Estimates of Dirichlet heat kernels for unimodal Lévy processes with low intensity of small jumps

Soobin Cho, Jaehoon Kang and Panki Kim

Abstract

In this paper, we study transition density functions for pure jump unimodal Lévy processes killed upon leaving an open set $D$. Under some mild assumptions on the Lévy density, we establish two-sided Dirichlet heat kernel estimates when the open set $D$ is $C^{1,1}$. Our result covers the case that the Lévy densities of unimodal Lévy processes are regularly varying functions whose indices are equal to the Euclidean dimension. This is the first result on two-sided Dirichlet heat kernel estimates for Lévy processes such that the lower scaling index of the Lévy densities is not necessarily strictly bigger than the Euclidean dimension.

1. Introduction and main results

1.1. Introduction

The transition density of a Lévy process killed upon leaving an open set $D$ (called the Dirichlet heat kernel of the process in $D$) is the fundamental solution of the equation $\partial_t u = Lu$ with zero exterior condition on $D^c$, where $L$ is the infinitesimal generator of the Lévy process. When the sample paths of the Lévy process are discontinuous, such generator is a non-local operator. Hence, transition densities of killed Lévy processes with jumps play significant roles in the study of non-local operators with killings. However, except for a few special cases, it is impossible to find an explicit expression of the Dirichlet heat kernel. Thus, obtaining sharp two-sided estimates on the Dirichlet heat kernels for discontinuous Lévy processes is a fundamental problem both in probability theory and analysis.

The first result on this topic was done in [14]. In [14], the third-named author, jointly with Chen and Song, established sharp two-sided small time estimates on the Dirichlet heat kernel of an isotropic $\alpha$-stable process ($0 < \alpha < 2$) killed upon leaving a $C^{1,1}$ open set $D$ in $\mathbb{R}^d$. They also obtained large time estimates on the Dirichlet heat kernel when $D$ is bounded. After [14], much has been developed on the Dirichlet heat kernel estimates for discontinuous Markov processes. In [20], the authors obtained global Dirichlet heat kernel estimates for an isotropic $\alpha$-stable process ($0 < \alpha < 2$) in a half-space like or an exterior $C^{1,1}$ open set $D$ in $\mathbb{R}^d$. Then in [6], the authors succeeded in extending that result for a $\kappa$-fat open subset $D$ in $\mathbb{R}^d$, and suggested a factorization formula for the Dirichlet heat kernel. Very recently, in [21], the first- and third-named authors, jointly with Song and Vondraček, obtained a general factorization formula for the Dirichlet heat kernel in a metric measure space. We refer to [12, 15, 19] for the Dirichlet heat kernel estimates for isotropic Lévy processes with non-vanishing...

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Gaussian component, [16, 17] for relativistic stable processes (see also [22, 27]), and mixed \(\alpha\)-stable processes in [8, 18]. Note that, every aforementioned result on the Dirichlet heat kernel estimates cover neither processes with high intensity of small jumps nor processes with low intensity of small jumps, namely, whose Lévy measure has the form \(\nu(dx) = |x|^{-d-2}\ell(|x|^{-1})dx\) or \(\nu(dx) = |x|^{-d}\ell(|x|^{-1})dx\) for a function \(\ell\) slowly varying at infinity (in Karamata’s sense). The case of high intensity of small jumps was treated for subordinate Brownian motions in [30] by the third named author jointly with Mimica using the result in [36]. After that, the third-named author and Bae extended that result to a subordinate Brownian motion with non-vanishing Gaussian component in [1].

The purpose of this paper is treating the other case, low intensity of small jumps, in the study of the Dirichlet heat kernel estimates. We consider the Dirichlet heat kernel estimates on isotropic unimodal Lévy processes without Gaussian components whose Lévy measure has a radially non-increasing density which is comparable to \(|x|^{-d}\ell(|x|^{-1})\) for a function \(\ell\) satisfying weak scaling conditions at infinity with possibly non-positive lower scaling index (see Definition 1.1 for the notion of the weak scaling condition). Typical examples of such processes are geometric stable processes and iterated geometric stable processes. (See, for example, [4, p. 112] for the definitions of these processes.) We refer to [24, 28, 29] for the scale invariant version of Harnack inequality and the Green function estimates for these processes. To the authors’ best knowledge, our result (Theorems 1.5 and 1.6) is the first result on the Dirichlet heat kernel estimates for Lévy processes with low intensity of small jumps both in small time and large time. Our paper is motivated by the recent paper [25] where sharp two-sided estimates on the heat kernel in the whole space \(\mathbb{R}^d\) is established for pure jump isotropic unimodal Lévy processes (without killing) having the Lévy measure in the form \(\nu(dx) = |x|^{-d}\ell(|x|^{-1})dx\) for a bounded function \(\ell\) slowly varying at infinity. Unlike [25], in this paper, we allow the function \(\ell\) to be unbounded and not slowly varying at infinity. Hence, our result is even new for the whole space case.

In this paper, we first derive heat kernel estimates for small time and the whole space by using the results and methods from [25]. Our heat kernel estimates in \(\mathbb{R}^d\) have two forms depending on whether \(\ell\) is bounded or unbounded. If the lower scaling index of \(\ell\) is positive, then our results can be written in the form of \(c_1(p(t,0) \land t\nu(x)) \leq p(t,x) \leq c_2(p(t,0) \land t\nu(x))\), which coincides with the main result in [7]. Hereafter, \(p(t,x)\) denotes the transition density function which is also called as the heat kernel.

Next, we study behaviours of the process near the boundary of \(D\). To do this, we use the boundary Harnack principle and gradient estimates on harmonic functions for pure jump isotropic Lévy processes. These results were obtained in [23, 35], respectively, under some mild assumptions. Under a set of conditions that give the boundary Harnack principle and gradient estimates (see the condition (B)), we obtain two-sided estimates on the mean exit time from the intersection of an open ball and \(D\), and survival probability in \(D\).

Using heat kernel estimates in the whole space and boundary behaviours of the process, we establish small time two-sided Dirichlet heat kernel estimates for isotropic unimodal Lévy processes in \(C^{1,1}\) open sets (see Theorem 1.5). Even with the heat kernel estimates and the precise boundary behaviour of the mean exit time and the survival probability on hand, it is highly non-trivial to obtain Dirichlet heat kernel estimates because the lower scaling index of \(\ell\) can be 0 and the heat kernel can be unbounded.

For bounded \(C^{1,1}\) open sets in \(\mathbb{R}^d\), we also obtain large time estimates (see Theorem 1.6). Since the killed semigroup \(\{P_t^D, t > 0\}\) may not be compact operators for all \(t > 0\) even for a bounded open set \(D \subset \mathbb{R}^d\), our method is different from ones for obtaining large time Dirichlet heat kernel estimates of stable processes or mixed stable processes.

Then we obtain two-sided estimates on the Green function in a bounded \(C^{1,1}\) open subset in \(\mathbb{R}^d\) (see Theorem 1.7). This result partially extends the result in [29] where the Green function estimates on subordinate Brownian motions, whose Lévy–Khintchine exponent possibly has the lower scaling index 0, are treated (see Remark 1.8 below).
The paper ends with an explicit example on the Dirichlet heat kernel estimates and the Green function estimates for some Lévy processes including geometric stable processes and iterated geometric stable processes.

Notations: We will use the symbol ‘:=’ to denote a definition, which is read as ‘is defined to be’. For $a, b \in \mathbb{R}$, we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For two functions $f, g$ and a constant $c > 0$, the notation $f \asymp g$ for $x > c$ means that there are strictly positive constants $c_1$ and $c_2$ such that $c_1g(x) \leq f(x) \leq c_2g(x)$ for all $x > c$. We denote an open ball by $B(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$ and the diagonal set by $\text{diag} := \{(x, x) : x \in \mathbb{R}^d\}$. For an open set $D$ in $\mathbb{R}^d$, we denote $\delta_D(x) := \text{dist}(x, \partial D)$.

Upper case letters with subscripts $C_i, i = 0, 1, 2, \ldots$, and the constants $\kappa_1, \kappa_2, \alpha_1$ and $\alpha_2$ will remain the same throughout this paper. Lower case letters $c$ without subscripts denote strictly positive constants whose values are unimportant and which may change even within a line, while values of lower case letters with subscripts $c_i, i = 0, 1, 2, \ldots$, are fixed in each proof, and the labelling of these constants starts anew in each proof. The notation $c = c(a, b, \ldots)$ denotes a constant depending on $a, b, \ldots$.

1.2. Setup

To describe our results, we introduce the notions of the weak scaling conditions, almost monotonicity and some geometric properties of subsets of $\mathbb{R}^d$.

**Definition 1.1.** Let $f : (0, \infty) \to (0, \infty)$ be a given (Lebesgue) measurable function.

(i) For $\alpha_1 \in \mathbb{R}$ and $c_1 > 0$, we say that $f$ satisfies $\text{WLS}^\infty(\alpha_1, c_1)$ (respectively, $\text{WLS}^0(\alpha_1, c_1)$) if there exists $c > 0$ such that

$$f(R) \geq c \left(\frac{R}{r}\right)^{\alpha_1} \quad \text{for all } c_1 < r \leq R \quad (\text{respectively } 0 < r \leq R \leq c_1).$$

(1.1)

Similarly, for $\alpha_2 \in \mathbb{R}$ and $c_2 > 0$, we say that $f$ satisfies $\text{WUS}^\infty(\alpha_2, c_2)$ (respectively, $\text{WUS}^0(\alpha_2, c_2)$) if there exists $c > 0$ such that

$$f(R) \leq c \left(\frac{R}{r}\right)^{\alpha_2} \quad \text{for all } c_2 < r \leq R \quad (\text{respectively } 0 < r \leq R \leq c_2).$$

(1.2)

If $f$ satisfies (1.1) (respectively, (1.2)), then we call $\alpha_1$ (respectively, $\alpha_2$) the lower scaling index (respectively, the upper scaling index) of the function $f$. If $f$ satisfies both $\text{WLS}^\infty(\alpha_1, c_3)$ and $\text{WUS}^\infty(\alpha_2, c_3)$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$ and $c_3 > 0$, we say that $f$ satisfies $\text{WS}^\infty(\alpha_1, \alpha_2, c_3)$.

(ii) We say that $f$ is almost increasing if there exists $c_0 > 0$ such that

$$f(x) \asymp \sup_{y \in [c_0, x]} f(y) \quad \text{for all } x > c_0.$$

Similarly, we say that $f$ is almost decreasing if there exists $c_0 > 0$ such that

$$f(x) \asymp \inf_{y \in [c_0, x]} f(y) \quad \text{for all } x > c_0.$$

**Definition 1.2.** (i) Let $d \geq 2$. An open set $D$ in $\mathbb{R}^d$ is said to be a (uniform) $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a $C^{1,1}$ function $\Gamma = \Gamma_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\Gamma(0) = 0$, $\nabla \Gamma(0) = (0, \ldots, 0)$, $|\nabla \Gamma|_\infty \leq \Lambda$, $|\nabla \Gamma(x) - \nabla \Gamma(\tilde{w})| \leq \Lambda |\tilde{x} - \tilde{w}|$ for $\tilde{x}, \tilde{w} \in \mathbb{R}^{d-1}$ and an orthonormal coordinate system $CS_Q : y = (\tilde{y}, y_d)$ with its origin at $Q$ such that

$$B(Q, R_0) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R_0) \cap D \subset CS_Q : y_d > \Gamma(\tilde{y})\}.$$
An open set $D$ in $\mathbb{R}$ is said to be a $C^{1,1}$ open set if there exists a localization radius $R_0 > 0$ such that $D$ is a union of open intervals of length at least $R_0$ and distanced one from another at least $R_0$.

A bounded set $D$ in $\mathbb{R}^d$ is said to be of scale $(r_1, r_2)$ if there exist $x_1, x_2 \in \mathbb{R}^d$ such that $B(x_1, r_1) \subset D \subset B(x_2, r_2)$.

Let $Y = (Y_t, t \geq 0)$ be a Lévy process in $\mathbb{R}^d$ with the Lévy–Khintchine exponent $\psi$. Then,

$$E[\exp(i\langle \xi, Y_t \rangle)] = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} p(t, dx) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

where $p(t, dx)$ is the transition probability of $Y$. If $Y$ is a pure jump symmetric Lévy process with Lévy measure $\nu$, then $\psi$ is of the form

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\langle \xi, x \rangle))\nu(dx), \quad \xi \in \mathbb{R}^d,$$

where $\int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(dx) < \infty$.

A measure $\mu(dx)$ is isotropic unimodal if it is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$ with a radial and radially non-increasing density. A Lévy process $Y$ is isotropic unimodal if $p(t, dx)$ is isotropic unimodal for all $t > 0$. This is equivalent to the condition that the Lévy measure $\nu(dx)$ of $Y$ is isotropic unimodal if $Y$ is pure jump Lévy process. (See [39].)

Throughout this paper, we always assume that $Y$ is a pure jump isotropic unimodal Lévy process with the Lévy–Khintchine exponent $\psi$. With a slight abuse of notation, we will use the notations $\psi(|x|) = \psi(x)$ and $\nu(dx) = \nu(x)dx = \nu(|x|)dx$ for $x \in \mathbb{R}^d$. Then, throughout this paper, we also assume the following condition (A) holds. A smooth function $\varphi : [0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^{n-1}\varphi^{(n)}(\lambda) \geq 0$ for all $n \geq 1$ and $\lambda > 0$.

(A) The Lévy measure $\nu$ on $\mathbb{R}^d$ is infinite and there exist constants $\kappa_1, \kappa_2 > 0$ and a continuous function $\ell : (0, \infty) \to (0, \infty)$ satisfying WS$^\infty(\alpha_1, \alpha_2, 1)$ for some $-d < \alpha_1 \leq \alpha_2 < 2$ such that

$$\kappa_1 r^{-d}\ell(r^{-1}) \leq \nu(r) \leq \kappa_2 r^{-d}\ell(r^{-1}) \quad \text{for all } r > 0. \quad (1.3)$$

If $d > 1$, then we assume further that either $\alpha_1 > -1$ or $\psi(\xi) = \varphi(|\xi|^2)$ for a Bernstein function $\varphi$.

Note that since we allow the constant $\alpha_1$ to be negative, the map $r \mapsto \ell(r^{-1})$ can be increasing near zero.

Here, we enumerate other main conditions which we will assume later.

(B) $\nu(r)$ is absolutely continuous such that $r \mapsto -\nu'(r)/r$ is non-increasing on $(0, \infty)$ and there exists a constant $c_0 > 1$ such that $\nu(r) \leq c_0 \nu(r+1)$ for all $r \geq 1$.

(C) $\ell(r)$ satisfies WUS$^\infty(\gamma, 1)$ for some $\gamma < 2$.

(S-1) $\lim_{r \to \infty} \ell(r) < \infty$.

(S-2) $\limsup_{r \to \infty} \ell(r) = \infty$ and $\ell(r)$ is almost increasing.

(L-1) $\liminf_{r \to \infty} \ell(r) = 0$ and $\ell(r)$ is almost decreasing.

(L-2) $0 < \liminf_{r \to \infty} \ell(r) \leq \limsup_{r \to \infty} \ell(r) < \infty$.

(D) If $d = 1$, then $\alpha_2 < 1$ where $\alpha_2$ is the constant in (A).

Remark 1.3. Let $B = (B_t, t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ and $S = (S_t, t \geq 0)$ be a driftless subordinator independent of $B$. The process $X = (X_t : t \geq 0)$ defined by $X_t = B_{S_t}$ is called a subordinate Brownian motion (SBM). Every SBM is an isotropic unimodal Lévy process. Let $\varphi$ be the Laplace exponent of the subordinator $S$, namely,

$$E[\exp(-\lambda S_t)] = \exp(-t\varphi(\lambda)), \quad \lambda \geq 0.$$
It is known that the Laplace exponent \( \varphi \) is a Bernstein function with \( \varphi(0) = 0 \). Since \( S \) has no drift, \( \varphi \) has the representation \( \varphi(\lambda) = \int_0^\infty (1 - e^{\lambda t}) \mu(dt) \) where \( \mu \) is a measure on \((0, \infty)\), satisfying \( \int_0^\infty (1 \wedge t) \mu(dt) < \infty \), called the Lévy measure of \( \varphi \). Note that the characteristic exponent of \( X \) is \( \varphi(|\xi|^2) \).

A function \( f : (0, \infty) \to [0, \infty) \) is said to be completely monotone, if \( (-1)^n f^{(n)} \geq 0 \) on \((0, \infty)\) for every \( n \geq 0 \). A Bernstein function is said to be a complete Bernstein function, if its Lévy measure has a completely monotone density.

(i) Suppose that \( \varphi \) is a complete Bernstein function such that \( \lim_{\lambda \to \infty} \varphi(\lambda) = \infty \) and \( \varphi' \) satisfies WUS\( ^{\infty}(\delta, 1) \) for some \( \delta \in (\frac{1}{2}, 1] \). Suppose further that, if \( d = 2 \), then \( \varphi' \) satisfies WLS\( ^{\infty}(\delta_0, 1) \) for some \( \delta_0 \in (0, 2) \), and if \( d = 1 \), then \( \varphi' \) satisfies WLS\( ^{\infty}(\frac{1}{2}\delta_0, 1) \) for some \( \delta_0 \in (\frac{1}{2}, 2\delta - \frac{1}{2}) \). Then according to [29, Proposition 2.6], an SBM with the characteristic exponent \( \varphi(|\xi|^2) \) satisfies (A) with \( \ell \) such that \( \ell(r) \asymp r^2 \varphi'(r^2) \) for \( r \geq 1 \).

(ii) Let \( X \) be an SBM with the characteristic exponent \( \varphi(|\xi|^2) \) for a complete Bernstein function \( \varphi \). Then by [22, Remark 1.4] and [5, Lemma 7.4], it satisfies (B). (See also the proof of [31, Proposition 3.5(b)].)

Remark 1.4. (i) (A) implies that \( \nu(r) \) satisfies WLS\( ^{0}(-d - \alpha_2, 1) \). Therefore, under (A), for every \( R > 0 \), there exists \( c > 0 \) such that

\[
\nu(2r) \geq cr \quad \text{for all } r \in (0, R].
\]

(1.4)

On the other hand, (C) implies that \( \nu(r) \) satisfies WLS\( ^{\infty}(-d - \gamma, 1) \) for some \( \gamma < 2 \). Thus, (C) implies that for every \( R > 0 \), there exists \( c > 0 \) such that

\[
\nu(2r) \geq cr \quad \text{for all } r \in [R, \infty).
\]

(1.5)

(ii) If (A) holds with \( \alpha_1 > 0 \), then (S-2) holds. (See, [3, Section 1.5].)

1.3. **Main results**

We define for \( r > 0 \),

\[
K(r) := r^{-2} \int_0^r s \ell(s^{-1})ds, \quad L(r) := \int_r^\infty s^{-1} \ell(s^{-1})ds,
\]

\[
h(r) := K(r) + L(r).
\]

Since (A) holds, we see that

\[
K(r) \asymp r^{-2}\int_{|y| \leq r} |y|^2 \nu(y)dy \quad \text{and} \quad L(r) \asymp \int_{|y| > r} \nu(y)dy,
\]

which are the functions introduced in [37]. We also define

\[
\ell^\ast(r) := \sup_{u \in [1, r]} \ell(u) \quad \text{for } r \geq 1
\]

and denote by \( \ell^{-1} \) the right continuous inverse of \( \ell^\ast \), that is,

\[
\ell^{-1}(t) := \inf\{r \geq 1 : \ell^\ast(r) > t\} \quad \text{for } t > 0.
\]

(1.6)

Now, we are ready to state our main results. Recall that \( \delta_D(x) = \text{dist}(x, \partial D) \).

Theorem 1.5. Suppose that \( Y \) is a pure jump isotropic unimodal Lévy process satisfying (A) and (B). Let \( D \) be a \( C^{1,1} \) open set in \( \mathbb{R}^d \) with characteristics \((R_0, \Lambda)\). If \( D \) is unbounded, we further assume that (C) holds. Then, the following estimates hold.
(i) If (S-1) holds, then for every \( T > 0 \), there exist positive constants \( c_1 = c_1(d, \psi, T, R_0, \Lambda) \), \( c_2 = c_2(d, \psi, T) \) and \( c_3 = c_3(d, \psi, T, R_0, \Lambda) > 1 \) such that

\[
c_3^{-1} \left( 1 \wedge \frac{1}{tL(\delta_D(x))} \right)^{1/2} \left( 1 \wedge \frac{1}{tL(\delta_D(y))} \right)^{1/2} \nu(|x - y|) \exp(-c_1 t h(|x - y|))
\]

\[
\leq p_D(t,x,y) \leq c_3 \left( 1 \wedge \frac{1}{tL(\delta_D(x))} \right)^{1/2} \left( 1 \wedge \frac{1}{tL(\delta_D(y))} \right)^{1/2} \nu(|x - y|) \exp(-c_2 t h(|x - y|)),
\]

for all \((t,x,y) \in (0,T] \times (D \times D \setminus \text{diag})\).

