsets of \( \Omega \) such that \( f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2 \). Then,

\[
\mathcal{G} \text{-deg}(f, \Omega) = \mathcal{G} \text{-deg}(f, \Omega_1) + \mathcal{G} \text{-deg}(f, \Omega_2).
\]
(Homotopy) If \( h : [0, 1] \times V \to V \) is an \( \Omega \)-admissible \( G \)-homotopy, then
\[
G\text{-deg}(h_t, \Omega) = \text{constant}.
\]

(Normalization) Let \( \Omega \) be a \( G \)-invariant open bounded neighborhood of \( 0 \) in \( V \). Then,
\[
G\text{-deg}(\text{Id}, \Omega) = (G).
\]

(Multiplicativity) For any \( (f_1, \Omega_1), (f_2, \Omega_2) \in M^G \),
\[
G\text{-deg}(f_1 \times f_2, \Omega_1 \times \Omega_2) = G\text{-deg}(f_1, \Omega_1) \cdot G\text{-deg}(f_2, \Omega_2),
\]
where the multiplication ‘\( \cdot \)’ is taken in the Burnside ring \( A(G) \).

(Recurrence Formula) For an admissible \( G \)-pair \( (f, \Omega) \), the \( G \)-degree can be computed using the following Recurrence Formula:
\[
n_H = \frac{\deg(f^H, \Omega^H) - \sum_{(K) \geq (H)} n_K n(H, K) |W(K)|}{|W(H)|},
\]
where \(|X|\) stands for the number of elements in the set \( X \) and \( \deg(f^H, \Omega^H) \) is the Brouwer degree of the map \( f^H := f|_{\Omega^H} \) on the set \( \Omega^H \subset V^H \).

The \( G\text{-deg}(f, \Omega) \) is called the \( G \)-equivariant Brouwer degree of \( f \) in \( \Omega \).

(c) Computation of Brouwer equivariant degree. Put \( B(V) := \{ x \in V : |x| < 1 \} \). For each irreducible \( G \)-representation \( V_i \), \( i = 0, 1, 2, \ldots \), define
\[
\deg_{V_i} := G\text{-deg}(-\text{Id}, B(V_i)),
\]
and call it the basic degree.

Consider a \( G \)-equivariant linear isomorphism \( T : V \to V \) and assume that \( V \) has a \( G \)-isotypic decomposition \([58]\). Then, by the Multiplicativity property,
\[
G\text{-deg}(T, B(V)) = \prod_{i=0}^r G\text{-deg}(T_i, B(V_i)) = \prod_{i=0}^r \prod_{\mu \in \sigma_-(T)} (\deg_{V_i})^m_i(\mu)
\]  
where \( T_i = T|_{V_i} \) and \( \sigma_-(T) \) denotes the real negative spectrum of \( T \), i.e., \( \sigma_-(T) = \{ \mu \in \sigma(T) : \mu < 0 \} \).

Notice that the basic degrees can be effectively computed from \([60]\):
\[
\deg_{V_i} = \sum_{(H)} n_H(H),
\]
where
\[
n_H = \frac{(-1)^{\dim V_i^H} - \sum_{H<K} n_K n(H, K) |W(K)|}{|W(H)|}.
\]

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