AVERAGING PRINCIPLE FOR STOCHASTIC KURAMOTO-SIVASHINSKY EQUATION WITH A FAST OSCILLATION

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Abstract. This work concerns the problem associated with averaging principle for a stochastic Kuramoto-Sivashinsky equation with slow and fast time-scales. This model can be translated into a multiscale stochastic partial differential equations. Stochastic averaging principle is a powerful tool for studying qualitative analysis of stochastic dynamical systems with different time-scales. To be more precise, under suitable conditions, we prove that there is a limit process in which the fast varying process is averaged out and the limit process which takes the form of the stochastic Kuramoto-Sivashinsky equation is an average with respect to the stationary measure of the fast varying process. Finally, by using the Khasminskii technique we can obtain the rate of strong convergence for the slow component towards the solution of the averaged equation, and as a consequence, the system can be reduced to a single stochastic Kuramoto-Sivashinsky equation with a modified coefficient.

1. Introduction. The Kuramoto-Sivashinsky equation

$$u_t + \gamma u_{xxxx} + \beta u_{xx} + uu_x = 0$$

is a one-dimensional model for turbulence and wave propagation in reaction-diffusion systems, where $\gamma$ and $\beta$ are coefficients accounting for the long-wave instability and the short-wave dissipation respectively, was derived in various physical contexts.

In order to consider a more realistic model our problem, it is sensible to consider some kind of stochastic perturbation represented by a noise term in the equations. Stochastic Kuramoto-Sivashinsky equation is a important equation, a large amount of work has been devoted to the study of the stochastic Kuramoto-Sivashinsky equation: [3, 4, 13, 14, 22, 42, 43].

Throughout the paper, we will take

$$\gamma = \beta = 1$$

for the sake of simplicity. All the results can be extended without difficulty to any constants $\gamma > 0, \beta$.

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1.1. Motivation and problem. In this paper, we will be concerned with the following stochastic Kuramoto-Sivashinsky equation with slow and fast time-scales:

\[
\begin{aligned}
&du^\varepsilon + (u^\varepsilon_{xx} + u^\varepsilon_x + \varepsilon^2 u^\varepsilon_t)dt = f(u^\varepsilon, v^\varepsilon)dt + dW_1 \\
&dv^\varepsilon + \frac{1}{\varepsilon}(-v^\varepsilon_x)dt = \frac{1}{\varepsilon}g(u^\varepsilon, v^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}dW_2
\end{aligned}
\]

in \( Q \) and 
\[
\begin{aligned}
u^\varepsilon(0, t) &= 0 = u^\varepsilon(1, t) \\
u^\varepsilon_x(0, t) &= 0 = u^\varepsilon_x(1, t) \\
u^\varepsilon(x, 0) &= u_0(x) \\
v^\varepsilon(x, 0) &= v_0(x)
\end{aligned}
\]

in \( I \),

where \( T > 0, I = (0, 1), Q = I \times (0, T), u^\varepsilon, v^\varepsilon \) are two real functions and \( \{W_1(t)\}_{t \geq 0} \) and \( \{W_2(t)\}_{t \geq 0} \) are \( L^2(I) \)-valued mutually independent \( Q_1 \) and \( Q_2 \) Wiener processes, the terms \( f(u, v) \) and \( g(u, v) \) are external forces depending on \( u \) and \( v \).

The first motivation of considering (1) is that (1) can be seen as a stochastic Kuramoto-Sivashinsky equation with a fast oscillating perturbation

\[
\begin{aligned}
&du^\varepsilon + (u^\varepsilon_{xxx} + u^\varepsilon_{xx} + u^\varepsilon u^\varepsilon_t)dt = f(u^\varepsilon(t), v^\varepsilon(t))dt + dW_1 \\
u^\varepsilon(0, t) &= 0 = u^\varepsilon(1, t) \\
u^\varepsilon_x(0, t) &= 0 = u^\varepsilon_x(1, t) \\
u^\varepsilon(x, 0) &= u_0(x) \\
v^\varepsilon(x, 0) &= v_0(x)
\end{aligned}
\]

where \( v^\varepsilon(t) \) is governed by the stochastic reaction-diffusion equation

\[
\begin{aligned}
&dv^\varepsilon + \frac{1}{\varepsilon}(-v^\varepsilon_x)dt = \frac{1}{\varepsilon}g(u^\varepsilon, v^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}dW_2
\end{aligned}
\]

in \( Q \) and 
\[
\begin{aligned}
v^\varepsilon(0, t) &= 0 = v^\varepsilon(1, t) \\
v^\varepsilon(x, 0) &= v_0(x)
\end{aligned}
\]

in \( I \).

System (*) is a model for phase turbulence in reaction-diffusion systems(see[27, 28, 29]) in a random environment with stochastic perturbation \( v^\varepsilon \) which denotes the dramatically varying temperature, it also can be used as a model for plane flame propagation(see[37]) with stochastic perturbation in a random environment, describing the combined influence of diffusion and thermal conduction of the gas on stability of a plane flame front in a random environment. More generally, this model also describes incipient instabilities in a variety of physical and chemical systems(see[10, 25, 30]) in a random environment. This model also arises in the modeling of the flow of a thin film of viscous liquid falling down on an inclined plane subject to an applied electric field(see[24]) in a random environment.

The second motivation of considering (1) is that the nonlinear coupled stochastic Kuramoto-Sivashinsky-heat equations (1) with fast and slow time scales may describe the surface waves on multilayered liquid films in a random environment. Indeed, in order to combine dissipative and dispersive features, and to simultaneously support stable solitary-pulse, a model consisting of a Kuramoto-Sivashinsky equation, linearly coupled to an extra linear dissipative equation, is proposed in [32] under the name of the stabilized Kuramoto-Sivashinsky system. The model applies to a description of surface waves on multilayered liquid films. Almost all physical systems have a certain hierarchy in which not all components evolve at the same rate, i.e., some of components vary very rapidly, while others change very slowly, see [36], so we consider multiscale stochastic partial differential equations (1).

The nonlinear coupled stochastic Kuramoto-Sivashinsky-heat equations (1) with fast and slow time scales may describe the surface waves on multilayered liquid films in a random environment, two real wave fields \( u, v \) evolve at the different rates. Then (1) is a multiscale stochastic partial differential equations. Multiscale
stochastic partial differential equations arise as models for various complex systems, such as, describing multiscale phenomena in, for example, nonlinear oscillations, material sciences, automatic control, fluids dynamics, chemical kinetics and in other areas leading to mathematical description involving “slow” and “fast” phase variables.

According to the above motivations, the asymptotic study of the behavior $\varepsilon \to 0$ of (1) is of great interest. Very often, one needs to simulate or predict the time evolution of the slow component of (1) without solving the full system of equations, then a reduced system which governs the slow motion over a long time scale is highly desirable. In this respect, the question of how the physical effects at large time scales influence the dynamics of (1) is arisen. This mathematical question arises naturally which are important from the point of view of dynamical systems from both physical and mathematical standpoints. We focus on this question and by using averaging principle, we show that, under some dissipative conditions on fast variable equation, the complexities effects at large time scales to the asymptotic behavior of the slow component can be omitted or neglected in some sense. More precisely, the slow process $u^\varepsilon$ converges to $\bar{u}$ in the strong way

$$\lim_{\varepsilon \to 0} \mathbb{E} \sup_{0 \leq t \leq T} \| u^\varepsilon(t) - \bar{u}(t) \|^p = 0,$$

where $\bar{u}$ is the solution of the following reduced problem

$$\begin{cases}
  d\bar{u} + (\bar{u}_{xxxx} + \bar{u}_{xx} + \bar{u}_x)dt = \bar{f}(\bar{u})dt + dW_1 & \text{in } Q \\
  \bar{u}(0,t) = 0 = \bar{u}(1,t) & \text{in } (0,T) \\
  \bar{u}_{xx}(0,t) = 0 = \bar{u}_{xx}(1,t) & \text{in } (0,T) \\
  \bar{u}(x,0) = u_0(x) & \text{in } I.
\end{cases}$$

Thus, with the help of averaging principle,

- we can establish an effective approximation for slow process $u^\varepsilon$ with respect to the limit $\varepsilon \to 0$, this can predicting the time evolution of the slow component $u^\varepsilon$.
- we can extract effective dynamic system (2) from complex system (1), it provides an effective tool to analyze qualitative behaviors of (1). It makes the interaction between nonlinearity, uncertainty and multiple scales of (1) more clear.
- it enormously reduce the computational load of (1) although computer technology is highly efficient nowadays.

It follows from the above facts that averaging principle can help us understand and investigate the physical phenomenon described by (1). On the one hand, the averaging principle for (1) shows that the dramatically varying temperature $v^\varepsilon$ does not affect the phase turbulence and plane flame propagation. On the other hand, it can be seen that when the propagation speed of the interface waves is high enough, the surface waves are not affected by the interface waves.

The theory of averaging principle has a long and rich history, which has been applied in many fields, such as, celestial mechanics, wireless communication, signal processing, oscillation theory and radio physics. The averaging principle in the stochastic ordinary differential equations setup was first considered by Khasminskii [26] which proved that an averaging principle holds in weak sense, and has been an active research field on which there is a great deal of literature. In recent years, there are many interesting results for stochastic system in infinite dimensional space: [1, 6, 7, 8, 9, 12, 18, 16, 20, 17, 19, 34, 38, 41, 39, 40].
1.2. Mathematical setting and assumptions. We introduce the following mathematical setting:

- Throughout the paper, the letter $C$ denotes positive constants whose value may change in different occasions. We will write the dependence of constant on parameters explicitly if it is essential.
- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space. Let $Y$ be a Banach space, and let $C([0, T]; Y)$ be the Banach space of all $Y$-valued strongly continuous functions defined on $[0, T]$. We denote by $L^p_T(0, T; Y)(1 \leq p < +\infty)$ the Banach space consisting of all $Y$-valued functions $\mathcal{F}_t \mapsto X(t)$ such that $E(\|X(\cdot)\|^p_{L^p_T(0, T; Y)}) < \infty$; by $L^2_T(\Omega; C([0, T]; Y))$ the Banach space consisting of all $Y$-valued functions $\{\mathcal{F}_t\}_{t \geq 0} \mapsto X(t)$ such that $E(\|X(\cdot)\|_{L^2_T(0, T; Y)}^2) < \infty$. All the above spaces are endowed with the canonical norm.
- We denote by $L^2(I)$ the space of all Lebesgue square integrable functions on $I$. The inner product on $L^2(I)$ is

$$(u, v) = \int_I u v dx,$$

for any $u, v \in L^2(I)$. The norm on $L^2(I)$ is

$$\|u\| = (u, u)^{\frac{1}{2}},$$

for any $u \in L^2(I)$.

$L^2(I), H^s(I)(s \geq 0)$ are the classical Sobolev spaces of functions on $I$. The definition of $H^s(I)$ can be found in [31], the norm on $H^s(I)$ is $\| \cdot \|_{H^s}$.

- We set the semigroups $\{S_1(t)\}_{t \geq 0}, \{S_2(t)\}_{t \geq 0}$ are generated by the operators

$$L_1 u = -u_{xxxx},$$

$$L_2 v = v_{xx},$$

with

$$\mathcal{D}(L_1) = \{u \in H^4(I) \mid u(0) = 0 = u(1), \ u_{xx}(0) = 0 = u_{xx}(1) \},$$

$$\mathcal{D}(L_2) = H^2(I) \cap H^3_0(I).$$

The eigenvalues of $L_1$ are $\lambda_k = k^4 \pi^4 (k = 1, 2, \ldots), \epsilon_k (k = 1, 2, \ldots)$ are eigenfunctions of $L_1$ with eigenvalue $\lambda_k$.

Through this paper, we make the following assumption (H):

1. There exist constants $L_f, L_g$ such that $f$ and $g$ satisfy

$$|f(u, v)| \leq L_f (1 + |u| + |v|),$$

$$|g(u, v)| \leq L_g (1 + |u| + |v|)$$

for any $u, v \in \mathbb{R}$.

$f$ and $g$ are Lipschitz continuous, that is

$$|f(u_1, v_1) - f(u_2, v_2)| \leq L_f(|u_1 - u_2| + |v_1 - v_2|),$$

$$|g(u_1, v_1) - g(u_2, v_2)| \leq L_g(|u_1 - u_2| + |v_1 - v_2|)$$

for any $u_1, v_1, u_2, v_2 \in \mathbb{R}$.

We assume that $\alpha \triangleq \dot{\lambda} - 2L_g > 0$, where $\lambda > 0$ is the smallest constant such that the following inequality holds

$$\|u_x\|^2 \geq \lambda \|u\|^2,$$

where $u \in H^3_0(I)$ or $\int_I u dx = 0$. 
2. \{W_1(t)\}_{t \geq 0} and \{W_2(t)\}_{t \geq 0} are \(L^2(I)\)-valued mutually independent \(Q_1\) and \(Q_2\) Wiener processes. \{W_1(t)\}_{t \geq 0} is given by
\[
W_1(t) = \sum_{k=1}^{\infty} \sqrt{\alpha_k^1 \beta_k^1(t)} e_k^1, \quad t \geq 0,
\]
where \(\alpha_k^1 \geq 0\) satisfies \(Q_1 e_k^1 = \alpha_k^1 e_k^1\) with \(Tr Q_1 = \sum_{k=1}^{\infty} \alpha_k^1 < +\infty\), and \(\{\beta_k^1\}\) is a sequence of mutually independent standard Brownian motions.

We assume there exists \(\beta \in (0, \frac{1}{2})\) such that
\[
\sum_{k=1}^{\infty} \alpha_k^1 \lambda_k^{1+2\beta} < +\infty.
\]

\{W_2(t)\}_{t \geq 0} is given by
\[
W_2(t) = \sum_{k=1}^{\infty} \sqrt{\alpha_k^2 \beta_k^2(t)} e_k^2, \quad t \geq 0,
\]
where \(\alpha_k^2 \geq 0\) satisfies \(Q_2 e_k^2 = \alpha_k^2 e_k^2\) with \(Tr Q_2 = \sum_{k=1}^{\infty} \alpha_k^2 < +\infty\), and \(\{\beta_k^2\}\) is a sequence of mutually independent standard Brownian motions.

1.3. **Main result.** Now, we are in a position to present the main result in this paper.

**Theorem 1.1.** Suppose that the hypothesis \((H)\) holds and \(u_0 \in H^2(I) \cap \mathbb{H}_0(I), v_0 \in L^2(I), (w^\varepsilon, v^\varepsilon) \) is the solution of (1) and \(\bar{u}\) is the solution of the effective dynamics equation (2), then for any \(T > 0\), any \(p > 0\), we have

\[
\lim_{\varepsilon \to 0} \mathbb{E} \sup_{0 \leq t \leq T} \|u^\varepsilon(t) - \bar{u}(t)\|^{2p} = 0,
\]

where
\[
\tilde{f}(u) = \int_{L^2(I)} f(u, v) \mu^u(dv)
\]
and \(\mu^u\) is an invariant measure for the fast motion with frozen slow component

\[
\begin{cases}
   dv = [v_{xx} + g(u, v)]dt + dW_2 & \text{in } Q \\
   v(0, t) = 0 = v(1, t) & \text{in } (0, T) \\
   v(x, 0) = v_0(x) & \text{in } I,
\end{cases}
\]

where \(u \in L^2(I)\).

