A curious differential calculus on the quantum disc and cones

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Abstract. A non-classical differential calculus on the quantum disc and cones is constructed and the associated integral is calculated.

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1. Introduction

The aim of this note is to present a two-dimensional differential calculus on the quantum disc algebra, which has no counterpart in the classical limit, but admits a well-defined (albeit different from the one in [2]) integral, and restricts properly to the quantum cone algebras. In this way the results of [3] are extended to other classes of non-commutative surfaces and to higher forms. The presented calculus is associated to an orthogonal pair of skew-derivations, which arise as a particular example of skew-derivations on generalized Weyl algebras constructed recently in [1]. It is also a fundamental ingredient in the construction of the Dirac operator on the quantum cone [6] that admits a twisted real structure in the sense of [5].

The reader unfamiliar with non-commutative differential geometry notions is referred to [4].

2. A differential calculus on the quantum disc

Let $0 < q < 1$. The coordinate algebra of the quantum disc, or the quantum disc algebra $\mathcal{O}(D_q)$ [8] is a complex $*$-algebra generated by $z$ subject to

$$z^*z - q^2zz^* = 1 - q^2. \tag{2.1}$$

To describe the algebraic contents of $\mathcal{O}(D_q)$ it is convenient to introduce a self-adjoint element $x = 1 - zz^*$, which $q^2$-commutes with the generator of $\mathcal{O}(D_q)$, $xz = q^2zx$. A linear basis of $\mathcal{O}(D_q)$ is given by monomials $x^kz^l, x^kz^l*$. We view $\mathcal{O}(D_q)$ as a $\mathbb{Z}$-graded algebra, setting $\deg(z) = 1,$
$\deg(z^*) = -1$. Associated with this grading is the degree-counting automorphism $\sigma : \mathcal{O}(D_q) \to \mathcal{O}(D_q)$, defined on homogeneous $a \in \mathcal{O}(D_q)$ by $\sigma(a) = q^{2\deg(a)}a$. As explained in [1] there is an orthogonal pair of skew-derivations $\partial, \bar{\partial} : \mathcal{O}(D_q) \to \mathcal{O}(D_q)$ twisted by $\sigma$ and given on the generators of $\mathcal{O}(D_q)$ by

$$\partial(z) = z^*, \quad \partial(z^*) = 0, \quad \bar{\partial}(z) = 0, \quad \bar{\partial}(z^*) = q^2z,$$

(2.2)

and extended to the whole of $\mathcal{O}(D_q)$ by the (right) $\sigma$-twisted Leibniz rule. Therefore, there is also a corresponding first-order differential calculus $\Omega^1(D_q)$ on $\mathcal{O}(D_q)$, defined as follows.

As a left $\mathcal{O}(D_q)$-module, $\Omega^1(D_q)$ is freely generated by one forms $\omega, \bar{\omega}$. The right $\mathcal{O}(D_q)$-module structure and the differential $d : \mathcal{O}(D_q) \to \Omega^1(D_q)$ are defined by

$$\omega a = \sigma(a) \omega, \quad \bar{\omega} a = \sigma(a) \bar{\omega}, \quad d(a) = \partial(a) \omega + \bar{\partial}(a) \bar{\omega}. \quad (2.3)$$

In particular,

$$dz = z^* \omega = q^2 z \omega^*, \quad dz^* = q^2 z^* \bar{\omega} = \bar{\omega} z,$$

(2.4)

and so, by the commutation rules (2.3),

$$\omega = \frac{q^{-2}}{1-q^2} (dzz - q^4 zdz), \quad \bar{\omega} = \frac{q^{-2}}{1-q^2} (z^* dz^* - q^2 dz^* z^*). \quad (2.5)$$

Hence $\Omega^1(D_q) = \{ \sum_i a_i db_i \mid a_i, b_i \in \mathcal{O}(D_q) \}$, i.e. $(\Omega^1(D_q), d)$ is truly a first-order differential calculus not just a degree-one part of a differential graded algebra. The appearance of $q^2 - 1$ in the denominators in (2.5) indicates that this calculus has no classical (i.e. $q = 1$) counterpart.

