Robust Principal Component Analysis Using a Novel Kernel Related with the $L_1$-Norm

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Abstract — We consider a family of vector dot products that can be implemented using sign changes and addition operations only. The dot products are energy-efficient as they avoid the multiplication operation entirely. Moreover, the dot products induce the $\ell_1$-norm, thus providing robustness to impulsive noise. First, we analytically prove that the dot products yield symmetric, positive semi-definite generalized covariance matrices, thus enabling principal component analysis (PCA). Moreover, the generalized covariance matrices can be constructed in an Energy Efficient (EEF) manner due to the multiplication-free property of the underlying vector products. We present image reconstruction examples in which our EEF PCA method result in the highest peak signal-to-noise ratios compared to the ordinary $\ell_2$-PCA and the recursive $\ell_1$-PCA.

Index Terms — Principal Component Analysis (PCA), $\ell_1$-norm kernel, robust PCA, multiplication-free methods.

I. INTRODUCTION

In data analysis problems with a large number of input variables, dimension reduction methods are very useful to reduce the size of the input by decreasing the complexity of the problem while sacrificing negligible accuracy. Principal Component Analysis (PCA) and related methods are widely used in data analysis field as dimension reduction techniques [1]–[4]. In most problems, the lower dimensional subspaces that are obtained using the eigenvectors effectively capture the nature of the input data structure. As a result, PCA can be also used in a variety of applications including novelty detection [5], [6], data clustering [7]–[13], denoising [14]–[17] and outlier detection [18]–[21].

Although the conventional PCA based on the regular dot-product and the $\ell_2$-norm has successfully solved many problems, it is sensitive to outliers in data because the effects of the outliers are not suppressed by the $\ell_2$-norm. It turns out that $\ell_1$-PCA is more robust to outliers and it can be iteratively solved in $O(NrK-K+1)$ for $D$ dimensional vectors, where $N$ is the number of data vectors, $1 \leq K < r =$ (rank of the $N \times D$ data matrix) [22]. Therefore, researchers proposed iterative methods to compute $\ell_1$-PCA to achieve robustness against outliers in data [22, 23]. The recursive $\ell_1$-PCA method requires some parameters to be properly adjusted. On the other hand, the proposed kernel based approach does not need any hyperparameters to be adjusted. This is because we construct a sample covariance matrix using the kernel and obtain the eigenvalues and eigenvectors to define the orthogonal linear transformation instead of solving an optimization problem.

We recently introduced a family of operators related with $\ell_1$-norm to extract features from image regions and to design Additive neural Networks (AddNet) in a wide range of computer vision applications [24]–[27]. We call the new family of operators Energy-Efficient (EEF) operators because they do not require any multiplications which consume more energy compared to additions and binary operations in most processors. Instead of a multiplication, the operators use the sign of multiplication and either sum the absolute values of operands, or calculate the minimum or maximum of operands. When we construct dot-product like operations from the EEF operators they induce the $\ell_1$-norm. Details of the EEF-operator are provided in Section II.

In this paper, we define three multiplication-free dot products and construct the corresponding multiplication-free covariance matrices. The fact that the underlying dot product is not an ordinary Euclidean inner product implies that the covariance matrix is not necessarily symmetric and positive semi-definite. Nevertheless, we analytically prove that two of our vector products yield symmetric and positive semi-definite covariances. Correspondingly, we find the eigenvalues and eigenvectors of the matrices as in regular $\ell_2$-PCA. The resulting eigenvectors are orthogonal to each other and one can perform orthogonal projection onto the subspace formed by the eigenvectors to reduce the dimension, perform denoising and other similar PCA applications used in data analysis. In addition, the dot products defined by the operators can be computed without performing any multiplications. Consequently, the matrices of the new kernels can be computed in an energy efficient manner because the new kernels are based on sign operations, binary operations and additions.

II. ENERGY-EFFICIENT (EEF) VECTOR PRODUCTS

In this section, we motivate and introduce the family of multiplication-free dot products and establish their relationship to the $\ell_1$-norm.

