Decimated generalized Prony systems

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Abstract

We continue studying robustness of solving algebraic systems of Prony type (also known as the exponential fitting systems), which appear prominently in many areas of mathematics, in particular modern “sub-Nyquist” sampling theories. We show that by considering these systems at arithmetic progressions (or “decimating” them), one can achieve better performance in the presence of noise. We also show that the corresponding lower bounds are closely related to well-known estimates, obtained for similar problems but in different contexts.

1 Introduction

The system of equations

\[ m_k = \sum_{j=1}^{K} a_j z_j^k, \quad a_j, z_j \in \mathbb{C}, \ k \in \mathbb{N} \quad (1.1) \]

appeared originally in the work of Baron de Prony [33] in the context of fitting a sum of exponentials to observed data samples. He showed that the unknowns \( a_j, z_j \) can be recovered explicitly from \( \{m_0, \ldots, m_{2K-1}\} \) by what is known today as “Prony’s method”. This “Prony system” appears in areas such as frequency estimation, Padé approximation, array processing, statistics, interpolation, quadrature, radar signal detection, error correction codes, and many more. The literature on this subject is huge (for instance, the bibliography on Prony’s method from [2] is some 50+ pages long). Original Prony’s solution method can be found in many places, e.g. [34].

Our interest in (1.1) and more general systems of “Prony type”, defined in Table 1 below, originates from their central role in the so-called algebraic sampling approach to problems of signal reconstruction. The basic idea there is to model the unknown signal by a small number of parameters, and subsequently reconstruct these parameters from the given small number of noisy measurements – in fact, from a number which is much smaller than would be required by the classical Shannon-Nyquist-Kotel’nikov-Whittaker sampling theorem (hence the name “sub-Nyquist sampling”). In many cases of interest, Prony-type systems appear precisely as the equations relating the parameters to the measurements. Let us give some examples.

Arguably the simplest sparse signal model, which is essentially non-bandlimited (and therefore inaccessible to the classical sampling theory), is given by a linear combination of a finite number of Dirac \( \delta \)-distributions (“spikes”, or simply “Diracs”):

\[ f(x) = \sum_{j=1}^{K} a_j \delta(x - \xi_j), \quad a_j \in \mathbb{R}, \ \xi_j \in \mathbb{R}. \quad (1.2) \]
Such \( f(x) \) is a useful model for many types of natural signals, as well as in ranging and wideband communication \[28\]. While Shannon’s sampling theorem is inapplicable in this case (it would require an infinitely fast sampling rate), it has been shown in \[43\] that such signals can be perfectly reconstructed from just \( 2K \) samples of the low-pass filtered version of \( f(x) \), for an appropriately chosen convolution kernel. After some algebraic manipulations, the algorithm amounts to recovering (1.2) from its Fourier samples \[
\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt = \sum_{j=1}^{K} \hat{a}_j e^{-i\xi_j^k}, \tag{1.3} \]
which leads precisely to the powersum fitting problem \[14\] with nodes on the unit circle, i.e. \(|z_j| = 1\).

The function (1.2) is a special case of what the authors of \[43\] called a signal with finite rate of innovation (FRI). Informally, such signals can be described as having a finite number of degrees of freedom per unit of time. Many of the FRI sampling architectures proposed since the appearance of the paper \[43\], ultimately reduce to the powersum fitting problem \[28\]. The underlying idea is that for perfect reconstruction, it is sufficient to sample such signals at their rate of innovation, and not at their Nyquist rate. Many types of signals have been shown to be perfectly reconstructable by FRI techniques, in particular nonuniform splines and piecewise polynomials \[16\]. In the latter case, the following generalization of the model (1.2) is considered:

\[
f(x) = \sum_{j=1}^{K} \sum_{\ell=0}^{\ell_j-1} a_{\ell,j} \delta^{(\ell)}(x - \xi_j), \quad a_{\ell,j} \in \mathbb{R}, \ \xi_j \in \mathbb{R}. \tag{1.4} \]

In this case, (1.3) becomes, after a change of variables, the following polynomial Prony system

\[
m_k = \sum_{j=1}^{K} \sum_{\ell=0}^{\ell_j-1} \bar{a}_{\ell,j} k^\ell, \quad a_{\ell,j} \in \mathbb{C}, \ |z_j| = 1. \tag{1.5} \]

Yet another generalization of (1.1) is the “confluent Prony” system

\[
m_k = \sum_{j=1}^{K} \sum_{\ell=0}^{\ell_j-1} a_{\ell,j}(k) \ell_j k^\ell, \quad z_j \in \mathbb{C} \setminus \{0\}, \ a_{\ell,j} \in \mathbb{C}, \tag{1.6} \]

where \((k)_{\ell}\) is the Pochhammer symbol for the falling factorial

\[
(k)_{\ell} \overset{\text{def}}{=} k(k - 1) \cdots (k - \ell + 1). \]

It appears for example in the problem of reconstructing quadrature domains from their moments \[22\,23\].

In the remainder of this paper we call (1.1), (1.5) and (1.6) by the name Prony-type systems. For convenience, we put all the formulas together into Table 1.

In this work (and also in the literature) the unknowns \(\{z_j\}\) (or the corresponding angles \(\xi_j = \pm \arg z_j\)) are frequently called “poles”, “nodes” or “jumps”, while the linear coefficients \(\{a_{\ell,j}\}\) are called “magnitudes”.

We denote the number of unknown coefficients \(a_{\ell,j}\) by \(C \overset{\text{def}}{=} \sum_{i=1}^{K} \ell_i\), and the overall number of unknown parameters by \(R \overset{\text{def}}{=} C + K\).

For more details on the algebraic sampling approach we refer to \[36\,35\,43\].

An important problem for the applicability of the algebraic reconstruction techniques is one of stability.

**Problem 1.1** (Robust Prony solution). How robustly can the parameters \(\{a_{\ell,j}, z_j\}\) be recovered from the noisy data \(\{\tilde{m}_k = m_k + \delta_k\}_{k=0}^{N}\)?
| Type         | Formula                                                                 | Assumptions |
|--------------|-------------------------------------------------------------------------|-------------|
| Basic        | $m_k = \sum_{j=1}^{K} a_j z_j^k$                                        | $a_j, z_j \in \mathbb{C}$ |
| Confluent    | $m_k = \sum_{j=1}^{\ell_j} \sum_{\ell=0}^{\ell_j-1} a_{\ell,j}(k) \ell z_j^{k-\ell}$ | $a_{\ell,j}, z_j \in \mathbb{C}$ |
| Polynomial   | $m_k = \sum_{j=1}^{K} \sum_{\ell=0}^{\ell_j-1} a_{\ell,j} k^\ell z_j^k$   | $a_{\ell,j} \in \mathbb{C}, |z_j| = 1$ |

Table 1: Prony type systems

Even more interesting and widely open problem is the following special case of Problem 1.1.

**Problem 1.2** (Problem of superresolution). How robustly can two closely spaced poles \(\{z_1, z_2\}\) be recovered from the noisy data \(\{\tilde{m}_k = m_k + \delta_k\}_{k=0}^N\)?

Stable solution of Prony-type systems turns out to be a difficult problem, and in recent years many algorithms have been devised for this task (e.g. \([1, 2, 3, 4, 5, 10, 13, 24, 26, 30, 31, 32, 34, 40, 42]\)). The basic idea underlying most of them is some kind of separation of variables – it turns out that the nonlinear parameters \(\{z_j\}\) can be found independently of the linear ones \(\{a_{\ell,j}\}\) by a kind of “elimination”.

Following \([34, 41, 28]\) and \([13]\), we roughly divide the various methods into several groups as follows. Note that hybrid approaches such as \([13]\) also exist.

1. **Prony-like methods** (polynomial root finding/annihilating filter, Pisarenko’s method, Approximate Prony method \([32]\)). These are based on the original Prony’s method \([33]\) of constructing a Hankel matrix \(H\) from the samples \(\{m_k\}_{k=0}^{2C-1}\), finding a vector \(v\) in the nullspace of \(H\) and constructing a polynomial from the entries of \(v\) whose roots are the unknown \(\{z_j\}_{j=1}^K\) with the corresponding multiplicites \(\{\ell_j\}_{j=1}^K\). These methods are considered to be numerically unstable.

2. **Subspace-based methods** (matrix pencils, MUSIC, ESPRIT and generalized ESPRIT). These methods utilize the special structure of the signal subspace (manifested in the so-called rotational invariance property), and in general have superior performance for large number of samples \(N \gg 1\) and Gaussian noise.

3. **Least squares based methods** (Nonlinear Least Squares, Total Least Squares, Separable Nonlinear Least Squares). Given a good initial approximation, these methods perform well in many cases, see \([21]\).

4. **Algebraic methods** (Cornell’s method \([12]\) and its generalizations, Eckhoff’s elimination method \([17]\)). These methods require solving extremely nontrivial nonlinear equations, explicit formulas are feasible only for small number of nodes, and they tend to be numerically unstable \([28]\).

