Discrete coherent states for higher Landau levels

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Abstract

We consider the quantum dynamics of a charged particle evolving under the action of a constant homogeneous magnetic field, with emphasis on the discrete subgroups of the Heisenberg group (in the Euclidean case) and of the \( SL(2, \mathbb{R}) \) group (in the Hyperbolic case). We prove completeness properties for discrete coherent states associated with higher order Euclidean and hyperbolic Landau levels, extending classic results of Perelomov and of Bargmann, Butera, Girardello and Klauder. In the Euclidean case, our results follow from identifying the completeness problem with known results from the theory of Gabor frames. The results for the hyperbolic setting follow by using a combination of methods from coherent states, time-scale analysis and the theory of Fuchsian groups and their associated automorphic forms.

Key words: Coherent states, Landau Levels, Quantization, Weyl-Heisenberg group, Affine group, discrete groups.
1 Introduction

In this paper we consider the quantum dynamics of a charged particle evolving under the action of a constant homogeneous magnetic field, first in the Euclidean and then in the hyperbolic setting. The goal is to construct discrete coherent states associated with the evolution of the particle when higher Landau levels are formed and to obtain conditions on the completeness of such coherent states. This extends well known results of Perelomov [27] and of Bargmann, Butera, Girardello and Klauder [9]. In the first part of the paper, we consider a constant magnetic field acting on the Euclidean space realized as the complex plane \( \mathbb{C} \), leading to the formation of a discrete spectrum known as the Euclidean Landau Levels. In the second part of the paper, we let a constant magnetic field act on the open hyperbolic plane realized as the Poincaré upper half-plane \( \mathbb{C}^+ = \{ z \in \mathbb{C}, \text{Im}z > 0 \} \), leading to the formation of a mixed spectrum, with a discrete part corresponding to bound states (hyperbolic Landau levels) and a continuous part corresponding to scattering states.

The concept of a set of states on a lattice in phase space was first considered by von Neumann in the Euclidean case. It became physically very attractive because they contain the fundamental commutation relations of quantum mechanics. Indeed, lattices have an underlying unit cell (fundamental domain) related to the size of the Plank constant.

![Fig. 1. An Euclidean lattice and a fundamental domain. See section 2.3.](image)

The situation in the first Landau Level has been clarified in [9] and [27] because it can be related to the structure of zeros of analytic functions, where classical methods from complex analysis can be used. However, in higher Landau Levels, even the case of the Euclidean Landau levels is not yet fully understood. In both the Euclidean and Hyperbolic setting, one has to deal with spaces of polyanalytic functions [1], [21], [3]. Since polyanalytic functions have a much more complicated structure of zeros [8], several essential tools from complex analysis cannot be applied. However, in recent years, important
progress has been made by combining analytic function theory with methods from time-frequency analysis [19], [1], [11]. In the first part of this paper we will translate these results from time-frequency analysis to the setting of coherent states attached to higher Landau Levels. This has a twofold purpose: to bring the results to the attention of researchers in physics and mathematical physics and to motivate the results on the hyperbolic setting of the subsequent section, where time-scale (wavelet) theory replaces time-frequency (Gabor) analysis. Indeed, our main object of study in the paper is the quantum dynamics of a charged particle evolving on the open hyperbolic plane under the action of a constant magnetic field. While previous work on this problem has been concerned with the spectral properties of the corresponding Landau hamiltonian [12], [20] and their associated continuous coherent states [7], the investigation of the associated discrete coherent states labeled by discrete subgroups of $SL(2, \mathbb{R})$, which are called Fuchsian groups, seems to have been overlooked. Important examples of Fuchsian groups are provided by the modular group $SL(2, \mathbb{Z})$ and by the congruence groups of order $n$. The geometry is quite rich when compared to the Euclidean case because one can explore much more symmetries in the hyperbolic plane. This is a remarkable instance of the usefulness of analytic number theory in a physical problem. The idea of using Fuchsian groups as a replacement for the Euclidean lattices seems to have first been used by Perelomov, who provides a full analysis of the first hyperbolic Landau level in [28, Chapter 14], where the analysis is done in the disc. In the present paper we make the corresponding analysis for the higher hyperbolic Landau levels. These discrete coherent states are relevant for the understanding of the hyperbolic setting because the nontrivial dynamics is induced by the tesselation of the Poincaré plane by Fuchsian groups. As the unit cell of the model one considers a fundamental domain for the group. For instance, the following set $D$ is a fundamental domain for the modular group $SL(2, \mathbb{Z})$:

$$D = \left\{ z \in \mathbb{C}^+ : |z| \geq 1 \text{ and } |\Re z| \leq \frac{1}{2} \right\}.$$

The next image illustrates the case of the modular group $SL(2, \mathbb{Z})$. The shadowed area represents the above fundamental domain.

Fig. 2. The modular group $SL(2, \mathbb{Z})$. See section 3.5
Let us fix the coherent states terminology to be used along the paper. A functional Hilbert space $\mathcal{H}$ has a system $\{f_g\}$ of coherent states, labelled by elements $g$ of a locally compact group $G$ if:

(i) There is a representation $T : g \to T_g$ of $G$ labelled by unitary operators $T_g$ on $\mathcal{H}$

(ii) There is a vector $f_0 \in \mathcal{H}$ such that for $f_g = T_g [f_0]$ and for arbitrary $f \in \mathcal{H}$ we have:

$$\langle f, f \rangle_{\mathcal{H}} = \int_G |\langle f, f_g \rangle|^2 \, d\nu(g),$$

where $d\nu$ stands for the left Haar measure of $G$.

The core of the paper is organized in two sections and an appendix with the more technical proofs. Section 2 deals with Euclidean Landau levels and section 3 with their hyperbolic analogues. In each of the sections, after providing some background on the mathematical and physical model, we first construct the coherent states associated with the higher levels and then investigate their discrete counterparts. We finish with a short conclusion including some remarks about the theoretical methodology, highlighting the interaction between physical and signal analysis which has made possible the investigations carried out in this paper.