A. Motivation

Let $w = [w_1 \cdots w_n]^T \in \mathbb{R}^{D \times 1}$ and $x = [x_1 \cdots x_n]^T \in \mathbb{R}^{D \times 1}$ be two $D$-dimensional column vectors. The standard
Euclidean inner product is defined as
\[ \langle w, x \rangle = w^T x = \sum_{i=1}^{D} w_i x_i \] (1)

Note that because the product \( \langle \cdot, \cdot \rangle \) induces the \( \ell_2 \)-norm in the sense that for any \( x \), we have \( \langle x, x \rangle = \| x \|^2 = \sum_{i=1}^{D} |x_i|^2 \).

The \( D \) multiplication operations that appear in the inner product Eq. (1) may be costly in terms of energy consumption and time. The existence of multiplications are also undesirable in the presence of outliers: For example, if a component is an outlier with a relatively large magnitude, multiplication will further amplify its effect, making the result of the inner product unreliable. In this context, it has been recently observed that in many applications, \( \ell_1 \)-based methods outperform \( \ell_2 \)-based methods thanks to their better resilience against outliers or impulse-type noise. These observations motivate us to define the new dot products that induce the \( \ell_1 \)-norm. The new dot products should avoid multiplications both for the sake of computational and energy efficiency as well as robustness.

B. Multiplication-Free (MF) Dot Products

In this work, we will evaluate the performance of three different MF operators, described in what follows. Given a real number \( a \in \mathbb{R} \), let
\[ \text{sign}(a) = \begin{cases} -1, & a < 0, \\ 0, & a = 0, \\ 1, & a > 0, \end{cases} \] (2)
denote the sign of \( a \). Unlike [26] where we define \( \text{sign}(0) = 1 \) or \( \text{sign}(0) = -1 \) to take advantage of bit-wise operations, we utilize the standard signum function for better precision here.

First, we introduce our original MF dot product [24], [25]. It is defined as
\[ w^T \oplus_{mf} x = \sum_{i=1}^{D} \text{sign}(w_i x_i)(|w_i| + |x_i|) \] (3)

Note that the only multiplication operations that appears in Eq. 3 correspond to sign changes and can be implemented with very low complexity. For this reason, we do not count the sign changes towards multiplication operations and thus call Eq. (3) an MF dot product. It can easily be verified that the product in Eq. (3) induces a scaled version of the \( \ell_1 \)-norm as
\[ x^T \oplus_{mf} x = \sum_{i=1}^{n} |x_i| + |x_i| = 2\| x \|_1 \] (4)

Notice that the original MF dot product conducts scale of \( 2 \), we are seeking another \( \ell_1 \)-norm based method without any scaling. We then define a min-based MF dot product:
\[ w^T \odot x = \sum_{i=1}^{D} \text{sign}(w_i x_i) \min(|w_i|, |x_i|) \] (5)

and its variation:
\[ w^T \odot_m x = \sum_{i=1}^{D} \mathbf{1}(\text{sign}(w_i) = \text{sign}(x_i)) \min(|w_i|, |x_i|) \] (6)

Here, \( \mathbf{1}(\cdot) \) is the indicator function. The variant is related to the XX similarity measure [28]. In Eq. (6), components of opposite sign \( \text{sign}(w_i) \neq \text{sign}(x_i) \) have no contribution towards the dot product, while in Eq. (5), they contribute as a subtractive term. Both of them induce \( \ell_1 \)-norm as
\[ x^T \odot x = \sum_{i=1}^{n} \min(|x_i|, |x_i|) = \| x \|_1 \] (7)
\[ x^T \odot_m x = \sum_{i=1}^{n} \min(|x_i|, |x_i|) = \| x \|_1 \] (8)

