Magnetic field-dependent inhomogeneities and their effect on the magnetoresponse of 2D superconductors

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We show that inhomogeneities in the spatial distribution of Cooper pairs and in the phase of the local superconducting order parameter in the vicinity of a superconductor-normal state transition (SNT) in two dimensions can be highly sensitive to a perpendicular magnetic field. We focus on the role of orbital effects in the field-dependence of local superfluid stiffness and superconducting phase disorder in homogeneously-disordered two-dimensional superconductor thin films. The relative importance of these orbital effects is analyzed in different physical regimes dominated by Coulomb blockade, thermal phase fluctuations and Aharonov-Bohm phase disorder respectively. Following this approach, we obtain explicit expressions for the field dependence of magnetoresistance and superfluid stiffness near the SNT, and attempt an understanding of some recent experimental findings.

One of the most challenging problems in strongly disordered superconductors relates to understanding the nature of the magnetic field-induced superconductor-normal state transition (SNT). Experimental and theoretical studies over the past two decades have opened a large number of puzzling questions such as the origin of the giant non-monotonous magnetic field dependence of the resistivity [1–7], flux quantization in the insulating state [8] and the universality class governing the field-induced SNT [9,10,11]. The two-dimensional (2D) case has in particular attracted intense theoretical attention and it is the focus of this work. In the absence of a magnetic field, it is well-known that strong homogeneous disorder introduces granularity in the form of superconducting islands embedded in an insulating matrix [12–16]. However the role of an external magnetic field on the SNT through its effect on the distribution of such islands [17] and on the associated phase frustration brought in by the Aharonov-Bohm (AB) phases of the Cooper pairs tunneling across the islands [17] is not well-understood and is a topic of considerable current debate.

Mean-field analyses of the field-sensitivity of the distribution of superconducting regions go back nearly two decades for weakly-disordered metals [19,20], and more recently, for strongly-disordered insulators. Standard, perturbative approaches fail in the strongly-disordered regime but numerical mean-field solutions of the appropriate Bogoliubov – de Gennes (BdG) equations [4] reveal a picture of shrinking superconducting regions in increasing fields, a downward shift of the distribution of the local superconducting gaps, and through the Ambegaokar-Baratoff relation [21], a corresponding decrease in the Josephson couplings \( J \) between neighboring grains. To understand the physical origin of these effects, we study a phenomenological model of repulsive bosons (Cooper pairs) subjected to a disordered potential and a perpendicular magnetic field. The approach is reminiscent of earlier work on Lifshitz states [22] in disordered Bose systems [3,23,24]. We obtain the typical size and separation of the superconducting islands and show that wave function shrinking in the presence of a magnetic field suppresses the Josephson couplings as \( J(B) \sim \exp[-(B/B_J)^2] \).

To understand magnetoresponse of these 2D granular superconductors, we study the standard Josephson-junction (XY) model,

\[
L = J \sum_i (\partial_r \phi_i)^2 - \sum_{<ij>} J_{ij}(B) \cos(\phi_{ij} + A_{ij}),
\]

where \( E_c \) represents the Coulomb blockade scale, \( \phi_{ij} = \phi_i - \phi_j \) is the superconducting phase difference between neighboring grains at positions \( i \) and \( j \) respectively, and \( A_{ij} = (2e/\hbar) \int_{ij} A \cdot dr \) are the AB phases acquired by the hopping Cooper pairs. Disregarding the contribution of normal quasi particles means the model can provide a good description of the magnetoresponse only at lower fields where Cooper pair breaking is not important. Spatial disorder in the grain positions introduces randomness in the Josephson couplings as well as the AB phases. Studies of the 2D classical limit of Eq. (1) in the \( B = 0 \) limit [4] have shown that strong disorder in \( J \) does not alter the universality class of the SNT from the homogeneous case (where it is known to be of Kosterlitz-Thouless (KT) type) but is nevertheless dominated by a percolating backbone of paths with the largest local superfluid stiffnesses. Likewise the transition in the quantum 1D disordered counterpart at \( B = 0 \) also falls in the KT universality class [23,24]. Therefore for simplicity we will work with the typical value of \( J \) ignoring its spatial disorder.

