Automorphisms of Chevalley groups of type $F_4$ over local rings with $1/2$

E. I. Bunina

Abstract.

In the given paper we prove that every automorphism of a Chevalley group of type $F_4$ over a commutative local ring with $1/2$ is standard, i.e., it is a composition of ring and inner automorphisms.

Introduction

An associative commutative ring $R$ with a unit is called local, if it contains exactly one maximal ideal (that coincides with the radical of $R$). Equivalently, the set of all non-invertible elements of $R$ is an ideal.

We describe automorphisms of Chevalley groups of type $F_4$ over local rings with $1/2$. Note that for the root system $F_4$ there exists only one weight lattice, that is simultaneously universal and adjoint, therefore for every ring $R$ there exists a unique Chevalley group of type $F_4$, that is $G(R) = G_{ad}(F_4, R)$. Over local rings universal Chevalley groups coincide with their elementary subgroups, consequently the Chevalley group $G(R)$ is also an elementary Chevalley group.

Theorem 1 for the root systems $A_l$, $D_l$, and $E_l$ was obtained by the author in [12], in [14] all automorphisms of Chevalley groups of given types over local rings with $1/2$ were described.

Theorem 1 for the root systems $B_2$ and $G_2$ is proved in [13].

Similar results for Chevalley groups over fields were proved by R. Steinberg [52] for the finite case and by J. Humphreys [37] for the infinite case. Many papers were devoted to description of automorphisms of Chevalley groups over different commutative rings, we can mention here the papers of Borel–Tits [10], Carter–Chen Yu [17], Chen Yu [18–22], A. Klyachko [41]. E. Abe [11] proved that all automorphisms of Chevalley groups under Noetherian rings with $1/2$ are standard.

The case $A_l$ was completely studied by the papers of W.C. Waterhouse [65], V.M. Petechuk [45], Fuan Li and Zunxian Li [40], and also for rings without $1/2$. The paper of I.Z. Golubchik and A.V. Mikhalev [31] covers the case $C_l$, that is not considered in the present paper. Automorphisms and isomorphisms of general linear groups over arbitrary associative rings were described by E.I. Zelmanov in [70] and by I.Z. Golubchik, A.V. Mikhalev in [32].

We generalize some methods of V.M. Petechuk [46] to prove Theorem 1.

The author is thankful to N.A. Vavilov, A.A. Klyachko, A.V. Mikhalev for valuable advices, remarks and discussions.

1The work is supported by the Russian President grant MK-2530.2008.1 and by the grant of Russian Fond of Basic Research 08-01-00693.
1. Definitions and main theorems.

We fix the root system $\Phi$ of the type $F_4$ (detailed texts about root systems and their properties can be found in the books [38], [15]). Let $e_1, e_2, e_3, e_4$ be an orthonorm basis of the space $\mathbb{R}^4$. Then we numerate the roots of $F_4$ as follows:

$$\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$$

are simple roots;

$$\begin{align*}
\alpha_5 &= \alpha_1 + \alpha_2 = e_2 - e_4, \\
\alpha_6 &= \alpha_2 + \alpha_3 = e_3, \\
\alpha_7 &= \alpha_3 + \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 + e_4), \\
\alpha_8 &= \alpha_1 + \alpha_2 + \alpha_3 = e_2, \\
\alpha_9 &= \alpha_2 + \alpha_3 + \alpha_4 = \frac{1}{2}(e_1 - e_2 + e_3 - e_4), \\
\alpha_{10} &= \alpha_2 + 2\alpha_3 = e_3 + e_4, \\
\alpha_{11} &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4), \\
\alpha_{12} &= \alpha_1 + \alpha_2 + 2\alpha_3 = e_2 + e_4, \\
\alpha_{13} &= \alpha_2 + 2\alpha_3 + \alpha_4 = \frac{1}{2}(e_1 - e_2 + e_3 + e_4), \\
\alpha_{14} &= \alpha_1 + 2\alpha_2 + 2\alpha_3 = e_2 + e_3, \\
\alpha_{15} &= \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = \frac{1}{2}(e_1 + e_2 - e_3 + e_4), \\
\alpha_{16} &= \alpha_2 + 2\alpha_3 + 2\alpha_4 = e_1 - e_2, \\
\alpha_{17} &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4), \\
\alpha_{18} &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 = e_1 - e_3, \\
\alpha_{19} &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4), \\
\alpha_{20} &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 = e_1 - e_4, \\
\alpha_{21} &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = e_1, \\
\alpha_{22} &= \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 = e_1 + e_4, \\
\alpha_{23} &= \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = e_1 + e_3, \\
\alpha_{24} &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = e_1 + e_2
\end{align*}$$

are other positive roots.

Suppose now that we have a semisimple complex Lie algebra $L$ of type $F_4$ with Cartan subalgebra $H$ (detailed information about semisimple Lie algebras can be found in the book [38]).
Then in the algebra $\mathcal{L}$ we can choose a Chevalley basis $\{h_i \mid i = 1, \ldots, 4; x_\alpha \mid \alpha \in \Phi\}$ so that for every two elements of this basis their commutator is an integral linear combination of the elements of the same basis.

Namely,

1) $[h_i, h_j] = 0$;
2) $[h_i, x_\alpha] = (\alpha_i, \alpha)x_\alpha$;
3) if $\alpha = n_1\alpha_1 + \cdots + n_4\alpha_4$, then $[x_\alpha, x_{-\alpha}] = n_1h_1 + \cdots + n_4h_4$;
4) if $\alpha + \beta \notin \Phi$, then $[x_\alpha, x_\beta] = 0$;
5) if $\alpha + \beta \in \Phi$, and $\alpha, \beta$ are roots of the same length, then $[x_\alpha, x_\beta] = c[x_{\alpha + \beta}]$;
6) if $\alpha + \beta \in \Phi$, $\alpha$ is a long root, $\beta$ is a short root, then $[x_\alpha, x_\beta] = ax_{\alpha + \beta} + bx_{\alpha + 2\beta}$.

Take now an arbitrary local ring with $1/2$ and construct an elementary adjoint Chevalley group of type $F_4$ over this ring (see, for example [51]). For our convenience we briefly put here the construction.

In the Chevalley basis of $\mathcal{L}$ all operators $(x_\alpha)^k/k!$ for $k \in \mathbb{N}$ are written as integral (nilpotent) matrices. An integral matrix also can be considered as a matrix over an arbitrary commutative ring with $1$. Let $R$ be such a ring. Consider matrices $n \times n$ over $R$, matrices $(x_\alpha)^k/k!$ for $\alpha \in \Phi$, $k \in \mathbb{N}$ are included in $\text{M}_n(R)$.

Now consider automorphisms of the free module $R^n$ of the form

$$\exp(tx_\alpha) = x_\alpha(t) = 1 + tx_\alpha + t^2(x_\alpha)^2/2 + \cdots + t^k(x_\alpha)^k/k! + \ldots$$

Since all matrices $x_\alpha$ are nilpotent, we have that this series is finite. Automorphisms $x_\alpha(t)$ are called elementary root elements. The subgroup in $\text{Aut}(R^n)$, generated by all $x_\alpha(t)$, $\alpha \in \Phi$, $t \in R$, is called an elementary adjoint Chevalley group (notation: $E_{\text{ad}}(\Phi, R) = E_{\text{ad}}(R)$).

In an elementary Chevalley group there are the following important elements:

$-w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$, $\alpha \in \Phi$, $t \in R^*$;

$h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$.

The action of $x_\alpha(t)$ on the Chevalley basis is described in [16], [64], we write it below (see Section 3).

Over local rings for the root system $F_4$ all Chevalley groups coincide with elementary adjoint Chevalley groups $E_{\text{ad}}(R)$, therefore we do not introduce Chevalley groups themselves in this paper. In this paper we denote our Chevalley groups by $G(R)$, since they depend only of a ring $R$.

We will work with two types of standard automorphisms of a Chevalley group $G(R)$ and with one unusual, “temporary” type of automorphisms.

Ring automorphisms. Let $\rho : R \to R$ be an automorphism of the ring $R$. The mapping $x \mapsto \rho(x)$ from $G(R)$ onto itself is an automorphism of the group $G(R)$, that is denoted by the same letter $\rho$ and is called a ring automorphism of the group $G(R)$. Note that for all $\alpha \in \Phi$ and $t \in R$ an element $x_\alpha(t)$ is mapped to $x_\alpha(\rho(t))$.

Inner automorphisms. Let $g \in G(R)$ be an element of a Chevalley group under consideration. Conjugation of the group $G(R)$ with the element $g$ is an automorphism of $G(R)$, that is denoted by $\iota_g$ and is called an inner automorphism of $G(R)$.

These two types of automorphisms are called standard. There are central and graph automorphisms, which are also standard, but in our case (root system $F_4$) they can not appear.
Therefore we say that an automorphism of the group $G(R)$ is standard, if it is a composition of ring and inner automorphisms.

Besides that, we need also to introduce temporarily one more type of automorphisms:

**Automorphisms–conjugations.** Let $V$ be a representation space of the Chevalley group $G(R)$, $C \in \text{GL}(V)$ be a matrix from the normalizer of $G(R)$:

$$CG(R)C^{-1} = G(R).$$

Then the mapping $x \mapsto CxC^{-1}$ from $G(R)$ onto itself is an automorphism of the Chevalley group, which is denoted by $i$ and is called an *automorphism–conjugation* of $G(R)$, *induced by the element* $C$ *of the group* $\text{GL}(V)$.

In Section 5 we will prove that in our case all automorphisms–conjugations are inner, but the first step is the proof of the following theorem:

**Theorem 1.** Let $G(R)$ be a Chevalley group of type $F_4$, where $R$ is a commutative local ring with $1/2$. Then every automorphism of $G(R)$ is a composition of a ring automorphism and an automorphism–conjugation.

Sections 2–4 are devoted to the proof of Theorem 1.

2. **Changing the initial automorphism to a special isomorphism, images of $w_{\alpha_i}$**

Since in the papers [12] and [13] the root system in there second sections was arbitrary, we can suppose all results of these sections to be proved also for our root system $F_4$.

Namely, by the fixed automorphism $\varphi$ we can construct a mapping $\varphi' = i_{g^{-1}}\varphi$, which is an isomorphism of the group $G(R) \subset \text{GL}_n(R)$ onto some subgroup of $\text{GL}_n(R)$ with the property that its image under factorization $R$ by $J$ (the radical of $R$) coincides with a ring automorphism $\overline{\varphi}$.

Besides, from sections 2 of the same papers we know that the image of any involution (a matrix of order 2) under such an isomorphism is conjugate to this involution in the group $\text{GL}_n(R)$.

These are the main facts that we need to know.

The order of roots we have fixed in the previous section.

The basis of the space $V$ (52-dimensional) we numerate as $v_i = x_{\alpha_i}$, $v_{-i} = x_{-\alpha_i}$, $V_1 = h_1$, \ldots, $V_4 = h_4$. 
Consider the matrices \( h_{\alpha_1}(-1), \ldots, h_{\alpha_4}(-1) \) in our basis. They have the form

\[
\begin{align*}
   h_{\alpha_1}(-1) &= \text{diag}[1, 1, -1, -1, 1, 1, 1, -1, -1, -1, 1, 1, -1, -1, -1, -1, -1, -1, -1], \\
   h_{\alpha_2}(-1) &= \text{diag}[-1, -1, 1, 1, -1, -1, 1, -1, -1, -1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1], \\
   h_{\alpha_3}(-1) &= \text{diag}[1, 1, 1, 1, -1, -1, 1, -1, -1, -1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1], \\
   h_{\alpha_4}(-1) &= \text{diag}[1, 1, 1, -1, -1, 1, 1, -1, -1, -1, -1, -1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1].
\end{align*}
\]

As we see, for all \( i \) we have \( h_{\alpha_i}(-1)^2 = 1 \).

