On entanglement-assisted classical capacity

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We give a modified proof of the recent result of C. H. Bennett, P. W. Shor, J. A. Smolin and A. V. Thapliyal concerning entanglement-assisted classical capacity of quantum channel and discuss relation between entanglement-assisted and unassisted classical capacities.
I. INTRODUCTION

The classical capacity of a quantum channel is the capacity of transmission of classical information through the channel. It is well known that the classical capacity can be increased if there is an additional resource in the form of an entangled state shared between the input $A$ and the output $B$ of the channel. While entanglement itself cannot serve for transmission of information from $A$ to $B$, it may enhance the transmission provided there is a quantum channel connecting the systems. If the channel is ideal (i.e., the identity map $\text{Id}$ from $A$ to $B$) then the entanglement-assisted capacity is twice as great as the unassisted classical capacity, the enhancement being realized by the dense coding protocol [1].

Recently C. H. Bennett, P. W. Shor, J. A. Smolin and A. V. Thapliyal (BSST) studied the effect of the shared entanglement on the classical capacity of quantum non-ideal (noisy) channel [2,3] and obtained a remarkably simple formula for the entanglement-assisted capacity in terms of the maximal mutual quantum information between $A$ and $B$. The proof, which is by no means trivial, involves in particular a rather tricky derivation of an important continuity property of quantum entropy. In this paper we give a modified proof of the BSST theorem, including a more transparent proof of this property; moreover, we make further simplifications by using heavily properties of conditional quantum entropy rather than the underlying strong subadditivity.

In Ref. [2] it was shown that the enhancement in the classical capacity can achieve arbitrarily large values. To this end the case of $d$-dimensional depolarizing channel in the limit of strong noise ($p \to 1$) was considered; we remark that the enhancement is even greater for the extreme case $p = \frac{d^2}{d^2 - 1}$. Moreover, we derive a general inequality between entanglement-assisted and unassisted capacities which may be relevant to the additivity problem in quantum information theory.

II. THE BSST THEOREM

We refer the reader to Refs. [4,5] for some basic definitions and results of quantum information theory used in this paper.

Consider the following protocol for the classical information transmission through a quantum channel $\Phi$. Systems $A$ and $B$ of the same dimension share an entangled (pure) state $S_{AB}$. $A$ does some encoding $i \to E^i_A$ depending on a classical signal $i$ with probabilities $\pi_i$ and sends its part of this shared state through the channel $\Phi$ to $B$. Thus $B$ gets the states $(\Phi \otimes \text{Id}_B)[S^i_{AB}]$, with probabilities $\pi_i$, and $B$ is trying to extract the maximal classical information by doing measurements on these states. This is similar to the dense coding, but instead of the ideal channel, $A$ uses a noisy channel $\Phi$. We now look for the classical capacity of this protocol, which is called entanglement-assisted classical capacity of the channel $\Phi$.

The maximum over measurements of $B$ can be evaluated using the coding theorem for the classical capacity [6].

First we have the one-shot entanglement-assisted classical capacity

$$C_{ea}(\Phi) = \max_{\pi_i, E^i_A, S_{AB}} \left[ H \left( \sum_i \pi_i (\Phi \otimes \text{Id}_B) \left[ S^i_{AB} \right] \right) - \sum_i \pi_i H \left( (\Phi \otimes \text{Id}_B) \left[ S^i_{AB} \right] \right) \right],$$

where $H(S)$ denotes the von Neumann entropy of the density operator $S$. Using the channel $n$ times and allowing entangled measurements on $B$’s side, one gets

$$C_{ea}(\Phi) = C_{ea}(\Phi^\otimes n).$$

The full entanglement-assisted classical capacity is then

$$C_{ea}(\Phi) = \lim_{n \to \infty} \frac{1}{n} C_{ea}(\Phi^\otimes n).$$

The following result was announced in Ref. [2], and a proof was given in Ref. [3]:

$$C_{ea}(\Phi) = \max_{S_A} I(S_A; \Phi),$$

where

$$I(S_A; \Phi) = H(S_A) + H(\Phi(S_A)) - H(S_A; \Phi)$$

(5)
is the quantum mutual information, with \( H(S_A; \Phi) \) denoting the entropy exchange (see Refs. [4,5]). Below we give a simplified proof of this remarkable formula.

