Geometry and Conservation Laws for a Class of Second-Order Parabolic Equations I: Geometry

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Abstract

I consider the geometry of the class of scalar parabolic equations using Cartan’s method of equivalence. After defining the exterior differential systems that correspond to general second order parabolic equations in arbitrarily many variables, I define local invariants of parabolic equations up to contact transformation. The first family of invariants determine when a parabolic system has a deprolongation to a parabolic Monge-Ampère system. The second family of invariants determine when a general parabolic equation has a local choice of coordinates putting it in evolutionary form. In addition to intrinsic interest, another motivation is to study the conservation laws of parabolic equations. The invariants developed in this paper are crucial to the results on conservation laws in Part II of this 2 part series of papers.

Introduction

0.1. Geometry

The goal of this paper is to study the geometry of differential equations of parabolic type. I define parabolic systems—exterior differential systems that are (locally) equivalent to parabolic scalar PDE—in arbitrarily many variables, and make progress on their equivalence problem. The local invariants established here provide insight into the geometry of parabolic systems and their solutions. In addition to intrinsic interest, another motivation is to study the conservation laws of parabolic equations. The invariants developed in this paper are crucial to the results on conservation laws in Part II of this 2 part series of papers.

The primary tool used here is Cartan’s method of equivalence, to study the geometry of parabolic systems. The method of equivalence is a general approach to solving the equivalence problem for geometric structures: if one is handed a class of geometric structure on real-analytic manifolds (eg. Riemannian, almost complex, CR), the equivalence method prescribes how to classify the local invariants of the geometry, which are special functions that distinguish non-isomorphic structures (eg. Riemannian curvature, the Nijenhuis tensor, the Levi form, respectively). Two good references, full of examples, are Gardner’s *Method of Equivalence and its Applications* [5] and Bryant,

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\[\text{\textsuperscript{1}The primary tool used to classify these invariants is Cartan-Kähler Theory, which depends on real analyticity. However, the real-analytic invariants typically describe smooth invariants as well.}\]
Griffiths and Grossman’s excellent *Exterior Differential Systems and Euler-Lagrange Partial Differential Equations* [2].

To treat differential equations as geometric objects, I consider them as exterior differential systems:

**Definition 0.1.** An exterior differential system \((M, I)\) is a smooth manifold \(M\) and a graded, differentially closed ideal \(I\) in \(\Omega^*(M)\).

A submanifold \(\iota: \Sigma \hookrightarrow M\) is an integral manifold of \((M, I)\) if the pullback \(\iota^* I\) is identically zero, or equivalently, if \(\phi|_{T_x \Sigma} = 0\) for all \(\phi \in I\) and \(x \in \Sigma\).

An integral element at a point \(x \in M\) is a subspace \(E \subset T_x M\) for which \(\phi|_E = 0\) for all \(\phi \in I\).

Every sufficiently non-degenerate system of partial differential equations corresponds naturally (in fact, functorially) to an exterior differential system. Furthermore, under this correspondence, the graphs of solutions to a PDE are naturally identified with integral submanifolds of the associated EDS.

Very briefly, the correspondence is given by sending a sufficiently non-degenerate differential equation

\[
F \left( x^i, u^a, \frac{\partial u^a}{\partial x^i}, \frac{\partial^2 u^a}{\partial x^i \partial x^j}, \ldots \right) = 0 \tag{0.1}
\]

of order \(k\), on \(s\) functions \(u^a\) of \(n\) independent variables \(x^i\) to the submanifold

\[M = F^{-1}(0) \subset J^k(\mathbb{R}^n, \mathbb{R}^s),\]

where is \(F\) considered as a function on the jet space \(J^k(\mathbb{R}^n, \mathbb{R}^s)\). The jet space has a canonical Pfaffian ideal \(C\), and if \(I\) is the pullback of \(C\) to \(M\), then the exterior differential system \((M, I)\) has integral submanifolds that correspond to the graphs of solutions to \(F\). (To be precise, the \(n\)-dimensional integral submanifolds that submerge onto \(\mathbb{R}^n\) are locally the graphs of solutions.)

Exterior differential systems form a category, where a morphism from \((M, I)\) to \((M', I')\) is a smooth map \(f: M \to M'\) that pulls back \(I'\) to a subset of \(I\). Within this category, an integral manifold \(\Sigma\) of \((M, I)\) is simply an EDS-embedding \(\varphi: (\Sigma, \{0\}) \to (M, I)\). It is occasionally useful to drop the condition that integral manifolds are embeddings. Then morphisms are characterized by the condition that they push forward solutions of \((M, I)\) to solutions of \((M', I')\). The isomorphisms in this category are exactly the maps that preserve the structure of solutions.

**Definition 0.2.** An equivalence of exterior differential systems \((M, I)\) and \((M', I')\) is a diffeomorphism \(f: M \to M'\) for which \(f^* I' = I\).

One class of example of EDS equivalences is given by the point transformations—equivalences induced by changes of variables of a PDE. More precisely, a change of coordinates

\[x^i, u^a \quad \longrightarrow \quad \tilde{x}^i(x^j), \tilde{u}^a(x^j, u^a)\]
transforms $F$ as in equation (0.1) into a new equation $\tilde{F}$. Then the following diagram is functorial:

$$
F(x^i, u^a, \frac{\partial u^a}{\partial x^i}, \frac{\partial^2 u^a}{\partial x^i \partial x^j}, \ldots) = 0 \xrightarrow{\text{EDS functor}} (M, \mathcal{I})
$$

Change of Variables \hspace{1cm} \text{EDS equivalence}

$$
\tilde{F}(\tilde{x}^i, \tilde{u}^a, \frac{\partial \tilde{u}^a}{\partial \tilde{x}^i}, \frac{\partial^2 \tilde{u}^a}{\partial \tilde{x}^i \partial \tilde{x}^j}, \ldots) = 0 \xrightarrow{\text{EDS functor}} (\tilde{M}, \tilde{\mathcal{I}})
$$

A change of variables on the left takes any solution of $F$ to one of $\tilde{F}$ and the corresponding EDS equivalence maps the graph of the solution to $F$ to the graph of the solution of $\tilde{F}$.

Not all EDS equivalences come from point transformations. For example, consider the map from $J^1(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ to itself given in coordinates by

$$
\varphi(x^i, u, p_i^i) = (p^1, x^i p^1 - u, x^i).
$$

It is straightforward to check that $\varphi$ pulls back the contact ideal $\{du - p_i \, dx^i\}$ to itself, and so induces an EDS automorphism of $(J^1(\mathbb{R}^n, \mathbb{R}), \{du - p_i \, dx^i\})$. As a consequence, $\varphi$ induces a morphism from any first PDE to a new exterior differential system in such a way that solutions are taken to solutions. For example, $\varphi$ takes the EDS induced by the equation $\frac{\partial u}{\partial x^1} \frac{\partial u}{\partial x^1} = x^i x^i$ to the EDS induced by the equation $\frac{\partial u}{\partial x^1} = x^i x^i$. Solutions to the second equation can be pushed forward by $\varphi^{-1}$ to give solutions to the first. Note that this transformation switches position and derivative variables, so cannot come from any change of coordinates.

