Extensions of Peripheric Extended Twists and Inhomogeneous Lie Algebras

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Abstract

Simple extensions of peripheric extended twists, introduced recently by Lyakhovsky and Del Olmo, are presented. Explicit form of twisting elements are given and it is shown that the new twists as well as peripheric extended twists are suitable to deform inhomogeneous Lie algebras such as $isu(n)$, $iso(n)$, $(1+n)$ dimensional Schrödinger algebras and Poincaré algebra.

Keywords: Quantum algebras, twist, inhomogeneous Lie algebras
1 Introduction

It is no doubt that inhomogeneous Lie algebras play a crucial role in physics. One can readily imagine the Poincaré algebra, the Galilei algebra, the Schrödinger algebra and so on. If we expect that quantizations of these Lie algebras also play an important role in theoretical physics, various possibilities of quantization should be studied. In this article, we consider quantizations by means of Drinfel’d twist \cite{1}, that preserves the triangularity of Lie algebras, along the line of recent developments \cite{2-8}. Quantization of a Lie algebra $g$ by Drinfel’d twist is defined by an invertible element $F \in U(g) \otimes U(g)$ subject to the relations

$$\begin{align*}
F_{12}(\Delta \otimes \text{id})(F) &= F_{23}(\text{id} \otimes \Delta)(F), \\
(\epsilon \otimes \text{id})(F) &= (\text{id} \otimes \epsilon)(F) = 1,
\end{align*}$$

(1.1)

where $\Delta$ and $\epsilon$ denote the coproduct and the counit of $U(g)$, respectively. The quantized Lie algebra $U_F(g)$ has the same commutation relations as $g$ and deformed coproducts $\Delta_F = F \Delta F^{-1}$. Explicit forms of the twist elements $F$ are important because not only the coproducts but also deformation of all other quantities are caused by the twisting element. Especially, the universal $R$-matrix for $U_F(g)$ is given by $R_F = F_{21}F^{-1}$ Note that irreducible representations for $U_F(g)$ are exactly the same as $g$ because of undeformed commutation relations. Therefore once an explicit form of twist element is obtained, matrix solutions of quantum Yang-Baxter equation are immediately calculated, so that it is an easy exercise to construct a quantum group dual to $U_F(g)$, differential calculi covariant under the quantum group and so on. Some physical application of twisting are discussed in \cite{9, 10}.

There are not so many literatures (to the author’s knowledge) discussing twist deformation of inhomogeneous Lie algebras. Twist deformation of the Poincaré algebra based on an abelian subalgebra is discussed in \cite{11}. In Ref.\cite{12}, the Jordanian twists \cite{13, 14} are generalized to multidimension and application to the Poincaré algebra is considered. Other approaches to obtain triangular deformation of inhomogeneous Lie algebras are found in \cite{15-19} (see also the references therein). We construct, in this article, explicit forms of twisting elements that are applicable to various inhomogeneous Lie algebras including the Poincaré algebra. Our twisting elements are extensions of the peripheric extended twists (PET). The PET was introduced in \cite{3} as nontrivial limits of extended Jordanian twists \cite{2}. Four dimensional subalgebras $h \subset g$ are considered in both the PET and the extended Jordanian twists. Then, regarding $F \in U(h) \otimes U(h)$ as a twisting element for $U(g)$, a twisted algebra $U_F(g)$ is constructed by the $F$.

The plan of this article is as follows: In the next section, after a brief review of PET, we extend the PET by adding one more generator to the four dimensional subalgebra of PET. As a result, we obtain four different twisting elements. In §3, the twisting elements are applied to $isu(n)$, $iso(n)$, Schrödinger algebras and Poincaré algebra. §4 is a conclusion.

2 Peripheric extended twists and their extensions

2.1 Peripheric extended twists

The Jordanian twists for a Lie algebra $g$ is a quantization using a Borel subalgebra $\{H, E\} \in g$ \cite{13, 14}. By adding two more elements $A, B$ to the Borel subalgebra, the subalgebra used in
extended Jordanian twists is obtained [2]. The extended Jordanian twists have two nontrivial limits specified by subalgebras $\mathbf{L}^c$ and $\mathbf{L}'^c$ [3]. The elements of subalgebras $\mathbf{L}^c$ and $\mathbf{L}'^c$ are denoted again by $\{H, E, A, B\}$. The algebra $\mathbf{L}^c$ is defined by the commutation relations

\[
[H, E] = \delta E, \quad [H, A] = 0, \quad [H, B] = \delta B,
\]

\[
[A, B] = \gamma E, \quad [E, A] = [E, B] = 0,
\]

