The bicanonical map of surfaces
with \( p_g = 0 \) and \( K^2 \geq 7 \)

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Abstract

A minimal surface of general type with \( p_g(S) = 0 \) satisfies \( 1 \leq K^2 \leq 9 \) and it is known that the image of the bicanonical map \( \varphi \) is a surface for \( K_S^2 \geq 2 \), whilst for \( K_S^2 \geq 5 \), the bicanonical map is always a morphism. In this paper it is shown that \( \varphi \) is birational if \( K_S^2 = 9 \) and that the degree of \( \varphi \) is at most 2 if \( K_S^2 = 7 \) or \( K_S^2 = 8 \).

By presenting two examples of surfaces \( S \) with \( K_S^2 = 7 \) and 8 and bicanonical map of degree 2, it is also shown that this result is sharp. The example with \( K_S^2 = 8 \) is, to our knowledge, a new example of a surface of general type with \( p_g = 0 \).

The degree of \( \varphi \) is also calculated for two other known surfaces of general type with \( p_g = 0 \), \( K_S^2 = 8 \). In both cases the bicanonical map turns out to be birational.

1 Introduction

Many examples of complex surfaces of general type with \( p_g = q = 0 \) are known, but a detailed classification is still lacking, despite much progress in the theory of algebraic surfaces. Surfaces of general type are often studied using properties of their canonical curves. If a surface has \( p_g = 0 \), then there are of course no such curves, and it seems natural to look instead at the bicanonical system, which is not empty.

Minimal surfaces \( S \) of general type with \( p_g(S) = 0 \) satisfy \( 1 \leq K_S^2 \leq 9 \). By a result of Xiao Gang [15], the image of the bicanonical map is a surface

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for $K_S^2 \geq 2$, while for $K_S^2 \geq 5$, by Reider’s theorem \cite{14}, the bicanonical map is always a morphism. Xiao Gang \cite{16} showed that the degree of the bicanonical map is $\leq 2$ for surfaces of general type, with a limited number of possible exceptions. Surfaces with $p_g = 0$ are among the exceptional cases, and \cite{14} gives practically no information on the possible degrees of their bicanonical maps.

The first author \cite{9} showed that if $K_S^2 \geq 3$ and the bicanonical map is a morphism, its degree is $\leq 4$. There are examples due to Burniat \cite{3} (see also \cite{13}) with $3 \leq K^2 \leq 6$ having bicanonical map of degree 4; indeed, \cite{10} gives a precise description of the surfaces with $K_S^2 = 6$, $p_g(S) = 0$ and bicanonical map of degree 4. Here we refine the result of \cite{9} by proving the following results:

**Theorem 1.1** Let $S$ be a minimal surface of general type defined over $\mathbb{C}$ with $p_g(S) = 0$, and $\varphi : S \rightarrow \Sigma \subset \mathbb{P}^{K_S^2}$ its bicanonical map, with image $\Sigma$.

(i) If $K_S^2 = 9$ then $\varphi$ is birational;

(ii) if $K_S^2 = 7, 8$ then $\varphi$ has degree $\leq 2$.

**Proposition 1.2** There exist minimal surfaces of general type with $p_g(S) = 0$, $K_S^2 = 7, 8$ and bicanonical map of degree 2.

We prove this by giving two examples of surfaces $S$ with $K_S^2 = 7$ and 8 and bicanonical map of degree 2. The example with $K_S^2 = 7$ is due to Inoue \cite[Remark 6]{8}, who constructed it as a quotient of a complete intersection in the product of four elliptic curves by a free action of $\mathbb{Z}_5^2$. Here we give an alternative description as a $\mathbb{Z}_2^2$-cover of a singular rational surface that allows us to describe the bicanonical map and compute its degree. The example with $K_S^2 = 8$ is obtained by applying a construction of Beauville (\cite[p. 123, Exercise 4]{1}, and cf. \cite{7}). To the best of our knowledge, it is a new example of a surface of general type with $p_g = 0$.

Only a few examples of surfaces with $p_g = 0$, $K_S^2 = 8$ are known. It is a nontrivial exercise to compute the degree of the bicanonical map for an explicit surface, and for interest, we include the computation for two other known surfaces of general type with $p_g = 0$, $K_S^2 = 8$, for both of which the bicanonical map turns out to be birational.

The paper is organized as follows: Section 2 recalls some facts on irregular double covers, one of the main ingredients of the proof of the theorem; in
Section 3 we prove the main result. For $K_S^2 = 7, 8$ the proof consists of using the methods of Section 2 to exclude the possibility that the bicanonical map has degree 4; for $K_S^2 = 9$ the result is obtained by combining Reider’s theorem with an analysis of the Picard group of $S$. In the final Section 4 we present the two examples to prove Proposition 1.2, and we also compute the degree of the bicanonical map of two other surfaces with $p_g = 0$ and $K_S^2 = 8$.

