TRACE FORMULA FOR TWO VARIABLES

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Abstract. A natural generalization of Krein’s theorem to a pair of commuting tuples \((H_0^0, H_0^1)\) and \((H_1, H_2)\) of bounded self-adjoint operators in a separable Hilbert space \(\mathcal{H}\) with \(H_j - H_0^j = V_j \in \mathcal{B}_2(\mathcal{H})\)(set of all Hilbert-Schmidt operators on \(\mathcal{H}\)) for \(j = 1, 2\), leads to a Stokes-like formula under trace. A major ingredient in the proof is the finite-dimensional approximation result for commuting self-adjoint n-tuples of operators, a generalization of Weyl-von Neumann-Berg’s theorem.

1. Introduction

In the following, \(\mathcal{H}\) will denote the separable Hilbert space we work in; \(\mathcal{B}(\mathcal{H}), \mathcal{B}_1(\mathcal{H}), \mathcal{B}_2(\mathcal{H}), \mathcal{B}_p(\mathcal{H})\) the set of bounded, trace class, Hilbert-Schmidt class and Schatten-p class operators in \(\mathcal{H}\) respectively with \(\|\cdot\|, \|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p\) as the associated norms. Let \(H\) be a self-adjoint operator in \(\mathcal{H}\) with \(\sigma(H)\) as the spectra and \(E_H(\lambda)\) as the spectral family and let \(\text{Tr}A\) be the trace of a trace class operator \(A\). Furthermore, \(C(X)\) will be the set of all continuous functions on a compact set \(X\) and \(L^p(Y, dx)\) \((1 \leq p \leq \infty)\) be the standard Lebesgue spaces, where \(Y = [a, b]^2\) or \([a, b]^4\) for the real interval \([a, b]\) and \(dx\) is the relevant Lebesgue measure.

Let \(H\) and \(H_0\) be two possibly unbounded self-adjoint operators in a separable Hilbert space \(\mathcal{H}\) such that \(V = H - H_0 \in \mathcal{B}_1(\mathcal{H})\). Then Krein proved that there exists a unique real-valued \(L^1(\mathbb{R})\)-function \(\xi\) with support in the interval \([a, b]\) such that

\[
\text{Tr} \left[ \phi(H) - \phi(H_0) \right] = \int_a^b \phi'(\lambda)\xi(\lambda) d\lambda,
\]

for a large class of functions \(\phi\) (where \(a = \min\{\inf \sigma(H), \inf \sigma(H_0)\}\) and \(b = \max\{\sup \sigma(H), \sup \sigma(H_0)\}\)). The function \(\xi\) is known as Krein’s spectral shift function and the relation (1) is called Krein’s trace formula. The original proof of Krein \([13]\) uses analytic function theory. But in 1985, Voiculescu approached the trace formula (1) from a different direction. If \(H\) and \(H_0\) are bounded, then Voiculescu \([20]\) proved that

\[
\text{Tr} \left[ p(H) - p(H_0) \right] = \lim_{n \to \infty} \text{Tr} \left[ p(H_n) - p(H_{0,n}) \right],
\]

where \(p\) is a polynomial and \(H_n, H_{0,n}\) are finite-dimensional approximation of \(H\) and \(H_0\) respectively (constructed by adapting Weyl-von Neumann theorem). Then one constructs the spectral shift function in the finite dimensional case and finally the formula is extended to the infinite dimensional case. Later Sinha and Mohapatra \([18, 17]\) used a similar method to get the same result for the unbounded self-adjoint case. If on the other hand \(H - H_0 = V \in \mathcal{B}_2(\mathcal{H})\) is a bounded self-adjoint operator, then the result can be obtained by adapting Weyl-von Neumann-Berg’s theorem.

2010 Mathematics Subject Classification. 47A13, 47A55, 47A56.

Key words and phrases. Trace formula, Spectral integral, Stokes formula, Multiple spectral integral.
\( B_2(\mathcal{H}) \), the difference \( \phi(H) - \phi(H_0) \) is no longer of trace-class and one has to consider instead \( \phi(H) - \phi(H_0) - D^{(1)}\phi(H_0)(V) \), where \( D^{(1)}\phi(H_0)(V) \) denotes the Fréchet derivative of \( \phi \) at \( H_0 \) acting on \( V \) (see [2]) and find a trace formula for the above expression. Under the above hypothesis, Koplienko’s formula [12] asserts that there exists a unique function \( \eta \in L^1(\mathbb{R}) \) such that

\[
\text{Tr}\{\phi(H) - \phi(H_0) - D^{(1)}\phi(H_0)(V)\} = \int_{-\infty}^{\infty} \phi''(\lambda) \eta(\lambda) d\lambda
\]

for rational functions \( \phi \) with poles off \( \mathbb{R} \). Gesztesy, Pushnitski and Simon [11] gave an alternative proof of the formula (3) for the bounded case and Dykema and Skripka [10] and earlier Boyadzhiev [6] obtained the formula (3) in the semi-finite von Neumann algebra setting. The present authors used the finite-dimensional approximation idea to prove the Koplienko formula [7] as well as the third-order trace formula for both bounded and unbounded cases [8]. More recently, Potapov, Skripka and Sukochev [15] has proven the trace-formula for all orders, obtaining a kind of Taylor’s theorem under trace. In fact they have shown in [15], the existence of \( \eta_n \in L^1(\mathbb{R}) \) for \( n \in \mathbb{N} \) such that

\[
\text{Tr} \left( \phi(H_0 + V) - \sum_{k=0}^{n-1} \frac{1}{k!} D^{(k)}\phi(H_0)(V,V,\ldots,V) \right) = \int_{\mathbb{R}} \phi^{(n)}(\lambda) \eta_n(\lambda) d\lambda,
\]

for every sufficiently smooth function \( \phi \), where \( H_0 \) is a self-adjoint operator defined on a Hilbert space \( \mathcal{H} \), \( V \) is a self-adjoint operator such that \( V \in B_n(\mathcal{H}) \), \( D^{(k)}\phi(H_0)(V,V,\ldots,V) \) denotes the \( k \)-th order Fréchet derivative of \( \phi \) at \( H_0 \) acting on \( (V,V,\ldots,V) \) (see [2]) and \( \phi^{(n)} \) denotes the \( n \)-th order derivative of the function \( \phi \). It is natural to ask similar questions for a pair of commuting self-adjoint \( n \)-tuples, particularly an appropriate adaptation of Krein’s formula [11] to two and higher dimensions. Here our aim is to formulate a relevant question for a pair of commuting bounded self-adjoint tuples and use the idea of finite dimensional approximation to obtain Stokes-like formula under trace. In this context, it should be mentioned that recently Skripka [19] has studied a related problem for commuting contractions.

The Section 2 addresses the issue of finite-dimensional approximation for \( n \)-tuples of commuting bounded self-adjoint operators by adapting Berg’s [9] extension of Weyl-von Neumann theorem. Section 3 deals with the spectral integrals of operator functions and reducing the problem into a finite dimensional case. Finally in Section 4 we have established Stokes-like formula for a class of operator functions under trace.

## 2. Approximation Results

The main result in this section (Theorem 2.2) is an adaptation from the proof of Weyl-von Neumann-Berg theorem [9] for proving a finite-dimensional approximation in suitable Schatten-von Neumann \( B_p \)-ideal norm for commuting \( n \)-tuples of bounded self-adjoint operators. First we need a known simple lemma, the proof of which is given for the sake of completeness.
Lemma 2.1. Let \( A \in \mathcal{B}(\mathcal{H}) \) be such that \( 0 \leq A \leq I \). Now consider the spectral projections \( E_k = E_A \left( \bigcup_{j=1}^{2^k-1} (2^{-k}(2j-1), 2^{-k}(2j)) \right) \) for \( k \geq 1 \). Then

\[
A = \sum_{k=1}^{\infty} 2^{-k} E_k,
\]

where the right hand side of (4) converges in operator norm.

Proof. We want to show that

\[
A = \sum_{k=1}^{\infty} 2^{-k} E_k = \sum_{k=1}^{\infty} 2^{-k} E_A \left( \bigcup_{j=1}^{2^k-1} (2^{-k}(2j-1), 2^{-k}(2j)) \right)
= \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{2^k-1} E_A (2^{-k}(2j-1), 2^{-k}(2j)),
\]

since \((2^{-k}(2i-1), 2^{-k}(2i)) \cap (2^{-k}(2j-1), 2^{-k}(2j)) = \emptyset\) for \( i \neq j\) and \(1 \leq i, j \leq 2^k-1\). Let

\[
S_N = \sum_{k=1}^{N} 2^{-k} E_k = \sum_{k=1}^{N} 2^{-k} \sum_{j=1}^{2^k-1} E_A (2^{-k}(2j-1), 2^{-k}(2j)).
\]

Next by applying principle of mathematical induction on \( N \), we prove that

\[
S_N = \sum_{m=1}^{2^N-1} m 2^{-N} E_A (2^{-N} m, 2^{-N}(m+1)).
\]

For \( N = 1 \), \( S_1 = 2^{-1} E_A (2^{-1}, 2^{-1}) = 1.2^{-1} E_A (2^{-1}, 2^{-1}(1+1)) \) and therefore the equation (5) is true for \( N = 1 \).

