BASIC FORMS AND ORBIT SPACES: A DIFFEOLOGICAL APPROACH

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ABSTRACT. If a Lie group acts on a manifold freely and properly, pulling back by the quotient map gives an isomorphism between the differential forms on the quotient manifold and the basic differential forms upstairs. We show that this result remains true for actions that are not necessarily free nor proper, as long as the identity component acts properly, where on the quotient space we take differential forms in the diffeological sense. Moreover, we show that this isomorphism is also a diffeomorphism with respect to the functional diffeologies on these spaces of differential forms.

1. Introduction

Let $G$ be a Lie group acting on a smooth manifold $M$. A basic differential form on $M$ is a form that is $G$-invariant and horizontal; the latter means that evaluating the form on any vector that is tangent to a $G$-orbit yields 0. Basic differential forms constitute a subcomplex of the de Rham complex. If $G$ acts properly and with a constant orbit-type, then the quotient $M/G$ is a manifold, and, denoting the quotient map by $\pi: M \to M/G$, the pullback by this map gives an isomorphism of the de Rham complex on $M/G$ with the complex of basic forms on $M$:

$$\pi^*: \Omega^k(M/G) \to \Omega^k_{\text{basic}}(M).$$

Even if $M/G$ is not a manifold, if $G$ acts properly, then the cohomology of the complex of basic forms is isomorphic to the singular cohomology of $M/G$ with real coefficients; this was shown by Koszul in 1953 [15] for compact group actions, and extended to proper group actions by Palais in 1961 [19]. In light of these facts, some authors define the de Rham complex on $M/G$ to be the complex of basic forms on $M$.

There is another, intrinsic, definition of a de Rham complex on $M/G$, which comes from viewing $M/G$ as a diffeological space. This de Rham complex agrees with the usual one when $M/G$ is a manifold. In this paper we show that this de Rham complex agrees with the complex of basic forms even if $M/G$ is not a manifold, if the identity component of $G$ acts properly. We do not require the quotient space to be Hausdorff (see, for example, the irrational torus in Example 6.13).

Diffeology was developed by Jean-Marie Souriau (see [26]) around 1980, following earlier work of Kuo-Tsai Chen (see, e.g., [3, 6]). Our primary reference for this theory is the text [12] by Iglesias-Zemmour. A smooth manifold $M$, and its quotient $M/G$ by a group action, are examples of diffeological spaces. There is an unambiguous definition of a differential form on a diffeological space. We can take the exterior derivative of a differential form, the wedge product of differential forms, and the pullback of a differential form under a smooth map.
Spaces of differential forms, such as $\Omega^k(M)$, $\Omega^k_{\text{basic}}(M)$, and $\Omega^k(M/G)$ (the latter being in the diffeological sense), are themselves diffeological spaces too.

Returning to an action of a Lie group $G$ on a manifold $M$ with quotient map $\pi: M \to M/G$, we have a pullback map

$$\pi^*: \Omega^k(M/G) \to \Omega^k_{\text{basic}}(M),$$

and we say that quotient diffeological forms agree with basic forms for this action if, for all $k$, this pullback map is an isomorphism. Here, we can take “isomorphism” in its strongest sense: an isomorphism of differential graded algebras, and also a diffeomorphism with respect to the diffeologies on $\Omega^k(M/G)$ and on $\Omega^k_{\text{basic}}(M)$. The main result of this paper is that quotient diffeological forms agree with basic forms if the action of the identity component of $G$ is proper (in particular, if $G$ is compact). However, there are examples where basic forms on a manifold agree with diffeological forms on the quotient, but the identity component of the group does not act properly (see Example 6.14).

Remark. Another common way to define a “smooth structure” on a quotient $M/G$ is to equip it with the sheaf of real valued functions whose pullback to $M$ is smooth. (See [11, 2, 22, 7].) This structure is determined by the diffeology on $M/G$ but it is weaker in the sense that it does not always determine the diffeology on $M/G$. For example, for each positive integer $n$, these structures on the quotients $\mathbb{R}^n/\text{SO}(n)$ are all isomorphic, but the quotient diffeology remembers the integer $n$ (see Exercise 50 of [12] with solution at the end of the book). Unlike in the case of diffeology, from a sheaf of real valued functions one obtains several inequivalent notions of “differential forms” (see [25] and [28]). We do not know if any of these notions agree with basic forms.

This paper is structured as follows. Section 2 contains background on diffeology. Section 3 contains background on differential forms on diffeological spaces; again, see [12] for more details. Section 4 contains a proof that the pullback map from the de Rham complex on the orbit space $M/G$ of a Lie group action to the de Rham complex on the manifold $M$ is an injection into the subcomplex of basic forms. Section 5 contains a technical lemma: “quotient in stages”. Section 6 is the technical heart of the paper. It contains a proof that the pullback map surjects onto the subcomplex of basic forms, if the identity component of the group acts properly. Section 7 contains a proof that the pullback map is a diffeomorphism with its image with respect to the so-called standard functional diffeologies on the spaces of differential forms. We summarise our results in Theorem 7.5:

Let $G$ be a Lie group acting on a manifold $M$. Let $\pi: M \to M/G$ be the quotient map. Then the pullback map $\pi^*: (\Omega^*(M/G), d) \to (\Omega^*(M), d)$ is one-to-one, and, as a map to its image, it is an isomorphism of differential graded algebras and a diffeological diffeomorphism. If the restriction of the action to the identity component of $G$ is proper, then the image of the pullback map is the space of basic forms: $\pi^*: (\Omega^*(M/G), d) \to (\Omega^*_\text{basic}(M), d)$.

Our appendices contain two applications of the case of a finite group action. In Appendix A we show that, on an orbifold, the notion of a diffeological differential form agrees with the usual notion of a differential form on the orbifold. In Appendix B we show that, on a regular
symplectic quotient (which is also an orbifold), the notion of a diffeological differential form also agrees with Sjamaar’s notion of a differential form on the symplectic quotient.

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2. Background: Diffeological Spaces

Definition 2.1 (Diffeology). Let $X$ be a set. A parametrisation on $X$ is a function $p: U \to X$ where $U$ is an open subset of $\mathbb{R}^n$ for some $n$. A diffeology $\mathcal{D}$ on $X$ is a set of parametrisations that satisfies the following three conditions.

1. (Covering) For every point $x \in X$ and every nonnegative integer $n \in \mathbb{N}$, the constant function $p: \mathbb{R}^n \to \{x\} \subseteq X$ is in $\mathcal{D}$.

2. (Locality) Let $p: U \to X$ be a parametrisation such that for every point in $U$ there exists an open neighbourhood $V$ in $U$ such that $p|_V \in \mathcal{D}$. Then $p \in \mathcal{D}$.

3. (Smooth Compatibility) Let $p: U \to X$ be a plot in $\mathcal{D}$. Then for every $n \in \mathbb{N}$, every open subset $V \subseteq \mathbb{R}^n$, and every smooth map $F: V \to U$, we have $p \circ F \in \mathcal{D}$.

A set $X$ equipped with a diffeology $\mathcal{D}$ is called a diffeological space and is denoted by $(X, \mathcal{D})$. When the diffeology is understood, we drop the symbol $\mathcal{D}$. The elements of $\mathcal{D}$ are called plots.

Example 2.2 (Standard Diffeology on a Manifold). Let $M$ be a manifold. The standard diffeology on $M$ is the set of all smooth maps to $M$ from open subsets of $\mathbb{R}^n$ for all $n \in \mathbb{N}$.

Definition 2.3 (Diffeologically Smooth Maps). Let $X$ and $Y$ be two diffeological spaces, and let $F: X \to Y$ be a map. We say that $F$ is (diffeologically) smooth if for any plot $p: U \to X$ of $X$ the composition $F \circ p: U \to Y$ is a plot of $Y$. Denote by $C^\infty(X, Y)$ the set of all smooth maps from $X$ to $Y$. Denote by $C^\infty(X)$ the set of all smooth maps from $X$ to $\mathbb{R}$, where $\mathbb{R}$ is equipped with its standard diffeology.

Remark 2.4 (Plots). A parametrisation is diffeologically smooth if and only if it is a plot.

Remark 2.5 (Smooth Maps Between Manifolds). A map between two manifolds is diffeologically smooth if and only if it is smooth in the usual sense. In particular, if $M$ is a manifold then $C^\infty(M)$ is the usual ring of smooth real valued functions.

Remark 2.6. Diffeological spaces, along with diffeologically smooth maps, form a category. It is shown in [1, Thm. 3.2] that this category is a complete and cocomplete quasi-topos. In particular, it is closed under passing to arbitrary quotients, subsets, function spaces, products, and coproducts.
Definition 2.7 (Quotient Diffeology). Let $X$ be a diffeological space, and let $\sim$ be an equivalence relation on $X$. Let $Y = X/\sim$ be the quotient set, and let $\pi: X \to Y$ be the quotient map. We define the quotient diffeology on $Y$ to be the diffeology for which the plots are those maps $p: U \to Y$ such that for every point in $U$ there exist an open neighbourhood $V \subseteq U$ and a plot $q: V \to X$ such that $p|_V = \pi \circ q$.