Notations and conventions We work over $\mathbb{C}$; all varieties are assumed to be compact and algebraic. We do not distinguish between line bundles and divisors on a smooth variety, and use additive and multiplicative notation interchangeably. Linear equivalence is denoted by $\equiv$ and numerical equivalence by $\sim$. The remaining notation is standard in algebraic geometry.

2 Irregular double covers and fibrations

We describe here the key facts used in some proofs in this paper.

Let $S$ be a smooth complex surface, $D \subset S$ a curve (possibly empty) with at worst ordinary double points, and $M$ a line bundle on $S$ with $2M \equiv D$. It is well known that there exists a normal surface $Y$ and a finite degree 2 map $\pi: Y \to S$ branched over $D$ such that $\pi_*\mathcal{O}_Y = \mathcal{O}_S \oplus M^{-1}$. The singularities of $Y$ are $A_1$ points and occur precisely above the singular points of $D$; thus it makes sense to speak of the canonical divisor, the geometric genus, the irregularity and the Albanese map of $Y$. We refer to $Y$ as the double cover defined by the relation $2M \equiv D$. The invariants of $Y$ are:

$$
\begin{align*}
K_Y^2 &= 2(K_S + M)^2, \\
\chi(\mathcal{O}_Y) &= 2\chi(\mathcal{O}_S) + \frac{1}{2}M(K_S + M), \\
p_g(Y) &= p_g(S) + h^0(S, K_S + M).
\end{align*}
$$

If $p_g(S) = q(S) = 0$, the existence of a double cover $\pi: Y \to S$ with $q(Y) > 0$ forces the existence of a fibration $f: S \to \mathbb{P}^1$ such that $\pi^{-1}$ of the general fibre of $f$ is disconnected. More precisely we have:

**Proposition 2.1 (De Franchis)** Let $S$ be a smooth surface with $p_g(S) = q(S) = 0$ and $\pi: Y \to S$ a double cover with at most $A_1$ points; if $q(Y) > 0$, then
(i) the Albanese image of $Y$ is a curve $B$;

(ii) let $\alpha : Y \to B$ be the Albanese fibration. Then there exists a fibration $g : S \to \mathbb{P}^1$ and a degree 2 map $p : B \to \mathbb{P}^1$ such that $p \circ \alpha = g \circ \pi$.

The possibility of existence of such a double cover often leads to a contradiction, using the following result:

**Corollary 2.2** Let $S$ be a minimal surface of general type with $p_g(S) = q(S) = 0$ and $K_S^2 \geq 3$, and $\pi : Y \to S$ a double cover with at most $A_1$ points. Then $K_Y^2 \geq 16(q(Y) - 1)$.

Proposition 2.1 is an old result of De Franchis [6], explained and generalized in several ways by Catanese and Ciliberto [5]. Proposition 2.1 and Corollary 2.2 are both stated and proved in [10] for smooth $Y$, but the proof extends verbatim to the case of $A_1$ points.

### 3 Proof of Theorem 1.1

Under the assumptions of Theorem 1.1, the image of the bicanonical map is a surface by [12], and the bicanonical map is a morphism by Reider’s theorem [12]. Moreover, since $4K_S^2 = \deg \varphi \deg \Sigma$ and $\Sigma$ is a nondegenerate surface in $\mathbb{P}^{K_S^2}$, the possible values of $\deg \varphi$ are 1, 2, 4 for $K_S^2 = 7, 8$ and 1, 2, 3, 4 for $K_S^2 = 9$.

We prove the theorem by analysing separately the cases $K_S^2 = 7, 8, 9$. In each case we argue by contradiction.

#### 3.1 The case $K_S^2 = 7$

By the above remark, it is enough to show that $\deg \varphi = 4$ does not occur. Assume that $\varphi$ has degree 4. The bicanonical image $\Sigma$ is a linearly normal surface of degree 7 in $\mathbb{P}^7$ and its nonsingular model has $p_g = q = 0$. By [12], Theorem 8], $\Sigma$ is the image of the blowup $\mathbb{P}$ of $\mathbb{P}^2$ at two points $P_1, P_2$ under its anticanonical map $f : \mathbb{P} \hookrightarrow \mathbb{P}^7$. If $P_1 \neq P_2$, then $f$ is an embedding, while if $P_2$ is infinitely near to $P_1$ (say) then $\Sigma$ has an $A_1$ singularity. In either case, the hyperplane section of $\Sigma$ can be written as $H = 2l + l_0$, where $l$ is the image on $\Sigma$ of a general line of $\mathbb{P}^2$ and $l_0$ is the image on $\Sigma$ of the strict transform of the line through $P_1$ and $P_2$. Notice that $l_0$ is contained
in the smooth part of $\Sigma$. Thus we have $2K_S \equiv 2L + L_0$, where $L = \varphi^*l$ and $L_0 = \varphi^*l_0$.

**Lemma 3.1** $L_0$ satisfies one of the following possibilities:

(i) there exists an effective divisor $D$ on $S$ such that $L_0 = 2D$; or

(ii) $L_0$ is a smooth rational curve with $L_0^2 = -4$; or

(iii) there exist smooth rational curves $A$ and $B$ with $A^2 = B^2 = -3$, $AB = 1$, and $L_0 = A + B$.

