Point Configurations and Cayley-Menger Varieties

Ciprian S. Borcea

Abstract

Equivalence classes of \( n \)-point configurations in Euclidean, Hermitian, and quaternionic spaces are related, respectively, to classical determinantal varieties of symmetric, general, and skew-symmetric bilinear forms. Cayley-Menger varieties arise in the Euclidean case, and have relevance for mechanical linkages, polygon spaces and rigidity theory. Applications include upper bounds for realizations of planar Laman graphs with prescribed edge-lengths and examples of special Lagrangians in Calabi-Yau manifolds.

Introduction. We are concerned, initially, with configurations of \( n \) labeled (or ordered) points in the Euclidean space \( \mathbb{R}^d \), up to equivalence under congruence and similarity (rescaling). We require at least two points to be distinct and denote by \( C_n(\mathbb{R}^d) \) the resulting configuration space, made of such equivalence classes.

Cayley-Menger varieties appear when point configurations are looked upon as encoded in the information given by the squared distances between any pair of points (up to proportionality).

Cayley expressed the necessary relations between these squared distances as the vanishing of certain determinants, and Menger found sufficient conditions for a set of solutions to actually represent the mutual squared distances of a point configuration. These conditions amount to sign requirements on determinants of the same kind [3] [4].

Here, we look at the matter not so much in terms of distance geometry, as envisaged e.g. in [3] [10], but rather in terms of algebraic geometry. In fact, our medium will be mostly that of complex algebraic-geometry, since we are also interested in certain complexifications of configuration spaces (cf. [5]).

Thus, we define the Cayley-Menger variety \( CM^{d,n}(C) \) as the Zariski-closure of \( \text{im} C_n(\mathbb{R}^d) \subset P_{n-1}(\mathbb{C}) \).
This approach will be useful with respect to the geometry of the real points as well, and we wish to emphasize that the real slices of our varieties (or rather the “realistic” parts thereof, which correspond to configurations) are objects of interest in other areas: rigidity theory, robot arm motion planning, mechanical linkages, or molecular conformations.

We explore first the planar case: \( d = 2 \), which is privileged because of the identification \( \mathbb{R}^2 = \mathbb{C} \), and relates (via linear sections of codimension \( n-1 \)) to planar polygon spaces and their Calabi-Yau complexifications, as studied in [5]. Then we generalize to arbitrary \( d \leq n-1 \). Note that \( n \) points in some affine space may span a subspace of dimension at most \( (n-1) \).

Indeed, for a better perspective on configuration spaces, the associated Cayley-Menger varieties \( CM^{d,n}(C) \) should be considered as a full family:

\[
CM^{1,n}(C) \subset CM^{2,n}(C) \subset \ldots \subset CM^{n-1,n}(C) = P_{(\frac{n}{2})-1}(C)
\]
as resulting from natural inclusions: \( R \subset R^2 \subset \ldots \subset R^{n-1} \).

The key fact is that the above series of inclusions can be identified with the natural stratification by rank of the projective space \( P_{(\frac{n}{2})-1}(C) = P(Sym^2(C^{n-1})) \) of symmetric \((n-1) \times (n-1)\) complex matrices. Under this identification, the configuration space of \( n \) points in \( R^d \) corresponds with real symmetric matrices of rank at most \( d \) and with non-negative eigenvalues (up to scalars).

In a certain sense, the most important configuration space is that of \( n \) points on a line \( (d = 1) \), since it leads to the quadratic Veronese embedding of \( P_{n-2} \), which rules the geometry of our varieties for all higher dimensions \( d \).

In fact, there’s a larger picture which encompasses the one above as its \( \mathbb{Z}_2 \)-invariant part.

If we replace \( R^d \) and its inner product with \( C^d \) and its Hermitian inner product, we may consider configuration spaces \( C_n(C^d) \) of \( n \) points in \( C^d \) (with at least two points distinct), up to equivalence under translations, unitary transformations, and rescaling.

All considerations and constructions related to \( C_n(R^d) \) have their counterpart for \( C_n(C^d) \), and one obtains what might be called (if a name be needed) Hermite-Gram varieties:

\[
HG^{1,n}(C) \subset HG^{2,n}(C) \subset \ldots \subset HG^{n-1,n}(C) = P_{(n-1)^2-1}(C)
\]

In any event, they are readily identified with the stratification of the projective space of \((n-1) \times (n-1)\) complex matrices by rank.

The real structure to be considered on \( HG^{d,n}(C) \) is the one corresponding to the anti-holomorphic involution \( M \mapsto M^* \) on matrices, where \( M^* \) is the adjoint (i.e. conjugate transpose) of \( M \).
$HG^{1,n}$ is the image of a Segre embedding $(P_{n-2})^2 \to P_{(n-1)^2-1}$, and it rules the geometry for higher dimensions $d$, much in the same way the quadratic Veronese image $CM^{1,n}$ does with respect to $CM^{*,n}$.

Clearly, the former scenario is the $Z_2$-invariant part of the latter one (for the $Z_2$-action given by transposition on matrices).

At this point, it becomes mandatory (cf. [1] [2]) to investigate the quaternionic (or hyper-Hermitian) case. It leads to the stratification of skew-symmetric (or Pfaffian) forms in $2(n-1)$ variables by rank $2d$, $d = 1, \ldots, n-1$.

From the point of view of the self-adjoint matrices involved, the triadic series corresponding to $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$ have an octonionic analogue for $d = 1, 2, 3$ and $n \leq 4$. This goes no further because of Desargues’ theorem, which requires associativity from (projective) dimension three on.

For specific applications, we return to the Euclidean planar case, or, with orientation, the Hermitian $d = 1$ case, and interpret the degree of the corresponding Cayley-Menger variety as an upper bound for the number of realizations of generically rigid graphs with $2n - 3$ edges of given length.

We also complement the approach of [3] on polygon spaces and special Lagrangians in Calabi-Yau manifolds.

The last section regroups the main aspects in a tetradic summary.

Our presentation is ordered as follows:

1. Planar configurations
2. Cayley-Menger varieties $CM^{2,n} = CM_{2n-4}$
3. A realization with symmetric forms
4. Cayley coordinates and Gram coordinates
5. The cone of positive semi-definite symmetric forms
6. A Hermitian analogue
7. A quaternionic analogue
8. An octonionic enclave
9. Mechanical linkages and linear sections
10. Polygon spaces and Calabi-Yau manifolds
11. Summary ($\text{Tεραλογια}$)
1 Planar configurations

We start with the collection \( F_n(\mathbb{R}^2) \) of all possible choices of \( n \) distinct points \((p_i)_{1 \leq i \leq n} \) in \( \mathbb{R}^2 \). The space \( F_n(\mathbb{R}^2) \) is naturally identified with the complement of the “thick diagonal” in \((\mathbb{R}^2)^n\):

\[
F_n(\mathbb{R}^2) = (\mathbb{R}^2)^n - \bigcup_{i \neq j} \{ p = (p_i)_{1 \leq i \leq n} \mid p_i = p_j \}
\]

For various purposes, this space can be variously compactified. Here, because we want to retain the metric aspect of a configuration and consider two of them as equivalent if one can be turned into the other by a Euclidean displacement (i.e. isometry) and rescaling (i.e. similarity), we will have to consider the orbit space of \( F_n(\mathbb{R}^2) \) under the (diagonal action) of the group consisting of all these Euclidean transformations in the plane.

Equivalence under translations can be eliminated by choosing the origin to be one particular point (say \( p_1 = 0 \)). This choice of representatives makes the natural(permutation) action of the symmetric group \( S_n \) on \( n \)-point configurations less manifest, but has other advantages, and we shall see this contrast reflected again between Cayley coordinates and Gram coordinates (cf. section 5). The choice converts \( n \) points into \((n-1)\) vectors: \( p_j - p_1, \ j = 2, \ldots, n \).

Equivalence under similarities is now simply expressed by passing to the projective space \( P((\mathbb{R}^2)^{n-1}) = P(R^{2n-2}) = P_{2n-3}(\mathbb{R}), \) and we need to exclude only the most degenerate case, namely when all points coincide.

Thus, the configuration (or moduli) space we are going to investigate is the orbit space:

\[
C_n(\mathbb{R}^2) = P((\mathbb{R}^2)^{n-1})/O(2, \mathbb{R}) = P_{2n-3}(\mathbb{R})/O(2, \mathbb{R})
\]

where \( O(2, \mathbb{R}) \) stands for the group of orthogonal transformations in \( \mathbb{R}^2 \), and the action is induced from the diagonal action on \((\mathbb{R}^2)^{n-1}\).

Obviously, \( O(2, \mathbb{R}) \) consists of rotations or rotations followed by reflection in a line through the origin; in other words: it has two connected components, each, topologically, a circle \( S^1 \), and the quotient \( C_n(\mathbb{R}^2) \) should be a space of dimension \( 2n - 4 \).

Using the identification \( \mathbb{R}^2 = C \), we can give a much more explicit description of this orbit space \( C_n(\mathbb{R}^2) \).

With the choice \( p_1 = 0 \), a configuration is described by \((p_2, \ldots, p_n)\), which we write as \((z_2, \ldots, z_n) \in C^{n-1} \) when we want to emphasize that we think of the points \( p_i = z_i \) as complex numbers.
Note now that multiplication with non-zero complex numbers in $C = \mathbb{R}^2$ means precisely a similarity followed by some rotation in $\mathbb{R}^2$, while conjugation in $C$, amounts to reflecting in the real (first) axis.

Thus the passage from $C^{n-1}$ to the complex projective space $P(C^{n-1}) = P_{n-2}(C)$ corresponds precisely to eliminating the configurations with all points coinciding and accounting for rescalings and rotations. All that remains to “factor out” is the equivalence under reflection in a line, that is under conjugation. This establishes:

**Proposition 1.1** The configuration space $C_n(\mathbb{R}^2)$ can be identified with the quotient of a complex projective space $P_{n-2}(C)$ by conjugation:

$$C_n(\mathbb{R}^2) = P_{n-2}(C)/\text{conj}$$

Note that the subscript in $P_{n-2}(C)$ indicates complex dimension $n - 2$, that is real dimension $2(n - 2)$. The fixed points of the conjugation in $P_{n-2}(C)$, make up precisely the real projective space $P_{n-2}(\mathbb{R}) \subset P_{n-2}(C)$, and clearly they represent the equivalence classes of configurations with all points collinear. Thus, the configuration space $C_n(\mathbb{R}^2)$ is a smooth manifold of real dimension $2(n - 2) = 2n - 4$ away from collinear configurations, which make up a “bad” locus parametrized by $P_{n-2}(\mathbb{R})$.

How “bad” this locus is depends on $n$, since locally along it the topology is the product of (germs at zero): $R^{n-2} \times (R^{n-2}/\mathbb{Z}_2)$, with $R^{n-2}/\mathbb{Z}_2$ denoting the quotient of $R^{n-2}$ under $x \mapsto -x$.

In particular, for $n = 4$, the quotient space remains non-singular, since $R^2/\mathbb{Z}_2 = C/\mathbb{Z}_2 \approx R^2 = C$ (by $z \mapsto z^2$).