Proof of the inequality

\[
C_{ea}(\Phi) \geq \max_{S_A} I(S_A; \Phi). \tag{6}
\]

It is shown in [3] by generalizing the dense coding protocol that

\[
C_{ea}^{(1)}(\Phi^\otimes n) \geq I\left(\frac{P}{\dim P}; \Phi^\otimes n\right) \tag{7}
\]

for arbitrary projection \( P \) in \( \mathcal{H}_A^\otimes n \). We give this proof for completeness here. Indeed, let \( P = \sum_{k=1}^{m} |e_k\rangle\langle e_k| \), where \( \{e_k; k = 1, \ldots, m = \dim P\} \) is an orthonormal system. Define unitary operators in \( \mathcal{H}_A \) acting as

\[
V|e_k\rangle = \exp\left(\frac{2\pi i k}{m}\right)|e_k\rangle; \quad U|e_k\rangle = |e_{k+1(\mod m)}\rangle; \quad k = 1, \ldots, m,
\]

\[
W_{\alpha\beta} = U^\alpha V^\beta; \quad \alpha, \beta = 1, \ldots, m
\]

on the subspace generated by \( \{e_k\} \), and as the identity onto its orthogonal complement. The operators \( W_{\alpha\beta} \) are a finite-dimensional version of the Weyl-Segal operators for Boson systems (see e. g. [5]). Let

\[
|\psi_{AB}\rangle = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} |e_k\rangle \otimes |e_k\rangle.
\]

Then it is easy to show that

1) \( (W_{\alpha\beta} \otimes I_B)|\psi_{AB}\rangle; \quad \alpha, \beta = 1, \ldots, m, \) is an orthonormal system in \( \mathcal{H}_A \otimes \mathcal{H}_B \); in particular, if \( m = \dim \mathcal{H}_A \), it is a basis;

2) \( \sum_{\alpha,\beta=1}^{m} (W_{\alpha\beta} \otimes I_B)|\psi_{AB}\rangle \langle \psi_{AB}| (W_{\alpha\beta} \otimes I_B)^* = P \otimes P. \)

Thus operators \( \{W_{\alpha\beta}; \quad \alpha, \beta = 1, \ldots, m\} \) play a role similar to Pauli matrices in the dense coding protocol for qubits.

Take the classical signal to be transmitted as \( i = (\alpha, \beta) \) with equal probabilities \( 1/m^2 \), the entangled state \( |\psi_{AB}\rangle \langle \psi_{AB}| \) and the unitary encodings \( E_i[S] = W_{\alpha\beta} S W_{\alpha\beta}^* \). Then we have

\[
C_{ea}^{(1)}(\Phi^\otimes n) \geq H\left(\frac{1}{m^2} \sum_{\alpha,\beta} (\Phi \otimes I_B)|S_{AB}\rangle \langle S_{AB}|\right) - \frac{1}{m^2} \sum_{\alpha,\beta} H\left(\left(\Phi \otimes I_B\right)|S_{AB}^\alpha_{\beta}\right),
\]

where \( S_{AB}^{\alpha_{\beta}} = (W_{\alpha\beta} \otimes I_B)|\psi_{AB}\rangle \langle \psi_{AB}| (W_{\alpha\beta} \otimes I_B)^* \). Then by the property 2) the first term in the right hand side is equal to \( H\left(\left(\Phi \otimes I_B\right)\left(\frac{P}{\dim P}\right)\right) = H\left(\frac{P}{\dim P}\right) + H\left(\Phi\left(\frac{P}{\dim P}\right)\right) \). Since \( S_{AB}^{\alpha_{\beta}} \) is a purification of \( \frac{P}{\dim P} \) in \( \mathcal{H}_B \), the entropies in the second term are all equal to \( H\left(\frac{P}{\dim P}\right) \). By the expression for quantum mutual information [3] this proves [6]. For future use, note that the last term in the quantum mutual information – the entropy exchange \( H(S_A; \Phi) \) – is equal to the final environment entropy \( H(\Phi_E[S_A]) \), where \( \Phi_E \) is a channel from the system space \( \mathcal{H}_A \) to the environment space \( \mathcal{H}_E \) the actual form of which we need not to know (see [6]).