Remark 2.8 (Quotient map). Let $X$ be a diffeological space and $\sim$ an equivalence relation on $X$. Then the quotient map $\pi: X \to X/\sim$ is smooth. ⊗

A special case that is important to us is the quotient of a manifold by the action of a Lie group.

Definition 2.9 (Subset Diffeology). Let $X$ be a diffeological space, and let $Y$ be a subset of $X$. The subset diffeology on $Y$ consists of those plots of $Y$ whose composition with the inclusion map $Y \to X$ are plots of $X$.

Definition 2.10 (Product Diffeology). Let $X$ and $Y$ be two diffeological spaces. The product diffeology on the set $X \times Y$ is defined as follows. Let $\text{pr}_X: X \times Y \to X$ and $\text{pr}_Y: X \times Y \to Y$ be the natural projections. A parametrisation $p: U \to X \times Y$ is a plot if $\text{pr}_X \circ p$ and $\text{pr}_Y \circ p$ are plots of $X$ and $Y$, respectively.

Definition 2.11 (Standard Functional Diffeology on Maps). Let $Y$ and $Z$ be diffeological spaces, and let $X = C^\infty(Y,Z)$. The standard functional diffeology on $X$ is defined as follows. A parametrisation $p: U \to X$ is a plot if the map

$$U \times Y \to Z \text{ given by } (u,y) \mapsto p(u)(y),$$

is smooth.

3. Background: Differential Forms on Diffeological Spaces

Definition 3.1 (Differential Forms). Let $(X,\mathcal{D})$ be a diffeological space. A (diffeological) differential $k$-form $\alpha$ on $X$ is an assignment to each plot $(p: U \to X) \in \mathcal{D}$ a differential $k$-form $\alpha(p) \in \Omega^k(U)$ satisfying the following smooth compatibility condition: for every open subset $V$ of some Euclidean space and every smooth map $F: V \to U$,

$$\alpha(p \circ F) = F^*(\alpha(p)).$$

Denote the set of differential $k$-forms on $X$ by $\Omega^k(X)$.

Definition 3.2 (Pullback Map). Let $X$ and $Y$ be diffeological spaces, and let $F: X \to Y$ be a diffeologically smooth map. Let $\alpha$ be a differential $k$-form on $Y$. Define the pullback $F^*\alpha$ to be the $k$-form on $X$ that satisfy the following condition: for every plot $p: U \to X$,

$$F^*\alpha(p) = \alpha(F \circ p).$$

Example 3.3. Let $\alpha$ be a differential form on a manifold $M$. Then $(p: U \to M) \mapsto p^*\alpha$ defines a diffeological differential form on $M$. In this way, we get an identification of the ordinary differential forms on $M$ with the diffeological differential forms on $M$.

Let $X$ be a diffeological space, $\alpha$ a differential form on $X$, and $p: U \to X$ a plot. The above identification of ordinary differential forms on $U$ with diffeological differential forms on $U$ gives $\alpha(p) = p^*\alpha$. Henceforth, we may write $p^*\alpha$ instead of $\alpha(p)$.
Example 3.4. The space $\Omega^0(X)$ of diffeological 0-forms is identified with the space $C^\infty(X)$ of smooth real valued functions, by identifying the function $f$ with the 0-form $(p: U \to X) \mapsto f \circ p$. See [12, Sect. 6.31]. With this identification, the pullback of 0-forms by a smooth map $F$ becomes the precomposition of smooth real-valued functions by $F$.

Remark 3.5 (Pullback is linear). The space of differential forms on a diffeological space is naturally a linear vector space: for $\alpha, \beta \in \Omega^k(X)$ and $a, b \in \mathbb{R}$, we define $a\alpha + b\beta: (p: U \to X) \mapsto a\alpha(p) + b\beta(p)$. If $F: X \to Y$ is a smooth map of diffeological spaces, then the pullback map $F: \Omega^k(Y) \to \Omega^k(X)$ is linear.

4. The Pullback Injects into Basic Forms

In this section we show that, for a Lie group $G$ acting on a manifold $M$ with quotient map $\pi: M \to M/G$, the pullback map $\pi^*$ is an injection from the set of diffeological differential forms on $M/G$ into the set of basic forms on $M$.

We begin by showing that the pullback $\pi^*$ is an injection:

Lemma 4.1. Let a Lie group $G$ act on a manifold $M$, and let $\pi: M \to M/G$ be the quotient map. Then the pullback map on forms, $\pi^*: \Omega^k(M/G) \to \Omega^k(M)$, is an injection.

More generally, let $X$ be a diffeological space, $\sim$ an equivalence relation on $X$, and $\pi: X \to X/\sim$ the quotient map. Then the pullback map on forms, $\pi^*: \Omega^k(X)/\sim \to \Omega^k(X)$, is an injection.

Proof. By Remark 3.5 it is enough to show that the kernel of the pullback map

$$
\pi^*: \Omega^k(X)/\sim \to \Omega^k(X)
$$

is trivial. Let $\alpha \in \Omega^k(X)/\sim$ be such that $\pi^*\alpha = 0$. Then, for any plot $p: U \to X$, we have $p^*\pi^*\alpha = 0$. By the definition of the quotient diffeology, this implies that for any plot $q: U \to X/\sim$ we have $q^*\alpha = 0$. This shows that $\pi^*$ is injective. □

We recall a few facts about Lie group actions on manifolds.

Lemma 4.2. Let a Lie group $G$ act on a manifold $M$. Let $x$ be a point in $M$ and $H$ its stabiliser. Then

- The quotient $G/H$ (equipped with the quotient diffeology induced from $G$) is a manifold.
- The orbit $G \cdot x$ (equipped with the subset diffeology induced from $M$) is a manifold.
- The orbit map $a \mapsto a \cdot x$ descends to a diffeomorphism from $G/H$ to $G \cdot x$ (as diffeological spaces). In particular, it is an injective immersion of $G/H$ into $M$.
- The tangent space $T_x(G \cdot x)$ is the space of vectors $\xi_M|_x$ for $\xi \in \mathfrak{g}$, where $\xi_M$ is the vector field on $M$ that is induced by the Lie algebra element $\xi$. This space is also the image of the differential at the identity of the orbit map $a \mapsto a \cdot x$ from $G$ to $M$.

Proof. See [12, §2 Paragraph 1]. □

Definition 4.3 (Basic Forms). Let $G$ be a Lie group acting on a manifold $M$. A differential form $\alpha$ on $M$ is horizontal if for any $x \in M$ and $v \in T_x(G \cdot x)$ we have

$$
v \perp \alpha = 0.
$$

A form that is both horizontal and $G$-invariant is called basic. When the $G$ action is understood, we denote the set of basic $k$-forms on $M$ by $\Omega^k_{\text{basic}}(M)$. 


**Remark 4.4.** The space of basic differential forms on a $G$-manifold $M$ is closed under linear combinations, wedge products, and exterior derivatives.

Indeed, let $\alpha$ and $\beta$ be basic differential forms on $M$ and let $a, b \in \mathbb{R}$. Because $\alpha$ and $\beta$ are $G$-invariant, so are $a\alpha + b\beta$, $d\alpha$, and $\alpha \wedge \beta$. For any $\xi \in \mathfrak{g}$, let $\xi_M$ be the induced vector field on $M$. We have $\xi_M (a\alpha + b\beta) = a \xi_M \alpha + b \xi_M \beta$ and $\xi_M (\alpha \wedge \beta) = (\xi_M \alpha) \wedge \beta \pm \alpha \wedge (\xi_M \beta)$. In either case, both summands on the right vanish because $\alpha$ and $\beta$ are horizontal. Hence, $a\alpha + b\beta$ and $\alpha \wedge \beta$ are horizontal, and hence basic. Next, $\xi_M d\alpha = \xi_M \pi^* \beta$; the first term on the right vanishes by $G$-invariance, and the second term vanishes because $\alpha$ is horizontal. This shows that $d\alpha$ is horizontal, and hence basic. \hfill $\Box$

**Proposition 4.5 (Pullbacks from the Orbit Space are Basic).** Let a Lie group $G$ act on a manifold $M$. Let $\alpha = \pi^* \beta$ for some $\beta \in \Omega^k(M/G)$. Then $\alpha$ is basic.

