Lengths of maximal green sequences for tame path algebras

Ryoichi Kase$^{1*}$ and Ken Nakashima$^2$

Abstract
In this paper, we study the maximal length of maximal green sequences for quivers of type $\tilde{D}$ and $\tilde{E}$ by using the theory of tilting mutation. We show that the maximal length does not depend on the choice of the orientation and determine it explicitly. Moreover, we give a program which counts all maximal green sequences by length for a given Dynkin/extended Dynkin quiver.

Keywords: Maximal green sequence, ($\tau$-)tilting theory, Quiver representation, Tame path algebra, Support ($\tau$-)tilting poset

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1 Introduction
Maximal green sequences were introduced by B. Keller to obtain quantum dilogarithm identities and refined Donaldson–Thomas invariants [15]. These are maximal sequences of “green quiver mutations.” If we consider a finite acyclic quiver and its path algebra, then maximal green sequences induce maximal chains of torsion classes in the module category [16, 17, 19] and maximal paths in the Hasse quiver of support ($\tau$-)tilting poset [5, 6]. By using these connections, T. Brüstle, G. Dupont, and M. Pérotin showed the finiteness of maximal green sequences for (simply laced) Dynkin/extended Dynkin quivers. Moreover, they presented a conjecture for lengths of maximal green sequences so-called no gap conjecture.

1.1 Lengths of maximal green sequences and no gap conjecture
Let $Q$ be an acyclic quiver and $A = KQ$ its path algebra over an algebraically closed field $K$. We denote by $sr$-tilt $A$ the poset of support $\tau$-tilting modules for $A$ and by $\overline{H}(sr$-tilt$A)$ its Hasse quiver (see Sect. 2). Then, in terms of ($\tau$-)tilting theory, the no gap conjecture is presented as follows.

Conjecture 1.1 ([5]) For each finite-dimensional path algebra $A$, possible lengths of maximal paths in $\overline{H}(sr$-tilt$A)$ form an interval in $\mathbb{Z}$.

We remark that no gap conjecture does not hold if we consider arbitrary finite-dimensional algebras. It was shown by Garver–McConville that no gap conjecture holds for cluster
tilted algebras of type $A$ [9]. Hermez–Igusa extended this result to cluster-tilted algebras of finite type and path algebras of tame type [11]. Therefore, for each Dynkin or extended Dynkin quiver $Q$, there are integers $\ell(Q)$ and $\ell'(Q)$ such that

$$\ell\text{MGS}(A) = [\ell'(Q), \ell(Q)] := \{\ell \in \mathbb{Z} \mid \ell'(Q) \leq \ell \leq \ell(Q)\},$$

where $\ell\text{MGS}(A)$ denotes the set of possible lengths of paths in $\overline{\mathcal{H}}$ (sr-tilt$A$). If we regard $\ell'(Q)$ as the minimal number in $\ell\text{MGS}(A)$ for an arbitrary acyclic quiver $Q$, then it is well-known that $\ell'(Q)$ is equal to the number of vertices of $Q$. In particular, we have the following equations.

$$\ell'(Q) = \#Q_0$$
$$\quad = \#\{\text{simple modules}\}/ \simeq$$
$$\quad = \#\{\text{indecomposable projective modules}\}/ \simeq$$
$$\quad = \#\{\text{indecomposable injective modules}\}/ \simeq$$

Moreover, the minimal length $\ell'(Q)$ was given in [7, 10] for each quiver $Q$ mutation equivalent to type $A$, $D$, or $\tilde{A}$.

For a quiver $Q$ of type $A_n$, $D_n$, $E_6$, $E_7$, $\tilde{A}_n$, the maximal length $\ell(Q)$ is also calculated.

**Theorem 1.2**

1. If $Q$ is a (simply-laced) Dynkin quiver, then we have

$$\ell(Q) = \#\{\text{indecomposable modules}\}/ \simeq .$$

2. If $Q$ is a quiver of type $\tilde{A}_{a,b}$, then we have

$$\ell(Q) = \frac{n(n+1)}{2} + ab.$$ 

In particular, $\ell(Q)$ is an invariant for sink or source mutations [2].

Further, Brüstle–Dupont–Pérotin computed $\ell(Q)$ for certain quivers $Q$ of type $\tilde{D}_n$ ($n \leq 7$), $\tilde{E}_6$, $\tilde{E}_7$. They also checked that $\ell(Q)$ (hence $\ell\text{MGS}(KQ)$) does not depend on the choice of the orientation of a quiver $Q$ of type $\tilde{D}_1$ ([5] and its arXiv version [arXiv:1205.2050v1]).

**1.2 Aim of this paper**

Theorem 1.2 and Brüstle–Dupont–Pérotin’s calculation give us the following question.

**Question 1.3** Let $Q$ be a finite acyclic quiver and $A = KQ$.

1. Let $\mu_i Q$ be the quiver given by mutating $Q$ at $i$. Does the equation $\ell(Q) = \ell(\mu_i Q)$ hold for each sink or source vertex $i$ of $Q$?

2. What is $\ell(Q)$? (If $Q$ is of type $A$, $D$, $E$, or $\tilde{A}$, then we know $\ell(Q)$ explicitly. Moreover, in the case of Dynkin quivers, it is the number of (isomorphism classes of) indecomposable $KQ$-modules.)

In this paper, we consider the above questions in the case of $\tilde{D}$ and $\tilde{E}$.

**Main Theorem 1** Let $Q$ be a quiver of type $\tilde{D}$ or $\tilde{E}$ and $A = KQ$.

1. $\ell(Q)$ does not depend on the choice of orientation.
2. $\ell(Q)$ is given by the following table.
In addition, we have created a program which counts all maximal green sequences of the path algebra $KQ$ by length for a given tame quiver $Q$. The program is available at the following URL and can be freely used by anyone for research purposes.

https://hfipy3.github.io/MGS/en.html

Remark 1.4 In [2], Apruzzese and Igusa also discuss the connection to the stability conditions. It is interesting that a similar result holds for the cases $\tilde{D}$ and $\tilde{E}$.

Notation

Throughout this paper, we use the following notation.

1. Algebras are finite-dimensional over an algebraically closed field $K$.
2. For an algebra $A$, we denote by $\text{mod } A$ (resp. $\text{proj } A$) the category of finite-dimensional right $A$-modules (resp. finite-dimensional projective right $A$-modules), and denote by $\tau = \tau_A$ the Auslander–Reiten translation of $\text{mod } A$ (refer to [3, 4] for definition and properties).
3. Let $Q$ be a finite quiver. For a path $w$ from $i$ to $j$ in $Q$, we define $s(w) := i$ and $t(w) := j$. For arrows $i \xrightarrow{a} j$ and $j \xrightarrow{b} k$ in $Q$, the product $ab$ in $KQ$ is the path $i \xrightarrow{a} j \xrightarrow{b} k$. For a vertex $i$ of $Q$, we denote by $\deg(i)$ the degree of $i$, i.e., the number of arrows that are connected to $i$.
4. Let $(\mathbb{P}, \leq)$ be a poset.
   - $\overrightarrow{H}(\mathbb{P})$ denotes the Hasse quiver of $\mathbb{P}$.
   - For $a, b \in \mathbb{P}$, we set $\mathbb{P}_{\geq a}, \mathbb{P}_{\leq b}$, and $[a, b]$ as follows.
     \[
     \begin{align*}
     \mathbb{P}_{\geq a} & := \{x \in \mathbb{P} \mid x \geq a\} \\
     \mathbb{P}_{\leq b} & := \{x \in \mathbb{P} \mid x \leq b\} \\
     [a, b] & := \mathbb{P}_{\geq a} \cap \mathbb{P}_{\leq b} = \{x \in \mathbb{P} \mid a \leq x \leq b\}
     \end{align*}
     \]
5. For a quiver $\overrightarrow{H}$, we denote by $\overrightarrow{H}_0$ (resp. $\overrightarrow{H}_1$) the set of all vertices (resp. arrows) of $\overrightarrow{H}$.
6. For an acyclic quiver $\overrightarrow{H}$ and a vertex $a$ of $\overrightarrow{H}$, we define $\text{suc}(a)$, $\text{dsuc}(a)$, $\text{pre}(a)$, $\text{dpre}(a)$ as follows.
   \[
   \begin{align*}
   \text{suc}(a) & := \{b \in \overrightarrow{H}_0 \mid \text{there is a (nontrivial) path from } a \text{ to } b \text{ in } \overrightarrow{H}\} \\
   \text{dsuc}(a) & := \{b \in \overrightarrow{H}_0 \mid \text{there is an arrow from } a \text{ to } b \text{ in } \overrightarrow{H}\} \\
   \text{pre}(a) & := \{b \in \overrightarrow{H}_0 \mid \text{there is a (nontrivial) path from } b \text{ to } a \text{ in } \overrightarrow{H}\} \\
   \text{dpre}(a) & := \{b \in \overrightarrow{H}_0 \mid \text{there is an arrow from } b \text{ to } a \text{ in } \overrightarrow{H}\}
   \end{align*}
   \]
7. For an \( m \times n \)-matrix \( M \) with coefficients in \( K \), we denote by \( f_M \) the \( K \)-linear map given by
\[
K^n \ni \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in K^m.
\]
Then we often write \( K^n \overset{M}{\longrightarrow} K^m \) instead of \( K^n f_M \overset{}{\longrightarrow} K^m \).

2 Preliminary

2.1 Support \( \tau \)-tilting modules

In this subsection, we recall the definitions and basic properties of support \( \tau \)-tilting modules. Let \( A \) be a basic algebra, and let \( Q \) be its Gabriel quiver.

2.1.1 Definition of support \( \tau \)-tilting modules

For a module \( M \), we denote by \( |M| \) the number of non-isomorphic indecomposable direct summands of \( M \) and by \( \text{Supp}(M) := \{ i \in Q_0 \mid Me_i \neq 0 \} \) the support of \( M \), where \( e_i \) is a primitive idempotent corresponding to a vertex \( i \in Q_0 \). We put \( e_M := \sum_{i \in \text{Supp}(M)} e_i \).

A module \( M \in \text{mod} A \) is said to be \( \tau \)-rigid if it satisfies \( \text{Hom}_A(M, \tau M) = 0 \). If a \( \tau \)-rigid module \( T \) satisfies \( |T| = \# \text{Supp}(T) \) (resp. \( |T| = n \)), then we call \( T \) a support \( \tau \)-tilting module (resp. \( \tau \)-tilting module).

We call a pair \( (M, P) \in \text{mod} A \times \text{proj} A \) a \( \tau \)-rigid pair (resp. support \( \tau \)-tilting pair) if \( M \) is \( \tau \)-rigid (resp. support \( \tau \)-tilting) and \( \text{add} P \subseteq \text{add}(1 - e_M)A \) (resp. \( \text{add} P = \text{add}(1 - e_M)A \)).

Remark 2.1 If \( A \) is hereditary, then it follows from the Auslander–Reiten duality that the notion of support \( \tau \)-tilting \( A \)-modules coincides with that of support tilting \( A \)-modules introduced by Ingalls–Thomas in [12].

In the rest of this paper, we use the following notation.

- We say that \( (M, P) \) is isomorphic to \( (M', P') \) if \( M \simeq M' \) and \( P \simeq P' \) hold. In this case, we write \( (M, P) \simeq (M', P') \).
- We say that \( (M, P) \) is basic if \( M \) and \( P \) are basic.
- We set \( (M, P) \oplus (M', P') := (M \oplus M', P \oplus P') \).
- We say that \( (N, R) \) is a direct summand of \( (M, P) \) if \( N \) is a direct summand of \( M \) and \( R \) is a direct summand of \( P \).
- We denote by \( \text{add}(M, P) \) the set of (isomorphism classes of) direct summands of finite direct sums of \( (M, P) \).
- We set \( |(M, P)| := |M| + |P| \).
- For a \( \tau \)-rigid pair \( (M, P) \), we denote by \( \text{Fac}(M, P) \) the category of factor modules of finite direct sums of \( M \).
- We identify \( M \in \text{mod} A \) with \( (M, 0) \in \text{mod} A \times \text{proj} A \).
- For \( P \in \text{proj} A \), we set \( P^- := (0, P) \).
- We denote by \( \tau \)-tilt \( A \) the set of (isomorphism classes of) basic support \( \tau \)-tilting pairs.

Remark 2.2 If \( M \) is a \( \tau \)-rigid module, then we have \( |M| \leq \# \text{Supp}(M) \) (see [1, Proposition 1.3]). In particular, a \( \tau \)-rigid pair \( T \) is support \( \tau \)-tilting if and only if \( |T| = |A| \).
The following proposition gives us a connection between $\tau$-rigid modules of $A$ and that of a factor algebra of $A$.

**Proposition 2.3** ([1, Lemma 2.1]) Let $M$ and $N$ be $A/J$-modules, where $J$ is a two-sided ideal of $A$. If $\text{Hom}_A(M, \tau N) = 0$, then $\text{Hom}_{A/J}(M, \tau A/J N) = 0$. Moreover, if $J = (e)$ is a two-sided ideal generated by an idempotent $e$, then the converse holds.

By the above proposition, we regard $s\tau$-tilt $A/(e)$ as a subset of $s\tau$-tilt $A$. More precisely, we have

$$s\tau\text{-tilt } A/(e) = \{ T \in s\tau\text{-tilt } A \mid (eA)^- \in \text{add } T \}.$$

### 2.1.2 Torsion classes induced by support $\tau$-tilting modules

A full subcategory $\mathcal{F}$ of $\text{mod } A$ which is closed under taking factor modules and extensions is called a **torsion class** of $\text{mod } A$. We say that $\mathcal{F}$ is **functorially finite** if for any $M \in \text{mod } A$, there are $f \in \text{Hom}_A(X, M)$ and $g \in \text{Hom}_A(M, Y)$ with $X, Y \in \mathcal{F}$ such that $\text{Hom}_A(N, f) : \text{Hom}_A(N, X) \to \text{Hom}_A(N, M)$ and $\text{Hom}_A(g, N) : \text{Hom}_A(Y, N) \to \text{Hom}_A(M, N)$ are surjective for all $N \in \mathcal{F}$. We denote by $f\text{-tors } A$ the set of functorially finite torsion classes of $\text{mod } A$.

Then the following theorem gives the connection between support $\tau$-tilting pairs and functorially finite torsion classes.

**Theorem 2.4** ([1, Theorem 2.7]) An assignment $M \mapsto \text{Fac } M$ implies a bijection $s\tau$-tilt $A \to f\text{-tors } A$.

### 2.2 Partial order on $s\tau$-tilt $A$

By using the connection in Theorem 2.4, we can define a partial order on $s\tau$-tilt $A$.

**Definition 2.5** ([1, Lemma 2.25]) For support $\tau$-tilting pairs $T$ and $T'$, we write $T \geq T'$ if $\text{Fac } T \supseteq \text{Fac } T'$. Then the following are equivalent.

1. $(M, P) \geq (M', P')$.
2. $\text{Hom}_A(M', \tau M) = 0$ and $P$ is a direct summand of $P'$.

Moreover, $\geq$ gives a partial order on $s\tau$-tilt $A$.

### 2.2.1 The mutation and the Hasse quiver of $s\tau$-tilt $A$

A $\tau$-rigid pair $M$ is said to be **almost complete support $\tau$-tilting** provided it satisfies $|M| = |A| - 1$. Then the mutation of support $\tau$-tilting pairs is formulated by the following theorem.

**Theorem 2.6**

1. [1, Theorem 2.18] Let $M$ be a basic almost complete support $\tau$-tilting pair. Then there are exactly two basic support $\tau$-tilting pairs $T$ and $T'$ such that $M \in \text{add } T \cap \text{add } T'$.
2. [1, Corollary 2.34] Let $T$ and $T'$ be basic support $\tau$-tilting pairs. Then $T$ and $T'$ are connected by an arrow of $\text{H}(s\tau$-tilt $A)$ if and only if $T$ and $T'$ have a common basic almost complete $\tau$-tilting pair as a direct summand. In particular, $s\tau$-tilt $A$ is $|A|$-regular.
Let \( T, T' \in \mathsf{s}_\tau\text{-tilt} A \). If \( T \prec T' \), then there is a direct predecessor \( U \) of \( T \) (resp. a direct successor \( U' \) of \( T' \)) in \( \overrightarrow{\mathcal{H}} (\mathsf{s}_\tau\text{-tilt} A) \) such that \( U \leq T' \) (resp. \( T \leq U' \)).

(4) [1, Corollary 2.38] If \( \overrightarrow{\mathcal{H}} (\mathsf{s}_\tau\text{-tilt} A) \) has a finite connected component \( \mathcal{C} \), then \( \mathcal{C} = \overrightarrow{\mathcal{H}} (\mathsf{s}_\tau\text{-tilt} A) \).

Remark 2.7 Let \( T \in \mathsf{s}_\tau\text{-tilt} A \) and \( P \) is an indecomposable projective module. By Theorem 2.6, we have the following statements.

- If \( P \in \text{add} T \), then there exists a direct successor \( T' \) of \( T \) in \( \overrightarrow{\mathcal{H}} (\mathsf{s}_\tau\text{-tilt} A) \) satisfying \( P/\in \text{add} T' \).
- If \( P^- \in \text{add} T \), then there exists a direct predecessor \( T' \) of \( T \) in \( \overrightarrow{\mathcal{H}} (\mathsf{s}_\tau\text{-tilt} A) \) satisfying \( P^-/\in \text{add} T' \).

2.2.2 An anti-isomorphism between \( \mathsf{s}_\tau\text{-tilt} A \) and \( \mathsf{s}_\tau\text{-tilt} A^{\text{op}} \)

The following proposition gives a relationship between the support \( \tau \)-tilting poset of \( A \) and that of \( A^{\text{op}} \).

**Proposition 2.8** ([1, Theorem 2.14, Proposition 2.27]) Let \( (M, P) = (M_{np} \oplus M_{pr}, P) \) be a \( \tau \)-rigid pair with \( M_{pr} \) being a maximal projective direct summand of \( M \). We put \( (M, P)^\dagger := (\text{Tr} M_{np} \oplus P^*, M_{pr}^*) \), where \( (-)^* = \text{Hom}_A (-, A) : \text{proj} A \to \text{proj} A^{\text{op}} \). Then \( (M, P)^\dagger \) is \( \tau \)-rigid. Moreover, \( (-)^\dagger \) gives a poset anti-isomorphism from \( \mathsf{s}_\tau\text{-tilt} A \) to \( \mathsf{s}_\tau\text{-tilt} A^{\text{op}} \) with \( ( (-)^\dagger )^\dagger = \text{id} \).

2.2.3 \( \tau \)-rigid pairs are determined by their \( g \)-vectors

Let \( X := (N, U) \) be a \( \tau \)-rigid pair and \( P' \to P \to N \to 0 \) be a minimal projective presentation of \( N \). Then the **\( g \)-vector** \( g^X := (g^X_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0} \) of \( X \) is defined by the following equation in the Grothendieck group \( K_0(\text{proj} A) \) of \( \text{proj} A \).

\[
[P] - [P'] - [U] = \sum_{i \in Q_0} g^X_i [P_i]
\]

Then we have the following theorem.

**Theorem 2.9** ([1, Theorem 5.5]) The assignment \( X \mapsto g^X \) induces an injection from the set of isomorphism classes of \( \tau \)-rigid pairs of \( A \) to \( \mathbb{Z}^{Q_0} \).

2.3 Maximal green sequences

In this subsection, we recall the definition of maximal green sequences.

**Definition 2.10** Let \( Q \) be a finite connected quiver.

1. \( Q \) is a cluster quiver if \( Q \) does not admit loops or oriented 2-cycles.
2. An ice quiver is a pair \( (Q, F) \) where \( Q \) is a cluster quiver and \( F \) is a subset of \( Q_0 \) such that there is no arrow between two vertices of \( F \). For an ice quiver \( (Q, F) \), we call a vertex in \( F \) a frozen vertex.

**Definition 2.11** Let \( (Q, F) \) be an ice quiver and \( k \in Q_0 \setminus F \). We define a new ice quiver \( (\mu_k Q, F) \) from \( (Q, F) \) by applying the following 4-steps.
Theorem 2.5 that we may identify maximal green sequences with paths from $Q$ in the case that $Q$ is acyclic and $A = KQ$.

Definition 2.12 Let $(Q, F)$ be ice quivers with $Q_0 = Q_0'$. We call $(\mu_k Q, F)$ the mutation of $(Q, F)$ at a non-frozen vertex $k$.

Definition 2.13 The framed quiver associated with $Q$ is the quiver $\hat{Q}$ defined as follows:
- $\hat{Q}_0 := Q_0 \cup Q_0'$
- $\hat{Q}_1 := Q_1 \cup \{i \rightarrow c(i) \mid i \in Q_0\}$

The coframed quiver associated with $Q$ is the quiver $\check{Q}$ defined as follows:
- $\check{Q}_0 := Q_0 \cup Q_0'$
- $\check{Q}_1 := Q_1 \cup \{c(i) \rightarrow i \mid i \in Q_0\}$

Note that $(\check{Q}, Q_0')$ and $(\hat{Q}, Q_0')$ are ice quivers. We denote by $\text{Mut}(\hat{Q})$ the mutation-equivalence class $\text{Mut}(\hat{Q}, Q_0')$ of $(\hat{Q}, Q_0')$.

Definition 2.14 Let $(R, Q_0') \in \text{Mut}(\hat{Q})$ and $i$ be a non-frozen vertex.

1. $i$ is said to be a green vertex if $[\alpha \in R_1 \mid s(\alpha) \in Q_0' \text{ and } t(\alpha) = i] = \emptyset$.
2. $i$ is said to be a red vertex if $[\alpha \in R_1 \mid s(\alpha) = i \text{ and } t(\alpha) \in Q_0'] = \emptyset$.

It is shown in [5] that each non-frozen vertex in $(R, Q_0') \in \text{Mut}(\hat{Q})$ is either green or red. Moreover, $(R, Q_0') \in \text{Mut}(\hat{Q})$ has no green vertices if and only if $(R, Q_0')$ is isomorphic to $(\check{Q}, Q_0')$. Then a maximal green sequence is defined as follows.

Definition 2.15 A green sequence for a cluster quiver $Q$ is a sequence $i = (i_1, \ldots, i_\ell)$ of $Q_0$ such that $i_1$ is green in $\hat{Q}$ and for any $2 \leq k \leq \ell$, $i_k$ is green in $\mu_{i_{k-1}} \cdots \mu_{i_1} \check{Q}$. In this case, $\ell$ is called the length of $i$. A green sequence $i = (i_1, i_2, \ldots, i_\ell)$ is said to be maximal if $\mu_{i_\ell} \cdots \mu_{i_1} \check{Q}$ has no green vertices.

In the case that $Q$ is acyclic and $A = KQ$, there exists a length-preserving bijection between the set of maximal green sequences for $Q$ and the set of paths from $\text{mod} A$ to $[0]$ in $\overline{H}_f(t-{\text{tors}A})$ [5, Section 6], [17, Theorem 5.2]). Therefore, it follows from Definition-Theorem 2.5 that we may identify maximal green sequences with paths from $A(= (A, 0))$...
to 0(= A−) in $\mathcal{H}(\text{sr-tilt} A)$ via the above bijection. Then we use the following notation.

\[
\begin{align*}
\text{MGS}(A) & \coloneqq \text{the set of paths from } A \text{ to } 0 \text{ in } \mathcal{H}(\text{sr-tilt} A) \\
\ell(\omega) & \coloneqq \text{the length of } \omega \in \text{MGS}(A) \\
\ell(Q) & \coloneqq \max \{ \ell(\omega) \mid \omega \in \text{MGS}(A) \} \\
\text{MGS}(A)_{\max} & \coloneqq \{ \omega \in \text{MGS}(A) \mid \ell(\omega) = \ell(Q) \}.
\end{align*}
\]

**Remark 2.16** In the rest of this paper, we often use the following fact.

\[\ell(Q) = \ell(Q^{\text{op}})\]

### 3 Key tools

In this section, we prepare useful tools to show our main theorem. From here on, we use the following notation.

- $A_i := A/(e_i)$ for each $i \in Q_0$.
- When we treat another basic algebra $B$ with Gabriel quiver $Q'$, we set $P_j^B = e_jB$, $S_j^B = P_j^B / \text{rad } P_j^B$, $I_j^B = D(Be_j)$ for each $j \in Q'_0$.
- For a basic $\tau$-rigid pair $R$, we define the subposet $\text{sr-tilt}_R A$ of $\text{sr-tilt} A$ as follows.

\[\text{sr-tilt}_R A := \{ T \in \text{sr-tilt} A \mid R \in \text{add } T \}\]

- For $T \in \text{sr-tilt} A$, we denote by $\text{MGS}(A, T)$ the set of all maximal green sequences which factors through $T$ and

\[\text{MGS}(A, T)_{\max} := \text{MGS}(A)_{\max} \cap \text{MGS}(A, T)\].

**Remark 3.1** If $A = KQ$, then we often regard $A_i$ as a path algebra $K(Q \setminus \{i\})$ and $P_j^A$ as an $A$-module.

### 3.1 Dimension vectors of indecomposable modules of path algebras of type ADE

For a module $X \in \text{mod } KQ$, we denote by $\underline{\text{dim}}(X) := (\dim_K X e_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$ the dimension vector of $X$. If $Q$ is a Dynkin quiver, then $\underline{\text{dim}}(\text{ind } KQ) := \{ \underline{\text{dim}}(X) \mid X \in \text{ind } KQ \}$ is given by Gabriel’s Theorem ([8]). In the rest of this paper, we often use this classification.

**Example 3.2** Here, we give $\underline{\text{dim}}(\text{ind } KQ)$ for a quiver $Q$ of type A or D. Readers also get $\underline{\text{dim}}(\text{ind } KQ)$ for a quiver $Q$ of type E from the following URL.

https://hfipy3.github.io/ADE/en.html

We put $e_i = (\delta_{ij})_{j \in Q_0} \in \mathbb{Z}^{Q_0}$, where $\delta$ is the Kronecker delta.

**Type A** If the underlying graph of $Q$ is $1 - 2 - \cdots - n$, then we have

\[\underline{\text{dim}}(\text{ind } A) = \{ e_i + \cdots + e_j \mid 1 \leq i \leq j \leq n \}.\]
Type D Let $Q$ be a quiver of type $D_n$ with the following underlying graph.

$$
\begin{array}{c}
1 \rightarrow 2 \rightarrow \cdots \rightarrow n-2 \\
\end{array}
\begin{array}{c}
n-1 \\

\end{array}
\begin{array}{c}
n \\
\end{array}
$$

Then we have

$$\dim(\text{ind } A) = \{ \sum_{i \in Q_0} e_i \ | \ Q' : \text{a connected full subquiver of } Q \} \cup \{ e_i + \cdots + e_{j-1} + 2(e_j + \cdots + e_{n-2}) + e_{n-1} + e_n \ | \ 1 \leq i < j \leq n-2 \}. $$

3.2 Permutation on $\tau$-tilt $KQ$ induced by Auslander–Reiten translation

In this subsection, we assume $Q$ is acyclic and $A = KQ$. We define a correspondence $\text{mod } A \times \text{proj } A \rightarrow \text{mod } A \times \text{proj } A$ by

$$
\begin{cases}
X & \mapsto \tau X \ (X \in \text{ind } A \setminus \text{proj } A) \\
P & \mapsto P_i^- \ (P \simeq P_i) \\
P^- & \mapsto I_i \ (P \simeq P_i)
\end{cases}
$$

As with the Auslander–Reiten translation, we write this correspondence by $\tau = \tau_A$. We also define a correspondence $\text{mod } A \times \text{proj } A \rightarrow \text{mod } A \times \text{proj } A$ by

$$
\begin{cases}
X & \mapsto \tau^{-1} X \ (X \in \text{ind } A \setminus \text{inj } A) \\
I & \mapsto P_i^- \ (I \simeq I_i) \\
P^- & \mapsto P_i \ (P \simeq P_i)
\end{cases}
$$

and denote it by $\tau^{-1}$.

**Lemma 3.3** For $X \in \text{mod } A \times \text{proj } A$, the following statements are equivalent.

(a) $X$ is $\tau$-rigid.
(b) $\tau X$ is $\tau$-rigid.
(c) $\tau^{-1} X$ is $\tau$-rigid.

In particular, $\tau, \tau^{-1} : \text{mod } A \times \text{proj } A \rightarrow \text{mod } A \times \text{proj } A$ give permutations on $\tau$-tilt $A$.

**Proof** Since $\tau^{-1} \tau X \simeq X \simeq \tau \tau^{-1} X$, it is sufficient to check

$X$ is $\tau$-rigid $\iff \tau X$ is $\tau$-rigid.

Let $X = M \oplus P^-$ with $\text{mod } A \ni M = N \oplus U \oplus \tau^{-1} U'$ such that $U, U' \in \text{proj } A$ and $\text{add } N \cap \text{add } (A \oplus \tau^{-1} A) = \{ 0 \}$, then we have

$$
\tau(X) = \tau(M \oplus P^-) = v P \oplus \tau N \oplus U' \oplus U^- \quad (v \text{ is the Nakayama functor}).
$$

Hence, we obtain

$$
X \text{ is } \tau\text{-rigid} \iff \text{Hom}_A(N \oplus \tau^{-1} U' \oplus U, \tau N \oplus U') = \text{Hom}_A(P, N \oplus \tau^{-1} U' \oplus U) = 0
$$

$$
\iff \text{Hom}_A(\tau N \oplus U', \tau^2 N) = \text{Hom}_A(U, \tau N \oplus U') = \text{Hom}_A(N \oplus \tau^{-1} U' \oplus U, vP) = 0
$$
In the rest of this paper, for an algebra $A$ with $e_k B$ (resp. $(\cdot)_k B$), $\sim$ follows from $vP$ is injective and $\tau (\tau N \oplus vP)$ is not injective.

Therefore, we have the assertion. \hfill $\Box$

### 3.3 Rotation property via Ladkani’s result

In this subsection, we assume $A$ is an algebra with $\mu_i A$ the APR-tilting module corresponding to $i$, i.e.,

$$\mu_i A := (\bigoplus_{k \neq i} P_k) \oplus \tau^{-1} P_i.$$ 

In this setting, we have an isomorphism $B \cong \text{End}_A(\mu_i A)$ with $e_k B \simeq \text{Hom}_A(\mu_i A, P_k)$ ($k \neq i$) and $e_i B \simeq \text{Hom}_A(\mu_i A, \tau^{-1} P_i)$. We denote by $F_i^+$ (resp. $F_i^-$) the BGP reflection functor $\text{Hom}_A(\mu_i A, -) : \text{mod} A \to \text{mod} B$ (resp. $- \otimes \mu_i A : \text{mod} B \to \text{mod} A$). It is well-known that $F_i^+$ and $F_i^-$ induce the following equivalence (see [3, Chap VII, Theorem 5.2] for example).

$$\mathcal{A}_i := \{ X \in \text{mod} A \mid P_i \notin \text{add} X \} \overset{\text{F}_i^+}{\cong} \{ Y \in \text{mod} B \mid I_i^n \notin \text{add} Y \} := \mathcal{B}_i$$

$$\{ X \in \text{mod} A \mid \text{Ext}_A^1(\mu_i A, X) = 0 \} \overset{\text{F}_i^+}{\cong} \{ Y \in \text{mod} B \mid \text{Tor}_1^B(Y, \mu_i A) = 0 \}$$

For a module $M \in \text{mod} A$ and an arrow $\alpha$ in $Q$, we define a $K$-linear map $M_\alpha : M_{\delta(\alpha)} \to M_{\tau(\alpha)}$ as follows.

$$M_\alpha : M_{\delta(\alpha)} \ni m_{\delta(\alpha)} \mapsto m\alpha \in M_{\tau(\alpha)}$$

Let $\tilde{M}_i := \bigoplus_{\alpha \in Q_i} M_{\delta(\alpha)}$. Then $F_i^+(M)$ is isomorphic to $N \in \text{mod} B$ given by the following (see [3, Chap VII, Prop. 5.6] for example).

$$N_{\delta(k)} = \begin{cases} M_{\delta(k)} & (k \neq i) \\ \text{Ker} \left( \sum_{i \in Q_i, \delta(i) = k} \left( M_i \to M_{\delta(i)} \right) \right) & (k = i) \end{cases}$$

$$[-, \beta] : N_{\delta(\beta)} \to N_{\tau(\beta)} \begin{cases} M_{\beta} & (\beta \in Q_1 \cap Q_i') \\ \text{Ker} \left( M_{\delta(\beta)} \to M_{\delta(\beta)} \right) & (\beta = \alpha^* \in Q_i' \setminus Q_i) \end{cases}$$

**Remark 3.4** If $k \neq i$, then $\dim_K \text{Hom}_A(P_k, M) = \dim_K \text{Hom}_B(p_k^B, F_i^+(M))$.

In the rest of this paper, for an algebra $A = KQ$ and a sink vertex $i$ (resp. a source vertex $j$) of $Q$, we simply denote by $(\text{sr-tilt} A)_{\leq \mu_i A}$ (resp. $(\text{sr-tilt} A)_{\geq S_j}$) instead of $(\text{sr-tilt} A)_{\leq (\mu_i A, 0)}$ (resp. $(\text{sr-tilt} A)_{\geq (S_j, 1 - e_j A)}$). Similarly, we use $\text{MGS}(A, \mu_i A)$ (resp. $\text{MGS}(A, S_j)$) instead of $\text{MGS}(A, (\mu_i A, 0))$ (resp. $\text{MGS}(A, (S_j, 1 - e_j A)))$. 


Proposition 3.5 \textsuperscript{[18, Proposition 4.3]} Let $Q$ be an acyclic quiver, $i$ be a sink vertex of $Q$. $Q' := \mu_i Q$ the quiver mutation of $Q$ at $i$, $A = KQ$ and $B = KQ'$. Then the assignment

$$
X \mapsto \begin{cases}
(P^B_i)^- & (X \simeq P_i) \\
F_i^+(X) & (X \in \text{ind} A \setminus \text{add} P_i)
\end{cases}
$$

$$
P_k^- \mapsto \begin{cases}
\chi^B_i(k = i) \\
(P^B_k)^- & (k \neq i)
\end{cases}
$$

induces a poset isomorphism

$$
\psi : (\text{sr-tilt} A)_{\leq \mu_i A} \sim (\text{sr-tilt} B)_{\geq \ell_i}.
$$

In particular, for any integer $\ell$, the following two statements are equivalent.

\begin{itemize}
\item There exists $\omega \in \text{MGS}(A, S_i)$ with $\ell(\omega) = \ell$.
\item There exists $\omega' \in \text{MGS}(B, \mu_i B)$ with $\ell(\omega') = \ell$.
\end{itemize}

For the reader's convenience, we give a proof.

Proof Let $X, Y \in \text{ind} A \setminus \text{add} P_i$ and $Q_0 \ni k \neq i$. Since $\mathcal{A}_i$ and $\mathcal{B}_i$ are closed under extensions, we obtain

$$
\text{Ext}_A^1(X, Y) = 0 \iff \text{Ext}_B^1(F_i^+(X), F_i^+(Y)) = 0.
$$

Let $0 \to F_i^+(X) \to \bigoplus_{k \in Q_0} (P^B_k)^k \to \bigoplus_{k \in Q_0} (F^B_k)^k \to 0$ be a minimal injective copresentation of $F_i^+(X)$. Note that we have an exact sequence

$$
0 \to \bigoplus_{k \in Q_0} (P^B_k)^k \to \bigoplus_{k \in Q_0} (F^B_k)^k \to \tau_i^{-1} F_i^+(X) \to 0.
$$

Since $I_k^B$ is simple injective, we have the following equations.

$$
\dim K F_i^+(X) e_j = \begin{cases}
\sum_{k \in Q_0} n_k \dim K I_k^B e_j - \sum_{i \neq k \in Q_0} m_k \dim K I_k^B e_j & (j \neq i) \\
\sum_{k \in Q_0} n_k \dim K I_k^B e_i - \sum_{k \in Q_0} m_k \dim K I_k^B e_i & (j = i)
\end{cases}
$$

Therefore, it follows from $P_i \notin \text{add} X$ and $i$ is source in $Q'$ that

$$
\text{Hom}_A(P_i, X) = 0 \iff \dim K F_i^+(X) e_i = \sum_{Q_0 \ni a \to \to i} \dim K F_i^+(X) e_{(a)}
$$

$$
\iff \dim K F_i^+(X) e_i = \sum_{Q_0 \ni a \to \to i} \left( \sum_{i \neq k \in Q_0} n_k \dim K I_k^B e_{(a)} - \sum_{i \neq k \in Q_0} m_k \dim K I_k^B e_{(a)} \right)
$$

$$
\iff \dim K F_i^+(X) e_i = \sum_{k \in Q_0 \setminus \{i\}} n_k \left( \sum_{Q_0 \ni a \to \to i} \dim K I_k^B e_{(a)} \right)
$$

$$
- \sum_{k \in Q_0 \setminus \{i\}} m_k \left( \sum_{Q_0 \ni a \to \to i} \dim K I_k^B e_{(a)} \right)
$$

$$
\iff \dim K F_i^+(X) e_i = \sum_{k \in Q_0 \setminus \{i\}} n_k \dim K I_k^B e_i - \sum_{k \in Q_0 \setminus \{i\}} m_k \dim K I_k^B e_i
$$

$$
\iff \sum_{k \in Q_0} n_k \dim K I_k^B e_i - \sum_{k \in Q_0 \setminus \{i\}} m_k \dim K I_k^B e_i = \sum_{k \in Q_0} n_k \dim K I_k^B e_i - \sum_{k \in Q_0 \setminus \{i\}} m_k \dim K I_k^B e_i
$$

$$
\iff n_i = m_i
$$

$$
\iff \tau_i^{-1} F_i^+(X) e_i = 0
$$
\[ \Leftrightarrow \text{Hom}_B(F_1^+(X), \tau B_1^R) = 0 = \text{Hom}_B(I_1^R, \tau F_1^+(X)) \]

We also have

\[ \text{Hom}_A(P_k, X) = 0 \Leftrightarrow \text{Hom}_B(P_k^R, F_1^X) = 0 \]

by Remark 3.4. Then we can easily check that the assignment induces a poset isomorphism

\[ \{T \in \text{sr-tilt}A \mid P_i \notin \text{add}T\} \sim \{T' \in \text{sr-tilt}B \mid (P_i^R)^- \notin \text{add}T'\}. \]

It remains to show the following statements.

\[ (\text{sr-tilt}A)_{\leq} = \{T \in \text{sr-tilt}A \mid P_i \notin \text{add}T\} \]

\[ (\text{sr-tilt}B)_{\leq} = \{T' \in \text{sr-tilt}B \mid (P_i^R)^- \notin \text{add}T'\} \]

Take \( T = (M, P) \in \text{sr-tilt}A \). Then the first equation follows from

\[ \text{Hom}_A(M, \tau_{\mu_i}A) = 0 \Leftrightarrow \text{Hom}_A(M, P_i) = 0 \Leftrightarrow P_i = S_i \notin \text{add}T, \]

and the second equation follows from

\[ (S_i^R, (1 - e_i)B) = (I_i^R, (1 - e_i)B) \leq T \Leftrightarrow \text{Supp}(S_i^R) \subset \text{Supp}(M) \Leftrightarrow (P_i^R)^- \notin \text{add}T. \]

This finishes the proof. \( \square \)

### 3.4 Jasso’s reduction theorem and elementary polygonal deformations

#### 3.4.1 Jasso’s reduction theorem

Here, we recall the \( \tau \)-tilting reduction theorem by Jasso [13]. Let \( R = (U, eA) \) be a basic \( \tau \)-rigid pair (or equivalently, \( U \) is a basic \( \tau \)-rigid \( A/(e) \)-module). By [1, Theorem 2.10], there exists a basic \( \tau \)-tilting \( (A/(e)) \)-module \( T \) satisfying

\[ \text{Fac}T = \{X \in \text{mod} A/(e) \mid \text{Hom}_{A/(e)}(X, \tau_{A/(e)}U) = 0\} \]

\[ = \{X \in \text{mod} A \mid \text{Hom}_A(X, \tau U) = 0, \text{Hom}_A(eA, U) = 0\}. \]

Then \( (T, eA) \in \text{sr-tilt}A \) is said to be the Bongartz completion of \( R \). By definition, \( (T, eA) \) is the Bongartz completion of \( R = (U, eA) \) if and only if it is maximum in \( \text{sr-tilt}A \). Let \( R = (U, eA) \) be a basic \( \tau \)-rigid pair of \( A \) and \( (T, eA) = (X \oplus U, eA) \) the Bongartz completion of \( R \). We set \( B = \text{End}_A(T) = \text{End}_{A/(e)}(T) \) and \( C = B/(eU) \), where \( eU \) is an idempotent of \( B \) corresponding to a projective module \( \text{Hom}_{A/(e)}(T, U) \).

