WELL-POSEDNESS AND BLOW-UP PHENOMENA FOR A GENERALIZED CAMASSA-HOLM EQUATION

JINLU LI
Department of Mathematics, Sun Yat-sen University
Guangzhou 510275, China

ZHAOYANG YIN
Department of Mathematics, Sun Yat-sen University
Guangzhou 510275, China
and
Faculty of Information Technology
Macau University of Science and Technology, Macau, China

(Communicated by Adrian Constantin)

Abstract. We first establish the local existence and uniqueness of strong solutions for the Cauchy problem of a generalized Camassa-Holm equation in nonhomogeneous Besov spaces by using the Littlewood-Paley theory. Then, we prove that the solution depends continuously on the initial data in the corresponding Besov space. Finally, we derive a blow-up criterion and present a blow-up result and a blow-up rate of the blow-up solutions to the equation.

1. Introduction. Recently, Novikov in [39] proposed the following integrable quasi-linear scalar evolution equation of order 2:

\[(1 - \epsilon^2 \partial_x^2) u_t = (2 - \epsilon \partial_x)(1 + \epsilon \partial_x) u_x]^2,\]

where \(\epsilon \neq 0\) is a real constant. He showed [39] that (1.1) possesses a hierarchy of local higher symmetries.

Eq. (1.1) belongs to the following class [39]:

\[(1 - \partial_x^2) u_t = F(u, u_x, u_{xx}, u_{xxx}),\]

which has attracted much attention on the possible integrable members of (1.2).

The first well-known integrable member of (1.2) is the Camassa-Holm (CH) equation [4]

\[(1 - \partial_x^2) u_t = -(3 u u_x - 2 u_x u_{xx} - u u_{xxx}).\]

The CH equation can be regarded as a shallow water wave equation [4, 5, 18]. It is completely integrable [4, 8, 19, 16], has a bi-Hamiltonian structure [7, 28], and admits exact peaked solitons of the form \(ce^{-|x-ct|}, c > 0\), which are orbitally stable [20]. It is worth mentioning that the peaked solitons present the characteristic for
the traveling water waves of greatest height and largest amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, cf. [10, 14, 15, 40]. The local well-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces was discussed in [11, 12, 21]. It was shown that there exist global strong solutions to the CH equation [9, 11, 12] and finite time blow-up strong solutions to the CH equation [9, 11, 12, 13]. The global conservative and dissipative solutions of CH equation were discussed in [2, 3].

The second well-known integrable member of (1.2) is the Degasperis-Procesi (DP) equation [24]:
\[(1 - \partial_x^2)u_t = -(4u u_x - 3u_x u_{xx} - u u_{xxx}).\]
The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the CH shallow water equation [25]. The DP equation is integrable and has a bi-Hamiltonian structure [23]. An inverse scattering approach for computing $n$-peakon solutions to the DP equation was presented in [38, 17]. Its traveling wave solutions were investigated in [33, 41]. The local well-posedness of the Cauchy problem of the DP equation in Sobolev spaces and Besov spaces were established in [29, 30, 47]. Similar to the CH equation, the DP equation has also global strong solutions [35, 48, 50] and finite time blow-up solutions [26, 27, 35, 36, 47, 48, 49, 50]. On the other hand, it has global weak solutions [6, 26, 49, 50]. Although the DP equation is similar to the CH equation in several aspects, these two equations are truly different. One of the novel features of the DP different from the CH equation is that it has not only peakon solutions [23] and periodic peakon solutions [49], but also shock peakons [37] and the periodic shock waves [27].

The third well-known integrable member of (1.2) is the Novikov equation [31]
\[(1 - \partial_x^2)u_t = 3uu_x u_{xx} + u^2 u_{xxx} - 4u^2 u_x.\]
The most difference between the Novikov equation and the CH and DP equations is that the former one has cubic nonlinearity and the latter ones have quadratic nonlinearity. It was showed that the Novikov equation is integrable, possesses a bi-Hamiltonian structure, and admits exact peakon solutions $u(t, x) = \pm \sqrt{ce^{x - ct}}$, $c > 0$ [31]. The local well-posedness for the Novikov equation in Sobolev spaces and Besov spaces was studied in [43, 44, 45, 46]. The global existence of strong solutions was established in [43] under some sign conditions and the blow-up phenomena of the strong solutions were shown in [46]. The global weak solutions for the Novikov equation were discussed in [42].

To our best knowledge, the Cauchy problem of Eq.(1.1) has not been studied yet. In this paper, we mainly study the Cauchy problem of Eq.(1.1). Since Eq.(1.1) has the similar structure with the Camassa-Holm equation, we call it as a generalized Camassa-Holm equation. Letting $v(t, x) = u(\epsilon^2 t, \epsilon x)$, then one can transform the Cauchy problem of Eq.(1.1) into the following equivalent form:
\[
\begin{cases}
  u_t - u_{txx} = (2 - \partial_x)(1 + \partial_x) u_x^2, & t > 0, \ x \in \mathbb{R}, \\
  u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]
which is equivalent to
\[
\begin{cases}
  m_t + 2(m - u_x - u)m_x = 0, & t > 0, \ x \in \mathbb{R}, \\
  m = (1 - \partial_x^2)u_t, & t > 0, \ x \in \mathbb{R}, \\
  m_0 = m(0, x) = (1 - \partial_x^2)u_0(x), & x \in \mathbb{R}.
\end{cases}
\]
In this paper, using the Littlewood-Paley theory, we establish the local existence and uniqueness of solutions to (1.4) in nonhomogeneous Besov spaces. For the stability of the solution, lots of papers just proved it in lower regularity Besov spaces \( B_{p,r}^{s'} \), \( s' < s \) when the initial data \( u_0 \in B_{p,r}^s \). We use a priori estimate of solutions to transport equation in Besov spaces \( B_{p,r}^{1+\frac{1}{p}} \) \([34]\) and apply the method introduced by Danchin \([22]\) to get the continuity with respect to initial data. By the structure of the equation (1.4) and Gronwall’s inequality, we present a blow-up result provided the initial data \( m_0 \) satisfies some conditions and give a blow-up rate for the blow-up solutions of (1.4).

Our paper is organized as follows. In Section 2, we give some preliminaries which will be used in the sequel. In Section 3, we establish the local existence and uniqueness of solutions of the Cauchy problem (1.4) in Besov spaces. In Section 4, we show the continuity of solutions to (1.4) with respect to initial data. In Section 5, we present a blow-up result to (1.4) and give a blow-up rate for the blow-up solutions of (1.4).

**Notation.** In the following, we denote by * the convolution and by \( \langle \cdot, \cdot \rangle \) the action between \( S'(\mathbb{R}) \) and \( S(\mathbb{R}) \). Given a Banach space \( X \), we denote its norm by \( \| \cdot \|_X \).

2. Preliminaries. In this section, we will recall some facts on the Littlewood-Paley decomposition, the nonhomogeneous Besov spaces and their some useful properties. We will also recall the transport equation theory, which will be used in our work. For more details, the readers can refer to \([1, 21]\).

**Proposition 2.1.** \([1]\) (Littlewood-Paley decomposition) There exists a couple of smooth functions \((\chi, \varphi)\) valued in \([0,1]\), such that \( \chi \) is supported in the ball \( B \triangleq \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3} \} \), and \( \varphi \) is supported in the ring \( C \triangleq \{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \} \). Moreover,

\[
\forall \ \xi \in \mathbb{R}^d, \ \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1,
\]

and

\[
\text{supp } \varphi(2^{-q}\cdot) \cap \text{supp } \varphi(2^{-q'}\cdot) = \emptyset, \text{ if } |q - q'| \geq 2,
\]

\[
\text{supp } \chi(\cdot) \cap \text{supp } \varphi(2^{-q}\cdot) = \emptyset, \text{ if } q \geq 1.
\]

Then for all \( u \in S'(\mathbb{R}^d) \), we can define the nonhomogeneous dyadic blocks as follows. Let

\[
\Delta_q u \triangleq 0, \text{ if } q \leq -2,
\]

\[
\Delta_{-1} u \triangleq \chi(D)u = \mathcal{F}^{-1}(\chi\mathcal{F}u),
\]

\[
\Delta_q u \triangleq \varphi(2^{-q}D)u = \mathcal{F}^{-1}(\varphi(2^{-q}\cdot)\mathcal{F}u), \text{ if } q \geq 0.
\]

Hence,

\[
u = \sum_{q \in \mathbb{Z}} \Delta_q u \text{ in } S'(\mathbb{R}^d),
\]

where the right-hand side is called the nonhomogeneous Littlewood-Paley decomposition of \( u \).

