Free boundary problems of a mutualist model with nonlocal diffusions

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Abstract

A mutualist model with nonlocal diffusions and a free boundary is first considered. We prove that this problem has a unique solution defined for $t \geq 0$, and its dynamics are governed by a spreading-vanishing dichotomy. Some criteria for spreading and vanishing are also given. Of particular importance is that we find that the solution of this problem has quite rich longtime behaviors, which vary with the conditions satisfied by kernel functions and are much different from those of the counterpart with local diffusion and free boundary. At last, we extend these results to the model with nonlocal diffusions and double free boundaries.

Keywords: Nonlocal diffusion; free boundary; mutualist model; longtime behaviors

AMS Subject Classification (2000): 35K57, 35R09, 35R20, 35R35, 92D25

1 Introduction

We investigate the following mutualist model with nonlocal diffusions and a free boundary

$$
\begin{cases}
  u_t = d_1 \int_0^\infty J_1(x-y)u(t,y)dy - d_1 j_1(x)u + f_1(u,v), & t > 0, \ x \in \mathbb{R}^+, \\
  v_t = d_2 \int_0^{h(t)} J_2(x-y)v(t,y)dy - d_2 j_2(x)v + f_2(u,v), & t > 0, \ x \in [0,h(t)), \\
  v(t,x) = 0, & t > 0, \ x \in [h(t), \infty), \\
  h'(t) = \mu \int_0^{h(t)} \int_0^\infty J_2(x-y)v(t,y)dydx, & t > 0, \\
  h(0) = h_0 > 0; \ u(0,x) = u_0(x), \ x \in \mathbb{R}^+; \ v(0,x) = v_0(x), \ x \in [0,h_0]
\end{cases}
$$

(1.1)

with

$$f_1(u,v) = r_1 u \left( a - u - \frac{u}{1+bv} \right), \quad f_2(u,v) = r_2 v \left( 1 - v - \frac{v}{1+cu} \right),$$

where all parameters are positive, $j_i(x) = \int_0^\infty J_i(x-y)dy$ for $i = 1, 2$, $0 < u_0 \in C(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, $v_0 \in C([0,h_0])$ and $v_0(h_0) = 0 < v_0(x)$ in $[0,h_0)$. The kernel function $J_i$ for $i = 1, 2$ satisfy

$$(J) \ J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}), \ J \geq 0, \ J(0) > 0, \ \int_{\mathbb{R}} J(x)dx = 1, \ J \ is \ even.$$

Such mutualist model with random diffusions and the Stefan type boundary condition has been proposed and studied in \cite{1, 2}. The existence and uniqueness of global solution, longtime behaviors as well as criteria for spreading and vanishing have been established by a series of classical analysis. Like this, the application of the Stefan type problem in ecology has recently attracted much attention from many researchers over the past decades since the pioneering work of \cite{3}. For
example, one can refer to [11, 5] for problems in high dimensional space, [6, 7, 8] for the model in time or space periodic setting, and [9, 10] for two species model.

Our interest in this paper is to study the dynamics of nonlocal diffusion problem (1.1). As is known to us, one of the advantages of nonlocal diffusion over the random diffusion is that it can be used to describe the movements of organisms between not only adjacent spatial locations but nonadjacent locations, which is expected to bring about some diverse dynamics; see, e.g. [11, 12].

To check the difference between dynamics of local and nonlocal diffusion models with free boundary, the authors of [13] and [14] proposed the following nonlocal diffusion model with free boundaries

\[
\begin{aligned}
    &u_t = d \int_{g(t)}^{h(t)} J(x - y)u(t, y)\,dy - du + f(u), \quad t > 0, \quad g(t) < x < h(t), \\
    &u(t, x) = 0, \quad t > 0, \quad x \notin (g(t), h(t)), \\
    &h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^\infty J(x - y)u(t, x)\,dy\,dx, \quad t > 0, \\
    &g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x - y)u(t, x)\,dy\,dx, \quad t > 0, \\
    &h(0) = -g(0) = h_0 > 0, \quad u(0, x) = u_0(x), \quad |x| \leq h_0,
\end{aligned}
\]

where \( J \) satisfies (J). The reaction term \( f(u) \) in [13] is of the Fisher-KPP type, while that in [14] is identical to zero, which results in very different dynamics between these two models.

Since the above two works, lots of related researches have recently emerged. Du et.al [15] considered the spreading speed of free boundaries in [13] when spreading happens. They found an important condition on \( J \).

(J1) \[ \int_0^\infty xJ(x)\,dx < \infty, \]

so that there is a unique finite spreading speed for the free boundaries if and only if (J1) holds for \( J \). More precisely, they obtained that when \( \lim_{t \to \infty} -g(t) = \lim_{t \to \infty} h(t) = \infty, \)

\[ \frac{-g(t)}{t} = \frac{h(t)}{t} = \begin{cases} C_0 & \text{if (J1) holds for } J, \\ \infty & \text{if (J1) is not satisfied}, \end{cases} \]

where \( C_0 \) is uniquely given by the semi-wave problem

\[
\begin{aligned}
    d \int_{-\infty}^0 J(x - y)\phi(y)\,dy - d\phi + c\phi' + f(\phi) &= 0, \quad -\infty < x < 0, \\
    \phi(-\infty) = U^*, \quad \phi(0) = 0, \quad c &= \mu \int_{-\infty}^0 \int_0^\infty J(x - y)\phi(x)\,dy\,dx, \\
\end{aligned}
\]

with \( U^* \) denoting the unique positive zero of \( f(u) \). Moreover, Du and Ni [16] not only extended the above results to monostable cooperative system, but also gave some accurate estimates on free boundary. In particular, if \( J \approx |x|^{-\gamma} \) for \( |x| \gg 1 \) and some \( \gamma \in (1, 2] \), that is, \( C_1 |x|^{-\gamma} \leq J(x) \leq C_2 |x|^{-\gamma} \) for some \( C_1, C_2 > 0 \) and \( |x| \gg 1 \), then \( -g(t), h(t) \approx t^{1/(\gamma - 1)} \) for \( \gamma \in (1, 2) \) and \( -g(t), h(t) \approx t \ln t \) for \( \gamma = 2 \). The high dimension and radial symmetry version was also considered in [17], where new difficulties arising from the kernel function \( J(|x|) \) have been solved by using some
delicate techniques. Among other things, there are other works about nonlocal diffusion problem with free boundaries; please see e.g. [18, 19, 20, 21, 22, 23] and references therein.

Inspired by the deduction of model (1.2), very recently Li and Wang [24] put forward the following nonlocal diffusion model with a free boundary

$$
\begin{align*}
    u_t &= d J(x-y)u(t,y)dy - d \left( \int_0^\infty J(x-y)dy \right) u + f(u), \\
    u(t, h(t)) &= 0, \\
    h'(t) &= \mu \int_0^{h(t)} \int_t^\infty J(x-y)u(t,x)dydx, \\
    h(0) &= h_0, \quad u(0, x) = u_0(x), \quad x \in [0, h_0].
\end{align*}
$$

They proved that this problem has the similar dynamics with those in [13, 15]. Later on, Li and Wang [25] obtained the following sharp estimates on solution of (1.4) when spreading happens.

$$
\begin{align*}
    \lim_{t \to \infty} \max_{x \in [0, ct]} |u(t, x) - u^*| &= 0 \text{ for any } c \in (0, c_0), \quad \text{if J satisfies (J1)}, \\
    \lim_{t \to \infty} \max_{x \in [0, ct]} |u(t, x) - u^*| &= 0 \text{ for any } c > 0, \quad \text{if J violates (J1)}, \\
    \lim_{t \to \infty} \max_{x \in [0, s(t)]} |u(t, x) - u^*| &= 0 \text{ for any } 0 \leq s(t) = t^{\gamma} o(1) \quad \text{if J} \approx |x|^{-\gamma} \text{ for } \gamma \in (1, 2), \\
    \lim_{t \to \infty} \max_{x \in [0, s(t)]} |u(t, x) - u^*| &= 0 \text{ for any } 0 \leq s(t) = (t \ln t) o(1) \quad \text{if J} \approx |x|^{-2}.
\end{align*}
$$

2 Longtime behaviors of (1.1)

First of all, the following well-posedness of (1.1) can be obtained by the similar arguments in proof of [13] Theorem 2.1 and [26] Theorem 2.2, and thus its proof is omitted here.

**Theorem 2.1.** Problem (1.1) has a unique global solution $(u, v, h)$, and $u, u_t \in C([0, \infty) \times \mathbb{R}^+)$, $v, v_t \in C([0, h(t)) \times \mathbb{R}^+)$ and $h \in C^1((0, \infty))$. Moreover, the following estimates hold:

$$
0 \leq u \leq K_1 := \max\{\|u_0\|_{L^\infty(\mathbb{R}^+)}, \ a\}, \quad 0 \leq v \leq K_2 := \max\{\|v_0\|_{C((0, h_0))}, \ 1\}.
$$

By the equation of $h'$, $h(t)$ is increasing in $t > 0$. Thus $h_\infty := \lim_{t \to \infty} h(t) \in (0, \infty]$. The case $h_\infty < \infty$ is called by vanishing case, while the other case $h_\infty = \infty$ is described as spreading case. The upcoming study of longtime behaviors of solution to (1.1) will be divided into these two cases.

2.1 Vanishing case: $h_\infty < \infty$

Let $a_0$ and $l$ be positive constants. It is well-known that eigenvalue problem

$$
(\mathcal{L}_{(0,l)} + a_0)\phi := d_2 \int_0^l J_2(x-y)\phi(y)dy - d_2 J_2(x)\phi + a_0 \phi = \lambda \phi \quad \text{in } [0, l]
$$

where $\mathcal{L}_{(0,l)} = \int_0^l J_2(x-y)\phi(y)dy$.
has a unique principal eigenvalue, denoted by \( \lambda_p(\mathcal{L}_{(0, l)} + a_0) \), whose corresponding eigenfunction is positive in \([0, l]\). Please see [24, Lemma 2.3] for some properties of \( \lambda_p(\mathcal{L}_{(0, l)} + a_0) \).

Moreover, it follows from [15, Proposition 1.3] that if \( J_1 \) satisfies the condition

\[
(J_2) \quad \int_0^\infty J(x)e^{tx}dx < \infty \text{ for some } \lambda > 0,
\]

then one can find a unique \( c_* > 0 \), usually called by the minimal wave speed, so that problem

\[
\begin{align*}
\phi(-\infty) &= a/2, \quad \phi(\infty) = 0, \quad \phi'(x) \leq 0 \\
\end{align*}
\]

has a unique solution \( \phi_c \) if and only if \( c \geq c_* \). Below is the main result of this subsection.

**Theorem 2.2.** If \( h_\infty < \infty \), then \( \lambda_p(\mathcal{L}_{(0, h_\infty)} + r_2) \leq 0 \), \( \lim_{t \to \infty} \| v(t, \cdot) \|_{C([0, h(t)])} = 0 \) and

\[
\begin{align*}
\lim_{t \to \infty} \max_{x \in [0, c_t]} |u(t, x) - a/2| = 0 & \text{ for any } c \in (0, c_*), \text{ if } J_1 \text{ satisfies } (J_2), \\
\lim_{t \to \infty} \max_{x \in [0, c_t]} |u(t, x) - a/2| = 0 & \text{ for any } c > 0, \text{ if } J_1 \text{ violates } (J_2), \\
\lim_{t \to \infty} \max_{x \in [0, s(t)]} |u(t, x) - a/2| = 0 & \text{ for any } 0 \leq s(t) = t^{\gamma} o(1) \text{ if } J_1 \approx |x|^{-\gamma} \text{ for } \gamma \in (1, 2), \\
\lim_{t \to \infty} \max_{x \in [0, s(t)]} |u(t, x) - a/2| = 0 & \text{ for any } 0 \leq s(t) = (t \ln t) o(1) \text{ if } J_1 \approx |x|^{-2}.
\end{align*}
\]

**Proof.** **Step 1: The proof of** \( \lambda_p(\mathcal{L}_{(0, h_\infty)} + r_2) \leq 0 \). Assume indirectly that \( \lambda_p(\mathcal{L}_{(0, h_\infty)} + r_2) > 0 \). It follows from (J) that there exist \( \varepsilon_0 \) and \( \sigma_0 > 0 \) such that \( J(x) > \sigma_0 \) for \( |x| \leq \varepsilon_0 \). Then, by continuity, there exist \( \varepsilon \in (0, \varepsilon_0/2) \) and \( T > 0 \) such that \( \lambda_p(\mathcal{L}_{(0, h_\infty-\varepsilon)} + r_2) > 0 \) and \( h(t) > h_\infty - \varepsilon \) for \( t \geq T \). Let \( w \) be a solution of

\[
\begin{align*}
\begin{cases}
w_t = d_2 \int_0^{h_\infty-\varepsilon} J_2(x-y)w(t,y)dy - dj_2(x)w + r_2w(1-2w), & t > T, \quad 0 \leq x \leq h_\infty - \varepsilon, \\
w(T, x) = v(T, x) & 0 \leq x \leq h_\infty - \varepsilon.
\end{cases}
\end{align*}
\]

From a simple comparison argument, \( v \geq w \) for \( t \geq T \) and \( x \in [0, h_\infty - \varepsilon] \). In view of \( \lambda_p(\mathcal{L}_{(0, h_\infty-\varepsilon)} + r_2) > 0 \) and \( [24, \text{Lemma 2.6}] \), we have that \( w \) converges to a positive steady state \( W(x) \) uniformly in \([0, h_\infty - \varepsilon]\) as \( t \to \infty \). Thus there exists \( T_1 > T \) such that \( v(t,x) \geq W(x)/2 \) for \( t \geq T_1 \) and \( x \in [0, h_\infty - \varepsilon] \). Furthermore, we see that for \( t > T_1 \),

\[
h'(t) = \mu \int_0^{h(t)} \int_0^{h(t)} J_2(x-y)v(t,y)dydx \geq \mu \int_{h_\infty-\varepsilon}^{h_\infty-\varepsilon} \int_{h_\infty-\varepsilon}^{h_\infty+\varepsilon} J_2(x-y)v(t,y)dydx
\]

\[
\geq \mu \int_{h_\infty-\varepsilon}^{h_\infty-\varepsilon} \int_{h_\infty-\varepsilon}^{h_\infty+\varepsilon} \frac{W(x)}{2}dydx > 0,
\]

which contradicts to \( h_\infty < \infty \). Consequently, \( \lambda_p(\mathcal{L}_{(0, h_\infty)} + r_2) \leq 0 \).