In regular lattices, the AB phase is associated with flux threading the plaquettes, and depending on the amount of frustration \( f \) (measured as a fraction of a flux quantum), leads to oscillations in properties such as the critical current and the resistance [27,28]. Such matching (commensuration) effects are absent in the disordered case as there is random flux penetration in different
In a phenomenally work, Carpenter and Le Doussal [24,30] studied phase transitions in the classical quenched random phase XY model on a square lattice close to integer $f$. The presence of disorder results in rare favorable regions for the occurrence of vortices at low temperatures. At sufficiently low temperatures, they found that the disorder-induced phase transition is not in the KT universality class. Very similar results were also obtained earlier [26] in a study of the Anderson localization in one-dimensional Luttinger-liquids subjected to quenched phase disorder. The similarity is puzzling since quenched disorder in 1D is equivalent to columnar disorder in the two-dimensional case. Quantum Monte Carlo studies [18] of the interplay of phase frustration and Coulomb blockade suggest a zero temperature field-driven SNT with dynamic exponent $z \approx 1.3$, placing the transition in a different universality class from 3D XY.

In this Letter we study the effect of three dominant mechanisms governing loss of phase coherence and their specific signatures on the magnetoresistance and superfluid stiffness. These are (a) quantum phase fluctuations originating from Coulomb blockade, (b) thermal fluctuations of the phase and (c) frustration effects due to disorder in AB phases. We show that Coulomb blockade effects impart a specific signature to the magnetoresistance, $\rho(B) \sim \exp[(B/B_0)^2]$. Where the SNT is driven by thermal fluctuations, we find a KT transition, with $\rho(B) \sim \exp[-1/(B - B_{KT})]$ in the critical region. In the AB phase frustration dominated regime, we find a new, non KT critical behavior, $\rho(B) \sim \exp[-1/(B - B_{AB})]$. The field-dependent superfluid stiffness $\Upsilon$ also shows a surprising behavior: at small fields, we find that phase frustration effects on $\Upsilon$ are more significant than the field dependence of Josephson couplings. In the Coulomb blockade regime away from the critical region, our predicted magnetoresistance is in excellent accord with experimental data [1,2]. However in the critical scaling region, existing experimental data is somewhat less clear, and while there is some evidence for mechanism (c) for the field-tuned SNT in oxide heterostructures [31], further study is needed and we propose additional probes to distinguish between the two.

We now analyze the effect of a transverse magnetic field on the distribution of the SC islands in the granular superconductor. Consider a model of repulsive bosons (Cooper pairs) with average density $n$ subjected to a random potential with a Gaussian white noise distribution:

$$ H = \sum_p \frac{\Pi^2}{2m} a_p^\dagger a_p + \int \left[ \frac{\hbar}{2} |\Psi(r)|^4 + U(r) |\Psi(r)|^2 \right], \quad (2) $$

where $\Psi(r) = \frac{1}{\sqrt{N}} \sum_p a_p \exp[i(p \cdot r)/\hbar], \Pi = (p - qA), U(r)$ is the random potential, $\langle U(r) \rangle = 0$ and $\langle U(r) U(r') \rangle = \kappa^2 \delta(r - r'), q = 2e$ is the boson charge and $g$, parametrizes the boson repulsion. We choose the gauge $A = i/2 (B \times r)$ with the field in the transverse $z$ direction. This model is equivalent to earlier studied (for $B = 0$) Ginzburg-Landau models with disorder in critical temperature [12]. The important length scales in the model are the single particle localization length $L = \hbar^2/(m \kappa)$ characterizing the disorder, and the magnetic length $l_B \sim \sqrt{e/(2\pi \hbar B)}$. We are specifically interested in the regime $L/l_B \ll 1$. At low densities, the interplay of disorder and interparticle repulsion leads to the formation of disconnected islands of localized bosons [24] whose typical size and separation may be estimated as follows. The optimal potential fluctuation that has a bound state at energy $E < 0$ is found by minimizing $\frac{1}{2} \int U^2 d\mathbf{r} + \lambda(E - H)$, where $\lambda$ is a Lagrange multiplier. We choose $\Psi$ to be real, assuming a spherical fluctuation and zero angular momentum bound state. Varying with respect to $U$, we obtain $U = \lambda \Psi^2$: thus the size $R$ of the optimal potential well is also of the same order as the wave function. The energy of a particle in an island, in the mean-field approximation, is thus of the order of $-\frac{\hbar^2}{2m} \sum_b |\Psi_b|^2 + g n_p/(\pi R^2)$, where $n_p$ is the number of bosons in the island. The density $n_w = n/N_p$ of these islands is determined by the Gaussian factor, $\exp[-(\pi R^2)/(1/\pi R^2)] = (1/\pi R^2) \exp[-(L/\kappa)^2]$. Minimizing the energy with respect to $R$, the size of the typical island, to logarithmic precision, is

