Non-Universal Fractional Quantum Hall States in a Quantum Wire

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Abstract

The ground state as well as low-lying excitations in a 2D electron system in strong magnetic fields and a parabolic potential is investigated by the variational Monte Carlo method. Trial wave functions analogous to the Laughlin state are used with the power-law exponent as the variational parameter. Finite size scaling of the excitation energy shows that the correlation function at long distance is characterized by a non-universal exponent in sharp contrast to the standard Laughlin state. The Laughlin-type state becomes unstable depending on strength of the confining potential.
The fractional quantum Hall (FQH) effect occurs in two-dimensional electron systems under strong magnetic field [1]. The ground state is described as an incompressible quantum liquid with an energy gap [4]. On the edge of the FQH state, however, gapless excitations exist. Hence the edge excitations play an important role in low-energy response of the FQH system. In a large system, the edge excitations are described as the chiral Tomonaga-Luttinger (TL) liquid. It has been discussed that the long-distance behavior in the edge state is universal in the sense that the exponent is determined by the filling alone and is independent of other details of the system [3].

In a mesoscopic system where the length scale can be made comparable to the radius of the cyclotron motion, the FQH state should be affected by the finite dimension of the system. In this paper we study the FQH state in a quantum wire which can be fabricated as a conducting channel in the semiconductor heterojunction. If the channel is wide enough, the chiral nature of the edge excitation should remain. In a narrow channel, on the contrary, two edges are not independent objects. Then the universality of the asymptotic behavior might be lost. The unique property of the quantum wire is that one can control the strength of interaction between the edges, and hence the chiral nature. This property is not shared with more popular geometries like a disk or a semi-infinite plane.

The quantum wire in a strong magnetic field has been studied theoretically by Yoshioka with use of the exact diagonalization method [4]. He has shown that a FQH state is realized under suitable condition with respect to the strength of a confinement potential and the Coulomb repulsion. The exact diagonalization can deal with only small systems containing several electrons. As an alternative we use in this paper the variational Monte Carlo (VMC) method [5,6] to investigate much larger systems with $O(10^2)$ electrons. We aim at deriving explicitly the non-universal property of the FQH state in the quantum wire, and show how the property depends on the size of the system. For this purpose we investigate not only the ground state but low-lying excitations for the Laughlin-type wave function [2].

Consider a two-dimensional electron system under a strong magnetic field and a parabolic potential in the $y$-direction. There is no potential in the $x$-direction for which we impose the
periodic boundary condition with length $L_x$. The magnetic field $B$ is applied antiparallel to the $z$-direction. Then the one-electron Hamiltonian in the Landau gauge is written as

$$\mathcal{H}_0 = \frac{1}{2m_e} \left[ (p_x + \frac{e}{c} B y)^2 + p_y^2 \right] + \frac{1}{2} m_e \omega_0^2 y^2, \quad (1)$$

where $m_e$ is the electron mass and $\omega_0$ is the strength of confinement potential. The Hamiltonian is rewritten for later convenience as

$$\mathcal{H}_0 = -\frac{\hbar}{2} \Omega \left[ \frac{1}{\mu} (\lambda \partial_x + i \frac{y}{\lambda} \partial_y)^2 + \mu \lambda^2 \partial_y^2 \right] - \frac{\hbar^2}{2m_e} \frac{\omega_0^2}{\Omega^2} \lambda^2 \partial_x^2. \quad (2)$$

We have introduced $\mu = \ell/\lambda = \sqrt{1 + (\omega_0/\omega_c)^2}$ and $\Omega = \sqrt{\omega_c^2 + \omega_0^2}$, where $\ell = \sqrt{\hbar c/eB}$ and $\omega_c = eB/m_e c$. $\Omega$ and $\lambda$ give the effective cyclotron frequency and the Larmor radius in the presence of the parabolic potential. The eigenstate $\phi_{n,k}$ and the eigenvalue $E_{n,k}$ of this Hamiltonian are obtained as

$$\phi_{n,k}(r) = \exp \left[ ikx - \frac{1}{2\mu} \left( \frac{y}{\lambda} + \lambda k \right) \right] H_n \left( \frac{y + \lambda^2 k}{\lambda \sqrt{\mu}} \right), \quad (3)$$

$$E_{n,k} = \hbar \Omega (n + \frac{1}{2}) + \frac{\hbar^2}{2m_e} \frac{\omega_0^2}{\Omega^2} k^2. \quad (4)$$

Here $k = 2\pi m/L_x$ ($m$ is integer) is the momentum along the $x$-direction, $n$ is the Landau level index and $H_n$ is the Hermite polynomial.