**Remark 2.2.** \([1]\) (1) The low frequency cut-off operator \( S_q \) is defined by

\[
S_q u \triangleq \sum_{p=-1}^{q-1} \Delta_p u = \chi(2^{-q}D)u = \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u), \ \forall \ q \in \mathbb{N}.
\]
(2) The Littlewood-Paley decomposition is quasi-orthogonal in $L^2$ in the following sense:
\[
\Delta_p \Delta_q u \equiv 0, \quad \text{if } |p - q| \geq 2, \\
\Delta_q (S_{p-1} u \Delta_p v) \equiv 0, \quad \text{if } |p - q| \geq 5,
\]
for all $u, v \in S'(\mathbb{R}^d)$.

(3) Thanks to Young’s inequality, we get
\[
\|\Delta_q u\|_{L^p(\mathbb{R}^d)}, \quad \|S_q u\|_{L^p(\mathbb{R}^d)} \leq C\|u\|_{L^p(\mathbb{R}^d)}, \quad \forall \ 1 \leq p \leq \infty,
\]
where $C$ is a positive constant independent of $p$ and $q$.

**Definition 2.3.** [1] (Besov spaces) Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^{s}_{p,r}(\mathbb{R}^d)$ is defined by
\[
B^{s}_{p,r}(\mathbb{R}^d) \triangleq \{ f \in S'(\mathbb{R}^d) : \|f\|_{B^{s}_{p,r}(\mathbb{R}^d)} < \infty \},
\]
where
\[
\|f\|_{B^{s}_{p,r}(\mathbb{R}^d)} \triangleq \|2^{qs}\Delta_q f\|_{L^r(\mathbb{R}^d)} = \|(2^{qs}\Delta_q f)\|_{L^r(\mathbb{R}^d)} 1 \leq s \leq \infty.
\]
If $s = \infty$, $B^\infty_{p,r}(\mathbb{R}^d) \triangleq \bigcap_{s \in \mathbb{R}} B^{s}_{p,r}(\mathbb{R}^d)$.

In the following lemma, we list some important properties of Besov spaces.

**Lemma 2.4.** [1] Suppose that $s \in \mathbb{R}$, $1 \leq p, r, p_i, r_i \leq \infty$, $i = 1, 2$. We have

(1) Topological properties: $B^{s}_{p,r}(\mathbb{R}^d)$ is a Banach space which is continuously embedded in $S'(\mathbb{R}^d)$.

(2) Density: $C^\infty_\mathbb{R}$ is dense in $B^{s}_{p,r}(\mathbb{R}^d) \iff 1 \leq p, r < \infty$.

(3) Embedding: $B^{s}_{p_1,r_1}(\mathbb{R}^d) \hookrightarrow B^{s-d(\frac{1}{p_1} - \frac{1}{p})}_{p_2,r_2}(\mathbb{R}^d)$, if $p_1 \leq p_2$ and $r_1 \leq r_2$.

(4) Algebraic properties: $\forall s > 0$, $B^{s}_{p,r}(\mathbb{R}^d) \bigcap L^\infty(\mathbb{R}^d)$ is an algebra. Moreover, $B^{s}_{p,r}(\mathbb{R}^d)$ is an algebra, provided that $s > \frac{d}{p}$ or $s = \frac{d}{p}$ and $r = 1$.

(5) Complex interpolation: $\forall f \in B^{s}_{p,r}(\mathbb{R}^d) \bigcap B^{s}_{p,r}(\mathbb{R}^d)$,
\[
\|f\|_{B^{s}_{p,r}(\mathbb{R}^d) \bigcap B^{s}_{p,r}(\mathbb{R}^d)} \leq \|f\|_{B^{s}_{p,r}(\mathbb{R}^d) \bigcap B^{s}_{p,r}(\mathbb{R}^d)} \|f\|_{B^{s}_{p,r}(\mathbb{R}^d) \bigcap B^{s}_{p,r}(\mathbb{R}^d)}, \quad \forall \theta \in [0, 1].
\]

(6) Fatou’s lemma: if $(u_n)_{n \in \mathbb{N}}$ is bounded in $B^{s}_{p,r}(\mathbb{R}^d)$ and $u_n \to u \in S'(\mathbb{R}^d)$, then
\[
\|u\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq \liminf_{n \to \infty} \|u_n\|_{B^{s}_{p,r}(\mathbb{R}^d)}.
\]

(7) Let $m \in \mathbb{R}$ and $f$ be an $S^m$-multiplier (i.e., $f : \mathbb{R}^d \to \mathbb{R}$ is smooth and satisfies that $\forall \alpha \in \mathbb{N}^d$, exists a constant $C_\alpha$, such that $\partial^\alpha f(\xi) \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}$ for all $\xi \in \mathbb{R}^d$). Then the operator $f(D)$ is continuous from $B^{s}_{p,r}(\mathbb{R}^d)$ to $B^{s-m}_{p,r}(\mathbb{R}^d)$.

**Lemma 2.5.** [1] Product laws:

(i) For $s > 0$,
\[
\|fg\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq C\|f\|_{B^{s}_{p,r}(\mathbb{R}^d)} \|g\|_{L^\infty(\mathbb{R}^d)} + \|g\|_{B^{s-m}_{p,r}(\mathbb{R}^d)} \|f\|_{L^\infty(\mathbb{R}^d)}.
\]

(ii) For $\forall s_1 \leq \frac{1}{p} < s_2$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > \max(0, \frac{2}{p} - 1)$, we have
\[
\|fg\|_{B^{s_1}_{p,r}(\mathbb{R}^d)} \leq C\|f\|_{B^{s_1}_{p,r}(\mathbb{R}^d)} \|g\|_{B^{s_2}_{p,r}(\mathbb{R}^d)}.
\]
Remark 2.6. We will see that, the product estimate (2.2) is one of the keys to our work. We mention here the simpler case, say, if Remark 2.6.

Proposition 2.7. [1] For all \(1 \leq p, r \leq \infty\) and \(s \in \mathbb{R}\),
\[
\begin{cases}
B^s_{p,r}(\mathbb{R}^d) \times B^{-s}_{p,r}(\mathbb{R}^d) \to \mathbb{R}, \\
(u, \phi) \to \sum_{|j| \leq 1} (\Delta_j u, \Delta_j \phi),
\end{cases}
\]
defines a continuous bilinear functional on \(B^s_{p,r}(\mathbb{R}^d) \times B^{-s}_{p,r}(\mathbb{R}^d)\). Denote by \(Q^{-s}_{p,r}(\mathbb{R}^d)\) the set of functions \(\phi\) in \(S(\mathbb{R}^d)\) such that \(\|\phi\|_{B^{-s}_{p,r}(\mathbb{R}^d)} \leq 1\). If \(u\) is in \(S'(\mathbb{R}^d)\), then we have
\[
\|u\|_{B^s_{p,r}(\mathbb{R}^d)} \leq C \sup_{\phi \in Q^{-s}_{p,r}(\mathbb{R}^d)} \langle u, \phi \rangle.
\]

Now we state some useful results in the transport equation theory, which are crucial to the proofs of our main theorems later.

