Dissipation and fluctuations in elongated bosonic Josephson junctions

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We investigate the dynamics of bosonic atoms in elongated Josephson junctions. We find that these systems are characterized by an intrinsic coupling between the Josephson mode of macroscopic quantum tunneling and the sound modes. This coupling of Josephson and sound modes gives rise to a damped and stochastic Langevin dynamics for the Josephson degree of freedom. From a microscopic Lagrangian, we deduce and investigate the damping coefficient and the stochastic noise, which includes thermal and quantum fluctuations. Finally, we study the time evolution of relative-phase and population-imbalance fluctuations of the Josephson mode and their oscillating thermalization to equilibrium.

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I. INTRODUCTION

In the last forty years dissipative quantum systems under the effect of random noise have been extensively investigated both theoretically and experimentally [1–5]. An important tool in the study of dissipative systems is the Caldeira-Leggett model [1], where a bath of harmonic oscillators is coupled to the main system. This bath leads to damping and noise in the main system and the dynamics is governed by the generalized Langevin equation [1,2]. One can find such dynamics in many different physical systems, for instance in superconducting Josephson circuits [6,7].

Recently, the dynamics of ultracold atoms in a double-well potential (bosonic Josephson junction) has attracted marked attention [8–10]. Ref. [9] has revealed the presence of intrinsic coupling between the Josephson mode and sound modes leading to damping of oscillations of population imbalance in a one-dimensional bosonic Josephson junction. The peculiarity of this system is the intrinsic coupling between the Josephson mode and the bath which is distinct from a system in contact with an external bath like the Caldeira-Leggett model [1]. Namely, the one-dimensional Bose Josephson junction exhibits quantum Langevin dynamics in the Josephson mode without any external reservoir. Furthermore, the typical approach of a Caldeira-Leggett model consists in the introduction of an external bath in a phenomenological way, as well as for the interaction between the system and the environment. Here instead we manage to pull out an intrinsic phonon bath (interacting with the Josephson modes) straightforwardly from the model.

In this paper we start from a model of two one-dimensional quasi-condensates with a Josephson coupling in the head-to-tail configuration discussed in Refs. [9,10]. We treat phase and number density as classical fields to obtain the effective Lagrangian of relative phase. By adopting this different procedure with respect to Ref. [9], we show that this system involves the intrinsic coupling between the Josephson mode and other infinite numbers of sound modes. Consequently, the dynamics of the Josephson mode is described by the quantum Langevin equation with damping and noise that includes thermal and quantum fluctuations. We find an analytic formula for the damping coefficient moreover, we derive analytic expressions for the fluctuations of the Josephson relative phase and Josephson population imbalance in the high-temperature regime, extending the results of Ref. [9]. Finally, we analyze numerically the fluctuations of the Josephson mode in the general case, in particular in the low-temperature regime where quantum fluctuations play a crucial role. Remarkably, in this low-temperature regime, an ultraviolet cutoff for the sound modes is needed. In this way, we determine the thermalization to the equilibrium of the Josephson fluctuations.

II. BOSONS WITH JOSEPHSON TUNNELLING

We start from the following Lagrangian density which consists of two weakly-interacting Bose-Einstein quasi-condensates ($j = 1,2$) made of atoms with mass $m$ in one dimension

$$\mathcal{L} = \sum_{j=1}^{2} \left[ \frac{i\hbar \psi_j^\ast \partial_t \psi_j - \frac{\hbar^2}{2m} |\partial_x \psi_j|^2 - \frac{g}{2} |\psi_j|^4} \right] + \frac{J(x)}{2} \left[ \psi_1^\ast \psi_2 + \psi_2^\ast \psi_1 \right],$$

where $\hbar$ is the Planck constant. Here $g$ is the strength of the inter-atomic potential in the contact-interaction approximation and $J(x)$ is the space dependent tunneling-
energy coupling. The complex field \( \psi_j(x,t) \) of the \( j \)-th quasi-condensate can be rewritten by means of the Madelung representation

\[
\psi_j(x,t) = \sqrt{\rho_j(x,t)} e^{i\phi_j(x,t)},
\]

where \( \rho_j(x,t) = |\psi_j(x,t)|^2 \) is its atomic density. Substituting the expression of Eq. (2) into the Lagrangian in Eq. (1), we obtain

\[
\mathcal{L} = \sum_{j=1}^{2} \left[ \frac{i\hbar}{2} \partial_t \rho_j - \hbar \rho_j \partial_t \phi_j - \frac{\hbar^2}{2m} \left( \frac{1}{4\rho_j} (\partial_x \rho_j)^2 + \rho_j (\partial_x \phi_j)^2 \right) - \frac{g}{2\rho_j^2} \right] + J(x)\sqrt{\rho_1 \rho_2} \cos(\phi_1 - \phi_2)
\]