**Theorem 3.6** ([13, Theorem 3.13 and Corollary 3.18]) *In the above setting, we set the functor \( F := \text{Hom}_{A/(e)}(T, -) : \text{mod} A/(e) \to \text{mod} B \) and \( U^\perp = \{N \in \text{mod} A \mid \text{Hom}_A(U, N) = 0\}. \*)

1. The assignment \( \varphi : T' \mapsto F(\text{Fac}(T') \cap U^\perp \cap \text{mod} A/(e)) \) induces a poset isomorphism

\[ \text{sr-tilt}_RA \sim \text{f-tors}C. \]

2. If \( A \) is hereditary, then \( C \) is also hereditary.

**Remark 3.7** Let \( R \) be a basic \( \tau \)-rigid pair. As we already mentioned, \( (T, P) \) is the Bongartz completion of \( R \) if and only if it is maximum in \( \text{sr-tilt}_RA \). Then Theorem 3.6 gives that the following statements are equivalent.

- \( (T, P) \) is the Bongartz completion of \( R \).
\( (T, P) \) is maximum in \( \tau \)-tilt \( RA \).

- \( \#(T', P') \in \tau \)-tilt \( RA \) \( \vdash (T', P') \) in \( \overline{H}(\tau \)-tilt \( A \)) = \( |A| - |R| \)

In the rest of this paper, we use the above characterization frequently.

**Remark 3.8** Let \( M = (U, eA) \) be a basic \( \tau \)-rigid pair such that its Bongartz completion can be written by \( M \oplus X \oplus P \). If \( i \) is a sink vertex, then \( P_i \in U^\perp \). Therefore, for any \( T \in \tau \)-tilt \( MA \) such that \( P_i \in \text{add} \ T \), we have

\[
e_{P_i}C \simeq e_{P_i}B/(e_U) \simeq \text{Hom}_{\overline{A}/(e)}(U \oplus X \oplus P_i \oplus P_i)/[U]
= \text{Hom}_{\overline{A}/(e)}(U \oplus X \oplus P_i \oplus P_i)
= F(P_i) \in F(\text{Fac}(T) \cap U^\perp \cap \text{mod} \ A/(e)) = \varphi(T).
\]

### 3.4.2 Elementary polygonal deformations

We recall the notion of elementary polygonal deformations defined in [11].

Let \( M \) be a basic \( \tau \)-rigid pair. Assume that \( |M| = |A| - 2 \) and \( \tau \)-tilt \( MA = [T_M, T_M] \) is a finite poset. In this case, Jasso’s reduction theorem (Theorem 3.6) implies that there exist precisely two maximal paths \( w_M \) and \( w'_M \) in \( \tau \)-tilt \( MA \). Then, for \( T, S \in \tau \)-tilt \( A \) and a path \( \omega : T \to \cdots \to T'' \to \cdots \to T_M \to \cdots \to S \) in \( \overline{H}(\tau \)-tilt \( A \)), we say that \( d(\omega) : T \to \cdots \to T'' \to \cdots \to T_M \to \cdots \to S \) in \( \overline{H}(\tau \)-tilt \( A \)) the elementary polygonal deformation of \( \omega \) by \( M \).

**Example 3.9** Let \( A = 1 \to 2 \to 3 \) and \( M = S_2 \). Then \( \tau \)-tilt \( S_2 A \) is given by

\[
\begin{align*}
P_1 & \oplus P_2 \oplus S_2 \\
P_1^- & \oplus P_2 \oplus S_2 \\
P_1^- & \oplus P_3^- \oplus S_2 \\
P_1^- & \oplus P_2 \oplus S_2 \\
P_1 & \oplus I_2 \oplus S_2 \\
P_3^- & \oplus I_2 \oplus S_2 \\
P_3^- & \oplus P_3 \oplus S_2
\end{align*}
\]

Therefore, we have the following.

\[
[A \to P_1 \oplus P_2 \oplus S_2 \to P_1^- \oplus P_2 \oplus S_2 \to P_1^- \oplus P_2^- \oplus S_2 \to P_1^- \oplus P_2^- \oplus P_3^-]
\]

Then, by applying Jasso’s reduction theorem (Theorem 3.6), we have the following statement.

**Lemma 3.10** Let \( A = KQ \) be a finite-dimensional path algebra and \( i \) be a sink vertex of \( Q \). Consider a path

\[
T_1 \to T_2 \to T_3
\]
in $\overline{H}(\text{sr-tilt}A)$ such that $P_i \in \text{add}T_2 \setminus \text{add}T_3$. Assume that $M \in \text{add}T_1 \cap \text{add}T_2 \cap \text{add}T_3$ with $|M| = |A| - 2$. Then $\text{sr-tilt}_MA$ has one of the following forms.

In particular, if $|\text{sr-tilt}_MA| < \infty$, then we have

$$\ell(\omega) \leq \ell(\text{d}_MA)$$

for each $\omega$ which contains $T_1 \to T_2 \to T_3$ as a subpath.

**Proof** Let $T' \in \text{dsuc}(T_1)$ such that $P_i \notin \text{add}T'$. Then $T_1, T_2, T_3,$ and $T'$ are in $\text{sr-tilt}_MA$, $T_1$ is the Bongartz completion of $M$.

We write $M = (N, eA)$ and $T_1 = (T, eA) = M \oplus X \oplus P_i$. Since $T_1$ is the Bongartz completion of $M$, we have $X \in \text{mod}A$ and $T = N \oplus X \oplus P_i$. In fact, if $X = P_j$ for some $j \in \mathbb{Q}_0$, then there exists $T'' \in \text{sr-tilt}_MA \cap \text{dpre}(T_1)$. This contradicts the maximality of $T_1$ in $\text{sr-tilt}_MA$.

Then, by Jasso’s reduction theorem (Theorem 3.6), we have $\text{sr-tilt}_MA \simeq \text{sr-tilt}C$ for the basic hereditary algebra $C := \text{End}_{A/(e)}(T)/(e_N)$, where $e_N$ be an idempotent of $B := \text{End}_{A/(e)}(T)$ corresponding to $N$. Let $e_P$ and $e_X$ be idempotents of $B$ such that $e_P B \simeq \text{Hom}_{A/(e)}(T, P_i)$ and $e_X B \simeq \text{Hom}_{A/(e)}(T, X)$ hold. Then $C$ has the following two indecomposable projective modules:

$$U = \frac{e_P B}{e_P B e_N B}, \quad V = \frac{e_X B}{e_X B e_N B}.$$

Let $f \in \text{Hom}_C(V, U) = \text{Hom}_B(V, U)$. Since $e_X B$ is a projective $B$-module, there exists $g \in \text{Hom}_B(e_X B, e_P B)$ such that the following diagram commutes.

$$\begin{array}{ccc}
V & \xrightarrow{f} & U \\
\downarrow & & \downarrow \\
e_X B & \xrightarrow{g} & e_P B
\end{array}$$

Note that we have

$$\text{Hom}_B(e_X B, e_P B) \simeq \text{Hom}_B(\text{Hom}_{A/(e)}(T, X), \text{Hom}_{A/(e)}(T, P_i)) \simeq \text{Hom}_{A/(e)}(X, P_i).$$

Since $P_i$ is a simple projective $A$-module, we have $\text{Hom}_{A/(e)}(X, P_i) = 0$. This shows $g = 0$ and $f = 0$. In particular, we have $\text{Hom}_C(V, U) = 0$. Therefore, $\text{sr-tilt}C$ has one of the
following forms.

(i) \[ U \oplus V \quad U \oplus V - \quad U - \oplus V - \]

(ii) \[ U \oplus V \quad U \oplus V - \quad U - \oplus V - \quad \]

(iii) \[ U \oplus V \quad U \oplus V - \quad U - \oplus V - \quad \cdots \]

Since \( P_i \in \text{add}T_2 \), we also have that an indecomposable projective \( C \)-module \( U \simeq e_{P_i}C \) is in \( \varphi(T_2) \) by Remark 3.8. Hence, by poset isomorphisms

\[ \text{sr-tilt}_A \xrightarrow{\varphi} \text{f-tors}_C \xrightarrow{\text{fac}} \text{sr-tilt}_C \]

the path \( T_1 \to T_2 \to T_3 \) is corresponding to the path \( U \oplus V \to U \oplus V^- \to U^- \oplus V^- \) in \( \overline{H} (\text{sr-tilt}_C) \).

\[ \square \]

3.5 Technical propositions for a proof of Main Theorem (1)

In this subsection, we prove two claims (Proposition 3.14 and Proposition 3.15) which are useful to show Main Theorem (1).

3.5.1 Two classes in \( \text{mod} A \)

To prove Main Theorem (1), we only check the following statement.

\[ \ell (Q) \leq \ell (\mu_iQ) \]

Then the following two classes in \( \text{mod} A \) play an important role.

\[ \mathcal{X}_i = \mathcal{X}_i(A) := \{ X \in \text{ind} A \mid \text{Hom}_A(P_i, X) = 0, \dim_K \text{Hom}_A(\tau^{-1}P_i, X) \geq 2 \} \]

\[ \mathcal{X}_i' = \mathcal{X}_i'(A) := \{ X' \in \text{ind} A \mid \text{Hom}_A(P_i, \tau X') = 0, \dim_K \text{Hom}_A(P_i, X') \geq 2 \} \]

We note that if \( Q \) is a tree, then

\[ \mathcal{X}_i \cap \text{proj} A = \emptyset = \mathcal{X}_i' \cap \text{proj} A \]

and the transpose \( \text{Tr} \) induces a bijection

\[ \mathcal{X}_i(A) \xleftarrow{\text{Tr}} \mathcal{X}_i'(A^{\text{op}}) . \]

Remark 3.11 Assume \( Q \) is a tree. For an indecomposable projective \( A \)-module \( P_i \) and a non-projective \( A \)-module \( X \), by considering an almost split sequence

\[ 0 \to P_i \to ( \bigoplus_{j \in \text{dsvc}(i)} \tau^{-1}P_j ) \oplus ( \bigoplus_{k \in \text{dpre}(i)} P_k ) \to \tau^{-1}P_i \to 0 , \]

we obtain the following statement.

\[ \dim_K \text{Hom}_A(\tau^{-1}P_i, X) = \sum_{j \in \text{dsvc}(i)} \dim_K \text{Hom}_A(\tau^{-1}P_j, X) + \sum_{k \in \text{dpre}(i)} \dim_K \text{Hom}_A(P_k, X) - \dim_K \text{Hom}_A(P_i, X) \cdots (\ast) \]

By repeating above statement (\( \ast \)), we can calculate \( \dim_K \text{Hom}_A(\tau^{-1}P_i, X) \) from the dimension vector of \( X \). For example, consider the following quiver \( Q \) and its path algebra \( A = KQ \).

\[
\begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 8 \\
\end{array}
\]
Let \( X \) be a non-projective indecomposable \( A \)-module with \( \dim X = t(x_1, x_2, \ldots, x_8) \). We calculate \( \dim_K \text{Hom}_A(\tau^{-1}P_1, X) \). Applying above statement \((*)\) to \( i = 1 \), we have
\[
\dim_K \text{Hom}_A(\tau^{-1}P_1, X) = \dim_K \text{Hom}_A(\tau^{-1}P_2, X) - \dim_K \text{Hom}_A(P_1, X).
\]
Continuing to apply above statement \((*)\) to \( i = 2, 3, 4, 5, 7 \), we have
\[
\begin{align*}
\dim_K \text{Hom}_A(\tau^{-1}P_2, X) - \dim_K \text{Hom}_A(P_1, X) &= \dim_K \text{Hom}_A(\tau^{-1}P_3, X) - \dim_K \text{Hom}_A(P_2, X) \\
&= \dim_K \text{Hom}_A(\tau^{-1}P_4, X) - \dim_K \text{Hom}_A(P_3, X) \\
&= \dim_K \text{Hom}_A(\tau^{-1}P_5, X) - \dim_K \text{Hom}_A(P_4, X) \\
&= \dim_K \text{Hom}_A(P_5, X) + \dim_K \text{Hom}_A(\tau^{-1}P_7, X) - \dim_K \text{Hom}_A(P_5, X) \\
&= \dim_K \text{Hom}_A(P_5, X) + \dim_K \text{Hom}_A(P_8, X) - \dim_K \text{Hom}_A(P_7, X) \\
&= x_6 + x_8 - x_7.
\end{align*}
\]
To find \( X \in \mathcal{X}_i \), the following lemma is useful.

**Lemma 3.12** Let \( Q \) be a quiver of type \( \mathcal{D} \) or \( \mathcal{E} \), and \( i \) be a source vertex of \( Q \). For \( X \in \text{ind} A_i \subset \text{ind} A \), we define the triple
\[
\left( \mathcal{C}, \mathcal{C}_+, j \right) = \left( \mathcal{C}(i), \mathcal{C}(i, X), j(i, X) \right)
\]
as follows.
- \( \mathcal{C} \) is the unique connected component of \( Q \setminus \{i\} \) containing \( \text{Supp} X \).
- \( \mathcal{C}_+ \) is the full subquiver of \( X \) having \( \mathcal{C}_0 \cup \{i\} \) as the vertex set.
- \( j \) is the unique neighbor of \( i \) which is in \( \mathcal{C}_+ \).

1. If \( X \in \mathcal{X}_i \), then \( X \) is not projective, \( \tau X \in \text{ind} K \mathcal{C}_+ \), and
\[
\dim_K \text{Hom}_A(P_\tau \tau X) = \dim_K \text{Hom}_A(\tau^{-1}P_\tau X) \geq 2.
\]
2. If \( X \in \mathcal{X}_i \), then \( \mathcal{C}_+ \) is neither of type \( \mathcal{A} \) nor \( \mathcal{D} \), and \( \mathcal{C} \) is not of type \( \mathcal{A} \).
3. If \( \text{deg} i \neq 1 \), then \( \mathcal{X}_i = \emptyset \).

**Proof**
(1). If \( X \) is projective, then we have \( \text{Hom}_A(\tau^{-1}P_\tau X) = 0 \). This is a contradiction. Therefore, \( X \) is not projective. Let \( 0 \rightarrow \bigoplus_{k=1}^4 P_{j_k} \rightarrow \bigoplus_{k=1}^4 P_{j_k} \rightarrow X \rightarrow 0 \) be a minimal projective presentation of \( X \). Since \( \text{Supp} X \subset \mathcal{C}_0 \) and \( i \) is a source vertex, \( i_1, \ldots, i_4 \) and \( j_1, \ldots, j_k \) are in \( \mathcal{C}_0 \). Note that there exists an exact sequence
\[
0 \rightarrow \tau X \rightarrow \bigoplus_{k=1}^\ell I_{j_k} \rightarrow \bigoplus_{k=1}^\ell I_{j_k} \rightarrow 0.
\]
Hence, we obtain
\[
\dim_K \text{Hom}_A(P_\tau \tau X) = 0
\]
for each \( j' \in Q_0 \setminus (\mathcal{C}_0 \cup \{i\}) \). In particular, we have \( \tau X \in \text{ind} K \mathcal{C}_+ \). Furthermore, by the argument in Remark 3.11, we have
\[
\dim_K \text{Hom}_A(P_\tau \tau X) = \dim_K \text{Hom}_A(\tau^{-1}P_\tau X) = \dim_K \text{Hom}_A(\tau^{-1}P_\tau X) \geq 2.
\]
(2). Let \( X \in \mathcal{X}_i \), \( x_6 = \dim_K \text{Hom}_A(P_\tau X) \), and \( x_8' = \dim_K \text{Hom}_A(\tau^{-1}P_\tau X) \). Since the degree of \( i \) in \( \mathcal{C}_+ \) is equal to 1, \( \mathcal{C}_+ \) is neither of type \( \mathcal{A} \) nor \( \mathcal{D} \) by (1) and the classification
of indecomposable modules for path algebras of type $A, D$. Hence, we check that $\overline{C}$ is not of type $A$. Suppose that $\overline{C}$ is of type $A$. Since $\overline{C}_+$ is of type $A$, $\overline{C}_+$ has the following form.

\[
\begin{array}{c}
\vdots \\
i \\
\vdots \\
\rightarrow t' \\
\rightarrow t = t_p \\
\rightarrow \cdots \\
\rightarrow t_0 = j = s_0 \\
\rightarrow \cdots \\
\rightarrow s_q = s \\
\rightarrow s' \\
\rightarrow \cdots
\end{array}
\]

Here, we admit the case that $t' (s')$ does not exist.

Assume $p = q = 0$. By Remark 3.11, we have the following equations.

\[
\begin{align*}
x'_i &= x'_j - x_i \\
   &= x_{t'} + x_{s'} - x_j
\end{align*}
\]

Assume $p = 0$ and $q > 0$. By Remark 3.11, we have the following equations.

\[
\begin{align*}
x'_i &= x'_j - x_i \\
   &= x_{t'} + x'_{t_1} - x_{s_0} \\
   &= x_{t'} + x'_{t_2} - x_{s_1} \\
   &\vdots \\
   &= x_{t'} + x'_{t_q} - x_{s_{q-1}} \\
   &= x_{t'} + x_{s'} - x_{s_q} \\
   &= x_{t'} + x_{s'} - x_s
\end{align*}
\]

Assume $p > 0$ and $q > 0$. By Remark 3.11, we have the following equations.

\[
\begin{align*}
x'_i &= x'_j - x_i \\
   &= x'_{t_1} + x'_{s_1} - x_j \\
   &= (x'_{t_1} - x_{t_0}) + (x'_{s_1} - x_{s_0}) + x_j \\
   &= (x'_{t_2} - x_{t_1}) + (x'_{s_2} - x_{s_1}) + x_j \\
   &\vdots \\
   &= (x'_{t_{p-1}} - x_{t_{p-2}}) + (x'_{s_{q-1}} - x_{s_{q-2}}) + x_j \\
   &= (x_{t'} - x_t) + (x_{s'} - x_s) + x_j
\end{align*}
\]

Here, we assume $x_{t'} = 0$ (resp. $x_{s'} = 0$) if $t'$ (resp. $s'$) does not exist.

Therefore, for each $(p, q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, we have

\[
x'_i = (x_{t'} - x_t) + (x_{s'} - x_s) + x_j.
\]

Then the classification of indecomposable modules for path algebras of type $A$ implies

\[
\dim_K \text{Hom}_A (P_\tau X) = \dim_K \text{Hom}_A (\tau^{-1} P_\tau X) = (x_{t'} - x_t) + (x_{s'} - x_s) + x_j \leq 1.
\]

In fact, each entry of the dimension vector $\dim X$ is in $\{0, 1\}$. Therefore, if $(x_{t'} - x_t) + (x_{s'} - x_s) + x_j \geq 2$, then one of the following statements holds.

- $(x_{t'}, x_t, x_{s'}, x_s, x_j) = (1, 0, 1, 0, 1)$
- $(x_{t'}, x_t, x_{s'}, x_s, x_j) = (1, 0, 1, 1, 1)$
- $(x_{t'}, x_t, x_{s'}, x_s, x_j) = (1, 1, 0, 1, 1)$
\[ (x_t', x_t, x_j, x_s, x_s') = (0, 0, 1, 0, 1) \]
\[ (x_t', x_t, x_j, x_s, x_s') = (1, 0, 1, 0, 0) \]
\[ (x_t', x_t, x_j, x_s, x_s') = (1, 0, 0, 0, 1) \]

This contradicts \( X \in \text{ind} A \). In fact, the full subquiver of \( Q \) whose vertex set is \( \text{Supp} X \) is connected. Hence, we have the following statements.

\[ x_t' = x_j = 1 \Rightarrow x_t = 1 \]
\[ x_j = x_s' = 1 \Rightarrow x_s = 1 \]
\[ x_t' = x_s' = 1 \Rightarrow x_j = 1 \]

(3). Suppose \( \mathscr{X}_i \neq \emptyset \). We take \( X \in \mathscr{X}_i \) and let \( x_v := \dim_K \text{Hom}_A(P_v, X) \).

If \( Q \) is of type \( \tilde{D} \) and \( \deg(i) \geq 2 \), then \( \overrightarrow{C}_+ \) is of type \( A, D \). This is a contradiction. Hence, it follows from (2) that we may assume \( Q \) is of type \( \tilde{E}_6, \tilde{E}_7 \), or \( \tilde{E}_8 \).

Assume that \( Q \) is of type \( \tilde{E}_6 \). By (2), \( \overrightarrow{C}_+ \) has one of the following forms.

\[
\begin{array}{cccc}
\downarrow & \downarrow & & \\
 i & p & q & j \leftrightarrow r \leftrightarrow s
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & \downarrow & & \\
 i & p & q & j \leftrightarrow r \leftrightarrow s
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & \downarrow & & \\
 i & p & q & j \leftrightarrow r \leftrightarrow s
\end{array}
\]

Therefore, we have that \( \overrightarrow{C}_+ \) is of type \( A \). This contradicts (2). Assume that \( Q \) is of type \( \tilde{E}_7 \).

By (2), we may assume \( \overrightarrow{C}_+ \) has the following form.

\[
\begin{array}{cccc}
\downarrow & \downarrow & & \\
 i & p & q & j \leftrightarrow r \leftrightarrow s
\end{array}
\]

where \( \epsilon_1, \ldots, \epsilon_5 \in \{\pm\} \) and, for a pair \( (a < b) \), \( a \leftrightarrow b \) means \( a \rightarrow b \) and \( a \leftarrow b \) means \( a \leftarrow b \).

Then we have

\[
\dim_K \text{Hom}_A(\tau^{-1}P_v, X) = \begin{cases} 
 x_1 - x_0 & (\epsilon_1 = -) \\
 x_5^+ - (x_1 - x_2) & (\epsilon_1 = +, \epsilon_2 = -) \\
 x_5^+ - (x_2 - x_3) & (\epsilon_1 = \epsilon_2 = +, \epsilon_3 = -) \\
 x_5^+ - (x_3 - x_4) & (\epsilon_1 = \epsilon_2 = \epsilon_3 = +, \epsilon_4 = -) \\
 x_5^+ - x_4 & (\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = +),
\end{cases}
\]

where \( x_5^\pm \) is given by the following.

\[
x_5^+ := \dim_K \text{Hom}_A(\tau^{-1}P_v, X) = x_1 - x_5.
\]
\[
x_5^- := \dim_K \text{Hom}_A(P_v, X) = x_5
\]

Therefore, it follows from \( X_i \in \text{ind} K \overrightarrow{C} \) and \( \overrightarrow{C} \) is of type \( D_6 \), we have

\[
\dim_K \text{Hom}_A(\tau^{-1}P_v, X) \leq 1.
\]
This is a contradiction.
Assume that $Q$ is of type $\widetilde{E}_8$. By (2), we may assume $\widetilde{C}_+$ has the following forms.

\[
\begin{array}{c}
 4 \\
 i \rightarrow j = 0 \\ 
\end{array}
\]

(i)

\[
\begin{array}{c}
 5 \\
 i \rightarrow j = 0 \\
\end{array}
\]

(ii)

\[
\begin{array}{c}
 6 \\
 i \rightarrow j = 0 \\
\end{array}
\]

(iii)

Suppose that $\widetilde{C}_+$ has the form (i) or (ii). Then, by the classification of indecomposable modules for path algebras of type $E_6$, $E_7$ and (1), we have

\[
\dim K \text{Hom}_A(\tau^{-1}P_0, X) = \dim K \text{Hom}_A(P_0 \tau X) \leq 1.
\]

This is a contradiction, and we may assume $\widetilde{C}_+$ has the form (iii). Then we obtain

\[
\dim K \text{Hom}_A(\tau^{-1}P_0, X) = \begin{cases} 
  x_1 - x_0 & (\epsilon_1 = -) \\
  x_2 - x_1 & (\epsilon_1 = +, \epsilon_2 = -) \\
  x_3 - x_2 & (\epsilon_1 = \epsilon_2 = +, \epsilon_3 = -) \\
  x_6^+ - (x_3 - x_4) & (\epsilon_1 = \epsilon_2 = \epsilon_3 = +, \epsilon_4 = -) \\
  x_6^- - (x_4 - x_5) & (\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = +, \epsilon_5 = -) \\
  x_6^- - x_5 & (\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = +) 
\end{cases}
\]

where $x_6^\pm$ is given by the following.

\[
\begin{align*}
x_6^+ & := \dim K \text{Hom}_A(\tau^{-1}P_0, X) = x_3 - x_6 \\
x_6^- & := \dim K \text{Hom}_A(P_0 \tau X) = x_6 
\end{align*}
\]

Since $X \in \text{ind } K \widetilde{C} \subset \text{ind } A$, it follows from the classification of the indecomposable modules of $K \widetilde{C}$ that

\[
\dim K \text{Hom}_A(\tau^{-1}P_0, X) \leq 1.
\]

This is a contradiction. Therefore, we have the assertion. 

\[\square\]

3.5.2 Technical propositions

For a path algebra $A = KQ$, we denote by $\mathcal{P}/\mathcal{R}/\mathcal{I}$, the set of (isomorphism classes of) indecomposable preprojective/regular/preinjective modules, respectively. We also set $\mathcal{J} := \mathcal{J} \cup \{P^-_i \mid i \in Q_0\}$. In addition, for $\mathcal{M} = \mathcal{P}$, $\mathcal{R}$, $\mathcal{I}$, or $\mathcal{J}$, we use the following notation.

\[
\text{add. } \mathcal{M} := \{X \in \text{mod } A \times \text{proj } A \mid \text{each indec. direct summand of } X \text{ is in } \mathcal{M}\}
\]

Let $Q$ be a quiver of type $\widetilde{D}$ or $\widetilde{E}$, $A = KQ$. For $\text{MGS}(A) \ni \omega : T_0 \rightarrow \cdots \rightarrow T_\ell$, we define

\[
s^{(r)}_\omega := \max\{r \mid P_{r+1} \in \text{add } T_{r+1}\},
\]

\[
t^{(l)}_\omega := \min\{r \mid P^-_r \in \text{add } T_{r-1}\}, \text{ and } t^{(l)}_\omega := \max\{t^{(l)}_\omega \mid \omega \in \text{MGS}(A)\}.
\]

First we show the following lemma.
Lemma 3.13  Let Q be a quiver of type $\tilde{D}$ or $\tilde{E}$, $i$ be a source vertex of Q and $i'$ be a sink vertex of Q, then the following statements hold.

(1) Let $\omega = (T_0 \to T_1 \to \cdots \to T_{\ell(Q)}) \in \text{MGS}(A)_{\text{max}}$ with $t = t(i_{\omega}) < \ell(Q) + 1$. If we write

$$T_t = M \oplus X_i \oplus P_i', \quad T_{t-1} = M \oplus X_i \oplus P_i',$$

then one of the following statements holds.

(a) $X_i \in X_i$ and $M \in \text{add}(R)$.

(b) There exists a maximal green sequence

$$T_0 \to \cdots \to T_{t-2} \to T_{t-1}' \to T_t \to \cdots \to T_{\ell(Q)}$$

with $P_i' \notin \text{add}(T_{t-1}')$.

(2) Let $\omega = (T_0 \to T_1 \to \cdots \to T_{\ell(Q)}) \in \text{MGS}(A)_{\text{max}}$ with $s = s(i_{\omega}) > -1$. If we write

$$T_s = M' \oplus X_i' \oplus P_i, \quad T_{s+1} = M' \oplus X_i' \oplus P_i,$$

then one of the following statements holds.

(a) $X_i' \in X_i'$ and $M' \in \text{add}(R)$.

(b) There exists a maximal green sequence

$$T_0 \to \cdots \to T_s \to T_{s+1}' \to T_{s+2} \to \cdots \to T_{\ell(Q)}$$

with $P_i' \notin \text{add}(T_{s+1}')$.

(3) $\text{MGS}(A, S_i) \neq \emptyset$ (resp. $\text{MGS}(A, \mu_i A) \neq \emptyset$) implies

$$\ell(Q) \leq \ell(\mu_i Q)$$

Proof  Let $Q' := Q^\text{op}$, $A' := A^\text{op} = \text{KQ}'$, $P_i' = e_{\delta A'}$, and $B = K(\mu_i Q)$.

(1) Let $T_{t-1}' \in \text{dpre}(T_t)$ such that $P_i' \notin T_{t-1}'$. Then $T_t, T_{t-1}, T_{t-2},$ and $T_{t-1}'$ are in $\sigma \tau - \text{tilt}_M A$. By applying the anti-poset isomorphism $\dagger : \sigma \tau - \text{tilt}_M A \rightarrow \sigma \tau - \text{tilt}_{A'}$, we have

\[
\begin{array}{c}
\text{\textbullet} \\
\text{T}^{\dagger}_{t-1} \\
\text{T}^{\dagger}_{t-2} \\
\text{\textbullet}
\end{array}
\]

in $\sigma \tau - \text{tilt}_{M^\dagger} A'$ with $P_i' \in \text{add}(T_{t-1}') \setminus \text{add}(T_{t-2}')$. In particular, $T_t^{\dagger}$ is the Bongartz completion of $M^\dagger$ and we can apply Lemma 3.10 to $T_t^{\dagger} \to T_{t-1}^{\dagger} \to T_{t-2}^{\dagger}$. Since $\omega \in \text{MGS}(A)_{\text{max}}$, $\sigma \tau - \text{tilt}_{M^\dagger} A'$ has the form (i) or (iii) in Lemma 3.10.

Note that if $\sigma \tau - \text{tilt}_M A$ is finite (or equivalently, $\sigma \tau - \text{tilt}_{M^\dagger} A'$ is finite), then $\sigma \tau - \text{tilt}_{M^\dagger} A'$ has the form (i) of Lemma 3.10. This implies $\sigma \tau - \text{tilt}_M A$ has the following form and (b) stands.
Now we assume that either \( X_i \notin \mathcal{X}_i \) or \( M \notin \text{add} \mathcal{R} \) holds. If \( M \notin \text{add} \mathcal{R} \), then there exists a pair \((r,a) \in \mathbb{Z} \times Q_0 \) such that
\[
\tau'' P_a^- \in \text{add} M.
\]
Then we have
\[
\text{sr-tilt}_{M} A \subset \text{sr-tilt}_{\tau'' P_a} A
\]
\[
\xrightarrow{r''} \text{sr-tilt}_{P_a} A = \text{sr-tilt} A/(e_a).
\]
Since \( A \) is a tame hereditary algebra, \( A/(e_a) \) is representation-finite. In particular, \( \text{sr-tilt}_{M} A \) is finite and we obtain (b).

Hence, we may assume \( X_i \notin \mathcal{X}_i \) and \( M \in \text{add} \mathcal{R} \). Then one of the following three cases occurs.

- \( X_i \in \text{add} A^\perp \).
- \( X_i \) is projective or equivalently, \( X_i^\perp \in \text{add}(A')^- \).
- \( X_i^\perp \in \text{ind} A' \setminus \mathcal{X}_i(A') \) such that \( \text{Hom}_A(P_{i'}, \tau A; X_i^\perp) = 0 \).

Since \( (P_{i'})^\perp \in \text{proj} A' \), the first case is included in the third case. The second case implies that \( X_i \ominus T_{i-1} \) should be \( r \)-rigid. This is a contradiction.

Therefore, we may assume the third case occurs.

Since \( i \) is a sink vertex of \( Q' \) and \( X_i^\perp \not\subset P_{i'} \), we have
\[
\dim_k \text{Hom}_A(X_i^\perp, P_{i'}) = 0, \quad \dim_k \text{Hom}_A(P_{i'}, X_i^\perp) \leq 1.
\]
By Jessou’s reduction theorem (Theorem 3.6), we can take an acyclic quiver \( \Delta \) with two vertices \( v, w \) such that
\[
\text{Ker}_\Delta \cong \frac{\text{End}_{A'}(X_i^\perp \oplus P_{i'} \oplus M^\perp)}{[M^\perp]} \cong \frac{\text{End}_{A'}(X_i^\perp \oplus P_{i'})}{[M^\perp]}, \quad \text{sr-tilt} \text{Ker}_\Delta \cong \text{sr-tilt} M^\perp A'.
\]
Thus, we have \( \dim_k e_v(K \Delta) e_w, \dim_k e_v(K \Delta) e_u \leq 1 \) and \( \# \Delta \leq 1 \). This shows that \( \text{Ker}_\Delta \) is a representation-finite algebra. In particular, \( \text{sr-tilt} M^\perp A' \) is a finite poset and we obtain (b). Therefore, we have the assertion (1).

(2) Let \( \omega = (T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_i) \in \text{MGS}(A) \) with \( s = \ell_\omega^{(r)} > -1 \). Then applying the anti-poset isomorphism \( \dagger : \text{sr-tilt} A \rightarrow \text{sr-tilt} A' \) to \( \omega \), we obtain \( \omega' = \omega^\perp \in \text{MGS}(A') \) satisfying \( s = \ell_{\omega'}^{(r')}, \ell(Q) + 1 = \ell(Q') + 1 \). Since \( Q' \) is a quiver of type \( \tilde{D} \) or \( \tilde{E} \) and \( r' \) is a source vertex of \( Q' \), we can apply (1) to \( \omega' \), i.e., one of the following statements holds.

(a') \( X_i^\perp \in \mathcal{X}_i(A') \) and \( M^\perp \in \text{add} \mathcal{R}(A') \).
(b') There exists a maximal green sequence
\[
T_0^\perp \rightarrow \cdots \rightarrow T_{s-2}^\perp \rightarrow (T_{s-1})^\perp \rightarrow T_s^\perp \rightarrow \cdots \rightarrow T_{(Q')}^\perp
\]
with \( (P_{i'}^A)^- \not\subset \text{add}(T_{s-1})^\perp \).
Since Tr induces the bijection between $\mathcal{X}_i(A')$ and $\mathcal{X}_i'(A)$, the assertion (2) follows from the definition of $\dagger$.

(3) Consider $\operatorname{MGS}(A, S_i) \ni \omega : T_0 \rightarrow \cdots \rightarrow T_{\ell(Q)}$. Since $T_{\ell(Q)} = S_i$, by applying the poset isomorphism $\psi^{-1} : (\sigma \text{-tilt} A)_{\leq S_i} \cong (\sigma \text{-tilt} B)_{\leq \mu B}$ in Theorem 3.5, we obtain a maximal green sequence

$$\omega' : B \rightarrow \mu_i B = \psi^{-1}(T_0) \rightarrow \cdots \rightarrow \psi^{-1}(T_{\ell(Q)-2}) \rightarrow \psi^{-1}(T_{\ell(Q)-1}) = 0$$

of $B$ with $\ell(\omega') = \ell(Q)$. This gives the assertion (3).

\[ \square \]

**Proposition 3.14** Let $Q$, $i$, and $i'$ be as in Lemma 3.13.

(1) If $i$ is a source (resp. sink) vertex and $\mathcal{X}_i = \emptyset$ (resp. $\mathcal{X}_i' = \emptyset$), then we have

$$\ell(Q) \leq \ell(\mu_i Q).$$

In particular, if $\deg i \geq 2$, then we have

$$\ell(Q) = \ell(\mu_i Q).$$

(2) If there exists a quiver automorphism $\sigma$ of $Q$ satisfying $\sigma(i) \neq i$ (resp. $\sigma(i') \neq i'$), then

$$\operatorname{MGS}(A, S_i) \neq \emptyset \ (\text{resp. } \operatorname{MGS}(A, \mu_i A) \neq \emptyset).$$

In particular,

$$\ell(Q) \leq \ell(\mu_i Q) \ (\text{resp. } \ell(Q) \leq \ell(\mu_i' Q)).$$

(3) $\operatorname{MGS}(A, S_i) = \emptyset \ (\text{resp. } \operatorname{MGS}(A, \mu_i A) = \emptyset)$ if and only if $t^{(i)} \leq \ell(Q)$ (resp. $s^{(i')} \geq 0$). In particular, if $t^{(i)} \leq \ell(Q) + 1$ (resp. $s^{(i')} = -1$), then we have

$$\ell(Q) \leq \ell(\mu_i Q) \ (\text{resp. } \ell(Q) \leq \ell(\mu_i' Q)).$$

**Proof** (1). Assume $\mathcal{X}_i = \emptyset$. Let $\omega \in \operatorname{MGS}(A)$ with $t := t^{(i)} = t^{(i)}$. If $\operatorname{MGS}(A, S_i) = \emptyset$, then we have $t < \ell(Q) + 1$. By Lemma 3.13 (1) and $\mathcal{X}_i' = \emptyset$, there exists a maximal green sequence

$$T_0 \rightarrow \cdots \rightarrow T_{i-2} \rightarrow T_{i-1}' \rightarrow T_i \rightarrow \cdots \rightarrow T_{\ell(Q)}$$

with $P_i' \notin \operatorname{add} T_{i-1}$. This contradicts the maximality of $t$.

Therefore, we have $\operatorname{MGS}(A, S_i) \neq \emptyset$, and it follows from Lemma 3.13 (3) that $\ell(Q) \leq \ell(\mu_i Q)$. Then the remaining assertion follows from Lemma 3.12 (3). In fact, we have

$$\ell(Q) \leq \ell(\mu_i Q) \quad \text{(Lemma 3.12(3) and (1))}$$

$$= \ell((\mu_i Q)^{\oplus})$$

$$\leq \ell(\mu_i(\mu_i Q)^{\oplus}) \quad \text{(Lemma 3.12(3) and (1))}$$

$$= \ell(Q^{\oplus})$$

$$= \ell(Q)$$

(2). Take $\omega = (T_0 \rightarrow \cdots \rightarrow T_{\ell(Q)}) \in \operatorname{MGS}(A)$ satisfying the following two conditions.

- $t := t^{(i)} = t^{(i)}$.
- $t' := t^{(i')}$ = $\max\{t^{(i')}_\omega \mid \omega' \in \operatorname{MGS}(A) \text{ with } t^{(i)}_\omega = t^{(i)}\}$. 
If \( t = \ell(Q) + 1 \), then we have nothing to show. Therefore, we may assume \( t \leq \ell(Q) \). We write
\[
T_i = M \oplus X \oplus P_i^-, \quad T_{i-1} = M \oplus X \oplus P_i^-, \quad T_{i'} = M' \oplus X' \oplus P_{\sigma(i)}^-, \quad T_{i'-1} = M' \oplus X' \oplus P_{\sigma(i)}^-.
\]
If \( t > t' \), then \( P_{\sigma(i)}^- \in \text{add}\, M \oplus X \). This implies that either \( M \not\in \text{add}\, \mathcal{R} \) or \( X \not\in \mathcal{R}_t \) holds. Then it follows from Lemma 3.13(1) that there exists \( \omega' \in \text{MGS}(A) \) such that \( l_{\omega'}^{(i)} > t \). This contradicts the maximality of \( t \).

If \( t < t' \leq \ell(Q) \), then \( P_i^- \in \text{add}\, M' \oplus X' \). This implies that either \( M' \not\in \text{add}\, \mathcal{R} \) or \( X' \not\in \mathcal{R}_{\sigma(i)} \) holds. Then it follows from Lemma 3.13(1) that there exists \( \omega' \in \text{MGS}(A) \) such that \( l_{\omega'}^{(i)} > t' \) and \( l_{\omega'}^{(i)} = t \). This contradicts the maximality of \( t' \).

Hence, we have \( t < t' = \ell(Q) + 1 \). In particular,
\[
\text{MGS}(A, S_{\sigma(i)}) \neq \emptyset.
\]
Then the isomorphism \( A \cong A \) given by \( \sigma^{-1} \) implies
\[
\text{MGS}(A, S_i) \neq \emptyset.
\]
By considering the poset anti-isomorphism \( \tau : \text{sr-tilt} A \rightarrow \text{sr-tilt} A^{op} \), we also have
\[
\text{MGS}(A, \mu_{\ell} A) \cong \text{MGS}(A^{op}, S_i^{op}) \neq \emptyset.
\]
This finishes the proof.

(3). This assertion follows from Lemma 3.13(3).

Let \((i, i := (i_1, \ldots, i_m), L)\) be a triple satisfying the following conditions.