Moreover, if \(p > \frac{5}{4}\), there exists a positive constant \(C(p)\) such that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|u^\varepsilon(t) - \bar{u}(t)\|^{2p} \right) \leq C(p)\left(\frac{1}{-\ln \varepsilon}\right)^{\frac{p}{4}};
\]
if \(0 < p \leq \frac{5}{4}\), for any \(\kappa > 0\), there exists a positive constant \(C(p, \kappa)\) such that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|u^\varepsilon(t) - \bar{u}(t)\|^{2p} \right) \leq C(p, \kappa)\left(\frac{1}{-\ln \varepsilon}\right)^{\frac{p}{4(2 + \kappa)}}.
\]

The rest of this paper is structured as follows. In Section 2, we gather all the necessary tools. Section 3 is devoted to the proof of Theorem 3.2. Section 4 is devoted to the proof of Theorem 1.1.
2. Preliminaries. This section is devoted to some preliminaries for the proof of Theorem 1.1.

2.1. Some useful inequalities.

Lemma 2.1. If $a, b \in \mathbb{R}$, $p > 0$, it holds that
\[
(|a| + |b|)^p \leq \begin{cases} 
|a|^p + |b|^p & 0 < p \leq 1, \\
2^{p-1}(|a|^p + |b|^p) & p > 1.
\end{cases}
\]

Lemma 2.2. (Young inequality) Let $a, b \in [0, +\infty)$ and $\varepsilon > 0$, then we have
\[
ab \leq \varepsilon^{-p} a^p + \varepsilon^q b^q,
\]
where $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.3. Let $y(t)$ be a nonnegative function, if $y' \leq -ay + f$, we have
\[
y(t) \leq y(s)e^{-a(t-s)} + \int_s^t e^{-a(t-\tau)} f(\tau) d\tau.
\]

2.2. Some useful estimates.

Proposition 1. \cite{21, 33} \{${S}_1(t)\}_{t \geq 0}$ and \{${S}_2(t)\}_{t \geq 0}$ are analytic on $L^p(I)$ for all $1 < p < \infty$ and enjoy the following properties
\[
\|D^j S_1(t) \varphi\|_{L^p} \leq C t^{-\frac{j}{2}} \|\varphi\|_{L^p},
\]
\[
\|D^j S_2(t) \varphi\|_{L^p} \leq C t^{-\frac{j}{2}} \|\varphi\|_{L^p},
\]
where $D^j$ denotes the $j$–th order derivative with respect to the spatial variable.

Proposition 2. For any $p, T > 0$, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| \int_0^t S_1(t-s) dW_1 \|_{H^2}^{2p} \leq C(p, T).
\]

Proof. By the same method as in \cite[Lemma 4.2]{12}, we can prove this proposition. \qed

2.3. The fast motion equation. We first consider the frozen equation associate to fast motion for fixed slow component
\[
\begin{cases}
 dv = (v_{xx} + g(u, v)) dt + dW_2 & \text{in } Q \\
v(0, t) = 0 = v(1, t) & \text{in } (0, T) \\
v(x, 0) = X(x) & \text{in } I.
\end{cases}
\]

We denote $v^{u,X}$ the solution to (5), we now discuss the asymptotic behavior of the fast equation (5).

By the same method as in \cite{2}, we can obtain the existence of the invariant measure for (5), namely, we have

Proposition 3. For $u, X, Y \in L^2(I)$, let $v^{u,X}$ be the solution of
\[
\begin{cases}
 dv = (v_{xx} + g(u, v)) dt + dW_2 & \text{in } I \times (0, +\infty) \\
v(0, t) = 0 = v(1, t) & \text{in } (0, +\infty) \\
v(x, 0) = X(x) & \text{in } I.
\end{cases}
\]
1) There exists a positive constant $C$ such that $v^{u,X}$ satisfies:
\[
\begin{align*}
E\|v^{u,X}(t)\|^2 &\leq e^{-2\alpha t}\|X\|^2 + C\|u\|^2 + 1, \\
E\|v^{u,X}(t) - v^{u,Y}(t)\|^2 &\leq \|X - Y\|^2 e^{-2\alpha t},
\end{align*}
\]
for $t \geq 0$.

2) There is unique invariant measure $\mu$ for the Markov semigroup $P^u_t$ associated with the system (6) in $L^2(\Omega)$. Moreover, we have
\[
\int_{L^2(\Omega)} \|z\|^2 \mu^u(dz) \leq C(1 + \|u\|^2).
\]

3) There exists a positive constant $C$ such that $v^{u,X}$ satisfies:
\[
\|Ef(u, v^{u,X}) - \bar{f}(u)\|^2 \leq C(1 + \|X\|^2 + \|u\|^2)e^{-2\alpha t}
\]
for $t \geq 0$.

Proof. 1) • By applying the generalized Itô formula with $\frac{1}{2}\|v^{u,X}\|^2$, we can obtain that
\[
\frac{1}{2}\|v^{u,x}\|^2 = \frac{1}{2}\|X\|^2 + \int_0^t (v^{u,x}, v^{u,x}_x + g(u, v^{u,x}))ds + \int_0^t (v^{u,x}, dW_2) + \frac{t}{2}TrQ_2
\]
Taking mathematical expectation from both sides of above equation, we have
\[
\frac{1}{2}E\|v^{u,x}\|^2 = \frac{1}{2}E\|X\|^2 - \int_0^t E\|v^{u,x}_x\|^2 ds + \int_0^t E(v^{u,x}, g(u, v^{u,x}))ds + \frac{t}{2}TrQ_2,
\]

namely,
\[
\frac{d}{dt}E\|v^{u,x}\|^2 = -2E\|v^{u,x}_x\|^2 + 2E(v^{u,x}, g(u, v^{u,x})) + TrQ_2.
\]

It follows from Young’s inequality that
\[
\frac{d}{dt}E\|v^{u,x}\|^2
\]
\[
\leq -2E\|v^{u,x}_x\|^2 + 2E\|v^{u,x}, g(u, v^{u,x})\| + TrQ_2
\]
\[
\leq -2E\|v^{u,x}_x\|^2 + 2E \int_\Omega \frac{|u^{u,x}|}{\|u^{u,x}\|} |g(u, v^{u,x})| dx + C
\]
\[
\leq -2\lambda E\|v^{u,x}\|^2 + 2L_gE\|v^{u,x}\|^2 + 2L_gE \int_\Omega |v^{u,x}| |u| dx + CL_gE \int_\Omega |v^{u,x}| dx + C
\]
\[
\leq -2\lambda E\|v^{u,x}\|^2 + 4L_gE\|v^{u,x}\|^2 + C\|u\|^2 + C
\]
\[
= -2\alpha E\|v^{u,x}\|^2 + C\|u\|^2 + C.
\]

Hence, by applying Lemma 2.3 with $E\|v^{u,X}(t)\|^2$, we have
\[
E\|v^{u,X}(t)\|^2 \leq e^{-2\alpha t}\|X\|^2 + C(\|u\|^2 + 1).
\]
• It is easy to see
\[
\begin{align*}
\frac{d}{dt}(v^u_X - v^u_Y) &= [(v^u_X - v^u_Y)_{xx} + g(u, v^u_X) - g(u, v^u_Y)] \quad \text{in } Q \\
(v^u_X - v^u_Y)(0, t) &= 0 = (v^u_X - v^u_Y)(1, t) \quad \text{in } (0, T) \\
(v^u_X - v^u_Y)(x, 0) &= X(x) - Y(x) \quad \text{in } I,
\end{align*}
\]

thus, it follows from the energy method that
\[
\begin{align*}
\frac{1}{2} \|v^u_X - v^u_Y\|^2 = \frac{1}{2} \|X - Y\|^2 + \int_0^t (v^u_X - v^u_Y, (v^u_X - v^u_Y)_{xx} + g(u, v^u_X) - g(u, v^u_Y)) ds \\
= \frac{1}{2} \|X - Y\|^2 + \int_0^t \|v^u_X - v^u_Y\|_x^2 ds \\
+ \int_0^t (v^u_X - v^u_Y, g(u, v^u_X) - g(u, v^u_Y)) ds,
\end{align*}
\]

namely,
\[
\begin{align*}
\frac{d}{dt} \|v^u_X - v^u_Y\|^2 = -2 \|v^u_X - v^u_Y\|_x^2 + 2(v^u_X - v^u_Y, g(u, v^u_X) - g(u, v^u_Y)).
\end{align*}
\]

Thus, we have
\[
\begin{align*}
\frac{d}{dt} \|v^u_X - v^u_Y\|^2 &\leq -2 \|v^u_X - v^u_Y\|_x^2 + 2Lg\|v^u_X - v^u_Y\|^2 \\
&\leq -2\lambda \|v^u_X - v^u_Y\|^2 + 2Lg\|v^u_X - v^u_Y\|^2 \\
&\leq -2\lambda \|v^u_X - v^u_Y\|^2 + 4Lg\|v^u_X - v^u_Y\|^2 \\
&= -2(\lambda - 2Lg)\|v^u_X - v^u_Y\|^2 \\
&= -2\alpha \|v^u_X - v^u_Y\|^2,
\end{align*}
\]

this yields
\[
\|v^u_X - v^u_Y\|^2 \leq \|X - Y\|^2 e^{-2\alpha t}.
\]

Thus, we have
\[
\mathbb{E}\|v^u_X - v^u_Y\|^2 \leq \|X - Y\|^2 e^{-2\alpha t}.
\]

2) (7) implies that for any \(u \in L^2(I)\) that there is unique invariant measure \(\mu^u\) for the Markov semigroup \(P_t^u\) associated with the system (6) in \(L^2(I)\) such that
\[
\int_{L^2(I)} P_t^u \varphi d\mu^u = \int_{L^2(I)} \varphi d\mu^u, \quad t \geq 0
\]

for any \(\varphi \in B_b(L^2(I))\) the space of bounded functions on \(L^2(I)\).

Then by repeating the standard argument as in [8, Proposition 4.2] and [9, Lemma 3.4], the invariant measure satisfies
\[
\int_{L^2(I)} \|z\|_u^2 \mu^u(dz) \leq C(1 + \|u\|^2).
\]
3) According to the invariant property of \( \mu^u \), (2) and (7), we have

\[
\| \mathbb{E}f(u, v^{u,X}) - \bar{f}(u) \|^2 = \| \mathbb{E}f(u, v^{u,X}) - \int_{L^2(I)} f(u, Y) \mu^u(dY) \|^2 \\
= \| \int_{L^2(I)} \mathbb{E}[f(u, v^{u,X}) - f(u, v^{u,Y})] \mu^u(dY) \|^2 \\
\leq C \int_{L^2(I)} \mathbb{E} \| v^{u,X} - v^{u,Y} \|^2 \mu^u(dY) \\
\leq C \int_{L^2(I)} \| X - Y \|^2 e^{-2\alpha t} \mu^u(dY) \\
\leq C(1 + \| X \|^2 + \| u \|^2) e^{-2\alpha t}.
\]

\[\Box\]

3. Well-posedness and some a priori estimates of (1). Let us explain what we mean by a solution of (1)

**Definition 3.1.** A pair of functions \((u^\varepsilon, v^\varepsilon)\) is called a mild solution of (1) on \([0, T]\), if for almost each \( \omega \in \Omega \) and \( t \in [0, T] \), \((u^\varepsilon, v^\varepsilon)\) satisfies the following Itô integral form

\[
u^\varepsilon(t) = S_1(t)u_0 + \int_0^t S_1(t-s) (-u^\varepsilon_{xx} - u^\varepsilon_{x} + f(u^\varepsilon, v^\varepsilon)) ds + \int_0^t S_1(t-s) dW_1, \\
v^\varepsilon(t) = S_2(t)\varepsilon v_0 + \frac{1}{\varepsilon} \int_0^t S_2(t-s) g(u^\varepsilon, v^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t S_2(t-s) dW_2.
\]

**Theorem 3.2.** Let \( T > 0 \). For any \( \varepsilon \in (0, 1) \), if \((u_0, v_0) \in H_0^1(I) \times L^2(I)\), (1) admits a unique mild solution \((u^\varepsilon, v^\varepsilon)\) \( \in L^2(\Omega; C([0, T]; H^1(I)) \times L^2(\Omega; C([0, T]; L^2(I))) \) and

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|^2_{H^1} + \mathbb{E} \sup_{0 \leq t \leq T} \| v^\varepsilon(t) \|^2 \leq C(1 + \| u_0 \|^2_{H^1} + \| v_0 \|^2),
\]

where \( C \) is a positive constant depending on \( \varepsilon, T, Q_1, Q_2 \).

**Remark 1.** The main idea of Proof of Theorem 3.2 comes from [21, Theorem 4.1].

3.1. Local existence. In this section, we will take

\[\varepsilon = 1\]

for the sake of simplicity. All the results can be extended without difficulty to the general case.

We set

\[X_t = L^2(\Omega; C([0, \tau]; H^1(I)) \times L^2(\Omega; C([0, \tau]; L^2(I))).\]

By the same method as in [31], let \( \rho \in C_0^\infty(\mathbb{R}) \) be a cut-off function such that \( \rho(r) = 1 \) for \( r \in [0, 1] \) and \( \rho(r) = 0 \) for \( r \geq 2 \). For any \( R > 0, y \in X_t \) and \( t \in [0, T] \), we set

\[
\rho_R(y)(t) = \rho\left( \frac{\| y \|_{C([0, t]; H^1(I))}}{R} \right).
\]
The truncated equation corresponding to (1) is the following stochastic partial differential equations:

\[
\begin{aligned}
    du + (u_{xxx} + u_{xx} + \rho_R(u)uu_x)dt &= f(u, v)dt + dW_1 & \text{in } Q \\
    dv - u_{xx}dt &= g(u, v)dt + dW_2 & \text{in } Q \\
    u(0, t) &= 0 = u(1, t), & \text{in } (0, T) \\
    u_{xx}(0, t) &= 0 = u_{xx}(1, t), & \text{in } (0, T) \\
    v(0, t) &= 0 = v(1, t), & \text{in } (0, T) \\
    u(x, 0) &= u_0(x), & \text{in } I \\
    v(x, 0) &= v_0(x) & \text{in } I.
\end{aligned}
\]

(8)

**Proposition 4.** For any \((u_0, v_0) \in H^1_0(I) \times L^2(I), (8)\) admits a unique mild solution \((u, v) \in X_{\tau_\infty}, \) where \(\tau_\infty\) is stopping time. Moreover, if \(\tau_\infty < +\infty\), then \(P\)-a.s.

\[
\lim_{t \to \tau_\infty} \| (u, v) \|_{X_t} = +\infty.
\]

**Proof.** We define

\[
\Phi_R(u, v) = \begin{pmatrix}
    \Phi_R^1(u, v) \\
    \Phi_R^2(u, v)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    S_1(t)u_0 + \int_0^t S_1(t-s)(-u_{xx} - \rho_R(u)uu_x + f(u, v))ds + \int_0^t S_1(t-s)dW_1 \\
    S_2(t)v_0 + \int_0^t S_2(t-s)g(u, v)ds + \int_0^t S_2(t-s)dW_2
\end{pmatrix}.
\]

It is easy to see that for any \(T_0 > 0\), operator \(\Phi_R(u, v)\) maps \(X_{T_0}\) into itself.

