Towards a Consistent SUSY QFT
in Extra Dimensions

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Abstract
We consider N=1 SUSY gauge theory in six dimensions in components and show that provided the Dynkin indices of the matter fields representations satisfy the relation \[ \sum T(R) = C_2(G), \] the gauge sector is completely one-loop finite. In the matter sector the UV divergences form several invariant structures some of which are cancelled in physical amplitudes. Thus, the theory which is in general non-renormalizable may be consistent and even finite. Consequences for the SUSY GUT models in the bulk are briefly discussed.

1 Introduction
It became nowadays popular to consider theories in extra dimensions as possible candidates for models of physics beyond the Standard Model. They provide new scenarios for the coupling unification as well as are able to elegantly solve some problems like doublet-triplet splitting, suppression of proton decay, SUSY breaking, etc. (See e.g. Ref. [1] and references therein.) Usually, for the sake of simplicity one considers one extra dimension and then assumes compactification on the orbifold. Particular models with \( S^1/(Z_2 \times Z_2') \) compactification are shown to possess some interesting properties and may serve as a basis for the Grand Unified Theories [1, 2]. In this case, one has N=1 supersymmetry in a five-dimensional bulk which is equivalent to N=2 SUSY at a four dimensional brane. The field content of the resulting theory at the brane depends on the compactification prescription and on quantum numbers with respect to the orbifold symmetries adjusted to the fields. Thus, below the compactification scale (which might be the GUT scale), one has the resulting D=4 theory on the brane with specific properties and above this scale one has a full N=1 D=5 theory in the bulk.

One may wonder whether this extra dimensional theory can be considered as a consistent QFT in any sense. Since by general power counting it is non-renormalizable, it looks hardly possible. However, there is a chance that all the UV divergences cancel each other, like it takes place in N=4, 2 and even N=1 SUSY theories in D=4 [3], and one has a consistent theory.

One way to consider an extra dimensional theory is the Kaluza-Klein approach. In this case, one takes the Fourier transform over the extra dimensions and obtains an infinite tower of states with quantized masses. Then one has to sum over all the states. This sum is usually divergent and a special prescription is needed to regularize it. Following this approach divergences in D=5
SUSY theory have been studied in [6, 7, 8] for the scalar effective potential. Some cancellations of UV divergences have been found.

The detailed structure of the K-K modes depends on the compactification pattern. Provided that in the zero mode sector the divergences cancel each other, one may wonder if this is also possible at each floor of an infinite tower. This way one may get a finite theory.

In what follows, we investigate the other possibility and consider explicitly D=6 N=1 SUSY gauge theory. We show that indeed under certain circumstances UV divergences may cancel each other and one can have a totally finite consistent quantum field theory in extra dimensions. These models are very distinguished by their properties and may serve as a basis for SUSY GUT models mentioned above. Below we discuss some of their properties.

2 The Model

We consider D=6 gauge QFT with on shell N=1 supersymmetry formulated in components. D=6 is chosen for simplicity as the lowest even dimension. It has the same N=1 supersymmetry as the D=5 one (equivalent to N=2 SUSY in D=4), but the integration in even dimensions is more familiar (in odd dimensions there are no one loop divergences in dimensional regularization which we are going to apply in the calculations). We do not take any particular compactification pattern, since the UV divergences do not depend on it; they reflect the small distance properties where locally one has the flat Minkowski metric. We assume that the theory is regularized in a SUSY invariant way but for practical purposes we take the dimensional regularization (or reduction), since there is no difference in the one loop order as concerns the UV divergences.

In what follows, we take the usual gauge invariant Lagrangian for the gauge and matter fields, and choose the background field gauge as being more useful for the calculations. Of course, the superfield formalism [9] would be most appropriate for our purposes and one should try to apply it. However, in this paper we confine ourselves to the component approach as a more familiar one. Then, the gauge and N=1 SUSY invariant Lagrangian in D=6 is

\[ \mathcal{L} = -\frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} + i\bar{\lambda} \hat{D}\lambda + i\bar{\psi} \hat{D}\psi + (D_\mu \phi_1^\dagger (D_\mu \phi_1) + (D_\mu \phi_2^\dagger (D_\mu \phi_2) ) \]

\[ + i\sqrt{2} g [ (\bar{\psi} \lambda \phi_1 + \bar{\lambda} \psi \phi_1^\dagger) + (\bar{\psi} \lambda \phi_2 + \bar{\lambda} \psi \phi_2^\dagger) ] - \frac{g^2}{2} |\phi_1^\dagger T^a \phi_1 - \phi_2^\dagger T^a \phi_2|^2 + 2g^2 |\phi_1^\dagger T^a \phi_2|^2 \]

and contains the following set of fields: one gauge field, \( A_\mu^a \), one Weyl gaugino field, \( \lambda \), a set of chiral matter fields in a representation \( R \), \( \psi \), and the corresponding complex scalar fields \( \phi_1 \) and \( \phi_2 \).

