THE FROBENIUS CHARACTER OF THE ORLIK-TERAO ALGEBRA OF TYPE A

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ABSTRACT. We provide a new virtual description of the symmetric group action on the cohomology of ordered configuration space on SU_2 up to translations. We use this formula to prove the Moseley-Proudfoot-Young conjecture. As a consequence we obtain the graded Frobenius character of the Orlik-Terao algebra of type An.

1. Introduction

The Orlik-Terao algebra OT_n is the subalgebra of rational functions on C^n generated by 1/(x_i - x_j) for all i ≠ j. It has been intensively studied in [Ter02, PS06, ST09, Ber10, Sch11, DGT14, Le14, Liu16, EPW16, MPY17, MMPR21]. Only recently, has an attempt to describe the symmetric group action on OT_n been made by Moseley, Proudfoot, and Young [MPY17]. They provided a recursive algorithm for computing the graded Frobenius character of the OT_n. That algorithm is based on a surprising relation between the Orlik-Terao algebra and the intersection cohomology ring M_n of a certain hypertoric variety constructed from the root system of type A_n [BP09, MP15].

Computation of M_n using the aforementioned algorithm has suggested the following conjecture. Let D_n be the cohomology algebra of the configuration spaces of n ordered points in SU_2 up to translations.

Conjecture 1.1 ([MPY17, Conjecture 2.10]). For each n, there exists an isomorphism of graded S_n-representations D_n ≃ M_n.

It has been verified for n ≤ 10 in [MPY17] and for n ≤ 22 in [MMPR21].

The algebra D_n has an independent interest, indeed each graded piece is the Whitehouse lift of Eulerian S_n-representation up to a sign (D_n^k = sgn_n ⊗ F_{n-1-k}^{n-1-k}) see [GS87, Han90, Whi97, ER19]). The Eulerian representations appear also in the study of the free Lie algebra [Reu93]. These representations are used to decompose the Hochshild Cohomology and Cyclic Cohomology in simpler pieces [Whi97]. Moreover, D_n

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appears in the Hochschild-Pirashvili homology of a wedge of circles and in the weight-zero compactly supported cohomology of $\mathcal{M}_{2,n}$ \cite{GH22}. Some tentatives to prove the Moseley-Proudfoot-Young conjecture failed for two reason: firstly the only known formula describing $D_n$ is

$$C_n = (V_n \oplus qV_{n-1,1}) \otimes D_n,$$

where $V_\lambda$ is the Schur representation and $C_n$ is the cohomology of the configuration space of $\mathbb{R}^3$. Although there is an explicit formula for $C_n$ involving plethysm (Proposition 2.6), inverting the Kronecker (tensor) product is very difficult. The second issue is that the recursive formula of \cite{MPY17} for $M_n$ is complicate and involves plethysm, Kronecker product and the character of $C_n$. We overcome the first problem providing a new virtual formula for the graded Frobenius character of $D_n$ (Theorem 3.1) by using the Cohen–Taylor-Totaro-Kriz spectral sequence \cite{CT78, Tot96, Kri94}. Instead of working on the recursive formula \cite[Theorem 3.2]{MPY17}, we use the isomorphism of graded $S_n$-representations

$$OT_n \simeq M_n \otimes R_n$$

provided in \cite[Proposition 7]{PS06}, where $R_n$ is the symmetric algebra on $V_{n-1,1}$. Then we virtually invert $R_n$ (Lemma 4.2) with respect the Kronecker product and we prove the conjecture by induction on $n$ (Theorem 4.7) relying on a certain subspace $T_n$ of $OT_n$ (Lemma 4.6). Finally, we obtain an explicit formula for the character of $OT_n$ (Corollary 4.8) and the generating functions for the characters of $D_n$ and of $OT_n$ (Corollary 4.11).

2. DEFINITIONS

We introduce the main objects of study and some notations. The Orlik-Terao algebra was introduced in \cite{Ter02}, in type $A_{n-1}$ the definition specializes as follow.

