PARTON SATURATION AT SMALL $x$ AND IN LARGE NUCLEI

A.H. Mueller
Department of Physics, Columbia University
New York, New York 10027

Abstract

Quark and gluon distributions in the light-cone wavefunction of a high energy hadron or nucleus are calculated in the saturation regime. One loop calculations are performed explicitly using the equivalence between the parton distribution in the light-cone wavefunction and the production distribution of that parton in a current-nucleon (nucleus) scattering. We argue that, except for some overall numerical factors, the Weizsäcker-Williams wavefunction correctly gives the physics of the gluon distribution in a light-cone wavefunction.

1 Introduction

The idea of parton saturation\textsuperscript{1} in QCD is at the heart of the interest in small-$x$ hadron and nuclear physics. This idea has its simplest and most intuitive statement in terms of the light-cone wavefunction of a high energy hadron or nucleus. Saturation of quark and antiquark densities\textsuperscript{1, 2} is the statement that the density of quarks per unit area and per unit of two-dimensional transverse momentum, that is per unit of true transverse phase space, is limited by a constant times the number of colors so long as the quark momentum is below some momentum $Q_s$, the saturation momentum. Above $Q_s$ the quark distribution becomes perturbative. The result is stated precisely in (29). While the value of $Q_s$ can depend on the particular hadron whose wavefunction is being considered and on the longitudinal momentum

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fraction of the quark, the statement of saturation, as given in (29), contains no knowledge of QCD dynamics or of the hadron in question. The analogous statement of gluon saturation[1, 2, 3, 4] is somewhat different. Gluon densities can be much larger than quark densities. The gluon density naturally has a term $\frac{N_c^2 - 1}{N_c}$ as a factor in its ultimate limit and a \( \ell nQ_s/\ell^2 \) also appears leading to the expression (64) for the gluon density per unit of phase space in the light-cone wavefunction. Eqs.(29) and (64) are based on one-loop quark and gluon calculations, respectively, and for a large nucleus with an additional use of BFKL dynamics in the gluon case. However, because (29) and (64) are so directly related, in general, to the scattering of a quark-antiquark pair and a gluon pair, respectively, on a nucleus or on a high momentum hadron we believe that these results are quite general except for the possibility of a pure number, not depending on the hadron in question, as a multiplicative factor on the right-hand sides of (29) and (64).

In the case of the gluon density in a large nucleus in the saturation region there is a very nice model suggested by McLerran and Venugopalan[5] where the valence quarks of the nucleons of the nucleus are treated as the sources of Weizsäcker-Williams gluons[3, 5, 6] which make up the small-x gluon distribution of the nucleus. This model leads to gluon saturation as expressed in (6) and (8). Except for an overall constant factor we believe these are general results and thus that the Weizsäcker-Williams model is a good picture of the gluon saturation regime of the high energy hadron or nucleus. In the Weizsäcker-Williams approximation the saturated gluon distribution is a pure gauge field and this again is believed to be a general result[7]. The fact that the saturated gluons are pure gauge fields does not mean that they do not have a direct physical interpretation. Indeed, our procedure for calculating the gluon distribution in the nucleus is to note that it is the same as the spectrum of produced gluons[4], a physical object. However, the fact the Weizsäcker-Williams gluons are pure gauge gluons is what allows an exact calculation of their distribution.

The dynamics that leads to parton saturation is BFKL evolution[8, 9] because the increasing number of gluons that appear in a hadron’s wavefunction due to longitudinal momentum or x-evolution occupy a common region in transverse phase space and so can naturally lead to the large values of \( A_{\mu} \) (see (8)) required for saturation. It is possible that the turnover in \( \frac{\partial F_2}{\partial \ln Q^2} \) observed recently at HERA[10, 11] at small \( x \) as \( Q^2 \) is lowered below \( 2 GeV^2 \) may already indicate a saturation of the quark distribution[12, 13, 14]. The
fact that diffractive production, $\gamma^* + P \rightarrow x + P$, indicates an energy dependence much stronger than that suggested by soft physics [14] may also be an indication that saturation has been reached at HERA [15, 16]. Deep inelastic scattering on nuclear targets at HERA should be a very good place to look for saturation. If the turnover in $\frac{dF}{d\ln Q^2}$ really is due to saturation one can expect that the turnover occur at $Q^2 > 5 GeV^2$ in deep inelastic scattering off large nuclei. Another place where saturation effects may be important is in the very early stages of relativistic heavy ion collisions. At RHIC one expects the gluon saturation momentum to be about one GeV with minijets in that regime contributing most of the freed energy. At LHC the saturation momentum should be 2-3 GeV bringing saturation dynamics into play as a major determinant in the very early stages, well before equilibration, of heavy ion collisions. Finally, when potentials as large as $A \sim 1/g$ are reached one has entered a whole new regime of nonperturbative QCD where, for example, instanton effects can become important. Since the large potentials in the wavefunctions are pure gauge fields perturbation theory remains valid in describing the light-cone wavefunctions [7]. However, in the very early stages of the central region of a head-on heavy ion collision these gauge fields are freed, and over the time of the freeing of the gluons large field strengths, $F_{\mu\nu} \sim 1/g$, appear whose dynamics should be genuinely nonperturbative.

In Sec.2, we indicate how one can determine the quark and gluon distributions in the light-cone wavefunction by looking at quark and gluon production. The essence of the argument is that in a particular light-cone gauge, described in detail in Ref.[4], final state interactions are absent allowing parton production to be an indicator of the wavefunction of the hadron. In Sec.2, the Weizsäcker-Williams result for the wavefunction is briefly reviewed.

In Sec.3, we calculate the quark momentum distribution in a large nucleus. This calculation is equivalent to the one-quark-loop fluctuations in the Weizsäcker-Williams background field of the nucleus. However, our method of doing the calculation, following Ref.[2], emphasizes the relationship between the quark distribution of the nucleus and the cross section for scattering a quark-antiquark pair on the nucleus. Saturation then corresponds to blackness in the scattering of the quark-antiquark pair on the nucleus.

In Sec.4, we determine the gluon distribution at the one-loop level. Here the saturated distribution includes a factor of $\ell n 1/x$ replacing $\frac{1}{\alpha_s} \ell n Q_s^2/\ell^2$ from the Weizsäcker-Williams result.

In Sec.5, we interpret the one-loop results in terms of unitarity limits and
the quantum mechanical shadow term from a black disc.

Finally, in Sec.6, we argue that higher corrections, beyond the one-loop correction should simply replace, up to a constant factor, the $\ell n1/x$ factor found in the one-loop calculation by $\frac{1}{a}\ell nQ^2/\ell^2$ leading again to the semiclassical result.

