Typical differentiability within an exceptionally small set.

Michael Dymond*
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We verify the existence of a purely unrectifiable set in which the typical Lipschitz function has a large set of full differentiability points. The example arises from a construction, due to Csörnyei, Preiss and Tišer, of a universal differentiability set in which a certain Lipschitz function has only a purely unrectifiable set of differentiability points.

1. Introduction.

Whilst Rademacher’s Theorem asserts that any set of points of non-differentiability of a Lipschitz function on Euclidean space is null, the sets most negligible from the point of view of differentiability problems are, as described in the work [1] of Alberti, Csörnyei and Preiss, those sets in which some Lipschitz function fails to have a single directional derivative. In this paper we show that even these most exceptional sets can nonetheless provide surprisingly many points of full differentiability for surprisingly many Lipschitz functions.

In [1] it is established that the negligible sets referred to above are precisely the class of uniformly purely unrectifiable sets. A subset \( P \) of Euclidean space is said to be purely unrectifiable if \( P \) intersects every \( C^1 \) curve in a set of one-dimensional Lebesgue measure zero. The class of uniformly purely unrectifiable sets are defined according to a formally stronger condition (see [11, Definition 1.4 and Remark 1.7]) and for a significant time it remained an open question whether these two classes coincide. However, a recent announcement of Máthe answers this question positively for Borel sets ([11, Remark 1.7]). In the present work, we adopt the convention of restricting both notions to Borel sets, that is, we add Borel as a condition to the definitions of pure and uniform pure unrectifiability. Thus, the notions of pure and uniform pure unrectifiability coincide and we will, from this point onwards, refer only to purely unrectifiable sets.

Current investigations of purely unrectifiable sets have established that these sets are most exceptional with respect to differentiability, not only in the sense of non-availability

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of directional derivatives. Preiss and Maleva prove in [11, Theorem 1.13] that any purely unrectifiable set is contained in a set points where non-differentiability of some Lipschitz function occurs in its strongest possible form. Any purely unrectifiable set $P \subseteq \mathbb{R}^d$ admits a $1$-Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that for every $x \in P$ every linear mapping $\mathbb{R}^d \to \mathbb{R}$ with norm at most one masquerades as the derivative of $f$ at $x$. More precisely,

$$\liminf_{r \to 0} \sup_{\|y\| \leq r} \frac{|f(x + y) - f(x) - \langle e, y \rangle|}{r} = 0$$

holds for all $x \in P$ and $e \in \mathbb{R}^n$ with $\|e\| \leq 1$. Further, Merlo [13] proves that for a given $F_\sigma$ purely unrectifiable set $P$, the typical Lipschitz mapping in an appropriate space has no directional derivatives inside $P$.

Merlo’s result and the notion of a typical Lipschitz mapping require further explanation. In what follows we consider for a compact metric space $K$ the space $\text{Lip}_1(K, \mathbb{R}^l)$ of Lipschitz mappings $f : K \to \mathbb{R}^l$ with $\text{Lip}(f) \leq 1$. When $l = 1$, as it will be for almost all of this work, we shorten the notation to $\text{Lip}_1(K)$. We view $\text{Lip}_1(K, \mathbb{R}^l)$ as a complete metric space equipped with the supremum metric

$$d_\infty(f, g) := \|f - g\|_\infty,$$

$f, g \in \text{Lip}_1(K)$.

The word typical is used in this paper in the sense of the Baire Category Theorem. Thus, we say that typical functions in $\text{Lip}_1(K)$ have a certain property if the set of those functions having that property is a residual subset of $\text{Lip}_1(K)$.

Differentiability of typical Lipschitz functions on the interval, i.e. in the function space $\text{Lip}_1([0, 1])$, is well understood, due to the work [17]. Here Preiss and Tišer characterise subsets of the interval $[0, 1]$ in which the typical function $f \in \text{Lip}_1([0, 1])$ is nowhere differentiable; they prove that the sets with this property are precisely those contained in an $F_\sigma$ set of Lebesgue measure zero. The aforementioned result of Merlo [13] extends this statement to higher dimensional settings. For any dimensions $1 \leq d \leq l$, Merlo characterises the subsets of $[0, 1]^d$ where typical mappings in the space $\text{Lip}_1([0, 1]^d, \mathbb{R}^l)$ have no directional derivatives. Such sets are precisely those contained in some $F_\sigma$, purely unrectifiable set.

The results of Preiss and Tišer in [17], whilst focusing on the space of Lipschitz functions on the interval $[0, 1]$, have surprising implications for questions of typical differentiability in higher dimensional spaces. An important observation in [17] asserts that the typical function $f \in \text{Lip}_1([0, 1])$ satisfies

$$\limsup_{y \to x} \left| \frac{f(y) - f(x)}{y - x} \right| = 1 \quad \text{for all } x \in [0, 1]. \tag{1}$$

If we consider a $C^1$ curve $\gamma : [0, 1] \to (0, 1)^d$ and use the mapping $f \mapsto f \circ \gamma$ to transfer between the spaces $\text{Lip}_1([0, 1]^d)$ and $\text{Lip}_1([0, 1])$, the conclusions of [17] concerning full differentiability of the typical $f \in \text{Lip}_1([0, 1])$ at points of $[0, 1]$ translate naturally to statements of directional differentiability of the typical $f \in \text{Lip}_1([0, 1]^d)$ at points on the curve $\gamma$ in the tangent direction of $\gamma$. With the additional use of the observation [11] and
a classical result of Fitzpatrick [9, Corollary 2.6], these translated statements actually refer to full differentiability of the typical \( f \in \text{Lip}([0,1]^d) \) at points on the curve \( \gamma \).

In particular, the result [17, Lemma 2] permits, via the arguments described above, examples of purely unrectifiable sets inside \((0,1)^d\) in which the typical \( f \in \text{Lip}_1([0,1]^d) \) has a point of full differentiability. Indeed, any relatively residual and null subset of some line segment in \((0,1)^d\) would provide such an example. This a particularly surprising outcome: Purely unrectifiable sets are so tiny that they see only the most terrible occurrences of non-differentiability of some Lipschitz function. However, as we have argued above, these exceptional sets may nonetheless capture points of full differentiability of very many Lipschitz functions. The situation is rather different for \( F_\sigma \) purely unrectifiable sets. Merlo [13, Proposition 33, Remark 34.] verifies that inside any given \( F_\sigma \) purely unrectifiable subset of \([0,1]^d\) the typical \( \text{Lip}_1([0,1]^d) \) function has no directional derivatives.

Although, as explained above, the results of Preiss, Tišer [17] may be used to verify existence of purely unrectifiable sets capturing a point of differentiability of the typical Lipschitz function, they do not allow for any non-trivial bound on the size of the set of captured differentiability points. Due to the fundamental Besicovitch-Federer Projection Theorem, one-dimensional Hausdorff measure is an important means of distinction between purely unrectifiable sets. The theorem implies that any purely unrectifiable set of \( \sigma \)-finite one-dimensional Hausdorff measure has projections of Lebesgue measure zero on almost every one-dimensional subspace. Our main result verifies the existence of a purely unrectifiable set in which the typical Lipschitz function has a particularly large set of differentiability points, where large is understood in the sense of the Besicovitch-Federer Projection Theorem.

**Theorem 1.1.** There exists a purely unrectifiable set \( P \subseteq [0,1]^2 \) such that the typical function \( f \in \text{Lip}_1([0,1]^2) \) has points of differentiability in \( P \) and moreover the set \( \text{Diff}(f) \cap P \) of these points is large in the following senses:

(a) \( \text{Diff}(f) \cap P \) has non-\( \sigma \)-finite one dimensional Hausdorff measure.

(b) \( \text{Diff}(f) \cap P \) projects in every direction to a set of positive Lebesgue measure, that is,

\[
\mathcal{L}(\langle \text{Diff}(f) \cap P, e \rangle) > 0
\]

for every \( e \in S^1 \).

Note that part (a) actually follows from (b) via the Besicovitch-Federer Projection Theorem.

The proof of Theorem [13] is based on the modern theory of universal differentiability sets which originates from the natural question of whether the classical Rademacher’s Theorem for Lipschitz mappings admits a converse and the first negative answer to this question given by Preiss [15]. The natural converse to Rademacher’s Theorem proposes that any Lebesgue null set \( E \subseteq \mathbb{R}^d \) is contained in the set of non-differentiability points of some Lipschitz mapping \( f : \mathbb{R}^d \to \mathbb{R}^l \). Whilst [13] provides a counterexample for the case of real valued functions on the plane, i.e. the case \( d = 2, l = 1 \), major breakthroughs
In the last decade have now completely resolved the question for general dimensions. The converse is valid if and only if \( l \geq d \), that is, if the dimension of the target space is at least that of the domain.

Thus, if \( 1 \leq l < d \), the Euclidean space \( \mathbb{R}^d \) contains Lebesgue null sets which capture a point of differentiability of every Lipschitz mapping \( \mathbb{R}^d \to \mathbb{R}^l \). Sets with the latter property are given the name universal differentiability sets, first proposed in [5]. These surprising objects have attracted much new research attention and have been studied in an array of different settings, for example Euclidean spaces (\[4\], \[5\], \[8\], \[7\], \[16\]), Banach spaces [6], and metric groups (\[14\], \[10\]).

For a given universal differentiability set \( E \subseteq \mathbb{R}^d \) it is natural to ask how large the sets

\[
E \cap \text{Diff}(f), \quad f : \mathbb{R}^d \to \mathbb{R}, \text{ Lipschitz},
\]

are as subsets of \( E \). Previous work [7] of the author verifies that these sets are large in a topological sense. Any universal differentiability set can be reduced to a ‘kernel’ in which the set of differentiability points of any given Lipschitz function form a dense subset. However, an example provided by Csörnyei, Preiss and Tišer [3], demonstrates that these sets of captured differentiability points can be surprisingly tiny subsets of \( E \) in a measure theoretic sense, namely they can be purely unrectifiable. Recall from previous discussion in this introduction that purely unrectifiable sets are very far away from being universal differentiability sets, hence purely unrectifiable subsets of \( E \) can be thought of as small subsets.

The aforementioned example of Csörnyei, Preiss and Tišer [3] and its construction provide the basis of the proof of Theorem 1.1. The construction produces a universal differentiability set \( E \subseteq \mathbb{R}^2 \), a purely unrectifiable subset \( P \subseteq E \) and a Lipschitz function \( h : \mathbb{R}^2 \to \mathbb{R} \) so that all differentiability points of \( h \) in the set \( E \) are contained in \( P \). By modification of the construction, we ensure that the purely unrectifiable set \( P \) additionally captures a point of differentiability of the typical Lipschitz function. Our argument stems from the idea that most points of non-differentiability of \( h \) are preserved for the function \( g + h \) for typical \( g \in \text{Lip}_1([0,1]^2) \). However, there are rather too many such points in the \( G_\delta \), dense set \( E \) given by [3] in order to preserve non-differentiability at all of them. Thus, we crucially pass to a compact universal differentiability set \( D \subseteq E \), given by a construction of Doré and Maleva in [5]. In this much smaller set we are able to preserve non-differentiability of \( h \) everywhere in the set \( D \setminus P \) for functions \( g + h \) for the typical \( g \in \text{Lip}_1([0,1]^2) \). Since \( D \) is a universal differentiability set, this leads to the conclusion that \( g + h \) has points of differentiability in \( P \). In other words, \( P \) captures a point of differentiability of the typical Lipschitz function in the shifted space \( h + \text{Lip}_1([0,1]^d) \). However, since differentiability of a sum \( g + h \) does not imply differentiability of \( g \), this is not enough to verify Theorem 1.1. Moreover, we caution that the typical behaviour in a shifted \( \text{Lip}_1 \) space can be very different to that in the natural space; Lemma 2.2 of the present work may be used to produce examples demonstrating this. To verify that \( P \) additionally captures a point of differentiability of the typical function in \( \text{Lip}_1([0,1]^d) \), we adapt the construction of [3] so that the function \( h \) is differentiable at almost all points of the set \( P \). Differentiability of \( g + h \) at such points then implies differentiability of \( g \).
The conclusions (a) and (b) of Theorem 1.1 come from the observation that the differentiability points of the typical \( g \in \text{Lip}_1([0,1]^d) \) inside of \( P \) correspond to the differentiability points of the function \( g + h \) inside the (necessarily much larger) universal differentiability set \( D \). The latter set of points is large in the sense of (b) due to [8, Lemma 2.1].

A further objective of this work is to provide a simplification of the argument in [3], based on recent advances in the theories of universal differentiability and uniformly purely unrectifiable sets. There are two main tools in the simplification: Firstly, we make use of the recently announced result of Máté, that the notions of pure unrectifiability and uniform pure unrectifiability coincide. Since the condition for pure unrectifiability is significantly easier to verify, this immediately removes much of the complexity of the argument in [3]. The second main way in which we achieve a simplification is in a more special choice of the universal differentiability set \( E \). We take \( E \) as a universal differentiability set of the form described in [11, Example 4.4]: A \( G_δ \) set containing all lines from a dense subset of the set of all lines with directions inside a small cone.

Whilst we aspire to provide a more accessible proof of the result in [3], we additionally obtain a stronger statement. We show that inside a universal differentiability set in \( \mathbb{R}^2 \) even directional derivatives of a Lipschitz function may be rather scarce.

**Theorem 1.2.** For every \( \alpha > 0 \) there exists a universal differentiability set \( E \subseteq \mathbb{R}^2 \) with the following property. There exists a Lipschitz function \( h : \mathbb{R}^2 \to \mathbb{R} \), and a double sided cone \( \hat{C} \subseteq S^1 \) of width at most \( \alpha \) such that the set of points in \( E \) where \( h \) has a directional derivative in any direction in \( S^1 \setminus \hat{C} \) is contained in a purely unrectifiable set.