This example also demonstrates that linearity of PDE is not preserved by EDS transformations\(^2\). In the category of exterior differential systems, an equation can at best be said to be \textit{linearizable}.

Now consider the following.

\textbf{Question 0.3.} When are two partial differential equations $F$ and $\tilde{F}$ related by a change of variables?

The correspondence between PDEs and EDSs suggests the following more geometric question, whose answer essentially answers Question 0.3.

\textbf{Question 0.4.} When are two given exterior differential systems related by an EDS equivalence?

The ideal $\mathcal{I}$ is extra geometric structure on $M$ that encodes the structure of solutions, so study of the geometry of $(M, \mathcal{I})$ provides insight into this question. This is where the method of equivalence comes in. Knowledge of the geometry of a differential equation also informs our understanding of the solutions of the differential

\(^2\)This is already true for the most general point transformations.
equation. Bryant, Griffiths and Hsu give a good general overview of this philosophy in the monograph *Toward a Geometry of Differential Equations*, [3].

The symbol of a differential equation is a well known example of this. For example, in studying the geometry of the EDS \((M,\mathcal{L})\) associated to a (non-linear) second-order differential equation, the first invariant one finds is the geometric principal symbol, which can locally be given by a symmetric-matrix valued function on \(M\). The signature of this symbol matrix controls the behaviour of solutions, as is well known for the classical division common of linear second-order equations into elliptic, hyperbolic and parabolic classes.

In this paper I provide some answers to Question 0.4 in the specific case of systems arising from scalar parabolic equations. I begin by studying the most general class of scalar 2\(^{nd}\)-order equations that could fairly be called parabolic, the class of PDE given by a single equation of the form

\[
F\left(x^i, u, \frac{\partial u}{\partial x^i}, \frac{\partial^2 u}{\partial x^i \partial x^j}\right) = 0 \quad i, j = 0, \ldots n \tag{0.2}
\]

whose geometric principal symbol is positive semi-definite with 1-dimensional kernel. In more concrete terms, a PDE such as Equation 0.2 is weakly parabolic if its linearization at any 2-jet of a solution is parabolic in the classical sense. If a parabolic equation is furthermore evolutionary, i.e. of the form

\[
\frac{\partial u}{\partial x^0} = F\left(x^0, x^i, u, \frac{\partial u}{\partial x^i}, \frac{\partial^2 u}{\partial x^i \partial x^j}\right) \quad i, j = 1, \ldots n, \tag{0.3}
\]

then it is strongly parabolic or evolutionary parabolic.

The starting point of this paper is the class of exterior differential systems that model weakly parabolic equations. These look pointwise like the exterior differential system modelling the heat equation

\[
\frac{\partial u}{\partial x^0} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x^i \partial x^i},
\]

but are typically more ‘curved.’ Indeed, the generic weakly parabolic equation is neither linear nor evolutionary for any choice of coordinates, even locally.

Even though weakly parabolic equations may not have all of the properties one is used to, this full generality is the geometrically natural place to start. Then the classical cases are picked out in a geometric manner, depending on the values of certain local invariants. This paper, part one of two, is devoted to introducing and studying these invariants.

One important feature of evolutionary parabolic systems is the existence of a natural foliation of solutions by characteristic hypersurfaces. These ‘constant time’ slices are defined by the restriction of the function \(x^0\) to any solution. In contrast to this, weakly parabolic systems still have characteristic hyperplane distributions on each solution, but these are generically not integrable to a foliation. Nonetheless, these distributions are also induced by a hyperplane distribution defined on the entire parabolic
system. Theorem 5.1 states that the characteristic distribution is Frobenius if and only if certain local invariants (the Goursat invariants, Section 5) vanish. This in turn holds if and only if the parabolic system can be written locally in evolutionary form.

The Goursat invariants give a partial answer to Question 0.4, allowing one to decide if a given parabolic equation has an appropriate change of variables that puts it into evolutionary form.

There is another geometrically interesting class of parabolic equations, those of Monge-Ampère form. A second order differential equation for one function of $n + 1$ variables is Monge-Ampère if it is quasi-linear in the minor-determinants of the Hessian, so that it can be written in the form

$$F(x^a, u, p_a, p_{ab}) = \sum_{|I|=|J|} A_{I,J}(x^a, u, p_a) H_{I,J} = 0,$$

(0.4)

where the $I, J$ range over subsets of $\{0, \ldots, n\}$ and $H_{I,J}$ stands for the minor determinant of the hessian matrix

$$H = \left( \frac{\partial^2 u}{\partial x^a \partial x^b} \right)$$

with rows $I$ and columns $J$ deleted.

The second family of invariants that I introduce for parabolic systems are the Monge-Ampère invariants. In Theorem 4.4 I prove that these invariants vanish exactly for Monge-Ampère equations.

This result was already known for 2 and 3 dimensional parabolic systems (Bryant/Griffiths [4] and Clelland [6] respectively). In higher dimensions the proof proceeds through linear-type Monge-Ampère systems, which I introduce in Section 4. These correspond to a particularly simple class of Monge-Ampère equations. My work here generalizes the previously studied cases in two directions: I study parabolic systems in more than 3 variables and I study non-evolutionary equations.

0.2. Conservation Laws

I apply the results of this paper in Part II to study the conservation laws of parabolic systems.

First, I calculate the auxiliary differential equation whose solutions are in bijection with conservation laws of $(M, I)$. This result holds for fully general scalar parabolic equations, including ones not of evolutionary form.

Second, any conservation law of an evolutionary parabolic equation is defined by a Jacobian function that depends on at most second derivatives of solutions. In other words, the KdV type phenomenon cannot happen for strongly parabolic systems—there are no hierarchies of conservation laws depending on increasing numbers of derivatives of solutions. An interesting phenomenon in its own right, this result means that the problem of classifying conservation laws is far more tractable than for general PDE.

Finally, any evolutionary parabolic system with at least one non-trivial conservation law is necessarily a Monge-Ampère equation.

All three of the results have analogues which were proved in $1 + 1$ dimensions by Bryant & Griffiths [4] and in $2 + 1$ dimensions by Clelland [6].
1 Parabolic Systems

1.1. Definition and Examples  The main object studied in this paper are parabolic systems, which are the exterior differential systems associated to 2\textsuperscript{nd}-order scalar parabolic differential equations.

Definition 1.1. A (weakly) parabolic system in \( n + 1 \) variables is a
\[
2n + 2 + (n + 1)(n + 2)/2
\]
dimensional\(^3\) exterior differential system \((M, \mathcal{I})\) such that any point has a neighborhood equipped with a spanning set of 1-forms
\[
\theta_0, \theta_a, \omega^a, \pi_{ab} = \pi_{ba}, a, b = 0, \ldots n \quad (1.1)
\]
that satisfy:

1. The forms \(\theta_0, \theta_a\) generate \(\mathcal{I}\) as a differential ideal.
2. The structure equations
\[
\begin{align*}
\text{d}\theta_0 &\equiv \sum_{a=0}^{n} -\theta_a \wedge \omega^a \quad \text{(mod } \theta_0) \\
\text{d}\theta_a &\equiv \sum_{b=0}^{n} -\pi_{ab} \wedge \omega^b \quad \text{(mod } \theta_0, \theta_b) \quad a, b = 0, \ldots, n.
\end{align*}
\]
3. The parabolic symbol relation
\[
\sum_{i=1}^{n} \pi_{ai} \equiv 0 \quad \text{(mod } \theta_0, \theta_i, \omega^a) \quad a = 0, \ldots, n.
\]

Any such (extended) coframing of \((M, \mathcal{I})\) is called 0-adapted.