$$ R(B) \sim \frac{L}{\sqrt{\ln[n_c(B)/n]}} \quad (3) $$

where $n_c(B) \approx \frac{\hbar^2}{2m} \left(1 + (g B L^2/\hbar^2) \ln^2(\ln(0)/n) \right)$ for small fields, is the critical density for percolation of the islands and $n_c(B)/n > 1$. For future convenience we introduce $w(B) = n_c(B)/n$. Clearly the magnetic field shrinks the islands but the field-dependence is very different from a simple expectation from wave function shrinking of a localized noninteracting particle. The distance $D \sim 1/\sqrt{m_w}$ between the islands can be estimated as $D(B) \sim R(B) e^{\frac{1}{2} (L^2/R(B))^2} \sim L^2 w(B)$, whence, on account of the exponential dependence of inter-island tunneling probability, the typical inter-island Josephson coupling behaves as $J \sim e^{-2(D(B)/R(B))} \quad (4)$

Note that even when at small magnetic fields, $L/l_B \ll 1$, the exponent in Eq. (1) can be large at low boson densities, $w(B) \gg 1$. For such fields we have $J(B)/J(0) \sim e^{-(B/B_0)^2},$ where $B_0 \approx \frac{\hbar}{m} (\hbar l_B^2/2)$. We now analyze the effects of the three different mechanisms that lead to loss of global phase coherence in their regimes of dominance which are determined by the dimensionless parameters $E_c/J, T/J$ and $\sigma$, with the latter a measure of disorder in the fluxes through elementary plaquettes. Figure [3] shows the phase diagram and the regimes of our study.
The above behavior shows the insulating nature of the superconductor-normal state transition, with the XY phase of dimensionless temperature $T/J, \rho_s/J$ and the Aharonov-Bohm (AB) phase disorder $\sigma$ for the model described in Eq. (4). Here $J$ is the (field-dependent) Josephson coupling estimated in the paper, (a) refers to the Coulomb blockade dominated regime. The shaded regions (b) and (c) denote transitions driven by AB phase frustration and thermal phase fluctuation respectively. The dotted line on the surface separates these two different critical scaling regimes. The critical disorder at low $T$ and $E_c$ is independent of $T$ and scaling of the correlation length is not of the Kosterlitz-Thouless type.

(a) Quantum phase fluctuations dominated insulating regime ($E_c/J, E_c/T \gg 1$): We treat the Josephson term in Eq. (1) as a perturbation, and calculate the conductivity using the Kubo formula [32, 33]. Transport in this model proceeds through Arrhenius activation and incoherent sequential hopping of charges between neighboring islands - this leads to a resistivity of the form

$$\rho(B) \sim J(B)^{-2} e^{E_c(B)/T} \sim e^{[4\sqrt{w(B)}+(q^2/L_T)\sqrt{\ln w(B)}]}.$$  

The above behavior shows the insulating nature of the normal state. For small fields, the magnetoresistance obeys the law $\rho(B)/\rho(0) \sim \exp[(B/B_0)^2]$, where $B_0^{-2} \approx (qL_T^2/2\pi)^2 \left[ \sqrt{w(0)} + (q^2/L_T)\sqrt{\ln w(0)} \right]$. More accurately, one must also take into account the renormalization of the charging energy by Josephson coupling [32, 33], $E_c \rightarrow E_c - J$. It is interesting to note that a similar field dependence of resistivity $\rho(B) \sim e^{(B/B_0)^2}$ has been obtained in the context of a superconductor to Hall insulator transition [34].

