GENERALIZED BINOMIAL DISTRIBUTION IN PHOTON STATISTICS

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Abstract

Photon-number distribution among two parts of a given volume is found in case of an arbitrary photon statistics. This problem is related to the interaction of light beam with a macroscopic device, for example a diaphragm, that separates the photon flux into two parts with known probabilities. To solve this problem, a Generalized Binomial Distribution (GBD) is derived that is applicable to an arbitrary photon statistics satisfying probability convolution equations. It is shown that if photons obey the Poisson statistics then the GBD is reduced to the ordinary binomial distribution whereas in case of Bose-Einstein statistics the GBD is reduced to the Polya distribution. In this case, the photon spatial distribution depends on the phase-space volume occupied by the photons. This result involves a photon bunching effect, or such collective behavior of photons that sharply differs from the behavior of classical particles. It is shown that the photon bunching effect looks similar to the quantum interference effect.

Keywords: Bose-Einstein statistics, Polya distribution, photon bunching, quantum interference

1 Introduction

This article examines spatial distribution of photons in a light beam of arbitrary photon statistics. If photons obey the Bose-Einstein (BE) statistics then the photon spatial distribution is found to exhibit certain features that are termed here as the photon bunching effect, or such collective behavior of photons that differs sharply from the behavior of classical particles.

The term photon bunching is sometimes related to the Brown-Twiss effect, which is explained by intensity fluctuations in the light beam [1]. In this paper, the term photon bunching is used in an entirely different sense, since the effects considered in this article are not associated with the intensity fluctuations. As shown in Sections 4–5, the photon bunching effect in the BE statistics is rather similar to the quantum interference effect, which was first observed in [2].

The statement of the problem is presented in Section 2 together with basic formula for the flux of classical particles obeying Poisson statistics and for the flux of quantum particles obeying the BE statistics.

To solve the stated problem, a generalized binomial distribution (GBD) is derived in Section 3 for arbitrary photon statistics. In the BE statistics, the GBD is reduced to the Polya distribution, which predicts the photon bunching effect that is discussed in Section 4.

The equivalence of various statistical problems is discussed in Section 5 implying the possibility to apply the results obtained to other equivalent problems involving interaction of photon flux with a beamsplitter, photodetector, or neutral filter. The standard solution of the stated problem is considered in Section 6, where it is proved that the standard solution has very limited domain of applicability. Main conclusions of this work are summarized in Section 7.

A theoretical approach developed in this paper allows one to study subtle features of spatial distribution of particles in the BE statistics, as well as in any other photon statistics. In the limit of large phase volumes the results obtained coincide with the known classical solution while in the limit of small phase volumes the results are consistent with the experiment [3], which examined the phenomenon of quantum interference between two photons incident on the same input port of a beamsplitter.

2 Interaction of light beam with a diaphragm from the viewpoint of photon statistics

2.1 Statement of the problem

Consider a beam of light that has uniform distribution of statistical properties over its cross-section. Let the beam cross-section s be divided into two arbitrary parts, a and b. Photon statistics in part a is measured during certain sampling time τ, i.e. in sampling volume A = acτ, where c is the speed of light. Capital characters A, B, and S will
be used in this paper to designate sampling volumes corresponding to light beam cross-section areas \( a, b \) and \( s \) (Fig. 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Light beam cross-section: sampling volumes \( A \) and \( S \) correspond to possible detector apertures in photon statistics measurement.}
\end{figure}

If a sampling volume is normalized to coherence volume then it will correspond to a certain phase-space volume because a coherence volume corresponds to a single phase-space cell. Therefore, volumes of \( A, B \) and \( S = A + B \) may be considered as dimensionless phase-space volumes consisting of arbitrary number of cells.

Since the light beam is supposed to have uniform properties through its cross-section, probabilities \( \alpha \) and \( \beta \) for a photon to be in volume \( A \) or \( B \), correspondingly, are given by

\[
\alpha = \frac{A}{A + B}, \quad \beta = \frac{B}{A + B},
\]

so that \( \alpha + \beta = 1 \).

Photon statistics in an arbitrary volume will be discussed in this article. Under photon statistics in volume \( A \), an infinite series \( p_n (A) \) is understood, where \( n = 0, 1, 2, \ldots \) and \( p_n (A) \) is the probability that \( n \) photons are in volume \( A \).

The following problem is considered in this work:

Assume photon statistics in \( S = A + B \) is known. Also, probabilities \( \alpha \) and \( \beta \) for a photon to be in volumes \( A \) and \( B \), correspondingly, are given. Then what is the probability distribution \( W(k, n - k) \) for \( k \) photons to be in \( A \) and \( n - k \) photons to be in \( B \)? Both classical and quantum solutions to this problem are discussed in this work. The results obtained for quantum statistics, as shown below, include photon bunching effect that looks similar to the quantum interference effect.

