Matrix Subspaces of $L_1$ *

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Abstract

If $E = \{e_i\}$ and $F = \{f_i\}$ are two 1-unconditional basic sequences in $L_1$ with $E$ $r$-concave and $F$ $p$-convex, for some $1 \leq r < p \leq 2$, then the space of matrices $\{a_{i,j}\}$ with norm $\|\{a_{i,j}\}\|_{E(F)} = \|\sum k \sum l a_{k,l}f_l e_k\|$ embeds into $L_1$. This generalizes a recent result of Prochno and Schütz.

1 Introduction

Recall that a basis $E = \{e_i\}_{i=1}^N$ of a finite ($N < \infty$) or infinite ($N = \infty$) dimensional real or complex Banach space is said to be $K$-unconditional if $\|\sum a_i e_i\| \leq K \|\sum b_i e_i\|$ whenever $|a_i| = |b_i|$ for all $i$. Given a finite or infinite 1-unconditional basis, $E = \{e_i\}_{i=1}^N$, and a sequence of Banach spaces $\{X_i\}_{i=1}^N$ denote by $(\sum \bigoplus X_i)_E$ the space of sequences $x = (x_1, x_2, \ldots)$, $x_i \in X_i$, for which the norm $\|x\| = \|\sum_i \|x_i\| e_i\|$ is finite.

If $X$ has a 1-unconditional basis $F = \{f_j\}$ then $(\sum \bigoplus X)_E$ can be represented as a space of matrices $A = \{a_{i,j}\}$, denoted $E(F)$, with norm

$$\|A\|_{E(F)} = \|\sum_i \|\sum_j a_{i,j}f_j e_i\|_1\|.$$

In [PS], Prochno and Schütz gave a sufficient condition for bases $E$ and $F$ of two Orlicz sequence spaces which assure that $E(F)$ embeds into $L_1$. Here we generalize this result by giving a sufficient condition on two unconditional

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bases $E, F$, which assure that $E(F)$ embeds into $L_1$. As we shall see this condition is also “almost” necessary.

Recall that an unconditional basis $\{e_i\}$ is said to be $p$-convex (resp. $r$-concave) with constant $K$ provided that for all $n$ and all $x_1, x_2, \ldots, x_n$ in the span of $\{e_i\}$,

$$
\| \sum_{i=1}^{n} (|x_i|^p)^{1/p} \| \leq K (\sum_{i=1}^{n} \|x_i\|^p)^{1/p}
$$

(resp.

$$
(\sum_{i=1}^{n} \|x_i\|^r)^{1/r} \leq K (\sum_{i=1}^{n} (|x_i|^r)^{1/r} \| )
$$

Here, for $x = \sum x(j)e_j$ and a positive $\alpha$, $|x|^\alpha = \sum |x(j)|^\alpha e_j$.

$L_p$ will denote here $L_p([0,1], \lambda)$, $\lambda$ being the Lebesgue measure. As is known and quite easy to prove, any 1-unconditional basic sequence in $L_p$, $1 \leq p \leq 2$ (resp. $2 \leq p < \infty$), is $p$-convex (resp. $p$-concave) with constant depending only on $p$. It is also worthwhile to remind the reader that any $K$-unconditional basic sequence in $L_p$ is equivalent, with a constant depending only on $p$ and $K$ to a 1-unconditional basic sequence in $L_p$. It is due to Maurey [Ma] (see also [Wo, III.H.10]), that for every $1 \leq r < p \leq 2$, the span of every $p$-convex 1-unconditional basic sequence in $L_1$ embeds into $L_p$ and also embeds into $L_r$ after change of density; i.e., there exists a probability measure $\mu$ on $[0,1]$ so that this span is isomorphic (with constants depending on $r, p$ and the $p$-convexity constant only) to a subspace of $L_r([0,1], \mu)$ on which the $L_r(\mu)$ and the $L_1(\mu)$ norms are equivalent.

If $M$ is an Orlicz function then the Orlicz space $\ell_M$ embeds into $L_p$ if and only if $M(t)/t^p$ is equivalent to an increasing function and $M(t)/t^2$ is equivalent to a decreasing function. This happens if and only if the natural basis of $\ell_M$ is $p$-convex and 2-concave.

Theorem 1 below states in particular that if $E$ and $F$ are two 1-unconditional basic sequences in $L_1$ with $E$ $r$-concave and $F$ $p$-convex for some $1 \leq r < p \leq 2$ then $E(F)$ embeds into $L_1$. When specializing to Orlicz spaces, this implies the main result of [PS].

2 The main result

Theorem 1 Let $E = \{e_i\}$ be a 1-unconditional basic sequence in $L_1$ with $\{e_i\}$ $r$-concave with constant $K_1$ and let $X$ be a subspace of $L_1([0,1], \mu)$ for
some probability measure \( \mu \) satisfying \( \|x\|_{L_r([0,1],\mu)} \leq K_2 \|x\|_{L_1([0,1],\mu)} \) for some constant \( K_2 \) and all \( x \in X \). Then \((\bigoplus X_E)\) embeds into \( L_1 \) with a constant depending on \( K_1, K_2 \) and \( r \) only.

Consequently, if \( E = \{e_i\} \) and \( F = \{f_i\} \) are two 1-unconditional basic sequences in \( L_1 \) with \( E \) \( r \)-concave with constant \( K_1 \) and \( F \) \( p \)-convex with constant \( K_2 \), for some \( 1 \leq r < p \leq 2 \), then the space of matrices \( A = \{a_{k,i}\} \) with norm
\[
\|A\|_{E(F)} = \| \sum_k \| \sum_l a_{k,l} f_l \| e_k \|
\]
embeds into \( L_1 \) with a constant depending only on \( r, p, K_1 \) and \( K_2 \).