The first-order calculus $(\Omega^1(D_q), d)$ is a $*$-calculus in the sense that the $*$-structure extends to the bimodule $\Omega^1(D_q)$ so that $(\nu b)^* = b^* \nu^* a^*$ and $(da)^* = d(a^*)$, for all $a, b \in \mathcal{O}(D_q)$ and $\nu \in \Omega^1(D_q)$, provided $\omega^* = \bar{\omega}$ (this choice of the $*$-structure justifies the appearance of $q^2$ in the definition of $\bar{\partial}$ in equation (2.2)). From now on we view $(\Omega^1(D_q), d)$ as a $*$-calculus, which allows us to reduce by half the number of necessary checks.

Next we aim to show that the module of 2-forms $\Omega^2(D_q)$ obtained by the universal extension of $\Omega^1(D_q)$ is generated by the anti-self-adjoint 2-form

$$\nu = \frac{q^{-6}}{q^2 - 1} (\omega^* \omega + q^8 \omega \omega^*), \quad \nu^* = -\nu \quad (2.6)$$

and to describe the structure of $\Omega^2(D_q)$. By (2.3), for all $a \in \mathcal{O}(D_q)$,

$$va = \sigma^2(a) v. \quad (2.7)$$

Combining commutation rules (2.3) with the relations (2.4) we obtain

$$z^* dz = q^2 dzz^*, \quad dzz - q^4 zdz = q^2 (1-q^2) \omega, \quad (2.8)$$

\footnote{One should remember that the $*$-conjugation takes into account the parity of the forms; see [3].}
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and their $*$-conjugates. The differentiation of the first of equations (2.8) together with (2.3) and (2.1) yield

$$\omega^* \omega = (1 - x)v, \quad \omega \omega^* = q^6(q^2x - 1)v, \quad (2.9)$$

which means that $\omega^* \omega$ and $\omega \omega^*$ are in the module generated by $v$. Next, by differentiating $\omega z^* = q^{-2}z^* \omega$ and $\omega z = q^2z \omega$ and using (2.4) and (2.3) one obtains

$$d\omega z^* = q^{-2}z^* dw + z(\omega^* \omega + q^4 \omega \omega^*), \quad d\omega z = q^2zd \omega + (q^2 + q^{-2})z^* \omega^2. \quad (2.10)$$

The differentiation of $dz = z^* \omega$ yields

$$z^* dw = -q^2z^* \omega. \quad (2.11)$$

Multiplying this relation by $z$ from left and right, and using commutation rules (2.1) and (2.3) one finds that $(1 - x)d \omega = q^{-4}z^* d \omega z$. Developing the right hand side of this equality with the help of the second of equations (2.10) we find

$$d \omega = 1 + \frac{q^{-4}}{q^2 - 1} z^* \omega^2. \quad (2.12)$$

Combining (2.10) with (2.12) we can derive

$$z^3 \omega^2 = -z \frac{q^8}{q^4 + 1}(\omega^* \omega + q^4 \omega \omega^*). \quad (2.13)$$

The multiplication of (2.13) by $z^3$ from the left and right and the usage of (2.1), (2.3) give

$$(1 - q^{2}x)(1 - q^{-4}x)(1 - q^{-6}x)\omega^2 = -\frac{q^8}{q^4 + 1}z^4 (\omega^* \omega + q^4 \omega \omega^*), \quad (2.14a)$$

$$(1 - q^{4}x)(1 - q^2x)(1 - q^6x)\omega^2 = -\frac{q^8}{q^4 + 1}z^4 (\omega^* \omega + q^4 \omega \omega^*). \quad (2.14b)$$

Comparing the left hand sides of equations (2.14), we conclude that

$$x \omega^2 = 0 = \omega^2 x \quad \text{and, by } *\text{-conjugation,} \quad x \omega^* \omega = 0 = \omega^* \omega x, \quad (2.15)$$

and hence in view of either of (2.14)

$$\omega^2 = -\frac{q^8}{q^4 + 1}z^4 (\omega^* \omega + q^4 \omega \omega^*). \quad (2.16)$$

By (2.9), the right hand side of (2.16) is in the module generated by $v$, and so is $\omega^2$ and its adjoint $\omega^* \omega^2$. Thus, the module $\Omega^2(D_q)$ spanned by all products of pairs of one-forms is indeed generated by $v$.