Lemma 2.8. [1] (A priori estimates in Besov spaces) Let \(1 \leq p, r \leq \infty\) and \(\sigma > -\min(\frac{1}{p}, 1 - \frac{1}{r})\). Assume that \(f_0 \in B^\sigma_{p,r}(\mathbb{R})\), \(F \in L^1(0, T; B^\sigma_{p,r}(\mathbb{R}))\), and \(\partial_x v\) belongs to \(L^1(0, T; B^{-1}_{p,r}(\mathbb{R}))\) if \(\sigma > 1 + \frac{1}{r}\) or to \(L^1(0, T; B^{\frac{1}{2}}_{p,r}(\mathbb{R}) \cap L^\infty(\mathbb{R}))\) otherwise. If \(f \in L^\infty(0, T; B^\sigma_{p,r}(\mathbb{R})) \cap C([0, T]; S'(\mathbb{R}))\) solves the following 1-D linear transport equation:
\[
(T) \quad \begin{cases}
\partial_t f + v \partial_x f = F, \\
f|_{t=0} = f_0,
\end{cases}
\]
then there exists a constant \(C\) depending only on \(p, r\) and \(\sigma\), such that the following statements hold:
(1) If \(r = 1\) or \(\sigma \neq 1 + \frac{1}{p}\),
\[
\|f(t)\|_{B^\sigma_{p,r}(\mathbb{R})} \leq e^{CV(t)} \left(\|f_0\|_{B^\sigma_{p,r}(\mathbb{R})} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B^\sigma_{p,r}(\mathbb{R})} d\tau\right),
\]
with
\[
V(t) = \begin{cases}
\int_0^t \|\partial_x v(\tau)\|_{B^\sigma_{p,r}(\mathbb{R}) \cap L^\infty(\mathbb{R})} d\tau, & \text{if } \sigma < 1 + \frac{1}{p}, \\
\int_0^t \|\partial_x v(\tau)\|_{B^{-1}_{p,r}(\mathbb{R})} d\tau, & \text{if } \sigma > 1 + \frac{1}{r} (\text{or } \sigma = 1 + \frac{1}{p}, \ r = 1).
\end{cases}
\]
(2) If \(f = v\), then for all \(\sigma > 0\), (1) holds true with \(V(t) = \int_0^t \|\partial_x v(\tau)\|_{L^\infty(\mathbb{R})} d\tau\).
(3) If \(r < \infty\), then \(f \in C([0, T]; B^\sigma_{p,r}(\mathbb{R}))\). If \(r = \infty\), then \(f \in C([0, T]; B^{\sigma'}_{p,1}(\mathbb{R}))\) for all \(\sigma' < \sigma\).

Lemma 2.9. [1] (Existence and uniqueness) Let \(p, r, \sigma, f_0\) and \(F\) be as in the statement of Lemma 2.8. Assume that \(v \in L^\rho(0, T; B^{-M}_{\infty,\infty}(\mathbb{R}))\) for some \(\rho > 1\) and \(M > 0\), and \(\partial_x v \in L^1(0, T; B^{-1}_{\infty,\infty}(\mathbb{R}))\) if \(\sigma > 1 + \frac{1}{p}\) or \(\sigma = 1 + \frac{1}{p}\) and \(r = 1\), and \(\partial_x v \in L^1(0, T; B^{\frac{1}{2}}_{p,r}(\mathbb{R}) \cap L^\infty(\mathbb{R}))\) if \(\sigma < 1 + \frac{1}{p}\). Then (T) has a unique solution \(f \in L^\infty(0, T; B^\sigma_{p,r}(\mathbb{R})) \cap \bigcap_{\sigma' < \sigma} C([0, T]; B^{\sigma'}_{p,1}(\mathbb{R}))\) and the inequalities of Lemma 2.8 can hold true. Moreover, if \(r < \infty\), then \(f \in C([0, T]; B^\sigma_{p,r}(\mathbb{R}))\).
Lemma 2.10. \[32\] If \( s > 0, f \in H^s(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}), \ g \in H^{s-1}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and denote that \( \Lambda^s = (1 - \Delta)^{\frac{s}{2}} \), then
\[
\| \Lambda^s(fg) - f\Lambda^s g \|_{L^2(\mathbb{R})} \leq C \left( \| \Lambda^s f \|_{L^2(\mathbb{R})} \| g \|_{L^\infty(\mathbb{R})} + \| f\|_{L^\infty(\mathbb{R})} \| \Lambda^{s-1} g \|_{L^2(\mathbb{R})} \right).
\]

Lemma 2.11. \[34\] Let \( \sigma > 1 + \frac{1}{p} \) or \( \sigma = 1 + \frac{1}{p} \) and \( r = 1 \). Suppose \( f_0 \in B^\sigma_{p,r}(\mathbb{R}), \ g \in C([0,T]; B^{\sigma-1}_{p,r}(\mathbb{R})) \) and \( v \in C([0,T]; B^\sigma_{p,r}(\mathbb{R})) \). If \( f \in C([0,T]; B^\sigma_{p,r}(\mathbb{R})) \) is a solution of 1-D linear transport equation \((T)\), then we have \( f \) belongs to \( C^1([0,T]; B^{\sigma-1}_{p,r}(\mathbb{R})) \).

We introduce a lemma to give a priori estimate for the solutions of 1-D linear transport equation \((T)\) in Besov spaces \( B^{1+\frac{1}{p}}_{p,r} \).

Lemma 2.12. \[34\] Let \( 1 \leq p, r \leq \infty \) and \( \sigma = 1 + \frac{1}{p} \). Assume that \( f_0 \in B^\sigma_{p,r}(\mathbb{R}), \ F \in L^1(0,T; B^\sigma_{p,r}(\mathbb{R})) \) and \( v \in C([0,T]; S'(\mathbb{R})) \). If \( f \in L^\infty(0,T; B^\sigma_{p,r}(\mathbb{R})) \cap C([0,T]; S'(\mathbb{R})) \) solves the 1-D linear transport equation \((T)\), then there exists a constant \( C \) depending only on \( p, r \) and \( \sigma \), such that the following statement holds:
\[
\| f(t) \|_{B^\sigma_{p,r}(\mathbb{R})} \leq Ce^{CV(t)} \left( \| f_0 \|_{B^\sigma_{p,r}(\mathbb{R})} + \int_0^t e^{-CV(\tau)} \| F(\tau) \|_{B^\sigma_{p,r}(\mathbb{R})} d\tau \right),
\]
with \( V(t) = \int_0^t \| v(\tau) \|_{B^{\sigma+1}_{p,r}(\mathbb{R})} d\tau \).

**Notation.** In the following, since all spaces of functions are over \( \mathbb{R} \), for simplicity, we drop \( \mathbb{R} \) in our notations of function spaces if there is no ambiguity.

3. Local existence and uniqueness. In the section, we will establish the local existence and uniqueness of solutions for the Cauchy problem (1.4) in Besov spaces. Since \( m = u - uu_x \), we can rewrite (1.4) as follows:
\[
\begin{cases}
  m_t - 2(uu_x + u_x)m_x = 0, & t > 0, x \in \mathbb{R}, \\
  m = (1 - \partial_x^2)u, & t > 0, x \in \mathbb{R}, \\
  m_0(x) := m(0,x) = (1 - \partial_x^2)u_0(x), & x \in \mathbb{R}.
\end{cases}
\]

We now give a definition and show the local existence and uniqueness result.

**Definition 3.1.** Let \( T > 0, s \in \mathbb{R} \) and \( 1 \leq p \leq \infty \). Set
\[
E^s_{p,r}(T) \triangleq C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}), \quad \text{if} \ r < \infty,
\]
\[
E^s_{p,\infty}(T) \triangleq L^\infty([0,T]; B^s_{p,\infty}) \cap \text{Lip} ([0,T]; B^{s-1}_{p,\infty}), \quad \text{if} \ r = \infty.
\]

**Theorem 3.2.** Let \( 1 \leq p, r \leq \infty, \ s > 1 + \frac{1}{p} \), (or \( s = 1 + \frac{1}{p} \), \( r = 1 \), \( 1 \leq p < \infty \)) and \( m_0 \in B^s_{p,r} \). There exists a time \( T > 0 \) such that (3.1) has a unique solution \( m \in E^s_{p,r}(T) \).

We use four steps to prove the local existence and one proposition to prove the uniqueness of the solution to (3.1).