### 2.2 A flux of classical particles

For the beginning, let beam \( S \) consist of classical non-interacting particles that appear in the selected volume independently of each other, so that there are no correlations between the particles. In this case, statistics \( p_n (S) \) in volume \( S = A + B \) is related to the relevant statistics in \( A \) and \( B \) by classical equation of probability convolution

\[
p_n (A + B) = \sum_{k=0}^{n} p_k (A)p_{n-k} (B),
\]

Equation (2) is known to be applicable to random processes in areas of \( A \) and \( B \) only if there are no correlations between these random processes [4]. In addition, (2) takes into account the conservation of energy, or the number of particles \( n = k + (n - k) \), if light beam \( S \) is separated into two parts \( A \) and \( B \).

Classical non-interacting particles also obey Poisson statistics in arbitrary volume \( A \)

\[
p_k (A) = \frac{(wA)^k}{k!} e^{-wA},
\]

where \( w \) is the average number of particles per unit volume. The Poisson statistics is known to hold for photons in a coherent state of radiation field [5], i.e. in case of amplitude stabilized laser radiation.

Substituting (3) in (2) yields

\[
\frac{(A + B)^n}{n!} = \sum_{k=0}^{n} \frac{A^k B^{n-k}}{k!(n-k)!}.
\]

This equation is an identity, which after multiplying both sides by \( n! \) turns into the binomial theorem. Therefore, the Poisson statistics (3) satisfies the probability convolution equation (2).

All the above are well-known facts that are presented here for the sake of convenience in comparing classical and quantum solutions.

### 2.3 A flux of quantum particles satisfying the Bose-Einstein statistics

Now let beam \( S \) be a blackbody radiation so that the photons in beam \( S \) satisfy the BE statistics:

\[
p_k = \frac{w^k}{(1 + w)^{k+1}},
\]

where \( w \) is the degeneracy parameter, or the average number of particles in a coherence volume. Writing photon statistics in this form, one assumes that the sampling volume in the statistics measurement coincides with the coherence volume, i.e. a single phase-space cell.

The BE statistics in an arbitrary phase-space volume \( A \) is given by:

\[
p_k (A) = \frac{A(A+1)...(A+k-1)}{k!} \frac{w^k}{(1 + w)^{k+1}}.
\]

This formula was derived by Leonard Mandel from combinatorial considerations [6]. If \( A = 1 \) then (6) becomes the usual expression for the BE statistics in a single cell of the phase-space.

Statistics (6) can be conveniently presented as

\[
p_k (A) = \frac{A^k}{k!} \frac{w^k}{(1 + w)^{k+1}},
\]

where

\[
A^k = A(A+1)...(A+k-1)
\]
is the rising factorial, or Pochhammer’s symbol.

We saw above that the Poisson statistics satisfies probability convolution equations (2). Let us check whether the BE statistics does satisfy (2) or not. Substituting (7) in (2) after obvious reductions yields

\[
\frac{(A + B)^n}{n!} = \sum_{k=0}^{n} \frac{A^k B^{n-k}}{k! (n-k)!}.
\]

(9)

This is the well-known Vandermonde’s identity \(\text{[7]}\), which is a generalization of the binomial theorem \(\text{[4]}\) for rising factorials. A proof of this identity is given in Appendix 1.

Thus, the BE statistics \(\text{[7]}\) satisfies the probability convolution equations (2). This is due to the fact that the BE statistics ignores intensity fluctuations in the light beam, and hence does not take into account correlations of photon numbers in the volumes of \(A\) and \(B\), which is the well known fact that was proven, among other sources, in \(\text{[5]}\).

Along with the BE statistics, the Glauber’s statistics \(\text{[9]}\) for a homogeneously broadened spectral line also satisfies \(\text{[2]}\), which can be shown by direct verification (see Appendix 2).

Generally speaking, any proper photon statistics\[^1\] should satisfy the probability convolution equation (2) because photon statistics cannot take into account photon correlations. This conclusion is explained in the density matrix approach by the fact that photon correlations are described by nondiagonal elements while photon statistics is represented by the diagonal elements of the density matrix. In semiclassical approach, to account for photon correlations one should examine the intensity correlation function rather than photon statistics. This issue is discussed in detail in \(\text{[5]}\).

3 Generalized Binomial Distribution

Dividing \(n\)-th equation in (2) by \(p_n(A + B)\) yields

\[
\sum_{k=0}^{n} W(k, n-k) = 1,
\]

(10)

where

\[
W(k, n-k) = \frac{p_k(A) p_{n-k}(B)}{p_n(A + B)}.
\]

(11)

The idea behind this expression is that \(W(k, n-k)\) is the probability that \(k\) photons are in volume \(A\) on condition that \(n-k\) photons are in \(B\). The denominator in (11) guarantees that this probability is correctly normalized according to (10). In other words, eq. (11) gives the probability distribution of photons among parts \(A\) and \(B\) of volume \(S\) given it contains \(n\) photons. Equation (11) is valid for an arbitrary photon statistics satisfying (2).

\[^1\]The term photon statistics only applies to random processes and does not apply to controlled processes, such as a light beam with given amplitude modulation, sub-Poissonian processes, etc., because in controlled processes photon statistics cannot be uniquely defined as it depends on the choice of starting points for the sampling intervals.