**Proof** Remark first that $K_S L_0 = 2$, $L_0^2 = -4$, and $L_0 = 2(K_S - L)$ is divisible by 2 in Pic $S$. Let $\theta$ be a $-2$-curve of $S$; then $\theta$ is contracted by $\varphi$ and thus $L_0 \theta = 0$. Since $L$ and $L_0$ are independent elements of the 3-dimensional space $H^1(S)$, $S$ contains at most one $-2$-curve. We write $L_0 = C + a\theta$, where $C$ is the strict transform of $L_0$, $\theta$ is a $-2$-curve and $a \geq 0$ (we set $a = 0$ if $S$ has no $-2$-curve). The equalities $\theta L_0 = 0$ and $L_0^2 = -4$ imply

$$\theta C = 2a, \quad \text{and} \quad C^2 = -4 - 2a^2. \quad (3.1)$$

If $C$ is irreducible, then $K_SC = 2$ implies $C^2 \geq -4$ and thus $a = 0$ and case (ii) holds. If $C$ is reducible, then $C = A + B$, with $A$ and $B$ irreducible curves such that $K_SA = K_SB = 1$. If $A = B$, then $AL_0 = 2A^2 + a\theta A = 2A^2 + a^2$ is even, because $L_0$ is divisible by 2, and thus $a$ is even and we are in case (i). If $A \neq B$, then $AB \geq 0$ and $A^2, B^2 \geq -3$; by parity considerations and (3.1) we get $A^2 = B^2 = -3$ and either $AB = 1$, $a = 0$ or $AB = 0$, $a = 1$. The first case corresponds to (iii), while the second does not occur. In fact the intersection matrix of $A, B, \theta$ would be negative definite, contradicting the index theorem, since $h^{1,1}(S) = 3$. ⊤

In cases (ii) or (iii) of Lemma 3.1, let $\pi : Y \rightarrow S$ be the double cover given by $2(K_S - L) \equiv L_0$; then the formulas (2.1) give $\chi(Y) = 2$ and $K_Y^2 = 16$. Since the bicanonical map $\varphi$ maps $L$ onto a twisted cubic, $h^0(S, O_S(2K_S - L)) = 4$ and thus $p_g(Y) = p_g(S) + h^0(S, O_S(2K_S - L)) = 4$; we thus obtain $q(Y) = 3$, contradicting Corollary 2.2.

In case (i) of Lemma 3.1, consider the étale double cover $\pi : Y \rightarrow S$ given by $2(K_S - L - D) \equiv 0$; arguing as above, we get that the invariants of $Y$ are

$$K_Y^2 = 14, \quad \chi(O_Y) = 2, \quad p_g(Y) = p_g(S) + h^0(S, O_S(2K_S - L - D)) = 3,$$
so that \( q(Y) = 2 \) and we again obtain a contradiction to Corollary 2.2. Hence \( \deg \varphi \neq 4 \) and we have proved Theorem 1.1 in case \( K_S^2 = 7 \).

### 3.2 The case \( K_S^2 = 8 \)

As in case \( K_S^2 = 7 \), it is enough to show that \( \deg \varphi = 4 \) does not occur. If \( \varphi \) has degree 4, then the bicanonical image \( \Sigma \) is a linearly normal surface of degree 8 in \( \mathbb{P}^8 \) whose nonsingular model has \( p_g = q = 0 \). By [11, Theorem 8], \( \Sigma \) is either the Veronese embedding in \( \mathbb{P}^8 \) of a quadric \( Q \subset \mathbb{P}^3 \) or the image of the blowup \( \hat{P} \) of \( \mathbb{P}^2 \) at a point \( P \) under its anticanonical map \( f: \hat{P} \hookrightarrow \mathbb{P}^8 \).

In the first case \( 2K_S \equiv 2A \), where \( A \) is the hyperplane section of \( Q \). Then \( \eta = K_S - A \) is a nontrivial 2-torsion element in \( \text{Pic} \, S \), since \( p_g(S) = 0 \). By \( \chi(Y) = 2, K_Y^2 = 16 \). Moreover, \( p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(A)) = 4 \), so that \( q(Y) = 3 \).

Since \( K_Y^2 = 16 \), this contradicts Corollary 2.2, and therefore \( \Sigma \) is not the Veronese embedding of a quadric.

If the bicanonical image \( \Sigma \) is the image of \( \hat{P} \) via the map induced by \( |−K_{\hat{P}}| \), then the hyperplane section of \( \Sigma \) can be written as \( H \equiv 2l + l_0 \), where \( l \) is the image on \( \Sigma \) of a general line of \( \mathbb{P}^2 \) and \( l_0 \) is the image on \( \Sigma \) of the strict transform of a general line through \( P \). Thus \( 2K_S \equiv 2L + L_0 \), where \( L = \varphi^*l \) and \( L_0 = \varphi^*l_0 \), and \( L_0 = \varphi^*l_0 \) is smooth by Bertini’s theorem. Consider now the double cover \( \pi: Y \rightarrow S \) given by \( 2(2K_S - L) \equiv L_0 \); the formulas (2.1) give \( \chi(Y) = 3 \) and \( K_Y^2 = 24 \). Since \( p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(2K_S - L)) = 0 + h^0(S, \mathcal{O}_S(L + L_0)) = 5 \), we get \( q(Y) = 3 \), contradicting Corollary 2.2. Thus \( \Sigma \) is also not the image of \( \hat{P} \).

