n-dimensional links, their components, and their band-sums

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Abstract. We prove the following results (1) (2) (3) on relations between n-links and their components.

(1) Let \( L = (L_1, L_2) \) be a \((4k + 1)\)-link \((4k + 1 \geq 5)\). Then we have
\[
\text{Arf} \ L = \text{Arf} L_1 + \text{Arf} L_2.
\]

(2) Let \( L = (L_1, L_2) \) be a \((4k + 3)\)-link \((4k + 3 \geq 3)\). Then we have
\[
\sigma L = \sigma L_1 + \sigma L_2.
\]

(3) Let \( n \geq 1 \). Then there is a nonribbon \( n \)-link \( L = (L_1, L_2) \) such that \( L_i \) is a trivial knot.

We prove the following results (4) (5) (6) (7) on band-sums of n-links.

(4) Let \( L = (L_1, L_2) \) be a \((4k + 1)\)-link \((4k + 1 \geq 5)\).
Let \( K \) be a band-sum of \( L \). Then we have
\[
\text{Arf} K = \text{Arf} L_1 + \text{Arf} L_2.
\]

(5) Let \( L = (L_1, L_2) \) be a \((4k + 3)\)-link \((4k + 3 \geq 3)\).
Let \( K \) be a band-sum of \( L \). Then we have
\[
\sigma K = \sigma L_1 + \sigma L_2,
\]
The above (4)(5) imply the following (6).

(6) Let \( 2m + 1 \geq 3 \). There is a set of three \((2m + 1)\)-knots \( K_0, K_1, K_2 \) with the following property: \( K_0 \) is not any band-sum of any \( n \)-link \( L = (L_1, L_2) \) such that \( L_i \) is equivalent to \( K_i \) \((i = 1, 2)\).

(7) Let \( n \geq 1 \). Then there is an \( n \)-link \( L = (L_1, L_2) \) such that \( L_i \) is a trivial knot \((i = 1, 2)\) and that a band-sum of \( L \) is a nonribbon knot.

We prove a 1-dimensional version of (1).

(8) Let \( L = (L_1, L_2) \) be a proper 1-link. Then
\[
\text{Arf} L = \text{Arf} L_1 + \text{Arf} L_2 + \frac{1}{2} \{ \beta^*(L) \mod 4 \{ \frac{1}{2} \text{lk}(L) \} \}
\]
\[
= \text{Arf} L_1 + \text{Arf} L_2 + \mod 2 \{ \lambda(L) \},
\]
where \( \beta^*(L) \) is the Saito-Sato-Levine invariant and \( \lambda(L) \) is the Kirk-Livingston invariant.

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1 Introduction and main results

We work in the smooth category.

An (oriented) m-component n-(dimensional) link is a smooth, oriented submanifold \( L = \{ L_1, ..., L_m \} \subset S^{n+2} \), which is the ordered disjoint union of \( m \) manifolds, each PL homeomorphic to the \( n \)-sphere. If \( m = 1 \), then \( L \) is called a knot.

We say that \( n \)-links \( L \) and \( L' \) are equivalent if there exists an orientation preserving diffeomorphism \( f : S^{n+2} \to S^{n+2} \) such that \( f(\partial L) = \partial L' \) and \( f|_L : L \to L' \) is an orientation and order preserving diffeomorphism. If \( n \)-knot \( K \) bounds a \((n+1)\)-ball \( \subset S^{n+2} \), then \( K \) is called a trivial \((n-)knot\).

We say that \( m \)-component \( n \)-dimensional links, \( L \) and \( L' \), are said to be (link-)concordant or (link-)cobordant if there is a smooth oriented submanifold \( C = \{ C_1, ..., C_m \} \subset S^{n+2} \times [0,1] \), which meets the boundary transversely in \( \partial C \), is PL homeomorphic to \( L \times [0,1] \) and meets \( S^{n+2} \times \{ 0 \} \) in \( L \) (resp. \( S^{n+2} \times \{ 1 \} \) in \( L' \)). An \( n \)-link \( L \) is called a slice link if \( L \) is cobordant to a trivial link.

We prove:

**Theorem 1.1.** (1) Let \( 4k + 1 \geq 5 \). Let \( L = (L_1, L_2) \) be a \((4k+1)\)-link. Then we have

\[
\text{Arf} L = \text{Arf} L_1 + \text{Arf} L_2.
\]

(2) Let \( 4k + 3 \geq 3 \). Let \( L = (L_1, L_2) \) be a \((4k+3)\)-link. Then we have

\[
\sigma L = \sigma L_1 + \sigma L_2.
\]

**Note.** In §2 we review the Arf invariant and the signature.

Furthermore we prove 1-dimensional version of Theorem 1.1.

**Proposition 1.2.** Let \( L = (L_1, L_2) \) be a 1-link. Suppose that the Arf invariants of 2-component 1-links are defined, that is, that the linking numbers are even.

(1) \( \text{Arf} L = \text{Arf} L_1 + \text{Arf} L_2 + \frac{1}{2} \beta'(L) \mod \{ 4 \} \}

where \( \beta'(L) \) is the Saito-Sato-Levine invariant.

(2) \( \text{Arf} L = \text{Arf} L_1 + \text{Arf} L_2 + \mod 2 \lambda(L) \}

where \( \lambda(L) \) is the Kirk-Livingston invariant.

**Note.** The Saito-Sato-Levine invariant is defined in \([3]\). The Kirk-Livingston invariant is defined in \([13]\).

In order to continue to state our main results, we need some more definitions. An \( n \)-link \( L = (L_1, ..., L_m) \) is called a ribbon \( n \)-link if \( L \) satisfies the following properties.

(1) There is a self-transverse immersion \( f : D^{n+1}_1 \cup ... \cup D^{n+1}_m \to S^{n+2} \) such that \( f(\partial D^{n+1}_i) = L_i \).

(2) The singular point set \( C (\subset S^{n+2}) \) of \( f \) consists of double points.

\( C \) is a disjoint union of \( n \)-discs \( D^n_i (i = 1, ..., k) \).

(3) Put \( f^{-1}(D^n_j) = D^n_{jb} \cup D^n_{js} \). The \( n \)-disc \( D^n_{js} \) is trivially embedded in the interior \( \text{Int} D^{n+1}_a \) of a \((n+1)\)-disc component \( D^{n+1}_a \). The circle \( \partial D^{n+1}_j \) is trivially embedded in the boundary \( \partial D^{n+1}_b \) of an \((n+1)\)-disc component \( D^{n+1}_b \). The \( n \)-disc \( D^n_{jb} \) is trivially embedded in the \((n+1)\)-disc component \( D^{n+1}_b \). (Note that there are two cases, \( \alpha = \beta \) and \( \alpha \neq \beta \).

It is well-known that it is easy to prove that all ribbon \( n \)-links are slice.

It is natural to consider the following.
Problem A. (1) Is there a nonribbon \( n \)-link \( L = (L_1, L_2) \) such that \( L_i \) is a ribbon knot \((i = 1, 2)\)?
(2) Is there a nonribbon \( n \)-link \( L = (L_1, L_2) \) such that \( L_i \) is a trivial knot \((i = 1, 2)\)?

The \( n = 1 \) case holds because the Hopf link is an example. In [2], the author gave the affirmative answer to the \( n = 2 \) case. In this paper we give the affirmative answer to the \( n \geq 3 \) case. We prove:

Theorem 1.3. (1) Let \( n \geq 1 \). Then there is a nonribbon \( n \)-link \( L = (L_1, L_2) \) such that \( L_1 \) is a trivial knot.