2 Euclidean Landau levels

2.1 Definitions

The Hamiltonian operator describing the dynamics of a particle of charge $e$ and mass $m_\star$ on the Euclidean $xy$-plane, while interacting with a perpendicular constant homogeneous magnetic field, is given by the operator

$$H := \frac{1}{2m_\star} \left( i \hbar \nabla - \frac{e}{c} A \right) \left( i \hbar \nabla - \frac{e}{c} A \right)^2,$$

(2.1)

where $\hbar$ denotes Planck’s constant, $c$ is the light velocity and $i$ the imaginary unit. Denote by $B > 0$ the strength of the magnetic field and select the symmetric gauge

$$A = -\frac{r}{2} \times B = \left( -\frac{B}{2} y, \frac{B}{2} x \right),$$
where \( r = (x, y) \in \mathbb{R}^2 \). For simplicity, we set \( m_* = e = c = \hbar = 1 \) in (2.1), leading to the Landau Hamiltonian

\[
H_B^L := \frac{1}{2} \left( \left( i \frac{\partial}{\partial x} - B y \right)^2 + \left( i \frac{\partial}{\partial y} + B x \right)^2 \right)
\]

acting on the Hilbert space \( L^2(\mathbb{R}^2, dx dy) \). The spectrum of the Hamiltonian \( H_B^L \) consists of infinite number of eigenvalues with infinite multiplicity of the form

\[
\epsilon_n^B = \left( n + \frac{1}{2} \right) B, \quad n = 0, 1, 2, \ldots
\]

These eigenvalues are called \textit{Euclidean Landau levels}. Denote the eigenspace of \( H_B^L \) corresponding to the eigenvalue \( \epsilon_n^B \) in (2.3) by

\[
A_{B,n}(\mathbb{R}^2) = \{ \varphi \in L^2(\mathbb{R}^2, dx dy), H_B^L[\varphi] = \epsilon_n^B \varphi \}.
\]

In the next section we recall a characterization theorem of the eigenspace \( A_{B,n}(\mathbb{R}^2) \) as the range of a suitable coherent states transform of the Hilbert space \( L^2(\mathbb{R}) \), originally obtained in [25].

### 2.2 Coherent states for Euclidean Landau levels

Define the Heisenberg group \( \mathbb{H} \) as the Lie group whose underlying manifold is \( \mathbb{R}^3 \) together with the group operation

\[
(x, y, r)(x', y', r') = \left( x + x', y + y', r + r' + \frac{1}{2} (xy' - x'y) \right).
\]

The continuous unitary irreducible representations of \( \mathbb{H} \) are well known [17]. Here we consider the Schrödinger representation \( T_B \) of \( \mathbb{H} \) on the Hilbert space \( L^2(\mathbb{R}, dt) \) [33] defined as

\[
T_{B,(x,y,t)} [\psi] (t) = \exp \left( i \left( Bt - \sqrt{B} y \xi + \frac{B}{2} xy \right) \right) \psi \left( t - \sqrt{B} x \right)
\]

for \((x, y, r) \in \mathbb{H}, B > 0, \psi \in L^2(\mathbb{R}, dt) \) and \( t \in \mathbb{R} \). This representation is square integrable modulo the center \( \mathbb{R} \) of \( \mathbb{H} \) and the Borel section \( \sigma_0 \) of \( \mathbb{H} \) over \( \mathbb{H} / \mathbb{R} = \mathbb{R}^2 \) which is given by \( \sigma_0(x, y) = (x, y, 0) \). Further, the following identity holds

\[
\int_{\mathbb{R}^2} \langle \psi_1, T_{B,\sigma_0(x,y)} [\phi_1] \rangle \langle T_{B,\sigma_0(x,y)} [\phi_2], \psi_2 \rangle d\mu(x, y) = \langle \psi_1, \psi_2 \rangle \langle \phi_1, \phi_2 \rangle
\]

for all \( \psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}) \). Displacing the reference state

\[
\langle t \mid 0 \rangle_{B,n} = (\sqrt{\pi} 2^n n!)^{-\frac{1}{2}} e^{-\frac{t^2}{4}} H_n(t), \quad t \in \mathbb{R},
\]
where $H_n(\cdot)$ is the Hermite polynomial

$$H_n(t) = \sum_{k=0}^{[n/2]} \frac{n! (-1)^k (2t)^{n-2k}}{k! (n-2k)!},$$

via the representation operator $T_{B,\sigma_0(x,y)}$, one obtains a set of coherent states denoted by the kets vectors $| (x, y), B, n \rangle$, with wave functions

$$\langle t | (x, y), B, n \rangle = (\sqrt{\pi} 2^n n!)^{-\frac{1}{2}} \exp \left(-i \sqrt{B} ty + \frac{B}{2} xy - \frac{1}{2} \left(t - \sqrt{B} x\right)^2\right) H_n \left(t - \sqrt{B} x\right).$$

The following resolution of the identity

$$1_{L^2(\mathbb{R})} = \int_{\mathbb{R}^2} | (x, y), B, n \rangle \langle (x, y), B, n | \; d\mu (x, y)$$

holds as a consequence of (2.5). Thus the construction of coherent states is justified by the square integrability of representation $T_B$ modulo the subgroup $\mathbb{R}$ and the section $\sigma_0$. For $n = 0$ (the lowest Euclidean Landau level), the states $| (x, y), B, 0 \rangle$ coincide with the canonical coherent states of the harmonic oscillator. The coherent states (2.6) are associated with the coherent state transform

$$V_{B,n} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2, dxdy)$$

such that, given $\varphi \in L^2(\mathbb{R})$,

$$V_{B,n} [\varphi] (x, y) := \int_{\mathbb{R}} <t | (x, y), B, n \rangle \varphi(t) \; dt.$$

Thanks to the square integrability of $T_B$, the transform $V_{B,n}$ is an isometrical map whose range is exactly the eigenspace in (2.4):

$$V_{B,n} \left[ L^2(\mathbb{R}) \right] = \mathcal{A}_{B,n} (\mathbb{R}^2).$$

Another realization of this eigenspace can be obtained by intertwining the Landau Hamiltonian (2.1.3) as follows

$$\Delta_B := e^{\frac{B^2}{2} \bar{z} z} \left(\frac{1}{2} H^L_{2B} - \frac{B}{2}\right) e^{-\frac{B^2}{2} \bar{z} z} = -\frac{\partial^2}{\partial z \partial \bar{z}} + B \bar{z} \frac{\partial}{\partial \bar{z}}.$$

The space $\mathcal{A}_{B,n} (\mathbb{R}^2)$ then becomes

$$\mathcal{A}_{B,n} (\mathbb{C}) := \left\{ \varphi \in L^2(\mathbb{C}, e^{-B^2 \bar{z} z} d\mu) , \Delta_B \varphi = n B \varphi \right\}. \quad (2.7)$$

The following functions form an orthogonal basis for $\mathcal{A}_{B,n} (\mathbb{C})$ [21]:

$$\begin{align*}
e_{i,n}^1(z) &= \sqrt{\frac{n!}{(n-i)!}} B^{\frac{i-1}{2}} z^i L_n^{(i)}(B \bar{z} z), \quad 0 \leq i, \\
e_{j,n}^2(z) &= \sqrt{\frac{j!}{(j+n)!}} B^{\frac{n-j}{2}} \bar{z}^j L_n^{(j)}(B \bar{z} z), \quad 0 \leq j, \quad (2.8)
\end{align*}$$
where the Laguerre polynomial is defined as
\[
L_n^{(\alpha)}(t) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{t^k}{k!}, \quad \alpha > -1.
\]

**Remark 1** In his book [28, pag. 35], Perelomov points out that the basis (2.8) had been used by Feynman and Schwinger in a somewhat different form in order to obtain an explicit expression for the matrix elements of the displacement operator. The functions (2.8) are also related to the complex Hermite polynomials. They occur naturally in several problems and different representations are used. For instance, they have recently found applications in quantization [19], [10], [13], time-frequency analysis [1], partial differential equations [18] and planar point processes [21].