**Definition 2.1.** The Orlik-Terao algebra of type $A_{n-1}$ is the ring $OT_n = \mathbb{Q}[e_{ij}]/I_n^{OT}$ generated by $e_{i,j}$ for distinct $i,j \in [n]$ and relations $I_n^{OT}$ given by:

- $e_{ij} + e_{ji} = 0$ for all $i,j$ distinct,
- $e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} = 0$ for all $i,j,k$ distinct.

**Definition 2.2.** Let $C_n^\ast := H^\ast(\text{Conf}_n(\mathbb{R}^3); \mathbb{Q})$ be the cohomology algebra of the ordered configuration space of $n$ points in $\mathbb{R}^3$.

The ring $C_n$ can be presented as quotient of $OT_n$ by the equations

- $e_{ij}^2 = 0$ for all $i,j$ distinct.

The above presentation was studied for the first time in \cite{OT94}.
Definition 2.3. Let $D_n^\bullet := H^{\bullet}(\text{Conf}_n(SU_2)/SU_2; \mathbb{Q})$ be the cohomology algebra of the ordered configuration space of $n$ points in $SU_2$ up to translations.

The algebra $D_n$ can be presented as $\mathbb{Q}[e_{ij}]/I_n^D$ generated by $e_{ij}$ for distinct $i, j \in [n]$ and relations $I_n^D$ given by:

- $e_{ij} + e_{ji} = 0$ for all $i, j$ distinct,
- $(e_{ij} + e_{jk} + e_{ki})^2 = 0$ for all $i, j, k$ distinct,
- $\sum_{j \neq i} e_{ij} = 0$ for all $i \in [n]$.

This presentation is due to Matherne, Miyata, Proudfoot, and Ramos [MMPR21, Theorem A4].

Definition 2.4. Let $M_n = OT_n/I_n^M$ be the quotient of the Orlik-Terao algebra by the relations:

- $\sum_{j \neq i} e_{ij} = 0$ for all $i \in [n]$.

The algebra $M_n$ was originally defined in a geometric way.

Theorem 2.5 ([MMPR21, Theorem A.6.]). The algebra $M_n^\bullet$ is isomorphic to $IH^{\bullet}(X_n; \mathbb{Q})$, the intersection cohomology of a hypertoric variety $X_n$ associated with the root system of the Lie algebra $\mathfrak{sl}_n$.

We use the standard notation for symmetric polynomial: let $h_\lambda, e_\lambda, s_\lambda, p_\lambda$ for $\lambda \vdash n$ a partition of $n$ be the complete homogeneous, elementary, Schur, and power sum symmetric polynomials, respectively. Given a graded $S_n$-representation $V$ we consider the graded Frobenius character $ch_V(q)$, frequently will omit the dependence on $q$. As an example if $V_\lambda$ is the irreducible Schur representation in degree zero, then $ch_{V_\lambda} = s_\lambda$.

We denote the plethysm of symmetric functions $f, g$ by $f[g]$. For $W$ a representation of $S_j$ we denote $\overline{W} = W^{\otimes m}$ the representation of the wreath product $S_j \wr S_m = (S_j)^{\times m} \rtimes S_m$, where $S_j^{\times m}$ acts coordinatewise and $S_m$ by permuting the coordinates. Let $V$ be a representation of $S_m$ and $V \otimes \overline{W}$ be the representation of $S_j \wr S_m$ where $S_j^{\times m}$ acts only on $\overline{W}$ and $S_m$ on both factors. The group $S_j \wr S_m$ is naturally a subgroup of $S_{jm}$, the main property of the plethysm is

$$ch_{\text{Ind}_{S_j \wr S_m}^{S_{jm}} V \otimes \overline{W}} = ch_V[ch_{\overline{W}}].$$

Let $\text{Lie}_n$ be the submodule of the multilinear part of the free Lie algebra on $n$ generators. As $S_n$ representation $\text{Lie}_n = \text{Ind}_{S_n}^{\mathbb{Z}_n} \zeta_n$ where $Z_n$ is the cyclic group generated by an $n$-cycle in $S_n$ and $\zeta_n$ is a primitive root of the unity. We denote by $l_j$ its character, cf. Remark 4.10 for an explicit description. The following result is due to Sundaram and Welker [SW97, Theorem 4.4(iii)], see also [HR15, Theorem 2.7].
Proposition 2.6. The graded character of $C_n$ is
\[ \text{ch}_{C_n} = \sum_{\lambda \vdash n} q^{n-\ell(\lambda)} \prod_{j \geq 1} h_{m_j} [l_j], \]
where $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ in the exponential notation and $\ell(\lambda) = \sum_j m_j$ is the number of blocks.

