Random bases for coprime linear groups

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Abstract. The minimal base size \( b(G) \) for a permutation group \( G \) is a widely studied topic in permutation group theory. Z. Halasi and K. Podoski [Every coprime linear group admits a base of size two, *Trans. Amer. Math. Soc.* **368** (2016), no. 8, 5857–5887] proved that \( b(G) \leq 2 \) for coprime linear groups. Motivated by this result and the probabilistic method used by T. Burness, M. W. Liebeck and A. Shalev, it was asked by L. Pyber [Personal communication, Bielefeld, 2017] whether or not, for coprime linear groups \( G \leq GL(V) \), there exists a constant \( c \) such that the probability that a random \( c \)-tuple is a base for \( G \) tends to 1 as \( |V| \to \infty \). While the answer to this question is negative in general, it is positive under the additional assumption that \( G \) is primitive as a linear group. In this paper, we show that almost all 11-tuples are bases for coprime primitive linear groups.

1 Introduction

For a finite permutation group \( G \), a subset \( B \) of \( \Omega \) is called a base for \( G \) if the pointwise stabiliser of \( B \) in \( G \) is trivial. The concept of base plays a fundamental role in the development of algorithms for permutation groups, and these algorithms are significantly faster if the size of the base is small (see the book of Á. Seress [21]). The minimal size of a base for \( G \) acting on \( \Omega \) is denoted by \( b(G) \). L. Pyber [17] showed that there exists a universal constant \( c > 0 \) such that almost all subgroups \( G \) of \( \text{Sym}(n) \) satisfy \( b(G) > cn \).

On the other hand, there are several important classes of groups where the minimal base size \( b(G) \) can be bounded by a fixed constant \( c \). Á. Seress [20] showed that \( b(G) \leq 4 \) for a solvable primitive permutation group \( G \). For an almost simple primitive permutation group \( G \), a famous conjecture of P.J. Cameron and W.M. Kantor [3] states that there exists an absolute constant \( c \) such that \( b(G) \leq c \) for all non-standard primitive permutation groups \( G \). In [2], P.J. Cameron suggested that \( c \) can be chosen to be 6 except for the Mathieu group \( M_{24} \) with its
natural action, where the minimal base size is 7. The Cameron–Kantor conjecture was proved by M. W. Liebeck and A. Shalev in [14]. However, this was an existence result for \( c \), using probabilistic methods without yielding any explicit value for this constant. Finally, T. C. Burness, M. W. Liebeck and A. Shalev [1] proved that if \( G \) is a finite almost simple group in a primitive faithful non-standard action, then \( b(G) \leq 7 \), with equality if and only if \( G \) is the Mathieu group \( M_{24} \) in its natural action of degree 24.

Furthermore, they proved that if \( G \) is a finite almost simple group and \( \Omega \) is a primitive faithful non-standard \( G \)-set, then the probability that a random 6-tuple in \( \Omega \) is a base for \( G \) tends to 1 as \( |G| \) tends to infinity.

For a finite vector space \( V \), a linear group \( G \leq GL(V) \) is called coprime if \( (|G|, |V|) = 1 \). D. Gluck and K. Magaard [6] proved that if \( G \) is such a group, then \( b(G) \leq 94 \). Z. Halasi and K. Podoski [7] improved this result by showing that \( b(G) \leq 2 \), and this estimation is sharp. (In particular, it follows that \( |G| \leq |V|^2 \) in this case. This fact will be used frequently in this paper.) Based on this result and the random base result of T. C. Burness, M. W. Liebeck and A. Shalev, L. Pyber [18] asked whether or not, for a coprime linear group \( G \leq GL(V) \), there exists an absolute constant \( c \) such that the probability that a random \( c \)-tuple in \( V \) is a base for \( G \) tends to 1 as \( |V| \to \infty \). While the answer to this question is negative in general (see Remark 1.3 (2)), it is positive under the additional assumption that \( G \) is primitive as a linear group. Recall that a linear group \( G \leq GL(V) \) is called primitive if there is no direct sum decomposition \( V = V_1 \oplus V_2 \oplus \cdots \oplus V_t \) of \( V \) (with \( t > 1 \)) which is preserved by \( G \). Note that if \( G \leq GL(V) \) is coprime and primitive, then \( G \) is irreducible on \( V \).

We prove the following.

**Theorem 1.1.** Let \( V \) be a finite vector space and \( G \leq GL(V) \) a coprime primitive linear group, i.e. \((|G|, |V|) = 1\). Then the probability that a random 11-tuple in \( V \) is a base for \( G \) tends to 1 as \( |V| \to \infty \).

In fact, we also give lower bounds for this probability in terms of the base field and the dimension of \( V \). Our bounds depend highly on the structure of \( G \). As a part of our argument, we give a general structure theorem for maximal coprime primitive linear groups in Theorem 5.1.

For any positive integer \( c \), let us define the probability

\[
Pb(c, G, V) := \frac{|\{(v_1, \ldots, v_c) \in V^c : \{v_1, \ldots, v_c\} \text{ is a base for } G\}|}{|V|^c}.
\]

Thus, \( Pb(c, G, V) \) is the probability that a random \( c \)-tuple \((v_1, \ldots, v_c)\) is a base for \( G \), where \( v_1, \ldots, v_c \) are chosen uniformly and independently from \( V \).

The main goal of this paper is to prove the following.
Theorem 1.2. Let $V$ be an $n$-dimensional vector space over the finite field $\mathbb{F}_q$, and let $G \leq GL(V)$ be a coprime primitive linear group. Then, for any $c \geq 11$, the probability $Pb(c, G, V)$ is close to zero if $|V|$ is large enough. More precisely, one of the following holds.

1. $Pb(c, G, V) \geq 1 - \frac{3}{q^{(\frac{n}{2}-5)/\sqrt{n}}}$.

2. There is an $\mathbb{F}_q^k$ vector space structure on $V$ for some field extension $\mathbb{F}_q^k$ with $1 \leq \dim(U) < \dim(V_1) \leq n/k$ such that $G \leq \Gamma L(\mathbb{F}_q^k, n/k)$ and $H = G \cap GL(\mathbb{F}_q^k, n/k)$ preserves this tensor product decomposition. Furthermore, $H = H_1 \otimes H_2$ with $H_1 \leq GL(V_1), H_2 \leq GL(U)$ are absolutely irreducible linear groups, and $S_1 = \text{Soc}(H_1 / Z(H_1))$ is a non-Abelian simple group.

   a. If $S_1$ is not an alternating group, then
      $$Pb(c, G, V) \geq 1 - \left( \frac{1}{q^{(c-4)/\sqrt{n}}} + \frac{2}{|V|(c-10)/80} \right).$$

   b. If $S_1 \simeq A_m$ for some $m$ and $V_1$ is not an irreducible component of the natural permutation $\mathbb{F}_q^k A_m$-module, then
      $$Pb(c, G, V) \geq 1 - \frac{3}{q^{c-10}/\sqrt{n}}.$$

   c. If $S_1 \simeq A_m$ for some $m$ and $V_1$ is the non-trivial irreducible component of the natural permutation $\mathbb{F}_q^k A_m$-module, then
      $$Pb(c, G, V) \geq 1 - \frac{3}{nc-2}.$$

Remark 1.3.

1. Let $Z = Z(GL(V)) \simeq \mathbb{F}_q^\times$ denote the group of scalar transformations on $V$. If $G \leq GL(V)$ is a coprime linear group on $V$, then so is $GZ \geq G$, and we have $Pb(c, G, V) \geq Pb(c, GZ, V)$. Therefore, for the rest of this paper, we will always assume that $G$ contains $Z$.

2. The assumption “primitive” is necessary here. To see this, let $H \leq GL(n, q)$ be the group of all invertible diagonal matrices, so we have $H \simeq (\mathbb{F}_q^\times)^n$. Then $v_1, \ldots, v_c \in \mathbb{F}_q^n$ is a base for $H$ if and only if, for each $1 \leq i \leq n$, the $i$-th component of some $v_j$ is non-zero. For any fixed $i$, this has probability $(1 - 1/q^c)$, so we have
   $$Pb(c, H, \mathbb{F}_q^n) = \left(1 - \frac{1}{q^c}\right)^n.$$
which is close to zero for any fixed $c$ and big enough $n$. If $(q,n) = 1$, then one can add the regular permutation action of $C_n$ on the components of $\mathbb{F}_q^n$ to get the coprime irreducible linear group $G = H \rtimes C_n \leq GL(n,q)$ satisfying $\lim_{n \to \infty} Pb(c, G, \mathbb{F}_q^n) = 0$.