**Step 2:** The proof of \( \lim_{t \to \infty} \| v(t, \cdot) \|_{C([0, h(t)])} = 0 \). Consider the following problem

\[
\begin{align*}
\begin{cases}
V_t = d_2 \int_0^{h_\infty} J_2(x-y)V(t,y)dy - dj_2(x)V + r_2V(1-V), & t > 0, \quad 0 \leq x \leq h_\infty, \\
V(0, x) = K_2 & 0 \leq x \leq h_\infty.
\end{cases}
\end{align*}
\]
The comparison principle indicates that \( v \leq V \) for \( t \geq 0 \) and \( 0 \leq x \leq h(t) \). By \( \lambda_p(C_{(0,h)} + r_2) \leq 0 \), we see from [24, Lemma 2.6] that \( \lim_{t \to \infty} V(t,x) = 0 \) in \( C([0,h]) \). Thus, we complete this step.

**Step 3:** The proof of (2.2). Clearly, for any small \( \varepsilon > 0 \) there is \( T > 0 \) such that \( v \leq \varepsilon \) for \( t \geq T \) and \( x \geq 0 \). Thus \( u \) satisfies

\[
\begin{aligned}
    u_t &\leq d_1 \int_0^\infty J_1(x-y)u(t,y)dy - d_1 j_1(x)u + f_1(u, \varepsilon), \quad t > T, \ x \in \mathbb{R}^+.
    \\
u(T,x) &\leq K_1, \quad x \in \mathbb{R}^+.
\end{aligned}
\]

A simple comparison principle arrives at \( \limsup_{t \to \infty} u(t,x) \leq a(1+b\varepsilon)/(2+b\varepsilon) \) uniformly in \( \mathbb{R}^+ \). Thanks to the arbitrariness of \( \varepsilon \), we have

\[
\limsup_{t \to \infty} u(t,x) \leq a/2 \quad \text{uniformly in } \mathbb{R}^+.
\] (2.3)

Let \((u,l)\) be the unique solution of the following problem

\[
\begin{aligned}
u_t &= d_1 \int_0^{l(t)} J_1(x-y)u(t,y)dy - d_1 j_1(x)u + r_1 u(a - 2u), \quad t > 0, \ 0 \leq x < l(t), \\
u(t,l(t)) &= 0, \quad t > 0,
\end{aligned}
\]

\[
\begin{aligned}
l'(t) &= \kappa \int_0^{l(t)} \int_0^{l(t)} J_1(x-y)u(t,x)dydx, \quad t > 0, \\
l(0) &= l_0 > 0, \ u(0,x) = \bar{u}_0(x) \leq u_0(x), \ x \in [0,l_0],
\end{aligned}
\]

with \( \kappa > 0, \bar{u}_0 \in C([0,l_0]) \) and \( \bar{u}_0(x) > 0 = \bar{u}_0(l_0) \) in \([0,l_0]\). A comparison argument yields \( u \geq \bar{u} \) in \([0,\infty) \times [0,\infty) \). By [24, Theorem 4.3] and [25, Theorems 2.2 and 4.1], for large \( \kappa > 0 \),

\[
\begin{aligned}
\lim_{t \to \infty} \max_{x \in [0,ct]} |u(t,x) - a/2| &= 0 \quad \text{for } c \in (0,c_0^\kappa), \quad \text{if (J1) is satisfied}, \\
\lim_{t \to \infty} \max_{x \in [0,ct]} |u(t,x) - a/2| &= 0 \quad \text{for } c > 0, \quad \text{if (J1) is violated}, \\
\lim_{t \to \infty} \max_{s(t)} |u(t,x) - a/2| &= 0 \quad \text{for } s(t) = t^{-\gamma} o(1), \quad \text{if } J_1 \approx |x|^{-\gamma} \quad \text{with } \gamma \in (1,2), \\
\lim_{t \to \infty} \max_{s(t)} |u(t,x) - a/2| &= 0 \quad \text{for } s(t) = (t \ln t) o(1), \quad \text{if } J_1 \approx |x|^{-2}.
\end{aligned}
\] (2.4)

where \( c_0^\kappa \) is uniquely given by (1.3) but with \((d,J,f,\mu)\) replaced by \((d_1,J_1,r_1\phi(a - 2\phi),\kappa)\).

If (J2) holds true for \( J_1 \), by [15, Theorem 5.1], we know \( \lim_{t \to \infty} c_0^\kappa = c_* \). Thus for any \( c < c_* \) one can take \( \kappa \) large enough such that \( c < c_0^\kappa \). Together with comparison principle and the first result in (2.4), we immediately derive

\[
\liminf_{t \to \infty} u(t,x) \geq a/2 \quad \text{uniformly in } [0,ct].
\] (2.5)

If \( J_1 \) does not satisfies (J2) but meets (J1), we learn from [15, Theorem 5.2] that \( \lim_{t \to \infty} c_0^\kappa = \infty \). For any \( c > 0 \), we choose \( \kappa > \gamma > 1 \) such that \( c < c_0^\kappa \). Similarly, (2.5) holds. If \( J_1 \) violates (J1), we can easily derive (2.5) by a comparison argument and the second result in (2.4). If \( J_1 \approx |x|^{-\gamma} \) for \( \gamma \in (1,2) \), by the last two conclusions in (2.4) we have \( \liminf_{t \to \infty} u(t,x) \geq a/2 \) uniformly in \([0,s(t)]\). Then due to (2.3), we get our desired conclusions. This finishes the proof.
2.2 Spreading case: $h_{\infty} = \infty$

In this subsection, we discuss the longtime behaviors of solution components $u$ and $v$ of \((1.1)\) when spreading occurs. The following lemma is crucial to our discussion.

**Lemma 2.3.** Assume that $s(t)$ is continuous in $[0, \infty)$ and strictly increasing to $\infty$, $s(0) = s_0 > 0$ and $P(x)$ satisfies \((J)\). Let $d, \alpha, \beta$ be positive constants, and $w$ be a unique solution of

\[
\begin{aligned}
&\frac{w_t}{t} = d \int_0^{s(t)} P(x - y)w(t, y)dy - dp(x)w + w(\alpha - \beta w), \quad t > 0, \quad 0 \leq x < s(t), \\
&w(t, s(t)) = 0, \quad t > 0, \\
&w(0, x) = w_0(x), \quad 0 \leq x \leq s_0,
\end{aligned}
\]

where $w_0 \in C([0, s_0])$, $w_0(x) > 0 = w_0(0)$ in $[0, s_0)$ and $p(x) = \int_0^{\infty} P(x-y)dy$. Then the followings hold true:

1. \(\lim_{t \to \infty} w(t, x) = \alpha/\beta\) locally uniformly in $\mathbb{R}^+$. \hfill (1)

2. Suppose that $P$ satisfies \((J1)\) and $\lim_{t \to \infty} s(t)/t = \xi \in (0, \infty]$. Then

\[
\begin{aligned}
&\lim_{t \to \infty} \max_{x \in [0, ct]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } c \in (0, \min\{\xi, C_*\}), \quad \text{if } P \text{ satisfies } (J2), \\
&\lim_{t \to \infty} \max_{x \in [0, ct]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } c \in (0, \xi), \quad \text{if } P \text{ violates } (J2),
\end{aligned}
\]

where $C_*$ is the minimal speed of the traveling wave problem \((2.1)\) with $(d_1, J, r_1\phi(a - 2\phi))$ replaced by $(d, P, \phi(\alpha - \beta\phi))$.

3. Suppose that $P$ does not satisfy \((J1)\) and $\lim_{t \to \infty} s(t)/t = \xi \in (0, \infty]$. Then

\[
\lim_{t \to \infty} \max_{t \in [0, ct]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } c \in (0, \xi).
\]

4. Suppose that there exist $C_1, C_2 > 0$, $\gamma \in (1, 2)$ and $\lambda \in (1, 1/(\gamma - 1)]$ such that $P(x) \geq C_1|x|^{-\gamma}$ for $|x| \geq 1$ and $s(t) \geq C_2 t^{1/\lambda}$ for $t \gg 1$. Then

\[
\lim_{t \to \infty} \max_{t \in [0, ct]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } \theta \in (1, \lambda).
\]

5. Suppose that there exist $C_1, C_2 > 0$ and $0 < \eta \leq 1$ such that $P(x) \geq C_1|x|^{-2}$ for $|x| \geq 1$ and $s(t) \geq C_2 t^{(\ln t)^\eta}$ for $t \gg 1$. Then

\[
\lim_{t \to \infty} \max_{t \in [0, t(\ln(t+1))^{-\eta}]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } \omega \in (0, \eta).
\]

**Proof.** (1) This proof is similar to that of [13, Theorem 3.9], and the details are thus omitted here.

2. Firstly, a comparison argument shows that $\limsup_{t \to \infty} w(t, x) \leq \alpha/\beta$ uniformly in $\mathbb{R}^+$. It is thus enough to show that

\[
\liminf_{t \to \infty} w(t, x) \geq \alpha/\beta \quad \text{uniformly in } [0, ct]. \quad (2.6)
\]

**Step 1:** Proof of the assertion with \((J2)\) satisfied by $P$. 


We first assume that $P$ has a compact support, say $\text{supp}P \subset [-K, K]$ for some $K > 0$. By the assumption and conclusion (1), for small $\varepsilon > 0$ and $c < c_1 < \min\{\mathfrak{g}, C_*\}$, there is $T > 0$ such that $s(t) > c_1 t + 2K$ and $w(t, x) \geq (1 - \varepsilon)\alpha/\beta$ for $t \geq T$ and $x \in [0, 2K]$. Then $w$ satisfies

$$
\begin{cases}
  w_t \geq d \int_0^{c_1 t + 2K} P(x-y)w(t, y)dy - dp(x)w + w(\alpha - \beta w), & t > T, \quad 0 \leq x \leq c_1 t + 2K, \\
  w(t, c_1 t + 2K) > 0, & t > T, \\
  w(T, x) > 0, & 0 \leq x \leq c_1 T + 2K.
\end{cases}
$$

Define $\tilde{w}(t, x) = w(t + T, x)$ for $t > 0$ and $0 \leq x \leq c_1 t + 2K$. Then

$$
\begin{cases}
  \tilde{w}_t \geq d \int_0^{c_1 t + 2K} P(x-y)\tilde{w}(t, y)dy - dp(x)\tilde{w} + \tilde{w}(\alpha - \beta \tilde{w}), & t > 0, \quad 0 \leq x \leq c_1 t + 2K, \\
  \tilde{w}(t, c_1 t + 2K) > 0, & t > 0, \\
  \tilde{w}(0, x) > 0, & 0 \leq x \leq 2K.
\end{cases}
$$

Since $P$ satisfies (J2), by [15, Theorems 1.2 and 5.1], the semi-wave problem (1.3) with $(P, \phi(\alpha - \beta\phi), \zeta)$ in place of $(J, f, \mu)$ has a unique solution pair $(c^\zeta, \phi, \zeta)$ for any $\zeta > 0$ and $\lim_{\zeta \to \infty} c^\zeta = C_*$. Thus $c_1 < \min\{\mathfrak{g}, c^\zeta\}$ by choosing $\zeta$ large sufficiently. For such $\zeta$, we set

$$
w = (1 - \varepsilon)\phi_{c_1}(x - c_1 t - 2K), \quad t > 0, \quad 0 \leq x \leq c_1 t + 2K.
$$

We shall check that the following inequalities hold true:

$$
\begin{cases}
  w_t \leq d \int_0^{c_1 t + 2K} P(x-y)w(t, y)dy - dp(x)w + w(\alpha - \beta w), & t > 0, \quad K \leq x \leq c_1 t + 2K, \\
  w(t, c_1 t + 2K) = 0, & t > 0, \\
  w(t, x) \leq \tilde{w}(t, x), & t > 0, \quad 0 \leq x \leq K, \\
  w(0, x) \leq \tilde{w}(0, x), & 0 \leq x \leq 2K.
\end{cases}
$$

Once it is done, then $w(t, x) \leq \tilde{w}(t, x) = w(t + T, x)$ for $t > 0$ and $0 \leq x \leq c_1 t + 2K$ by the comparison principle. On the other hand,

$$
\max_{x \in [0, ct]} |w(t, x) - (1 - \varepsilon)\alpha/\beta| = (1 - \varepsilon) \left( \alpha/\beta - \phi_{c_1}(ct - c_1 t - 2K) \right) \to 0 \text{ as } t \to \infty.
$$

Thus, $\liminf_{t \to \infty} w(t, x) \geq (1 - \varepsilon)\alpha/\beta$ uniformly in $[0, ct]$. The arbitrariness of $\varepsilon$ implies (2.6).

Now we verify (2.7). Obviously, $w(t, c_1 t + 2K) = 0$ and $w(t, x) \leq (1 - \varepsilon)\alpha/\beta \leq w(t + T, x) = \tilde{w}(t, x)$ for $t \geq 0$ and $0 \leq x \leq K$, and $w(0, x) \leq \tilde{w}(0, x)$ for $0 \leq x \leq 2K$. It thus remains to verify the first inequality of (2.7). As $c_1 < c^\zeta$, it follows that, for $t > 0$ and $K \leq x \leq c_1 t + 2K$,

$$
w_t = -(1 - \varepsilon)c_1 \phi_{c_1}(x - c_1 t - 2K) \leq -(1 - \varepsilon)c_1 \psi_{c_1}(x - c_1 t - 2K) = (1 - \varepsilon) \left( d \int_{-\infty}^{c_1 t + 2K} P(x-y)\phi_{c_1}(y - c_1 t - 2K)dy - dp(x)\phi_{c_1}(x - c_1 t - 2K) + \phi_{c_1}(\alpha - \beta \phi_{c_1}) \right) \leq d \int_0^{c_1 t + 2K} P(x-y)w(t, y)dy - dp(x)w + w(\alpha - \beta w).
$$
Now we turn to the case where \( P \) does not have a compact support. Define
\[
\xi(x) = 1 \quad \text{for} \ |x| \leq 1, \quad \xi(x) = 2 - |x| \quad \text{for} \ 1 \leq |x| \leq 2, \quad \xi(x) = 0 \quad \text{for} \ |x| \geq 2,
\]
and \( P_n(x) = \xi(\frac{x}{n})P(x) \). Clearly, \( P_n \) are supported compactly and nondecreasing in \( n \), \( P_n(x) \leq P(x) \), \( P_n(x) \to P(x) \) in \( L^1(\mathbb{R}) \) and locally uniformly in \( \mathbb{R} \) as \( n \to \infty \). Direct calculations show that
\[
p_n(x) := \int_{-x}^{\infty} P_n(y)dy \to p(x) = \int_{-x}^{\infty} P(y)dy \quad \text{uniformly in} \ \mathbb{R} \ as \ n \to \infty.
\]
Hence, for any \( \delta \in (0, \alpha) \), there is \( N_\delta > 0 \) such that \( d(p_n(x) - p(x)) + \delta > 0 \) for \( x \geq 0 \) and \( n \geq N_\delta \).