Next we assume that the equation (6) is true for \( N = l \) i.e.

\[
S_l = \sum_{k=1}^{l} 2^{-k} \sum_{j=1}^{2^k-1} E_A (2^{-k}(2j-1), 2^{-k}(2j)) = \sum_{m=1}^{2^l-1} m 2^{-l} E_A (2^{-l} m, 2^{-l}(m+1)).
\]
Therefore by using equation (7) and changing summation index appropriately we get

\begin{equation}
S_{l+1} = \sum_{k=1}^{l+1} 2^{-k} \sum_{j=1}^{2^{k-1}} E_A \left( 2^{-k}(2j - 1), 2^{-k}(2j) \right)
\end{equation}

\begin{align*}
&= \sum_{k=1}^{l} 2^{-k} \sum_{j=1}^{2^{k-1}} E_A \left( 2^{-k}(2j - 1), 2^{-k}(2j) \right) + 2^{-l+1} \sum_{j=1}^{2^l} E_A \left( 2^{-l+1}(2j - 1), 2^{-l+1}(2j) \right) \\
&= \sum_{m=1}^{2^l-1} m 2^{-l} E_A \left( 2^{-l} m, 2^{-l}(m + 1) \right) + 2^{-l+1} \sum_{m=1}^{2^l} E_A \left( 2^{-l+1}(2m - 1), 2^{-l+1}(2m) \right) \\
&= \sum_{m=1}^{2^l-1} m 2^{-l} E_A \left( 2^{-l} m, 2^{-l}(m + 1) \right) + 2^{-l+1} E_A \left( 2^{-l+1}, 2^{-l} \right) \\
&\quad + \sum_{m=2}^{2^l} 2^{-l+1} E_A \left( 2^{-l+1}(2m - 1), 2^{-l+1}(2m) \right) \\
&= \sum_{m=1}^{2^l-1} m 2^{-l} E_A \left( 2^{-l} m, 2^{-l}(m + 1) \right) + 2^{-l+1} E_A \left( 2^{-l+1}, 2^{-l} \right) \\
&\quad + \sum_{m=1}^{2^l-1} 2^{-l+1} E_A \left( 2^{-l+1}(2m + 1), 2^{-l+1}(2m + 2) \right) .
\end{align*}

But on the other hand the first summation in the equation (8) gives us

\begin{align*}
\sum_{m=1}^{2^l-1} m 2^{-l} E_A \left( 2^{-l} m, 2^{-l}(m + 1) \right) &= \sum_{m=1}^{2^l-1} 2m 2^{-l+1} E_A \left( 2^{-l+1} 2m, 2^{-l+1}(2m + 2) \right) \\
&= \sum_{m=1}^{2^l-1} 2m 2^{-l+1} E_A \left\{ \left( 2^{-l+1} 2m, 2^{-l+1}(2m + 1) \right) \cup \left( 2^{-l+1}(2m + 1), 2^{-l+1}(2m + 2) \right) \right\} \\
&= \sum_{m=1}^{2^l-1} 2m 2^{-l+1} E_A \left( 2^{-l+1} 2m, 2^{-l+1}(2m + 1) \right) \\
&\quad + \sum_{m=1}^{2^l-1} 2m 2^{-l+1} E_A \left( 2^{-l+1}(2m + 1), 2^{-l+1}(2m + 2) \right) .
\end{align*}
Combining (8) and (9), we conclude that

\[
S_{l+1} = 2^{-(l+1)} E_A \left(2^{-(l+1)}, 2^{-l}\right) + \sum_{m=1}^{2^l - 1} 2m 2^{-(l+1)} E_A \left(2^{-(l+1)} 2m, 2^{-(l+1)} (2m + 1)\right) \\
+ \sum_{m=1}^{2^l - 1} (2m + 1) 2^{-(l+1)} E_A \left(2^{-(l+1)} (2m + 1), 2^{-(l+1)} (2m + 2)\right) \\
= \sum_{m=1}^{2^{(l+1)-1}} m 2^{-(l+1)} E_A \left(2^{-(l+1)} m, 2^{-(l+1)} (m + 1)\right).
\]

Therefore the equation (6) is true for \( N = l + 1 \), completing the induction. Thus for \( f \in \mathcal{H} \),

\[
\left(\sum_{k=1}^{\infty} 2^{-k} E_k\right) f = \lim_{N \to \infty} \left(\sum_{k=1}^{N} 2^{-k} E_k\right) f \\
= \lim_{N \to \infty} \left(\sum_{m=1}^{2^N - 1} m 2^{-N} E_A \left(m2^{-N}, (m + 1)2^{-N}\right)\right) f = \int \lambda E_A(d\lambda)f,
\]

by using the definition of spectral integral of \( A \) (see [1]). Thus

\[
A = \int \lambda E_A \left(d\lambda\right) = \sum_{k=1}^{\infty} 2^{-k} E_k.
\]

A result due to Weyl and von Neumann [14] proves that for a self-adjoint operator \( A \) that given \( \epsilon > 0 \), \( \exists K \in \mathcal{B}_2(\mathcal{H}) \) such that \( \|K\|_2 < \epsilon \) and \( A + K \) has pure point spectrum. Later Berg extended this to an \( n \)-tuples of bounded commuting self-adjoint operators \( (A_1, A_2, \ldots, A_n) \), which says that given \( \epsilon > 0 \), \( \exists \{K_j\}_{j=1}^{n} \) of compact operators such that \( \|K_j\| < \epsilon \ \forall j \) and \( \{A_j - K_j\}_{j=1}^{n} \) is a commuting family of bounded self-adjoint operators with pure point spectra. We extend in the next theorem the ideas of the proof of Berg’s result as given in [9]. It is worth mentioning that Voiculescu [21] had earlier obtained related (though not the same) results.

**Theorem 2.2.** Let \( \{A_i\}_{1 \leq i \leq n} \) be a commuting family of bounded self-adjoint operators in an infinite-dimensional separable Hilbert space \( \mathcal{H} \). Then there exists a sequence \( \{P_N\} \) of finite-rank projections such that \( \{P_N\} \uparrow I \) as \( N \to \infty \) and such that there exists a commuting family of bounded self-adjoint operators \( \{B_i^{(N)}\}_{1 \leq i \leq n} \) with the properties that for \( p \geq n \) and for each \( i \) (1 \( \leq i \leq n \)), as \( N \to \infty \),

(i) \( P_N B_i^{(N)} P_N = B_i^{(N)} P_N \),

(ii) \( \left\|A_i - B_i^{(N)}\right\|_p \to 0 \),

(iii) \( \|[A_i, P_N]\|_p \to 0 \),

(iv) \( \left\|P_N A_i P_N - B_i^{(N)} P_N\right\|_p \to 0 \) and

(v) \( \{B_i^{(N)}\} \uparrow A_i \).
Proof. One can assume without loss of generality that 0 ≤ A_i ≤ I for all 1 ≤ i ≤ n, and therefore for each i, by Lemma 2.1,

\[ A_i = \sum_{k=1}^{\infty} 2^{-k} E_k^{(i)}, \]

where \( E_k^{(i)} = E_{A_i} \left( \bigcup_{j=1}^{2^{k-1}} (2^{-k}(2j-1), 2^{-k}(2j]) \right) \) with \( E_{A_i} \) the spectral measure associated to the bounded self-adjoint operator \( A_i \). Next set for \( N \in \mathbb{N} \) (the set of natural numbers),

\[ \mathcal{L}_N \equiv \text{span}\{ \prod_{k=1}^{N} \prod_{i=1}^{n} \left( E_k^{(i)} \right)^\epsilon \ f_j \mid 1 \leq j \leq N; \ \epsilon = \pm 1 \}, \]

where \( \{ f_1, f_2, \ldots, f_N, \ldots \} \) be a countable orthonormal basis of \( \mathcal{H} \) and \( \left( E_k^{(i)} \right)^1 = E_k^{(i)} \) and \( \left( E_k^{(i)} \right)^{-1} = I - E_k^{(i)} \). Thus \( \mathcal{L}_N \) is a finite dimensional subspace of \( \mathcal{H} \) and it has the following properties:

(a) \( \mathcal{L}_N \subseteq \mathcal{L}_{N+1} \), (b) \( \left( \bigcup_{N=1}^{\infty} \mathcal{L}_N \right) = \mathcal{H} \) and (c) \( \dim(\mathcal{L}_N) \leq N \left( 2^n - 1 \right)^N + N \).