This paper is organised as follows. In Section 2, we define the concepts of minimal support and maximal character ratio, and we show how they can be used to get bounds for $Pb(c, G, V)$. Then we give some bounds for the minimal support of normal subgroups and tensor products of linear groups in Section 3. In Section 4, we work with quasisimple linear groups. Here the main result is Theorem 4.8, which says that if $G \leq GL(V)$ is a quasisimple linear group over the $q$-element field, then $\log_q(G)$ is bounded by a linear function of the minimum support of $G$, with the exceptional case, when $\text{Soc}(G) = A_m$ is an alternating group and $V$ is the fully deleted permutation $\mathbb{F}_q A_m$-module. We note that sporadic groups are handled by using the GAP system [23], while the case when $G$ is a quasisimple group of Lie type, then we use known bounds on maximal character degrees for $G$ proven by Gluck [5]. So, most of Section 4 deals with the representations of the symmetric groups and their covers. After that, in Section 5, we give a structure theorem for maximal primitive coprime linear groups, very similar to a known description of maximal primitive solvable linear groups. In Section 6, we use this structure theorem and the bounds given in Section 3 to reduce the proof of Theorem 1.2 to the case when $G$ is close to being a quasisimple linear group, which is solved by using Theorem 4.8. The exceptional case, when $V$ is the fully deleted permutation $\mathbb{F}_q A_m$-module is handled separately in Theorem 6.1 and in Corollary 6.2.

2 Bounds on $Pb(c, G, V)$ in terms of supports and character ratios

In order to prove Theorem 1.2, our primary tool will be the concept of support for elements of a linear group.

**Definition 2.1.** For a linear group $G \leq GL(V)$ and a $g \in G$, the fixed-point space and the support of $g$ are defined as

$$\text{Fix}(g) := \{v \in V \mid g(v) = v\} \quad \text{and} \quad \text{Supp}(g) := \dim(V) - \dim(\text{Fix}_V(g)).$$

Furthermore, let the minimal support of $G$ be defined as

$$\text{MinSupp}(G) := \min_{1 \neq g \in G} \text{Supp}(g).$$

We use the notation $\text{Fix}_V(g)$, $\text{Supp}_V(g)$ and $\text{MinSupp}_V(G)$ if we also want to highlight the vector space on which the group acts.
Remark 2.2. If $G$ strictly contains $Z$, then $\text{MinSupp}(G)$ equals
\[
\min_{g \in G \setminus Z} \left( \dim(V) - \max_{\lambda \in \mathbb{F}_q^\times} (\dim(\ker(g - \lambda \cdot \text{id}_V))) \right).
\]

In order to get bounds for $\text{MinSupp}_V(G)$ in the case where $G \leq GL(V)$ is a quasisimple coprime linear group, we will use results from character ratios of complex irreducible characters of such groups.

Definition 2.3. For a finite group $G$ and $\chi \in \text{Irr}(G)$ with $\chi(1) \neq 1$, let us define the maximal character ratios
\[
\text{mr}(G, \chi) := \max_{g \notin Z(\chi)} \left| \chi(g) \right| \quad \text{and} \quad \text{mr}(G) := \max_{\chi \in \text{Irr}(G), \chi(1) \neq 1} \text{mr}(G, \chi).
\]

Clearly, $\text{mr}(G) < 1$ for every finite group $G$.

The connection between minimal support and maximal character ratio is described in the following lemma.

Lemma 2.4. Let $V$ be an $n$-dimensional vector space over the finite field $\mathbb{F}_q$, and let $G \leq GL(V)$ be a non-Abelian coprime irreducible linear group. Then we have
\[
\text{MinSupp}_V(G) \geq \frac{\dim(V)}{2}(1 - \text{mr}(G)).
\]

Moreover, if $\chi \in \text{Irr}(G)$ is any irreducible component of the Brauer character associated to $V$, then
\[
\text{MinSupp}_V(G) \geq \frac{1}{2} \left( \chi(1) - \max_{g \notin Z(\chi)} |\chi(g)| \right).
\]

Proof. Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$, and let $\overline{V} = V \otimes \overline{\mathbb{F}}_q$ be the $\overline{\mathbb{F}}_q G$-module arising from the $\mathbb{F}_q G$-module $V$. Let $\overline{V} = V_1 \oplus \cdots \oplus V_t$ be the decomposition of $\overline{V}$ into irreducible $\overline{\mathbb{F}}_q G$-modules. Then the corresponding representations $G \mapsto GL(V_i)$ form a single Galois conjugacy class by [9, Theorem 9.21], so $\text{Supp}_{V_i}(g) = \frac{1}{t} \text{Supp}_V(g)$ holds for every $g \in G$. Let $\chi_i: G \mapsto \mathbb{C}$ be the irreducible Brauer character associated to $V_i$ for each $1 \leq i \leq t$. Since $(q, |G|) = 1$, we get $\chi_i \in \text{Irr}(G)$ by [9, Theorem 15.13]. Furthermore,
\[
\chi_i(1) = \dim(V_i) \quad \text{and} \quad [\chi_i(g), 1_i(g)] = \dim(\text{Fix}_{V_i}(g)).
\]

For any $g \in G$, we have
\[
\chi_1(g) = (\chi_1(1) - k) \cdot 1 + \varepsilon_1 + \cdots + \varepsilon_k.
\]
where $k = \text{Supp}_{V_1}(g)$ and $\varepsilon_1, \ldots, \varepsilon_k$ are $\sigma(g)$-th root of unity. Then

$$|\chi_1(g)| \geq \chi_1(1) - 2k = \chi_1(1) - 2\text{Supp}_{V_1}(g)$$

holds, so

$$2\min\text{Supp}_{V_1}(G) \geq \chi_1(1) - \max_{g \not\in Z(\chi_1)} |\chi_1(g)|.$$

(Note that the assumption that $G$ is non-Abelian implies that the $\chi_i$ are non-linear characters. Furthermore, if $1 \neq g \in Z(\chi_1)$, then $\text{Supp}_V(g) = \dim(V)$, so $\min\text{Supp}_V(G) = \min_{g \not\in Z(\chi_1)} \text{Supp}_V(g)$ must hold.)

It follows that

$$2\min\text{Supp}_V(G) = 2t\min\text{Supp}_{V_1}(G) \geq t(\chi_1(1) - \max_{g \not\in Z(\chi_1)} |\chi_1(g)|)$$

$$= t\chi_1(1)(1 - \text{mr}(G, \chi_1)) \geq \dim(V)(1 - \text{mr}(G)).$$

Now, the first inequality proves the second claim, while the second inequality proves the first claim. $\square$

**Lemma 2.5.** Using the hypothesis of Lemma 2.4, we have

$$P_b(c, G, V) \geq 1 - \sum_{1 \neq g \in G} \frac{1}{q^{c-\text{Supp}(g)}}$$

$$\geq 1 - \frac{\left|G\right|}{q^{c-\text{MinSupp}(G)}} \geq 1 - \frac{1}{|V|^{c(1 - \text{mr}(G))/2 - 2}}.$$

In particular, $P_b(c, G, V) \geq 1 - \frac{1}{|V|^c}$ for $c \geq \frac{4 + 2\varepsilon}{1 - \text{mr}(G)}$.

**Proof.**

$$P((v_1, \ldots, v_c) \in V^c \text{is not a base for } G)$$

$$\leq \sum_{1 \neq g \in G} P(g(v_i) = v_i \text{ for all } 1 \leq i \leq c)$$

$$= \sum_{1 \neq g \in G} \left(\frac{|\text{Fix}(g)|}{|V|}\right)^c = \sum_{1 \neq g \in G} \frac{1}{q^{c-\text{Supp}(g)}}$$

$$\leq \frac{|G|}{q^{c-\text{MinSupp}(G)}} \leq \frac{|V|^2}{(q^n)^{c(1 - \text{mr}(G))/2}} = \frac{1}{|V|^{c(1 - \text{mr}(G))/2 - 2}},$$

and the claim follows. $\square$
3 Some bounds on minimal supports

In this small section, we give some bounds on minimal supports of normal sub-

groups and tensor products of linear groups. These bounds will be used in the final

proof of Theorem 1.2.

Lemma 3.1. Let $G$ be a group, $K$ a field and $V$ an arbitrary finite-dimensional

$KG$-module.

(1) For any $g, h \in G$, we have $\text{Supp}([g, h]) \leq 2 \text{Supp}(g)$.

(2) If $N \vartriangleleft G$ such that $V$ is absolutely irreducible as a $KN$-module, then we have

$\text{MinSupp}(N) \leq 2 \text{MinSupp}(G)$.

Proof. Let us consider the subspaces $U = \text{Fix}(g)$ and $W = \text{Fix}(h^{-1}gh)$ of $V$.

Then we have

$$\dim(U) + \dim(W) - \dim(U \cap W) = \dim(U + W) \leq \dim(V).$$

Using that $\dim(U) = \dim(W) = \dim(V) - \text{Supp}(g)$, we get

$$\dim(U \cap W) \geq \dim(V) - 2 \text{Supp}(g).$$

On the other hand, $U \cap W \leq \text{Fix}([g, h])$ holds trivially, so

$$\text{Supp}([g, h]) = \dim(V) - \dim(\text{Fix}([g, h])) \leq \dim(V) - \dim(U \cap W)$$

$$\leq \dim(V) - (\dim(V) - 2 \text{Supp}(g)) = 2 \text{Supp}(g),$$

and part (1) follows.

For part (2), let $1 \neq g \in G$ be any element. If $[g, N] = 1$, then $g$ acts as a scalar

transformation on $V$ by [9, Theorem 9.2], so $\text{Supp}(g) = \dim(V) \geq \text{MinSupp}(N)$.

