Symmetry operators and separation of variables in the
$(2+1)$-dimensional Dirac equation with external electromagnetic
field

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Abstract
We obtain and analyze equations determining first-order differential symmetry operators with
matrix coefficients for the Dirac equation with an external electromagnetic potential in a $(2+1)$-
dimensional Riemann (curved) spacetime. Nonequivalent complete sets of mutually commuting
symmetry operators are classified in a $(2+1)$-dimensional Minkowski (flat) space. For each of the
sets we carry out a complete separation of variables in the Dirac equation and find a corresponding
electromagnetic potential permitting separation of variables.

Keywords: $(2+1)$-dimensional Dirac equation; symmetry operators; separation of variables.
I. INTRODUCTION

In this paper, we consider a complete separation of variables for the Dirac equation in a (2 + 1)-dimensional spacetime (the (2 + 1)-dimensional Dirac equation) with an external electromagnetic field in a Minkowski spacetime with the use of mutually commuting first-order differential symmetry operators with matrix coefficients. This problem was solved for the Dirac equation in the usual (3 + 1)-dimensional Minkowski spacetime in the early 1970s [1, 2] on the basis of the theory of separation of variables in linear systems of first-order partial differential equations (PDEs) and in second-order scalar PDEs [2, 4].

This advancement in theory gave rise to a systematic search for new exact solutions to the Dirac and Klein–Gordon equations with external electromagnetic fields and to a classification of the fields permitting separation of variables [3].

A comprehensive review and detailed analysis of the known solutions obtained by separation of variables was carried out by Bagrov and Gitman in the book [5].

The Dirac equation in (2 + 1) dimensions has been actively studied for the past three decades by many researchers. We do not intend here to provide an exhaustive literature review related to physical problems where solutions of the (2 + 1)-dimensional Dirac equation are essential and refer only to some articles to illustrate a variety of contexts where these solutions can be applied. The topical problem in relativistic quantum theory is the self-adjoint extension of the Dirac Hamiltonian in external singular potentials. The Dirac equation for a magnetic solenoid field is the basis of the theory of the Aharonov–Bohm effect both in (3 + 1) and in (2 + 1) dimensions [6–8]. Khalilov [9] considered the Dirac equation in (2 + 1) dimensions for a relativistic charged zero-mass fermion in Coulomb and Aharonov-Bohm potentials in the context of a self-adjoint extension problem. The self-adjoint extension problem in quantum mechanics was investigated in detail by Gitman, et al. [10].

Exact solutions of the (2 + 1)-dimensional Dirac equation for an external field in the presence of a minimal length were studied by Menculini et al. [11, 12].

The Dirac equation is of interest in studying planar gravity and BTZ black holes [13, 14] and also in cosmology [15, 16].

Another motivation in studying the (2 + 1)-dimensional Dirac equation in curved spacetime is that, although the (2 + 1)-dimensional gravity is a toy model for a regular Einstein theory in (3 + 1) dimensions, it preserves some significant properties of regular gravity being mathematically simpler (see, e.g., [18]).

Finally, it should be noted that in condensed matter physics, the (2 + 1)-dimensional Dirac equation with an external electromagnetic potential is used to study theoretically the electronic properties of graphene (see [19, 21]), graphene quantum dots [22], etc.

Summarizing, we can see that the development of mathematical techniques to find new exact solutions for the (2 + 1)-dimensional Dirac equation is important in view of extending applications of relativistic quantum equations to topical problems of quantum theory.

The paper is organized as follows. In Section 2, basic definitions regarding the (2 + 1)-dimensional Dirac equation with an external electromagnetic potential in pseudo-Riemannian spacetime are given. In Section 3, we obtain and analyze in detail the equations determining a first-order symmetry operator with matrix coefficients for the Dirac equation in the (2 + 1)-dimensional Riemann spacetime. In Section 4, the focus is on classifying complete sets of symmetry operators in (2 + 1)-dimensional Minkowski space. Separation of variables in the (2 + 1)-dimensional Dirac equation is carried out in Section 5 with the use of the complete sets of symmetry operators in (2 + 1)-dimensional Minkowski space. The separa-
able coordinates and the corresponding electromagnetic potentials permitting separation of variables are presented.

II. THE DIRAC EQUATION IN (2 + 1)-DIMENSIONAL RIEMANN SPACETIME

A. A (2 + 1)-dimensional Riemann spacetime

We introduce the notation used in the Dirac equation in a curved spacetime that is described as a (2 + 1)-dimensional pseudo-Riemannian manifold \( \mathcal{M}(g) \). A pseudo-Riemannian metric \( (g_{\mu \nu}) \) on the manifold \( \mathcal{M} \) is given by its covariant components \( g_{\mu \nu}(x) \) in local coordinates \( x = (x^\alpha) = (x^0, x^1, x^2) \) on \( \mathcal{M} \); \( \mu, \nu, \alpha, \cdots = 0, 1, 2 \). The contravariant components of the metric tensor are \( g^{\mu \nu}, g^{\mu \nu} g_{\alpha \nu} = \delta^\mu_\alpha \). Here \( \delta^\mu_\alpha \) is the the Kronecker delta \( (\delta^\mu_\mu = 1 \text{ and } otherwise) \).

The Levi-Civita connection on the tangent bundle \( T \mathcal{M} \) is described by the Christoffel symbols

\[
\Gamma^\mu_{\nu \alpha} = \frac{1}{2} g^{\mu \beta} \left( \partial_\nu g_{\beta \alpha} + \partial_\alpha g_{\beta \nu} - \partial_\beta g_{\nu \alpha} \right), \quad \partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{2.1}
\]

The Riemann curvature tensor is given in terms of the Christoffel symbols as

\[
- R^\sigma_{\alpha \nu \mu} = \partial_\nu \Gamma^\sigma_{\alpha \mu} - \partial_\mu \Gamma^\sigma_{\alpha \nu} + \Gamma^\sigma_{\beta \nu} \Gamma^\beta_{\alpha \mu} - \Gamma^\sigma_{\beta \mu} \Gamma^\beta_{\alpha \nu} \tag{2.2}
\]

(we follow here the notation of \([23]\)). The Ricci tensor \( R_{\mu \nu} \) and curvature \( R \) are

\[
R_{\nu \mu} = R^\alpha_{\nu \alpha \mu}, \quad R = R^\mu_{\mu}. \tag{2.3}
\]

The curvature tensor \( (2.2) \) in the three-dimensional manifold \( \mathcal{M} \) is uniquely determined through the metric and the Ricci tensor \( (2.3) \) as

\[
R_{\nu \mu \alpha \beta} = R_{\nu \alpha \beta \mu} - R_{\nu \beta \alpha \mu} + R_{\mu \beta \alpha \nu} - R_{\mu \alpha \beta \nu} + \frac{R}{2} (g_{\nu \beta} g_{\mu \alpha} - g_{\nu \alpha} g_{\mu \beta}) .
\]

Covariant differentiation of a contravariant vector \( V^\mu \) and a covariant vector \( V_\mu \) yields

\[
\nabla_\nu V^\mu = V_\mu^{\nu} = V^{\mu \nu} + \Gamma^{\mu}_{\nu \alpha} V^\alpha \quad \text{and} \quad \nabla_\nu V_\mu = V_{\nu \mu} - \Gamma^{\alpha}_{\nu \mu} V_\alpha \quad \text{respectively}. \quad \text{Here the semicolon denotes the covariant derivative with respect to coordinate indices, and the comma implies a partial derivative. The connection} \quad \Gamma^\mu_{\nu \alpha} \quad \text{is metric compatible, i.e.} \quad g_{\alpha \beta \mu} = 0.
\]

To define spin connection on the (2+1)-dimensional spacetime, we need local frame fields \( e^a_\alpha(x) \) that diagonalize the metric:

\[
g_{\mu \nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab} , \tag{2.4}
\]

where \( \eta_{ab} \) is the Minkowski metric, \( (\eta_{ab}) = \text{diag}(1, -1, -1) \). Here, Latin letters denote the local frame indices, \( a, b = 0, 1, 2 \). Also, \( e^\alpha_\alpha(x) = g^{\mu \nu} e^\alpha_\nu(x), \quad e^\mu_\alpha(x) = g^{\mu \nu} e^\nu_\alpha(x) \eta_{ab}, \quad \eta^{ac} \eta_{ab} = \delta^a_b \).

In (3 + 1)-general relativity, a frame field is also called a tetrad or a vierbein field; in (2 + 1) spacetime we use the term triad for \( e^a_\alpha(x) \), and then \( a, b, c \) are triad indices.

In what follows, we will need the Levi–Civita antisymmetric tensor \( e_{\mu \nu \alpha}(x) \) on the (2+1)-dimensional manifold \( (\mathcal{M}, g) \), which is defined as

\[
e_{\mu \nu \alpha}(x) = \det (e^a_\mu(x)) e_{\mu \nu \alpha}. \tag{2.5}
\]

By \( e_{\mu \nu \alpha} \) we denote the Levi–Civita antisymmetric symbol, which is defined as \( e_{012} = 1 \). We also note that

\[
e^{\alpha \beta \gamma} e_{\alpha \beta \gamma} = 6, \quad e^{\alpha \beta \gamma} e_{\mu \beta \gamma} = 2 \delta^\alpha_\mu, \quad e^{\alpha \mu \nu} e_{\alpha \beta \gamma} = \delta^\mu_\beta \delta^\nu_\gamma - \delta^\mu_\gamma \delta^\nu_\beta. \tag{2.6}
\]

\(^1\) The Einstein rule of summation is used here and below.
B. The Dirac equation

The Dirac equation in \((2 + 1)\)-dimensional spacetime is determined by introducing the corresponding Dirac matrices, the spin connection, and the generalized momentum operator.

The Dirac matrices

We follow the notation of [5] for \((2 + 1)\) \(\gamma\)-matrices:

\[
\gamma^\mu(x) = e^\mu_a(x) \hat{\gamma}^a,
\]

where

\[
\hat{\gamma}^0 = \sigma_3, \quad \hat{\gamma}^1 = i\sigma_1, \quad \hat{\gamma}^2 = \sigma_2,
\]

and \(\sigma_1, \sigma_2, \sigma_3\) are the Pauli spin matrices \(^2\) The four matrices

\[
\gamma^\mu(x), \ I
\]

form a basis for the set of \(2 \times 2\) matrices, where \(I\) is the identity matrix.

