Quantitative weighted bounds for Calderón commutators with rough kernels

by

YANPING CHEN (Beijing) and JI LI (Sydney)

Abstract. We obtain a quantitative weighted bound for the Calderón commutator $C_\Omega$ which is a typical example of a non-convolution Calderón–Zygmund operator under the condition $\Omega \in L^\infty(S^{n-1})$; this is the best known quantitative result for this class of rough operators.

1. Introduction. Calderón commutators (see [C65, C78]) originate from a representation of linear differential operators by means of singular integral operators, which is an approach to the uniqueness of the Cauchy problem for partial differential equations (see [C58]). The first version was introduced by Calderón [C78]:

$$\left[ b, H \frac{d}{dx} \right] f(x) := \text{p.v.} \int_{-\infty}^{\infty} \left( \frac{-1}{x-y} \right) \left( \frac{b(x) - b(y)}{x-y} \right) f(y) \, dy.$$ 

It also plays an important role in the theory of Cauchy integrals along Lipschitz curves in $\mathbb{C}$ and the Kato square root problem on $\mathbb{R}$ (see [C58, F74, M90, MC91] for the details).

A more general version is a Calderón commutator with rough kernel, $C_\Omega f(x) = \lim_{\epsilon \to 0^+} C_\epsilon f(x)$, a.e. $x \in \mathbb{R}^n$, defined initially for $f \in C_0^\infty(\mathbb{R}^n)$, where $C_\epsilon f$ is the truncated Calderón commutator of $f$:

$$C_\epsilon f(x) := \int_{|x-y| > \epsilon} \left( \frac{\Omega(x-y)}{|x-y|^{n+1}} \right)(b(x) - b(y)) f(y) \, dy, \quad \forall x \in \mathbb{R}^n,$$

where $\Omega$ is homogeneous of degree zero, integrable on $S^{n-1}$ (the unit sphere

2020 Mathematics Subject Classification: Primary 42B20; Secondary 42B25.
Key words and phrases: quantitative weighted bound, Calderón commutator, rough kernel.
Received 13 February 2021; revised 12 June 2021.
Published online 15 November 2021.

DOI: 10.4064/sm210213-12-7
in $\mathbb{R}^n$) and satisfies the cancellation condition on the unit sphere:

$$\int_{\mathbb{S}^{n-1}} \Omega(x')(x'_k)^N \, d\sigma(x') = 0, \quad \forall (k, N) \in \{1, \ldots, n\} \times \{0, 1\}. \quad (1.3)$$

Using the method of rotation, Calderón [C65] proved the boundedness of $C_\Omega$ for $\Omega$ in $L \log L(S^{n-1})$ and $b \in \text{Lip}(\mathbb{R}^n)$, and then obtained the boundedness of the operators $[b, T]\nabla$ and $\nabla[b, T]$, where $T$ is a homogeneous singular integral operator with some symbol $K$ which can be defined similarly to $C_\Omega$, that is,

$$\begin{cases}
Tf(x) = \lim_{\epsilon \to 0^+} T_\epsilon f(x), \text{ a.e. } x \in \mathbb{R}^n, \\
T_\epsilon f(x) = \int_{|y| > \epsilon} K(y) f(x - y) \, dy,
\end{cases} \quad (1.4)$$

where $f \in C_0^\infty(\mathbb{R}^n)$, and the kernel $K$ is homogeneous of degree $-n$, belongs to $L^1_{\text{loc}}(\mathbb{R}^n)$ and enjoys cancellation on the unit sphere,

$$\int_{\mathbb{S}^{n-1}} K(y') \, d\sigma(y') = 0. \quad (1.5)$$

Later, many authors made important progress on Calderón commutators; see [CS1, CM78, CM75, M14, M86, H90, Y95, Ta15, MW71, GH12, CDH16] and the references therein for these developments and applications.

We first recall the definition and some properties of $A_p$ weights on $\mathbb{R}^n$. Let $w$ be a non-negative locally integrable function defined on $\mathbb{R}^n$. For $1 < p < \infty$, we say that $w \in A_p$ if there exists a constant $C > 0$ such that

$$[w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} \, dx \right)^{p-1} \leq C, \quad (1.6)$$

where the supremum is taken over all cubes with edges parallel to the coordinate axes. We will adopt the following definition for the $A_\infty$ constant for a weight $w$, introduced by Fujii [F78], and later by Wilson [W87]:

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w(x)\chi_Q)(x) \, dx. \quad (1.7)$$

Here $w(Q) := \int_Q w(x) \, dx$, and the supremum is taken over all cubes with edges parallel to the coordinate axes. When the supremum is finite, we will say that $w$ belongs to the $A_\infty$ class. Note that $A_\infty = \bigcup_{p \geq 1} A_p$. The weights $w$ for which the usual operators like the Hardy–Littlewood maximal operator, the Hilbert transform, and Calderón–Zygmund operators act boundedly on $L^p(w)$ were identified in the 1970’s in the works of Muckenhoupt, Hunt, Wheeden, Coifman and Fefferman [CF74, MW71, HMW73]; later for singular integrals with rough kernels in the works of Watson and Duoandikoetxea [D93, W90]; and for Calderón commutators with rough kernels in the work of Hofmann [H90].
The first to study the optimal quantitative bounds
\[ \|Mf\|_{L^2} \lesssim [w]_{A_2} \|f\|_{L^2} \tag{1.7} \]
was Buckley \cite{B93} for the usual Hardy–Littlewood maximal function on \( \mathbb{R}^n \). However, there has been a strong impetus for finding such precise dependence for more singular operators after the work of Astala, Iwaniec and Saksman \cite{AIS01} due to connections with sharp regularity results for solutions to the Beltrami equation. For example, see Petermichl and Volberg \cite{PV02}, Petermichl \cite{P07, P08} and Lacey \cite{L17} for the Hilbert transform and the Riesz transforms.

Let \( T_\Omega \) be the homogeneous singular integral operator defined by
\[ T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x - y) \, dy, \tag{1.8} \]
where \( y' = y/|y| \), \( \Omega \) is homogeneous of degree zero, integrable on \( S^{n-1} \) and satisfies the cancellation condition \( \int_{S^{n-1}} \Omega(y') \, d\sigma(y') = 0 \). Among these quantitative weighted bounds for \( T_\Omega \), we highlight that Hytönen–Roncal–Tapiola \cite{HRT17} first proved that when \( \Omega \in L^\infty(S^{n-1}) \),
\[ \|T_\Omega\|_{L^2(w) \to L^2(w)} \leq C_n \|\Omega\|_{L^\infty} [w]_{A_2}^2. \tag{1.9} \]
They introduced a two-step technique involving pointwise sparse domination for Dini-type Calderón–Zygmund kernels, a Littlewood–Paley decomposition along the lines of \cite{DR86} and interpolation with change of measure \cite{SW58}. Later, Conde-Alonso–Culiuc–Di Plinio–Ou \cite{CCDO17} proved a sparse domination for the bilinear form associated with \( T_\Omega \) with \( \Omega \in L^q(S^{n-1}) \) for \( 1 < q \leq \infty \) satisfying the cancellation conditions, which leads to quantitative weighted bounds for \( T_\Omega \) with \( \Omega \in L^\infty(S^{n-1}) \) (see \cite{CCDO17, (1.5) in Corollary A.1}), extending a previous result of \cite{HRT17}. Later on, Li–Pérez–Rivera–Roncal \cite{LPRR19} improved the known \( A_p \) type estimates for rough homogeneous singular integrals at the critical index. Inequality \eqref{1.9} was also extended to maximal singular integrals \( T_\Omega^* \) by Di Plinio, Hytönen and Li \cite{DHL17} and Lerner \cite{L18} via sparse domination.

All the rough operators considered in the cited papers have nice symmetry properties, for instance, a semigroup structure or dilation invariance. As far as we know, it has been unknown whether the same quantitative weighted bounds hold for Calderón commutators with rough kernels, which are typical examples of non-convolution Calderón–Zygmund operators. In the present paper, we give the problem a positive answer and give quantitative weighted bounds for Calderón commutators with rough kernels. We do not know whether this is sharp, but it is the best known quantitative result for this class of operators.
Denote
\begin{align}
(w)_{A_p} & := \max \{[w]_{A_\infty}, [w^{1-p'}]_{A_\infty}\}, \\
\{w\}_{A_p} & := [w]_{A_p}^{1/p} \max \{[w^{1/p'}]_{A_\infty}, [w^{1-p'}]_{A_\infty}\}.
\end{align}
Recall from [HRT17] that
\begin{align}
(w)_{A_p} \leq \{w\}_{A_p} \leq [w]_{A_p}^{\max\{1,1/(p-1)\}}.
\end{align}

Our main result can be stated as follows.