**2. Preliminaries and Notation.**

We use the term \( C^{1} \)-curve to refer to a \( C^{1} \) mapping \( \gamma \) from a closed interval \( I \subseteq \mathbb{R} \) to \( \mathbb{R}^d \) satisfying \( \gamma'(t) \in S^{d-1} \) for all \( t \in I \). Here \( \gamma'(t) \) denotes the derivative of \( \gamma \) at the point \( t \) (or the one-sided derivative if \( t \) is an endpoint). We identify this derivative with an element of \( \mathbb{R}^d \) (or in this case \( S^{d-1} \)) in the standard way. A Borel set \( P \subseteq \mathbb{R}^d \) is said to be purely unrectifiable if for every \( C^{1} \)-curve \( \gamma : I \to \mathbb{R}^d \) the set \( \gamma^{-1}(P) \) has Lebesgue measure zero.

For \( w \in S^{d-1} \) and \( \alpha \in [0,1] \) we define a set

\[
C(w, \alpha) := \left\{ v \in S^{d-1} : \langle v, w \rangle \geq 1 - \alpha \right\},
\]

and refer to this set as the cone around \( w \) of width \( \alpha \). We additionally define

\[
\hat{C}(w, \alpha) = \left\{ v \in S^{d-1} : |\langle v, w \rangle| \geq 1 - \alpha \right\}
\]

and call this set the double sided cone around \( w \) of width \( \alpha \). Observe that \( \hat{C}(w, \alpha) = C(w, \alpha) \cup C(-w, \alpha) = C(w, \alpha) \cup -C(w, \alpha) \).

For a function \( f : \mathbb{R}^d \to \mathbb{R} \) and \( x \in \mathbb{R}^d \) we write \( Df(x) \) for the derivative of \( f \) at the point \( x \) if it exists and we identify this with the unique element of \( \mathbb{R}^d \) satisfying
$Df(x) = \langle Df(x), \cdot \rangle$. To detect non-differentiability of $f$, we utilise the following test quantities. Given a point $z \in \mathbb{R}^2$ a direction $e \in S^1$ and $\varepsilon > 0$ we consider the quantity

$$\zeta(f, z, \varepsilon, e) := \sup \left| \frac{f(x + te) - f(x)}{t} - \frac{f(y + se) - f(y)}{s} \right|,$$

where the supremum is taken over all segments of the form $[x, x + te] \cap [y, y + se]$ satisfying $z \in [x, x + te] \cap [y, y + se]$ and $s, t \in [-\varepsilon, \varepsilon] \setminus \{0\}$. We further consider the related quantity $\Upsilon(f, z, \varepsilon)$ where the variable $e \in S^1$ is ’moved inside the supremum’, that is

$$\Upsilon(f, z, \varepsilon) := \sup_{e \in S^1} \zeta(f, z, \varepsilon, e).$$

Roughly speaking, both quantities $\zeta(f, z, \varepsilon, e)$ and $\Upsilon(f, z, \varepsilon)$ reflect non-differentiability of $f$ at $z$ at scale $\varepsilon$. Severity of non-differentiability of $f$ at $z$ is sharply quantified by their limiting behaviour as $\varepsilon \to 0$.

**Proposition 2.1.** Let $f: \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function, $z \in \mathbb{R}^d$ and $e \in S^{d-1}$. Then,

(a) $\lim_{\varepsilon \to 0} \zeta(f, z, \varepsilon, e) = 0 \iff f$ has a directional derivative at $z$ in direction $e$.

(b) $\limsup_{\varepsilon \to 0} \Upsilon(f, z, \varepsilon) > 0 \implies \exists u \in S^{d-1}$ such that $\limsup_{\varepsilon \to 0} \zeta(f, z, \varepsilon, u) > 0$.

The proof of Proposition 2.1 is a standard exercise. We postpone it until the Appendix.

The next lemma plays a key part in the proof of Theorem 1.1. It allows us to preserve non-differentiability of a Lipschitz function $h$ at many points after adding a typical function $g \in \text{Lip}_1([0, 1])$.

**Lemma 2.2.** Let $K \subseteq [0, 1]^d$ be an $F_\sigma$ set, $\sigma > 0$ and $h: [0, 1]^d \to \mathbb{R}$ be a Lipschitz function satisfying

$$\limsup_{\varepsilon \to 0} \Upsilon(h, z, \varepsilon) \geq \sigma$$

for all $z \in K$. Then for typical $g \in \text{Lip}_1([0, 1]^d)$ we have

$$\limsup_{\varepsilon \to 0} \Upsilon(h + g, z, \varepsilon) \geq \sigma$$

for all $z \in K$.

**Proof.** We may assume that $K$ is compact. Let $0 < \lambda < \lambda' < \lambda'' < \sigma$. It suffices to prove that typical $g \in \text{Lip}_1([0, 1]^d)$ satisfy the assertion of the lemma with $\sigma$ replaced by $\lambda$.

We describe a winning strategy for Player II in the relevant Banach-Mazur game in $\text{Lip}_1([0, 1]^d)$. In response to the $n$-th play $U_n$ of Player I, Player II chooses a smooth function $g_n \in U_n$ and $\delta_n \in (0, 2^{-n})$ so that $B(g_n, \delta_n) \subseteq U_n$. Next, Player II chooses for each point $z \in K$ a direction $e(z) \in S^{d-1}$, points $x(z), y(z) \in [0, 1]^d$ and numbers $s(z), t(z) \in [-\delta_n, \delta_n] \setminus \{0\}$ witnessing that

$$\gamma(z, h + g_n, \delta_n) > \lambda''.$$  

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Given \( z' \in B(z, \eta) \) we have for \( w(z') := z' - z \) that \( z' \in [x+w, x+w+te] \cap [y+w, y+w+se] \) and
\[
\left| \frac{(h + g_n)(x + w + te) - (h + g_n)(x + w)}{t} \right| - \left| \frac{(h + g_n)(y + w + se) - (h + g_n)(y + w)}{s} \right| \geq \lambda'' - 4(\text{Lip}(h) + 1)\eta \min\{s,t\}.
\]

Let now \( \eta(z) \) be sufficiently small so that
\[
\left| \frac{(h + g_n)(x + w + te) - (h + g_n)(x + w)}{t} \right| - \left| \frac{(h + g_n)(y + w + se) - (h + g_n)(y + w)}{s} \right| > \lambda'
\]
for all points \( z' \in B(z, \eta(z)) \). The collection \( (B(z, \eta(z)))_{z \in K} \) is an open cover of the compact set \( K \). Player II extracts a finite subcover \( (B(z_i, \eta(z_i)))_{i=1}^N \) and returns the open set \( V_n := B(g_n, \theta_n) \) for \( \theta_n \) chosen sufficiently small based on the data corresponding to the points \( z_1, \ldots, z_N \) and, in particular, small enough so that \( V_n \subseteq U_n \). The precise remaining condition on \( \theta_n \) that we require will be determined later in the proof.

Let us now verify that Player II wins the Banach Mazur game following the above strategy. Let \( g \in \bigcap_{n=1}^\infty V_n \) and \( z \in K \). We need to prove \( \limsup_{\varepsilon \to 0} \Upsilon(h + g, z, \varepsilon) \geq \lambda \). Fixing \( \varepsilon > 0 \) we verify that \( \Upsilon(h + g, z, \varepsilon) \geq \lambda \). Let \( n \in \mathbb{N} \) be large enough so that \( \delta_n < \varepsilon \) and let \( z_i \) be one of the points corresponding to the ball \( B(z_i, \eta(z_i)) \) chosen by Player II in the \( n \)-th round of the Banach-Mazur game such that \( z \in B(z_i, \eta(z_i)) \). Then for \( w := z - z_i \) and \( (x, y, s, t) = (x(z_i), y(z_i), s(z_i), t(z_i)) \) we have that
\[
z \in [x+w, x+w+te] \cap [y+w, y+w+se], \quad s, t \in [-\delta_n, \delta_n] \setminus \{0\} \subseteq [-\varepsilon, \varepsilon] \setminus \{0\}
\]
and (2) holds. Since \( g \in B(g_n, \theta_n) \), the same inequality holds with \( g \) replaced by \( g_n \) and \( \lambda' \) replaced by \( \lambda' - \frac{4\theta}{\min\{s, t\}} \). Thus, we obtain
\[
\zeta(h + g, z, \varepsilon) \geq \lambda \text{ with the condition }
\]
\[
\theta_n \leq \frac{(\lambda' - \lambda) \min_{1 \leq i \leq N} \{s(z_i), t(z_i)\}}{4}
\]
imposed on \( \theta \). Here the minimum is taken over all points \( z_1, \ldots, z_N \in K \) chosen by Player II in the \( n \)-th round of the game.

\[\square\]

3. Construction of a Universal Differentiability Set.

We present a construction of a universal differentiability set \( E \subseteq \mathbb{R}^2 \) and a Lipschitz function \( h \) having very few differentiability points in \( E \). This will serve both the proof of Theorem 1.2 and the proof of Theorem 1.1. The construction is based heavily on that of [3], but contains a few new modifications. Crucially for the proof of Theorem 1.1, we modify the construction in order to distinguish points of the set \( E \) where \( h \) is differentiable.
3.1. The Set $E$.

Let $E \subseteq \mathbb{R}^2$ be a set of the form

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B(L_k, \rho_k)$$

where $(L_k)_{k=1}^{\infty}$ is a sequence of lines $L_k \subseteq \mathbb{R}^2$ and $(\rho_k)_{k=1}^{\infty}$ is a sequence of positive numbers $\rho_k$ which converges to zero sufficiently fast, in particular so that $\sum_{k=1}^{\infty} \rho_k < \infty$. Further conditions will be imposed on these sequences in the course of the proof.

We define a sequence of functions $(k_p)_{p=0}^{\infty}$ on $\mathbb{R}^2$ whose purpose is to record for each point $z \in \mathbb{R}^2$ the possibly empty subsequence of $k \in \mathbb{N}$ for which $z \in B(L_k, \rho_k)$. Setting $k_0 = 0$ on the whole plane $\mathbb{R}^2$ we define $k_p$ inductively by

$$k_p(z) = \inf \{k > k_{p-1}(z) : z \in B(L_k, \rho_k)\},$$

where we interpret the infimum of the empty set as $\infty$. We impose an additional constraint on the set $E$, namely, that the directions $e_k$ of each line $L_k$ lie in a cone around a fixed vector $w \in S^1$. For a parameter $\eta \in (0, 1]$ we demand that $e_k \in C(w, \eta)$ for all $k \in \mathbb{N}$. (3)

For each line $L_k$ we fix a point $x_k \in L_k$ so that $L_k = x_k + \mathbb{R}e_k$. We can now formulate a sufficient condition for $E$ to be a universal differentiability set.

Lemma 3.1. If the sequence of lines $(L_k = x_k + \mathbb{R}e_k)_{k=1}^{\infty}$ is chosen so that the sequence of pairs $((x_k, e_k))_{k=1}^{\infty}$ is dense in $\mathbb{R}^2 \times C(w, \eta)$ then the set $E$ is a universal differentiability set. Moreover, $E$ contains a (possibly different) sequence of lines $(\tilde{L}_k = \tilde{x}_k + \mathbb{R}\tilde{e}_k)_{k=1}^{\infty}$ for which the sequence of pairs $((\tilde{x}_k, \tilde{e}_k))_{k=1}^{\infty}$ is dense in $\mathbb{R}^2 \times C(w, \eta)$.

Proof. For the first part see [11, Example 4.4]. The moreover part is proved by a Baire Category argument given in [3, p. 362].

We demand that the sequence $(L_k)_{k=1}^{\infty}$ of lines satisfies the condition of Lemma 3.1 so that $E$ is a universal differentiability set. The next lemma represents a key step in the proof of Theorem 1.1. It is not needed for the proof of Theorem 1.2.

Lemma 3.2. There is a compact universal differentiability set $D \subseteq E \cap [0, 1]^2$.

Proof. The $G_\delta$ set $E$ contains a sequence of lines $(\tilde{L}_k = \tilde{x}_k + \mathbb{R}\tilde{e}_k)_{k=1}^{\infty}$ which is dense in $\mathbb{R}^2 \times C(w, \eta)$ in the sense of Lemma 3.1. We follow the construction of [10] to produce a family of compact sets inside of $E$. The construction provides families of sets of the form

$$M_k(\lambda) = \bigcup_{k \leq n \leq (1+\lambda)k} \overline{B}_{\lambda w_n}(R_n) \subseteq [0, 1]^2, \quad \lambda \in (0, 1],$$
where the sets \( R_n \) are increasing, finite unions of line segments. In our modified construction the line segments of \( R_n \) will always be chosen inside the lines \( L_k \subseteq E \). The universal differentiability sets produced by \( \delta \) take the form

\[
T_\lambda = \bigcap_{k=1}^{\infty} M_k(\lambda) \subseteq [0,1]^2, \quad \lambda \in (0,1],
\]

and the construction ensures that each set \( T_\lambda \) fits inside a \( G_\delta \) set fixed at the start containing all lines added to the sets \( R_n \). We take \( E \) as this \( G_\delta \) set and so we obtain compact sets \( T_\lambda \subseteq E \). Further note that the sets \( (T_\lambda)_{\lambda \in (0,1]} \) are nested in the sense that \( T_{\lambda_1} \subseteq T_{\lambda_2} \) whenever \( \lambda_1 \leq \lambda_2 \).