(ii) If (S-2) holds, then for every \( T > 0 \) and \( \eta > 0 \), there exist positive constants \( a_0 = a_0(d, \psi) \), \( c_4 = c_4(d, \psi, T, R_0, \Lambda) \), \( c_5 = c_5(d, \psi) \) and \( c_6 = c_6(d, \psi, T, \eta, R_0, \Lambda) > 1 \) such that

\[
c_6^{-1} \left( 1 \wedge \frac{1}{tL(\delta_D(x))} \right)^{1/2} \left( 1 \wedge \frac{1}{tL(\delta_D(y))} \right)^{1/2} \nu(\theta_\eta(|x - y|, t)) \exp(-c_4 t h(\theta_\eta(|x - y|, t)))
\]

\[
\leq p_D(t,x,y)
\]

\[
\leq c_6 \left( 1 \wedge \frac{1}{tL(\delta_D(x))} \right)^{1/2} \left( 1 \wedge \frac{1}{tL(\delta_D(y))} \right)^{1/2} \nu(\theta_{a_0}(|x - y|, t)) \exp(-c_5 t h(\theta_{a_0}(|x - y|, t))),
\]

for all \((t,x,y) \in (0,T] \times D \times D \) where \( \theta_a(r,t) := r \vee |\ell^{-1}(a/t)|^{-1} \) and \( \ell^{-1} \) is defined as (1.6).

If we further assume that \( D \) is bounded, then we can obtain the large time estimates for the Dirichlet heat kernel and the Green function estimates under some mild assumptions.

**Theorem 1.6.** Suppose that \( Y \) is a pure jump isotropic unimodal Lévy process satisfying (A) and (B). Let \( D \) be a bounded \( C^{1,1} \) open set in \( \mathbb{R}^d \) with characteristics \((R_0, \Lambda)\) of scale \((r_1, r_2)\). Then, the following estimates hold.

(i) If (L-1) holds, then for every \( T > 0 \), there exist positive constants \( c_1 = c_1(d, \psi) \), \( c_2 = c_2(d, \psi) \) and \( c_3 = c_3(d, \psi, T, R_0, \Lambda, r_1, r_2) > 1 \) such that

\[
c_3^{-1} L(\delta_D(x))^{-1/2} L(\delta_D(y))^{-1/2} \nu(|x - y|) \exp(-c_1 t h(|x - y|)) + \exp(-c_2 t h(r_1/2))
\]

\[
\leq p_D(t,x,y)
\]

\[
\leq c_3 L(\delta_D(x))^{-1/2} L(\delta_D(y))^{-1/2} \left( \nu(|x - y|) \exp(-c_2 t h(|x - y|)) + \exp\left(-\frac{\kappa_1 C_5}{2} t h(r_2)\right)\right),
\]

for all \((t,x,y) \in [T, \infty) \times (D \times D \setminus \text{diag})\) where \( \kappa_1 \) and \( \kappa_2 \) are the positive constants in (A) and \( C_4 \) and \( C_5 \) are positive constants which only depend on the dimension \( d \).

(ii) If (L-2) holds, then there exist \( T_1 \geq 0 \) and \( \lambda_1 = \lambda_1(\psi, D) > 0 \) such that for every fixed \( T > T_1 \), there exists \( c_4 = c_4(d, \psi, T, R_0, \Lambda, r_1, r_2) > 1 \) such that

\[
c_4^{-1} e^{-\lambda_1 t} L(\delta_D(x))^{-1/2} L(\delta_D(y))^{-1/2} \leq p_D(t,x,y) \leq c_4 e^{-\lambda_1 t} L(\delta_D(x))^{-1/2} L(\delta_D(y))^{-1/2},
\]

for all \((t,x,y) \in [T, \infty) \times D \times D \). Moreover, we have

\[
\frac{\kappa_1 C_5}{2} h(r_2) \leq \lambda_1 \leq \kappa_2 C_4 h(r_1/2).
\]

(iii) If (S-2) holds, then the estimates in (ii) holds with \( T_1 = 0 \). Moreover, the constant \(-\lambda_1 < 0\) is the largest eigenvalue of the generator of \( Y^D \).
For a Borel subset $D \subset \mathbb{R}^d$, the Green function $G_D(x,y)$ of $Y$ in $D$ is defined by

$$G_D(x,y) := \int_0^\infty p_D(t,x,y)dt \quad \text{for } x,y \in D.$$ 

**Theorem 1.7.** Suppose that $Y$ is a pure jump isotropic unimodal Lévy process satisfying (A), (B) and (D). Let $D$ be a bounded $C^{1,1}$ open subset in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$ of scale $(r_1, r_2)$. Then, the Green function $G_D(x,y)$ of $Y$ in $D$ satisfies the following two-sided estimates: for every $x,y \in D$,

$$G_D(x,y) \asymp \left( 1 \wedge \frac{L(|x-y|)}{\sqrt{L(\delta_D(x))L(\delta_D(y))}} \right) \frac{\ell(|x-y|^{-1})}{|x-y|^dL(|x-y|^2)},$$

(1.7)

where the comparison constants depend only on $d, \psi, R_0, \Lambda$ and $r_2$.

**RemarK 1.8.** (i) One can obtain (1.7) just by integrating the estimates for $p_D(t,x,y)$ given in Theorems 1.5 and 1.6 (for example, [30, Theorem 7.3]). However, to use Theorems 1.5 and 1.6, we need conditions more than (A), (B) and (D). By adopting arguments from [29] instead of integrating the Dirichlet heat kernel, we obtained the Green function estimates in more general situations.

(ii) It is established in [29, Theorem 1.2] that for a large class of transient subordinate Brownian motions, the Green function $G_D(x,y)$ in a bounded $C^{1,1}$ open set $D$ enjoys the following sharp two-sided estimates:

$$G_D(x,y) \asymp \left( 1 \wedge \frac{\varphi(|x-y|^{-2})}{\sqrt{\varphi(\delta_D(x)^{-2})\varphi(\delta_D(y)^{-2})}} \right) \frac{\varphi'(|x-y|^{-2})}{|x-y|^{d+2}\varphi(|x-y|^{-2})^2}.$$  

(1.8)

An important novelty of this result is that it was the first explicit Green function estimates even if the lower scaling index of the Lévy–Khintchine exponent can be 0. Note that, in view of Remark 1.3 and [28, Lemma 4.1], when the lower scaling index of the Lévy–Khintchine exponent can be 0, assumptions (A-1)–(A-5) in [29, Theorem 1.2] imply the following.

1. The Lévy–Khintchine exponent $\psi(\xi) = \varphi(|\xi|^2)$ for a complete Bernstein function $\varphi$. Thus, (B) holds (see Remark 1.3(ii));
2. (A) holds with $\ell$ such that $\ell(r) \asymp r^2\varphi'(r^2)$ for $r \geq 1$, and constants $\alpha_1 \in (-d,1) \cap [-2,1)$ and $\alpha_2 < 1$. Thus, (D) holds.

Therefore, by Lemma 2.1 and (2.4), we see that Theorem 1.7 recovers (1.8). Here, we note that Theorem 1.7 does not assume the transience of the process unlike [29, Theorem 1.2].

2. Heat kernel estimates in $\mathbb{R}^d$

Recall that under (A), we have

$$K(r) = r^{-2} \int_0^r s\ell(s^{-1})ds \asymp r^{-2} \int_{|y| \leq r} |y|^2 \nu(y)dy,$$

$$L(r) = \int_r^\infty s^{-1}\ell(s^{-1})ds \asymp \int_{|y| > r} \nu(y)dy,$$

$$h(r) = K(r) + L(r) \asymp r^{-2} \int_{\mathbb{R}^d} (r^2 \wedge |y|^2) \nu(y)dy.$$
Clearly, \(L(r)\) is decreasing. Moreover, we see that \(h'(r) = -2r^{-1}K(r) \leq 0\) for all \(r > 0\) and hence \(h(r)\) is also decreasing. Since the underlying process \(Y\) is isotropic unimodal, there are a number of general properties related to these functions. (See, [7, 9 23].)

First, since \(\nu(r)\) is non-increasing, we have

\[
K(r) \geq c\nu(r)r^{-2} \int_0^r s^{d+1}ds = cr^d\nu(r) \quad \text{for all } r > 0. \tag{2.1}
\]

On the other hand, by Karamata’s Tauberian-type theorem, the opposite inequality \(K(r) \leq cr^d\nu(r)\) holds for \(0 < r \leq 1\) if and only if \(\ell(r)\) satisfies WUS\(^\infty\)(\(\gamma, 1\)) for some \(\gamma < 2\). Similarly, we have \(K(r) \leq cr^d\nu(r)\) for \(r \geq 1\) if and only if \(\ell(r)\) satisfies WUS\(^0\)(\(\gamma', 1\)) for some \(\gamma' < 2\). (See, [23, Appendix A].) In particular, (A) implies

\[
K(r) \asymp r^d\nu(r) \asymp \ell(r^{-1}) \quad \text{for } 0 < r \leq 1 \tag{2.2}
\]

and (C) implies

\[
K(r) \asymp r^d\nu(r) \asymp \ell(r^{-1}) \quad \text{for } r \geq 1. \tag{2.3}
\]

Next, by [7, (6) and (7)], there exist positive constants \(C_0\) and \(C_1\) which only depend on the dimension \(d\) and \(\kappa_1\) and \(\kappa_2\) in (1.3) such that for all \(r > 0\),

\[
C_0h(r) \leq \psi(r^{-1}) \leq C_1h(r). \tag{2.4}
\]

Under (A), we can extend this relations by including \(L(r)\) if \(r\) is small.

**Lemma 2.1.** There exists a constant \(c_1 > 0\) such that

\[
L(r) \leq h(r) \leq c_1L(r) \quad \text{for all } 0 < r \leq 1.
\]

**Proof.** From the definitions of \(L\) and \(h\), the first inequality is obvious. To prove the second inequality, it suffices to show that there exists \(c > 0\) such that \(L(r) \geq cK(r)\) for \(0 < r \leq 1\). Since (A) holds, by (1.4) and (2.2), we have \(\nu(r) \asymp \nu(2r)\) and \(K(r) \asymp r^d\nu(r)\) for \(0 < r \leq 1\). Thus, for \(0 < r \leq 1\), we get

\[
L(r) \geq c \int_r^{2r} s^{d-1}\nu(s)ds \geq cr^d\nu(r) \geq cK(r). \tag*{□}
\]

By Lemma 2.1 and (2.4), we deduce that \(L(r) \asymp \psi(r^{-1})\) for small \(r\). In view of this relation, to make some computations easier, we define \(\Phi: [0, \infty) \to [0, \infty)\) by

\[
\Phi(r) := L(r^{-1}) = \int_{r^{-1}}^{\infty} u^{-1}\ell(u^{-1})du = \int_0^r s^{-1}\ell(s)ds.
\]

We used the change of variables \(u = s^{-1}\) in the last equality.

**Lemma 2.2.** (i) \(\Phi(r)\) satisfies WUS\(^\infty\)(\(\alpha_1, \alpha_2 \lor \frac{1}{2}, 1\)).

(ii) We have that

\[
C_0\Phi(r) \leq \psi(r) \quad \text{for all } r \geq 0. \tag{2.5}
\]

Moreover, there exists a constant \(C_2 > 0\) such that

\[
C_2\Phi(r) \geq h(r^{-1}) \quad \text{for all } r \geq 1. \tag{2.6}
\]
Proof. (i) Let \( \alpha_2' = \alpha_2 \sqrt{\frac{1}{2}} \). By the change of variables and (A), we have that
\[
c_1 K^{\alpha_1} \Phi(r) \leq \Phi(kr) = \int_1^r s^{-1} \ell((ks)\ell(s)) ds + \Phi(k) \leq c_2 K^{\alpha_2'} \Phi(r) + \Phi(k) \quad \text{for all } k, r > 1. \tag{2.7}
\]
The first inequality in (2.7) shows that \( \Phi \) satisfies WLS\((\alpha_1, 1)\).

Now, we prove that \( \Phi \) satisfies WUS\((\alpha_2', 1)\). Choose any \( k > 1 \) and \( r \geq 2 \). Let \( n \) be the smallest integer satisfying \( r^n \geq k \). By applying the latter inequality in (2.7) \( n \) times, since \( \Phi(r) \) is increasing, we obtain
\[
\Phi(kr) \leq c_2 K^{\alpha_2'} \Phi(r) + \Phi(\frac{kr}{r^n}) \leq c_2 K^{\alpha_2'} (1 + r^{-\alpha_2'}) \Phi(r) + \Phi(\frac{kr}{r^n}) \\
\leq \cdots \leq c_2 K^{\alpha_2'} (1 + r^{-\alpha_2'} + \cdots + r^{-(n-1)\alpha_2'}) \Phi(r) + \Phi(\frac{kr}{r^n}) \\
\leq \left(1 - 2^{-1/2}\right)^{-1} c_2 + 1 \right) K^{\alpha_2'} \Phi(r).
\]
Besides, for any \( k > 1 \) and \( 1 < r < 2 \), since \( \Phi \) is increasing, we see from the above inequalities that \( \Phi(kr)/\Phi(r) \leq (\Phi(2)/\Phi(1)) \cdot (\Phi(k)/\Phi(2)) \leq c_3 K^{\alpha_2'} \). Hence, we get the desired result.

(ii) It follows from the definition of \( \Phi \), Lemma 2.1 and (2.4).

Let \( C_\infty(\mathbb{R}^d) \) be the set of all continuous functions which vanish at infinity. In [26], Hartman and Wintner proved sufficient conditions in terms of the Lévy exponent \( \psi \) under which the transition density \( p(t, \cdot) \) of \( Y \) is in \( C_\infty(\mathbb{R}^d) \). Then, in [34], Knopova and Schilling improve that result and they also give some necessary conditions. Using (2.5) and (2.6), we can formulate these conditions in terms of \( \Phi \). Since the underlying process \( Y \) is isotropic unimodal, these conditions determine whether \( p(t, 0) < \infty \) or \( p(t, 0) = \infty \).

**Proposition 2.3.** Let \( p(t, \cdot) \) be the transition density of \( Y \). Suppose
\[
\liminf_{r \to \infty} \frac{\Phi(r)}{\log(1 + r)} = c_1 \in [0, \infty], \quad \limsup_{r \to \infty} \frac{\Phi(r)}{\log(1 + r)} = c_2 \in [0, \infty].
\]
Then, the following are true.

(i) If \( c_1 = \infty \), then \( p(t, 0) < \infty \) for all \( t > 0 \).
(ii) If \( c_2 = 0 \), then \( p(t, 0) = \infty \) for all \( t > 0 \).
(iii) If \( 0 < c_1 \leq c_2 < \infty \), then there exist \( T_2 \geq T_1 > 0 \) such that \( p(t, 0) = \infty \) for \( 0 < t \leq T_1 \) and \( p(t, 0) < \infty \) for \( t > T_2 \).

In particular, by l’Hospital’s rule, the following are true.

(iv) If \( \liminf_{r \to \infty} \ell(r) = \infty \), then \( p(t, 0) < \infty \) for all \( t > 0 \).
(v) If \( \limsup_{r \to \infty} \ell(r) = 0 \), then \( p(t, 0) = \infty \) for all \( t > 0 \).
(vi) If \( 0 < \liminf_{r \to \infty} \ell(r) \leq \limsup_{r \to \infty} \ell(r) < \infty \), then there exist \( T_2 \geq T_1 > 0 \) such that \( p(t, 0) = \infty \) for \( 0 < t \leq T_1 \) and \( p(t, 0) < \infty \) for \( t > T_2 \).

Proof. By (2.5) and (2.6), the first two assertions follow from Part II in [26] and the third one follows from [34, Lemma 2.6].

Here, we introduce some general estimates which are established in [25]. Note that the following estimates hold no matter \( p(t, 0) < \infty \) or \( p(t, 0) = \infty \).

**Proposition 2.4** [25, Proposition 5.3]. There are constants \( b_0, c_0 > 0 \), which only depend on the dimension \( d \) and \( \kappa_2 \) in (1.3) such that for all \( (t, x) \in (0, \infty) \times \mathbb{R}^d \),
\[
p(t, x) \geq c_0 t \nu(|x|) \exp(-b_0 t h(|x|)).
\]
PROPOSITION 2.5 [25, Theorem 5.4]. There is a constant $c_1 > 0$, which only depends on the dimension $d$ and $\kappa_2$ in (1.3) such that for all $t > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$p(t,x) \leq c_1 t |x|^{-d} K(|x|).$$

The following lemma will be used several times to obtain heat kernel upper bounds for the whole space. (Cf. [25, Lemma 4.1 and Corollary 4.4].)

**Lemma 2.6.** For every $\lambda > 1$, there exists a constant $c = c(\lambda) > 0$ such that

$$\sup_{1 < k \leq \lambda} |\psi(kr) - \psi(r)| \leq c \ell(r) \quad \text{for all } r \geq 1. \quad (2.8)$$

**Proof.** Recall the condition (A). We first assume that either $d = 1$ or $\alpha_1 > -1$. For $y > 0$, set $\nu_1(y) = \nu(y)$ if $d = 1$, and

$$\nu_1(y) := \int_{\mathbb{R}^{d-1}} \nu \left( \left( y^2 + |z|^2 \right)^{1/2} \right) dz \quad \text{if } d > 2.$$

We claim that there exists a constant $c_1 > 0$ such that

$$\nu_1(y) \leq c_1 y^{-1} \ell(y^{-1}) \quad \text{for all } y \in (0,1], \quad (2.9)$$

If $d = 1$, then (2.9) follows from (1.3). Hence, we assume $\alpha_1 > -1$ and $d \geq 2$. Since $\ell$ is continuous and satisfies $WS^{\infty}(\alpha_1, \alpha_2, 1)$, it also satisfies $WS^{\infty}(\alpha_1, \alpha_2, 1/2)$. Hence, according to (1.3) and the change of the variables, we have that, for any $y \in (0,1]$,

$$\frac{1}{y^{-1} \ell(y^{-1})} \int_0^1 \nu \left( \left( y^2 + k^2 \right)^{1/2} \right) k^{d-2} dk \leq c_2 \int_0^1 \frac{k^{d-2}}{(1 + k^2)^{1/2} \ell(k)} dk \leq c_2 \int_0^1 \frac{k^{d-2}}{(1 + k^2)^{(\alpha_1 + d)/2}} dk = c_2(2 + \alpha_1) \frac{1}{1 + \alpha_1}.$$

Besides, since $\nu$ is non-increasing and a Lévy measure, we also have that for any $y \in (0,1]$,

$$\frac{1}{y^{-1} \ell(y^{-1})} \int_1^\infty \nu \left( \left( y^2 + k^2 \right)^{1/2} \right) k^{d-2} dk \leq \frac{1}{y^{-1} \ell(y^{-1})} \int_1^\infty \nu_1(k) k^{d-2} dk \leq \frac{c_4}{y^{-1} \ell(y^{-1})} \ell(1) \int_1^\infty \nu_1(\xi) d\xi \leq c_5 y^{1 + \alpha_1} \leq c_5.$$

Therefore, we obtain (2.9) with $c_1 = c_2(2 + \alpha_1) / (1 + \alpha_1) + c_5$.

Observe that for $r > 0$,

$$\psi(r) = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^d} (1 - \cos(rz_1)) \nu \left( \left( z_1^2 + |z|^2 \right)^{1/2} \right) d\tilde{z} dz_1 = 2 \int_0^\infty \left( 1 - \cos(ry) \right) \nu_1(y) dy.$$

Hence, we see that for any $1 < k \leq \lambda$ and $r \geq 1$,

$$|\psi(kr) - \psi(r)| = 2 \left| \int_0^\infty (\cos(ry) - \cos(kry)) \nu_1(y) dy \right| \leq 2r^{-1} \left| \int_0^1 (\cos(y) - \cos(ky)) \nu_1(y/r) dy + \int_1^\infty (\cos(y) - \cos(ky)) \nu_1(y/r) dy \right| \leq 2r^{-1} \int_0^1 |\cos(y) - \cos(ky)| \nu_1(y/r) dy.$$
\begin{align*}
+2r^{-1} \left| \int_1^\infty \cos(y)\nu_1(y/r)dy \right| + 2r^{-1} \left| \int_1^\infty \cos(ky)\nu_1(y/r)dy \right| \\
=: I_1 + I_2 + I_3.
\end{align*}

By Taylor expansion of the cosine function, (2.9) and the assumption that \( \ell \) satisfies \( WUS^\infty(\alpha_2,1) \) with \( \alpha_2 < 2 \), we have

\[
I_1 \leq 2\lambda^2 r^{-1} \int_0^1 y^2 \nu_1(y/r)dy \leq 2c_1 \lambda^2 \ell(r) \int_0^1 \frac{y(\ell(y/r))}{\ell(r)} dy \leq c\lambda^2 \ell(r) \int_0^1 y^{1-\alpha_2} dy = c\lambda^2 \ell(r).
\]

Next, to bound \( I_2 \) and \( I_3 \), we use a trick from the proof of [25, Theorem 3.5]. Since \( y \to \nu_1(y) \) is non-increasing, there exists a measure \( -d\nu_1 \) on \((0,\infty)\) such that \( \nu_1(y) = \int_y^\infty (-d\nu_1(z)) \) for \( y > 0 \). Then by Fubini theorem and (2.9), we obtain

\[
I_2 = 2r^{-1} \left| \int_1^\infty \int_{y/r}^\infty \cos(y)(-d\nu_1(z))dy \right| = 2r^{-1} \left| \int_{1/r}^\infty \int_1^\infty \cos(y)dy(-d\nu_1(z)) \right| \\
\leq 4r^{-1} \left| \int_{1/r}^\infty (-d\nu_1(z)) \right| = 4r^{-1} \nu_1(1/r) \leq 4c_1 \ell(r).
\]

Similarly, we also have that \( I_3 \leq 4c_1 \ell(r) \). Therefore, we get (2.8) in this case.