**First Step. constructing approximate solutions.** Starting from \( m^0 \triangleq 0 \), we define by induction a sequence \( (m^n)_{n \in \mathbb{N}} \) of smooth functions by solving the following linear system:
\[
(T_n) \begin{cases}
  (\partial_t - 2(uu_x + u_x)\partial_x) m^{n+1} = 0, & t > 0, x \in \mathbb{R}, \\
  m^n = (1 - \partial_x^2)u^n, & t > 0, x \in \mathbb{R}, \\
  m^n(0,x) = S_{n+1}m_0(x), & x \in \mathbb{R}.
\end{cases}
\]
Since \( m_0 \in B_{p,r}^s \), then all initial data \( S_{n+1}m_0 \in B_{p,r}^\infty \) and \( \|S_{n+1}m_0\|_{B_{p,r}^s} \leq C\|m_0\|_{B_{p,r}^s} \). Apply Lemma 2.9 and by induction, for every \( n \geq 1 \), \( T_{n-1} \) has a unique solution \( m^n \) in \( C^1([0,T];B_{p,r}^s) \). Obviously, \( m^n \) belongs to \( E_{p,r}(T) \) for all positive \( T \).

**Second Step. uniform bounds.** Using the embedding relations, product laws (Lemmas 2.4 – 2.5 ) and Lemma 2.8, we see that for all \( n \in \mathbb{N} \),

\[
\|m^{n+1}\|_{B_{p,r}^s} \leq C e^{C\|u^n\|_{B_{p,r}^s}} \|S_{n+1}m_0\|_{B_{p,r}^s},
\]

where \( U^n(t) = \int_0^t e^{(t-s)\|u^n\|_{B_{p,r}^s}}dt \) and \( C \geq 1 \). We now fix a \( T > 0 \) such that \( C^2\|m_0\|_{B_{p,r}^s} T < 1 \). By induction, we gain

\[
\forall t \in [0, T], \quad \|m^n\|_{B_{p,r}^s} \leq \frac{C\|m_0\|_{B_{p,r}^s}}{1 - C^2\|m_0\|_{B_{p,r}^s} t}, \quad \forall n \in \mathbb{N}.
\]

In fact, suppose it is valid for \( n \), by (3.2) and (3.3), we obtain

\[
\|m^{n+1}(t)\|_{B_{p,r}^s} \leq C e^{C\|u^n\|_{B_{p,r}^s}} \|S_{n+1}m_0\|_{B_{p,r}^s} \\
\leq C \exp \left\{ - \int_0^t \frac{d(1 - C^2t\|m_0\|_{B_{p,r}^s})}{1 - C^2t\|m_0\|_{B_{p,r}^s}} \right\} \|m_0\|_{B_{p,r}^s} \\
\leq \frac{C\|m_0\|_{B_{p,r}^s}}{1 - C^2t\|m_0\|_{B_{p,r}^s}} \leq \frac{C\|m_0\|_{B_{p,r}^s}}{1 - C^2T\|m_0\|_{B_{p,r}^s}}.
\]

Therefore, \((m^n)_{n \in \mathbb{N}}\) is bounded in \( L^\infty([0,T];B_{p,r}^s) \). This entails that \((u_{xx}^n + u_x^n)_{mx}^{n+1}\) is bounded in \( L^\infty([0,T];B_{p,r}^{s-1}) \). Then we can conclude that \((m^n)_{n \in \mathbb{N}}\) is bounded in \( E_{p,r}^s(T) \).

**Third Step. convergence.** We will prove that: \((m^n)_{n \in \mathbb{N}}\) is a Cauchy sequence of \( C([0,T];B_{p,r}^{s-1}) \). Indeed, for all \( n \in \mathbb{N} \), we have

\[
\left\{ \begin{array}{l}
(\partial_t - 2(u_{xx}^n + u_x^n)\partial_x)(m^{n+1} - m^n) \\
= 2(u_{xx}^n - u_{xx}^n + u_x^n - u_x^n)\partial_x m^{n+1} + \]
\end{array} \right.
\]

\[
(m^{n+1} - m^n)|_{t=0} = S_{n+1}m_0(x) - S_{n+1}m_0(x).
\]

From the definition of operator \( S_n \), we get

\[
(S_{n+1} - S_{n+1})m_0(x) = \sum_{q=n+1}^{n+l} \Delta_q m_0(x).
\]

Thanks to Remark 2.2 and Definition 2.3, we readily have

\[
\| \sum_{q=n+1}^{n+l} \Delta_q m_0\|_{B_{p,r}^{s-1}} = (\sum_{k \geq 1}^{n+l} 2^{k(s-1)r}\|\Delta_k (\sum_{q=n+1}^{n+l} \Delta_q m_0)\|_{L^r})^{\frac{1}{s}} \\
\leq C (\sum_{k=n+1}^{n+l} 2^{-kr}2^{k sr}\|\Delta_k m_0\|_{L^r})^{\frac{1}{s}} \leq C 2^{-n}\|m_0\|_{B_{p,r}^s}.
\]

Using Lemmas 2.4 – 2.5 and Lemmas 2.8 – 2.12, we see that, for all \( t \) in \([0,T]\),

\[
\|(m^{n+1} - m^n)(t)\|_{B_{p,r}^{s-1}} \leq C e^{C\|u^n\|_{B_{p,r}^{s-1}}} (\|S_{n+1}m_0 - S_{n+1}m_0\|_{B_{p,r}^{s-1}} + C \int_0^t e^{-C\|u^n\|_{B_{p,r}^{s-1}}} m^{n+1} - m^n\|_{B_{p,r}^{s-1}} m_{n+1} + m^n\|_{B_{p,r}^{s-1}} dt).
\]
Let \( b_n(t) \triangleq \|(m^{n+1} - m^n)(t)\|_{B^s_{p,r}} \). So we can find a positive \( C_T \) independent of \( n, m \) such that
\[
\begin{align*}
\int b_{n+1}^t(t) &\leq C_T(2^{-n} + \int_0^t b_n^t(\tau)d\tau), \quad \forall t \in [0, T], \\
b_1^t(t) &\leq C_T(1 + t), \quad \forall t \in [0, T].
\end{align*}
\]
Arguing by induction with respect to the index \( n \), we can obtain
\[
b_{n+1}^t(t) \leq (C_T \sum_{k=0}^n \frac{(2tC_T)^k}{k!})2^{-n} + \frac{(tC_T)^{n+1}}{(n+1)!}
\leq (C_T \sum_{k=0}^n \frac{(2tC_T)^k}{k!})2^{-n} + \frac{(tC_T)^{n+1}}{(n+1)!} \to 0, \quad \text{as } n \to \infty.
\]
Therefore, we deduce that \((m^n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \( C([0,T]; B^s_{p,r}^{-1}) \) and converges to a limit function \( m \in C([0,T]; B^s_{p,r}^{-1}) \).

**Fourth Step. conclusion.** We will show that \( m \) belongs to \( E^s_{p,r}(T) \) and satisfies \((3.1)\). Since \((m^n)_{n \in \mathbb{N}}\) is bounded in \( L^\infty(0,T; B^s_{p,r}) \), the Fatou property for Besov spaces guarantees that \( m \) also belongs to \( L^\infty(0,T; B^s_{p,r}) \). Now, as \((m^n)_{n \in \mathbb{N}}\) converges to \( m \in C([0,T]; B^s_{p,r}^{-1}) \), an interpolation argument (in Lemma 2.4) ensures that the convergence actually holds true in \( C([0,T]; B^s_{p,r}^{-1}) \) for any \( s' < s \). It is then easy to pass to the limit in \( (T_n) \) and to conclude that \( m(t, x) \) is a solution to \((3.1)\). Indeed, for any test function \( \varphi \in C^1([0,T]; S), \forall t \in [0,T] \), besides the easier terms, the most complicated terms in the convergence of the approximation are \( \int_0^t \langle u_{xx}^n, m_{xx}^n, \varphi \rangle dt' \) and \( \int_0^t \langle u_{xx}^n m_{xx}^{n+1}, \varphi \rangle dt' \). For the sake of conciseness, we only check the first one here. Letting \( s' = s - \frac{1}{4} \), then we have \( s' > \max\{\frac{1}{p}, \frac{1}{2}\} \). According to Lemmas 2.4 − 2.5, we obtain
\[
\begin{align*}
| \int_0^t \langle u_{xx}^n, m_{xx}^{n+1}, \varphi \rangle dt' - \int_0^t \langle u_{xx}^n, \varphi \rangle dt' |
\leq & \left| \int_0^t \langle (u_{xx}^n - u_{xx}) m_{xx}^{n+1}, \varphi \rangle dt' \right| + \left| \int_0^t \langle u_{xx}^n (m^{n+1} - m), \varphi \rangle dt' \right| \\
\leq & C \int_0^t \|(u_{xx}^n - u_{xx}) m_{xx}^{n+1}\|_{B^{-r}_{p,r}} \|\varphi\|_{B^{s'}_{p',r'}} dt' \\
+ & C \int_0^t \|u_{xx}^n (m^{n+1} - m), \varphi\|_{B^{-r}_{p,r}} \|\varphi\|_{B^{s'}_{p',r'}} dt' \\
\leq & C \left\{ \|m^n - m\|_{L^p_x(B_{p,r}^{s'}; L^p_x(B_{p,r}^{s'}))} \|m^{n+1}\|_{L^p_x(B_{p,r}^{s'})} \|\varphi\|_{L^p_x(B_{p,r}^{s'})} \\
+ & \|m\|_{L^p_x(B_{p,r}^{s'})} \|m^{n+1} - m\|_{L^p_x(B_{p,r}^{s'})} \|\varphi\|_{L^p_x(B_{p,r}^{s'})} \right\},
\end{align*}
\]
which ensures
\[
\int_0^t \langle u_{xx}^n m_{xx}^{n+1}, \varphi \rangle dt' \to \int_0^t \langle u_{xx} m_{xx}, \varphi \rangle dt', \quad (\text{as } n \to +\infty).
\]
Finally, because \( m \) belongs to \( L^\infty(0,T; B^s_{p,r}) \), in view of Lemma 2.9, the solution \( m \) belongs to \( C([0,T]; B^s_{p,r}) \) (resp., \( C_w([0,T]; B^s_{p,r}) \)) if \( r < \infty \) (resp., \( r = \infty \)). Applying Lemma 2.11, we see that \( \partial_t u \) is in \( C([0,T]; B^{s-1}_{p,r}) \) if \( r \) is finite, and in \( L^\infty([0,T]; B^{s-1}_{p,r}) \) otherwise. Hence we conclude that the solution \( m \in E^s_{p,r}(T) \).