To establish that each of the sets \( T_\lambda \) is a universal differentiability set, the paper \( \delta \) proves that the family \( (T_\lambda)_{\lambda \in (0,1]} \) possesses the ‘wedge approximation property’ described in \( \delta \) Lemma 3.5] and \( \delta \) Lemma 3.1]. In our modified construction, we only add line segments to the sets \( R_n \) with directions inside the cone \( C(w, \eta) \). Accordingly, we obtain sets \( (T_\lambda)_{\lambda \in (0,1]} \) with a weaker form of the wedge approximation property. Namely, the identical approximation property restricted only to wedges \( [x,y] \cup [y,z] \) in which the two line segments \( [x,y] \) and \( [y,z] \) are both parallel to some direction in the cone \( C(w, \eta) \).

It now remains to argue that this restricted wedge approximation property is sufficient for universal differentiability. Given a Lipschitz function \( f_0 : \mathbb{R}^2 \to \mathbb{R} \) we follow the proof of \( \delta \) Theorem 3.1] in order to find a point of differentiability of \( f_0 \) inside say \( T_1 \). To begin, we fix some \( \lambda_0 < \lambda_1 \in (0,1) \) and find a pair \((x_0,e_0)\) with \( x_0 \in T_{\lambda_0} \) and \( e_0 \in S^1 \) such that the directional derivative \( f'(x_0,e_0) \) exists. Since the set \( T_{\lambda_0} \) contains line segments in \( R_1 \), parallel to some direction in the cone \( C(w, \eta) \) we may additionally prescribe here that the direction \( e_0 \) is taken inside \( C(w, \eta) \). Given this starting data, the proof of \( \delta \) Theorem 3.1] constructs a Lipschitz function \( f : \mathbb{R}^2 \to \mathbb{R} \) which differs from \( f_0 \) only by a linear function and a sequence of point-direction pairs \((x_n,e_n)\in T_{\lambda_1} \times S^1 \) converging to a pair \((x,e)\in T_{\lambda_0} \times S^1 \) such that the directional derivative \( f'(x,e) \) exists and satisfies a very delicate ‘almost locally maximal’ condition defined in the statement of \( \delta \) Theorem 3.1]. In the iterative construction of the sequence \((x_n,e_n)\) the new direction \( e_{n+1} \) may always be chosen arbitrarily close to the previous one \( e_n \); in this proof the inequality \( ||e_{n+1} - e_n|| \leq \sigma_n \) is satisfied at each step where the only meaningful condition on the sequence \((\sigma_n)_{n=0}^{\infty} \) is that \( \sigma_n \in (0,\sigma_{n-1}/4) \). Hence, by choosing these \( \sigma_n \) sufficiently small, we may ensure that the limit direction \( e \) lies inside the cone \( C(w, \eta) \).

Finally, having arrived at a pair \((x,e)\) in \( T_{\lambda_1} \times C(w, \eta) \) for which the directional derivative \( f'(x,e) \) is almost locally maximal, we argue that \( f \) and therefore also \( f_0 \) is differentiable at \( x \). In what follows the point \( z \in \mathbb{R}^2 \) is denoted by \( \lambda \) in the referred literature. We change the notation in this instance in order to avoid confusion with the index \( \lambda \) of the sets \( T_\lambda \), but otherwise we use the same notation as the referred literature. If \( f \) is not differentiable at \( x \) then we follow the argument of \( \delta \) Lemma 4.3] and use \( \delta \) Lemma 4.2] to show that on arbitrarily small wedges of the form

\[
[x - se, x + z] \cup [x + z, x + se] \subseteq \mathbb{R}^2,
\]

and on all sufficiently good approximations of such wedges we may find points \( x' \) admitting a direction \( e' \) for which the directional derivative \( f'(x', e') \) exists and is greater,
in a technical sense, than \( f'(x,e) \). If such wedges can be found inside the sets \( T_\alpha \) with \( \alpha \) greater than but arbitrarily close to \( \lambda_1 \) then we obtain a contradiction to the almost locally maximal condition on \( f'(x,e) \), which completes the proof. In [8] this is ensured by the wedge approximation property of the sets \( (T_\lambda)_{\lambda \in (0,1]} \). The point \( x + z \) appearing in (4) may be taken arbitrarily close to the line segment \( [x - se, x + se] \) relative to the scale \( s > 0 \); see [4] (4.4), Lemma 4.2. Therefore, the directions of the two segments \( [x - se, x + z] \) and \( [x + z, x + se] \) may be taken arbitrarily close to \( e \in C(w,\eta) \). In particular, it suffices to consider only wedges in which the two component line segments are parallel to directions in \( C(w,\eta) \). This means that the restricted wedge approximation property present in our sets \( (T_\lambda)_{\lambda \in (0,1]} \) is enough.

**Remark 3.3.** The argument used in the proof of Lemma 3.2 also shows that there exist compact universal differentiability sets of arbitrarily small cone width in the sense of [11, Definition 1.1].

### 3.2. \( C^1 \) curves meeting \( E \).

Recall that our ultimate goal is to construct a function \( h : \mathbb{R}^2 \to \mathbb{R} \) whose set of differentiability points inside of \( E \) intersect every \( C^1 \) curve in a set of measure zero. The objective of the present section is to investigate how \( C^1 \) curves intersect the whole set \( E \). The results that follow depend entirely on the geometry of the set \( E \) and in particular rely on the thinness of the strips \( B(L_k, \rho_k) \). They have nothing to do with the function with a small set of differentiability points that we will construct later on.

**Lemma 3.4.** For every \( C^1 \) curve \( \gamma : I \to \mathbb{R}^2 \) satisfying

\[
\gamma'(t) \notin \hat{C}(w,\eta) \quad \text{for all } t \in I,
\]

it holds that \( L(\gamma^{-1}(E)) = 0 \).

**Proof.** The function

\[
I \times C(w,\eta) \to \mathbb{R}, \quad (t,e) \mapsto |\langle \gamma'(t), e \rangle|,
\]

is continuous and defined on a compact set. Therefore, it attains its maximum, which must be greater than zero, at some pair \((t_0, e_0) \in I \times C(w,\eta)\). Since \( \gamma'(t_0) \notin \hat{C}(w,\eta) \) and \( e_0 \in C(w,\eta) \) we have

\[
0 < c_0 := |\langle \gamma'(t_0), e_0 \rangle| < 1.
\]

Setting \( \delta_0 := 1 - \sqrt{1 - c_0^2} \), we deduce that

\[
\gamma'(t) \in \hat{C}(e^\perp, \delta_0) \quad \text{for all } t \in I,
\]

for all \( e \in C(w,\eta) \) and in particular for all \( e = e_k, k \in \mathbb{N} \). Now for each \( k \in \mathbb{N} \), elementary geometric reasoning leads to

\[
\gamma^{-1}(B(L_k, \rho_k)) \leq \frac{2\rho_k}{1 - \delta_0}.
\]
More precisely, we obtain the above inequality by applying Lemma A.1 of Appendix A.2 with \( W = B(L_k, \rho_k) \), \( v = e_k^L \) and \( \delta = \delta_0 \). Since, for arbitrary \( N \in \mathbb{N} \) the set \( \bigcup_{k=N}^\infty B(L_k, \rho_k) \) covers \( E \), we have

\[
\mathcal{L}(\gamma^{-1}(E)) \leq \frac{2}{1 - \delta_0} \sum_{k=N}^\infty \rho_k \quad \text{for all} \ N \in \mathbb{N},
\]

and hence \( \mathcal{L}(\gamma^{-1}(E)) = 0 \).

The remaining results of the present section share a common hypothesis.

**Hypothesis 3.5.** Let \( \delta \in (0, 1) \) be sufficiently small, \( \gamma : I \to \mathbb{R}^2 \) be a \( C^1 \) curve and suppose that \( \gamma'(t) \in C_w(\delta) \quad \text{for all} \ t \in I \). For each \( p \geq 0 \) let \( \Sigma_p \) be the smallest \( \sigma \)-algebra on \( I \) with respect to which the functions \( k_q \circ \gamma, \quad q = 0, 1, 2, \ldots, p \) are measurable. Furthermore, we define for each \( p \geq 0 \) a mapping \( \beta_p : I \to \mathbb{R}^2 \) by

\[
\beta_p = E[\gamma' | \Sigma_p]
\]

and consider the corresponding sets

\[
D_p := \{ t \in I : k_p(\gamma(t)) < \infty, \quad \left| \langle \beta_p(t), e_k^L(\gamma(t)) \rangle \right| > 2^{-p} \}, \quad D := \bigcap_{n=1}^\infty \bigcup_{p=n}^\infty D_p.
\]

**Lemma 3.6** (Under Hypothesis 3.5). The set \( D \subseteq I \) has Lebesgue measure zero.

The proof of Lemma 3.6 is based on the following observation:

**Lemma 3.7** (Under Hypothesis 3.5). Let \( k \in \mathbb{N} \) and \( P \) be a component of

\[
B(L_k, \rho_k) \setminus \bigcup_{1 \leq j < k} \partial B(L_j, \rho_j)
\]

for which \( k_p(z) = k \) for all \( z \in P \). Then

\[
\int_{\gamma^{-1}(P)} \left| \langle \beta_p(t), e_k^L \rangle \right| \, dt \leq 12 \rho_k.
\]

**Proof.** Note that \( \langle w, e_k \rangle \geq 1 - \eta \geq \frac{1}{\sqrt{2}} \), where the final inequality is a condition on \( \eta \). Thus, for all \( t \in I \) we have

\[
\langle \gamma'(t), e_k \rangle \geq (1 - \delta) \frac{1}{\sqrt{2}} - \sqrt{\delta(2 - \delta)} \frac{1}{\sqrt{2}} \geq \frac{1}{2\sqrt{2}}
\]

for \( \delta \) sufficiently small. Hence, viewing \( \mathbb{R}^2 \) with the coordinate system \( (e_k, e_k^L) \), \( \gamma \) is a curve which moves strictly from left to right. Moreover, \( P \) is an open, convex set contained in the horizontal strip \( B(L_k, \rho_k) \) of width \( 2\rho_k \). These considerations imply a
Hence, \( \gamma P \) also constant on \( \int T \). To complete the proof we show that the quantity on the left hand side above is equal to \( \int_{\gamma^{-1}(P)} |\beta_p(t), e_k^+| \) dt. For any fixed \( z_0 \in P \) the set \( P \) has the precisely the form

\[
P = \bigcap_{q=0}^p \{ z \in \mathbb{R}^2 : k_q(z) = k_q(z_0) \}.
\]

Hence, \( \gamma^{-1}(P) \in \Sigma_p \), all functions \( k_q \circ \gamma, 0 \leq q \leq p \) are constant on \( \gamma^{-1}(P) \) and \( \beta_p \) is also constant on \( P \). It follows that

\[
\int_{\gamma^{-1}(P)} |\langle \beta_p(t), e_k^+ \rangle| \ dt = \int_{\gamma^{-1}(P)} |\langle \beta_p(t), e_k^+ \rangle| \ dt = \int_{\gamma^{-1}(P)} |\langle \gamma'(t), e_k^+ \rangle| \ dt.
\]

We are now ready to give the proof of Lemma 3.6:

**Proof of Lemma 3.6.** It suffices to prove that the sequence \((\mathcal{L}(D_p))_{p=1}^{\infty}\) is summable. The set \( D_p \) can be expressed as the union of all sets

\[
D_{p,k} := \{ t \in [0,1] : k_p(\gamma(t)) = k, \quad |\langle \beta_p(t), e_k^+ \rangle| > 2^{-p} \}
\]

for \( k \geq p \). We observe that

\[
D_{p,k} \subseteq \bigcup_{\gamma^{-1}(P)}
\]

where the union is taken over all components \( P \) of \( B(L_k, \rho_k) \setminus \bigcup_{j < k} B(L_j, \rho_k) \) for which \( k_p(z) = k \) for all \( z \in P \). Using the bound given by Lemma 3.7 and the fact that there are at most \( 3^k \) such components \( P \) we deduce

\[
\int_{D_{p,k}} |\langle \beta_p(t), e_k^+ \rangle| \ dt \leq 3^k \cdot 12 \rho_k.
\]

Summing this inequality over \( k \geq p \) we obtain

\[
\int_{D_p} |\langle \beta_p(t), e_{k_p(\gamma(t))}^+ \rangle| \ dt \leq 12 \sum_{k=p}^{\infty} 3^k \rho_k \leq 4^{-p} \tag{5}
\]

The last inequality is a condition that we impose on the sequence \((\rho_k)_{k=1}^{\infty}\). For the random variable \( X_p : [0,1] \to \mathbb{R} \) defined by

\[
X_p(t) = \langle \beta_p(t), e_{k_p(\gamma(t))}^+ \rangle \chi_{D_p}, \quad t \in [0,1],
\]

...
Applying Markov’s Inequality, we conclude

\[ \mathcal{L}(D_p) \leq \mathcal{L}(\{ t : |X_p(t)| > 2^{-p} \}) \leq \frac{E[|X_p|]}{2^{-p}} < \frac{4^{-p}}{2^{-p}} = 2^{-p}. \]

When studying $\gamma^{-1}(E)$ later on, Lemma 3.8 will allow us to discard the sets $D_p$. In the remaining set we have that $\beta_p(t)$ is very close to the direction $e_{kp}(\gamma(t))$ of the $p$-th strip containing $\gamma(t)$. The form of this approximation that we will require is recorded in the following lemma.

