Thermal fluctuations correction to magnetization and specific heat of vortex solids in type II superconductors.

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Abstract

A systematic calculation of magnetization and specific heat contributions due to fluctuations of vortex lattice in strongly type II superconductors to precision of 1% is presented. We complete the calculation of the two loop low temperature perturbation theory by including the umklapp processes. Then the gaussian variational method is adapted to calculation of thermodynamic characteristics of the 2D and the 3D vortex solids in high magnetic field. Based on it as a starting point for a perturbation theory we calculate the leading correction providing simultaneously an estimate of precision. The results are compared to existing nonperturbative approaches.

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I. INTRODUCTION

Existence of vortex lattice in type II superconductors in magnetic field was predicted by Abrikosov and subsequently observed in various types of such superconductors ranging from metals to high $T_c$ cuprates. In the original treatment mean field Ginzburg - Landau (GL) theory which neglects thermal fluctuations of the vortex matter was used. Thermal fluctuations are expected to play much larger role in high $T_c$ superconductors than in the low temperature ones because the Ginzburg parameter $G\ell$ characterizing fluctuations is much larger \[1\]. In addition the presence of strong magnetic field and strong anisotropy in superconductors like BSCCO effectively reduces their dimensionality thereby further enhancing effects of thermal fluctuations. Under these circumstances fluctuations make the lattice softer influencing various physical properties like magnetization and specific heat and eventually lead to vortex lattice melting into vortex liquid far below the mean field phase transition line \[2,4\] clearly seen in both magnetization \[3\] and specific heat experiments \[4\]. To develop a theory of these fluctuations even in the case of lowest Landau level (LLL) corresponding to regions of the phase diagram “close” to $H_{c2}$ is a nontrivial task and several different approaches were developed.

At high temperature end (and thereby in the ”vortex liquid” phase) Thouless and Ruggeri \[5,6\] proposed a perturbative expansion around a homogeneous (liquid) state in which all the ”bubble” diagrams are resummed. It was shown in field theory that summation of bubble diagrams is equivalent to the gaussian variational approach \[7\]. In this approach one searches for a ”gaussian” state having the lowest energy. The series provides accurate results at high temperatures, but become inapplicable for LLL dimensionless temperature $a_T \sim (T - T_{mf}(H))/(TH)^{1/2}$ smaller than 2 in 2D and for $a_T \sim (T - T_{mf}(H))/(TH)^{2/3}$ smaller than 1 in 3D both quite far above the melting line. Generally, attempts to extend the theory to lower temperature by Pade extrapolation were not successful \[8\]. It is in fact doubtful whether the perturbative results based on gaussian approximation assuming translational invariant liquid state should be attempted at low $a_T$.

In ref. \[9\], it was shown that below $a_T < -5$ different gaussian states which are no longer translationally invariant have lower energy. We will present the detailed calculation here (optimized perturbation theory was used to study for liquid state, see refs. \[9,10\]). It is in general a very nontrivial problem to find an inhomogeneous solutions of the corresponding ”gap equation” (see section IV). However using previous experience with low temperature perturbation theory \[11,12\], the problem can be significantly simplified and solved using rapidly convergent ”modes” expansion. A consistent perturbation theory should start from these states \[13\]. We then generalize the approach of ref. \[1\] by setting up a perturbation theory around the gaussian Abrikosov lattice state.

Magnetization and specific heat contributions due to vortex lattices are calculated in perturbation theory around this state to next to leading order. This allows to estimate the precision of the calculation. It is the worst, about 1%, near the melting point at $a_T = -10$ and becomes better for lower $a_T$. At low temperature the result is consistent with the first principles low temperature perturbation theory advanced recently to the two loop order \[11,12\]. The previous two loop calculation is completed by including the umklapp processes. One can make several definitive qualitative conclusions using the improved accuracy of the results. The LLL scaled specific heat monotonously rises from its mean
field value of $1/\beta_A$ at $a_T = -\infty$ to a slightly higher value of $1.05/\beta_A$ where $\beta_A = 1.16$ is Abrikosov parameter. This is at variance with theory ref. [14] which uses completely different ideas and has a freedom of arbitrarily choosing certain parameters on a 2% precision level. Although we calculate the contribution of the LLL only, corrections due to higher Landau levels calculated earlier in ref. [15,16] using less sophisticated method can be included.

The paper is organized as follows. The model is defined in section II. In section III a brief summary of existing results and umklapp corrected two loop perturbative results in both 2D and 3D are given. Gaussian approximation and the mode expansion used is described in section IV. The basic idea of expansion around the best gaussian state is explained in section V. The leading corrections are calculated. Results are presented and compared with perturbation theory and other theories in section VI together with conclusions.

II. MODELS

To describe fluctuations of order parameter in thin films or layered superconductors one can start with the Ginzburg-Landau free energy:

$$F = L_z \int d^2x \frac{\hbar^2}{2m_{ab}} |D\psi|^2 - a|\psi|^2 + \frac{b'}{2}|\psi|^4,$$

(1)

where $\mathbf{A} = (By,0)$ describes a constant magnetic field (considered nonfluctuating) in Landau gauge and covariant derivative is defined by $D \equiv \nabla - i\frac{2\pi}{\Phi_0} \mathbf{A}$. $\Phi_0 \equiv \frac{hc}{e}$. For strongly type II superconductors like the high $T_c$ cuprates ($\kappa \sim 100$) and not too far from $H_{c2}$ (this is the range of interest in this paper, for the detailed discussion of the range of applicability see [15]) magnetic field is homogeneous to a high degree due to superposition from many vortices. For simplicity we assume $a(T) = \alpha T_c(1-t)$, $t \equiv T/T_c$, although this temperature dependence can be easily modified to better describe the experimental $H_{c2}(T)$. The thickness of a layer is $L_z$.

