BI-ININVARIANT DIFFERENTIAL OPERATORS ON THE GALILEAN GROUP

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ABSTRACT. The Galilean group is the group of symmetries of Newtonian mechanics, with Lie algebra \( \mathfrak{gal}(n) \). We find algebraically independent generators for the center of the universal enveloping algebra of \( \mathfrak{gal}(n) \) using coadjoint orbits.

1. Introduction

The Galilean group \( \text{Gal}(n) \) is the Lie group of transformations between reference frames in Newtonian mechanics in \( n \) dimensional space. It is generated by spatial rotations, translations in space and time, and boosts, which correspond to changes in velocity (see [1]). Elements of \( \text{Gal}(n) \) can be represented as \((n + 2) \times (n + 2)\) matrices

\[
\begin{pmatrix}
\rho & v_1 & x_1 \\
\vdots & \vdots & \vdots \\
0 & \cdots & 1 \\
0 & x_0 & 1
\end{pmatrix}
\]

where \( \rho \) is an element of \( \text{O}(n) \), \( \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \) the spatial translation, \( x_0 \) the time shift, and \( \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \) is the boost.

To get the action on \((n + 1)\)-dimensional spacetime, we first identify affine \((n + 1)\)-space with the plane \( x_{n+2} = 1 \) in \( \mathbb{R}^{n+2} \). Then the action on a point \( (X_1, \ldots, X_n, T) \in \mathbb{R}^{n+1} \) is:

\[
\begin{pmatrix}
\rho & v_1 & x_1 \\
\vdots & \vdots & \vdots \\
0 & \cdots & 1 \\
0 & x_0 & 1
\end{pmatrix}
\begin{pmatrix}
X_1 \\
\vdots \\
X_n \\
T
\end{pmatrix}
= 
\begin{pmatrix}
X'_1 \\
\vdots \\
X'_n \\
T'
\end{pmatrix}
\]

Taking derivatives, the Lie algebra \( \mathfrak{gal}(n) \) of \( \text{Gal}(n) \) is identified with the set of matrices of the form

\[
\begin{pmatrix}
K & v_1 & x_1 \\
\vdots & \vdots & \vdots \\
0 & \cdots & 1 \\
0 & x_0 & 1
\end{pmatrix}
\]

where \( K \) is an element of \( \text{so}(n) \).

The Galilean group is an important example of a contraction. Given a Lie algebra \( \mathfrak{g} = (V, [\cdot, \cdot]) \), and a subalgebra \( \mathfrak{h} = (U, [\cdot, \cdot]) \), choose a complementary subspace \( W \) so that \( V = \)

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$U \oplus W$. For $\epsilon > 0$, let $T_\epsilon$ be the linear operator

$$T_\epsilon(u + w) = u + \epsilon w \quad (u \in U, w \in W)$$

One can define a new bracket $[, \cdot, \cdot]_\epsilon$ on $V$ by

$$[X, Y]_\epsilon = T_\epsilon^{-1}[T_\epsilon X, T_\epsilon Y]$$

The Lie algebra $(V, [, \cdot, \cdot]_\epsilon)$ is isomorphic to $\mathfrak{g}$. As long as $\mathfrak{h}$ is a subalgebra, we may define $[, \cdot, \cdot]_0 = \lim_{\epsilon \to 0} [\cdot, \cdot]_\epsilon$. The resulting Lie algebra $\mathfrak{g}_0 = (V, [, \cdot, \cdot]_0)$ is in general not isomorphic to $\mathfrak{g}$, and is known as an Inönü-Wigner contraction [2]. Note that the subalgebra $\mathfrak{h}_0 = (U, [, \cdot, \cdot]_0)$ is isomorphic to $\mathfrak{h}$, and that $(W, [, \cdot, \cdot]_0)$ is abelian. The contraction process can be seen as ‘flattening’ $W$.

The Galilean group is a contraction of the Poincaré group, and this reflects the fact that special relativity becomes Newtonian mechanics in the limit as the speed of light goes to infinity. The Poincaré group is itself a contraction, whose bi-invariant differential operators were found in [6] and [3].