A compact description of the system is reached when we introduce the relative phase and the population imbalance

\[
\begin{align*}
\phi &= \phi_1 - \phi_2, \\
\zeta &= \frac{\rho_1 - \rho_2}{2\bar{\rho}},
\end{align*}
\]

with \( \bar{\rho} = (\rho_1 + \rho_2)/2 \) the average atomic density. Introducing also a total phase \( \bar{\phi} = \phi_1 + \phi_2 \), one can express the Lagrangian in Eq. (4) in terms of these quantities. In addition, let us work in a canonical ensemble (no particle leaves or enters the system) for which \( \partial_t \bar{\rho} = 0 \) holds, and assuming it also for the total phase \( \partial_t \bar{\phi} = 0 \), we obtain a new Lagrangian density

\[
\mathcal{L} = -\hbar \bar{\rho} \ddot{\bar{\phi}} - \frac{\hbar^2 \bar{\rho}}{4m} (\partial_x \bar{\phi})^2 - g(\rho^2 + \bar{\rho}^2 \zeta^2) + J(x)\bar{\rho} \sqrt{1 - \zeta^2} \cos(\phi),
\]

in which we also assumed space homogeneity \( \partial_x \bar{\rho} = \partial_x \bar{\phi} = 0 \) and neglected space variations of the population imbalance \( \partial_x \zeta = 0 \) (see also, for instance, Refs. [10, 11]). It is reasonable (also with the experimental setup) to work with small values of the population imbalance, such that \( |\zeta| \ll 1 \). In this case we obtain the equation of motion

\[
\zeta(x,t) = -\frac{\hbar \ddot{\phi}(x,t)}{2g\bar{\rho} + J(x) \cos(\phi)}.
\]

Setting the Josephson regime \( 2g\bar{\rho} \gg J(x) \), we can simplify the expression of Eq. (7) and the Lagrangian density only for the relative phase can be written as

\[
\mathcal{L} = \frac{\hbar}{4g} \ddot{\phi}^2 - \frac{\hbar^2 \bar{\rho}}{4m} (\partial_x \phi)^2 + J(x)\bar{\rho} \cos(\phi).
\]

In the rest of the paper, we shall adopt the head-to-tail configuration, as depicted in Fig. 1, where

\[
J(x) = J_0 \delta(x).
\]

with \( J_0 \) a constant tunneling coupling, \( x \in [0, L] \) and \( L \) the length of the two elongated Bose-Einstein quasi-condensates. This head-to-tail configuration is equivalent to a system where one quasi-condensate is confined in the region \([-L, 0]\) while the other quasi-condensate is confined in the region \([0, L]\), and the tunneling barrier is located at \( x = 0 \). This is the same configuration considered in Ref. [9]. As we shall see, the Dirac delta function gives rise to the so-called boundary sine-Gordon model [12]. More generally, the idea of approximating a realistic finite-range tunneling energy \( J(x) \) with a zero-range one \( J_0 \delta(x) \) implies that \( J_0 = \int_{-L}^{L} J(x) dx = J(k = 0) \), where \( J(k) \) is the Fourier transform of \( J(x) \). This approximation is justified if the characteristic range of \( J(x) \), localized around \( x = 0 \), is much smaller than the characteristic lengths of the problem under consideration. Within this approximation one can obtain the tunneling coupling given by Eq. (9) starting from the Lagrangian density of a single bosonic field \( \psi(x,t) \), with \( x \in [-L, L] \), setting \( \psi(x,t) = \psi_1(x,t) + \psi_2(x,t) \) where \( \psi_1(x,t) \) is mainly localized on one side of a narrow potential barrier \( U(x) \) with maximum at \( x = 0 \) and \( \psi_2(x,t) \) is mainly localized on the other side of \( U(x) \). In this case one gets \( U(x) |\psi(x,t)|^2 \approx -J_0 \delta(x) |\psi_1^*(x,t)\psi_2(x,t) + \psi_1(x,t)\psi_2^*(x,t)|^2 \), with \( U(x) \approx -J_0 \delta(x) \). Alternatively, the delta-function tunneling term of Eq. (9) can be interpreted as due to a local internal Rabi coupling. Experimentally one can induce a Rabi coupling in a small region of a Bose-Bose mixture.

It is important to stress that the case of a uniform tunneling coupling, i.e. \( J(x) = J_{\text{unif}} \), which is relevant for the experiment of Pigneur et al. [8], was theoretically considered by Bouchou [13] and also by Grisins and Mazets [14].

### III. QUASI-PARTICLE DESCRIPTION

We will now introduce a quasi-particle description for the phase, based on the following mode expansion

\[
\phi(x,t) = \frac{1}{\sqrt{L}} \sum_{n=0}^{+\infty} q_n(t) \Phi_n(x),
\]

where \( q_n(t) \) are coordinates, and \( \Phi_n(x) \) are real eigenfunctions satisfying \( -\hbar^2/(2m) \partial_x^2 \Phi_n(x) = \epsilon_n \Phi_n(x) \) where