- The estimate of

\[
\mathbb{E} \sup_{0 \leq t \leq T_0} \| (\Phi_R^1(u_1, v_1) - \Phi_R^1(u_2, v_2))(t) \|^2_{H^1}.
\]

Indeed, we set \(F(u) = uu_x\). Noting the fact for

\[
\| u_1 u_{1x} - u_2 u_{2x} \| = \| u_1 (u_1 - u_2)_x + (u_1 - u_2)u_{2x} \|
\]

\[
\leq (\| u_1 \|_{H^1} + \| u_2 \|_{H^1}) \| u_1 - u_2 \|_{H^1}.
\]

Set

\[
F_R(u) = F(u)\rho_R(u).
\]

We have without any loss of generality, assume that \(\| u_1 \|_{H^1} \geq \| u_2 \|_{H^1},
\]

\[
\| F_R(u_1) - F_R(u_2) \|
\]

\[
= \| (F(u_1) - F(u_2))\rho_R(u_1) + (\rho_R(u_1) - \rho_R(u_2))F(y_2) \|
\]

\[
\leq \| (F(u_1) - F(u_2))\rho_R(u_1) \| + \| (\rho_R(u_1) - \rho_R(u_2))F(y_2) \|
\]

\[
= \| (F(u_1) - F(u_2))\rho_R(u_1) \| + \| (\rho_R(u_1) - \rho_R(u_2))F(u_2) \| \chi_{\{u_2 \leq 2R\}}
\]

\[
\leq \| (F(u_1) - F(u_2))\rho_R(u_1) \| + \| \rho_R \|_{L^\infty} \frac{1}{R} \| u_1 - u_2 \| F(u_2) \| \chi_{\{u_2 \leq 2R\}}
\]

\[
\leq C \| u_1 - u_2 \|_{H^1} (\| u_1 \|_{H^1} + \| u_2 \|_{H^1}) \chi_{\{u_2 \leq 2R\}}
\]

\[
+ \frac{1}{R} \| \rho_R \|_{L^\infty} \| u_1 - u_2 \|_{L^\infty} \| F(u_2) \| \chi_{\{u_2 \leq 2R\}}
\]

\[
\leq CR \| u_1 - u_2 \|_{H^1} + \frac{1}{R} \| \rho_R \|_{L^\infty} \| u_1 - u_2 \|_{H^1} \| u_2 u_{2x} \| \chi_{\{u_2 \leq 2R\}}
\]

\[
\leq CR \| u_1 - u_2 \|_{H^1} + \frac{1}{R} \| \rho_R \|_{L^\infty} \| u_1 - u_2 \|_{H^1} \| u_2 \|_{H^1} \| \chi_{\{u_2 \leq 2R\}}
\]

\[
\leq CR \| u_1 - u_2 \|_{H^1}.
\]
By taking \( p = q = 2, j = 1 \) in (3), we have
\[
E \sup_{0 \leq t \leq T_0} \left\| \int_0^t S_1(t-s)(f_R(u_1) - f_R(u_2))(s) ds \right\|_{H^s}^2 \\
\leq C E \sup_{0 \leq t \leq T_0} \left( \int_0^t (t-s)^{-\frac{3}{4}} \|f_R(u_1) - f_R(u_2)(s)\| ds \right)^2 \\
\leq C \sup_{0 \leq t \leq T_0} \left( \int_0^t (t-s)^{-\frac{3}{4}} R\|u_1 - u_2\|_{H^s} ds \right)^2 \\
\leq C R^2 \sup_{0 \leq t \leq T_0} \left( \int_0^t (t-s)^{-\frac{3}{4}} ds \right)^2 E \sup_{0 \leq t \leq T_0} \|u_1 - u_2(t)\|^2_{H^s} \\
\leq C R^2 T_0^3 E \sup_{0 \leq t \leq T_0} \|u_1 - u_2(t)\|^2_{H^s}.
\]

and
\[
E \sup_{0 \leq t \leq T_0} \left\| \int_0^t S_1(t-s)(f(u_1, v_1) - f(u_2, v_2)) ds \right\|_{H^s(t)}^2 \\
\leq E \sup_{0 \leq t \leq T_0} \left( \int_0^t \|S_1(t-s)(f(u_1, v_1) - f(u_2, v_2))\|_{H^s} ds \right)^2 \\
\leq C \sup_{0 \leq t \leq T_0} \left( \int_0^t (t-s)^{-\frac{3}{4}} \|f(u_1, v_1) - f(u_2, v_2)\| ds \right)^2 \\
\leq C \sup_{0 \leq t \leq T_0} \left( \int_0^t (t-s)^{-\frac{3}{4}} ((u_1 - u_2)(s)) + \|v_1 - v_2(s)\| ds \right)^2 \\
\leq C \sup_{0 \leq t \leq T_0} \left( \int_0^t (t-s)^{-\frac{3}{4}} ds \right)^2 \left( E \sup_{0 \leq t \leq T_0} \|u_1 - u_2(t)\|^2 + E \sup_{0 \leq t \leq T_0} \|v_1 - v_2(t)\|^2 \right) \\
\leq C T_0^2 \left( E \sup_{0 \leq t \leq T_0} \|u_1 - u_2(t)\|^2 + E \sup_{0 \leq t \leq T_0} \|v_1 - v_2(t)\|^2 \right). \tag{10}
\]

By taking \( p = q = 2, j = 3 \) in (3), we have
\[
E \sup_{0 \leq t \leq T_0} \left\| \int_0^t S_1(t-s)(u_{1xx} - u_{2xx})(s) ds \right\|_{H^s}^2 \\
\leq C E \sup_{0 \leq t \leq T_0} \left( \int_0^t (t-s)^{-\frac{3}{4}} \|u_1 - u_2(t)\| ds \right)^2 \\
\leq C \sup_{0 \leq t \leq T_0} \left( \int_0^t (t-s)^{-\frac{3}{4}} \|u_1 - u_2(t)\|_{H^s} ds \right)^2 \\
\leq C \sup_{0 \leq t \leq T_0} \left( \int_0^t (t-s)^{-\frac{3}{4}} ds \right)^2 E \sup_{0 \leq t \leq T_0} \|u_1 - u_2(t)\|^2_{H^s} \\
\leq C T_0^2 E \sup_{0 \leq t \leq T_0} \|u_1 - u_2(t)\|^2_{H^s}. \tag{11}
\]

Finally, collecting the above estimates (9)-(11), we get
\[
E \sup_{0 \leq t \leq T_0} \|\Phi_R^j(u_1, v_1) - \Phi_R^j(u_2, v_2)(t)\|^2_{H^s}.
\]
\[ \leq C(T^\frac{1}{2} + T^\frac{3}{2} + R^2T^\frac{3}{2})(\mathbb{E} \sup_{0 \leq t \leq T_0} \|(u_1 - u_2)(t)\|_{H^1}^2 + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(v_1 - v_2)(t)\|_2^2). \]

(12)

- The estimate of
\[
\mathbb{E} \sup_{0 \leq t \leq T_0} \|((\Phi^2_R(u_1, v_1)) - (\Phi^2_R(u_2, v_2)))(t)\|^2.
\]

Indeed, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T_0} \int_0^t S_2(t - s)(g(u_1, v_1) - g(u_2, v_2))ds \leq \mathbb{E} \sup_{0 \leq t \leq T_0} \int_0^t \|S_2(t - s)(g(u_1, v_1) - g(u_2, v_2))\|ds^2.
\]
\[
\leq \mathbb{E} \sup_{0 \leq t \leq T_0} \int_0^t \|g(u_1, v_1) - g(u_2, v_2)\|ds^2 \leq \mathbb{E} \sup_{0 \leq t \leq T_0} (\int_0^t (\|u_1 - u_2\|(s) + \|v_1 - v_2\|(s))ds)^2.
\]
\[
\leq C \int_0^t (\mathbb{E} \sup_{0 \leq t \leq T_0} \|(u_1 - u_2)(t)\|_2^2 + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(v_1 - v_2)(t)\|_2^2) dt \leq CT_0(\mathbb{E} \sup_{0 \leq t \leq T_0} \|(u_1 - u_2)(t)\|_2^2 + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(v_1 - v_2)(t)\|_2^2).
\]

This implies that
\[
\mathbb{E} \sup_{0 \leq t \leq T_0} \|((\Phi^2_R(u_1, v_1)) - (\Phi^2_R(u_2, v_2)))(t)\|^2 \leq CT_0(\mathbb{E} \sup_{0 \leq t \leq T_0} \|(u_1 - u_2)(t)\|_2^2 + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(v_1 - v_2)(t)\|_2^2).
\]

(13)

- The estimate of
\[
\|\Phi_R(u_1, v_1) - \Phi_R(u_2, v_2)\|_{X_{T_0}}.
\]

Indeed, it follows from (12) and (13) that
\[
\mathbb{E} \sup_{0 \leq t \leq T_0} \|((\Phi^1_R(u_1, v_1)) - (\Phi^1_R(u_2, v_2)))(t)\|^2 H_1 + \mathbb{E} \sup_{0 \leq t \leq T_0} \|((\Phi^2_R(u_1, v_1)) - (\Phi^2_R(u_2, v_2)))(t)\|^2 \leq C(T^\frac{1}{2} + T^\frac{3}{2} + R^2T^\frac{3}{2})(\mathbb{E} \sup_{0 \leq t \leq T_0} \|(u_1 - u_2)(t)\|_2^2 + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(v_1 - v_2)(t)\|_2^2),
\]

namely, we have
\[
\|\Phi_R(u_1, v_1) - \Phi_R(u_2, v_2)\|_{X_{T_0}} \leq C(T^\frac{1}{2} + T^\frac{3}{2} + R^2T^\frac{3}{2})(\mathbb{E} \sup_{0 \leq t \leq T_0} \|(u_1 - u_2)(t)\|_2^2 + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(v_1 - v_2)(t)\|_2^2).
\]

(14)

For a sufficiently small \(T_0\), \(\Phi_R(u, v)\) is a contraction mapping on \(X_{T_0}\).

Hence, by applying the Banach contraction principle, \(\Phi_R(u, v)\) has a unique fixed point in \(X_{T_0}\), which is the unique local solution to (8) on the interval \([0, T_0]\). Since \(T_0\) does not depend on the initial value \((u_0, v_0)\), this solution may be extended to the whole interval \([0, T]\).

We denote by \((u_R, v_R)\) this unique mild solution and let
\[
\tau_R = \inf\{t \geq 0 : \|(u_R, v_R)\|_{X_t} \geq R\},
\]

with the usual convention that \(\inf \emptyset = \infty\).
Since \( R_1 \leq R_2, \tau_{R_1} \leq \tau_{R_2} \), we can put \( \tau_{\infty} = \lim_{R \to +\infty} \tau_R \). Set \( \tau = \tau_{R_1} \land \tau_{R_2} \). We define a local solution to (1) as follows

\[
    u(t) = u_R(t), \ \forall t \in [0, \tau], \\
    v(t) = v_R(t), \ \forall t \in [0, \tau].
\]

Indeed, for any \( t \in [0, \tau] \)

\[
    u_{R_1}(t) - u_{R_2}(t) = \int_0^t S_1(t - s)(-u_{R_1,xx} + u_{R_2,xx} - \rho_{R_1}(u_{R_1})u_{R_1}u_{R_1,x} \\
    + \rho_{R_2}(u_{R_2})u_{R_2}u_{R_2,x} + f(u_{R_1}, v_{R_1}) - f(u_{R_2}, v_{R_2}))ds,
\]

\[
    v_{R_1}(t) - v_{R_2}(t) = \int_0^t S_2(t - s)[g(u_{R_1}, v_{R_1}) - g(u_{R_2}, v_{R_2})]ds.
\]

Proceeding as in the proof of (14), we can obtain

\[
    \| (u_{R_1}, v_{R_1}) - (u_{R_2}, v_{R_2}) \|_{X_t} \leq C(t)\| (u_{R_1}, v_{R_1}) - (u_{R_2}, v_{R_2}) \|_{X_t},
\]

where \( C(t) \) is a monotonically increasing function and \( C(0) = 0 \). If we take \( t \) sufficiently small, we can obtain

\[
    u_{R_1}(t) = u_{R_2}(t), \\
    v_{R_1}(t) = v_{R_2}(t).
\]

Repeating the same argument for the interval \( [t, 2t] \) and so on yields

\[
    u_{R_1}(t) = u_{R_2}(t), \\
    v_{R_1}(t) = v_{R_2}(t).
\]

for the whole interval \([0, \tau]\). According to this, we can know the above definition of local solution to (1) is well defined.

If \( \tau_{\infty} < +\infty \), the definition of \((u, v)\) yields \( \mathbb{P} \)-a.s.

\[
    \lim_{t \to \tau_{\infty}} \| (u, v) \|_{X_t} = +\infty,
\]

which shows that \((u, v)\) is a unique local solution to (1) on the interval \([0, \tau_{\infty}]\).

This completes the proof of Proposition 4. \( \square \)

3.2. Some a priori estimates of \((u^\varepsilon, v^\varepsilon)\).

**Proposition 5.** If \( u_0, v_0 \in L^2(\Omega) \), for \( \varepsilon \in (0, 1) \) and \( T > 0 \), \((u^\varepsilon, v^\varepsilon)\) is the unique solution to (1), then for any \( p > 0 \), there exists a constant \( C \) such that the solutions \((u^\varepsilon, v^\varepsilon)\) satisfy

\[
    \sup_{\varepsilon \in (0, 1)} \sup_{0 \leq t \leq T} \mathbb{E}\| u^\varepsilon(t) \|^{2p} \leq C, \\
    \sup_{\varepsilon \in (0, 1)} \sup_{0 \leq t \leq T} \mathbb{E}\| v^\varepsilon(t) \|^{2p} \leq C,
\]

where \( C \) is dependent of \( p, T, u_0, v_0 \) but independent of \( \varepsilon \in (0, 1) \).

**Proof.** For simplicity, we will omit the index \( \varepsilon \).

It is also suffice to prove this inequality holds when \( p \) is large enough.

\( \star \) We apply the Ito formula (see \([11, 35]\)) with \( \| v^\varepsilon \|^{2p} (p \geq 2) \) and obtain that

\[
    dv^2 = \frac{2p}{\varepsilon} \| v \|^{2p-2} v_{xx} + g(u, v)dv + \frac{p}{\varepsilon} \| v \|^{2p-2}TrQ_2 dt \\
    + \frac{2p(p - 1)}{\varepsilon} \| v \|^{2p-4}(v, dW_2)^2 + \frac{2p}{\sqrt{\varepsilon}} \| v \|^{2p-2}(v, dW_2),
\]

