An approximation method for electromagnetic wave models based on fractional partial derivative

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Abstract

The present article is devoting a numerical approach for solving a fractional partial differential equation (FPDE) arising from electromagnetic waves in dielectric media (EMWDM). The truncated Bernoulli and Hermite wavelets series with unknown coefficients have been used to approximate the solution in both the temporal and spatial variables. The basic idea for discretizing the FPDE is wavelet approximation based on the Bernoulli and Hermite wavelets operational matrices of integration and differentiation. The resulted system of a linear algebraic equation has been solved by the collocation method. Moreover, convergence and error analysis have been discussed. Finally, several numerical experiments with different fractional-order derivatives have been provided and compared with the exact analytical solutions to illustrate the accuracy and efficiency of the method.

Keywords: Bernoulli wavelets, Hermite wavelets, Kronecker multiplication, Collocation scheme, Riemann-Liouville fractional derivative, Operational matrix.

1. Introduction

In the last two-three decade, the hypothesis of fractional derivative administrators has been growing principally as an open field of science valuable just for mathematicians. These days, a few scientists discovered that the fractional order models are more appropriate than the integer-order. Fractional differential equations (FDEs) provide a powerful and flexible tool for modeling and describing the behavior of real materials [1], finance [2], viscoelastic fluid [3], signal and image processing [4], biological systems [5], control theory [6], electrochemical process [7], electromagnetic waves [8] and so on. Moreover, the investigative arrangements of uncommon FDEs are as the trigonometric arrangement and hard to compute. Subsequently, the numerical solution of FDEa has turned into an important and significant research topic. Fractional derivatives give an

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incredible instrument to the depiction of memory and genetic properties of different materials and procedures. A lot of exertion has extended in the course of the most recent 15 years or so in endeavoring to discover hearty and stable numerical & scientific strategies for unraveling FPDEs of physical intrigue. These motivate us to scan for new and effective numerical techniques for understanding FPDEs.

Some numerical strategies for time, space, and space-time FPDEs have been proposed for explaining the diverse definitions by some authors. Liu et al. [9] examined a guess of the Levy-Feller shift in weather conditions scattering process by utilizing a random walk and limited distinction strategy. Liu et al. [10]-[12] additionally talked about the solidness and joining of the distinctive techniques for space fractional Fokker-Planck equation, the altered irregular subdiffusion condition, and the space-time FPDEs. Meerschaert [13] utilized the limited contrast technique and got numerical answers for the Caputo space fractional advection-dispersion flow equation. Shen and Liu [14], Liu et al. [15] inferred numerical solutions for space fractional diffusion equation and the Riesz FPDEs separately.

These days, FPDEs have pulled in remarkable consideration in [16] and [17]. The FPDEs have exhibited as a helpful approach for the portrayal of transport progression in complex frameworks that are represented by peculiar dispersion and non-exponential unwinding designs (see for moment [18]). The time, space, and time-space FPDEs have utilized to communicated imperative physical wonders. And, which emerge in shapeless, glassy, colloid and porous media, in fractals and permeation groups, comb structures, dielectrics, and organic frameworks, semiconductors, polymers, irregular & disarranged media and geophysical & geographical processes (see [20]-[23]).

Nowadays, broad numerical strategies have just created for FPDEs, as limited, finite element method (see [24]), finite difference (see [25]), and spectral techniques [26]. Due to the nonlocal properties of fractional derivative operators, the essential issue for numerical calculation of FPDEs is how to diminish the calculation costs. A few strategies for the lessening have just been composed, such as alternating-direction implicit methods (ADI) [13], multigrid techniques [27] and appropriate iterative strategies [28]. In any case, as indicated by the ebb and flow learning of the specialists, there are few examinations on the numerical treatment of the FPDEs for time derivatives, effective numerical techniques, stability, and convergence analysis, which are still limited.

In this article, we present and analyze a precise numerical technique for tackling an FPDE emerging from EMWDM (see [29]). It is outstanding that the dielectric unwinding in solids portrayed by the complex frequency-dependent dielectric sensitivity build up the complete power-law dependence. And Here, an outcome acquired FPDE for EMWDM with the assistance of Maxwell condition. It is a power law reliance in the recurrence space brings about the association between the electric field and the polarization thickness detailed as a feebly solitary Caputo integral and, thus, the field conditions take a type of
FPDE [29]:

\[(cD_\alpha t^\beta)\xi(t,x) - \lambda_1 (cD_\beta t^\beta)\xi(t,x) - \lambda_2 \nabla^2 \xi(t,x) = f(t,x), x \in \Omega, t \in (0,T] \tag{1}\]

subject to initial condition

\[
\begin{align*}
\xi(0,x) &= g(x), \\
\left[\frac{\partial \xi(t,x)}{\partial t}\right]_{t=0} &= h(x),
\end{align*}
\tag{2}
\]

where \(\xi(t,x)\) is magnetic field induction, \(f(t,x)\) is the current density of free charges, \(\Omega = [0,L]\), the constant coefficient \(\lambda_1\) and \(\lambda_2\) depend on the frequency independent properties of a medium, \(1 < \beta < \alpha < 2\), \(cD_\alpha t^\beta\) & \(cD_\beta t^\beta\) both fractional derivative in Eq. (1) are defined in the Caputo derivative operator sense, and \(\nabla^2\) is Laplace operator with space variable \(x\) and \(u\) is unknown. For numerical solution, we have taken space domain at \(L = 1\) and temporal domain at \(T = 1\) throughout the article.

As in the wake of actualizing Caputo fractional derivative operator on proposed problem Eq. (1), Eq. (1) converted into an FPIDE then this FPIDE is said to be Volterra type integro-differential equation. Even though the hypothesis of Volterra FPIDE has experienced fast advancement amid the most recent decades, along these lines, it stays open for additionally advance progress.

The scalar Eq. (1) can be analyzed as a speculation of the alleged Szabo condition [30]. It has a more endorsed frame in the examination with Eq. (1), since the case \(1 \leq \beta < \alpha < 2\) has viewed as and widely considered by [31]-[34] as it were. The FPDE has proposed for EMWDM by the Grunwald-Lenikov discretization scheme [35] only. Therefore, this article introduced a numerical scheme for Eq. (1) based on wavelets operational matrices.

Wavelets are a relatively ongoing advancement in connected science. Their name itself has begun around 20 years prior, for instance, Morlet, Bernoulli, Hermite, Legendre, Chebyshev, and so on. As we know, Wavelets are an intense device which has utilized as a part of numerical procedures which concede the precise portrayal of an assortment of capacities and administrators. And One can be noticed that wavelets operational matrices are disentangling the proposed issue as well as speed-up the calculation. These days, wavelets have been discovered their area in more applications (see for moment [36, 37, 38]). In specially, wavelets effectively utilized as a part of signal investigation [39]. It has demonstrated that wavelets are the ground-breaking device to look at new heading for settling any FPDEs. Likewise, we know that Wavelets are localized functions [40], which are the reason for energy-bounded functions [41] and specifically for \(L^2(R)\).

Consequently, The article depends on two orthogonal wavelets (Bernoulli and Hermite) where the fundamental thought of utilizing an orthogonal basis is that the issue under consideration converted into a system linear or non-linear algebraic equations by using operational matrices. It is possible by the truncated series of orthogonal basis function to the solution of the issue in light of operational matrices (see for moment [42, 43, 44]). In this article, we present and
analyze the Bernoulli and Hermite wavelet approximation based on operational matrices for solving the problem (1)–(2). The numerical approach constructed by using an approximation in the time-space domain and its operational matrices. The rest of the article is organized into 8 sections. Section 2 introduced the preliminaries of Wavelets, Hermite polynomials, Hermite wavelets, Bernoulli polynomials, Bernoulli wavelets, and function approximation. Theoretical aspects of the function approximations have been discussed. And, fractional derivatives in the Riemann Liouville sense and Kronecker multiplication have been introduced. In section 3, Operational matrices of differentiation and integration based on Bernoulli wavelet approximation (BWA) and Hermite wavelet approximation (HWA) have been proposed. And also, almost operational matrices have been constructed. A numerical approach for the solution of (1) has been discussed by using operational matrices in sectional 4. In sections 5 to 6, convergence and error analysis have been estimated. And finally, few numerical examples have illustrated the efficiency of the proposed numerical approach in section 7.

