The numerical evaluation of the Riesz function

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Abstract
The behaviour of the generalised Riesz function defined by
\[ S_{m,p}(x) = \sum_{k=0}^{\infty} \frac{(-)^{k-1}x^k}{k! \zeta(mk+p)} \quad (m \geq 1, \ p \geq 1) \]
is considered for large positive values of \( x \). A numerical scheme is given to compute this function which enables the visualisation of its asymptotic form. The two cases \( m = 2, \ p = 1 \) and \( m = p = 2 \) (introduced respectively by Hardy and Littlewood in 1918 and Riesz in 1915) are examined in detail. It is found on numerical evidence that these functions appear to exhibit the \( x^{-1/4} \) and \( x^{-3/4} \) decay, superimposed on an oscillatory structure, required for the truth of the Riemann hypothesis.

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1. Introduction

The generalised Riesz function is defined by the sum
\[ S_{m,p}(x) = \sum_{k=0}^{\infty} \frac{(-)^{k-1}x^k}{k! \zeta(mk+p)} \quad (x \geq 0), \quad (1.1) \]
where \( m \geq 1, \ p \geq 1 \) and \( \zeta(s) \) is the Riemann zeta function. The original function considered by Riesz \[8\] took the form
\[ \sum_{k=1}^{\infty} \frac{(-)^{k-1}x^k}{(k-1)! \zeta(2k)}, \]
which corresponds to a case of (1.1) with \( m = p = 2 \) since it is easily seen to equal \(-xS_{2,2}(x)\). A similar function corresponding to \( m = 2, \ p = 1 \) was discussed in the famous memoir by Hardy and Littlewood \[3\]. The interest in both these cases of (1.1) results from the fact that a necessary and sufficient condition for the truth of the Riemann hypothesis is that \[9, p. 382\]
\[ S_{2,1}(x) = O(x^{-1/4+\epsilon}), \quad S_{2,2}(x) = O(x^{-3/4+\epsilon}) \quad (1.2) \]
as \( x \to +\infty \), where \( \epsilon \) is an arbitrarily small positive quantity. The results in (1.2) are superficially attractive as they are derived from sums containing only values of \( \zeta(s) \) at positive integer values of \( s \).

The sum in (1.1) can also be viewed as an example of a perturbation of the exponential series for \( e^{-x} \) in the form \( \sum_{k \geq 0} a_k(-x)^k/k! \), where \( a_k \) are coefficients that possess the property
as \(a_k \to 1\) as \(k \to \infty\); in the case of the Riesz function we have \(a_k = 1/\zeta(mk + p)\). The growth of this series for large (complex) \(x\) is found to depend sensitively on the decay of the perturbing coefficients \(a_k\). A discussion of this problem, together with several examples, is given in [6].

This paper is partly based on the earlier report by the author in [5]. We present a computational scheme for the numerical evaluation of \(S_{m,p}(x)\) for large positive values of \(x\). In particular, we concentrate on the Hardy-Littlewood case of \(m = 2, p = 1\) and also on the Riesz case \(m = p = 2\) and determine numerically their large-\(x\) behaviour. Based on our numerical results, we conclude that these cases are characterised by a damped oscillatory structure for sufficiently large positive \(x\) with an amplitude that corresponds to the estimates in [12].

2. A computational scheme

We use the result [9, p. 3]

\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad (\Re(s) > 1),
\]

where \(\mu(n)\) is the Möbius function defined by \(\mu(n) = (-)^r\) if \(n\) has \(r\) distinct primes (with \(\mu(1) = 1\) and \(\mu(n) = 0\) otherwise. Then we obtain the expansion for the case \(m = 2, p \geq 1\)

\[
S_{2,p}(x) = \sum_{k=0}^{\infty} \frac{(-)^{k-1} \mu(k)}{k! \zeta(2k + p)}
\]

\[
= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^p} \left(1 - e^{-X_n}\right) - \frac{1}{\zeta(p)}, \quad X_n := \frac{x}{n^2}, \quad (2.1)
\]

The factor \(1 - e^{-X_n}\) has a “cut-off” when \(n = N = [x^{1/2}]\); for \(n > N\), the decay of the terms in this series is slow with late terms eventually behaving like \(\mu(n)/n^{p+2}\).