5. **\(\ell_1\)-/total variation minimization \([11]\)**. This recent approach uses ideas from compressed sensing for reconstructing signals of the form \(\{m_k\}\) from the first \(N\) noisy Fourier samples in \([1.1]\). Under an explicit node separation assumption of the form

\[
\Delta \overset{\text{def}}{=} \min_{i \neq j} |z_i - z_j| \geq \frac{2}{N},
\]
a stable recovery is possible, while the $\ell_1$-norm of the error satisfies  
\[ \| \tilde{f} - f \|_1 \lesssim N^{-2} \varepsilon \]

where $\varepsilon$ is the $L_2$-error in the input.

Turning to best possible performance of any method whatsoever, we are aware of two general results – the Cramer-Rao bounds (CRB) for the Polynomial Amplitude Complex Exponential (PACE) model [4], and Donoho’s lower bounds for recovery of sparse measures [15]. The former approach gives fairly elaborate estimates, which are unfortunately not directly applicable to many problems of algebraic sampling where no assumptions can be made on the noise except an absolute bound on its magnitude. The latter bounds are of different kind, however they are not entirely satisfactory either, since e.g. they provide only $L_2$-norm estimates, while in many applications a bound for $|\Delta z_j|$ is required. We provide more details on these two results in Section 5 below.

In [10] we considered the problem of estimating the best possible accuracy of solving the confluent Prony system (1.6) from the noisy measurements $\{\tilde{m}_k\}_{k \in S}$ for the index subset $S = \{0, 1, \ldots, R - 1\}$. The assumption that the number of equations equals the number of unknowns is not unreasonable in applications, and furthermore there exist indications such as the work [31], that increasing the number of measurements might actually result in deterioration in stability of solution. Under this assumption, we defined the local stability as the Lipschitz constant of the “inverse Prony map”, and estimated this constant at each point in the measurement space $\mathbb{C}^R$ where the inverse is defined. We have shown that in this case, if the noise is bounded in $\ell_\infty$ norm by $\varepsilon$, then the local accuracy is bounded as follows:

\[ |\Delta a_{\ell,j}| \leq C_1 \varepsilon \left( 1 + \frac{|a_{\ell-1,j}|}{|a_{\ell,j-1}|} \right) \quad 0 \leq \ell \leq \ell_j - 1, \quad j = 1, \ldots, K; \]

\[ |\Delta z_j| \leq C_1 \varepsilon, \quad j = 1, \ldots, K; \]

where $C_1$ is the maximal row sum norm of the inverse confluent Vandermonde matrix defined on the nodes $\{z_1, \ldots, z_K\}$ with corresponding multiplicities $\{\ell_1 + 1, \ldots, \ell_K + 1\}$. In fact (see also Remark 2.13 below), in Subsection 3.2 below we show that this constant can be essentially bounded by $\delta^{-R}$ where $\delta$ is the node separation

\[ \delta \overset{\text{def}}{=} \min_{i<j} |z_i - z_j|. \]

The “Prony map” method certainly cannot be applied in the case of oversampling, i.e. when taking $S = \{0, 1, \ldots, N - 1\}$ for some large $N \gg R$ and solving the resulting system in some least-squares sense. While oversampling is certainly justified in the case of noise with a known statistical distribution, it is not a-priori clear that it would provide any increase in robustness (and as we pointed out, there are indications to the contrary). That said, it is natural to assume that one can somehow “utilize” the additional information and obtain a better accuracy of reconstruction, while staying with small number of measurements.

Our main goal in this paper is to show that such a utilization is indeed possible. In particular, we extend the analysis of [10] to Prony type systems on evenly spaced sampling sets with starting index $t$ and step size $p$:

\[ S_{t,p} = \{t, t + p, t + 2p, \ldots, t + (R - 1)p\}. \]

(1.7)

Such “decimation” turns out to retain the essential structure of the problem, while reducing the Lipschitz constant of the inverse Prony map. In particular, denoting

\[ \delta_p \overset{\text{def}}{=} \min_{i<j} |z_i^p - z_j^p|, \]
the error amplification is shown to satisfy (see Theorems 2.7 and 2.8)

\[ |\Delta z_j| \lesssim p^{-\ell_j} \delta_p^{-R} \varepsilon \]
\[ |\Delta a_{\ell,j}| \lesssim p^{-\ell} \delta_p^{-R+\ell-\ell_j} \varepsilon. \]

Consequently, decimation provides an improvement in accuracy of the order \( p^{\ell_j} \) (see Corollary 2.15 for the precise statement). In qualitative terms, these bounds are immediately seen to be very similar to the CRB bounds for the PACE model (see Subsection 5.1). Numerical experiments, presented in Section 4, suggest that indeed decimation leads to improvement in performance of algorithms for solving Prony type systems. Furthermore, for closely spaced nodes we have \( \delta_p \gtrsim \frac{p^2}{2} \) for moderate values of \( p \) (see Lemma 2.16), and therefore by Corollary 2.17 in this case decimation provides improvement in accuracy by the overall factor of \( p^{R+\ell_j} \). This effect can be considered as a type of “superresolution”. In fact, we show that the Prony stability bounds in this case are asymptotically of the same order as Donoho’s bounds – see Subsection 5.2 for details.

A method very similar to decimation, called “subspace shifting”, or interleaving, was proposed by Maravic & Vetterli in [29] in the context of FRI sampling in the presence of noise. Their idea was to interleave the rows of the Hankel matrix used in subspace estimation methods, effectively increasing the separation of closely spaced nodes. They confirmed this idea with numerical experiments. The results of this paper can be informally considered as another justification of their approach.

The system (1.5) is of central importance in Eckhoff’s approach to overcoming the Gibbs phenomenon, and has been a subject of considerable interest to us in this context. In particular, K.Eckhoff conjectured that the discontinuity locations of a piecewise-smooth function, having \( d \) continuous derivatives between the jumps, can be reconstructed from its first \( N \) Fourier coefficients with accuracy \( \sim N^{-d-2} \), by solving a particular instance of the noisy system (1.5) with sufficiently high accuracy. The main problem which remained unsolved was: is this high accuracy indeed achievable, given the assumptions on the noise? We have recently provided a solution to this problem in [6, 9]. In Section 6 we explain how those results can be reinterpreted in the framework of stability of Prony type systems, and in particular the decimation technique.

2 Decimation and optimal recovery

2.1 Definitions

Now we consider Problem 1.1 for sampling sets \( S_{t,p} \) as in (1.7) on the preceding page.

**Definition 2.1.** A Prony-type system is called **decimated** if it is considered with \( S_{t,p} \) where \( p > 1 \), and **non-decimated** if \( p = 1 \).

**Definition 2.2.** A Prony-type system is called **shifted** if it is considered with \( S_{t,p} \) where \( t > 0 \), and **non-shifted** if \( t = 0 \).

**Remark 2.3.** Consider the system (1.5) in the case \( t = 0 \) and \( p \gtrsim 1 \). It is easy to see that by making the change of variables

\[
\begin{align*}
  b_{i,j} &= a_{i,j} p^i, \\
  w_j &= z_j^p, \\
  n_k &= m_{kp},
\end{align*}
\]

we arrive at the non-decimated system

\[
n_k = \sum_{j=1}^{\infty} w_j^k \sum_{i=0}^{\ell_j-1} b_{i,j} k^i, \quad k = 0, 1, 2, \ldots.
\]
For studying stable recovery, we introduced in [10] the following framework.

**Definition 2.4.** Let \( P : \mathbb{C}^R \to \mathbb{C}^R \) be some differentiable mapping, and let \( x \in \mathbb{C}^R \) be a regular point of \( P \). Assume that \( \varepsilon \) is small enough so that that the inverse function \( N = P^{-1} \) exists in \( \varepsilon \)-neighborhood of \( y = P(x) \). For every \( 1 \leq r \leq R \), let \( [v]_r \) denote the \( r \)-th component of the vector \( v \in \mathbb{C}^R \). The best possible local point-wise accuracy of inverting \( P \) with each noise component bounded above by \( \varepsilon \) at the point \( x \) with respect to the component \( r \) is

\[
\text{ACC}_{LOC}^{(P)}(x, \varepsilon, r) = \sup_{y \in B(y, \varepsilon)} |[J_N(y)](\hat{y} - y)\rangle_r|
\]

where \( J_N(y) \) is the Jacobian of \( N \) at the point \( y \).

**Definition 2.5.** The measurement mapping \( P_{t,p}^{(P)} : \mathbb{C}^R \to \mathbb{C}^R \) (resp. \( P_{t,p}^{(C)} \)) is defined by the Prony equations (1.5) (resp. (1.6)) on \( S_{t,p} \):

\[
P_{t,p}^{(P)}([z_j],\{a_{i,j}\}) = (m_t, m_{t+p}, \ldots, m_{t+(R-1)p}), \quad m_k = \sum_{j=1}^{K} \sum_{i=0}^{\ell_j-1} a_{i,j} k^i;
\]

\[
P_{t,p}^{(C)}([z_j],\{a_{i,j}\}) = (m_t, m_{t+p}, \ldots, m_{t+(R-1)p}), \quad m_k = \sum_{j=1}^{K} a_{i,j}(k)z_j^{k-i}.
\]

### 2.2 Main results

By factorizing the Jacobians of \( P_{t,p}^{(P)} \) and \( P_{t,p}^{(C)} \), we get the following results. The proofs are rather technical and they are presented in Section 3.

