A few reflections on protective measurements and more.

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Abstract. Quantum measurement represents the most severe and charming problem of foundations of quantum mechanics. Several issues ranging from the collapse of the wave function to the ontic meaning of the wave function remain unclarified. Here we briefly summarize some recent ideas about this problem, with particular attention to protective measurements and their meaning.

1. Introduction
Quantum measurement represents the most severe and charming problem of foundations of quantum mechanics. Several issues ranging from the collapse of the wave function to the ontic meaning of the wave function remain unclarified [1].

The recent research in this sense has, on the one hand, pointed to define more precisely the formalism and, on the other hand, to look deeper in quantum measurement mechanisms for better understanding the meaning of the wave function and its collapse.

In the first line of research a very successful attempt is the one based on monoidal categories [2].

In summary, a symmetric monoidal category is defined from objects, $A$, forming a commutative monoid with the tensor product $\otimes$ and identity $I$. Objects are interpreted as system types, as qubits, more qubits, classical data, ... When every object has a dual (the adjoint) the category is compact.

Morphisms $I \rightarrow A$ represent states, through their preparation. Defined classical objects $X$, a non-demolitive measurement is represented by $M : A \rightarrow X \otimes A$. The point is how defining classical states. This is done defining a classical state as a state coming together with a copying operation $\delta : X \rightarrow X \otimes X$. Formally a classical object in a compact category is defined through a Frobenius algebra.

The concept of projector is captured by the one of idempotent morphism, i.e. by the fact that repeated projective measurements of the same observable on the same state give the same result, $A \overset{M}{\rightarrow} X \otimes A \overset{1_X \otimes i}{\rightarrow} X \otimes X \otimes A \overset{\delta \otimes i}{\rightarrow} X \otimes A \overset{M}{\rightarrow} A$.

Nevertheless, describing a genuine quantum measurement requires a further step for including decoherence. This is achieved by defining a genuine quantum measurement

$$Meas := (1_B \otimes \delta \circ \delta \otimes 1_b) \circ (M \otimes M).$$

(1)
This can then be extended to demolitive measurements \( m : A \to X \) by \((\delta \circ \delta) \circ (m \otimes m_a)\).

This new formulation of quantum mechanics in terms of monoidal categories can indeed provide a new compact definition of quantum mechanics, connecting it to the last developments in mathematics, but, anyway, does not provide any new physical insight on the physics of quantum measurements.

For achieving new insights on the physical meaning of quantum measurement, in the second line of research we mentioned, a significant research has been addressed to study quantum measurement in weak coupling regime.

In 1988 Aharonv, Albert and Vaidman introduced \([3]\) weak value measurements that represent a new paradigm of quantum measurement where so little information is extracted from a single measurement that the state does not collapse. In little more detail, the weak value of an observable \( \hat{A} \) is defined as \( \langle \hat{A} \rangle_w = \langle \psi_f | \hat{A} | \psi_i \rangle \), where the key role is symmetrically played by the pre-selected \((|\psi_i\rangle)\) and post-selected \((|\psi_f\rangle)\) quantum states. In category calculus this requires an accordingly modification of Eq.1. When the pre- and post-selected states are equal, the weak value is just the expectation value of \( \hat{A} \).

If we consider a von Neumann coupling between the observable \( \hat{A} \) and a pointer observable \( \hat{P} \), according to the unitary transformation \( \hat{U} = \exp(-ig\hat{A} \otimes \hat{P}) \), in the weak interaction regime the evolution of this system is

\[
\langle \psi_f | e^{-ig\hat{A} \otimes \hat{P}} | \psi_i \rangle \simeq \langle \psi_f | \psi_i \rangle (1 - ig \langle \hat{A} \rangle_w \hat{P}).
\]

(2)

Thus, weak values are measurable quantities and provide some information on the system. The interpretation of this provided information is anyway not trivial. Weak values being normalised transition amplitudes are neither conventional probabilities nor expectation values/ eigenvalues of (Hermitian) operators.

Furthermore, weak values present non-classical properties assuming anomalous values (imaginary, negative, unbounded values), i.e. "the result of a measurement of a component of a spin 1/2 particle can turn to be 100" \([3]\), and when sequential weak measurements are considered one can find paradoxical situations where for two projection operators \( P_a \) and \( P_b \) one has \( \langle P_a \rangle_{\text{weak}} > 0 \) and \( \langle P_b \rangle_{\text{weak}} > 0 \), while \( \langle P_a \cdot P_b \rangle_{\text{weak}} \leq 0 \) or \( \langle P_a + P_b \rangle_{\text{weak}} \leq 0 \), a property now also demonstrated experimentally \([6]\). This situation represents a real conundrum \([10]\) since, on the one hand, at variance with projective measurements, weak values (due to the weak coupling) do not collapse the wave function and they can be performed sequentially on the same systems and, on the other hand, for the previous description in terms of von Neumann coupling demonstrates the measurement of the projection operator \( P_a \) can trigger the meter of the pointer only if the system before postselection had a non zero probability corresponding to \( P_a \).

Therefore, the attribution of a clear meaning to weak values is still missing. Eventually it can be searched in categorical formalism of quantum mechanics.

For these amazing properties, weak values in the recent years have been subject of a large interest since their non-classical properties allow investigating fundamental aspects of quantum mechanics \([5, 6]\). Furthermore, the possibility of amplifying the measurement of small parameters by expolring weak values paved the way to significant applications in quantum metrology & sensing \([4]\) and other quantum technologies \([7]\).