(2.1)

and $\mathbf{L}'^c$ by

\[
[H, E] = \delta E, \quad [H, A] = \delta A, \quad [H, B] = 0,
\]

\[
[A, B] = \gamma E, \quad [E, A] = [E, B] = 0,
\]

(2.2)

where $\gamma$ and $\delta$ are arbitrary nonzero complex numbers. One can say that $\mathbf{L}^c$ and $\mathbf{L}'^c$ are isomorphic, since

\[
H \leftrightarrow H, \quad E \leftrightarrow E, \quad A \leftrightarrow -B, \quad B \leftrightarrow A,
\]

(2.3)

give the isomorphism. However, it turns out that $\mathbf{L}^c$ and $\mathbf{L}'^c$ give different twists as limits of extended Jordanian twists [3]. We thus treat these two cases separately.

In the case of $\mathbf{L}^c$, the twisting element of PET is given by

\[
\mathcal{F}_P = \Phi_P \Phi_j, \quad \Phi_P = \exp(A \otimes B e^{-\delta \sigma}), \quad \Phi_j = e^{H \otimes \sigma},
\]

(2.4)

where

\[
\sigma = \frac{1}{\delta} \ln(1 + \gamma E),
\]

(2.5)

and $\Phi_j$ is the twisting element of the Jordanian twist. The twisting element $\mathcal{F}_P$ leads to the algebra $U_P(\mathbf{L}^c)$ with the twisted coproduct $\Delta_P \equiv \mathcal{F}_P \Delta F_P^{-1}$:

\[
\Delta_P(H) = H \otimes e^{-\delta \sigma} + 1 \otimes H - \delta A \otimes B e^{-2\delta \sigma},
\]

\[
\Delta_P(A) = A \otimes e^{-\delta \sigma} + 1 \otimes A,
\]

\[
\Delta_P(B) = B \otimes e^{\delta \sigma} + e^{\delta \sigma} \otimes B,
\]

\[
\Delta_P(E) = E \otimes e^{\delta \sigma} + 1 \otimes E.
\]

(2.6)

It is important to note that there exist two primitive elements in $U_P(\mathbf{L}^c)$, namely, $\sigma$ and $B e^{-\delta \sigma}$.

In the case of $\mathbf{L}'^c$, the twisting element is given by

\[
\mathcal{F}_{P'} = \Phi_{P'} \Phi_j, \quad \Phi_{P'} = e^{A \otimes B},
\]

(2.7)

where $\Phi_j$ is same as (2.4). We denote the twisted algebra by $U_{P'}(\mathbf{L}'^c)$ and the twisted coproducts by $\Delta_{P'} \equiv \mathcal{F}_{P'} \Delta F_{P'}^{-1}$:

\[
\Delta_{P'}(H) = H \otimes e^{-\delta \sigma} + 1 \otimes H - \delta A \otimes B e^{-\delta \sigma},
\]

\[
\Delta_{P'}(A) = A \otimes 1 + 1 \otimes A,
\]

\[
\Delta_{P'}(B) = B \otimes 1 + e^{\delta \sigma} \otimes B,
\]

\[
\Delta_{P'}(E) = E \otimes e^{\delta \sigma} + 1 \otimes E.
\]

(2.8)
Again there exist two primitive elements in $U_P(L^c)$, namely, $\sigma$ and $A$.

Both twisting elements $F_P$ and $F_{P'}$ satisfy the relations

$$\Delta \otimes \text{id}(F) = F_{13}F_{23}, \quad (\text{id} \otimes \Delta_F)(F) = F_{12}F_{13}. \quad (2.9)$$

These relations guarantee that the twisting elements satisfy (1.1). It is also possible to regard $\Phi_P$ and $\Phi_{P'}$ as twisting elements for the Jordanian deformed algebras, that is, we start with algebras with twisted coproduct $\Delta_j = \Phi_j \Delta \Phi_j^{-1}$, then consider the twisting by $\Phi_P$ or $\Phi_{P'}$. In such situation, it turns out that $\Phi_P$ and $\Phi_{P'}$ have different factorizable properties stems from the fact that $U_P(L^c)$ and $U_P(L^{\infty})$ have different primitive elements. The twisting element $\Phi_P$ satisfies the same relation as (2.9), while $\Phi_{P'}$ satisfies

$$\Delta_{P'} \otimes \text{id}(\Phi_{P'}) = (\Phi_{P'})_{13}(\Phi_{P'})_{23}, \quad (\text{id} \otimes \Delta_j)(\Phi_{P'}) = (\Phi_{P'})_{12}(\Phi_{P'})_{13}. \quad (2.10)$$

Although this is not mentioned in [3], these PET are appropriate for twisting inhomogeneous Lie algebras as we shall see later.

