Freudenthal ranks: GHZ versus W

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Abstract

The Hilbert space of three-qubit pure states may be identified with a Freudenthal triple system. Every state has an unique Freudenthal rank ranging from 1 to 4, which is determined by a set of automorphism group covariants. It is shown here that the optimal success rates for winning a three-player non-local game, varying over all local strategies, are strictly ordered by the Freudenthal rank of the shared three-qubit resource.

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1. Introduction

It is by now well known that, under the paradigm of stochastic local operations and classical communication (SLOCC), three-qubits\(^1\) can be entangled in four physically distinct ways: (1) separable \(A–B–C\), (2) biseparable \(A–BC\), (3) totally entangled W states, (4) totally entangled GHZ states (Greenberger–Horne–Zeilinger) [2]. The most important and interesting aspect of this classification is the appearance of two inequivalent forms of totally entangled states, W and GHZ. It is not enough to simply declare a state is totally entangled, one must also specify how it is totally entangled.

This three-qubit SLOCC classification may be elegantly captured by identifying the three-qubit state space with a particular Freudenthal triple system (FTS) defined over a cubic Jordan algebra [3]. In the present work, this construction is reformulated in section 2 using the axiomatic FTS, which dispenses with the underlying Jordan algebra. This FTS framework is not limited to three-qubits\(^2\) and has been extended to a number of more exotic multipartite systems including mixtures of bosonic and fermionic qudits [9–13].

An important feature common to all FTS is the universal notion of rank. Any element of a given FTS has a unique rank 1, 2, 3 or 4. For the three-qubit FTS these ranks are nothing but

\(^1\) Here and throughout we restrict our attention to pure states. An FTS perspective on three-qubit mixed state entanglement can be found in [1].

\(^2\) Remarkably, the seemingly unrelated concept of Freudenthal duality, introduced in the context of supergravity [4–7], also has a qubit significance [8].
the SLOCC entanglement classes: rank (1) separable $A-B-C$, rank (2) biseparable $A-BC$, rank (3) totally entangled W states, rank (4) totally entangled GHZ states.

The labelling of the ranks 1 through 4 is not incidental; they are so ordered by implication through the defining rank conditions (5). This suggests, from the perspective of the FTS, that W and GHZ are not merely inequivalent, but in fact ordered; GHZ is both differently and more entangled than W, in some precise sense.

This particular mathematical ordering of the entanglement classes naturally raises the question of its physical significance. What set-up would lead three experimenters, with no knowledge of SLOCC, to conclude unequivocally that a black-box secretly containing a rank 4 state is more non-local than one containing a rank 3 state? Is there a single experiment which separates out all the FTS ranks?

It turns out that the obvious guess, Mermin’s elegant three-party GHZ experiment [14], is also the correct guess. To make the logic as clear as possible we adopt a reformulation of Mermin’s set-up in terms of a non-local cooperative game of incomplete information introduced in [15]. In this language the contradiction with local realism exposed by Mermin translates into the existence of a local strategy utilizing the GHZ state that wins the game with certainty.

Specifically, it is shown here that the algebraic Freudenthal rank conditions alone imply that there is no local strategy utilizing a rank 3 state that wins the game with certainty, in contrast to the rank 4 GHZ case. In fact, the optimal success rates are strictly ordered according as the rank:

$$\frac{3}{4} = p(\text{rank } 1) < p(\text{rank } 2) < p(\text{rank } 3) < p(\text{rank } 4) = 1,$$

where $p(\text{rank } n)$ denotes the greatest possible probability of winning using a rank $n$ state. On this basis we argue that the physical significance of the three-qubit Freudenthal ranks is most naturally expressed in terms of this three-player non-local game.

## 2. Freudenthal SLOCC classification

### 2.1. The Freudenthal triple system

In 1954 Freudenthal [16, 17] found that the 133-dimensional exceptional Lie group $E_7$ could be understood in terms of the automorphisms of a construction based on the minuscule 56-dimensional $E_7$-module built from the exceptional Jordan algebra of $3 \times 3$ Hermitian octonionic matrices. Today this construction goes by the name of the FTS, reflecting the special role played by its triple product.

Following Freudenthal, Meyberg [18] and Brown [19] axiomatized the ternary structure underlying the FTS. The $E_7$-module is just one of a class of modules of ‘groups of type $E_7$’, a set of (semi)simple Lie groups sharing common structural/geometrical properties as encapsulated by the FTS axioms.

**Definition 1** (FTS [19]). An FTS is axiomatically defined as a finite dimensional vector space $\mathcal{F}$ over a field $\mathbb{F}$ (not of characteristic 2 or 3), such that:

(i) $\mathcal{F}$ possesses a non-degenerate anti-symmetric bilinear form $\{ x, y \}$.

(ii) $\mathcal{F}$ possesses a symmetric four-linear form $q(x, y, z, w)$ which is not identically zero.