We can find some \( \delta_n \) such that \( \delta_n \to 0 \) as \( \delta \to 0 \). Set \( \alpha = \beta \), \( \gamma = 0 \), \( \delta \) and \( \alpha = \beta \) in the above analysis, one can easily deduce
\[
\lim_{n \to \infty} w_{N_\delta}(t, x) = \gamma \quad \text{uniformly in} \ [0, \alpha] \ \text{for} \ \delta \in (0, \alpha).
\]

Step 2: Proof of the result with (J2) violated by \( P \).

Similarly, we are going to construct an adequate lower solution by using the solution of a semi-wave problem. Define \( w_n \) and \( c_{n, \delta} \) as above. Noting \( \lim_{n \to \infty} c_{n, \delta} = c_{0, \delta} \) which is determined uniquely by problem (2.8) with \( (\hat{d}_n, \hat{P}_n) \) replaced by \( (d, P) \). Moreover, since \( \lim_{\delta \to 0} c_{0, \delta} = c_0 \) and \( \lim_{\zeta \to \infty} c_{0, \delta} = C_\zeta \), there are large \( \zeta > 0 \) and small \( \delta_0 < \alpha \) such that \( c < c_1 < \min\{c, C_\zeta\} \) for any \( \delta \in (0, \delta_0) \). Clearly, for such \( \zeta \) and \( \delta \), we can find some \( N_1 > N_\delta \) such that the desired result holds.

(3) It clearly suffices to show (2.6). We first deal with the case \( \liminf_{t \to \infty} s(t)/t < \infty \). Since \( P \) does not satisfies (J1), similarly to the proof of [17, Proposition 5.1] we have \( \lim_{n \to \infty} c_{n, \delta} \to \infty \) for any \( \delta \in (0, \alpha) \). Let \( N_\delta > 0 \) such that \( \delta \leq c_{N_1, \delta} \). As in the proof of conclusion (2), we obtain \( \liminf_{t \to \infty} w_{N_\delta}(t, x) \geq (\alpha - \delta)/\beta \quad \text{uniformly in} \ [0, \alpha] \ \text{for} \ c \in (0, \infty) \). As before, (2.6) holds.

Now we handle the case \( \lim_{t \to \infty} s(t)/t = \infty \). Obviously, for any \( c_1 > 0 \) there exists \( T > 0 \) such that \( s(t) > c_1 t + s_0 \) for \( t \geq T \). Define \( \bar{w}(t, x) = w(t + T, x) \) with \( t \geq 0 \) and \( 0 \leq x \leq c_1 t + s_0 \). Then,
\[
\begin{align*}
\bar{w}(t, x) & \geq \int_0^{c_1 t + s_0} P(x - y)\bar{w}(t, y)dy - dp(x)\bar{w} + \bar{w}(\alpha - \beta \bar{w}), \quad t > 0, \ 0 \leq x \leq c_1 t + s_0, \\
\bar{w}(t, c_1 t + s_0) & > 0, \quad t > 0, \\
\bar{w}(0, x) & > 0, \quad 0 \leq x \leq s_0.
\end{align*}
\]
Let \( w \) be a solution of
\[
\begin{cases}
w_t = d \int_0^{c_1 t + s_0} P(x - y)w(t, y)dy - dp(x)w + w(\alpha - \beta w), & t > 0, \ 0 \leq x \leq c_1 t + s_0, \\
w(t, c_1 t + s_0) = 0, & t > 0, \\
w(0, x) = w_0(x) \leq \tilde{w}(0, x), & 0 \leq x \leq s_0.
\end{cases}
\]

By the above conclusion with \( \varepsilon \in (0, \infty) \) and a comparison argument, we get (2.0).

(4) Take \( \theta_1 \in (\theta, \lambda) \). We shall verify that there exist constants \( \varepsilon, K_1, K_2, T_0 > 0 \) such that functions defined by \( l(t) = (K_1 t + K_2)^{\theta_1} \) and \( w(t, x) = K_\varepsilon \min\{1, 2(l(t) - x)/l(t)\} \) with \( K_\varepsilon = \alpha/\beta - \sqrt{\varepsilon} \) satisfy, for any \( T > T_0 \),
\[
\begin{cases}
\tilde{w} \leq d \int_0^{l(t)} P(x - y)\tilde{w}(t, y)dy - dp(x)\tilde{w} + \tilde{w}(\alpha - \beta \tilde{w}), & t > 0, \ x \in [0, l(t)) \setminus \{l(t)/2\}, \\
\tilde{w}(t, l(t)) \leq 0, & t > 0, \\
\tilde{w}(0, x) \leq w(T, x), & x \in [0, K_2^{\theta_1}].
\end{cases}
\]

Once (2.9) is obtained, as \( \lambda > \theta_1 \) and \( s(t) \geq Ct^\lambda \) for \( t \gg 1 \), we can find \( T_1 > T_0 \) such that \( s(t) > l(t) \) for \( t \geq T_1 \). Then the function \( \tilde{w}(t, x) = w(t + T_1, x) \) satisfies
\[
\begin{cases}
\tilde{w}_t \geq d \int_0^{l(t)} P(x - y)\tilde{w}(t, y)dy - dp(x)\tilde{w} + \tilde{w}(\alpha - \beta \tilde{w}), & t > 0, \ 0 \leq x \leq l(t), \\
\tilde{w}(t, l(t)) > 0, & t > 0, \\
\tilde{w}(0, x) = w(T_1, x) \geq \tilde{w}(0, x), & 0 \leq x \leq K_2^{\theta_1}.
\end{cases}
\]

The comparison principle indicates \( \tilde{w}(t, x) \geq w(t, x) \) for \( t \geq 0 \) and \( 0 \leq x \leq l(t) \). Since
\[
\max_{x \in [0, t^\theta]} |w(t, x) - \alpha/\beta + \sqrt{\varepsilon}| = \frac{(\alpha/\beta - \sqrt{\varepsilon}) \left(1 - \min\left\{1, 2(l(t) - t^\theta)/l(t)\right\}\right)}{\gamma} \to 0 \quad \text{as} \quad t \to \infty,
\]
it follows that \( \liminf_{t \to \infty} w(t, x) \geq \alpha/\beta - \sqrt{\varepsilon} \) uniformly in \([0, t^\theta]\). Thanks to the arbitrariness of \( \varepsilon \), we easily derive the desired result.

Thus it remains to show that (2.9) holds. Although the following analysis as well as that of conclusion (5) are similar to those of [23, Theorem 4.1], we give the details for the convenience of readers. Beginning with proving the first inequality of (2.9), we easily see \( w(t, x) \geq K_\varepsilon (1 - x/l(t)) \) for \( t \geq 0 \) and \( x \in [0, l(t)] \). For \( x \in [l(t)/4, l(t)] \), we have that, when \( K_2 \) is large,
\[
\int_0^{l(t)} P(x - y)w(t, y)dy = \int_{-x}^{l(t) - x} P(y)w(t, x + y)dy \\
\geq \int_{-l(t)/4}^{-l(t)/8} P(y)w(t, x + y)dy \\
\geq K_\varepsilon \int_{-l(t)/4}^{-l(t)/8} \frac{C_1}{|y|^\gamma} \left(1 - \frac{x + y}{l(t)}\right)dy \\
\geq K_\varepsilon \frac{l(t)}{|l(t)/4|} \int_{-l(t)/4}^{-l(t)/8} \frac{C_1}{|y|^\gamma} (-y)dy \geq \tilde{C}_1 K_\varepsilon l^{1 - \gamma}(t),
\]
where \( \tilde{C}_1 \) depends only on \( P \). For \( x \in [0, l(t)/4] \), we have
\[
\int_0^{l(t)} P(x - y)w(t, y)dy \geq \int_0^{l(t)/2} P(x - y)w(t, y)dy
\]
\[ K_\varepsilon \int_0^{l(t)/2} P(x - y)dy \]
\[ \geq K_\varepsilon (p(x) - \varepsilon) + K_\varepsilon \left( \varepsilon - \int_{l(t)/4}^\infty P(y)dy \right) \]
\[ \geq K_\varepsilon (p(x) - \varepsilon) = (p(x) - \varepsilon)w(t, x) \]

provided that \( K_2 \gg 1 \). Moreover, for \( x \in [0, l(t)/4] \),
\[ w(\alpha - \beta w) \geq \frac{\alpha}{2} \min\{w, \frac{\alpha}{\beta} - w\} \geq \frac{\alpha}{2} \min\{K_\varepsilon, \frac{\alpha}{\beta} - w\} \geq \frac{\alpha}{2} \sqrt{\varepsilon} \quad \text{if } \varepsilon \leq \frac{\alpha^2}{4\beta^2} \quad (2.10) \]

From [25, Proposition 4.3], by sending \( L_2 = l(t) \) and \( L_1 = l(t)/2 \) with \( K_2 \gg 1 \), we obtain
\[ \int_0^{l(t)} P(x - y)w(t, y)dy \geq (1 - \varepsilon^2)w(t, x) \quad \text{for } t > 0, \ x \in [l(t)/4, l(t)]. \]

Additionally, it is easy to see
\[ w(\alpha - \beta w) \geq \frac{\alpha}{2} \min\{w, \frac{\alpha}{\beta} - w\} \geq \frac{\alpha}{2} \varepsilon w \quad \text{if } \varepsilon \leq \min\{1, \left(\frac{\beta}{\alpha}\right)^2\} \quad (2.11) \]

Therefore, we can obtain that for \( x \in [0, l(t)/4] \),
\[ d \int_0^{l(t)} P(x - y)w(t, y)dy - dp(x)w + w(\alpha - \beta w) \geq -d\varepsilon w + \frac{\alpha}{2} \sqrt{\varepsilon} \geq -d\varepsilon \frac{\alpha}{\beta} + \frac{\alpha}{2} \sqrt{\varepsilon} \geq 0 \]
if \( \varepsilon \leq \left(\frac{\beta}{\alpha}\right)^2 \), and for \( x \in [l(t)/4, l(t)] \),
\[ d \int_0^{l(t)} P(x - y)w(t, y)dy - dp(x)w + w(\alpha - \beta w) \]
\[ \geq d \int_0^{l(t)} P(x - y)w(t, y)dy - \left( d - \frac{\alpha}{2} \right) w(t, x) \]
\[ = \left( \min\left\{ \frac{\alpha \varepsilon}{4}, d \right\} + \left\{ d - \frac{\alpha \varepsilon}{4} \right\}^+ \right) \int_0^{l(t)} P(x - y)w(t, y)dy \]
\[ \geq \min\left\{ \frac{\alpha \varepsilon}{4}, d \right\} \tilde{C}_1 K_\varepsilon l^{1-\gamma}(t) + \left\{ d - \frac{\alpha \varepsilon}{4} \right\}^+ (1 - \varepsilon^2)w(t, x) - \left( d - \frac{\alpha \varepsilon}{2} \right) w(t, x) \]
\[ \geq \min\left\{ \frac{\alpha \varepsilon}{4}, d \right\} \tilde{C}_1 K_\varepsilon l^{1-\gamma}(t) \]
if \( \varepsilon \leq \frac{\alpha \varepsilon}{4} \). On the other hand, we have \( w_2(t, x) = 0 \) for \( t > 0 \) and \( x \in [0, l(t)/2] \). For \( t > 0 \) and \( x \in (l(t)/2, l(t)) \), noticing \( \theta_1 < 1/(\gamma - 1) \) and \( K_2 \gg 1 \), we obtain
\[ w_1(t, x) = 2K_\varepsilon \frac{x^l(t)}{l^2(t)} \leq 2\theta_1 K_1 K_\varepsilon K_2 \frac{x^l(t)}{K_1 t + K_2} \leq \min\left\{ \frac{\alpha \varepsilon}{4}, d \right\} \tilde{C}_1 K_\varepsilon \frac{K_1 x^l(t)}{(K_1 t + K_2)^{\theta_1(\gamma - 1)}} \quad \text{if } K_1 \leq \frac{\tilde{C}_1 \min\{\alpha \varepsilon, d\}}{2\theta_1} \]

The first inequality of (2.29) holds. The second inequality is obvious. Now we focus on the last one. For \( \varepsilon \) and \( K_2 \) as chosen above, by conclusion (1), there is \( T_0 > 0 \) such that \( w(0, x) \leq K_\varepsilon = \alpha/\beta - \sqrt{\varepsilon} \leq w(T, x) \) for \( T \geq T_0 \) and \( 0 \leq x \leq K_2 \). Thus conclusion (4) follows.

(5) For \( 0 < \omega < \omega_1 < \eta \) and small \( \varepsilon > 0 \), we define \( l(t) = K_1(t + K_2) \left[ \ln(t + K_2) \right]^{\omega_1} \) and \( w(t, x) = K_\varepsilon \min\{1, (l(t) - x)/(t + K_2)^{1/2}\} \) with \( K_\varepsilon = \alpha/\beta - \sqrt{\varepsilon} \) and \( K_1, K_2 > 0 \). We prove that
there are $K_1, K_2$ and $T_0 > 0$ such that

\[
\begin{cases}
  \omega \leq d \int_0^{l(t)} P(x - y) \omega(t, y) \, dy - dp(x) \omega + \omega(\alpha - \beta \omega), \\
  \omega(t, l(t)) = 0, \\
  \omega(0, x) \leq \omega(T, x), \quad T > T_0, \ x \in [0, K_1 K_2 (\ln K_2)^{\omega_1}].
\end{cases}
\]

(2.12)

Once (2.12) is obtained, by the assumption on $s(t)$ we can find a $T_1 > T_0$ such that $s(t) > l(t)$ for $t \geq T_1$. We again define $\tilde{\omega}(t, x) = \omega(t + T_1, x)$, and then $\tilde{\omega}$ satisfies

\[
\begin{cases}
  \tilde{\omega}_t \geq d \int_0^{l(t)} P(x - y) \tilde{\omega}(t, y) \, dy - dp(x) \tilde{\omega} + \tilde{\omega}(\alpha - \beta \tilde{\omega}), \\
  \tilde{\omega}(t, l(t)) > 0, \\
  \tilde{\omega}(0, x) = \omega(T_1, x) \geq \omega(0, x),
\end{cases}
\]

for $t > 0$, $0 \leq x \leq l(t)$. Moreover, as $t \to \infty$,

\[
\max_{x \in [0, t \ln(t + 1)^{\omega}]} |\omega(t, x) - \alpha/\beta + \sqrt{\epsilon}| = (\alpha/\beta - \sqrt{\epsilon}) (1 - \min \{1, 2 (l(t) - t \ln(t + 1)^{\omega}) / l(t)\}) \to 0,
\]

which, as well as the arbitrariness of $\epsilon$, implies $\lim_{t \to \infty} \omega(t, x) \geq \alpha/\beta$ uniformly in $[0, t \ln(t + 1)^{\omega}]$.