**Proof:** The \( p \)-convexity of \( \{f_i\} \) implies that after a change of density the \( L_1 \) and \( L_r \) norms are equivalent on the span of \( \{f_i\} \). See [Ma]. That is, there is a probability measure \( \mu \) on \([0,1]\) and a constant \( K_3 \), depending only on \( r, p \) and \( K_2 \) such that \( \| \sum_j f_j \|_{L_r([0,1],\mu)} \leq K_3 \| \sum_j f_j \|_{L_1([0,1],\mu)} \) for some sequence \( \{f_j\} \) 1-equivalent, in the relevant \( L_1 \) norm, to \( \{f_j\} \), and for all coefficients \( \{a_i\} \). It thus follows that the second part of the theorem follows from the first part.

To prove the first part, in \( L_1([0,1] \times [0,1], \lambda \times \mu) \) consider the tensor product of the span of \( \{e_i\} \) and \( X \), that is the space of all functions of the form \( \sum_i e_i \otimes x_i, x_i \in X \) for all \( i \), where \( e_i \otimes x_i(s,t) = e_i(s)x_i(t) \). Then, by the 1-unconditionality of \( \{e_i\} \) and the triangle inequality,
\[
\| \sum_i e_i \otimes x_i \|_1 = \int \| \sum_i |x_i(t)|e_i \|_{L_1([0,1],\lambda)} d\mu(t) \\
\geq \| \sum_i (\int |x_i(t)| d\mu(t)) e_i \|_{L_1([0,1],\lambda)} \\
= \| \sum_i \|x_i\| e_i \|.
\]

On the other hand, by the 1-unconditionality and the \( r \)-concavity with con-
stant $K_1$ of $\{e_i\}$ (used in integral instead of summation form),
\[
\| \sum_i e_i \otimes x_i \|_1 = \int \int \left| \sum_i |x_i(t)|e_i(s)|d\lambda(s)d\mu(t) \right|
\leq \left( \int \left| \sum_i |x_i(t)|e_i(s)|d\lambda(s) \right|^r d\mu(t) \right)^{1/r}
\leq K_1 \left( \int |x_i(t)|e_i(s)\|_L_{1([0,1],\lambda)}^r d\mu(t) \right)^{1/r}
\leq K_1 K_2 \left( \sum_i \int |x_i(t)|d\mu(t)e_i\|_L_{1([0,1],\lambda)} \right)
\leq K_1 K_2 \left( \sum_i \|x_i\|e_i \right)
\]

As is explained in the introduction the main result of [PS] follows as corollary.

**Corollary 1** If $M$ and $N$ are Orlicz functions such that $M(t)/t^r$ is equivalent to a decreasing function, $N(t)/t^p$ is equivalent to an increasing function and $N(t)/t^2$ is equivalent to a decreasing function then $\ell_M(\ell_N)$ embeds into $L_1$.

**Remark 1** The role of $L_1$ in Theorem 1 can easily be replaced with $L_s$ for any $1 \leq s \leq r$.

**Remark 2** If the bases $E$ and $F$ are infinite, say, and the smallest $r$ such that $E$ is $r$-concave is larger than the largest $p$ such that $F$ is $p$-convex, then $E(F)$ doesn’t embed into $L_1$. This follows from the fact that in this case it is known that $\ell_r^n$ uniformly embed as blocks of $E$ and $\ell_p^n$ uniformly embed as blocks of $F$, for some $r > p$, while it is known that in this case $\ell_r^n(\ell_p^n)$ do not uniformly embed into $L_1$.

This still leaves the case $r = p$, which is not covered in Theorem 1 open:

- If $E$ and $F$ are two 1-unconditional basic sequences in $L_1$ with $E$ $r$-concave and $F$ $r$-convex, does $E(F)$ embed into $L_1$?
In the case that $E$ is an Orlicz space the problem above has a positive solution. We only sketch it. By the factorization theorem of Maurey mentioned above ([Wo, III.H.10] is a good place to read it), and a simple compactness argument (to pass from the finite to the infinite case), it is enough to consider the case that $F$ is the $\ell_r$ unit vector basis. If the basis of $\ell_M$ is $r$-concave, then the $2/r$-convexification of $\ell_M$ (which is the space with norm $\|\{a_i^{2/r}\}\|^{r/2}_{\ell_M}$) embeds into $L_{2/r}$. This is again an Orlicz space, say, $\ell_{M'\ell}$. Now, tensoring with the Rademacher sequence (or a standard Gaussian sequence) we get that $\ell_{M'\ell}(\ell_2)$ embeds into $L_{2/r}$. We now want to $2/r$ concavify back, staying in $L_1$, so as to get that $\ell_M(\ell_r)$ embeds into $L_1$. This is known to be possible (and is buried somewhere in [MS]): If $\{x_i\}$ is a 1-unconditional basic sequence in $L_s$, $1 < s \leq 2$ then its $s$-concavification (which is the space with norm $\|\{a_i^{1/s}\}\|_{\ell_M^s}$) embeds into $L_1$. Indeed, Let $\{f_i\}$ be a sequence of independent $2/s$ symmetric stable random variables normalized in $L_1$ and consider the span of the sequence $\{f_i \otimes |x_i|^s\}$ in $L_1$.

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