For the case \( \psi(\xi) = \varphi(\xi^2) \) for a Bernstein function \( \varphi \), we use [25, Lemma 5.13] and (2.2), and obtain that for any \( 1 < k \leq \lambda \) and \( r \geq 1 \),

\[
|\psi(kr) - \psi(r)| = \int_{kr}^{\infty} \varphi(u)du \leq kr\varphi(0) \leq c_6 \lambda^d \ell(\lambda r) \leq c_7 \lambda^{d+\alpha_2} \ell(r).
\]

This completes the proof. \( \square \)

Now, we first consider the case when (S-2) holds. Recall that \( \ell^*(r) := \sup_{u \in [1,r]} \ell(u) \) and \( \ell^{-1} \) is the right continuous inverse of \( \ell^* \) (see (1.6)). Since (S-2) holds, we get that \( \lim_{r \to \infty} \ell^*(r) = \infty \) and there exists a constant \( C_3 \geq 1 \) such that

\[
\ell(r) \leq \ell^*(r) \leq C_3 \ell(r) \quad \text{for all } r > 2.
\]

(2.10)

Note that in this case, by Proposition 2.3, \( p(t,0) < \infty \) for all \( t > 0 \). Here, we give the small time estimates for \( p(t,0) \) under (S-2).

**Lemma 2.7.** Assume that (S-2) holds. Then, there exists a constant \( c_1 > 0 \) such that

\[
p(t,x) \leq c_1 \left[ \ell^{-1}(a_1/t) \right]^d \exp \left( -b_1 th(\ell^{-1}(a_1/t)^{-1}) \right),
\]

for all \( 0 < t \leq t_1 \) and \( x \in \mathbb{R}^d \) where \( a_1 := 2dC_3/C_0 \), \( b_1 := C_0/(4C_2C_3) \) and \( t_1 := a_1/\ell^*(3) \).

**Proof.** Let \( a_1 := 2dC_3/C_0 \) and \( t_1 := a_1/\ell^*(3) \). Then, \( \ell^{-1}(a_1/t) \geq 3 \) for all \( t \in (0,t_1] \). By Fourier inversion theorem, (2.5), integration by parts and the change of variables \( s = \Phi(r) \), we have that for all \( t \in (0,t_1] \),

\[
p(t,x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t\varphi(\xi)} d\xi \leq c \int_0^\infty e^{-\Phi(r) \ell(r)} r^{-d-1} dr \\
\leq ct \int_0^\infty r^d e^{-C_0 \Phi(r)} \Phi'(r) dr = ct \int_0^\infty \Phi^{-1}(s) d e^{-C_0 ts} ds \\
\leq ct + ct \int_{\Phi(1)}^{\Phi(\ell^{-1}(a_1/t))} \Phi^{-1}(s) d e^{-C_0 ts} ds + ct \int_{\Phi(\ell^{-1}(a_1/t))}^\Phi e^{-C_0 ts} ds \\
=: ct + I_1 + I_2.
\]
Observe that for $\Phi(2) < v \leq u$, we have
\[
u - v = \Phi(\Phi^{-1}(u)) - \Phi(\Phi^{-1}(v)) = \int_{\Phi^{-1}(v)}^{\Phi^{-1}(u)} k^{-1} \ell(k) dk \]
\[\geq C_3^{-1} \int_{\Phi^{-1}(v)}^{\Phi^{-1}(u)} k^{-1} \ell^*(k) dk \geq C_3^{-1} \ell^*(\Phi^{-1}(v)) \log \frac{\Phi^{-1}(u)}{\Phi^{-1}(v)}.
\]
Thus, for all $\Phi(2) < v \leq u$, we have that (cf. Section 3.10 in [3])
\[
\frac{\Phi^{-1}(u)}{\Phi^{-1}(v)} \leq \exp \left( C_3 \frac{u - v}{\ell^*(\Phi^{-1}(v))} \right).
\] (2.11)

Then, by (2.11) and the definition of $a_1$, we get
\[
I_2 = ct \left[ \ell^{-1}(a_1/t) \right] d \int_0^\infty \Phi^{-1}(s) \left( \Phi^{-1}(s) \right) d \exp \left( -C_0 t s \right) ds
\]
\[\leq c \left[ \ell^{-1}(a_1/t) \right] d \int_0^\infty \Phi^{-1}(s) \left( \Phi^{-1}(s) \right) d \exp \left( -C_0 t s \right) ds
\]
\[\leq c \left[ \ell^{-1}(a_1/t) \right] d \exp \left( -C_0 t \ell^{-1}(a_1/t) \right) \int_0^\infty \Phi^{-1}(s) \left( \Phi^{-1}(s) \right) d \exp \left( -C_0 t s \right) ds
\]
\[\leq c \left[ \ell^{-1}(a_1/t) \right] d \exp \left( -C_0 t \ell^{-1}(a_1/t) \right).
\]

On the other hand, define $g(r) := r^d \exp(-C_0 t \ell (r))$ for $r \geq 1$. Then, we have
\[
g'(r) = \left( d - \frac{C_0}{2C_3} t \ell(r) \right) r^{d-1} \exp \left( -\frac{C_0}{2C_3} t \ell(r) \right).
\]
It follows that $g$ is strictly increasing on $[1, \ell^{-1}(a_1/t)]$. Therefore, we obtain
\[
I_1 \leq ct \int_{\Phi(1)}^{\Phi^{-1}(a_1/t)} \Phi^{-1}(s) ds \leq 2ct \int_{\Phi(1)}^{\Phi^{-1}(a_1/t)} \Phi^{-1}(s) ds
\]
\[\leq c \left[ \ell^{-1}(a_1/t) \right] d \int_{\Phi(1)}^{\Phi^{-1}(a_1/t)} \left( \Phi^{-1}(s) \right) d \exp \left( -\frac{C_0}{2C_3} t s \right) ds
\]
\[\leq c \left[ \ell^{-1}(a_1/t) \right] d \exp \left( -\frac{C_0}{4C_3} t \ell^{-1}(a_1/t) \right).
\]

We also have that
\[
I_1 \geq ct \int_{\Phi(1)}^{\Phi(3)} \Phi^{-1}(s) d \exp(-C_0 ts) ds \geq ct.
\]

Finally, we deduce the result from (2.6). \qed

**Lemma 2.8.** Assume that (S-2) holds. Let $a_1, b_1$ and $t_1$ be the positive constants in Lemma 2.7. Then, there exists a constant $c_1 > 0$ such that
\[
p(t,x) \leq c_1 t |x|^{-d} \ell^* (|x|^{-1}) \exp (-b_1 t h(|x|)),
\]
for all $0 < t \leq t_1$ and $x \in \mathbb{R}^d$ satisfying $[\ell^{-1}(a_1/t)^{-1} \leq |x| \leq 1/2$. 
Proof. Fix $x \in \mathbb{R}^d$ satisfying $[\ell^{-1}(a_1/t)]^{-1} \leq |x| \leq 1/2$ and let $r = |x|$. By [25, (5.4)], the mean value theorem, (2.5) and Lemma 2.6, for $0 < t \leq t_1$, we have

$$r^d p(t, x) \leq c \int_{\mathbb{R}^d} \left( e^{-t\psi(|z|/r)} - e^{-t\psi(2|z|/r)} \right) e^{-|z|^2/4} \, dz$$

$$\leq ct \int_{\mathbb{R}^d} \sup_{|z| \leq 2|z|} e^{-t\psi(y/r)} |\psi(2|z|/r) - \psi(|z|/r)| e^{-|z|^2/4} \, dz$$

$$\leq c t r^d + ct \int_r^1 e^{-C_0 t \Phi(u/r)} \ell(u/r) u^{-1} du + ct \int_1^\infty e^{-C_0 t \Phi(u/r)} \ell(u/r) e^{-u^2/4} u^{-1} du$$

$$=: c t r^d + I_1 + I_2. \tag{2.12}$$

Since $\ell$ satisfies WUS$^\infty(\alpha_2, 1)$ and $\Phi$ is increasing, we have

$$I_2 \leq c t \ell(1/r) \exp(-C_0 t \Phi(1/r)) \int_r^\infty e^{-u^2/4} u^{-1+\alpha_2} du \leq c t \ell^*(1/r) \exp(-C_0 t \Phi(1/r)).$$

On the other hand, define $m(u) := u^{d-1/2} \exp(-\frac{C_0}{4C_3} t \Phi(u/r))$ for $r > 0$. Then, for all $u \in (r, 1)$, since $1/r \leq \ell^{-1}(a_1/t)$ and $a_1 = 2dC_3/C_0$, we get

$$m'(u) \exp\left(\frac{C_0}{4C_3} t \Phi(u/r)\right) = \left(d - \frac{1}{2} - \frac{C_0}{4C_3} t \ell(u/r)\right) u^{d-3/2} \geq \left(d - \frac{C_0}{4C_3} t \ell^*(1/r)\right) u^{d-3/2} \geq 0.$$
Lemma 2.7. Else if \([\ell^{-1}(a_1/t)]^{-1} \leq |x| \leq 1/2\), then (2.15) follows from (2.1), (1.3), (2.10) and Lemma 2.8. Otherwise, if \(|x| > 1/2\), then since \(r \mapsto h(r)\) is decreasing, we see that
\[
t\frac{K(\theta_{a_1}(|x|),t)}{[\theta_{a_1}(|x|),t]^d} \exp (-b_1 t h(\theta_{a_1}(|x|),t))) \asymp \frac{t K(|x|)}{|x|^d}.
\]
Thus, we get (2.15) from Proposition 2.5.

Now, suppose \(t \in (t_1,T)\). In this case, we have that \(\ell^{-1}(a_1/t) \times 1\). Therefore, if \(|x| \geq [\ell^{-1}(a_1/t)]^{-1}\), then we get the result from Proposition 2.5. Otherwise, if \(|x| < [\ell^{-1}(a_1/t)]^{-1}\), then by the semigroup property, Lemma 2.7 and (2.2), we have that
\[
p(t,x) = \int_{\mathbb{R}^d} p(t_1/2, x-z)p(t-t_1/2, z)dz \leq p(t_1/2, 0) \int_{\mathbb{R}^d} p(t,t_1/2, z)dz \leq c + t \frac{K(\theta_{a_1}(|x|),t)}{[\theta_{a_1}(|x|),t]^d} \exp (-b_1 t h(\theta_{a_1}(|x|),t))).
\]
This completes the proof. \(\Box\)

By combining Propositions 2.4 and 2.9, we obtain the following two-sided heat kernel estimates under (S-2).

**Corollary 2.10.** Assume that (S-2) holds. For all \(T > 0\), there exists a constant \(c_1 > 1\) such that for every fixed \(\delta > 0\), we have that for all \((t,x) \in (0,T) \times \mathbb{R}^d\),
\[
c_1^{-1} t \nu(\theta_\delta(|x|,t)) \exp (-b_0 t h(\theta_\delta(|x|,t))) \leq p(t,x) \leq c_1 t \frac{K(\theta_{a_1}(|x|,t))}{[\theta_{a_1}(|x|,t)]^d} \exp (-b_1 t h(\theta_{a_1}(|x|,t))),
\]
where \(b_0\) is the constant in Proposition 2.4, and \(a_1\) and \(b_1\) are the constants in Lemma 2.7.

**Proof.** The upper bound follows from Proposition 2.9. On the other hand, since \(p(t,\cdot)\) is radially non-increasing and \(\theta_\delta(|x|,t) \geq |x|\) for all \(\delta, t > 0\) and \(x \in \mathbb{R}^d\), we deduce the lower bound from Proposition 2.4. \(\Box\)

**Remark 2.11.** If \(\ell\) satisfies WLS\(\infty(\alpha,1)\) for some \(\alpha > 0\), then \(\ell(r) \asymp \Phi(r)\) for \(r \geq 1\). (See, [3, Theorem 2.6.1].) Therefore, when \(\ell\) satisfies WLS\(\infty(\alpha,1)\) for some \(\alpha > 0\), we see that the estimate (2.16) can be expressed as follows: For every \((t,x) \in (0,T) \times \mathbb{R}^d\),
\[
c_1^{-1} \Phi^{-1}(1/t)^d \wedge t \nu(|x|) \leq p(t,x) \leq c_1 \Phi^{-1}(1/t)^d \wedge t \frac{K(|x|)}{|x|^d}.
\]
Hence, if (C) further holds, then we see from (2.2) and (2.3) that \(p(t,x) \asymp \Phi^{-1}(1/t)^d \wedge t \nu(|x|)\) for \((t,x) \in (0,T) \times \mathbb{R}^d\). In view of (2.4), (2.5) and (2.6), this coincides with the main result in [7].

In the rest of this section, we assume that (S-1) holds. Then, by Proposition 2.3, we have that \(p(t,0) = \infty\) for all sufficiently small \(t\). Recently, some general estimates for such type of heat kernels were established in [25]. Using those results, we obtain the heat kernel estimates in analogous form to (2.16).

**Proposition 2.12.** Assume that (S-1) holds. Then, there exist constants \(t_0, c_1 > 0\) such that for all \((t,x) \in (0,t_0) \times (\mathbb{R}^d \setminus \{0\})\),
\[
p(t,x) \leq c_1 t |x|^{-d} K(|x|) \exp \left(-t \psi(|x|^{-1})\right).
\]

for all 

In this section, we investigate the boundary behaviour of the process via the renewal function by Proposition 2.5. Suppose $|x| \geq 1$, we have that $e^{-\tau \psi(|x|^{-1})} \asymp 1$. Then, we get the result from Proposition 2.5.

**Corollary 2.13.** Assume that (S-1) holds. For all $T > 0$, there exist constants $c_1, c_2 > 0$ such that

$$c_1^{-1} tv(|x|) \exp (-b_0 t h(|x|)) \leq p(t, x) \leq c_1 t |x|^{-d} K(|x|) \exp (-b_2 t h(|x|)),$$

for all $(t, x) \in (0, T] \times (\mathbb{R}^d \setminus \{0\})$ where $b_0$ is the constant in Proposition 2.4.

**Proof.** By Propositions 2.4 and 2.12, (2.4) and induction, it suffices to prove the upper bound in (2.18) for $t \in (t_0, 2t_0]$ and $x \in \mathbb{R}^d \setminus \{0\}$, where $t_0$ is the constant in Proposition 2.12. If $|x| \geq 1$, then $\exp(-c t h(|x|)) \asymp 1$ for each fixed constant $c > 0$ so that the assertion holds by Proposition 2.5. Suppose $|x| < 1$. Without loss of generality, we may assume that $2b_2 \leq b_0$. Then, by the semigroup property, (2.2), the induction hypothesis, monotonicity of $p(t, \cdot)$ and Proposition 2.5, we get

$$p(t, x) = \int_{B(x, 1)} p(t/2, x - z)p(t/2, z)dz + \int_{\mathbb{R}^d \setminus B(x, 1)} p\left(\frac{b_2}{b_0} t, x - z\right) p\left(\frac{b_0 - b_2}{b_0} t, z\right)dz$$

$$\leq c_1 \int_{B(x, 1)} t^2 \nu(|x - z|) \exp\left(-\frac{b_0 b_2}{2b_0} t h(|x - z|)\right) \nu(|z|) \exp\left(-\frac{b_0 b_2}{2b_0} t h(|z|)\right)dz$$

$$+ \left(\frac{b_2}{b_0} t, 1\right) \int_{\mathbb{R}^d} p\left(\frac{b_0 - b_2}{b_0} t, z\right)dz$$

$$\leq c_2 \int_{\mathbb{R}^d} p\left(\frac{b_2}{b_0} t, x - z\right) p\left(\frac{b_2}{2b_0} t, z\right)dz + \left(\frac{b_2}{b_0} t, 1\right)$$

$$\leq (c_2 + 1)p\left(\frac{b_2}{b_0} t, x\right) \leq c_3 t |x|^{-d} K(|x|) \exp\left(-\frac{b_2}{b_0} t h(|x|)\right). \quad \square$$

3. Boundary Harnack principle with explicit decay

In this section, we investigate the boundary behaviour of the process via the renewal function $V$ of $Y$ and the tail of its Lévy measure. Throughout this section, we assume that (B) holds. For an open set $D \subset \mathbb{R}^d$, the first exit time is denoted by $\tau_D := \inf\{t > 0 : Y_t \notin D\}$. We give the probabilistic definition of a (regular) harmonic function.

**Definition 3.1.** (i) A function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be harmonic in an open set $D \subset \mathbb{R}^d$ with respect to $Y$ if for every open set $B$ whose closure is a compact subset of $D$, $\mathbb{E}_x[|u(Y_{\tau_B})|] < \infty$ and $u(x) = \mathbb{E}_x[u(Y_{\tau_B})]$ for every $x \in B$.

(ii) A function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be regular harmonic in an open set $D \subset \mathbb{R}^d$ with respect to $Y$ if $\mathbb{E}_x[|u(Y_{\tau_D})|] < \infty$ and $u(x) = \mathbb{E}_x[u(Y_{\tau_D})]$ for every $x \in D$.

Here, we provide the precise definition of the renewal function $V$ of $Y$. Let $Y^d$ be the last coordinate of $Y$, $M_t = \sup_{s \leq t} Y^d_s$ and $Z_t$ be the local time at 0 for $M_t - Y^d_t$, the last coordinate
of $Y$ reflected at the supremum. Define the ascending ladder-height process as $H_t = Y_{\mathcal{L}^{-1}}^d = M_{\mathcal{L}^{-1}}$ where $\mathcal{L}^{-1}$ is the right continuous inverse of $\mathcal{L}$. Then, the renewal function $V$ is defined as
\[
V(s) = \int_0^\infty \mathbb{P}(H_t \leq s) \, dt, \quad s \in \mathbb{R}.
\]
Since the process $Y$ is isotropic unimodal, there are several known properties for the renewal function. (See, [38, Theorem 1.2], [2, p. 74] and [8, Section 1.2].)

**Lemma 3.2.** (i) $V$ is strictly increasing, $V(s) = 0$ if $s < 0$ and $\lim_{s \to \infty} V(s) = \infty$.
(ii) $V$ is subadditive; that is,
\[
V(s + r) \leq V(s) + V(r) \quad \text{for all } s, r \in \mathbb{R}.
\]
(iii) $V$ is absolutely continuous and harmonic on $(0, \infty)$ for the process $Y^d_i$. Also, $V'$ is a positive harmonic function for $Y^d_i$ on $(0, \infty)$.

According to [9, Proposition 2.4], the relation (2.4) can be extended to include the renewal function. That is, there exist comparison constants which are only depend on the dimension $d$ and $\kappa_1$ and $\kappa_2$ in (1.3) such that $h(r) \asymp \psi(r^{-1}) \asymp [V(r)]^{-2}$ for all $r > 0$. Then, by Lemmas 2.1 and 2.2, we have that
\[
L(r) \asymp h(r) \asymp \psi(r^{-1}) \asymp \Phi(r^{-1}) \asymp [V(r)]^{-2} \quad \text{for all } 0 < r \leq 1. \tag{3.1}
\]
In particular, by (3.1) and Lemma 2.2, there are constants $c_1, c_2, c_3, c_4 > 0$ such that
\[
c_1 \left( \frac{R}{r} \right)^{\alpha_1/2} \leq \frac{V(R)}{V(r)} \leq c_2 \left( \frac{R}{r} \right)^{(\alpha_2/2) \sqrt{(1/4)}} \quad \text{for all } 0 < r \leq R \leq 1. \tag{3.2}
\]
and
\[
c_3 \left( \frac{R}{r} \right)^{\alpha_1} \leq \frac{L(r)}{L(R)} \leq c_4 \left( \frac{R}{r} \right)^{\alpha_2 \sqrt{(1/2)}} \quad \text{for all } 0 < r \leq R \leq 1. \tag{3.3}
\]

**Proposition 3.3.** The renewal function $V$ is twice-differentiable on $(0, \infty)$, and there exists $c_1 > 0$ such that
\[
|V''(r)| \leq c_1 \frac{V'(r)}{r} \wedge 1 \quad \text{and} \quad V'(r) \leq c_1 \frac{V(r)}{r} \wedge 1, \quad r > 0.
\]

**Proof.** Since (A) and (B) hold, the scale-invariant Harnack inequality holds for $Y$. (See, [23, Theorem 1.9].) Then, the results follows from [35, Theorem 1.1] and Lemma 3.2(iii). $\square$

Define $w(x) := V((x_d)^+) = x_d^+$ for $x \in \mathbb{R}^d$ and let $\mathbb{H} := \{ x = (\bar{x}, x_d) \in \mathbb{R}^d : x_d > 0 \}$ the upper half-space. Since the renewal function $V$ is harmonic on $(0, \infty)$ for $Y^d$, by the strong Markov property, $w$ is harmonic in $\mathbb{H}$ with respect to $Y$.

**Proposition 3.4.** For all $\lambda > 0$, there exists $c_1 = c_1(d, \lambda) > 0$ such that for any $r > 0$,
\[
\sup_{\{ x \in \mathbb{R}^d : 0 < x_d \leq \lambda r \}} \int_{B(x, r)^c} w(y) \nu(|x - y|) \, dy \leq c_1 V(r)^{-1}.
\]

**Proof.** See, the proof of [22, Proposition 3.2]. $\square$
Denote \( C^2_\infty(\mathbb{R}^d) \) by the set of all twice-differentiable functions in \( \mathbb{R}^d \) vanishing at infinity. We define an operator \( \mathcal{L}_Y \) as follows: for \( \varepsilon > 0 \) and \( x \in \mathbb{R}^d \),

\[
\mathcal{L}_Y^\varepsilon f(x) := \int_{B(x,\varepsilon)^c} (f(y) - f(x))\nu(|x-y|)dy,
\]

\[
\mathcal{L}_Y f(x) := \text{P.V.} \int_{\mathbb{R}^d} (f(y) - f(x))\nu(|x-y|)dy = \lim_{\varepsilon \downarrow 0} \mathcal{L}_Y^\varepsilon f(x),
\]

\[
\mathcal{D}(\mathcal{L}_Y) := \left\{ f \in C^2_\infty(\mathbb{R}^d) : \text{P.V.} \int_{\mathbb{R}^d} (f(y) - f(x))\nu(|x-y|)dy \text{ exists and is finite.} \right\}.
\]

**Theorem 3.5.** For any \( x \in \mathbb{H} \), \( \mathcal{L}_Y w(x) \) is well defined and \( \mathcal{L}_Y w(x) = 0 \).