(iii) If the ternary product $T(x, y, z)$ is defined on $\mathcal{F}$ by $T(x, y, z), w = q(x, y, z, w)$, then

$$3[T(x, x, y), T(y, y, y)] = \{ x, y \} q(x, y, y, y).$$

Groups of type $E_7$ are defined in terms of the ‘automorphisms’ of the triple product.
Definition 2 (Automorphism group [19]). The automorphism group of an FTS is defined as the subset of invertible $F$-linear transformations preserving the quartic and quadratic forms:

$$\text{Aut}(\mathfrak{g}) := \{ \sigma \in \text{Iso}_F(\mathfrak{g}) | [\sigma x, \sigma y] = [x, y], \quad q(\sigma x) = q(x) \}. \quad (3)$$

Note, the conditions $[\sigma x, \sigma y] = [x, y]$ and $q(\sigma x) = q(x)$ immediately imply $\sigma$ acts as an automorphism of the triple product,

$$T(\sigma x, \sigma y, \sigma z) = \sigma T(x, y, z), \quad (4)$$

hence the name.

The conventional concept of matrix rank may be generalized to FTSs in a natural and $\text{Aut}(\mathfrak{g})$-invariant manner.

Definition 3 (The FTS rank [20, 21]). The rank of an arbitrary element $x \in \mathfrak{g}$ is defined by:

$$\text{rank}(x) = 0 \iff \begin{cases} x = 0 \end{cases}$$

$$\text{rank}(x) = 1 \iff \begin{cases} x \neq 0 \\ \Upsilon_x(y) = 0 \forall y \end{cases}$$

$$\text{rank}(x) = 2 \iff \begin{cases} \exists y \text{ s.t. } \Upsilon_x(y) \neq 0, \\ T(x, x, x) = 0 \end{cases}$$

$$\text{rank}(x) = 3 \iff \begin{cases} T(x, x, x) \neq 0 \\ q(x) = 0 \end{cases}$$

$$\text{rank}(x) = 4 \iff \{ q(x) \neq 0 \}

where we have defined $\Upsilon_x(y) := 3T(x, y, x) + x[x, y]x$.

The ranks partition $\mathfrak{g}$ and are manifestly invariant under $\text{Aut}(\mathfrak{g})$. Note, they are self-consistent and ordered in the sense that,

$$x = 0 \Rightarrow \Upsilon_x(y) = 0;$$

$$\Upsilon_x(y) = 0 \Rightarrow T(x, x, x) = 0;$$

$$T(x, x, x) = 0 \Rightarrow q(x) = 0. \quad (6)$$

The rank condition can be understood in terms of the representation theory of $\text{Aut}(\mathfrak{g})$. Recall, $\mathfrak{g}$ constitutes an $\text{Aut}(\mathfrak{g})$-module. Define,

$$\Upsilon : \mathfrak{g} \times \mathfrak{g} \rightarrow \text{Hom}_F(\mathfrak{g}, \mathfrak{g})$$

$$(x, y) \mapsto \Upsilon_{x,y} \quad (7)$$

where

$$\Upsilon_{x,y}(z) := 3T(x, y, z) + \frac{1}{2}[x, z]y + \frac{1}{2}[y, z]x. \quad (8)$$

Then $\Upsilon$ belongs to $\text{Lie}(\mathfrak{g})$, the Lie algebra of $\text{Aut}(\mathfrak{g})$. That is, $\Upsilon$ is the projection onto the adjoint in $\text{Sym}^2(\mathfrak{g})$. This follows from the observation that $\text{Lie}(\mathfrak{g})$ is given by all $\phi \in \text{Hom}_F(\mathfrak{g}, \mathfrak{g})$ such that $q(\phi x, x, x) = 0$ and $[\phi x, y] + \{ x, \phi y \} = 0$ for all $x, y \in \mathfrak{g}$, as is easily verified [22].

Lemma 1. The $F$-linear map $\Upsilon_x : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$\Upsilon_x(y) = 3T(x, y, y) + [x, y]y \quad (9)$$

is in $\text{Aut}(\mathfrak{g})$. Linearizing (9) with respect to $x$ implies that $\Upsilon_{x,y} : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$\Upsilon_{x,y}(z) = 6T(x, y, z) + [x, z]y + [y, z]x \quad (10)$$

is in $\text{Aut}(\mathfrak{g})$. 


Note, \( \Upsilon_{x,y} \) is a manifestly \( \text{Aut}(\mathfrak{F}) \)-covariant expression for the Freudenthal product \( x \wedge y \) given in [23]. To establish this simple result, note

\[
\{ \Upsilon_{x}(x), y \} + \{ x, \Upsilon_{x}(y) \} = 0
\]

(11)

follows directly from the anti-symmetry and symmetry of \( \{ x, y \} \) and \( q(x, y, z, w) = \{ T(x, y, z), w \} \), respectively. The second condition, \( q(\phi x, x, x) = 0, \forall \phi \in \text{Aut}(\mathfrak{F}) \), is satisfied since

\[
q(x, x, x, \Upsilon_{x}(x)) = 3[T(x, x, x), T(z, z, z)] + [z, x]T(x, x, x, z)
\]

(12)

vanishes due to the defining FTS relation (2).