2.1 The gauge sector

In the gauge sector, due to the background field gauge invariance the divergent structures in the one loop order can take one of the following forms:

\[ I_1 = \text{Tr} D_\nu F_{\mu\nu} D_\mu F_{\mu\nu} , \]

\[ I_2 = \text{Tr} D_\mu F_{\mu\nu} D_\rho F_{\nu\rho} , \]

\[ I_3 = \text{Tr} D_\mu F_{\mu\nu} D_\mu F_{\nu\rho} , \]

\[ I_4 = \text{Tr} F_{\mu\nu} F_{\nu\mu} F_{\mu\nu} . \]
However, these invariants are not independent. Due to the relation $[D_{\mu}, D_{\nu}] = F_{\mu\nu}$, and the Bianchi identity $D_\mu F_{\nu\rho} + D_\rho F_{\mu\nu} + D_\nu F_{\rho\mu} = 0$ one has only 2 independent structures and can choose any of them. We take the first two. Then calculating the diagrams and extracting the contribution to two independent Lorentz structures one can find the coefficients in front of them.

The structures written above contain 2-, 3-, 4-, 5- and 6-leg diagrams. For simplicity, we consider 2- and 3-point functions. We use the Feynman rules from Ref.[11]. The calculation of the two-point function can be performed exactly in an arbitrary dimension. The result is proportional to the transverse tensor and the proper power of momenta. The coefficient functions are summarized in the table (in Feynman gauge).

| The Diagram              | Group factor | D-dim Expression                                                                 | $D = 4 - 2\varepsilon$ | $D = 4 - 2\varepsilon$ |
|--------------------------|--------------|----------------------------------------------------------------------------------|-------------------------|-------------------------|
| Gauge loop               | $C_2(G)$     | $(-)^{[D/2]}\Gamma(2-D/2)\Gamma^2(D/2)}\frac{7D-8}{D-2}$                      | $-\frac{10}{3\varepsilon}$ | $-\frac{17}{30\varepsilon}$ |
| Ghost loop               | $C_2(G)$     | $(-)^{[D/2]}\Gamma(2-D/2)\Gamma^2(D/2)}\frac{4}{D-2}$                        | $\frac{1}{3\varepsilon}$  | $\frac{1}{30\varepsilon}$  |
| (Complex) Scalar         | $T(R)$       | $(-)^{[D/2]}\Gamma(2-D/2)\Gamma^2(D/2)}\frac{4}{D-2}$                        | $\frac{1}{3\varepsilon}$  | $\frac{1}{30\varepsilon}$  |
| Majorana (Weyl) fermion  | $T(R)$       | $(-)^{[D/2]}2^{[D/2]}\Gamma^2(D/2)}\frac{2}{D-2}$                            | $\frac{2}{3\varepsilon}$  | $\frac{8}{30\varepsilon}$  |

Table 1: The two-point gauge function

It is instructive to consider separately the dimension of integration $D$ and the space-time dimension $D'$. The latter corresponds to Lorentz algebra. Then the formulas in the table are modified as follows: in the first line one has to change

$$7D - 8 \Rightarrow 8(D - 1) - D' + \frac{(D - 4)(D - 1)\alpha(8 - \alpha)}{2(D - 2)},$$

where we give the result in an arbitrary $\alpha$-gauge ($\alpha = 0$ corresponds to Feynman gauge), and in the last line one has to change $2^{[D/2]}$ by $2^{[D'/2]}$.

Let us first take the $D = D' = 4, \ N = 1$ case. It corresponds to a gauge and a ghost field, one Majorana spinor in adjoint representation, and a number of supermultiplets which contain a Weyl spinor and a complex scalar in representation R. Then, the pole term is

$$-\frac{11}{3}C_2(G) + \frac{2}{3}C_2(G) + \frac{2}{3}T(R) + \frac{1}{3}T(R) = -3C_2(G) + T(R),$$

i.e., cancellation of divergences is possible if

$$\sum T(R) = 3C_2(G).$$  \hspace{1cm} (4)

In the $D = D' = 4, \ N = 2$ case, one has to take one supermultiplet in the adjoint representation and add a mirror partner to any matter supermultiplet. The pole term here is

$$-\frac{11}{3}C_2(G) + \frac{2}{3}C_2(G) + \frac{2}{3}C_2(G) + \frac{1}{3}C_2(G) + 2(\frac{2}{3}T(R) + \frac{1}{3}T(R)) = -2C_2(G) + 2T(R).$$
Hence, one has instead of (4)
\[ \sum T(R) = C_2(G). \] (5)
This case corresponds to the zero modes of D=6 theory in the K-K approach. So for zero modes
the UV divergences in the gauge sector cancel if eq.(3) is satisfied. If the other modes follow
the same pattern, one can get a finite theory.