The algebraic properties of \((2 + 1)\) gamma-matrices \((2.7), (2.8)\) are

\[
\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2g^{\alpha\beta}, \quad \gamma^\alpha \gamma^\mu = g^\alpha_\mu - ise^\alpha_\mu_\sigma \gamma_\sigma, \quad [\gamma^\mu, \gamma^\nu] = -2ise^\mu_\nu_\sigma \gamma_\sigma.
\]

where \([A, B] = AB - BA\) is the commutator of \(A\) and \(B\).

The spinor connection

The spinor connection \(\Gamma^\mu\) provides covariant differentiation of spinors: if \(\psi\) is a spinor, then \(\nabla_\mu \psi + \Gamma^\mu \psi = \psi_\mu + \Gamma^\mu \psi\) is also a spinor. The matrices \(\Gamma^\mu\) are assumed to be traceless:

\[
\text{tr}(\Gamma^\mu) = 0.
\]

For the spinor-covariant derivative of the Dirac matrices \(\gamma^\mu\) we have

\[
[\nabla^\alpha + \Gamma^\alpha, \gamma^\mu] = 0 \quad \text{or} \quad \gamma^\mu_\alpha = -[\Gamma^\alpha, \gamma^\mu].
\]

It is easy to verify that

\[
\Gamma^\nu = -\frac{1}{4} \gamma^\alpha_\mu \gamma^\mu_\alpha.
\]

Indeed, since \(\Gamma^\nu\) are traceless \((2.11)\), we can expand \(\Gamma^\nu\) in terms of the basis \((2.9)\) as \(\Gamma^\nu = a^\nu_\alpha \gamma^\alpha\) with constant expansion coefficients \(a^\nu_\alpha\). From \((2.12)\) it follows that \(\gamma^\mu_\nu + a^\nu_\alpha (\gamma^\alpha \gamma^\mu - \gamma^\mu \gamma^\alpha) = 0\). Multiplying this equation by \(\gamma^\mu\) and summing over \(\mu\), we get \(\gamma^\mu_\nu \gamma^\mu + 4a^\nu_\alpha \gamma^\alpha = 0\) and consequently \((2.13)\).

\(^2\) The Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

are Hermitian, traceless, and possessing the properties:

\[
\sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = i\sigma_3, \quad \sigma_2 \sigma_3 = -\sigma_3 \sigma_2 = i\sigma_1, \quad \sigma_3 \sigma_1 = -\sigma_1 \sigma_3 = i\sigma_2.
\]
The generalized momentum operator

The components $\mathcal{P}_\mu$ of the generalized momentum operator are defined as

$$\mathcal{P}_\nu = p_\nu - A_\nu, \quad p_\nu = i(\nabla_\nu + \Gamma_\nu),$$

where $A_\nu$ are the components of the vector potential of the external electromagnetic field. The components of the electromagnetic tensor are $F_{\nu\mu} = \nabla_\nu A_\mu - \nabla_\mu A_\nu = A_{\mu,\nu} - A_{\nu,\mu}$.

**Theorem 1** The commutators of $\mathcal{P}_\nu$ read

$$[\mathcal{P}_\nu, \mathcal{P}_\mu] = -[\nabla_\nu, \nabla_\mu] - \frac{i}{4} s R_{\alpha\beta\sigma}^{\gamma\beta} e^{\alpha\beta\sigma} \gamma^\sigma - i F_{\nu\mu}.$$  

**Proof**

$$[\mathcal{P}_\nu, \mathcal{P}_\mu] = -[\nabla_\nu, \nabla_\mu] - (\Gamma_{\mu;\nu} - \Gamma_{\nu;\mu}) - [\Gamma_\nu, \Gamma_\mu] - i F_{\nu\mu},$$  

where

$$\Gamma_{\mu;\nu} - \Gamma_{\nu;\mu} = -\frac{1}{4} \gamma^\alpha R_{\alpha\nu\mu}^\beta.$$  

Now we define an operator $\hat{G}_{\nu\mu} = - (\Gamma_{\mu;\nu} - \Gamma_{\nu;\mu} + [\Gamma_\nu, \Gamma_\mu])$ and rewrite equation (2.15) as

$$[\mathcal{P}_\nu, \mathcal{P}_\mu] = -[\nabla_\nu, \nabla_\mu] + \hat{G}_{\nu\mu} - i F_{\nu\mu}.  \quad (2.16)$$

Then it is easy to see that the tracelessness property of $\Gamma_\mu$ in (2.11) immediately leads to

$$\text{tr}(\hat{G}_{\mu\nu}) = 0. \quad (2.17)$$

Note that $[[\mathcal{P}_\nu, \mathcal{P}_\mu], \gamma^\alpha] = 0$ is the obvious consequence of (2.12). Hence, we have

$$- [[\nabla_\nu, \nabla_\mu], \gamma^\alpha] + [\hat{G}_{\nu\mu}, \gamma^\alpha] = 0. \quad (2.18)$$

On the other hand,

$$[[\nabla_\nu, \nabla_\mu], \gamma^\alpha] = \gamma^\alpha_{;\mu\nu} - \gamma^\alpha_{;\nu\mu} = R_{\alpha\beta;\mu\nu}^\gamma \gamma^\beta.$$  

Then (2.18) yields

$$R_{\alpha\beta;\mu\nu}^\gamma + [\hat{G}_{\nu\mu}, \gamma^\alpha] = 0. \quad (2.19)$$

Substituting in (2.19) the expansion formula

$$\hat{G}_{\nu\mu} = a_{\nu\mu\beta} \gamma^\beta$$

for $\hat{G}_{\nu\mu}$ in terms of the basis (2.9) with the use of (2.17), we get

$$R_{\alpha\beta;\mu\nu}^\gamma + a_{\nu\mu\beta} [\gamma^\beta, \gamma^\alpha] = 0. \quad (2.20)$$

Then, applying the commutation relation (2.10) to (2.21), we have

$$R_{\alpha\beta;\mu\nu}^\gamma - 2 i a_{\nu\mu\beta} e^{\beta\alpha\sigma} \gamma^\sigma = 0, \quad (2.22)$$

and using (2.6) we find that

$$a_{\nu\mu\alpha} = -\frac{i}{4} s e^{\alpha\beta\sigma} R_{\sigma;\nu\mu}^\beta.$$  

Finally, substituting (2.22) and (2.20) in equation (2.16), we obtain (2.15).

Now we can write the Dirac equation for a Dirac particle of mass $m$ in the $(2 + 1)$-dimensional pseudo-Riemannian manifold $\mathcal{M}$ with external electromagnetic potential $A_\nu$ as follows:

$$H \psi = 0, \quad H = \gamma^\nu \mathcal{P}_\nu - m. \quad (2.23)$$
III. THE DETERMINING EQUATIONS

Separation of variables in the Dirac equation (2.23) involves a matrix-valued first-order differential symmetry operator $X$,

$$X = X^\nu P_\nu + \chi,$$

mapping each solution of (2.23) in another solution of this equation. Here $X^\nu$ and $\chi$ are matrix functions of $x$.

The determining equation for the symmetry operator $X$ can be written as

$$[X, H] = \Psi H.$$  \hfill (3.2)

Here $\Psi$ is a Lagrange operator multiplier having the form

$$\Psi = \Psi^\nu P_\nu + \bar{\Psi},$$  \hfill (3.3)

where $\Psi^\nu$ and $\bar{\Psi}$ are matrix functions of $x$.

The left and right sides of equation (3.2) act on smooth scalar functions of $x$ from the general domain of definition of the operators $X$ and $H$.

Operators (3.1) that satisfy equations (3.2) form a Lie algebra $g$. It is clear, that any $Y = RH$ is a solution of (3.2) with $\Psi = [R, H]$, where $R$ is a linear differential operator. The set of operators $Y$ forms an ideal $h$ in the Lie algebra $g$:

$$[Y, X] = ([R, X] - R\Psi) H.$$  \hfill (3.4)

The operators $Y$ belonging to $h$ do not contain any information about solutions of equation (2.23). These symmetry operators will be called trivial. We are interested in the elements of the quotient algebra $\mathfrak{l} = g/h$, which we call non-trivial symmetry operators.

Substituting expressions (3.1) and (3.3) in (3.2) and equating coefficients of the same powers of $P_\mu$, we come to the following result:

**Lemma 1** The determining equation (3.2) is equivalent to the following system of equations for $X^\nu$, $\chi$ and $\Psi^\nu$, $\bar{\Psi}$:

$$[\gamma^\mu, X^\nu] + [\gamma^\nu, X^\mu] = \Psi^\nu \gamma^\mu + \Psi^\mu \gamma^\nu,$$  \hfill (3.4)

$$[\gamma^\mu, \chi] + \gamma^\nu [P_\nu, X^\mu] = \bar{\Psi} \gamma^\mu + m \Psi^\mu,$$  \hfill (3.5)

$$\gamma^\nu [P_\nu, \chi] + \frac{1}{2} \{ \gamma^\nu, X^\mu \} + [P_\nu, P_\mu] = \frac{1}{2} \Psi^\mu \gamma^\nu [P_\nu, P_\mu] + m \bar{\Psi}.$$  \hfill (3.6)

Note that Lemma 1 is true not only for the $(2 + 1)$ Dirac equation (2.23) but also for the $(3 + 1)$ one.

Expand the matrix functions $X^\nu$, $\chi$, $\Psi^\nu$, $\bar{\Psi}$ in terms of the basis (2.9):

$$X^\nu = \xi^\nu + \gamma^\alpha X_{\alpha}^\nu, \quad \chi = \varphi + \varphi^\alpha \gamma_\alpha, \quad \Psi^\nu = \Psi^\nu_\alpha \gamma^\alpha + \bar{\Psi}^\nu, \quad \bar{\Psi} = \bar{\Psi}_0 \gamma_0 + \bar{\Psi}_0,$$  \hfill (3.7)

where

$$\xi^\nu, \quad X_{\alpha}^\nu, \quad \varphi, \quad \varphi^\alpha, \quad \Psi^\nu_\alpha, \quad \bar{\Psi}^\nu, \quad \bar{\Psi}_0, \quad \bar{\Psi}_0$$

are smooth scalar functions of $x$.