Similarly, \( T \) is the projection onto \( \mathfrak{F} \) in \( \text{Sym}^4(\mathfrak{F}) \), as confirmed by (4), while \( q \) is by definition the singlet in \( \text{Sym}^4(\mathfrak{F}) \).

2.2. The three-qubit FTS

Consider the three-qubit pure states,

\[
|\psi\rangle = a_{ABC}|ABC\rangle, \quad A, B, C = 0, 1
\]

(13)

in \( \mathcal{H}_{ABC} = C^2 \otimes C^2 \otimes C^2 \). For notational clarity, we will use both \( e^{ABC} \) and \( |ABC\rangle \) interchangeably to denote the computational basis vectors.

**Definition 4** (Three-qubit FTS). The FTS of three-qubits is defined by,

\[
\mathcal{E}^{ABC} = e^{ABC} \delta_{BF} \varepsilon_{CC}
\]

(14)

and

\[
q(e^{A_1B_1C_1}, e^{A_2B_2C_2}, e^{A_3B_3C_3}, e^{A_4B_4C_4}) := \sum_{\text{perms}(1, 2, 3, 4)} e^{A_1A_2A_3A_4B_1B_2B_3B_4C_1C_2C_3C_4}
\]

(15)

Here \( \varepsilon^{ABC} \) is the \( \text{SL}_3(2, \mathbb{C}) \)-invariant anti-symmetric \( 2 \times 2 \) tensor, where \( \varepsilon^{01} = 1 \). With these definitions \( \mathcal{H}_{ABC} \) forms an FTS, as can be verified by checking (2). In fact, this FTS is based on an underlying Jordan algebra \( \mathfrak{J}_{ABC} \equiv C \oplus C \oplus C \). For a detailed discussion of this construction the reader is referred to [3].

The automorphism group is

\[
\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rtimes S_3,
\]

(16)

where \( S_3 \) denotes the three-qubit permutation group. The automorphism invariant rank conditions of (5) are given explicitly by the following tensors.

For a state \( |\psi\rangle = a_{ABC}|ABC\rangle \), the quartic norm \( q(|\psi\rangle) \) is given by

\[
q(|\psi\rangle) = 2 \det \gamma^A = 2 \det \gamma^B = 2 \det \gamma^C = -2 \text{Det} \omega_{ABC}.
\]

(17)
where \( \text{Det}_{a_{ABC}} \) is Cayley’s hyperdeterminant \([24, 25]\) and we have introduced the three symmetric matrices \( \gamma^A, \gamma^B, \) and \( \gamma^C \) defined by,
\[
\begin{align*}
(\gamma^A)_{A_1 A_2} &= \varepsilon^{B_1 B_2 B_3} a_{A_1 B_1 C_1} a_{A_2 B_2 C_2}, \\
(\gamma^B)_{B_1 B_2} &= \varepsilon^{C_1 C_2 C_3} a_{A_1 B_1 C_1} a_{A_2 B_2 C_2}, \\
(\gamma^C)_{C_1 C_2} &= \varepsilon^{A_1 A_2 A_3} a_{A_1 B_1 C_1} a_{A_2 B_2 C_2},
\end{align*}
\]
transforming respectively as a \((3, 1, 1), (1, 3, 1), (1, 1, 3)\) under \( \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \). Explicitly,
\[
\begin{align*}
\gamma^A &= \begin{pmatrix}
2(a_0 a_3 - a_1 a_2) & a_0 a_7 - a_1 a_6 + a_2 a_3 - a_5 a_2 \\
 a_0 a_7 - a_1 a_6 + a_2 a_3 - a_5 a_2 & 2(a_0 a_7 - a_5 a_6)
\end{pmatrix}, \\
\gamma^B &= \begin{pmatrix}
2(a_0 a_5 - a_4 a_1) & a_0 a_7 - a_4 a_3 + a_5 a_4 - a_6 a_1 \\
 a_0 a_7 - a_4 a_3 + a_5 a_4 - a_6 a_1 & 2(a_0 a_7 - a_6 a_3)
\end{pmatrix}, \\
\gamma^C &= \begin{pmatrix}
2(a_0 a_6 - a_2 a_4) & a_0 a_7 - a_2 a_5 + a_4 a_6 - a_3 a_3 \\
 a_0 a_7 - a_2 a_5 + a_4 a_6 - a_3 a_3 & 2(a_0 a_7 - a_3 a_3)
\end{pmatrix},
\end{align*}
\]
where we have made the decimal-binary conversion \( 0, 1, 2, 3, 4, 5, 6, 7 \) for \( 000, 001, 010, 011, 100, 101, 110, 111 \). In the same notation the hyperdeterminant is,
\[
\text{Det}_a = a_0^2 a_1^2 + a_1^2 a_2^2 + a_2^2 a_3^2 - 2(a_0 a_1 a_6 a_7 + a_0 a_2 a_3 a_7 + a_0 a_4 a_5 a_7 + a_1 a_2 a_5 a_6 + a_1 a_3 a_4 a_6 + a_2 a_3 a_4 a_5) + 4(a_0 a_1 a_3 a_6 + a_1 a_2 a_4 a_7).
\]