We know that every matrix \( h_i = \varphi'(h_{\alpha_i}(-1)) \) in some basis is diagonal with \( \pm 1 \) on its diagonal, and the number of 1 and \( -1 \) coincides with their number for the matrix \( h_{\alpha_i}(-1) \). Since all matrices \( h_i \) commute, then there exists a basis, where all \( h_i \) has the same form as \( h_{\alpha_i}(-1) \) in the initial basis from weight vectors. Suppose that we came to this basis with the help of the matrix \( g_1 \). Clear that \( g_1 \in \text{GL}_n(R, J) = \{ X \in \text{GL}_n(R) \mid X - E \in M_\mathbb{R}(J) \} \). Consider the mapping \( \varphi_1 = \bar{g}_1^{-1} \varphi' \). It is also an isomorphism of the group \( G(R) \) onto some subgroup of \( \text{GL}_n(R) \) such that its image under factorization \( R \) by \( J \) is \( \overline{\rho} \), and \( \varphi_1(h_{\alpha_i}(-1)) = h_{\alpha_i}(-1) \) for all \( i = 1, \ldots, 4 \).

Instead of \( \varphi' \) we now consider the isomorphism \( \varphi_1 \).

Every element \( w_i = w_{\alpha_i}(1) \) moves by conjugation \( h_i \) to each other, therefore its image has a block-monomial form. In particular, this image can be rewritten as a block-diagonal matrix, where the first block is \( 48 \times 48 \), and the second is \( 4 \times 4 \).

Consider the first basis vector after the last basis change. Denote it by \( e \). The Weil group \( W \) acts transitively on the set of roots of the same length, therefore for every root \( \alpha_i \) of the same length as the first one, there exists such \( w^{(\alpha_i)} \in W \), that \( w^{(\alpha_i)}\alpha_1 = \alpha_i \). Similarly, all roots of the second length are also conjugate under the action of \( W \). Let \( \alpha_k \) be the first root of the length that is not equal to the length of \( \alpha_1 \), and let \( f \) be the \( k \)-th basis vector after the last basis change. If \( \alpha_j \) is a root conjugate to \( \alpha_k \), then let us denote by \( w_{(\alpha_j)} \) an element of \( W \) such that \( w_{(\alpha_j)}\alpha_k = \alpha_j \). Consider now the basis \( e_1, \ldots, e_{48}, e_{49}, \ldots, e_{52} \), where \( e_1 = e, e_k = f \), for \( 1 < i \leq 48 \) either \( e_i = \varphi_1(w^{(\alpha_i)})e \), or \( e_i = \varphi_1(w^{(\alpha_i)})f \) (it depends of the length of \( \alpha_k \)); for \( 48 < i \leq 52 \) we do not move \( e_i \). Clear that the matrix of this basis change is equivalent to the unit modulo radical. Therefore the obtained set of vectors also is a basis.

Clear that a matrix for \( \varphi_1(w_i) \) \( (i = 1, \ldots, 4) \) in the basis part \( \{ e_1, \ldots, e_{2n} \} \) coincides with the matrix for \( w_i \) in the initial basis of weight vectors. Since \( h_i(-1) \) are squares of \( w_i \), then there images are not changed in the new basis.

Besides, we know that every matrix \( \varphi_1(w_i) \) is block-diagonal up to decomposition of basis in the first 48 and last 4 elements. Therefore the last part of basis consisting of 4 elements, can be changed independently.

Initially (in the basis of weight vectors) \( w_i \) in this basis part are
\[
\begin{align*}
    w_1 : & \begin{pmatrix}
    -1 & 1 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    \end{pmatrix}, & \\
    w_2 : & \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    1 & -1 & 1 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    \end{pmatrix}, & \\
    w_3 : & \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 2 & -1 & 1 \\
    0 & 0 & 0 & 1 \\
    \end{pmatrix}, & \\
    w_4 : & \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 1 & -1 \\
    \end{pmatrix}.
\end{align*}
\]

We have the following conditions for these elements (on the given basis part):

1) for all \( i \) \( w_i^2 = E \);
2) \( w_i \) and \( w_j \) commute for \(|i - j| > 1\);
3) \( w_1 w_2 \) and \( w_3 w_4 \) have order 3, \( w_2 w_3 \) has order 2.

Therefore the images \( \varphi_1(w_i) \) satisfy the same conditions. Besides, we know, that these images are equivalent to the initial \( w_1 \) modulo \( \phi_1 \).

Let us make the basis change with the matrix, which is a product of (commuting with each other) matrices
\[
\begin{pmatrix}
    1 & 1/2 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{pmatrix} \times \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 1 & 1 & 1 \\
    0 & 0 & 0 & 2 \\
\end{pmatrix}.
\]

In this basis \( w_1 = \text{diag} [-1, 1, 1, 1] \), \( w_3 = \text{diag} [1, 1, -1, 1] \),
\[
\begin{pmatrix}
    1/2 & 1/4 & -1/2 & -1/2 \\
    1 & 1/2 & 1 & 1 \\
    -1 & 1/2 & 0 & -1 \\
    0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    -1/2 & 1/2 & 3/2 \\
    0 & 1/2 & 1/2 & -1/2 \\
\end{pmatrix}.
\]

Consider now the images of \( \varphi_1(w_i) \) in the changed basis. All these images are involutions, and every of them has exactly one \(-1\) in its diagonal form, also \( \varphi_1(w_1) \) and \( \varphi_1(w_3) \) commute. Hence we can choose such a basis (equivalent to the previous one modulo \( J \)), where \( \varphi_1(w_1) \) and \( \varphi_1(w_3) \) have a diagonal form with one \(-1\) on the corresponding places.

Consider now where \( w_4 \) can move under this basis change.

Since \( \varphi_1(w_4) \) commutes with \( \varphi_1(w_1) \), has order two and is equivalent to \( w_4 \) modulo radical, we have
\[
\varphi_1(w_4) = \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & a & b & c \\
    0 & d & e & f \\
    0 & g & h & i \\
\end{pmatrix}.
\]

Use the facts that \( \varphi_1(w_4)^2 = E \), \( \varphi_1(w_3 w_4) \) has order 3. Then we obtain
\[
\begin{cases}
    ad + de + fg = 0, \\
    ad - de + fg = -d,
\end{cases}
\]
therefore \( 2de = d \), and since \( d \equiv 1/2 \mod J \), we have \( e = 1/2 \). Moreover,
\[
\begin{cases}
    ag + dh + gi = 0, \\
    ag - dh + gi = g
\end{cases}
\]
consequently $2g(a + i) = g$, i.e., $a + i = 1/2$. Make now a basis change with the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & \frac{a-1}{g} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

This change does not move the elements $\varphi_1(w_1)$ and $\varphi_1(w_3)$, and $\varphi_1(w_4)$ now has the form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & b & c \\
0 & d & 1/2 & f \\
0 & g & h & -1/2
\end{pmatrix}.
$$

Using the above conditions, we obtain the equation $bg + eh + hi = 0$, consequently $bg = 0$, i.e., $b = 0$. In this case from $a^2 + bd + cg = 1$ it follows $c = 0$. All other conditions gives the system

$$
\begin{cases}
f g = -3/2d, \\
d h = -g/2, \\
f h = -1/4.
\end{cases}
$$

Clear that with a diagonal basis change (which does not move $\varphi_1(w_1)$ and $\varphi_1(w_3)$) we can come to a basis, where $\varphi_1(w_4)$ has the same form as $w_4$ after the first our basis change. Making now the inverse basis change, we obtain that $\varphi_1(w_1), \varphi_1(w_3)$ and $\varphi(w_4)$ have the same form as $w_1, w_3, w_4$, respectively. Look at $\varphi_1(w_2)$.

Since $\varphi_1(w_2)$ commutes with $\varphi_1(w_4)$, we have

$$
\varphi_1(w_2) = \begin{pmatrix}
a & b & c & 0 \\
d & e & f & 0 \\
g & i & h & 0 \\
g/2 & i/2 & k & h - 2k
\end{pmatrix}.
$$

Since $(h - 2k)^2 = 1$, we have $h - 2k = 1$. Now similarly to the consideration of $\varphi_1(w_4)$, we take the conditions for $\varphi_1(w_2)$. After suitable diagonal change we get $\varphi_1(w_i) = w_i$ in the new last basis.

Therefore we can now come from the isomorphism $\varphi_1$ under consideration to an isomorphism $\varphi_2$ with all properties of $\varphi_1$ and such that $\varphi_2(w_i) = w_i$ for all $i = 1, \ldots, 4$.

We suppose now that an isomorphism $\varphi_2$ with all these properties is given.
3. Images of $x_\alpha(1)$ and diagonal matrices.

Let us write the matrices $w_i$, $i = 1, \ldots, 4$:

\[
w_1 = -e_{\alpha_1, \alpha_1} - e_{-\alpha_1, \alpha_1} + e_{\alpha_2, \alpha_5} + e_{-\alpha_2, \alpha_5} - e_{\alpha_5, \alpha_2} - e_{-\alpha_5, \alpha_2} + e_{\alpha_3, \alpha_3} + e_{-\alpha_3, \alpha_3} + e_{\alpha_4, \alpha_4} + e_{-\alpha_4, \alpha_4} + e_{\alpha_6, \alpha_6} + e_{-\alpha_6, \alpha_6} - e_{\alpha_8, \alpha_8} - e_{-\alpha_8, \alpha_8} + e_{\alpha_7, \alpha_7} + e_{-\alpha_7, \alpha_7} + e_{\alpha_9, \alpha_9} + e_{-\alpha_9, \alpha_9} + e_{\alpha_{10}, \alpha_{12}} + e_{-\alpha_{10}, \alpha_{12}} - e_{\alpha_{12}, \alpha_{10}} - e_{-\alpha_{12}, \alpha_{10}} + e_{\alpha_{13}, \alpha_{15}} + e_{-\alpha_{13}, \alpha_{15}} - e_{\alpha_{15}, \alpha_{13}} - e_{-\alpha_{15}, \alpha_{13}} + e_{\alpha_{14}, \alpha_{14}} + e_{-\alpha_{14}, \alpha_{14}} + e_{\alpha_{16}, \alpha_{18}} + e_{-\alpha_{16}, \alpha_{18}} - e_{\alpha_{18}, \alpha_{16}} - e_{-\alpha_{18}, \alpha_{16}} + e_{\alpha_{17}, \alpha_{17}} + e_{-\alpha_{17}, \alpha_{17}} + e_{\alpha_{19}, \alpha_{19}} + e_{-\alpha_{19}, \alpha_{19}} + e_{\alpha_{20}, \alpha_{20}} + e_{-\alpha_{20}, \alpha_{20}} + e_{\alpha_{21}, \alpha_{21}} + e_{\alpha_{22}, \alpha_{22}} + e_{-\alpha_{22}, \alpha_{22}} + e_{\alpha_{23}, \alpha_{24}} + e_{-\alpha_{23}, \alpha_{24}} - e_{\alpha_{24}, \alpha_{23}} - e_{-\alpha_{24}, \alpha_{23}} - e_{h_1, h_1} + e_{h_1, h_2} + e_{h_2, h_2} + e_{h_3, h_3} + e_{h_4, h_4} ;
\]

\[
w_2 = -e_{\alpha_2, \alpha_2} - e_{-\alpha_2, \alpha_2} + e_{\alpha_1, \alpha_5} + e_{-\alpha_1, \alpha_5} - e_{\alpha_5, \alpha_1} - e_{-\alpha_5, \alpha_1} - e_{\alpha_3, \alpha_6} - e_{-\alpha_3, \alpha_6} + e_{\alpha_4, \alpha_4} + e_{-\alpha_4, \alpha_4} + e_{\alpha_7, \alpha_7} + e_{-\alpha_7, \alpha_7} + e_{\alpha_9, \alpha_9} + e_{-\alpha_9, \alpha_9} + e_{\alpha_{10}, \alpha_{10}} + e_{-\alpha_{10}, \alpha_{10}} + e_{\alpha_{11}, \alpha_{11}} + e_{-\alpha_{11}, \alpha_{11}} + e_{\alpha_{12}, \alpha_{12}} + e_{-\alpha_{12}, \alpha_{12}} + e_{\alpha_{13}, \alpha_{13}} + e_{-\alpha_{13}, \alpha_{13}} + e_{\alpha_{14}, \alpha_{14}} + e_{-\alpha_{14}, \alpha_{14}} + e_{\alpha_{15}, \alpha_{15}} + e_{-\alpha_{15}, \alpha_{15}} + e_{\alpha_{16}, \alpha_{16}} + e_{-\alpha_{16}, \alpha_{16}} + e_{\alpha_{17}, \alpha_{17}} + e_{-\alpha_{17}, \alpha_{17}} + e_{\alpha_{18}, \alpha_{18}} + e_{-\alpha_{18}, \alpha_{18}} + e_{\alpha_{19}, \alpha_{19}} + e_{-\alpha_{19}, \alpha_{19}} + e_{\alpha_{20}, \alpha_{20}} + e_{-\alpha_{20}, \alpha_{20}} + e_{\alpha_{21}, \alpha_{21}} + e_{\alpha_{22}, \alpha_{22}} + e_{-\alpha_{22}, \alpha_{22}} + e_{\alpha_{23}, \alpha_{23}} + e_{-\alpha_{23}, \alpha_{23}} + e_{\alpha_{24}, \alpha_{24}} + e_{-\alpha_{24}, \alpha_{24}} + e_{h_1, h_1} + e_{h_2, h_1} + e_{h_3, h_2} + e_{h_4, h_3} + e_{h_4, h_4} ;
\]