3.1 Classical statistics

In classical case, all the probabilities in (11) are given by the Poisson statistics \(\text{[5]}\). Substituting (4) in (11) with \(\text{[5]}\) taken into account gives

\[
W(k, n-k) = \frac{n!}{k!(n-k)!} \alpha^k \beta^{n-k}.
\]

(12)

This is the binomial distribution that occurs when a flux of classical non-interacting particles is separated into two parts with probabilities \(\alpha\) and \(\beta\).

Obviously, should any nonpoissonian statistics be substituted into (11) then the binomial distribution (12) would not be obtained. That means that the binomial distribution is valid only if particles in beam \(S\) obey the Poisson statistics. For any other photon statistics (11) should produce a probability distribution that differs from the binomial distribution. Therefore, (11) can be regarded as a Generalized Binomial Distribution, which is valid for arbitrary statistics \(p_k(A)\) that satisfies the probability convolution equations (2).

3.2 Quantum statistics

In quantum case, all the probabilities in (11) are given by eq. (7) for the BE statistics in an arbitrary volume. This is true because the BE statistics, as well as the Poisson statistics, satisfies (2) as shown in Section 2.3. Therefore, substituting (7) in (11) yields

\[
W(k, n-k) = \frac{p_k(A) p_{n-k}(B)}{p_n(S)} = \frac{n!}{k!(n-k)!} \frac{A^k B^{n-k}}{S^n} = \frac{C^n_k A(A+1)\ldots(A+k-1)B(B+1)\ldots(B+n-k-1)}{S(S+1)\ldots(S+n-1)}.
\]

(13)

This probability distribution is known as the Polya distribution. Eq. (13) determines the probabilities of different photon-number distributions \((k, n-k)\) among volumes \(A\) and \(B\) if \(n\) photons in \(S\) obey the BE statistics. Therefore, (13) is a replacement for the classical binomial distribution in case the incident photons obey quantum statistics.

It is known that the Polya distribution takes into account aftereffects that are alien to the Bernoulli process \(\text{[4]}\). This property of Polya distribution is shown below to produce the photon bunching effect.

Given \(A = \alpha S\) and \(B = \beta S\), eq. (13) may be written using probability \(\alpha\) that a photon is in \(A\) and probability \(\beta\) that a photon is in \(B\):

\[
W(k, n-k) = \frac{(\alpha S)^k (\beta S)^{n-k}}{k! (n-k)!} \frac{n!}{S^n}.
\]

(14)

This form of the generalized binomial distribution in BE statistics will be convenient for further calculations.
If \( S \to \infty \) then the Polya distribution \([14]\) becomes the classical binomial distribution:

\[
\lim_{S \to \infty} W(k, n-k) = \lim_{S \to \infty} \frac{\binom{n}{k} \alpha^k \beta^{n-k}}{\frac{1}{S^{n-k}} \cdot \frac{1}{S^k}} = \left( \frac{n}{k} \right)^{\alpha k^2 - k},
\]

which means that the classical binomial distribution \([12]\) is applicable not only in case of Poisson statistics (as noted above), but also in the BE statistics in the limit of low particle density \( n \ll S \). In this limit the mean distance between photons is much greater than the coherence length and photons behave on average as independent classical particles that do not tend to bunch. In the opposite limit as \( S \to 0 \) photons show nonclassical properties, which will be discussed below.

Information on the degeneracy parameter \( w \) is missing from \([14]\), which means that the probability distribution \([14]\) does not depend on the radiation temperature and the frequency range selected to study photon statistics. However, probabilities \( W(k, n-k) \) depend on the phase space volume \( S \) occupied by the photons. That is the fundamental difference between \([14]\), which is valid in the BE statistics, and the classical binomial distribution \([12]\), which is valid in the Poisson statistics.

The Polya distribution \([14]\) gives the exact quantum statistical solution to the problem of interaction of an arbitrary number of photons of thermal radiation occupying arbitrary phase-space volume \( S \), with a classical device that separates the photon flux into two parts with known probabilities \( \alpha = A/s \) and \( \beta = B/s \).

In this section it was shown that, in case of BE statistics, the generalized binomial distribution \([11]\) takes the form of Polya distribution that is known to occur in random processes that differ from the Bernoulli process \([4]\).

### 4 Photon bunching in Bose-Einstein statistics

This Section presents several examples of the generalized binomial distribution in the BE statistics. Probabilities \( W(k, n-k) \) of different photon-number states \((k, n-k)\) for \( n \) photons in volume \( S \) are calculated on the basis of \([14]\) in some simple cases, the number of photons in \( A \) being designated as \( k \) while the number of photons in \( B \) being \( n-k \).