Finally, we define $R_n = S^* V_{(n-1,1)}$ and $\Lambda_n = \Lambda^* V_{(n-1,1)}$ be the symmetric (resp. alternating) algebra on the standard representation of $S_n$. We regard $V_{(n-1,1)}$ in degree one, hence $\text{ch}_{\Lambda_n} = \sum_{i=0}^{n-1} q^i s_{n-i,1}$. See Remark 4.10 for an expression of $\text{ch}_{R_n}$ in term of Schur polynomials.

3. Graded Frobenius character of $D_n$

In this section we provide a virtual formula for $\text{ch}_{D_n}$ that will be used in the proof of Theorem 4.7. We denote by $\text{ch}_V(-q)$ for $V$ a graded $S_n$-representation. Let $P_n$ be the $S_n$-representation by permutations, i.e. $P_n = V_{(n-1,1)} \oplus V(1^n)$. For a partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n}) \vdash n$ let $S_\lambda$ be the subgroup of $S_n$ stabilizing $\lambda$, i.e. $S_\lambda = \prod_{j \geq 1} S_j \wr S_{m_j}$.

Theorem 3.1. The graded character of $D_n$ is:
\[ \text{ch}_{D_n}(q) = \sum_{\lambda \vdash n} q^{n-\ell(\lambda)} \prod_{j \geq 1} \text{ch}_{P_{m_j}}[l_j]. \]

Proof. We consider the Cohen–Taylor-Totaro-Kriz spectral sequence $E_\bullet(SU_2)$ [CT78, Tot96, Kri94] that converge to $H^\bullet(\text{Conf}_n(SU_2))$. In our case since $SU_2$ is 3-dimensional and has nonzero cohomology only in degree 0 and 3, we have that $E_2^{p,q} = 0$ if $3 \nmid p$ and $2 \nmid q$. The $S_n$-representation on the second page is described in [AAB14, Theorem 3.15]:
\[ E_2^{3p,2q} = \bigoplus_{\ell(\lambda)=n-q} \text{Ind}_{S_\lambda}^{S_n} \left( \bigotimes_j (\text{Ind}_{S_j}^{S_n} \zeta_j)^{2m_j} \otimes \text{Res}_{W_\lambda}^{S(\lambda)} \Lambda^p P_{\ell(\lambda)} \right). \]

Since $\text{Res}_{W_\lambda}^{S(\lambda)} P_{\ell(\lambda)} = \bigoplus_{j \geq 1} P_{m_j}$ we have
\[ \text{ch}_{E_2}(s,t) = \sum_{\lambda \vdash n} t^{2(n-\ell(\lambda))} \prod_{j \geq 1} \text{ch}_{P_{m_j}}(s^3)[l_j]. \]