Hence \( \deg \varphi \neq 4 \) and the proof of Theorem 1.1, (ii) is complete.

### 3.3 The case \( K_S^2 = 9 \)

If \( K_S^2 = 9 \), then by Poincaré duality, \( H^2(S, \mathbb{Z}) \) is generated up to torsion by the class of a line bundle \( L \) with \( L^2 = 1 \); thus every divisor on \( S \) is numerically a multiple of \( L \), and in particular \( K_S \sim 3L \).

Assume by contradiction that \( \varphi \) is not birational; then by Reider’s theorem (cf. [2, Theorem 2.1]), for every pair of points \( x_1, x_2 \in S \) with \( \varphi(x_1) = \varphi(x_2) \) there exists an effective divisor \( C \) containing \( x_1, x_2 \) such that \( K_SC - 2 \leq C^2 < \frac{1}{2}K_SC < 2 \). Since \( K_S \sim 3L \), the only possibility is that \( C \sim L \). We can assume that, as \( x_1 \) and \( x_2 \) vary, the divisor \( C \) varies in an irreducible system of curves, which is linear by the regularity of \( S \). Every curve of \( |C| \)
is irreducible, since the class of $C$ generates $H^2(S, \mathbb{Z})$ up to torsion, and the general curve of $|C|$ is smooth by Bertini’s theorem, since $C^2 = 1$. Therefore $|C|$ is a linear pencil of curves of genus 3 with one base point. For a general $C \in |C|$ we consider the exact sequence:

$$0 \to \mathcal{O}_S(2K_S - C) \to \mathcal{O}_S(2K_S) \to \mathcal{O}_C(2K_S) \to 0.$$ (3.2)

Since $2K_S - C \sim K_S + 2L$, Kodaira vanishing gives $H^1(S, \mathcal{O}_S(2K_S - C)) = 0$, and the map $H^0(S, \mathcal{O}_S(2K_S)) \to H^0(C, \mathcal{O}_C(2K_S))$ induced by the sequence (3.2) is surjective. So the map $f : C \to \mathbb{P}^3$ given by $|\mathcal{O}_C(2K_S)|$ is not birational; it follows that $f$ maps $C$ two-to-one onto a twisted cubic, and thus $C$ is hyperelliptic. If we denote by $\Delta$ the $g_1^2$ of $C$, then $2K_S|_C \equiv 3\Delta$ and also, by the adjunction formula, $K_S + C|_C \equiv 2\Delta$. So $\eta \equiv 4K_S - (3K_S + 3C) \equiv K_S - 3C$ is trivial when restricted to $C$. Moreover $\eta \sim 0$ and so $\eta$ is a torsion element of $\text{Pic} S$. Since $p_g(S) = 0$, $\eta$ is nonzero. Consider the connected étale cover $\pi : Y \to S$ associated to $\eta$. Because $\eta|_C = 0$, the cover $\pi|_{\pi^{-1}(C)} : \pi^{-1}(C) \to C$ is trivial and thus $\pi^{-1}(C)$ is a smooth disconnected curve with each component of self-intersection 1. This contradicts the Index theorem and we have thus proved Theorem 1.1, (i).

4 Examples

This section calculates the degree of the bicanonical map in 4 interesting examples, as discussed in the introduction.

Example 4.1 Starting from the quadrilateral $P_1P_2P_3P_4$ in $\mathbb{P}^2$ of Figure 4, let $P_5$ be the intersection point of the lines $P_1P_2$ and $P_3P_4$ and $P_6$ the intersection point of $P_1P_4$ and $P_2P_3$. Write $\Sigma \to \mathbb{P}^2$ for the blowup of $P_1, \ldots, P_6$, and $e_i$ for the exceptional curves of $\Sigma$ over $P_i$. Denote by $l$ the pullback of a line.

Write $S_1, \ldots, S_4$ for the strict transforms on $\Sigma$ of the sides $P_iP_{i+1}$ of the quadrilateral $P_1P_2P_3P_4$ (we take subscripts modulo 4); these are the only $-2$-curves of $\Sigma$. The morphism $f : \Sigma \to \mathbb{P}^3$ given by $|-K_\Sigma|$ has image a cubic surface $V \subset \mathbb{P}^3$, and $f$ is an isomorphism on $\Sigma \setminus \bigcup S_i$, and contracts each $S_i$ to an $A_1$ point.