If \( B = \pi \) and \( n = 0 \) the space (2.7) is precisely the Fock-Bargmann space of entire square integrable functions with respect to the Gaussian measure on \( \mathbb{C} \). For \( n > 0 \), the characterization takes the form
\[
\mathcal{V}_{2\pi,n} \left[ L^2(\mathbb{R}) \right] = A_{\pi,n}(\mathbb{C})
\]
where the coherent state transform is given explicitly by
\[
\mathcal{V}_{2\pi,n} [\varphi] (z) = e^{\frac{1}{2} \pi z^2} \circ V_{2\pi,n} [\varphi] (z) = (-1)^n B_n [\varphi] \left( \sqrt{\pi} z \right)
\]
where
\[
B_n [\varphi] (w) = (-1)^n c_n \int_{\mathbb{R}} \varphi(t) \exp \left( -\frac{1}{2} t^2 + \sqrt{2} tw - \frac{1}{2} w^2 \right) H_n \left( t - \frac{w + \omega}{\sqrt{2}} \right) dt
\]
The transform \( \mathcal{V}_{2\pi,n} \) is precisely the true polyanalytic Bargmann transform and the space \( A^2_{\pi,n}(\mathbb{C}) \) is the true-polyanalytic space of index \( n \), see [35], [1], [21].

### 2.3 Completeness theorem

We want to understand the completeness properties of the coherent states constructed in the previous section once they are labeled by a lattice \( \Lambda \subset \mathbb{C} \). The key observation is the fact that their completeness and basis properties are equivalent to the completeness and basis properties of Gabor systems with Hermite functions [19] and to sampling and uniqueness sets in true-polyanalytic spaces [1]. Consider the lattice
\[
\Lambda = \Lambda(\omega_1, \omega_2) := \{ m_1 \omega_1 + m_2 \omega_2; m_1, m_2 \in \mathbb{Z} \} \subset \mathbb{C}
\]
spanned by the periods \( \omega_1 \) and \( \omega_2 \in \mathbb{C} \) with \( \Im(\omega_1/\omega_2) > 0 \). The size of the lattice \( \Lambda \) is the area of the parallelogram spanned by \( \omega_1 \) and \( \omega_2 \). Identifying \( \mathbb{R}^2 \)
with \( \mathbb{C} \) we can write \( \Lambda = \Omega \mathbb{Z}^2 \), where \( \Omega = [\omega_1, \omega_2] \) is an invertible \( 2 \times 2 \) matrix. The size of the lattice can now be defined as \( s(\Lambda) = |\det \Omega| \). We say that \( \Lambda \) is a set of sampling for the space \( \mathcal{A}_{B,n}(\mathbb{C}) \) if there exist constants \( C_1, C_2 > 0 \) such that for all \( F \in \mathcal{A}_{B,n}(\mathbb{C}) \),

\[
C_1 \| F \|_{\mathcal{A}_{B,n}(\mathbb{C})}^2 \leq \sum_{\lambda \in \Lambda} |F(\lambda)|^2 e^{-B|\lambda|^2} \leq C_2 \| F \|_{\mathcal{A}_{B,n}(\mathbb{C})}^2 .
\]

Given a point \((q, p)\) in the phase space \( \mathbb{R}^2 \), the corresponding time-frequency shift is

\[
\pi_{(q, p)} [f](t) = e^{2\pi i pt} f(t - q), t \in \mathbb{R}.
\]

Let \( h_n(t) \) denote a Hermite function. The set \( G(h_n, \Lambda) := \{ \pi_{(q,p)} h_n, (q,p) \in \mathbb{R} \} \) is a Gabor frame or a Weyl-Heisenberg frame in \( L^2(\mathbb{R}) \) whenever there exist constants \( C_1, C_2 > 0 \) such that

\[
C_1 \| f \|_{L^2(\mathbb{R})}^2 \leq \sum_{(q,p) \in \Lambda} \left| \left\langle f, \pi_{(q,p)} h_n \right\rangle_{L^2(\mathbb{R})} \right|^2 \leq C_2 \| f \|_{L^2(\mathbb{R})}^2 .
\]

It follows from the lower inequality that if \( G(h_n, \Lambda) \) is a frame then \( G(h_n, \Lambda) \) is complete. For simplicity, we consider the square lattice \( \Lambda_\omega := \omega (\mathbb{Z}+i\mathbb{Z}) \), \( \omega \in \mathbb{R} \). In this case \( s(\Lambda_\omega) = \omega^2 \). For \( B = \pi \), it was proved that the lattice \( \Lambda_\omega \) is a set of sampling for \( \mathcal{A}_{\pi,n}(\mathbb{C}) \) if and only if \( G(h_n, \Lambda_\omega) \) is a Gabor frame, see [1]. The following result is a consequence of combining this identification with relatively recent results from time-frequency analysis.

**Theorem 1.** Let \( (| (x,y), \pi, n >)_{(x,y) \in \mathbb{R}^2} \) be a system of coherent states attached to the nth Landau level defined in (2.6). Then, the following holds:

(i) If \( \omega^2 < \frac{1}{n+1} \) then the system \( (| (x,y), \pi, n >)_{(x,y) \in \Lambda_\omega} \) is complete.

(ii) If \( \omega^2 > 1 \) then the system \( (| (x,y), \pi, n >)_{(x,y) \in \Lambda_\omega} \) is incomplete.

**Proof.** The completeness property (i) follows from the fact that if \( \omega^2 < \frac{1}{n+1} \), then \( G(h_n, \Lambda) \) is a Gabor frame [19], therefore complete. The property (ii) is a consequence of the fact that, if \( \omega^2 > 1 \), then a Gabor system cannot be complete [20].

