On the dynamic instability of a rectangular plate with one free edge

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Abstract. The problem of dynamic stability of a rectangular plate is considered. It is supposed that two opposite sides of the plate are freely supported, the third side is either hingedly supported or rigidly clamped, and the fourth side is free and carries a finite mass. The plate is compressed by a tracking load applied to the last side. The dependence of the critical load on the relative length of the plate, i.e., the distance between sides with different conditions, is obtained.

1. Introduction
A. Yu. Ishlinskii [1] investigated the stability problem of a thin rectangular plate with two opposite free and two freely supported edges when a compressive load is applied to the freely supported sides. In fact, this work for the first time showed the possibility of the existence of a localized instability near the free edges of the plate. Later, this question was studied in a number of articles [2–6]. In the case where a compressive load is applied to the freely supported sides, there is a complete analogy with the problem of plate vibrations [7]. The equations determining the critical loads and the vibration frequencies of localized instability and localized vibrations coincide. But when the load is applied to a free edge, there is no analogy with the problem of vibrations. This raises the natural question of determining the conditions for the onset of localized instability in the case where a compressive load is applied to free edges. In this aspect, we can, for example, refer to [8]. In [9], the problem of localized instability of a rectangular plate is considered, where a compressive load is applied to the free side, the two opposite sides of the plate are freely supported, and the fourth side is either freely supported or rigidly clamped. It is shown that if a uniformly distributed tracking load acts on the free side, then no localized instability can arise in any of the cases considered.

In this paper, we consider the problem stated in [9], where the compressive load is tracking and the possibility of the appearance of dynamic instability of an elastic rectangular plate is investigated in Bolotin’s formulation [10].

2. Statement of the problem
Let a thin elastic plate of constant thickness $2h$ in the Cartesian coordinate system $(x, y, z)$ occupy the area $0 \leq x \leq a, 0 \leq y \leq b, -h \leq z \leq h$. According to [10], the equation of plate stability can be written as

$$
\Delta^2 w(x, y, t) + \alpha^2 \frac{\partial^2 w(x, y, t)}{\partial x^2} = 0, \quad \alpha^2 = \frac{P}{D}.
$$

(1)
Here \( w(x, y, t) \) is the deflection whose dependence on the time \( t \) is contained only in the boundary conditions, \( P \) is the intensity of uniformly distributed compressive tracking load, and \( D \) is the bending rigidity of the plate.

Assuming that the plate edges \( y = 0, b \) are freely supported, we have the following boundary conditions on this edges:

\[
\begin{align*}
  w &= 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{for} \quad y = 0, b. 
\end{align*}
\]

(2)

On the free edge \( (x = 0) \), where a uniform distributed mass \( m \) is assumed to be located, we have the conditions

\[
\begin{align*}
  &\frac{\partial^2 w(x, y, t)}{\partial x^2} + \nu \frac{\partial^2 w(x, y, t)}{\partial y^2} = 0,
  \\
  &\frac{\partial}{\partial x} \left[ \frac{\partial^2 w(x, y, t)}{\partial x^2} + (2 - \nu) \frac{\partial^2 w(x, y, t)}{\partial y^2} \right] = -\frac{m}{D} \frac{\partial^2 w(x, y, t)}{\partial t^2}.
\end{align*}
\]

(3)

The last conditions are written in the case where the external load is tracking. The conditions on the edge \( x = a \) will be given below.

The solution of equation (1), satisfying conditions (2) and taking into account the presence of an inertial term in (3), can be written as

\[
\begin{align*}
  w(x, y, t) &= e^{i\omega t} \sum_{n=1}^{\infty} f_n(x) \sin \lambda_n y, \quad \lambda_n = \frac{n\pi}{b}.
\end{align*}
\]

(4)

Substituting (4) in equation (1), we obtain a sequence of ordinary differential equations for the functions \( f_n(x) \), whose solutions can be written in the form [9]

\[
\begin{align*}
  f_n(x) &= [A_n \sin(\lambda_n s_2 x) + B_n \cos(\lambda_n s_2 x)] \sinh(\lambda_n s_1 x) \\
  &\quad + [D_n \sin(\lambda_n s_2 x) + C_n \cos(\lambda_n s_2 x)] \cosh(\lambda_n s_1 x),
\end{align*}
\]