Now we are ready to prove the first inequality of (2.12). Clearly, $l(t) (t + K_2)^{-1/2} \to \infty$ uniformly in $t \geq 0$ as $K_2 \to 0$, and $\omega(t, x) \geq K_\epsilon (l(t) - x) / (t + K_2)^{1/2}$ for $x \in [l(t) - 2 (t + K_2)^{1/2}, l(t)]$. Thus when $K_2$ is large enough, we see that, for $x \in [(l(t) - (t + K_2)^{1/2}) / 2, l(t)]$,

\[
\begin{align*}
\int_0^{l(t)} P(x - y) \omega(t, y) \, dy & = \int_{-x}^{l(t) - x} P(y) \omega(t, x + y) \, dy \\
& \geq K_\epsilon \int_{-(t + K_2)^{1/2}}^{-(t + K_2)^{1/2}} P(y) \frac{l(t) - x - y}{(t + K_2)^{1/2}} \, dy \\
& \geq K_\epsilon \int_{-(t + K_2)^{1/2}}^{-(t + K_2)^{1/2}} P(y) \frac{-y}{(t + K_2)^{1/2}} \, dy \\
& \geq K_\epsilon C_1 \int_{-(t + K_2)^{1/2}}^{-(t + K_2)^{1/2}} \frac{(-y)^{-1}}{(t + K_2)^{1/2}} \, dy = \frac{K_\epsilon C_1 \ln(t + K_2)}{8(t + K_2)^{1/2}}.
\end{align*}
\]

Similarly, we also can obtain that, for $x \in [(l(t) - (t + K_2)^{1/2}) / 2, l(t) - (t + K_2)^{1/2}]$,

\[
\begin{align*}
\int_0^{l(t)} P(x - y) \omega(t, y) \, dy & = \int_{-x}^{l(t) - x} P(y) \omega(t, x + y) \, dy \\
& \geq K_\epsilon \int_{-(t + K_2)^{1/2}}^{-(t + K_2)^{1/2}} P(y) \, dy \\
& \geq K_\epsilon C_1 \int_{-(t + K_2)^{1/2}}^{-(t + K_2)^{1/2}} (-y)^{-2} \, dy \geq \frac{K_\epsilon C_1 \ln(t + K_2)}{4(t + K_2)^{1/2}}.
\end{align*}
\]

Moreover, it can be seen from [25] Proposition 4.3] with $L_2 = l(t)$ and $L_1 = (t + K_2)^{1/2}$ that

\[
\int_0^{l(t)} P(x - y) \omega(t, y) \, dy \geq (1 - \varepsilon^2) \omega(t, x) \quad \text{for } t > 0, \ x \in [(l(t) - (t + K_2)^{1/2}) / 2, l(t)].
\]
By similar arguments in the proof of conclusion (4) we have that, for $x \in [0, (l(t) - (t + K_2)^{1/2})/2]$,\[
\int_0^{l(t)} P(x-y)w(t,y)dy \geq (p(x) - \varepsilon)w(t,x),
\]
which, together with (2.10), leads to\[
d\int_0^{l(t)} P(x-y)w(t,y)dy - dp(x)w + w(\alpha - \beta w) \geq -d\varepsilon \frac{\alpha}{\beta} + \frac{\alpha}{2}\sqrt{\varepsilon} \geq 0 \ 	ext{if} \ \varepsilon \leq \left(\frac{\beta}{2d}\right)^2.
\]
For $x \in [(l(t) - (t + K_2)^{1/2})/2, l(t)]$, we have\[
d\int_0^{l(t)} P(x-y)w(t,y)dy - dp(x)w + w(\alpha - \beta w) \\
\geq d\int_0^{l(t)} P(x-y)w(t,y)dy - \left(d - \frac{\alpha\varepsilon}{2}\right)w(t,x) \\
= \left[\left(\min\left\{\frac{\alpha\varepsilon}{4}, d\right\} + \left(d - \frac{\alpha\varepsilon}{4}\right)^+\right)\right]\int_0^{l(t)} P(x-y)w(t,y)dy - \left(d - \frac{\alpha\varepsilon}{2}\right)w(t,x) \\
\geq \min\left\{\frac{\alpha\varepsilon}{4}, d\right\} \frac{K_c C_1 \ln(t + K_2)}{8(t + K_2)^{1/2}} + \left(d - \frac{\alpha\varepsilon}{4}\right)^+(1 - \varepsilon^2)w(t,x) - \left(d - \frac{\alpha\varepsilon}{2}\right)w(t,x) \\
\geq \min\left\{\frac{\alpha\varepsilon}{4}, d\right\} \frac{K_c C_1 \ln(t + K_2)}{8(t + K_2)^{1/2}} \ 	ext{if} \ \varepsilon \leq \frac{\alpha}{4d}.
\]
On the other hand, we have $w(t,x) = 0$ for $t > 0$ and $x \in [0, l(t) - (t + K_2)^{1/2})$, and for $x \in (l(t) - (t + K_2)^{1/2}, l(t)]$,\[
w(t,x) \leq \frac{\ln(t + K_2)}{(t + K_2)^{1/2}}(K_1K_x + K_1K_{\omega_1}) \leq \min\left\{\frac{\alpha\varepsilon}{4}, d\right\} \frac{K_c C_1 \ln(t + K_2)}{8(t + K_2)^{1/2}}.
\]
Thus we obtain the first inequality of (2.12) if $K_1 \leq \frac{C_1 \min\{\frac{\alpha\varepsilon}{4}, d\}}{8(1+\omega_1)}$. Moreover, for $K_1$ and $K_2$ as above, by conclusion (1) we can choose $T_0 > 0$ such that $w(0,x) \leq K_x = \alpha/\beta - \sqrt{\varepsilon} \leq w(T,x)$ with $T \geq T_0$ and $0 \leq x \leq K_1K_2(\ln K_2)^{\omega_1}$. Therefore, all assertions are proved. □

Let $(u^*, v^*)$ be the unique positive root of $f_1(u,v) = 0$ and $f_2(u,v) = 0$. By [15] Theorem 1.2, the semi-wave problem [13], where $(d,J,f)$ is replaced with $(d_2,J_2,r_2\phi(1 - 2\phi))$, has a unique solution pair $(c_0, \phi_0)$ with $c_0 > 0$ and $\phi_0$ nonincreasing in $(-\infty, 0]$ if and only if $J_2$ satisfies (J1).

**Theorem 2.4.** If $h_\infty = \infty$, then we have the following conclusions:

(1) $\lim_{t \to \infty} u(t,x) = u^*$ and $\lim_{t \to \infty} v(t,x) = v^*$ locally uniformly in $\mathbb{R}^+$. 

(2) If $J_1$ satisfies (J2) and $J_2$ satisfies (J1), then
$$
\lim_{t \to \infty} \max_{x \in [0,c_t]} \{u(t,x) - u^* + v(t,x) - v^*\} = 0 \text{ for any } c \in (0,\min\{c_*, c_0\}).
$$

(3) If $J_1$ violates (J2) and $J_2$ satisfies (J1), then
$$
\lim_{t \to \infty} \max_{x \in [0,c_t]} \{u(t,x) - u^* + v(t,x) - v^*\} = 0 \text{ for any } c \in (0,c_0).
$$

(4) If $J_1$ meets (J2) and $J_2$ violates (J1), then
$$
\lim_{t \to \infty} \max_{x \in [0,c_t]} \{u(t,x) - u^* + v(t,x) - v^*\} = 0 \text{ for any } c \in (0,c_*).$$
(5) If $J_1$ violates (J2) and (J1) does not hold for $J_2$, then
\[
\lim_{t \to \infty} \max_{x \in [0, ct]} \{|u(t, x) - u^*| + |v(t, x) - v^*|\} = 0 \text{ for any } c > 0.
\]

(6) If there is $C > 0$ such that $\min\{J_1(x), J_2(x)\} \geq C|x|^{-\gamma}$ for $|x| \gg 1$ and $\gamma \in (1, 2)$. Then
\[
\lim_{t \to \infty} \max_{x \in [0, ct]} \{|u(t, x) - u^*| + |v(t, x) - v^*|\} = 0 \text{ for any } 1 < \theta < 1/(\gamma - 1).
\]

(7) If there is $C > 0$ such that $\min\{J_1(x), J_2(x)\} \geq C|x|^{-2}$ for $|x| \gg 1$. Then
\[
\lim_{t \to \infty} \max_{x \in [0, t[|t|^{-\omega}]} \{|u(t, x) - u^*| + |v(t, x) - v^*|\} = 0 \text{ for any } 0 < \omega < 1.
\]

Proof. (1) This conclusion can be proved by arguing as in the proof of [23, Theorem 4.11] with some obvious modifications, and thus the details are omitted.

(2) Let $(U, V)$ be the unique solution of problem
\[
U_t = f_1(U, V), \quad V_t = f_2(U, V); \quad U(0) = K_1, \quad V(0) = K_2.
\]

The comparison principle implies $u(t, x) \leq U(t)$ and $v(t, x) \leq V(t)$ for $t \geq 0$ and $x \geq 0$. It follows from a simple phase-plane analysis that $\lim_{t \to \infty} U(t) = u^*$ and $\lim_{t \to \infty} V(t) = v^*$. Thus
\[
\limsup_{t \to \infty} u(t, x) \leq u^* \quad \text{and} \quad \limsup_{t \to \infty} v(t, x) \leq v^* \quad \text{uniformly in } \mathbb{R}^+.
\]

Since $r_1 u(a - u - \frac{u^*}{t+1}) \geq r_1 u(a - 2u)$, we can derive from comparison principle and [24, Theorems 4.3 and 4.4] that $\lim_{t \to \infty} h(t)/t \geq c_0$. For any $c \in (0, \min\{c_1, c_0\})$, we choose a strictly decreasing sequence $\{c_n\}$ with $c_n \in (c, \min\{c_1, c_0\})$. Clearly, $u$ satisfies
\[
\begin{cases}
  u_t \geq d_1 \int_0^{h(t)} J_1(x-y)u(t,y)dy - d_1 j_1(x)u + r_1 u(a - 2u), & t > 0, \quad 0 \leq x < h(t), \\
  u(t, h(t)) > 0, & t > 0, \\
  u(0, x) = u_0(x) > 0, & 0 \leq x \leq h_0.
\end{cases}
\]

By a comparison consideration and the conclusion (2) of Lemma [23, 2] we obtain $\liminf_{t \to \infty} u(t, x) \geq a/2 := u_{10}$ uniformly in $[0, c_1 t]$. For small $\varepsilon > 0$, there is $T > 0$ such that $h(t) > c_2 t + h_0$ and $u(t, x) \geq u_{10} - \varepsilon$ for $t \geq T$ and $x \in [0, c_2 t + h_0]$. Hence $v$ satisfies
\[
\begin{cases}
  v_t \geq d_2 \int_0^{c_2 t + h_0} J_2(x-y)v(t,y)dy - d_2 j_2(x)v + f_2(u_{10} - \varepsilon, v), & t > T, \quad x \in [0, c_2 t + h_0], \\
  v(t, c_2 t + h_0) > 0, & t > T, \\
  v(T, x) > 0, & x \in [0, c_2 T + h_0].
\end{cases}
\]

Define $\tilde{v}(t, x) = v(t + T, x)$ with $t \geq 0$ and $0 \leq x \leq c_2(t + T) + h_0$. Then
\[
\begin{cases}
  \tilde{v}_t \geq d_2 \int_0^{c_2 t + h_0} J_2(x-y)\tilde{v}(t,y)dy - d_2 j_2(x)\tilde{v} + f_2(u_{10} - \varepsilon, \tilde{v}), & t > 0, \quad x \in [0, c_2 t + h_0], \\
  \tilde{v}(t, c_2 t + h_0) > 0, & t > 0, \\
  \tilde{v}(0, x) > 0, & x \in [0, h_0].
\end{cases}
\]
Again by the conclusion (2) of Lemma 2.13 and comparison argument we get \( \liminf_{t \to \infty} v(t, x) \geq \frac{1 + c(u_1 - \varepsilon)}{2 + c(u_1 - \varepsilon)} \) uniformly in \([0, c_3 t]\). The arbitrariness of \( \varepsilon \) implies \( \liminf_{t \to \infty} v(t, x) \geq \frac{1 + c u_1}{2 + c u_1} := v_1 \) uniformly in \([0, c_3 t]\). For small \( \varepsilon > 0 \), there is \( T > 0 \) such that \( v(t, x) \geq v_1 - \varepsilon \) in \([T, \infty) \times [0, c_4 t + h_0] \). Then \( u \) satisfies

\[
\begin{cases}
  u_t \geq d_1 \int_0^{c_4 t + h_0} J_1(x - y) u(t, y) dy - d_1 j_1(x) u + f_1(u, v_1 - \varepsilon), & t > T, \ x \in [0, c_4 t + h_0], \\
  u(t, c_4 t + h_0) > 0, & t > T, \\
  u(T, x) > 0, & x \in [0, c_4 T + h_0].
\end{cases}
\]

Similar to the above we can deduce \( \liminf_{t \to \infty} u(t, x) \geq \frac{a(1 + b u_1)}{2 + b u_1} := u_0 \) uniformly in \([0, c_5 t]\). Repeating the above arguments we can find two sequences \( \{u_n\} \) and \( \{v_n\} \) such that \( \liminf_{t \to \infty} u(t, x) \geq u_n \), \( \liminf_{t \to \infty} v(t, x) \geq v_n \) uniformly in \([0, c t]\), and \( \{u_n\}, \{v_n\} \) satisfy

\[
\begin{align*}
  u_1 &= \frac{a}{2}, & v_1 &= \frac{1 + c u_1}{2 + c u_1}, & u_{n+1} &= \frac{a(1 + b v_n)}{2 + b v_n}, & n &= 1, 2, \ldots.
\end{align*}
\]

Moreover, it is easy to show that \( u_n \) and \( v_n \) are both increasing in \( n \) and \( u_n \leq a \) and \( v_n \leq 1 \). Thus we can define \( u_\infty = \lim_{n \to \infty} u_n \) and \( v_\infty = \lim_{n \to \infty} v_n \) with

\[
\begin{align*}
  v_\infty &= \frac{1 + c u_\infty}{2 + c u_\infty}, & u_\infty &= \frac{a(1 + b v_\infty)}{2 + b v_\infty}.
\end{align*}
\]

By the uniqueness of positive root, we have \( u_\infty = u^* \) and \( v_\infty = v^* \). So assertion (2) is proved.

(3) Similarly to the above analysis we can make use of conclusions (2) and (3) of Lemma 2.13 to construct the same sequences \( \{u_n\} \) and \( \{v_n\} \), so that for any \( c \in (0, c_0) \), \( \liminf_{t \to \infty} u(t, x) \geq u_n \) and \( \liminf_{t \to \infty} v(t, x) \geq v_n \) uniformly in \([0, c t]\). By (2.13) and the above discussion on \( \{u_n\} \) and \( \{v_n\} \), we complete the proof of conclusion (3).