(a): Now \( \left[ \prod_{k=1}^{N} \prod_{i=1}^{n} \left( E_k^{(i)} \right)^\epsilon \right] f_j \in \mathcal{L}_N \) for 1 ≤ j ≤ N,

i.e. \( \left( E_N^{(n)} \right)^{\epsilon} \left[ \prod_{k=1}^{N-1} \prod_{i=1}^{n-1} \left( E_k^{(i)} \right)^\epsilon \right] f_j \in \mathcal{L}_N \) for 1 ≤ j ≤ N,

i.e. \( \left( I - E_N^{(n)} \right) \left[ \prod_{k=1}^{N-1} \prod_{i=1}^{n-1} \left( E_k^{(i)} \right)^\epsilon \right] f_j \in \mathcal{L}_N \) and hence

\( \left( E_N^{(n)} \right)^{-\epsilon} \left[ \prod_{k=1}^{N-1} \prod_{i=1}^{n-1} \left( E_k^{(i)} \right)^\epsilon \right] f_j \in \mathcal{L}_N \) for 1 ≤ j ≤ N,

which implies that \( \left[ \prod_{k=1}^{N-1} \prod_{i=1}^{n-1} \left( E_k^{(i)} \right)^\epsilon \right] f_j \in \mathcal{L}_N \) for 1 ≤ j ≤ N. By repeating the above argument we conclude that \( \left( E_1^{(1)} \right)^{\epsilon} f_j \in \mathcal{L}_N \) for 1 ≤ j ≤ N, i.e. \( E_1^{(1)} f_j, \left( I - E_1^{(1)} \right) f_j \in \mathcal{L}_N \) for 1 ≤ j ≤ N and hence \( \{ f_1, f_2, f_3, \ldots, f_N \} \subset \mathcal{L}_N \), proving that \( \left( \bigcup_{N=1}^{\infty} \mathcal{L}_N \right) = \mathcal{H}. \)

(b): Again \( \left[ \prod_{k=1}^{N+1} \prod_{i=1}^{n} \left( E_k^{(i)} \right)^\epsilon \right] f_j \in \mathcal{L}_{N+1} \) for 1 ≤ j ≤ N + 1. But on the other hand

\[ \left[ \prod_{k=1}^{N+1} \prod_{i=1}^{n} \left( E_k^{(i)} \right)^\epsilon \right] f_j = \prod_{i=1}^{n} \left( E_N^{(i)} \right)^{\epsilon} \left[ \prod_{k=1}^{N} \prod_{i=1}^{n} \left( E_k^{(i)} \right)^\epsilon \right] f_j \]
\[ = (E^{(n)}_{N+1})^\epsilon \left[ \prod_{i=1}^{n-1} (E^{(i)}_{N+1})^\epsilon \right] \left[ \prod_{k=1}^N \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right] f_j \] and hence

\[ \left( I - E^{(n)}_{N+1} \right) \left[ \prod_{i=1}^{n-1} (E^{(i)}_{N+1})^\epsilon \right] \left[ \prod_{k=1}^N \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right] f_j, \]

\[ \left( I - E^{(n)}_{N+1} \right) \left[ \prod_{i=1}^{n-1} (E^{(i)}_{N+1})^\epsilon \right] \left[ \prod_{k=1}^N \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right] f_j \in \mathcal{L}_{N+1} \text{ for } 1 \leq j \leq N + 1. \]

Therefore, \[ \prod_{i=1}^{n-1} (E^{(i)}_{N+1})^\epsilon \left[ \prod_{k=1}^N \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right] f_j \in \mathcal{L}_{N+1} \text{ for } 1 \leq j \leq N + 1 \text{ and hence } \mathcal{L}_N \subseteq \mathcal{L}_{N+1}. \]

(c): According to the definition of \( E^{(i)}_k \), it follows that for each fixed \( k \in \mathbb{N} \),

\[ \sum_{\epsilon = \pm 1} \prod_{i=1}^n (E^{(i)}_k)^\epsilon = I. \]

We claim that for any fixed vector \( f \in \mathcal{H} \), the span \( \left\{ \prod_{k=1}^N \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right\} f : \epsilon = \pm 1 \right\} \) contains at most \((2^n - 1)^N\) linearly independent vectors, without counting \( f \). We prove our claim by induction on \( N \). For \( N = 1 \), because of the identity \((10)\) we conclude that the span \( \left\{ \prod_{i=1}^n (E^{(1)}_1)^\epsilon \right\} f : \epsilon = \pm 1 \right\} \) contains at most \((2^n - 1)\) linearly independently vectors besides \( f \). Since \{\( A_j \}_{1 \leq i \leq n} \) is a commuting family, we have the following:

\[ \text{span} \left\{ \prod_{k=1}^{N+1} \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right\} f : \epsilon = \pm 1 \} := \text{span} \left\{ \prod_{i=1}^n (E^{(i)}_{N+1})^\epsilon \left[ \prod_{k=1}^N \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right] f : \epsilon = \pm 1 \right\} \]

and thus by the induction hypothesis, span \( \left\{ \prod_{k=1}^N \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right\} f : \epsilon = \pm 1 \right\} \) contains at most \((2^n - 1)^N\) linearly independent vectors, other than \( f \). Therefore using the equation \((10)\) we conclude that the span \( \left\{ \prod_{k=1}^{N+1} \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right\} f : \epsilon = \pm 1 \right\} \) contains at most

\[ 2^n (2^n - 1)^N - (2^n - 1)^N = (2^n - 1)^{N+1} \]

number of linearly independent vectors, other than \( f \) itself, completing the induction. Hence for any fixed vector \( f \in \mathcal{H} \), the span \( \left\{ \prod_{k=1}^N \prod_{i=1}^n (E^{(i)}_k)^\epsilon \right\} f : \epsilon = \pm 1 \right\} \) contains the maximum of possible \((2^n - 1)^{N+1}\) linearly independent vectors, including the vector \( f \). This implies that \( \mathcal{L}_N \) contains at most \( N \{(2^n - 1)^{N+1}\} \) number of linearly independent vectors and therefore
\[ \dim(\mathcal{L}_N) \leq N(2^n - 1)^N + N. \]

Now we set \( P_N \) to be the finite rank projection associated with the finite dimensional subspace \( \mathcal{L}_N \). Then by (a) and (b) the sequence \( \{P_N\} \) increases to \( I \).

Next we define

\[ B_i^{(N)} = \sum_{k=1}^{N} 2^{-k} E_k^{(i)} + \sum_{k=N+1}^{\infty} 2^{-k} E_k^{(i)} (I - P_k), \]

and observe that since \( \{E_k^{(i)}\}_{1 \leq k \leq N; 1 \leq i \leq n} \) is a commuting family and since each member of that family for fixed \( k \) commutes with \( P_i \) for \( 1 \leq k \leq l \), it is easy to verify that

\[ E_k^{(i)} (I - P_k) E_k^{(i)} (I - P_{k'}) = (I - P_k) (I - P_{k'}) E_k^{(i)} E_k^{(i)} \]

\[ = (I - P_{k'}) E_k^{(i)} E_k^{(i)} = E_k^{(i)} (I - P_{k'}) E_k^{(i)} (I - P_k), \]

where we have assumed without loss of generality that \( k \leq k' \). Thus \( \{B_i^{(N)}\}_{1 \leq i \leq n} \) is a commuting family of positive self-adjoint contractions and since \( (I - P_k) P_N = 0 \) for \( k \geq N + 1 \), it follows that

\[ P_N B_i^{(N)} P_N = B_i^{(N)} P_N = \sum_{k=1}^{N} 2^{-k} E_k^{(i)} P_N, \]

and hence \( B_i^{(N)} P_N \), a finite dimensional self-adjoint operator in the Hilbert space \( P_N \mathcal{H} \). Furthermore, \( A_i - B_i^{(N)} = \sum_{k=N+1}^{\infty} 2^{-k} E_k^{(i)} P_k \) and

\[ \left\| A_i - B_i^{(N)} \right\|_n \leq \sum_{k=N+1}^{\infty} 2^{-k} \left\| P_k \right\|_n \leq \sum_{k=N+1}^{\infty} 2^{-k} \left[ k \left\{ 1 + (2^n - 1)^k \right\} \right]^{\frac{1}{n}} \]

\[ = \sum_{k=N+1}^{\infty} k^{\frac{1}{n}} \left[ 2^{-nk} + (1 - 2^{-n})^k \right]^{\frac{1}{n}} \]

\[ \leq \sum_{k=N+1}^{\infty} k^{\frac{1}{n}} 2^{-k} + \sum_{k=N+1}^{\infty} k^{\frac{1}{n}} \left[ (1 - 2^{-n}) \right]^{\frac{k}{n}}, \]

where we have used that for \( a, b > 0, (a + b)^{\frac{1}{n}} \leq (a^{\frac{1}{n}} + b^{\frac{1}{n}}) \). Since for fixed \( n, (1 - 2^{-n})^{\frac{k}{n}} < 1 \), and since \( \sum_{k=1}^{\infty} k^{\frac{1}{n}} \alpha^k < \infty \) for \( \alpha < 1 \), it follows that for each \( i(1 \leq i \leq n), \left\| A_i - B_i^{(N)} \right\|_n \rightarrow 0 \) as \( N \rightarrow \infty \). Therefore for any \( p \geq n \) we get

\[ \left\| A_i - B_i^{(N)} \right\|_p \leq \left( \left\| A_i \right\| + \left\| B_i^{(N)} \right\| \right)^{\left(1 - \frac{1}{p}\right)} \left\| A_i - B_i^{(N)} \right\|_n \]

\[ \leq 2^{\left(1 - \frac{1}{p}\right)} \left\| A_i - B_i^{(N)} \right\|_n \rightarrow 0 \]

as \( N \rightarrow \infty \). Thus

\[ \left\| A_i, P_N \right\|_p = \left\| A_i - B_i^{(N)}, P_N \right\|_p \leq 2 \left\| A_i - B_i^{(N)} \right\|_p \rightarrow 0 \text{ as } N \rightarrow \infty, \]