**Example 1.1:** For $n = 3$ the space of all triangles in $\mathbb{R}^2$, up to isometry and similarity is a closed hemisphere, the boundary corresponding to degenerated triangles (with vertices alligned, but not all three confounded).

**Example 1.2:** For $n = 4$, the configuration space of quadrilaterals $C_4(\mathbb{R}^2)$ will be, as remarked above, a non-singular fourfold. Actually,

$$C_4(\mathbb{R}^2) = P_2(C)/\text{conj} \approx S^4$$

but the isomorphism with the four dimensional sphere is not that obvious, although the Betti numbers are clearly those of a sphere [19] [22] [2]. We outline an argument in section 5.

A simple way to remain in the smooth realm is to use $P_{n-2}(C)$ for computations and rephrase the proposition as follows:

*Identification of conjugate points gives a double covering:*

$$P_{n-2}(C) \rightarrow C_n(\mathbb{R}^2)$$

ramified over $P_{n-2}(\mathbb{R}) \subset P_{n-2}(C)$
2 Cayley-Menger varieties \( CM^{2,n} = CM_{2n-4} \)

The fact which leads to what we will call Cayley-Menger varieties is that if we retain (up to proportionality) only the mutual (squared) distances between the points \( p_i \) we obtain the same \( \binom{n}{2} \) projective coordinates for equivalent configurations. (We use squared distances in order to have (quadratic) polynomial expressions and remain in the realm of algebraic geometry.) In other words, we have a natural map:

\[
C_n(R^2) \to P_{\binom{n}{2}-1}(\mathbb{R}) \quad (p_i) \to (s_{ij} = d_{ij}^2 = |p_i - p_j|^2)
\]

At this point we allow complex coordinates in the target space, and envisage \( P_{\binom{n}{2}-1}(\mathbb{R}) \) simply as the real points of the complex projective space \( P_{\binom{n}{2}-1}(\mathbb{C}) \). Thus, the image of the configuration space will be considered as a subset of \( P_{\binom{n}{2}-1}(\mathbb{C}) \), and we enter into complex algebraic geometry with:

**Definition 2.1:** The complex Cayley-Menger variety \( CM^{2,n}(\mathbb{C}) = CM_{2n-4}(\mathbb{C}) \) is defined as the Zariski-closure of the image of the configuration space \( C_n(R^2) \) in \( P_{\binom{n}{2}-1}(\mathbb{C}) \).

Recall that, for a given subset in a projective space, the Zariski-closure means the vanishing locus of all homogeneous polynomials which vanish on the given subset.

**Definition 2.2:** The real Cayley-Menger variety \( CM_{2n-4}(\mathbb{R}) \) is the intersection of \( CM_{2n-4}(\mathbb{C}) \) with the real locus \( P_{\binom{n}{2}-1}(\mathbb{R}) \), in other words: the fixed locus of conjugation on \( CM_{2n-4}(\mathbb{C}) \).

Obviously \( \text{im}(C_n(R^2)) \) is a part of \( CM_{2n-4}(\mathbb{R}) \), but only a part: there are points in \( CM_{2n-4}(\mathbb{R}) \) which do not correspond with mutual squared distances of a configuration (see Corollary 2.6 and section 5 below ). Menger’s inequalities, expressing sign conditions for various Cayley-Menger determinants, are one way of distinguishing the “realistic part” \( \text{im}(C_n(R^2)) \) (which is semi-algebraic) from the full algebraic real part \( CM_{2n-4}(\mathbb{R}) \). In section 5 we’ll see the general distinction expressed as that between positive semi-definite symmetric forms and real symmetric forms of possibly other signatures (and rank at most \( d \))- up to sign.

**Example 2.1:** For \( n = 3 \), \( CM_2(C) = P_2(C) \), and \( CM_2(R) = P_2(R) \) strictly contains the closed disc \( \text{im}(C_3(R^2)) \).

We do now a rank computation which gives a better idea of the image involved in our definition. We use the double covering described in section 1:

\[
P_{n-2}(C) \to C_n(R^2)
\]
which continues with the map:

\[ C_n(R^2) \rightarrow P_{(2)}^{n-1} \]

With complex (homogeneous) coordinates \((z_2 : \ldots : z_n)\) for \(P_{n-2}(C)\), the composition:

\[ P_{n-2}(C) \rightarrow P_{(2)}^{n-1} \]

reads:

\[(z_2 : \ldots : z_n) \mapsto (z_2 \bar{z}_2 : \ldots : z_n \bar{z}_n : \ldots : (z_i - z_j)(\bar{z}_i - \bar{z}_j) : \ldots)\]

**Proposition 2.1** The differential (i.e. tangent map) of \(P_{n-2}(C) \rightarrow P_{(2)}^{n-1}\) has rank \(2(n-2)\) on \(P_{n-2}(C) - P_{n-2}(R)\), and rank \((n-2)\) on \(P_{n-2}(R)\).

**Proof:** We consider the map at the affine level: \(C^{n-1} \rightarrow R_{(2)}^{n}\). Using partial derivatives \(\partial/\partial z_i, \partial/\partial \bar{z}_i\) we obtain a matrix of the form:

\[
\begin{pmatrix}
\bar{z}_2 & 0 & \ldots & 0 & \bar{z}_2 - \bar{z}_3 & \ldots & \bar{z}_2 - \bar{z}_n \\
z_2 & 0 & \ldots & 0 & z_2 - z_3 & \ldots & z_2 - z_n \\
0 & * & **
\end{pmatrix}
\]

where * stands for the corresponding matrix for variables \(z_3, \ldots, z_n\). If we add the first column to each of the last \((n-2)\) columns, and then change their sign, and move the first in front of them, we obtain:

\[
\begin{pmatrix}
0 & 0 & \bar{z}_2 & \bar{z}_3 & \ldots & \bar{z}_n \\
0 & 0 & z_2 & z_3 & \ldots & z_n \\
* & ***
\end{pmatrix}
\]

which makes plain that the rank of * increases by two, unless all \(z_i\)'s are collinear, when it increases by one (excepting the origin). The statement follows by induction. \(\square\)

We’ll see now that virtually the same computation allows us to give a description of \(CM_{2n-4}(C)\), even before we say anything about defining equations.

Indeed, the above context “complexifies” naturally as follows:

- we use coordinates \((u, v)\) on \(P_{n-2}(C) \times P_{n-2}(C)\), with \(u = (u_2 : \ldots : u_n)\), \(v + (v_2 : \ldots : v_n)\) labeled in agreement with our labeling in Proposition 2.1, and consider the map:

\[ P_{n-2}(C) \times P_{n-2}(C) \rightarrow P_{(2)}^{n-1}(C) \]
\[(u, v) \mapsto (u_2v_2 : \ldots : u_nv_n : \ldots : (u_i - u_j)(v_i - v_j) : \ldots)\]
i.e. \(u\) takes the role of \(z\) and \(v\) that of \(\bar{z}\), but with \(u\) and \(v\) independent complex homogeneous coordinates, and with a complex target, the map is regular (holomorphic). It is straightforward to see it’s everywhere defined.

**Proposition 2.2** The (holomorphic) differential of the above morphism of complex projective manifolds:

\[P_{n-2}(C) \times P_{n-2}(C) \rightarrow P_{(z)}^{-1}(C)\]

has rank \(2(n - 2)\) away from the diagonal in \((P_{n-2}(C))^2\), and rank \((n - 2)\) on this diagonal, i.e. on

\[\text{im}[ P_{n-2}(C) \rightarrow (P_{n-2}(C))^2 \mid w \mapsto (w, w) ]\]

**Proof:** As above, with \(\partial/\partial u_i, \partial/\partial v_i\) instead of \(\partial/\partial z_i, \partial/\partial \bar{z}_i\). \(\square\)

**Proposition 2.3** The complex Cayley-Menger variety \(CM_{2n-4}(C)\) is the image of

\[P_{n-2}(C) \times P_{n-2}(C) \rightarrow P_{(z)}^{-1}(C)\]

The map:

\[(P_{n-2}(C))^2 \rightarrow CM_{2n-4}(C) = (P_{n-2})^2/Z_2\]

is a double covering branched along the diagonal \(P_{n-2}(C) \subset (P_{n-2}(C))^2\).

**Proof:** \((P_{n-2}(C))^2\) contains the double covering \(P_{n-2}(C)\) of the configuration space \(C_n(R^2)\) via the embedding:

\[P_{n-2}(C) \rightarrow (P_{n-2}(C))^2, \ z \mapsto (z, \bar{z})\]

that is, as the fixed ocus of the anti-holomorphic involution of \((P_{n-2}(C))^2 : (u, v) \mapsto (\bar{v}, \bar{u})\).

The map to \(P_{(z)}^{-1}(C)\) clearly takes \((u, v)\) and \((v, u)\) to the same image, and thus the image of \((P_{n-2}(C))^2\) contains the image of \(C_n(R^2)\) used in the definition of the Cayley-Menger variety.

Referring now to an elementary fact about uniqueness of holomorphic extensions of real power series, we see that the vanishing of a polynomial on \(imC_n(R^2)\) implies its vanishing on \(im(P_{n-2}(C))^2\), because of vanishing (after composition) on the real points of \((P_{n-2}(C))^2\) under the anti-holomorphic involution \((u, v) \mapsto (\bar{v}, \bar{u})\).
The fact that \( \text{im}(P_{n-2}(C))^2 \) is irreducible concludes the identification:

\[
\text{im}(P_{n-2}(C))^2 = CM_{2n-4}(C)
\]

The fact that this image is precisely the \( \mathbb{Z}_2 \) quotient of \( (P_{n-2}(C))^2 \) under the involution \( (u, v) \mapsto (v, u) \) follows from the rank computation which gives a generic immersion, and the \( n = 3 \) case:

\[
(P_1(C))^2 \rightarrow (P_1(C))^2/\mathbb{Z}_2 = P_2(C) = CM_2(C)
\]

which says that, given:

\[
(s_{12} = u_2v_2 : s_{23} = (u_2 - u_3)(v_2 - v_3) : s_{31} = u_3v_3) \in P_2(C)
\]

there are (counting multiplicities) two solutions in \( (P_1(C))^2 \) (the intersection of two \( \mathcal{O}(1,1) \) divisors), obviously symmetric under the \( \mathbb{Z}_2 \) action. \( \square \)

**Corollary 2.4** The complex Cayley-Menger variety \( CM_{2n-4}(C) \subset P_{n-1}(\mathbb{C}) \) is an irreducible projective subvariety of complex dimension \( 2n - 4 \) and degree

\[
D^{2,n} = \deg(CM_{2n-4}(C)) = \frac{1}{2} \binom{2n-4}{n-2}
\]

It is swept-out by an \( (n-2) \)-parameter family of linear subspaces \( P_{n-2}(C) \) corresponding to the two \( P_{n-2}(C) \) fibrations of \( (P_{n-2}(C))^2 \) (identified under the \( \mathbb{Z}_2 \) action).