This slow decay in the tail can be accelerated as follows. We write

\[
S_{2,p}(x) = \sum_{n=1}^{N-1} \frac{\mu(n)}{n^p} \left(1 - e^{-X_n}\right) + T_N(p; x) - \frac{1}{\zeta(p)}, \quad T_N(p; x) := \sum_{n=N}^{\infty} \frac{\mu(n)}{n^p} \left(1 - e^{-X_n}\right), \quad (2.2)
\]

where \(N\) is chosen as above. In the tail \(T_N(p; x)\) (where \(X_n \leq 1\)) we put

\[
1 - e^{-X_n} = X_n \left(1 - \frac{X_n}{2!} + \frac{X_n^2}{3!} - \cdots\right) = X_n \sum_{k=1}^{\infty} \frac{X_n^k}{k!} F_1(1; 2; -X_n)
\]

\[
= X_n e^{-X_n} F_1(1; 2; X_n) = X_n f_1(X_n),
\]

where

\[
f_k(z) := e^{-z} F_1(1; k + 1; z) \quad (k \geq 0). \quad (2.3)
\]

Here \(F_1\) denotes the confluent hypergeometric function and Kummer’s transformation [4, p. 325] has been used to change the argument from \(-X_n\) to \(X_n\).

From (A.2) in the appendix the following lemma is established:

**Lemma 1.** The function \(zf_1(z)\) satisfies the recursion formula

\[
z f_1(z) = 1 - \frac{1}{e_k(z)} + \frac{z^k f_k(z)}{k! e_k(z)} \quad (k \geq 1),
\]

where \(e_k(z)\) denotes the sum of the first \(k\) terms of the exponential series

\[
e_k(z) := \sum_{r=0}^{k-1} \frac{z^r}{r!} \quad (2.5)
\]
From the definition (2.5) it follows that
\[ \frac{1}{e_k(z)} = e_k(-z) + \frac{z^k g_k(z)}{k! e_k(z)}, \] (2.6)
where \( g_k(z) \) is a polynomial of degree \( k - 2 \). Some routine algebra shows that
\[ g_2(z) = 2, \quad g_3(z) = -\frac{3}{2}z, \quad g_4(z) = 2 + \frac{2}{3}z^2, \]
\[ g_5(z) = -\frac{5}{3}z - \frac{5}{24}z^3, \quad g_6(z) = 2 + \frac{3}{4}z^2 + \frac{1}{20}z^4, \ldots. \]

Then, using (2.4)–(2.6), the tail of the series can be written in the form
\[ T_N(p; x) = T_N^{(1)}(p; x) + T_N^{(2)}(p; x), \] (2.7)
where
\[ T_N^{(1)}(p; x) = \frac{1}{k!} \sum_{n=N}^{\infty} \frac{\mu(n)}{np} \frac{X_n^k \Delta_k(X_n)}{e_k(X_n)}, \quad \Delta_k(X_n) := f_k(X_n) - g_k(X_n) \]
and
\[ T_N^{(2)}(p; x) = \sum_{n=N}^{\infty} \frac{\mu(n)}{np} \sum_{r=1}^{k-1} \frac{(-1)^{r-1} X_n^r}{r!} = \sum_{r=1}^{k-1} \frac{(-X_N^r)}{r!} \lambda_r \] (2.9)
with
\[ X_N := \frac{x}{N^2}, \quad \lambda_r := N^{2r} \left\{ \sum_{n=1}^{N-1} \frac{1}{n^{2r+p}} - \frac{1}{\zeta(2r+p)} \right\}. \]

Since \( X_n \to 0 \) as \( n \to \infty \), we see that the decay of the late terms in \( T_N^{(1)}(x) \) is now controlled by \( \mu(n)/n^{2k+p} \), which for \( k \geq 2 \) represents a modest improvement in the rate of convergence of the series.

Collecting together the results in (2.2), (2.7)–(2.9), we have
\[ S_{2,p}(x) = \sum_{n=1}^{N-1} \frac{\mu(n)}{np} (1 - e^{-X_n}) - \frac{1}{\zeta(p)} + T_N^{(1)}(p; x) + T_N^{(2)}(p; x) \] (2.10)
for \( p \geq 1 \).