- \( i \) is a source vertex of \( Q \).
- \( i_p \neq i_q \) if \( p \neq q \).
- \( Q = Q(0) \rightarrow Q(1) \rightarrow \cdots \rightarrow Q(m) \) is a sink mutation sequence, i.e., \( i_k \) is sink in \( Q(k-1) \) for each \( k \in \{1, \ldots, m\} \).
- \( L \leq \{i_1, \ldots, i_m\} \).
- \( L \in \text{ind} \, A \).

For such a triple \((i, i, L)\), we consider the following assumption.

**Assumption 1** Let \((i, i, L)\) be as above, \( A = A(0) = \text{KQ}, A(p) = \text{KQ}(p) \) (\( 0 \leq p \leq m - 1 \)).

A1. \( \mathcal{X}_i(A) \) contains a unique module \( X_i \) up to isomorphism.
A2. If \( M \oplus X_i \oplus P_i^- \in \text{sr-tilt} A \) with \( M \in \text{add}\, \mathcal{R} \), then \( L \in \text{add}\, M \).
A3. \( \text{Hom}_{A(p-1)} \left( P_{A(p-1)}^{+}, \tau_{A(p-1)}(F_{A(p-1)}^{+} \circ \cdots \circ F_{X_i(p)}^{+} (L)) \right) \neq 0 \) for each \( p \in \{1, \ldots, m\} \).
A4. For each \( p \in \{1, \ldots, m\} \), one of the following statements holds.

(i) \( \mathcal{X}_i'(A(p-1)) = \emptyset \).
(ii) \( \mathcal{X}_i'(A(p-1)) \) contains a unique module \( X_i' \) in \( \text{mod} A(p-1) \) up to isomorphism. In addition, if \( M' \oplus X' \oplus P_{A(p-1)}^{+} \in \text{sr-tilt} A(p-1) \) with \( M' \in \text{add} \, \mathcal{R}(A(p-1)) \), then \( F_{X_i(p)}^{+} \circ \cdots \circ F_{X_i(p)}^{+} (L) \in \text{add}\, \mathcal{R}(A(p-1), M') \).
A5. \( \ell(\mu_p, \cdots, \mu_i, Q) \leq \ell(Q) \) holds for each \( p \in \{1, \ldots, m - 1\} \).
Proposition 3.15 Let $Q$ be a quiver of type $\tilde{D}$ or $\tilde{E}$ and $B = A^{(m-1)}$. Assume that $(i, i, L)$ satisfies Assumption 1 and one of the following conditions holds.

(i) $\ell(Q) \leq \ell(\mu_1 Q)$.
(ii) There exists a maximal green sequence

\[
\text{MGS}(A) \ni \omega : A = T_0 \to T_1 \to \cdots \to T_{\ell(Q)} = 0
\]

such that $T_{t_0} = M \oplus X_i \oplus P_i^-$, $T_{t_{i-1}} = M \oplus X_i \oplus P_i^-$ with $M \in \text{add } \mathcal{A}$.

Then we have the following statements.

(1) There exists a maximal green sequence

\[
\text{MGS}(B) \ni \omega^B : B = T^B_0 \to T^B_1 \to \cdots \to T^B_{\ell(Q^{(m-1)})} = 0
\]

such that $b^B_{t_m} \notin \text{add } T^B_1$.

(2) Let $\mu_4 Q := \mu_{i_m} \mu_{i_{m-1}} \cdots \mu_{i_1} Q$. We have

\[
\ell(Q) = \ell(\mu_{i_1} Q) = \cdots = \ell(\mu_{i_{m-1}} \cdots \mu_{i_1} Q) \leq \ell(\mu_{i_1} Q).
\]

(3) If $(i, i', (i_1, \ldots, i_{m-1}, i'), L')$ also satisfies Assumption 1 and $\ell(\mu_{i_1} Q) \leq \ell(Q)$, then we have

\[
\ell(Q) \leq \ell(\mu_{i_1} Q).
\]

Proof Let $\ell_p := \ell(\mu_{i_p} \cdots \mu_{i_1} Q)$, $p^{\mu}_p := p^{\mu}_p$, and $F^+_p := F^+_i \circ \cdots \circ F^+_1$ for each $p \in \{0, \ldots, m-1\}$.

We first show the following claim.

Claim For any $p \in \{0, 1, \ldots, m-1\}$ and for any $M \in \text{add } \mathcal{A}$ with $|M| = n - 2$, we define $\Omega_p(M) \subset \text{MGS}(A^{(p)})$ as follows: $\omega^{(p)} = (T^{(p)}_0 \to \cdots \to T^{(p)}_{t^{(p)}}) \in \Omega_p(M)$ if and only if we can write

\[
T^{(p)}_{t} = F^+_p(M) \oplus F^+_p(X_i) \oplus (P^{(p)}_i)^-, \quad T^{(p)}_{t-1} = F^+_p(M) \oplus F^+_p(X_i) \oplus (P^{(p)}_i)^-
\]

where $t := t^{(p)}_{\omega^{(p)}}$. We also define $\Omega_p^{\text{max}}(M) \subset \text{MGS}(A^{(p)})$ by

\[
\Omega_p^{\text{max}}(M) := \Omega_p(M) \cap \text{MGS}(A^{(p)}).
\]

Then, for each $p \in \{0, \ldots, m\}$, we have the following conditions $c_p$ and $c'_p$.

- $c_p$: $\ell_{p-1} \leq \ell_p$. (Here, we put $\ell_{-1} := \ell_0$ and $\ell_m := \ell_{m-1}$.)
- $c'_p$: $0 \leq p \leq m - 1$: There exists $M \in \text{add } \mathcal{A}$ such that $M \oplus X_i \oplus P_i^- \in \text{sr-tilt } A$ and $\Omega_p^{\text{max}}(M) \neq \emptyset$.
- $c'_m$: There are $M \in \text{add } \mathcal{A}$ and a maximal green sequence

\[
\omega^B : B = T^B_0 \to T^B_1 \to \cdots \to T^B_{\ell(Q^{(m-1)})} = 0,
\]

such that $\omega^B$ is in $\Omega_p^{\text{max}}(M)$ and $p^B_{t_m} \notin \text{add } T^B_1$.

Proof of Claim. We show the assertion by using induction on $p$. Let $\ell := \ell_0 = \ell(Q)$.

$(p = 0)$: If the condition (ii) holds, then we have nothing to show. Thus, we may assume $\ell(Q) \leq \ell(\mu_{i_1} Q)$. Take

\[
\text{MGS}(A) \ni \omega : A = T_0 \to \cdots \to T_1 = 0
\]
such that $t := t^{(i)}_\omega = \max\{t_\omega^{(i)} : \omega' \in \text{MGS}(A)\}$. If $t = \ell + 1$, then it follows from Lemma 3.13 (3) that

$$\ell(Q) \leq \ell(\mu, Q).$$

This is a contradiction. Hence, we have $t \leq \ell$. Then Lemma 3.13 (1) and the maximality of $t$ imply

$$T_t = M \oplus X_t \oplus P^-_t, \quad T_{t-1} = M \oplus \overline{X}_t \oplus P^-_t$$

for some $M \in \text{add} \mathscr{R}$.

$(p > 0)$ : Note that $c_0, \ldots, c_{p-1}$, and $A5$ imply

$$\ell = \ell_{p-1}.$$

Then take $\omega_{p-1} = \left(\frac{T^{(p-1)}_0 \rightarrow \cdots \rightarrow T^{(p-1)}_t}{(p-1)}\right) \in \Omega^\text{max}_{p-1}(M)$ such that

$$s := s_{\omega_{p-1}} = \min\{s_{\omega'} : \omega' \in \Omega^\text{max}_{p-1}(M)\},$$

and put $t := t_i^{(i)}_{\omega_{p-1}}$.

Suppose that $t - 1 \leq s + 2$. Then we have $T^{(p-1)}_{t-2} \geq T^{(p-1)}_{s+1}$. By definition, we can write

$$T^{(p-1)}_{t-2} = F^{p-1}_p (M) \oplus F^{p-1}_{p-1} (X_i) \oplus (P_i^{(p-1)})^\perp.$$

Therefore, we obtain $F^{p-1}_p (M) \in \text{add} T^{(p-1)}_{t-2}$ and

$$\text{Hom}_{A^{(p-1)}} (P^{(p-1)}_p, \tau_{A^{(p-1)}} (F^{p-1}_p (M))) = 0.$$

On the other hand, it follows from $A2$ and $A3$ that

$$F^{p-1}_p (L) \in \text{add} \left(F^{p-1}_p (M) \right) \text{ and } \text{Hom}_{A^{(p-1)}} (P^{(p-1)}_p, \tau_{A^{(p-1)}} (F^{p-1}_p (L))) \neq 0.$$

This is a contradiction. Therefore, we obtain $s + 2 \leq t - 2$.

Now we write

$$T^{(p-1)}_s = M' \oplus X' \oplus P^{(p-1)}_i, \quad T_{s+1} = M' \oplus \overline{X}' \oplus P^{(p-1)}_i.$$

Suppose that $s \geq 0$. Then, by applying Lemma 3.13 (2), one of the following conditions holds.

(a) $X' \in \mathscr{R}'_{ip}(A^{(p-1)})$ and $M' \in \text{add} \mathscr{R}(A^{(p-1)})$.

(b) There exists a maximal green sequence

$$T^{(p-1)}_0 \rightarrow \cdots \rightarrow T^{(p-1)}_s \rightarrow T'_{s+1} \rightarrow T^{(p-1)}_{s+2}$$

$$\rightarrow \cdots \rightarrow T^{(p-1)}_{t-2} \rightarrow T^{(p-1)}_{t-1} \rightarrow T^{(p-1)}_t$$

$$\rightarrow \cdots \rightarrow T^{(p-1)}_\ell$$

with $P^{(p-1)}_i \notin \text{add} T'_{s+1}$.

The condition (b) contradicts the minimality of $s$, and if the condition $A4(i)$ in Assumption 1 holds for $p$, then the condition (a) does not occur. Therefore, the condition (a) and the condition $A4(ii)$ in Assumption 1 holds for $p$. In particular, we may assume $X' = X'_i$,

$M' \in \text{add} \mathscr{R}(A^{(p-1)})$ and obtain

$$0 \neq F^{p-1}_p (L) \in \text{add} F^{p-1}_p (M) \cap \text{add} \tau_{A^{(p-1)}} M'.$$
This contradicts $T_i \leq T_s$. Hence, we have $s = -1$. Note that this shows
\[ c_{m-1} \land c'_{m-1} \Rightarrow c'_m. \]
If $p \leq m - 1$, then, by applying $\psi^p : \text{str-tilt}_{\leq \mu_p} A^{(p-1)} A \simeq \text{str-tilt}_{\leq S_p} A^{(p)}$ to $\omega_{p-1}$, we obtain a maximal green sequence of length $\ell = \ell_{p-1}$ and it is in $\Omega_p(M)$. In particular, we have
\[ c_{p-1} \land c'_{p-1} \Rightarrow c'_p \text{ for each } p \in \{1, \ldots, m\}. \]
Furthermore, if $p < m$, then A5 implies $\ell_p = \ell$. This shows
\[ c_{p-1} \land c'_{p-1} \Rightarrow c'_p \text{ for each } p \in \{1, \ldots, m - 1\}. \]
Thus, Claim holds.

The assertion (1) follows from Claim. Then the assertion (2) follows from (1), Claim, and Lemma 3.13 (3).

We prove (3). We set
\[ \Omega_{m-1}^{\max}(M, \mu_{\text{in}} B) := \Omega_{m-1}^{\max}(M) \cap \text{MGS}(B, \mu_{\text{in}} B). \]
By Claim, we can take $\omega' = (T'_{0} \rightarrow \cdots \rightarrow T'_{\ell}) \in \Omega_{m-1}^{\max}(M, \mu_{\text{in}} B)$ such that
\[ s := s_{\omega'}^{(r)} = \min \{ s_{\omega'}^{(r)} | \omega' \in \Omega_{m-1}^{\max}(M, \mu_{\text{in}} B) \}. \]
Suppose that $s \geq 1$. Then the same argument used in the proof of Claim ($p > 0$) implies a contradiction. Therefore, we have
\[ s = 0. \]
Since $\ell(\mu_{\text{in}}(Q)) \leq \ell(Q)$, it follows from (2) that
\[ \ell(Q) = \ell(\mu_{\text{in}} Q). \]
Then the statement (3) follows from Proposition 3.5.

4 A computational approach

In this section, we give a program which counts all maximal green sequences of the path algebra $KQ$ by length for a given tame quiver $Q$. We define
\[ \overline{H}_{\text{fin}}(\text{str-tilt} A) := \overline{H}(\text{suc}(A) \cap \text{pre}(0)). \]
Note that $\overline{H}_{\text{fin}}(\text{str-tilt} A)$ is the full subquiver of $\overline{H}(\text{str-tilt} A)$ given by the support $r$-tilting modules which appear in $\text{MGS}(A)$.

4.1 Indecomposable modules which may appear in MGS

For a maximal green sequence
\[ \omega : (T_0 \rightarrow \cdots \rightarrow T_\ell) \in \text{MGS}(A), \]
we put
\[ \text{ind} \omega := \text{ind} \left( \bigcup_{k=0}^{\ell} \text{add} T_k \right) \]
and define
\[ \text{ind} \text{MGS}(A) := \bigcup_{\omega \in \text{MGS}(A)} \text{ind} \omega. \]
In this subsection, we give a finite subset \( A = A_A \) of \( \text{ind} \, A \cup \{ P_i^- \mid i \in Q_0 \} \) satisfying
\[
\text{ind} \, \text{MGS}(A) \subseteq A.
\]
The following proposition is the key to this purpose.

**Proposition 4.1** ([20, Sect. 4.2(8)]) Let \( A \) be a tame hereditary algebra with \( |A| = n \). If \( R \in \text{add} \, \mathcal{R} \) is \((\tau \cdot \cdot \cdot)\)-rigid, then \( |R| \leq n - 2 \). In particular, each \( T \in \mathcal{r} \)-tilt \( A \) has at least two indecomposable direct summands in \( \mathcal{P} \cup \mathcal{F} \).

First, consider the candidates for regular modules which may appear in \( \text{MGS} \).

**Lemma 4.2** We define \( \mathcal{R} := \{ X \in \mathcal{R} \mid \tau \cdot \cdot \cdot X \text{ is nonsincere for some } r \in \mathbb{Z} \} \), then each indecomposable \( \tau \cdot \cdot \cdot \)-rigid regular module is in \( \mathcal{R} \). In particular, the following inclusion relation holds.
\[
\mathcal{R} \cap \text{ind} \, \text{MGS}(A) \subseteq \mathcal{R}
\]

**Proof** Let \( X \) be in \( \mathcal{R} \cap \text{ind} \, \text{MGS}(A) \). By Proposition 4.1, there exists \( Y \in \mathcal{P} \cup \mathcal{F} \) such that \( X \oplus Y \) is \( \tau \cdot \cdot \cdot \)-rigid. In the case \( Y \in \mathcal{P} \), we have \( Y \cong \tau \cdot \cdot \cdot P_i \) for some \( r \in \mathbb{Z}_{>0} \) and \( i \in Q_0 \).

Then \( \text{Hom}_A(\tau \cdot \cdot \cdot P_i, \tau \cdot \cdot \cdot X) = 0 \), that is, \( \text{Hom}_A(P_i, \tau \cdot \cdot \cdot X) = 0 \) holds, so \( \tau \cdot \cdot \cdot X \) is nonsincere. In the case \( Y \in \mathcal{F} \), we similarly obtain \( \text{Hom}_A(\tau \cdot \cdot \cdot X, \tau \cdot \cdot \cdot i) = 0 \) for some \( r \in \mathbb{Z}_{>0} \) and \( i \in Q_0 \), namely, \( \tau \cdot \cdot \cdot (r \cdot \cdot \cdot)X \) is nonsincere. Either way, we have \( X \in \mathcal{R} \).

For each \( i \in Q_0 \), we define \( p_i := \min\{r \in \mathbb{Z}_{>0} \mid \tau \cdot \cdot \cdot P_i \text{ is sincere for all } s \geq r \} \) and \( q_i := \min\{r \in \mathbb{Z}_{>0} \mid \tau \cdot \cdot \cdot i \text{ is sincere for all } s \geq r \} \). Further, we define \( m := \max\{p_i, q_i \mid i \in Q_0 \} \).

**Lemma 4.3** We define \( \mathcal{B} := \bigcup_{i \in Q_0} \{ \tau \cdot \cdot \cdot P_i \mid 0 \leq r < m + p_i \} \), then the following inclusion relation holds.
\[
\mathcal{P} \cap \text{ind} \, \text{MGS}(A) \subseteq \mathcal{B}
\]

**Proof** For some \( \tau \cdot \cdot \cdot P_i \in \mathcal{P} \cap \text{ind} \, \text{MGS}(A) \), we suppose \( m + p_i \leq r \). Then, there exists a maximal green sequence \( \omega : \mathcal{P} = T_0 \to T_1 \to \cdots \to T_k = 0 \) such that \( \tau \cdot \cdot \cdot P_i \in \text{ind} \, \omega \).

Now, let \( \tau \cdot \cdot \cdot P_i \) be in \( \text{add} \, T_i \) \((0 \leq t \leq \ell)\), the following claim holds.

**Claim** Let \( j \in Q_0, r' \in \mathbb{Z}_{>0} \) and \( t' \in \mathbb{Z}, \). Then, \( \tau \cdot \cdot \cdot P_j \in \text{add} \, T_{t'} \) implies \( m \leq r' \).

**Proof (proof of Claim)** Once we suppose \( r' \leq m - 1 \). By \( t \leq t' \), \( \text{Hom}_A(T_{t'}, \tau \cdot \cdot \cdot T_t) = 0 \) holds.

Thus, we have
\[
\text{Hom}_A(\tau \cdot \cdot \cdot P_j, \tau (\tau \cdot \cdot \cdot P_i)) = 0.
\]

This implies the following equation.
\[
\text{Hom}_A(\tau \cdot \cdot \cdot P_j, \tau \cdot \cdot \cdot (r \cdot \cdot \cdot - 1)P_i) = 0
\]

Since \( r' \leq m - 1 \) and \( m + p_i \leq r \) hold, we have \( p_i \leq r - r' - 1 \). However, this means that \( \tau \cdot \cdot \cdot (r \cdot \cdot \cdot - 1)P_i \) is sincere, that is, a contradiction.

Let \( \tau \cdot \cdot \cdot P_k \) be the last preprojective direct summand to disappear in the sequence \( \omega \). Then \( m \leq s \) holds by the above claim. By Proposition 4.1, there are integers \( s' \geq -1 \) and \( k' \in Q_0 \) such that \( \tau \cdot \cdot \cdot P_k \oplus \tau \cdot \cdot \cdot ^{s'} I_{k'} \) is \( \tau \cdot \cdot \cdot \)-rigid. If \( s' = -1 \), then \( (\tau \cdot \cdot \cdot P_k) \circ e_{k'} = 0 \). In particular, \( \tau \cdot \cdot \cdot P_k \) is not sincere. This contradicts \( s \geq m \). Therefore, \( s \geq 0 \) and \( \text{Hom}_A(\tau \cdot \cdot \cdot P_k, \tau \cdot \cdot \cdot (r \cdot \cdot \cdot I_{k'})) = 0 \) holds.

Then we have \( \text{Hom}_A(P_k, \tau \cdot \cdot \cdot (r \cdot \cdot \cdot + 1)I_{k'}) = 0 \). Since \( s + s' + 1 \geq m \), we have \( \tau \cdot \cdot \cdot (s + s' + 1)I_{k'} \) is sincere. This is a contradiction. \( \square \)
As a dual of Lemma 4.3, the following lemma also stands.

**Lemma 4.4** We define \( \mathcal{I} := \bigcup_{i \in Q_0} \{ \tau^r I_i \mid 0 \leq r < m + q_i \} \), then the following inclusion relation holds.

\[ \mathcal{I} \cap \text{ind MGS}(A) \subseteq \mathcal{I} \]

Then we define \( A \) by

\[ \mathcal{P} \cup \mathcal{R} \cup \mathcal{I} \cup \{ P_i^- \mid i \in Q_0 \} \]

and put

\[ \text{sr-tilt} \ A \mid A := \{ T \in \text{sr-tilt} \ A \mid \text{add}T \subseteq \text{add}A \} \]

\[ \text{suc}(A) \mid A := \text{the subposet of} \ \text{sr-tilt} \ A \ \text{given by} \ \text{suc}(A) \cap \text{sr-tilt} \ A \mid A \].

### 4.2 Pseudocode

In this subsection, we give a pseudocode of our program.

Let \( A = KQ \) be a finite-dimensional path algebra. We assume that \( A \) is either a rep-finite algebra or an indecomposable tame algebra.

**Definition 4.5** We define a relation \( \preceq \) on \( \text{ind} A \cup \{ P_i^- \mid i \in Q_0 \} \) as follows:

**(finite case)** Assume that \( A \) is rep-finite. In this case, each indecomposable module is preprojective. Therefore, we can write

\[ \text{ind} A = \mathcal{P} = \bigcup_{i \in Q_0} \{ \tau^- P_i \mid 0 \leq r \leq k_i \} \]

Then we define \( \prec \) as follows:

- For \( i, j \in Q_0, r \in \{0, \ldots, k_i\} \) and \( s \in \{0, \ldots, k_j\} \),

  \[ \tau^- s P_j \prec \tau^- r P_i :\Leftrightarrow [r \leq s] \text{ or } [r > s \text{ and } (\tau^- (r-s-1) P_i) e_j = 0] \].

- \( P_i^- \prec X \) for each \( i \in Q_0 \) and \( X \in \text{ind} A \cup \{ P_j^- \mid j \in Q_0 \} \).

- If \( X \in \text{ind} A \), then

  \( X \prec P_i^- :\Leftrightarrow X e_i = 0 \).

**(tame case)** Assume that \( A \) is an indecomposable tame algebra. In this case, \( A_i := A_i = K(Q \setminus \{ i \}) \) is rep-finite for each vertex \( i \) of \( Q \). Then the condition for \( Y \prec X \) is given as follows. (In the following table, we assume \( r, s \in \mathbb{Z}_{\geq 0} \).)

| \( Y \setminus X \) | \( \tau^- r P_i \) | \( \mathcal{P} \) | \( \tau^s P_j^- \) |
|---------------------|-----------------|-----------------|-----------------|
| \( \tau^- s P_j \)  | (i)             | (ii)            | (iii)           |
| \( \mathcal{P} \)    | ALWAYS          | (iii)           | (iv)            |
| \( \tau^s P_j^- \)  | ALWAYS          | ALWAYS          |                 |

(i) Either \( r \leq s \) or \( r > s \) and \( (\tau^- (r-s-1) P_i) e_j = 0 \) holds.

(ii) One of the following equivalent conditions holds.
– (τ^{r+s}I_i)e_j = 0
– (τ^{r+s}P_i)e_i = 0

(iii) There is (i, r) ∈ Q_0 × Z_{≥0} such that τ^{r}(X), τ^{r}(Y) ∈ \text{ind}_A i and τ^{r}X ≺ τ^{r}Y.

(iv) Either (r ≥ s) or (r < s and (τ^{s-r-1}I_i)e_j = 0) holds.

Lemma 4.6 Let X, Y ∈ \text{ind} A ∪ \{P^{-i} \mid i ∈ Q_0\}.

(1) If Y ∈ \text{ind} A and (X, Y) ∉ R × R, then we have the following statements.

\[
\begin{align*}
Y ≺ X \iff & \quad \text{Hom}_A(Y, \tau X) = 0 \quad (Y ∈ \text{ind} A) \\
& \quad Ye_i = 0 \quad (X = P^{-i})
\end{align*}
\]

In particular, if A is rep-finite, then X ⊕ Y is τ-rigid if and only if both X ≺ Y and Y ≺ X hold.

(2) If X, Y ∈ R, then we have

Y ≺ X ⇒ \text{Hom}_A(Y, \tau X) = 0.

(3) X ⊕ Y is τ-rigid if and only if X ≺ Y and Y ≺ X.

(4) Let T = M ⊕ X and T' = M ⊕ Y be in sτ-tiltA with |M| = |A| − 1. Then there is an arrow \(T \to T'\) in \(\overline{H}(sτ\text{-tilt}A)\) if and only if Y ≺ X.

Proof (1). We first assume A is rep-finite and Y = τ^{-s}P_i with 0 ≤ s ≤ k_j. If X ∈ \text{ind} A, then we can write X = τ^{-r}P_i with 0 ≤ r ≤ k_i. Then we have the following statements.

\[
\begin{align*}
\text{Hom}_A(Y, \tau X) = 0 \quad \text{always holds} & \quad (r ≤ s) \\
\text{Hom}_A(Y, \tau X) = 0 \iff & \quad \text{Hom}_A(P_j, \tau^{s-r-1}P_i) = 0 \quad (r > s)
\end{align*}
\]

If X = P^{-i}, then we have

\[
Y ≺ X \iff Ye_i = 0
\]

by definition. Therefore, the assertion holds for the case that A is representation-finite.

We assume A is an indecomposable (representation-infinite) tame algebra. Assume Y ∈ \(\mathcal{P}\) and put

\[
Y = τ^{-s}P_i.
\]

Then we have

\[
\begin{align*}
\text{Hom}_A(Y, \tau X) = 0 \quad \text{always holds} & \quad (X ≃ τ^{-r}P_i \text{ with } 0 ≤ r ≤ s) \\
\text{Hom}_A(Y, \tau X) = 0 \iff & \quad \text{Hom}_A(P_j, τ^{s-r-1}X) = 0 \text{ (otherwise)}.
\end{align*}
\]

Hence, if Y is preprojective, then we have

\[
\text{Hom}_A(Y, \tau X) = 0 \iff Y ≺ X.
\]

Similarly, if X is in \(\mathcal{P}\), then we can check

\[
\text{Hom}_A(Y, \tau X) = 0 \iff Y ≺ X.
\]

If X = P^{-i}, then we have

\[
Y ≺ X \iff Ye_i = 0
\]

by definition.
Hence, we may assume \((X, Y) \in (\mathcal{P} \times (\mathcal{P} \cup \mathcal{I})) \cup ((\mathcal{P} \cup \mathcal{R}) \times \mathcal{I})\). In this case, 
\(\text{Hom}_A(Y, \tau X) = 0\) always holds. In particular, we have
\[
\text{Hom}_A(Y, \tau X) = 0 \iff Y < X.
\]

(2). Assume \(X, Y \in \mathcal{R}\) and \(Y < X\). Then \(\tau' Y < \tau' X\) holds for some \((i, r) \in \mathcal{A}_i\). Since \(\mathcal{A}_i\) is representation finite hereditary algebra, it follows from (1) that 
\(\text{Hom}_{\mathcal{A}_i}(\tau' Y, \tau_{\mathcal{A}_i}(\tau' X)) = 0\). By Theorem 2.3, we have
\[
\text{Hom}_{\mathcal{A}_i}(\tau' Y, \tau(\tau' X)) = 0.
\]
This shows
\[
\text{Hom}_{\mathcal{A}_i}(Y, \tau X) = 0.
\]

(3). By (1), we may assume either \((X, Y) \in \mathcal{R} \times \mathcal{R}\) or \(X = P_i^−\). If \(X = P_i^−\), then it is easy to check the assertion. Therefore, we assume \((X, Y) \in \mathcal{R} \times \mathcal{R}\). Note that \(M\) is \(\tau\)-rigid if and only if \(\tau\text{-tilt}_M\mathcal{A} \neq \emptyset\). Then it follows from Proposition 4.1 that
\[
X \oplus Y \text{ is } \tau\text{-rigid} \iff X \oplus Y \oplus \tau' P_i^− \text{ is } \tau\text{-rigid for some } (i, r) \in Q_0 \times \mathbb{Z}
\]
\[
\iff \tau' (X \oplus Y) \oplus P_i^− \text{ is } \tau\text{-rigid for some } (i, r) \in Q_0 \times \mathbb{Z}
\]
\[
\iff \tau' (X \oplus Y) \text{ is } \tau\text{-rigid in mod} \mathcal{A}_i \text{ for some } (i, r) \in Q_0 \times \mathbb{Z}
\]

Then, by applying (1) to the rep-finite algebra \(\mathcal{A}_i\), we have the assertion.

(4). First assume there is an arrow \(T \rightarrow T'\). Then we have \(X \notin \{P_i^− \mid i \in Q_0\}\). If \(Y \in \{P_i^− \mid i \in Q_0\}\), then we have \(Y < X\). Furthermore, if \((X, Y) \notin \mathcal{R} \times \mathcal{R}\), then \(Y < X\) follows from (1). Hence, we may assume \((X, Y) \in \mathcal{R} \times \mathcal{R}\). In this case, it follows from Proposition 4.1 that \(\tau P_i^− \in \text{add}M\) for some \((i, r) \in Q_0 \times \mathbb{Z}\). Then we have
\[
\text{Hom}_A(Y, \tau X) = 0 \Rightarrow \text{Hom}_{\mathcal{A}_i}(\tau' Y, \tau(\tau' X)) = 0
\]
\[
\Rightarrow \text{Hom}_{\mathcal{A}_i}(\tau' Y, \tau_{\mathcal{A}_i}(\tau' X)) = 0 \text{ (Theorem 2.3)}
\]
\[
\Rightarrow Y < X.
\]

Next we assume \(Y < X\). Suppose \(X \in \{P_i^− \mid i \in Q_0\}\). Then \(X < Y\) holds. In particular, \(M \oplus X \oplus Y\) is \(\tau\)-rigid with \(|M \oplus X \oplus Y| = n + 1\). This is a contradiction. If \(Y \in \{P_i^− \mid i \in Q_0\}\), then there is an arrow \(T \rightarrow T'\). Therefore, we may assume \(X, Y \in \text{ind} \mathcal{A}\). Then it follows from (1) and (2) that there is an arrow \(T \rightarrow T'\) in \(\overline{\mathcal{H}}(\tau\text{-tilt}A)\). \(\square\)

We are going to present a procedure for counting up the elements of MGS(\(A\)) by length with some pseudocodes. From here on, let \(A\) be an indecomposable tame algebra \(KQ\) with \(\mathcal{Q} = \{0, \cdots, n\}\), \(\dim(X) = \dim_A(X) = (\dim_K X e_0, \ldots, \dim_K X e_n)\) the dimension vector of \(X \in \text{mod} \mathcal{A}\), \(C_A = \left(\begin{array}{ccc}
\dim P_0 & \dim P_1 & \cdots \\
\dim P_n
\end{array}\right)\) the Cartan matrix of \(A\), and \(\Phi_A = -^{-1}C_A C_A^{-1}\) the Coxeter matrix of \(A\). Then, for each \(X \in \text{ind} A \setminus \mathcal{P}\), we have the following equation (see [1, Corollary IV–2.9] for example).
\[
\dim(\tau X) = \Phi_A \dim(X)
\]
In addition, we use the following notation.
**Definition 4.7** For each vertex \( i \in Q_0 \), let \( \overrightarrow{Q}_i \) be the quiver made by erasing all arrows directly connected to \( i \), and we define the Cartan matrix \( C_{A_i} \) of \( A_i \) on \( A \) as follows.

\[
(C_{A_i})_{j,k} := \begin{cases} 
1 & \text{if there exists a path from } k \text{ to } j \text{ on } \overrightarrow{Q}_i \ (j, k \in \{0, \ldots, n\}) \\
0 & \text{(otherwise)} 
\end{cases}
\]

This gives us the Coxeter matrix \( \Phi_{A_i} := -^tC_{A_i}C_{A_i}^{-1} \) of \( A_i \) on \( A \).

**Remark 4.8** For each \( i \in Q_0 \) and \( i \neq k \in Q_0 \), the \( k \)-th column of \( C_{A_i} \) is equal to the dimension vector of \( P^A_i \) as an \( A \)-module.

Given a tame quiver \( Q \) as an input, we can easily compute matrices \( C_{A}, \Phi_{A}, C_{A_i}, \Phi_{A_i} \), and their inverse matrices for each \( i \in Q_0 \). Further, we can also compute dimension vectors \( \dim P_i \) and \( \dim I_i \) for each \( i \in Q_0 \). Therefore, in programs, we treat these as global variables.

The following pseudocode is a procedure for constructing the set \( S \) which consists of all the dimension vectors of the non-sincere indecomposable \( A \)-modules.

**Algorithm 1** construction of the set \( S \)

```plaintext
1: function getNonsincereModules
2: \( S \leftarrow \) an empty set (not a list)
3: for \( i \in Q_0 \) do
4:     for \( i \neq j \in Q_0 \) do
5:         \( v \leftarrow \) the \( j \)-th column of \( C_{A_i} \)
6:         while \( v \) has no negative component do
7:             Add the vector \( v \) to the set \( S \)
8:             \( v \leftarrow \Phi_{A_i}^{-1}v \)
9:         end while
10:     end for
11: end for
12: return \( S \)
13: end function
```

**4.2.1 Determine \( \Lambda \)**

Using the set \( S \) obtained by Algorithm 1, we construct a finite set \( \Lambda \), which contains \( \text{ind MGS}(A) \) as a subset. First, we are going to explain how we treat support \( \tau \)-tilting \( A \)-modules and indecomposable \( A \)-modules as data.

In programs, an indecomposable \( A \)-module is denoted by a triple of nonnegative integers \((a, b, c)\) as follows.

- \((0, b, c)\) denotes the preprojective \( A \)-module \( \tau^cP_b \).
- \((1, b, c)\) denotes the regular \( A \)-module \( R \) which satisfies \( \tau^b(R) \cong R \) and has numbering \( c \).
- \((2, b, c)\) denotes the preinjective \( A \)-module \( \tau^bI_b \).

In addition, the object \( P^{-}_b \) corresponding to the \( \tau \)-rigid pair \((0, P_b)\) is denoted by \((2, b, -1)\) and treated as if it were an \( A \)-module. Therefore, a support \( \tau \)-tilting \( A \)-module can be represented as a sequence of \( n + 1 \) triples, for example, the support \( \tau \)-tilting \( A \)-module...
\[ A = \bigoplus_{i=0}^n P_i \] is represented as the sequence
\[ ((0, 0, 0), (0, 1, 0), \cdots, (0, n, 0)). \]

For each support \( \tau \)-tilting \( A \)-module, this sequence is uniquely determined by setting a linear order (e.g., a lexicographic order) of the triples in advance. That is, we can judge whether two support \( \tau \)-tilting \( A \)-modules are isomorphic by judging whether they are equal as a sequence.

For each \( X \in \mathcal{P} \cup \mathcal{R} \cup \mathcal{I} \), we set \( M(X) := \dim X \). Algorithm 2 shows a procedure for constructing the set \( \mathcal{R} \) and the restriction \( M|_\mathcal{R} : \mathcal{R} \rightarrow \mathbb{Z}^{n+1} \) where the two maps \( \text{isPrj} : \mathbb{Z}^{n+1} \rightarrow \mathbb{Z} \) and \( \text{isInj} : \mathbb{Z}^{n+1} \rightarrow \mathbb{Z} \) are defined by
\[
\text{isPrj}(u) := \begin{cases} 
  k & \text{if } u = \dim P_k \\
  -1 & \text{(otherwise)}
\end{cases}
\text{isInj}(u) := \begin{cases} 
  k & \text{if } u = \dim I_k \\
  -1 & \text{(otherwise)}
\end{cases}
\]
respectively. For each \( i \in Q_0 \), we also define a map \( \text{isPrj}_i : \mathbb{Z}^{n+1} \rightarrow \mathbb{Z} \) as follows.
\[
\text{isPrj}_i(u) := \begin{cases} 
  k & \text{if } u = \dim_A (P^k_i) \\
  -1 & \text{(otherwise)}
\end{cases}
\]

For each \( u \in \mathcal{S} \), only one of the following (i)-(iii) holds.

(i) There exists an integer \( b \geq 0 \) such that \( \text{isPrj}(\Phi^b_A u) \geq 0 \) holds.
(ii) There exists an integer \( b \geq 0 \) such that \( \text{isInj}(\Phi^{-b}_A u) \geq 0 \) holds.
(iii) There exists an integer \( b \geq 1 \) such that \( \Phi^b_A u = u \) holds.

Let \( X \) be the indecomposable \( A \)-module satisfying \( \dim X = u \). These cases (i)-(iii) correspond to the cases where \( X \) belongs to \( \mathcal{P} \), \( \mathcal{I} \), and \( \mathcal{R} \), respectively. In the construction of \( \mathcal{R} \) and \( M|_\mathcal{R} \), it is the case (iii) that should be considered. By Lemma 4.2, then all modules on the \( \tau \)-orbit of \( X \) belong to \( \mathcal{R} \). In Algorithm 2, the dimension vectors of them are given as a list \( \mathcal{T} \), in particular, inside the if statement in lines 22-30 of the program, \( \mathcal{T} = (u, \Phi_A u, \cdots, \Phi_A^{b-1} u) \) holds. Note that the set \( \mathcal{L} \) corresponds to the image of \( \mathcal{M} \) at that moment. It is used to avoid duplicate counting elements of \( \mathcal{R} \) that have already been found.
Algorithm 2 construction of $\mathcal{R}$ and $\mathcal{M}|_{\mathcal{R}}$
\begin{verbatim}
1: $\mathcal{R} \leftarrow$ an empty set, $\mathcal{M} \leftarrow$ an empty map
2: $p_i \leftarrow 0, q_i \leftarrow 0$ for each $i \in \{0, 1, \cdots, n\}$
3: $\mathcal{S} \leftarrow \text{getNonsincereModules}()$
4: $\mathcal{L} \leftarrow$ an empty set
5: $c \leftarrow 1$
6: for $u \in \mathcal{S}$ do
7: if $u \in \mathcal{L}$ then
8: continue
9: end if
10: $\mathcal{F} \leftarrow$ an empty list
11: $v \leftarrow u, w \leftarrow u$
12: for $b = 0, 1, 2, \cdots$ do
13: $j \leftarrow \text{isPrj}(v), k \leftarrow \text{isInj}(w)$
14: if $j \geq 0$ then
15: $p_j \leftarrow \max(p_j, b + 1)$
16: break
17: end if
18: if $k \geq 0$ then
19: $q_k \leftarrow \max(q_k, b + 1)$
20: break
21: end if
22: if $b \geq 1$ and $u = v$ then
23: for $i = 0, \cdots, b - 1$ do
24: Append the triple $(1, b, c)$ to the list $\mathcal{R}$
25: Define $\mathcal{M}((1, b, c)) :=$ the $i$-th element of list $\mathcal{F}$.
26: Append the vector $\mathcal{M}((1, b, c))$ to the set $\mathcal{L}$. $c \leftarrow c + 1$
27: end for
28: break
29: end if
30: end for
31: Append the vector $v$ to the end of list $\mathcal{F}$
32: $v \leftarrow \Phi_A v, w \leftarrow \Phi_A^{-1} w$
33: end for
34: end for
35: $m \leftarrow \max(p_0, \cdots, p_n, q_0, \cdots, q_n)$
\end{verbatim}

In Algorithm 2, the $\{p_i\}_{0 \leq i \leq n}$, $\{q_i\}_{0 \leq i \leq n}$ and $m$ which appear in Lemma 4.3 and Lemma 4.4 are also obtained at the same time as $\mathcal{R}$ and $\mathcal{M}|_{\mathcal{R}}$. Thus, we can further construct $\mathcal{P}$, $\mathcal{I}$ and $\mathcal{M}$ by performing the following procedure.
Algorithm 3 construction of $\mathfrak{P}$, $\mathfrak{I}$ and $\mathfrak{M}$

1: for $i = 0, \ldots, n$ do
2: Append the triple $(0, i, 0)$ to the list $\mathfrak{P}$.
3: Define $\mathfrak{M}((0, i, 0)) := \dim P_i$
4: for $j = 1, \ldots, m + p_i - 1$ do
5: Append the triple $(0, i, j)$ to the list $\mathfrak{P}$.
6: Define $\mathfrak{M}((0, i, j)) := \Phi^{-1} \mathfrak{M}((0, i, j - 1))$
7: end for
8: Append the triple $(2, i, 0)$ to the list $\mathfrak{I}$.
9: Define $\mathfrak{M}((2, i, 0)) := \dim I_i$
10: for $j = 1, \ldots, m + q_i - 1$ do
11: Append the triple $(2, i, j)$ to the list $\mathfrak{I}$.
12: Define $\mathfrak{M}((2, i, j)) := \Phi \mathfrak{M}((2, i, j - 1))$
13: end for
14: Define $\mathfrak{M}((2, i, -1)) := -\dim P_i$
15: end for
16: $\Lambda \leftarrow \mathfrak{P} \cup \mathfrak{R} \cup \mathfrak{I} \cup \{(2, 0, -1), \ldots, (2, n, -1)\}$

4.2.2 Check whether $X \prec Y$

In this subsection, we are going to present an algorithm to determine whether $X \prec Y$ for all $X = (X_0, X_1, X_2)$ and $Y = (Y_0, Y_1, Y_2) \in \Lambda$. We first discuss the case where $X$ and $Y$ are both regular. In this case, the following statement is useful.