(b) AB phase frustration dominated regime ($E_c/J \ll 1, T/J \ll 1, \sigma/\sigma_c \sim 1$): To study this regime, it is useful to consider the Coulomb gas representation of the model in Eq. (1). Following earlier works [26, 36] we assume a Gaussian white noise distribution for the AB phases on the links, reckoned from a background average corresponding to a typical separation of islands, $D$. In the Coulomb gas representation, such disorder translates to a random flux threading elementary plaquettes, corresponding to an external potential $V_r$ acting on the "charges" (vortices) with a Gaussian distribution $\langle (V_r - V_r')^2 \rangle = 4\sigma J^2 \ln |r - r'| + O(1)$. Denoting the plaquette area fluctuation by $(\delta D)^2$, we identify $\sigma \sim B^2(\delta D)^4$. It is crucial that the random background potential has long-range (logarithmic) correlations. In the continuum description of the model with a lower cutoff scale $a_0$, $V_r$ has a local part $V_r : \langle (V_r - V_r')^2 \rangle \sim J^2$ and a long-range correlated part $V_r^\gamma$ with no cross-correlation between these two parts. The Coulomb gas Hamiltonian then reads

$$H = -J \sum_{r \neq r'} n_{r'} \ln \left( \frac{|r - r'|}{a_0} \right) - \sum_r \left[ n_r V_r^\gamma - \ln Y[n_r, r] \right],$$  

where $n_r$ represents integer charge at $r$ and the spatially dependent fugacities have the bare value, $\ln Y[n_r, r] = \gamma J q r^2 + n_r e r$, and $\gamma$ is a constant of order unity. We have dropped the background term as it just sets the chemical potential of the vortices and does not affect the scaling equations [37].

In the absence of disorder, the usual RG procedure consists of (i) increasing the short scale cutoff, $a_0 \rightarrow a_0 + dl$, and eliminating all dipoles in the annulus of thickness $dl$, and (ii) disregard all configurations that increase the net charge within the cutoff region. The RG procedure is perturbatively controlled by small dipole fugacities. For the disordered case, we follow Ref. [29] and introduce replicas which allows us to perform the average over Gaussian disorder. The lowest excitations continue to carry charges $0, \pm 1$ but now the $n_r^\alpha$ also carry a replica index $\alpha$. An important difference from the RG procedure of the disorder-free case is that now when the cutoff is increased, one must, apart from considering annihilation of replica charges, also take into account "fusion" of unit charges in different replicas (see appendix). Another important difference that invalidates the usual perturbative expansion in small dipole fugacities is that the random potential creates favorable regions for single vortex formation. Hence we study the scale dependence of the single vortex fugacity distribution identifying the density of rare favorable regions, $\rho_0^\nu$, for the occurrence of vortices as the perturbation parameter. By studying the scaling of $\rho_0^\nu$, two distinct regimes can be identified for $T/J \ll 1$: (a) an XY phase phase at sufficiently low bare disorder where $\rho_0^\nu$ scales to zero, and (b) a disordered phase beyond a critical bare disorder where $\rho_0^\nu$ diverges (see appendix for details). In the disordered phase, the phase correlation length has a surprising non-KT behavior, $\xi \sim e^{1/[(\sigma - \sigma_c)]}$, which in our context translates to a field dependence $\xi \sim e^{1/(B - B_{AB})}$, with $B_{AB} \sim h/\sqrt{\ln(\delta D)^2}$. Such a non-KT behavior is a direct consequence of the
logarithmic scaling of the disorder potential correlations. Another peculiarity is that over a range of low temperatures up to a scale of order \( J \), the critical disorder \( \sigma_c \) is independent of the temperature [29].

We obtain the magnetic field dependence of the superfluid stiffness by solving the scaling equations in the critical region at low temperatures for the coupling constant \( J_1 \) and the effective disorder \( \sigma_1 \). Taking the ratio of the scaling equations for \( J_1 \) and \( \sigma_1 \) obtained in Ref. [29], we get

\[
\frac{\partial_{\mu} J_{1}^{-1}}{\partial_{\mu} \sigma_{1}} \sim \frac{1}{J_{1} \sqrt{\sigma_{1}}},
\]

and from the solution \( J_{1} \sim e^{-2\sqrt{\sigma}} \) it follows that the superfluid stiffness \( \Upsilon(B) \) has the behavior

\[
\Upsilon(B) \sim J(B)e^{-2\sqrt{\sigma(B)}} \sim e^{-(B/B_1)-(B/B_2)^2},
\]

where \( B_1 \) is of the order of \( B_{AB} \). Phase frustration effects thus play a more important role in determining the low-field dependence of superfluid stiffness in the AB phase-frustration dominated regime compared to the effect coming from orbital shrinking.