![Fig. 1. Head-to-tail configuration of the system under investigation. The tunneling of bosons occurs with strength \( J_0 \) only at \( x = 0 \). Here \( J_0 \) has the dimensions of energy times length.](image-url)
\( \epsilon_n = \hbar^2 k_n^2 / (2m) \) and \( k_n = \pi n / L \), and constituting an orthonormal basis \( f_n^L \Phi_n(x)\Phi_m(x) dx = \delta_{n,m} \). This mode expansion in Eq. (10) leads to

\[
L = \int_0^L \mathcal{L} \, dx = \frac{M}{2} \sum_n \dot{q}_n^2 - \frac{M}{2} \sum_n \omega_n^2 \dot{q}_n^2 + J_0 \rho \cos \left( \frac{L}{L} \sum_n q_n(t) \right).
\]

(11)

Here we have defined the effective mass

\[
M = \frac{\hbar}{2gL},
\]

(12)

and the mode dispersion

\[
\omega_n = c_s k_n,
\]

(13)

where

\[
c_s = \sqrt{\frac{\rho g}{m}},
\]

(14)

is the speed of sound, as known from the description of superfluids, and that \( J_0 \) and \( g \) have the dimension of energy times a unit of length.

We can distinguish between the Josephson mode we are interested in \((n = 0)\) and a phonon bath (we mention phonons due to the acoustic spectrum above) made of infinite contributions, that affect the evolution of the oscillations in the Josephson mode. Separating the Josephson mode from bath modes, one obtains

\[
L = \frac{M}{2} \dot{q}_0(t)^2 + \frac{M}{2} \sum_{n=1}^{+\infty} \dot{q}_n(t)^2 - \frac{M}{2} \sum_{n=1}^{+\infty} \omega_n^2 q_n(t)^2 + J_0 \rho \cos \left( \frac{\dot{q}_0(t)}{L} \right) + \frac{1}{L} \sum_{n=1}^{+\infty} q_n(t).
\]

(15)

This is the Lagrangian describing the Josephson mode coupled to a bath composed of independent harmonic oscillators.

IV. DAMPED DYNAMICS

The Legendre transformation \( \mathcal{H} = \sum_{n=0}^{+\infty} \dot{q}_n(t)p_n(t) - \mathcal{L} \) with \( p_n(t) = M \dot{q}_n(t) \) results in the following Hamiltonian

\[
\mathcal{H} = \sum_{n=0}^{+\infty} \frac{p_n^2}{2M} + \frac{M \omega_n^2}{2} q_n^2 - J_0 \rho \cos \left( \frac{1}{L} \sum_{n=0}^{+\infty} q_n(t) \right).
\]

(16)

Now we perform a canonical transformation with new coordinates and momenta

\[
Q_0(t) = q_0(t) + \sum_{n=1}^{+\infty} q_n(t),
\]

(17)

\[
Q_n(t) = q_n(t) \quad \text{for} \quad n \neq 0,
\]

(18)

\[
P_0(t) = p_0(t),
\]

(19)

\[
P_n(t) = p_n(t) - p_0(t) \quad \text{for} \quad n \neq 0.
\]

(20)

Notice that this is the analogue of the unitary transformation used in Ref. [9] for the quantum operators. In this way, we obtain the transformed Hamiltonian

\[
\mathcal{H} = \frac{P_0^2}{2M} + \sum_{n=1}^{+\infty} \left[ \frac{(P_n + P_0)^2}{2M} + \frac{1}{2} M \omega_n^2 Q_n^2 \right] - J_0 \rho \cos \left( \frac{Q_0}{L} \right).
\]

(21)

It is worth noting that the canonical transformation introduced an intrinsic coupling in the harmonic oscillators of the bath, between the zero-mode and the excited modes. The Hamiltonian of Eq. (21) corresponds to the velocity-coupling model in which the coupling is through the momentum \( \mathbf{J} \). For the sake of completeness, we stress that in the case of two parallel tubes with uniform tunneling energy, i.e., with \( J(x) = J_{\text{unit}} \), only the inclusion of anharmonic terms gives rise to a coupling between the Josephson mode and the bath of elementary excitations.

Hamilton equations provided by the Hamiltonian in Eq. (21) are

\[
\dot{Q}_0(t) = \frac{P_0(t)}{M} + \sum_{n=1}^{+\infty} \frac{P_n(t) + P_0(t)}{M},
\]

(22)

\[
\dot{Q}_n(t) = \frac{P_0(t) + P_n(t)}{M},
\]

(23)

\[
\dot{P}_0(t) = -\frac{J_0 \rho}{L} \sin \left( \frac{Q_0(t)}{L} \right),
\]

(24)

\[
\dot{P}_n(t) = -M \omega_n^2 Q_n(t).
\]

(25)

These equations of motion lead to

\[
Q_n(t) = \cos(\omega_n t)Q_n(0) + \frac{\sin(\omega_n t)}{\omega_n} \dot{Q}_n(0) - \frac{J_0 \rho}{ML \omega_n^2} \left[ \sin \left( \frac{Q_0(t)}{L} \right) - \cos(\omega_n t) \sin \left( \frac{Q_0(0)}{L} \right) \right.
\]

\[
- \int_0^t dt' \cos(\omega_n(t-t')) \cos \left( \frac{Q_0(t')}{L} \right) \dot{Q}_0(t') \right] \dot{Q}_0(t).
\]

