Spectral conditions for graph rigidity in the Euclidean plane

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Abstract

Rigidity is the property of a structure that does not flex. It is well studied in discrete geometry and mechanics and has applications in material science, engineering and biological sciences. A bar-and-joint framework is a pair $(G, p)$ of graph $G$ together with a map $p$ of the vertices of $G$ into the Euclidean $d$-space. We view the edges of $(G, p)$ as bars and the vertices as universal joints. The vertices can move continuously as long as the distances between pairs of adjacent vertices are preserved. The framework is rigid if any such motion preserves the distances between all pairs of vertices. In 1970, Laman obtained a combinatorial characterization of rigid graphs in the Euclidean plane. In 1982, Lovász and Yemini discovered a new characterization and proved that every 6-connected graph is rigid in the Euclidean plane. Consequently, if Fiedler’s algebraic connectivity is at least 6, then $G$ is rigid. In this paper, we show that if $G$ has minimum degree $\delta \geq 6$ and algebraic connectivity greater than $2 + \frac{1}{\delta-1}$, then $G$ is rigid. We prove a more general result giving a necessary spectral condition for packing $k$ edge-disjoint spanning rigid subgraphs. The same condition implies that a graph contains $k$ edge-disjoint 2-connected spanning subgraphs. This result extends previous spectral conditions for packing edge-disjoint spanning trees.

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Key words: eigenvalue, algebraic connectivity, rigid graph, connectivity

1 Introduction

In this paper, we consider finite undirected simple graphs. Throughout the paper, $k$ denotes a positive integer and $G$ denotes a simple graph with vertex set $V(G)$ and edge set $E(G)$.

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Rigidity is the property of a structure that does not flex. It is well studied in discrete geometry and mechanics [1, 20, 22, 25] and has applications in material science, engineering and biological sciences (see [6, 8, 26] for example). A \textit{d-dimensional framework} is a pair \((G, p)\), where \(G\) is a graph and \(p\) is a map from \(V(G)\) to \(\mathbb{R}^d\). Roughly speaking, it is a straight line realization of \(G\) in \(\mathbb{R}^d\). Two frameworks \((G, p)\) and \((G, q)\) are \textit{equivalent} if \(||p(u)−p(v)|| = ||q(u)−q(v)||\) holds for every edge \(uv \in E(G)\), where \(||\cdot||\) denotes the Euclidean norm in \(\mathbb{R}^d\). Two frameworks \((G, p)\) and \((G, q)\) are \textit{congruent} if \(||p(u)−p(v)|| = ||q(u)−q(v)||\) holds for every \(u, v \in V(G)\). A framework \((G, p)\) is \textit{generic} if the coordinates of its points are algebraically independent over the rationals. The framework \((G, p)\) is \textit{rigid} if there exists an \(\varepsilon > 0\) such that if \((G, p)\) is equivalent to \((G, q)\) and \(||p(u)−q(u)|| < \varepsilon\) for every \(u \in V(G)\), then \((G, p)\) is congruent to \((G, q)\). A generic realization of \(G\) is rigid in \(\mathbb{R}^d\) if and only if every generic realization of \(G\) is rigid in \(\mathbb{R}^d\). Hence the generic rigidity can be considered as a property of the underlying graph. A graph is called \textit{rigid} in \(\mathbb{R}^d\) if every/some generic realization of \(G\) is rigid in \(\mathbb{R}^d\).

Laman [20] obtained the following combinatorial characterization of rigid graphs in \(\mathbb{R}^2\). For a subset \(X \subseteq V(G)\), let \(G[X]\) be the subgraph of \(G\) induced by \(X\) and \(E(X)\) denote the edge set of \(G[X]\). A graph \(G\) is \textit{sparse} if \(|E(X)| \leq 2|X|−3\) for every \(X \subseteq V(G)\) with \(|X| \geq 2\). By definition, any sparse graph is simple. If in addition \(|E(G)| = 2|V(G)|−3\), then \(G\) is called \((2, 3)\)-tight or \textit{minimally rigid} or \textit{Laman graph}. Laman [20] proved that a graph \(G\) is rigid if \(G\) contains a spanning \((2, 3)\)-tight subgraph. Lovász and Yemini [22] gave a new proof of Laman’s result using matroid theory and showed that any 6-connected graph is rigid. These authors also constructed infinitely many 5-connected graphs that are not rigid. Lovász and Yemini also obtained a useful characterization for rigidity to determine the rank function of rigidity matroid of a graph. In Section 2, we give more details about rigid graphs (see also [3, 11, 16, 18, 22]).