### 2.2 Extensions of PET

The existence of primitive elements except $\sigma$ allows us to extend the PET with an extra Jordanian like factor. To this end, we need one more generator $J$ and consider five dimensional subalgebras: $L = L^c \cup \{J\}$ and $L' = L^{\infty} \cup \{J\}$. Additional commutation relations to define $L$ are given by

$$[J, H] = [J, E] = 0, \quad [J, A] = -\mu A, \quad [J, B] = \mu B, \quad (2.11)$$

and for $L'$

$$[J, H] = [J, E] = 0, \quad [J, A] = \mu A, \quad [J, B] = -\mu B, \quad (2.12)$$

where $\mu$ is an arbitrary complex constant. We define for $L$

$$\Phi = e^{J \otimes \rho}, \quad \rho = \frac{1}{\mu} \ln(1 + \mu Be^{-\delta A}), \quad (2.13)$$

and for $L'$

$$\Phi' = e^{J \otimes \rho'}, \quad \rho' = \frac{1}{\mu} \ln(1 + \mu A). \quad (2.14)$$

Note that $\{J, Be^{-\delta A}\}$ in (2.13) and $\{J, A\}$ in (2.14) satisfy the commutation relation of Borel subalgebra, respectively. Then it can be verified that followings are twisting elements for $L$:

$$F = \Phi F_P, \quad \tilde{F} = \Phi_{21}F_P, \quad (2.15)$$

and for $L'$:

$$F' = \Phi' F_{P'}, \quad \tilde{F}' = \Phi'_{21}F_{P'}. \quad (2.16)$$

To show this, we first calculate PET for $J$. It is easy to see that $J$ is primitive with respect to both twisting by $L^c$ and $L^{\infty}$:

$$\Delta_P(J) = \Delta_{P'}(J) = J \otimes 1 + 1 \otimes J. \quad (2.17)$$
We next calculate twisted coproduct for the elements of \( L \) and \( L' \) by \( \mathcal{F}(\tilde{F}) \) and \( \mathcal{F}'(\tilde{F}') \), respectively. Twisted coproducts of \( L \) by \( \mathcal{F} \) are given by

\[
\begin{align*}
\Delta_\mathcal{F}(J) &= J \otimes e^{-\mu} + 1 \otimes J, \\
\Delta_\mathcal{F}(H) &= H \otimes e^{-\delta} + 1 \otimes H - J \otimes e^{-\mu} \{ \delta - 1 \} Be^{-\delta} + Be^{-2\delta} \\
\Delta_\mathcal{F}(A) &= A \otimes e^{-\delta - \mu} + 1 \otimes A - \gamma J \otimes e^{-\delta - \mu}, \\
\Delta_\mathcal{F}(B) &= B \otimes e^{\delta + \mu} + e^{\delta} \otimes B, \\
\Delta_\mathcal{F}(E) &= E \otimes e^{\delta} + 1 \otimes E.
\end{align*}
\]

Twisted coproducts of \( L \) by \( \tilde{F} \) are given by

\[
\begin{align*}
\Delta_{\tilde{F}}(J) &= J \otimes 1 + e^{-\mu} \otimes J, \\
\Delta_{\tilde{F}}(A) &= A \otimes e^{-\delta} + e^{-\mu} \otimes A - \gamma J \otimes e^{-\delta - \mu} \otimes Je^{-\sigma}, \\
\Delta_{\tilde{F}}(B) &= B \otimes e^{\delta} + e^{\delta + \mu} \otimes B, \\
\Delta_{\tilde{F}}(E) &= E \otimes e^{\delta} + 1 \otimes E.
\end{align*}
\]

It seems to be difficult to have a closed form of \( \Delta_{\tilde{F}}(H) \) because of the last term of \( \Delta_\mathcal{F}(H) \). We therefore do not give it here. Twisted coproducts of \( L' \) by \( \mathcal{F}' \) are given by

\[
\begin{align*}
\Delta_{\mathcal{F}'}(J) &= J \otimes e^{-\mu'} + 1 \otimes J, \\
\Delta_{\mathcal{F}'}(A) &= A \otimes e^{\mu'} + 1 \otimes A, \\
\Delta_{\mathcal{F}'}(B) &= B \otimes e^{-\mu'} + e^{\delta} \otimes B + \gamma J e^{\delta} \otimes E e^{-\mu'}, \\
\Delta_{\mathcal{F}'}(E) &= E \otimes e^{\delta} + 1 \otimes E.
\end{align*}
\]