**Proof.** We first show that $\alpha$ is $G$-invariant. Let $g \in G$. Then

$$g^* \alpha = g^* \pi^* \beta$$

$$= \pi^* \beta \quad \text{because } \pi \circ g = \pi$$

$$= \alpha.$$

If $\alpha$ is a zero-form (that is, a smooth function) then $\alpha$ is automatically horizontal and we are done. Next, we assume that $\alpha$ is a differential form of positive degree, and we show that $\alpha$ is horizontal. By Lemma 4.2 and Definition 4.3 what we need to show is that for all $x \in M$ we have $A_x^* \alpha = 0$, where $A_x$ is the orbit map $a \mapsto a \cdot x$. This, in turn, follows from the following commutative diagram.

\[
\begin{array}{ccc}
G & \xrightarrow{A_x} & M \\
\downarrow{e} & & \downarrow{\pi} \\
\{\star\} & \xrightarrow{j} & M/G
\end{array}
\]

Indeed, $A_x^* \alpha = A_x^* \pi^* \beta = c^* j^* \beta = c^* 0 = 0$. \hfill $\Box$

**Corollary 4.6.** Let a Lie group $G$ act on a manifold $M$, let $\pi: M \to M/G$ be the quotient map. Then, for each $k \geq 0$, the pullback map $\pi^*$ gives an injection

$$\pi^*: \Omega^k(M/G) \to \Omega^k_{\text{basic}}(M).$$

**5. Quotient in stages**

In this section we give a technical result that we use in the next section.

**Lemma 5.1 (Quotient in stages).** Let a Lie group $G$ act on a manifold $X$. Let $K$ be a (not necessarily closed) Lie subgroup of $G$ that is normal in $G$. Also consider the induced action of $G$ on the quotient $X/K$.

(i) There exists a unique map $e: X/G \to (X/K)/G$ such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_K} & X/K \\
\downarrow{\pi_G} & & \downarrow{\pi_{G/K}} \\
X/G & \xrightarrow{e} & (X/K)/G
\end{array}
\]
(ii) The map $e$ is a diffeomorphism.

(iii) The pullback map

$$\pi_K^* : \Omega^*(X/K) \to \Omega^*(X)$$

restricts to a bijection from $\text{Image } \pi_{G/K}^*$ onto $\text{Image } \pi_G^*$.

(iv) Suppose that $K$ acts on $X$ properly and with a constant orbit-type, so that $X/K$ is a manifold and $X \to X/K$ is a fibre bundle. Then the pullback map $\pi_K^*$ also restricts to a bijection from $\Omega_{\text{basic}}^*(X/K)$ onto $\Omega_{\text{basic}}^*(X)$.

Consequently, every $G/K$-basic form on $X/K$ is in $\text{Image } \pi_{G/K}^*$ if and only if every $G$-basic form on $X$ is in $\text{Image } \pi_G^*$.

Remark 5.2. The $G$ action on $X/K$ splits through an action of $G/K$. The quotients $(X/K)/G$ and $(X/K)/(G/K)$ coincide, as they are quotients of $X/K$ by the same equivalence relation. If $K$ is closed in $G$ (so that $G/K$ is a Lie group) and $X/K$ is a manifold, then $G$-basic forms coincide with $(G/K)$-basic forms on $X/K$.

Proof of Lemma 5.1. Because $\pi_K$ is $G$-equivariant, the map $e$ exists. Because $\pi_G$ is onto, such a map $e$ is unique. Because the preimage under $\pi_K$ of a $G$-orbit in $X/K$ is a single $G$-orbit in $X$, the map $e$ is one-to-one. Because the maps $\pi_{G/K}$ and $\pi_K$ are onto, the map $e$ is onto.

Thus, the map $e$ is a bijection. To show that it is a diffeomorphism, it remains to show, for every parametrisation $p : U \to X/G$, that $p$ is a plot of $X/G$ if and only if $e \circ p$ is a plot of $(X/K)/G$.

Fix a parametrisation,

$$p : U \to X/G.$$ 

Suppose that $p$ is a plot of $X/G$. Let $u \in U$. Then there exist an open neighbourhood $W$ of $u$ in $U$ and a plot $q : W \to X$ such that $p|_W = \pi_G \circ q$. The composition $\pi_K \circ q$ is a plot of $X/K$, and

$$\pi_{G/K} \circ \pi_K \circ q = e \circ \pi_G \circ q = e \circ p|_W.$$ 

Because $u \in U$ was arbitrary, this shows that $e \circ p$ is a plot of $(X/K)/G$.

Conversely, suppose that $e \circ p$ is a plot of $(X/K)/G$. Let $u \in U$. By applying the definition of the quotient diffeology at $\pi_{G/K}$ and then at $\pi_K$, we obtain an open neighbourhood $W$ of $u$ in $U$ and a plot $r : W \to X$ such that

$$e \circ p|_W = \pi_{G/K} \circ \pi_K \circ r.$$ 

We have

$$e \circ \pi_G \circ r = \pi_{G/K} \circ \pi_K \circ r \quad \text{because } e \circ \pi_G = \pi_{G/K} \circ \pi_K$$

$$= e \circ p|_W \quad \text{by the choice of } r.$$ 

Because $e$ is one-to-one, this implies that $\pi_G \circ r = p|_W$. Because $u \in U$ was arbitrary, this shows that $p$ is a plot of $X/G$. This completes the proof that $e$ is a diffeomorphism.
Because the diagram (5.1) commutes, \( \pi_K^* \) takes Image \( \pi_{G/K}^* \) into Image \( \pi_G^* \). Because \( e \) is a diffeomorphism, we can consider its inverse. From the commuting diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_K} & X/K \\
\pi_G & & \pi_{G/K} \\
X/G & \xleftarrow{e^{-1}} & (X/K)/G
\end{array}
\]

we see that \( \pi_K^* \) takes Image \( \pi_{G/K}^* \) onto Image \( \pi_G^* \). Because \( \pi_K^* \) is one-to-one (by Lemma 4.1), we have a bijection \( \pi_K^* : \text{Image} \pi_{G/K}^* \mapsto \text{Image} \pi_G^* \).

Now suppose that \( K \) acts on \( X \) properly and with a constant orbit-type, so that \( X/K \) is a manifold and \( \pi_K \) is a fibre bundle. Then we know that \( \pi_K^* \) is a bijection from the differential forms on \( X/K \) to the \( K \)-basic differential forms on \( X \).

Because \( \pi_K^* \) is \( G \)-equivariant, \( \pi_K^* \) takes \( G \)-invariant forms on \( X \) to \( G \)-invariant forms on \( X \) and \( G \)-horizontal forms on \( X/K \) to \( G \)-horizontal forms on \( X \). So we have an injection \( \pi_K^* : \Omega^\ast_{\text{basic}}(X/K) \rightarrow \Omega^\ast_{\text{basic}}(X) \).

Let \( \alpha \) be a \( G \)-basic form on \( X \). In particular \( \alpha \) is \( K \)-basic, so there exists a form \( \beta \) on \( X/K \) such that \( \alpha = \pi_K^* \beta \). Because \( \pi_K^* \) is one-to-one and \( \alpha \) is \( G \)-invariant, \( \beta \) is \( G \)-invariant. Because \( \pi_K \) is \( G \)-equivariant and \( \alpha \) is \( G \)-horizontal, \( \beta \) is \( G \)-horizontal. This completes the proof that the map \( \pi_K^* : \Omega^\ast_{\text{basic}}(X/K) \rightarrow \Omega^\ast_{\text{basic}}(X) \) is a bijection. \( \square \)

6. The Pullback Subjects onto Basics Forms

In this section we give conditions on a group action under which every basic form is the pullback of some diffeological form on the quotient. We use the following simple criterion for being such a pullback:

**Proposition 6.1 (Pullbacks of Quotient Diffeological Forms).** Let \( G \) be a Lie group, acting on a manifold \( M \), and let \( \pi : M \rightarrow M/G \) be the quotient map. Then a differential form \( \alpha \) on \( M \) is in the image of \( \pi^* \) if and only if, for every two plots \( p_1 : U \rightarrow M \) and \( p_2 : U \rightarrow M \) such that \( \pi \circ p_1 = \pi \circ p_2 \), we have

\[
p_1^* \alpha = p_2^* \alpha.
\]

**Proof.** This result is a special case of [12, Sect. 6.38]. \( \square \)

In the setup of Proposition 6.1, the assumption \( \pi \circ p_1 = \pi \circ p_2 \) means that for each \( u \in U \) there is some \( g \in G \) such that \( p_2(u) = g \cdot p_1(u) \). If \( g \) can be chosen to be a smooth function of \( u \), then it is easy to conclude that \( p_1^* \alpha = p_2^* \alpha \) if \( \alpha \) is basic:

**Lemma 6.2.** Let a Lie group \( G \) act on a manifold \( M \). Let \( p_1, p_2 : U \rightarrow M \) be plots. Suppose that \( p_2(u) = a(u) \cdot p_1(u) \) for some smooth function \( a : U \rightarrow G \). Then for every \( \alpha \in \Omega^k_{\text{basic}}(M) \) we have \( p_1^* \alpha = p_2^* \alpha \).

**Proof.** Pick a point \( u \in U \) and a tangent vector \( v \in T_uU \). Let \( \xi_1 = (p_1)_* v \ (\in T_{p_1(u)}M) \) and \( \xi_2 = (p_2)_* v \ (\in T_{p_2(u)}M) \). Let \( g = a(u) \). The directional derivative \( D_v a|_u \) of \( a(\cdot) \) in the direction of \( v \) has the form \( \eta \cdot g \ (\in T_gG) \) for some Lie algebra element \( \eta \) (i.e., it is the right translation of \( \eta \) by \( g \)). We then have that

\[
\xi_2 = g \cdot \xi_1 + \eta^g|_{p_2(u)}
\]
where \( g \cdot \xi_1 \) is the image of \( \xi_1 \) under the differential (push-forward) map \( g_\ast : T_{p_1(u)}M \to T_{p_2(u)}M \), and where \( \eta^z \) is the vector field on \( M \) that corresponds to \( \eta \); in particular \( \eta^z \) is everywhere tangent to the \( G \) orbit.