(4) and (5) Consider the following problem

\[
\begin{align*}
  w_t &= d_2 \int_0^{r(t)} J_2(x - y) w(t, y) dy - d_2 j_2(x) w + r_2 w(1 - 2w), & t > 0, & 0 \leq x < r(t), \\
  w(t, r(t)) &= 0, & t > 0, \\
  r'(t) &= \mu \int_0^{r(t)} \int_{r(t)}^{\infty} J_2(x - y) w(t, x) dy dx, & t > 0, \\
  r(0) &= h(T), \ w(0, x) = v(T, x), & 0 \leq x \leq r(0).
\end{align*}
\]

By [24] Theorem 4.3, spreading happens for \((w, r)\) if \( T \) is large enough. Note that \( J_2 \) violates (J1). Using [24] Theorem 4.4 we know \( \lim_{t \to \infty} r(t)/t = \infty \). Moreover, by \( f_2(u, v) \geq r_2 v(1 - 2v) \) and the comparison principle, one has \( h(t + T) \geq r(t) \) for \( t \geq 0 \), and thus \( \lim_{t \to \infty} h(t)/t = \infty \). Then by following the similar lines as above we can finish the proof, and the details are omitted here.

(6) For any \( \theta \in (1, 1/(\gamma - 1)) \), we choose a strictly decreasing sequence \( \theta_n \in (\theta, 1/(\gamma - 1)) \). Then \( u \) satisfies

\[
\begin{align*}
  u_t &\geq d_1 \int_0^{t^{\theta_1} + h_0} J_1(x - y) u(t, y) dy - d_1 j_1(x) u + r_1 u(a - 2u), & t > 0, & 0 \leq x < t^{\theta_1} + h_0, \\
  u(t, t^{\theta_1} + h_0) &> 0, & t > 0, \\
  u(0, x) &> 0, & 0 \leq x \leq h_0.
\end{align*}
\]
Lemma 3.1

Proof. We first claim that \( \bar{u} \geq 0 \) and \( \bar{v} \geq 0 \) in \( [0, T] \times \mathbb{R}^+ \). In fact, this is a direct consequence of maximum principle if \( 1 + \bar{b}\bar{v} > 0 \) in \( [0, T] \times [0, \bar{h}(t)] \). Assume on the contrary that there is \( (t_1, x_1) \in (0, T] \times [0, \bar{h}(t_1)] \) such that \( 1 + \bar{b}\bar{v}(t_1, x_1) \leq 0 \). By virtue of \( 1 + \bar{b}\bar{v}(0, x) \geq 1 \) and the continuity, there exists \( (t_*, x_*) \in (0, t_1] \times [0, \bar{h}(t_1)] \) such that \( 1 + \bar{b}\bar{v}(t_*, x_*) = 0 \) and \( 1 + \bar{b}\bar{v} > 0 \) in \( [0, t_*) \times [0, \bar{h}(t_1)] \). Apply the maximum principle to the equation of \( \bar{u} \) to deduce that \( \bar{u} \geq 0 \) in \( [0, s] \times \mathbb{R}^+ \) for any \( s \in (0, t_*) \). Hence, \( \bar{u} \geq 0 \) in \( [0, t_*) \times \mathbb{R}^+ \). Applying the maximum principle to \( \bar{v} \) we have \( \bar{v} \geq 0 \) in \( [0, t_*) \times [0, \bar{h}(t_1)] \). This contradiction implies our claim.

3 Criteria for spreading and vanishing

In this section, we investigate the criteria for spreading and vanishing. The following comparison principle will be used to prove our results. In this section, let \( (u, v, h) \) be a solution of (1.1).

Lemma 3.1 (Comparison principle). Assume that \( \bar{u}, \bar{u}_t \in C([0, T] \times [0, t^\beta]) \), \( \bar{u} \in L^\infty([0, T] \times \mathbb{R}^+) \), \( \bar{v}, \bar{v}_t \in C([0, T] \times [0, \bar{h}(t)]) \) and \( \bar{h} \in C^1([0, T]) \). If \( (\bar{u}, \bar{v}, \bar{h}) \) satisfies

\[
\begin{aligned}
\bar{u}_t &\geq d_2 \int_0^{t^\beta + h_0} J_2(x-y)\bar{v}(t,y)dy - d_2 J_2(x)\bar{v} + f_2(\bar{u}, \bar{v}), \quad t \in (0, T], \quad x \in \mathbb{R}^+, \\
\bar{v}_t &\geq d_2 \int_0^{\bar{h}(t)} J_2(x-y)\bar{v}(t,y)dy - d_2 J_2(x)\bar{v} + f_2(\bar{u}, \bar{v}), \quad t \in (0, T], \quad x \in [0, \bar{h}(t)), \\
\bar{v}(t, x) &\geq 0, \quad t \in (0, T], \quad x \in [\bar{h}(t), \infty) \quad (3.1) \\
\bar{h}'(t) &\geq \mu \int_0^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J_2(x-y)\bar{v}(t,x)dydx, \quad t \in (0, T], \\
\bar{u}(0, x) &\geq u_0(x), \quad x \in \mathbb{R}^+; \quad \bar{h}(0) \geq h_0 > 0, \quad \bar{v}(0, x) \geq v_0(x) \quad x \in [0, \bar{h}(0)],
\end{aligned}
\]

then we have

\[
\bar{h}(t) \geq h(t), \quad t \in [0, T], \quad \bar{u}(t, x) \geq u(t, x) \quad \bar{v}(t, x) \geq v(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^+.
\]

Proof. We first claim that \( \bar{u} \geq 0 \) and \( \bar{v} \geq 0 \) in \( [0, T] \times \mathbb{R}^+ \). In fact, this is a direct consequence of maximum principle if \( 1 + \bar{b}\bar{v} > 0 \) in \( [0, T] \times [0, \bar{h}(t)] \). Assume on the contrary that there is \( (t_1, x_1) \in (0, T] \times [0, \bar{h}(t_1)] \) such that \( 1 + \bar{b}\bar{v}(t_1, x_1) \leq 0 \). By virtue of \( 1 + \bar{b}\bar{v}(0, x) \geq 1 \) and the continuity, there exists \( (t_*, x_*) \in (0, t_1] \times [0, \bar{h}(t_1)] \) such that \( 1 + \bar{b}\bar{v}(t_*, x_*) = 0 \) and \( 1 + \bar{b}\bar{v} > 0 \) in \( [0, t_*) \times [0, \bar{h}(t_1)] \). Apply the maximum principle to the equation of \( \bar{u} \) to deduce that \( \bar{u} \geq 0 \) in \( [0, s] \times \mathbb{R}^+ \) for any \( s \in (0, t_*) \). Hence, \( \bar{u} \geq 0 \) in \( [0, t_*) \times \mathbb{R}^+ \). Applying the maximum principle to \( \bar{v} \) we have \( \bar{v} \geq 0 \) in \( [0, t_*) \times [0, \bar{h}(t_1)] \). This contradiction implies our claim.
For $0 < \varepsilon \ll 1$, we define $u_0^\varepsilon(x) = (1 - \varepsilon)u_0(x)$, $h_0^\varepsilon = (1 - \varepsilon)h_0$ and $\mu^\varepsilon = (1 - \varepsilon)\mu$. Choose $v_0^\varepsilon \in C([0, h_0^\varepsilon])$ satisfying $v_0^\varepsilon(h_0^\varepsilon) = 0 < v_0^\varepsilon(x)$ in $[0, h_0^\varepsilon]$ and $v_0^\varepsilon \rightarrow v_0$ in $C([0, h_0^\varepsilon])$ as $\varepsilon \rightarrow 0$. Let $(\bar{u}_0, v_0, h_0^\varepsilon)$ be the unique solution of \eqref{11} with $(u_0^\varepsilon, v_0^\varepsilon, h_0^\varepsilon, \mu^\varepsilon)$ in place of $(u_0, v_0, h_0, \mu)$.

We claim $\bar{h} > h^\varepsilon$ in $[0, T]$. Clearly, it holds true when $t$ is small. If it is not true, there is the smallest $t_0 \in [0, T]$ such that $\bar{h}(t_0) = h^\varepsilon(t_0)$ and $\bar{h} > h^\varepsilon$ in $[0, t_0)$. Obviously, $\bar{h}'(t_0) \leq (h^\varepsilon)'(t_0)$. We now compare $(\bar{u}, \bar{v})$ and $(u^\varepsilon, v^\varepsilon)$ in $[0, t_0] \times \mathbb{R}^+$. Define $w = \bar{u} - u^\varepsilon$ and $z = \bar{v} - v^\varepsilon$. Then

$$
\begin{cases}
    w_t \geq d_1 \int_0^\infty J_1(x-y)w(t,y)dy - d_1j_1(x)w + c_{11}w + c_{12}z, & t \in (0, t_0], \ x \geq 0, \\
    z_t \geq d_2 \int_0^{h^\varepsilon(t)} J_2(x-y)z(t,y)dy - d_2j_2(x)z + c_{21}w + c_{22}z, & t \in (0, t_0], \ 0 \leq x \leq h^\varepsilon(t), \\
    z(t, x) \geq 0, \ 0 \leq t \leq t_0, \ x \geq h^\varepsilon(t), \ w(0, x) > 0, \ x \geq 0; \ z(0, x) > 0, \ 0 \leq x \leq h_0^\varepsilon,
\end{cases}
$$

where $c_{ij} \in L^\infty([0, t_0] \times \mathbb{R}^+)$ for $i, j = 1, 2$ and $c_{12}, c_{21} \geq 0$ in $[0, t_0] \times \mathbb{R}^+$. Define $\bar{w} = w + \delta e^{kt}$ and $\bar{z} = z + \delta e^{kt}$ with small $\delta > 0$ and $k > 2(d_1 + d_2 + \sum_{i,j=1}^2 \|c_{ij}\|_{\infty} + 1)$. Then we have

$$
\begin{cases}
    \tilde{w}_t \geq d_1 \int_0^\infty J_1(x-y)\tilde{w}(t,y)dy - d_1j_1(x)\tilde{w} + c_{11}\tilde{w} + c_{12}\tilde{z} + A_1, & t \in (0, t_0], \ x \geq 0, \\
    \tilde{z}_t \geq d_2 \int_0^{h^\varepsilon(t)} J_2(x-y)\tilde{z}(t,y)dy - d_2j_2(x)\tilde{z} + c_{21}\tilde{w} + c_{22}\tilde{z} + A_2, & t \in (0, t_0], \ 0 \leq x \leq h^\varepsilon(t), \\
    \tilde{z}(t, x) > 0, \ 0 < t \leq t_0, \ x \geq h^\varepsilon(t); \ \tilde{w}(0, x) \geq \delta, \ x \geq 0; \ \tilde{z}(0, x) \geq \delta, \ x \in [0, h_0^\varepsilon),
\end{cases}
$$

where $A_i(x, t) = \delta e^{kt}(k - c_{i1} - c_{i2}), i = 1, 2$. We now prove that $\bar{w}, \tilde{z} \geq 0$ in $[0, t_0] \times \mathbb{R}^+$. Argue indirectly, and suppose that there exists $(t, x) \in [0, t_0] \times \mathbb{R}^+$ such that $\bar{w}(t, x) < 0$ or $\tilde{z}(t, x) < 0$. Since $\bar{w}(0, x) \geq \delta$ and $\bar{w}_0$ has a uniform lower bound, there exists the smallest $\tau \in (0, t_0)$ such that $\inf_{x \in \mathbb{R}^+} \bar{w}(\tau, x) = 0$, and $\bar{w} > 0$, $\tilde{z} \geq 0$ in $[0, \tau) \times \mathbb{R}^+$; or $\tilde{z}(\tau, x_0) = 0$ for some $x_0 \in [0, h^\varepsilon(\tau))$, and $\bar{w} \geq 0$, $\tilde{z} > 0$ in $[0, \tau) \times \mathbb{R}^+$. For the first case, we define $\bar{w} = \bar{w} - \delta$, and then

$$
\begin{cases}
    \tilde{w}_t \geq d_1 \int_0^\infty J_1(x-y)\tilde{w}(t,y)dy - d_1j_1(x)\tilde{w} + c_{11}\tilde{w} + A_1 + \delta c_{11}, & t \in (0, \tau), \ x \in \mathbb{R}^+, \\
    \tilde{w}(0, x) \geq 0, & x \in \mathbb{R}^+.
\end{cases}
$$

As $A_1 + \delta c_{11} = \delta e^{kt}(k - c_{11} - c_{12} + c_{11}e^{-kt}) \geq 0$, the maximum principle gives $\tilde{w} \geq 0$, and so $\bar{w} \geq \delta$ in $[0, \tau] \times \mathbb{R}^+$. This contradiction indicates that the first case cannot happen. Similarly, we can show that the second case also cannot happen. Hence $\bar{w}, \tilde{z} \geq 0$ in $[0, t_0] \times \mathbb{R}^+$. By the arbitrariness of $\delta$ and maximum principle, $w > 0$ in $[0, t_0] \times \mathbb{R}^+$ and $z > 0$ in $[0, t_0] \times [0, h^\varepsilon(t))$. Thus we have

$$
0 \geq \bar{h}'(t_0) - h^\varepsilon(t_0) \geq \mu \int_0^{h(t_0)} \int_0^{h(t_0)} J_2(x-y)v(t, x)dydx - \mu^\varepsilon \int_0^{h(t_0)} \int_0^{h(t_0)} J_2(x-y)v^\varepsilon(t, x)dydx \\
> \mu^\varepsilon \int_0^{h(t_0)} \int_0^{h(t_0)} J_2(x-y)(v(t, x) - v^\varepsilon(t, x)) dydx > 0.
$$

This contradiction implies $\bar{h} > h^\varepsilon$ in $[0, T]$. It then follows from the above analysis that $\bar{u} \geq u^\varepsilon$, $\bar{v} \geq v^\varepsilon$ in $[0, T] \times \mathbb{R}^+$. By the continuous dependence of solution of \eqref{11} on $\varepsilon$, the proof is ended. \hfill \square
From the above comparison principle, we can see that the solution \((u, v, h)\) of (1.1) is monotonically increasing in \(\mu > 0\). If \(r_2 < d_2/2\), from [24, Lemma 2.3] there is a unique \(\ell^* > 0\) such that 
\[
\lambda_p(L_{(0,\ell^*)} + r_2) = 0 \quad \text{and} \quad \lambda_p(L_{(0,l)} + r_2)(l - \ell^*) > 0 \quad \text{for any} \ l \neq \ell^*.
\]

**Theorem 3.2.** Then we have the following conditions governing spreading and vanishing of (1.1).

1. If \(r_2 \geq d_2/2\), then spreading happens;
2. If \(r_2 < d_2/2\), then spreading happens if \(h_0 \geq \ell^*\). Moreover, if \(h_\infty < \infty\), then \(h_\infty \leq \ell^*\);
3. If \(r_2 < d_2/2\) and \(h_0 < \ell^*\), then there exists a unique \(\mu^* > 0\) such that spreading happens if \(\mu > \mu^*\), and vanishing occurs if \(0 < \mu \leq \mu^*\).