Topologically $SU_2 \cong S^3$ is a formal orientable manifold, the only nonzero differential of $E_\bullet(SU_2)$ is $d_3$ as observed in [Pet20, §1.10] and in [Get99, Section 2]. The differential $d_3$ is compatible with the $S_n$-action by the functoriality property of the spectral sequence. It follows
\[ \text{ch}_{E_2}(-q^2, q^3) = \text{ch}_{E_\infty}(-q^2, q^3), \]
because this is the right evaluation that simplifies the coimage of $d_3$ with its image.
Consider the map \( f: (\mathbb{R}^3)^{n-1} \to (SU_2)^n \) defined by \((x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, e)\) where \(e\) is the identity of \(SU_2\) and \(\mathbb{R}^3\) is identified with \(SU_2 \setminus \{e\}\). It has a retraction defined by
\[
(g_1, g_2, \ldots, g_n) \mapsto (g_1^{-1} g_1, g_2^{-1} g_2, \ldots, g_n^{-1} g_n-1).
\]
Both maps restrict to the subspaces \(Conf_{\mathbb{R}^3} \to Conf_n(SU_2)\) and this implies that \(E_\bullet(\mathbb{R}^3)\) is a direct addendum of \(E_\bullet(SU_2)\). Notice that \(Conf_{\mathbb{R}^3} \times SU_2 \simeq Conf_n(SU_2)\) via the map \(((x_1, \ldots, x_{n-1}), g) \mapsto g \cdot f(x)\), hence \(E_\infty(SU_2) = E_\infty(\mathbb{R}^3) \otimes H^\bullet(SU_2)\) as graded vector spaces. Since \(E_2(\mathbb{R}^n)\) is supported on the column \(p = 0\), so is \(E_\infty(\mathbb{R}^n)\). Therefore \(E_\infty(SU_2)\) is supported only on the column \(p = 0\) and \(p = 3\), indeed the even cohomology of \(Conf_n(SU_2)\) is supported in degrees \((0, 2q)\) and the odd one in degrees \((3, 2q)\). So
\[
\text{ch}_{E_\infty}(s, t) = \text{ch}_{H^{\text{even}}(Conf_n(SU_2))}(t) + s^3 t^{-3} \text{ch}_{H^{\text{odd}}(Conf_n(SU_2))}(t).
\]
Let \(\pi: Conf_n(SU_2) \to Conf_n(SU_2)/SU_2\) be the natural projection, it is a \(S_n\)-equivariant fiber bundle. The Leray-Hirsch theorem for rational cohomology asserts that \(H(Conf_n(SU_2))\) is a free \(H(Conf_n(SU_2)/SU_2)\)-module with basis given by \(1, \omega\) for any nonzero \(\omega \in H^3(Conf_n(SU_2))\). The module structure is given by \(\pi^*\) so it is \(S_n\)-equivariant. We observe that \(S_n\) acts trivially on \(H^0(Conf_n(SU_2))\) and on \(H^3(Conf_n(SU_2))\), because the latter is a 1-dimensional quotient of \(E_2^{3,0}(SU_2) \cong P_n\). Therefore
\[
\text{ch}_{H^{\text{even}}(Conf_n(SU_2))}(t) = \text{ch}_H(Conf_n(SU_2)/SU_2)(t) = \text{ch}_{D_n}(t^2),
\]
\[
\text{ch}_{H^{\text{odd}}(Conf_n(SU_2))}(t) = t^3 \text{ch}_H(Conf_n(SU_2)/SU_2)(t) = t^3 \text{ch}_{D_n}(t^2).
\]
We have \(\text{ch}_{E_\infty}(s, t) = (1 + s^3) \text{ch}_{D_n}(t^2)\) and together with eq. (3) and (4) they imply
\[
(1 - q^6) \text{ch}_{D_n}(q^6) = \sum_{\lambda \vdash n} q^{6(n - \ell(\lambda))} \prod_{j \geq 1} \text{ch}_{\Lambda_{m_j}}(q^6)[l_j].
\]
That is our claim. \(\square\)

**Remark 3.2.** The formula (1) has \((1-q)\) in the denominator and seems to be an infinite series. However it can be written as a polynomial in \(q\) of degree \(n - 1\):
\[
\text{ch}_{D_n}(q) = \sum_{\lambda \vdash n} q^{n - \ell(\lambda)} (1 - q)^{c_\lambda - 1} \prod_{j \geq 1} \text{ch}_{\Lambda_{m_j}}[l_j],
\]
where \(c_\lambda = |\{ j \mid m_j \neq 0\}|\). Furthermore, since the left hand side is a polynomial in \(q\) of degree \(n - 2\), the coefficient of \(q^{n-1}\) in the right hand side must be zero.
4. Proof of the MPY Conjecture

Now we prove the conjecture and provide a new formula for the character of the Orlik-Terao algebra. The Kronecker product of two symmetric function $f \ast g$ is the linear extension of the tensor product for representation, i.e. $\text{ch}_{V \otimes W} = \text{ch}_V \ast \text{ch}_W$.