Applying this to vectors \( v^{(1)}, \ldots, v^{(k)} \in T_u U \), we get that
\[
(p^*_2 \alpha)|_u (v^{(1)}, \ldots, v^{(k)}) = \alpha|_{p_2(u)} \left( \xi_2^{(1)}, \ldots, \xi_2^{(k)} \right) \quad \text{where } \xi_2^{(j)} := (p_2)_\ast v^{(j)}
\]
\[
= \alpha|_{p_2(u)} \left( g \cdot \xi_1^{(1)} + \eta^{(1)} t, \ldots, g \cdot \xi_1^{(k)} + \eta^{(k)} t \right)
\]
where \( \xi_1^{(j)} := (p_1)_\ast v^{(j)} \) and \( \eta^{(j)} g = D_v a|_u \)
\[
= \alpha|_{p_2(u)} \left( g \cdot \xi_1^{(1)}, \ldots, g \cdot \xi_1^{(k)} \right) \quad \text{because } \alpha \text{ is horizontal}
\]
\[
= \alpha|_{p_1(u)} \left( \xi_1^{(1)}, \ldots, \xi_1^{(k)} \right) \quad \text{because } \alpha \text{ is invariant}
\]
\[
= (p^*_1 \alpha)|_u (v^{(1)}, \ldots, v^{(k)}) \quad \text{because } \xi_1^{(j)} = (p_1)_\ast v^{(j)}.
\]

\[\square\]

Unfortunately, in the setup of Proposition 6.1, it might be impossible to choose \( g \) to be a smooth function of \( u \):

**Example 6.3** \((\mathbb{Z}_2 \circ \mathbb{R})\). Let \( M = \mathbb{R} \) and let \( G = \mathbb{Z}/2\mathbb{Z} \) with \((\pm 1) \cdot x = \pm x\). Consider the two plots \( p_1 : \mathbb{R} \to \mathbb{R} \) and \( p_2 : \mathbb{R} \to \mathbb{R} \) defined as follows:

\[
p_1(t) := \begin{cases} 
-e^{-1/t^2} & t < 0, \\
0 & t = 0, \\
e^{-1/t^2} & t > 0,
\end{cases}
\]
and

\[
p_2(t) := \begin{cases} 
-e^{-1/t^2} & t \neq 0, \\
0 & t = 0.
\end{cases}
\]

If \( \pi : M \to M/G \) is the quotient map, then \( \pi \circ p_1 = \pi \circ p_2 \); however, for \( t < 0 \) we have \( p_1(t) = 1 \cdot p_2(t) \), whereas for \( t > 0 \) we have \( p_1(t) = -1 \cdot p_2(t) \), and so the two plots do not differ by a continuous function \( g : \mathbb{R} \to \mathbb{Z}/2\mathbb{Z} \) about \( t = 0 \).

\[\checkmark\]

Our proofs use the following consequence of the Baire category theorem.

**Lemma 6.4.** Let \( U \subseteq \mathbb{R}^n \) be an open set. Let \( \{C_i\} \) be a (finite or) countable collection of relatively closed subsets of \( U \) whose union is \( U \). Then the union of the interiors, \( \bigcup_i \text{int}(C_i) \), is open and dense in \( U \).

**Proof.** The union \( \bigcup_i \text{int}(C_i) \) is clearly open. To show that it is dense, we fix an arbitrary open subset \( V \) of \( U \), and we would like to show that the intersection \( \left( \bigcup_i \text{int}(C_i) \right) \cap V \) is nonempty.

We have
\[
V = \bigcup_i (C_i \cap V).
\]

By the Baire category theorem, the open subset \( V \) of \( \mathbb{R}^n \) cannot be a countable union of nowhere dense subsets. So there exists some \( j \) such that \( C_j \cap V \) is not nowhere dense. For
such a $j$, the closure of $C_j \cap V$ has a nonempty interior. But $C_j$ is closed, and so $\text{int}(C_j \cap V)$ is nonempty. We conclude that $(\bigcup_i \text{int}(C_i)) \cap V \neq \emptyset$, as required. □

We begin with the case of a finite group action. We note that, in this case, basic differential forms are simply invariant differential forms, as the tangent space to an orbit at any point is trivial.

**Proposition 6.5 (Case of a Finite Group).** Let $G$ be a finite group, acting on a manifold $M$. Then every basic form on $M$ is the pullback of a diffeological form on $M/G$.

**Proof.** Fix a basic differential $k$-form $\alpha$ on $M$. By Proposition 6.1, it is enough to show the following: if $p_1 : U \to M$ and $p_2 : U \to M$ are plots such that $\pi \circ p_1 = \pi \circ p_2$, then $p_1^* \alpha = p_2^* \alpha$ on $U$. Fix two such plots $p_1 : U \to M$ and $p_2 : U \to M$. For each $g \in G$ let

$$C_g := \{ u \in U \mid g \cdot p_1(u) = p_2(u) \}. $$

By continuity, $C_g$ is closed for each $g$. By our assumption on $p_1$ and $p_2$,

$$U = \bigcup_{g \in G} C_g. $$

By Lemma 6.4, the set $\bigcup_{g \in G} \text{int}(C_g)$ is open and dense in $U$. Thus, by continuity, it is enough to show that $p_1^* \alpha = p_2^* \alpha$ on $\text{int}(C_g)$ for each $g \in G$.

Fix $g \in G$. Since $g \circ p_1|_{\text{int}(C_g)} = p_2|_{\text{int}(C_g)}$, we have that

$$(p_1^* g^* \alpha)|_{\text{int}(C_g)} = (p_2^* \alpha)|_{\text{int}(C_g)}. $$

Since $\alpha$ is invariant, we have $g^* \alpha = \alpha$. Thus, $p_1^* \alpha = p_2^* \alpha$ on the open subset $\text{int}(C_g)$. This completes the proof. □

Our next result, contained in Lemma 6.7 and preceded by Lemma 6.6, is a generalisation of the case of a finite group action: it shows that the property that interests us holds for a Lie group action if it holds for the action of the identity component of that Lie group, assuming that the action of the identity component is proper.

**Lemma 6.6.** Let $G$ be a Lie group, and let $G_0$ be its identity component. Assume that $G$ acts on a manifold $M$ such that the restricted action of $G_0$ on $M$ is proper. Then, for any $\gamma \in G/G_0$, and for any two plots $p_1 : U \to M$ and $p_2 : U \to M$, the set

$$C_\gamma := \{ u \in U \mid \exists g \in \gamma \text{ such that } g \cdot p_1(u) = p_2(u) \}$$

is (relatively) closed in $U$.

**Proof.** Since $G_0$ acts on $M$ properly, the quotient space $M/G_0$, equipped with the quotient topology, is Hausdorff. Thus, the diagonal $\Delta \subseteq (M/G_0) \times (M/G_0)$ is closed. Denote the equivalence classes in $M/G_0$ by $[x]_0$ for $x \in M$. Note that $G/G_0$ admits a topological group action on $M/G_0$ where for $g \in G/G_0$, we have $\gamma \cdot [x]_0 = [g \cdot x]_0$. The reader may check that this action is well-defined and continuous. Fix $\gamma \in G/G_0$. Let $\Psi_\gamma : U \to (M/G_0) \times (M/G_0)$ be the continuous map $\Psi_\gamma(u) = (\gamma \cdot [p_1(u)]_0, [p_2(u)]_0)$. Then, $\Psi_\gamma^{-1}(\Delta)$ is a closed subset of $U$. However,

$$C_\gamma = \Psi_\gamma^{-1}(\Delta). $$

This completes the proof. □
Lemma 6.7. Let $G$ be a Lie group. Let $G_0$ be the identity component of $G$. Fix an action of $G$ on a manifold $M$. Suppose that the restricted $G_0$-action is proper, and suppose that every $G_0$-basic differential form on $M$ is the pullback of a diffeological form on $M/G_0$. Then every $G$-basic differential form on $M$ is the pullback of a diffeological form on $M/G$.

Proof. Fix a $G$-basic form $\alpha$ on $M$. Let $\pi: M \to M/G$ be the quotient map, and let $p_1: U \to M$ and $p_2: U \to M$ be plots such that $\pi \circ p_1 = \pi \circ p_2$. Fix $g \in G/G_0$ and $v \in \gamma$, and define $C_\gamma$ as in Lemma 6.6. Define $\tilde{\gamma}: U \to M$ as the composition $g \circ p_1$.