**Lemma 3.8 (Under Hypothesis 3.3).** Let $t \in \gamma^{-1}(E) \setminus D_p$. Then, writing $k_p$ for $k_p(\gamma(t))$,

\[
\frac{\langle w, e_{k_p} \rangle}{\langle w, e_{k_p} \rangle} - \frac{\langle \beta_p(t), w^{-1} \rangle}{\langle \beta_p(t), w \rangle} \leq \frac{2^{-p}}{(1 - \eta)(1 - \delta)}.
\]

**Proof.** We rewrite the considered expression as

\[
\left| \frac{\langle \beta_p(t), w \rangle \langle e_{k_p}, w^{-1} \rangle - \langle \beta_p(t), w^{-1} \rangle \langle e_{k_p}, w \rangle}{\langle w, e_{k_p} \rangle \langle \beta_p(t), w \rangle} \right|.
\]

The numerator above is precisely the determinant of the $2 \times 2$ matrix with columns $\beta_p(t)$ and $e_{k_p}$, which is given in absolute value by $\left| \langle \beta_p(t), e_{k_p} \rangle \right| \leq 2^{-p}$. The denominator is bounded below in absolute value by $(1 - \eta)(1 - \delta)$. }

**3.3. A function with small set of differentiability points inside $E$.**

Our aim is now to construct a Lipschitz function $h$ having only a very small set of differentiability points in $E$. The function $h$ will have the form

\[ h(z) = \sum_{k=1}^{\infty} 2^{-m_{k-1}(z)} \sigma_{k-1}(z) \varphi_k(z) = \lim_{k \to \infty} h_k(z) \]

where $(m_k)_{k=0}^{\infty}$, $(\sigma_k)_{k=0}^{\infty}$, $(\varphi_k)_{k=1}^{\infty}$ and $(h_k)_{k=1}^{\infty}$ are functions to be constructed.

**Definition and properties of $\varphi_k$.** The construction of the functions $\varphi_k$ will be intertwined with that of the lines $L_k$, widths $\rho_k$ and additional sequences of sets $T_k \subseteq \mathbb{R}^2$ and numbers $\delta_k > 0$. Thus, we prescribe here, that the sequences $(L_k)_{k=1}^{\infty}$ and $(\rho_k)_{k=1}^{\infty}$ introduced previously, are in fact constructed according to the following procedure. It is a trivial matter to adapt the procedure described below so that the sequence of lines $(L_k)_{k=1}^{\infty}$ it produces satisfies the existing conditions of Section 3.1, namely (1) and the hypothesis of Lemma 3.1. We spare the details of this.

The construction begins by setting $T_0 = \emptyset$. Now for $k \geq 1$ and $T_{k-1}$ already defined as a finite union of lines, we choose the line $L_k \subseteq \mathbb{R}^2$ so that the set $S_k := L_k \cap T_{k-1}$ is
finite. The number $\delta_k > 0$ is then chosen small compared to the cardinality of $S_k$ and then $\rho_k > 0$ is chosen sufficiently small compared to $\delta_k$; these conditions will be made precise later. We let $\tilde{\varphi}_k : \mathbb{R}^2 \to \mathbb{R}$ be the function uniquely determined by the following conditions:

(A) $\tilde{\varphi}_k$ is constant along all lines parallel to $e_k$.

(B) Along each line parallel to $w^\perp$ the function $\tilde{\varphi}_n$ is constant 0 in the lower component (with respect to the direction $w^\perp$) of $\mathbb{R}^2 \setminus B(L_k, \rho_k)$, grows with slope 1 inside the strip $B(L_k, \rho_k)$ and is constant $\frac{2\rho_k}{\langle w, e_k \rangle}$ on the upper component of $\mathbb{R}^2 \setminus B(L_k, \rho_k)$.

Next we define $\varphi_k : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi_k(z) = \min \left\{ \tilde{\varphi}_k(z), 2^{-k} \text{dist}(z, T_{k-1}) \right\}$$

and define $T_k$ as the minimal union of lines in $\mathbb{R}^2$ which contains $T_{k-1} \cup \partial B(L_k, \rho_k)$ and for which $\varphi_k$ is affine on each component of $\mathbb{R}^2 \setminus T_k$. This completes the construction.

The next lemma records the important properties of the functions $(\varphi_k)_{k=1}^\infty$:

**Lemma 3.9.** For each $k \in \mathbb{N}$ the function $\varphi_k : \mathbb{R}^2 \to \mathbb{R}$ has the following properties:

(a) $\varphi_k$ is affine on each component of $\mathbb{R}^2 \setminus T_k$.

(b) $\|\varphi_k\|_\infty \leq \|\tilde{\varphi}_k\|_\infty \leq \frac{2\rho_k}{\langle w, e_k \rangle} \leq \frac{2\rho_k}{1-\eta}$.

(c) For each point $z \in B(L_k, \rho_k) \setminus B(S_k, \delta_k)$ we have

(i) $B(z, \frac{5\rho_k}{\sqrt{\eta}}) \cap T_{k-1} = \emptyset$,

(ii) $\varphi_k = \tilde{\varphi}_k$ on $B(z, \frac{5\rho_k}{\sqrt{\eta}})$, and

(iii) $D\varphi_k(z) = w^\perp + \frac{\langle w, e^+_k \rangle}{\langle w, e_k \rangle} \cdot w$.

(d) For each point $z \in \mathbb{R}^2 \setminus B(L_k, \rho_k)$ at which the derivative of $\varphi_k$ exists we have

$$\|D\varphi_k(z)\| \leq 2^{-k}.$$

(e) $\|D\varphi_k\|_\infty \leq \sqrt{1 + \left(\frac{1}{1-\eta}\right)^2}$.

(f) For each point $z \in B(L_k, \rho_k) \setminus B(S_k, \delta_k)$ and each direction $v \in S^1 \setminus \hat{C}(w, 3\sqrt{\eta})$ there exist a point $u \in \mathbb{R}^2$, and numbers $\rho_k \leq t_1 \leq t_2 \leq \frac{2\rho_k}{\sqrt{\eta}}$ such that $z \in [u, u + t_1 v]$ and

$$\left| \frac{\varphi_k(u + t_1 v) - \varphi_k(u)}{t_1} - \frac{\varphi_k(u + t_2 v) - \varphi_k(u)}{t_2} \right| \geq \frac{\sqrt{\eta}}{2}.$$
Proof. Properties (a) and (b) are immediate from the construction. For (c) we need to impose a condition on $\rho_k$ relative to $\delta_k$. Since $T_{k-1}$ is a finite union of lines and $L_k \setminus B(S_k, \delta_k)$ is a finite union of closed line segments and half-rays not intersecting $T_{k-1}$ the quantity  
\[ c_k := \inf \{ \text{dist}(x, y) : x \in T_{k-1}, y \in L_k \setminus B(S_k, \delta_k) \} \]

is positive. Let $\theta_k > 0$ be a small parameter to be determined later and let $0 < \rho_k < \theta_k c_k$. Then for all $z \in B(L_k, \rho_k) \setminus B(S_k, \delta_k)$ we have  
\[ \text{dist}(z, T_{k-1}) \geq c_k - \rho_k < \frac{1 - \theta_k}{\theta_k} \cdot \rho_k > \frac{5\rho_k}{\sqrt{\eta}}, \]

where the last inequality is a condition we impose on $\theta_k$. This proves (ci). Given $z \in B(L_k, \rho_k) \setminus B(S_k, \delta_k)$ and $z' \in B(z, \frac{5\rho_k}{\sqrt{\eta}})$ we have  
\[ \text{dist}(z', T_{k-1}) \geq \text{dist}(z, T_{k-1}) - \frac{5\rho_k}{\sqrt{\eta}} \geq \left( 1 - \frac{\theta_k}{\theta_k} - \frac{5}{\sqrt{\eta}} \right) \rho_k > \frac{2^{k+1} \rho_k}{1 - \eta} \geq 2^k \| \varphi_k \|_\infty \]

where the penultimate inequality is a further condition on $\theta_k$. We deduce that $\varphi_k(z') < 2^{-k} \text{dist}(z', T_{k-1})$. Hence, $\varphi_k(z') = \hat{\varphi}_k(z')$. This proves (ciii), after which (civ) derives easily from the defining properties (A) and (B) of $\hat{\varphi}_k$.

For (d) and (e) we observe that the plane $\mathbb{R}^2$ may be decomposed as a union of finitely many (possibly unbounded) polygons, that is finite intersections of half-spaces, on each of which $\varphi_k$ is affine and either $\varphi_k = \hat{\varphi}_k$ or $\varphi_k = 2^{-k} \text{dist}(\cdot, T_{k-1})$. The inequalities of (d) and (e) are readily verified for both cases.

Finally we verify (f): Given $z \in B(L_k, \rho_k) \setminus B(S_k, \delta_k)$ and $v \in S^1 \setminus \hat{C}(w, 3\sqrt{\eta})$ we choose $u \in L_k$ and so that $z \in u + \mathbb{R}v$. We assume, without loss of generality that $z \in u + [0, \infty)v$ and let $t_1$ and $t_2$ be defined by the conditions  
\[ u + t_j v \in \partial B(L_k, j\rho_k), \quad j = 1, 2. \]

Clearly $t_1 \geq \rho_k$, $t_2 = 2t_1$ and $z \in [u, u + t_1 v]$. From elementary geometric considerations and the conditions $e_k \in C(w, \eta)$ and $v \in S^{d-1} \setminus \hat{C}(w, 3\sqrt{\eta})$ we derive  
\[ \left| \langle v, e_k \rangle \right| = \frac{\rho_k}{t_1}, \quad \left| \langle v, e_k \rangle \right| < (1 - 3\sqrt{\eta}) + \sqrt{\eta(2 - \eta)} < 1 - \sqrt{\eta}. \]

Together with the identity $|\langle v, e_k \rangle|^2 + |\langle v, e_k \rangle|^2 = 1$, this leads to  
\[ t_1 < \frac{\rho_k}{(2\sqrt{\eta} - \eta)^{1/2}} \leq \frac{\rho_k}{\sqrt{\eta}}. \]

Now, from the definition of $\hat{\varphi}_k$ it is clear that  
\[ |\hat{\varphi}_k(u + t_1 v) - \hat{\varphi}_k(u)| \geq \rho_k, \quad \text{and} \quad \hat{\varphi}_k(u + 2t_1 v) = \hat{\varphi}_k(u + t_1 v). \]
Moreover, we note that \([u, u + 2t_1v] \subseteq B(z, \frac{5\rho_k}{\sqrt{n}})\). Therefore, using part (iii) we have that 
\(\varphi_k = \tilde{\varphi}_k\) on \([u, u + 2t_1v]\). We deduce

\[
\left| \frac{\varphi_k(u + t_1v) - \varphi_k(u)}{t_1} - \frac{\varphi_k(u + 2t_1v) - \varphi_k(u)}{2t_1} \right| = \left| \frac{\tilde{\varphi}_k(u + t_1v) - \tilde{\varphi}_k(u)}{t_1} - \frac{\tilde{\varphi}_k(u + 2t_1v) - \tilde{\varphi}_k(u)}{2t_1} \right| \geq \frac{\rho_k}{2t_1} \geq \frac{\sqrt{n}}{2}.
\]

**Definition and properties of \(\sigma_k\).** For each \(k \in \mathbb{N}\) we define \(\sigma_k : \mathbb{R}^2 \to \mathbb{R}\) by

\[
\sigma_k(z) = (-1)^p
\]
where \(p \in \mathbb{N}\) is the unique integer satisfying \(k_{p-1}(z) \leq k < k_p(z)\).

**Lemma 3.10.** For each \(k \in \mathbb{N}\), \(\sigma_k\) is constant on each component of \(\mathbb{R}^2 \setminus \bigcup_{j=1}^{k} \partial B(L_j, \rho_k)\).

**Proof.** It is clear that \(\sigma_0 \equiv -1\). Let \(k \geq 1\) and suppose that \(\sigma_{k-1}\) is constant on each component of \(\mathbb{R}^2 \setminus \bigcup_{j=1}^{k-1} \partial B(L_j, \rho_j)\). Given \(z \in \mathbb{R}^2\), let \(p \in \mathbb{N}\) be the unique integer with

\[
k_{p-1}(z) = k - 1 < k_p(z),
\]

determining that \(\sigma_{k-1}(z) = (-1)^p\). The inequalities above express that the point \(z\) belongs to precisely \(p-1\) strips \(B(L_j, \rho_j)\) with index \(j \in \{1, \ldots, k-1\}\). Hence, \(k_p(z) = k\) if \(z \in B(L_k, \rho_k)\) and \(k_p(z) > k\) otherwise. From this consideration it follows that

\[
\sigma_k(z) = \begin{cases} 
(-1)^{p+1} & \text{if } z \in B(L_k, \rho_k), \\
(-1)^p = \sigma_{k-1}(z) & \text{otherwise}.
\end{cases}
\]

This completes the induction step, proving the lemma. 

**Definition and properties of \(m_k\) and \(h_k\).** The functions \(m_k\) and \(h_k\) are defined for each \(k \in \mathbb{N}\) inductively as follows. Set \(m_0 = h_0 = 0\) on the whole of \(\mathbb{R}^2\). If \(k \geq 1\) and the functions \(m_{k-1}\) and \(h_{k-1}\) are already defined, we let

\[
h_k(z) = h_{k-1}(z) + 2^{-m_{k-1}(z)} \sigma_{k-1}(z) \varphi_k(z), \quad z \in \mathbb{R}^2.
\]

Finally, whenever \(h_0, \ldots, h_k\) and \(m_0, \ldots, m_k\) are already defined we let

\[
j_k(z) := \max \{ j < k : m_j(z) \neq m_{j-1}(z) \},
\]

where we interpret the maximum as zero if the set considered is empty. For \(z \in \mathbb{R}^2\) let

\[
m_k(z) = \begin{cases} m_{k-1}(z) + 1 & \text{if } z \in \mathbb{R}^2 \setminus T_k \text{ and } \| Dh_k(z) - Dh_{j_k(z)}(z) \| > \varepsilon(m_{j_k(z)}(z)), \\
m_{k-1}(z) & \text{otherwise},
\end{cases}
\]

where \((\varepsilon(n))_{n=0}^{\infty}\) is a sequence of positive real numbers, the necessary conditions on which we specify at the relevant point later on.