If $A \subset \{P_1, \ldots, P_6\}$ consists of 4 points no three of which are collinear, then the linear system of conics through the points of $A$ gives rise to a free pencil on $\Sigma$; we denote by $f_1$ the strict transform of a general conic through
we have introduced satisfy the following relations:

(i) \(-K_\Sigma \equiv \Delta_1 + \Delta_2 + \Delta_3\);

(ii) \(f_i \equiv \Delta_{i+1} + \Delta_{i+2}\) for all \(i \in \mathbb{Z}_3\);

(iii) \(\Delta_i S_j = 0\) for all \(i, j\);

(iv) \(\Delta_i f_j = 2\delta_{ij}\) for \(1 \leq i, j \leq 3\).

Denote by \(\gamma_1, \gamma_2, \gamma_3\) the nonzero elements of \(\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2\) and by \(\chi_i \in \Gamma^*\) the nontrivial character orthogonal to \(\gamma_i\); by [12, Propositions 2.1 and 3.1], to define a smooth \(\Gamma\)-cover \(\pi: X \to \Sigma\), we specify:

(I) smooth divisors \(D_i\) for \(i = 1, 2, 3\) such that \(D = D_1 + D_2 + D_3\) is a normal crossing divisor,

(II) line bundles \(L_1, L_2\) satisfying \(2L_1 \equiv D_2 + D_3, 2L_2 \equiv D_1 + D_3\).
The branch locus of $\pi$ is $D$. More precisely, $D_i$ is the image of the divisorial part of the fixed locus of $\gamma_i$ on $S$. We have
\[
\pi_*\mathcal{O}_S = \mathcal{O}_\Sigma \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1},
\]
where $L_3 = L_1 + L_2 - D_3$, and $\Gamma$ acts on $L_i^{-1}$ via the character $\chi_i$.

Here we set:

(I) $D_1 = \Delta_1 + f_2 + S_1 + S_2$, $D_2 = \Delta_2 + f_3$, $D_3 = \Delta_3 + f_1 + f'_3 + S_3 + S_4$;
where $f_1, f'_3 \in |f_1|$, $f_2 \in |f_2|$, $f_3 \in |f_3|$ are general curves;

(II) $L_1 = 5l - e_1 - 2e_2 - e_3 - 3e_4 - 2e_5 - 2e_6$, and
$L_2 = 6l - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 3e_5 - 3e_6$

and we obtain $L_3 = 4l - 2e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6$. For $i = 1, \ldots, 4$, the (set theoretic) inverse image of $S_i$ in $X$ is the disjoint union of two $-1$-curves $E_{ij}, E_{ij'}$; contracting these 8 exceptional curves on $X$ and contracting the $S_i$ on $\Sigma$, we obtain a smooth $\mathbb{Z}_2^2$-cover $p: S \to V$. The map $p$ is branched on the four singular points of $V$ and on the image $\overline{D}$ of $D$, which is contained in the smooth locus of $V$. The bicanonical divisor $2K_X$ is equal to $\pi^*(2K_\Sigma + D) = \pi^*(-K_\Sigma + f_1 + S_1 + S_2 + S_3 + S_4) = \pi^*(-K_\Sigma + f_1) + 2 \sum E_{ij}$, and thus the bicanonical divisor $2K_S$ is equal to $\pi^*(-K_V + \overline{f_1})$, where $\overline{f_1}$ is the image of $f_1$ in $V$. So $2K_S$ is ample, since it is the pullback of an ample line bundle by a finite map, $S$ is minimal and of general type, and $K_S^2 = \frac{1}{4}(K_V + \overline{f_1})^2 = 7$.

To compute the geometric genus of $S$, recall that $p_g(X) = p_g(\Sigma) + \sum h^0(\Sigma, K_\Sigma + L_i)$ (cf. [4] or [12, Lemma 4.2]). We have
\[
\begin{align*}
K_\Sigma + L_1 &= 2l - e_2 - 2e_4 - e_5 - e_6, \\
K_\Sigma + L_2 &= 3l - e_1 - e_2 - e_3 - e_4 - 2e_5 - 2e_6, \\
K_\Sigma + L_3 &= l - e_1 - e_2 - e_3.
\end{align*}
\]