**Lemma 2.6.** The point \( x = ([a_{i,j}], [z_j]) \in \mathbb{C}^R \) is a regular point of \( P_{t,p}^{(P)} \) (resp. \( P_{t,p}^{(C)} \)) if and only if

1. \( z_j^p \neq z_i^p \) for \( i \neq j \), and
2. \( a_{\ell_j-1,j} \neq 0 \) for all \( j = 1, \ldots, K \).

**Proof.** Use Proposition 3.19 and Proposition 3.20 as well as Proposition 3.9.

**Theorem 2.7.** Let \( x = ([a_{i,j}], [z_j]) \in \mathbb{C}^R \) be a regular point of \( P_{t,p}^{(P)} \). Denote the minimal \( p \)-separation by

\[
\delta_p \overset{\text{def}}{=} \min_{i \neq j} |z_i^p - z_j^p| > 0.
\]

Then

\[
\text{ACC}_{LOC}^{(P_{t,p}^{(P)})}(x, \varepsilon, a_{\ell,j}) = C_1(\ell, \ell_j) \left( \frac{2}{\delta_p} \right)^R \left( \frac{1}{2} + \frac{R}{\delta_p} \right)^{\ell_j-\ell} \times \left( 1 + \frac{|a_{\ell-1,j}|}{|a_{\ell-1,j}|} \right) \max \{1, t\ell_j-\ell\} p^\ell \varepsilon,
\]

\[
\text{ACC}_{LOC}^{(P_{t,p}^{(C)})}(x, \varepsilon, z_j) = \frac{2}{\ell_j} \left( \frac{2}{\delta_p} \right)^R \frac{1}{|a_{\ell-1,j}|} p^\ell \varepsilon,
\]

where \( C_1(\ell, \ell_j) \) is an explicit constant defined in 3.14 on page 17 below.
Theorem 2.8. Let \( x = (\{a_{ij}\}, \{z_i\}) \in \mathbb{C}^R \) be a regular point of \( P_{t,p}^{(C)} \). Assume that the nodes \( \{z_j\}_{j=1}^K \) of the confluent Prony system (1.6) on page 3 satisfy the condition (2.1) on the previous page, and also that

\[
0 < |z_j| \leq 1 \quad \text{for} \quad j = 1, \ldots, K.
\]

Then

\[
\text{ACC}_{\text{LOC}} \left( P_{t,p}^{(C)} \right) (x, \varepsilon, a_{\ell,j}) \leq C_2 (\ell, \ell_j) \left( \frac{2}{\delta_p} \right)^R \left( \frac{1}{2} + \frac{R}{\delta_p} \right)^{\ell_j - \ell} |z_j|^{\ell_j - t - p \ell_j} \times \left( 1 + \frac{|a_{\ell-1,j}|}{|a_{\ell_j-1,j}|} \right) \max \left\{ \left| \frac{1}{p^\ell_j} \right| \right\} \varepsilon,
\]

\[
\text{ACC}_{\text{LOC}} \left( P_{t,p}^{(C)} \right) (x, \varepsilon, z_j) \leq \frac{2}{\ell_j!} \left( \frac{2}{\delta_p} \right)^R |z_j|^{\ell_j - t - p \ell_j} \frac{|a_{\ell_j-1,j}|}{|a_{\ell_j-1,j}|} \varepsilon,
\]

where \( C_2 (\ell, \ell_j) \) is an explicit constant defined in (3.16) on page 17 below.

Remark 2.9. Note that the bounds of Theorem 2.7 and Theorem 2.8 coincide in the case of \( \ell_1 = \cdots = \ell_K = 1 \) and \( |z_j| = 1 \), in which case we just have the original Prony system (1.1) on page 4.

Remark 2.10. The noise amplification for recovering the nodes does not depend on the initial index \( t \), while the accuracy of recovering \( |a_{\ell,j}| \) actually deteriorates with increasing \( t \) (if keeping \( p \) constant).

Remark 2.11. If \( t = 0 \), then one can make the change of variables described in Remark 2.3 above and derive Theorem 2.7 from the results of [10] on non-shifted and non-decimated Prony system.

Remark 2.12. If \( p > 1 \) (non-trivial decimation) we have a problem of “aliasing” or non-uniqueness. In effect, instead of \( z_j \) one recovers \( z_{p \ell_j} \). Therefore the decimation cannot be used on general Prony systems without having a good a-priori approximation to the nodes (at least with accuracy \( \sim o \left( p^{-1} \right) \)).

Remark 2.13. Taking Theorem 2.8 with \( t = 0 \), \( p = 1 \) we obtain a refinement of the main result of [10].

2.3 Improvement gained by decimation

We subsequently consider the following question: \textit{how much improvement can one get by decimation?}\n
To be more specific, we take \( t = 0 \) (non-shifted system), and consider the quantity \( \text{ACC}_{\text{LOC}} \left( P_{0,p}^{(C)} \right) (x, \varepsilon, z_j) \) as function of the decimation \( p \), for \( 1 \leq p \leq \frac{N}{R-1} \) (so that we always have \( S_{0,p} \subset [0, N] \)). Keeping \( \varepsilon \) constant, we have essentially

\[
\text{ACC}_{\text{LOC}} \left( P_{0,p}^{(C)} \right) (x, \varepsilon, z_j) \sim \delta_p^{-R} p^{-\ell_j}.
\]

Definition 2.14. The “improvement function” is defined as the ratio between non-decimated and decimated error amplification, i.e.

\[
\rho(x, z_j, p) \overset{\text{def}}{=} \frac{\text{ACC}_{\text{LOC}} \left( P_{0,1}^{(C)} \right) (x, \varepsilon, z_j)}{\text{ACC}_{\text{LOC}} \left( P_{0,p}^{(C)} \right) (x, \varepsilon, z_j)} = \left( \frac{\delta_p}{\delta} \right)^R p^{\ell_j}.
\]
In particular, accuracy is increased whenever \( \rho (x, z_j, p) > 1 \), or
\[
\frac{p^{-\ell_j}}{\delta_p R} < \frac{1}{\delta R}. 
\] (2.4)

In order to keep things simple but still nontrivial, we shall deal exclusively with the case \( K = 2 \).

**Corollary 2.15.** Let \( x = \{ \{ a_{ij} \}, \{ z_i \} \} \in \mathbb{C}^R \) be a regular point of \( P_{0,1}^{(p)} \). Then there exists a constant \( C_3 = C_3 (x) \) such that for any \( M \in \mathbb{N} \) there exists \( p = p (M, x) > M \) for which
\[
\rho (x, z_j, p) \geq C_3 p^{\ell_j}.
\]

**Proof.** Without loss of generality, let the jumps be \( z_1 = 1, z_2 = e^{- i \xi} \). We have
\[
\rho (x, z, p) = \left( \frac{\delta_p}{\delta} \right)^R p^{\ell_j} = \left| \frac{1 - e^{- i \xi}}{1 - e^{- i \xi}} \right| p^d = \left( \frac{\sin \frac{\xi}{2}}{\sin \frac{\xi}{2}} \right)^R p^{\ell_j}.
\]

Let \( \alpha \overset{\text{def}}{=} \xi \), and let \( n \in \mathbb{N} \) be arbitrary. Clearly, we have
\[
\exists p_0 = p_0 (n, \xi) > n : |\sin p_0 \alpha| > \frac{1}{2}.
\] (2.5)

Thus, for this \( p_0 \) we have
\[
\rho (x, z, p_0) \geq \frac{p_0^{\ell_j}}{(2 \sin \frac{\xi}{2})^R}
\]
which proves the claim. \( \square \)

### 2.4 Superresolution

We return to Problem 1.2 on page 9. In this section we estimate the effect of decimation on closely spaced nodes. For simplicity, let us consider the case of polynomial Prony system (1.5) on page 2 with only two nodes. Below we show that for moderate values of the decimation parameter \( p \), accuracy is improved by an additional factor \( p^R \).

**Lemma 2.16.** Consider the case of polynomial Prony system (1.5) with two nodes \( z_1 = e^{i \xi_1} \) and \( z_2 = e^{i \xi_2} \), so that \( \xi_2 = \xi_1 + \delta \) and \( 0 < \delta \ll 1 \). Fix some \( 0 < r_0 < 2 \pi \). For any \( p \) satisfying
\[
p \delta < r_0 \iff p < \left[ \frac{r_0}{\delta} \right]
\]
we have
\[
|\delta_p| > \alpha (r_0) p \delta,
\]
where
\[
\alpha (r) \overset{\text{def}}{=} \sqrt{2 \left( 1 - \cos r \right)}.
\]

**Proof.** We have
\[
|\delta_p|^2 = |e^{i \xi_2 p} - e^{i \xi_1 p}|^2 = |1 - e^{i \delta p}|^2 = 2 (1 - \cos (p \delta)) = \frac{2 (1 - \cos (p \delta)) (p \delta)^2}{(p \delta)^2} (p \delta)^2 = [\alpha (p \delta)]^2 (p \delta)^2.
\]
Since the function \( \alpha (r) \) is monotonically decreasing in \( 0 < r < 2 \pi \) (see Figure 1 on page 9), and since \( p \delta < r_0 \), we have immediately \( \alpha (p \delta) > \alpha (r_0) \), which completes the proof. \( \square \)
Corollary 2.17. Under the conditions of Lemma 2.16

\[ \rho(x, z_j, p) > \alpha(r_0)^{R} R R^{R + \ell_j}, \quad j = 1, 2. \]

![Figure 1: The function \( \alpha(r) \).](image)

3 Proofs of main results

This section contains the proofs of the theorems stated in Section 2. Most of the technical propositions regarding matrix factorizations are straightforward, using nothing more than some elementary algebra and binomial identities. Therefore, we have omitted most of these calculations, confident that the reader would reproduce them without any difficulty.