Equations (3.4) — (3.6) for the matrix coefficients $X^\nu$, $\chi$ and $\Psi^\nu$, $\bar{\Psi}$ of the symmetry operator (3.1) and of the Lagrange multiplier, respectively, lead to the corresponding equations for scalar functions (3.9) which are given below in terms of the following lemmas.
Lemma 2  
Equation (3.4) gives
\[
\begin{align*}
\Psi^{\mu
u} &= 2X^{\mu
u} - \frac{2}{3} \text{Sp}(X)g^{\mu
u}, \quad \bar{\Psi} = 0, \\
X^{\mu\nu} + X^{\nu\mu} &= \frac{2}{3} \text{Sp}(X)g^{\mu\nu}, \quad \text{Sp}(X) = X_{\mu}^{\mu}.
\end{align*}
\]

Proof  
Substituting (3.7) and (3.8) in equation (3.4) and expanding the latter in terms of the basis (2.9) with the use of formulas (2.10), we get
\[
- \frac{i}{4} \varepsilon^{\alpha\beta}(2X^{\mu}_{\alpha} - \Psi^{\mu}_{\alpha}) + \frac{i}{4} \varepsilon^{\mu\nu}(2X^{\nu}_{\alpha} - \Psi^{\nu}_{\alpha}) \gamma_{\alpha} = - \bar{\Psi}^{\mu} + \Psi^{\nu\mu} + \bar{\Psi}^{\mu} \gamma^{\nu} + \bar{\Psi}^{\nu} \gamma^{\mu}.
\]
From (3.12), we immediately obtain \( \Psi^{\mu\nu} = - \Psi^{\nu\mu} \), and setting \( \nu = \mu \) in (3.12), we get
\[
- \frac{i}{4} \varepsilon^{\alpha\beta}(2X^{\mu}_{\alpha} - \Psi^{\mu}_{\alpha}) = \bar{\Psi}^{\mu} g^{\mu\alpha}.
\]
For \( \alpha = \sigma \) we have \( \bar{\Psi} = 0 \), and for \( \mu \neq \alpha \) it follows that
\[
\Psi^{\mu}_{\alpha} = 2X^{\mu}_{\alpha}.
\]
For arbitrary indices \( \mu \) and \( \alpha \), we can write (3.13) as (3.10). Substituting (3.10) into the condition \( \Psi^{\mu\nu} = - \Psi^{\nu\mu} \), we obtain equation (3.11).

Taking Lemma 2 into account, we find that symmetry operator (3.7) takes the form
\[
X = \left( \xi^{\alpha} + \frac{1}{2} \Psi^{\alpha}_{\beta} \gamma^{\beta} \right) \hat{P}_{\alpha} + \frac{1}{3} \text{Sp}(X) \hat{H} + \left( \chi + \frac{1}{3} \text{Sp}(X)m \right).
\]

Since we are interested in nontrivial symmetry operators, we can factorize them into elements of the ideal \( \mathfrak{h} \). Then, without loss of generality, we can set
\[
\text{Sp}(X) = 0.
\]

Next, in view of (3.14), we can write the symmetry operator in the form
\[
X = \left( \xi^{\alpha} - \frac{i}{2} X_{\mu\nu} \varepsilon^{\mu\nu\alpha} \right) \hat{P}_{\alpha} + \left( \frac{i}{2} X_{\mu\nu} \varepsilon^{\mu\nu\alpha} \gamma^{\alpha} \right) \hat{H} + \left[ \chi + m \frac{i}{2} X_{\mu\nu} \varepsilon^{\mu\nu\alpha} \gamma^{\alpha} \right]
\]
Thereby, without loss of generality, we can explore only those symmetry operators (3.1) whose coefficients of the derivatives do not include any other matrix except the identity matrix. In other words, we can set in (3.7): \( X_{\mu\nu} = 0 \).

Lemma 3  
Equation (3.5) provides
\[
\begin{align*}
\bar{\Psi} = - \frac{i}{3} \xi^{\mu}_{\mu}, \quad \Psi_{0\mu} &= 0, \\
\varphi_{\alpha} &= - \frac{s}{4} \varepsilon^{\mu\nu} \varepsilon^{\sigma\alpha}_{\nu}, \\
\xi^{\mu\sigma} + \xi^{\sigma\mu} &= \frac{2}{3} \xi^{\alpha}_{\alpha} g^{\mu\sigma}.
\end{align*}
\]
Proof Substitute (3.7) and (3.8) in equation (3.5) and expand it in terms of the basis (2.9). Then we obtain the system of equations

\[ \Psi_0^\mu = 0, \quad i se_{\alpha\mu} (2\varphi_\alpha - \Psi_{0\alpha}) = -i\xi^{\muhad}_{\nu\sigma}g_{\nu\sigma}. \]  (3.18)

Contraction of equation (3.18) with \( g_{\mu\sigma} \) gives (3.15), and its contraction with \( e_{\tau\mu\sigma} \) results in (3.16). From (3.15), (3.16) and (3.18), in view of (3.14), we get (3.17).

Substituting (3.7) and (3.8) in equation (3.6), and taking into account Lemmas 2 and 3 and conditions (3.14), we obtain the following lemma:

**Lemma 4** From equation (3.6) it follows that

\[ \varphi,_{\beta} = \frac{i}{3} \xi^\nu_{;\nu\beta} + F_{\beta\mu} \xi^\mu, \]  (3.19)
\[ m \xi^\mu_{;\mu} = 0. \]  (3.20)

Let us summarize the results obtained in the form of a theorem.

**Theorem 2** The symmetry operator (3.1) of the Dirac equation (2.23) is of the form

\[ \hat{X} = \xi^\mu \hat{P}_\mu - \frac{s}{4} \xi_{\mu\nu} e_{\mu\nu\alpha} \gamma^\alpha + \varphi, \]  (3.21)

where the vector field \( \xi^\mu \) is determined by the system of equations (3.17) and (3.20). The function \( \varphi \) is found from equation (3.19).

It is natural to consider the symmetry operators \( X \) as analytical functions of the real mass parameter \( m \) entering into the Dirac equation in a neighborhood of the point \( m = 0 \). Then condition (3.20) leads to exploration separately the massive case \( (m \neq 0) \) and the massless one \( (m = 0) \).

In the present work, we deal with the massive case. Then from Lemma 4 it follows that

\[ \hat{\Psi} = -i\xi^\mu_{;\mu} = 0. \]  (3.22)

We find that \( \xi^\mu \) is a Killing vector field:

\[ \xi_{\nu;\mu} + \xi_{\mu;\nu} = 0. \]  (3.23)

Note that the symmetry operators \( X \) with \( \hat{\Psi} = 0 \) in (3.2) were used for separation of variables in the \((3 + 1)\)-dimensional Dirac equation in Minkowski spacetime [1] (see also [2]).

In other words, for the \((2 + 1)\)-dimensional Dirac equation, the first-order symmetry operator \( X \) commuting with the Dirac operator \( H \) of the form (2.23) has only scalar coefficients of the derivatives, which are Killing vector fields. This situation is different from the case of the \((3 + 1)\)-dimensional Dirac equation when the symmetry operator has both scalar and matrix coefficients of the first derivatives. Here, by a scalar coefficient we mean a scalar function multiplied by the identity matrix.
IV. MINKOWSKI SPACETIME

Consider symmetry operators (3.21) in (2 + 1)-dimensional Minkowski spacetime with the metric \((g_{\mu\nu}) = \text{diag}(1,1,-1,-1)\) in Cartesian coordinates \((x) = (x^\mu)\).

The solution of the Killing equations in a plane space \(R^3_{\nu\mu\varepsilon} = 0\) is well known. Using the designation \(S = \xi_\nu a^\nu\), where \(a^\nu\) are arbitrary real parameters, we have from (3.23): 
\(S_\nu a^\nu = 0\).

Solving this equation with the use of characteristics, we find \(S = 2A^\alpha x_\alpha a^\nu + B^\nu a_\nu\), where \(A^\nu = -A^\alpha\) and \(B^\nu\) are arbitrary real constants of integration, \(x_\mu = g_{\mu\nu}x^\nu\). Then the Killing vector field \(\xi^\nu\) reads (3.7):

\[
\xi^\nu = 2A^\alpha x_\alpha + B^\nu.
\]

In Cartesian coordinates, the spinor connection \(\Gamma^L_{\nu\mu}\) where we use the designation (2.14), takes the form
\[
P_{\nu} = p_\nu - A_\nu, \quad p_\nu = i\partial_\nu.
\]

As a result, we can write (3.21) as

\[
X = X(x, P) = A^\alpha L_{\alpha\nu} + B^\nu P_\nu + \varphi + \varphi_\alpha \gamma^\alpha,
\]

where we use the designation \(L_{\alpha\nu} = x_\alpha P_\nu - x_\nu P_\alpha\). Equation (3.19) takes the form

\[
\varphi = F^\mu_{\nu} \xi^\nu.
\]

It is important to note that the Killing equation (3.23) does not include the potential \(A_\nu\) and therefore it determines a symmetry operator \(X\) of the free \((A_\nu = 0)\) Dirac equation (2.23).

Equation (4.3) shows that a field \(F_{\nu\mu}\) (or a potential \(A_\nu\)) permitting a symmetry operator \(X\) of the form (4.2) is determined by the Killing vector field \(\xi^\nu\).

This allows us to use the properties of invariance of the free Dirac equation for classification of symmetry operators \(X\) and fields \(F_{\nu\mu}\). Below we follow the definitions of equivalence given in [1-3].

The free Dirac equation (2.23) is invariant with respect to the (2+1)-dimensional Poincare group \(\mathcal{P}(1, 2)\) (translations and the Lorentz group \(SO(1, 2)\)) of transformations of (2 + 1)-dimensional Minkowski spacetime and a transformation of the wave function \(\psi\). Denote by \(\mathfrak{p}(1, 2)\) the Poincare algebra of the Poincare group \(\mathcal{P}(1, 2)\).

