STABILITY OF DELAY DIFFERENTIAL EQUATIONS
WITH FADING STOCHASTIC PERTURBATIONS
OF THE TYPE OF WHITE NOISE AND POISSON’S JUMPS

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ABSTRACT. Delay differential equation is considered under stochastic perturbations of
the type of white noise and Poisson’s jumps. It is shown that if stochastic perturbations
fade on the infinity quickly enough then sufficient conditions for asymptotic stability of the
zero solution of the deterministic differential equation with delay provide also asymptotic
mean square stability of the zero solution of the stochastic differential equation. Stability
conditions are obtained via the general method of Lyapunov functionals construction and the
method of Linear Matrix Inequalities (LMIs). Investigation of the situation when stochastic
perturbations do not fade on the infinity or fade not enough quickly is proposed as an
unsolved problem.

1. Introduction

Let \( \{ \Omega, \mathcal{F}, P \} \) be a complete probability space, \( \{ \mathcal{F}_t \}_{t \geq 0} \) be a nondecreasing family
of sub-\( \sigma \)-algebras of \( \mathcal{F} \), i.e., \( \mathcal{F}_s \subset \mathcal{F}_t \) for \( s < t \), \( P \{ \cdot \} \) be the probability of an event
enclosed in the braces, \( E \) be the mathematical expectation, \( H_2 \) be the space of \( \mathcal{F}_0 \)-adapted stochastic processes \( \varphi(s) \), \( s \leq 0 \),
\( \| \varphi \|_2 = \sup_{s \leq 0} E|\varphi(s)|^2 \).

Following Gikhman and Skorokhod [1], we will consider the stochastic delay
differential equation
\[
dx(t) = (Ax(t) + Bx(t - h))dt + C(t)x(t)dw(t) + \int G(t, u)x(t)\tilde{\nu}(dt, du), \quad t \geq 0,
\]
\[
x(s) = \phi(s) \in H_2, \quad s \in [-h, 0], \tag{1.1}
\]

where \( x(t) \in \mathbb{R}^n \), \( A, B, C(t), G(t, u) \) are \( n \times n \)-matrices, \( h > 0 \), \( w(t) \) is the scalar
standard Wiener process on a probability space \( \{ \Omega, \mathcal{F}, P \} \), \( \nu(t, A) \) is the Poisson
measure with the parameter \( t \Pi(A) \), \( \tilde{\nu}(t, A) = \nu(t, A) - t \Pi(A) \) [1].

To explain the idea of the proposed investigation note that the second moment
\( y(t) = E x^2(t) \) of the solution \( x(t) \) of the scalar stochastic differential equation
\[
dx(t) = -ax(t)dt + \sigma(t)x(t)dw(t) + \int \gamma(t, u)x(t)\tilde{\nu}(dt, du) \tag{1.2}
\]
satisfies the differential equation \( \dot{y}(t) = (-2a + \rho(t))y(t) \), where [1]
\[
\rho(t) = \sigma^2(t) + \int \gamma^2(t, u)\Pi(du), \tag{1.3}
\]

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and can be represented in the form

\[ y(t) = y(0) \exp \left\{ \int_0^t (-2\alpha + \rho(s))ds \right\} \]

\[ = y(0) \exp \{-(2\alpha - \mu(t))t\} \]

(1.4)

with

\[ \mu(t) = \frac{1}{t} \int_0^t \rho(s)ds. \]

(1.5)

Via (1.4) we obtain the following

**Statement 1.1.** Put \( \mu = \limsup_{t \to \infty} \mu(t) \). If \( 2\alpha > \mu \) then \( \lim_{t \to \infty} y(t) = 0 \), i.e., the zero solution of the equation (1.2) is asymptotically mean square stable. In particular,

- (i) if \( \lim_{t \to \infty} \rho(t) = 0 \) and \( \int_0^\infty \rho(s)ds < \infty \), i.e., stochastic perturbations fade on the infinity quickly enough, then \( \mu = 0 \) and stability condition takes the form \( \alpha > 0 \);

- (ii) if stochastic perturbations do not fade on the infinity or fade not very quickly, i.e., \( \int_0^\infty \rho(s)ds = \infty \), then stability condition depends on the value of \( \mu \). For instance, if \( \sigma^2(s) = \sigma^2 \) and \( \gamma^2(s, u) = \gamma^2(u) \) then stability condition takes the form \( 2\alpha > \sigma^2 + \int \gamma^2(u)\Pi(du) \).