Vector dot products described above can be extended to matrix multiplications as follows: Let \( W \in \mathbb{R}^{n \times m} \) and \( X \in \mathbb{R}^{p \times n} \) be arbitrary matrices. We then define
\[ W^T \oplus X \triangleq \begin{bmatrix} w_1^T \oplus x_1 & w_1^T \oplus x_2 & \ldots & w_1^T \oplus x_p \\ w_2^T \oplus x_1 & w_2^T \oplus x_2 & \ldots & w_2^T \oplus x_p \\ \vdots & \vdots & \cdots & \vdots \\ w_m^T \oplus x_1 & w_m^T \oplus x_2 & \ldots & w_m^T \oplus x_p \end{bmatrix} \] (9)

where \( \oplus \in \{ \oplus_{mf}, \odot, \odot_m \} \), \( w_i \) is the \( i \)th column of \( W \) for \( i = 1, 2, \ldots, m \) and \( x_j \) is the \( j \)th column of \( X \) for \( j = 1, 2, \ldots, p \). In brief, the definition is similar to the matrix production \( W^T X \) by only changing the element-wise product to element-wise MF-operation or element-wise min-operation.

III. ROBUST PRINCIPAL COMPONENT ANALYSIS

Suppose that we collect members of a \( D \)-dimensional dataset \( \{x_1, \ldots, x_N\} \) to a \( D \times N \) matrix \( X = [x_1 \ x_2 \ \ldots \ x_N] \in \mathbb{R}^{D \times N} \). The well-known \( \ell_2 \)-PCA method relies on investigating the eigendecomposition of the sample covariance matrix
\[ C = XX^T. \] (10)

We have omitted normalization by the number of elements \( N \) of the dataset as it will not change the final eigenvectors and the order of eigenvalues. Elementary linear algebra guarantees that \( C \) has non-negative eigenvalues (i.e. \( C \) is positive semi-definite) and thus the eigenvector corresponding to the \( i \)th largest eigenvalue becomes the \( i \)th principal vector.

In this work, we propose to investigate the analogue of Eq. (10) for MF operators. In other words, we consider the eigendecomposition of
\[ A = X \oplus X^T, \] (11)
where \( \oplus \in \{ \oplus_{mf}, \odot, \odot_m \} \). Matrix \( A \) is called as MF-covariance matrix. Note that the ordinary matrix product in Eq. (10) is replaced by the MF product in Eq. (11). On the other hand, since \( A \) is no longer constructed using \( \ell_2 \)-products, it is not guaranteed to be symmetric or positive semi-definite. Still, we have the following result.

**Theorem 1.** Let \( \odot \in \{ \odot, \odot_m \} \). Then, \( A = X \odot X^T \) is symmetric and positive semi-definite for any \( X \).

The proof can be found in the appendix. In particular, the theorem shows that \( \odot \) and \( \odot_m \) describe Mercer-type kernels.
Algorithm 1 Algorithm for $L_1$ PCA using MF operators

| Input: $X = [x_1 \ x_2 \ ... \ x_N] \in \mathbb{R}^{D \times N}$ | Output: $W \in \mathbb{R}^{D \times K}$ |
|---------------------------------------------------------------|----------------------------------|
| 1: Construct the MF covariance matrix $A$ of $X$ based on Eq. 1  | 2: $[W, D] = \text{eigs}(A, K)$ |
| 3: return $W$. Comment: Step 2 represents eigendecomposition of $A$ and returns a subset of diagonal matrix $D$ of $K$ largest eigenvalues and matrix $W$ whose columns are the corresponding right eigenvectors, so that $AW = WD$. Compared with the conventional $L_2$-PCA Algorithm, we can see that the only difference is at Step 1. We replace the standard covariance matrix by the multiplication-free covariance matrix. |

Theorem [1] paves the way for extending PCA to multiplication-free operators $\odot$ and $\odot_m$, as shown via Algorithm [1].

The conclusions of Theorem [1] do not hold for the $\odot_{mf}$ operator. A counterexample is provided by the dataset $x_1 = [1 \ 2]^T$, $x_2 = [-1 -2]^T$, which yields a generalized covariance matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$ with a negative determinant, and thus not positive semi-definite.