**Proof:** The degree formula comes from a volume computation on \( (P_{n-2}(C))^2 \). Let \( h_1 \) and \( h_2 \) denote the hyperplane classes on the two factors. The hyperplane class on \( P_{n-1}(\mathbb{C}) \) pulls back to \( (h_1 + h_2) \), and:

\[
D^{2,n} = \deg(CM_{2n-4}(C)) = \frac{1}{2} \int_{(P_{n-2}(C))^2} (h_1 + h_2)^{2(n-2)} = \frac{1}{2} \binom{2n-4}{n-2}
\]

A similar computation yields the sectional genus:

**Corollary 2.5** For \( n \geq 4 \), a generic linear section of codimension \( 2n - 5 \) cuts \( CM_{2n-4}(C) \) along a smooth curve of genus:

\[
g^{2,n} = 1 + \frac{n-4}{2} D^{2,n} = 1 + \frac{n-4}{4} \binom{2n-4}{n-2}
\]
Corollary 2.6 The real Cayley-Menger variety $CM_{2n-4}(R) \subset P_{n-1}^{(2)}$ is the union of the two $Z_2$-quotients: $P_{n-2}(C)/\text{conj} = imC_n(R^2)$ and $(P_{n-2}(R))^2/Z_2$, glued along their common $P_{n-2}(R)$ ramification locus:

$$CM_{2n-4}(R) = P_{n-2}(C)/Z_2 \bigcup_{P_{n-2}(R)} (P_{n-2}(R))^2/Z_2$$

Thus, the “threshold” between the “realistic” i.e. configuration part $P_{n-2}(C)/\text{conj} = imC_n(R^2)$, and the “fake” remnant $(P_{n-2}(R))^2/Z_2$, is made of (the image of) collinear configurations. We’ll see in the sequel that these collinear configurations are geometrically paramount for arbitrary dimension $d \leq n - 1$.

Example 2.1. revisited: For $n = 3$, we saw that $P_1(C)/\text{conj} = imC_3(R^2)$ is a closed hemisphere or, equivalently, a closed disc. On the other hand, $(P_2(R))^2/Z_2$ is a Möbius band. Gluing the two along their boundary $S^1 = P_1(R)$ yields $CM_2(R) = P_2(R)$.

Example 1.2. revisited: For $n = 4$, we anticipated the result that $P_3(C)/\text{conj} = imC_4(R^2)$ is the 4-sphere $S^4$. It is somewhat simpler to obtain: $(P_2(R))^2/Z_2 \approx P_4(R)$ [22]. Thus:

$$CM_4(R) = S^4 \bigcup_{P_2(R)} P_4(R)$$

is made topologically of the realistic part $S^4$, and the closed “fake” part $P_4(R)$, glued along their common $P_2(R)$. See end of section 5 for details.

3 A realization with symmetric forms

If we observe that the map with image $CM^{2,n}(C) = CM_{2n-4}(C)$:

$$(P_{n-2}(C))^2 \rightarrow P_{n-1}^{(2)}$$

is given by the linear system of all symmetric divisors in

$$|\mathcal{O}(1,1)| = H^0((P_{n-2}(C))^2, \mathcal{O}(1,1))$$

we are led to the following intrinsic realization.

Let $V$ be a vector space (of dimension $n - 1$), and $P(V)$ the associated projective space (of dimension $n - 2$). For our purposes $V$ should be a complex vector space defined over the real field i.e. $V = V_C = V_R \otimes_R C$, with $V_R$ a real vector space. Essentially: $V_R = R^{n-1}$ and $V = V_C = C^{n-1}$. Consider:

$$(P(V))^2 \rightarrow P(V \otimes V) \cdots \rightarrow P(\text{Sym}(V \otimes V))$$
where the first map is the Segre embedding, given by \((u, v) \mapsto u \otimes v\), and the second rational map corresponds to the linear projection on symmetric tensors along skew-symmetric tensors:

\[
V \otimes V = \text{Sym}(V \otimes V) \oplus \bigwedge^2(V) \rightarrow \text{Symm}(V \otimes V) = (V \otimes V)/(V \wedge V)
\]

\[u \otimes v \mapsto \frac{1}{2}(u \otimes v + v \otimes u)\]

Note, in particular, that the restriction of the Segre map \((P(V))^2 \rightarrow P(V \otimes V)\) to the diagonal \(P(V) \subset (P(V))^2\) gives the quadratic Veronese embedding

\[
\nu_2 : P(V) \rightarrow P(\text{Sym}(V \otimes V))
\]

\[w \mapsto \nu_2(w) = w \otimes w\]

As announced earlier, \(\nu_2(P(V)) \subset P(\text{Sym}(V \otimes V))\), that is: the image of this quadratic Veronese embedding of \(P(V)\) for \(V\) of dimension \(n - 1\), will be the key object (with its real, respectively complex points) in our description of configuration spaces and Cayley-Menger varieties:

\[\text{im}C_n(R^d) \subset CM^{d,n}(R) \subset CM^{d,n}(C)\]

for arbitrary \(d \leq n - 1\).

For the moment, we make explicit the situation pertaining to \(d = 1, 2\).

Considering the identification \(V \otimes V = \text{Hom}(V^*, V)\), we may speak of symmetric: \(M = tM\), and skew-symmetric transformations: \(M = -tM\). With \(V = C^{n-1}\) identified with \(V^* = C^{n-1}\) via the standard bilinear form, we have actually symmetric and skew-symmetric matrices with complex entries.

Thus, we let \(V \otimes V = \text{Hom}(C^{n-1}, C^{n-1}) = \mathcal{M}(n - 1)\), and \(\text{Sym}(V \otimes V) = \text{Sym}(V \otimes V) = \text{Sym}^2(n - 1)\) be our abbreviations for the space of all \((n-1) \times (n-1)\) matrices with complex entries, respectively all symmetric matrices. With these standard coordinates, the above maps read:

\[(P_{n-2}(C))^2 \rightarrow P(\mathcal{M}(n - 1)) = P_{n(n-2)}(C) \cdots \rightarrow P(\text{Sym}^2(n - 1))\]

with the Segre map corresponding to \((u, v) \mapsto u \cdot t v\) : the multiplication of the column vector \(u\) with the row vector \(t v\), and the composition of the two maps becoming:
Clearly, the first map covers the locus of matrices of rank $\leq 1$, which meets the skew-symmetric locus only in zero, and thus projectively maps further onto the locus of symmetric matrices of rank $\leq 2$. This gives our realization:

**Proposition 3.1** The complex Cayley-Menger varieties:

$$CM^{1,n}(C) \subset CM^{2,n}(C) = CM_{2n-4}(C) \subset P_{(2)}^{n-1}(C)$$

can be identified by a linear change of coordinates in $P_{(2)}^{n-1}(C)$ with the loci:

$$R^1(n-1) \subset R^2(n-1) \subset P(Sym^2(n-1))$$

defined by symmetric $(n-1) \times (n-1)$ complex matrices of rank one, respectively at most two.

The change of coordinates is defined over $R$, hence the corresponding statement for real points holds true.

It is fairly transparent now, what the general result ($d \leq n-1$) should be. Thus, Cayley-Menger varieties are recognized as familiar objects of algebraic geometry: cf. [15] [16] [17] [18] [12] [13]. In particular, we have:

**Corollary 3.2** The projective dual of the Cayley-Menger variety $CM^{1,n}(C)$ can be identified with the determinantal hypersurface given by all singular symmetric $(n-1) \times (n-1)$ matrices:

$$CM^{1,n}(C)^* \approx P\{A = ^tA \mid detA = 0\} \subset P(Sym^2(n-1)) = P_{(2)}^{n-1}(C)$$

**Corollary 3.3** The Cayley-Menger variety $CM^{2,n}(C)$ is the secant (or chordal) variety of $CM^{1,n}(C)$.

**Proof:** This is immediate from the correspondence with symmetric matrices (Prop. 3.1). A line through two points of $R^1(n-1)$ consists of linear combinations of two symmetric matrices of rank one, which combinations have rank at most two, and therefore lie in $R^2(n-1)$. Thus, our secant variety $S(R^1(n-1)$ is contained in $R^2(n-1)$.

The fact that we get all of $R^2(n-1)$ is clear for real symmetric matrices (by orthogonal diagonalization) and follows over $C$ by permanence of algebraic (analytic) identities.  $\square$
As might be expected, this result generalizes to $CM^{d,n}(C)$, which is the variety of secant $(d - 1)$ planes to $CM^{1,n}(C)$. This brings forth the central role of $CM^{1,n}$ i.e. of the quadratic Veronese embedding (of $P_{n-2}$).

Our secant variety $S(CM^{1,n}(C)) = CM^{2,n}(C) = CM_{2n-4}(C)$ is clearly defective, in the sense that it only doubles the dimension of $CM^{1,n}(C)$ (without the usual +1), and we have therefore:

**Corollary 3.4** Let $Tan(CM^{1,n}(C))$ denote the variety of projective tangent spaces to $CM^{1,n}(C) \subset P_{(n)_-1}(C)$. Then:

$$CM^{2,n}(C) = S(CM^{1,n}(C)) = Tan(CM^{1,n}(C)) \subset P_{(n)_-1}(C)$$

### 4 Cayley coordinates and Gram coordinates

The change of coordinates which gives the realization of the Cayley-Menger varieties by symmetric matrices (cf. Prop.3.1) is a simple passage from Cayley coordinates to Gram coordinates.

We adopt here the general setting for $n$-point configurations $p_1, ..., p_n \in R^d$.

The **Cayley coordinates** are our familiar homogeneous coordinates $s_{ij}$, $1 \leq i < j \leq n$, for $P_{(n)_-1}(C)$, and for an $n$-point configuration in $R^d$ we have:

$$s_{ij} = s_{ij}(p) = \left| p_i - p_j \right|^2 = < p_i - p_j, p_i - p_j >$$

with $< , >$ denoting the usual inner product in $R^d$.

In order to relate Gram coordinates to configurations, we have to choose one of the points as origin, so that with the choice $p_1 = 0$ for example, the Gram coordinates $a_{ij}$, $2 \leq i \leq j \leq n$ corresponding to a configuration would be:

$$a_{ij} = a_{ij}(p) = < p_i - p_1, p_j - p_1 > = < p_i, p_j >$$

It is important to observe that permuting the $n$ points of a configuration amounts, in Cayley coordinates, to a corresponding permutation, but in Gram coordinates, if the permutation affects the chosen origin, we do not have a permutation of these coordinates any more.

With this caveat, and our choice decided for $p_1 = 0$, we proceed to relating the two sets of coordinates for configurations in $C_n(R^d)$. This is simply the cosine theorem:

$$a_{ij} = \frac{1}{2} (s_{ii} + s_{jj} - s_{ij}), \ 2 \leq i \leq j \leq n \quad (C-G)$$

where $s_{ij} = 0$ for $i = j$, that is: $a_{ii} = s_{ii}$. 