**Theorem 1.1.** Let $1 < p < \infty$, $w \in A_p$, and $b \in \text{Lip}(\mathbb{R}^n)$. Suppose that $\mathcal{C}_\Omega$ with $\Omega \in L^{\infty}(\mathbb{S}^{n-1})$ satisfies (1.3). Then
\begin{align}
\|\mathcal{C}_\Omega f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty} \{w\}_{A_p}^{1/p} \|\nabla b\|_{L^\infty} \|f\|_{L^p(w)},
\end{align}
where $(w)_{A_p}$, $\{w\}_{A_p}$ are given by (1.10), (1.11) and $[w]_{A_r}$ ($1 < r \leq \infty$) is the $A_r$ characteristic of $w$, given in (1.6), and the implicit constant is independent of $b$, $f$ and $w$. In particular,
\begin{align*}
\|\mathcal{C}_\Omega f\|_{L^2(w)} \lesssim \|\Omega\|_{L^\infty} [w]^{2}_{A_2} \|\nabla b\|_{L^\infty} \|f\|_{L^2(w)}.
\end{align*}

As a direct application of Theorem 1.1, we obtain the following quantitative weighted bounds of Calderón-type operators. For $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a linear operator $A$ on some measurable function space, the commutator between $A$ and $b$ is defined by $[b, Af](x) := b(x)Af(x) - Af(b(x))$. Thus $[b, T_\Omega]f(x) := b(x)T_\Omega f(x) - T_\Omega (bf)(x)$.

For $1 \leq j \leq n$, denote $f_j(x) = \frac{\partial}{\partial x_j} f(x)$. Then
\begin{align*}
T_\Omega (\nabla f)(x) = (T_\Omega f_1, \ldots, T_\Omega f_n)
\end{align*}
and
\begin{align*}
[b, T_\Omega] (\nabla f)(x) = ([b, T_\Omega] f_1, \ldots, [b, T_\Omega] f_n).
\end{align*}
Also,
\begin{align*}
\| [b, T_\Omega] (\nabla f)\|_{L^p} = \left( \sum_{j=1}^{n} \| [b, T_\Omega] f_j\|_{L^p}^2 \right)^{1/2}.
\end{align*}
Finally,
\begin{align*}
\nabla [b, T_\Omega] f(x) = \left( \frac{\partial}{\partial x_1} ([b, T_\Omega] f)(x), \ldots, \frac{\partial}{\partial x_n} ([b, T_\Omega] f)(x) \right)
\end{align*}
and
\begin{align*}
\| \nabla [b, T_\Omega] f\|_{L^p} = \left( \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x_j} ([b, T_\Omega] f) \right\|_{L^p}^2 \right)^{1/2}.
\end{align*}

**Theorem 1.2.** Suppose $1 < p < \infty$, $w \in A_p$, $b \in \text{Lip}(\mathbb{R}^n)$ and $f \in C_0^1(\mathbb{R}^n)$. Let $T_\Omega$ be as in (1.8). Suppose that $\Omega$ has locally integrable first-order derivatives, and $\Omega$ and its partial derivatives belong locally to $L^\infty$. 


Then

$$
\| [b, T_\Omega](\nabla f) \|_{L^p(w)} \lesssim (w)_{A_p} \{ w \} \| \nabla b \|_{L^\infty} \| f \|_{L^p(w)}.
$$

Furthermore, if $[b, T_\Omega] f$ has first-order derivatives in $L^p(w)$, then

$$
\| \nabla [b, T_\Omega] f \|_{L^p(w)} \lesssim (w)_{A_p} \{ w \} \| \nabla b \|_{L^\infty} \| f \|_{L^p(w)}.
$$

NOTATION. Throughout the paper, $p' = p/(p - 1)$ represents the conjugate index of $p \in [1, \infty)$; $X \lesssim Y$ stands for $X \leq CY$ for a constant $C > 0$ which is independent of the essential variables living on $X \& Y$; and $X \approx Y$ means $X \lesssim Y \lesssim X$.

2. Fundamental lemmas. Let us begin by presenting some auxiliary lemmas, which will play a key role in proving Theorems 1.1 and 1.2.

We first recall some definitions. A modulus of continuity is a function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ that is subadditive in the sense that

$$
u \leq t + s \implies \omega(u) \leq \omega(t) + \omega(s).
$$

Substituting $s = 0$ one sees that $\omega(u) \leq \omega(t)$ for all $0 \leq u \leq t$. Note that the composition and sum of two moduli of continuity is again a modulus of continuity. In particular, if $\omega(t)$ is a modulus of continuity and $\theta \in (0, 1)$, then $\omega(t^\theta)$ and $\omega(t^\theta)$ are also moduli of continuity. The Dini norm of a modulus of continuity is defined by setting

$$
\| \omega \|_{\text{Dini}} := \int_0^1 \omega(t) \frac{dt}{t} \leq \infty. \tag{2.1}
$$

For any $c > 0$ the integral can be equivalently (up to a $c$-dependent multiplicative constant) replaced by the sum over $\omega(2^{-j/c})$ with $j \in \mathbb{N}$. The basic example is $\omega(t) = t^\theta$.

Let $T$ be a bounded linear operator on $L^2(\mathbb{R}^n)$ represented as

$$
Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy, \quad \forall x \notin \text{supp}(f). \tag{2.2}
$$

We say that $T$ is an $\omega$-Calderón–Zygmund operator if the kernel $K$ satisfies the following size and smoothness conditions:

- for any $x, y \in \mathbb{R}^n \setminus \{ 0 \}$,

$$
|K(x, y)| \leq \frac{C_K}{|x - y|^n}, \quad x \neq y; \tag{2.3}
$$

- for any $h \in \mathbb{R}^n$ with $2|h| \leq |x - y|$,

$$
|K(y, x + h) - K(y, x)| + |K(x + h, y) - K(x, y)| \leq C \omega(|h|/|x - y|) \frac{|x - y|^n}{|x - y|^n}. \tag{2.4}
$$
Lemma 2.1 ([HRTT7 Theorem 1.3]). Let $T$ be an $\omega$-Calderón–Zygmund operator with $\omega$ satisfying the Dini condition. Then for $1 < p < \infty$ and $w \in A_p$,

$$
\|Tf\|_{L^p(w)} \leq C_{n,p}(\|T\|_{L^2\to L^2} + C_K + \|\omega\|_{\text{Dini}})\|f\|_{L^p(w)}.
$$

Lemma 2.2 ([CD16 Lemma 2.3]). Let $b \in \text{Lip}(\mathbb{R}^n)$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function such that $\text{supp} (\hat{\psi}) \subset \{1/2 \leq |\xi| \leq 2\}$. Define the multiplier operator $\Delta_j$ by $\hat{\Delta}_j f (\xi) = \hat{\psi}(2^j \xi) \hat{f}(\xi)$ for $j \in \mathbb{Z}$. Then

$$
\left\| \left( \sum_{j \in \mathbb{Z}} 2^{-2j} |[b, \Delta_j]f|^2 \right)^{1/2} \right\|_{L^2} \lesssim \|\nabla b\|_{L^\infty} \|f\|_{L^2}.
$$

Lemma 2.3. Let $(k, j) \in \mathbb{Z} \times \mathbb{Z}$, $b \in \text{Lip}(\mathbb{R}^n)$, and $m_{k,j} \in C^\infty(\mathbb{R}^n)$. Suppose that $\hat{T}_{k,j} f (\xi) = m_{k,j}(\xi) \hat{f}(\xi)$ and $[b, T_{k,j}]f(x) := b(x) T_{k,j} f(x) - T_{k,j}(bf)(x)$. If for all $k,j$ the function $m_{k,j}$ satisfies the conditions

$$
\begin{cases}
\|m_{k,j}\|_{L^\infty} \lesssim 2^{-k} \min \{2^{-\beta j}, 2^j\}, & \text{where } \beta > 0 \text{ is independent of } j,k; \\
\|\partial^\alpha m_{k,j}\|_{L^\infty} \lesssim 2^k \text{ for any fixed multi-indices } \alpha \text{ with } |\alpha| = 2,
\end{cases}
$$

then there exists a constant $0 < \lambda < 1$ such that

$$
\|[b, T_{k,j}]f\|_{L^2} \lesssim \min \{2^{-\beta \lambda j}, 2^{2\lambda j}\} \|\nabla b\|_{L^\infty} \|f\|_{L^2}.
$$

Proof. Let $\varphi \in C^\infty_0(\mathbb{R}^n)$ be a radial function with $\text{supp}(\varphi) \subset \{1/2 \leq |x| \leq 2\}$. Let $\sum_{l \in \mathbb{Z}} \varphi(2^{-l} x) = 1$ for $|x| > 0$. We set $\varphi_0(x) = \sum_{l=-\infty}^0 \varphi(2^{-l} x)$, $\varphi_l(x) = \varphi(2^{-l} x)$ for $l \in \mathbb{N}$. Denote by $K_{k,j}(x) = m_{k,j}^{-1}(x)$ the inverse Fourier transform of $m_{k,j}$. Then decompose each $K_{k,j}$ as follows:

$$
K_{k,j}(x) = K_{k,j}(x) \varphi_0(x) + \sum_{l=1}^\infty K_{k,j}(x) \varphi_l(x) =: \sum_{l=0}^\infty K_{k,j}^l(x),
$$

where

$$
\hat{K}_{k,j}^l(x) = \int_{\mathbb{R}^n} m_{k,j}(x-y) \hat{\varphi}_l(y) \, dy.
$$