\[
w_3 = e_{\alpha_1, \alpha_1} + e_{-\alpha_1, \alpha_1} + e_{\alpha_2, \alpha_10} + e_{-\alpha_2, \alpha_10} + e_{\alpha_{10}, \alpha_2} + e_{-\alpha_{10}, \alpha_2} - e_{\alpha_3, \alpha_3} - e_{-\alpha_3, \alpha_3} + e_{\alpha_4, \alpha_4} - e_{-\alpha_4, \alpha_4} - e_{\alpha_5, \alpha_5} - e_{-\alpha_5, \alpha_5} + e_{\alpha_6, \alpha_6} - e_{-\alpha_6, \alpha_6} - e_{\alpha_7, \alpha_7} - e_{-\alpha_7, \alpha_7} + e_{\alpha_8, \alpha_8} - e_{-\alpha_8, \alpha_8} + e_{\alpha_9, \alpha_9} - e_{-\alpha_9, \alpha_9} - e_{\alpha_{10}, \alpha_{10}} - e_{-\alpha_{10}, \alpha_{10}} - e_{\alpha_{11}, \alpha_{11}} - e_{-\alpha_{11}, \alpha_{11}} + e_{\alpha_{12}, \alpha_{12}} + e_{-\alpha_{12}, \alpha_{12}} + e_{\alpha_{13}, \alpha_{13}} + e_{-\alpha_{13}, \alpha_{13}} + e_{\alpha_{14}, \alpha_{14}} + e_{-\alpha_{14}, \alpha_{14}} + e_{\alpha_{15}, \alpha_{15}} + e_{-\alpha_{15}, \alpha_{15}} + e_{\alpha_{16}, \alpha_{16}} + e_{-\alpha_{16}, \alpha_{16}} + e_{\alpha_{17}, \alpha_{17}} + e_{-\alpha_{17}, \alpha_{17}} + e_{\alpha_{18}, \alpha_{18}} + e_{-\alpha_{18}, \alpha_{18}} + e_{\alpha_{19}, \alpha_{19}} + e_{-\alpha_{19}, \alpha_{19}} + e_{\alpha_{20}, \alpha_{20}} + e_{-\alpha_{20}, \alpha_{20}} + e_{\alpha_{21}, \alpha_{21}} + e_{\alpha_{22}, \alpha_{22}} + e_{-\alpha_{22}, \alpha_{22}} + e_{\alpha_{23}, \alpha_{23}} + e_{-\alpha_{23}, \alpha_{23}} + e_{\alpha_{24}, \alpha_{24}} + e_{-\alpha_{24}, \alpha_{24}} + e_{h_1, h_1} + e_{h_2, h_2} + e_{h_3, h_2} + e_{h_4, h_3} + e_{h_4, h_4} ;
\]

\[
w_4 = e_{\alpha_1, \alpha_1} + e_{-\alpha_1, \alpha_1} + e_{\alpha_2, \alpha_2} + e_{-\alpha_2, \alpha_2} + e_{\alpha_3, \alpha_7} - e_{-\alpha_3, \alpha_7} + e_{\alpha_7, \alpha_3} + e_{-\alpha_7, \alpha_3} - e_{\alpha_4, \alpha_4} + e_{-\alpha_4, \alpha_4} + e_{\alpha_5, \alpha_5} + e_{-\alpha_5, \alpha_5} + e_{\alpha_6, \alpha_6} + e_{-\alpha_6, \alpha_6} + e_{\alpha_9, \alpha_9} + e_{-\alpha_9, \alpha_9} + e_{\alpha_{10}, \alpha_{10}} + e_{-\alpha_{10}, \alpha_{10}} + e_{\alpha_{11}, \alpha_{11}} + e_{-\alpha_{11}, \alpha_{11}} + e_{\alpha_{12}, \alpha_{12}} + e_{-\alpha_{12}, \alpha_{12}} + e_{\alpha_{13}, \alpha_{13}} + e_{-\alpha_{13}, \alpha_{13}} + e_{\alpha_{15}, \alpha_{15}} + e_{-\alpha_{15}, \alpha_{15}} + e_{\alpha_{16}, \alpha_{16}} + e_{-\alpha_{16}, \alpha_{16}} + e_{\alpha_{17}, \alpha_{17}} + e_{-\alpha_{17}, \alpha_{17}} + e_{\alpha_{18}, \alpha_{18}} + e_{-\alpha_{18}, \alpha_{18}} + e_{\alpha_{19}, \alpha_{19}} + e_{-\alpha_{19}, \alpha_{19}} + e_{\alpha_{20}, \alpha_{20}} + e_{-\alpha_{20}, \alpha_{20}} + e_{\alpha_{21}, \alpha_{21}} + e_{\alpha_{22}, \alpha_{22}} + e_{-\alpha_{22}, \alpha_{22}} + e_{\alpha_{23}, \alpha_{23}} + e_{-\alpha_{23}, \alpha_{23}} + e_{\alpha_{24}, \alpha_{24}} + e_{-\alpha_{24}, \alpha_{24}} + e_{h_1, h_1} + e_{h_2, h_2} + e_{h_3, h_2} + e_{h_4, h_3} + e_{h_4, h_4} .
\]
Besides that, \( x_{\alpha_1}(t) = E + tX_1 + t^2X_1^2/2 \), where

\[
X_1 = 2e_{\alpha_1,h_1} - e_{\alpha_1,h_2} - e_{h_1,-\alpha_1} + e_{\alpha_5,a_2} - e_{-\alpha_2,-a_5} + e_{\alpha_8,a_6} - e_{-a_6,-a_8} +
+ e_{\alpha_{11},a_9} - e_{-a_9,-a_{11}} + e_{\alpha_{12},a_{10}} - e_{-a_{10},-a_{12}} + e_{\alpha_{15},a_{13}} - e_{-a_{13},-a_{15}} +
+ e_{\alpha_{18},a_{16}} - e_{-a_{16},-a_{18}} + e_{\alpha_{24},a_{23}} - e_{-a_{23},-a_{24}};
\]

\[
x_{\alpha_3}(t) = E + tX_3 + t^2X_3^2/2,
\]

where

\[
X_3 = -2e_{\alpha_3,h_2} + 2e_{\alpha_3,h_3} - e_{\alpha_3,h_4} - e_{h_3,-\alpha_3} + e_{\alpha_7,\alpha_4} - e_{-\alpha_4,-\alpha_7} +
+ e_{\alpha_{13},a_9} - e_{-a_9,-a_{13}} + e_{\alpha_{15},a_{11}} - e_{-a_{11},-a_{15}} + e_{\alpha_{19}a_{17}} - e_{-a_{17},-a_{19}} -
- 2e_{\alpha_6,a_2} + e_{-\alpha_2,-a_6} - e_{\alpha_{10},a_6} + 2e_{-a_6,-a_{10}} - 2e_{\alpha_8,a_5} + e_{-a_5,-a_8} -
- e_{\alpha_{12},a_8} + 2e_{-a_8,a_{12}} - 2e_{\alpha_2,a_{20}} + e_{-a_20,-a_{21}} - e_{\alpha_{22},a_{21}} + 2e_{-a_{21},-a_{22}}.
\]

We are interested in images of \( x_{\alpha}(t) \). Let \( \varphi_2(x_{\alpha}(1)) = x_1 = (y_{i,j}) \). Since \( x_1 \) commutes with all \( h_{\alpha_i}(-1) \), \( i = 1, 3, 4 \), and also with \( w_3, w_4, \) and \( w_{14}, \) then by direct calculus we obtain:

1. The matrix \( x_1 \) can be decomposed into following eight diagonal blocks:

\[
B_1 = \{v_1, v_{-1}, v_{14}, v_{-14}, v_{20}, v_{-20}, v_{22}, v_{-22}, V_1, V_2, V_3, V_4\};
\]

\[
B_2 = \{v_2, v_{-2}, v_5, v_{-5}, v_{10}, v_{-10}, v_{16}, v_{-16}, v_{18}, v_{-18}, v_{23}, v_{-23}, v_{24}, v_{-24}\};
\]

\[
B_3 = \{v_3, v_{-3}, v_{21}, v_{-21}\};
\]

\[
B_4 = \{v_4, v_{-4}, v_{17}, v_{-17}\};
\]

\[
B_5 = \{v_6, v_{-6}, v_8, v_{-8}\};
\]

\[
B_6 = \{v_7, v_{-7}, v_{19}, v_{-19}\};
\]

\[
B_7 = \{v_9, v_{-9}, v_{11}, v_{-11}\};
\]

\[
B_8 = \{v_{13}, v_{-13}, v_{15}, v_{-15}\}.
\]

2. On the block \( B_1 \) the matrix \( x_1 \) has the form

\[
\begin{pmatrix}
 y_1 & y_2 & -y_3 & y_5 & -y_7 & y_7 & -y_8 & y_8 & 0 & 0 \\
 y_5 & y_6 & -y_7 & y_7 & -y_7 & y_7 & -2y_8 & y_8 & 0 & 0 \\
y_9 & y_{10} & y_{11} & -y_{12} & y_{13} & -y_{13} & y_{13} & -2y_{14} + 2y_{15} & y_{14} & 0 & -y_{15} \\
-y_9 & -y_{10} & y_{12} & y_{11} & -y_{13} & -y_{13} & -y_{13} & 2y_{14} - 2y_{15} & -y_{14} + 2y_{15} & 0 & y_{15} \\
y_9 & y_{10} & -y_{13} & y_{11} & y_{12} & -y_{13} & y_{13} & 2(-y_{14} + y_{15}) & y_{14} & -y_{15} & y_{15} \\
-y_9 & -y_{10} & y_{13} & -y_{13} & y_{12} & -y_{13} & y_{13} & 2(y_{14} - y_{15}) & -y_{14} + 2y_{15} & -y_{15} & y_{15} \\
y_9 & y_{10} & -y_{13} & -y_{13} & y_{13} & y_{12} & y_{11} & 2(-y_{14} + y_{15}) & -y_{14} + 2y_{15} & y_{15} & 0 \\
-y_9 & -y_{10} & y_{13} & -y_{13} & -y_{13} & y_{12} & y_{11} & 2(y_{14} - y_{15}) & -y_{14} & y_{15} & 0 \\
y_{16} & -y_{17} & y_{18} & -y_{18} & y_{18} & -y_{18} & y_{18} & y_{19} - 2y_{20} & y_{20} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{20} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{20} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{20}
\end{pmatrix}
\]
3. On the block \( B_2 \) it is

\[
\begin{pmatrix}
  y_{21} & y_{22} & -y_{23} & -y_{24} & -y_{25} & y_{26} & y_{27} & y_{28} & -y_{25} & -y_{26} & y_{27} & y_{28} & y_{29} & y_{27} & y_{26} & y_{25} \\
y_{29} & y_{30} & -y_{31} & y_{32} & -y_{33} & -y_{34} & y_{35} & y_{36} & y_{37} & y_{38} & y_{39} & y_{40} & y_{41} & y_{42} & y_{43} & y_{44} & y_{45} & y_{46} \\
y_{47} & y_{48} & -y_{49} & y_{50} & -y_{51} & y_{52} & y_{53} & y_{54} & y_{55} & y_{56} & y_{57} & y_{58} & y_{59} & y_{60} & y_{61} & y_{62} & y_{63} & y_{64} \\
y_{65} & y_{66} & -y_{67} & y_{68} & -y_{69} & y_{70} & y_{71} & y_{72} & y_{73} & y_{74} & y_{75} & y_{76} & y_{77} & y_{78} & y_{79} & y_{80} & y_{81} & y_{82} \\
\end{pmatrix}
\]