Consider now the case \( D = 6, \ N = 1 \) case. This corresponds to one Weyl spinor in the
adjoint representation and matter supermultiplets in R representation which contain a Weyl
spinor and two complex scalars. The pole term, which corresponds to logarithmic divergence,
is (in Feynman gauge)
\[-\frac{17}{30} C_2(G) - \frac{1}{30} C_2(G) + \frac{8}{30} C_2(G) + \frac{8}{30} T(R) + \frac{2}{30} T(R) = -\frac{1}{3} C_2(G) + \frac{1}{3} T(R),\]
i.e. one has the same relation (5) as in the D=4 case.

However, there is also a quadratic divergence in D=6. The corresponding gauge invariant
operator in this case is simply \( Tr F_{\mu\nu} F_{\mu\nu} \). It can be reproduced in dimensional regularization as
the residue at the pole at \( \varepsilon = 1 \), i.e one has to take \( D = 4 \). Substituting in eq.(3) \( D' = 6, D = 4 \) one gets for the quadratic divergence
\[ \frac{1}{6} [-(18 - 2 + 8) C_2(G) + 12 T(R)] = 2[-C_2(G) + T(R)], \]
i.e. one has again eq.(5), but the gauge dependence disappears here from the result. It is gauge
invariant.

Though we are interested here in D=6 theory, it is interesting to apply eq.(5) to D=10
case, since it corresponds to N=4 SUSY theory in D=4. One has here besides logarithmic
also quadratic, quartic and sextic divergences. They can be reproduced from eq.(5) by allowing
\( D' = 10 \) and \( D = 10, 8, 6 \) and 4, respectively. In this case one has no matter, but one Majorana-
Weyl spinor in adjoint representation. The pole term is proportional to (for any D)
\[ C_2(G)[-16 + 16 + \frac{(D - 4)(D - 1)\alpha(8 - \alpha)}{2(D - 2)}], \]
i.e. all divergences cancel in Feynman gauge and the highest divergence is gauge invariant and
vanishes in any gauge.

Consider now the three-point vertices. The divergent part of the 3-point diagrams in the
\( D = 6, \ N = 1 \) case is (in Feynman gauge):
\[
\frac{T(R) - C_2(G)}{6} \left\{ \frac{4}{3} k_{1\mu} k_{1\nu} k_{1\rho} - \frac{4}{3} k_{2\mu} k_{2\nu} k_{2\rho} + \frac{2}{3} k_{1\rho} k_{1\mu} k_{2\nu} - \frac{2}{3} k_{1\mu} k_{2\rho} k_{2\nu} \\
+ \frac{4}{3} k_{1\mu} k_{1\nu} k_{2\rho} - \frac{4}{3} k_{2\mu} k_{2\nu} k_{1\rho} \\
+ g_{\mu\nu} k_{1\rho} \left[ \frac{2}{3} k_{1}^2 + 2 k_{2}^2 + \frac{4}{3} (k_{1} k_{2}) \right] - g_{\mu\rho} k_{2\nu} \left[ \frac{2}{3} k_{2}^2 + 2 k_{1}^2 + \frac{4}{3} (k_{1} k_{2}) \right] \\
- g_{\mu\nu} k_{1\nu} \left[ \frac{8}{3} k_{1}^2 + \frac{8}{3} k_{2}^2 + \frac{8}{3} (k_{1} k_{2}) \right] - g_{\mu\rho} k_{2\nu} \left[ \frac{8}{3} k_{2}^2 + \frac{4}{3} k_{1}^2 \right] \\
+ g_{\nu\rho} k_{1\mu} \left[ \frac{8}{3} k_{2}^2 + \frac{4}{3} k_{2}^2 \right] + g_{\nu\rho} k_{2\mu} \left[ \frac{8}{3} k_{1}^2 + \frac{8}{3} k_{2}^2 + \frac{8}{3} (k_{1} k_{2}) \right] \right\}.
\]
This leads to the following terms (after reduction)

\[ \frac{T(R) - C_2(G)}{3} f^{abc} \left( 2 \partial_\mu \partial_\nu A_\mu^a A_\nu^b A_\rho^c + 2 \partial^2 \partial_\nu A_\mu^a A_\mu^b A_\nu^c + 6 \partial_\nu A_\mu^a \partial_\nu A_\mu^b A_\nu^c + 4 \partial_\nu \partial_\rho A_\mu^a \partial_\nu A_\mu^b A_\nu^c \right) \]

