Conditions for superradiant instability of the Kerr-Newman black holes

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Abstract

We find two conditions for superradiant instability of Kerr-Newman black holes under a charged massive scalar perturbation by analyzing asymptotic scalar potential and far-region wave function. Actually, they correspond to the condition for getting a trapping well. Also, we obtain the conditions for superradiant stability of Kerr-Newman black holes which states that there is no trapping well. The analysis is applied to Kerr black holes to find a condition for superradiant instability.

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1 Introduction

If a light boson exist with proper mass, gravitational bound states are formed around rotating charged black holes. These bound states could extract continuously electromagnetic or rotational energy from black holes [1]. It is a superradiance phenomenon in the black holes [2]. The existence of superradiant modes can be converted into an instability of the black hole background if a mechanism of mirror to trap these modes is installed near the black hole. This is called superradiant instability. If a scalar has a mass $\mu$, its mass would act as a reflecting mirror [3]. The superradiant instability of the Kerr black hole was found for $M\mu \gg 1$ [4], $M\mu \ll 1$ [5], and $M\mu \leq 0.5$ [6]. A scalar potential including a trapping well is essential for generating a quasibound state [4, 1, 7] whose wave function is peaked far outside the ergo region. If there is no trapping well in the potential, it may correspond to the superradiant stability with bound state.

In the Kerr-Newman black hole (KNBH) with mass $M$, charge $Q$, and angular momentum $J$, the superradiant instability condition for a charged massive scalar with mass $\mu$ and charge $q$ was firstly obtained as $qQ < \mu M$ which may be a condition for a trapping well [8]. However, $qQ < \mu M$ is not satisfied simultaneously if one imposes the superradiance condition ($\omega < \omega_c$ with $\omega_c = m\Omega_H + q\Phi_H$) and thus, it is a condition for bound states [9]. In other words, the condition of $qQ < \mu M$ corresponds to the Newton-Coulomb requirement for the gravitational force to exceed the electrostatic force. Also, it is noted that their effective potential $V_{\text{eff}}(r)$ seems not to be a correct form. Scalar clouds with $\omega = \omega_c$ and $\omega < \mu$ were obtained in [10, 11]. The absorption cross section was recently found to give a negative absorption cross section for co-rotating spherical waves [12].

On the other hand, there was an approach to analyzing superradiant instability based on the scalar potentials [13]. We stress that the appearance/disappearance of a trapping well is a decisive condition for superradiant instability/stability. Recently, it was shown that the superradiant stability of KNBHs under a charged massive scalar perturbation can be achieved if $qQ > \mu M$ and $r_-/r_+ \leq 1/3$ are satisfied [13], in addition to $\omega < \omega_c$ and the bound state condition ($\omega < \mu$). However, their potential based on the analysis seems to be incorrect because $\Psi_{\ell m} = \sqrt{\Delta}R_{\ell m}$ is used and a tortoise coordinate $r_*$ defined by $dr_* = (r^2 + a^2)dr/\Delta$ is not used by following Ref. [15]. So, it suggests that $qQ > \mu M$ is not a condition for superradiant stability. The superradiant stability of a charged massive scalar based on a desirable potential was discussed in the KNBH background [16].
In this work, we wish to find two conditions for getting a trapping well of KNBH under a charged massive scalar perturbation by analyzing asymptotic scalar potential $V_{aaKN}(r)$ and far-region wave function $U[p, s; cr]$. They are given by $V'_{aaKN}(r) > 0$ ($M\mu^2 > qQ\omega$) and $U'[p, s; cr] > 0$ ($p < 0$). In addition, the conditions for no trapping well are $V'_{aaKN}(r) < 0$ ($M\mu^2 < qQ\omega$) and $U'[p, s; cr] < 0$ ($p > 0$). We apply the same analysis to Kerr black hole under a massive scalar propagation to find the condition for a trapping well.