(12)
for any \( p \geq n \). Moreover,
\[
\left\| P_N A_i P_N - P_N B_i^{(N)} P_N \right\|_p = \left\| P_N \left( A_i - B_i^{(N)} \right) P_N \right\|_p \leq \left\| A_i - B_i^{(N)} \right\|_p \to 0
\]
as \( N \to \infty \) for any \( p \geq n \). Now from (11) we have
\[
B_i^{(N+1)} = \sum_{k=1}^{N+1} 2^{-k} E_k^{(i)} P_{N+1}
\]
\[
= \sum_{k=1}^{N} 2^{-k} E_k^{(i)} (P_{N+1} - P_N) + \sum_{k=1}^{N} 2^{-k} E_k^{(i)} P_N + 2^{-(N+1)} E_N^{(i)} P_{N+1}.
\]
Thus
\[
B_i^{(N+1)} - B_i^{(N)} = \sum_{k=1}^{N} 2^{-k} (P_{N+1} - P_N) E_k^{(i)} (P_{N+1} - P_N) + 2^{-(N+1)} P_N E_N^{(i)} P_{N+1} \geq 0,
\]
since \( \{P_N\} \uparrow I \). Finally by using (13) and the fact that \( P_N A_i P_N \to A_i \), strongly as \( N \to \infty \), we conclude that \( B_i^{(N)} P_N \to A_i \), strongly as \( N \to \infty \). This completes the proof. \( \square \)

**Remark 1.** The choice that \( 0 \leq A_i \leq I \) does not materially affect the calculations of Theorem 2.2. For if \( C_i \in \mathcal{B}(\mathcal{H}) \) \( (1 \leq i \leq n) \), then we can set
\[
A_i = (2\|C_i\|)^{-1} C_i + \frac{1}{2} I
\]
so that \( 0 \leq A_i \leq I \) and thus \( C_i = 2\|C_i\| (\sum 2^{-k} E_k^{(i)} - \frac{1}{2} I) \). Thus choosing
\[
B_i^{(N)} = 2\|C_i\| \left\{ \sum_{k=1}^{N} 2^{-k} E_k^{(i)} + \sum_{k=N+1}^{\infty} (I - P_k) E_k^{(i)} - \frac{1}{2} I \right\}
\]
one has \( \|[C_i, B_i^{(N)}]\|_p = 2\|C_i\|\|[A_i, B_i^{(N)}]\|_p \to 0 \) as \( N \to \infty \) for \( p \geq n \).

3. **Spectral Integrals of Operator Functions and Stokes-like Formula**

In this section we are going to define spectral integrals of operator functions in the next few lemmas.

**Lemma 3.1.** Let \( H \) be a bounded self-adjoint operator in \( \mathcal{H} \) with spectrum in \([a, b]\) and let \( A : [a, b] \to \mathcal{B}(\mathcal{H}) \) be operator norm Hölder continuous with Hölder index \( k > \frac{1}{2} \), that is,
\[
\|A(\alpha_1) - A(\alpha_2)\| \leq C|\alpha_1 - \alpha_2|^k,
\]
where \( C \) is some positive constant and \( k > \frac{1}{2} \). Then
\[
\int_a^b A(\alpha) E_H(d\alpha)
\]
is well-defined as a operator norm Riemann-Stieltjes integral, where \( E_H(.) \) is the spectral measure corresponding to the bounded self-adjoint operator \( H \).
Proof. (see also [8]) Let $P \equiv \{\Delta_i\}_{i=1}^n$ be a partition of the interval $[a, b]$. Let $P' \equiv \{\Delta_{ij}\}_{1 \leq i \leq n; 1 \leq j \leq m}$ be another partition of $[a, b]$, which is finer than $P$ (i.e. $P' \supseteq P$) and is obtained by dividing each interval $\Delta_i$ of $P$ into equal number (say $m$) of subintervals $\{\Delta_{ij}\}$ (i.e. $\Delta_i = \bigcup_{j=1}^m \Delta_{ij}$). In particular one can use diadic partitions. For $f \in \mathcal{H}$, consider the following Riemann-Stieltjes sums
\[
\Sigma_P f \equiv \sum_{i=1}^n A(\xi_i) E_H(\Delta_i)f \quad \text{and} \quad \Sigma_{P'} f = \sum_{i=1}^n \sum_{j=1}^m A(\xi_{ij}) E_H(\Delta_{ij})f,
\]
where $\xi_i \in \Delta_i$ and $\xi_{ij} \in \Delta_{ij}$. Hence, if we write $|\Delta_i| = \text{length of the interval } \Delta_i$, we get that
\[
(\Sigma_P - \Sigma_{P'}) f = \sum_{i=1}^n \sum_{j=1}^m [A(\xi_i) - A(\xi_{ij})] E_H(\Delta_{ij})f.
\]
Therefore by hypothesis and applying triangle-inequality, Cauchy-Schwartz inequality and using the fact that $|\xi_i - \xi_{ij}| \leq |\Delta_i|$, we conclude that
\[
\| (\Sigma_P - \Sigma_{P'}) f \| \leq \sum_{i=1}^n \sum_{j=1}^m \| A(\xi_i) - A(\xi_{ij}) \| \| E_H(\Delta_{ij})f \| \leq C \sum_{i=1}^n \sum_{j=1}^m |\xi_i - \xi_{ij}| \| E_H(\Delta_{ij})f \|
\]
\[
\leq C \left( \sum_{i=1}^n \sum_{j=1}^m |\xi_i - \xi_{ij}|^2 \right)^{1/2} \left( \sum_{i=1}^n \sum_{j=1}^m \| E_H(\Delta_{ij})f \|^2 \right)^{1/2} = C \| f \| \left( \sum_{i=1}^n \sum_{j=1}^m |\xi_i - \xi_{ij}|^2 \right)^{1/2}
\]
\[
\leq C \| f \| \left( \sum_{i=1}^n \sum_{j=1}^m |\Delta_i|^2 \right)^{1/2} = C \| f \| \| P \| \frac{1}{2} (b - a)^{1/2},
\]
where $\| P \|$ is the norm ($\equiv \max_{1 \leq i \leq n} |\Delta_i|$) of the partition $P \equiv \{\Delta_i\}_{i=1}^n$. Hence
\[
\| \Sigma_P - \Sigma_{P'} \| = \sup_{f \in \mathcal{H} : f \neq 0} \frac{\| (\Sigma_P - \Sigma_{P'}) f \|}{\| f \|} \leq C m^{1/2} \| P \|^{k - \frac{1}{2}} (b - a)^{1/2} \to 0 \text{ as } \| P \| \to 0,
\]
proving the existence of the integral $\int_a^b A(\alpha) E_H(d\alpha)$ as an operator norm Riemann-Stieltjes integral. \qed

Lemma 3.2. Let $A, B, C$ be three bounded self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that $\sigma(A), \sigma(B), \sigma(C) \subseteq [a, b]$. Let $\phi : [a, b]^2 \to \mathbb{C}$ be a bounded measurable function. Then the symbol $\int_A^B \phi(x, C) dx$, defined as:
\[
\int_A^B \phi(x, C) dx \equiv \int_a^b \left( \int_a^x \phi(x, C) dx \right) [E_B(d\alpha) - E_A(d\alpha)],
\]
(where $E_A(\cdot), E_B(\cdot)$ are the spectral measures of the operators $A, B$ respectively), exists as a bounded operator.
Proof. Since \( \phi \) is bounded and measurable on \([a,b]^2\), then by spectral theorem we write
\[
\phi(x, C) = \int_a^b \phi(x, y) E_C(dy),
\]
where \( E_C(.) \) is the spectral measure corresponding to the self-adjoint operator \( C \) and hence \( \phi(x, C) \) is a bounded operator with operator norm
\[
\| \phi(x, C) \| = \sup\{|\phi(x, y)| : y \in [a, b]\} = \| \phi \|_{\infty}
\]
for all \( x \in [a, b] \). Now the map \([a, b] \ni x \mapsto \phi(x, C) \in \mathcal{B}(\mathcal{H}) \) is a \( \mathcal{B}(\mathcal{H}) \)-valued bounded measurable function and hence the integral \( \int_a^b \phi(x, C) dx \) exists as a Bochner integral for each fixed \( \alpha \in [a, b] \). Moreover, the map \( T : [a, b] \rightarrow \mathcal{B}(\mathcal{H}) \) defined by
\[
T(\alpha) = \int_a^\alpha \phi(x, C) dx
\]
is operator norm Hölder continuous with Hölder index \( 1 \left( > \frac{1}{2} \right) \). i.e.
\[
\| T(\alpha_1) - T(\alpha_2) \| \leq \| \phi \|_{\infty}|\alpha_1 - \alpha_2| \text{ for } \alpha_1, \alpha_2 \in [a, b].
\]
Thus by Lemma 3.1, the integrals \( \int_a^b T(\alpha) E_A(d\alpha) \) and \( \int_a^b T(\alpha) E_B(d\alpha) \) exist as a operator norm Riemann-Stieltjes integral. Hence
\[
\int_a^b \left( \int_a^\alpha \phi(x, C) dx \right) [E_B(d\alpha) - E_A(d\alpha)] \text{ is well-defined as a bounded operator and we denote it by the symbol } \int_A^B \phi(x, C) dx.
\]

We derive two formulae for the trace of a Stokes-like expression, one in terms of a spectral function and the other in terms of divided differences. First we need a simple lemma.