IV. EXPERIMENTAL RESULTS

In this section, we carry out an image reconstruction and denoising experiment using the EEF kernel based PCAs, $\ell_2$-PCA and the recursive $\ell_1$-PCA to illustrate the robustness of the EEF kernel introduced in Section III. Image reconstruction example is the same as the experiment in [22]. The source code of [22] is available in [29], so we only set the tolerance parameter of the recursive $\ell_1$-PCA method as $1 \times 10^{-8}$ as suggested by the author P. Markopoulos. For convenience, we name our method based on Eq. (3) as "MF-$\ell_1$"-PCA, method based on Eq. (5) as "min-$\ell_1$-PCA-1" and method based on Eq. (6) as "min-$\ell_1$-PCA-2", respectively, in Table I and Table II.

In the first row of Fig. 1 we have three $128 \times 128 = 16384$ "clean" gray-scaled images ($I \in \{1, \frac{1}{255}, ..., \frac{255}{255}\}^{128 \times 128}$). We assume that the image $I$ is not available but we have $N = 10$ occluded versions $I_1, I_2, ..., I_{10}$, are available as shown in the second row of Fig. 1a and Fig. 1b. The occluded images are created by partitioning the original image $I$ into sixteen tiles of size $32 \times 32$ and replacing three arbitrarily selected tiles by $32 \times 32$ gray-scale-noise patches. The noise patches are in the uniformly random distribution in the interval $(0, 1)$.

In the second experiment, we add salt and pepper noise to images and restore the original images using various PCA methods. We assume that the image $I$ is not available but we have $N = 10$ corrupted versions $I_1, I_2, ..., I_{10}$, are available as shown in the third column (Fig. 1c) and the forth column (Fig. 1d) of Fig. 1 respectively. The corrupted images are created by adding salt and pepper noise to the original image $I$ with noise density 0.1. In other words, this affects 10% pixels by making them either 0 or 1 assuming that the image pixel values are in the range of $[0, 1]$.

We perform PCA on the set of $V = [v_1 \ v_2 \ ... \ v_{10}]$, where $v_i = \text{vec}(I_i), i = 1, 2, ..., 10$, is the vector form of $I_i$. In this way, we obtain the eigenvector matrix $W \in \mathbb{R}^{16384 \times 2}$ of the covariance or the MF-covariance matrices of $(V - \tilde{V})$. Then, we recover the image $I$ as

$$\hat{v}_i = WW^T(v_i - \tilde{v}) + \tilde{v} \quad (12)$$

$$\hat{I} = \text{mat}(\hat{v})_i \quad (13)$$

![Image 1](image1.png)

Fig. 1: Samples of image reconstruction results. Images in each columns are ordered as the original image (1st row), the noise patches occluded image (2nd row, 1st and 2nd columns) or salt-and-pepper noise corrupted image (2nd row, 3rd and 4th columns), results of $\ell_2$-PCA (3rd row), recursive $\ell_1$-PCA (4th row), MF-$\ell_1$-PCA (5th row), min-$\ell_1$-PCA-1 (6th row) and min-$\ell_1$-PCA-2 (7th row), respectively.
where \( \bar{v} \in \mathbb{R}^{16384 \times 1} \) is the mean value of \([v_1, v_2, \ldots, v_{10}]\), 0.5 or 0, \( \mathbf{I}_i \) is an arbitrary occluded image, and \( \text{mat}(\cdot) \) is the inverse transform of vec(\cdot) that reshapes a vector back to the matrix form. We calculate \( \hat{\mathbf{v}}_i \) in the method that returns the largest peak signal-to-noise-ratio (PSNR).

PSNR between the reconstructed image \( \hat{\mathbf{I}} \) and the original image \( \mathbf{I} \) as the following equations is used for evaluation in Table I and Table II.

\[
\text{MSE} = \text{mean}(\langle \hat{\mathbf{I}} - \mathbf{I} \rangle^2), \quad (14)
\]
\[
\text{PSNR} = 10 \log_{10} \left( \frac{\text{peakval}^2}{\text{MSE}} \right), \quad (15)
\]
where \((\cdot)^2\) is the element-wise square and “peakval” is the peak signal value. The higher the value of PSNR is, the better the reconstruction result is.