**Example 1.1.** Put in the equation (1.2) \( \rho(t) = \beta \sin^2(t) \), \( \beta > 0 \). So, stochastic perturbations do not fade on the infinity and \( \mu = \limsup_{t \to \infty} \frac{1}{t} \int_0^t \rho(s)ds = \frac{\beta}{2} \). Therefore, the condition of asymptotic mean square stability for the zero solution of the equation (1.2) takes the form \( \alpha > \frac{\beta}{4} \). In Figure 1.1 50 trajectories (blue) of the equation (1.2) solution are shown for \( x(0) = 2 \), \( \alpha = 0.8 \), \( \sigma(t) = \sqrt{3} \sin(t) \), \( \gamma(t, u) = 0 \) together with \( \rho(t) = 3 \sin^2(t) \) (red). The condition \( \alpha = 0.8 > \frac{\beta}{4} = \frac{3}{4} \) holds, therefore, the zero solution of the equation (1.2) is asymptotically mean square stable and all trajectories converge to zero.

**Example 1.2.** Suppose that in the equation (1.2) \( \rho(t) = \frac{\beta}{(t+1)^\alpha} \), \( \alpha > 0 \), \( \beta > 0 \). Then \( \lim_{t \to \infty} \rho(t) = 0 \) and

\[ \int_0^t \rho(s)ds = \begin{cases} \frac{\beta}{1-\alpha} \left( \frac{1}{(t+1)^{1-\alpha}} - 1 \right) & \text{if } \alpha \in (0, 1), \\ \frac{\beta}{\alpha-1} \left( 1 - \frac{1}{(t+1)^{\alpha-1}} \right) & \text{if } \alpha > 1. \end{cases} \]

(1.6)

In this case \( \int_0^\infty \rho(s)ds < \infty \) for \( \alpha > 1 \) and \( \int_0^\infty \rho(s)ds = \infty \) for \( \alpha \in (0, 1) \). From (1.5), (1.6) it follows that \( \mu = \limsup_{t \to \infty} \mu(t) = 0 \) for all \( \alpha > 0 \) and the condition of asymptotic mean square stability for the zero solution of the equation (1.2) takes the form \( \alpha > 0 \). In Figure 1.2 50 trajectories (blue) of the equation (1.2) solution are shown for \( x(0) = 2 \), \( \alpha = 0.8 \), \( \sigma(t) = \frac{\sqrt{3}}{t+1} \), \( \gamma(t, u) = 0 \) together with \( \rho(t) = \frac{3}{t+1} \) (red). The condition \( \alpha > 0 \) holds, \( \alpha = 1 \), \( \beta = 3 \), via (1.6) \( \int_0^\infty \rho(s)ds = \infty \), the zero solution of the equation (1.2) is asymptotically mean square stable and all trajectories converge to zero.
Figure 1.1. 50 trajectories (blue) of the equation (1.2) solution, 
\[ x(0) = 2, \ a = 0.8, \ \sigma(t) = \sqrt{3\sin(t)}, \ \gamma(t, u) = 0, \ \rho(t) = 3\sin^2(t) \] (red).

Figure 1.2. 50 trajectories (blue) of the equation (1.2) solution, 
\[ x(0) = 2, \ a = 0.8, \ \sigma(t) = \sqrt{3t + 1}, \ \gamma(t, u) = 0, \ \rho(t) = \frac{3}{t + 1} \] (red).

Figure 1.3. 50 trajectories (blue) of the equation (1.2) solution, 
\[ x(0) = 2, \ a = 0.8, \ \sigma(t) = \frac{\sqrt{3}}{t + 1}, \ \gamma(t, u) = 0, \ \rho(t) = \frac{3}{(t + 1)^2} \] (red).
In Figure 1.3 50 trajectories (blue) of the equation (1.2) solution are shown for $x(0) = 2$, $a = 0.8$, $\sigma(t) = \frac{\sqrt{3}}{t + 1}$, $\gamma(t, u) = 0$ together with $\rho(t) = \frac{3}{(t + 1)^2}$ (red). The condition $a > 0$ holds, $\alpha = 2$, $\beta = 3$. Via (1.6) $\int_0^\infty \rho(s)ds < \infty$, the zero solution of the equation (1.2) is asymptotically mean square stable and all trajectories converge to zero.

Below Statement 1.1 is generalized on the equation (1.1) and one unsolved problem in this direction is formulated too. Note also that in [5] a similar research was considered for stochastic difference equations.