(5)

where \( A_n, B_n, C_n, D_n \) are arbitrary constants,

\[
\begin{align*}
  s_1 &= \sqrt{1 - \frac{\alpha_n^2}{2}}, \quad s_2 = \frac{\alpha_n}{\sqrt{2}}, \quad \alpha_n^2 = \frac{P}{2D\lambda_n^2}.
\end{align*}
\]

Conditions (3) for the functions \( f_n(x) \) become

\[
\begin{align*}
  &f_n''(x) - \nu \lambda_n^2 f_n(x) = 0,
  \\
  &f_n''(x) - (2 - \nu) \lambda_n^2 f_n'(x) - \frac{m\omega^2}{D} f_n(x) = 0.
\end{align*}
\]

Substituting representation (5) in these conditions, we obtain

\[
\begin{align*}
  &\lambda_n^2 [C_n(s_1^2 - s_2^2) + 2A_n s_1 s_2] - \nu \lambda_n^2 C_n = 0,
  \\
  &B_n s_1(s_1^2 - 3s_2^2) - D_n s_2(s_2^2 - 3s_1^2) - (2 - \nu)(B_n s_1 + D_n s_2) - \eta C_n = 0, \quad \eta = \frac{m\omega^2}{D\lambda_n^3}.
\end{align*}
\]

(6)

From (6) we express \( A_n \) and \( D_n \) through \( B_n \) and \( C_n \) to write the desired function in the form

\[
\begin{align*}
  f_n(x) &= B_n \left[ \cos(\lambda_n s_2 x) \sinh(\lambda_n s_1 x) + \frac{s_1(1 - \nu + 4s_2^2)}{s_2(1 + \nu - 4s_2^2)} \sin(\lambda_n s_2 x) \cosh(\lambda_n s_1 x) \right] \\
  &\quad + C_n \left\{ -\frac{1 - \nu - 2s_2^2}{2s_1 s_2} \sin(\lambda_n s_2 x) \sinh(\lambda_n s_1 x) \\
  &\quad + \left[ \frac{\eta}{s_2(1 + \nu - 4s_2^2)} \sin(\lambda_n s_2 x) + \cos(\lambda_n s_2 x) \right] \cosh(\lambda_n s_1 x) \right\}.
\end{align*}
\]

(7)
Thus, we obtain a general solution satisfying the boundary conditions on three edges of the plate and containing two arbitrary constants whose choice allows us to satisfy the conditions on the fourth side. According to the title of the paper, different boundary conditions will be considered on this side, and hence we consider each of these cases separately.

3. Freely supported edge

In this case, on the edge $x = a$, we have the conditions

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad \text{for} \quad x = a. \quad (8)$$

Substituting expression (7) in conditions (8), we obtain a homogeneous system of linear algebraic equations. For the existence of a nontrivial solution, i.e., nonzero values of the constants $B_n$ and $C_n$, it is necessary that the determinant of the matrix of the obtained system be equal to zero. Satisfying this condition, we come to the equation

$$-s_1[1 + \beta - (2 + \beta)\nu + \nu^2 - 4s_2^2]\sin(2\zeta_2)$$
$$-s_2[3 + \beta - (2 - \beta)\nu - \nu^2 - 4s_2^2]\sinh(2\zeta_1) + \eta\{2s_1s_2[\cosh(2\zeta_1) - \cos(2\zeta_2)]\} = 0,$$

where the following notation is adopted

$$\zeta_1 = \lambda_n s_1 a, \quad \zeta_2 = \lambda_n s_2 a.$$

From here we derive

$$\eta = \frac{s_1[(1 - \nu)^2 - 4s_2^2\nu]\sin(2\zeta_2) - s_2[(1 + \nu)^2 - 4 - 4s_2^2\nu]\sinh(2\zeta_1)}{2s_1s_2[\cosh(2\zeta_1) - \cos(2\zeta_2)]}. \quad (9)$$

If the representation $\eta = m\omega^2/(D\lambda_n^3)$ is taken into account, then the implementation of the condition $\eta > 0$ is necessary for the dynamic stability of the plate. It is obvious that if the external load is absent ($\alpha_n = 0$), then the plate is stable and, consequently, as the parameter $\alpha_n$ increases, the loss of stability can occur when either the numerator or the denominator in expression (9) vanishes for the first time. In this case, the condition that the numerator is zero coincides with the equation for determining the critical load for the static loss of stability, which was obtained in [9].