**Remark 3.1.** In the case \( n = 0 \) it is a classical result [27], [9] that the systems are complete if \( \omega^2 \leq 1 \) and incomplete if \( \omega^2 > 1 \). The above result is an extension of these results to coherent states attached to higher Euclidean Landau levels \( \epsilon_n^\pi, n = 1, 2, 3, ... \).
3 Hyperbolic Landau levels

3.1 Hyperbolic Landau levels

In the hyperbolic setting, the configuration space is now the Poincaré upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C}, \Im z > 0\}$. It is a complete two-dimensional simply connected Riemannian manifold of constant negative curvature $R = -1$, endowed with the metric $ds^2 = y^{-2} (dx^2 + dy^2)$, where $z = x + iy$. A constant homogeneous magnetic field on $\mathbb{C}^+$ is given by a 2-form $d\mu_B$ defined as

$$d\mu_B = \frac{2B}{y^2} dx dy$$

where $B$ is the field intensity. The form $d\mu_B$ is exact and any 1-form $A$ such that $d\mu_B = dA$ is called a vector potential related to $d\mu_B$. For our purposes it is convenient to choose $A = 2By^{-1}dx$. In suitable units and up to an additive constant, the Schrödinger operator describing the dynamics of a charged particle moving on $\mathbb{C}^+$ under the action of the magnetic field $B$ is given by

$$H_B := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2iBy \frac{\partial}{\partial x}$$

Different aspects of the spectral analysis of the operator $H_B$ have been studied by many authors, (see [20], [12] or, for a more mathematical approach, [26]). We list here the following important properties.

(i) $H_B$ is an elliptic densely defined operator on the Hilbert space $L^2(\mathbb{C}^+, d\mu_B)$, with a unique self-adjoint realization that we denote also by $H_B$.

(ii) The spectrum of $H_B$ in $L^2(\mathbb{C}^+, d\mu_B)$ consists of two parts: a continuous part $[1/4, +\infty[$, corresponding to scattering states and a finite number of eigenvalues with infinite degeneracy (hyperbolic Landau levels) of the form

$$\epsilon_n^B := (B - n) (1 - B + n), \quad n = 0, 1, 2, \ldots, \lfloor B - \frac{1}{2} \rfloor.$$  \hfill (3.1)

The finite part of the spectrum exists provided $2B > 1$. The notation $\lfloor a \rfloor$ stands for the greatest integer not exceeding $a$.

(iii) For each fixed eigenvalue $\epsilon_n^B$, we denote by

$$\mathcal{E}_n^B(\mathbb{C}^+) = \{ \Phi \in L^2(\mathbb{C}^+, d\mu_B), H_B\Phi = \epsilon_n^B \Phi \}$$  \hfill (3.2)
the corresponding eigenspace. Its reproducing kernel is given by

\[ K_{n,B}(z, \zeta) = \frac{(-1)^n \Gamma(2B-n)}{n! \Gamma(2B-2n)} \left( \frac{|z - \zeta|^2}{4 \text{Im} z \text{Im} \zeta} \right)^{-B+m} \left( \frac{\zeta - z}{z - \zeta} \right)^B \times \, _2F_1 \left( -2B - m, -m, 2B - 2m, \frac{4 \text{Im} z \text{Im} \zeta}{|z - \zeta|^2} \right) \]

where \( _2F_1 \) is the Gauss hypergeometric function.

**Remark 2** The condition \( 2B > 1 \) ensuring the existence of the discrete eigenvalues means that the magnetic field has to be strong enough to capture the particle in a closed orbit. If this condition is not fulfilled the motion will be unbounded and the particle will escape to infinity. More precisely, the orbit of the particle will intercept the upper half-plane boundary whose points stand for points at infinity \([12, \text{pg. 189}]. To the eigenvalues in (3.1) below the continuous spectrum correspond eigenfunctions which are called bound states. This terminology is due to the fact that the particle in such a bound state cannot leave the system without additional energy.

**Remark 3** For \( n = 0 \), the reproducing kernel reduces to

\[ K_{0,B}(z, \zeta) = e^{i\pi B} 4^B \frac{(\text{Im} z \text{Im} \zeta)^B}{(z - \zeta)^{2B}}. \]

This is the reproducing kernel for the \((2B-1)\)-weighted Bergman space \( A_{2B-1}(\mathbb{C}^+) \) constituted of analytic functions \( f \) on the upper half-plane with finite norm

\[ \int_0^{+\infty} \int_0^\infty |f(x + iy)|^2 \frac{dy}{y} 2^{B-1} \frac{dx}{x} < +\infty \]

That is \( \mathcal{E}_0^B(\mathbb{C}^+) \) coincides with \( A_{2B-1}(\mathbb{C}^+) \). Note also that for a general weight \( \nu \), the Bergman space \( A_\nu(\mathbb{C}^+) \) is also connected to the classical Hardy space \( \mathcal{H}_1^+ \) of analytic functions \( h \) on \( \mathbb{C}^+ \) with finite norm \( \sup_{0<y<\infty} \int_0^\infty |h(x + iy)|^2 \, dx \) by the integral transform

\[ \text{Ber}_\nu : \mathcal{H}_1^+ \rightarrow A_\nu(\mathbb{C}^+) \]

defined as

\[ \text{Ber}_\nu [h](z) = \int_0^{+\infty} t^{\frac{1}{2} \nu + \frac{1}{2}} h(t) e^{itz} \, dt \quad (3.3) \]

see, for instance [2].
3.2 The affine group acting on the Poincaré half-plane

For our purposes we will recall a characterization theorem of $\mathcal{E}_n^B (\mathbb{C}^+) \subseteq L^2 (\mathbb{R}^+, t^{-1}dt)$. We start with the identification of the Poincaré upper half-plane $\mathbb{C}^+$ with the affine group $G = \mathbb{R} \times \mathbb{R}^+$, by setting $z = x + iy \equiv (x, y)$. The group law of $G$ is $(x, y) \cdot (x', y') = (x + yx', yy')$. $G$ is a locally compact unimodular group with the left Haar measure $d\mu (x, y) = y^{-2} dxdy$ and modular function $\Delta (x, y) = y^{-1}$. By this identification the space $L^2 (G, d\mu)$ coincides with the space $L^2 (\mathbb{C}^+, d\mu_B)$. We shall consider one of the two inequivalent infinite dimensional unitary irreducible representations of the affine group $G$, denoted $\pi_+$, realized on the Hilbert space $H := L^2 (\mathbb{R}^+, t^{-1}dt)$. The representation is square integrable since it is easy to find a vector $\varphi_0 \in H$ such that the function $(x, y) \mapsto \langle \pi_+ (x, y) \varphi_0, \varphi_0 \rangle$ belongs to $L^2 (G, d\mu)$. This condition can also be expressed by saying that the self-adjoint operator $K : H \to H$ defined as $K [\varphi] (t) = t^{-1/2} \varphi (t)$ gives

$$\int_G d\mu (x, y) \langle \varphi_1, \pi_+ (x, y) \psi_1 \rangle \langle \pi_+ (x, y) \psi_2, \varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle \left\langle K^{1/2} [\varphi_1], K^{1/2} [\varphi_2] \right\rangle,$$

for all $\psi_1, \psi_2, \varphi_1, \varphi_2 \in H$. The operator $K$ is unbounded because $G$ is not unimodular [15]. We will also use the notation

$$\pi_+^1 (x, y) \varphi (t) := y^{1/2} \exp \left( \frac{1}{2} ixt \right) \varphi (yt), \quad \varphi \in H, \quad t \in \mathbb{R}^+$$

such that

$$\pi_+ (x, y) \left( \cdot \right)^{1/2} \varphi (\cdot) (t) = t^{1/2} \pi_+^1 (x, y) \left( \varphi (\cdot) \right) (t) \quad (3.4)$$