Since $\text{supp}(\varphi) \subset \{1/2 \leq |x| \leq 2\}$, we see that

$$
\int_{\mathbb{R}^n} \hat{\varphi}(y) \, dy = \varphi(0) = 0
$$

and for each multi-index $\vartheta$ such that $|\vartheta| = 1$,

$$
(2.5) \quad \int_{\mathbb{R}^n} \hat{\varphi}(y) y^\vartheta \, dy = \frac{1}{2\pi i} \partial^\vartheta \varphi(0) = 0.
$$
By using the above cancellation conditions, along with Taylor’s expansion of $m_{k,j}(x - y)$ around $y$, we get

$$
\|\hat{K}_{l}^{k,j}\|_{L^{\infty}} \leq \sum_{|\alpha|=2} |\alpha| \|\partial^{\alpha} m_{k,j}\|_{L^{\infty}} \int_{\mathbb{R}^{n}} |y|^{2} |\hat{\varphi}_{l}(y)| \, dy
$$

$$
= \sum_{|\alpha|=2} |\alpha| \|\partial^{\alpha} m_{k,j}\|_{L^{\infty}} \int_{\mathbb{R}^{n}} |2^{-l}y|^{2} |\hat{\varphi}(y)| \, dy
$$

$$
\lesssim 2^{k} 2^{-2l} \int_{\mathbb{R}^{n}} |y|^{2} |\hat{\varphi}(y)| \, dy \lesssim 2^{-2l} 2^{k}.
$$

On the other hand, by the Young inequality we have

$$
\|\hat{K}_{l}^{k,j}\|_{L^{\infty}} = \|m_{k,j} \ast \hat{\varphi}_{l}\|_{L^{\infty}} \leq \|m_{k,j}\|_{L^{\infty}} \|\hat{\varphi}_{l}\|_{L^{1}}
$$

$$
\lesssim 2^{-k} \min\{2^{-\beta j}, 2^{2j}\}.
$$

Therefore, interpolating between (2.6) and (2.7) gives, for any $0 < \theta < 1$,

$$
\|\hat{K}_{l}^{k,j}\|_{L^{\infty}} \lesssim 2^{-2\theta l} 2^{k(2\theta-1)} \min\{2^{-(1-\theta)\beta j}, 2^{2(1-\theta)j}\}.
$$

Denote $T_{k,j}^{l} f(x) = K_{l}^{k,j} \ast f(x)$. We now estimate $[b, T_{k,j}^{l}]$, the commutator of the operator $T_{k,j}^{l}$. Decompose $\mathbb{R}^{n}$ into a grid of non-overlapping cubes with side lengths $2^{l}$:

$$
\mathbb{R}^{n} = \bigcup_{d=\infty}^{\infty} Q_d,
$$

and set $f_{d} := f\chi_{Q_d}$. Then

$$
f(x) = \sum_{d=\infty}^{\infty} f_{d}(x) \quad \text{for a.e. } x \in \mathbb{R}^{n}.
$$

From the support condition of $T_{k,j}^{l}$, it is obvious that

$$
\text{supp}([b, T_{k,j}^{l}] f_{d}) \subset 2nQ_d,
$$

and hence the supports of $\{[b, T_{k,j}^{l}] f_{d}\}_{d=\infty}$ have bounded overlaps. So we get the following almost orthogonality property:

$$
\|\|b, T_{k,j}^{l}\|_{L^{2}}^{2} \lesssim \sum_{d=\infty}^{\infty} \|b, T_{k,j}^{l} f_{d}\|_{L^{2}}^{2}.
$$

Thus, we may assume that $\text{supp}(f) \subset Q$ for some cube $Q$ with $\ell(Q) = 2^{l}$. Choose $\varphi \in C_{0}^{\infty}(\mathbb{R}^{n})$ satisfying $0 \leq \varphi \leq 1$, $\text{supp}(\varphi) \subset 100nQ$, and $\varphi(x) = 1$ for $x \in 30nQ$. Denote $\tilde{Q} = 200nQ$ and $\tilde{b} = (b(x) - b_{\tilde{Q}}) \varphi(x)$. We find that

$$
\|b, T_{k,j}^{l}\|_{L^{2}} \leq \sum_{l \geq 0} \|b, T_{k,j}^{l}\|_{L^{2}} \leq \sum_{l \geq 0} \|\tilde{b} T_{k,j}^{l}\|_{L^{2}} + \sum_{l \geq 0} \|T_{k,j}^{l}(\tilde{b} f)\|_{L^{2}}.
$$
By using (2.8) with $1/2 < \theta_1 < 1$ and with $0 < \theta_2 < 1/2$, and applying the fact that $\|b\|_{L^\infty} \leq 2^l \|\nabla b\|_{L^\infty}$, we obtain

$$
\sum_{l \geq 0} \|\tilde{b} T_{k,j}^l f\|_{L^2} 
\leq \sum_{l \geq k} \|\tilde{b}\|_{L^\infty} \|T_{k,j}^l f\|_{L^2} + \sum_{l < k} \|\tilde{b}\|_{L^\infty} \|T_{k,j}^l f\|_{L^2}
\lesssim \sum_{l \geq k} 2^l \|\nabla b\|_{L^\infty} 2^{-2\theta_1 l} 2^{k(2\theta_1 - 1)} \min \left\{2^{-(1 - \theta_1)\beta_j}, 2^{2(1 - \theta_1)j}\right\} \|f\|_{L^2}
+ \sum_{l < k} 2^l \|\nabla b\|_{L^\infty} 2^{-2\theta_2 l} 2^{k(2\theta_2 - 1)} \min \left\{2^{-(1 - \theta_2)\beta_j}, 2^{2(1 - \theta_2)j}\right\} \|f\|_{L^2}
\approx \|\nabla b\|_{L^\infty} \min \left\{2^{-(1 - \theta_1)\beta_j}, 2^{2(1 - \theta_1)j}\right\} \|f\|_{L^2} \sum_{l \geq k} 2^{(1 - \theta_1)(l-k)}
+ \|\nabla b\|_{L^\infty} \min \left\{2^{-(1 - \theta_2)\beta_j}, 2^{2(1 - \theta_2)j}\right\} \|f\|_{L^2} \sum_{l < k} 2^{(k-l)(2\theta_2 - 1)}
\lesssim \|\nabla b\|_{L^\infty} \|f\|_{L^2} (\min \left\{2^{-(1 - \theta_1)\beta_j}, 2^{2(1 - \theta_1)j}\right\} + \min \left\{2^{-(1 - \theta_2)\beta_j}, 2^{2(1 - \theta_2)j}\right\})
\approx \min \left\{2^{-\beta \lambda j}, 2^{2\lambda j}\right\} \|\nabla b\|_{L^\infty} \|f\|_{L^2},
$$

where $\lambda := \min \{1 - \theta_1, 1 - \theta_2\} = 1 - \theta_1$.

Similarly, we can get

$$
\sum_{l \geq 0} \|T_{k,j}(\tilde{b} f)\|_{L^2} \lesssim \min \left\{2^{-\beta \lambda j}, 2^{2\lambda j}\right\} \|\nabla b\|_{L^\infty} \|f\|_{L^2}.
$$

As a consequence, we obtain

$$
\|b, T_{k,j} f\|_{L^2} \lesssim \min \left\{2^{-\beta \lambda j}, 2^{2\lambda j}\right\} \|\nabla b\|_{L^\infty} \|f\|_{L^2}.
$$

The proof of Lemma 2.3 is complete. ■

3. Proof of Theorem 1.1 Recall the definition of the operator $C_\Omega$ given in the introduction:

$$
(3.1) \quad C_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \left( \frac{\Omega(x-y)}{|x-y|^{n+1}} \right) (b(x) - b(y)) f(y) \, dy, \quad \forall x \in \mathbb{R}^n.
$$

It can be written as

$$
C_\Omega f = [b, T_{\Omega}^1] f = b(x) T_{\Omega}^1 f(x) - T_{\Omega}^1 (bf)(x),
$$

where

$$
T_{\Omega}^1 f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} f(y) \, dy.
$$
Write

\[ T^1_\Omega = \sum_{k \in \mathbb{Z}} T_k f = \sum_{k \in \mathbb{Z}} K_k \ast f, \quad K_k = \frac{\Omega(x')}{|x|^{n+1}} \chi\{2k < |x| \leq 2k+1\}. \tag{3.2} \]

To prove Theorem 1.1, we borrow the idea from [HRT17] via using a two-step approach involving the known result for Dini-type Calderón–Zygmund kernels. However, the technique in [HRT17] of decomposition of \( T_\Omega \) is not applicable to the Calderón commutator \( C_\Omega \) since the symbol \( b \in \text{Lip}(\mathbb{R}^n) \) is also involved in \( C_\Omega \), thus the operator is non-convolution. To overcome this problem, we consider the following partition of unity. Let \( \phi \) also involved in applicable to the Calderón commutator \( C_\Omega \) since the symbol \( b \in \text{Lip}(\mathbb{R}^n) \) is also involved in \( C_\Omega \), thus the operator is non-convolution. To overcome this problem, we consider the following partition of unity. Let \( \phi \in S(\mathbb{R}^n) \) and let \( \hat{\phi} \in S(\mathbb{R}^n) \) be a radial function satisfying \( \hat{\phi}(\xi) = 1 \) for \( |\xi| \leq \frac{1}{2} \) and \( \hat{\phi}(\xi) = 0 \) for \( |\xi| \geq 1 \). Let us also define \( \psi \) by \( \hat{\psi}(\xi)^3 = \hat{\phi}(\xi) - \hat{\phi}(2\xi) \in S(\mathbb{R}^n) \). Then, with this choice of \( \hat{\psi} \), it is supported on \( \{1/2 \leq |\xi| \leq 2\} \). We write \( \varphi_j(x) = \frac{1}{2\pi} \varphi \left( \frac{x}{2^j} \right) \), and \( \psi_j(x) = \frac{1}{2\pi} \psi \left( \frac{x}{2^j} \right) \). We now define the partial sum operators \( S_j \) by \( S_j(f) = f \ast \varphi_j \). Recall that \( \Delta_j f(x) = \psi_j \ast f(x) \). The differences are given by

\[ S_j(f) - S_{j+1}(f) = f \ast \psi_j \ast \psi_j \ast \psi_j = \Delta^3_j f. \tag{3.3} \]

