COMPUTATIONAL EXPERIMENTS WITH ABS ALGORITHMS
FOR OVERDETERMINED LINEAR SYSTEMS

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1 Introduction

In this report, we present numerical experiments with two particular ABS algorithms:

1. The Huang or modified Huang algorithm,
2. The implicit QR algorithm,

These methods are used for solving the following problem:

(a) Finding the least-squares solution of an overdetermined linear system $Ax = b$, where $A \in \mathbb{R}^{m,n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $m \geq n$.

The considered algorithms belong to the scaled ABS class of algorithms for solving linear systems, that is defined by the following scheme:

(A) Let $x_1 \in \mathbb{R}^n$ be arbitrary and $H_1 \in \mathbb{R}^{n,n}$ be nonsingular arbitrary. Set $i = 1$.

(B) Compute the residual $r_i = Ax_i - b$. If $r_i = 0$, then stop, $x_i$ solves the problem. Otherwise compute $s_i = H_iA^Tv_i$, where $v_i \in \mathbb{R}^n$ is arbitrary, save that $v_1, \ldots, v_i$ are linearly independent. If $s_i \neq 0$, then go to (C). If $s_i = 0$ and $r_i^Tv_i = 0$, then set $x_{i+1} = x_i$, $H_{i+1} = H_i$ and go to (F). If $s_i = 0$ and $r_i^Tv_i \neq 0$, then stop, the system is incompatible.

(C) Compute the search vector $p_i$ by

$$p_i = H_i^Tz_i,$$

where $z_i \in \mathbb{R}^n$ is arbitrary, save that $z_i^TH_iA^Tv_i \neq 0$.

(D) Update the estimate of the solution by

$$x_{i+1} = x_i - \alpha_ip_i,$$

where the stepsize $\alpha_i$ is given by

$$\alpha_i = r_i^Tv_i/p_i^TA^Tv_i.$$

(E) Update the Abaffian matrix by

$$H_{i+1} = H_i - H_iA^Tvw_i^TH_i/w_i^TH_iA^Tv_i,$$

where $w_i \in \mathbb{R}^n$ is arbitrary, save that $w_i^TH_iA^Tv_i \neq 0$.

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(F) If \( i = m \), then stop, \( x_{i+1} \) solves the problem. Otherwise increment the index \( i \) by one and go to (B).

From (E), it follows by induction that \( H_{i+1}^T V_i = 0 \) and \( H^T_{i+1} K_i = 0 \), where \( V_i = [v_1, \ldots, v_i] \) and \( W_i = [w_1, \ldots, w_i] \). One can show that the implicit factorization \( V_i^T P_i = L_i \) holds, where \( P_i = [p_1, \ldots, p_i] \) and \( L_i \) is nonsingular lower triangular. Moreover the general solution of the scaled subsystem \( V_i^T A x = V_i^T b \) can be expressed in the form

\[
x = x_i + H_i^T q,
\]

where \( q \in \mathbb{R}^m \) is arbitrary (see 2 for the proof).

The basic ABS class is the subclass of the scaled ABS class obtained by taking \( v_i = e_i, e_i \) being the \( i \)-th unitary vector (\( i \)-th column of the unit matrix). In this case, residual \( r_i = A x_i - b \) need not be computed in (B), \( i \)-th element \( r_i^T e_j = a_i^T x_i - b_i \) suffices.

This report is organized as follows. In Section 2, a short description of individual algorithms is given. Section 3 contains some details concerning test matrices and numerical experiments and discusses the numerical results. The Appendix contains listing of all numerical results. For other numerical results on ABS methods see [1,11]. For a listing of the used ABS codes see [12].

2 The tested ABS algorithms for overdetermined linear systems

In this report, we deal with two basic algorithms, one belonging to the basic ABS class and one belonging to the scaled ABS class. To simplify description, we will assume that \( A \in \mathbb{R}^{m,n} \), \( m \leq n \), has full row rank so that \( s_i \neq 0 \) in (B).

The Huang algorithm is obtained by the parameter choices \( H_1 = I, v_i = e_i, z_i = a_i, w_i = a_i \). Therefore

\[
p_i = H_i a_i
\]

and

\[
H_{i+1} = H_i - p_i P_i^T / a_i^T p_i,
\]

From (2) and (3), it follows by induction that \( p_i \in Range(A_i) \) and

\[
H_{i+1} = I - P_i D_i^{-1} P_i^T.
\]

where \( A_i = [a_1, \ldots, a_i] \), \( P_i = [p_1, \ldots, p_i] \) and \( D_i = diag(a_i^T p_1, \ldots, a_i^T p_i) \). Moreover \( H_i \) is symmetric, positive semidefinite and idempotent (it is the orthogonal projection matrix into \( Null(A_{i-1}) \)). Since the requirement \( p_i \in Null(A_{i-1}) \) is crucial, we can improve orthogonality by iterative refinement \( p_i = H_i^T a_i, j > 1 \) (usually \( j = 2 \)), obtaining the modified Huang algorithm.