Now let \( S_A = S \) be an arbitrary state in \( \mathcal{H}_A \), and let \( P^{n,\delta} \) be the typical projection of the state \( S^\otimes n \) in \( \mathcal{H}_A^\otimes n \). It was suggested in [2] that for arbitrary channel \( \Psi \) from \( \mathcal{H}_A \) to possibly other Hilbert space \( \mathcal{H} \)

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} H\left(\Psi^\otimes n \left(\frac{P^{n,\delta}}{\dim P^{n,\delta}}\right)\right) = H(\Psi(S)),
\]

which would imply, by the expressions for the mutual information and the entropy exchange, that

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} I\left(\frac{P^{n,\delta}}{\dim P^{n,\delta}}; \Phi^\otimes n\right) = I(S; \Phi), \tag{8}
\]

and hence, by [6], the required inequality [3]. We shall prove [3] with \( P^{n,\delta} \) being the strongly typical projection of the state \( S^\otimes n \).
Let us fix small positive $\delta$, and let $\lambda_j$ be the eigenvalues, $|e_j\rangle$ the eigenvectors of the density operator $S$. Then the eigenvalues and eigenvectors of $S_{\otimes n}$ are $\lambda_j = \lambda_{j_1} \cdots \lambda_{j_n}$, $|e_j\rangle = |e_{j_1}\rangle \otimes \ldots \otimes |e_{j_n}\rangle$ where $J = (j_1,\ldots,j_n)$. The sequence $J$ is called strongly typical if the numbers $N(j|J)$ of appearance of the symbol $j$ in $J$ satisfy the condition

$$\left| \frac{N(j|J)}{n} - \lambda_j \right| < \delta, \quad j = 1,\ldots,d,$$

and $N(j|J) = 0$ if $\lambda_j = 0$. Let us denote the collection of all strongly typical sequences as $B^{n,\delta}$, and let $P^n$ be the probability distribution given by the eigenvalues $\lambda_j$. Then by the Law of Large Numbers, $P^n(B^{n,\delta}) \to 1$ as $n \to \infty$. It is shown in \[7\] that the size of $B^{n,\delta}$ satisfies

$$2^n[H(S) - \Delta_n(\delta)] < |B^{n,\delta}| < 2^n[H(S) + \Delta_n(\delta)],$$

where $H(S) = -\sum_{j=1}^d \lambda_j \log \lambda_j$, and $\lim_{\delta \to 0} \lim_{n \to \infty} \Delta_n(\delta) = 0$.

For arbitrary function $f(j), j = 1,\ldots,d$, and $J = (j_1,\ldots,j_n) \in B^{n,\delta}$ we have

$$\left| \frac{f(j_1) + \ldots + f(j_n)}{n} - \sum_{j=1}^d \lambda_j f(j) \right| < \delta \max f.$$

In particular, any strongly typical sequence is (entropy) typical: taking $f(j) = -\log \lambda_j$ gives

$$n[H(S) - \delta_1] < -\log \lambda_j < n[H(S) + \delta_1],$$

where $\delta_1 = \delta \max \lambda_{j_i} > 0(-\log \lambda_j)$. The converse is not true – not every typical sequence is strongly typical.

The strongly typical projector is defined as the following spectral projector of $S_{\otimes n}$:

$$P^{n,\delta} = \sum_{\lambda \in B^{n,\delta}} |e_{n,\lambda}\rangle \langle e_{n,\lambda}|.$$

We denote $d_n,\delta = \dim P^{n,\delta}$ and $\bar{S}^{n,\delta} = \frac{P^{n,\delta}}{d_n,\delta}$ and we are going to prove that

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} H(\Psi^{\otimes n}(\bar{S}^{n,\delta})) = H(\Psi(S))$$

(12)

for arbitrary channel $\Psi$.