3.3 Coherent states for higher hyperbolic Landau levels

Now, as in [24], we consider a set of coherent states denoted by the ket vectors $| (x, y), B, n >$ and obtained by displacing, via the representation operator $\pi_+ (x, y)$, the reference state vector $| 0 >_{B, n}$ in the Hilbert space $H$ with wave function given by

$$< t | 0 >_{B, n} = \left( \frac{\Gamma (2B - n)}{n!} \right)^{-1/2} t^{B-n} e^{-\frac{1}{2}t} L_n^{(2B-2n-1)} (t).$$

Precisely,

$$| (x, y), B, n > := \pi_+ (x, y) | 0 >_{B, n}. \quad (3.5)$$
The wave functions of the coherent states \(| (x,y), B, n >\) are given by

\[
<t | (x,y), B, n > = \left( \frac{\Gamma (2B - n)}{n!} \right)^{-\frac{1}{2}} (ty)^{B-n} e^{-\frac{1}{2}t(y-ix)} L_n^{(2B-2n-1)} (ty).
\]

These coherent states are completely justified by the square integrability of the unitary irreducible representation \(\pi_+\) and we have a resolution of the identity for the space \(\mathcal{H} = L^2 (\mathbb{R}^+, t^{-1}dt)\)

\[
1_{\mathcal{H}} = c_{B,n}^{-1} \int_G d\mu (x,y) | (x,y), B, n > < (x,y), B, n | ,
\]

where \(c_{B,n} = (2(B - n) - 1)^{-1}\). The coherent states \((3.5)\) are associated with the coherent state transform

\[
W_{B,n} [\varphi] (x,y) = c_{B,n}^{-\frac{1}{2}} \int_{\mathbb{R}^+} < t | (x,y), B, n > \varphi (t) \frac{dt}{t}.
\]

The range of the map \(W_{B,n} : L^2 (\mathbb{R}^+, t^{-1}dt) \rightarrow L^2 (\mathbb{C}^+, d\mu_B)\) is the eigenspace \((3.2)\):

\[
W_{B,n} L^2 (\mathbb{R}^+, t^{-1}dt) = E^B_n (\mathbb{C}^+)
\]

for every \(n \in \mathbb{Z}_+ \cap \left[0, B - \frac{1}{2}\right]\) provided that \(2B > 1\).

Remark 4 Note that for \(n = 0\), the lowest hyperbolic Landau level, the states \(| (x,y), B, 0 >\) coincide with the well known affine coherent states [23].

3.4 Wavelet transforms with Laguerre functions

In this subsection we write the coherent states of the previous section in terms of wavelet transforms with analyzing wavelets \(\Phi^\alpha_n\) defined via the Fourier transforms in terms of Laguerre polynomials \(L^\alpha_n(.)\) as

\[
\mathcal{F} \Phi^\alpha_n (t) = t^{\frac{\alpha+1}{2}} e^{-t} L^\alpha_n (2t).
\]

Some of the structural properties of \(\Phi^\alpha_n\) that will be key in our approach are a consequence of their explicit formula, which displays \(\Phi^\alpha_n\) as linear combinations of \(\{\Phi_n^{\alpha+2k}\}_{k=0}^\infty\):

\[
\Phi^\alpha_n (t) = \sum_{k=0}^n \frac{(-2)^k}{k!} \binom{n + \alpha}{n - k} \Phi_0^{\alpha+2k} (t).
\]
Now, let \( \varphi \in L^2(\mathbb{R}^+, t^{-1}dt) \). Combining (3.7) and (3.6) gives

\[
W_{B,n} [\varphi](x, y) = \frac{c_{B,n}}{n!} \left( \frac{\Gamma (2B-n)}{n!} \right)^{-\frac{1}{2}} \int_{\mathbb{R}^+} (ty)^{B-n} e^{-\frac{1}{2}t(y+ix)} L_n^{(2B-2n-1)}(ty) \varphi(t) \frac{dt}{t}.
\]

With \( z = x + iy \), we have that \( -\frac{1}{2}t(y + ix) = \frac{t}{2} \xi iz \). Set \( \gamma_{B,n} = c_{B,n} (n!)^{-1} \Gamma (2B-n) \) and rewrite the above as

\[
W_{B,n} [\varphi](x, y) = \gamma_{B,n}^{-\frac{1}{2}} \int_{\mathbb{R}^+} \varphi(t) \left( (ty)^{B-n} e^{\frac{1}{2}t \xi iz} L_n^{(2B-2n-1)}(ty) \right) \frac{dt}{t} \tag{3.9}
\]

Since \( \pi_+ (x, y) \left( (\cdot)^{\frac{1}{2}} I_n^{(2B-2n-1)}(\cdot) \right) (t) = (ty)^{B-n} e^{\frac{1}{2}t \xi iz} L_n^{(2B-2n-1)}(ty) \), then (3.9) becomes

\[
W_{B,n} [\varphi](x, y) = \gamma_{B,n}^{-\frac{1}{2}} \int_{\mathbb{R}^+} \varphi(t) \left( \pi_+ (x, y) \left( (\cdot)^{\frac{1}{2}} I_n^{(2B-2n-1)}(\cdot) \right) \right) \frac{dt}{t} \tag{3.10}
\]

Since \( \pi_+ \left( (\cdot)^{\frac{1}{2}} \phi(\cdot) \right) (t) = t^{\frac{1}{2}} \pi_+ \left[ \phi(\cdot) \right] (t) \), then (3.10) becomes

\[
W_{B,n} [\varphi](x, y) = \gamma_{B,n}^{-\frac{1}{2}} \left( (\cdot)^{-\frac{1}{2}} \varphi, \pi_+^1 (z) \left[ I_n^{(2B-2n-1)}(\cdot) \right] \right)_{L^2(\mathbb{R}^+, dt)}. \tag{3.11}
\]