Furthermore we have the following.
(2) Let \( 2m + 1 \geq 1 \). Then there is a nonslice \((2m + 1)\)-link \( L = (L_1, L_2) \) such that \( L_1 \) is a trivial knot. (Note that \( L \) is nonribbon since \( L \) is nonslice.)
(3) Let \( n \geq 2 \). Then there is a slice and nonribbon \( n \)-link \( L = (L_1, L_2) \) such that \( L_1 \) is a trivial knot.

We need some more definitions. Let \( L = (L_1, L_2) \) be an \( n \)-link. An \( n \)-knot \( K \) is called a band-sum (of the components \( L_1 \) and \( L_2 \)) of the 2-link \( L \) along a band \( h \) if we have:
(1) There is an \((n+1)\)-dimensional 1-handle \( h \), which is attached to \( L \), embedded in \( S^4 \).
(2) There are a point \( p_1 \in L_1 \) and a point \( p_2 \in L_2 \). We attach \( h \) to \( L_1 \sqcup L_2 \) along \( p_1 \) \( \sqcup p_2 \). \( h \cap (L_1 \cup L_2) \) is the attach part of \( h \). Then we obtain an \( n \)-knot from \( L_1 \) and \( L_2 \) by this surgery. The \( n \)-knot is \( K \).

The set \((K_0, K_1, K_2)\) is called a triple of \( n \)-knots if \( K_i \) is an \( n \)-knot. A triple \((K_0, K_1, K_2)\) of \( n \)-knots is said to be band-realizable if there is an \( n \)-link \( L = (L_1, L_2) \) such that \( K_1 \) (resp. \( K_2 \)) is equivalent to \( L_1 \) (resp. \( L_2 \)) and that \( K_0 \) is a band-sum of \( L \).

Note: Suppose that a triple \((K_0, K_1, K_2)\) of \( n \)-knots is band-realizable. Then \([K_0] = [K_1] + [K_2]\), where \([X]\) represents an element in the homotopy sphere group \( \Theta_n \). See [11] for \( \Theta_n \).

It is natural to consider the following.

Problem B. Let \( K_0, K_1, K_2 \) be arbitrary \( n \)-knots. Then is the triple \((K_0, K_1, K_2)\) of \( n \)-knots band-realizable?

By using the results in [3, 12, 13], we can prove: we have the affirmative answer to the \( n = 1 \) case. By using [51], we can prove: if \( K_0, K_1, K_2 \) are ribbon \( n \)-knots \((n \geq 2)\), we have the affirmative answer. In [2], the author proved: there is a nonribbon 2-link \( L = (L_1, L_2) \) such that \( L_i \) is the trivial knot and that a band-sum of \( L \) is a nonribbon knot. ‘Nonribbon case’ of the \( n \)-dimensional version \((n \geq 2)\) is not solved completely. Thus, in this paper, we consider the following problems C, D, which are the special cases of Problem B.

Problem C. Is there a set of three \( n \)-knots \( K_0, K_1, K_2 \) such that the triple \((K_1, K_2, K_3)\) is not band-realizable?

In this paper we give the affirmative answer when \( n \) is odd and \( n \geq 3 \). (Theorem 1.4, 1.5.)

Problem D. (1) Is there a set of one nonribbon \( n \)-knot \( K_0 \) and two ribbon \( n \)-knots \( K_1, K_2 \) such that the triple \((K_1, K_2, K_3)\) is band-realizable?
(2) Is there a set of one nonribbon \( n \)-knot \( K_0 \) and two trivial \( n \)-knots \( K_1, K_2 \) such that the triple \((K_1, K_2, K_3)\) is band-realizable?
In [22], the author gave the affirmative answer to the \( n = 2 \) case. In this paper we give the affirmative answer to the \( n \geq 3 \) case. (Theorem 1.6.)

**Theorem 1.4.** Let \( 2m + 1 \geq 3 \). There is a set of three \((2m + 1)\)-knots \( K_0, K_1, K_2 \) such that the triple \((K_0, K_1, K_2)\) is not band-realizable.

Theorem 1.4 is deduced from Theorem 1.5.

**Theorem 1.5.**

(1) Let \( 4k + 1 \geq 5 \). Let \( L = (L_1, L_2) \) be a \((4k + 1)\)-link. Let \( K \) be a band-sum of \( L \). Then we have

\[
\text{Arf}K = \text{Arf} L_1 + \text{Arf} L_2.
\]

(2) Let \( 4k + 3 \geq 3 \). Let \( L = (L_1, L_2) \) be a \((4k + 3)\)-link. Let \( K \) be a band-sum of \( L \). Then we have

\[
\sigma K = \sigma L_1 + \sigma L_2.
\]

**Theorem 1.6.**

(1) Let \( n \geq 1 \). Let \( T \) be a trivial \( n \)-knot. Then there is a nonribbon \( n \)-knot \( K \) such that the triple \((K, T, T)\) is band-realizable. Furthermore we have the following.

(2) Let \( 2m + 1 \geq 1 \). Let \( T \) be a trivial \((2m + 1)\)-knot. Then there is a nonslice \((2m + 1)\)-knot \( K \) such that the triple \((K, T, T)\) is band-realizable. (Note that \( K \) is nonribbon if \( K \) is nonslice.)

(3) Let \( 2m + 1 \geq 3 \). Let \( T \) be a trivial \((2m + 1)\)-knot \( T \). Then there is a slice and nonribbon \((2m + 1)\)-knot \( K \) such that the triple \((K, T, T)\) is band-realizable.

**Note.** All even dimensional knots are slice. ([10].)

[26] includes the announcement of this paper.

Our organization is as follows:

\( \S 2 \) Seifert matrices, the signature and the Arf invariant of \( n \)-knots (resp. \( n \)-links)

\( \S 3 \) Some properties of band-sums

\( \S 4 \) Proof of Theorem 1.1.(1)

\( \S 5 \) Proof of Theorem 1.1.(2)

\( \S 6 \) Proof of Proposition 1.2

\( \S 7 \) Proof of Theorem 1.5

\( \S 8 \) Proof of Theorem 1.4

\( \S 9 \) Proof of Theorem 1.3.(2)

\( \S 10 \) Proof of Theorem 1.3.(3)

\( \S 11 \) Proof of Theorem 1.3.(1)

\( \S 12 \) Proof of Theorem 1.6.(2)

\( \S 13 \) Proof of Theorem 1.6.(3)

\( \S 14 \) Proof of Theorem 1.6.(1)

\( \S 15 \) Open problems

In \( \S 9 \), we give an alternative proof of one of the main theorems of [8] and that of [3]. In \( \S 10 \), we give a short proof of the main theorem of [8].

2 Seifert matrices, the signature and the Arf invariant of \( n \)-knots (resp. \( n \)-links)

See the \( n = 1 \) case [8] [24] [27]. See the \( n \geq 2 \) case [10] [17].
Let $K$ be a $(2m+1)$-knot $(2m + 1 \geq 1)$. Let $V$ be a connected Seifert hypersurface of $K$. Note the orientation of $V$ is compatible with that of $K$. Let $x_1, \ldots, x_\mu$ be $(m+1)$-cycles in $V$ which are basis of $H_{m+1}(V; \mathbb{Z})/\text{Tor}$. Push $x_i$ to the positive direction of the normal bundle of $V$. Call it $x^+_i$. A Seifert matrix of $K$ associated with $V$ represented by basis $x_1, \ldots, x_\mu$ is a matrix $A = (a_{ij}) = (\langle k(x_i, x^+_j) \rangle)$. Then we have: $A - (-1)^m A$ represents the map $\{H(V; \mathbb{Z})/\text{Tor}\} \times \{H(V; \mathbb{Z})/\text{Tor}\} \to \mathbb{Z}$, which is defined by the intersection product.