It also seems to be difficult to have a closed form of \( \Delta_{\tilde{F}'}(H) \) because of the last term of \( \Delta_{\mathcal{F}'}(H) \). We do not give it here. Twisted coproducts of \( L' \) by \( \tilde{F}' \) are given by

\[
\begin{align*}
\Delta_{\tilde{F}'}(J) &= J \otimes 1 + e^{-\mu} \otimes J, \\
\Delta_{\tilde{F}'}(H) &= H \otimes e^{-\delta} + 1 \otimes H + \frac{\delta}{\mu}(e^{-\mu'} - 1) \otimes J e^{-\delta} - \delta A e^{-\mu'} \otimes B e^{-\delta}, \\
\Delta_{\tilde{F}'}(A) &= A \otimes 1 + e^{\mu} \otimes A, \\
\Delta_{\tilde{F}'}(B) &= B \otimes e^{-\mu'} \otimes B + \gamma J e^{-\mu'} \otimes J, \\
\Delta_{\tilde{F}'}(E) &= E \otimes e^{\delta} + 1 \otimes E.
\end{align*}
\]

From these relations, it is seen that \( \sigma, \rho \) and \( \rho' \) are primitive in all cases. Together with (2.17), it follows that

\[
(\Delta_\alpha \otimes id)(\Psi) = \Psi_{13} \Psi_{23}, \quad (id \otimes \Delta_f)(\Psi) = \Psi_{12} \Psi_{13},
\]

for \((\alpha, f, \Psi) = (P, \mathcal{F}, \Phi), (P', \mathcal{F}', \Phi')\) and

\[
(\Delta_f \otimes id)(\Psi) = \Psi_{13} \Psi_{23}, \quad (id \otimes \Delta_\alpha)(\Psi) = \Psi_{12} \Psi_{13},
\]

for \((\alpha, f, \Psi) = (P, \tilde{F}, \Phi_{21}), (P', \tilde{F}', \Phi_{21})\). The relations (2.22) and (2.23) imply that \( \Phi \) and \( \Phi_{21} \) (resp. \( \Phi' \) and \( \Phi_{21}' \)) are twisting elements for \( U_P(L) \) (resp. \( U_{P'}(L') \)). It is, indeed, easy to verify the second relation of (1.1). Thus the statement has been proved.
We can introduce a deformation parameter \( z \) in the above expressions by the following replacement

\[
\begin{align*}
E &\rightarrow zE, \quad B \rightarrow zB, \quad \text{for } L, \\
E &\rightarrow zE, \quad A \rightarrow zA, \quad \text{for } L'.
\end{align*}
\] (2.24)

Undeformed algebras are recovered in the limit of \( z = 0 \). Let us consider the classical \( r \)-matrices obtained from the universal \( R \)-matrices given by twisting elements (2.15) and (2.16). Classical \( r \)-matrices are obtained by keeping up to the first order in \( z \) in the expansion of the universal \( R \)-matrices. The classical \( r \)-matrices obtained from \( R = F_{21}F^{-1} \) and \( \tilde{R} = \tilde{F}_{21}\tilde{F}^{-1} \) are

\[
\pm J \wedge B + A \wedge B + \frac{\gamma}{\delta} H \wedge E,
\] (2.25)

where + (−) corresponds to \( R \) (\( \tilde{R} \)). The classical \( r \)-matrices obtained from \( R' = F'_{21}F'^{-1} \) and \( \tilde{R}' = \tilde{F}'_{21}\tilde{F}'^{-1} \) are

\[
\pm J \wedge A + A \wedge B + \frac{\gamma}{\delta} H \wedge E,
\] (2.26)

where + (−) corresponds to \( R' \) (\( \tilde{R}' \)). These classical \( r \)-matrices (2.25) and (2.26) solve the classical Yang-Baxter equation.

3 Twist deformation of inhomogeneous Lie algebras

In this section, we show that many inhomogeneous Lie algebras can be twisted by the twisting elements (2.15) and (2.16). The algebras considered are \( isu(n) \), \( iso(n) \), the Shrödinger algebra in \((1 + n)\) spacetime. We also treat the Poincaré algebra in \((1 + 3)\) spacetime separately because of its physical importance. What is necessary is to show that these algebras have the subalgebras \( L \) and \( L' \). It is, however, enough to show the existence of \( L \), because the subalgebra \( L' \) is obtained by making use of the replacement (2.3). Note that \( L \) and \( L' \) contain the subalgebra \( L^c \) and \( L'^c \) of PET so that inhomogeneous Lie algebras considered in this section admit PET, too.