where...
by taking mathematical expectation from both sides of above equation, we have
\[
E\|v(t)\|^{2p} = E\|v(0)\|^{2p} + \frac{2p}{\varepsilon} E\int_0^t \|v(s)\|^{2p-2}(v_{xx}, v) ds + \frac{2p}{\varepsilon} E\int_0^t \|v(s)\|^{2p-2}(g(u, v), v) ds + \frac{p}{\varepsilon} E\int_0^t \|v(s)\|^{2p-2}TrQ_2 ds + 2p(p-1)E\int_0^t \|v(s)\|^{2p-4}\|Q_2^{1/2}v\|^2 ds.
\]
Then,
\[
\frac{d}{dt}E\|v(t)\|^{2p} = \frac{2p}{\varepsilon} E\|v(t)\|^{2p-2}(v_{xx}, v) + \frac{2p}{\varepsilon} E\|v(t)\|^{2p-2}(g(u, v), v)
\]
\[
+ \frac{p}{\varepsilon} E\|v(t)\|^{2p-2}TrQ_2 + \frac{2p(p-1)}{\varepsilon} E\|v(t)\|^{2p-4}\|Q_2^{1/2}v\|^2
\]
\[
= - \frac{2p}{\varepsilon} E\|v(t)\|^{2p-2}(v_x, v_x) + \frac{2p}{\varepsilon} E\|v(t)\|^{2p-2}(g(u, v), v)
\]
\[
+ \frac{p}{\varepsilon} E\|v(t)\|^{2p-2}TrQ_2 + \frac{2p(p-1)}{\varepsilon} E\|v(t)\|^{2p-4}\|Q_2^{1/2}v\|^2,
\]
it follows from the property of \(g\) that
\[
(g(u, v), v) \leq L_g \|v\|^2 + C|u|^2 + C,
\]
thus, we have
\[
\frac{d}{dt}E\|v(t)\|^{2p}
\]
\[
\leq - \frac{2p}{\varepsilon} E\|v(t)\|^{2p-2}(v_x, v_x) + \frac{2p}{\varepsilon} L_g E\|v(t)\|^{2p-2}\|v\|^2 + \frac{C}{\varepsilon} E\|v(t)\|^{2p-2}\|u(t)\|^2
\]
\[
+ \frac{C}{\varepsilon} E\|v(t)\|^{2p-2} + \frac{p}{\varepsilon} E\|v(t)\|^{2p-2}TrQ_2 + \frac{2p(p-1)}{\varepsilon} E\|v(t)\|^{2p-4}\|Q_2^{1/2}v\|^2
\]
\[
\leq - \frac{2p\lambda}{\varepsilon} E\|v(t)\|^{2p} + \frac{p}{\varepsilon} L_g E\|v(t)\|^{2p} + \frac{C}{\varepsilon} E\|v(t)\|^{2p-2}\|u(t)\|^2 + \frac{C}{\varepsilon} E\|v(t)\|^{2p-2},
\]
then, by using the Young inequality, we have
\[
\frac{d}{dt}E\|v(t)\|^{2p}
\]
\[
\leq - \frac{2p\lambda}{\varepsilon} E\|v(t)\|^{2p} + \frac{2p}{\varepsilon} L_g E\|v(t)\|^{2p} + \frac{2p}{\varepsilon} L_g E\|v(t)\|^{2p} + \frac{C(p, L_g)}{\varepsilon} E\|u(t)\|^{2p} + \frac{C}{\varepsilon}
\]
\[
= - \frac{2p\lambda}{\varepsilon} E\|v(t)\|^{2p} + \frac{4p}{\varepsilon} L_g E\|v(t)\|^{2p} + \frac{C}{\varepsilon} E\|u(t)\|^{2p} + \frac{C}{\varepsilon}
\]
\[
\leq - \frac{p\lambda}{\varepsilon} E\|v(t)\|^{2p} + \frac{C}{\varepsilon} E\|u(t)\|^{2p} + \frac{C}{\varepsilon}
\]
\[
\leq - \frac{p\lambda}{\varepsilon} E\|v(t)\|^{2p} + \frac{C}{\varepsilon} E\|u(t)\|^{2p} + \frac{C}{\varepsilon},
\]
hence, by comparison theorem
\[
E\|v(t)\|^{2p} \leq E\|v(0)\|^{2p} e^{-\frac{p\lambda t}{\varepsilon}} + \frac{C}{\varepsilon} \int_0^t e^{-\frac{p\lambda (t-s)}{\varepsilon}} (E\|u(s)\|^{2p} + 1) ds. \tag{15}
\]

\* We apply the Itô formula (see [11, 35]) with \(\|u^c\|^{2p}\) and obtain that
\[
d\|u\|^{2p} = 2p\|u\|^{2p-2}(-u_{xxx} - u_{xx} - u_{x} + f(u, v), u) dt + p\|u\|^{2p-2}TrQ_1 dt + 2p(p-1)\|u\|^{2p-4}(u, dW_1)^2 + 2p\|u\|^{2p-2}(u, dW_1),
\]
by taking mathematical expectation from both sides of above equation, we have
\[ E\|u(t)\|^{2p} = E\|u(0)\|^{2p} + 2pE \int_0^t \|u(s)\|^{2p-2}(-u_{xxxx} - u_{xx} + f(u, v), u)ds \]
\[ + pE \int_0^t \|u(s)\|^{2p-2}TrQ_1ds + 2p(p-1)E \int_0^t \|u(s)\|^{2p-4}\|Q_1^2 u\|^2 ds \]
\[ = E\|u(0)\|^{2p} + 2pE \int_0^t \|u(s)\|^{2p-2}(-u_{xxxx} - u_{xx} + f(u, v), u)ds \]
\[ + pE \int_0^t \|u(s)\|^{2p-2}TrQ_1ds + 2p(p-1)E \int_0^t \|u(s)\|^{2p-4}\|Q_1^2 u\|^2 ds, \]
then
\[ \frac{d}{dt} E\|u(t)\|^{2p} = 2pE\|u(t)\|^{2p-2}(-u_{xxxx} - u_{xx} + f(u, v), u) \]
\[ + pE\|u(t)\|^{2p-2}TrQ_1 + 2p(p-1)E\|u(t)\|^{2p-4}\|Q_1^2 u\|^2 \]
\[ = 2pE\|u(t)\|^{2p-2}[-\|u_{xx}\|^2 + \|u_x\|^2 + (f(u, v), u)] \]
\[ + pE\|u(t)\|^{2p-2}TrQ_1 + 2p(p-1)E\|u(t)\|^{2p-4}\|Q_1^2 u\|^2 \]
\[ \leq 2pE\|u(t)\|^{2p-2}(-\|u_{xx}\|^2 + \frac{1}{2}\|u_x\|^2 + \|u\|^2 + \|v\|^2 + C) \]
\[ + pE\|u(t)\|^{2p-2}TrQ_1 + 2p(p-1)E\|u(t)\|^{2p-4}\|Q_1^2 u\|^2 \]
\[ \leq 2pE\|u(t)\|^{2p-2}(-\|u_{xx}\|^2 + \|u\|^2 + \|v\|^2 + C) \]
\[ + pE\|u(t)\|^{2p-2}TrQ_1 + 2p(p-1)E\|u(t)\|^{2p-4}\|Q_1^2 u\|^2, \]
thus, it follows from Young inequality that
\[ \frac{d}{dt} E\|u(t)\|^{2p} \leq C(\|u(t)\|^{2p} + E\|v(t)\|^{2p} + 1), \]
hence, by comparison theorem
\[ E\|u(t)\|^{2p} \leq e^{Ct}E\|u(0)\|^{2p} + C \int_0^t e^{C(t-s)}(E\|v(s)\|^{2p} + 1)ds. \]
Plug this inequality into (15), we have
\[ E\|v(t)\|^{2p} \leq E\|v(0)\|^{2p}e^{-\frac{\alpha t}{2}} + C \int_0^t e^{-\frac{\alpha}{2}(t-s)}(E\|u(s)\|^{2p} + 1)ds \]
\[ \leq C(1 + \|v(0)\|^{2p}) + C \int_0^t e^{-\frac{\alpha}{2}(t-s)}E\|u(s)\|^{2p} ds \]
\[ \leq C(1 + \|v(0)\|^{2p}) \]
\[ + \frac{C}{\alpha} \int_0^t e^{-\frac{\alpha}{2}(t-s)}[e^{Cs}\|u(0)\|^{2p} + C \int_0^s e^{C(s-\tau)}(E\|v(\tau)\|^{2p} + 1)d\tau]ds \]
\[ \leq C(1 + \|u(0)\|^{2p} + \|v(0)\|^{2p}) + \frac{C}{\alpha} \int_0^t e^{-\frac{\alpha}{2}(t-s)} \int_0^s E\|v(\tau)\|^{2p} d\tau ds \]
\[ \leq C(1 + \|u(0)\|^{2p} + \|v(0)\|^{2p}) + \frac{C}{\alpha p} \int_0^t (1 - e^{-\frac{\alpha}{2}(t-\tau)})E\|v(\tau)\|^{2p} d\tau \]
\[ \leq C(1 + \|u(0)\|^{2p} + \|v(0)\|^{2p}) + C \int_0^t E\|v(\tau)\|^{2p} d\tau, \]
it follows from Gronwall inequality that
\[ \sup_{0 \leq t \leq T} \mathbb{E} \| v(t) \|^{2p} \leq C(\| u_0 \|^{2p} + \| v_0 \|^{2p} + 1), \]
thus, we have
\[ \sup_{0 \leq t \leq T} \mathbb{E} \| u(t) \|^{2p} \leq C(\| u_0 \|^{2p} + \| v_0 \|^{2p} + 1). \]

\[ \square \]

**Proposition 6.** Let \( \tau = \tau_{\infty} \wedge T. \) For any \( \varepsilon \in (0, 1), \) if \( u_0 \in H_0^1(I), v_0 \in L^2(I), \)
\((u^\varepsilon, v^\varepsilon)\) is the unique solution to (1), then for any \( p > 0, \) there exist constants \( C_1, C_2 \)
such that the solution \((u^\varepsilon, v^\varepsilon)\) satisfies
\[ \mathbb{E} \sup_{0 \leq t \leq \tau} \| u^\varepsilon(t) \|_{H^1}^{2p} \leq C_1, \]
\[ \mathbb{E} \sup_{0 \leq t \leq \tau} \| v^\varepsilon(t) \|^{2p} \leq C_2, \]
where \( C_1 \) is dependent of \( p, T, u_0, v_0 \) and \( C_2 \) is dependent of \( \varepsilon, p, T, u_0, v_0. \)

**Proof.** It is also suffice to prove these inequalities hold when \( p \) is large enough.
1) Noting
\[ u(t) = S_1(t)u_0 + \int_0^t S_1(t-s)(-u_{xx}(s) - u(s)u_x(s) + f(u(s), v(s)))ds + \int_0^t S_1(t-s)dW_1. \]
\[ \text{• Estimate} \]
\[ \mathbb{E} \sup_{0 \leq t \leq \tau} \| S_1(t)u_0 \|_{H^1}^{2p} \leq C(T)\| u_0 \|_{H^1}^{2p}. \]
Indeed,
\[ \mathbb{E} \sup_{0 \leq t \leq \tau} \| S_1(t)u_0 \|_{H^1}^{2p} \leq C(T)\mathbb{E}\| u_0 \|_{H^1}^{2p} = C(T)\| u_0 \|_{H^1}^{2p}. \]
\[ \text{• Estimate} \]
\[ \mathbb{E} \sup_{0 \leq t \leq \tau} \| \int_0^t S_1(t-s)dW_1 \|_{H^1}^{2p} \leq C(p, T). \]
Indeed, this can be obtained from Proposition 2.
\[ \text{• Estimate} \]
\[ \mathbb{E} \sup_{0 \leq t \leq \tau} \| \int_0^t S_1(t-s)(-u(s)u_x(s))ds \|_{H^1}^{2p} \]
\[ \leq C(T)(1 + \| u_0 \|_{L^6}^{6p} + \| v_0 \|_{L^6}^{6p} + \mathbb{E} \int_0^\tau \| u(s) \|_{H^1}^{2p}ds) \]
Indeed, it follows from the Agmon's inequality, Gagliardo-Nirenberg inequality that
\[ \| \int_0^t S_1(t-s)(-u(s)u_x(s))ds \|_{H^1}^{2p} \leq \int_0^t \| S_1(t-s)(-u(s)u_x(s)) \|_{H^1}^{2p}ds \]
\[ \leq \int_0^t (t-s)^{-\frac{1}{2}} \| u^2(s) \|_{L^2}ds)^{2p} \]
\[ \begin{align*}
&\leq (\int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^p} ds)^{2p} \\
&\leq (\int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{H_1} \|u(s)\|_{H_1} ds)^{2p} \\
&= (\int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{H_1} \|u(s)\|_{H_1} ds)^{2p}, \\
\end{align*} \]

thus,

\[ \begin{align*}
\mathbb{E} \sup_{0 \leq t \leq \tau} \| \int_0^t S_1(t-s)(-u(s)u_x(s)) ds \|_{H_1}^{2p} \leq \mathbb{E} \sup_{0 \leq t \leq \tau} (\int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{H_1} \|u(s)\|_{H_1} ds)^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq \tau} \left( \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{H_1} \|u(s)\|_{H_1} \|u(s)\|_{H_1} ds \right) \\
&\leq C(T) \mathbb{E} \int_0^\tau \|u(s)\|_{H_1}^{2p} ds + \mathbb{E} \int_0^\tau \|u(s)\|_{H_1}^{6p} ds \leq C(T) \left( 1 + \|u_0\|^{2p} + \|v_0\|^{2p} + \int_0^\tau \|u(s)\|_{H_1}^{2p} ds \right). \\
\end{align*} \]

**Estimate**

\[ \begin{align*}
\mathbb{E} \sup_{0 \leq t \leq \tau} \| \int_0^t S_1(t-s)(-u_{xx}(s)) ds \|_{H_1}^{2p} \leq C(T) \left( 1 + \|u_0\|^{2p} + \mathbb{E} \int_0^\tau \|u(s)\|_{H_1}^{2p} ds \right). \\
\end{align*} \]

Indeed,

\[ \begin{align*}
\| \int_0^t S_1(t-s)(-u_{xx}(s)) ds \|_{H_1}^{2p} &\leq (\int_0^t \|S_1(t-s)(-u_{xx}(s))\|_{H_1} ds)^{2p} \\
&\leq C \left( \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{H_1} ds \right)^{2p},
\end{align*} \]

thus, it follows from this that

\[ \begin{align*}
\mathbb{E} \sup_{0 \leq t \leq \tau} \| \int_0^t S_1(t-s)(-u_{xx}(s)) ds \|_{H_1}^{2p} &\leq C(T) \left( 1 + \|u_0\|^{2p} + \mathbb{E} \int_0^\tau \|u(s)\|_{H_1}^{2p} ds \right).
\end{align*} \]

**Estimate**

\[ \begin{align*}
\mathbb{E} \sup_{0 \leq t \leq \tau} \| \int_0^t S_1(t-s)f(u(s), v(s)) ds \|_{H_1}^{2p} &\leq C(T) \left( 1 + \|u_0\|^{2p} + \|v_0\|^{2p} \right).
\end{align*} \]
Indeed,

\[
\| \int_0^t S_1(t-s)f(u(s), v(s))ds \|_{H^1}^{2p} \leq \left( \int_0^t \| S_1(t-s)f(u(s), v(s)) \|_{H^1} ds \right)^{2p} \\
\leq \left( \int_0^t (t-s)^{-\frac{1}{4}} \| f(u(s), v(s)) \| ds \right)^{2p} \\
\leq C \left( \int_0^t (t-s)^{-\frac{1}{4}} (1 + \| u(s) \| + \| v(s) \|) ds \right)^{2p},
\]

thus,

\[
E \sup_{0 \leq t \leq \tau} \| \int_0^t S_1(t-s)f(u(s), v(s))ds \|_{H^1}^{2p} \\
\leq C E \sup_{0 \leq t \leq \tau} \left( \int_0^t (t-s)^{-\frac{1}{4}} (1 + \| u(s) \| + \| v(s) \|) ds \right)^{2p} \\
\leq C E \left[ \sup_{0 \leq t \leq \tau} \left( \int_0^t (t-s)^{-\frac{1}{4}} \frac{2p}{2p-1} ds \right)^{2p-1} \cdot \sup_{0 \leq t \leq \tau} \left( \int_0^t (1 + \| u(s) \| + \| v(s) \|)^{2p} ds \right) \right] \\
\leq C \sup_{0 \leq t \leq \tau} \left( \int_0^t (t-s)^{-\frac{1}{4}} \frac{2p}{2p-1} ds \right)^{2p-1} \cdot E \left[ \sup_{0 \leq t \leq \tau} \left( \int_0^t (1 + \| u(s) \|^{2p} + \| v(s) \|^{2p}) ds \right) \right] \\
\leq C(1 + E \int_0^\tau \| u(s) \|^{2p} ds + E \int_0^\tau \| v(s) \|^{2p} ds) \\
\leq C(1 + \| u_0 \|^{2p} + \| v_0 \|^{2p}).
\]