Now, we turn to the uniqueness of the solution to \((3.1)\). In fact, it is a straightforward corollary of the following proposition.
Proposition 3.3. Let \( 1 \leq p, r \leq \infty, s > 1 + \frac{1}{p} \) (or \( s = 1 + \frac{1}{p}, \) \( r = 1, \) \( 1 \leq p < \infty \)). Suppose we are given \( m, \tilde{m} \in L^\infty([0, T]; B^s_{p,r}) \) two solutions of (3.1) with initial data \( m_0, \tilde{m}_0 \in B^s_{p,r} \). Then there exists a constant \( C = C(p, r, s) \) such that, for all \( t \in [0, T], \)

\[
|\tilde{m}(t) - \tilde{m}(t)|_{B^s_{p,r}} \leq C|\tilde{m}_0 - \tilde{m}_0|_{B^s_{p,r}} \exp \left( C \int_0^t (|m|_{B^s_{p,r}} + |\tilde{m}|_{B^s_{p,r}}) \, dt \right).
\]

Proof. Let \( v = \tilde{m} - m. \) Since \( m \) and \( \tilde{m} \) satisfy Eq. (3.1), then we have

\[
\begin{cases}
\partial_t v - 2(\tilde{u}_{xx} + \tilde{u}_x)v_x = 2(\partial_x^2(1 - \partial_x^2)^{-1}v + \partial_x(1 - \partial_x^2)^{-1}v)m_x, \\
v_0 = \tilde{m}_0 - m_0.
\end{cases}
\]

Applying Lemmas 2.4 - 2.5 and Lemmas 2.8 - 2.12, we obtain

\[
|v(t)|_{B^s_{p,r}} \leq C e^{C f_0^t (|\tilde{m}|_{B^s_{p,r}} + |m|_{B^s_{p,r}}) \, dt'}
\]

By Gronwall’s inequality, we get

\[
|v(t)|_{B^s_{p,r}} \leq C|v_0|_{B^s_{p,r}} e^{C f_0^t (|\tilde{m}|_{B^s_{p,r}} + |m|_{B^s_{p,r}}) \, dt'}.
\]

4. Continuity with respect to initial data. In this section, we will prove the solution of (1.4) guaranteed by Theorem 3.2 depends continuously on the initial data. First, we introduce a useful lemma in the proof of the next theorem.

Lemma 4.1. [34] Let \( 1 \leq p \leq \infty, 1 \leq r < \infty, \sigma > 1 + \frac{1}{p} \) (or \( \sigma = 1 + \frac{1}{p}, \) \( r = 1, \) \( 1 \leq p < \infty \)). Denote \( \tilde{\mathbb{N}} = \mathbb{N} \cup \{ \infty \}. \) Let \( (v^n)_{n \in \tilde{\mathbb{N}}} \) be a sequence of functions belonging to \( C([0, T]; B^\sigma_{p,r}) \). Assume that \( v^n \) is the solution to

\[
\partial_t v^n + a^n \partial_x v^n = f, \quad v^n|_{t=0} = v_0,
\]

with \( v_0 \in B^\sigma_{p,r} \), \( f \in L^1(0, T; B^\sigma_{p,r}) \) and that, for some \( \alpha(t) \in L^1(0, T), \)

\[
\sup_{n \in \tilde{\mathbb{N}}} \|a^n(t)\|_{B^\sigma_{p,r}} \leq \alpha(t).
\]

If \( a^n \) tends to \( a^\infty \) in \( L^1(0, T; B^\sigma_{p,r}) \), then \( v^n \) tends to \( v^\infty \) in \( C([0, T]; B^\sigma_{p,r}) \).

Theorem 4.2. Let \( 1 \leq p, r \leq \infty, s > 1 + \frac{1}{p} \) (or \( s = 1 + \frac{1}{p}, \) \( r = 1, \) \( 1 \leq p < \infty \)). Denote \( \tilde{\mathbb{N}} = \mathbb{N} \cup \{ \infty \}. \) Let \( (m^n)_{n \in \tilde{\mathbb{N}}} \) be the corresponding solution of (1.4) guaranteed by Theorem 3.2 with initial data \( m^n_0(x) \in B^s_{p,r}. \) If \( m^n_0 \) tends to \( m^\infty_0 \) in \( B^s_{p,r}, \) then we have \( m^n(t, x) \) tends to \( m^\infty(t, x) \) in \( C([0, T]; B^s_{p,r}) \) (resp., \( C_w([0, T]; B^s_{p,r}) \) if \( r < \infty \) (resp., \( r = \infty \)) with \( C^2 \sup_{n \in \tilde{\mathbb{N}}} \|m^n_0\|_{B^s_{p,r}, T} < 1. \)

Proof. By Proposition 3.3, we get \( \|m^n - m^\infty\|_{L^\infty(0, T; B^s_{p,r})} \) tends to zero as \( n \to \infty. \)

So we only need to prove \( \|\partial_x m^n - \partial_x m^\infty\|_{L^\infty(0, T; B^s_{p,r})} \) tends to zero as \( n \to \infty \) where \( r \) is finite. According to Theorem 3.2, we can find \( M > 0 \) such that for all \( n \in \mathbb{N}, \)

\[
\sup_{n \in \tilde{\mathbb{N}}} \|m^n\|_{L^\infty(B^s_{p,r})} \leq M.
\]
Denote $v^n = \partial_x m^n$. Note that $v^n$ solves the following linear transport equation:

$$\partial_t v^n + a^n \partial_x v^n = f^n, \quad v^n|_{t=0} = \partial_x m_0^n,$$

where

$$a^n := -2(u^n_{xx} + u^n_x), \quad f^n := -2(m^n_x - u^n_{xx} - u^n_x)m^n_x.$$

We decompose $v^n$ into $v^n = z^n + w^n$ such that

$$\begin{cases}
\partial_t z^n + a^n \partial_x z^n = f^n - f^\infty, \\
\partial_t w^n + a^n \partial_x w^n = f^\infty,
\end{cases} \quad \text{and} \quad \frac{\partial_t w^n}{|w^n|_{t=0}} = \partial_x m_0^n.$$

Note that $|w^n|_{t=0}$ is uniformly bounded in $C([0, T]; B^{s-1}_{p, r})$ and $f^n - f^\infty = -2(m^n_x - u^n_{xx} - u^n_x)(m^n_x - m^\infty_x) - 2(m^n_x - m^\infty_x) - (u^n_{xx} - u^\infty_x) - (u^n_x - u^\infty_x)m^\infty_x$.

