Real C*-Algebras, United $KK$-Theory, and the Universal Coefficient Theorem

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March 29, 2022

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Abstract

We define united $KK$-theory for real C*-algebras $A$ and $B$ such that $A$ is separable and $B$ is $\sigma$-unital, extending united $K$-theory in the sense that $KK^{CRT}(\mathbb{R}, B) = K^{CRT}(B)$. United $KK$-theory contains real, complex, and self-conjugate $KK$-theory; but unlike unaugmented real $KK$-theory, it admits a universal coefficient theorem. For all separable $A$ and $B$ in which the complexification of $A$ is in the bootstrap category, $KK^{CRT}(A, B)$ can be written as the middle term of a short exact sequence whose outer terms involve the united $K$-theory of $A$ and $B$. As a corollary, we prove that united $K$-theory classifies $KK$-equivalence for real C*-algebras whose complexification is in the bootstrap category.

1 Introduction

The Universal Coefficient Theorem for the $K$-theory of complex C*-algebras, proven by Rosenberg and Schochet ([?]) in 1987, states the existence of an unnaturally split, short exact sequence

$$0 \to \text{Ext}_{K_*}^{(C)}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^{C}(A, B) \xrightarrow{\gamma} \text{Hom}_{K_*}^{(C)}(K_*(A), K_*(B)) \to 0$$

for any complex separable C*-algebras $A$ and $B$ such that $A$ is in the bootstrap category $\mathcal{N}$. Recall that $\mathcal{N}$ is the smallest subcategory of complex, separable, nuclear C*-algebras which contains the separable type I C*-algebras; which is closed under the operations of taking inductive limits, stable isomorphisms, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$; and which satisfies the two out of three rule for short exact sequences (i.e. if $0 \to A \to B \to C \to 0$ is exact and two of $A$, $B$, $C$ are in $\mathcal{N}$, then the third is also in $\mathcal{N}$).

One immediate and powerful corollary of this theorem states that $KK$-equivalence is completely characterized by $K$-theory for algebras in the bootstrap class, $\mathcal{N}$ (see Section 7 of [?], Section 23.10 of [?], and
Section 2.4 of [?]. The role of the Universal Coefficient Theorem in the standard toolkit in the subject of $K$-theory of complex $C^*$-algebras is considerable. In particular, it is used essentially in the proofs of $K$-theoretic classification theorems for simple $C^*$-algebras, such as those found in [?] and [?] in the purely infinite case; in [?] in the stably finite, real rank zero case; and in [?] in the tracial rank zero case.

In this paper, we consider real $C^*$-algebras. Recall that a real $C^*$-algebra $A$ is a Banach $*$-algebra over $\mathbb{R}$ which satisfies the $C^*$-equation $\|a^*a\| = \|a\|^2$ as well as the axiom that $1 + a^*a$ is invertible in the unitization $\tilde{A}$ for all $a \in A$. This is equivalent (by [?]; see also Chapter 15 of [?]) to saying that $A$ is isomorphic to a norm-closed adjoint-closed algebra of operators on a Hilbert space over $\mathbb{R}$. Note that every complex $C^*$-algebra can be considered a real $C^*$-algebra by forgetting the complex structure. Conversely, given any real $C^*$-algebra $A$, we can form the complexification $\mathbb{C} \otimes A = A_c$.

Except for the classification of complex AF-algebras in [?], which was repeated for real AF-algebras in [?], none of the major work in classifying complex $C^*$-algebras has been carried over to real $C^*$-algebras. One reason for this is that there is no universal coefficient theorem for real $C^*$-algebras. While Kasparov [?] considered $KK$-theory in a very general setting, Rosenberg and Schochet proved the universal coefficient theorem only for complex $C^*$-algebras as they explained, “For reasons pointed out already by Atiyah, there can be no good Künneth Theorem or Universal Coefficient Theorem for the KKO groups of real $C^*$-algebras; this explains why we deal only with complex $C^*$-algebras.” The reference to Atiyah is [?] wherein a counter-example is given; the essential obstruction lies in the homological algebra, namely that $K_\ast(A)$ does not have projective or injective dimension one in the category of $K_\ast(\mathbb{R})$-modules. In fact, the projective dimension may be infinite.

In the present paper, we remedy this situation by proving a universal coefficient theorem for real $C^*$-algebras using united $K$-theory and united $KK$-theory. This paper is a sequel to our earlier paper [?] in which we extended united $K$-theory from its original topological setting [?] to the setting of real $C^*$-algebras. In the present paper, we further extend these constructions, defining united $KK$-theory for real $C^*$-algebras. That is, we define $KK^{CRT}(A, B)$ for real $C^*$-algebras $A$ and $B$ such that $A$ is separable and $B$ is $\sigma$-unital. This functor combines real $KK$-theory, complex $KK$-theory, and self-conjugate $KK$-theory as well as the natural transformations among the three. Thus we will show that it takes values in the category of so-called $CRT$-modules (also referred to as the category $CRT$), which is a much more hospitable setting than the category of $K_\ast(\mathbb{R})$-modules (the main source of information on the category $CRT$ and its hospitality is [?]).

In [?], we took advantage of the fact that the object $K^{CRT}_\ast(A)$ has projective dimension one in $CRT$ to prove a Künneth formula for real $C^*$-algebras. In the current paper, we take advantage of the fact that the same object has injective dimension one to prove the following universal coefficient theorem.

**Theorem 1.1 (Main theorem).** Let $A$ and $B$ be real separable $C^*$-algebras such that $\mathbb{C} \otimes A$ is in the
bootstrap category $\mathcal{N}$. Then there is a short exact sequence

$$0 \to \text{Ext}_{\text{CRT}}(K^\text{CRT}(A), K^\text{CRT}(B)) \xrightarrow{\kappa} KK^\text{CRT}(A, B) \xrightarrow{\gamma} \text{Hom}_{\text{CRT}}(K^\text{CRT}(A), K^\text{CRT}(B)) \to 0$$

where $\kappa$ has degree $-1$. As in the Künneth sequence for united $K$-theory, this short exact sequence does not split in general.

If Rosenberg and Schochet say that no good UCT is possible for real $C^*$-algebras, we believe that ours is the best possible. Indeed our universal coefficient theorem does exactly what a universal coefficient theorem is supposed to do: it expresses $KK^\text{CRT}(A, B)$ (and thus $KK_*(A, B)$, since united $KK$-theory contains real $KK$-theory) in terms of $K$-theoretic data involving $A$ and $B$; although the data required is more than just $K_*(A)$ and $K_*(B)$.

Furthermore, as a corollary to our universal coefficient theorem, we find that two real separable $C^*$-algebras whose complexifications are in the bootstrap category are $KK$-equivalent if and only if their united $K$-theories are isomorphic. If the development of new mathematical machinery is justified by its ability to answer questions that exist independently of that machinery, then this corollary justifies the development of united $K$-theory. Indeed, united $K$-theory exactly captures $KK$-equivalence, a notion which involves only elements in real $KK$-theory and is thus prior to the development of united $K$-theory or united $KK$-theory. This also strengthens our case that united $K$-theory is the right functor to look at for real $C^*$-algebras, especially when considering the possibility of a classification of real simple $C^*$-algebras.

The work of Hewitt in [?] on $CRT$-modules suggests that the same information can be retained in a somewhat smaller invariant. On the other hand, Bousfield has made us aware of examples of acyclic $CRT$-modules which are not determined by their real part. Together with our result that every acyclic $CRT$-module can be realized as the united $K$-theory of a real $C^*$-algebra (which will be written and published elsewhere), this implies that real $K$-theory alone is certainly not sufficient to determine $KK$-equivalence.

Our proof of the universal coefficient theorem will be found in the final section of the paper, and follows the same line of attack as in [?]. We first prove that $\gamma$ is an isomorphism in case $K^\text{CRT}(B)$ is an injective $CRT$-module. We then tackle the general case by using a geometric injective resolution of $K^\text{CRT}(B)$. The fact that $K^\text{CRT}(B)$ has injective dimension one is a necessary condition for the existence of such a geometric injective resolution.

In the intervening sections we build up the required machinery. In Section 2, we define united $KK$-theory and work out its main properties. Section 3 is entirely algebraic, defining $\text{Hom}_{\text{CRT}}(M, N)$ in the category of $CRT$-modules and working out its main properties.
2 United $KK$-theory

In this section, we define united $KK$-theory and the intersection product for united $KK$-theory. In this paper, each $K$-group or $KK$-group is considered as $\mathbb{Z}$-graded module with a periodicity automorphism of degree 8 (or degree 2 if the C*-algebras are complex). This periodicity automorphism is implicit in the $K_*(\mathbb{R})$-module structure described below.

2.1 The United $KK$-Theory Bifunctor

Recall from [?] that $T = \{ f : [0,1] \to \mathbb{C} \mid f(0) = \overline{f(1)} \}$ is the self-conjugate algebra used in the definition of self-conjugate $K$-theory for real C*-algebras. We will use it here to define self-conjugate $KK$-theory.

**Definition 2.1.** For any pair of real C*-algebras $A$ and $B$ such that $A$ is separable and $B$ is $\sigma$-unital, we define real $KK$-theory, complex $KK$-theory, and self-conjugate $KK$-theory as follows:

1. $KKO_*(A,B) = KK_*(A,B)$
2. $KKU_*(A,B) = KK_*(A,C \otimes B)$
3. $KKT_*(A,B) = KK_*(A,T \otimes B)$

Because of the intersection product, real $KK$-theory takes values in the category of modules over the ring $K_*(\mathbb{R})$, given in degrees 0 through 8 by

$$K_*(\mathbb{R}) = \mathbb{Z} \quad \mathbb{Z}_2 \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Z}$$

(see page 23 in [?] or Table 1 in [?]). The generators are the elements $\eta_0 \in K_0(\mathbb{R})$, $\xi \in K_4(\mathbb{R})$, and the invertible element $\beta_0 \in K_8(\mathbb{R})$. Complex $KK$-theory takes values in the category of modules over the ring $K_*(\mathbb{C})$, which is the free polynomial ring generated by $\beta_U \in K_2(\mathbb{C})$:

$$K_*(\mathbb{C}) = \mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z}$$

(see Table 2 in [?]). The action of $\beta_U$ on $KK_*(A,C \otimes B)$ is actually implemented by first letting $\beta_U$ pass through the natural homomorphisms

$$K_2(\mathbb{C}) = KK_2(\mathbb{R}, \mathbb{C}) \to KK_2(\mathbb{C}, \mathbb{C} \otimes \mathbb{C}) \to KK_2(\mathbb{C}, \mathbb{C})$$

and then applying the intersection product. We let $\beta_U$ also denote the element of $KK_2(\mathbb{C}, \mathbb{C})$ which more directly implements the natural transformation on $KK_*(A,C \otimes B)$. Finally, self conjugate $KK$-theory takes values in the category of modules over the ring $K_*(T) = KK_*(\mathbb{R}, T)$

$$K_*(T) = \mathbb{Z} \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}$$
(see Table 3 in [?]) with generators $\eta_T$ in degree 1, $\omega$ in degree 3, and the invertible element $\beta_T$ is degree 4. As in the complex case, the action of these elements on $KK(A, T \otimes B)$ is implemented by intersection product with corresponding elements in $KK_*(T, T)$.

Furthermore, there are natural transformations $c_n : KKO_n(A, B) \to KKU_n(A, B)$, $r_n : KKU_n(A, B) \to KKO_n(A, B)$, $\varepsilon_n : KKO_n(A, B) \to KKT_n(A, B)$, $\zeta_n : KKT_n(A, B) \to KKU_n(A, B)$, $(\psi_U)_n : KKU_n(A, B) \to KKU_n(A, B)$, $(\psi_T)_n : KKT_n(A, B) \to KKT_n(A, B)$, $\gamma_n : KKU_n(A, B) \to KKT_{n-1}(A, B)$, $\tau_n : KKT_n(A, B) \to KKO_{n+1}(A, B)$ among the three variants of $KK$-theory given by intersection product with the $KK$-elements

\[ c \in KKO_0(\mathbb{R}, \mathbb{C}) \quad r \in KKO_0(\mathbb{C}, \mathbb{R}) \]
\[ \varepsilon \in KKO_0(\mathbb{R}, T) \quad \zeta \in KKO_0(T, \mathbb{C}) \]
\[ \psi_U \in KK_0(\mathbb{C}, \mathbb{C}) \quad \psi_T \in KKO_0(T, T) \]
\[ \gamma \in KK_{-1}(\mathbb{C}, T) \quad \tau \in KKO_0(T, \mathbb{R}) \]

These elements are obtained as follows. In Section 1 of [?], the operations $c$, $r$, $\varepsilon$, $\zeta$, $\psi_U$, $\psi_T$, and $\gamma$ are defined on united $K$-theory. Each is induced by a homomorphism among the C*-algebras $\mathbb{R}$, $\mathbb{C}$, and $T$ (or suspensions thereof or matrix algebras thereover). Hence each produces a $KK$-element which implements the natural transformation. Also in Section 1 of [?], the operation $\tau$ is defined in terms of two homomorphisms $\sigma_i : T \to C(S^1, M_i(\mathbb{R}))$ (for $i = 1, 2$) such that $\pi \circ \sigma_1 = \pi \circ \sigma_2$ where $\pi : C(S^1, M_i(\mathbb{R})) \to M_i(\mathbb{R})$ is the base-point evaluation map. Therefore there are $KK$-elements $\sigma_i \in KKO_0(T, C(S^1, \mathbb{R}))$ such that $\sigma_1 - \sigma_2$ lies in the kernel of the homomorphism $KK_0(T, C(S^1, \mathbb{R})) \to KKO_0(T, \mathbb{R})$. Hence $\sigma_1 - \sigma_2$ lifts to produce a unique element $\tau \in KKO_0(T, S\mathbb{R}) = KKO_1(T, \mathbb{R})$.

**Definition 2.2 (United $KK$-Theory).** For any pair of real C*-algebras $A$ and $B$ such that $A$ is separable and $B$ is $\sigma$-unital, we define the united $KK$-theory to be the triple

\[ KK^{\text{CRT}}_*(A, B) := \{ KKO_*(A, B), KKU_*(A, B), KKT_*(A, B) \} \]

consisting of graded modules over $K_*(\mathbb{R})$, $K_*(\mathbb{C})$, and $K_*(T)$ (respectively) together with the operations \{ $c, r, \varepsilon, \zeta, \psi_U, \psi_T, \gamma, \tau$ \}.
Theorem 2.3. United $KK$-theory is a natural bifunctor; covariant in the second argument and contravariant in the first argument; additive, homotopy invariant, stable, and has period 8 in both arguments; and it satisfies the following exactness properties:

Assume that $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ is a semisplit short exact sequence of $\sigma$-unital C*-algebras (semisplit means that $\beta$ has a completely positive cross-section with norm no more than 1). If $D$ is separable, there is a long exact sequence

$$\cdots \to KK^{\text{CRT}}(D,A) \overset{\alpha^*}{\to} KK^{\text{CRT}}(D,B) \overset{\beta^*}{\to} KK^{\text{CRT}}(D,C) \overset{\delta}{\to} KK^{\text{CRT}}(D,A) \to \cdots.$$ 

If $B$ is separable, there is a long exact sequence

$$\cdots \to KK^{\text{CRT}}(A,D) \overset{\alpha^*}{\leftarrow} KK^{\text{CRT}}(B,D) \overset{\beta^*}{\leftarrow} KK^{\text{CRT}}(C,D) \overset{\delta}{\leftarrow} KK^{\text{CRT}}(A,D) \leftarrow \cdots.$$ 

In either case, $\delta$ is a homomorphism of degree $-1$ implemented by an element $\delta \in KK_{-1}(C,A)$.