Let \(g\) be the generic element of the transformation group \(\mathcal{P}(1, 2)\):

\[
x^{\mu} = (g \cdot x)^{\mu} = a^{\mu}_{\nu} x^{\nu} + b^{\mu},
\]

where \(b^{\mu}\) are real translation parameters, and \(a^{\mu}_{\nu}\) is a \(3 \times 3\) real pseudo-orthogonal matrix defined by \(a^\alpha_{\nu\beta} g_{\alpha\beta} a^\beta_{\mu} = g_{\nu\mu}\). The inverse transformation is

\[
x^{\nu} = (g^{-1} \cdot x)^{\nu} = a^{\nu}_{\alpha} (x^{\alpha} - b^{\alpha}),
\]

where we use the notation: \((a^{-1})^\alpha_{\nu} = a^\nu_{\alpha}\), \(a^\alpha_{\nu} a^\nu_{\beta} = \delta^\alpha_{\beta}\).

Let \(\psi(x)\) be a solution of the Dirac equation (2.23):

\[
\hat{H}(x)\psi(x) = 0, \quad \hat{H}(x) = \gamma^\nu \mathcal{P}_\nu - m.
\]

Let us transform equation (4.6) by using the change of variables (4.5) and the function

\[
\psi(x) = S(g)\tilde{\psi}(x'),
\]

(4.7)
where the matrix $S(g)$ is determined by

$$S^{-1}(g)\alpha^\nu.\gamma^\alpha S(g) = \gamma^\nu. \tag{4.8}$$

As a result, equation $(4.6)$ takes the form

$$\hat{H}(x')\bar{\psi}(x') = 0, \quad \hat{H}(x') = \gamma^\nu \hat{P}_\nu - m, \tag{4.9}$$

$$\hat{P}_\nu = p'_\nu - A^\nu(x'), \quad p'_\nu = i \frac{\partial}{\partial x'^\nu},$$

$$A_{(\nu)}(x') = a_\nu^\alpha A_\alpha = a_\nu^\alpha A_\alpha(g^{-1}x'). \tag{4.10}$$

**Definition 1** The potentials $A_\nu(x)$ and $A_{(\nu)}(x)$ bound by condition $(4.10)$ are called equivalent with respect to the group $\mathcal{P}(1,2)$.

From $(4.6)$, $(4.10)$, and $(4.10)$ it follows that if a function $\psi(x)$ is a solution of the Dirac equation $(4.6)$ with potential $A_\nu(x)$, then $\bar{\psi}(x)$ will be a solution of the Dirac equation with potential $A_{(\nu)}(x)$.

For this reason, we can consider only non-equivalent potentials.

After the change of variables $(4.4)$ and transformation of the wave function $(4.7)$, the symmetry operator $X$ in $(4.2)$ takes the form

$$\bar{X} = \bar{X}(x',\mathcal{P'}) = S^{-1}(g)X(x,\mathcal{P})S(g) =$$

$$= a_\nu^\alpha a_\mu^\beta A^{\mu\nu}(L'_{\alpha\beta} - b_\alpha \mathcal{P}_\beta + b_\beta \mathcal{P}_\alpha) + a_\nu^\alpha B^{\nu\beta} \mathcal{P}_\alpha + \varphi(g^{-1}x') + \varphi_\alpha(g^{-1}x') \gamma^\alpha,$$

where $L'_{\alpha\beta} = x'_{\alpha} \mathcal{P}'_{\beta} - x'_{\beta} \mathcal{P}'_{\alpha}$.

**Definition 2** We call the symmetry operators $X(x,\mathcal{P})$ and $\bar{X}(x,\mathcal{P})$ in $(4.11)$ equivalent with respect to the group $\mathcal{P}(1,2)$.

Then it is sufficient to consider only nonequivalent operators $X$.

Let us show that classification of the symmetry operators $X$ can be reduced to classification of one-dimensional subalgebras of the Poincare algebra $\mathfrak{p}(1,2)$ under the adjoint action of the Poincare group $\mathcal{P}(1,2)$.

From $(4.12)$ it follows that there must be a one-to-one correspondence between the symmetry operator $X$ and an element $\eta$ of the Poincare algebra $\mathfrak{p}(1,2)$:

$$X = A^{\alpha\nu}L_{\alpha\nu} + B^{\nu\beta} \mathcal{P}_\nu + \varphi + \varphi_\alpha \gamma^\alpha \Leftrightarrow \eta = A^{\alpha\nu}l_{\alpha\nu} + B^{\nu\beta} p_\nu. \tag{4.12}$$

where $l_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$. From $(4.4)$, $(4.5)$, $(4.7)$, $(4.8)$, and $(4.11)$ it is seen that one-to-one correspondence $(4.11)$ is also valid for $\bar{X}$ in $(4.11)$ and $\bar{\eta}$,

$$\bar{X} \Leftrightarrow \bar{\eta} = \bar{\eta}(x',\mathcal{P'}) = S^{-1}(g)\eta(x,\mathcal{P})S(g) =$$

$$= a_\nu^\alpha a_\mu^\beta A^{\mu\nu}(l'_{\alpha\beta} - b_\alpha p'_\beta + b_\beta p'_\alpha) + a_\nu^\alpha B^{\nu\beta} p'_\alpha. \tag{4.13}$$

This correspondence indicates that classification of the symmetry operators $X$ is equivalent to classification of the elements $\eta$ of the Poincare algebra $\mathfrak{p}(1,2)$ under transformations of the form $(4.13)$ or, in other words, classification of orbits in the Lie algebra $\mathfrak{p}(1,2)$ of the Lie group $\mathcal{P}(1,2)$.
Let us first assume that $A^{\alpha\nu} \neq 0$ in $\eta$ in (4.12) and classify $\xi = A^{\mu\nu}l_{\mu\nu}$ under transformation (4.13). This implies seeking for the orbits in the Lie algebra $\mathfrak{so}(1,2)$.

Nonequivalent representatives of orbits are well known (e.g., [1–3] and [24, 25]) and can be written up to a constant factor as

$$l_{01}, \ l_{21}, \ l_{21} + l_{01}. \quad (4.14)$$

For each element of $\mathfrak{so}(1,2)$ in (4.14), we have an element of the Poincare algebra $\mathfrak{p}(1,2)$:

$$l_{01} + B^a p_\alpha, \ l_{21} + B^a p_\alpha, \ l_{21} + l_{01} + B^a p_\alpha$$

where $B^a$ are arbitrary constants. Then, using translations $x^\alpha = x^\alpha + b^\alpha$, we can simplify these elements to $l_{01} + ap_2, \ l_{21} + ap_0, \ l_{21} + l_{01} + a(p_0 - p_2)$, where $a$ is an arbitrary constant.

If $A^{\alpha\nu} = 0$ in (4.12), then $\eta = B^a p_\alpha$. Nonequivalent $\eta$ under adjoint action of $SO(1,2)$ are obtained directly: $p_0, \ p_1, \ p_0 + p_1$.

Summarising, we can write the representatives of orbits in the Poincare algebra $\mathfrak{p}(1,2)$:

$$p_0, \ p_1, \ p_0 + p_1, \ l_{21} + ap_0, \ l_{01} + ap_2, \ l_{01} + l_{21} + a(p_0 - p_2). \quad (4.15)$$

For separation of variables in the Dirac equation (2.23), we need sets of pairs of mutually commuting symmetry operators $\{X_1, X_2\}$ of the form (4.12), $[X_1, X_2] = 0$.

**Definition 3** We call the complete set of symmetry operators for the Dirac equation (2.23) a pair $\{X_1, X_2\}$ of mutually commuting linearly independent symmetry operators $X_1$ and $X_2$ nonequivalent with respect to the group $\mathfrak{p}(1,2)$.

**Definition 4** Consider two sets of symmetry operators $\{X_1, X_2\}$ and $\{\tilde{X}_1, \tilde{X}_2\}$. We call these sets equivalent if

$$\tilde{X}_j = c^k_j \tilde{X}_k(x, \mathcal{P}) + c_j, \quad j, k = 1, 2,$$

where $\tilde{X}_k(x, \mathcal{P})$ are defined by equation (4.11), and $c^k_j, c_j$ are constants, $\det(c^k_j) \neq 0$.

Taking into account the one-to-one correspondence (4.12), we can classify the pairs $\{\eta_1, \eta_2\}$ of elements of the Poincare algebra $\mathfrak{p}(1,2)$ commuting as vector fields on Minkowski spacetime.

**Definition 5** Two sets $\{\tilde{\eta}_1, \tilde{\eta}_2\}$ and $\{\eta_1, \eta_2\}$ are regarded as equivalent if

$$\tilde{\eta}_j(x, p) = c^k_j \tilde{\eta}_k(x, p) + c_j,$$

where $\tilde{\eta}_k(x, p)$ are defined by equation (4.13).

From (4.15) we directly obtain nonequivalent sets of commuting vector fields $\{\eta_1, \eta_2\}$, which are presented in the table [3]

|   | $\eta_1$          | $\eta_2$          |
|---|-------------------|-------------------|
| 1 | $p_0$             | $p_1$             |
| 2 | $p_1$             | $p_2$             |
| 3 | $p_0$             | $l_{21}$          |
| 4 | $p_1$             | $l_{02}$          |
| 5 | $p_1$             | $\frac{1}{2}(p_0 + p_2)$ |
| 6 | $\frac{1}{2}(p_0 + p_1)$ | $l_{21} + l_{01} + a(p_0 - p_2)$ |
For each pair \( \{ \eta_1, \eta_2 \} \) in (4.16), according to one-to-one correspondence (4.12), we immediately obtain a pair \( \{ X_1, X_2 \} \).

In the next section, we use the nonequivalent sets of symmetry operators \( \{ X_1, X_2 \} \) for separation of variables in the Dirac equation (2.23).

V. SEPARATION OF VARIABLES

We start with a simple lemma.

**Lemma 5** Let the Dirac equation (2.23) permits a symmetry operator \( X \) of the form (4.2), i.e. conditions (3.19) and (3.23) hold. Then, in a coordinate system where \( \xi^\mu = \delta^\mu_0 \), we have \( \partial_0 F_{\alpha\beta} = 0 \).

