CONGRUENCES FOR APÉRY NUMBERS \( \beta_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \)

HUI-QIN CAO, YURI MATIYASEVICH, AND ZHI-WEI SUN

Abstract. In this paper we establish some congruences involving the Apéry numbers \( \beta_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \). For example, we show that
\[
\sum_{k=0}^{n-1} (11k^2 + 13k + 4) \beta_k \equiv 0 \pmod{2n^2}
\]
for any positive integer \( n \), and
\[
\sum_{k=0}^{p-1} (11k^2 + 13k + 4) \beta_k \equiv 4p^2 + 4p^7 B_{p-5} \pmod{p^8}
\]
for any prime \( p > 3 \), where \( B_{p-5} \) is the \((p-5)\)th Bernoulli number. We also present certain relations between congruence properties of the two kinds of Apéry numbers, \( \beta_n \) and \( A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \).

1. Introduction

In 1979 R. Apéry (see [1] and [19]) established the irrationality of \( \zeta(3) = \sum_{n=1}^{\infty} 1/n^3 \) by using the Apéry numbers
\[
A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad (n \in \mathbb{N} = \{0, 1, 2, \ldots\}).
\]
His method also allowed him to prove the irrationality of \( \zeta(2) = \pi^2/6 \) via another kind of Apéry numbers
\[
\beta_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \quad (n \in \mathbb{N}).
\]

In 2012, the third author [15] proved that
\[
\sum_{k=0}^{n-1} (2k+1) A_k \equiv 0 \pmod{n} \quad \text{for all } n \in \mathbb{Z}_+ = \{1, 2, 3, \ldots\}
\]

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and
\[\sum_{k=0}^{p-1} (2k + 1)A_k \equiv p + \frac{7}{6}p^4B_{p-3} \pmod{p^5} \quad \text{for any prime } p > 3,\]

where \(B_0, B_1, B_2, \ldots\) are the Bernoulli numbers defined by
\[B_0 = 1, \quad \text{and } \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad \text{for all } n \in \mathbb{Z}^+.\]

(For the basic properties of Bernoulli numbers, see [8, pp. 228-248].) The third author also conjectured that
\[\sum_{k=0}^{n-1} (2k + 1)(-1)^k A_k \equiv 0 \pmod{n} \quad \text{for all } n \in \mathbb{Z}^+\]

and
\[\sum_{k=0}^{p-1} (2k + 1)(-1)^k A_k \equiv p \left( \frac{p}{3} \right) \pmod{p^3} \quad \text{for any prime } p > 3,\]

which was confirmed by V.J.W. Guo and J. Zeng [5]. In 2016 the third author [17] showed that
\[\sum_{k=0}^{n-1} (6k^3 + 9k^2 + 5k + 1)(-1)^k A_k \equiv 0 \pmod{n^3} \quad \text{for all } n \in \mathbb{Z}^+.\]

In view of this, it is natural to ask whether the Apéry numbers \(\beta_n\) also have such kind of congruence properties. This is the main motivation of this paper.

Another motivation is due to a theorem of E. Rowland and R. Yassawi [12] about \(A_n\) (\(n \in \mathbb{N}\)) modulo 16. We will extend their result to \(A_n\) modulo 32; quite unexpectedly, this requires consideration of \(\beta_n\) modulo 8.

Now we state our main results which were first found via Mathematica.

**Theorem 1.1.** For any positive integer \(n\), we have
\[\sum_{k=0}^{n-1} (11k^2 + 13k + 4)\beta_k \equiv 0 \pmod{2n^2}, \tag{1.1}\]
\[\sum_{k=0}^{n-1} (11k^2 + 9k + 2)(-1)^k \beta_k \equiv 0 \pmod{2n^2}, \tag{1.2}\]
\[\sum_{k=0}^{n-1} (11k^3 + 7k^2 - 1)(-1)^k \beta_k \equiv 0 \pmod{n^2}. \tag{1.3}\]
Theorem 1.2. Let $p > 3$ be a prime. Then
\begin{align*}
\sum_{k=0}^{p-1}(11k^2 + 13k + 4)\beta_k &\equiv 4p^2 + 4p^7B_{p-5} \pmod{p^8}, \quad (1.4) \\
\sum_{k=0}^{p-1}(11k^2 + 9k + 2)(-1)^k\beta_k &\equiv 2p^2 + 10p^3H_{p-1} - p^7B_{p-5} \pmod{p^8}, \quad (1.5)
\end{align*}
and
\begin{align*}
\sum_{k=0}^{p-1}(11k^3 + 7k^2 - 1)(-1)^k\beta_k
\equiv& 2p^3 - 3p^2 + (10p - 5)p^3H_{p-1} - \frac{3}{2}p^7B_{p-5} \pmod{p^8}, \quad (1.6)
\end{align*}
where $H_{p-1}$ denotes the harmonic number $\sum_{k=1}^{p-1}\frac{1}{k}$.

Theorem 1.3. Let $n$ be a nonnegative integer. If
\begin{equation}
A_n \equiv 8j + 4i + 1 \pmod{16} \quad (1.7)
\end{equation}
with $\{i, j\} \subseteq \{0, 1\}$, then
\begin{equation}
\beta_n \equiv 2(i + j) + 1 \pmod{4}. \quad (1.8)
\end{equation}

Remark 1.4. For any $n \in \mathbb{N}$, we clearly have
\begin{equation*}
A_n = 1 + 4 \sum_{0 < k \leq n} \binom{n+k}{2k}^2 \binom{2k-1}{k-1}^2 \equiv 1 \pmod{4}.
\end{equation*}

Theorem 1.5. Let $n$ be a nonnegative integer. If
\begin{equation*}
A_n \equiv 16k + 8j + 4i + 1 \pmod{32} \text{ and } \beta_n \equiv 4m + 2l + 1 \pmod{8}
\end{equation*}
with $\{i, j, k, l, m\} \subseteq \{0, 1\}$, then
\begin{equation}
L_2(n) \equiv 4(k + m) + 2j + i \pmod{8}, \quad (1.9)
\end{equation}
where $L_2(n)$ (the binary run-length of $n$) is defined as the number of blocks of consecutive 0’s and 1’s in the binary expansion of $n$.