Proof. All properties follow from the corresponding properties enjoyed by $KK$-theory for real C*-algebras (see Sections 2.3-2.5 in [?]).

For exactness, assume that $0 \to A \to B \to C \to 0$ is semisplit. Recall Lemma 1.23 in [?] which states that a short exact sequence is semisplit if and only if its complexification

$$0 \to \mathbb{C} \otimes A \to \mathbb{C} \otimes B \to \mathbb{C} \otimes C \to 0$$

is also semisplit. Let $s$ be a completely positive section of Extension ?? with norm at most 1. If we apply the functor $C(S^1,-)$ to this last sequence we obtain the sequence

$$0 \to C(S^1, \mathbb{C} \otimes A) \to C(S^1, \mathbb{C} \otimes B) \to C(S^1, \mathbb{C} \otimes C) \to 0$$

which is, we claim, semisplit with section $s_*: f \mapsto s \circ f$. It is easily seen that $\|s_*\| \leq 1$ since $\|s\| \leq 1$. That $s_*$ is completely positive follows from the fact that a function $f \in C(X,D)$ is positive if and only if $f(x)$ is positive for all $x \in X$. Therefore Sequence ?? is semisplit.

Now recall that $T = \{ f \in C(S^1, \mathbb{C}) \mid f(z) = \overline{f(-z)} \}$. Then $\mathbb{C} \otimes T = C(S^1, \mathbb{C})$. Hence Sequence ?? is the complexification of the sequence

$$0 \to T \otimes A \to T \otimes B \to T \otimes C \to 0.$$ 

Therefore, by Lemma 1.23 in [?] again, Sequence ?? is semisplit.

To summarize, if Sequence ?? is semisplit, so are Sequences ?? and ???. Therefore by Theorem 2.5.6 in [?], there are long exact sequences in each of the three parts of united $KK$-theory. The maps in these long exact sequences commute with the transformation maps of united $KK$-theory and therefore we have long exact sequences in united $KK$-theory as in the statement of the Theorem. □
Proposition 2.4. If $A$ and $B$ are real $C^*$-algebras such that $A$ is separable and $B$ is $\sigma$-unital, then $KK(A,B)$ is an acyclic CRT-module.

The statement that $KK(A,B)$ is a CRT-module (see Section 2 of [?] or Section 1.3 of [?]) means that $KK_*(A,B)$ is a module over $KO_*(\mathbb{R}) = KK_*(\mathbb{R},\mathbb{R})$; that $KK_*(A,\mathbb{C} \otimes B)$ is a module over $KU_*(\mathbb{R}) = KK_*(\mathbb{R},\mathbb{C})$; that $KK_*(A,T \otimes B)$ is a module over $KT_*(\mathbb{R}) = KK_*(\mathbb{R},T)$; and that the following relations hold among the operations:

\[
\begin{align*}
rc &= 2 & \psi_v \beta_v &= - \beta_v \psi_v & \xi &= r\beta_v^2c \\
cr &= 1 + \psi_v & \psi_T \beta_T &= \beta_T \psi_T & \omega &= \beta_T \gamma \zeta \\
r &= \tau \gamma & \varepsilon \beta_o &= \beta_T^2 \varepsilon & \beta_T \varepsilon \tau &= \varepsilon \tau \beta_T + \eta \beta_T \\
c &= \zeta \varepsilon & \zeta \beta_o &= \beta_T^2 \zeta & \varepsilon \tau \zeta &= 1 + \psi_T \\
(\psi_v)^2 &= 1 & \gamma \beta_v^2 &= \beta_T \gamma & \gamma \tau &= 1 - \psi_T \\
(\psi_T)^2 &= 1 & \tau \beta_T^2 &= \beta_o \tau & \tau &= \tau \psi_T \\
\psi_T \varepsilon &= \varepsilon & \gamma &= \gamma \psi_v & \tau \beta_T \varepsilon &= 0 \\
\zeta \gamma &= 0 & \eta_o &= \tau \varepsilon & \varepsilon \xi &= 2 \beta_T \varepsilon \\
\zeta &= \psi_v \zeta & \eta_T &= \gamma \beta_o \zeta & \xi \tau &= 2 \tau \beta_T .
\end{align*}
\]

The statement that $KK(A,B)$ is acyclic means that the following sequences are exact:

\[
\cdots \to KK_n(A,B) \xrightarrow{\eta_0} KK_{n+1}(A,B) \xrightarrow{\zeta} KK_{n+1}(A,\mathbb{C} \otimes B) \xrightarrow{r\beta_v^{-1}} KK_{n-1}(A,B) \to \cdots \quad (5)
\]

\[
\cdots \to KK_n(A,B) \xrightarrow{\eta_o} KK_{n+2}(A,B) \xrightarrow{\zeta} KK_{n+2}(A,T \otimes B) \xrightarrow{\tau \beta_T^{-1}} KK_{n-1}(A,B) \to \cdots \quad (6)
\]

\[
\cdots \to KK_{n+1}(A,\mathbb{C} \otimes B) \xrightarrow{\zeta} KK_n(A,T \otimes B) \xrightarrow{\zeta} KK_n(A,\mathbb{C} \otimes B) \xrightarrow{1-\psi_v} KK_n(A,\mathbb{C} \otimes B) \to \cdots \quad (7)
\]

Proof of Proposition 2.4. We first show that $KK(A,B)$ is acyclic. In Section 1.4 of [?] we produced two semisplit short exact sequences which were homotopy equivalent to

\[
0 \to S^{-1}\mathbb{R} \to \mathbb{R} \xrightarrow{\zeta} \mathbb{C} \to 0 \quad (8)
\]

and

\[
0 \to S^{-2}\mathbb{R} \to \mathbb{R} \xrightarrow{\zeta} T \to 0 \quad (9)
\]

In the first extension, it was proven that the homomorphism $S^{-1}\mathbb{R} \to \mathbb{R}$ is represented by the $KK$-element $\eta_o \in KK_1(\mathbb{R},\mathbb{R})$ and the connecting homomorphism in the associated long exact sequence in $K$-theory is
represented by \( r\beta_0^{-1} \in KK_{-2}(\mathbb{C}, \mathbb{R}) \). For the second it was proven that the homomorphism \( S^{-2}\mathbb{R} \to \mathbb{R} \) is represented by \( \eta_0^2 \in KK_2(\mathbb{R}, \mathbb{R}) \) and the connecting homomorphism is represented by \( r\beta_0^{-1} \in KK_{-3}(T, \mathbb{R}) \).

When we tensor these sequences by \( B \) and apply part (5) of Theorem 2.5 we obtain Sequences 3 and 4 as desired.

Sequence 2 is similarly derived from the split exact sequence
\[
0 \to SC \overset{\gamma}{\to} T \overset{\zeta}{\to} C \to 0
\]
where \( \gamma \) is defined by inclusion and \( \zeta \) is defined by evaluation at one endpoint. (See Theorem 1.5 of [2].)

Now we show that the corresponding classes in \( KK_\ast(M, N) \) are in Hom\(_{KK_\ast(\mathbb{R})}(K_\ast(M), K_\ast(N)) \) is injective when \( M \) and \( N \) are in \( \{ \mathbb{R}, \mathbb{C}, T \} \).

We first need to compute \( KK_\ast(M, N) \) for all possible combinations of \( M \) and \( N \). If \( M = \mathbb{R} \), we refer to Table 1 in [2] since \( KK_\ast(\mathbb{R}, N) \cong K_\ast(N) \). We determine \( KK_\ast(\mathbb{C}, \mathbb{R}) \) and \( KK_\ast(T, \mathbb{R}) \) using the sequences
\[
KK_{n+1}(\mathbb{R}, \mathbb{R}) \overset{R\mu}{\leftarrow} KK_n(\mathbb{R}, \mathbb{R}) \overset{r\beta_0^{-1}}{\leftarrow} KK_{n+2}(\mathbb{R}, \mathbb{R}) \overset{\eta_0^2}{\leftarrow} KK_{n+1}(\mathbb{R}, \mathbb{R})
\]
and
\[
KK_{n+2}(\mathbb{R}, \mathbb{R}) \overset{\eta_0^2}{\leftarrow} KK_n(\mathbb{R}, \mathbb{R}) \overset{r\beta_0^{-1}}{\leftarrow} KK_{n+3}(\mathbb{R}, \mathbb{R}) \overset{\eta_0^2}{\leftarrow} KK_{n+1}(\mathbb{R}, \mathbb{R})
\]
which are derived from ?? and ??.

By Proposition 1.9 in [2] the CRT-relations hold for united \( K \)-theory. To show that they hold on the level of \( KK \)-elements it is enough to show that the homomorphism
\[
\alpha: KK_\ast(M, N) \to \text{Hom}_K(K_\ast(M), K_\ast(N))
\]
is injective when \( M \) and \( N \) are in \( \{ \mathbb{R}, \mathbb{C}, T \} \).

Now consider \( KK_\ast(\mathbb{C}, \mathbb{R}) \). The class \( r \in KK_0(\mathbb{C}, \mathbb{R}) \) induces a non-trivial homomorphism of infinite degree from \( K_0(\mathbb{C}) \cong \mathbb{Z} \) to \( K_0(\mathbb{R}) \cong \mathbb{Z} \). Therefore the image of \( \alpha \) in Hom\(_{KK_0(\mathbb{R})}(K_\ast(\mathbb{C}), K_\ast(\mathbb{R})) \) contains an infinite cyclic subgroup in degree 0. Since \( KK_0(\mathbb{C}, \mathbb{R}) \cong \mathbb{Z} \), it follows that \( \alpha \) must be injective in degree 0.

More generally, the class \( r\beta_0^\ast \in KK_{2n}(\mathbb{C}, \mathbb{R}) \) induces a non-trivial homomorphism from \( K_0(\mathbb{C}) \) to \( K_{2n}(\mathbb{R}) \) showing that \( \alpha \) must be injective on \( KK_{2n}(\mathbb{C}, \mathbb{R}) \).
Table 1: Some KK-groups

| n  | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| K K₄(ℝ, ℝ) | Z   | Z₂  | Z₂  | 0   | Z   | 0   | 0   | 0   | Z   |
| K K₄(ℝ, ℂ) | Z   | 0   | Z   | 0   | Z   | 0   | Z   | 0   | Z   |
| K K₄(ℝ, T) | Z   | Z₂  | 0   | Z   | Z   | Z₂  | 0   | Z   | Z   |
| K K₄(ℂ, ℝ) | Z   | 0   | Z   | 0   | Z   | 0   | Z   | 0   | Z   |
| K K₄(ℂ, ℂ) | Z ⊕ Z | 0   | Z ⊕ Z | 0   | Z ⊕ Z | 0   | Z ⊕ Z | 0   | Z ⊕ Z |
| K K₄(ℂ, T) | Z   | Z   | Z₂  | 0   | Z   | Z   | Z₂  | 0   | Z   |
| K K₄(T, ℝ) | Z   | Z   | Z₂  | 0   | Z   | Z   | Z₂  | 0   | Z   |
| K K₄(T, ℂ) | Z   | Z   | Z   | Z₂  | Z   | Z   | Z   | Z   | Z   |
| K K₄(T, T) | Z ⊕ Z | Z ⊕ Z₂ | Z₂  | Z   | Z ⊕ Z | Z ⊕ Z₂ | Z₂  | Z   | Z ⊕ Z |

According to Table 1 in [?], ψᵥ: K₂(ℂ) → K₂(ℂ) is multiplication by −1. Therefore the elements βᵥⁿ and ψᵥβᵥⁿ induce two independent homomorphisms from K₄(ℂ) to itself, which show that ω is injective for K K₂n(ℂ, ℂ).

The rest of the proof proceeds in the same way. Using Table 1 in [?] we study the behavior of the operations among the K-theory of ℝ, ℂ, and T. The elements εrβᵣⁿ and γβᵣⁿ show that ω is injective for K K₄(ℂ, T); rζβᵣⁿ, τβᵣⁿ, and ηᵣτβᵣⁿ for K K₄(T, ℝ); βᵥⁿεr and βᵥⁿτ for K K₄(T, ℂ); and finally, the elements βᵣⁿ, ψᵥβᵣⁿ, τᵣβᵣⁿ, εrβᵣⁿ, γr, and ωβᵣⁿ for K K₄(T, T).

□

2.2 The Intersection Product

In this section we will describe an intersection product for united KK-theory and we will prove that this product is a CRT-pairing. Throughout this section, we assume that all C*-algebras are separable and that enough of the C*-algebras are nuclear so that tensor products are unique.

The intersection product (see Section 2.4 of [?]) for real KK-theory is a pairing

\[ \alpha: K K_i(A_1, B_1 ⊗ D) ⊗ K K_j(D ⊗ A_2, B_2) → K K_{i+j}(A_1 ⊗ A_2, B_1 ⊗ B_2) \]

which is natural, bilinear, and associative. The associativity implies that the pairing is bilinear over the ring K O₄(ℝ) = K K₄(ℝ, ℝ). Hence we have a K O₄(ℝ)-module pairing

\[ \alpha_o: K K O_4(A_1, B_1 ⊗ D) ⊗ K K O_4(D ⊗ A_2, B_2) → K K O_4(A_1 ⊗ A_2, B_1 ⊗ B_2) . \]

We also have a K U₄(ℝ)-module pairing

\[ \alpha_u: K K U_4(A_1, B_1 ⊗ D) ⊗ K K U_4(D ⊗ A_2, B_2) → K K U_4(A_1 ⊗ A_2, B_1 ⊗ B_2) \]

which arises by composing the intersection product with multiplication C ⊗ C → C.

Similarly, since T is commutative, the homomorphism T ⊗ T → T induces a K T₄(ℝ)-module pairing

\[ \alpha_r: K K T_4(A_1, B_1 ⊗ D) ⊗ K K T_4(D ⊗ A_2, B_2) → K K T_4(A_1 ⊗ A_2, B_1 ⊗ B_2) . \]
Proposition 2.5. The pairings $\alpha_O$, $\alpha_U$, and $\alpha_T$ together form a natural associative CRT-pairing

$$\alpha : KK^{\text{CRT}}(A_1, B_1 \otimes D) \otimes_{\text{CRT}} KK^{\text{CRT}}(D \otimes A_2, B_2) \to KK^{\text{CRT}}(A_1 \otimes A_2, B_1 \otimes B_2)$$

Proof. We must show that the CRT-pairing relations of Definition 3.1 in [?]. These relations are

$$(1) \ c(m_O \cdot_O n_O) = c(m_O) \cdot_O c(n_O)$$

$$(2) \ r(c(m_O) \cdot_U n_U) = m_O \cdot_O r(n_U)$$

$$(3) \ r(m_U \cdot_U c(n_O)) = r(m_U) \cdot_O n_O$$

$$(4) \ v(m_O \cdot_O n_O) = (m_O) \cdot_T v(n_O)$$

$$(5) \ z(m_T \cdot_T n_T) = z(m_T) \cdot_U z(n_T)$$

$$(6) \ u(m_U \cdot_U n_U) = u(m_U) \cdot_U u(n_U)$$

$$(7) \ u(m_T \cdot_T n_T) = u(m_T) \cdot_T u(n_T)$$

$$(8) \ g(\gamma(m_U \cdot_U (n_T))) = \gamma(m_U) \cdot_T (n_T)$$

$$(9) \ G(\gamma(m_T \cdot_T (n_U))) = (-1)^{|m_U|} \gamma(m_T) \cdot_T (n_U)$$

$$(10) \ t(m_T \cdot_T \eta(n_O)) = t(m_T) \cdot_O \eta(n_O)$$

$$(11) \ t(\eta(m_O) \cdot_T (n_T)) = (-1)^{|m_O|} \eta(m_O) \cdot_O \tau(n_T)$$

$$(12) \ e\tau(m_U \cdot_U (n_T)) = e\tau(m_U) \cdot_U (n_T) + (-1)^{|m_T|} \eta_T \cdot_U e\tau(n_U) + \eta_T(m_T \cdot_T n_T) .$$

where $\cdot_O$, $\cdot_U$, and $\cdot_T$ represent the product under the pairings $\alpha_O$, $\alpha_U$, and $\alpha_T$ respectively.