This paper focuses on the study of rigid graphs from spectral graph theory viewpoint. We describe the matrices and the eigenvalues of our interest below. If \(G\) is an undirected simple graph with \(V(G) = \{v_1, v_2, \cdots, v_n\}\), its \textit{adjacency matrix} is the \(n\) by \(n\) matrix \(A(G)\) with entries \(a_{ij} = 1\) if there is an edge between \(v_i\) and \(v_j\) and \(a_{ij} = 0\) otherwise, for \(1 \leq i, j \leq n\). Let \(D(G) = (d_{ij})_{1 \leq i, j \leq n}\) be the \textit{degree matrix} of \(G\), that is, the \(n\) by \(n\) diagonal matrix with \(d_{ii}\) being the degree of vertex \(v_i\) in \(G\) for \(1 \leq i \leq n\). The matrices \(L(G) = D(G) − A(G)\) and \(Q(G) = D(G) + A(G)\) are called the \textit{Laplacian matrix} and the \textit{signless Laplacian matrix} of \(G\), respectively. For \(1 \leq i \leq n\), we use \(\lambda_i(G)\) and \(q_i(G)\) to denote the \(i\)-th largest eigenvalue of \(A(G)\) and \(Q(G)\), respectively. Also, \(\mu_i(G)\) denotes the \(i\)-th smallest eigenvalue of \(L(G)\). It is not difficult to see that \(\mu_1(G) = 0\). The second smallest eigenvalue of \(L(G)\), \(\mu_2(G)\), is known
as the algebraic connectivity of $G$.

Fiedler [7] proved that the vertex-connectivity of $G$ is at least $\mu_2(G)$. Thus, the theorem of Lovász and Yemini [22] that every 6-connected graph is rigid implies that if $\mu_2(G) \geq 6$, then $G$ is rigid. In this paper, we will improve this sufficient condition to “$\mu_2(G) > 2 + \frac{1}{6-1}$”, as stated in Corollary 1.4. Actually we obtain stronger and more general sufficient spectral conditions for packing edge-disjoint spanning rigid subgraphs in Theorem 1.1 and Corollary 1.2.

**Theorem 1.1.** Let $G$ be a graph with minimum degree $\delta(G) \geq 6k$. If

1. $\mu_2(G) > \frac{6k - 1}{\delta(G) + 1}$,
2. $\mu_2(G - u) > \frac{4k - 1}{\delta(G - u) + 1}$ for every $u \in V(G)$, and
3. $\mu_2(G - v - w) > \frac{2k - 1}{\delta(G - v - w) + 1}$ for every $v, w \in V(G)$,

then $G$ contains at least $k$ edge-disjoint spanning rigid subgraphs.

Theorem 1.1 has the following weaker, but neater corollary.

**Corollary 1.2.** Let $G$ be a graph with minimum degree $\delta \geq 6k$. If

$$\mu_2(G) > 2 + \frac{2k - 1}{\delta - 1},$$

then $G$ contains at least $k$ edge-disjoint spanning rigid subgraphs.

When $k = 1$, we obtain the following sufficient spectral conditions for a graph to be rigid.

**Corollary 1.3.** Let $G$ be a graph with minimum degree $\delta(G) \geq 6$. If

1. $\mu_2(G) > \frac{5}{\delta(G) + 1}$,
2. $\mu_2(G - u) > \frac{3}{\delta(G - u) + 1}$ for every $u \in V(G)$, and
3. $\mu_2(G - v - w) > \frac{1}{\delta(G - v - w) + 1}$ for every $v, w \in V(G)$,

then $G$ is rigid.