2 Determining quark and gluon densities in the light-cone wavefunction

The key observation allowing one to measure and calculate quark and gluon distributions in the light-cone wavefunction is that these distributions are directly related to quark and gluon production in hard scattering reactions initiated by currents coupling to quarks and gluons. It is a familiar result in deep inelastic electron-proton scattering that the structure function $F_2$ gives a measure of the quark distributions

$$F_2(x, Q^2) = \sum_f e_f^2 [xq_f(x, Q^2) + x\bar{q}_f(x, Q^2)],$$

at least in a first-order QCD formalism. We are now searching a stronger result. In (1) the transverse momentum of the struck quark is integrated over the range $0 \leq \ell^2_\perp \leq Q^2$. What we now wish to determine is the quark and gluon distributions in a proton or nucleus for a definite value of the parton’s transverse momentum, $\ell_\perp$. These unintegrated quark and gluon distributions are determined by the cross section for producing quarks and gluons at a definite $\ell_\perp$ in a deep inelastic reaction. The idea is to choose the boundary conditions, the $\iota\epsilon$’s, of the light-cone gauge so that there are no final state interactions thus guaranteeing that the struck parton appears in the final state with unchanged momentum.

2.1 The quasi-classical approximation

This analysis has already been carried out in some detail in a quasi-classical calculation of gluon production off a large nucleus in a deep inelastic reaction initiated by the “current” $j = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$ which couples directly to gluons.

The result found for the spectrum of produced gluons, $\frac{dN}{d\ell^2}$, is most easily expressed in terms of the quantity
\[ \tilde{N}(x) = \int d^2 \ell e^{-i \ell \cdot x} \frac{dN}{d^2 \ell} \]  
which is given as \cite{3,4} \[ \tilde{N}(x) = \int d^2 b \frac{N_e^2 - 1}{\pi^2 \alpha N_c x^2} (1 - e^{-x^2 Q_s^2/4}). \]  
In (2) the saturation momentum \( Q_s \) is given by 
\[ Q_s^2 = \frac{8\pi^2 \alpha N_c}{N_c^2 - 1} \sqrt{R^2 - b^2} \rho xG(x, Q^2) \]  
where \( \rho \) is the nuclear density and \( xG \) is the gluon distribution for a nucleon with \( 1/x^2 = Q^2 \), the scale at which gluons are measured. \( b \) is the impact parameter of the current-nucleus interaction while \( R \) is the radius of the nucleus. Eq.2 is valid in a quasi-classical limit in which \( Q^2 \frac{\partial}{\partial Q^2} xG(x, Q^2) \) has neither \( Q^2 \)-dependence nor \( \ln 1/x \) factors at small \( x \). Two limits of (2) are noteworthy. (i) At small values of \( x^2 \) 
\[ \tilde{N}(x) \rightarrow x^2 \text{small} xG_A(x, x^2) \]  
reflecting independent scattering on the various nucleons of the nucleus. (ii) For very large \( R \) 
\[ \tilde{N}(x) \rightarrow R \text{large} \frac{N_e^2 - 1}{\pi \alpha N_c} \frac{R^2}{x^2} \]  
reflecting the saturation of the number of gluons per unit area. In momentum space, but neglecting the logarithmic \( x^2 \) dependence of \( Q_s^2 \), 
\[ \frac{dN}{d^2bd^2\ell} = \frac{N_e^2 - 1}{4\pi^3 \alpha N_c} \int_1^\infty \frac{dt}{t} e^{-tQ_s^2/4} \rightarrow t^2/Q_s^2 \ll 1 \frac{N_e^2 - 1}{4\pi^3 \alpha N_c} \ln Q_s^2/\ell^2. \]  
This interpretation of saturation is made sharper by noting that 
\[ - \int d^2 b < A_{\mu}^{1+}(b) A_{\mu}^{1+}(b + x) > = \pi \tilde{N}(x) \]  
so that (6) can be written as \cite{3} 
\[ - < A_{\mu}^{i+}(b) A_{\mu}^{i+}(b + x) > = \frac{(N_e^2 - 1)}{\pi \alpha N_c} \frac{1}{x^2} \]
showing that for gluons of transverse size $\Delta x_\perp$ the maximum value of $\Delta x_\perp^2 A^2_\mu$ is of order $1/\alpha$. This is gluon saturation. The $<>$ in (7) and (8) indicates averages in the light-cone wavefunction of the nucleus. We note that saturation is different than shadowing\[4\] since (4) indicates that there is no shadowing in the quasi-classical approximation. Finally, comparing (1) and (7) we note that the number density of gluons in the light-cone wavefunction as given by (7) is equivalent to the distribution of produced gluons in deep inelastic scattering.

In Ref.[4], detailed arguments were given that, with a proper choice of light-cone denominators, final state interactions are absent in light-cone gauge thus allowing gluon production, at a given transverse momentum, to directly reflect the transverse and longitudinal momentum distributions of gluons in the light-cone wavefunction. In Ref.[4], these arguments were given in the context of a quasi-classical (Weizsäcker-Williams) approximation, however the result appears to be more general as illustrated in Appendix A of this paper. In the next two sections of this paper we calculate first the produced quark distribution and then the produced gluon distribution in deep inelastic scatterings off a nucleus in the one-loop approximation. The results can then be identified with the quark and gluon distributions in a nucleus in the one-loop approximation.

3 Quark distributions

In this section, we calculate the produced quark distribution in deep inelastic scattering off a large nucleus. Though the calculation is done at the one-loop level we shall argue that the result is quite general and valid also for scattering off protons at very small values of $x$. The calculation we are about to perform is not so far different from what has previously been done\[2\] for the deep inelastic cross section. The new element which is added here is the determination of the transverse momentum of the leading quark which then gives the quark distribution in the light-cone wavefunction.

3.1 The lowest order

In order to set normalizations we begin by calculating deep inelastic scattering off a single nucleon and in the one-loop approximation. The relevant graphs are shown in Fig.1. We choose a frame where
Figure 1: Virtual Compton Scattering off a Nucleon in the One-quark loop Approximation.