2 Potentials around KNBHs

Let us introduce the Boyer-Lindquist coordinates to represent a KNBH with mass $M$, charge $Q$, and angular momentum $J$.

$$ds^2_{KN} = \bar{g}_{\mu \nu} dx^\mu dx^\nu = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dt]^2$$

with

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \text{and} \quad a = \frac{J}{M}.$$  \hspace{1cm} (1)

We choose the electromagnetic potential as

$$\bar{A}_\mu = \frac{Qr}{\rho^2} (-1, 0, 0, a \sin^2 \theta).$$

The outer and inner horizons are obtained from $\Delta = (r - r_+)(r - r_-) = 0$ ($\bar{g}^{rr} = 0$) as

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}.$$  \hspace{1cm} (2)

One describes a charged massive scalar perturbation $\Phi$ on the background of KNBHs by adapting the perturbed scalar equation

$$(\nabla^\mu - iq\bar{A}^\mu)(\nabla_\mu - iq\bar{A}_\mu)^*\Phi - \mu^2 \Phi = 0.$$  \hspace{1cm} (5)

Considering the static and axis-symmetric background [1], it is plausible to separate the scalar perturbation into modes

$$\Phi(t, r, \theta, \phi) = \sum_{lm} e^{-i\omega t + im\phi} S_{lm}(\theta)R_{lm}(r),$$

$$3$$
where $S_{\ell m}(\theta)$ is spheroidal harmonics with $-m \leq \ell \leq m$ and $R_{\ell m}(r)$ describes a radial part of the wave function. Substituting (6) into (5), we have the angular equation for $S_{\ell m}(\theta)$ and the Teukolsky equation as [10]

$$
\frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta S_{\ell m}(\theta) \right) + \left[ \lambda_{\ell m} + a^2(\mu^2 - \omega^2) \sin^2 \theta - \frac{m^2}{\sin^2 \theta} \right] S_{\ell m}(\theta) = 0, \quad (7)
$$

$$
\Delta \partial_r \left( \Delta \partial_r R_{\ell m}(r) \right) + U(r) R_{\ell m}(r) = 0, \quad (8)
$$

where

$$
U(r) = \left[ \omega(r^2 + a^2) - am - qQr \right]^2 + \Delta [2am\omega - \mu^2(r^2 + a^2) - \lambda_{\ell m}]. \quad (9)
$$

We note that Eq.(8) could be used directly for computing absorption cross section, quasi-normal modes of the scalar, and scalar clouds. At this stage, we introduce the tortoise coordinate $r_*$ defined by $dr_* = \frac{r^2 + a^2}{\Delta} dr$ to derive the Schrödinger-type equation. In this case, an interesting region of $r \in [r_+, \infty)$ could be mapped into the whole region of $r_* \in (-\infty, \infty)$. Then, the radial equation (8) takes the Schrödinger-type equation when choosing $\Psi_{\ell m} = \sqrt{a^2 + r^2} R_{\ell m}$

$$
\frac{d^2 \Psi_{\ell m}(r_*)}{dr_*^2} + \left[ \omega^2 - V_{KN}(r) \right] \Psi_{\ell m}(r_*) = 0, \quad (10)
$$

where the potential $V_{KN}(r)$ is found to be [11]

$$
\begin{align*}
V_{KN}(r) &= \omega^2 - \frac{3\Delta^2 r^2}{(a^2 + r^2)^4} + \frac{\Delta[\Delta + 2r(r - M)]}{(a^2 + r^2)^3} \\
&\quad + \frac{\Delta \mu^2}{a^2 + r^2} - \left[ \omega - \frac{am}{a^2 + r^2} - \frac{qQr}{a^2 + r^2} \right]^2 \\
&\quad - \frac{\Delta}{(a^2 + r^2)^2} \left[ 2am\omega - \lambda_{\ell m} \right]. \quad (11)
\end{align*}
$$