**Proof.** Conclusions (1) and (2) are easily proved by [24, Lemma 2.3] and Theorem 2.2, and we omit the details. Next we focus on the proof of conclusion (3). We first claim that there is a \(\mu > 0\) such that vanishing happens if \(\mu < \mu^*\). Let \((\nu, h)\) be the unique solution of (1.4) with \((d_2, J_2, r_2u(1 - u), v_0(x))\) in place of \((d, J, f(u), u_0(x))\). By a comparison consideration, we have \(h(t) \leq \overline{h}(t)\) for \(t \geq 0\). By virtue of [24, Theorem 4.3], there is a \(\underline{\mu} > 0\) such that \(\lim_{t \to \infty} \overline{h}(t) < \infty\) if \(\mu \leq \underline{\mu}\). Hence our claim is proved.

Then we assert that there is a \(\overline{\mu} > 0\) such that spreading happens if \(\mu > \overline{\mu}\). Let \((\nu, h)\) be the unique solution to (1.4) with \((d, J, f(u), u_0(x))\) replaced by \((d_2, J_2, r_2u(1 - 2u), v_0(x))\). By a comparison argument again, we see \(h(t) \leq \overline{h}(t)\) for \(t \geq 0\). Moreover, it follows from [24, Theorem 4.3] that there is a \(\overline{\mu} > 0\) such that \(\lim_{t \to \infty} \overline{h}(t) = \infty\) if \(\mu > \overline{\mu}\). Thus our assertion follows.

Based on the above analysis, we can define \(\mu^* = \inf \{\Lambda > 0: \text{spreading happens if} \ \mu > \Lambda\}\). Clearly, \(\mu^* \in (0, \infty)\). Moreover, spreading occurs if \(\mu > \mu^*\), and vanishing happens if \(\mu < \mu^*\). To our purpose, it remains to show that vanishing happens if \(\mu = \mu^*\). To stress the dependence on \(\mu\), we denote the solution of (1.1) by \((u_\mu, v_\mu, h_\mu)\). Arguing indirectly, we assume that \(\lim_{t \to \infty} h_\mu(t) = \infty\). Then there exists a \(T > 0\) such that \(h_\mu(T) > \ell^*\). By continuity, there is a small \(\varepsilon > 0\) such that \(h_\mu(T) > \ell^*\) when \(\mu^* - \varepsilon < \mu < \mu^*\), which contradicts the definition of \(\mu^*\). So the proof is end. \(\square\)

## 4 Double free boundaries model

In this section, we are going to check the dynamics of (1.1) but with double free boundaries, and to prove that there are analogous conclusions holding true for this problem.

\[
\begin{align*}
    u_t &= d_1 \int_{-\infty}^{\infty} J_1(x - y)u(t, y)dy - d_1 u + f_1(u, v), & t > 0, \ x \in \mathbb{R}, \\
    v_t &= d_2 \int_{g(t)}^{h(t)} J_2(x - y)v(t, y)dy - d_2 v + f_2(u, v), & t > 0, \ x \in (g(t), h(t)), \\
    v(t, x) &= 0, & t > 0, \ x \notin (g(t), h(t)), \\
    g'(t) &= -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_2(x - y)v(t, x)dydx, & t > 0, \\
    h'(t) &= \mu \int_{h(t)}^{\infty} \int_{h(t)}^{\infty} J_2(x - y)v(t, x)dydx, & t > 0, \\
    u(0, x) &= u_0(x) > 0, \ x \in \mathbb{R}; \ -g(0) = h(0) = h_0; \ v(0, x) = v_0(x), \ x \in [-h_0, h_0],
\end{align*}
\]

where \(u_0(x) \in C([\mathbb{R}] \cup L^\infty(\mathbb{R}), v_0(x) \in C([-h_0, h_0])\) and \(v_0(\pm h_0) = 0 < v_0(x)\) in \((-h_0, h_0)\). Below is firstly the well-posedness result of it.
Theorem 4.1. Problem (4.1) has a unique solution \((u, v, g, h)\) defined for \(t \geq 0\). Moreover, \(u, u_t \in C(\mathbb{R}^+ \times \mathbb{R})\), \(v, v_t \in C(\mathbb{R}^+ \times [g(t), h(t)])\) and \(g(t), h(t) \in C^1([0, \infty))\). The following estimates hold

\[
0 \leq u \leq K_1 := \max\{\|u_0\|_{L^\infty(\mathbb{R})}, a\}, \quad 0 \leq v \leq K_2 := \max\{\|v_0\|_{C([\alpha, \beta])}, 1\}.
\]

4.1 Longtime behaviors of solution

It is easy to see that \(-g(t)\) and \(h(t)\) are increasing in \(t \geq 0\). So setting \(g_\infty = \lim_{t \to \infty} g(t)\) and \(h_\infty = \lim_{t \to \infty} h(t)\), we can define \(h_\infty - g_\infty < \infty\) as vanishing case, and \(h_\infty - g_\infty = \infty\) as spreading case. For any \(a_0 > 0\), it is well-known that problem

\[
(\tilde{L}_{(l_1, l_2)} + a_0)\phi := d_2 \int_{l_1}^{l_2} J_2(x - y)\phi(y)dy - d_2\phi + a_0\phi = \lambda\phi \quad \text{in} \quad (l_1, l_2)
\]

has a unique principal eigenvalue with a positive eigenfunction. Please see [13, Proposition 3.4] for more details about this eigenvalue problem. Denote \((u, v, g, h)\) a solution of (4.1) in this section.

4.1.1 Vanishing case: \(h_\infty - g_\infty < \infty\)

Theorem 4.2. If \(h_\infty - g_\infty < \infty\), then \(\lambda_p(\tilde{L}_{(g_\infty, h_\infty)} + r_2) \leq 0\), \(\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0\) and

\[
\begin{align*}
\lim_{t \to \infty} & \max_{x \in [-ct, ct]} |u(t, x) - a/2| = 0 \quad \text{for any} \quad c \in (0, c_\ast), \quad \text{if} \ J_1 \ \text{satisfies} \ (J2), \\
\lim_{t \to \infty} & \max_{x \in [-ct, ct]} |u(t, x) - a/2| = 0 \quad \text{for any} \quad c > 0, \quad \text{if} \ J_1 \ \text{violates} \ (J2).
\end{align*}
\]

Proof. Since it can be proved by arguing as in the proof of Theorem 2.2 we omit the details. \(\square\)

Remark 4.3. We believe that in vanishing case, the followings hold true for problem (4.1).

\[
\begin{align*}
\lim_{t \to \infty} & \max_{x \in [-s(t), s(t)]} |u(t, x) - a/2| = 0 \quad \text{for any} \quad 0 \leq s(t) = t^{\gamma - 2}o(1) \quad \text{if} \ J_1 \approx |x|^{-\gamma} \quad \text{for} \ \gamma \in (1, 2), \\
\lim_{t \to \infty} & \max_{x \in [-s(t), s(t)]} |u(t, x) - a/2| = 0 \quad \text{for any} \quad 0 \leq s(t) = (t \ln t)o(1) \quad \text{if} \ J_1 \approx |x|^{-2}.
\end{align*}
\]

In fact, by taking advantage of the arguments in the proof of [16, Lemmas 6.4 and 6.6] and [23, Theorem 4.1] one can prove these results.

4.1.2 Spreading case: \(h_\infty - g_\infty = \infty\)

Lemma 4.4. Assume that \(-s_1(t), s_2(t)\) are continuous in \([0, \infty)\) and strictly increasing to \(\infty\), \(-s_1(0) = s_2(0) = s_0 > 0\) and \(P\) satisfies (J). Let \(s = \min\{\lim_{t \to \infty} \frac{-s_1(t)}{t}, \lim_{t \to \infty} \frac{s_2(t)}{t}\}\), and constants \(d, \alpha, \beta\) be positive. Let \(w\) be the unique solution of

\[
\begin{align*}
w_t &= d \int_{s_1(t)}^{s_2(t)} P(x - y)w(t, y)dy - dw + w(\alpha - \beta w), \quad t > 0, \quad s_1(t) < x < s_2(t), \\
w(t, s_i(t)) &= 0, \quad t > 0, \quad i = 1, 2, \\
w(0, x) &= w_0(x), \quad -s_0 \leq x \leq s_0,
\end{align*}
\]

where \(w_0(x) \in C([-s_0, s_0])\), \(w_0(x) > 0 = w_0(\pm s_0)\ in \ (-s_0, s_0)\). Then the followings hold true:

1. \(\lim_{t \to \infty} w(t, x) = \alpha/\beta \text{ locally uniformly in } \mathbb{R}\).
(2) Suppose that \( P \) satisfies (J1) and \( \underline{s} \in (0, \infty] \). Then
\[
\lim_{t \to \infty} \max_{x \in [-ct, ct]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } c \in (0, \min\{\underline{s}, C_*\}), \text{ if } P \text{ satisfies (J2)},
\]
\[
\lim_{t \to \infty} \max_{x \in [-ct, ct]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } c \in (0, \underline{s}), \text{ if } P \text{ violates (J2)},
\]
where \( C_* \) is defined as in Lemma 2.3.

(3) Suppose that \( P \) does not satisfy (J1) and \( \underline{s} \in (0, \infty] \). Then
\[
\lim_{t \to \infty} \max_{x \in [-ct, ct]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } c \in (0, \underline{s}).
\]

(4) Suppose that there exist \( C_1, C_2 > 0 \) such that \( P(x) \geq C_1|x|^{-\gamma} \) for \( |x| \gg 1 \) and \( \gamma \in (1, 2) \), and \( \min\{-s_1(t), s_2(t)\} \geq C_2 t^\lambda \) for \( t \gg 1 \) and some \( \lambda \in (1, 1/(\gamma - 1)] \). Then
\[
\lim_{t \to \infty} \max_{x \in [-ct, ct]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } \theta_1, \theta_2 \in (1, \lambda).
\]

(5) Suppose that there exist \( C_1, C_2 > 0 \) such that \( P(x) \geq C_1|x|^{-2} \) for \( |x| \gg 1 \), and \( \min\{-s_1(t), s_2(t)\} \geq C_2 t (\ln t)^\eta \) for \( t \gg 1 \) and some \( 0 < \eta \leq 1 \). Then
\[
\lim_{t \to \infty} \max_{x \in [-ct, ct]} |w(t, x) - \alpha/\beta| = 0 \quad \text{for any } \omega_1, \omega_2 \in (0, \eta).
\]

\textbf{Proof.} (1) Since it is easy to prove this conclusion, the details are omitted here.

(2) Clearly, by a simple comparison argument we easily see \( \lim \sup_{t \to \infty} w(t, x) \leq \alpha/\beta \) uniformly in \( \mathbb{R} \). Thus it remains to study the lower limitation of \( w \) as \( t \to \infty \). Assume that \( P \) satisfies (J2). For any \( c < c_1 \in (0, \min\{\underline{s}, C_*\}) \), there exist a large \( \zeta > 0 \) and a small \( \delta > 0 \) such that \( c_1 < \min\{\underline{s}, c_0^\zeta - \delta\} \) with \( c_0^\zeta \) defined as in Lemma 2.3. For simplicity, let \( F(w) = w(\alpha - \beta w) \). Then by letting \( \varepsilon > 0 \) sufficiently small, we have \( F'(w) < 0 \) for \( w \in [\alpha/\beta(1 - \varepsilon/2)(1 - \varepsilon), \alpha/\beta] \),
\[
1 - \frac{\varepsilon}{4} > (1 - \frac{\varepsilon}{2})(1 - \varepsilon) \text{ and } 2(1 - \varepsilon) F(\alpha/\beta(1 - \frac{\varepsilon}{4})) < F(\alpha/\beta(1 - \varepsilon)). \tag{4.3}
\]
Moreover, we may choose \( M > 0 \) large enough such that \( \phi\zeta_0(-M) > \alpha/\beta(1 - \varepsilon/4) \). Define \( \varepsilon_0 = \min_{x \in [-M, 0]} \{ |\phi\zeta_0'(x)| \} > 0 \) and \( M_0 = \max_{w \in [-\alpha/\beta, \alpha/\beta]} |F'(w)| \) and \( \tilde{\varepsilon} = \delta \varepsilon_0(1 - \varepsilon)/2M_0 \). Also we can find a large \( L > M \) such that \( \phi\zeta_0(-2L + M) \geq \alpha/\beta - \tilde{\varepsilon} \).

Let \( \underline{s}(t) = c_1 t + L \) for \( t \geq 0 \). By conclusion (1) and \( c_1 < \underline{s} \), there is \( T > 0 \) such that \( w(t, x) \geq (1 - \varepsilon)\alpha/\beta \) and \( \min\{-s_1(t), s_2(t)\} \geq c_1 t + L \) for \( t \geq T \) and \( -L \leq x \leq L \). Define \( \hat{w}(t, x) = w(t, x + T, x) \). Clearly, \( \hat{w} \) satisfies
\[
\begin{align*}
\hat{w}_t &\geq d \int_{-\underline{s}(t)}^{\underline{s}(t)} P(x - y) \hat{w}(t, y) dy - d \hat{w} + F(\hat{w}), & t > 0, & -\underline{s}(t) < x < \underline{s}(t), \\
\hat{w}(t, \pm \underline{s}(t)) &> 0, & t > 0, \\
\hat{w}(0, x) &\geq (1 - \varepsilon)\alpha/\beta, & -L \leq x \leq L.
\end{align*}
\]

Let \( \bar{w}(t, x) = (1 - \varepsilon)[\phi\zeta_0(x - \underline{s}(t)) + \phi\zeta_0(-x - \underline{s}(t)) - \alpha/\beta] \), \( t \geq 0 \), \( x \in [-\underline{s}(t), \underline{s}(t)] \).
We are going to show that the following inequalities hold true:

\[
\begin{align*}
\max_{x \in [-ct,ct]} [w(t, x) - (1 - \varepsilon)\alpha/\beta] & \leq 2(1 - \varepsilon)[\alpha/\beta - \phi_{c_0}(ct - c_1 t - L)] \\
& \to 0 \text{ as } t \to \infty,
\end{align*}
\]

which, together with the arbitrariness of \( \varepsilon \), yields our desired result.