At the same time, expansion of the invariants over the fields up to the third order gives

\[
I_1 = 2 A_\mu^a \partial^2 (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A_\nu^a + f^{abc} (4 \partial^2 \partial_\nu A_\mu^a A_\mu^b A_\nu^c - 4 \partial_\mu \partial_\nu A_\mu^a \partial_\nu A_\mu^b A_\nu^c + 4 \partial_\mu \partial_\nu A_\mu^a \partial_\nu A_\mu^b A_\nu^c), \\
I_2 = A_\mu^a \partial^2 (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A_\nu^a + f^{abc} (2 \partial^2 \partial_\nu A_\mu^a A_\nu^b A_\nu^c - 6 \partial^2 A_\mu^a \partial_\nu A_\mu^b A_\nu^c + 4 \partial_\mu \partial_\nu A_\mu^a \partial_\nu A_\mu^b A_\nu^c \\
+ 2 \partial_\mu \partial_\nu A_\mu^a \partial_\nu A_\mu^b A_\nu^c).
\]

Adding them together one finds

\[
x I_1 + y I_2 = A_\mu^a \partial^2 (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A_\nu^a (2x + y) \\
+ 2 f^{abc} \partial^2 \partial_\nu A_\mu^a A_\mu^b A_\nu^c (2x + y) - 2 f^{abc} \partial_\mu \partial_\nu A_\mu^a \partial_\nu A_\mu^b A_\nu^c (2x - 2y) \\
- 2 f^{abc} \partial^2 A_\mu^a \partial_\nu A_\mu^b A_\nu^c (3y) + 2 f^{abc} \partial_\mu \partial_\nu A_\mu^a \partial_\nu A_\mu^b A_\nu^c (2x + y) - 2 f^{abc} \partial_\mu \partial_\nu A_\mu^a \partial_\nu A_\mu^b A_\nu^c (2x).
\]

Comparing this with eq.(5), one gets \((C_2(G) = C_A)\)

\[
2x + y = \frac{T_R - C_A}{3}, \quad 2x - 2y = -2 \frac{T_R - C_A}{3}, \quad 3y = T_R - C_A, \quad 2x = 0.
\]

Thus, one has the one-loop logarithmic divergences in the form

\[
\frac{T_R - C_A}{3} Tr D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}.
\]

One finds that the result for ALL the structures is proportional to \(\sum T(R) - C_2(G)\), like for the two-point functions, and vanishes if eq.(5) is satisfied. Due to the fact that all the structures vanish we claim that all the one loop divergences in the gauge sector cancel for \(\sum T(R) = C_2(G)!\)

The situation is not that simple in an arbitrary \(\alpha\)-gauge. Equation (7) in accordance with (3) in this case looks like

\[
\frac{T_R - C_A(1 + \alpha - \alpha^2/8)}{3} Tr D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}.
\]

The cancellation is not obvious anymore. One has to consider the proper combination of the Green functions to observe the cancellation of the gauge dependence and associated cancellation of the UV divergences. We come back to this problem when considering the matter sector.

### 2.2 The matter sector

In the matter sector, in four dimensions one has both the propagators and the vertices to diverge and only the proper combination of them is finite. The situation is even more complicated in \(D=6\), since here one has extra powers of momenta in the diagrams and the usual cancellation does not work. Still one can try to find a proper combination of vertices that gives finite matrix elements. We consider some examples of this cancellation below but first we calculate the one-loop diagrams with the matter fields.
The spinor fields

Consider the fermions first. Restricting oneself to the diagrams with two fermion legs, in the one-loop order one has the following invariants:

\[ J_1 = \bar{\Psi} \gamma^\nu D_\mu D_\nu \Psi, \]
\[ J_2 = \bar{\Psi} \gamma^\nu D_\mu D_\nu D_\rho \Psi, \]
\[ J_3 = \bar{\Psi} \gamma^\nu D_\rho D_\mu D_\nu \Psi, \]
\[ J_4 = \bar{\Psi} \gamma^\mu \gamma^\nu \gamma^\rho D_\mu D_\nu D_\rho \Psi. \] (9)