We will prove Theorem 1.1 in the next section. We are going to provide some lemmas in Section 3 and show Theorem 1.2 in Section 4. Theorems 1.3 and 1.5 will be proved in Section 5.
2. Proof of Theorem 1.1

Lemma 2.1. For any \( n \in \mathbb{Z}^+ \), we have

\[
\sum_{k=0}^{n-1} (11k^2 + 13k + 4) \beta_k = -\sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} a_1(n,k), \quad (2.1)
\]

\[
\sum_{k=0}^{n-1} (11k^2 + 9k + 2)(-1)^{n-k} \beta_k = \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} a_2(n,k), \quad (2.2)
\]

\[
\sum_{k=0}^{n-1} (11k^3 + 7k^2 - 1)(-1)^{n-k} \beta_k = \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} a_3(n,k), \quad (2.3)
\]

where

\[
a_1(n,k) = n^3 - n^2k - nk^2 - 2n^2 - 5nk + k^2 - 4n + 2k + 1,
\]

\[
a_2(n,k) = n(3n - k - 1),
\]

\[
a_3(n,k) = 4n^3 - 2n^2k - nk^2 - 4n^2 - 3nk + k^2 - 2n + 2k + 1. \quad (2.4)
\]

Proof. Note that

\[
\sum_{k=0}^{n-1} (11k^2 + 13k + 4) \beta_k = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} F(k,l)
\]

is a double sum with

\[
F(k,l) = (11k^2 + 13k + 4) \binom{k}{l}^2 \binom{k+l}{l}.
\]

Using the method in Mu and Sun’s paper [9], we find

\[
F_1(k,l) = -\frac{(k-l)^2a_1(k,l)}{(l+1)^2(11k^2 + 13k + 4)} F(k,l)
\]

and

\[
F_2(k,l) = \frac{l(3kl - l^2 + 2k - 2l)}{11k^2 + 13k + 4} F(k,l)
\]

so that

\[
F(k,l) = (F_1(k+1,l) - F_1(k,l)) + (F_2(k,l+1) - F_2(k,l))
\]

which can be verified directly. This allows us to reduce the double sum \( \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} F(k,l) \) to the right-hand side of (2.1).

Identities (2.2) and (2.3) can be deduced similarly; in fact,

\[
(11k^2 + 9k + 2)(-1)^k \binom{k}{l}^2 \binom{k+l}{l} = (G_1(k+1,l) - G_1(k,l)) + (G_2(k,l+1) - G_2(k,l))
\]
and
\[
(11k^3 + 7k^2 - 1)(-1)^k \binom{k}{l}^2 \binom{k+l}{l}
\]
\[= (H_1(k+1, l) - H_1(k, l)) + (H_2(k, l + 1) - H_2(k, l)),
\]
where
\[
G_1(k, l) = - \frac{(k-l)^2a_2(k, l)}{(l+1)^2} (-1)^k \binom{k}{l}^2 \binom{k+l}{l},
\]
\[
G_2(k, l) = - l(6k + l + 3)(-1)^k \binom{k}{l}^2 \binom{k+l}{l},
\]
\[
H_1(k, l) = - \frac{(k-l)^2a_3(k, l)}{(l+1)^2} (-1)^k \binom{k}{l}^2 \binom{k+l}{l},
\]
\[
H_2(k, l) = - l(8k^2 + l^2 + 4k + 2l + 2)(-1)^k \binom{k}{l}^2 \binom{k+l}{l}.
\]

This ends our proof. \(\square\)

**Lemma 2.2.** For any positive even integer \(n\), we have
\[
\sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{2k} \binom{2k}{k+1} \equiv 1 \pmod{2}.
\] (2.5)

**Proof.** Clearly,
\[
\sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{2k} \binom{2k}{k+1} = \sum_{0 \leq k < n} \binom{n-1}{k} \binom{n+k}{k+1} \binom{n-1}{k-1}.
\]

For \(k \in \{0, \ldots, n-1\}\), if \(2 \mid k\) then \(n+k\) is even and hence
\[
\binom{n+k}{k+1} = \frac{n+k}{k+1} \binom{n+k-1}{k} \equiv 0 \pmod{2};
\]
if \(2 \nmid k\) then
\[
\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k} = k \binom{n-1}{k-1} \equiv 0 \pmod{2}.
\]

Therefore,
\[
\sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{2k} \binom{2k}{k+1} \equiv \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k+1} \binom{n-1}{k}
\]
\[\equiv \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{k+1} \binom{n+k}{k+1}
\]
\[= (-1)^n \pmod{2},
\]
where we have applied (in the last step) the Chu-Vandermonde identity (cf. [4, (3.1)]). This proves (2.5).

Proof of Theorem 1.1. For each $k = 0, \ldots, n - 1$, clearly
\[
\binom{n}{k+1}^2 (k+1)^2 = n^2 \binom{n-1}{k}^2
\]
and
\[
\binom{n}{k+1} \binom{n+k}{k} (n(k+1)) = n^2 \binom{n-1}{k} \binom{n+k}{2k} (n-k)C_k,
\]
where $C_k$ is the Catalan number $\frac{2^k k^k}{k+1} = \frac{2^k k^k - 1}{k+1}$.

(i) As
\[
a_1(n, k) = n^2(n-k-2) - n(k+1)(k+4) + (k+1)^2,
\]
we have
\[
\frac{1}{n^2} \sum_{k=0}^{n-1} \binom{n}{k+1}^2 \binom{n+k}{k} a_1(n, k)
\]
\[
= \sum_{k=0}^{n-1} \binom{n}{k+1}^2 \binom{n+k}{k} (n-k-2)
\]
\[
- \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{2k} (n-k)(k+4)C_k + \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{n+k}{k}.
\]

Clearly $11k^2 + 13k + 4 \equiv 0 \pmod{2}$ for all $k = 0, \ldots, n-1$. So, by Lemma 2.1 and the above, (1.1) holds when $2 \nmid n$.