**Lemma 3.3.** Let \( \psi \in L^\infty([a,b]^2) \). Then there exists two measurable functions \( \phi_1, \phi_2 \) on \([a, b] \times [a, b] \) such that \( \phi_1 \) and \( \phi_2 \) are differentiable (almost everywhere) with respect to the second and first variable respectively with bounded derivatives such that,
\[
\frac{\partial \phi_2}{\partial x}(x, y) - \frac{\partial \phi_1}{\partial y}(x, y) = \psi(x, y).
\]
Moreover \( \phi_1 \) and \( \phi_2 \) are Lipschitz in the second and first variable respectively, uniformly with respect to the other variable. Conversely, if \( \phi_1 \) and \( \phi_2 \) are two measurable functions differentiable with respect to the second and first variable respectively with bounded measurable derivatives, then
\[
\psi(x, y) = \frac{\partial \phi_2}{\partial x}(x, y) - \frac{\partial \phi_1}{\partial y}(x, y) \in L^\infty([a,b]^2).
\]

**Proof.** Let \( \psi \in L^\infty([a,b]^2) \) and \( \phi_1, \phi_2 \) be defined as:
\[
\phi_1(x, y) = -\frac{1}{2} \int_a^y \psi(x, t) dt + \psi_1(x) = \tilde{\phi}_1(x, y) + \psi_1(x) \text{ and }
\]
\[
\phi_2(x, y) = \frac{1}{2} \int_a^x \psi(t, y) dt + \psi_2(y) = \tilde{\phi}_2(x, y) + \psi_2(y),
\]
(14)
where \(\psi_1, \psi_2\) are two measurable functions on \([a, b]\). Thus from the definition of \(\phi_1\) and \(\phi_2\) it follows that \(\phi_1, \phi_2\) are two measurable functions, differentiable with respect to the second and first variable respectively with \(\frac{\partial \phi_1}{\partial y}(x, y) = -\frac{1}{2}\psi(x, y)\) and \(\frac{\partial \phi_2}{\partial x}(x, y) = \frac{1}{2}\psi(x, y)\) almost everywhere and hence \(\frac{\partial \phi_2}{\partial x}(x, y) - \frac{\partial \phi_1}{\partial y}(x, y) = \psi(x, y)\). Moreover, \(\left|\frac{\partial \phi_2}{\partial x}(x, y)\right|, |\frac{\partial \phi_2}{\partial x}(x, y)| \leq \frac{1}{2}\|\psi\|_\infty\). Again from the definition of \(\phi_1\), we conclude that \(\phi_1(x, y_1) - \phi_1(x, y_2) = \int \psi(x, t)dt\), which implies that \(|\phi_1(x, y_1) - \phi_1(x, y_2)| \leq \|\psi\|_\infty|y_1 - y_2|\) and hence \(\phi_1\) is Lipschitz in second variable uniformly with respect to the first variable. By repeating the above argument we conclude that \(\phi_2\) is also Lipschitz in first variable uniformly with respect to the second variable.

Converse part follows from the hypotheses.

The following theorem tells us about the trace formula for two variables in finite dimension.

**Theorem 3.4.** Let \(P\) and \(Q\) be two finite dimensional projections in \(\mathcal{H}\) and let \((H_0^1, H_0^2)\) and \((H_1, H_2)\) be two commuting pairs of self-adjoint operators acting in the reducing subspaces \(PH\) and \(QH\) respectively. Let

\[
\psi \in L^\infty([a, b]^2),
\]

where

\[
\sigma(H_1), \sigma(H_2), \sigma(H_0^1), \sigma(H_0^2) \subseteq [a, b].
\]

Then

\[
\mathcal{I} \equiv \text{Tr}\{\int_{H_0^1} P\phi_1(x, H_0^2) Qdx + \int_{H_0^2} Q\phi_2(H_1, y) Pdy
\]

\[
+ \int_{H_0^1} P\phi_1(x, H_2) Qdx + \int_{H_2} Q\phi_2(H_1, y) Pdy\}
\]

(15)

\[= \text{Tr}\{\int_{H_0^1} P[\phi_1(x, H_0^2) - \phi_1(x, H_2)] Qdx \]

\[+ \int_{H_2} Q[\phi_2(H_1, y) - \phi_2(H_0^2, y)] Pdy\}

\[= \int_a^b \int_a^b \left[\frac{\partial \phi_2(x, y)}{\partial x} - \frac{\partial \phi_1(x, y)}{\partial y}\right] \xi(x, y)dxdy = \int_a^b \int_a^b \psi(x, y)\xi(x, y)dxdy,
\]

where

\[
\xi(x, y) = \text{Tr}\{Q[E_{H_1}(x) - E_{H_0^1}(x)] P[E_{H_2}(y) - E_{H_0^2}(y)] Q\}
\]

and \(E_{H_1}(\cdot), E_{H_2}(\cdot), E_{H_1^0}(\cdot), E_{H_0^2}(\cdot)\) are the spectral measures of the operators \(H_1, H_2, H_0^1, H_0^2\) respectively and \(\phi_1, \phi_2\) are same as in (13).
Proof. Let $\psi \in L^\infty([a,b]^2)$ and $\phi_1, \phi_2$ be defined as in (14). We note that

$$\phi_1(x, H_2^0) - \phi_1(x, H_2) = \widetilde{\phi}_1(x, H_2^0) - \widetilde{\phi}_1(x, H_2),$$

$$\phi_2(H_1, y) - \phi_2(H_1^0, y) = \widetilde{\phi}_2(H_1, y) - \widetilde{\phi}_2(H_1^0, y);$$

and that

$$\frac{\partial \phi_2}{\partial x}(x, y) - \frac{\partial \phi_1}{\partial y}(x, y) = \frac{\partial \widetilde{\phi}_2}{\partial x}(x, y) - \frac{\partial \widetilde{\phi}_1}{\partial y}(x, y) = \psi(x, y)$$

almost everywhere.

By Lemma 3.3, $\widetilde{\phi}_1(x, \cdot)$ is Lipschitz uniformly with respect to $x$ and $\widetilde{\phi}_2(\cdot, y)$ is Lipschitz uniformly with respect to $y$ and we note that by Lemma 3.1, the integrals

$$\int_{H_1} \Phi \left[ \phi_1(x, H_2^0) - \phi_1(x, H_2) \right] Q \, dx$$

and

$$\int_{H_2^0} Q \left[ \phi_2(H_1, y) - \phi_2(H_1^0, y) \right] P \, dy$$

exist in $B(H)$. Thus

$$\text{Tr} \left\{ \int_{H_1} \Phi \left[ \phi_1(x, H_2^0) - \phi_1(x, H_2) \right] Q \, dx \right\} = \text{Tr} \left\{ \int_{H_1} \Phi \left[ \widetilde{\phi}_1(x, H_2^0) - \widetilde{\phi}_1(x, H_2) \right] Q \, dx \right\}$$

(16)

$$= \text{Tr} \left\{ \int_a^b \left( \int_a^x \Phi \left[ \phi_1(x, H_2^0) - \phi_1(x, H_2) \right] Q \, dx \right) E_{H_1}(\alpha) - E_{H_1^0}(\alpha) \right\}.$$
above expression is equal to

\[
\begin{align*}
&= -\text{Tr} \{ \int_a^b d\alpha \left( \int_a^b \tilde{\phi}_1(\alpha, \beta) P \left[ E_{H_2^0}(d\beta) - E_{H_2}(d\beta) \right] Q \right) \left[ E_{H_1}(\alpha) - E_{H_1^0}(\alpha) \right] P \} \\
&= -\text{Tr} \int_a^b d\alpha \left( \int_a^b \tilde{\phi}_1(\alpha, \beta) P \left[ E_{H_2^0}(\beta) - E_{H_2}(\beta) \right] Q \right) \left[ E_{H_1}(\alpha) - E_{H_1^0}(\alpha) \right] P \\
&= -\text{Tr} \int_a^b d\alpha \left( \int_a^b \frac{\partial \tilde{\phi}_1}{\partial \beta}(\alpha, \beta) P \left[ E_{H_2^0}(\beta) - E_{H_2}(\beta) \right] Q \right) \left[ E_{H_1}(\alpha) - E_{H_1^0}(\alpha) \right] P d\beta \\
&= -\int_a^b \int_a^b \frac{\partial \tilde{\phi}_1}{\partial \beta}(\alpha, \beta) \text{Tr} \{ Q \left[ E_{H_1}(\alpha) - E_{H_1^0}(\alpha) \right] P \left[ E_{H_2}(\beta) - E_{H_2^0}(\beta) \right] Q \} d\alpha d\beta,
\end{align*}
\]

since by Fubini’s theorem, the iterated integral is equal to the double integral in this case. Thus we have

\[
\begin{align*}
\text{Tr} \left\{ \int_{H_1^0}^{H_1} P \left[ \phi_1(x, H_2^0) - \phi_1(x, H_2) \right] Q dx \right\}
&= -\int_a^b \int_a^b \frac{\partial \tilde{\phi}_1}{\partial \beta}(x, y) \text{Tr} \{ Q \left[ E_{H_1}(x) - E_{H_1^0}(x) \right] P \left[ E_{H_2}(y) - E_{H_2^0}(y) \right] Q \} dxdy.
\end{align*}
\]

(17)