We summarise the important properties of the functions \(h_k\) and \(m_k\):
Lemma 3.11. (a) For each $k$ and on each component of $\mathbb{R}^2 \setminus T_k$ we have that $h_k$ is affine and $m_k$ is constant.

(b) For all $k$ the function $m_k$ is lower semi-continuous.

(c) For all $l \geq k$ and all $z \in \mathbb{R}^2 \setminus T_l$ we have

$$
\| Dh_l(z) - Dh_k(z) \| \leq K(\eta) \sum_{j=m_k(z)}^{\infty} 2^{-j} + \varepsilon(j),
$$

where $K(\eta)$ denotes a constant depending only on $\eta$.

(d) For all $l \geq k$ and all $z \in \mathbb{R}^2 \setminus T_l$ we have

$$
\| Dh_l(z) - Dh_k(z) \| \leq 2^{-m_k} + \left\| \sum_{\{s: k < k_s \leq l\}} 2^{-m_{k_s-1}}\sigma_{k_s-1}D\varphi_{k_s} \right\|.
$$

Proof. The statement (a) is trivially valid for $m_0 \equiv h_0 \equiv 0$. Assume now that (a) holds for the objects $T_j$, $m_j$ and $h_j$ for all $j < k$. Then by Lemma 3.10 and the construction of the sets $T_j$ we deduce that $h_k$ is affine on each component of $\mathbb{R}^2 \setminus T_k$. In other words, $Dh_k$ is constant on each component of $\mathbb{R}^2 \setminus T_k$. Moreover, we observe that the function $j_k$ is constant on each component of $\mathbb{R}^2 \setminus T_k$. Referring to the definition of $m_k$ above, we conclude that the set of points where $m_k \neq m_{k-1}$ (meaning $m_k = m_{k-1} + 1$), is a union of components of $\mathbb{R}^2 \setminus T_k$. Applying the induction hypothesis, the proof of (a) complete.

A simple induction argument based on (a) also verifies part (b).

We turn our attention to part (c). Let $l \geq k$ and $z \in \mathbb{R}^2 \setminus T_l$. Then both derivatives $Dh_l(z)$ and $Dh_k(z)$ exist. In what follows we use the fact that all functions $\sigma_t$, $m_t$, $j_t$ and $Dh_l$ with index $t \leq l$ are constant on the component of $\mathbb{R}^2 \setminus T_l$ containing $z$. Since we are only concerned with a neighbourhood of $z$, we will sometimes omit the argument of such functions. We also allow the constant $K(\eta)$ to change in each occurrence. Let $(r_n)_{n \geq 0}$ be the finite sequence of minimal indices $r_n$ satisfying $m_{r_n}(z) = n$ and $r_n \leq l$. By the definition of the functions $j_k(z)$ we have

$$
j_k(z) = r_{n-1} \quad r_{n-1} < k \leq r_n, \quad n \geq 1.
$$

Now, combining Lemma 3.9 (c) and the rule (6) governing the growth of the sequence $(m_k(z))_{k=0}^{\infty}$, we deduce

$$
\| Dh_{r_n} - Dhr_{n-1} \| \leq \left\| 2^{-(n-1)}\sigma_{r_n-1}D\varphi_{r_n} \right\| + \left\| Dhr_{n-1} - Dhr_{n-1} \right\| \leq K(\eta)2^{-(n-1)} + \varepsilon(n-1),
$$

for all $n \geq 1$. Choose $s$ and $t$ maximal with $r_s \leq l$ and $r_t \leq k$. Then $m_l(z) = m_{r_s}(z) = s$ and $m_k(z) = m_{r_t}(z) = t$. Moreover, from (b) and the bound above we get

$$
\| Dh_l(z) - Dh_k(z) \| \leq \| Dh_l - Dh_r \| + \| Dhr_s - Dhr_t \| + \| Dhr_t - Dh_k \|
\leq \varepsilon(s) + K(\eta)\sum_{j=t}^{s-1}(2^{-j} + \varepsilon(j)) + \varepsilon(t) \leq K(\eta)\sum_{j=m_k(z)}^{\infty} 2^{-j} + \varepsilon(j).
$$
This proves (c). For (d) we note that
\[
\|Dh_l(z) - Dh_k(z)\| = \left\| \sum_{j=k+1}^l 2^{-m_{j-1}} \sigma_{j-1} D\varphi_j(z) \right\|. 
\]

The above sum can be split into two parts: firstly the sum over those indices \( j \) for which \( z \not\in B(L_j, \rho_j) \) and secondly, the sum over those indices \( j \) of the form \( k_\alpha(z) \). The inequality of (d) is obtained simply by leaving the second sum unchanged and bounding the first sum by \( 2^{-m_k} \) using Lemma 3.9, part (d).

**Properties of \( h \).** We bring together all the components and derive the important properties of the function
\[
h(z) = \sum_{k=1}^\infty 2^{-m_{k-1}(z)} \sigma_{k-1}(z) \varphi_k(z) = \lim_{k \to \infty} h_k(z), \quad z \in \mathbb{R}^2.
\]

**Lemma 3.12.** The function \( h \) is well-defined and Lipschitz.

**Proof.** We will first verify that \( h \) is well-defined and continuous. Lemmas 3.10 and 3.11 part (a), the continuity of \( \varphi_k \) and the fact that \( \varphi_k = 0 \) on each line in the family \( T_{k-1} \) ensure that each summand \( 2^{-m_{k-1}} \sigma_{k-1} \varphi_k \) is continuous. Using \( \|\varphi_k\|_\infty \leq \frac{2\rho_k}{\varepsilon_k} \) (Lemma 3.9, part (b)), the sequence of partial sums \( h_k \) is easily seen to converge uniformly to \( h \) and so \( h \) is well-defined and continuous as well. To show that \( h \) is Lipschitz, it suffices to show that the functions \( h_k \) are Lipschitz with uniformly bounded Lipschitz constants. Since the functions \( h_k \) are continuous and piecewise affine, it suffices to verify that their derivatives \( Dh_k \) are uniformly bounded. This is implied by Lemma 3.11, part (b), when we prescribe that the sequence \( (\varepsilon(n))_{n=0}^\infty \) is summable.

**Sets \( G \), \( H \) and \( F_m \).** We now introduce two sets \( G, H \) which will be shown to cover the set of points inside \( E \) where \( h \) has a directional derivative in any direction outside of a small double sided cone. We let
\[
G := \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty B(S_k, \delta_k), \quad H := \left\{ z \in E : \lim_{k \to \infty} m_k(z) = \infty \right\}.
\]

Note that the complement of \( G \cup H \) inside of \( E \) may be covered by the sets
\[
F_m := \left\{ z \in \mathbb{R}^2 : \lim_{k \to \infty} m_k(z) \leq m \right\} \setminus G, \quad m \in \mathbb{N}.
\]

The topological properties of the sets \( G \), \( H \) and \( F_m \) will be important later on. We note that \( G \) and \( H \) are both \( G_\delta \) sets. For \( G \) this is clear; for \( H \) it follows easily from the fact that \( E \) is \( G_\delta \) and Lemma 3.11, (b). Using Lemma 3.11, (b) again, we deduce that each set \( F_m \) is \( F_\sigma \).

In the next lemma we show that \( h \) is nowhere differentiable inside each set \( E \cap F_m \). Moreover, we obtain a uniform bound on the degree of non-differentiability.
Lemma 3.13. Let \( m \in \mathbb{N} \), \( z \in E \cap F_m \) and \( v \in S^1 \setminus \hat{C}(w, 3\sqrt{\eta}) \). Then
\[
\limsup_{\varepsilon \to 0} \zeta(h, z, \varepsilon, v) \geq \frac{2^{-m} \sqrt{\eta}}{4}.
\]
Hence, in the set \( E \setminus (G \cup H) = E \cap \bigcup_{m=1}^{\infty} F_m \) we have that \( h \) is nowhere differentiable and has no directional derivatives in any direction from \( \hat{C}(w, 3\sqrt{\eta}) \).

Proof. Fixing \( \varepsilon > 0 \), we need to find two line segments passing through \( z \), parallel to \( v \) and of length at most \( \varepsilon \) on which \( h \) has slopes differing by at least \( 2^{-m} \sqrt{\eta}/4 \). Since \( z \in E \), the numbers \( k_p := k_p(z) \) are finite and \( z \in B(L_k, \rho_k) \) for all \( p \in \mathbb{N} \). Since \( z \notin G \), we may choose \( p \) sufficiently large so that for \( k := k_p \) we have \( z \notin B(S_k, \delta_k) \). We additionally choose \( p \) sufficiently large so that \( \frac{2\rho_p}{\sqrt{\eta}} < \varepsilon \) and
\[
m_{k-1}(z) = \max_{j \in \mathbb{N}} m_j =: \hat{m} \leq m.
\]
By Lemma 3.9, Lemma 3.10 and Lemma 3.11, the functions \( \sigma_{k-1} \) and \( m_{k-1} \) are constant on \( B \left( z, \frac{5\rho_p}{\sqrt{\eta}} \right) \). Therefore, for all \( y \in B \left( z, \frac{5\rho_p}{\sqrt{\eta}} \right) \) we have
\[
h(y) = h_{k-1}(y) + 2^{-\hat{m}} \sigma_{k-1}(z) \varphi_k(y) + \sum_{j=k+1}^{\infty} 2^{-\hat{m}} \sigma_{j-1}(y) \varphi_j(y).
\]
Moreover, by Lemma 3.11, the function \( h_{k-1} \) is affine on \( B \left( z, \frac{5\rho_p}{\sqrt{\eta}} \right) \).

Let \( u, t_1, t_2 \) be given by the conclusion of Lemma 3.9 part (f) for \( \varphi_k, z \) and \( v \). Then the segments \([u, u + t_1 v]\) and \([u, u + t_2 v]\) both contain \( z \), have length at most \( t_2 \leq \frac{2\rho_p}{\sqrt{\eta}} < \varepsilon \) and are therefore contained in \( B \left( z, \frac{5\rho_p}{\sqrt{\eta}} \right) \). Hence \( h_{k-1} \) restricted to \([u, u + t_2 v]\) is affine and
\[
\left| \frac{h_{k-1}(u + t_1 v) - h_{k-1}(u)}{t_1} - \frac{h_{k-1}(u + t_2 v) - h_{k-1}(u)}{t_2} \right| = 0.
\]
The corresponding difference for the tail sum in (7) may be bounded above using \( |\sigma_j| = 1 \), \( \| \varphi_j \|_{\infty} \leq \frac{2\rho_p}{1-\eta} \) and \( t_1, t_2 \geq \rho_k \), leading to
\[
\left| \frac{(h - h_k)(u + t_1 v) - (h - h_k)(u)}{t_1} - \frac{(h - h_k)(u + t_2 v) - (h - h_k)(u)}{t_2} \right| \leq \frac{4}{\rho_k} \sum_{j=k+1}^{\infty} 2^{-\hat{m}} \cdot \frac{2\rho_k}{1-\eta} \leq \frac{2^{-\hat{m}} \sqrt{\eta}}{16},
\]
where the last inequality imposes a condition on the sequence \( (\rho_j)_{j=1}^{\infty} \). Now combining the two bounds above with that of Lemma 3.9 part (f) we obtain
\[
\left| \frac{h(u + t_1 v) - h(u)}{t_1} - \frac{h(u + t_2 v) - h(u)}{t_2} \right| \geq \frac{2^{-\hat{m}} \sqrt{\eta}}{2} - \frac{2^{-\hat{m}} \sqrt{\eta}}{16} \geq \frac{2^{-m} \sqrt{\eta}}{4},
\]
which completes the proof. \( \square \)
We now prove that \( h \) is differentiable everywhere in the set \( H \setminus G \).

**Lemma 3.14.** Let \( z \in H \setminus G \). Then \( h \) is differentiable at \( z \).

**Proof.** Since \( z \in H \setminus G \subseteq E \setminus G \) we have \( k_p := k_p(z) < \infty \) for all \( p \in \mathbb{N} \) and that \( z \in B(L_{k_p}, \rho_{k_p}) \setminus B(S_{k_p}, \delta_{k_p}) \) for all sufficiently large \( p \in \mathbb{N} \). By Lemma 3.31 part (c) and Lemma 3.11 part (c) there is, for each such \( p \), a neighbourhood \( B_p := B(z, \frac{5p_{k_p}}{\sqrt{\eta}}) \) of \( z \) on which the function \( h_{k_p-1} \) is affine. In particular each function \( h_{k_p-1} \) is differentiable at \( z \). Set \( g_p = h_{k_p-1} \).

By Lemma 3.11 (c) we have that \( m_{k_p-1} \) is constant on the set \( B_p \). Hence, from the inequality of Lemma 3.11 part (c), we may derive

\[
\text{Lip}((g_q - g_p)|_{B_p}) \leq K(\eta) \sum_{j=m_{k_p-1}(z)}^{\infty} 2^{-j} + \varepsilon(j)
\]

for \( q \geq p \). Since this bound is independent of \( q \geq p \) and the functions \( g_q \) converge uniformly to \( h \) as \( q \to \infty \), we obtain

\[
\text{Lip}((h - g_p)|_{B_p}) \leq K(\eta) \sum_{j=m_{k_p-1}(z)}^{\infty} 2^{-j} + \varepsilon(j).
\]

As \( p \to \infty \) the lower index \( m_{k_p-1}(z) \) in the sums above tends to \( \infty \), because \( z \in H \). We conclude that

\[
\lim_{p \to \infty} \sup_{q \geq p} \text{Lip}((g_q - g_p)|_{B_p}) = \lim_{p \to \infty} \text{Lip}((h - g_p)|_{B_p}) = 0.
\]

Moreover, for any \( q \geq p \) we have \( \|Dg_q(z) - Dg_p(z)\| \leq \text{Lip}((g_q - g_p)|_{B_p}) \). Therefore, the sequence \( (Dg_p(z)) \) is a Cauchy sequence. Let \( L \in \mathbb{R}^2 \) denote its limit.