**Proof** The proof of this lemma is quite straightforward. Consider equation (4.3). For \( \xi^\mu = \delta^\mu_0 \) we have \( \varphi_\mu = F_{\mu\nu} \xi^\nu = F_{\mu\nu} \). Then, from the compatibility condition, \( \varphi_{,01} = \varphi_{,10} \), it follows that \( \partial_0 F_{01} = 0 \), and from \( \varphi_{,02} = \varphi_{,20} \) we obtain \( \partial_0 F_{02} = 0 \). The relation \( \varphi_{,12} = \varphi_{,21} \) yields \( \partial_2 F_{10} = \partial_1 F_{20} \). Taking into account the Maxwell equation \( \partial_2 F_{01} + \partial_0 F_{12} + \partial_1 F_{20} = 0 \), we arrive at the following relation: \( \partial_0 F_{12} = 0 \).

From this lemma we get the following corollary. Since \( F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu \) and \( A_\mu = A_\mu(x^1, x^2) \), we find that \( \varphi_\mu = \partial_\mu A_0 \) and then we get \( \varphi = A_0 \).

Therefore, the symmetry operator (3.21) takes the form

\[
X = \xi^\nu \partial_\nu + \varphi + \varphi_\alpha \gamma^\alpha = \xi^\nu (p_\nu - A_\nu) + \varphi + \varphi_\alpha \gamma^\alpha = \delta_0^\mu (p_\nu - A_\nu) + A_0 + \varphi_\alpha \gamma^\alpha = p_0 + \varphi_\alpha \gamma^\alpha.
\]

From (3.16) and \( \xi^\nu = \delta^\nu_0 \) we have that \( \varphi_\alpha = 0 \) and \( X = p_0 \).

Our next step is finding the potentials permitting the sets of symmetry operators \( \{ X_1, X_2 \} \) corresponding to pairs \( \{ \eta_1, \eta_2 \} \) of the form (4.16) and separation of variables in the Dirac equation (2.23).

The results can be formulated in the form of the following theorem.

**Theorem 3** Each complete set \( \{ \hat{X}_1, \hat{X}_2 \} \) of the symmetry operators leads to complete separation of variables in the Dirac equation in \( (2 + 1) \)-dimensional Minkowski spacetime.

The proof of this theorem consists in the actual separation of variables with complete sets of symmetry operators.

In the following subsections we carry out complete separation of variables in the \( (2 + 1) \)-dimensional Dirac equation (4.6) with each complete set of symmetry operators listed in to table 4.16. Also, we find electromagnetic potentials permitting separation of variables.

Below we use curvilinear coordinates denoted by \( (u) = (u_0, u_1, u_2) \) with lower indices used for convenience; the partial derivatives are denoted by \( \partial/\partial u_\mu = \partial_u \).

Separable solutions of the Dirac equation in separable coordinates (curvilinear in general case) are found from the system

\[
X_1 \psi = \lambda_1 \psi, \quad X_2 \psi = \lambda_2 \psi, \quad H \psi = 0.
\]

(5.1)

Note that if we select the local frame as

\[
e_\alpha^\mu(x) = \partial_\alpha u_\mu, \quad e_\mu^a(u) = \partial_{u_\mu} x^a(u),
\]

(5.2)
then the spinor connection (2.13) is zero ($\Gamma_\nu = 0$) and the Dirac equation in curvilinear coordinates ($u$) is

$$H \psi = \left\{ \gamma^\nu \left( i \partial_\nu - A_\nu (u) \right) - m \right\} \psi = 0.$$  

(5.3)

The solution $\psi$ of system (5.1) and the solution $\phi_C$ of the Dirac equation in Cartesian coordinates ($x$),

$$H_C \phi_C = \left\{ \hat{\gamma}^\alpha \left( i \partial_\alpha - A_{(C)\alpha}(x) \right) - m \right\} \phi_C = 0,$$

(5.4)

are bound by means of an operator $\hat{S}$

$$\phi_C = \hat{S} \psi,$$

(5.5)

where $A_{(C)\alpha}$ are the Cartesian components of the potential, and $\hat{\gamma}^\nu$ are given by (2.8).

A. The set $\{p_0, p_1\}$

For the set $\{p_0, p_1\}$ (see table 4.16), in view of one-to-one correspondence (4.12), symmetry operators $\{X_1, X_2\}$ of the form (4.2) can be written as

$$X_1 = P_0 + \varphi + \varphi_1 \gamma^\alpha, \quad X_2 = P_1 + \varphi + \varphi_2 \gamma^\alpha.$$  

(5.6)

Here, according to (4.1), (4.3) and (3.16), we have, in Cartesian coordinates ($x^0, x^1, x^2$), that

$$\xi_1^\nu = \delta_1^\nu, \quad \xi_2^\nu = \delta_2^\nu,$$

and

$$\varphi_1^\nu = -\frac{1}{4} g^{\sigma\alpha} \sigma_{0,\alpha}^\mu = 0, \quad \varphi_2^\nu = -\frac{1}{4} g^{\sigma\alpha} \sigma_{1,\alpha}^\mu = 0,$$

$$\varphi_1 \gamma^\mu = F_{\mu 0}, \quad \varphi_2 \gamma^\mu = F_{\mu 1}.$$  

(5.7)

From (5.7), in view of Lemma 5, we obtain $F_{\mu\nu} = F_{\mu\nu}(x^2)$ and

$$A_\nu = A_{\nu}(x^2), \quad \varphi_1 = A_0, \quad \varphi_2 = A_1,$$

where $A_\nu(x^2)$ are arbitrary functions of $x^2$.

Then from (5.6) it follows that

$$X_1 = i \partial_0, \quad X_2 = i \partial_1.$$

Separable solutions to the Dirac equation (2.23) are obtained from system (5.1), where $H = H_C$, and the eigenvalues $\lambda_1$ and $\lambda_2$ play the role of arbitrary constants of separation of variables. Integrating equations (5.1), we can write $\psi$ as

$$\psi = \phi_C = e^{-i\lambda_1 x^0 - i\lambda_2 x^1} \tilde{\psi}(x^2), \quad \tilde{\psi}(x^2) = \left( \begin{array}{c} \tilde{\psi}_1(x^2) \\ \tilde{\psi}_2(x^2) \end{array} \right).$$  

(5.8)

Note that, in the case under consideration, the Cartesian coordinates are separable and $\psi = \phi_C$. Substituting (5.8) in the Dirac equation (5.4), we have

$$\left\{ \hat{\gamma}_0^0 (\lambda_1 - A_0(x^2)) + \hat{\gamma}_1^1 (\lambda_2 - A_1(x^2)) + \\
+ \hat{\gamma}_2^2 \left( i \left( \frac{d}{dx^2} - A_2(x^2) \right) - m \right) \right\} \tilde{\psi}(x^2) = 0,$$

(5.9)

where (5.9) is a system of ordinary differential equations (ODEs).
B. The set \{p_1, p_2\}

Separation of variables with the set \{p_1, p_2\} (see table 4.16) is quite similar to that considered in subsection V.A.

Therefore, we omit intermediate details and present the main results.

The symmetry operators \{X_1, X_2\} corresponding to \{p_1, p_2\}, in Cartesian coordinates \(x^0, x^1, x^2\), are

\[
X_1 = \mathcal{P}_1 + \varphi + \varphi \alpha \gamma^\alpha, \quad X_2 = \mathcal{P}_2 + \varphi + \varphi \alpha \gamma^\alpha. \tag{5.10}
\]

Here, in accordance with (4.1), we find \(\xi^\nu = \delta_1^\nu, \xi^\nu = \delta_2^\nu\). From (3.16), (4.3) it follows that \(\varphi_{,\nu} = \varphi_{,\nu} = 0\) and \(\varphi_{;\mu} = F_{\mu 1}, \varphi_{;\mu} = F_{\mu 2}\). According to Lemma 5 we obtain \(F_{\mu \nu} = F_{\mu \nu}(x^0)\), and a potential admitting the set of symmetry operators (5.10) is

\[
A_\nu = A_\nu(x^0), \quad \varphi = A_1, \quad \varphi = A_1,
\]

where \(A_\nu = A_\nu(x^0)\) are arbitrary functions of \(x^0\). From (5.10) it follows that

\[
X_1 = i\partial_1, \quad X_2 = i\partial_2. \tag{5.11}
\]

Separable solutions \(\psi\) to the Dirac equation obtained from system (5.1) (where \(H = H_C\)) with operators (5.11) are

\[
\psi = \phi_C = e^{-i\lambda_1 x^1 - i\lambda_2 x^2} \tilde{\psi}(x^0), \quad \tilde{\psi}(x^0) = \begin{pmatrix} \tilde{\psi}_1(x^0) \\ \tilde{\psi}_2(x^0) \end{pmatrix},
\]

\[
\begin{cases}
\tilde{\psi}_1(i \frac{d}{dx^0} - A_0(x^0)) + \tilde{\psi}_1(\lambda_1 - A_1(x^0)) + \tilde{\psi}_2(\lambda_2 - A_2(x^0)) - m \end{cases} \tilde{\psi}(x^0) = 0.
\]

C. The set \{p_0, l_{21}\}

Consider the set \{p_0, l_{21}\} (see table 4.16). Similar to the above cases, in accordance with (4.12), we come to symmetry operators \{X_1, X_2\} of the form

\[
X_1 = \mathcal{P}_0 + \varphi + \varphi \alpha \gamma^\alpha, \quad X_2 = L_{21} + \varphi + \varphi \alpha \gamma^\alpha. \tag{5.12}
\]

Then we can write the Killing vector fields (4.1) in Cartesian coordinates \(x^0, x^1, x^2\) as \(\xi_1^\nu = \delta_0^\nu, \xi_2^\nu = x^1 \delta_0^\nu - x^2 \delta_1^\nu\), and \(\varphi_{,\nu}, \varphi_{,\nu}\) are found from equations (3.16) and (4.3), respectively.