**Theorem 4.1** ([PS06, Proposition 7]). For each $n$ the equation

$$\text{ch}_{OT_n} = \text{ch}_{M_n} \ast \text{ch}_{R_n}$$

holds.

**Lemma 4.2.** Let $V$ be any representation of the symmetric group $S_n$. We have:

$$\text{ch}_{S \ast V} \ast \text{ch}'_{\Lambda \ast V} = s_n.$$

**Proof.** The Koszul complex for the ring $S \ast V$ is a free resolution of $Q = S \ast V/(V)$. The bigraded character of the Koszul complex is $\text{ch}_{S \ast V}(s) \ast \text{ch}_{\Lambda \ast V}(t)$, hence by exactness we have $\text{ch}_{S \ast V}(q) \ast \text{ch}_{\Lambda \ast V}(-q) = s_n$. □

It follows that $\text{ch}_{R_n}$ is invertible with respect to the Kronecker product, whose inverse is $\text{ch}'_{\Lambda_n}$.

**Lemma 4.3.** Let $g$ be a symmetric function of degree $j$ and $m$ a positive integer. We have

$$\text{ch}'_{\Lambda \ast P_m}[g] = h_m[(1 - q)g].$$

**Proof.** Using the identity $h_{n-k}e_k = s_{n-k,1^k} + s_{n-k+1,1^{k-1}}$ we obtain

$$\text{ch}'_{\Lambda \ast P_n} = (1 - q) \sum_{k=0}^{n-1} (-q)^k s_{n-k,1^k} = \sum_{k=0}^{n} (-q)^k h_{n-k}e_k.$$

Recall the subtraction formula (see for example in [LR11, §3.3])

$$h_m[f - g] = \sum_{i=0}^{m} (-1)^k h_{m-k} e_k[f] e_k[g],$$

we obtain

$$h_m[(1 - q)g] = \sum_{k=0}^{m} (-1)^k h_{m-k} [g] e_k[gg]$$

$$= \sum_{k=0}^{m} (-q)^k (h_{m-k} e_k)[g]$$

$$= \text{ch}'_{\Lambda \ast P_n}[g].$$ □

Using the Lemma above we can rewrite the character of $D_n$ as follow.

**Corollary 4.4.** The graded character of $D_n$ is

$$\text{ch}_{D_n}(q) = \frac{1}{1 - q} \sum_{\lambda \vdash n} \prod_{j \geq 1} h_{m_j} [q^{j-1}(1 - q)l_j].$$

(5)
Proof. It follows from Theorem 3.1 and Lemma 4.3.

Lemma 4.5. Let \( \lambda = (1^{m_1}, 2^{m_2}, \ldots) \) be a partition of \( n \) and \( f_{m_j} \) be symmetric functions of degree \( j \) and \( m_j \) respectively. We have:

\[
\text{ch}_{\Lambda^{\bullet} P_n} \ast \prod_{j \geq 1} f_{m_j}[g_j] = \prod_{j \geq 1} f_{m_j}[g_j \ast \text{ch}_{\Lambda^{\bullet} P_j}].
\]

Proof. Firstly observe that

\[
\text{Res}_{S_n}^S \prod_{j \geq 1} S_{jm_j} P_n = \bigoplus_{j \geq 1} P_{jm_j},
\]

and so

\[
\text{Res}_{S_n}^S \prod_{j \geq 1} S_{jm_j} \Lambda^{\bullet} P_n = \bigotimes_{j \geq 1} \Lambda^{\bullet} P_{jm_j}.
\]

Using the projection formula (sometimes called Frobenius reciprocity) we obtain:

\[
\text{ch}_{\Lambda^{\bullet} P_n} \ast \prod_{j \geq 1} f_{m_j}[g_j] = \prod_{j \geq 1} \text{ch}_{\Lambda^{\bullet} P_{jm_j}} \ast f_{m_j}[g_j].
\]