Remark 1.2. These exterior differential systems model scalar, parabolic, 2\textsuperscript{nd}-order PDE. The first two conditions exhibit \((M, \mathcal{I})\) as locally equivalent to a 2\textsuperscript{nd}-order differential equation. The third condition shows that the principal symbol is everywhere parabolic.

Parabolic systems are Pfaffian systems, see for example [1], chapter IV. Several of the results there are invaluable in the following.

Remark 1.3. The existence (or not) of an extended 0-adapted coframing near a point of \(M\) is determined entirely by the structure of the ideal \(\mathcal{I}\). If they exist, 0-adapted coframings are compatible with the geometry of the parabolic system \((M, \mathcal{I})\).

\(^3\)This is 1 less than the dimension of \(J^2(\mathbb{R}^{n+1}, \mathbb{R})\), corresponding to the fact that a parabolic equation is defined by a single differential relation on \(J^2(\mathbb{R}^{n+1}, \mathbb{R})\).
Note that the forms in Equation 1.1 are not independent, so do not quite define a coframing on \( M \). One can choose an independent subset of the forms to get a coframing, but there is no geometrically privileged choice. For this reason, it is convenient to work with the extended coframing of Equation 1.1.

It will be also be useful to define a \( G \)-structure \( B_0 \) on \( M \) whose local sections are 0-adapted coframings. In fact, for a particular Lie group \( G_0 \) defined below, the space of parabolic structure \( \mathcal{I} \) on \( M \) is equivalent to the space of \( G_0 \)-structures that satisfy a further torsion condition. This is done in Section 2.1.

**Example 1.4** (Jet formulation of a parabolic equation). Consider the 2-jet bundle \( J^2 = J^2(\mathbb{R}^{n+1}, \mathbb{R}) \) over \( \mathbb{R}^{n+1} \), with jet-coordinates \( x^a, u, p_a, \) and \( p_{ab} = p_{ba} \), where the \( p_a \) correspond to the first derivatives of \( u \) with respect to \( x^a \) and \( p_{ab} \) to the second derivatives. These coordinates may be used to define the contact forms

\[
\hat{\theta}_\emptyset = du - p_a \, dx^a \\
\hat{\theta}_a = dp_a - p_{ab} \, dx^b,
\]

as well as the canonical contact ideal

\[ C = \{ \hat{\theta}_\emptyset, \hat{\theta}_a \}_{EDS} = \{ \hat{\theta}_\emptyset, \hat{\theta}_a, d\hat{\theta}_\emptyset, d\hat{\theta}_a \} \]
on \( J^2 \). The pair \( (J^2, \mathcal{C}) \) is an exterior differential system\(^5\).

An \((n + 1)\)-dimensional submanifold \( \Sigma \) of \( J^2 \) is locally the 2-jet graph of a function \( u \) if and only if

1. the ‘independence condition’ \( dx^0 \wedge \ldots \wedge dx^n \) is nonzero when pulled back to \( \Sigma \) and
2. \( \Sigma \) is an integral manifold of \( (J^2, \mathcal{C}) \)

This fact allows one to represent any second order differential equation as an exterior differential system \( (M, \mathcal{I}) \). A given non-degenerate 2nd order PDE

\[
F \left( x^a, u, \frac{\partial u}{\partial x^a}, \frac{\partial^2 u}{\partial x^a \partial x^b} \right) = 0 \quad a, b = 0, \ldots n \quad (1.2)
\]
defines a function on \( J^2 \), and \( M \) is the zero locus

\[
M = F^{-1}(0).
\]

\(^4\)These are not contact forms in the sense of contact geometry, in which a contact form defines a totally non-integrable hyperplane distribution. However, the concepts are related. In particular, the form \( \hat{\theta}_\emptyset \) can be defined on the space of 1-jets, where it does define a maximally non-integrable distribution.

\(^5\)The geometric structure of this EDS is independent of the choice of coordinates on \( J^2 \) made above. In fact, \( \mathcal{C} \) can be defined intrinsically: Let \( C_x \subset T_x(J^2) \) be the subspace spanned by the tangent planes of all 2-jet graphs which pass through \( x \in J^2 \). This defines the \( n + n(n+1)/2 \) dimensional contact distribution \( C \) on \( J^2 \), and \( \mathcal{C} \) is the differential ideal generated by \( C^\perp \subset \Omega^1(M) \).
Then $\mathcal{I}$ is the pullback of $\mathcal{C}$ to $M$. By construction, integral manifolds of $(M, \mathcal{I})$ for which
\[ dx^0 \wedge \ldots \wedge dx^n|_{\Sigma} \neq 0 \]
are in local bijection with the 2-jet graphs of solutions to $F$.

If $F$ is a parabolic equation, then the EDS $(M, \mathcal{I})$ is a parabolic system. This can be seen by first noting that the coframing\(^6\)
\[ \hat{\theta}_0, \hat{\theta}_a, \hat{\omega}^a = dx^a, \hat{\pi}_{ab} = dp_{ab} \] 
(1.3)
of $M$ is partially adapted to $\mathcal{I}$, in that
\[ \mathcal{I} = \{ \hat{\theta}_0, \hat{\theta}_a \} \]
and conditions 1 and 2 of Definition 1.1 hold. Now, for any GL($\mathbb{R}^{n+1}$)-valued function $(B^a_b)$ on $M$, the coframing
\[ \theta_0 = \hat{\theta}_0, \quad \theta_a = B^a_b \hat{\theta}_b, \quad \omega^a = (B^{-1})^a_b \hat{\omega}^b, \quad \pi_{ab} = B^c_a \hat{\pi}_{cd} B^d_b \]
will also satisfy conditions 1 and 2. On the other hand, there is a non-trivial relation
\[ 0 = dF \equiv \frac{\partial F}{\partial p_{ab}} dp_{ab} \quad (\text{mod } \theta_0, \theta_a, \omega^a), \]
on $M$. By definition, $F$ is parabolic when the ‘symbol’ matrix $\left( \frac{\partial F}{\partial p_{ab}} \right)$ is positive semidefinite. In this case, an appropriate choice for $B^a_b$ will diagonalize the symbol as in condition 3 of Definition 1.1. Thus we have a coframing of $M$ that satisfies conditions 1, 2, and 3.

Any small enough neighborhood of a parabolic system is EDS equivalent to one equipped with such an embedding into $J^2$. See [1] Theorem 5.10 for example, or the proof of Theorem 5.1 for details. However, there are parabolic systems which don’t have a global embedding into $J^2$, such as the Mean Curvature Flow, taken up in Example 1.6.

**Example 1.5 (Heat equation).** Consider the heat equation
\[ \frac{\partial u}{\partial x^0} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x^i \partial x^i} \]
and the corresponding exterior differential system $M$, given by the submanifold $\{ p_0 = p_{ij} \}$ of $J^2(\mathbb{R}^{n+1}, \mathbb{R})$. The coframing given in Equation 1.3 already restricts to a 0-adapted extended coframing of $M$.