\[ q_\mu = (q_+, q_-, q_\perp) = (-\frac{q^2}{2q_-}, q_-, 0) \]

and

\[ (q - \ell)_\mu = (\frac{\ell^2}{2(1 - z)q_-}, (1 - z)q_-, -\ell) \] (9)

and, in addition, we always suppose that \( q_- \gg Q^2/2q_- \) so that the scattering, say in a covariant gauge calculation, takes the form of the virtual (transverse) photon breaking up into a quark-antiquark pair which then scatters off the proton. The vertical lines running through the graphs of Fig.1 indicate that the imaging part of the forward Compton amplitude is taken. It is straightforward to write

\[ xq + x\bar{q} = \frac{2\alpha Q^2}{\pi} \sum_\lambda \int \frac{d^2\ell}{4\pi^2} dz \left[ z^2 + (1 - z)^2 \right] \]

\[ |\ell^\lambda \cdot \left( \frac{\ell}{\ell^2 + Q^2 z(1 - z)} - \frac{(\ell + k)}{(\ell + k)^2 + Q^2 z(1 - z)} \right) |^2 \cdot \frac{d^2 k}{k^2 |k^2|^2} \frac{\partial xG(x, k^2)}{\partial k^2} \] (10)

where

\[ \epsilon^\lambda_\mu = (\epsilon^\lambda_+, \epsilon^\lambda_-, \epsilon^\lambda_\perp) = (0, 0, \mathcal{E}^\lambda) \] (11)

is the polarization of the virtual photon while \( xG \) is the gluon distribution of the target nucleon. It is not difficult to see that (10) leads to the usual expression for the quark sea. In the logarithmic approximation \( k^2 \ll \ell^2 \) the term \( |^2 \) in (10) becomes
\[ \sum_{\lambda} | |^2 = \frac{\ell^2}{[\ell^2 + Q^2z(1-z)]^2} [k^2 - \frac{4(k \cdot \ell)^2 Q^2 z(1-z)}{[\ell^2 + Q^2z(1-z)]^2}], \]

which after angular averaging in \( k \) becomes

\[ \sum_{\lambda} | |^2 = \frac{k^2}{[\ell^2 + Q^2z(1-z)]^4} \{ (\ell^2)^2 + [Q^2z(1-z)]^2 \}, \]  

(13)

Again, in the logarithmic approximation \( \ell^2 << Q^2 \) and \( z << 1 \), with \( \ell^2 \) of the same size as \( Q^2z(1-z) \), so that using (13) one finds from (10)

\[ x(q(x, Q^2) + \bar{q}(x, Q^2)) = \frac{\alpha}{3\pi} \int_0^{Q^2} \frac{d\ell^2}{\ell^2} xG(x, \ell^2) \]

(14)

which is the correct leading logarithmic form of the DGLAP\[17, 18, 19\] equation thus confirming our normalization in (10).

From (10) we an get the differential distribution in transverse momentum as

\[ \frac{dxq(x, Q^2)}{d^2\ell} = \frac{\alpha Q^2}{2\pi^3} \sum_{\lambda} \int_0^1 dz [z^2 + (1-z)^2] \]

\[ |\ell^\lambda \cdot \left( \frac{\ell}{\ell^2 + Q^2z(1-z)} - \frac{\ell + k}{(\ell + k)^2 + Q^2z(1-z)} \right) |^2 \frac{d^2k}{k^2} \frac{\partial xG}{\partial R^2}. \]  

(15)

It is convenient to go from \( \ell \) to the conjugate coordinate \( x \) by using

\[ \frac{\ell \cdot \ell}{\ell^2 + Q^2z(1-z)} = \int \frac{d^2x}{4\pi} e^{-i\ell \cdot x} \]

\[ \frac{i \cdot \nabla x}{\sqrt{Q^2x^2z(1-z)}} K_0(\sqrt{Q^2x^2z(1-z)}) \]

(16)

which gives

\[ |\ell^\lambda \cdot \left( \frac{\ell}{\ell^2 + Q^2z(1-z)} - \frac{\ell + k}{(\ell + k)^2 + Q^2z(1-z)} \right) |^2 = \int \frac{d^2x_1}{16\pi^2} e^{-i\ell \cdot (x_1 - x_2)} (1 - e^{-i\ell \cdot x_1})(1 - e^{i\ell \cdot x_2}) \nabla x_1 K_0(\sqrt{Q^2x^2z(1-z)}) \nabla x_2 K_0(\sqrt{Q^2x^2z(1-z)}). \]

(17)

Thus,

\[ \frac{dxq}{d^2\ell} = \frac{\alpha Q^2}{2\pi^3} \int dz \frac{d^2x_1 d^2x_2}{16\pi^2} [(1 - e^{-i\ell \cdot x_1})(1 - e^{i\ell \cdot x_2}) - (1 - e^{-i\ell \cdot (x_1 - x_2)}]. \]
\[
d\frac{d^2 k}{k^2} \frac{\partial x G}{\partial k^2} \left[ z^2 + (1 - z)^2 \right] \nabla x_1 K_0(\sqrt{Q^2 x_1^2 z(1 - z)}) \cdot \nabla x_2 K_0(\sqrt{Q^2 x_2^2 z(1 - z)}) e^{-i\ell \cdot (x_1 - x_2)}. \tag{18}
\]

In the logarithmic approximation the exponential factors involving \( k \) in (18) can be expanded through second order with the resulting integration over \( k \) limited by the corresponding \( x \)-factor. Thus,

\[
\int (1 - e^{-k \cdot x_1}) d\frac{d^2 k}{k^2} \frac{\partial x G}{\partial k^2} = \frac{\pi}{4} x_1^2 x G(x, x_1^2) \tag{19}
\]

where \( x_1^2 G(x, x_1^2) \) is an abbreviation for \( x_1^2 G(x, Q^2 = 1/x_1^2) \), leading to

\[
\frac{dx q}{d\ell^2} = \frac{\alpha Q^2}{128\pi^4} \int d^2 x_1 d^2 x_2 dz \left[ x_1^2 x G(x, x_1^2) + x_2^2 x G(x, x_2^2) - (x_1 - x_2)^2 x G(x, (x_1 - x_2)^2) \right] \cdot \left[ z^2 + (1 - z)^2 \right] \nabla x_1 K_0 \cdot \nabla x_2 K_0 e^{-i\ell \cdot (x_1 - x_2)}. \tag{20}
\]

### 3.2 The quark distribution for a large nucleus

Now consider (20) as the single scattering approximation for quark production on a large nucleus. Introducing the nuclear density, \( \rho \), and the impact parameter corresponding to the nucleon in the nucleus, \( b \), we may write (20) as

\[
\frac{dx q}{d\ell^2} = \frac{Q^2 N_c}{64\pi^6} \int d^2 b d^2 x_1 d^2 x_2 \left[ \frac{x_1^2 \tilde{v}}{2\lambda} \sqrt{R^2 - b^2} C_F \frac{N_c}{N_c} + \frac{x_2^2 \tilde{v}}{2\lambda} \sqrt{R^2 - b^2} C_F \frac{N_c}{N_c} \right.
\]

\[
- \frac{(x_1 - x_2)^2}{2\lambda} \sqrt{R^2 - b^2} C_F \frac{N_c}{N_c}],
\]

\[
e^{-i\ell \cdot (x_1 - x_2)} \cdot [z^2 + (1 - z)^2] d\nabla x_1 K_0(\sqrt{Q^2 x_1^2 z(1 - z)}) \cdot \nabla x_2 K_0(\sqrt{Q^2 x_2^2 z(1 - z)}).
\tag{21}
\]

In arriving at (21) we have used \([20]\)

\[
\frac{x_i^2 \tilde{v}}{\lambda} = \frac{4\pi^2 \alpha N_c}{N_c^2 - 1} \rho x_i^2 x G(x, x_i^2)
\tag{22}
\]

where

\[
2 \int d^2 b \rho \sqrt{R^2 - b^2} = A
\tag{23}
\]
with $A$ the atomic number of the nucleus.