Replacing $\lambda_{\ell m}$ by $\tilde{\lambda}_{\ell m} + a^2(\omega^2 - \mu^2)$ with $\tilde{\lambda}_{\ell m} = l(l + 1) + \cdots$, one finds a familiar angular equation

$$
\frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta S_{\ell m}(\theta) \right) + \left[ \tilde{\lambda}_{\ell m} + a^2(\omega^2 - \mu^2) \cos^2 \theta - \frac{m^2}{\sin^2 \theta} \right] S_{\ell m}(\theta) = 0. \quad (12)
$$

Before we proceed, we would like to mention a superradiant scattering of the scalar of the KNBHs. We find two limits such that $V_{KN}(r \to \infty) \to \mu^2$ and $V_{KN}(r \to r_+) \to$
\( \omega^2 - (\omega - \omega_c)^2 \). In this case, we have scattering forms of plane waves as \[2\]

\[
\Psi_{lm} \sim e^{-i\sqrt{\omega^2 - \mu^2} r_\ast} (\leftarrow) + R e^{+i\sqrt{\omega^2 - \mu^2} r_\ast} (\rightarrow), \quad r_\ast \to +\infty (r \to \infty),
\]

(13)

\[
\Psi_{lm} \sim T e^{-i(\omega-\omega_c) r_\ast} (\leftarrow), \quad r_\ast \to -\infty (r \to r_+)
\]

(14)

with the \( T(R) \) the transmission (reflection) amplitudes. A wrongskian \( W(\Psi, \Psi^*) \) condition of \( i \frac{d}{dr_\ast} W(\Psi, \Psi^*) = 0 \) leads to

\[
|R|^2 = 1 - \frac{\omega - \omega_c}{\sqrt{\omega^2 - \mu^2}} |T|^2,
\]

(15)

which implies that outgoing waves with \( \omega > \mu \) propagate to infinity and the superradiant scattering occurs (\( |R|^2 > |I|^2 \)) whenever the superradiance condition is satisfied

\[
\omega < \omega_c.
\]

(16)

Curiously, the superradiance is associated to having a negative absorption cross section \[12\]. For a KNBH, the total absorption cross section becomes negative for co-rotating spherical waves at low frequencies.

Now, we wish to describe the superradiant instability briefly. The two boundary conditions imply an exponentially decaying wave (bound state) away from trapping well and a purely outgoing wave (superradiance) near the outer horizon under Eq.(16):

\[
\Psi \sim e^{-\sqrt{\mu^2 - \omega^2} r_\ast}, \quad r_\ast \to \infty (r \to \infty), \quad (17)
\]

\[
\Psi \sim e^{-i(\omega-\omega_c) r_\ast}, \quad r_\ast \to -\infty (r \to r_+). \quad (18)
\]

From Eq.(17), one needs the bound state condition to obtain an exponentially decaying mode

\[
\omega < \mu.
\]

(19)

Furthermore, the superradiant instability/stability is determined by a shape of the potential, in addition to Eqs.(16) and (19). Importantly, a key condition for superradiant instability is to include a positive trapping well in the potential. If there is no trapping well in the potential, it implies a superradiant stability.

To find out the condition of a trapping well, we have to consider a potential \( V_{aKN}(r) \) obtained when expanding \( V_{KN}(r) \) in the far-region

\[
V_{aKN}(r) = \mu^2 - \frac{2(M\mu^2 - qQ\omega)}{r} + \frac{\lambda_{lm} + Q^2(\mu^2 - q^2)}{r^2}. \quad (20)
\]