4. On the blocks \( B_3, B_4, B_6 \) it has the form

\[
\begin{pmatrix}
y_{36} & y_{37} & y_{38} & y_{39} \\
y_{37} & y_{36} & y_{38} & y_{39} \\
y_{38} & y_{38} & y_{36} & y_{37} \\
y_{39} & y_{38} & y_{37} & y_{36} \\
\end{pmatrix}
\]

5. Finally, on the blocks \( B_5, B_7, B_8 \) it is

\[
\begin{pmatrix}
y_{39} & y_{40} & y_{41} & y_{42} \\
y_{40} & y_{41} & y_{42} & y_{43} \\
y_{41} & y_{42} & y_{43} & y_{44} \\
y_{42} & y_{43} & y_{44} & y_{45} \\
y_{43} & y_{44} & y_{45} & y_{46} \\
y_{44} & y_{45} & y_{46} & y_{47} \\
y_{45} & y_{46} & y_{47} & y_{48} \\
y_{46} & y_{47} & y_{48} & y_{49} \\
\end{pmatrix}
\]

Let now \( \varphi_2(x_{a_4}(1)) = x_4 = (z_{i,j}) \). Since \( x_4 \) commutes with all \( h_{a_i}(-1) \), \( i = 1, 2, 4, \) and \( w_1, w_2 \), and also for \( w_{13} \) we have \( w_{13}x_4w_{13}^{-1} = x_4^{-1} = h_{a_3}(-1) \) then by direct calculation we obtain:

1. The matrix \( x_4 \) can be decomposed into following eight diagonal blocks:

\[
B'_1 = \{v_4, v_{-4}, V_1, V_2, V_3, V_4\};
\]

\[
B'_2 = \{v_1, v_{-1}, v_{14}, v_{-14}, v_{17}, v_{-17}, v_{20}, v_{-20}, v_{22}, v_{-22}\};
\]

\[
B'_3 = \{v_2, v_{-2}, v_{10}, v_{-10}, v_{13}, v_{-13}, v_{16}, v_{-16}, v_{24}, v_{-24}\};
\]

\[
B'_4 = \{v_5, v_{-5}, v_{12}, v_{-12}, v_{15}, v_{-15}, v_{18}, v_{-18}, v_{23}, v_{-23}\};
\]

\[
B'_5 = \{v_6, v_{-6}, v_9, v_{-9}\};
\]

\[
B'_6 = \{v_3, v_{-3}, v_7, v_{-7}\};
\]

\[
B'_7 = \{v_8, v_{-8}, v_{11}, v_{-11}\};
\]

\[
B'_8 = \{v_{19}, v_{-19}, v_{21}, v_{-21}\}.
\]
2. On the first block the matrix \( x_4 \) has the form

\[
\begin{pmatrix}
z_1 & z_2 & 0 & 0 & z_3 & -2z_3 \\
z_4 & z_5 & 0 & 0 & z_6 & -2z_6 \\
0 & 0 & z_7 & 0 & 0 & 0 \\
0 & 0 & 0 & z_7 & 0 & 0 \\
0 & 0 & 0 & 0 & z_7 & 0 \\
z_8 & z_9 & 0 & 0 & z_{10} & z_7 - 2z_{10}
\end{pmatrix}.
\]

3. On the second, third and fourth blocks it is

\[
\begin{pmatrix}
z_{11} & z_{12} & -z_{13} & -z_{14} & z_{15} & z_{15} & z_{14} & z_{13} & -z_{16} & z_{16} \\
z_{12} & z_{11} & z_{13} & z_{14} & -z_{15} & -z_{15} & -z_{14} & -z_{13} & z_{16} & -z_{16} \\
-z_{17} & z_{17} & z_{18} & z_{19} & z_{20} & z_{21} & z_{22} & z_{23} & z_{17} & -z_{17} \\
z_{24} & -z_{24} & z_{25} & z_{26} & z_{27} & z_{28} & z_{29} & z_{30} & -z_{24} & z_{24} \\
-z_{31} & z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} & z_{37} & z_{31} & -z_{31} \\
-z_{31} & z_{31} & -z_{37} & -z_{36} & z_{35} & z_{34} & -z_{33} & -z_{32} & z_{31} & -z_{31} \\
-z_{24} & z_{24} & z_{30} & z_{29} & -z_{28} & -z_{27} & z_{26} & z_{25} & z_{24} & -z_{24} \\
z_{17} & -z_{17} & z_{23} & z_{22} & -z_{21} & -z_{20} & z_{19} & z_{18} & -z_{17} & z_{17} \\
-z_{16} & z_{16} & z_{13} & z_{14} & -z_{15} & -z_{15} & -z_{14} & -z_{13} & z_{11} & z_{12} \\
z_{16} & -z_{16} & -z_{13} & -z_{14} & z_{15} & z_{15} & z_{14} & z_{13} & z_{12} & z_{11}
\end{pmatrix}.
\]

4. On all other blocks \( x_4 \) has the form

\[
\begin{pmatrix}
z_{38} & z_{39} & z_{40} & z_{41} \\
z_{42} & z_{43} & z_{44} & z_{45} \\
-z_{45} & -z_{44} & z_{43} & z_{42} \\
-z_{41} & -z_{40} & z_{39} & z_{38}
\end{pmatrix}.
\]

Therefore, we have 85 variables \( y_1, \ldots, y_{40}, z_1, \ldots, z_{45} \), where \( y_1, y_6, y_{11}, y_{20}, y_{21}, y_{30}, y_{32}, y_{36}, y_{39}, y_{44}, z_1, z_5, z_7, z_{11}, z_{18}, z_{26}, z_{28}, z_{30}, z_{38}, z_{43}, z_{45} \) are 1 modulo radical, \( y_2, y_4, y_{17}, y_{46}, z_2, z_3, z_9 \) are \(-1\) modulo radical, \( z_{32} \) is \(-2\) modulo radical, all other variables are from radical.

We apply step by step four basis changes, commuting with each other and with all matrices \( w_i \). These changes are represented by matrices \( C_1, C_2, C_3, C_4 \). Matrices \( C_1 \) and \( C_2 \) are block-diagonal, where first 24 blocks have the size \( 2 \times 2 \), the last block is \( 4 \times 4 \). On all \( 2 \times 2 \) blocks, corresponding to short roots, the matrix \( C_1 \) is unit, on all \( 2 \times 2 \) blocks, corresponding to long roots, it is

\[
\begin{pmatrix}
1 & -y_{16}/y_{17} \\
y_{16}/y_{17} & 1
\end{pmatrix}.
\]

On the last block it is unit.

Similarly, \( C_2 \) is unit on the blocks corresponding to long roots, and on the last block. On the blocks corresponding to the short roots, it is

\[
\begin{pmatrix}
1 & -z_8/z_9 \\
-z_8/z_9 & 1
\end{pmatrix}.
\]

Matrices \( C_3 \) and \( C_4 \) are diagonal, identical on the last \( 4 \times 4 \) block, the matrix \( C_3 \) is identical on all places, corresponding to short root, and scalar with multiplier \( a \) on all places corresponding
to long roots. In the contrary, the matrix $C_4$, is identical on all places, corresponding to long roots, and is scalar with multiplier $b$ on all places, corresponding to short roots.

Since all these four matrices commutes with all $w_i$, $i = 1, 2, 3, 4$, then after basis change with any of these matrices all conditions for elements $x_1$ and $x_4$ still hold.

At the beginning we apply basis changes with the matrices $C_1$ and $C_2$. After that new $y_{16}$ in the matrix $x_1$ and $z_8$ in the matrix $x_4$ are equal to zero (for the convenience of notations we do not change names of variables). Then we choose $a = -1/y_{17}$ (it is new $y_{17}$) and apply the third basis change. After it $y_{17}$ in the matrix $x_1$ becomes to be $-1$. Clear that $y_{16}$ is still zero.

Finally, apply the last basis change with $b = -1/z_9$ (where $z_9$ is the last one, obtained after all previous changes). We have that $y_{16}, y_{17}, z_8$ are not changed, and $z_9$ is now $-1$.

Now we can suppose that $y_{16} = 0, y_{17} = -1, z_8 = 0, z_9 = -1$, we have now just 81 variables.

From the fact that $x_1$ and $x_4$ commute (Cond. 1), it directly follows $y_{37} = y_{38} = 0, y_{36} = y_{20}$. From the condition $h_{\alpha_2}(-1)x_1h_{\alpha_2}(-1)x_1 = E$ (Cond. 2, its position $(52, 52)$) follows that $y_{20}^2 = 1$, consequently $y_{20} = 1$.

From the condition $w_2x_1w_2^{-1}x_1 = x_1w_2(1)x_1w_2(1)^{-1}$ (Cond. 3, the position $(50, 10)$) it follows $y_{21} = 1$, from its position $(49, 10)$ it follows $y_{10} = 0$.

The condition $w_2w_3w_2^{-1}w_3^{-1}w_{1}x_1 = x_1w_2w_3w_2^{-1}w_3^{-1}w_{1}^{-1}w_{1}^{-1}$ (Cond. 4, the position $(51, 52)$) implies $y_{15} = 0$.

Again from Cond. 3 (the position $(18, 13)$) we have $y_{46}(y_{45} + y_{42}) = 0$, whence $y_{45} = -y_{42}$. From Cond. 2 (the positions $(11, 12)$ and $(12, 11)$) we obtain $y_{40}(y_{39} + y_{41}) = 0$ and $y_{43}(y_{39} + y_{41}) = 0$, therefore $y_{40} = y_{43} = 0$. After that in the same condition the position $(12, 16)$ gives $y_{44} = y_{39}$. The position $(12, 16)$ of Cond. 3 now gives us $y_{46}(y_{39} - 1) = 0 \Rightarrow y_{39} = 1$.

In the condition $h_{\alpha_3}(-1)x_1h_{\alpha_3}(-1)x_4 = E$ (Cond. 5) the position $(8, 7)$ gives $z_4 = 0$, the position $(7, 7)$ gives $z_1 = 1$; $(51, 51)$ gives $z_7 = 1$;

In the condition $w_3x_4x_3^{-1}x_4 = x_4w_3x_4x_3^{-1}$ (Cond. 6) the position $(51, 5)$ gives $z_{41} = 0$, the position $(51, 6)$ gives $z_{40} = 0$, the position $(52, 7)$ gives $z_{39} = 0$, the position $(51, 8)$ gives $z_{10} = 0$, the position $(52, 8)$ gives $z_{38} = 1$.

Again from Cond. 5 (positions $(52, 52), (52, 8), (7, 8)$) we obtain $z_6 = 0, z_5 = 1, z_2 = z_3$.

Returning to Cond. 6, from $(13, 51)$ we have $z_{43} = 1$, from $(5, 51)$ we have $z_{44} = 0$, from $(5, 14)$ we have $z_{42} = 0$, from $(12, 17)$ we have $z_{35} = 0$, from $(12, 18)$ we have $z_{34} = 1$, from $(12, 19) - z_37 = -z_{31}$, from $(12, 20) - z_{36} = z_{31}$, from $(9, 15) - z_{20} = -z_{15}$, and from $(10, 15) - z_{27} = z_{15}$.

The position $(11, 22)$ of Cond. 1 now gives us $y_{42} = 0$, and the position $(11, 11)$ of Cond. 2 gives $y_{41} = 0$.

Considering \(x_{1+2} = \varphi_2(x_{\alpha_1+\alpha_2}(1)) = w_2x_1w_2^{-1}, x_2 = \varphi(x_{\alpha_2}(1)) = w_1x_{1+2}w_1\) and Cond. 7: \(x_1x_2 = x_{1+2}x_2x_1\) (the position $(6, 16)$), we obtain $y_{46} = -1$.

Similarly, considering \(x_{3+4} = \varphi_2(x_{\alpha_3+\alpha_4}(1)) = w_3x_4w_3^{-1}, x_3 = \varphi(x_{\alpha_3}(1)) = w_4x_{3+4}w_4^{-1}\), and Cond. 8: \(x_3x_4 = x_{3+4}x_4x_3\) (applying positions $(51, 14), (13, 52), (12, 11), (29, 9), (15, 35), (15, 36), (16, 36), (12, 19), (12, 20), (11, 25), (12, 26), (10, 30), (47, 11), (1, 2), (1, 1), (4, 4), (3, 4), (3, 18), (3, 17), (4, 17), (4, 3), (3, 3), (18, 3))\), we obtain $z_{45} = 1, z_{3} = -1, z_{31} = 0, z_{32} = -2, z_{14} = 0, z_{13} = 0, z_{30} = 1, z_{25} = 0, z_{26} = 1, z_{15} = 0, z_{28} = 1, z_{24} = 0, z_{16} = 0, z_{12} = 0, z_{11} = 1, z_{17} = 0, z_{19} = 0, z_{21} = 0, z_{22} = 0, z_{29} = 0, z_{23} = 0, z_{18} = 1, z_{33} = 0$, respectively.