3.1 Common definitions

We start by defining the matrices which will be used throughout the subsequent calculations.

**Definition 3.1.** Denote by \( A_j \) the following \( \ell_j \times \ell_j \) block:

\[
A_j \overset{\text{def}}{=} \begin{bmatrix}
    a_{0,j} & a_{1,j} & \cdots & \cdots & a_{\ell_j-1,j} \\
    a_{1,j} & (\ell_j-2)a_{\ell_j-1,j} & 0 & \cdots & 0 \\
    \cdots & \cdots & 0 & \cdots & 0 \\
    (\ell_j-1)a_{\ell_j-1,j} & 0 & \cdots & 0 \\
    a_{\ell_j-1,j} & 0 & \cdots & 0 \\
\end{bmatrix}.
\]
Definition 3.2. Let $Q_{t,r}$ denote the $r \times r$ square matrix with entries:

$$(Q_{t,r})_{m,n} = (-t)^{n-m} \binom{n-1}{n-m}$$

Example 3.3. For $r = 5$ we have

$$Q_{t,5} = \begin{pmatrix}
1 & -t & t^2 & -t^3 & t^4 \\
0 & 1 & -2t & 3t^2 & -4t^3 \\
0 & 0 & 1 & -3t & 6t^2 \\
0 & 0 & 0 & 1 & -4t \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Definition 3.4. For every $x \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{N}$ let $T_{x,c}$ denote the $c \times c$ matrix

$$T_{x,c} \overset{\text{def}}{=} \text{diag}\{1, x, x^2, \ldots, x^{c-1}\}.$$ 

Obviously,

$$T_{x,c}^{-1} = T_{x^{-1},c}.$$ 

In addition we need the following auxiliary matrices.

Definition 3.5. For every $j = 1, \ldots, K$ let us denote by $E_j$ and $D_j$ the following $(\ell_j + 1) \times (\ell_j + 1)$ blocks

$$E_j \overset{\text{def}}{=} \begin{bmatrix}
1 & 0 & 0 & 0 & a_{0,j} \\
0 & 1 & 0 & -z_j & a_{0,j} \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -z_j & a_{0,j} \\
0 & 0 & 0 & \cdots & a_{\ell_j-1,j}
\end{bmatrix}, \quad (3.1)$$

$$D_j \overset{\text{def}}{=} \begin{bmatrix}
1 & 0 & 0 & 0 & a_{0,j} \\
0 & 1 & 0 & a_{0,j} & a_{0,j} \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & a_{0,j} & a_{0,j} \\
0 & 0 & 0 & \cdots & a_{\ell_j-1,j}
\end{bmatrix}. \quad (3.2)$$

Direct calculation gives

$$E_j^{-1} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & a_{0,j} \\
0 & 1 & 0 & \cdots & -a_{0,j} & a_{0,j} \\
0 & 0 & 1 & \cdots & -z_j & a_{0,j} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & a_{\ell_j-1,j} & z_j \\
0 & 0 & 0 & \cdots & 1 & a_{\ell_j-1,j}
\end{bmatrix}, \quad (3.3)$$

$$D_j^{-1} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & a_{0,j} \\
0 & 0 & 1 & \cdots & a_{0,j} \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & a_{\ell_j-1,j} \\
0 & 0 & 0 & \cdots & 1 + a_{\ell_j-1,j}
\end{bmatrix}. \quad (3.4)$$

\(^1\text{Note that the matrices } Q_{t,r} \text{ are unipotent, i.e. all their eigenvalues are 1.}\)
The Stirling numbers of the first and second kind are denoted by \( \binom{n}{k} \) and \( \{ n \atop k \} \), respectively [II Section 24.1].

**Definition 3.6.** Let \( S_m \) denote the \( m \times m \) upper triangular matrix

\[
S_m \overset{\text{def}}{=} \begin{bmatrix}
0 & 1 & 2 & \cdots & m-1 \\
0 & 0 & 2 & \cdots & m-1 \\
o & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
o & 0 & 0 & \cdots & m-1 \\
0 & 0 & 0 & \cdots & m-1
\end{bmatrix}.
\]

It is well-known that

\[
S_m^{-1} = \begin{bmatrix}
0 & 1 & 2 & \cdots & m-1 \\
0 & 0 & 2 & \cdots & m-1 \\
o & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
o & 0 & 0 & \cdots & m-1 \\
0 & 0 & 0 & \cdots & m-1
\end{bmatrix}.
\]

### 3.2 Generalized Vandermonde matrices

**Definition 3.7.** The shifted decimated Pascal-Vandermonde matrix is

\[
U_{t,p}(z_1, \ell_1, \ldots, z_K, \ell_K) = U_{t,p} \overset{\text{def}}{=} \begin{bmatrix}
u_0^{(t,p)}(z_1, \ell_1) & u_0^{(t,p)}(z_2, \ell_2) & \cdots & u_0^{(t,p)}(z_K, \ell_K) \\
u_1^{(t,p)}(z_1, \ell_1) & u_1^{(t,p)}(z_2, \ell_2) & \cdots & u_1^{(t,p)}(z_K, \ell_K) \\
\vdots & \vdots & \ddots & \vdots \\
u_{C-1}^{(t,p)}(z_1, \ell_1) & u_{C-1}^{(t,p)}(z_2, \ell_2) & \cdots & u_{C-1}^{(t,p)}(z_K, \ell_K)
\end{bmatrix},
\]

where

\[
u_k^{(t,p)}(z_j, \ell_j) \overset{\text{def}}{=} z_j^{t+kp} \begin{bmatrix} 1 & (t+kp) & (t+kp)^2 & \cdots & (t+kp)^{\ell_j-1} \end{bmatrix}.
\]

The non-shifted Pascal-Vandermonde matrix on the nodes \( \{z_1^p, \ldots, z_K^p\} \) is denoted by

\[
U_p^\# \overset{\text{def}}{=} U_{0,1}^\#(z_1^p, \ell_1, \ldots, z_K^p, \ell_K).
\]

**Definition 3.8.** The shifted decimated confluent Vandermonde matrix is

\[
V_{t,p}(z_1, \ell_1, \ldots, z_K, \ell_K) = V_{t,p} \overset{\text{def}}{=} \begin{bmatrix}
v_0^{(t,p)}(z_1, \ell_1) & v_0^{(t,p)}(z_2, \ell_2) & \cdots & v_0^{(t,p)}(z_K, \ell_K) \\
v_1^{(t,p)}(z_1, \ell_1) & v_1^{(t,p)}(z_2, \ell_2) & \cdots & v_1^{(t,p)}(z_K, \ell_K) \\
\vdots & \vdots & \ddots & \vdots \\
v_{C-1}^{(t,p)}(z_1, \ell_1) & v_{C-1}^{(t,p)}(z_2, \ell_2) & \cdots & v_{C-1}^{(t,p)}(z_K, \ell_K)
\end{bmatrix},
\]
where
\[ v_k^{(t,p)}(z_j, \ell_j) \overset{\text{def}}{=} \left[ z_j^{t+kp}, (t+k)z_j^{t+kp-1}, \ldots, (t+kp)\ell_j-1z_j^{t+kp-\ell_j+1} \right]. \]

The non-shifted confluent Vandermonde matrix on the nodes \( \{z_1^p, \ldots, z_K^p\} \) is denoted by
\[ V_{0,1}^{\#} \overset{\text{def}}{=} V_{0,1}^{p} (z_1^p, \ell_1, \ldots, z_K^p, \ell_K). \]

The confluent Vandermonde matrix \( V_{0,1} \) is well-studied in numerical analysis due to its central role in polynomial interpolation.

Let us start with the well-known fact about these matrices.

**Proposition 3.9.** The matrix \( V_{0,1} (z_1, \ell_1, \ldots, z_K, \ell_K) \) is invertible if and only if the nodes \( \{z_j\}_{j=1}^K \) are pairwise distinct.

Of particular interest to us are estimates on the row-wise norm of \( V_{0,1}^{-1} \).

**Theorem 3.10.** Let \( \{x_1, \ldots, x_n\} \) be pairwise distinct complex numbers with \( |x_j| \leq 1 \), satisfying the separation condition \( |x_i - x_j| \geq \delta > 0 \) for \( i \neq j \). Further, let \( \ell_1, \ldots, \ell_n \) be a vector of natural numbers such that \( \ell_1 + \ell_2 + \cdots + \ell_n = N \). Denote by \( u_{j,k} \) the row with index \( \ell_1 + \cdots + \ell_{j-1} + k \) of \( \left[V_{0,1} (x_1, \ell_1, \ldots, x_n, \ell_n)\right]^{-1} \) (for \( k = 0, 1, \ldots, \ell_j - 1 \)). Then the \( \ell_1 \)-norm of \( u_{j,k} \) satisfies
\[ \|u_{j,k}\|_1 \leq \left( \frac{2}{\delta} \right)^N \frac{2}{k!} \left( 1 + \frac{N}{\delta} \right)^{\ell_j - 1 - k}. \] (3.5)

The proof of this theorem (see below) combines original Gautschi’s technique [20] and the well-known explicit expressions for the entries of \( V_{0,1}^{-1} \) from [37], plus a technical lemma (Lemma 3.12).