Now it is straightforward to allow the quark-antiquark pair coming from the virtual photon to scatter on an arbitrary number of nucleons in the nucleus. One simply makes the replacement

$$
\frac{x^2 \tilde{v}}{2\lambda} \sqrt{R^2 - b^2} \frac{C_F}{N_c} \rightarrow 1 - \exp\left[\frac{x^2 \tilde{v}}{2\lambda} \sqrt{R^2 - b^2} \frac{C_F}{N_c}\right]. \tag{24}
$$

Introducing the saturation momentum, $Q_s$, by

$$
Q_s^2 = \frac{2\tilde{v}}{\lambda} \sqrt{R^2 - b^2} \frac{C_F}{N_c}, \tag{25}
$$

one finds

$$
dx q = \frac{Q_s^2 N_c}{64\pi^6} \int d^2 b d^2 x_1 d^2 x_2 \left(1 + e^{-\bar{q}(\bar{x}_1 - \bar{x}_2)^2/4} - e^{-\bar{q}^2/4} - e^{-\bar{q}^2/4} - e^{-\bar{q}^2/4}\right)
\cdot \nabla_{x_1} K_0(\sqrt{Q_s^2 x_1^2 (1 - z)}) \cdot \nabla_{x_2} K_0(\sqrt{Q_s^2 x_2^2 z (1 - z)}).
\tag{26}
$$

We suppose $Q^2 >> Q_s^2$. Then the dominant contribution to (24) comes from the region $z << 1$. It is convenient to define a scaled variable $y = Q^2 z$ in terms of which

$$
dx q = \frac{N_c}{64\pi^6} \int d^2 x_1 d^2 x_2 \left(1 + e^{-\bar{q}(\bar{x}_1 - \bar{x}_2)^2/4} - e^{-\bar{q}^2/4} - e^{-\bar{q}^2/4} - e^{-\bar{q}^2/4}\right) e^{-i\ell \cdot (\bar{x}_1 - \bar{x}_2)}
\cdot dy \nabla_{x_1} K_0(\sqrt{Q_s^2 y}) \cdot \nabla_{x_2} K_0(\sqrt{Q_s^2 y}). \tag{27}
$$

It appears difficult to give a closed form for all the integrals in (27). However, it is rather simple to evaluate (27) either when $\ell^2 >> Q_s^2$ or when $\ell^2 << Q_s^2$. In these cases

$$
dx q = \frac{N_c}{64\pi^6} \frac{Q_s^2}{\ell^2} \text{ for } \ell^2 >> Q_s^2 \tag{28}
$$

and

$$
dx q = \frac{N_c}{2\pi^4} \text{ for } \ell^2 << Q_s^2. \tag{29}
$$

In arriving at (29) we have used

$$
\int d^2 x_1 d^2 x_2 e^{-i\ell \cdot (\bar{x}_1 - \bar{x}_2)} dy \nabla_{x_1} K_0(\sqrt{x_1^2 y}) \cdot \nabla_{x_2} K_0(\sqrt{x_2^2 y}) = 16\pi^2 \tag{30}
$$
\[ \int d^2x_1 d^2x_2 e^{-(x_1^2 - x_2^2)/Q^2} dy \nabla_{x_1} K_0(\sqrt{x_1^2 y}) \cdot \nabla_{x_2} K_0(\sqrt{x_2^2 y}) = 16\pi^2 \] (31)

while the other two terms in (27) are negligible when \( \ell^2 << Q_x^2 \). The integral (31) coming from the second term on the right-hand side of (27) corresponds to the multiple scattering of the observed quark by the medium in both the amplitude and in the complex conjugate amplitude. The integral (30) coming from the first term on the right-hand side of (27) is the shadow term for (31) corresponding to complete blackness of the scattering of the quark-antiquark pair on the nucleus in the region \( \ell^2 << Q_x^2 \).

Eq.(29) reflects saturation of the quark density in the nucleus for \( \ell^2 << Q_x^2 \). There is, up to a constant, one quark per unit phase space in the saturation limit. Unfortunately, we are unable to give a physical interpretation of the constant appearing on the right-hand side of (28). As will be discussed in more detail for the gluon case we believe that, except for the constant factor, (29) is completely general and that it does not depend on the one-loop approximation which we have used or on the fact that we have considered a large nucleus rather than a hadron with the quark at a very small value of \( x \).

### 4 Gluon distributions

Now we turn to determining the gluon distribution in our large nucleus. This has earlier been done in the quasi-classical approximation. Here, we do the calculation at the one-loop level, a calculation from which we shall be able to extract a general result in Sec.6. The calculation we are about to perform is related to that done some time ago [2], however, here we focus on the transverse momentum distribution of the leading gluon, a quantity related to the light-cone quantized distribution of gluons in the target.

#### 4.1 The lowest order

In order to set normalizations we begin with the lowest order calculation. We use the “current”

\[ j(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}_i \] (32)
which produces gluons off a nucleon at lowest order from the graphs illustrated in Fig.2. We parametrize $q$ and $\ell$ as in (9) while the polarizations of the produced gluons are written as

$$\epsilon^{\lambda_1}(q-\ell) = (\epsilon_{1+1}^{\lambda_1}, \epsilon_{1-1}^{\lambda_1}, \xi_1^{\lambda_1}) = \left( -\frac{\xi_1^{\lambda_1} \cdot \ell}{(q-\ell)_-}, 0, \xi_1^{\lambda_1} \right)$$

$$\epsilon^{\lambda_2}(\ell + k) = \left( \frac{\xi_2^{\lambda_2} \cdot (\ell + k)}{\ell_-}, 0, \xi_2^{\lambda_2} \right)$$

and for shortness of notation we shall often write $\xi_1^{\lambda_1} = \xi_1$ and $\xi_2^{\lambda_2} = \xi_2$.