Now suppose that $n$ is even. By Lemma 2.1 and the above, we have
\[
\frac{1}{n^2} \sum_{k=0}^{n-1} (11k^2 + 13k + 4)\beta_k
\]
\[
\equiv \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} + \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{2k} kC_k
\]
\[
+ \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \pmod{2}.
\]
With the help of the Chu–Vandermonde identity,

\[
\sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} k = \sum_{k=0}^{n-1} \binom{n}{k-1} \binom{n-1}{k} (-1)^k \equiv (-n-1) \sum_{k=1}^{n-1} \binom{n}{k-1} \binom{n-2}{k-1} \\
\equiv \left( -\frac{2}{n-2} \right) = (-1)^n(n-1) \equiv 1 \pmod{2}
\]

and

\[
\sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} = \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n-1}{k} (-1)^k \\
\equiv \sum_{k=0}^{n-1} \binom{n-1}{n-1-k} \binom{n-1}{k} \\
\equiv \left( -\frac{2}{n-1} \right) = (-1)^n n \equiv 0 \pmod{2}.
\]

In view of (2.5), we also have

\[
\sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{2k} kC_k \equiv 1 \pmod{2} \tag{2.6}
\]

since \(kC_k = \binom{2k}{k+1}\). Therefore (1.1) holds.

(ii) As \(a_2(n,k) = 3n^2 - (k+1)n\), we have

\[
= \frac{1}{n^2} \sum_{k=0}^{n-1} \binom{n}{k+1}^2 \binom{n+k}{k} a_2(n,k) \\
= 3 \sum_{k=0}^{n-1} \binom{n}{k+1}^2 \binom{n+k}{k} - \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{2k} (n-k)C_k.
\]

Clearly \(11k^2 + 9k + 2 \equiv 0 \pmod{2}\) for all \(k = 0, \ldots, n-1\). So, by Lemma 2.1 and the above, (1.2) holds when \(2 \nmid n\).

Now suppose that \(n\) is even. By Lemma 2.1 and the above, we have

\[
= \frac{1}{n^2} \sum_{k=0}^{n-1} (11k^2 + 9k + 2)(-1)^k \beta_k \\
\equiv \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} + \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{2k} kC_k \pmod{2}.
\]
By the Chu–Vandermonde identity,
\[ \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} = \sum_{k=0}^{n-1} \binom{n}{n-1-k} \binom{-n-1}{k} = \binom{-1}{n-1} \equiv 1 \pmod{2}. \]
Combining this with (2.6) we obtain (1.2).

(iii) Since
\[ a_3(n, k) = 2n^2(2n - k - 2) - n(k + 1)(k + 2) + (k + 1)^2, \]
we have
\[ \frac{1}{n^2} \sum_{k=0}^{n-1} \binom{n}{k+1}^2 \binom{n+k}{k} a_3(n, k) = 2 \sum_{k=0}^{n-1} \binom{n}{k+1}^2 \binom{n+k}{k} (2n - k - 2) \]
\[ - \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{2k} (n-k)(k+2)C_k + \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{n+k}{k} \]
and hence (1.3) follows from (2.3).

In view of the above, we have completed the proof of Theorem 1.1. □

3. SOME LEMMAS

For each \( s \in \mathbb{Z}^+ \), we define
\[ H_{n,s} := \sum_{0<k\leq n} \frac{1}{k^s} \quad \text{for } n = 0, 1, 2, \ldots, \]
and call such numbers *harmonic numbers of order* \( s \). Those \( H_n := H_{n,1} \) (\( n \in \mathbb{N} \)) are usually called *harmonic numbers*.

**Lemma 3.1.** For any \( n \in \mathbb{Z}^+ \), we have
\[ \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} = H_n, \quad (3.1) \]
\[ \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} H_k = H_{n,2}, \quad (3.2) \]
\[ \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} H_{k,2} = \frac{H_n}{n}. \quad (3.3) \]

**Remark 3.2.** The identities (3.1)-(3.3) are known. See [4, (1.45)] for (3.1), [6] for (3.2), and the proof of [16, Lemma 3.1] for simple proofs of (3.2) and (3.3).
Let $p > 3$ be a prime. In 1900 J.W.L. Glaisher \[2, 3\] refined Wolstenholme’s work \[20\] on congruences by showing that
\[
H_{p-1} \equiv -\frac{p^2}{3} B_{p-3} \pmod{p^3}
\] (3.4)
and
\[
H_{p-1,s} \equiv \frac{s}{s+1} pB_{p-1-s} \pmod{p^2} \quad \text{for all } s = 1, \ldots, p-2.
\] (3.5)

**Lemma 3.3.** (X. Zhou and T. Cai \[22, p. 1332\]) Let $r, s \in \mathbb{Z}^+$. For any prime $p > rs + 2$, we have
\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq p-1} \frac{1}{i_1^{s_1} \cdots i_r^{s_r}} \equiv \left\{ \begin{array}{ll}
(-1)^r \frac{s(rs+1)}{2(rs+2)} p^2 B_{p-rs-2} \pmod{p^3} & \text{if } 2 \nmid rs, \\
(-1)^r \frac{s}{rs+1} pB_{p-1} - s \pmod{p^2} & \text{if } 2 \mid rs.
\end{array} \right.
\]

**Lemma 3.4.** (i) (J. Zhao \[21, Theorems 3.1 and 3.2\]) Let $s, t \in \mathbb{Z}^+$ and let $p \geq s + t$ be a prime. Then
\[
\sum_{1 \leq j < k \leq p-1} \frac{1}{j^s k^t} \equiv \left( \frac{-1}{s+t} \right) \frac{s+t+1}{s} B_{p-s-t} \pmod{p}.
\]
If $s \equiv t \pmod{2}$ and $p > s + t + 1$, then
\[
\sum_{1 \leq j < k \leq p-1} \frac{1}{j^s k^t} \equiv (-1)^s t \left( \frac{s+t+1}{s} \right) - (-1)^t s \left( \frac{s+t+1}{t} \right) - s - t
\times \frac{pB_{p-s-t-1}}{2(s+t+1)} \pmod{p^2}.
\]