The interaction between the electrons should also satisfy the periodic boundary condition. We introduce the following function:

$$V_c(r) \equiv \lambda \left[ \left( \frac{L_x}{\pi} \sin \frac{\pi}{L_x} x \right)^2 + y^2 \right]^{-1/2},$$

which is periodic along the $x$-direction, and reduces to the usual Coulomb form $\lambda/|r|$ in the limit of $x/L_x \to 0$. Then the interaction Hamiltonian $\mathcal{H}_{\text{int}}$ is given by

$$\mathcal{H}_{\text{int}} = \frac{e^2}{\varepsilon \lambda} \sum_{i<j} V_c(r_{ij}), \quad r_{ij} = r_i - r_j, \quad (5)$$

where $\varepsilon$ is the dielectric constant.
From now on, we assume that the magnetic field is so strong that it is sufficient to consider only the lowest Landau level ($n=0$), with complete spin polarization. The Laughlin-type wave function is constructed as follows:

$$\Psi_p(z_1, \ldots, z_N) = \prod_{i<j} \sin \frac{\pi}{L_x} (z_i - z_j) \prod_j \exp \left( -\frac{y_j^2}{2\mu \lambda^2} \right),$$

(6)

where is $z = x + iy/\mu$ and $p$ is an odd integer. $\Psi_{p=1}$ is the Slater determinant of the non-interacting system, and is the eigenstate of the $N$-body sum of eq.(2). $\Psi_p$ with $p \neq 1$ is not an eigen function of the non-interacting system in the presence of the parabolic potential. However, the state belongs to the lowest Landau level. Thus as long as $\omega_0/\omega_c \ll 1$, $\Psi_p$ is expected to be a good trial function to account for the repulsive correlation. We take the number $N$ of electrons odd to avoid degeneracy in the ground state. Then the maximum momentum of the one-body state is $\pi(N-1)p/L_x \sim \pi \rho p$ where $\rho = N/L_x$ is the line density. Since the wave function with momentum $k$ is localized around $-\lambda^2 k$ in the $y$-direction, the width of the channel is about $p$ times of that with $p = 1$. Thus the Laughlin-type state corresponds to the filling $\nu = 1/p$ of the lowest Landau level.

The total energy of the trial state $\Psi_p$ is written as

$$E_0 \sum_j \frac{\langle \Psi_p | -\lambda^2 \partial^2_{x_i} | \Psi_p \rangle}{\langle \Psi_p | \Psi_p \rangle} + U \sum_{i<j} \frac{\langle \Psi_p | V_c(r_{ij}) | \Psi_p \rangle}{\langle \Psi_p | \Psi_p \rangle},$$

(7)

where $E_0 = (\hbar^2/2m_e \lambda^2)(\omega_0^2/\Omega^2) = \hbar \omega_0^2/(2\omega_c)$ and $U = e^2/\varepsilon \lambda$ are constants characterizing the magnitude of the confinement potential and the Coulomb repulsion, respectively. The term with $E_0$ in eq.(7) corresponds to the final term in eq.(4). We neglect the zero point oscillation energy $\hbar \Omega N/2$ because it only shifts the origin of energy. We calculate numerically these expectation values by the VMC method taking $E_0/U (\ll 1)$ as the parameter to control the confinement. For simplicity we put $\mu = 1$, which is justified in the case of $\hbar \omega_0 \ll e^2/(\varepsilon \ell) \ll \hbar \omega_c$.

We take the odd integer $p$ as the variational parameter and optimize it for various values of $E_0/U$ and the line density $\rho$. The results for $N=45$ is shown in Fig.1. The unit of length is taken to be $\lambda = \ell$ here and from now on. In varying $p$ we keep $N$ constant, so $\rho$ also
represents the ratio of width to length of the system. Hence the system with larger \( \rho \) favors smaller \( p \) as in Fig. 4 because the effect of the external potential becomes larger. Our result supports qualitatively the phase diagram obtained by Yoshioka. In our case, however, more complex FQH states such as \( \nu = 2/3 \) or \( 2/5 \) have been neglected. The presence of such states is indeed signaled as an instability of the simplest Laughlin-type state as will be discussed later. Our primary concern in this paper, however, is not the phase diagram itself.