Expanding them up to the third order over the gauge fields one has

\[ J_1 = \bar{\Psi} \gamma^\mu \left( \partial^2 \partial_\mu + 2A_\mu \partial_\nu \partial_\mu + \partial_\nu A_\mu \partial_\mu + \partial^\nu (A_\mu \Psi) \right) \]
\[ \Rightarrow p^2 \hat{p} + (2 \hat{p}_1 p_1^\mu + \hat{p}_1 p_1^\mu + p_2^2 \gamma^\mu) A_\mu, \]
\[ J_2 = \bar{\Psi} \gamma^\mu \left( \partial^2 \partial_\mu + A_\mu \partial_\nu \partial_\mu + \partial_\nu A_\mu \partial_\mu + A_\mu \partial_\nu \partial_\mu + \partial_\nu \partial_\mu (A_\nu \Psi) \right) \]
\[ \Rightarrow p^2 \hat{p} + (\hat{p}_1 p_1^\mu + \hat{p}_2 p_2^\mu + p_3^2 \gamma^\mu) A_\mu, \]
\[ J_3 = \bar{\Psi} \gamma^\mu \left( \partial^2 \partial_\mu + A_\mu \partial_\nu \partial_\mu + \partial_\nu A_\mu \partial_\mu + 2A_\nu \partial_\nu \partial_\mu + \partial_\nu \partial_\mu A_\nu + \partial_\nu A_\nu A_\mu \right) \Psi \]
\[ \Rightarrow p^2 \hat{p} + (2 \hat{p}_3 p_1^\mu + \hat{p}_1 p_1^\mu + \hat{p}_2 p_2^\mu + \hat{p}_1 p_2^\mu + p_1^2 \gamma^\mu) A_\mu, \]
\[ J_4 = \bar{\Psi} \gamma^\mu \left( \partial^2 \partial_\mu + \partial^2 (A_\mu + \partial_\nu A_\mu \partial_\nu) \right) \Psi \]
\[ \Rightarrow p^2 \hat{p} + (p_2^2 \gamma^\mu - \hat{p}_2 \gamma^\nu \hat{p}_1 + p_1^2 \gamma^\mu) A_\mu. \]

Adding these expressions together one finds \((p_1 + p_2 + p_3 = 0)\)

\[ x J_1 + y J_2 + z J_3 + t J_4 \]
\[ = \hat{p} p^2 (x + y + z + t) + A_\mu \left[ \hat{p}_1 p_1^\mu (x + y) \right. \]
\[ + \left. \hat{p}_2 p_2^\mu (y + z) + \hat{p}_3 p_3^\mu (-x) + \hat{p}_1 p_1^\mu (z + t) \right \] \[ \rightarrow \hat{p}_2 \gamma^\mu \hat{p}_1 (-t) \].

At the same time, the calculation of two- and three-point diagrams gives (we take here an arbitrary \(\alpha\)-gauge)

\[ - \hat{p} p^2 \alpha \frac{C_R}{3} + \frac{C_A}{6} \left[ (-6 + \alpha) \hat{p}_2 \gamma^\mu \hat{p}_1 + \left( 2 + \frac{5}{2} \alpha - \frac{1}{4} \right) (\hat{p}_1 p_1^\mu + \hat{p}_2 p_2^\mu) + \left( 8 + \frac{5}{2} \alpha - \frac{1}{4} \right) (\hat{p}_2 p_1^\mu + \hat{p}_2 p_1^\mu + \hat{p}_1 p_1^\mu) \right. \]
\[ + \left. \left( 8 + \frac{5}{2} \alpha - \frac{1}{4} \right) \hat{p}_1 p_1^\mu + \gamma^\mu \left[ (-4 - \frac{7}{2} \alpha + \frac{1}{4} \right) (p_1^2 + p_2^2) + (-10 - 5 \alpha + \frac{1}{2} \right) p_1 p_2 \right] A_\mu \]
\[ + \frac{C_R - C_A/2}{3} \left[ \alpha \hat{p}_2 \gamma^\mu \hat{p}_1 - 2(\hat{p}_1 p_1^\mu + \hat{p}_2 p_2^\mu) - 2(\hat{p}_2 p_1^\mu + \hat{p}_1 p_2^\mu) + (2 - \alpha) \gamma^\mu (p_1^2 + p_2^2) + 4 \gamma^\mu p_1 p_2 \right] A_\mu. \]

Comparing these expressions one gets

\[ x = z = \frac{2}{3} \alpha R - \frac{C_A}{3} \left( \frac{5}{4} \alpha - \frac{1}{8} \alpha^2 \right), \quad y = -\frac{4}{3} \alpha R + \alpha - \frac{1}{4} \alpha^2, \quad t = \frac{C_A}{3} (3) - \frac{C_R}{3} \alpha. \]

