COMBINATORIAL IDENTITIES FOR INCOMPLETE TRIBONACCI POLYNOMIALS

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Abstract

The incomplete tribonacci polynomials, denoted by $T^{(s)}_n(x)$, generalize the usual tribonacci polynomials and were introduced in [10], where several algebraic identities were shown. In this paper, we provide a combinatorial interpretation for $T^{(s)}_n(x)$ in terms of weighted linear tilings involving three types of tiles. This allows one not only to supply combinatorial proofs of the identities for $T^{(s)}_n(x)$ appearing in [10] but also to derive additional identities. In the final section, we provide a formula for the ordinary generating function of the sequence $T^{(s)}_n(x)$ for a fixed $s$, which was requested in [10]. Our derivation is combinatorial in nature and makes use of an identity relating $T^{(s)}_n(x)$ to $T_n(x)$.

Keywords: tribonacci numbers, incomplete tribonacci polynomials, combinatorial proof

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1. Introduction

The tribonacci numbers $t_n$ are defined by the recurrence relation $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ if $n \geq 3$, with initial values $t_0 = 0$ and $t_1 = t_2 = 1$. See sequence A000073 in OEIS [11]. The tribonacci numbers are given equivalently by the explicit formula

$$t_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} B(n-i,i), \quad n \geq 0,$$

where $B(n,i) = \sum_{j=0}^{i} \binom{i}{j} (n-j)^i$, as shown in [2]. The number $B(n,i)$ is the $n$-th row, $i$-th column entry of the tribonacci triangle (see [1]).

The tribonacci polynomials $T_n(x)$ were introduced in [6] and are defined by the recurrence $T_n(x) = x^2 T_{n-1}(x) + x T_{n-2}(x) + T_{n-3}(x)$ if $n \geq 3$, with initial values $T_0(x) = 0$, $T_1(x) = 1$, and $T_2(x) = x^2$. In analogy to (1.1), the tribonacci polynomials are given by the following explicit formula (see [10]):

$$T_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-i-j}{i} x^{2n-3(i+j)}.$$
The incomplete tribonacci polynomials $T_n^{(s)}(x)$ were considered in [10] and are defined as

\[(1.3)\quad T_{n+1}^{(s)}(x) = \sum_{i=0}^{s} \sum_{j=0}^{i} \binom{i}{j} \left( \frac{n-j}{i} \right) x^{2n-3(i+j)}, \quad 0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor.\]

Note that the incomplete tribonacci polynomials generalize the ordinary ones and reduce to them when $s = \left\lfloor \frac{n}{2} \right\rfloor$. The incomplete tribonacci number, denoted by $t_n^{(s)}$, is defined as the value of $T_n^{(s)}(x)$ at $x = 1$. Incomplete Fibonacci numbers and polynomials have also been considered and are defined in a comparable way; see, e.g., [5, 8]. Some combinatorial identities for the incomplete Fibonacci numbers were given in [3] and a bi-periodic generalization was studied in [9].

In [10], several identities were derived for the incomplete tribonacci numbers and polynomials using various algebraic methods. In this paper, we supply combinatorial proofs of these identities using a weighted tiling interpretation of $T_n^{(s)}(x)$ (described in Theorem 2.1 below). In some cases, a further generalization of an identity can be given. In addition, using our interpretation, one also can find other relations not given in [10] that are satisfied by $T_n^{(s)}(x)$. In the final section, we provide an explicit formula for the generating function of $T_n^{(s)}(x)$, as requested in [10]. Our derivation is combinatorial in nature and makes use of some identities involving $T_n(x)$.

2. Combinatorial interpretation for $T_n^{(s)}(x)$

We will use the following terminology. By a square, domino, or tromino, we will mean, respectively, a $1 \times 1$, $2 \times 1$, or $3 \times 1$ rectangular tile. A (linear) tiling of length $n$ is a covering of the numbers $1, 2, \ldots, n$ written in a row by squares, dominos, and trominos, where tiles of the same kind are indistinguishable. Let $T_n$ denote the set of all tilings of length $n$. It is well known that $T_n$ has cardinality $t_{n+1}$ (see, e.g., [21] p. 36)). We will often represent squares, dominos, and trominos by the letters $r$, $d$, and $t$, respectively. Thus, a member of $T_n$ may be regarded as a word in the alphabet $\{r, d, t\}$ in which there are $n-2i-3j$, $i$, and $j$ occurrences of the letters $r$, $d$, and $t$, respectively, for some $i$ and $j$.