We show that $h^0(\Sigma, K_\Sigma + L_2) = 0$. Assume by contradiction that there exists $D \in |K_\Sigma + L_2|$ and consider the image $C$ of $D$ in $\mathbb{P}^2$; $C$ is a cubic containing $P_1, \ldots, P_6$ which has a double point at $P_5$ and $P_6$. By Bezout’s theorem, $\Delta_3$ is contained in $C$ and thus $C = \Delta_3 + Q$, where $Q$ is a conic containing $P_1, \ldots, P_6$, which is impossible. By similar (easier) arguments, one shows that $h^0(\Sigma, K_\Sigma + L_1) = h^0(\Sigma, K_\Sigma + L_3) = 0$, and thus $p_g(S) = p_g(X) = 0$. By the projection formula for a finite flat morphism,
\[ H^0(X, 2K_X) = \]
\[ H^0(\Sigma, -K_{\Sigma} + f_1 + \sum S_j) \oplus \left( \bigoplus_i H^0(\Sigma, -K_{\Sigma} + f_1 + \sum S_j - L_i) \right), \]
and \( \Gamma \) acts on \( H^0(\Sigma, -K_{\Sigma} + f_1 + \sum S_j - L_i) \) via the character \( \chi_i \). We have \( h^0(\Sigma, -K_{\Sigma} + f_1 + \sum S_j) = h^0(\Sigma, -K_{\Sigma} + f_1) \), since
\[
S_j(-K_{\Sigma} + f_1 + \sum S_j) = -2 \quad \text{for } i = 1, \ldots, 4;
\]
in addition, \( h^0(\Sigma, -K_{\Sigma} + f_1) = 7 \), since \( \Sigma \) is rational, \( 2f_1 + f_2 + f_3 \) has arithmetic genus 7, and \(-K_{\Sigma} + f_1 = K_{\Sigma} + 2f_1 + f_2 + f_3 \). Since \( p_2(S) = 8 \), there is a value \( i \in \{1, 2, 3\} \) such that \( h^0(\Sigma, -K_{\Sigma} + \sum S_j + f_1 - L_i) = 1 \) and
\[
h^0(\Sigma, -K_{\Sigma} + \sum S_j + f_1 - L_k) = 0 \quad \text{for } k \neq i.\]
Actually, an argument similar to that used for computing \( p_2(S) \) shows that
\[
h^0(-K_{\Sigma} + \sum S_j + f_1 - L_1) = h^0(\sum S_j + e_4) = 1, \]
\[
h^0(-K_{\Sigma} + \sum S_j + f_1 - L_2) = h^0(3l - e_1 - 2e_2 - e_3 - 2e_4 - e_5 - e_6) = 0, \]
\[
h^0(-K_{\Sigma} + \sum S_j + f_1 - L_3) = h^0(5l - e_1 - 2e_2 - e_3 - 3e_4 - 3e_5 - 3e_6) = 0. \]

It follows that the bicanonical map \( \varphi: S \to \mathbb{P}^7 \) is composed with the involution \( \gamma_1 \) but not with \( \gamma_2 \) and \( \gamma_3 \). Since \( |2K_S| \supseteq \pi^*| -K_{\Sigma}| \) and the map \( \Sigma \to \mathbb{P}^3 \) induced by \( |-K_{\Sigma}| \) is birational, it follows that \( \varphi \) has degree 2.

The remaining examples are obtained using the following construction due to Beauville (see [4, p. 123, Ex. 4] and cf. [7]). Let \( C_1, C_2 \) be curves of genus \( g_1, g_2 \), and assume that a group \( G \) of order \( (g_1 - 1)(g_2 - 1) \) acts on \( C_1, C_2 \) so that \( C_i/G \) is isomorphic to \( \mathbb{P}^1 \) for \( i = 1, 2 \); write \( p_i: C_i \to \mathbb{P}^1 \) for the projections onto the quotients and \( p: C_1 \times C_2 \to \mathbb{P}^1 \times \mathbb{P}^1 \) for the product of \( p_1 \) and \( p_2 \). Thus \( p \) is a Galois cover with group \( G \times G \). Assume in addition that there exists an automorphism \( \psi \in \operatorname{Aut}G \) whose graph \( \Gamma = \Gamma_{\psi} \subset G \times G \) acts freely on \( C_1 \times C_2 \). Then set \( S = (C_1 \times C_2)/\Gamma \) and denote by \( q: C_1 \times C_2 \to S \) the quotient map and by \( \pi: S \to \mathbb{P}^1 \times \mathbb{P}^1 \) the map induced by \( p \). If \( G \) is Abelian, then \( \pi \) is a \( G \)-cover. The surface \( S \) is minimal and of general type since \( C_1 \times C_2 \) is minimal of general type and \( q \) is étale. Since \( \Gamma \) acts freely, \( \chi(\mathcal{O}_{C_1 \times C_2}) = |G|\chi(\mathcal{O}_S) \) and \( K_{C_1 \times C_2}^2 = |G|K_S^2 \), namely \( \chi(\mathcal{O}_S) = 1, K_S^2 = 8 \). The irregularity \( q(S) \) equals the dimension of the \( \Gamma \)-invariant subspace of
$H^0(C_1 \times C_2, \Omega^1_{C_1 \times C_2}) \cong H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2})$. Since $C_1/G$ and $C_2/G$ are both rational and $\psi$ is an automorphism, it follows that $q(S) = 0$, and thus $p_\psi(S) = 0$.

