THE $H^\infty$-CALCULUS AND SUMS OF CLOSED OPERATORS

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Abstract. We develop a very general operator-valued functional calculus for operators with an $H^\infty$-calculus. We then apply this to the joint functional calculus of two commuting sectorial operators when one has an $H^\infty$-calculus. Using this we prove theorem of Dore-Venni type on sums of commuting sectorial operators and apply our results to the problem of $L_p$-maximal regularity. Our main assumption is the R-boundedness of certain sets of operators, and therefore methods from the geometry of Banach spaces are essential here. In the final section we exploit the special Banach space structure of $L_1$-spaces and $C(K)$-spaces, to obtain some more detailed results in this setting.

1. Introduction

In recent years the notion of an $H^\infty$-calculus for sectorial operators on a Banach space has played an important role in spectral theory for unbounded operators and its applications to differential operators and evolution equations. We recall that a sectorial operator of type $0 \leq \omega < \pi$ satisfies a “parabolic” estimate of the type

\begin{equation}
\| \zeta R(\zeta, A) \| \leq C_\sigma \quad |\arg \zeta| \geq \sigma
\end{equation}

for every $\omega < \sigma < \pi$. This estimate allows a definition of $f(A)$ as a bounded operator for functions $f$ which are bounded and analytic on the sector $\Sigma_\sigma = \{ \zeta : |\arg \zeta| < \sigma \}$ and which obey a condition of the type $|f(\zeta)| \leq C(|\zeta|/(1 + |\zeta|^2)\epsilon)$ for some $\epsilon > 0$. This is described in detail in [3] and [28] and we give a somewhat different approach in Section 4 below. If we then have an estimate

$$
\| f(A) \| \leq C \| f \|_{H^\infty(\Sigma_\sigma)}
$$

it is possible to extend the definition of $f(A)$ to any $f \in H^\infty(\Sigma_\sigma)$ and we say that $f$ has an $H^\infty(\Sigma_\sigma)$-calculus. It is, by now, well-known that many systems of parabolic differential operators, Schrödinger operators and pseudodifferential operators do have an $H^\infty$-calculus (cf. [22], [4], [3] and [13]) and this has proved useful in applications.

Of particular importance are two closely related problems:

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• the maximal $L_p$-regularity of the Cauchy problem
\[ y'(t) + Ay(t) = f(t), \quad y(0) = 0 \]
for a sectorial operator of type $\omega < \frac{\pi}{2}$
• the question whether the sum $A + B$ with domain $\mathcal{D}(A) \cap \mathcal{D}(B)$ of two commuting sectorial operators is closed.

In fact the first problem can be reduced to the second, and the latter problem is essentially the question whether one can construct a bounded operator $B(A + B)^{-1}$. This then is a special case of the problem of constructing a joint functional calculus of $A, B$. In the case of Hilbert spaces and some related situations it was shown in [20], [28] and [30] that one can construct an operator-valued functional calculus associated to an operator with $H^\infty$-calculus and this permits a solution; however, it was also shown that such an approach cannot work in general Banach spaces and additional conditions are therefore needed.

We now describe the main results of this paper. In Section 2 we describe a method of setting up the joint functional calculus of $n$ commuting sectorial operators and an operator-valued extension. In Section 3 we recall the notion of Rademacher-boundedness (or R-boundedness) of families of operators. This implicitly goes back to work of Bourgain [6] and has recently been studied in [4], [10], [11] and [41] in connection with vector-valued multiplier theorems. We also introduce some weaker notions and study their relationship to certain Banach space properties of the underlying space.

Using these ideas in Section 4 we prove a very general result on the existence of an operator-valued functional calculus for operators with an $H^\infty$-calculus. Roughly speaking this permits us to replace boundedness of the range of the function by Rademacher-boundedness (or even the weaker concept of U-boundedness introduced in Section 3).

We then study the relationship between R-boundedness and the functional calculus for general sectorial operators. Of particular importance is the notion of R-sectoriality when the boundedness condition (1.1) is replaced by an R-boundedness condition. Using this in Theorem 6.3 we prove a general result on sums which can be regarded as an extension of the Dore-Venni Theorem [18]. We show that if $A, B$ are commuting sectorial operators such that $A$ has an $H^\infty(\Sigma_{\sigma})$-calculus and $B$ is R-sectorial of type $\sigma'$ when $\sigma + \sigma' < \pi$ then $A + B$ (with domain $\mathcal{D}(A) \cap \mathcal{D}(B)$) is closed. One practical advantage of our result is that it is easier to check R-sectoriality than the boundedness of imaginary powers (see [27], [42] and [11]). We also give applications to the joint $H^\infty$-functional calculus (cf. [30]) and show how Banach space properties such as UMD, analytic UMD and property $(\alpha)$ of Pisier relate to our results.

It might be added that our results on the $H^\infty$-calculus emphasize the fact that an $H^\infty$-calculus really induces an unconditional expansion of the identity...
on the underlying Banach space. We feel our development of the theory here is somewhat simpler than preceding approaches (even for Hilbert spaces).

Finally in Section 2 we use this observation to show how classical results on unconditional bases due to Lindenstrauss and Pełczyński [34] can be recast as results on operators with an $H^\infty$-calculus on $L_1$ and $C(K)$-spaces. In these cases we get very strong conclusions, but they are mitigated by the observation that there are in practice very few examples of such operators on spaces of this type.

2. AN OPERATOR-VALUED FUNCTIONAL CALCULUS FOR SECTORIAL OPERATORS

In this section we sketch a method of setting up an operator-valued functional calculus for a sectorial operator and a joint operator-valued functional calculus for finite collections of commuting sectorial operators. For an alternative construction of the $H^\infty$-calculus based on McIntosh’s approach see [30].

Let us first introduce some notation. Suppose $0 < \sigma < \pi$. Then we denote by $\Sigma_\sigma$ the sector $\{z : |\arg z| < \sigma, |z| > 0\}$ and by $\Gamma_\sigma$ the contour $\{|t|e^{i\sgn t}\sigma : -\infty < t < \infty\}$, where $\sgn t = 1$ if $t > 0$ and $\sgn t = -1$ if $t < 0$. We denote by $H^\infty(\Sigma_\sigma)$ the space of all bounded analytic functions on $\Sigma_\sigma$. We define $H^\infty_0(\Sigma_\sigma)$ to be the subspace of all $f \in H^\infty(\Sigma_\sigma)$ which obey an estimate of the form $|f(z)| \leq C(|z|/(1 + |z|^2))^\epsilon$ with $\epsilon > 0$. Let us extend this to dimension $m$. In $\mathbb{C}^m$ if $\sigma = (\sigma_1, \ldots, \sigma_m)$ where $0 < \sigma_k < \pi$ for $1 \leq k \leq m$ we define $\Sigma_\sigma = \prod_{k=1}^m \Sigma_{\sigma_k}$ and $\Sigma_\sigma = \prod_{k=1}^m \Gamma_{\sigma_k}$. If $\sigma, \nu \in \mathbb{R}^m$ we write $\sigma > \nu$ if $\sigma_k > \nu_k$ for $1 \leq k \leq m$. We denote by $H^\infty(\Sigma_\sigma)$ the space of all bounded analytic functions on $\Sigma_\sigma$. We define $H^\infty_0(\Sigma_\sigma)$ to be the subspace of all $f \in H^\infty(\Sigma_\sigma)$ which obey an estimate of the form $|f(z)| \leq C\prod_{k=1}^m (|z_k|/(1 + |z_k|^2))^\epsilon$ with $\epsilon > 0$ where $z = (z_1, \ldots, z_m)$.

Next we introduce some corresponding vector-valued spaces. Now suppose $X$ is a Banach space and $A$ is a subalgebra of $\mathcal{L}(X)$, which is closed for the strong-operator topology. If $\sigma = (\sigma_1, \ldots, \sigma_m)$ as above, we define $H^\infty(\Sigma_\sigma; A)$ the space of all bounded functions $F : \Sigma_\sigma \to A$, so that for every $x \in X$ the map $z \to F(z)x$ is analytic (i.e. $F$ is analytic for the strong-operator topology). We consider the scalar space $H^\infty(\Sigma_\sigma)$ as a subspace of $H^\infty(\Sigma_\sigma; A)$ via the identification $f \to fI$. We shall say that $F_n$ converges boundedly to $F$ in $H^\infty(\Sigma_\sigma; A)$ if $\sup_n \sup_{z \in \Sigma_\sigma} \|F_n(z)\| < \infty$ and $F_n(z)x \to F(z)x$ for every $z \in \Sigma_\sigma$, and $x \in X$. We define $H^\infty_0(\Sigma_\sigma, A)$ the subspace of all $F \in H^\infty(\Sigma_\sigma, A)$ which obey an estimate of the form $\|F(z)\| \leq C\prod_{k=1}^m (|z_k|/(1 + |z_k|^2))^\epsilon$ with $\epsilon > 0$ where $z = (z_1, \ldots, z_m)$.

We next consider the space of germs of such functions. Fix $0 \leq \omega_k < \pi$ for $1 \leq k \leq m$. We consider the space $\mathcal{H}(\omega; A) = \bigcup_{\sigma > \omega} H^\infty(\Sigma_\sigma; A)$ where $(F, G)$ are identified if there exists $\sigma > \omega$ with $F(z) = G(z)$ for all $z \in \Sigma_\sigma$. 

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\( \mathcal{H}(\omega, A) \) is then an algebra. In \( \mathcal{H}(\omega, A) \) we define a notion of sequential convergence \( \tau \) by \( F_n \to F \) if there exists \( \sigma > \omega \) so that each \( F_n, F \in H^\infty(\Sigma_\sigma; A) \), \( \sup_n \sup_{z \in \Sigma_\sigma} \| F_n(z) \| < \infty \) and \( F_n(z)x \to F(z)x \) for all \( z \in \Sigma_\sigma \) and all \( x \in X \).