If \( \varphi \in L^2(\mathbb{R}^+, t^{-1}dt) \), then \( \mathfrak{S}^{-1} \left( t^{-\frac{1}{2}} \varphi \right) \in \mathfrak{S}^+ \) and the scalar product above may also be written as

\[
\langle t^{-\frac{1}{2}} \varphi, \pi_+^1 (z) \left[ I_n^{(2B-2n-1)}(\cdot) \right] \rangle_{L^2(\mathbb{R}^+, dt)} = \mathcal{W}_{\Phi_2^{2(\beta-n)-1}} \left[ \mathfrak{S}^{-1} \left( (\cdot)^{-\frac{1}{2}} \varphi \right) \right](z),
\]

where \( \mathcal{W}_{\Phi_2^{2(\beta-n)-1}} \) stands for the wavelet transformation \[14\], defined as

\[
\mathcal{W}_{\Phi} [\varphi](x, y) = \langle \varphi, \pi_z \Phi \rangle_{L^2(\mathbb{R})}, \quad z = x + iy, \quad y > 0,
\]

where \( \pi_z \Phi(t) = y^{-\frac{1}{2}} \Phi(y^{-1}(t - x)) \) and \( \mathfrak{S} \Phi \in L^2(\mathbb{R}^+, t^{-1}dt) \). The two transforms are related as follows

\[
W_{B,n} [\varphi](x, y) = \gamma_{B,n}^{-\frac{1}{2}} \mathcal{W}_{\Phi_2^{2(\beta-n)-1}} \left[ \mathfrak{S}^{-1} \left( (\cdot)^{-\frac{1}{2}} \varphi \right) \right](z). \tag{3.11}
\]

**Remark 5** This also means that we have another realization of the bound states space \( \mathcal{E}_n^B(\mathbb{C}^+) \) in (3.3) as the image of the Hardy space \( \mathfrak{S}^+ \) under the wavelet transform \( \mathcal{W}_{\Phi_2^{2(\beta-n)-1}} \).

With the help of the transform \( Ber_{\nu} \) in (3.3) we will be able to express the transform \( W_{B,n} [\varphi] \) of any function \( \varphi \in L^2(\mathbb{R}^+, t^{-1}dt) \) as a combination of derivatives of an analytic function.
Proposition 1 If \( f \in L^2(\mathbb{R}^+, t^{-1} dt) \), then

\[
W_{B,n}[f](z) = \gamma_{B,n}^{-\frac{1}{2}} \sum_{k=0}^{n} \frac{(2i)^k}{k!} \binom{n + \alpha}{n - k} y^{\frac{1}{2}n + k} F^{(k)}(z),
\]

where \( F(z) = \text{Ber}_\nu \left( \delta^{-1} \left( t^{-\frac{1}{2}} f \right) \right) \) belongs to the weighted Bergman space \( A_\nu(\mathbb{C}^+) \).

Proof Take \( f \in L^2(\mathbb{R}^+, t^{-1} dt) \). Then \( u = \delta^{-1} \left( t^{-\frac{1}{2}} f \right) \in \mathcal{F}^+ \). Denote by \( \mathcal{W} := \text{Ber}_\nu \{ u \} \) where \( \nu = 2(B-n) - 1 \). In [2, pg. 256], it is shown that the wavelet transform of \( u \) decomposes in terms of derivatives of the analytic function \( F \in A_{2(B-n)-1}(\mathbb{C}^+) \) as

\[
\mathcal{W}_{2^{B-n-1}} u(z) = \sum_{k=0}^{n} \frac{(2i)^k}{k!} \binom{2B-n-1}{n-k} y^{B-n-\frac{1}{2} + k} F^{(k)}(z). \tag{3.12}
\]

Recalling the relation (3.11) between the two transforms, we may rewrite (3.12) as

\[
W_{B,n} f(x,y) = \gamma_{B,n}^{-\frac{1}{2}} \sum_{k=0}^{n} \frac{(2i)^k}{k!} \binom{2B-n-1}{n-k} y^{B-n-\frac{1}{2} + k} F^{(k)}(z). \tag{3.13}
\]

This completes the proof.

3.5 Fuchsian goups and their automorphic forms

Let \( I_2 \) be the identity matrix. Since one can identify the Poincaré half-plane \( \mathbb{C}^+ \) with the quotient group

\[
PSL(2, \mathbb{R}) := SL(2, \mathbb{R}) / \{ \pm I_2 \},
\]

also known as the group of Möbius transformations, the subgroups of \( PSL(2, \mathbb{R}) \), known as Fuchsian groups, describe the isometries of the hyperbolic metric of \( \mathbb{C}^+ \). Since the nontrivial dynamics of a particle in the upper half-plane is induced by its tessellation by discrete subgroups, we want to understand the completeness properties of the coherent states introduced in the previous section, once they are labelled by Fuchsian groups. Thus, we need to recall some basic facts about Fuchsian groups and their associated automorphic forms. Consider the group \( SL(2, \mathbb{R}) \) of real \( 2 \times 2 \) matrices with determinant one,
acting on \( \mathbb{C}^+ \) according to the rule

\[
g.z = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).
\]

Notice that \( g \) and \(-g\) have the same action on \( \mathbb{C}^+ \).

A Fuchsian group \( G \) is a discrete subgroup of \( PSL(2, \mathbb{R}) \). The most important example is the modular group \( SL(2, \mathbb{Z}) \),

\[
SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.
\]

Any subgroup of \( SL(2, \mathbb{Z}) \) will also be Fuchsian. An important class of subgroups of \( SL(2, \mathbb{Z}) \) is provided by the congruence groups of order \( n \), \( G(n) \),

\[
G(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm I \pmod{n} \right\}.
\]

Further terminology will be required. The \( G \)-orbit \( Gz \) of a point \( z \in \mathbb{C}^+ \) under the action of the group \( G \) is

\[
Gz = \{gz : g \in G\}.
\]

A fundamental domain for a Fuchsian group \( G \) is a closed set \( D \subset \mathbb{C}^+ \) such that \( D \) is the closure of its interior \( D^0 \), no two points of \( D^0 \) lie in the same \( G \)-orbit and the images of \( D \) under \( G \) cover \( \mathbb{C}^+ \). For instance, a fundamental domain for \( G = SL(2, \mathbb{Z}) \) is given by

\[
D = \left\{ z \in \mathbb{C}^+ : |z| \geq 1 \text{ and } |\Re z| \leq \frac{1}{2} \right\}.
\]