The signature $\sigma(K)$ of $K$ is the signature of the matrix $A + A^t$. Therefore, we have:

**Claim.** If $2m + 1 = 4k + 3 \geq 3$, the signature of $K$ coincides with the signature of $V$, where $V$ is the closed oriented manifold which we obtain by attaching a $(4k+4)$-dimensional 0-handle to $\partial V$.

Let $K$ be a $(4k+1)$-knot $(4k+1 \geq 1)$. We regard naturally $(H_{2k+1}(V; \mathbb{Z})/\text{Tor}) \otimes \mathbb{Z}_2$ as a subgroup of $H_{2k+1}(V; \mathbb{Z}_2)$. Then we can take basis $x_1, \ldots, x_\nu, y_1, \ldots, y_\nu$ of $(H_{2k+1}(V; \mathbb{Z})/\text{Tor}) \otimes \mathbb{Z}_2$ such that $x_i \cdot x_j = 0, y_i \cdot y_j = 0, x_i \cdot y_j = \delta_{ij}$ for any pair $(i, j)$, where $\cdot$ is the intersection product. The Arf invariant of $K$ is mod 2

$$\Sigma_{i=1}^\nu \cdot \text{lk}(x^+_i, x^+_i).$$

Let $L = (K_1, K_2)$ be a $(2m+1)$-link $(2m + 1 \geq 1)$. Let $V$ be a Seifert hypersurface of $L$. We define $x_i, x^+_i, A, \sigma L$ in the same manner. If $2m + 1 = 4k + 3 \geq 3$, then $\sigma L$ is the signature of the closed oriented manifold $\hat{V}$, where $\hat{V}$ is the closed oriented manifold which we obtain by attaching $(4k+4)$-dimensional 0-handles to $\partial V$.

Let $L = (L_1, L_2)$ be a $(4k+1)$-link $(4k+1 \geq 1)$. We define the Arf invariant of $L = (L_1, L_2)$ $(4k+1 \geq 1)$. There are two cases.

1. Let $4k + 1 \geq 5$. The Arf invariant of $L$ is defined in the same manner as the knot case.

2. Let $4k + 1 = 1$. The Arf invariant of $L = (L_1, L_2)$ is defined only if the linking number $\text{lk}(L_1, L_2)$ of $L$ is even. Then we can take basis $x_1, \ldots, x_\mu, y_1, \ldots, y_\nu, z$ of $H_1(V; \mathbb{Z})/\text{Tor}$ such that $x_1 \cdot x_j = 0, x_\mu \cdot y_1 = \delta_{ij}, y_\nu \cdot y_\nu = 0, x_1 \cdot z = 0, y_\nu \cdot z = 0, z \cdot z = 0$. The Arf invariant of $L$ is mod $2 \Sigma_{i=1}^\nu \cdot \text{lk}(x^+_i, x^+_i) \cdot \text{lk}(y_i, y_i^+)$. See e.g. Appendix of [14].

### 3 Some properties of band-sums

In our proof of main results we use the following properties of band-sums.

**Proposition 3.1.** Let $L = (L_1, L_2)$ be an $n$-link. Let $K$ be a band-sum of $L$ along a band $h$.

1. $\text{Arf}K = \text{Arf}L$. ($n = 4k + 1 \geq 5$.)
2. $\sigma K = \sigma L$. ($n = 4k + 3 \geq 3$.)
3. Knot cobordism class of $K$ is independent of the choice of $h$. ($n \geq 2$.)
4. The following two equivalent conditions hold. ($n \geq 1$.)
   - (i) If $L$ is slice, then $K$ is slice.
   - (ii) If $K$ is nonslice, $L$ is nonslice.
5. The following two equivalent conditions hold. ($n \geq 1$.)
   - (i) If $L$ is ribbon, then $K$ is ribbon.
   - (ii) If $K$ is nonribbon, $L$ is nonribbon.

**Proof of (1)(2)(3).** We need a lemma.
Lemma 3.2. There is a Seifert hypersurface $V$ for $L$ such that $V \cap h$ is the attach part of $h$.

Proof of Lemma 3.2. Let $h \times [-1, 1]$ be a tubular neighborhood of $h \subset S^{n+2}$. Suppose $h \times [-1, 1] \cap L$ is the attach part of $h$. Then we have $[L] = 0 \in H_n(S^{n+2} - (h \times [-1, 1]); \mathbb{Z})$. By the following Claim 3.3, the above Lemma 3.2 holds. Claim 3.3 is proved by an elementary obstruction theory. (The author gave a proof in Appendix of [24].)

Claim 3.3. Let $X$ be a compact oriented $(n + 2)$-manifold with boundary. Let $M$ be a closed oriented $n$-submanifold $\subset X$. We do not suppose that $M \cap X = \emptyset$ nor that $M \cap X \neq \emptyset$. Let $[M] = 0 \in H_n(X; \mathbb{Z})$. Then there is a compact oriented $(n + 1)$-manifold $W$ such that $\partial W = M$.

Suppose that $n$ is odd and that $n \geq 3$. By Lemma 3.2, a Seifert matrix of $L$ is a Seifert matrix of $K$. By [10], all even dimensional knots are slice. Hence Proposition 3.1.(3) holds when $n$ is odd and $n \geq 3$.

Suppose that $n$ is even. By [10], all even dimensional knots are slice. Hence Proposition 3.1.(3) holds when $n$ is even.

Proof of (5). Proposition 3.1.(5) holds by the definition of ribbon links.

Proof of (4). If $L = \{L_1, ..., L_m\} \subset S^{n+2} = \partial B^{n+3}$ is a slice $n$-link, then there is a disjoint union of embedded $(n + 1)$-discs, $	ilde{D} = \{D_1, ..., D_m\} \subset B^{n+3}$, such that $\tilde{D}$ meets the boundary transversely in $\partial \tilde{C}$ and that $\partial D_i = L_i$. $\tilde{D}$ is called a set of slice discs for $L$. If $L$ is a knot, $\tilde{D} = D_1$ is called a slice disc for $L = L_1$.

We prove (i). Let $L = \{L_1, L_2\}$ be embedded in $S^{2m+3} = \partial B^{2m+4} = B^{2m+4}$. Let $D_1^{2m+2} \sqcup D_2^{2m+2} \subset B^{2m+4}$ be a set of slice discs for $L$. Note that $D_1^{2m+2} \cap D_2^{2m+2} = \emptyset$. Then we can regard $h$ is a $(2m + 2)$-dimensional 1-handle which is attached to $D_1^{2m+2} \sqcup D_2^{2m+2}$. Put $D = h \cup D_1^{2m+2} \cup D_2^{2m+2}$. Then we can make a slice disc for $K$ from $D$.

4 Proof of Theorem 1.1.(1)

Theorem 1.1. (1) Let $4k + 1 \geq 5$. Let $L = \{L_1, L_2\}$ be a $(4k + 1)$-link. Then we have

$$\text{Arf}L = \text{Arf}L_1 + \text{Arf}L_2.$$ 

Proof. We prove:

Lemma 4.1. Let $K$ be a $(4k + 1)$-knot $\subset S^{4k+3} = \partial B^{4k+4} \subset B^{4k+4}$ $(4k + 1 \geq 5)$.

Suppose that there is a compact $(4k + 2)$-manifold $M$ which is embedded in $B^{4k+4}$ with the following properties.

(1) $M \cap \partial B = \partial M = K$.
(2) $M$ intersects $\partial B^{4k+4}$ transversely.
(3) $H_i(M; \mathbb{Z}) \cong H_i(\xi S^1 \times S^{4k+1} - D^{4k+2}; \mathbb{Z})$ for each $i$, where $\xi$ is a nonnegative integer and $\xi S^1 \times S^{4k+1} = S^{4k+2}$.