Throughout most of the paper will use the coherence length $\xi = \sqrt{\frac{\hbar^2}{2m_{ab}a_T}}$ as a unit of length and $\frac{dH_{c2}(T)}{dT}T_c = \frac{\Phi_0}{2\pi \xi}$ as a unit of magnetic field. After the order parameter field is rescaled as $\psi^2 \rightarrow \alpha T_c b' \psi^2$, the dimensionless free energy (the Boltzmann factor) is:

$$\frac{F}{T} = \frac{1}{\omega} \int d^2x \left[ \frac{1}{2} |D\psi|^2 - \frac{1-t}{2} |\psi|^2 + \frac{1}{2} |\psi|^4 \right].$$

(2)

The dimensionless coefficient describing the strength of fluctuations is

$$\omega = \sqrt{2Gi\pi^2}t = \frac{m_{ab}b'}{2\hbar^2\alpha L_z}t, \quad Gi \equiv \frac{1}{2} \left( \frac{32\pi e^2 \hbar^2 \xi^2 T_c}{c^2 \hbar^2 L_z} \right)^2$$

(3)

where $Gi$ is the Ginzburg number in 2D. When $\frac{1-t}{125} << 1$, the lowest Landau level (LLL) approximation can be used [15]. The model then simplifies due to the LLL constraint, $-\frac{D^2}{2}\psi = \frac{b}{2}\psi$, rescaling $x \rightarrow x/\sqrt{b}, y \rightarrow y/\sqrt{b}$ and $|\psi|^2 \rightarrow |\psi|^2 \sqrt{\frac{b\omega}{4\pi}}$, one obtains

$$f = \frac{1}{4\pi} \int d^2x \left[ a_T |\psi|^2 + \frac{1}{2} |\psi|^4 \right].$$

(4)
where the 2D LLL reduced temperature
\[ a_T \equiv -\sqrt{\frac{4\pi}{b\omega}} \frac{1-t-b}{2} \] (5)
is the only parameter in the theory [17,5].

For 3D materials with asymmetry along the z axis the GL model takes a form:
\[ F = \int d^3x \frac{\hbar^2}{2m_{ab}} \left( \nabla - \frac{ie^*}{\hbar c} A \right)^2 \psi + \frac{\hbar^2}{2m_c} |\partial_z \psi|^2 + a|\psi|^2 + \frac{b'}{2} |\psi|^4 \] (6)
which can be again rescaled into
\[ f = \frac{F}{T} = \frac{1}{\omega} \int d^3x \left[ \frac{1}{2} |D\psi|^2 + \frac{1}{2} |\partial_z \psi|^2 - \frac{1-t}{4} |\psi|^2 + \frac{1}{2} |\psi|^4 \right], \] (7)
by \( x \to \xi x, y \to \xi y, z \to \frac{\xi^* z}{\sqrt{3}}, \) \( \psi^2 \to \frac{2\alpha T}{\sqrt{3}} \psi^2, \) where \( \gamma \equiv m_c/m_{ab} \) is anisotropy. The Ginzburg number is now given by
\[ Gi \equiv \frac{1}{2} \left( \frac{32\pi e^2 k^2 \xi T_c \gamma^{1/2}}{c^2 \hbar^2} \right)^2 \] (8)

Within the LLL approximation, and rescaling
\( x \to x/\sqrt{b}, y \to y/\sqrt{b}, z \to z \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{-1/3}, \) \( \psi^2 \to \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{2/3} \psi^2, \) the dimensionless free energy becomes:
\[ f = \frac{1}{4\pi \sqrt{2}} \int d^3x \left[ \frac{1}{2} |\partial_z \psi|^2 + a_T |\psi|^2 + \frac{1}{2} |\psi|^4 \right]. \] (9)

The 3D reduced temperature is:
\[ a_T = - \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{-2/3} \frac{1-t-b}{2}. \] (10a)

From now on we work with rescaled quantities only and related them to measured quantities in section V.

III. PERTURBATION THEORIES AND EXISTING NONPERTURBATIVE RESULTS

A variety of perturbative as well as nonperturbative methods have been used to study this seemingly simple model. There are two phases. Neglecting thermal fluctuations, one obtains the lowest energy configuration \( \psi = 0 \) for \( a_T > 0 \) and
\[ \psi = \sqrt{\frac{a_T}{\beta_A}} \varphi; \]
\[ \varphi = \sqrt{\frac{2\pi}{\sqrt{\pi a_\Delta}}} \sum_{l=-\infty}^{\infty} \exp \left\{ i \left[ \frac{\pi l(l-1)}{2} + \frac{2\pi}{a_\Delta} lx \right] - \frac{1}{2} \left( y - \frac{2\pi}{a_\Delta} l \right)^2 \right\} \] (11)
for \( a_T < 0, \) where \( a_\Delta = \sqrt{\frac{4\pi}{\sqrt{3}}} \) is the lattice spacing in our units and \( \beta_A = 1.16. \)
A. High temperature expansion in the liquid phase

Homogeneous "vortex liquid" phase (which is not separated from the "normal" phase by a transition) has been studied using high temperature perturbation theory by Thouless and Ruggeri [5]. Unfortunately this asymptotic series (even pushed to a very high orders [18]) are applicable only when $a_T > 2$. In order to extend the results to lower $a_T$ attempts have been made to Pade resum the series [6] imposing a constraint that the result matches the Abrikosov mean field as $a_T \to -\infty$. However, if no matching to the limit of is imposed, the perturbative results cannot be significantly improved [8]. Experiments [3,4], Monte Carlo simulations [19] and nonperturbative Bragg chain approximation [20] all point out that there is a first order melting transition around $a_T = -12$. If this is the case it is difficult to support such a constraint.

B. Low temperature perturbation theory in the solid phase. Umklapp processes

Recently a low temperature perturbation theory around Abrikosov solution eq. (11) was developed and shown to be consistent up to the two loop order [11,12,16]. Since we will use in the present study the same basis and notations and also will compare to the perturbative results we recount here few basic expressions. The order parameter field $\psi$ is divided into a nonfluctuating (mean field) part and a small fluctuation

$$\psi(x) = \sqrt{-a_T/\beta_A} \phi(x) + \chi(x).$$

(12)

The field $\chi$ can be expanded in a basis of quasi - momentum eigen functions on LLL in 2D:

$$\varphi_k = \sqrt{2\pi a_\Delta} \sum_{l=-\infty}^{\infty} \exp \left\{ i \left[ \pi l(l-1)/2 + 2\pi(x-k_y)l - xk_x \right] - 1/2(y+k_x - 2\pi/\Delta l)^2 \right\}.$$ 

(13)