Otherwise, there is an element $n \in N$ such that $[g, n] \neq 1$. Then we have

$$\text{MinSupp}(N) \leq \text{Supp}([g, n]) \leq 2 \text{Supp}(g).$$

Thus,

$$\text{MinSupp}(N) \leq 2 \text{Supp}(g) \quad \text{for every } 1 \neq g \in G,$$

which proves that $\text{MinSupp}(N) \leq 2 \text{MinSupp}(G)$. \qed

Lemma 3.2. Let $V_1, \ldots, V_k$ be finite-dimensional vector spaces over the field $\mathbb{F}_q$, and let

$$Z < G_1 \leq GL(V_1), \ldots, Z < G_k \leq GL(V_k)$$

be coprime linear groups. Consider the group $G := G_1 \otimes \cdots \otimes G_k$ acting on the
tensor product $V := V_1 \otimes \cdots \otimes V_k$ in a natural way.
(1) Let \( g = g_1 \otimes \cdots \otimes g_k \in G \) with \( g_j \in G_j \) for each \( j \), and let us assume that \( g_i \notin Z \) for some \( i \). Then either
\[
\text{Supp}_V(g) \geq \text{MinSupp}_{V_i}(G_i) \cdot \frac{\dim(V)}{\dim(V_i)}
\]
or
\[
\text{Supp}_V(g) \geq \frac{1}{2} \dim(V).
\]

(2) As a consequence
\[
\text{MinSupp}_V(G) = \min_i \left\{ \text{MinSupp}_{V_i}(G_i) \cdot \frac{\dim(V)}{\dim(V_i)} \right\},
\]
or
\[
\text{MinSupp}_V(G) \geq \frac{1}{2} \dim(V).
\]

Proof. To prove part (1), first we consider the case \( k = 2 \). Let \( n_1 = \dim(V_1) \), \( n_2 = \dim(V_2) \), so \( n = \dim(V) = n_1n_2 \). Furthermore, let
\[ 1 \neq g = g_1 \otimes g_2 \in G_1 \otimes G_2 \]
be an element of \( G \) with \( g_1 \notin Z \). Since the action is coprime, \( g_1 \) and \( g_2 \) are diagonalisable over \( \mathbb{F}_q \). Let \( \alpha_1, \ldots, \alpha_s \in \mathbb{F}_q \) be the different eigenvalues of \( g_1 \) with multiplicity \( k_1, k_2, \ldots, k_s \). We can assume that \( k_1 \) is the largest among the \( k_i \). Let \( l_1, \ldots, l_s \) be the multiplicities of \( \alpha_1^{-1}, \ldots, \alpha_s^{-1} \) in the characteristic polynomial of \( g_2 \) (some of which may be zero). Then
\[
\text{Supp}_V(g) = \text{Supp}_V(g_1 \otimes g_2) = n - \dim(\text{Fix}_V(g_1 \otimes g_2))
\]
\[ = n - \sum_{i=1}^{s} k_i l_i \geq n - \sum_{i=1}^{s} k_1 l_i \geq (n_1 - k_1)n_2. \]

If \( \alpha_1 \in \mathbb{F}_q \), then we can substitute \( g_1 \) by \( \alpha_1^{-1}g_1 \) and \( g_2 \) by \( \alpha_1g_2 \) (since both \( G_1 \) and \( G_2 \) contains all the scalar transformations), so we can assume that \( \alpha_1 = 1 \). Now, since \( g_1 \neq 1 \), we get
\[
\text{Supp}_V(g) \geq (n_1 - k_1)n_2 = \text{Supp}_{V_1}(g_1)n_2 \geq \text{MinSupp}_{V_1}(G_1) \cdot \dim(V_2).
\]

Now, let us assume that \( \alpha_1 \notin \mathbb{F}_q \). Then there is an algebraic conjugate element of \( \alpha_1 \) (different from \( \alpha_1 \)) under the action of \( \text{Gal}(\mathbb{F}_q, \mathbb{F}_q) \) which is also an eigenvalue of \( g_1 \) with the same multiplicity as \( \alpha_1 \). In particular, \( k_1 \leq n_1/2 \). Thus,
\[
\text{Supp}_V(g) \geq (n_1 - k_1)n_2 \geq (n_1/2)n_2 = \frac{\dim(V)}{2}.
\]

By interchanging the role of \( g_1 \) and \( g_2 \) in the proof and by using induction on \( k \), we get the claim of part (1).
Finally, if \( \text{Supp}_{V_i}(g_i) = \text{MinSupp}_{V_i}(G_i) \) for some \( g_i \notin Z \), then
\[
\text{Supp}_V(1 \otimes \cdots \otimes 1 \otimes g_i \otimes 1 \otimes \cdots \otimes 1) = \text{MinSupp}_{V_i}(G_i) \cdot \frac{\dim(V)}{\dim(V_i)},
\]
so part (2) follows by part (1).

\[\square\]

4 Bounds for character ratios and for minimal supports of quasisimple linear groups

The goal of this section is to give lower bounds for minimal supports of coprime quasisimple groups \( G \leq GL(V) \) in terms of \( |G| \) and \( \dim(V) \).

First we handle the case when \( G \) is a sporadic group or a finite quasisimple group of Lie type. For such groups, we use bounds for their maximal character ratios \( \text{mr}(G) \).

**Theorem 4.1.** Let \( G \) be a finite quasisimple group such that \( G/Z(G) \) is not an alternating group.

1. If \( G/Z(G) \) is a sporadic simple group, then \( \text{mr}(G) < 0.54 \).
2. If \( G = G(r) \) is a finite quasisimple group of Lie type over the field \( \mathbb{F}_r \), then
\[
\text{mr}(G) \leq \begin{cases} 
\max \left( \frac{1}{\sqrt{r-1}}, \frac{9}{r} \right) & \text{if } r > 9, \\
\frac{19}{20} & \text{if } r \leq 9.
\end{cases}
\]

**Proof.** We checked part (1) for the covering groups of the sporadic simple groups by using the GAP [23] character table library and also the undeveloped GAP package FUtil to turn cyclotomic complex numbers into floating ones in order to be able to compare the values of \( |\chi(g)| \) for various \( g \) and \( \chi \). We found that if \( G \) is the universal covering group of a sporadic simple group, \( \chi \in \text{Irr}(G), \chi \neq 1_G \) and \( g \in G \) is a non-central element of \( G \), then \( |\chi(g)|/\chi(1) \leq 0.539 \ldots \), where equality is attained for two faithful characters of \( 2.J_2 \).

Regarding part (2), this is a simplified version of a result of Gluck [5]. (For a summary of his results, see also [13, Theorem 2.4].)

**Remark 4.2.** For simple groups of alternating type, there is no general upper bound for \( \text{mr}(G) \) smaller than 1. Moreover, it can be shown that, for every \( \varepsilon > 0 \), the number of irreducible characters \( \chi \in \text{Irr}(S_m) \) (or \( \chi \in \text{Irr}(A_m) \)) that satisfy \( \text{mr}(S_m, \chi) > 1 - \varepsilon \) is not bounded if \( m \) is large enough.
Corollary 4.3. Let $V$ be a vector space over the finite field $\mathbb{F}_q$, and let

$$G = Z \cdot G_0 \leq GL(V),$$

where $G_0$ is a coprime quasisimple irreducible linear group which is not of alternating type. Then $\text{MinSupp}_V(G) \geq \frac{1}{40} \dim(V)$.

Proof. By Theorem 4.1, we have $\text{mr}(G) = \text{mr}(G_0) \leq \frac{19}{20}$, so the claim follows from Lemma 2.4. \qed

Now, we handle the case when $\text{Soc}(G/Z(G))$ is an alternating group.

Theorem 4.4. Let $G = S_m$ and $\chi = \chi^{(\lambda)} \in \text{Irr}(G)$ corresponding to the partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ of $m$. Then $\chi^\lambda(1) - \chi^\lambda((123)) \geq \frac{1}{m-1} \chi^\lambda(1)$ unless $\lambda \in \{(m); (1, \ldots, 1)\}$.

Proof. First, we introduce some notation. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ be a partition of $m$, different from the two exceptional ones given in the theorem. For any natural numbers $i_1, \ldots, i_k$, let $\chi^{\lambda - \{i_1, \ldots, i_k\}}$ be the character of $S_{m-k}$ corresponding to the Young diagram obtained from the diagram of $\lambda$ by deleting the last cells of the $i_1$-th, $\ldots$, $i_k$-th row in that order with the assumption that $\lambda - \{i_1, \ldots, i_s\}$ is a valid Young diagram for each $1 \leq s \leq k$. Otherwise, we define $\chi^{\lambda - \{i_1, \ldots, i_k\}}$ as the constant zero function on $S_{m-k}$.

By the Murnaghan–Nakayama rule (see [10, Theorem 21.1]),

$$\chi^\lambda((123)) \leq \sum_{\nu \in \{\lambda - rh(3)\}} \chi^\nu(1) + \sum_{\nu \in \{\lambda - rh(1,1,1)\}} \chi^\nu(1)$$

$$= \sum_{\nu \in \{\lambda - rh(3)\}} \chi^\nu(1) + \sum_{\nu \in \{\lambda - rh(3)\}} \chi^\nu(1),$$

where $\{\lambda - rh(*)\}$ denotes the set of partitions of $m - 3$ which we can get from the Young diagram of $\lambda$ by removing a rim 3-hook of type (*) such that the remaining cells form a valid Young diagram.