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra can be identified with the algebra of left-invariant differential operators on the corresponding Lie group $D(G)$. If $S(\mathfrak{g})$ is the symmetric algebra on $\mathfrak{g}$, there is a linear bijection $\lambda : S(\mathfrak{g}) \to D(G)$ defined by

$$(\lambda(P)f)(g) = P(\partial_1, \ldots, \partial_n)f(g \exp(t_1X_1 + \cdots + t_nX_n))$$

where $\partial_i = \frac{\partial}{\partial t_i}$, and $X_1, \ldots, X_n$ is a basis of $\mathfrak{g}$. [4, ch II, Theorem 4.3]

This bijection commutes with the adjoint action of $G$ on $S(\mathfrak{g})$ and $D(G)$, and so we have, for $P \in S(\mathfrak{g})^G$:

$$\text{Ad}(g)\lambda(P) = \lambda(\text{Ad}(g)P) = \lambda(P)$$

which shows that the image of an $\text{Ad}(G)$ invariant polynomial is itself $\text{Ad}(G)$ invariant. Since the $\text{Ad}(G)$ invariant elements of $\mathcal{U}(\mathfrak{g})$ are exactly $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$, we have identified $S(\mathfrak{g})^G$ with $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$.

Unfortunately, while $\lambda$ is a linear bijection, it isn’t an isomorphism of algebras, and so in general, $\lambda(PQ) \neq \lambda(P)\lambda(Q)$. However we do have that:

$$(3) \quad \deg(\lambda(PQ) - \lambda(P)\lambda(Q)) < \deg(\lambda(PQ))$$

This allows us to show inductively that if $\{P_1, \ldots, P_m\}$ generate $S(\mathfrak{g})^G$, then $\{\lambda(P_1), \ldots, \lambda(P_m)\}$ generate $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$. The degree zero case is trivial. If $D \in \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$, then we can write

$$D = \lambda(q(P_1, \ldots, P_m))$$

for some polynomial $q$. Then

$$D - q(\lambda(P_1), \ldots, \lambda(P_m))$$

is a central element whose degree is less than $\deg(D)$ by [3]. By induction, $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ is generated by $\{\lambda(P_1), \ldots, \lambda(P_m)\}$.

Elements of $S(\mathfrak{g})$ can be viewed as polynomials on $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$. With this identification, $\text{Ad}(g)P(X^*) = P(\text{Ad}^*(g^{-1})X^*)$, where $\text{Ad}^*$ is the coadjoint representation of $G$ on $\mathfrak{g}^*$. A polynomial which is invariant under the adjoint representation will be constant on coadjoint orbits. This allows us to use the tools of classical invariant theory to solve the problem.
2. Restriction of the Problem

We would like to find a subspace $S$ of $g^*$ for which the set $\{\text{Ad}^*(g)s | g \in G, s \in S\}$ is Zariski dense in $g^*$. In such a case, any polynomial which is invariant under the coadjoint action is defined by its values on that subspace. In particular, we would like to find an $S$ which is transversal to the coadjoint orbits. Since in general orbits will intersect the transversal subspace multiple times, the restriction of an invariant function will satisfy some further restrictions, namely that it be invariant under the the action of the subgroup of $G$ which fixes the subspace.

Any polynomial function on $g^*$ will restrict to a polynomial function on a subspace, but the converse is not true. Therefore, after identifying candidate polynomials, we must eliminate those which don’t extend to a polynomial function.

For a semisimple Lie group, the Killing form provides a nondegenerate inner product with which we can identify the Lie algebra and its dual. Because the Killing form is invariant with respect to the adjoint action, the coadjoint action is essentially the same as the adjoint action. Unfortunately, $\text{Gal}(n)$ does not have such a convenient inner product. Instead, we use the usual matrix inner product $\langle A, B \rangle = \text{tr}(A^T B)$.

3. The Coadjoint Action

Recalling the matrix forms of the Galilean group and its Lie algebra given in (1) and (2), we must now describe the dual space $\text{gal}^*$ and the coadjoint action of $\text{Gal}(n)$ on $\text{gal}^*$.