This result is similar in spirit and motivated by the work of Jackson and Jordán [16], in which they proved that a simple graph $G$ is rigid if $G$ is 6-edge-connected, $G - v$ is 4-edge-connected for every $v \in V(G)$ and $G - \{u, v\}$ is 2-edge-connected for every $u, v \in V(G)$. Corollary 1.3 involves several conditions and we can show that the condition “$\mu_2(G) > \frac{5}{\delta(G) + 1}$” is essentially best possible. A family of examples will be constructed in the last section of the paper.

As before, we can also obtain the following weaker, but easier to state and verify condition for a graph to be rigid.
Corollary 1.4. Let $G$ be a graph with minimum degree $\delta \geq 6$. If

$$\mu_2(G) > 2 + \frac{1}{\delta - 1},$$

then $G$ is rigid.

Corollary 1.4 gives a simple spectral condition for rigidity, but we do not know if it is best possible. It would be interesting to see how large can $\mu_2(G)$ for non-rigid graphs. Another problem of interest would be obtain a spectral condition for a graph to contain a spanning $(a,b)$-tight subgraph for other values of $a$ and $b$.

By Courant-Weyl inequalities (on page 29 of [2]), it is not hard to see that $\mu_2 + \lambda_2 \geq \delta$ and $\delta + \lambda_2 \leq q_2$. Thus all results involving $\mu_2$ in the paper will imply sufficient conditions involving $\lambda_2$ and $q_2$. For example, by Corollary 1.4, it follows that if $\lambda_2(G) < \delta - 2 - \frac{1}{\delta - 1}$, then $G$ is rigid. Similarly, if $q_2(G) < 2\delta - 2 - \frac{1}{\delta - 1}$, then $G$ is rigid. If $G$ is a connected $d$-regular graph, then let $\lambda = \max_{2 \leq i \leq n} |\lambda_i| = \max\{|\lambda_2|, |\lambda_n|\}$. It is known that a $d$-regular graph on $n$ vertices with small $\lambda$ has edge distribution similar to the random graph of same edge density, namely it is a pseudo-random graph (see [19] for more details). By the above remark, the results in this paper give spectral conditions for the rigidity of pseudo-random graphs.

Since every rigid graph with at least 3 vertices is 2-connected, by Corollary 1.2, we immediately have the following result on packing edge-disjoint 2-connected spanning subgraphs. This result can be seen as a spectral analogue of Jordán’s combinatorial sufficient condition [18] for packing edge-disjoint 2-connected spanning subgraphs. It also extends the spectral conditions for vertex-connectivity of [4,7,19], and the spectral conditions for packing connected subgraphs (edge-disjoint spanning trees) of [4,5,10,12,21], to packing edge-disjoint 2-connected spanning subgraphs.

Corollary 1.5. Let $G$ be a graph with minimum degree $\delta \geq 6k$. If

$$\mu_2(G) > 2 + \frac{2k - 1}{\delta - 1},$$

then $G$ has at least $k$ edge-disjoint 2-connected spanning subgraphs.

In Sections 2 and 3, we present our basic tools involving rigid subgraph packing theorems and eigenvalue interlacing. The proof of the main result will be presented in Section 4. A family of examples will be constructed in Section 5 to show the best possible bound of $\mu_2(G)$ in Corollary 1.3.
2 Packing rigid subgraphs