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Its first derivative
\[ V'_{aKN}(r) = \frac{2(M \mu^2 - qQ \omega)}{r^2} - \frac{2[\lambda_m + Q^2(\mu^2 - q^2)]}{r^3}, \] (21)
where the mass term \( \mu^2 \) in Eq. (20) is missed. We note that there is no restrictions on parameters in deriving \( V'_{aKN}(r) \). Here, one tempts to say that the condition for trapping well (no trapping well) is given by \( V'_{aKN}(r) > 0 \) \( (V'_{aKN}(r) < 0) \). Nevertheless, it is difficult to find any analytic condition from \( V'_{aKN}(r) > 0 \) \( (V'_{aKN}(r) < 0) \). To this end, we consider an asymptotic potential \( V_{aaKN}(r) \) obtained when expanding \( V_{KN}(r) \) at \( r = \infty \) and its first derivative
\[ V_{aaKN}(r) = \mu^2 - \frac{2(M \mu^2 - qQ \omega)}{r}, \quad V'_{aaKN}(r) = \frac{2(M \mu^2 - qQ \omega)}{r^2}. \] (22)
Here, one requires \( V'_{aaKN}(r) > 0 \) \( (M \mu^2 > qQ \omega) \) for arising a trapping well, whereas \( V'_{aaKN}(r) > 0 \) \( (M \mu^2 < qQ \omega) \) is demanded for no trapping well. However, this is not a sufficient condition for getting a trapping well. In the next section, we will find the other condition by analyzing far-region scalar function. The other condition for trapping well (no trapping well) will be given by \( U'[p, s; cr] > 0 \) \( (U'[p, s; cr] < 0) \) where \( U[p, s, cr] \) is the confluent hypergeometric function.

At this stage, we display two potentials, their far-region and asymptotic potentials where one includes a trapping well with \( M \mu^2 > qQ \omega \) [(Left) Fig. 1] and the other does not include a trapping well but it matches up with \( M \mu^2 > qQ \omega \) [(Right) Fig. 1]. Fig. 2 indicates a potential with a trapping well for \( q > \mu \) and \( M \mu^2 > qQ \omega \), showing a feature of a charged massive scalar propagating around the KNBH. It is worth noting that all have \( V'_{aaKN}(r) > 0 \). One needs to find the other condition to have no trapping well.

Finally, we describe two conditions for superradiant instability and stability under a charged massive scalar propagating around the KNBHs:
(i) superradiant instability \( \rightarrow \omega < \omega_c \) and \( \omega < \mu \) with a positive trapping well.
(ii) superradiant stability \( \rightarrow \omega < \omega_c \) and \( \omega < \mu \) without a positive trapping well.

### 3 Condition for trapping well

One needs to observe far-region scalar functions to distinguish between trapping well and no trapping well. For this purpose, we wish to derive a scalar equation in the far-region.
Figure 1: (Left) Potential with trapping well $V_{KN}(r)$, its far-region $V_{aKN}(r)$, and asymptotic potential $V_{aaKN}(r)$ as functions of $r \in [r_{-}, 1.06, 100]$ with $M = 1, Q = 0.01, \omega = 3.85, a = 0.998, m = 13, q = 0.2, \lambda_{lm} = 180, \mu = 4$. $V_{KN}(r)$ has a trapping well located at $r = 9.61$. We check $\omega < \omega_{c}(= 6.11)$ and $\omega < \mu$ as two conditions for superradiant instability.  (Right) Potential without trapping well $V_{KN}(r)$ as function of $r \in [r_{+} = 1.741, 100]$ with $M = 1, Q = 0.6, \omega = 0.02, a = 0.3, m = 1, q = 0.1, \lambda_{lm} = 12, \mu = 0.05$. We check two conditions of $\omega < \omega_{c}(= 0.13)$ and $\omega < \mu$.

Figure 2: (Left) Potential with trapping well $V_{KN}(r)$, its far-region $V_{aKN}(r)$, and asymptotic potential $V_{aaKN}(r)$ as functions of $r \in [r_{+} = 1.39, 100]$ with $M = 1, Q = 0.01, \omega = 2.95, a = 0.9, m = 13, q = 20, \lambda_{lm} = 180, \mu = 3$. Here, we have $q > \mu$, showing a feature of KNBH. $V_{KN}(r)$ has a trapping well located at $r = 19.3$. We check that two conditions of $\omega < \omega_{c}(= 4.38)$ and $\omega < \mu$ are satisfied for superradiant instability.