Now we analyze magnetoresistance in the disordered phase at low temperatures and close to the field-induced transition. Following Halperin and Nelson [38] we estimate the electrical resistivity (which is essentially the vortex conductivity) as \( \rho(B) = \nu_v n(B) \), where \( \nu_v \) is the temperature and field-dependent mobility of the vortices, and \( n(B) \sim 1/\xi^2 \) is the vortex density. We make an assumption that \( \nu_v(B) \) is well-behaved near \( B = B_{AB} \), which allows us to neglect its field dependence in comparison to the singular behavior of \( \xi(B) \). The temperature dependence of resistivity is governed by the temperature dependence of the mobility, and we believe it shows an activated behavior given the logarithmic Coulomb interaction of the vortices [30]. The magnetoresistance in this AB phase frustration dominated regime thus grows as

\[
\rho(B) \sim \nu_v(T)e^{-1/(B-B_{AB})}.
\]

\textbf{(c) Thermal phase fluctuations dominated KT regime} \((E_c/J \ll 1, \sigma/\sigma_c \ll 1, T/J(B) \sim 1)\): In this regime, the transition is brought about by the proliferation of thermally activated vortices. The superfluid stiffness now has a field dependence \( \Upsilon(B) \sim J(B) \sim e^{-(B/B_1)^2} \) arising from orbital shrinking of the superconducting islands. For the resistivity we again consider the correlation length in the disordered phase, which has the well-known form, \( \xi \sim e^{1/\sqrt{B-B_{KT}}} \), with \( T_{KT} \propto J(B) \). Near the transition, this is equivalent to a field-dependent correlation length, \( \xi \sim e^{1/\sqrt{B-B_{KT}}} \). Thus the resistivity in this regime has the form

\[
\rho(B) \sim \nu_v(T)e^{-1/\sqrt{B-B_{KT}}},
\]

For regimes (b) and (c), the normal state has a “metallic” temperature dependence since enhancement of vortex mobilities at higher temperatures translates to higher resistivity.

\textbf{Relation to experiments:} Figure 2 shows the low-temperature and low-field magnetoresistance of disordered InO$_x$ thin films extracted from two different experiments [1, 2]. The positive magnetoresistance data is very well-described by Eq. (7) which places these samples in our Coulomb blockade dominated regime. Deviation from the Coulomb blockade prediction is seen near the magnetoresistance peak and we believe this is due to the quasi particle transport channel opening up. In samples with lower disorder [2], unsurprisingly, Coulomb blockade does not adequately explain the data; however, the other critical scaling regimes (AB phase frustration and KT) show better agreement even though we were unable to distinguish between the two (see appendix). In a recent study of the field-tuned SNT at 2D interfaces of gated oxide heterostructures [31], it was reported that for certain gate voltages, the critical magnetic field at low temperatures was independent of the temperature, suggestive of the phase frustration driven SNT mechanism. Finally, our predictions for superfluid stiffness in the XY regime can possibly be tested through studies of field-dependent ac conductivity [40] and may provide an independent means for distinguishing between the two regimes in the XY phase.

In summary, we studied the field-dependence of the distribution of SC islands in strongly disordered superconductors and constructed an effective Josephson-junction model with field-dependent parameters. Analyzing the model in different physical regimes - dominated by Coulomb blockade, thermal phase fluctuations or Aharonov-Bohm phase fluctuations - we obtained the field-dependence of resistivity and superfluid stiffness.
In the Coulomb blockade regime, available experimental data is in excellent agreement with our prediction \( \rho(B) \sim e^{(B/B_0)^2} \), while in the critical scaling region, available magnetoresistance data \(^2\) is insufficient to distinguish between KT and AB phase frustration regimes.