3. Numerical results for \( m = 2 \) and \( p = 1, 2 \)

We have employed the scheme (2.10) with \( k = 6 \) to compute the Hardy-Littlewood case \( m = 2 \)
\( p = 1 \) for \( x \geq 0 \) up to \( x = 10^6 \). The results are shown in the sequence of plots in Fig. 1. It is found that \( S(x) = S_{2,1}(x) \) decreases once \( x \geq 3 \) down to values of the order \( 10^{-6} \), whereupon the graph commences to oscillate about the zero line. The oscillations appear to be regular and have a decreasing amplitude.

The successive maxima \( (x_{2,1}^+)^k \) and minima \( (x_{2,1}^-)^k \) in the oscillatory region are determined by calculating the zeros of the derivative \( S'_{2,1}(x) \), where
\[ S'_{2,1}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^3} e^{-X_n} - \frac{1}{\zeta(3)} - \sum_{n=1}^{\infty} \frac{\mu(n)}{n^3} (1 - e^{-X_n}), \]
by using (2.10) with \( p = 3 \). The results of these calculations together with the corresponding values of \( S(x_{2,1}^k) \) are shown in Table 1. Plots of \( \log S_{2,1}(x_{2,1}^k) \) against \( \log x \) are shown in Fig. 2,
Figure 1: The graph of $S(x) \equiv S_{2,1}(x)$ against $\log_{10} x$ for different ranges of $x$.

Table 1: Values of the maxima and minima $\log_{10} x_k^+$ and the corresponding values of $S_{2,1}(x_k^+)$.

| $k$ | $\log_{10} x_k^+$ | $S_{2,1}(x_k^+)$ | $\log_{10} x_k^-$ | $S_{2,1}(x_k^-)$ |
|-----|------------------|------------------|------------------|------------------|
| 1   | 4.83284          | 2.62044 × 10^{-6}| 4.65573          | -4.78520 × 10^{-7} |
| 2   | 5.22278          | 1.65298 × 10^{-6}| 5.03476          | -1.20507 × 10^{-6} |
| 3   | 5.61033          | 1.21568 × 10^{-6}| 5.41918          | -1.18328 × 10^{-6} |
| 4   | 5.99669          | 9.38573 × 10^{-7}| 5.80406          | -1.00735 × 10^{-6} |
| 5   | 6.38315          | 7.47310 × 10^{-7}| 6.19039          | -8.20165 × 10^{-7}  |
| 6   | 6.76905          | 5.93349 × 10^{-7}| 6.57596          | -6.62603 × 10^{-7}  |
| 7   | 7.15545          | 4.76797 × 10^{-7}| 6.96249          | -5.30007 × 10^{-7}  |
| 8   | 7.54124          | 3.80007 × 10^{-7}| 7.34817          | -4.26057 × 10^{-7}  |

where it is seen that they reveal a linear variation to a good approximation. The dashed lines in these figures have slope equal to $-0.25$, thereby numerically confirming the estimate in (1.2).

The case of the original Riesz function has $m = p = 2$ and from (2.1) we have

$$S_{2,2}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} (1 - e^{-X_n}) - \frac{1}{\zeta(2)},$$

where $\zeta(2) = \frac{\pi^2}{6}$. The results are shown in Fig. 3 for the range $0 \leq x \leq 10^8$ obtained using the scheme (2.10) with $p = 2$. This function presents a similar behaviour with its value decreasing until about $10^{-8}$ before commencing to oscillate about the zero line. The first zero has the value $x = 1.15671$. A plot of $xS_{2,2}(x)$ is also given in [2]. The successive maxima and minima in the oscillatory region are computed as the zeros of the derivative

$$S_{2,2}'(x) = \frac{1}{\zeta(4)} - \sum_{n=1}^{\infty} \frac{\mu(n)}{n^4} (1 - e^{-X_n}).$$

We commence the enumeration of the maxima and minima from the point where the graph of $S(x)$ first becomes negative.
Asymptotics of $F_{n,\sigma}(x; \mu)$

Figure 2: Plots of (a) $\log_{10} S(x^+_k)$ and (b) $\log_{10} S(x^-_k)$ against $\log_{10} x$ (where $S(x) \equiv S_{2,1}(x)$). The dashed lines have slope $-0.25$.

Figure 3: The graph of $S(x) \equiv S_{2,2}(x)$ against $\log_{10} x$ for different ranges of $x$.