Then we diagonalize the quadratic term of free energy eq. (4) to obtain the spectrum. Instead of the complex field $\chi$, two "real" fields $O$ and $A$ will be used:

$$\chi(x) = \frac{1}{\sqrt{2}} \int_k \frac{\exp[-i\theta_k/2] \varphi_k(x)}{(\sqrt{2\pi})^2} (O_k + iA_k)$$

$$\chi^*(x) = \frac{1}{\sqrt{2}} \int_k \frac{\exp[i\theta_k/2] \varphi_k^*(x)}{(\sqrt{2\pi})^2} (O_{-k} - iA_{-k})$$

where $\gamma_k = |\gamma_k| \exp[i\theta_k]$ and definition of $\gamma_k$ (and all other definitions of functions) can be found in Appendix A. The eigenvalues found by Eilenberger in ref. [21] are

$$\epsilon_A(k) = -a_T \left( -1 + 2 \beta_k - 1/\beta_A |\gamma_k| \right)$$

$$\epsilon_O(k) = -a_T \left( -1 + 2 \beta_k + 1/\beta_A |\gamma_k| \right),$$

(15)

where $\beta_k$ is defined in Appendix A. In particular, when $k \to 0$ [22]
\[ \epsilon_A \approx 0.12 |a_T| |k|^4. \]  

The second excitation mode \( \epsilon_O \) has a finite gap. The free energy to the two loop order was calculated in [12], however the umklapp processes were not included in some two loop corrections. These processes correspond to momentum nonconserving (up to integer times inverse lattice constant) four leg vertices (see Appendix A eq. (A2)). We therefore recalculated these coefficients. The result in 2D is (see Fig.1a,b):

\[ f_{\text{eff}}^{2D} = -\frac{a_T^2}{2 \beta_A} + 2 \log \left| \frac{a_T}{4\pi^2} \right| - \frac{19.9}{a_T^2} + c_v. \]  

where \( c_v = \langle \log \left( \frac{\epsilon_A(k) \epsilon_O(k)}{a_T^2} \right) \rangle = -2.92 \). In 3D, similar calculation (extending the one carried in ref. [11] to umklapp processes) gives:

\[ f_{\text{eff}}^{3D} = -\frac{a_T^2}{2 \beta_A} + 2.848 |a_T|^{1/2} + \frac{2.4}{a_T}. \]  

C. Nonperturbative methods

Few nonperturbative methods have been attempted. Tesanovic and coworkers [14] developed a method based on an approximate separation of the two energy scales. The larger contribution (98%) is the condensation energy, while the smaller one (2%) describes motion of the vortices. The result for energy in 2D is:

\[ f_{\text{eff}} = -\frac{a_T^2 U^2}{4} - \frac{a_T^2 U}{2} \sqrt{\frac{U^2}{4} + \frac{2}{a_T^2}} + 2 \arcsinh \left[ \frac{a_T U}{2 \sqrt{2}} \right] \]  

\[ U = \frac{1}{2} \left[ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{\beta_A}} + \tanh \left[ \frac{a_T}{4 \sqrt{2}} + \frac{1}{2} \right] \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{\beta_A}} \right) \right] \]

Corresponding expressions in 3D were also derived.

No melting phase transition is seen since it belongs to the 2% which cannot be accounted for within this approach. There exist several Monte Carlo (MC) simulations of the system [19]. The expression eq. (13) agrees quite well with high temperature perturbation theory and MC simulations and has been used to fit both magnetization and specific heat experiments [1], but only to mean field level agrees with low temperature perturbation theory. Expanding eq. (13) in \( 1/a_T \), one obtains an opposite sign of the one loop contribution, see Fig.1a and discussion in section VI.

Other interesting nonperturbative methods include the \( 1/N \) expansion [23,24] and the "Bragg chain fluctuation approximation" [20].

IV. GAUSSIAN VARIATIONAL APPROACH

A. General Anzatz

Gaussian variational approach originated in quantum mechanics and has been developed in various forms and areas of physics [23,26]. In quantum mechanics it consists of choosing a
gaussian wave function which has the lowest energy expectation value. When fermionic fields are present the approximation corresponds to BCS or Hartree- Fock variational state. In scalar field theory one optimizes the quadratic part of the free energy

$$f = \int -\frac{1}{2} \phi^a D^{-1} \phi^a + V(\phi^a)$$

$$= \int \frac{1}{2} (\phi^a - v^a) G^{-1ab} (\phi^b - v^b) + \bar{V}(G, \phi^a)$$

$$= K + \bar{V}$$ (20)

To obtain ”shift” $v^a$ and ”width of the gaussian” $G$, one minimizes the gaussian effective free energy [26], which is an exact upper bound on the energy (see proof in [25]). The result of the gaussian approximation can be thought of as resummation of all the ”cacti” or loop diagrams [5,7]. Further corrections will be obtained in section V by inserting this solution for $G$ and taking more terms in the expansion of $Z$.

B. 2D Abrikosov vortex lattice

In our case of one complex field one should consider the most general quadratic form

$$K = \int_{x,y} (\psi^*(x) - v^*(x)) G^{-1}(x, y) (\psi(y) - v(y))$$

$$+ (\psi - v(x)) H^* (\psi - v(x)) + (\psi^* - v^*(x)) H (\psi^* - v^*(x))$$ (21)

Assuming hexagonal symmetry (a safe assumption for the present purpose), the shift should be proportional to the mean field solution eq. (12), $v(x) = v\phi(x)$, with a constant $v$ taken real thanks global $U(1)$ gauge symmetry. On LLL, as in perturbation theory, we will use variables $O_k$ and $A_k$ defined in eq. (14) instead of $\psi(x)$

$$\psi(x) = v\phi(x) + \frac{1}{\sqrt{2\pi}} \int_k \exp \left[ -\frac{i\theta_k}{2} \right] \varphi_k(x) (O_k + iA_k).$$ (22)