This is a plot of $M$, and for any $u \in \text{int}(C_\gamma)$ we have $p_2(u) = h \cdot \tilde{\gamma}(u)$ for some $h \in G_0$. Consider the restricted action of $G_0$ on $M$. Let $\pi_0: M \to M/G_0$ be the corresponding quotient map. Then the restrictions $\pi_0 \circ \tilde{\gamma}$ and $p_2|\text{int}(C_\gamma)$ are plots of $M$, and they satisfy $\pi_0 \circ \tilde{\gamma} = \pi_0 \circ p_2|\text{int}(C_\gamma)$. By hypothesis, and because $\alpha$ is $G_0$-basic as it is $G$-basic, $\alpha$ is a pullback of a diffeological form on $M/G_0$. By Proposition 6.1, $\tilde{\gamma}^*\alpha = p_2^*\alpha$ on $\text{int}(C_\gamma)$.

But on $\text{int}(C_\gamma)$ we have $\tilde{\gamma}^*\alpha = p_2^*\gamma^*(\alpha)$ (since $\alpha$ is $G$-invariant), and so $p_1^*\alpha = p_2^*\alpha$ on $\text{int}(C_\gamma)$. Since $\gamma \in G/G_0$ is arbitrary, and $\bigcup_{\gamma \in G/G_0} \text{int}(C_\gamma)$ is open and dense in $U$ by Lemma 6.6 and Lemma 6.4, from continuity we have that $p_1^*\alpha = p_2^*\alpha$ on all of $U$. Finally, by Proposition 6.1, $\alpha$ is the pullback of a form on $M/G$. \qed

We proceed with two technical lemmas that we will use to address points with non-trivial stabilisers.

Lemma 6.8. Let $G$ be a compact connected Lie group acting orthogonally on some Euclidean space $V = \mathbb{R}^N$. Let $g \in G$ and $\eta \in \mathfrak{g}$ be such that $\exp(\eta) = g$. Let $v \in V$. Then there exists $v' \in V$ such that $|v'| \leq |v|$ and $g \cdot v - v = \eta \cdot v'$.

Proof. Since $V$ is a vector space, we identify tangent spaces at points of $V$ with $V$ itself.

\[
g \cdot v - v = \exp(t\eta) \cdot v \bigg|_0^1 = \int_0^1 \left( \frac{d}{dt} \exp(t\eta) \cdot v \right) dt = \int_0^1 (\eta \cdot \exp(t\eta) \cdot v) dt = \eta \cdot \int_0^1 (\exp(t\eta) \cdot v) dt.
\]

So define $v' := \int_0^1 (\exp(t\eta) \cdot v) dt$. Finally,

\[
|v'| = \left| \int_0^1 (\exp(t\eta) \cdot v) dt \right| \leq \int_0^1 |\exp(t\eta) \cdot v| dt = \int_0^1 |v| dt \quad \text{because the action is orthogonal} \quad = |v|.
\]

This completes the proof. \qed

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Lemma 6.9. Let $G$ be a compact connected Lie group acting orthogonally on some Euclidean space $V = \mathbb{R}^N$. Let $\gamma_1$ and $\gamma_2$ be smooth curves from $\mathbb{R}$ into $V$ such that $\gamma_1(0) = \gamma_2(0) = 0$ and such that for every $t \in \mathbb{R}$ there exists $g_t \in G$ satisfying $\gamma_2(t) = g_t \cdot \gamma_1(t)$. Let $\xi_1 = \dot{\gamma}_1(0)$ and $\xi_2 = \dot{\gamma}_2(0)$. Then, for every basic form $\alpha$ on $V$, we have $(\xi_2 - \xi_1) \cdot \alpha = 0$.

Note that, in the assumptions of this lemma, $t \mapsto g_t$ is not necessarily continuous.

Proof. First, we claim that there exists a sequence of real numbers $t_n$ converging to 0, a sequence of vectors $v'_n$ converging to 0 in $V$, and a sequence $\mu_n$ in $\mathfrak{g}$, such that

$$\xi_2 - \xi_1 = \lim_{n \to \infty} \mu_n \cdot v'_n.$$ 

Let $(t_n)$ be any sequence in $\mathbb{R}$ that converges to 0. For each $n$ choose $\eta_n \in \mathfrak{g}$ such that $\exp(\eta_n) = g_{t_n}$. Since we are working on a vector space, we can subtract the curves and consider $\gamma_2(t) - \gamma_1(t)$. We have

$$\xi_2 - \xi_1 = \frac{d}{dt}\Big|_{t=0} (\gamma_2(t) - \gamma_1(t))$$

$$= \lim_{t \to 0} \left( \frac{\gamma_2(t) - \gamma_1(t)}{t} \right)$$

$$= \lim_{n \to \infty} \left( \frac{\gamma_2(t_n) - \gamma_1(t_n)}{t_n} \right)$$

$$= \lim_{n \to \infty} \left( \frac{g_{t_n} \cdot \gamma_1(t_n) - \gamma_1(t_n)}{t_n} \right)$$

$$= \lim_{n \to \infty} \left( \frac{\eta_n \cdot v'_n}{t_n} \right)$$

for some $v'_n \in V$ that satisfy $|v'_n| \leq |\gamma_1(t_n)|$ for each $n$; the last equality is a result of Lemma 6.9. Because $|v'_n| \leq |\gamma_1(t_n)| \to |\gamma_1(0)| = 0$, the claim holds with $\mu_n := \eta_n/t_n$.

We now have

$$(\xi_2 - \xi_1) \cdot \alpha|_0 = \lim_{n \to \infty} \left( (\mu_n \cdot v'_n) \cdot (\alpha|_{v'_n}) \right)$$

and

$$= \lim_{n \to \infty} \left( (\mu_n)_{V} \cdot \alpha|_{v'_n} \right)$$

where $(\mu_n)_V$ is the vector field on $V$ induced by $\mu_n \in \mathfrak{g}$. Because $\alpha$ is basic, the last line above vanishes.

The final ingredient that we need is the Slice Theorem, along with some properties of the models that appear in this theorem.

Theorem 6.10 (Slice Theorem). Let $G$ be a Lie group acting properly on a manifold $M$. Fix $x \in M$. Let $H$ be the stabiliser of $x$, and let $V = T_x M/(T_x (G \cdot x))$ be the normal space to the orbit $G \cdot x$ at $x$, equipped with the linear $H$ action that is induced by the linear isotropy action of $H$ on $T_x M$. Then there exist a $G$-invariant open neighbourhood $U$ of $x$ and a $G$-equivariant diffeomorphism $F : U \to G \times_H V$. 

Koszul [15] proved the above Slice Theorem for compact Lie group actions; Palais [19] proved it for proper Lie group actions. The proof is described in Theorem 2.3.3 of [8] and in Appendix B of [9].

Let $G$ be a Lie group, $H$ a compact subgroup, and $V$ a vector space with a linear $H$ action. The equivariant vector bundle $G \times_H V$ over $G/H$ is obtained as the quotient of $G \times V$ by the anti-diagonal $H$-action $h \cdot (g, v) = (gh^{-1}, h \cdot v)$.

Consider the map

$$i : V \to G \times_H V, \quad v \mapsto [e, v],$$

where $e$ is the identity element. Also consider the quotient maps

$$\pi_V : V \to V/H \quad \text{and} \quad \pi : G \times_H V \to (G \times_H V)/G.$$

**Lemma 6.11.** Suppose that every $H$-basic form on $V$ is the pullback of a diffeological form on $V/H$. Then every $G$-basic form on $G \times_H V$ is the pullback of a diffeological form on $(G \times_H V)/G$.

**Proof.** Consider the pullback map

$$i^* : \Omega^*(G \times_H V) \to \Omega^*(V).$$

It is enough to show the following two facts.

(a) The map $i^*$ restricts to a bijection from $\text{Image} \, \pi^*$ onto $\text{Image} \, \pi_V^*$.

(b) The map $i^*$ restricts to a bijection from the space of $H$-basic forms on $V$ onto the space of $G$-basic forms on $(G \times_H V)$.

Let $G \times H$ act on $G \times V$ where $G$ acts by left multiplication on the first factor and where $H$ acts by the anti-diagonal action $h : (g, v) \mapsto (gh^{-1}, g \cdot v)$. We have a commuting diagram

$$
\begin{array}{ccc}
G \times V & \xrightarrow{i} & G \times_H V \\
\downarrow{\pi_H} & & \downarrow{\pi} \\
V & & (G \times_H V)/G \\
\end{array}
$$

(6.1)

Here, the map $\pi_H : G \times V \to G \times_H V$ is the quotient by the $H$ action, and the projection to the second factor $\text{pr}_2 : G \times V \to V$ can be identified with the quotient by the $G$ action.