Recall that a closed densely defined operator \( A \) on a Banach space \( X \) is a **sectorial operator of type** \( 0 \leq \omega = \omega(A) < \pi \) if \( A \) is one-one with dense range, the resolvent \( R(\lambda, A) \) is defined and bounded for \( \lambda = re^{i\theta} \) where \( r > 0 \) and \( \omega < |\theta| \leq \pi \) and satisfies an estimate \( \| \lambda R(\lambda, A) \| \leq C_\sigma \) for \( \omega < \sigma \leq |\theta| \).

Suppose \( (A_1, \ldots, A_m) \) is a commuting family of sectorial operators where \( A_k \) is of type \( \omega_k \) for \( 1 \leq k \leq m \), and let \( \omega = (\omega_1, \ldots, \omega_m) \). Define the resolvent for \( |\arg \lambda| > \omega \) by \( R(\lambda, A_1, \ldots, A_m) = \prod_{k=1}^m R(\lambda_k, A_k) \). Let \( \mathcal{A} \) be the closed subalgebra of \( \mathcal{L}(X) \) of all operators \( T \) so that \( T \) commutes with \( R(\lambda, A_k) \) for every \( k \) and every \( \lambda \) with \( |\arg \lambda| > \omega_k \).

If \( F \in \mathcal{H}(\omega, A) \) is of the form \( F(z) = \prod_{k=1}^m (\lambda_k - z_k)^{-p_k}S \) where \( p_k \in \mathbb{N} \cup \{0\} \) and \( S \in \mathcal{A} \) we define \( F(A_1, \ldots, A_m) = \prod_{k=1}^m R(\lambda_k, A_k)^{P_k}S \) and then this definition can be extended by linearity to the linear span of such functions, which we call the *rational functions*, denoted \( \mathcal{R}(\omega, A) \), in \( \mathcal{H}(\omega, A) \).

To extend this definition further we use the following device. Consider the algebra of all \( (F, F(A_1, \ldots, A_m)) \) for \( F \in \mathcal{R}(\omega, A) \) as a subset of \( \mathcal{H}(\omega, A) \times \mathcal{A} \). Denote by \( \tau^* \) the sequential convergence \( (F_n, T_n) \to (F, T) \) if \( F_n \to F(\tau) \) and \( T_n \to T \) in the strong-operator topology. Let \( \mathcal{B} \) be the \( \tau^* \)-closure of this set (i.e. the smallest set which is closed under sequential convergence and contains it). Notice that this construction might involve taking infinitely many iterations of sequential limits, but our construction actually shows that two iterations suffice. It is clear that \( \mathcal{B} \) is an algebra. Our next task is to show that if \( F \in \mathcal{H}(\omega, A) \) there is at most one choice of \( T \in \mathcal{A} \) so that \( (F, T) \in \mathcal{B} \), this will enable us to define \( F(A_1, \ldots, A_m) \) unambiguously.

Consider the function on \( \mathbb{C} \)

\[
\varphi_n(z) = \frac{n}{n + z} - \frac{1}{1 + nz}
\]

and then define on \( \mathbb{C}^m \), \( \psi_n(z) = \prod_{k=1}^m \varphi_n(z_k) \) so that \( \psi_n \in H_0^\infty(\Sigma_\sigma) \) for every \( \sigma > 0 \). Then

\[
\psi_n(A_1, \ldots, A_m) = \prod_{k=1}^m (\frac{1}{n} R(-\frac{1}{n}, A_k) - nR(-n, A_k)) = V_n
\]

is an approximate identity in the sense that \( \sup \| V_n \| < \infty \) and \( V_n x \to x \) for every \( x \in X \).

If \( F \in \mathcal{H}(\omega, A) \) then if \( F \in H^\infty(\Sigma_\sigma, A) \) we can define

\[
L_n(F)x = \left(\frac{-1}{2\pi i}\right)^m \int_{\Gamma_{\nu}} \psi_n(\zeta) F(\zeta) R(\zeta, A_1, \ldots, A_m) x \, d\zeta,
\]

as long as \( \omega < \nu < \sigma \). (Note that we are using short-hand and this is really a multiple contour integral.) An application of Cauchy’s Theorem shows that
$L_n$ is independent of the choice of $\nu$. By the Lebesgue Dominated Convergence Theorem $L_n : \mathcal{H}(\omega, A) \to A$ is $\tau$–continuous if $A$ is equipped with the strong-operator topology.

If $F$ is rational then we have by a standard contour integration,

$$L_n(F)x = F(A_1, \ldots, A_m)V_n x \quad x \in X. \tag{2.3}$$

Now the map $(F, T) \to L_n(F) - T$ is continuous for $\tau^*$ and the strong-operator topology. We conclude that if $(F, T) \in \mathcal{B}$,

$$L_n(F)x = TV_n x \quad x \in X.$$

Since $V_n x \to x$ for all $x \in X$, this shows that $T$ is uniquely determined by $F$. Hence we can define $\mathcal{H}(A_1, \ldots, A_m; A)$ to be the set of $F \in \mathcal{H}(\omega, A)$ such that for some $T$ we have $(F, T) \in \mathcal{B}$ and then we can define $T = F(A_1, \ldots, A_m)$ for $F \in \mathcal{H}(A_1, \ldots, A_m, A)$. The space $\mathcal{H}(A_1, \ldots, A_m; A)$ is an algebra and $F \to F(A_1, \ldots, A_m)$ is an algebra homomorphism. For $F \in \mathcal{H}(A_1, \ldots, A_m; A) \cap H^\infty(\Sigma_\sigma; A)$ and $\sigma > \nu > \omega$ then (2.2) and (2.3) can be rewritten as:

$$F(A_1, \ldots, A_m)V_n x = \left(\frac{-1}{2\pi i}\right)^m \int_{\Gamma_\nu} \psi_n(\zeta) F(\zeta) R(\zeta, A_1, \ldots, A_m) x d\zeta \tag{2.4}$$

If $F \in H^\infty(\Sigma_\sigma; A)$ then the integrals in (2.4) converge as $n \to \infty$. We can show by approximating the integral by Riemann sums that $F \in \mathcal{H}(\omega, A)$ and then we have:

$$F(A_1, \ldots, A_m)x = \left(\frac{-1}{2\pi i}\right)^m \int_{\Gamma_\nu} F(\zeta) R(\zeta, A_1, \ldots, A_m) x d\zeta \quad x \in X. \tag{2.5}$$

It now follows that if $F \in \mathcal{H}(\omega, A)$ then $(\psi_k F) \in \mathcal{H}(A_1, \ldots, A_m; A)$ for each $k \in \mathbb{N}$. Furthermore if $F_n \to F(\tau)$ we have $(\psi_k F_n)(A_1, \ldots, A_m) \to (\psi_k F)(A_1, \ldots, A_m)$ in the strong-operator topology for each fixed $k$. From this it follows that if $F_n \in \mathcal{H}(A_1, \ldots, A_m; A)$ and $\sup \|F_n(A_1, \ldots, A_m)\| < \infty$ then $F \in \mathcal{H}(A_1, \ldots, A_m)$ and $F_n(A_1, \ldots, A_m) \to F(A_1, \ldots, A_m)$ in the strong-operator topology (indeed we have convergence on each $V_n x$). In particular it follows that $F \in \mathcal{H}(A_1, \ldots, A_m; A)$ if and only if $\sup_n \|(\psi_k F)(A_1, \ldots, A_m)\| < \infty$.

If we consider the scalar functions in $\mathcal{H}(A_1, \ldots, A_m) \subset \mathcal{H}(A_1, \ldots, A_m; A)$ then we have defined the joint functional calculus for $(A_1, \ldots, A_m)$. We recall that a single operator $A$ has an $H^\infty(\Sigma_\sigma)$–calculus if $H^\infty(\Sigma_\sigma) \subset \mathcal{H}(A)$. The collection $(A_1, \ldots, A_m)$ has a joint $H^\infty(\Sigma_\sigma)$–calculus if $H^\infty(\Sigma_\sigma) \subset \mathcal{H}(A_1, \ldots, A_m)$.

3. Rademacher-boundedness and related ideas

We recall ([10], [11]) that a family $\mathcal{F}$ of bounded operators on a Banach space $X$ is called Rademacher-bounded or $R$-bounded with $R$-boundedness constant
if letting \((\epsilon_k)_{k=1}^{\infty}\) be a sequence of independent Rademachers on some probability space then for every \(x_1, \ldots, x_n \in X\) and \(T_1, \ldots, T_n \in \mathcal{F}\) we have:

\[
(\mathbb{E}\| \sum_{k=1}^{n} \epsilon_k T_k x_k \|^2)^{\frac{1}{2}} \leq C (\mathbb{E}\| \sum_{k=1}^{n} \epsilon_k x_k \|^2)^{\frac{1}{2}}.
\]

It is important to note that this definition and the associated constant \(C\) are unchanged if we require \(T_1, \ldots, T_n\) to be distinct in (3.1) (see e.g. [14], Lemma 3.3). The same remark applies to each of the following definitions.

We will introduce two related weaker notions. Let us say that \(\mathcal{F}\) is weakly Rademacher-bounded or WR-bounded with WR-boundedness constant \(C\) if for every \(x_1, \ldots, x_n \in X, x_1^*, \ldots, x_n^* \in X^*\) and \(T_1, \ldots, T_n \in \mathcal{L}(X)\) we have:

\[
\sum_{k=1}^{n} |\langle T_k x_k, x_k^* \rangle| \leq C (\mathbb{E}(\| \sum_{k=1}^{n} \epsilon_k x_k \|^2)^{\frac{1}{2}} (\mathbb{E}(\| \sum_{k=1}^{n} \epsilon_k x_k^* \|^2)^{\frac{1}{2}}.
\]

Finally we say that \(\mathcal{F}\) is unconditionally bounded or U-bounded with U-boundedness constant \(C\) if for every \(x_1, \ldots, x_n \in X, \alpha_1, \ldots, x_n^* \in X^*\) and \(T_1, \ldots, T_n \in \mathcal{F}\) we have

\[
\sum_{k=1}^{n} |\langle T_k x_k, x_k^* \rangle| \leq C \max_{\epsilon_k=\pm 1} (\mathbb{E}\| \sum_{k=1}^{n} \epsilon_k x_k \|) \max_{\epsilon_k=\pm 1} (\mathbb{E}\| \sum_{k=1}^{n} \epsilon_k x_k^* \|).
\]

The following Lemma is recorded for future reference:

**Lemma 3.1.** Let \(\mathcal{F}\) be a subset of \(\mathcal{L}(X)\). Then for \(\mathcal{F}\), R-bounded \(\Rightarrow\) WR-bounded \(\Rightarrow\) U-bounded. If \(X\) has nontrivial Rademacher type then WR-bounded \(\Rightarrow\) R-bounded.