Since \( S_j f \to f \) as \( j \to -\infty \), for any sequence \( \{N(j)\}_{j=0}^\infty \) of integer numbers with \( 0 = N(0) < N(1) < \cdots < N(j) \to \infty \) we have the identity

\[ T_k = T_k S_k + \sum_{j=1}^\infty T_k (S_{k-N(j)} - S_{k-N(j)-1}). \tag{3.4} \]

Such a decomposition with respect to the \( \{N(j)\}_{j} \) is due to [HRT17]. Next, by writing

\[ S_k = \sum_{j=1}^\infty (S_{k+N(j)-1} - S_{k+N(j)}), \]

we obtain

\[ T_k = \sum_{j=1}^\infty T_k (S_{k+N(j)-1} - S_{k+N(j)}) + \sum_{j=1}^\infty T_k (S_{k-N(j)} - S_{k-N(j)-1}). \]

This gives

\[ T^1_\Omega = \sum_{j=1}^\infty T^N_{1,j} + \sum_{j=1}^\infty T^N_{2,j}, \tag{3.5} \]

where for \( j \geq 1 \),

\[ T^N_{1,j} := \sum_{k \in \mathbb{Z}} T_k (S_{k-N(j)} - S_{k-N(j)-1}), \tag{3.6} \]

\[ T^N_{2,j} := \sum_{k \in \mathbb{Z}} T_k (S_{k+N(j)-1} - S_{k+N(j)}). \tag{3.7} \]
Thus
\[ C_\Omega f = [b, T_{\Omega}^1]f = \sum_{j=1}^{\infty} [b, T_{1,j}^N + T_{2,j}^N]f = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} [b, T_{m,j}^N]f. \]

Therefore, for \(1 < p < \infty\), and \(w \in A_p\),
\[ \|C_\Omega f\|_{L^p(w)} \leq 2 \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \|[b, T_{m,j}^N]f\|_{L^p(w)}. \]

To begin with, we need the following lemma.

**Lemma 3.1.** Let \(b \in \text{Lip}(\mathbb{R}^n)\) and \(\Omega \in L^\infty(S^{n-1})\). The following inequality holds for \(m = 1, 2\), and some \(\delta \in (0, 1)\):
\[ \|[b, T_{m,j}^N]f\|_{L^2} \lesssim \|\Omega\|_{L^\infty} \|
abla b\|_{L^\infty} 2^{-\delta N(j-1)} \|f\|_{L^2}. \]

**Proof.** Recall that
\[ [b, T_{1,j}^N]f = \sum_{k \in \mathbb{Z}} [b, T_k(S_{k-N(j)} - S_{k-N(j-1)})]f, \]
\[ [b, T_{2,j}^N]f = \sum_{k \in \mathbb{Z}} [b, T_k(S_{k+N(j-1)} - S_{k+N(j)})]f, \]
where
\[ S_{k-N(j)} - S_{k-N(j-1)} = \sum_{i=N(j-1)+1}^{N(j)} (S_{k-i} - S_{k-i+1}) = \sum_{i=N(j-1)+1}^{N(j)} \Delta_{k-i}^3, \]
\[ S_{k+N(j-1)} - S_{k+N(j)} = \sum_{i=-N(j)+1}^{-N(j)} (S_{k-i} - S_{k-i+1}) = \sum_{i=-N(j)+1}^{-N(j)} \Delta_{k-i}^3. \]

Then
\[ \|[b, T_{1,j}^N]f\|_{L^2} \leq \sum_{i=N(j-1)+1}^{N(j)} \left\| \sum_{k \in \mathbb{Z}} [b, T_k \Delta_{k-i}^3]f \right\|_{L^2}, \]
\[ \|[b, T_{2,j}^N]f\|_{L^2} \leq \sum_{i=-N(j)+1}^{-N(j)} \left\| \sum_{k \in \mathbb{Z}} [b, T_k \Delta_{k-i}^3]f \right\|_{L^2}. \]

First, we estimate the \(L^2\)-norm of \(\sum_{k \in \mathbb{Z}} [b, T_k \Delta_{k-i}^3]f\). For any \(i, k \in \mathbb{Z}\), we define \(T_k^i f(x) := T_k \Delta_{k-i}^3 f(x)\). Recall \(T_k\) and \(\Delta_{k-i}\) are convolution type operators, hence \(T_k \Delta_{k-i} = \Delta_{k-i} T_k\). Then we write
\[ [b, T_k \Delta_{k-i}^3]f = \Delta_{k-i} T_k^i ([b, \Delta_{k-i}]f) + \Delta_{k-i} ([b, T_k^i] \Delta_{k-i} f) + [b, \Delta_{k-i}] (T_k^i \Delta_{k-i} f). \]
We get
\[
\left\| \sum_{k \in \mathbb{Z}} [b, T_k \Delta_{k-i}^3] f \right\|_{L^2} \leq \left\| \sum_{k \in \mathbb{Z}} \Delta_{k-i} T_k^i [b, \Delta_{k-i}] f \right\|_{L^2} + \left\| \sum_{k \in \mathbb{Z}} \Delta_{k-i} [b, T_k^i \Delta_{k-i}] f \right\|_{L^2} + \left\| \sum_{k \in \mathbb{Z}} [b, \Delta_{k-i}] T_k^i \Delta_{k-i} f \right\|_{L^2} =: I_1 + I_2 + I_3.
\]

Denote
\[
\hat{T}_k^i f(\xi) = \hat{K}_k(\xi) \hat{\psi}(2^{k-i} \xi) \hat{f}(\xi) =: m_{i,k}(\xi) \hat{f}(\xi).
\]

We now provide pointwise estimates for \(m_{i,k}(\xi)\) and its derivative. By the cancellation condition \((1.3)\) and integrability of \(\Omega\), we have
\[
|\hat{K}_k(\xi)| = \left| \int_{\mathbb{S}^{n-1}} \int_{2^k \mathbb{S}^{n-1}} \Omega(y') (e^{-2\pi iy' \cdot \xi} + 2\pi iy' \cdot \xi - 1) \, d\sigma(y') \frac{dr}{r^2} \right|
\]
\[
= \frac{1}{2^k} \left| \int_{1 \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \Omega(y') (e^{-2\pi iy' \cdot 2^k \xi} + 2\pi iy' \cdot 2^k \xi - 1) \, d\sigma(y') \frac{dr}{r^2} \right|
\]
\[
\leq \frac{1}{2^k} \int_{1 \mathbb{S}^{n-1}} |\Omega(y')| \left| e^{-2\pi iy' \cdot 2^k \xi} + 2\pi iy' \cdot 2^k \xi - 1 \right| \, d\sigma(y') \frac{dr}{r^2}.
\]

Then by noting that
\[
|e^{-2\pi iy' \cdot 2^k \xi} + 2\pi iy' \cdot 2^k \xi - 1| \lesssim |2^k \xi|^2,
\]
we get
\[
|\hat{K}_k(\xi)| \lesssim 2^{-k} \|\Omega\|_{L^\infty} |2^k \xi|^2.
\]

On the other hand,
\[
|\hat{K}_k(\xi)| = \left| \int_{\mathbb{S}^{n-1}} \int_{2^k \mathbb{S}^{n-1}} \Omega(x') \int_{2^k} e^{-2\pi i r x' \cdot \xi} \frac{dr}{r^2} \, d\sigma(x') \right|
\]

Note that, by van der Corput’s lemma, for any \(0 < \beta < 1\),
\[
\left| \int_{2^k} e^{-2\pi i r x' \cdot \xi} \frac{dr}{r^2} \right| \lesssim 2^{-k} |x' \cdot \xi'|^{-\beta} |2^k \xi|^{-\beta}.
\]

Hence
\[
|\hat{K}_k(\xi)| \lesssim 2^{-k} |2^k \xi|^{-\beta} \|\Omega\|_{L^\infty} \int_{\mathbb{S}^{n-1}} |x' \cdot \xi'|^{-\beta} \, d\sigma(x') \lesssim 2^{-k} |2^k \xi|^{-\beta} \|\Omega\|_{L^\infty}.
\]

Thus, there exists \(0 < \beta < 1\) such that
\[
|\hat{K}_k(\xi)| \lesssim \|\Omega\|_{L^\infty} 2^{-k} \min \{|2^k \xi|^{-\beta}, |2^k \xi|^2\}.
\]
Secondly, for any multi-index \( \eta \) with \(|\eta| \leq 2\), we get
\[
|\partial^\eta \hat{K}_k(\xi)| \lesssim \left| \int \int_{S^{n-1}} \Omega(y')y'^\eta e^{-2\pi i \xi \cdot y'} d\sigma(y') r^{|\eta|-2} \, dr \right|
\lesssim \|\Omega\|_{L^1} 2^k |\eta|-1| 2^k |\xi|^{2-|\eta|}.
\]
Since \( \hat{\psi} \in \mathcal{S}(\mathbb{R}^d) \) with \( \text{supp}(\hat{\psi}) \subset \{ 1/2 \leq |\xi| \leq 2 \} \), we see that \( 2^k |\xi| \approx 2^i \).