2. Auxiliary statements and definitions

Consider a functional $V(t, \varphi) : [0, \infty) \times H_2 \to \mathbb{R}_+$ that can be represented in the form $V(t, \varphi(0), \varphi(s))$, $s < 0$, and for $\varphi = x_1$ put [3]

$$V_{\varphi}(t, x) = V(t, \varphi) = V(t, x_1) = V(t, x, x(t + s)),\]

$$x = \varphi(0) = x(t), \quad s < 0.$$ (2.1)

Let $D$ be the set of the functionals, for which the function $V_{\varphi}(t, x)$ defined by (2.1) has a continuous derivative with respect to $t$ and two continuous derivatives with respect to $x$. The generator $L$ of the equation (1.1) is defined on the functionals from $D$ as follows [1, 3, 4]

$$LV(t, x_t) = \frac{\partial V_{\varphi}(t, x(t))}{\partial t} + \nabla V'_{\varphi}(t, x(t))(Ax(t) + Bx(t - h)) + \frac{1}{2}x'(t)C'(t)\nabla^2 V_{\varphi}(t, x(t))C(t)x(t) + \int \left[V_{\varphi}(t, x(t) + G(t, u)x(t)) - V_{\varphi}(t, x(t)) - \nabla V'_{\varphi}(t, x(t))G(t, u)x(t)\right]\Pi(du).$$ (2.2)

Definition 2.1. The zero solution of the equation (1.1) is called:
- mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $E|x(t, \phi)|^2 < \varepsilon$, $t \geq 0$, provided that $\|\phi\|^2 < \delta$;
- asymptotically mean square stable if it is mean square stable and $\lim_{t \to \infty} E|x(t, \phi)|^2 = 0$ for each initial function $\phi$.

Theorem 2.1. [3] Let there exist a functional $V(t, \varphi) \in D$, positive constants $c_1$, $c_2$, $c_3$, such that the following conditions hold:

$$EV(t, x_t) \geq c_1 E|x(t)|^2, \quad EV(0, \phi) \leq c_2 \|\phi\|^2, \quad ELV(t, x_t) \leq -c_3 E|x(t)|^2.$$ Then the zero solution of the equation (1.1) is asymptotically mean square stable.

Schur complement [2]. The symmetric matrix $\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times m}$, is negative definite if and only if $C$ and $A - BC^{-1}B'$ are both negative definite.
3. Main result

Note that in the case of an autonomous system \( (C(t) = C, \ G(t, u) = G(u)) \) conditions of asymptotic mean square stability for the equation (1.1) zero solution can be obtained in the form of the matrix Riccati inequality [3]

\[
A'P + PA + C'PC + \int G'(u)PG(u)\Pi(du) + R + PBR^{-1}B'P < 0,
\]

\[P > 0, \quad R > 0,
\]

which via Schur complement can be reformulated also in the form of LMI [4]

\[
\begin{bmatrix} A'P + PA + C'PC + \int G'(u)PG(u)\Pi(du) + R & PB \\ BR & -R \end{bmatrix} < 0.
\]

Below we will consider non-autonomous system and will suppose that stochastic perturbations in the equation (1.1) quickly enough fade on the infinity, i.e., the function \( \rho(t) \) is integrable on \([0, \infty)\).

**Theorem 3.1.** Let there exist positive definite \( n \times n \)-matrices \( P, R \) and the functions \( \sigma(t) \) and \( \gamma(t, u) \) such that the following inequalities hold:

\[
C'(t)PC(t) \leq \sigma^2(t)P, \quad G'(t, u)PG(t, u) \leq \gamma^2(t, u)P,
\]

\[
\Phi = \begin{bmatrix} A'P + PA + R & PB \\ BR & -R \end{bmatrix} < 0, \quad \int_0^\infty \rho(t)dt < \infty,
\]

(3.1)

where \( \rho(s) \) is defined in (1.3). Then the zero solution of the equation (1.1) is asymptotically mean square stable.

**Proof.** Following the general method of Lyapunov functionals construction [3] we will construct the Lyapunov functional \( V \) for the equation (1.1) in the form \( V = V_1 + V_2 \), where \( V_1(x(t)) = e^{-\int_0^t \rho(s)ds}x'(t)Px(t) \).