(ii) (J. Zhao \[21, Theorem 3.5\]) Let $r, s, t \in \mathbb{Z}^+$ with $r + s + t$ odd. For any prime $p > r + s + t$, we have
\[
\sum_{1 \leq i < j < k \leq p-1} \frac{1}{i^r j^s k^t} \equiv \left( (-1)^r \left( \frac{r+s+t}{r} \right) - (-1)^t \left( \frac{r+s+t}{t} \right) \right)
\times \frac{B_{p-(r+s+t)}}{2(r+s+t)} \pmod{p}.
\]

(iii) (J. Zhao \[21, Corollary 3.6\]) For any prime $p > 5$, we have
\[
\sum_{1 \leq i < j < k \leq p-1} \frac{1}{i^2 j^2 k} \equiv \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j} \equiv \sum_{1 \leq i < j \leq p-1} \frac{1}{ijk} \equiv 0 \pmod{p}.
\]

**Lemma 3.5.** For any prime $p > 5$, we have
\[
\sum_{k=1}^{p-1} \sum_{1 \leq i < j \leq k-1} \frac{1}{i^2 j^2} \equiv \frac{2}{5} pB_{p-5} - \frac{3H_{p-1}}{p^2} \pmod{p^2} \quad \text{(3.6)}
\]
and
\[ \sum_{k=1}^{p-1} \sum_{1 \leq i < j \leq k-1} \frac{H_{k-1}}{i^2 j^2} \equiv \frac{3H_{p-1}}{p^2} \pmod{p}. \] (3.7)

**Proof.** (i) Clearly,
\[ \sum_{k=1}^{p-1} \sum_{1 \leq i < j \leq k-1} \frac{1}{i^2 j^2} = \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j^2} \]
\[ \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j^2} = \sum_{1 \leq i < j \leq p-1} \frac{p-j}{i^2 j^2} \]
and hence
\[ \sum_{k=1}^{p-1} \sum_{1 \leq i < j \leq k-1} \frac{1}{i^2 j^2} = (p-1) \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j^2} - \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j^2}. \] (3.8)

Note that \( H_{p-1,3} \equiv 0 \pmod{p^2} \) by (3.5) or Lemma 3.3 and
\[ \sum_{k=1}^{p-1} \frac{H_{k-1}}{k^2} \equiv -3 \frac{H_{p-1}}{p^2} \pmod{p^2} \] (3.9)
by R. Tauraso [18, Theorem 1.3]. Thus
\[ \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j} = \sum_{i=1}^{p-1} \frac{H_{p-1} - H_i}{i^2} \]
\[ = H_{p-1} H_{p-1,2} - H_{p-1,3} - \sum_{i=1}^{p-1} \frac{H_{i-1}}{i^2} \equiv 3 \frac{H_{p-1}}{p^2} \pmod{p^2}. \]

By Lemma 3.2,
\[ \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j^2} \equiv -\frac{2}{5} p B_{p-5} \pmod{p^2}. \] (3.10)

Combining these with (3.8), we immediately obtain (3.6).

(ii) As \( H_{p-1} \equiv 0 \pmod{p} \) and
\[ \sum_{s=1}^{p-1} H_s = \sum_{1 \leq r \leq s \leq p-1} \frac{1}{r} = \sum_{r=1}^{p-1} \frac{p-r}{r} = p H_{p-1} - (p-1) \equiv 1 \pmod{p}, \]
we have
\[
\sum_{k=1}^{p-1} \sum_{1 \leq i < j \leq k-1} \frac{H_{k-1}}{i^2 j^2}
\]
\[
\equiv \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j^2} \left(1 - \sum_{0 \leq r \leq j} \frac{1}{r}\right)
\]
\[
= \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j^2} \left(1 - \sum_{0 \leq r < j} \frac{j-r}{r}\right) = \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j} \left(1 - \sum_{0 \leq r < j} \frac{1}{r}\right)
\]
\[
= \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j} - \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j} - \sum_{1 \leq i \leq r < j \leq p-1} \frac{1}{r i^2 j} - \sum_{1 \leq i \leq r < j \leq p-1} \frac{1}{i^2 r j}
\]
\[
\equiv \frac{(-1)^1}{3} \frac{(3)}{(2)} B_{p-3} - \frac{(-1)^1}{4} \frac{(4)}{(3)} B_{p-4} - 0 - 0 \equiv \frac{3H_{p-1}}{p^2} \pmod{p}
\]
with the help of Lemma 3.4 and (3.4). This proves (3.7).

The proof of Lemma 3.5 is now complete. □

Lemma 3.6. Let \( p > 5 \) be a prime. Then
\[
\sum_{k=1}^{p-1} \binom{p}{k} \frac{(-1)^k}{k} H_{k-1,2} \equiv \frac{9}{10} p^2 B_{p-5} \pmod{p^3}. \tag{3.11}
\]

Proof. As
\[
(-1)^k \binom{p-1}{k-1} = \prod_{0 \leq j < k} \left(1 - \frac{p}{j}\right) \equiv 1 - p H_{k-1} \pmod{p^2}
\]
for all \( k = 1, \ldots, p - 1 \), we have
\[
\sum_{k=1}^{p-1} \binom{p}{k} \frac{(-1)^k}{k} H_{k-1,2} \equiv p \sum_{k=1}^{p-1} \binom{p-1}{k-1} \frac{(-1)^k}{k^2} H_{k-1,2}
\]
\[
\equiv p \sum_{k=1}^{p-1} \frac{p H_{k-1} - 1}{k^2} H_{k-1,2} \pmod{p^3}.
\]
By (3.10),
\[
\sum_{k=1}^{p-1} \frac{H_{k-1,2}}{k^2} \equiv - \frac{2}{5} p B_{p-5} \pmod{p^3}.
\]
Note also that
\[
\sum_{k=1}^{p-1} \frac{H_{k-1}H_{k-1,2}}{k^2} = \sum_{1 \leq j < k \leq p-1} \sum_{1 \leq i < k} \frac{1}{ij^2k^2}
\]
\[
= \sum_{1 \leq j < k \leq p-1} \frac{1}{ij^2k^2} + \sum_{1 \leq i < k} \frac{1}{ij^2k^2} + \sum_{1 \leq i < j < k \leq p-1} \frac{1}{j^2ik^2}
\]
\[
\equiv \frac{(-1)^2}{3+2} \left( 3 + \frac{2}{3} \right) B_{p-5} + \left( (-1)^1 \left( \frac{5}{1} \right) - (-1)^2 \left( \frac{5}{2} \right) \right) \frac{B_{p-5}}{2 \times 5} + 0
\]
\[
\equiv 2B_{p-5} - \frac{3}{2} B_{p-5} = \frac{B_{p-5}}{2} \pmod{p}
\]
with the help of parts (i) and (iii) of Lemma 3.4. Combining the above, we get the desired congruence (3.11). □