The results of these calculations together with the corresponding values of $S(x^+_k)$ are shown in Table 2. Plots of $\log S_{2,2}(x^+_k)$ against $\log x$ are shown in Fig. 4, where it is seen that they reveal a linear variation to a good approximation. The dashed lines in these figures have slope equal to $-0.75$, thereby numerically confirming the estimate in (1.2).

Table 2: Values of the maxima and minima $\log_{10} x^+_k$ and the corresponding values of $S_{2,2}(x^+_k)$.

| $k$ | $\log_{10} x^+_k$          | $S_{2,2}(x^+_k)$ | $\log_{10} x^-_k$ | $S_{2,2}(x^-_k)$ |
|-----|--------------------------|-----------------|------------------|-----------------|
| 1   | 4.48752                  | 4.97204 $\times 10^{-8}$ | 4.31797        | -6.46896 $\times 10^{-9}$ |
| 2   | 4.87969                  | 1.97351 $\times 10^{-8}$ | 4.69479        | -1.67414 $\times 10^{-8}$ |
| 3   | 5.26779                  | 9.09065 $\times 10^{-9}$ | 5.07699        | -1.09071 $\times 10^{-8}$ |
| 4   | 5.65449                  | 4.53355 $\times 10^{-9}$ | 5.46263        | -5.99878 $\times 10^{-9}$ |
| 5   | 6.04080                  | 2.28418 $\times 10^{-9}$ | 5.84775        | -3.14719 $\times 10^{-9}$ |
| 6   | 6.42706                  | 1.17580 $\times 10^{-9}$ | 6.23435        | -1.62554 $\times 10^{-9}$ |
| 7   | 6.81307                  | 5.98676 $\times 10^{-10}$ | 6.61979        | -8.37753 $\times 10^{-10}$ |
| 8   | 7.19932                  | 3.07392 $\times 10^{-10}$ | 7.00651        | -4.29573 $\times 10^{-10}$ |
It has been shown in [1] that $S_{2,2}(x)$ has an infinite number of zeros. It is very probable that $S_{2,1}(x)$ also has an infinite number of zeros.

4. An asymptotic expansion

An integral representation of $S_{m,p}(x)$ in the form of a Mellin-Barnes integral is given by

$$S_{m,p}(x) = -\frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{\Gamma(s)x^{-s}}{\zeta(p-ms)} \, ds \quad (0 < c < c_0),$$

(4.1)

where $c_0 = (p - \frac{1}{2})/m$. The integrand possesses simple poles at $s = 0, -1, -2, \ldots$ and at the trivial zeros of the zeta function at $s = (p+2k)/m, k = 1, 2, \ldots$. On the assumption of the Riemann hypothesis, there is also an infinite number of (simple) poles on the line $\Re(s) = c_0$ given by $s = c_0 \pm i\gamma_k/m \, (k = 1, 2, \ldots)$, where $\zeta(\frac{1}{2} \pm i\gamma_k) = 0$. Assuming that it is permissible to displace the integration path past this line, we obtain the result

$$S_{m,p}(x) = -\frac{2x^{-c_0}}{m} \Re \sum_{k=1}^{\infty} \frac{\Gamma(c_0 - i\gamma_k/m)}{\zeta(\frac{1}{2} + i\gamma_k)} x^{i\gamma_k/m} + O(x^{-(p+2)/m})$$

as $x \to +\infty$. The details of the case $m = 2, p = 1$ are discussed in [3, §2.5]; see also the account presented in [7, p. 143].

If we now set

$$A_k = \frac{\Gamma(c_0 - i\gamma_k/m)}{\zeta(\frac{1}{2} + i\gamma_k)}, \quad \psi_k = \pi + \arg A_k,$$

we find that

$$S_{m,p}(x) = \frac{2x^{-c_0}}{m} \sum_{k=1}^{\infty} |A_k| \cos \left( \frac{\gamma_k}{m} \log x + \psi_k \right) + O(x^{-(p+2)/m}).$$

(4.2)

The convergence of the sum (4.2) is difficult to establish. The gamma function present in the coefficients $A_k$ decays very rapidly for increasing $k$, since from Stirling’s formula it contains the exponential factor $\exp[-\pi\gamma_k/(2m)]$ for large $\gamma_k$. The magnitude of $\zeta(\frac{1}{2} + i\gamma_k)$ (which is non-zero on the assumption that the non-trivial zeros are all simple) generally increases with $k$, but it is possible that there are zeros for which this quantity could become small.