The phase defined after eq. (14) is quite important for simplification of the problem and was introduced for future convenience. The most general quadratic form is

$$K = \frac{1}{8\pi} \int_k O_k G^{-1}_{OO}(k) O_{-k} + A_k G^{-1}_{AA}(k) A_{-k} +$$

$$O_k G^{-1}_{OA}(k) A_{-k} + A_k G^{-1}_{OA}(k) O_{-k},$$ (23)

with matrix of functions $G(k)$ on Brillouin zone to be determined together with the constant $v$ by the variational principle. The corresponding gaussian free energy is

$$f_{gauss} = a_T v^2 + \frac{\beta_A}{2} v^4 - 2 - \langle \log \left[ (4\pi)^2 \det(G) \right] \rangle_k +$$

$$\langle a_T (G_{OO}(k) + G_{AA}(k)) + v^2 [2\beta_k + |\gamma_k|] G_{OO}(k) + (2\beta_k - |\gamma_k|) G_{AA}(k) \rangle_k$$

$$\langle \beta_{k-l} [G_{OO}(k) + G_{AA}(k)] [G_{OO}(l) + G_{AA}(l)] \rangle_{k,l} +$$

$$\frac{1}{2\beta_A} \{ (|\gamma_k| (G_{OO}(k) - G_{AA}(k)))^2 + 4 (|\gamma_k| G_{OA}(k))^2 \}$$ (24)
where \((\ldots)_k\) denotes average over Brillouin zone. The minimization equations are:

\[
v^2 = -\frac{a_T}{\beta_A} - \frac{1}{\beta_A} \langle (2\beta_k + |\gamma_k|) G_{OO}(k) + (2\beta_k - |\gamma_k|) G_{AA}(k) \rangle_k
\]

(25)

\[
\left[ G(k)^{-1} \right]_{OO} = a_T + v^2 (2\beta_k + |\gamma_k|) + \left\langle \left( 2\beta_{k-l} + \frac{|\gamma_k| |\gamma_l|}{\beta_A} \right) G_{OO}(l) + \left( 2\beta_{k-l} - \frac{|\gamma_k| |\gamma_l|}{\beta_A} \right) G_{AA}(k) \right\rangle_l
\]

(26)

\[
\left[ G(k)^{-1} \right]_{AA} = a_T + v^2 (2\beta_k - |\gamma_k|) + \left\langle \left( 2\beta_{k-l} + \frac{|\gamma_k| |\gamma_l|}{\beta_A} \right) G_{AA}(l) + \left( 2\beta_{k-l} - \frac{|\gamma_k| |\gamma_l|}{\beta_A} \right) G_{OO}(k) \right\rangle_l
\]

(27)

\[
\left[ G(k)^{-1} \right]_{OA} = -\frac{G_{OA}(k)}{G_{OO}(k)G_{AA}(k) - G_{OA}(k)^2} = 4\frac{|\gamma_k|}{\beta_A} \langle |\gamma_l| G_{OA}(l) \rangle_l
\]

(28)

These equations look quite intractable, however they can be simplified. The crucial observation is that after we have inserted the phase \(\exp \left[ -i\theta_k/2 \right] \) in eq. (22) using our experience with perturbation theory, \( G_{AO} \) appears explicitly only on the right hand side of the last equation. It also implicitly appears on the left hand side due to a need to invert the matrix \( G \). Obviously \( G_{OA}(k) = 0 \) is a solution and in this case the matrix diagonalizes. However general solution can be shown to differ from this simple one just by a global gauge transformation. Subtracting eq. (23) from eq. (27) and using eq. (28), we observe that matrix \( G^{-1} \) has a form:

\[
G^{-1} \equiv \begin{pmatrix} E_O(k) & E_{OA}(k) \\ E_{OA}(k) & E_A(k) \end{pmatrix} = \begin{pmatrix} E(k) + \Delta_1 |\gamma_k| & \Delta_2 |\gamma_k| \\ \Delta_2 |\gamma_k| & E(k) - \Delta_1 |\gamma_k| \end{pmatrix}
\]

where \(\Delta_1, \Delta_2\) are constants. Substituting this into the gaussian energy one finds that it depends on \(\Delta_1, \Delta_2\) via the combination \(\Delta = \sqrt{\Delta_1^2 + \Delta_2^2}\) only. Therefore without loss of generality we can set \(\Delta_2 = 0\), thereby returning to the \( G_{OA} = 0 \) case [27].

Using this observation the gap equations significantly simplify. The function \( E(k) \) and the constant \(\Delta\) satisfy:

\[
E(k) = a_T + 2v^2\beta_k + 2 \left\langle \beta_{k-l} \left( \frac{1}{E_O(l)} + \frac{1}{E_A(l)} \right) \right\rangle_l + \beta_A \Delta = a_T - 2 \left\langle \beta_k \left( \frac{1}{E_O(k)} + \frac{1}{E_A(k)} \right) \right\rangle_k.
\]

(29)

(30)

The gaussian energy becomes:

\[
f = v^2 a_T + \frac{\beta_A}{2} v^4 + f_1 + f_2 + f_3
\]

\[
f_1 = \left\langle \log \left[ \frac{E_O(k)}{4\pi^2} \right] + \log \left[ \frac{E_A(k)}{4\pi^2} \right] \right\rangle_k
\]

(31)

\[
f_2 = -2 + \left\langle a_T \left( \frac{1}{E_O(k)} + \frac{1}{E_A(k)} \right) \right\rangle + v^2 \left[ (2\beta_k + |\gamma_k|) \frac{1}{E_O(k)} + (2\beta_k - |\gamma_k|) \frac{1}{E_A(k)} \right] \right\rangle_k
\]

\[
f_3 = \left\langle \beta_{k-l} \left[ \frac{1}{E_O(k)} + \frac{1}{E_A(k)} \left( \frac{1}{E_O(l)} + \frac{1}{E_A(l)} \right) \right] \right\rangle_{k,l} + \frac{1}{2\beta_A} \left\langle |\gamma_k| \left( \frac{1}{E_O(k)} - \frac{1}{E_A(k)} \right) \right\rangle_k^2
\]

(32)
Using eq. (29), a formula

$$\beta_k = \sum_{n=0}^{\infty} \chi^n \beta_n(k)$$

$$\beta_n(k) \equiv \sum_{|X| = na_\Delta} \exp[i k \cdot X]$$

derived in Appendix A and the hexagonal symmetry of the spectrum, one deduces that $E(k)$ can be expanded in "modes"

$$E(k) = \sum E_n \beta_n(k) \quad (33)$$

The integer $n$ determines the distance of a point on reciprocal lattice from the origin, see Fig. 4. and $\chi \equiv \exp[-a_\Delta^2/2] = \exp[-2\pi/\sqrt{3}] = 0.0265$. One estimates that $E_n \simeq \chi^n a_T$, therefore the coefficients decrease exponentially with $n$. Note (see Fig.4) that for some integers, for example $n = 2, 5, 6$, $\beta_n = 0$. Retaining only first $s$ modes will be called "the $s$ mode approximation". We minimized numerically the gaussian energy by varying $v, \Delta$ and first few modes of $E(k)$. The sample results for various $a_T$ and number of modes are given in Table 1.