13
Normally, we look at the Gram coordinates as arranged in a symmetric \((n - 1) \times (n - 1)\) matrix \(A\) with entries \(a_{ij} = a_{ji}\), while the Cayley coordinates are arranged in a (bordered) symmetric \((n + 1) \times (n + 1)\) matrix \(S\) with entry indices running from zero to \(n\), and:

\[
s_{kk} = 0, \ s_{0i} = s_{i0} = 1, \text{ and } s_{ij} = s_{ji} \text{ for } 1 \leq i < j \leq n
\]

**Lemma 4.1** Let a Cayley matrix \(S\) and a Gram matrix \(A\) be related by (C-G). Then:

\[
\text{rk}(S) = 2 + \text{rk}(A), \text{ and } \det(S) = (-1)^n 2^{n-1} \det(A)
\]

**Proof:** Subtract column \((\ast, 1)\) in \(S\) from columns \((\ast, j)\), \(j = 2, \ldots, n\), then subtract row \((1, \ast)\) from rows \((i, \ast)\), \(i = 2, \ldots, n\), to obtain \(-2A\) in the lower right corner. The lemma becomes obvious on this form. \(\square\)

**Lemma 4.2** Let \(p_1 = 0, p_2, \ldots, p_n \in \mathbb{R}^d\) represent a point in \(C_n(\mathbb{R}^d)\). Let \(P\) denote the \(d \times (n - 1)\) matrix with columns \(p_2, \ldots, p_n\).

Then, the Gram matrix \(A(p)\) of the configuration is given by \(A(p) = \cdot P \cdot P\), and \(A(p)\) is consequently positive semi-definite of rank at most \(d\).

Conversely, if \(A\) is a (non-zero) real symmetric \((n - 1) \times (n - 1)\) matrix which is positive semi-definite of rank at most \(d\), there is a configuration \(p_1 = 0, p_2, \ldots, p_n \in \mathbb{R}^d\) (with at least two distinct points), such that \(A = A(p)\).

**Proof:** The first part amounts to observing that:

\[
a_{ij}(p) = \langle p_i; p_j \rangle = \cdot p_j \cdot p_i
\]

Clearly \(\cdot P \cdot P\) is positive semi-definite of rank at most \(\text{rk}(P) \leq d\).

For the converse, we use an orthogonal diagonalization of \(G\): for some orthogonal matrix \(T = \cdot T^{-1} \in O(n - 1, \mathbb{R})\) we obtain a diagonal matrix \(D = T \cdot A \cdot T^{-1}\) with at most \(d\) non-zero eigenvalues which are positive, and we may suppose the eigenvalues listed in decreasing order along the diagonal.

\(D\) has a “square root” \(D^{1/2}\), with: \(D = D^{1/2} \cdot D^{1/2}\) (and which commutes with all matrices commuting with \(D\)). \(D^{1/2}\) is diagonal, with positive square roots for the corresponding positive eigenvalues in \(D\) as the only non-zero eigenvalues.

Then:

\[
A = T^{-1}DT = \cdot T \cdot D^{1/2} \cdot D^{1/2}T
\]

We may retain the first \(d\) rows in \(D^{1/2}T\) (the remaining rows being obviously zero), and call this \(d \times (n - 1)\) matrix \(P\). Then \(A = \cdot P \cdot P\) and \(p_1 = 0\), together with the columns of \(P\) give the required configuration. \(\square\)
Remark: This lemma shows how to retrieve a representative for the equivalence class of a configuration in $C_n(R^d)$ when given the mutual squared distances between its points i.e. the Cayley matrix $S$: one produces the Gram matrix $A$ by $(C-G)$ and finds $P$ as above. This clearly applies to molecular conformations, in which context it appears as: the EMBED algorithm [10] (6.3, pg. 303)

We can establish now our anticipated correspondence in full generality.

Recall that the Cayley-Menger variety $CM^{d,n}(C)$ is defined as the Zariski-closure of the image of the configuration space $C_n(R^d)$ in the complex projective space $P_{(d-1)}$ by the map $p \mapsto S(p)$, given by all squared distances between the points.

**Theorem 4.3** The linear transformation $A : P_{(d-1)} ightarrow P_{(d-1)}(C)$ defined by passage from Cayley coordinates $S$ to Gram coordinates $A(S)$, that is:

$$S \mapsto A(S), \quad a_{ij}(S) = \frac{1}{2}(s_{ii} + s_{jj} - s_{ij}), \quad 2 \leq i \leq j \leq n \quad (C-G)$$

identifies the family of Cayley-Menger varieties:

$$CM^{1,n}(C) \subset CM^{2,n}(C) \subset \cdots CM^{n-1,n}(C) = P_{(d)}(C)$$

with the determinantal varieties in $P_{(d-1)}(C) = P(Sym^2(C^{n-1}))$ given by symmetric $(n-1) \times (n-1)$ complex matrices of rank at most $d$, $d = 1,..., n-1$, that is:

$$R^1(n-1) \subset R^2(n-1) \subset \cdots R^{n-1}(n-1) = P_{(d)}(C)$$

with $A(CM^{d,n}(C)) = R^d(n-1)$.

Consequently, $CM^{d,n}(C)$ is the variety $S_{d-1}(CM^{1,n}(C))$ of secant $(d-1)$-planes to $CM^{1,n}(C)$.

The (complex) dimension of $CM^{d,n}(C)$ is: $dn - \binom{d+1}{2} - 1$, and its degree (for $d \leq n-2$) is given by the formula:

$$D^{d,n} = deg(CM^{d,n}(C)) = \prod_{k=0}^{n-d-2} \frac{n-d-1-k}{(2k+1)}$$

Proof: We know that $A(imC_n(R^d)) \subset R^d(n-1)$ consists precisely of real symmetric $(n-1) \times (n-1)$ matrices of rank at most $d$ and with non-negative eigenvalues. We have to show that any homogeneous polynomial vanishing on such matrices necessarily vanishes on all symmetric matrices of rank at most $d$. 

15
It will be enough (by permanence of algebraic identities) to prove this for real symmetric matrices. By orthogonal diagonalization, every such matrix of rank at most $d$ is a linear combination of (diagonal) rank one positive semi-definite symmetric matrices (i.e. lies in the $(d - 1)$-plane spanned by $d$ points on $imC_n(R) \subset CM^{1,n}(C)$). The convex hull of these $d$ matrices clearly determines (in the projective picture) points in $imC_n(R^d)$, and a polynomial vanishing on the latter must vanish on the whole linear span. Thus $A(CM^{d,n}(C) = R^d(n - 1)$.

The argument above also proves that $CM^{d,n}(C) = S_{d-1}(CM^{1,n}(C)$, since clearly $S_{d-1}(R^1(n - 1)) \subset R^d(n - 1)$, and the image contains the real points, hence the equality.

The dimension formula will be apparent from the next proposition on resolving the singularities of $R^d(n - 1)$. It agrees, of course, with the “naive” count of real parameters for $C_n(R^d)$:

$$(n - 1)d - dim_RO(d, R) - 1 = dn - \binom{d + 1}{2} - 1$$

where (with $p_1 = 0$) we need $(n - 1)d$ parameters for $p_2, ..., p_n$, and we factor out orthogonal transformations and rescaling.

The general degree formula is more elaborate, and we refer to [17][18][12]. One may verify by induction that for $d = 2$ this yields our $D^2,n$ in Corollary 2.4. Henceforth, we freely substitute $R^d(n - 1)$ for $CM^{d,n}(C)$, or conversely.

**Corollary 4.4** The Cayley-Menger variety $CM^{d,n}(C)$ can be defined by the family of homogeneous polynomials of degree $d + 1$ expressing the vanishing of all $(d + 1)$-minors in the Gram matrix.

**Remark:** In view of Lemma 4.1., another possibility would be to choose as defining equations the vanishing of all $(d + 3)$-minors in the Cayley matrix. These would be homogeneous polynomials of degree $d + 1, d + 2$, and $d + 3$.

**Proposition 4.5** The singular locus of $CM^{d,n}(C)$ is $CM^{d-1,n}(C)$.

A resolution of singularities for $CM^{d,n}(C)$ can be presented as a $P^{d-1}_{d+1}$-bundle over the Grassmann manifold $G(n - d - 1, n - 1)$ of codimension $d$ subspaces in $C^{n-1}$.

In particular, all Cayley-Menger varieties are rational.

**Proof:** First, we remark that the Grassmann manifold $G = G(n - d - 1, n - 1)$ has dimension $d(n - d - 1)$, and the proposition yields the dimension formula in the theorem above.

We consider the incidence variety:
\[ T^d(n-1) = \{(\Lambda, A) \in G \times \mathcal{R}^d(n-1) \mid \Lambda \subset \text{Ker}A\} \]

which projects onto the Grassmannian with fibers:

\[ T^d_A(n-1) = P(Sym(\Lambda^\perp \otimes \Lambda^\perp)) \cong P((-1)^{d+1})^{-1}(C) \]

The projection on \( \mathcal{R}^d(n-1) \) gives the resolution of the latter, with the exceptional divisor projecting onto \( \mathcal{R}^{d-1}(n-1) \).

Corollary 3.2. generalizes to:

**Proposition 4.6** The projective dual of \( CM^{d,n}(C) \subset P(\frac{2n}{n-1})(C) \) can be naturally identified with \( CM^{n-d-1,n}(C) \).

In terms of symmetric matrices, that is: *quadrics*, one uses the pairing

\[ (A, B) \mapsto Tr(AB) \]

and the description of the projective tangent space at a non-singular point \( A \in \mathcal{R}^d(n-1) \) as all quadrics vanishing on \( \text{Ker}A \).

\[ \Box \]

5 **The cone of positive semi-definite symmetric forms**

A positive semi-definite symmetric form on \( \mathbb{R}^{n-1} \) is a bilinear symmetric form \( A : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R} \) with \( A(x,x) \geq 0 \) for any \( x \in \mathbb{R}^{n-1} \). Symmetric \( (n-1) \times (n-1) \) real matrices with non-negative eigenvalues are identified with positive semi-definite symmetric forms by \( A(x,y) = \langle Ax, y \rangle \), for the usual inner product \( \langle , \rangle \).

We have seen above that, in Gram coordinates, or, in other words, under the isomorphism \( CM^{d,n} \cong \mathcal{R}^d(n-1) \), the image of the configuration space \( C_n(\mathcal{R}^d) \) is made precisely of positive semi-definite symmetric forms of rank at most \( d \).

If we let \( d \) run from 1 to \( n-1 \), we have a sequence of inclusions:

\[ \text{im}C_n(R) \subset \text{im}C_n(R^2) \subset ... \subset \text{im}C_n(R^{n-1}) \]

with the last term identified with the (projective image of) the *convex cone of all (non-zero) positive semi-definite symmetric forms on \( \mathbb{R}^{n-1} \).*

**Proposition 5.1** The extremal rays of this cone correspond with \( \text{im}C_n(R) \) i.e. with collinear configurations.
Proof: Every non-zero positive semi-definite symmetric form of rank $d$ is the barycenter of a $(d - 1)$-simplex of such forms of rank 1 (as argued in the proof of Theorem 4.3). □

It will be convenient here to carry on our considerations in the vector space of all real symmetric matrices rather than the associated (real) projective space.

We consider on $\text{Sym}^2(R^{n-1})$ the Lorentzian form:

$$L(A, B) = \text{Tr}(AB) - \text{Tr}(A)\text{Tr}(B)$$

Indeed, $L$ is symmetric and has signature $(n - 2, 1)$, since positive on traceless forms and negative on multiples of the identity.