When $m = p = 2$, we obtain the expansion

$$S_{2,2}(x) = x^{-3/4} \sum_{k=1}^{\infty} |A_k| \cos \left( \frac{\gamma_k}{2} \log x + \psi_k \right) + O(x^{-2})$$

(4.3)

with $A_k = \frac{\Gamma(\frac{3}{4} - \frac{1}{2}i\gamma_k)}{\zeta(\frac{1}{2} + i\gamma_k)}$. The graph of $S_{2,2}(x)$ against $\log_{10} x$ compared with the expansion (4.3) truncated after $k = 5$ terms is shown in Fig. 5. It is seen that for $x \gtrsim 10^6$ the curves are indistinguishable on the scale of the figure.
5. Concluding remarks

The numerical results obtained in Section 3 are indicative only. It appears from the numerical investigation – without any reference to the Riemann hypothesis – that the large-

x behaviour of the functions $S_{2,1}(x)$ and $S_{2,2}(x)$ possesses respectively the $x^{-1/4}$ and $x^{-3/4}$ decay superimposed on an oscillatory structure. A striking feature of the plots in Figs. 1 and 3 is the fact that the final decaying oscillatory structure is not obtained until $x$ has attained the value of approximately $10^5$. This is an unusual occurrence since most special functions begin to exhibit their asymptotic structure for often surprisingly modest values of the variable.

The computation of $S_{m,p}(x)$ for $m \geq 3$ is made easier since the cut-off value $N$ then scales like $x^{1/m}$ and the rate of decay of the various series in (2.10) is correspondingly more rapid. As an example, the case $m = 3, p = 2$ is shown in [5]. The behaviour is found to be similar to that depicted in Figs. 1 and 3, with the maxima and minima following an approximate $x^{-1/2}$ scaling predicted by (4.2).

Appendix: Derivation of Lemma 1

From the contiguous relation satisfied by the confluent hypergeometric function $\mathbf{1} \mathbf{F}_1(a; b, z)$ [4 (13.3.2)]

$$b(b-1) \mathbf{1} \mathbf{F}_1(a; b-1; z) + b(1-b-z) \mathbf{1} \mathbf{F}_1(a; b; z) + z(b-a) \mathbf{1} \mathbf{F}_1(a; b+1; z) = 0,$$

we obtain, with $a = 1, b = k + 2$, the recursion formula satisfied by $f_k(z) := e^{-z} \mathbf{1} \mathbf{F}_1(1; k+1; z)$ in the form

$$f_k(z) - \left(1 + \frac{z}{k+1}\right) f_{k+1}(z) + \frac{z}{k+2} f_{k+2}(z) = 0 \quad (k \geq 0).$$

Repeated use of this result, combined with $f_0(z) = 1$ and the partial sum of the exponential series $e_k(z)$ defined in (2.5), shows successively that

$$f_1(z) = \frac{1}{e_2(z)} + \frac{ze_1(z)}{2e_2(z)} f_2(z),$$

$$f_2(z) = \frac{1}{e_3(z)} + \frac{ze_2(z)}{3e_3(z)} f_3(z),$$

and, in general,

$$f_{k-1}(z) = \frac{1}{e_k(z)} + \frac{ze_{k-1}(z)}{ke_k(z)} f_k(z) \quad (k \geq 1). \quad (A.1)$$
Then, from (A.1) we find that

$$zf_1(z) = \frac{z}{e_2(z)} + \frac{z^2 f_2(z)}{2e_2(z)} = 1 - \frac{1}{e_2(z)} + \frac{z^2 f_2(z)}{2e_2(z)}$$

$$= 1 - \frac{1}{e_2(z)} + \frac{z^2}{2e_2(z)} \left( \frac{1}{e_3(z)} + \frac{ze_2(z)}{3e_3(z)} f_3(z) \right)$$

$$= 1 - \frac{1}{e_3(z)} + \frac{z^3 f_3(z)}{3e_3(z)},$$

where we have used $e_1(z) = 1$, $e_2(z) = 1 + z$ and have written $z^2/2! = e_3(z) - e_2(z)$. This procedure can be continued to produce the final result

$$zf_1(z) = 1 - \frac{1}{e_k(z)} + \frac{z^k f_k(z)}{k!e_k(z)} \quad (k \geq 1). \quad (A.2)$$

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