Then we have $\text{Arf}(K) = 0$.

Proof of Lemma 4.1. We first prove:

Claim. In order to prove Lemma 4.1, it suffices to prove the case where $K$ is a simple knot.
**Note.** See [17] for simple knots. Recall: If an \((2w + 1)\)-knot \(K\) is a simple knot, then there is a Seifert hypersurface \(V\) for \(K\) with the following propositions. (1) \(\pi_i(V) = 0 \ i \leq w\). (2) There are embedded spheres in \(V\) such that the set of the homology classes of the spheres is a set of generators of \(H_{w+1}(V; \mathbb{Z})\).

**Proof of Claim.** Take a collar neighborhood of \(S^{4k+3} = \partial B^{4k+4} \subset B^{4k+4}\). Call it \(S^{4k+3} \times [0, 1]\). Push \(M \cap (S^{4k+3} \times [0, 1])\) into the inside.

By [17], there is an embedding \(f : S^{4k+1} \times [0, 1] \hookrightarrow S^{4k+3} \times [0, 1]\) with the following properties.

1. \(f(S^{4k+1} \times \{1\})\) in \(S^{4k+3} \times \{1\}\) is \(K\).
2. \(f(S^{4k+1} \times \{0\})\) in \(S^{4k+3} \times \{0\}\) is a simple knot \(K'\).

Then \(\text{Arf}\(K\) = \text{Arf}\(K'\) and \(M \cup f(S^{4k+1} \times [0, 1])\) satisfies (1), (2), and (3) in Lemma 4.1. This completes the proof of the above Claim.

We prove Lemma 4.1 in the case where \(K\) is a simple knot. There is a Seifert hypersurface \(V\) for \(K\) with the following properties: (1) \(\pi_i(V) = 0 (1 \leq i \leq 2k)\). (2) There are embedded spheres in \(V\) such that the set of the homology classes of the spheres is a set of generators of \(H_{2k+1}(V; \mathbb{Z})\).

Then we have:

\[
H_i(V; \mathbb{Z}) \cong \begin{cases} 0 & \text{for } i \neq 2k + 1, \\ \mathbb{Z}^{2k} & \text{for } i = 2k + 1, \end{cases}
\]

\[
H_{2k+1}(V; \mathbb{Z}) \otimes \mathbb{Z}_2 \cong H_{2k+1}(V; \mathbb{Z}_2),
\]

\[
H_{2k+1}(V \cup M; \mathbb{Z}) \otimes \mathbb{Z}_2 \cong H_{2k+1}(V \cup M; \mathbb{Z}_2),
\]

and

\[
H_i(V \cup M; \mathbb{Z}_2) \cong \begin{cases} 0 & \text{for } i = 2k + 2 \\ H_{2k+1}(V; \mathbb{Z}_2) & \text{for } i = 2k + 1 \\ 0 & \text{for } i = 2k. \end{cases}
\]

By Claim 3.3, there is a compact oriented \((4k + 3)\)-submanifold \(W \subset B^{4k+4}\) such that \(\partial W = V \cup M\).

Take the Meyer-Vietoris exact sequence:

\[
H_i(V \cup M; \mathbb{Z}_2) \to H_i(W; \mathbb{Z}_2) \to H_i(W, V \cup M; \mathbb{Z}_2)
\]

Consider the following part of the above sequence:

\[
H_{2k+2}(V \cup M; \mathbb{Z}_2) \to H_{2k+2}(W; \mathbb{Z}_2) \to H_{2k+2}(W, V \cup M; \mathbb{Z}_2) \to H_{2k+1}(V \cup M; \mathbb{Z}_2) \to H_{2k+1}(V \cup M; \mathbb{Z}_2).
\]

Therefore we have

\[
0 \to H_{2k+2}(W; \mathbb{Z}_2) \to H_{2k+2}(W, V \cup M; \mathbb{Z}_2) \to \mathbb{Z}_2^{2n} \to H_{2k+1}(W; \mathbb{Z}_2) \to H_{2k+1}(W, V \cup M; \mathbb{Z}_2) \to 0.
\]

By using the Poincaré duality and the universal coefficient theorem, we have

\[
H_{2k+2}(W; \mathbb{Z}_2) \cong H_{2k+1}(W, V \cup M; \mathbb{Z}_2) \quad \text{and} \quad H_{2k+1}(W; \mathbb{Z}_2) \cong H_{2k+2}(W, V \cup M; \mathbb{Z}_2).
\]

Hence there is a set of basis \(x_1, \ldots, x_\mu, y_1, \ldots, y_\mu \in H_{2k+1}(V; \mathbb{Z}_2)\) with the following properties.

1. \(x_i \cdot x_j = 0, \ y_i \cdot y_j = 0, \ x_i \cdot y_j = \delta_{ij}\), where \(\cdot\) denote the intersection product.
2. Let \(f\) be the above map \(H_{2k+1}(V \cup M; \mathbb{Z}_2) \to H_{2k+1}(W; \mathbb{Z}_2)\). Then \(f(x_i) = 0\).
3. \(x_i\) is represented by an embedded \((2k+1)\)-sphere in \(V\).
We prove:

**Lemma.** If \( \text{mod } 2 \lk(x^+_i, x_i) = 0 \) for each \( i \), then \( \Arf K = 0 \), where \( x^+_i \) is one in \( S^2 \).

**Proof.** Put \( p : H_{2k+1}(V; \mathbb{Z}) \to H_{2k+1}(V; \mathbb{Z}_2) \). There is a basis \( x_1, \ldots, x_\mu, y_1, \ldots, y_\mu \) \( \in H_{2k+1}(V; \mathbb{Z}_2) \) with the following properties.

1. \( \bar{x}_i \cdot x_j = 0, \quad \bar{y}_i \cdot y_j = 0, \quad \bar{x}_i \cdot y_j = \delta_{ij} \), where \( \cdot \) denote the intersection product.
2. \( \bar{x}_i = p(x_i), \quad \bar{y}_i = p(y_i) \).

Then \( \lk(\bar{x}_i^+, \bar{x}_i) \equiv \lk(x^+_i, x_i) \mod 2 \) and \( \lk(\bar{y}_i^+, \bar{y}_i) \equiv \lk(y^+_i, y_i) \mod 2 \).

\( \Arf K = \mod 2 \sum_{i=1}^{\mu} \lk(\bar{x}_i^+, \bar{x}_i) \cdot \lk(\bar{y}_i^+, \bar{y}_i) = \mod 2 \sum_{i=1}^{\mu} \lk(x^+_i, x_i) \cdot \lk(y^+_i, y_i) \)

Hence the above Lemma holds.

Let \( \alpha \) be a \( \mathbb{Z}_2 \)-\((2k+2)\)-chain in \( W \) which bounds \( x_i \). Let \( \beta \) be a \( \mathbb{Z}_2 \)-\((2k+2)\)-chain in \( S^{4k+3} \) which bounds \( x_i \). Then \( \gamma = \alpha \cup \beta \) is a \( \mathbb{Z}_2 \)-\((2k+2)\)-cycle in \( B^{4k+4} \).

We prove:

**Claim.** The \( \mathbb{Z}_2 \)-intersection product \( \gamma \cdot \gamma \) in \( B^{4k+4} \) is \( \mod 2 \lk(x^+_i, x_i) \).

**Proof.** Push off \( \alpha \) to the positive direction of the normal bundle of \( W \) in \( X \). Call it \( \alpha^+ \). Note \( \alpha^+ \) bounds \( x^+_i \). By considering the collar neighborhood \( S^{4k+3} \times [0, 1] \), we have that the \( \mathbb{Z}_2 \)-intersection product \( \gamma \cdot \gamma \) is the mod 2 number of the points \( \alpha^+ \cap \beta \).