To represent $\text{gal}(n)^*$, we use the usual matrix inner product $A^*(B) = \text{tr}(A^T B)$. We can then identify $\text{gal}(n)^*$ with $\mathbb{R}^{(n+2)\times(n+2)}$ modulo $\text{gal}(n)^\perp$, where

$$\text{gal}(n)^\perp = \left\{ \begin{pmatrix} S & 0 0 \vdots \\ 0 & \ddots \\ 0 & 0 & A \end{pmatrix} | S \text{ symmetric, } A \in \mathbb{R}^{2\times n}, a, b, c \in \mathbb{R} \right\}$$

Using the fact that the trace is invariant under cyclic permutations,

$$(\text{Ad}^*(g)A^*)(B) = A^*(\text{Ad}(g^{-1}B))
= A^*(g^{-1}Bg)
= \text{tr}(A^T g^{-1}B g)
= \text{tr}(gA^T g^{-1}B)
= (\text{tr}(gA^T g^{-1}B))^T \ast (B)$$

Every element of $\text{Gal}(n)$ can be written as $\tau \rho$, where $\rho$ is a spatial rotation and $\tau$ is a space-time translation and boost. Because of this, we may consider the actions of rotations separately from boosts and translations.

If

$$\rho = \begin{pmatrix} \rho & 0 0 \\ 0 & 1 0 \\ 3 & 0 1 \end{pmatrix}$$
and

\[
A^* = \begin{pmatrix} K^* & v_1^* x_1^* \\ \vdots & \vdots \\ v_n^* x_n^* & 0 \\ 0 & x_0 0 \\ 0 & 0 \end{pmatrix}
\]

then a straightforward calculation shows:

\[
\rho^{-1} = \begin{pmatrix} \rho^{-1} & 0 0 \\ 0 & 1 1 \end{pmatrix}
\]

and

\[
(\rho A^T \rho^{-1})^T = \begin{pmatrix} \rho K^* \rho^{-1} & \rho \begin{pmatrix} v_1^* x_1^* \\ \vdots \\ v_n^* x_n^* \end{pmatrix} \\ 0 & 0 \end{pmatrix}
\]

For translations and boosts,

\[\tau = \begin{pmatrix} I & v_1 x_1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 & 0 \\ 0 & 1 \end{pmatrix} \]

\[\tau^{-1} = \begin{pmatrix} I & -v_1 x_0 - x_1 \\ \vdots & \vdots \\ 0 & 1 - x_0 \end{pmatrix} \begin{pmatrix} x_0 & 0 \\ 0 & 1 \end{pmatrix} \]

\[(\tau A^* \tau^{-1})^T = \begin{pmatrix} K^* + \begin{pmatrix} v_1^* x_1^* \\ \vdots \\ v_n^* x_n^* \end{pmatrix} (v_1 \ldots v_n) \end{pmatrix} \begin{pmatrix} x_0 + x_0 x_1^* x_1^* \\ \vdots \\ x_0 + x_0 x_1^* x_1^* \end{pmatrix} \]

\[\equiv \begin{pmatrix} K^* + \frac{1}{2} \begin{pmatrix} v_1^* x_1^* \\ \vdots \\ v_n^* x_n^* \end{pmatrix} (v_1 \ldots v_n) - \frac{1}{2} (v_1 \ldots v_n) \end{pmatrix} (v_1^* x_1^*) \begin{pmatrix} x_0 + x_0 x_1^* x_1^* \\ \vdots \\ x_0 + x_0 x_1^* x_1^* \end{pmatrix} \]

where the final congruence is modulo \( \mathfrak{g} \mathfrak{a} \mathfrak{i}(n)^\perp \).

To find a transversal subspace of \( \mathfrak{g} \mathfrak{a} \mathfrak{i}(n)^* \), let \( A^* \) be a generic element written as as (4).

We would like to find a \( g \) which can take \((g^{-1} A^* T g)^T \) to our subspace.