To straighten out the Killing vector field \(\xi_j^\nu\), we introduce polar coordinates \((x^0, r, \varphi)\), where

\[
x^1 = r \cos \varphi, \quad x^2 = r \sin \varphi, \quad r = \sqrt{(x^1)^2 + (x^2)^2}, \quad \varphi = \arctan \left(\frac{x^2}{x^1}\right). \tag{5.13}
\]

In coordinates (5.13), the components of the metric tensor are \((g_{\mu \nu}) = \text{diag}(1, -1, -r^2)\), the Killing vectors \(\xi_1^\nu\) take the Kronecker delta form \(\xi_1^\nu = \delta_0^\nu, \xi_2^\nu = \delta_2^\nu\), and the non-zero
Christoffel symbols read $\Gamma_{22}^1 = \Gamma_{rr} = -r$, $\Gamma_{12}^2 = \Gamma_{\varphi r} = r^{-1}$. Let us define local frame fields \cite{2.4} as 
\[
(e_\mu^a(x)) = (\delta_\nu^a, \delta_\nu^r, r^{-1}\delta_\nu^r), \quad (e_\nu^a(x)) = (\delta_\nu^0, \delta_\nu^1, r\delta_\nu^2).
\]
Then the spinor connection (2.13) in polar coordinates takes the form
\[
\Gamma_0 = \Gamma_1 = 0, \quad \Gamma_2 = \frac{1}{4}[\hat{\gamma}^1, \hat{\gamma}^2] = -\frac{i}{2}s\hat{\gamma}^0 = -\frac{i}{2}s\sigma_3. \tag{5.14}
\]
From (3.16) and (4.3) we have $\varphi_\nu = 0$, $\varphi_\nu = -(s/2)\delta_{0\nu}$, where we have used that $e_{\mu\nu} = \sqrt{g}e_{\mu\nu} = r\epsilon_{\mu\nu}$ according to (2.5). Taking into account Lemma 5, we find
\[
A_\nu = A_\nu(r), \quad \varphi = A_0(r), \quad \varphi = A_2(r). \tag{5.15}
\]
Here $A_\nu(r)$ are arbitrary functions of $r$. In view of (5.13)–(5.15), we can write the symmetry operators (5.12) and the Dirac operator (2.23) in polar coordinates (5.13) as
\[
X_1 = i\partial_\theta, \quad X_2 = i\partial_\phi, \tag{5.16}
\]
\[
H = \hat{\gamma}^0\left(i\partial_\theta - A_0(r)\right) + \hat{\gamma}^1\left(i\partial_\phi - A_1(r)\right) + \frac{1}{r}\hat{\gamma}^2\left(i\partial_\phi + s\hat{\gamma}^0 - A_2(r)\right) - m. \tag{5.17}
\]
Separable solutions $\psi$ to the Dirac equation in polar coordinates are obtained from the system (5.1) with operators (5.16) and (5.17) in the form
\[
\psi = e^{-i\lambda_1 x^0 - i\lambda_2 \varphi_1}\tilde{\psi}(r), \quad \tilde{\psi}(r) = \begin{pmatrix} \tilde{\psi}_1(r) \\ \tilde{\psi}_2(r) \end{pmatrix}, \tag{5.18}
\]
\[
\left\{ \hat{\gamma}^0(\lambda_1 - A_0(r)) + \hat{\gamma}^1\left(i\frac{d}{dr} - A_1(r)\right) + \frac{1}{r}\hat{\gamma}^2\left(\lambda_2 + \frac{s}{2}\hat{\gamma}^0 - A_2(r)\right) - m \right\}\tilde{\psi}(r) = 0.
\]
**Cartesian coordinates.** In Cartesian coordinates $(x^0, x^1, x^2)$, the solution $\phi_C$ satisfies the Dirac equation (5.4) and is connected with $\psi$ from (V C), according to (5.3), as follows:
\[
\phi_C = \hat{S}\psi = \exp\left(\frac{s\varphi_1}{2i}\hat{\gamma}^0\right)\psi.
\]
The Cartesian components $A^{(C)a}$ of the potential are
\[
A^{(C)0} = A_0(r), \quad A^{(C)1} = \frac{x^1}{r}A_1(r) - \frac{x^2}{r^2}A_2(r), \quad A^{(C)2} = \frac{x^2}{r}A_1(r) + \frac{x^1}{r^2}A_2(r).
\]
**D. The set $\{p_1, l_{02}\}$**

Separation of variables with the set $\{p_1, l_{02}\}$ (see table 4.16) is similar to the set $\{p_0, l_{21}\}$ considered in previous subsection (V C).

In accordance with (4.12), we get the symmetry operators $\{X_1, X_2\}$ in Cartesian coordinates $(x^0, x^1, x^2)$ as
\[
X_1 = \mathcal{P}_1 + \varphi + \varphi_1 \alpha \gamma^0, \quad X_2 = L_{02} + \varphi + \varphi_2 \alpha \gamma^0. \tag{5.18}
\]
The Killing vector fields (4.1) take the form $\xi^\nu = \delta_1^\nu$, $\xi^\nu = x^0\delta_2^\nu + x^2\delta_0^\nu$, and $\varphi_\nu$, $\varphi_\nu$, $\varphi_\nu$, $j = 1, 2$, are found from equations (3.16) and (4.3), respectively.

We carry out the separation of variables with operators (5.18) in two domains: $|x^0| > |x^2|$ and $|x^0| < |x^2|$.
From (3.16) we have
\[ \phi \] and the non-zero Christoffel symbols take the form permitting the set of symmetry operators (5.18) as
\[ W \] we specify the local frame fields (2.4) according to (5.2):
\[ \text{In coordinates (5.19), the components of the metric tensor} \ g \ \text{are necessary to put} \ \gamma \ \text{from (5.1), (5.3) with operators (5.23) and can be written as} \ A \ \text{in the form} \]
\[ \text{We introduce a coordinate system} \ (u_0, u_1, u_2) \ \text{as} \]
\[ \psi = 1 \quad \text{then we can write the symmetry operators (5.18) in the form} \]
\[ \text{The Killing vectors in coordinates (5.19) take the Kronecker delta form} \ \xi^\nu = \delta_1^\nu, \ \xi^\nu = \delta_2^\nu. \]
\[ \text{In coordinates (5.19), the components of the metric tensor} \ g_{\mu\nu} \ \text{are} \ (g_{\mu\nu}) = \text{diag}(1, -1, -u_0^2), \]
\[ \text{and the non-zero Christoffel symbols take the form} \]
\[ \Gamma_{22}^0 = u_0, \ \Gamma_{02}^2 = u_0^{-1}. \]
\[ \text{We specify the local frame fields (2.4) according to (5.2):} \]
\[ \epsilon_\nu = (\partial_\nu u_0) = \epsilon (\cosh u_2 \delta_0^\nu - u_0^{-1} \sinh u_2 \delta_2^\nu), \ \epsilon_1 = (\partial_1 u_\nu) = \delta_1^\nu, \]
\[ \epsilon_2 = (\partial_2 u_\nu) = \epsilon (-\sinh u_2 \delta_0^\nu + u_0^{-1} \cosh u_2 \delta_2^\nu). \]
\[ \text{From (3.16) we have} \ \varphi_\nu = 0, \ \varphi_\nu = -(s/2) \delta_\nu. \]
\[ \text{Following Lemma 5, we find the potential permitting the set of symmetry operators (5.18) as} \]
\[ A_\nu = A_\nu(u_0) \]
\[ \text{and} \ \varphi = A_1, \ \varphi = A_2. \]
\[ \text{Here} \ A_\nu(u_0) \ \text{are arbitrary functions of} \ u_0. \]
\[ \text{Then we can write the symmetry operators (5.18) in the form} \]
\[ \text{The Dirac equation (2.23) in coordinates (5.19), (5.20) takes the form (5.3), where it is necessary to put} \ \gamma^1 = \hat{\gamma}^1. \]
\[ \text{Separable solutions to the Dirac equation (2.23) in coordinates (5.19) are found from system (5.1), (5.3) with operators (5.23) and can be written as} \]
\[ \psi = \frac{1}{\sqrt{u_0}} \exp \left( -i\lambda_1 u_1 - i\lambda_2 u_2 - \frac{s}{2} \hat{\gamma}^1 u_2 \right) \tilde{\psi}(u_0), \quad \tilde{\psi}(u_0) = \left( \begin{array}{c} \tilde{\psi}_1(u_0) \\ \tilde{\psi}_2(u_0) \end{array} \right), \]
\[ \left\{ \epsilon \hat{\gamma}^0 \left( i \frac{d}{du_0} - A_0(u_0) \right) + \hat{\gamma}^1 \left( \lambda_1 - A_1(u_0) \right) + \right. \]
\[ \left. \quad + \epsilon u_0^{-1} \hat{\gamma}^2 \left( \lambda_2 - A_2(u_0) \right) - m \right\} \psi(u_0) = 0. \]
\[ \textbf{Cartesian coordinates.} \] The solution \( \phi_C \) of the Dirac equation (5.4) in Cartesian coordinates \( (x^0, x^1, x^2) \) and the solution \( \psi \) from (5.1) in the curvilinear separable coordinates (5.19), (5.20) are \( \phi_C = \psi \).
\[ \text{The Cartesian components} \ A_{(C)0} \ \text{of the potential are} \]
\[ A_{(C)0} = \frac{x^0}{u_0} A_0(u_0) - \frac{x^2}{u_0^2} A_2(u_0), \quad A_{(C)1} = A_1(u_0), \]
\[ A_{(C)2} = -\frac{x^2}{u_0} A_0(u_0) + \frac{x^0}{u_0} A_2(u_0). \]
The domain $|x^0| < |x^2|$.