In what follows we shall always work in alogarithmic approximation for longitudinal momentum so that we may assume $\ell_- << q_-$. The sum of the graphs shown in Fig.2 have already been evaluated in Ref.[2] with result, when $Q^2 >> \ell^2, (\ell + k)^2$ and when $\ell^2, (\ell + k)^2 << Q^2$

$$\Gamma^{\lambda_1 \lambda_2} = \xi_1^{\lambda_1} \cdot \ell \left[ \frac{\ell + k}{(\ell + k)^2} - \frac{\ell}{\ell^2} \right] \cdot \xi_2^{\lambda_2}$$

If the graphs of Fig.2 are interpreted as light-cone perturbation theory graphs, in contrast to Feynman graphs, and evaluated in $A_- = 0$ gauge then graph c is zero while the sum of graphs a and b is

$$\Gamma^{a}_{\lambda_1 \lambda_2} + \Gamma^{b}_{\lambda_1 \lambda_2} = \frac{\epsilon_{1}^{\cdot} \cdot (\ell + k) \epsilon_{2}^{\cdot} \cdot (\ell + k)}{(\ell + k)^2} - \frac{\epsilon_{1}^{\cdot} \cdot \ell \epsilon_{2}^{\cdot} \cdot \ell}{\ell^2}.$$
\[
\sum_{\lambda_1, \lambda_2} |\Gamma_{\lambda_1\lambda_2}|^2 = \frac{k^2}{(\ell + k)^2}
\]  
(37)

while

\[
\sum_{\lambda_1, \lambda_2} |\Gamma^a_{\lambda_1\lambda_2} + \Gamma^b_{\lambda_1\lambda_2}|^2 = 2\frac{\ell^2 k^2 - (\ell \cdot k)^2}{\ell^2(\ell + k)^2}
\]

(38)

so that (37) and (38) are identical when \(k^2/\ell^2 << 1\) after an angular average over directions of \(k\). Thus in what follows we consider only graphs (a) + (b), which terms have an interpretation as the scattering of a color neutral two-gluon system on the target.

For scattering of the current \(j\) off a single nucleon we again have four graphs exactly as in Fig.1 but with \(j\) replacing the electromagnetic current \(j_\mu\) and with the lines \(\ell, q - \ell, \cdots\) now referring to gluons rather than quarks. Then analogous to (10) one can write

\[
x_G(xQ^2) = \frac{4\alpha N_c}{\pi} \sum_{\lambda_1, \lambda_2} \int \frac{d^2 \ell}{4\pi^2} \frac{dz}{z} \int \frac{d^2 k}{4\pi^2} \frac{\partial}{\partial k^2} x_G(x, \ell^2)
\]

\[
\frac{1}{\ell^2} \epsilon_1 \cdot \ell \epsilon_2 \cdot \ell - \frac{\epsilon_1 \cdot (\ell + k) \epsilon_2 \cdot (\ell + k)}{(\ell + k)^2} 2\frac{d^2 k}{k^2} \frac{\partial x_G(x, k^2)}{\partial k^2}
\]

(39)

where, again, we work in an approximation where there are no loops in \(x_G(x_1 k^2)\) so that \(k^2 \frac{\partial}{\partial k^2} x_G(x, k^2)\) is a constant, both in \(\ell n1/x\) and in \(k^2\). The \(Q^2\) dependence on the left-hand side of (39) comes from a cutoff \(\ell^2 < Q^2\) which is understood. Using (38) in (39) one finds

\[
x_G(x, Q^2) = \int Q^2 \frac{\alpha N_c}{\pi} \frac{d\ell^2}{\ell^2} x_G(x, \ell^2)
\]

(40)

which is correct in the leading double logarithmic limit, thus checking our normalization in (39).

Going back to (39) one can write the differential distribution, the unintegrated gluon distribution, as

\[
\frac{dx_G}{d^2 \ell} = \frac{\alpha N_c}{\pi^3} \sum_{\lambda_1, \lambda_2} \int_{x \ell^2/Q^2} \frac{dz}{z} \frac{\epsilon_1 \cdot \ell \epsilon_2 \cdot \ell}{\ell^2} - \frac{\epsilon_1 \cdot (\ell + k) \epsilon_2 \cdot (\ell + k)}{(\ell + k)^2} 2\frac{d^2 k}{k^2} \frac{\partial x_G}{\partial k^2}
\]

(41)
We note that $\frac{dxG}{d^2\ell}$ is identical to what we called $\frac{dN}{d^2\ell}$ in Sec. 2, and in Ref. [4]. Writing

$$\mathbf{\ell} \cdot \mathbf{\epsilon}_2 - \frac{2\mathbf{\ell}_1 \cdot \mathbf{\ell} \mathbf{\epsilon}_2}{\ell^2} = -\int d^2 x \ e^{-i\mathbf{\ell} \cdot \mathbf{x}} \frac{1}{\pi x^2} \left( \mathbf{\ell}_1 \cdot \mathbf{\epsilon}_2 - \frac{2\mathbf{\ell}_1 \cdot \mathbf{x} \mathbf{\epsilon}_2 \cdot \mathbf{x}}{x^2} \right)$$

(42)

one finds

$$\frac{dxG}{d^2\ell} = \frac{\alpha N_c}{2\pi^3} \int \frac{dz}{z} \ \frac{d^2x_1d^2x_2}{x_1^2x_2^2} \ e^{-i\mathbf{\ell} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} \left( \frac{2(\mathbf{x}_1 \cdot \mathbf{x}_2)^2}{x_1^2x_2^2} - 1 \right)$$

$$\cdot \left( 1 + e^{-i\mathbf{\ell} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} - e^{-i\mathbf{\ell} \cdot \mathbf{x}_1} - e^{i\mathbf{\ell} \cdot \mathbf{x}_2} \right) \frac{d^2k}{k^2} \ \frac{\partial xG}{\partial k^2}.$$  

(43)

Duplicating the steps that led from (18) to (26) leads to

$$\frac{dxG}{d^2bd^2\ell} = \frac{N_c^2 - 1}{8\pi^6} \int \frac{dz}{z} \ \frac{d^2x_1d^2x_2}{x_1^2x_2^2} \ e^{i\mathbf{\ell} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} \left( \frac{2(\mathbf{x}_1 \cdot \mathbf{x}_2)^2}{x_1^2x_2^2} - 1 \right)$$

$$\left( 1 + e^{-(\mathbf{x}_1 - \mathbf{x}_2)^2Q_s^2/4} - e^{-\mathbf{x}_1^2Q_s^2/4} - e^{-\mathbf{x}_2^2Q_s^2/4} \right)$$

(44)

where now

$$Q_s^2 = \frac{2\tilde{v}}{\lambda} \sqrt{R^2 - b^2}$$

(45)

which is identical to (25) except for the absence of the $C_F/N_c$ factor and $\frac{dxG}{d^2\ell} \equiv \frac{dN}{d^2\ell}$ with $\frac{dN}{d^2\ell}$ as defined in Sec. 2.

Again, it is easy to evaluate (44) either when $\ell^2 >> Q_s^2$ or when $\ell^2 << Q_s^2$.