The Huang algorithm can be used for finding the minimum-norm solution to the compatible underdetermined system \( A x = b \), i.e. for minimizing \( \| x \| \) s.t. \( A x = b \). To see this, we use the Lagrangian function and convexity of \( \| x \| \). Then \( x \) is a required solution if and only if \( x = A^T u \) for some \( u \in \mathbb{R}^m \) or \( x \in Range(A^T) \). But

\[
x_{m+1} = x_1 - \sum_{i=1}^{m} \alpha_i p_i
\]

by (D) and \( p_i \in Range(A_i) \subset Range(A^T) \), so that if \( x_1 = 0 \), then \( x_{m+1} \in Range(A^T) \).

A short description of two versions of the Huang and modified Huang algorithms follows.
Algorithm 1
(Huang and modified Huang, formula (3)).
Set \( x_1 = 0 \) and \( H_1 = I \).
For \( i = 1 \) to \( m \) do
Set \( p_i = H_i a_i \) (Huang) or
\( p_i = H_i (H_i a_i) \) (modified Huang),
\( d_i = a_i^T p_i \) and \( x_{i+1} = x_i - ((a_i^T x_i - b_i)/d_i)p_i \).
If \( i < m \), then set \( H_{i+1} = H_i - p_i p_i^T / d_i \).
end do

Algorithm 2
(Huang and modified Huang, formula (4)).
Set \( x_1 = 0 \) and \( P_0 \) empty.
For \( i = 1 \) to \( m \) do
Set \( p_i = (I - P_{i-1} D_{i-1}^{-1} P_{i-1}^T) a_i \) (Huang) or
\( p_i = (I - P_{i-1} D_{i-1}^{-1} P_{i-1}^T)(I - P_{i-1} D_{i-1}^{-1} P_{i-1}^T) a_i \) (modified Huang),
\( d_i = a_i^T p_i \) and \( x_{i+1} = x_i - ((a_i^T x_i - b_i)/d_i)p_i \).
If \( i < m \), then set \( P_i = [P_{i-1}, p_i] \).
end do

The implicit QR algorithm is obtained by the parameter choices \( H_1 = I, v_i = A p_i, z_i = e_i, \)
\( w_i = e_i \). Since \( z_i = e_i \) and \( w_i = e_i \), the matrices \( H_i \) and \( P_i \) have the same structure as that in
the implicit LU algorithm. Using the implicit factorization property \( V_i^T A P_i = L_i \), we can see
that \( V_{i-1}^T v_i = V_{i-1}^T A p_i = 0 \) so that the vectors \( v_j, j = 1, \ldots, i \), are mutually orthogonal.

The implicit QR algorithm can be used for finding the least-squares solution of an overde-
termined linear system \( Ax = b \), where \( A \in \mathbb{R}^{m,n}, b \in \mathbb{R}^m, x \in \mathbb{R}^n \) and \( m \geq n \), which is obtained
after at most \( n \) steps. To see this, we recall that algorithms from the scaled ABS class solve
the scaled system \( V_n^T Ax = V_n^T b \). Since \( V_n = A P_n \), where \( P_n \) is square nonsingular, the condi-
tion \( P_n^T A^T Ax = P_n^T A^T b \) implies \( A^T Ax = A^T b \), which defines the least-squares solution of the
overdetermined system \( Ax = b \).

A short description of the implicit QR algorithm follows.

Algorithm 3
(Implicit QR).
Set \( x_1 = 0, r_1 = -b \) and \( H_1 = I \).
For \( i = 1 \) to \( m \) do
Set \( p_i = H_i^T e_i, v_i = A p_i \)
(only \( i(i-1)(n-i+1) \) nonzero elements of \( H_i \) is used),
\( \alpha_i = r_i^T v_i / v_i^T v_i, x_{i+1} = x_i - \alpha_i p_i \) and \( r_{i+1} = r_i - \alpha_i v_i \).
(only \( i \) nonzero elements of \( x_i \) are updated).
If \( i < m \), then set \( s_i = H_i^T A^T v_i \) and \( H_{i+1} = H_i - s_i p_i^T / v_i^T v_i \)
(only \( i(n-i) \) nonzero elements of \( H_{i+1} \) are updated).
end do

We now consider the problem of solving overdetermined linear systems, in the least squares
sense. Consider the linear system \( Ax = b \), where \( A \in \mathbb{R}^{m,n}, b \in \mathbb{R}^m, x \in \mathbb{R}^n \) and \( m \geq n \). This
system can be solved in the least squares sense in \( n \) steps by the implicit QR algorithm as was
shown above. Another possibility is based on the application of the Huang algorithm. Since the least-squares solution has to satisfy the normal equation
\[ A^T Ax = A^T b, \]
it can be obtained from the solution of the following augmented system.
\[ Ax = y, \quad (5) \]
\[ A^T y = A^T b. \quad (6) \]
Since \( y \in \text{Range}(A) \) by (3), it is a minimum-norm solution of (3) and it can be obtained by the Huang algorithm. Having \( y \), the compatible system (2) can be solved by any ABS algorithm. Moreover, using the implicit factorization property \( A^T P_n = L_n \), we can write
\[ L_n^T x = P_n^T Ax = P_n^T y = \tilde{b} \]
which as a system with a lower triangular matrix can be easily solved by the back substitution. The lower triangular matrix \( L_n \) can be obtained as a by-product of the Huang algorithm. If we use formula (3), then \( d_i \) is \( i \)-th diagonal element of \( L_n \) and \( P_i^T \tilde{a}_i \) contains the other \( i \)-th-column elements of \( L_n \) (\( \tilde{a}_i \) is \( i \)-th column of the matrix \( A \)).