The triple product is given by
\[
|T(\psi)\rangle = T(a)_{ABC} |ABC\rangle,
\]
where \( T(a)_{ABC} \) may be written in three equivalent ways
\[
\begin{align*}
T_{A_1 B_1 C_1} &= \varepsilon^{A_1 A_2 A_3} a_{A_1 B_1 C_1} (\gamma^A)_{A_2 A_3}, \\
T_{A_1 B_1 C_1} &= \varepsilon^{B_1 B_2 B_3} a_{A_1 B_1 C_1} (\gamma^B)_{B_2 B_3}, \\
T_{A_1 B_1 C_1} &= \varepsilon^{C_1 C_2 C_3} a_{A_1 B_1 C_1} (\gamma^C)_{C_2 C_3},
\end{align*}
\]
each of which makes the identity
\[
q(\psi) = \{T(\psi), \psi\}
\]
manifest. Finally, \( \Upsilon(\phi) \) for an arbitrary state \( |\phi\rangle = b_{ABC} |ABC\rangle \) is given by
\[
|\Upsilon(\phi)\rangle = \Upsilon_{ABC} |ABC\rangle
\]
where
\[
\Upsilon_{ABC} = -\varepsilon^{A_1 A_2} b_{A_1 B_1 C_1} (\gamma^A)_{A_2 A_3} - \varepsilon^{B_1 B_2} b_{A_1 B_1 C_1} (\gamma^B)_{B_2 B_3} - \varepsilon^{C_1 C_2} b_{A_1 B_1 C_1} (\gamma^C)_{C_2 C_3}.
\]

2.3. SLOCC entanglement classification

The concept of SLOCC equivalence was introduced in \([2, 26]\). Two states lie in the same SLOCC-equivalence class if and only if they may be transformed into one another with some non-zero probability using LOCC operations. For more on LOCC operations and entanglement the reader is referred to \([27–29]\) and the references therein. The crucial observation is that since LOCC cannot create entanglement any two SLOCC-equivalent states must necessarily possess the same entanglement, irrespective of the particular measure used. It is this property which makes the SLOCC paradigm so suited to the task of classifying entanglement.

Restricting our attention to pure states, two \( n \)-qubit states are SLOCC-equivalent if and only if they are related by an element of \( \text{SL}_1(2, \mathbb{C}) \times \text{SL}_2(2, \mathbb{C}) \times \ldots \times \text{SL}_n(2, \mathbb{C}) \) \([2]\), which
Table 1. Three-qubit entanglement classification as according to the FTS rank system.

| Class    | Rank | Representative state | FTS rank condition | Vanishing | Non-vanishing |
|----------|------|----------------------|--------------------|-----------|---------------|
| Null     | 0    | −                    | −                  | Ψ         | −             |
| A–B–C    | 1    | [000]                | 3T(ψ,ψ,ψ)+{ψ,ψ}ψ | Ψ         |
| A–BC     | 2a   | [000]+[011]          | T(ψ,ψ,ψ)         | γ^A       |
| B–CA     | 2b   | [000]+[101]          | T(ψ,ψ,ψ)         | γ^B       |
| C–AB     | 2c   | [000]+[110]          | T(ψ,ψ,ψ)         | γ^C       |
| W        | 3    | [011]+[101]+[110]   | q(ψ)              | T(ψ,ψ,ψ) |
| GHZ      | 4    | a[000]−[011]−[101]−[110] | −                  | q(ψ)      |

will be referred to as the SLOCC-equivalence group. The Hilbert space is partitioned into equivalence classes or orbits under the SLOCC-equivalence group. Hence, for the n-qubit system the space of SLOCC-equivalence classes is given by,

\[
\frac{\mathbb{C}^2 \otimes \mathbb{C}^2 \cdots \otimes \mathbb{C}^2}{\text{SL}_1(2, \mathbb{C}) \times \text{SL}_2(2, \mathbb{C}) \times \cdots \times \text{SL}_n(2, \mathbb{C})}.
\] (26)

This is the space of physically distinct entanglement classes; the SLOCC entanglement classification amounts to understanding (26).