On the other hand, by using the branching rule (three times), one gets

$$\chi^\lambda(1) = \sum_{i,j,k} \chi^{\lambda - \{i,j,k\}}(1).$$

Let $\nu \in \{\lambda - rh(3)\}$. Then $\nu = \lambda - \{i,i,i\}$ for some (unique) $i$. Now, there is a $j \neq i$ such that $\tau = \lambda - \{i,i,j\}$ is a valid Young diagram. Then both induced characters $(\chi^\tau)^{S_{m-2}}$ and $(\chi^\nu)^{S_{m-2}}$ contain $\chi^{\lambda - \{i,i\}}$ as a component which results
\[ \chi^\nu(1) \leq \chi^{\lambda-(i,i)}(1) \leq (m-2)\chi^\tau(1). \]
The same argument can be applied to any \( \nu \in \{\lambda - rh(3)\} \). It follows that
\[
\chi^\lambda(1) \geq \sum_i \chi^{\lambda-(i,i,i)}(1) \left( 1 + \frac{1}{m-2} \right) + \sum_i \chi^{\lambda-(i,i,i)}(1) \left( 1 + \frac{1}{m-2} \right) 
\]
\[
\geq \frac{m-1}{m-2} \chi^\lambda((123)).
\]
Hence, \( \chi^\lambda(1) - \chi^\lambda((123)) \geq \frac{1}{m-1} \chi^\lambda(1) \), which proves the claim. \( \square \)

This result will be adequate for our purposes only if the degree of \( \chi \) is large enough. In order to get an overall picture about the form of Young diagrams defining characters of small degree, we will use a result of Rasala [19]. In what follows, we use the terminology from Rasala’s paper. For any partition \( \lambda \) of \( m \), let \( \lambda^* \) be the partition of \( m \) dual to \( \lambda \). The partition \( \lambda \) is called primary, if \( \lambda \geq \lambda^* \), where \( \geq \) denotes the standard ordering on partitions. If \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_s) \) is a partition of \( k \) and \( m \geq \lambda_k + k \), then let \( m/\lambda \) denote the partition of \( m \) defined as \( m/\lambda = (m-k \geq \lambda_1 \geq \cdots \geq \lambda_s) \), and let \( \varphi_\lambda(m) := \chi^{m/\lambda}(1) \) be the degree of the character of \( S_m \) associated to \( m/\lambda \). (Note that \( \varphi_\lambda(m) \) is a polynomial in \( m \) by [19, Theorem A].) For any set \( P \) of partitions of \( k \) and for \( m \) large enough, let \( L(P,m) := \{ \varphi_\lambda(m) \mid \lambda \in P \} \), and let \( \delta(P,m) \) be the largest degree in \( L(P,m) \). Then \( P \) is said to be \( m \)-minimal if, for every primary partition \( \mu \) of \( m \), either \( \chi^\mu(1) > \delta(L,P) \) or \( \mu = m/\lambda \) for some \( \lambda \in P \).

By [19, Main Theorem 1] (for \( k = 3 \)), we have the following.

**Theorem 4.5.** Let \( P_3 \) be the set of all partitions of order at most 3, that is,
\[
P_3 = \{ \emptyset; (1); (2); (1,1); (3); (2,1); (1,1,1) \}.
\]
Then \( P_3 \) is \( m \)-minimal for every \( m \geq 15 \).

Thus, by using the hook length formula and the Murnaghan–Nakayama rule, we can calculate the exact values of \( \chi^\lambda(1) \) and \( \chi^\lambda((123)) \) when \( \chi^\lambda(1) \) is among the first seven smallest character degrees of \( S_m \) for \( m \geq 15 \). Otherwise, we get a reasonably large lower bound for \( \chi^\lambda(1) \). (Note that \( \lambda \) or \( \lambda^* \) is primary and \( \chi^\lambda(1) = \chi^{\lambda^*}(1), \chi^\lambda((123)) = \chi^{\lambda^*}((123)) \) holds for every partition \( \lambda \) of \( m \).)

**Corollary 4.6.** Let \( \lambda \) be a partition of \( m \) for \( m \geq 15 \), and let \( \chi^\lambda \in \text{Irr}(S_m) \) be the character of \( S_m \) associated to \( \lambda \). Then \( \chi^\lambda(1) \) and \( \chi^\lambda((123)) \) are as given in Table 1 or \( \chi^\lambda(1) > \frac{1}{3} m(m-2)(m-4) \).

Now, we give an analogue of Corollary 4.3 for alternating-type groups.
Table 1. Character values of $S_m$ when the degree is small.

| $\lambda$ or $\lambda^*$ | $\chi^\lambda(1) = \chi^{\lambda^*}(1)$ | $\chi^\lambda((123)) = \chi^{\lambda^*}((123))$ |
|--------------------------|----------------------------------------|----------------------------------------|
| (m)                      | 1                                      | 1                                      |
| (m − 1, 1)               | $m − 1$                                | $m − 4$                                |
| (m − 2, 2)               | $\frac{1}{2} m(m − 3)$                | $\frac{1}{2} (m − 3)(m − 6)$           |
| (m − 2, 1, 1)            | $\frac{1}{2} (m − 1)(m − 2)$          | $\frac{1}{2} (m − 4)(m − 5)$           |
| (m − 3, 3)               | $\frac{1}{6} m(m − 1)(m − 5)$         | $\frac{1}{6} (m − 3)(m − 4)(m − 8) + 1$ |
| (m − 3, 2, 1)            | $\frac{1}{3} m(m − 2)(m − 4)$         | $\frac{1}{3} (m − 3)(m − 5)(m − 7) − 1$ |
| (m − 3, 1, 1, 1)         | $\frac{1}{6} (m − 1)(m − 2)(m − 3)$   | $\frac{1}{6} (m − 4)(m − 5)(m − 6) + 1$ |

Corollary 4.7. Let $V$ be a vector space over the finite field $\mathbb{F}_q$, and let

$$G = Z \cdot G_0 \leq GL(V),$$

where $G_0$ is a coprime quasisimple irreducible linear group and $G_0/Z(G_0) \simeq A_m$ for some $m \geq 5$. Let us assume that $V$ is not a component of the natural permutation $\mathbb{F}_q A_m$-module. Then $\text{MinSupp}_V(G) \geq \frac{1}{16} \sqrt{\dim(V)}$.

Proof. As in the proof of Lemma 2.4, $\text{MinSupp}_V(G) = t \cdot \text{MinSupp}_{V_1}(G)$ and $\dim(V) = t \cdot \dim(V_1)$, where $V_1$ is an (absolutely) irreducible component of $\mathbb{F}_q G$-module $V \otimes \mathbb{F}_q$. Then the claim clearly follows if we prove that

$$\text{MinSupp}_{V_1}(G) \geq \frac{1}{16} \sqrt{\dim(V_1)}.$$

In other words, we can assume that $V$ is absolutely irreducible. First let us assume that $G_0 \simeq A_m$ for some $m \geq 9$. Let $\varphi \in \text{Irr}(A_m)$ be the Brauer character associated to $V$ and $\chi \in \text{Irr}(S_m)$ above $\varphi$, i.e. $[\chi_{A_m}, \varphi] \neq 0$. Then either $\chi_{A_m} = \varphi$ (if $\chi$ is not self-dual) or $\chi_{A_m} = \varphi + \varphi^{(12)}$ (if $\chi$ is self-dual). In the latter case, $\varphi((123)) = \chi((123))/2$ since the conjugacy class $(123)^{S_m}$ does not split in $A_m$. Let $\epsilon$ be 1 or $\frac{1}{2}$ according to these cases, so $\varphi(1) = \epsilon \chi(1)$, $\varphi((123)) = \epsilon \chi((123))$. By [19, Result 2], we have $\dim(V) = \epsilon \chi(1) \geq \frac{1}{2} m(m − 3)$. If $\varphi((123)) < 0$, then $\text{Supp}_V((123)) \geq \frac{1}{4} \sqrt{\dim(V)}$ holds trivially. Otherwise, using Lemma 2.4 and Theorem 4.4, we get

$$\begin{align*}
\text{Supp}_V((123)) & \geq \frac{1}{2} (\varphi(1) − |\varphi((123))|) = \frac{\epsilon}{2} (\chi(1) − \chi((123))) \geq \frac{\epsilon \chi(1)}{2(m − 1)} \\
& = \frac{\dim(V)}{2(m − 1)} \geq \frac{\sqrt{m(m − 3)/2} \sqrt{\dim(V)}}{2(m − 1)} \geq \frac{1}{4} \sqrt{\dim(V)}.
\end{align*}$$
Let \( g \in G \) be such that \( g \not\in Z \), so \( g = a g_0 \) for some \( a \in \mathbb{Z} \) and \( 1 \neq g_0 \in A_m \). Then there are \( x, y \in A_m \) such that \([g, x, y] = [g_0, x, y]\) is a three-cycle. Applying Lemma 3.1 twice, we get \( \text{Supp}_V(g) \geq \frac{1}{4} \text{Supp}_V((123)) \geq \frac{1}{16} \sqrt{\dim(V)} \).