Multiplying (2.12) and (2.11) by $x$ and using relations (2.15) we obtain

$$xz \omega^* \omega = 0 = \omega^* \omega xz. \quad (2.17)$$

Following the same steps but now starting with the differentiation of $dz^* = q^2z^* \omega$ (see (2.4)), we obtain the complementary relation

$$xz \omega \omega^* = 0 = \omega \omega^* xz. \quad (2.18)$$

In view of the definition of $v$, (2.17) and (2.18) yield $xzv = 0 = vzx$. Next, the multiplication of, say, the first of these equations from the left and right
by \( z^* \) and the use of (2.1) yield \( x(1-x)v = 0 \) and \( x(1-q^2x)v = 0 \). The subtraction of one of these equations from the suitable scalar multiple of the other produces the necessary relation

\[
xv = 0 = vx,
\]

(2.19)

which fully characterises the structure of \( \Omega^2(D_q) \) as an \( \mathcal{O}(D_q) \)-module generated by \( v \). In the light of (2.19), the \( \mathbb{C} \)-basis of \( \Omega^2(D_q) \) consists of elements \( vz^n, vz^{*m} \), and hence, for all \( w \in \Omega^2(D_q) \), \( wx = xw = 0 \), i.e., \( \Omega^2(D_q) \) is a torsion (as a left and right \( \mathcal{O}(D_q) \)-module). Since \( \mathcal{O}(D_q) \) is a domain and \( \Omega^2(D_q) \) is a torsion, the dual of \( \Omega^2(D_q) \) is the zero module, hence, in particular \( \Omega^2(D_q) \) is not projective. Again by (2.19), the annihilator of \( \Omega^2(D_q) \),

\[
\text{Ann}(\Omega^2(D_q)) := \{ a \in \mathcal{O}(D_q) \mid \forall w \in \Omega^2(D_q), aw = wa = 0 \},
\]

is the ideal of \( \mathcal{O}(D_q) \) generated by \( x \). The quotient \( \mathcal{O}(D_q)/\text{Ann}(\Omega^2(D_q)) \) is the Laurent polynomial ring in one variable, i.e. the algebra \( \mathcal{O}(S^1) \) of coordinate functions on the circle. When viewed as a module over \( \mathcal{O}(S^1) \), \( \Omega^2(D_q) \) is free of rank one, generated by \( v \). Thus, although the module of 2-forms over \( \mathcal{O}(D_q) \) is neither free nor projective, it can be identified with sections of a trivial line bundle once pulled back to the (classical) boundary of the quantum disc.

With (2.19) at hand, equations (2.9), (2.16), (2.12) and their \( * \)-conjugates give the following relations in \( \Omega^2(D_q) \)

\[
d\omega = q^8 z^2 \omega, \quad d\omega^* = -z^{2*} \omega, \quad \omega^* \omega = q^{6} \omega, \quad \omega^2 = q^{12} z^4 \omega, \quad \omega^{*2} = q^{-4} z^{4*} \omega.
\]

(2.20a)

(2.20b)

One can easily check that (2.20), (2.19) and (2.7) are consistent with (2.3) with no further restrictions on \( v \). Setting \( \Omega^n(D_q) = 0 \), for all \( n > 2 \), we thus obtain a 2-dimensional calculus on the quantum disc.

