Large N Limit on Langevin Equation:
Two-Dimensional Nonlinear Sigma Model

Riuji Mochizuki and Kazuhiro Yoshida

Department of Physics, Chiba University,
1-33 Yayoi-cho, Chiba 260, Japan

1. Abstract

We study the stochastic quantization of two-dimensional nonlinear sigma model in the large $N$ limit. Our main tool is the effective Langevin equation with which we investigate nonperturbative phenomena and derive the results which are same as the path integral approach gives.
Recently, the stochastic quantization method\textsuperscript{[1][2]} has been used to study spontaneous (dynamical) breakdown of symmetry,\textsuperscript{[3]} for example, Nambu-Jona-Lasinio model.\textsuperscript{[4]} Their main tool for investigation is the stochastic generating functional from which the effective potential is derived. On the contrary, the Langevin equation, which is very useful in perturbative calculation, loses its position because the dynamical symmetry breaking appears as a nonperturbative phenomenon. Nevertheless if we can use it even for nonperturbative calculation, we may expect that computation will become easier and the relation between perturbative and nonperturbative phenomena will become clearer. In this paper we introduce the effective Langevin equation for two-dimensional nonlinear sigma model in the large $N$ limit. With the help of it we derive some nonperturbative results which, of course, coincide with the results of the path-integral quantization.

We consider the two-dimensional nonlinear sigma model whose Lagrangian in Euclidean space-time is

$$L = \frac{1}{2} \partial_{\mu} \phi_i \partial_{\mu} \phi_i$$

\hspace{1cm} $i = 1, \cdots, N$ \hspace{1cm} $\mu = 1, 2$

with the constraint

$$\phi_i \phi_i = \frac{N}{g^2}.$$  \hspace{1cm} (2)

Introducing a Lagrange multiplier field $\sigma$ and taking account of the constraint, we rewrite the Lagrangian as

$$L_c = \frac{1}{2} \partial_{\mu} \phi_i \partial_{\mu} \phi_i + \frac{1}{2} \sigma (\phi_i \phi_i - \frac{N}{g^2}).$$ \hspace{1cm} (3)

In the stochastic quantization scheme fields depend on the fictitious time $t$ and their development along the fictitious time obeys the Langevin equation

$$\dot{\phi}_i(x, t) = -\frac{\partial L_c}{\partial \phi_i} |_{\phi=\phi(x,t)} + \eta_i(x, t),$$ \hspace{1cm} (4)

where a dot denotes fictitious-time derivative and $\eta_i(x, t)$ is a noise field whose corre-
lations are

\[ \langle \eta_i(x,t) \rangle_\eta = 0, \quad (5.a) \]

\[ \langle \eta_i(x,t) \eta_j(x',t') \rangle_\eta = 2 \delta_{ij} \delta(x-x') \delta(t-t'). \quad (5.b) \]

Here the average manipulation \( \langle \cdots \rangle_\eta \) is defined as

\[ \langle \cdots \rangle_\eta \equiv \frac{1}{Z} \int D\eta \langle \cdots \rangle \exp \left\{ -\frac{1}{4} \int \! d^2x \! dt \eta_i(x,t) \eta_i(x,t) \right\}, \quad (6.a) \]

\[ Z \equiv \int D\eta \exp \left\{ -\frac{1}{4} \int \! d^2x \! dt \eta_i(x,t) \eta_i(x,t) \right\}. \quad (6.b) \]

The Langevin equation for the field \( \phi_i \) with the Lagrangian (3) becomes

\[ \dot{\phi}_i(x,t) = \partial^2 \phi_i(x,t) - \sigma(x,t) \phi_i(x,t) + \eta_i(x,t). \quad (7) \]

The equation (7) is studied perturbatively.\(^5\) In that case the fields fluctuate around a point on the \((N-1)\)-dimensional sphere \( \langle \phi_i \phi_i \rangle = \frac{N}{g^2} \), so that the vacuum loses the original O(\(N\)) symmetry and, as a result, \((N-1)\) independent \( \phi \)'s become massless Nambu-Goldstone bosons. Nevertheless these particles suffer from infrared divergence in two dimensional space time. On the other hand it is well-known that if we study the model nonperturbatively in the large \( N \) limit by ordinary quantization methods, the O(\(N\)) symmetry recovers and the field gains a dynamical mass.\(^6\) In the following we introduce the effective Langevin equation for the two-dimensional nonlinear sigma model in the large \( N \) limit and investigate nonperturbative phenomena. The method used here is applicable to other models with dynamical symmetry breaking.\(^7\)