We apply Lemma 5.1 in two ways: taking the quotient by $G$ and then by $H$, and taking the quotient by $H$ and then by $G$. This gives the following commuting diagram:

$$
\begin{array}{ccc}
G \times V & \xrightarrow{\text{pr}_2} & G \times V \\
\downarrow{\pi_V} & & \downarrow{\pi_G \times H} \\
V/H & \xrightarrow{\pi_V} & (G \times V)/(G \times H) \\
\end{array}
\quad \begin{array}{ccc}
G \times_H V & \xrightarrow{\pi} & (G \times_H V)/G \\
\downarrow{\pi} & & \downarrow{\pi} \\
V/H & \xrightarrow{\pi} & (G \times_H V)/G \\
\end{array}
$$

where $e$ and $e'$ are diffeomorphisms.
By Lemma 5.1, \( \text{pr}_2^* \) restricts to a bijection from \( \text{Image} \, \pi_V \) onto \( \text{Image} \, \pi^*_V \times H \), and \( \pi^*_H \) restricts to a bijection from \( \text{Image} \, \pi^* \) onto \( \text{Image} \, \pi^*_V \times H \). By the commuting diagram (6.1), it follows that \( i^* \) restricts to a bijection from \( \text{Image} \, \pi^* \) onto \( \text{Image} \, \pi^*_V \), as required.

Also, by Lemma 5.1 and Remark 5.2, \( \text{pr}_2^* \) restricts to a bijection from the space of \( H \)-basic forms on \( V \) onto the space of \((G \times H)\)-basic forms on \( G \times V \), and \( \pi^*_H \) restricts to a bijection from the space of \( G \)-basic forms on \( G \times H V \) onto the space of \((G \times H)\)-basic forms on \( G \times V \). By the commuting diagram [6.1], it follows that \( i^* \) restricts to a bijection from the space of \( G \)-basic forms on \( G \times H V \) onto the space of \( H \)-basic forms on \( V \), as required. \( \square \)

We are now ready to prove the main result of this section.

**Proposition 6.12 (Pullback Subjects to Basic Forms).** Let a Lie group \( G \) act on a manifold \( M \). Assume that the identity component of \( G \) acts properly. Let \( \pi : M \to M/G \) be the quotient map. Then every basic form on \( M \) is the pullback of a diffeological form on \( M/G \).

**Proof.** By Lemma 6.7, we may assume that \( G \) is connected and that the action is proper.

If the dimension of \( G \) is 0, then \( G \) is the trivial group, and the result holds trivially.

We proceed inductively on the dimension of \( G \). Assume that for all Lie groups \( K \) with \( \dim K < \dim G \) and all \( K \)-manifolds \( N \), every \( K \)-basic form on \( N \) is the pullback of a diffeological form on \( N/K \).

Fix a \( G \)-basic form \( \alpha \) on \( M \). By Proposition 6.1, we need to show, for any two plots \( p_1 : W \to M \) and \( p_2 : W \to M \) for which \( \pi \circ p_1 = \pi \circ p_2 \), that we have \( p_1^* \alpha = p_2^* \alpha \). Let \( p_1 \) and \( p_2 \) be two such plots. Fix \( u \in W \). We would like to show that \( p_1^* \alpha|_u = p_2^* \alpha|_u \).

Let \( x = p_2(u) \). Let \( H \) be the stabiliser of \( x \). By the [Slice Theorem](#), there exists a \( G \)-invariant open neighbourhood \( U \) of \( x \) and an equivariant diffeomorphism \( F : U \to G \times H V \) where \( V = T_x M / T_x (G \cdot x) \). Because \( F \) is an equivariant diffeomorphism and \( \alpha \) is \( G \)-basic, \((F^{-1})^* \alpha \) is \( G \)-basic on \( G \times H V \).

First, suppose that \( x \) is a fixed point. Then \( p_1(u) = p_2(u) = x \), and \( F \) identifies \( U \) with \( V = T_x M \), sending \( x \) to \( 0 \in V \). Fixing a \( G \)-invariant Riemannian metric, we have that \( G \) acts linearly and orthogonally on \( V \). Let \( v \in T_0 V \). By the equation (6.9), to the curves \( \gamma_1(t) := F(p_1(u + tv)) \) and \( \gamma_2(t) := F(p_2(u + tv)) \) in \( V \) and to the basic form \((F^{-1})^* \alpha \) on \( V \), we obtain that \( \gamma_1(0) \cdot (F^{-1})^* \alpha = \gamma_2(0) \cdot (F^{-1})^* \alpha \). This, in turn, implies that \( v \cdot p_1^* \alpha = v \cdot p_2^* \alpha \). Because \( v \in T_0 W \) was arbitrary, we conclude that \( p_1^* \alpha|_u = p_2^* \alpha|_u \), as required.

Next, suppose that \( x \) is not a fixed point. Then the stabiliser \( H \) of \( x \) is a proper subgroup of \( G \). Since \( G \) is connected, we must have that \( \dim H < \dim G \). By the induction hypothesis, every \( H \)-basic form on \( V \) is the pullback of a diffeological form on \( V/H \). By Lemma 6.11, every \( G \)-basic form on \( G \times H V \) is the pullback of a diffeological form on \((G \times H V)/G \). Because \( F \) is an equivariant diffeomorphism, every \( G \)-basic form on \( U \) is the pullback of a diffeological form on \( U/G \). So \( \alpha|_U \) is the pullback of a diffeological form on \( U/G \). This implies that \( p_1^* \alpha|_u = p_2^* \alpha|_u \), as required. \( \square \)

**Example 6.13 (Irrational Torus, First Construction).** Fix an irrational number \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). The corresponding *irrational torus* is

\[
T_\alpha := \mathbb{R}/(\mathbb{Z} + \alpha \mathbb{Z});
\]

note that it is not Hausdorff. It is obtained as the quotient of \( \mathbb{R} \) by the \( \mathbb{Z}^2 \) action \((m, n) \cdot x = x + m + n \alpha \). The basic differential forms on \( \mathbb{R} \) with respect to this action are the constant
functions and the constant coefficient one-forms $cdx$. By Proposition 6.12, each of these is the pullback of a differential form on $T_\alpha$.

We note that these “boring” differential forms do not capture the richness of the diffeology on $T_\alpha$. In fact, $T_\alpha$ and $T_\beta$ are diffeomorphic if and only if there exist integers $a, b, c, d$ such that $ad - bc = \pm 1$ and $\alpha = \frac{a\beta + b}{c + d}$. See Exercise 4 and Exercise 105 of [12] with solutions at the end of the book.

Example 6.14 (Irrational Torus, Second Construction). Fix an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Consider the quotient $T^2/S_\alpha$ of $T^2 := \mathbb{R}^2/\mathbb{Z}^2$ by the irrational solenoid $S_\alpha := \{[t, \alpha t] \mid t \in \mathbb{R}\} \subset T^2$. It is obtained as the quotient of $T^2$ by the $\mathbb{R}$ action $t \cdot [x, y] = [x + t, y + \alpha t]$. The basic forms on $T^2$ with respect to this action are the constant functions and the constant multiples of the one-form $\alpha dx - dy$. The quotient $T^2/S_\alpha$ is diffeomorphic to the irrational torus $T_\alpha$ of Example 6.13; see Exercise 31 of [12] (with solution at the end of the book).

In fact, consider the action of $\mathbb{R} \times \mathbb{Z}^2$ on $\mathbb{R}^2$ that is given by $(t, m, n) \cdot (x, y) = (x + m + t, y + n + \alpha t)$. Taking the quotient first by $\mathbb{R}$ and then by $\mathbb{Z}^2$ (and identifying the first of these quotients with $\mathbb{R}$ through the map $(x, y) \mapsto y - \alpha x$) yields $T_\alpha$. Taking the quotient first by $\mathbb{Z}^2$ and then by $\mathbb{R}$ yields $T^2/S_\alpha$. Applying Lemma 5.1 twice, we get the following commuting diagram.

\[\begin{array}{c}
\mathbb{R}^2 \\
\downarrow \downarrow \\
T^2 \\
\downarrow \downarrow \\
T^2/S_\alpha
\end{array}\]

where $e$ and $e'$ are diffeomorphisms. As noted in Example 6.13, every basic form on $\mathbb{R}$ is the pullback of a diffeological differential form on $T_\alpha$. By Lemma 5.1 this implies that every basic form on $\mathbb{R}^2$ is the pullback of a diffeological differential form on $\mathbb{R}^2/(\mathbb{R} \times \mathbb{Z}^2)$. Again by Lemma 5.1, we conclude that every basic form on $T^2$ is the pullback of a diffeological form on $T^2/S_\alpha$. Thus, the $\mathbb{R}$-action on $T^2$ through $S_\alpha$ satisfies the conclusion of Proposition 6.12 although it does not satisfy the assumption of Proposition 6.12; this $\mathbb{R}$-action is not proper.

7. Stronger Form of “Isomorphism”

Let a Lie group $G$ act on a manifold $M$. Let $\pi : M \to M/G$ be the quotient map. By Remark 3.5 and Corollary 4.6 for every $k \geq 0$, the pullback map

$$\pi^* : \Omega^k(M/G) \to \Omega^k(M)$$

is linear, is one-to-one, and its image is contained in the space of basic $k$-forms on $M$.