We note that the only really non-trivial part of the Lemma is the last sentence and this follows easily from Pisier’s characterization of spaces with non-trivial type as those in which the Rademacher projection is bounded [37].

We shall also need some related Banach space concepts. Suppose \((\epsilon_k)_{k=1}^{\infty}\) and \((\eta_k)_{k=1}^{\infty}\) are two mutually independent sequences of Rademachers. We say that \(X\) has property \((\alpha)\) (see [36] and [28]) if there is a constant \(C\) so that for any \((x_{jk})_{j,k=1}^{n} \subset X\) and any \((\alpha_{jk})_{j,k=1}^{n} \subset \mathbb{C}\) we have

\[
(\mathbb{E}\| \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} \epsilon_j \eta_k x_{jk} \|^2)^{\frac{1}{2}} \leq C \max_{j,k} |\alpha_{jk}| (\mathbb{E}\| \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_j \eta_k x_{jk} \|^2)^{\frac{1}{2}}.
\]

We say that \(X\) has property \((A)\) [30] if there is a constant \(C\) such that for any \((x_{jk})_{j,k=1}^{n} \subset X\) and for any \((x_{jk}^*)_{j,k=1}^{n} \subset X^*\) we have:

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} |\langle x_{jk}, x_{jk}^* \rangle| \leq C (\mathbb{E}\| \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_j \eta_k x_{jk} \|^2)^{\frac{1}{2}} (\mathbb{E}\| \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_j \eta_k x_{jk}^* \|^2)^{\frac{1}{2}}.
\]
Clearly $(\alpha)$ implies $(A)$ and the converse holds if $X$ has nontrivial Rademacher type; this is a fairly simple deduction from the boundedness of the Rademacher projection \[37\]. Any subspace of a Banach lattice with nontrivial cotype has property $(\alpha)$ while any Banach lattice has property $(A)$. It is also observed in \[28\] that $L_1/H_1$ has $(\alpha)$. The Schatten ideals $\mathcal{C}_p$ when $1 \leq p \leq \infty$ fail to have $(A)$.

We shall say that $X$ has property $(\Delta)$ if there is a constant $C$ so that for any $(x_{jk})_{j,k=1}^n \in X$

$$\mathbb{E}\left\| \sum_{j=1}^n \sum_{k=1}^j \epsilon_j \eta_k x_{jk} \right\| ^2 \leq C \mathbb{E}\left\| \sum_{j=1}^n \sum_{k=1}^j \epsilon_j \eta_k x_{jk} \right\| ^2.$$

(3.6)

It is clear that $(\Delta)$ is a weaker property than $(\alpha)$. It is in fact shared by all spaces with (UMD) and even analytic UMD. We recall \([5]\) that $X$ has analytic UMD if every $L_1$-bounded analytic martingale has unconditional martingale differences.

**Proposition 3.2.** Suppose $X$ has analytic UMD. Then $X$ has property $(\Delta)$.

**Proof.** Let $(\tilde{\epsilon}_k)_{k=1}^\infty$ and $(\tilde{\eta}_k)_{k=1}^\infty$ be two mutually independent sequences of Steinhaus variables (i.e. each is complex-valued and uniformly distributed on the unit circle). By applying the unconditionality of the Rademachers and the Khintchine-Kahane inequality it is sufficient to show the existence of a constant $C$ so that for any $(x_{jk})_{j,k=1}^n$ we have:

$$\mathbb{E}\left\| \sum_{j=1}^n \sum_{k=1}^j \tilde{\epsilon}_j \tilde{\eta}_k x_{jk} \right\| \leq C \mathbb{E}\left\| \sum_{j=1}^n \sum_{k=1}^j \tilde{\epsilon}_j \tilde{\eta}_k x_{jk} \right\|.$$

To see this we define $f_j$ for $1 \leq j \leq 2n - 1$ by $f_{2r-1} = \sum_{j\leq r} \sum_{k\leq r} \tilde{\epsilon}_j \tilde{\eta}_k x_{jk}$ and $f_{2r} = \sum_{j\leq r+1} \sum_{k\leq r} \tilde{\epsilon}_j \tilde{\eta}_k x_{jk}$. Let $f_0 = 0$. Then $(f_j)$ is an analytic martingale and so for a suitable constant $C$ depending only on $X$ we have:

$$\mathbb{E}\left\| \sum_{r=0}^{n-1} (f_{2r+1} - f_{2r}) \right\| \leq C \mathbb{E}\left\| f_{2n-1} \right\|.$$

This yields the desired inequality.

Since any space with (UMD) has analytic (UMD) this shows that (UMD)-spaces have $(\Delta)$; actually a direct proof using Rademacher in place of Steinhaus variables in the above argument is possible for this case. Thus the Schatten classes $\mathcal{C}_p$ have property $(\Delta)$ as long as $1 < p < \infty$. However Haagerup and Pisier \[22\] show that $\mathcal{C}_1$ (which has cotype 2) fails analytic UMD and their argument actually shows it fails property $(\Delta)$. This implies that $C(K)$-spaces of infinite dimension also fail $(\Delta)$ since $\mathcal{C}_1$ is finitely representable in any such space.
We now come to an important result relating the above properties to Rademacher-boundedness. Some similar results are shown in [10].

**Theorem 3.3.** Suppose \((U_k)_{k=1}^\infty\) and \((V_k)_{k=1}^\infty\) are two sequences of operators in \(\mathcal{L}(X)\) satisfying
\[
\sup_n \sup_{\epsilon_k = \pm 1} \left\| \sum_{k=1}^n \epsilon_k U_k \right\| \leq M < \infty
\]
and
\[
\sup_n \sup_{\epsilon_k = \pm 1} \left\| \sum_{k=1}^n \epsilon_k V_k \right\| \leq M < \infty.
\]
Suppose further \(F \subset \mathcal{L}(X)\) is a family of operators which commutes with each \(U_k\) and each \(V_k\) and is \(R\)-bounded with constant \(R\). Then:

1. The sequence \((U_n)_{n=1}^\infty\) is \(R\)-bounded with constant \(M\).
2. If \(X\) has property \((\alpha)\) the collection \(\{\sum_{k=1}^n \alpha_k T_k U_k V_k : n \in \mathbb{N}, |\alpha_k| \leq 1, T_k \in F\}\) is \(R\)-bounded with constant \(CM^2\) where \(C\) depends only on \(X\). In particular the family \(\{\sum_{k=1}^n \alpha_k U_k V_k : n \in \mathbb{N}, |\alpha_1|, \ldots, |\alpha_n| \leq 1\}\) is \(R\)-bounded with constant \(CM^2\) where \(C\) depends only on \(X\).
3. If \(X\) has property \((A)\) then the family \(\{\sum_{k=1}^n \alpha_k U_k V_k : n \in \mathbb{N}, |\alpha_1|, \ldots, |\alpha_n| \leq 1\}\) is \(WR\)-bounded with constant \(CM^2\) where \(C\) depends only on \(X\).
4. If \(X\) has property \((\Delta)\) then the set \(\{\sum_{k=1}^n U_k V_k : n \in \mathbb{N}\}\) is \(R\)-bounded with constant \(CM^2\) where \(C\) depends only on \(X\).

**Proof.** (1) We use the remark that it is enough to establish (3.1) for distinct operators \(T_1, \ldots, T_n\). If \(x_1, \ldots, x_n \in X\) and \(\alpha_k = \pm 1\) then
\[
\mathbb{E}(\left( \sum_{k=1}^n \epsilon_k U_k \right) \left( \sum_{k=1}^n \epsilon_k \alpha_k x_k \right)) = \sum_{k=1}^n \alpha_k U_k x_k
\]
and hence
\[
\sup_{\alpha_k = \pm 1} \left\| \sum_{k=1}^\infty \alpha_k U_k x_k \right\| \leq M(\mathbb{E}(\left\| \sum_{k=1}^n \epsilon_k x_k \right\|^2)^\frac{1}{2}.
\]
This proves (1) and indeed a rather stronger result.

(2) Let \(S_j = \sum_{k=1}^\infty \alpha_{jk} T_j U_k V_k\) where \(T_{jk} \in F\) and \((\alpha_{jk})\) is a finitely nonzero collection of complex numbers with \(|\alpha_{jk}| \leq 1\). Suppose \(x_1, \ldots, x_n \in X\). We first note that
\[
\left\| \sum_{k=1}^n U_k V_k x_k \right\| = \mathbb{E} \left( \left\| \sum_{k=1}^n \epsilon_k U_k \right\| \left( \sum_{k=1}^n \epsilon_k V_k x_k \right) \right) \leq M\mathbb{E}(\left\| \sum_{k=1}^n \epsilon_k V_k x_k \right\|^2)^\frac{1}{2}.
\]
We will also use the fact (Lemma 3.13 of [10]) that there is a constant $C$ depending only on $X$ so that for $(x_{jk})_{j,k=1}^n \in X$ we have from property $(\alpha)$,

$$\left(\mathbb{E}_c \mathbb{E}_{\eta} \left\| \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} \epsilon_j \eta_k T_{jk} x_{jk} \right\|^2 \right)^{\frac{1}{2}} \leq CR \left(\mathbb{E}_c \mathbb{E}_{\eta} \left\| \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_j \eta_k x_{jk} \right\|^2 \right)^{\frac{1}{2}}.$$ 

Hence

$$\left(\mathbb{E}_c \left\| \sum_{j=1}^{n} \epsilon_j S_j x_j \right\|^2 \right)^{\frac{1}{2}} = \left(\mathbb{E}_c \left\| \sum_{k=1}^{\infty} U_k V_k \sum_{j=1}^{n} \alpha_{jk} \epsilon_j \eta_k T_{jk} x_j \right\|^2 \right)^{\frac{1}{2}} \leq M \left(\mathbb{E}_c \mathbb{E}_{\eta} \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \alpha_{jk} \epsilon_j \eta_k V_k T_{jk} x_j \right\|^2 \right)^{\frac{1}{2}} \leq CRM \left(\mathbb{E}_c \mathbb{E}_{\eta} \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \epsilon_j \eta_k V_k x_j \right\|^2 \right)^{\frac{1}{2}} \leq CRM^2 \left(\mathbb{E}_c \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\|^2 \right)^{\frac{1}{2}}.$$

This proves (2).