By an identical set of computations as above, we have that

\[
\begin{align*}
\text{Tr} \left\{ \int_{H_2^0}^{H_2} Q \left[ \phi_2(H_1, y) - \phi_2(H_1^0, y) \right] P dy \right\}
&= \text{Tr} \left\{ \int_a^b \left( \int_a^\alpha Q \left[ \tilde{\phi}_2(H_1, y) - \tilde{\phi}_2(H_1^0, y) \right] P dy \right) \left[ E_{H_2}(d\alpha) - E_{H_2^0}(d\alpha) \right] \right\} \\
&= \int_a^b \int_a^b \frac{\partial \tilde{\phi}_2}{\partial x}(x, y) \text{Tr} \{ Q \left[ E_{H_1}(x) - E_{H_1^0}(x) \right] P \left[ E_{H_2}(y) - E_{H_2^0}(y) \right] Q \} dxdy.
\end{align*}
\]

(18)
Combining (17) and (18), we get

\[ I \equiv \text{Tr} \left\{ \int_{H_1^0} H_1 P \left[ \phi_1 (x, H_2^0) - \phi_1 (x, H_2) \right] Q dx \right. \]
\[ + \left. \int_{H_2^0} Q \left[ \phi_2 (H_1, y) - \phi_2 (H_1^0, y) \right] P dy \right\} \]
\[ = \int_{a}^{b} \int_{a}^{b} \left[ \frac{\partial \tilde{\phi}_2}{\partial x} (x, y) - \frac{\partial \tilde{\phi}_1}{\partial y} (x, y) \right] \xi(x, y) dxdy = \int_{a}^{b} \int_{a}^{b} \psi(x, y) \xi(x, y) dxdy, \]

where \( \xi(x, y) = \text{Tr} \left\{ Q \left[ E_{H_1} (x) - E_{H_1^0} (x) \right] P \left[ E_{H_2} (y) - E_{H_2^0} (y) \right] \right\} \).

The following theorem finds another formula for the above Stokes-like expression \( I \) of operator functions under trace in terms of divided differences, which is useful to control the measure generated by \( \xi \).

**Theorem 3.5.** Under the hypotheses of Theorem 3.4,

\[ I = \int_{[a,b]^2} \int_{[a,b]^2} \psi(x, y) dxdy \]
\[ \int_{[a,b]^2} \int_{[a,b]^2} \left[ \frac{\partial \tilde{\phi}_2}{\partial x} (x, y) - \frac{\partial \tilde{\phi}_1}{\partial y} (x, y) \right] \xi(x, y) dxdy = \int_{a}^{b} \int_{a}^{b} \psi(x, y) \xi(x, y) dxdy, \]

where \( H_0 = (H_0^1, H_0^2), H = (H_1, H_2) \) and \( E_{H_0^0} (.) \) and \( E_H (.) \) are the spectral measures of the operators tuples \( H_0 \) and \( H \) respectively on the Borel sets of \( [a,b]^2 \) and where \( \langle ., . \rangle_2 \) denotes the inner product of the Hilbert space \( \mathcal{B}_2 (\mathcal{H}) \).

**Proof.** In finite dimensional space \( \mathcal{H} \), using the ideas of double spectral integrals ([4],[5]), we get that

\[ P \left[ \tilde{\phi}_1 (x, H_2^0) - \tilde{\phi}_1 (x, H_2) \right] Q = \int_{a}^{b} \left[ \tilde{\phi}_1 (x, y_1) - \tilde{\phi}_1 (x, y_2) \right] P E_{H_2^0} (dy_1) E_{H_2} (dy_2) Q \]
\[ = - \int_{a}^{b} \int_{a}^{b} \tilde{\phi}_1 (x, y_1) - \tilde{\phi}_1 (x, y_2) \frac{PE_{H_2} (dy_1) (H_2 - H_2^0) E_{H_2} (dy_2)}{y_1 - y_2} \]
\[ = \frac{1}{2} \int_{[a,b]^2} \psi(x, t) dt \]
\[ \int_{[a,b]^2} \frac{PE_{H_2} (dy_1) (H_2 - H_2^0) E_{H_2} (dy_2)}{y_1 - y_2} Q. \]
Therefore
\[
\int_a^b \left( \int_a^\alpha P \left[ \tilde{\phi}_1(x, H_2^0) - \tilde{\phi}_1(x, H_2) \right] Q dx \right) \left[ E_{H_1}(d\alpha) - E_{H_1^0}(d\alpha) \right]
\]
\[
= \frac{1}{2} \int_a^b \left( \int_a^\alpha \int_{[y_1, y_2]} \frac{\int_y^1 \psi(x, t) dt}{y_1 - y_2} P E_{H_2^0}(dy_1) \left( H_2 - H_2^0 \right) E_{H_2}(dy_2) Q dx \right) \left[ E_{H_1}(d\alpha) - E_{H_1^0}(d\alpha) \right]
\]
\[
= \frac{1}{2} \left[ \int_{[y_1, y_2]} \frac{\int_y^1 \psi(x, t) dt}{y_1 - y_2} \right. \\
\left. \cdot \int_{[y_1, y_2]} \int_a^\alpha \int_a^\beta \frac{\psi(x, t) dt}{y_1 - y_2} \left[ E_{H_1}(d\alpha) \left( H_1 - H_1^0 \right) E_{H_1^0}(d\beta) \right] \right]
\]

where we have used Fubini’s theorem to interchange the order of the integration and utilized the ideas of double spectral integrals ([4], [5]). Thus by using trace properties, Fubini’s theorem and the fact that \((H_1, H_2^0), (H_1, H_2)\) are two commuting pairs of self-adjoint operators, we conclude that

\[
(19)
\]
\[
\text{Tr}\left\{ \int_{H_1^0}^H P \left[ \tilde{\phi}_1(x, H_2^0) - \tilde{\phi}_1(x, H_2) \right] Q dx \right\}
\]
\[
= \frac{1}{2} \text{Tr}\left\{ \int_{[y_1, y_2]} \frac{\int_y^1 \psi(x, t) dt}{y_1 - y_2} P E_{H_2^0}(dy_1) \left( H_2 - H_2^0 \right) E_{H_2}(dy_2) Q \right\} \cdot \\
\int_{(x_1, x_2) \in [a, b]^2} \frac{\int \int \psi(x, t) dt dx}{x_1 y_1 (x_1 - x_2)(y_1 - y_2)} \left[ E_{H_1}(dx_1) \left( H_1 - H_1^0 \right) E_{H_1^0}(dx_2) \right] \right\}
\]
The conclusion of the theorem follows by combining the equations (19) and (20).
The next theorem shows how Theorem 2.2 can be used for the reduction to finite dimension. We deal here with the simpler case of \( n = 2 \). In the statement of the theorem below we apply Theorem 2.2 to the pairs \((H_1^0, H_2^0)\) and \((H_1, H_2)\) to get two commuting pairs of finite dimensional self-adjoint operators \( (H_1^{0(N)}, H_2^{0(N)}) \) and \( (H_1^{(N)}, H_2^{(N)}) \) in \( P_N^0 \mathcal{H} \) and \( P_N \mathcal{H} \) respectively, such that

\[
(21) \quad \| [H_j^0, P_N^0] \|_p, \| P_N^0 H_j^0 P_N^0 - H_j^{0(N)} P_N^0 \|_p \to 0 \text{ as } N \to \infty \text{ for } p \geq 2, \ j = 1, 2 \text{ and }
\]

\[
(22) \quad \| [H_j, P_N] \|_p, \| P_N H_j P_N - H_j^{(N)} P_N \|_p \to 0 \text{ as } N \to \infty \text{ for } p \geq 2, \ j = 1, 2,
\]

where \( P_N^0, P_N \) are projections increasing to \( I \) (i.e. \( P_N^0, P_N \uparrow I \)).

**Theorem 3.6.** Let \((H_1^0, H_2^0)\) and \((H_1, H_2)\) be two commuting pairs of bounded self-adjoint operators in a separable Hilbert space \( \mathcal{H} \) such that \( H_j - H_j^0 \equiv V_j \in \mathcal{B}_2(\mathcal{H}) \) and such that \( \sigma(H_j), \sigma(H_j^0) \subseteq [a, b] \) for \( j = 1, 2 \). Furthermore let

\[
p_1(x, y) = \sum_{0 \leq i+j \leq n} c(i, j)x^i y^j, \quad p_2(x, y) = \sum_{0 \leq r+s \leq m} d(r, s)x^r y^s
\]

be two polynomials in \([a, b]^2\) with complex coefficients. Then

\[
\text{Tr}\{ \int_{H_1^0}^{H_1} p_1(x, H_2^0) dx + \int_{H_2^0}^{H_2} p_2(H_1, y) dy \} + \int_{H_1^0}^{H_1} p_1(x, H_2) dx + \int_{H_2^0}^{H_2} p_2(H_1, y) dy
\]

\[
= \lim_{N \to \infty} \text{Tr}\{ \int_{H_1^{0(N)}}^{H_1^{(N)}} P_N^0 \left[ p_1 \left( x, H_2^{0(N)} \right) - p_1 \left( x, H_2^{(N)} \right) \right] P_N dx
\]

\[
+ \int_{H_2^{0(N)}}^{H_2^{(N)}} P_N \left[ p_2 \left( H_1^{(N)}, y \right) - p_2 \left( H_1^{0(N)}, y \right) \right] P_N^0 dy \}
\]

\[
= \lim_{N \to \infty} \int_{a}^{b} \int_{a}^{b} \left[ \frac{\partial p_2}{\partial x}(x, y) - \frac{\partial p_1}{\partial y}(x, y) \right] \xi_N(x, y) dx dy,
\]

where \( \xi_N(x, y) = \text{Tr}\{P_N \left[ E_{H_1^{0(N)}}(x) - E_{H_1^{0(N)}}(x) \right] P_N^0 \left[ E_{H_2^{0(N)}}(y) - E_{H_2^{0(N)}}(y) \right] P_N \} \) and \( E_{H_1^{0(N)}}(\cdot), E_{H_2^{0(N)}}(\cdot), E_{H_1^{(N)}}(\cdot), E_{H_2^{(N)}}(\cdot) \) are the spectral measures of the operators \( H_1^{0(N)}, H_2^{0(N)}, H_1^{(N)}, H_2^{(N)} \) respectively and \( P_N^0, P_N \) are the projections obtained by applying Theorem 2.2 to the pairs \((H_1^0, H_2^0)\) and \((H_1, H_2)\) respectively, as mentioned above.