In order to determine the critical load leading to the loss of stability, we equate the numerator and the denominator of expression (9) to zero to obtain the system of equations

$$s_1[(1 - \nu)^2 - 4s_2^2\nu]\sin(2\zeta_2) + s_2[4 - (1 + \nu)^2 + 4s_2^2\nu]\sinh(2\zeta_1) = 0, \quad (10)$$
$$\cosh(2\zeta_1) - \cos(2\zeta_2) = 0. \quad (11)$$

It is easy to verify that none of the equations has roots satisfying the condition $\alpha_n^2 < 2$, and therefore the localized instability can not occur. For $\alpha_n^2 > 2$, equations (10)–(11) become

$$\sqrt{\frac{\alpha_n^2}{2}} - 1[(1 - \nu)^2 - 2\alpha_n^2\nu]\sin(\sqrt{2\pi n}a^*\alpha_n)$$
$$+ \frac{\alpha_n}{\sqrt{2}}[4 - (1 + \nu)^2 + 2\alpha_n^2\nu]\sin \left(\frac{2\pi n}{\sqrt{2}}\alpha_n^* \sqrt{\frac{\alpha_n^2}{2} - 1}\right) = 0, \quad (12)$$
$$\cos \left(\frac{2\pi n}{\sqrt{2}}a^* \sqrt{\frac{\alpha_n^2}{2} - 1}\right) - \cos(\sqrt{2\pi n}a^*\alpha_n) = 0, \quad (13)$$

where $a^* = a/b$. 

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Figure 1. Dependence of the critical tracking load on the relative length of the plate: curve 1 corresponds to the first form of the loss of stability, and curve 2 corresponds to the second form of the loss of stability.

The solution of equation (13) is the sequence

$$\frac{\alpha_n}{\sqrt{2}} = \frac{k^2 + (a^*n)^2}{2a^*nk} \quad (k = 1, 2, 3 \ldots),$$

which is not arranged in ascending order, and the ordinal number of its smallest term depends essentially on the parameter $a^*$.

The roots of equation (12) exceed the roots (14).

We introduce a dimensionless parameter $\delta \equiv \sqrt{Pb^2/(4\pi^2D)} = (\alpha_n n)/\sqrt{2}$ characterizing the external load. Then, from (14), for each form ($n = 1, 2, 3 \ldots$), we have

$$\delta = \min_k k^2 + (a^*n)^2 \quad 2a^*k.$$

It is obvious that we have the smallest critical load at $n = 1$, but, depending on $a^*$, for different values of the parameter $k$.

The graphs of the dependence of $\delta$ on the parameter $a^*$ for the first two forms of the loss of stability are shown in figure 1.

It is easy to verify that the transition from curve $k$ to curve $k + 1$ occurs at the following values of the parameter $a^*$:

$$a^* = \frac{\sqrt{k(k + 1)}}{n} \quad (k, n = 1, 2, 3 \ldots),$$

and at these points, we have

$$\delta = \frac{2k + 1}{2\sqrt{k(k + 1)}n}.$$

It is interesting to note that, for $n = 1$, the value $a^*$ is the geometric mean of two successive natural numbers, and $\delta$ is the ratio of their arithmetic to the geometric mean.
4. Rigidly clamped edge