In the case of three-qubits the SLOCC-equivalence group coincides with the three-qubit FTS automorphism group and the space of entanglement classes (26) is determined by the ranks as in table 1 [3]. All states of a given rank 1, 2 or 3 are SLOCC-equivalent while the set of rank 4 states constitute a dimC = 1 family of equivalent states parameterized by q(ψ). More specifically, the entanglement classes and their (unnormalized) representative states are as follows:

(i) (rank 1) totally separable states A–B–C,

\[ |000⟩ \]

(ii) (rank 2) biseparable states A–BC, B–CA, C–AB,

\[
\begin{align*}
&|000⟩ + |011⟩ \\
&|000⟩ + |101⟩ \\
&|000⟩ + |110⟩
\end{align*}
\] (28)

(iii) (rank 3) totally entangled W states,

\[
|011⟩ + |101⟩ + |110⟩
\] (29)

(iv) (rank 4) one-parameter family of totally entangled GHZ states

\[
a|000⟩ − |011⟩ − |101⟩ − |110⟩
\] (30)

where \( q(ψ) = 8a \).

Since the rank conditions are ordered by implication, so are the entanglement classes. The rank 4 GHZ class is regarded as maximally entangled in the sense that it has non-vanishing quartic norm. Note that the three rank 2 classes collapse into a single class, since the three-matrices \( γ^A, h, c \), given in (18), are rotated into each other under the three-qubit permutation group.
3. Non-local games

A non-local game, as introduced in [30], consists of players (Alice, Bob, Charlie...), who act cooperatively in order to win, and a referee who coordinates the game. The players may collectively decide on a strategy before the game commences. Once it has begun they may no longer communicate. Whether or not the players win is determined by the referee. To begin the referee randomly selects one question, from a known fixed set $Q$, to be sent to each player. The players know only their own questions. Each player must then send back a response from the set of answers, denoted $A$. The referee determines whether the players win using the set of sent questions and received answers according to some predetermined rules. These rules are known to the players before the game gets under way so that they may attempt to devise a winning strategy.

For the three-player game [15] the questions sent to Alice, Bob and Charlie, denoted respectively by $r, s$ and $t$, are taken from the set $Q = \{0, 1\}$. However, the referee ensures that $rst \in \{000, 110, 101, 011\}$ and the players are aware of this. The answers $a, b, c$, sent back by Alice, Bob and Charlie, are elements of $A = \{0, 1\}$. The players win if $r \lor s \lor t = a \oplus b \oplus c$, where $\lor$ and $\oplus$ respectively denote disjunction and addition mod 2, i.e for question sets $rst = 000, 011, 101$ and $110$ the answer set $abc$ must satisfy $a \oplus b \oplus c = 0, 1, 1$ and 1, respectively.

In the quantum version, Alice, Bob and Charlie each possess a qubit, which they may manipulate locally. Any entanglement shared by the three-qubits can potentially be used to the players advantage. However, before examining how this works let us consider first how well the players can do classically, unassisted by entanglement.

What is the best possible classical deterministic strategy? A deterministic strategy amounts to specifying three functions, one for each player, from the question set $Q$ to the answer set $A$,

$$
a : Q \rightarrow A; \quad r \mapsto a(r),
$$

$$
b : Q \rightarrow A; \quad s \mapsto b(s),
$$

$$
c : Q \rightarrow A; \quad t \mapsto c(t). \quad (31)
$$

The condition that the players win may then be written as,

$$
a(0) \oplus b(0) \oplus c(0) = 0,
$$

$$
a(1) \oplus b(1) \oplus c(0) = 1,
$$

$$
a(1) \oplus b(0) \oplus c(1) = 1,
$$

$$
a(0) \oplus b(1) \oplus c(1) = 1. \quad (32)
$$

This implies that the best one can do is win 75% of the time; the four equations cannot be simultaneously satisfied as can be seen by adding them mod 2 [15]. On the other hand, the simple strategy that ‘everyone always answers 1’ satisfies three of the four equations so that the 75% upper bound is actually met.

Can this be bettered when equipped with an entangled resource? The answer is a resounding yes: by sharing a GHZ state,

$$
|\Psi\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle), \quad (33)
$$

they can always win [15].

---

3 We need only consider this case here since, for non-local games, the best winning probability possible using a deterministic strategy is an upper bound on the best winning probability possible using a probabilistic strategy [15, 30].
The winning quantum strategy is remarkably simple. If a player receives the question ‘0’ they measure their qubit in the computational basis \( \{ |0\rangle, |1\rangle \} \). If a player receives the question ‘1’ they measure their qubit in the Hadamard basis \( \left\{ \left( |0\rangle + |1\rangle \right) / \sqrt{2}, \left( |0\rangle - |1\rangle \right) / \sqrt{2} \right\} \). The measurement outcome is sent back as their answer. By symmetry we need only consider the two cases \( rst = 000 \) and \( rst = 011 \).