We now note that the pullback map $\pi^*$ is also an isomorphism with its image as differential graded algebras:
Remark 7.1 (Wedge Product and Exterior Derivative). Let $X$ be a diffeological space. Define the wedge product of $\alpha \in \Omega^k(X)$ and $\beta \in \Omega^l(X)$ to be the $(k+l)$-form $\alpha \wedge \beta: (p: U \to X) \mapsto p^*\alpha \wedge p^*\beta$. Define the exterior derivative of $\alpha$ to be the $k+1$ form $d\alpha: (p: U \to X) \mapsto d(p^*\alpha)$. Then $\Omega^*(X) = \bigoplus_{k=0}^{\infty} \Omega^k(X)$ is a differential graded algebra. In particular, $\Omega^*(X)$ is an exterior algebra, and $(\Omega^*(X),d)$ is a complex. If $F: X \to Y$ is a smooth map of diffeological spaces, then the pullback map $F^*: \Omega^*(Y) \to \Omega^*(X)$ is a morphism of differential graded algebras, that is, it intertwines the wedge products and the exterior derivatives.

The pullback map $\pi^*$ is an isomorphism with its image in an even stronger sense: it is a diffeomorphism with respect to the so-called standard functional diffeologies on the spaces of differential forms. We will now give the relevant definitions and the proof of this property of the pullback map $\pi^*$.

Definition 7.2 (Standard Functional Diffeology on Forms). Let $(X,D)$ be a diffeological space. The standard functional diffeology on $\Omega^k(X)$ is defined as follows. A parametrisation $p: U \to \Omega^k(X)$ is a plot if for every plot $(q: V \to X) \in D$, where $V$ is open in $\mathbb{R}^n$, the map $U \times V \to \bigwedge^k \mathbb{R}^n$ sending $(u,v)$ to $q^*(p(u))|_v$ is smooth. See [12, sect. 6.29] for a proof that this is indeed a diffeology.

Remark 7.3. Let $X$ be a diffeological space. We have the following facts.

1. Under the identification of Example 3.4 of the space of diffeological 0-forms with the space of smooth real valued functions, Definitions 2.11 and 7.2 of the standard functional diffeology on these spaces agree.
2. If $F: X \to Y$ is a smooth map to another diffeological map, then the pullback map $F^*: \Omega^k(Y) \to \Omega^k(X)$ is smooth with respect to the standard functional diffeologies on the sets of differential forms. See [12, Sect. 6.32].
3. The exterior derivative $d: \Omega^k(X) \to \Omega^{k+1}(X)$ and the wedge product $\Omega^k(X) \times \Omega^l(X) \to \Omega^{k+l}(X)$ are smooth, and they both commute with pullbacks. See Sections 6.32, 6.34, and 6.35 of [12].

Proposition 7.4 (Pullbacks via Quotient Maps). Let a Lie group $G$ act on a manifold $M$, and let $\pi: M \to M/G$ be the quotient map. Then the pullback map

$$\pi^*: \Omega^k(M/G) \to \pi^*\Omega^k(M/G)$$

is a diffeomorphism, where the target space is equipped with the subset diffeology induced from $\Omega^k(M)$.

Proof. More generally, let $X$ be a diffeological space, let $\sim$ be an equivalence relation on $X$, and let $\pi: X \to X/\sim$ be the quotient map. We will show that the pullback map

$$\pi^*: \Omega^k(X/\sim) \to \pi^*\Omega^k(X/\sim)$$

is a diffeomorphism, where the target space is equipped with the subset diffeology induced from $\Omega^k(X)$.

Clearly, $\pi^*$ is surjective to its image. For the injectivity of $\pi^*$, see Lemma 4.1. By Part 2 of Remark 7.3, we know that $\pi^*$ is smooth. We wish to show that the inverse map

$$(\pi^*)^{-1}: \pi^*\Omega^k(X/\sim) \to \Omega^k(X/\sim)$$
space that is locally diffeomorphic to finite linear quotients of $\mathbb{R}^n$ spaces of differential forms, we need to show, given any plot $G$, Proposition 6.12, if the identity component of $G$ is an isomorphism of differential graded algebras and a diffeological diffeomorphism. By Remark 7.1 and Proposition 7.4, as a map to its image, the pullback $(\pi^* \circ \phi)^* = \phi^* 
abla^* \pi^* (\phi^* (\pi^* \circ \phi)^* (u))$ is a map to $\bigwedge^k \mathbb{R}^n$ that is smooth in $(u, w) \in U \times W$. Here, $(\pi^*)^{-1}$ is restricted to the image of $\pi^*$, on which it is well defined because $\pi^*$ is injective.

It is enough to show smoothness locally. For any point $w \in W$ there exist an open neighbourhood $V \subseteq W$ of $w$ and a plot $q: V \to X$ such that $r|_V = \pi \circ q$. For all $v \in V$, we have

$$r^*((\pi^*)^{-1} \circ p)(u)) = q^* (p(u))|_v,$$

which is smooth in $(u, v) \in U \times V$ by the definition of the standard functional diffeology on $\Omega^k(X)$. And so we are done. \qed

**Theorem 7.5 (Main Theorem).** Let $G$ be a Lie group acting on a manifold $M$. Let $\pi: M \to M/G$ be the quotient map. Then the pullback map $\pi^*: (\Omega^*(M/G), d) \to (\Omega^*(M), d)$ is one-to-one, and, as a map to its image, it is an isomorphism of differential graded algebras and a diffeological diffeomorphism. If the restriction of the action to the identity component of $G$ is proper, then the image of the pullback map is the space of basic forms:

$$\pi^*: (\Omega^*(M/G), d) \xrightarrow{\cong} (\Omega^*_\text{basic}(M), d).$$

**Proof.** By Remark 3.5 and Corollary 4.6 the pullback is a linear bijection into the space of basic forms. By Remark 7.1 and Proposition 7.4 as a map to its image, the pullback is an isomorphism of differential graded algebras and a diffeological diffeomorphism. By Proposition 6.12 if the identity component of $G$ acts properly, the image is the space of basic forms. \qed

**Appendix A. Orbifolds**

Let $X$ be a Hausdorff, second countable topological space. Fix a positive integer $n$.

The following definition is based on Haefliger, [10, Sect. 4].

1. An $n$ dimensional orbifold chart on $X$ is a triplet $(\tilde{U}, \Gamma, \phi)$ where $\tilde{U} \subseteq \mathbb{R}^n$ is an open ball, $\Gamma$ is a finite group of diffeomorphisms of $\tilde{U}$, and $\phi: \tilde{U} \to X$ is a $\Gamma$-invariant map onto an open subset $U$ of $X$ that induces a homeomorphism $\tilde{U}/\Gamma \to U$.

2. Two orbifold charts on $X$, $(\tilde{U}, \Gamma, \phi)$ and $(\tilde{V}, \Gamma, \psi)$, are compatible if for every two points $u \in \tilde{U}$ and $v \in \tilde{V}$ such that $\phi(u) = \psi(v)$ there exist a neighbourhood $O_u$ of $u$ in $\tilde{U}$ and a neighbourhood $O_v$ of $v$ in $\tilde{V}$ and a diffeomorphism $g: O_u \to O_v$ that takes $u$ to $v$ and such that $\psi \circ g = \phi$.

3. An orbifold atlas on $X$ is a set of orbifold charts on $X$ that are pairwise compatible and whose images cover $X$. Two orbifold atlases are equivalent if their union is an orbifold atlas.

The following definition was introduced in [14]: A diffeological orbifold is a diffeological space that is locally diffeomorphic to finite linear quotients of $\mathbb{R}^n$. 

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These two definitions are equivalent in the following sense. Given an orbifold atlas on $X$, there exists a unique diffeology on $X$ such that all the homeomorphisms $\tilde{U}/\Gamma \to U$ are diffeomorphisms. With this diffeology, $X$ becomes a diffeological orbifold. Two orbifold atlases are equivalent if and only if the corresponding diffeologies are the same. Finally, every diffeological orbifold structure on $X$ can be obtained in this way. For details, see [14, Sec. 8].

Let $\{(\tilde{U}, \Gamma, \psi)\}$ be an orbifold atlas on $X$. An orbifold differential form on $X$ is given by, for each chart $(\tilde{U}, \Gamma, \psi)$ in the atlas, a $\Gamma$ invariant differential form $\alpha_{\tilde{U}}$ on the domain $\tilde{U}$ of the chart. We require the following compatibility condition. For every two charts $(\tilde{U}, \Gamma, \phi)$, $(V', \Gamma', \psi)$, and every two points $u \in \tilde{U}$ and $v \in V$ with $\phi(u) = \psi(v)$, there exist a diffeomorphism $g: O_u \to O_v$ from a neighbourhood of $u$ to a neighbourhood of $v$ that takes $u$ to $v$, such that $\psi \circ g = \phi$, and such that $g^*(\alpha_{\tilde{U}}|_{O_u}) = \alpha_{V'}|_{O_v}$. Two such collections $\{\alpha_{\tilde{U}}\}$ of differential forms, defined on the domains of the charts in two equivalent orbifold atlases, represent the same orbifold differential form if their union still satisfies the compatibility condition.

Every diffeological differential form $\alpha$ on $X$ determines an orbifold differential form by associating to every chart $(\tilde{U}, \Gamma, \psi)$ the pullback $\psi^*\alpha$. Proposition 6.5 implies that this gives a bijection between diffeological differential forms and orbifold differential forms.