Hence,
\[
(3.9) \quad |m_{i,k}(\xi)| \leq |\hat{K}_k(\xi)| |\hat{\psi}(2^{k-i}\xi)| \lesssim 2^{-k} \min \{ 2^{-\beta i}, 2^{2i} \} \|\Omega\|_{L^\infty}.
\]

On the other hand, for any multi-index \( \alpha \), we have
\[
\partial^\alpha m_{i,k}(\xi) = \partial^\alpha (\hat{K}_k(\xi)\hat{\psi}(2^{k-i}\xi)) = \sum_{\eta} C_{\eta_1}^\alpha \ldots C_{\eta_n}^\alpha (\hat{\eta}\hat{K}_k(\xi))(\hat{\partial^\alpha \eta}\hat{\psi}(2^{k-i}\xi)),
\]
where \( C_{\eta_i}^\alpha, i = 1, \ldots, n \), are positive constants and the sum is taken over all multi-indices \( \eta \) with \( 0 \leq \eta_i \leq \alpha_i \) for \( 1 \leq \ell \leq n \). By taking \( \alpha \) with \(|\alpha| = 2\), we obtain
\[
(3.10) \quad \|\partial^\alpha m_{i,k}(\xi)\| \lesssim \sum_{0 \leq |\eta| \leq |\alpha|} 2^{(k-i)(|\alpha|-|\eta|)} |\partial^\eta \hat{K}_k(\xi)|
\lesssim \sum_{0 \leq |\eta| \leq |\alpha|} 2^{(k-i)(|\alpha|-|\eta|)} 2^{(-1+|\eta|)k} 2^{(2-|\eta|)i} \|\Omega\|_{L^1}
\lesssim \sum_{0 \leq |\eta| \leq |\alpha|} 2^{-i(|\alpha|-|\eta|)} 2^{(-1+|\alpha|)k} 2^{(2-|\eta|)i} \|\Omega\|_{L^1} \lesssim 2^{k} \|\Omega\|_{L^1}.
\]

As a consequence, after combining (3.9) and (3.10), we can apply Lemma 2.3 to \([b, T_k^i] f\) to deduce that there exists some constant \( 0 < \lambda < 1 \) such that
\[
(3.11) \quad \|[b, T_k^i] f\|_{L^2} \lesssim \min \{ 2^{-\beta \lambda_i}, 2^{2\lambda_i} \} \|\Omega\|_{L^\infty} \|\nabla b\|_{L^\infty} \|f\|_{L^2}, \quad i \in \mathbb{Z}.
\]

By using the Plancherel theorem and (3.9), we have
\[
(3.12) \quad \|T_k^i f\|_{L^2} \lesssim 2^{-k} \min \{ 2^{-\beta i}, 2^{2i} \} \|\Omega\|_{L^\infty} \|f\|_{L^2}.
\]

We now estimate \( I_1, I_2 \) and \( I_3 \). By using Littlewood–Paley theory, (3.12) and Lemma 2.2, we get
\[
I_1 \lesssim \left( \sum_{i \in \mathbb{Z}} \|T_k^i([b, \Delta_{k-i}] f)\|_{L^2}^2 \right)^{1/2}
\lesssim \min \{ 2^{-(1+\beta i)}, 2^{i} \} \|\Omega\|_{L^\infty} \left( \sum_{i \in \mathbb{Z}} 2^{-2(k-i)} \|[b, \Delta_{k-i}] f\|_{L^2}^2 \right)^{1/2}
\lesssim \min \{ 2^{-(1+\beta i)}, 2^{i} \} \|\Omega\|_{L^\infty} \|\nabla b\|_{L^\infty} \|f\|_{L^2}.
\]
By using (3.11) and Littlewood–Paley theory, we get
\[
I_2 \lesssim \left( \sum_{k \in \mathbb{Z}} \|[b, T_k^i] (\Delta_{k-i} f)\|_{L^2}^2 \right)^{1/2} 
\lesssim \min \left\{ 2^{-\beta \lambda_i}, 2^{2\lambda_i} \right\} \|\Omega\|_{L^\infty} \|
abla b\|_{L^\infty} \left( \sum_{k \in \mathbb{Z}} \|\Delta_{k-i} f\|_{L^2}^2 \right)^{1/2} 
\lesssim \min \left\{ 2^{-\beta \lambda_i}, 2^{2\lambda_i} \right\} \|\Omega\|_{L^\infty} \|
abla b\|_{L^\infty} \|f\|_{L^2}.
\]
Finally, by duality and by the estimate of \(I_1\), we obtain
\[
I_3 \lesssim \min \left\{ 2^{-\beta \lambda_i}, 2^{2\lambda_i} \right\} \|\Omega\|_{L^\infty} \|
abla b\|_{L^\infty} \|f\|_{L^2}.
\]
Combining the estimates of \(I_1, I_2\) and \(I_3\) above, we find that there exists some constant \(0 < \alpha < 1\) such that
\[
(3.13) \quad \left\| \sum_{k \in \mathbb{Z}} [b, T_k^i] f \right\|_{L^2} \lesssim 2^{-\alpha |i|} \|
abla b\|_{L^\infty} \|\Omega\|_{L^\infty} \|f\|_{L^2}.
\]
Thus for \(\delta_1, \delta_2 \in (0, 1)\) we get
\[
\|[b, T_{1,j}^N] f\|_{L^2} \lesssim \sum_{i=-N(j-1)+1}^{N(j)} 2^{-\alpha i} \|
abla b\|_{L^\infty} \|\Omega\|_{L^\infty} \|f\|_{L^2}
\lesssim 2^{-\delta_1 N(j-1)} \|
abla b\|_{L^\infty} \|\Omega\|_{L^\infty} \|f\|_{L^2},
\]
so
\[
\|[b, T_{2,j}^N] f\|_{L^2} \lesssim \sum_{i=-N(j-1)+1}^{N(j)} 2^{\alpha i} \|
abla b\|_{L^\infty} \|\Omega\|_{L^\infty} \|f\|_{L^2}
\lesssim 2^{-\delta_2 N(j-1)} \|
abla b\|_{L^\infty} \|\Omega\|_{L^\infty} \|f\|_{L^2}.
\]
The proof of Lemma 3.1 is complete. ■

**Lemma 3.2.** Let \(b \in \text{Lip}(\mathbb{R}^n)\). If \(\Omega \in L^\infty(S^{n-1})\), then the operator
\([b, T_{m,j}^N], m = 1, 2,\) is a Calderón–Zygmund operator with kernel \(K_{m,j}^N(x, y)\),
\(m = 1, 2,\) that satisfies
\[
|K_{m,j}^N(x, y)| \lesssim \frac{\|\Omega\|_{L^\infty} \|
abla b\|_{L^\infty}}{|x - y|^n},
\]
and for \(2|h| \leq |x - y|\),
\[
|K_{m,j}^N(x, y + h) - K_{m,j}^N(x, y)| + |K_{m,j}^N(x, y) - K_{m,j}^N(x + h, y)| \lesssim \frac{\omega_j(|h|/|x - y|)}{|x - y|^n},
\]
where \(\omega_j(t) = \|\Omega\|_{L^\infty} \|
abla b\|_{L^\infty} \min \{1, 2^{N(j)}t\}\).

**Proof.** Recall that \(S_k f(x) = \varphi_k * f(x)\), \(k \in \mathbb{Z}\), where \(\varphi_k(x) = 2^{-kn} \varphi(2^{-k}x)\),
\(\varphi \in \mathcal{S}(\mathbb{R}^n)\) and \(\int_{\mathbb{R}^n} \varphi(x) \, dx = 1\). Recall the definition of the kernel of \([b, T_{m,j}^N], m = 1, 2\):
\[ K_{1,j}^N(x, y) = \sum_{k \in \mathbb{Z}} K_k \ast \varphi_{k-N(j)}(x-y)(b(x) - b(y)) \]

\[ - \sum_{k \in \mathbb{Z}} K_k \ast \varphi_{k-N(j-1)}(x-y)(b(x) - b(y)) , \]

\[ K_{2,j}^N(x, y) = \sum_{k \in \mathbb{Z}} K_k \ast \varphi_{k+N(j-1)}(x-y)(b(x) - b(y)) \]

\[ - \sum_{k \in \mathbb{Z}} K_k \ast \varphi_{k+N(j)}(x-y)(b(x) - b(y)). \]

Since \( \varphi \in \mathcal{S} (\mathbb{R}^n) \), for any multi-index \( \beta \) we have

\[ |\partial^{\beta} \varphi (x)| \lesssim \frac{1}{(|x| + 1)^{n+\gamma}} \]

for any fixed \( \gamma > 0 \). Thus

\[ |\partial^{\beta} \varphi_k(x)| \lesssim 2^{-|\beta|} \frac{2^{k\gamma}}{(|x| + 2^k)^{n+\gamma}} \]

for any fixed \( \gamma > 0 \).

Now, we study the size and smoothness of the kernel \( K_{m,j}^N(x, y) \), \( m = 1, 2 \). Again, it suffices to verify the pointwise bound for \( |K_k \ast \varphi_{k\mp N(j)}(x)| \) and \( |\nabla K_k \ast \varphi_{k\mp N(j)}(x)| \). Note that

\[ |K_k \ast \varphi_{k\mp N(j)}(x)| = \left| \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y'|^n} 1_{2^k < |y| < 2^{k+1}} 2^{-(k \mp N(j))n} \varphi \left( \frac{x-y}{2^{k \mp N(j)}} \right) dy \right| . \]

We consider two cases: \( |x| > 2^{k+2} \) and \( |x| \leq 2^{k+2} \).