**Definition 3.11.** For \( j = 1, \ldots, n \) let
\[ h_j(x) = \prod_{i \neq j} (x - x_i)^{-\ell_i}. \] (3.6)

**Lemma 3.12** ([14]). For any natural \( k \), the \( k \)-th derivative of \( h_j \) at \( x_j \) satisfies
\[ \left|h_j^{(k)} (x_j)\right| \leq N(N+1) \cdots (N+k-1)\delta^{-N-k}. \]

**Proof.** We proceed by induction on \( k \). For \( k = 0 \) we have immediately \( |h_j (x_j)| \leq \delta^{-N} \). Now
\[ h_j'(x) = h_j(x) \sum_{i \neq j} \frac{-\ell_i}{x - x_i}. \] (3.7)

Now we apply the Leibniz rule and get
\[
\begin{align*}
\left(h_j^{(k)}(x)\right) &= \left(h_j' h_j\right)^{(k-1)} \\
&= \sum_{r=0}^{k-1} \binom{k-1}{r} h_j^{(r)}(x) \left(h_j' h_j\right)^{(k-1-r)} \\
&= \sum_{r=0}^{k-1} \binom{k-1}{r} h_j^{(r)}(x) \sum_{i \neq j} \frac{(-1)^{k-r-1}(k-r-1)!\ell_i}{(x - x_i)^{k-r}},
\end{align*}
\]
hence

\[ |h_j^{(k)}(x_j)| \leq \sum_{r=0}^{k-1} \binom{k-1}{r} \frac{(k-r-1)!}{\delta^{N+r}} \cdot \frac{(k-r-1)! N}{\delta^{k-r}} \]

This implies, together with the induction hypothesis, that

\[ |h_j^{(k)}(x_j)| \leq \frac{N (N+1) \cdots (N+k-1)}{\delta^{N+k}}. \]

By a well-known binomial identity (proof is immediate by induction and Pascal’s rule) we have

\[ \sum_{r=0}^{k-1} \binom{N-1+r}{r} = \binom{N+k-1}{k-1}. \]

Therefore

\[ |h_j^{(k)}(x_j)| \leq \frac{N (N+1) \cdots (N+k-1)}{\delta^{N+k}}, \]

as required. \[ \square \]

**Proof of Theorem 3.10** By using a generalization of the Hermite interpolation formula (39), it is shown in [37] that the components of the row \( u_{j,k} \) are just the coefficients of the polynomial

\[ \frac{1}{k!} \sum_{t=0}^{\ell_j-1-k} \frac{t!}{t!} h_j^{(t)}(x_j)(x-x_j)^{k+t} \prod_{i \neq j} (x-x_i)^{\ell_i} \]

where \( h_j(x) \) is given by (39). By \([19]\) Lemma, the sum of absolute values of the coefficients of the polynomials \((x-x_j)^{k+t} \prod_{i \neq j} (x-x_i)^{\ell_i}\) is at most

\[ (1 + |x_j|)^{k+t} \prod_{i \neq j} (1 + |x_i|)^{\ell_i} \leq 2^{N-(\ell_j-k-t)}. \]

Therefore

\[ \|u_{j,k}\|_1 \leq \frac{1}{k!} \sum_{t=0}^{\ell_j-1-k} \frac{t!}{t!} \frac{1}{2^{N+t}} \frac{N(N+1) \cdots (N+t-1)}{\delta^{N+t}} 2^{N-\ell_j+k+t} \]

\[ = \left( \frac{2}{\delta} \right)^N \frac{1}{2^{\ell_j-k}k!} \sum_{t=0}^{\ell_j-1-k} \binom{\ell_j-1-k}{t} \frac{N(N+1) \cdots (N+t-1)}{(\ell_j-k-t) \cdots (\ell_j-k-2) (\ell_j-k-1)} \left( \frac{2}{\delta} \right)^t \]

\[ \leq \left( \frac{2}{\delta} \right)^N \frac{1}{2^{\ell_j-k}k!} \left( 1 + \frac{2N}{\delta} \right)^{\ell_j-1-k} \]

\[ = \left( \frac{2}{\delta} \right)^N \frac{2}{k!} \left( \frac{1}{2} + \frac{N}{\delta} \right)^{\ell_j-1-k} \]

which completes the proof. \[ \square \]
Corollary 3.13. Assume that $|z_j| \leq 1$, and $p \geq 1$ is chosen such that
\[
\delta_p \overset{\text{def}}{=} \min_{i \neq j} |z_i^p - z_j^p| > 0.
\]

Denote by $u_{j,k}$ the row with index $\ell_1 + 1 + \cdots + \ell_{j-1} + 1 + k + 1$ of $\{V_p^\# (z_1, \ell_1 + 1, \ldots, z_K, \ell_K + 1)\}^{-1}$ (for $k = 0, 1, \ldots, \ell_j$). Then the $\ell_1$-norm of $u_{j,k}$ satisfies
\[
\|u_{j,k}\|_1 \leq \left( \frac{2}{\delta_p} \right) \frac{R}{k!} \left( \frac{1}{2} + \frac{R}{\delta_p} \right)^{\ell_j - k}
\]
where $R = \sum_{j=1}^K (\ell_j + 1) = C + K$.

Proposition 3.14. For the Pascal-Vandermonde matrix, we have
\[
U_{t,p} = U_{0,p} \text{diag} \left\{ z_j^t Q_{t,\ell_j}^{-1} \right\}_{j=1}^K
\]
\[
U_{0,p} = U_p^\# \text{ diag} \left\{ T_{p,\ell_j} \right\}_{j=1}^K.
\]

Proposition 3.15. If $K = 1$ then in addition to the above factorizations we have
\[
U_p^\# (z, \ell) = T_{z^{-1}, \ell} U_{0,1} (z, \ell).
\]

In addition, we have the following identity.

Proposition 3.16. The confluent Vandermonde and Pascal-Vandermonde matrices satisfy
\[
U_{t,p} (z_1, \ell_1, \ldots, z_K, \ell_K) = V_{t,p} (z_1, \ell_1, \ldots, z_K, \ell_K) \text{ diag} \left\{ T_{z_j,\ell_j} S_{\ell_j} \right\}_{j=1}^K,
\]
and therefore also
\[
U_p^\# = V_p^\# \text{ diag} \left\{ T_{z_j^{-1},\ell_j} S_{\ell_j} \right\}_{j=1}^K.
\]

3.3 Data matrices

Definition 3.17. The data matrix $\mathcal{M}^{(P)}_{t,p}$ (resp. $\mathcal{M}^{(C)}_{t,p}$) for the system (1.5) (resp. (1.6)) is the Hankel matrix
\[
\begin{bmatrix}
    m_t & m_{t+p} & \cdots & m_{t+(C-1)p} \\
    m_{t+p} & m_{t+2p} & \cdots & m_{t+Cp} \\
    \vdots & \vdots & \ddots & \vdots \\
    m_{t+(C-1)p} & m_{t+Cp} & \cdots & m_{t+(2C-2)p}
\end{bmatrix}
\]
where $m_k$ are given by (1.5) (resp. (1.6)).

Proposition 3.18. The data matrices are factorized as follows:
\[
\mathcal{M}^{(P)}_{t,p} = V_{t,p} \text{ diag} \left\{ A_j \right\}_{j=1}^K V_{0,p}^T,
\]
\[
\mathcal{M}^{(C)}_{t,p} = U_{t,p} \text{ diag} \left\{ A_j \right\}_{j=1}^K U_{0,p}^T.
\]
3.4 Jacobian factorizations

Recall the definition of the maps \( P_{t,p}^{(P)} \) and \( P_{t,p}^{(C)} \) given by Definition \[2.5\]

Again, the following two results can be shown via direct calculations, therefore we omit the proofs.

**Proposition 3.19.** The Jacobian matrix of the map \( P_{t,p}^{(P)} \) can be decomposed as follows:

\[
\mathcal{J}_{P_{t,p}^{(P)}} = U_{t,p} (z_1, \ell_1 + 1, \ldots, z_K, \ell_K + 1) \text{diag} \{ E_j \}_{j=1}^K
\]

\[
= U_p^# (z_1, \ell_1 + 1, \ldots, z_K, \ell_K + 1) \text{diag} \left\{ z_j^T T_{p,\ell_j + 1} Q_{p,\ell_j + 1}^{-1} E_j \right\}_{j=1}^K
\]

\[
= V_p^# (z_1, \ell_1 + 1, \ldots, z_K, \ell_K + 1) \text{diag} \left\{ z_j^T T_{p,\ell_j + 1} S_{\ell_j + 1} T_{p,\ell_j + 1} Q_{p,\ell_j + 1}^{-1} E_j \right\}_{j=1}^K.
\]

**Proposition 3.20.** The Jacobian matrix of the map \( P_{t,p}^{(C)} \) can be decomposed as follows:

\[
\mathcal{J}_{P_{t,p}^{(C)}} = V_{t,p} (z_1, \ell_1 + 1, \ldots, z_K, \ell_K + 1) \text{diag} \{ D_j \}_{j=1}^K
\]

\[
= U_{t,p} (z_1, \ell_1 + 1, \ldots, z_K, \ell_K + 1) \text{diag} \left\{ S_{\ell_j + 1} T_{p,\ell_j + 1}^{-1} D_j \right\}_{j=1}^K
\]

\[
= V_p^# (z_1, \ell_1 + 1, \ldots, z_K, \ell_K + 1) \text{diag} \left\{ z_j^T T_{p,\ell_j + 1} S_{\ell_j + 1} T_{p,\ell_j + 1} Q_{p,\ell_j + 1}^{-1} D_j \right\}_{j=1}^K.
\]

3.5 Auxiliary lemma

We’ll need the following elementary computation.