First, use a rotation (5) to put \( x^* \) in the form \((A0 \ldots 0)^T \) and \( v^* \) in the form \((CB0 \ldots 0)^T \).

We can then use an \( x_0 \)-only translation element (equation (6) with \( x_0 = -\frac{x_1^*}{2} \)) to zero out the first element of \( v^* \). Thus we can restrict our attention to matrices of the form

\[
\begin{pmatrix} K^* & 0 & A \\ B & 0 & 0 \\ 0 & x_0 & 0 \end{pmatrix}
\]
Let’s now turn to the top-left quadrant. Referring again to (6),

\[
\frac{1}{2} \begin{pmatrix}
0 & A \\
B & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 & \ldots & v_n \\
x_1 & \ldots & x_n
\end{pmatrix}
- \frac{1}{2} \begin{pmatrix}
0 & A \\
B & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 & \ldots & v_n \\
x_1 & \ldots & x_n
\end{pmatrix}
\]

\[
= \frac{1}{2}
\begin{pmatrix}
0 & Ax_2 - Bv_1 & Ax_3 & Ax_4 & \ldots & Ax_n \\
Bv_1 - Ax_2 & 0 & Bv_3 & Bv_4 & \ldots & Bv_n \\
-Ax_3 & -Bv_3 & 0 & 0 & \ldots & 0 \\
-Ax_4 & -Bv_4 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-Ax_n & -Bv_n & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

Appropriate choices for the \(x_i\)'s and \(v_i\)'s allow us to zero out the topmost two rows and leftmost two columns of \(K^*\). In particular, for \(i \geq 3\), we set \(x_i = -\frac{2K_i^*}{A}\) and \(v_i = -\frac{2K_i^*}{B}\). It is important to note that there is a remaining degree of freedom when zeroing \(K^*_{12}\), as this will allow us to clear \(x_0^*\) by setting \(v_1 = -\frac{x_0^*}{x_1^*}\) in (6).

We are free to choose a rotation from the subgroup of \(O(n)\) which fixes the first two coordinates. We can use this freedom to conjugate the upper-left block to an element of a maximal torus in so\((n - 2)\).

Our transversal subspace is made of matrices of the form

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & A \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

where \(A\) and \(B\) are real numbers, and \(K\) is an element of a so\((n - 2)\) maximal torus.

4. Case \(n = 1\)

When \(n = 1\), the upper-left hand block is always zero. We use \(x_0\) to zero out the \(v_1^*\) and \(v_1\) to zero out \(x_0^*\). Thus, a generic element

\[
\begin{pmatrix}
0 & v_1^* & x_1^* \\
0 & 0 & x_0^* \\
0 & 0 & 0
\end{pmatrix}
\]

of \(gal(1)^*\) can be conjugated to

\[
\begin{pmatrix}
0 & 0 & x_1^* \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Since \((-1) \in O(1)\), the algebra of Ad\(^*-\)invariant polynomials is generated by \(X_1^2\).

5. Case \(n = 2, 3\)

**Theorem 1.** For \(n \in \{2, 3\}\), \(S(gal(n))^\text{Gal(n)}\) is generated by \(\sum X_i^2\) and

\[
\left(\sum X_i^2\right) = \left(\sum V_i^2\right) - \left(\sum X_i V_i\right)^2
\]
Proof. When \( n = 2 \) or \( 3 \), \( K^* \) may not be zero, but we can still zero out the upper-left block entirely. If
\[
Ξ = \begin{pmatrix}
K^* & v^* & x^*
0 & 0 & t^*
0 & 0 & 0
\end{pmatrix}
\]
is a matrix representing an element of \( \text{gal}(n)^* \), then then there exists an element \( g \in \text{Gal}(n) \) such that
\[
\text{Ad}(g)^*Ξ ≡ \begin{pmatrix}
0 & 0 & A
B & 0 & 0
0 & 0 & 0
0 & 0 & 0
\end{pmatrix}
\]
where \( A = \|x^*\| \) and \( B = \|\text{proj}_{x^*⊥}(v^*)\| \). Conjugating by an element of \( O(n) \) can switch the signs of \( A \) and \( B \) independently, so the invariant polynomials on the transverse manifold are generated by \( A^2 \) and \( B^2 \). Thus any polynomial \( P \) which is invariant under the coadjoint action of \( \text{Gal}(n) \) restricts to a polynomial \( \overline{P} = F(A^2, B^2) \) defined on the transverse manifold.