| $a_T$  | 1 mode      | 2 modes     | 3 modes     |
|-------|-------------|-------------|-------------|
| -1000 | -446023.8395 | -431171.9948 | -431171.9757 |
| -300  | -40131.29217  | -38796.0277  | -38796.0297  |
| -100  | -4450.41636   | -4303.28685  | -4303.28593  |
| -50   | -1106.51575   | -1070.63806  | -1070.63791  |
| -20   | -171.678045   | -166.690727  | -166.690827  |
| -10   | -39.292885    | -38.433571   | -38.433645   |
| -5    | -7.3153440    | -7.2237197   | -7.2237422   |

We see that in the interesting region of not very low temperatures the energy converges extremely fast. In practice two modes are quite enough. The results for the gaussian energy are plotted on Fig.1 and will be compared with other approaches in section VI. Furthermore one can show that around $a_T < -4.6$, the gaussian liquid energy is larger than the gaussian solid energy. So naturally when $a_T < -4.6$, one should use the gaussian solid to set up a perturbation theory. For $a_T > -4.2$, there is no solution for the gap equations.

**C. 3D Abrikosov vortex lattice**

In 3D, we expand in bases of plan waves in the third direction times previously used quasi - momentum function:

$$\psi(x, z) = v \varphi(x) + \frac{1}{\sqrt{2}} \frac{(2\pi)^{3/2}}{2} \int_{k, k_z} \exp \left[-\frac{i \theta_k}{2}\right] \varphi_k(x) \exp i (k_z \cdot z) (O_k + iA_k). \quad (34)$$
The quadratic form is
\[
K = \frac{1}{8\pi\sqrt{2}} \int J_k \left( O_{k} G_{O}^{-1}(k) O_{-k} + A_{k} G_{A}^{-1}(k) A_{-k} \right)
\]  
(35)

where integration over \( k \) is understood as integration over Brillouin zone and over \( k_z \). Most of the derivation and important observations are intact. The modifications are following
\[
G_{O}^{-1}(k) = \frac{k^2}{2} + E_{O}(k)
\]
\[
G_{A}^{-1}(k) = \frac{k^2}{2} + E_{A}(k).
\]

The corresponding gaussian free energy density (after integration over \( k_z \)) is:
\[
f = v^2 a_T + \frac{\beta_A}{2} v^4 + f_1 + f_2 + f_3
\]
\[
f_1 = \left\langle \frac{1}{\sqrt{E_{O}(k)}} + \frac{1}{\sqrt{E_{A}(k)}} \right\rangle_k + \left\langle v^2 \left[ (2\beta_k + |\gamma_k|) \frac{1}{\sqrt{E_{O}(k)}} + (2\beta_k - |\gamma_k|) \frac{1}{\sqrt{E_{A}(k)}} \right] \right\rangle_k
\]
\[
f_2 = a_T \left\langle \frac{1}{\sqrt{E_{O}(k)}} + \frac{1}{\sqrt{E_{A}(k)}} \right\rangle_k + \left\langle v^2 \left[ (2\beta_k + |\gamma_k|) \frac{1}{\sqrt{E_{O}(k)}} + (2\beta_k - |\gamma_k|) \frac{1}{\sqrt{E_{A}(k)}} \right] \right\rangle_k
\]
\[
f_3 = \left\langle \beta_{k-l} \left[ \frac{1}{\sqrt{E_{O}(k)}} + \frac{1}{\sqrt{E_{A}(k)}} \right] \left[ \frac{1}{\sqrt{E_{O}(l)}} + \frac{1}{\sqrt{E_{A}(l)}} \right] \right\rangle_{k,l} + \frac{1}{2\beta_A} \left\langle |\gamma_k| \left( \frac{1}{\sqrt{E_{O}(k)}} - \frac{1}{\sqrt{E_{A}(k)}} \right) \right\rangle_k^2 .
\]
(36)

Minimizing the above energy, gap equations similar to that in 2D can be obtained. One finds that
\[
E_{O}(k) = E(k) + \Delta |\gamma_k| ,
\]
\[
E_{A}(k) = E(k) - \Delta |\gamma_k| .
\]

\( E(k) \) can be solved by modes expansion two. We minimized numerically the gaussian energy by varying \( v, \Delta \) and first few modes of \( E(k) \). The sample results of free energy density for various \( a_T \) with 3 modes are given in Table 2.

| Table 2. |
| --- |
| Mode expansion 3D. |
| \( a_T \) | -300 | -100 | -50 | -30 | -20 | -10 | -5.5 |
| \( f \) | -38757.2294 | -4283.2287 | -1057.6453 | -372.2690 | -159.5392 | -33.9873 | -6.5103 |

In practice two modes are also quite enough in 3D. As in the case of 2D, one can show that around \( a_T < -5.5 \), the gaussian liquid energy is larger than the gaussian solid energy. So naturally when \( a_T < -5.5 \), one should use the gaussian solid to set up a perturbation theory in 3D. When around \( a_T > -5 \), there is no solution for the gap equations.
V. CORRECTIONS TO THE GAUSSIAN APPROXIMATION

In this section, we calculate the lowest order correction to the gaussian approximation (that will be called postgaussian correction), which will determine the precision of the gaussian approximation. This is necessary in order to fit experiments and compare with low temperature perturbation theory and other nonperturbative methods. First we review a general idea behind calculating systematic corrections to the gaussian approximation [25]. The procedure is rather similar to calculating corrections to the Hartree-Fock approximations used in fermionic systems. Gaussian variational principle provided us with the best (in a certain sense) quadratic part of the free energy from which the "steepest descent" corrections can be calculated. The free energy is divided into the quadratic part and a "small" perturbation $\tilde{V}$. For a general scalar theory defined in eq. (20) it takes a form:

$$f = K + \alpha \tilde{V}$$

$$K = \frac{1}{2} \phi^a G^{-1ab} \phi^b$$

$$\tilde{V} = \frac{1}{2} \phi^a D^{-1} \phi^a + V(\phi^a) - \frac{1}{2} \phi^a G^{-1ab} \phi^b.$$  (38)

Here the auxiliary parameter $\alpha$ was introduced to set up a perturbation theory. It will be set to one at the end of calculation. Expanding logarithm of the statistical sum in powers of $\alpha$

$$Z = \int \mathcal{D}\phi^a \exp(-K) \exp(-\alpha \tilde{V}) = \int \mathcal{D}\phi^a \sum_{n=0}^{1} \frac{1}{n!} \left(\alpha \tilde{V}\right)^n \exp(-K),$$  (39)

one retains only first few terms. It was shown in refs. [26] that generally only two - particle irreducible diagrams contribute to the postgaussian correction. The gaussian approximation corresponds to retaining only first two terms, $n = 0, 1$, while the postgaussian correction retains in addition the contribution of order $\alpha^2$.