Now, let us assume that \( m > 7 \) and \( G_0 \) is the universal covering group of \( A_m \), so \( G_0 \cong 2.A_m \). Let \( z \in G_0 \) be the generator of \( Z(G_0) \cong C_2 \), and let \( \tilde{g} \in A_m \) denote the image of any \( g \in G_0 \) under the natural surjection by \( G_0 \twoheadrightarrow A_m \). Then \( z \) acts on \( V \) as a scalar transformation \( z(v) = -v \) for all \( v \in V \), so \( \text{Supp}_V(z) = \dim(V) \).

Let \( t \in G_0 \) such that \( \tilde{t} = (12)(34) \). By Theorem [8, Theorem 3.9], \( t \) and \( tz \) are conjugate, so \( z = [h, t] \) for some \( h \in G_0 \). It follows by Lemma 3.1 that

\[
\text{Supp}_V(t) \geq \frac{1}{2} \text{Supp}_V(z) = \frac{\dim(V)}{2}.
\]

(In fact, applying this argument to \( tz \) instead of \( t \), one can prove equality here.)

Again, let \( g \in G \setminus Z \) of the form \( g = a g_0 \), where \( a \in \mathbb{Z} \) and \( g_0 \in G_0 \setminus Z \). Then one can choose \( x, y \in G \) such that \([\tilde{g}_0, \tilde{x}, \tilde{y}]\) is conjugate to \( \tilde{t} \), which results that \([g, x, y]\) is conjugate to \( t \). Using again Lemma 3.1 twice, we get

\[
\text{Supp}_V(g) \geq \frac{1}{4} \text{Supp}_V(t) = \frac{1}{8} \dim(V) \geq \frac{1}{16} \sqrt{\dim(V)}.
\]

For the remaining cases,

\[
\dim(V) \leq \sqrt{|G_0|} < 16^2,
\]

so \( \frac{1}{16} \sqrt{\dim(V)} < 1 \leq \text{MinSupp}_V(G) \) follows.

The next result gives a bound to the order of most coprime quasisimple linear groups similar to that of \( |G| \leq |V|^2 = q^{2 \dim(V)} \) but using the minimal support \( \text{MinSupp}_V(G) \) instead of \( \dim(V) \).

**Theorem 4.8.** Let \( V \) be a vector space over the finite field \( \mathbb{F}_q \), and let

\[
G = Z \cdot G_0 \leq GL(V),
\]

where \( G_0 \) is a coprime quasisimple irreducible linear group.

Then one of the following holds.

1. \( \log_q |G| \leq d \cdot \text{MinSupp}_V(G) \) with \( d = 5 \).
2. \( G_0 \cong A_m \), and \( V \) is the non-trivial irreducible component of the natural permutation module of \( A_m \) over \( \mathbb{F}_q \).
3. \( G_0 = G_0(r) \) is a finite quasisimple group of Lie type over the finite field \( \mathbb{F}_r \), with \( r \leq 43 \), and \( |V| \) is bounded by an absolute constant.
Proof. For any sporadic group $S$, let $\hat{S}$ be its universal covering group, and let $q(S)$ be the smallest prime not dividing the order of $S$. By using GAP [23], we checked that, for every $\chi \in \text{Irr}(\hat{S})$, the inequality
\[
\log_{q(S)}|\hat{S}| < d \cdot (\chi(1) - \max_{g \in \hat{S} - Z(\chi)} |\chi(g)|)/2
\]
holds with $d > 4.22$. (The largest value is attained for $2.J_2$.) Now, if $G \leq GL(V)$ is any finite quasisimple group with sporadic simple quotient $S = G/Z(G)$, then $G$ is a homomorphic image of $\hat{S}$, and we can view $V$ as an irreducible $\mathbb{F}_q\hat{S}$-module (where $q \geq q(S)$). Now, if $\chi \in \text{Irr}(\hat{S})$ is any irreducible component of the Brauer character corresponding to $V \otimes \mathbb{F}_q$, then
\[
\log_q|G| \leq \log_{q(S)}|\hat{S}| < d \cdot (\chi(1) - \max_{g \in \hat{S} - Z(\chi)} |\chi(g)|)/2
\]
\[
\leq d \cdot \text{MinSupp}_V(\hat{S}) \leq d \cdot \text{MinSupp}_V(G)
\]
also holds with $d > 4.22$ by Lemma 2.4.

Next, let $G_0 \simeq A_m$ for some $m \geq 15$. Then we have $m < q$ by the coprime assumption. Let us assume that $V$ is not a component of the natural permutation $\mathbb{F}_qA_m$-module. Let $\varphi \in \text{Irr}(A_m)$ be an irreducible component of the Brauer character associated to $V$ and $\chi \in \text{Irr}(S_m)$ above $\varphi$. By the proof of Corollary 4.7, we have $\varphi(1) = \varepsilon\chi(1)$, and $\varphi((123)) = \varepsilon\chi((123))$, where $\varepsilon$ is $\frac{1}{2}$ or $1$ if $\chi$ is self-dual or not.

If $\chi$ is one of the characters given in Table 1, then $\chi$ is not self-dual. In that case, we have
\[
\text{Supp}_V((123)) \geq \frac{1}{2}(\chi(1) - |\chi(123)|) \geq \frac{3}{2}(m - 3)
\]
by using Lemma 2.4 and the last five rows of Table 1. Otherwise,
\[
\chi(1) > \frac{1}{3}m(m - 2)(m - 4),
\]
so
\[
\text{Supp}_V((123)) \geq \frac{1}{2}(\varphi(1) - |\varphi(123)|) \geq \frac{1}{4}(\chi(1) - |\chi(123)|)
\]
\[
\geq \frac{\chi(1)}{4(m - 1)} > \frac{m(m - 2)(m - 4)}{12(m - 1)} \geq m - 3
\]
holds if $\varphi(123) \geq 0$. However, if $\varphi(123) < 0$, then
\[
\text{Supp}_V((123)) \geq \frac{1}{2}\dim(V) \geq m - 3
\]
Random bases for coprime linear groups

holds trivially. Thus, \( \text{Supp}_V((123)) \geq m - 3 \) holds in any case. Now, for any element \( g \in G \setminus Z \), there are \( x, y \in A_m \) such that \([g, x, y]\) is a three-cycle. Applying Lemma 3.1 twice, we get that \( \text{Supp}_V(g) \geq \frac{1}{4} \text{Supp}_V((123)) \geq \frac{m-3}{4} \) holds for any \( 1 \neq g \in A_m \). Thus,

\[
d \cdot \text{MinSupp}_V(G) \geq \frac{d(m-3)}{4} \geq m \geq \log_m(m!) \geq \log_q |G|
\]

holds for \( d \geq 5 \).

Now, let \( m \geq 12 \), and let \( G_0 \) be the universal covering group of \( A_m \), so we have \( G_0 \cong 2.A_m \). By the proof of Corollary 4.7, we have \( \text{MinSupp}_V(G) \geq \frac{1}{8} \dim(V) \). Using [12, Main Theorem] we get that

\[
d \cdot \text{MinSupp}_V(G) \geq \frac{d}{8} \dim(V) \geq \frac{d}{8} \min\{\chi(1) \mid \chi \in \text{Irr}(G), \chi(z) \neq \chi(1)\}
\]

\[
\geq d \cdot 2^{[m/2]-4} \geq m \geq \log_q |G|
\]

holds for \( d \geq 3.25 \). For the remaining alternating groups and their covers (i.e. for \( A_m, 12 \leq m \leq 14 \)), for \( 2.A_m (m = 5 \text{ or } 8 \leq m \leq 11) \) and for \( 6.A_6, 6.A_7 \), we use the same algorithm as for sporadic groups.

Finally, let \( G_0 = G_0(r) \) be a quasisimple group of Lie type over a finite field \( \mathbb{F}_r \) with \((r, q) = 1 \). First, suppose that \( r \geq 47 \). By part (2) of Theorem 4.1, we have

\[
\text{mr}(G) \leq \max\left(\frac{1}{\sqrt{r} - 1}, \frac{9}{r}\right) < \frac{1}{5}.
\]

By [16, Theorem 1] and Lemma 2.4,

\[
\log_q |G| \leq 2n = 5 \cdot \frac{2n}{5} \leq 5 \cdot \text{MinSupp}_V(G).
\]

For the rest of the proof, suppose that \( r \leq 43 \). Since \( \chi(1) - 2 \text{Supp}(g) \leq |\chi(g)| \) for any \( \chi \in \text{Irr}(G) \), we have

\[
\frac{1}{2} \chi(1) \left(1 - \frac{|\chi(g)|}{\chi(1)}\right) \leq \text{Supp}(g).
\]

Using that \( \text{mr}(G) \leq \frac{19}{20} \) also holds for all quasisimple groups of Lie type by part (2) of Theorem 4.1, we obtain

\[
\frac{\dim(V)}{8} \leq 5 \text{MinSupp}_V(G)
\]

using Lemma 2.4 again. Since (see [11, Table 5.3.A]) \( \dim(V) \geq r^{\Theta(m)} \) (where \( m \) denotes the rank of \( G_0(r) \)) and \( \log_q |G| = \Theta(m^2 \log r) \), there exist only finitely many possible pairs \((m, r)\) such that \( \log_q |G| > 5 \text{MinSupp}_V(G) \). Furthermore, for any fixed \((m, r)\), the inequality \( \log_q |G| \leq 5 \text{MinSupp}_V(G) \) still holds provided that \(|V|\) is large enough. \( \square \)
5 A structure theorem for maximal coprime primitive linear groups

Throughout this section, let $V$ be an $n$-dimensional vector space over the finite field $\mathbb{F}_q$, and let $G \leq GL(V) = GL(n, q)$ be a coprime primitive linear group, which is maximal, i.e. there is no coprime subgroup $L \leq GL(V)$ strictly containing $G$. In the following, we give a structure theorem for such groups, very similar to a result about maximal solvable primitive linear groups (see [20, Lemma 2.2] and [22, §§ 19–20]). Our proof uses ideas similar to those found in [4, 7, 22]. For the convenience of the reader, we give a self-contained proof here.