Define polynomials \( Q_1 : \text{gal}(n)^* \to \mathbb{R} \) and \( Q_2 : \text{gal}(n)^* \to \mathbb{R} \), which take an element of \( \text{gal}(n)^* \) to the values of \( A^2 \) and \( A^2B^2 \) on the intersection of its \( \text{Ad}^* \) orbit with the transversal submanifold. If \( Ξ \) is a matrix representing an element of \( \text{gal}(n)^* \), we have:
\[
Q_1(Ξ) = \|x^*\|^2 = \left( \sum_i X_i^2 \right)(Ξ)
\]
\[
Q_2(Ξ) = \|x^*\|^2 \|\text{proj}_{x^*⊥} v^*\|^2 = \left( \left( \sum_i X_i^2 \right) \left( \sum_i V_i^2 \right) - \left( \sum_i X_iV_i \right)^2 \right)(Ξ)
\]
Because the restriction is injective on the space of \( \text{Ad}^* \) invariant polynomials, we know that
\[
P(Ξ) = F\left( Q_1(Ξ), \frac{Q_2(Ξ)}{Q_1(Ξ)} \right)
\]
(7)
\[
= F_1(Q_1(Ξ), Q_2(Ξ)) + \frac{F_2(Q_1(Ξ), Q_2(Ξ))}{Q_1(Ξ)^{ℓ}}
\]
(8)
where \( F_2 \) is indivisible by its first argument.

While we are considering only real Lie algebras, if \( P \) is a polynomial, it must extend to a polynomial on the complexification of \( \text{gal}(n)^* \). With this in mind, consider
\[
Ξ = \begin{pmatrix}
0 & 0 & 0 & i
0 & 0 & z & 1
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{pmatrix}
\]
for \( n = 2 \) or
\[
Ξ = \begin{pmatrix}
0 & 0 & 0 & 0 & i
0 & 0 & 0 & z & 1
0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
for \( n = 3 \).
when \( n = 3 \). In either case, \( Q_1(\Xi) = 0 \), and \( Q_2(\Xi) = z^2 \). (8) then becomes

\[
P(\Xi) = F_1(0, z^2) + \frac{F_2(0, z^2)}{0^\ell}
\]

Since \( P \) was assumed to be a polynomial, either \( \ell = 0 \), or \( F_2(0, z^2) = 0 \) for all \( z \in \mathbb{C} \). Since \( F_2 \) was assumed to be indivisible by its first element, this implies that either \( \ell = 0 \) or \( F_2 \) is the constant function 0. Therefore, \( P \) is a polynomial in \( Q_1 \) and \( Q_2 \).

6. Case \( n > 3 \)

**Theorem 2.** When \( n > 3 \), \( S(\text{gal}(n))^{\text{Gal}(n)} \) is generated by the sums of determinants of \( 2k \times 2k \) submatrices formed by taking the symmetric minors of

\[
\begin{pmatrix}
K^* & v_1^* x_1^* \\
-\nu^* T & 0
\end{pmatrix}
\]

which include the last two rows and columns.

**Proof.** When \( n > 3 \), the coadjoint action can no longer zero out the entire upper-left block, only the uppermost two rows and leftmost two columns. The transversal subspace \( S \) is made of matrices of the form

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & A \\
0 & 0 & \cdots & B & 0 \\
\vdots & \vdots & \cdots & 0 & 0 \\
K & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

where \( K \) is an \( \text{so}(n - 2) \) maximal torus. Conjugating by elements of \( O(n) \) can change the signs of \( A \) and \( B \) independently, as well as permuting and changing signs of the elements of \( K \). The polynomials on \( S \) invariant under the subgroup of \( \text{Gal}(n) \) which fixes \( S \) are generated by \( A^2 \), \( B^2 \), and the coefficients of the characteristic polynomial of \( K \) [5].