At very low temperatures, the critical behavior in the vicinity of the quantum critical point \((E_c/J(B) \sim 1)\) is expected to be that of the 3D XY universality class. For the field-tuned transition in systems with homogeneous potential disorder, the rapid decrease of the Josephson coupling \(J(B)\) with field implies that the likely experimental trajectories in the \(T/J\) vs.\(E_c/J\) plane rapidly move out of the quantum critical region into the Coulomb-blockade dominated region where \(E_c/J \gg 1\). In contrast, in systems such as nanopatterned superconducting proximity arrays, the fabrication technique is such that the separation of superconducting regions (and thus \(J(B)\)) is not as field-sensitive. Such systems look attractive from the point of view of studying the critical behavior near the field-tuned SNT, especially in the Coulomb-blockade dominated region where \(E_c/J \gg 1\).

1. RG equations and phase diagram of the disordered XY model

In this section we show the essential steps followed for obtaining the phase diagram of the two-dimensional XY model with phase disorder. A comprehensive study can be found in Ref. \(^{29}\).

The partition function of the replicated Coulomb gas with \(m\)-vector charges after averaging over the bare disorder is

\[
\mathcal{Z}^m = 1 + \sum_{p=2}^{\infty} \sum_{n_1,\ldots,n_p} \int_{|r_i-r_j|>a_0} \exp(-\beta H^{(m)}[\mathbf{n}, \mathbf{r}]),
\]

where the sum is over all distinct neutral configurations and

\[
\beta H^{(m)} = \sum_{i \neq j} K_{ab} n_i^a \ln \left(\frac{|r_i-r_j|}{a_0}\right) n_j^b + \sum_i \ln Y[\mathbf{n}_i].
\]

Here, \(Y[\mathbf{n}] = \exp(-n^a \gamma K_{ab} n^b)\), where \(K_{ab} = \beta J \delta_{ab} - \sigma \beta^2 J^2\). Significant contribution to the partition function only comes from charges \(\pm 1, 0\) and hence we restrict to these. We increase the hard core cutoff \(a_0 \rightarrow a_0 e^{(dl)}\) and retain the original form of the partition function in terms of scale dependent coupling constants \((K_i)_{ab}\) and fugacities \(Y_i[\mathbf{n}]\). To \(O(Y[\mathbf{n}]^2)\), we obtain the following RG flow equations \(^{29}\):

\[
\partial_t(K^{-1})_{ab} = 2\pi n^a n^b Y[\mathbf{n}] Y[-\mathbf{n}]
\]

\[
\partial_t Y[\mathbf{n} \neq 0] = (2 - n^a K_{ab} n^b) Y[\mathbf{n}] + \sum_{n' \neq 0, \mathbf{n}} \pi Y[\mathbf{n}'] Y[\mathbf{n} - \mathbf{n}']
\]

Equation \(10\) comes from the annihilation of dipoles of opposite vector charges in the annulus \(a_0 < |r_i-r_j| < a_0 e^{(dl)}\). It gives the renormalization of the interaction and of the disorder. Simple rescaling gives the first part of equation \(11\). The second part comes from the possibility of fusion of two replica vector charges upon coarse graining. Some examples of fusion are given below.

\[
\begin{pmatrix}
+1 & 0 & +1 & 0 & +1 & 0 & +1 & +1 & -1 & 0 \\
+1 & 0 & +1 & 0 & +1 & 0 & +1 & +1 & 0 & +1 & +1 & 0 & +1 & 0 & +1 & 0 & +1 & 0 & +1
\end{pmatrix}
\]
Replica permutation symmetry, which we will assume here and which is preserved by the RG, together with \( n^a = 0, \pm 1 \) implies that \( Y[n] \) depends only on the numbers \( n_+ \) and \( n_- \) of \(+1/−1\) components of \( n \). We parameterize \( Y[n] \) by introducing a function of two arguments \( \Phi(z_+, z_-) \), where \( z_+ (r) = \exp(\pm i q r) \), such that:

\[
Y[n] = \langle z_+^{n_+} z_-^{n_-} \rangle_\Phi \tag{12}
\]

where, \( O = \beta J (2 + z_+ \partial_{z_+} + z_- \partial_{z_-}) + \sigma(\beta J)^2(z_+ \partial_{z_+} - z_- \partial_{z_-})^2 \). The \( m \to 0 \) limit of eq(10) similarly yields,

\[
T \frac{dJ}{dl} = 8 \left( \frac{z_{+}^{z_{+}''} + z_{-}^{z_{-}''} + 4 z_{+} z_{-} z_{+}'' z_{-}''}{(1 + z_{+}' z_{+}' + z_{-}' z_{-}')^2} \right) \tag{14}
\]

\[
\frac{d\sigma}{dl} = 8 \left( \frac{z_{+} z_{-}'' + z_{-} z_{+}'' + 4 z_{+} z_{-} z_{+}'' z_{-}''}{(1 + z_{+}' z_{+}' + z_{-}' z_{-}')^2} \right) \tag{15}
\]

Equations (13), (14) and (15) form the complete set of RG equations.