2. Preliminaries

2.1. Wavelets

Wavelets have been connected broadly for signal processing in correspondences and science inquire about, and have ended up being an awesome numerical instrument. Wavelets can be utilized for arithmetical controls in the arrangement of conditions got which prompts the better coming about the framework. Wavelets constitute a family of functions developed from widening and interpretation of single function called the mother wavelet. At the point when the dilation parameter $a$ and the translation parameter $b$ fluctuate persistently, we have the accompanying group of ceaseless wavelets $L^2$:

$$\Phi_{a,b}^L(t) = |a|^{-\frac{1}{2}} \Phi^L\left(\frac{t-b}{a}\right), \ a, b \in \mathbb{R}, \ a \neq 0.$$

If the parameter $a$ and $b$ are restricted to the discrete values as $a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0, n$ and $k$ are positive integers, from the above Eq. we have the following family of discrete wavelets:

$$\Phi_{k,n}^L(t) = |a_0|^{-\frac{1}{2}} \Phi^L(a_0^k t - nb_0),$$

where, $\Phi_{k,n}^L(t)$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\Phi_{k,n}^L(t)$ form an orthonormal basis.

2.2. Bernoulli wavelet

Bernoulli wavelet $\Phi_{n,m}^B(t) = \Phi^B(k, \hat{n}, m, t)$ have four arguments; $\hat{n} = n - 1, k \in \mathbb{N}, n = 1, 2, ..., 2^{k-1}$, and $m$ is the degree of the Bernoulli polynomial of the
fist kind and $t$ is the normalized time i.e. $t \in [0,1)$. They are defined on the interval $[0,1)$ as (see [48]):

\[
\Phi_{n,m}^B(t) = \Phi^B(k, \hat{n}, m, t) = \begin{cases} 
2^{(k-1)/2}\hat{B}_m(2^{k-1}t - \hat{n}) & \text{if } \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k};\\
0 & \text{if otherwise.}
\end{cases}
\] (3)

with

\[
\hat{B}_m(t) = \begin{cases} 
\frac{1}{\sqrt{(-1)^{m-1}(m!)^2\alpha_{2m}}} B_m(t) & \text{if } m = 0;\\
\sqrt{\frac{2m!}{(-1)^{m-1}(m!)^2\alpha_{2m}}} B_m(t) & \text{if } m > 0.
\end{cases}
\] (4)

where, $m = 0, 1, \ldots, M-1$, and $n = 1, 2, \ldots, 2^{k-1}$. The coefficient $\sqrt{\frac{2m!}{(-1)^{m-1}(m!)^2\alpha_{2m}}}$ is for normality, the dilation parameter is $g = 2^{-(k-1)}$ and the translation parameter $h = \hat{n}2^{-(k-1)}$. Here, $B_m(t)$ are the well-known Bernoulli polynomial of order $m$ which can be defined as follows:

\[
B_m(t) = \sum_{i=0}^{m} \frac{m}{i!} \alpha_{m-1} t^i,
\]

where, $\alpha_i$, $i = 0, 1, \ldots, m$ are Bernoulli numbers. These numbers are a sequence of signed rational numbers which arise in the series expansion of trigonometric function and can be defined by

\[
\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.
\]

Here, Bernoulli polynomials are as follows:

- For $m = 0$, $B_0(t) = 1$,
- For $m = 1$, $B_1(t) = t - \frac{1}{2}$,
- For $m = 2$, $B_2(t) = t^2 - t + \frac{1}{6}$,
- For $m = 3$, $B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2} t$, and so on.

### 2.3. Two-dimensional Bernoulli wavelet

Two-dimensional Bernoulli wavelet can be written in the product of one-dimensional Bernoulli wavelet as follows:

\[
\Phi_{n,m,n',m'}^B(a,b) = \Phi_{n,m}^B(a)\Phi_{n',m'}^B(b),
\] (5)

where, $\Phi_{n,m}^B(a)$ and $\Phi_{n',m'}^B(b)$ are defined as Eq.(5), $n' = 1, 2, \ldots, 2^{k'-1}$ and $m' = 0, 1, 2, \ldots, M' - 1$. 

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2.4. Hermite wavelets

Hermite wavelet \( \Phi_{n,m}^H(t) = \Phi(k, \hat{n}, m, t) \) have four arguments \( \hat{n} = 2n - 1, n = 1, 2, \ldots, 2^{k-1}, k \in \mathbb{Z}^+ \), \( m \) is the order of Hermite polynomials and \( t \) is normalized time. They are defined on \([0, 1)\) as follows (see [37]):

\[
\Phi_{n,m}^H(t) = \Phi(k, \hat{n}, m, t) = \begin{cases} 
\sqrt{\frac{1}{n!2^n \sqrt{\pi}}} \frac{2^k}{\sqrt{\pi}} H_m(2^k t - \hat{n}), & \text{if } \frac{n-1}{2^k} \leq t \leq \frac{n+1}{2^k}; \\
0, & \text{otherwise.}
\end{cases}
\]

where, the coefficient \( \sqrt{\frac{1}{n!2^n \sqrt{\pi}}} \) is for orthonormality and Hermite polynomials are as following

- For \( m = 0 \), we have \( H_0(t) = 1 \),
- For \( m = 1 \), we have \( H_0(t) = 2t \),
- \( H_{m+1}(t) = 2tH_m(t) - 2mH_{m-1}(t) \), where \( m = 1, 2, \ldots \)

2.5. Two-dimensional Hermite wavelets

The set of two-dimensional Hermite wavelet functions from \( L^2 \) over \( \Omega = [0, 1] \times [0, 1] \) can be easily expressed in terms of 1D Hermite wavelet functions applying the tensor product scheme as follows:

\[
\Phi_{n,m,n',m'}^H(t,x) = \begin{cases} 
\Phi_{n,m}^H(t)\Phi_{n',m'}^H(x), & \text{for } \frac{(n-1)}{2^k} \leq t \leq \frac{n}{2^k}, \\
\frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, & \text{otherwise,}
\end{cases}
\]

where for a fixed \( k \)

\[
\Phi_{n,m}^H(t) = \sqrt{\frac{1}{n!2^n \sqrt{\pi}}} \frac{2^k}{\sqrt{\pi}} H_m(2^k t - \hat{n}),
\]

\[
\Phi_{n',m'}^H(x) = \sqrt{\frac{1}{n'!2^{n'} \sqrt{\pi}}} \frac{2^{k'}}{\sqrt{\pi}} H_{m'}(2^{k'} x - \hat{n}),
\]

Here \( m = 0, 1, 2, \ldots, M-1, m' = 0, 1, 2, \ldots, M'-1, n = 1, 2, 3, \ldots, 2^{k-1}, n' = 1, 2, 3, \ldots, 2^{k-1}; H_m \) and \( H_{m'} \) are the Hermite polynomials of orders \( m \) and \( m' \) defined over the interval \([0, 1]\).
2.6. Kronecker multiplication

It is a generalization of the outer product from vectors to matrices. Definition of Kronecker multiplication of matrices and also some important property which is related to Kronecker multiplications defined (see [47]) as following remarks:

**Remark 1.** Suppose that \( A \) and \( B \) are two dimensions \( m \times n \) and \( p \times q \), respectively, then the Kronecker multiplication of \( A \) and \( B \) is denoted by \( A \otimes B = \text{kron}(A,B) \) and is defined in the following form:

\[
G_{m \times n} \otimes H_{p \times q} = \begin{pmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{pmatrix},
\]

where the above matrix of order \( mp \times nq \).

**Remark 2.** If matrices \( A,B,V \) and \( W \) with appropriate dimensions then following interesting property is satisfied:

\[
(AB) \otimes (VW) = (A \otimes V)(B \otimes W).
\]

2.7. Function approximation

This system of basis functions is orthonormal over \( \Omega \) and, therefore, can be applied to the decomposition of a function \( f(t,x) \) approximated in \( L^2(\Omega) \) via the series as follows:

\[
f(t,x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{2^{k-1}} \sum_{n'=1}^{2^{k'-1}} \sum_{m'=0}^{2^{k'-1}} f_{nmn'm'} \Phi_{nmn'm'}(t,x).
\]  

(10)

Note that we again do not need put \( k \to \infty \) to obtain the convergence of this series to the function \( f(x,y) \) that can be easily proven in an explicit way.
Theorem 1. Let \( f(t, x) \) be the continuous function and \( \Phi(t, x) \) be the 2-D Bernoulli/Hermite wavelets. Then, the series

\[
\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{2^{k'-1}} \sum_{n'=1}^{M-1} \sum_{m'=0}^{M'-1} c_{nn'm'm'} \Phi(t, x)
\]

approximation of \( f(t, x) \) converges uniformly to \( f(t, x) \) as \( n \to \infty \).

Proof. See [49].