For \( rst = 000 \): all players measure in the computational basis. From (33) it is clear that only an odd number of 0’s can appear \( \Rightarrow a \oplus b \oplus c = 0 \). Always win.

For \( rst = 011 \): Alice measures in the computational basis, while Bob and Charlie measure in the Hadamard basis. Consulting the locally rotated state, \( 1 \otimes H \otimes H |\Psi\rangle = \frac{1}{2} \left( |001\rangle + |010\rangle - |100\rangle + |111\rangle \right) \), (34) where \( H \) is the Hadamard matrix, it is clear that only an even number of 0’s can appear \( \Rightarrow a \oplus b \oplus c = 1 \). Always win. Hence, using the GHZ entangled resource (33) Alice, Bob and Charlie can win certainty, outdoing the best possible classical strategy.

4. Freudenthal ranks: GHZ versus W

We will now show that the FTS rank conditions imply that there is no local strategy utilizing a rank 3 state that wins with certainty. Similarly, the optimal rank 2 state strategy falls short of the rank 3 case implying that the winning probabilities are ordered by rank.

A local strategy corresponds to choosing six unitary rotations, \( R_r, S_s, T_t \), where \( r, s, t = 0, 1 \), one pair each for Alice, Bob and Charlie. Let \( |\psi^{rst}\rangle = \psi^{rst}_{ABC} |ABC\rangle = R_r \otimes S_s \otimes T_t |\psi\rangle \), (35) where \( |\psi\rangle \) is the initial shared state. Note, for notational convenience will shall use the decimal expression for both \( rst \) and \( ABC \), so that, for example, the amplitudes of \( |\psi^{000}\rangle \) are \( \psi^0_0, \psi^0_1, \ldots, \psi^0_7 \).

Since it is assumed \( |\psi\rangle \) is a rank 3 state we have \( T(\psi) \neq 0, \text{Det}(\psi) = 0 \), which implies \( T(\psi^{rst}) \neq 0, \text{Det}(\psi^{rst}) = 0 \). (36)

Let us now assume that there is in fact a strategy that wins with certainty. For \( rts = 0 \) this implies \( \psi_0^{0,1,2,4} = 0 \). (37)

Hence

\[ \text{Det} \psi^0 = 4\psi_0^0 \psi_3^0 \psi_5^0 \psi_6^0 \] (38)

and

\[ \gamma^A(\psi^0) = \left( \begin{array}{cc} 2\psi_0^0 \psi_3^0 & 0 \\ 0 & -2\psi_5^0 \psi_6^0 \end{array} \right), \] (39a)

\[ \gamma^B(\psi^0) = \left( \begin{array}{cc} 2\psi_0^0 \psi_5^0 & 0 \\ 0 & -2\psi_3^0 \psi_6^0 \end{array} \right), \] (39b)

\[ \gamma^C(\psi^0) = \left( \begin{array}{cc} 2\psi_0^0 \psi_6^0 & 0 \\ 0 & -2\psi_3^0 \psi_5^0 \end{array} \right), \] (39c)

which together imply that one and only one of \( \psi_0^0, \psi_3^0, \psi_5^0, \psi_6^0 \) must be vanishing in order that the rank condition \( T(\psi^0) \neq 0, \text{Det}(\psi^0) = 0 \) be satisfied.

Similarly, for \( rts = i = 3, 5, 6 \) we have

\[ \psi_0^{0,3,5,6} = 0, \] (40)
\[ \text{Det} \psi^i = 4\psi^j \psi^j \psi^j \psi^j \]  

(41)

and

\[ \gamma^A (\psi') = \begin{pmatrix} -2\psi^j \psi^j & 0 \\ 0 & 2\psi^j \psi^j \end{pmatrix}, \]  

(42a)

\[ \gamma^B (\psi') = \begin{pmatrix} -2\psi^j \psi^j & 0 \\ 0 & 2\psi^j \psi^j \end{pmatrix}, \]  

(42b)

\[ \gamma^C (\psi') = \begin{pmatrix} -2\psi^j \psi^j & 0 \\ 0 & 2\psi^j \psi^j \end{pmatrix}, \]  

(42c)

which, again, imply that one and only one of \( \psi^j_{1,2,4} \) must be vanishing.