Lemma 4.9 ([20, Sect. 3.6 (5)]) Let $X, Y \in \mathcal{R}$. If $\# \{\tau^r X \mid r \in \mathbb{Z}\} \neq \# \{\tau^r Y \mid r \in \mathbb{Z}\}$, then we have

$$\text{Hom}_A(X, Y) = 0 = \text{Hom}_A(Y, X).$$

Thus, if $X_1 \neq Y_1$ stands, it can be determined immediately without tedious calculations. Otherwise, as shown in Definition 4.5, we check if there exists $(i, r) \in Q_0 \times \mathbb{Z}_{\geq 0}$ such that $\tau^r X, \tau^r Y \in \text{ind}_A$ and $\tau^r X \prec \tau^r Y$ are satisfied. In the following program, this is achieved by checking all possible $(i, r)$ by means of a double for statement.
Algorithm 4 check whether $X \prec Y$

1: function caseOfRegularPair($X, Y$)
2: if $X_1 \neq Y_1$ then
3: return True
4: end if
5: for $i \in Q_0$ do
6: for $r = 0, \ldots, X_1 - 1$ do
7: $u \leftarrow \Phi^r_A M(X)$, $v \leftarrow \Phi^r_A M(Y)$
8: if $u_i \neq 0$ or $v_i \neq 0$ then
9: continue
10: end if
11: $j \leftarrow \text{isPrj}_i(u)$, $k \leftarrow \text{isPrj}_i(v)$
12: while $j = -1$ and $k = -1$ do
13: $u \leftarrow \mathfrak{T}_A u$, $v \leftarrow \mathfrak{T}_A v$
14: $j \leftarrow \text{isPrj}_i(u)$, $k \leftarrow \text{isPrj}_i(v)$
15: end while
16: if $k \geq 0$ or $(\mathfrak{T}_A v)_j = 0$ then
17: return True
18: end if
19: end for
20: end for
21: return False
22: end function

The above has solved the most troublesome case. The other cases can also be judged as per Definition 4.5 by the following procedure. For a non-negative integer $a$ and a positive integer $b$, the remainder of $a$ divided by $b$ is denoted by $a \% b$.

Precalculating comparison $(X, Y)$ for all pairs $(X, Y) \in \Lambda \times \Lambda$ can be done in a short enough time because the number $\# \Lambda$ is only about 750, even for a case $Q = \tilde{E}_8$. This makes it possible to determine whether $X \prec Y$ with a computational complexity of $O(1)$.

4.2.3 Construct the Hasse quiver of $\text{suc}(A) \cap \text{pre}(0)$

In $\overline{\overline{H}}(s_r\text{-tilt}A)$, the two vertices $T$ and $T'$ connected by an arrow are mutations of each other. In order to treat the mutations of support $\tau$-tilting modules, it is useful to compute in advance the set

$\text{COEXIST}(X) := \{Y \in \Lambda \mid X \neq Y \text{ and } X \oplus Y: \tau\text{-rigid}\}$

for all $X \in \Lambda$. With Lemma 4.6(2) and the precalculations performed in the previous subsection, this task can be completed in a sufficiently short time. The following proposition is clear by Theorem 2.6([1, Theorem 2.18]).
Algorithm 5 check whether $X \prec Y$

1: function comparison($X, Y$)
2: if $X_0 = 0$ and $Y_0 = 0$ then
3: if $X_2 \geq Y_2$ or $\mathfrak{M}((0, Y_1, Y_2 - X_2 - 1))X_1 = 0$ then
4: return True
5: else
6: return False
7: end if
8: end if
9: if $X_0 = 0$ and $Y_0 = 1$ then
10: if $(\Phi^X_{X_2 + 1}Y_1^X \mathfrak{M}(Y))X_1 = 0$ then
11: return True
12: else
13: return False
14: end if
15: end if
16: if $X_0 = 0$ and $Y_0 = 2$ then
17: if $Y_2 + X_2 + 1 < qY_1$ and $\mathfrak{M}((2, Y_1, Y_2 + X_2 + 1))X_1 = 0$ then
18: return True
19: else
20: return False
21: end if
22: end if
23: if $X_0 = 1$ and $Y_0 = 1$ then
24: return caseOfRegularPair($X, Y$)
25: end if
26: if $X_0 = 1$ and $Y_0 = 2$ then
27: if $(\Phi^X_{Y_2 + 1}X_1 \mathfrak{M}(X))Y_1 = 0$ then
28: return True
29: else
30: return False
31: end if
32: end if
33: if $X_0 = 2$ and $Y_0 = 2$ then
34: if $X_2 \leq Y_2$ or $\mathfrak{M}((2, X_1, X_2 - Y_2 - 1))Y_1 = 0$ then
35: return True
36: else
37: return False
38: end if
39: end if
40: return True
41: end function

Proposition 4.10 Let $T = \bigoplus_{i=0}^{n} T_i$ be a support $\tau$-tilting module satisfying $T_i \in \Lambda$ for all $i \in \{0, \cdots, n\}$. Now take a number $j \in \{0, \cdots, n\}$ and construct the subset $\mathcal{M}(T, j)$ of $\Lambda$ as follows.

$$\mathcal{M}(T, j) := \bigcap_{j \neq i \in \{0, \cdots, n\}} \text{COEXIST}(T_i)$$

Then $\# \mathcal{M}(T, j)$ is equal to 1 or 2.
It is obvious from the way \( \mathcal{M}(T, j) \) is constructed that \( T_j \) always belongs to the set \( \mathcal{M}(T, j) \). In a case \( \# \mathcal{M}(T, j) = 2 \), let \( T'_j \) be the one of the two elements of \( \mathcal{M}(T, j) \) that is not \( T_j \), and \( T' \) be the support \( \tau \)-tilting module obtained by replacing the direct summand \( T_j \) of \( T \) with \( T'_j \), then there exists an arrow from \( T \) to \( T' \) in \( \vec{\mathbb{H}}(\text{sr-tilt}A) \) if and only if \( T'_j \prec T_j \) holds by Lemma 4.6(3).

**Algorithm 6** mutation of a support \( \tau \)-tilting module \( T = \bigoplus_{i=0}^{n} T_i \) on \( T_j \)

1: function mutation\((T, j)\)
2: \( \mathcal{M} \leftarrow \mathcal{M}(T, j) \)
3: if \( \# \mathcal{M} \neq 2 \) then
4: return False
5: end if
6: \( T'_j \leftarrow \) the one of the two elements of \( \mathcal{M} \) that is not \( T_j \)
7: if \( T'_j \prec T_j \) then
8: \( T' \leftarrow \bigoplus_{i=0}^{n} T'_i \) where \( T'_i := T_i \) for all \( i \neq j \)
9: return SORT\((T')\)
10: else
11: return False
12: end if
13: end function

This means that given a vertex \( u \), it is possible to enumerate all vertices \( v \) that are connected by an arrow from \( u \) in \( \vec{\mathbb{H}}(\text{sr-tilt}A |_A) \). Therefore, we can construct \( \vec{\mathbb{H}}(\text{suc}(A) |_A) \) by the following breadth-first search algorithm.

**Algorithm 7** construction of Hasse quiver of \( \text{sr-tilt}A |_A \)

1: \( \mathcal{L} \leftarrow \) the list \( \{ A = (\bigoplus_{i=0}^{n} P_i = ((0, 0, 0), \ldots, (0, n, 0))) \} \)
2: \( \mathcal{V} \leftarrow \) the list \( \{ 0 \} \), \( \mathcal{E} \leftarrow \) an empty list.
3: Define \( \text{SEEN}(A) := 0 \) and \( \text{SEEN}(T) := -1 \) for any support \( \tau \)-tilting module \( T \neq A \).
4: for \( j = 0 \ldots \# \mathcal{L} - 1 \) do
5: \( T \leftarrow \) the \( j \)-th element of list \( \mathcal{L} \)
6: for \( i = 0 \ldots n \) do
7: \( T' \leftarrow \text{mutation}(T, i) \)
8: if \( T' = \text{False} \) then
9: continue
10: end if
11: \( k \leftarrow \text{SEEN}(T') \)
12: if \( k < 0 \) then
13: Append a new vertex \( \# \mathcal{L} \) to the end of list \( \mathcal{V} \).
14: Append a new edge \((j, \# \mathcal{L})\) to the end of list \( \mathcal{E} \).
15: Redefine \( \text{SEEN}(T') := \# \mathcal{L} \).
16: Append the new support \( \tau \)-tilting module \( T' \) to the end of list \( \mathcal{L} \).
17: else
18: Append a new edge \((j, k)\) to the end of list \( \mathcal{E} \).
19: end if
20: end for
21: end for
The map \( \text{SEEN} \) in the above algorithm can be interpreted as the following map. In short, \( \text{SEEN} \) is a way to remember which vertex corresponds to each support \( \tau \)-tilting module once appeared.

\[
\text{SEEN}(T) = \begin{cases} 
  j & \text{if } T \text{ is isomorphic to the } j \text{-th element of the list } \mathcal{L} \\
  -1 & \text{(otherwise)}
\end{cases}
\]

Then \( \overrightarrow{H} := (\mathcal{V}, \mathcal{E}) \) obtained by Algorithm 7 is none other than \( \overrightarrow{H} (\text{suc}(A) | A) \). However, \( \overrightarrow{H} \) has also some vertices which have no path of finite length to the vertex corresponding to \( 0 = \bigoplus_{i=0}^{n} P_i \). These "useless" vertices can be removed by a simple task with a computational complexity of \( O(\#\mathcal{E}) \). By putting this finishing touch to \( \overrightarrow{H} \), we obtain \( \overrightarrow{H}_{\text{fin}}(\text{sr-tilt}A) \).

### 4.2.4 Counting maximal green sequences

Finally, we explain how to count all elements of \( \text{MGS}(A) \) by length.

Note that \( \overrightarrow{H}_{\text{fin}}(\text{sr-tilt}A) \) is a directed acyclic graph (DAG). In general, if a quiver \( \overrightarrow{H} := (\mathcal{V}, \mathcal{E}) \) is a DAG then \( \overrightarrow{H} \) has a topological ordering, that is, an ordering of the vertices such that if there is an arrow from \( u \) to \( v \), then the vertex \( u \) comes before \( v \) in the ordering. It is known that this sorting can be completed by Kahn’s algorithm [14] with a computational complexity of \( O(\#\mathcal{V} + \#\mathcal{E}) \).

The following proposition is obvious.

**Proposition 4.11** Let \( \overrightarrow{H} = \overrightarrow{H}_{\text{fin}}(\text{sr-tilt}A) \). For vertices \( v \) and \( s \) of \( \overrightarrow{H} \) and each number \( \ell \in \{0, \cdots, \#\overrightarrow{H}_{0} - 1\} \), we define

\[
\text{PATH}_s(v, \ell) := \text{the number of paths of length } \ell \text{ from } s \text{ to } v \text{ in } \overrightarrow{H}.
\]

Then, the following equation holds.

\[
\text{PATH}_s(v, \ell + 1) = \sum_{u \in \text{dpre}(v)} \text{PATH}_s(u, \ell)
\]

Therefore, our goal in this section is accomplished by the following procedure.
Algorithm 8 Counting maximal green sequences

1: \((\mathcal{Y}, \mathcal{E}) \leftarrow \overline{H}_{\text{fin}}(s\tau\text{-tilt}A)\)
2: \(L[u] \leftarrow \text{the list } [0] \text{ for each } u \in \mathcal{Y}\)
3: \(\text{NEXT}(u) \leftarrow \text{an empty list for each } u \in \mathcal{Y}\)
4: for \((u_1, u_2) \in \mathcal{E}\) do
5: Append the vertex \(u_2\) to the end of list \(\text{NEXT}(u_1)\).
6: \end for
7: Sort the elements of \(\mathcal{Y}\) in topological order.
8: \(s \leftarrow \text{the first vertex of } \mathcal{Y}, t \leftarrow \text{the last vertex of } \mathcal{Y}\)
9: \(L[s] \leftarrow \text{the list } [1]\)
10: for \(i = 0 \ldots \#\mathcal{Y} - 1\) do
11: \(v \leftarrow \text{the } i\text{-th vertex of } \mathcal{Y}\)
12: for \(w \in \text{NEXT}(v)\) do
13: \(\text{while } \#L[w] \leq \#L[v] \text{ do}\)
14: Append the number 0 to the end of list \(L[w]\).
15: \(\text{end while}\)
16: for \(j = 0 \ldots \#L[v] - 1\) do
17: Add the \(j\)-th number of list \(L[v]\) to the \((j + 1)\)-th number of list \(L[w]\).
18: \(\text{end for}\)
19: \(\text{end for}\)
20: \(\text{end for}\)

In line 7, the vertices are topologically sorted, so in line 8, \(s\) is the vertex corresponding to \(A = \bigoplus_{i=0}^{n} P_i\) and \(t\) is the vertex corresponding to \(0 = \bigoplus_{i=0}^{n} P_i^{-}\). That is, the goal of Algorithm 8 is to calculate \(\text{PATH}(\ell) := \text{PATH}_s(t, \ell)\) for each \(\ell\). Eventually, \(\text{PATH}(\ell)\) can be found as the \(\ell\)-th number of list \(L[t]\).

4.3 Demonstration and examples

The following is a demonstration of the methods described so far in this section and runs in a web browser.

https://hfipy3.github.io/MGS/en.html

Here are some examples of the results by this tool.

Example 4.12

\[
\begin{tikzpicture}
  \node (0) at (0,0) {0};
  \node (1) at (1,0) {1};
  \node (2) at (2,1) {2};
  \node (3) at (2,2) {3};
  \node (4) at (2,-1) {4};
  \draw (0) -- (2);
  \draw (2) -- (1);
  \draw (2) -- (3);
\end{tikzpicture}
\]

In this case, the quiver \(\overline{H}_{\text{fin}}(s\tau\text{-tilt}A)\) consists of 314 vertices and 743 arrows. The total numbers of MGS by length are below.

| \(\ell\) | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------|----|----|----|----|----|----|
| \(\text{PATH}(\ell)\) | 4  | 24 | 40 | 168| 144| 272|
|           | 11 | 12 | 13 | 14 | 15 | 16 |
|           | 400| 1144| 1720| 1792| 2912| 4928|
|           | 17 | 18 | 19 | 20 | 21 | 22 |
|           | 8192| 9984| 12672| 31104| 72576| 62208| 210284 |

total
The results in the above table are consistent with those presented by Brüstle–Dupont–Pérotin (arXiv:1205.2050v1). Please try the other examples presented in their paper.

Example 4.13  As a larger case, we present an example of type $\tilde{E}_8$.

$$ Q := \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array} $$

In this case, the quiver $\overline{H}_{\text{fin}}(\text{st-tilt}A)$ consists of 528510 vertices and 2353207 arrows. Since the number $\#\text{MGS}(A)$ is very huge, we present some excerpts.

| $\ell$ | PATH($\ell$) | 4224 | 36884 | $3.543 \ldots \times 10^{188}$ | $3.758 \ldots \times 10^{187}$ | $2.546 \ldots \times 10^{192}$ |
|-------|-------------|-------|--------|--------------------------------|--------------------------------|--------------------------------|

The calculation of this example takes less than 4 minutes using a standard home computer. It is possible in less than a day that we check whether the no gap conjecture is true for all cases of type $\tilde{E}$. Hence, this demonstration gives us an “elephant proof” of the no gap conjecture for type $\tilde{E}$.

## 5 A proof of Main Theorem (1): the case $\tilde{D}$

The purpose of this section is to show the following statement.

**Theorem 5.1**  Let $Q$ be a quiver of type $\tilde{D}_n$. Then $\ell(Q)$ does not depend on the choice of the orientation of $Q$.

We already know that the statement in Theorem 5.1 holds for the case $n = 4$ ([5]). Hence, we assume $n \geq 5$ in the rest of this section.

### 5.1 Settings

In this section, for $k \in \{2, 3, \ldots, n-2\}$ and $\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) \in \{\pm\} \times \{\pm\} \times \{\pm\} \times \{\pm\}$, $\mathcal{Q}(k, \epsilon)$ denotes the set of quivers having the following form:

0 $\xleftarrow{\epsilon_0}^\epsilon 1$ $\cdots$ $2$ $\cdots$ $k$ $\cdots$ $k+1$ $\cdots$ $n-2$ $\xrightarrow{\epsilon_2}^{n-1}$ $\cdots$ $\xrightarrow{\epsilon_3}^n$

with $\# \{t \in \{2, 3, \ldots, n-3\} \mid t \leftarrow t + 1 \} = k - 2$. We define a quiver $Q(k, \epsilon) \in \mathcal{Q}(k, \epsilon)$ as follows:

0 $\xleftarrow{\epsilon_0}^\epsilon 1$ $\cdots$ $2$ $\cdots$ $k$ $\cdots$ $k+1$ $\cdots$ $n-2$ $\xrightarrow{\epsilon_2}^{n-1}$ $\cdots$ $\xrightarrow{\epsilon_3}^n$

where, for $(p, q) \in \{(0, 2), (1, 2), (n-2, n-1), (n-2, n)\}$,

$$ p \xleftarrow{\epsilon} q = \begin{cases} p \rightarrow q & (\epsilon = +) \\ p \leftarrow q & (\epsilon = -) \end{cases}.$$

We say that $Q$ is of type $(k, \epsilon)$ if $Q$ is isomorphic to a quiver in $\mathcal{Q}(k, \epsilon)$. 
Lemma 5.2 Let \( Q \in \mathcal{D}(k, \varepsilon) \). Then there exists a sink mutation sequence

\[
Q \rightarrow \mu_{k_1} Q \rightarrow \cdots \rightarrow \mu_{k_t} \cdots \mu_{k_1} Q = Q(k, \varepsilon)
\]

with \( k_1, \ldots, k_t \in \{3, 4, \ldots, n-3\} \).

Proof If \( \{k \in \{3, 4, \ldots, n-3\} \mid k - 1 \rightarrow k \leftarrow k + 1 \} = \emptyset \), then \( Q \) is \( Q(k, \varepsilon) \) and there is nothing to show. Assume that \( \{k \in \{3, 4, \ldots, n-3\} \mid k - 1 \rightarrow k \leftarrow k + 1 \} \neq \emptyset \). We take \( k_1 = \min\{k \in \{3, 4, \ldots, n-3\} \mid k - 1 \rightarrow k \leftarrow k + 1 \} \) and consider the new quiver \( \mu_{k_1} Q \).

By repeating the same arguments, we obtain a desired sink mutation sequence

\[
Q \rightarrow \mu_{k_1} Q \rightarrow \cdots \rightarrow \mu_{k_t} \cdots \mu_{k_1} Q = Q(k, \varepsilon).
\]

This finishes the proof. \( \square \)

Lemma 5.3 Assume that \( Q \in \mathcal{D}(k, \varepsilon) \), \( i \) is a source vertex of \( Q \), and \( i' \) is a sink vertex of \( Q \).

(1) If the degree of \( i \) is equal to 1, then

\[
\ell(Q) = \ell(Q(k, \varepsilon)), \quad \ell(\mu_i Q) = \ell(\mu_i Q(k, \varepsilon)).
\]

(2) If the degree of \( i' \) is equal to 1, then

\[
\ell(Q) = \ell(Q(k, \varepsilon)), \quad \ell(\mu_{i'} Q) = \ell(\mu_{i'} Q(k, \varepsilon)).
\]

Proof The assertion follows from Proposition 3.14 (1) and Lemma 5.2. \( \square \)

In the rest of this section, we use the following notation.

\( A = KQ \)
\( A_v := A/(e_v) \cong K(Q \setminus \{v\}) \)
\( A' := A^{op} \)

5.2 Case: \( Q = Q(k, +, -, -, +) \)

In this subsection, we treat the case that \( Q = Q(k, +, -, -, +) \). Let \( L \) be an indecomposable module given by

\[
\begin{array}{ccccccc}
0 & & K & \leftarrow & K & \leftarrow & \cdots \leftarrow & K & \rightarrow & \cdots & \rightarrow & K & \leftarrow & 0 \\
& & 0 & & & & & & & & & & & 0
\end{array}
\]

We will show the following proposition.

Proposition 5.4 Let \( Q \) and \( L \) be as above.

(1) \( (0, n, L) \) and \( (0, 1, L) \) satisfy Assumption 1.

(2) \( \ell(Q) = \ell(\mu_0 Q) = \ell(\mu_1 Q) = \ell(\mu_{n-1} Q) = \ell(\mu_n Q) \).

5.2.1 \( \mathcal{D}_0 \) and \( \mathcal{D}_n' \)

Here, we determine \( \mathcal{D}_0 \) and \( \mathcal{D}_n' \).

Let \( X \) be an indecomposable non-projective module with \( X e_0 = 0 \). We set

\[
x_t := \dim_K \text{Hom}_A(P_t, X), \quad x'_t := \dim_K \text{Hom}_A(\tau^{-1} P_t, X),
\]

for any \( t \in \{0, 1, \ldots, n\} \).
If \( k = 2 \), then we have
\[
\begin{align*}
x'_0 &= x'_2 \\
&= x'_1 + x'_3 - x_2 \\
&= (x_2 - x_1) + (x'_4 - x_3) \\
&\vdots \\
&= (x_2 - x_1) + (x'_{n-2} - x_{n-3}) \\
&= (x_2 - x_1) + (x_{n-1} + x'_n - x_{n-2}) \\
&= (x_2 - x_1) + (x_{n-1} - x_n)
\end{align*}
\]

If \( k > 2 \), then we have
\[
\begin{align*}
x'_0 &= x'_2 \\
&= x'_1 + x_3 - x_2 \\
&= x_3 - x_1
\end{align*}
\]

Therefore, by the classification of indecomposable modules of \( A_0 \), \( \mathcal{X}_0 \) has a unique module \( X_0 \) (up to isomorphism) and it is given by the following quiver representation.

\[
\begin{align*}
\text{If } k &= 2, \text{ then we have} \\
\begin{cases}
K &\overset{\text{id}}{\longrightarrow} K \\
0 &\overset{\text{id}}{\longrightarrow} 0
\end{cases}
\quad (k = 2)
\end{align*}
\]

\[
\begin{align*}
\text{Similarly, let } X' \text{ be an indecomposable non-projective module with } (\text{Tr } X')e_n = 0. \text{ We set } y_t := \dim K \text{ Hom}_A(P^A_t, \text{Tr } X') \text{ and } y'_t := \dim K \text{ Hom}_A(\tau^{-1}_A P^A_t, \text{Tr } X').
\end{align*}
\]

If \( k = 2 \), then we have
\[
\begin{align*}
y'_n &= y'_{n-2} - y_n \\
&= y'_{n-1} + y'_{n-3} - y_{n-2} \\
&= (y_{n-2} - y_{n-1}) + (y'_{n-3} - y_{n-2}) \\
&= (y_{n-2} - y_{n-1}) + (y'_{n-4} - y_{n-3}) \\
&\vdots \\
&= (y_{n-2} - y_{n-1}) + (y'_2 - y_3) \\
&= (y_{n-2} - y_{n-1}) + (y'_0 + y_1 - y_2). \\
&= (y_{n-2} - y_{n-1}) + (y_2 - y_0 + y_1 - y_2). \\
&= (y_{n-2} - y_{n-1}) + (y_1 - y_0).
\end{align*}
\]

If \( 2 < k < n - 2 \), then we have
\[
\begin{align*}
y'_n &= y'_{n-2} - y_n \\
&= y'_{n-1} + y'_{n-3} - y_{n-2} \\
&= (y_{n-2} - y_{n-1}) + (y'_{n-3} - y_{n-2}) \\
&= (y_{n-2} - y_{n-1}) + (y'_{n-4} - y_{n-3})
\end{align*}
\]
\[= (yn-2 - yn-1) + (yk' - yk+1)\]

\[= (yn-2 - yn-1) + (yk-1 - yk).\]

If \(k = n - 2\), then we have

\[y_n' = y_{n-2} - y_n\]

\[= y_{n-1} + y_{n-3} - y_{n-2}\]

\[= (y_{n-2} - y_{n-1}) + y_{n-3} - y_{n-2}\]

\[= y_{n-3} - y_{n-1}.\]

Hence, we obtain that \(\mathcal{X}'_n = \text{Tr}(\mathcal{X}_n(A'))\) has a unique module \(X'_n\) (up to isomorphisms) and \(Y_n := \tau X'_n = D(\text{Tr} X'_n)\) is given by the following quiver representation.

\[
\begin{array}{cccccccc}
0 & \xrightarrow{\text{id}} & K & \xrightarrow{\text{id}} & K & \cdots & \xrightarrow{\text{id}} & K & \xleftarrow{\text{id}} & 0 \\
K & & & & & & & & \\
K & \xrightarrow{(0 \ 1)} & K^2 & \xrightarrow{\text{id}} & \cdots & \xrightarrow{\text{id}} & K^2 & \xrightarrow{\text{id}} & 0 & \xleftarrow{\text{id}} & (k > 2) \\
K & \xrightarrow{(1 \ 1)} & K & \xrightarrow{\text{id}} & \cdots & \xrightarrow{\text{id}} & K & \xrightarrow{\text{id}} & 0 & \xleftarrow{\text{id}} & (k = 2) \\
\end{array}
\]

**Lemma 5.5** Let \(X_0\) and \(Y_n\) be as above. Then we have the following statements.

1. \(\tau_{A_0}^{-(k-1)}X_0 = P_{n-2}^-\)
2. \(\tau_{A_0}^{k-1}Y_n = P_n^-\).
3. \(\tau_{A_0}^{k-1}P_{n-1}^-\) and \(\tau_{A_0}^{k-1}P_n\) are not in \(\mathcal{R}\).
4. \(\tau_{A_n}^{-(k-1)}X_0\) and \(\tau_{A_n}^{-(k-1)}P_n\) are not in \(\mathcal{R}\).
5. If \(M \oplus X_0 \oplus P_0^- \in \text{sr-tilt}A\) with \(M \in \text{add}\mathcal{R}\) (resp. \(M' \oplus X'_n \oplus P_n \in \text{sr-tilt}A\) with \(M' \in \text{add}\mathcal{R}\)), then
   
   \[L \in \text{add}M\] (resp. \(L \in \text{add}M'\)).
6. Both \((0, n, L)\) and \((0, 1, L)\) satisfy A1, A2, and A4 in Assumption 1.

**Proof** (1) and (2). If \(k = 2\), then the assertion is obvious. Therefore, we assume \(k > 2\) and set \(X(t) \in \text{ind}A_0 \subset \text{ind}A\) for each \(t \in \{2, 3, \ldots, k-1\}\) as follows.

\[
\begin{array}{cccccccc}
0 & \xleftarrow{\text{id}} & \cdots & \xleftarrow{\text{id}} & 0 & \xleftarrow{\text{id}} & K & \xrightarrow{(1 \ 0)} & K^2 & \xleftarrow{\text{id}} & \cdots & \xleftarrow{\text{id}} & K^2 & \xrightarrow{(1 \ 0)} & K & \xleftarrow{\text{id}} & 0 \\
0 & \xleftarrow{\text{id}} & 0 & \xleftarrow{\text{id}} & 0 & \xleftarrow{\text{id}} & K & \xleftarrow{t-1 \ 1} & K & \xleftarrow{t \ 1} & K & \xleftarrow{t+1 \ 1} & K & \xleftarrow{t+2 \ 1} & K \\
0 & \xleftarrow{\text{id}} & t & \xleftarrow{\text{id}} & t & \xleftarrow{\text{id}} & t+1 & \xleftarrow{\text{id}} & t+2 & \xleftarrow{\text{id}} & (t \in \{2, 3, \ldots, k-1\}) \\
\end{array}
\]

We also define \(Y(t) \in \text{ind}A_n \subset \text{ind}A\) as follows.

\[
\begin{array}{cccccccc}
K & \xrightarrow{(0 \ 1)} & K & \xleftarrow{\text{id}} & \cdots & \xleftarrow{\text{id}} & K^2 & \xleftarrow{0 \ 1} & K & \xleftarrow{0} & \cdots & \xleftarrow{0} & 0 \\
K & \xrightarrow{(1 \ 1)} & K & \xleftarrow{t \ 1} & K & \xleftarrow{t \ 1} & K & \xleftarrow{t+2 \ 1} & K & \xleftarrow{0} & \cdots & \xleftarrow{0} & (t < k-1) \\
\end{array}
\]
For each $k < n - 2$, we have the following exact sequence.

$$
0 \to X(t) \to I_{t}^{A_0} \oplus I_{t+1}^{A_0} \oplus I_{t+2}^{A_0} \oplus I_{t}^{A_0} \to \left( I_{k}^{A_0} \right)^{\oplus 2} \oplus I_{n-1}^{A_0} \to 0 \quad (t < k - 1 < n - 3)
$$

$$
0 \to X(t) \to I_{t}^{A_0} \oplus I_{t+1}^{A_0} \oplus I_{t+2}^{A_0} \oplus I_{t}^{A_0} \to I_{n-2}^{A_0} \oplus I_{n-1}^{A_0} \to 0 \quad (t < k - 1 = n - 3)
$$

$$
0 \to X(k - 1) \to I_{n-3}^{A_0} \oplus I_{n-2}^{A_0} \oplus I_{n-1}^{A_0} \to I_{n-1}^{A_0} \to 0 \quad (t = k - 1 < n - 3)
$$

$$
0 \to X(k - 1) \to I_{n-3}^{A_0} \oplus I_{n-2}^{A_0} \oplus I_{n-1}^{A_0} \to I_{n-1}^{A_0} \to 0 \quad (t = k - 1 = n - 3)
$$

Note that the above exact sequence gives a minimal injective copresentation of $X(t)$ in mod $A_0$. Since $A_0$ is hereditary, we have the following exact sequence.

$$
0 \to P_{t}^{A_0} \oplus P_{t+1}^{A_0} \oplus P_{t+2}^{A_0} \oplus P_{t}^{A_0} \to \left( P_{k}^{A_0} \right)^{\oplus 2} \oplus P_{n-1}^{A_0} \to \tau_{A_0}^{-1} X(t) \to 0 \quad (t < k - 1 < n - 3)
$$

$$
0 \to P_{t}^{A_0} \oplus P_{t+1}^{A_0} \oplus P_{n}^{A_0} \to P_{n-2}^{A_0} \oplus P_{n-1}^{A_0} \to \tau_{A_0}^{-1} X(t) \to 0 \quad (t < k - 1 = n - 3)
$$

$$
0 \to P_{k-1}^{A_0} \oplus P_{n-2}^{A_0} \oplus P_{n-1}^{A_0} \to P_{k}^{A_0} \oplus P_{n-1}^{A_0} \to \tau_{A_0}^{-1} X(k - 1) \to 0 \quad (t = k - 1 < n - 3)
$$

$$
0 \to P_{n-3}^{A_0} \oplus P_{n}^{A_0} \to P_{n-1}^{A_0} \to \tau_{A_0}^{-1} X(k - 1) \to 0 \quad (t = k - 1 = n - 3)
$$

Then, by comparing dimension vectors, we obtain the following statement.

$$
\tau_{A_0}^{-1} X(t) = \begin{cases} 
X(t+1) & (t < k - 1) \\
\tau_{A_0}^{A_0} & (t = k - 1)
\end{cases}
$$

Similarly, we obtain the following statement.

$$
\tau_{A_0} Y(t) = \begin{cases} 
Y(t-1) & (2 < t) \\
P_{A_0}^{A_0} & (t = 2)
\end{cases}
$$

(3). We can easily check that $\tau_{A_0}^{t} P_{n-1}^{A_0} = \tau_{A_0}^{t-1} I_{n-1}^{A_0}$ is equal to $\tau_{A_0}^{t-1} I_{n-1}$ for any $1 \leq t \leq k - 1$ and given by

$$
\begin{cases} 
I_{n-1}^{A_0} = I_{n-1} = S_{n-1} & (t = 1) \\
0 & (t \geq 2)
\end{cases}
$$

where $V_t = \begin{cases} 
K & \text{if } t \text{ is odd} \\
0 & \text{if } t \text{ is even}
\end{cases}$

We can also check that $\tau_{A_0}^{t} P_{n} = \tau_{A_0}^{t-1} P_{n}$ is equal to $\tau_{A_0}^{t-1} P_{n}$ for any $1 \leq t \leq k - 1$ and given by

$$
\begin{cases} 
P_{n} & (t = 1) \\
0 & (t \geq 2)
\end{cases}
$$
where \( V_t = \begin{cases} K & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even} \end{cases} \)

(4). We can easily check that \( \tau_{A_n}^{-t} p_{1} = \tau_{A_n}^{-(t-1)} p_{A_n}^{1} \) is equal to \( \tau_{-}(t-1) P_{1} \) for any \( 1 \leq t \leq k - 1 \) and given by

\[
V_t = \begin{cases} P_{A_n}^{1} = P_{1} = S_{1} & (t = 1) \\ V_{t-1} \end{cases} \]

where \( V_t = \begin{cases} K & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even} \end{cases} \)

We can also check that \( \tau_{A_n}^{-t} f_{0} = \tau_{A_n}^{-(t-1)} P_{0}^{-} \) is equal to \( \tau_{-}(t-1) P_{0}^{-} \) for any \( 1 \leq t \leq k - 1 \) and given by

\[
V_{t-1} \begin{cases} P_{0}^{-} & (t = 1) \\ V_{t-1} \end{cases} \]

where \( V_t = \begin{cases} K & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even} \end{cases} \)

(5). Assume that \( M \oplus X_{0} \oplus P_{0}^{-} \in \text{sr-tilt} A \) with \( M \in \text{add} R \). Then it follows from \( M \oplus X_{0} \in \text{sr-tilt} A_{0} \) that

\[
\tau_{A_0}^{-(k-1)}(M \oplus X_{0}) \equiv \tau_{A_0}^{-(k-1)} M \oplus P_{n-2}^{-} \in \text{sr-tilt} A_{0}. \]

Thus, either \( P_{n-1}^{-} \) or \( S_{n-1} \) is in \( \text{add} \tau_{A_0}^{-(k-1)} M \), and either \( P_{n}^{-} \) or \( S_{n} \) is in \( \text{add} \tau_{A_0}^{-(k-1)} M \).

Suppose \( P_{n-1}^{-} \in \text{add} \tau_{A_0}^{-(k-1)} M \). Then (3) implies

\[
R \not\ni \tau_{A_0}^{-k-1} P_{n-1}^{-} \in \text{add} M. \]

This contradicts \( M \in \text{add} R \). Similarly, if \( S_{n} \in \text{add} \tau^{-(k-1)} M \), then we reach a contradiction. In particular, we have

\[
\tau_{A_0}^{-k-1} S_{n-1} \oplus \tau_{A_0}^{-k-1} P_{n}^{-} \in \text{add} M. \]

Next we assume \( M' \oplus X_{n}^{'} \oplus P_{n} \in \text{sr-tilt} A \) with \( M' \in \text{add} R \). Let \( N = \tau M' \). Since \( \tau(M' \oplus X_{0}) = N \oplus Y_{n} \in \text{sr-tilt} A_{n} \), we have

\[
\tau_{A_0}^{-k-1}(N \oplus Y_{n}) \equiv \tau_{A_0}^{-k-1} N \oplus P_{2}^{-} \in \text{sr-tilt} A_{n}. \]

In particular, either \( P_{0}^{-} \) or \( S_{0} \) is in \( \text{add} \tau_{A_0}^{k-1} N \) and either \( P_{1}^{-} \) or \( S_{1} \) is in \( \text{add} \tau^{k-1} N \). Suppose \( S_{0} = \tau_{A_n}^{k} S_{0} \in \text{add} \tau^{k-1} N \). Then (4) implies

\[
R \not\ni \tau_{A_0}^{-(k-1)} S_{0} \in \text{add} M \subset \text{add} M' \subset R. \]
This contradicts \( M \in \text{add}\mathcal{A} \). Similarly, if \( P_1^- \in \text{add}\tau^{k-1}N \), we reach a contradiction. In particular, we have

\[
\tau_{A_n}^{-(k-1)}P_0^- \oplus \tau_{A_n}^{-(k-1)}S_1 \in \text{add}N.
\]

Therefore, it is sufficient to show

\[
L \in \text{add}(\tau_{A_0}^{k-1}S_{n-1} \oplus \tau_{A_0}^{k-1}P_n^-) \cap \text{add}(\tau_{A_n}^{-(k-1)}P_0^- \oplus \tau_{A_n}^{-(k-1)}S_1).
\]

We show

\[
L \in \text{add}(\tau_{A_0}^{k-1}S_{n-1} \oplus \tau_{A_0}^{k-1}P_n^-).
\]

As we already checked, either \( \tau_{A_0}^{k-1}P_{n-1}^- = \tau_{A_0}^{k-2}S_{n-1} \) or \( \tau_{A_0}^{k-1}S_n = \tau_{A_0}^{k-2}P_n^- \) has the following form.

\[
\begin{array}{cccccc}
& 0 & \rightarrow & 0 & \rightarrow & K & \rightarrow & \ldots & \rightarrow & K & \rightarrow & \cdots & \rightarrow & K \\
0 & \rightarrow & K & \leftarrow & \cdots & \leftarrow & K & \rightarrow & \ldots & \rightarrow & K & \leftarrow & \cdots & \leftarrow & K
\end{array}
\]

In particular, either \( \tau_{A_0}^{k-1}S_{n-1} \) or \( \tau_{A_0}^{k-1}P_n^- \) is isomorphic to \( L \). Similarly, we can check

\[
L \in \text{add}(\tau_{A_n}^{-(k-1)}P_0^- \oplus \tau_{A_n}^{-(k-1)}S_1).
\]

(6). We already checked \( \mathcal{A}_0 \) has a unique module \( X_0 \) (up to isomorphism). Then it follows from (5) that both \((0, n, L)\) and \((0, 1, L)\) satisfy the assumptions \( A1 \) and \( A2 \). (5) also implies that \((0, n, L)\) satisfies \( A4 \). Moreover, by considering the correspondence

\[
0 \leftrightarrow n - 1 \\
1 \leftrightarrow n \\
v \leftrightarrow n - v \ (v \in \{2, \ldots, n - 2\}) \\
k \leftrightarrow n - k,
\]

we have that \((n - 1, 1, L)\) satisfies Assumption \( 1 \) with \( \mathcal{A}'_{1} \neq \emptyset \). In particular, the following statements hold.

- \( \mathcal{A}'_{1} \) has a unique module \( X'_{1} \) up to isomorphism.
- If \( M' \oplus X'_{1} \oplus P_1 \in \text{sr-tilt}\mathcal{A} \) with \( M' \in \text{add}\mathcal{A} \), then

\[
L \in \text{add}\tau M'.
\]

This shows that \((0, 1, L)\) satisfies \( A4 \).

\[\square\]

5.2.2 A proof of Proposition 5.4

Here we prove Proposition 5.4.

Proof (1). By Lemma 5.5(6), \((0, n, L)\) and \((0, 1, L)\) satisfy \( A1 \), \( A2 \), and \( A4 \). Moreover, \( A5 \) follows from Proposition 3.14(2). Therefore, it is sufficient to check \( A3 \).

By applying Hom\(_\mathcal{A}(-, L)\) to the almost split sequence

\[
0 \rightarrow P_n \rightarrow P_{n-2} \rightarrow \tau^{-1}P_n \rightarrow 0,
\]

we obtain an exact sequence

\[
0 \rightarrow \text{Hom}_\mathcal{A}(\tau^{-1}P_m, L) \rightarrow \text{Hom}_\mathcal{A}(P_{n-2}, L) \rightarrow \text{Hom}_\mathcal{A}(P_m, L) \rightarrow \text{Ext}_\mathcal{A}^1(\tau^{-1}P_m, L).
\]
Since \(L\) is not projective, we have \(\text{Ext}^1_A(\tau^{-1}P_n, L) = 0\). This shows
\[
\dim_K \text{Hom}_A(P_n, \tau L) = \dim_K \text{Hom}_A(\tau^{-1}P_n, L) = \dim_K \text{Hom}_A(P_{n-2}, L) - \dim_K \text{Hom}_A(P_n, L) = 1.
\]

Hence, \((0, n, L)\) satisfies \(A_3\).