**Lemma 3.7.** For any prime \( p > 5 \), we have
\[
\left( \frac{2p-1}{p-1} \right) \equiv 1 + 2pH_{p-1} - \frac{4}{5} p^5 B_{p-5} \pmod{p^6}.
\]

**Proof.** It is known (cf. [18, Theorem 2.4]) that
\[
\left( \frac{2p-1}{p-1} \right) \equiv 1 + 2pH_{p-1} + \frac{2}{3} p^3 H_{p-1,3} \pmod{p^6}.
\]

By Lemma 3.3,
\[
H_{p-1,3} \equiv - \frac{3 \times 4}{2 \times 5} p^2 B_{p-5} = - \frac{6}{5} p^2 B_{p-5} \pmod{p^3}.
\]
(3.12)

So, the desired congruence follows. □

**Lemma 3.8.** For any prime \( p > 7 \), we have
\[
H_{p-1} + \frac{p}{2} H_{p-1,2} + \frac{p^2}{6} H_{p-1,3} \equiv 0 \pmod{p^6}.
\]
(3.13)

**Proof.** Clearly,
\[
\sum_{k=1}^{p-1} \frac{1}{p-k} = \sum_{k=1}^{p-1} \frac{p^6 - k^6}{p-k} \left( -1 \right) \left( \frac{1}{k^6} \right) = - \sum_{k=1}^{p-1} \frac{p^5 + p^4k + p^3k^2 + p^2k^3 + pk^4 + k^5}{k^6}
\]
\[
\equiv - p^5 H_{p-1,6} - p^4 H_{p-1,5} - p^3 H_{p-1,4} - p^2 H_{p-1,3} - p H_{p-1,2} - H_{p-1}
\]
\[
\equiv - p^3 H_{p-1,4} - p^2 H_{p-1,3} - p H_{p-1,2} - H_{p-1} \pmod{p^6}
\]
since \( H_{p-1,6} \equiv 0 \pmod{p} \) and \( H_{p-1,5} \equiv 0 \pmod{p^2} \) by (3.5) or Lemma 3.3. Note that
\[
H_{p-1,4} \equiv 4p \left( \frac{B_{2p-6}}{2p-6} - \frac{B_{p-5}}{p-5} \right) \equiv - \frac{2}{3p} H_{p-1,3} \pmod{p^3}
\]
CONGRUENCES FOR APÉRY NUMBERS $\beta_n = \sum_{k=0}^n \binom{n+k}{k}^2$

by [14, Theorem 5.1 and Remark 5.1]. Therefore

$$2H_{p-1} + pH_{p-1,2} \equiv \frac{2}{3}p^2H_{p-1,3} - p^2H_{p-1,3} \pmod{p^6}$$

and hence (3.13) follows. \hfill \qed

**Remark 3.9.** Let $p > 3$ be a prime. If $p > 7$ then (3.13) has the following equivalent form

$$2H_{p-1} + pH_{p-1,2} \equiv \frac{2}{3}p^4B_{p-5} \pmod{p^5} \quad (3.14)$$

in view of (3.12). It is easy to see that (3.14) also holds for $p = 5, 7$.

4. PROOF OF THEOREM 1.2

To prove Theorem 1.2, we need an auxiliary theorem.

**Theorem 4.1.** Let $p > 5$ be a prime. Then

$$\sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p + k - 1}{k - 1} \equiv pH_{p-1} + \frac{9}{10}p^5B_{p-5} \pmod{p^6}, \quad (4.1)$$

$$\sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p + k - 1}{k - 1}k \equiv -3p^2H_{p-1} + \frac{21}{10}p^6B_{p-5} \pmod{p^7}, \quad (4.2)$$

and

$$\sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p + k - 1}{k - 1}k^2 \equiv -p^2 - (4p^5 + 4p^4 + 2p^3)H_{p-1} + \frac{4}{5}p^7B_{p-5} \pmod{p^8}. \quad (4.3)$$

**Proof.** For each $k = 1, \ldots, p - 1$, we clearly have

$$\frac{1}{p} \binom{p}{k} = \frac{(-1)^{k-1}}{k} \prod_{0<j<k} \left( 1 - \frac{p}{j} \right) \equiv \frac{(-1)^{k-1}}{k} \left( 1 - pH_{k-1} + p^2 \sum_{1 \leq i < j \leq k-1} \frac{1}{ij} \right) \pmod{p^3} \quad (4.4)$$

and

$$(-1)^{k-1} \frac{p - 1}{k} \binom{p + k - 1}{k - 1} \binom{p + k - 1}{k - 1} \equiv \prod_{0<j<k} \left( 1 - \frac{p^2}{j^2} \right) \equiv 1 - p^2H_{k-1,2} + p^4 \sum_{1 \leq i < j \leq k-1} \frac{1}{i^2j^2} \pmod{p^6}. \quad (4.5)$$
Let $r \in \{0, 1, 2\}$ and
\[
\sigma_r := \sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p+k-1}{k-1} k^r.
\] (4.6)