It holds that \( \mod 2 \lk(x^+, x) \) is the mod 2 number of the points \( \alpha^+ \cap \beta \). Hence \( \gamma \cdot \gamma = \mod 2 \lk(x^+, x) \).

**Claim.** The \( \mathbb{Z}_2 \)-intersection product \( \gamma \cdot \gamma \) in \( B^{4k+4} \) is zero.

**Proof.** \( H_{2k+2}(B; \mathbb{Z}_2) = 0 \). Hence \( \gamma \cdot \gamma = 0 \).

This completes the proof of Lemma 4.1.

In [23], the author proved the following. [24] includes the announcement.

**Theorem.** (See [25] [26]...) Let \( L_0 = (L_{0a}, L_{0b}) \) be a \((4k+1)\)-link \((4k + 1 \geq 5)\). Then there is a boundary link \( L_1 = (L_{1a}, L_{1b}) \) and a compact oriented submanifold \( P \sqcup Q \subset S^{4k+3} \times [0, 1] \) with the following properties.

1. \( P = S^{4k+1} \times [0, 1], \quad \text{Put } \partial P = P_0 \sqcup P_1 \). \( Q = (S^1 \times S^{4k+1}) - B^{4k+2} - B^{4k+2} \). \( \text{Put } \partial Q = Q_0 \sqcup Q_1 \).
2. \( P \) (resp. \( Q \)) is transverse to \( S^{4k+3} \times \{0, 1\} \).
3. \( (f(P_i), f(Q_i)) \text{ in } (S^{4k+1} \times \{i\}) \) is a link \( L_i \) \((i = 0, 1)\), where \( (P \sqcup Q) \cap (S^{4k+3} \times \{i\}) \) is \((f(P_i), f(Q_i))\).

In order to prove Theorem 1.1.(1), it is suffices to prove that \( \Arf L_0 = \Arf L_{0a} + \Arf L_{0b} \).

Since \( L_{0a} \) is cobordant to \( L_{1a} \), we have \( \Arf L_{0a} = \Arf L_{1a} \).

Take \( L_{0a} \sqcup L_{1b} \). By using \( Q \), we can make a manifold like \( M \) in Lemma 4.1 for \( L_{0a} \sqcup L_{1b} \). By Lemma 4.1, we have \( \Arf L_{0b} = \Arf L_{1b} \).

Since \( L_1 \) is a boundary link, there is a Seifert surface \( V_{1a} \) for \( L_{1a} \) (resp. \( V_{1b} \) for \( L_{1b} \)) such that \( V_{1a} \cap V_{1b} = \emptyset \). Let \( K_1 \) be a band-sum of \( L_1 \) along a band \( h \) such that \( h \cap \{V_{1a} \sqcup V_{1b}\} \) is the attach part of \( h \). By considering \( V_{1a}, V_{1b}, \) and \( h \), we have \( \Arf K_1 = \Arf L_{1a} + \Arf L_{1b} \).

Let \( K_0 \) be a band-sum of \( L_0 \). By Proposition 3.1.(1), \( \Arf K_0 = \Arf L_0 \). Take \( L_0 \) and \( -L_1^+ \) in \( S^{4k+3} \) such that \( L_0 \) is embedded in a ball \( B^{4k+3} \) and that \(-L_1^+ \) is embedded in \( S^{4k+3} - B^{4k+3} \). Make \( K_0 \) in \( B^{4k+3} \). Make \(-K_1^+ \) in \( S^{4k+3} - B^{4k+3} \). Take a connected-sum \( K_0 \# (-K_1^+) \). By using \( P \sqcup Q \) and band-
Proposition 1.2. Proof of Proposition 1.2 of 2-component 1-links are defined, that is, that the linking numbers are even.

5 Proof of Theorem 1.1.(2)

Theorem 1.1. (2) Let $4k + 3 \geq 3$. Let $L = (L_1, L_2)$ be a $(4k + 3)$-link. Then we have

$$
\sigma L = \sigma L_1 + \sigma L_2.
$$

Proof. Let $L = (L_+, L_-)$ be a $(4k + 3)$-link $(4k + 3 \geq 3) \subset S^{4k+5}$. Let $V$ (resp. $V_+, V_-$) be a Seifert hypersurface of $L$ (resp. $L_+, L_-$). Take $S^{4k+5} \times \{-1, 1\}$.

Regard $L = (L_+, L_-)$ as in $S^{4k+5} \times \{0\}$.

Take $L_+ \times [0, 1]$ in $S^{4k+5} \times [0, 1]$ so that $L_+ \times \{t\}$ is embedded in $S^{4k+5} \times \{t\}$ and that $L_+ \times \{0\}$ coincides with $L_+ \times \{0\}$ in $S^{4k+5} \times \{0\}$.

Take $L_- \times [-1, 0]$ in $S^{4k+5} \times [-1, 0]$ so that $L_- \times \{t\}$ is embedded in $S^{4k+5} \times \{t\}$ and that $L_- \times \{0\}$ coincides with $L_- \times \{0\}$ in $S^{4k+5} \times \{0\}$.

Then it holds that $(L_+ \times \{0\}, L_- \times \{0\})$ in $S^{4k+5} \times \{0\}$ is $L$. Take $V$ (resp. $V_+, V_-$) in $S^{4k+5} \times \{0\}$ (resp. $S^{4k+5} \times \{1\}$, $S^{4k+5} \times \{-1\}$). Put $W = V_+ \cup (L_+ \times [0, 1]) \cup (-V) \cup (L_- \times [-1, 0]) \cup V_-$. Note $W \supset L$.

By Claim 4.2, there is a compact oriented $(4k+5)$-submanifold $X \subset S^{4k+5} \times [-1, 1]$ such that $\partial X = W$. Hence

$$
\sigma(W) = 0 — (i).
$$

By the definition of $W$,

$$
\sigma(W) = \sigma(V_+) + \sigma(-V) + \sigma(V_-) = \sigma(V_+) - \sigma(V) + \sigma(V_-) — (ii).
$$

By (i)(ii), $\sigma(V) = \sigma(V_+) + \sigma(V_-)$. Hence $\sigma(L) = \sigma(L_+) + \sigma(L_-)$.

6 Proof of Proposition 1.2

Proposition 1.2. Let $L = (L_1, L_2)$ be a 1-link. Suppose that the Arf invariants of 2-component 1-links are defined, that is, that the linking numbers are even.

(1) $\text{Arf}L = \text{Arf}L_1 + \text{Arf}L_2 + \frac{1}{2}\{\sigma(L) + \text{mod}4\{\frac{1}{2}\text{lk}(L)\}\}$,

where $\sigma(L)$ is the Saito-Sato-Levine invariant.

(2) $\text{Arf}L = \text{Arf}L_1 + \text{Arf}L_2 + \text{mod}2\{\lambda(L)\}$,

where $\lambda(L)$ is the Kirk-Livingston invariant.

Proof. Put the Conway polynomial $\nabla_L(z)$ of $L = (L_1, L_2)$ to be $\nabla_L(z) = c_1 \cdot z + c_3 \cdot z^3 + \ldots$. By Lemma 3.6 of

$c_1(L) = \text{lk}(L)$ — (i).
The Saito-Sato-Levine invariant $\beta(\ ) \in \mathbb{Z}_4$ is defined in [30] for $L = (L_1, L_2)$ whose linking number is even. It is a generalization of the Sato-Levine invariant $\in \mathbb{Z}_2$ in [29].