(26)

Details for the derivation of Eq. (26) are summarized in Appendix A. One can easily show that the Josephson mode is related to the relative phase at \( x = 0 \) as

\[
\phi_0(t) = \phi(x = 0, t) = \frac{Q_0(t)}{L}.
\]

(27)

Eventually, we reach the equation of motion for a damped harmonic oscillator with respect to the relative phase

\[
\ddot{\phi}_0(t) + \int_0^t dt' \gamma(t-t') \dot{\phi}_0(t') + \Omega_0^2 \sin(\phi_0(t)) = \xi_\phi(t),
\]

(28)

in which the damping is ruled by the damping kernel

\[
\gamma(t-t') = \Omega_0^2 \sum_{n=1}^{+\infty} \cos(\omega_n(t-t')) \cos(\phi_0(t')),
\]

(29)

proportional to the Josephson frequency defined as

\[
\Omega_0 = \sqrt{\frac{\rho J_0}{ML^2}}.
\]

(30)
Now let us focus on the case of a small relative phase $|\phi_0(t)| \ll 1$. In this case $\cos(\phi_0(t')) \sim 1$ holds. The expression of Eq. (29) can be calculated explicitly by moving to the continuum limit as

$$
\gamma(t - t') = \gamma_0 \delta(t - t'),
$$

where

$$
\gamma_0 = \frac{\rho J_0}{M L c_s},
$$

is the damping constant that rules the relaxation in time of the phase. Using the damping constant, one can write Eq. (28) as

$$
\ddot{\phi}_0(t) + \gamma_0 \dot{\phi}_0(t) + \Omega_0^2 \phi_0(t) = \xi_\phi(t).
$$

This is the main equation in the paper. On the left-hand side of this equation, it involves the Josephson frequency $\Omega_0$, given by Eq. (30), and the damping constant, given by Eq. (32) with the speed of sound $c_s$ of Eq. (14).

The right-hand side of Eq. (33) is a noise term denoted by $\xi_\phi(t)$, composed of the infinite contributions of the bath modes and dependent only on the initial conditions of coordinates and momenta

$$
\xi_\phi(t) = -\sum_{n=1}^{+\infty} \left[ \omega_n^2 \cos(\omega_n t) \frac{Q_n(0)}{L} + \omega_n \sin(\omega_n t) \frac{\dot{Q}_n(0)}{L} \right]
- \gamma_0 \delta(t) \sin(\phi_0(0)).
$$

The last term has the nature of a transient, and in order to avoid its complications, we will get rid of the transient term by setting the initial condition $\phi_0(0) = 0$.

### A. Deterministic dynamics of the Josephson mode

As an homogeneous solution of Eq. (33) without the noise, one gets

$$
\phi_0(t) = e^{-\gamma_0 \Omega_0 t} \frac{\dot{\phi}_0(0)}{\Omega_0 \gamma_J} \sin(\gamma_J \Omega_0 t),
$$

where we defined

$$
\gamma_Q = \frac{\gamma_0}{2 \Omega_0},
$$

and

$$
\gamma_J = \sqrt{1 - \gamma_Q^2}.
$$

We can confirm that the dimensionless damping constant $\gamma_Q$ corresponds to $\sqrt{E_J/K}$ in Ref. [2] in the weak-coupling limit where $K \sim 1/\sqrt{g}$ denotes the Luttinger parameter in Luttinger liquid theory [9]. This indicates that our current analysis under the quasi-particle description recovers the analysis within Luttinger liquid theory.

We can see that the deterministic dynamics of the Josephson mode can be calculated explicitly by setting the initial condition $\dot{\phi}_0(0) = 0$. In the weak-coupling limit $K \sim 1/\sqrt{g}$, in the following, only the underdamped regime $\gamma_Q < 1$ is analyzed for simplicity. It reflects the most similar behaviour with respect to the experimental observations of the relative phase and population imbalance [8]. The other two regimes, overdamped and critical damped, are respectively obtained in the case of $\gamma_Q > 1$ and $\gamma_Q = 1$.

In panel (a) of Fig. 2 we plot the dynamics of the relative phase $\phi_0(t)$, without the effect of noise $\xi_\phi(t)$ and setting the initial condition $\phi_0(0)/\Omega_0 = 1$ for simplicity. In panel (b) of Fig. 2 we plot instead the corresponding population imbalance $\zeta_\phi(t)$, with initial conditions $\zeta_\phi(0) = 0$ and $\dot{\zeta}_\phi(0)/\Omega_0 = 0$. The initial conditions for $\zeta_\phi(0)/\Omega_0$ is dependent on the choice of $\phi_0(0)$ through Eq. (24). On the other hand, the choice of $\zeta_\phi(0)$ is necessary in order to start from a configuration out of equilibrium, in which the number of atoms in the two wells is not balanced. It illustrates quite intuitive behaviours. As one increases the damping coefficient $\gamma_0$, the amplitude
od oscillation is suppressed and both of \( \dot{\phi}_0(t) \) and \( \zeta_0(t) \) vanish for \( t \to \infty \).