### 3. Differential calculus on the quantum cone

The quantum cone algebra \( \mathcal{O}(C_q^N) \) is a subalgebra of \( \mathcal{O}(D_q) \) consisting of all elements of the \( \mathbb{Z} \)-degree congruent to 0 modulo a positive natural number \( N \). Obviously \( \mathcal{O}(C_q^1) = \mathcal{O}(D_q) \), the case we dealt with in the preceding section, so we may assume \( N > 1 \). \( \mathcal{O}(C_q^N) \) is a \( * \)-algebra generated by the self-adjoint \( x = 1 - zz^* \) and by \( y = z^N \), which satisfy the following commutation rules

\[
xy = q^{2N}yx, \quad yy^* = \prod_{l=0}^{N-1} (1 - q^{-2l}x), \quad y^*y = \prod_{l=1}^{N} (1 - q^{2l}x).
\]

(3.1)

The calculus \( \Omega(C_q^N) \) on \( \mathcal{O}(C_q^N) \) is obtained by restricting of the calculus \( \Omega(D_q) \), i.e. \( \Omega^n(C_q^N) = \{ \sum a_i^0 d(a_1^1) \cdots d(a_n^1)a_{n+1} \mid a_i^j \in \mathcal{O}(C_q^N) \} \). Since \( d \) is a degree-zero map \( \Omega(C_q^N) \) contains only these forms in \( \Omega(D_q) \), whose \( \mathbb{Z} \)-degree is a multiple of \( N \). We will show that all such forms are in \( \Omega(C_q^N) \). Since \( \deg(\omega) = 2, \deg(\omega^*) = -2 \) and \( \deg(\nu) = 0 \), this is equivalent to

\[
\Omega^1(C_q^N) = \mathcal{O}(D_q) \omega \oplus \mathcal{O}(D_q) \omega^*, \quad \Omega^2(C_q^N) = \mathcal{O}(C_q^N) \nu,
\]
where \( O(D_q)_{\mathbb{Z}} = \{ a \in O(D_q) \mid \deg(a) \equiv s \mod N \} \).

As an \( O(C_q^N) \)-module, \( O(D_q)_{\mathbb{Z}} \) is generated by \( z^{N-2} \) and \( z^* \), hence to show that \( O(D_q)_{\mathbb{Z}} \omega \subseteq \Omega^1(C_q^N) \) suffices it to prove that \( z^{N-2} \omega, z^* \omega \in \Omega^1(C_q^N) \). Using the Leibniz rule one easily finds that

\[
dy = \left([N; q^2] - q^{-2N+4} [N; q^4] x\right) z^{N-2} \omega, \tag{3.2a}
\]

where \([n; s] := \frac{s^n - 1}{s - 1}\). Hence, in view of (2.1) and (2.3),

\[
y^* dy = [N; q^2] \left(1 - q^4 \frac{[N; q^4]}{[N; q^2]} x\right) \prod_{l=3}^{N} (1 - q^{2l} x) z^{*2} \omega, \tag{3.2b}
\]

The polynomial in \( x \) on the right hand side of (3.2a) has roots in common with the polynomial on the right hand side of (3.2b) if and only if there exists an integer \( k \in [-2N + 2, -N - 1] \cup [2, N - 1] \) such that

\[q^{2k}(q^{2N} + 1) = q^2 + 1. \tag{3.3}\]

Equation (3.3) is equivalent to \( q^{2} [N + k - 1; q^2] + [k; q^2] = 0 \), with the left hand side strictly positive if \( k > 0 \) and strictly negative if \( k \leq -N \). So, there are no solutions within the required range of values of \( k \). Hence the polynomials (3.2a), (3.2b) are coprime, and so there exists a polynomial (in \( x \)) combination of the left hand sides of equations (3.2) that gives \( z^{*2} \omega \). This combination is an element of \( \Omega^1(C_q^N) \) and so is \( z^{*2} \omega \). Next,

\[
z^{*2} \omega y = q^{2N} (1 - q^2 x)(1 - q^4 x) z^{N-2} \omega, \\
y z^{*2} \omega = (1 - q^{-2N+4} x)(1 - q^{-2N+2} x) z^{N-2} \omega,
\]

so again there is an \( x \)-polynomial combination of the left hand sides (which are already in \( \Omega^1(C_q^N) \)) giving \( z^{N-2} \omega \). Therefore, \( O(D_q)_{\mathbb{Z}} \omega \subseteq \Omega^1(C_q^N) \). The case of \( O(D_q)_{\mathbb{Z}} \) follows by the \( * \)-conjugation.