We have

$$n H(\Psi(S)) - H(\Psi^{\otimes n}(\bar{S}^{n,\delta})) = H(\Psi(S)^{\otimes n}) - H(\Psi^{\otimes n}(\bar{S}^{n,\delta}))$$

$$= H(\Psi^{\otimes n}(\bar{S}^{n,\delta}) | \Psi^{\otimes n}(S^{\otimes n})) + \text{Tr} \log \Psi(S)^{\otimes n} \left( \Psi^{\otimes n}(\bar{S}^{n,\delta}) - \Psi(S)^{\otimes n} \right),$$

(13)

where $H(\cdot | \cdot)$ is relative entropy. Strictly speaking, this formula is correct if the density operator $\Psi(S)^{\otimes n}$ is nondegenerate, which we assume for a moment. Later we shall show how the argument can be modified to the general case.

For the first term we have the estimate by the fundamental property of monotonicity of the relative entropy

$$H(\Psi^{\otimes n}(\bar{S}^{n,\delta}) | \Psi^{\otimes n}(S^{\otimes n})) \leq H(\bar{S}^{n,\delta} | S^{\otimes n})$$

with the right hand side computed explicitly as

$$H(\bar{S}^{n,\delta} | S^{\otimes n}) = \sum_{J \in B^{n,\delta}} \frac{1}{d_{n,\delta}} \log \frac{1}{d_{n,\delta} \lambda_j} = -\log d_{n,\delta} - \sum_{J \in B^{n,\delta}} \frac{1}{d_{n,\delta}} \log \lambda_j,$$

which is less than or equal to $n (\delta_1 + \Delta_n(\delta))$ by (11), (4), giving sufficient estimate.

By using the identity

$$\log \Psi(S)^{\otimes n} = \log \Psi(S) \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes \log \Psi(S),$$

(4)
and introducing the operator $F = \Psi^*(\log \Psi(S))$ where $\Psi^*$ is the dual channel, we can rewrite the second term as

$$n \text{Tr} \left( \frac{F \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes F}{n} \right) (S^{n,\delta} - S^{\otimes n})$$

$$= \frac{n}{d_{n,\delta}} \sum_{j \in B^{n,\delta}} \left[ \frac{f(j_1) + \ldots + f(j_n)}{n} - \sum_{j=1}^{d} \lambda_j f(j) \right],$$

where $f(j) = \langle e_j | F | e_j \rangle$, which is evaluated by $n\delta \max f$ via (10). This establishes (12) in the case of a nondegenerate $\Psi(S)$.

Coming back to the general case, let us denote $P_{\Psi}$ the supporting projector of $\Psi(S)$. Then the supporting projector of $\Psi(S)^{\otimes n}$ is $P_{\Psi}^{\otimes n}$, and the support of $\Psi^{\otimes n} (S^{n,\delta})$ is contained in the support of $\Psi(S)^{\otimes n} = \Psi^{\otimes n}(S^{\otimes n})$, because the support of $S^{n,\delta}$ is contained in the support of $S^{\otimes n}$. Thus the second term in (13) should be understood as

$$\text{Tr} P_{\Psi}^{\otimes n} \log [P_{\Psi}^{\otimes n} \Psi(S)^{\otimes n} P_{\Psi}^{\otimes n}] \left( S^{n,\delta} - \Psi(S)^{\otimes n} \right),$$

where we now have log of a nondegenerate operator in $P_{\Psi}^{\otimes n} H_{A}^{\otimes n}$. We can then repeat the argument with $F$ defined as $\Psi^*(P_{\Psi} \log P_{\Psi} \Psi(S) P_{\Psi}) P_{\Psi}$. This fulfills the proof of (10), from which (8) follows.

**Proof** of the inequality

$$C_{ca}(\Phi) \leq \max_{S_A} I(S_A, \Phi). \quad (14)$$

We first prove that

$$C_{ca}^{(1)}(\Phi) \leq \max_{S_A} I(S_A, \Phi). \quad (15)$$

The proof is a modification of that from (3), using properties of conditional quantum entropy which are known to follow from the strong subadditivity of the entropy (see e.g. (3), (4)), rather than the strong subadditivity itself.