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\(^6\)Omitting pullbacks from the notation, which will be done without further comment.
On the other hand, the degenerate form of this equation,

\[
0 = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x^i \partial x^i},
\]

has parabolic principal symbol, but the sub-principal symbol is trivial in the characteristic direction. It will be shown in Section 5 that such degenerate equations can be ruled out geometrically using the Goursat invariant.

**Example 1.6 (Mean Curvature Flow).** Fix an \(n\)-dimensional manifold \(N\). A family \(u: N \times \mathbb{R} \to \mathbb{E}^{n+1}\) of immersions of \(N\) into Euclidean space, paramaterized by \(t \in \mathbb{R}\), satisfies *mean curvature flow* if for each \(t\) and each point \(x \in N\),

\[
\frac{\partial u}{\partial t}(x, t) \cdot \hat{n} = H,
\]

where \(H\) is the mean curvature of \(N\) at \(u(x, t)\) and \(\hat{n}\) the unit normal. I now describe an exterior differential system whose integral manifolds are equivalent to solutions of the mean curvature flow. This system is equivalent to a parabolic system after certain symmetries (technically: Cauchy characteristic directions) are taken into account.

Let

\[
M = \mathbb{R} \times \mathcal{F}(\mathbb{E}^{n+1}) \times S^2(\mathbb{R}^{n+1}),
\]

where \(\mathcal{F}(\mathbb{E}^{n+1})\) is the bundle of orthonormal coframes of \(\mathbb{E}^{n+1}\) and \(S^2(\mathbb{R}^{n+1})\) is the symmetric square. Denote by \(t\) the projection of \(M\) onto the first factor and \(h = (h_{ab})\) the projection onto the last factor. Further, let

\[
H = \sum_{i=1}^{n} h_{ii}
\]

denote the partial trace of \(h\). Since \(\mathbb{E}^{n+1}\) is a space form, \(\mathcal{F}(\mathbb{E}^{n+1})\) may be identified with the Lie group \(\mathbb{R}^{n+1} \rtimes SO(n + 1)\) of Euclidean motions. The components of the Maurer-Cartan form on \(SO(n + 1)\) extend the tautological form \(\eta = (\eta^a)\) of \(\mathcal{F}(\mathbb{E}^{n+1})\) to a unique \(\mathbb{R}^{n+1} \rtimes so(n + 1)\)-valued coframing \(\eta^a, \eta^{ab} = -\eta^{ba}\) so that

\[
d\eta^a = -\eta^{ab} \wedge \eta^b, \quad d\eta^{ab} = -\eta^{ac} \wedge \eta^c.
\]

These can be used to define a partial coframing of \(M\):

\[
\begin{align*}
\theta_\emptyset &= \eta^0 - H dt \\
\theta_j &= dH - h_{0i} \eta^i - h_{00} dt \\
\theta^0 &= dt \\
\zeta_{ij} &= dh_{ij} - h_{ik} \eta^k - h_{jk} \eta^k - H h_{ik} h_{kj} dt \\
\bar{\zeta}_{0i} &= dh_{0i} - h_{0j} \eta^j - H h_{ij} h_{0j} dt \\
\bar{\zeta}_{00} &= dh_{00}
\end{align*}
\]

(1.4)
Let \( \mathcal{I} \) be the ideal generated by \( \theta_a, \theta_\emptyset \).

Consider a solution manifold \( \Sigma \) of the exterior differential system \((M, \mathcal{I})\) for which the form \( \omega^0 \wedge \ldots \wedge \omega^n \) does not vanish. In particular, \( \omega^0 \) is not zero on \( \Sigma \), so each point is contained in a small neighborhood foliated by diffeomorphic level surfaces of \( t \). Without loss of generality then, suppose that

\[
\Sigma = (-1, 1) \times N \subset M,
\]

and let

\[
\Sigma_t := \Sigma \cap \{ t = \text{const.} \} \cong N.
\]

Both \( \theta_\emptyset \) and \( dt \) vanish on a given \( \Sigma_t \), so the form \( \eta^0 \) pulls back to zero on \( \Sigma_t \) as well. Therefore, for each \( x \in \Sigma_t \), the tautological coframing \( \eta^0, \ldots, \eta^n \) of \( E^{n+1} \) is adapted to \( \Sigma_t \), in that

\[
\eta^0 \big|_{T_x \Sigma_t} = 0 \quad \text{and} \quad \eta^0(\hat{n}) = 1.
\]

Since

\[
0 = d\eta^0 = -\eta^0_i \wedge \eta^i
\]
on \( \Sigma_t \), Cartan’s lemma implies that there is a \( S^2(\mathbb{R}^n) \)-valued function \( (\tilde{h}_{ij}) \) so that

\[
\eta^0_i = \tilde{h}_{ij} \eta^j.
\]

The function \( (\tilde{h}_{ij}) \) is the second fundamental form of \( \Sigma_t \) in the orthonormal coframing \( \eta^i \). On the other hand, the \( \theta_a \) vanish on \( \Sigma_t \), so the value of \( h_{ij} \) at each point of \( \Sigma_t \) agrees with \( \tilde{h}_{ij} \). In particular, \( H \) restricts to the mean curvature function on \( \Sigma_t \). Thus, for any parameterization \( u \) of \( \Sigma_t \),

\[
\frac{\partial u}{\partial t}(x) \cdot \hat{n} = \eta^0 \left( \frac{\partial u}{\partial t}(x) \right) = H dt \left( \frac{\partial u}{\partial t}(x) \right) = H,
\]

and the manifolds \( \Sigma_t \) satisfy mean curvature flow.

Conversely, given a solution \( N_t \) to the Mean Curvature Flow, and a fixed orthonormal coframing of \( N \), the graph of \( N_t \) in \( \mathbb{E}^{n+1} \) has a unique lift to and integral manifold of \((M, \mathcal{I})\).

The Coframing (1.4) has been chosen so that

\[
d\theta_\emptyset \equiv -\theta_\emptyset \wedge \omega^a \quad \text{mod} \ \theta_\emptyset
\]

\[
d\theta_a \equiv -\pi_{ab} \wedge \omega^a \quad \text{mod} \ \theta_\emptyset, \theta_a
\].

These structure equations show that the Cartan system of \( \mathcal{I} \) is given by

\[
\mathcal{J} = \{ \theta_\emptyset, \theta_a, \omega^a, \pi_{ab} \}.
\]

It follows from the theory of Cauchy characteristics (see, for example, [1] Chapter 2.2) that:
1. The ideal \( J \) is Frobenius.

2. In any small neighborhood of \( M \) for which the leaf space is a manifold \( M' \), there is an ideal \( \mathcal{I}' \) on \( M' \) that pulls back to \( \mathcal{I} \). In other words, the quotient map \( q: (M, \mathcal{I}) \rightarrow (M', \mathcal{I}') \) is an EDS morphism (so pushes solutions down to solutions).

3. For an integral manifold \( \Sigma \in M' \), the inverse image \( q^{-1}(\Sigma) \) is an integral manifold of \( M \). This defines a bijection between maximal integral manifolds in \( M \) and \( M' \).