Thus,

$$\frac{dxG}{d^2bd^2\ell} = \frac{N_c^2 - 1}{4\pi^4} \ \frac{Q_s^2}{\ell^2} \int \frac{dz}{z} = \frac{N_c^2 - 1}{4\pi^4} \ \frac{\ell n 1/x \ Q_s^2}{\ell^2} \ \text{for} \ \ell^2 >> Q_s^2.$$  

(46)

When $\ell^2 << Q_s^2$ only the 1 and $e^{-(\mathbf{x}_1 - \mathbf{x}_2)^2Q_s^2}$ terms contribute to (44) exactly as happened in going from (27) to (29). Using

$$\int \frac{d^2x_1d^2x_2}{x_1^2x_2^2} \left( 2\frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{x_1^2x_2^2} - 1 \right) e^{i\mathbf{\ell} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} = \pi^2$$

(47)

and (see Appendix B).
we arrive at

\[ \int \frac{d^2x_1 d^2x_2}{x_1^2 x_2^2} (2 \frac{(x_1 \cdot x_2)^2}{x_1^2 x_2^2} - 1) e^{-Q_s^2(x_1 - x_2)^2/4} = \pi^2 \]  

we arrive at

\[ \frac{dxG}{d^2bd^2\ell} = \frac{N_c^2 - 1}{4\pi^4} \int_{xQ^2_{RM}/Q^2}^{Q^2/Q^2} \frac{dz}{z} \approx \frac{N_c^2 - 1}{4\pi^4} \ln \frac{1}{x} \]  

where the limits of the \( z \)-integration are given by assuming the transverse momentum of each of the gluons approaching the nuclear target is of order \( Q_s \) and by requiring the gluonic system have a coherence length \( \geq R \). \( M \) is the nucleon mass. As in the fermion case the integral (48) corresponds to absorption of the two-gluon state as it passes over the nucleus while the contribution (47) can be viewed as the quantum mechanical shadow of that absorption and, as usual, the shadow and the absorption terms are equal when the target is completely absorptive (black). The new element here is the longitudinal momentum integral, the \( \ln 1/x \) factor in (49). In Sec.6, we shall focus on what happens when the calculation is done beyond the one-loop level and what happens to the \( \ln 1/x \) factor in that case.

5 Interpreting the one-loop results

Now, however, let’s try to understand the significance of the rather simple results contained in (46) and (49) as well as in (28) and (29). Eqs.(28) and (46) of course are straightforward and represent a process which is hard enough so that only a single nucleon in each nucleus is effective. Thus, (28) and (29) give quark and gluon number densities which are, after integrating over the impact parameter \( b \), just \( A \) times production off an isolated nucleon. It is the region where \( \ell^2 \ll Q_s^2 \) which is more interesting. Refer for a moment to (27) where \( \frac{da_k}{d^2\ell} \) is given as a sum of four terms on the right-hand side of that equation. The coordinate \( \xi_1 \) refers to the transverse position of the observed quark in the amplitude while \( \xi_2 \) refers to the same quantity in the complex conjugate amplitude. The second term on the right-hand side of (27), the \( e^{-Q_s^2(x_1 - x_2)^2/4} \) term, corresponds to the S-matrix for the quark-antiquark pair coming from the virtual photon and interacting with nucleons in the nucleus, both elastic and inelastic interactions, as it passes over the nucleus. Only the interactions with the observed quark do not cancel between real and virtual (inelastic and elastic) reactions. The fact
that \((x_1 - x_2)^2 \leq 1/Q_s^2\) shows that typically the measured quark will have transverse momentum on the order of \(Q_s^2\). Thus the contribution when \(\ell^2 << Q_s^2\) is determined by the probability that a quark which gets many random “kicks” be found with relatively small transverse momentum. It is natural that \(d\sigma \propto d^2\ell\) the phase space be available to the quark. Thus, except for normalization this is purely a statistical problem for \(\ell^2 << Q_s^2\). The fact that \(\frac{d\sigma}{d^2\ell}\) is independent of both \(Q^2\) and \(Q_s^2\) comes from the fact that the quarks which dominate the process are those having transverse momentum on the order of \(Q_s\) before the quark-antiquark pair passes over the nucleus. Quarks having transverse momentum much greater than \(Q_s\), before reaching the nucleus, are not freed while passing over the nucleus while quarks having transverse momentum much less than \(Q_s\), before reaching the nucleus, are few in number and can be neglected. Since the total number of “effective” quarks is proportional to \(Q_s^2\) and distributed according to phase space the functional form, a constant, of (29) follows with one-half of that constant given by the second term on the right-hand side of (27).

Still in the region \(\ell^2 << Q_s^2\), the first term on the right-hand side of (27) corresponds to no scattering whatsoever of the quark-antiquark pair by the nucleons of the nucleus. It can be viewed as the quantum mechanical shadow of the term described just above. When a quark-antiquark pair having relative transverse momentum \(2\ell\) impinges on the nucleus, and if \(\ell^2 << Q_s^2\), this pair always interacts with the nucleus with the momentum of the quark and antiquark getting distorted far from \(\ell\). The destruction of this part of the wavefunction is accompanied by a “shadow” term where the quark again has momentum \(\ell\). This is the first term on the right-hand side of (27).

For the gluon-loop (44),(46) and (49) are direct analogies to (27), (28) and (29) with the new element being the longitudinal phase space of the gluons. When the “current” \(j\) breaks up into a gluon-gluon pair there is a large phase space for one of the gluons (the observed gluon) to carry almost all the current’s longitudinal momentum while the other gluon carries a small amount which, however, because of the vector nature of the gluon gives a logarithmic integral in the probability that the current break into a gluon-gluon pair. It might seem that this gives an arbitrarily large factor as \(\ell n1/x\) becomes large, but this is not quite so as we shall now see in the next section.

Finally, we note that when \(\ell^2 \leq Q_s^2\) \(\frac{d\sigma}{d\ell}\) and \(\frac{d\sigma}{d\ell}\) have an interpretation as quark and gluon densities in a light-cone wavefunction but they do not have the interpretation as quark and gluon distributions in terms of an operator
product expansion. This is most easily seen in the discussion of Sec.2 where $\frac{dN}{d\tau^2} \equiv \frac{d\sigma_G}{d\tau}$ has significant $A$-dependence, but where there is no shadowing whatsoever and no nontrivial $A$-dependence in terms coming from the operator product expansion.