Matrix \( L_n \) has not to be stored since it can be reconstructed from columns of the matrix \( P_n \). However, the back substitution has to be realized in a slightly different way in this case.

A short description of two versions of the Huang algorithm for least-squares solution of overdetermined systems follows.

**Algorithm 4**
(Huang and modified Huang for least-squares with stored \( L_n \)).

Set \( x_1 = 0 \) and \( P_0 \) empty.

**For** \( i = 1 \) **to** \( n \) **do**

If \( i > 1 \), then compute \( g_i = P_i^T \tilde{a}_i \) and copy it into the lower triangular matrix \( L_n \).

Set \( p_i = \tilde{a}_i - P_{i-1} D_{i-1}^{-1} g_i \) (Huang) or
\[ p_i = (I - P_{i-1} D_{i-1}^{-1} P_i^T) (\tilde{a}_i - P_{i-1} D_{i-1}^{-1} g_i) \] (modified Huang).

Set \( d_i = \tilde{a}_i^T p_i \) and copy it into the lower triangular matrix \( L_n \). Set \( \tilde{b}_i = b^T p_i \).

If \( i < n \), then set \( P_i = [P_{i-1}, p_i] \).

**end do**

Solve the triangular system \( L_n^T x = \tilde{b} \).

**Algorithm 5**
(Huang and modified Huang for least-squares without stored \( L_n \)).

Set \( x_1 = 0 \) and \( P_0 \) empty.

**For** \( i = 1 \) **to** \( n \) **do**

Set \( p_i = (I - P_{i-1} D_{i-1}^{-1} P_i^T) \tilde{a}_i \) (Huang) or
\[ p_i = (I - P_{i-1} D_{i-1}^{-1} P_i^T) (I - P_{i-1} D_{i-1}^{-1} P_i^T) \tilde{a}_i \] (modified Huang).

Set \( d_i = \tilde{a}_i^T p_i \).

If \( i < n \), then set \( P_i = [P_{i-1}, p_i] \).

**end do**

Set \( f_n = b \).

**For** \( i = n \) **to** \( 1 \) **do**

Set \( x_i = p_i^T f_i / d_i \).

If \( i > 1 \), then set \( f_{i-1} = f_i - x_i \tilde{a}_i \)

**end do**
3 The computational experiments

Performance of ABS algorithms has been tested by using several types of ill-conditioned matrices. These matrices can be classified in the following way. The first letter distinguishes matrices with integer 'I' and real 'R' elements, both actually stored as reals in double precision arithmetic. The second letter denotes randomly generated matrices 'R' or matrices determined by an explicit formula 'D'. For randomly generated matrices, a number specifying the interval for the random number generator follows, while the name of matrices determined by the explicit formula contains the formula number (F1, F2, F3). The last letter of the name denotes a way for obtaining ill-conditioned matrices: 'R' - matrices with nearly dependent rows, 'C' - matrices with nearly dependent columns, 'S' - nearly singular symmetric matrices, 'B' - both matrices in KKT system ill-conditioned. More specifically:

IR500 Randomly generated matrices with integer elements uniformly distributed in the interval \([-500,500]\).
IR500R Randomly generated matrices with integer elements uniformly distributed in the interval \([-500,500]\) perturbed in addition to have two rows nearly dependent.
IR500C Randomly generated matrices with integer elements uniformly distributed in the interval \([-500,500]\) perturbed in addition to have two columns nearly dependent.
RR100 Randomly generated matrices with real elements uniformly distributed in the interval \([-100,100]\).
IDF1 Matrices with elements \(a_{ij} = |i - j|, 1 \leq i \leq m, 1 \leq j \leq n\) (Micchelli-Fiedler matrix).
IDF2 Matrices with elements \(a_{ij} = |i - j|^2, 1 \leq i \leq m, 1 \leq j \leq n\).
IDF3 Matrices with elements \(a_{ij} = |i + j - (m + n)/2|, 1 \leq i \leq m, 1 \leq j \leq n\).
IR50 Randomly generated matrices with integer elements uniformly distributed in the interval \([-50,50]\).

Matrices with linearly dependent rows were obtained in the following way. The input data contain four integers which specify two row indices \(i_1, i_2\), one column index \(i_3\) and one exponent \(i_4\). Then the row \(a_{i_1}\) is copied into \(a_{i_2}\). Furthermore \(a_{i_1i_3}\) is set to zero and \(a_{i_2i_3}\) to \(2^{-i_4}\). Similar procedures are used for columns and symmetric matrices.