2. The protective measurements

Another interesting example of non conventional quantum measurement is provided by protective measurements. They represent a new paradigm of quantum measurement \([8]\) allowing the possibility of measuring directly the expectation value of a quantum observable. In little more detail, let us consider the standard von Neumann quantum measurement of an observable \( A \): it
involves an interaction Hamiltonian coupling the observable \( H \) to the pointer momentum \( P \)

\[
H_A = g(t)PA
\]  

In a standard impulsive measurement \( g(t) \neq 0 \) in a very short interval, leading to a shift of the pointer proportional to the eigenvalue \( a_n \) of the observable \( A \) corresponding to the eigenstate \( |a_n\rangle \) in which the state is collapsed. Thus, disposing of a single state no information on the original state is acquired. On the other hand, in a protective measurement \( g(t) \sim 1/\tau \) is smooth enough that the state is not changed (i.e. a ”protection” of the state is realised). The evolution of the initial product state describing the system state (in the eigenvalue \( |\nu\rangle \) of system Hamiltonian \( H_S \)) and the pointer state, \( |\Psi(0)\rangle = |\nu\rangle|\phi(r_0)\rangle \), in terms of the system energy eigenstates \( |\mu\rangle \) is

\[
|\Psi(T)\rangle = \sum_{n,\mu} e^{\frac{i}{\hbar}(P\langle A\rangle + \langle \mu|H_S|\mu\rangle)T}|a_n\rangle|\mu\rangle|\nu\rangle|\phi(r_0)\rangle \approx \sum_{\mu} e^{\frac{i}{\hbar}\langle P\langle A\rangle + \langle \mu|H_S|\mu\rangle\rangle}\phi(r_0)\rangle = e^{\frac{i}{\hbar}\langle [\mu|H_S|\mu]\rangle\phi(r_0)\rangle}
\]

which is valid in the adiabatic limit when the energy eigenstates remain unchanged. Thus, at the end of the measurement the pointer shift is given by the average value \( \langle A \rangle \) and the system, in its initial state \( |\nu\rangle \), is not entangled with the pointer. The essential point is that the average value of an observable can be evaluated directly by a single system, eventually even not an eigenstate of the corresponding operator, and not as a statistical average of eigenvalues for an ensemble. By measuring the averages of a sufficiently large number of variables the wave function of the system can eventually be reconstructed to any desired precision.

In practice, the protection of the state can be performed in two ways. In a first case when it is a nondegenerate eigenstate of system Hamiltonian (as above). In a second case by exploiting Zeno effect, i.e. through frequent measurements of a variable for which the state is a nondegenerate eigenstate. Of course in both cases the protection of the state requires some ”a priori” knowledge of the state itself [16]; in category language this means that we are now considering a map \( X \otimes A \rightarrow X \otimes A \). Anyway, for practical applications one has to distinguish between the information available to the party protecting the state and the one measuring the state.

In particular, protective measurements represent an interesting element in the debate about the ontic vs. epistemic role of the wavefunction. The quantum wavefunction is the fundamental element of quantum mechanics. Originally Schrödinger attributed to the wavefunction an ontological meaning, i.e. that it really exists regardless of our knowledge. However, after Born promoted the epistemological view of the wavefunction, i.e. seeing it as a statistical tool depending on our present knowledge of the system, this opinion prevailed up to recent years when the discussion re-emerged [9].

The relevance of protective measurements in this debate stems from the fact that protective measurements would allow inferring the properties of a single quantum state of a single system and even the (protected) wavefunction itself only negligibly disturbing it during the process, while in general the state of an unknown system, according with the ”no cloning” theorem, should remain undetectable.

Nevertheless, of course protective measurements require an ”a priori” knowledge of the state (eventually regarding a subspace of Hilbert space if the Hamiltonian evolution in Eq. 3 does not involve the whole space), for casting the protection, and therefore their meaning in the previous discussion can be questioned [14, 15, 16, 17, 18, 19, 20].

However, protective measurements have found several interesting theoretical applications in foundations of quantum mechanics, among which we can mention exploring the meaning...
of the wavefunction [20], examining empiricist vs. realist aspects of quantum mechanics [11] or the discussion about Bohmian interpretation of quantum mechanics [12, 13, 21]. An application to quantum state measurements has also been proposed [22]: here the optimization of protective measurement is used for minimising the state disturbance when performing a protective measurement of a single system. As a further interesting study, we can mention the extension of the concept of protection to include protection in pre- and post-selected systems [23, 20].

Finally, protective measurements also represent an interesting quantum information protocol: Alice prepares a single particle in a well-defined (pure) state. She then sends this "particle" to Bob who has to find its state. Bob knows the no-cloning theorem, which renders this task impossible and hence asks Alice to send also a proper protection mechanism for the specific state she prepared. Alice agrees, sending him the protection apparatus in the form of a black box. Bob can nevertheless employ this black box for performing his set of weak measurements to find what is the state of the particle without altering it. A protocol that can find applications, for instance, in testing state preparation, and that provides the most significant experimental motivation for this kind of measurement, that is currently being realised in INRIM (see Fig.1).

3. Conclusions
Drawing some conclusion, the field of quantum measurement remains one of the most fascinating fields of investigation, involving the very basis of the most fundamental and successful theory we dispose of. Protective measurements are an interesting tool in this debate and may also represent a useful protocol for quantum technologies.

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