We turn to excitation spectra from the Laughlin-type ground state, and describe the results in terms of the TL liquid theory. Since the one-body state in our model is characterized by a single momentum along the \( x \)-direction, the system can be regarded as quasi-1D. The effective 1D field operator in the system is defined by

\[
\Psi_1(x) = \frac{1}{\sqrt{2\pi\lambda}} \int_{-\infty}^{\infty} dy \Psi(r) \exp \left[ -\frac{ix}{\lambda^2} (y - \frac{ix}{2}) \right], \tag{8}
\]

where \( \Psi(r) \) is the field operator in the lowest Landau level. The integral corresponds to the average along the \( y \)-direction with account of the cyclotron motion. The Fourier transform of \( \Psi_1(x) \) gives the annihilation operator of electron with momentum \( k \). The low-energy excitations in a spinless 1D Fermion system are classified into three kinds \[7\]: (i) the particle-hole excitation giving rise to density fluctuation; (ii) the number excitation associated with the change of the chemical potential; (iii) the current excitation with momentum twice the Fermi momentum. Each excitation is gapless and is characterized by the velocities \( v_s, v_N \) and \( v_J \), respectively. In terms of these velocities, the elementary excitation energies are given by (i) \( 2\pi v_s m/L_x \) with \( m \) positive integer, (ii) \( \pi(\Delta N)^2 v_N/2L_x \), and (iii) \( \pi J^2 v_J/2L_x \). Here \( J = N_R - N_L \) and \( \Delta N = N' - N \) where \( N_R \) (\( N_L \)) is the number of right-mover (left-mover) giving \( N = N_R + N_L \), and where \( N' \) is the particle number in the excited state. If one can calculate the excitation energies, these velocities are derived by analysis of their size dependence.

We shall derive the exponent which characterizes the asymptotic behavior of the density matrix as \( \langle \Psi_1^\dagger(x)\Psi_1(0) \rangle \sim |x|^{-\theta-1} \cos(k_F x) \) where \( k_F \) is the Fermi momentum and \( \theta = (K + 1/K - 2)/2 \). It is known that the exponent \( K \) is derived from \( K = \sqrt{v_J/v_N} = v_J/v_s \).
Of these velocities, $v_N$ involves the change of $N$ and keeping the charge neutrality is a subtle procedure. Therefore we use $v_J$ and $v_s$ to derive $K$.

Let us consider the following wave functions:

$$
\Psi_p^{(ph)}(z_1, \ldots, z_N) = \sum_j \exp \left( i \frac{2\pi}{L_x} z_j \right) \Psi_p(z_1, \ldots, z_N), \quad (9)
$$

$$
\Psi_p^J(z_1, \ldots, z_N) = \prod_j \exp \left( i \frac{2\pi}{L_x} z_j \right) \Psi_p(z_1, \ldots, z_N). \quad (10)
$$

$\Psi_p^{(ph)}$ describes the density excitation with momentum $2\pi/L_x$ and gives the lowest excitation of the type (i), while $\Psi_p^J$ gives the lowest excitation of the type (iii) with $J=2$. Both $\Psi_p^{(ph)}$ and $\Psi_p^J$ are orthogonal to $\Psi_p$, namely $\langle \Psi_p^{(ph)} | \Psi_p \rangle = 0$ and $\langle \Psi_p^J | \Psi_p \rangle = 0$. It is convenient to divide the excitation energy of a particle-hole pair from the Laughlin-type state eq.(6) into two parts: $\Delta E^{(ph)} = \Delta E_{\text{edge}}^{(ph)} + \Delta E_{\text{int}}^{(ph)}$. Here $\Delta E_{\text{edge}}^{(ph)}$ comes from the Hamiltonian part with $E_0$ and $\Delta E_{\text{int}}^{(ph)}$ from that with $U$. Fig.2 shows the VMC results in the case of $\rho = 3.0$. The size dependence is consistent with the gapless linear mode expected from the TL theory. The negative slope for the interaction part $\Delta E_{\text{int}}^{(ph)}$ means that the wider quantum wire reduces the Coulomb repulsion. We can obtain numerically the velocity $v_s$ from the slopes of the lines in Fig.2. The results are summarized in Table I. On the other hand, $v_J$ can be calculated analytically as $v_J = 2\pi \rho E_0$ which is in fact independent of $p$ and $U$.