In the following, we extend the vector space structure on $V$ by defining multiplication on $V$ with elements from a (possibly) larger field $\mathbb{F}_q^k$ for some $k \mid n$. In that way, $V$ will be both an $\mathbb{F}_q$-vector space and an $\mathbb{F}_q^k$-vector space at the same time.

We will use the notation $V = V_n(q)$, $V = V_d(q^k)$ or $V = V(q^k)$ if we like to highlight the base field and/or the dimension of $V$.

**Theorem 5.1.** Let $V = V_n(q)$ be an $n$-dimensional vector space over the finite field $\mathbb{F}_q$, and let $G \leq GL(V)$ be a maximal coprime primitive linear group. Then the following statements hold.

1. There is a unique maximal Abelian subgroup $Z \leq GL(V)$, which is normalised by $G$. Moreover, $Z$ is contained in $G$.
2. $Z$ is cyclic, and $Z \cup \{0\} \simeq \mathbb{F}_q^k$ for some $k \mid n$.
3. There is a (unique and maximal) $\mathbb{F}_q^k$ vector space structure $V = V_d(q^k)$ on $V$ for $d = n/k$ such that $G \leq GL(d, q^k)$.
4. Let $H := G \cap GL(d, q^k)$. Then we have $Z \leq H = C_G(Z) \triangleleft G$. Furthermore, $Z = Z(GL(d, q^k))$ is the group of scalar transformations on $V_d(q^k)$, and $G/H$ is included into the Galois group $Gal(\mathbb{F}_q^k, \mathbb{F}_q)$.
5. Let $N = F^*(H)$ be the generalised Fitting subgroup of $H$. Then $N/Z$ is the socle of $H/Z$. Furthermore, $V_d(q^k)$ is an absolutely irreducible $\mathbb{F}_q^k N$-module.
6. Let $N_1, \ldots, N_t$ be the set of minimal normal subgroups of $H$ above $Z$. Then there is an absolutely irreducible $\mathbb{F}_q^k N_i$-module $V_i$ for every $i$ such that $V \simeq V_1 \otimes_{\mathbb{F}_q^k} \cdots \otimes_{\mathbb{F}_q^k} V_t$. Furthermore,

$$N = N_1 \otimes N_2 \otimes \cdots \otimes N_t \quad \text{and} \quad H = H_1 \otimes H_2 \otimes \cdots \otimes H_t,$$

where $N_i \triangleleft H_i \leq GL(V_i(q^k))$ for every $i$. 

(7) If $N_i/Z$ is Abelian, then $N_i = ZR_i$, where $R_i \leq N_i$ is an extraspecial $r_i$-group for some prime $r_i$ of order $r_i^{2l_i} + 1$. Furthermore, $|N_i/Z| = r_i^{2l_i}$ and $\dim_{\mathbb{F}_q}^k(V_i) = r_i^{l_i}$.

(8) If $N_i/Z$ is a direct product of $s$ many isomorphic non-Abelian simple groups, then there is a tensor product decomposition $V_i = W_1 \otimes \cdots \otimes W_s$ preserved by $N_i$. Then we have $N_i = K_1 \otimes \cdots \otimes K_s$, where $K_i = S_i Z$ for each $i$, and the $S_i \leq GL(W_i)$ are isomorphic quasisimple absolutely irreducible groups.

Finally, $H_i$ permutes the $K_i$ and the $W_i$ transitively.

Proof. Let $A \leq GL(V)$ be any Abelian subgroup normalised by $G$ and $P$ is the (unique) Sylow-$p$ subgroup of $A$ for $p = \text{char}(\mathbb{F}_q)$. Then $P$ is normalised by $G$. Then $0 \neq \text{Fix}_V(P) = \bigcap_{p \in P} \text{Fix}_V(p) \leq V$ is $G$-invariant. Since $V$ is an irreducible $\mathbb{F}_q G$-module, we get $P = 1$, so $|A|$ is coprime to $|V|$. Therefore, $GA \geq G$ is a coprime linear group, so $A \leq G$ by the maximality of $G$, and part of (1) is proved.

Let $Z \leq GL(V)$ be a maximal Abelian subgroup normalised by $G$. By the previous paragraph, $Z \triangleleft G$. Since $G \leq GL(V)$ is primitive linear, $V$ is a homogeneous $\mathbb{F}_q Z$-module. If $V = V_1 \oplus \cdots \oplus V_d$ is a decomposition of $V$ into (isomorphic) irreducible $\mathbb{F}_q Z$-modules, then $Z \simeq ZV_j \leq \text{End}_{\mathbb{F}_q} Z(V_j) \simeq \mathbb{F}_q^k$ for some $k \geq 1$ by Schur’s lemma. Then $(Z)_{\mathbb{F}_q^*}$ (the subalgebra of $\text{End}(V)$ generated by $Z$) is isomorphic to the field $\mathbb{F}_q^k$, and it is invariant under conjugation by elements of $G$. It follows that $(Z)_{\mathbb{F}_q^*} \setminus \{0\} \simeq \mathbb{F}_q^*$ is an Abelian subgroup of $GL(V)$ normalised by $G$. Therefore, (2) follows by the maximality of $Z$.

Identifying $Z \cup 0 \leq \text{End}(V)$ with $\mathbb{F}_q^k$ defines an $\mathbb{F}_q^k$ vector space structure on $V$. The conjugation action of $G$ on $Z \cup \{0\} = \mathbb{F}_q^k$ defines a homomorphism $\sigma : G \mapsto \text{Gal}(\mathbb{F}_q, \mathbb{F}_q)$. Now, for any $g \in G$, $\alpha \in \mathbb{F}_q^k$ and $v \in V$, we have

$$g(\alpha v) = (g\alpha q^{-1})g(v) = \alpha^{\sigma(g)}(v),$$

so $G$ is included into the semilinear group $\Gamma L(V_d(q^k)) = \Gamma L(d, q^k)$. The subgroup $H$ is just the kernel of $\sigma$, so (4) and part of (3) follows.

Let $B \triangleleft G$ be any Abelian normal subgroup, $\alpha \in Z$ a generator of $Z$ and $b \in B$. Then $bab^{-1} = \alpha^{\sigma(b)} = \alpha^q$ for some $0 \leq s < k$, so $[b, \alpha] = \alpha^{q^s-1} \in B$ is centralised by $b$. Changing $b$ to $b^{-1}$ if necessary, we can assume that $0 \leq s \leq \frac{k}{2}$. This means $(\alpha^{q^s-1})q^s = \alpha^{q^s-1}$, so $q^k - 1 | (q^s - 1)^2 < q^k - 1$. Therefore, $s = 0$. Thus, $B \leq C_G(Z)$, so $BZ \geq Z$ is an Abelian normal subgroup in $G$. By the maximality of $Z$, we get $B \leq Z$, which completes the proof of both (1) and (3).

Let $M = F(H)$ be the Fitting subgroup of $H$. Then $Z(M)$ is an Abelian normal subgroup of $G$, so $Z(M) = Z$ by the maximality of $Z$. Let $n$ be the nilpotency
class of $M$. If $n = 1$ then $M = Z$. Otherwise, we claim that $n = 2$. Assuming that $n \geq 3$, we have $1 \neq \gamma_n(M) \leq Z$ and

$$\left[\gamma_{n-1}(M), \gamma_{n-1}(M)\right] \leq \left[\gamma_2(M), \gamma_{n-1}(M)\right] \leq \gamma_{n+1}(M) = 1,$$

so $\gamma_{n-1}(M)$ is an Abelian normal subgroup of $G$, so it must be contained in $Z$. This forces $\gamma_n(M) = 1$, a contradiction. Therefore, $n \leq 2$, i.e. $M/Z$ is Abelian.

Let $R$ be a Sylow-$r$-subgroup of $M$ for some prime $r$ dividing $|M/Z|$. The commutator map defines a symplectic bilinear function from $R/Z$ into $Z(R) = R \cap Z$. Therefore, for any $x, y \in R$, we have $[x^r, y^r] = [x, y]^{r^2} = [x^{r^2}, y]$. If $r^s$ is the exponent of $R/(R \cap Z)$ for some $s \geq 2$, then $R^{r^{s-1}} \cap Z$ is an Abelian normal subgroup of $G$, so $R^{r^{s-1}} \leq Z$, a contradiction. Thus, we see that $R/(R \cap Z)$ is an elementary Abelian $r$-group. Using this and the above commutator identity, it also follows that $R' \leq Z$ is of exponent $r$. It follows that $R = (R \cap Z)R_0$ for some extraspecial $r$-group $R_0$.