(3) Let $S_j = \sum_{k=1}^{\infty} \alpha_{jk} U_k V_k$ where $(\alpha_{jk})$ is a finitely nonzero matrix with $|\alpha_{jk}| \leq 1$. In this case if $x_1, \ldots, x_n \in X$ and $x_1^\ast, \ldots, x_n^\ast \in X^*$ we note that:

$$\sum_{j=1}^{n} |\langle S_j x_j, x_j^\ast \rangle| \leq \sum_{j=1}^{n} \sum_{k=1}^{\infty} |\langle V_k x_j, U_k^\ast x_j^\ast \rangle| \leq C \left(\mathbb{E}_c \mathbb{E}_{\eta} \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \epsilon_j \eta_k V_k x_j \right\|^2 \right)^{\frac{1}{2}} \left(\mathbb{E}_c \mathbb{E}_{\eta} \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \epsilon_j \eta_k U_k^\ast x_j^\ast \right\|^2 \right)^{\frac{1}{2}} \leq CM^2 \left(\mathbb{E}_c \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\|^2 \right)^{\frac{1}{2}} \left(\mathbb{E}_c \left\| \sum_{j=1}^{n} \epsilon_j x_j^\ast \right\|^2 \right)^{\frac{1}{2}} \left(\mathbb{E}_c \mathbb{E}_{\eta} \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \epsilon_j \eta_k V_k x_j \right\|^2 \right)^{\frac{1}{2}} \left(\mathbb{E}_c \mathbb{E}_{\eta} \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \epsilon_j \eta_k U_k^\ast x_j^\ast \right\|^2 \right)^{\frac{1}{2}} \leq CM^2 \left(\mathbb{E}_c \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\|^2 \right)^{\frac{1}{2}} \left(\mathbb{E}_c \left\| \sum_{j=1}^{n} \epsilon_j x_j^\ast \right\|^2 \right)^{\frac{1}{2}}.$$

(4) We use the proof of (2). This time we again use the fact it suffices to consider the operators without repetition. So we consider $S_j = \sum_{k=1}^{j} U_k V_k$ and repeat the proof of (2) with $\alpha_{jk} = 1$ if $k \leq j$ and 0 otherwise and replace each $T_{jk}$ by the identity. Using (3,4) in place of (3,3) gives the desired conclusion.

We conclude this section with a useful Lemma. In fact in the case of R-boundedness, this result is found in [11].

**Lemma 3.4.** Suppose $0 < \sigma < \pi$ and $F \in H^\infty(\Sigma_\sigma, \mathcal{L}(X))$. Suppose $0 < \sigma_0 < \nu < \sigma$ and for some $M < \infty$ and $a > 1$, and for each $t \in \mathbb{R}$ the set $\{F(a^k te^{\pm i\nu})\}_{k \in \mathbb{Z}}$ is U-bounded (respectively, WR-bounded; respectively, R-bounded) with constant bounded by $M$ (independent of $t$). Then the family
\{F(z) : z \in \Sigma_{\sigma_0}\} is U-bounded, (respectively, WR-bounded; respectively, R-bounded).

Proof. We give the proof in the U-boundedness case, the others being similar. We first make the observation that it suffices to consider the case when \(\nu = \frac{\pi}{2}\) as one can make the transformation \(z = w^{2\nu/\pi}\). In this case we have the formula

\[
F(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(it) \Re(z-it)^{-1} dt.
\]

We write

\[
F_1(z) = \frac{1}{\pi} \int_{0}^{\infty} F(it) \Re(z-it)^{-1} dt
\]

and

\[
F_2(z) = \frac{1}{\pi} \int_{0}^{\infty} F(-it) \Re(z+it)^{-1} dt
\]

so that \(F(z) = F_1(z) + F_2(z)\).

Note that for a suitable constant \(C\) we have an estimate \(0 \leq \Re(z \pm it)^{-1} \leq C|z| \min(t^{-2},|z|^{-2})\) whenever \(z \in \Sigma_{\sigma_0}\).

Now suppose \(x_1, \ldots, x_n \in X, x_1^*, \ldots, x_n^* \in X^*\). Suppose \(z_1, \ldots, z_n \in \Sigma_{\sigma_0}\). Let us suppose that \(m_j \in \mathbb{Z}\) are chosen so that \(a^{m_j} \leq |z_j| \leq a^{m_j+1}\). We have:

\[
\sum_{j=1}^{n} |\langle F_1(z_j)x_j, x_j^* \rangle| \leq \frac{1}{\pi} \sum_{j=1}^{n} \int_{0}^{\infty} |\langle F(ia^{m_j}t)x_j, x_j^* \rangle| \Re(z_j-ia^{m_j}t)^{-1}a^{m_j} dt
\]

\[
\leq \frac{aC}{\pi} \sum_{j=1}^{n} \int_{0}^{\infty} |\langle F(ia^{m_j}t)x_j, x_j^* \rangle| \min(1,t^{-2}) dt
\]

\[
\leq C' \max_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\| \max_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^{n} \epsilon_j x_j^* \right\|
\]

for a suitable constant \(C'\). A similar argument can be done for \(F_2\). \qed

4. Functional calculus for operator-valued functions

Let us suppose \(A\) is sectorial of type \(\omega\) and \(\sigma > \omega\). We let \(\mathcal{A}\) denote as in Section 3 the algebra of all bounded operators which commute with \(A\).

Before we prove our basic estimate for an operator-valued functional calculus, we will describe in Lemma 4.1 and Proposition 4.3 the connection between the \(H^\infty\) calculus and unconditional expansions in the underlying Banach space.
Lemma 4.1. Suppose that $A$ admits an $H^\infty(\Sigma_\sigma)$-calculus, and that $f \in H^\infty_0(\Sigma_\sigma)$. Then there is a constant $C$ so that for any $t > 0$ and any finitely nonzero sequence $(\alpha_k)_{k \in \mathbb{Z}}$ we have:

$$\left\| \sum_{k \in \mathbb{Z}} \alpha_k f(2^k tA) \right\| \leq C \max_{k \in \mathbb{Z}} |\alpha_k|.$$

Furthermore for every $x \in X$ and $t > 0$ the series $\sum_{k \in \mathbb{Z}} f(2^k tA)x$ converges unconditionally in $X$.

Proof. We can assume $\max_{k \in \mathbb{Z}} |\alpha_k| \leq 1$. For a suitable constants $C, C'$ and $\epsilon > 0$ we have

$$\left\| \sum_{k \in \mathbb{Z}} \alpha_k f(2^k tA) \right\| \leq C \sup_{z \in \Sigma} \sum_{k \in \mathbb{Z}} |f(2^k z)|$$

$$\leq CC' \sup_{z \in \Sigma} \sum_{k \in \mathbb{Z}} \left( \frac{2^k |z|}{1 + 2^{2k} |z|^2} \right)^\epsilon$$

and the last quantity is finite.

For the last part observe that for any bounded sequence $(\alpha_k)_{k \in \mathbb{Z}}$ and $t > 0$, the series $\sum_{k \in \mathbb{Z}} \alpha_k f(2^k tA)x$ must converge to $g(A)x$ where $g(z) = \sum_{k \in \mathbb{Z}} \alpha_k f(2^k t z) \in H^\infty(\Sigma_\sigma)$.

Proposition 4.2. Suppose $F \in H^\infty_0(\Sigma_\sigma, A)$. Then for any $\omega < \nu < \sigma$, $0 < s < 1$, and any $x \in X$, we have

$$F(A)x = \frac{-1}{2\pi i} \int_{\Gamma_\nu} \zeta^{-s} F(\zeta) A^s R(\zeta, A)x \ d\zeta. \tag{4.1}$$

Proof. First note that $A^s R(\lambda, A)$ is a bounded operator for $\lambda \in \Gamma_\nu$ which is given by the integral

$$A^s R(\lambda, A)x = \frac{-1}{2\pi i} \int_{\Gamma_{\nu'}} \zeta^s (\lambda - \zeta)^{-1} R(\zeta, A)x \ d\zeta,$$

if $\omega < \nu' < \nu$. This gives an estimate $\|A^s R(\lambda, A)\| \leq C_s |\lambda|^{-s-1}$ and shows that the integral in (4.1) converges as a Bochner integral. It is clear that we only need establish the formula if $x = \varphi_n(A)y$ (see (2.1)) for some $y \in X$. To do
this we compute
\[ F(A)\varphi_n(A)x = F(A)\varphi_n^2(A)y = \]
\[ = \left( A^s \varphi_n(A) \right) \left( A^{1-s} \varphi_n(A) \right) y = \]
\[ = \frac{1}{2\pi i} \left( A^s \varphi_n(A) \right) \int_{\Gamma_\nu} \zeta^{-s} \varphi_n(\zeta) F(\zeta) R(\zeta, A) y d\zeta = \]
\[ = \frac{1}{2\pi i} \int_{\Gamma_\nu} \zeta^{-s} \varphi_n(\zeta) F(\zeta) (A^s \varphi_n(A)) R(\zeta, A) y d\zeta = \]
\[ = \frac{1}{2\pi i} \int_{\Gamma_\nu} \zeta^{-s} \varphi_n(\zeta) F(\zeta) (A^s R(\zeta, A)) x d\zeta. \]