By the same argument as in the \( n = 2 \) case, \( B^2 \) must be multiplied by \( A^2 \) to clear the denominator. This gives invariant polynomials \( A^2 = \sum X_i^2 \) and \( A^2 B^2 = (\sum X_i^2)(\sum V_i^2) - (\sum X_i V_i)^2 \)

We now turn to the polynomials which depend on the \( K \) part. When conjugating to \( S \), one step was to zero out the rows and columns corresponding to \( x^* \) and \( v^* \). This is equivalent to pre- and post- multiplying by the matrix \( P \), the orthogonal projection to \( \text{span}\{x^*, v^*\}^\perp \). The resulting element of \( \text{so}(n-2)^* \) is subject to a \( \text{O}(n-2) \) action, and it is well-known that the resulting invariant polynomials generated by the coefficients of the characteristic polynomial. Thus we are looking for \( \text{charpoly}(PK^*P) \).

We will also make use of some facts about exterior algebras. Suppose that \( e_1, \ldots, e_m \) and \( f_1, \ldots, f_n \) are bases for vector spaces \( V_1 \) and \( V_2 \) respectively. If \( A : V_1 \to V_2 \) is a linear function, then define \( \bigwedge^k A : \bigwedge^k V_1 \to \bigwedge^k V_2 \) to be the map \( x_1 \wedge x_2 \wedge \cdots \wedge x_k \mapsto Ax_1 \wedge Ax_2 \wedge \cdots \wedge Ax_k \). Then the \( (e_{i_1} \wedge \cdots \wedge e_{i_k}, f_{j_1} \wedge \cdots \wedge f_{j_k}) \) element of the matrix of \( \bigwedge^k A \) is the determinant of the \( k \times k \) matrix formed by taking elements in rows \( i_1, \ldots, i_k \) and columns \( j_1, \ldots, j_k \). In particular, the coefficient of the \( x^{n-k} \) term of \( \text{charpoly}(A) \) is \( \text{tr}(\bigwedge^k A) \).
Let
\[ \omega = \frac{v \wedge x}{\|v \wedge x\|} = \frac{1}{AB} v \wedge x \]
(the sign of \(AB\) is ambiguous, WLOG we may assume it’s positive), and let
\[ K' = \begin{pmatrix} K^* & v^*_1 x^*_1 \\ \vdots & \vdots \\ v^*_n x^*_n \\ -v^*T \\ -x^*T \end{pmatrix} \]

Let \( e_i \) be the vector (written vertically) with a 1 in the \( i \)’th row and zeros elsewhere.

If \( y \in \text{span}\{e_1, \ldots, e_n\}^\perp \):

\[ K'y = K^*y - (y \cdot v^*)e_{n+1} - (y \cdot x^*)e_{n+2} \]

And we also have the following:

\[ \left( \bigwedge^2 K' \right) e_{n+1} \wedge e_{n+2} = v^* \wedge x^* = AB\omega \]

and

\[ \left( \bigwedge^2 K' \right) \omega = \frac{1}{AB} \left( \bigwedge^2 K' \right) v^* \wedge x^* = \frac{1}{AB} (K^*v^* - (v^* \cdot v^*)e_{n+1} - (v^* \cdot x^*)e_{n+2}) \wedge (K^*x^* - (x^* \cdot v^*)e_{n+1} - (x^* \cdot x^*)e_{n+2}) \]

\[ = \frac{1}{AB} [K^*v^* \wedge (K^*x^* - (x^* \cdot v^*)e_{n+1} - (x^* \cdot x^*)e_{n+2}) + (K^*v^* - (v^* \cdot v^*)e_{n+1} - (v^* \cdot x^*)e_{n+2}) \wedge K^*x^* + (x^* \cdot x^*)(v^* \cdot v^*) - (x^* \cdot v^*)^2] e_{n+1} \wedge e_{n+2} \]

In particular, note that the \( e_{n+1} \wedge e_{n+2} \) term of \( \left( \bigwedge^2 K' \right) \omega \) is \( AB e_{n+1} \wedge e_{n+2} \).