By a longer piece within a member of $T_n$, we will mean one that is either a domino or a tromino. Given $0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$, let $T_n^{(s)}$ denote the subset of $T_n$ whose members contain at most $s$ longer pieces. For example, if $n = 5$ and $s = 1$, then $T_5^{(1)} = \{r^5, dr^3, rrd, r^2dr, r^3d, tr^2, rtt, r^2tr, r^3t^2\}$. Note that $T_n^{(s)}$ is all of $T_n$ when $s = \left\lfloor \frac{n}{2} \right\rfloor$. By a square-and-domino tiling, we will mean a member of $T_n$ that contains no trominos.

Given $\pi \in T_n^{(s)}$, let $\delta(\pi)$ and $\nu(\pi)$ record the number of squares and dominos, respectively, in $\pi$. We now provide a combinatorial interpretation of $T_n^{(s)}(x)$ in terms of linear tilings.

**Theorem 2.1.** The polynomial $T_{n+1}^{(s)}(x)$ is the distribution for the statistic $2\delta + \mu$ on $T_n^{(s)}$.

**Proof.** First note that $T_{n+1}^{(s)}(x)$ may be written as

\[(2.1)\quad T_{n+1}^{(s)}(x) = \sum_{i=0}^{s} B(n-i,i)(x),\]

where $B(n,i)(x) = \sum_{j=0}^{i} \binom{i}{j} (n-j)x^{2n-i-3j}$. We next observe that when $x = 1$, the polynomial $B(n,i)(x)$ gives the cardinality of the set $B_{n,i}$ consisting of square-and-domino tilings of length $n$ in which the squares come in two colors, black and white, and containing $i$ dominos and white squares...
combined. To see this, note that members of $B_{n,i}$ containing exactly $j$ dominos are in one-to-one correspondence with words in the alphabet $\{D, W, B\}$ containing $j$ $D$’s, $i - j$ $W$’s, and $n - i - j$ $B$’s and thus have cardinality

$$\binom{n - j}{j, i - j, n - i - j} = \frac{(n - j)!}{j!(i - j)!(n - i - j)!} = \binom{n - j}{i, j}.$$ 

Summing over $j$ gives

$$|B_{n,i}| = \sum_{j=0}^{i} \binom{i}{j} \binom{n - j}{i}.$$

Given $\pi \in B_{n,i}$, let $\delta_1(\pi)$ and $\delta_2(\pi)$ record the number of black and white squares, respectively. Thus, if $\pi \in B_{n,i}$ has $j$ dominos, then

$$2\delta_1(\pi) + \delta_2(\pi) = 2(n - i - j) + i - j = 2n - i - 3j.$$ 

Considering all $j$, this implies $B(n, i)(x)$ is the distribution on $B_{n,i}$ for the statistic $2\delta_1(\pi) + \delta_2(\pi)$. Suppose now $\lambda \in B_{n-1,i}$ is given and contains $j$ dominos for some $j$, where $0 \leq i \leq s$. We replace each domino of $\lambda$ with a tromino and each white square with a domino. The resulting tiling $\lambda'$ belongs to $\mathcal{T}_n^{(s)}$ and has $j$ trominos, $i - j$ dominos, and $n - 2i - j$ squares. Thus we have

$$2\delta_1(\lambda') + \nu(\lambda') = 2\delta_1(\lambda) + \delta_2(\lambda)$$

for all $\lambda \in B_{n-1,i}$. By (2.1), it follows that $T_{n+1}^{(s)}(x)$ is the distribution on $\bigcup_{i=0}^{s} B_{n-1,i}$ for $2\delta_1 + \delta_2$, equivalently, for the distribution of $2\delta + \nu$ on $\mathcal{T}_n^{(s)}$. \hfill \Box

Remark: Taking $x = 1$ in the prior theorem shows that the cardinality of $\mathcal{T}_n^{(s)}$ is $t_n^{(s)}$. Taking $s = \lfloor \frac{n}{2} \rfloor$ shows that $T_{n+1}(x)$ is the distribution for $2\delta + \mu$ on all of $\mathcal{T}_n$. 

Using our interpretation for $T_n^{(s)}(x)$, one obtains the following recurrence formula from [10] as a corollary.