The first eleven relations hold for intersection products in $KK$-theory for the same reason that they hold for external products in K-theory: diagrams on the level of C*-algebras which commute up to homotopy (see Lemmas 1.7 and 1.10 in [?]). For example, the diagram

$$\begin{array}{c}
S^T \otimes S^T & \overset{10\zeta}{\longrightarrow} & S^T \otimes S^T \\
\gamma \otimes 1 & \downarrow & \gamma \\
T \otimes T & \longrightarrow & T
\end{array}$$

commutes up to homotopy, as we have seen in the proof of Lemma 1.7 of [?]. Putting in suspensions, we have the diagram

$$\begin{array}{c}
S^m \otimes S^n & \overset{10\zeta}{\longrightarrow} & S^m \otimes S^n \\
\gamma \otimes 1 & \downarrow & \gamma \\
S^{m-1} \otimes S^n & \longrightarrow & S^{m+n-1}
\end{array}$$
which commutes up to homotopy (recall that our convention is that the homomorphism \( \gamma : S^n C \to S^{n-1} T \) uses the leftmost suspension) and establishes the relation \( \gamma(x \cdot \zeta(y)) = \gamma(x) \cdot \gamma(y) \) for \( x \in KK_m(A_1, C \otimes B_1 \otimes D) \) and \( y \in KK_n(D \otimes A_2, T \otimes B_1) \).

The relation \( \gamma(\zeta(x) \cdot y) = (-1)^m x \cdot \gamma(y) \) for \( x \in KK_T(A_1, B_1 \otimes D) \) and \( y \in KK_U(D \otimes A_2, B_2) \) is established similarly starting from the diagram

\[
\begin{array}{ccc}
T \otimes SC & \xrightarrow{\zeta \otimes 1} & C \otimes SC \\
1 \otimes \gamma & \downarrow & \downarrow \gamma \\
T \otimes T & \rightarrow & T
\end{array}
\]

which commutes up to homotopy. Putting in suspensions we get the diagram

\[
\begin{array}{ccc}
S^m T \otimes S^n C & \xrightarrow{\zeta \otimes 1} & S^m C \otimes S^n C \\
1 \otimes \gamma & \downarrow & \downarrow \gamma \\
S^m T \otimes S^{n-1} T & \rightarrow & S^{m+n-1} T
\end{array}
\]

which commutes up to homotopy and up to a factor of \( \iota^n \) where \( \iota \) is the involution on \( S^2 \) which interchanges the two suspension factors. This factor of \( \iota \) arises because \( \gamma \) is required to always use the outermost suspension. Since \( \iota \) induces multiplication by \(-1\) on \( KK \)-theory, the relation is established.

The rest of the first eleven CRT-pairing relations can also be established in this way based directly on the commutative diagrams used in establishing Lemmas 1.7 and 1.10 in [?].

For the twelfth relation

\[
\varepsilon \tau(x \cdot y) = \varepsilon \tau(x) \cdot \tau(y) + (-1)^n x \cdot \varepsilon \tau(y) \eta \tau(x \cdot \tau(y))
\]

(10)

for \( x \in KK_T(A_1, B_1 \otimes D) \) and \( y \in KK_U(D \otimes A_2, B_2) \) we take a different tack. Let \( \phi \) be the \( KK \)-element corresponding to the \( C^* \)-algebra homomorphism \( T \otimes T \to T \). The operation \( x \otimes y \mapsto \varepsilon \tau(x \cdot y) \) is performed by composing the product homomorphism

\[
KK_m(A_1, T \otimes B_1 \otimes D) \otimes KK_n(D \otimes A_1, T \otimes B_2) \to KK_{m+n}(A_1 \otimes A_2, T \otimes T \otimes B_1 \otimes B_2)
\]

by the homomorphism

\[
KK_{m+n}(A_1 \otimes A_2, T \otimes T \otimes B_1 \otimes B_2) \to KK_{m+n+1}(A_1 \otimes A_2, T \otimes B_1 \otimes B_2)
\]

that is induced by the \( KK \)-element \( \varepsilon \tau \phi \in KK_1(T \otimes T, T) \).

Meanwhile the right side of Equation (10) is the image of \( x \otimes y \) under an operation which is obtained in exactly the same way, except that the relevant \( KK \)-element is \( \phi(\varepsilon \tau \otimes 1) + (-1)^n \phi(1 \otimes \varepsilon \tau) + \eta \phi \). We will show that these two \( KK \)-elements are identical.
We know that these two operations agree in the case $A_1 = A_2 = B_1 = B_2 = D = R$ by Proposition 4.1 of \[?\]. Therefore, it is enough to show that the map

$$\alpha: KK_\ast(T \otimes T, T) \to \text{Hom}_\ast(K_\ast(T \otimes T), K_\ast(T))$$

is injective. Lemma \[??\], which follows, implies the existence of a split exact sequence

$$0 \to ST \to T \otimes T \to T \to 0$$

which allows us to make the decompositions

$$KK_\ast(T \otimes T, T) \cong KK_\ast(ST, T) \oplus KK_\ast(T, T) \cong \text{Hom}_\ast(K_\ast(ST), K_\ast(T)) \oplus \text{Hom}_\ast(K_\ast(T), K_\ast(T)).$$

Since $\alpha$ respects this direct sum decomposition, it suffices to show that the map

$$KK_\ast(T, T) \to \text{Hom}(K_\ast(T), K_\ast(T))$$

is injective. But this fact was established in the proof of Proposition \[??\].

**Lemma 2.6.** There is a $C^\ast$-algebra isomorphism $T \otimes T \cong C(S^1, T)$.

**Proof.** Let $I = [0, 1]$. Then $T \cong \{ f: I \to \mathbb{C} \mid f(0) = \overline{f(1)} \}$ so we have

$$T \otimes T \cong \{ f: I^2 \to \mathbb{C} \otimes \mathbb{C} \mid f(0,t) = \overline{f(1,t)} \text{ and } f(s,0) = f(s,1) \}$$

where the straight bar represents conjugation in the first factor of $\mathbb{C} \otimes \mathbb{C}$ and the wavy bar represents conjugation in the second factor.

Now, recall from \[?] the isomorphism $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ given by $(\lambda_1, \lambda_2) \mapsto \frac{1}{2}(\lambda_1 + \lambda_2) \otimes 1 + \frac{1}{2}(\lambda_1 - \lambda_2)i \otimes i$. Under this isomorphism, conjugation in the first factor of $\mathbb{C} \otimes \mathbb{C}$ corresponds to the involution $(\lambda_1, \lambda_2) \mapsto (\overline{\lambda_2}, \overline{\lambda_1})$ while conjugation in the second factor corresponds to the involution $(\lambda_1, \lambda_2) \mapsto (\lambda_2, \lambda_1)$.

Therefore,

$$T \otimes T \cong \{ f, g: I^2 \to \mathbb{C} \mid f(0,t) = \overline{g(1,t)}, g(0,t) = f(1,t), f(s,0) = g(s,1), \text{ and } g(s,0) = f(s,1) \}.$$

Next, we use the gluing operation $(f, g) \mapsto h$ defined by

$$h(s,t) = \begin{cases} f(s, 2t) & t \leq 1/2 \\ g(s, 2t - 1) & t \geq 1/2 \end{cases}$$

which gives us an isomorphism

$$T \otimes T \cong \{ h: I^2 \to \mathbb{C} \mid h(s,0) = h(s,1) \text{ and } h(0, t) = h(1, t + \frac{1}{2}) \}.$$
where the addition $t + \frac{1}{2}$ is done modulo 1.

Finally, we untwist this algebra using the operation $h \mapsto h'$ defined by $h'(s,t) = h(s,t + s/2)$ where, again, addition is modulo 1. Hence we finally have

$$T \otimes T \cong \{ h: I^2 \to \mathbb{C} \mid h(s,0) = h(s,1) \text{ and } h(0,t) = h(1,t) \} \cong C(S^1, T)$$

3 The Hom functor in CRT

This section is entirely algebraic, wherein we define the functor $\text{Hom}_{\text{CRT}}(M,N)$ and work out the properties of this functor necessary for our development of the universal coefficient theorem. Note that $\text{Hom}_{\text{CRT}}(M,N)$ does not refer to the graded group of $\text{CRT}$-morphisms from $M$ to $N$, which we denote $[M,N]_*$. Instead, we will (in Section ??) define $\text{Hom}_{\text{CRT}}(M,N)$ to be a $\text{CRT}$-module $\text{Hom}_{\text{CRT}}(M,N)$ such that the adjoint relationship

$$[L, \text{Hom}_{\text{CRT}}(M,N)]_* \cong [L \otimes_{\text{CRT}} M, N]_*$$

holds for all $\text{CRT}$-modules $L$, $M$, and $N$. (See section 3 of [?] for tensor products in $\text{CRT}$.)

In Section ??, we prove three propositions which characterize $\text{Hom}_{\text{CRT}}(M,N)$ when $M$ is a monogenic free $\text{CRT}$-module. These theorems help us get a more concrete handle on the Homfunctor in $\text{CRT}$. Indeed, based on these theorems, one strategy to compute $\text{Hom}_{\text{CRT}}(M,N)$ when $M$ is acyclic is to build a free resolution of $M$ and apply the functor $\text{Hom}_{\text{CRT}}(\_,N)$ to the resolution. Finally, in Section ??, we prove a couple propositions giving sufficient conditions for acyclicity of $\text{Hom}_{\text{CRT}}(M,N)$.

3.1 Defining $\text{Hom}_{\text{CRT}}(M,N)$

In what follows, for any $i \in \mathbb{Z}$ and any $X \in \{O,U,T\}$, let $F(b,i,X)$ be the free $\text{CRT}$-module with a single generator $b \in F(b,i,X)^X$. These modules are described in Section 2.4 of [?] and in Section 2.1 of [?].

Definition 3.1. Given two $\text{CRT}$-modules $M$ and $N$, we define the $\text{CRT}$-module $\text{Hom}_{\text{CRT}}(M,N)$ as follows

1. For $i \in \mathbb{Z}$ and $X \in \{O,U,T\}$, we define $\text{Hom}_{\text{CRT}}(M,N)^X_i$ to be the group of $\text{CRT}$-module homomorphisms of degree 0 from $F(b,i,X) \otimes_{\text{CRT}} M$ to $N$. That is,

$$\text{Hom}_{\text{CRT}}(M,N)^X_i = [F(b,i,X) \otimes_{\text{CRT}} M, N]_0$$

2. Let $\theta: L^X_i \to L^Y_j$ be a $\text{CRT}$-operation and let $\mu_\theta$ be the function $F(c,j,Y) \to F(b,i,X)$ which carries $c$ to $\theta(b) \in F(b,i,X)^Y$. Then for any $\alpha \in \text{Hom}_{\text{CRT}}(M,N)^X_i = [F(b,i,X) \otimes M, N]_0$, define $\theta(\alpha) \in \text{Hom}_{\text{CRT}}(M,N)^Y_j = [F(c,j,Y) \otimes_{\text{CRT}} M, N]_0$ by the formula $\theta(\alpha) = \alpha \circ (\mu_\theta \otimes 1)$. This describes the operation

$$\theta: \text{Hom}_{\text{CRT}}(M,N)^X_i \to \text{Hom}_{\text{CRT}}(M,N)^Y_j$$
**Proposition 3.2.** \( \text{Hom}_{\text{CRT}}(-,-) \) is a bifunctor with values in CRT which is contravariant in the first argument and covariant in the second.

**Proof.** The only statement that is not clear is that \( \text{Hom}_{\text{CRT}}(M,N) \) is a CRT-module if \( M \) and \( N \) are. To show that the CRT-relations are all satisfied, it suffices to show that if \( \phi \circ \psi = \theta \) is a general CRT-relation, then it holds in \( \text{Hom}_{\text{CRT}}(M,N) \). Suppose that \( \psi: L^X_i \to L^Y_j \); that \( \phi: L^Y_j \to L^Z_k \); and that \( \theta: L^X_i \to L^Z_k \). Then the CRT-morphisms

\[
\mu_{\psi}: F(c,j,Y) \to F(b,i,X)
\]
\[
\mu_{\phi}: F(d,k,Z) \to F(c,j,Y)
\]
\[
\mu_{\theta}: F(d,k,Z) \to F(b,i,X)
\]
defined as above satisfy \( \mu_{\theta} = \mu_{\psi} \circ \mu_{\phi} \). Hence if \( \alpha \) is any element of \( [F(b,i,X) \otimes_{\text{CRT}} M, N]_0 \) then \( (\phi \circ \psi)(\alpha) = \theta(\alpha) \).

**Proposition 3.3.** If \( L, M, \) and \( N \) are CRT-modules, then there is a natural isomorphism of graded groups

\[
[L \otimes_{\text{CRT}} M, N]_* \cong [L, \text{Hom}_{\text{CRT}}(M,N)]_*
\]

**Proof.** For simplicity, we will consider only homomorphisms of degree 0. The isomorphism in arbitrary degree can be established analogously (or by suspending algebras).

We will define natural transformations

\[
\Gamma: [L \otimes_{\text{CRT}} M, N]_0 \to [L, \text{Hom}_{\text{CRT}}(M,N)]_0
\]
\[
\Delta: [L, \text{Hom}_{\text{CRT}}(M,N)]_0 \to [L \otimes_{\text{CRT}} M, N]_0
\]
which are inverse to each other.

First we define \( \Gamma \). Let \( \alpha: L \otimes_{\text{CRT}} M \to N \) and let \( l \in L^X_i \). Then \( \Gamma(\alpha)(l) \in \text{Hom}_{\text{CRT}}(M,N)^X_i \) is defined to be the composite

\[
F(b,i,X) \otimes M \xrightarrow{\mu^l_\alpha \otimes 1} L \otimes M \xrightarrow{\Delta} N
\]

where \( \mu^l_\alpha \) is the CRT-homomorphism which carries the generator \( b \in F(b,i,X) \) to the element \( l \in L^X_i \). The formula for \( \Gamma \) is \( \Gamma(\alpha)(l) = \alpha \circ (\mu^l_\alpha \otimes 1) \).