Now consider any diagonal element \( \left( \bigwedge^{k+4} K' \right)_I \) for which \( I = (i_1, \ldots, i_{k+2}, n+1, n+2) \).

By [10], \( \left( \bigwedge^{k+4} K' \right)_I \wedge e_{n+1} \wedge e_{n+2} \in \bigwedge^{k+2} \mathbb{R}^{n+2} \wedge \omega \). Since we’re only considering diagonal elements, this means we can restrict our attention to the subspace

\[ \left( \bigwedge^k \mathbb{R}^n \right) \wedge \omega \wedge e_{n+1} \wedge e_{n+2} \]
where by $\mathbb{R}^n$ refers to span\{$e_1, \ldots, e_n$\} $\subseteq \mathbb{R}^{n+2}$. Recall that $P$ projects to span\{$x^*, v^*$\}$\perp$. The we have, for $y_1, \ldots, y_k \in \mathbb{R}^n$:

\[
\left(\bigwedge^{(k+4)} K'\right) y_1 \wedge \cdots \wedge y_k \wedge \omega \wedge e_{n+1} \wedge e_{n+2} = \left(\bigwedge^{(k+4)} K'\right) Py_1 \wedge \cdots \wedge Py_k \wedge \omega \wedge e_{n+1} \wedge e_{n+2}
\]

\[
=A^2 B^2 K^* P y_1 \wedge \cdots \wedge K^* P y_k \wedge \omega \wedge e_{n+1} \wedge e_{n+2} + \text{terms without } \omega \wedge e_{n+1} \wedge e_{n+2}
\]

\[
=A^2 B^2 PK^* P y_1 \wedge \cdots \wedge PK^* P y_k \wedge \omega \wedge e_{n+1} \wedge e_{n+2} + \text{terms without } \omega \wedge e_{n+1} \wedge e_{n+2}
\]

By (12),

\[
\sum_{I=(i_1, \ldots, i_k+2, n+1, n+2)} \left(\bigwedge^{k+4} K'\right)_{(I,I)} = A^2 B^2 \text{tr} \left(\bigwedge^k PK^* P\right)
\]

which implies that the sum of $(k+4) \times (k+4)$ subdeterminants which include the last two rows and columns is the $x^{n-k}$ coefficient of $A^2 B^2 \text{charpoly}(PK^* P)$. Because $PK^* P$ is skew-symmetric, the odd-$k$ terms will be zero.

To summarize, we have the polynomials $Q_1 = x^* \cdot x^*$, $Q_2 = (v^* \cdot v^*)(x^* \cdot x^*) - (x^* \cdot v^*)^2$, and $Q_3, \ldots, Q_{2+[n/2]}$, which are $Q_2$ times the characteristic polynomial coefficients of $PK^* P$.

Finally, we must show that the $Q_i$ generate the invariant polynomials. Suppose that $F(X)$ is an invariant polynomial not generated by the $Q_i$. Restricting to the transversal subspace $S$, $F$ is generated by the restrictions of $Q_1, Q_2, Q_3, \ldots, Q_{2+[n/2]}$. By the injectivity of restriction to $S$, this means that

\[
F(X) = F'(Q_1(X), Q_2(X), Q_3(X), \ldots, \frac{Q_{2+[n/2]}}{Q_2}(X))
\]

\[
= F''(Q_1(X), Q_2(X), \ldots, Q_{2+[n/2]}(X)) + \frac{\bar{F}(Q_1(X), Q_2(X), \ldots, Q_{2+[n/2]}(X))}{Q_1(X)^k Q_2(X)^\ell}
\]

where $\bar{F}$ is assumed not to be divisible by its first or second arguments. To show that $F$ is generated by the $Q_i$, we must show that $k = \ell = 0$.