Now using the Dominated Convergence Theorem we obtain (4.1). □

Let us rewrite (4.1) by using the parameterization \( \zeta = |t| e^{i(\text{sgn} t)\nu} \) for \( -\infty < t < \infty \). We introduce the notation \( h^\nu_s(z) = z^s (e^{i\rho} - z)^{-1} \). Then for \( F \in H_0^\infty(\Sigma_{\sigma, A}) \) where \( \sigma > \nu > \omega \),

\[ F(A)x = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{i(1-s)(\text{sgn} t)\nu} |t|^{-s} F(|t| e^{i(\text{sgn} t)\nu}) A^s R(|t| e^{i(\text{sgn} t)\nu}, A) x dt \]
\[ = \frac{e^{i(1-s)\nu}}{2\pi i} \int_{0}^{\infty} F(te^{i\nu}) h^\nu_s(t^{-1} A)x \frac{dt}{t} + \frac{e^{-i(1-s)\nu}}{2\pi i} \int_{0}^{\infty} F(te^{-i\nu}) h^{-\nu}_s(t^{-1} A)x \frac{dt}{t} \]

This can then be reformulated as:

(4.2) \[ F(A)x = \frac{1}{2\pi i} \int_{1}^{2} \left( M_+(t) + M_-(t) \right) \frac{dt}{t} \]

where

(4.3) \[ M_{\pm}(t) = e^{\pm i(1-s)\nu} \sum_{k \in \mathbb{Z}} F(2^{-k} t^{-1} e^{\pm i\nu}) h_{\pm}^{\nu}(2^k t A)x. \]

We first make an essentially trivial deduction characterizing the \( H^\infty \)-calculus.

**Proposition 4.3.** Suppose \( \nu > \omega \) and \( 0 < s < 1 \). Consider the conditions:

(4.4) \[ \sup_{t > 0} \sup_{N} \sup_{\epsilon_k = \pm 1} \left\| \sum_{k = -N}^{N} \epsilon_k (2^k t)^{(1-s)} A^s R(2^k t e^{\pm i\nu}, A) \right\| < \infty, \]

Then (4.4) is necessary for \( A \) to admit an \( H^\infty(\Sigma_{\sigma}) \)-calculus for some \( \sigma < \nu \) and sufficient for \( A \) to admit an \( H^\infty(\Sigma_{\sigma}) \)-calculus for every \( \sigma > \nu \).
Proof. Necessity follows immediately from Lemma 4.1 for the functions $h_s^{\pm \nu}$. Conversely, by (4.4), if $f \in H(\sigma)$ where $\sigma > \nu$ we obtain by (4.2) and (4.3)
\[ \| (\varphi_n f)(A) \| \leq C \]
independent of $n$. This implies that $f \in H(A)$.

Our main result is also easy from (4.2) and (4.3).

**Theorem 4.4.** Suppose $A$ admits an $H(\Sigma_{\sigma})−$calculus and $F \in H(\Sigma_{\rho}; A)$ for some $\rho > \sigma$. Suppose further that the set \{\( F(z) : z \in \Sigma_{\rho} \)\} is U-bounded. Then $F \in H(A, A)$.

**Remarks.** (1) Of course the Theorem holds if we assume the stronger property that \{\( F(z) : z \in \Sigma_{\rho} \)\} is WR-bounded or R-bounded.

(2) For Hilbert spaces and certain operators on $L_2(X)$ such an operator-valued functional calculus is constructed in [28] Theorem 5.2 and [30]. These are cases when the U-boundedness condition is satisfied automatically. See also [20]. In [11] there are constructions based on transference results which work for generators of bounded $c_0$-groups on UMD-spaces and some other special cases.

Proof. As before we consider $\varphi_n F = F_n$ so that $F_n \in H(A, A)$. It suffices to show sup $\| F_n(A) \| < \infty$. Referring to (4.2) and (4.3) with some fixed $0 < s < 1$ and $\rho > \nu > \sigma$ for $x \in X$, $x^* \in X^*$ with $\| x \|, \| x^* \| \leq 1$, we obtain the estimate for $1 \leq t \leq 2$:
\[ |\langle M_{\pm}(t)x, x^* \rangle| \leq \sum_{k \in \mathbb{Z}} |\langle F_n(2^{-k}t^{-1}e^{\pm i\nu})g(2^k tA)x, g(2^k tA)^*x^* \rangle| \]
where $g(z) = (h_s^{\pm \nu}(z))^{1/2}$. Suppose $C$ is the U-boundedness constant of \{\( F(z) : z \in \Sigma_{\sigma} \)\}. Then
\[ |\langle M_{\pm}(t)x, x^* \rangle| \leq C \sup_{\epsilon_\pm = \pm 1} \sup_N \sum_{|k| \leq N} \epsilon_k g(2^k tA)^2. \]
Hence by Lemma 4.1 we have
\[ \sup_n \| F_n(A) \| < \infty. \]

Let us apply this to the case of two commuting operators:

**Theorem 4.5.** Suppose $A, B$ are commuting sectorial operators, such that $A$ admits a $H(\Sigma_{\sigma})−$calculus and $\omega(B) < \sigma'$. Suppose $f \in H(\Sigma_{\rho} \times \Sigma_{\sigma'})$ where $\sigma < \rho < \pi$ is such that \{\( f(w, \cdot) : w \in \Sigma_{\rho} \)\} is contained in $H(B)$. Suppose further the set \{\( f(w, B) : w \in \Sigma_{\rho} \)\} is U-bounded. Then $f \in H(A, B)$ (i.e. $f(A, B)$ is a bounded operator).
Proof. We define $F(w) = f(w, B)$ and note that $F \in H^\infty(\Sigma_\sigma; A)$; this follows easily from the integral representation (2.3). Our conditions and Theorem 4.4 ensure that $F \in \mathcal{H}(A; A)$. It is only necessary to check that this implies $f \in \mathcal{H}(A, B)$ and of course $F(A) = f(A, B)$. But this follows directly from (2.3), (2.4) and the remarks thereafter. \hfill \Box

Example. Let us show by example that Theorem 4.5 is close to the best possible. Let $B$ be a sectorial operator on $X$. Suppose $0 < \sigma < \pi$ and consider the space $L_2\{\{-1, 1\}^{\Sigma_\sigma}; X\}$ where $\{-1, 1\}^{\Sigma_\sigma}$ has the usual product measure. Denote by $\epsilon_z$ the co-ordinate maps for $z \in \Sigma_\sigma$. Let $\text{Rad} X$ denote the closed linear span of the functions $\{\epsilon_z \otimes x : z \in \Sigma_\sigma, x \in X\}$. We define $\tilde{B} = I \otimes B$ on $L_2(X)$ and restrict it to the subspace $\text{Rad} X$ which is invariant. We define $A$ on $\text{Rad} X$ by

$$A(\sum_{z \in \Sigma_\sigma} \epsilon_z x_z) = \sum_{z \in \Sigma_\sigma} z \epsilon_z x_z$$

with domain consisting of all $\sum_{z \in \Sigma_\sigma} \epsilon_z x_z \in L_2$ so that $\sum_{z \in \Sigma_\sigma} z \epsilon_z x_z \in L_2$.

Clearly $A$ has an $H^\infty(\Sigma_\sigma)-$calculus and $f(A, \tilde{B})$ is bounded if and only if the family $\{f(z, B) : z \in \Sigma_\sigma\}$ is R-bounded.

We remark that the reader who prefers separable spaces can easily modify this example when $X$ is separable to replace $\text{Rad} X$ by a separable countable subset of $\Sigma_\sigma$.

5. R-boundedness and the functional calculus

We now consider strengthenings of the boundedness conditions in the definition of sectoriality. Let $A$ be a sectorial operator and let $\omega(A)$ denote the infimum of all $\sigma$ so that $A$ is of type $\sigma$. We will say that $A$ is $R$-sectorial, (respectively WR-sectorial, respectively U-sectorial) if there exists $0 < \sigma < \pi$ so that the family of operators $\{\lambda R(\lambda, A) : |\arg \lambda| > \sigma\}$ is R-bounded (respectively WR-bounded, respectively U-bounded). We then define $\omega_R(A)$, (respectively $\omega_{WR}(A)$, respectively $\omega_U(A)$) to be the infimum of all such $\sigma$. We will say $A$ is $H^\infty-$sectorial (respectively, $RH^\infty-$sectorial, respectively $WRH^\infty-$sectorial) if there exists a $0 < \sigma < \pi$ so that $A$ admits an $H^\infty(\Sigma)-$calculus (respectively, such that the set $\{f(A) : \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\}$ is R-bounded, respectively such that the set $\{f(A) : \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\}$ is WR-bounded). The infimum of all such $\sigma$ is denoted $\omega_H(A)$ (respectively $\omega_{RH}(A)$, respectively $\omega_{WRH}(A)$).

There are certain obvious and trivial relationships between these concepts. Clearly R-sectorial implies WR-sectorial implies U-sectorial and whenever these concepts are defined, $\omega_R(A) \geq \omega_{WR}(A) \geq \omega_U(A) \geq \omega(A)$. Similarly $RH^\infty-$sectorial implies $WRH^\infty-$sectorial implies $H^\infty-$sectorial and $\omega_{RH}(A) \geq \omega_{WRH}(A) \geq \omega_H(A) \geq \omega(A)$.

We now turn to less trivial observations:
Proposition 5.1. Suppose $A$ is $H^\infty$-sectorial and $U$-sectorial. Then $\omega_H(A) \leq \omega_U(A)$.

Proof. Let us assume that $\{\lambda R(\lambda, A) : |\arg \lambda| \geq \nu\}$ is U-bounded with constant $K$ where $\nu > \omega(A)$, and that $\sigma > \nu$. We will show that $A$ admits an $H^\infty(\Sigma_\sigma)$-calculus. We use Proposition 4.3. Fix some $0 < s < 1$. We can assume that there exists $\rho > \sigma$ so that

$$\sup_N \sup_{\epsilon = \pm 1} \sup_{t > 0} \left\| \sum_{|k| \leq N} \epsilon_k h_s^{2\rho}(A) \right\| = M < \infty$$

and so that $A$ admits an $H^\infty(\Sigma_\tau)$-calculus for some $\tau < \rho$.