Similarly, we have an exact sequence
\[
0 \to \text{Hom}_A(\tau^{-1}P_1, L) \to \text{Hom}_A(P_2, L) \to \text{Hom}_A(P_1, L) \to \text{Ext}^1_A(\tau^{-1}P_n, L).
\]

Since \(L\) is not projective, we have \(\text{Ext}^1_A(\tau^{-1}P_n, L) = 0\). This shows
\[
\dim_K \text{Hom}_A(P_1, \tau L) = \dim_K \text{Hom}_A(\tau^{-1}P_1, L) = \dim_K \text{Hom}_A(P_2, L) - \dim_K \text{Hom}_A(P_1, L) = 1.
\]

Hence, \((0, 1, L)\) satisfies \(A_3\).

(2). By (1), \((0, n, L)\) satisfies Assumption 1. Suppose that \(\ell(Q) \not\leq \ell(\mu_0Q)\). Then it follows from Proposition 3.15(2) that
\[
\ell(Q) \leq \ell(\mu_nQ).
\]

Since \((\mu_nQ)_{\text{op}}\) is isomorphic to the quiver
\[
\begin{array}{cccccccc}
0 & \xleftarrow{} & 2 & \cdots & n-k & \xleftarrow{} & \cdots & n-2
\end{array}
\begin{array}{cccccccc}
\xrightarrow{} & n-1 & \xleftarrow{} & \cdots & \xleftarrow{} & \cdots & \xleftarrow{} & 1
\end{array}
\]

in \(\mathcal{Q}(k, -, -, +)\). Hence, we have
\[
\ell(Q) \leq \ell(\mu_nQ) = \ell((\mu_nQ)_{\text{op}}) \leq \ell(Q(k, -, -, +)) = \ell(\mu_0Q).
\]

This is a contradiction. Therefore, we have
\[
\ell(Q) \leq \ell(\mu_0Q) \overset{\text{Prop. 3.14(2)}}{\leq} \ell(Q).
\]

By the correspondence
\[
0 \leftrightarrow n
\]
\[
1 \leftrightarrow n - 1
\]
\[
v \leftrightarrow n - v (v \in \{2, \ldots, n - 2\}),
\]

we get a quiver \(Q' \in \mathcal{Q}(k, +, -, -, +)\) such that
\[
Q'_{\text{op}} \simeq Q', \mu_nQ'_{\text{op}} \simeq \mu_0Q'.
\]

Therefore, it follows from Lemma 5.3 and \(\ell(Q) = \ell(\mu_0Q)\) that
\[
\ell(Q) = \ell(\mu_0Q) = \ell(Q(k, -, -, +)) = \ell(\mu_0Q) = \ell(\mu_nQ'_{\text{op}}) = \ell((\mu_nQ')_{\text{op}}) = \ell(\mu_nQ).
\]

Then, by considering the correspondence
\[
0 \leftrightarrow n - 1
\]
\[
1 \leftrightarrow n
\]
\[
v \leftrightarrow n - v (v \in \{2, \ldots, n - 2\})
\]
\[
k \leftrightarrow n - k,
\]
we also obtain
\[ \ell(Q) = \ell(\mu_1 Q) = \ell(\mu_{n-1} Q). \]
This shows the assertion.

5.3 Case: \( Q = Q(k, +, -, +, +) \)
In this subsection, we treat the case that \( Q = Q(k, +, -, +, +) \) and show the following proposition.

**Proposition 5.6** We have \( \ell(Q) = \ell(\mu_0 Q) \).

**Proof** Let \( B = K(\mu_{n-1} Q) \) and suppose that \( \ell := \ell(Q) \not\leq \ell(\mu_0 Q) \).

Since \( \mu_{n-1} Q = Q(k, +, -, +, +) \), we have \( \ell(Q) = \ell(\mu_{n-1} Q) \) by Proposition 5.4 (2). Then Proposition 3.5 implies that
\[ \text{MGS}(B, S_{n-1}^B) \neq \emptyset \iff \text{MGS}(A, \mu_{n-1} A) \neq \emptyset. \]
Since \( \text{MGS}(A, \mu_{n-1} A) \neq \emptyset \) by Proposition 3.14 (2), we obtain \( \text{MGS}(B, S_{n-1}^B) \neq \emptyset \).

We take \( \omega = (T_0 \to \cdots \to T_\ell) \in \text{MGS}(B, S_{n-1}^B) \) such that
\[ t := t^{(0)}_\omega = \max\{ t^{(0)}_{\omega'} | \omega' \in \text{MGS}(B, S_{n-1}^B) \}. \]
Then we show
\[ T_\ell = M \oplus X_0 \oplus P_0, \quad T_{\ell-1} = M \oplus X_0 \oplus P_0^- \]
with \( X_0 \in \mathcal{R}_0^B \) and \( M \in \text{add} \mathcal{R}(B) \). Otherwise, Lemma 3.13 (1) implies that one of the following two cases occurs.

(i) \( t = \ell. \)
(ii) There exists \( \omega' \in \text{MGS}(B, S_{n-1}^B) \) with \( t^{(0)}_{\omega'} > t. \)

The second case contradicts the maximality of \( t \). Thus, \( t = \ell \) holds. Therefore, we obtain
\[ T_{\ell-2} = S_0^B \oplus S_{n-1}^B = (S_0^B \oplus S_{n-1}^B, (1 - e_0 - e_{n-1}B)). \]
Then, by applying Proposition 3.5 to the source vertex \( n-1 \) in \( \mu_{n-1} Q \), we get the following maximal green sequence with length \( \ell \).
\[ A = T'_0 \to T'_1 = \mu_{n-1} A \to \cdots \to T'_{\ell-1} = S_0 \to T'_\ell = 0 \]
Thus, \( \text{MGS}(A, S_0) \neq \emptyset \), and the following inequality follows from Lemma 3.13 (3).
\[ \ell(Q) \leq \ell(\mu_0 Q) \]
This is a contradiction. Therefore, we have
\[ X_0 \in \mathcal{R}_0(B), \quad M \in \text{add} \mathcal{R}(B). \]
Take \( L \in \text{mod} B \) as in subsection 5.2 (\( B = K(Q(k, +, -, -, +, +)) \)). By Proposition 5.4(1), we can apply Proposition 3.15 (3) to \( (\mu_{n-1} Q, (0, n, L), (0, 1, L)) \) and obtain
\[ \ell(Q) = \ell(\mu_{n-1} Q) \leq \ell(\mu_1 \mu_n \mu_{n-1} Q). \]
Note that $Q' := (\mu_1 \mu_n \mu_{n-1} Q)^{\text{op}}$ is the following quiver.

In particular, $Q'$ is of type $(k, -, -, +, +)$. Then it follows from Lemma 5.3 that $\ell(Q') = \ell((Q(k, -, -, +, +)) = \ell(\mu_0 Q)$. This shows

$$\ell(Q) = \ell(\mu_{n-1} Q) \leq \ell((\mu_1 \mu_n \mu_{n-1} Q)) = \ell(\mu_0 Q).$$

This is a contradiction and hence we have $\ell(Q) \leq \ell(\mu_0 Q)$. Then it follows from Proposition 5.3 that $\ell(Q)$ is of type $(k, -, -, +, +)$. Therefore, we have the assertion. □

5.4 Case: $Q = Q(k, +, -, -, -)$

In this subsection, we treat the case that $Q = Q(k, +, -, -, -)$ and show the following equality.

$$\ell(Q) = \ell(\mu_0 Q).$$

If $k \neq n - 2$, then we have

$$\ell(Q) = \ell(\mu_{n-2} Q) \quad \text{(Proposition 3.14(1))}$$

$$= \ell(\mu_0 \mu_{n-2} Q) \quad \text{(Lemma 5.3 and Proposition 5.6)}$$

$$= \ell(\mu_{n-2} \mu_0 \mu_{n-2} Q) \quad \text{(Proposition 3.14(1))}$$

$$= \ell(\mu_0 Q).$$

Hence we may assume $k = n - 2$.

We put $Q = Q(n - 2, +, -, -, -), i = (1, 2, \ldots, n - 2, n - 1), i' = (1, 2, \ldots, n - 2, n)$, and define two indecomposable modules $L_{n-1}$ and $L_n$ as follows:

$$L_{n-1} : 0 \xleftarrow{K} K \xleftarrow{K} \cdots \xleftarrow{K} K \xleftarrow{K} 0$$

$$L_n : 0 \xleftarrow{K} K \xleftarrow{K} \cdots \xleftarrow{K} K \xleftarrow{K} 0.$$

We will show the following proposition.

**Proposition 5.7** Let $Q, i, i', L_{n-1}$, and $L_n$ be as above.

1. $(0, i, L_{n-1})$ and $(0, i', L_n)$ satisfy Assumption 1.
2. We have $\ell(Q) = \ell(\mu_0 Q).

5.4.1 First step

Here we check that $(0, (1, 2, \ldots, n - 2), L_{n-1})$ and $(0, (1, 2, \ldots, n - 2), L_n)$ satisfy Assumption 1.

Let $Q = Q(n - 2, +, -, -, -)$ and $A = KQ$. We assume $X \in \text{ind} A_0 \subset \text{ind} A$ and set $x_i := \dim_K \text{Hom}_A(P_i, X), x_i' := \dim_K \text{Hom}_A(\tau^{-1} P_i, X)$. Then we have

$$x_0' = x_2' - x_0 = x_1' + x_3 - x_2 = x_3 - x_1.$$
Therefore, it follows from the classification of indecomposable modules of $A_0$ that $\mathcal{P}_0$ has a unique module $X_0$ (up to isomorphism) and it has the following form.

$$
\begin{array}{c}
0 \\
K \\
K^2 \\
\vdots \\
K^2 \\
K \\
(1)
\end{array}
\begin{array}{c}
\text{id} \\
\text{id} \\
\text{id} \\
\text{id} \\
\text{id} \\
(1)
\end{array}
\begin{array}{c}
K \\
K^2 \\
\vdots \\
K^2 \\
K \\
(0)
\end{array}
$$

Let $A' = A^{op}$, $Y' \in \text{ind} A'/e_1 \subset \text{ind} A'$. We put $y_i := \dim_K \text{Hom}_{A'}(P_i^{A'}, Y')$ and $y'_i := \dim_K \text{Hom}_{A'}(\tau_{i-1}^{-1}P_i^{A'}, Y')$. Then we have

$$
y'_1 = y'_2 - y_1 = y'_0 + y'_3 - y_2 = y_2 - y_0 + y'_3 - y_2 = y_2 - y_0 + y'_4 - y_3 \\
\vdots \\
y_2 - y_0 + y'_{n-2} - y_{n-3} = y_2 - y_0 + y'_{n-1} + y'_n - y_{n-2} = y_2 - y_0 + y_{n-2} - y_{n-1} - y_n
$$

Since $y_1 = 0$ and $A'/e_1$ is isomorphic to a path algebra of type $D_n$, the classification of indecomposable modules of $A'/e_1$ implies that $\mathcal{P}_1(A')$ has a unique module $Y'_1$ (up to isomorphism) and it has the following form.

$$
\begin{array}{c}
0 \\
K \\
K \\
\vdots \\
K \\
K \\
0
\end{array}
$$

Since $\text{Tr}$ induces a bijection between $\mathcal{P}_1(A')$ and $\mathcal{P}_1'(A')$, $\mathcal{P}_1'(A)$ has a unique module $X'_1 = \text{Tr} Y'_1$. Moreover, we have

$$
\tau X'_1 = D \text{Tr} (\text{Tr} Y'_1) \simeq D Y'_1 \simeq P_{n-2}^{A_1}.
$$

**Lemma 5.8** Let $Q, X_0, X'_1$ be as above.

1. $\tau_{A_0}^3 X_0 = P_{n-2}^{-} \text{ and } \tau_{A_1}^3 X'_1 \simeq P_{n-2}^{A_1}$.
2. $\tau_{A_0}^3 P_{n-1}^{-} \text{ and } \tau_{A_1}^3 P_{n}^{-} \text{ are preinjective}.$
3. Let $L_{n-1}$ and $L_n$ be indecomposable modules having the following forms:

$$
\begin{array}{c}
0 \\
K \\
K \\
\vdots \\
K \\
K \\
0
\end{array}
$$

Then $\tau_{A_0}^{-3} S_{n-1} = \tau_{A_1}^{-1} P_{n}^{-} = L_{n-1}$ and $\tau_{A_0}^{-3} S_{n} = \tau_{A_1}^{-1} P_{n-1}^{-} = L_n$.

4. If $T = M \oplus X_0 \oplus P_0^{-} \in \text{sr-tilt} A$ (resp. $T' = M' \oplus X'_1 \oplus P_1^{-} \in \text{sr-tilt} A$) with $M \in \text{add} \mathcal{R}$ (resp. $M' \in \text{add} \mathcal{R}$). Then we have

$$
L_{n-1} \oplus L_n \in \text{add} M \text{ (resp. } L_{n-1} \oplus L_n \in \text{add} M')
$$
(5) Both \((0, 1, 2, \ldots, n - 2), L_{n-1}\) and \((0, 1, 2, \ldots, n - 2), L_n\) satisfy Assumption 1.

Proof (1). We already checked \(\tau X_1' \simeq P_{n-1}^{A_1}\). Thus, we only check
\[ \tau_{A_0}^3 X_0 = P_{n-2}^A \]
A minimal projective presentation of \(X_0\) in mod \(A_0 \subset \text{mod} A\) is given by
\[ 0 \rightarrow P_{1}^{A_0} \oplus P_{2}^{A_0} \rightarrow P_{n-1}^{A_0} \oplus P_{n}^{A_0} \rightarrow X_0 \rightarrow 0. \]
Since \(A_0\) is hereditary, we have the following exact sequence.
\[ 0 \rightarrow \tau_{A_0} X_0 \rightarrow I_{1}^{A_0} \oplus I_{2}^{A_0} \rightarrow I_{n-1}^{A_0} \oplus I_{n}^{A_0} \rightarrow 0. \]
Therefore, \(\tau_{A_0} X_0 \in \text{ind} A_0 \subset \text{ind} A\) is given by the following quiver representation.

Then a minimal projective presentation of \(\tau_{A_0} X_0\) in mod \(A_0 \subset \text{mod} A\) is given by
\[ 0 \rightarrow P_{1}^{A_0} \rightarrow P_{n-1}^{A_0} \oplus P_{n}^{A_0} \rightarrow X_0 \rightarrow 0, \]
and the same argument as above implies that \(\tau_{A_0}^2 X_0\) is given by the following quiver representation.

In particular, we have
\[ \tau_{A_0}^2 X_0 \simeq P_{n-2}^{A_0}, \quad \tau_{A_0}^3 X_0 = P_{n-2}^A. \]
In particular, we have
\[ \tau_{A_0}^2 X_0 \simeq P_{n-2}^{A_0}, \quad \tau_{A_0}^3 X_0 = P_{n-2}^A. \]
(2) and (3). Since \(S_n = I_n = I_n^{A_0}\), we have \(\tau_{A_0}^{-1} S_n = P_n\) and \(\tau_{A_0}^{-1} P_n = P_{n}^{A_0}\). Then a minimal injective co-presentation of \(P_{n}^{A_0}\) in mod \(A_0 \subset \text{mod} A\) is given by
\[ 0 \rightarrow P_{1}^{A_0} \rightarrow I_{1}^{A_0} \rightarrow I_{n-1}^{A_0} \rightarrow 0. \]
Hence we have an exact sequence
\[ 0 \rightarrow P_{1}^{A_0} \rightarrow P_{n-1}^{A_0} \rightarrow \tau_{A_0}^{-1} P_{n}^{A_0} \rightarrow 0. \]
This gives us that \(\tau_{A_0}^{-1} P_{n}^{A_0} = \tau^{-3} S_n\) is given by the following quiver representation.

In particular, \(\tau_{A_0}^{-3} S_n = L_n\).
Similarly, we can check that \(M_n := \tau_{A_0}^{-3} P_n = \tau_{A_0}^{-1} P_n\) is given by
and \( \tau^{-1}M_n \) is given by
\[
\begin{pmatrix}
0 \\

\vdots \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
V_{t+1} \\

\vdots \\
K \\
K
\end{pmatrix}
\begin{pmatrix}
0 \\

\vdots \\
0 \\
0
\end{pmatrix}
(t \leq n - 5)
\]

where \( V_t = \begin{cases} 
0 & \text{if } t \text{ is odd} \\
K & \text{if } t \text{ is even.}
\end{cases} \)

Hence we obtain that \( \tau_{A_0}^{-3}P_n^- \) is preinjective and \( \tau_{A_0}^{-3}S_n = L_n \). Similarly, we can check that \( \tau_{A_0}^{-3}P_{n-1}^- \) is preinjective and \( \tau_{A_0}^{-3}S_{n-1} = L_{n-1} \). Then the remaining assertions \( \tau_{A_1}^{-1}P_n^- = L_n - 1 \) and \( \tau_{A_1}^{-1}P_{n-1} = L_{n-1} \) follow form \( L_{n-1} \simeq P_{n-1}^{A_1} \) and \( L_n \simeq P_{n-1}^{A_1} \).

(4). Assume that \( M \oplus X_0 \oplus P_0^* \in \text{sr-tilt} A_0 \) and \( M \in \text{add} \mathcal{R} \). Then we have \( M \oplus X_0 \in \text{sr-tilt} A_0 \). Thus, \( \tau_{A_0}^{3}M \oplus \tau_{A_0}^{3}X_0 \) is also in \( \text{sr-tilt} A_0 \). Then it follows from (1) that
\[
\tau_{A_0}^{3}M \oplus \tau_{A_0}^{3}P_{n-2}^* \in \text{sr-tilt} A_0.
\]

In particular, either \( S_{n-1} \) or \( P_{n-1}^- \) is in \( \text{add} \tau_{A_0}^{3}M \). If \( P_{n-1}^- \in \text{add} \tau_{A_0}^{3}M \), then \( \tau_{A_0}^{-3}P_{n-1} \in \text{add} M \). Therefore, (2) implies that \( M \) has a non-regular indecomposable direct summand. This is a contradiction. Hence, we obtain \( S_{n-1} \in \text{add} \tau_{A_0}^{3}M \). Then \( L_{n-1} \in \text{add} M \) follows from (3). Similarly, we can check \( L_n \in \text{add} M \).

Next we assume that \( M' \oplus X'_1 \oplus P_1 \in \text{sr-tilt} A_1 \) and \( M' \in \text{add} \mathcal{R} \). Let \( N := \tau M \). Since \( N \oplus \tau X'_1 \in \text{sr-tilt} A_1 \), it follows from (1) that
\[
\tau_{A_1}^{3}N \oplus \tau_{A_1}^{3}P_{n-2}^* \in \text{sr-tilt} A_1.
\]

In particular, either \( S_{n-1} \) or \( P_{n-1}^- \) is in \( \text{add} \tau_{A_1}^{3}N \). If \( S_{n-1} = I_{n-1} = I_{n-1}^{A_1} \in \text{add} \tau_{A_1}^{3}N \), then \( \tau_{A_1}^{3}S_{n-1} = L_{n-1} \) and \( \tau_{A_1}^{3}P_{n-1} = P_{n-1}^{A_1} \). This contradicts \( M' \in \text{add} \mathcal{R} \). Hence, we obtain \( P_{n-1}^- \in \text{add} \tau_{A_1}^{3}N \). Therefore, \( L_n = P_{n-1}^{A_1} = \tau_{A_1}^{-1}P_{n-1}^- \in \text{add} N = \text{add} M' \) by (3). Similarly, we can check \( L_{n-1} \in \text{add} M' \).

(5). As we already checked, \( X_0 \) is a unique module in \( \mathcal{R}_1 \) and \( X'_1 \) is a unique module in \( \mathcal{R}'_1 \). Then \( A_1, A_2, \) and \( A_4 \) follow from (4) and Lemma 3.12 (3). We can also check \( A_5 \). In fact, it follows from Proposition 3.14 that
\[
\ell(Q) = \ell(\mu_1 \mu_1 Q) \\
\geq \ell(\mu_1 Q) \quad \text{(Proposition 3.14 (2))} \\
= \ell(\mu_2 \mu_1 Q) \quad \text{(Proposition 3.14 (1))} \\
\vdots \\
= \ell(\mu_{n-3} \cdots \mu_1 Q) \quad \text{(Proposition 3.14 (1)).}
\]

Therefore, it is sufficient to check \( A_3 \).

For each \( p \in \{2, \ldots, n - 1\} \), we set \( L_{n-1}^{(p-1)} := F_{p-1}^+ \circ \cdots \circ F_1^+ (L_{n-1}) \) and \( L_{n}^{(p-1)} := F_{p-1}^+ \circ \cdots \circ F_1^+ (L_n) \). Then it is easy to check the following equations.

\[
\dim \left( L_{n-1}^{(p-1)} \right) = \ell(011 \cdots 101) \\
\dim \left( L_{n}^{(p-1)} \right) = \ell(011 \cdots 110)
\]
By applying Hom$_A(-, L_{n-1})$ to an almost split sequence
\[ 0 \to P_1 \to P_2 \to \tau^{-1}P_1 \to 0, \]
we obtain the exact sequence
\[ 0 \to \text{Hom}(\tau^{-1}P_1, L_{n-1}) \to \text{Hom}(P_2, L_{n-1}) \to \text{Hom}(P_1, L_{n-1}) \to \text{Ext}_A^1(\tau^{-1}P_1, L_{n-1}). \]
Since $L_{n-1}$ is not projective, we have Ext$_A^1(\tau^{-1}P_1, L_{n-1}) = 0$. In particular, we obtain the following equations.
\[
\begin{align*}
\dim K \text{Hom}(P_1, \tau L_{n-1}) &= \dim K \text{Hom}(\tau^{-1}P_1, L_{n-1}) \\
&= \dim K \text{Hom}(P_2, L_{n-1}) - \dim K \text{Hom}(P_1, L_{n-1}) \\
&= 1 \\
\dim K \text{Hom}(P_1, \tau L_n) &= \dim K \text{Hom}(\tau^{-1}P_1, L_n) \\
&= \dim K \text{Hom}(P_2, L_n) - \dim K \text{Hom}(P_1, L_n) \\
&= 1
\end{align*}
\]
Similarly, we can verify
\[
\begin{align*}
\dim K \text{Hom}(p_{\mu_{n-1}}^{-1}(P_p^{A_{(p-1)}}, \tau q_{(p-1)}(L_{p-1}^{(p-1)}))) &= 1 \\
\dim K \text{Hom}(p_{\mu_{n-1}}^{-1}(P_p^{A_{(p-1)}}, \tau q_{(p-1)}(L_{p-1}^{(p-1)}))) &= 1
\end{align*}
\]
by using the equations $(*)$. Therefore, we have the assertion. \hfill \Box

### 5.4.2 Second step
Let $B = K(\mu_{n-2} \cdots \mu_1 Q), B_{n-1} = B/(e_{n-1}), B_n = B/(e_n), B' := B^{\text{op}}$. For a module $Y' \in \text{ind } B'/(e_{n-1}) \subset \text{ind } B'$, we have
\[
y'_{n-1} = y'_{n-2} - y_{n-1} = y_{n-3} + y_n - y_{n-2}
\]
where $y_i := \dim K \text{Hom}_{B'}(p_i^{B'}, Y')$ and $y_i' := \dim K \text{Hom}_{B'}(\tau_i^{-1}p_i^{B'}, Y')$.

Since $B'/(e_{n-1})$ is isomorphic to a path algebra of type $D_n$, the classification of indecomposable modules of $B'/(e_{n-1})$ implies that $\mathcal{X}_{n-1}(B')$ has a unique module $Y'_{n-1}$ and it has the following form.
\[
\begin{array}{c}
K \\
\downarrow \\
K^2 \\
\downarrow \\
\vdots \\
\downarrow \\
K^2 \\
\downarrow \\
K
\end{array}
\begin{array}{c}
0 \\
\leftarrow \\
K \\
\leftarrow \\
K
\end{array}
\]
In particular, $X'_{n-1} := \text{Tr } Y'_{n-1}$ is a unique module in $\mathcal{X}_{n-1}'(B)$ (up to isomorphism). Moreover, we have that $Y_{n-1} := \tau_B X'_{n-1} = D \text{Tr } X'_{n-1} = DY'_{n-1}$ has the following form.
\[
\begin{array}{c}
K \\
\downarrow \\
K^2 \\
\downarrow \\
\vdots \\
\downarrow \\
K^2 \\
\downarrow \\
K
\end{array}
\begin{array}{c}
0 \\
\leftarrow \\
K \\
\leftarrow \\
K
\end{array}
\]
The same argument implies that $\mathcal{X}_n'(B)$ has a unique module $X'_n$ (up to isomorphism) and $Y_n := \tau_B X'_n$ has the following form.
\[
\begin{array}{c}
K \\
\downarrow \\
K^2 \\
\downarrow \\
\vdots \\
\downarrow \\
K^2 \\
\downarrow \\
K
\end{array}
\begin{array}{c}
0 \\
\leftarrow \\
K \\
\leftarrow \\
K
\end{array}
\]
Lemma 5.9 Let $Y_{n-1}$ and $Y_n$ be as above.

1. If $\tau_{B_{n-1}}^{-3} Y_{n-1} = (p_{B_n}^B)^{-}$ and $\tau_{B_n}^{-3} Y_n = (p_{B_n}^B)^{-}$.
2. If $M_{n-1} \oplus X_{n-1} \oplus (P_{n-1}^B) \in \mathcal{SB}$ with $M_{n-1} \in \text{add} \mathcal{Y}(B)$, then
   $F_{n-2}^+ \circ \cdots \circ F_1^+(L_{n-1}) \in \text{add} \tau_B M_{n-1}$.
3. If $M_n \oplus X_n' \oplus P_n^B \in \tau - \mathcal{SB}$ with $M_n \in \text{add} \mathcal{Y}(B)$, then
   $F_{n-2}^+ \circ \cdots \circ F_1^+(L_n) \in \text{add} \tau_B M_n$.
4. $\ell(Q) = \ell(\mu_1 Q) = \cdots = \ell(\mu_{n-2} \cdots \mu_1 Q)$.
5. Both $(0, i, L_{n-1})$ and $(0, i', L_n)$ satisfy Assumption 1.

Proof (1). A minimal projective presentation of $I_{B_{n-1}}^{B_{n-1}}$ in mod $B_{n-1}$ is given by

$$0 \to P_0^{B_{n-1}} \oplus P_1^{B_{n-1}} \oplus P_n^{B_{n-1}} \to \tau_1^{B_{n-1}} \to I_{B_{n-1}}^{B_{n-1}} \to 0.$$

Therefore, we have an exact sequence

$$0 \to \tau_{B_{n-1}}^{-2} \to \tau_1^{B_{n-1}} \to I_{B_{n-1}}^{-2} \to 0.$$

In particular, $\tau_{B_{n-1}}^{-2} \oplus I_{B_{n-1}}^{-2}$ has the following form.

$$\begin{array}{cccccc}
K & K & K & K & K & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
K & K & K & K & K & \\
\end{array}$$

Then a minimal projective presentation of $\tau_{B_{n-1}}^{-2} \oplus I_{B_{n-1}}^{-2}$ in mod $B_{n-1}$ is given by

$$0 \to P_0^{B_{n-1}} \oplus P_1^{B_{n-1}} \oplus P_n^{B_{n-1}} \to \tau_1^{B_{n-1}} \to I_{B_{n-1}}^{-2} \to 0.$$

Therefore, we have an exact sequence

$$0 \to \tau_1 \to I_{B_{n-1}}^{-2} \to 0.$$

In particular, $\tau_1 \oplus I_{B_{n-1}}^{-2}$ has the following form.

$$\begin{array}{cccccc}
K & K & K & K & K & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
K & K & K & K & K & \\
\end{array}$$

Hence, we have

$$Y_{n-1} \simeq \tau_{B_{n-1}}^{-2} \oplus I_{B_{n-1}}^{-2}.$$

The same argument implies

$$Y_n \simeq \tau_{B_n}^{-2} \oplus I_{B_n}^{-2}.$$

In particular, we obtain the assertions (1).

(2) and (3). Let $N_{n-1} = \tau_B M_{n-1}$, $L_{n-1} = F_{n-2}^+ \circ \cdots \circ F_1^+(L_{n-1})$.

We have $N_{n-1} \oplus Y_{n-1} \oplus (P_{n-1}^B)^{-} = \tau_B (M_{n-1} \oplus X_{n-1} \oplus P_{n-1}^B) \in \tau - \mathcal{SB}$. In particular,

$$N_{n-1} \oplus Y_{n-1} \in \tau - \mathcal{SB}_{n-1}.$$

Then (1) implies

$$\tau_{B_{n-1}}^{-3} N_{n-1} \in \tau - \mathcal{SB}_{n-1} / (e_2).$$
This shows that either $S_0$ or $(P_0^B)^-\tau_{B_{n-1}}^{-1}$ is in $\text{add} \tau_{B_{n-1}}^{-3} N_{n-1}$. Hence, either $\tau_{B_{n-1}}^3 S_0$ or $\tau_{B_{n-1}}^3 (P_0^B)^- = \tau_{B_{n-1}}^4 S_0$ is in $\text{add} N_{n-1}$. We can easily verify

\[
\begin{array}{c}
  S_0^B \xrightarrow{\tau_{B_{n-1}}} (P_0^B)^- \xrightarrow{\tau_{B_{n-1}}} I_0 \xrightarrow{\tau_{B_{n-1}}} L'_{n-1} = \ \begin{array}{c}
  K \leftarrow \cdots \leftarrow K \leftarrow K \leftarrow K \leftarrow 0 \\
  K \leftarrow \cdots \leftarrow K \leftarrow 0 \leftarrow \cdots \leftarrow 0 \\
  0 \leftarrow \cdots \leftarrow K \leftarrow 0 \leftarrow \cdots \leftarrow 0
  \end{array}
\end{array}
\]

By using induction, we can also check that

\[
\tau_B^q \left( \tau_{B_{n-1}}^4 S_0 \right) = \begin{cases}
  K & 0 \\
  K & n - 2 - q \\
  K & 0 \\
  K & n - 2 - q
\end{cases}
\]

for each $q \in \{1, 2, \ldots, n - 4\}$. In particular, $\tau_B^{n-4} \left( \tau_{B_{n-1}}^4 S_0 \right)$ is either $P_0^B$ or $P_1^B$. Thus, $\tau_{B_{n-1}}^4 S_0$ is preprojective and not in $\text{add} N_{n-1} = \text{add} \tau_{B_{n-1}}^{-1} R(B)$. Therefore, we have

\[
\tau_{B_{n-1}}^3 S_0 = L'_{n-1} \in \text{add} \tau_{B_{n-1}}^{-1} R(B).
\]

This shows the assertion (2). Then the same argument implies the assertion (3).

(4). By Proposition 3.15 (2) and Lemma 5.8 (5), we have

\[
\ell(Q) = \ell(\mu_1 Q) = \cdots = \ell(\mu_{n-3} \cdots \mu_1 Q) \leq \ell(\mu_{n-2} \mu_{n-3} \cdots \mu_1 Q).
\]

Then the following equality follows from Proposition 3.14 (1).

\[
\ell(\mu_{n-3} \cdots \mu_1 Q) = \ell(\mu_{n-2} \mu_{n-3} \cdots \mu_1 Q)
\]

Therefore, we have the assertion (4).

(5). By applying $\text{Hom}_B(-, L'_{n-1})$ to an almost split sequence

\[
0 \rightarrow P_{n-1}^B \rightarrow P_{n-2}^B \rightarrow \tau_B^{-1} P_{n-1} \rightarrow 0,
\]

we obtain the exact sequence

\[
0 \rightarrow \text{Hom}_B(\tau_B^{-1} P_{n-1}^B, L'_{n-1}) \rightarrow \text{Hom}_B(P_{n-2}^B, L'_{n-1}) \rightarrow \text{Hom}_B(P_{n-1}^B, L'_{n-1}) \rightarrow \text{Ext}_B^1(\tau_B^{-1} P_{n-1}^B, L'_{n-1}).
\]

Since $L'_{n-1}$ is not projective, we have $\text{Ext}_B^1(\tau_B^{-1} P_{n-1}^B, L'_{n-1}) = 0$. In particular, we obtain

\[
\dim_K \text{Hom}_B(P_{n-1}^B, \tau_B L'_{n-1})
\]
= \dim K \text{Hom}_B(\tau_B^{-1}P_{n-1}^B, L'_{n-1})

= \dim K \text{Hom}_B(P_{n-2}^B, L'_{n-1}) - \dim K \text{Hom}_B(P_{n-1}^B, L'_{n-1})

= 1.

Then it follows from (2), (4), and Lemma 5.8 (5) that \((0, i, L_{n-1})\) satisfies Assumption 1. Similarly, \((0, i', L_n)\) also satisfies Assumption 1. \qed

5.4.3 A proof of Proposition 5.7

Proposition 5.7 (1) is proved in Lemma 5.9 (5). Thus, it is sufficient to show the following equation.

\[ \ell(Q) = \ell(\mu_i Q). \]

Suppose \(\ell(Q) \neq \ell(\mu_i Q)\). Since \(\mu_i Q \simeq Q(2, +, -, +, +)\) and \(n-1\) is a unique source vertex of \(\mu_i Q\), it follows from Proposition 5.6 and Lemma 5.9 (4) that

\[ \ell(Q) = \ell(\mu_{n-2} \cdots \mu_1 Q) = \ell(\mu_{n-1} \mu_i Q) \geq \ell(\mu_i Q). \]

Then, by Lemma 5.8 (5), we can apply Proposition 3.15 (3) and obtain

\[ \ell(Q) \leq \ell(\mu_n \mu_{n-1} \cdots \mu_1 Q) = \ell(\mu_0 Q). \]

This is a contradiction. Hence, we have

\[ \ell(Q) \leq \ell(\mu_i Q). \]

On the other hand, we have \(\ell(Q) \geq \ell(\mu_0 Q)\) by Proposition 3.14 (2). This shows

\[ \ell(Q) = \ell(\mu_0 Q). \]

Therefore, we have Proposition 5.7.

5.5 A proof of Theorem 5.1

We prove Theorem 5.1. If \(Q\) and \(Q'\) are quivers of type \(\widetilde{D}_n\), then there exists a source mutation sequence

\[ Q \rightarrow \mu_i \mu_{i'} Q \rightarrow \cdots \rightarrow \mu_{i_m} \cdots \mu_{i_1} Q \simeq Q'. \]

Hence, it is sufficient to show that if \(Q\) is a quiver of type \(\widetilde{D}_n\) and \(i\) is a source vertex of \(Q\), then

\[ \ell(Q) \leq \ell(\mu_i Q). \]

Assume that \(Q \in \mathcal{Q}(k, \xi)\). If \(i \neq 0, 1, n-1, n\), then it follows from Proposition 3.14 (1) that \(\ell(Q) = \ell(\mu_i Q)\). Therefore, by Proposition 3.14 (1), (2) and Lemma 5.3, we may assume \(i = 0\) and \(Q\) is either \(Q(k, +, -, +, +), Q(k, +, -, -, +), \) or \(Q(k, +, -, -, -)\). Then the assertion follows from Proposition 5.4 (2), Proposition 5.6, and Proposition 5.7 (2).

6 A proof of Main Theorem (1): the case \(\widetilde{E}_6\)

The main result of this section is as follows.
Theorem 6.1  Let $Q$ be a quiver of type $\tilde{E}_6$ and $i$ be a source of $Q$. Then we have $\ell(Q) \leq \ell(\mu_i Q)$. In particular, $\ell(Q)$ does not depend on the choice of the orientation of $Q$.

Theorem 6.1 follows from the following two lemmas.

Lemma 6.2  Let $Q$ be a quiver of type $\tilde{E}_6$, $i$ be a source of $Q$ and $i'$ be a sink of $Q$.

1. If $\deg i \neq 1$, then $\ell(Q) = \ell(\mu_i Q)$.
2. If $\deg i' \neq 1$, then $\ell(Q) = \ell(\mu_{i'} Q)$.

Proof  This follows from Proposition 3.14 (1). \qed

Lemma 6.3  Let $Q$ be a quiver of type $\tilde{E}_6$, $i$ be a source vertex of $Q$. If the degree of $i$ is equal to 1, then we have $\ell(Q) \leq \ell(\mu_i Q)$.

In the rest of this section, we prove Lemma 6.3.

6.1 A proof of Lemma 6.3

Here we give a proof of Lemma 6.3. Assume that $i$ is a source vertex of $Q$ with $\deg i = 1$. We rewrite $Q$ as follows.

```
ii'           jj'           kk''           ii''
\downarrow    \downarrow    \downarrow    \downarrow
j' j''        j'' k''      j'' i''
```

We set $C_A := \left( \dim P_i \ \dim P_j \ \dim P_{i'} \ \dim P_{j'} \ \dim P_{j''} \ \dim P_{j'''} \ \dim P_k \right)$ the Cartan matrix of $A$.

6.1.1 Case: $j \to k$

Here, we consider the case $j \to k$.

Proposition 6.4  Assume that there is an arrow from $j$ to $k$. Then we have $\ell(Q) \leq \ell(\mu_i Q)$. If either $i' \to j' \to k$ or $i'' \to j'' \to k$ holds, then it follows from Proposition 3.14 (2) that $\ell(Q) \leq \ell(\mu_i Q)$.

Thus, we may assume $Q$ is one of the following quivers.

```
(a) i \to j \to k \to j' \to i'  (b) i \to j \to k \to j'' \to i''  (c) i \to j \to k \to j' \to i''
```

Lemma 6.5  Assume that $Q$ is the quiver $(b')$. Then we have $\ell(Q) = \ell(\mu_i Q) = \ell(\mu_{ij} Q)$.

Proof  We can check this assertion by a computational approach. □

Lemma 6.6  Assume that $Q$ is the quiver $(b)$. Then we have $\ell(Q) = \ell(\mu_{ij} Q)$.

Proof  Note that $\mu_{ij} Q$ is the quiver $(b')$. By Proposition 3.14(1) and Lemma 6.5, we obtain

\[ \ell(Q) = \ell(\mu_{ij} Q) = \ell(\mu_i \mu_{ij} Q) = \ell(\mu_{ij} \mu_i Q). \]

Then the assertion follows from $\mu_{ij} \mu_i Q = \mu_i Q$. □

Lemma 6.7  Assume that $Q$ is the quiver $(a)$. Then we have $\ell(Q) \leq \ell(\mu_i Q)$.

Proof  Note that $\mu_i Q$ is the quiver $(b')$. Thus, Proposition 3.14 and Lemma 6.5 imply

\[ \ell(Q) \leq \ell(\mu_i Q) \quad \text{(Proposition 3.14 (2))} \]
\[ = \ell(\mu_{ij} \mu_i Q) \quad \text{(Lemma 6.5)} \]
\[ = \ell(\mu_{ij} \mu_i \mu_{ij} Q) \quad \text{(Proposition 3.14 (1))} \]
\[ = \ell(\mu_{ij} \mu_i \mu_{ij} Q) \quad \text{(Proposition 3.14 (1))} \]
\[ = \ell(\mu_{ij} \mu_i \mu_{ij} Q) \quad \text{(Proposition 3.14 (1))} \]

Then the assertion follows from $\mu_i \mu_i \mu_{ij} Q = \mu_i Q$. □

Lemma 6.8  Assume that $Q$ is the quiver $(c)$. Then we have $\ell(Q) = \ell(\mu_i Q)$.

Proof  We can check this assertion by a computational approach. □

Lemma 6.9  Assume that $Q$ is either the quiver $(c')$ or $(c'')$. Then we have

\[ \ell(Q) \leq \ell(\mu_i Q). \]

Proof  If $Q$ is the quiver $(c')$, then $\mu_{ij} Q$ is the quiver $(c)$. Hence, Proposition 3.14 (1) and Lemma 6.8 imply

\[ \ell(Q) = \ell(\mu_{ij} Q) = \ell(\mu_i \mu_{ij} Q) = \ell(\mu_{ij} \mu_i Q) = \ell(\mu_i Q). \]

If $Q$ is the quiver $(c'')$, then $\mu_{ij} Q$ is the quiver $(c')$. Hence, Proposition 3.14 (1) and Lemma 6.2 imply

\[ \ell(Q) = \ell(\mu_{ij} Q) = \ell(\mu_i \mu_{ij} Q) = \ell(\mu_{ij} \mu_i Q) = \ell(\mu_i Q). \]

Thus, we obtain the assertion. □

By Lemma 6.5, Lemma 6.6, Lemma 6.7, Lemma 6.8, and Lemma 6.9, we obtain Proposition 6.4.
6.1.2 case: \( j \leftarrow k \)

Next we show the following proposition.