**Proposition 5.2** The cone of positive semi-definite symmetric forms on $R^{n-1}$ lies within the negative cone of the Lorentzian form $L$, except for its extremal rays (collinear configurations), which lie on the “light cone” $L(A, A) = 0$.

*Proof:* The form $L$ is clearly invariant under the action of the orthogonal group $O(n - 1, R)$ on symmetric forms: $A \mapsto TAT^{-1}$, and the proposition is obvious for diagonal forms. □

**Corollary 5.3** The Lorentzian form $L$ determines a hyperbolic metric on the projective image of the negative cone, which becomes a model of a hyperbolic $[(n - 1)^2 - 1]$-dimensional space. This induces a Riemann metric on all smooth configuration strata: $\text{im}C_n(R^{d+1}) - \text{im}C_n(R^d)$, $d \geq 1$. □

**Remark:** The form $L$ and the induced metrics depend on the choice of Gram coordinates and are not invariant under the full $S_n$-action on configurations.

**Example 1.2. once more:** We can present now an argument (cf. [19]; see also [22][3]) for the topological result announced in Example 1.2:

$$C_4(R^2) = P_2(C)/\text{conj} \approx S^4$$

We use an affine chart $\text{Tr}(A) = 1$, and picture $C_4(R^2) = \text{im}C_4(R^2)$ as the boundary of the convex hull of $\text{im}C_4(R) \subset S^4 = \{\text{Tr}(A^2) = 1\}$. This boundary is topologically a 4-sphere as well.

The other claim (cf. end of section 2), concerning the closure of the “fake” part, namely:

$$(P_2(R))^2/Z_2 \approx P_4(R)$$

also becomes transparent at this stage. One can use the fact that $CM_4$ is a cubic hypersurface in $P_5$, and, choosing a point in the interior of the convex hull of $\text{im}C_4(R)$ as “center” for the 4-sphere of realistic points, map a pair of antipodal realistic points to the “fake” one given by the third intersection of the “diameter” with $CM_4(R)$. This gives the (closed) “fake” part as: $S^4/Z_2 = P_4(R)$. □
6 A Hermitian analogue

As outlined in the introduction, one may consider configuration spaces \( C_n(C^d) \) for equivalence classes of \( n \) points in \( C^d \), where equivalence, this time, is under translations, unitary transformations and rescaling. Again, we require that at least two points be distinct.

In this section we let \( < , > \) denote the standard Hermitian inner product of \( C^d \):

\[
< z, w > = \sum_{k=1}^{d} z_k \bar{w_k}
\]

An analogue of Cayley coordinates would still record only the Euclidean information, but a Hermitian Gram matrix will take account of the symplectic imaginary part. Thus, upon choosing the origin at the first point \( p_1 = 0 \) of a configuration, we put:

\[
\alpha_{ij}(p) = < p_i, p_j >, \quad 2 \leq i, j \leq n
\]

defining a Hermitian matrix \( A(p) = P^* \cdot P \), where \( P \) denotes the \( d \times (n-1) \) complex matrix with columns \( p_2, ..., p_n \).

This gives a map:

\[
A : C_n(C^d) \rightarrow P_{(n-1)^2-1}(C)
\]

and we define \( HG^{d,n}(C) \) to be the Zariski-closure of \( \text{im}C_n(C^d) \).

\( P_{(n-1)^2-1}(C) \) is to be conceived as the projective space associated to the vector space of \( (n-1) \times (n-1) \) complex matrices on which we have the anti-holomorphic involution: \( M \mapsto M^* \). The fixed points of this involution (i.e. the real points for this real structure) are precisely the Hermitian matrices \( H = H^* \). We'll put \( P_{(n-1)^2-1}(\mathbb{R}) \) for these real points, when we need to emphasise the distinction from the ordinary real points \( P_{(n-1)^2-1}(\mathbb{R}) \).

Replacing orthogonal diagonalization with unitary diagonalization, all the arguments in Theorem 4.6 carry through and give:

**Theorem 6.1** The family of projective varieties:

\[
HG^{1,n}(C) \subset HG^{2,n}(C) \subset ... \subset HG^{n-1,n}(C) = P_{(n-1)^2-1}(C)
\]

coincides with the stratification by rank of the projective space of \( (n-1) \times (n-1) \) complex matrices, with \( HG^{d,n}(C) \) corresponding with matrices of rank at most \( d \).

The image of the configuration space \( \text{im}C_n(C^d) \subset HG^{d,n}(C) \) consists precisely of Hermitian positive semi-definite matrices of rank at most \( d \).
$HG^{d,n}(C)$ is the variety of $(d-1)$-planes secant to $HG^{1,n}(C)$. The latter space is the image of the Segre embedding:

$$(P_{n-2}(C))^2 \to P_{(n-1)^2-1}(C), \quad (u, v) \mapsto u \otimes v$$

The (complex) dimension of $HG^{d,n}(C)$ is: $2dn - d(d+2) - 1$, and its degree (for $d \leq n-2$) is given by the formula:

$$D^{d,n} = \deg(HG^{d,n}(C)) = \prod_{k=0}^{n-d-2} \frac{(n-1+k)}{(d+k)}$$

The Cayley-Menger varieties $CM^{d,n}(C)$ can be identified (via the same choice for Gram coordinates) with the fixed points of the corresponding varieties $HG^{d,n}(C)$ under the holomorphic involution given by transposition on matrices:

$$CM^{d,n}(C) = HG^{d,n}(C)^{Z_2}$$

Again, the dimension formula will be apparent from the statement below on resolution of singularities, and agrees with the “naive” count of real parameters for $C_n(C^d)$:

$$(n-1)2d - \dim U(d) - 1 = 2d(n-1) - d^2 - 1 = 2dn - d(d+2) - 1$$

where $U(d)$ stands for the unitary group in $C^d$.

For the degree formula, we refer to [12] 14.4.11.

**Remark:** The real structures induced on $CM^{d,n}(C)$ by conjugation and adjunction are clearly the same.

**Proposition 6.2** The singular locus of $HG^{d,n}(C)$ is $HG^{d-1,n}(C)$.

A resolution of singularities for $HG^{d,n}(C)$ can be presented as a $P_{d(n-1)-1}$-bundle over the Grassmann manifold $G(n-d-1,n-1)$ of codimension $d$ subspaces in $C^{n-1}$.

In particular, all varieties $HG^{d,n}(C)$ are rational.

**Proposition 6.3** The symmetric bilinear form: $(A, B) \mapsto Tr(A \cdot B)$ identifies the projective dual of $HG^{d,n}(C)$ with $HG^{n-d-1,n}(C)$. 

20
These are classical results: cf. [16] [13].

If we let $d$ run from 1 to $n - 1$, we have a sequence of inclusions:

$$imC_n(C) \subset imC_n(C^2) \subset \ldots \subset imC_n(C^{n-1})$$

with the last term identified with the (projective image of) the convex cone of all (non-zero) positive semi-definite Hermitian forms on $C^{n-1}$.

**Proposition 6.4** The extremal rays of this cone correspond with $imC_n(C)$ i.e. with $C$-collinear configurations.

We may define a Hermitian form on the space of $(n - 1) \times (n - 1)$ complex matrices (i.e. linear operators on $C^{n-1}$) by:

$$L(A, B) = Tr(AB^*) - Tr(A)Tr(B^*)$$

$L$ restricts to a real Lorentzian form on Hermitian operators (and restricts further to $L$ on real symmetric forms).

The analogue of Proposition 5.2 now reads:

**Proposition 6.5** The cone of positive semi-definite Hermitian forms on $C^{n-1}$, i.e. $imC_n(C^{n-1})$, lies within the negative cone of the Lorentzian form $L$ (on Hermitian operators), except for its extremal rays ($C$-collinear configurations), which lie on the “light cone” $L(A, A) = 0$.

### 7 A quaternionic analogue

For Hamilton’s quaternions $H$, we use the standard description:

$$x = a + bi + cj + dk, \quad i^2 = j^2 = k^2 = ijk = -1$$

We put $x^* = a - bi - cj - dk$ for the conjugate, and this gives:

$$Re(x) = Re(x^*) = \frac{1}{2}(x + x^*) = a$$

$$|x|^2 = |x^*|^2 = xx^* = x^*x = a^2 + b^2 + c^2 + d^2$$

$C$ has an $S^2$ family of embeddings in $H$, since any quaternion of square $-1$ can be used to represent $i \in C$. The choice which makes $i \in C$ be $i \in H$, and takes 1 and $j$ as a $C$-basis (for scalar multiplication on the right), gives an identification:

$$H = C^2 \quad x \mapsto (u, v)$$
via the matrix description of left multiplication:

\[
x = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}
\]

\[u = a + bi, \quad v = c - di, \quad x = a + bi + cj + dk = u + jv\]

and then \(x^*\) is genuinely the adjoint of \(x\).

This choice is meant to relate to a right vector space structure on \(H^d\) (i.e. scalar multiplication by \(x \in H\) is to the right), which allows then the familiar description of \(H\)-linear maps \(H^d \to H^d\) as \(d \times d\) matrices with entries in \(H\). The expression over \(C \subset H\) (with the \(1,j\) basis in each factor) replaces each quaternionic entry with its corresponding \(2 \times 2\) block. Adjunction, that is transposition and conjugation, is then consistent in the \(H\)-version and \(C\)-version.

Consider now the hyper-Hermitian inner product on \(H^d\):

\[< x, y > = \sum_{i=1}^{d} y_i^* x_i \in H\]

The \(H\)-linear transformations preserving this inner product make-up a compact Lie group traditionally denoted \(Sp(d)\), but perhaps more suggestively described as the group of hyper-unitary transformations. We have:

\[Sp(d) = \{ T = (t_{ij})_{1 \leq i,j \leq d} \mid T^* T = I_d \}\]

and \(\text{dim}_R Sp(d) = d(2d+1)\), since the Lie algebra is given by:

\[sp(d) = \{ \Theta \mid \Theta^* + \Theta = 0 \}\]

and \(3d + 4(d^2) = 3d + 2d(d - 1) = d(2d + 1)\).

We may consider now the configuration space \(C_n(H^d)\) which consists of equivalence classes of \(n\) ordered points in \(H^d\), modulo translations, hyper-unitary transformations, and rescaling. As before, the “naive” count of parameters proposes the real dimension:

\[(n - 1)4d - \text{dim}_R Sp(d) - 1 = 4d(n - 1) - d(2d + 1) - 1\]

Again, by translation, we make the first point in a configuration become the origin: \(p_1 = 0\), and arrange the column vectors \(p_2, ..., p_n\) in a \(d \times (n - 1)\) matrix \(P\) with quaternionic entries. We then have an associated Gram matrix: \(A(p) = P^* \cdot P\), which is self-adjoint, with columns spanning an \(H\)-subspace of dimension at most \(d\), and positive semi-definite:

\[< A(p)x, y > = < x, A(p)y >, \quad < A(p)x, x > = < Px, Px > \geq 0\]
The analogy with symmetric (R), and Hermitian (C) matrices continues in the sense that quaternionic self-adjoint matrices become diagonal in a suitable hyper-unitary basis, and with this, all previous considerations have their quaternionic avatar.