Solution vectors were generated randomly with integer or real elements uniformly distributed in the interval \([-10,10]\). Right hand sides for overdetermined systems were obtained by the following way. First, vector \(\tilde{b}\) was generated randomly with integer or real elements uniformly distributed in the interval \([-10,10]\). Then its first element together with elements in the first row of the matrix \(A\) were redefined by the formulas \(\tilde{b}_1 = -1\) and \(a_{ij} = \sum_{i=2}^{n} a_{ij}\tilde{b}_j\) so that \(A^T\tilde{b} = 0\). The right-hand side vector was determined by the formula \(b = \tilde{b} + Ax^*\). Since \(A^T Ax^* = A^T(b - \tilde{b}) = A^T\tilde{b}\), the normal equation is satisfied and \(x^*\) is a least squares solution of the system \(Ax = b\). Vector \(\tilde{b}\) was generated nonzero, so that the system \(Ax = b\) is incompatible.

We have tested the following particular algorithms:

impl.qr5 The implicit QR algorithm: Algorithm 5 (subroutine alg5d.f).
huang6 The Huang algorithm for overdetermined systems: Algorithm 6 (subroutine alg6d.f).
mod.huang6 The modified Huang algorithm for overdetermined systems: Algorithm 6 (subroutine alg6d.f).
huang7 The Huang algorithm for overdetermined systems: Algorithm 7 (subroutine alg7d.f).
mod.huang7 The modified Huang algorithm for overdetermined systems: Algorithm 7 (subroutine alg7d.f).
qr lapack Subroutine DGELS from the LAPACK package.
svd lapack  Subroutine DGELSS from the LAPACK package.
gqr lapack  Subroutine DGELSX from the LAPACK package.

Notice that ABS algorithms were implemented in their basic form without partitioning into block or other special adjustments serving for speed increase as done in the LAPACK software.

Detailed results of computational experiments are presented in the Appendix. For each selected problem, the type of matrix and the dimension is given. Furthermore, both the solution and the residual errors together with the detected rank and the computational time are given for each tested algorithm. Computational experiments were performed on a Digital Unix Workstation in the double precision arithmetic (machine epsilon equal to about $10^{-16}$).

We have tested overdetermined systems with $n >> m/2$, $n = m/2$ and $n << m/2$. These systems were solved by using Algorithms 5 - Algorithm 7 together with explicit QR and SVD decomposition based methods taken from the LAPACK package.

The following tables give synthetic results for the 21 tested problems, the number at the intersection of the $i$-th row with the $k$-th column indicating how many times the algorithm at the heading of the $i$-th row gave a lower error than the algorithm at the heading of the $k$-th row (in case there is a second number, this counts the number of cases when difference was less than one percent).

| solution error - 21 overdetermined linear systems |
|--------------------------------------------------|
| methods                                           | huang mod. huang mod. impl. qr svd gqr |
| "       | huang 7 huang qr5 lap lap lap |
| "       | 6 7 6 7 |
| huang6  | 4 0/21 4 5 9 2 2 26/21 |
| mod.huang6 | 17 17 1/12 13/6 16 11 8 83/18 |
| huang7  | 0/21 4 4 5 9 2 2 26/21 |
| mod.huang7 | 17 8/12 17 14/6 18 11 10 95/18 |
| impl.qr5 | 16 2/6 16 1/6 10 8 6 59/12 |
| qr lapack | 12 5 12 3 11 6/4 9 58/4 |
| svd lapack | 19 10 19 10 13 11/4 4/12 86/16 |
| gqr lapack | 19 13 19 11 15 12 5/12 94/12 |
residual error - 21 overdetermined linear systems

| methods    | huang | mod. | huang | mod. | impl. | qr  | svd | gqr |
|------------|-------|------|-------|------|-------|-----|-----|-----|
|            | 6 huang | 7 huang | qr5   | lap  | lap  | lap | lap | lap | total |
| huang6     | 11     | 4    | 7     | 17   | 16   | 9   | 13  |     | 77    |
| mod.huang6 | 10     | 6/2  | 9     | 15/1 | 13   | 10  | 11/1|     | 74/4  |
| huang7     | 17     | 13/2 | 9     | 16   | 16   | 12  | 12  |     | 95/2  |
| mod.huang7 | 14     | 12   | 12    | 17   | 14/1 | 13  | 13  |     | 95/1  |
| impl.qr5   | 4      | 5/1  | 5     | 4    | 8    | 7   | 7   |     | 40/1  |
| qr lapack  | 5      | 8    | 5     | 6/1  | 13   | 9   | 9   |     | 55/1  |
| svd lapack | 12     | 11   | 9     | 8    | 14   | 12  | 13  |     | 79    |
| gqr lapack | 8      | 9/1  | 9     | 8    | 14   | 12  | 8   |     | 68/1  |

From the results in the Appendix and the above tables we can state the following conclusions:

(1) When solving well-conditioned overdetermined systems, the explicit QR algorithms based on the Householder orthogonalization are usually faster than ABS methods tested (especially if $n >> m/2$). Therefore, we cannot recommend the later for solving standard problems.