The results in Fig. 2 is, however, missing the relevant role of the noise, which produces fluctuations around the mean-field behaviour shown in the figure. In general, these fluctuations have both quantum and thermal components. We investigate these effects of the noise in the next Sections.

**B. Dynamics of the Josephson mode in the presence of noise**

We refer to \( \xi_\phi(t) \) as a quasi-stochastic noise due to the analogy of Eq. (33) with a generalized Langevin equation. The stochastic nature of this term, however, lies only in the initial conditions \( Q_n(0) \) and \( Q_n(0) \), then \( \xi_\phi(t) \) is not a random quantity in time. Thanks to this property, we do not need special tools like stochastic calculus in order to integrate Eq. (33) in time, but we will find the particular solution using the method of variation of parameters. The solution takes the form of

\[
\phi_0(t) = e^{-\gamma_0 \Omega_0 t} \frac{\sin(\gamma_0 \Omega_0 t)}{\gamma_0} + \int_0^t dt' \chi(t-t') \xi_\phi(t'),
\]

where we have defined the following retarded function

\[
\chi(t-t') = \frac{2}{\omega_D} e^{-\gamma_0 \Omega_0 (t-t')} \sin \left( \frac{\omega_D}{2} (t - t') \right) \theta(t-t'),
\]

with \( \theta(t) \) the Heaviside step function and the oscillation frequency

\[
\omega_D = \sqrt{4\Omega_0^2 - \gamma_0^2}.
\]

**V. THE POPULATION IMBALANCE**

Using Eq. (7) in the Josephson regime, one can make a quasi-particle description also for the population imbalance \( \zeta(x, t) \) as

\[
\zeta(x, t) = \frac{\sqrt{L}}{\hbar \rho} \sum_{n=0}^{+\infty} p_n(t) \Phi_n(x)
\]

\[
= \frac{P_0(t)}{\hbar \rho} + \frac{\sqrt{L}}{\hbar \rho} \sum_{n=1}^{+\infty} \left[ P_0(t) + P_n(t) \right] \Phi_n(x).
\]

We can identify the Josephson mode as the first term

\[
\zeta_0(t) = \frac{P_0(t)}{\hbar \rho}.
\]

Equations of motion for the momenta \( P_0(t) \) and \( P_n(t) \) read

\[
\ddot{P}_0(t) + \Omega_0^2 P_0(t) = \Omega_0^2 \sum_{n=1}^{+\infty} \frac{\dot{P}_n(t)}{\omega_n^2},
\]

\[
\ddot{P}_n(t) + \omega_n^2 P_n(t) = -\omega_n^2 P_0(t).
\]

They give the damped behaviour of the Josephson mode

\[
\ddot{\zeta}_0(t) + \gamma_0 \dot{\zeta}_0(t) + \Omega_0^2 \zeta_0(t) = \xi_\zeta(t).
\]

Here we have defined the noise term for the population imbalance as

\[
\xi_\zeta(t) = -\frac{\Omega_0^2}{\hbar \rho} \sum_{n=1}^{+\infty} \left[ \cos(\omega_n t) P_n(0) + \frac{\sin(\omega_n t)}{\omega_n} \dot{P}_n(0) \right],
\]

where we have set the initial condition \( P_0(0) = 0 \). The meaning of this choice will be stressed in the next section. The time evolution of \( \zeta_0(t) \), without the effect of the noise \( \xi_\zeta(t) \), is displayed in Fig. 2(b).

**VI. QUANTUM AND THERMAL PROPERTIES OF THE NOISE**

Interesting quantities regarding the noise, which allow us to investigate the quantum and thermal fluctuations of \( \phi_0(t) \) due to the presence of \( \xi_\phi(t) \), are its bath average \( \langle \xi_\phi(t) \rangle \) and, in particular, its correlation function \( \langle \xi_\phi(t) \xi_\phi(t') \rangle \). We will perform a calculation already done in Ref. [20] in the context of a Caldeira-Leggett model [1]. In order to evaluate the average over the environment, we have to identify the bath Hamiltonian. From Eq. (21), denoting the bath part by \( \mathcal{H}_B \), we have

\[
\mathcal{H}_B = \sum_{n=1}^{+\infty} \left[ \frac{P_0^2}{2M} + \frac{M \omega_n^2}{2} Q_n^2 \right].
\]

As we previously pointed out, this Hamiltonian is composed of infinite harmonic oscillators which are intrinsically coupled to the system. However, it is reasonable to assume that the size of the phonon bath is huge such that the system does not affect it. For this reason one can impose \( P_0(0) = 0 \), namely the system is completely decoupled from the bath at \( t = 0 \). Hence, in this model, we deal with a bath of independent harmonic oscillators [2] at initial time \( t = 0 \) as

\[
\mathcal{H}_B = \sum_{n=1}^{+\infty} \left[ \frac{P_n^2}{2M} + \frac{M \omega_n^2}{2} Q_n^2 \right]
\]

which is used to evaluate the ensemble average.