We are now ready to verify the differentiability of \( h \) at \( z \) with \( Dh(z) = L \). Let \( \varepsilon \in S^1 \) and \( \varepsilon > 0 \). Now choose \( p \) large enough so that \( \text{Lip}((h - g_p)|_{B_p}) < \varepsilon/3 \) and \( \|Dg_p(z) - L\| \leq \varepsilon/3 \). Choose \( \delta_p > 0 \) small enough so that \( B(z, \delta_p) \subseteq B_p \). In particular, this ensures that \( g_p \) is affine on the ball of radius \( \delta_p \) around \( z \). Now, for all \( t \in (-\delta_p, \delta_p) \) we have

\[
|h(z + te) - h(z) - tL(e)| \leq |(h - g_p)(z + te) - (h - g_p)(z)| + |g_p(z + te) - g_p(z) - tDg_p(z)(e)| + |tDg_p(z)(e) - tL(e)|
\]

\[
\leq \frac{\varepsilon}{3} |t| + 0 + \frac{\varepsilon}{3} |t| < \varepsilon |t|.
\]

\[\Box\]

### 3.4. Pure Unrectifiability

To complete the proof of Theorem 3.2 we show that the set \( G \cup H \) is purely unrectifiable.
**Lemma 3.15.** Let $\gamma : I \to \mathbb{R}^2$ be a $C^1$ curve, $v \in S^{d-1}$ and $\theta \in (0, 1)$ such that

$$\gamma'(t) \in C(v, \theta) \quad \text{for all } t \in I.$$  

Then $L(\gamma^{-1}(G)) = 0$. Moreover, for any one dimensional subspace $U \subseteq \mathbb{R}^2$ the projection $\pi_U(G)$ has 1-dimensional Lebesgue measure zero.

**Proof.** Fix $\varepsilon > 0$. Imposing the condition

$$\sum_{k=1}^{\infty} |S_k| \delta_k < \infty$$

on the sequence $(\delta_k)_{k=1}^{\infty}$, we can choose $n \in \mathbb{N}$ sufficiently large so that

$$\sum_{k=n}^{\infty} |S_k| \delta_k < \frac{(1 - \theta)\varepsilon}{2}.$$  

Then for each $k \geq n$ and each point $x \in S_k$, we may apply Lemma A.1 with $W = B(x, \delta_k)$ to get that

$$L(\gamma^{-1}(B(x, \delta_k))) \leq \frac{2}{1 - \theta} \sum_{k=n}^{\infty} |S_k| \delta_k < \varepsilon.$$  

For the ‘moreover’ part, it suffices to observe that for any $n \geq 1$ the sum $\sum_{k=n}^{\infty} |S_k| \cdot 2\delta_k$ is an upper bound on the one-dimensional Lebesgue measure of any projection $\pi_U(G)$. $\square$

Lemma 3.15 clearly implies that the set $G$ is purely unrectifiable. Thus, we are left needing to prove the pure unrectifiability of $H \setminus G$. For a given $C^1$ curve $\gamma : I \to \mathbb{R}^2$ with some mild restrictions we will show that the set of points $t \in I$ for which $\gamma(t) \in H \setminus G$ may be modelled by the set of points at which some martingale (see Definition 3.17) associated to $\gamma$ becomes large. We then appeal to martingale theory to argue that such a set is small in measure. The quantity considered in the next lemma for points $z = \gamma(t) \in E$ will be well approximated, as a consequence of Lemma 3.8, by the aforementioned martingale.

**Lemma 3.16.** Let $z \in H \setminus G$. Then, writing $k_s := k_s(z)$, we have

$$\sup_{q \in \mathbb{N}} \left| \sum_{s=0}^{2p-1} (-1)^s \frac{\langle w, e_{k_s}^\perp \rangle}{\langle w, e_{k_s} \rangle} \right| = \infty.$$  

Let us explain informally the idea behind the present lemma. Since $z \notin G$ all derivatives $D\varphi_{k_r}$ with $r$ sufficiently large have the form of Lemma 3.9, part (c),. Hence they all have component 1 in the $w^\perp$ direction. In the summands $2^{-m_{k_s}} \sigma_{k_s-1} \varphi_{k_s}$ of $h$, the alternating factor $\sigma_{k_s-1} = (-1)^s$ ensures that the sum of these derivative components in the $w^\perp$ direction is alternating and therefore cannot get large. On the other hand, $z$ being in $H$ requires that $m_k(z)$ to grows to infinity. The growth of $m_k(z)$ is induced by growth of derivative of the partial sums $h_k$. With the derivative of these sums in the $w^\perp$
direction staying small, we conclude that their derivative in the $w$ direction must become large and so we derive a lower bound on the sum of the derivative components in the $w$ direction, i.e. the quantity $\sum (-1)^s \frac{\langle w, e_{k_s}^+ \rangle}{\langle w, e_{k_s} \rangle}$.

We now present this argument formally.

**Proof of Lemma 3.16.** It is sufficient to prove

$$\sup_{p<q} \left| \sum_{p<s \leq q} (-1)^s \frac{\langle w, e_{k_s}^+ \rangle}{\langle w, e_{k_s} \rangle} \right| = \infty. \quad (8)$$

Let $(r_n)_{n=1}^{\infty}$ be the sequence of minimal indices $r_n = r_n(z)$ with $m_{r_n}(z) = n$. The rule (6) governing the growth of $m_k(z)$ implies that all derivatives $Dh_{r_n}(z)$ exist and

$$\|Dh_{r_{n+1}}(z) - Dh_{r_n}(z)\| > \varepsilon(m_{r_n}(z)) = \varepsilon(n)$$

for all $n$. Since $z \in E \setminus G$ we have that $z \in B(L_{k_s}, \rho_{k_s}) \setminus B(S_{k_s}, \rho_{k_s})$ for all sufficiently large $k_s$. Hence for all sufficiently large $k_s$ we have an expression for the derivative $D\varphi_{k_s}(z)$ given by Lemma 3.9 part (iii). This allows for refinement of the inequality of Lemma 3.11 part (i). For all sufficiently large $n \in \mathbb{N}$, we get namely

$$\|Dh_{r_{n+1}}(z) - Dh_{r_n}(z)\|$$

$$\leq 2^{-n} + 2^{-n} \left| \sum_{\{s: r_n < k_s \leq r_{n+1}\}} (-1)^s \frac{\langle w, e_{k_s}^+ \rangle}{\langle w, e_{k_s} \rangle} \right| + 2^{-n} \sum_{\{s: r_n < k_s \leq r_{n+1}\}} (-1)^s \frac{\langle w, e_{k_s}^+ \rangle}{\langle w, e_{k_s} \rangle}$$

$$\leq 2^{-n} \left( 2 + \sum_{\{s: r_n < k_s \leq r_{n+1}\}} (-1)^s \frac{\langle w, e_{k_s}^+ \rangle}{\langle w, e_{k_s} \rangle} \right).$$

Combining the upper and lower bounds on $\|Dh_{r_{n+1}}(z) - Dh_{r_n}(z)\|$ derived above, we deduce

$$\left| \sum_{\{s: r_n < k_s \leq r_{n+1}\}} (-1)^s \frac{\langle w, e_{k_s}^+ \rangle}{\langle w, e_{k_s} \rangle} \right| \geq 2^n \varepsilon(n) - 2.$$

Up until now we have only required the sequence $(\varepsilon(n))_{n=1}^{\infty}$ to be summable. Therefore, we may now prescribe that $\varepsilon(n) = \frac{1}{n}$. The latter expression above is then unbounded for $n \in \mathbb{N}$ and provides a lower bound for the supremum in (8).

We recall the definition of a martingale; see for example [19] p. 94].

**Definition 3.17.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration on $(\Omega, \mathcal{F})$. A sequence $(X_n)_{n=0}^{\infty}$ of measurable functions $X_n : \Omega \to \mathbb{R}$ is called a martingale with respect to $(\mathcal{F}_n)_{n=0}^{\infty}$ and $\mu$ if it satisfies the following conditions:

(i) $X_n \in L^1(\Omega, \mathcal{F}_n, \mu)$ for each $n$. In particular, $X_n$ is $\mathcal{F}_n$-measurable for each $n$.  

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(ii) \( \mathbb{E}[X_{n+1}|F_n] = X_n \) for each \( n \).

If, in (ii), the equality is weakened to the inequality \( \geq \) then we call \( (X_n)_{n=0}^{\infty} \) a submartingale with respect to \((F_n)_{n=0}^{\infty}\) and \( \mu \).

**Proposition 3.18.** Let \( v \in S^1 \), \( c > 0 \) and \( \gamma: I \to \mathbb{R}^2 \) be a \( C^1 \) curve with

\[
\langle \gamma'(t), v \rangle \geq c, \quad \text{for all } t \in I.
\]

Let \( (\Sigma_p)_{p=0}^{\infty} \) be a filtration on \( I \) and \( \beta_p := \mathbb{E}[\gamma'|\Sigma_p] \) for each \( p \geq 0 \). Then the sequence of functions

\[
I \to \mathbb{R}, \quad t \mapsto \frac{\langle \beta_p(t), v^+ \rangle}{\langle \beta_p(t), v \rangle}, \quad p \geq 0,
\]

is a martingale with respect to the filtration \( (\Sigma_p)_{p=0}^{\infty} \) and probability measure

\[
\mu^v(A) := \frac{\int_A \langle \gamma', v \rangle \, d\mathcal{L}}{\int_I \langle \gamma', v \rangle \, d\mathcal{L}} \quad A \subseteq I.
\]

Moreover

\[
\left\| \frac{\langle \beta_p, v^+ \rangle}{\langle \beta_p, v \rangle} \right\|_{L^2(\mu^v)} \leq \frac{K(\gamma)}{c} \quad \text{for all } p \geq 0,
\]

where \( K(\gamma) \) is a constant depending only on \( \gamma \).

**Proof.** In what follows we will assume \( \int_I \langle \gamma', v \rangle \, d\mathcal{L} = 1 \), which simplifies the expression for the measure \( \mu^v \). Accordingly all computations are correct up to multiplication by a fixed constant \( K(\gamma) \) depending only on \( \gamma \). From elementary properties of the conditional expectation we get that (9) implies

\[
\langle \beta_p(t), v \rangle \geq c \quad \text{for all } t \in I.
\]

Hence the mappings \( \frac{\langle \beta_p, v^+ \rangle}{\langle \beta_p, v \rangle} \) are bounded, which trivially implies \( \frac{\langle \beta_p, v^+ \rangle}{\langle \beta_p, v \rangle} \in L^1(I, \Sigma_p, \mu^v) \) for every \( p \geq 0 \). Hence property (iii) of Definition 3.17 is satisfied. We turn now to property (ii). Given \( A \in \Sigma_p \) we have

\[
\int_A \frac{\langle \beta_{p+1}, v^+ \rangle}{\langle \beta_{p+1}, v \rangle} \, d\mu^v = \int_A \frac{\langle \beta_{p+1}, v^+ \rangle}{\langle \beta_{p+1}, v \rangle} \cdot \langle \gamma', v \rangle \, d\mathcal{L} = \int_A \mathbb{E} \left[ \frac{\langle \beta_{p+1}, v^+ \rangle}{\langle \beta_{p+1}, v \rangle} \cdot \langle \gamma', v \rangle | \Sigma_{p+1} \right] \, d\mathcal{L}.
\]

Now we use a standard property of the conditional expectation (see [13] 22.(i), p. 54) to deduce

\[
\mathbb{E} \left[ \frac{\langle \beta_{p+1}, v^+ \rangle}{\langle \beta_{p+1}, v \rangle} \cdot \langle \gamma', v \rangle | \Sigma_{p+1} \right] = \frac{\langle \beta_{p+1}, v^+ \rangle}{\langle \beta_{p+1}, v \rangle} \cdot \mathbb{E} [\langle \gamma', v \rangle | \Sigma_{p+1}] = \langle \beta_{p+1}, v^+ \rangle,
\]

and similarly

\[
\mathbb{E} \left[ \frac{\langle \beta_p, v^+ \rangle}{\langle \beta_p, v \rangle} \cdot \langle \gamma', v \rangle | \Sigma_p \right] = \langle \beta_p, v^+ \rangle.
\]
Hence
\[
\int_A \frac{\langle \beta_{p+1}, v^\perp \rangle}{\langle \beta_{p+1}, v \rangle} d\mu^v = \int_A \langle \beta_{p+1}, v^\perp \rangle d\mathcal{L} = \int_A \langle \gamma', v^\perp \rangle d\mathcal{L} = \int_A \langle \beta_p, v^\perp \rangle d\mathcal{L} = \int_A \langle \beta_p, v \rangle d\mu^v.
\]

The bound on the $L^2(\mu^v)$ norm follows trivially from a bound on the $L^\infty$ norm:
\[
\left\| \frac{\langle \beta_p, v^\perp \rangle}{\langle \beta_p, v \rangle} \right\|_\infty \leq \frac{1}{c} \Rightarrow \left\| \frac{\langle \beta_p, v^\perp \rangle}{\langle \beta_p, v \rangle} \right\|_{L^2(\mu^v)} \leq \frac{K(\gamma)}{c}.
\]

The proof of the next lemma can be given as an exercise; we include it in the appendix.