#### 4.1 One photon

If one photon is in volume \( S = A + B \) then the probabilities of photon-number states \((1,0)\) and \((0,1)\) designating a photon in \( A \) or in \( B \), correspondingly, are given by

\[
W(1,0) = \frac{\alpha S^k}{\alpha} \frac{1}{\alpha !} \frac{1}{S^k} = \alpha,
\]

\[
W(0,1) = \frac{\beta S^{n-k}}{\beta !} \frac{1}{S^{n-k}} = \beta.
\]

Probabilities \([16]\) turn out to be independent of volume \( S \). One photon is found in \( A \) with probability \( \alpha \) or in \( B \) with probability \( \beta \) as it should be according to \([1]\).

#### 4.2 Two photons

If there are two photons in \( S \) then the probabilities of different photon-number distributions between volumes \( A \) and \( B \), according to \([14]\), are:

\[
W(0,2) = \frac{\alpha (\alpha S + 1)}{S + 1} \quad \rightarrow \alpha^2,
\]

\[
W(1,1) = \frac{\alpha \beta S}{S + 1} \quad \rightarrow 2\alpha \beta,
\]

\[
W(2,0) = \frac{\beta (\beta S + 1)}{S + 1} \quad \rightarrow \beta^2.
\]

In this case probabilities \( W(k, n-k) \) depend on volume \( S \) occupied by the photons. The limit values of corresponding probabilities as \( S \to \infty \) are shown in the right-hand side of \([17]\). These limit values coincide with the well known classical results based on the binomial distribution. Probabilities \([17]\) are shown in Fig. 2 as functions of volume \( S \) occupied by the photons in case \( A = B \) \((\alpha = \beta = 1/2)\).

![Fig. 2: Non-classical probabilities of two-photon distribution among two halves of volume \( S \) in Bose-Einstein statistics. Volume \( S \) is measured in the units of coherence volume.](image-url)

Decreasing volume \( S \) makes classical probabilities invalid because in quantum statistics, as follows from Fig. 2, photons tend to bunch together if they are located in a small area.
Photon bunching is manifested by an abnormally high probability of states (2,0) and (0,2) that designate both photons occupying only one half of volume $S$ if that volume becomes small, e.g. less than 2-3 coherence volumes. Note that $S = sc \tau$, therefore, $S$ may be varied by changing either the beam cross-section $s$ or sampling time $\tau$.

### 4.3 Three photons

If three photons are in volume $S$ then using (14) one obtains probabilities of different photon-number configurations against volume $S$:

\[
\begin{align*}
W(3,0) &= \alpha^3, \\
W(2,1) &= 3\alpha^2 \beta, \\
W(1,2) &= 3\alpha \beta^2, \\
W(0,3) &= \beta^3.
\end{align*}
\]

Basic features of three-photon distribution among volumes $A$ and $B$ are the same as in the previous example of two photons. In the limit of large volumes $S \to \infty$, shown on the right-hand side of (18), photons behave as if they were independent classical particles obeying the binomial distribution. In contrast to such classical behavior, photons occupying a small volume of about several bins exhibit space distribution that notably deviates from the predictions of classical binomial distribution.

Probabilities versus volume $S$ occupied by three photons are shown in Fig. 3 for a non-symmetrical division of $S$ into two parts ($\alpha = 0.55$). It is obvious from the plots that for small volumes $S$, less than 3-4 coherence volumes, probabilities $W(k,3-k)$ notably differ from their classical limits that are obtained as $S \to \infty$. This, again, is the manifestation of photon bunching in quantum statistics.

### 4.4 Fifty photons

As an example of bunching of a large number of photons in quantum statistics, consider fifty photons in volume $S$ that is divided into two equal parts ($\alpha = \beta = 1/2$). Fig. 4 presents probabilities $W(k,n-k)$ that $k$ photons are in part $A$ while $n-k$ photons are in part $B$ given total number of photons in $S$ is $n = 50$. Different curves correspond to different values of phase-space volume $S$ occupied by the photons.

A histogram for $S = 10^4$ is a good approximation to the limit of $S \to \infty$ because in this case each photon occupies, on average, a volume of 200 bins, and probability distribution $W(k,n-k)$, according to (15), looks like classical binomial distribution.

Fifty photons occupying a single bin (curve $S = 1$) exhibit substantially non-classical properties because the probability distribution in this quantum state displays pronounced maximums at $k = 0$ and $k = n$. This is a manifestation of photon bunching, which is a tendency to collective behavior in quantum statistics. In this case the probability for a group of photons to be separated into two almost equal parts is minimal while in classical case
described by the binomial formula this probability is maximal (curve $S = 10^4$). In the limit $S \to 0$, a photon bunch looks like a single quantum entity that is not inclined to dissolve into smaller groups of photons.

According to the above, in the Poisson statistics (coherent radiation) photons behave like classical particles in accordance with the classical binomial distribution, while in quantum statistics (blackbody radiation) photons exhibit different features showing a tendency to bunching, or collective behavior. In Fig. 3 curve $S = 10^4$ actually describes the behavior of classical particles while other curves show the deviation of quantum particles from classical behavior that becomes more pronounced for smaller phase-space volumes occupied by the particles.