(2) Interesting results were obtained for extremely ill-conditioned problems. The modified Huang and the implicit QR algorithms failed to solve systems with the matrix IDF2. Such systems was not solved by simple explicit QR methods as well. On the other hand, the modified Huang and the implicit QR algorithms found solutions of systems with the matrix IDF3 extremely fast and, moreover, they were able to determine the numerical rank correctly. Performance of these algorithms in that particular case is remarkable, they are at least 20 times faster then the best LAPACK routines.

(3) The LAPACK routine DGELSS based on the SVD decomposition is extremely slow (especially if $n >> m/2$), not suitable for solving our problems (it can be substituted by the much faster routine DGELSX).

(4) In term of solution error mod-huang7 and gqr are the best; in term of residual error huang7 and mod.huang7 are the best.

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## 4 Appendix: Test results on overdetermined linear systems

| matrix | dimension | method     | solution error | residual error | rank | time |
|--------|-----------|------------|----------------|----------------|------|------|
| IR500  | 1050 950  | huang6     | 0.17D-01       | 0.23D-14       | 950  | 32.00|
| IR500  | 1050 950  | mod.huang6 | 0.78D-10       | 0.20D-15       | 950  | 63.00|
| IR500  | 1050 950  | huang7     | 0.17D-01       | 0.16D-14       | 950  | 30.00|
| IR500  | 1050 950  | mod.huang7 | 0.79D-10       | 0.85D-15       | 950  | 60.00|
| IR500  | 1050 950  | impl.qr5   | 0.51D-09       | 0.12D-13       | 950  | 57.00|
| IR500  | 1050 950  | qr lapack  | 0.84D-09       | 0.83D-14       | 950  | 16.00|
| IR500  | 1050 950  | svd lapack | 0.31D-08       | 0.80D-14       | 950  | 119.00|
| IR500  | 1050 950  | gqr lapack | 0.28D-08       | 0.28D-14       | 950  | 26.00|

**condition number:** $0.42D+09$

| IR500  | 1400 700  | huang6     | 0.13D-02       | 0.51D-14       | 700  | 23.00|
| IR500  | 1400 700  | mod.huang6 | 0.41D-11       | 0.12D-13       | 700  | 49.00|
| IR500  | 1400 700  | huang7     | 0.13D-02       | 0.82D-14       | 700  | 24.00|
| IR500  | 1400 700  | mod.huang7 | 0.39D-11       | 0.15D-14       | 700  | 47.00|
| IR500  | 1400 700  | impl.qr5   | 0.33D-10       | 0.31D-13       | 700  | 40.00|
| IR500  | 1400 700  | qr lapack  | 0.54D-10       | 0.56D-14       | 700  | 15.00|
| IR500  | 1400 700  | svd lapack | 0.54D-10       | 0.18D-13       | 700  | 67.00|
| IR500  | 1400 700  | gqr lapack | 0.23D-09       | 0.77D-14       | 700  | 23.00|