With the help of the above estimates, we arrive at

\[
E \sup_{0 \leq t \leq \tau} \| u(t) \|_{H^1}^{2p} \leq C(p, T)(1 + \| u_0 \|_{H^1}^{2p} + \| u_0 \|_{H^1}^{6p} + \| v_0 \|_{H^1}^{6p} + E \int_0^\tau \| u(s) \|_{H^1}^{2p} ds),
\]

by applying Gronwall inequality and Young inequality, we have

\[
E \sup_{0 \leq t \leq \tau} \| u(t) \|_{H^1}^{2p} \leq C(p, T)(1 + \| u_0 \|_{H^1}^{2p} + \| u_0 \|_{H^1}^{6p} + \| v_0 \|_{H^1}^{6p}).
\]

2) Noting

\[
v(t) = S_2(\frac{t}{\varepsilon})v_0 + \frac{1}{\varepsilon} \int_0^t S_2(\frac{t-s}{\varepsilon})g(u(s), v(s))ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t S_2(\frac{t-s}{\varepsilon})dW_2.
\]

- Estimate

\[
E \sup_{0 \leq t \leq \tau} \| S_2(\frac{t}{\varepsilon})v_0 \|^{2p} \leq C(T) \| v_0 \|^{2p}.
\]

- Estimate

\[
E \sup_{0 \leq t \leq \tau} \| \frac{1}{\sqrt{\varepsilon}} \int_0^t S_2(\frac{t-s}{\varepsilon})dW_2 \|^{2p} \leq C(\varepsilon, p, T).
\]

- Estimate

\[
E \sup_{0 \leq t \leq \tau} \| \frac{1}{\varepsilon} \int_0^t S_2(\frac{t-s}{\varepsilon})g(u(s), v(s))ds \|^{2p} \leq C(\varepsilon, p, T)(1 + \| u_0 \|^{2p} + \| v_0 \|^{2p}).
\]
Indeed,
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \frac{1}{\varepsilon} \int_0^t S_2 \left( \frac{t-s}{\varepsilon} \right) g(u(s), v(s)) ds \right\|^{2p} \leq C(\varepsilon, p, T) \mathbb{E} \sup_{0 \leq t \leq T} (1 + \|u(s)\| + \|v(s)\|) ds^{2p} \leq C(\varepsilon, p, T)(1 + \|u_0\|^{2p} + \|v_0\|^{2p}).
\]

With the help of the above estimates, we arrive at
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|v(t)\|^{2p} \leq C(\varepsilon, p, T)(1 + \|u_0\|^{2p} + \|v_0\|^{2p}).
\]

This completes the proof of Proposition 6.

\[\square\]

3.3. **Proof of Theorem 3.2.** Now, we prove Theorem 3.2.

**Proof of Theorem 3.2.** By the Chebyshev inequality, Proposition 6 and the definition of \((u, v)\), we have
\[
\mathbb{P} \{\omega \in \Omega| \tau_\infty(\omega) < +\infty\} = \lim_{T \to +\infty} \mathbb{P} \{\omega \in \Omega| \tau_\infty(\omega) \leq T\} = \lim_{T \to +\infty} \lim_{R \to +\infty} \mathbb{P} \{\omega \in \Omega| \tau R(\omega) \leq \tau(\omega)\}\]
\[
= \lim_{T \to +\infty} \lim_{R \to +\infty} \mathbb{P} \{\omega \in \Omega| |(u^\varepsilon, v^\varepsilon)|\|X_\varepsilon \geq \|u^\varepsilon, v^\varepsilon\||X_\varepsilon \geq R\}\]
\[
= \lim_{T \to +\infty} \lim_{R \to +\infty} \frac{\mathbb{E} |(u^\varepsilon, v^\varepsilon)|^2}{R^2} = 0,
\]

this shows that
\[
\mathbb{P} \{\omega \in \Omega| \tau_\infty(\omega) = +\infty\} = 1,
\]

namely, \(\tau_\infty = +\infty\) \(P\)-a.s. \[\square\]

4. **Proof Theorem 1.1.** The proof of Theorem 1.1 is divided into several steps.

4.1. **Some a priori estimates of** \(u^\varepsilon\). By the same method as in Proposition 6, we can obtain:

**Proposition 7.** For any \(u_0 \in H^2(I) \cap H_0^1(I), v_0 \in L^2(I), T > 0\) and \(p > 0\), there exists a positive constant \(C(p, T)\) independent of \(\varepsilon\), such that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{H^2}^{2p} \leq C(p, T)(1 + \|u_0\|_{H^2}^{2p} + \|u_0\|_{H^2}^{6p} + \|v_0\|_{H^2}^{6p}).
\]

4.2. **Well-posedness for the averaged equation (2).** By the same method as in Theorem 3.2 and Proposition 6, we can obtain the following proposition.

**Proposition 8.** If \(u_0 \in H^2(I) \cap H_0^1(I)\), the averaged equation

\[
\left\{ \begin{array}{ll}
\dot{u}(t) = (\mathbf{a}_{xxx} - \mathbf{a}_{xx} - \mathbf{a}_{x} + \mathbf{f}(\mathbf{u})) dt + dW_1 & \text{in } Q \\
\mathbf{u}(0, t) = \mathbf{u}(1, t) & \text{in } (0, T) \\
\mathbf{u}(0, t) = \mathbf{u}(1, t) & \text{in } (0, T) \\
\mathbf{u}(x, 0) = \mathbf{u}_0(x) & \text{in } I,
\end{array} \right.
\]

solves the averaged equation.
Proposition 9.

For any $\bar{u} \in L^2(\Omega, C([0, T]; H^2(I) \cap H^1_0(I)))$, where

$$\bar{u}(t) = \int_{L^2(I)} f(u, v)dv, \quad u \in L^2(I).$$

Moreover, for any $p > 0$, there exists a constant $C$ such that the solution $\bar{u}$ satisfies

$$E\sup_{0 \leq t \leq T} \|\bar{u}(t)\|^{2p}_{H^2} \leq C,$$

where $C$ dependent of $p, T, u_0$.

4.3. Hölder continuity of time variable for $u^\varepsilon$. Next, we are going to provide a Hölder continuity of time variable for $u^\varepsilon$.

**Proposition 9.** For any $u_0 \in H^2(I) \cap H^1_0(I), v_0 \in L^2(I)$, there exists a constant $C(p, T)$ such that

$$E\|u^\varepsilon(t + h) - u^\varepsilon(t)\|^{2p} \leq C(p, T)h^p$$

for any $t \in [0, T], h > 0$.

**Proof.** Let us write

$$u^\varepsilon(t + h) - u^\varepsilon(t) = (S_1(h) - Id)u^\varepsilon(t) + \int_t^{t+h} S_1(t + h - s)(-u^\varepsilon_{xx} - u^\varepsilon u^\varepsilon_x + f(u^\varepsilon, v^\varepsilon))ds$$

$$+ \int_t^{t+h} S_1(t + h - s)dW_1,$$

here $Id$ denotes the identity operator.

* Due to [33], there is a $C$ such that for all $x \in H^2(I)$,

$$\|(S_1(h) - Id)x\| \leq Ch^2\|x\|_{H^2},$$

and then, according to the above estimate and Proposition 7, we have

$$E\|(S_1(h) - Id)u^\varepsilon(t)\|^{2p} \leq Ch^pE\|u^\varepsilon(t)\|^{2p}_{H^2} \leq Ch^p.$$

* Noting that

$$E\|\int_t^{t+h} S_1(t + h - s)u^\varepsilon_{xx}(s)ds\|^{2p} \leq CE\left(\int_t^{t+h} \|S_1(t + h - s)u^\varepsilon_{xx}(s)\|ds\right)^{2p}$$

$$\leq CE\left(\int_t^{t+h} \|u^\varepsilon_{xx}(s)\|ds\right)^{2p}$$

$$\leq CE\left(\int_t^{t+h} \|u^\varepsilon(s)\|_{H^2}ds\right)^{2p}$$

$$\leq CE\left(\int_t^{t+h} 1ds\right)^{2p-1} \cdot \int_t^{t+h} \|u^\varepsilon(s)\|^{2p}_{H^2}ds$$

$$= C(\int_t^{t+h} 1ds)^{2p-1} \cdot E\left[\int_t^{t+h} \|u^\varepsilon(s)\|^{2p}_{H^2}ds\right]$$

$$= Ch^{2p-1} \cdot \int_t^{t+h} E\|u^\varepsilon(s)\|^{2p}_{H^2}ds$$

$$\leq Ch^{2p}.$$
**4.4. Auxiliary process** $(\hat{u}^\varepsilon, \hat{v}^\varepsilon)$. Next, we introduce an auxiliary process $(\hat{u}^\varepsilon, \hat{v}^\varepsilon) \in L^2(I) \times L^2(I)$ by Khasminskii in [26].

Fix a positive number $\delta$ and do a partition of time interval $[0, T]$ of size $\delta$. We construct a process $\hat{v}^\varepsilon \in L^2(I)$ by means of the equation

$$
\hat{v}^\varepsilon(t) = v^\varepsilon(k\delta) + \frac{1}{\varepsilon} \int_{k\delta}^{t} (\hat{v}^\varepsilon_{ss}(s) + g(u^\varepsilon(k\delta), \hat{v}^\varepsilon(s)))ds + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{t} dW_2
$$

* Noting that

$$
\mathbb{E}\| \int_{t}^{t+h} S_1(t + h - s)(u^\varepsilon_{u^\varepsilon})(s)ds\|^{2p}
\leq \mathbb{E}(\int_{t}^{t+h} \|S_1(t + h - s)((u^\varepsilon_{u^\varepsilon})(s))\|ds)^{2p}
\leq \mathbb{E}(\int_{t}^{t+h} \|(u^\varepsilon_{u^\varepsilon})(s)\|ds)^{2p}
\leq \mathbb{E}(\int_{t}^{t+h} 1ds)^{2p-1} \cdot \int_{t}^{t+h} \|(u^\varepsilon_{u^\varepsilon})(s)\|^{2p}ds
\leq h^{2p-1}\mathbb{E}(\int_{t}^{t+h} \|u^\varepsilon(s)\|^2 ||u^\varepsilon(s)\|_{L^\infty}^{2p}ds)
\leq h^{2p-1}\mathbb{E}(\int_{t}^{t+h} \|u^\varepsilon(s)\|_{H^1}^2 \|u^\varepsilon(s)\|_{H^1}^p \|u^\varepsilon(s)\|^p ds)
\leq h^{2p-1}h^{6p} \mathbb{E}(\int_{t}^{t+h} \|u^\varepsilon(s)\|_{H^1}^{6p}ds + \int_{t}^{t+h} \|u^\varepsilon(s)\|^{2p}ds)
\leq Ch^{2p}.

* Noting that

$$
\mathbb{E}\| \int_{t}^{t+h} S_1(t + h - s)f(u^\varepsilon, v^\varepsilon)(s)ds\|^{2p}
\leq \mathbb{E}(\int_{t}^{t+h} \|S_1(t + h - s)f(u^\varepsilon, v^\varepsilon)(s)\|ds)^{2p}
\leq \mathbb{E}(\int_{t}^{t+h} \|f(u^\varepsilon, v^\varepsilon)(s)\|ds)^{2p}
\leq \mathbb{E}(\int_{t}^{t+h} 1ds)^{2p-1} \cdot \int_{t}^{t+h} \|f(u^\varepsilon, v^\varepsilon)(s)\|^{2p}ds
\leq Ch^{2p-1} \mathbb{E}(\int_{t}^{t+h} (1 + \|u^\varepsilon\|^{2p} + \|v^\varepsilon\|^{2p})ds)
\leq Ch^{2p}.

* In view of the Burkholder-Davis-Gundy inequality and Hölder inequality, it yields

$$
\mathbb{E}\| \int_{t}^{t+h} S_1(t + h - s)dW_1\|^{2p} \leq Ch^{p}.
$$

* With the help of the above estimates, we arrive at (16). □
for $t \in [k\delta, \min\{(k + 1)\delta, T\})$, $k \geq 0$. Also define the process $\hat{u}^\varepsilon \in L^2(I)$ by

$$\hat{u}^\varepsilon(t) = u_0 + \int_0^t (-\hat{u}^\varepsilon_{xxx} - \hat{u}^\varepsilon_{xx} - u^\varepsilon(t_s)u^\varepsilon_x(t_s) + f(u^\varepsilon(t_s), \hat{v}^\varepsilon))ds + \int_0^t dW_1$$

for $t \in [0, T]$, where $t_s = s(\delta) = \lfloor \frac{s}{\delta} \rfloor \delta$ is the nearest breakpoint proceeding $s$ and $[\cdot]$ is the integer function.

Thus $(\hat{u}^\varepsilon, \hat{v}^\varepsilon)$ satisfies

\[
\begin{align*}
&\begin{cases} 
    d\hat{u}^\varepsilon(t) = (-\hat{u}^\varepsilon_{xxx} - \hat{u}^\varepsilon_{xx} - u^\varepsilon(t_s)u^\varepsilon_x(t_s) + f(u^\varepsilon(t_s), \hat{v}^\varepsilon))dt + dW_1 \\
    \hat{u}^\varepsilon(0, t) = 0 = \hat{\hat{u}}^\varepsilon(1, t) \\
    \hat{u}^\varepsilon(t, 0) = 0 = \hat{\hat{u}}^\varepsilon(t, 1) \\
    \hat{v}^\varepsilon(t, 0) = 0 = \hat{\hat{v}}^\varepsilon(t, 1) \\
    \hat{u}^\varepsilon(x, k\delta) = u(x, k\delta) \quad & \text{in } I
\end{cases} \\
&\begin{cases} 
    d\hat{v}^\varepsilon(t) = \frac{1}{\varepsilon} (\hat{v}^\varepsilon_{xx} + g(\varepsilon^2(k\delta), \hat{v}^\varepsilon))dt + \frac{1}{\sqrt{\varepsilon}} dW_2 \\
    \hat{v}^\varepsilon(0, t) = 0 = \hat{\hat{v}}^\varepsilon(1, t) \\
    \hat{v}^\varepsilon(t, 0) = 0 = \hat{\hat{v}}^\varepsilon(t, 1) \\
    \hat{v}^\varepsilon(x, k\delta) = v(x, k\delta) \quad & \text{in } I
\end{cases}
\end{align*}
\]

(17)

By the same method as in Proposition 5, Proposition 6 and Proposition 7, it holds that

**Proposition 10.** If $u_0 \in H^2(I) \cap H^1_0(I)$, $v_0 \in L^2(I)$, for $\varepsilon \in (0, 1)$, $(\hat{u}^\varepsilon, \hat{v}^\varepsilon)$ is the unique solution to (17), then for any $p > 0$, there exists a constant $C$ such that the solution $(\hat{u}^\varepsilon, \hat{v}^\varepsilon)$ satisfies

$$\sup_{\varepsilon \in (0, 1)} \sup_{0 \leq t \leq T} \mathbb{E}\|\hat{u}^\varepsilon(t)\|^{2p} \leq C,$$

$$\sup_{\varepsilon \in (0, 1)} \sup_{0 \leq t \leq T} \mathbb{E}\|\hat{v}^\varepsilon(t)\|^{2p} \leq C,$$

$$\sup_{\varepsilon \in (0, 1)} \mathbb{E} \sup_{0 \leq t \leq T} \|\hat{u}^\varepsilon(t)\|^{2p}_{H^2} \leq C,$$

where $C$ is dependent of $p, T, u_0, v_0$ but independent of $\varepsilon \in (0, 1)$.