Let $\text{lk}(L)$ be even. By Theorem 4.1 of [30],
\[
\beta^*(L) = \text{mod} 4\{2c_3(L) - \frac{1}{2}\text{lk}(L)\} \quad \text{(ii)}.
\]
By (i) and (ii),
\[
\beta^*(L) = \text{mod} 4\{2c_3(L) - \frac{1}{2}\text{lk}(L)\} \quad \text{mod} 4\{\frac{1}{2}\text{lk}(L)\} \quad \text{— (iii)}.
\]

By (iii),
\[
\text{mod} 2\{c_3(L)\} = \frac{1}{2}\{\beta^*(L) + \text{mod} 4\{\frac{1}{2}\text{lk}(L)\}\} \quad \text{— (iv)}.
\]

Note. The first $\frac{1}{2}$ in the right side make sense. We can regard the right side as an element in $\mathbb{Z}_2$.

The Kirk-Livingston invariant $\lambda(\ )$ is defined in [15]. By the definition of $\lambda(\ )$ and Theorem 6.3 of [15], it holds that: If $\text{lk}(L)$ is even,
\[
\text{mod} 4\{\lambda(L)\} = \text{mod} 4\{c_3(L)\} \quad \text{— (v)}.
\]
By (v),
\[
\text{mod} 2\{\lambda(L)\} = \text{mod} 2\{c_3(L)\} \quad \text{— (vi)}.
\]
By [20], it holds that: If $\text{lk}(L)$ is even,
\[
\text{mod} 2\{c_3(L)\} = \text{Arf}L + \text{Arf}L_1 + \text{Arf}L_2 \quad \text{— (vii)}.
\]

By (vi)(vii), Proposition 1.2.(2) holds. By (iv)(vii) Proposition 1.2.(1) holds.

Note. Let $\text{lk}(L)$ be even. Then, by (iii) and (v), we have
\[
\beta^*(L) = \text{mod} 4\{2\lambda(L) - \frac{1}{2}\text{lk}(L)\}.
\]
It is written in Addenda of [KL] that the author proved this result.

7 Proof of Theorem 1.5

Theorem 1.5.  (1) Let $4k + 1 \geq 5$. Let $L = (L_1, L_2)$ be a $(4k + 1)$-link. Let $K$ be a band-sum of $L$. Then we have
\[
\text{Arf}K = \text{Arf}L_1 + \text{Arf}L_2.
\]
(2) Let $4k + 3 \geq 3$. Let $L = (L_1, L_2)$ be a $(4k + 3)$-link. Let $K$ be a band-sum of $L$. Then we have
\[
\sigma K = \sigma L_1 + \sigma L_2.
\]

Proof of (1). By Proposition 3.1(1), $\text{Arf}K = \text{Arf}L$. By Theorem 1.1.(1), $\text{Arf}L = \text{Arf}L_1 + \text{Arf}L_2$. Hence $\text{Arf}K = \text{Arf}L_1 + \text{Arf}L_2$.

Proof of (1). By Proposition 3.1.(2), $\sigma K = \sigma L$. By Theorem 1.1.(2), $\sigma L = \sigma L_1 + \sigma L_2$. Hence $\sigma K = \sigma L_1 + \sigma L_2$.

8 Proof of Theorem 1.4

Theorem 1.4. Let $2m + 1 \geq 3$. There is a set of three $(2m + 1)$-knots $K_0$, $K_1$, $K_2$ such that the triple $(K_0, K_1, K_2)$ is not band-realizable.

Proof of the $2m + 1 = 4k + 1 \geq 5$ case. There is a $(4k + 1)$-knot $(4k + 1 \geq 5)$ whose $\text{Arf}$ invariant is zero (resp. nonzero).

Proof of the $2m + 1 = 4k + 3 \geq 3$ case. There is a $(4k + 3)$-knot $(4k + 3 \geq 3)$ whose signature is zero (resp. nonzero).
9 Proof of Theorem 1.3.(2)

Theorem 1.3.(2) Let $2m + 1 \geq 1$. Then there is a nonslice $(2m + 1)$-link $L = (L_1, L_2)$ such that $L_i$ is a trivial knot.

Proof of the $2m + 1 = 4k + 1 (\geq 1)$ case. We prove:

Proposition 9.1. There is a nonslice $(4k + 1)$-link $L = (L_1, L_2)$ $(4k + 1 \geq 1)$ such that $L_i$ is a trivial knot $(i = 1, 2)$.

**Proof.** Let $V_i$ be a Seifert surface for $L_i$. Let $V_i \cong (\mathbb{S}^{2k+1} \times \mathbb{S}^{2k+1}) - B^{4k+2}$. Suppose $V_1 \cap V_2 = \emptyset$. Let $a, b$ be basis of $H_{2k+1}(V_1, \mathbb{Z})$. Let $c, d$ be basis of $H_{2k+1}(V_2, \mathbb{Z})$.

Let $K$ be a band-sum of $L$ along a band $h$. Suppose $h \cap V$ is the attach part of $h$. Put $V = V_1 \cup V_2 \cup h$.

We can suppose that a Seifert matrix of $L_1$ associated with $V_1$ represented by basis $a, b$ is:

$$
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
\end{pmatrix}.
$$

We can suppose that a Seifert matrix of $L_2$ associated with $V_2$ represented by basis $c, d$ is:

$$
\begin{pmatrix}
0 & 1 \\
0 & 1 \\
\end{pmatrix}.
$$

We can suppose that a Seifert matrix of $K$ associated with $V$ represented by basis $a, b, c, d$ is:

$$
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
$$

One way of construction of $K$ is the following one: Take a ball $B^{4k+3} \subset \mathbb{S}^{4k+3}$. Take a submanifold $V'_1$ in $B^{4k+3}$ which is equivalent to $V_1$. Take a submanifold $V'_2$ in $\mathbb{S}^{4k+3} - B^{4k+3}$ which is equivalent to $V_2$. Let $L'_i$ be $\partial V'_i$. Take a connected-sum $L'_1 \sharp L'_2$. By using pass-moves, we can make $K$ from $L'_1 \sharp L_2$.

(Pass-moves for 1-knots are defined in [8]. Pass-moves for $(2n + 1)$-knots are defined by the author in [24], $(2n + 1 \geq 3)$.)

we can make $K$ from $L'_1 \sharp L_2$.

We have

$$
\det (A + t^t A) = \begin{pmatrix}
2 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 \\
\end{pmatrix} = -15.
$$

$A + t^t A$ is a $(4 \times 4)$-matrix. Hence $\sigma (A + t^t A) \neq 0$. Hence $K$ is nonslice. By Proposition 3.1.(4), $L$ is nonslice. This completes the proof when $2m + 1 = 4k + 1 (\geq 5)$.

Proof of the $2m + 1 = 4k + 3 (\geq 3)$ case. We prove:

Proposition 9.2. There is a nonslice $(4k + 3)$-link $L = (L_1, L_2)$ $(4k + 3 \geq 3)$ such that $L_i$ is a trivial knot $(i = 1, 2)$.

**Proof.** Let $V_i$ be a Seifert surface for $L_i$. Let $V_i \cong (\mathbb{S}^{2k+2} \times \mathbb{S}^{2k+2}) - B^{4k+4}$. Suppose $V_1 \cap V_2 = \emptyset$.

Let $a, b$ be basis of $H_{2k+2}(V_1, \mathbb{Z})$. Let $c, d$ be basis of $H_{2k+2}(V_2, \mathbb{Z})$. 

Let \( K \) be a band-sum of \( L \) along a band \( h \). Suppose \( h \cap V \) is the attach part of \( h \). Put \( V = V_1 \cup V_2 \cup h \).