Therefore we obtain that \(x_4 = x_{\alpha_4}(1)\).
Directly from the first condition we now have $y_3 = y_7 = y_{27} = y_{25} = y_{34} = y_{26} = y_{33} = y_{28} = y_{35} = y_{22} = y_{24} = y_{29} = y_{31} = y_{12} = y_9 = y_{10} = y_{23} = y_{18} = y_{14} = 0$, $y_{30} = y_{32} = y_{11} = 1$.

Finally, from Cond. 3 we get $y_5 = 0$, $y_6 = 1$, $y_1 = 1$, $y_8 = 0$, $y_4 = -1$, from Cond. 2 we get $y_2 = -1$.

Now $x_1 = x_{\alpha_1}(1)$, it is what we needed.

Since all long (and all short) roots are conjugate under the action of Weil group, it means that $\varphi_2(x_{\alpha}(1)) = x_{\alpha}(1)$ for all $\alpha \in \Phi$.

Consider now the matrix $d_t = \varphi_2(h_{\alpha_4}(t))$.

**Lemma 1.** The matrix $d_t$ is $h_{\alpha_4}(s)$ for some $s \in R^*$.

**Proof.** Since the matrix $d_t$ commutes with $h_{\alpha}(-1)$ for all $\alpha \in \Phi$, then $d_t$ is decomposed to the following diagonal blocks:

$$D_1 = \{v_1, v_{-1}, v_{14}, v_{-14}, v_{20}, v_{-20}, v_{22}, v_{-22}\},$$
$$D_2 = \{v_2, v_{-2}, v_{10}, v_{-10}, v_{16}, v_{-16}, v_{24}, v_{-24}\},$$
$$D_3 = \{v_3, v_{-3}\}, \quad D_4 = \{v_4, v_{-4}\},$$
$$D_5 = \{v_5, v_{-5}, v_{12}, v_{-12}, v_{18}, v_{-18}, v_{23}, v_{-23}\},$$
$$D_6 = \{v_6, v_{-6}\}, \quad D_7 = \{v_7, v_{-7}\},$$
$$D_8 = \{v_8, v_{-8}\}, \quad D_9 = \{v_9, v_{-9}\},$$
$$D_{10} = \{v_{11}, v_{-11}\}, \quad D_{11} = \{v_{13}, v_{-13}\},$$
$$D_{12} = \{v_{15}, v_{-15}\}, \quad D_{13} = \{v_{17}, v_{-17}\},$$
$$D_{14} = \{v_{19}, v_{-19}\}, \quad D_{15} = \{v_{21}, v_{-21}\},$$
$$D_{16} = \{V_1, V_2, V_3, V_4\}.$$

Using the fact that $d_t$ commutes with $w_1, w_2, w_{13}$ and $x_1$, we obtain that on the blocks $D_1, D_2, D_5$ the matrix $d_t$ has the form

$$\begin{pmatrix}
    t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & t_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & t_8 & 0 & t_9 & 0 & 0 & 0 \\
    0 & 0 & 0 & t_{10} & 0 & t_{11} & 0 & 0 \\
    0 & 0 & t_{11} & 0 & t_{10} & 0 & 0 & 0 \\
    0 & 0 & 0 & t_9 & 0 & t_8 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & t_1 + 2t_{13} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1
\end{pmatrix},$$

on the blocks $D_3, D_6, D_8, D_{14}$ it is $\text{diag}[t_2, t_3]$; on the blocks $D_7, D_9, D_{10}, D_{15}$ it is $\text{diag}[t_3, t_2]$, on the block $D_4$ it is

$$\begin{pmatrix}
    t_4 & t_5 \\
    t_6 & t_7
\end{pmatrix},$$
on the blocks $D_{11}, D_{12}, D_{13}$ it has the form $\text{diag}[t_{12}, t_{12}]$; and on the last block it is

$$
\begin{pmatrix}
t_1 & 0 & 0 & 0 \\
0 & t_1 & 0 & 0 \\
0 & 0 & t_1 & 0 \\
0 & 0 & t_{13} & t_1 - 2t_{13}
\end{pmatrix}.
$$

Using the condition $w_3d_1w_4^{-1}d_1 = E$, we obtain: from the position $(1,1)$ it follows $t_1^2 = 1$, consequently $t_1 = 1$, from $(52, 52)$ it follows $(1 - 2t_{13})^2 = 1$, therefore $t_{13} = 0$; $(5, 5)$ implies $t_3 = 1/t_2$; $(7, 8)$ implies $t_7(t_5 + t_6) = 0$, whence $t_4 = -t_5$; from $(24, 36)$ we have $t_8(t_9 + t_{11}) = 0$, therefore $t_{11} = -t_9$; from $(26, 26)$ we have $t_{12}^2 = 1$, and then $t_{12} = 1$.

Now consider the condition $w_3d_1w_3^{-1} = d_1w_3d_1w_3^{-1}w_4^{-1}$. Its position $(13, 14)$ gives $t_5 = 0$, the position $(5, 5)$ gives $t_4 = 1/t_2$; $(6, 6)$ gives $t_7 = t_{12}^2$; $(3, 19)$ gives $t_9 = 0$; $(19, 19)$ gives $t_{10} = 1/t_8$.

Finally, introduce $\varphi_2(h_{\alpha_3}(t)) = w_1w_3d_1w_3^{-1}w_4^{-1}$, $\varphi_2(h_{\alpha_6}(t)) = w_2\varphi_2(h_{\alpha_3}(t))w_4^{-1}$, $\varphi_2(h_{\alpha_{10}}(t)) = \varphi_2(h_{\alpha_4}(t))\varphi_2(h_{\alpha_3}(t))$. Since $\varphi_2(h_{\alpha_{10}}(t))$ commutes with $x_{\alpha_8}(1)$, we obtain (the position $(9, 6)$) that $t_8 = t_{12}^2$.

Therefore, $\varphi_2(h_{\alpha_4}(t)) = h_{\alpha_4}(1/t_2)$, and the lemma is proved. □

Clear, that this lemma holds also for images of all $h_{\alpha}(t)$, $\alpha \in \Phi$.

4. Images of $x_{\alpha}(t)$, proof of theorem 1.

We have shown that $\varphi_2(h_{\alpha}(t)) = h_{\alpha}(s)$, $\alpha \in \Phi$. Denote the mapping $t \mapsto s$ by $\rho : R^* \to R^*$. Note that for $t \in R^*$ $\varphi_2(x_1(t)) = \varphi_2(h_{\alpha_2}(t^{-1})x_1(1)h_{\alpha_2}(t)) = h_{\alpha_2}(s^{-1})x_1(1)h_{\alpha_2}(s) = x_1(s)$. If $t \notin R^*$, then $t \in J$, i.e., $t = 1 + t_1$, where $t_1 \in R^*$. Then $\varphi_2(x_1(t)) = \varphi_2(x_1(1)x_1(t_1)) = x_1(1)x_1(\rho(t_1)) = x_1(1 + \rho(t_1))$. Therefore if we extend the mapping $\rho$ to the whole $R$ (by the formula $\rho(t) := 1 + \rho(t - 1)$, $t \in R$), we obtain $\varphi_2(x_1(t)) = x_1(\rho(t))$ for all $t \in R$. Clear that $\rho$ is injective, additive, and also multiplicative on all invertible elements. Since every element of $R$ is a sum of two invertible elements, we have that $\rho$ is an isomorphism from the ring $R$ onto some its subring $R'$. Note that in this situation $CG(R)C^{-1} = G(R')$ for some matrix $C \in \text{GL}(V)$. Let us show that $R' = R$.

Denote matrix units by $E_{ij}$.

Lemma 2. The Chevalley group $G(R)$ generates the matrix ring $M_n(R)$.

Proof. The matrix $(x_{\alpha_1}(1) - 1)^2$ has a unique nonzero element $-2 \cdot E_{12}$. Multiplying it to suitable diagonal matrices, we can obtain an arbitrary matrix of the form $\lambda \cdot E_{12}$ (since $-2 \in R^*$ and $R^*$ generates $R$). Since the Weil group acts transitively on all roots of the same length, i.e., for every long root $\alpha_k$ there exists such $w \in W$, that $w(\alpha_1) = \alpha_k$, and then the matrix $\lambda E_{12} \cdot w$ has the form $\lambda E_{12}$, and the matrix $w^{-1} \cdot \lambda E_{12}$ has the form $\lambda E_{2k-1, 2}$. Besides, with the help of the Weil group element, moving the first root to the opposite one, we can get the matrix unit $E_{2,1}$. Taking now different combinations of the obtained elements, we can get an arbitrary element $\lambda E_{ij}, 1 \leq i, j \leq 48$, indices $i, j$ correspond to the numbers of long roots.

The matrix $(x_{\alpha_4}(1) - 1)^2$ is $-2E_{7, 8} + 2E_{20, 32} + 2E_{24, 36} + 2E_{28, 40} + 2E_{31, 19} + 2E_{35, 23} + 2E_{39, 27}$. All matrix units in this sum, except the first one, are already obtained, therefore we can subtract them and get $E_{7, 8}$. Similarly to the longs roots, using the fact that all short roots are also
Lemma 3. If for some $\lambda E_{ij}$, $1 \leq i, j \leq 48$, indices $i, j$ correspond to the short roots.

Now subtract from the matrix $x_{ai}(1) - 1$ suitable matrix units and obtain the matrix $E_{49,2} - 2E_{1,49} + E_{1,50}$. Multiplying it (from the right side) to $E_{2,i}$, $1 \leq i \leq 48$, where $i$ corresponds to a long root, we obtain all $E_{49,i}$, $1 \leq i \leq 48$ for $i$ corresponding to the long roots. Multiplying these last elements from the left side to $w_2$, we obtain $E_{50,i}$, $1 \leq i \leq 48$ for $i$, corresponding to the long roots; then by multiplying them from the left side to $w_3$ we obtain all $E_{51,i}$, $1 \leq i \leq 48$ for $i$, corresponding to the long roots, and, similarly, $E_{52,i}$. Therefore, now we have all $E_{i,j}$, $49 \leq i \leq 52$, $1 \leq j \leq 52$, where $j$ correspond to the long roots.

Then $A = 1/8(h_{a_1}(-1) + E) \cdots (h_{a_4}(-1) + E) = E_{49,49} + E_{50,50} + E_{51,51} + E_{52,52}$, $B = A(w_1 + \cdots + w_4)A + 2A = E_{49,50} + E_{50,49} + E_{50,51} + 2E_{51,50} + E_{51,52} + E_{52,51}$, $C = B^2 - A = E_{49,51} + 2E_{50,50} + E_{50,52} + 2E_{51,49} + 2E_{51,51} + 2E_{52,50} + C^2 - B^2 = 2E_{52,50}$. So we have $E_{52,50}$ and then all $E_{i,j}$, $48 < i, j \leq 52$, therefore all $E_{i,j}$, $1 \leq i \leq 48$, $48 < j \leq 52$, where $i$ corresponds to the long roots.

Then, taking the matrix $x_{a_4}(t)$ and multiplying it from the left and right side to some suitable matrix units $E_{i,j}$, we can obtain $E_{i,j}$, where $i$ corresponds to the long root, $j$ corresponds to the short one. After that it becomes clear, how to get all matrix units $E_{i,j}$, $1 \leq i, j \leq 48$ with the help of the Weil group. Finally, as above, we can obtain all $E_{i,j}$, $1 \leq i \leq 48$, $48 < j \leq 52$, where $i$ correspond to the short roots, and so all matrix units.

\[ \square \]

Lemma 3. If for some $C \in \text{GL}_n(R)$ we have $CG(R)C^{-1} = G(R')$, where $R'$ is a subring of $R$, then $R' = R$.

Proof. Suppose that $R'$ is a proper subring of $R$.