Now suppose $x \in X$ and $x^* \in X^*$. Then for any $N$ and $\epsilon_j = \pm 1$ we have

$$\left| \langle \sum_{|k| \leq N} \epsilon_k h_s^{\nu}(2k^2A)x, x^* \rangle \right| \leq M \|x\| \|x^*\| + \sum_{|k| \leq N} \left| \langle (h_s^{\rho}(2k^2A) - h_s^{\rho}(2k^2A))x, x^* \rangle \right|.$$

By the resolvent equation,

$$h_s^{\nu}(2k^2A) - h_s^{\rho}(2k^2A) = (e^{i(\rho - \nu)} - 1)2^{-k^2}e^{i\nu}R(2^{-k^2}e^{i\nu}, A)h_s^{\rho}(A).$$

Since $A$ has an $H^\infty(\Sigma_\tau)$-calculus we can define $g(z) = (h_s^{\rho}(z))^{1/2}$ and note that

$$\sup_N \sup_{\epsilon = \pm 1} \left\| \sum_{|k| \leq N} \epsilon_k g(2k^2A) \right\| \leq C$$

where $C$ is independent of $t$. Thus, by the U-boundedness of $\{\lambda R(\lambda, A) : |\arg \lambda| = \nu\}$

$$\sum_{|k| \leq N} \left| \langle (2^{-k^2}e^{i\nu}R(2^{-k^2}e^{i\nu}, A)g(2k^2A)x, g(2k^2A)x^* \rangle \right| \leq KC \|x\| \|x^*\|.$$ 

It follows that

$$\left| \langle \sum_{|k| \leq N} \epsilon_k h_s^{\nu}(2k^2A)x, x^* \rangle \right| \leq (M + 2KC) \|x\| \|x^*\|$$

and this gives

$$\sup_N \sup_{\epsilon = \pm 1} \sup_{t > 0} \left\| \sum_{|k| \leq N} \epsilon_k h_s^{\nu}(A) \right\| \leq M + 2KC < \infty.$$ 

Combined with a similar estimate for $-\nu$ we obtain the result by using Proposition 4.3. □

In order to study an analytic semigroup with generator $(-A)$ it is of particular interest to know that $\omega_H(A) < \pi/2$. Therefore we use Proposition 5.1 to improve on a result in [24].

Corollary 5.2. Let $(-A)$ generate an analytic contractive and positive semigroup on $L_p(\Omega, \mu)$ for some $1 < p < \infty$. Then $\omega_H(A) < \pi/2$. 
Theorem 5.3. Suppose $A$ is an $H^\infty$–sectorial operator on a Banach space $X$. Then:

1. If $X$ has property $(\alpha)$ then $A$ is $RH^\infty$–sectorial and $\omega_H(A) = \omega_{RH}(A) = \omega_R(A) = \omega_U(A)$.
2. If $X$ has property $(\alpha)$ then $A$ is $WRH^\infty$–sectorial and $\omega_H(A) = \omega_{WRH}(A) = \omega_W(A) = \omega_U(A)$.
3. If $X$ has property $(\Delta)$ then $A$ is $R$–sectorial and $\omega_H(A) = \omega_R(A) = \omega_U(A)$.

Proof. (1) Assume that $A$ admits an $H^\infty(\Sigma_\sigma)$–calculus. Suppose $\sigma < \nu < \pi$. Suppose $0 < s < 1$ and let $g_\pm(z) = (h_s^{\pm \nu}(z))^{\frac{1}{2}}$. We then can argue by Lemma 4.1 that

$$\sup_{N} \sup_{\epsilon_k = \pm 1} \| \sum_{k = -N}^{N} \epsilon_k g_\pm(2^k t A) \| \leq M < \infty$$

independent of $t$. Hence by Lemma 3.3 the family $\{ \sum_{|k| \leq N} \alpha_k h_s^{\pm \nu}(2^k t A) \}$ is $R$-bounded with constant bounded independent of $t$. Now by (4.2) and (4.3) it follows that if $\sigma' > \nu$ then $\{ f(A) : \| f \|_{H^\infty(\Sigma_\sigma')} \leq 1 \}$ is Rademacher-bounded. Indeed for $f_k \in H_0^\infty(\Sigma_\sigma')$ and $x_k \in X$ for $1 \leq k \leq n$, we have

$$(\mathbb{E}) \left\| \sum_{k = 1}^{n} \epsilon_k f_k(A) x_k \right\|^2 \leq 4 \max_{\pm} \sup_{t > 0, N \in \mathbb{N}} \left( \mathbb{E} \right) \left\| \sum_{k = 1}^{n} \epsilon_k (\sum_{|j| \leq N} f_n(e^{\pm \nu} t^{-1} 2^{-j}) h_s^{\pm \nu}(2^j t A) x_k) \right\|^2.$$ 

It follows that $\omega_{RH}(A) = \omega_H(A)$. Now clearly $\omega_U(A) = \omega_R(A) \leq \omega_{RH}(A)$ and so (1) follows from Proposition 5.1.

(2) is very similar and we omit it.

(3) Here we use Lemma 3.4. Suppose $A$ admits an $H^\infty(\Sigma_\sigma)$–calculus and suppose $\sigma' > \nu > \sigma$. We show that the sequence $\{ 2^k t R(2^k t e^{\pm \nu}) : k \in \mathbb{Z} \}$ is Rademacher-bounded with constant independent of $t$. To do this we note that if $N_1 > N_2$

$$2^{N_1} t R(2^{N_1} t e^{\pm \nu}, A) - 2^{N_2} t R(2^{N_2} t e^{\pm \nu}, A) = - \sum_{j = N_2 + 1}^{N_1} t^{2j - 1} AR(2^j t e^{\pm \nu}, A) R(2^{j-1} t e^{\pm \nu}, A).$$
Let $k(z) = z(e^{i\nu} - z)^{-1}(e^{i\nu} - 2z)^{-1}$. Let $u(z) = (k(z))^{\frac{1}{2}} \in H^\infty(\Sigma_\sigma)$. We observe that
\[
\sup_{N_1 > N_2} \sup_{\epsilon_j = \pm 1} \| \sum_{j=N_2+1}^{N_1} \epsilon_j u(2^{-j}t^{-1}A) \| \leq M < \infty
\]

independent of $t$ by Lemma 4.1. Applying Lemma 3.3 yields that
\[
\{ \sum_{j=N_2+1}^{N_1} k(2^{-j}t^{-1}A) : N_1 > N_2 \}
\]
is Rademacher-bounded with constant independent of $t$. But this implies that
\[
\{ 2^{N_1}tR(2^{N_1}te^{i\nu}, A) - 2^{N_2}tR(2^{N_2}te^{i\nu}, A) : N_1 > N_2 \}
\]
is also Rademacher-bounded with constant independent of $t$ and hence (taking limits) so is $\{ 2^n tR(2^n te^{i\nu}, A) : n \in \mathbb{Z} \}$. A similar argument for $-\nu$ and an application of Lemma 3.4 shows that $\omega_R(A) \leq \nu$. Hence $\omega_R(A) \leq \omega_H(A)$. The proof is finished as in (1).

As a Corollary to the proof of Theorem 5.3 we obtain some additional information on the operator-valued calculus considered in Theorem 4.4:

Corollary 5.4. Assume that $X$ has property $(\alpha)$ and let $F \subset L(X)$ be an $R$-bounded set. If $A$ is $H^\infty$-sectorial then for any $\sigma > \omega_H(A)$ the set $\{ F(A) : F \in H^\infty(\Sigma_\sigma, A), F(\zeta) \in F \forall \zeta \in \Sigma_\sigma \}$ is $R$-bounded.

Proof. Adapt the proof of Theorem 5.3 (1) using the fact that the set
\[
\{ \sum_{|k| \leq N} T_k h^{\frac{1}{2}}(2^k A) : T_k \in \mathcal{F} \cap \mathcal{A} \}
\]
is $R$-bounded, again by Lemma 3.3. \qed

6. The joint $H^\infty$-calculus and sums of closed operators

First we consider the joint functional calculus.

Theorem 6.1. Suppose $A$ and $B$ are commuting $H^\infty$-sectorial operators such that $B$ is $WRH^\infty$-sectorial. Then for any $\sigma > \omega_H(A)$ and $\sigma' > \omega_{WRH}(B)$ the pair $(A, B)$ has a joint $H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$-calculus.

Proof. We need only observe that if $f \in H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$ then the family $\{ f(z, B) : z \in H^\infty(\Sigma_\sigma) \}$ is WR-bounded and then apply Theorem 4.4. \qed

We can now apply Theorem 5.3 to obtain a result of Lancien, Lancien and Le Merdy [28] (see also [1]). Note that their argument depends on the quite technical discretization developed in [20].

Corollary 6.2. (Lancien, Lancien and Le Merdy [28]) If $X$ has property $(A)$ then if $A$ and $B$ are commuting $H^\infty$-sectorial operators, for any $\sigma > \omega_H(A)$ and $\sigma' > \omega_H(B)$ the pair $(A, B)$ has a joint $H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$-calculus.
If $A$ and $B$ are commuting sectorial operators on a Banach space $X$ with $\omega(A) + \omega(B) < \pi$ then one can show that the closure $A + B$ of $A + B$ with $\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ is a sectorial operator with $\omega(A + B) \leq \max(\omega(A), \omega(B))$ (see [13]). However, in many applications one needs to show that $A + B$ is already closed on $\mathcal{D}(A) \cap \mathcal{D}(B)$. We now give a criterion for this.