To see that \( \Gamma(\alpha) \) is a CRT-morphism, let \( \theta \) be a CRT-operation sending \( l \in L^X_i \) to \( \theta(l) \in L^Y_j \). Let
b ∈ F(b, i, X) and let c ∈ F(c, j, Y). Then

\[
\Gamma(\alpha)(\theta(l)) = \alpha \circ (\mu_c^{\theta(l)} \otimes 1) \\
= \alpha \circ (\mu_b \otimes 1) \circ (\mu_c^{\theta(b)} \otimes 1) \\
= \alpha \circ (\mu_b \otimes 1) \circ (\mu \otimes 1) \\
= \Gamma(\alpha)(l) \circ (\mu \otimes 1) \\
= \theta(\Gamma(\alpha)(l)) .
\]

Now we define \( \Delta \). Let \( \beta : L \rightarrow \text{Hom}_{CRT}(M, N) \) be given. For any \( l \in L_i^X \), the element \( \beta(l) \) is in \( \text{Hom}_{CRT}(M, N)_i^X = [F(b, i, X) \otimes M, N]_0 \). So we can define \( \Delta(\beta) \in [L \otimes M, N]_0 \) by the formula

\[
\Delta(\beta)(l \otimes m) = \beta(l)(b \otimes m) .
\]

on the pure tensors of \( L \otimes M \). To show that \( \Delta(\beta) \) is an element of \([L \otimes M, N]_0\), we must show that it is a CRT-pairing. Most of the CRT-pairing relations are of the form

\[
\theta_1(\Delta(\beta)(\theta_2(l) \otimes \theta_3(m))) = \psi_1(\Delta(\beta)(\psi_2(l) \otimes \psi_3(m)))
\]

for certain CRT-operations \( \theta_i \) and \( \psi_i \). To show this holds, we use the fact that \( \beta(l) \) is a CRT-pairing for any \( l \in L_i^X \). Let \( \theta_2(l) \in L_j^Y \) and let \( c \) be the generator of \( F(c, j, Y) \). Then

\[
\theta_1(\Delta(\beta)(\theta_2(l) \otimes \theta_3(m))) = \theta_1(\beta(\theta_2(l))(c \otimes \theta_3(m))) \\
= \theta_1(\beta(\theta_2(l)(c \otimes \theta_3(m)))) \\
= \theta_1(\beta(l)(\mu_{\theta_2}(c) \otimes \theta_3(m))) \\
= \theta_1(\beta(l)(\theta_2(b) \otimes \theta_3(m))) \\
= \psi_1(\beta(l)(\psi_2(b) \otimes \psi_3(m))) \\
= \psi_1(\Delta(\beta)(\psi_2(l) \otimes \psi_3(m)))
\]

The most general form of the CRT-pairing relations is

\[
\theta_1(\alpha(\theta_2(l) \otimes \theta_3(m))) = \sum_{i=1}^n (\psi_{1,1}(\alpha(\psi_{1,2}(l) \otimes \psi_{1,3}(m))))
\]

which can be demonstrated similarly.

Finally, we need to show that \( \Gamma \) and \( \Delta \) are inverses. To show \( \Gamma \circ \Delta = 1 \), let \( \beta \in [L, \text{Hom}_{CRT}(M, N)]_0 \) be
given. For any \( l \in L^X_1 \) and for any arbitrary pure tensor \( \theta(b) \otimes m \in F(b, i, X) \otimes M \) we have

\[
((\Gamma \circ \Delta)\beta)(l)(\theta(b) \otimes m) = \Gamma(\Delta(\beta))(l)(\theta(b) \otimes m) \\
= (\Delta(\beta) \circ (\mu^b_1 \otimes 1))(\theta(b) \otimes m) \\
= \Delta(\beta)(\theta(l) \otimes m) \\
= \beta(\theta(l))(c \otimes m) \\
= \theta(\beta(l))(c \otimes m) \\
= \beta(l)(\mu^\theta(c) \otimes m) \\
= \beta(l)(\theta(b) \otimes m) .
\]

To show \( \Delta \circ \Gamma = 1 \), let \( \alpha \in [L \otimes M, N]\) be given. Then

\[
(\Delta \circ \Gamma)(\alpha)(l \otimes m) = \Delta(\Gamma(\alpha))(l \otimes m) \\
= \Gamma(\alpha)(l)(b \otimes m) \\
= \alpha \circ (\mu^b_1 \otimes 1)(b \otimes m) \\
= \alpha(l \otimes m) .
\]

Proposition 3.4. If \( M \) and \( N \) are CRT-modules, then

\[
\Hom_{\text{CRT}}(M, N)^O \cong [M, N]_* .
\]

Proof.

\[
\Hom_{\text{CRT}}(M, N)^O = [F(b, i, \mathbb{R}) \otimes_{\text{CRT}} M, N]_0 \cong [M, N]_i
\]

Proposition 3.5. If \( M \) and \( N \) are CRT-modules, then

\[
\Hom_{\text{CRT}}(M, N)^U \cong \Hom_{\text{KU}^*}(M^U, N^U) .
\]
Proof. By use of suspensions, the problem is reduced to that of showing that the two graded groups are isomorphic in graded degree zero. By definition, $\text{Hom}_{\text{CRT}}(M, N)^U = [F(b, 0, \mathbb{C}) \otimes M, N]_0$. Recall from Proposition 3.6 of [?] that

$$
F(b, 0, \mathbb{C}) \otimes_{\text{CRT}} M \cong \{ \{ r(b \otimes m_U) \mid m_U \in M_U \}, \\
\{ b \otimes m_1 + \psi_U(b \otimes m_2) \mid m_i \in M_U \}, \\
\{ \gamma(b \otimes m_1) + \varepsilon r(b \otimes m_2) \mid m_i \in M_U \} \}
$$

Any degree-0, CRT-module homomorphism from $F(b, 0, \mathbb{C}) \otimes M$ to $N$ restricts to a degree-0, $KU_*(\mathbb{R})$-module homomorphism from $M$ to $N$ by restricting to elements of the form $b \otimes m$ for $m \in M$. This gives us a natural transformation

$$
\Gamma: \text{Hom}_{\text{CRT}}(M, N)^U_0 \to \text{Hom}_{KU_*(\mathbb{R})}(M^U, N^U)_0
$$

To define an inverse

$$
\Delta: \text{Hom}_{KU_*(\mathbb{R})}(M^U, N^U)_0 \to \text{Hom}_{\text{CRT}}(M, N)^U_0
$$

let $\phi: M^U \to N^U$ be a degree-0, $KU_*(\mathbb{R})$-module homomorphism. Then define $\Gamma(\phi) \in [F(b, 0, \mathbb{C}) \otimes M, N]$ by

$$
\Delta(\phi)r(b \otimes m) = r\phi(m) \\
\Delta(\phi)(b \otimes m) = \phi(m) \\
\Delta(\phi)\phi_U(b \otimes m) = \psi_U\phi(m) \\
\Delta(\phi)\gamma(b \otimes m) = \gamma\phi(m) \\
\Delta(\phi)\varepsilon r(b \otimes m) = \varepsilon r\phi(m)
$$

It is clear that $\Gamma$ and $\Delta$ are inverses once we are convinced that $\Delta(\phi)$ is an honest CRT-module homomorphism. We must show that all of the CRT-operations commute with $\Delta(\phi)$. This is fairly straightforward but involves many tedious computations. As a representative example we will show that the operation $\omega$ commutes with $\Delta(\phi)$, when operating on elements of the form $\gamma(b \otimes m)$ and $\varepsilon r(b \otimes m)$.

$$
\Delta(\phi)\omega\gamma(b \otimes m) = \Delta(\phi)\beta r\gamma\zeta\eta(b \otimes m) \\
= 0 \\
= \beta r\gamma\zeta\eta\phi(m) \\
= \omega\gamma\phi(m) \\
= \omega\Delta(\phi)\gamma(b \otimes m)
$$
\[
\Delta(\phi) \omega \varepsilon r (b \otimes m) = \Delta(\phi) \beta_r \gamma \varepsilon r (b \otimes m)
\]
\[
= \Delta(\phi) \beta_r \gamma (1 + \psi_U) (b \otimes m)
\]
\[
= \Delta(\phi) 2 \gamma \beta_r^2 (b \otimes m)
\]
\[
= \Delta(\phi) \gamma (b \otimes 2 \beta_r^2 m)
\]
\[
= \gamma \phi (2 \beta_r^2 m)
\]
\[
= 2 \beta_r \gamma \phi (m)
\]
\[
= \beta_r \gamma (1 + \psi_U) \phi (m)
\]
\[
= \beta_r \gamma \varepsilon r \phi (m)
\]
\[
= \omega \Delta(\phi) \varepsilon r (b \otimes m).
\]

### 3.2 Hom and free objects

Recall that if \( M \) is a \( \mathbb{Z} \)-graded object, then \( \Sigma M \) is the \( \mathbb{Z} \)-graded object defined by \( (\Sigma M)_n = M_{n+1} \). Now, given a CRT-module \( N \), let \( \mathfrak{C}(N) \) be the CRT-module defined to be the triple of graded groups, \( \{ N^v, N^v \oplus N^u, \Sigma N^v \oplus N^u \} \) with CRT-operations

\[
\varepsilon(n) = (0, n)
\]
\[
\zeta(n_1, n_2) = (n_2, n_2)
\]
\[
\psi_U(n_1, n_2) = (n_2, n_1)
\]
\[
\psi_r(n_1, n_2) = (-n_1, n_2)
\]
\[
\gamma(n_1, n_2) = (n_1 + n_2, 0)
\]
\[
\tau(n_1, n_2) = n_1
\]

with \( c \) and \( r \) defined by \( c = \zeta \varepsilon \) and \( r = \tau \gamma \). It is convenient to write the elements of this CRT-module formally as

\[
\mathfrak{C}(N) = \{ r(n_U) | n_U \in N^v \},
\]
\[
\{ n_U | n_U \in N^v \} \oplus \{ \psi_U(n_U) | n_U \in N^v \},
\]
\[
\{ \gamma(n_U) | n_U \in N^v \} \oplus \{ \varepsilon r(n_U) | n_U \in N^v \} \}
\]

Also, \( \mathfrak{T}(N) \) is defined to be the triple \( \{ N^T, N^v \oplus \Sigma N^u, N^T \oplus \Sigma N^T \} \) with operations as given below:

\[
\varepsilon(n) = (n, (-1)^{|n|} \varepsilon \tau(n))
\]
\( \zeta(n_1, n_2) = (\zeta(n_1), \zeta(n_2)) \)
\( \psi_U(n_1, n_2) = (\psi_U(n_1), \psi_U(n_2)) \)
\( \psi_T(n_1, n_2) = (\psi_T(n_1) + (-1)^{|n_2|+1})\beta^{-1}\omega(n_2), \psi_T(n_2)) \)
\( \gamma(n_1, n_2) = (\gamma(n_1), \gamma(n_2)) \)
\( \tau(n_1, n_2) = \varepsilon\tau(n_1) + \eta_T(n_1) + (-1)^{|n_2|}n_2 \)

In these formulas, the notation \(|n|\) refers to the inherent graded degree of the element \( n \in N \).

**Proposition 3.6.** Let \( N \) be an arbitrary CRT-module. Then there is a natural isomorphism

\[ \text{Hom}_{\text{CRT}}(F(b, 0, \mathbb{R}), N) \cong N \]

**Proposition 3.7.** Let \( N \) be an arbitrary CRT-module. Then there is a natural isomorphism

\[ \text{Hom}_{\text{CRT}}(F(b, 0, \mathbb{C}), N) \cong \mathcal{F}(N) \]

**Proposition 3.8.** Let \( N \) be an arbitrary CRT-module. Then there is a natural isomorphism

\[ \text{Hom}_{\text{CRT}}(F(b, 0, T), N) \cong \mathfrak{S}(N) \]

**Proof of Proposition 3.** Let \( F = F(b, 0, \mathbb{R}) \). For any \( i \in \mathbb{Z} \) and \( X \in \{O, U, T\} \) we have an isomorphism

\[ N^X_i \cong [F(c, i, X), N]_0 \cong [F(c, i, X) \otimes F, N]_0 = \text{Hom}_{\text{CRT}}(F, N)^X_i \]

given by \( \Gamma: f \mapsto f(c \otimes b) \) (or simply \( f \mapsto f(b) \)). So \( N \) and \( \text{Hom}_{\text{CRT}}(F, N) \) are isomorphic as graded groups. To show that the CRT-operations coincide, it suffices to show that \( \Gamma \) commutes with a typical CRT-operation \( \theta: N^X_i \rightarrow N^Y_j \). Given any \( f \in [F(c, i, X), N]_0 \):

\[ \Gamma(\theta(f)) = (\theta(f))(c) = (f \circ \mu_\theta)(c) = f(\theta(c)) = \theta(f(c)) = \theta(\Gamma(f)) . \]
Proof of Proposition ?? \[ \text{Let } F = F(b, 0, \mathbb{C}). \text{ Then for any integer } i \text{ we have} \]

\[
\begin{align*}
\text{Hom}_{\text{CRT}}(F, N)_i^O &= [F(c, i, \mathbb{R}) \otimes F, N]_0 \\
&= [F(x, i, \mathbb{C}), N]_0 \\
&= N_i^U \\
\text{Hom}_{\text{CRT}}(F, N)_i^U &= [F(c, i, \mathbb{C}) \otimes F, N]_0 \\
&= [F(y, i, \mathbb{C}) \oplus F(z, i, \mathbb{C}), N]_0 \\
&= N_i^U \oplus N_i^U \\
\text{Hom}_{\text{CRT}}(F, N)_i^T &= [F(c, i, T) \otimes F, N]_0 \\
&= [F(u, i, \mathbb{C}) \oplus F(v, i + 1, \mathbb{C}), N]_0 \\
&= N_i^U \oplus \Sigma N_i^U
\end{align*}
\]

since \( F(c, i, \mathbb{R}) \otimes F(b, 0, \mathbb{C}) \) has free generator \( x = c(c) \otimes b \); the \( \text{CRT}\)-module \( F(c, i, \mathbb{C}) \otimes F(b, i, \mathbb{C}) \) has free generators \( y = c \otimes b \) and \( z = \psi_U(c) \otimes b \); and \( F(c, i, T) \otimes F(b, i, \mathbb{C}) \) has free generators \( u = \zeta(c) \otimes b \) and \( v = cr(c) \otimes b \) (using Propositions 3.5, 3.6, and 3.7 of ??). Again, we have the isomorphism we want on the level of graded groups. We will show that the isomorphism respects the \( \text{CRT}\)-operations.

For any \( n \in N_i^U \), let \( n^x \in [F(c, i, \mathbb{R}) \otimes F, N]_0 = \text{Hom}_{\text{CRT}}(F, N)_i^O \) be the \( \text{CRT}\)-homomorphism which carries \( x \) to \( n \). In \([F(c, i, \mathbb{C}) \otimes F, N]_0 = \text{Hom}_{\text{CRT}}(F, N)_i^U \), let \( n^y \) (respectively \( n^z \)) be the \( \text{CRT}\)-homomorphism which carries \( y \) (respectively \( z \)) to \( n \) and \( z \) (respectively \( y \)) to \( 0 \). For \( n \in N_i^U \), let \( n^u \) (respectively \( n^v \)) be the element of \([F(c, i, T) \otimes F, N]_0 = \text{Hom}_{\text{CRT}}(F, N)_i^T \) which carries \( u \) (respectively \( v \)) to \( n \) and vanishes on \( v \) (respectively \( u \)). Finally, for \( n \in N_i^U + 1 \), let \( n^v \in [F(c, i, T) \otimes F, N] = \text{Hom}_{\text{CRT}}(F, N)_i^T \) be the \( \text{CRT}\)-homomorphism which carries \( v \) to \( n \) and vanishes on \( u \).