**Proof.** By Propositions 3.3 and 3.4, using [11, Lemma 2.3, Theorem 2.11], the proof is essentially the same as the one given in [22, Theorem 3.3]. Hence, we omit it. \( \Box \)

**Lemma 3.6.** Let \( D \) be a \( C^{1,1} \) open set in \( \mathbb{R}^d \) with characteristics \( (R_0, \Lambda) \). For any \( Q \in \partial D \) and \( r > 0 \), we define

\[
h_r(y) = h_{r,Q}(y) := V(\delta_D(y))\mathbf{1}_{D \cap B(Q,r)}(y).
\]

Then, there exist \( R_1 = R_1(R_0, \Lambda, \psi, d) \in (0, (R_0 \wedge 1)/2) \) and \( c_1 = c_1(R_0, \Lambda, \psi, d) > 1 \) independent of \( Q \) such that for every \( r \in (0,R_1) \), \( \mathcal{L}_Y h_r \) is well defined in \( D \cap B(Q,r/4) \) and

\[
|\mathcal{L}_Y h_r(x)| \leq \frac{c_1}{V(r)} \leq c_1^2 L(r)^{1/2} \quad \text{for all } x \in D \cap B(Q,r/4).
\]

**Proof.** Since the case of \( d = 1 \) is easier, we only give the proof for \( d \geq 2 \). Fix \( Q \in \partial D \), \( r \in (0, (R_0 \wedge 1)/2) \) and \( x \in D \cap B(Q,r/4) \). Let \( z \in \partial D \) be the point satisfying \( \delta_D(x) = |x-z| \) and denote \( \Gamma_z \) and \( CS_z \) by the \( C^{1,1} \) function and orthonormal coordinate system determined by \( z \), respectively. (See, Definition 1.2.) Henceforth, we use the coordinate system \( CS_z \). Hence, we have \( z = 0 \), \( x = (0,x_d) \) and \( D \cap B(z,R_0) = \{ y = (\tilde{y}, y_d) \in B(0,R_0) \in CS_z : y_d > \Gamma_z(\tilde{y}) \} \). Since \( D \) is a \( C^{1,1} \) open set, it satisfies the inner and outer ball conditions. Thus, we may assume that

\[
A_1 := \{ y = (\tilde{y}, y_d) \in CS_z : |y| < R_0, y_d > \phi(\tilde{y}) \} \subset D,
\]

and

\[
A_2 := \{ y = (\tilde{y}, y_d) \in CS_z : |y| < R_0, y_d < -\phi(\tilde{y}) \} \subset D^c,
\]

where \( \phi : \mathbb{R}^{d-1} \to \mathbb{R} \) is defined by \( \phi(\tilde{y}) := 1 - \sqrt{1 - |\tilde{y}|^2} \).

Let \( E := \{ y = (\tilde{y}, y_d) : |\tilde{y}| < r/2, |y_d| < r/2 \}, E_1 := \{ y \in E : y_d > 2\phi(\tilde{y}) \} \) and \( E_2 := \{ y \in E : y_d < -2\phi(\tilde{y}) \} \). We also let \( w_z(y) := V((y_d)^+) \). By Theorem 3.5, we get \( \mathcal{L}_Y w_z(x) = 0 \). Since \( h_r(x) = w_z(x) \) and \( h_r(y) = w_z(y) = 0 \) for \( y \in E_2 \), we have

\[
|\mathcal{L}_Y h_r(x)| = |\mathcal{L}_Y (h_r - w_z)(x)|
\]

\[
= \lim_{\varepsilon \downarrow 0} \int_{|y-x| > \varepsilon} \frac{(h_r(y) - w_z(y)) - (h_r(x) - w_z(x))\nu(|x-y|)}{|x-y|}dy
\]

\[
= \lim_{\varepsilon \downarrow 0} \int_{|y-x| > \varepsilon} (h_r(y) - w_z(y))\nu(|x-y|)dy
\]

\[
\leq \limsup_{\varepsilon \downarrow 0} \left( \int_{E_1 \setminus |y-x| > \varepsilon} + \int_{E \setminus (E_1 \cup E_2) \setminus |y-x| > \varepsilon} + \int_{E \setminus E_1 \setminus |y-x| > \varepsilon} |h_r(y) - w_z(y)|\nu(|x-y|)dy \right)
\]

\[
=: I_1 + I_2 + I_3.
\]

(3.4)
First, since $|h_r(y)| \leq V(r)$, using Lemma 2.1, (3.1), (3.2) and Proposition 3.4, we have

$$I_3 \leq \int_{B(x,r/2)} (|h_r(y)| + |w_z(y)|)\nu(|x - y)|dy \leq cV(r)L(r/2) + cV(r)^{-1} \leq cV(r)^{-1}.$$

Next, we note that for $y \in E \setminus (E_1 \cup E_2)$,

$$\delta_D(y) \leq \delta_{A_2}(y) \leq |y_d + \phi(\bar{y})| \leq 3\phi(\bar{y}) \leq 3|\bar{y}|^2 \leq 3|\bar{y}|,$$

and hence by subadditivity of $V$, we obtain

$$|h_r(y)| + |w_z(y)| \leq V(3|\bar{y}|) + V(2|\bar{y}|) \leq 5V(|\bar{y}|).$$

Since $1 - \sqrt{1 - l^2} \leq l^2$ for $0 \leq l < 1$, we have for $0 < s < 1$,

$$m_{d-1}\{y \in E \setminus (E_1 \cup E_2) : |\bar{y}| = s\} \leq m_{d-1}\{y : |\bar{y}| = s, -2|\bar{y}|^2 \leq y_d \leq 2|\bar{y}|^2\} \leq cs^d,$$

where $m_{d-1}(dx)$ is the $(d - 1)$-dimensional Lebesgue measure. From these observations, using (1.4), (2.1), the definitions of $K$ and $h$, (3.1) and (3.2), since $\alpha_2 < 2$, we get

$$I_2 \leq c \int_0^r \int_{|\bar{y}| = s, y \in E \setminus (E_1 \cup E_2)} m_{d-1}(dy)V(s)\nu(s)ds \leq c \int_0^r V(s)\nu(s)s^d ds \leq c \int_0^r \frac{V(s)}{s} \frac{(\alpha_2/2)^{(1/4)}}{s} ds \leq cV(r)^{-1}.$$

Lastly, to estimate $I_1$, we first claim that

$$\delta_D(y) \approx y_d, \quad |\delta_D - y_d| \leq 2|\bar{y}|^2 \quad \text{for all } y \in E_1. \quad (3.5)$$

Indeed, for any $y \in E_1$, if $0 < y_d \leq \delta_D(y)$, then we have

$$\delta_D(y) - y_d \leq \delta_{A_2}(y) - y_d \leq \phi(\bar{y}) \leq (y_d/2) \wedge (2|\bar{y}|^2).$$

Otherwise, if $\delta_D(y) < y_d$, then we have

$$y_d - \delta_D(y) \leq y_d - \delta_{A_1}(y) = y_d - 1 + \sqrt{|\bar{y}|^2 + (1 - y_d)^2} = \frac{|\bar{y}|^2}{(1 - y_d) + \sqrt{|\bar{y}|^2 + (1 - y_d)^2}}.$$

Hence, since $|\bar{y}|, |y_d| < r/2 < 1/4$, we get $y_d - \delta_D(y) \leq (2/3)|\bar{y}|^2 \leq (4/3)\phi(\bar{y}) < (2/3)y_d$. Therefore, (3.5) holds.

Recall that by Lemma 3.2(iii), $V'$ is a harmonic function for $Y^d$ on $(0, \infty)$. Since the scale-invariant Harnack inequality holds for $Y$ (see, [23, Theorem 1.9]), by (3.5), we deduce that for every $y \in E_1$,

$$|h_r(y) - w_z(y)| \leq \left(\sup_{\delta_D(y) \wedge y_d \leq l \leq \delta_D(y) \wedge y_d} V'(l)\right)|\delta_D(y) - y_d| \leq cV'(y_d)|\bar{y}|^2,$$

for some constant $c > 0$ independent of choice of $Q, r$ and $x$. Hence, we obtain

$$I_1 \leq c \int_0^r \int_{2\phi(k)}^k V'(y_d)\nu\left(\sqrt{k^2 + (y_d - x_d)^2}\right)k^d dy_d dk + c \int_0^r \int_k^r V'(y_d)\nu\left(\sqrt{k^2 + (y_d - x_d)^2}\right)k^d dy_d dk =: I_{1,1} + I_{1,2}.$$

By the monotonicity of $\nu$, (2.1), the definition of $h$, (3.1) and (3.2), since $\alpha_2 < 2$, we have

$$I_{1,1} \leq c \int_0^r \int_0^k V'(y_d)dy_d\nu(k)k^d dk \leq c \int_0^r V(k)h(k)dk \leq cV(r)^{-1} \int_0^r \frac{V(r)}{V(k)} dk \leq cV(r)^{-1}.$$
Besides, set $\rho := 4^{-1}(2 - \alpha_2) \land 1 \in (0, 1/4]$. By Proposition 3.3, (2.1), the definition of $h$, (3.1) and (3.2), since $\sqrt{a^2 + b^2} \geq (a + b)/\sqrt{2}$ for all $a, b \geq 0$, we have that

$$I_{1,2} \leq c \int_0^r \int_k^r V(y_d)^{k - \rho} \frac{1}{y_d^{1 - \rho}} V(u) (k + |y_d - x_d|)^{-\rho} dy_d dk,$$

$$\leq c \int_0^r \int_k^r \left( \sup_{u \in [y_d,1)} V(u) \right) (k + |y_d - x_d|)^{-\rho} dy_d dk,$$

By (3.2), since $1 - \rho > (\alpha_2/2) \land (1/4)$, we see that for all $u \in [y_d,1)$,

$$V(u) \leq cV(y_d) \left( \frac{u}{y_d} \right)^{-(1 - \rho) + (\alpha_2/2) \land (1/4)} \leq cV(y_d) y_d^{1 - \rho}.$$

Since $s \mapsto \sup_{u \in [y,1)} V(u)^{u - \rho}$ and $s \mapsto s^{-\rho} V(s)^{-2}$ are non-increasing, by (3.6), Lemma 3.7, (3.7) and (3.2), we obtain that

$$I_{1,2} \leq c \int_0^{3r/2} \int_k^r V(u) s^{1 - \rho} ds \frac{u^{-\rho}}{V(u)^2} du \leq c \int_0^{3r/2} \int_k^r ds \frac{s^{-\rho}}{V(u)} du \leq cV(r)^{-1} \int_0^{3r/2} \frac{V(r)}{V(u)} du \leq cV(r)^{-1}.$$

Finally, we get from (3.4) that $|\mathcal{L}_Y h_r(x)| \leq cV(r)^{-1}$. By (3.1), this finishes the proof. \hfill \Box

For $l > 0$, we define $D_{\text{int}}(l) := \{ y \in D : \delta_D(y) > l \}$.

**Lemma 3.7.** Let $D$ be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$ and $R_1$ be the constant in Lemma 3.6. Then, there exist constants $R_2 = R_2(R_0, \Lambda, \psi, d) \in (0, R_1/16]$ and $c_1 = c_1(R_0, \Lambda, \psi, d) > 1$ such that for every $r \in (0, R_2]$ and $x \in D$ with $\delta_D(x) < r/2$,

$$\frac{c_1^{-1}}{L(\delta_D(x))^{1/2} L(r)^{1/2}} \leq \mathbb{E}_{x}[\tau_{D \cap B(z,r)}] \leq \frac{c_1}{L(\delta_D(x))^{1/2} L(r)^{1/2}},$$

and

$$\mathbb{P}_x(Y_{\tau_{D \cap B(z,r)}} \in D_{\text{int}}(r/4)) \geq c_1^{-1} \left( \frac{L(r)}{L(\delta_D(x))} \right)^{1/2},$$

where $z \in \partial D$ is the point satisfying $\delta_D(x) = |x - z|$.

**Proof.** Let $R_1$ be the constant in Lemma 3.6. Fix $r \in (0, R_2]$ and $x \in D$ with $\delta_D(x) < r/2$ where the constant $R_2 \in (0, R_1/16]$ will be selected later. Let $z \in \partial D$ be the point satisfying $\delta_D(x) = |x - z|$. As in Lemma 3.6, we denote by $\Gamma_z : \mathbb{R}^{d-1} \to \mathbb{R}$ and $CS_z$ for a $C^{1,1}$ function and coordinate system with respect to $z$, respectively, and hereinafter we use the coordinate system $CS_z$.

Denote by $U(s) := D \cap B(0, s)$ for $s > 0$. Then, we define

$$u(y) = V(\delta_D(y)) 1_{U(R_1)}(y).$$

Using Dynkin’s formula and approximation argument, (see, [30, Proposition 4.7],) by Lemma 3.6, there exists a positive constant $a$ independent of choice of $R_2$ and $x$ such that

$$\mathbb{E}_x[u(Y_{\tau_W})] - aL(R_1)^{1/2} \mathbb{E}_x[\tau_W] \leq u(x) \leq \mathbb{E}_x[u(Y_{\tau_W})] + aL(R_1)^{1/2} \mathbb{E}_x[\tau_W],$$

for every open subset $W \subset U(R_1/8)$. 

Let
\[ C_1 := \{(\tilde{y}, y_d) : 2\Lambda|\tilde{y}| < y_d, 0 < |y| < R_0\} \quad \text{and} \quad C_2 := \{(\tilde{y}, y_d) : 4\Lambda|\tilde{y}| < y_d, 0 < |y| < R_0\}. \]

We claim that \( C_2 \subset C_1 \subset D \). Indeed, the first inclusion is obvious. Moreover, for all \( y \in C_1 \), \( y_d - \Gamma(z(\tilde{y})) \geq y_d - \Lambda|\tilde{y}| \geq y_d/2 > 0 \). Hence, the second inclusion holds.

Observe that for \( 0 < s \leq R_1 \) and \( y \in C_2 \cap \partial U(s) \), we have
\[ s \geq \delta_D(y) \geq \delta_{C_1}(y) \geq c_0 s, \quad (3.11) \]
for some constant \( c_0 \) which only depends on \( \Lambda \). By the Lévy system, (3.11), integration by parts, Lemma 2.1, (3.1), (3.3) and monotonicity of \( V \), for \( 0 < 4s < R_1 \),
\[ E_x [u(Y_{\tau_U(s)})] \geq E_x [u(Y_{\tau_U(s)}) : Y_{\tau_U(s)} \in C_2 \cap (U(R_1) \setminus U(2s))] \]
\[ = E_x \left[ \int_0^{\tau_U(s)} \int_{C_2 \cap (U(R_1) \setminus U(2s))} V(\delta_D(y))\nu(|Y_k - y|)dydk \right] \]
\[ \geq c E_x [T_U(s)] \int_{2s}^{R_1} V(k)\nu(k)k^{d-1}dk = c_1 E_x [T_U(s)] \int_{2s}^{R_1} (-L'(k))V(k)dk \]
\[ = c_1 E_x [T_U(s)] \left( L(2s)V(2s) - L(R_1)V(R_1) + \int_{2s}^{R_1} L(k)V'(k)dk \right) \]
\[ \geq c_1 E_x [T_U(s)] \left( c_2 L(s)^{1/2} - c_3 L(R_1)^{1/2} \right), \quad (3.12) \]
for some constants \( c_1, c_2, c_3 > 0 \) independent of \( s \). Moreover, by the similar argument, we also have that
\[ P_x \left( Y_{\tau_U(r)} \in D_{\text{int}}(r/4) \right) \geq E_x \left[ \int_0^{\tau_U(s)} \int_{C_2 \cap (U(R_1) \setminus U(2r))} \nu(|Y_k - y|)dydk \right] \]
\[ \geq c E_x [T_U(r)] \int_{2r}^{R_1} \nu(k)k^{d-1}dk = c_1 E_x [T_U(r)] \int_{2r}^{R_1} (-L'(k))dk \]
\[ \geq c_1 E_x [T_U(r)] (c_4 L(r) - c_5 L(R_1)) \quad (3.13) \]
and
\[ E_x [u(Y_{U(r)}) : Y_{U(r)} \in D_{\text{int}}(2r)] \leq E_x \left[ \int_0^{\tau_U(s)} \int_{U(R_1) \setminus U(2r)} V(\delta_D(y))\nu(|Y_k - y|)dydk \right] \]
\[ \leq c E_x [T_U(r)] \int_{2r}^{R_1} V(k)\nu(k/2)k^{d-1}dk \leq c E_x [T_U(r)] \int_{2r}^{R_1} L(k)^{-1/2}(-L'(k))dk \]
\[ \leq c E_x [T_U(r)] L(r)^{1/2} \leq c E_x [T_U(r)] V(r)^{-1}. \quad (3.14) \]

We used (1.4) in the third inequality.

For selected constants \( a, c_1, c_2, c_3, c_4 \) and \( c_5 \) in (3.10), (3.12) and (3.13), we set
\[ R_2 = V^{-1}\left( \frac{c_1 c_2}{2(a + c_1 c_3)} V(R_1) \right) \land V^{-1}\left( \frac{c_4}{2c_5} V(R_1) \right) \land \frac{R_1}{4}. \]
Then, by (3.10) and (3.12), we get
\[
L(\delta_D(x))^{-1/2} \asymp V(\delta_D(x)) = u(x) \geq \mathbb{E}_x[u(Y_{\tau_U(r)})] - aL(R_1)^{1/2}\mathbb{E}_x[\tau_U(r)] \\
\geq \left(c_1c_2L(r)^{1/2} - (c_1c_3 + a)L(R_1)^{1/2}\right)\mathbb{E}_x[\tau_U(r)] \geq 2^{-1}c_1c_2L(r)^{1/2}\mathbb{E}_x[\tau_U(r)].
\]
This proves the upper bound of (3.8).

On the other hand, by (3.13), we get
\[
\mathbb{P}_x(\tau_{U(r)} \in D_{int}(r/4)) \geq 2^{-1}c_1c_2\mathbb{E}_x[\tau_U(r)]L(r).
\]
By [9, Lemma 2.1] and (3.1), there exists $c_6 > 0$ such that
\[
\mathbb{P}_x(\tau_{U(r)} \in D) \leq c_6\mathbb{E}_x[\tau_U(r)]V(r)^{-2}.
\]
Then, by (3.10), (3.14), (3.16), and the monotonicity of $V$, we get
\[
V(\delta_D(x)) \leq \mathbb{E}_x[u(Y_{U(r)}): Y_{U(r)} \in \mathcal{D}(2r)] + \mathbb{E}_x[u(Y_{U(r)}): Y_{U(r)} \in D \setminus \mathcal{D}(2r)] + c\mathbb{E}_x[\tau_U(r)] \\
\leq c\mathbb{E}_x[\tau_U(r)]V(r)^{-1} + cV(2r\sqrt{1 + \Lambda^2})\mathbb{P}_x(Y_{U(r)} \in D \setminus \mathcal{D}(2r)) + c\mathbb{E}_x[\tau_U(r)] \\
\leq c\mathbb{E}_x[\tau_U(r)]\left(V(r)^{-1} + V(2r\sqrt{1 + \Lambda^2})V(r)^{-2} + 1\right) \leq c\mathbb{E}_x[\tau_U(r)]V(r)^{-1}.
\]
This proves the lower bound of (3.8) in view of (3.1). Finally, we get (3.9) from (3.15). \qed

4. Estimates of survival probability

In this section, we obtain two-sided estimates for the survival probability $\mathbb{P}_x(\tau_D > t)$ which play a crucial role in factorization of the Dirichlet heat kernel. We first state the general two-sided estimates for the survival probability in balls which are recently established in [25].

**Proposition 4.1** [25, Proposition 5.2]. There exist positive constants $c_1, c_2, C_4$ and $C_5$ which only depend on the dimension $d$ such that for all $t, r > 0$,
\[
c_1 \exp(-\kappa_2 C_4 th(r)) \leq \mathbb{P}_x(\tau_B(x,r) > t) \leq \sup_{z \in B(x,r)} \mathbb{P}_z(\tau_B(x,r) > t) \leq c_2 \exp(-\kappa_1 C_5 th(r)),
\]
where $\kappa_1$ and $\kappa_2$ are constants in (A). As a consequence, for all $r > 0$,
\[
\mathbb{E}_x[\tau_B(x,r)] = \int_0^\infty \mathbb{P}_x(\tau_B(x,r) > s)ds \asymp h(r)^{-1}.
\]

Note that the last inequality in (4.1), and (4.2) were obtained for a large class of Feller processes in $\mathbb{R}^d$. See [10, Corollaries 5.3, 5.8 and Theorem 5.9].