**Lemma 3.21.** Let \( A_1, \ldots, A_k, B \) be \( n \times n \) upper triangular matrices over \( \mathbb{C} \), and \( c = (c_1, \ldots, c_n)^T \in \mathbb{C}^n \) some \( n \times 1 \) vector. Denote the entries of \( A_k \) by \( a_{i,j}^{(k)} \) \( i,j=1,\ldots,n \), and those of \( B \) by \( b_{i,j} \). Let the vector \( d \in \mathbb{C}^n \) be defined as

\[
d = BA_k A_{k-1} \cdots A_1 c.
\]

(3.11)

Fix \( 1 \leq j \leq n \). Assume that exactly \( 0 \leq m \leq k \) matrices among the \( A_1, \ldots, A_k \) are strictly diagonal. Then the \( j \)-th component of the vector \( d \) satisfies

\[
|d_j| \leq (n-j)^{k-m} \max_{i \geq j} \|c_i\| \left( \sum_{i \geq j} |b_{j,i}| \right) \left( \prod_{\ell=1}^{k} \alpha_j^{(\ell)} \right),
\]

where

\[
\alpha_j^{(\ell)} = \max_{i,r \geq j} |a_{i,r}^{(\ell)}|.
\]

**Proof.** By induction on \( k \). Consider first \( d = Bc \), the conclusion is obvious. In the induction step, take \( B = BA_k \). If \( A_k \) is strictly diagonal, then the corresponding entries of \( B \) satisfy

\[
\sum_{i \geq j} |\tilde{b}_{j,i}| \leq \left( \sum_{i \geq j} |b_{j,i}| \right) \max_{i \geq j} |a_{i,i}^{(k)}|.
\]

In the general case, both \( B, A_k \) are upper triangular, and so clearly

\[
\sum_{i \geq j} |\tilde{b}_{j,i}| \leq (n-j) \left( \sum_{i \geq j} |b_{j,i}| \right) \max_{i,r \geq j} |a_{i,r}^{(k)}|.
\]

This proves the claim for \( k \). \( \square \)
3.6 Proof of Theorem 2.7

The proof of Theorem 2.7 boils down to estimating the corresponding entries in the vector

$$\begin{bmatrix}
\Delta a_{0,1} \\
\vdots \\
\Delta a_{\ell-1,1} \\
\Delta z_1 \\
\vdots \\
\Delta x
\end{bmatrix} \sim \mathcal{J}_{p_{1,p^{-1}}^{j}} \begin{bmatrix}
O(\varepsilon) \\
O(\varepsilon)
\end{bmatrix}. \quad (3.12)
$$

From Proposition 3.19 we immediately get:

$$\mathcal{J}_{p_{1,p^{-1}}^{j}} = \text{diag}\left\{ z_j^{-1}E_j^{-1}Q_{t,\ell,j+1}T_p^{-1,\ell,\ell,j+1}\mathcal{S}_{\ell,j+1}^{-1}T_{\ell,j}^{-1,p,\ell,j+1} \right\}_{j=1}^{K} (V_p^#)^{-1}. \quad (3.13)$$

Now we represent (3.12) in the form (3.11).

First denote

$$w = (V_p^#)^{-1} \Delta m.$$

Fix $j = 1, \ldots, K$ and some $\ell = 0, \ldots, \ell_j$. Let $\gamma(j, \ell) = \ell_1 + 1 + \cdots + \ell_{j-1} + 1 + \ell + 1$. If $u_{j,\ell}$ is the $\gamma$-th row of $(V_p^#)^{-1}$, then

$$|w_{\gamma}| \leq \|u_{j,\ell}\|_1 \varepsilon.$$

Finally, notice that all the blocks $B_j = (b_{m,n}^{(j)})$ in the big matrix multiplying $(V_p^#)^{-1}$ are upper triangular.

Then we write

$$\begin{bmatrix}
\Delta a_{0,j} \\
\vdots \\
\Delta a_{\ell-1,j} \\
\Delta z_j
\end{bmatrix} = z_j^{-1}E_j^{-1}Q_{t,\ell,j+1}T_p^{-1,\ell,\ell,j+1}\mathcal{S}_{\ell,j+1}^{-1}T_{\ell,j}^{-1,p,\ell,j+1} \begin{bmatrix}
w_{\gamma}(j,0) \\
w_{\gamma}(j,1) \\
\vdots \\
w_{\gamma}(j,\ell_j)
\end{bmatrix}. \quad (3.13)$$

and it is left to evaluate each entry of the vector in the left-hand side using Lemma 3.21

Recall that $|z_j| = 1$. For the last entry we clearly have

$$|\Delta z_j| \leq \frac{1}{|a_{\ell-1,j}|} \cdot \frac{1}{p^{\ell_j}} \cdot \frac{1}{p^{\ell_j}} \cdot |w_{\gamma}(j,\ell_j)| = \frac{1}{p^{\ell_j} |a_{\ell-1,j}|} \|u_{j,\ell_j}\|_1 \varepsilon.$$

Combining this with (3.13) proves the claim for $|\Delta z_j|$.

In order to evaluate $|\Delta a_{\ell,j}|$ for $\ell = 0, \ldots, \ell_j - 1$, note that only $E_j^{-1}$, $Q_{t,\ell,j+1}$ and $\mathcal{S}_{\ell,j+1}^{-1}$ are non-diagonal. Therefore by Lemma 3.21, (3.3) and (3.13)

$$|\Delta a_{\ell,j}| \leq (\ell_j - \ell)^3 \max_{i \geq \ell} \left\{ w_{\gamma}(j,i) \right\} \left( 1 + \frac{|a_{\ell-1,j}|}{|a_{\ell-1,j}|} \right) \times$$

$$\max_{\ell \geq \ell} \left\{ (Q_{t,\ell,j+1})_{\ell,i} \right\} \max_{\ell \geq \ell} \left\{ (S_{\ell,j+1})_{\ell,i} \right\}.$$
which proves the claim with

\[
C_1(\ell, \ell_j) = \frac{2}{\ell!} (\ell_j - \ell)^3 \max \left\{ 1, \left( \frac{\ell_j - 1}{\ell_j - \ell} \right) \right\} \max \left\{ 1, \left( \frac{\ell_j}{\ell} \right) \right\}.
\]  

(3.14)

### 3.7 Proof of Theorem 2.8

Proceeding exactly as in Subsection 3.6 we obtain instead of (3.13) the following identity:

\[
\begin{bmatrix}
\Delta a_{0,j} \\
\vdots \\
\Delta a_{\ell_j-1,j}
\end{bmatrix} = z_j^{-\ell} D_j^{-1} T_{z_j,\ell_j+1} S_{\ell_j+1} Q_{\ell_j+1} T_{p^{-1},\ell_j+1} S_{\ell_j+1}^{-1} T_{z_j^{-1},\ell_j+1} \begin{bmatrix}
w_{\gamma}(j,0) \\
w_{\gamma}(j,1) \\
\vdots \\
w_{\gamma}(j,\ell_j)
\end{bmatrix}.
\]  

(3.15)

For the last entry in the left-hand side we have by Lemma 3.21

\[
|\Delta z_j| \leq \frac{1}{|z_j|} \cdot 1 \cdot \left| \alpha_{\ell_j-1,j} \right| \cdot 1 \cdot \frac{1}{p^j} \cdot 1 \cdot \frac{1}{|z_j|} |w_{\gamma}(j,\ell_j)|
\]

\[
= \frac{1}{|z_j|^{\ell_j - t - p^j}} \left| \alpha_{\ell_j-1,j} \right| \frac{1}{p^j} \|u_{j,\ell_j}\| \|1\| \varepsilon
\]

\[
\leq \frac{1}{|z_j|^{\ell_j - t - p^j}} \frac{2}{\ell_j} \left( \frac{2}{\delta_p} \right) \frac{2}{\ell_j} \varepsilon,
\]

proving the claim for |\Delta z_j|.