To evaluate \( \sigma(x,t) \) in the Langevin equation (7) we regard it as a functional of \( \phi_i(x,t) \) and insert the unity

\[ 1 = \int D\phi \ | J | \delta \{ \dot{\phi}_i(x,t) - \partial^2 \phi_i(x,t) + \sigma(x,t) \phi_i(x,t) - \eta_i(x,t) \} \]

into the generating functional (6.b) and integrate over the noise field \( \eta \). Here \( | J | \)
denotes the Jacobian factor which is a divergent constant\(^\dagger\). We obtain

\[
Z = \int D\phi \mid J \mid \exp\left\{-\frac{1}{2} \int d^2x dt L\right\}, \tag{8.a}
\]

where

\[
L = \frac{1}{2}(\dot{\phi}_i(x, t) - \partial^2 \phi_i(x, t) + \sigma(x, t)\phi_i(x, t))(\dot{\phi}_i(x, t) - \partial^2 \phi_i(x, t) + \sigma(x, t)\phi_i(x, t)). \tag{8.b}
\]

We thus derive a field equation for \(\sigma(x, t)\)

\[
\frac{\partial L}{\partial \sigma} = \phi_i(\dot{\phi}_i - \partial^2 \phi_i + \sigma \phi_i) = 0. \tag{9}
\]

Restoring the noise fields as the integration variables of the generating functional (8), we can obtain the Langevin equation (7) again,\(^8\) in which \(\sigma(x, t)\) is the solution of the field equation (9)

\[
\sigma(x, t) = -g^2 N \phi_i(x, t)(\dot{\phi}_i(x, t) - \partial^2 \phi_i(x, t)), \tag{10}
\]

where the constraint (2) is taken into account. In large \(N\) expansion, since \(\delta_{ii} = N\), we may expand \(\sigma(x, t)\) as

\[
\sigma(x, t) = \langle \sigma \rangle_\eta(t) + O(1/N) + \cdots. \tag{11}
\]

To observe dependence of \(\langle \sigma \rangle_\eta\) on the stochastic time, we call the Langevin equation for it;

\[
\dot{\sigma}(x, t) = -\frac{\partial L_c}{\partial \sigma} \mid_{\phi = \phi(x, t)} + \xi
= - (\phi_i(x, t)\phi_i(x, t) - \frac{N}{g^2}) + \xi.
\]

Noting that the expectation value of one noise should vanish and taking account of the constraint (2), the expectation value of the above Langevin equation vanish

\[
\langle \dot{\sigma} \rangle_{\eta, \xi} = 0,
\]

which holds at any fictitious time. Consequently \(\langle \sigma \rangle_\eta\) does not depend on the fictitious time and we will evaluate it at \(t \to \infty\) later. In the following we study the Langevin equation (7) exclusively in the large \(N\) limit \(N \to \infty\), which enables us to treat it nonperturbatively since the Langevin equation (7) no longer has any interaction terms.

\(^\dagger\) We use Ito’s calculation rule in this paper.
To investigate the dynamical breaking of the $O(N)$ symmetry, we decompose
\( \phi_i(x, t) \) into the expectation value
\( \langle \phi(x, t) \rangle_\eta \equiv \Phi_i(t) \), which is the order parameter of the $O(N)$ symmetry, and a fluctuating field \( \phi'_i(x, t) \). Then the Langevin equation (7) becomes
\[
\dot{\phi}'_i(x, t) = (\partial^2 - \langle \sigma \rangle_\eta)\phi'_i(x, t) - \{\dot{\Phi}_i(t) + \langle \sigma \rangle_\eta \Phi_i(t)\} + \eta_i(x, t).
\] (12)

\( \langle \sigma \rangle_\eta \) plays a role of square of dynamical mass in the equation (12). By definition of \( \phi'_i(k = 0, t) \), its expectation value must vanish and consequently
\[
\dot{\Phi}_i(t) = -\langle \sigma \rangle_\eta \Phi_i(t),
\] (13)

which explicitly shows a relation between the order parameter \( \Phi_i \) and the square of the dynamical mass \( \langle \sigma \rangle_\eta \). If \( \langle \sigma \rangle_\eta = 0 \), the equation (13) imposes no restrictions on \( \Phi_i \). Nevertheless it is easily known that the massless bosons cause infrared divergence. On the other hand if \( \langle \sigma \rangle_\eta > 0 \), \( \Phi_i \) vanishes when \( t \to \infty \), namely, the $O(N)$ symmetry does not break down.