The Lie algebra \( isu(n) \) is defined by

\[
\begin{align*}
[U^a_b, U^c_d] &= U^a_d\delta_b^c - U^c_b\delta_d^a, \\
[U^a_b, P^c] &= P^a\delta_b^c, & [U^a_b, P_c] &= -P_b\delta^a_c, \\
[P^a, P^b] &= [P^a, P_b] &= 0,
\end{align*}
\] (3.1)

where \( a, b, c, d \in \{1, 2, \cdots, n\} \). The subalgebra \( L \) exists for \( n \geq 4 \) and given by

\[
\begin{align*}
J &= \sum_{2 \leq k \leq n/2} (U^k_{n - k} - U^{n - k}_k), & H &= U^1_1 - U^n_n, & E &= P^1 + P_n, \\
A &= \sum_{2 \leq k \leq n/2} \alpha_k \{P^k - P_k - i(P^{n - k} - P_{n - k})\}, \\
B &= \sum_{2 \leq k \leq n/2} \beta_k \{U^1_{n - k} + U^k_n + i(U^1_{n - k} + U^{n - k}_n)\},
\end{align*}
\] (3.2)
where $\alpha_k$ and $\beta_k$ are arbitrary constant satisfying the relation
\[
\gamma = -2 \sum_{2 \leq k \leq n/2} \alpha_k \beta_k.
\]  
(3.3)

Other parameters in $L$ are $\delta = 1$, $\mu = i$.

The Lie algebra $iso(n)$ is generated by $Y_{ab} = -Y_{ba}$ and $P_a$ satisfying the relations
\[
[Y_{ab}, Y_{cd}] = Y_{ad}\delta_{bc} + Y_{bc}\delta_{ad} - Y_{ac}\delta_{bd} - Y_{bd}\delta_{ac},
\]
\[
[Y_{ab}, P_c] = P_a\delta_{bc} - P_b\delta_{ac},
\]
\[
[P_a, P_b] = 0,
\]
\[
\text{where } a, b, c, d \in \{1, 2, \cdots, n\}. \text{ The subalgebra } L \text{ is found for } n \geq 4 \text{ and given by }
\]
\[
J = \sum_{2 \leq k \leq n/2} Y_{k,n-k+1}, \quad H = Y_{1n}, \quad E = P_1 + iP_n,
\]
\[
A = \sum_{2 \leq k \leq n/2} \alpha_k(P_k - iP_{n-k+1}),
\]
\[
B = \sum_{2 \leq k \leq n/2} \beta_k(Y_{1k} - iY_{kn} + iY_{1n-k+1} + Y_{n-k+1}n),
\]
where $\alpha_k$ and $\beta_k$ are arbitrary constant satisfying the same relation as (3.3). Other parameters in $L$ are $\delta = \mu = i$.

We next consider the centrally extended Schrödinger algebra in $(1+n)$ spacetime [21]. It is generated by $\frac{1}{2}n(n+3) + 4$ elements, namely, $P_t$: time translation, $P_a$: space translations, $G_a$: Galilei transformations, $J_{ab}$: rotations, $K$: conformal transformation, $D$: dilatation and $M$: center (mass). All suffixes range from 1 to $n$. The nonvanishing commutation relations are given by
\[
[P_t, D] = 2P_t,
\]
\[
[P_a, G_a] = P_a,
\]
\[
[P_t, K] = D,
\]
\[
[D, K] = 2K,
\]
\[
[P_a, G_b] = \delta_{ab}M,
\]
\[
[J_{ab}, J_{cd}] = \delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{ad}J_{bc} - \delta_{bc}J_{ad}.
\]
(3.6)

The subalgebra $L$ is found for $n \geq 2$:
\[
J = i \sum_{<k>} J_{k,k+1} + D, \quad H = i \sum_{<k>} J_{k,k+1} + D, \quad B = P_t,
\]
\[
A = \sum_{<k>} \alpha_k(G_k + iG_{k+1}), \quad E = \sum_{<k>} \alpha_k(P_k + iP_{k+1}),
\]
(3.7)

where $\sum_{<k>}$ means that $k$ takes odd integers ranging from 1 to $n - 1$ (or $n - 2$) for even $n$ (odd $n$) and $\alpha_k$’s are arbitrary nonzero constants. The parameters appearing in $L$ are $\delta = 2$, $\gamma = -1$ and $\mu = -2$.