The coordinate system $(u_0, u_1, u_2)$ for this domain is as follows:

$$u_0 = \sqrt{(x^2)^2 - (x^0)^2}, \quad u_2 = \frac{1}{2} \log \left( \frac{x^2 + x^0}{x^2 - x^0} \right), \quad u_1 = x^1. \quad (5.25)$$

The inverse coordinate transformation is

$$x^0 = \varepsilon u_0 \sinh u_2, \quad x^2 = \varepsilon u_0 \cosh u_2, \quad \varepsilon = sign(x^2 + x^0), \quad u_0 > 0. \quad (5.26)$$

The Killing vectors in coordinates (5.25), (5.26) take the Kronecker delta form $\xi^\nu = \delta^\nu_1$, $\xi^\nu = \delta^\nu_2$. The metric tensor $(g_{\mu\nu})$ is $g_{\mu\nu} = \text{diag}(-1, -1, u_0^2)$. The non-zero Christoffel symbols are the same as in the previous case and are given by (5.21). Choose local frame fields (2.4) in the form

$$e_\nu^\nu = (\partial_0 u_\nu) = \varepsilon \left( - \sinh u_2 \delta_0^\nu + u_0^{-1} \cosh u_2 \delta_2^\nu \right), \quad e^\nu_1 = (\partial_1 u_\nu) = \delta^\nu_1,$$

$$e^\nu_2 = (\partial_2 u_\nu) = \varepsilon \left( \cosh u_2 \delta_0^\nu - u_0^{-1} \sinh u_2 \delta_2^\nu \right). \quad (5.27)$$

Relation (5.22) also remains valid. From (5.23)–(5.27) we obtain that symmetry operators (5.18) and the Dirac operator in coordinates (5.25), (5.26) take the form (5.23) and (5.3), respectively.

Separable solutions to the Dirac equation (2.23) in coordinates (5.25), (5.26) are given by (5.24), where $\hat{\psi}(u_0)$ satisfies the equation

$$\left\{ \varepsilon \hat{\gamma}^2 \left( i \frac{d}{du_0} - A_0(u_0) \right) + \hat{\gamma}^1 \left( \lambda_1 - A_1(u_0) \right) + \varepsilon u_0^{-1} \hat{\gamma}^0 \left( \lambda_2 - A_2(u_0) \right) - m \right\} \hat{\psi}(u_0) = 0. \quad (5.28)$$

**Cartesian coordinates.** The solution $\phi_C$ of the Dirac equation (5.4) in Cartesian coordinates $(x^0, x^1, x^2)$ and the solution $\psi$ from (5.1) in the curvilinear separable coordinates (5.25), (5.26) are identical: $\phi_C = \psi$. The Cartesian components $A_{(C)\alpha}$ of the potential are

$$A_{(C)0} = -\frac{x^0}{u_0} A_0(u_0) + \frac{x^2}{u_0^2} A_2(u_0), \quad A_{(C)1} = A_1(u_0),$$

$$A_{(C)2} = \frac{x^2}{u_0} A_0(u_0) - \frac{x^0}{u_0^2} A_2(u_0).$$

E. The set $\{p_1, \frac{1}{2}(p_0 + p_2)\}$

Consider now the set $\{p_1, \frac{1}{2}(p_0 + p_2)\}$ (see table 4.16). Although this set leads to a non-orthogonal separable coordinate system, separation of variables in the Dirac equation (2.23) goes like in the case of the set $\{p_0, p_1\}$ considered in subsection V.A. Therefore, we omit here technical details and just give the main results.

According to (4.12), we find the symmetry operators $\{X_1, X_2\}$ as

$$X_1 = P_1 + \varphi + \varphi_\alpha \gamma^\alpha, \quad X_2 = \frac{1}{2} (P_0 + P_2) + \varphi + \varphi_\alpha \gamma^\alpha. \quad (5.29)$$
The Killing vectors (4.1) are \( \xi_1^\nu = \delta_1^\nu, \xi_2^\nu = (\delta_0^\nu + \delta_2^\nu)/2 \). From (3.16) we find \( \varphi_1 = \varphi_2 = 0 \).

The vector \( \xi_2^\nu \) is straightened out in a coordinate system \((u_0, u_1, u_2)\),

\[
\begin{align*}
  u_0 &= x^0 - x^2, \quad u_1 = x^1, \quad u_2 = x^0 + x^2. \\
\end{align*}
\]

In coordinates (5.30), the components of the metric tensor are

\[
(g_{\mu\nu}) = \begin{pmatrix}
  0 & 0 & 2 \\
  0 & 0 & 1 \\
  2 & 0 & 0
\end{pmatrix}, \quad (g'^{\mu\nu}) = \begin{pmatrix}
  0 & 0 & 1/2 \\
  0 & 1 & 0 \\
  1/2 & 0 & 0
\end{pmatrix}.
\]

The Killing vectors take the Kronecker delta form \( \xi_1^\nu = \delta_1^\nu, \xi_2^\nu = \delta_2^\nu \). From (2.1) and (2.13) it follows that \( \Gamma_{\mu\alpha}^\nu = 0 \).

From (5.31) and Lemma 5, we find a potential permitting the set of symmetry operators (5.29) in coordinates (5.30) as

\[ A_\nu = A_\nu(u_0), \text{ where } A_\nu(u_0) \text{ are arbitrary functions of } u_0. \]

In addition, we find that \( \varphi_1 = A_1 \) and \( \varphi_2 = A_2 \), and then

\[
X_1 = i\partial_{u_1}, \quad X_2 = i\partial_{u_2}.
\]

Separable solutions to the Dirac equation (2.23) are obtained from system (5.1) with operators (5.32) and have the form

\[
\psi = e^{-i\lambda_1 u_1- i\lambda_2 u_2} \tilde{\psi}(u_0), \quad \tilde{\psi}(u_0) = \begin{pmatrix}
  \tilde{\psi}_1(u_0) \\
  \tilde{\psi}_2(u_0)
\end{pmatrix},
\]

\[
\left\{ \begin{align*}
  (\gamma^0 - \gamma^2) \left( i\frac{d}{du_0} - A_0(u_0) \right) + (\gamma^0 + \gamma^2) \left( \lambda_2 - A_2(u_0) \right) + \\
  + \gamma^1 (\lambda_1 - A_1(u_0)) - m \right\} \tilde{\psi}(u_0) = 0.
\end{align*}
\]

Catresian coordinates. The solution \( \phi_C \) (5.4) and the solution \( \psi \) from (5.1) in the curvilinear separable coordinates (5.30) are identical: \( \phi_C = \psi \). The Cartesian components \( A_{(C)\alpha} \) of the potential are

\[
A_{(C)0} = A_0(u_0) + A_2(u_0), \quad A_{(C)1} = A_1(u_0), \quad A_{(C)2} = -A_0(u_0) + A_2(u_0).
\]

F. The set \( \left\{ (p_0 + p_1)l_{21} + l_{01} + a(p_0 - p_2) \right\} \)

For the set \( \left\{ (p_0 + p_1)l_{21} + l_{01} + a(p_0 - p_2) \right\} \) (see table 4.16) in accordance with (4.12), the corresponding set of symmetry operators \( \{X_1, X_2\} \) in Cartesian coordinates is written as

\[
X_1 = \frac{1}{2}(P_0 + P_2) + \varphi + \varphi_1 \alpha \gamma^\alpha, \quad X_2 = L_{21} + L_{01} + a(P_0 - P_2) + \varphi + \varphi_2 \alpha \gamma^\alpha.
\]
The corresponding Killing vectors (4.1) are
\[
\xi^\nu = \frac{1}{2}(\delta_0^\nu + \delta_2^\nu), \quad \xi^\nu = (x^1 + a)\delta_0^\nu + (x^0 - x^2)\delta_1^\nu + (x^1 - a)\delta_2^\nu. \tag{5.34}
\]

The Killing vectors (5.34) can be straightened out in a coordinate system \((u_0, u_1, u_2)\):
\[
\begin{align*}
    u_0 &= \frac{1}{2}(x^0 - x^2)^2 - 2ax^1, \\
    u_1 &= x^0 + x^2 + \frac{1}{a}(x^0 - x^2) \left[\frac{(x^0 - x^2)^2}{6a} - x^1\right], \tag{5.35} \\
    u_2 &= \frac{1}{2a}(x^0 - x^2).
\end{align*}
\]

The inverse coordinate transformation is
\[
\begin{align*}
    x^0 &= \frac{1}{2}u_1 - \frac{1}{2a}u_0u_2 + \frac{1}{3}au_2^3 + au_2, \\
    x^1 &= au_2^2 - \frac{1}{2a}u_0, \tag{5.36} \\
    x^2 &= \frac{1}{2}u_1 - \frac{1}{2a}u_0u_2 + \frac{1}{3}u_2^3 - au_2.
\end{align*}
\]

Here we assume that \(a \neq 0\). In coordinates (5.35), (5.36) the metric tensor takes the form
\[
(g^{\mu\nu}) = \begin{pmatrix}
-4a^2 & 0 & 0 \\
0 & \frac{2a^2}{a^2} & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad (g_{\mu\nu}) = \begin{pmatrix}
-\frac{1}{4a^2} & 0 & 0 \\
0 & 0 & a \\
0 & a & -2u_0
\end{pmatrix}.
\]

The Killing vectors (5.34) take the Kronecker delta form \(\xi^\nu = \delta_1^\nu, \xi^\nu = \delta_2^\nu\). The non-zero Christoffel symbols are \(\Gamma_{22}^0 = -4a^2, \Gamma_{20}^1 = -1/a\). Choose local frame fields \((e^\nu_\alpha(x)) (2.4)\) as
\[
\begin{align*}
    (e^\nu_0) &= (\partial_0 u_\nu) = \left(2au_2, 1 + \frac{1}{2a^2}u_0 + u_2^2, \frac{1}{2a}\right), \\
    (e^\nu_1) &= (\partial_1 u_\nu) = (-2a, -2u_2, 0), \\
    (e^\nu_2) &= (\partial_2 u_\nu) = \left(-2au_2, 1 - \frac{1}{2a^2}u_0 - u_2^2, -\frac{1}{2a}\right). \tag{5.37}
\end{align*}
\]