Then $CM_n(R)C^{-1} = M_n(R')$, since the group $G(R)$ generates the whole ring $M_n(R)$ (the previous lemma), and the group $G(R') = CG(R)C^{-1}$ generated the ring $M_n(R')$. It is impossible, since $C \in \text{GL}_n(R)$.

Proof of Theorem 1. We have just proved that $\rho$ is an automorphism of the ring $R$. Consequently, the composition of the initial automorphism $\varphi$ and some basis change with a matrix $C \in \text{GL}_n(R)$, (mapping $G(R)$ into itself) is a ring automorphism $\rho$. It proves Theorem 1. \[ \square \]

5. Theorem about normalizers and Main Theorem

To prove the main theorem of this paper (see Theorem \[ \square \] in the end of this section), we need to obtain the following important fact (that has proper interest):

Theorem 2. Every automorphism–conjugation of a Chevalley group $G(R)$ of type $F_4$ over a local ring $R$ with $1/2$ is an inner automorphism.

Proof. Suppose that we have some matrix $C = (c_{i,j}) \in \text{GL}_{52}(R)$ such that

$$C \cdot G \cdot C^{-1} = G.$$ 

If $J$ is the radical of $R$, then $M_n(J)$ is the radical in the matrix ring $M_n(R)$, therefore

$$C \cdot M_n(J) \cdot C^{-1} = M_n(J),$$
consequently,

\[ C \cdot (E + M_n(J)) \cdot C^{-1} = E + M_n(J), \]

i.e.,

\[ C \cdot G(R, J) \cdot C^{-1} = G(R, J), \]

since \( G(R, J) = G \cap (E + M_n(J)) \).

Thus, the image \( \overline{C} \) of the matrix \( C \) under factorization \( R \) by \( J \) gives us an automorphism–conjugation of the Chevalley group \( G(k) \), where \( k = R/J \) is a residue field of \( R \).

But over a field every automorphism–conjugation of a Chevalley group of type \( F_4 \) is inner (see [51]), therefore a conjugation by \( C \) (denote it by \( i_C \)) is \( i_C = i_g \), where \( g \in G(k) \).

Since over a field our Chevalley group (of type \( F_4 \)) coincides with its elementary subgroup, every its element is a product of some set of unipotents \( x_\alpha(t) \) and the matrix \( g \) can be decomposed into a product \( x_{\alpha_1}(Y_1) \ldots x_{\alpha_N}(Y_N), \ Y_1, \ldots, Y_N \in k. \)

Since every element \( Y_1, \ldots, Y_N \) is a residue class in \( R \), we can choose (arbitrarily) elements \( y_1 \in Y_1, \ldots, y_N \in Y_N, \) and the element

\[ g' = x_{\alpha_1}(y_1) \ldots x_{\alpha_N}(y_N) \]

satisfies \( g' \in G(R) \) and \( \overline{g'} = g \).

Consider the matrix \( C' = g'^{-1} \circ d^{-1} \circ C \). This matrix also normalizes the group \( G(R) \), and also \( \overline{C'} = E \). Therefore, from the description of the normalizer of \( G(R) \) we come to the description of all matrices from this normalizer equivalent to the unit matrix modulo \( J \).

Therefore we can suppose that our initial matrix \( C \) is equivalent to the unit modulo \( J \).

Our aim is to show that \( C \in \lambda G(R) \).

Firstly we prove one technical lemma that we will need later.

**Lemma 4.** Let \( X = \lambda t_{\alpha_1}(s_1) \ldots t_{\alpha_4}(s_4)x_{\alpha_1}(t_1) \ldots x_{\alpha_24}(t_{24})x_{-\alpha_1}(u_1) \ldots x_{-\alpha_24}(u_{24}) \in \lambda G(R, J). \) Then the matrix \( X \) has such 53 coefficients (precisely described in the proof of lemma), that uniquely define all \( s_1, \ldots, s_4, t_1, \ldots, t_{24}, u_1, \ldots, u_{24}, \lambda. \)
Proof. Consider the sequence of roots:

\[
\begin{align*}
\gamma_1 &= \alpha_1, \\
\gamma_2 &= \alpha_5 = \alpha_1 + \alpha_2, \\
\gamma_3 &= \alpha_8 = \alpha_1 + \alpha_2 + \alpha_3, \\
\gamma_4 &= \alpha_{12} = \alpha_1 + \alpha_2 + 2\alpha_3, \\
\gamma_5 &= \alpha_{15} = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \\
\gamma_6 &= \alpha_{17} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \\
\gamma_7 &= \alpha_{19} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \\
\gamma_8 &= \alpha_{21} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \\
\gamma_9 &= \alpha_{22} = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \\
\gamma_{10} &= \alpha_{23} = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \\
\gamma_{11} &= \alpha_{24} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.
\end{align*}
\]

All roots of \(F_4\), except \(\alpha_{14}\) and \(\alpha_{18}\), are differences between two distinct roots of this sequence (or its member).

Besides, \(\gamma_1\) is a simple root, \(\gamma_{11}\) is a maximal root of the system, every root of the sequence is obtained from the previous one by adding some simple root.

Consider in the matrix \(X\) some place \((\mu, \nu)\), \(\mu, \nu \in \Phi\).

To find an element on this position we need to define all sequences of roots \(\beta_1, \ldots, \beta_p\), satisfying the following properties:

1. \(\mu + \beta_1 \in \Phi, \mu + \beta_1 + \beta_2 \in \Phi, \ldots, \mu + \beta_1 + \cdots + \beta_i \in \Phi, \ldots, \mu + \beta_1 + \cdots + \beta_p = \nu\).

2. In the initial numerated sequence \(\alpha_1, \ldots, \alpha_{24}, -\alpha_1, \ldots, -\alpha_{24}\) the roots \(\beta_1, \ldots, \beta_k\) are replaced strictly from right to left.

Finally in the matrix \(X\) on the position \((\mu, \nu)\) there is the sum of all products \(\pm \beta_1 \cdot \beta_2 \cdots \beta_p\) by all sequences with these two properties, multiplying to \(d_\mu = \lambda s_1^{(\alpha_1, \mu)} \cdots s_4^{(\alpha_4, \mu)}\). If \(\mu = \nu\), we must add 1 to the sum.

We will find the obtained elements \(s_1, \ldots, s_4, t_1, \ldots, t_m, u_1, \ldots, u_m\) step by step.

Firstly we consider in the matrix \(X\) the position \((-\gamma_{11}, -\gamma_{11})\). We can not add to the root \(-\gamma_{11}\) any negative root to obtain a root in the result. If in a sequence \(\beta_1, \ldots, \beta_p\) the first root is positive, then all other roots must be positive. Thus, this position contains an element \(1 \cdot d_\nu\).

So we know \(d_{-\gamma_{11}}\). By the previous arguments if we consider the position \((-\gamma_{11}, -\gamma_{10})\), the suitable sequence is only \(\alpha_1 = \gamma_{11} - \gamma_{10}\). Since there is \(d_{-\gamma_{11}}t_1\) on this position and we already know \(d_{-\gamma_{11}}\), we can find \(t_1\) on the position \((-\alpha_{24}, -\alpha_{23})\). Considering the positions \((-\gamma_{10}, -\gamma_{10})\) and \((-\gamma_{10}, -\gamma_{11})\), we see that by similar reasons there are \(d_{-\gamma_{10}}(1 \pm u_1t_1)\) and \(\pm d_{-\gamma_{10}}u_1\) there. So we find \(d_{-\gamma_{10}}\) and \(u_1\).

Now we come to the second step. As we have written above, in the matrix \(X\) on the position \((-\gamma_{10}, -\gamma_{9})\) there is \(d_{-\gamma_{10}}(\pm t_2 \pm u_1 t_5)\); on the position \((-\gamma_{9}, -\gamma_{10})\) there is \(d_{-\gamma_{9}}(\pm u_2 \pm u_5 t_1)\); on the position \((-\gamma_{11}, -\gamma_{9})\) there is \(\pm d_{-\gamma_{11}}t_5\) (the second summand is absent, since \(\alpha_1\) is staying earlier than \(\alpha_2\)); on the position \((-\gamma_{9}, -\gamma_{11})\) there is \(d_{-\gamma_{9}}(\pm u_5 \pm u_2 u_1)\); finally, on the position \((-\gamma_{9}, -\gamma_{9})\) there is \(d_{-\gamma_{9}}(1 \pm u_5 t_5 \pm u_2 t_2)\). From the position \((-\gamma_{11}, -\gamma_{9})\) we find \(t_5\), then
from the position \((-\gamma_{10}, -\gamma_9)\) we find \(t_2\), and from other three positions together we can know \(u_2, u_5, d_{-\gamma_9}\). Therefore, now we know \(t_1, t_2, t_5, u_1, u_2, u_5, d_{-\gamma_9}, d_{-\gamma_{10}}, d_{-\gamma_{11}}\).

On the third step we consider the positions \((-\gamma_9, -\gamma_8)\) with \(d_{-\gamma_9}(\pm t_3 \pm u_2 t_6 \pm u_5 t_8), (-\gamma_8, -\gamma_9)\) with \(d_{-\gamma_9}(\pm u_3 \pm t_2 u_6 \pm t_5 u_8), (-\gamma_{10}, -\gamma_8)\) with \(d_{-\gamma_{10}}(\pm t_6 \pm u_1 t_8), (-\gamma_8, -\gamma_{10})\) with \(d_{-\gamma_8}(\pm u_6 \pm u_2 u_3 \pm t_1 u_8), (-\gamma_{11}, -\gamma_8)\) with \(d_{-\gamma_{11}}(\pm t_8 \pm t_5 t_3), (-\gamma_8, -\gamma_{11})\) with \(d_{-\gamma_8}(\pm u_8 \pm u_3 u_2 u_1 \pm u_6 u_1)\), and \((-\gamma_8, -\gamma_8)\) with \(d_{-\gamma_8}(\pm u_3 t_3 \pm u_5 t_5 \pm u_8 t_5 t_3)\). From these seven equations with seven unknown variables (all of them from radical) we can find all variables \(d, t\) and \(s\). So we know all \(d\) the known element \(d\), and all other elements are known and lie in radical; and all this sum is multiplying to \(\pm\) uniform) linear equations as the number of roots of the form \((-\gamma_8, -\gamma_8)\) are from radical, for distinct equations such variables are different. Clear that such a system \(\gamma_{q_1}(\alpha_i \gamma_{q_2})\) we have \(\gamma_{q_2}(\alpha_i \gamma_{q_1})\), \((-\gamma_{11}, -\gamma_8)\) with \(d_{-\gamma_{11}}(\pm t_8 \pm t_5 t_3), (-\gamma_8, -\gamma_{11})\) with \(d_{-\gamma_8}(\pm u_8 \pm u_3 u_2 u_1 \pm u_6 u_1)\), and \((-\gamma_8, -\gamma_8)\) with \(d_{-\gamma_8}(\pm u_3 t_3 \pm u_5 t_5 \pm u_8 t_5 t_3)\). From these seven equations with seven unknown variables (all of them from radical) we can find all variables \(t_3, t_6, u_6, t_8, u_8\) and \(d_{-\gamma_8}\).

Similarly on the next step we consider the positions \((-\gamma_8, -\gamma_7)\), \((-\gamma_7, -\gamma_8)\), \((-\gamma_9, -\gamma_7)\), \((-\gamma_7, -\gamma_9)\), \((-\gamma_{10}, -\gamma_7)\), \((-\gamma_7, -\gamma_{10})\), \((-\gamma_7, -\gamma_{11})\), \((-\gamma_{11}, -\gamma_7)\), \((-\gamma_7, -\gamma_{11})\), \((-\gamma_{11}, -\gamma_{11})\), and find \(t_4, t_9, \ldots\), all other elements are known and lie in radical; and all this sum is multiplying to the known element \(d_{-\gamma_7}\). The same situation is on the positions \((-\gamma_s, -\gamma_i)\), \(1 \geq i > s\), but there is not \(t_p\), but \(u_p\) without multipliers here. Therefore, we have exactly the same number of (not uniform) linear equations as the number of roots of the form \(\pm (\gamma_i - \gamma_s)\), with the same number of variables, in every equation exactly on variable has invertible coefficient, other coefficients are from radical, for distinct equations such variables are different. Clear that such a system has the solution, and it is unique. Consequently, we have made the induction step and now we know elements \(t_i, u_j\) for all indices, corresponding to the roots \(\gamma_p - \gamma_q\), \(11 \geq p, q \geq s\).