**Example 4.2** As far as we know, this is a new example. In this case $G = \mathbb{Z}_2^3$, $g_1 = 5$, $g_2 = 3$. We denote by $\gamma_1, \gamma_2, \gamma_3$ the standard generators of $G$ and by $\chi_1, \chi_2, \chi_3$ the dual basis of the group of characters $G^*$. To construct the $G$-cover $p_i : C_i \to \mathbb{P}^1$ we have to specify (cf. [12, Propositions 2.1 and 3.1]):

(i) a divisor $D_\gamma$ on $\mathbb{P}^1$ for each nonzero $\gamma \in G$;

(ii) line bundles $L_1, L_2, L_3$ on $\mathbb{P}^1$ satisfying

$$2L_i \equiv \sum_\gamma \varepsilon_i(\gamma)D_\gamma,$$

where

$$\begin{cases} 
\varepsilon_i(\gamma) = 0 & \text{if } \gamma \in \ker \chi_i, \\
\varepsilon_i(\gamma) = 1 & \text{otherwise}.
\end{cases}$$

To construct $p_1 : C_1 \to \mathbb{P}^1$, we choose distinct points $P_1, \ldots, P_6 \in \mathbb{P}^1$ and set $D_{\gamma_1} = P_1 + P_2$, $D_{\gamma_2} = P_3 + P_4$, $D_{\gamma_3} = P_5 + P_6$, $D_\gamma = 0$ for $\gamma \neq \gamma_i$, and $L_1 = L_2 = L_3 = \mathcal{O}_{\mathbb{P}^1}(1)$. The curve $C_1$ is smooth connected of genus 5. To construct $p_2 : C_2 \to \mathbb{P}^1$, we choose distinct points $Q_1, \ldots, Q_5 \in \mathbb{P}^1$ and set $D_{\gamma_1} = Q_1$, $D_{\gamma_2} = Q_2$, $D_{\gamma_1 + \gamma_2} = Q_3$, $D_{\gamma_3} = Q_4 + Q_5$, $D_\gamma = 0$ for the remaining nonzero elements of $G$, and $L_1 = L_2 = L_3 = \mathcal{O}_{\mathbb{P}^1}(1)$. The curve $C_2$ is smooth connected of genus 3. Define $\psi \in \text{Aut } G$ by

$$\gamma_1 \mapsto \gamma_1 + \gamma_3, \quad \gamma_2 \mapsto \gamma_2 + \gamma_3, \quad \gamma_3 \mapsto \gamma_1 + \gamma_2 + \gamma_3.$$

In the above notation, $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is a $G$-cover and $2K_S = \pi^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1)$. By the projection formula, we have

$$H^0(S, 2K_S) = \bigoplus_{\chi \in \Gamma^\perp} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1) \otimes M_\chi^{-1}),$$

where $M_\chi^{-1}$ is the eigensheaf of $\pi_* \mathcal{O}_S$ corresponding to $\chi \in \Gamma^\perp \cong G^*$, and $(G \times G)/\Gamma \cong G$ acts on $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1) \otimes M_\chi^{-1})$ via $\chi$. The $M_\chi$ are line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ that can be determined using $[12, (2.15)]$. One checks that $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1) \otimes M_\chi^{-1})$ is nonzero only for the elements of $\Gamma^\perp$ that are orthogonal to $(0, \gamma_3) \in G \times G$. It follows that the bicanonical map of $S$ is composed with the involution induced on $S$ by $(0, \gamma_3)$, and thus it has degree 2 by Theorem [14].
Example 4.3 As for Example 4.1, this is due to Inoue [8, p. 317], and arises as the quotient of a complete intersection in the product of 4 elliptic curves by a free group action. Here we give a construction in the style of Beauville as explained above which is more suitable for our purpose. Let $\gamma_1, \ldots, \gamma_4$ be a basis of $G = \mathbb{Z}_2^4$ and $\chi_1, \ldots, \chi_4$ the dual basis of $G^*$; set $\gamma_0 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. We construct $C_i$ as $G$-covers of $\mathbb{P}^1$ for $i = 1, 2$. As in Example 4.2, for this, we specify (cf. [12, Propositions 2.1 and 3.1]):

(i) a divisor $D_\gamma$ of $\mathbb{P}^1$ for every nonzero $\gamma \in G$;

(ii) line bundles $L_1, \ldots, L_4$ of $\mathbb{P}^1$ satisfying

\[
2L_i \equiv \sum_\gamma \varepsilon_i(\gamma) D_\gamma, \quad \text{where} \quad \begin{cases} 
\varepsilon_i(\gamma) = 0 & \text{if } \gamma \in \ker \chi_i \\
\varepsilon_i(\gamma) = 1 & \text{otherwise}.
\end{cases}
\]

Choose distinct points $P_0, \ldots, P_4 \in \mathbb{P}^1$ and set $D_\gamma = P_i$ for $i = 0, \ldots, 4$, $D_\gamma = 0$ if $\gamma \neq \gamma_i$, and $L_i = \mathcal{O}_{\mathbb{P}^1}(1)$ for $i = 1, \ldots, 4$. We write $p_1 : C_1 \to \mathbb{P}^1$ for the corresponding $G$-cover. Then $C_1$ is a smooth connected curve of genus 5. We construct the curve $C_2$ in the same way, starting from points $Q_0, \ldots, Q_4 \in \mathbb{P}^1$.