In view of the above, we have
\[
\sigma_r = p \sum_{k=1}^{p-1} \binom{p}{k} k^{r-1} \binom{p-1}{k-1} \binom{p+k-1}{k-1}
\equiv p \sum_{k=1}^{p-1} \binom{p}{k} k^{r-1} (-1)^{k-1} \left( 1 - p^2 H_{k-1,2} + p^4 \sum_{1 \leq i < j \leq k-1} \frac{1}{i^2 j^2} \right)
\equiv p \sum_{k=1}^{p-1} \binom{p}{k} k^{r-1} (-1)^{k-1} + p^3 \sum_{k=1}^{p-1} \binom{p}{k} k^{r-1} (-1)^k H_{k-1,2} + p^5 \sum_{k=1}^{p-1} \frac{p}{k} (1 - p H_{k-1}) k^{r-1} \sum_{1 \leq i < j \leq k-1} \frac{1}{i^2 j^2} \pmod{p^8}.
\]

**Case 1.** $r = 0$.

In this case,
\[
p \sum_{k=1}^{p-1} \binom{p}{k} k^{r-1} (-1)^{k-1} = p H_p - 1 = p H_{p-1}
\]
by (3.1), and hence
\[
\sigma_0 \equiv p H_{p-1} + p^3 \sum_{k=1}^{p-1} \binom{p}{k} \frac{(-1)^k}{k} H_{k-1,2} \equiv p H_{p-1} + \frac{9}{10} p^5 B_{p-5} \pmod{p^6}
\]
with the help of Lemma 3.6. This proves (4.1).

**Case 2.** $r = 1$.

In this case,
\[
p \sum_{k=1}^{p-1} \binom{p}{k} k^{r-1} (-1)^{k-1} = -p \sum_{k=1}^{p} \binom{p}{k} (-1)^k = -p (1 - 1)^p = 0,
\]
and also
\[
\sum_{1 \leq i < j < k \leq p-1} \frac{1}{i^2 j^2 k^2} \equiv \left( (-1)^2 \binom{5}{2} - (-1)^1 \binom{5}{1} \right) \frac{B_{p-5}}{2 \times 5} = \frac{3}{2} B_{p-5} \pmod{p}
\]
by Lemma 3.4. Thus
\[
\sigma_1 \equiv p^5 \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k H_{k-1,2} + \frac{3}{2} p^6 B_{p-5} \pmod{p^7}. \tag{4.7}
\]
In view of (3.3) and (4.4), we have

\[
\sum_{k=1}^{p-1} \left( \binom{p}{k} \right) (-1)^k H_{k-1,2} = \sum_{k=1}^{p} \left( \binom{p}{k} \right) (-1)^k H_{k,2} + H_{p,2} - \sum_{k=1}^{p-1} \left( \binom{p}{k} \right) \frac{(-1)^k}{k^2}
\]

\[
\equiv - \frac{H_p}{p} + H_{p,2} + p \sum_{k=1}^{p-1} \frac{1 - pH_{k-1} + p^2 \sum_{1 \leq i, j \leq k-1} \frac{1}{ij}}{k^3}
\]

\[
\equiv - \frac{H_{p-1}}{p} + H_{p-1,2} + pH_{p-1,3} - p^2 \sum_{1 \leq j, k \leq p-1} \frac{1}{jk^3}
\]

\[
+ p^3 \sum_{1 \leq i, j, k \leq p-1} \frac{1}{ijk^3} \pmod{p^4}.
\]

Note that

\[
H_{p-1,2} \equiv 2 \frac{p^3 B_{p-5}}{5} - 2 \frac{p}{p} H_{p-1} \pmod{p^4} \text{ and } H_{p-1,3} \equiv - 6 \frac{p^2 B_{p-5}}{3} \pmod{p^3}
\]

by (3.14) and (3.12). In view of Lemma 3.4

\[
\sum_{1 \leq j, k \leq p-1} \frac{1}{jk^3} \equiv \left( -3 \binom{5}{1} - (-1)^3 \binom{5}{3} \right) \frac{pB_{p-5}}{2 \times 5} = - \frac{9}{10} pB_{p-5} \pmod{p^2}
\]

and

\[
\sum_{1 \leq i, j, k \leq p-1} \frac{1}{ijk^3} \equiv \left( (-1)^1 \binom{5}{1} - (-1)^3 \binom{5}{3} \right) \frac{B_{p-5}}{2 \times 5} = \frac{B_{p-5}}{2} \pmod{p}.
\]

Therefore

\[
\sum_{k=1}^{p-1} \left( \binom{p}{k} \right) (-1)^k H_{k-1,2} \equiv - 3 \frac{H_{p-1}}{p} + \frac{2}{5} p^3 B_{p-5} - \frac{6}{5} p^3 B_{p-5} + \frac{9}{10} p^3 B_{p-5} + \frac{p^3}{2} B_{p-5}
\]

\[
\equiv - \frac{3}{p} H_{p-1} + \frac{3}{5} p^3 B_{p-5} \pmod{p^4}.
\]

Combining this with (4.7), we see that

\[
\sigma_1 \equiv p^3 \left( - \frac{3}{5} p H_{p-1} + \frac{3}{5} p^3 B_{p-5} \right) + \frac{3}{2} p^6 B_{p-5} = -3p^2 H_{p-1} + 21 \frac{1}{10} p^6 B_{p-5} \pmod{p^7}.
\]

This proves (4.2).

**Case 3.** \( r = 2 \).

In this case,

\[
p \sum_{k=1}^{p-1} \left( \binom{p}{k} \right) k^{r-1} (-1)^{k-1} = p^2 \sum_{k=1}^{p-1} \left( \binom{p-1}{k-1} \right) (-1)^{k-1} = p^2 ((1 - 1)^{p-1} - 1) = -p^2
\]
and hence

\[ \sigma_2 \equiv -p^2 + p^4 \sum_{k=1}^{p-1} \left( \frac{p-1}{k-1} \right) (-1)^k H_{k-2} \]

\[ + p^6 \sum_{k=1}^{p-1} \sum_{1 \leq i < j \leq k-1} 1 \mathbf i j - p^7 \sum_{k=1}^{p-1} \sum_{1 \leq i < j \leq k-1} \frac{H_{k-1}}{i^2 j^2} \]

\[ \equiv -p^2 + p^4 \left( \frac{H_{p-1}}{p-1} + H_{p-1,2} \right) + p^6 \left( \frac{2}{5} p B_{p-5} - 3 \frac{H_{p-1}}{p^2} \right) - p^7 3 H_{p-1} \]

\[ \equiv -p^2 + p^4 \left( -(p+1) H_{p-1} + \frac{2}{5} p B_{p-5} - \frac{2}{p} H_{p-1} \right) \]

\[ + \frac{2}{5} p^7 B_{p-5} - 3 p^4 H_{p-1} - 3 p^5 H_{p-1} \pmod{p^8} \]

with the help of (3.3), (3.14) and Lemma 3.5. This yields the desired (4.3).