**condition number:** $0.65D+08$
### Test results for overdetermined linear systems - continued

| matrix | dimension | method    | solution error | residual error | rank | time |
|--------|-----------|-----------|----------------|----------------|------|------|
| IR500  | 2000 400  | huang6    | 0.35D-03       | 0.95D-15       | 400  | 12.00|
| IR500  | 2000 400  | mod.huang6 | 0.96D-12       | 0.78D-15       | 400  | 21.00|
| IR500  | 2000 400  | huang7    | 0.35D-03       | 0.41D-15       | 400  | 11.00|
| IR500  | 2000 400  | mod.huang7 | 0.87D-12       | 0.33D-15       | 400  | 21.00|
| IR500  | 2000 400  | impl.qr5  | 0.55D-11       | 0.11D-13       | 400  | 18.00|
| IR500  | 2000 400  | qr lapack | 0.18D-10       | 0.46D-13       | 400  | 7.00 |
| IR500  | 2000 400  | svd lapack | 0.18D-10       | 0.54D-13       | 400  | 20.00|
| IR500  | 2000 400  | gqr lapack | 0.65D-10       | 0.10D-12       | 400  | 12.00|
|        |           |           |                |                |      |      |
| IR500C | 1050 950  | huang6    | 0.23D-01       | 0.15D-14       | 950  | 33.00|
| IR500C | 1050 950  | mod.huang6 | 0.11D-01       | 0.19D-14       | 950  | 63.00|
| IR500C | 1050 950  | huang7    | 0.23D-01       | 0.36D-16       | 950  | 30.00|
| IR500C | 1050 950  | mod.huang7 | 0.11D-01       | 0.11D-14       | 950  | 60.00|
| IR500C | 1050 950  | impl.qr5  | 0.12D-01       | 0.40D-12       | 950  | 57.00|
| IR500C | 1050 950  | qr lapack | 0.45D-02       | 0.36D-14       | 950  | 17.00|
| IR500C | 1050 950  | svd lapack | 0.69D+00       | 0.41D-14       | 949  | 114.00|
| IR500C | 1050 950  | gqr lapack | 0.69D+00       | 0.19D-14       | 949  | 26.00|
|        |           |           |                |                |      |      |
| IR500C | 1400 700  | huang6    | 0.27D+00       | 0.11D-14       | 700  | 24.00|
| IR500C | 1400 700  | mod.huang6 | 0.21D-05       | 0.34D-15       | 700  | 49.00|
| IR500C | 1400 700  | huang7    | 0.27D+00       | 0.62D-15       | 700  | 24.00|
| IR500C | 1400 700  | mod.huang7 | 0.21D-05       | 0.23D-15       | 700  | 47.00|
| IR500C | 1400 700  | impl.qr5  | 0.12D+01       | 0.65D-14       | 700  | 29.00|
| IR500C | 1400 700  | qr lapack | 0.35D-06       | 0.11D-15       | 700  | 15.00|
| IR500C | 1400 700  | svd lapack | 0.35D-06       | 0.80D-15       | 700  | 66.00|
| IR500C | 1400 700  | gqr lapack | 0.11D-05       | 0.32D-14       | 700  | 22.00|
|        |           |           |                |                |      |      |
| IR500C | 2000 400  | huang6    | 0.11D-01       | 0.31D-14       | 400  | 11.00|
| IR500C | 2000 400  | mod.huang6 | 0.33D-03       | 0.18D-14       | 400  | 21.00|
| IR500C | 2000 400  | huang7    | 0.11D-01       | 0.50D-15       | 400  | 11.00|
| IR500C | 2000 400  | mod.huang7 | 0.33D-03       | 0.37D-15       | 400  | 21.00|
| IR500C | 2000 400  | impl.qr5  | 0.80D-03       | 0.23D-13       | 400  | 18.00|
| IR500C | 2000 400  | qr lapack | 0.10D-02       | 0.37D-15       | 400  | 7.00 |
| IR500C | 2000 400  | svd lapack | 0.70D+00       | 0.55D-15       | 399  | 19.00|
| IR500C | 2000 400  | gqr lapack | 0.70D+00       | 0.19D-13       | 399  | 11.00|

condition number:

- IR500: $0.24D+08$
- IR500C: $0.52D+13$
- IR500C 1050 950: $0.25D+14$
- IR500C 1400 700: $0.51D+12$
- IR500C 2000 400: $0.52D+13$

condition number:

- IR500C 1400 700: $5.1D+12$
Test results for overdetermined linear systems - continued

| matrix dimension | method     | solution error | residual error | rank | time |
|------------------|------------|----------------|----------------|------|------|
| RR100 1050 950   | huang6     | 0.59D-10       | 0.21D-14       | 950  | 32.00|
| RR100 1050 950   | mod.huang6| 0.96D-14       | 0.25D-14       | 950  | 63.00|
| RR100 1050 950   | huang7     | 0.59D-10       | 0.93D-15       | 950  | 30.00|
| RR100 1050 950   | mod.huang7| 0.85D-14       | 0.58D-15       | 950  | 61.00|
| RR100 1050 950   | impl.qr5  | 0.82D-14       | 0.31D-14       | 950  | 57.00|
| RR100 1050 950   | qr lapack  | 0.79D-14       | 0.19D-15       | 950  | 16.00|
| RR100 1050 950   | svd lapack | 0.20D-13       | 0.30D-14       | 950  | 158.00|
| RR100 1050 950   | gqr lapack | 0.80D-14       | 0.26D-14       | 950  | 26.00|
| condition number: |           |                |                |      | 0.35D+04|
| RR100 1400 700   | huang6     | 0.25D-11       | 0.47D-12       | 700  | 24.00|
| RR100 1400 700   | mod.huang6| 0.44D-14       | 0.99D-12       | 700  | 48.00|
| RR100 1400 700   | huang7     | 0.25D-11       | 0.99D-12       | 700  | 24.00|
| RR100 1400 700   | mod.huang7| 0.24D-14       | 0.85D-13       | 700  | 47.00|
| RR100 1400 700   | impl.qr5  | 0.35D-14       | 0.54D-12       | 700  | 40.00|
| RR100 1400 700   | qr lapack  | 0.28D-14       | 0.49D-12       | 700  | 15.00|
| RR100 1400 700   | svd lapack | 0.86D-14       | 0.91D-12       | 700  | 84.00|
| RR100 1400 700   | gqr lapack | 0.36D-14       | 0.16D-11       | 700  | 22.00|
| condition number: |           |                |                |      | 0.49D+03|
| RR100 2000 400   | huang6     | 0.18D-11       | 0.40D-14       | 400  | 11.00|
| RR100 2000 400   | mod.huang6| 0.55D-14       | 0.49D-15       | 400  | 22.00|
| RR100 2000 400   | huang7     | 0.18D-11       | 0.41D-14       | 400  | 11.00|
| RR100 2000 400   | mod.huang7| 0.21D-14       | 0.12D-14       | 400  | 21.00|
| RR100 2000 400   | impl.qr5  | 0.60D-14       | 0.33D-15       | 400  | 18.00|
| RR100 2000 400   | qr lapack  | 0.30D-14       | 0.59D-14       | 400  | 8.00 |
| RR100 2000 400   | svd lapack | 0.84D-14       | 0.74D-14       | 400  | 22.00|
| RR100 2000 400   | gqr lapack | 0.27D-14       | 0.15D-13       | 400  | 11.00|
| condition number: |           |                |                |      | 0.26D+03|
| IDF1 1050 950    | huang6     | 0.17D-03       | 0.25D-14       | 950  | 32.00|
| IDF1 1050 950    | mod.huang6| 0.44D-10       | 0.23D-16       | 950  | 63.00|
| IDF1 1050 950    | huang7     | 0.17D-03       | 0.98D-15       | 950  | 30.00|
| IDF1 1050 950    | mod.huang7| 0.24D-10       | 0.23D-17       | 950  | 60.00|
| IDF1 1050 950    | impl.qr5  | 0.32D-07       | 0.42D-12       | 950  | 58.00|
| IDF1 1050 950    | qr lapack  | 0.23D-09       | 0.14D-14       | 950  | 17.00|
| IDF1 1050 950    | svd lapack | 0.22D-09       | 0.16D-14       | 950  | 128.00|
| IDF1 1050 950    | gqr lapack | 0.18D-09       | 0.52D-14       | 950  | 29.00|
| condition number: |           |                |                |      | 0.15D+08|
Test results for overdetermined linear systems - continued