**Case 1:** \( |x| > 2^{k+2} \). We get \( |x - \theta y| \geq |x| - \theta |y| \geq |x| - |x|/2 \geq |x|/2 \) for any \( \theta \in (0, 1) \). By using the cancellation condition of \( \Omega \) as in (1.3) and Taylor’s expansion of \( \varphi_{k\mp N(j)}(x-y) \) around \( y \), we get

\[ |\partial^\alpha \varphi_k(x)| \lesssim 2^{-|\alpha|} \frac{2^{k\gamma}}{(|x| + 2^k)^{n+\gamma}}, \quad k \in \mathbb{Z}, \]

with \( |\alpha| = 2 \) and \( \gamma = 2 \), therefore

\[ |K_k \ast \varphi_{k\mp N(j)}(x)| \]

\[ \lesssim 2^k \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y'|^{n+2}} 1_{2^k < |y| < 2^{k+1}}(y) \partial^\alpha \varphi_{k\mp N(j)}(x-\theta y) |y|^2 dy \]

\[ = \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y'|^{n+1}} 1_{2^k < |y| < 2^{k+1}}(y) \frac{1}{2^2(k+N(j))} \frac{2^{2(k+N(j))}|y|^2}{(2^{k+N(j)} + |x-\theta y|)^{n+2}} dy \]

\[ \lesssim \frac{2^k}{|x|^{n+2}} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y'|^n} 1_{2^k < |y| < 2^{k+1}}(y) dy \lesssim \|\Omega\|_{L^1} \frac{2^k}{|x|^{n+2}} . \]
On the other hand, by noting that
\[ |\varphi_k(x)| \lesssim \frac{2^{2k}}{(|x| + 2^k)^{n+2}}, \quad k \in \mathbb{Z}, \]
we get
\[
|K_k \ast \varphi_{k-N(j)}(x)| \leq \int_{\mathbb{R}^n} \frac{|\Omega(y')|}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}}(y) \frac{2^{2(k-N(j))}}{(2^{k-N(j)} + |x-y|)^{n+2}} dy \\
\lesssim \frac{2^{2(k-N(j))}}{|x|^{n+2}} \int_{\mathbb{R}^n} \frac{|\Omega(y')|}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}}(y) dy \lesssim \|\Omega\|_{L^1} \frac{2^k}{|x|^{n+2}}.
\]

CASE 2: \(|x| \leq 2^{k+2}\). We get
\[
|K_k \ast \varphi_{k+N(j)}(x)| \leq \int_{\mathbb{R}^n} \frac{|\Omega(y')|}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}}(y) 2^{-(k+N(j))n} \varphi\left(\frac{x-y}{2^{k+N(j)}}\right) dy \\
\lesssim \|\Omega\|_{L^\infty} 2^{-k(n+1)} \int_{\mathbb{R}^n} 2^{-(k+N(j))n} \varphi\left(\frac{x-y}{2^{k+N(j)}}\right) dy \\
\lesssim \|\Omega\|_{L^\infty} 2^{-k(n+1)} \|\varphi\|_{L^1} \lesssim \|\Omega\|_{L^\infty} 2^{-k(n+1)}.
\]

Combining the above two cases yields
\[
\sum_{k \in \mathbb{Z}} |K_k \ast \varphi_{k+N(j)}(x-y)| \\
\lesssim \frac{\|\Omega\|_{L^\infty}}{|x-y|^{n+2}} \sum_{k \in \mathbb{Z}} 2^{k} 1_{2^{k+2} < |x-y|}(x-y) \\
+ \|\Omega\|_{L^\infty} \sum_{k \in \mathbb{Z}} 2^{-k(n+1)} 1_{2^{k+2} \geq |x-y|}(x-y) \\
\lesssim \frac{\|\Omega\|_{L^\infty}}{|x-y|^{n+2}} |x-y| + \|\Omega\|_{L^\infty} |x-y|^{-(n+1)} \lesssim \frac{\|\Omega\|_{L^\infty}}{|x-y|^{n+1}},
\]
and hence
\[
\sum_{k \in \mathbb{Z}} |(b(x) - b(y))K_k \ast \varphi_{k+N(j)}(x-y)| \lesssim \frac{\|\Omega\|_{L^\infty} \|
abla b\|_{L^\infty}}{|x-y|^{n}}.
\]

This gives, for \(m = 1, 2\),
\[
|K_{m, j}^{N}(x, y)| \lesssim \frac{\|\Omega\|_{L^\infty} \|\nabla b\|_{L^\infty}}{|x-y|^{n}}.
\]

We now estimate \(\nabla K_k \ast \varphi_{k+N(j)}(x)\). Since \(\varphi \in \mathcal{S}(\mathbb{R}^n)\), for any multi-index \(\beta\) we get
\[
|\partial^\beta (\nabla \varphi)_k(x)| \lesssim 2^{-k|\beta|} \frac{2^{k\gamma}}{(|x| + 2^k)^{n+\gamma}}
\]
for any fixed \(\gamma > 0\). We again consider two cases: \(|x| > 2^{k+2}\) and \(|x| \leq 2^{k+2}\).
CASE 1: $|x| > 2^{k+2}$. We get $|x - \theta y| \geq |x - \theta|^2 \geq |x|/2 \geq |x|/2$ for $0 < \theta < 1$. By using the cancellation condition for $\Omega$ and $|\partial^\beta(\nabla \varphi)_k(x)| \lesssim 2^{-k|\beta|}2^{k\gamma}/(|x| + 2^k)^{n+\gamma}$ with $k \in \mathbb{Z}$, $|\beta| = 1$ and $\gamma = 3$, we see that

$$\nabla (K_k \ast \varphi_{k-N(j)})(x) = |K_k \ast \nabla \varphi_{k-N(j)}(x)|$$

$$= \left| \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) 2^{-N(j)-k} (\nabla \varphi)_{k-N(j)}(y-x) \, dy \right|$$

$$\lesssim \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) 2^{-N(j)-k} \nabla \varphi_{k-N(j)}(y-x) \, dy$$

$$\lesssim \frac{2^{k+N(j)}}{|x|^{n+3}} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) \, dy \lesssim \|\Omega\|_{L^1} \frac{2^{k}}{|x|^{n+3}}.$$

On the other hand, by $|\partial^\beta(\nabla \varphi)_k(x)| \lesssim 2^{-k|\beta|}2^{k\gamma}/(|x| + 2^k)^{n+\gamma}$ with $k \in \mathbb{Z}$, $|\beta| = 0$ and $\gamma = 3$, we have

$$\nabla (K_k \ast \varphi_{k-N(j)})(x) = |K_k \ast \nabla \varphi_{k-N(j)}(x)|$$

$$= \left| \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) 2^{N(j)-k} (\nabla \varphi)_{k-N(j)}(y-x) \, dy \right|$$

$$\lesssim \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) 2^{N(j)-k} \nabla \varphi_{k-N(j)}(y-x) \, dy$$

$$\lesssim \frac{2^{2(k-N(j))}}{|x|^{n+3}} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) \, dy \lesssim \|\Omega\|_{L^1} \frac{2^{k}}{|x|^{n+3}}.$$

CASE 2: $|x| \leq 2^{k+2}$. We get

$$\nabla (K_k \ast \varphi_{k-N(j)})(x) = \left| \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) 2^{N(j)-k} (\nabla \varphi)_{k-N(j)}(y-x) \, dy \right|$$

$$\lesssim \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) 2^{N(j)-k} \nabla \varphi_{k-N(j)}(y-x) \, dy$$

$$\lesssim \frac{2^{N(j)}}{2^{k(n+2)}} \|\Omega\|_{L^\infty} \int_{\mathbb{R}^n} |(\nabla \varphi)_{k-N(j)}(x-y)| \, dy$$

$$\lesssim \frac{2^{N(j)}}{2^{k(n+2)}} \|\Omega\|_{L^\infty} \|\nabla \varphi\|_{L^1} \lesssim \frac{2^{N(j)}}{2^{k(n+2)}} \|\Omega\|_{L^\infty}.
On the other hand,

\[
|\nabla (K_k \ast \varphi_{k+N(j)}) (x)|
\]

\[
= \left| \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) 2^{-N(j)-k} (\nabla \varphi)_{k+N(j)} (x-y) \, dy \right|
\]

\[
\lesssim \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n+1}} 1_{2^k < |y| < 2^{k+1}} (y) 2^{-N(j)-k} |(\nabla \varphi)_{k+N(j)} (x-y)| \, dy
\]

\[
\lesssim 2^{-N(j)} \| \Omega \|_{L^\infty} \frac{1}{2^{k(n+2)}} \int_{\mathbb{R}^n} |(\nabla \varphi)_{k+N(j)} (x-y)| \, dy
\]

\[
\lesssim \frac{\| \Omega \|_{L^\infty}}{2^{k(n+2)}} \| \nabla \varphi \|_{L^1} \lesssim \frac{\| \Omega \|_{L^\infty}}{2^{k(n+2)}}.
\]