The four relations

\[
\begin{align*}
r(n^y) &= n^x \\
\gamma(n^y) &= n^v \\
\varepsilon(n^x) &= n^u \\
\psi_U(n^y) &= n^z
\end{align*}
\]

are demonstrated below and are sufficient to establish that the two \( \text{CRT}\)-modules under consideration are
isomorphic.

\[
\begin{align*}
 r(n^y)(x) &= n^y \circ (\mu_r \otimes 1)(c(c) \otimes b) \\
 &= n^y(c r(c) \otimes b) \\
 &= n^y((1 + \psi_U)(c) \otimes b) \\
 &= n^y(y + z) \\
 &= n \\
 \gamma(n^y)(u) &= n^y(\mu_\gamma \otimes 1)(\zeta(c) \otimes b) \\
 &= n^y(\zeta(\gamma(c)) \otimes b) \\
 &= 0 \\
 \gamma(n^y)(v) &= n^y(\mu_\gamma \otimes 1)(c r(c) \otimes b) \\
 &= n^y(c R\gamma(c) \otimes b) \\
 &= n^y((1 + \psi_\gamma)(c) \otimes b) \\
 &= n^y(y + z) \\
 &= n \\
 \varepsilon(n^y)(u) &= n^y(\mu_\varepsilon \otimes 1)(\zeta(c) \otimes b) \\
 &= n^y(\zeta(\varepsilon(c)) \otimes b) \\
 &= n^y(c(c) \otimes b) \\
 &= n \\
 \varepsilon(n^y)(v) &= n^y(\mu_\varepsilon \otimes 1)(c r(c) \otimes b) \\
 &= n^y(c R\varepsilon(c) \otimes b) \\
 &= 0 \\
 \psi_U(n^y)(y) &= n^y(\mu_\psi \otimes 1)(c \otimes b) \\
 &= n^y(\psi_U(c) \otimes b) \\
 &= n^y(z) \\
 &= 0 \\
 \psi_U(n^y)(z) &= n^y(\mu_\psi \otimes 1)(\psi_U(c) \otimes b) \\
 &= n^y(y) \\
 &= n
\end{align*}
\]
Proof of Proposition ?? Let $F = F(b, 0, T)$. Then we have

$$\begin{align*}
\text{Hom}_{\text{CRT}}(F, N)^{O}_i & = [F(c, i, \mathbb{R}) \otimes F, N] \\
& = [F(x, i, T), N] \\
& = N^T_i
\end{align*}$$

$$\begin{align*}
\text{Hom}_{\text{CRT}}(F, N)^{U}_i & = [F(c, i, \mathbb{C}) \otimes F, N] \\
& = [F(y, i, \mathbb{C}) \oplus F(z, i + 1, \mathbb{C}), N] \\
& = N^U_i \oplus \Sigma N^U_i
\end{align*}$$

$$\begin{align*}
\text{Hom}_{\text{CRT}}(F, N)^{T}_i & = [F(c, i, T) \otimes F, N] \\
& = [F(u, i, T) \oplus F(v, i + 1, T), N] \\
& = N^T_i \oplus \Sigma N^T_i
\end{align*}$$

since, again referring to Section 3 of [?], $F(c, i, \mathbb{R}) \otimes F(b, 0, T)$ has a free generator $x = \varepsilon(c) \otimes b$; the CRT-module $F(c, i, \mathbb{C}) \otimes F(b, 0, T)$ has free generators $y = c \otimes \zeta b$ and $z = c \otimes c \tau b$; and $F(c, i, T) \otimes F(b, 0, T)$ has free generators $u = c \otimes b$ and $v = c \otimes \varepsilon \tau b$.

As in the previous proof, we designate the following elements: $n^x \in \text{Hom}_{\text{CRT}}(F, N)^{O}_i$ for any $n \in N^T_i$; $n^y \in \text{Hom}_{\text{CRT}}(F, N)^{U}_i$ for $n \in N^U_i$; $n^z \in \text{Hom}_{\text{CRT}}(F, N)^{U}_i$ for any $n \in N^U_{i+1}$; $n^u \in \text{Hom}_{\text{CRT}}(F, N)^{T}_i$ for any $n \in N^T_i$; and finally $n^v \in \text{Hom}_{\text{CRT}}(F, N)^{T}_i$ for any $n \in N^T_{i+1}$.

To show that we have an isomorphism of CRT-modules we need to show that the relations in the definition of $\Xi(N)$ hold for these elements defined in the previous paragraph.

$$\begin{align*}
\varepsilon(n^x) & = n^u + (-1)^i(\varepsilon \tau n)^v = n^u + (-1)^{|n|}(\varepsilon \tau n)^v \\
\zeta(n^u) & = (\zeta n)^y \\
\zeta(n^v) & = (\zeta n)^z \\
\psi_U(n^y) & = (\psi_U n)^y \\
\psi_U(n^z) & = (\psi_U n)^z \\
\psi_T(n^u) & = (\psi_T n)^u \\
\psi_T(n^v) & = (-1)^i\beta^{-1}_n \omega n^u + (\psi_T n)^v = (-1)^{|n|+1}\beta^{-1}_n \omega n^u + (\psi_T n)^v \\
\gamma(n^y) & = (\gamma n)^y \\
\gamma(n^z) & = (\gamma n)^z \\
\tau(n^u) & = (\varepsilon \tau n + \eta_T n)^x \\
\tau(n^v) & = (-1)^{i+1} n^x = (-1)^{|n|} n^x
\end{align*}$$
We will work out a few of these, leaving the rest for the reader. The calculations
\[
\varepsilon(n^x)(u) = n^x(\varepsilon(c) \otimes b) = n
\]
\[
\varepsilon(n^x)(v) = n^x(\varepsilon(c) \otimes \varepsilon \tau b) \\
= \varepsilon n^x(c \otimes \tau b) \\
= (-1)^i \varepsilon n^x(\varepsilon(c) \otimes b) \\
= (-1)^i \varepsilon \tau(n)
\]
\[
\tau(n^u)(x) = n^y(\varepsilon \tau(c) \otimes b) \\
= n^y(\varepsilon \tau(c \otimes b) + (-1)^{i+1} c \otimes \varepsilon \tau b + \eta \tau c \otimes b) \\
= \varepsilon \tau n + \eta \tau n
\]
\[
\tau(n^v)(x) = n^y(\varepsilon \tau(c) \otimes b) \\
= n^y(\varepsilon \tau(c \otimes b) + (-1)^{i+1} c \otimes \varepsilon \tau b + \eta \tau c \otimes b) \\
= (-1)^{i+1} n
\]
show that \(\varepsilon(n^x) = n^u + (-1)^i(\varepsilon \tau n)^v; \tau(n^u) = (\varepsilon \tau n + \eta \tau n)^x\); and \(\tau(n^v) = (-1)^{i+1} n^x\).

3.3 Hom and acyclic objects

**Proposition 3.9.** Let \(M\) be a free CRT-module and \(N\) be an acyclic CRT-module. Then \(\text{Hom}_{\text{CRT}}(M, N)\) is acyclic.

**Proof.** First we consider the case that \(M\) is monogenic. If \(M = F(b, 0, \mathbb{R})\), then by Proposition ??, \(\text{Hom}_{\text{CRT}}(M, N) \cong N\) which is acyclic.

If \(M = F(b, 0, \mathbb{C})\), then \(\text{Hom}_{\text{CRT}}(M, N) \cong \mathcal{C}(N)\) by Proposition ??. We claim that \(\mathcal{C}(N)\) is acyclic. In fact, by comparing to Proposition 3.6 in [?], the CRT-module \(\mathcal{C}(N)\) is isomorphic to \(F(b, 0, \mathbb{C}) \otimes N\), which is acyclic by Lemma 3.11 in [?].

If \(M = F(b, 0, T)\), then \(\text{Hom}_{\text{CRT}}(M, N) \cong \mathfrak{T}(N)\) by Proposition ??, By Proposition 3.7 in [?],

\[
F(b, 0, T) \otimes_{\text{CRT}} N \cong \{ \{ \tau(b \otimes n_T) \mid n_T \in N_T \}, \\
\{ \zeta b \otimes n_U \mid n_U \in N_U \} \oplus \{ cb \otimes n_U \mid n_U \in N_U \}, \\
\{ b \otimes n_T \mid n_T \in N_T \} \oplus \{ \varepsilon \tau b \otimes n_T \mid n_T \in N_T \} \} \\
\cong \{ \Sigma^{-1} N_T, N_U \oplus \Sigma^{-1} N_U, N_T \oplus \Sigma^{-1} N_T \}.
\]
It can easily be shown that the defining relations of $\mathfrak{S}(N)$ are satisfied by the elements

$$
(-1)^{|n_T|\tau} (b \otimes n_T) \in (F(b,0,T) \otimes_{CRT} N)^o
$$

$$
(-1)^{|n_U|} crb \otimes n_U \in (F(b,0,T) \otimes_{CRT} N)^v
$$

$$
\zeta b \otimes n_U \in (F(b,0,T) \otimes_{CRT} N)^v
$$

$$
(-1)^{|n_T|} (\varepsilon \tau b \otimes n_T + \eta_T (b \otimes n_T)) \in (F(b,0,T) \otimes_{CRT} N)^\tau
$$

$$
b \otimes n_T \in (F(b,0,T) \otimes_{CRT} N)^\tau
$$

of $F(b,0,T) \otimes_{CRT} N$. Therefore, $\mathfrak{S}(N)$ is isomorphic to $F(b,0,T) \otimes_{CRT} N$ (with a shift); and so Lemma 3.13 in [?] implies that $\mathfrak{S}(N)$ is acyclic.

Now the suspension formula

$$
\text{Hom}_{CRT}(\Sigma M, N) = \Sigma^{-1} \text{Hom}_{CRT}(M, N)
$$

follows directly from the definition and implies that $\text{Hom}_{CRT}(M, N)$ is acyclic if we take $M$ to be any free monogenic $CRT$-module.

Finally, we can write an arbitrary free $CRT$-module as a direct sum of monogenic free $CRT$-modules (see Section 2.4 of [?]). Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$. Then we claim that $\text{Hom}_{CRT}(M, N) \cong \prod_{\lambda \in \Lambda} \text{Hom}_{CRT}(M_\lambda, N)$. It then follows that $\text{Hom}_{CRT}(M, N)$ is acyclic. To prove the claim, we have

$$
\text{Hom}_{CRT} \left( \bigoplus_{\lambda \in \Lambda} M_\lambda, N \right)_i^X \cong \left[ F(b,i,X) \otimes \bigoplus_{\lambda \in \Lambda} M_\lambda, N \right]
$$

$$
\cong \left[ \bigoplus_{\lambda \in \Lambda} (F(b,i,X) \otimes M_\lambda), N \right] \quad \text{by Lemma IV.1.6 in [?]}
$$

$$
\cong \prod_{\lambda \in \Lambda} [F(b,i,X) \otimes M_\lambda, N] \quad \text{by Exercise A.1.4 in [?]}
$$

$$
\cong \prod_{\lambda \in \Lambda} \text{Hom}_{CRT}(M_\lambda, N)_i^X.
$$

Since these isomorphisms are natural, they commute with respect to any homomorphism $\mu_\theta : F(c,j,Y) \to F(b,i,X)$ and so we have a $CRT$-module homomorphism as desired.

**Proposition 3.10.** Let $M$ be an acyclic $CRT$-module and let $N$ be an injective $CRT$-module. Then $\text{Hom}_{CRT}(M, N)$ is acyclic.

**Proof.** Because $M$ is acyclic, it has projective dimension at most 1 (Theorem 3.4 in [?]) so we can form a resolution of $M$

$$
0 \to F_1 \to F_0 \to M \to 0
$$

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where $F_0$ and $F_1$ are projective. Then by Theorem 3.2 in [?] $F_0$ and $F_1$ are free. Now, applying $\text{Hom}_{\text{CRT}}(-, N)$ to this resolution, we obtain

$$\text{Ext}_{\text{CRT}}(M, N) \leftarrow \text{Hom}_{\text{CRT}}(F_1, N) \leftarrow \text{Hom}_{\text{CRT}}(F_0, N) \leftarrow \text{Hom}_{\text{CRT}}(M, N) \leftarrow 0.$$ 

Since $N$ is injective, $\text{Ext}_{\text{CRT}}(M, N) = 0$ so we have a short exact sequence of CRT-modules.

By Theorem 3.3 in [?], the injectivity of $N$ implies it is acyclic. Then by Proposition ??, $\text{Hom}_{\text{CRT}}(F_0, N)$ and $\text{Hom}_{\text{CRT}}(F_1, N)$ are acyclic. Finally, the Snake Lemma (1.3.2 in [?]) implies $\text{Hom}_{\text{CRT}}(M, N)$ is acyclic. □

4 The Universal Coefficient Theorem

By Proposition ??, the CRT-pairing

$$\alpha: KK^{\text{CRT}}(\mathbb{R}, A) \otimes_{\text{CRT}} KK^{\text{CRT}}(A, B) \rightarrow KK^{\text{CRT}}(\mathbb{R}, B)$$

described in Section ?? implies the existence of a natural CRT-homomorphism

$$\gamma: KK^{\text{CRT}}(A, B) \rightarrow \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B))$$

which is the adjoint of $\alpha$. We note that we are also implicitly making use of the isomorphism between $KK^{\text{CRT}}(\mathbb{R}, A) \otimes_{\text{CRT}} KK^{\text{CRT}}(A, B)$ and $KK^{\text{CRT}}(A, B) \otimes_{\text{CRT}} KK^{\text{CRT}}(\mathbb{R}, A)$.

Theorem 4.1. Let $A$ and $B$ be real separable C*-algebras such that $\mathbb{C} \otimes A$ is in the bootstrap category $\mathcal{N}$. Then there is a short exact sequence

$$0 \rightarrow \text{Ext}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B)) \xrightarrow{\kappa} KK^{\text{CRT}}(A, B) \xrightarrow{\gamma} \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B)) \rightarrow 0$$

where $\kappa$ has degree $-1$.

As in Rosenberg and Schochet in [?], the proof of this theorem will proceed in two steps. The first step (in Section ??) addresses the special case in which $K^{\text{CRT}}(B)$ is injective and the second step (in Section ??) uses a geometric injective resolution of $K^{\text{CRT}}(B)$ to extend the result to arbitrary separable $B$. Our debt to Rosenberg and Schochet goes beyond analogy, however, since we will actually reduce our proof in the special case to an application of the universal coefficient theorem for complex C*-algebras.

In section ??, we will discuss the most immediate ramifications of Theorem ?? . Throughout the rest of this paper, all C*-algebras are assumed to be separable.

4.1 Proof of the UCT — Special Case

In the special case that $K^{\text{CRT}}(B)$ is injective, the CRT-module $\text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B))$ vanishes. Therefore the universal coefficient theorem collapses to the following.
Proposition 4.2. Let $A$ and $B$ be real C*-algebras such that $\mathbb{C} \otimes A$ is in $\mathcal{N}$ and $K^{\text{CRT}}(B)$ is an injective CRT-module. Then

$$\gamma: KK^{\text{CRT}}(A, B) \to \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B))$$

is an isomorphism.

Before we prove this proposition, we need two lemmas relating $KK$-theory for real C*-algebras to $KK$-theory for complex C*-algebras. Up to this point, we have been dealing entirely with real $KK$-theory which for real C*-algebras $A$ and $B$ is defined in terms of real Kasparov bimodules as in [?], Section 2.3. Recall that a real Kasparov $(A,B)$-bimodule is a triple $(E,\phi,T)$ consisting of a Hilbert $B$-module $E$, a graded $*$-homomorphism $\phi: A \to \mathcal{L}_B(E)$, and a degree one operator $T \in \mathcal{L}_B(E)$ such that the elements $(T - T^*) \phi(a)$, $(T^2 - 1) \phi(a)$, and $[T, \phi(a)]$ all lie in $\mathcal{K}_B(E)$ for all $a \in A$. In particular, even if $A$ and $B$ happen to be complex C*-algebras, the homomorphism $\phi$ is only required to be real. Note that if $B$ is a complex C*-algebra, then $E$ (and hence $\mathcal{L}_B(E)$) inherits a complex structure using the approximate identity of $B$ together with the fact that the approximate identity for $B$ is an approximate identity for the action of $B$ on $E$ (Lemma 1.1.4 of [?]).