In the rest of this section, we assume that (B) holds. Fix $T > 0$ and $D$ a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$. Let $R_2$ be the constant in Lemma 3.7. For $t \in (0, T]$, we set
\[
r_t = r_t(T, R_0, \Lambda, \psi, d) := \frac{L^{-1}(1/t)}{L^{-1}(1/T)}R_2.
\]
For $x \in D$ with $\delta_D(x) < r_t/2$, we define an open neighborhood $U(x,t)$ of $x$ and an open ball $W(x,t) \subset D \setminus U(x,t)$ as follows:

Find $z_x \in \partial D$ satisfying $\delta_D(x) = |x - z_x|$ and let $v_x := z_x + 2r_t(x - z_x)/|x - z_x|$. Then, we have $\delta_D(v_x) \geq r_t/\sqrt{1 + \Lambda^2}$. We define
\[
U(x,t) := D \cap B(z_x, r_t) \quad \text{and} \quad W(x,t) := B\left(v_x, \frac{r_t}{2\sqrt{1 + \Lambda^2}}\right) \subset D.
\]
Note that by the construction, we have that for all \( u \in U(x,t) \) and \( w \in W(x,t) \),
\[
|u - w| \geq |z_x - v_x| - |u - z_x| - |v_x - w| \geq 2r_t - r_t - r_t/2 \geq r_t/2
\]
and
\[
|u - w| \leq |z_x - v_x| + |u - z_x| + |v_x - w| \leq 4r_t.
\]
It follows that
\[
|u - w| \asymp r_t \quad \text{for all } u \in U(x,t), \quad w \in W(x,t).
\] (4.4)

**Proposition 4.2.** Let \( D \) be a \( C^{1,1} \) open set in \( \mathbb{R}^d \) with characteristics \((R_0, \Lambda)\). Let \( r_t \) and \( U(x,t) \) be defined as in just before the Proposition. For all \( T > 0 \) and \( M \geq 1 \), we have that for every \( t \in (0, T) \) and \( x \in D \) with \( \delta_D(x) < r_t/2 \),
\[
\mathbb{P}_x(\tau_D > t) \asymp \mathbb{P}_x(\tau_D > Mt) \asymp \mathbb{P}_x(\tau_{U(x,t)} \in D) \asymp t^{-1} \mathbb{E}_x[\tau_{U(x,t)}] \asymp (tL(\delta_D(x)))^{-1/2},
\]
where the comparison constants depend only on \( T, M, \psi, R_0, \Lambda \) and \( d \).

**Proof.** Recall that \( z_x \in \partial D \) is the point satisfying \( \delta_D(x) = |x - z_x| \). Let
\[
o_x = z_x + \frac{r_t(x - z_x)}{2|x - z_x|} \in D.
\]
Indeed, we have \( r_t/(2\sqrt{1 + \Lambda^2}) \leq \delta_D(o_x) \leq r_t/2 \). By (A) and (B), we see that assumptions in [23, Theorem 1.9] hold and hence by that theorem, the (scale-invariant) boundary Harnack principle holds. Therefore, we get
\[
\mathbb{P}_x(\tau_{U(x,t)} \in D) \leq c\mathbb{P}_x(\tau_{U(x,t)} \in W(x,t)) \mathbb{E}_{\delta_{o_x}}(\tau_{U(x,t)} \in W(x,t)) \mathbb{P}_{\alpha_x}(\tau_{U(x,t)} \in D) \leq c\mathbb{P}_x(\tau_{U(x,t)} \in W(x,t)) \mathbb{E}_{\delta_{o_x}}(\tau_{U(x,t)} \in W(x,t)),
\] (4.5)
where \( W(x,t) \) is the subset of \( D \) defined as in just before the Proposition. By the Lévy system, (1.4), (4.4) and Lemma 3.7, we have
\[
\mathbb{P}_x(\tau_{U(x,t)} \in W(x,t)) = \mathbb{E}_x \left[ \int_0^{\tau_{U(x,t)}} \int_{W(x,t)} \nu(y, w) dw ds \right]
\]
\[
\asymp \mathbb{E}_x[\tau_{U(x,t)}] \nu(r_t) r_t^d \asymp L(r_t)^{-1/2} L(\delta_D(x))^{-1/2} \nu(r_t) r_t^d.
\]
Similarly, we also have that \( \mathbb{E}_{\delta_{o_x}}(\tau_{U(x,t)} \in W(x,t)) \asymp \mathbb{E}_{\delta_{o_x}}[\tau_{U(x,t)}] \nu(r_t) r_t^d \asymp L(r_t)^{-1/2} \nu(r_t) r_t^d \).

Then, by the strong Markov property, Chebyshev’s inequality, (4.5) and Lemma 3.7, since \( L(r_t) \asymp t^{-1} \), we obtain
\[
\mathbb{P}_x(\tau_D > t) \leq \mathbb{P}_x(\tau_D > t) \leq \mathbb{P}_x(\tau_{U(x,t)} > t) \leq t^{-1} \mathbb{E}_x[\tau_{U(x,t)}] + cL(r_t)^{1/2} L(\delta_D(x))^{-1/2}
\]
\[
\leq t^{-1} L(r_t)^{-1/2} L(\delta_D(x))^{-1/2} + cL(r_t)^{1/2} L(\delta_D(x))^{-1/2} \leq ct^{-1/2} L(\delta_D(x))^{-1/2}.
\]

On the other hand, for any \( a > 0 \), by the strong Markov property, (4.1), (3.1), Lemma 3.7 and Chebyshev’s inequality,
\[
\mathbb{P}_x(\tau_D > at)
\]
\[
\geq \mathbb{P}_x(\tau_{U(x,t)} < at, \tau_{U(x,t)} \in D_{\text{int}}(r_t/4), |Y_{U(x,t)} - Y_{U(x,t)} + s| \leq r_t/4 \text{ for all } 0 < s < at)
\]
\[
\geq \mathbb{P}_x(\tau_{U(x,t)} < at, \tau_{U(x,t)} \in D_{\text{int}}(r_t/4)) \mathbb{P}_0(\tau_{B(0, r_t/4)} > at)
\]
\[
\geq c_1 \left( \mathbb{P}_x(\tau_{U(x,t)} \in D_{\text{int}}(r_t/4)) - \mathbb{P}_x(\tau_{U(x,t)} \geq at) \right)
\]
\[
\geq c_1 \left( ct^{-1} \mathbb{E}_x[\tau_{U(x,t)}] - a^{-1} t^{-1} \mathbb{E}_x[\tau_{U(x,t)}] \right).
\]
Take $a = (2c_2^{-1}) \vee M$. By Lemma 3.7 and the fourth line in the above inequalities, we obtain
\[
\mathbb{P}_x(\tau_D > Mt) \geq \mathbb{P}_x(\tau_D > at) \geq c_1^{-1} \frac{1}{2} \mathbb{P}_x(Y_{\tau_D(x,t)} \in D_{\text{int}}(r_t/4)) \geq ct^{-1/2}L(\delta_D(x))^{-1/2}.
\]
This completes the proof. \[\square\]

**Corollary 4.3.** Let $D$ be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$. For all $T > 0$, there exists a constant $c_1 = c_1(d, T, \psi, R_0, \Lambda) > 1$ such that for every $t \in (0, T]$ and $x \in D$,
\[
c_1^{-1} \left(1 \wedge \frac{1}{t \wedge 2}L(\delta_D(x))\right)^{1/2} \leq \mathbb{P}_x(\tau_D > t) \leq c_1 \left(1 \wedge \frac{1}{t \wedge 2}L(\delta_D(x))\right)^{1/2}.
\]

**Proof.** We use the same notations as those in Proposition 4.2. If $\delta_D(x) < r_t/2$, then the result follows from Proposition 4.2. If $\delta_D(x) \geq r_t/2$, then $tL(\delta_D(x)) \leq tL(r_t/2) \leq c$. Also, by (4.1) and (3.1), we get $1 \geq \mathbb{P}_x(\tau_D > t) \geq \mathbb{P}_x(\tau_B(x, r_t/2) > t) \geq c$. \[\square\]

**Corollary 4.4.** Let $D$ be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$ of scale $(r_1, r_2)$. Then, there exists constants $c_1 = c_1(R_0, \Lambda, \psi, d) > 0$ such that for all $t > 0$ and $x \in D$,
\[
c_1^{-1} \left(1 \wedge \frac{1}{(t \wedge 2) L(\delta_D(x))}\right)^{1/2} \exp(-\kappa_2 C_4 t h(r_1/2)) \leq \mathbb{P}_x(\tau_D > t) \leq c_1 \left(1 \wedge \frac{1}{(t \wedge 2) L(\delta_D(x))}\right)^{1/2} \exp(-\kappa_1 C_5 t h(r_2)),
\]
where $\kappa_1, \kappa_2$ are constants in (A) and $C_4, C_5$ are constants in (4.1).

**Proof.** Fix $(t, x) \in (0, \infty) \times D$. If $t \leq 2$, then the assertion follows from Corollary 4.3. Hence, we assume that $t > 2$. Let $x_1, x_2 \in \mathbb{R}^d$ be the points satisfying $B(x_1, r_1) \subset D \subset B(x_2, r_2)$. By the semigroup property, (4.1) and Corollary 4.3, we get
\[
\mathbb{P}_x(\tau_D > t) = \int_D p_D(t, x, y) dy \leq \int_D \int_D p_D(1, x, z)p_B(x_2, r_2)(t-1, z, y) dz dy
\]
\[
\leq \mathbb{P}_x(\tau_D > 1) \sup_{z \in D} \mathbb{P}_z(\tau_B(x_2, r_2) > t - 1) \leq \frac{c}{L(\delta_D(x))^{1/2}} \exp(-\kappa_1 C_5 t h(r_2)).
\]
To prove the lower bound, we first assume that $\delta_D(x) < R_2/2$ where $R_2$ is the constant in Lemma 3.7. Without loss of generality, we may assume that $R_2 \leq r_1/2$. Let $z \in \partial D$ be the point satisfying $\delta_D(x) = |x - z|$ and $\theta$ be the shift operator defined as $Y_t \circ \theta_s = Y_{s+t}$. Then, by the strong Markov property, (3.9), the Lévy system and (4.1), we hav
\[
\mathbb{P}_x(\tau_D > t)
\]
\[
\geq \mathbb{E}_x\left[\mathbb{P}_{Y_{\tau_D}(x,R_2)} \in D_{\text{int}}(R_2/4), \ Y_{B(y, R_2/4)} \circ \theta_{\tau_D(x,R_2)} \in B(x_1, r_1/2), \ \tau_D \circ \theta_{B(y, R_2/4)} \circ \theta_{\tau_D(x,R_2)} > t\right]
\]
\[
\geq \frac{c L(R_2)^{1/2}}{L(\delta_D(x))^{1/2}} \inf_{w \in D_{\text{int}}(R_2/4)} \mathbb{P}_w\left(Y_{B(w, R_2/4)} \in B(x_1, r_1/2)\right) \inf_{y \in B(x_1, r_1/2)} \mathbb{P}_y(\tau_B(x_1, r_1) > t)
\]
\[
\geq \frac{c}{L(\delta_D(x))^{1/2}} \exp(-\kappa_2 C_4 t h(r_1/2)).
\]
Indeed, on $\{Y_{\tau_D(x,R_2)} \in D_{\text{int}}(R_2/4)\}$, we have $B(Y_{\tau_D(x,R_2)}, R_2/4) \subset D$. Also, since $R_2 \leq r_1/2$, we can always find $A \subset B(x_1, r_1/2) \setminus B(Y_{\tau_D(x,R_2)}, R_2/2)$ such that $|A| \geq c_1 > 0$ for
some constant $c_1 > 0$. Then, by the Lévy system and \((4.2)\), we obtain
\[
\mathbb{P}_x(Y_{B(t \mathbb{1})} \circ \theta_{B(t \mathbb{1})} \in B(x, r_1(2)) \bigg) \geq \mathbb{E}_0 \left[ \int_{0}^{\tau_{B(t \mathbb{1})} \circ \theta_{B(t \mathbb{1})}} \nu(|Y_s - y|)dy \right] \geq c > 0.
\]
Similarly, if $\delta_D(x) \geq R_2/2$, then we have
\[
\mathbb{P}_x(\tau_D > t) \geq \mathbb{E}_x[Y_{B(t \mathbb{1})} \circ \theta_{B(t \mathbb{1})} \in B(x, r_1(2)), \tau_D \circ \theta_{B(t \mathbb{1})} > t] \geq c \inf_{y \in B(x, r_1/2)} \mathbb{P}_y(\tau_{B(t \mathbb{1})} > t) \geq c \exp(-\kappa_2 C_4 \theta_D(r_1/2)).
\]

5. Small time Dirichlet heat kernel estimates in $C^{1,1}$ open set

In this section, we provide the proof of Theorem 1.5. Let $T > 0$ be a fixed constant and $D$ be a fixed $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$. We assume that (B) holds. If $D$ is unbounded, then we further assume that (C) holds. Then, by (1.4) and (1.5), we have
\[
\nu(|x - y|) \sim \nu(2|x - y|) \quad \text{for all } x, y \in D.
\]
By (2.2), (2.3), Corollary 2.10 and Corollary 2.13, we have the following heat kernel estimates for small $t$. Let $b_0$ be the constant in Proposition 2.4.

(1) If (S-1) holds, then there exist constants $c_1 > 1$ and $b_2 > 0$ such that
\[
c_1^{-1} t^\nu(|x|) \exp(-b_2 t h(|x|)) \leq p(t, x) \leq c_1 t^\nu(|x|) \exp(-b_2 t h(|x|)),
\]
for all $(t, x) \in (0, T) \times (D \setminus \{0\})$.

(2) If (S-2) holds, then there exist a constant $c_2 > 1$ such that
\[
c_2^{-1} t^\nu(\theta_{a_1}(|x|, t)) \exp(-b_2 t h(\theta_{a_1}(|x|, t))) \leq p(t, x) \leq c_2 t^\nu(\theta_{a_1}(|x|, t)) \exp(-b_1 t h(\theta_{a_1}(|x|, t))),
\]
for all $(t, x) \in (0, T) \times D$ and $\eta > 0$ where $a_1$ and $b_1$ are the constants in Lemma 2.7, and $\theta_{a_1}(r, t) = r \vee \lfloor t^{-1}(a/t) \rfloor^\nu$ is defined as (2.14).

Before proving Theorem 1.5, we obtain a lower bound of $p_D(t, x, y)$ without (S-1) and (S-2). This result will be used later to obtain Green function estimates.

**Proposition 5.1.** For every $T > 0$, there exist positive constants $c_1 = c_1(d, \psi, T, R_0, \Lambda)$ and $c_2 = c_2(d, \psi, T, R_0, \Lambda)$ such that for all $(t, x, y) \in (0, T) \times (D \times D \setminus \text{diag})$,
\[
p_D(t, x, y) \geq c_2 \left(1 \wedge \frac{1}{t L(\delta_D(x))} \right)^{1/2} \left(1 \wedge \frac{1}{t L(\delta_D(y))} \right)^{1/2} t^\nu(|x - y|) \exp(-c_1 t h(|x - y|)).
\]

**Proof.** Let $R_2$ be the constant in Lemma 3.7. Fix $(t, x, y) \in (0, T) \times (D \times D \setminus \text{diag})$ and set
\[
r_t = \frac{L^{-1}(t)}{L^{-1}(T)} R_2 \quad \text{and} \quad l_t(x, y) = r_t \wedge \frac{|x - y|}{4}.
\]
Note that by (3.1), (3.2) and (3.3), we have $V(r_t) \asymp t^{1/2}$ and $L(r_t) \asymp h(r_t) \asymp t^{-1}$. Let $z_x, z_y \in \partial D$ be the points satisfying $\delta_D(x) = |x - z_x|$ and $\delta_D(y) = |y - z_y|$. By (3.2), there exists a constant $m > 1$ such that
\[
m V(\delta k) \geq \delta V(k) \quad \text{for all } 0 < \delta \leq 1, 0 < k \leq 1.
\]
Case 1. Suppose $|x - y| \leq R_2$. Define open neighbourhoods of $x$ and $y$ as follows:

$$
O(x) = \begin{cases} 
B \left( x, V^{-1} \left[ \frac{1}{8m} V(|x-y|) \right] \right), & \text{if } 8mV(\delta_D(x)) \geq V(|x-y|); \\
D \cap B \left( z_x, \frac{1}{3} |x-y| \right), & \text{if } 8mV(\delta_D(x)) < V(|x-y|),
\end{cases}
$$

and

$$
O(y) = \begin{cases} 
B \left( y, V^{-1} \left[ \frac{1}{8m} V(|x-y|) \right] \right), & \text{if } 8mV(\delta_D(y)) \geq V(|x-y|); \\
D \cap B \left( z_y, \frac{1}{3} |x-y| \right), & \text{if } 8mV(\delta_D(y)) < V(|x-y|).
\end{cases}
$$

Then, we see that $x \in O(x) \subset D$, $y \in O(y) \subset D$ and

$$|u-w| \asymp |x-y| \quad \text{for all } u \in O(x), \ w \in O(y).$$

Thus, by the strong Markov property and (5.1), we have (cf. [17, Lemma 3.3]),

$$p_D(t,x,y) \geq tP_x(\tau_{O(x)} > t)P_y(\tau_{O(y)} > t) \inf_{u \in O(x), w \in O(y)} \nu(|u-w|)$$

$$\geq c t \nu(|x-y|)P_x(\tau_{O(x)} > t)P_y(\tau_{O(y)} > t). \quad (5.6)$$

To calculate the survival probability $P_x(\tau_{O(x)} > t)$, we first assume that $8mV(\delta_D(x)) \geq V(|x-y|)$. In this case, we see from (4.1) and (3.1) that

$$P_x(\tau_{O(x)} > t) \geq c \exp \left( -c_1 t V(|x-y|)^{-2} \right) \geq c \exp \left( -c_2 t h(|x-y|) \right). \quad (5.7)$$

Next, assume $8mV(\delta_D(x)) < V(|x-y|)$. Note that by the monotonicity of $V$ and (5.5), we get $|x-y| > 8\delta_D(x)$ in this case. Let $\rho := V^{-1}(\varepsilon V(l_t(x,y)))$ where $\varepsilon \in (0, (8m)^{-1})$ will be chosen later. Then, we see from (3.1) and (3.2) that

$$V(\rho) \asymp V(l_t(x,y)) \asymp t^{1/2} \wedge V(|x-y|) \quad \text{and} \quad h(\rho) \asymp h(l_t(x,y)) \asymp t^{-1} \wedge h(|x-y|). \quad (5.8)$$

Note that we cannot expect that $\rho \asymp l_t(x,y)$ in general.

If $8\delta_D(x) \geq \rho$, then by (4.1) and (5.8), we have

$$P_x(\tau_{O(x)} > t) \geq P_x(\tau_{B(x,\rho/8)} > t) \geq c \exp \left( -\kappa_2 C_4 t h(\rho/8) \right) \geq c \exp \left( -c_3 t h(|x-y|) \right). \quad (5.9)$$

Indeed, by Lemma 2.2(i) and (3.1), we see that $h(\rho/8) \asymp h(4\rho)$. Thus, if $l_t(x,y) = |x-y|/4$, then we get (5.9). Otherwise, if $l_t(x,y) = r_t$, then $P_x(\tau_{O(x)} > t) \asymp 1 \asymp \exp(-c_3 t h(|x-y|))$ and hence (5.9) holds.

If $8\delta_D(x) < \rho$, then we can find a piece of annulus $A(x) \subset \{ w \in O(x) : \rho < |w - z_x| < |x-y|/4 \}$ such that $\text{dist}(A(x), \partial O(x)) > \rho/8$. Recall that $\theta$ is shift operator. Then, by the strong Markov property, the Lévy system, (4.1), (3.8), (3.1) and (3.2), we have

$$P_x(\tau_{O(x)} > t) \geq P_x(Y_{\tau_{B(z_x,\rho/2)} \cap D} \in A(x), \ \tau_{O(x)} \circ \theta_{B(z_x,\rho/2) \cap D} > t)$$

$$\geq P_x(Y_{\tau_{B(z_x,\rho/2)} \cap D} \in A(x)) \inf_{z \in A(x)} P_x(\tau_{O(x)} > t)$$

$$\geq c E_x \left[ \int_0^{\tau_{B(z_x,\rho/2)} \cap D} \int_{A(x)} \nu(|Y_s - w|) dw ds \right] P_0(\tau_{B(0,\rho/8)} > t)$$

$$\geq c E_x [\tau_{B(z_x,\rho/2) \cap D}] \int_{\rho}^{(|x-y|)/4} (-L'(k)) dk \exp(-c_4 t h(|x-y|))$$
\[
\geq c(L(\rho) - L(|x - y|/4))L(\delta_D(x))^{-1/2}L(\rho/2)^{-1/2} \exp(-c_4 th(|x - y|))
\]
\[
\geq c(c_5^{-1} V(\rho)^{-2} - c_5 V(|x - y|)^{-2})L(\delta_D(x))^{-1/2} V(\rho) \exp(-c_4 th(|x - y|)),
\]
where \(c_5 > 1\) is a constant independent of choice of \(\varepsilon\). Now, we choose \(\varepsilon = (2c_5)^{-1} \wedge (16m)^{-1}\). Then, we get from (5.8) that
\[
P_x(\tau_{O(x)} > t) \geq c V(\rho)^{-1} L(\delta_D(x))^{-1/2} \exp(-c_4 th(|x - y|))
\]
\[
\geq c t^{-1/2} L(\delta_D(x))^{-1/2} \exp(-c_4 th(|x - y|)).
\]
Finally, by combining the above inequality with (5.7) and (5.9), we deduce that
\[
P_x(\tau_{O(x)} > t) \geq c \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \exp(-c_5 th(|x - y|)).
\]
By the same way, we get \(P_y(\tau_{O(y)} > t) \geq c(1 \wedge \frac{1}{tL(\delta_D(y))})^{1/2} \exp(-c_5 th(|x - y|))\). Then, (5.6) yields the desired lower bound.