| matrix dimension | method      | solution error | residual error | rank | time |
|------------------|-------------|----------------|----------------|------|------|
| m n              | error       | error          |                |      |      |

| IDF1 1400 700 | huang6      | 0.53D-03       | 0.21D-14       | 700  | 23.00|
| IDF1 1400 700 | mod.huang6  | 0.61D-10       | 0.11D-14       | 700  | 47.00|
| IDF1 1400 700 | huang7      | 0.53D-03       | 0.11D-14       | 700  | 24.00|
| IDF1 1400 700 | mod.huang7  | 0.23D-10       | 0.45D-15       | 700  | 47.00|
| IDF1 1400 700 | impl.qr5    | 0.28D-07       | 0.30D-12       | 700  | 43.00|
| IDF1 1400 700 | qr lapack    | 0.24D-09       | 0.23D-14       | 700  | 15.00|
| IDF1 1400 700 | svd lapack   | 0.23D-09       | 0.89D-15       | 700  | 63.00|
| IDF1 1400 700 | gqr lapack   | 0.23D-09       | 0.33D-14       | 700  | 23.00|

Condition number: 0.18D+08

| IDF1 2000 400 | huang6      | 0.12D-02       | 0.17D-14       | 400  | 11.00|
| IDF1 2000 400 | mod.huang6  | 0.14D-09       | 0.90D-15       | 400  | 22.00|
| IDF1 2000 400 | huang7      | 0.12D-02       | 0.21D-15       | 400  | 11.00|
| IDF1 2000 400 | mod.huang7  | 0.31D-10       | 0.98D-15       | 400  | 21.00|
| IDF1 2000 400 | impl.qr5    | 0.20D-07       | 0.18D-12       | 400  | 17.00|
| IDF1 2000 400 | qr lapack    | 0.36D-09       | 0.97D-15       | 400  | 8.00 |
| IDF1 2000 400 | svd lapack   | 0.36D-09       | 0.87D-15       | 400  | 17.00|
| IDF1 2000 400 | gqr lapack   | 0.52D-09       | 0.82D-16       | 400  | 12.00|

Condition number: 0.33D+08

| IDF2 1050 950 | huang6      | 0.65D+03       | 0.18D-12       | 950  | 29.00|
| IDF2 1050 950 | mod.huang6  | 0.83D+10       | 0.86D-05       | 4    | 1.00 |
| IDF2 1050 950 | huang7      | 0.65D+03       | 0.76D-13       | 950  | 30.00|
| IDF2 1050 950 | mod.huang7  | 0.83D+10       | 0.17D-05       | 4    | 0.00 |
| IDF2 1050 950 | impl.qr5    | --- break-down --- |            |      |      |
| IDF2 1050 950 | qr lapack    | 0.11D+14       | 0.11D-05       | 950  | 16.00|
| IDF2 1050 950 | svd lapack   | 0.10D+01       | 0.36D-15       | 3    | 138.00|
| IDF2 1050 950 | gqr lapack   | 0.10D+01       | 0.31D-15       | 3    | 25.00|