Let us denote $E^i_A$ the encodings used by $A$. Let $S_{AB}$ be the pure state initially shared by $A$ and $B$, then the state of the system $AB$ (resp. $A$) after the encoding is

$$S_{AB}^i = (E^i_A \otimes I_B)[S_{AB}], \quad \text{resp.} \quad S_A^i = E^i_A[S_A]. \quad (16)$$

Note that the partial state of $B$ does not change after the encoding, $S_B^i = S_B$. We are going to prove that

$$H \left( \sum_i \pi_i (\Phi \otimes I_B)[S_{AB}^i] \right) - \sum_i \pi_i H \left( (\Phi \otimes I_B)[S_{AB}^i] \right)$$

$$\leq I \left( \sum_i \pi_i S_A^i; \Phi \right). \quad (17)$$

By the quantum coding theorem, the maximum of the left hand side with respect to all possible $\pi_i, E^i_A$ is just $C_{ca}^{(1)}(\Phi)$, whence (15) will follow.

By using subadditivity of quantum entropy, we can evaluate the first term in the left hand side of (17) as

$$H \left( \sum_i \pi_i \Phi[S_A^i] \right) + H(S_B) = H \left( \Phi \left( \sum_i \pi_i S_A^i \right) \right) + \sum_i \pi_i H(S_B).$$

Here the first term already gives the output entropy from $I \left( \sum_i \pi_i S_A^i; \Phi \right)$. Let us proceed with evaluation of the remainder

$$\sum_i \pi_i [H(S_B) - H \left( (\Phi \otimes I_B)[S_{AB}^i] \right)].$$

We first show that the term in squared brackets does not exceed $H(S_A^i) - H \left( (\Phi \otimes I_{R^i})[S_{AR}^i] \right)$, where $R^i$ is the purifying (reference) system for $S_A^i$, and $S_{AR}^i$ is the purified state. To this end consider the unitary extension of
the encoding $\mathcal{E}_A^i$ with the environment $E_i$, which is initially in a pure state. From (16) we see that we can take $R_i = BE_i$ (after the unitary interaction which involves only $AE_i$). Then, again denoting with primes the states after the application of the channel $\Phi$, we have
\[
H(S_A) - H((\Phi \otimes \text{Id}_R)(S_{AB}^i)) = H(S_B) - H(S_{AB}^i) = -H_i(A^i|B),
\]
where the lower index $i$ of the conditional entropy points out to the joint state $S_{AB}^i$. Similarly
\[
H(S_A^i) - H((\Phi \otimes \text{Id}_R)(S_{AR}^i)) = H(S_R) - H(S_{AR}^i)
\]
which is greater or equal than (18) by monotonicity of the conditional entropy.
Using the concavity of the function $S_A \rightarrow H(S_A) - H((\Phi \otimes \text{Id}_R)(S_{AR}^i))$ to be shown below, we get
\[
\sum_i \pi_i [H(S_A^i) - H((\Phi \otimes \text{Id}_R)(S_{AR}^i))] \leq H \left( \sum_i \pi_i S_A^i \right) - H \left( \Phi \otimes \text{Id}_R(S_{AR}) \right),
\]
where $S_{AR}$ is the state purifying $\sum_i \pi_i S_A^i$ with a reference system $R$.
To complete this proof it remains to show the above concavity. By introducing the environment $E$ for the channel $\Phi$, we have
\[
H(S_A) - H((\Phi \otimes \text{Id}_R)(S_{AR})) = H(S_R) - H(S_{AR})
\]
The conditional entropy $H(A'|E')$ is a concave function of $S_{A'E'}$. The map $S_A \rightarrow S_{A'E'}$ is affine and therefore $H(A'|E')$ is a concave function of $S_A$.
Applying the same argument to the channel $\Phi^{\otimes n}$ gives
\[
C^{(n)}_{ca}(\Phi) \leq \max_{S_A^{n}} I(S_A^{n};\Phi^{\otimes n}).
\]
Then from subadditivity of quantum mutual information [3], we have
\[
\max_{S_{12}} I(S_{12};\Phi_1 \otimes \Phi_2) = \max_{S_1} I(S_1;\Phi_1) + \max_{S_2} I(S_2;\Phi_2),
\]
implying the remarkable additivity property
\[
\max_{S_A^{n}} I(S_A^{n};\Phi^{\otimes n}) = n \max_{S_A} I(S_A;\Phi).
\]
Therefore, finally we obtain (14).