In the far-region where taking $r_{*} \sim r$ approximately, we obtain an equation from Eqs.
with (20) as
\[
\left[ \frac{d^2}{dr^2} + \omega^2 - V_{aKN}(r) \right] \Psi_{lm}(r) = 0. \tag{23}
\]
The above equation could be rewritten as
\[
\left[ \frac{d^2}{dr^2} - A^2 + \frac{B}{r} - \frac{C}{r^2} \right] \Psi_{lm}(r) = 0, \tag{24}
\]
where three coefficients are given by
\[
A = \sqrt{\mu^2 - \omega^2}, \quad B = 2(M\mu^2 - qQ\omega), \quad C = \lambda_{lm} + Q^2(\mu^2 - q^2). \tag{25}
\]
The solution is given exactly by the Whittaker function \(W[p, s; cr]\) and the confluent hypergeometric function \(U[p, s; cr]\) as
\[
\Psi_{lm}(r) = c_1 W\left[\frac{B}{2A}, k; 2Ar\right] \tag{26}
\]
\[
= c_1 e^{-Ar}(2Ar)^{k+\frac{1}{2}} U\left[k + \frac{1}{2} - \frac{B}{2A}, 1 + 2k; 2Ar\right], \tag{27}
\]
where
\[
k = \frac{1}{2}\sqrt{1 + 4C}. \tag{28}
\]
Here, we find a bound state of \(e^{-\sqrt{\mu^2 - \omega^2}r}\) appeared in (17) and \(M\mu^2 > qQ\omega\) corresponding to \(B > 0\) \((B < 0)\). In the asymptotic region, one has an asymptotic wave function
\[
\Psi_{lm}^A(r) \sim e^{-\sqrt{\mu^2 - \omega^2}r} \left(2\sqrt{\mu^2 - \omega^2}r\right)^{\frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}}}, \tag{29}
\]
which always leads to zero as \(r \to \infty\) for \(\sqrt{\mu^2 - \omega^2} > 0\).

Let us observe a radial mode \(\Psi_{lm}(r)\) for a trapping well [see (Left) Fig. 1], implying superradiant instability. As is shown in (Left) Fig. 3, Eq.(27) shows a quasibound state whose wave function is a peak located far from the outer horizon. A similar picture is found from (Left) Fig. 4 which is based on Fig. 2 with \(q > \mu\). This case indicates a feature of a charged massive scalar propagating around the KNBH.

Contrastively, we consider a radial mode for a potential without trapping well [shown in (Right) Fig. 1 with \(V_{aKN} > 0(M\mu^2 > qQ\omega)\)], implying superradiant stability. As is shown in Fig. 5, Eq.(27) shows an exponentially decaying mode. This case explains why one needs to find the other condition for no trapping well.
At this stage, it is useful to introduce the asymptotic form of $U[p, s; cr]$ as \[ U[p, s; cr \to \infty] \to (cr)^p \left[ 1 - \frac{p(1 + p - s)}{cr} + O\left(\frac{1}{cr}\right)^2 \right]. \] (30)

Here, one observes an increasing function $U[p, s; cr]$ for a negative $p$ [(Right) Fig. 3 and (Right) Fig. 4], while one finds a decreasing function for a positive $p$ (Fig. 5). Furthermore, considering the first derivative of $U[p, s; cr]$ with respect to $r$ ($c > 0$) as

\[ U'[p, s; cr] = -pcU[1 + p, 1 + s; cr], \] (31)

it implies that the condition for a trapping well is

\[ U'[p, s; cr] > 0 \to p < 0, \] (32)
Figure 5: Bound state function $\Psi_{13}(r)$ as $r \in [5, 100]$ without trapping well. Confluent hypergeometric function $U[3.97, 8; 0.09r]$ represents a decreasing function of $r$ and its derivative $U'[3.97, 8; 0.09r]$ is negative. All parameters go together with (Right) Fig. 1.

whereas the condition for no trapping well is given by

$$U'[p, s; cr] < 0 \rightarrow p > 0.$$  \hspace{1cm} (33)

Therefore, the quasibound state with a peak could be found when $p$ is negative as

$$p < 0 \rightarrow \sqrt{1 + 4C} < \frac{B}{A} - 1 \rightarrow \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}} > k + \frac{1}{2},$$  \hspace{1cm} (34)

which is the other condition for a trapping well, in addition to $M\mu^2 > qQ\omega$ ($V'_{aaKN}(r) > 0$).