Thus, the one loop divergences for the matter spinor fields have the following form:

\[ \bar{\Psi} \left\{ \gamma^\nu [D_\mu D_\nu D_\rho - 2D_\mu D_\nu D_\rho + D_\nu D_\rho D_\mu] \frac{2}{3} \alpha \left( \frac{C_R}{3} - \alpha \right) \left[ \frac{C_A}{3} (3) - \alpha \frac{C_R}{3} \gamma^\nu \gamma^\mu \gamma^\rho D_\mu D_\nu D_\rho \right] \Psi \right\}. \] (10)
For the gaugino field they are slightly different

\[
\bar{\lambda} \left\{ \gamma^\mu [D_\mu D_\mu D_\nu - 2 D_\mu D_\nu D_\mu + D_\nu D_\mu D_\mu] \left[ \frac{C_A}{3} \left( -\frac{5}{4} \alpha + \frac{\alpha^2}{8} \right) - T_R \right] \right. \\
- [\gamma^\nu D_\mu D_\nu D_\rho - \gamma^\mu \gamma^\nu \gamma^\rho D_\mu D_\nu D_\rho] \frac{C_A}{3} + \frac{2 T_R}{3} + \frac{C_A(1 - \alpha) - T_R \gamma^\mu \gamma^\nu \gamma^\rho D_\mu D_\nu D_\rho}{3} \left. \right\} \lambda .
\]

(11)

In the case when \( \sum T(R) = C_A \) one has

\[
\bar{\lambda} \left\{ \gamma^\mu [D_\mu D_\mu D_\nu - 2 D_\mu D_\nu D_\mu + D_\nu D_\mu D_\mu] \frac{C_A}{3} \left( -3 - \frac{5}{4} \alpha + \frac{\alpha^2}{8} \right) \right. \\
- [\gamma^\nu D_\mu D_\nu D_\rho - \gamma^\mu \gamma^\nu \gamma^\rho D_\mu D_\nu D_\rho] \frac{C_A}{3} (3) - \frac{\alpha}{3} C_A \gamma^\mu \gamma^\nu \gamma^\rho D_\mu D_\nu D_\rho \left. \right\} \lambda,
\]

and for \( \alpha = 0 \)

\[
- C_A \bar{\lambda} \left\{ \gamma^\mu [D_\mu D_\mu D_\nu - D_\mu D_\nu D_\mu + D_\nu D_\mu D_\mu] - \gamma^\mu \gamma^\nu \gamma^\rho D_\mu D_\nu D_\rho \right\} \lambda .
\]

(13)

No wonder, these expressions look complicated. The same is in D=4. However there, if one takes the product of the 3-point function and the square roots of the propagators for each leg, the resulting invariant charge is finite and does not depend on the gauge. One can try the same combination in D=6, but apparently it does not work due to the complicated momentum structure of the 3-point function.

Since our final goal is to get finite and gauge invariant observables, we consider the amplitude of a physical process. As an example we take the Compton scattering. In this case one has the combination of 2-, 3- and 4-point functions shown in Fig.1. Taking the contribution to the diagrams on the r.h.s. of Fig.1 from eq.(10) one gets the resulting amplitude. Since there are three independent structures in eq.(10), we consider their contribution separately. The simplest is the last one which comes with the coefficient proportional to the gauge parameter and is expected to disappear from the final answer. Indeed, one has the following divergent contribution coming from this structure

\[
\begin{align*}
\text{Figure 1: The Compton scattering amplitude. The bulbs in the r.h.s. denote the one-particle irreducible graphs.}
\end{align*}
\]
should disappear in this case as well. We have not found yet the way how it actually happens.

The scalar fields

The situation with the scalar fields is similar. Restricting oneself again to the diagrams with
two scalar legs one has the following invariants in the one-loop order:

\[
S_1 = (D_\mu D_\nu \Phi)^\dagger (D_\nu D_\nu \Phi), \\
S_2 = (D_\mu D_\nu \Phi)^\dagger (D_\mu D_\nu \Phi), \\
S_3 = (D_\mu D_\nu \Phi)^\dagger (D_\nu D_\nu \Phi).
\]