Lovász and Yemini [22] proved that a graph $G$ is rigid iff $\sum_{X \in \mathcal{G}} (2|V(X)| - 3) \geq 2|V| - 3$ for every collection $\mathcal{G}$ of induced subgraphs of $G$ whose edges partition $E(G)$. Packing spanning rigid subgraphs has been studied in several papers (see [3, 11, 18] for example). Jordán [18] showed that every $6k$-connected graph contains $k$ edge-disjoint spanning rigid subgraphs. Cheriyan, Durand de Gevigney and Szigeti [3] proved that a simple graph $G$ contains $k$ edge-disjoint spanning rigid subgraphs if $G - Z$ is $(6k - 2|Z|)$-edge-connected for every $Z \subset V(G)$. Motivated by this work and the spanning tree packing theorem of Nash-Williams [23] and Tutte [24], the second author [11] has recently obtained a partition condition for packing spanning rigid subgraphs which we describe below. For any partition $\pi$ of $V(G)$, $e_G(\pi)$ denotes the number of edges of $G$ whose ends lie in two different parts of $\pi$. A part of $\pi$ is trivial if it consists of a single vertex. Let $Z \subset V(G)$ and $\pi$ be a partition of $V(G - Z)$ with $n_0$ trivial parts $u_1, u_2, \ldots, u_{n_0}$. We define $n_Z(\pi)$ to be $\sum_{1 \leq i \leq n_0} |Z_i|$ where $Z_i$ is the set of vertices in $Z$ that are adjacent to $u_i$ for $1 \leq i \leq n_0$. If $Z = \emptyset$, then $n_Z(\pi) = 0$.

**Theorem 2.1** (Gu [11]). A graph $G$ contains $k$ edge-disjoint spanning rigid subgraphs if for every $Z \subset V(G)$ and every partition $\pi$ of $V(G - Z)$ with $n_0$ trivial parts and $n'_0$ nontrivial parts, $e_{G-Z}(\pi) \geq k(3 - |Z|)n'_0 + 2kn_0 - 3k - n_Z(\pi)$.

Note that the condition above is always true when $|Z| \geq 3$ (see the proof of Theorem 4.1 in Section 4 for details).

3 Eigenvalue Interlacing

In this section, we present some of the eigenvalue interlacing results that we will use in the next section to prove our main results. For a square matrix $M$ with real eigenvalues, $tr(M)$ denotes the trace of $M$ and $\theta_i(M)$ denotes the $i$-th largest eigenvalue of $M$.

Given two sequences of real numbers $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$, the second sequence is said to interlace the first one if $\xi_i \geq \eta_i \geq \xi_{n-m+i}$, for every $1 \leq i \leq m$. When we say the eigenvalues of a matrix $B$ interlace the eigenvalues of a matrix $A$, it means the non-increasing eigenvalue sequence of $B$ interlaces that of $A$.

**Theorem 3.1** (Cauchy Interlacing, Corollary 2.5.2 in [2]). Let $A$ be a real symmetric matrix and $B$ be a principal submatrix of $A$. Then the eigenvalues of $B$ interlace the eigenvalues of $A$. 

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Fiedler [7] applied Cauchy interlacing to the Laplacian matrix and obtained the following result (see also [2, Secton 1.7] and [9, Thm. 13.5.1]).

**Theorem 3.2** (Fiedler [7]). If $S$ is a subset of vertices of the graph $G$, then $\mu_2(G) \leq \mu_2(G - S) + |S|$.

Given a partition $\pi = \{X_1, X_2, \cdots, X_s\}$ of the set $\{1, 2, \cdots, n\}$ and a matrix $A$ whose rows and columns are labeled with elements in $\{1, 2, \cdots, n\}$, $A$ can be expressed as the following partitioned matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1s} \\ \vdots & \ddots & \vdots \\ A_{s1} & \cdots & A_{ss} \end{bmatrix}$$

with respect to $\pi$. The **quotient matrix** $A_\pi$ of $A$ with respect to $\pi$ is the $s$ by $s$ matrix $(b_{ij})_{1\leq i,j\leq s}$ such that each entry $b_{ij}$ is the average row sum of $A_{ij}$. Haemers [13] obtained the following interlacing result in his Ph.D. thesis (see also [14]).

**Theorem 3.3** (Haemers [13]). The eigenvalues of any quotient matrix of a real symmetric matrix $A$ interlace the eigenvalues of $A$.

It is not hard to obtain the following corollary.