The kernel of \( J \) is the space of Cauchy characteristic directions, in that any maximal integral manifold will contain them in its tangent sub-bundle. The map \( q \) quotients down these irrelevant directions.

Finally, it is clear from the Coframing (1.4) that \( M' \) is a parabolic system.

### 1.2. Independence Conditions for Parabolic Systems

As demonstrated by the examples, it is often useful to restrict attention to integral manifolds satisfying a given transversality condition. This is clearest for integral manifolds described in example 1.4, where integral manifolds for which \( dx^0 \wedge \ldots \wedge dx^n \neq 0 \) are in bijection with classical solutions of the given PDE. There are integral manifolds that don’t satisfy this condition, corresponding to generalized solutions. These are also of interest, but it is clearly important to distinguish the two classes.

The condition that the form \( dx^1 \wedge \ldots \wedge dx^n \) restrict to be non-zero is appropriately generalized in the following definition. For clarity, consider an EDS \((M, \mathcal{I})\) that satisfies the constant rank condition that the degree 1 grade of \( \mathcal{I} \),

\[
\mathcal{I}^1 = \mathcal{I} \cap \Omega^1(M),
\]

has constant rank \( s \). This assumption typically holds in examples of interest, including parabolic systems.

**Definition 1.7.** An independence condition for \((M, \mathcal{I})\) is a locally free \( C^\infty(M)\)-submodule \( J \) in \( \Omega^1(M) \) of rank \( n + s \) so that (i) \( \mathcal{I}^1 \subset J \) and (ii) \( J \) has everywhere a local basis

\[
\theta_1, \ldots, \theta_s, \omega^1, \ldots, \omega^n
\]

for which

\[
\mathcal{I}^1 = \{ \theta_1, \ldots, \theta_s \} \quad \text{and} \quad \omega^1 \wedge \ldots \wedge \omega^n \notin \mathcal{I}.
\]

An \( n \)-dimensional integral manifold \( \Sigma \) of \( M \) satisfies the independence condition if \( J|_\Sigma \) has rank \( n \), or equivalently, if

\[
\omega^1 \wedge \ldots \wedge \omega^n|_\Sigma \neq 0
\]

everywhere.

A solution manifold is an integral manifold that satisfies a given independence condition, in accordance with Example 1.4.
Parabolic systems have a natural choice of independence condition, given by the Cauchy system of $\theta_0$. It follows immediately from the structure equations that the Cauchy system of $\theta_0$ is given by the ideal

$$J = \{\theta_0, \theta_a, \omega^a\}_{alg}.$$

By the general theory of Cauchy systems, the ideal $J$ is Frobenius.

2 The Equivalence Problem for Parabolic Systems

2.1. $G_0$-structures; Intrinsic definition of parabolic systems  

I now turn to the description of the $G_0$-structure of a parabolic system.

Let

$$\tilde{V} = \mathbb{R} \oplus W \oplus W^\vee \oplus S^2 W,$$

where $W = \mathbb{R} \oplus \mathbb{R}^n$ and consider the subspace

$$V = \mathbb{R} \oplus W \oplus W^\vee \oplus (S^2 W)_0,$$

where

$$(S^2 W)_0 = \mathbb{R} \oplus \mathbb{R}^n \oplus S^2_0 \mathbb{R}^n$$

is the “spatially-traceless” symmetric product of $W$. It is clear that a 0-adapted coframing on $M$ may be considered as a coframing on $M$ with values in $\tilde{V}$. Furthermore, at each point $x$ of $M$, the image of $T_x M$ is $V$.

One may check that any two 0-adapted coframes at a point of $M$ differ by the action of a matrix $g \in GL(\tilde{V})$ of the form

$$g = \begin{pmatrix}
    k_\varnothing & 0 & 0 & 0 \\
    \tilde{k} & B & 0 & 0 \\
    * & tB^{-1}S & k_\varnothing tB^{-1} & 0 \\
    * & D & BT & C_B/k_\varnothing
\end{pmatrix}$$

for which

$$k_\varnothing \in \mathbb{R}^\times, \quad \tilde{k} = (k_a) \in W$$

$$B = \begin{pmatrix}
    B_0^0 & B_0^0 \\
    0 & bB'
\end{pmatrix} \in \left( \begin{array}{cc}
    \mathbb{R}^\times & \mathcal{L}(\mathbb{R}^n) \\
    0 & CO(n)
\end{array} \right) \subset GL(W), \quad b \in \mathbb{R}^\times, \quad B' \in SO(n).$$

$$S = \begin{pmatrix}
    S_0^{00} & S_0^{0j} \\\n    S_0^{ij} & S_0^{ij}
\end{pmatrix} \in S^2 W \subset \text{Hom}(W^\vee, W),$$

$$T \in S^2 W^\vee \subset \text{Hom}(S^2 W, W^\vee),$$

$$D \in \text{Hom}(S^2 W, W),$$
and finally, the matrix $C_B$ is induced by conjugate transpose action of $B$ on $S^2 W$, so that
\[ C_B(v \circ w) = Bv \circ Bw \quad \forall v, w \in W. \]
The components labeled with a $*$ are unrestricted.

Let $G_0$ be the subgroup of matrices in $GL(V)$ of this form. The canonical representation of $G_0$ on $V$ contains $V$ as a subrepresentation, so we may consider the $V$-valued coframe bundle $\mathcal{F}(M)$ over $M$, whose fiber at each $x \in M$ consists of all of the linear isomorphisms
\[ u: T_x M \longrightarrow V. \]
Then each 0-adapted coframing of $M$ gives a local section of $\mathcal{F}(M)$. The set of coframes in $\mathcal{F}(M)$ that come from 0-adapted coframings defines a $G_0$-structure $\mathcal{B}_0$ on $M$—a $G_0$-principal subbundle of the coframe bundle.

The block solvable structure of $G_0$ corresponds to the following filtration of ideals, which are adapted to the geometry of the parabolic system $(M, I)$ and are well defined independent of any particular choice of coframing:
\[
\begin{align*}
\{\theta_0\} & \subset \{\theta_0, \theta_i\} \subset \{\theta_0, \theta_i, \omega^0\} \subset \{\theta_0, \theta_i, \omega^0, \omega_i\} \\
& \quad \subset \{\theta_0, \theta_i, \omega^0, \omega_i, \zeta_{ij}\} \subset \{\theta_0, \theta_i, \omega^0, \omega_i, \zeta_{ij}, \zeta_{ab}\} = \Omega^*(M).
\end{align*}
\]

Now recall the tautological 1-form of a $G$-structure, which can be used to calculate properties of adapted coframings in a uniform way.

**Definition 2.1.** On the coframe bundle $\pi: \mathcal{F}(M) \rightarrow M$, the tautological form $\eta \in \Omega^1(\mathcal{F}(M), V)$ is defined by
\[ \eta_u(X) = (\pi^* u)(X) \]
for all $u \in \mathcal{F}(M)$ and $X \in T_u \mathcal{F}(M)$.

The tautological form of a $G$-structure $\mathcal{B} \subset \mathcal{F}(M)$ is the pullback of $\eta$ to $\mathcal{B}$.

The tautological form is uniquely characterized by its reproducing property, the property that
\[ \eta^* \eta = \eta \]
for any section $\eta$ of $\mathcal{F}(M)$. For this reason, the tautological form may be thought of as a ‘universal’ choice of coframing for $M$.