In order to evaluate ensemble averages, we consider \( Q_n(0) \) and \( P_n(0) \) to be quantum operators. By means of the Hamiltonian in Eq. (48), we can adopt the annihilation and creation operators \( a_n \) and \( a_n^\dagger \) as

\[
Q_n(0) = \sqrt{\frac{\hbar}{2M \omega_n}} (a_n + a_n^\dagger),
\]

\[
P_n(0) = -i \frac{\hbar M \omega_n}{2} (a_n - a_n^\dagger),
\]
and then evaluate the average, in the case of $\xi_\phi(t)$ for instance, as
\[
\langle \xi_\phi(t) \rangle = \frac{\text{Tr} \left[ e^{-\beta H_0} \xi_\phi(t) \right]}{\text{Tr} \left[ e^{-\beta H_0} \right]},
\]
where $\beta = 1/(k_B T)$ with $k_B$ the Boltzmann constant and $T$ the absolute temperature of the bath of oscillators.

One readily obtains the vanishing averages for both of Eqs. (53) and (54)
\[
\langle \xi(t) \rangle = 0,
\]
while we find the two-point correlation functions as
\[
\langle \xi(t) \xi(t') \rangle = \sum_{n=1}^{+\infty} \frac{h \omega_n^3}{2ML^2} \left[ \coth \left( \frac{2}{\beta h \omega_n} \right) \cos \left( \omega_n (t - t') \right) - i \sin \left( \omega_n (t - t') \right) \right],
\]
resulting in the following correlation functions
\[
\langle \xi(t) \xi(t') \rangle = -\frac{\gamma_0}{M \Omega_0^2} k_B T \frac{d^2}{dt^2} \delta(t - t'),
\]
\[
\langle \xi(t) \xi(t') \rangle = \frac{\gamma_0 M \Omega_0^2}{h^2 \rho^2} k_B T \delta(t - t').
\]
It is worth noting the fact that $\xi_\phi(t)$ is a delta-correlated noise at high temperature likewise the classical fluctuation-dissipation relation, while $\xi_\phi(t)$ is slightly different from a white noise.

In the case of a generic temperature, one has to take into account that Eqs. (53) and (54) are complex valued quantities. In particular it is convenient to separate the real part from the purely imaginary one, by rewriting the correlators as
\[
\langle \xi(t) \xi(t') \rangle = \frac{1}{2} \left( \langle \xi(t) \xi(t') \rangle + \langle \xi(t) \xi(t') \rangle \right).
\]
The first term, which involves the anti-commutator $\{ , \}$ is related to the real part of Eq. (53), while the second one, which involves the commutator $[ , ]$, describes the purely imaginary term of the noise. The anti-symmetric parts are, however, found to result in no contribution to the variance of phase or population imbalance because they are odd functions of $\omega$. The Fourier transformed correlators of the symmetric part are given by
\[
\int_{-\infty}^{+\infty} dt \langle \{ \xi(t) \xi(0) \} \rangle e^{-i\omega t} = \frac{2 \gamma_0 h \omega^3}{ML^2 \Omega_0^2} \coth \left( \frac{\beta h \omega}{2} \right),
\]
\[
\int_{-\infty}^{+\infty} dt \langle \{ \xi(t) \xi(0) \} \rangle e^{-i\omega t} = \frac{2 M \Omega_0^2 \gamma_0 h \omega}{h^2 \rho^2} \coth \left( \frac{\beta h \omega}{2} \right).
\]

\section{VII. Fluctuations}

In order to understand the effect of the noise in the dynamics of $\phi_0(t)$ and $\zeta_0(t)$, we focus on the calculation of the variance $\Delta \phi_0(t)^2 = \langle \phi_0(t)^2 \rangle - \langle \phi_0(t) \rangle^2$ and $\Delta \zeta_0^2 = \langle \zeta_0(t)^2 \rangle - \langle \zeta_0(t) \rangle^2$. It is obvious from Eq. (38) that they will depend only on the correlators of the noise.

\subsection{A. High temperature}

Let us start from the relative phase fluctuations at high temperature with Eqs. (56) and (57). Eqs. (38), (56), and (57) give
\[
\Delta \phi_0(t)^2 = \Delta \phi_0(\infty)^2 \left[ 1 + e^{-2\gamma_0 \Omega_0 t} \left[ \frac{\gamma_0^2 - \gamma_J^2}{\gamma_J} - \frac{\gamma_J^2}{\gamma_J} \cos(2\gamma_J \Omega_0 t) - \frac{\gamma_Q}{\gamma_J} \sin(2\gamma_J \Omega_0 t) \right] \right],
\]
\[
\Delta \zeta_0^2 = \Delta \zeta_0(\infty)^2 \left[ 1 + e^{-2\gamma_0 \Omega_0 t} \left[ \frac{\gamma_0^2 - \gamma_J^2}{\gamma_J} - \frac{\gamma_J^2}{\gamma_J} \cos(2\gamma_J \Omega_0 t) - \frac{\gamma_Q}{\gamma_J} \sin(2\gamma_J \Omega_0 t) \right] \right].
\]
\[
\Delta \zeta(t)^2 = \Delta \zeta(\infty)^2 \left[ 1 - e^{-2\gamma Q \Omega_0 t} \left( \frac{1}{\gamma J} - \frac{\gamma_Q}{\gamma J} \cos(2\gamma J \Omega_0 t) + \frac{\gamma_Q}{\gamma J} \sin(2\gamma J \Omega_0 t) \right) \right],
\]
where the asymptotic values of these variances for \( t \to +\infty \) are given by
\[
\Delta \phi(\infty) = \frac{1}{\Omega_0} \sqrt{\frac{k_B T}{2 M \Omega_0^2}},
\]
\[
\Delta \zeta(\infty) = \sqrt{\frac{M k_B T}{2 \hbar^2 \rho^2}}.
\]