Remark: As it is proven in theorem 1,

\[ \| f(t, x) - f_{N_1}(t, x) \| \to 0, \text{ as } N \to \infty, \]

but in practical computations the infinite series (14) should be truncated with \( m = M - 1, m' = M' - 1, \) and the approximation can be represented in the following form

\[
f(t, x) \approx \sum_{n=1}^{2^{k-1}2^{k'-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{M'-1} f_{nn'm'm'} \Phi_{nn'm'm'}(t, x) = F^T \Phi(t, x),
\]

where \( C \) and \( \Phi \) are \( 2^{k-1}2^{k'-1}MM' \times 1 \) vectors.

and

\[
f_{nn'm'm'} = \langle f(t, x), \Phi_{nn'm'm'}(t, x) \rangle
\]

\[
= \left( \int_{\frac{n}{2^{k-1}}}^{\frac{n+1}{2^{k-1}}} f(t) \Phi(t) dt \right) \left( \int_{\frac{n}{2^{k-1}}}^{\frac{n+1}{2^{k-1}}} f(x) \Phi(x) dx \right)
\]

\[
= \left( \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \Phi(t) dt \right) \left( \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \Phi(x) dx \right)
\]

\[
= (F^T_1 \Phi) \otimes (F^T_2 \Phi)
\]

\[
= (F^T_1 \otimes F^T_2) (\Phi(t) \otimes \Phi(x))
\]

\[
= F^T \Phi(t, x).
\]

i.e.

\[
f_{nn'm'm'} = F^T \Phi(t, x),
\]

here, \( F^T_1 \) and \( F^T_2 \) are operational matrix (for formulation see section 3) of \( \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t) \Phi(t) dt \) and \( \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x) \Phi(x) dx \) respectively.

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2.8. The fractional derivative in the Caputo senses

\((c^\alpha D_t^\alpha \xi)(t, x)\) and \((c^\beta D_t^\beta \xi)(t, x)\) denote the Caputo fractional derivative of order \(\alpha\) and \(\beta\) respectively with respect to time \(t\) and defined by (14):

\[
(c^\alpha D_t^\alpha \xi)(t, x) = \frac{1}{\Gamma(2 - \alpha)} \int_0^t \frac{u_{ss}(s, x)}{(t-s)^{\alpha-1}} ds, \quad \alpha \in (1, 2),
\]

and

\[
(c^\beta D_t^\beta \xi)(t, x) = \frac{1}{\Gamma(2 - \beta)} \int_0^t \frac{\xi_{ss}(s, x)}{(t-s)^{\beta-1}} ds, \quad \beta \in (1, 2).
\]

3. Operational matrices

3.1. Operational matrix of differentiation

Let \(\Phi(t, x)\) be 2-D wavelets (Hermite and Bernoulli) vector defined in subsection 2.2 -2.5, then we have

\[
\frac{d}{dt} \Phi(t, x) = \begin{bmatrix}
\Phi_{1,0}(t) \\
\Phi_{1,1}(t) \\
\Phi_{1,2}(t) \\
\vdots \\
\Phi_{1,M-1}(t) \\
\Phi_{2k-1,0}(t) \\
\Phi_{2k-1,1}(t) \\
\Phi_{2k-1,2}(t) \\
\vdots \\
\Phi_{2k-1,M-1}(t)
\end{bmatrix}
\otimes
\Phi(x) = \begin{bmatrix}
\frac{d}{dt} \Phi_{1,0}(t) \\
\frac{d}{dt} \Phi_{1,1}(t) \\
\frac{d}{dt} \Phi_{1,2}(t) \\
\vdots \\
\frac{d}{dt} \Phi_{1,M-1}(t) \\
\frac{d}{dt} \Phi_{2k-1,0}(t) \\
\frac{d}{dt} \Phi_{2k-1,1}(t) \\
\frac{d}{dt} \Phi_{2k-1,2}(t) \\
\vdots \\
\frac{d}{dt} \Phi_{2k-1,M-1}(t)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
a_{1,0}(t) \\
a_{1,1}(t) \\
a_{1,2}(t) \\
\vdots \\
a_{1,M-1}(t) \\
a_{2k-1,0}(t) \\
a_{2k-1,1}(t) \\
a_{2k-1,2}(t) \\
\vdots \\
a_{2k-1,M-1}(t)
\end{bmatrix}
\otimes
\Phi(x) = (D\Phi(t)) \otimes (I\Phi(x))
\]

\[
= (D \otimes I) (\Phi(t) \otimes \Phi(x))
\]

\[
=D_t \Phi(t, x)
\]
\[
\frac{d}{dt} \Phi(t, x) \approx D \Phi(t, x)
\]  

where, \( D \) is the matrix of order \( 2^{k-1} M \) (see for Bernoulli wavelet in [48] and for Hermite wavelets in ) and also \( I \) is the identity matrix. We call \( D \), namely as Bernoulli/Hermite wavelet operational matrix of differentiation. Trivially \( \Phi^{(r)}(t) = D^r \Phi(t) \) for all positive integers \( r \), where \( \Phi^{(r)}(t) \) is the \( r \)th derivative of \( \Phi(t) \).

### 3.2. Operational matrix of Integration

Let \( \Phi(t, x) \) be 2-D wavelets (Hermite and Bernoulli) vector defined in subsection 2.2 -2.5, then we have

\[
\int_0^t \Phi(s, x) ds = \int_0^t (\Phi(s) \otimes \Phi(x)) ds
\]

\[
= \int_0^t \Phi(s) ds \otimes \Phi(x)
\]

\[
\begin{pmatrix}
\Phi_{1,0}(s) \\
\Phi_{1,1}(s) \\
\Phi_{1,2}(s) \\
\vdots \\
\Phi_{1,M-1}(s) \\
\vdots \\
\Phi_{2^{k-1},0}(s) \\
\Phi_{2^{k-1},1}(s) \\
\Phi_{2^{k-1},2}(s) \\
\vdots \\
\Phi_{2^{k-1},M-1}(t)
\end{pmatrix}
= \int_0^s ds \otimes \Phi(x) = \begin{pmatrix}
\int_0^t \Phi_{1,0}(s) ds \\
\int_0^t \Phi_{1,1}(s) ds \\
\int_0^t \Phi_{1,2}(s) ds \\
\vdots \\
\int_0^t \Phi_{1,M-1}(s) ds \\
\vdots \\
\int_0^t \Phi_{2^{k-1},0}(s) ds \\
\int_0^t \Phi_{2^{k-1},1}(s) ds \\
\int_0^t \Phi_{2^{k-1},2}(s) ds \\
\vdots \\
\int_0^t \Phi_{2^{k-1},M-1}(s) ds
\end{pmatrix}
\otimes \Phi(x)
\]

\[
= \begin{pmatrix}
p_{1,0}(t) \\
p_{1,1}(t) \\
p_{1,2}(t) \\
\vdots \\
p_{1,M-1}(t) \\
\vdots \\
p_{2^{k-1},0}(t) \\
p_{2^{k-1},1}(t) \\
p_{2^{k-1},2}(t) \\
\vdots \\
p_{2^{k-1},M-1}(t)
\end{pmatrix}
\otimes \Phi(x) \approx (P \Phi(t)) \otimes (I \Phi(x))
\]
\[ (P \otimes I)(\Phi(t) \otimes \Phi(x)) = P_t \Phi(t, x) \]

so,
\[
\int_0^t \Phi(s, x) ds \approx P_t \Phi(t, x), \quad (18)
\]

where, \( I \) is \( 2^{k-1}M \times 2^{k-1}M \) identity matrix and \( P_t = P \otimes I \) is a \( 2^{k-1}2^k - 1 MM' \times 2^{k-1}2^k - 1 MM' \) matrix (see [50]).