Case 2. Suppose \(|x - y| > R_2\). In this case, we let \(D_x := D \cap B(x, R_2/4)\) and \(D_y := D \cap B(y, R_2/4)\). By the same argument as (5.6), (5.1) and Corollary 4.3, we get
\[
p_D(t, x, y) \geq tP_x(\tau_{D_x} > t)P_y(\tau_{D_y} > t) \inf_{w \in D_x, v \in D_y} \nu([u - w])
\]
\[
\geq c \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} tv(|x - y|).
\]
This completes the proof. \(\Box\)

Now, we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Fix \((t, x, y) \in (0, T] \times (D \times D \setminus \text{diag})\) and continue using the notation \(r_t, i_t(x, y)\) in (5.4).

(i) Since we have proved the lower bound in Proposition 5.1, it suffices to show that there exist \(c_1 > 0, b_3 \in (0, b_0]\) such that for all \((t, x, y) \in (0, T] \times (D \times D \setminus \text{diag})\),
\[
p_D(t, x, y) \leq c_1 \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} tv(|x - y|) \exp(-b_3 th(|x - y|)),
\]
where \(b_0\) is the constant in Proposition 2.4. Indeed, if (5.10) holds, then by the semigroup property and (5.2), we get
\[
p_D(t, x, y) = \int_D p_D(t/2, x, z)p_D(t/2, y, z)dz
\]
\[
\leq c \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} \int_D p \left(\frac{b_1}{2b_0}, t, x, z\right)p \left(\frac{b_1}{2b_0}, t, y, z\right)dz
\]
\[
\leq c \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} tv(|x - y|) \exp\left(-\frac{b_2b_3}{b_0} th(|x - y|)\right),
\]
which yields the result.

Now, we prove (5.10). If \(\delta_D(x) \geq r_t/2\), then (5.10) is a consequence of (5.2) and the trivial bound that \(p_D(t, x, y) \leq p(t, x - y)\). Hence, we assume \(\delta_D(x) < r_t/2\). By (3.3), there exists a
Thus, if $ML(16k) \geq L(k)$ for all $k \leq 1/16$. (5.11)

Observe that by the semigroup property, monotonicity of $p(t, \cdot)$ and Proposition 4.2, we have

\[
p_D(t, x, y) = \left( \int_{\{z \in D : |y-z| > |x-y|/2\}} p_D(t/2, x, z) p_D(t/2, z, y) dz \right) + \int_{\{z \in D : |y-z| \leq |x-y|/2\}} p_D(t, x, z) p_D(t, z, y) dz
\]

\[
\leq \left( \int_{\{z \in D : |y-z| > |x-y|/2\}} p_D(t/2, x, z) p_D(t/2, z, y) dz \right) + p(t/2, |x-y|/2) \left( \frac{t}{L} \right)^{-1/2} + \mathbb{P}_y(\tau_D > t/2)
\]

\[
\leq c p(t/2, |x-y|/2) \left( \frac{t}{L} \right)^{-1/2} + \mathbb{P}_y(\tau_D > t/2)
\]

Thus, if $ML(\delta_D(y)) \geq L(\delta_D(x))$, then we get (5.10). Therefore, we assume $ML(\delta_D(y)) < L(\delta_D(x))$. Since $L$ is strictly decreasing, it follows from (5.11) that $\delta_D(y) < 16\delta_D(x)$ and hence $|x - y| > |y - z_x| - |z_x - x| \geq \delta_D(y) - \delta_D(x) > 15\delta_D(x)$ where $z_x \in \partial D$ is the point satisfying $\delta_D(x) = |x - z_x|$. Define

\[
W_1 := D \cap B(z_x, l_s(x, y)), \quad W_3 := \{ w \in D : |w - y| \leq |x - y|/2 \}
\]

and $W_2 := D \setminus (W_1 \cup W_3) = \{ w \in D \setminus W_1 : |w - y| > |x - y|/2 \}$. Then, for $u \in W_1$ and $w \in W_3$, we obtain

\[
|u - w| \geq |x - y| - |z_x - x| - |u - z_x| - |y - w| \geq \left( 1 - \frac{1}{15} - \frac{1}{4} - \frac{1}{2} \right) |x - y| > \frac{|x - y|}{6}.
\]

(5.12)

Observe that by the strong Markov property,

\[
p_D(t, x, y) = \mathbb{E}_x[p_D(t - \tau_{W_1}, Y_{\tau_{W_1}}, y : \tau_{W_1} < t)]
\]

\[
= \mathbb{E}_x[p_D(t - \tau_{W_1}, Y_{\tau_{W_1}}, y : \tau_{W_1} < t, \tau_{\tau_{W_1}} \in W_3)]
\]

\[
+ \mathbb{E}_x[p_D(t - \tau_{W_1}, Y_{\tau_{W_1}}, y : \tau_{W_1} \in (0, 2t/3], Y_{\tau_{W_1}} \in W_2)]
\]

\[
+ \mathbb{E}_x[p_D(t - \tau_{W_1}, Y_{\tau_{W_1}}, y : \tau_{W_1} \in (2t/3, t), Y_{\tau_{W_1}} \in W_2)]
\]

\[
= : I_1 + I_2 + I_3.
\]

(5.13)

First, by the Lévy system and (5.12), we get

\[
I_1 = \int_0^t \int_{W_3} \int_{W_1} p_{W_1}(s, x, u) \nu(|w - u|) p_D(t - s, w, y) dw du ds
\]

\[
\leq \nu(|x - y|/6) \int_0^t \mathbb{P}_x(\tau_{W_1} > s) \int_{W_3} p(t - s, y - w) dw ds.
\]

(5.14)

By (5.2) and Lemma 2.1, for all $s \in (0, T)$ and $l \in (0, 2r_s]$, we have

\[
\int_{B(y, l)} p(s, y - w) dw \leq c \int_0^l -sL'(k) \exp(-c_2sL(k)) dk \leq c \exp(-c_3sl(l)).
\]

(5.15)
It follows that for all $s \in (0, t]$,

$$
\int_{W_3} p(s, y-w)dw \leq c \left\{ \begin{array}{ll}
\exp(-c_3sh(|x-y|)), & \text{if } |x-y| \leq 2r_t; \\
1, & \text{if } |x-y| > 2r_t
\end{array} \right.
$$

\leq c \exp(-c_3sh(|x-y|)). \quad (5.16)

Indeed, if $|x-y| > 2r_t$, then we have $sh(|x-y|) \leq sh(2r_t) \propto st^{-1} \leq 1$. Moreover, by the semigroup property, Proposition 4.2, (5.15) and monotonicity of $h$, we get

$$
P_x(\tau_{W_1} > 2t/3) = \int_{W_1} \int_{W_1} p_{W_1}(t/3, x, v) p_{W_1}(t/3, v, u) dv du
$$

$$
\leq P_x(\tau_D > t/3) \int_{B(0, 2l_s(x,y))} p(t/3, u) du \leq ct^{-1/2}L(\delta_D(x))^{-1/2} \exp(-3^{-1}c_3th(2l_s(x,y)))
$$

$$
\leq ct^{-1/2}L(\delta_D(x))^{-1/2} \exp(-3^{-1}c_3th(|x-y|)). \quad (5.17)
$$

Then, using (5.14), (5.1), (5.15), (5.17) and Proposition 4.2, we obtain

$$
I_1 \leq c \nu(|x-y|) \int_0^t P_x(\tau_{W_1} > s) \int_{W_3} p(t-s, y-w)dw ds
$$

$$
\leq c \nu(|x-y|) \exp(-c_3th(|x-y|)/3) \int_0^{2t/3} P_x(\tau_D > s) ds
$$

$$
+ c \nu(|x-y|)P_x(\tau_{W_1} > 2t/3) \int_0^{t/3} \exp(-c_3sh(|x-y|)) ds
$$

$$
\leq cL(\delta_D(x))^{-1/2} \nu(|x-y|) \exp(-c_3th(|x-y|)/3) \left( \int_0^{2t/3} s^{-1/2}ds + t^{-1/2} \int_0^{t/3} ds \right)
$$

$$
= ct^{-1/2}L(\delta_D(x))^{-1/2} \nu(|x-y|) \exp(-3^{-1}c_3th(|x-y|)). \quad (5.18)
$$

Second, by monotonicity of $p(t, \cdot)$, (5.2), (5.1) and Proposition 4.2, we get

$$
I_2 \leq cP_x(Y_{\tau_{W_1}} \in W_2) \sup_{s \in [t/3, t), |x-y|/2} p(s, l) = cP_x(Y_{\tau_{W_1}} \in W_2) \sup_{s \in [t/3, t)} p(s, |x-y|/2)
$$

$$
\leq cP_x(Y_{\tau_{W_1}} \in W_2) \nu(|x-y|) \left( \sup_{s \in [t/3, t]} s \exp(-b_2sh(|x-y|)) \right)
$$

$$
\leq c \left\{ \begin{array}{ll}
L(r_t)^{-1/2}L(\delta_D(x))^{-1/2} \nu(|x-y|) \exp(-3^{-1}b_2th(|x-y|)), & \text{if } |x-y| \geq 4r_t; \\
L(|x-y|)^{-1/2}L(\delta_D(x))^{-1/2} \nu(|x-y|) \exp(-3^{-1}b_2th(|x-y|)), & \text{if } |x-y| < 4r_t
\end{array} \right.
$$

$$
\leq ct^{-1/2}L(\delta_D(x))^{-1/2} \nu(|x-y|) \exp(-4^{-1}b_2th(|x-y|)). \quad (5.19)
$$

In the last inequality, we used (3.1), $L(r_t) \approx t^{-1}$ and the fact that $e^x \geq 2e\sqrt{x}$ for all $x > 0$ and $h(x) \geq L(r)\geq L(r)\geq L(r)$ for all $r > 0$.

Lastly, we note that $t \mapsto te^{-at}$ is increasing on $(0, 1/a)$ and decreasing on $(1/a, \infty)$. Thus, using similar calculation as the one given in (5.17), by monotonicity of $p(t, \cdot)$, (5.2), (5.1),
Proposition 4.2 and (3.1), we have

\[
I_3 \leq c \mathbb{P}_x(\tau_{W_1} > 2t/3) \nu(|x - y|) \left( \sup_{s \in (0,t/3)} s \exp \left( -b_2 h(|x - y|) \right) \right)
\]

\[
\leq c \left\{ \begin{array}{ll}
\mathbb{P}_x(\tau_{W_1} > 2t/3) \nu(|x - y|) h(|x - y|)^{-1}, & \text{if } b_2 h(|x - y|) \geq 3;
\mathbb{P}_x(\tau_D > 2t/3) \nu(|x - y|) t \exp \left( -3^{-1} b_2 h(|x - y|) \right), & \text{if } b_2 h(|x - y|) < 3.
\end{array} \right.
\]

\[
\leq c \left\{ \begin{array}{ll}
t^{-1/2} L(\delta_D(x))^{-1/2} t \nu(|x - y|) \exp \left( -3^{-1} c_3 h(|x - y|) \right), & \text{if } b_2 h(|x - y|) \geq 3;
\end{array} \right.
\]

Combining the above inequality with (5.18), (5.19) and (5.13), we get (5.10).

(ii) We use the same notations as in the proof of (i) and follow the proof of (i).

(Upper bound) By the semigroup property and (5.3), it suffices to show that there exist positive constants \(c_1\) and \(b_2\) such that

\[
p_D(t, x, y) \leq c_1 \left( 1 + \frac{1}{t L(\delta_D(x))} \right)^{1/2} t \nu(\theta_{3a_1}(|x - y|, t)) \exp \left( -b_2 h(\theta_{3a_1}(|x - y|, t)) \right) \tag{5.20}
\]

By the similar argument to the one given in the proof of (i), we may assume \(\delta_D(x) < r_t/2\). Moreover, observe that for every \(u, v \in D\), by the triangle inequality, \(\max\{ |x - u|, |u - v|, |v - y| \} \geq |x - y|/3\). Thus, by the semigroup property, monotonicity of \(p(t, \cdot)\) and Proposition 4.2, we have that

\[
p_D(t, x, y) = \int_{u \in D, |x - u| \geq |x - y|/3} \int_D p_D(t/3, x, u)p_D(t/3, u, v)p_D(t/3, v, y)dvdu + \int_{u \in D, |x - u| < |x - y|/3} \int_{v \in D, |u - v| \geq |x - y|/3} p_D(t/3, x, u)p_D(t/3, u, v)p_D(t/3, v, y)dvdu + \int_{u \in D, |x - u| < |x - y|/3} \int_{v \in D, |u - v| < |x - y|/3} p_D(t/3, x, u)p_D(t/3, u, v)p_D(t/3, v, y)dvdu \leq p(t/3, |x - y|/3) \int_D p_D(t/3, v, y) \int_D p_D(t/3, u, v)dvdu + p(t/3, |x - y|/3) \int_D p_D(t/3, x, u) \int_D p_D(t/3, v, y)dvdu + p(t/3, |x - y|/3) \int_D p_D(t/3, x, u) \int_D p_D(t/3, u, v)dvdu \leq 2p(t/3, |x - y|/3)(\mathbb{P}_x(\tau_D > t/3) + \mathbb{P}_y(\tau_D > t/3)) \leq cp(t/3, |x - y|/3) \left( t^{-1/2} L(\delta_D(x))^{-1/2} + t^{-1/2} L(\delta_D(y))^{-1/2} \right).
\]

Therefore, if \(ML(\delta_D(y)) \leq L(\delta_D(x))\) for some \(M > 0\), then we get (5.20) from (5.3). Hence, we may assume that \(\delta_D(y) > 16\delta_D(x)\) by the same argument as the one given in the proof of (i).

To prove (5.20), we first assume that \(|x - y| \leq \lfloor t^{-1}(3a_1/t) \rfloor^{-1}\). In this case, we have that \(\theta_{a_1}(|x - y|, t/3) = \theta_{3a_1}(|x - y|, t) = \lfloor t^{-1}(3a_1/t) \rfloor^{-1}\). Then, by the semigroup property, (5.3) and
Proposition 4.2, we get
\[ p_D(t, x, y) = \int_D p_D(2t/3, x, z)p_D(t/3, z, y)dz \leq cP_x(\tau_D > 2t/3)p(t/3, 0) \]
\[ \leq ct^{-1/2}L(\delta_D(x))^{-1/2}tv(\theta_{3a_1}(|x - y|, t)) \exp \left( -3^{-1}b_1t h(\theta_{3a_1}(|x - y|, t)) \right). \]

Now, suppose \(|x - y| > \ell^{-1}(3a_1/t)|^{-1}\). In this case, we use (5.13) and find upper bounds for \(I_1, I_2\) and \(I_3\). Observe that for all \(s \in (0, T)\) and \(l \in (0, 2r_1)\), by (5.3) and the similar calculation to the one given in (5.15),
\[ \int_{B(y, l)} p(s, y - w)dw \leq c\left\{L^d[\ell^{-1}(a_1/s)]^d \exp \left( -b_1sh(\ell^{-1}(a_1/s))^{-1} \right), \right. \]
\[ \left. \exp \left( -c_2sh(l) \right), \right. \]
\[ \leq c \exp \left( -c_3sh(\theta_{a_1}(l, s)) \right). \] (5.21)

Then, by using (5.21) instead of (5.15), we have that for all \(0 < s \leq T\),
\[ \mathbb{P}_x(\tau_{W_1} > s) = \int_{W_1} \int_{W_1} p_{W_1}(s/3, x, u)p_{W_1}(2s/3, u, v)dudv \]
\[ \leq cs^{-1/2}L(\delta_D(x))^{-1/2}tv(|x - y|) \exp \left( -c_4sh(|x - y|, 2s/3) \right). \]

Hence, by the similar arguments to the ones given in (5.16) and (5.18), we obtain
\[ I_1 \leq ct^{-1/2}L(\delta_D(x))^{-1/2}tv(|x - y|) \exp \left( -c_5th(|x - y|) \right). \]

Next, by (5.3), (5.1), monotonicity of \(h\), we have
\[ \sup_{s \in [t/3, t]} p(s, |x - y|/2) \leq ct \sup_{s \in [t/3, t]} \left[ \nu(\theta_{a_1}(|x - y|, s)) \exp \left( -3^{-1}b_1t h(\theta_{a_1}(|x - y|, s)) \right) \right]. \]

Let \(f(r) := r^{-d} \exp \left( -c_7th(r) \right)\) where the constant \(c_7 \in (0, b_1/3)\) will be chosen later. Then, by (2.2), there exists a constant \(c_6 > 0\) such that for \(r \in (0, [\ell^{-1}(a_1/t)]^{-1})\),
\[ r^{d+1} \exp \left( c_7th(r) \right)f'(r) = -d + 2c_7tK(r) \leq -d + c_6c_7t\ell(r^{-1}). \]

Set \(c_7 = d/(3a_1c_6) \land b_1/3\). Then, we see that \(f\) is decreasing on \([\ell^{-1}(3a_1/t)^{-1}, \ell^{-1}(a_1/t)^{-1}]\). Using this fact, since \(\ell\) is almost increasing, we deduce that
\[ \sup_{s \in [t/3, t]} \left[ \nu(\theta_{a_1}(|x - y|, s)) \exp \left( -3^{-1}b_1th(\theta_{a_1}(|x - y|, s)) \right) \right] \]
\[ \leq cv(\theta_{a_1}(|x - y|, t/3)) \exp \left( -c_7th(\theta_{a_1}(|x - y|, t/3)) \right) \exp \left( -c_7th(|x - y|) \right). \]

It follows that by the same argument as in the one given in (5.19),
\[ I_2 \leq ct^{-1/2}L(\delta_D(x))^{-1/2}tv(|x - y|) \exp \left( -c_8th(|x - y|) \right). \]

Lastly, we note that since \(|x - y| > [\ell^{-1}(3a_1/t)]^{-1}\),
\[ \sup_{s \in (0, t/3)} \left[ s\nu(\theta_{a_1}(|x - y|, s)) \exp \left( -b_1sh(\theta_{a_1}(|x - y|, s)) \right) \right] \]
\[ = \sup_{s \in (0, t/3)} \left[ s\nu(|x - y|) \exp \left( -b_1sh(|x - y|) \right) \right]. \]

Therefore, by the same proof as in the one given in (i), we obtain
\[ I_3 \leq ct^{-1/2}L(\delta_D(x))^{-1/2}tv(|x - y|) \exp \left( -c_9th(|x - y|) \right). \]

This finishes the proof for the upper bound.
(Lower bound) Fix $\eta > 0$. By Proposition 5.1, it remains to prove the lower bound when $|x - y| < \ell^{-1}(\eta/t)|^{-1} \wedge R_2$, where $R_2$ is the constant in Lemma 3.7. Let $\zeta_\ell := \ell^{-1}(\eta/t)|^{-1} \wedge R_2$ and define open neighborhoods of $x$ and $y$ as follows. Recall that $z_x, z_y \in \partial D$ are the points satisfying $\delta_D(x) = |x - z_x|$ and $\delta_D(y) = |y - z_y|$. We define

$$
\mathcal{U}(x) = \begin{cases} 
B(x, V^{-1}(\frac{1}{8m}V(\zeta_\ell))), & \text{if } 8mV(\delta_D(x)) \geq V(\zeta_\ell), \\
B(z_x, \frac{1}{2}\zeta_\ell) \cap D, & \text{if } 8mV(\delta_D(x)) < V(\zeta_\ell),
\end{cases}
$$

and

$$
\mathcal{U}(y) = \begin{cases} 
B(y, V^{-1}(\frac{1}{8m}V(\zeta_\ell))), & \text{if } 8mV(\delta_D(y)) \geq V(\zeta_\ell), \\
B(z_y, \frac{1}{2}\zeta_\ell) \cap D, & \text{if } 8mV(\delta_D(y)) < V(\zeta_\ell),
\end{cases}
$$

where $m$ is the constants in (5.5). Then, we can see that $x \in \mathcal{U}(x) \subset D$ and $y \in \mathcal{U}(y) \subset D$.

We claim that there exist a constant $c_4 > 0$ independent of the choice of $\eta$, and a constant $c_3 > 0$ such that

$$
\mathbb{P}_x(\tau_{\mathcal{U}(x)} > t) \geq c_3 \left(1 + \frac{1}{tL(\delta_D(x))}\right)^{1/2} \exp(-c_4th(\zeta_\ell)). \tag{5.22}
$$

Indeed, if $8mV(\delta_D(x)) \geq V(\zeta_\ell)$, then by (4.1) and (3.1), we have

$$
\mathbb{P}_x(\tau_{\mathcal{U}(x)} > t) \geq c \exp(-c_3th(\zeta_\ell)). \tag{5.23}
$$

Suppose $8mV(\delta_D(x)) < V(\zeta_\ell)$. If $\zeta_\ell = R_2$, then by Corollary 4.3, we get

$$
\mathbb{P}_x(\tau_{\mathcal{U}(x)} > t) \geq c \left(1 + \frac{1}{tL(\delta_D(x))}\right)^{1/2}. \tag{5.24}
$$

Otherwise, if $\zeta_\ell = [\ell^{-1}(\eta/t)]^{-1} < R_2$, then by the similar argument to the one given in the proof of Proposition 5.1,

$$
\mathbb{P}_x(\tau_{\mathcal{U}(x)} > t) \geq cL(\zeta_\ell)^{1/2}L(\delta_D(x))^{-1/2} \exp(-c_2th(\zeta_\ell)) \geq ct^{-1/2}L(\delta_D(x))^{-1/2} \exp(-c_2th(\zeta_\ell)). \tag{5.25}
$$

In the second inequality, we used $L(\zeta_\ell) \geq cK(\zeta_\ell) \asymp \ell(\zeta_\ell^{-1}) \asymp t^{-1}$ which follows from the proof of Lemma 2.1 and (2.2). By combining (5.23), (5.24) and (5.25), we obtain (5.22).