Note, by the covariance of the rank condition we have

\[ \gamma^A (\psi'^m) = e^{i(\theta_0 + \lambda_0)} R_{\gamma^A (\psi)} R^T_{\gamma^A (\psi)} , \]  

(43a)

\[ \gamma^B (\psi'^m) = e^{i(\theta_0 + \lambda_0)} S_{\gamma^B (\psi)} S^T_{\gamma^B (\psi)} , \]  

(43b)

\[ \gamma^C (\psi'^m) = e^{i(\theta_0 + \lambda_0)} T_{\gamma^C (\psi)} T^T_{\gamma^C (\psi)} , \]  

(43c)

where \( \text{det} R_{\gamma^A (\psi)} = e^{i\theta_0} \), \( \text{det} S_{\gamma^B (\psi)} = e^{i\phi_0} \), \( \text{det} T_{\gamma^C (\psi)} = e^{i\lambda_0} \), so that

\[ e^{-i(\theta_0 + \lambda_0)} \gamma^A (\psi'^m) = e^{-i(\theta_0 + \lambda_0)} \gamma^A (\psi'^m) , \]  

(44a)

\[ e^{-i(\theta_0 + \lambda_0)} \gamma^B (\psi'^m) = e^{-i(\theta_0 + \lambda_0)} \gamma^B (\psi'^m) , \]  

(44b)

\[ e^{-i(\theta_0 + \lambda_0)} \gamma^C (\psi'^m) = e^{-i(\theta_0 + \lambda_0)} \gamma^C (\psi'^m) , \]  

(44c)

Hence, from equations (39a) through (39c) and (42a) through (42c) we obtain the following set of 12 conditions:

\[ |\psi^0_0||\psi^0_0| = |\psi^2_0||\psi^2_0| , \]  

(45a)

\[ |\psi^0_0||\psi^0_0| = |\psi^2_0||\psi^2_0| , \]  

(45b)

\[ |\psi^2_0||\psi^0_0| = |\psi^2_0||\psi^2_0| , \]  

(45c)

\[ |\psi^2_0||\psi^0_0| = |\psi^2_0||\psi^2_0| , \]  

(45d)

\[ |\psi^0_0||\psi^0_0| = |\psi^1_0||\psi^1_0| , \]  

(45e)

\[ |\psi^0_0||\psi^0_0| = |\psi^1_0||\psi^1_0| , \]  

(45f)

\[ |\psi^1_0||\psi^1_0| = |\psi^1_0||\psi^1_0| , \]  

(45g)

\[ |\psi^1_0||\psi^1_0| = |\psi^1_0||\psi^1_0| , \]  

(45h)

\[ |\psi^0_0||\psi^0_0| = |\psi^2_0||\psi^2_0| , \]  

(45i)

\[ |\psi^0_0||\psi^0_0| = |\psi^2_0||\psi^2_0| , \]  

(45j)
the players win with probability 7/8, as a quick calculation will confirm. Second, the rank 2
not immediately obvious, observations. First, adopting the optimal GHZ strategy for a
particular insight beyond the previous case, the argument does rely on two simple, but perhaps
symmetry of the game) the
implies that the state is biseparable. Let us consider without loss of generality (by the
general (46)

\[ \left| \psi_1 \right| \left| \psi_2 \right| = \left| \psi_3 \right| \left| \psi_4 \right| \text{.} \tag{45k} \]

\[ \left| \psi_3 \right| \left| \psi_4 \right| = \left| \psi_5 \right| \left| \psi_6 \right| \text{.} \tag{45l} \]

Under the rank condition that one and only one of each of \( \psi_{0,3,5,6} \) and \( \psi_{1,2,4} \)
must be vanishing this set of equations has no solution, yielding a contradiction. Hence,
\( p(\text{rank } 3) < p(\text{rank } 4) = 1 \) as claimed.

Using a similar logic one can show that \( p(\text{rank } 2) < p(\text{rank } 3) \). While the details offer no
particular insight beyond the previous case, the argument does rely on two simple, but perhaps
not immediately obvious, observations. First, adopting the optimal GHZ strategy for a W state
the players win with probability 7/8, as a quick calculation will confirm. Second, the rank 2
conditions imply that for any rank 2 state and only one of \( \gamma_{A,B,C} \) is non-vanishing. This
follows from the identity,
\[ (\gamma^A)^{A_i} \gamma^C)^{C_i} = e^{B_2} A_i B_i C_i T_{A_i B_i C_i} + e^{B_2} A_i B_i C_i T_{A_i B_i C_i}, \tag{46} \]

which implies that if \( T_{A_i B_i C_i} = 0 \) then there is at most one non-vanishing \( \gamma \), while the non-
vanishing of \( \gamma \) implies at least one non-zero \( \gamma \) as can be seen from (25).