III. RELATION BETWEEN ENTANGLEMENT-ASSISTED AND UNASSISTED CAPACITIES

The definition of $C^{(1)}_{ca}(\Phi)$ and hence of $C_{ca}(\Phi)$ can be formulated without explicit introduction of the encoding operations $\mathcal{E}_A^i$, namely
\[
C^{(1)}_{ca}(\Phi) = \max_{\pi_i,\{S_{AB}^i\} \in \Sigma_B} \left[ \sum_i \pi_i H((\Phi \otimes \text{Id}_B)(S_{AB}^i)) \right],
\]
where $\Sigma_B$ is the collection of families of the states $\{S_{AB}^i\}$ satisfying the condition that their partial states $S_B^i$ do not depend on $i$, $S_B^i = S_B$. This follows from
Lemma. Let \( \{ S_{AB}^i \} \) be a family of the states satisfying the condition \( S_{AB}^i = S_B \). Then there exits a pure state \( S_{AB} \) and encodings \( \mathcal{E}_A^i \) such that

\[
S_{AB}^i = (\mathcal{E}_A^i \otimes \text{Id}_B)[S_{AB}].
\]  

(21)

Proof. For simplicity assume that \( S_B \) is nondegenerate. Then

\[
S_B = \sum_{k=1}^d \lambda_k |e_k^B \rangle \langle e_k^B |,
\]

where \( \lambda_k > 0 \) and \( \{ |e_k^B \rangle \} \) is an orthonormal basis in \( \mathcal{H}_B \). Let \( \{ |e_k^A \rangle \} \) be an orthonormal basis in \( \mathcal{H}_A \). For a vector \( |\psi^A\rangle = \sum_{k=1}^d c_k |e_k^A \rangle \) we denote \( |\bar{\psi}^B\rangle = \sum_{k=1}^d \bar{c}_k |e_k^B \rangle \). The map \( |\psi^A\rangle \to |\bar{\psi}^B\rangle \) is anti-isomorphism of \( \mathcal{H}_A \) and \( \mathcal{H}_B \). Put

\[
|\psi_{AB}\rangle = \sum_{k=1}^d \sqrt{\lambda_k} |e_k^A \rangle \otimes |e_k^B \rangle,
\]

so that \( S_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}| \) and define encodings by the relation

\[
\mathcal{E}_A^i |\psi^A\rangle \langle \phi^A| = |\bar{\psi}^B\rangle S_{AB}^{-1/2} S_{AB}^i S_{AB}^{-1/2} |\phi^B\rangle, \quad |\psi^A, \phi^A\rangle \in \mathcal{H}_A.
\]

Then one can check that \( \mathcal{E}_A^i \) are indeed channels fulfilling the formula (21).

In the case \( S_B \) is degenerate, the above construction should be modified by replacing \( S_{AB}^{-1/2} S_{AB}^i S_{AB}^{-1/2} \) in the formula above with \( \sqrt{S_B^{-1} S_B^i S_B^{-1} + P_B^0} \) where \( S_B^{-1} \) is the generalized inverse of \( S_B \) and \( P_B^0 \) is the projection onto the null subspace of \( S_B \).

We now observe an inequality relating the asymptotic entanglement-assisted and unassisted capacities. Apparently,

\[
C_{ea}^{(1)}(\Phi) \leq \max_{\pi_i \sim S_{AB}} H \left( \sum_i \pi_i (\Phi \otimes \text{Id}_B) \left[ S_{AB}^i \right] \right) - \sum_i \pi_i H \left( (\Phi \otimes \text{Id}_B) \left[ S_{AB}^i \right] \right),
\]

(22)

where \( S_{AB}^i \) are already arbitrary states, not necessarily of the form (21). The quantity on the right hand side is nothing but the one-shot classical capacity \( C^{(1)}(\Phi \otimes \text{Id}_B) \) of the channel \( \Phi \otimes \text{Id}_B \). It was shown in [14] that \( C^{(1)}(\Phi \otimes \text{Id}_B) = C^{(1)}(\Phi) + C^{(1)}(\text{Id}_B) = C^{(1)}(\Phi) + \log d \). Applying the same argument to \( \Phi^{\otimes n} \) instead of \( \Phi \), we have

\[
C_{ea}^{(1)}(\Phi^{\otimes n}) \leq C^{(1)}(\Phi^{\otimes n}) + n \log d.
\]