In this case, on the edge $x = a$, we have the conditions

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \text{for } x = a,$$

Satisfying the last conditions, we obtain the homogeneous system of linear algebraic equations

$$B_n \left[ \sinh \zeta_1 \cos \zeta_2 + \frac{s_1}{s_2} \frac{1 - \nu + 4s_2^2}{1 + \nu - 4s_2^2} \cosh \zeta_1 \sin \zeta_2 \right]$$

$$+ C_n \left[ \frac{s_2}{s_2(1 + \nu - 4s_2^2)} \cosh \zeta_1 \sin \zeta_2 - \frac{1 - \nu - 2s_2^2}{2s_1s_2} \sinh \zeta_1 \sin \zeta_2 \right] = 0,$$

$$B_n \left[ \frac{2(1 - \nu + 2s_2^2)}{1 + \nu - 4s_2^2} \sin \zeta_2 \sinh \zeta_1 + \frac{4s_1s_2}{1 + \nu - 4s_2^2} \cos \zeta_2 \cos \zeta_1 \right] + C_n \left[ (1 + \nu) \frac{s_2}{s_1} \cos \zeta_2 \sinh \zeta_1 \right.$$

$$- (1 - \nu) \sin \zeta_2 \cosh \zeta_1 + \frac{2\eta s_2}{1 + \nu - 4s_2^2} \cos \zeta_2 \cosh \zeta_1 + \frac{s_1}{s_2} \sinh \zeta_1 \sin \zeta_1 \left. \right] = 0.$$

From the condition that the determinant of the matrix of this system is zero we obtain the equation

$$2\eta s_1 s_2 [s_1 \sin (2\zeta_2) - s_2 \sinh (2\zeta_1)] + (1 - \nu)^2 + 4s_2^2(1 - 2s_2^2)$$

$$- s_1^2 [(1 - \nu)^2 - 4s_2^2\nu] \cos (2\zeta_2) - s_2^2 [(1 + \nu)^2 - 4s_2^2\nu] \cosh (2\zeta_1) = 0.$$

As in the previous case, to determine the critical load leading to the loss of stability, we obtain the system of equations

$$(1 - \nu)^2 + 4s_2^2(1 - 2s_2^2) - s_1^2 [(1 - \nu)^2 - 4s_2^2\nu] \cos (2\zeta_2) - s_2^2 [(1 + \nu)^2 - 4s_2^2\nu] \cosh (2\zeta_1) = 0, \quad (15)$$

$$s_2 \sinh (2\zeta_1) - s_1 \sin (2\zeta_2) = 0.$$

The first equation in (15) is a dispersion equation in static formulation and has a unique positive root $\alpha_n = \sqrt{2}$ which, as is easy to verify, leads only to the trivial, zero solution of the problem under study. The second equation in (15) does not have roots satisfying the condition $\alpha_n^2 < 2$; under the assumption that $\alpha_n^2 > 2$, it becomes an equation that, as an expression for the parameter $\delta$, has the form

$$\delta \sin (2\pi a^* \sqrt{\delta^2 - n^2}) - \sqrt{\delta^2 - n^2} \sin (2\pi a^* \delta) = 0.$$

The dependence of the parameter $\delta$ on the relative length $a^*$ was found by solving this equation numerically. For a comparative analysis, in figure 2, this dependence for the first form $n = 1$ is shown together with a similar curve in the case of free support of the edge $x = a$.

As can be seen from the graphs in figure 2, the critical value of the compressive load in the case where the fourth side is rigidly clamped, as expected, exceeds the analogous value for the freely supported side, and this excess is most significant at small values of the relative length. Of special interest are the values of the relative length $a^*$ equal to the geometric mean of two successive natural numbers, when the critical values of the compressive load coincide regardless of the type of the fourth side fixation.
Figure 2. Dependence of the critical tracking load on the relative length of the plate: curve 1 corresponds to the free support, and curve 2 corresponds to the rigid clamping of the edge $x = a$.

Conclusions
The possibility of the occurrence of instability of a rectangular plate is investigated in dynamic formulation. It is assumed that one side of the plate is not supported, two opposite sides are freely supported, and various fixation conditions can be posed on the fourth side, – free support or rigid clamping. It is shown that when a uniformly distributed compressive tracking load acts on the free edge, the loss of stability occurs in both cases, while the critical value of the compressive load in the second case exceeds the corresponding value in the first case, and these values coincide only for certain values of the relative length. The minimum values of the relative length of the plate, at which the loss of stability may begin, are determined by the strength characteristics of the plate material.

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