While all of the preceeding work was done over $\mathbb{R}$, any polynomial extends to a polynomial on $\mathbb{C}$. Consider matrices of the form

\[
X\Xi = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & i \\
0 & 0 & \Xi & \cdots & 0 & 0 & 0 \\
0 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\]

where $\Xi$ is an $(n-2) \times (n-2)$ skew-symmetric matrix.
For any $\Xi$, $Q_1(X_{\Xi}) = 0$ and $Q_2(X_{\Xi}) = 1$. For $3 \leq i \leq 2 + \lfloor n/2 \rfloor$, we get the nonconstant characteristic polynomial coefficients for $\Xi$, which are known to be algebraically independent. Because of this algebraic independence, and that $\tilde{F}$ was assumed not to be divisible by its first argument, there exists a $\Xi$ such that the numerator of (13) is nonzero, while $Q_1(X_{\Xi})$ is zero. Therefore, $k = 0$.

To show that $\ell = 0$, let

$$
Y_{\Xi} = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \Xi & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}
$$

For any $\Xi \in \text{so}(n-2)$, $Q_1(Y_{\Xi}) = 1$ and $Q_2(Y_{\Xi}) = 0$. For $j \geq 3$, $Q_j(Y_{\Xi})$ is the sum of all $2j \times 2j$ subdeterminants of

$$
Y'_{\Xi} = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \Xi & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & -1 & -i & \ldots & 0 & 0 & 0 \\
-1 & 0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}
$$

which include the leftmost two columns and bottom two rows.

Consider a skew-symmetric submatrix of $Y'_{\Xi}$ which includes the last two rows and columns. If the submatrix doesn’t also include the first row and column, its leftmost column, and thus its determinant, will be zero. Similarly, any nonzero such minor must also include at least one of the second and third rows.

If the submatrix includes the second row and column, the corresponding minor is

$$
\det \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \tilde{\Xi} & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & -1 & -\alpha & \ldots & 0 & 0 & 0 \\
-1 & 0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix} = \det(\tilde{\Xi})
$$

where $\tilde{\Xi}$ is a $(2j - 4) \times (2j - 4)$ submatrix of $\Xi$, and $\alpha$ is either 0 or $i$ depending on whether the third column is included.
If the submatrix does not include the second row and column (which can only happen for $n > 4$), it must include the third, and the minor is instead

\[
\begin{vmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & \xi_{1k_1} & \ldots & \xi_{1k_{2k-4}} & i & 0 \\
0 & -\xi_{1k_1} & 0 & \ldots & 0 & : & : \\
: & : & \hat{\Xi} & : & : & 0 & 0 \\
0 & -\xi_{1k_{2k-4}} & 0 & \ldots & 0 & 0 & 0 \\
-1 & 0 & 0 & \ldots & 0 & 0 & 0
\end{vmatrix}
= -\det(\hat{\Xi})
\]

where now $\hat{\Xi}$ is a $(2j - 4) \times (2j - 4)$ submatrix of $\Xi$ which does not include the first row and column. Summing all of the $2j \times 2j$ minors of $Y_{\hat{\Xi}}'$ then yields the sum of $(2j - 4) \times (2j - 4)$ minors of $\Xi$ which include the first row.

Suppose $\Xi$ is of the form:

\[
\Xi = \begin{pmatrix}
0 & \xi_1 & 0 & \ldots & 0 \\
-\xi_1 & 0 & \vdots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \xi_1 & 0 \\
0 & \cdots & \cdots & \cdots & \xi_1
\end{pmatrix}
\]

Then $\text{charpoly}(\Xi) = \lambda \cdot \text{charpoly}(\Xi') - \xi_1^2 \cdot \text{charpoly}(\Xi')$ Since the nonconstant coefficients of the characteristic polynomial are the sums of the minors, and are algebraically independent on $\text{so}(m)$, this tells us that the sums of minors which include the first row and column are also algebraically independent. Putting everything together, this tells us that for any $\tilde{F}$ in (13), we can find a $Y_{\Xi}$ such that $Q_2(Y_{\Xi}) = 0$ and $\tilde{F}(Y_{\Xi}) \neq 0$. Therefore, $\ell = 0$ in (13), and so every polynomial on $\text{gal}(n)^\ast$ which invariant on the coadjoint orbits is generated by our $Q_j$’s.

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