Notice that Eq. (62) is equivalent to the one in Ref. [9].

By using Eqs. (61) and (62), we plot the time evolution of the fluctuations in the relative phase and population imbalance in Fig. 3. They exhibit strong similarities in the behaviour of these two quantities, which are ruled by the damping parameter \( \gamma_Q \). One can see that curves tend to overlap as much as one lowers \( \gamma_Q \).

### B. Generic temperature

Let us focus on the generic temperature case to see effects of quantum fluctuations in addition to thermal ones. We already mentioned the fact that the noise correlators in the most general case are complex valued quantities. Then we use Eq. (58) for both \( \langle \xi(t) \xi(t') \rangle \) and \( \langle \chi(t) \chi(t') \rangle \) and we obtain
\[
\Delta \phi(\infty) = \frac{1}{\Omega_0} \sqrt{\frac{k_B T}{2 M \Omega_0^2}}.
\]
\[
\Delta \zeta(\infty) = \sqrt{\frac{M k_B T}{2 \hbar^2 \rho^2}}.
\]

Eq. (59) provides
\[
\Delta \phi(\infty)^2 = \frac{\gamma}{M \Omega_0} \int_{-\infty}^{\infty} d\omega \omega^3 \text{coth} \left( \frac{\beta \hbar \omega}{2} \right) 
\times \int dt_1 \chi(t_1) e^{-i \omega t_1} \int dt_2 \chi(t_2) e^{i \omega t_2}.
\]

where we changed the time variables \( t_1 = t - s \) and \( t_2 = t - s' \). Introducing a new adimensional frequency \( \tilde{\omega} = \omega/\Omega_0 \), we obtain an expression for the variance of the phase
\[
\Delta \phi(\infty)^2 = \Gamma \gamma Q \int_{-\infty}^{\infty} d\tilde{\omega} \tilde{\omega}^3 \text{coth} \left( \frac{\beta \hbar \tilde{\omega}}{2} \right) G_{\tilde{\omega}}(\tilde{\omega}) G_{-\tilde{\omega}}(\tilde{\omega}),
\]
where
\[
\Gamma = \frac{2\hbar}{M \Omega_0},
\]
and
\[
G_{\tilde{\omega}}(\tilde{\omega}) = \frac{e^{-(\tilde{\omega}+\gamma_Q)\Omega_0 t}}{(i \tilde{\omega} + \gamma_Q)^2 - \gamma_J^2} \left[ \gamma_J e^{(i \tilde{\omega} + \gamma_Q)(\Omega_0 t)} - \gamma_J \cos(\gamma_J \Omega_0 t) - (i \tilde{\omega} + \gamma_Q) \sin(\gamma_J \Omega_0 t) \right].
\]

What we have just seen can be identically translated into the calculation of Eq. (66) as
\[
\Delta \zeta(\infty)^2 = \eta \frac{\gamma Q}{\gamma_J} \int_{-\infty}^{\infty} d\tilde{\omega} \tilde{\omega} \text{coth} \left( \frac{\beta \hbar \tilde{\omega}}{2} \right) G_{\tilde{\omega}}(\tilde{\omega}) G_{-\tilde{\omega}}(\tilde{\omega}),
\]
where
\[
\eta = \frac{M \Omega_0}{\hbar \rho^2}.
\]
A damped harmonic oscillator in thermal equilibrium. Obtained in Ref. [21] for the quadratic fluctuations of $k_n$ for three values of the rescaled temperature $Q$. We have verified that the same happens with the asymptotic value $\Delta \zeta$ for $Q$ also strongly depends on the damping parameter $\gamma$. The variance $\Delta \zeta$ grows by increasing the temperature $Q$, leads to a lower asymptotic value. The variance $\Delta \zeta$ of the population imbalance of the Josephson mode in panel (b) of Fig. 4 also oscillates in time, reaching asymptotically a finite value. Here the asymptotic value also strongly depends on the damping parameter $\gamma Q$ and decreasing $\gamma Q$ leads to a larger asymptotic value. For the sake of completeness, in Fig. 5 we plot $\Delta \zeta(t)^2$ for three values of the rescaled temperature $k_B T/(\hbar \Omega_0)$ of the phonon bath. The figure shows that the asymptotic value $\Delta \zeta(\infty)^2$ grows by increasing the temperature. We have verified that the same happens with $\Delta \phi(\infty)^2$. These results are fully consistent with the ones obtained in Ref. [21] for the quadratic fluctuations of the damped harmonic oscillator in thermal equilibrium.