Applying Lemmas 2.8 – 2.10, product laws and embedding relations, we get

$$
\|z^n(t)\|_{B^{s-1}_{p, r}} \leq C e^{C \int_0^t \|m^n(t')\|_{B^{s-1}_{p, r}} dt'} \left(\|\partial_x m^n_0 - \partial_x m_0^\infty\|_{B^{s-1}_{p, r}}ight)
+ \int_0^t \left(\|m^n(t')\|_{B^{s-1}_{p, r}} + \|m^\infty(t')\|_{B^{s-1}_{p, r}}\right)\|\partial_x m^n(t') - \partial_x m^\infty(t')\|_{B^{s-1}_{p, r}} dt'.
$$

Since the sequence $(a^n)_{n \in \mathbb{N}}$ is uniformly bounded in $C([0, T]; B^{s-1}_{p, r})$ and tends to $a^\infty$ in $C([0, T]; B^{s-1}_{p, r})$, Lemma 4.1 tells us that $w^n$ tends to $w^\infty = \partial_x m^\infty$ in $C([0, T]; B^{s-1}_{p, r})$. Let $\epsilon > 0$. Since $m^n$ tends to $m^\infty$ in $C([0, T]; B^{s-1}_{p, r})$, the last term is less than $\frac{1}{2} \epsilon$ for large $n$. Combining the above convergence results and Lemma 2.8, one concludes that for large enough $n \in \mathbb{N}$,

$$
\|\partial_x m^n(t) - \partial_x m^\infty(t)\|_{B^{s-1}_{p, r}} \leq \epsilon + CM e^{C Mt} \left(\|\partial_x m^n_0 - \partial_x m_0^\infty\|_{B^{s-1}_{p, r}}ight)
+ \int_0^t \|\partial_x m^n(t') - \partial_x m^\infty(t')\|_{B^{s-1}_{p, r}} dt'.
$$

Hence, thanks to Gronwall’s inequality, we get

$$
\|\partial_x m^n(t) - \partial_x m^\infty(t)\|_{L^\infty([0, T]; B^{s-1}_{p, r})} \leq \tilde{C}(\epsilon + \|\partial_x m^n_0 - \partial_x m_0^\infty\|_{B^{s-1}_{p, r}}),
$$

for some constant $\tilde{C}$ depending only on $s, p, r M$ and $T$. This completes the proof in the case $r < \infty$.

When $r = \infty$, for fixed $\phi \in B^{s-1}_{p, 1}$, we write

$$
\langle m^n(t) - m^\infty(t), \phi \rangle = \langle S_j[m^n(t) - m^\infty(t)], \phi \rangle + \langle (I - S_j)[m^n(t) - m^\infty(t)], \phi \rangle
= \langle m^n(t) - m^\infty(t), S_j \phi \rangle + \langle m^n(t) - m^\infty(t), (I - S_j) \phi \rangle.
$$

Applying Lemma 2.4, we have

$$
\|\langle m^n(t) - m^\infty(t), (I - S_j) \phi \rangle\| \leq CM \|\phi - S_j \phi\|_{B^{s-1}_{p, 1}}, \quad \text{(4.1)}
$$

and

$$
\|\langle m^n(t) - m^\infty(t), S_j \phi \rangle\| \leq C \|m^n - m^\infty\|_{L^\infty([0, T]; B^{s-1}_{p, 1})} \|S_j \phi\|_{B_{p, 1}^{s-1}}. \quad \text{(4.2)}
$$

Note that $\|\phi - S_j \phi\|_{B^{s-1}_{p, 1}}$ tends to zero as $j \to \infty$ and $\|m^n - m^\infty\|_{L^\infty([0, T]; B^{s-1}_{p, 1})}$ tends to zero as $n \to \infty$. Then (4.1) can be made arbitrarily small for $j$ large enough. For
fixed \( j \), we then let \( n \) tend to infinity so that (4.2) tends to zero, and we conclude that \( \langle m^\infty (t) - m^\infty (\phi) \rangle \) tends to zero. Then this completes the proof in the case \( r = \infty \).

**Remark 4.3.** Note that if \( p = r = 2 \), then \( H^s = B^s_{2,2} \). By Theorem 3.2 and Theorem 4.2, we can get the local well-posedness result of (3.1) in \( C([0, T]; H^s) \cap C^1([0, T]; H^{s-1}) \), \( s > \frac{3}{2} \).

5. **Blow-up and blow-up rate.** In this section, we first establish a blow-up criterion for the solutions to (1.4) in Sobolev spaces. Using this blow-up criterion, we then present a sufficient condition on the initial data that ensures the corresponding strong solution of (1.4) to blow up in finite time. Finally, we give a blow-up rate for the blow-up solutions to (1.4).

**Lemma 5.1.** Suppose \( m_0 \in H^s \), \( s > \frac{3}{2} \). Let \( m \in C([0, T]; H^s) \) be the corresponding solution to (1.4) with the initial data \( m_0 \). Then there exists a constant \( C = C(s) \) such that, for all \( t \in [0, T) \),

\[
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} e^{\int_0^t \|m_x\|_{L^\infty} + \|m_0\|_{L^\infty} dt'}.
\]

**Proof.** Set \( \Lambda = (1 - \partial_x^2) \frac{1}{2} \). Applying \( \Lambda^s \) to both sides of (1.4) and taking the \( L^2 \) inner product with \( \Lambda^s m \), we have

\[
\frac{d}{dt} \int_{\mathbb{R}} (\Lambda^s m)^2 dx = -4 \int_{\mathbb{R}} \Lambda^s (m_x (m - u_x - u)) \Lambda^s m dx
\]

\[
= -4 \int_{\mathbb{R}} \Lambda^s (m_x m) \Lambda^s m dx + 4 \int_{\mathbb{R}} \Lambda^s (m_x u_x) \Lambda^s m dx
\]

\[
+ 4 \int_{\mathbb{R}} \Lambda^s (m_x u) \Lambda^s m dx.
\]

For the last term of the right-hand side of (5.2), applying Lemma 2.10, we obtain

\[
\int_{\mathbb{R}} \Lambda^s (m_x u) \Lambda^s m dx = \int_{\mathbb{R}} [\Lambda^s (m_x u) - u \Lambda^s m_x] \Lambda^s m dx - \frac{1}{2} \int_{\mathbb{R}} u_x (\Lambda^s m)^2 dx
\]

\[
\leq C (\|m_x\|_{L^\infty} + \|u_x\|_{L^\infty}) \|\Lambda^s m\|_{L^2}^2
\]

\[
\leq C (\|m_x\|_{L^\infty} + \|u_x\|_{L^\infty}) \|m\|_{H^s}^2.
\]

The similar argument allows us to show that

\[
\int_{\mathbb{R}} \Lambda^s (m_x m) \Lambda^s m dx \leq C (\|m_x\|_{L^\infty} \|m\|_{H^s}^2),
\]

\[
\int_{\mathbb{R}} \Lambda^s (m_x u_x) \Lambda^s m dx \leq C (\|m_x\|_{L^\infty} + |u_x|_{L^\infty}) \|m\|_{H^s}^2.
\]

Plugging (5.3), (5.4) and (5.5) into (5.2), we have

\[
\frac{d}{dt} \|m\|_{H^s}^2 \leq C (\|m_x\|_{L^\infty} + |u_x|_{L^\infty} + |u|_{L^\infty} + \|m\|_{L^\infty}) \|m\|_{H^s}^2,
\]

which implies that

\[
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} e^{\int_0^t \left( \|m_x\|_{L^\infty} + |u_x|_{L^\infty} + |u|_{L^\infty} + \|m\|_{L^\infty} \right) dt'}.
\]

Now, let us consider the following initial value problem
\[
\begin{cases}
q_t = 2(m - u_x - u)(t, q), \ t \in [0, T), \\
q(0, x) = x, \ x \in \mathbb{R}.
\end{cases}
\] (5.7)

By (1.4), one easily has \(m(t, q(t, x)) = m_0\), which leads to \(\|m(t)\|_{L^\infty} = \|m_0\|_{L^\infty}\).