By the previous two paragraphs, $F(H)/Z$ is exactly the direct product of the minimal Abelian normal subgroups of $H/Z$, and therefore $F(H)/Z$ is contained in $\text{Soc}(H/Z)$. Since $N = F^*(H)$ is the central product of $F(H)$ and the layer $E(H)$, where $E(H)/Z$ is the direct product of the minimal non-Abelian normal subgroups of $H/Z$, it follows that $N/Z = \text{Soc}(H/Z)$ as claimed. By [4, Lemma 12.1], $V_d(q^k)$ is an absolutely irreducible $\mathbb{F}_{q^k} H$-module. If the irreducible $\mathbb{F}_{q^k} N$-components of $V_d(q^k)$ were not absolutely irreducible, then

$$Z(C_{\text{GL}(V_d(q^k))}(N))$$

would be the multiplicative group of a proper field extension of $\mathbb{F}_{q^k}$ normalised by $G$, which again contradicts the maximality of $Z$. Now, let us assume that $V_d(q^k) = U \oplus \cdots \oplus U$ is a direct sum of $s$ many isomorphic absolutely irreducible $\mathbb{F}_{q^k} N$-modules for some $s \geq 2$. By [11, Lemma 4.4.3 (ii)–(iv)], there is a tensor product decomposition $U \otimes_{\mathbb{F}_{q^k}} W$ of $V_d(q^k)$ such that

$$N \leq GL(U) \otimes 1_W \leq GL(U) \otimes GL(W),$$

$$G \leq N_{\Gamma L(V)}(N) \leq N_{\Gamma L(V)}(GL(U) \otimes GL(W)).$$

It follows that $H \leq GL(U) \otimes GL(W)$. Let

$$L = \{1_U \otimes h_W \mid \text{there exists } h_U \in GL(U) \text{ such that } h_U \otimes h_W \in H\}.$$

If $L = Z$, then $V_d(q^k)$ is not irreducible as an $\mathbb{F}_{q^k} H$-module, a contradiction. We have that $L \leq GL(V)$ is a coprime linear group normalised by $G$, so $LG \leq GL(V)$ is coprime. Using the maximality of $G$, we get $L < G$. But then $Z < L \leq H$ clearly centralises $N = F^*(H)$, a contradiction. So, $V_d(q^k)$ is an absolutely irre-
ducible \( \mathbb{F}_{q^n} N \)-module, and (5) is proved. Now, (6) follows by a combined use of [15, Corollary 18.2 (a)] and [11, Lemma 4.4.3 (iii)].

If \( N_i / Z \) is Abelian, then it is a minimal Abelian normal subgroup of \( H / Z \), so it is an elementary Abelian \( r_i \)-group for some prime \( r_i \). Using the same argument as in paragraph 6 of this proof, one can find the extraspecial \( r_i \)-group \( R_i \) by taking the full inverse image of a maximal non-degenerate subspace of \( R / R' \), where \( R \) is the Sylow-\( r_i \) subgroup of \( N_i \). For this subgroup, it clearly follows that \( N_i = Z R_i \), and \( |R_i| = r_i^{2l_i + 1} \) for some integer. Furthermore, since \( V_i \) is an absolutely irreducible \( \mathbb{F}_{q^n} N_i \)-module, it must be an absolutely irreducible \( \mathbb{F}_{q^n} R_i \)-module. It is well known that an extraspecial \( r_i \)-group of order \( r_i^{2l_i + 1} \) has a unique faithful absolutely irreducible ordinary representation, and this representation has degree \( r_i^{l_i} \), which finishes the proof of (7).

Finally, (8) can again be deduced from [15, Corollary 18.2 (a)] and from the fact that \( N_i / Z \) is a minimal normal subgroup in \( H_i / Z \). \( \square \)

### 6 Proof of Theorem 1.2

We start this section by giving a direct proof for Theorem 1.2 for the case when \( V \) is the natural permutation module for \( A_n \) or \( S_n \), which is case (2) in Theorem 4.8. Since in this case \( \text{MinSupp}_V(G) \) is bounded, we cannot use bounds for \( P \cdot P.b(c, G, V) \) in terms of supports given in Section 2.

**Theorem 6.1.** Let \( U \) be an \( m \)-dimensional vector space over \( \mathbb{F}_q \), and let \( G = S_m \) with its natural permutation action on \( U \). Assuming that \( (|G|, |U|) = 1 \), we have

\[
P(\text{random } u \in U^c \text{ is a base for } G) > 1 - \frac{1}{m^{c-2}} \quad \text{for any } c \geq 3.
\]

Hence, three random vectors form a base for \( G \) with high probability if \( m \) is large.

**Proof.** The \( \mathbb{F}_q G \)-module \( V^c \) can be naturally identified with \( M^{m \times c}(q) \), the space of \( m \times c \)-matrices over \( \mathbb{F}_q \). Under this identification, \( G \) acts on \( M^{m \times c}(q) \) by permuting the rows of each element of \( M^{m \times c}(q) \) in a natural way. Hence, a matrix \( a \in M^{m \times c}(q) \) is a base for \( G \) if and only if the rows of \( a \) are pairwise different elements of \( M^{1 \times c}(q) \), the space of \( c \)-dimensional row vectors over \( \mathbb{F}_q \). Thus, the probability in question is equal to the probability that \( m \) random elements of \( M^{1 \times c}(q) \) are pairwise different, which is

\[
\prod_{i=0}^{m-1} \frac{q^c - i}{q^c} > \left( \frac{q^c - q}{q^c} \right)^m \geq \left( 1 - \frac{1}{m^{c-1}} \right)^m \geq 1 - \frac{1}{m^{c-2}},
\]

where the first and second inequalities follow since \( m < q \) by the coprime assumption. The claim follows. \( \square \)
Corollary 6.2. Let \( V \) be an \( n \)-dimensional vector space over the finite field \( \mathbb{F}_q \), and let \( G = Z \cdot G_0 \leq GL(V) \) be a coprime linear group, where \( G_0 \simeq S_m \) or \( G_0 \simeq A_m \) and \( V \) is the non-trivial irreducible component of the natural \( \mathbb{F}_q G_0 \)-module. Then we have
\[
P(\text{random } \underline{v} \in V^c \text{ is a base for } G) > 1 - \frac{1}{n^{c-2}} \quad \text{for any } c \geq 3.
\]

Proof. First, note that \( \text{Fix}(g) = 0 \) for every \( g \in G \setminus G_0 \), so a \( \underline{v} \in V^c \) is a base for \( G \) if and only if it is a base for \( G_0 \). Second, let \( U = V \oplus U_0 \), where \( U_0 \) is the trivial module for \( G_0 \). For any random vectors \( u_1, \ldots, u_c \in U \), let \( v_i \) be the projection of \( u_i \) to \( V \) along \( U_0 \). Then \( u_1, \ldots, u_c \) is a base for \( G_0 \) if and only if \( v_1, \ldots, v_c \) is a base for \( G_0 \), so the claim follows from Theorem 6.1.

Now, we are ready to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. Let \( V \) be an \( n \)-dimensional vector space over the finite field \( \mathbb{F}_q \), and let \( G \leq GL(V) \) be a coprime primitive linear group. Without loss of generality, we can assume that \( G \) is maximal among such subgroups of \( GL(V) \). Let \( Z \) be the unique maximal Abelian subgroup in \( GL(V) \) which is normalised by \( G \), and let \( H \) be the intersection of \( G \) and \( GL(d, q^k) \) as in Theorem 5.1. If \( g \in G \setminus H \), then there is a \( z \in Z \) such that \([g, z] \neq 1\). By Lemma 3.1,
\[
\text{Supp}_V(g) \geq \frac{1}{2} \text{Supp}_V([g, z]) = \frac{1}{2} \dim(V).
\]
Therefore, if \( c > 4 \), then
\[
\sum_{g \in G \setminus H} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \leq \frac{|G \setminus H|}{q^{\frac{c}{2}} \dim(V)} \leq \frac{|V|^2}{|V|^c} \leq \frac{1}{|V|^c - 2}.
\]