Thus, for any \( x, y, z \) satisfying \( 2|y-z| \leq |x-y| \),

\[
(3.17) \quad \sum_{k \in \mathbb{Z}} |K_k \ast \varphi_{k \mp N(j)} (x-y) - K_k \ast \varphi_{k \mp N(j)} (x-z)|
\]

\[
\lesssim \sum_{k \in \mathbb{Z}} | \nabla K_k \ast \varphi_{k \mp N(j)} ((1-\theta)(x-y) + \theta(x-z))| |y-z|
\]

\[
\lesssim \| \Omega \|_{L^\infty} \left( \frac{2^{N(j)}}{|x-y|^{n+3}} \sum_{k \in \mathbb{Z}} 2^k 1_{2^k < |x-y|} (x-y) + \sum_{k \in \mathbb{Z}} \frac{1|y-z|<2^{k+2}|x-y|}{2^{k(n+2)}} \right) x |y-z|
\]

\[
\lesssim \| \Omega \|_{L^\infty} \left( \frac{2^{N(j)}}{|x-y|^{n+3}} |x-y| + \frac{1}{|x-y|^{n+2}} \right) |y-z|
\]

\[
\lesssim \| \Omega \|_{L^\infty} 2^{N(j)} \frac{|y-z|}{|x-y|^{n+2}}.
\]

From (3.17) and (3.14), for any \( x, y, z \) satisfying \( 2|y-z| \leq |x-y| \) we get

\[
(3.18) \quad \sum_{k \in \mathbb{Z}} |(b(x) - b(y)) K_k \ast \varphi_{k \mp N(j)} (x-y) - (b(x) - b(z)) K_k \ast \varphi_{k \mp N(j)} (x-z)|
\]

\[
\lesssim |b(x) - b(y)| \sum_{k \in \mathbb{Z}} |K_k \ast \varphi_{k \mp N(j)} (x-y) - K_k \ast \varphi_{k \mp N(j)} (x-z)|
\]

\[
+ |b(y) - b(z)| \sum_{k \in \mathbb{Z}} |K_k \ast \varphi_{k \mp N(j)} (x-z)|
\]

\[
\lesssim \| \Omega \|_{L^\infty} \| \nabla b \|_{L^\infty} 2^{N(j)} \frac{|y-z|}{|x-y|^{n+1}} + \| \Omega \|_{L^\infty} \| \nabla b \|_{L^\infty} \frac{|y-z|}{|x-z|^{n+1}}
\]

\[
\lesssim 2^{N(j)} \| \Omega \|_{L^\infty} \| \nabla b \|_{L^\infty} \frac{|y-z|}{|x-y|^{n+1}}.
\]
Combining inequalities (3.15) and (3.18), we deduce that for any \( x, y, z \) satisfying \( 2|y - z| \leq |x - y| \),

\[
\sum_{k \in \mathbb{Z}} |(b(x) - b(y))K_k \ast \varphi_{k + N(j)}(x - y) - (b(x) - b(z))K_k \ast \varphi_{k + N(j)}(x - z)| \leq \omega_j \left( \frac{|y - z|}{|x - y|} \right) \frac{1}{|x - y|^n},
\]

where

\[
\omega_j(t) = \|\Omega\|_{L^\infty} \|\nabla b\|_{L^\infty} \min \{1, 2^{N(j)}t\}.
\]

This shows that for \( m = 1, 2 \) and \( x, y, z \) satisfying \( 2|y - z| \leq |x - y| \),

\[
|K_{m,j}^N(y, x) - K_{m,j}^N(z, x)| + |K_{m,j}^N(x, y) - K_{m,j}^N(x, z)| \lesssim \omega_j \left( \frac{|y - z|}{|x - y|} \right).
\]

The proof of Lemma 3.2 is complete. ■

In fact, applying Lemmas 3.1, 3.2 and 2.1 to \( T = [b, T_{m,j}^N], m = 1, 2 \), we find that for \( 1 < p < \infty \) and \( w \in A_p \),

\[
[b, T_{m,j}^N]f \lesssim \|\Omega\|_{L^\infty} \|\nabla b\|_{L^\infty} (1 + N(j)) \{w\}_{A_p} \|f\|_{L^p(w)}.
\]

Interpolating between (3.8) and (3.19) with \( w = 1 \), we deduce that for \( m = 1, 2, \) and \( 1 < p < \infty \),

\[
[b, T_{m,j}^N]f \lesssim \|\nabla b\|_{L^\infty} \|\Omega\|_{L^\infty} (1 + N(j)) 2^{-\theta(j)} \|f\|_{L^p}.
\]

Let \( \varepsilon := \frac{1}{2} \frac{c_n}{(w)_{A_p}} \). By the estimate (3.19) and [HRT17] Corollary 3.18, for this \( \varepsilon \) we obtain

\[
[b, T_{m,j}^N]f \lesssim \|\nabla b\|_{L^\infty} \|\Omega\|_{L^\infty} (1 + N(j)) \{w^{1+\varepsilon}\}_{A_p} \|f\|_{L^p(w^{1+\varepsilon})}.
\]

**Lemma 3.3** (Stein–Weiss, [SW58]). Assume that \( 1 \leq p_0, p_1 \leq \infty \), that \( w_0 \) and \( w_1 \) are positive weights, and that \( T \) is a sublinear operator satisfying

\[
T : L^{p_i}(w_i) \to L^{p_i}(w_i), \quad i = 0, 1,
\]

with quasi-norms \( M_0 \) and \( M_1 \), respectively. Then

\[
T : L^p(w) \to L^p(w)
\]

with quasi-norm \( M \leq M_0^\lambda M_1^{1-\lambda} \), where

\[
\frac{1}{p} = \frac{\lambda}{p_0} + \frac{1 - \lambda}{p_1}, \quad w = w_0^{p \lambda / p_0} w_1^{(1-\lambda) / p_1}.
\]

With (3.20) and (3.21), we now apply Lemma 3.3 to \( T = [b, T_{m,j}^N], m = 1, 2 \) with \( p_0 = p_1 = p, w_0 = w^0 = 1, w_1 = w^{1+\varepsilon} \) and \( \lambda = \varepsilon/(1 + \varepsilon) \). Thus
there exist some $\theta, \gamma > 0$ such that
\[
\| [b, T_{m,j}] \|_{L^p(w) \to L^p(w)} \lesssim \| \nabla b \|_{L^\infty} \| \Omega \|_{L^\infty} (1 + N(j)) 2^{-\theta N(j-1)\varepsilon/(1+\varepsilon)} \{ w \}_{A_p}
\lesssim \| \nabla b \|_{L^\infty} \| \Omega \|_{L^\infty} (1 + N(j)) 2^{-\gamma N(j-1)/(w)} \{ w \}_{A_p}
\]
This gives
\[
\sum_{j=1}^{\infty} \| [b, T_{m,j}] \|_{L^p(w) \to L^p(w)} \lesssim \| \nabla b \|_{L^\infty} \| \Omega \|_{L^\infty} \{ w \}_{A_p} \sum_{j=1}^{\infty} (1 + N(j)) 2^{-\gamma N(j-1)/(w)} \{ w \}_{A_p}.
\]
Thus, it suffices to choose a suitable increasing sequence $\{N(j)\}$ to get the quantitative estimate. We choose $N(j) = 2^j$ for $j \geq 1$. Similarly, we have
\[
\sum_{j=1}^{\infty} (1 + N(j)) 2^{-\gamma N(j-1)/(w)} \{ w \}_{A_p} \lesssim (w)_{A_p}.
\]
As a consequence,
\[
\sum_{j=1}^{\infty} \| [b, T_{m,j}^N]f \|_{L^p(w)} \lesssim \| \nabla b \|_{L^\infty} \| \Omega \|_{L^\infty} \{ w \}_{A_p} (w)_{A_p} \| f \|_{L^p(w)}.
\]
The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2. For $1 \leq j \leq n$, let $f_j(x) := \frac{\partial}{\partial x_j} f(x)$. Recall that
\[
T_\Omega(\nabla f)(x) = (T_{\Omega} f_1, \ldots, T_{\Omega} f_n)
\]
and
\[
[b, T_{\Omega}](\nabla f)(x) = ([b, T_{\Omega}] f_1, \ldots, [b, T_{\Omega}] f_n).
\]
Write
\[
[b, T_{\Omega}] \nabla f(x) = -T_{\Omega} [b, \nabla] f(x) + [b, \nabla T_{\Omega}] f(x).
\]
For the first term, since
\[
[b, \nabla] f = -(\nabla b) f,
\]
applying [HRT17, Theorem 1.4] yields, for $1 < p < \infty$ and $w \in A_p$,
\[
(4.1) \quad \| T_{\Omega} [b, \nabla] f \|_{L^p(w)} \lesssim \{ w \}_{A_p} (w)_{A_p} \| \Omega \|_{L^\infty} \| (\nabla b) f \|_{L^p(w)}
\lesssim \{ w \}_{A_p} (w)_{A_p} \| \Omega \|_{L^\infty} \| \nabla b \|_{L^\infty} \| f \|_{L^p(w)}.
\]
For the second term, write $K(x) = \Omega(x')/|x|^n$; the commutator satisfies
\[
[b, \nabla T_\Omega] f(x) = \text{p.v.} \int_{\mathbb{R}^n} \nabla K(x-y)(b(x) - b(y)) f(y) \, dy.
\]

It is easy to verify that $\nabla K(x)$ is homogeneous of degree $-n - 1$. A trivial computation gives
\[
||\nabla K||_{L^\infty(S^{n-1})} \lesssim ||\Omega||_{L^\infty} + ||\nabla \Omega||_{L^\infty}.
\]