**Theorem 6.3.** Suppose $A$ and $B$ are commuting sectorial operators such that $A$ is $H^\infty$-sectorial and $B$ is $R$-sectorial and $\omega_H(A) + \omega_R(B) < \pi$. Then $A + B$ is closed on the domain $\mathcal{D}(A) \cap \mathcal{D}(B)$, there is a constant $C$ such that

$$(6.1) \quad \|Ax\| + \|Bx\| \leq C\|Ax + Bx\| \quad x \in \mathcal{D}(A) \cap \mathcal{D}(B)$$

and $(A + B)$ is invertible if either $A$ or $B$ is invertible. Furthermore if $X$ has property $(\alpha)$ then $A + B$ is again $R$-sectorial with $\omega_R(A + B) \leq \max(\omega(A), \omega(B))$.

**Remarks.** (1) Let us compare this theorem with the well-known Dore-Venni Theorem. It is shown in [18] that $(6.1)$ holds if $X$ is a UMD-space and $A, B$ both have bounded imaginary powers (BIP), with

$$\|A^{is}\| \leq Ce^{\theta_A|s|}, \quad \|B^{is}\| \leq Ce^{\theta_B|s|}$$

where $\theta_A + \theta_B < \pi$. In a UMD-space, R-sectoriality is weaker than (BIP) (see [11]) and $H^\infty$-sectoriality is stronger than (BIP). But in many applications $A$ is an operator known to have an $H^\infty$-calculus, e.g. $A = -\Delta$ or $A = d/dt$ on $L_p(X)$ where $X$ is UMD and $1 < p < \infty$. Thus the weaker assumption on $B$ does lead to more general results, see e.g. Theorem 6.3 below.

(2) Some special cases of Theorem 5.3 where shown in [11] Theorem 5.2, [12] and more recently in [11] (e.g. if $A$ is the generator of a strongly continuous group in a UMD-space).

(3) An extension to non-commuting sums will be given in a forthcoming paper [34].

**Proof.** Choose $\sigma, \sigma'$ with $\omega_H(A) < \sigma, \omega_R(B) < \sigma'$ and $\sigma + \sigma' < \pi$. The function $f(w, z) = w(w + z)^{-1}$ is in $H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$ and the set $f(w, B) = -wR(-w, B)$ for $w \in \Sigma_\sigma$ is an R-bounded family. Applying Theorem 1.5 we have $f \in \mathcal{H}(A, B)$. We can see this implies $(6.1)$ either by applying Proposition 2.7 in [30] or by the following simple direct argument based on our construction of the functional calculus. Defining $\varphi_n$ as in $(2.1)$ we note that $R^2 \varphi_n(\varphi_n)^2$ and $B^2 \varphi_n(B)^2$ are bounded operators since $\varphi_n(z)^2 \in H^\infty_0(\Sigma_\tau)$ for any $\tau < \pi$. Now if $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ we have

$$f(A, B)(A + B) \varphi_n(A)^2 \varphi_n(B)^2 x = \varphi_n(A)^2 \varphi_n(B)^2 Ax.$$

Thus

$$\|\varphi_n(A)^2 \varphi_n(B)^2 Ax\| \leq C\|\varphi_n(A)^2 \varphi_n(B)^2 (A + B) x\|$$

where $C = \|f(A, B)\|$. Letting $n \to \infty$ yields the result.
Now assume $X$ has property $(\alpha)$. For $\max(\sigma, \sigma') < \rho < \pi$ and $|\arg \mu| \geq \rho$ consider the functions $f_{\mu}(w, z) = \mu(\mu - w - z)^{-1} \in H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$. Note that

$$f_{\mu}(w, B) = \frac{\mu}{\mu - w}((\mu - w)R(\mu - w, B)).$$

Since $\mu(\mu - w)^{-1}$ is bounded uniformly for $|\arg \mu| \geq \rho$ and $w \in \Sigma_\sigma$ and also $|\arg(\mu - w)| \geq \sigma'$ this collection of operators is R-bounded. Now, Corollary 5.4 yields that the set $f_{\mu}(A, B) = \mu R(\mu, A + B)$ is R-bounded for $|\arg \mu| \geq \rho$.

Applying Theorem 5.3 gives:

**Corollary 6.4.** Suppose $X$ has property $(\Delta)$ (e.g. if $X$ has analytic (UMD)). Suppose $A$ and $B$ are commuting $H^\infty$–sectorial operators such that $\omega_H(A) + \omega_H(B) < \pi$. Then $A + B$ is closed on the domain $D(A) \cap D(B)$ and (6.1) holds.

**Example.** We now show by example that both Corollaries 5.2 and 5.4 are nearly optimal. To do this we let $(\epsilon_j)$ and $(\eta_k)$ be as before two sequence of mutually independent Rademachers on some probability space $(\Omega, \mathbb{P})$. We define $\operatorname{Rad}_2(X)$ to be the subspace of $L^2(\Omega; X)$ spanned by functions of the form $\epsilon_j \eta_k x$ for $j, k \in \mathbb{N}$ and $x \in X$. Let $A$ be defined by

$$A(\sum_{j, k} \epsilon_j \eta_k x_{jk}) = \sum_{j, k} (2j + 1)! \epsilon_j \eta_k x_{jk}$$

with the natural domain and let

$$B(\sum_{j, k} \epsilon_j \eta_k x_{jk}) = \sum_{j, k} (2k)! \epsilon_j \eta_k x_{jk}$$

with its natural domain.

Both $A$ and $B$ are $H^\infty$–sectorial with $\omega_H(A) = \omega_H(B) = 0$. Clearly $f \in \mathcal{H}(A, B)$ if and only if

$$\sum_{j, k} \epsilon_j \eta_k x_{jk} \rightarrow \sum_{j, k} f((2j + 1)!, (2k)!) \epsilon_j \eta_k x_{jk}$$

is a bounded operator.

If $(A, B)$ has a joint $H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$–calculus then since the map $f \rightarrow (f((2j + 1)!, (2k)!)_{j, k}$ maps $H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$ onto $\ell_\infty(\mathbb{N}^2)$ we must have that $X$ has property $(\alpha)$ (and hence so does $\operatorname{Rad}_2(X)$.)

If $f(w, z) = w(w + z)^{-1}$ defines a bounded operator then by a limiting argument the map

$$\sum_{j, k} \epsilon_j \eta_k x_{jk} \rightarrow \sum_{k \leq j} \epsilon_j \eta_k x_{jk}$$

is also bounded, i.e. $X$ (and $\operatorname{Rad}_2(X)$) has property $(\Delta)$.
Let us note that we can recapture the main result of [41] on maximal regularity. Suppose \(-A\) is the generator of an analytic semigroup i.e. \(\omega(A) < \frac{\pi}{2}\). Then \(A\) has maximal \(L_p\)–regularity for \(1 < p < \infty\) if the Cauchy problem
\[
y'(t) + Ay(t) = f(t), \quad t > 0, \quad y(0) = 0
\]
has for every \(f \in L_p(\mathbb{R}_+, X)\) a solution \(y: \mathbb{R}_+ \to X\) which satisfies the estimate
\[
\|y'\|_{L_p(\mathbb{R}_+, X)} + \|Ay\|_{L_p(\mathbb{R}_+, X)} \leq C\|f\|_{L_p(\mathbb{R}_+, X)}.
\]
If we denote by \(\tilde{B}\) the derivative \(d/dt\) on \(\tilde{X} = L_p(\mathbb{R}_+, X)\) and by \(\tilde{A}\) the extended operator \(\tilde{A}f(t) = Af(t)\) then (6.3) is equivalent to:
\[
\|\tilde{A}y\|_{\tilde{X}} + \|\tilde{B}y\|_{\tilde{X}} \leq C\|\tilde{A} + \tilde{B}\)y\|_{\tilde{X}}.
\]
Thus we can apply Theorem 6.3:

**Theorem 6.5.** Suppose \(X\) is a Banach space with (UMD) and suppose \(A\) is an \(R\)-sectorial operator on \(X\) with \(\omega_R(A) < \frac{\pi}{2}\). Then \(A\) has maximal \(L_p\)–regularity for \(1 < p < \infty\).

**Proof.** Since \(X\) has (UMD), we have that \(\tilde{B}\) is \(H^\infty\)–sectorial and \(\omega_H(\tilde{B}) = \frac{\pi}{2}\) (see e.g. [24]). It is easy to see that \(\omega_R(\tilde{A}) = \omega_R(A)\) so the result is a consequence of Theorem 6.3. \(\square\)

**Remarks.** (1) If \(X\) has (UMD) and \(A\) is \(H^\infty\)–sectorial then \(\omega_H(A) < \frac{\pi}{2}\) implies maximal \(L_p\)–regularity by Corollary 6.4.

(2) It is shown in [23] that any non-Hilbertian Banach space with an unconditional basis admits a sectorial operator \(A\) with \(\omega(A) < \frac{\pi}{2}\) but failing maximal \(L_p\)–regularity.

(3) It is shown in [41] that the condition \(\omega_R(A) < \frac{\pi}{2}\) actually characterizes \(L_p\)–maximal regularity. The operator approach to maximal regularity has a long history, see e.g. [15], [16], [21] and [42].

7. \(L_1\)-spaces and \(C(K)\)-spaces

We recall that a GT-space is a Banach space \(X\) so that the Grothendieck theorem is valid, i.e. if \(T: X \to \ell_2\) is bounded then \(T\) is absolutely summing and for some \(C\) independent of \(T\), \(\pi_1(T) \leq C\|T\|\); see [38] for a full discussion. Examples of such spaces are \(L_1\)–spaces, their quotients by reflexive spaces and \(L_1/H_1\) ( [1], [2] and [38]).