3.3. Almost operational matrix of singular integral

Lemma 1. Let \( \Phi(t) \) be one-dimensional wavelets (Hermite and Bernoulli) vector defined in subsection 2.2 -2.5, and \( 0 < \alpha, \beta < 1 \) then,
\[
\int_0^t \frac{\Phi(s)}{(t-s)^{\alpha - 1}} ds = Q_1^T \Phi(t)
\]

and
\[
\int_0^t \frac{\Phi(s)}{(t-s)^{\beta - 1}} ds = Q_2^T \Phi(t)
\]

Proof. Let \( \Phi(t) \) be one-dimensional wavelets (Hermite and Bernoulli) vector defined in subsection 2.2 -2.5, and \( 0 < \alpha < 1 \) then,
\[
\int_0^t \frac{\Phi(s)}{(t-s)^{\alpha - 1}} ds = \int_0^t \frac{\Phi(s)}{(t-s)^{\alpha - 1}} ds = \int_0^t \frac{\Phi(s)}{(t-s)^{\alpha}} ds
\]

\[
= \int_0^t \begin{pmatrix}
\frac{\Phi_{10}(s)}{(t-s)^{\alpha}} \\
\frac{\Phi_{11}(s)}{(t-s)^{\alpha}} \\
\frac{\Phi_{12}(s)}{(t-s)^{\alpha}} \\
\vdots \\
\frac{\Phi_{1M-1}(s)}{(t-s)^{\alpha}} \\
\frac{\Phi_{20}(s)}{(t-s)^{\alpha}} \\
\frac{\Phi_{21}(s)}{(t-s)^{\alpha}} \\
\frac{\Phi_{22}(s)}{(t-s)^{\alpha}} \\
\vdots \\
\frac{\Phi_{2k-10}(s)}{(t-s)^{\alpha}} \\
\frac{\Phi_{2k-11}(s)}{(t-s)^{\alpha}} \\
\frac{\Phi_{2k-12}(s)}{(t-s)^{\alpha}} \\
\vdots \\
\frac{\Phi_{2k-1M-1}(s)}{(t-s)^{\alpha}} \\
\end{pmatrix} ds
\]

\[
= \begin{pmatrix}
\int_0^t \frac{\Phi_{10}(s)}{(t-s)^{\alpha}} ds \\
\int_0^t \frac{\Phi_{11}(s)}{(t-s)^{\alpha}} ds \\
\int_0^t \frac{\Phi_{12}(s)}{(t-s)^{\alpha}} ds \\
\vdots \\
\int_0^t \frac{\Phi_{1M-1}(s)}{(t-s)^{\alpha}} ds \\
\int_0^t \frac{\Phi_{20}(s)}{(t-s)^{\alpha}} ds \\
\int_0^t \frac{\Phi_{21}(s)}{(t-s)^{\alpha}} ds \\
\int_0^t \frac{\Phi_{22}(s)}{(t-s)^{\alpha}} ds \\
\vdots \\
\int_0^t \frac{\Phi_{2k-10}(s)}{(t-s)^{\alpha}} ds \\
\int_0^t \frac{\Phi_{2k-11}(s)}{(t-s)^{\alpha}} ds \\
\int_0^t \frac{\Phi_{2k-12}(s)}{(t-s)^{\alpha}} ds \\
\vdots \\
\int_0^t \frac{\Phi_{2k-1M-1}(s)}{(t-s)^{\alpha}} ds \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
q_{10}(t) \\
q_{11}(t) \\
q_{12}(t) \\
\vdots \\
q_{2^k-10}(t) \\
q_{2^k-11}(t) \\
q_{2^k-12}(t) \\
q_{2^k-1M-1}(t)
\end{pmatrix}
\begin{pmatrix}
\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} f_{nm}^{10} \Phi_{nm}(t) \\
\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} f_{nm}^{11} \Phi_{nm}(t) \\
\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} f_{nm}^{12} \Phi_{nm}(t) \\
\vdots \\
\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} f_{nm}^{1(M-1)} \Phi_{nm}(t) \\
\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} f_{nm}^{2^k-10} \Phi_{nm}(t) \\
\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} f_{nm}^{2^k-11} \Phi_{nm}(t) \\
\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} f_{nm}^{2^k-12} \Phi_{nm}(t) \\
\vdots \\
\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} f_{nm}^{2^k-1M-1} \Phi_{nm}(t)
\end{pmatrix}
\approx Q_1 \Phi(t)
\]

i.e.
\[
\int_0^t \Phi(s) \frac{1}{(t-s)^{d-1}} ds \approx Q_1 \Phi(t).
\]

Here, \(Q_1^T = \)
\[
\begin{pmatrix}
 f_{10}^{10} & q_{11}^{10} & f_{12}^{10} & \cdots & f_{1M-1}^{10} & f_{2^k-10}^{10} & f_{2^k-11}^{10} & f_{2^k-12}^{10} & \cdots & f_{2^k-1M-1}^{10} \\
 f_{11}^{10}(t) & f_{11}^{11} & f_{12}^{11} & \cdots & f_{1M-1}^{11} & f_{2^k-10}^{11} & f_{2^k-11}^{11} & f_{2^k-12}^{11} & \cdots & f_{2^k-1M-1}^{11} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 f_{1M-1}^{10} & f_{1M-1}^{11} & f_{1M-1}^{12} & \cdots & f_{1M-1}^{1M-1} & f_{2^k-10}^{1M-1} & f_{2^k-11}^{1M-1} & f_{2^k-12}^{1M-1} & \cdots & f_{2^k-1M-1}^{1M-1} \\
 f_{2^k-10}^{10} & f_{2^k-11}^{10} & f_{2^k-12}^{10} & \cdots & f_{2^k-10}^{1M-1} & f_{2^k-11}^{1M-1} & f_{2k-11}^{1M-1} & f_{2k-12}^{1M-1} & \cdots & f_{2k-1M-1}^{1M-1} \\
 f_{2^k-10}^{11} & f_{2^k-11}^{11} & f_{2^k-12}^{11} & \cdots & f_{2k-10}^{1M-1} & f_{2k-11}^{1M-1} & f_{2k-11}^{1M-1} & f_{2k-12}^{1M-1} & \cdots & f_{2k-1M-1}^{1M-1} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 f_{2^k-10}^{1M-1} & f_{2^k-11}^{1M-1} & f_{2k-12}^{1M-1} & \cdots & f_{2^k-10}^{1M-1} & f_{2k-11}^{1M-1} & f_{2k-11}^{1M-1} & f_{2k-12}^{1M-1} & \cdots & f_{2k-1M-1}^{1M-1}
\end{pmatrix}
\]

where,
\[
f_{2^k-1M-1}^{2^k-1M-1} = \int_{\frac{1}{2^k-1}}^{\frac{1}{2^k-1}} q_{2^k-1M-1}(t) \Phi_{nm}(t) dt.
\]

Similarly,
\[
\int_0^t \Phi(s) \frac{1}{(t-s)^{d-1}} ds \approx Q_2 \Phi(t).
\]

\begin{proof}
\end{proof}

**Theorem 2.** Let \(\Phi(t, x)\) be two-dimensional wavelets (Hermite and Bernoulli) vector defined in subsection 2.2 -2.5, and \(0 < \alpha, \beta < 1\) then,
\[
\int_0^t \Phi(s, x) \frac{1}{(t-s)^{\alpha-1}} ds = Q_\alpha^T \Phi(t, x),
\]
and
\[ \int_0^t \frac{\Phi(s, x)}{(t-s)^{\alpha-1}} ds = Q_1^{T} \Phi(t, x). \]

where \( Q_1^{T} = Q_1 \otimes I \) and \( Q_2^{T} = Q_2 \otimes I \).

Proof. Let \( \Phi(t, x) \) be two-dimensional wavelets (Hermite and Bernoulli) vector defined in subsection 2.2 -2.5, and \( 0 < \alpha < 1 \) then,

\[ \int_0^t \frac{\Phi(s, x)}{(t-s)^{\alpha-1}} ds = \int_0^t \frac{\Phi(s) \otimes \Phi(x)}{(t-s)^{\alpha-1}} ds 
\approx (Q_1 \Phi(t)) \otimes (I \Phi(x)) \quad \text{ (using lemma 1)} 
= (Q_1 \otimes I)(\Phi(t) \otimes \Phi(x)) 
= Q_\alpha \Phi(t, x) \]

so,
\[ \int_0^t \frac{\Phi(s, x)}{(t-s)^{\alpha-1}} ds \approx Q_\alpha \Phi(t, x) \quad (19) \]

where, \( Q_\alpha = Q_1 \otimes I, I \) is identity matrix, and \( Q_1 \) is defined in lemma 1.