From the results of $v_s$ and $v_J$, we can obtain the exponent $K$ using the relationship $K = v_J/v_s$. Fig.3 shows the dependence of $K$ on $E_0/U$ in the case of $\rho = 3.0$ and $p = 3$. There appear three regions in Fig.3: (I) $0.0013 < E_0/U < 0.0031$ ($K < 0$), (II) $0.0031 < E_0/U < 0.0047$ ($K > 1$) and (III) $0.0047 < E_0/U < 0.0085$, ($0 < K < 1$). The excitation energy is negative in the case of $K < 0$. This means that $\nu = 1/3$ Laughlin-type state is unstable in that region (I). Unfortunately it is impossible to obtain the true ground state in our method. In the region of (II) and (III), the excitation energy is positive, and the system behaves as a TL liquid. In the region (II) the pairing fluctuation is dominant with $K > 1$. One might interpret this as follows: The Coulomb repulsion is overcompensated in making the Laughlin-type state, and the remaining correlation becomes attractive. It is
however more likely that the real ground state in that region is not the $\nu = 1/3$ state. The most interesting region is (III) where $K$ approaches $1/3$ from above as $U$ gets smaller. This behavior is in contrast with the universal exponent $K = 1/p$ in a large FQH droplet with $\nu = 1/p$ [3][5]. The deviation is interpreted in terms of interaction between the edges. As the interaction becomes insignificant with the decrease of $U$, the exponent approaches the universal value. If $U$ becomes too small, however, the $1/3$ state becomes unstable. We have confirmed that the $\nu = 1/5$ state has a similar character with respect to $K$ as the $\nu = 1/3$ state. In the limit of large width, the density matrix should decay as $|x|^{-1/\nu}$ according to the chiral Luttinger liquid theory [3]. Here the distance $|x|$ should be smaller than the width of the system, and the field operator consists of right- and left-going components representing both edges.

In summary we have shown that the $\nu = 1/p$ Laughlin-type state in a quantum wire has the edge excitation which has a non-universal exponent dependent on details of the system. We have demonstrated that competition between the Coulomb repulsion and the harmonic confinement determines the value of $p$ in the ground state. The next problem to be investigated is how the non-universality appears in the response of the system such as the Hall coefficient and in the temperature dependence of the conductance.

The authors would like to thank H. Yokoyama and Y. Kato for helpful discussion. One of the authors (S.T) gratefully acknowledges the guidance of H. Ebisawa and K. Yonemitsu.
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FIGURES

FIG. 1. Phase diagram of the system with \( N = 45 \). This result is an average over \( 5 \times 10^4 \) samples.

FIG. 2. (a) \( \Delta E_{\text{edge}}^{(ph)}/L_x E_0 \) and (b) \( \Delta E_{\text{int}}^{(ph)}/L_x U \) as a function of \( 1/L_x^2 \) for particle–hole excitation from \( \nu = 1/p \) Laughlin state. The line density is fixed to be \( \rho = 3.0 \), so that \( N \) is varied up to 75. The result of \( \nu = 1, 1/3 \) and \( 1/5 \) are shown with open circle, square and triangle respectively. These excitation energies are obtained by averaging over \( 1 \times 10^5 \) samples.

FIG. 3. The exponent \( K \) of TL-liquid by as a function of \( E_0/U \) for \( \rho = 3.0 \) and \( \nu = 1/3 \). The range of \( E_0/U \) is taken so that the \( \nu = 1/3 \) Laughlin-type state is the ground state for \( N = 45 \). The regions (I), (II) and (III) have \( K < 0 \), \( K > 1 \) and \( 0 < K < 1 \), respectively. \( K = 1/3 \) is the value expected for the chiral TL liquid.
TABLE I. Velocity of the particle-hole excitation from $\nu = 1/p$ Laughlin state.

| $\nu$ | $v_s$ |
|-------|-------|
| 1     | $1.844 \times 10^1 E_0 - 5.134 \times 10^{-1} U$ |
| 1/3   | $5.574 \times 10^1 E_0 - 1.727 \times 10^{-1} U$ |
| 1/5   | $9.163 \times 10^1 E_0 - 9.996 \times 10^{-2} U$ |
$K = 2 \pm 2^{1/3} E_0/U$
\[ \frac{\alpha}{\gamma} \frac{E_{\text{edge}}(\phi \hbar)}{L_x E_0} \]

\[ \frac{\alpha}{\gamma} \frac{E_{\text{int}}(\phi \hbar)}{L_x U} \]

(a) 

(b)