It will be useful to employ the vector notation
\[ \theta_0, \quad \Theta = \begin{pmatrix} \theta_0 \\ \theta_i \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega^0 \\ \omega^i \end{pmatrix}, \quad \Pi = \begin{pmatrix} \pi_{00} & \pi_{i0} \\ \pi_{i0} & \pi_{ij} \end{pmatrix} \]
for the components of the tautological form on $\mathcal{B}_0$. With analogous notation for a 0-adapted coframing, the structure equations can be written more concisely as
\[
\begin{align*}
\frac{d\theta_0}{d\Theta} & \equiv -\Theta \wedge \Omega \pmod{\theta_0} \\
\frac{d\Theta}{d\Omega} & \equiv -\Pi \wedge \Omega \pmod{\theta_0, \theta_a}.
\end{align*}
\]
The ideal of semi-basic forms of $B$, denoted $\Omega^*_{sb}$, is generated as a $C^\infty(B)$-module by the component 1-forms of $\eta$.

Cartan’s first structure equation states that on a given $G_0$-structure $B_0$, there is a pseudo-connection

$$\varphi \in \Omega^1(B_0, g_0)$$

(for $g_0$ the Lie algebra of $G_0$) and a torsion map

$$T: B_0 \rightarrow \text{Hom}(\Lambda^2 V, V)$$

so that

$$d\eta = -\varphi \wedge \eta + T(\eta \wedge \eta).$$

Roughly, $\varphi$ measures the variation of $\eta$ in the fiber direction and $T$ measures the first order twisting between fibers. In vector notation,

$$d\begin{pmatrix} \theta_\emptyset \\ \Theta \\ \Omega \\ \Pi \end{pmatrix} = \begin{pmatrix} \kappa_\emptyset & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ * & \sigma & \kappa_\emptyset - \kappa_\emptyset & 0 \\ * & \gamma & \tau & C_\beta - \kappa_\emptyset \end{pmatrix} \wedge \begin{pmatrix} \theta_\emptyset \\ \Theta \\ \Omega \\ \Pi \end{pmatrix} + \begin{pmatrix} T_\theta_\emptyset \\ T_\Theta \\ T_\Omega \\ T_\Pi \end{pmatrix},$$

where the torsion terms $T_\theta_\emptyset, T_\Theta, T_\Omega,$ and $T_\Pi$ are semi-basic. The matrix valued one-forms in Equation (2.4) have components as follows:

$$\beta = \begin{pmatrix} \beta_0^0 & \beta_0^i \\ 0 & \beta_i^i \end{pmatrix} \in \Omega^1(B_0, \begin{pmatrix} \mathbb{R} & \mathbb{R}^n \\ 0 & \mathfrak{so}(n) \end{pmatrix})$$

and

$$\sigma = \begin{pmatrix} \sigma^{00} & \sigma^{0i} \\ \sigma^{ij} & \sigma^{ij} \end{pmatrix} \in \Omega^1(B_0, S^2 W).$$

Note that

$$\beta_i^j = -\beta_j^i$$

for $i \neq j$. Denote the conformal trace component $\beta_{\text{tr}}$ of $\beta$ by

$$\beta_{\text{tr}} := \beta_1^1 = \ldots = \beta_n^n$$

and the components of $\gamma$ by

$$\gamma = \gamma_{bc}.$$

Finally, the action of $C_\beta$ is given by left and right multiplication by the matrix $\beta$.

The reproducing property of the tautological form immediately determines some of the torsion forms. Because (2.3) holds for any 0-adapted coframing $\eta$,

$$\eta^*(T_{\theta_\emptyset}) \equiv \eta^* d\theta_\emptyset = d\theta_\emptyset \equiv -\Theta \wedge \Omega \equiv \text{mod} \theta_\emptyset,$$

and thus

$$T_{\theta_\emptyset} = -\Theta \wedge \Omega + \xi_\emptyset \wedge \theta_\emptyset.$$
for a semi-basic 1-form $\xi$. Adding $\xi$ to $\kappa$ will not affect Cartan’s structure equation, but will absorb the torsion. Doing so simplifies the first component of (2.4) to

$$d\theta = -\kappa \wedge \theta - \theta \wedge \Omega.$$  

It is clear that no other modification of $\kappa$ can be made to absorb the remaining torsion, and that $\kappa$ is uniquely defined up to a multiple of $\theta$.

An analogous calculation shows that there is a matrix of semi-basic 1-forms

$$\xi = \begin{pmatrix} \xi_0^0 & \xi_0^i \\ \xi_i^0 & \xi_i^i \end{pmatrix}$$

so that

$$d\Theta = -\xi \wedge \theta - \kappa \wedge \Omega - \zeta \wedge \Theta.$$  

From this equation it is clear that semi-basic forms may be added to $\beta$ in such a way that the torsion $\xi$ reduces to

$$\xi = \begin{pmatrix} 0 & 0 \\ \xi_i^0 & \xi_i^i \end{pmatrix},$$

where furthermore

$$\xi_i^i = \xi_i^j \quad \text{and} \quad \xi_i^i = 0.$$  

The torsion forms $\xi$ also control the torsions of the forms $\omega^a$. Indeed, from the structure equations,

$$0 = d^2 \theta = d(-\kappa \wedge \theta - \kappa \wedge \omega) + \kappa \wedge \Omega,$$

$$\equiv \kappa \wedge (-\xi \wedge \Omega + T_\Omega) \quad (\text{mod } \theta).$$

An application of the generalized Cartan’s lemma shows that

$$T_\Omega \equiv H \wedge \Theta + \xi \wedge \Omega \quad (\text{mod } \theta),$$

where $H$ is a semi-basic, $S^2 W$-valued 1-form. Thus

$$d\omega \equiv -\sigma \wedge \Theta - (\kappa - \xi) \wedge \Omega + \xi \wedge \Omega \quad (\text{mod } \theta).$$

By modifying $\sigma$ accordingly, all of the torsion $H$ may be absorbed.

Finally, I remark without proof that $\xi$ also determines some of the torsion of $\Pi$. For example, by considering $d^2 \theta_i = 0$, one finds that

$$T_{\pi_{ij}} \equiv -\xi^a \wedge \pi_{aj} + \pi_{ia} \wedge \xi^i \quad (\text{mod } \mathcal{F}).$$

To summarize the structure equations so far,

$$d\theta = -\kappa \wedge \theta - \theta \wedge \omega,$$

$$d\theta_i = -\kappa \wedge \theta - \beta^i \wedge \theta - \xi^i \wedge \theta - \xi_i \wedge \theta - \pi_{ia} \wedge \omega,$$

$$d\theta_0 = -\beta_0 \wedge \theta - \pi_{0a} \wedge \omega,$$

$$d\omega^0 \equiv -\sigma^0 \wedge \theta - (\kappa - \beta_0) \wedge \omega + \xi_0 \wedge \omega \quad (\text{mod } \theta),$$

$$d\omega_i \equiv -\sigma^i \wedge \theta - (\delta^i \kappa - \beta^i) \wedge \omega + \beta_i \wedge \omega + \xi_i \wedge \omega \quad (\text{mod } \theta),$$

$$d\pi_{ab} \equiv -\gamma_{ab} \wedge \theta - \tau_{abc} \wedge \omega + \kappa \wedge \pi_{ab} - \beta_{ab} \wedge \pi_{cb} + \pi_{ac} \wedge \beta_{bc} + T_{\pi_{ab}} \quad (\text{mod } \theta).$$
3 First Invariants of parabolic systems

3.1. Primary Monge-Ampère Invariants

Because the forms $\xi_0^i$ are semi-basic, there are functions $U_j^k, U_0^i, U_i$ on $B_0$ so that

$$\xi_i \equiv U_i^{jk} \pi_{jk} + U_i^j \pi_{j0} + U_i^0 \pi_{00} \quad (\text{mod} \ \theta, \theta_a, \omega^a).$$

The components of these functions that cannot be absorbed comprise the primary Monge-Ampère invariants. Instead of treating them all at once, it is simpler to filter $\xi$ into pieces using the flag (2.2). Then the functions $U_i^*$ split into three levels, and each level defines relative invariants if the previous levels vanish identically.