On the last step we know elements \(t_i, u_j\) for all indices, corresponding to the roots \(\gamma_p - \gamma_q\), \(11 \geq p, q \leq 1\). Consider now in \(X\) the positions \((-\gamma_{11}, h_{\gamma_{11}})\), \((h_{\gamma_{11}}, -\gamma_{11})\), \((-\gamma_{10}, h_{\gamma_{10}})\), \((h_{\gamma_{10}}, -\gamma_{10})\), \(\ldots\), \((-\gamma_1, h_{\gamma_1})\), \((h_{\gamma_1}, -\gamma_1)\). Similarly to the previous arguments we can find all \(t\) and \(u\), corresponding to the roots \(\pm \gamma_1, \ldots, \pm \gamma_k\).

We have not found yet the obtained coefficients for two pairs of roots: \(\pm \alpha_{14}\) and \(\pm \alpha_{18}\). Note that \(\alpha_{14} + \alpha_{18} = \alpha_{24}\).

Consider in \(X\) the positions \((-\alpha_{24}, -\alpha_{14})\), \((-\alpha_{14}, -\alpha_{24})\), \((-\alpha_{24}, -\alpha_{18})\), \((-\alpha_{18}, -\alpha_{24})\). On these positions there are sums of \(t_{18}\) (respectively, \(u_{18}, t_{14}, u_{14}\)), and products of elements \(t_i, u_j\), corresponding to roots of smaller heights. Since for all heights smaller than the height of \(\alpha_{14}\), we know \(t, u\), then we can directly find the obtained coefficients.

Therefore, lemma is completely proved.

Now return to our main proof. Recall that we work with a matrix \(C\), equivalent to the unit matrix modulo radical, and normalizing Chevalley group \(G(R)\).

For every root \(\alpha \in \Phi\) we have

\[
(1) \quad C x_{\alpha}(1) C^{-1} = x_{\alpha}(1) \cdot g_{\alpha}, \quad g_{\alpha} \in G(R, J).
\]
Every $g_\alpha \in G(R, J)$ can be decomposed into a product

\[(2) \quad t_{\alpha_1}(1 + a_1) \cdots t_{\alpha_4}(1 + a_4)x_{\alpha_1}(b_1) \cdots x_{\alpha_{24}}(b_{24})x_{\alpha_{-1}}(c_1) \cdots x_{\alpha_{-24}}(c_{24}),\]

where $a_1, \ldots, a_4, b_1, \ldots, b_{24}, c_1, \ldots, c_{24} \in J$ (see, for example, \([2]\)).

Let $C = E + X = E + (x_{i,j})$. Then for every root $\alpha \in \Phi$ we can write a matrix equation with variables $x_{i,j}, a_1, \ldots, a_4, b_1, \ldots, b_{24}, c_1, \ldots, c_{24}$, every of them is from radical.

Let us change these equations. We consider the matrix $C$ and “imagine”, that it is some matrix from Lemma \([4]\) (i.e., it is from $\lambda G(R)$). Then by some its concrete 53 positions we can “define” all coefficients $\lambda, s$ corresponding to the vectors to zeros.

This equation can be written for every $\alpha \in \Phi$ (naturally, with another $a_j, b_j, c_j$), and can be written only for generating roots: for $\alpha_1, \ldots, \alpha_4, -\alpha_1, \ldots, -\alpha_4$. Then the linearized system has the form

\[Zx_{\alpha}(1) - x_{\alpha}(1)(Z + a_1T_1 + \cdots + a_4T_4 + b_1X_{\alpha_1} + \cdots + c_{24}X_{\alpha_{24}}) = 0.\]
\[
\begin{align*}
& \quad Zx_{\alpha_1}(1) - x_{\alpha_1}(1)(Z + a_{1,1}T_1 + \cdots + a_{4,1}T_4 + b_{1,1}X_{\alpha_1} + b_{2,1}X_{\alpha_2} + \cdots + b_{24,1}X_{\alpha_{24}} + c_{1,1}X_{-\alpha_1} + \cdots + c_{24,1}X_{-\alpha_{24}}) = 0; \\
& \quad \vdots \\
& \quad Zx_{\alpha_4}(1) - x_{\alpha_4}(1)(Z + a_{1,4}T_1 + \cdots + a_{4,4}T_4 + b_{1,4}X_{\alpha_1} + \cdots + X_{\alpha_2} b_{24,1}X_{\alpha_{24}} + c_{1,4}X_{-\alpha_1} + \cdots + c_{24,4}X_{-\alpha_{24}}) = 0; \\
& \quad \vdots \\
& \quad Zx_{-\alpha_1}(1) - x_{-\alpha_1}(1)(Z + a_{1,5}T_1 + \cdots + a_{4,5}T_4 + b_{1,5}X_{\alpha_1} + \cdots + b_{24,5}X_{\alpha_{24}} + c_{1,5}X_{-\alpha_1} + \cdots + c_{24,5}X_{-\alpha_5}) = 0; \\
& \quad \vdots \\
& \quad Zx_{-\alpha_1}(1) - x_{-\alpha_1}(1)(Z + a_{1,8}T_1 + \cdots + a_{4,8}T_4 + b_{1,8}X_{\alpha_1} + \cdots + b_{24,8}X_{\alpha_{24}} + c_{1,8}X_{-\alpha_1} + \cdots + c_{24,8}X_{-\alpha_{24}}) = 0.
\end{align*}
\]

The matrix $T_1$ is

\[
diag [2, -2, -1, 1, 0, 0, 0, 1, -1, -1, 1, 0, 1, -1, -1, 1, 1, -1, -1, 1, 0, 0, 0, 0];
\]

$T_2$ is $w_1w_2T_1w_2^{-1}w_1^{-1}$; $T_3$ is

\[
diag [0, 0, -2, 2, 2, -2, -1, -1, -2, 2, 0, 0, 1, -1, 0, 1, -1, -1, 1, 2, -2, -1, 1, 2, -2, 0, 0, 0, 0, 0, 0, 0];
\]

the matrix $T_4$ is $w_3w_4T_3w_4^{-1}w_3^{-1}$.

The matrices $X_{\alpha_1}$, $X_{\alpha_3}$ were written above. Besides them, $X_{-\alpha_1} = w_1X_{\alpha_1}w_1^{-1}$, $X_{-\alpha_3} = w_3X_{\alpha_3}w_3^{-1}$. Other matrices $X_{\alpha}$ are obtained as follows: $X_{\pm\alpha_5} = w_2X_{\pm\alpha_5}w_2^{-1}$, $X_{\pm\alpha_9} = w_1X_{\pm\alpha_9}w_1^{-1}$, $X_{\pm\alpha_{10}} = w_3X_{\pm\alpha_{10}}w_3^{-1}$, $X_{\pm\alpha_{12}} = w_1X_{\pm\alpha_{12}}w_1^{-1}$, $X_{\pm\alpha_{14}} = w_2X_{\pm\alpha_{14}}w_2^{-1}$, $X_{\pm\alpha_{16}} = w_4X_{\pm\alpha_{16}}w_4^{-1}$, $X_{\pm\alpha_{18}} = w_2X_{\pm\alpha_{18}}w_2^{-1}$, $X_{\pm\alpha_{20}} = w_3X_{\pm\alpha_{20}}w_3^{-1}$, $X_{\pm\alpha_{22}} = w_2X_{\pm\alpha_{22}}w_2^{-1}$, $X_{\pm\alpha_{24}} = w_1X_{\pm\alpha_{24}}w_1^{-1}$, $X_{\pm\alpha_7} = w_2X_{\pm\alpha_7}w_2^{-1}$, $X_{\pm\alpha_9} = w_3X_{\pm\alpha_9}w_3^{-1}$, $X_{\pm\alpha_{11}} = w_4X_{\pm\alpha_{11}}w_4^{-1}$, $X_{\pm\alpha_{13}} = w_3X_{\pm\alpha_{13}}w_3^{-1}$, $X_{\pm\alpha_{15}} = w_2X_{\pm\alpha_{15}}w_2^{-1}$.

From Lemma 4 we obtain that the following positions of $Z$ are zeros: $(48, 48), (48, 46), (46, 48), (46, 44), (44, 44), (44, 46), (44, 42), (42, 42), (42, 44), (42, 38), (38, 38), (38, 42), (44, 44), (44, 48), (46, 42), (42, 38), (38, 44), (48, 42), (48, 46), (48, 38), (48, 48), (24, 2), (24, 48), (48, 38), (48, 48), (49, 24), (46, 34), (34, 46), (48, 36), (36, 48), (48, 44), (44, 44), (48, 24), (24, 48), (48, 16), (16, 48), (48, 10), (10, 48), (48, 2), (2, 48), (48, 49), (49, 48).

Suppose that we fixed the obtained uniform linear system of equation. Recall that our aim is to show that all values $z_{i,j}$, $a_{s,t}$, $b_{s,t}$, $c_{s,t}$ are equal to zero.

Consider the first condition. It implies $a_{4,1} = 0$ (pos. $(42, 42)$); $a_{1,1} = 0$ (pos. $(48, 48)$); $a_{3,1} = 0$ (pos. $(38, 38)$); $a_{2,1} = 0$ (pos. $(39, 39)$). Therefore, $T_1, T_2, T_3, T_4$ do not entry to this condition. Later, $c_{1,1} = 0$ (pos. $(3, 9)$); $b_{2,1} = 0$ (pos. $(3, 51)$); $c_{2,1} = 0$ (pos. $(46, 44)$); $b_{3,1} = 0$ (pos. $(5, 51)$); $c_{3,1} = 0$ (pos. $(6, 51)$); $b_{4,1} = 0$ (pos. $(7, 51)$); $c_{4,1} = 0$ (pos. $(8, 51)$); $b_{5,1} = 0$ (pos. $(44, 48)$); $c_{5,1} = 0$ (pos. $(10, 51)$); $b_{6,1} = 0$ (pos. $(3, 6)$); $c_{6,1} = 0$ (pos. $(46, 42)$); $b_{7,1} = 0$ (pos. $(13, 51)$).
\[c_{7,1} = 0 \text{ (pos. (14, 51))}; b_{8,1} = 0 \text{ (pos. (42, 48))}; c_{8,1} = 0 \text{ (pos. (16, 52))}; b_{9,1} = 0 \text{ (pos. (17, 51))}; c_{9,1} = 0 \text{ (pos. (46, 38))}; b_{10,1} = 0 \text{ (pos. (19, 51))}; b_{11,1} = 0 \text{ (pos. (38, 48))}; c_{11,1} = 0 \text{ (pos. (22, 51))}; c_{12,1} = 0 \text{ (pos. (24, 51))}; b_{13,1} = 0 \text{ (pos. (25, 51))}; c_{13,1} = 0 \text{ (pos. (46, 34))}; b_{14,1} = 0 \text{ (pos. (27, 52))}; c_{14,1} = 0 \text{ (pos. (28, 51))}; b_{15,1} = 0 \text{ (pos. (34, 48))}; c_{15,1} = 0 \text{ (pos. (30, 51))}; b_{16,1} = 0 \text{ (pos. (31, 52))}; c_{16,1} = 0 \text{ (pos. (46, 28))}; b_{17,1} = 0 \text{ (pos. (33, 51))}; c_{17,1} = 0 \text{ (pos. (34, 51))}; b_{18,1} = 0 \text{ (pos. (20, 44))}; c_{18,1} = 0 \text{ (pos. (36, 52))}; b_{19,1} = 0 \text{ (pos. (37, 51))}; c_{19,1} = 0 \text{ (pos. (38, 51))}; b_{20,1} = 0 \text{ (pos. (39, 51))}; c_{20,1} = 0 \text{ (pos. (40, 51))}; b_{21,1} = 0 \text{ (pos. (41, 52))}; c_{21,1} = 0 \text{ (pos. (42, 52))}; b_{22,1} = 0 \text{ (pos. (43, 51))}; c_{22,1} = 0 \text{ (pos. (44, 51))}; b_{23,1} = 0 \text{ (pos. (3, 44))}; c_{24,1} = 0 \text{ (pos. (10, 43))}.