Since \( z^2 \omega^* \), \( z^* \omega \) are elements of \( \Omega^1(C_q^N) \),

\[
\Omega^2(C_q^N) \ni z^2 \omega^* z^{*2} \omega = q^{-4}(1 - x)(1 - q^{-2} x) \omega^* \omega = -q^2 \nu, \tag{3.4}
\]

by the quantum disc relations and (2.20) and (2.19). Consequently, \( \nu \in \Omega^2(C_q^N) \). Therefore, \( \Omega(C_q^N) \) can be identified with the subspace of \( \Omega(D_q) \), of all the elements whose \( \mathbb{Z} \)-degree is a multiple of \( N \).

### 4. The integral

Here we construct an algebraic integral associated to the calculus constructed in Section 2. We start by observing that since \( \sigma \) preserves the \( \mathbb{Z} \)-degrees of elements of \( O(D_q) \) and \( \partial \) and \( \partial \) satisfy the \( \sigma \)-twisted Leibniz rules, the definition (2.2) implies that \( \partial \) lowers while \( \partial \) raises degrees by 2. Hence, one can equip \( \Omega^1(D_q) \) with the \( \mathbb{Z} \)-grading so that \( d \) is the degree zero map,
provided \( \deg(\omega) = 2, \deg(\omega^*) = -2 \). Furthermore, in view of the definition of \( \sigma \), one easily finds that
\[
\sigma^{-1} \circ \partial \circ \sigma = q^4 \partial, \quad \sigma^{-1} \circ \bar{\partial} \circ \sigma = q^{-4} \bar{\partial},
\]
i.e. \( \partial \) is a \( q^4 \)-derivation and \( \bar{\partial} \) is a \( q^{-4} \)-derivation. Therefore, by [2], \( \Omega(D_q) \) admits a divergence, for all right \( \Omega(D_q) \)-linear maps \( f : \Omega^1(D_q) \rightarrow \Omega(D_q) \), given by
\[
\nabla_0(f) = q^4 \partial (f(\omega)) + q^{-4} \bar{\partial} (f(\omega^*)).
\]  
Since the \( \Omega(D_q) \)-module \( \Omega^2(D_q) \) has a trivial dual, \( \nabla_0 \) is flat. Recall that by the integral associated to \( \nabla_0 \) we understand the cokernel map of \( \nabla_0 \).

**Theorem 4.1.** The integral associated to the divergence (4.2) is a map \( \Lambda : \Omega(D_q) \rightarrow \mathbb{C} \), given by
\[
\Lambda(x^k z^l) = \lambda \frac{(k+1; q^2)}{[k+1; q^4]} \delta_{l,0}, \quad \text{for all } k \in \mathbb{N}, \ l \in \mathbb{Z},
\]
where, for \( l < 0 \), \( z^l \) means \( z^{*-l} \) and \( \lambda \in \mathbb{C} \).