Thus it is enough to show

\[
\text{ch}_{\Lambda^{\bullet} P_{jm_j}} \ast f[g] = f[g \ast \text{ch}_{\Lambda^{\bullet} P_j}].
\]

This last equality is linear and multiplicative in the entry \( f \): the linearity is trivial and the multiplicativity follow from the argument above

\[
\text{ch}_{\Lambda^{\bullet} P_{jm_j}} \ast (f_1 f_2)[g] = \text{ch}_{\Lambda^{\bullet} P_{jm_j}} \ast (f_1[g] f_2[g]) = (\text{ch}_{\Lambda^{\bullet} P_{jm_1}} \ast f_1[g]) (\text{ch}_{\Lambda^{\bullet} P_{jm_2}} \ast f_2[g]).
\]

Therefore we may assume \( f = p_m \). Again \( \text{ch}_{\Lambda^{\bullet} P_{jm_j}} \ast p_m[g] = p_m[g \ast \text{ch}_{\Lambda^{\bullet} P_j}] \) is linear and multiplicative in the entry \( g \) and so we reduce to the case \( g = p_j \).

It remains to prove that \( \text{ch}_{\Lambda^{\bullet} P_{jm_j}} \ast p_{jm_j} = p_m[p_j \ast \text{ch}_{\Lambda^{\bullet} P_j}] \). Since \( (p_\lambda)_\lambda \) are orthogonal idempotent with respect to the Kronecker product

\[
\text{ch}_{\Lambda^{\bullet} P_n} \ast p_n = \chi_{\Lambda^{\bullet} P_n}(c_n)p_n
\]

where \( \chi_V(\sigma) \) is the graded character of \( \sigma \in S_n \) with \( q \) replaced by \( -q \) and \( c_n \in S_n \) be an \( n \)-cycle. It is easy to see that

\[
\chi_{\Lambda^{\bullet} P_n}(c_n) = 1 + (-1)^{n-1}(-q)^n = 1 - q^n
\]

on the canonical base of \( \Lambda^{\bullet} P_n \): let \( (v_i)_i \) the standard base of \( P_n \), the product of some \( v_j \) is invariant for \( c_n \) if and only if each generator appears a fixed number of times (i.e. 0 or 1 times). Finally the equalities

\[
p_m[p_j \ast \text{ch}_{\Lambda^{\bullet} P_j}] = p_m[(1 - q^j)p_j] = (1 - q^{jm})p_{jm} = \text{ch}_{\Lambda^{\bullet} P_{jm_j}} \ast p_{jm_j}
\]

conclude the proof. □
For each monomial \( m = \prod_k e_{i_k,j_k} \in \mathbb{Q}[e_{i,j}] \) we define the support of \( m \) as the finest set partition \( B(m) \vdash [n] \) such that for all \( k \) \( i_k \) and \( j_k \) belong to the same block of \( B(m) \). We also define the type of \( m \) as the partition \( \lambda(m) \vdash n \) collecting the size of blocks of \( B(m) \). Notice that the relations defining \( OT_n \) (Definition 2.1) are sum of monomials with the same support, hence the notion of support and type are well defined in \( OT_n \). Moreover, monomials with different supports are linearly independent.

For \( B \vdash [n] \) a set partition let \( T_B \subset OT_n \) be the vector space generated by all monomials \( m \) such that \( B(m) = B \). For \( S \subseteq [n] \) we define \( T_S = T_B \) where \( B \) is the finest set partition of \([n]\) with a block equal to \( S \). Given two monomials \( m, n \) such that \( mn \neq 0 \) in \( OT_n \), we have that \( B(mn) \) is the finest set partition coarsening both \( B(m) \) and \( B(n) \), hence

\[
T_B \cong \bigotimes_{i=1}^{\lvert B \rvert} T_{B_i}
\]