Numerical study [29] of the RG equations indicates the existence of an XY phase at low temperatures and below some critical disorder. Guided by the RG flow observed numerically within and near the boundaries of the XY phase, we can approximate the full RG equations by a simpler equation involving only the single fugacity distribution, \( P_l(z) = \int dz_+ P_l(z_+, z) = \int dz_- P_l(z, z_-) \). In the low \( T \) regime, the distribution \( P_l(z_+, z_-) \) is broad and the physics is dominated by rare favorable regions \((z_+ \sim 1 \text{ or } z_- \sim 1) \). Here we identify a parameter that allows to organise perturbation theory as: \( P_l(1) \equiv P_l(z_+ \sim 1) \sim P_l(z_+ \sim 1, z_- \sim 0) = P_l(z_+ \sim 0, z_- \sim 1) \). We also observe that \( P_l(1, 1) \equiv P_l(z_+ \sim 1, z_- \sim 1) \sim P_l(1)^2 \). Using these we can see schematically the RG equation (13) as a correction to \( P_l(1) \) of order \( P_l(1)^2 \) by the first term and order \( P_l(1)^2 \) by the second term; in RG equation (14), (15) as a correction to order \( P_l(1)^2 \) to \( J_l \) and \( \sigma_l \). Again working to order \( P_l(1)^2 \), we see that the denominators in the delta functions in (13) could be neglected. This approximation also simplifies equations (14) and (15).

Introducing

\[
G_l(x) = 1 - \int_{-\infty}^{\infty} du \hat{P}_l(u) \exp(-e^{\beta(u-x+\epsilon_l)}), \tag{16}
\]

where \( u = 1/\beta \) \( \ln(z) \) and \( \epsilon_l = \int_0^l J(\ell') d\ell' \), we see that eq(14) can be written as \( \frac{1}{2} \partial^2 G = -\frac{\sigma^2}{2} \partial^2 G + G(1-G) \). If \( \sigma \) and \( J \) are \( l \) independent we identify the above with Kolmogorov-Petrovskii-Piscounov (KPP) equation, whose general form is \( \frac{1}{2} \partial_u G = D \partial^2_u G + f(G) \), where \( D \) is a constant and \( f \) satisfies \( f(0) = f(1) = 0 \), \( f \) positive between 0 and 1 and \( f'(0) = 1 \), \( f'(1) \leq 1 \) between 0 and 1. Since at large \( l \), both \( J \) and \( \sigma \) converge and effectively becomes \( l \) independent, we see that we can use results from the study of KPP equation in our case at large \( l \).

For a large class of initial conditions, the solutions of the KPP equation are known to converge uniformly towards traveling wave solutions of the form: \( G_l(x) = h(x - m_l) \). The velocity of the wave is given by \( c = \lim_{l \to \infty} \partial_x m_l \). A theorem due to Branson [11] shows that the asymptotic traveling wave is determined by the behavior at \( x \to \infty \) of the initial condition \( G_{l=0}(x) \) in the following manner. If \( G_{l=0}(x) \) decays faster than \( e^{-\mu x} \) where \( \mu = 1/\sqrt{D} \), then \( c = \sqrt{D} \). If \( G_{l=0}(x) \) decays slower than \( e^{-\mu x} \) where \( \mu < 1/\sqrt{D} \), then \( c = 2(D\mu + \mu^{-1}) \). The parameterization(16) implies that the distribution \( \hat{P}_l(u) \) itself converges to a traveling front solution

\[
\hat{P}_l(u) \to_{l \to \infty} \hat{p}(u - X_l) , \quad X_l = m_l - E_l. \tag{17}
\]

Since \( \partial_x E_l \to_{l \to \infty} J_R \), we see that the asymptotic velocity of the front of \( \hat{P}_l(u) \) is \( c - J_R \); \( c \) is the KPP front velocity. The center of the front corresponds to the maximum of the distribution \( \hat{P}_l(u) \).