**Lemma 3.19.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $(\mathcal{F}_n)_{n=0}^\infty$ be a filtration on $\Omega$ and $(X_n)_{n=0}^\infty$ be a martingale with respect to the filtration $(\mathcal{F}_n)_{n=0}^\infty$ and measure $\mu$. Then the sequence of alternating sums
\[
\sum_{n=0}^{2N-1} (-1)^n X_n, \quad N \in \mathbb{N},
\]
is a martingale with respect to the filtration $(\mathcal{F}_{2N-1})_{N=1}^\infty$ and measure $\mu$ with
\[
\left\| \sum_{n=0}^{2N-1} (-1)^n X_n \right\|_{L^2(\mu)} \leq 2 \sup_{n \geq 0} \|X_n\|_{L^2(\mu)}.
\]

Together Proposition 3.18 and Lemma A.3 admit the following corollary:

**Corollary 3.20.** With the hypothesis of Proposition 3.18 we have for every $\lambda > 0$
\[
\mathcal{L} \left( \left\{ t \in I : \sup_{p \in \mathbb{N}} \left| \sum_{q=0}^{2p-1} (-1)^q \frac{\langle \beta_q(t), v^\perp \rangle}{\langle \beta_q(t), v \rangle} \right| > \lambda \right\} \right) \leq \frac{16K(\gamma)}{\lambda^2 c^3}.
\]

**Proof.** By combining Proposition 3.18 and Lemma A.3 we deduce that the sequence of functions
\[
\alpha^v_p : I \to \mathbb{R}, \quad t \mapsto \sum_{q=0}^{2p-1} (-1)^q \frac{\langle \beta_q(t), v^\perp \rangle}{\langle \beta_q(t), v \rangle}, \quad p \geq 1,
\]
is a martingale with respect to the filtration $(\mathcal{F}_{2p-1})_{p=1}^\infty$ and measure $\mu^v$ with
\[
\left\| \alpha^v_p \right\|_{L^2(\mu^v)} \leq \frac{2}{c} \quad \text{for all } p \geq 1.
\]
Now, making use of Doob’s $L^2$ inequality [18, p. 60], we derive
\[
\lambda^2 \mu^v \left( \left\{ t \in I : \sup_{p \in \mathbb{N}} \sum_{q=0}^{2p-1} (-1)^q \frac{\langle \beta_q(t), v^\perp \rangle}{\langle \beta_q(t), v \rangle} > \lambda \right\} \right)
\leq \left\| \sup_{p \in \mathbb{N}} |\alpha_p^v| \right\|_{L^2(\mu^v)}^2 \leq 2^2 \sup_{q \in \mathbb{N}} \|\alpha_q^v\|^2_{L^2(\mu^v)} \leq \frac{16}{c^2},
\]
after which a simple rearrangement and application of the inequality $L \leq K(\gamma) \mu^v$ verifies the corollary.

\[\square\]

**Lemma 3.21.** Let $\gamma : I \to \mathbb{R}^2$ be a $C^1$ curve with
\[\gamma'(t) \in C(w, 2\eta) \quad \text{for all } t \in I.\]
Then $\mathcal{L}(\gamma^{-1}(H \setminus G)) = 0$.

**Proof.** At this point we prescribe that $\eta$ is sufficiently small so that the conditions of Hypothesis 3.5 are satisfied for $\delta = 2\eta$ and the conditions of Proposition 3.18 are satisfied for $c = 1 - 2\eta$ and $v = w$. Let the filtration $(\Sigma_p)_{p=0}^\infty$, the conditional expectations $\beta_p := E[\gamma' \mid \Sigma_p]$ and the set $D \subseteq I$ be defined according to Hypothesis 3.5. In view of Lemma 3.6 it suffices to show that the set
\[Z := \gamma^{-1}(H \setminus G) \setminus D\]
has Lebesgue measure zero. Let $t \in Z$. Applying Lemma 3.16 with $z = \gamma(t) \in H \setminus G$ we get, writing $k_s$ for $k_s(\gamma(t))$,
\[\sup_{q \in \mathbb{N}} \left| \sum_{s=0}^{2q-1} (-1)^s \frac{\langle w, e^\perp \rangle}{\langle w, e_s \rangle} \right| = \infty,
\]
which together with Lemma 3.8 and $t \notin D$ implies
\[\sup_{q \in \mathbb{N}} \left| \sum_{s=0}^{2q-1} (-1)^s \frac{\langle \beta_s(t), w^\perp \rangle}{\langle \beta_s(t), w \rangle} \right| = \infty.
\]
To summarise, we have shown that
\[Z \subseteq \left\{ t \in I : \sup_{q \in \mathbb{N}} \left| \sum_{s=0}^{2q-1} (-1)^s \frac{\langle \beta_s(t), w^\perp \rangle}{\langle \beta_s(t), w \rangle} \right| = \infty \right\},
\]
and the latter set has measure zero by Corollary 3.20. \[\square\]

The following statement is the final piece in the proof of Theorem 1.2.

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Lemma 3.22. The set of points in $E$ where $h$ has a directional derivative in any direction in $S^1 \setminus \hat{C}(w,3\sqrt{m})$ is contained in a purely unrectifiable set.

Proof. We show that $G \cup H$ is the required purely unrectifiable set. By Lemma 3.13, $G \cup H$ covers the set of points in $E$ where $h$ has a directional derivative in any direction in $S^1 \setminus \hat{C}(w,3\sqrt{m})$. Thus, we just need to verify the pure unrectifiability of $G \cup H$. Both $G$ and $H$ are $G_\delta$ sets, hence $G \cup H$ is Borel. Let $\gamma : [0,1] \to \mathbb{R}^2$ be a $C^1$ curve. To complete the proof we verify that the set $\gamma^{-1}(G \cup H)$ has Lebesgue measure zero. We may cover $[0,1]$ by countably many intervals $I$ so that each restriction $\gamma : I \to \mathbb{R}^2$ satisfies (possibly with orientation reversed) either

$$\gamma'(t) \notin \hat{C}(w,\eta) \quad \text{for all } t \in I,$$

or, $\gamma'(t) \in C(w,2\eta)$ for all $t \in I$.

It now suffices to argue that each such restriction of $\gamma$ intersects $G \cup H$ in a set of measure zero. The curves for which the first condition holds intersect $E \supset G \cup H$ in a set of measure zero, by Lemma 3.3. The curves of the second type intersect $H \setminus G$ in a set of measure zero, by Lemma 3.21 and $G$ in a set of measure zero by Lemma 3.15. □

3.5. Proof of Theorem 1.1

Referring to the above construction, we present a proof of Theorem 1.1.

Proof of Theorem 1.1. We begin with the universal differentiability set $E$ and perform the following trimmings. First we appeal to Lemma 3.2 to replace $E$ with a compact universal differentiability set $D \subseteq E \cap [0,1]^2$. Next, we remove the set $G$ and argue that $G$ is sufficiently negligible so that we again retain a universal differentiability set. This is justified by [8, Lemma 2.1] and the fact, of Lemma 3.15, that $G$ projects in any direction to a set of 1-dimensional Lebesgue measure zero. Thus, in the end, we are left with a universal differentiability set

$$\tilde{D} := D \setminus G \subseteq [0,1]^2.$$

In light of Lemmas 3.13 and 3.14 we have that $h$ is non-differentiable at all points of $\tilde{D} \setminus H \subseteq E \setminus (G \cup H)$ and differentiable at all points of $\tilde{D} \cap H \subseteq H \setminus G$. We verify that the purely unrectifiable set $P := \tilde{D} \cap H$ has the properties asserted in Theorem 1.1.

Observe that $\tilde{D} \setminus H \subseteq D \setminus (G \cup H) = D \cap \bigcup_{m=1}^\infty F_m$. Since $D$ is compact and each $F_m$ is $F_\sigma$, the sets $D \cap F_m$ are $F_\sigma$. Moreover, for each $m$ Lemma 3.13 ensures the conditions of Lemma 2.2 are satisfied for $K = D \cap F_m$, $E$, $\sigma = \frac{2^{-m}\sqrt{m}}{1}$ and $h$. Intersecting the residual subsets of $\text{Lip}_1([0,1]^2)$ obtained by applying Lemma 2.2 to each $D \cap F_m$, we obtain a residual set in which all functions $g$ have the property that $g + h$ is nowhere differentiable in $D \cap \bigcup_{m=1}^\infty F_m \supseteq \tilde{D} \setminus H$. Since $\tilde{D}$ is a universal differentiability set, it follows that

$$\emptyset \neq \text{Diff}(g + h) \cap \tilde{D} \subseteq \tilde{D} \cap H = P$$

for typical $g \in \text{Lip}_1([0,1]^2)$. But $h$ is differentiable at all points of $P = \tilde{D} \cap H$, so we conclude that

$$\text{Diff}(g) \cap P \supseteq \text{Diff}(g + h) \cap \tilde{D}$$
for typical \( g \in \text{Lip}_1([0,1]^2) \). The latter sets have all one-dimensional projections of positive measure by [8, Lemma 2.1] and so, due to the Besicovitch-Federer Projection Theorem [12], they must also be of non-\( \sigma \)-finite one-dimensional Hausdorff measure. \( \square \)

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\section{A. Appendix}

\subsection{A.1. Differentiability}

**Proposition 2.1.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a Lipschitz function, \( z \in \mathbb{R}^d \) and \( e \in S^{d-1} \). Then,

(a) \( \lim_{\varepsilon \to 0} \zeta(f, z, \varepsilon, e) = 0 \iff f \) has a directional derivative at \( z \) in direction \( e \).

(b) \( \limsup_{\varepsilon \to 0} T(f, z, \varepsilon) > 0 \Rightarrow \exists u \in S^{d-1} \) such that \( \limsup_{\varepsilon \to 0} \zeta(f, z, \varepsilon, u) > 0 \).

**Proof.** We begin with part (a). If the directional derivative of \( f \) at \( z \) does not exist then there are null sequences of non-zero real numbers \( (t_k)_{k=1}^\infty \) and \( (s_k)_{k=1}^\infty \) for which

\[
\lim_{k \to \infty} \frac{f(z + t_k e) - f(z)}{t_k} \neq \lim_{k \to \infty} \frac{f(z + s_k e) - f(z)}{s_k}.
\]

The corresponding sequences \( [(z, z + t_k e)]_{k=1}^\infty \) and \( [(z, z + s_k e)]_{k=1}^\infty \) of line segments witness that \( \limsup_{\varepsilon \to 0} \zeta(f, z, \varepsilon, e) > 0 \). Conversely, if \( f'(z, e) \) does exist then given \( \varepsilon > 0 \) we may choose \( \delta > 0 \) such that

\[
r \in (-\delta, \delta) \Rightarrow |f(z + re) - f(z) - r f'(z, e)| < \varepsilon |r|.
\]

Let \([x, x + te]\) be a line segment containing \( z \) with \( t \in (-\delta/2, \delta/2) \setminus \{0\} \) and fix \( \theta \in [0, 1] \) with \( x = z - \theta t e \). Then,

\[
\left| \frac{f(x + te) - f(x)}{t} - f'(z, e) \right| = \frac{|(f(z + ((1 - \theta)t)e) - f(z)) - (f(z - \theta te) - f(z))|}{t} \leq \frac{|(1 - \theta) t f'(z, e) + \theta t f'(z, e) - f'(z, e)|}{|t|} + \frac{\varepsilon(1 - \theta) |t| + \theta |t|}{|t|} = \varepsilon.
\]

It follows from the above that \( \zeta(f, z, \delta/2, e) \leq 2\varepsilon \). Thus, the argument proves that \( \limsup_{\varepsilon \to 0} \zeta(f, z, \varepsilon, e) = 0 \) and completes the proof of (a).

Now we move to part (b). Set \( c := \limsup_{\varepsilon \to 0} T(f, z, \varepsilon) \). Let \((e_k)_{k=1}^\infty\) and \((\varepsilon_k)_{k=1}^\infty\) be sequences in \( S^{d-1} \) and \((0, \infty)\) respectively with \( \lim_{k \to \infty} \varepsilon_k = 0 \) and \( \zeta(f, z, e_k, \varepsilon_k) \geq c/2 \).
for all \( k \in \mathbb{N} \). By passing to subsequences if necessary we may assume that the sequence \((e_k)_{k=1}^\infty\) converges to a direction \( e \in S^1 \). We show that \( f \) is non-differentiable at \( z \) in direction \( e \). For each \( k \in \mathbb{N} \) we choose \( x_k, y_k \in \mathbb{R}^2 \) and \( s_k, t_k \in [-\varepsilon_k, \varepsilon_k] \setminus \{0\} \) for which \( z \in [x_k, x_k + t_k e_k] \cap [y_k, y_k + s_k e_k] \) and

\[
\frac{|f(x_k + t_k e_k) - f(x_k)|}{t_k} \geq \frac{|f(y_k + s_k e_k) - f(y_k)|}{s_k} \geq \frac{c}{4}.
\]

To obtain a lower estimate on \( \limsup_{\varepsilon \to 0} \zeta(f, z, e, \varepsilon) \), we approximate each of the line segments \([x_k, x_k + t_k e_k], [y_k, y_k + s_k e_k]\) by line segments passing through \( z \) parallel to \( e \). More precisely, for each \( k \in \mathbb{N} \) we let

\[
\tilde{x}_k := z + (x_k - z)e, \quad \tilde{y}_k := z + (y_k - z)e.
\]

Then the line segments \([\tilde{x}_k, \tilde{x}_k + t_k e_k]\) and \([\tilde{y}_k, \tilde{y}_k + s_k e_k]\) pass through \( z \) and we have

\[
\|\tilde{x}_k - x_k\| = \|z + (x_k - z, e) - x_k\| = \|(x_k - z, e) - (x_k - z, e_k)\| = 0
\]

and the corresponding estimate with \( x \) replaced by \( y \) and \( t \) by \( s \). Now for each \( k \in \mathbb{N} \) we observe

\[
\frac{|f(x_k + t_k e_k) - f(x_k)|}{t_k} \geq \frac{|f(y_k + s_k e_k) - f(y_k)|}{s_k} \geq \frac{c}{4} - \text{Lip}(f) \|e_k - e\| - \frac{2 \text{Lip}(f) \|\tilde{x}_k - x_k\|}{|t_k|} - \frac{2 \text{Lip}(f) \|\tilde{y}_k - y_k\|}{|s_k|} - \text{Lip}(f) \|e_k - e\|
\]

\[
\geq \frac{c}{4} - 10 \text{Lip}(f) \|e_k - e\| \to \frac{c}{4} \quad \text{as } k \to \infty.
\]

We conclude that \( \limsup_{\varepsilon \to 0} \zeta(z, f, e, \varepsilon) \geq \frac{c}{4} > 0 \).

\[\square\]

A.2. Geometry of curves

For \( W \subseteq \mathbb{R}^2 \) and \( v \in S^1 \) we define a quantity

\[
diam_v(W) := \sup \{ \langle y - x, v \rangle : x, y \in W \}.
\]

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Lemma A.1. Let $W \subseteq \mathbb{R}^2$, $v \in S^1$, $\delta \in (0,1)$ and $\gamma: I \to \mathbb{R}^2$ be a $C^1$ curve satisfying $\gamma'(t) \in \hat{C}(v, \delta)$ for all $t \in I$.