Let $w_x := z_x + 4\zeta_\ell(x - z_x)/|x - z_x| \in D$ and define

$$
\mathcal{W}_0 := B\left(w_x, \frac{\zeta_\ell}{2\sqrt{1 + \Lambda^2}}\right) \quad \text{and} \quad \mathcal{W} := B\left(w_x, \frac{\zeta_\ell}{\sqrt{1 + \Lambda^2}}\right) \subset D.
$$

Then, for all $u \in \mathcal{U}(x)$ and $v \in \mathcal{W}$, we have $|u - v| \asymp \zeta_\ell$. Moreover, since $|x - y| \asymp \zeta_\ell$, we also have $|u' - v'| \asymp \zeta_\ell$ for all $u' \in \mathcal{U}(y)$ and $v \in \mathcal{W}$. Thus, for every $v \in \mathcal{W}_0$, by the similar argument to (5.6), (4.1) and (5.22), we have

$$
p_D(t/2, x, v) \geq ctv(\zeta_\ell)\mathbb{P}_x(\tau_{\mathcal{U}(x)} > t/2)\mathbb{P}_v(\tau_{B(v, \zeta_\ell/(2\sqrt{1 + \Lambda^2})) > t/2}) \geq c \left(1 + \frac{1}{tL(\delta_D(x))}\right)^{1/2} tv(\zeta_\ell) \exp(-c_5th(\zeta_\ell)).
$$

Similarly, we also have that

$$
p_D(t/2, v, y) \geq c \left(1 + \frac{1}{tL(\delta_D(y))}\right)^{1/2} tv(\zeta_\ell) \exp(-c_5th(\zeta_\ell)).
$$
It follows that by the semigroup property and (A),

\[ p_D(t, x, y) \geq \int_W p_D(t/2, x, v)p_D(t/2, v, y)dv \]

\[ \geq c \left( 1 \wedge \frac{1}{tL(\delta_D(x))} \right)^{1/2} \left( 1 \wedge \frac{1}{tL(\delta_D(y))} \right)^{1/2} t^2 |W| \nu(\zeta_t)^2 \exp(-c_5\kappa th(\zeta_t)) \]

\[ \geq c \left( 1 \wedge \frac{1}{tL(\delta_D(x))} \right)^{1/2} \left( 1 \wedge \frac{1}{tL(\delta_D(y))} \right)^{1/2} t^2 \ell(\zeta_t^{-1}) \nu(\zeta_t) \exp(-c_5\kappa th(\zeta_t)). \]

If \( \zeta_t = [\ell^{-1}(\eta/t)]^{-1} \), then since \( \ell \) is almost increasing, we get \( \ell(\zeta_t^{-1}) \asymp t^{-1} \). Hence, we are done. Otherwise, if \( \zeta_t = R_2 \), then we have \( t \asymp 1 \) and hence \( t^2 \ell(\zeta_t^{-1}) \nu(\zeta_t) \exp(-c_5\kappa th(\zeta_t)) \asymp t \nu([\ell^{-1}(\eta/t)]^{-1}) \exp(-c\kappa th([\ell^{-1}(\eta/t)]^{-1})) \asymp 1 \). This completes the proof. □

6. Large time estimates

In this section, we give the proof of Theorem 1.6. Let \( D \) be a fixed bounded \( C^{1,1} \) open subset in \( \mathbb{R}^d \) of scale \((r_1, r_2)\) and \( x_1, x_2 \in \mathbb{R}^d \) be the fixed points satisfying \( B(x_1, r_1) \subset D \subset B(x_2, r_2) \). We mention that under condition \((L-1)\), the transition semigroup \( \{P^D_t, t \geq 0\} \) of \( Y^D_t \) may not be compact operators in \( L^2(D; dx) \), though \( D \) is bounded. (See, Proposition 2.3.) Hence, in that case, we need some lemmas to obtain the large time estimates instead of the general spectral theory.

**Lemma 6.1.** There exists a constant \( c_1 > 0 \) which only depend on the dimension \( d \) such that for all \((t, x, y) \in (0, \infty) \times (D \times D \setminus \text{diag})\),

\[ p_D(t, x, y) \leq c_1 p(t/2, |x - y|/2) \exp(\frac{1}{2} \kappa c_5\kappa th(r_2)). \]

**Proof.** By the semigroup property, we have

\[ p_D(t, x, y) = \int_{\{z \in D: |x - z| > |x - y|/2\}} \int_{\{z \in D: |y - z| \leq |x - y|/2\}} p_D(t/2, x, z)p_D(t/2, z, y)dz \]

\[ \leq t^{-d} \left( \int_{\{z \in D: |x - z| > |x - y|/2\}} \int_{\{z \in D: |y - z| > |x - y|/2\}} p_D(t/2, x, z)p_D(t/2, z, y)dz \right). \]

Hence, we obtain the result from (4.1). □

Define for \( r \geq 1 \),

\[ \hat{\ell}(r) := \sup_{s \in [1, r]} \frac{1}{\ell(s)} \quad \text{and} \quad \hat{\Phi}(r) := \int_1^r \frac{1}{k\ell(k)}dk. \]

Note that if \((L-1)\) holds, we have that

\[ \hat{\ell}(r)^{-1} \asymp \ell(r) \quad \text{for all} \quad r \geq 2. \quad (6.1) \]

Moreover, by the same argument as the one given in the proof of Lemma 2.2, there exist positive constants \( C_6 \) and \( C_7 \) which only depend on the dimension \( d \) and \( \kappa_1 \) and \( \kappa_2 \) in (1.3) such that

\[ C_6 \hat{\Phi}(r) \leq \psi(r) \quad \text{and} \quad h(r^{-1}) \leq C_7 \hat{\Phi}(r) \quad \text{for all} \quad r \geq 2. \quad (6.2) \]
We also note that $\hat{\Phi}$ satisfies $\text{WS}^{\infty}(0,2,1)$. Here, we get the large time on-diagonal estimates for $p(t,x)$ under condition (L-1).

**Lemma 6.2.** Assume that (L-1) holds. Then, there exists a positive constant $b_5 = b_5(d,\psi,r_2)$ such that for any $T > 0$, there exist $c_1, c_2 > 0$ such that for all $t \in [T, \infty)$ and $|x| \leq 2r_2$,

$$p(t,x) \leq c_1 + c_2 \nu(|x|) \exp(-b_5 \text{th}(|x|)).$$  \hfill (6.3)

**Proof.** Fix $x \in \mathbb{R}^d$ satisfying $|x| \leq 2r_2$ and let $r = |x|$. By [25, (5.4)], the mean value theorem, Lemma 2.6, (6.1) and (6.2), we have that, for all $t > 0$ (cf. (2.12)),

$$r^d p(t,x) \leq c \int_{\mathbb{R}^d} \left( e^{-t\psi(|z|/r)} - e^{-t\psi(2|z|/r)} \right) e^{-|z|^2/4} dz$$

$$\leq c r^d + c t \int_{|z| > 2r} \sup_{|y| \leq 2|z|} e^{-t\psi(y/r)} \left| \psi(2|z|/r) - \psi(|z|/r) \right| e^{-|z|^2/4} dz$$

$$\leq c r^d + c t \int_{2r}^{4r_2} e^{-C_6 t \hat{\Phi}(u/r)} \frac{u^{d-1}}{\ell(u/r)} du + c t \int_{4r_2}^{\infty} e^{-C_6 t \hat{\Phi}(u/r)} \frac{u^{d-1}}{\ell(u/r)} e^{-u^2/4} du$$

$$=: c r^d + I_1 + I_2. \hfill (6.4)$$

First, by (6.1), and the monotonicity and the scaling properties of $\hat{\ell}$, $\ell$ and $\hat{\Phi}$, we have

$$I_2 \leq c t \left[ \hat{\ell}(4r_2/r) \right]^{-1} \exp \left( -C_6 t \hat{\Phi}(4r_2/r) \right) \int_{4r_2}^{\infty} u^{d-1} e^{-u^2/4} du$$

$$\leq c t \ell(4r_2/r) \exp \left( -c_1 t \hat{\Phi}(1/r) \right) \leq c t \ell(1/r) \exp \left( -2^{-1} c_1 \hat{\Phi}(1/r) \right). \hfill (6.5)$$

In the last inequality above, we used the facts that $e^x \geq x$ for $x > 0$ and $\hat{\Phi}(1/r) \geq \hat{\Phi}(1/(2r_2))$.

On the other hand, set $q_{\gamma,k}(u) = q_{\gamma,k}(u,t) := u^\gamma \exp(-kt\hat{\Phi}(u))$ for $u \geq 2$ and $\gamma, k > 0$. We observe that for any $\gamma, k > 0$,

$$\frac{d}{du} q_{\gamma,k}(u) = \left( \gamma - kt\hat{\ell}(u)^{-1} \right) q_{\gamma-1,k}(u).$$

Since $\hat{\ell}$ is increasing, it follows that there exists $u_0 \in [2, \infty)$ such that $q$ is decreasing on $[2, u_0]$ and increasing on $[u_0, \infty)$. Thus, for any $[a,b] \subset [2, \infty)$ and $\gamma, k > 0$, we have that

$$\sup_{u \in [a,b]} q_{\gamma,k}(u) = q_{\gamma,k}(a) \vee q_{\gamma,k}(b). \hfill (6.6)$$

Choose any constant $\varepsilon \in (0,1)$ such that $(1-\varepsilon)\alpha_1 > d$. This is possible since $\alpha_1 > -d$. Then we set $\rho := ((1-\varepsilon)d + \alpha_1)/(1+\varepsilon) \in (0,d + \alpha_1)$ so that

$$\varepsilon = (d + \alpha_1 - \rho)/(d + \rho). \hfill (6.7)$$

First, suppose $q_{d+\rho,C_6}(2) \geq q_{d+\rho,C_6}(4r_2/r)$. Then by the change of variables and (6.6), since $\hat{\ell}$ is increasing, we have

$$I_1 = c t r^d \int_{2r}^{4r_2/r} \frac{u^{d-1}}{\ell(u)} e^{-C_6 t \hat{\Phi}(u)} du = c t r^d \int_{2r}^{4r_2/r} \frac{q_{d+\rho,C_6}(u)}{u^{1+\rho} \hat{\ell}(u)} du$$

$$\leq c t r^d \frac{q_{d+\rho,C_6}(2)}{\ell(2)} \int_{2r}^{4r_2/r} \frac{du}{u^{1+\rho}} \leq c t r^d \exp \left( -C_6 \hat{\Phi}(2) \right) \leq \frac{c}{C_6 \hat{\Phi}(2)} r^d.$$
Now, we suppose \( q_{d+p, C_0}(2) < q_{d+p, C_0}(4r_2/r) \). Note that by (6.1) and the scaling property of \( \ell \), it holds that for any \( 2 < u \leq s \),

\[
\hat{\ell}(s) \leq c \ell(u) \leq c \left( \frac{u}{s} \right)^\alpha_1 \ell(u), \quad \text{that is,} \quad \frac{1}{u^{\alpha_1} \hat{\ell}(u)} \leq \frac{c}{s^{\alpha_1} \hat{\ell}(s)}.
\]

We also note that for any \( \gamma, k > 0 \) and \( u \geq 2 \), the map \( q_{\gamma, k}(u) \leq q_{\gamma, k}(u)^s = q_{\gamma, k}(u)^s \) for all \( s > 0 \). Thus, by the change of variables, (6.1), (6.6), (6.7) and the scaling properties of \( \ell \) and \( \hat{\Phi} \), since we have assumed \( q_{d+p, C_0}(2) < q_{d+p, C_0}(4r_2/r) \), we get that

\[
I_1 = \text{c} t r^d \int_2^{4r_2/r} \frac{q_{d+p, C_0}(u)}{u^{1-p} \ell(u)} du \leq \frac{\text{c} tr^d}{(4r_2)\alpha_1 \ell(4r_2/r)} \int_2^{4r_2/r} \frac{q_{d+p, C_0}(u)}{u^{1-p}} du
\]

\[
\leq \text{c} tr^{d+\alpha_1} \ell(1/r)(q_{d+p, C_0}(2) \vee q_{d+p, C_0}(4r_2/r)) \int_2^{4r_2/r} \frac{q_{d+p, C_0}(u)}{u^{1-p}} du
\]

\[
\leq \text{c} tr^{d+\alpha_1} \ell(1/r)(q_{d+p, C_0}(2) \vee q_{d+p, C_0}(4r_2/r))^{(d+\alpha_1-p)/(d+p)}
\]

\[
= \text{c} tr^{d+\alpha_1} \ell(1/r)(q_{d+p, C_0}(2) \vee q_{d+p, C_0}(4r_2/r)) = \text{c} t \ell(1/r) \exp \left( -\varepsilon C_6 \hat{\Phi}(4r_2/r) \right)
\]

\[
\leq \ell(1/r) \exp \left( -2^{\frac{1}{2}} \varepsilon C_6 \hat{\Phi}(4r_2/r) \right) \leq \ell(1/r) \exp \left( -c_2 \hat{\Phi}(1/r) \right).
\]

Then we get (6.3) by using (6.4), (6.5), (1.3) and (6.2) again. \( \square \)

Now, we give the proof of Theorem 1.6.

**Proof of Theorem 1.6.** Let \( a(x, y) := L(\delta_D(x))^{-1/2}L(\delta_D(y))^{-1/2} \).

(i) Choose \((t, x, y) \in [T, \infty) \times (D \times D \setminus \text{diag})\) and let \( x_1 \in D \) be the point satisfying \( B(x_1, r_1) \subset D \). By the semigroup property, Theorem 1.5(i), (5.2) and (4.1), we have

\[
p_D(t, x, y) \geq \int_{B(x_1, r_1/4) \times B(x_1, 3r_1/4)} p_D(T/4, x, u) p_D(t - T/2, u, v) p_D(T/4, v, y) du dv
\]

\[
\geq \inf_{u \in B(x_1, r_1/4)} \mathbb{P}_u(\tau_{B(x_1, 3r_1/4)} > T - T/2)
\]

\[
\geq \inf_{u \in B(x_1, r_1/4)} \mathbb{P}_u(\tau_{B(x_1, 3r_1/4)} > T - T/2)
\]

(6.8)

On the other hand, since \( D \) is a bounded set, one can follow the proof of Proposition 5.1, after changing the definition of \( l_t(x, y) \) therein from \( r_t \wedge (|x - y|/4) \) to \( |x - y|/4 \), and see that

\[
p_D(t, x, y) \geq c \left( 1 \wedge \frac{L(|x - y|)}{L(\delta_D(x))} \right)^{1/2} \left( 1 \wedge \frac{L(|x - y|)}{L(\delta_D(x))} \right)^{1/2} t \nu(|x - y|) \exp(-c_1 t h(|x - y|))
\]

\[
\geq c T \left( 1 \wedge \frac{L(2r_2)}{L(\delta_D(x))} \right)^{1/2} \left( 1 \wedge \frac{L(2r_2)}{L(\delta_D(x))} \right)^{1/2} \nu(|x - y|) \exp(-c_1 t h(|x - y|))
\]

(6.9)

In the last inequality above, we used the fact that \( L(2r_2)/L(\delta_D(z)) \leq L(2r_2)/L(r_2) \leq c \) for all \( z \in D \), which comes from the monotonicity of \( L \) and (3.3). By combining (6.8) with (6.9), we get the desired lower bound.
Now, we prove the upper bound. By the semigroup property, Theorem 1.5(i), Corollary 2.13, Lemma 6.1 and Lemma 6.2, we get
\[
p_D(t, x, y) = \int_{D \times D} p_D(T/4, x, u)p_D(t - T/2, u, v)p_D(T/4, v, y)du dv
\leq ca(x, y) \exp\left(-2^{-1} \kappa_1 C_5 th(r_2)\right)
\times \int_{D \times D} p(cT/4, |x - u|/2)p(t/2 - T/4, |u - v|/2)p(cT/4, |v - y|/2)du dv
\leq ca(x, y)p(t/2 - cT, |x - y|/2) \exp\left(-2^{-1} \kappa_1 C_5 th(r_2)\right)
\leq ca(x, y)\left[c + cv(|x - y|)\right] \exp\left(-2^{-1} \kappa_1 C_5 th(r_2)\right),
\]
which yields the upper bound.

(ii) & (iii) Since the proof of (iii) is similar and easier, we only provide the proof of (ii).
By Proposition 2.3, there exist \(T_0 > 0\) such that the transition semigroup \(\{P^D_t : t \geq T_0\}\) of \(Y^D_t\) consists of compact operators. Let \(0 < \mu_1 < 1\) be the largest eigenvalue of the operator \(P^D_{T_0}\) and \(\phi_1 \in L^2(D; dx)\) be the corresponding eigenfunction with unit \(L^2\)-norm. For \(n \geq 1\), we denote by \(\{\mu_{n,k}, k \geq 1\} \subset (0, 1)\) the discrete spectrum of \(P^D_{nT_0}\), arranged in decreasing order and repeated according to their multiplicity and \(\{\phi_{n,k}, k \geq 1\}\) be the corresponding eigenfunctions with unit \(L^2\)-norm. Then, by the semigroup property, we have \(\mu_{n,1} = \mu_1^n\) and \(\phi_{n,1} = \phi_1\) for all \(n \geq 1\).
From the eigenfunction expansion of \(p_D(nT_0, \cdot, \cdot)\) and Parseval’s identity, we have for \(n \geq 1\),
\[
\int_{D \times D} p_D(nT_0, x, y)dx dy = \sum_{k=1}^{\infty} \mu_{n,k} \left( \int_D \phi_{n,k}(y)dy \right)^2 \leq \sup_{k} \mu_{n,k} \int_D 1^2 dy = \mu_1^n |D|. \quad (6.10)
\]
On the other hand, for all \(s > 0\) and \(x \in D\), since \(p(T_0, 0) \leq c_0 < \infty\), we have
\[
\phi_1(x) \leq \int_{D \times D} p_D(s, x, z)p_D(T_0, z, y)\phi_1(y)dz dy \leq c_0 \mathbb{P}_x(\tau_D > s) \int_D \phi_1(y)dy
\leq c_0 \mathbb{P}_x(\tau_D > s)\|\phi_1\|_{L^2(D)} \left( \int_D 1^2 dy \right)^{1/2} = c_0 |D|^{1/2} \mathbb{P}_x(\tau_D > s).
\]
Thus, we obtain for all \(0 < s \leq T_0\) and \(n \geq 1\),
\[
\int_{D \times D} \mathbb{P}_x(\tau_D > s)p_D(nT_0, x, y)\mathbb{P}_y(\tau_D > s)dx dy
\geq \mu_1^n \left( \int_D \mathbb{P}_y(\tau_D > s)\phi_1(y)dy \right)^2 \geq \mu_1^n \left( \int_D c_0^{-2} |D|^{-1/2} \phi_1(y)^2 dy \right)^2 \geq c_0^{-2} |D|^{-1} \mu_1^n. \quad (6.11)
\]
For \(t \geq 4T_0\) and \(x, y \in D\), we let \(n := |(t - 3T_0)/T_0| \geq 1\) and \(s := t - (n + 2)T_0 \in [T_0, 2T_0)\).
Recall \(a(x, y) = L(\delta_D(x))^{-1/2}L(\delta_D(y))^{-1/2}\). By (6.10) and Corollary 4.3, we have
\[
p_D(t, x, y) = \int_{D \times D \times D \times D} p_D(s/2, x, z_1)p_D(T_0, z_1, z_2)p_D(nT_0, z_2, z_3)
\times p_D(T_0, z_3, z_4)p_D(s/2, z_4, y)dz_1dz_2dz_3dz_4
\leq c_0^2 \int_D p_D(s/2, x, z_1)dz_1 \int_D p_D(nT_0, z_2, z_3)dz_2dz_3 \int_D p_D(s/2, z_4, y)dz_4
\leq c_0^2 |D|\mathbb{P}_x(\tau_D > s/2)\mathbb{P}_y(\tau_D > s/2)\mu_1^n \leq ca(x, y) e^{-\lambda_1 t},
\]
where \( \lambda_1 := T_0^{-1} \log(\mu_1^{-1}) \). Moreover, by Theorem 1.5, Corollary 4.3 and (6.11), we get
\[
p_D(t, x, y) = \int_{D \times D} p_D(s/2, x, z_1)p_D((n + 2)T_0, z_1, z_2)p_D(s/2, z_2, y) \, dz_1 \, dz_2
\geq ca(x, y) \int_{D \times D} \mathbb{P}_{z_1}(\tau_D > s/2)p_D((n + 2)T_0, z_1, z_2)\mathbb{P}_{z_2}(\tau_D > s/2) \, dz_1 \, dz_2 \geq ca(x, y)e^{-\lambda_1 t}.
\]
This completes the proof. 

7. Green function estimates

In this section, we provide the proof of Theorem 1.7. Throughout this section, we assume that (D) holds.