Our experiment is summarized in Algorithm 2. Results of these PCA methods are shown in Fig. 1 for four test images and their statistics are provided in Table I and Table II. Although which method works the best depends on the images, our three methods return larger PSNR than \( \ell_2 \)-PCA and the recursive \( \ell_1 \)-PCA in both experiments, and the two min-\( \ell_1 \)-PCAs are better than the MF-\( \ell_1 \)-PCA, globally. For example, the min-\( \ell_1 \)-PCA produces about 1.4dB better than the recursive \( \ell_1 \)-PCA in the salt-and-pepper noise removal experiment.

### Algorithm 2 Image Reconstruction Experiment

**Input:** \( N \) corrupted images \( \mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_N \in \mathbb{R}^{D \times D} \).

**Output:** Reconstructed image \( \hat{\mathbf{I}} \).

1: \( \text{for } i = 1, 2, \ldots, N \text{ do} \)
2: \( \mathbf{v}_i = \text{vec}(\mathbf{I}_i) \in \mathbb{R}^{D^2 \times 1}; \)
3: \( \text{end for} \)
4: \( \mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N] \in \mathbb{R}^{D^2 \times N}; \)
5: \( \bar{\mathbf{v}} = 0.5 \text{ or mean}(\mathbf{V}) \in \mathbb{R}^{D^2 \times 1}; \)
6: Run PCA on \( (\mathbf{V} - \bar{\mathbf{v}}) \) to obtain \( K \)-dominant eigenvector matrix \( \mathbf{W} = [w_1, w_2, \ldots, w_K] \in \mathbb{R}^{D^2 \times K}; \)
7: \( \hat{\mathbf{v}}_i = \mathbf{W}^T (\mathbf{v}_i - \bar{\mathbf{v}}) + \bar{\mathbf{v}} \in \mathbb{R}^{D^2 \times 1}; \)
8: \( \hat{\mathbf{I}} = \text{mat}(\hat{\mathbf{v}}_i) \in \mathbb{R}^{D \times D}; \)
9: \( \text{return } \hat{\mathbf{I}}. \)

Comment: In this experiment, \( N = 10, D = 128 \) and \( K = 2 \). Function mean(\cdot) is the mean of each row, so it returns a column vector. Function vec(\cdot) reshapes a matrix into the column vector form, and function mat(\cdot) is its inverse transform that reshapes a column vector back to the matrix form. \( (\mathbf{V} - \bar{\mathbf{v}}) \) is defined as \([\mathbf{v}_1 - \bar{\mathbf{v}} \mathbf{v}_2 - \bar{\mathbf{v}} \ldots \mathbf{v}_N - \bar{\mathbf{v}}]\).
We also compared the computational cost of the PCA algorithms to reconstruct an image in MATLAB. As it is shown in Table III, $\ell_2$-PCA is the fastest algorithm, while our proposed kernel methods are slightly slower than $\ell_2$-PCA but significantly faster than the recursive $\ell_1$-PCA. The recursive $\ell_1$-PCA is the slowest because it obtains the result by recursion, while $\ell_2$-PCA and our three methods return the result straight-forwardly. The reason why our kernel PCAs run a little slower than $\ell_2$-PCA is that, the time to construct an MFD-covariance matrix is slightly slower compared to the sample covariance matrix, which is optimized in MATLAB. The computational cost of eigenvalue-eigenvector computations are the same in both $\ell_2$-PCA and the proposed kernel-PCAs.

| Image Size | $L_2$-PCA | Recursive $L_1$-PCA [22] | Our PCAs$^a$ |
|------------|-----------|-------------------------|-----------|
| 32 x 32    | 0.02      | 3.57                    | 0.02      |
| 48 x 48    | 0.07      | 4.80                    | 0.09      |
| 64 x 64    | 0.21      | 6.50                    | 0.25      |
| 80 x 80    | 0.49      | 8.04                    | 0.50      |
| 96 x 96    | 1.03      | 10.81                   | 1.10      |
| 112 x 112  | 1.81      | 14.02                   | 1.91      |
| 128 x 128  | 3.24      | 20.38                   | 3.38      |

$^a$ Proposed kernel PCAs are comparable to the regular PCA. Due to space limitation, we list them in one column.