In the hyperbolic model, the orbit of an element \( z \in D \) will replace the role of the lattice \( \Lambda(\omega_1, \omega_2) \) in the Euclidean model of the previous section, while the fundamental domain \( D \) replaces the role of the paralelogram spanned by \( \omega_1 \) and \( \omega_2 \). We will restrict to Fuchsian groups such that \( D \) has finite hyperbolic area. In this case \( D \) can be chosen as a polygon with an even number \( 2k \) of sides. The sides, grouped in pairs, are equivalent with respect to the action of \( G \). The vertices of the polygon are joined in cycles of vertices which are equivalent to each other. If the region is a polygon with vertices lying on the boundary of \( \mathbb{C}^+ \), the cycle is called parabolic (often referred to in the literature as cusps), otherwise it is called elliptic. Let \( r \) be the total number of cycles and \( e_1, \ldots, e_r \) be the orders of the inequivalent elliptic points of \( G \). Joining equivalent vertices and cycles, leads to the construction of the
Riemann surface $G \setminus \mathbb{C}^+$, whose genus $\mathcal{G}$ is given by $2\mathcal{G} = 1 + k - r$. The set $(\mathcal{G}, r, e_1, \ldots, e_r)$ is called the signature of the group $G$. It contains information to compute the area $S_G$ of the fundamental domain $D$:

$$S_G = \mathcal{G} - 1 + \frac{1}{2} \sum_{i=1}^{r} \left[ (1 - \frac{1}{e_i}) \right].$$

Now we introduce the notion of an automorphic form associated with $G$. For all $m \in \mathbb{Z}$, $z \in \mathbb{C}^+$ and any function $f$ with domain $\mathbb{C}^+$, let

$$(f \mid_m g)(z) = (cz + d)^{-2m} f(g.z), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

An automorphic form of weight $m$ with respect to a Fuchsian group $G$ is a meromorphic function $f$ on $\mathbb{C}^+$ such that

$$(f \mid_m g) = f,$$

for all $g \in G$. The number $N$ of zeros of $f$ inside the fundamental domain $D$ of the group $G$ is given by Poincaré’s formula

$$N = 2m \frac{S_G}{\pi}.$$

Thus, if $G$ admits an automorphic form with a single zero inside $D$, then $S_G = \frac{\pi}{2m}$ and we can choose the automorphic form of weight $m_0 = \frac{\pi}{2S_G}$. The set of all automorphic forms of weight $m$ is denoted by $\Omega^m_G(\mathbb{C}^+)$. Consider also $\mathfrak{Hol}^m_G(\mathbb{C}^+)$, the set of functions $f \in \Omega^m_G(\mathbb{C}^+)$ holomorphic on $\mathbb{C}^+$ (including all cusps of $G$). We write $\mathcal{C}^m_G(\mathbb{C}^+)$ for the set of functions $f \in \Omega^m_G(\mathbb{C}^+)$ which are zero at all cusps of $G$ (the so-called cusp forms). The inclusions among these spaces are the following:

$$\mathcal{C}^m_G(\mathbb{C}^+) \subset \mathfrak{Hol}^m_G(\mathbb{C}^+) \subset \Omega^m_G(\mathbb{C}^+).$$

The dimension $\dim \mathfrak{Hol}^m_G(\mathbb{C}^+)$ is known explicitly [31, p. 46, Theorem 2.23] in terms of $m$, the genus $\mathcal{G}$ of the Riemann surface $G \setminus \mathbb{C}^+$, the orders of the inequivalent elliptic points of $G$ and the number $s$ of inequivalent cusps of $G$. 

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Then
\[ \dim \mathcal{H}_G^m (\mathbb{C}^+) = \begin{cases} 
(2m - 1) (G - 1) + ms + \sum_{l=1}^r \left[ \frac{2m(e_l - 1)}{2e_l} \right], & m > 1 \\
G + s - 1, & m = 1, s > 0 \\
G, & m = 1, s = 0 \\
1, & m = 0 \\
0, & m < 0 
\end{cases} \] (3.14)

Here \([x]\) denotes the integer part of \(x\).

### 3.6 Completeness theorem

The next results (see the Appendix for proofs) provide necessary conditions for the completeness of the discrete coherent states indexed by Fuchsian groups.

**Theorem 1** Let \(\{ | z, B, n \rangle \}_{z \in \mathbb{C}^+} \) be a system of coherent states attached to the \(n\)th hyperbolic Landau level. If the subsystem \(\{ | z, B, n \rangle \}_{g \in G} \) indexed by the Fuchsian group \(G\) associated with the automorphic form \(f_0\) of weight \(m_0\), vanishing at one point of \(\mathbb{C}^+\) is complete, then

\[ m_0 \geq \frac{1}{2} \frac{B - n}{1 + n}. \]

If \(G\) admits an automorphic form of weight with a single zero inside \(D\), then \(S_G = \pi \frac{m_0}{2m}\) and we can choose the automorphic form of weight \(m_0 = \pi \frac{m_0}{2S_G}\), where \(S_G\) is the area of the fundamental domain. Thus we can rephrase the above theorem as a necessary upper bound on the area of the fundamental domain.

**Corollary 1** Let \(\{ | z, B, n \rangle \}_{z \in \mathbb{C}^+} \) be a system of coherent states attached to the \(n\)th hyperbolic Landau level. If the subsystem \(\{ | z, B, n \rangle \}_{g \in G} \) indexed by the Fuchsian group \(G\) if \(G\) admits an automorphic form of weight with a single zero inside \(D\), then In terms of the area \(S_G\) of the fundamental domain

\[ S_G \leq \frac{1}{\pi} \frac{B - n}{1 + n}. \]

We remark that if the automorphic form \(f_0 \in \mathcal{C}_G^m (\mathbb{C}^+)\), one can always find an automorphic form vanishing at a point of \(\mathbb{C}^+\).

**Corollary 2** Let \(G = SL(2, \mathbb{Z})\). If the subsystem \(\{ | z, B, n \rangle \} \) connected with
the subgroup $SL(2, \mathbb{Z})$ is complete, then

$$\text{Min} \left\{ m \geq \dim \text{Hol}_G^m (\mathbb{C}^+) \right\} \geq \frac{1}{2} \frac{B - n}{1 + n}.$$  

**Corollary 3** Let $G = SL(2, \mathbb{Z})$. If the subsystem $\{| z, B, n >\}$ connected with the subgroup $SL(2, \mathbb{Z})$ is a frame, then $m_0 > \frac{1}{2} \frac{B - n}{1 + n}$.