It is worthwhile noting that the average number of photons per one cell in BE statistics is defined by the temperature and frequency of radiation:

$$w = \frac{1}{e^{\frac{E}{kT}} - 1}. \quad (19)$$

Therefore, 50 photons in one coherence volume could be observed with notable probability if $kT \approx 50h\nu$, i.e. either in a low-frequency range of blackbody radiation or for a black body of very high temperature.

The Polya distribution (13) and Fig. 2 present the quantum solution to the problem stated in Section 2.1 regarding the distribution of $n$ photons among volumes $A$ and $B$ in case of BE statistics.

### 5 Statistical equivalence of different problems

The problem stated in Section 2.1 is directly related to the interaction of light beam with a diaphragm as the distribution $W(k, n - k)$ defines the probability that $k$ photons pass through the hole $A$ while $n - k$ photons are absorbed by the screen $B$ (see Fig. 1). This problem is also related to the following three problems:

1. Photon number distribution at the output ports of a beamsplitter of transmittance $\alpha = \frac{A}{S}$;
2. Statistics of photon detection by a photodetector of quantum efficiency $\alpha$.
3. Photon statistics behind a neutral filter of transmittance $\alpha$.

In all of the above problems, a photon flux is separated into two parts with given probabilities $\alpha$ and $\beta = 1 - \alpha$. Although very different physical processes are involved in these problems, with regard to photon statistics they are equivalent and described by the same formulas provided the light beam has the same properties in all problems. This is because in statistics the probability of event is important, not the nature of the event. That is the well-known principle that is widely used in statistics.

For example, photon statistics in part $A$ of light beam $S$ (Fig. 1) should coincide with photon statistics produced by the same light beam $S$ after a beamsplitter of transmittance $\alpha = \frac{A}{S}$ because the probability for a photon to be in volume $A$ equals the probability that a photon is transmitted through the beamsplitter. Likewise, if probability of photodetection is $\alpha$, the same as the probability for a photon to pass through the beamsplitter, then statistics of photo-absorptions should be the same as photon statistics behind the beamsplitter. A neutral filter actually may be considered as a beamsplitter with the reflected radiation being absorbed.

Consequently, the above problems are equivalent with respect to photon statistics provided the light beams involved in these problems have identical properties. Therefore, any statistical result obtained for one problem is sure to be valid for other equivalent problems as well. In particular, the Polya distribution (14) and photon bunching effects shown in Fig. 2 must hold in all of the above problems if thermal photons are involved.

Given such interpretation of the results, Fig. 2 shows non-classical probabilities of photon-number states at the output ports of a beamsplitter versus the volume occupied by the two photons before interacting with the beamsplitter provided photons obey the BE statistics and have entered the same input port of the beamsplitter. Such photon behavior like shown in Fig. 2 is not something unusual. Indeed, similar results were observed in quantum interference experiment 3, in which two indistinguishable photons entered the same input port of a beamsplitter. The probability of output states $(2,0)$ and $(0,2)$ was measured in this experiment against the distance between two input photons and was found to be maximum for zero delay, just like shown in Fig. 2 (for details refer to 3). Regardless of the fact that non-thermal photons were used in this experiment, the experimental results match well the curves presented in Fig. 2. Qualitative coincidence of experimental results and theoretical curves obtained with the help of Polya distribution indicates that photon bunching effect considered in this work turns out to be analogous to the quantum interference effect that was examined in the above experiment.

Applicability of the results shown in Fig. 2 to the interaction of photons with a diaphragm, photodetector, and neutral filter suggests that the quantum interference phenomenon (or photon bunching effect) may manifest itself in situations where no beamsplitters are involved.

## 6 Critical analysis of standard solution

### 6.1 Standard solution

The problem formulated in Section 2.1 has the well-known standard solution that, being widely recognized, is found in many textbooks and monographs on quantum optics, for example in 10,5,8, in connection with the discussion of such different problems as statistics of photo absorptions or the influence of a beamsplitter on photon statis-
tics. The standard solution differs from the solution presented in this article. It will be shown below that the standard solution inevitably leads to insurmountable contradictions and, therefore, is incorrect. First, we present the commonly accepted derivation of the standard solution.

If the probability for a single photon in beam $S$ to be in volume $A$ is $\alpha$ then the probability that $k$ photons are in $A$ is proportional to $\alpha^k$. Similarly, the probability that $k$ photons are in $A$ while $n - k$ photons are in $B$ is proportional to $\alpha^k \beta^{n-k}$. This probability can be realized in a number of ways that is taken into account by the “$n$ choose $k$” combinatorial factor $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Thus we arrive at the probability that $k$ photons are in $A$ in case there are $n$ photons in $S$:

$$W(k, n-k) = \binom{n}{k} \alpha^k \beta^{n-k}. \quad (20)$$

This is the classical binomial distribution for the probability of $k$ successes in $n$ trials in the Bernoulli experiment. Probability (20) holds when there are $n$ photons in volume $S$, which happens, according to the statement of the problem, with known probability $p_n(S)$. Therefore, to obtain the resultant photon statistics in $A$ we must multiply (20) by $p_n(S)$ and sum over $n$:

$$p_k(A) = \sum_{n=k}^{\infty} p_n(S) W(k, n-k). \quad (21)$$

Equations (20) and (21) provide the standard solution to the problem formulated in Section 2.1. This standard solution, as stated in [10, 5, 8], is applicable to an arbitrary volume $S$.