4.5. The errors of $u^\varepsilon - \hat{u}^\varepsilon$ and $v^\varepsilon - \hat{v}^\varepsilon$.

**Proposition 11.** There exists a constant $C$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}\|\hat{v}^\varepsilon(t) - \hat{v}^\varepsilon(t)\|^{2p} \leq C \frac{\delta^{p+1}}{\varepsilon},$$

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\hat{u}^\varepsilon(t) - \hat{u}^\varepsilon(t)\|^{2p} \leq C(\frac{\delta^{p+1}}{\varepsilon} + \delta^p).$$

**Proof.** We prove the first inequality.

Indeed, we have

\[
\begin{align*}
&\begin{cases} 
    du^\varepsilon = \frac{1}{\varepsilon} (v^\varepsilon_{xx} + g(u^\varepsilon, v^\varepsilon))dt + \frac{1}{\sqrt{\varepsilon}} dW_2 \\
    dv^\varepsilon = \frac{1}{\varepsilon} (\hat{v}^\varepsilon_{xx} + g(\varepsilon^2(k\delta), \hat{v}^\varepsilon))dt + \frac{1}{\sqrt{\varepsilon}} dW_2,
\end{cases} \quad & \text{in } Q
\end{align*}
\]

it is easy to see that $v^\varepsilon - \hat{v}^\varepsilon$ satisfies the following SPDE

\[
\begin{align*}
&\begin{cases} 
    (v^\varepsilon - \hat{v}^\varepsilon) = \frac{1}{\varepsilon} (v^\varepsilon_{xx} - \hat{v}^\varepsilon_{xx} + g(u^\varepsilon, v^\varepsilon) - g(u^\varepsilon(k\delta), \hat{v}^\varepsilon))dt \\
    (v^\varepsilon - \hat{v}^\varepsilon)(0, t) = 0 = (\hat{v}^\varepsilon - \hat{v}^\varepsilon)(1, t) \\
    (v^\varepsilon - \hat{v}^\varepsilon)(x, 0) = 0 \\
\end{cases} \quad & \text{in } (0, T)
\end{align*}
\]

(18)
For $t \in [0, T]$ with $t \in [k\delta, (k+1)\delta)$, applying Itô formula to (18)

$$
\|(v^\varepsilon - \hat{v}^\varepsilon)(t)\|^2_p
$$

$$
= 2p \int_{k\delta}^{t} \|(v^\varepsilon - \hat{v}^\varepsilon)(s)\|^{2p-2}(v^\varepsilon - \hat{v}^\varepsilon, \frac{1}{\varepsilon}[(v^\varepsilon - \hat{v}^\varepsilon)_x + g(u^\varepsilon, v^\varepsilon) - g(u^\varepsilon(k\delta), \hat{v}^\varepsilon)])ds
$$

$$
= 2p \int_{k\delta}^{t} \|(v^\varepsilon - \hat{v}^\varepsilon)(s)\|^{2p-2}(v^\varepsilon - \hat{v}^\varepsilon, (v^\varepsilon - \hat{v}^\varepsilon)_x + g(u^\varepsilon, v^\varepsilon) - g(u^\varepsilon(k\delta), \hat{v}^\varepsilon))ds
$$

$$
= - \frac{2p}{\varepsilon} \int_{k\delta}^{t} \|(v^\varepsilon - \hat{v}^\varepsilon)(s)\|^{2p-2}(v^\varepsilon - \hat{v}^\varepsilon)_x^2 ds
$$

By taking mathematical expectation from both sides of above equation, we have

$$
\mathbb{E}\|(v^\varepsilon - \hat{v}^\varepsilon)(t)\|^2_p
$$

$$
= - \frac{2p}{\varepsilon} \mathbb{E} \int_{k\delta}^{t} \|(v^\varepsilon - \hat{v}^\varepsilon)(s)\|^{2p-2}(v^\varepsilon - \hat{v}^\varepsilon)_x^2 ds
$$

thus, we have

$$
\frac{d}{dt} \mathbb{E}\|(v^\varepsilon - \hat{v}^\varepsilon)(t)\|^2_p
$$

$$
= - \frac{2p}{\varepsilon} \mathbb{E}\|(v^\varepsilon - \hat{v}^\varepsilon)(t)\|^{2p-2}(v^\varepsilon - \hat{v}^\varepsilon)_x^2
$$

$$
+ \frac{2p}{\varepsilon} \mathbb{E}\|(v^\varepsilon - \hat{v}^\varepsilon)(t)\|^{2p-2}(v^\varepsilon - \hat{v}^\varepsilon, g(u^\varepsilon, v^\varepsilon) - g(u^\varepsilon(k\delta), \hat{v}^\varepsilon))
$$

$$
\leq - \frac{2p}{\varepsilon} \mathbb{E}\|(v^\varepsilon - \hat{v}^\varepsilon)(t)\|^{2p-2}(v^\varepsilon - \hat{v}^\varepsilon)_x^2
$$

$$
+ \frac{2p}{\varepsilon} \mathbb{E}\|(v^\varepsilon - \hat{v}^\varepsilon)(t)\|^{2p-2}\int_{t}^{1} |v^\varepsilon - \hat{v}^\varepsilon| : L_\delta(|u^\varepsilon - u^\varepsilon(k\delta)| + |v^\varepsilon - \hat{v}^\varepsilon|)dx
$$

$$
\leq - \frac{2p\lambda}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^2 + \frac{2pL_\delta}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^{2p-2}\|v^\varepsilon - \hat{v}^\varepsilon\|(|u^\varepsilon - u^\varepsilon(k\delta)| + \|v^\varepsilon - \hat{v}^\varepsilon\|)
$$

$$
\leq - \frac{2p\lambda}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^2 + \frac{2pL_\delta}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^{2p-1}(|u^\varepsilon - u^\varepsilon(k\delta)| + \|v^\varepsilon - \hat{v}^\varepsilon\|)
$$

$$
= - \frac{2p\lambda}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^2 + \frac{2pL_\delta}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^{2p} + \frac{2pL_\delta}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^{2p-1}\|u^\varepsilon - u^\varepsilon(k\delta)\|
$$

$$
= - \frac{2p\lambda}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^2 + \frac{4pL_\delta}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^{2p} + \frac{C}{\varepsilon} \mathbb{E}\|u^\varepsilon - u^\varepsilon(k\delta)\|^{2p}
$$

$$
\leq - \frac{p(2\lambda - 4L_\delta)}{\varepsilon} \mathbb{E}\|v^\varepsilon - \hat{v}^\varepsilon\|^2 + \frac{C}{\varepsilon} \mathbb{E}\|u^\varepsilon - u^\varepsilon(k\delta)\|^{2p}
$$

Due to Proposition 9, it holds that

$$
\frac{d}{dt} \mathbb{E}\|(v^\varepsilon - \hat{v}^\varepsilon)(t)\|^2_p \leq - \frac{\alpha p}{\varepsilon} \mathbb{E}\|(v^\varepsilon - \hat{v}^\varepsilon)(t)\|^2_p + \frac{C}{\varepsilon} \delta^p.
$$
Hence, by applying Lemma 2.3 with $E$ as in (18), we have
\[
\mathbb{E} \left( \| (v^\varepsilon - \tilde{v}^\varepsilon)(t) \|^{2p} \right) \leq \int_{k\delta}^{t} e^{-\frac{2p}{p}(t-s)} \frac{C}{\varepsilon} \delta^p ds
\]
\[
= \frac{C}{\varepsilon} \delta^p \int_{k\delta}^{t} e^{-\frac{2p}{p}(t-s)} ds
\]
\[
= \frac{C}{\varepsilon} \delta^p (t-k\delta) \leq \frac{C}{\varepsilon} \delta^{p+1}.
\]

We prove the second inequality. Indeed, we have
\[
\left\{ \begin{array}{l}
du^\varepsilon + (u_{xx}^\varepsilon + u_{x}^\varepsilon + u_{x}^\varepsilon)dt = f(u^\varepsilon, v^\varepsilon) dt + dW_1, \\
d\hat{u}^\varepsilon + (\hat{u}_{xxx}^\varepsilon + \hat{u}_{xx}^\varepsilon + \hat{u}(t_s)u_{x}^\varepsilon(t_s))dt = f(u^\varepsilon(t_s), \hat{v}^\varepsilon)) dt + dW_1,
\end{array} \right.
\]

then,
\[
(u^\varepsilon(t) - \hat{u}^\varepsilon(t)) = \int_{0}^{t} S_1(t-s)[-(u_{xx}^\varepsilon - \hat{u}_{xx}^\varepsilon) - u_x^\varepsilon + u_x^\varepsilon(t_s)u_{x}^\varepsilon(t_s) + f(u^\varepsilon, v^\varepsilon) - f(u^\varepsilon(t_s), \hat{v}^\varepsilon)] ds.
\]

It follows from Proposition 1 that
\[
\| \int_{0}^{t} S_1(t-s)[-u_x^\varepsilon - \hat{u}_x^\varepsilon]ds \|^{2p} \leq (\int_{0}^{t} \| S_1(t-s)[-u_x^\varepsilon - \hat{u}_x^\varepsilon] \| ds)^{2p}
\]
\[
\leq (\int_{0}^{t} (t-s)^{-\frac{1}{2}} \| u^\varepsilon - \hat{u}^\varepsilon \| ds)^{2p}
\]
\[
\leq (\int_{0}^{t} (t-s)^{-\frac{1}{2}} \| u^\varepsilon - \hat{u}^\varepsilon \| ds)^{2p-1} \cdot \int_{0}^{t} \| u^\varepsilon - \hat{u}^\varepsilon \|^{2p} ds
\]
\[
\leq C(T) \int_{0}^{t} \| u^\varepsilon - \hat{u}^\varepsilon \|^{2p} ds.
\]

It follows from (3) that
\[
\| \int_{0}^{t} S_1(t-s)[-u_x^\varepsilon + u_x^\varepsilon(t_s)u_{x}^\varepsilon(t_s)]ds \|^{2p}
\]
\[
\leq (\int_{0}^{t} \| S_1(t-s)[-u_x^\varepsilon + u_x^\varepsilon(t_s)u_{x}^\varepsilon(t_s)] \| ds)^{2p}
\]
\[
= C(\int_{0}^{t} \| S_1(t-s)[(u^\varepsilon)^2 - (u^\varepsilon)^2(t_s)] \| ds)^{2p}
\]
\[
\leq C(\int_{0}^{t} (t-s)^{-\frac{1}{2}} \| (u^\varepsilon)^2 - (u^\varepsilon)^2(t_s) \| ds)^{2p}
\]
\[
\leq C(\int_{0}^{t} (t-s)^{-\frac{1}{2}} \| u^\varepsilon - u^\varepsilon(t_s) \| \| u^\varepsilon + u^\varepsilon(t_s) \|_{L^\infty} ds)^{2p}
\]

Thus, it follows from the Gronwall inequality that

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_{0}^{t} \left( \left\| u^{\varepsilon} - u^{\varepsilon}(t) \right\| \right) ds \right)^{2p} \leq C \left( \int_{0}^{T} \left( \left\| \nabla u^{\varepsilon} \right\| \right) ds \right)^{p} \]  

(21)



\[ \leq C(T) \mathbb{E} \sup_{0 \leq t \leq T} \left( \left\| u^{\varepsilon} - u^{\varepsilon}(t) \right\| \right) \left( \int_{0}^{T} \left( \left\| \nabla u^{\varepsilon} \right\| \right) ds \right)^{p} \]

\[ \leq C(T) \mathbb{E} \left( \left\| u^{\varepsilon} - u^{\varepsilon}(t) \right\| \right) \left( \int_{0}^{T} \left( \left\| \nabla u^{\varepsilon} \right\| \right) ds \right)^{p} \]

\[ \leq C \left( \left\| u^{\varepsilon} - u^{\varepsilon}(t) \right\| \right) \left( \int_{0}^{T} \left( \left\| \nabla u^{\varepsilon} \right\| \right) ds \right)^{p} \]

\[ \leq C(\delta^{p+1} \varepsilon + \delta^{p}). \]

Thus,

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left( \left\| u^{\varepsilon} - \hat{u}^{\varepsilon} \right\| \right) \leq C(T) \mathbb{E} \sup_{0 \leq s \leq T} \left( \left\| u^{\varepsilon} - \hat{u}^{\varepsilon} \right\| \right) + C \left( \delta^{p+1} \varepsilon + \delta^{p} \right), \]

it follows from the Gronwall inequality that

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left( \left\| u^{\varepsilon} - \hat{u}^{\varepsilon} \right\| \right)^{2p} \leq C \left( \frac{\delta^{p+1}}{\varepsilon} + \delta^{p} \right). \]

This completes the proof of Proposition 11.
4.6. The error of $\hat{u}^\varepsilon - \bar{u}$. Next we prove strong convergence of the auxiliary process $\hat{u}^\varepsilon$ to the averaging solution process $\bar{u}$.

**Proposition 12.** If $p > \frac{2}{3}$, there exists a constant $C(p, T)$ such that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|\hat{u}^\varepsilon(t) - \bar{u}(t)\|^{2p} \leq C(\varepsilon^1 + \varepsilon^2 + \varepsilon^{\frac{p-1}{2}} + \varepsilon^{\frac{p-1}{3}})\varepsilon^{-\frac{1}{3}} + C(p, T)(\frac{1}{\ln \varepsilon})^{\frac{1}{p}}.
\]

**Proof.** It is easy to see
\[
\begin{aligned}
\hat{u}^\varepsilon &= (-\hat{\eta}^\varepsilon_{xx} - \hat{\eta}^\varepsilon_x - u^\varepsilon(t)u^\varepsilon_x(t) + f(u^\varepsilon(t), \dot{v}^\varepsilon))dt + dW_1 \quad \text{in } Q \\
\hat{u}^\varepsilon(0, t) &= 0 = \hat{u}^\varepsilon(1, t) \quad \text{in } (0, T) \\
\hat{\eta}^\varepsilon_{xx}(0, t) &= 0 = \hat{\eta}^\varepsilon_{xx}(1, t) \quad \text{in } (0, T) \\
\hat{u}^\varepsilon(x, 0) &= u_0(x) \quad \text{in } I,
\end{aligned}
\]

it follows from Proposition 8 that the averaged equation
\[
\begin{aligned}
d\bar{u} &= (-\bar{\eta}_{xx} - \bar{\eta}_x - \bar{u} + \bar{f}(\bar{u}))dt + dW_1 \quad \text{in } Q \\
\bar{u}(0, t) &= 0 = \bar{u}(1, t) \quad \text{in } (0, T) \\
\bar{u}_{xx}(0, t) &= 0 = \bar{u}_{xx}(1, t) \quad \text{in } (0, T) \\
\bar{u}(x, 0) &= u_0(x) \quad \text{in } I,
\end{aligned}
\]

has a unique solution $\bar{u} \in L^2(\Omega, C([0, T]; H^2(I) \cap H^1_0(I)))$, where
\[
\bar{f}(u) = \int_{L^2(I)} f(u, v)d\mu^u(dv), \ u \in L^2(I).
\]

In mild sense, we introduce the following decomposition
\[
\hat{u}^\varepsilon(t) - \bar{u}(t)
= \int_0^t S_1(t-s)[-\hat{\eta}^\varepsilon_{xx} + \bar{\eta}_x - \bar{u} + \bar{f}(\bar{u})]dt + \int_0^t S_1(t-s)[f(u^\varepsilon(t), \dot{v}^\varepsilon) - \bar{f}(\bar{u})]ds
= \int_0^t S_1(t-s)[-\hat{\eta}^\varepsilon_{xx} + \bar{\eta}_x]ds + \int_0^t S_1(t-s)[f(u^\varepsilon(t), \dot{v}^\varepsilon)]ds
+ \int_0^t S_1(t-s)[-u^\varepsilon(t)u^\varepsilon_x(t) + \bar{u}u_x]ds
\triangleq J_1 + J_2 + J_3.
\]

We define the stopping time
\[
\tau_n^\varepsilon = \inf\{t > 0 : \|\hat{u}^\varepsilon(t)\|_{H^1} + \|\bar{u}(t)\|_{H^1} > n\}
\]
for any $n \geq 1$, and $\varepsilon > 0$.