**Corollary 2.2.** If $n \geq 2s + 1$, then

$$T_{n+3}^{(s)}(x) = x^2 T_{n+2}^{(s)}(x) + x T_{n+1}^{(s)}(x) + T_n^{(s)}(x) - (xB(n - s, s)(x) + B(n - 1 - s, s)(x)).$$

**Proof.** We will show that the right-hand side of (2.2) gives the weighted sum of all the members of $\mathcal{T}_{n+2}^{(s)}$ with respect to the statistic $2\delta + \nu$ by considering the final piece. The first term clearly accounts for all tilings ending in a square. On the other hand, if a member of $\mathcal{T}_{n+2}^{(s)}$ ends in a longer piece, then there can be at most $s - 1$ additional longer pieces. From the proof of Theorem 2.1 above, we have for each $m$ that $B(m - s, s)(x)$ gives the weight of all members of $\mathcal{T}_m^{(s)}$ containing exactly $s$ longer pieces. Note that addition of a longer piece to the end of a tiling already containing $s$ longer pieces is not allowed. Thus, by subtraction, the total weight of all members of $\mathcal{T}_{n+2}^{(s)}$ ending in a domino is given by $x(T_{n+2}^{(s)}(x) - B(n - s, s)(x))$ and the weight of those ending in a tromino by $T_n^{(s)}(x) - B(n - 1 - s, s)(x)$, which completes the proof. \hfill \Box
3. Some identities of $T_n^{(s)}(x)$

In this section, we provide combinatorial proofs of some identities involving the incomplete tribonacci polynomials that generalize those shown in [10] using algebraic methods. We also consider some further identities that can be obtained from the combinatorial interpretation given in Theorem 2.1.

In this section and the next, by the weight of a subset $S$ of $T_n$ or $T_n^{(s)}$, we will mean the sum $\sum_{\lambda \in S} x^{2\ell(\lambda) + \nu(\lambda)}$.

The $x = 1$ case of the following identity was shown in [10] by an inductive argument.

**Identity 3.1.** If $h \geq 1$ and $n \geq 2s + 2$, then

$$\sum_{i=0}^{h-1} x^{2(h-i-1)} T_n^{(s)}(x) = \frac{1}{1 + x^3} \left( T_n^{(s+1)}(x) - x^{2h} T_n^{(s+1)}(x) + x^{2h+1} T_n^{(s)}(x) - x T_n^{(s)}(x) \right).$$

**Proof.** We show equivalently

$$T_n^{(s+1)}(x) = (1 + x^3) \sum_{i=0}^{h-1} x^{2(h-i-1)} T_n^{(s)}(x) + x^{2h} T_n^{(s+1)}(x) + x T_n^{(s)}(x) - x^{2h+1} T_n^{(s)}(x).$$

For this, we’ll argue that the right-hand side gives the total weight of all the members of $T_{n+h+1}^{(s+1)}$. First note that $x^{2h} T_{n+h+2}^{(s+1)}(x)$ gives the weight of the members of $T_{n+h+1}^{(s+1)}$ in which positions $n + 2$ through $n + h + 1$ are covered by squares (i.e., the right-most longer piece ends at position $n + 1$ or before). On the other hand, the weight of all members of $T_{n+h}^{(s+1)}$ whose right-most longer piece starts at position $n + i - 1$ for some $0 \leq i \leq h - 1$ is given by $(x^{2(h-i)} + x^{2(h-i-1)}) T_n^{(s)}(x)$ since such tilings $\lambda$ are of the form $\lambda = \lambda' d r^{h-i}$ or $\lambda = \lambda' t r^{h-i-1}$ for some tiling $\lambda'$ of length $n + i - 1$ where $r^m$ denotes a sequence of $m$ squares. Note that $\lambda' \in T_{n+i-1}^{(s)}$ since the number of longer pieces in $\lambda'$ is limited to $s$. Summing over $0 \leq i \leq h - 1$ gives the indexed sum on the right-hand side. Next, the term $x T_n^{(s)}(x)$ accounts for all members of $T_{n+h+1}^{(s+1)}$ whose final piece is a domino which were missed in the sum. Finally, members of $T_{n+i-1}^{(s+1)}$ of the form $\lambda' d r^h$, where $\lambda'$ has length $n - 1$, were accounted for by both the $x^{2h} T_{n+h+2}^{(s+1)}(x)$ term and by the $i = 0$ term of the indexed sum; hence, we must subtract their weight, $x^{2h+1} T_n^{(s)}(x)$, to correct for this double count. Combining all of the cases above completes the proof. \qed

The following identity from [10] gives a formula for the sum of all the incomplete tribonacci polynomials of a fixed order.