In order to evaluate |\Delta a_{\ell_j}| for \( \ell = 0, \ldots, \ell_j - 1 \), note that only \( D_j^{-1}, S_{\ell_j+1}, Q_{\ell,\ell_j+1} \) and \( S_{\ell_j+1}^{-1} \) are non-diagonal. Therefore by Lemma 3.21, (5.3) and (5.14) we have

\[
|\Delta a_{\ell,j}| \leq |z_j|^{-\ell_j} (\ell_j - \ell)^4 \max_{t \geq \ell_j} |w_{\gamma}(j,t)| \left( 1 + \frac{|a_{\ell_j-1,j}|}{|a_{\ell_j-1,j}|} \right) \max_{\ell_j \geq \ell} \left\{ 1, \left( \frac{\ell_j - 1}{\ell_j - \ell} \right) \right\} \times
\]

\[
\max_{\ell_j \geq \ell} \left\{ 1, \left( \frac{\ell_j - 1}{\ell_j - \ell} \right) \right\} \frac{2}{\ell_j} \max_{\ell_j \geq \ell} \left\{ 1, \left( \frac{\ell_j}{\ell} \right) \right\} \frac{1}{p^j} \|u_{j,\ell_j}\| \|1\| \varepsilon
\]

\[
\leq C_2(\ell, \ell_j) \frac{1}{\ell_j} \left( 1 + \frac{|a_{\ell_j-1,j}|}{|a_{\ell_j-1,j}|} \right) \frac{2}{\ell_j} \left( \frac{2}{\delta_p} \right) \frac{2}{\ell_j} \varepsilon,
\]

with

\[
C_2(\ell, \ell_j) = \frac{2}{\ell_j} (\ell_j - \ell)^4 \max_{t \geq \ell_j} \left\{ 1, \left( \frac{\ell_j - 1}{\ell_j - \ell} \right) \right\} \max_{\ell_j \geq \ell} \left\{ 1, \left( \frac{\ell_j}{\ell} \right) \right\} \max_{\ell_j \geq \ell} \left\{ 1, \left( \frac{\ell_j - 1}{\ell_j - \ell} \right) \right\}.
\]  

(3.16)

This completes the proof.

### 4 Numerical experiments

Recalling Remark 2.3 it is immediate that all known solution methods for non-decimated systems are directly transferrable to decimated setting. Indeed, taking any algorithm for the standard Prony-type system, one just needs to make the modifications described in Algorithm 1.
Algorithm 1 Implementing decimation for existing algorithms

For simplicity we consider only the recovery of the nodes \( \{z_j\} \).

1. Obtain initial approximations to the nodes.
2. Choose the decimation parameter \( p \) such that \( \delta_p \) is not too small.
3. Feed the original algorithm with the decimated measurements \( m_0, m_p, m_{2p}, \ldots \), and obtain the estimated node \( w_j \).
4. Take \( z_j = \sqrt[w_j] \).

We have tested the decimation technique according to Algorithm 1 on two well-known algorithms for Prony systems - generalized ESPRIT [5] and nonlinear least squares (LS, implemented by MATLAB’s \texttt{lsqnonlin}).

In the first experiment, we fixed the number of measurements to be 66, and changed the decimation parameter \( p \), while keeping the noise level constant. The accuracy of recovery increased with \( p \) – see Figure 2 on page 18. In the second experiment, we fixed the highest available measurement to be \( M = 1600 \), and changed the decimation from \( p = 1 \) to \( p = 100 \) (thereby reducing the number of measurements from 1600 to just 16). The accuracy of recovery stayed relatively constant – see Figure 3 on page 19. Note that such a reduction leads to a corresponding decrease in the running time (calculating singular-value decomposition of large matrices, as in ESPRIT, is a time-consuming operation).

![Graph](image)

Figure 2: Reconstruction error as a function of the decimation with fixed number of measurements \((M = 66)\). The signal has two nodes with distance \( \delta = 10^{-2} \) between each other. Notice that ESPRIT requires significantly higher Signal-to-Noise Ratio in order to achieve the same performance as LS.

5 Relation to known lower bounds

5.1 Cramer-Rao Lower Bounds

The polynomial Prony system [10] is equivalent to the so-called PACE (Polynomial-Amplitude-Complex-Exponentials) model [3, 4] known from signal processing literature. The Cramer-Rao bound (CRB) (which
Figure 3: Reconstruction error as a function of the decimation, reducing number of measurements from $M = 1600$ to $M = 16$. The signal has two nodes with distance $\delta = 10^{-2}$ between each other. The reconstruction accuracy remains almost constant.

gives a lower bound for the variance of any unbiased estimator, see [27]) of the PACE model in white Gaussian noise is as follows (note that the original expressions have been appropriately modified to match the notations of this paper).

**Theorem 5.1** ([3] Propositions III.1, III.3]). *Let the noise have variance $\sigma^2$, then for small number of samples $N$ we have*\[ CRB \{z_j\} \approx \frac{\sigma^2}{|a_{\ell_j-1,j}|^2}, \] \[ (5.1) \]

\[ CRB \{a_{\ell,j}\} \approx \sigma^2 \left( C_1 \left| \frac{a_{\ell-1,j}}{a_{\ell_j-1,j}} \right|^2 + C_2 R \left\{ \frac{a_{\ell-1,j}}{a_{\ell_j-1,j}} \right\} + 1 \right), \quad \ell = 1, 2, \ldots, \ell_j - 1. \] \[ (5.2) \]

*When the number of samples $M \gg 1$, then the asymptotic CRB bounds satisfy*

\[ CRB \{z_j\} \approx \frac{\sigma^2}{|a_{\ell_j-1,j}|^2 M^{2\ell_j+1}}, \] \[ (5.3) \]

\[ CRB \{a_{\ell,j}\} \approx \frac{\sigma^2}{M^{2\ell+1}}, \quad \ell = 1, 2, \ldots, \ell_j - 1. \] \[ (5.4) \]

In the context of algebraic reconstruction, these bounds are not always applicable, since they assume a particular statistical distribution for the error terms $\varepsilon_k$ (see e.g. Section 3).

However, we notice the clear analogy between the above bounds and our main results. In particular, applying Theorem 2.7 in the case $t = 0, p = 1$, we see that (5.1), (5.2) and, respectively, (2.3), (2.2) have the same qualitative dependence on the parameters – in particular, inverse proportionality w.r.t the highest coefficient $a_{\ell_j-1,j}$. Furthermore, in the asymptotic setting $N \gg 1$, considering decimation with $p = \frac{N}{M}$, we see that

\[ \text{Here } \Re(\cdot) \text{ denotes the real part.} \]
and, respectively, (2.3), (2.2), have similar asymptotic dependence on $N$. As we noted in [10], the reason for such similarity is not a-priori clear (although it could be partially attributed to the fact that both methods require calculation of the partial derivatives of the measurements with respect to the parameters), and it certainly prompts for further investigation.

5.2 Donoho’s lower bounds

Perhaps the most general lower bound for the performance of any method whatsoever was given in the work of Donoho [15]. We consider this kind of bound to be very important, and elaborate it further immediately below.

Assume that one wants to recover a measure

$$\mu = \sum_{k=-\infty}^{\infty} a_k \delta_{k \Delta}, \quad (5.5)$$

supported on a lattice $\{k \Delta\}_{k=-\infty}^{\infty}$, where $\Delta \ll 1$, from noisy measurements

$$y(\omega) = \hat{\mu}(\omega) + z(\omega), \quad |\omega| \leq \Omega$$

$$\hat{\mu}(\omega) = \sum_{k=-\infty}^{\infty} a_k e^{-i k \omega \Delta}. \quad (5.5)$$

An essential requirement is that the frequency cutoff is much smaller than the Nyquist threshold, $\Omega \ll \frac{\pi}{\Delta}$, while the measure $\mu$ is assumed to be sparse in the following sense.

**Definition.** Let $S(R, \Delta)$ denote the set of measures of the form (5.5), such that

1. The sequence $\{a_k\}$ is in $\ell_1$;
2. the Rayleigh index of $\text{supp}(\mu)$, defined for any set $S$ by

$$R^*(S) \overset{\text{def}}{=} \min \left\{ r : r \geq \sup_t \# (S \cap [t, t + r]) \right\},$$

satisfies

$$R^*(\text{supp}(\mu)) \leq R.$$

**Definition.** For every discrete measure $\mu$, denote

$$\|\mu\|_2 = \left( \sum_{t \in \text{supp}(\mu)} |\mu(t)|^2 \right)^{\frac{1}{2}}. \quad (5.6)$$

**Definition.** The modulus of continuity is

$$\Lambda(\varepsilon, S(R, \Delta), \Omega) \overset{\text{def}}{=} \sup \left\{ \|\mu_1 - \mu_2\|_2 : \mu_1, \mu_2 \in S(R, \Delta), \|\hat{\mu}_1 - \hat{\mu}_2\|_{L_2[-\Omega, \Omega]} \leq \varepsilon \right\}.$$
Theorem (Superresolution theorem). Let $\Omega > 2\pi$ and $\Delta_0 \in (0, 1)$. Then there exist constants $C_1 (R, \Omega, \Delta_0)$ and $C_2 (R, \Omega)$ such that for every $\Delta < \Delta_0$

$$C_1 (R, \Omega, \Delta_0) \left( \frac{1}{\Delta} \right)^{2R-1} \varepsilon \leq \Lambda (\varepsilon, S (R, \Delta), \Omega) \leq C_2 (R, \Omega) \left( \frac{1}{\Delta} \right)^{2R+1} \varepsilon. \quad (5.7)$$

As $\Delta_0 \to 0$, the lower bound can be approximated by

$$\Lambda (\varepsilon) \gtrsim \sqrt{\frac{4R - 2}{2R - 1}} \frac{\pi}{\Omega} \frac{4R - 1}{\Omega \Delta} \left( \frac{1}{\Delta} \right)^{2R-1} \varepsilon, \quad (5.8)$$

and so the ratio $\frac{1}{\Omega \Delta}$ can be called the “superresolution factor”.