We can suppose that a Seifert matrix of \( L_1 \) associated with \( V_1 \) represented by basis \( a, b \) is
\[
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix}.
\]

We can suppose that a Seifert matrix of \( L_2 \) associated with \( V_2 \) represented by basis \( c, d \) is
\[
\begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix}.
\]

We can suppose that a Seifert matrix of \( K \) associated with \( V \) represented by basis \( a, b, c, d \) is
\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

We can construct \( K \) by a similar way to the way of construction of the knot \( K \) in Proof of Proposition 9.1.

By §25, 26 of [17], \( A \) is not a Seifert matrix of any slice knot. Hence \( K \) is nonslice. By Proposition 3.1.(4), \( L \) is nonslice. This completes the proof when \( 2m + 1 = 4k + 3(\geq 3) \).

This completes the proof of Theorem 1.3.(2).

Note. (1) The above \((4k + 3)\)-knots \( K \) are discussed in [17], [19].

(2) By using the above link \( L \), we can give a short alternative proof to one of the main results of [2], [9]. The theorem is that there is a boundary \((2m + 1)\)-link \((2m + 1 \geq 1)\) which is not cobordant to any split link. Proof: If the above link \( L \) is concordant to a split link, then \( L \) is slice. Therefore \( L \) is a boundary link which is not cobordant to any split link.

[19] prove a theorem which is close to this theorem but different from this theorem.

(3) We give a question: Do we give some answers to Problems in §1 by using [3], [4], [18]?

10 Proof of Theorem 1.3.(3)

Theorem 1.3.(3) Let \( n \geq 2 \). Then there is a slice and nonribbon \( n \)-link \( L = (L_1, L_2) \) such that \( L_i \) is a trivial knot.

Proof of the \( n \geq 3 \) case. Recall that the following facts hold by Theorem 4.1 of [3] or by using the Mayer-Vietoris exact sequence. See, e.g., §14 of [17] for the Alexander polynomials. See, e.g., p.160 of [Rolfsen] and [16] for the Alexander invariant. Let \( \widetilde{X}_K \) denote the canonical infinite cyclic covering of the complement of the knot \( K \).

**Theorem 10.1. (known)** Let \( K \) be a simple \((2k + 1)\)-knot \((k \geq 1)\). Let \( \Delta_K(t) \) be the Alexander polynomial of \( K \). Suppose the \((k+1)\)-Alexander invariant \( H_{k+1}(\widetilde{X}_K; \mathbb{Q}) \cong ((\mathbb{Q}[t, t^{-1}]/\delta_K^1(t)) \oplus \cdots \oplus ((\mathbb{Q}[t, t^{-1}]/\delta_K^p(t))). \) Then \( \Delta_K(t) = a \cdot t^b \cdot \delta_K^1(t) \cdot \cdots \cdot \delta_K^p(t) \) for a rational number \( a \) and an integer \( b \) and we can put \( \Delta_K(1) = 1 \).

**Theorem 10.2. (known)** Let \( K^{(n+1)} \) be the spun knot of \( K^{(n)} \) \((n \geq 1)\). Let \( H_k(\widetilde{X}_K^{(n)}; \mathbb{Q}) \) (resp. \( H_k(\widetilde{X}_K^{(n+1)}; \mathbb{Q}) \)) denote the \( k \)-Alexander invariant of
If $K$ is not a Seifert hypersurface, then $H_2(X; \mathbb{Z}) \cong 0$. Then $H_2(X; \mathbb{Q}) \cong H_2(X; \mathbb{Q})$.

**Proposition 10.3. (known)** Let $K^{(n+1)}$ be the spun knot of $K^{(n)}$ ($n \geq 1$). If $K^{(n)}$ has a simply connected Seifert hypersurface, then $K^{(n+1)}$ has a simply connected Seifert hypersurface.

We prove:

**Proposition 10.4.** Let $K$ be a ribbon $n$-knot $\subset S^{n+2}$ ($n \geq 1$). Then $H_2(\bar{X}_K; \mathbb{Q})$ does not have $\mathbb{Q}[t, t^{-1}]-$torsion.

**Proof.** Since $K$ is ribbon, there is a Seifert hypersurface $V$ which is diffeomorphic to $S^1 \times S^n - D^{n+1}$. It holds that $H_i(V; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } i = 1, n \\ 0 & \text{for } i \neq 1, n. \end{cases}$

Let $N(K)$ be a tubular neighborhood of $K$ in $S^{n+2}$. Put $X = S^{n+2} - N(K)$. The submanifold $V \cap X$ is called $V$ again. Let $N(V)$ be a tubular neighborhood of $V$ in $X$. Put $Y = X - N(V)$. By using the Mayer-Vietoris exact sequence, it holds that $H_i(Y; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } i = 1, n \\ 0 & \text{for } i \neq 1, n. \end{cases}$

Let $p : X_K \to X$ be the canonical projection map. Put $p^{-1}(N(V)) = \Pi_{j=-\infty}^\infty V_j$. Put $p^{-1}(Y) = \Pi_{j=-\infty}^\infty \tilde{V}_j$. Suppose $\partial Y_j \subset V_j \cup Y_{j+1}$. Put $Y_j = V_j \cup Y_j \cup Y_{j+1}$. Then there is the Mayer-Vietoris exact sequence:

$$H_i(\Pi_{j=\infty}^\infty V_j; \mathbb{Q}) \to H_i(\Pi_{j=-\infty}^\infty Y_j; \mathbb{Q}) \to H_i(\bar{X}_K; \mathbb{Q}).$$

Consider the following part: $H_2(\Pi_{j=-\infty}^\infty Y_j; \mathbb{Q}) \to H_2(X_K; \mathbb{Q}) \to H_2(\Pi_{j=-\infty}^\infty V_j; \mathbb{Q}).$

Hence $0 \to H_2(X_K; \mathbb{Q}) \to \oplus \mu \mathbb{Q}[t, t^{-1}]$ is exact, where $\mu$ is a nonnegative integer. Therefore Proposition 10.4 holds.

Take a 3-link $L = (L_1, L_2) \subset S^5$ in the proof of Theorem 1.3.(2).

Suppose $L \subset \mathbb{R}^5 \subset S^5$. Let $\alpha : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4 \times \mathbb{R}$ be the map defined by $(x, y) \mapsto (x, -y)$.

Suppose that $L \subset \mathbb{R}^4 \times \{y|y \geq 0\}$, that $L_{i} \cap (\mathbb{R}^4 \times \{y|y = 0\})$ is a 3-disc $D_i$. Note $D_1 \cap D_2 = \phi$. The link $L$ is called $-L' = (-L_1, -L_2)$. The link $\{(L_1 \cup (-L_2)) - D_1, (K_2 \cup (-K_2)) - D_2\}$ is called $\tilde{L} = (\tilde{L}_1, \tilde{L}_2)$.

We prove:

**Claim.** $\tilde{L}$ is slice and nonribbon.

**Proof.** Firstly we prove that $\tilde{L}$ is slice. Take $\mathbb{R}^4 \times \mathbb{R} \times \{z|z \geq 0\}$. Regard $\mathbb{R}^4 \times \mathbb{R}$ as $\mathbb{R}^4 \times \mathbb{R} \times \{z|z = 0\}$. Put $F_0 = \mathbb{R}^4 \times \{(y, z)|y = r \cdot \cos \theta, z = r \cdot \sin \theta, r \geq 0, \theta : \text{fix.}\}$, where $0 \leq \theta < \pi$. Regard $\mathbb{R}^4 \times \mathbb{R} \times \{z|z \geq 0\}$ as the rotating of $F_0$ around the axis $\mathbb{R}^4 \times \{0\} \times \{0\}$. When rotating $F_0$, we rotate $\hat{L} - D_1 - D_2$ as well. The result of rotating $\hat{L} - D_1 - D_2$ is a set of slice discs for $\tilde{L}$. Hence $\tilde{L}$ is slice.