**Identity 3.2.** If $n \geq 1$, then

$$\sum_{s=0}^{\ell} T_n^{(s)}(x) = (\ell + 1) T_n^{(s)}(x) - \sum_{i=0}^{\ell} \sum_{j=0}^{i} \binom{i}{j} \binom{n - i - j}{i} x^{2n - 3(i+j)},$$

where $\ell = \left\lfloor \frac{n}{2} \right\rfloor$.

**Proof.** Let $\lambda \in \mathcal{T}_n$ and suppose that it contains exactly $k$ longer pieces, where $0 \leq k \leq \ell$. Then the weight of $\lambda$ is counted by each summand of $\sum_{s=0}^{\ell} T_n^{(s)}(x)$ such that $s \geq k$. That is, the tiling $\lambda$ is
counted $\ell + 1 - k$ times by this sum. The proof of Theorem 2.1 above shows that the total weight of all members of $T_n$ containing exactly $k$ longer pieces is given by

$$\sum_{j=0}^{k} \binom{k}{j} \binom{n-k-j}{k} x^{2n-3(k+j)},$$

upon considering the number $j$ of dominos. Thus, the only inner sum in the double sum on the right-hand side of (3.2) in which $\lambda$ is counted occurs when $i = k$ and here it is counted $k$ times (due to the extra factor of $i = k$). Since $\lambda$ is clearly counted $\ell + 1$ times by the term $(\ell + 1)T_{n+1}(x)$, we have by subtraction that $\lambda$ is counted $\ell + 1 - k$ times by the right-hand side of (3.2) as well. Since $\lambda$ was arbitrary, the identity follows. 

□

The $x = 1$ case of the following identity was conjectured in [10] and follows from the generating function proof given in [7].

**Identity 3.3.** If $n \geq 1$, then

$$\sum_{s=0}^{\ell} T_{n+1}^{(s)}(x) = (\ell + 1)T_{n+1}(x) - \sum_{j=1}^{n-1} (xT_j(x) + T_{j-1}(x))T_{n-j}(x),$$

where $\ell = \left\lfloor \frac{n}{2} \right\rfloor$.

**Proof.** Suppose $\lambda \in T_n$ has exactly $k$ longer pieces. By the proof of the preceding identity, we need only show that the weight of $\lambda$ is counted $k$ times by the sum on the right-hand side of (3.3). Note that $xT_j(x)T_{n-j}(x)$ gives the weight of all members of $T_n$ in which a domino covers positions $j$ and $j+1$, while $T_{j-1}(x)T_{n-j}(x)$ gives the weight of those in which a tromino covers positions $j-1$, $j$, and $j+1$. Thus, for each longer piece of $\lambda$, there is a term in the sum that counts the weight of $\lambda$, which implies that $\lambda$ is counted $k$ times by the sum, as desired. □

**Remark:** Comparing the $x = 1$ cases of the preceding two identities, it follows that

$$\sum_{n \geq 1} a_n z^n = \frac{z^2 + z^3}{(1 - z - z^2 - z^3)^2},$$

where

$$a_n = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{i}{j} \binom{n-i-j}{i},$$

which can also be shown directly using the methods of [12, Section 4.3] (see [7]).

The next three identities follow from the combinatorial interpretation of $T_n^{(s)}(x)$ given in Theorem 2.1 and do not occur in [10].

**Identity 3.4.** If $n \geq 2s + 1$, then

$$T_{n+1}^{(s)}(x) = \sum_{i=0}^{s} (x^{i+2}T_{n-2i}^{(s-i)}(x) + x^{i}T_{n-2i-2}^{(s-i-1)}(x)).$$
The incomplete Fibonacci polynomials introduced in [8] are given as

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Proof. Suppose that there are

Identity 3.5. If \( n \geq 3s + 1 \), then

\[
T_n(s)(x) = \sum_{i=0}^{s} \sum_{j=0}^{s-i} \binom{s-i}{i, j, s-i-j} x^{2s-i-2j} T_{n-s-i-2j}(x).
\]

Proof. Suppose that there are \( i \) dominos and \( j \) trominos among the final \( s \) tiles within a member of \( T_{n-1}(s) \), where \( n \geq 3s + 1 \). Then there are \( \binom{s}{i, j, s-i-j} \) ways to arrange these tiles, which contribute \( x^{3(s-i-j)+i} \) towards the weight, with the remaining tiles forming a member of \( T_{n-s-i-2j-1}(s-i-j) \). Considering all possible \( i \) and \( j \) gives [3.3].

Our next formula relates the incomplete tribonacci polynomials to the trinomial coefficients.