Similarly, we can construct operational matrices for \( \int_0^t \frac{\Phi(s, x)}{(t-s)^{\beta-1}} ds \) as follows
\[ \int_0^t \frac{\Phi(s, x)}{(t-s)^{\beta-1}} ds \approx Q_\beta \Phi(t, x), \quad (20) \]

where, \( Q_\beta = Q_2 \otimes I, I \) is identity matrix, and \( Q_2 \) is defined in lemma 1. \( \square \)

4. Numerical method of solution

Let us consider Eq.(1) as follows
\[ (\partial^\alpha D_t^\alpha \xi)(t, x) - \lambda_1 (\partial^\beta D_t^\beta \xi)(t, x) - \lambda_2 \nabla^2 \xi(t, x) = f(t, x) \quad (21) \]

Grouping Eqs.(14 – 15) and (21), we get
\[ \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\xi_s(s, x)}{(t-s)^{\alpha-1}} ds - \frac{\lambda_1}{\Gamma(2-\beta)} \int_0^t \frac{\xi_s(s, x)}{(t-s)^{\beta-1}} ds - \lambda_2 \xi_{xx}(t, x) = f(t, x). \quad (22) \]

Now for numerical solution of Eq.(22), unknown \( u \) is approximated as follows
\[ \xi_u(t, x) \approx \sum_{n=1}^{2^{k-1} M-1} \sum_{m=0}^{2^{k-1} M'-1} \sum_{n'=1}^{M-1} \sum_{m'=0}^{M'-1} c_{nmn'm'} \Phi(t, x) = C^T \Phi(t, x), \quad (23) \]
where $C^T$ is unknown which will be determined.

Integrate Eq. (23) w.r.t. $t$ as follows

$$\xi_t(t, x) - \xi_t(0, x) = C^T P_t \Phi(t, x), \quad (24)$$

again integrate Eq. (24) w.r.t. $t$ as follows

$$\xi(t, x) - \xi(0, x) - \xi_t(0, x) = C^T P_t^2 \Phi(t, x). \quad (25)$$

Now using (2) and Eq. (25) we get

$$\xi(t, x) = g(x) + h(x) + C^T P_t^2 \Phi(t, x). \quad (26)$$

Approximation of $g(x)$ and $h(x)$ are as follows:

$$g(x) \approx G^T \Phi(t, x),$$
$$h(x) \approx H^T \Phi(t, x). \quad (27)$$

So using (26) and (27) we get

$$\xi(t, x) = \left(G^T + H^T + C^T P_t^2\right) \Phi(t, x). \quad (28)$$

Differentiate (28) twice w.r.t. $x$ as follows

$$\xi_{xx}(t, x) = \left(G^T + H^T + C^T P_t^2\right) D_x^2 \Phi(t, x). \quad (29)$$

Grouping (11), (22), (23), (28), and (29), we get

$$\frac{1}{\Gamma(2 - \alpha)} \int_0^t C^T \Phi(s, x) \frac{ds}{(t-s)^{\alpha-1}} - \frac{\lambda_1}{\Gamma(2 - \beta)} \int_0^t C^T \Phi(s, x) \frac{ds}{(t-s)^{\beta-1}} \Phi(t, x)$$
$$- \lambda_2 \left(G^T + H^T + C^T P_t^2\right) D_x^2 \Phi(t, x) = F^T \Phi(t, x). \quad (30)$$

or,

$$\frac{C^T}{\Gamma(2 - \alpha)} \int_0^t \Phi(s, x) \frac{ds}{(t-s)^{\alpha-1}} - \frac{\lambda_1 C^T}{\Gamma(2 - \beta)} \int_0^t \Phi(s, x) \frac{ds}{(t-s)^{\beta-1}} \Phi(t, x)$$
$$- \lambda_2 \left(G^T + H^T + C^T P_t^2\right) D_x^2 \Phi(t, x) = F^T \Phi(t, x). \quad (30)$$

Again grouping (19), (20) and (30), we get

$$\left(\frac{C^T Q_\alpha}{\Gamma(2 - \alpha)} - \frac{\lambda_1 C^T Q_\beta}{\Gamma(2 - \beta)} - \lambda_2 \left(G^T + H^T + C^T P_t^2\right) D_x^2\right) \Phi(t, x) = F^T \Phi(t, x),$$

or,

$$C^T \left(\frac{Q_\alpha}{\Gamma(2 - \alpha)} - \frac{\lambda_1 Q_\beta}{\Gamma(2 - \beta)} - \lambda_2 D_x^2\right) \Phi(t, x) = \left(F^T + \lambda_2 \left(G^T + H^T\right) D_x^2\right) \Phi(t, x). \quad (31)$$
Hence, we have
\[
C^T \left( \frac{Q_\alpha}{\Gamma(2-\alpha)} - \frac{\lambda_1 Q_\beta}{\Gamma(2-\beta)} - \lambda_2 D_x^2 \right) = F^T + \lambda_2 \left( G^T + H^T \right) D_x^2,
\]
or,
\[
C^T = \left( F^T + \lambda_2 \left( G^T + H^T \right) D_x^2 \right) \left( \frac{Q_\alpha}{\Gamma(2-\alpha)} - \frac{\lambda_1 Q_\beta}{\Gamma(2-\beta)} - \lambda_2 D_x^2 \right)^{-1}. \tag{32}
\]
Substituting Eq. (32) back into Eq. (28), we finally have the approximate solution for Eq. (1).

5. Convergence analysis

**Lemma 2.** Let \( X = L^2(I \times I) \), \( I \subset \mathbb{R} \) and \( K \) be Volterra integral operator with square summable kernel \( K(t,x) \) i.e. \( \int_0^1 \int_0^1 k(t,x)dt\,dx = S^2 \), where \( S \) is constant and \( L \) is defined by \( L(g(t,x)) = \int_0^t k(s,x)g(s,x)ds \) then \( L \) is bounded. i.e.
\[
\| L(g(t,x)) \| \leq S \| g(t,x) \|.
\]

**Proof.** See [51]. \( \square \)

**Lemma 3.** [51] Let \( L \) is a linear operators on \( X \) and defined for \( (t,x) \in I \times I \) and \( \Phi \in L^2(I \times I) \) as
\[
L(\xi(t,x)) = \frac{\partial^2 \xi(t,x)}{\partial t^2}
\]
then by Lemma 1, both the operator \( L \) is bounded if \( L \) is 1-1, onto then \( L^{-1} \) is bounded.

**Lemma 4.** [51] Let \( T_\alpha \) and \( T_\beta \) are two linear operators on \( X \) and defined for \( (t,x) \in I \times I \) and \( u \in L^2(I \times I) \) as
\[
T_\alpha(\xi) = \int_0^t \frac{L(\xi(s,x))}{(t-s)^\alpha} ds, \tag{34}
\]
and
\[
T_\beta(\xi) = \int_0^t \frac{L(\xi(s,x))}{(t-s)^\beta} ds, \tag{35}
\]
then by Lemma 1, both the operator \( T_\alpha \) and \( T_\beta \) are bounded if \( T_\alpha \) is 1-1, onto then \( T_\alpha^{-1} \) is bounded.
6. Error estimations

Let $\|\xi_{xx} - (\xi_{xx})_N\|$ and $\|\xi_{xx} - (\xi_{xx})_N\|$ be bounded then using lemma 2 and lemma 3, Eq.(1) takes the form as follows

\[
\frac{1}{\Gamma(2-\alpha)} T_\alpha (\xi(t, x)) - \frac{\lambda_1}{\Gamma(2-\beta)} (T_\beta \xi(t, x)) = f(t, x),
\]

or,

\[
\xi(t, x) - \frac{\lambda_1 \Gamma(2-\alpha)}{\Gamma(2-\beta)} (T_\alpha^{-1} (T_\beta \xi(t, x))) - \lambda_2 \Gamma(2-\alpha) T_\alpha^{-1} \xi_{xx}(t, x) = \Gamma(2-\alpha) T_\alpha^{-1} (f(t, x)).
\]

Let $\xi_N(t, x)$ be the $N^{th}$ Bernoulli and Hermite wavelets approximation of $\xi(t, x)$ and approximate solution of Eq.(36) as follows

\[
\xi_N(t, x) - \frac{\lambda_1 \Gamma(2-\alpha)}{\Gamma(2-\beta)} (T_\alpha^{-1} (T_\beta \xi_N(t, x))) - \lambda_2 \Gamma(2-\alpha) T_\alpha^{-1} (\xi_{xx}(t, x))_N = \Gamma(2-\alpha) T_\alpha^{-1} (f_N(t, x)).
\]

Subtracting Eq.(37) from Eq.(36), we get

\[
e_N(t, x) - \frac{\lambda_1 \Gamma(2-\alpha)}{\Gamma(2-\beta)} (T_\alpha^{-1} (T_\beta e_N(t, x))) = \lambda_2 \Gamma(2-\alpha) T_\alpha^{-1} (\xi_{xx} - (\xi_{xx})_N)
\]

\[
+ \Gamma(2-\alpha) T_\alpha^{-1} (f - f_N).
\]

where $e_N = u - \xi_N$.