**Proposition 6.10** Assume that there is an arrow from \( k \) to \( j \), then we have

\[
\ell(Q) \leq \ell(\mu_i Q).
\]

If either \( j' \) or \( j'' \) is sink, then it follows from Proposition 3.14 (2) that

\[
\ell(Q) \leq \ell(\mu_i Q).
\]

If \( j' \) is source, then Proposition 3.14 gives

\[
\ell(Q) = \ell(\mu_{j'} Q) \quad \text{(Proposition 3.14 (1))}
\]
\[
\leq \ell(\mu_{j'} \mu_i Q) \quad \text{(Proposition 3.14 (2))}
\]
\[
= \ell(\mu_{j'} \mu_i Q) \quad \text{(Proposition 3.14 (1))}
\]
\[
= \ell(\mu_i Q).
\]

Similarly, if \( j'' \) is source, then we have

\[
\ell(Q) = \ell(\mu_{j''} Q) \leq \ell(\mu_{i} \mu_{j''} Q) = \ell(\mu_i Q).
\]

If \( k \) is a source vertex, then Proposition 3.14 (1) and Proposition 6.4 gives

\[
\ell(Q) = \ell(\mu_{k} Q) \leq \ell(\mu_{i} \mu_{k} Q) = \ell(\mu_{i} \mu_{k} Q) = \ell(\mu_i Q).
\]

Thus, we may assume \( Q \) is one of the following quivers.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
i' \\
\uparrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
j'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
j
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
j''
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\uparrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i''
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

(a) \hspace{1cm} (b)

Then we can check the assertion by a computational approach.

## 7 A proof of Main Theorem (1): the case \( \vec{E}_7 \)

In this section, we prove the following statement.

**Theorem 7.1** Let \( Q \) be a quiver of type \( \vec{E}_7 \) and \( i \) be a source vertex of \( Q \). Then we have

\[
\ell(Q) \leq \ell(\mu_i Q).
\]

In particular, \( \ell(Q) \) does not depend on the orientation.

Throughout this section, we assume that the underlying graph of \( Q \) has the following form.

\[
\begin{array}{cccccccc}
* & & & & & & & \\
1 & 
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 & \longrightarrow & 6 & \longrightarrow & 7
\end{array}
\]
Assume that $i = *$. Then it follows from Lemma 3.12 (2) and Proposition 3.14 (1) that

$$
\ell(\mu_* Q) \leq \ell(Q) \text{ and } \ell((\mu_* Q)^{op}) \leq \ell(\mu_* (\mu_* Q)^{op}) = \ell(Q^{op}) = \ell(Q).
$$

In particular, we have the following lemma.

**Lemma 7.2** We have $\ell(Q) = \ell(\mu_* Q)$.

Then by Proposition 3.14 (1) and Lemma 7.2, we may assume $i = 1$ and Theorem 7.1 follows from the following two lemmas.

**Lemma 7.3** Assume that 1 is a source vertex and 7 is a sink vertex. Then we have $\ell(Q) = \ell(\mu_1 Q)$.

**Lemma 7.4** Assume that both 1 and 7 are source vertices. Then we have $\ell(Q) = \ell(\mu_1 Q)$.

### 7.1 A proof of Lemma 7.3

In this subsection, we show Lemma 7.3. Let $\mathcal{A}$ be the set of quivers having the following form.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,0) {$2$};
\node (3) at (2,0) {$3$};
\node (4) at (3,0) {$4$};
\node (5) at (4,0) {$5$};
\node (6) at (5,0) {$6$};
\node (7) at (6,0) {$7$};
\node (anchor) at (3,1) {$*$};
\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (3) -- (4);
\draw[->] (4) -- (5);
\draw[->] (5) -- (6);
\draw[->] (6) -- (7);
\end{tikzpicture}
\end{center}

To prove Lemma 7.3, we divide $\mathcal{A}$ into the following five classes.

(Class 1)

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,0) {$2$};
\node (3) at (2,0) {$3$};
\node (4) at (3,0) {$4$};
\node (5) at (4,0) {$5$};
\node (6) at (5,0) {$6$};
\node (7) at (6,0) {$7$};
\node (anchor) at (3,1) {$*$};
\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (3) -- (4);
\draw[->] (4) -- (5);
\draw[->] (5) -- (6);
\end{tikzpicture}
\end{center}

(Class 2)

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,0) {$2$};
\node (3) at (2,0) {$3$};
\node (4) at (3,0) {$4$};
\node (5) at (4,0) {$5$};
\node (6) at (5,0) {$6$};
\node (7) at (6,0) {$7$};
\node (anchor1) at (3,1) {$*$};
\node (anchor2) at (5,1) {$*$};
\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (3) -- (4);
\draw[->] (4) -- (5);
\draw[->] (5) -- (6);
\draw[->] (6) -- (7);
\end{tikzpicture}
\end{center}

(Class 3)

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,0) {$2$};
\node (3) at (2,0) {$3$};
\node (4) at (3,0) {$4$};
\node (5) at (4,0) {$5$};
\node (6) at (5,0) {$6$};
\node (7) at (6,0) {$7$};
\node (anchor1) at (3,1) {$*$};
\node (anchor2) at (5,1) {$*$};
\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (3) -- (4);
\draw[->] (4) -- (5);
\draw[->] (5) -- (6);
\draw[->] (6) -- (7);
\end{tikzpicture}
\end{center}

(Class 4)

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,0) {$2$};
\node (3) at (2,0) {$3$};
\node (4) at (3,0) {$4$};
\node (5) at (4,0) {$5$};
\node (6) at (5,0) {$6$};
\node (7) at (6,0) {$7$};
\node (anchor1) at (3,1) {$*$};
\node (anchor2) at (5,1) {$*$};
\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (3) -- (4);
\draw[->] (4) -- (5);
\draw[->] (5) -- (6);
\draw[->] (6) -- (7);
\end{tikzpicture}
\end{center}

(Class 5)

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,0) {$2$};
\node (3) at (2,0) {$3$};
\node (4) at (3,0) {$4$};
\node (5) at (4,0) {$5$};
\node (6) at (5,0) {$6$};
\node (7) at (6,0) {$7$};
\node (anchor1) at (3,1) {$*$};
\node (anchor2) at (5,1) {$*$};
\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (3) -- (4);
\draw[->] (4) -- (5);
\draw[->] (5) -- (6);
\draw[->] (6) -- (7);
\end{tikzpicture}
\end{center}
We note that if $Q$ and $Q'$ belong the same class, then there exists sink or source mutation sequence

$$Q \rightarrow \mu_{i_1} Q \rightarrow \cdots \rightarrow \mu_{i_k} \cdots \mu_{i_1} Q = Q'$$

with $i_1, \ldots, i_k \in \{3, 4, 5, *\}$. Therefore, it follows from Lemma 3.14 (1) and Lemma 7.2 that

$$\ell(Q) = \ell(Q'), \quad \ell(\mu_1 Q) = \ell(\mu_1 Q').$$

Now put $Q(1), Q(2), Q(3), Q(4), Q(5)$ as follows.

$$Q(1) = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7$$

$$Q(2) = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \leftarrow 6 \rightarrow 7$$

$$Q(3) = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \leftarrow 6 \rightarrow 7$$

$$Q(4) = 1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow 6 \rightarrow 7$$

$$Q(5) = 1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow 6 \rightarrow 7$$

Then the following lemma gives Lemma 7.3.

**Lemma 7.5** Let $Q = Q(z) \ (z = 1, 2, 3, 4, 5)$. Then we have

$$\ell(Q) = \ell(\mu_1 Q).$$

**Proof** We can check the assertion by a computational approach. \qed
7.2 A proof of Lemma 7.4
In this subsection, we prove Lemma 7.4. Assume that 1 and 7 are source vertices and put $Q' = \mu_1 Q$. Then we have
\[ \ell(Q') = \ell((Q')^{op}) = \ell(\mu_1(Q')^{op}) = \ell(Q^{op}) = \ell(Q). \]
by Lemma 7.3. This finishes the proof.

8 A proof of Main Theorem (1): the case $\tilde{E}_8$
In this section, we prove the following statement.

**Theorem 8.1** Let $Q$ be a quiver of type $\tilde{E}_8$ and $i$ be a source vertex of $Q$. Then we have
\[ \ell(Q) \leq \ell(\mu_i Q). \]
In particular, $\ell(Q)$ does not depend on the orientation.

Throughout this section, we assume that the underlying graph of $Q$ has the following form.

```
\begin{array}{cccccccc}
9 & & & & & & & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
```

Then the following lemma gives Theorem 8.1.

**Lemma 8.2** Let $Q$ be a quiver of type $\tilde{E}_8$, $i$ be a source of $Q$. If $i \neq 8$, then $\mathcal{X}_i = \emptyset$.

In fact, if $i \neq 8$, then $\ell(Q) \leq \ell(\mu_i Q)$ follows from Proposition 3.14 (1) and Lemma 8.2. Thus, we may assume $i = 8$. In this case, there exists a source mutation sequence
\[ Q^{op} \to \mu_1 Q^{op} \to \cdots \to \mu_{i_8} \mu_{i_7} \cdots \mu_{i_1} Q = \mu_8 Q^{op} \]
with $[i_1, \ldots, i_8] = Q_0 \setminus \{8\}$. Then it follows from Proposition 3.14 (1) and Lemma 8.2 that
\[ \ell(Q) = \ell(Q^{op}) \leq \ell(\mu_{i_8} \mu_{i_7} \cdots \mu_{i_1} Q^{op}) = \ell(\mu_8 Q^{op}) = \ell(\mu_8 Q). \]
Therefore, it is sufficient to show Lemma 8.2.

8.1 A proof of Lemma 8.2
By Lemma 3.12 (3), it is sufficient to check that the following implication.
\[ 1 \text{ is a source vertex } \Rightarrow \mathcal{X}_1 = \emptyset. \]
Assume $i = 1$ is a source vertex and $X \in \mathcal{X}_1$. Then $C^{-1}_+ = Q$ is given by the following.

```
\begin{array}{cccccccc}
9 & & & & & & & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
```

Then we obtain
\[
\dim_k \text{Hom}_A(\tau^{-1}P_1, X_1) = \begin{cases} 
  x_3 - x_2 & (\epsilon_3 = -) \\
  x_6^\epsilon - (x_3 - x_4) & (\epsilon_3 = +, \epsilon_4 = -) \\
  x_5^\epsilon - (x_4 - x_5) & (\epsilon_3 = +, \epsilon_5 = -) \\
  x_6^\epsilon - (x_5 - x_6) & (\epsilon_3 = +, \epsilon_5 = +, \epsilon_6 = -) \\
  x_7^\epsilon - (x_6 - x_7) & (\epsilon_3 = +, \epsilon_5 = +, \epsilon_6 = +, \epsilon_7 = -) \\
  x_8^\epsilon - (x_7 - x_8) & (\epsilon_3 = +, \epsilon_5 = +, \epsilon_6 = +, \epsilon_7 = +, \epsilon_8 = -) \\
  x_9^\epsilon - x_8 & (\epsilon_3 = +, \epsilon_5 = +, \epsilon_6 = +, \epsilon_7 = +, \epsilon_8 = +) \\
\end{cases}
\]
where \( x_k := \dim_K \text{Hom}_A(P_k, X) \) and
\[
\begin{align*}
x_3^+ & := \dim_K \text{Hom}_A(\tau^{-1}P_9, X_1) = x_3 - x_9 \\
x_9^- & := \dim_K \text{Hom}_A(P_9, X_i) = x_9.
\end{align*}
\]
Since \( X \in \text{ind} \, K \subset \text{ind} \, A \), it follows from the classification of the indecomposable modules of \( K \) that
\[
\dim_K \text{Hom}_A(\tau^{-1}P_1, X_1) \leq 1.
\]
This is a contradiction. Therefore, we have the assertion.

9 Length of maximal green sequences for quivers of type \( \tilde{D} \) and \( \tilde{E} \)
In this section, we show Main theorem (2).

**Theorem 9.1** Let \( Q \) be a quiver of type \( \tilde{D} \) or \( \tilde{E} \). Then \( \ell(Q) \) is given by the following table.

| Type of \( Q \) | \( \tilde{D}_n \) | \( \tilde{E}_6 \) | \( \tilde{E}_7 \) | \( \tilde{E}_8 \) |
|----------------|-----------------|----------------|----------------|----------------|
| \( \ell(Q) \) | \( 2n^2 - 2n - 2 \) | 78 | 159 | 390 |

For the case that \( Q \) is of type \( \tilde{D}_4 \) or type \( \tilde{E}_{6,7,8} \), we can check the assertion by using the computer program (see Sect. 4). Therefore, we assume \( Q \) is of type \( \tilde{D}_n \) with \( n \geq 5 \).

For a proof, we consider the following quiver \( Q \) and its path algebra \( A \).

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & \cdots & n - 2 & n - 1 & n
\end{array}
\]

9.1 Nonsincere indecomposable preprojective/preinjective modules and indecomposable regular \( \tau \)-rigid modules
In this subsection, we classify nonsincere indecomposable preprojective/preinjective modules and indecomposable regular \( \tau \)-rigid module of \( A \). Note that if \( M \) is a nonsincere indecomposable module of \( A \), then there is \( v \in \{0, 1, n - 1, n\} \) such that
\[
M \in \text{ind} \, A_v \subset \text{ind} \, A \quad \text{(recall that, for each} \ i \in Q_0, \ \text{we set} \ A_i := A/(e_i)).
\]
Note that \( A_v \) is isomorphic to a path algebra of type \( D_n \). In particular, each indecomposable module is \( \tau \)-rigid by Proposition 2.3. Furthermore, each indecomposable module whose \( \tau \)-orbit contains a nonsincere module is \( \tau \)-rigid by the Auslander–Reiten duality. We also note that indecomposable \( \tau \)-rigid modules are determined by their \( g \)-vectors (see Theorem 2.9).

To the end of this subsection, we put \( e_i = (\delta_{ij})_{i \in Q_0} \in \mathbb{Z}^{Q_0} \), where \( \delta \) is the Kronecker delta. We recall that \( g^X \) denotes the \( g \)-vector of \( \tau \)-rigid pair \( X \). We also denote by Odd (resp. Even) the set of odd (resp. even) integers.
Lemma 9.2

(1) For each \( t \in \{2, \ldots, n - 1\} \), \( \tau^{-t+2} P_\epsilon \ (\epsilon \in \{0, 1\}) \) has the following form.

\[
\begin{align*}
\begin{cases}
0 & \quad (t \leq n - 2, \epsilon + t \in \text{Odd}) \\
K & \quad (t \leq n - 2, \epsilon + t \in \text{Even}) \\
0 & \quad (t = n - 1, \epsilon + n - 1 \in \text{Odd}) \\
0 & \quad (t = n - 1, \epsilon + n - 1 \in \text{Even})
\end{cases}
\end{align*}
\]

(2) For each \((s, t)\) satisfying \(1 \leq s < t \leq n - 2\), \( \tau^{-s+1} P_{t-s+1} \) has the following form.

\[
\begin{align*}
\begin{cases}
0 & \quad (s = 1) \\
K & \quad (s \geq 2)
\end{cases}
\end{align*}
\]

(3) For each \( t \in \{1, \ldots, n - 2\} \), \( \tau^{-t+1} P_{n-t} \ (\epsilon \in \{0, 1\}) \) has the following form.

\[
\begin{align*}
\begin{cases}
0 & \quad (t = 1, \epsilon = 1) \\
0 & \quad (t = 1, \epsilon = 0) \\
K & \quad (t \geq 2, \epsilon + t \in \text{Even}) \\
K & \quad (t \geq 2, \epsilon + t \in \text{Odd})
\end{cases}
\end{align*}
\]

Proof We denote by \( \alpha, \beta, \gamma \) the \( K \)-linear maps from \( K \) to \( K^2 \) given by \( \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \) and \( \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \), respectively.
(1). For each \( t \in \{2, \ldots, n-1 \} \) and \( \epsilon \in \{0, 1\} \), we define \( M_\epsilon(t) \) as follows.

\[
\begin{aligned}
&\begin{cases}
K \xrightarrow{\text{id}} K \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} K \xrightarrow{0} 0 \xrightarrow{0} 0 \\
0 & \quad \text{\( (t \leq n - 2, \epsilon = 0) \)} \\
0 & \quad \text{\( (t \leq n - 2, \epsilon = 1) \)} \\
K \xrightarrow{\text{id}} K \xrightarrow{\gamma} K \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} K \xrightarrow{0} 0 \xrightarrow{0} 0 \\
0 & \quad \text{\( (t = n - 1, \epsilon = 0) \)} \\
0 & \quad \text{\( (t = n - 1, \epsilon = 1) \)} \\
K \xrightarrow{\gamma} K^2 \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} K^2 \xrightarrow{0} 0 \xrightarrow{0} 0 \\
0 & \quad \text{\( (t = n - 1, \epsilon = 0) \)} \\
0 & \quad \text{\( (t = n - 1, \epsilon = 1) \)}
\end{cases}
\end{aligned}
\]

It is easy to check that \( M_0(t) \) and \( M_1(t) \) are indecomposable for any \( t \in \{2, \ldots, n-1\} \).

If \( 2 \leq t \leq n-3 \), then the minimal injective copresentation of \( M_0(t) \) and \( M_1(t) \) are given by

\[
0 \to M_0(t) \to I_2 \to I_1 \oplus I_{t+1} \to 0, \quad 0 \to M_1(t) \to I_2 \to I_0 \oplus I_{t+1} \to 0,
\]

respectively. Therefore, we have

\[
g^{\tau^{-1}M_0(t)} = g^{M_1(t+1)} = e_1 + e_{t+1} - e_2.
\]

In particular, we obtain

\[
\tau^{-1}M_0(t) \simeq M_1(t + 1).
\]

Similarly, we obtain

\[
\tau^{-1}M_1(t) \simeq M_0(t + 1).
\]

If \( t = n - 2 \), then the minimal injective copresentation of \( M_0(n - 2) \) and \( M_1(n - 2) \) are given by

\[
0 \to M_0(n - 2) \to I_2 \to I_1 \oplus I_{n-1} \oplus I_n \to 0, \quad 0 \to M_1(n - 2) \to I_2 \to I_0 \oplus I_{n-1} \oplus I_n \to 0,
\]

respectively. Therefore, we have

\[
g^{\tau^{-1}M_0(n-2)} = g^{M_1(n-1)} = e_1 + e_{n-1} + e_n - e_2.
\]

In particular, we obtain

\[
\tau^{-1}M_0(n - 2) \simeq M_1(n - 1).
\]

Similarly, we obtain

\[
\tau^{-1}M_1(n - 2) \simeq M_0(n - 1).
\]

Then the assertion follows from \( M_0(2) \simeq P_0 \) and \( M_1(2) \simeq P_1 \).
(2). For each \((s, t)\) satisfying \(1 \leq s < t \leq n - 2\), we define \(M(s, t)\) as follows.

\[
\begin{array}{c}
\begin{array}{cccccccc}
0 & \rightarrow & K & \rightarrow & \cdots & \rightarrow & K & \rightarrow & 0 \\
\end{array}
\end{array}
\]

\((s = 1)\)

\[
\begin{array}{c}
\begin{array}{cccccccc}
K & \rightarrow & \cdots & \rightarrow & K & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\end{array}
\]

\((s \geq 2)\)

It is easy to check that \(M(s, t) \in \text{ind } A\) for each \((s, t)\) satisfying \(1 \leq s < t \leq n - 2\).

If \(1 = s < t \leq n - 3\), then the minimal injective copresentation of \(M(1, t)\) has the following form.

\[
0 \rightarrow M(1, t) \rightarrow I_2 \rightarrow I_0 \oplus I_1 \oplus I_{t+1} \rightarrow 0.
\]

Therefore, we have

\[
g^{-1}M(1, t) = g^{M(2, t+1)} = e_0 + e_1 + e_{t+1} - e_2.
\]

In particular, we obtain

\[
\tau^{-1}M(1, t) \simeq M(2, t + 1).
\]

If \(2 \leq s < t \leq n - 3\), then the minimal injective copresentation of \(M(s, t)\) is given by

\[
0 \rightarrow M(s, t) \rightarrow I_{\beta^2}^2 \rightarrow I_0 \oplus I_1 \oplus I_{s+1} \oplus I_{t+1} \rightarrow 0.
\]

Therefore, we have

\[
g^{-1}M(s, t) = g^{M(s+1, t+1)} = e_0 + e_1 + e_{s+1} + e_{t+1} - 2e_2.
\]

In particular, we obtain

\[
\tau^{-1}M(s, t) \simeq M(s + 1, t + 1).
\]

Then the assertion follows from \(M(1, t) \simeq P_t\).

(3). For each \(t \in \{1, \ldots, n - 2\}\) and \(\epsilon \in \{0, 1\}\), we define \(M_{\eta-\epsilon}(t)\) as follows.

\[
\begin{array}{c}
\begin{array}{cccccccc}
0 & \rightarrow & K & \rightarrow & \cdots & \rightarrow & K & \rightarrow & 0 \\
\end{array}
\end{array}
\]

\((t = 1, \epsilon = 0)\)

\[
\begin{array}{c}
\begin{array}{cccccccc}
K & \rightarrow & \cdots & \rightarrow & K & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\end{array}
\]

\((t = 1, \epsilon = 1)\)

\[
\begin{array}{c}
\begin{array}{cccccccc}
K^2 & \rightarrow & \cdots & \rightarrow & K^2 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\end{array}
\]

\((2 \leq t \leq n - 3, \epsilon = 0)\)

\[
\begin{array}{c}
\begin{array}{cccccccc}
K^2 & \rightarrow & \cdots & \rightarrow & K^2 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\end{array}
\]

\((2 \leq t \leq n - 3, \epsilon = 1)\)

\[
\begin{array}{c}
\begin{array}{cccccccc}
K^2 & \rightarrow & \cdots & \rightarrow & K^2 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\end{array}
\]

\((t = n - 2, \epsilon = 0)\)

\[
\begin{array}{c}
\begin{array}{cccccccc}
K^2 & \rightarrow & \cdots & \rightarrow & K^2 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\end{array}
\]

\((t = n - 2, \epsilon = 1)\)
It is easy to check that $M_{n-\epsilon}(t) \in \text{ind} A$ for each $(\epsilon, t) \in \{0, 1\} \times \{1, \ldots, n-2\}$.

If $t = 1$, then the minimal injective copresentation of $M_n(1)$ and $M_{n-1}(1)$ are given by

$$
0 \to M_n(1) \to I_2 \to I_0 \oplus I_1 \oplus I_{n-1} \to 0, \\
0 \to M_{n-1}(1) \to I_2 \to I_0 \oplus I_1 \oplus I_n \to 0,
$$

respectively. Therefore, we have

$$g_{\tau^{-1}M_n(1)} = g_{M_{n-1}(2)} = e_0 + e_1 + e_{n-1} - e_2, \\
g_{\tau^{-1}M_{n-1}(1)} = g_{M_n(2)} = e_0 + e_1 + e_n - e_2.
$$

In particular, we obtain

$$\tau^{-1}M_n(1) \simeq M_{n-1}(2), \quad \tau^{-1}M_{n-1}(1) \simeq M_n(2).$$

If $2 \leq t \leq n-3$, then the minimal injective copresentation of $M_n(t)$ and $M_{n-1}(t)$ are given by

$$0 \to M_n(t) \to I_2^{\oplus 2} \to I_0 \oplus I_1 \oplus I_{t+1} \oplus I_{n-1} \to 0, \\
0 \to \tau M_{n-1}(t) \to I_2^{\oplus 2} \to I_0 \oplus I_1 \oplus I_{t+1} \oplus I_n \to 0,
$$

Therefore, we have

$$g_{\tau^{-1}M_n(t)} = g_{M_{n-1}(t+1)} = e_0 + e_1 + e_{t+1} + e_{n-1} - 2e_2.
$$

In particular, we obtain

$$\tau^{-1}M_n(t) \simeq M_{n-1}(t+1).$$

Similarly, we obtain

$$\tau^{-1}M_{n-1}(t) \simeq M_n(t+1).$$

Then the assertion follows from $M_{n-1}(1) \simeq P_{n-1}$ and $M_n(1) \simeq P_n$. $\square$

**Lemma 9.3**

(1) For each $t \in \{2, \ldots, n-1\}$, $\tau^{n-1-t}I_\epsilon \ (\epsilon \in \{0, 1\})$ has the following form.

(2) For each $(s, t)$ satisfying $2 \leq s \leq t \leq n-2$, $\tau^{n-2-t}I_{n-1-t+s} \ s_{\epsilon \in \{0, 1\}}$ has the following form.
(3) For each \( t \in \{1, \ldots, n - 2 \}, \tau^{n-2-t}I_{n-t} \) \((\epsilon \in \{0, 1\})\) has the following form.

![Diagram]

Proof. We denote by \( \alpha, \beta \) the \( K \)-linear maps from \( K \) to \( K^2 \) given by \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), respectively, and by \( \delta \) the \( K \)-linear map from \( K^2 \) to \( K \) given by \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

(1). For each \( t \in \{1, \ldots, n-2\} \) and \( \epsilon \in \{0, 1\} \), we define \( N_\epsilon(t) \) as follows.

![Diagram]

(We assume \( \alpha = \beta = \text{id} \) if \( t = n-2 \).) If \( t \leq n-3 \), then the minimal injective copresentation of \( M_0(t) \) and \( M_1(t) \) are given by

\[
\begin{align*}
0 &\rightarrow N_0(t) \rightarrow I_2 \oplus I_{t+1} \rightarrow I_1 \oplus I_{n-1} \oplus I_n \rightarrow 0, \\
0 &\rightarrow N_1(t) \rightarrow I_2 \oplus I_{t+1} \rightarrow I_0 \oplus I_{n-1} \oplus I_n \rightarrow 0,
\end{align*}
\]

respectively. Hence, we have

\[
\begin{align*}
g^{\tau^{-1}N_0(t)} &= e_1 + e_{n-1} + e_n - e_{t+1}, \\
g^{\tau^{-1}N_1(t)} &= e_0 + e_{n-1} + e_n - e_{t+1}.
\end{align*}
\]

In particular, we have

\[
\tau^{-1}N_0(t) \simeq N_1(t+1) \text{ and } \tau^{-1}N_1(t) \simeq N_0(t+1).
\]

If \( t = n-2 \), then the minimal injective copresentation of \( N_0(n-2) \) and \( N_1(n-2) \) are given by

\[
\begin{align*}
0 &\rightarrow N_0(n-2) \rightarrow I_2 \rightarrow I_1 \rightarrow 0, \\
0 &\rightarrow N_1(n-2) \rightarrow I_2 \rightarrow I_0 \rightarrow 0,
\end{align*}
\]
respectively. Hence, we have
\[
g^{-1}N_0(n-2) = gN_1(n-1) = e_1 - e_2
\]
\[
g^{-1}N_1(n-2) = gN_0(n-1) = e_0 - e_2.
\]
In particular, we have
\[
\tau^{-1}N_0(n-2) \cong N_1(n-1) \text{ and } \tau^{-1}N_1(n-2) \cong N_0(n-1).
\]
Then the assertion follows from \(N_0(n-1) \cong I_0\) and \(N_1(n-1) \cong I_1\).

(2). For each \((s, t)\) satisfying \(2 \leq s < t \leq n - 2\), we define \(N(s, t)\) as follows.
\[
\begin{array}{ccccccc}
0 & \cdots & 0 & \text{id} & K & \text{id} & \cdots & K & \text{id} & K^2 & \text{id} & \cdots & K^2 & \text{id} & K \\
0 & \cdots & 0 & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K \\
& & & s & & & & t & & & & & & &
\end{array}
\]
It is easy to check that \(N(s, t) \in \text{ind} A\) for each \((s, t)\) satisfying \(2 \leq s < t \leq n - 2\).
If \(t \leq n - 3\), then the minimal injective copresentation of \(N(s, t)\) is given by
\[
0 \to N(s, t) \to I_{s+1} \oplus I_{t+1} \to I_{n-1} \oplus I_n \to 0.
\]
Hence, we have
\[
g^{-1}N(s, t) = gN(s+1, t+1) = e_{n-1} + e_n - e_{s+1} - e_{t+1}.
\]
In particular, we obtain
\[
\tau^{-1}M(s, t) \cong M(s+1, t+1).
\]
Then the assertion follows from \(N(s, n-2) \cong I_{s+1}\).

(3). For each \(t \in \{1, \ldots, n-2\}\) and \(\epsilon \in \{0, 1\}\), we define \(N_{n-\epsilon}(t)\) as follows.
\[
\begin{array}{ccccccc}
K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & 0 \\
K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & 0 \\
K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & 0 \\
K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & 0 \\
0 & 0 & \cdots & 0 & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & 0 \\
0 & 0 & \cdots & 0 & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & 0 \\
0 & 0 & \cdots & 0 & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & 0 \\
0 & 0 & \cdots & 0 & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & \text{id} & \cdots & K & \text{id} & K & 0 \\
& & & s & & & & t & & & & & & &
\end{array}
\]
It is easy to check that \(N_{n-1}(t)\) and \(N_n(t)\) are indecomposable for each \(t\).
For any \(t \in \{1, \ldots, n-3\}\), the minimal injective copresentation of \(N_{n-1}(t)\) (resp. \(N_n(t)\))
is given by
\[
0 \to N_{n-1}(t) \to I_{t+1} \to I_n \to 0 \ (\text{resp. } 0 \to N_n(t) \to I_{t+1} \to I_{n-1} \to 0).
\]
Hence, we obtain the following equations.
\[
g^{-1}N_{n-1}(t) = gN_{n-1}(2) = e_n - e_{t+1}
\]
\[
g^{-1}N_n(t) = gN_{n-1}(2) = e_{n-1} - e_{t+1}
\]
In particular, we have
\[
\tau^{-1}N_{n-1}(t) \cong N_n(t+1) \text{ and } \tau^{-1}N_n(t) \cong N_{n-1}(t+1).
\]
Then the assertion follows from \(N_{n-1}(n-2) \cong I_{n-1}\) and \(N_n(n-2) \cong I_n\). \(\square\)
For each $t \in \{2, 3, \ldots, n - 2\}$, we define an indecomposable module $R_{t-1}$ as follows.

```
  K \xrightarrow{id} K \cdots \xrightarrow{id} K \xrightarrow{t} K \xrightarrow{id} \cdots \xrightarrow{id} K \xrightarrow{0} 0
```

**Lemma 9.4** Let $t \in \{2, \ldots, n - 2\}$. Then we have $\tau^{n-2}R_{t-1} \cong R_{t-1}$ and $\tau^{-k}R_{t-1}$ ($k \in \{1, \ldots, n - 3\}$) is isomorphic to $R_{t-1}(k)$ given by the following:

```
\begin{align*}
  &\begin{cases}
    0 & \quad (1 \leq k \leq n - t - 2) \\
    0 & \quad (k = n - t - 1) \\
    0 & \quad (n - t \leq k \leq n - 3)
  \end{cases} \\
  &\begin{cases}
    K & \quad (1 \leq k \leq n - t - 2) \\
    K & \quad (k = n - t - 1) \\
    K & \quad (n - t \leq k \leq n - 3)
  \end{cases}
\end{align*}
```

where $\alpha, \beta, \gamma$ are the $K$-linear maps from $K$ to $K^2$ given by $\left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$, $\left( \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right)$ and $\left( \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right)$, respectively, and $\delta$ is the $K$-linear map from $K^2$ to $K$ given by $(1, 1)$. (We assume $\alpha = \beta = \text{id}$ if $k = n - 3$.)

**Proof** We let $R_{t-1}(0) = R_{t-1}(n-2) := R_{t-1}$. We first show $R_{t-1}(k)$ is indecomposable for each $(t, k) \in \{2, \ldots, n - 2\} \times \{0, \ldots, n - 3\}$. It is easy to check that $R_{t-1}(k)$ is indecomposable for $0 \leq k \leq n - t - 1$. Hence we assume $n - t \leq k \leq n - 3$ and $R := R_{t-1}(k)$. Let $f \in \text{End}_A(R)$ and $f_j : R \to R$ be the $K$-linear map induced by $f$. Then, by the definition of $R$, we have

$$f_2 = \cdots = f_{k-t-n+2}, f_{k-t-n+3} = \cdots = f_{k+1}, f_{k+2} = \cdots = f_{n-2}.$$  

Now assume that $f_0, f_1, f_{n-1}, f_n$, and $f_{k-t-n+3} = f_{k+1} = \cdots = f_{k+2}$ are given by $a \in K$, $b \in K$, $c \in K$, $d \in K$, and $e \in K$, respectively. By $f_2 \circ \alpha = \alpha \circ f_0$ and $f_2 \circ \beta = \beta \circ f_1$, we obtain

$$f_2 = \cdots = f_{k-t-n+2} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$  

Similarly, we have

$$f_{k+2} = \cdots = f_{n-2} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$  

Then it follows from $f_{k+t-n+2} \circ \gamma = \gamma \circ f_{k+t-n+3}$ (resp. $f_{k+1} \circ \delta = \delta \circ f_{k+2}$) we obtain

$$\left( \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right) = \left( \begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix} \right) \quad (\text{resp.} (e \quad e) = (c \quad d)).$$

This shows that $a = b = c = d = e$ and $f = a \cdot \text{id}$. In particular, $\dim_K \text{End}_A(R) = 1$ and $R$ is indecomposable.
Then the minimal injective copresentation of $R_{t-1}(k)$ ($0 \leq k \leq n - 3$) is given by the following.

\[
\begin{align*}
0 \rightarrow R_{t-1}(k) & \rightarrow I_{k+2} \rightarrow I_{k+k+1} \rightarrow 0 & (0 \leq k \leq n - t - 3) \\
0 \rightarrow R_{t-1}(n-t-2) & \rightarrow I_{n-1} \rightarrow I_{n-1} \oplus I_{n} \rightarrow 0 & (k = n - t - 2) \\
0 \rightarrow R_{t-1}(n-t-1) & \rightarrow I_{2} \oplus I_{n-t+1} \rightarrow I_{0} \oplus I_{1} \oplus I_{n-1} \oplus I_{n} \rightarrow 0 & (t \geq 3, k = n - t - 1) \\
0 \rightarrow R_{t-1}(n-t) & \rightarrow I_{2} \oplus I_{k+t-n+3} \oplus I_{n-1} \oplus I_{n} \rightarrow 0 & (n-t \leq k \leq n - 4) \\
0 \rightarrow R_{t-1}(n-3-t) & \rightarrow I_{2} \rightarrow I_{0} \oplus I_{1} \rightarrow 0 & (t = 2, k = n - 3) \\
0 \rightarrow R_{t-1}(n-3-t) & \rightarrow I_{2} \rightarrow I_{0} \oplus I_{1} \oplus I_{t} \rightarrow 0 & (t \geq 3, k = n - 3)
\end{align*}
\]

Therefore, we obtain the following statement.

\[
g^{-1}R_{t-1}(k) = \begin{cases} 
\epsilon_{t+k+1} - \epsilon_{k+2} & (0 \leq k \leq n - t - 3) \\
\epsilon_{n-1} + \epsilon_{n} - \epsilon_{n-2} & (k = n - t - 2) \\
\epsilon_{0} + \epsilon_{1} + \epsilon_{n-1} + \epsilon_{n} - \epsilon_{2} - \epsilon_{n-t+1} & (t \geq 3, k = n - t - 1) \\
\epsilon_{0} + \epsilon_{1} + \epsilon_{k+t-n+3} + \epsilon_{n-1} + \epsilon_{n} - 2\epsilon_{2} - \epsilon_{k+2} & (n-t \leq k \leq n - 4) \\
\epsilon_{0} + \epsilon_{1} - \epsilon_{2} & (t = 2, k = n - 3) \\
\epsilon_{0} + \epsilon_{1} + \epsilon_{2} - 2\epsilon_{2} & (t \geq 3)
\end{cases}
\]

Then, by comparing the $g$-vectors, we have

\[\tau^{-1}R_{t-1}(k) \simeq R_{t-1}(k + 1).\]

This gives the assertion. □

**Lemma 9.5** Let $L_{0,n-1}$, $L_{0,n}$, $L_{1,n-1}$ and $L_{1,n}$ be indecomposable modules given by the following.

\[
\begin{align*}
L_{0,n-1} &= \begin{array}{c}
0 \\
K \\
K \\
K
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
0 \\
K \\
K \\
K
\end{array} \\
L_{0,n} &= \begin{array}{c}
0 \\
K \\
K \\
0
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
K \\
K \\
K
\end{array} \\
L_{1,n-1} &= \begin{array}{c}
K \\
0 \\
K \\
K
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
0 \\
K \\
K \\
K
\end{array} \\
L_{1,n} &= \begin{array}{c}
K \\
0 \\
K \\
K
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
K \\
K \\
K
\end{array}
\end{align*}
\]

(1) We have

\[
L_{0,n-1} \xrightarrow{\tau} L_{1,n} \xrightarrow{\tau} L_{0,n} \xrightarrow{\tau} L_{1,n-1}
\]

(2) Let $L, L' \in \text{add} \bigoplus_{\epsilon, \epsilon' \in \{0, 1\}} L_{\epsilon, n-\epsilon'}$ be indecomposable modules. Then $\text{Hom}_{A}(L, L') \neq 0$ if and only if $L \simeq L'$.

**Proof** (1). Let $\epsilon, \epsilon' \in \{0, 1\}$. Then the minimal injective copresentation of $L_{\epsilon, n-\epsilon'}$ is given by

\[
0 \rightarrow L_{\epsilon, n-\epsilon'} \rightarrow I_2 \rightarrow I_\epsilon \oplus I_{n-\epsilon'} \rightarrow 0.
\]
Hence, we have
\[ g^{-1}L_{\epsilon,n-\epsilon'} = g^{L_{1-\epsilon,n-(1-\epsilon')}} = e_\epsilon + e_{n-\epsilon'} - e_2. \]

In particular, we obtain
\[ \tau^{-1}L_{\epsilon,n-\epsilon'} \simeq L_{1-\epsilon,n-(1-\epsilon')}. \]

This shows (1).

(2). Let \( f \in \text{Hom}_A(L, L') \) and \( f_i : L e_i \to L' e_i \) the \( K \)-linear map induced by \( f \). We may assume \( L = L_{0,n} \) and \( L' = L_{\epsilon,n-\epsilon'} \) with \( \epsilon, \epsilon' \in \{0, 1\} \). Then we have
\[ f_2 = \cdots = f_{n-2}. \]

If \((\epsilon, \epsilon') = (0, 1)\), then \( f_0 = f_{n-1} = f_n = 0, f_1 = f_2, \) and \( f_{n-2} \) factors through 0. Therefore, we have \( f = 0 \). Similarly, we can check \( f = 0 \) for each \((\epsilon, \epsilon') \neq (0, 0)\). This shows the assertion. \( \square \)

We define \( T_{n-2} \) and \( T_2 \) as follows.
\[ T_{n-2} := \{ \tau^{-k}R_{t-1} \mid 2 \leq t \leq n-2, \ 0 \leq k \leq n-3 \} \]
\[ T_2 := \{ L_{0,n-1}, L_{0,n}, L_{1,n-1}, L_{1,n} \} \]

Then we have the following proposition.

**Proposition 9.6** Each nonsincere indecomposable module appears in Lemma 9.2, Lemma 9.3, Lemma 9.4, or Lemma 9.5. Moreover,
\[ T_{n-2} \cup T_2 \]
gives a complete set of representatives of isomorphism classes of indecomposable regular \( \tau \)-rigid modules.