**Proof.** First we need to calculate the image of \( \nabla_0 \). Using the twisted Leibniz rule and the quantum disc algebra commutation rules (2.1), one obtains
\[
\partial(x^k) = -q^{-2} [k; q^4] x^{k-1} z^{*2}.
\]  
Since \( \partial(z^*) = 0 \), (4.4) means that all monomials \( x^k z^t z^{*2} \) are in the image of \( \partial \) hence in the image of \( \nabla_0 \). Using the \(*\)-conjugation we conclude the \( x^k z^t z^{*2} \) are in the image of \( \bar{\partial} \) hence in the image of \( \nabla_0 \). So \( \Lambda \) vanishes on (linear combinations of) all such polynomials. Next note that
\[
\partial(z^2) = (q^2 + 1) - (q^4 + 1)x,
\]
hence
\[
\partial(z^* z^2 - q^4 z^2 z^*) = (1 - q^4)z^*, \quad \partial(z^* z^2 - q^2 z^2 z^*) = (1 - q^2)(1 + q^4)xz^*.
\]
This means that \( z^* \) and \( xz^* \) are in the image of \( \partial \), hence of \( \nabla_0 \). In fact, all the \( x^k z^* \) are in this image which can be shown inductively. Assume \( x^k z^* \in \text{Im}(\partial) \), for all \( k \leq n \). Then using the twisted Leibniz rule, (4.4) and (4.5) one finds
\[
\partial(x^n z^2) = -q^2 [N; q^4] x^{n-1} + (q^2 + 1) [n + 1; q^4] x^n - [n + 2; q^4] x^{n+1}.
\]  
Since \( \partial(z^*) = 0 \), equation (4.6) implies that \( \partial(z^n z^2 z^*) \) is a linear combination of monomials \( x^{n-1} z^*, x^n z^* \) and \( x^{n+1} z^* \). Since the first two are in the image of \( \partial \) by the inductive assumption, so is the third one. Therefore, all linear combinations of \( x^k z^* \) and \( x^k z \) (by the \(*\)-conjugation) are in the image of \( \nabla_0 \).

Put together all this means that \( \Lambda \) vanishes on all the polynomials \( \sum_{k,l=1}^{n} (c_{kl} x^k z^l + c'_{kl} x^k z^* l) \). The rest of the formula (4.3) can be proven by induction. Set \( \lambda = \Lambda(1) \). Since \( \Lambda \) vanishes on all elements in the image of \( \nabla_0 \), hence also in the image of \( \partial \), the application of \( \Lambda \) to the right hand side of (4.4) confirms (4.3) for \( k = 1 \). Now assume that (4.3) is true for all \( k \leq n \).
Then the application $\Lambda$ to the right hand side of (4.6) followed by the use of the inductive assumption yields

$$
[n + 2; q^4] \Lambda (x^{n+1}) = q^2 [N; q^4] \Lambda (x^{n-1}) - (q^2 + 1) [n + 1; q^4] \Lambda (x^n)
$$

$$
= \lambda ((q^2 + 1) [n + 1; q^2] - q^2 [n; q^2]) = \lambda [n + 2; q^2].
$$

Therefore, the formula (4.3) is true also for $n + 1$, as required.

The restriction of $\Lambda$ to the elements of $O(D_q)$, whose $\mathbb{Z}$-degree is a multiple of $N$ gives an integral on the quantum cone $O(C_q^N)$.

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**References**

[1] A. Almulhem & T. Brzeziński, *Skew derivations on generalized Weyl algebras*. arXiv:1610.03282 (2016).

[2] E.J. Beggs & S. Majid, *Spectral triples from bimodule connections and Chern connections*. J. Noncommut. Geom. to appear, arXiv:1508.04808v2, (2015).

[3] T. Brzeziński, *Non-commutative differential geometry of generalized Weyl algebras*. SIGMA Symmetry Integrability Geom. Methods Appl. 12 (2016), 059.

[4] T. Brzeziński, *Differential and integral forms on non-commutative algebras*. arXiv:1611.01016 (2016).

[5] T. Brzeziński, N. Ciccoli, L. Dąbrowski & A. Sitarz, *Twisted reality condition for Dirac operators*. Math. Phys. Anal. Geom. 19 (2016), 19:16.

[6] T. Brzeziński & L. Dąbrowski, *In preparation*.

[7] T. Brzeziński, L. El Kaoutit & C. Lomp, *Non-commutative integral forms and twisted multi-derivations*. J. Noncommut. Geom. 4 (2010), 281–312.

[8] S. Klimek & A. Lesniewski, *A two-parameter quantum deformation of the unit disc*. J. Funct. Anal. 115 (1993), 1–23.

[9] S.L. Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups)*. Comm. Math. Phys. 122 (1989), 125–170.

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