**Corollary 3.4** (Hong, Gu, Lai and Liu [15]). Suppose that $G$ is a simple graph and $\pi$ is a partition of $V(G)$ with $|\pi| = s$. Let $L_\pi$ be the quotient matrix of $L(G)$ with respect to $\pi$. Then $\mu_2 \leq \theta_{s-1}(L_\pi)$.

**Proof.** By Theorem 3.3, $\theta_{s-1}(L_\pi) \geq \theta_{n-s+(s-1)}(L(G))$ which implies that $\theta_{s-1}(L_\pi) \geq \mu_2(G)$. \hfill $\square$

For any subset $U \subset V(G)$, $\partial_G(U)$ or simply $\partial(U)$ denotes the set of edges in $G$, each of which has one end in $U$ and the other end in $V(G) \setminus U$. Liu, Hong, Gu and Lai [21, Lemma 3.2] used eigenvalue interlacing to obtain the following result.

**Lemma 3.5** (Liu, Hong, Gu and Lai [21]). Suppose that $X, Y \subset V(G)$ with $X \cap Y = \emptyset$. Let $e(X,Y)$ denote the number of edges with one end in $X$ and the other in $Y$. If $\mu_2(G) \geq \max\{|\partial(X)|, |\partial(Y)|\}$, then $[e(X,Y)]^2 \geq |X| \cdot |Y| \cdot \left(\mu_2(G) - \frac{|\partial(X)|}{|X|}\right) \cdot \left(\mu_2(G) - \frac{|\partial(Y)|}{|Y|}\right)$.

The following combinatorial lemma is not hard to obtained, but see [10,12] for a proof.

**Lemma 3.6.** Let $G$ be a graph with minimum degree $\delta$ and $U$ be a non-empty proper subset of $V(G)$. If $|\partial(U)| \leq \delta - 1$, then $|U| \geq \delta + 1$.
The proofs of main results

In this section, we present the proofs of Theorem 1.1 and Corollary 1.2. We first restate Theorem 1.1 as below and present its proof.

**Theorem 4.1.** Let $G$ be a graph with minimum degree $\delta(G) \geq 6k$. If

$$\mu_2(G - Z) > \frac{6k - 2k|Z| - 1}{\delta(G - Z) + 1},$$

for every $Z \subset V(G)$ with $|Z| \leq 2$, then $G$ has at least $k$ edge-disjoint spanning rigid subgraphs.

**Proof.** By Theorem 2.1, it suffices to show that for any partition $\pi$ of $V(G - Z)$ with $n_0$ trivial parts and $n_0'$ nontrivial parts,

$$e_{G - Z}(\pi) \geq k(3 - |Z|)n_0' + 2kn_0 - 3k - n_Z(\pi),$$

for every $Z \subset V(G)$.

We first prove that if $|Z| \geq 3$, then (1) is always true. Actually, for every trivial part (a single vertex) $u_j$, its degree $d(u_j)$ in $G - Z$ must satisfy the inequality $d(u_j) \geq \delta - |Z_j| \geq 6k - |Z_j|$, where $Z_j$ is the set of neighbors of $u_j$ in $Z$. Recall that $n_Z(\pi) = \sum_{1 \leq j \leq n_0} |Z_j|$. If $|Z| \geq 3$, then

$$e_{G - Z}(\pi) \geq \frac{1}{2} \sum_{1 \leq j \leq n_0} d(u_j)$$

$$\geq \frac{1}{2} \sum_{1 \leq j \leq n_0} \delta - \frac{1}{2} \sum_{1 \leq j \leq n_0} |Z_j|$$

$$\geq 3kn_0 - \frac{1}{2} n_Z(\pi)$$

$$\geq k(3 - |Z|)n_0' + 2kn_0 - 3k - n_Z(\pi).$$

We assume that $|Z| \leq 2$ from now on. If $V_1, V_2, \ldots, V_{n_0'}$ are the nontrivial parts in the partition $\pi$ of $G - Z$ and $u_1, u_2, \ldots, u_{n_0}$ are the trivial parts of $\pi$, then

$$\sum_{1 \leq j \leq n_0} d(u_j) \geq \sum_{1 \leq j \leq n_0} (\delta - |Z_j|) \geq 6kn_0 - n_Z(\pi).$$