Then

$$\mathcal{L}(\gamma^{-1}(W)) \leq \frac{\text{diam}_e(W)}{1 - \delta}.$$ 

Proof. Since $\gamma'$ is continuous, we either have $\gamma'(t) \in C(v, \delta)$ for all $t \in I$ or $\gamma'(t) \in C(-v, \delta)$ for all $t \in I$. We assume the former without loss of generality. Then the function $t \mapsto \langle \gamma(t), v \rangle$ is strictly increasing, implying that the set $\gamma^{-1}(W)$ is contained in the interval $[a, b]$, where $a, b \in \gamma^{-1}(W)$ are defined by the conditions

$$\langle \gamma(a), v \rangle = \min \{ \langle \gamma(t), v \rangle : t \in \gamma^{-1}(W) \}, \quad \langle \gamma(b), v \rangle = \max \{ \langle \gamma(t), v \rangle : t \in \gamma^{-1}(W) \}.$$ 

Now we have

$$\mathcal{L}(\gamma^{-1}(W)) \leq b - a \leq \frac{1}{1 - \delta} \int_a^b \langle \gamma'(t), v \rangle \, dt = \frac{\langle \gamma(b) - \gamma(a), v \rangle}{1 - \delta} \leq \frac{\text{diam}_e(W)}{1 - \delta}.$$  

Lemma A.2. Let $P \subseteq \mathbb{R}^2$ be an open and convex set, $e \in S^1$ be a direction and $\gamma: I \to \mathbb{R}^2$ be a $C^1$ curve with $\langle \gamma'(t), e \rangle \geq 0$ for all $t \in I$. Then

$$\left| \int_{\gamma^{-1}(P)} \langle \gamma'(t), e^\perp \rangle \, dt \right| \leq 6 \text{diam}_{e^\perp}(P).$$

Proof. Let $a := \inf \{ \langle z, e \rangle : z \in P \}$ and $b := \sup \{ \langle z, e \rangle : z \in P \}$. As a convex and open set, $P$ admits functions $\psi^-, \psi^+: (a, b) \to \mathbb{R}$ with $\psi^-$ convex and $\psi^+$ concave, $\psi^- < \psi^+$ and so that $\partial P \cap \{ z \in \mathbb{R}^d : a < \langle z, e \rangle < b \}$ is the union of the graphs of $\psi^-$ and $\psi^+$ in the co-ordinate system $(e, e^\perp)$. For points $z \in \mathbb{R}^2$ with $a < \langle z, e \rangle < b$ and $\psi \in \{ \psi^+, \psi^- \}$ we will let, for example, $z \geq \psi$ signify that, with respect to the coordinate system $(e, e^\perp)$, the point $z$ lies on or above the graph of $\psi: (a, b) \to \mathbb{R}$. With this notation we have

$$P \cap \left\{ z \in \mathbb{R}^d : a < \langle z, e \rangle < b \right\} = \left\{ z : \psi^- < z < \psi^+ \right\}.$$ 

The condition $\langle \gamma'(t), e \rangle \geq 0$ guarantees that the segment $\gamma(I) \cap \{ z \in \mathbb{R}^2 : a \leq \langle z, e \rangle \leq b \}$ is connected. This leads to the simple observation that whenever $s < t$ with $\gamma(s) \geq \psi$ and $\gamma(t) < \psi$ there must be a point $r \in [s, t]$ with $\gamma(r) \in \text{Graph}(\psi)$. We make frequent use of this observation in the argument that follows.

The open set $\gamma^{-1}(P)$ can be written as a countable union of intervals $(a_{2i-1}, a_{2i}) \subset I$, $i = 1, 2, \ldots$ with $\gamma(a_j) \in \partial P$ for all $j$. We choose $N \in \mathbb{N}$ sufficiently large so that

$$\sum_{i \geq N+1} (a_{2i} - a_{2i-1}) \leq \text{diam}_{e^\perp}(P).$$
By relabelling if necessary, we may assume that \( a_1 < a_2 \leq a_3 < a_4 \leq \ldots \leq a_{2N-1} < a_{2N} \). In what follows we say that the point \( a_j \) is of type +, respectively of type −, if \( \gamma(a_j) \in \text{Graph } \psi^+ \), respectively if \( \gamma(a_j) \in \text{Graph } \psi^- \). We argue that the finite sequence \( \left( (a_{2j-1}, a_{2j}) \right)_{j=1}^N \) may be extended to a finite sequence of components of \( \gamma^{-1}(P) \) ordered with respect to \( \leq \) in which the points \( a_{2j+1} \) and \( a_{2j} \) have the same type for every \( j \). Let \( j \in \{1, \ldots, N - 1\} \) be an index for which \( a_{2j} \) and \( a_{2j+1} \) have different types. Without loss of generality, we may assume that \( \gamma(a_{2j}) \in \text{Graph } \psi^+ \) and \( \gamma(a_{2j+1}) \in \text{Graph } \psi^- \). Then the interval \( [a_{2j-1}, a_{2j}] \) must contain a component \( (a_{2k-1}, a_{2k}) \) of \( \gamma^{-1}(P) \) with \( k \leq N + 1 \), \( a_{2k-1} \in \text{Graph } \psi^+ \) and \( a_{2k} \in \text{Graph } \psi^- \). For example, the points \( a_{2k-1} \) and \( a_{2k} \) may be defined by

\[
a_{2k} := \inf \{ t > a_{2j} : \gamma(t) \leq \psi^- \} \leq a_{2j+1}, \quad a_{2k-1} := \sup \{ t < a_{2k} : \gamma(t) \geq \psi^+ \} \geq a_{2j}.
\]

For each index \( j \in \{1, \ldots, N - 1\} \) for which \( a_{2j} \) and \( a_{2j+1} \) have different types, we let \( k_j \geq N + 1 \) be the index defined by the above discussion. By inserting the intervals \( (a_{2k_j-1}, a_{2k_j}) \) in between the relevant terms of the original sequence \( (a_{2i-1}, a_{2i}) \) and relabelling the components of \( \gamma^{-1}(P) \) we obtain an extended finite sequence \( \left( (a_{2i-1}, a_{2i}) \right)_{i=1}^M \) of components of \( \gamma^{-1}(P) \) ordered with respect to \( \leq \) with the desired property.

Thus, for each \( i \in \{1, \ldots, M - 1\} \) either \( \gamma(a_{2i}) \) and \( \gamma(a_{2i+1}) \) both lie on the graph of \( \psi^+ \) or they both lie on the graph of \( \psi^- \). The sum of the quantities

\[
\langle \gamma(a_{2i+1}) - \gamma(a_{2i}), e^\perp \rangle = \psi^- \langle \gamma(a_{2i+1}), e \rangle - \psi^- \langle \gamma(a_{2i}), e \rangle
\]

over \( 1 \leq i \leq M \) for which the first case occurs is bounded above by \( 2 \text{diam}_{e^\perp}(P) \), because \( \psi^- \) is convex and oscillates at most \( \text{diam} \{ \langle z, e^\perp \rangle : z \in P \} = \text{diam}_{e^\perp}(P) \). The same estimate holds for the corresponding sum over the second case indices. This gives us

\[
\left| \int_{\gamma^{-1}(P)} \langle \gamma'(t), e^\perp \rangle \, dt \right| \leq \sum_{i=1}^M \left| \langle \gamma(a_{2i}) - \gamma(a_{2i-1}), e^\perp \rangle \right| + \sum_{i=M+1}^M (a_{2i} - a_{2i-1})
\]

\[
\leq \left| \langle \gamma(a_{2M}) - \gamma(a_1), e^\perp \rangle \right| + \sum_{i=1}^{M-1} \left| \langle \gamma(a_{2i+1}) - \gamma(a_{2i}), e^\perp \rangle \right| + \text{diam}_{e^\perp}(P)
\]

\[
\leq \text{diam}_{e^\perp}(P) + 4 \text{diam}_{e^\perp}(P) + \text{diam}_{e^\perp}(P) = 6 \text{diam}_{e^\perp}(P).
\]

\[\Box\]

A.3. Martingale Theory

Lemma A.3. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space, \((\mathcal{F}_n)_{n=0}^\infty\) be a filtration on \(\Omega\) and \((X_n)_{n=0}^\infty\) be a martingale with respect to the filtration \((\mathcal{F}_n)_{n=0}^\infty\) and measure \(\mu\). Then the sequence of alternating sums

\[
\sum_{n=0}^{2N-1} (-1)^n X_n, \quad N \in \mathbb{N},
\]

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is a martingale with respect to the filtration \((\mathcal{F}_{2N-1})_{N=1}^\infty\) and measure \(\mu\) with
\[
\left\| \sum_{n=0}^{2N-1} (-1)^n X_n \right\|_{L^2(\mu)} \leq 2 \sup_{n \geq 0} \|X_n\|_{L^2(\mu)}.
\]

**Proof.** For an arbitrary set \(A \in \mathcal{F}_{2N-1}\) we have
\[
\int_A \sum_{n=0}^{2N+1} (-1)^n X_n \, d\mu = \int_A \sum_{n=0}^{2N-1} (-1)^n X_n \, d\mu + \int_A X_{2N} \, d\mu - \int_A X_{2N+1} \, d\mu
\]
\[
= \int_A \sum_{n=0}^{2N-1} (-1)^n X_n \, d\mu
\]
This proves
\[
\mathbb{E} \left[ \sum_{n=0}^{2N+1} (-1)^n X_n | \mathcal{F}_{2N-1} \right] = \sum_{n=0}^{2N-1} (-1)^n X_n, \quad N \in \mathbb{N}.
\]
This establishes the martingale part. To get the bound on the \(L^2\) norm we note that for \(n \geq m\) we have
\[
\langle X_m, X_n \rangle = \int_\Omega X_m X_n \, d\mu = \int_\Omega \mathbb{E}[X_m X_n | \mathcal{F}_m] \, d\mu = \int_\Omega X_m \mathbb{E}[X_n | \mathcal{F}_m] \, d\mu = \int_\Omega X_m^2 \, d\mu,
\]
where \(\langle - , - \rangle\) denotes the standard inner product on \(L^2(\Omega, \mathcal{F}, \mu)\). The third equality above makes use of a standard property of the conditional expectation [IS, 22.(i), p 54].

We may now compute the \(L^2\) norm of the alternating sum as follows
\[
\left\| \sum_{n=0}^{2N-1} (-1)^n X_n \right\|_{L^2}^2 = \sum_{0 \leq m, n \leq 2N-1} (-1)^{m+n} \langle X_m, X_n \rangle
\]
\[
= 2 \sum_{0 \leq m, n \leq 2N-1} (-1)^{m+n} \langle X_m, X_n \rangle - \sum_{0 \leq n \leq 2N-1} (-1)^n \langle X_n, X_n \rangle
\]
\[
= 2 \sum_{m=0}^{2N-1} (-1)^m \langle X_m, X_m \rangle \sum_{n=m}^{2N-1} (-1)^n - \sum_{m=0}^{2N-1} \langle X_m, X_m \rangle
\]
\[
= 2 \sum_{m \text{ odd}} (-1)^{2m} \langle X_m, X_m \rangle - \sum_{m=0}^{2N-1} \langle X_m, X_m \rangle
\]
\[
= \sum_{m=0}^{2N-1} (-1)^{m+1} \langle X_m, X_m \rangle
\]
\[
= \sum_{m=0}^{2N-1} (-1)^{m+1} \int_\Omega X_m^2 \, d\mu.
\]
The sequence $\left( X_m^2 \right)_{m=1}^\infty$ is a submartingale; hence the inequality
\[ \int_\Omega X_{2n-1}^2 \, d\mu \leq \int_\Omega X_{2n}^2 \, d\mu, \quad n \geq 1 \]
holds. Applying this inequality to the final expression above we deduce
\[ \left\| \sum_{n=0}^{2N-1} (-1)^n X_n \right\|_{L^2}^2 \leq \int_\Omega X_{2N-1}^2 \, d\mu - \int_\Omega X_0^2 \, d\mu \leq 2 \sup_{n \in \mathbb{N}} \| X_n \|_{L^2}^2. \]

\[ \square \]

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