In fact, if we look at the quadratic form \(< Ax, x >\) associated to a self-adjoint operator \(A\), we see that it takes values in \(R\), and with \(H = C^2 = R^4\), we have natural inclusions:

\[ h\text{Her}(H^{n-1}, H^{n-1}) \subset \text{Her}(C^{2(n-1)}, C2(n-1)) \subset \text{Sym}(R^{4(n-1)}, R^4(n-1)) \]

where, as emphasised in [2], the first space corresponds to real quadratic forms invariant under the action of \(S^3 = Sp(1) = SU(2)\) \(< Axu, xu >= u^*(x^*Ax)u = (x^*Ax)u^*u \subset Ax, x >, for any quaternion \(u\) of norm one), and the second space corresponds to real quadratic forms invariant under the action of \(S^1 = U(1) = SO(2, R)\).

Obviously, this gives inclusions:

\[ C_n(H^d) \subset C_{2n-1}(C^{2d}) \subset C_{4n-3}(R^{4d}) \quad (H - C - R) \]

and one can rely on the notion of Cayley-Menger variety \(CM^{d,4n-3}(C)\), which is the Zariski-closure of the last term in the projective space of complex symmetric matrices, in order to obtain corresponding complexifications and Zariski-closures for the first two terms.

However, we saw in the previous section a more direct way to define \(HG^{d,n}(C)\), and we want to present a similar approach in the quaternionic case. Since we need some notation for the envisaged varieties, we propose \(PG^{d,n}\), which suggests both Pfaff-Gram and Plücker-Grassmann. The reason for these associations will become apparent presently.

Indeed, one can establish a correspondence between hyper-Hermitian (i.e. quaternionic self-adjoint) matrices acting on \(H^{n-1}\) and (a certain real slice of) skew-symmetric complex matrices acting on \(C^{2(n-1)}\).

We have describe above the identification \(H = C^2\) given by:

\[ x = a + bi + cj + dk = u + jv = (u, v) = (a + bi, c - di) \]

and we consider now an order four \(R\)-linear transformation:

\[ \sigma : H = C^2 \rightarrow H = C^2 \quad x = (u, v) \mapsto \sigma(x) = (\bar{v}, -u) \]

When we consider left multiplication by \(x\), and look upon \(x\) as a \(2 \times 2\) matrix, the effect of \(\sigma\) can be described as a rotation by \(\pi/2\), followed by change of sign for the first column.
Lemma 7.1 Let $A = (a_{ij})_{1 \leq i,j \leq n-1}$ be a hyper-Hermitian matrix of rank $d$, that is: $a_{ij}^* = a_{ji}$, and the $H$-subspace generated by the columns of $A$ (with right scalar multiplication) is $d$-dimensional.

Let $\sigma(A) = (\sigma(a_{ij}))_{ij}$, and consider $\sigma(A)$ as a complex $2(n-1) \times 2(n-1)$ matrix, corresponding to $H = C^2$, that is: replace each quaternionic entry by the corresponding $2 \times 2$ block.

Then, $\sigma(A)$ is skew-symmetric of rank $2d$.

Proof: The fact that $\sigma(A)$ turns out skew-symmetric is straightforward, and for the rank comparison, we use:

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \tilde{\sigma}(A) = \begin{pmatrix} \sigma(A) & 0 \\ 0 & \sigma(A) \end{pmatrix}$$

where $A$ is also considered as a $2(n-1) \times 2(n-1)$ complex matrix. Clearly, as complex matrices, the first has rank $4d$, and the second $2 \cdot \text{rk}[\sigma(A)]$. Thus, we have to show that the two matrices have equal rank.

But this is clear when we rotate the second matrix with $\pi/2$ around its center (which is the same as a transposition and a permutation of rows corresponding to reflecting in the horizontal mid-axis), and change signs for every other column. This yields precisely:

$$\begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} \quad \Box$$

The operation just described shows at the same time how to introduce an anti-holomorphic involution on the projective space of complex skew-symmetric matrices, with fixed point locus exactly the image by $\sigma$ of hyper-Hermitian matrices. Expressed at the level of the $2 \times 2$ blocks indexed by $ij$, $1 \leq i,j \leq n-1$, this involution amounts to: rotating each block around its center by $\pi$, changing signs along the second diagonal in each block, and ending by conjugation.

With this prepared background, we may consider now the configuration space $C_n(H^d)$ as embedded in the projective space of complex skew-symmetric matrices, that is:

$$C_n(H^d) \hookrightarrow P(\mathbb{H}^2C^{2(n-1)}) = P(\mathbb{H}^{2n-2})_{-1}(C), \quad p \mapsto \sigma(A(p))$$

where $A(p)$ is the Gram matrix for $p_1, p_2, ..., p_n \in H^d$.

Definition 7.1: The variety $PG^{d,n}(C)$ is the Zariski-closure of

$$C_n(H^d) \subset P(\mathbb{H}^{2n-2})_{-1}(C)$$
with real points $PG^{d,n}(R)$ defined according to the above real structure on skew-symmetric matrices.

We have now a string of results, analogous to the orthogonal (R), and unitary (C) context:

**Theorem 7.2** The family of projective varieties:

$$
PG^{1,n}(C) \subset PG^{2,n}(C) \subset \ldots \subset PG^{n-1,n}(C) = P(\wedge^{n-2}2n-2)_{-1}(C)
$$

coincides with the stratification by (even) rank of the projective space of skew-symmetric $2(n-1) \times 2(n-1)$ complex matrices, with $PG^{d,n}(C)$ corresponding with matrices of rank at most $2d$.

$PG^{d,n}(C)$ is the variety of $(d-1)$-planes secant to $PG^{1,n}(C)$. The latter space is the image of the Grassmann-Plücker embedding:

$$
G(2n-4,2n-2) \hookrightarrow P(\wedge^{2n-4}C2n-2).
$$

The (complex) dimension of $PG^{d,n}(C)$ is: $4d(n-1) - d(2d+1) - 1$, and its degree (for $d \leq n-2$) is given by the formula:

$$
\text{deg}(PG^{d,n}(C)) = \frac{1}{2^{2n-2d-3}} \prod_{i=0}^{2n-2d-4} \frac{(2n-2+i)}{(2d+2i+1)} = \prod_{1 \leq i \leq j \leq 2n-2d-3} \frac{2d + i + j}{i + j}.
$$

The identification $PG^{1,n}(C) = G(2n-4,2n-2) \approx G(2,2n-2)$ can be made more explicit as follows:

(i) first, we have an identification of the configuration space $C_n(H)$ with the quaternionic projective $(n-2)$-space with respect to left scalar multiplication, denoted here $HP_{n-2};$

(ii) then, $HP_{n-2}$ can be identified with the quaternionic Grassmannian $G_H(n-2,n-1)$ of codimension one $H$-subspaces of $H^{n-1}$ with respect to right scalar multiplication;

(iii) then, $G_H(n-2,n-1) \subset G(2n-4,2n-2)$ from our identification $H = C^2$;

(iv) and finally $G(2n-4,2n-2) = PG^{1,n}(C)$ from the correspondence between skew-symmetric matrices of rank two and their kernels.

Note that the real structure introduced above on skew-symmetric matrices gives as real points on $G(2n-4,2n-2)$ precisely the $H$-subspaces, i.e. $G_H(n-2,n-1)$. In this sense, the complexification $HP_{n-2}(C)$ of $HP_{n-2} = HP_{n-2}(R)$ is the complex Grassmannian $G(2n-4,2n-2) \approx G(2,2n-2)$.

For the degree formula, see: [14] [17] [18].
Proposition 7.3 The singular locus of $PG^{d,n}(C)$ is $PG^{d-1,n}(C)$.

A resolution of singularities for $PG^{d,n}(C)$ can be presented as a $P_{\binom{d-1}{2}}$-bundle over the Grassmann manifold $G(2(n-d-1), 2(n-1))$ of codimension 2d subspaces in $C^{2n-2}$.

In particular, all varieties $PG^{d,n}(C)$ are rational.

Proposition 7.4 The symmetric bilinear form: $(A, B) \mapsto Tr(AB)$ identifies the projective dual of $PG^{d,n}(C)$ with $PG^{n-d-1,n}(C)$.

This follows from the description of the projective tangent space at a non-singular point $S \in PG^{d,n}(C)$ as the projective subspace of all skew-symmetric matrices $T$ which, as two-forms, restrict to zero on $Ker(S)$, that is: $\forall x, y \in Ker(S), xy^\top = 0$.

In relation with the sequence of inclusions of configuration spaces:

$C_n(H) \subset C_n(H^2) \subset \ldots \subset C_n(H^{n-1})$

it will be convenient to revert to their description in terms of Gram matrices i.e. quaternionic self-adjoint operators, and then the last term is identified with the (projective image of) the \textit{convex cone of all (non-zero) positive semi-definite hyper-Hermitian forms} on $H^{n-1}$.

Corresponding to propositions 5.1 and 6.4, we have:

Proposition 7.5 The extremal rays of this cone are given by $C_n(H)$, that is: $H$-collinear configurations.

Similarly,

$$\mathcal{L}(A, B) = Tr(\frac{1}{2}(AB + BA)) - Tr(A)Tr(B)$$

is well defined and Lorentzian on hyper-hermitian operators, and allows one to picture non-collinear quaternionic configurations as plunged in a hyperbolic space of dimension $2n^2 - 5n + 2$.

8 An octonionic enclave

In this section we mention the varieties related to octonions, from the perspective developed for $R, C$, and $H$. They are, mercifully\footnote{After three long dramas, a short satyr play.}, restricted to $d = 1, 2, 3$ and $n \leq 4$.\footnote{After three long dramas, a short satyr play.}
The Graves-Cayley octaves, or octonions $\mathbb{O}$, can be described in terms of quaternions $H^2 = \mathbb{O}$ by:

\[ e^2 = -1, \quad x = x_1 + x_2 e, \quad y = y_1 + y_2 e \]

\[ xy = (x_1 y_1 - y_2 x_2) + (x_2 y_1 + y_2 x_1) e \]

There is a conjugation: \( \tilde{x} = x_1^* - x_2 e \), with:

\[ |x|^2 = \tilde{x} x = x_1^* x_1 + x_2^* x_2 = |x_1|^2 + |x_2|^2 \]

The associator: \([x, y, z] = (xy)z - x(yz)\) vanishes whenever two arguments are equal or conjugate.

We give now a brief description of varieties denoted $OG_d^n$ for \((d, n) = (1, 3), (2, 3)\) and \((1, 4), (2, 4), (3, 4)\), which represent the octonionic analogue of previous constructions.