(2)

Without loss of generality, we may assume that $|\partial(V_1)| \leq |\partial(V_2)| \leq \cdots \leq |\partial(V_{n_0'})|$ (Here $\partial$ means $\partial_{G - Z}$). If $|\partial(V_2)| \geq 6k - 2k|Z|$, then

$$e_{G - Z}(\pi) = \frac{1}{2} \left( \sum_{1 \leq i \leq n_0'} |\partial(V_i)| + \sum_{1 \leq j \leq n_0} d(u_j) \right)$$

$$\geq \frac{1}{2} \left( (6k - 2k|Z|)(n_0' - 1) + 6kn_0 - n_Z(\pi) \right)$$

$$\geq k(3 - |Z|)n_0' + 2kn_0 - 3k - n_Z(\pi).$$
done. Thus, we assume that $|\partial(V_2)| \leq 6k - 2k|Z| - 1$.

Let $q$ be the largest index such that $|\partial(V_q)| \leq 6k - 2k|Z| - 1$. Then $2 \leq q \leq n'_0$. Therefore,

$$|\partial(V_i)| \geq 6k - 2k|Z|, \text{ for } q < i \leq n'_0, \quad (3)$$

whenever such an $i$ exists.

For $1 \leq i \leq q$, since $|\partial(V_i)| \leq 6k - 2k|Z| - 1 \leq \delta - 2k|Z| - 1 \leq \delta(G - Z) - 1$, Lemma 3.6 implies that $|V_i| \geq \delta(G - Z) + 1$. As $\mu_2(G - Z) > \frac{6k - 2k|Z| - 1}{\delta(G - Z) + 1}$, it follows that $|V_i|\mu_2(G - Z) > 6k - 2k|Z| - 1$ for $1 \leq i \leq q$. By Lemma 3.5, for $2 \leq i \leq q$,

$$e(V_1, V_i)^2 \geq |V||V_i|\left(\frac{\mu_2(G - Z) - |\partial(V_i)|}{|V_i|}\right)\left(\frac{\mu_2(G - Z) - |\partial(V_i)|}{|V_i|}\right)$$

$$= (|V_i|\mu_2(G - Z) - |\partial(V_i)|) (|V_i|\mu_2(G - Z) - |\partial(V_i)|)$$

$$> (6k - 2k|Z| - 1 - |\partial(V_i)|)(6k - 2k|Z| - 1 - |\partial(V_i)|)$$

$$\geq (6k - 2k|Z| - 1 - |\partial(V_i)|)^2.$$ 

Thus $e(V_1, V_i) > 6k - 2k|Z| - 1 - |\partial(V_i)|$, and so $e(V_1, V_i) \geq 6k - 2k|Z| - |\partial(V_i)|$. We get that

$$|\partial(V_i)| \geq \sum_{2 \leq i \leq q} e(V_1, V_i) \geq (6k - 2k|Z|)(q - 1) - \sum_{2 \leq i \leq q} |\partial(V_i)|,$$

and thus

$$\sum_{1 \leq i \leq q} |\partial(V_i)| = |\partial(V_1)| + \sum_{2 \leq i \leq q} |\partial(V_i)| \geq (6k - 2k|Z|)(q - 1). \quad (4)$$

Using (2), (3) and (4), we obtain that

$$e_{G-Z}(\pi) = \frac{1}{2}\left(\sum_{1 \leq i \leq n'_0} |\partial(V_i)| + \sum_{1 \leq j \leq n_0} d(u_j)\right)$$

$$= \frac{1}{2}\left(\sum_{1 \leq i \leq q} |\partial(V_i)| + \sum_{q < i \leq n'_0} |\partial(V_i)| + \sum_{1 \leq j \leq n_0} d(u_j)\right)$$

$$\geq \frac{1}{2}\left((6k - 2k|Z|)(q - 1) + (6k - 2k|Z|)(n'_0 - q) + 6kn_0 - n_Z(\pi)\right)$$

$$\geq k(3 - |Z|)n'_0 + 2kn_0 - 3k - n_Z(\pi),$$

which completes the proof. 