To see how the $U_i^*$ vary in each fiber, consider the exterior derivative of Cartan’s structure equation. In indices,

$$d\theta_i \equiv -\beta_j^i \theta_j - \xi_j^0 \theta_j - \xi_i^0 \theta_0 - \pi_{ij} \omega^j - \pi_{i0} \omega^0 \quad (\text{mod} \ \theta, \theta_a, \omega^a).$$

Taking the exterior derivative,

$$0 \equiv (d \xi_0^i + \beta_0^j \theta_j + \beta_i^0 \theta_0 + \pi_{ij} \sigma^j + \pi_{i0} \sigma^0 + \xi_j^0 \beta_i^0) \wedge \theta_0 \quad (\text{mod} \ \theta, \theta_a, \omega^a, \Omega_{sb}^2),$$

which in turn shows that

$$0 \equiv d(\xi_0^0) + \delta_i^0 \xi_0^j + \sigma_{ij} \beta_i^0 \pi_{i0} \wedge \sigma_{00} \wedge \xi_j^0 \wedge \beta_i^0 \quad (\text{mod} \ \mathcal{J}, \Omega_{sb}^2). \quad (3.1)$$

At the ‘highest weight’, after plugging in for $\xi_0^i$, equation (3.1) simplifies to

$$0 \equiv (dU_i + \beta_i^0 U_j - 3\beta_0^0 U_i + \kappa_\omega U_i) \wedge \pi_{00} \quad (\text{mod} \ \mathcal{J}, \pi_{ij}, \pi_{i0}, \Omega_{sb}^2).$$

By an application of Cartan’s lemma,

$$dU_i \equiv -\beta_i^j U_j + (3\beta_0^0 - \kappa_\omega) U_i \quad (\text{mod} \ \Omega_{sb}^2).$$

Integrating this, one finds that the vector-valued function $(U_i)$ on $B_0$ is $G_0$-equivariant\(^7\).

Indeed, for a 0-coframe $u$ and a matrix $g$ as in (2.1),

$$(U_i(g \cdot u)) = (B_0^0)^{\mathcal{J}}(B')^{-1}(U_i(u)).$$

In other words, $(U_i)$ is a relative invariant. This is the highest weight primary Monge-Ampère invariant.

If the function $(U_i)$ vanishes identically on $B_0$, then (3.1) reduces to

$$0 \equiv (dU_i^j + \beta_i^k U_k^j - U_i^k \beta_0^j + U_i^j \kappa_\omega - 2\beta_0^j - \delta_i^j \sigma^0) \wedge \pi_{j0} \quad (\text{mod} \ \mathcal{J}, \pi_{ij}, \Omega_{sb}^2),$$

\(^7\)To be precise, this argument only works for the identity component of $G_0$. However, one can check the variation of $(U_i)$ for one element in each component of $G_0$ to see that it really is a relative invariant.
which implies that
\[ d(U^i_j) \equiv -\beta^k_i U^i_k + U^i_k \beta_k^j + (2\beta^0_0 - \kappa_\phi) U^i_j + \delta^j_i \sigma^{00} \pmod{\Omega^{1}_{sb}} \]
for all \( i \) and \( j \). Integrating, the function \((U^i_j)\) varies by the rule
\[
(U^i_j(g \cdot u)) = \left( \frac{B^0_0}{k_\phi} \right)^{-1} (U^i_j(u)) B' + S^{00} \delta^j_i.
\]
Due to the last term, there are choices of coframe for which \( U^i_j \) is traceless. The subbundle of such coframes has structure group consisting of matrices as in (2.1) such that \( S^{00} = 0 \). When restricted to this reduced \( G \)-structure, the remaining component of \((U^i_j)\) is a relative invariant, the next level of the primary Monge-Ampère invariant. Note that the pseudo-connection form \( \sigma^{00} \) becomes semi-basic when restricted to \( B_0 \).

Finally, suppose that \((U^i_i)\) and \((U^i_j)\) vanish identically and the coframe reduction has been carried out. Then (3.1) simplifies to
\[
0 \equiv \left( dU^j_i + \beta^i_j U^i_k - \beta^j_i U^i_k + \beta^i_k U^j_i + (\kappa_\phi - \beta^0_0)U^j_k + \frac{1}{2} \delta^j_i \sigma^{0k} - \frac{1}{2} \delta^j_i \sigma^{0j} \right) \wedge \pi_{jk} \pmod{J},
\]
so that
\[
dU^j_i \equiv -\beta^j_i U_k^i + \beta^i_k U^j_i + \beta^i_l U^j_k + (\beta^0_0 - \kappa_\phi)U^j_k + \frac{1}{2} \delta^j_i \sigma^{0k} + \frac{1}{2} \delta^j_i \sigma^{0j} \pmod{\Omega^{1}_{sb}}.
\]
Integrating, \((U^j_i)\) transforms as
\[
(U^j_i(g \cdot u)) = \left( \frac{B^0_0}{k_\phi} B' \right) (U^j_i(u)) - \frac{1}{2} \delta^j_i S^{0k} - \frac{1}{2} \delta^j_i S^{0j}
\]
in each fiber, where \( B' \) acts by the tensor product representation on \((\mathbb{R}^n)^\vee \otimes S^2_0 \mathbb{R}^n\). A coframe adaptation may be made to absorb the trace components of this representation, so that
\[
U^j_i \equiv 0
\]
for each \( k \). Such coframes are called 1-adapted. After this coframe adaptation, the remaining components of \((U^j_i)\) are relative invariants, the lowest weight piece of the primary Monge-Ampère invariant.

The subbundle of 1-adapted coframes \( B_1 \) has structure group \( G_1 \), which consists of matrices as in (2.1) so that
\[
S^{00} = S^{0i} = 0.
\]
Upon restricting to \( B_1 \), the pseudo-connection forms \( \sigma^{0a} \) become semi-basic, contributing new non-absorbable torsion.