Secondly we prove that $\tilde{L}$ is nonribbon. Take Seifert hypersurfaces $V_1$ and $V_2$ in the proof of Theorem 1.3.(2) for $L$. Suppose $V_1, V_2 \subset \mathbb{R}^4 \times \{y|y = 0\}$. Suppose that $V_i \cap (\mathbb{R}^4 \times \{y|y = 0\}) = D_i$. Put $\tilde{V}_i = V_i \cup \alpha(V_i)$.

Let $\tilde{K}$ be a band-sum of $\tilde{L} = (\tilde{K}_1, \tilde{K}_2)$ along a band $h$. Suppose $h \cap \tilde{V}_i$ is the attach part of $h$. Then a Seifert matrix of the 3-knot $\tilde{K}$ is

$$P = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix},$$

where $X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. 

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Then \( \det(X + t^iX) \) is the Alexander polynomial of \( \tilde{K} \) (see [16, 17]). By Theorem 10.1, \( H_{m+1}(X_K; \mathbb{Q}) \) has \( \mathbb{Q}[t, t^{-1}] \)-torsion. By Proposition 10.4, \( \tilde{K} \) is nonribbon. By Proposition 3.1.(5), \( \bar{L} \) is nonribbon.

Next we prove the \( n \geq 3 \) case. Let \( L(3) = (L_1^{(3)}, L_2^{(3)}) \) be the above 3-link \( \bar{L} = (\bar{L}_1, \bar{L}_2) \). For any band-sum \( K(3) \) of \( L(3) \), \( H_2(X_{K(3)}; \mathbb{Q}) \) has \( \mathbb{Q}[t, t^{-1}] \)-torsion.

Let \( L(n+1) = (L_1^{(n+1)}, L_2^{(n+1)}) \) be a spun link of \( L(n) = (L_1^{(n)}, L_2^{(n)}) \) \( (n \geq 3) \). We can take \( h(n) \) and \( h(n+1) \) so that the band-sum \( K(n+1) \) of \( L(n+1) \) along \( h(n+1) \) is a spun knot of \( K(n) \). (Put the core of the band in the axis of the rotation.)

By Theorem 10.2 and Proposition 10.3, \( H_2(X_{K(n)}; \mathbb{Q}) \cong H_2(\tilde{X}_{K(n)}; \mathbb{Q}) \) \( (n \geq 3) \). Hence \( H_2(X_{K(n)}) \) has \( \mathbb{Q}[t, t^{-1}] \)-torsion. By Proposition 10.4, \( K(n) \) is nonribbon \( (n \geq 3) \). By Proposition 3.1.(5), \( L(n) \) is nonribbon.

Since \( L(n) \) is a spun link, \( L(n) \) is slice \( (n \geq 4) \). Hence \( L(n) \) is slice \( (n \geq 3) \). This completes the proof of the \( n \geq 3 \) case.

**Proof of the \( n = 2 \) case** In [22] the author made a nonribbon 2-link as follows:

Let \( K \) be a 2-knot. Let \( N(K) \) be a tubular neighborhood. We made a way to construct a 2-link \( \bar{L} \) of \( (L_1^K, L_2^K) \) in \( N(K) \). We proved that there is a 2-knot \( K' \) such that \( \bar{L}^{K'} \) is nonribbon.

We prove: \( \bar{L}^{K'} \) is slice. Because: Let \( K \subset S^4 = \partial B^5 = B^5 \). Take a slice disc \( D^K \subset B^5 \) for \( K \). Take a tubular neighborhood \( N(D^K) \) of \( D^K \). Note \( N(D^K) \cap S^4 = N(K) \). Suppose that \( K \) is a trivial knot and that \( D^K \) is embedded trivially in \( B^5 \). Then we can make a set of slice discs \( (D_i^K, D_j^K) \) for \( (L_1^K, L_2^K) \) such that \( (D_i^K, D_j^K) \) is embedded in \( N(D^K) \). Take a diffeomorphism map \( f : N(D^K) \to N(D^K) \) such that \( f(D_i^K) = L_i^K \). The submanifold \( f(D_i^K) \) is called \( D_i^{K'} \). Then \( (D_i^{K'}, D_j^{K'}) \) is a set of slice discs for \( L^{K'} \).

**Note.** (1) By using this section we can give a short alternative proof of the main theorem of \( K \): there is a nonribbon and slice \( n \)-knot \( (n \geq 3) \).

(Nonribbon 2-knots and nonribbon 1-knots are known before \( \bar{K} \) is written as \( \bar{K} \) quoted.)

(2) In Proposition 10.4, furthermore, we can prove that \( H_i(\tilde{X}_K; \mathbb{Z}) = 0 \) for \( 2 \leq i \leq n-1 \).

11 Proof of Theorem 1.3.(1)

**Theorem 1.3.(1)** Let \( n \geq 1 \). Then there is a nonribbon \( n \)-link \( L = (L_1, L_2) \) such that \( L_i \) is a trivial knot.

**Proof.** The \( n = 1 \) case holds because the Hopf link is an example. The \( n \geq 2 \) case follows from Theorem 1.3.(2), (3). This completes the proof.

12 Proof of Theorem 1.6.(2)

**Theorem 1.6.(2)** Let \( 2m + 1 \geq 1 \). Let \( T \) be a trivial \((2m + 1)\)-knot. Then there is a nonslice \((2m + 1)\)-knot \( K \) such that the triple \((K, T, T)\) is band-realizable.

**Proof.** \( K \) and \( L = (L_1, L_2) \) in the proof of Theorem 1.3.(2) give examples.
13 Proof of Theorem 1.6.(3)

Theorem 1.6.(3) Let $2m + 1 \geq 3$. Let $T$ be a trivial $(2m + 1)$-knot $T$. Then there is a slice and nonribbon $(2m + 1)$-knot $K$ such that the triple $(K, T, T)$ is band-realizable.

Proof. $K$ and $L = (L_1, L_2)$ in the proof of Theorem 1.3.(3) give examples.

14 Proof of Theorem 1.6.(1)

Theorem 1.6.(1) Let $n \geq 1$. Let $T$ be a trivial $n$-knot. Then there is a nonribbon $n$-knot $K$ such that the triple $(K, T, T)$ is band-realizable.

Proof. $K$ and $L = (L_1, L_2)$ in the proof of Theorem 1.3.(3) give examples for the $n \geq 3$ case. $K$ and $L = (L_1, L_2)$ in the proof of Theorem 1.3.(2) give examples for the case where $n \geq 1$ and $n$ is odd. The $n = 2$ case follows from [22]. This completes the proof.

15 Open problems

Even dimensional case of Problem C in §1 is open. If the answer to the following problem is positive, then the $n = 2$ case of Problem C is positive.

Problem 15.1 Let $L = (K_1, K_2)$ be a 2-link. Do we have: $\mu(L) = \mu(K_1) + \mu(K_2)$?

See [28] for the $\mu$-invariant of 2-knots. See [22] for the $\mu$-invariant of 2-links.

In [22], the author proved: if $L$ is a SHB link, the answer to Problem 15.1 is positive.

If the answer to Problem 15.1 is negative, then the answer to the following problem is positive.

Problem 15.2 Is there a non SHB link?

We can define an invariant for $(4k+2)$-knots corresponding to the $\mu$ invariant for 2-knots. We use this invariant and make a similar problem to Problem 15.1.

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