Corollary 1.2 follows directly from Theorem 1.1 and the following lemma.

**Lemma 4.2.** Let $G$ be a graph with minimum degree $\delta \geq 6k$. If $\mu_2(G) > 2 + \frac{2k-1}{\delta}$, then $\mu_2(G - Z) > \frac{6k - 2k|Z| - 1}{\delta(G - Z) + 1}$ for every $Z \subset V(G)$ with $|Z| \leq 2$. 

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Proof. Notice that $\delta(G - u) \geq \delta - 1$ for every $u \in V(G)$ and $\delta(G - v - w) \geq \delta - 2$ for every $v, w \in V(G)$. It suffices to show that $\mu_2(G) > \frac{6k-1}{\delta+1}, \mu_2(G-u) > \frac{4k-1}{\delta}$ and $\mu_2(G-v-w) > \frac{2k-1}{\delta}$. Because $\delta \geq 6k$, it is not hard to verify that $2 + \frac{2k-1}{\delta+1} \geq \frac{6k-1}{\delta+1}$ and $1 + \frac{2k-1}{\delta} \geq \frac{4k-1}{\delta}$. Thus $\mu_2(G) > 2 + \frac{2k-1}{\delta+1} \geq \frac{6k-1}{\delta+1}$. By Theorem 3.2, $\mu_2(G-u) \geq \mu_2(G) - 1 > 1 + \frac{2k-1}{\delta} \geq \frac{4k-1}{\delta}$ and $\mu_2(G-v-w) \geq \mu_2(G) - 2 > \frac{2k-1}{\delta}$. \qed

5 Examples

In this section, we construct a family of graphs to show that the condition “$\mu_2(G) > \frac{5}{\delta(G)+1}$” in Corollary 1.3 is essentially best possible.

Figure 1: An example of $H_d$ when $d = 10$

The family of graphs was initially constructed in [5]. Let $d \geq 6$ be an integer and let $H_1, H_2, H_3, H_4, H_5$ be 5 vertex-disjoint copies of a graph obtained from $K_{d+1}$ by deleting two disjoint edges. Suppose that the deleted edges are $a_i b_i$ and $c_i d_i$ in $H_i$ for $1 \leq i \leq 5$. Let $H_d$ be
the $d$-regular graph whose vertex set is $\bigcup_{i=1}^{5} V(H_i)$ and whose edge set is the union of $\bigcup_{i=1}^{5} E(H_i)$ with the set $F = \{b_1a_2, b_2a_3, b_3a_4, b_4a_5, b_5a_1, c_1d_3, c_3d_5, c_5d_2, c_2d_4, c_4d_1\}$. An example is shown in Figure 1 when $d = 10$. By the computation in [5], it follows that $\frac{5}{d+3} < \mu_2(\mathcal{H}_d) \leq \frac{5}{d+1}$ for $d \geq 6$. However, we can show that $\mathcal{H}_d$ is not rigid as below.

Let $X_i = V(H_i)$ for $1 \leq i \leq 5$, and for $6 \leq i \leq 15$, $X_i$ be the vertex set induced by a single edge in $F$. Clearly $\{E(X), X \in \mathcal{G}\}$ partitions $E(G)$. Then

$$\sum_{X \in \mathcal{G}} (2|X| - 3) = 5(2(d + 1) - 3) + 10(2 \times 2 - 3) = 10d + 5.$$  

Notice that $|V(G)| = 5d + 5$ and it follows that $2|V| - 3 = 10d + 7$, which violates the characterization of Lovász and Yemini [22]. Thus $\mathcal{H}_d$ is not rigid.

References

[1] L. Asimov and B. Roth, The rigidity of graphs, *Trans. Amer. Math. Soc.* 245 (1978), 279-289.

[2] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer Universitext, 2012.