Let \(p = \frac{1}{2}e^{-|x|}\). As \(u = p \ast m\) and \(u_x = \partial_x p \ast m\), then we have
\[
\|u(t)\|_{L^\infty} \leq \|m(t)\|_{L^\infty} = \|m_0\|_{L^\infty}, \ \|u_x(t)\|_{L^\infty} \leq \|m(t)\|_{L^\infty} = \|m_0\|_{L^\infty}.
\] (5.8)

Using Gronwall’s inequality and (5.6), we get
\[
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} e^{C \int_0^T (\|m_x\|_{L^\infty} + \|m_0\|_{L^\infty}) \, dt'}.
\]
This completes the proof of Lemma 5.1.

\[\text{Theorem 5.2.} \text{ Let } m_0 \in H^s, \ s > \frac{5}{2}. \text{ Then the corresponding solution } m = m(t, x) \text{ to } (1.4) \text{ blows up in finite time } T \text{ if and only if } \liminf_{t \uparrow T} \inf_{x \in \mathbb{R}} \{m_x(t, x)\} = -\infty.\]

\[\text{Proof.} \text{ On the one hand, by Lemma 5.1 and Sobolev’s imbedding theorem, it is clear that if the slope of the solution tends to minus infinity in finite time, then } T < \infty.\]

On the other hand, we first multiply (1.4) by \(m\) and integrate by parts to get
\[
\frac{d}{dt} \int_{\mathbb{R}} m^2 \, dx = -4 \int_{\mathbb{R}} (m - u_x - u)m_x m \, dx = -2 \int_{\mathbb{R}} (u_{xx} + u_x)m^2 \, dx. \quad (5.9)
\]

Then, differentiating (1.4) with respect to the spatial variable \(x\), multiplying the obtained equation by \(m_x\) and integrating by parts, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}} m_x^2 \, dx = -4 \int_{\mathbb{R}} \partial_x [(m - u_x - u)m_x] m_x \, dx \\
= 4 \int_{\mathbb{R}} (m - u_x - u)m_x m_{xx} \, dx \\
= -2 \int_{\mathbb{R}} (m_x - u_{xx} - u_x)m_x^2 \, dx. \quad (5.10)
\]

Next, differentiating (1.4) twice with respect to the spatial variable \(x\), multiplying the obtained equation by \(m_{xx}\) and integrating by parts, we have
\[
\frac{d}{dt} \int_{\mathbb{R}} m_{xx}^2 \, dx = -4 \int_{\mathbb{R}} \partial_x^2 [(m - u_x - u)m_x] m_{xx} \, dx \quad (5.11)
\]
\[
= -6 \int_{\mathbb{R}} (m_x - u_{xx} - u_x)m_{xx}^2 \, dx - 4 \int_{\mathbb{R}} \partial_x^2 (m - u_x - u)m_x m_{xx} \, dx \\
= -10 \int_{\mathbb{R}} m_x m_{xx}^2 \, dx + 6 \int_{\mathbb{R}} (u_{xx} + u_x)m_{xx}^2 \, dx + 2 \int_{\mathbb{R}} (m_x - u_{xx} - u_x)m_x^2 \, dx.
\]

Combining (5.9), (5.10) and (5.11), we get
\[
\frac{d}{dt} \|m\|_{H^2}^2 \leq C \|m\|_{H^2}^2 - 10 \int_{\mathbb{R}} m_x m_{xx}^2 \, dx.
\]

If \(m_x\) is bounded from below on \([0, T) \times \mathbb{R}\), i.e., if there exists \(M > 0\) such that
\[
-m_x(t, x) \leq M \quad \text{on} \quad [0, T) \times \mathbb{R},
\]
then we deduce
\[
\frac{d}{dt} \|m\|^2_{L^2} \leq (C + 10M)\|m\|^2_{L^2}.
\]
Applying Gronwall’s inequality and Lemma 5.1, we conclude that \(\|u(t)\|_{H^s}\) is uniformly bounded in \(t \in [0, T]\) which contradicts the fact that the solution blows up in finite time \(T\). This completes the proof of Theorem 5.2. \(\square\)

**Theorem 5.3.** Let \(m_0 \in H^s, \ s > \frac{5}{2}\). Assume there exists a point \(x_0 \in \mathbb{R}\) such that \(m_x(x_0) < -3\|m_0\|_{L^\infty}\). Then the solution \(m(t, x)\) of (1.4) blows up in finite time \(T\).

**Proof.** Take \(K = 3\|m_0\|_{L^\infty}\). From (1.4) and (5.8), we see that
\[
\partial_t m_x + 2(m - u_x - u)m_{xx} = -2m_x^2 + 2(u_{xx} + u_x)m_x \leq -m_x^2 + K^2, \quad (5.12)
\]
which associating with (5.7) yields
\[
\partial_t m_x(t, q(t, x)) \leq -m_x^2(t, q(t, x)) + K^2. \quad (5.13)
\]
Set \(y(t) = m_x(t, q(t, x_0))\). According to the assumption that \(m_x(x_0) < -K\), the inequality (5.13) at \(x_0\) becomes \(y(t) \leq -y^2(t) + K^2\). Then we claim: \(y(t) < -K\).

In fact, let \(T^* = \sup\{t \in [0, T]\} |y(t) < -K\}\). If \(T^* < T\), then we have \(y(T^*) = -K\) and \(y(t) < -K\), for \(t \in (0, T^*]\). However, we get \(y'(t) \leq -y^2(t) + K^2 \leq 0\) for \(\forall t \in [0, T^*]\), which implies that \(y(T^*) \leq m_x(x_0) < -K\). This contradicts \(y(T^*) = -K\). So we have \(T^* = T\). From the inequality \(y'(t) \leq -y^2(t) + K^2\), we obtain
\[
\frac{y(0) + K}{y(0)} e^{2Kt} - 1 \leq \frac{2K}{y(t) - K} < 0.
\]
Due to 0 < \(\frac{y(0) + K}{y(0)} - K\) < 1, there exists \(0 < T < \frac{1}{2K} \ln(\frac{y(0) + K}{y(0)} - K)\), such that \(\lim_{t \to T} y(t) = -\infty\). This completes the proof of Theorem 5.3. \(\square\)

**Lemma 5.4.** [13] Let \(T > 0\) and \(w \in C^1([0, T); H^s(\mathbb{R})]\) with \(s > \frac{3}{2}\). Then for every \(t \in [0, T]\), there exists at least one point \(\xi \in \mathbb{R}\) with
\[
m(t) := \inf_{x \in \mathbb{R}} w_x(x, t) = w_x(\xi(t), t).
\]
The function \(m(t)\) is absolutely continuous on \((0, T)\) with
\[
\frac{dm}{dt} = w_x(\xi(t), t), \quad \text{a.e. on } (0, T).
\]

**Theorem 5.5.** Let \(m_0 \in H^s, \ s > \frac{5}{2}\) and let \(T > 0\) be the maximal existence time of the corresponding solution \(m(t, x)\) to (1.4) with the initial data \(m_0\). If \(T < \infty\), then we have
\[
\lim_{t \to T} (T - t) \min_{x \in \mathbb{R}} m_x(t, x) = -\frac{1}{2}. \quad (5.14)
\]

**Proof.** We already know by Theorem 5.2 that
\[
\liminf_{t \uparrow T} \{ \inf_{x \in \mathbb{R}} |m_x(t, x)| \} = -\infty.
\]
Denote now \(m(t) = \min_{x \in \mathbb{R}} m_x(t, x)\) and let \(\xi(t) \in \mathbb{R}\) be a point where this minimum is attained. Clearly \(m_{xx}(t, \xi(t)) = 0\) since \(m(t, \cdot) \in H^3 \subset C^2\). Evaluating (5.12) at \((t, \xi(t))\), we obtain
\[
\frac{d}{dt} m = -2m_x^2 + 2m \cdot (u_{xx} + u_x)(t, \xi(t)), \quad \text{a.e. } t \in [0, T), \quad (5.15)
\]
which implies \(m'(t) \leq -m_x^2(t) + K^2\) for almost all \(t \in [0, T)\).
For any $\epsilon \in (0, \frac{1}{2})$, since $\liminf_{t \to T} m(t) = -\infty$, there exists $\delta > 0$ such that $m(T - \delta) < -\frac{2K}{\epsilon}$. By the standard bootstrap argument, we get

$$m(t) < -\frac{2K}{\epsilon}, \quad \forall \ t \in (T - \delta, T).$$

According to (5.15) and (5.4), we have

$$\frac{d}{dt} \left( \frac{1}{m} \right) = 2 - \frac{g(t)}{m(t)}, \quad \text{with} \quad g(t) := 2(u_{xx} + u_x)(t, \xi(t)),$$

which yields

$$2 + \epsilon \geq \frac{d}{dt} \left( \frac{1}{m} \right) \geq 2 - \epsilon, \quad \text{a.e.} \quad t \in (T - \delta, T). \quad (5.16)$$