Since
\[
\hat{\frac{\partial f}{\partial x_k}}(\xi) = (-2\pi i x_k f(x))^{\wedge}(\xi) \quad \text{and} \quad \left(\hat{\frac{\partial f}{\partial x_k}}\right)^{\wedge}(\xi) = 2\pi i \xi_k \hat{f}(\xi),
\]
we have
\[
(x_k \nabla K(x))^{\wedge}(\xi) = \frac{i}{2\pi} \frac{\partial \hat{\nabla K}(\xi)}{\partial \xi_k} = i \frac{\partial}{\partial \xi_k} (i \xi_1 \hat{K}(\xi), \ldots, i \xi_n \hat{K}(\xi)).
\]

Moreover,
\[
\frac{\partial}{\partial \xi_k} (\xi_j \hat{K}(\xi)) = \begin{cases} \hat{K}(\xi) + \xi_j \frac{\partial \hat{K}(\xi)}{\partial \xi_k} & \text{if } j = k, \\ \xi_j \frac{\partial \hat{K}(\xi)}{\partial \xi_k} & \text{if } j \neq k. \end{cases}
\]

Hence
\[
(x_k \nabla K(x))^{\wedge}(0) = 0, \quad \forall k \in \{1, \ldots, n\}.
\]

Additionally, from $\hat{\nabla K}(\xi) = i \xi \hat{K}(\xi)$ we see that $\hat{\nabla K}(0) = 0$.

This implies that
\[
\int_{S^{n-1}} (x')^\gamma \nabla K(x') \, d\sigma(x') = 0, \quad \forall k \in \{1, \ldots, n\}, \quad \gamma \in \{0, 1\}.
\]

Since $|\nabla K(x')| \in L^\infty(S^{n-1})$, by using Theorem 1.1, we see that
\[
(4.2) \quad ||[b, \nabla T_\Omega] f||_{L^p(w)} \lesssim \{w\}_{A_p(w) A_p} ||\nabla K||_{L^\infty(S^{n-1})} ||\nabla b||_{L^\infty} ||f||_{L^p(w)},
\]

Combining the estimates (4.1) and (4.2), we get
\[
||[b, T_\Omega] \nabla f||_{L^p(w)} \lesssim \{w\}_{A_p(w) A_p} (||\Omega||_{L^\infty} + ||\nabla \Omega||_{L^\infty}) ||\nabla b||_{L^\infty} ||f||_{L^p(w)},
\]

thereby reaching the first part of Theorem 1.2.

Moreover, regarding $\nabla [b, T_\Omega] f$ we have
\[
\nabla [b, T_\Omega] f(x) = -[b, \nabla] T_\Omega f(x) + [b, \nabla T_\Omega] f(x) = - (\nabla b(x) T_\Omega f(x) + [b, \nabla T_\Omega] f(x).
\]

In a similar way, we obtain
\[
||\nabla [b, T_\Omega]||_{L^p(w)} \lesssim \{w\}_{A_p(w) A_p} (||\Omega||_{L^\infty} + ||\nabla \Omega||_{L^\infty}) ||\nabla b||_{L^\infty} ||f||_{L^p(w)}.
\]

The proof of Theorem 1.2 is complete.
Acknowledgements. The authors would like to express their gratitude to the referees for several valuable suggestions, which have greatly improved the exposition.

Yanping Chen is supported by NNSF of China (No. 11871096). Ji Li is supported by DP 170101060.

References

[AIS01] K. Astala, T. Iwaniec and E. Saksman, Beltrami operators in the plane, Duke Math. J. 107 (2001), 27–56.
[B93] S. M. Buckley, Estimates for operator norms on weighted spaces and reverse Jensen inequalities, Trans. Amer. Math. Soc. 340 (1993), 253–272.
[C58] A. P. Calderón, Uniqueness in the Cauchy problem for partial differential equations, Amer. J. Math. 80 (1958), 16–36.
[C65] A. P. Calderón, Commutators of singular integrals, Proc. Nat. Acad. Sci. USA 53 (1965), 1092–1099.
[C78] A. P. Calderón, Commutators, singular integrals on Lipschitz curves and applications, in: Proc. Int. Congress of Mathematicians (Helsinki, 1978), Acad. Sci. Fenn., Helsinki, 1980, 85–96.
[CD16] Y. Chen and Y. Ding, Necessary and sufficient conditions for the bounds of the Calderón type commutator for the Littlewood–Paley operator, Nonlinear Anal. 130 (2016), 279–297.
[CDH16] Y. Chen, Y. Ding and G. Hong, Commutators with fractional differentiation and new characterizations of BMO-Sobolev spaces, Anal. PDE 9 (2016), 1497–1522.
[C81] J. Cohen, A sharp estimate for a multilinear singular integral in $\mathbb{R}^d$, Indiana Univ. Math. J. 30 (1981), 693–702.
[CF74] R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. 51 (1974), 241–250.
[CM75] R. R. Coifman and Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Trans. Amer. Math. Soc. 212 (1975), 315–331.
[CM78] R. R. Coifman et Y. Meyer, Au-delà des opérateurs pseudo-différentiels, Astérisque 57 (1978).
[CCDO17] J. M. Conde-Alonso, A. Culiuc, F. Di Plinio and Y. Ou, A sparse domination principle for rough singular integrals, Anal. PDE 10 (2017), 1255–1284.
[DHL17] F. Di Plinio, T. P. Hytönen and K. Li, Sparse bounds for maximal rough singular integrals via the Fourier transform, Ann. Inst. Fourier (Grenoble) 70 (2020), 1871–1902.
[D93] J. Duoandikoetxea, Weighted norm inequalities for homogeneous singular integrals, Trans. Amer. Math. Soc. 336 (1993), 869–880.
[DR86] J. Duoandikoetxea and J. L. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, Invent. Math. 84 (1986), 541–561.
[F74] C. Fefferman, Recent progress in classical Fourier analysis, in: Proc. Int. Congress of Mathematicians (Vancouver, 1974), Canad. Math. Congress, Montréal, 1975, 95–118.
[F78] N. Fujii, Weighted bounded mean oscillation and singular integrals, Math. Japon. 22 (1977/1978), 529–534.
L. Grafakos and P. Honzík, A weak-type estimate for commutators, Int. Math. Res. Notices 2012, 4785–4796.

S. Hofmann, Weighted inequalities for commutators of rough singular integrals, Indiana Univ. Math. J. 39 (1990), 1275–1304.

R. Hunt, B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc. 176 (1973), 227–251.

T. P. Hytönen, L. Roncal and O. Tapiola, Quantitative weighted estimates for rough homogeneous singular integrals, Israel J. Math. 218 (2017), 181–195.

M. T. Lacey, An elementary proof of the $A_2$ bound, Israel J. Math. 217 (2017), 133–164.

A. K. Lerner, A note on weighted bounds for rough singular integrals, C. R. Math. Acad. Sci. Paris 356 (2018), 77–80.

K. Li, C. Pérez, I. P. Rivera-Ríos and L. Roncal, Weighted norm inequalities for rough singular integral operators, J. Geom. Anal. 29 (2019), 2526–2564.

Y. Meyer, Ondelettes et Opérateurs, Vol. II, Hermann, Paris, 1990.

Y. Meyer et R. R. Coifman, Ondelettes et Opérateurs, Vol. III, Hermann, Paris, 1991.

B. Muckenhoupt and R. L. Wheeden, Weighted norm inequalities for singular and fractional integrals, Trans. Amer. Math. Soc. 161 (1971), 249–258.

T. Murai, Boundedness of singular integral operators of Calderón type (V), Adv. Math. 59 (1986), 71–81.

C. Muscalu, Calderón commutators and the Cauchy integral on Lipschitz curves revisited II. The Cauchy integral and its generalizations, Rev. Mat. Iberoamer. 30 (2014), 1089–1122.

S. Petermichl, The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical $A_p$ characteristic, Amer. J. Math. 129 (2007), 1355–1375.

S. Petermichl, The sharp weighted bound for the Riesz transforms, Proc. Amer. Math. Soc. 136 (2008), 1237–1249.

S. Petermichl and A. Volberg, Heating of the Ahlfors–Beurling operator: weakly quasiregular maps on the plane are quasiregular, Duke Math. J. 112 (2002), 281–305.

E. M. Stein and G. Weiss, Interpolation of operators with change of measures, Trans. Amer. Math. Soc. 87 (1958), 159–172.

M. E. Taylor, Commutator estimates for Hölder continuous and bmo-Sobolev multipliers, Proc. Amer. Math. Soc. 143 (2015), 5265–5274.

D. K. Watson, Weighted estimates for singular integrals via Fourier transform estimates, Duke Math. J. 60 (1990), 389–399.

J. M. Wilson, Weighted inequalities for the dyadic square function without dyadic $A_\infty$, Duke Math. J. 55 (1987), 19–50.

A. Youssfi, Regularity properties of commutators and BMO-Triebel–Lizorkin spaces, Ann. Inst. Fourier (Grenoble) 45 (1995), 795–807.

Yanping Chen
School of Mathematics and Physics
University of Science and Technology Beijing
100083 Beijing, China
E-mail: yanpingch@126.com

Ji Li
Department of Mathematics and Statistics
Macquarie University
Sydney NSW 2109, Australia
E-mail: ji.li@mq.edu.au