When one considers $\mathbb{O}$-Hermitian matrices, the $2 \times 2$ case has a straightforward determinant:

\[ \det \begin{pmatrix} \alpha & x \\ \tilde{x} & \beta \end{pmatrix} = \alpha \beta - x \tilde{x} \in R, \quad \text{for } \alpha, \beta \in R, \ x \in \mathbb{O} \]

Thus, upon complexification ($\otimes_R \mathbb{C}$), one has:

\[ OG^{1,3}(\mathbb{C}) = P\{ \begin{pmatrix} a & x_c \\ \tilde{x}_c & b \end{pmatrix} \mid ab = x_c \tilde{x}_c \} \approx Q_8 \subset P_9(\mathbb{C}) = OG^{2,3}(\mathbb{C}) \]

For $3 \times 3$ $\mathbb{O}$-Hermitian matrices:

\[ A = \begin{pmatrix} \alpha & z & y \\ \tilde{z} & \beta & x \\ \tilde{y} & \tilde{x} & \gamma \end{pmatrix}, \quad \alpha, \beta, \gamma \in R, \ x, y, z \in \mathbb{O} \]

one considers the Jordan algebra structure on the complexification, with commutative product of matrices:

\[ A_c \cdot B_c = \frac{1}{2}(A_c B_c + B_c A_c) \]

and determinant:

\[ \det(A_c) = \frac{1}{3}(TrA_c^3) - \frac{1}{2}(TrA_c)(TrA_c^2) + \frac{1}{6}(TrA_c)^3 \]

One recognizes on the right hand side the expression of the product of three variables (the 'eigenvalues') in terms of three basic symmetric functions: sum, sum of squares, and sum of cubes.
Then:

\[ OG^{2,4}(C) = P\{A_c \mid \text{det}A_c = 0\} \subset P_{26}(C) = OG^{3,4}(C) \]

\[ OG^{1,4}(C) = \text{sing}(OG^{2,4}(C)) \subset OG^{2,4}(C) \]

\(OG^{1,4}(C)\) has an interpretation as “rank one” matrices, and is better known as the (complexified) octonionic projective plane, with the rational homogeneous space description: \(E_6/P\), where \(E_6\) is the exceptional complex simple Lie group of that type, and \(P\) is the maximal parabolic subgroup corresponding to the first (or last) root in the \(E_6\) graph. \([11]\) \([21]\)

As expected, \(OG^{2,4}(C)\) is, on the one hand, the secant variety of \(OG^{1,4}(C)\), and can be identified, on the other hand, with the projective dual of \(OG^{1,4}(C)\).

9 Mechanical linkages and linear sections

It was suggested in the introduction that the formalism of Cayley-Menger varieties would be useful with respect to mechanical linkages. Indeed, mechanical linkages are point configurations with constraints expressed as prescription of (squared) distances between certain pairs of points (and visualised as rigid bars connecting those points).

Cayley coordinates become particularly relevant in this context, since the configuration space of a linkage with \(n\) vertices and \(k\) bars in \(R^d\) is simply a linear section of codimension \((k-1)\) in \(P_{\binom{n}{2}}^{-1}(C)\) intersecting \(\text{im}C_n(R^d) \subset CM^{d,n}(R) \subset CM^{d,n}(C)\) with equations involving only the Cayley coordinates corresponding to the specified rigid bars.

We wish to emphasise that, for many purposes, linkage linear sections may not qualify as generic among all conceivable linear sections of a given dimension. In fact, it is rather the peculiar character of such sections that gives distinctiveness to this study and the related rigidity theory.

Definition 9.1: A mechanical linkage \((\Gamma^{d,n}(\sigma), p)\) in a Euclidean space of dimension \(d\) is a connected graph \(\Gamma\) on \(n\) vertices (labeled from 1 to \(n\)), together with an assignment of non-negative numbers \(\{ij\} \mapsto \sigma_{ij}\) for all edges \(\{ij\} \in \Gamma\), and a realization \(p\) as a configuration of \(n\) labeled points \(p_1, ..., p_n \in R^d\) (at least two of them distinct) with the prescribed squared distances:

\[ |p_i - p_j|^2 = p_i - p_j, p_i - p_j >= \sigma_{ij}, \text{ for all edges } \{ij\} \in \Gamma \]

We let \(|\Gamma|\) stand for the cardinality of \(\Gamma\) i.e. the number of edges in the graph.
Obviously, the set of admissible \( \sigma = (\sigma_{ij})_{(ij) \in \Gamma} \) may be constrained by the nature of the graph, \( d \), and \( n \). Note that we want \( p \) to define a point in the configuration space \( C_n(R^d) \) where we do not allow all points to be one and the same point. Thus, the choice \( \sigma = 0 \), will be deemed to have no configuration realization.

**Definition 9.2:** The configuration space \( C(\Gamma^{d,n}(\sigma)) \) of a mechanical linkage \((\Gamma^{d,n}(\sigma), p)\) is the space of all congruence classes of realizations by \( n \) labeled points in \( R^d \).

Since one can choose any non-zero bar as unit of measurement, we see that we have a canonical embedding:

\[
C(\Gamma^{d,n}(\sigma)) \subset C_n(R^d)
\]

and with that, a Zariski-closure in the Cayley-Menger variety \( CM^{d,n}(C) \), which is clearly a linear section:

**Proposition 9.1** The equations defining the Zariski-closure of the configuration space \( C(\Gamma^{d,n}(\sigma)) \) in \( CM^{d,n}(C) \subset P(\binom{n}{2}-1(C)) \), with respect to Cayley coordinates \( (s_{ij}) \), are simply:

\[
\frac{s_{ij}}{\sigma_{ij}} = \frac{s_{kl}}{\sigma_{kl}} \quad \text{for all edges } \{ij\}, \{kl\} \in \Gamma
\]

Of course, the equations are to be understood as:

\[
\sigma_{kl}s_{ij} = \sigma_{ij}s_{kl} \quad \{ij\}, \{kl\} \in \Gamma
\]

and the projective subspace \( L\Gamma_{\sigma} \) they define in \( P(\binom{n}{2}-1(C)) \) will have codimension \(|\Gamma| - 1\). \( \square \)

**Definition 9.3:** The projective variety \( L\Gamma_{\sigma} \cap CM^{d,n}(C) \) will be called the linkage variety associated to the机械 linkage \((\Gamma^{d,n}(\sigma), p)\). It contains, as its “realistic” points, the linkage configuration space \( C(\Gamma^{d,n}(\sigma)) \).

It is our contention, to be pursued in [7], that linkage varieties are instrumental in understanding configuration spaces of mechanical linkages. Here, we illustrate this point by obtaining an upper bound for the number of realizations of a generic planar Laman linkage.

**Definition 9.4:** A planar Laman linkage is a mechanical linkage \((\Gamma^{2,n}(\sigma), p)\) in \( R^2 \), with a (connected) graph \( \Gamma \) on \( n \) vertices satisfying:

(i) \( \Gamma \) has \( 2n - 3 \) edges;

(ii) for any subset of \( k \) vertices, there are at most \( 2k - 3 \) edges in \( \Gamma \) connecting them.
Remark: These Laman graphs characterize, in dimension two, the mechanical linkages which, for generic $\sigma$, are locally rigid (with a minimum number of bars) \[^{[20]}\]. Since $\dim_R(CM^2,n(R)) = 2n - 4$, it is immediate that one needs at least $2n - 3$ bars in order to obtain isolated points in a generic linkage configuration space.

**Proposition 9.2** For generic $\sigma$, the number of possible realizations (up to congruence) of a planar Laman linkage $(\Gamma^{2,n}(\sigma), p)$ is bounded by $\frac{1}{2}(2n-4)^n$.

*Proof:* Our bound is the degree of the Cayley-Menger variety $CM^2,n(C) = CM_{2n-4}(C)$ (cf. Corollary 2.4), and the claim follows from the (refined) Bézout theorem, considering that, for generic $\sigma$, all possible realizations are infinitesimally rigid i.e. isolated not only as points in the configuration space $C(\Gamma^{2,n}(\sigma))$, but as points in the linkage variety as well.  

See also \[^{[6]}\] for more details and the case of arbitrary dimensions.

## 10 Polygon spaces and Calabi-Yau manifolds

Presently, we are going to use the Cayley-Menger varieties $CM^2,n = CM_{2n-4}$ (and $HG^{1,n}$) in relation to planar polygonal linkages and their Calabi-Yau complexifications.

The approach here complements the one in \[^{[5]}\], which emphasizes toric geometry and expands on matters related to mirror symmetry. The latter context motivates the study of special Lagrangian submanifolds in Calabi-Yau manifolds, with particular emphasis on special Lagrangian tori.

For the limited purposes envisaged in this section, we may adopt the (fairly inclusive) terminology which calls Calabi-Yau a complex projective manifold with vanishing canonical class modulo torsion.

Since all considerations here relate to real points of Calabi-Yau manifolds defined over $\mathbb{R}$, we need not dwell on the notion of special Lagrangian submanifold, normally defined for the finite étale coverings where one actually has a trivial canonical bundle i.e. a holomorphic volume form. When there’s a real structure in this more restricted setting, the real points do yield a special Lagrangian submanifold - for some adequate Calabi-Yau metric\[^{2}\].

We begin with a simple observation, valid because of the equality $U(1) = SO(2, \mathbb{R})$ between the unitary group and the special orthogonal group in the plane $C = \mathbb{R}^2$.

\[^{2}\]While this seems, in the absence of explicit Calabi-Yau metrics, an “easy” way to get hold of some special Lagrangians, it faces nevertheless the serious challenge of real algebraic geometry: assessing the topology of the real slice.
Lemma 10.1 There's a natural double covering map:

\[ HG^{1,n}(C) \rightarrow CM^{2,n}(C) = CM_{2n-4}(C) \]

ramified over \( CM^{1,n}(C) \). and extending the natural double covering:

\[ \text{im}C_n(C) \rightarrow \text{im}C_n(R^2) \]

Of course, this is essentially the double covering described in Proposition 2.3, considering that \( HG^{1,n}(C) \) is the image of the Segre embedding of \( (P_{n-2}(C))^2 \), and \( CM^{2,n}(C) \) the further projection on symmetric tensors (cf. section 3). In terms of Gram matrices the map is:

\[ A \mapsto \frac{1}{2}(A + A^T), \text{ for } A \text{ Hermitian of rank one} \]

This double covering redresses the loss of orientation for planar configurations identified by reflection.

Accordingly, planar polygon spaces for unoriented \( n \)-gons will be realized in \( CM^{2,n}(R) \), while their pull-back to \( HG^{1,n}(R) \) by the above double covering will give realizations of the corresponding spaces for oriented \( n \)-gons.