In view of the above, we have completed the proof of Theorem 4.1. □

**Proof of Theorem 1.2** It is easy to see that (1.4)-(1.6) hold for \( p = 5 \). Below we assume \( p > 5 \).

For \( i = 1, 2, 3 \) let

\[ S_i := \sum_{k=0}^{p-1} \left( \frac{p}{k+1} \right)^2 p^k a_i(p, k) \]

with \( a_i(n, k) \) given by Lemma 2.1. In view of Lemma 2.1 it suffices to show the following three congruences:

\[ S_1 \equiv -4 p^2 - 4 p^7 B_{p-5} \pmod{p^8}, \]  

\[ S_2 \equiv 2 p^2 + 10 p^3 H_{p-1} + p^7 B_{p-5} \pmod{p^8}, \]  

\[ S_3 \equiv 2 p^3 - 3 p^2 + (10 p^4 - 5 p^3) H_{p-1} - \frac{3}{2} p^7 B_{p-5} \pmod{p^8}, \]

Clearly,

\[ S_i = \sum_{j=1}^{p} \left( \frac{p}{j} \right)^2 \left( \frac{p+j-1}{j-1} \right) a_i(p, j-1) \]

\[ = \left( \frac{2p-1}{p-1} \right) a_i(p, p-1) + \sum_{k=1}^{p-1} \left( \frac{p}{k} \right)^2 \left( \frac{p+k-1}{k-1} \right) a_i(p, k-1). \]

(4.11)

Note that

\[ a_1(p, p-1) = -p^3 - 3 p^2, \ a_2(p, p-1) = 2 p^2 \] and \( a_3(p, p-1) = p^3 - 2 p^2 \).
In view of Lemma 3.7, we have
\[
\binom{2p - 1}{p - 1} a_i(p, p - 1)
= \left( 1 + 2pH_{p-1} - \frac{4}{5}p^5 B_{p-5} \right) a_i(p, p - 1)
\equiv \begin{cases} 
-p^3 - 2p^4 H_{p-1} - 3p^2 - 6p^3 H_{p-1} + \frac{12}{5}p^7 B_{p-5} \pmod{p^8} & \text{if } i = 1, \\
2p^2 + 4p^3 H_{p-1} - \frac{8}{5}p^7 B_{p-5} \pmod{p^8} & \text{if } i = 2, \\
p^3 + 2p^4 H_{p-1} - 2p^2 - 4p^3 H_{p-1} + \frac{8}{5}p^7 B_{p-5} \pmod{p^8} & \text{if } i = 3.
\end{cases}
\] (4.12)

For \( r = 0, 1, 2 \) let \( \sigma_r \) be defined as in (4.6). Noting
\[
a_1(p, k - 1) = (p^3 - p^2) - (p^2 + 3p)k - (p - 1)k^2
\]
and applying Theorem 4.1 we get
\[
\sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p + k - 1}{k - 1} a_1(p, k - 1)
= (p^3 - p^2)\sigma_0 - (p^2 + 3p)\sigma_1 - (p - 1)\sigma_2
\equiv (p^3 - p^2) \left( pH_{p-1} + \frac{9}{10}p^5 B_{p-5} \right) - (p^2 + 3p) \left( -3p^2 H_{p-1} + \frac{21}{10}p^6 B_{p-5} \right)
- (p - 1) \left( -p^2 - (4p^5 + 4p^4 + 2p^3) H_{p-1} + \frac{4}{5}p^7 B_{p-5} \right)
\equiv p^3 - p^2 + (2p^4 + 6p^3) H_{p-1} - \frac{32}{5}p^7 B_{p-5} \pmod{p^8}.
\]
Combining this with (4.11) and (4.12), we get (4.8).

Similarly,
\[
S_2 \equiv 2p^2 + 4p^3 H_{p-1} - \frac{8}{5}p^7 B_{p-5} + 3p^2 \sigma_0 - p\sigma_1
\equiv 2p^2 + 4p^3 H_{p-1} - \frac{8}{5}p^7 B_{p-5} + 3p^2 \left( pH_{p-1} + \frac{9}{10}p^5 B_{p-5} \right)
- p \left( -3p^2 H_{p-1} + \frac{21}{10}p^6 B_{p-5} \right)
\equiv 2p^2 + 10p^3 H_{p-1} - p^7 B_{p-5} \pmod{p^8}.
\]
This proves (4.9).

As
\[
a_3(p, k - 1) = 4p^3 - 2p^2 - (2p^2 + p)k - (p - 1)k^2,
\]
we have
\[ \sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p+k-1}{k-1} a_3(p, k-1) \]
\[ = (4p^3 - 2p^2) \sigma_0 - (2p^2 + p) \sigma_1 - (p - 1) \sigma_2 \]
\[ \equiv (4p^3 - 2p^2) \left( pH_{p-1} + \frac{9}{10} p^5 B_{p-5} \right) - (2p^2 + p) \left( -3p^2 H_{p-1} + \frac{21}{10} p^6 B_{p-5} \right) \]
\[ - (p - 1) \left( -p^2 - (4p^5 + 4p^4 + 2p^3) H_{p-1} + \frac{4}{5} p^7 B_{p-5} \right) \]
\[ \equiv p^3 - p^2 + (8p^4 - p^3) H_{p-1} - \frac{31}{10} p^7 B_{p-5} \pmod{p^8}. \]
Combining this with (4.11) and (4.12), we get (4.10).