On the other hand, the bound state could be found for a positive $p$ as

$$p > 0 \rightarrow \sqrt{1 + 4C} > \frac{B}{A} - 1 \rightarrow \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}} < k + \frac{1}{2},$$  \hspace{1cm} (35)

which denotes the other condition for no trapping well, in addition to $M\mu^2 < qQ\omega$ ($V'_{aaKN}(r) < 0$). We could not find a condition of $M\mu/qQ < 1$ for superradiant stability \[14\]. However, we obtain one condition from $V'_{aaKN}(r) < 0$ as

$$\frac{M\mu}{qQ} < \mu < 1.$$  \hspace{1cm} (36)

Finally, we consider the case of $p \simeq 0$ to denote a boundary state between quasibound and bound states. Here, we display a corresponding potential in Fig. 6 which seems to have a trapping well. The asymptotic wave function shows a half of peak [(Left) Fig. 7] and thus, it does not represent a quasibound state completely. Its confluent hypergeometric function $U[p, 29.5; 2.17]$ with $p = -5.9 \times 10^{-9} \simeq 0$ is a nearly constant [(Right) Fig. 7], which implies $U'[p, 29.5; 2.17] \simeq 0$. This completes our classification: $p < 0, \ p = 0, \ p > 0$.  

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Figure 6: Potential with trapping well $V_{KN}(r)$, its far-region $V_{aKN}(r)$, and asymptotic potential $V_{aaKN}(r)$ as functions of $r \in [r_+ = 1.062, 100]$ with $M = 1, Q = 0.01, \omega = 3.85, a = 0.998, m = 13, q = 0.2, \lambda_{lm}, \mu = 4$. Here, we choose $\lambda_{lm} = 202.461$ to find $a = -5.9 \times 10^{-9} \sim 0$ in $U'[p, s; cr]$. $V_{KN}(r)$ has a trapping well located at $r = 10.8$.

Figure 7: (Left) Radial mode $\Psi(r)$ showing a half of a peak as a function of $r \in [12.66, 100]$ with trapping well. (Right) Confluent hypergeometric function $U[-5.9 \times 10^{-9} \simeq 0, 29.5; 2.17r]$ represents a constant and its derivative $U'[-5.9 \times 10^{-9} \simeq 0, 29.5; 2.17r]$ is nearly zero. All coefficients go together with Fig. 6.

4 Condition for trapping well in Kerr black holes

Now, we are in a position to study the superradiant instability/stability of Kerr black holes under a massive scalar propagation. We obtain this case when choosing $q = Q = 0$. Its far-region equation is given by Eq. (24) with $A = \sqrt{\mu^2 - \omega^2}, B = 2\mu^2, C = l(l + 1)$, and $k = l + 1/2$. Here, one should find the other condition for no trapping well because $B > 0 [V'_{aaK}(r) > 0]$ implies that a trapping well necessarily arises. In this case, the solution
is given by \[18\]

\[
\Psi^K_{lm}(r) = c_2 e^{-\sqrt{\mu^2-\omega^2}r} (2\sqrt{\mu^2-\omega^2})^{l+1} U\left[ l + 1 - \frac{M\mu^2}{\sqrt{\mu^2 - \omega^2}}, 2l + 2; 2\sqrt{\mu^2 - \omega^2}r \right].
\] (37)