6 What happens to the $\ell n 1/x$ factor?

In this section we shall argue that when higher quantum corrections are included the $\ell n1/x$ factor in (49) gets modified so as to lead to an expression essentially identical to (6). To see what happens to the $\ell n 1/x$ factor in (49) we must go beyond the simple small-$x$ dynamics we have used so far in our discussion. We continue to find it useful to imagine the scattering of the current $j$ on a large nucleus where we choose an unusual frame in order to explain, heuristically, what is the essential dynamics. Thus, choose the kinematics representing the system before the collision occurs, and illustrated in Fig.3, to be

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{Kinematics of the current-nucleus scattering just before the collision.}
\end{figure}

\begin{align}
q &= \left( -\frac{Q^2}{2q_-}, q_-, 0 \right) \\
k &= \left( \frac{k^2}{2zq_-}, zq_-, k \right) \\
p &= \left( p_+, \frac{m^2}{2p_+}, 0 \right)
\end{align}

(50) (51) (52)
with \( p \) the momentum per nucleon of the nucleus. With \( s = 2q_-p_+ \) fixed choose \( q_- \) large enough so that, for a given \( z \), \( k_-/k_+ \) is significantly larger than one but not too large. That is, take \( q_- = N_0 \cdot \frac{1}{z} \) with \( N_0 \) a fixed, and moderately large, number. The idea here is to put most of the longitudinal momentum into \( p \) leaving just enough in \( q \) so that the process may be viewed as a two-gluon system, \( q - k \) and \( k \), colliding with a highly evolved wavefunction of the nucleus. By restricting the longitudinal momentum of the left-moving two-gluon system coming initially from \( j \) we need not consider further evolution in that system, at least in a leading logarithmic approximation in longitudinal momenta. All logarithms except the \( dz/z \) integration, which is our focus, are included in the wavefunction of the nucleus. Now if \( z \) is decreased one must correspondingly increase \( q_- \) and decrease \( p_+ \) in order to keep \( q_- = N_0k/z \) and \( s = q_-p_+ \) fixed. By decreasing \( p_+ \) we limit the range of useful \( x \)-evolution in the nucleus and in so doing decrease the saturation momentum \( Q_s \). But we must guarantee that \( k^2 \leq Q_s^2 \) in order that a reaction occur with reasonable probability, and this determines the lower limit of \( z \) in the \( dz/z \) integration. Our task then is to determine the \( x \)-dependence of \( Q_s^2 \). In the quasi-classical approximation \( Q_s^2 \), given by (3) or (45), has no \( x \)-dependence. However, once we go beyond the quasi-classical approximation we expect an \( x \)-dependence. Indeed, (3) gives an \( x \)-dependence through \( xG(x, Q^2) \) and it is to the determination of the \( x \)-dependence of this quantity that we now turn.

For dynamics we use the fixed coupling BFKL equation which incorporates leading logarithmic \( x \)-evolution, including the small-\( x \) approximation to DGLAP evolution. Let \( xG(\mathbb{b}, x, Q^2) \) be the gluon number density for the nucleus at momentum fraction \( x \), at scale \( Q \), and at impact parameter \( b \). Our normalization is such that

\[
\int d^2b \ xG(\mathbb{b}, x, Q^2) = xG_A(x, Q^2) \tag{53}
\]

the normal gluon distribution of the nucleus. It is \( xG(\mathbb{b}, x, Q^2) \) that we expect to replace \( 2\sqrt{R^2 - b^2} \rho xG(x, Q^2) \) in (3) and (45). Then, in the BFKL approximation

\[
xG(\mathbb{b}, x, Q^2) = \alpha \int N_0(\mathbb{b}, \lambda, Q_0^2) e^{2\int \frac{\alpha N_0(\lambda)Y + \lambda n}{2\pi i} \frac{Q^2}{Q_0^2} d\lambda} \tag{54}
\]

where we expect \( N_0 \) to be slowly varying in \( \lambda \). In Eq.(54)
\[ \chi(\lambda) = \psi(1) - \frac{1}{2} \psi(\lambda) - \frac{1}{2} \psi(1 - \lambda) \]  

(55)

where the \( \lambda \)-integration goes along the imaginary axis, while \( Y = \ell \, n 1/x \). The saddle point of the \( \lambda \)-integration is determined by

\[ \chi'(\lambda_0) = -\frac{\ell n \, Q^2/Q_0^2}{2\alpha N_c} \]  

(56)

giving

\[ xG(b, x, Q^2) = \frac{\alpha N_0(b, \lambda_0, Q_0^2)}{\sqrt{4\alpha N_c \chi''(\lambda_0)Y}} \left( \frac{Q^2}{Q_0^2} \right)^{\lambda_0} e^{\frac{2\alpha N_c}{\pi} \chi(\lambda_0)Y}. \]  

(57)

We expect (57) to be valid so long as \( Q^2 >> Q_s^2 \) with \( Q_s^2 \) being determined by

\[ xG(b, x, Q_s^2) = \frac{c}{\alpha} \left( \frac{Q_s^2}{Q_0^2} \right) \]  

(58)

where \( xG(b, x, Q_s^2) \), approached from the perturbative regime \( Q^2 > Q_s^2 \) agrees with (49) when \( \ell^2 \rightarrow Q_s^2 \) from the lower momentum side, \( \ell^2 < Q_s^2 \). We allow \( c \) to have weak (logarithmic) \( x \) and \( Q_s^2 \) dependences. Using (57) and (58) one finds

\[ (1 - \lambda_0)\ell n \, Q_s^2/Q_0^2 = \ell n\{ \frac{N_0\alpha^2}{c\sqrt{4\alpha N_c\chi''Y}} \} + \frac{2\alpha N_c \chi(\lambda_0)}{\pi} Y. \]  

(59)

Using (56) in (58) gives

\[ [1 - \lambda_0 + \frac{\chi(\lambda_0)}{\chi'(\lambda_0)}] \ell n \, Q_s^2/Q_0 = \ell n\{ \frac{N_0\alpha^2}{c\sqrt{4\alpha N_c\chi''Y}} \}. \]  

(60)

The right-hand side of (60) is slowly varying in \( Y \) and in \( \lambda_0 \). Thus, for very large \( Y \), leading to very large \( Q_s^2 \), \( \lambda_0 \) is determined by

\[ 1 - \lambda_0 + \frac{\chi(\lambda_0)}{\chi'(\lambda_0)} = 0 \]  

(61)

a value of \( \lambda_0 \) which is not too far from \( \lambda = 1/x \) so that \( \ell n \, 1/x \) evolution dominates \( Q^2 \)-evolution indicating that our approach to the problem should be reasonable.
Turning to (59) and using the fact that the first term on the right-hand side of that equation is slowly varying in $Y$ one can determine that

$$\frac{1}{Q_s^2} \frac{dQ_s^2(Y)}{dY} = \frac{2\alpha N_c \chi(\lambda_0)}{\pi(1 - \lambda_0)}$$

(62)

giving the dependence of the saturation momentum on $Y$.

Now we are in a position to answer the question of what happens to the $\ln 1/x$ factor in (49) when higher quantum corrections are included. Let $Y = \ell n s/Q^2$ and let $Y(\ell)$ be that rapidity such $Q_s^2(Y(\ell)) = \ell^2$. Then the $z-$integral in (49) becomes

$$\int_{x_0}^{\ell^2/Q^2} \frac{dz}{z} = \ln 1/x_0 = \ell n \frac{Q_s^2(Y)}{Q_s^2(Y(\ell))}$$

(63)

where $\ln 1/x_0 = Y - Y(\ell)$ so that, using (63) one gets (49) to become

$$\frac{dxG}{d^2bd'^2\ell} = \frac{N_c^2 - 1}{4\pi^3\alpha N_c} \frac{1 - \lambda_0}{2\chi(\lambda_0)} \ell n \frac{Q_s^2(Y)}{\ell^2}$$

(64)

which, apart from the $\frac{1 - \lambda_0}{2\chi}$ factor, is identical to (6). We feel that the essential factors in (64), the $\frac{N_c^2 - 1}{4\pi^3\alpha N_c}$ and the $\ell n Q_s^2/\ell^2$ factors, are general results in QCD for the light-cone wavefunction. The overall constant in (64) we do not trust.