Let $T_\alpha^{-1}$ and $T_\beta$ are bounded and bounded by $\Delta_1$ & $\Delta_2$ respectively then taking $L^2 - \text{norm}$ both side as follows

\[
\|e_N(t, x) - \frac{\lambda_1 \Gamma(2-\alpha)}{\Gamma(2-\beta)} (T_\alpha^{-1} (T_\beta e_N(t, x)))\|_2 = \|\lambda_2 \Gamma(2-\alpha) T_\alpha^{-1} (\xi_{xx} - (\xi_{xx})_N)
\]

\[
+ \Gamma(2-\alpha) T_\alpha^{-1} (f - f_N)\|_2.
\]

or,

\[
\left\|e_N(t, x)\right\| - \frac{\lambda_1 \Gamma(2-\alpha)}{\Gamma(2-\beta)} \left\|T_\alpha^{-1}\left\|T_\beta\right\|e_N(t, x)\right\| \leq \left\|\lambda_2 \Gamma(2-\alpha) T_\alpha^{-1} (\xi_{xx} - (\xi_{xx})_N)
\]

\[
+ \Gamma(2-\alpha) T_\alpha^{-1} (f - f_N)\right\|.
\]

or,

\[
\left\|e_N(t, x)\right\| 1 - \frac{\lambda_1 \Gamma(2-\alpha)}{\Gamma(2-\beta)} \left\|T_\alpha^{-1}\left\|T_\beta\right\| \leq \left\|\lambda_2 \Gamma(2-\alpha) T_\alpha^{-1} (\xi_{xx} - (\xi_{xx})_N)
\]

\[
+ \left\|\Gamma(2-\alpha) T_\alpha^{-1} (f - f_N)\right\|.
\]
or,
\[
\|e_N(t,x)\| \leq \frac{\Delta_1 \lambda_2 \Gamma(2-\alpha)}{1 - \frac{\lambda_1 \Gamma(2-\alpha)}{\Gamma(2-\beta)} \Delta_1 \Delta_2} \|\xi_{xx} - (\xi_{xx})_N\| + \Delta_1 \Gamma(2-\alpha)\|f - f_N\|.
\]
(39)

So, the proposed scheme can be applied to solve Eq.(39) approximately for \(e_N(t,x)\).

7. Numerical examples

In this section, we provide numerical examples, which allow for discussing differences in the numerical solutions of the considered restricted FPDE for different number of subdivisions. To illustrate the description above and to test the scheme developed here, and also article shown some numerical solution to Eq.(1). Eq.(1) is solved numerically for different values of \(\alpha, \beta\) and \(N\) and the article has compared the results with some well known Bernoulli and Hermite wavelets. Numerical results of absolute errors provided in Tables 1–2,4-5 and 7-8. And also comparison of \(l^2 - \text{norm}\) errors and \(l^\infty - \text{norm}\) between exact and approximate solutions with schemes are provided in Table 3, 6 and 9. Moreover, the history of errors are depicted in Figs. 1-11. Since, the proposed technique is a numerical wavelets approach based on the operational matrices, one can reach to the exact solutions if the solutions of the considered EMWDM are in polynomial forms. Error functions are defined as

**Absolute error**

\[|e_N(t,x)| = |\xi(t,x) - \xi_N(t,x)|,\]

**\(l^2\) norm error**

\[\|e_N(t,x)\|_2 = \left(\sum_{i=1}^{N} \sum_{j=1}^{N} |\xi(t_i,x_j) - \xi_N(t_i,x_j)|^2\right)^{\frac{1}{2}},\]

and

**\(l^\infty\) norm error**

\[\|e_N(t,x)\|_\infty = \max_{1 \leq i,j \leq N} |\xi(t_i,x_j) - \xi_N(t_i,x_j)|.\]

**Example 1.** Consider the following FPDE:

\[
(_cD_0^\alpha \xi)(t,x) - \lambda_1 (_cD_0^\beta \xi)(t,x) - \lambda_2 \nabla^2 \xi(t,x) = f(t,x),
\]

with homogeneous initial conditions; where \(\Omega = [0,1]\), \(1 < \beta < \alpha < 2\) and we have the exact solution

\[\xi(t,x) = t^{(\alpha)+\beta},\]

where, \{\alpha\} and \{\beta\} are fractional part of \(\alpha\) and \(\beta\) respectively. And also, we have different values of \(f(t,x)\) for different values of \(\alpha\) and \(\beta\).
Behavior of exact and numerical solutions of Example 1 at time $T = 1$ for $\alpha = 1.5, \beta = 1.1$, $\alpha = 1.7, \beta = 1.5$ and $\alpha = 1.9, \beta = 1.3$ using Bernoulli and Hermite wavelets both at small number of basis $k = k' = 1, M = M' = 3$ are show a good similarity between exact and numerical solutions obtained by proposed numerical scheme.

Figure 1: The spatiotemporal distribution of an error between the exact solution and its numerical solution utilizing the Bernoulli and Hermite wavelets approximation for $\alpha = 1.5$ and $\beta = 1.1$ at $k = k' = 1, M = M' = 3$ shown in Fig 1(a) and Fig 1(b) respectively.

Figure 2: The spatiotemporal distribution of an error between the exact solution of Example 1 and its numerical solution utilizing the Bernoulli and Hermite wavelets approximation for $\alpha = 1.7$ and $\beta = 1.3$ at $k = k' = 1, M = M' = 3$ shown in Fig 2(a) and Fig 2(b) respectively.
Figure 3: The spatiotemporal distribution of an error between the exact solution of Example 1 and its numerical solution utilizing the Bernoulli and Hermite wavelets approximation for $\alpha = 1.9$ and $\beta = 1.5$ at $k = k' = 1, M = M' = 3$ shown in Fig 3(a) and Fig 3(b) respectively.

At time $T = 1$ and for different values of $\alpha$ & $\beta$, if we fixed the domain $k = k' = 1$ then the absolute, $L^2$ and $L^\infty$ errors for example 1 are given in Table 1-3 respectively.

**Example 2.**
Consider the following FPDE:

$$(cD_t^\alpha \xi)(t, x) - \lambda_1 (cD_t^\beta \xi)(t, x) - \lambda_2 \nabla^2 \xi(t, x) = f(t, x),$$

with homogeneous initial conditions; where $\Omega = [0, 1], 1 < \beta < \alpha < 2$ and we have the exact solution

$$\xi(t, x) = t^{\{\alpha\} + \{\beta\}} x(t^{\alpha} + t^{\beta} - 1)(x - 1).$$

where, $\{\alpha\}$ and $\{\beta\}$ are fractional part of $\alpha$ and $\beta$ respectively. And also, we have different values of $f(t, x)$ for different values of $\alpha$ and $\beta$.

Behavior of exact and numerical solutions of Example 2 at time $T = 1$ for $\alpha = 1.5, \beta = 1.1$, $\alpha = 1.7, \beta = 1.5$ and $\alpha = 1.9, \beta = 1.3$ using Bernoulli and Hermite wavelets both at small number of basis $k = k' = 1, M = M' = 3$ are show a good similarity between exact and numerical solutions obtained by proposed numerical scheme.
Figure 4: The spatiotemporal distribution of an error between the exact solution of Example 2 and its numerical solution utilizing the Bernoulli and Hermite wavelets approximation for $\alpha = 1.5$ and $\beta = 1.1$ at $k = k' = 1, M = M' = 3$ shown in Fig 4(a) and Fig 4(b) respectively.

Figure 5: The spatiotemporal distribution of an error between the exact solution of Example 2 and its numerical solution utilizing the Bernoulli and Hermite wavelets approximation for $\alpha = 1.7$ and $\beta = 1.3$ at $k = k' = 1, M = M' = 3$ shown in Fig 5(a) and Fig 5(b) respectively.
Figure 6: The spatiotemporal distribution of an error between the exact solution of Example 2 and its numerical solution utilizing the Bernoulli and Hermite wavelets approximation for α = 1.9 and β = 1.5 at k = k’ = 1, M = M’ = 3 shown in Fig 6(a) and Fig 6(b) respectively.

At time $T = 1$ and for different values of $\alpha$ & $\beta$, if we fixed the domain $k = k’ = 1$ then the absolute, $L^2$ and $L^\infty$ errors for example 2 are given in Table 4-6 respectively.

**Example 3.**

Consider the following FPDE:

$$(cD^\alpha_t \xi)(t,x) - \lambda_1(cD^\beta_t \xi)(t,x) - \lambda_2 \nabla^2 \xi(t,x) = f(t,x),$$

with homogeneous initial conditions; where $\Omega = [0, 1], 1 < \beta < \alpha < 2$ and we have the exact solution

$$\xi(t,x) = t^{\alpha+\beta}\sin(\pi x).$$

And also, we have different values of $f(t,x)$ for different values of $\alpha$ and $\beta$.