Expanding them up to the third order over the gauge fields one gets

\[
S_1 = \partial^2 \Phi^\dagger \partial^2 \Phi + \partial^2 \Phi^\dagger \partial_\sigma A_\nu \Phi + 2 \partial^2 \Phi^\dagger A_\nu \partial_\nu \Phi - \Phi^\dagger \partial_\mu A_\mu \partial^2 \Phi - 2 \partial_\mu \Phi^\dagger A_\mu \partial_\mu \Phi \\
\Rightarrow p^4 + (2 \hat{p}_{1}^2 \hat{p}_{3} - 2 \hat{p}_{2}^2 \hat{p}_{3}^2 - 2 \hat{p}_{1}^2 \hat{p}_{2}^2) \mu_{\nu}^{\mu} = p^4 + [p_{1}^2 (p_{1}^2 - p_{2}^2) - p_{2} (p_{1}^2 - p_{2}^2)] \mu_{\nu}^{\mu},
\]

\[
S_2 = \partial_\mu \Phi^\dagger \partial_\sigma \partial_\nu \Phi + \partial_\mu \partial_\nu \partial_\sigma \Phi + 2 \partial_\mu \partial_\nu \partial_\sigma \Phi - \Phi^\dagger \partial_\mu A_\mu \partial_\nu \partial_\sigma \Phi \\
\Rightarrow p^4 + [2 \hat{p}_{1} p_{2} \hat{p}_{3}^2 + p_{2} p_{3} \hat{p}_{1}^2 - 2 \hat{p}_{1} \hat{p}_{2} \hat{p}_{3} \hat{p}_{4}^2] \mu_{\nu}^{\mu} = p^4 + [p^{\mu}_{1} (p_{1}^2 - p_{2}^2) - p^{\mu}_{2} (p_{1}^2 - p_{2}^2)] \mu_{\nu}^{\mu},
\]

\[
S_3 = \partial_\mu \partial_\nu \Phi^\dagger \partial_\sigma \partial_\tau \Phi + \partial_\mu \partial_\nu \partial_\sigma \Phi + 2 \partial_\mu \partial_\nu \partial_\sigma \Phi - \Phi^\dagger \partial_\mu A_\mu \partial_\nu \partial_\sigma \Phi \\
\Rightarrow p^4 + [2 \hat{p}_{1} p_{2} \hat{p}_{3}^2 + p_{2} p_{3} \hat{p}_{1}^2 - 2 \hat{p}_{1} \hat{p}_{2} \hat{p}_{3} \hat{p}_{4}^2] \mu_{\nu}^{\mu} = p^4 + [p^{\mu}_{1} (p_{1}^2 - p_{2}^2) - p^{\mu}_{2} (p_{1}^2 - p_{2}^2)] \mu_{\nu}^{\mu},
\]
Thus, the one-loop divergences in the scalar sector have the form

\[ xS_1 + yS_2 + zS_3 = p^4(x + y + z) + A_\mu[p_1^\mu(p_1^2(x + y + z) - p_1p_2(y + z) + p_2^2(x)) - p_2^\mu(...)]. \]

At the same time, the calculation of the 2- and 3-point functions gives

\[
\begin{align*}
- p^4\frac{\alpha}{2}C_R \\
+ A_\mu \left\{ p_1^\mu \left[ \frac{C_A}{2} (p_1^2 \frac{3 - 4\alpha}{12} + p_1p_2 \frac{5 - 12\alpha}{6} - p_2^2 \frac{5 + 15\alpha}{6}) \right] \\
+ (C_R - \frac{C_A}{2}) (p_1^2 \frac{3 - \alpha}{6} + \frac{4}{3} p_1p_2 + p_2^2 \frac{8 - 3\alpha}{6}) - (2C_R - \frac{C_A}{2}) p_1^2 \frac{3 + 2\alpha}{12} - p_2^\mu(...). \right\}
\end{align*}
\]

Comparing the above two expressions one gets

\[ x + y + z = -\frac{\alpha}{2}C_R, \quad y + z = \left( \frac{1}{4} + \alpha \right) C_A - \frac{4}{3} C_R, \quad x = -\left( \frac{1}{4} + \alpha \right) C_A + \left( \frac{4}{3} - \frac{\alpha}{2} \right) C_R. \]

Thus, the one-loop divergences in the scalar sector have the form

\[
- \left[ \left( \frac{1}{4} + \alpha \right) C_A - \left( \frac{4}{3} - \frac{\alpha}{2} \right) C_R \right] (D_\mu D_\nu \Phi)^\dagger (D_\nu D_\nu \Phi) + y (D_\mu D_\nu \Phi)^\dagger (D_\mu D_\nu \Phi) \\
+ \left[ \left( \frac{1}{4} + \alpha \right) C_A - \frac{4}{3} C_R - y \right] (D_\mu D_\nu \Phi)^\dagger (D_\nu D_\mu \Phi). \tag{15}
\]

To find the value of \( y \), one has to calculate the 4-point function.