**Proof** By the classification of indecomposable modules of path algebra of type \( D \), we can check that each indecomposable module in \( \text{ind} A_0 \cup \text{ind} A_1 \cup \text{ind} A_{n-1} \cup \text{ind} A_n \subset \text{ind} A \) appears in either Lemma 9.2, Lemma 9.3, Lemma 9.4, or Lemma 9.5. Then the assertion follows from Lemma 4.2. \( \square \)

### 9.2 A proof of the inequality \( \ell(Q) \leq 2n^2 - 2n - 2 \)

In this subsection, we show the following inequality.
\[ \ell(Q) \leq 2n^2 - 2n - 2 \]

Take a maximal green sequence
\[ \omega : T_0 \to \cdots \to T_\ell \]
with length \( \ell \) and set
\[ \text{add}\omega = \text{add} \bigoplus_{k=0}^{\ell} T_k \]
\[ \mathcal{P}_\omega = \text{add}\omega \cap \mathcal{P} \]
\[ \mathcal{J}_\omega = \text{add}\omega \cap \mathcal{J} \]
\[ \mathcal{F}_\omega = \text{add}\omega \cap \mathcal{T} \]
\[ \mathcal{R}_\omega = \text{add}\omega \cap \mathcal{R} \]
\[ t = \max \{ k \mid \#(\text{add} T_k \cap \mathcal{P}) \geq 1 \} \]
\[ s = \min \{ k \mid \#(\text{add} T_k \cap \mathcal{F}) \geq 1 \} \]

Then we assume
\[ \tau^{-p} P_t \in \text{add} T_t \text{ and } \tau^q P_t^- \in \text{add} T_t \text{ with } p, q \geq 0. \]

**Lemma 9.7** In the above setting, the following statements hold.

1. \( t \geq s. \)
2. \( \tau^{-p-q} P_t \in \text{mod} A_j. \)
3. \( \# \mathcal{P}_\omega \leq (n + 1)(p + 1) + \#(\mathcal{P} \cap \text{ind} A_i). \)
4. \( \# \mathcal{F}_\omega \leq (n + 1)q + \#(\mathcal{F} \cap \text{ind} A_i). \)
5. \[ \# \mathcal{P}_\omega \leq \#(\mathcal{P} \cap \text{ind} A_i) + \#(\mathcal{P} \cap \text{ind} A_j) - \# \left\{ R \in (\mathcal{P} \cap \text{ind} A_i) \mid \tau^{-(p+q+1)} R \in \text{ind} A_j \right\}. \]

**Proof** (1). Suppose \( t < s. \) Then the maximality of \( t \) and the minimality of \( s \) give
\[ \#(\text{add} T_t \cap (\mathcal{P} \cup \mathcal{F})) = 1. \]
This contradicts Proposition 4.1.

(2). By (1), we have \( T_t \leq T_s. \) Therefore, we have
\[ \begin{cases} \tau^{-p} P_t \in \text{mod} A_j & (q = 0) \\ \text{Hom}_A(\tau^{-p} P_t, \tau^q I_j) = 0 & (q \geq 1) \end{cases} \]
In particular, we obtain \( \tau^{-p-q} P_t \in \text{mod} A_j. \)

(3) and (4). By Lemma 4.6 (1) and the maximality of \( t \) (resp. the minimality of \( s \)), each module \( \tau^{-p} P_k \) in \( \mathcal{P}_\omega \) (resp. \( \tau^q P_k^- \) in \( \mathcal{F}_\omega \)) satisfies
\[ \tau^{-p} P_t < \tau^{-p} P_k \text{ (resp. } \tau^q P_k^- < \tau^q P_t^- \text{).} \]
Hence one of the following conditions holds.

(i) \( 0 \leq r \leq p \) (resp. \( 1 \leq r \leq q \))
(ii) \( r > p \) and \( \tau^{p+1}(\tau^{-p} P_k) \in \mathcal{P} \cap \text{ind} A_i \text{ (resp. } r > q \text{ and } \tau^{-q}(\tau^q P_k^-) \in \mathcal{F} \cap \text{ind} A_i \))

Since the condition (ii) is equivalent to \( \tau^{-p} P_k \in \tau^{-(p+1)}(\mathcal{P} \cap \text{ind} A_i) \) (resp. \( \tau^q P_k^- \in \tau^q(\mathcal{F} \cap \text{ind} A_i) \)), we obtain
\[ \# \mathcal{P}_\omega \leq (n + 1)(p + 1) + \#(\mathcal{P} \cap \text{ind} A_i) \text{ (resp. } \# \mathcal{F}_\omega \leq (n + 1)q + \#(\mathcal{F} \cap \text{ind} A_i) \)). \]

(5). Assume \( R \in \mathcal{P}_\omega \cap \text{add} T_k. \) If \( k \leq t \) (resp. \( k \geq s \)), then we have \( R \oplus \tau^{-p} P_t \) (resp. \( R \oplus \tau^q P_t^- \)) is \( \tau \)-rigid. In particular, we have
\[ \tau^{-p} P_t < R \quad (k \leq t) \]
\[ R < \tau^q P_t^- \quad (k \geq s) \]
by Lemma 4.6 (3). Therefore, it follows from (1) that
\[ \mathcal{P}_\omega \subset \{ R \in \mathcal{P} \mid \tau^{p+1} R \in \text{ind} A_i \} \cup \{ R \in \mathcal{P} \mid \tau^{-q} R \in \text{ind} A_i \}. \]

Therefore, if we set \( \mathcal{A} = \{ R \in \mathcal{P} \mid \tau^{p+1} R \in \text{ind} A_i \} \) and \( \mathcal{B} = \{ R \in \mathcal{P} \mid \tau^{-q} R \in \text{ind} A_i \}, \) then we have
\[ \# \mathcal{P} \leq \# \mathcal{A} + \# \mathcal{B} - \#(\mathcal{A} \cap \mathcal{B}). \]
Thus the assertion follows from the following bijections.

\[ A \leftrightarrow R \cap \text{ind } A_i \]
\[ B \leftrightarrow R \cap \text{ind } A_j \]
\[ A \cap B \leftrightarrow \left\{ R \in (R \cap \text{ind } A_i) \mid \tau^{-p+q+1}R \in \text{ind } A_j \right\} \]

This finishes the proof. \( \Box \)

Then we give \# \((R \cap \text{ind } A_i)\), \# \((J \cap \text{ind } A_i)\), and \# \((R \cap \text{ind } A_i)\) explicitly.

**Lemma 9.8** We have the following equations.

1. \# \((R \cap \text{ind } A_i)\) = \[
\begin{cases}
2n - 3 & (i \in \{0, 1\}) \\
\frac{1}{2}(i^2 + i - 6) & (i \in \{2, \ldots, n - 2\}) \\
\frac{1}{2}(n^2 + n - 10) & (i \in \{n - 1, n\})
\end{cases}
\]

2. \# \((J \cap \text{ind } A_i)\) = \[
\begin{cases}
\frac{1}{2}(n^2 - n - 4) & (i \in \{0, 1\}) \\
\frac{1}{2}(i^2 - (2n + 1)i + n^2 + n + 2) & (i \in \{2, \ldots, n - 2\}) \\
& (i \in \{n - 1, n\})
\end{cases}
\]

3. \# \((\mathbb{T}_{n-2} \cup \mathbb{T}_2) \cap \text{ind } A_i)\) = \[
\begin{cases}
\frac{1}{2}(n - 3)(n - 2) + 2 & (i \in \{0, 1, n - 1, n\}) \\
\frac{1}{2}(n - 2)(n^2 - 3n + 4) & (2 \leq i \leq n - 2)
\end{cases}
\]

**Proof** (1). By Lemma 9.2 and Proposition 9.6, \(R \cap \text{ind } A_i\) is given by the following.

\[ \{ \tau^{-i+2}P_t \mid 2 \leq t \leq n - 1, \epsilon \in \{0, 1\}, \epsilon + t \in \text{Odd} \} \]
\[ \cup \{ P_t \mid 1 < t \leq n \} \quad (i = 0) \]

\[ \{ \tau^{-i+2}P_t \mid 2 \leq t \leq n - 1, \epsilon \in \{0, 1\}, \epsilon + t \in \text{Even} \} \]
\[ \cup \{ P_t \mid 1 < t \leq n \} \quad (i = 1) \]

\[ \{ \tau^{-i+2}P_t \mid 2 \leq t \leq i - 1, \epsilon \in \{0, 1\} \}
\[ \cup \{ \tau^{-s+1}P_t \mid 1 \leq s < t \leq i - 1 \} (2 \leq i \leq n - 2) \]

\[ \{ \tau^{-i+2}P_t \mid 2 \leq t \leq n - 2, \epsilon \in \{0, 1\} \}
\[ \cup \{ \tau^{-s+1}P_t \mid 1 \leq s < t \leq n - 2 \} (i = n - 1) \]
\[ \cup \{ \tau^{-s+1}P_{n-\epsilon} \mid 1 \leq t \leq n - 2, \epsilon \in \{0, 1\}, \epsilon + t \in \text{Odd} \} \]

\[ \{ \tau^{-i+2}P_t \mid 2 \leq t \leq n - 2, \epsilon \in \{0, 1\} \} \]
\[ \cup \{ \tau^{-s+1}P_t \mid 1 \leq s < t \leq n - 2 \} (i = n) \]
\[ \cup \{ \tau^{-s+1}P_{n-\epsilon} \mid 1 \leq t \leq n - 2, \epsilon \in \{0, 1\}, \epsilon + t \in \text{Even} \} \]

If \(i \in \{0, 1\}\), then \# \((R \cap \text{ind } A_i)\) is given by

\[(n - 2) + (n - 1) = 2n - 3.\]

If \(i \in \{2, \ldots, n - 2\}\), then \# \((R \cap \text{ind } A_i)\) is given by

\[2(i - 2) + \sum_{t=2}^{i-1} (t - 1) = 2i - 4 + \frac{1}{2}(i - 2)(i - 1) = \frac{1}{2}(i^2 + i - 6).\]
If $i \in \{n-1, n\}$, then $\# (\mathcal{P} \cap \text{ind } A_i)$ is given by
\[
2(n-3) + \sum_{t=2}^{n-2} (t-i) + (n-2) = 3n - 8 + \frac{1}{2} (n-3)(n-2) = \frac{1}{2} (n^2 + n - 10).
\]
Therefore, the assertion (1) holds.

(2). By Lemma 9.3 and Proposition 9.6, $\mathcal{I} \cap \text{ind } A_i$ is given by the following.
\[
\begin{align*}
\{ & \tau^{n-1-i} \mathcal{I}_k \mid 2 \leq t \leq n-1, \epsilon \in \{0,1\}, n-1 - t + \epsilon \in \text{Odd} \\
\cup & \{ \tau^{n-2-i} \mathcal{I}_{n-1-t+s} \mid 2 \leq s < t \leq n-2 \} & (i = 0) \\
\cup & \{ \tau^{n-2-i} \mathcal{I}_{n-\epsilon} \mid 2 \leq t \leq n-2, \epsilon \in \{0,1\} \}
\end{align*}
\]
\[
\begin{align*}
\{ & \tau^{n-1-i} \mathcal{I}_k \mid 2 \leq t \leq n-1, \epsilon \in \{0,1\}, n-1 - t + \epsilon \in \text{Even} \\
\cup & \{ \tau^{n-2-i} \mathcal{I}_{n-1-t+s} \mid 2 \leq s < t \leq n-2 \} & (i = 1) \\
\cup & \{ \tau^{n-2-i} \mathcal{I}_{n-\epsilon} \mid 2 \leq t \leq n-2, \epsilon \in \{0,1\} \}
\end{align*}
\]
\[
\begin{align*}
\{ & \mathcal{I}_0, \mathcal{I}_1 \} \\
\cup & \{ \tau^{n-2-i} \mathcal{I}_{n-1-t+s} \mid i \leq s < t \leq n-2 \} & (2 \leq i \leq n-2) \\
\cup & \{ \tau^{n-2-i} \mathcal{I}_{n-\epsilon} \mid i \leq t \leq n-2, \epsilon \in \{0,1\} \}
\end{align*}
\]
\[
\begin{align*}
\{ & \mathcal{I}_0, \mathcal{I}_1 \} \\
\cup & \{ \tau^{n-2-i} \mathcal{I}_{n-\epsilon} \mid 1 \leq t \leq n-2, \epsilon \in \{0,1\}, n-2 - t + \epsilon \in \text{Even} \} & (i = n-1) \\
\cup & \{ \tau^{n-2-i} \mathcal{I}_{n-\epsilon} \mid 1 \leq t \leq n-2, \epsilon \in \{0,1\}, n-2 - t + \epsilon \in \text{Odd} \} & (i = n)
\end{align*}
\]
If $i \in \{0,1\}$, then $\# (\mathcal{I} \cap \text{ind } A_i)$ is given by
\[
\begin{align*}
(n-2) + \sum_{s=2}^{n-3} (n-2-s) + 2(n-3) = 3n - 8 + \frac{1}{2} (n-4)(n-3) \\
= \frac{1}{2} (n^2 - n - 4).
\end{align*}
\]
If $i \in \{2, \ldots, n-2\}$, then $\# (\mathcal{I} \cap \text{ind } A_i)$ is given by
\[
2 + \sum_{s=1}^{n-3} (n-2-s) + 2(n-1-i) = 2n - 2i + \frac{1}{2} (n-2-i)(n-1-i) \\
= \frac{1}{2} (i^2 - (2n+1)i + n^2 + n + 2).
\]
If $i \in \{n-1, n\}$, then $\# (\mathcal{I} \cap \text{ind } A_i)$ is given by
\[
2 + (n-2) = n.
\]
Therefore, the assertion (2) holds.

(3). Assume that $\tau^{-k} R_{t-1} \in \mathbb{G}_{n-2}$ ($t = 2, 3, \ldots, n-2$) is in mod $A_i$.
First we consider the case $2 \leq t \leq n-2$. By Lemma 9.4 and the definition of $R_{t-1}$, we have $0 \leq k \leq n-t-2$ and
\[
i \in [2, k + 1] \cup [t + k + 1, n-2].
\]
Equivalently, we obtain
\[
k \in [0, i - t - 1] \cup [i - 1, n - t - 2].
\]
In particular, the number of elements in \( \{ R_{t-1}^{-k} \mid 0 \leq k \leq n - 3 \} \cap \text{mod } A_i \) is given by
\[
\begin{cases}
  (i-t)+(n-t-i) & (t < i, n-t > i) \\
  i-t & (t < i, n-t \leq i) \\
  n-t-i & (t \geq i, n-t > i) \\
  0 & (t \geq i, n-t \leq i)
\end{cases}
\]
Therefore, we have the following equations.
\[
\#(T_{n-2} \cap \text{ind } A_i) = \frac{1}{2} (i-2)(i-1) + (n-i-2)(n-i-1) = i^2 - ni + 1/2(n^2 - 3n + 4).
\]
Next we consider the case \( i \in \{0, 1, n-1, n\} \). By Lemma 9.4 and the definition of \( R_{t-1} \), we obtain
\[
k \in \begin{cases}
  [1, n-t-1] & (i = 0, 1) \\
  [0, n-t-2] & (i = n-1, n).
\end{cases}
\]
Thus the number of elements in \( \{ R_{t-1}^{-k} \mid 0 \leq k \leq n - 3 \} \cap \text{mod } A_i \) is given by
\[
n - 1 - t.
\]
Therefore, we obtain the following equations.
\[
\#(T_{n-2} \cap \text{ind } A_i) = \frac{1}{2} (n-3)(n-2).
\]
Then the assertion follows from
\[
\#(T_2 \cap \text{ind } A_i) = \begin{cases}
  0 & (i \in \{2, \ldots, n-2\}) \\
  2 & (i \in \{0, 1, n-1, n\}).
\end{cases}
\]
This finishes the proof. \( \square \)

In the rest of this subsection, we give an upper bound for \( \ell \).

\textbf{9.2.1 (}2 \leq i \leq n - 2, 2 \leq j \leq n - 2\)

We assume \( 2 \leq i \leq n - 2 \) and \( 2 \leq j \leq n - 2 \). By Lemma 9.2, Proposition 9.6, and Lemma 9.7 (2), there exists \((s, t)\) satisfying the following conditions.

\[(i)\quad 1 \leq s < t < j\]
\[(ii)\quad \tau^{-p-q}P_1 \simeq \tau^{-s+1}P_{i-s+1}\]

The condition (ii) implies \( s = p + q + 1 \) and \( t = i + s - 1 = i + p + q \). Thus it follows from (i) that
\[
j > i \quad \text{and} \quad 0 \leq p + q < j - i.
\]

\textbf{Lemma 9.9} We have the following inequalities.

\((1)\quad \#P_{\omega} \leq \frac{1}{2}(i^2 + i - 6) + (n + 1)(p + 1)\)
(2) \( \# \mathcal{F}_\omega \leq \frac{1}{2}(i^2 - (2n + 1)j + n^2 + n + 2) + (n + 1)q = \frac{1}{2}(n-j)(n-j+1) + (n+1)q + 1 \)

(3) \( \# \mathcal{R}_\omega \leq i^2 + j^2 - n(i+j) + n^2 - 3n + 4 \)

**Proof** The inequality (1) follows from Lemma 9.7 (3) and Lemma 9.8 (1). The inequality (2) follows from Lemma 9.7 (4) and Lemma 9.8 (2). The inequality (3) follows from Lemma 9.7 (5) and Lemma 9.8 (3).

Let \( j' = n - j \) and \( x = i + j' \). Then we have \( 0 \leq p + q < j - i = n - x \) and

\[
eq \frac{3}{2}X^2 - \frac{1}{6}X + 2n^2 - 2n - 10 - \frac{1}{24}(4n + 1)^2.
\]

Since \( n > x = i + j' \geq 4 > 0 \), we have

\[
eq \max \left\{ 2n^2 - 2n - 10, \frac{1}{2}(3n^2 - 5n) - 10 \right\} < 2n^2 - 2n - 2.
\]

9.2.2 (2 \( \leq i \leq n - 2, j \in \{0, 1\} \))

Assume \( 2 \leq i \leq n - 2 \) and \( j \in \{0, 1\} \). By Lemma 9.2, Proposition 9.6, and Lemma 9.7 (2), there exists \( t \in \{2, 3, \ldots, n - 2\} \) such that

\[
\tau^{-p-a}P_t \simeq P_t.
\]

Hence \( p = q = 0 \) and \( i = t \).

**Lemma 9.10** We have the following inequalities.

(1) \( \# \mathcal{P}_\omega \leq \frac{1}{2}(i^2 + i - 6) + (n + 1) \)

(2) \( \# \mathcal{F}_\omega \leq \frac{1}{2}(n^2 - n - 4) \)

(3) \( \# \mathcal{R}_\omega \leq i^2 - ni + n^2 - 4n + 7 \)

**Proof** The inequality (1) follows from Lemma 9.7 (3) and Lemma 9.8 (1). The inequality (2) follows from Lemma 9.7 (4) and Lemma 9.8 (2). The inequality (3) follows from Lemma 9.7 (5) and Lemma 9.8 (3).
Then we have
\[ \ell(Q) = \# \mathcal{P}_\omega + \# \mathcal{I}_\omega + \# \mathcal{R}_\omega \]
\[ \leq \frac{3}{2} i^2 - \frac{1}{2} (2n - 1)i + \frac{1}{2} (3n^2 - 7n) + 3. \]

We set
\[ P(X) := \frac{3}{2} X^2 - \frac{1}{2} (2n - 1)X + \frac{1}{2} (3n^2 - 7n) + 3 \]
\[ = \frac{3}{2} \left( X - \frac{1}{6} (2n - 1) \right)^2 + \frac{1}{2} (3n^2 - 7n) + 3 - \frac{1}{24} (2n - 1)^2. \]

Since \( 0 < i < n \) and \( n \geq 5 \), we obtain
\[ \ell(Q) < \max \{ P(0), P(n) \} = \max \left\{ \frac{3}{2} n^2 - \frac{7}{2} n + 3, 2n^2 - 3n + 3 \right\} \leq 2n^2 - 2n - 2. \]

9.2.3 (2 \leq i \leq n - 2, j \in \{ n - 1, n \})

We assume \( 2 \leq i \leq n - 2 \) and \( j \in \{ n - 1, n \} \). By Lemma 9.2, Proposition 9.6, and Lemma 9.7 (2), there exists \( (s, t) \) satisfying the following conditions.

(i) \( 1 \leq s < t \leq n - 2 \).
(ii) \( \tau^{-p-q} P_t \simeq \tau^{-s+1} P_{t-s+1} \).

The condition (ii) implies \( s = p + q + 1 \) and \( t = i + s - 1 = i + p + q \). Thus it follows from (i) that
\[ 0 \leq p + q \leq n - 2 - i. \]

**Lemma 9.11** We have the following inequalities.

1. \( \# \mathcal{P}_\omega \leq \frac{1}{2} (i^2 + i - 6) + (n + 1)(p + 1) \)
2. \( \# \mathcal{I}_\omega \leq n + (n + 1)q \)
3. \( \# \mathcal{R}_\omega \leq i^2 - ni + n^2 - 4n + 7 \)

**Proof** The inequality (1) follows from Lemma 9.7 (3) and Lemma 9.8 (1). The inequality (2) follows from Lemma 9.7 (4) and Lemma 9.8 (2). The inequality (3) follows from Lemma 9.7 (5) and Lemma 9.8 (3). \( \square \)

Then we have the following inequalities.
\[ \ell(Q) = \# \mathcal{P}_\omega + \# \mathcal{I}_\omega + \# \mathcal{R}_\omega \]
\[ \leq \frac{3}{2} i^2 - \frac{1}{2} (2n - 1)i + (n + 1)(p + q + 1) + n^2 - 3n + 4 \]
\[ \leq \frac{3}{2} i^2 - \frac{1}{2} (2n - 1)i + (n + 1)(n - i - 1) + n^2 - 3n + 4 \]
\[ = \frac{3}{2} i^2 - \frac{1}{2} (4n + 1)i + 2n^2 - 3n + 3 \]

We set
\[ P(X) := \frac{3}{2} X^2 - \frac{1}{2} (4n + 1)X + 2n^2 - 3n + 3 \]
\[
= \frac{3}{2} \left( X - \frac{1}{6} (4n + 1) \right)^2 + 2n^2 - 3n + 3 - \frac{1}{24} (4n + 1)^2.
\]

Since \(0 < i < n\) and \(5 \leq n\), we obtain
\[
\ell(Q) < \max \{ P(0), P(n) \} = \max \left\{ 2n^2 - 3n + 3, \frac{3}{2} n^2 - \frac{7}{2} n + 3 \right\} \leq 2n^2 - 2n - 2.
\]

9.2.4 \((i \in \{0, 1\}, 2 \leq j \leq n - 2)\)

We assume \(i \in \{0, 1\}\) and \(2 \leq j \leq n - 2\). Without loss of generality, we may assume \(i = 0\).

By Lemma 9.2, Proposition 9.6, and Lemma 9.7 (2), there exists \(t \in \{2, \ldots, j - 1\}\) such that
\[
\tau^{-p-q} P_0 \simeq \tau^{-t+2} P_0.
\]

Hence, we have \(j \geq 3\) and
\[
0 \leq t - 2 = p + q \leq j - 3.
\]

**Lemma 9.12** We have the following inequalities.

\begin{enumerate}
  \item \(# \mathcal{P}_\omega \leq 2n - 3 + (n + 1)(p + 1)\)
  \item \(# \mathcal{I}_\omega \leq \frac{1}{2} j^2 - (2n + 1) j + n^2 + n + 2 + (n + 1)q\)
  \item \(# \mathcal{R}_\omega \leq j^2 - nj + n^2 - 4n + 7\)
\end{enumerate}

**Proof** The inequality (1) follows from Lemma 9.7 (3) and Lemma 9.8 (1). The inequality (2) follows from Lemma 9.7 (4) and Lemma 9.8 (2). The inequality (3) follows from Lemma 9.7 (5) and Lemma 9.8 (3).

Then we have the following inequalities.

\[
\ell(Q) = \# \mathcal{P}_\omega + \# \mathcal{I}_\omega + \# \mathcal{R}_\omega
\]
\[
\leq \frac{3}{2} j^2 - \frac{1}{2} (4n + 1) j + (n + 1)(p + q + 1) + \frac{3}{2} n^2 - \frac{3}{2} n + 5
\]
\[
\leq \frac{3}{2} j^2 - \frac{1}{2} (4n + 1) j + (n + 1)(j - 2) + \frac{3}{2} n^2 - \frac{3}{2} n + 5
\]
\[
= \frac{3}{2} j^2 - \frac{1}{2} (2n - 1) j + \frac{3}{2} n^2 - \frac{7}{2} n + 3.
\]

We set
\[
P(X) := \frac{3}{2} X^2 - \frac{1}{2} (2n - 1)X + \frac{3}{2} n^2 - \frac{7}{2} n + 3
\]
\[
= \frac{3}{2} \left( X - \frac{1}{6} (2n - 1) \right)^2 + \frac{3}{2} n^2 - \frac{7}{2} n + 3 - \frac{1}{24} (2n - 1)^2.
\]

Since \(0 < j < n\) and \(5 \leq n\), we obtain
\[
\ell(Q) < \max \{ P(0), P(n) \} = \max \left\{ \frac{3}{2} n^2 - \frac{7}{2} n + 3, 2n^2 - 3n + 3 \right\} \leq 2n^2 - 2n - 2.
\]

9.2.5 \((i,j \in \{0, 1\})\)

We assume \(i,j \in \{0, 1\}\). Without loss of generality, we may assume \(i = 0\). Then, by Lemma 9.2, Proposition 9.6, and Lemma 9.7 (2), there exists \(t \in \{2, \ldots, n - 1\}\) such that
\[
\tau^{-p-q} P_0 \simeq \tau^{-t+2} P_0\] and \(t \equiv j + 1 \pmod{2}\).
Hence, we have

\[ 0 \leq p + q = t - 2 \leq n - 3 \quad \text{and} \quad p + q = t - 2 \equiv j + 1 \pmod{2}. \]

**Lemma 9.13** Let \( r = p + q \). Then we have the following inequalities.

1. \( \# \mathcal{P}_o \leq 2n - 3 + (n + 1)(p + 1) \)
2. \( \# \mathcal{I}_o \leq \frac{1}{2}(n^2 - n - 4) + (n + 1)q \)
3. \( \# \mathcal{R}_o \leq \frac{1}{2}(n^2 - 3n + 4) - (r^2 + (4 - n)r) \)

**Proof** (1) and (2). The inequality (1) follows from Lemma 9.7 (3) and Lemma 9.8 (1). The inequality (2) follows from Lemma 9.7 (4) and Lemma 9.8 (2).

(3). We calculate

\[ \# \{ R \in (T_{n-2} \cap \text{ind } A_0) \mid \tau^{-(r+1)}R \in \text{ind } A_j \} \]

Assume that \( R := \tau^{-k}R_{u-1} \) (\( 0 \leq k \leq n - 3, 2 \leq u \leq n - 2 \)) satisfies the following conditions.

- \( R \in \text{ind } A_0 \)
- \( \tau^{-(r+1)} \in \text{ind } A_j \)

Since \( r + 1 \leq n - 2 \) and \( R \in T_{n-2} \), \( R \) satisfies the above conditions if and only if

\[ k \in [1, n - 2 - r - u] \cup [n - 2 - r, n - 1 - u]. \]

Then we have

\[
\begin{align*}
\# [1, n - 2 - r - u] &= n - 2 - r - u & (2 \leq u \leq n - 2 - r) \\
&= 0 & \text{(otherwise)} \\
\# [n - 2 - r, n - 1 - u] &= r + 2 - u & (2 \leq u \leq r + 1) \\
&= 0 & \text{(otherwise)} 
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
\# \left\{ R \in (T_{n-2} \cap \text{ind } A_0) \mid \tau^{-(r+1)}R \in \text{ind } A_j \right\} &= \sum_{u=2}^{n-2-r} (n - 2 - r - u) + \sum_{u=2}^{r+1} (r + 2 - u) \\
&= \frac{1}{2}(n - 4 - r)(n - 3 - r) + \frac{1}{2}(r)(r + 1) \\
&= r^2 + (4 - n)r + \frac{1}{2}(n^2 - 7n + 12). 
\end{align*}
\]

Furthermore, it follows from Lemma 9.5 (1) and \( r + 1 \equiv j \pmod{2} \), we have

\[
\# \left\{ R \in (T_2 \cap \text{ind } A_0) \mid \tau^{-(r+1)}R \in \text{ind } A_j \right\} = 2. 
\]

Therefore, it follows from Lemma 9.7 (5) and Lemma 9.8 (3) that

\[
\# \mathcal{R}_o \leq n^2 - 5n + 10 - \# \left\{ R \in (\mathcal{R} \cap \text{ind } A_0) \mid \tau^{-(r+1)}R \in \text{ind } A_j \right\} \\
= \frac{1}{2}(n^2 - 3n + 4) - (r^2 + (4 - n)r). 
\]

This finishes the proof. \( \Box \)
Let \( r = p + q \). Then \( 0 \leq r \leq n - 3 \) and we have

\[
\ell(Q) = \# \mathcal{P}_\omega + \# \mathcal{I}_\omega + \# \mathcal{R}_\omega \\
\leq n^2 - 3 + (n + 1)(r + 1) - (r^2 + (4 - n)r) \\
= n^2 + n - 2 - (r^2 - (2n - 3)r).
\]

We set

\[
P(X) := n^2 + n - 2 - (X^2 - (2n - 3)X) \\
= n^2 + n - 2 - \left( X - \left( n - \frac{3}{2} \right) \right)^2 - \left( n^2 - 3n + \frac{9}{4} \right).
\]

Since \( r \leq n - 3 \), we obtain

\[
\ell(Q) \leq P(n - 3) = 2n^2 - 2n - 2.
\]

9.2.6 \((i \in \{0, 1\}, j \in \{n - 1, n\})\)

We assume \( i \in \{0, 1\} \) and \( j \in \{n - 1, n\} \). Without loss of generality, we may assume \( i = 0 \) and \( j = n \). By Lemma 9.2, Proposition 9.6, and Lemma 9.7 (2), there exists \( t \in \{2, \ldots, n - 2\} \) such that

\[
\tau^{-p-q}p_0 \simeq \tau^{-t+2}p_0.
\]

Hence we have

\[
0 \leq p + q = t - 2 \leq n - 4.
\]

**Lemma 9.14** We have the following inequalities.

1. \( \# \mathcal{P}_\omega \leq 2n - 3 + (n + 1)(p + 1) \)
2. \( \# \mathcal{I}_\omega \leq n + (n + 1)q \)
3. \( \# \mathcal{R}_\omega \leq n^2 - 5n + 10 \)

**Proof** The inequality (1) follows from Lemma 9.7 (3) and Lemma 9.8 (1). The inequality (2) follows from Lemma 9.7 (4) and Lemma 9.8 (2). The inequality (3) follows from Lemma 9.7 (5) and Lemma 9.8 (3). \( \square \)

Let \( r = p + q \). Then \( 0 \leq r \leq n - 4 \). Since \( 5 \leq n \), we have the following inequalities.

\[
\ell(Q) = \# \mathcal{P}_\omega + \# \mathcal{I}_\omega + \# \mathcal{R}_\omega \\
\leq n^2 - 2n + 7 + (n + 1)(r + 1) \\
\leq n^2 - 2n + 7 + (n + 1)(n - 3) \\
= 2n^2 - 4n + 4 \\
< 2n^2 - 2n - 2
\]

9.2.7 \((i \in \{n - 1, n\}, j \in \{2, \ldots, n - 2\})\)

By Lemma 9.2 and Proposition 9.6, we have

\[
\{ \tau^{-t}p_i \mid i \in \{n - 1, n\}, \ t \in \mathbb{Z}_{\geq 0} \} \cap \left( \text{mod } A_j \right) = \emptyset
\]

for all \( j \in \{2, \ldots, n - 2\} \). Then it follows from Lemma 9.7 (2) that

\((i, j) \notin \{n - 1, n\} \times \{2, \ldots, n - 2\}\).
9.2.8 \((i \in \{n - 1, n\}, j \in \{0, 1\})\)

We assume \(i \in \{n - 1, n\}\) and \(j \in \{0, 1\}\). Without loss of generality, we may assume \(i = n\) and \(j = 0\). By Lemma 9.2, Proposition 9.6, and Lemma 9.7 (2), we have

\[
\tau^{-p-q}P_n \simeq P_n.
\]

Hence we have

\[p = q = 0.\]

**Lemma 9.15** We have the following inequalities.

1. \(\#\mathcal{P}_\omega \leq \frac{1}{2}(n^2 + n - 10) + (n + 1)\)
2. \(\#\mathcal{I}_\omega \leq \frac{1}{2}(n^2 - n - 4)\)
3. \(\#\mathcal{R}_\omega \leq \frac{1}{2}n^2 - 5n + 10\)

**Proof** The inequality (1) follows from Lemma 9.7 (3) and Lemma 9.8 (1). The inequality (2) follows from Lemma 9.7 (4) and Lemma 9.8 (2). The inequality (3) follows from Lemma 9.7 (5) and Lemma 9.8 (3).

Since \(5 \leq n\), we have the following inequalities.

\[
\ell(Q) = \#\mathcal{P}_\omega + \#\mathcal{I}_\omega + \#\mathcal{R}_\omega \\
\leq 2n^2 - 4n + 4 \\
< 2n^2 - 2n - 2
\]

9.2.9 \((i, j \in \{n - 1, n\})\)

We assume \(i, j \in \{n - 1, n\}\). Without loss of generality, we may assume \(i = n\). By Lemma 9.2, Proposition 9.6, and Lemma 9.7 (2), there exists \(t \in \{1, \ldots, n - 2\}\) such that

\[
\tau^{-p-q}P_n \simeq \tau^{-t+1}P_n \text{ and } t \equiv n - j \pmod{2}.
\]

Hence, we have

\[0 \leq p + q = t - 1 \leq n - 3 \text{ and } p + q = t - 1 \equiv n - j + 1 \pmod{2}.\]

**Lemma 9.16** Let \(r = p + q\). Then we have the following inequalities.

1. \(\#\mathcal{P}_\omega \leq \frac{1}{2}(n^2 + n - 10) + (n + 1)(p + 1)\)
2. \(\#\mathcal{I}_\omega \leq n + (n + 1)q\)
3. \(\#\mathcal{R}_\omega \leq \frac{1}{2}n^2 - \frac{3}{2}n + 2 - (r^2 + (4 - n)r)\)

**Proof** (1) and (2). The inequality (1) follows from Lemma 9.7 (3) and Lemma 9.8 (1). The inequality (2) follows from Lemma 9.7 (4) and Lemma 9.8 (2).

(3). We calculate

\[
\left\{ \begin{array}{l}
R \in (\mathbb{T}_{n-2} \cap \text{ind } A_n) \mid \tau^{-(r+1)}R \in \text{ind } A_j \end{array} \right\}
\]

Assume that \(R := \tau^{-k}R_{u-1} \ (0 \leq k \leq n - 3, 2 \leq u \leq n - 2)\) satisfies the following conditions.

- \(R \in \text{ind } A_n\)
- \(\tau^{-(r+1)}R \in \text{ind } A_j\)
Since \( r + 1 \leq n - 2 \) and \( R \in \mathbb{T}_{n-2} \), \( R \) satisfies the above conditions if and only if
\[
k \in [0, n - u - 2 - (r + 1)] \cup [n - 2 - (r + 1), n - u - 2]
\[
= [0, n - 3 - r - u] \cup [n - 3 - r, n - 2 - u].
\]
Then we have
\[
\#[0, n - 3 - r - u] = \begin{cases} n - 2 - r - u & (2 \leq u \leq n - 2 - r) \\ 0 & \text{(otherwise)} \end{cases}
\]
\[
\#[n - 3 - r, n - 2 - u] = \begin{cases} r + 2 - u & (2 \leq u \leq r + 1) \\ 0 & \text{(otherwise)} \end{cases}
\]
Then we have
\[
\# \left\{ R \in (\mathbb{T}_{n-2} \cap \text{ind} \ A_n) \mid \tau^{-(r+1)}R \in \text{ind} A_j \right\}
\]
\[
= \sum_{u=2}^{n-2-r} (n - 2 - r - u) + \sum_{u=2}^{r+1} (r + 2 - u)
\]
\[
= \frac{1}{2} (n - 4 - r)(n - 3 - r) + \frac{1}{2} (r)(r + 1)
\]
\[
= r^2 + (4 - n)r + \frac{1}{2}(n^2 - 7n + 12).
\]
Furthermore, it follows from Lemma \ref{lemma:9.5} (1) and \( r + 1 \equiv n - j \mod 2 \), we have
\[
\# \left\{ R \in (\mathbb{T}_{n-2} \cap \text{ind} \ A_n) \mid \tau^{-(r+1)}R \in \text{ind} A_j \right\} = 2.
\]
Therefore, it follows from Lemma \ref{lemma:9.7} (5) and Lemma \ref{lemma:9.8} (3) that
\[
\# \mathcal{R}_\omega \leq n^2 - 5n + 10 - \# \left\{ R \in (\mathcal{A} \cap \text{ind} \ A_n) \mid \tau^{-(r+1)}R \in \text{ind} A_j \right\}
\]
\[
= \frac{1}{2} n^2 - \frac{3}{2} n + 2 - (r^2 + (4 - n)r)
\]
This finishes the proof. \(\square\)

Let \( r = p + q \). Then \( 0 \leq r \leq n - 3 \) and we have
\[
\ell(Q) = \# \mathcal{P}_\omega + \# \mathcal{J}_\omega + \# \mathcal{R}_\omega
\]
\[
\leq n^2 - 3 + (n + 1)(r + 1) - (r^2 + (4 - n)r)
\]
\[
= n^2 + n - 2 - (r^2 - (2n - 3)r).
\]
We set
\[
P(X) := n^2 + n - 2 - (X^2 - (2n - 3)X)
\]
\[
= n^2 + n - 2 - \left( \left( X - \left( \frac{n - \frac{3}{2}}{2} \right) \right)^2 - \left( n^2 - 3n + \frac{9}{4} \right) \right).
\]
Since \( r \leq n - 3 \), we obtain
\[
\ell(Q) \leq P(n - 3) = 2n^2 - 2n - 2.
\]

### 9.3 Construction of a maximal green sequence for \( Q \) with length \( 2n^2 - 2n - 2 \)

In this subsection, we construct a maximal green sequence for \( Q \) with length \( 2n^2 - 2n - 2 \) as follows.

1. Construct a path \( \omega^{(0)} \) with length \( n^2 - 2n - 3 \) such that \( s(\omega^{(0)}) = A \).
2. Construct a path \( \omega^{(1)} \) with length \( \frac{1}{2}(n^2 + n - 12) \) such that \( s(\omega^{(1)}) = t(\omega^{(0)}) \).
Lemma 9.17 We denote by $(\omega^{(2)})$ with length $n - 2$ such that $s(\omega^{(2)}) = t(\omega^{(1)})$.
4. Construct a path $\omega^{(3)}$ with length 6 such that $s(\omega^{(3)}) = t(\omega^{(2)})$.
5. Construct a path $\omega^{(4)}$ with length $\frac{1}{2}(n^2 - 7n + 12)$ such that $s(\omega^{(4)}) = t(\omega^{(3)})$.
6. Construct a path $\omega^{(5)}$ with length $n - 3$ such that $s(\omega^{(5)}) = t(\omega^{(4)})$.
7. Construct a path $\omega^{(6)}$ with length $n - 3$ such that $s(\omega^{(6)}) = t(\omega^{(5)})$.
8. Construct a path $\omega^{(7)}$ with length 3 such that $s(\omega^{(7)}) = t(\omega^{(6)})$ and $t(\omega^{(7)}) = A^{-}$.
9. Concatenate $\omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(7)}$.

At the end of this paper, we give an example of the above construction for the case that $n = 5$. This example would be helpful for reference when going through the construction.

In this subsection, $\frac{T}{X}$ means $T'$ for a module $T = T' \oplus X$.

9.3.1 Some $(\tau)$-rigid modules in $\text{add} \mathcal{R}$

Before construct $\omega^{(0)}, \ldots, \omega^{(7)}$, we define some $(\tau)$-rigid modules in $\text{add} \mathcal{R}$.

Denote by $C$ the following quiver.

$$C_0 := \{1, \ldots, n - 3\} \times (\mathbb{Z}/(n - 2)\mathbb{Z})$$

$$C_1 := \{a_{(t,k)} : (t, k) \to (t + 1, k) \mid 1 \leq t \leq n - 4, k \in \mathbb{Z}/(n - 2)\mathbb{Z}\}$$

$$\cup \{b_{(t,k)} : (t, k) \to (t - 1, k + 1) \mid 2 \leq t \leq n - 3, k \in \mathbb{Z}/(n - 2)\mathbb{Z}\}$$

We denote by $\mathcal{J}$ the set of connected full subquivers of $C$ defined as follows: $\Sigma \in \mathcal{J}$ if and only if for any $t \in \{1, \ldots, n - 3\}$, there exists a unique $k \in \mathbb{Z}/(n - 2)\mathbb{Z}$ such that $(t, k) \in \Sigma_0$.