By contrast, if $A$ and $B$ are complex C*-algebras, let $KK_c(A,B)$ denote the complex $KK$-theory comprised of equivalence classes of complex Kasparov bimodules $(E,\phi,T)$. This is the $KK$-theory of Rosenberg and Schochet’s Universal Coefficient Theorem. The difference is that now $\phi$ is required to respect the complex structures of $A$ and $\mathcal{L}_B(E)$. Since every complex homomorphism is a real homomorphism, there is a natural transformation $\rho: KK_c(A,B) \to KK(A,B)$ which forgets the complex structures. The next lemma describes another natural transformation connecting the two versions of $KK$-theory.

Lemma 4.3. Let $A$ be a real C*-algebra and let $B$ be a complex C*-algebra. There is a natural isomorphism $\nu: KK_c(\mathbb{C} \otimes A, B) \to KK(A,B)$.

Proof. Let $(E,\phi,T)$ be a complex Kasparov $(\mathbb{C} \otimes A, B)$-bimodule representing an element of $KK_c(\mathbb{C} \otimes A, B)$. Then by restricting $\phi$ to $\phi|_A: A \to \mathcal{L}_B(E)$, we obtain a real Kasparov $(A,B)$-bimodule $(E,\phi|_A,T)$. This mapping induces a homomorphism $\nu$ which has an inverse: given a real Kasparov $(A,B)$-bimodule $(E,\phi,T)$, the homomorphism $\phi: A \to \mathcal{L}_B(E)$ extends uniquely to a complex homomorphism $\phi_c: \mathbb{C} \otimes A \to \mathcal{L}_B(E)$ using the complex structure of $\mathcal{L}_B(E)$. \hfill \Box

Now, we recall the mechanics of the intersection product described for the general case in [?]. Other expositions include Section 2.4 in [?] for real C*-algebras; and Section 18.3 in [?] and Section 2.2 in [?] for complex C*-algebras. Let $A$, $B$, and $C$ be real C*-algebras; let $x \in KK(A, B)$ be represented by an $(A, B)$-bimodule $(E,\phi,T)$; and let $y \in KK(B, C)$ be represented by a $(B, C)$-bimodule $(F,\psi,S)$. Then the product $\alpha(x \otimes y) \in KK(A, C)$ is given by an $(A, C)$-bimodule $(E \otimes_{\phi} F, \phi \otimes_{\psi} 1, T \# S)$. The operator $T \# S$
The same construction also defines a pairing

$$\alpha_c : KK_c(A,B) \otimes KK_c(B,C) \rightarrow KK_c(A,C)$$

if $A$, $B$, and $C$ are complex $C^*$-algebras (not to be confused with $\alpha_v$ described in Section ??).

We can also effect a pairing

$$\alpha' : KK(A,C \otimes B) \otimes KK(B,C \otimes C) \rightarrow KK(A,C \otimes C)$$

by $\alpha' = \alpha \circ (1 \otimes \mu_\tau) \circ (1 \otimes \tau_c)$ where $\mu : C \otimes C \otimes C \rightarrow C \otimes C$ is given by complex multiplication and the $\tau_c$ is the homomorphism

$$KK_c(B,C \otimes C) \rightarrow KK_c(C \otimes B,C \otimes C \otimes C)$$

defined by $(E,\phi,T) \mapsto (C \otimes E,1 \otimes \phi,1 \otimes T)$ (according to the external tensor product construction in Section 2.1.6 of [??]).

**Lemma 4.4.** Let $A$, $B$, and $C$ be real $C^*$-algebras. Then the following diagram commutes:

$$\begin{array}{ccc}
KK_c(C \otimes A,C \otimes B) \otimes KK_c(C \otimes B,C \otimes C) & \xrightarrow{\alpha_c} & KK_c(C \otimes A,C \otimes C) \\
\downarrow{\nu \circ \nu} & & \downarrow{\nu} \\
KK(A,C \otimes B) \otimes KK(B,C \otimes C) & \xrightarrow{\alpha'} & KK(A,C \otimes C)
\end{array}$$

**Proof.** We claim first that $\mu_\tau \circ \tau_c \circ \nu = \rho : KK_c(C \otimes B,C \otimes C) \rightarrow KK_c(C \otimes B,C \otimes C)$. Let $x$ be an element of $KK_c(C \otimes B,C \otimes C)$ represented by a complex Kasparov $(C \otimes B,C \otimes C)$-bimodule $(F,\psi,S)$. Then $\nu(x)$ is represented by $(F,\psi|_B,S)$ and $(\tau_c \circ \nu)(x) \in KK(C \otimes B,C \otimes C \otimes C)$ is represented by $(C \otimes F,1 \otimes \psi|_B,1 \otimes S)$. Finally $\mu_\tau \nu(x)$ is given by $(\tilde{F},\mu_\tau(1 \otimes \psi_B),\mu_\tau(1 \otimes S))$ where $\tilde{F}$ is the Hilbert $C \otimes C$-module

$$\tilde{F} = \mu_\tau(C \otimes F) = (C \otimes F)/\{x|\mu_\tau(x,x) = 0\}$$

(according to the push-out construction of Section 2.1.5 of [??]). There is a Hilbert $C \otimes C$-module homomorphism $\mu_F : \tilde{F} \rightarrow F$ defined by $\mu_F[\lambda \otimes f] = \lambda f$. Since $\langle \mu_F(x),\mu_F(y) \rangle = \mu(x,y)$, this is well-defined. Define
\[ \mu^{-1}_F(f) = [1 \otimes f]. \] Clearly \( \mu_F \circ \mu^{-1}_F = 1. \) To show that \( \mu^{-1}_F \circ \mu_F = 1, \) it is straightforward to calculate that \( \mu(x, x) = 0 \) if \( x = \lambda \otimes f - 1 \otimes \lambda f. \) Therefore, \( \mu_F \) is an isomorphism. Under this isomorphism, the homomorphism \( \mu_*(1 \otimes \psi_B) \) corresponds to \( \psi \) and the operator \( \mu_*(1 \otimes S) \) corresponds to \( S. \) This proves the claim.

Thus the diagram that we wish to commute can be redrawn as

\[
\begin{array}{ccc}
KK_c(C \otimes A, C \otimes B) \otimes KK_c(C \otimes B, C \otimes C) & \overset{\alpha c}{\longrightarrow} & KK_c(C \otimes A, C \otimes C) \\
\nu \otimes \rho & & \nu \\
KK(A, C \otimes B) \otimes KK(C \otimes B, C \otimes C) & \overset{\tilde{\alpha}}{\longrightarrow} & KK(A, C \otimes C)
\end{array}
\]

To prove that this commutes, let \( x \) and \( y \) be elements of \( KK_c(C \otimes A, C \otimes B) \) and \( KK_c(C \otimes B, C \otimes C) \) respectively. We must show that \( \alpha(\nu(x) \otimes \rho(y)) = \nu \alpha_c(x \otimes y). \) Let \( x \) be represented by a Kasparov \((C \otimes A, C \otimes B)\)-bimodule \((E, \phi, T)\) as before and let \( y \) be represented by a Kasparov \((C \otimes B, C \otimes C)\)-bimodule \((F, \psi, S)\).

Now, \( \nu(x) = (E, \phi_A, T) \) and \( \rho(y) = (F, \psi, S) \) are both real Kasparov bimodules, as is the product \( \alpha(\nu(x) \otimes \rho(y)) = (E \otimes \psi F, \phi_A \otimes 1, T \# S). \) On the other hand, \( x = (E, \phi, T) \) and \( y = (F, \psi, S) \) are both complex Kasparov bimodules, as is the product \( \alpha_c(x \otimes y) = (E \otimes \psi F, \phi \otimes 1, T \# S). \) However, since \( \psi \) respects the complex structures of \( C \otimes B \) and \( L_{C \otimes C}(C \otimes E), \) the Hilbert \((C \otimes C)\)-module \( E \otimes \psi F \) is the same in either case. Furthermore, the operator \( T \# S \in L_{C \otimes C}(E \otimes \psi F) \) can be taken to be the same in either case since any operator which satisfies the formulae given in (??) for all \( a \in C \otimes A \) will certainly satisfy the same formulae for all \( a \in A. \) Finally, since the restriction of \( \phi \otimes 1: C \otimes A \to L_{C \otimes C}(E \otimes \psi F) \) to \( A \) is \( \phi|_A \otimes 1, \) it follows that

\[
\nu(\alpha_c(x \otimes y)) = \alpha(\nu(x) \otimes \rho(y)).
\]

\[ \Box \]

**Proof of Proposition ??**. According to Propositions ?? and ??, both \( KK^{\text{CRT}}(A, B) \) and \( \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B)) \) are acyclic CRT-modules. Hence by Section 2.3 of [?], to show that

\[
\gamma: KK^{\text{CRT}}(A, B) \to \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B))
\]

is an isomorphism, it suffices to show that the complex part

\[
KK^{\text{CRT}}(A, B)^\vee \to \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B))^{\vee}
\]

is an isomorphism. Using the definition of united \( KK \)-theory and Proposition ??, this is equivalent to showing that the homomorphism

\[
\gamma_U: KK(A, C \otimes B) \to \text{Hom}_{KU, (\mathbb{R})}(K(C \otimes A), K(C \otimes B))
\]

(which is the adjoint of \( \alpha_U \)) is an isomorphism.
Table 2: $K^{CRT}(N_1)$

| n  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|----|---|---|---|---|---|---|---|---|---|
| $KO_*(N_1)$ | Q | 0 | 0 | 0 | Q | 0 | 0 | 0 | Q |
| $KU_*(N_1)$ | Q | 0 | Q | 0 | Q | 0 | Q | 0 | Q |
| $KT_*(N_1)$ | Q | 0 | Q | 0 | Q | 0 | Q | 0 | Q |
| $e_n$ | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 1 |   |
| $r_n$ | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |   |
| $\varepsilon_n$ | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 |
| $\zeta_n$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $(\psi_U)_n$ | 1 | 0 | -1 | 0 | 1 | 0 | -1 | 0 | 1 |
| $(\psi_T)_n$ | 1 | 0 | 0 | -1 | 1 | 0 | 0 | -1 | 1 |
| $\gamma_n$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $\tau_n$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 |

The following diagram commutes by Lemma ??.

\[
\begin{array}{ccc}
KK(C \otimes A, C \otimes B) & \xrightarrow{\gamma_C} & \text{Hom}_{KU_*(\mathbb{R})}(KK(C, C \otimes A), KK(C, C \otimes B)) \\
\downarrow \nu & & \downarrow \text{Hom}(\nu, \nu) \\
KK(A, C \otimes B) & \xrightarrow{\gamma_U} & \text{Hom}_{KU_*(\mathbb{R})}(K(C \otimes A), K(C \otimes B))
\end{array}
\]

Since $K^{CRT}(B)$ is an injective CRT-module, the complex part $K(C \otimes B) = KK(C \otimes C \otimes B)$ consists of divisible groups (Theorem 3.3 in [?]). Hence, Theorem 2.1 of [?] implies that $\gamma_C$ is an isomorphism and therefore so is $\gamma_U$. 

\[\square\]

4.2 Proof of the UCT — General Case

**Lemma 4.5.** There exists a real C*-algebra $N$ containing a projection $p \in N$ such that 1) $K^{CRT}(N)$ is an injective CRT-module and 2) for all real C*-algebras such that $K^{CRT}(F)$ is free, the homomorphism $F \to F \otimes N$ defined by $a \mapsto a \otimes p$ induces a monomorphism of united K-theory.

The C*-algebra $N$ will play a role analogous to the role in Theorem 3.2 of [?] played by the complex UHF-algebra whose K-theory is $\mathbb{Q}$ in even degrees. In Table ??, we have the united K-theory of the real UHF-algebra $N_1$ with $K_0(N_1) \cong \mathbb{Q}$. However, because of the presence of 2-torsion in free CRT-modules, this algebra will not suffice. To capture this 2-torsion, we will also involve a real C*-algebra $N_2$ whose united K-theory is given in Table ??.

**Proof.** We will first construct the algebra $N_2$ as a limit of a directed system of matrix algebras over real Cuntz algebras. We begin by proving the claim that for any positive integers $n$ and $m$ there is a unital
Since the system of $\text{CRT}_M$ where each connecting map is $S(\text{algebra homomorphism} \phi)$, Now, we define elements $S_1, S_2, \ldots, S_n$ with orthogonal ranges and which satisfy $S_i S_i = 1$ and $\sum_{i=0}^{n} S_i S_i^* = 1$. Let $\mathcal{O}_{m+1}$ similarly be generated by the isometries $T_0, T_{ij}$ for $1 \leq i \leq n, 1 \leq j \leq m$ which satisfy similar relations.

Now, we define elements $S'_0, S'_1, \ldots, S'_n \in M_m(\mathcal{O}_{m+1})$ by

$$S'_0 = \begin{pmatrix} T_0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad S'_i = \begin{pmatrix} T_{i1} & T_{i2} & \cdots & T_{im} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ for } i \geq 1.$$

It can be checked easily that these elements satisfy $(S'_i S'_i = 1$ and $\sum_{i=0}^{n} S'_i (S'_i)^* = 1$ and so there is a C*-algebra homomorphism $\phi$ which takes each $S_i$ to $S'_i$. The homomorphism $\phi$ is unital since $\phi(1) = \phi(S'_1 S_1) = (S'_1)^* S'_1 = 1$. This proves the claim.

Then we define the algebra $N_2$ to be the limit of the system

$$\mathcal{O}_{4+1} \to M_2(\mathcal{O}_{8+1}) \to M_4(\mathcal{O}_{16+1}) \to M_8(\mathcal{O}_{32+1}) \to \ldots$$

where each connecting map is $M_{2k-2}(\phi)$ with $n = 2^k$ and $m = 2$. The united $K$-theory of $N_2$ is the limit of the system of $\text{CRT}$-modules $\{K^{\text{CRT}}(\mathcal{O}_{2k+1}), M_{2k-2}(\phi)\}$. The united $K$-theory of $\mathcal{O}_{2k+1}$ is given in Table ??.

Since $\phi$ induces multiplication by 2 on $KO_0(\mathcal{O}_{2k+1})$, we obtain $KO_0(N_2) = \mathbb{Z}(2^\infty)$, as in all the gradings of $K^{\text{CRT}}(N_2)$ in which $K^{\text{CRT}}(\mathcal{O}_{2^k+1}) \cong \mathbb{Z}_{2^k}$.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_0(N_2)$ | $\mathbb{Z}(2^\infty)$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}(2^\infty)$ | 0 | 0 | 0 | $\mathbb{Z}(2^\infty)$ |
| $KU_0(N_2)$ | $\mathbb{Z}(2^\infty)$ | 0 | $\mathbb{Z}(2^\infty)$ | 0 | $\mathbb{Z}(2^\infty)$ | 0 | $\mathbb{Z}(2^\infty)$ | 0 | $\mathbb{Z}(2^\infty)$ |
| $KT_0(N_2)$ | $\mathbb{Z}(2^\infty)$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}(2^\infty)$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}(2^\infty)$ | 0 | $\mathbb{Z}(2^\infty)$ |
| $c_n$ | 1 | 0 | \(\frac{1}{2}\) | 0 | 2 | 0 | 0 | 0 | 1 |
| $r_n$ | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| $\varepsilon_n$ | 1 | 0 | 1 | \(\frac{1}{2}\) | 2 | 0 | 0 | 0 | 1 |
| $\zeta_n$ | 1 | 0 | \(\frac{1}{2}\) | 0 | 1 | 0 | \(\frac{1}{2}\) | 0 | 1 |
| $(\psi_U)_n$ | 1 | 0 | -1 | 0 | 1 | 0 | -1 | 0 | 1 |
| $(\psi_T)_n$ | 1 | 0 | 1 | -1 | 1 | 0 | 1 | -1 | 1 |
| $\tau_n$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $\tau_n$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 2 | 0 |
Now $\phi_* = 0$ on $KO_1(\mathcal{O}_{2^k+1}) \cong \mathbb{Z}_2$ since the nonzero element of that group is $\eta_0(1)$. The same is true for $KT_2(\mathcal{O}_{2^k+1})$ (with nonzero element $\varepsilon_0(1)$), the first summand of $KO_2(\mathcal{O}_{2^k+1}) (\eta_0^2(1))$, and $KT_5(\mathcal{O}_{2^k+1}) (\beta_7 \varepsilon_0(1))$. Therefore, these groups vanish when we pass to the limit.