Since the second and third inequalities of (4.4) are obvious, we next focus on the first one. Direct calculations show

\[
\begin{align*}
[w(t, x)] & = -(1 - \varepsilon)c_1 \left[ \phi_c'(x - s(t)) + \phi_c'(x - s(t)) \right] \\
& \leq -(1 - \varepsilon)(c^\zeta_0 - \delta)[\phi_c'(x - s(t)) + \phi_c'(x - s(t))] \\
& \leq (1 - \varepsilon)\delta[\phi_c'(x - s(t)) + \phi_c'(x - s(t))] + (1 - \varepsilon) \left[ d \int_{-\infty}^{s(t)} P(x - y)[\phi_c'(y - s(t))] dy \\
& - d\phi_c'(x - s(t)) + d \int_{-\infty}^{\infty} P(x - y)[\phi_c'(y - s(t))] dy - d\phi_c'(x - s(t)) \\
& + (1 - \varepsilon) \left[ F(\phi_c'(x - s(t))) + F(\phi_c'(x - s(t))) \right] \\
& = (1 - \varepsilon)\delta[\phi_c'(x - s(t)) + \phi_c'(x - s(t))] + (1 - \varepsilon) \left[ d \int_{-\infty}^{s(t)} P(x - y)w(t, y) dy - dw \right] \\
& + (1 - \varepsilon)d \left[ \int_{-\infty}^{s(t)} P(x - y)[\phi_c'(y - s(t))] - \frac{\alpha}{\beta} \right] dy + \int_{s(t)}^{\infty} P(x - y)[\phi_c'(y - s(t))] - \frac{\alpha}{\beta} \right] dy \\
& + (1 - \varepsilon) \left[ F(\phi_c'(x - s(t))) + F(\phi_c'(x - s(t))) \right] \\
& \leq d \int_{-\infty}^{s(t)} P(x - y)w(t, y) dy - dw + A(t, x),
\end{align*}
\]

where

\[
A(t, x) = (1 - \varepsilon) \left\{ \delta[\phi_c'(x - s(t)) + \phi_c'(x - s(t))] + F(\phi_c'(x - s(t))) + F(\phi_c'(x - s(t))) \right\}.
\]

We next verify \( A \leq F(w) \) for \( t \geq 0 \) and \( x \in [-s(t), s(t)] \). Firstly, we deal with the case \( x \in [s(t) - M, s(t)] \). For such \((t, x)\), from our choice of \( L \) we have \( 0 > \phi_{c_0}(-x - s(t)) - \alpha/\beta \geq -\hat{\varepsilon} \). It then follows from the mean value theorem that

\[
\begin{align*}
F(w) & \geq F((1 - \varepsilon)\phi_{c_0}'(x - s(t))) - M_0(1 - \varepsilon)\hat{\varepsilon}, \\
F(\phi_{c_0}'(x - s(t))) & = F(\phi_{c_0}'(x - s(t))) - F(\hat{\varepsilon}) \leq M_0\hat{\varepsilon}. \\
\end{align*}
\]

Therefore,

\[
A(t, x) - F(w) \leq -(1 - \varepsilon)\delta\varepsilon_0 + (1 - \varepsilon) \left[ F(\phi_{c_0}'(x - s(t))) + M_0\hat{\varepsilon} \right]
\]
\[-F((1-\varepsilon)\phi_{c_0}(x - \underline{s}(t))) + M_0(1-\varepsilon)\hat{\varepsilon} \leq -(1-\varepsilon)\delta \varepsilon_0 + 2M_0(1-\varepsilon)\hat{\varepsilon} < 0.\]

Moreover we have \(A(t, x) \leq F(w)\) for \(t \geq 0\) and \(x \in [\underline{s}(t), -\underline{s}(t)+M]\) since \(A\) and \(w\) are both even in \(x\). It then remains to check the case \(x \in [-\underline{s}(t)+M, \underline{s}(t)-M]\). Clearly, in this case we have
\[
w(t, x) \in \left(\frac{\alpha}{\beta}(1-\varepsilon)(1-\frac{\varepsilon}{2}), (1-\varepsilon)\frac{\alpha}{\beta}\right).
\]

So, by (4.3), we have
\[
A(t, x) - F(w) \leq (1-\varepsilon)2F\left(\frac{\alpha}{\beta}(1-\frac{\varepsilon}{4})\right) - F\left(\frac{\alpha}{\beta}(1-\varepsilon)\right) < 0.
\]

Consequently, we prove (4.4). The assertion with \(P\) satisfying (J2) follows.

As for the case \(P\) violating (J2), we can argue as above to derive the conclusion. In fact, since \(\lim_{\zeta \to \infty}c_0^\zeta = \infty\), for any \(c \in (0, \underline{s})\) we can choose \(\zeta > 0\) large sufficiently such that \(c < c_0^\zeta\). Then a similar lower solution can be constructed to verify this case, and the details are omitted.

(3) We first consider the case \(\underline{s} \in (0, \infty)\). Define \(P_n\) as in the proof of Lemma 2.3. Let \(w_n\) be a solution of (4.1) with \(P\) replaced by \(P_n\), and \(C_n\) be the minimal speed of problem
\[
\begin{cases}
\hat{d}_n \int_{-\infty}^{\infty} \hat{P}_n(x - y) \phi(y)dy - \hat{d}_n \phi + c (\phi' - d + \alpha - \beta \phi) = 0, & -\infty < x < \infty, \\
\phi(-\infty) = (\hat{d}_n - d + \alpha)/\beta, & \phi(\infty) = 0, \phi'(x) \leq 0,
\end{cases}
\]
where \(n\) is large sufficiently such that \(\hat{d}_n - d + \alpha > 0\). Obviously, from a comparison consideration we have \(w(t, x) \geq w_n(t, x)\) for \(t \geq 0\) and \(s_1(t) \leq x \leq s_2(t)\). Since \(P\) does not satisfy (J1), it is easy to see (13) that \(\lim_{n \to \infty} C_n = \infty\). Thus for any \(0 < c < \underline{s}\), we can choose \(N\) large enough such that \(c < \min\{\underline{s}, C_N\}\). Then by conclusion (2) we have for any large \(n\), \(\liminf_{t \to \infty} w_n(t, x) \geq (\hat{d}_n - d + \alpha)/\beta\) uniformly in \([-ct, ct]\), and so \(\liminf_{t \to \infty} w(t, x) \geq (\hat{d}_n - d + \alpha)/\beta\) uniformly in \([-ct, ct]\). Sending \(n \to \infty\), we obtain \(\liminf_{t \to \infty} w(t, x) \geq \alpha/\beta\) uniformly in \([-ct, ct]\). Together with \(\limsup_{t \to \infty} w(t, x) \leq \alpha/\beta\) uniformly in \(\mathbb{R}\), we finish the proof of this case.

When \(\underline{s} = \infty\), for any \(0 < c < c_1 < \infty\) there is \(T > 0\) such that \(\min\{-s_1(t), s_2(t)\} > c_1 t + s_0 := \bar{s}(t)\) for \(t \geq T\). Define \(\bar{w}(t, x) = w(t + T, x)\) for \(t \geq 0\) and \(x \in [-\bar{s}(t), \bar{s}(t)]\). Then \(\bar{w}(t, x)\) satisfies
\[
\begin{cases}
\bar{w}_t \geq d \int_{-\bar{s}(t)}^{\bar{s}(t)} P(x - y)\bar{w}(t, y)dy - d\bar{w} + \bar{w}(\alpha - \beta \bar{w}), & t > 0, \quad -\bar{s}(t) < x < \bar{s}(t), \\
\bar{w}(t, \pm \bar{s}(t)) > 0, & t > 0, \\
\bar{w}(0, x) > 0, & -s_0 \leq x \leq s_0.
\end{cases}
\]

Let \(w\) be a solution of
\[
\begin{cases}
w_t = d \int_{-\bar{s}(t)}^{\bar{s}(t)} P(x - y)w(t, y)dy - dw + w(\alpha - \beta w), & t > 0, \quad -\bar{s}(t) < x < \bar{s}(t), \\
w(t, \pm \bar{s}(t)) = 0, & t > 0, \\
w(0, x) = \tilde{w}_0(x) = \bar{w}(0, x), & -s_0 \leq x \leq s_0.
\end{cases}
\]

By the conclusion of the case \(\underline{s} \in (0, \infty)\) and comparison principle, we have \(\liminf_{t \to \infty} w(t, x) \geq \alpha/\beta\) uniformly in \([-ct, ct]\). So we get the conclusion (3).
(4) For any \( \theta_1, \theta_2 \in (1, \lambda) \), we choose \( \theta \in (\max\{\theta_1, \theta_2\}, \lambda) \). For small \( \varepsilon > 0 \), define

\[
\tilde{s}(t) = (K_1t + K_2)^\theta, \quad \tilde{w}(t, x) = K_\varepsilon(\tilde{s}(t) - |x|)/\tilde{s}(t)
\]

with \( K_\varepsilon = \alpha/\beta - \varepsilon \), \( K_1 \) and \( K_2 \) to be determined later.

We next show that there exist adequately small \( \varepsilon, K_1 > 0 \) and large \( K_2, T_0 > 0 \) such that

\[
\begin{align*}
&\left\{ \begin{array}{ll}
\tilde{w}_t \leq d \int_{-\tilde{s}(t)}^{\tilde{s}(t)} P(x-y)\tilde{w}(t,y)\,dy - d\tilde{w} + \tilde{w}(\alpha - \beta \tilde{w}), & t > 0, \quad x \in (-\tilde{s}(t), \tilde{s}(t)), \\
\tilde{w}(t, \pm \tilde{s}(t)) = 0, & t > 0, \\
\tilde{w}(0, x) \leq w(T, x), & T \geq T_0, \quad x \in [-K^0_2, K^0_2].
\end{array} \right.
\]

If (4.5) is obtained, for such \( K_1 \) and \( K_2 \), by our assumptions there is \( T_1 > T_0 \) such that \( \min\{-s_1(t), s_2(t)\} > \tilde{s}(t) \) for \( t \geq T_1 \). Define \( \tilde{w}(t, x) = w(t + T_1, x) \) for \( t \geq 0 \) and \( x \in [-\tilde{s}(t), \tilde{s}(t)] \). Then

\[
\begin{align*}
&\left\{ \begin{array}{ll}
\tilde{w}_t \geq d \int_{-\tilde{s}(t)}^{\tilde{s}(t)} P(x-y)\tilde{w}(t,y)\,dy - d\tilde{w} + \tilde{w}(\alpha - \beta \tilde{w}), & t > 0, \quad -\tilde{s}(t) < x < \tilde{s}(t), \\
\tilde{w}(t, \pm \tilde{s}(t)) > 0, & t > 0, \\
\tilde{w}(0, x) = w(T_1, x) \geq w(0, x), & -K^\theta_2 \leq x \leq K^\theta_2.
\end{array} \right.
\]

Employing a comparison method, we have \( w(t + T_1, x) = \tilde{w}(t, x) \geq \tilde{w}(t, x) \) for \( t \geq 0 \) and \( x \in [-\tilde{s}(t), \tilde{s}(t)] \). In addition, it is easy to see that \( \lim_{t \to \infty} \tilde{w}(t, x) = K_\varepsilon \) uniformly in \([-t^\theta_1, t^\theta_2]\). Our assertion then follows from the arbitrariness of \( \varepsilon \).

Now we are in the position to verify (4.5). We first claim that there is a positive constant \( \tilde{C}_1 \) depending only on \( P \) and \( \gamma \) such that

\[
\int_{-\tilde{s}(t)}^{\tilde{s}(t)} P(x-y)w(t,y)\,dy \geq K_\varepsilon \tilde{C}_1 \tilde{s}^{1-\gamma}(t).
\]

For \( x \in [\tilde{s}(t)/4, \tilde{s}(t)] \), we have that if \( K_2 \) is large enough,

\[
\int_{-\tilde{s}(t)}^{\tilde{s}(t)} P(x-y)w(t,y)\,dy = \int_{-\tilde{s}(t)/4}^{\tilde{s}(t)/4} P(y)w(t, x + y)\,dy \\
\geq \int_{-\tilde{s}(t)/4}^{\tilde{s}(t)/8} P(y)w(t, x + y)\,dy \\
\geq \frac{K_\varepsilon \tilde{C}_1}{\tilde{s}(t)} \int_{-\tilde{s}(t)/4}^{\tilde{s}(t)/8} (-y)^{1-\gamma}\,dy \geq \tilde{C}_1 K_\varepsilon \tilde{s}^{1-\gamma}(t).
\]

For \( x \in [0, \tilde{s}(t)/4] \), by following the similar method as above we see

\[
\int_{-\tilde{s}(t)}^{\tilde{s}(t)} P(x-y)w(t,y)\,dy \geq K_\varepsilon \int_{\tilde{s}(t)/8}^{\tilde{s}(t)/4} P(y) \left( \frac{\tilde{s}(t) - x - y}{\tilde{s}(t)} \right) \,dy \geq \tilde{C}_1 K_\varepsilon \tilde{s}^{1-\gamma}(t).
\]

Then noting that \( P \) and \( w \) are both symmetric in \( x \), our claim holds true.

Moreover, from [16, Lemma 6.3] and (2.11) we have that for small \( \varepsilon > 0 \) and large \( K_2 > 0 \),

\[
d \int_{-\tilde{s}(t)}^{\tilde{s}(t)} P(x-y)w(t,y)\,dy - dw + w(\alpha - \beta w)
\]
\[
\begin{align*}
&\geq (d - \frac{\alpha \varepsilon}{4}) \int_{-\theta(t)}^{\theta(t)} P(x - y) \omega(t, y)dy - d\omega + \frac{\alpha \varepsilon}{2}\omega + \frac{\alpha \varepsilon}{4} \int_{-\theta(t)}^{\theta(t)} P(x - y) \omega(t, y)dy \\
&\geq \left[(d - \frac{\alpha \varepsilon}{4})(1 - \varepsilon^2) - d + \frac{\alpha \varepsilon}{2}\right]\omega + \frac{\alpha \varepsilon}{4} \int_{-\theta(t)}^{\theta(t)} P(x - y) \omega(t, y)dy \\
&\geq \frac{\alpha \varepsilon}{4} \int_{-\theta(t)}^{\theta(t)} P(x - y) \omega(t, y)dy \geq \frac{\alpha \varepsilon \tilde{C}_1 K_\varepsilon \xi^{1-\gamma}(t)}{4}.
\end{align*}
\]

Since \( \theta < 1/(\gamma - 1) \), we have that for \( t > 0 \) and \( x \in (-\xi(t), \xi(t)) \),
\[
\omega(x) = K_\varepsilon |x| \xi(t) \leq K_\varepsilon \theta K_1 \leq \frac{K_\varepsilon \theta K_1}{(K_1 t + K_2)\theta(\gamma-1)} = K_\varepsilon \theta K_1 \xi^{1-\gamma}(t) \leq \frac{\alpha \varepsilon \tilde{C}_1 K_\varepsilon \xi^{1-\gamma}(t)}{4}
\]
with \( K_1 < \frac{\alpha \varepsilon \tilde{C}_1}{\theta} \). So we have proved the first inequality of (4.5). The second identity is obvious. For such \( K_2 \) and \( \varepsilon \) as chosen above, from conclusion (1) there is \( T_0 > 0 \) such that \( w(T, x) \geq \alpha/\beta - \varepsilon \geq w(0, x) \) for \( T \geq T_0 \) and \( x \in [-K_2^{\theta}, K_2^{\theta}] \). So the proof of assertion (4) is complete.