Feynman rules in our case are shown on Fig.5. We have two propagators for fields $A$ and $O$ and three and four leg vertices. Using these rules the postgaussian correction is presented on Fig. 6 as a set of two and three loop diagrams. The corresponding expressions are given in Appendix B. The Brillouin zone averages were computed numerically using the three modes gaussian solution of the previous section. Now we turn to discussion of the results.

VI. RESULTS, COMPARISON WITH OTHER APPROACHES AND CONCLUSION.

Results for LLL scaled energy, magnetization and specific heat in 2D are presented on Fig. 1, 2 and 3 respectively.

A. Energy

The gaussian energy provides a rigorous upper bound on free energy [25]. Fig.1a shows the 2D gaussian energy (the dash - dotted line), which in the range of $a_T$ from $-30$ to $-10$ is
just above the mean field (the solid gray line). This is because it correctly accounts for the (positive) logarithmic one loop correction of eq. (17). In contrast the results of the theory by Tesanovic et al [14] (the dashed gray line) are lower than the mean field. This reflects the fact that although the correct large $|a_T|$ limit is built in, the expansion of the expression eq. (19) gives negative coefficient of the log $|a_T|$ term. This is inconsistent with both the low temperature perturbation theory and the gaussian approximation. The difference between this theory and our result is smaller than 2% only when $a_T < 30$ or at small $a_T$ below the 2D melting line (which occurs at $a_T = -13$ according to Monte Carlo [19] and phenomenological estimates [18,1]) where the lines become closer again. It never gets larger than 10% though. To effectively quantitatively study the model one has to subtract the dominant mean field contribution. This is done in the inset of Fig. 1a. We plot the gaussian result (the dash-dotted line), the one loop perturbative result (the solid line) and eq. (19) (the dashed gray line) in an expanded region $-100 < a_T < -10$. The gaussian approximation is a bit higher than the one loop. To determine the precision of the gaussian approximation and compare with the perturbative two loop result, we further subtracted the one loop contribution on Fig.1b. As expected the postgaussian result is lower than the gaussian though higher than the two loop. The difference between the gaussian and the postgaussian approximation in the region shown is about $|\Delta f| = 0.2$, which translates into 0.2% at $a_T = -30, 0.4$% at $a_T = -20$ and 2% at $a_T = -12$. The fit for the gauss and postgaussian energy in the region $-30 < a_T < -6$ are

$$f_{g2D} = -\frac{a_T^2}{2\beta_A} + 2\log|a_T| + 0.119 - \frac{19.104}{a_T} - \frac{60.527\log|a_T|}{a_T^2} + \frac{36.511}{a_T^2} + c_v$$

$$f_{pg2D} = -\frac{a_T^2}{2\beta_A} + 2\log|a_T| + 0.068 - \frac{11.68}{a_T} - \frac{60.527\log|a_T|}{a_T^2} + \frac{38.705}{a_T^2} + c_v.$$  

In 3D, similarly one found that

$$f_{g3D} = -\frac{a_T^2}{2\beta_A} + 2.84835|a_T|^{1/2} + \frac{3.1777}{a_T} - \frac{0.8137\log^2[-a_T]}{a_T} \quad (40)$$

### B. Magnetization

The dimensionless LLL magnetization is defined as

$$m(a_T) = -\frac{df_{eff}(a_T)}{da_T} \quad (41)$$

and the measure magnetization is

$$4\pi M = -\frac{e^*\hbar}{cm_{ab}} \langle |\psi|^2 \rangle = -\frac{e^*\hbar}{cm_{ab}} |\psi_r|^2 - \frac{b'}{2\alpha Tc} \sqrt{\frac{b\omega}{4\pi}} \quad (42)$$

where $\psi$ is the order parameter of the original model, and $\psi_r$ is the rescaled one, which is equal to $\frac{df_{eff}(a_T)}{da_T}$. Thus
\[ 4\pi M = \frac{e^* h}{cm_{ab}} b' \sqrt{\frac{b\omega}{4\pi}} m(a_T). \] (43)

We plot the scaled magnetization in region \(-30 < a_T < -6\). Again, the mean field contribution dominates, so we subtract it in Fig.2. The solid line is the one loop approximation, while the gray line is the two loop approximation. At small negative \(a_T\) the postgaussian (the upper gray dash-dotted line) is very close to the two loop result, while the gaussian approximation (the dash-dotted line) is a bit lower. All of these lines are above mean field. On the other hand, the result of ref. [14] (the gray dashed line) is below the mean field. Magnetization jump at the melting point is smaller than our precision of 2% at \(a_T = -12\). Our result for the gaussian magnetization and the postgaussian correction in this range can be conveniently fitted with

\[
m_{g2D} = \frac{a_T}{\beta_A} - \frac{2}{a_T^2} - \frac{19.10}{a_T^3} + \frac{133.55}{a_T^4} - \frac{121.05 \log[-a_T]}{a_T^3},
\]

\[
\Delta m_{pg2D} = \frac{7.525}{a_T^2} - \frac{59.15}{a_T^3} + \frac{43.64 \log[-a_T]}{a_T^3}
\]

respectively.

Similar discussion for the case of 3D can be deduced from eqs. (41), eq. (40) and

\[ 4\pi M = -\frac{e^* h}{cm_{ab}} \langle |\psi|^2 \rangle = -\frac{e^* h}{cm_{ab}} |\psi_r|^2 \left( \frac{b\omega}{2\alpha T_c} \right) \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{2/3}
\]

\[ = \frac{e^* h}{cm_{ab}} b' \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{2/3} m(a_T), \] (44)

where the gaussian scaled magnetization can be obtained by differentiation of eq. (41). We didn’t attempt to calculate the postgaussian correction in 3D.