Consider a partition \( \lambda \vdash n \), let \( T_{\lambda} \) be the vector space generated by all monomials of type \( \lambda \). Choose a set partition \( B_{\lambda} \vdash [n] \) whose blocks \( B_i \) are of length \( \lambda_i \) and let \( S_{B_{\lambda}} \) be the subgroup of \( S_n \) stabilizing \( B_{\lambda} \), if \( \lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n}) \) then \( S_{B_{\lambda}} \cong \prod_{j \geq 1} S_j^{\lambda_j} \). We have

\[
T_{\lambda} \cong \text{Ind}^{S_n}_{S_{B_{\lambda}}} T_{B_{\lambda}}
\]

as representation of \( S_n \), where \( S_{B_i} \) acts on the factor \( T_{B_i} \) of \( T_{B_{\lambda}} = \bigotimes_{i=1}^{\lvert B_{\lambda} \rvert} T_{B_i} \) and \( S_{m_j} \) permutes the \( m_j \) factors of size \( j \). For the sake of notation we set \( T_n = T_{(n)} \).

**Lemma 4.6.** We have

\[
\text{ch}_{OT_n} = \sum_{\lambda \vdash n} \prod_{j \geq 1} h_{m_j} \left[ \text{ch}_{T_{\lambda}} \right].
\]

**Proof.** The Orlik-Terao algebra decomposes

\[
OT_n = \bigoplus_{B \vdash [n]} T_B
\]

\[
= \bigoplus_{B \vdash [n]} \bigotimes_{i=1}^{\lvert B \rvert} T_{B_i}
\]

\[
= \bigoplus_{\lambda \vdash n} \text{Ind}_{S_{B_{\lambda}}}^{S_n} \bigotimes_{i=1}^{\ell(\lambda)} T_{B_i}
\]

\[
= \bigoplus_{\lambda \vdash n} \text{Ind}_{S_{B_{\lambda}}}^{S_n} \bigotimes_{i=1}^{\ell(\lambda)} T_{B_i}
\]

\[
= \bigoplus_{\lambda \vdash n} \prod_{j \geq 1} S_{m_j} \left( \bigotimes_{j \geq 1} \text{Ind}_{S_j^{\lambda_j} S_{m_j}}^{S_{T_j}} \right)
\]

as \( S_n \)-representation. Taking the character we obtain the claimed relation. \( \square \)
Theorem 4.7. We have
\[ \text{ch}_{D_n} = \text{ch}_{M_n} \]
and
\[ \text{ch}_{T_n} = q^{n-1}l_n \ast \text{ch}_{R_n}. \]

Proof. We prove both equality by induction on \( n \). The base case \( n = 1 \) is trivial. For the inductive step we consider:

\[ \text{ch}_{M_n} = \text{ch}_{OT_n} \ast \text{ch}_{\Lambda_n}' \]
\[ = \frac{1}{(1-q)} \sum_{\lambda \vdash n} \prod_{j \geq 1} h_{m_j} [\text{ch}_{T_j} \ast \text{ch}_{\Lambda_P j}'] \]
\[ = \text{ch}_{T_n} \ast \text{ch}_{\Lambda_n}' + \frac{1}{(1-q)} \sum_{\lambda \vdash n} \prod_{\lambda \neq (n)} h_{m_j} [q^{j-1}(1-q)l_j]. \]

The first equality follows from Theorem 4.1 and Lemma 4.2. The second one follows from Lemma 4.6 and Lemma 4.5 together with the identity \( \text{ch}_{\Lambda_P j}' = (1-q) \text{ch}_{\Lambda_j}' \). The last one follows from the inductive hypothesis and Lemma 4.2. We have proven the identity

\[ \text{ch}_{M_n} - \text{ch}_{T_n} \ast \text{ch}_{\Lambda_n}' = \frac{1}{(1-q)} \sum_{\lambda \vdash n} \prod_{\lambda \neq (n)} h_{m_j} [q^{j-1}(1-q)l_j] = \text{ch}_{D_n} - q^{n-1}l_n, \]

where the last equality is given by Corollary 4.4. Since \( \text{ch}_{D_n} \) and \( \text{ch}_{M_n} \) has degree less than \( n-1 \) and \( \text{ch}_{T_n} \ast \text{ch}_{\Lambda_n}' \) bigger than \( n-2 \), \( \text{ch}_{M_n} = \text{ch}_{D_n} \) and \( \text{ch}_{T_n} \ast \text{ch}_{\Lambda_n}' = q^{n-1}l_n \) hold. Therefore \( \text{ch}_{T_n} = q^{n-1}l_n \ast \text{ch}_{R_n} \).