Dividing by \( n \) and taking limit \( n \to \infty \), we obtain

\[
C_{ea}(\Phi) \leq C(\Phi) + \log d.
\]

One can expect that a similar inequality

\[
C_{ea}(\Phi) \leq C^{(1)}(\Phi) + \log d
\]

holds generally for the one-shot classical capacity; if it breaks for some channel \( \Phi \), then for this channel \( C^{(1)}(\Phi) < C(\Phi) \), which would imply negative answer to the long-standing question concerning additivity of the classical capacity.

It is not difficult to check that the inequality indeed holds for all unital qubit channels and for \( d \)-depolaring channel

\[
\Phi[S] = (1 - p) S + p \frac{I}{d} \text{Tr} S.
\]

(23)

Here \( \text{dim} \mathcal{H} = d \) and the parameter \( p \) should lie in the range \( 0 \leq p \leq \frac{d^2 - 1}{d^2} \), as can be seen from the Kraus representation

\[
\Phi[S] = \left( 1 - p \frac{d^2 - 1}{d^2} \right) S + p \frac{1}{d^2} \sum_{\alpha, \beta \neq d} W_{\alpha \beta} S W_{\alpha \beta}^*,
\]

(24)

with \( W_{\alpha \beta} ; \alpha, \beta = 1, \ldots, d \) built upon arbitrary orthonormal basis in \( \mathcal{H} \).
The quantity $C_{ca}(\Phi)$ can be computed by using unitary covariance of the depolarizing channel and concavity of the function $S \rightarrow I(S; \Phi)$. It follows that it achieves the maximum at the chaotic state $\bar{S} = \frac{I}{d}$. We have $H(\bar{S}) = H(\Phi[\bar{S}]) = \log d$. The entropy exchange $H(\bar{S}; \Phi)$ can be computed by as the entropy of the matrix $[\text{Tr}\bar{S}A_{\alpha,\beta}^*A_{\alpha,\beta}]$, where $A_{\alpha,\beta} = \frac{\sqrt{d}}{d}W_{\alpha,\beta}; \alpha, \beta \neq d; A_{dd} = \sqrt{1 - p^2d^{-2}}I$ are the Kraus operators from the representation (24). We thus obtain

$$C_{ca}(\Phi) = \log d^2 + \left(1 - p \frac{d^2 - 1}{d^2}\right) \log \left(1 - p \frac{d^2 - 1}{d^2}\right) + p \frac{d^2 - 1}{d^2} \log \frac{p}{d^2}.$$  \hspace{1cm} (25)

This should be compared with the unassisted classical capacity, which is equal to

$$C^{(1)}(\Phi) = \log d + \left(1 - p \frac{d - 1}{d}\right) \log \left(1 - p \frac{d - 1}{d}\right) + p \frac{d - 1}{d} \log \frac{p}{d},$$  \hspace{1cm} (26)

and is achieved for an ensemble of equiprobable pure states taken from an orthonormal basis in $\mathcal{H}$. One then sees \cite{2} that $\frac{C_{ca}(\Phi)}{C^{(1)}(\Phi)} \rightarrow d + 1$ in the limit of strong noise $p \rightarrow 1$ (note that both capacities tend to zero!)

Moreover, taking the maximal possible value $p = \frac{d^2}{d^2 - 1}$, we obtain

$$C_{ca} = \log \frac{d^2}{d^2 - 1},$$

$$C^{(1)} = \frac{1}{d + 1} \log \frac{d}{d + 1} + \frac{d}{d + 1} \log \frac{d^2}{d^2 - 1}.$$  

Here the ratio $\frac{C_{ca}}{C^{(1)}}$ monotonically increases from the value 5.0798 for $d = 2$, approaching tightly the asymptotic line $2(d + 1)$ as $d$ grows to infinity.

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