On the second summand of $KO_2(\mathcal{O}_{2^k+1})$, we claim that $\phi_*$ is an isomorphism. Indeed, let $\binom{1}{0}$ be the nonzero element of $KO_2(\mathcal{O}_{2^k+1})$ representing the second summand and let $y$ be a generator of $KU_2(\mathcal{O}_{2^k+1}) = \mathbb{Z}_2$. Then $(c \circ \phi_*)(\binom{1}{0}) = (\phi_*(c)) \binom{1}{0} = \phi_*(\frac{k}{2} \cdot y) = k \cdot y$. Therefore $\phi_*(\binom{1}{0}) = \binom{1}{0}$ where $\alpha = 0$ or 1. Using the naturality of the operations $\varepsilon$ and $\eta_0$, it also follows from this that $\phi_*$ is an isomorphism on $KT_2(\mathcal{O}_{2^k+1})$.

Therefore these groups persist in the direct limit and $N_2$ has the united $K$-theory as shown.

The $K$-theory element we want $p$ to represent is the non-zero element of $KO_2(N_2)$, so we will consider $S^2 N_2$ to shift that $K$-theory element down to degree 0. We get $N_1$ into the picture by the following unitization of $S^2 N_2$. For each positive integer $n$ let

$$\phi_n : M_n!(S^2 N_2^+) \to M_{(n+1)!}(S^2 N_2^+)$$

be given by

$$\phi_n(a \oplus \lambda) = \begin{pmatrix} a + \lambda & 0 & 0 & \ldots & 0 \\ 0 & \lambda & 0 & \ldots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda \end{pmatrix}$$

and define $N$ to be the limit of the system $\{M_n!(S^2 N_2^+), \phi_n\}$.

The split exact sequence

$$0 \to S^2 N_2 \to S^2 N_2^+ \to \mathbb{R} \to 0$$

passes to a split exact sequence

$$0 \to K \otimes S^2 N_2 \to N \to N_1 \to 0$$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^4$ | 0 | 0 | 0 | $\mathbb{Z}_2^8$ |
| $KU_n$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2^2$ | 0 | $\mathbb{Z}_2^4$ | 0 | $\mathbb{Z}_2^8$ | 0 | $\mathbb{Z}_2^8$ |
| $KT_n$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_2^8$ |

Table 4: $K^{cr2}(\mathcal{O}_{2^k+1})$ for $k \geq 2$
in the limit, so we have $K^{\text{CRT}}(N) \cong K^{\text{CRT}}(N_1) \oplus K^{\text{CRT}}(S^2N_2)$. According to Theorem 3.3 in [?], a CRT-module is injective if and only if it is acyclic and the groups in the complex part are divisible. Hence $K^{\text{CRT}}(N)$ is injective.

Let $p$ be a projection in $M_l(N)$ such that $[p] - [p_k] = 1 \oplus 1 \in KO_0(N) \cong KO_0(N_1) \oplus KO_2(N_2) \cong \mathbb{Q} \oplus \mathbb{Z}_2$. Then $[p] = (1 + k) \oplus 1$. By picking a new preferred generator of $\mathbb{Q}$, we may assume that $[p] = 1 \oplus 1$. For future reference, we also note that $c[p] = 1 \oplus \frac{1}{2} \in \mathbb{Q} \oplus \mathbb{Z}(2^\infty)$ and $\varepsilon[p] = 1 \oplus 1 \in \mathbb{Q} \oplus \mathbb{Z}_2$.

Now we must show that if $F$ is a C*-algebra such that $K^{\text{CRT}}(F)$ is free, then the homomorphism

$$\psi: K^{\text{CRT}}(F) \to K^{\text{CRT}}(F \otimes N)$$

induced by the inclusion map $a \mapsto a \otimes p$ is injective.

If we define $\phi: K^{\text{CRT}}(F) \to K^{\text{CRT}}(F) \otimes K^{\text{CRT}}(N)$ by

$$\phi(x) = x \otimes [p] \text{ for } x \in KO_*(F)$$

$$\phi(y) = y \otimes c[p] \text{ for } y \in KU_*(F)$$

$$\phi(z) = z \otimes \varepsilon[p] \text{ for } y \in KT_*(F)$$

and we let $\alpha$ be the pairing $K^{\text{CRT}}(F) \otimes K^{\text{CRT}}(N) \to K^{\text{CRT}}(F \otimes N)$, then $\alpha \circ \phi = \psi$. By the K"unneth formula for united $K$-theory (Theorem 4.2 of [?]), $\alpha$ is an isomorphism. So it suffices to show that the map $\phi$ is injective.

Since any free CRT-module can be written as a direct sum of monogenic free CRT-modules and since the homomorphism $\phi$ respects direct sums and suspensions, it is enough to show that $\phi$ is injective in case $F$ is one of the three free monogenic CRT-modules $F(b, 0, \mathbb{R})$, $F(b, 0, \mathbb{C})$, and $F(b, 0, T)$, shown in Table ?? (or see Tables I, II, and III in [?] for the tables of operations).

### Table 5: Free monogenic CRT-modules

| $n$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $F(b, 0, \mathbb{R})_n$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}$ | 0   | 0   | 0   | $\mathbb{Z}$ |
| $F(b, 0, \mathbb{R})_n$ | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ |
| $F(b, 0, \mathbb{R})_n$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_k$ |
| $F(b, 0, \mathbb{C})_n$ | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ |
| $F(b, 0, \mathbb{C})_n$ | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}_2$ |
| $F(b, 0, T)_n$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0   | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $F(b, 0, T)_n$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $F(b, 0, T)_n$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}$ |

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First assume $F = F(b,0,\mathbb{R})$. Then the homomorphism
\[
\phi: K^{\text{CRT}}(F) \to K^{\text{CRT}}(F) \otimes K^{\text{CRT}}(N)
\]
\[
\cong K^{\text{CRT}}(F) \otimes K^{\text{CRT}}(N_1) \oplus K^{\text{CRT}}(F) \otimes \Sigma^2 K^{\text{CRT}}(N_2)
\]
\[
\cong K^{\text{CRT}}(N_1) \oplus \Sigma^2 K^{\text{CRT}}(N_2)
\]
is the unique homomorphism which sends the generator $b$ to $b \otimes [p] = 1 \oplus 1 \in \mathbb{Q} \oplus \mathbb{Z}_2$ and is seen to be injective by checking that the CRT-module generated by $1 \oplus 1$ is isomorphic to $F(b,0,\mathbb{R})$.

In the case of $K^{\text{CRT}}(F) = F(b,0,\mathbb{C})$ we have the homomorphism
\[
\phi: K^{\text{CRT}}(F) \to K^{\text{CRT}}(F) \otimes K^{\text{CRT}}(N_1) \oplus K^{\text{CRT}}(F) \otimes \Sigma^2 K^{\text{CRT}}(N_2)
\]
Using Proposition 3.6 of [?], we compute $K^{\text{CRT}}(F) \otimes K^{\text{CRT}}(N_1)$ and $K^{\text{CRT}}(F) \otimes \Sigma^2 K^{\text{CRT}}(N_2)$ as shown in Table ???. The homomorphism $\phi$ sends the generator $b$ to $b \otimes c[p] = b \otimes (1 \oplus \frac{1}{2})$ which is the element $(\frac{1}{0}) \oplus (\frac{1/2}{0}) \in \mathbb{Q}^2 \oplus \mathbb{Z}(2^\infty)^2$. Again, we check that the CRT-module generated by this element is isomorphic to $F(b,0,\mathbb{C})$.

Finally, in the case of $K^{\text{CRT}}(F) = F(b,0,T)$ we have the homomorphism
\[
\phi: K^{\text{CRT}}(F) \to K^{\text{CRT}}(F) \otimes K^{\text{CRT}}(N_1) \oplus K^{\text{CRT}}(F) \otimes \Sigma^2 K^{\text{CRT}}(N_2)
\]
with the latter groups shown in Table ???, computed using Proposition 3.7 of [?]. The homomorphism $\phi$ is determined by sending $b$ to $b \otimes e[p] = b \otimes (\alpha \oplus 1)$ which is the element $\alpha \oplus 1 \in \mathbb{Q}^2 \oplus \mathbb{Z}_2$. Again, check that this element generates a CRT-submodule freely.

Let $\mathfrak{A} = (S^+)^8 A$ be the eightfold unital suspension of $A$ as utilized in Section 2.2 of [?].

**Proposition 4.6.** Let $A$ be a real unital C*-algebra. Then there exists a C*-algebra $D$ such that $K^{\text{CRT}}(D)$ is an injective CRT-module and a C*-algebra map $\nu: S\mathfrak{A} \to D$ such that the induced map $\nu_*$ on united $K$-theory is injective.
Table 7: $F(b, 0, T) \otimes K^{CRT}(N)$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $(F(b, 0, T) \otimes N_1)^{\otimes}$ | $Q$ | $Q$ | 0 | 0 | $Q$ | 0 | 0 | 0 | $Q$ |
| $(F(b, 0, T) \otimes N_1)_{\mu}^{|}$ | $Q$ | $Q$ | $Q$ | $Q$ | $Q$ | $Q$ | $Q$ | $Q$ | $Q$ |
| $(F(b, 0, T) \otimes N_1)^{\otimes}$ | $Q^{\otimes}$ | $Q$ | 0 | 0 | $Q^{\otimes}$ | 0 | 0 | 0 | $Q^{\otimes}$ |
| $(F(b, 0, T) \otimes N_2)^{\otimes}$ | $Z(2^\infty)$ | $Z(2^\infty)$ | 0 | $Z_2$ | $Z(2^\infty)$ | 0 | $Z_2$ | $Z(2^\infty)$ | $Z(2^\infty)$ |
| $(F(b, 0, T) \otimes N_2)_{\mu}^{|}$ | $Z(2^\infty)$ | $Z(2^\infty)$ | $Z(2^\infty)$ | $Z(2^\infty)$ | 0 | $Z(2^\infty)$ | $Z(2^\infty)$ | $Z(2^\infty)$ | $Z(2^\infty)$ |
| $(F(b, 0, T) \otimes N_2)^{\otimes}$ | $Z(2^\infty)^{\otimes}$ | $Z(2^\infty)^{\otimes}$ | $Z(2^\infty)$ | $Z(2^\infty)$ | $Z(2^\infty)^{\otimes}$ | $Z(2^\infty)$ | $Z(2^\infty)$ | $Z(2^\infty)$ | $Z(2^\infty)^{\otimes}$ |

Proof. By Proposition 2.1 in [?], there exists a $C^*$-algebra $F$ such that $K^{CRT}(F)$ is free and a homomorphism $\mu: F \to K \otimes \mathcal{G}A$ such that $\mu_{\ast}$ is surjective on united $K$-theory. Then as the proof of Theorem 2.2 in [?] (using the construction of Schochet in section 2 of [?]) let $F_0$ be the mapping cylinder of $\mu$ and let $F_1$ be the mapping cone of $\mu$ to form the sequence

$$0 \to F_1 \xrightarrow{\mu_1} F_0 \xrightarrow{\mu_0} K \otimes \mathcal{G}A \to 0,$$

where $K^{CRT}(F_i)$ is a free CRT-module for $i = 0, 1$ and the induced map $(\mu_0)_{\ast}$ is surjective on united $K$-theory.

Let $\phi: F_0 \to F_0 \otimes N$ be the homomorphism defined by $\phi(a) = a \otimes p$ and form the commutative square

$$\begin{array}{ccc}
F_1 & \xrightarrow{\mu_1} & F_0 \\
\downarrow{=} & & \downarrow{\phi} \\
F_1 & \xrightarrow{\phi_{\mu_1}} & F_0 \otimes N
\end{array}$$

Using the naturality of the mapping cone construction (see Proposition 2.9 of [?]) we form a diagram

$$\begin{array}{cccccccc}
SF_1 & \xrightarrow{S\mu_1} & SF_0 & \xrightarrow{C\mu_1} & F_1 & \xrightarrow{\mu_1} & F_0 \\
\downarrow{=} & & \downarrow{\gamma} & & \downarrow{=} & & \downarrow{\phi} \\
SF_1 & \xrightarrow{S(\phi_{\mu_1})} & SF_0 \otimes N & \xrightarrow{C(\phi \circ \mu_1)} & F_1 & \xrightarrow{\phi_{\mu_1}} & F_0 \otimes N
\end{array}$$

commuting up to homotopy. The horizontal sequences are mapping cone sequences which induce exact sequences on united $K$-theory. In fact, since $\mu_1$ and $\phi \circ \mu_1$ induce monomorphisms on united $K$-theory, the sequences in united $K$-theory are actually short exact sequences:

$$0 \to K^{CRT}(F_1) \xrightarrow{(\mu_1)_{\ast}} K^{CRT}(F_0) \xrightarrow{\phi_{\ast}} K^{CRT}(C\mu_1) \to 0,$$

Now since $\phi_{\ast}$ is a monomorphism, the snake lemma (Lemma 1.3.2 in [?]) shows that $\gamma_{\ast}$ is also a monomorphism. We will take $C(\phi \circ \mu_1)$ to be the sought-for $C^*$-algebra $D$. As the quotient of an injective CRT-module,
the CRT-module $K^{\text{CRT}}(D)$ is injective. Let
\[
\nu: S(K \otimes \mathfrak{S}A) \to C\mu_1
\]
be the homotopy equivalence of Proposition 2.4 in [?] and let
\[
j: S\mathfrak{S}A \to S(K \otimes \mathfrak{S}A)
\]
be the homomorphism induced by the choice of a rank one projection in $K$. We take the composition
\[
\gamma \circ \iota \circ j: S\mathfrak{S}A \to C(\phi \circ \mu_1) = D
\]
to be the sought-for homomorphism $\nu$. Since $\iota$ and $j$ induce isomorphisms on $K$-theory, $\nu$ induces a monomorphism on united $K$-theory as desired.

We now commence the proof proper of Theorem ??, beginning with the case that $B$ is unital.