\[ \]□

Corollary 4.8. We obtain the character of \( OT_n \):

\[ \text{ch}_{OT_n} = \sum_{\lambda \vdash n} q^{n-\ell(\lambda)} \prod_{j \geq 1} h_{m_j} [l_j \ast \text{ch}_{R_j}]. \]

(6)

Proof. It follows from Theorem 4.7 and Lemma 4.6.

An important object for the proof of Theorem 4.7 is the \( R_n \)-module \( T_n \). It is a submodule of the free module \( OT_n \) and its Frobenius character is equal to the one of the free module \( R_n \otimes \mathbb{Q} T_n^{n-1} \). This observations lead to the following conjecture:

Conjecture 4.9. The \( R_n \)-module \( T_n \) is free.

Remark 4.10. The formula (6) is completely explicit because \( \text{ch}_{R_j} \) and \( l_j \) are known. Indeed

\[ \text{ch}_{R_n} = (1-q) \sum_{\lambda \vdash n} s_{\lambda}(1, q, q^2, ...) s_{\lambda} = (1-q) h_n \left[ \frac{X}{1-q} \right] \]
by [Pro03, Section 5.6] or [Sta99, Exercise 7.73] where \( X = h_1 \). Moreover,

\[
l_n = \frac{1}{n} \sum_{d \mid n} \mu(d) p_d \frac{n}{d},
\]

by [Reu93, Theorem 8.3], \( l_n \) is known as the Lyndon symmetric function or as Gessel-Reutenauer symmetric function [GR93].

Let \( \text{Exp} \) be the plethystic exponential defined by

\[
\text{Exp}(f) = \exp \left( \sum_{k \geq 1} \frac{p_k[f]}{k} \right) = \sum_{k \geq 0} h_k[f],
\]

and \( \text{Log} \) be its inverse. We define the symmetric functions

\[
L = \sum_{n \geq 1} q^{n-1} t^n l_n = -\frac{\log(1 - qtX)}{q}
\]

and \( H = \sum_{k \geq 0} h_k \).

**Corollary 4.11.** The generating functions for \( \text{ch}_D \) and \( \text{ch}_{OT} \) are:

\[
\sum_{t \geq 1} \text{ch}_{D_n}(q) t^n = \frac{1}{1 - q} (\text{Exp}((1 - q)L) - 1),
\]

\[
\sum_{t \geq 1} \text{ch}_{OT_n}(q) t^n = \text{Exp} \left( (1 - q)L \ast H \left[ \frac{X}{1 - q} \right] \right) - 1.
\]

**Proof.** Let \( f \) be a symmetric function and call \( f_j \) be the homogeneous part of degree \( j \). Assume that \( f \) has zero constant term, i.e. \( f = \sum_{j \geq 1} f_j \); then

\[
\text{Exp}(f) = \prod_{j \geq 1} \text{Exp}(f_j)
\]

\[
= \prod_{j \geq 1} \sum_{m \geq 0} h_m[f_j]
\]

\[
= \sum_{\lambda} \prod_{j \geq 1} h_{m_j}[f_j],
\]

where the sum is taken over all partitions \( \lambda = (1^{m_1}, 2^{m_2}, \ldots) \). The corollary follows by taking \( f = (1 - q)L \) and \( f = (1 - q)L \ast H[(1 - q)^{-1}X] \). \( \square \)

Formulas of this paper are checked and implemented in SageMath [Sage]. The code is available at

[https://github.com/paga92/character_OT](https://github.com/paga92/character_OT).
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