For $\Sigma \in \mathcal{J}$, we define a basic module $R(\Sigma) \in \text{add} \mathcal{T}_{n-2}$ as follows.

$$R(\Sigma) := \bigoplus_{(t,k) \in \Sigma_0} \tau^{-k}R_t.$$ (Since $\tau^{n-2}R_t \simeq R_t$, we can naturally define $\tau^{-k}R_t$ for $k \in \mathbb{Z}/(n - 2)\mathbb{Z}$.)

Then we have the following statement.

Lemma 9.17 $R(\Sigma)$ is $(\tau)$-rigid for any $\Sigma \in \mathcal{J}$.

Proof (1). Let $\Sigma \in \mathcal{J}$ and $(t, k), (t', k') \in \Sigma_0$. We show

$$\tau^{-k}R_t \oplus \tau^{-k'}R_{t'}$$ is $(\tau)$-rigid.

We may assume that $t < t'$ and denote by $\Sigma'$ the full subquiver of $\Sigma$ such that

$$(t'', k'') \in \Sigma' \iff (t'', k'') \in \Sigma \text{ and } t \leq t'' \leq t'$$

(or equivalently, $\Sigma'$ is the minimal connected full subquiver of $\Sigma$ containing $(t, k)$ and $(t', k')$).

We set

$$n_a := \#(\mathcal{A} \cap \Sigma_1')$$
$$n_b := \#(\mathcal{B} \cap \Sigma_1')$$

where $\mathcal{A} := \{a_{(t,k)} \mid 1 \leq t \leq n - 4, k \in \mathbb{Z}/(n - 2)\mathbb{Z}\}$ and $\mathcal{B} := \{b_{(t,k)} \mid 2 \leq t \leq n - 3, k \in \mathbb{Z}/(n - 2)\mathbb{Z}\}$. Then $t' - t = n_a + n_b, k - n_b = k'$, and there is the following full subquiver of $C$.

$$(t, k) \xleftarrow{b_{n_b}} \cdots \xleftarrow{b_1} (t + n_b, k - n_b) \xrightarrow{a_1} \cdots \xrightarrow{a_{n_a}} (t + n_a + n_b, k - n_b) = (t', k').$$
For indecomposable regular modules $R$ and $R'$, the direct sum $R \oplus R'$ is $\tau$-rigid if and only if $\tau R \oplus \tau R'$ is $\tau$-rigid. Therefore, we may assume $k' = 0$, $k = n_b$, and $t' = t + n_a + n_b$. Then it is sufficient to show $\tau^{-n_b}R_t \oplus R_{t+n_a+n_b}$ is $\tau$-rigid.

It is easy to verify that $\text{Hom}_A(\tau^{-1}R_s, R_s') = 0$ for any $1 \leq s, s' \leq n - 3$. In particular, $R_s \oplus R_{s'}$ is $\tau$-rigid for any $1 \leq s, s' \leq n - 3$. Therefore, we may assume that $n_b \geq 1$.

Since $1 \leq t + n_a + n_b = t' \leq n - 3$, we have $n - (t + 1) - 2 - n_b = n - t - n_b - 3 \geq n_a \geq 0$. Hence we obtain

$$1 \leq n_b \leq n - (t + 1) - 2.$$

Then, by Lemma 9.4, $\tau^{-n_b}R_t$ is given by the following quiver representation.

$$\begin{array}{c}
\vdots & \vdots & \vdots \\
0 & K & 0 \\
n_b + 1 & \text{id} & t + 1 + n_b & 0 \\
0 & & & \\
\end{array}$$

It is sufficient to show

$$\text{Hom}_A(R_{t+n_a+n_b}, \tau(\tau^{-n_b}R_t)) = 0 = \text{Hom}_A(\tau^{-1}(\tau^{-n_b}R_t), R_{t+n_a+n_b}).$$

It follows from Lemma 9.4 that $\tau^{-n_b+1}R_t$ is given by the following.

$$\begin{array}{c}
\vdots & \vdots & \vdots & (n_b = 1) \\
0 & K & 0 \\
n_b & \text{id} & 0 & 0 \\
0 & & & \\
\end{array}$$

$$\begin{array}{c}
\vdots & \vdots & \vdots & (n_b \geq 2) \\
0 & K & 0 \\
0 & \text{id} & 0 & 0 \\
0 & & & \\
\end{array}$$

It also follows from Lemma 9.4 that $\tau^{-n_b-1}R_t$ is given by the following.

$$\begin{array}{c}
\vdots & \vdots & \vdots & (t + n_b = n - 3) \\
0 & K & 0 \\
n_b + 2 & \text{id} & 0 \\
0 & & & \\
\end{array}$$

$$\begin{array}{c}
\vdots & \vdots & \vdots & (t + n_b \leq n - 4) \\
0 & K & 0 \\
0 & \text{id} & 0 \\
0 & & & \\
\end{array}$$

Note that $R_{t+n_a+n_b}$ is given by the following quiver representation.

$$\begin{array}{c}
K & \text{id} & K \\
K & \text{id} & K \\
& & \\
\vdots & \vdots & \vdots \\
0 & \text{id} & 0 \\
0 & \text{id} & 0 \\
\vdots & \vdots & \vdots \\
0 & \text{id} & 0 \\
0 & \text{id} & 0 \\
\vdots & \vdots & \vdots \\
0 & \text{id} & 0 \\
0 & \text{id} & 0 \\
\end{array}$$

Then it follows from $t + n_a + n_b + 1 > t + n_b$ that

$$\text{Hom}_A(R_{t+n_a+n_b}, \tau(\tau^{-n_b}R_t)) = 0.$$  

Thus we want to show $\text{Hom}_A(\tau^{-1}(\tau^{-n_b}R_t), R_{t+n_a+n_b}) = 0$. If $t + n_b \leq n - 4$, then we can easily check the assertion. Therefore, we assume $t + n_b = n - 3$. In this case, we have $n_a = 0$ and $t + 1 + n_a + n_b = n - 2$. Let $f \in \text{Hom}_A(\tau^{-n_b-1}R_t, R_{n-3})$. For each $i \in Q_0$, we set $f_i : (\tau^{-n_b-1}R_t)e_i \rightarrow (R_{n-3})e_i$ the $K$-linear map induced by $f$.

If $t = 1$, then we have

$$f_0 = f_1 = f_{n-1} = f_n = 0, f_2 = \cdots = f_{n-2}, f_{n-2} \circ \text{id} = 0.$$

This shows $f = 0$.

If $t \geq 2$, then we have
\[ \begin{align*}
- f_0 &= f_1 = f_{n-1} = f_n = 0. \\
- f_2 &= \cdots = f_{n-t-1}. \\
- f_{n-t-1} \circ \delta &= f_{n-t}. \\
- f_{n-t} &= \cdots = f_{n-2}. \\
- f_{n-2} \circ \alpha &= 0, f_{n-2} \circ \beta = 0.
\end{align*} \]

By the fourth and last equations, we obtain
\[ f_{n-t} = \cdots = f_{n-2} = 0. \]

Then the second and third equations imply
\[ f_2 = \cdots = f_{n-t-1} = 0. \]

Hence \( f = 0. \) \( \square \)

The following lemma is a direct consequence of Lemma 4.9.

**Lemma 9.18** We have \( \text{Hom}_A(R, L) = 0 = \text{Hom}_A(L, R) \) for each \((R, L) \in \mathbb{T}_{n-2} \times \mathbb{T}_2. \) In particular, if \( X \in \text{add} \mathbb{T}_{n-2} \) and \( Y \in \text{add} \mathbb{T}_2, \) then \( X \oplus Y \) is \( \tau \)-rigid if and only if \( X \) and \( Y \) are \( \tau \)-rigid.

**9.3.2 Construct \( \omega^{(0)} \)**

Consider a sink mutation sequence \((i_1, i_2, \ldots, i_m, i_{n+1}) = (2, 3, \ldots, n-2, 0, 1, n-1, n)\) of \( Q. \) We define \( \mu_{i_k} \cdots \mu_{i_1} A \in \text{mod} A \) \((1 \leq k \leq n+1)\) by
\[ \tau^{-1} (P_{i_1} \oplus \cdots \oplus P_{i_k}) \oplus P_{i_{k+1}} \oplus \cdots \oplus P_{i_{m+1}}. \]

Then it is well-known that
\[ T_r^{(0)} := \tau^{-r} (\mu_{i_k} \cdots \mu_{i_1} A) \in \text{sr-tilt} A \]
for each \((r, k) \in \mathbb{Z}_{\geq 0} \times \{1, \ldots, n+1\}\) (see [20, Section 4.2] for example). Then we construct \( \omega^{(0)} \) as follows.

\[ \omega^{(0)}:\begin{bmatrix}
A & T_{0,1}^{(0)} & \cdots & T_{0,n+1}^{(0)} \\
& T_{1,1}^{(0)} & \cdots & T_{1,n+1}^{(0)} \\
& & \cdots & \\
& & & T_{r,n+1}^{(0)} \\
& & & \cdots \\
& & & & T_{n-4,1}^{(0)} & \cdots & T_{n-4,n+1}^{(0)}
\end{bmatrix} \]

**9.3.3 Construct \( \omega^{(1)} \)**

For each \((a, b)\) such that \(0 \leq a \leq b \leq n - 3\), we define a module \( T_{a,b}^{(1)} \) as follows.
\[ \tau^{-a} \left( \left( a_{x_{(a)}} \right) \oplus \left( b_{x_{(a)}} \right) \oplus \left( c_{x_{(a)}} \right) \right) \oplus \tau^{-a} (P_{n-2} \oplus P_0) \].

We also define \( T_{a,a,+}^{(1)}, T_{a,a,+}^{(1)} \) \((0 \leq a \leq n - 4)\) as follows.
\[ T_{a,a,+}^{(1)} = \frac{T_{a,a,+}^{(1)}}{\tau^{-(n-3+a)}} P_1 \oplus \tau^{-(n-2+a)} P_0 \]

\[ T_{a,a,+}^{(1)} = \frac{T_{a,a,+}^{(1)}}{\tau^{-(n-3+a)}} P_0 \oplus \tau^{-(n-2+a)} P_0 \]
Lemma 9.19 We have the following statements.

(1) $T^{(1)}_{a,b} \in \mathcal{sr}$-tilt $A$ for any $(a, b)$.
(2) For each $(a, b)$ satisfying $a < b$, there is an arrow $T^{(1)}_{a,b} \rightarrow T^{(1)}_{a,b-1}$ in $\overline{H} (\mathcal{sr}$-tilt $A)$.
(3) For each $a \leq n - 4$, there is a path

$$T^{(1)}_{a,a} \rightarrow T^{(1)}_{a,a+1} \rightarrow T^{(1)}_{a,a++} \rightarrow T^{(1)}_{a+a, n-3}$$

in $\overline{H} (\mathcal{sr}$-tilt $A)$.

Proof (1). We define $M_1$, $M_2$, and $M_3$ as follows.

$$M_1 := \bigoplus_{k=0}^{a} \tau^{-k} P_{n-\delta_{k, odd}}$$
$$M_2 := \left( \bigoplus_{k=a}^{b} \tau^{-a} P_{n-1-k} \right) \oplus \tau^{-d} (P_1 \oplus P_0)$$
$$M_3 := \bigoplus_{k=b}^{n-4} \tau^{-(a+1)} P_{n-2-k}$$

Since $T^{(1)}_{a,b} = \tau^{-(n-3)} (M_1 \oplus M_2 \oplus M_3)$, it is sufficient to show $M_1 \oplus M_2 \oplus M_3 \in \mathcal{sr}$-tilt $A$.

Then the assertion (1) follows from Lemma 4.6 (3) and Lemma 9.2.

(2) and (3). By definition, we have the following equations.

$$T^{(1)}_{a,a-1} = \frac{T^{(1)}_{a,b}}{\tau^{-(n-3)a} P_{n-1-b}} \oplus \tau^{-(n-2+a)} P_{n-1-b}$$
$$T^{(1)}_{a,a+1} = \frac{T^{(1)}_{a,a}}{\tau^{-(n-3)a} P_1} \oplus \tau^{-(n-2+a)} P_1$$
$$T^{(1)}_{a,a++} = \frac{T^{(1)}_{a,a+1}}{\tau^{-(n-3)a} P_0} \oplus \tau^{-(n-2+a)} P_0$$
$$T^{(1)}_{a+a, n-3} = \frac{T^{(1)}_{a,a++}}{\tau^{-(n-3)a} P_{n-1-a}} \oplus \tau^{-(n-2+a)} P_{n-\delta_{k+1, odd}}$$

Then the assertions (2) and (3) follow from Lemma 4.6 (3).

By Lemma 9.19 (2) and (3), we obtain a path

$$\omega^{(1)} : \begin{bmatrix}
T^{(0)}_{a+4, n+1} = & T^{(1)}_{a+4, 3} & \cdots & T^{(1)}_{a+4, 0} & T^{(1)}_{a+4, 1} & T^{(1)}_{a+4, 2} & T^{(1)}_{a+4, 3} \\
& T^{(1)}_{a+3, 1} & \cdots & T^{(1)}_{a+3, 0} & T^{(1)}_{a+3, 1} & T^{(1)}_{a+3, 2} & T^{(1)}_{a+3, 3} \\
& & \cdots & \vdots & \vdots & \vdots & \vdots \\
& T^{(1)}_{a-4, 0} & \cdots & T^{(1)}_{a-4, 0} & T^{(1)}_{a-4, 1} & T^{(1)}_{a-4, 2} & T^{(1)}_{a-4, 3} \\
& T^{(1)}_{a-3, 0} & \cdots & T^{(1)}_{a-3, 0} & T^{(1)}_{a-3, 1} & T^{(1)}_{a-3, 2} & T^{(1)}_{a-3, 3} \\
\end{bmatrix}$$

in $\overline{H} (\mathcal{sr}$-tilt $A)$.

9.3.4 Construct $\omega^{(2)}$

For each $c \in \{1, \ldots, n - 2\}$, we define a module $T^{(2)}_{c}$ as follows.

$$T^{(2)}_{c} = \tau^{-n+3} P_n \bigoplus_{k=c}^{n-2} \tau^{-(n-3+k)} (P_{n-\delta_{k, odd}}) \oplus \tau^{-2n+6} (P_1 \oplus P_0) \oplus R_1 \oplus \cdots \oplus R_{c-1}.$$
**Lemma 9.20**  \( T_{c}^{(2)} \in \mathfrak{s}\tau\text{-}\text{tilt}A \) for any \( c \in \{1, \ldots, n-2\}. \)

*Proof.* Let \( M = P_n \oplus \bigoplus_{k=c}^{n-2} \tau^{-k}(P_n - \delta_{k,\text{odd}}) \oplus \tau^{-n+3}(P_1 \oplus P_0) \) and \( R = R_1 \oplus \cdots \oplus R_{c-1}. \) Then we have
\[
T_{c}^{(2)} = \tau^{-n+3}M \oplus R.
\]

Then, by Lemma 4.6 (3) and Lemma 9.2, \( M \) is \( \tau \)-rigid. In particular, we have \( \tau^{-n+3}M \) is \( \tau \)-rigid. By Lemma 9.17, \( R \) is also \( \tau \)-rigid. Therefore, by Lemma 4.6 (2), it remains to check the following conditions.

(i) \( \tau^{-(n-3)}P_n \prec R_t \) for any \( t \in \{1, \ldots, c-1\}. \)

(ii) \( \tau^{-(n-3+k)}P_n - \delta_{k,\text{odd}} \prec R_t \) for any \( (k, t) \in \{c, \ldots, n-2\} \times \{1, \ldots, c-1\}. \)

(iii) \( \tau^{-(2n-6)}P_0 \prec R_t \) and \( \tau^{-(2n-6)}P_1 \prec R_t \) for any \( t \in \{1, \ldots, c-1\}. \)

Note that the condition (i) (resp. (ii), (iii)) is equivalent to the condition (i') (resp. (ii'), (iii')) below.

(i') \( \tau^{n-2}R_t e_n = R_t e_n = 0 \) for any \( t \in \{1, \ldots, c-1\}. \)

(ii') \( \tau^{n-2+k}R_t e_{n-\delta_{k,\text{odd}}} = \tau^{-(n-2-k)}R_t e_{n-\delta_{k,\text{odd}}} = 0 \) for any \( (k, t) \in \{c, \ldots, n-2\} \times \{1, \ldots, c-1\}. \)

(iii') \( \tau^{2n-5}R_t e_\epsilon = \tau^{-1}R_t e_\epsilon = 0 \) for any \( (\epsilon, t) \in \{0, 1\} \times \{1, \ldots, c-1\}. \)

Then the assertion follows from Lemma 9.4. In fact, (i') and (iii') are obvious and (ii') follows from \( 0 \leq n-2-k \leq n-(t+1)-2. \)

\[ \square \]

Note that
\[
T_{1}^{(2)} = \frac{T_{n-3,n-3}^{(1)}}{\tau^{-(2n-6)}P_2} \bigoplus \tau^{-(2n-6)}P_n - \delta_{\text{odd}}
\]
\[
T_{c+1}^{(2)} = \frac{T_{c}^{(2)}}{\tau^{-(n-3+c)}P_n - \delta_{\text{odd}}} \bigoplus R_c.
\]

Then it follows from Lemma 4.6 (4) and Lemma 9.20 that there exists a path
\[
\omega^{(2)} : T_{n-3,n-3} \to T_{1}^{(2)} \to \cdots \to T_{n-2}^{(2)}
\]
in \( \overline{H}(s\tau\text{-}\text{tilt}A) \).

**9.3.5 Construct** \( \omega^{(3)} \)

Let \( R = R_1 \oplus \cdots \oplus R_{n-4} \oplus R_{n-3}. \) For \( d \in \{1, 2, 3, 4, 5, 6\}, \) we define \( T_{d}^{(3)} \) as follows.
\[ T_1^{(3)} = \frac{T_1^{(2)}}{\tau^{-(n-3)}P_n} \oplus L_{0,n-\delta_{\text{odd}}} \]
\[ = \tau^{-(n-3)}P_n \oplus \tau^{-(2(n-5))P_{n-\delta}} L_{0,n-\delta_{\text{odd}}} \oplus \tau^{-2(n-3)}P_1 \oplus R \]

\[ T_2^{(3)} = \frac{T_2^{(3)}}{\tau^{-(n-3)}P_n} \oplus L_{1,n-\delta_{\text{odd}}} \]
\[ = \tau^{-(n-3)}P_n \oplus \tau^{-(2(n-5))P_{n-\delta}} L_{0,n-\delta_{\text{odd}}} \oplus L_{1,n-\delta_{\text{odd}}} \oplus R \]

\[ T_3^{(3)} = \frac{T_3^{(3)}}{\tau^{-(n-3)}P_n} \oplus P_{n-\delta_{\text{odd}}} \]
\[ = \tau^{-(n-3)}P_n \oplus P_{n-\delta_{\text{odd}}} \oplus L_{0,n-\delta_{\text{odd}}} \oplus L_{1,n-\delta_{\text{odd}}} \oplus R \]

\[ T_4^{(3)} = \frac{T_4^{(3)}}{\tau^{-(n-3)}P_n} \oplus \tau^{-3}I_n \]
\[ = \tau^{-(n-3)}P_n \oplus \tau^{-3}I_n \oplus P_{n-\delta_{\text{odd}}} \oplus L_{0,n-\delta_{\text{odd}}} \oplus L_{1,n-\delta_{\text{odd}}} \oplus R \]

\[ T_5^{(3)} = \frac{T_5^{(3)}}{L_{0,n-\delta_{\text{odd}}} \oplus I_0} \]
\[ = \tau^{-(n-3)}P_n \oplus P_{n-\delta_{\text{odd}}} \oplus I_0 \oplus L_{1,n-\delta_{\text{odd}}} \oplus R \]

\[ T_6^{(3)} = \frac{T_6^{(3)}}{L_{1,n-\delta_{\text{odd}}} \oplus I_1} \]
\[ = \tau^{-(n-3)}P_n \oplus P_{n-\delta_{\text{odd}}} \oplus I_0 \oplus I_1 \oplus R \]

Then we have the following lemma.

**Lemma 9.21** For any \( d \in \{1, 2, 3, 4, 5, 6\}, T_d^{(3)} \in \mathfrak{s}_r\text{-tilt}A and there is an arrow \( T_{d-1}^{(3)} \rightarrow T_d^{(3)} \) in \( \mathfrak{H}(\mathfrak{s}_r\text{-tilt}A) \), where we put \( T_0^{(3)} = T_{n-2}^{(2)} \).

**Proof** We put \( \delta = \delta_{\text{odd}} \) and \( \delta' = \delta_{\text{even}} \).

\( (d = 1) \). We show \( T_1^{(3)} \in \mathfrak{s}_r\text{-tilt}A \) and there is an arrow \( T_1^{(3)} \rightarrow T_1^{(3)} \) in \( \mathfrak{H}(\mathfrak{s}_r\text{-tilt}A) \). By Lemma 4.6 (3), (4) and Lemma 9.18, it is sufficient to check the following conditions.

(i) \( L_{0,n-\delta} < \tau^{-(n-3)}P_n, \tau^{-(2(n-5))P_{n-\delta}}, \tau^{-2(n-3)}P_1 \).

(ii) \( \tau^{-(n-3)}P_n, \tau^{-(2(n-5))P_{n-\delta}}, \tau^{-2(n-3)}P_1 < L_{0,n-\delta} \).

Then (i) and (ii) follow from the definition of \( \prec \) and (ii) follows from Lemma 9.5.

\( (d = 2) \). We show \( T_2^{(3)} \in \mathfrak{s}_r\text{-tilt}A \) and there is an arrow \( T_1^{(3)} \rightarrow T_2^{(3)} \) in \( \mathfrak{H}(\mathfrak{s}_r\text{-tilt}A) \). By Lemma 4.6 (3), (4), Lemma 9.5, and Lemma 9.18, it is sufficient to check the following conditions.

(i) \( L_{1,n-\delta} < \tau^{-(n-3)}P_n, \tau^{-(2(n-5))P_{n-\delta}} \).
(ii) $\tau^{-\left(n-3\right)} P_n, \tau^{-\left(2n-5\right)} P_{n-\delta} \prec L_{1,n-\delta}$.

(iii) $L_{1,n-\delta} \prec \tau^{-2\left(n-3\right)} P_1$.

Then (i) and (iii) follow from the definition of $\prec$ and (ii) follows from Lemma 9.5.

($d = 3$). We show $T_3^{(3)} \in s\tau\text{-tilt}A$ and there is an arrow

$$T_2^{(3)} \rightarrow T_3^{(3)}$$

in $\overline{H}$ ($s\tau\text{-tilt}A$). By Lemma 9.2, Lemma 9.3, Lemma 9.4, and Lemma 9.5, we have

$$\tau^{-\left(n-3\right)} P_n \oplus L_{0,n-\delta} \oplus L_{1,n-\delta} \oplus R \in \text{mod } A_{n-\delta}.$$

In particular, $T_3^{(3)} \in s\tau\text{-tilt}A$ and we have an arrow

$$T_2^{(3)} \rightarrow T_3^{(3)}$$

in $\overline{H}$ ($s\tau\text{-tilt}A$).

($d = 4$). We show $T_4^{(3)} \in s\tau\text{-tilt}A$ and there is an arrow

$$T_3^{(3)} \rightarrow T_4^{(3)}$$

in $\overline{H}$ ($s\tau\text{-tilt}A$). By Lemma 4.6 (3), (4) and Lemma 9.18, it is sufficient to check the following conditions.

(i) $L_{0,n-\delta}, L_{1,n-\delta} \prec \tau^{n-3} I_n$.

(ii) $R_t \prec \tau^{n-3} I_n$ for each $t \in \{1, \ldots, n-3\}$.

(iii) $\tau^{n-3} I_n \prec L_{0,n-\delta}, L_{1,n-\delta}$.

(iv) $\tau^{n-3} I_n \prec R_t$ for each $t \in \{1, \ldots, n-3\}$.

(v) $\tau^{n-3} I_n \in \text{mod } A_{n-\delta}$.

(vi) $\tau^{n-3} I_n \prec \tau^{-\left(n-3\right)} P_n$.

Then (iii), (iv), and (vi) follow from the definition of $\prec$, (v) follows from Lemma 9.3. Further, (ii) follows from Lemma 9.4 and (i) follows from Lemma 9.5.

($d = 5$). We show $T_5^{(3)} \in s\tau\text{-tilt}A$ and there is an arrow

$$T_4^{(3)} \rightarrow T_5^{(3)}$$

in $\overline{H}$ ($s\tau\text{-tilt}A$). By Lemma 4.6 (3), (4) and Lemma 9.18, it is sufficient to check the following conditions.

(i) $\tau^{n-3} I_n \prec I_0$.

(ii) $R_t \prec I_0$ for each $t \in \{1, \ldots, n-3\}$.

(iii) $L_{1,n-\delta} \prec I_0$.

(iv) $I_0 \prec \tau^{n-3} I_n$.

(v) $I_0 \prec L_{1,n-\delta}$.

(vi) $I_0 \prec R_t$ for each $t \in \{1, \ldots, n-3\}$.

(vii) $I_0 \in \text{mod } A_{n-\delta}$.

(viii) $I_0 \prec L_{0,n-\delta}$.

Then (iv), (v), (vi), and (viii) follow from the definition of $\prec$, and (vii) follows from Lemma 9.3. Further, (i) follows from Lemma 9.3, (ii) follows from Lemma 9.4, and (iii) follows from Lemma 9.5.

($d = 6$). We show $T_6^{(3)} \in s\tau\text{-tilt}A$ and there is an arrow

$$T_5^{(3)} \rightarrow T_6^{(3)}$$
in $\overline{H}$ (sr-tiltA). By Lemma 4.6 (3), (4) and Lemma 9.18, it is sufficient to check the following conditions.

(i) $\tau^{n-3}I_n < I_1$.
(ii) $R_t < I_1$ for each $t \in \{1, \ldots, n-3\}$.
(iii) $I_1 < \tau^{n-3}I_n$.
(iv) $I_1 < I_0 < I_1$.
(v) $I_1 < R_t$ for each $t \in \{1, \ldots, n-3\}$.
(vi) $I_1 \in \mod A_{n-\delta}$.
(vii) $I_1 < L_{1,n-\delta}$.

Then (iii), (iv), (v), and (vii) follow from the definition of $<$, and (vi) follows from Lemma 9.3. Further, (i) follows from Lemma 9.3 and (ii) follows from Lemma 9.4. □

By Lemma 9.21, there is a path

$$\omega^{(3)} : T_{n-2}^{(2)} \rightarrow T_1^{(3)} \rightarrow T_2^{(3)} \rightarrow T_3^{(3)} \rightarrow T_4^{(3)} \rightarrow T_5^{(3)} \rightarrow T_6^{(3)}.$$  

### 9.3.6 Construct $\omega^{(4)}$

For each $(e, f) \in \mathbb{Z} \times \mathbb{Z}$ satisfying

$$1 \leq e \leq n-4, 1 \leq f \leq n-4, e+f \leq n-3,$$

we define $R(e, f)$ as follows.

$$R(e, f) = \tau^{-e}(R_1 \oplus \cdots \oplus R_f) \oplus \tau^{-e+1}(R_{f+1} \oplus \cdots \oplus R_{n-2-e}) \oplus \left( \bigoplus_{t=n-1-e}^{n-3} \tau^{-(n-3-t)}R_t \right)$$

Let $\Sigma_{ef}$ the subquiver of $C$ (defined in Sect. 9.3.1) given by the following.

$$\omega = (1, e) \rightarrow (e, f) \rightarrow (f+1, e-1) \rightarrow \cdots \rightarrow (n-2-e, e-1) \rightarrow (n-1-e, e-2) \rightarrow \cdots \rightarrow (n-3, 0)$$

Then $R(e, f) = R(\Sigma_{ef})$ and it is $\tau$-rigid by Lemma 9.17.

### Lemma 9.22

Let $T_{ef}^{(4)} := R(e, f) \oplus \tau^{n-3}I_n \oplus I_1 \oplus I_0 \oplus P_{n-\delta_{n, odd}}^{-}$.

1. $T_{ef}^{(4)} \in \sr$-tiltA.
2. We have the following statements.
   (i) There is an arrow $T := T_{6}^{(3)} \rightarrow T_{e+1}^{(4)} \rightarrow T'$ in $\overline{H}(sr$-tiltA).
   (ii) For each $(e, f)$ with $e+f \leq n-4$, there is an arrow $T := T_{ef}^{(4)} \rightarrow T_{ef+1}^{(4)} \rightarrow T'$ in $\overline{H}(sr$-tiltA).
   (iii) For each $(e, n-3-e)$ with $e \leq n-5$, there is an arrow $T := T_{e,n-3-e}^{(4)} \rightarrow T_{e+1,n-3-e}^{(4)} \rightarrow T'$ in $\overline{H}(sr$-tiltA).

**Proof** (1). Note that $R(e, f)$ and $\tau^{n-3}I_n \oplus I_1 \oplus I_0 \oplus P_{n-\delta_{n, odd}}^{-}$ are $\tau$-rigid. Since $R(e, f) \in \add \mathcal{R}$, we have

$$\Hom_A((\tau^{n-3}I_n \oplus I_1 \oplus I_0), \tau R(e, f)) = 0.$$  

Therefore, it is sufficient to check that $R(e, f)$ satisfies the following conditions.

(i) $\Hom_A(R(e, f), \tau(\tau^{n-3}I_n \oplus I_1 \oplus I_0)) = 0$
(ii) $R(e, f)e_{n-\delta_{n, odd}} = 0$
Since \( \tau^{-(n-2)}R(e,f) \simeq R(e,f) \), it is sufficient to check
\[
R(e,f)e_{n-k_{\text{od}}e} + R(e,f)e_n = 0 \quad \text{and} \quad \tau^{-1}R(e,f)(e_0 + e_1) = 0.
\]
Note that \( \tau^{-1}R(e,f) \) is equal to
\[
\tau^{-(e+1)}(R_1 \oplus \cdots \oplus R_j) \oplus \tau^{-e}(R_{j+1} \oplus \cdots \oplus R_{n-2-e}) \oplus \left( \bigoplus_{l=n-1-e}^{n-3} \tau^{-(n-2-l)}R_l \right).
\]
Then the assertion (1) follows from the hypothesis for \((e,f)\) and Lemma 9.4.

(2). Note that there are \( M \) with \(|M| = n - 1 \) and \( R \in \mathbb{T}_{n-2} \) such that
\[
T = M \oplus R \quad \text{and} \quad T' = M \oplus \tau^{-1}R.
\]

Therefore, either \( T \to T' \) or \( T' \to T \) holds. Suppose that there is an arrow \( T' \to T \) in \( \overline{\mathbb{H}}(sr\text{-tilt}A) \). Then we have \( T' \geq T \). In particular, we have
\[
0 = \text{Hom}_A(R, \tau(\tau^{-1}R)) = \text{Hom}_A(R, R).
\]
This is a contradiction. Therefore, we have the assertion. \( \square \)

By Lemma 9.22 (2), we can construct \( \omega^{(4)} \) as follows.
\[
\omega^{(4)} : \begin{bmatrix}
T_6^{(3)} & T_1^{(4)} & T_{1,2} & \cdots & T_{1,n-4} \\
\vdots & & & & \\
& T_{e,1} & \cdots & T_{e,n-3-e} \\
\vdots & & & & \\
& T_{n-4,1}
\end{bmatrix}
\]

9.3.7 Construct \( \omega^{(5)} \)

For each \( s \in \{1, 2, \ldots, n - 3\} \), we define \( T_s^{(5)} \) as follows.
\[
= \frac{T_1^{(4)}}{R_n} + \tau^{n-4}I_{n-1}.
\]
\[
= \frac{T_s^{(5)}}{\tau^{-(s-1)}R_{n-2-s}} + \tau^{n-3-s}I_{n-k_{\text{od}}e} \quad (2 \leq s \leq n - 3).
\]

In other word, \( T_s^{(5)} \) has the following form.
\[
\left( \bigoplus_{k=1}^{n-4} \tau^{-k}R_{n-3-k} \right) \oplus \left( \bigoplus_{j=0}^{n-3-s} \tau^{n-3-j}I_{n-k_{\text{od}}e} \right) \oplus I_1 \oplus I_0 \oplus P_{n-k_{\text{od}}e}
\]

Lemma 9.23 For each \( s \in \{1, \ldots, n-3\} \), \( T_s^{(5)} \) is \( sr\text{-tilt}A \) and there is an arrow \( T_{s-1}^{(5)} \to T_s^{(5)} \) in \( \overline{\mathbb{H}}(sr\text{-tilt}A) \), where we put \( T_0^{(5)} = T_{n-4,1}^{(4)} \).

Proof By Lemma 4.6 (3), (4), and Lemma 9.22, it is sufficient to check the following conditions.

(i) \( \tau^{-k}R_{n-3-k} < \tau^{n-3-j}I_{n-k_{\text{od}}e} < \tau^{-k}R_{n-3-k} \) holds for each \((j, k) \in \{0, \ldots, s\} \times \{s, \ldots, n-4\}\).

(ii) \( \tau^{n-3-j}I_{n-k_{\text{od}}e} < \tau^{n-3-j'}I_{n-k'_{\text{od}}e} \) for each \( 0 \leq j, j' \leq s \).

(iii) \( I_e < \tau^{n-3-j}I_{n-k_{\text{od}}e} < I_e \) for each \((j, e) \in \{0, \ldots, n\} \times \{0, 1\}\).
Moreover, we have a path that
\[ M_t - \rightarrow \text{Lemma 9.3} \]
\[ \tau^{n-3-j} I_{n-\delta_{\text{odd}}} = 0 \text{ for each } 0 \leq j \leq s. \]
\[ \tau^{n-3-j} I_{n-\delta_{\text{odd}}} \prec \tau^{-(s-1)} R_{n-2}. \]

Note that \( \tau^r I_t = \tau^{r+1} P_1^{-} \).

(i) By the definition of \( \prec \), we have \( \tau^{n-3-j} I_{n-\delta_{\text{odd}}} < \tau^{-k} R_{n-3-k} \). By Lemma 9.4, we have
\[ (\tau^{-(k+n-2-j)} R_{n-3-k})(e_{n-1} + e_n) = (\tau^{-(k-j)} R_{n-3-k})(e_{n-1} + e_n) = 0. \]
This shows \( \tau^{-k} R_{n-3-k} < \tau^{n-3-j} I_{n-\delta_{\text{odd}}} \) and the condition (i) holds.

(ii) If \( j \geq j' \), the assertion follows from the definition of \( \prec \). Thus, we may assume \( j < j' \). In this case, it follows from \( 0 \leq j' - j - 1 \leq n - 4, (j' - j - 1) + \delta_{\text{odd}} \equiv j' - 1 \pmod{2} \), and Lemma 9.3 that
\[ \tau^{j'-j-1} (I_{n-\delta_{\text{odd}}})(e_{n-\delta_{j', \text{odd}}}) = 0. \]
Thus the condition (ii) holds.

(iii). If \( j = n - 3 \), then the assertion follows from the definition of \( \prec \). Therefore, we may assume \( j < n - 3 \). In this case, we have \( 0 \leq n - 4 - j \leq n - 4 \) and the condition (iii) follows from Lemma 9.3.

(iv). Since \( n - 3 - j + \delta_{\text{odd}} \equiv n - 1 \pmod{2} \), the condition (iv) follows from Lemma 9.3.

(v). The condition (v) follows from the definition of \( \prec \). \( \square \)

By Lemma 9.23, we can construct \( \omega^{(5)} \) as follows.
\[ \omega^{(5)} : T_{n-4,1}^{(4)} \rightarrow T_{1}^{(5)} \rightarrow \cdots \rightarrow T_{n-3}^{(5)}. \]

### 9.3.8 Construct \( \omega^{(6)} \)

Let \( M_t^{(6)} = \left( \bigoplus_{k=t-1}^{n-3} \tau^{n-3-k} I_{n-\delta_{\text{odd}}} \right) \oplus I_1 \oplus I_0 \oplus P_{n-\delta_{\text{odd}}}^{-} \) for each \( 1 \leq t \leq n - 2 \). Note that \( M_t^{(6)} \in \text{add} T_{n-3}^{(5)} \). In particular, it is \( \tau \)-rigid by Lemma 9.23. We also note that if \( t - 1 \leq k \leq n - 3 \), then \( \tau^{n-3-k} I_{n-\delta_{\text{odd}}} e_j = 0 \) for each \( j \in \{2, \ldots, t\} \). In fact, it follows from Lemma 9.3 (3) that
\[ \{2, \ldots, t\} \subset \{2, \ldots, k + 1\} \subset Q_0 \setminus \text{Supp}(\tau^{n-3-k} I_{n-\delta_{\text{odd}}}). \]
Since \( (I_1 \oplus I_0)(e_2 + \cdots + e_{n-2}) = 0 \), we have
\[ T_{t}^{(6)} := M_t^{(6)} \oplus P_{2}^{-} \oplus \cdots \oplus P_{t}^{-} \in \tau \text{-\text{tilt}} A. \]
Moreover, we have a path
\[ \omega^{(6)} : T_{n-3}^{(5)} = T_{1}^{(6)} \rightarrow \cdots \rightarrow T_{n-2}^{(6)}. \]
in \( \overline{\mathcal{H}} (\tau \text{-\text{tilt}} A) \).

### 9.3.9 Construct \( \omega^{(7)} \)

Note that
\[ T_{n-2}^{(6)} = I_{n-\delta_{\text{odd}}} - \tau - 3 \oplus I_1 \oplus I_0 \oplus P_{n-\delta_{\text{odd}}}^{-} \oplus P_{2}^{-} \oplus \cdots \oplus P_{n-2}^{-}. \]
Therefore, we have a path
\[ \omega^{(7)} : T_{n-2}^{(6)} \rightarrow I_1 \oplus I_0 \oplus \left( \bigoplus_{k=2}^{n} P_k^{-} \right) \rightarrow I_0 \oplus \left( \bigoplus_{k=1}^{n} P_k^{-} \right) \rightarrow A^{-} \]
in \( \overline{\mathcal{H}} (\tau \text{-\text{tilt}} A) \).
9.3.10 **Construct a path with length $2n^2 - 2n - 2$**

Now let $\omega$ be a path in $\overline{H}$ (6-$t$-tiltA) given by connecting $\omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \omega^{(4)}, \omega^{(5)}, \omega^{(6)}$, and $\omega^{(7)}$. Then we have

$$\ell(\omega) = \sum_{k=1}^{7} \ell(\omega_k)$$

$$= (n + 1)(n - 3) + \left(\frac{(n - 2)(n - 1)}{2} + 2(n - 3) - 1\right) + (n - 2) + 6$$

$$+ \left(\frac{(n - 4)(n - 3)}{2}\right) + (n - 3) + (n - 3) + 3$$

$$= 2n^2 - 2n - 2$$

This finishes the proof.

9.3.11 **An example**

Let $Q$ be the following quiver.

```
\begin{tikzpicture}[->,>=stealth,auto,thick,font=\normalfont, vertex/.style={circle,draw,minimum size=1cm},
                        edge/.style={very thick}]
\node[vertex] (0) at (0,0) {$0$};
\node[vertex] (2) at (2,-1) {$2$};
\node[vertex] (3) at (3,0) {$3$};
\node[vertex] (4) at (5,0) {$4$};
\node[vertex] (1) at (2,2) {$1$};
\node[vertex] (5) at (4,2) {$5$};
\draw[edge] (0) to (2);
\draw[edge] (2) to (3);
\draw[edge] (3) to (1);
\draw[edge] (1) to (5);
\end{tikzpicture}
```

In the following, we give an example of a path in MGS(A) based on the construction in this subsection.

$$\omega = \rho_0 \rho_1 \rho_2 \rho_3 \rho_4 \rho_5 \rho_6$$

$$\begin{array}{cccccccccccccccccccc}
\rho_0 & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 \\
T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 \\
r_{\rho_0} & r_{\rho_1} & r_{\rho_2} & r_{\rho_3} & r_{\rho_4} & r_{\rho_5} & r_{\rho_6} \\
\end{array}$$

9.3.12 **Author details**

1 Faculty of Informatics, Okayama University of Science, 1-1 Ridaicho, Kita-ku, Okayama-shi 700-0005, Japan, 2 Center for Advanced Intelligence Project, RIKEN, Nihonbashi 1-chome Mitsui Building, 15th floor, 1-4-1 Nihonbashi, Chuo-ku, Tokyo 103-0027, Japan.

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