Here, the condition for a trapping well is obtained from the first argument \((p < 0)\) in \(U[p, s; cr]\) as

\[
\frac{M\mu^2}{\sqrt{\mu^2 - \omega^2}} > l + 1,
\] (38)

while the condition for no trapping well takes the form

\[
\frac{M\mu^2}{\sqrt{\mu^2 - \omega^2}} < l + 1.
\] (39)

Eq.\[38\] is satisfied for the potential with trapping well \([\text{similar to (Left) Fig. 1}]\) whose parameters are given by \(M = 1, \omega = 3.85, a = 0.998, m = 13, l = 13, \mu = 4,\) and \(\omega_c = 6.101 > \omega.\) We note that its confluent hypergeometric function \(U[-0.61, 28, 2.19r]\) is an increasing function of \(r.\) Also, we have checked the stability condition \[39\] for the potential without trapping well \([\text{similar to (Right) Fig. 1}]\) whose parameters are \(M = 1, \omega = 0.02, a = 0.3, m = 1, l = 3, \mu = 0.05,\) and \(\omega_c = 0.076 > \omega.\) As is expected, its confluent hypergeometric function \(U[3.95, 8, 0.09r]\) is a decreasing function of \(r.\)

For a complex \(\omega = 3.85 + 10^{-6}i\) \((|\omega| \ll \omega_R, \omega_R < m\Omega_H)\) with the same parameters \[18, 19\], one has its asymptotic solution

\[
\Psi_{13,13} \sim (51422 - 2.35i)e^{(-1.09 + 3.5 \times 10^{-6}i)r} U[-0.74 - 5 \times 10^{-5}i, 28, (2.17 - 7 \times 10^{-6}i)r]
\] (40)

whose real part takes the form of a peak as in (Left) Fig. 3. Here, one may rewrite \(p\) in \(U[p, s; cr]\) as

\[
p = -n - \delta\nu,
\] (41)

where \(n = 1\) and \(\delta\nu = -0.26 + 5 \times 10^{-5}i(|\delta\nu| = 0.26 < 1).\) The latter complex number might represent a deviation from the hydrogen wave functions. A state of superradiant instability could not be approximately described by a small shift of hydrogen energy level because \(|\delta\nu| = 0.26\) is not a small shift. This may be so because we use \(M\mu = 4 > 1\) (not ultralight bosons with \(\omega \sim \mu \ll 1/M)).\)
5 Discussions

First of all, we would like to mention the conditions for superradiant instability and stability. The superradiant instability can be achieved for \( \omega < \omega_c \) and \( \omega < \mu \) with a positive trapping well, whereas the superradiant stability can be found for \( \omega < \omega_c \) and \( \omega < \mu \) without a positive trapping well. The presence of a trapping well is regarded as a decisive condition for superradiant instability. If there is no trapping well, it corresponds to the superradiant stability.

In this work, we have firstly obtained two conditions for getting a trapping well of KNBHs under a charged massive scalar perturbation by analyzing asymptotic scalar potential \([V_{aaKN}(r)]\) and far-region wave function \((U[p, s; cr])\). They are given by \( V'_{aaKN}(r) > 0 \) \((M\mu^2 > qQ\omega)\) and \( U'[p, s; cr] > 0 \) \((p < 0)\). Also, the two conditions for no trapping well are \( V'_{aaKN}(r) < 0 \) \((M\mu^2 < qQ\omega)\) and \( U'[p, s; cr] < 0 \) \((p > 0)\). From the former, we have derived one condition of Eq.(36) for the superradiant stability.

We have carried out the same analysis for Kerr black hole under a massive scalar propagation to find the conditions for superradiant instability and stability. The superradiant instability condition is given by \( M\mu^2/\sqrt{\mu^2 - \omega^2} > l + 1 \), while the stability condition is \( M\mu^2/\sqrt{\mu^2 - \omega^2} < l + 1 \) because \( V'_{aaK}(r) > 0 \) \((M\mu^2 > 0)\) is always satisfied.

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