**FIG. 5.** Variance $\Delta \zeta(t)^2$ of the Josephson population imbalance as a function of time $t$ for three values of the temperature $T$ of the bath of phonons. The damping coefficient $\gamma_0$ is set to be $\gamma_0 = 1/10$. $\Omega_0$ is the Josephson frequency of Eq. (30) while $\eta$ is given by Eq. (72).

Both of Eqs. (68) and (71) are computed numerically, but it is worth noting that they have an ultraviolet divergence. For this reason it is necessary to set a cutoff that, taking into account the phonon dispersion relation we found previously, can be seen as $\omega_{max} = c_s \pi N/\Omega_0 L$. Using typical parameters like $N = 10^3$, $L = 10^{-6}$m, $\Omega_0 = 10^2$s$^{-1}$, $c_s = 10^{-3}$ms$^{-1}$ [8], we find $\omega_{max} = 10^4$.

In Fig. 4 we report the time evolution of the fluctuations for three values of $\gamma_0$ in the underdamped regime ($\gamma_0 < 1$). Panel (a) in Fig. 5 clearly shows that the variance $\Delta \phi(t)^2$, related to the relative phase of the Josephson mode, oscillates until it reaches an asymptotic value in the long-time limit, as shown in the inset. The huge initial oscillations are crucially dependent on the cutoff $\omega_{max}$, and the asymptotic value strongly depends on the choice of $\gamma_0$. It is worth noting that decreasing the value of $\gamma_0$ leads to a lower asymptotic value. The variance $\Delta \zeta(t)^2$ of the population imbalance of the Josephson mode in panel (b) of Fig. 4 also oscillates in time, reaching asymptotically a finite value. Here the asymptotic value also strongly depends on the damping parameter $\gamma Q$ and decreasing $\gamma Q$ leads to a larger asymptotic value.

For the sake of completeness, in Fig. 5 we plot $\Delta \zeta(t)^2$ for three values of the rescaled temperature $k_B T/(\hbar \Omega_0)$ of the phonon bath. The figure shows that the asymptotic value $\Delta \zeta(\infty)^2$ grows by increasing the temperature. We have verified that the same happens with $\Delta \phi(\infty)^2$. These results are fully consistent with the ones obtained in Ref. [21] for the quadratic fluctuations of the damped harmonic oscillator in thermal equilibrium.

**VIII. CONCLUSIONS**

We have found that in elongated Josephson junctions the damped dynamics of the Josephson mode is ruled by the damping constant $\gamma_0$, given by Eq. (27). Moreover, the Josephson dynamics is strictly dependent on the interaction between the Josephson mode and the quantum-thermal bath of phonons. We have studied the phase fluctuation $\Delta \phi_0(t)$ and the population-imbalance fluctuation $\Delta \zeta_0(t)$ of the Josephson mode. In our work, as well as in Ref. [9], both noise and damping have the same origin: the intrinsic coupling to the phonon bath. The damping of correlations is caused by the friction while nonzero variances are ascribed to the noise. In the high-temperature regime, where the thermal fluctuations dominate, we have derived analytic expressions for the Josephson fluctuations. The generic temperature case has also been taken into account. Here quantum fluctuations play an important role and we rely on numerical calculations to determine the time evolution of the Josephson fluctuations, which exhibit their thermalization to constant values after a transient characterized by oscillating dynamics.

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**Appendix A: Derivation of Eq. (26)**

In this section, we briefly explain how to obtain the solution of the bath coordinates in Eq. (26). Taking the derivative of the first two equations of motion in Eqs. (22) and (23) with respect to time, we reach two differential equations for the Josephson mode and the excited modes as

$$
\dot{Q}_n(t) + \omega_n^2 Q_n(t) = -\frac{J_0 \bar{\rho}}{ML} \sin \left( \frac{Q_0(t)}{L} \right), \quad (A1)
$$

$$
\dot{Q}_0(t) + \frac{J_0 \bar{\rho}}{ML} \sin \left( \frac{Q_0(t)}{L} \right) = \sum_{n=1}^{\infty} \dot{Q}_n(t). \quad (A2)
$$

In this way we can start by solving the first equation, finding $Q_n(t)$, and then we exploit what we found in order to study the second one. Eq. (A1) can be solved by taking the Laplace transformation on each member, and we end up with

$$
Q_n(t) = \cos(\omega_n t) Q_n(0) + \frac{\sin(\omega_n t)}{\omega_n} \dot{Q}_n(0) +
$$

$$
- \frac{J_0 \bar{\rho}}{ML \omega_n} \int_0^t dt' \sin [\omega_n(t - t')] \sin \left( \frac{Q_0(t')}{L} \right).
$$

(A3)
Integrating by parts the last term, we obtain Eq. [26].

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