Condition number: 0.11D+21

| IDF2 1400 700 | huang6      | 0.18D+04       | 0.97D-12       | 700  | 20.00|
| IDF2 1400 700 | mod.huang   | 0.23D+11       | 0.50D-04       | 4    | 1.00 |
| IDF2 1400 700 | huang7      | 0.18D+04       | 0.38D-12       | 700  | 20.00|
| IDF2 1400 700 | mod.huang7  | 0.23D+11       | 0.35D-04       | 4    | 0.00 |
| IDF2 1050 700 | impl.qr5    | --- break-down --- |            |      |      |
| IDF2 1400 700 | qr lapack    | 0.32D+13       | 0.41D-06       | 700  | 14.00|
| IDF2 1400 700 | svd lapack   | 0.10D+01       | 0.29D-15       | 3    | 74.00|
| IDF2 1400 700 | gqr lapack   | 0.10D+01       | 0.68D-15       | 3    | 21.00|

Condition number: 0.45D+20
Test results for overdetermined linear systems - continued

| m  | n  | method         | solution error | residual error | rank | time |
|----|----|----------------|----------------|----------------|------|------|
|    |    | m n            | m n            | m n            |      |      |
|    |    |                |                |                |      |      |
| m  | n  | method         | solution error | residual error | rank | time |
|----|----|----------------|----------------|----------------|------|------|
| IDF2 2000 400 huang6 0.10D+04 0.51D-12 400 9.00 | IDF2 2000 400 mod.huang6 0.76D+10 0.33D-05 4 0.00 | IDF2 2000 400 huang7 0.10D+04 0.50D-12 400 9.00 | IDF2 2000 400 mod.huang7 0.76D+10 0.45D-05 4 0.00 | IDF2 1050 700 impl.qr5 --- break-down --- | IDF2 2000 400 qr lapack 0.91D+12 0.22D-07 400 7.00 | IDF2 2000 400 svd lapack 0.10D+01 0.68D-15 3 18.00 | IDF2 2000 400 gqr lapack 0.10D+01 0.29D-14 3 11.00 | condition number: 0.17D+20 | condition number: 0.16D+21 | condition number: 0.27D+20 | condition number: 0.63D+19 |
Test results for overdetermined linear systems - continued

| matrix | dimension | method  | solution error | residual error | rank | time |
|--------|-----------|---------|----------------|----------------|------|------|
| IR50   | 1050 950  | huang6  | 0.10D+02       | 0.34D-15       | 950  | 33.00|
| IR50   | 1050 950  | mod.huang6 | 0.92D+01       | 0.35D-13       | 773  | 61.00|
| IR50   | 1050 950  | huang7  | 0.10D+02       | 0.66D-14       | 950  | 32.00|
| IR50   | 1050 950  | mod.huang7 | 0.92D+01       | 0.48D-13       | 773  | 58.00|
| IR50   | 1050 950  | impl.qr5 | 0.92D+01       | 0.68D-13       | 773  | 52.00|
| IR50   | 1050 950  | qr lapack | 0.44D+12       | 0.16D-01       | 950  | 17.00|
| IR50   | 1050 950  | svd lapack | 0.41D+00       | 0.13D-13       | 773  | 96.00|
| IR50   | 1050 950  | gqr lapack | 0.41D+00       | 0.17D-13       | 773  | 28.00|

condition number: 0.65D+21

| IR50   | 1400 700  | huang6  | 0.22D+02       | 0.29D-13       | 700  | 24.00|
| IR50   | 1400 700  | mod.huang6 | 0.64D+01       | 0.12D-12       | 618  | 48.00|
| IR50   | 1400 700  | huang7  | 0.22D+02       | 0.26D-13       | 700  | 24.00|
| IR50   | 1400 700  | mod.huang7 | 0.64D+01       | 0.18D-13       | 618  | 48.00|
| IR50   | 1400 700  | impl.qr5 | 0.64D+01       | 0.57D-13       | 618  | 37.00|
| IR50   | 1400 700  | qr lapack | 0.34D+12       | 0.87D-01       | 700  | 15.00|
| IR50   | 1400 700  | svd lapack | 0.36D+00       | 0.62D-13       | 618  | 50.00|
| IR50   | 1400 700  | gqr lapack | 0.36D+00       | 0.11D-12       | 618  | 22.00|

condition number: 0.43D+21

| IR50   | 2000 400  | huang6  | 0.41D+04       | 0.30D-11       | 400  | 10.00|
| IR50   | 2000 400  | mod.huang6 | 0.45D+00       | 0.12D-14       | 374  | 20.00|
| IR50   | 2000 400  | huang7  | 0.41D+04       | 0.71D-12       | 400  | 11.00|
| IR50   | 2000 400  | mod.huang7 | 0.45D+00       | 0.24D-14       | 374  | 21.00|
| IR50   | 2000 400  | impl.qr5 | 0.45D+00       | 0.43D-14       | 374  | 17.00|
| IR50   | 2000 400  | qr lapack | 0.80D+12       | 0.19D+00       | 400  | 7.00 |
| IR50   | 2000 400  | svd lapack | 0.33D+00       | 0.14D-13       | 374  | 18.00|
| IR50   | 2000 400  | gqr lapack | 0.33D+00       | 0.11D-12       | 374  | 11.00|

condition number: 0.11D+21