C. Specific heat

The scaled LLL specific heat is defined as

\[ c(a_T) = -\frac{d^2 f_{eff}(a_T)}{da_T^2}. \] (45)

and the original specific heat is related to the scaled specific heat \(c\) in 2D via

\[
C = \frac{1}{4\pi \xi^2 T} \left[ -b + \sqrt{\frac{\pi h^2 \alpha T_c}{2m^* b'T}} -3t - 1 + \frac{\pi h^2 \alpha T_c (-t - 1 + b)^2}{m^* b'T} \right] c(a_T)
\]

We plot the scaled specific heat divided by the mean field value \(c_m = 1/\beta_A\) in the range \(-30 < a_T < -6\) on Fig.3. The solid line is one loop approximation, while the gray line is the two loop approximation. At large \(|a_T|\) the postgaussian (gray dashed-dotted line) is very close to the one loop result. Finally the gaussian approximation (dash-dotted line) is a bit lower. All these lines are slightly above mean field. On the contrary, the result of
ref. [14] (dashed gray line) is below the mean field. Our gaussian result and its correction in this range can be conveniently fitted with:

\[
\frac{c_g}{c_{mf}} = 1 + \beta_A \left( \frac{2}{a_T^2} + \frac{38.2}{a_T^4} - \frac{521.7}{a_T^6} + \frac{363.2 \ln(-a_T)}{a_T^8} \right)
\]

\[
\frac{\Delta c_g}{c_{mf}} = \beta_A \left( -\frac{15.05}{a_T^2} + \frac{221.1}{a_T^4} - \frac{130.9 \ln(-a_T)}{a_T^6} \right).
\]

Qualitatively the gaussian specific heat is consistent with experiments [4] which show that the specific heat first raise before dropping sharply beyond the melting point.

D. Conclusions

In this paper, we applied the gaussian variational principle to the problem of thermal fluctuations in vortex lattice state. Then the correction to it was calculated perturbatively. This generalizes corresponding treatment of fluctuations in the homogeneous phase (vortex liquid) by Thouless and coworkers [5]. Also umklapp processes were included in the low temperature two loop perturbation theory expression. The results of gaussian perturbative and some nonperturbative approaches were compared. We hope that increased sensitivity of both magnetization and specific heat experiments will test the precision of the theory.

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APPENDIX A:

In this Appendix the basic definitions are collected. Brillouin zone averages of products of four quasi - momentum functions are defined by:

\[
\begin{align*}
\beta_k &= \langle |\varphi|^2 \varphi_{-k}^* \varphi_k^* \rangle \\
\gamma_k &= \langle (\varphi^*)^2 \varphi_{-k}^* \varphi_k^* \rangle \\
\gamma_{k,l} &= \langle \varphi_{k^*}^* \varphi_{-k}^* \varphi_{-l}^* \varphi_l^* \rangle.
\end{align*}
\]

We also need a more general product \(\langle \varphi_{k_1}^* \varphi_{k_2} \varphi_{k_3}^* \varphi_{k_4} \rangle\) in order to calculate postgaussian corrections. This is just a perturbative four - leg vertex:

\[
\langle \varphi_{k_1}^* \varphi_{k_2} \varphi_{k_3}^* \varphi_{k_4} \rangle = \exp \left[ \frac{i\pi^2}{2} \left( n_1^2 - n_1 \right) + \frac{2\pi i n_1 n_3 k_{3y}}{a_\Delta} \right] \\delta^q \left[ k_1 - k_2 + k_3 - k_4 \right] \lambda \left[ k_1 - k_2, k_2 - k_4 \right],
\]

\[
\lambda \left[ l_1, l_2 \right] = \sum_Q \exp \left[ -\frac{|l_1 + Q|^2}{2} \right] + i \left( l_{1x} + l_{2x} \right) Q_y - i \left( l_{1y} + l_{2y} \right) Q_x \right] \\
\times \exp \left[ i \left( l_{1x} + l_{2x} \right) l_{3y} \right],
\]

14
where are reciprocal lattice vectors:
\[ Q = m_1 \tilde{d}_1 + m_2 \tilde{d}_2. \] (A3)

Here \( \mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 = n_1 \tilde{\mathbf{d}}_1 + n_2 \tilde{\mathbf{d}}_2 \) is assumed and the basis of reciprocal lattice is
\[ \tilde{\mathbf{d}}_1 = \frac{2\pi}{a_\Delta} (1, -\frac{1}{\sqrt{3}}); \quad \tilde{\mathbf{d}}_2 = \left( 0, \frac{4\pi}{a_\Delta \sqrt{3}} \right), \quad a_\Delta = \frac{2\pi}{\sqrt{3}}. \] It is dual to the lattice \( \mathbf{e}_1 = (a_\Delta, 0), \) \( \mathbf{e}_2 = \left( \frac{2\pi}{\sqrt{3}}, \frac{2\pi \sqrt{3}}{\sqrt{3}} \right). \) When \( \mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 \neq n_1 \tilde{\mathbf{d}}_1 + n_2 \tilde{\mathbf{d}}_2 \) the quantity vanishes. The delta function differs from the Kroneker:
\[ \delta^n[k] = \sum_Q \delta[k + Q]. \]

From the above formula, one gets the following expansion of \( \beta_k \):
\[ \beta_k = \sum_{m_1, m_2} \exp \left[ -\frac{|\mathbf{X}|^2}{2} + i \mathbf{k} \cdot \mathbf{X} \right] \]
\[ = \sum_n \exp \left[ -\frac{a_\Delta^2}{2} n \right] \beta_n(k) \]

where \( \mathbf{X} = m_1 \mathbf{d}_1 + m_2 \mathbf{d}_2 \).

To simplify the minimization equations we used the following general identity. Any sixfold (\( D_6 \)) symmetric function \( F(k) \) (namely a function satisfying \( F(k) = F(k') \), where \( k, k' \) is related by a \( \frac{2\pi}{6} \) rotation) obeys:
\[ \int F(k) \gamma_{k,t} = \frac{\gamma_t}{\beta_A} \int |F(k)|^2. \] (A5)

This can be seen by expanding \( F \) in Fourier modes and symmetrizing.