Behavior of exact and numerical solutions of Example 3 at time $T = 1$ for $\alpha = 1.5, \beta = 1.1$, $\alpha = 1.7, \beta = 1.5$ and $\alpha = 1.9, \beta = 1.3$ using Bernoulli and Hermite wavelets both at small number of basis $k = k’ = 1, M = M’ = 7$ are a little bit slow as comparison to Example 1 and 2. But we can decrease error by extending a number of bases.
Figure 7: The spatiotemporal distribution of an error between the exact solution of Example 2 and its numerical solution utilizing the Bernoulli and Hermite wavelets approximation for $\alpha = 1.5$ and $\beta = 1.1$ at $k = k' = 1, M = M' = 7$ shown in Fig 7(a) and Fig 7(b) respectively.
Figure 8: The spatiotemporal distribution of an error between the exact solution of Example 2 and its numerical solution utilizing the Bernoulli and Hermite wavelets approximation for $\alpha = 1.7$ and $\beta = 1.3$ at $k = k' = 1$, $M = M' = 7$ shown in Fig 8(a) and Fig 8(b) respectively.
Figure 9: The spatiotemporal distribution of an error between the exact solution of Example 2 and its numerical solution utilizing the Bernoulli and Hermite wavelets approximation for $\alpha = 1.9$ and $\beta = 1.5$ at $k = k' = 1, M = M' = 7$ shown in Fig 9(a) and Fig 9(b) respectively.

At time $T = 1$ and for different values of $\alpha$ & $\beta$, if we fixed the domain $k = k' = 1$ then the absolute, $L^2$ and $L^\infty$ errors for example 1 are given in Table 7-9 respectively.
Figure 10: Absolute errors of Example 3 using BWA at time $T = 1$ for $k = k' = 1, M = M' = 7$ and different values of $\alpha$ & $\beta$

Figure 11: Absolute errors of Example 3 using HWA at time $T = 1$ for $k = k' = 1, M = M' = 7$ and different values of $\alpha$ & $\beta$
Table 1: Absolute Error of Example 1. via BWA at $k = k' = 1, M = M' = 3$

| (a,b)  | $\alpha = 1.5, \beta = 1.1$ | $\alpha = 1.7, \beta = 1.3$ | $\alpha = 1.9, \beta = 1.5$ |
|--------|-------------------------------|-------------------------------|-------------------------------|
| (0.1, 0.1) | $2.4822 \times e^{-15}$ | $1.8908 \times e^{-15}$ | $2.9776 \times e^{-13}$ |
| (0.2, 0.2) | $2.5396 \times e^{-15}$ | $2.1233 \times e^{-15}$ | $4.4077 \times e^{-13}$ |
| (0.3, 0.3) | $2.8380 \times e^{-15}$ | $2.6368 \times e^{-15}$ | $5.9331 \times e^{-13}$ |
| (0.4, 0.4) | $3.4556 \times e^{-15}$ | $3.4972 \times e^{-15}$ | $7.5508 \times e^{-13}$ |
| (0.5, 0.5) | $4.3854 \times e^{-15}$ | $4.7184 \times e^{-15}$ | $9.2565 \times e^{-13}$ |
| (0.6, 0.6) | $5.7732 \times e^{-15}$ | $6.3283 \times e^{-15}$ | $1.1046 \times e^{-12}$ |
| (0.7, 0.7) | $7.6050 \times e^{-15}$ | $8.4932 \times e^{-15}$ | $1.2914 \times e^{-12}$ |
| (0.8, 0.8) | $9.6589 \times e^{-15}$ | $1.0991 \times e^{-15}$ | $1.4853 \times e^{-12}$ |
| (0.9, 0.9) | $1.2546 \times e^{-14}$ | $1.4100 \times e^{-14}$ | $1.6857 \times e^{-12}$ |

Table 2: Absolute Error of Example 1. via HWA at $k = k' = 1, M = M' = 3$

| (a,b)  | $\alpha = 1.5, \beta = 1.1$ | $\alpha = 1.7, \beta = 1.3$ | $\alpha = 1.9, \beta = 1.5$ |
|--------|-------------------------------|-------------------------------|-------------------------------|
| (0.1, 0.1) | $1.7897 \times e^{-15}$ | $8.1137 \times e^{-15}$ | $1.5042 \times e^{-13}$ |
| (0.2, 0.2) | $1.8562 \times e^{-15}$ | $1.4884 \times e^{-14}$ | $2.6983 \times e^{-13}$ |
| (0.3, 0.3) | $2.1441 \times e^{-15}$ | $2.2302 \times e^{-14}$ | $3.9762 \times e^{-13}$ |
| (0.4, 0.4) | $2.7200 \times e^{-15}$ | $3.0323 \times e^{-14}$ | $5.3348 \times e^{-13}$ |
| (0.5, 0.5) | $3.5805 \times e^{-15}$ | $3.8858 \times e^{-14}$ | $6.7701 \times e^{-13}$ |
| (0.6, 0.6) | $4.8295 \times e^{-15}$ | $4.7906 \times e^{-14}$ | $8.2778 \times e^{-13}$ |
| (0.7, 0.7) | $6.4948 \times e^{-15}$ | $5.7176 \times e^{-14}$ | $9.8521 \times e^{-13}$ |
| (0.8, 0.8) | $8.5487 \times e^{-15}$ | $6.6035 \times e^{-14}$ | $1.1487 \times e^{-12}$ |
| (0.9, 0.9) | $1.1020 \times e^{-15}$ | $6.6272 \times e^{-14}$ | $1.3178 \times e^{-12}$ |

Table 3: $l^2$ and $l^\infty$ norm error of Example 1. at $k = 1, M = 3$

| Norm | $\alpha = \frac{4}{3}, \beta = \frac{5}{4}$ | $\alpha = \frac{3}{3}, \beta = \frac{5}{4}$ | $\alpha = \frac{7}{3}, \beta = \frac{9}{4}$ |
|------|------------------------------------------|------------------------------------------|------------------------------------------|
| $l^2$ for BWA | $1.6857 \times e^{-12}$ | $7.7938 \times e^{-14}$ | $1.2286 \times e^{-11}$ |
| $l^\infty$ for BWA | $1.0477 \times e^{-14}$ | $1.0477 \times e^{-14}$ | $1.0775 \times e^{-14}$ |
| $l^2$ for HWA | $7.1579 \times e^{-14}$ | $6.5935 \times e^{-13}$ | $1.9019 \times e^{-11}$ |
| $l^\infty$ for HWA | $1.1815 \times e^{-14}$ | $1.1593 \times e^{-13}$ | $1.1059 \times e^{-11}$ |
Table 4: Absolute Error of Example 2. via BWA at $k = k' = 1, M = M' = 3$

| (a,b) | $\alpha = 1.5, \beta = 1.1$ | $\alpha = 1.7, \beta = 1.3$ | $\alpha = 1.9, \beta = 1.5$ |
|-------|-----------------|-----------------|-----------------|
| (0.1,0.1) | $7.6328 \times e^{-16}$ | $4.1599 \times e^{-15}$ | $4.1947 \times e^{-14}$ |
| (0.2,0.2) | $7.9450 \times e^{-16}$ | $6.8522 \times e^{-15}$ | $6.5292 \times e^{-14}$ |
| (0.3,0.3) | $8.5348 \times e^{-16}$ | $9.7353 \times e^{-15}$ | $9.0213 \times e^{-14}$ |
| (0.4,0.4) | $9.0206 \times e^{-16}$ | $1.2858 \times e^{-15}$ | $1.1673 \times e^{-13}$ |
| (0.5,0.5) | $9.7145 \times e^{-16}$ | $1.6168 \times e^{-14}$ | $1.4481 \times e^{-13}$ |
| (0.6,0.6) | $1.0547 \times e^{-15}$ | $1.9706 \times e^{-14}$ | $1.7454 \times e^{-13}$ |
| (0.7,0.7) | $1.1449 \times e^{-15}$ | $2.3474 \times e^{-14}$ | $2.0580 \times e^{-13}$ |
| (0.8,0.8) | $1.2525 \times e^{-15}$ | $2.7461 \times e^{-14}$ | $2.3868 \times e^{-13}$ |
| (0.9,0.9) | $1.3583 \times e^{-15}$ | $3.1657 \times e^{-14}$ | $2.7312 \times e^{-13}$ |