Using these conditions it can be shown directly that \( p(\text{rank } 2) < p(\text{rank } 3) \). However, a
more illuminating way to proceed follows from the fact that one and only one non-vanishing \( \gamma \)
implies that the state is biseparable. Let us consider without loss of generality (by the
symmetry of the game) the A–B split with representative state \( \left| \psi \right| \left| \phi \right|_{BC} \). Defining the
observables \( A_r = R_i \sigma_i R_r, A_r = S_i \sigma_i S_r \) and \( C_i = T_i \sigma_i T_r \), we note that the expectation value of
\[ E := A_0 B_0 C_0 - A_0 B_1 C_1 - A_1 B_0 C_1 - A_1 B_1 C_0 \tag{47} \]
is four times the difference between the probability of winning and losing the game. In the
biseparable case we can therefore use a Tsirelson type argument [31]. Let
\[ S = \langle \psi | (\phi | E | \psi) \phi \rangle \]
\[ = \langle \phi | B_0 (a_0 C_0 - a_1 C_1) - B_1 (a_1 C_0 + a_0 C_1) | \phi \rangle \tag{48} \]
where \( a_r = \langle a | \psi \rangle |_0 \in [-1, 1] \). Then
\[ S \leq || B_0 (a_0 C_0 - a_1 C_1) - B_1 (a_1 C_0 + a_0 C_1) | | \langle \phi | || \]
\[ \leq || B_0 (a_1 C_1 - a_0 C_0) | | \langle \phi | || + || B_1 (a_1 C_0 + a_0 C_1) | | \langle \phi | || \]
\[ \leq || I \otimes (a_1 C_1 - a_0 C_0) | | \langle \phi | || + || I \otimes (a_1 C_0 + a_0 C_1) | | \langle \phi | || \]
\[ = || a_1 | \phi_1 | - a_0 | \phi_0 | || + || a_1 | \phi_0 | + a_0 | \phi_1 | || \tag{49} \]
where \( | \phi_i | = I \otimes C_i | \phi \). Since \( || \phi_1 || \leq 1 \) and \( a_r \in [-1, 1] \) we have
\[ S \leq \sqrt{2 - 2a_0 a_1 \text{Re} \langle \phi_0 | \phi_1 \rangle} + \sqrt{2 + 2a_0 a_1 \text{Re} \langle \phi_0 | \phi_1 \rangle} \tag{50} \]
which is just the usual Tsirelson bound \( S \leq 2 \sqrt{2} \). Hence,
\[ p(\text{rank } 2) \leq \frac{1}{2} + \frac{1}{2 \sqrt{2}} < p(\text{rank } 3). \tag{51} \]

Finally, as the above analysis suggests, played with a rank 2 biseparable state, \( |000 \rangle + |011 \rangle \)
say, where Alice decides to always answer ‘0’, the three-player game is equivalent (bit-flipping
the rules) to the Clauser, Horne, Shimony and Holt (CHSH) two-qubit game [30, 32]. Hence,
there is indeed a local strategy that wins with probability
\[ \frac{1}{2} + \frac{1}{2 \sqrt{2}} \]
and it is trivially true that \( p(\text{rank } 1) = 3/4 < p(\text{rank } 2) \).
5. Further work

We have shown that the optimal success rates when sharing a three-qubit resource are strictly ordered according as the rank of the state used:

\[
\frac{3}{4} = p(\text{rank } 1) < p(\text{rank } 2) < p(\text{rank } 3) < p(\text{rank } 4) = 1. \quad (52)
\]

We conclude that the inherent ordering of the Freudenthal rank conditions is reflected physically by the increasing advantage acquired with respect to the Freudenthal rank of the entangled state used. It would be interesting to understand to what extent this observation applies beyond the three-qubit case. Indeed, by reverting back to the Jordan algebraic perspective it is possible to generalize the basic features of the FTS to \(n\)-qubits [33]. The two-qubit case is rather simple: there are two ranks corresponding to a single orbit of separable states and a one-parameter family of entangled states. The CHSH game [30] somewhat trivially reflects the ordering of the ranks. In the four-qubit case, on the other hand, the complete set of ranks is not even known and, moreover, one would anticipate them to be only partially ordered, since there are four independent SLOCC-equivalence group invariants [35]. Nonetheless, it would be interesting, given a complete set of ranks, to attempt to identify a minimal set of non-local games that would ‘experimentally verify’ this expected partial order. This non-trivial task is very much left as an open problem.

Returning briefly now to the three-qubit case in hand, we remark that the non-local properties of the W and GHZ states may also be compared using the sheaf-theoretic framework of [43, 44]. In this case one applies in both instances the winning GHZ strategy described in section 3. The resulting GHZ model is shown to be strongly contextual, admitting no global section, while the W model is merely contextual [43, 44]. It would be interesting to understand to what extent this sheaf-theoretic take on non-locality, and its associated notions of strong contextuality etc, can be understood in terms of FTS ranks and more generally the conventional SLOCC perspective on entanglement classes.

Finally, we note in passing that the Freudenthal ranks determine the degree of supersymmetry preserved by the single-centre extremal black hole solutions of \(\mathcal{N} = 8\) supergravity [10, 41, 42], suggesting an admittedly tenuous link between non-local games and Killing spinor equations.

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Essentially this is simply a particular way of organizing the conventional SLOCC entanglement classification using SLOCC-equivalence group covariants. However, while much is known about the four-qubit case [34–40], the complete covariant SLOCC classification is only known up to three-qubits.

By ‘complete set’ we mean the minimal number of independent covariants required to characterize the entanglement classification.
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