Let $\psi \in \text{Aut } G$ be the automorphism:

\[
\gamma_1 \mapsto \gamma_1 + \gamma_3, \quad \gamma_2 \mapsto \gamma_2 + \gamma_4, \quad \gamma_3 \mapsto \gamma_1 + \gamma_4, \quad \gamma_4 \mapsto \gamma_1 + \gamma_3 + \gamma_4.
\]

In the above notation, $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is a $G$-cover and $2K_S = \pi^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$. Since $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ is very ample, it follows that the bicanonical map $\varphi$ of $S$ is birational if and only if it is not composed with an involution $\gamma$ of the Galois group $G$ of $\pi$. To check that this is indeed the case, we use the projection formula

\[
H^0(S, 2K_S) = \bigoplus_{\chi \in \Gamma^\perp} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \otimes M^{-1}_\chi),
\]

where $M^{-1}_\chi$ is the eigensheaf of $\pi_* \mathcal{O}_S$ corresponding to $\chi \in \Gamma^\perp \cong G^*$, and $(G \times G)/\Gamma \cong G$ acts on $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \otimes M^{-1}_\chi)$ via $\chi$. The $M_\chi$ are line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ that can be determined using [12, (2.15)]. It turns out that no $\gamma \in G \setminus \{0\}$ acts trivially on $H^0(S, 2K_S)$ and thus $\varphi$ is birational.
Example 4.4 This example is due to Beauville and appears in [1, p. 123, Ex. 4], where the group action is not described explicitly, and in [7]. The assertion concerning the group action in [7] is not correct, since the group action described is not free.

In this case $g = 6$, $G = \mathbb{Z}_5^2$, and $C_1 = C_2 = \{x^5 + y^5 + z^5 = 0\} \subset \mathbb{P}^2$ is the Fermat quintic. If $\varepsilon$ is a primitive 5th root of 1, then $(1, 0) \in G$ acts on $C$ by $(x : y : z) \mapsto (\varepsilon x : y : z)$ and $(0, 1)$ acts by $(x : y : z) \mapsto (x : \varepsilon y : z)$. Let $\psi$ be the automorphism of $G$ taking $(1, 0) \mapsto (1, -1)$ and $(0, 1) \mapsto (1, 2)$.

We compute the degree of the bicanonical map of $S$ by writing down an explicit basis of the $\Gamma$-invariant subspace of $H^0(C \times C, 2K_{C \times C})$. Take homogeneous coordinates $(x : y : z; x_1 : y_1 : z_1)$ on $\mathbb{P}^2 \times \mathbb{P}^2 \supset C \times C$; using the fact that a regular 1-form on $C$ is the residue of a rational form $\frac{g(x, y, z)}{x^i+y^j+z^k} \, dx \wedge dy \wedge dz$ for $g$ homogeneous of degree 2, we see that $(a, b) \in G$ acts on bicanonical forms on $C \times C$ by:

$$x^i y^j z^{4-i-j} x_1^a y_1^b z_1^{4-a-b} \mapsto \varepsilon^l x^i y^j z^{4-i-j} x_1^a y_1^b z_1^{4-a-b},$$

where $l = a(2 + i + \alpha - \beta) + b(3 + j + \alpha + 2\beta)$

Thus the following is a basis of $H^0(Y, 2K_Y)^{\text{inv}}$:

$$x^4 y_1 z_1^3, \quad y^3 z y_1^2 z_1^2, \quad x y z^2 y_1^3 z_1, \quad x^2 y z x_1^2 z_1^3, \quad z^4 x_1 y_1^3,$$

$$x z^2 x_1^2 z_1^2, \quad x^3 y x_1^2 y_1^2, \quad y^4 x_1^3 z_1, \quad x y^2 z x_1^3 y_1.$$

The subfield of $\mathbb{C}(S)$ generated by ratios of these monomials is the function field $\mathbb{C}(\Sigma)$ of the bicanonical image $\Sigma$ of $S$. The map $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ identifies $\mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$ with the subfield of $\mathbb{C}(S)$ generated by $x^5 z^{-5}$ and $x_1^5 z_1^{-5}$. The extension $\mathbb{C}(S) \supset \mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$ is Galois with Galois group $G = \mathbb{Z}_5^2$. We observe:

$$x^5 z^{-5} = (x^3 y x_1^2 y_1^2)(x^4 y_1 z_1^3)(x^2 y z x_1^3 z_1^3)^{-1}(z^4 x_1 y_1^3)^{-1}$$

and

$$x_1^5 z_1^{-5} = (z^4 x_1 y_1^3)^2(y^3 z y_1^2 z_1^2)(x^3 y x_1^2 y_1^2)^2(x^2 y z x_1^3 z_1^3)^{-1}(x y z^2 y_1^3 z_1)^{-4}.$$

It follows that $\mathbb{C}(\Sigma) \supset \mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$. Now one checks that no element of the Galois group $G = \mathbb{Z}_5^2$ of $\mathbb{C}(S)$ over $\mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$ acts trivially on $\mathbb{C}(\Sigma)$. It follows that $\mathbb{C}(\Sigma) = \mathbb{C}(S)$, namely $\varphi$ is birational.
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