[3] J. Cheriyan, O. Durand de Gevigney and Z. Szigeti, Packing of rigid spanning subgraphs and spanning trees, *J. Combin. Theory Ser. B* 105 (2014), 17-25.

[4] S. M. Cioabă and X. Gu, Connectivity, toughness, spanning trees of bounded degrees, and spectrum of regular graphs, special issue dedicated to Professor Miroslav Fiedler, *Czechoslovak Mathematical Journal* 66 (2016), 913–924.

[5] S. M. Cioabă and W. Wong, Edge-disjoint spanning trees and eigenvalues of regular graphs, *Linear Algebra Appl.* 437 (2012), 630–647.

[6] M. Cucuringu, A. Singer and D. Cowburn, Eigenvector synchronization, graph rigidity and the molecule problem, *Inf. Inference* 1 (2012), no. 1, 21–67.

[7] M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak Math. J.* 23(98) (1973), 298–305.

[8] M. Gáspár and P. Csermely, Rigidity and flexibility of biological networks, *Brief Funct Genomics* 11(6) (2012), 443–456.

[9] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.

10
[10] X. Gu, *Connectivity and Spanning Trees of Graphs*, PhD Dissertation, West Virginia University, 2013.

[11] X. Gu, Spanning rigid subgraph packing and sparse subgraph covering, *SIAM Journal on Discrete Mathematics* **32** (2018), 1305-1313.

[12] X. Gu, H.-J. Lai, P. Li and S. Yao, Edge-disjoint spanning trees, edge connectivity and eigenvalues in graphs, *Journal of Graph Theory* **81** (2016), 16-29.

[13] W.H. Haemers, *Eigenvalue Techniques in Design and Graph Theory*, Reidel, Dordrecht, 1980. Thesis (T.H. Eindhoven, 1979) = Math. Centr. Tract 121 (Amsterdam, 1980).

[14] W. H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226-228** (1995), 593-616.

[15] Y. Hong, X. Gu, H.-J. Lai and Q. Liu, Fractional spanning tree packing, forest covering and eigenvalues, *Discrete Applied Mathematics* **213** (2016), 219-223.

[16] B. Jackson and T. Jordán, A sufficient connectivity condition for generic rigidity in the plane, *Discrete Appl. Math.* **157** (2009), 1965-1968.

[17] D.J. Jacobs and B. Hendrickson, An Algorithm for Two-Dimensional RigidityPercolation: The Pebble Game, *Journal of Computational Physics* **137** (1997), 346–365.

[18] T. Jordán, On the existence of $k$ edge-disjoint 2-connected spanning subgraphs, *J. Combin. Theory Ser. B* **95** (2005), 257-262.

[19] M. Krivelevich and B. Sudakov, Pseudo-random graphs. More sets, graphs and numbers, 199–262, *Bolyai Soc. Math. Stud.*, 15, Springer, Berlin, 2006.

[20] G. Laman, On graphs and rigidity of plane skeletal structures, *J. Engrg. Math.* **4** (1970), 331–340.

[21] Q. Liu, Y. Hong, X. Gu and H.-J. Lai, Note on edge-disjoint spanning trees and eigenvalues, *Linear Algebra Appl.* **458** (2014), 128–133.

[22] L. Lovász and Y. Yemini, On generic rigidity in the plane, *SIAM J. Algebraic Discrete Methods* **3** (1982), 91–98.

[23] C. St. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961), 445–450.
[24] W. T. Tutte, On the problem of decomposing a graph into \( n \) factors, *J. London Math. Soc.* 36 (1961), 221–230.

[25] W. Whiteley, Some matroids from discrete applied geometry, *Matroid theory (Seattle, WA, 1995)*, 171-311, *Contemp. Math.*, 197, *Amer. Math. Soc.*, Providence, RI, 1996

[26] D. Zelazo, A. Franchi, H.H. Bülthoff and P.R. Giordano, Decentralized rigidity maintenance control with range measurements for multi-robot systems, *The International Journal of Robotics Research* 34(1) (2015), 105–128.