**APPENDIX B:**

In this Appendix we specify Feynman rules and collect expressions for diagrams. The solid line Fig.5a represents \( O \) and the dashed line Fig.5b represents \( A \). Fig.5c is a vertex with three \( O \). In the coordinate space, it is \( 2v \left[ \mathcal{O}O(O)^A + c.c. \right] \). And Fig.5e is
\[-2i \varphi^+ \varphi^2 A^+ - 4i \varphi \varphi^3 O^+ A^+ + c.c. \] Fig.5g is \( \frac{1}{2} |O(x)|^4 \). Fig.5h is \( 2OO^+ (AO^+ - OA^+) \). Fig.5i is \( 4OO^+ AA^+ - \left[ O^2 (A^+)^2 + c.c. \right] \).

Other vertices, for example, formulas for diagrams Fig.5e, f, j and k, can be obtained by substituting \( O \rightarrow i\mathbf{A}, A \rightarrow -i\mathbf{O} \) from formulas for diagrams Fig.5d, c, h and g respectively. The propagator in coordinate space can be written as
\[ \langle O^+ (x) O^+ (y) \rangle = 4\pi \int_k E_O (k) \varphi^*_k (x) \varphi^*_k (y) = 4\pi P^+_O (x, y), \]
\[ \langle O (x) O (y) \rangle = 4\pi \int_k E_O (k) \varphi_k (x) \varphi_k (y) = 4\pi P^-_O (x, y), \] (B1)
\[ \langle O (x) O^+ (y) \rangle = 4\pi \int_k E_O (k) \varphi_k (x) \varphi^*_k (y) = 4\pi P_O (x, y) \]
Functions $P_A^+(x, y), P_A^-(x, y), P_A(x, y)$ can be defined similarly. One finds three loops contribution to free energy from two $OOOO$ vertex contraction, see Fig. 6a, $-\frac{1}{16(2\pi)^5} \int_x \langle f_{oooo} \rangle_y$:

$$f_{oooo} = 4 \left( |P_O|^4 + |P_O^+|^4 + 4 |P_O P_O^+|^2 \right)_y.$$ (B2)

Coordinates are not written explicitly since all of them are the same $P_O(x, y)$ etc.

The contribution from the diagrams Fig. 6b is $-\frac{1}{16(2\pi)^5} \int_x \langle f_{oooa} \rangle_y$, and

$$f_{oooa} = |P_O|^2 \left( -16 P_O^+ P_A^- + 8 P_O^* P_A^* \right) + |P_O^+|^2 \left( -8 P_O^+ P_A^- + 16 P_O P_A^* \right) + \text{c.c.}$$ (B3)

The diagrams Fig. 6c is $-\frac{1}{16(2\pi)^5} \int_x \langle f_{ooaa} \rangle_y$ and

$$f_{ooaa} = 16 \left( |P_O|^2 + |P_O^+|^2 \right) \left( |P_A|^2 + |P_A^+|^2 \right) + 4 \left( |P_O^+|^2 [P_A^-]^2 + P_O^2 [P_A^+]^* + \text{c.c.} \right) - 32 \left( P_O P_O P_O^+ P_A + \text{c.c.} \right)$$ (B4)

The diagrams Fig. 6f is $-\frac{v^2}{16(2\pi)^5} \int_x \langle f_{oooo} \rangle_y$ and

$$f_{oooo} = |P_O|^2 \left( 16 P_O^+ \varphi (x) \varphi (y) + 8 P_O^* \varphi (x) \varphi^* (y) + \text{c.c.} \right) + |P_O^+|^2 \left( 8 P_O^+ \varphi (x) \varphi (y) + 16 P_O \varphi (x) \varphi^* (y) + \text{c.c.} \right)$$ (B5)

The diagrams Fig. 6h is $-\frac{v^2}{16(2\pi)^5} \int_x \langle f_{oooa} \rangle_y$ and

$$f_{oooa} = -8 \left( P_O^2 \right)^2 P_A^+ \varphi^* (x) \varphi^* (y) - 16 \left( |P_O|^2 + |P_O^+|^2 \right) \times \left( P_A^+ \varphi (x) \varphi (y) - P_A \varphi^* (x) \varphi (y) \right) + 8 P_O^2 P_A^+ \varphi (x) \varphi^* (y) + 8 P_O^2 P_A^+ \varphi^* (x) \varphi^* (y)$$

$$- 32 P_O \left[ P_A^+ \varphi^* (x) \varphi (y) - P_A^+ \varphi^* (x) \varphi^* (y) \right] + \text{c.c.}$$ (B6)

Other contributions, Fig. 6e, d, i, g can be obtained by substituting

$P_O \leftrightarrow P_A, P_A^+ \leftrightarrow -P_A^-, P_A^- \leftrightarrow -P_O^-$ in eq. (B2), eq. (B3), eq. (B5) and eq. (B6).
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[27] Note that similar simplification occurs in the large $N$ approach developed in ref. [23]. Consequently their Ansatz is in fact a rigorous solution to their gap equations.
Figure captions

Fig. 1a
Scaled free energy of vortex solid. From top to bottom, gaussian approximation (dash dotted line), mean field (solid line), theory ref. [14] (dashed line). Inset: corrections to mean field calculated using (from top to bottom) gaussian (dash - dotted line), one loop perturbation theory (dotted line) and theory ref. [14] (dashed line).

Fig. 1b
More refined comparison of different approximations to free energy. Mean field as well as the one loop perturbative contributions are subtracted.

Fig. 2
Thermal fluctuations correction to magnetization of vortex solid. From top to bottom, one and two loop perturbation theory (solid lines ”p1” and ”p2” respectively), gaussian and postgaussian approximation (dash dotted lines ”g” and ”pg” respectively), theory ref. [14] (dashed line ”t”).

Fig. 3
Scaled specific heat eq.(45) normalized by the mean field. One and two loop perturbation theory (solid lines ”p1” and ”p2” respectively), gaussian and postgaussian approximation (dash dotted lines ”g” and ”pg” respectively), theory ref. [14] (dashed line ”t”).

Fig. 4
Reciprocal hexagonal lattice points X belonging to three lowest order ”stars” in the mode expansion of $\beta_k$.

Fig. 5
Feynman rules of the low temperature perturbation theory. The solid line (a) denoted the O mode propagator, while the dashed line (b) denotes the A mode propagator. Various three leg and four leg vertices are presented on (c-f) and (g-k) respectively.

Fig. 6
Contributions to the postgaussian correction to free energy.