Identity 3.6. If \( n \geq 2s \), then

\[
T_n(s)(x) = x^{n/2} F_n^{(s)}(x^{3/2}) + \sum_{i=1}^{n-2} x^{i-1/2} \sum_{j=0}^{s-1} \left( T_{n-i-1}(s) - T_{n-i-1}(j) \right) F_{i+j}^{(s-j-1)}(x^{3/2}).
\]

Proof. First note that the weight of all members of \( T_n(s) \) that contain no trominos is given by

\[
\sum_{r=0}^{s} \binom{n-r-1}{r} x^{2n-3r} = x^{n/2} F_n^{(s)}(x^{3/2}).
\]

So assume a member of \( T_n(s) \) contains at least one tromino and that the left-most tromino covers positions \( i \) through \( i + 2 \). Suppose further that there are exactly \( r \) dominos to the left of the left-most tromino. Then the weight of all such members of \( T_n(s) \) is given by \( \binom{i-r-1}{r} x^{2i-3r-2} T_{n-i-1}(s-r-1) \). Summing over the possible \( i \) and \( r \) implies that the total weight of all the members of \( T_n(s) \) containing at least one tromino is

\[
\sum_{i=1}^{n-2} \sum_{r=0}^{s-1} \binom{i-r-1}{r} x^{2i-3r-2} T_{n-i-1}(s-r-1) \].
To obtain the expression in (3.16), we write $T_{n-i-1}^{(s-r-1)}$ as $\sum_{j=0}^{s-r-1}(T_{n-i-1}^{(j)} - T_{n-i-1}^{(j-1)})$, where $T_{n-i-1}^{(-1)} = 0$. We then obtain a total weight formula of

$$
\sum_{i=1}^{n-2} \sum_{r=0}^{s-1} \binom{i-r-1}{r} x^{2i-3r-2} \sum_{j=0}^{s-r-1} (T_{n-i-1}^{(j)} - T_{n-i-1}^{(j-1)}) \\
= \sum_{i=1}^{n-2} \sum_{j=0}^{s-1} (T_{n-i-1}^{(j)} - T_{n-i-1}^{(j-1)}) \sum_{r=0}^{s-j-1} \binom{i-r-1}{r} x^{2i-3r-2} \\
= \sum_{i=1}^{n-2} \sum_{j=0}^{s-1} (T_{n-i-1}^{(j)} - T_{n-i-1}^{(j-1)}) x^{(i-1)/2} F_i^{(s-j-1)}(x^{3/2}),
$$

which gives (3.16).

4. Generating function formula for $T_n^{(s)}(x)$

The generating function formula for the incomplete tribonacci numbers was found in [10] and a formula was requested for the tribonacci polynomials. The next result provides such a formula. We remark that our method is more combinatorial than that used in [10] in the case $x = 1$ and thus supplies an alternate proof in that case.

**Theorem 4.1.** Let $Q_s(z)$ be the generating function for the incomplete tribonacci polynomials $T_n^{(s)}(x)$, where $n \geq 2s + 1$. Then

$$
Q_s(z) = \frac{T_{2s+1}(x) + (T_{2s-1}(x) + xT_{2s-1}(x))z + T_{2s}(x)z^2 - z^2(\frac{x+s}{1-x})^{s+1}}{1 - x^2z - xz^2 - z^3}.
$$

**Proof.** Let $r_n = r_n(x)$ be given by

$$
r_n = \sum_{j=0}^{s} \binom{n+s-j-2}{s} x^{2n+s-3j-3} + \sum_{j=0}^{s} \binom{n+s-j-3}{s} x^{2n+s-3j-6}, \quad n \geq 3,
$$

with $r_0 = r_1 = 0$ and $r_2 = x^{s+1}$. We claim that $r_n(x)$ gives the total weight with respect to the statistic $2\delta + \nu$ of all the members of $T_{i+2s}$ containing exactly $s+1$ longer pieces and ending in a longer piece, the subset of which we will denote by $A$. To show this, first note that $r_i(x)$ evaluated at $x = 1$ is seen to give the number of square-and-domino tilings of length $i + s - 2$ or $i + s - 3$ in which squares are black or white and having exactly $s$ white squares and dominos combined. We then increase the length of each white square and each domino by one and add a domino to the end if the original tiling had length $i + s - 2$ and add a tromino to the end if it had length $i + s - 3$. This yields all members of $A$ in a bijective manner and thus implies $r_i(x)$ at $x = 1$ gives the cardinality of $A$. Note that members of $A$ ending in a domino contain $i - j - 2$ squares, $s - j + 1$ dominos, and $j$ trominos for some $0 \leq j \leq s$, while members of $A$ ending is a tromino contain $i - j - 3$ squares, $s - j$ dominos, and $j + 1$ trominos for some $j$. Summing over $j$ then implies that $r_i(x)$ is the distribution for the statistic $2\delta + \nu$ on $A$, as claimed.