From (5.35) — (5.37) and Lemma 5, we find a potential permitting the set of symmetry operators (5.33) in coordinates (5.35), (5.36) as \(A_\nu = A_\nu(u_0)\), where \(A_\nu(u_0)\) are arbitrary functions of \(u_0\). In addition, we find that \(\varphi = A_1\) and \(\varphi = A_2\), and from (3.16) it follows that \(\varphi_\nu = 0, \varphi_\nu = -as\delta_{2\nu}\). Then the symmetry operators (5.33) can be written as
\[
X_1 = i\partial_{u_1}, \quad X_2 = i\partial_{u_2} - as\gamma^2, \tag{5.38}
\]
and the Dirac equation (2.23) in coordinates (5.35), (5.36) takes the form (5.3), where it is necessary to put
\[
\begin{align*}
    \gamma^0 &= 2au_2(\dot{\gamma}^0 - \dot{\gamma}^2) - 2a\dot{\gamma}^1, \quad \gamma^2 = \frac{1}{2a}(\dot{\gamma}^0 - \dot{\gamma}^2), \\
    \gamma^1 &= \dot{\gamma}^0 + \dot{\gamma}^2 + \left(\frac{1}{2a^2}u_0 + u_2^2\right)(\dot{\gamma}^0 - \dot{\gamma}^2) - 2u_2\dot{\gamma}^1.
\end{align*}
\]
Separable solutions to the Dirac equation (2.23) in coordinates (5.35), (5.36) are found from system (5.1), (5.3) with operators (5.38) and can be written as

\[
\psi = \exp \left( -i\lambda_1 u_1 - i\lambda_2 u_2 - i\frac{s}{2}(\dot{\gamma}^0 - \dot{\gamma}^2)u_2 \right) \tilde{\psi}(u_0), \quad \tilde{\psi}(u_0) = \left( \begin{array}{c} \tilde{\psi}_1(u_0) \\ \tilde{\psi}_2(u_0) \end{array} \right),
\]

\[
\left\{ -2a\dot{\gamma}^1 \left( i\frac{\partial}{\partial u_0} - A_0(u_0) \right) + \frac{1}{2a}(\dot{\gamma}^0 - \dot{\gamma}^2)(\lambda_2 - A_2(u_0)) + \left( \dot{\gamma}^0 + \dot{\gamma}^2 + \frac{u_0}{2a^2}(\dot{\gamma}^0 - \dot{\gamma}^2) \right)(\lambda_1 - A_1(u_0)) - m \right\} \tilde{\psi}(u_0) = 0.
\]

**Cartesian coordinates.** The solution \(\phi_C\) (5.4) and the solution \(\psi\) from (5.3) in the curvilinear separable coordinates (5.35), (5.36) are identical: \(\phi_C = \psi\). The Cartesian components \(A_{(C)\alpha}\) of the potential are

\[
A_{(C)0} = (x^0 - x^2)A_0(u_0) + \left( 1 - \frac{1}{a}x^1 + \frac{1}{2a^2}(x^0 - x^2)^2 \right)A_1(u_0) + \frac{1}{2a}A_2(u_0),
\]

\[
A_{(C)1} = -2aA_0(u_0) - \frac{1}{a}(x^0 - x^2)A_1(u_0),
\]

\[
A_{(C)2} = -(x^0 - x^2)A_0(u_0) + \left( 1 + \frac{1}{a}x^1 - \frac{1}{2a^2}(x^0 - x^2)^2 \right)A_1(u_0) - \frac{1}{2a}A_2(u_0).
\]

**G. The set \(\{\frac{1}{2}(p_0 + p_1), l_{21} + l_{01}\}\)**

Now we consider the set \(\{\frac{1}{2}(p_0 + p_1), l_{21} + l_{01} + a(p_0 - p_2)\}\) (see table 4.16) when \(a = 0\). In accordance with (4.12), the corresponding set of symmetry operators \(\{X_1, X_2\}\) in Cartesian coordinates can be written as

\[
X_1 = \frac{1}{2}(\mathcal{P}_0 + \mathcal{P}_2) + \varphi + \varphi_{\alpha}\gamma^\alpha, \quad X_2 = L_{21} + L_{01} + \varphi + \varphi_{\alpha}\gamma^\alpha.
\]

The corresponding Killing vectors (4.11) are

\[
\xi^\nu_1 = \frac{1}{2}(\delta^\nu_0 + \delta^\nu_2), \quad \xi^\nu_2 = x^1\delta^\nu_0 + (x^0 - x^2)\delta^\nu_1 + x^1\delta^\nu_2.
\]

(5.40)

The Killing vectors (5.40) can be straightened out in a coordinate system \((u_0, u_1, u_2)\):

\[
u_0 = x^0 - x^2, \quad u_1 = x^0 + x^2 - \frac{(x^1)^2}{x^0 - x^1}, \quad u_2 = \frac{x^1}{x^0 - x^2}.
\]

(5.41)

The inverse coordinate transformation is

\[
x^0 = \frac{1}{2}(u_0 + u_1 + u_0u_2), \quad x^1 = u_0u_2, \quad x^2 = x^0 = \frac{1}{2}(-u_0 + u_1 + u_0u_2).
\]

(5.42)

In coordinates (5.41), the metric tensor is

\[
(g^{\mu\nu}) = \begin{pmatrix}
0 & 2 & 0 \\
2 & 0 & 0 \\
0 & 0 & -\frac{1}{u_0^2}
\end{pmatrix}, \quad (g_{\mu\nu}) = \begin{pmatrix}
0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 \\
0 & 0 & -u_0^2
\end{pmatrix}, \quad g = \det(g_{\nu\mu}) = \frac{1}{4}u_0^2.
\]
The Killing vectors (5.40) take the Kronecker delta form \( \xi^\nu_1 = \delta^\nu_1, \xi^\nu_2 = \delta^\nu_2 \). The non-zero Christoffel symbols are \( \Gamma^1_{22} = 2u_0, \Gamma^2_{20} = 1/u_0 \). Choose local frame fields \( (e^\nu_a(x)) \) (2.4) as

\[
(e^\nu_0) = (\partial_0 u_\nu) = \left(1, 1 + u_2^2, -\frac{u_2}{u_0}\right),
\]

\[
(e^\nu_1) = (\partial_1 u_\nu) = (0, -2u_2, \frac{1}{u_0}),
\]

\[
(e^\nu_2) = (\partial_2 u_\nu) = \left(-1, 1 - u_2^2, \frac{u_2}{u_0}\right).
\]

From (3.16) and Lemma 5, we find a potential permitting the set of symmetry operators (5.39) in coordinates (5.41), (5.42) as

\[
A = \hat{0} \gamma^0, \quad (C) = \exp \left(i \lambda_1 u_1 - i \lambda_2 u_2 - \frac{s}{2} (\hat{\gamma}^0 - \hat{\gamma}^2) u_2 \right) \tilde{\psi}(u_0), \quad \check{\psi}(u_0) = \left(\frac{\tilde{\psi}_1(u_0)}{\tilde{\psi}_2(u_0)}\right),
\]

\[
\left\{ (\hat{\gamma}^0 - \hat{\gamma}^2) \left(i \frac{d}{d u_0} + \frac{i}{2 u_0} - A_0(u_0) \right) + (\hat{\gamma}^0 + \hat{\gamma}^2) (\lambda_1 - A_1(u_0)) + \frac{1}{u_0} \hat{\gamma}^1 (\lambda_2 - A_2(u_0)) - m \right\} \tilde{\psi}(u_0) = 0.
\]

Cylindrical coordinates. The solution \( \phi_C \) (5.4) and the solution \( \psi \) from (5.3) in the curvilinear separable coordinates (5.41), (5.42) are identical: \( \phi_C = \psi \). The Cartesian components \( A_{(C)a} \) of the potential are

\[
A_{(C)0} = A_0(u_0) + \left(1 + \frac{(x^1)^2}{(x^0 - x^2)^2}\right) A_1(u_0) - \frac{x_1}{(x^0 - x^2)^2} A_2(u_0),
\]

\[
A_{(C)1} = -A_0(u_0) + \left(1 - \frac{(x^1)^2}{(x^0 - x^2)^2}\right) A_1(u_0) + \frac{x_1}{(x^0 - x^2)^2} A_2(u_0),
\]

\[
A_{(C)2} = \frac{2x^1}{x^0 - x^2} A_1(u_0) + \frac{1}{x^0 - x^2} A_2(u_0).
\]
VI. CONCLUSION

We have considered the problem of separation of variables in the \((2 + 1)\)-dimensional Dirac equation with external electromagnetic potential in Minkowski spacetime with the use of first-order differential symmetry operators with the derivatives having matrix coefficients.

To this end we explored the properties of symmetry operators in \((2 + 1)\)-dimensional pseudo-Riemannian space in terms of the determining equations for the operator.

The symmetry operators under consideration which are analytical (holomorphic) with respect to the mass parameter \(m\) entering into the Dirac equation, are shown to have different properties. In a massive case \((m \neq 0)\), a symmetry operator commutes with the operator of the Dirac equation and has scalar (non-matrix) coefficients of the derivatives. The complete set of symmetry operators provides separation of variables in the Dirac equation in \((2 + 1)\)-dimensional Minkowski spacetime like in the \((3 + 1)\)-dimensional case [5]. The complete sets have been classified and for each of them, separation of variables has been carried out.

In the curved pseudo-Riemann \((2 + 1)\)-dimensional space, as well as in the \((3 + 1)\)-dimensional space, the problem of separation of variables in the Dirac equation has specific features as compared to that in the plane Minkowski space and it is to be a subject of separate study.

From the point of view of developing a theory of separation of variables in the Dirac equation, the \((2 + 1)\)-dimensional case is especially attractive as it is mathematically simpler but includes all basic elements of the theory. Unlike \((3 + 1)\)-dimensional space, in a massive case, the symmetry operators for the \((2 + 1)\)-dimensional Dirac equation are presented in terms of the Killing vectors, and the spin operators with matrix coefficients of the derivatives can be removed from the symmetry operators without loss of generality.

In a massless case \((m = 0)\), the commutator (3.2) of the symmetry operator with the \((2 + 1)\)-dimensional Dirac operator \(H\) in (2.23) is proportional to the Dirac operator with a Lagrange multiplier having the form of a scalar function. The symmetry operator is presented in terms of a conformal Killing vector field \(\xi^\mu\). Therefore, the set of symmetry operators is wider in this case and the problem of separation of variables calls for a particular research.

In summary, we note that to use symmetry operators of both the first and the second order for separation of variables in the Dirac equation, one can turn to the squared Dirac equation [26]. However, this problem is beyond the scope of the present work.

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