Let \( L \) be the generator of the equation (1.1). Then via (2.2), two first inequalities (3.1) and (1.3) for \( V_1 \) we have

\[
LV_1(x(t)) = e^{-\int_0^t \rho(s)ds} \left[ -\rho(t)x'(t)Px(t) + 2x'(t)P(Ax(t) + Bx(t - h)) \\
+ x'(t)C'(t)PC(t)x(t) + \int x'(t)G'(t, u)PG(t, u)x(t)\Pi(du) \right]
\]

\[
\leq e^{-\int_0^t \rho(s)ds} \left[ x'(t)(A'P + PA)x(t) \\
+ x'(t)PBx(t - h) + x'(t - h)B'Px(t) \\
+ x'(t) \left( C'(t)PC(t) + \int G'(t, u)PG(t, u)\Pi(du) - \rho(t)P \right)x(t) \right]
\]

\[
\leq e^{-\int_0^t \rho(s)ds} \left[ x'(t)(A'P + PA)x(t) \\
+ x'(t)PBx(t - h) + x'(t - h)B'Px(t) \right].
\]
Using the additional functional $V_2(x_t) = \int_{t-h}^{t} e^{-\int_{t-h}^{\tau} \rho(\sigma) d\tau} x'(s) R x(s) ds$ with

$$LV_2(x_t) = e^{-\int_{0}^{\infty} \rho(\tau) d\tau} x'(t) R x(t) - e^{-\int_{0}^{\infty} \rho(\tau) d\tau} x'(t-h) R x(t-h)$$

$$\leq e^{-\int_{0}^{\infty} \rho(\tau) d\tau} \left( x'(t) R x(t) - x'(t-h) R x(t-h) \right),$$

for the Lyapunov functional $V(x_t) = V_1(x(t)) + V_2(x_t)$ via two last inequalities (3.1) we obtain

$$LV(x_t) \leq e^{-\int_{0}^{\infty} \rho(s) ds} \left( x'(t)(A'P + PA + R)x(t) + x'(t)PBx(t-h) + x'(t-h)B'Px(t-h) - x'(t-h)R x(t-h) \right)$$

$$= e^{-\int_{0}^{\infty} \rho(s) ds} [x'(t) x'(t-h)]' [\Phi(x'(t) x'(t-h))']$$

$$\leq e^{-\int_{0}^{\infty} \rho(s) ds} [x'(t) x'(t-h)] [\Phi_{xy} x'(t) x'(t-h)]'$$

$$\leq - |c x(t)|^2.$$

So, the constructed functional $V(x_t)$ satisfies the conditions of Theorem 2.1. Therefore, the zero solution of the equation (1.1) is asymptotically mean square stable. The proof is completed.

**Remark 3.1.** In the deterministic case ($C(t) \equiv 0, G(t, u) \equiv 0$) the LMI $\Phi < 0$ is a sufficient condition for asymptotic stability of the zero solution of the delay differential equation (1.1). Via Theorem 3.1 if the stochastic perturbations fade on the infinity quickly enough then the LMI $\Phi < 0$ is a sufficient condition also for asymptotic mean square stability of the zero solution of the delay differential equation (1.1) under stochastic perturbations.

**Remark 3.2.** Consider the scalar case of the equation (1.1) and put $A = -a, B = b, C(t) = \sigma(t), G(t, u) = \gamma(t, u)$. It is easy to check that the matrix $\Phi$ in the condition (3.1) is a negative definite matrix (i.e., the LMI $\Phi < 0$ holds) if and only if $a > |b|$. So, we obtain a generalization of the part (i) of Statement 1.1: if $a > |b|$ and $\int_{0}^{\infty} \rho(s) ds < \infty$ then the zero solution of the equation (1.1) is asymptotically mean square stable.

**Remark 3.3.** (About an unsolved problem) Note that the last condition (3.1) $\int_{0}^{\infty} \rho(s) ds < \infty$ means that the stochastic perturbations in the equation (1.1) fade on the infinity quickly enough and is used in the proof of Theorem 3.1. But as it is shown in Examples 1.1 and 1.2 this condition is not a necessary condition for asymptotic mean square stability of the zero solution of the equation (1.1). For the scalar differential equation (1.2) the opposite assumption $\int_{0}^{\infty} \rho(s) ds = \infty$ is included in the part (ii) of Statement 1.1 and the stability condition $2a > \mu \gamma$ which immediately follows from the representation (1.4). A generalization of this stability condition for the equation (1.1) on the case $\int_{0}^{\infty} \rho(s) ds = \infty$ requires a proof that is fundamentally different from the proof of Theorem 3.1 and is currently an unsolved problem.
Conclusions

The effect of fading stochastic perturbations of the type of white noise and Poisson’s jumps on asymptotically stable delay differential equation is investigated. It is shown that if stochastic perturbations fade on the infinity quickly enough then condition for asymptotic stability of the deterministic system provides also asymptotic mean square stability of the considered system under stochastic perturbations. Consideration of the situation with stochastic perturbations which do not fade on the infinity or fade not enough quickly is proposed as an unsolved problem.

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