Let $t \in (T - \delta, T)$. Integrating (5.16) on $(t, T)$, we obtain

$$(2 + \epsilon)(T - t) \geq -\frac{1}{m(t)} \geq (2 - \epsilon)(T - t),$$

that is,

$$\frac{1}{2 + \epsilon} \leq -m(t)(T - t) \leq \frac{1}{2 - \epsilon}.$$

By the arbitrariness of $\epsilon \in (0, \frac{1}{2})$, then we get (5.14). This completes the proof of Theorem 5.5.

**Acknowledgments.** The authors thank the referees for their valuable comments and suggestions.

**REFERENCES**

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften, Vol. 343, Berlin-Heidelberg-NewYork: Springer, 2011.

[2] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, *Arch. Ration. Mech. Anal.*, 183 (2007), 215–239.

[3] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, *Anal. Appl.*, 5 (2007), 1–27.

[4] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, 71 (1993), 1661–1664.

[5] R. Camassa, D. Holm and J. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.*, 31 (1994), 1–33.

[6] G. M. Coclite and K. H. Karlsen, On the well-posedness of the Degasperis-Procesi equation, *J. Funct. Anal.*, 233 (2006), 60–91.

[7] A. Constantin, The Hamiltonian structure of the Camassa-Holm equation, *Exposition. Math.*, 15 (1997), 53–85.

[8] A. Constantin, On the scattering problem for the Camassa-Holm equation, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 457 (2001), 953–970.

[9] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: A geometric approach, *Ann. Inst. Fourier (Grenoble)*, 50 (2000), 321–362.

[10] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.*, 166 (2006), 523–535.

[11] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26 (1998), 303–328.

[12] A. Constantin and J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math.*, 51 (1998), 475–504.

[13] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, 181 (1998), 229–243.

[14] A. Constantin and J. Escher, Particle trajectories in solitary water waves, *Bull. Amer. Math. Soc.*, 44 (2007), 423–431.
[15] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, Ann. of Math., 173 (2011), 559–568.
[16] A. Constantin, V. S. Gerdjikov and R. I. Ivanov, Inverse scattering transform for the Camassa-Holm equation, Inverse Problems, 22 (2006), 2197–2207.
[17] A. Constantin, R. I. Ivanov and J. Lenells, Inverse scattering transform for the Degasperis-Procesi equation, Nonlinearity 23 (2010), 2559–2575.
[18] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, Arch. Ration. Mech. Anal., 192 (2009), 165–186.
[19] A. Constantin and H. P. McKean, A shallow water equation on the circle, Comm. Pure Appl. Math., 52 (1999), 949–982.
[20] A. Constantin and W. A. Strauss, Stability of peakons, Comm. Pure Appl. Math., 53 (2000), 603–610.
[21] R. Danchin, A few remarks on the Camassa-Holm equation. Differential Integral Equations, 14 (2001), 953–988.
[22] R. Danchin, A note on well-posedness for Camassa-Holm equation, J. Differential Equations, 192 (2003), 429–444.
[23] A. Degasperis, D. D. Holm and A. N. W. Hone, A new integrable equation with peakon solutions, Theoret. and Math. Phys., 133 (2002), 1463–1474.
[24] A. Degasperis and M. Procesi, Asymptotic integrability. Symmetry and perturbation theory, World Sci. Publ., River Edge, NJ, 1999, 23–37.
[25] H. R. Dullin, G. A. Gottwald and D. D. Holm, On asymptotically equivalent shallow water wave equations, Phys. D, 190 (2004), 1–14.
[26] J. Escher, Y. Liu and Z. Yin, Global weak solutions and blow-up structure for the Degasperis-Procesi equation, J. Funct. Anal., 241 (2006), 457–485.
[27] J. Escher, Y. Liu and Z. Yin, Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation, Indiana Univ. Math. J., 56 (2007), 87–117.
[28] A. Fokas and B. Fuchssteiner, Symplectic structures, their B"acklund transformation and hereditary symmetries, Phys. D, 4 (1981/82), 47–66.
[29] G. Gui and Y. Liu, On the Cauchy problem for the Degasperis-Procesi equation, Quart. Appl. Math., 69 (2011), 445–464.
[30] A. A. Himonas and C. Holliman, The Cauchy problem for the Novikov equation, Nonlinearity, 25 (2012), 449–479.
[31] A. N. W. Hone and J. Wang, Integrable peakon equations with cubic nonlinearity, J. Phys. A, 41 (2008), 372002, 10 pp.
[32] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math., 41 (1988), 891–907.
[33] J. Lenells, Traveling wave solutions of the Degasperis-Procesi equation, J. Math. Anal. Appl., 306 (2005), 72–82.
[34] J. Li and Z. Yin, Well-posedness and global existence for a generalized Degasperis–Procesi equation, Nonlinear Anal. Real World Appl., 28 (2016), 72–90.
[35] Y. Liu and Z. Yin, Global Existence and Blow-up Phenomena for the Degasperis-Procesi Equation, Comm. Math. Phys., 267 (2006), 801–820.
[36] Y. Liu and Z. Yin, On the blow-up phenomena for the Degasperis-Procesi equation, Int. Math. Res. Not. IMRN, 23 (2007), Art. ID rnm117, 22 pp.
[37] H. Lundmark, Formation and dynamics of shock waves in the Degasperis-Procesi equation, J. Nonlinear Sci., 17 (2007), 169–198.
[38] H. Lundmark and J. Szmigielski, Multi-peakon solutions of the Degasperis-Procesi equation, Inverse Problems, 19 (2003), 1241–1245.
[39] V. Novikov, Generalization of the Camassa-Holm equation, J. Phys. A, 42 (2009), 342002, 14 pp.
[40] J. F. Toland, Stokes waves, Topol. Methods Nonlinear Anal., 7 (1996), 1–48.
[41] V. O. Vakhnenko and E. J. Parkes, Periodic and solitary-wave solutions of the Degasperis-Procesi equation, Chaos Solitons Fractals, 20 (2004), 1059–1073.
[42] X. Wu and Z. Yin, Global weak solutions for the Novikov equation, J. Phys. A, 44 (2011), 055202, 17pp.
[43] X. Wu and Z. Yin, Well-posedness and global existence for the Novikov equation, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 11 (2012), 707–727.
[44] X. Wu and Z. Yin, A note on the Cauchy problem of the Novikov equation, Appl. Anal., 92 (2013), 1116–1137.
[45] W. Yan, Y. Li and Y. Zhang, The Cauchy problem for the integrable Novikov equation, J. Differential Equations, 253 (2012), 298–318.

[46] W. Yan, Y. Li and Y. Zhang, The Cauchy problem for the Novikov equation, NoDEA Nonlinear Differential Equations Appl., 20 (2013), 1157–1169.

[47] Z. Yin, On the Cauchy problem for an integrable equation with peakon solutions, Illinois J. Math., 47 (2003), 649–666.

[48] Z. Yin, Global existence for a new periodic integrable equation, J. Math. Anal. Appl., 283 (2003), 129–139.

[49] Z. Yin, Global weak solutions for a new periodic integrable equation with peakon solutions, J. Funct. Anal., 212 (2004), 182–194.

[50] Z. Yin, Global solutions to a new integrable equation with peakons, Indiana Univ. Math. J., 53 (2004), 1189–1209.

Received November 2015; revised December 2015.

E-mail address: lijl29@mail2.sysu.edu.cn
E-mail address: mcsyzy@mail.sysu.edu.cn