- For $J_1$,
\[
\mathbb{E} \sup_{0 \leq t \leq T \land \tau_n^\varepsilon} \|J_1\|^{2p} = \mathbb{E} \sup_{0 \leq t \leq T \land \tau_n^\varepsilon} \left\| \int_0^t S_1(t-s)[-\hat{\eta}^\varepsilon_{xx} + \bar{\eta}_x]ds \right\|^{2p}
\leq \mathbb{E} \sup_{0 \leq t \leq T \land \tau_n^\varepsilon} \left( \int_0^t \left\| S_1(t-s)[-\hat{\eta}^\varepsilon_{xx} + \bar{\eta}_x] \right\| ds \right)^{2p}
\]

- For $J_2$,
\[
\begin{align*}
\leq & \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^*} \left( \int_{0}^{t} (t-s)^{-\frac{1}{2}} \| \hat{u}^\varepsilon - \bar{u} \| ds \right)^{2p} \\
\leq & \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^*} \left[ \int_{0}^{t} [(t-s)^{-\frac{1}{2}} \| \hat{u}^\varepsilon - \bar{u} \| ds \right]^{2p-1} \cdot \int_{0}^{t} \| \hat{u}^\varepsilon - \bar{u} \|^2 ds \\
\leq & C(T) \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^*} \int_{0}^{t} \| \hat{u}^\varepsilon - \bar{u} \|^2 ds \\
\leq & C \mathbb{E} \int_{0}^{T \wedge \tau_n^*} \| \hat{u}^\varepsilon - \bar{u} \|^2 ds \\
\leq & C \mathbb{E} \int_{0}^{T \wedge \tau_n^*} \sup_{0 \leq r \leq s \wedge \tau_n^*} \| \hat{u}^\varepsilon (r) - \bar{u}(r) \|^2 ds \\
\leq & C \int_{0}^{T} \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^*} \| \hat{u}^\varepsilon (r) - \bar{u}(r) \|^2 ds,
\end{align*}
\]

- For \( J_2 \),
\[
J_2 = \int_{0}^{t} S_1(t-s)[f(u^\varepsilon(t_s), \hat{v}^\varepsilon) - \bar{f}(u^\varepsilon)] ds \\
+ \int_{0}^{t} S_1(t-s)[\bar{f}(u^\varepsilon) - \bar{f}(\hat{u}^\varepsilon)] ds + \int_{0}^{t} S_1(t-s)[\bar{f}(\hat{u}^\varepsilon) - \bar{f}(\bar{u})] ds \\
= \sum_{k=0}^{m_r-1} \int_{k\delta}^{(k+1)\delta} S_1(t-s)[f(u^\varepsilon(k\delta), \hat{v}^\varepsilon(s)) - \bar{f}(u^\varepsilon(k\delta))] ds \\
+ \sum_{k=0}^{m_r-1} \int_{k\delta}^{(k+1)\delta} S_1(t-s)[\bar{f}(u^\varepsilon(k\delta)) - \bar{f}(\hat{u}^\varepsilon(s))] ds \\
+ \int_{m_r \delta}^{t} S_1(t-s)[f(u^\varepsilon(m_r \delta), \hat{v}^\varepsilon(s)) - \bar{f}(u^\varepsilon(s))] ds \\
+ \int_{0}^{t} S_1(t-s)[\bar{f}(u^\varepsilon(s)) - \bar{f}(\hat{u}^\varepsilon(s))] ds + \int_{0}^{t} S_1(t-s)[\bar{f}(\hat{u}^\varepsilon(s)) - \bar{f}(\bar{u}(s))] ds \\
= J_{21} + J_{22} + J_{23} + J_{24} + J_{25},
\]
where \( m_r = \lfloor \frac{t}{\delta} \rfloor \).

* For \( J_{21} \), by a time shift transformation, we can obtain that for any fixed \( p \) and \( t \in [0, \delta) \) that
\[
\hat{v}^\varepsilon(t + p\delta) = v^\varepsilon(p\delta) + \frac{1}{\varepsilon} \int_{p\delta}^{t+p\delta} \hat{v}^\varepsilon_{xx}(s) ds + \frac{1}{\varepsilon} \int_{p\delta}^{t+p\delta} g(u^\varepsilon(p\delta), \hat{v}^\varepsilon(s)) ds + \frac{1}{\sqrt{\varepsilon}} \int_{p\delta}^{t+p\delta} dW_2
\]
\[
= v^\varepsilon(p\delta) + \frac{1}{\varepsilon} \int_{0}^{t} \hat{v}^\varepsilon_{xx}(s + p\delta) ds + \frac{1}{\varepsilon} \int_{0}^{t} g(u^\varepsilon(s + p\delta), \hat{v}^\varepsilon(s + p\delta)) ds + \frac{1}{\sqrt{\varepsilon}} \int_{0}^{t} dW_2^*,
\]
where \( W_2^*(t) \) is the shift version of \( W_2(t) \) and hence they have the same distribution.

Let \( \bar{W}(t) \) be a Wiener process defined on the same stochastic basis and independent of \( \bar{W}_1(t) \) and \( \bar{W}_2(t) \). Construct a process \( \nu^\varepsilon(p\delta), \nu^\varepsilon(p\delta)(t) \in L^2(I) \) by means
of $v_u^v(p\delta), v_v^v(p\delta)(\frac{t}{\varepsilon})$:
\[
v_u^v(p\delta) + \int_0^\frac{t}{\varepsilon} v_x^x u_v^v(p\delta), v_v^v(p\delta) (s) ds + \int_0^\frac{t}{\varepsilon} g(u_v^v(p\delta), v_u^v(p\delta)) (s) ds + \int_0^\frac{t}{\varepsilon} \tilde{W}(t) \approx 2 \sup_{E} \sup_{E} T \]
\[
v_u^v(p\delta) + \frac{1}{\varepsilon} \int_0^\frac{t}{\varepsilon} v_x^x u_v^v(p\delta), v_v^v(p\delta) (\frac{s}{\varepsilon}) ds + \frac{1}{\varepsilon} \int_0^\frac{t}{\varepsilon} g(u_v^v(p\delta), v_u^v(p\delta)) (\frac{s}{\varepsilon}) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t \tilde{W}(t) \]

here $\tilde{W}(t)$ is the scaled version of $\tilde{W}(t)$.

By comparing the above two equations, we see that
\[
(u_v^v(p\delta), \dot{v}^v(t + p\delta)) \sim (u_v^v(p\delta), v_u^v(p\delta), \dot{v}^v(p\delta)(\frac{t}{\varepsilon})), \ t \in [0, \delta),
\]
where $\sim$ denotes a coincidence in distribution sense. Thus, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J_{21} \|^2
\]
\[
= \mathbb{E} \sup_{0 \leq t \leq T} \| \sum_{k=0}^{m_t-1} \int_{k\delta}^{(k+1)\delta} S_1(t-s)(f(u_v^v(k\delta), \dot{v}^v(s)) - \tilde{f}(u_v^v(k\delta))) ds \|^2
\]
\[
= \mathbb{E} \sup_{0 \leq t \leq T} \| \sum_{k=0}^{m_t-1} \int_{k\delta}^{(k+1)\delta} S_1(t-(k+1)\delta) S_1((k+1)\delta-s)(f(u_v^v(k\delta), \dot{v}^v(s)) - \tilde{f}(u_v^v(k\delta))) ds \|^2
\]
\[
= \mathbb{E} \sup_{0 \leq t \leq T} \| \sum_{k=0}^{m_t-1} S_1((k+1)\delta-s)(f(u_v^v(k\delta), \dot{v}^v(s)) - \tilde{f}(u_v^v(k\delta))) ds \|^2
\]
\[
\leq \mathbb{E} \sup_{0 \leq t \leq T} \{ m_t \sum_{k=0}^{m_t-1} \| \int_{k\delta}^{(k+1)\delta} S_1((k+1)\delta-s)(f(u_v^v(k\delta), \dot{v}^v(s)) - \tilde{f}(u_v^v(k\delta))) ds \|^2 \}
\]
\[
\leq \frac{T}{\delta} \mathbb{E} \{ \sum_{k=0}^{\frac{T}{\delta}-1} \| \int_{k\delta}^{(k+1)\delta} S_1((k+1)\delta-s)(f(u_v^v(k\delta), \dot{v}^v(s)) - \tilde{f}(u_v^v(k\delta))) ds \|^2 \}
\]
\[
= \frac{T}{\delta} \mathbb{E} \{ \int_0^\frac{T}{\delta} S_1(\delta-s)(f(u_v^v(k\delta), \dot{v}^v(s+\delta)) - \tilde{f}(u_v^v(k\delta))) ds \|^2 \}
\]
\[
= 2 \frac{T}{\delta} \mathbb{E} \{ \int_0^\frac{T}{\delta} S_1(\delta-s)(f(u_v^v(k\delta), \dot{v}^v(s+\delta)) - \tilde{f}(u_v^v(k\delta))) ds \|^2 \}
\]
\[
= 2 \frac{T}{\delta} \mathbb{E} \{ \int_0^\frac{T}{\delta} S_1(\delta-s)(f(u_v^v(k\delta), \dot{v}^v(s+\delta)) - \tilde{f}(u_v^v(k\delta))) ds \|^2 \}
\]
where
\[
\mathcal{J}_k(s, r) = \int_0^r \mathbb{E} \{ S_1(\delta-s)(f(u_v^v(k\delta), \dot{v}^v(s+\delta)) - \tilde{f}(u_v^v(k\delta))) \times S_1(\delta-r)(f(u_v^v(k\delta), \dot{v}^v(s+\delta)) - \tilde{f}(u_v^v(k\delta))) \} ds dr
\]

It follows from (22) and the property of semigroup $\{ S_1(t) \}_{t \geq 0}$ that
\[
\mathcal{J}_k(s, r) = \mathbb{E} \{ S_1(\delta-s)(f(u_v^v(k\delta), \dot{v}^v(s+\delta)) - \tilde{f}(u_v^v(k\delta))) \times S_1(\delta-r)(f(u_v^v(k\delta), \dot{v}^v(s+\delta)) - \tilde{f}(u_v^v(k\delta))) \} ds dr
\]
\[ S_1(\delta - r\varepsilon)(f(u^\varepsilon(k\delta), \delta^\varepsilon(r\varepsilon + k\delta)) - \bar{f}(u^\varepsilon(k\delta)))dx \]

\[ = E \int S_1(\delta - s\varepsilon)(f(u^\varepsilon(k\delta), v^{u^\varepsilon(k\delta), u^\varepsilon(k\delta)}(s)) - \bar{f}(u^\varepsilon(k\delta)))\times \]

\[ E \int S_1(\delta - r\varepsilon)(f(u^\varepsilon(k\delta), v^{u^\varepsilon(k\delta), u^\varepsilon(k\delta)}(r)) - \bar{f}(u^\varepsilon(k\delta)))dx \]

\[ \leq \left\{ E \int \left\{ E^{v^{u^\varepsilon(k\delta), u^\varepsilon(k\delta)}(r)}(S_1(\delta - r\varepsilon))(f(u^\varepsilon(k\delta), v^{u^\varepsilon(k\delta), u^\varepsilon(k\delta)}(s - r)) - \bar{f}(u^\varepsilon(k\delta))))\right\}^2 dx \right\}^{\frac{1}{2}} \]

\[ \leq \left\{ E \int \left\{ E^{v^{u^\varepsilon(k\delta), u^\varepsilon(k\delta)}(r)}(S_1(\delta - r\varepsilon))(f(u^\varepsilon(k\delta), v^{u^\varepsilon(k\delta), u^\varepsilon(k\delta)}(s - r)) - \bar{f}(u^\varepsilon(k\delta))))\right\}^2 dx \right\}^{\frac{1}{2}} \]

In view of the above inequality, Proposition 3 and the method in [18, 16, 20, 17, 41, 39, 40], there holds

\[ J_k(s, r) \leq C e^{-\mu(s - r)}, \]

where \( \mu > 0 \). Thus if choose \( \delta = \delta(\varepsilon) \) such that \( \frac{1}{\varepsilon} \) sufficiently large, we have

\[ E \sup_{0 \leq \varepsilon \leq T} \left\| J_{21} \right\|^{2p} \leq 2\left| \frac{T}{\delta} \right|^{2p} \max_{0 \leq \varepsilon \leq \frac{T}{\delta} - 1} \int_0^{\frac{T}{\delta}} \int_r^{\frac{T}{\delta}} J_k(s, r)dsdr \]

\[ \leq C \left| \frac{T}{\delta} \right|^{2p} e^{\mu(s - r)} \int_0^{\frac{T}{\delta}} \int_r^{\frac{T}{\delta}} e^{-\mu(s - r)}dsdr \]

\[ \leq C_{\frac{\varepsilon}{\delta}}. \]

On the other hand, it holds that

\[ E \sup_{0 \leq \varepsilon \leq T} \left\| J_{21} \right\|^{2p} \]

\[ = E \sup_{0 \leq \varepsilon \leq T} \left\| \sum_{k=0}^{m-1} \int_{k\delta}^{(k+1)\delta} S_1(t - s)(f(u^\varepsilon(k\delta), \delta^\varepsilon(s)) - \bar{f}(u^\varepsilon(k\delta)))ds \right\|^{2p} \]

\[ \leq E \sup_{0 \leq \varepsilon \leq T} \left( \int_0^{m\delta} \left\| S_1(t - s)(f(u^\varepsilon(k\delta), \delta^\varepsilon(s)) - \bar{f}(u^\varepsilon(k\delta)))ds \right\|^{2p} \right) \]

\[ \leq E \left( \int_0^T \left\| S_1(t - s)(f(u^\varepsilon(k\delta), \delta^\varepsilon(s)) - \bar{f}(u^\varepsilon(k\delta)))ds \right\|^{2p} \right) \]