The fallacy of such statement is easily demonstrated by considering specific examples. Suppose, for example, that $p_n(S)$ is the BE statistics in volume $S = 2$. Then, according to (4),

$$p_n(S = 2) = \frac{S(S + 1)...(S + n - 1)}{n!} \frac{w^n}{(1+w)^{n+S}}$$

$$= \frac{(n+1)w^n}{(1+w)^{n+2}}. \quad (22)$$

Substituting (22) and (20) in (21), by redefining the summation index one obtains

$$p_k(A) = \sum_{n=k}^{\infty} \binom{n+1}{k+1} \frac{w^n}{(1+w)^{n+2}} \frac{n!}{k!(n-k)!} \alpha^k \beta^{n-k}$$

$$= \frac{(\alpha \omega)^k}{(1+\omega)^{k+2}} \sum_{s=0}^{\infty} \frac{(k+s+1)!}{k!s!} \left( \frac{\beta \omega}{1+\omega} \right)^s. \quad (23)$$

Replacing $x = \frac{\beta \omega}{1+\omega}$ allows to take the last sum:

$$\sum_{s=0}^{\infty} \frac{(k+s+1)!}{k!s!} x^s = \frac{k+1}{(1-x)^{k+2}}. \quad (24)$$

The validity of this result can be verified by expanding (24) in the Maclaurin series.

Now using (23) and (24) yields for the statistics in volume $A$:

$$p_k(A) = \frac{(k+1)(\alpha \omega)^k}{(1+\alpha \omega)^{k+2}}. \quad (25)$$

If volume $S = 2$ is divided into two equal parts, i.e., if $\alpha = \beta = 0.5$, then (25) must give the BE statistics in a single phase-space cell. However, eq. (25) produces different result.

Thus, an attempt to use (20) with the BE statistics in (21) leads to contradictions. Therefore, using classical binomial distribution in quantum statistics is a mistake. The reason behind this mistake is the fact that in deriving (20) one tacitly assumes that photons enter volume $A$ (as well as volume $B$) independently of one another, which is true for the Poisson statistics only and not true for other statistics.

### 6.2 Other contradictions arising from the standard solution

Let us consider a few more examples in which standard solution (20) - (21) leads to contradictions (this is for those who deem previous example unconvincing).

While working with equations (20) and (21), it is convenient to use generating functions of photon statistics. By definition, generating function $P(z)$ of statistics $p_k$ is a power series

$$P(z) = \sum_{k=0}^{\infty} p_k z^k. \quad (26)$$

Using generating functions makes it possible to avoid direct calculation of complicated sums like (21) with arbitrary statistics $p_n(S)$. Eq. (21) that provides relation between statistics in $A$ and in $S$ yields simple relation between generating functions of these statistics

$$P_A(z) = P_S(\alpha z + \beta), \quad (27)$$

which can be easily verified by direct calculation using the generating function definition (26) and (20) - (21).

Let us consider the above example of BE statistics in volume $S = 2$ using generating functions. The BE statistics (7) in an arbitrary volume $A$ has generating function

$$P_A(z) = \frac{1}{(1+w-wz)^A}, \quad (28)$$

which can be verified by expanding (28) in Maclaurin series:

$$P_A(z) = \sum_{k=0}^{\infty} A_k \frac{w^k z^k}{k! (1+w)^{k+A}} = \sum_{k=0}^{\infty} p_k(A) z^k. \quad (29)$$
The generating function of BE statistics in volume $A = 1$ is obtained from (28):

$$P_A(z) = \frac{1}{1 + w - wz},$$  \hspace{1cm} (30)

while the generating function of BE statistics in volume $S = 2$ is

$$P_S(z) = \frac{1}{(1 + w - wz)^2}. \hspace{1cm} (31)$$

According to (27), for $\alpha = \beta = 0.5$ generating functions (30) and (31) must satisfy the following equation:

$$\frac{1}{1 + w - wz} = \frac{1}{(1 + w - w(\alpha + \beta))^2} \hspace{1cm} (32)$$

However, this equation is violated for all values of $z$, $\alpha$ and $\beta = 1 - \alpha$. Consequently, eq. (27) that was obtained from (20)-(21) does not provide necessary connection between generating functions in the BE statistics. Therefore, the standard solution (20)-(21) is wrong.

Any attempt to apply (20) to an arbitrary photon statistics inevitably leads not only to mathematical inconsistencies but also to physically absurd results. Consider the following example.