One can see that in the scalar sector, like in the spinor one, there is no simple cancellation of divergences. One has to consider again the proper combination of the Green functions. Since the usual product of the propagators and the 3-point vertices does not work here as well, we look for the physical amplitude.

Consider again the Compton-like amplitude but for the scalar fields and take the first invariant \( S_1 \) from eq.\,(14). Then, in full analogy with the fermion case, one has

\[
Amp = i \frac{\nu(p_1 + p_3 - p_2)\mu(2p_1 + p_3)^\nu}{(p_1 + p_3)^2} [p_1^2 + (p_1 + p_3)^2 + p_2^2] - i g^{\mu\nu} (p_1^2 + p_2^2) \\
- i \frac{(p_1 + p_3 - p_2)^\mu}{(p_1 + p_3)^2} [p_1^\mu(p_1^2 + (p_1 + p_3)^2) + (p_1 + p_3)^\nu((p_1 + p_3)^2 + p_2^2)] \\
- i [(p_1 + p_3)^\mu(p_2^2 + (p_1 + p_3)^2) - p_2^\mu(p_2^2 + (p_1 + p_3)^2)] \frac{(2p_1 + p_3)^\nu}{(p_1 + p_3)^2} \\
+ i [g^{\mu\nu}(p_1^2 + p_2^2) + 2p_1^\mu p_1^\nu + p_1^\mu p_1^\nu - 2p_2^\mu p_1^\nu - p_2^\mu p_3^\nu + 2p_3^\mu p_1^\nu + p_3^\mu p_3^\nu] = 0,
\]

i.e., this contribution drops from the amplitude. Again, like in the spinor case, this does not happen for the other invariants, though the gauge invariance arguments are still valid and one has to find out how the gauge dependence actually goes away.
2.3 The Yukawa sector

In the Yukawa sector in D=4 due to analyticity in superspace one has no renormalization of a superpotential. This means that all the vertex diagrams converge and divergences can only happen in chiral field propagators. This might also be true in D=6 in a superfield approach, but in components it looks more complicated. We have not studied this sector yet.

3 SUSY GUT Models in the Bulk

Taking eq.(5) seriously, one may wonder what kind of models in the bulk satisfy this requirement. It happens to be not so many possibilities bearing in mind that one should have at least three generations of the SM particles. Looking for the values of Dynkin indices one finds for instance that the SU(5) theory does not satisfy eq.(3), neither any other SU(N) or SO(2k+2), k > 2 theory does. There are only two viable models: SO(10) and E(6) with the following particle content:

| The Model | Gauge Group | Matter fields | Higgs fields |
|-----------|-------------|---------------|--------------|
| I         | SO(10)      | $4 \times 16$ | -            |
| II        | SO(10)      | $3 \times 16$ | $1 \times 16$ |
| III       | SO(10)      | $3 \times 16$ | $2 \times 10$ |
| IV        | E(6)        | $4 \times 27$ | -            |
| V         | E(6)        | $3 \times 27$ | $1 \times 27$ |

Table 2: Possible consistent N=1 D=6 models

Each SM particle in these models being projected to a 4-dimensional brane obtains the mirror partner in a conjugated representation $\bar{R}$, which has to be heavy enough not to be observed. This is the well-known problem in N=2 SUSY models in D=4. It can be solved in the brane world scenario by adjusting proper quantum numbers to all the particles with respect to an orbifold symmetry group. One can then remove some unwanted particles from the SM brane confining them to another brane, etc \cite{1, 2}. There may be complicated scenarios when some particles are in the bulk while the others are confined to the brane. We do not consider these questions here, but concentrate on the construction of a consistent QFT in extra dimensions.

4 Conclusion

We have demonstrated how UV divergences cancel each other in some cases even in non-renormalizable models. In Feynman gauge it is straightforward in the gauge sector but is rather tricky in the matter one.

The situation can be simplified when going on shell; however, since the equations of motion mix the matter fields due to the Yukawa type interactions, one should consider all the invariants

\footnote{The result depends of course on whether or not some particles propagate in the bulk or confined to the brane in the brane world scenario. Here for simplicity we assume that all the particles are in the bulk.}
together. This does not seem to be simple. On the other hand, as has already been mentioned, we would like to stay off shell to be able to go beyond one loop.

Unfortunately, we have not completed this task. We rely here on the superfield formalism which should simplify the situation drastically. On the other hand, the K-K approach can also be useful if cancellation of infinite towers level by level is really possible.

If the N=1 D=6 SUSY theory inherits some properties of N=2 D=4 one (see e.g. [12]), where the UV divergences occur only in one loop, one can hope that these results are valid in higher orders as well.

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