\[ \leq E \left( \int_0^T \left\| (f(u^\varepsilon(k\delta), \delta^\varepsilon(s)) - \bar{f}(u^\varepsilon(k\delta)))ds \right\|^{2p} \right). \]
\[
\begin{align*}
\leq & \mathbb{E}\left[ \int_0^T 1 ds \right]^{2p-1} \cdot \int_0^T \mathbb{E}\left[ \left( f\left( u^\varepsilon(k\delta), \hat{v}^\varepsilon(s) \right) - \bar{f}\left( u^\varepsilon(k\delta) \right) \right)^2 ds \right] \\
\leq & \left( \int_0^T 1 ds \right)^{2p-1} \cdot \mathbb{E}\left[ \int_0^T \left( f\left( u^\varepsilon(k\delta), \hat{v}^\varepsilon(s) \right) - \bar{f}\left( u^\varepsilon(k\delta) \right) \right)^2 ds \right] \\
\leq & CE \int_0^T \left[ 1 + \left\| u^\varepsilon(k\delta) \right\|^2 + \left\| \hat{v}^\varepsilon(s) \right\|^2 \right] ds \\
\leq & C(p,T),
\end{align*}
\]
thus,
\[
E \sup_{0 \leq t \leq T} \left\| J_{21} \right\|^{2p} \leq \left( E \sup_{0 \leq t \leq T} \left\| J_{21} \right\|^{2p-1} \right) \frac{1}{2} \left( E \sup_{0 \leq t \leq T} \left\| J_{21} \right\| \right)^{\frac{1}{2}} \\
\leq C(p,T) \sqrt{\frac{\varepsilon}{\delta}}.
\]
* For $J_{22}$, due to Proposition 9, it concludes that
\[
E \sup_{0 \leq t \leq T} \left\| J_{22} \right\|^{2p} \\
= E \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{m-1} \int_{k\delta}^{(k+1)\delta} S_1(t-s)\left[ f\left( u^\varepsilon(k\delta) \right) - \bar{f}\left( u^\varepsilon(s) \right) \right] ds \right\|^{2p} \\
= E \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{m-1} \int_{k\delta}^{(k+1)\delta} S_1(t-s)\left[ \hat{f}\left( u^\varepsilon(s) \right) \right] ds \right\|^{2p} \\
= E \sup_{0 \leq t \leq T} \left\| \int_{0}^{m\delta} S_1(t-s)\left[ \hat{f}\left( u^\varepsilon(s) \right) \right] ds \right\|^{2p} \\
\leq C \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_{0}^{m\delta} \left\| u^\varepsilon(s) \right\| ds \right)^{2p} \\
\leq C \mathbb{E} \left( \int_{0}^{T} 1 ds \right)^{2p-1} \cdot \int_{0}^{T} \left\| u^\varepsilon(s) \right\|^{2p} ds \\
\leq C \delta^p.
\]
* For $J_{23}$, according to Proposition 5 and the global Lipschitz property of $f$ and $\bar{f}$, we have
\[
E \sup_{0 \leq t \leq T} \left\| J_{23} \right\|^{2p} \\
= E \sup_{0 \leq t \leq T} \left\| \int_{m_t\delta}^{t} S_1(t-s)\left[ f\left( u^\varepsilon(m_t\delta), \hat{v}^\varepsilon(s) \right) - \bar{f}\left( u^\varepsilon(s) \right) \right] ds \right\|^{2p} \\
\leq E \sup_{0 \leq t \leq T} \left( \int_{m_t\delta}^{t} \left\| S_1(t-s)\left[ f\left( u^\varepsilon(m_t\delta), \hat{v}^\varepsilon(s) \right) - \bar{f}\left( u^\varepsilon(s) \right) \right] ds \right\|^2 ds \right)^{p} \\
\leq E \sup_{0 \leq t \leq T} \left( \int_{m_t\delta}^{t} \left\| f\left( u^\varepsilon(m_t\delta), \hat{v}^\varepsilon(s) \right) - \bar{f}\left( u^\varepsilon(s) \right) \right\| ds \right)^{2p}.
\]
\[
\begin{align*}
\leq & C \mathbb{E} \sup_{0 \leq t \leq T} \left[ \left( \int_{m_t \delta}^{t} 1 ds \right)^{2p-1} \cdot \left( \int_{m_t \delta}^{t} \| f(u^\varepsilon(m_t \delta), \dot{v}^\varepsilon(s)) - \bar{f}(u^\varepsilon(s)) \|^2 ds \right) \right] \\
\leq & C \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_{m_t \delta}^{t} 1 ds \right)^{2p-1} \sup_{0 \leq t \leq T} \left( \int_{m_t \delta}^{t} \| f(u^\varepsilon(m_t \delta), \dot{v}^\varepsilon(s)) - \bar{f}(u^\varepsilon(s)) \|^2 ds \right) \\
= & C \sup_{0 \leq t \leq T} (t - m_t \delta)^{2p-1} \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_{m_t \delta}^{t} \| f(u^\varepsilon(m_t \delta), \dot{v}^\varepsilon(s)) - \bar{f}(u^\varepsilon(s)) \|^2 ds \right) \\
\leq & C \delta^{2p-1} \cdot \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_{m_t \delta}^{t} (1 + \| u^\varepsilon(m_t \delta) \|^2 + \| \dot{v}^\varepsilon(s) \|^2 + \| u^\varepsilon(s) \|^2) ds \right) \\
\leq & C \delta^{2p-1} \cdot \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_{m_t \delta}^{t} (1 + \| u^\varepsilon(m_t \delta) \|^2 + \| \dot{v}^\varepsilon(s) \|^2 + \| u^\varepsilon(s) \|^2) ds \right) \\
\leq & C \delta^{2p-1} \cdot \mathbb{E} \int_{0}^{T} (1 + \| u^\varepsilon(m_t \delta) \|^2 + \| \dot{v}^\varepsilon(s) \|^2 + \| u^\varepsilon(s) \|^2) ds \\
\leq & C \delta^{2p-1}.
\end{align*}
\]

* For $J_{24}$, using the contractive property of semigroup, Lipschitz continuity of $\bar{f}$ and Proposition 11, we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J_{24} \|^2 = \mathbb{E} \sup_{0 \leq t \leq T} \left\{ \left( \int_{0}^{t} S_1(t-s) [\bar{f}(u^\varepsilon) - \bar{f}(\dot{u}^\varepsilon)] ds \right)^2 \right\} \\
\leq \mathbb{E} \left( \sup_{0 \leq t \leq T} \left( \int_{0}^{t} \| S_1(t-s) [\bar{f}(u^\varepsilon) - \bar{f}(\dot{u}^\varepsilon)] \| ds \right)^2 \right) \\
\leq \mathbb{E} \left( \sup_{0 \leq t \leq T} \left( \int_{0}^{t} \| u^\varepsilon - \dot{u}^\varepsilon \| ds \right)^2 \right) \\
\leq \mathbb{E} \left( \int_{0}^{T} \| u^\varepsilon - \dot{u}^\varepsilon \| ds \right)^2 \\
\leq C \delta^{p} + \frac{\delta^{p+1}}{\varepsilon}.
\]

* For $J_{25}$, using the contractive property of semigroup and the Lipschitz continuity of $\bar{f}$, we have

\[
\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \| J_{25} \|^2 = \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left\{ \left( \int_{0}^{t} S_1(t-s) [\bar{f}(\dot{u}^\varepsilon(s)) - \bar{f}(\bar{u}(s))] ds \right)^2 \right\} \\
\leq \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left( \int_{0}^{t} \| S_1(t-s) [\bar{f}(\dot{u}^\varepsilon(s)) - \bar{f}(\bar{u}(s))] \| ds \right)^2 \\
\leq \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left( \int_{0}^{t} \| \bar{f}(\dot{u}^\varepsilon(s)) - \bar{f}(\bar{u}(s)) \| ds \right)^2 \\
\leq \mathbb{E} \left( \int_{0}^{T \wedge \tau_n^\varepsilon} \| \bar{f}(\dot{u}^\varepsilon(s)) - \bar{f}(\bar{u}(s)) \| ds \right)^2 \\
\leq C \delta^{p} \cdot \frac{\delta^{p+1}}{\varepsilon}.
\]
For $J_3$, we have

$$J_3 = \int_0^T S_1(t-s)[-u^\varepsilon(t_s)u^\varepsilon_x(t_s) + u^\varepsilon(s)u^\varepsilon_x(s)]ds$$

$$+ \int_0^T S_1(t-s)[-u^\varepsilon(s)u^\varepsilon_x(s) + \tilde{u}^\varepsilon(s)\tilde{u}^\varepsilon_x(s)]ds$$

$$+ \int_0^T S_1(t-s)[-\tilde{u}^\varepsilon(s)\tilde{u}^\varepsilon_x(s) + \tilde{u}(s)\tilde{u}_x(s)]ds$$

$$\leq 4J_{31} + J_{32} + J_{33}.$$
In order to deal with the above estimate, we will use the skill of stopping times, as inspired from [12].

We have

\[
E \sup_{0 \leq t \leq T \wedge \tau_n^3} \|J_{33}\|^{2p} \leq E \sup_{0 \leq t \leq T \wedge \tau_n^3} \int_0^t S_1(t-s)[-\hat{u}^\varepsilon(s)\hat{u}_x(s) + \bar{u}\hat{u}_x]ds\|^{2p} \\
\leq E \sup_{0 \leq t \leq T \wedge \tau_n^3} \|\int_0^t S_1(t-s)[-\hat{u}^\varepsilon(s)\hat{u}_x(s) + \bar{u}\hat{u}_x]ds\|^{2p} \\
\leq E \sup_{0 \leq t \leq T \wedge \tau_n^3} \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{u}^\varepsilon - \bar{u}\|\|\hat{u}^\varepsilon\|_{H^1} + \|\bar{u}\|_{H^1})ds\|^{2p}.
\]

In order to deal with the above estimate, we will use the skill of stopping times, as inspired from [12].

We have

\[
E \sup_{0 \leq t \leq T \wedge \tau_n^3} \|J_{33}\|^{2p} \leq E \sup_{0 \leq t \leq T \wedge \tau_n^3} \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{u}^\varepsilon(s) - \bar{u}(s)\|\|\hat{u}^\varepsilon\|_{H^1} + \|\bar{u}\|_{H^1})ds\|^{2p} \\
\leq n^{2p} E \sup_{0 \leq t \leq T \wedge \tau_n^3} \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{u}^\varepsilon(s) - \bar{u}(s)\|ds\|^{2p} \\
\leq n^{2p} E \sup_{0 \leq t \leq T \wedge \tau_n^3} \left[ \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{u}^\varepsilon(s) - \bar{u}(s)\|^{2p}ds \right]^{\frac{1}{2p}} \\
\leq C n^{2p} \left( \sup_{0 \leq t \leq T} \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{u}^\varepsilon(s) - \bar{u}(s)\|^{2p}ds \right) \\
\leq C n^{2p} \int_0^{T \wedge \tau_n^3} \|\hat{u}^\varepsilon(s) - \bar{u}(s)\|^{2p}ds \\
\leq C n^{2p} \int_0^{T \wedge \tau_n^3} \sup_{0 \leq r \leq s \wedge \tau_n^3} \|\hat{u}^\varepsilon(r) - \bar{u}(r)\|^{2p}ds \\
\leq C n^{2p} \int_0^T \sup_{0 \leq r \leq s \wedge \tau_n^3} \|\hat{u}^\varepsilon(r) - \bar{u}(r)\|^{2p}ds \\
\leq C n^{2p} \int_0^T E \sup_{0 \leq r \leq s \wedge \tau_n^3} \|\hat{u}^\varepsilon(r) - \bar{u}(r)\|^{2p}ds.
\]

* With the help of the above estimates, we have

\[
E \sup_{0 \leq t \leq T \wedge \tau_n^3} \|\hat{u}^\varepsilon(t) - \bar{u}(t)\|^{2p} \leq C \left( \frac{\varepsilon}{\delta} + \delta^p + \delta^{2p-1} + \frac{\delta^{p+1}}{\varepsilon} + \frac{\delta^{2p+1}}{\sqrt{\varepsilon}} \right) + C n^{2p} \int_0^T E \sup_{0 \leq r \leq s \wedge \tau_n^3} \|\hat{u}^\varepsilon(r) - \bar{u}(r)\|^{2p}ds \\
+ C \int_0^T E \sup_{0 \leq r \leq s \wedge \tau_n^3} \|\hat{u}^\varepsilon(r) - \bar{u}(r)\|^{2p}ds \\
\leq C \left( \frac{\varepsilon}{\delta} + \delta^p + \delta^{2p-1} + \frac{\delta^{p+1}}{\varepsilon} + \frac{\delta^{2p+1}}{\sqrt{\varepsilon}} \right) + C n^{2p} \int_0^T E \sup_{0 \leq r \leq s \wedge \tau_n^3} \|\hat{u}^\varepsilon(r) - \bar{u}(r)\|^{2p}ds.
\]
By using the Gronwall inequality, we have
\[ E \sup_{0 \leq t \leq T \wedge \tau_n} \| \dot{u}^\varepsilon(t) - \bar{u}(t) \|^2 \leq C \left( \sqrt{\frac{\varepsilon}{\delta}} + \delta^{p+1} + \frac{\delta^{p+1}}{\varepsilon} + \frac{\varepsilon^{2p+1}}{\sqrt{\varepsilon}} \right) \epsilon \sqrt{n}, \]
this implies that
\[ E( \sup_{0 \leq t \leq T} \| \dot{u}^\varepsilon(t) - \bar{u}(t) \|^2 \cdot \chi(T \leq \tau_n)) \leq C \left( \sqrt{\frac{\varepsilon}{\delta}} + \delta^{p+1} + \frac{\delta^{p+1}}{\varepsilon} + \frac{\varepsilon^{2p+1}}{\sqrt{\varepsilon}} \right) \epsilon \sqrt{n} \epsilon \sqrt{n}. \]
On the other hand, due to Proposition 5 and Proposition 8, we have
\[ E( \sup_{0 \leq t \leq T} \| \dot{u}^\varepsilon(t) - \bar{u}(t) \|^2 \cdot \chi(T > \tau_n)) \leq \frac{C}{\sqrt{n}} \cdot (E(\chi(T > \tau_n))^2 \leq \frac{C}{\sqrt{n}}. \]
Hence, we have
\[ E( \sup_{0 \leq t \leq T} \| \dot{u}^\varepsilon(t) - \bar{u}(t) \|^2) \leq C(\sqrt{\frac{\varepsilon}{\delta}} + \delta^{p+1} + \frac{\delta^{p+1}}{\varepsilon} + \frac{\varepsilon^{2p+1}}{\sqrt{\varepsilon}}) \epsilon \sqrt{n} + \frac{C}{\sqrt{n}}. \]
if we take \( n = \sqrt{-\frac{1}{8\delta} \ln \varepsilon} \), \( \delta = \varepsilon^\frac{1}{2} \), we obtain
\[ E( \sup_{0 \leq t \leq T} \| \dot{u}^\varepsilon(t) - \bar{u}(t) \|^2) \leq C(\sqrt{\frac{\varepsilon}{\delta}} + \varepsilon^\frac{p}{2} + \varepsilon^{2p+1} + \frac{\varepsilon^{2p+1}}{\sqrt{\varepsilon}} + \frac{\varepsilon^{p+1}}{\sqrt{\varepsilon}} + \frac{\varepsilon^{p+1}}{\sqrt{\varepsilon}}) \varepsilon^{-\frac{1}{8}} \varepsilon^{-\frac{1}{8}} \ln \varepsilon + \frac{C}{\sqrt{n}}. \]
This completes the proof of Proposition 12.

4.7. Proof of Theorem 1.1. By taking \( \delta = \varepsilon^\frac{1}{2} \), we have
\[ E( \sup_{0 \leq t \leq T} \| u^\varepsilon(t) - \dot{u}^\varepsilon(t) \|^2) \leq C \varepsilon^\frac{p}{2} + C \varepsilon^{\frac{p+1}{2}}. \]
If \( p > \frac{5}{4} \), we have
\[ E( \sup_{0 \leq t \leq T} \| \dot{u}^\varepsilon(t) - \bar{u}(t) \|^2) \leq C(\frac{1}{\ln \varepsilon})^\frac{1}{p}, \]
thus, we have
\[ E( \sup_{0 \leq t \leq T} \| u^\varepsilon(t) - \bar{u}(t) \|^2) \leq C(\frac{1}{\ln \varepsilon})^\frac{1}{p}. \]
If \( 0 < p \leq \frac{5}{4} \), for any \( \kappa > 0 \), it holds that
\[ E( \sup_{0 \leq t \leq T} \| u^\varepsilon(t) - \bar{u}(t) \|^2) \leq (E( \sup_{0 \leq t \leq T} \| u^\varepsilon(t) - \bar{u}(t) \|^{\frac{p}{2} + \kappa})^{\frac{2p}{2 + \kappa}} \leq C(\kappa)(E( \sup_{0 \leq t \leq T} \| u^\varepsilon(t) - \bar{u}(t) \|^{\frac{p}{2} + \kappa})^{\frac{2p}{2 + \kappa}} \leq C(p, \kappa)(\varepsilon^{-\frac{1}{\ln \varepsilon}})^{\frac{p}{2 + \kappa}}. \]
This completes the proof of Theorem 1.1.

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