**Appendix B. Sjamaar Differential Forms; Case of Regular Symplectic Quotients.**

Let a Lie group $G$ act properly on a symplectic manifold $(M, \omega)$ with an (equivariant) momentum map $\Phi: M \to g^*$. Let $Z = \Phi^{-1}(0)$ be the zero level set and $i: Z \to M$ its inclusion map. Let

$$Z_{\text{reg}} = \{ z \in Z \mid \exists \text{ neighbourhood } U \text{ of } z \text{ in } Z \text{ such that, for all } z' \in U, $$

$$\text{the stabilisers of } z' \text{ and of } z \text{ are conjugate in } G \}.$$  

The set $Z_{\text{reg}}$, (with the subset diffeology induced from $M$ or, equivalently, from $Z$), is a manifold, and it is open and dense in $Z$ (see [24]). The quotient $Z_{\text{reg}}/G$, (with the quotient diffeology induced from $Z_{\text{reg}}$, or, equivalently, the subset diffeology induced from $M/G$), is also a manifold.

If $M$ is connected and $\Phi$ is proper, then $Z_{\text{reg}}$ and $Z_{\text{reg}}/G$ are connected. More generally, we allow a manifold to have connected components of different dimensions.

Denote by $i_{\text{reg}}: Z_{\text{reg}} \to M$ the inclusion map and by $\pi_{\text{reg}}: Z_{\text{reg}} \to Z_{\text{reg}}/G$ the quotient map.

The following definition was introduced (but not yet named) by Reyer Sjamaar in [23]:

**Definition B.1.** A Sjamaar differential l-form $\sigma$ on $Z/G$ is a differential l-form on $Z_{\text{reg}}/G$ (in the ordinary sense) such that there exists $\tilde{\sigma} \in \Omega^l(M)$ satisfying $i_{\text{reg}}^*\tilde{\sigma} = \pi_{\text{reg}}^*\sigma$.

A special case of a Sjamaar form is the reduced symplectic form, $\omega_{\text{red}}$, which satisfies $\pi_{\text{reg}}^*\omega_{\text{red}} = i_{\text{reg}}^*\omega$. The orbit type stratification on $M$ induces a stratification of the reduced space $Z/G$, and the Sjamaar differential forms naturally extend to the strata of $Z/G$. The extensions of $\omega_{\text{red}}$ to these strata exhibit $Z/G$ as a stratified symplectic space in the sense of Sjamaar and Lerman [24].
We denote the set of Sjamaar l-forms by $\Omega^l_{\text{sjamaar}}(Z/G)$. We define the exterior derivative $d$ of a Sjamaar form $\alpha$ as the usual exterior derivative of $\alpha$ as a differential form on $Z_{\text{reg}}/G$. We define the wedge product of Sjamaar forms $\alpha$ and $\beta$ as the usual wedge product of $\alpha$ and $\beta$ as differential forms on $Z_{\text{reg}}/G$. The space of Sjamaar forms is closed under wedge products and it forms a subcomplex of the de Rham complex $(\Omega^* (Z_{\text{reg}}/G), d)$. Sjamaar forms satisfies a Poincaré Lemma, Stokes’ Theorem, and a de Rham theorem.

For details, see Sjamaar’s paper [23].

The reduced space $Z/G$ comes equipped with the quotient diffeology inherited from $Z$, which equals the subset diffeology inherited from $M/G$. We call this the subquotient diffeology.

It is now natural to ask how Sjamaar forms on a symplectic quotient $Z/G$, which a-priori depend on the ambient symplectic manifold $M$, relate to the diffeological forms on $Z/G$, whose definition is intrinsic. More precisely, consider the inclusion map $J: Z_{\text{reg}}/G \to Z/G$. Then we have the pullback map on diffeological forms

$$J^*: \Omega^l(Z/G) \to \Omega^l(Z_{\text{reg}}/G),$$

and we identify the target space with the ordinary differential forms on $Z_{\text{reg}}/G$. We ask:

- Is the space of Sjamaar forms contained in the image of $J$?
- Is the image of $J^*$ contained in the space of Sjamaar forms?
- Is $J^*$ one-to-one?

If 0 is a regular value of the momentum map $\Phi$, then the answers to each of these questions is “yes”. If 0 is a critical value, then the answer to the first question is “yes”, and we do not know the answers to the other two questions. We refer the reader to Section 3.4 of the second author’s thesis [27] for details.

References

[1] J. C. Baez and A. E. Hoffnung, “Convenient categories of smooth spaces”, Trans. Amer. Math. Soc., 363 (2011), no. 11, 5789–5825.
[2] Glen E. Bredon, Introduction to compact transformation groups, Pure and Applied Mathematics volume 46, Academic Press, New York and London, 1972.
[3] K.-T. Chen, “Iterated integrals of differential forms and loop space homology”, Ann. of Math. 97 (1973), no. 2, 217–246.
[4] ———, “Iterated integrals, fundamental groups and covering spaces”, Trans. Amer. Math. Soc. 206 (1975), 83–98.
[5] ———, “Iterated path integral”, Bull. Amer. Math. Soc. 83 (1977), no. 5, 831–879.
[6] ———, “On differentiable spaces”, In: Categories in continuum physics (Buffalo, N.Y. 1982), Lecture Notes in Math., 1174 (1986), Springer-Verlag, Berlin, 138–142.
[7] R. Cushman and J. Śniatycki, “Differential structure of orbit spaces”, Canadian J. Math. 53 (2001), no. 4, 715–755.
[8] J. J. Duistermaat and J. A. C. Kolk, Lie Groups, Universitext, Spering-Verlag, Berlin (2000).
[9] V. Guillemin, V. Ginzburg, Y. Karshon, Moment Map, Cobordisms, and Hamiltonian Group Actions, Math. Surveys and Monographs, Amer. Math. Soc., 2002.
[10] A. Haefliger, “Groupoides d’holonomie et classifiants”, Asterisque 116 (1984), 70–97.
[11] Gerhard Paul Hochschild, The Structure of Lie Groups, Holden-Day, San-Francisco, California, 1965.
[12] P. Iglesias-Zemmour, Diffeology, Math. Surveys and Monographs, Amer. Math. Soc., 2013.
[13] P. Iglesias-Zemmour, Y. Karshon, “Smooth Lie group actions are parametrized diffeological subgroups”, Proc. Amer. Math. Soc. 140, no. 2 (2012), 731–739.
[14] P. Iglesias-Zemmour, Y. Karshon, and M. Zadka, “Orbifolds as diffeologies”, Trans. Amer. Math. Soc. 362 (2010), 2811–2831.

[15] J.-L. Koszul, “Sur certains groupes de transformations de Lie”, Géométrie Différentielle, Colloques Internationaux du Centre National Recherche Scientifique (Strasbourg, 1953), Centre National Recherche Scientifique, Paris (1953), 137–141.

[16] C. D. Marshall, “Calculus on subcartesian space”, J. Diff. Geom. 10 (1975), 551–574.

[17] _____, “Basic differential forms for actions of Lie groups”, Proc. Amer. Math. Soc. 124 (1996), no. 5, 1633–1642.

[18] _____, “Basic differential forms for actions of Lie groups II”, Proc. Amer. Math. Soc. 125 (1997), no. 7, 2175–2177.

[19] R. Palais, “On the existence of slices for actions of non-compact Lie groups”, Ann. of Math., 73 (1961), 295–323.

[20] I. Satake, “On a generalization of the notion of a manifold”, Proc. Nat. Acad. Sci. USA, 42 (1956), 359–363.

[21] _____, “The Gauss-Bonnet theorem for V-manifolds”, J. Math. Soc. Japan, 9 (1957), no. 4, 464–492.

[22] G. Schwarz, “Smooth functions invariant under the action of a compact Lie group”, Topology 14 (1975), 63–68.

[23] R. Sjamaar, “A de Rham theorem for symplectic quotients”, Pacific J. Math., 220 (2005), no. 1, 153–166.

[24] R. Sjamaar and E. Lerman, “Stratified symplectic spaces and reduction”, Ann. of Math., 134 (1991), no.2, 375–422.

[25] J. Śniatycki, “Differential Geometry of Singular Spaces and Reduction of Symmetry”, Cambridge University Press, New Mathematical Monographs (no. 23), 2013.

[26] J.-M. Souriau, “Groupes différentiels”, In: Differential geometric methods in mathematical physics (Proceedings of conference held in Aix-en-Provence Sept. 3-7, 1979 and Salamanca, Sept. 10-14, 1979) (P. L. Garcia, A. Pérez-Rendón, and J.-M. Souriau, eds.), Lecture Notes in Mathematics, 836 (1980), Springer, New York, 91–128.

[27] J. Watts, Diffeologies, Differential Spaces, and Symplectic Geometry, Ph.D. Thesis (2012), University of Toronto, Canada.

[28] J. Watts, The Calculus on Subcartesian Spaces, M.Sc. Thesis (2006), University of Calgary, Canada.