By the interpretation for $r_i(x)$ just described, the product $r_i(x)T_{n-2s-i}(x)$ gives the total weight of all members of $T_{n-1}$ containing at least $s + 1$ longer pieces in which the $(s + 1)$-st longer piece ends at position $i + 2s$ since the final $n - 2s - i - 1$ positions of such a member of $T_{n-1}$ may be covered by
any tiling of the same length. Summing over all possible $i$ then gives the total weight of all members of $T_{n-1}$ containing strictly more than $s$ longer pieces. Subtracting from $T_n(x)$ thus gives the weight of all members of $T_{n-1}$ containing at most $s$ longer pieces and implies the following identity:

$$T_n^{(s)}(x) = T_n(x) - \sum_{i=0}^{n-2s-1} r_i(x)T_{n-2s-i}(x), \quad n \geq 2s + 1. \tag{4.2}$$

In order to find a closed form expression for $Q_s(z)$ using (4.2), we express $T_n = T_n(x)$ as follows:

$$T_n = T_{n-2s}T_{2s+1} + T_{n-2s-1}(T_{2s-1} + xT_{2s}) + T_{n-2s-2}T_{2s}, \quad n \geq 2s + 1. \tag{4.3}$$

We provide a combinatorial proof of (4.3) as follows. Note that (4.3) is clearly true if $s = 0$ or if $n = 2s + 1$ since $T_0 = T_{-1} = 0$, so we may assume $s \geq 1$ and $n \geq 2s + 2$. Observe first that the $T_{n-2s}T_{2s+1}$ term gives the weight of all members of $T_{n-1}$ in which there is no piece covering the boundary between positions $2s$ and $2s + 1$. On the other hand, the total weight of the members of $T_{n-1}$ in which a domino covers this boundary is given by $xT_{n-2s-1}T_{2s}$. Finally, if a tromino covers the boundary between positions $2s$ and $2s + 1$, then that tromino covers either positions $2s - 1$, $2s$, and $2s + 1$ or positions $2s$, $2s + 1$, and $2s + 2$. In the former case, the weight of the corresponding members of $T_{n-1}$ is $T_{n-2s-1}T_{2s-1}$, while in the latter it would be $T_{n-2s-2}T_{2s}$. Combining all of the cases above gives (4.3).

Multiplying both sides of the equation

$$T_n^{(s)} = T_{n-2s}T_{2s+1} + T_{n-2s-1}(T_{2s-1} + xT_{2s}) + T_{n-2s-2}T_{2s} - \sum_{i=0}^{n-2s-1} r_iT_{n-2s-i}$$

by $z^n$ and summing over $n \geq 2s + 1$ yields

$$\frac{Q_s(z)}{z^{2s}} = \left( T_{2s+1} + (T_{2s-1} + xT_{2s})z + T_{2s}z^2 - \sum_{i\geq0} r_iz^i \right) \cdot \sum_{n\geq1} T_nz^n.$$

The proof is completed by noting

$$\sum_{i\geq0} r_iz^i = z^2 \left( \frac{x + z}{1 - x^2z} \right)^{s+1}$$

and

$$\sum_{n\geq1} T_nz^n = \frac{z}{1 - x^2z - xz^2 - z^3},$$

the former being computed by the methods given in [12, Section 4.3].

Taking $x = 1$ in the prior theorem yields the following result.

**Corollary 4.2.** Let $q_s(z)$ be the generating function for the incomplete tribonacci numbers $t_n^{(s)}$. Then

$$\frac{q_s(z)}{z^{2s+1}} = \frac{t_{2s+1} + (t_{2s-1} + t_{2s})z + t_{2s}z^2 - z^2 \left( \frac{1+z}{1-z} \right)^{s+1}}{1 - z - z^2 - z^3}. \tag{4.4}$$

**Remark:** Equation (4.4) appears as Theorem 8 of [10]. We note however that there was a slight misstatement of this theorem; in particular, there should be no $-2$ in the factor multiplying $z^2$ in the numerator on the right-hand side.
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