Now let \( g \) be an element of \( H = H_1 \otimes \cdots \otimes H_t \). So we have \( g = (g_1, \ldots, g_t) \), where \( g_i \in H_i \) for all \( i \in [t] \) and \( g \) preserves the tensor product decomposition \( V = V_1 \otimes \cdots \otimes V_t \) over \( \mathbb{F}_q^k \) as in Theorem 5.1 (6) and \( \dim_{\mathbb{F}_q^k} (V_i) = d_i \) for all \( i \) (therefore \( d = \dim_{\mathbb{F}_q^k} (V) = \prod_{i=1}^t d_i \) and \( \dim(V) = \dim_{\mathbb{F}_q} (V) = k \cdot \prod_{i=1}^t d_i \)). We can assume that in this decomposition the dimensions of the vector spaces are decreasing, i.e. \( d_1 \geq d_2 \geq \cdots \geq d_t \geq 2 \). Let \( g_i \notin Z \) for some \( i \neq 1 \). Then, by Lemma 3.2,
\[
\text{Supp}_V(g) \geq \text{MinSupp}_{V_i} (H_i) \cdot \frac{\dim(V)}{\dim(V_i)} \geq k \prod_{j \neq i} d_j.
\]
Since
\[ 2 \prod_{j \neq i} d_j \geq 2^{t-1} d_1 \geq \sum_{i=1}^{t} d_1 \quad \text{and} \quad k \prod_{j \neq i} d_j \geq k \prod_{j=1}^{t} d_j \geq \sqrt{\dim(V)}, \]
we get
\[ c \cdot \text{Supp}_V(g) \geq 2k \sum_{i=1}^{t} d_i + (c - 4)\sqrt{\dim(V)}. \]
Hence,
\[ \sum_{g \in H \setminus H_1} \frac{1}{q^{c\cdot \text{Supp}_V(g)}} \leq \frac{\prod_{i=1}^{t} |H_i|}{q^{2k \sum_{i=1}^{t} d_i +(c-4)\sqrt{\dim(V)}}} \leq \frac{q^{2k \sum_{i=1}^{t} d_i}}{q^{2k \sum_{i=1}^{t} d_i +(c-4)\sqrt{\dim(V)}}} = \frac{1}{q^{(c-4)\sqrt{\dim(V)}}}. \]
Now assume that \( g \in H_1 \). In this case \( \text{Supp}_V(g) = \text{Supp}_{V_1}(g) \cdot \frac{d}{d_i} \). By Theorem 5.1, \( Z \leq N_1 \leq H_1 \leq GL(V_1(q^k)) \), where \( N_1 \) is a minimal normal subgroup above \( Z \) and \( N_1/Z \) is characteristically simple. Therefore, it is either an elementary Abelian group, a direct product of non-Abelian simple groups, or a non-Abelian simple group.

First, if \( N_1/Z \) is elementary Abelian, then \( N_1 = Z \cdot P \), where \( P \) is an extraspecial \( r \)-group for a prime \( r \) with \( r \mid q^k - 1 \). Then \( V_1(q^k) \) is an absolutely irreducible \( \mathbb{F}_{q^k} P \)-module. If \( n \in P \setminus Z \), then \( n \) has exactly \( r \) different eigenvalues on \( V_1 \) (or on \( V \)) each with the same multiplicity. It follows that
\[ \text{MinSupp}_V(N_1) \geq \frac{r - 1}{r} \dim(V) \geq \frac{1}{2} \dim(V), \]
so \( \text{MinSupp}_V(H_1) \geq \frac{1}{4} \dim(V) \) by Lemma 3.1. In this case,
\[ \sum_{g \in H_1} \frac{1}{q^{c\cdot \text{Supp}_V(g)}} \leq \frac{|H_1|}{q^{\frac{c}{4} \dim(V)}} \leq \frac{|V|^2}{q^{2 \dim(V) + (\frac{c}{4} - 2) \dim(V)}} \leq \frac{1}{|V|^\frac{c}{4} - 2}. \]
Next, let \( N_1/Z \) be a direct product of \( s \geq 2 \) many isomorphic non-Abelian simple groups. By Theorem 5.1 (8), the action of \( N_1 = K_1 \otimes \cdots \otimes K_s \) on \( V_1 \) preserves a tensor product decomposition \( V_1 = W_1 \otimes \cdots \otimes W_s \) over \( \mathbb{F}_{q^k} \), where \( \dim_{\mathbb{F}_q}(W_i) = \sqrt[3]{d_1} \geq 2 \) for every \( i \). Using [16, Theorem 1], we get
\[ |N_1| \leq \prod_{i=1}^{s} |K_i| \leq \prod_{i=1}^{s} |W_i|^2 = q^{2ks} \sqrt[3]{d_1}. \]
On the other hand, $H_1/N_1$ acts faithfully on $\{W_1, \ldots, W_s\}$, and $|H_1/N_1|$ is coprime to $q$, so $|H_1/N_1| \leq q^s$ by [7, Corollary 2.4]. Therefore, $|H_1| \leq q^{2ks/\sqrt{d_1}+s}$. By Lemma 3.1 and by Lemma 3.2,

$$\text{Supp}_{V_1}(g) \geq \frac{1}{2} \text{MinSupp}_{V_1}(N_1) \geq \frac{k}{2} \cdot \frac{d_1}{\sqrt{d_1}}.$$  

Therefore,

$$c \cdot \text{Supp}_V(g) \geq 5kd_1^{(s-1)/s} + \left(\frac{c}{2} - 5\right)k \sqrt{d_1} \cdot \frac{d}{d_1} \geq 2ks \sqrt{d_1} + s + \left(\frac{c}{2} - 5\right) \sqrt{\text{dim}(V)}.$$

So,

$$\sum_{1 \neq g \in H_1} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \leq \frac{|H_1|}{q^{c \cdot \text{MinSupp}_V(H_1)}} \leq \frac{q^{2ks \sqrt{d_1}+s}}{q^{2ks \sqrt{d_1}+s+(\frac{c}{2} - 5)\sqrt{\text{dim}(V)}}} \leq \frac{1}{q^{(\frac{c}{2} - 5)\sqrt{\text{dim}(V)}}}.$$

Finally, let $N_1/Z$ be a non-Abelian simple group. If $d_1 \leq \sqrt{d}$, then we can use the same argument as in the previous paragraph to see that

$$\sum_{1 \neq g \in H_1} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \leq \frac{1}{q^{(c-2)\sqrt{\text{dim}(V)}}}.$$

Summarising the bounds given up to this point, we get

$$Ph(c, G, V) \geq 1 - \sum_{1 \neq g \in G} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \geq 1 - \left(\frac{1}{|V|^{\frac{c}{2}}} + \frac{1}{q^{(c-4)\sqrt{\text{dim}(V)}}} + \frac{1}{q^{\frac{c}{2} - 5)\sqrt{\text{dim}(V)}}} \right) \geq 1 - \frac{3}{q^{(\frac{c}{2} - 5)\sqrt{\text{dim}(V)}}},$$

which is case (1) of Theorem 1.2.

Now, let us assume that $d_1 \geq \sqrt{d}$. If $|V_1| = q^{k d_1}$ is bounded by the constant appearing in part (3) of Theorem 4.8, then $|V|$ is also bounded. Hence, we can assume that either part (1) or part (2) of Theorem 4.8 holds. By Lemma 3.1, we also have $\text{MinSupp}_V(H_1) \geq \frac{1}{2} \text{MinSupp}_V(N_1)$. 


If $N_1/Z$ is not an alternating group, then
\[ \text{MinSupp}_V(N_1) \geq \frac{1}{40} \dim(V) \quad \text{and} \quad 5 \cdot \text{MinSupp}_V(N_1) \geq \log_q |H_1| \]
by Corollary 4.3, Theorem 4.8 (1) and Lemma 3.2 (2). Thus, we have
\[ \sum_{1 \neq g \in H_1} \frac{1}{q^{c-\text{Supp}_V(g)}} \leq \frac{1}{|V|(c-10)/80}. \]
So, in this case, we get
\[
\begin{align*}
    Pb(c,G,V) &\geq 1 - \left( \frac{1}{|V|^{\frac{c}{2}-2}} + \frac{1}{q^{(c-4)\sqrt{\dim(V)}}} + \frac{1}{|V|(c-10)/80} \right) \\
    &\geq 1 - \left( \frac{1}{q^{(c-4)\sqrt{\dim(V)}}} + \frac{2}{|V|(c-10)/80} \right),
\end{align*}
\]
which is case (2) (a) of Theorem 1.2.

Finally, let $N_1/Z \simeq A_m$ for some $m$. If $V_1$ is not an irreducible component of the natural $\mathbb{F}_q^k A_m$ permutation module, then
\[ \text{MinSupp}_V(N_1) \geq \frac{1}{16} \sqrt{\dim(V)} \quad \text{and} \quad 5 \cdot \text{MinSupp}_V(N_1) \geq \log_q |H_1| \]
by Corollary 4.7, Theorem 4.8 (1) and Lemma 3.2 (2). Thus, we have
\[ \sum_{1 \neq g \in H_1} \frac{1}{q^{c-\text{Supp}_V(g)}} \leq \frac{1}{q^{c-10/\sqrt{\dim(V)}}} \]
and
\[
\begin{align*}
    Pb(c,G,V) &\geq 1 - \left( \frac{1}{|V|^{\frac{c}{2}-2}} + \frac{1}{q^{(c-4)\sqrt{\dim(V)}}} + \frac{1}{q^{c-10/\sqrt{\dim(V)}}} \right) \\
    &\geq 1 - \frac{3}{q^{c-10/\sqrt{\dim(V)}}}.
\end{align*}
\]
Finally, if $V_1$ is the non-trivial irreducible component of the natural $\mathbb{F}_q^k A_m$-module, then, with the use of Corollary 6.2, we get
\[
\begin{align*}
    Pb(c,G,V) &\geq 1 - \left( \frac{1}{|V|^{\frac{c}{2}-2}} + \frac{1}{q^{(c-4)\sqrt{\dim(V)}}} + 1 - Pb(c,H_1,V) \right) \\
    &\geq 1 - \frac{3}{n^{c-2}},
\end{align*}
\]
which completes the proof of Theorem 1.2.
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