Let the blackbody radiation be observed in a unit volume $S = 1$. The BE statistics in this case is given by (26). Substituting (5) and (20) in (21) yields

$$p_k(\alpha) = \frac{(\alpha w)^k}{(1 + \alpha w)^{k+1}}, \hspace{1cm} (33)$$

which, according to derivation, is the probability that a photon is in part $\alpha$ of unit volume. This well-known formula is often interpreted as the statistics of photo-counts, should the detector efficiency be $\alpha$, or photon statistics behind a beamsplitter, should its transmittance be $\alpha$ [10, 5, 8]. Such interpretation is, naturally, based on the principle of equivalence of different statistical problems that was discussed in Section 3.

Eq. (33) is the usual BE statistics [3] with mean occupation number $\bar{w} = \alpha w$. However, the mean occupation number in the BE statistics cannot depend on the probability $\alpha$ that a photon is found in certain volume because the mean occupation number is determined by eq. (19), which includes only the blackbody temperature and frequency chosen to observe the radiation.

It follows from (19) that, at a given frequency $\nu$, the mean occupation number may be changed only by changing the blackbody temperature $T$. However, it follows from (33) that the mean occupation number (and, therefore, the blackbody temperature) must depend on the diameter of the aperture in an opaque screen, through which the radiation is observed (see Fig. 1). The irrelevance of this conclusion testifies to incorrectness of (33) and standard solution (20)-(21), from which (33) was obtained. The observed blackbody temperature and its radiation spectrum cannot depend on the aperture diameter. Therefore, the mean occupation number should not depend on probability $\alpha$.

The correct solution to this problem is given by the Mandel’s formula (3) for $A = \alpha$:

$$p_k(\alpha) = \frac{\alpha(\alpha + 1)...(\alpha + k - 1)}{k!} \frac{w^k}{(1 + w)^{k+\alpha}}, \hspace{1cm} (34)$$

which should be used instead of (33) in order to avoid insurmountable contradictions.

### 6.3 Domain of applicability of standard solution

If photon statistics in volumes $A$, $B$, and $S = A + B$ satisfy probability convolution eq. (2) then the corresponding generating functions must satisfy [4]

$$P_{A+B}(z) = P_A(z) P_B(z). \hspace{1cm} (35)$$

The inverse proposition is also true [4]: if generating functions satisfy (35) then the corresponding statistics satisfy (2).

As noted above, the standard solution (20)-(21) yields eq. (27) for the corresponding generating functions. Since every photon statistics satisfies (2), corresponding generating function (27) must satisfy (35). Therefore, we substitute (27) in (35):

$$P_S(z) = P_S(\alpha z + \beta) P_S(\beta z + \alpha). \hspace{1cm} (36)$$

This can be viewed as the functional equation in $P_S(z)$. The solution of this equation may be found in the form

$$P(z) = e^{x z + y}, \hspace{1cm} (37)$$

where $x$ and $y$ are unknown constants. Substituting (37) in (36) yields

$$e^{x z + y} = e^{x(\alpha z + \beta) + y} e^{x(\beta z + \alpha) + y}. \hspace{1cm} (38)$$

Given $\alpha + \beta = 1$ from (38) one obtains $y = -x$. Therefore, generating function (37) has the form:

$$P(z) = e^{x(z - 1)}. \hspace{1cm} (39)$$

But that is the generating function of the Poisson statistics [4]. Thus, it is proven that the standard solution (20)-(21) is applicable only in case incident particles obey Poisson statistics. This result agrees with the conclusion drawn earlier in Section 3.

### 6.4 How does the generalized binomial distribution work

Let us check the compatibility of generalized binomial distribution [11] with formula (21). Substituting (11) in
one obtains:

\[
p_k(A) = \sum_{n=k}^{\infty} p_n(S) \frac{p_k(A)p_{n-k}(B)}{p_n(S)} = p_k(A) \sum_{n=k}^{\infty} p_{n-k}(B).
\]

The last sum in (40) is equal to unity due to the normalization of probability \(p_n(S)\). Consequently, (40) is satisfied identically for any photon statistics \(p_n(S)\).

In Sections 6.1-6.3 it was shown that if \(W(k,n-k)\) is the usual binomial distribution \([20]\) then (i) the standard solution applies only in case of Poisson statistics; (ii) in general case of nonpoissonian statistics contradictions are unavoidable. Equation (40) proves that if the generalized binomial distribution \([11]\) is taken for \(W(k,n-k)\) then no contradictions arise and (21) is satisfied identically for an arbitrary statistics \(p_n(S)\).

7 Conclusions

Major results of this work are derived from two basic facts: (a) probability convolution equations (2) for interaction of light beam with a macroscopic device

\[
p_n(A + B) = \sum_{k=0}^{n} p_k(A)p_{n-k}(B)
\]

and (b) the Mandel’s formula (3) for the Bose-Einstein statistics in an arbitrary phase-space volume \(A\)

\[
p_k(A) = \frac{A(A+1)...(A+k-1)}{k!} \frac{w^k}{(1+w)^{k+A}},
\]

where \(w\) is the degeneracy parameter of photon gas.