Table 5: Absolute Error of Example 2. via HWA at $k = k' = 1, M = M' = 3$

| (a,b) | $\alpha = 1.5, \beta = 1.1$ | $\alpha = 1.7, \beta = 1.3$ | $\alpha = 1.9, \beta = 1.5$ |
|-------|-----------------|-----------------|-----------------|
| (0.1,0.1) | $6.3838 \times e^{-15}$ | $5.7263 \times e^{-15}$ | $5.3971 \times e^{-14}$ |
| (0.2,0.2) | $8.3614 \times e^{-15}$ | $8.9026 \times e^{-15}$ | $8.1418 \times e^{-14}$ |
| (0.3,0.3) | $1.0478 \times e^{-15}$ | $1.2344 \times e^{-14}$ | $1.1061 \times e^{-13}$ |
| (0.4,0.4) | $1.3045 \times e^{-15}$ | $1.6064 \times e^{-14}$ | $1.4164 \times e^{-13}$ |
| (0.5,0.5) | $1.5543 \times e^{-15}$ | $2.0995 \times e^{-14}$ | $1.7440 \times e^{-13}$ |
| (0.6,0.6) | $1.9013 \times e^{-15}$ | $2.4356 \times e^{-14}$ | $2.0897 \times e^{-13}$ |
| (0.7,0.7) | $2.2274 \times e^{-15}$ | $2.8887 \times e^{-14}$ | $2.4535 \times e^{-13}$ |
| (0.8,0.8) | $2.5917 \times e^{-15}$ | $3.3713 \times e^{-14}$ | $2.8346 \times e^{-13}$ |
| (0.9,0.9) | $2.9577 \times e^{-15}$ | $3.8801 \times e^{-14}$ | $3.2338 \times e^{-13}$ |

Table 6: $l^2$ and $l^\infty$ norm error of Example 2. at $k = 1, M = 3$

| Norm | $\alpha = 1.5, \beta = 1.1$ | $\alpha = 1.7, \beta = 1.3$ | $\alpha = 1.9, \beta = 1.5$ |
|------|-----------------|-----------------|-----------------|
| $l^2$ for BWA | $1.8731 \times e^{-14}$ | $2.2419 \times e^{-13}$ | $1.9591 \times e^{-12}$ |
| $l^\infty$ for BWA | $1.3151 \times e^{-14}$ | $1.9091 \times e^{-13}$ | $1.6821 \times e^{-12}$ |
| $l^2$ for HWA | $2.1800 \times e^{-14}$ | $2.7644 \times e^{-13}$ | $2.3316 \times e^{-12}$ |
| $l^\infty$ for HWA | $3.6589 \times e^{-14}$ | $4.8529 \times e^{-13}$ | $4.0124 \times e^{-12}$ |
Table 7: Absolute Error of Example 3. via BWA at $k = k' = 1, M = M' = 3$

| (a,b) | $\alpha = 1.5, \beta = 1.1$ | $\alpha = 1.7, \beta = 1.3$ | $\alpha = 1.9, \beta = 1.5$ |
|-------|-----------------------------|-----------------------------|-----------------------------|
| (0.1,0.1) | $1.0270 \times 10^{-5}$ | $3.8663 \times 10^{-4}$ | $3.1000 \times 10^{-3}$ |
| (0.2,0.2) | $9.2100 \times 10^{-6}$ | $3.3100 \times 10^{-4}$ | $2.7000 \times 10^{-3}$ |
| (0.3,0.3) | $8.2782 \times 10^{-6}$ | $2.7828 \times 10^{-4}$ | $2.4000 \times 10^{-3}$ |
| (0.4,0.4) | $7.4088 \times 10^{-6}$ | $2.2861 \times 10^{-4}$ | $2.0000 \times 10^{-3}$ |
| (0.5,0.5) | $6.5372 \times 10^{-6}$ | $1.8212 \times 10^{-4}$ | $1.7000 \times 10^{-3}$ |
| (0.6,0.6) | $5.6000 \times 10^{-6}$ | $1.3890 \times 10^{-4}$ | $1.4000 \times 10^{-3}$ |
| (0.7,0.7) | $4.5356 \times 10^{-6}$ | $9.9038 \times 10^{-5}$ | $1.0000 \times 10^{-3}$ |
| (0.8,0.8) | $3.2847 \times 10^{-6}$ | $6.2580 \times 10^{-5}$ | $6.6918 \times 10^{-4}$ |
| (0.9,0.9) | $1.7907 \times 10^{-6}$ | $2.9563 \times 10^{-5}$ | $3.3224 \times 10^{-4}$ |

Table 8: Absolute Error of Example 3. via HWA at $k = k' = 1, M = M' = 3$

| (a,b) | $\alpha = 1.5, \beta = 1.1$ | $\alpha = 1.7, \beta = 1.3$ | $\alpha = 1.9, \beta = 1.5$ |
|-------|-----------------------------|-----------------------------|-----------------------------|
| (0.1,0.1) | $2.4825 \times 10^{-5}$ | $3.5759 \times 10^{-4}$ | $3.1000 \times 10^{-3}$ |
| (0.2,0.2) | $2.3537 \times 10^{-5}$ | $3.0423 \times 10^{-4}$ | $2.7000 \times 10^{-3}$ |
| (0.3,0.3) | $2.2016 \times 10^{-5}$ | $2.5412 \times 10^{-4}$ | $2.4000 \times 10^{-3}$ |
| (0.4,0.4) | $2.0197 \times 10^{-5}$ | $2.0736 \times 10^{-4}$ | $2.1000 \times 10^{-3}$ |
| (0.5,0.5) | $1.8020 \times 10^{-5}$ | $1.6403 \times 10^{-4}$ | $1.7000 \times 10^{-3}$ |
| (0.6,0.6) | $1.5429 \times 10^{-5}$ | $1.2420 \times 10^{-4}$ | $1.3000 \times 10^{-3}$ |
| (0.7,0.7) | $1.2374 \times 10^{-5}$ | $8.7886 \times 10^{-5}$ | $1.0000 \times 10^{-3}$ |
| (0.8,0.8) | $8.1000 \times 10^{-6}$ | $5.5101 \times 10^{-5}$ | $6.6207 \times 10^{-4}$ |
| (0.9,0.9) | $4.6964 \times 10^{-6}$ | $2.5821 \times 10^{-5}$ | $3.2758 \times 10^{-4}$ |

Table 9: $l^2$ and $l^\infty$ norm error of Example 3. at $k = 1, M = 3$

| Norm | $\alpha = 1.5, \beta = 1.1$ | $\alpha = 1.7, \beta = 1.3$ | $\alpha = 1.9, \beta = 1.5$ |
|------|-----------------------------|-----------------------------|-----------------------------|
| $l^2$ for BWA | $2.0552 \times 10^{-5}$ | $6.7513 \times 10^{-4}$ | $5.8000 \times 10^{-3}$ |
| $l^\infty$ for BWA | $1.0210 \times 10^{-5}$ | $3.8663 \times 10^{-4}$ | $3.1000 \times 10^{-3}$ |
| $l^2$ for HWA | $5.3651 \times 10^{-5}$ | $6.1781 \times 10^{-4}$ | $3.1000 \times 10^{-3}$ |
| $l^\infty$ for HWA | $2.4825 \times 10^{-5}$ | $3.5759 \times 10^{-4}$ | $3.1000 \times 10^{-3}$ |

8. Conclusion

The electromagnetic waves in dielectric media (EMWDM) are of special interest as they are used to understand various complex and physical phenomena. In this article, time-space Bernoulli and Hermite wavelets collocation method has been successfully computed the numerical approach based on time-dependent FPDE. In this study, we have used Bernoulli and Hermite wavelets operational matrices of integration and differentiation for discretizing both temporal and spatial variables that reduced the considered problems into
a system of linear algebraic equations. The obtained numerical solutions are in
good agreement with the exact analytical solutions. A small trivial conclusion
of this article is that a very accurate numerical solution may be achieved by
implementing the proposed numerical approach in a rarely less computational
time by using a small number of wavelet bases. Additionally, we examined
convergence and error analysis of the proposed numerical approach. At last,
from the considered examples, convergence analysis, and error analysis, it can
be effectively observed that the proposed numerical approach for FPDE arising
from EWDM has acquired the outcome as precise as could be expected under
the circumstances.

Acknowledgement

The first author acknowledge the financial support under National Post-
doctoral Fellowship from Science and Engineering Research Board, India, with
sanction file no. PDF/2019/001275. The author would like to thank the re-
viewers for the comment as very good and priority publishing.

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