It is important to point out that the bounds of the Superresolution theorem are theoretical, and no practical way to achieve them is known [11].

Also, these bounds do not estimate the error in the locations of the spikes since the norm (5.6) measures only the magnitudes of the linear coefficients $\{a_j\}$.

Now we would like to provide a coarse estimate on the performance of the algebraic method in this setting. Consider the Prony system

$$f (x) = \sum_{j=1}^{R} a_j \delta (x - \xi_j),$$

$$\hat{f} (k) = \sum_{j=1}^{R} a_j e^{-ik\xi_j}, \quad |k| \leq \Omega,$$

with actual measurements $m_k = \hat{f} (k) + z (k)$ and $|z (k)| \leq \varepsilon$. Define

$$\Delta = \min_{i \neq j} |e^{-i\xi_j} - e^{-i\xi_i}|.$$

First assume that we solve the Prony system taking only $k = 0, 1, \ldots, 2R - 1$. Then by Theorem 2.7 with $t = 0$, $p = 1$ we have

$$|\Delta a_j| \leq C_1 \left( \frac{2}{\Delta} \right)^{2R} \left( \frac{1}{2} + \frac{2R}{\Delta} \right) \varepsilon \leq C_1 R 4^{R+1} \left( \frac{1}{\Delta} \right)^{2R+1} \varepsilon. \quad (5.9)$$

Next, consider the case $\Delta \to 0$. We can apply the decimation technique: namely, take the samples

$$k = 0, \left\lfloor \frac{\Omega}{2R - 1} \right\rfloor, 2 \left\lfloor \frac{\Omega}{2R - 1} \right\rfloor, \ldots, \Omega.$$

The decimation parameter is therefore

$$p = \left\lfloor \frac{\Omega}{2R - 1} \right\rfloor.$$

Set $r_0 = \frac{1}{2}$ for definiteness, and let $C_2 \overset{\text{def}}{=} \alpha \left( \frac{1}{2} \right)$. Since $\Delta \to 0$, we can assume that $p \Delta < r_0$. By Lemma 2.16 we have

$$|\delta_p| > C_2 \Delta p.$$
Then by Theorem 2.7 the Prony accuracy is bounded by

$$|Δa_j| ≤ C_3 \left( \frac{2}{C_2 \cdot M^{1/2R-1}} \right)^{2R} \left( \frac{1}{2} + \frac{2R}{C_2 \cdot M^{1/2R-1}} \right) ε \leq C_3 R \left( \frac{4}{C_2} \right)^{R+1} (2R-1)^{2R+1} \left( \frac{1}{ΔΩ} \right)^{2R+1} ε.$$  

(5.10)

The clear similarity between the estimates (5.9), (5.10) and (5.7), (5.8), in particular their asymptotic growth rates as \( Δ \rightarrow 0 \), strongly suggests that the algebraic method, via solving Prony type systems, has the potential to achieve best possible results for superresolution.

6 Application to piecewise-smooth Fourier reconstruction

In this section we present an application of Prony decimation to the recent solution of the so-called Eckhoff’s conjecture, as elaborated in [6, 9]. Our goal here is simply to point out the strong connection of this problem with the main results of this paper.

Consider the problem of reconstructing an integrable function \( f : [-π, π] \rightarrow \mathbb{R} \) from a finite number of its Fourier coefficients \([1, 3]\). If \( f \) is \( C^d \)-periodic, then the truncated Fourier series \( ʃ_M(f) \defeq \sum_{|k|=0}^M c_k(f) e^{ikx} \) approximates \( f \) with error at most \( C \cdot M^{-d-1} \), which is optimal. If, however, \( f \) is not smooth even at a single point, the rate of accuracy drops to only \( M^{-1} \). This accuracy problem, also known as the Gibbs phenomenon, is very important in applications, such as calculation of shock waves in PDEs. It has received much attention especially in the last few decades - see e.g. a recent book [20].

The so-called “algebraic approach” to this problem, first suggested by K.Eckhoff [17], is as follows. Assume that \( f \) has \( K > 0 \) jump discontinuities \( \{ξ_j\}_{j=1}^K \), and \( f \in C^d \) in every segment \( (ξ_{j-1}, ξ_j) \). We say that in this case \( f \) belongs to the class \( PC(d, K) \). Denote the associated jump magnitudes at \( ξ_j \) by \( a_{ℓ,j} \defeq f^{(ℓ)}(ξ_j^+) - f^{(ℓ)}(ξ_j^-) \). Then write the piecewise smooth \( f \) as the sum \( f = Ψ + Φ \), where \( Ψ(x) \) is smooth and periodic and \( Φ(x) \) is a piecewise polynomial of degree \( d \), uniquely determined by \( \{ξ_j\}, \{a_{ℓ,j}\} \) such that it “absorbs” all the discontinuities of \( f \) and its first \( d \) derivatives. In particular, the Fourier coefficients of \( Φ \) have the explicit form

$$c_k(Φ) = \frac{1}{2π} \sum_{j=1}^K e^{-ikξ_j} \sum_{ℓ=0}^d (ik)^{d-1} a_{ℓ,j}, \quad k = 1, 2, \ldots.$$  

(6.1)

For \( k \gg 1 \), we have \( |c_k(Φ)| \sim k^{-1} \), while \( |c_k(Ψ)| \sim k^{-d-2} \). Consequently, Eckhoff suggested to pick large enough \( k \) and solve the approximate system of equations (1.3) where \( m_k = 2π (ik)^{d+1} c_k(f) \), \( z_j = e^{-ikξ_j} \) and \( a_{ℓ,j} = ℓ ! a_{d-ℓ,j} \). His proposed method of solution was to use the known values \( \{m_k\}_{k∈I} \) where

$$I = \{M - (d + 1)K + 1, M - (d + 1)K + 2, \ldots, M\}$$  

(6.2)

to construct an algebraic equation satisfied by the unknowns \( \{ξ_1, \ldots, ξ_K\} \), and solve this equation numerically. Based on some explicit computations for \( d = 1, 2 \), \( K = 1 \) and large number of numerical experiments, he conjectured that his method would reconstruct the jump locations with accuracy \( M^{-d-2} \), and the values of the function between the jumps with accuracy \( M^{-d-1} \).

Let us consider the problem in the framework of Prony type system \([1, 3]\). The error term is of magnitude \( |ε| \sim M^{-1} \). The index set \([6.2]\) is just \( I_{t,p} \) with \( t \sim M, \ p = 1 \) (i.e. no decimation). Therefore, by Theorem 2.7 we get accuracy only of order \( |Δξ_j| \sim M^{-1} \). This is indeed confirmed by our numerical experiments in [6]. In [9] we have partially overcome this difficulty by considering the Prony system \([6.1]\) whose order was only half the actual smoothness of the function. In effect, this corresponded to providing the error terms.
with additional structure, which eventually lead to some cancellations and improvement of the estimate $M^{-1}$ to $M^{-\lfloor \frac{d}{2} \rfloor - 2}$.

Now consider the decimated setting for this problem. By the above, we can approximate each jump $\xi_j$ up to accuracy $M^{-\lfloor \frac{d}{2} \rfloor - 1}$. Set

$$N = \left\lfloor \frac{M}{(d+2)k} \right\rfloor.$$

Now take the index set $I_{t,p}$ where $t=p=N$, i.e. $I_{N,N} = \{N, 2N, \ldots, M\}$. As before, $|\epsilon| \sim M^{-1}$, but now due to decimation we get accuracy $|\Delta \xi_j| \sim N^{-d-1}N^{-1} \sim M^{-d-2}$, and by Remark 2.12 the decimation can be applied. In [6] we have shown that, indeed, by “decimating” the algorithm of [9] in a certain sense, we get the full (best possible) accuracy $\sim M^{-d-2}$ for the jumps, and accuracy $\sim M^{-d-1}$ for values between the jumps.

Thus decimation for Prony systems has ultimately provided the solution to Eckhoff’s conjecture.

### 7 Discussion

In addition to the subspace shifting of [29], the decimation approach has another analogue in the literature – the so-called “arithmetic progression sampling”, developed by A.Sidi for convergence acceleration of Richardson extrapolation problems in numerical analysis [38]. It would be interesting to make this connection more elaborate and precise.

An important drawback of the “Prony map” approach is that it cannot handle oversampling, namely, considering Prony type systems for the measurements $\{m_k\}_{k \in S}$ where $|S| > R$. It remains to be seen whether this drawback can be eventually overcome.

As our results show, when two nodes collide, the Prony system has an algebraic singularity. It is possible that further study of these singularities, started in [7] using divided differences, may prove to have an important role for obtaining sharp bounds for the Prony stability problem.

An important research problem connected with Prony systems is non-uniform sampling. While formally the “Prony map” approach can be applied in the case when the set $S$ contains arbitrary $R$ integers between 0 and $N-1$, the explicit analysis of Jacobians seems to be rather difficult. In [8] we provided some estimates using discrete version of Turan-Nazarov inequalities for exponential polynomials from [18]. In these inequalities, a central role is played by a geometric invariant of the sampling set connected to its metric entropy. It can be shown that this invariant is minimal precisely for evenly spaced sampling sets, giving another justification for the decimation procedure. We intend to provide the details in a future publication.

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