**Proposition 7.1.** Suppose \(X\) is a GT-space of cotype 2 and that \(A\) is a \(H^\infty\)–sectorial operator on \(X\). Then if \(\omega_H(A) < \nu\) and \(0 < s < 1\) there is a constant \(C\) so that if \(x \in X\),
\[
C^{-1}\|x\| \leq \int_{\Gamma_\nu} \|A^* R(\zeta, A)x\| \frac{|d\zeta|}{|\zeta|^s} \leq C\|x\| \quad x \in X.
\]
Proof. This is just a version of a classical result of Lindenstrauss and Pełczyński [34] on uniqueness of unconditional bases in $\ell_1$. Let $g(z) = (h^\nu_s(z))^\frac{1}{2}$. Then for some constant $C_0$ independent of $x, t$ we have if $|\alpha_k| \leq 1$ for $|k| \leq N$,

$$\| \sum_{|k| \leq N} \alpha_k g(2^k t A) x \| \leq C_0 \|x\| \quad x \in X, t > 0$$

by Lemma 4.1. Now since $X$ has cotype 2, there exists a constant $C_1$ independent of $x, t$, so that

$$\left( \sum_{k \in \mathbb{Z}} \| g(2^k t A) x \|^2 \right)^{\frac{1}{2}} \leq C_1 \|x\| \quad x \in X, t > 0. \tag{7.2}$$

Now suppose $x \in X$ and $t > 0$ and choose by the Hahn-Banach theorem $x_k^* \in X^*$ with $\|x_k^*\| = 1$ and $\langle h^\nu_s(2^k t A)x, x_k^* \rangle = \|h^\nu_s(2^k t A)x\|$. Consider the map $S_x : X \to \ell_2(\mathbb{Z})$ defined by $S_x y = (\langle (2^k t A)y, x_k^* \rangle)_{k \in \mathbb{Z}}$. By (7.2) $S$ is bounded and $\|S_x\| \leq C_1$. Hence for some constant $C_2$ we have $\pi_1(S_x) \leq C_2$.

Now

$$\sum_{|k| \leq N} \| h^\nu_s(2^k t A)x \| = \sum_{|k| \leq N} \langle h^\nu_s(2^k t A)x, x_k^* \rangle$$

$$\leq \sum_{|k| \leq N} \| S_x g(2^k t A)x \|$$

$$\leq C_2 \sup_{|\alpha_k| \leq 1} \| \sum_{|k| \leq N} \alpha_k g(2^k t A)x \|$$

$$\leq C_2 C_0 \|x\|.$$

If we integrate for $1 \leq t \leq 2$ we obtain the right-half of (7.1) (cf. (4.2) and (4.3)). The left-half follows from the equation:

$$x = \lim_{n \to \infty} \frac{-1}{2\pi i} \int_{\Gamma_\nu} \varphi_n(\zeta) \zeta^{-s} A^s R(\zeta, A)x \, d\zeta. \quad \square$$

The same argument yields:

**Proposition 7.2.** Suppose $X^*$ is a GT-space of cotype 2 and that $A$ is a $H^\infty$--sectorial operator on $X$. Then if $\omega_H(A) < \nu$ and $0 < s < 1$ there is a constant $C$ so that if $x \in X$,

$$\frac{1}{C} \|x^*\| \leq \int_{\Gamma_\nu} \| (A^s R(\zeta, A))^* x^* \| \frac{d\zeta}{|\zeta|^s} \leq C \|x^*\| \quad x^* \in X^*.$$

**Remarks.** Let us point out that Proposition 7.1 implies that very few operators on $L_1$ can have an $H^\infty$--calculus. This statement can be made much more precise but since the techniques required are rather specialized we will
defer this to a later paper and instead note the following simple application, which effectively shows that no reasonable differential operator on $L_1$ can have an $H^\infty$—calculus.

**Proposition 7.3.** Suppose $X$ is a GT-space of cotype 2 and $A$ is an $H^\infty$—sectorial operator on $X$. If $Y$ is an infinite-dimensional closed reflexive subspace of $\mathcal{D}(A)$ (with the graph norm) then $A$ is bounded on $Y$ (and so $Y$ is closed in $X$).

**Proof.** We use the notation of Proposition [7.1]. In particular notice that (7.1) implies the existence of an isomorphic embedding $T : X \to L_1(\Gamma, |d\zeta|; X)$ defined by

$$Tx(\zeta) = |\zeta|^{-\frac{1}{2}}A^{\frac{1}{2}}R(\zeta, A)x.$$ 

Fix $0 \neq \lambda \in \Gamma$. Then $R(\lambda, A)$ maps $X$ isomorphically onto $\mathcal{D}(A)$ (with the graph norm). Let $Y_0 = R(\lambda, A)^{-1}Y$; then $Y_0$ is an infinite-dimensional reflexive subspace of $X$. Then (since $Y_0$ contains no copy of $\ell_1$) the set $\{\|Ty(\zeta)\| : \|y\| \leq 1, y \in Y_0\}$ is equi-integrable. We show that this implies that the (bounded) operator $A^{\frac{1}{2}}R(\lambda, A)$ satisfies a lower bound on $Y_0$. Indeed if not there exists a sequence $(y_n)$ in $Y_0$ so that $\|y_n\| = 1$ and $\|A^{\frac{1}{2}}R(\lambda, A)y_n\| \to 0$. But, by the resolvent equation, $\|A^{\frac{1}{2}}R(\zeta, A)y_n\| \to 0$ for all $\zeta \in \Gamma \setminus \{0\}$. Now by (7.1) and equi-integrability, we have $\|y_n\| \to 0$ which gives a contradiction. Now applying the same argument to $Y_1 = A^{\frac{1}{2}}R(\lambda, A)Y_0$ gives a lower bound on $AR(\lambda, A)^2$ on $Y_0$. Thus $R(\lambda, A)$ has a lower bound on $Y_0$. Since $Y = R(\lambda, A)Y_0$ this implies the result. 

Let us note, with respect to differential operators, that embeddings of Sobolev spaces into $L_1(\Omega)$ are Dunford-Pettis operators. In view of this comparative rarity of $H^\infty$—sectorial operators in this setting, it is not surprising that we may substantially improve the results of this paper for these special spaces. Our first deduction is that for these special spaces, Theorem [4.3] can be improved by removing the U-boundedness assumption.

**Theorem 7.4.** Suppose $X$ is a Banach space such that either $X$ or $X^*$ is a GT-space of cotype 2. Suppose $A, B$ are commuting sectorial operators such that $A$ admits an $H^\infty$—calculus. Suppose $f \in H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$ where $\sigma > \omega_H(A)$ and $\sigma' > \omega(B)$. Suppose that for each $z \in \Sigma_\sigma$, $f_z(w) = f(z, w) \in \mathcal{H}(B)$ and

$$\sup_{z \in \Sigma_\sigma} \|f(z, B)\| < \infty.$$ 

Then $f \in \mathcal{H}(A, B)$ (i.e. $f(A, B)$ is bounded).
Proof. Let us assume that $X$ is a GT-space of cotype 2, the other case is similar. By Proposition 4.2 if $f_n(w, z) = \varphi_n(w)\varphi_n(z)f(w, z)$ we can write
\[
f_n(A, B)x = -\frac{1}{2\pi i} \int_{\Gamma^\nu} \zeta^{-s} f_n(\zeta, B)A^s R(\zeta, A)x d\zeta
\]
and this leads immediately to $\sup_n \|f_n(A, B)\| < \infty$ which implies the boundedness of $f(A, B)$.

It is clear now if $A$ is $\mathcal{H}^\infty$–sectorial on $L_1$ with $\omega_H(A) < \pi/2$ then one can apply the above result to conclude that $A$ has $L_1$–maximal regularity. More generally we have the following result (suggested by a question of Gilles Lancien):

**Theorem 7.5.** Suppose either $X$ or $X^*$ is a GT-space of cotype 2 (e.g. if $X = L_1$, $C(K)$ or the disk algebra $A(D)$). If $A$ is an $\mathcal{H}^\infty$–sectorial operator on $X$ with $\omega_H(A) < \pi/2$ then $A$ has $L_p$–maximal regularity for $1 < p < \infty$.

Proof. Let us prove this for $X$ a GT-space of cotype 2 as the other case is dual. Suppose $\omega_H(A) < \nu < \pi/2$ and $0 < s < 1$. If $u > 0$ then for $x \in X$ by Proposition 4.2
\[
Ae^{-uA}x = -\frac{1}{2\pi i} \int_{\Gamma^\nu} \zeta^{1-s} e^{-u\zeta} A^s R(\zeta, A)x d\zeta.
\]
Now suppose $f \in L_p(\mathbb{R}, X)$. We will estimate the norm of $S_\delta$ where
\[
S_\delta f(v) = \int_\delta^\infty A e^{-uA} f(v - u) du.
\]
If we let $G(t) = A^s R(|t| e^{i\text{sgn} \ t}u, A)$ then we have an estimate
\[
\|Ae^{-uA}x\| \leq C_0 \int_{-\infty}^\infty |t|^{1-s} e^{-c|t|} \|G(t)x\| dt
\]
where $c = \cos \nu > 0$ and so
\[
\|S_\delta f(v)\| \leq C_0 \int_{-\infty}^\infty \int_{-\infty}^{v-\delta} |t|^{1-s} e^{-c|t|} \|G(t)f(u)\| du dt.
\]
Now if $g \in L_q(\mathbb{R})$ with $g \geq 0$ we have
\[
\int_{u+\delta}^{\infty} e^{c(u-v)|t|} |g(v)| dv \leq \frac{1}{c|t|} \int_{u}^{\infty} (v-u)e^{c(u-v)|t|} \frac{1}{v-u} \int_{u}^{v} g(w) dw dv.
\]
Hence the left-hand side is estimated by $C_1 |t|^{-1}(Mg)(t)$ where $M$ is the Hardy-Littlewood maximal function.
Hence
\[ \int_{-\infty}^{\infty} g(v) \| S_\delta f(v) \| dv \leq C_0 C_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |t|^{-s} \mathcal{M} g(u) \| G(t) f(u) \| du \, dt. \]

However Proposition 7.1 implies that we have an estimate
\[ \int_{-\infty}^{\infty} \| G(t) f(u) \| dt \leq C_2 \| f(u) \|. \]

Substituting in we have
\[ \int_{-\infty}^{\infty} g(v) \| S_\delta f(v) \| dv \leq C_0 C_1 C_2 \int_{-\infty}^{\infty} \mathcal{M} g(u) \| f(u) \| du \]

and since \( \mathcal{M} \) is bounded on \( L_q \) this establishes a uniform bound on the operators \( S_\delta \). Letting \( \delta \to 0 \) yields the result. \( \square \)

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