The proof of Theorem 1.2 is now complete. □

5. Proofs of Theorems 1.3 and 1.5

Clearly \( L_2(0) = 0 \), and for each \( n \in \mathbb{Z}^+ \) we have
\[ L_2(n) = L_2 \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + \begin{cases} 0 & \text{if } n \equiv 0, 3 \pmod{4}, \\ 1 & \text{if } n \equiv 1, 2 \pmod{4}. \end{cases} \]

E. Rowland and R. Yassawi discovered that the two least significant binary digits of \( L_2(n) \) are completely determined by the fourth and the third least significant digits of the Apéry number \( A_n \). (As \( A_n \equiv 1 \pmod{4} \), the two least significant digits of \( A_n \) are always 0 and 1, and hence these digits contain no information about \( n \).)

**Theorem 5.1.** [12] Theorem 3.28] Let \( n \in \mathbb{N} \). If
\[ A_n \equiv 8j + 4i + 1 \pmod{16} \]  
(5.1)
with \{i, j\} \subseteq \{0, 1\}, then
\[ L_2(n) \equiv 2j + i \pmod{4}. \]  
(5.2)

Theorem 1.3 shows that the fourth and the third least significant digits of \( A_n \) also determine the second least significant digit of \( \beta_n \) (all numbers \( \beta_n \) are odd, so the very least significant digit is always 1).

Theorem 1.5 shows that the knowledge of the five least significant digits of \( A_n \) by itself is not sufficient for determining the three least significant digits of \( L_2(n) \). This is because the third least significant digit of \( \beta_n \) can be 0 or 1. But the additional knowledge of this digit gives the missing
bit of information for determining the three least significant digits of $L_2(n)$ according to Theorem 1.5.

![Figure 1. Automaton $A_{16}$ calculating $A_n \mod 16$](image1)

![Figure 2. Automaton $B_4$ calculating $\beta_n \mod 4$](image2)

Theorems 1.3 and 1.5 can be proved by the technique used in [12] with the aid of software [10] implemented by E. Rowland.

Namely, Apéry numbers of both kinds are known to be the diagonal sequences of certain rational functions. In particular, A. Straub [13] proved that $A_n$ is the coefficient of $x_1^n x_2^n x_3^n x_4^n$ in the formal Taylor expansion of the function

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}. \quad (5.3)$$

Similar coefficient in the expansion of

$$\frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)-(1-x_1)x_1x_2x_3} \quad (5.4)$$

is, according to [12], equal to $\beta_n$. 
According to [12, Theorem 2.1], the above representations of Apéry numbers as the diagonal coefficients imply that for every prime \( p \) and its power \( q \) the sequences \( A_n \mod q \) and \( \beta_n \mod q \) are \( p \)-automatic. Moreover, [12] contains two algorithms for constructing corresponding automata; Mathematica implementations of these algorithms are given in [10].

To prove Theorem 1.3, we need to examine the sequence \( A_n \mod 16 \). Corresponding automaton \( \mathcal{A}_{16} \) was calculated in [11] and exhibited in [12], we reproduce it here in Figure 1 (with the initial state marked in light color). This automaton calculates \( A_n \mod 16 \) in the following way. Let

\[
    n = \sum_{k=0}^{m} n_k 2^k
\]

where \( \{ n_0, \ldots, n_m \} \subseteq \{0, 1\} \); start from the initial state and follow the oriented path formed by edges marked \( n_0, \ldots, n_m \); the final vertex is labeled by \( A_n \mod 16 \).

Function AutomaticSequenceReduce from [10], being applied to (5.4), returns the minimal automaton \( \mathcal{B}_4 \) calculating \( \beta_n \mod 4 \); this automaton is exhibited in Figure 2.

Let

\[
    f(1) = f(13) = 1 \quad \text{and} \quad f(5) = f(9) = 3.
\]

Theorem 1.3 asserts that

\[
    (\beta_n \mod 4) = f(A_n \mod 16).
\]

We can construct an automaton calculating \( f(A_n \mod 16) \) simply by applying function \( f \) to the labels of states in automaton \( \mathcal{A}_{16} \). The resulting automaton is not minimal, but we can minimize it using, for example, function AutomatonMinimize from [10]. The minimal automaton calculating \( f(A_n \mod 16) \) turns out to be equal to \( \mathcal{B}_4 \), which proves (5.6) and hence Theorem 1.3 is proved as well.

The minimal automaton \( \mathcal{A}_{32} \) calculating \( A_n \mod 32 \) was constructed in [11], it has 33 states; the minimal automaton \( \mathcal{B}_8 \) calculating \( \beta_n \mod 8 \) has 17 states. Taking into account Theorem 5.1 for proving Theorem 1.5 it is sufficient to work just with the fifth least significant digits of \( A_n \) and the third least significant digits of \( \beta_n \) and \( L_2(n) \), which allows us to deal with automata having fewer number of states.
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\[ \beta_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \]

Figure 3. Automaton \( A_{32,5} \) calculating the fifth digit of \( A_n \)

Figure 4. Automaton \( B_{8,3} \) calculating the third digit of \( \beta_n \)

Figures 3 and 4 exhibit the minimal automata \( A_{32,5} \) and \( B_{8,3} \) calculating respectively the fifth least significant digit of \( A_n \) and the third least significant digit of \( \beta_n \). These automata are easily constructed from \( A_{32} \) and \( B_8 \) by properly relabelling the states and minimizing resulting automata.
With function \texttt{AutomatonProduct} from \cite{10} we can easily construct automaton $C$ which is the product of $A_{32,5}$ and $B_{8,3}$. Each state of $C$ is labeled by two bits (corresponding to the fifth digit of $A_n$ and the third digit of $\beta_n$). Replacing each such pair of bits by their sum modulo 2 and performing minimization, we get the automaton exhibited on Figure 5. It is not difficult to see that this automaton calculates the third digit of $L_2(n)$, which proves Theorem 1.5.

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(HUI-QIN CAO) Department of Applied Mathematics, Nanjing Audit University, Nanjing 211815, People’s Republic of China
E-mail address: cachq@nau.edu.cn

(YURI MATIYASEVICH) St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023, St. Petersburg, Russia
E-mail address: yumat@pdmi.ras.ru

(ZHI-WEI SUN, CORRESPONDING AUTHOR) Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China
E-mail address: zwsun@nju.edu.cn