Obviously, planar polygon spaces will be particular cases of linkage configuration spaces, corresponding to a polygonal graph \( \Pi = \Pi^{2,n} \). We adopt the standard labeling of edges, namely:

\[ \Pi = \Pi^{2,n} = \{\{1, 2\}, \{2, 3\}, \ldots, \{i, i+1\}, \ldots, \{n-1, n\}, \{n, 1\}\} \]

We have a polygon (configuration) space \( C(\Pi^{2,n}(\sigma)) \) once we specify an admissible edge-length-vector \( q = (q_1, \ldots, q_n) \) and put:

\[ \sigma_{i,i+1} = q_i^2 \text{ for } 1 \leq i \leq n - 1, \text{ and } \sigma_{n,1} = q_n^2 \]

for the squared length of the bars. By definition, it consists (in the unoriented case) of all congruence classes of configurations \( p_1, \ldots, p_n \in R^2 \), such that:

\[ |p_j - p_i|^2 = \sigma_{ij} \text{ for all } \{ij\} \in \Pi \]

Since we may take the perimeter as our scale, we may suppose the edge-length-vector \( q \) standardized by: \( q_1 + \ldots + q_n = 1 \). Then \( q \) is admissible if \( 0 \leq q_i \leq 1/2, \text{ i.e. no edge is longer than the sum of the rest.} \)

From the previous section we have:
Proposition 10.2 The planar polygon space $C(\Pi^{2,n}(\sigma))$ can be embedded in the real Cayley-Menger variety $CM^{2,n}(R)$ as the intersection of its “realistic” part $imC_n(R^2)$ with the codimension $(n-1)$ linear section defined by the equations:

$$
\frac{s_{12}}{\sigma_{12}} = \frac{s_{23}}{\sigma_{23}} = \ldots = \frac{s_{n-1,n}}{\sigma_{n-1,n}} = \frac{s_{n1}}{\sigma_{n1}} \quad (L\Pi)$$

where $s_{ij}$ are Cayley coordinates in $P_{n-1}^{(n-2)}$.

Remark: The case $d = 2$ has a particularly simple way of describing the “realistic” part $imC_n(R^2) \subset CM^{2,n}(R)$: it is the part of $CM^{2,n}(R)$ contained in the closure of the negative cone of the Lorentzian form $L$ (cf. section 5).

As one would expect, the polygon space $C(\Pi^{2,n}(\sigma))$ has singularities only when the linear section $L\Pi$ meets the singular locus $CM^{1,n}(R) \subset CM^{2,n}(R)$, that is, when the edge-length-vector allows a degeneration of the polygon into a one-dimensional configuration. This amounts to a relation of the form:

$$\sum_{i=1}^{n} \epsilon_i \cdot q_i = 0 \quad \text{with } \epsilon_i = \pm 1$$

Thus, when $q$ avoids all “walls” of this form, the polygon configuration space $C(\Pi^{2,n}(\sigma(q)))$ is a smooth $(n-3)$-dimensional manifold, and its topology would change only when the edge-length-vector parameter $q$ “moves across a wall”.

The fact we want to retain here from [5] is that the topology of $C(\Pi^{2,n}(\sigma))$ can be investigated by separate means: Morse theory -first and foremost. This will ‘pre-empt’ the question about the nature of the real points, when we complexify.

Example: It is easy to see, intuitively, how to make $C(\Pi^{2,n}(\sigma))$ into a torus. When one cuts a small corner of an $(n-1)$-gon, and produces an $n$-gon with the new edge sufficiently small by comparison with the old edges, the configuration space will be the product of a circle with the old configuration space (since the small edge can assume any position around one end, and the polygon closes-up essentially as for a null new edge). Thus, one can start with a triangle, perform a succession of $(n-3)$ such small cuts, and obtain the edges for a $(S^1)^{n-3}$ polygon configuration space.

The complexification process we are about to consider will offer, in particular, an illustration for the distinction between linkage sections (cf. section 9) and more general linear sections.

Indeed, the analysis in [3] shows that the polygonal linkage variety $L\Pi \subset CM^{2,n}(R)$ is always singular (a contraction and $Z_2$ quotient of the resolved Darboux varieties considered there). However, we may perturb the linear section

---

3The area function, for generic $q$, is a Morse function.
(still with real coefficients) and obtain smooth intersections with $CM^{2,n}(C)$, while the real locus maintains a connected component diffeomorphic with the polygon configuration space $C(\Pi^{2,n}(\sigma))$:

**Theorem 10.3** Let $q$ be an admissible edge-length-vector away from the walls described above, and let $\sigma$ stand for the corresponding squared lengths.

A generic linear section of codimension $(n-1)$, defined over $R$, and sufficiently close to the linkage section $L_{I_{r}}$, intersects the Cayley-Menger variety $CM^{2,n}(C)$ along a smooth $(n-3)$-dimensional Calabi-Yau manifold defined over $R$ whose real points contain a connected component isomorphic with the polygon configuration space $C(\Pi^{2,n}(\sigma))$.

**Proof:** Generic linear sections of codimension $(n-1)$ will avoid the $(n-2)$-dimensional singular locus $CM^{1,n}(C)$ and meet $CM^{2,n}(C)$ transversely. This must also be the case for generic linear sections defined over $R$, since the parameter locus for singular sections is contained in a proper subvariety invariant under conjugation.

In order to see that the canonical class (modulo torsion) is trivial, one looks at the pull-back of the section under the double covering map (cf. lemma 10.1. above):

$$HG^{1,n}(C) \rightarrow CM^{2,n}(C)$$

But $HG^{1,n}(C) = (P_{n-2}(C))^2$ and the codimension $(n-1)$ section corresponds with a smooth intersection of $(n-1)$ hypersurfaces of bidegree $(1,1)$. □

**Remark:** The case of pentagons ($n=5$) yields Enriques surfaces defined over $R$. Renouncing the reality condition, one obtains the full family of Enriques surfaces constructed by Reye congruences [9].

We conclude this section with another construction, in the manner of [8], of special Lagrangian 3-tori in Calabi-Yau hypersurfaces of the 4-quadratic $G(2,4)$. Again, we look at real loci in particular cases defined over $R$.

We are going to use not the usual real structure of $G(2,4) = Q_{4} \subset P_{5}(C)$, but the one explained in Theorem 7.2, which presents the Grassmannian $G(2,2n-2) = PG^{1,n}(C)$ as the complexification of the quaternionic projective space $HP_{n-2} = C_{n}(H)$.

For $n = 3$, we have: $HP_{1} \approx S^{4}$: the one point compactification of $H$.

The Calabi-Yau threefolds under consideration are degree four sections of $G(2,4) = Q_{4} \subset P_{5}(C)$. Given that the embedding of $HP_{1}$ in the Grassmannian is quadratic, we should look for 3-tori defined in $H$ by the vanishing of an octic polynomial.
Examples of this kind can be produced as follows: consider the 2-torus \((S^1)^2\) defined in \(R^4 = H\) by:

\[x_1^2 + x_2^2 = 1, \quad x_3^2 + x_4^2 = 1\]

The normal bundle is trivial, hence the circle bundle of radius \(r\) in it defines a 3-torus \((S^1)^3\). For \(r < 1\) we have an embedding of this torus in \(H\) as:

\[(a, b, c, d) = (\lambda x_1, \lambda x_2, \mu x_3, \mu x_4)\]

with \(x_i\) as above and \((\lambda - 1)^2 + (\mu - 1)^2 = r^2\).

Elimination yields the octic (non-homogeneous) polynomial equation:

\[
((a^2 + b^2 + c^2 + d^2)^2 - 2r^2(a^2 + b^2 + c^2 + d^2) + (2 - r^2)^2)^2 = 8^2(a^2 + b^2)(c^2 + d^2)
\]

which vanishes on the intended image. This gives:

**Proposition 10.4** The family of Calabi-Yau threefolds given by degree four sections of a smooth quadric \(Q_4 \subset P_5(C)\) contains members which allow a real structure with real locus a 3-torus. \(\square\)

### 11 Summary

In this summary we uniformize the notation by allowing \(K\) to become \(R, C, H,\) or \(O\), that is: to designate the real, complex, quaternionic or octonionic numbers.

The common features of these algebraic structures are best expressed in a theorem of Hurwitz stating that a *finite dimensional real vector space* with:

(i) a positive definite inner product,

(ii) a (distributive) multiplication with \(|xy| = |x| \cdot |y|\), and

(iii) a unity

must be one of them.

Accordingly, \(C_n(K^d)\) will stand for the *configuration space* of \(n\) points in \(K^d\), and \(G_{K,n}^d(C) \subset P_k(n)(C)\) for its *Zariski-closure* in a complex projective space \(P_k(n)(C) = G_{K}^{n-1,n}(C)\). Recall that, for \(K = O\), this is only symbolic, and the pairs \((d, n)\) are restricted to \((1, 3), (2, 3), (1, 4), (2, 4), (3, 4)\).

Thus, with respect to previous notations:

\[
G_R^{d,n} = CM^{d,n}, \quad G_C^{d,n} = HG^{d,n}, \quad G_H^{d,n} = PG^{d,n}
\]

corresponding respectively with *determinantal varieties* of symmetric (R), general (C), and skew-symmetric (H) forms.
The equivalence relation on configurations is modulo translations, rescaling, and transformations of $K^d$ given by:

(R) the orthogonal group: $O(d, R)$, with $\dim_R O(d, R) = \binom{d}{2}$

(C) the unitary group: $U(d)$, with $\dim_R U(d) = d^2$

(H) the hyper-unitary group: $Sp(d)$, with $\dim_R Sp(d) = d(2d + 1)$.

The projective dual of $G_{d,n}^d(C)$ can be identified with $G_{d,n}^{d-1}(C)$.

For $d = 1$, $C_n(K) = G_{1,n}^1(R) \subset G_{1,n}^d(C)$, and for $n = 3, 4$ we have:

| K | $\dim_G K_{d,n}^d(C)$ | $\dim_G K_{d,n}^{d-1}(C) = k(n)$ |
|---|---|---|
| R | $d(n-1) - \binom{d}{2} - 1$ | $\binom{d}{2} - 1$ |
| C | $2d(n-1) - d^2 - 1$ | $(n-1)^2 - 1 = n(n-2)$ |
| H | $4d(n-1) - d(2d+1) - 1$ | $\binom{2n-2}{2} - 1 = (n-2)(2n-1)$ |

The projective dual of $G_{d,n}^d(C)$ can be identified with $G_{k}^{n-d-1,n}(C)$.

For $d = 1$, $C_n(K) = G_{1,n}^1(R) \subset G_{1,n}^d(C)$, and for $n = 3, 4$ we have:

| K | $\mathbb{C}_3(K)$ | $G_{d,n}^{d}(C)$ | $C_4(K)$ | $G_{d,n}^{d-1}(C)$ |
|---|---|---|---|---|
| R | $RP_1 = S^1$ | $Q_1 \subset P_2(C)$ | $RP_2$ | $P_2(C) \subset P_5(C)$ |
| C | $CP_1 = S^2$ | $Q_2 \subset P_3(C)$ | $CP_2$ | $(P_2(C))^2 \subset P_6(C)$ |
| H | $HP_1 = S^4$ | $Q_4 \subset P_5(C)$ | $HP_2$ | $G(2,6) \subset P_{14}(C)$ |
| O | $OP_1 = S^8$ | $Q_8 \subset P_9(C)$ | $OP_2$ | $E_6/P \subset P_{26}(C)$ |

One recognizes in the last column the four Severi varieties [21].

$G_{d,n}^d(C)$ is determined by $G_{1,n}^1(C)$ as its variety of $(d-1)$ secant planes.

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Rider University
Department of Mathematics
Lawrenceville, NJ 08648
U.S.A.

*E-mail address:*
borcea@rider.edu