The asymptotic velocity clearly decides the phase of the system: since we start with a distribution peaked at some small \( z \), if the velocity is positive, then \( P_l(1) \) will increase and this would imply that the system is in the disordered phase. On the other hand, negative velocity implies that the system is in the XY phase. The velocity vanishes at the phase boundary. By construction, the initial condition \( G_{l=0}(x) \) decays for large \( x \) as \( <z > \to \rho_0 e^{-\beta x} \). Hence we identify \( \mu = \beta \). Based on the results discussed above about the front velocity selection in KPP equation we can conclude the following about the phase diagram of the model:

(a) For \( T > T_g = J_R/\sqrt{\sigma_R/2} \), \( c = T \left( 2 + \frac{\sigma_R J_R^2}{2} \right) \). Thus
here the XY phase would exist for

\[ 2 - \frac{J_R}{T} + \frac{\sigma_R J_R^2}{T^2} < 0. \]  

(18)

(b) For \( T \leq T_g, \) \( c = J_R\sqrt{8\sigma_R}. \) Thus here the XY phase would exist for \( \sigma_R < \sigma_c = \frac{1}{3}. \)

**Critical behavior at zero temperature:** The zero temperature phase transition from the XY phase to the disordered phase occurs at \( \sigma_R = 1/8. \) The center of the front is located at \( u = X_1 \) near the transition. It follows from [41] that, \( X_1 \sim (4\sqrt{D} - J)/3 - 2\sqrt{D}\ln l + X_0. \) Hence in the critical region to leading order, we get,

\[ \partial_l X_1 \sim 4\sqrt{D} - J - \frac{3\sqrt{D}}{2l}. \]  

(19)

After some manipulations the RG equations for \( J \) and \( \sigma \) in the critical region reads,

\[ \partial_l (J^{-1}) = k \int du \tilde{p}_l(u - X_l)\tilde{p}_l(-u - X_l) \]

\[ \partial_l \sigma = k \int_{u + u' > -2X_l} \tilde{p}_l(u)\tilde{p}_l(u'), \]

where \( k \) is some constant. Using the asymptotic form of \( \tilde{p}_l(u)\) discussed in [41] and working up to leading order in \( (\sigma - \sigma_c), \) we can simplify the above equations to get,

\[ \partial_l (J^{-1}) \sim \frac{C}{\sqrt{D}} X_l^3 \exp \left( \frac{2X_l}{\sqrt{D}} \right) \]

\[ \partial_l \sigma \sim CX_l^3 \exp \left( \frac{2X_l}{\sqrt{D}} \right), \]

where \( C \) is a constant. To estimate the form of correlation length, we first introduce the small parameter, \( g_l = \exp(X_l/\sqrt{D}). \) Then eq. \([19] \) reads,

\[ \partial_l g \sim \left( 16(\sigma - \sigma_c) - \frac{3}{2l} \right) g \]

Now starting away from criticality, \( \epsilon = \sigma_c - \sigma_R > 0, \) we find, \( g_l \sim l^{-3/2} \exp(16\epsilon l). \) Identifying the correlation length \( \xi \) as when \( g_{\xi} \sim 1, \) we find,

\[ \xi \sim \exp \left( \frac{b}{\sigma - \sigma_c} \right), \]

where \( b \) is some constant. We then see that the universality class of this transition is clearly different from the KT universality class.

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2. Comparison of Kosterlitz-Thouless (KT) and non-KT scaling with experiments

In Fig. 3 we show the sheet resistance \( R_{\square} \) vs. magnetic field data near a field-driven SIT in a homogeneously-disordered InO\(_x\) thin film from Ref. [2], and attempt fits of this data to the Kosterlitz-Thouless (KT) behavior \( (R_{\square} = R_0 e^{-1/(B-B^*)} ) \) and the non-KT behavior \( (R_{\square} = R_0 e^{-1/(B-B_{\square})}) \) obtained in this Letter. It is difficult to say which of these two laws describes the data better; however, we argue that the non-KT fit might be a bit better on account of a more reasonable value for the high-field resistance \( R_0. \)

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