It is shown that the Mandel’s formula satisfies the convolution equations, which fact immediately yields the probability (13) of different photon-number states \((k,n-k)\) after interaction of photon flux with a macroscopic device:

\[
W(k,n-k) = \frac{p_k(A)p_{n-k}(B)}{p_n(S)} = C_n^k A^k B^{n-k} (A+B)^{-n},
\]

The last formula is the Polya distribution that, according to the analysis presented in Sections 3.4, describes the photon bunching effect (which looks like quantum interference) in the BE statistics.

Therefore, the main results of this work have been obtained as a direct consequence of first principles, i.e. the quantum statistics. No additional assumptions were made and no approximate methods were applied to obtain the results. For this reason, the Polya distribution (13) is the exact quantum statistical solution of the stated problem. Main conclusions of this work may be formulated as follows:

1. It is shown that the Bose-Einstein statistics satisfies the probability convolution equations, which is just another proof of the well-known fact that photon statistics is unable to take into account intensity fluctuations and related photon correlations.

2. For the probabilities of final photon-number states \((k,n)\) arising in the photon flux separated into two parts, a generalized binomial distribution is obtained

\[
W(k,n) = \frac{p_k(A)p_n(B)}{p_{k+n}(A+B)}
\]

that is applicable for an arbitrary photon statistics \(p_k(A)\) satisfying probability convolution equations.

3. In case of Bose-Einstein statistics, the generalized binomial distribution takes the form of Polya distribution, which presents the exact quantum solution of the problem of interaction of arbitrary number of thermal photons occupying arbitrary phase-space volume with a macroscopic device that separates the photon flux into two parts with known probabilities.

4. Due to statistical equivalence of different problems, theoretical results obtained for the interaction of photons of arbitrary statistics with a diaphragm can be applied to the interaction of photons with a beamsplitter, neutral filter, or photodetector.

5. It is shown that the binomial distribution correctly describes the probabilities of output photon-number states after interaction of photons with the macroscopic device in two cases only: (a) photons obey Poisson statistics; (b) the average distance between photons greatly exceeds the coherence length (in this case any photon statistics tends to Poisson statistics). Therefore, these two conditions determine the domain of applicability of the binomial formula.

6. It follows from Fig. 24 that photon bunching, or quantum interference in BE statistics takes place if the mean occupation number is large \(\omega \gtrsim 1\).

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Appendix 1: Proof of identity (9)

Equation (9) includes numerical sequence

\[
f_k(A) = \frac{A^k}{k!}.
\]
so that (9) can be written as

\[ f_n(A + B) = \sum_{k=0}^{n} f_k(A) f_{n-k}(B) \]  

(42)

The generating function of sequence (41) is

\[ F(A) = \frac{1}{(1 - z)^A} \]  

(43)

which can be verified by direct expansion of (43) in Maclaurin series:

\[ \frac{1}{(1 - z)^A} = \sum_{k=0}^{\infty} \frac{A^k}{k!} z^k = \sum_{k=0}^{\infty} f_k(A) z^k. \]  

(44)

Identity (42) is now validated by the following obvious relation between the generating functions:

\[ \frac{1}{(1 - z)^{A+B}} = \frac{1}{(1 - z)^A} \frac{1}{(1 - z)^B}, \]  

(45)

Q.E.D.

Appendix 2: The Glauber’s statistics satisfies the probability convolution equations

Denote as \( P(A) \) the generating function (GF) of statistics \( p_k(A) \) in an arbitrary phase-space volume \( A \). If this statistics satisfies eq. [2] then its GF obeys [4]

\[ P(A + B) = P(A)P(B). \]  

(46)

Functional equation (46) in \( P \) has the solution

\[ P(A) = P(1)^A, \]  

(47)

where \( P(1) \) is the GF of photon statistics in a unit volume \( A = 1 \). Obviously, any statistics satisfies [2] if its GF satisfies [47].

Recalling that by definition \( A = ac\tau \), eq. (47) may be written as

\[ P(A) = F(z)^A, \]  

(48)

where \( F(z) = P(1)^{ac} \).

Photon statistics obtained by Glauber [9] for Lorentzian spectral line has GF (in Glauber’s notations)

\[ Q(\lambda, \tau) = \exp \left\{ - \left[ (\gamma^2 + 2\gamma W \lambda)^{1/2} - \gamma \right] \tau \right\}, \]  

(49)

where \( \gamma \) is the half-width of the spectral line, \( W \) is the average number of photons per second, \( \tau \) is the sampling time in photon statistics measurement, \( \lambda = 1 - z \).

So, the GF of Glauber’s statistics (49) has the form \( Q(\lambda, \tau) = F(z)^\tau \), which coincides with (48). For this reason, the Glauber’s statistics satisfies [2], Q.E.D.

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