We need a preliminary result for the proof of the above theorem.
Lemma 3.7. Let $H_j^0, H_j, H_j^{(N)}, H_j^{(N)}$, $P_N^0, P_N$ be as above for $j = 1, 2$. Then

(i) $\left\| P_N^0 (H_j^0)^k - (P_N^0 H_j^0 P_N^0)^k \right\|_2$ and $\left\| P_N (H_j)^k - (P_N H_j P_N)^k \right\|_2 \rightarrow 0$ as $N \rightarrow \infty$ for $j = 1, 2$ and $k \geq 1$.

(ii) $\left\| (H_j^0)^k P_N^0 - (P_N H_j^0 P_N^0)^k \right\|_2$ and $\left\| (H_j)^k P_N - (P_N H_j P_N)^k \right\|_2 \rightarrow 0$ as $N \rightarrow \infty$ for $j = 1, 2$ and $k \geq 1$.

(iii) $\left\| P_N^0 \left[ (P_N H_j^0 P_N^0)^k - (P_N H_j P_N)^k \right] P_N \right\|_2$ is uniformly bounded in $N$ for $j = 1, 2$ and $k \geq 1$.

(iv) $\lim_{N \rightarrow \infty} \text{Tr} \left\{ P_N^0 \left[ (H_2^0)^k - (H_2)^k \right] P_N \left[ (H_1)^l - (H_2)^l \right] P_N^0 \right\} = \lim_{N \rightarrow \infty} \text{Tr} \left\{ P_N^0 \left[ (H_2^{(N)})^k - (H_2^{(N)})^l \right] P_N \left[ (H_1^{(N)})^l - (H_1^{(N)})^l \right] P_N^0 \right\}$ for $k, l \geq 1$.

(v) $\left\| P_N \left( H_j^{(N)} - H_j^{(N)} \right) P_N^0 - P_N V_j P_N^0 \right\|_2 \rightarrow 0$ as $N \rightarrow \infty$ for $j = 1, 2$.

Proof. (i) Note that

$$
\begin{align*}
P_N^0 (H_j^0)^k - (P_N^0 H_j^0 P_N^0)^k &= P_N^0 (H_j^0)^k P_N^0 - (P_N^0 H_j^0 P_N^0)^k + P_N^0 (H_j^0)^k (P_N^0)^\perp \\
&= -P_N^0 \sum_{l=0}^{k-1} (P_N^0 H_j^0 P_N^0)^{k-l-1} [P_N^0 H_j^0 P_N^0 - H_j^0] (H_j^0)^l P_N^0 + P_N^0 (H_j^0)^k (P_N^0)^\perp \\
&= -P_N^0 \sum_{l=0}^{k-1} (P_N^0 H_j^0 P_N^0)^{k-l-1} P_N^0 [H_j^0, P_N^0] (H_j^0)^l P_N^0 + P_N^0 \left[ P_N^0, (H_j^0)^k \right]
\end{align*}
$$

and therefore

$$
\left\| P_N^0 (H_j^0)^k - (P_N^0 H_j^0 P_N^0)^k \right\|_2 \leq k \left\| (H_j^0)^{k-1} \left[ H_j^0, P_N^0 \right] \right\|_2 + \left\| P_N^0, (H_j^0)^k \right\|_2 \rightarrow 0 \text{ as } N \rightarrow \infty,
$$

by using (21) for $j = 1, 2$. By similar computations as above and using the equation (22) we achieve the other conclusion of (i).

(ii) The proof of (ii) follows from (i) by taking the adjoint of the expressions involved.

(iii) By (i) and (ii) we conclude that

$$
\left\| P_N^0 \left[ (P_N H_j^0 P_N^0)^k - (P_N H_j P_N)^k \right] - \left[ (H_j^0)^k - (H_j)^k \right] P_N \right\|_2 \rightarrow 0
$$

as $N \rightarrow \infty$ and this completes the proof of (iii) since $(H_j^0)^k - (H_j)^k \in B_2(\mathcal{H})$ for $j = 1, 2$ and $k \geq 1$. 


\textbf{(iv)} By (i) and (ii) we have for \(k,l \geq 1\) that
\[
\lim_{N \to \infty} \text{Tr}\left\{ P_N^0 \left[ (H_2^0)^k - (H_2^0)^k \right] P_N \left[ (H_1)^j - (H_1)^j \right] P_N^0 \right\} \\
= \lim_{N \to \infty} \text{Tr}\left\{ P_N^0 \left[ (P_N H_2^0 P_N)^k - (P_N H_2 P_N)^k \right] P_N \left[ (P_N H_1 P_N)^j - (P_N H_1^0 P_N)^j \right] P_N^0 \right\}.
\]
The conclusion of (iv) follows from the above equality and using the equations (21) and (22).

\textbf{(v)} First note that
\[
P_N \left( H_j^{(N)} - H_j^{0(N)} \right) P_N^0 - P_N \left( H_j - H_j^0 \right) P_N^0 \\
= P_N \left( P_N H_j^{(N)} - P_N H_j P_N \right) P_N^0 - P_N \left[ P_N, H_j \right] P_N^0 - P_N \left( H_j^{0(N)} P_N^0 - P_N^0 H_j^0 P_N^0 \right) \\
+ P_N \left[ H_j^0, P_N^0 \right] P_N^0
\]
and therefore
\[
\left\| P_N \left( H_j^{(N)} - H_j^{0(N)} \right) P_N^0 - P_N \left( H_j - H_j^0 \right) P_N^0 \right\|_2 \\
\leq \left\| P_N H_j^{(N)} - P_N H_j P_N \right\|_2 + \left\| [P_N, H_j] \right\|_2 + \left\| H_j^{0(N)} P_N^0 - P_N^0 H_j^0 P_N^0 \right\|_2 + \left\| [H_j^0, P_N^0] \right\|_2,
\]
which converges to 0 as \(N \to \infty\) for \(j = 1, 2\) by using the equations (21) and (22). \(\Box\)

\textbf{Proof of Theorem 3.6} Note that
\[
\int_{H_1^0}^{H_1} \left[ p_1(x, H_2^0) - p_1(x, H_2) \right] dx \\
= \sum_{2 \leq i+j \leq n \ ; \ i,j \in \mathbb{N}} \frac{c(i, j)}{i+1} \left[ (H_2^0)^i - (H_2)^i \right] \left[ (H_1)^{i+1} - (H_1^0)^{i+1} \right],
\]
which is trace class by hypotheses. Similarly,
\[
\int_{H_2^0}^{H_2} \left[ p_2(H_1, y) - p_2(H_1^0, y) \right] dy \\
= \sum_{2 \leq r+s \leq m \ ; \ r,s \in \mathbb{N}} \frac{d(r, s)}{s + 1} \left[ (H_1)^r - (H_1^0)^r \right] \left[ (H_2)^{s+1} - (H_2^0)^{s+1} \right] \in B_1(\mathcal{H}).
\]
On the other hand, in the right hand side of (23),
\[
\int_{H_1^{(N)}}^{H_1^{0(N)}} P_N^0 \left[ p_1 \left( x, H_2^{0(N)} \right) - p_1 \left( x, H_2^{(N)} \right) \right] P_N dx \\
= \sum_{2 \leq i+j \leq n \ ; \ i,j \in \mathbb{N}} \frac{c(i, j)}{i+1} P_N^0 \left[ (H_2^{0(N)})^j - (H_2^{(N)})^j \right] P_N \left[ (H_1^{(N)})^{i+1} - (H_1^{0(N)})^{i+1} \right]
\]
and
\[
\sum_{j=1}^{H_2(N)} P_N \left[ p_2(H_1^{(N)}, y) - p_2(H_1^{(0)(N)}, y) \right] P_N^0 dy
\]
\[
= \sum_{2 \leq r+s \leq m, r,s \in \mathbb{N}} \frac{d(r,s)}{s+1} P_N \left[ (H_1^{(N)})^r - (H_1^{(0)(N)})^r \right] P_N^0 \left[ (H_2^{(N)})^{s+1} - (H_2^{(0)(N)})^{s+1} \right].
\]

In order to establish the first equality in (23), it is enough to note that by Lemma 3.7 (iv) for \( i, j \in \mathbb{N}, \)
\[
\text{Tr}\{[(H_0^0)^i - (H_2)^i] [(H_1)^i - (H_0^0)^i]\} = \lim_{N \to \infty} \text{Tr}\{P_N^0 [(H_0^0)^i - (H_2)^i] P_N [(H_1)^i - (H_0^0)^i] P_N^0\}
\]
\[
= \lim_{N \to \infty} \text{Tr}\{P_N^0 (H_0^0)^j - (H_2)^j \} P_N (H_1)^i - (H_0^0)^i \} P_N^0\}.
\]
The last equality in (23) follows immediately by applying Theorem 3.4 for the operator tuples \((H_1^{(0)(N)}, H_2^{(0)(N)})\) and \((H_1^{(N)}, H_2^{(N)})\) in the reducing subspaces \(P_N^0 \mathcal{H}\) and \(P_N \mathcal{H}\) respectively. This completes the proof. \(\square\)

4. Trace Formula for Two Variables in Infinite Dimension

In this section we prove our main result namely Stokes-like formula under trace for pairs of commuting bounded self-adjoint operators.