V. CONCLUSION

In this paper, we proposed three new robust PCA methods. We have reached the following conclusions: (i) Proposed novel kernel methods are more energy-efficient than $\ell_2$-PCA because their Gram matrices are computed without any multiplication operations. (ii) They do not suffer from outliers in the data as in $\ell_2$-PCA because they are based on the $\ell_1$-norm. (iii) They do not require any hyper-parameter optimization as in the recursive $\ell_1$-PCA [22] because their Gram matrices are straightforward to compute as described in Eq. (11).

We compared the new kernel-based methods with the $\ell_2$-PCA and the recursive $\ell_1$-PCA on an image reconstruction and salt-and-pepper noise removal tasks and found out that our min-$\ell_1$-PCAs returns the largest PSNR among these methods in most scenarios.

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APPENDIX

Let \( x, y \in \mathbb{R}^N \). We define the min-operator \( \oplus : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \) as following

\[
x \oplus y := \sum_{i=1}^{N} \text{sgn}(x_i y_i) \min(|x_i|, |y_i|) \quad (16)
\]

In the following we will show that the operator \( \oplus \) defines a valid kernel \( K(x, y) \). A symmetric function \( K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \) is a kernel iff

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j K(x_i, x_j) \geq 0 \quad (17)
\]

for any reals \( a_i, a_j \) and for any vectors \( x_i, x_j \in \mathbb{R}^N \). In our case, we are interested in proving that \( K(x_i, x_j) = x_i \oplus x_j \) satisfies Eq. (17).

Define a matrix \( K \in \mathbb{R}^{N \times N} \) such that \( K_{ij} = \text{sgn}(x_i x_j) \text{min}(|x_i|, |x_j|) \). Proving that \( K(., .) \) is a valid kernel is equivalent to proving that the matrix \( K \) is positive semi-definite.

We will use the following facts to construct our proof that \( \oplus \) is a kernel:

**Theorem 2 (Schur product theorem).** [30] Let \( A, B \in \mathbb{R}^{N \times N} \) be two positive semi-definite matrices, then their Hadamard product \( (A \odot B)_{ij} := A_{ij} B_{ij} \) is also positive semi-definite.

**Lemma 1.** [28] R. Nader, A. Bretto, B. Mourad and H. Abbas. "On the Positive Semi-definite Property of Similarity Matrices." Theoretical Computer Science Let \( x \in \mathbb{R}^N \) be a strictly positive vector. Then the matrix \( A_{ij} := \text{min}(x_i, x_j) \) is positive semi-definite.

Our claim is the following:

**Corollary 2.1.** Let \( x \in \mathbb{R}^N \). Then the matrix \( K_{ij} := \text{sgn}(x_i x_j) \text{min}(|x_i|, |x_j|) \) is positive semi-definite.

**Proof.** The matrix \( K_{ij} \) can be written as a hadamard product between matrix \( B_{ij} = \text{sgn}(x_i) \text{sgn}(x_j) \) and \( A_{ij} = \text{min}(|x_i|, |x_j|) \), the matrix \( B \) is a (rank-one) positive semi-definite matrix since it can be written as \( \text{sgn}(x) \text{sgn}(x)^T \). The matrix \( A \) is positive semi-definite according to Lemma 1. The Hadamard product \( K = A \odot B \) is positive semi-definite according to Theorem 1. Thus the \( \oplus \) operator defines a valid kernel. \( \square \)