**Lemma 7.1.** For all Borel set \( D \) and \( x, y \in D \), we have
\[
\left( 1 \wedge \frac{V(\delta_D(x))}{V(|x - y|)} \right) \left( 1 \wedge \frac{V(\delta_D(y))}{V(|x - y|)} \right) \asymp \left( 1 \wedge \frac{V(\delta_D(x))V(\delta_D(y))}{V(|x - y|)^2} \right). \tag{7.1}
\]
In particular, if \( D \) is bounded, then for all \( x, y \in D \)
\[
\left( 1 \wedge \frac{L(|x - y|)}{L(\delta_D(x))} \right)^{1/2} \left( 1 \wedge \frac{L(|x - y|)}{L(\delta_D(y))} \right)^{1/2} \asymp \left( 1 \wedge \frac{L(|x - y|)}{\sqrt{L(\delta_D(x))L(\delta_D(y))}} \right).
\]

**Proof.** Since \((1 \wedge p)(1 \wedge q) \leq 1 \wedge (pq)\) for every \( p, q > 0 \), one inequality in (7.1) is trivial. On the other hand, since \( 1 \wedge (p/q) \asymp p/(p + q) \) for every \( p, q > 0 \), it suffices to prove that
\[V(\delta_D(x)) + V(|x - y|)V(\delta_D(y)) + V(\delta_D(y)) \leq 2V(\delta_D(x))V(\delta_D(y)) + 4V(|x - y|)^2.
\]
By symmetry, we may assume that \( \delta_D(x) \leq \delta_D(y) \). According to the subadditivity of \( V \),
\[
\begin{align*}
V(\delta_D(x)) + V(|x - y|) & V(\delta_D(y)) + V(|x - y|) \\
& \leq V(\delta_D(x))V(\delta_D(y)) + V(|x - y|)^2 + 2V(|x - y|)V(\delta_D(y)) \\
& \leq V(\delta_D(x))V(\delta_D(y)) + V(|x - y|)^2 + 2V(\delta_D(x))(V(\delta_D(x)) + V(|x - y|)) \\
& \leq V(\delta_D(x))V(\delta_D(y)) + V(|x - y|)^2 + V(|x - y|)^2 + V(\delta_D(x))^2 + 2V(|x - y|)^2 \\
& \leq 2V(\delta_D(x))V(\delta_D(y)) + 4V(|x - y|)^2.
\end{align*}
\]
This proves (7.1). The second claim follows from (3.1). 

**Lemma 7.2.** It holds that
\[
\liminf_{r \to 0} \frac{\nu(r)}{L(r)} = \liminf_{r \to 0} \frac{\nu(r)}{L(r)^2} = \infty.
\]

**Proof.** Since the Lévy measure \( \nu \) is infinite, we have \( \lim_{r \to 0} L(r) = \infty \). Thus, it suffices to show that the second equality holds. By l'Hôpital's rule, \cite[Lemmas 3.1 and 3.2]{13}, (1.3) and (3.3), since \( \alpha_2 < d \), we have
\[
\liminf_{r \to 0} \frac{\nu(r)}{L(r)^2} \geq c \liminf_{r \to 0} \frac{r^{-1} \nu(r)}{2r^{-1} L(r) \ell^{-1}} \geq \frac{c}{L(1)} \liminf_{r \to 0} \frac{L(1) r^{-d}}{L(r)} \geq c \liminf_{r \to 0} r^{-d + (\alpha_2 \sqrt{2} - 1)} = \infty.
\]
Indeed, since \( r \to \nu(r^{-1}) \) satisfies \( W^\infty(d + \alpha_1, d + \alpha_2, 1) \) and \( d + \alpha_1 > 0 \), according to [13, Lemmas 3.1 and 3.2], there exists a function \( \tilde{\nu}(r) \) such that for all \( 0 < r < 1 \), \( \tilde{\nu}(r) \asymp \nu(r) \) and \( -\tilde{\nu}'(r) \asymp r^{-1} \tilde{\nu}(r) \asymp r^{-1} \nu(r) \). Hence, the first inequality in the display holds.

Recall that for a Borel subset \( D \subset \mathbb{R}^d \), the Green function \( G_D(x, y) \) is defined by

\[
G_D(x, y) := \int_0^\infty p_D(t, x, y)dt.
\]

Since the process \( Y \) can be recurrent, we cannot expect to obtain upper estimates for \( G_{\mathbb{R}^d}(x, y) \) in general. However, when \( D \) is bounded, we can establish a prior upper estimates for \( G_D(x, y) \) regardless of transience of \( Y \). By \( \text{diam}(D) \) we denote the diameter of \( D \).

**Lemma 7.3.** Let \( D \subset \mathbb{R}^d \) be a bounded Borel set. Then, there exists a constant \( c_1 = c_1(d, \psi, \text{diam}(D)) > 0 \) such that for all \( x, y \in D \),

\[
G_D(x, y) \leq \frac{c_1 \ell(|x - y|^{-1})}{|x - y|^d L(|x - y|)^2} \asymp \frac{\nu(|x - y|)}{L(|x - y|)^2}.
\]

**Proof.** Fix \( x, y \in D \) and let \( r = |x - y| \). If \( x = y \), by Lemma 7.2, there is nothing to prove. Hence, we assume \( r > 0 \).

By Lemma 6.1, (2.12) and Fubini’s Theorem, we have

\[
r^d G_D(x, y) \leq c \int_0^\infty r^d p(t/2, r/2) \exp(-2^{-1} \kappa_1 C_5 \theta(diam(D))) dt
\]

\[
\leq c \int_0^\infty c r^d \exp(-2^{-1} \kappa_1 C_5 \theta(diam(D))) dt + c \int_0^\infty t \int_r^\infty e^{-C_0 \Phi(u/r)} \ell(u/r) u^{d-1} e^{-u^2/4} du dt
\]

\[
\leq c r^d + c \int_r^\infty \ell(u/r) u^{d-1} e^{-u^2/4} \int_0^\infty te^{-C_0 \Phi(u/r)} dt du
\]

\[
\leq c r^d + c \int_r^\infty \frac{\ell(u/r) u^{d-1} e^{-u^2/4}}{C_0^2 \Phi(u/r)^2} du + c \int_1^\infty \frac{\ell(u/r) u^{d-1} e^{-u^2/4}}{C_0^2 \Phi(u/r)^2} du
\]

\[
=: c r^d + I_1 + I_2.
\]

First, by the change of the variables, we have

\[
I_1 = c r^d \int_1^{1/r} \frac{\ell(u) u^{d-1}}{\Phi(u)^2} du = c r^d \int_1^{1/r} \frac{\ell(u) u^{d-1}}{L(u^{-1})^2} du.
\]

Observe that by applying l’Hospital’s rule three times and the same argument as the one, coming from [13, Lemmas 3.1 and 3.2], given in the proof of Lemma 7.2, we obtain

\[
\limsup_{s \to 0} \frac{s^{d-1/2} \int_1^{1/s} \ell(u) u^{d-1} L(u^{-1})^{-2} du}{\ell(s^{-1}) L(s)^{-2}} = \limsup_{s \to 0} \frac{L(s)^2 \int_1^{1/s} \ell(u) u^{d-1} L(u^{-1})^{-2} du}{s^{d-1} \ell(s^{-1})}
\]

\[
\leq c + c \limsup_{s \to 0} \frac{s^{-d-1} \ell(s^{-1}) L(s) \int_1^{1/s} \ell(u) u^{d-1} L(u^{-1})^{-2} du}{s^{d-1} \ell(s^{-1})}
\]

\[
= c + c \limsup_{s \to 0} \frac{L(s) \int_1^{1/s} \ell(u) u^{d-1} L(u^{-1})^{-2} du}{s^{d-1}}
\]

\[
\leq c + c \limsup_{s \to 0} \frac{\ell(s^{-1}) L(s)^{-1} s^{-d-1}}{s^{d-1}} + c \limsup_{s \to 0} \frac{s^{-d-1} \ell(s^{-1}) \int_1^{1/s} \ell(u) u^{d-1} L(u^{-1})^{-2} du}{s^{d-1}}
\]
\[ \leq c + c \limsup_{s \to 0} \int_1^{1/s} \ell(u)u^{d-1}L(u^{-1})^{-2}du \]
\[ \leq c + c \limsup_{s \to 0} \frac{\ell(s^{-1})L(s)^{-2}s^{-d-1}}{s^{-d}/\ell(s^{-1})} = c + c \limsup_{s \to 0} \frac{\ell(s^{-1})^2}{L(s)^2} \leq c. \]

The fourth inequality above is valid, since we can assume that 
\[-(r^{-d}/\ell(r^{-1}))' \approx r^{-d-1}/\ell(r^{-1})\]
for \(0 < r < 1\) by the argument given in the proof of Lemma 7.2 because \(r \mapsto r^d/\ell(r)\) satisfies \(\mathcal{W}^\infty(d - \alpha_2, d - \alpha_1, 1)\) and \(\alpha_2 < d\). In the third and fifth inequalities above, we used the fact that \(\ell(r^{-1}) \leq cL(r)\) for \(0 < r < 1\), which follows from (2.2) and the proof of Lemma 2.1. Thus, since \(D\) is bounded, we get that \(I_1 \leq c\ell(r^{-1})L(r)^{-2}\).

On the other hand, by the scaling property of \(\ell\) and the monotonicity of \(\Phi\), we obtain
\[ I_2 \leq c \frac{\ell(r^{-1})}{\Phi(r^{-1})^2} \int_1^{\infty} \frac{\ell(u/r)}{\ell(1/r)} u^{d-1} e^{-u^2/4} du \leq c \frac{\ell(r^{-1})}{\Phi(r^{-1})^2} \int_1^{\infty} u^{d-1+\alpha_2} e^{-u^2/4} du = c \frac{\ell(r^{-1})}{L(r)^2}. \]

This completes the proof.

**Proof of Theorem 1.7.** Fix \(x, y \in D\) and set \(a(x, y) := L(\delta_D(x))^{-1/2}L(\delta_D(y))^{-1/2}\). By (1.3) and Lemma 2.1, it suffices to prove that
\[ G_D(x, y) = \int_0^\infty p_D(t, x, y)dt \asymp (1 \wedge [a(x, y)L(|x - y|)]) \frac{\nu(|x - y|)}{h(|x - y|)^2}. \]

**(Lower bound)** By Proposition 5.1, we have that
\[ G_D(x, y) \geq \int_0^1 p_D(t, x, y)dt \]
\[ \geq c \nu(|x - y|) \int_0^1 \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} t \exp(-c_1th(|x - y|))dt \]
\[ = \frac{c \nu(|x - y|)}{h(|x - y|)^2} \int_0^{h(|x - y|)} \left(1 \wedge \frac{h(|x - y|)}{sL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{h(|x - y|)}{sL(\delta_D(y))}\right)^{1/2} se^{-c_1s}ds \]
\[ \geq \frac{c \nu(|x - y|)}{h(|x - y|)^2} \int_0^1 \frac{L(|x - y|)}{L(\delta_D(x))} \left(1 \wedge \frac{L(|x - y|)}{L(\delta_D(y))}\right)^{1/2} \frac{h(2r)^1}{se^{-c_1s}}ds \]
\[ \geq c(1 \wedge [a(x, y)L(|x - y|)]) \frac{\nu(|x - y|)}{h(|x - y|)^2}. \quad (7.2) \]

In the above, we used the change of the variables \(s = th(|x - y|)\) in the third line, the fact that \(h(r) \geq L(r)\) for all \(r > 0\) in the fourth line, and Lemma 7.1 and (3.1) in the fifth line.

**(Upper bound)** Using boundary Harnack principle and Lemma 7.3, one can prove the upper bound following the proofs of [29, Theorem 1.2 and Theorem 6.4] and [32, Theorem 4.6] line by line. Thus, we provide the main steps of the proof only.

By the boundary Harnack principle (see, [23, Theorem 1.9]), Lemma 7.3 and (7.2), we can follow the proof of [29, Theorem 6.4] to obtain
\[ G_D(x, y) \leq c \frac{g_D(x)g_D(y) \nu(|x - y|)}{g_D(A)^2 \cdot h(|x - y|)^2}, \quad (7.3) \]
where \(g_D(z) := G_D(z, z_0) \land c_1\) for some fixed constant \(c_1 > 0, z_0 \in D\) is a fixed point in \(D\) and \(A \in B(x, y)\), where \(B(x, y)\) is given by [29, (6.7)]. Moreover, we can also follow the proof of [32, Theorem 4.6] to show that for all \(z \in D\),
\[ g_D(z) \asymp L(\delta_D(z))^{-1/2}. \quad (7.4) \]
Indeed, let $R_3 := \delta_D(z_0) \wedge R_2$, where $R_2$ is the constant in Lemma 3.7. If $\delta_D(z) \geq R_3/8$, then we get $L(\delta_D(z))^{-1/2} \geq L(R_3/8)^{-1/2} \geq c_1^{-1} L(R_3/8)^{-1/2} g_D(z)$. Moreover, by (7.2) and Lemma 7.2, we also get $g_D(z) \geq c \geq c L(r_2)^{-1/2} \geq c L(\delta_D(z))^{-1/2}$. Hence, (7.4) holds in this case.

Next, we assume $\delta_D(z) < R_3/8$. Then, we get that $|z - z_0| \geq \delta_D(z_0) - \delta_D(z) \geq 7R_3/8$. Therefore, by Lemma 7.2, $g_D(z) \simeq G_D(z, z_0)$. Choose $w_z \in \partial D$ satisfying $\delta_D(z) = |z - w_z|$. Let $z^* := w_z + R_3(z - w_z)/(4|z - w_z|) \in D$ and define $U(z, 1)$ as (4.3). Then, by the boundary Harnack principle, (3.9), Lemma 7.3, (7.2) and Proposition 4.2, we get

$$g_D(z) \simeq G_D(z, z_0) \simeq G_D(z^*, z_0) \simeq \mathbb{P}_z(Y_{\tau_U(z_1)} \in D) \simeq \mathbb{P}_z(Y_{\tau_U(z_1)} \in D) \simeq L(\delta_D(z))^{-1/2}.$$

Hence, we obtain (7.4).

We see from the definition of $B(x, y)$ that $\delta_D(A) \geq c|x - y|$ for some constant $c > 0$. Thus, by combining (7.3) and (7.4), we get from (3.3) that

$$G_D(x, y) \leq ca(x, y)L(\delta_D(A)) \frac{\nu(|x - y|)}{h(|x - y|)^2} \leq ca(x, y)L(|x - y|) \frac{\nu(|x - y|)}{h(|x - y|)^2}.$$

This together with Lemma 7.3 completes the proof. \hfill \Box

8. Example

In this section, we give an example that is covered by our results.

EXAMPLE 8.1. Let $Y = (Y_t : t \geq 0)$ be a pure jump isotropic unimodal Lévy process with Lévy measure $\nu$ satisfying (A) and (B), and $D$ be a $C^{1, 1}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$. Suppose that there exists $p \in [-1, \infty)$ such that

$$\nu(r) \asymp r^{-d}|\log r|^p \quad \text{for } 0 < r \leq 1/2. \quad (8.1)$$

Typical examples of isotropic unimodal Lévy processes satisfying (8.1) are geometric stable processes ($p = 0$) and iterated geometric stable processes ($p = -1$). The condition $p \geq -1$ is necessary to make the Lévy measure $\nu$ be infinite. We let

$$\log(r) := \log(e + r) \quad \text{for } r > 0.$$

Then, for every fixed $R > 0$, we have that for $0 < r \leq R$,

$$\ell(r) \asymp \log(r^{-1})^p \quad \text{and} \quad L(r) \asymp h(r) \asymp \begin{cases} \log(r^{-1})^{p+1}, & \text{if } p > -1; \\ \log \circ \log(r^{-1}), & \text{if } p = -1. \end{cases}$$

We first obtain the small time estimates for the Dirichlet heat kernel. Define for $p > -1$,

$$\mathfrak{B}_p(x, y) := \left(1 \wedge \frac{\log(\delta_D(x)^{-1/2} - (p+1)/2)}{\sqrt{t}}\right) \left(1 \wedge \frac{\log(\delta_D(y)^{-1/2} - (p+1)/2)}{\sqrt{t}}\right),$$

and

$$\mathfrak{B}_{-1}(x, y) := \left(1 \wedge \frac{[\log \circ \log(\delta_D(x)^{-1/2})]^{-1/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{[\log \circ \log(\delta_D(y)^{-1/2})]^{-1/2}}{\sqrt{t}}\right).

\quad (\text{Case } 1) \ p > 0.$$

In this case, (S-1) holds. Note that we do not need the condition (C) when we estimate $p_D(t, x, y)$ only for $|x - y| \leq 1$. Thus, according to Theorem 1.5(ii), for every $T > 0$, there are constants $c_0 > 1, c_1, \ldots, c_6 > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$ satisfying $|x - y| \leq 1$,

$$c_0^{-1} \mathfrak{B}_p(x, y) F_p(t, |x - y|, c_1, c_2, c_3) \leq p_D(t, x, y) \leq c_0 \mathfrak{B}_p(x, y) F_p(t, |x - y|, c_4, c_5, c_6),$$
where

\[
F_p(t,r,a_1,a_2,a_3) := \begin{cases} 
\exp (a_1 t^{-1/p}), & \text{if } r \leq \exp (-a_2 t^{-1/p}); \\
(t \log(r^{-1}) \exp \left((-d + a_2 t \log(r^{-1}) \log r)\right), & \text{if } r > \exp (-a_2 t^{-1/p}). 
\end{cases}
\]

(Case 2) $-1 < p \leq 0$.

Since (S-1) holds, by Theorem 1.5(i), for every $T > 0$, there are constants $c_0 > 1$, $c_1, c_2 > 0$ such that for all $(t, x, y) \in (0, T] \times (D \times D \setminus \text{diag})$ satisfying $|x - y| \leq 1$,

\[
c_0^{-1} \mathfrak{B}_p(x, y) t \log |x - y|^{-1} \exp \left((-d + c_1 t \log |x - y|^{-1}) \log |x - y|\right) \\
\leq p_D(t, x, y) \leq c_0 \mathfrak{B}_p(x, y) t \log |x - y|^{-1} \exp \left((-d + c_2 t \log |x - y|^{-1}) \log |x - y|\right).
\]

(Case 3) $p = -1$.

Since (S-1) holds, by Theorem 1.5(i), for every $T > 0$, there are constants $c_0 > 1$, $c_1, c_2 > 0$ such that for all $(t, x, y) \in (0, T] \times (D \times D \setminus \text{diag})$ satisfying $|x - y| \leq 1$,

\[
c_0^{-1} \mathfrak{B}_1(x, y) t |x - y|^{-d} \log(r^{-1})^{-1-c_1 t} \\
\leq p_D(t, x, y) \leq c_0 \mathfrak{B}_1(x, y) t |x - y|^{-d} \log(r^{-1})^{-1-c_2 t}.
\]

Now, we further assume that $D$ is bounded and of scale $(r_1, r_2)$. Then, we get the following large time estimates.

(Case 1) $p > 0$.

Since either (S-2) or (L-2) holds, by Theorem 1.6(ii) and (iii), there exist $T_1 > 0$ (if $p > 0$, then $T_1 = 0$) and $\lambda_1 > 0$ such that for every $T > T_1$, we have that for all $(t, x, y) \in [T, \infty) \times D \times D$,

\[
p_D(t, x, y) \asymp e^{-\lambda_1 t} \log(\delta_D(x)^{-1})^{-(p+1)/2} \log(\delta_D(y)^{-1})^{-(p+1)/2}.
\]

(Case 2) $-1 < p < 0$.

Since (L-1) holds, by Theorem 1.6(i), for every $T > 0$, there are constants $c_0, c_1, c_2, \lambda_2, \lambda_3 > 0$ such that for all $(t, x, y) \in [T, \infty) \times (D \times D \setminus \text{diag})$,

\[
c_0^{-1} \log(\delta_D(x)^{-1})^{-(p+1)/2} \log(\delta_D(y)^{-1})^{-(p+1)/2} \\
\times \left(|x - y|^{-d+c_1 t \log |x - y|^{-1} \log |x - y|}, e^{-\lambda_2 t}\right) \leq p_D(t, x, y) \\
\leq c_0 \log(\delta_D(x)^{-1})^{-(p+1)/2} \log(\delta_D(y)^{-1})^{-(p+1)/2} \\
\times \left(|x - y|^{-d+c_2 t \log |x - y|^{-1} \log |x - y|}, e^{-\lambda_3 t}\right).
\]

(Case 3) $p = -1$.

Since (S-1) holds, by Theorem 1.6(i), for every $T > 0$, there are constants $c_0, c_1, c_2, \lambda_2, \lambda_3 > 0$ such that for all $(t, x, y) \in [T, \infty) \times (D \times D \setminus \text{diag})$,

\[
c_0^{-1} \left[\log \circ \log(\delta_D(x)^{-1})\right]^{-1/2} \left[\log \circ \log(\delta_D(y)^{-1})\right]^{-1/2} \left(|x - y|^{-d} \log(r^{-1})^{-1-c_1 t} + e^{-\lambda_2 t}\right) \\
\leq p_D(t, x, y) \\
\leq c_0 \left[\log \circ \log(\delta_D(x)^{-1})\right]^{-1/2} \left[\log \circ \log(\delta_D(y)^{-1})\right]^{-1/2} \left(|x - y|^{-d} \log(r^{-1})^{-1-c_2 t} + e^{-\lambda_3 t}\right).
\]

Finally, we obtain the Green function estimates by Theorem 1.7. Let $D$ be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. Note that the Lévy measure $\nu$ satisfies (D). Hence, we have that, for all
and if \( p < -1 \), then

\[
G_D(x, y) \asymp \left( 1 \wedge \frac{\log(\delta_D(x)^{-1}) - (p+1)/2 \log(\delta_D(y)^{-1}) - (p+1)/2}{\log(|x - y|^{-1} - (p+1))} \right) \frac{\log(|x - y|^{-1} - (p+2))}{|x - y|^d}
\]

and if \( p = -1 \), then

\[
G_D(x, y) \asymp \left( 1 \wedge \frac{\log(\delta_D(x)^{-1}) - (p+1)/2 \log(\delta_D(y)^{-1}) - (p+1)/2}{\log(|x - y|^{-1} - (p+1))} \right) \frac{\log(|x - y|^{-1} - (p+2))}{|x - y|^d}
\]

\[
\times \log(|x - y|^{-1} - (p+1)) \frac{\log(|x - y| - (p+2))}{|x - y|^d}.
\]

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