**Theorem 4.1.** Let \( H_0^0 = (H_1^0, H_2^0) \) and \( H = (H_1, H_2) \) be two commuting tuples of bounded self-adjoint operators in a separable Hilbert space \( \mathcal{H} \) such that \( H_j - H_0^0 = V_j \in \mathcal{B}(\mathcal{H}) \) for \( j = 1, 2 \). Then there exists a unique complex Borel measure \( \mu \) on \([a,b]^2\) such that
\[
\text{Tr}\left\{ \int_{H_1^0}^{H_1} p_1(x, H_2^0) dx + \int_{H_2^0}^{H_2} p_2(H_1, y) dy + \int_{H_1}^{H_1^0} p_1(x, H_2) dx + \int_{H_2}^{H_2^0} p_2(H_1, y) dy \right\}
\]
\[
= \text{Tr}\left\{ \int_{H_1^0}^{H_1} [p_1(x, H_2^0) - p_1(x, H_2)] dx + \int_{H_2^0}^{H_2} [p_2(H_1, y) - p_2(H_1^0, y)] dy \right\}
\]
\[
= \int_{[a,b]^2} \left[ \frac{\partial p_2}{\partial x} (x, y) - \frac{\partial p_1}{\partial y} (x, y) \right] \mu(dx \times dy),
\]
where \( p_1 \) and \( p_2 \) are two polynomials in \([a,b]^2\) i.e. \( p_1(x, y) = \sum_{0 \leq i+j \leq n} c(i, j)x^i y^j \) and \( p_2(x, y) = \sum_{0 \leq r+s \leq m} d(r, s)x^r y^s \) with complex coefficients and \( \bigcup_{j=1}^2 \{ \sigma(H_j) \cup \sigma(H_j^0) \} \subseteq [a,b]. \)
Proof. By Theorem 3.6 corresponding to the tuples $H^0$, $H$, we conclude that

$$\text{Tr}\{ \int_{H^0} p_1(x, H^0) dx + \int_{H^2} p_2(H_1, y) dy \\ + \int_{H^0} p_1(x, H_2) dx + \int_{H^2} p_2(H_1^0, y) dy \}$$

(25)

$$= \lim_{N \to \infty} \int_a^b \int_a^b \left[ \frac{\partial p_2}{\partial x}(x, y) - \frac{\partial p_1}{\partial y}(x, y) \right] \xi_N(x, y) dxdy,$$

where $\xi_N(x, y) = \text{Tr}\{ P_N \left[ E_{H_1^N}(x) - E_{H_1^0}(x) \right] P_N^0 \left[ E_{H_2^N}(y) - E_{H_2^0}(y) \right] P_N \}.$

For a Borel subset $\Delta$ of $[a, b]^2$, set

$$\mu_N(\Delta) = \int_{\Delta} \xi_N(x, y) dxdy$$

and observe that $\|\xi_N\|_\infty \leq 4\|P_N\|_2\|P_N^0\|_2$ and therefore $\mu_N$ is a complex Borel measure on $[a, b]^2$. Next we want to show that there exists a complex Borel measure $\mu$ on $[a, b]^2$ such that for a suitable subsequence $\{N_k\}$, $\mu_{N_k}$ converges weakly to $\mu$ i.e.

$$\lim_{k \to \infty} \int_{[a, b] \times [a, b]} \psi(x, y) \mu_{N_k}(dx \times dy) = \int_{[a, b] \times [a, b]} \psi(x, y) \mu(dx \times dy)$$

for all $\psi(x, y) \in C([a, b]^2)$. Let $\psi(x, y) \in C([a, b]^2)$ and let $\phi_j (j = 1, 2)$ be given as in (14).

Then by applying Theorem 3.5 for the pairs $(H_1^N, H_2^N)$ and $(H_1^0, H_2^0)$, we have that

(26)

$$\mathcal{I}_N = \text{Tr}\{ \int_{H_1^0} P_N^0 \phi_1 \left( x, H_2^0 \right) P_N dx + \int_{H_2^0} P_N^0 \phi_2 \left( H_1^0, y \right) P_N dy \\ + \int_{H_1^0} P_N^0 \phi_1 \left( x, H_2^0 \right) P_N dx + \int_{H_2^0} P_N^0 \phi_2 \left( H_1^0, y \right) P_N dy \}$$

$$= \int_{[a, b]^2} \int_{[a, b]^2} \psi(x, y) dxdy$$

$$\left\langle P_N^0 \left( H_1^N - H_1^0 \right) P_N, P_N E_{H_1^N}(dx_2) E_{H_2^0}(dy_1) \left( H_2^N - H_2^0 \right) E_{H_1^N}(dx_1) E_{H_2^0}(dy_2) P_N \right\rangle_2.$$
Next we recall from Lemma 3.7 that \( \sup_{N} \left\| P_{N} \left( H_{j}^{(N)} - H_{j}^{0(N)} \right) P_{N}^{0} \right\|_{2} < C_{j} < \infty \) for \( j = 1, 2 \).

Thus by the property of a spectral measure in a Hilbert space, one has that

\[
\left\| P_{N}^{0} \left( H_{1}^{(N)} - H_{1}^{0(N)} \right) P_{N}, P_{N}^{0} E_{H_{1}^{0(N)}}(dx_{2}) E_{H_{2}^{0(N)}}(dy_{1}) \left( H_{2}^{(N)} - H_{2}^{0(N)} \right) E_{H_{1}^{(N)}}(dx_{1}) E_{H_{2}^{(N)}}(dy_{2}) P_{N} \right\|_{2} \leq \left\| P_{N}^{0} \left( H_{1}^{(N)} - H_{1}^{0(N)} \right) P_{N} \right\|_{2} \left\| P_{N}^{0} \left( H_{2}^{(N)} - H_{2}^{0(N)} \right) P_{N} \right\|_{2} < C_{1}C_{2}.
\]

On the other hand, by Theorem 3.4 we conclude that

\[
I_{N} = \int_{a}^{b} \int_{a}^{b} \left[ \frac{\partial \phi_{2}(x, y)}{\partial x} - \frac{\partial \phi_{1}}{\partial y}(x, y) \right] \xi_{N}(x, y) dy dx = \int_{[a, b]^{2}} \psi(x, y) \mu_{N}(dx \times dy),
\]

leading to the equality:

\[
\int_{[a, b]^{2}} \psi(x, y) \mu_{N}(dx \times dy)
= \int_{[a, b]^{2}} \int_{[a, b]^{2}} \psi(x, y) dxdy
\]

\[
\left\| P_{N}^{0} \left( H_{1}^{(N)} - H_{1}^{0(N)} \right) P_{N}, P_{N}^{0} E_{H_{1}^{0(N)}}(dx_{2}) E_{H_{2}^{0(N)}}(dy_{1}) \left( H_{2}^{(N)} - H_{2}^{0(N)} \right) E_{H_{1}^{(N)}}(dx_{1}) E_{H_{2}^{(N)}}(dy_{2}) P_{N} \right\|_{2},
\]

for all \( \psi(x, y) \in C([a, b]^{2}) \). Thus

\[
\left\| \int_{[a, b]^{2}} \psi(x, y) \mu_{N}(dx \times dy) \right\| < C_{1}C_{2} \| \psi\|_{\infty}.
\]

Since, by the Riesz’s theorem (page 251, [16]), \( C([a, b]^{2}) \) is separable in sup-norm, one can apply Helley’s theorem (page 171, [16]) to conclude that there exists a subsequence \( \mu_{N_{k}} \) of \( \mu_{N} \) such that \( \mu_{N_{k}} \) converges weakly to a unique complex Borel measure \( \mu \) on \( [a, b]^{2} \). i.e.

\[
\lim_{k \to \infty} \int_{[a, b]^{2}} \psi(x, y) \mu_{N_{k}}(dx \times dy) = \int_{[a, b]^{2}} \psi(x, y) \mu(dx \times dy) \forall \psi \in C([a, b]^{2}).
\]

This completes the proof, by applying this conclusion to the right hand side of the equation (23).

\( \Box \)

**Acknowledgment:** The second author acknowledges a conversation (nearly two decades old) with Voiculescu, which inspired this work, the primary aim of which is to obtain a two variable spectral shift function \( \xi \) (like \( \xi_{N} \) in Theorem 3.4 for finite dimensions) instead of the associated measure \( \mu \) in Theorem 3.1.
The first author is grateful to Jawaharlal Nehru Centre for Advanced Scientific Research and Indian Statistical Institute, Bangalore Centre for warm hospitality and also thanks Department of Atomic Energy, India through N.B.H.M Post Doctoral Fellowship for financial support. The second author thanks Jawaharlal Nehru Centre for Advanced Scientific Research for support.

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