We easily obtain the solution of the equation (12);
\[
\phi'_i(k, t) = \int_0^t d\tau e^{-(k^2 + \langle \sigma \rangle_\eta)(t-\tau)} \{\eta_i(k, \tau)\},
\] (14)

where
\[
\phi'_i(k, t) = \int \frac{d^2x}{2\pi} e^{ikx} \phi'_i(x, t),
\]
\[
\eta_i(k, t) = \int \frac{d^2x}{2\pi} e^{ikx} \eta_i(x, t).
\]

The correlations of \( \eta_i(k, t) \)'s are
\[
\langle \eta_i(k, t) \rangle_\eta = 0,
\] (15.a)
\[
\langle \eta_i(k, t) \eta_j(k', t') \rangle_\eta = 2\delta_{ij} \delta(k + k') \delta(t - t').
\] (15.b)

Let us derive a gap equation and compute \( \langle \sigma \rangle_\eta \). Using the relations (10) and (13),
\[ \langle \sigma \rangle_\eta \text{ is written as} \]

\[
\langle \sigma \rangle_\eta = \frac{g^2}{N} \lim_{t \to \infty} \langle \phi_i(x,t)(-\dot{\phi}_i(x,t) + \partial^2\phi_i(x,t)) \rangle_\eta \\
= \frac{g^2}{N} \lim_{t \to \infty} \langle \Phi_i(t) \langle \sigma \rangle_\eta \Phi_i(t) + \phi'_i(x,t) \partial^2\phi'_i(x,t) + N \delta^2(0) \rangle_\eta. \tag{16}
\]

In the second equality † we have taken account of \( \langle \phi'_i(k,t) \rangle_\eta = 0 \) and Ito’s calculation rule

\[
0 = \lim_{t \to \infty} \langle \dot{F}[\phi] \rangle_\eta = \lim_{t \to \infty} \left\{ \frac{\delta F}{\delta \phi_i} \dot{\phi}_i + \frac{\partial^2 F}{\partial \phi - i \partial \phi_i} \delta(0) \right\}. 
\]

We can easily compute the last term with the equations (14) and (15)

\[
\frac{g^2}{N} \lim_{t \to \infty} \langle \phi'_i(x,t) \partial^2\phi'_i(x,t) \rangle_\eta \\
= \frac{g^2}{N} \int \frac{d^2k d^2k'}{(2\pi)^2} e^{-i(k+k')x} \lim_{t \to \infty} \langle -\phi'_i(k',t)k^2 \phi'_i(k,t) \rangle_\eta \\
= - g^2 \int \frac{d^2k}{(2\pi)^2} \frac{k^2}{k^2 + \langle \sigma \rangle_\eta} \\
= \frac{g^2}{4\pi} \langle \sigma \rangle_\eta \ln\left( \frac{\Lambda^2}{\langle \sigma \rangle_\eta} \right) - \frac{g^2 \Lambda^2}{4\pi},
\]

where we have introduced a straight cutoff parameter \( \Lambda \). The two divergent terms in the above equation and in the equation (16) cancel out. Interested in non-zero solutions of \( \langle \sigma \rangle_\eta \) we divide the both sides of the equation (16) by it and obtain the gap-equation

\[
1 - \frac{g^2}{N} \Phi_i(\infty) \Phi_i(\infty) = \frac{g^2}{4\pi} \ln\left( \frac{\Lambda^2}{\langle \sigma \rangle_\eta} \right). \tag{17}
\]

Note that the above equation can be obtained by computing the expectation value of \( N/g^2 = \langle \phi_i(x,t)\phi_i(x,t) \rangle_\eta \).

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† The equation (16) can be directly derived from the equation (7).
If we define the renormalized coupling constant \( g_r \) at a renormalization point \( \mu \) as

\[
\frac{1}{g_r^2} \equiv \frac{1}{g^2} - \frac{1}{4\pi} \ln \left( \frac{\Lambda^2}{\mu^2} \right),
\]

the solution of the gap-equation (17) is

\[
\langle \sigma \rangle_\eta = \mu^2 \exp \left\{ -\frac{4\pi}{g_r^2} + \frac{4\pi \Phi_i(\infty)\Phi_i(\infty)}{N} \right\},
\]

which is independent of the renormalization point \( \mu \). Remembering the relation (13), we conclude that the \( O(N) \) symmetry is not broken, that is,

\[
\Phi_i(\infty) = 0
\]

and the \( n \) scalar particles gain a mass \( m \)

\[
m^2 = \mu^2 \exp \left\{ -\frac{4\pi}{g_r^2} \right\}.
\]

These results are same as the path integral approach gives.

In this paper we have not discussed stability of the vacuum. Nevertheless the methods developed here can be applied to, for example, the Nambu-Jona-Lasinio model\(^7\) whose symmetry spontaneously breaks down in the large \( N \) limit. In that case the Langevin equations for bound states are introduced and stability of the vacuum is considered convergency of them.

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