Consequently the right side of the condition contains only \(X_{\alpha_2}, X_{\alpha_4}, X_{-\alpha_10}, X_{-\alpha_23}\), the condition itself is simplified, many elements of \(Z\) are equal to zero. Firstly, these are elements on the positions \((i, j)\), \(i = 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 16, 17, 19, 22, 24, 25, 27, 28, 30, 31, 33, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 48, 50, 51, 52\), \(j = 1, 4, 9, 12, 15, 18, 20, 21, 23, 26, 29, 32, 35, 46, 47, 49\) (except \(z_{6,15} = c_{10,1}, z_{5,12} = b_{12,1}, z_{7,20} = c_{10,1}\), \(z_{8,26} = b_{12,1}, z_{24,49} = -c_{10,1}\), \(z_{28,35} = c_{23,1}, z_{27,32} = b_{24,1}, z_{33,26} = -b_{24,1}, z_{34,29} = -c_{23,1}, z_{37,18} = b_{24,1}, z_{38,21} = c_{23,1}, z_{38,18} = c_{10,1}, z_{39,47} = -c_{10,1}, z_{39,20} = b_{24,1}, z_{40,23} = c_{23,1}, z_{41,12} = -b_{24,1}, z_{42,15} = -c_{23,1}, z_{43,4} = -b_{24,1}, z_{44,9} = c_{23,1}, z_{54,49} = -b_{24,1}\).

When we make these elements equal to zero, we see that \(b_{12,1} = 0 \text{ (pos. (19, 2))}\), \(c_{10,1} = 0 \text{ (pos. (44, 36)), b_{24,1} = 0 \text{ (pos. (45, 2))}\), \(c_{23,1} = 0 \text{ (pos. (48, 2))\}, i.e., the condition now looks as \(x_{\alpha_1}(1)Z = Zx_{\alpha_1}(1)\). By similar way finally all our conditions become of the form \(x_{\pm \alpha_p}(1)Z = Zx_{\pm \alpha_p}(1)\), \(p = 1, \ldots, 4\). Since the centralizer of the given eight matrices consists of scalar matrices, and the matrix \(Z\) has a zero element \(z_{52,52}\), we have that \(Z = 0\), what we need.

Theorem 2 is proved.

From Theorems 1 and 2 directly follows the main theorem of the paper:

**Theorem 3.** Let \(G(R)\) be a Chevalley group with root system \(F_4\), where \(R\) is a local ring with 1/2. Then every automorphism of \(G(R)\) is standard, i.e., it is a composition of ring and inner automorphisms.

**REFERENCES**

[1] Abe E. Automorphisms of Chevalley groups over commutative rings. Algebra and Analysis, 5(2), 1993, 74–90.

[2] Abe E. Chevalley groups over local rings. Tohoku Math. J., 1969, 21(3), 474–494.

[3] Abe E. Chevalley groups over commutative rings. Proc. Conf. Radical Theory, Sendai, 1988, 1–23.

[4] Abe E. Normal subgroups of Chevalley groups over commutative rings. Contemp. Math., 1989, 83, 1–17.

[5] Abe E., Hurley J. Centers of Chevalley groups over commutative rings. Comm. Algebra, 1988, 16(1), 57–74.

[6] Abe E., Suzuki K. On normal subgroups of Chevalley groups over commutative rings. Tohoku Math. J., 1976, 28(1), 185-198.

[7] Bak A. Nonabelian K-theory: The nilpotent class of \(K_1\) and general stability. K-Theory, 1991, 4, 363–397.

[8] Bak A., Vavilov Normality of the elementary subgroup functors. Math. Proc. Cambridge Philos. Soc., 1995, 118(1), 35–47.

[9] Bleshhtysyn V.Ya. Automorphisms of general linear group over a commutative ring, not generated by zero divisors. Algebra and Logic, 1978, 17(6), 639–642.

[10] Borel A., Tits J. Homomorphismes “abstraits” de groupes algébriques simples. Ann. Math., 1973, 73, 499–571.

[11] Borel A. Properties and linear representations of Chevalley groups. Seminar in algebraic groups, M., 1973, 9–59.
Bunina E.I. Automorphisms of elementary adjoint Chevalley groups of types $A_l$, $D_l$, $E_l$ over local rings. Algebra and Logic, 2009, to appear (arXiv:math/0702046).

Bunina E.I. Automorphisms of adjoint Chevalley groups of types $B_2$ and $G_2$ over local rings. Journal of Mathematical Science, 2008, 155(6), 795–814.

Bunina E.I. Automorphisms and normalizers of Chevalley groups of types $A_l$, $D_l$, $E_l$ over local rings with 1/2. Fundamentalnaya i prikladnaya matematika, 2009, to appear.

Bourbaki N. Groupes et Algèbres de Lie. Hermann, 1968.

Carter R.W. Simple groups of Lie type, 2nd ed., Wiley, London et al., 1989.

Carter R.W., Chen Yu. Automorphisms of affine Kac–Moody groups and related Chevalley groups over rings. J. Algebra, 1993, 155, 44–94.

Chen Yu. Isomorphic Chevalley groups over integral domains. Rend. Sem. Mat. univ. Padova, 1994, 92, 231–237.

Chen Yu. On representations of elementary subgroups of Chevalley groups over algebras. proc. Amer. Math. Soc., 1995, 123(8), 2357–2361.

Chen Yu. Automorphisms of simple Chevalley groups over $\mathbb{Q}$-algebras. Tohoku Math. J., 1995, 348, 81–97.

Chen Yu. Isomorphisms of adjoint Chevalley groups over integral domains. Trans. Amer. Math. Soc., 1996, 348(2), 1–19.

Chen Yu. Isomorphisms of Chevalley groups over algebras. J. Algebra, 2000, 226, 719–741.

Chevalley C. Certain schemas des groupes semi-simples. Sem. Bourbaki, 1960–1961, 219, 1–16.

Chevalley C. Sur certains groupes simples. Tohoku Math. J., 1955, 2(7), 14–66.

Cohn P., On the structure of the $GL_2$ of a ring, Publ. Math. Inst. Hautes Et. Sci., 1966, 30, 365–413.

Demazure M., Gabriel P. Groupes algébriques. I. North Holland, Amsterdam et al., 1971, 1–654.

Demazure M., Grothendieck A. Schémas en groupes. I, II, III, Lecture Notes Math., 1971, 151, 1–564; 152, 1–654; 153, 1–529.

Diedonne J., On the automorphisms of classical groups, Mem. Amer. Math. Soc., 1951, 2.

Diedonne J. Geometry of classical groups, 1974.

Golubchik I.Z. Isomorphisms of the linear general group $GL_n(R)$, $n \geq 4$, over an associative ring. Contemp. Math., 1992, 131(1), 123–136.

Golubchik I.Z., Mikhailov A.V. Isomorphisms of unitary groups over associative rings. Zapiski nauchnyh seminarov LOMI, 1983, 132, 97–109 (in Russian).

Golubchik I.Z., Mikhailov A.V. Isomorphisms of the general linear group over associative ring. Vestnik MSU, ser. math., 1983, 3, 61–72 (in Russian).

Grothendieck A. Eléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné). IV. Etude locale des schémas et des morphismes de schémas, 1967, 32, Publ. Math. IHES, 5–361.

Hahn A.J., O’Meara O.T. The classical groups ans K-theory. Springer, Berlin et al., 1989.

Hazrat R., Vavilov N.A. $K_1$ of Chevalley groups are nilpotent. J. Pure Appl. Algebra, 2003, 179, 99–116.

Hua L.K., Reiner I. Automorphisms of unimodular groups, Trans. Amer. Math. Soc., 71, 1951, 331–348.

Humphreys J.F., On the automorphisms of infinite Chevalley groups, Canad. J. Math., 21, 1969, 908-911.

Humphreys J.E. Introduction to Lie algebras and representation theory. Springer–Verlag New York, 1978.

Jantzen J.C. Representations of algebraic group. Academic Press, N.Y., 1987.

Fuan Li, Zunxian Li. Automorphisms of $SL_3(R)$, $GL_3(R)$. Contemp. Math., 1984, 82, 47–52.

Klyachko Anton A. Automorphisms and isomorphisms of Chevalley groups and algebras. arXiv:math/0708.2256v3 (2007).

Matsumoto H. Sur les sous-groupes arithmétiques des groupes semi-simples deployés. Ann. Sci. Ecole Norm. Sup. 4ème sér., 1969, 2, 1–62.

McDonald B.R., Automorphisms of $GL_n(R)$, Trans. Amer. Math. Soc., 215, 1976, 145–159.

O’Meara O.T., The automorphisms of linear groups over any integral domain, J. reine angew. Math., 223, 1966, 56–100.

Petechuk V.M. Automorphisms of matrix groups over commutative rings. Mathematical Sbornik, 1983, 45, 527–542.
[46] Petechuk V.M. Automorphisms of groups $SL_n$, $GL_n$ over some local rings. Mathematical Notes, 1980, 28(2), 187–206.

[47] Petechuk V.M. Automorphisms of groups $SL_3(K)$, $GL_3(K)$. Mathematical Notes, 1982, 31(5), 657–668.

[48] Stein M.R. Generators, relations and coverings of Chevalley groups over commutative rings. Amer. J. Math., 1971, 93(4), 965–1004.

[49] Stein M.R. Surjective stability in dimension 0 for $K_2$ and related functors, Trans. Amer. Soc., 1973, 178(1), 165–191.

[50] Stein M.R. Stability theorems for $K_1$, $K_2$ and related functors modeled on Chevalley groups. Japan J. Math., 1978, 4(1), 77–108.

[51] Steinberg R. Lectures on Chevalley groups, Yale University, 1967.

[52] Steinberg R., Automorphisms of finite linear groups, Canad. J. Math., 121, 1960, 606–615.

[53] Suslin A.A., On a theorem of Cohn, J. Sov. Math. 17 (1981), N2, 1801–1803.

[54] Suzuki K., On the automorphisms of Chevalley groups over $p$-adic integer rings, Kumamoto J. Sci. (Math.), 1984, 16(1), 39–47.

[55] Swan R., Generators and relations for certain special linear groups, Adv. Math., 1971, 6, 1–77.

[56] Taddei G. Normalité des groupes élémentaire dans les groupes de Chevalley sur un anneau. Contemp. Math., Part II, 1986, 55, 693–710.

[57] Vaserstein L.N. On normal subgroups of Chevalley groups over commutative rings. Tohoku Math. J., 1986, 36(5), 219–230.

[58] Vavilov N.A. Structure of Chevalley groups over commutative rings. Proc. Conf. Non-associative algebras and related topics (Hiroshima – 1990). World Sci. Publ., London et al., 1991, 219–335.

[59] Vavilov N.A. An $A_2$-proof of structure theorems for Chevalley groups of types $E_6$ and $E_7$. J. Pure Appl. Algebra, 2007, 1-16.

[60] Vavilov N.A. Parabolic subgroups of Chevalley groups over commutative ring. Zapiski nauchnyh seminarov LOMI, 1982, 116, 20–43 (in Russian).

[61] Vavilov N.A., Gavrilovich M.R. $A_2$-proof of structure theorems for Chevalley groups of types $E_6$ and $E_7$. Algebra and Analisys, 2004, 116(4), 54–87.

[62] Vavilov N.A., Gavrilovich M.R., Nikolenko S.I. Structure of Chevalley groups: proof from the book. Zapiski nauchnyh seminarov LOMI, 2006, 330, 36–76 (in Russian).

[63] Vavilov N.A., Petrov V.A. On overgroups of $Ep(2l, R)$. Algebra and Analisys, 2003, 15(3), 72–114.

[64] Vavilov N.A., Plotkin E.B. Chevalley groups over commutative rings. I. Elementary calculations. Acta Applicandae Math., 1996, 45, 73–115.

[65] Waterhouse W.C. Introduction to affine group schemes. Springer-Verlag, N.Y. et al., 1979.

[66] Waterhouse W.C. Automorphisms of $GL_n(R)$. Proc. Amer. Math. Soc., 1980, 79, 347–351.

[67] Waterhouse W.C. Automorphisms of quotients of $\prod GL(n_i)$. Pacif. J. Math., 1982, 79, 221–233.

[68] Waterhouse W.C. Automorphisms of $det(X_{ij})$: the group scheme approach. Adv. Math., 1987, 65(2), 171–203.

[69] Zalesskiy A.E. Linear groups. Itogi Nauki. M., 1989, 114–228 (in Russian).

[70] Zelmanov E.I. Isomorphisms of general linear groups over associative rings. Siberian Mathematical Journal, 1985, 26(4), 49–67 (in Russian).