Our final example is the Poincaré algebra in $(1+3)$ spacetime. The $(1+3)$ Poincaré algebra is generated by $P_t$: time translation, $P = (P_1, P_2, P_3)$: space translations, $J = (J_1, J_2, J_3)$: rotations and $K = (K_1, K_2, K_3)$: boosts. Their commutation relations are given by
\[
[J, P_t] = 0, \quad [J, J] = J, \quad [J, P] = P, \quad [J, K] = K, \quad [P_t, P] = 0,
\]
\[
[P_t, K] = P, \quad [P, P] = 0, \quad [K, K] = -J, \quad [P, K] = P_t,
\]
(3.8)
where \([P, K] = P_t\) means \([P_a, P_b] = \delta_{ab} P_t\), \([J, J] = J\) means \([J_a, J_b] = \epsilon_{abc} J_c\) with antisymmetric tensor \(\epsilon_{abc}\) and so on. The subalgebra \(L\) is given by

\[
J = J_3, \quad H = K_3, \quad E = P_t - P_3, \\
A = P_1 + iP_2, \quad B = J_1 - iJ_2 + iK_1 + K_2,
\]

and other parameters are \(\delta = 1, \gamma = 2i\) and \(\mu = i\). The classical \(r\)-matrices are obtained from \((2.25)\)

\[
r = (\pm J_3 + P_+) \wedge (J_- + iK_-) + 2iK_3 \wedge (P_t - P_3),
\]

where \(J_- \equiv J_1 - iJ_2, \quad P_+ \equiv P_1 + iP_2\) and \(K_- \equiv K_1 - iK_2\).

Some remarks are in order: (i) One can deform inhomogeneous Lie algebras just by twisting their semi-simple or abelian part. In the above examples, however, the semi-simple part and abelian part are mixed in the each factor of the twisting element \(F\). This indicates that the deformation presented above is nontrivial in the sense that the deformation is not caused just by the semi-simple or abelian parts. (ii) Our deformation of Poincaré algebra is different from the ones in \([11, 12]\). The twisting elements of \([11]\) consist of only the abelian part, while the twisting element of \([12]\) is a product of Jordanian twisting factors. (iii) All possible classical \(r\)-matrices for the Poincaré algebra are classified in \([21]\). Our \(r\)-matrices \((3.10)\) may corresponds to the case of \(c = H \wedge X_+, \quad a = 0\) in Table 1 of \([21]\). (iv) We dealt with only Poincaré algebras in this article. A classification of quantum Poincaré groups is given in \([22]\). (v) The Poincaré generators of \((3.9)\) are expressed in a light-cone type basis. A deformation of the Poincaré algebra in light-cone type basis (null-plane basis) is discussed in \([16]\). The deformation presented in \([16]\) is simpler than the one given by \((3.9)\) in the sense that the classical \(r\)-matrix \((3.10)\) has more terms than the one in \([16]\). The peculiarity of the \(r\)-matrix \((3.10)\) is that it contains the term consisting of rotations : \(J_3 \wedge J_-\). Such term is not in the \(r\)-matrix in \([16]\).

4 Conclusion

The explicit form of twisting elements that are extensions of PET were constructed in this article. Each extensions have two primitive elements. It was also shown that our twisting elements as well as PET were suitable to deform various inhomogeneous Lie algebras. Some of the twisted algebras considered here may admit additional Reshetikhin twists \([23]\) that are twisting based on an abelian subalgebra. It is shown in \([4]\) that additional Reshetikhin twist to PET gives an equivalent result as extended twist for the case of \(sl(3)\). It may be interesting to consider such mechanism for the twisted inhomogeneous Lie algebras. We considered only one parametric deformation. However it is also interesting problem to consider possibilities of multiparametric deformation. There may exist two different approach to this problem: (i) seeking a possible subalgebra (such as \(L\) considered in this article) that admits multiparametric twisting, (ii) introducing additional twisting elements that commute with the ones considered here. The second approach may be easier. This will be a future work.
Acknowledgments

The author would like to express his sincere thanks to Prof. H.-D. Doebner for his warm hospitality at Technical University of Clausthal where this work was done. He also thanks Prof. J. Lukierski for helpful comments. This work was supported by the Ministry of Education, Science, Sports and Culture, Japan.

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