(5) For any \( \omega_1, \omega_2 \in (0, \eta) \), let \( \omega \in (\max\{\omega_1, \omega_2\}, \eta) \). For small \( \varepsilon > 0 \), define
\[
\xi(t) = K_1(t + K_2)[\ln(t + K_2)]^{\omega}, \quad \omega(t, x) = K_\varepsilon \min\{1, (\xi(t) - |x|)/(t + K_2)^{1/2}\}
\]
with \( K_\varepsilon = \alpha/\beta - \varepsilon, K_1 \) and \( K_2 \) to be determined later.

Similarly, we next prove that there are small \( \varepsilon, K_1 > 0 \) and large \( K_2, T_0 > 0 \) such that
\[
\begin{align*}
\omega(t, x) \leq d\int_{-\xi(t)}^{\xi(t)} P(x - y) \omega(t, y)dy - d\omega + \omega(\alpha - \beta \omega), \\
\omega(t, x) = 0, & \quad t > 0, \quad x \in (-\xi(t), \xi(t)) \setminus \{\pm (\xi(t) - (t + K_2)^{1/2})\}, \\
\omega(0, x) \leq w(T, x), & \quad t > T_0, \quad x \in [-\xi(0), \xi(0)].
\end{align*}
\]
If (4.6) is proved, for \( K_1 \) and \( K_2 \) as chosen above, there is \( T_1 > T_0 \) such that \( \min\{-s_1(t), s_2(t)\} > \xi(t) \) for \( t \geq T_1 \). Define \( \hat{w}(t, x) = w(t + T_1, x) \) for \( t \geq 0 \) and \( x \in (-\xi(t), \xi(t)) \). Clearly, \( \hat{w} \) satisfies
\[
\begin{align*}
\hat{w}(t, x) \leq d\int_{-\xi(t)}^{\xi(t)} P(x - y) \hat{w}(t, y)dy - d\hat{w} + \hat{w}(\alpha - \beta \hat{w}), & \quad t > 0, \quad -\xi(t) < x < \xi(t), \\
\hat{w}(t, \pm \xi(t)) > 0, & \quad t > 0, \\
\hat{w}(0, x) = w(T_1, x) \geq \omega(0, x), & \quad -\xi(0) \leq x \leq \xi(0).
\end{align*}
\]
It follows from a comparison consideration that \( \hat{w}(t, x) \geq \omega(t, x) \) for \( t \geq 0 \) and \( x \in (-\xi(t), \xi(t)) \). Moreover, direct calculations show that \( \lim_{t \to \infty} \omega(t, x) = K_\varepsilon \) uniformly in \( [-t[\ln(t + 1)]^{\omega_1}, t[\ln(t + 1)]^{\omega_2}] \). By the arbitrariness of \( \varepsilon \), we immediately obtain the desired result.

Now we are ready to prove (4.6). We first claim that when \( K_2 \) is large sufficiently, for \( x \in [-\xi(t), -\xi(t) + (t + K_2)^{1/2}] \cup \xi(t) - (t + K_2)^{1/2}, \xi(t)] \),
\[
\int_{-\xi(t)}^{\xi(t)} P(x - y) \omega(t, y)dy \geq \frac{K_\varepsilon C_1 \ln(t + K_2)}{2(t + K_2)^{1/2}}.
\]
For \( x \in [\xi(t) - 3(t + K_2)^{1/2}/4, \xi(t)] \), we have that if \( K_2 \) is large enough,
\[
\int_{-\xi(t)}^{\xi(t)} P(x - y) \omega(t, y)dy \geq K_\varepsilon \int_{\xi(t) - (t + K_2)^{1/2}}^{\xi(t)} P(x - y) \frac{\xi(t) - y}{(t + K_2)^{1/2}}dy
\]
For \( x \in [\xi(t) - 3(t + K_2)^{1/2}/4, \xi(t)] \), we have that if \( K_2 \) is large enough,
For \( x \in [\underline{s}(t) - (t + K_2)^{1/2}, \underline{s}(t) - 3(t + K_2)^{1/2}/4] \), we similarly get
\[
\int_{-\underline{s}(t)}^{\underline{s}(t)} P(x - y)\underline{w}(t, y)dy \geq \frac{K_\varepsilon}{(t + K_2)^{1/2}} \int_{(t + K_2)^{1/4}/4}^{(t + K_2)^{1/4}/4} P(y) (\underline{s}(t) - x - y) dy
\]
\[
\geq \frac{K_\varepsilon C_1}{(t + K_2)^{1/2}} \int_{(t + K_2)^{1/4}/4}^{(t + K_2)^{1/4}/4} y^{-1} dy = \frac{K_\varepsilon C_1 \ln(t + K_2)}{4(t + K_2)^{1/2}}.
\]
By the symmetry of \( P \) and \( w \) about \( x \), our claim is proved.

In view of Lemma 6.5 and (2.11), by choosing \( \varepsilon \) small and \( K_2 \) large enough we have for \( x \in [-\underline{s}(t), -\underline{s}(t) + (t + K_2)^{1/2}] \cup [\underline{s}(t) - (t + K_2)^{1/2}, \underline{s}(t)] \),
\[
d \int_{-\underline{s}(t)}^{\underline{s}(t)} P(x - y)\underline{w}(t, y)dy - d\underline{w} + \underline{w}(\alpha - \beta \underline{w})
\]
\[
\geq (d - \frac{\alpha \varepsilon}{4}) \int_{-\underline{s}(t)}^{\underline{s}(t)} P(x - y)\underline{w}(t, y)dy - d\underline{w} + \frac{\alpha \varepsilon}{2} \underline{w} + \frac{\alpha \varepsilon}{4} \int_{-\underline{s}(t)}^{\underline{s}(t)} P(x - y)\underline{w}(t, y)dy
\]
\[
\geq \left[(d - \frac{\alpha \varepsilon}{4})(1 - \varepsilon^2) - d + \frac{\alpha \varepsilon}{2}\right] \underline{w} + \frac{\alpha \varepsilon}{4} \int_{-\underline{s}(t)}^{\underline{s}(t)} P(x - y)\underline{w}(t, y)dy
\]
\[
\geq \frac{\alpha \varepsilon}{4} \int_{-\underline{s}(t)}^{\underline{s}(t)} P(x - y)\underline{w}(t, y)dy \geq \frac{K_\varepsilon \alpha \varepsilon C_1 \ln(t + K_2)}{16(t + K_2)^{1/2}}.
\]
On the other hand, \( x \in (-\underline{s}(t), -\underline{s}(t) + (t + K_2)^{1/2}) \cup (\underline{s}(t) - (t + K_2)^{1/2}, \underline{s}(t)) \),
\[
\underline{w}_t \leq \frac{\ln(t + K_2)}{(t + K_2)^{1/2}} [K_1 K_\varepsilon + K_1 \omega K_\varepsilon] \leq \frac{K_\varepsilon \alpha \varepsilon C_1 \ln(t + K_2)}{16(t + K_2)^{1/2}}
\]
provided that \( K_1 \leq \frac{\tilde{C}_3 \alpha \varepsilon}{16(1 + \omega)} \). For \( x \in (-\underline{s}(t) + (t + K_2)^{1/2}, \underline{s}(t) - (t + K_2)^{1/2}) \), we have
\[
d \int_{-\underline{s}(t)}^{\underline{s}(t)} P(x - y)\underline{w}(t, y)dy - d\underline{w} + \underline{w}(\alpha - \beta \underline{w}) \geq \frac{\alpha \varepsilon}{4} \int_{-\underline{s}(t)}^{\underline{s}(t)} P(x - y)\underline{w}(t, y)dy \geq 0 = \underline{w}_t.
\]
So we have proved the first inequality of (14.6). Obviously, the second identity holds true. For such \( K_1, K_2 \) and \( \varepsilon \) as chosen above, by conclusion (1) there is \( T_0 > 0 \) such that \( w(T, x) \geq \alpha/\beta - \varepsilon \geq w(0, x) \) for \( T \geq T_0 \) and \( x \in [-K_1 K_2 \ln K_2]^{\omega}, K_1 K_2 \ln K_2]^{\omega}] \). This completes the proof of conclusion (5). Consequently, we finish the proof of all assertions. \(\square\)

With the help of Lemma 4.4, it is not hard to prove the following theorem by arguing as in the proof of Theorem 2.4. The details are omitted here.

**Theorem 4.5.** If \( h_\infty - g_\infty = \infty \), then we have the following conclusions:

1. \( \lim_{t \to \infty} u(t, x) = u^* \) and \( \lim_{t \to \infty} v(t, x) = v^* \) locally uniformly in \( \mathbb{R} \).

2. If \( J_1, J_2 \) satisfy (J2) and (J1) respectively, for \( c_\ast \) and \( c_0 \) defined as in Theorem 2.4 we get
\[
\lim_{t \to \infty} \max_{\{x \in [-ct, ct]\}} |u(t, x) - u^*| + |v(t, x) - v^*| = 0 \quad \text{for any} \quad c \in (0, \min\{c_\ast, c_0\})
\]
(3) If $J_1$ violates (J2) and $J_2$ satisfies (J1), then
\[
\lim_{t \to \infty} \max_{x \in [-c_t, c_t]} \{|u(t, x) - u^*| + |v(t, x) - v^*|\} = 0 \text{ for any } c \in (0, c_0).
\]

(4) If (J2) is true for $J_1$ and $J_2$ violates (J1), then
\[
\lim_{t \to \infty} \max_{x \in [-c_t, c_t]} \{|u(t, x) - u^*| + |v(t, x) - v^*|\} = 0 \text{ for any } c \in (0, c_*)\).
\]

(5) If $J_1$ violates (J2) and $J_2$ does not satisfy (J1), then
\[
\lim_{t \to \infty} \max_{x \in [-c_t, c_t]} \{|u(t, x) - u^*| + |v(t, x) - v^*|\} = 0 \text{ for any } c > 0.
\]

(6) If there is $C > 0$ such that $\min\{J_1(x), J_2(x)\} \geq C|x|^{-\gamma}$ for $|x| \gg 1$ and $\gamma \in (1, 2)$, then
\[
\lim_{t \to \infty} \max_{x \in [-\theta_1, \theta_2]} \{|u(t, x) - u^*| + |v(t, x) - v^*|\} = 0 \text{ for any } 1 < \theta_1, \theta_2 < 1/(\gamma - 1).
\]

(7) If there is $C > 0$ such that $\min\{J_1(x), J_2(x)\} \geq C|x|^{-2}$ for $|x| \gg 1$, then
\[
\lim_{t \to \infty} \max_{x \in [-t|\ln(1 + t)|^{-1}, t|\ln(1 + t)|^{-2}]} \{|u(t, x) - u^*| + |v(t, x) - v^*|\} = 0 \text{ for any } 0 < \omega_1, \omega_2 < 1.
\]

### 4.2 Criteria for spreading and vanishing

Some criteria for spreading and vanishing of (4.1) will be given in this subsection. To this end, a comparison principle is first shown. We omit the proofs of them since they can be proved by employing the similar methods as in the previous arguments.

**Lemma 4.6** (Comparison principle). Assume that $\bar{u}, \bar{u}_t \in C([0, T] \times \mathbb{R})$, $\bar{u} \in L^\infty([0, T] \times \mathbb{R})$, $\bar{v}, \bar{v}_t \in C([0, T] \times [\bar{g}(t), \bar{h}(t)])$ and $\bar{g}(t), \bar{h}(t) \in C^1([0, T])$. If $(\bar{u}, \bar{v}, \bar{g}, \bar{h})$ satisfies

\[
\begin{cases}
\bar{u}_t \geq d_1 \int_{-\infty}^{\bar{g}(t)} J_1(x-y)\bar{u}(t, y)dy - d_1 \bar{u} + f_1(\bar{u}, \bar{v}), & t \in (0, T], \ x \in \mathbb{R},
\\
\bar{v}_t \geq d_2 \int_{\bar{g}(t)}^{\bar{h}(t)} J_2(x-y)\bar{v}(t, y)dy - d_2 \bar{v} + f_2(\bar{u}, \bar{v}), & t \in (0, T], \ x \in (\bar{g}(t), \bar{h}(t)),
\\
\bar{v}(t, x) \geq 0, & t \in (0, T], \ x \notin (\bar{g}(t), \bar{h}(t)),
\\
\bar{g}'(t) \leq -\mu \int_{\bar{g}(t)}^{\bar{h}(t)} \int_{-\infty}^{\bar{g}(t)} J_2(x-y)\bar{v}(t, x)dxdy, & t \in (0, T],
\\
\bar{h}'(t) \geq \mu \int_{\bar{g}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\bar{g}(t)} J_2(x-y)\bar{v}(t, x)dxdy, & t \in (0, T],
\\
\bar{u}(0, x) \geq u_0(x), \ x \in \mathbb{R}; \quad \bar{h}(0) \geq h_0 > 0; \quad \bar{v}(0, x) \geq v_0(x), \ x \in [-\bar{h}(0), \bar{h}(0)],
\end{cases}
\]

then

\[
\bar{h}(t) \geq h(t), \quad \bar{g}(t) \leq g(t), \quad t \in [0, T]; \quad \bar{u}(t, x) \geq u(t, x) \quad \bar{v}(t, x) \geq v(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}.
\]

The above comparison principle implies that solution $(u, v, g, h)$ of (4.1) is increasing in $\mu$. By Proposition 3.4, there is a unique $\hat{\ell} > 0$ such that $\lambda_p(\hat{L}(l_1, l_2) + r_2) = 0$ if $l_2 - l_1 = \hat{\ell}$ and $\lambda_p(\hat{L}(l_1, l_2) + r_2)(l_2 - l_1 - \hat{\ell}) > 0$ if $l_2 - l_1 \neq \hat{\ell}$.
Theorem 4.7. Let \((u, v, g, h)\) be a solution of \((4.1)\). Then the following conclusions hold true:

1. If \(r_2 \geq d_2\), then spreading happens.
2. If \(r_2 < d_2\) and \(2h_0 \geq \ell\), then spreading happens. Moreover, if \(h_\infty - g_\infty < \infty\), \(h_\infty - g_\infty \leq \tilde{\ell}\).
3. If \(r_2 < d_2\) and \(2h_0 < \ell\), then there exists a unique \(\tilde{\mu} > 0\) such that spreading happens when \(\mu > \tilde{\mu}\), and vanishing occurs when \(0 < \mu \leq \tilde{\mu}\).

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