4 Conclusion

We have constructed discrete coherent states associated with the evolution of the particle under the action of a constant magnetic field when higher Landau levels are formed, first in the Euclidean and then in the hyperbolic model. Both in the higher Euclidean and the hyperbolic Landau levels, one can construct discrete coherent states by indexing the continuous ones by the discrete subgroups that reflect the symmetries of the underlying geometry. The main conclusion is that, in both cases, the completeness of the coherent states depend explicitly on the size of the fundamental domain, on the order of the Landau Level and on the intensity of the magnetic field. The analysis of the hyperbolic case is based on the properties of the automorphic form of weight $m$ associated with the Fuchsian group $G$ of the hyperbolic plane. If $G$ admits an automorphic form of weight with a single zero inside $D$, then $S_G = \frac{\pi}{2m}$ and we can choose the automorphic form of weight $m_0 = \frac{\pi}{2S_G}$, where $S_G$ is the area of the fundamental domain. Then, the following restriction must be imposed for the completeness of the coherent states:

$$m_0 \geq \frac{1}{2} \frac{B - n}{1 + n}.$$  

In terms of the area $S_G$ of the fundamental domain

$$S_G \leq \frac{1}{\pi} \frac{B - n}{1 + n}.$$  

The methods used in this paper have their origins in several areas of mathematics, physics and signal analysis. It is not surprising that signal analysis and physics are strongly interrelated, since time-frequency (Gabor) analysis is the counterpart of the standard coherent states and time-scale (wavelet) analysis is the counterpart of affine coherent states and affine integral quantization [10]. But the arithmetic aspects connected to the hyperbolic geometry seem to have been somehow overlooked. Among the possible subgroups, only the Fuchsian group of dilations has been used in signal analysis [32], leading to the standard discretization of the half plane used in wavelet theory. We
speculate that the discrete coherent states introduced in this paper may be useful in the analysis of signals, due to the variety of the discrete groups of the upper half-plane. Finally, we would like to subscribe the last sentence of the conclusion of [10], since we believe it also applies to the current research: (...) mutual irrigations between quantum physics and signal analysis deserve a lot more attention in future investigations.

Appendix

Proof of Theorem 1. Let \( f \in L^2(\mathbb{R}^+, \frac{dt}{t}) \). Then we can use Proposition 1

\[
W_{B,n} [f] (x, y) = \gamma_{B,n}^\frac{1}{2} \sum_{k=0}^{n} \frac{(2i)^k}{k!} \binom{2B-n-1}{n-k} y^{B-n-\frac{1}{2}+k} f^{(k)}(z)
\]

with \( F \in A_{2(B-n)-1}(\mathbb{C}^+) \). Let \( f_m \) be a modular form of weight \( m \), that is, a function analytic on the upper-half plane such that

\[
f_m(z) = (cz + d)^{-2m} f_m \left( \frac{az + b}{cz + d} \right)
\]

If \( G \) admits an automorphic (vanishing at possible cusps) form \( f_m(z) \) vanishing at a point \( \zeta_0 \in \mathbb{C}^+ \), then from the functional equation it follows immediately that \( f_m(z) \) vanishes at all the points \( \zeta_k = \zeta_0 \cdot g_k, g_k \in G \). Since

\[
(\Im z)^{-1} \Im \left( \frac{az + b}{cz + d} \right) = (cz + d)^{-2}
\]

we have

\[
|f_m(z)| = (\Im z)^{-m} \left| \Im \left( \frac{az + b}{cz + d} \right) \right|^m \left| h \left( \frac{az + b}{cz + d} \right) \right|.
\]

Now, if \( f_m \) is a cusp form, we can write

\[
d_H(z, G) |\Im z|^{-m} \lesssim |h(z)| \lesssim |\Im z|^{-m}.
\]

The automorphic form \( f_m(z) \) satisfies

\[
|f_{m_0}(z)| \lesssim s^{-2m}.
\]

The condition \( 2m < \frac{B-n}{1+n} \) implies the existence of \( \epsilon > 0 \) such that \( m_0(n+1) = \frac{\alpha + 1 - \epsilon}{2} \), \( \alpha = 2(B-n) - 1 \). Now, set

\[
H(z) = (z + i)^\epsilon [f_{m_0}(z)]^{n+1}(z).
\]

The estimate (4.1) yields

\[
|H(z)| \lesssim |z + i|^\epsilon s^{-(n+1)m_0} = |z + i|^\epsilon s^{-\frac{n+1-\epsilon}{2}}.
\]
We now make the change of variables \( z = \frac{i w + 1}{1 - w} \) to write the integral in the unit disk. The detailed calculation follows

\[
z + i = \frac{2i}{1 - w}; \quad s = \frac{4(1 - |w|^2)}{|1 - w|^2}; \quad s^\alpha d\mu^+(z) = \frac{(1 - |w|^2)^\alpha}{(1 - w)^{2\alpha + 2}} dA(w).
\]

Thus (4.2) becomes

\[
\left| \frac{1}{(1 - w)^{\alpha + 1}} H(i \frac{w + 1}{1 - w})^{n + 1} \right| \lesssim (1 - |w|^2)^{-\frac{\alpha + 1 - \epsilon}{2}}.
\]

The integral can be estimated as follows.

\[
\int_{C^+} |H(z)|^2 s^\alpha d\mu^+(z) \simeq \int_D \left| \frac{1}{(1 - w)^{\alpha + 1 - \epsilon}} H(i \frac{w + 1}{1 - w}) \right|^2 (1 - |w|^2)^\alpha dA(w)
\lesssim \int_D (1 - |w|^2)^{-\alpha - 1 + \epsilon} (1 - |w|^2)^\alpha dA(w)
= \int_D (1 - |w|^2)^{-1 + \epsilon} dA(w) < \infty.
\]

The last inequality can easily be verified directly by definition of area measure or using the reproducing kernel equation for Bergman spaces in the unit disc.

We have thus constructed \( H(z) \in A_{\alpha=2(B-n)-1}(C^+) \) vanishing on \( G \) together with its derivatives. Since there exists \( f \in H^2(C^+) \) such that \( Ber f = H \), this contradicts the condition \( 2m < \frac{B-n}{1+n} \). As a result one must have \( m \geq \frac{1}{2} \frac{B-n}{1+n} \).

**Proof of Corollary 2.** If \( G = SL(2, \mathbb{Z}) \) we have \( G = 0 \). The cusps of \( G \) are exactly the points in \( \mathbb{Q} \cup \{ \infty \} \). Since all these cusps are equivalent we have \( s = 1 \). The elliptic points of \( G \) are of order 2 and 3. The points of order 2 are equivalent to \( i \) and and the points of order 3 are equivalent to \( j \) (with \( j^2 + j + 1 = 0 \) ) [31, pp.14-15]. Thus, we have \( r = 2 \) elliptic points with \( e_1 = 2 \) and \( e_2 = 3 \). Since \( G = 0 \), formula (3.14) provides

\[
\dim \mathcal{S}ol_G^m (C^+) = \left\{ \begin{array}{ll}
1 + \left\lfloor \frac{7m}{6} \right\rfloor - m, & m > 1 \\
0, & m = 1 \\
\ldots, & \ldots \\
1, & m = 0 \\
0, & m < 0
\end{array} \right.
\]

**Proof of Corollary 3.** Since a frame is a complete set, the inequality \( \nu \geq \frac{B-n}{1+n} \) is an obvious consequence of Theorem 1. To obtain the strict inequality we use a standard perturbation argument from wavelet theory [6], which assures
that small pseudohyperbolic perturbations of the index set of a wavelet frame keep the wavelet frame property.

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