**Proof of Theorem ??**. Let $A$ be a real C*-algebra whose complexification is in the complex bootstrap category and let $B$ be a real separable unital C*-algebra. Let $\nu: S\mathfrak{S}B \to D$ be given according to Proposition ??. Then using the natural mapping cone construction described in Proposition 2.5 of [?] there exist C*-algebras $I_0$ and $I_1$ and homomorphisms $\iota_0$ and $\iota_1$ such that
\[
0 \to S^2\mathfrak{S}B \xrightarrow{\iota_0} I_0 \xrightarrow{\iota_1} I_1 \to 0 \tag{12}
\]
is a semisplit (see Example 19.5.2(b) in [?]) exact sequence of C*-algebras. Furthermore, $I_0$ is homotopy equivalent to $SD$ and under this equivalence, the homomorphism $\iota_0$ corresponds to $S\nu$. Since $\nu_*$ is injective, when we apply united $K$-theory, we obtain a short exact sequence
\[
0 \to K^{\text{CRT}}(S^2\mathfrak{S}B) \xrightarrow{(\iota_0)_*} K^{\text{CRT}}(I_0) \xrightarrow{(\iota_1)_*} K^{\text{CRT}}(I_1) \to 0 \tag{13}
\]
Since $K^{\text{CRT}}(I_0)$ is an injective CRT-module and since $K^{\text{CRT}}(S^2\mathfrak{S}B)$ has injective dimension 1, we conclude that $K^{\text{CRT}}(I_1)$ is injective and we’re looking at an injective resolution.

If we instead apply the functor $KK(A, \cdot)$ to Sequence ?? we obtain the sequence
\[
\cdots \to KK^{\text{CRT}}(A, S^2\mathfrak{S}B) \xrightarrow{(1, \iota_0)_*} KK^{\text{CRT}}(A, I_0) \xrightarrow{(1, \iota_1)_*} KK^{\text{CRT}}(A, I_1) \xrightarrow{\delta} KK^{\text{CRT}}(A, S^2\mathfrak{S}B) \to \cdots
\]
which we unsplce to form the short exact sequence
\[
0 \to \ker(1, \iota_1)_* \to KK^{\text{CRT}}(A, S^2\mathfrak{S}B) \to \ker(1, \iota_1)_* \to 0 .
\]
To complete the construction of the universal coefficient exact sequence, it remains to make the identifications

$$\ker(1, \iota_1)_* \cong \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(SB))$$

$$\text{coker}(1, \iota_1)_* \cong \text{Ext}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(SB))$$

which we do by use of the following commutative diagram:

$$\begin{array}{cccc}
0 & \rightarrow & \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(S^2SB)) & \xrightarrow{\gamma(A, S^2\mathfrak{S} B)} & KK^{\text{CRT}}(A, S^2\mathfrak{S} B) \\
& & \downarrow \text{Hom}(1, (\iota_0)_*) & & \downarrow (1, \iota_0)_* \\
& \downarrow \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(I_0)) & \gamma(A, I_0) & KK^{\text{CRT}}(A, I_0) & \downarrow (1, \iota_0)_* \\
& \downarrow \text{Hom}(1, (\iota_1)_*) & \gamma(A, I_1) & KK^{\text{CRT}}(A, I_1) & \downarrow \delta \\
0 & \rightarrow & \text{Ext}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(S^2SB)) & \xrightarrow{\gamma(A, S^2\mathfrak{S} B)} & KK^{\text{CRT}}(A, S^2\mathfrak{S} B) \\
\end{array}$$

(14)

In this diagram, both vertical sequences are exact, the left one derived from the resolution given by Sequence ???. The horizontal homomorphisms $\gamma(A, I_0)$ and $\gamma(A, I_1)$ are isomorphisms since by Proposition ??.

The diagram shows that the kernel of $(1, \iota_1)_*$ is isomorphic to the kernel of $\text{Hom}(1, (\iota_1)_*)$ which, in turn, is isomorphic to $\text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(S^2SB))$. Furthermore, in this identification, the composition

$$KK^{\text{CRT}}(A, S^2\mathfrak{S} B) \rightarrow \ker(1, \iota)_* \rightarrow \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(S^2\mathfrak{S} B))$$

is $\gamma(A, S^2\mathfrak{S} B)$.

The diagram also shows that the cokernel of $(1, \iota_1)_*$ is isomorphic to the cokernel of $\text{Hom}(1, (\iota_1)_*)$ which is isomorphic to $\text{Ext}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(S^2\mathfrak{S} B))$. Composing this homomorphism with $\delta$ gives the homomorphism

$$\kappa(A, S^2\mathfrak{S} B) : \text{Ext}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(S^2\mathfrak{S} B)) \rightarrow KK^{\text{CRT}}(A, S^2\mathfrak{S} B)$$

which has degree $-1$.

This proves that the universal coefficient theorem holds for the pair $(A, S^2\mathfrak{S} B)$. In Proposition ?? we will show that this homomorphism $\kappa$ is natural with respect to homomorphisms $B \rightarrow B'$. In particular, consider the homomorphism $B \rightarrow 0$, which induces a homomorphism $\mathfrak{S} B \rightarrow \mathfrak{S} 0 = (S^+)^7 \mathbb{R}$. This homomorphism forms part of a split exact sequence

$$0 \rightarrow S^8B \rightarrow \mathfrak{S} B \rightarrow \mathfrak{S} 0 \rightarrow 0.$$
After suspending twice, we obtain the sequence

$$0 \to S^{10}B \xrightarrow{i} S^2 \otimes B \xrightarrow{\pi} S^2 \otimes 0 \to 0.$$  

Because this sequence is split, the vertical sequences of the following diagram are exact:

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Ext}_{\text{CRT}}(K_{\text{CRT}}(A), K_{\text{CRT}}(S^{10}B)) \\
0 & \longrightarrow & K_{\text{CRT}}(A, S^{10}B) \\
0 & \longrightarrow & \text{Hom}_{\text{CRT}}(K_{\text{CRT}}(A), K_{\text{CRT}}(S^{10}B)) \\
0 & \longrightarrow & 0
\end{array}
\]

Furthermore, we have shown that the two lower horizontal sequences are exact, the map $\gamma$ commutes with $i_*$ and with $\pi_*$, and the map $\kappa$ commutes with $\pi_*$ by the proof of Proposition 7. Therefore, a homomorphism

$$\kappa(A, S^{10}): \text{Ext}_{\text{CRT}}(K_{\text{CRT}}(A), K_{\text{CRT}}(S^{10}B)) \to K_{\text{CRT}}(A, S^{10}B)$$

can be defined to make the diagram commute and then by a diagram chase (Exercise 1.3.2 of [?]), the top horizontal sequence is exact. This proves that the universal coefficient theorem holds for the pair $(A, S^{10}B)$. By Lemma 7. below, it holds for the pair $(A, B)$.

Finally, an argument similar to that in the previous paragraph, based on the exact sequence

$$0 \to B \to B^+ \to \mathbb{R} \to 0$$

takes care of the general case in which $B$ is not necessarily unital.

\begin{lemma}
The universal coefficient theorem holds for a pair $(A, B)$ if and only if it holds for the pair $(A, S B)$.
\end{lemma}
Proof. Consider the following diagram.

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\text{Ext}_{\text{CRT}}(K_{\text{CRT}}(A), K_{\text{CRT}}(B)) & \longrightarrow & \text{Ext}_{\text{CRT}}(K_{\text{CRT}}(A), K_{\text{CRT}}(SB)) \\
\downarrow \kappa(A,B) & & \downarrow \kappa(A,SB) \\
\text{KK}_{\text{CRT}}(A,B) & \longrightarrow & \text{KK}_{\text{CRT}}(A,SB) \\
\downarrow \gamma(A,B) & & \downarrow \gamma(A,SB) \\
\text{Hom}_{\text{CRT}}(K_{\text{CRT}}(A), K_{\text{CRT}}(B)) & \longrightarrow & \text{Hom}_{\text{CRT}}(K_{\text{CRT}}(A), K_{\text{CRT}}(SB)) \\
\downarrow & & \downarrow \\
0 & & 0 
\end{array}
\]

The horizontal homomorphisms are isomorphisms of degree \(-1\). Therefore, there is a homomorphism \(\kappa(A,B)\) as shown in the diagram making the left vertical sequence exact if and only if there is a homomorphism \(\kappa(A,SB)\) making the right vertical sequence exact. \(\square\)

Proposition 4.8. The universal coefficient sequence is natural with respect to homomorphisms of \(A\) and \(B\).

Proof. It is clear that \(\gamma\) is natural. We must show that \(\kappa\) is natural in both arguments. First we show that \(\kappa\) doesn’t depend on the particular resolution of \(S^2\mathcal{B}\) chosen in its construction. For this suppose that we start with two alternatives

\[ \nu_1 : S\mathcal{B} \rightarrow D_1 \]

and

\[ \nu_2 : S\mathcal{B} \rightarrow D_2 \]

which satisfy the requirements described in Proposition ???. Then we construct yet another alternative

\[ \nu_3 : S\mathcal{B} \rightarrow D_3 \]

where \(D_3 = D_1 \oplus D_2\) and \(\nu_3 = \nu_1 \oplus \nu_2\). These three possibilities lead to three possible injective resolutions

\[
0 \rightarrow S^2\mathcal{B} \xrightarrow{\iota_{i,0}} I_{i,0} \xrightarrow{\iota_{i,1}} I_{i,1} \rightarrow 0
\]

for \(i \in \{1,2,3\}\) which lead to three possible constructions of \(\kappa\). Furthermore, for each \(i \in \{1,2\}\), the coordinate projection \(D_3 \rightarrow D_i\) commutes with the \(\nu\) homomorphisms, and thus induces a homomorphism of injective resolutions:

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
| & & | \\
\downarrow & & \downarrow \\
S^2\mathcal{B} & \longrightarrow & S^2\mathcal{B} \\
| & & | \\
\downarrow & & \downarrow \\
I_{i,0} & \longrightarrow & I_{i,0} \\
\downarrow & & \downarrow \\
I_{i,1} & \longrightarrow & I_{i,1} \\
\end{array}
\]

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In turn, this induces a homomorphism from the groups of Diagram ?? based on the resolution built from \( D_3 \) to the corresponding groups of the same diagram based on the resolution built from \( D_i \). The resulting three-dimensional diagram (which we do not attempt to represent) commutes and shows that the homomorphism \( \kappa \) is the same in either case.

Now if \( \phi: A \to A' \) is a homomorphism of real C*-algebras, then we choose a fixed geometric injective resolution of \( S^2 \mathfrak{B} \) to compute \( \kappa(A, S^2 \mathfrak{B}) \) and \( \kappa(A', S^2 \mathfrak{B}) \). Because \( \gamma \) is natural, there is a homomorphism induced by \( \phi \) from the groups of Diagram ?? to the groups of the same diagram with \( A \) replaced by \( A' \). Since \( \kappa \) is defined via this diagram in terms of \( \gamma \), it follows that \( \kappa \) commutes with \( \phi \).

Finally suppose that \( \phi: B \to B' \) is a homomorphism of real C*-algebras. First construct two homomorphisms

\[ \nu: S \mathfrak{B} \to D \]

and

\[ \nu': S \mathfrak{B}' \to D' \]

according to Proposition ??, Then consider the commutative diagram

\[
\begin{array}{ccc}
S \mathfrak{B} & \xrightarrow{\nu''} & D \oplus D' \\
\downarrow{\phi} & & \downarrow{(0,1)} \\
S \mathfrak{B}' & \xrightarrow{\nu'} & D'
\end{array}
\]

where \( \nu'' = \nu \oplus \nu' \circ S \mathfrak{B} \phi \). Again this gives us a homomorphism from one geometric injective resolution to another and thus we obtain a family of homomorphisms from the version of Diagram ?? based on the modified resolution of \( S^2 \mathfrak{B} \) to another the version of Diagram ?? based on the resolution of \( S^2 \mathfrak{B}' \). Since \( \kappa \) does not depend on the particular geometric resolution, it follows that \( \kappa \) commutes with \( \phi \).

\[ \square \]

4.3 Some Corollaries

If we focus on the real part of each CRT-module in the short exact sequence of Theorem ??, we obtain the following corollary. Recall that \([M,N]\) represents the graded group of CRT-morphisms and let \( \text{Ext}_{[-,\cdot]}(M,N) \) denote the associated derived functor.

**Corollary 4.9.** Let \( A \) and \( B \) be real separable C*-algebras such that \( \mathbb{C} \otimes A \) is in \( \mathcal{N} \). Then there is a short exact sequence

\[
0 \to \text{Ext}_{[-,\cdot]}(K^{\text{CRT}}(A),K^{\text{CRT}}(B)) \xrightarrow{\kappa_o} KK(A,B) \xrightarrow{\gamma_o} [K^{\text{CRT}}(A),K^{\text{CRT}}(B)] \to 0
\]

where \( \kappa_o \) has degree \(-1\) and the homomorphism \( \gamma_o \) is given by \( \gamma_o(x)(y) = \alpha_o(y \otimes x) \).
Proof. By definition, $KK^{\text{CRT}}(A, B)^\sigma = KK(A, B)$ and by Proposition ??

$$\text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B))^\sigma \cong [K^{\text{CRT}}(A), K^{\text{CRT}}(B)].$$

To identify the real part of $\gamma$ under these identifications, let $x \in KK_i(A, B)$. Then $\gamma(x) \in \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B))^\sigma = [K^{\text{CRT}}(A) \otimes_{\text{CRT}} F(b, i, R), K^{\text{CRT}}(B)]$ is defined by $\gamma(x) = \Gamma(\alpha(x) = \alpha(1 \otimes b_\sigma))$ (see the proof to Proposition ?? for the description of $\Gamma$) thus $\gamma_0(x)(y) = \alpha_0(1 \otimes b_\sigma)(y \otimes b) = \alpha_0(y \otimes x)$.

Finally, we claim that $\text{Ext}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B))^\sigma \cong \text{Ext}_{\text{CRT}}([K^{\text{CRT}}(A), K^{\text{CRT}}(B)$. Let

$$0 \to F_1 \overset{\mu_1}{\to} F_0 \overset{\mu_0}{\to} K^{\text{CRT}}(A) \to 0$$

be a free resolution of $K^{\text{CRT}}(A)$. We apply $\text{Hom}_{\text{CRT}}$ to obtain

$$0 \leftarrow \text{Ext}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B)) \leftarrow \text{Hom}_{\text{CRT}}(F_1, K^{\text{CRT}}(B)) \overset{\mu_1^*}{\leftarrow} \text{Hom}_{\text{CRT}}(F_0, K^{\text{CRT}}(B)) \overset{\mu_0^*}{\leftarrow} \text{Hom}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B)) \leftarrow 0$$

and then restrict to the real part to obtain

$$0 \leftarrow \text{Ext}_{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(B))^\sigma \leftarrow [F_1, K^{\text{CRT}}(B)] \overset{\mu_1^*}{\leftarrow} [F_0, K^{\text{CRT}}(B)] \overset{\mu_0^*}{\leftarrow} [K^{\text{CRT}}(A), K^{\text{CRT}}(B)] \leftarrow 0.$$

But the cokernel of $\mu_1^*$ is by definition $\text{Ext}_{\text{CRT}}([K^{\text{CRT}}(A), K^{\text{CRT}}(B))$, proving the claim. \hfill \Box

The final corollaries can be proven from Corollary ?? exactly as in the proof of Propositions 23.10.1 and 23.10.2 in [?].

Corollary 4.10. Let $A$ and $B$ be real separable $C^*$-algebras such that $\mathbb{C} \otimes A$ is in $\mathcal{N}$, and let $x \in KK(A, B)$ have the property that $\gamma(x) \in [K^{\text{CRT}}(A), K^{\text{CRT}}(B)]$ is an isomorphism. Then $x$ is a $KK$-equivalence.

Corollary 4.11. Let $A$ and $B$ be real separable $C^*$-algebras such that $\mathbb{C} \otimes A$ and $\mathbb{C} \otimes B$ are in $\mathcal{N}$. Then $A$ and $B$ are $KK$-equivalent if and only if $K^{\text{CRT}}(A) \cong K^{\text{CRT}}(B)$. 
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