It is perhaps useful to consider carefully why we claim that (49) is general, except for the issue of the $\ell n 1/x$ factor which we have discussed in some detail in this section. To that end turn to (44). Except for the four terms in parentheses at the end of the right-hand side of (44) all the other factors reflect the current $j$ breaking up into a gluon-gluon pair along with the Fourier transform going from transverse coordinate to transverse momentum space. Thus all the dynamics of the target is in the last factor. Of the four terms constituting the last factor the first, the 1, corresponds to no interactions of the gluon pair with the target and this term is universal and independent of the nature of the target. The third and fourth term correspond to possible interactions only in the amplitude and complex conjugate amplitudes, respectively. These two terms are functions only of $x_1^2$ and $x_2^2$, respectively and so must generally integrate to zero when $\ell^2 \ll Q_s^2$. The second term corresponding to $S(\underline{x}_1) S^*(\underline{x}_2)$, the product of the $S-$matrices in the amplitude and complex conjugate amplitude, need not in general take the form given in (44). This term in general must be 1 when $\underline{x}_1 = \underline{x}_2$ and it should be small when $(\underline{x}_1 - \underline{x}_2)^2 Q_s^2 >> 1$, but we have no argument as
to the exponential form of (44) being exact. It is this specific form which gives (47) and (48) the same value. Physically, the exponential form in (44) came from the many independent scatterers in the nucleus. While this is also natural in our more general circumstance we do not know how to prove it. In any case, the expectation that the term in question go to zero when \((x_1 - x_2)^2 Q_s^2 >> 1\) immediately gives a contribution which is \(\ell\)-independent when \(\ell^2 << Q^2\) since the \(\ell\)-dependence only comes in through \(e^{-i\ell \cdot (x_1 - x_2)}\). The real issue then is what constant replaces the \(\pi^2\) on the right-hand side of (48). We have written (64) as if that constant remains \(\pi^2\) while it may possibly be some other pure number. Thus the form given in (64) we feel must be true, but there may be an additional constant multiplying the right-hand side of that equation.

**Appendix A**

In this appendix we illustrate how, with a proper choice of \(i\epsilon\)'s in light-cone denominators, final state interactions can be suppressed. The example given here is similar to that given some time ago [21] although now we have a better physical interpretation of what is happening.

Consider the graph shown in Fig.4 where there is a final state interaction of the gluon \((k)\) with the struck quark. In light-cone gauge only the term \(\frac{i}{\ell_+ + i\epsilon} \frac{\eta_\mu k_\mu}{k_+} \) is important in the gluon propagator. \(p\) has a large + component of the momentum, and we suppose \(q\) has only + and \(-\) components with \(2q_+q_- = -Q^2\). We may assume that all lines in the upper blob have + components of their momentum greater or equal to \((\ell - k - q)_+ = xp_+\) while \(|(\ell - q)_-| << q_-\). Thus \((\ell - k - q)^2 \approx -(\ell - k)^2\) and \(k^2 \approx -k_+^2\) so that one need only consider \((\ell - k)^2 + i\epsilon\) and \(k_+\) as denominators possibly trapping the \(k_+\)-contour and thus limiting \(k_+\) to small values. It is only in case the \(k_+\)-contour is trapped at a value where \(|k_+| \leq \frac{k_+^2}{\ell_+}\) that final state interactions are important. Now \((k - \ell)^2 + i\epsilon \approx -2\ell_-( (k - \ell)_+ + (\frac{(k - \ell)^2}{2\ell_+} - i\epsilon)\). Thus, if the light-cone denominator is taken to be \([k_+ - i\epsilon]^{-1}\) there is no trapping of the \(k_+\) contour while any other choice leads to trapping. This choice of \(i\epsilon\) corresponds to the gauge potential extending to large negative \(x_-\) - values, but not to large positive \(x_-\) - values[4, 3], thus naturally avoiding final state interactions. It is straightforward to include additional gluons connecting to the struck quark. What is not so clear is whether or not there is a consistent definition of light-cone denominators which renders higher loops finite and
at the same time eliminates the gauge field for large positive values of $x_-$. This is an important technical problem yet to be resolved.

Figure 4: Potential final-state interaction which is absent with appropriate choice of boundary conditions for the light-cone gauge propagator.

**Appendix B**

In this appendix we outline how the integral in (48) can be evaluated.

$$I = \int \frac{d^2 x_1 d^2 x_2}{x_1^2 x_2^2} \left( 2 \frac{(x_1 \cdot x_2)^2}{x_1^2 x_2^2} - 1 \right) e^{-\left(x_1 - x_2\right)^2 Q_s^2/4}, \quad (B1)$$

then

$$I = \frac{1}{2} \int \frac{d^2 x_1 d^2 x_2}{(x_1^2)^2 (x_2^2)^2} \left[ 16 \frac{\partial^2}{\partial (Q_s^2)^2} + 8(x_1^2 + x_2^2) \frac{\partial}{\partial Q_s^2} + (x_1^2)^2 + (x_2^2)^2 \right] e^{-\left(x_1 - x_2\right)^2 Q_s^2/4} \quad (B2)$$

Using

$$\frac{1}{2\pi} \int_{0}^{2\pi} d\phi e^{\frac{1}{2}x_1 x_2 \cos \phi} Q_s^2 = I_0 \left( \frac{1}{2} Q_s^2 x_1 x_2 \right). \quad (B3)$$

One finds

$$I = \pi^2 \int_{0}^{\infty} \frac{dx_1^2 dx_2^2}{x_1^2 x_2^2} I_2 \left( \frac{1}{2} Q_s^2 x_1 x_2 \right) e^{-\left(x_1^2 + x_2^2\right) Q_s^2/4}. \quad (B4)$$

Now (See formula 19 on page 197, Ref. [22]).

$$\int_{0}^{\infty} \frac{dx_2^2}{x_2^2} I_2 \left( \frac{1}{2} Q_s^2 x_1 x_2 \right) e^{-x_2^2 Q_s^2/4} = \frac{1}{Q_s^2 x_1^2} e^{x_1^2 Q_s^2/4} \gamma(2, Q_s^2 x_1^2/4)$$
where $\gamma(\alpha, x)$ is the incomplete $\gamma-$function. We find

$$I = \pi^2 \int_0^\infty \frac{dz}{z^2} \gamma(2, z) = \pi^2. \quad (B5)$$

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