All of the primary Monge-Ampère invariants vanish if and only if there are coframings so that
\[
\xi^0_i \equiv 0 \pmod{\mathcal{J}},
\]
\[
\xi^0_i \equiv 0 \pmod{\mathcal{J}},
\]
in which case
\[ 0 = d^2 \theta_i \equiv (\sigma^{0j} \land \pi_{ij} + \sigma^{00} \land \pi_{i0}) \land \theta_0 \pmod{\theta_0, \theta_i, \omega^a}. \]
When \( n \geq 3 \), multiple applications of Cartan’s lemma show that
\[ \sigma^{0j} \equiv \sigma^{00} \equiv 0 \pmod{J}. \]

**3.2. Secondary Monge-Ampère Invariants**

For a parabolic system with vanishing primary Monge-Ampère invariants, there are functions \( V_{ij}^j, V_{ijk}^j, V_{ijkl}^j \) on \( B_1 \) so that
\[ \xi_i^j \equiv V_{ij}^j \pi_{00} + V_{ijk}^j \pi_{k0} + V_{ijkl}^j \pi_{kl} \pmod{\theta_0, \theta_i, \omega^a}. \]
The non-absorbable components of these functions are the secondary Monge-Ampère invariants. In this case,
\[ d\theta_i \equiv -\xi_i^j \land \theta_j - \beta_i^j \land \theta_j - \pi_{ij} \land \omega^j - \pi_{i0} \land \omega^0 \pmod{\theta_0, \Lambda^2 J}, \]
and thus
\[ 0 \equiv \left( d(\xi_i^j) + \xi_i^k \land \beta_k^j + \beta_i^j \land \xi_k^j + \pi_{ik} \land \sigma^{kj} + d\beta_i^j + \beta_i^k \land \beta_k^j \right) \land \theta_j \pmod{\theta_0, \omega^a, \Lambda^2 J, \Omega^3_{sa}} \]

By an application of Cartan’s lemma,
\[ 0 \equiv d(\xi_i^j) + \xi_i^k \land \beta_k^j + \beta_i^j \land \xi_k^j + \pi_{ik} \land \sigma^{kj} + d\beta_i^j + \beta_i^k \land \beta_k^j \pmod{J, \Omega^2_{sa}} \quad (3.2) \]
One would have to prolong the \( G_1 \)-structure to properly understand the \( d\beta + \beta \land \beta \) term, but that won’t be necessary here. Instead, consider just the component of (3.2) that is symmetric and traceless in \( i \) and \( j \),
\[ 0 \equiv d(\xi_i^j) + \xi_i^k \land \beta_k^j + \beta_i^j \land \xi_k^j + \frac{1}{2}(\pi_{ik} \land \sigma^{kj} + \pi_{jk} \land \sigma^{ki}) \pmod{J, \Omega^2_{sa}}. \]
This equation determines the variation of the secondary Monge-Ampère invariants in each fiber.

Because the argument is very similar to the one used for the primary Monge-Ampère invariants, I simply state the results: Let \( u \) be a coframe in \( B_1 \) and \( g \in G_1 \). The highest weight secondary Monge-Ampère invariants vary by the rule
\[ \left( V_i^j(g \cdot u) \right) = \frac{(B_0^0)^2}{K_0} (B')^{-1} \left( V_i^j(u) \right) B'. \]
If \( (V_i^j) \) vanishes identically, then at the next level
\[ \left( V_i^{jk}(g \cdot u) \right) = \frac{B_0^0}{K_0} B' \cdot \left( V_i^{jk}(u) \right). \]
Here $B'$ acts by the tensor product representation on $S^2_0 \mathbb{R}^n \otimes \mathbb{R}^n$. Finally, if $(V^{jk}_i)$ vanishes, then
\[
\left( V^{jkl}_i (g \cdot u) \right) = \frac{1}{k_\emptyset} B' \cdot \left( V^{jkl}_i (u) \right) + \frac{1}{2} \delta^k_i S_{lj} + \frac{1}{2} \delta^l_i S_{kj},
\]
where $B'$ acts by the tensor product representation on $S^2_0 \mathbb{R}^n \otimes S^2_0 \mathbb{R}^n$.

There is a reduction of coframes so that $V^{jik}_i \equiv 0$ for all $j$ and $k$, which reduces $B_1$ to a $G_{MA}$-structure $B_{MA}$, where $G_{MA}$ is the subgroup of matrices as in (2.1) for which
\[
S = 0.
\]
The pseudo-connection forms $\sigma^{ij}$ are semi-basic when restricted to $B_{MA}$.

Now consider a parabolic system $M$, all of whose Monge-Ampère invariants vanish. When restricted to $B_{MA}$, the torsion form $\xi$ satisfies
\[
\xi_i \equiv \xi^j_i \equiv 0 \pmod{J},
\]
and, when $n \geq 3$, the torsion $\sigma$ satisfies
\[
\sigma^{00} \equiv \sigma^{i0} \equiv \sigma^{ij} \equiv 0 \pmod{J}.
\]
Indeed,
\[
d\theta_i \equiv -\beta^j_i \wedge \theta_j - \pi_{ij} \wedge \omega^j - \pi_{i0} \wedge \omega^0 \pmod{\theta_2, \Lambda^2 J},
\]
and the $ij$-symmetric component of $d^2 \theta^i = 0$ is
\[
0 \equiv \pi_{ij} \wedge \sigma^{jk} \wedge \theta_k \pmod{\theta_2, \omega^a, \Lambda^2 J}.
\]
Several applications of Cartan’s lemma show that
\[
\sigma^{ij} \equiv 0 \pmod{J}.
\]

4 Monge-Ampère Systems

4.1. Definition Parabolic systems whose Monge-Ampère invariants vanish are closely related to a special class of non-linear differential equations. Recall that a Monge-Ampère equation is one of the form
\[
F(x^a, u, p_a, p_{ab}) = \sum_{I,J \subseteq \{0, \ldots, n\}} A_{I,J}(x^a, u, p_a) H_{I,J} = 0, \quad (4.1)
\]
where $H_{I,J}$ range over minors of the Hessian of $u$. This special structure allows one to ‘deprolong’ the associated exterior differential system to a simpler Monge-Ampère system. The following definition is from [2], in the context of Euler-Lagrange equations.
Definition 4.1. A Monge–Ampère system in \( n + 1 \) variables is a \( 2n + 3 \) dimensional exterior differential system \((M, \mathcal{I})\) such that \( \mathcal{I} \) is locally generated by a 1-form \( \theta_\emptyset \) and an \((n + 1)\)-form \( \Upsilon \) satisfying:

1. \( \theta_\emptyset \) is maximally non-integrable:

\[
\theta_\emptyset \wedge (d\theta_\emptyset)^{n+1} \neq 0.
\]

2. \( \Upsilon \) is not in the differential ideal generated by \( \theta_\emptyset \), \( d\theta_\emptyset \).

Equivalently, such systems are defined by local coframings

\[
\theta_\emptyset, \omega^a, \pi_a
\]

so that

\[
d\theta_\emptyset \equiv -\pi_a \wedge \omega^a \quad (\text{mod } \theta_\emptyset)
\]

and

\[
\Upsilon = \sum_{I,J \subseteq \{0, \ldots, n\} \mid |I|=|J|} A_{I,J} \pi_I \wedge \omega_J \neq 0 \quad (\text{mod } \theta_\emptyset, d\theta_\emptyset) \quad (4.2)
\]

for functions \( A_{I,J} \) on \( M \). Here

\[
\pi_I = \prod_{a \in I} \pi_a
\]

and \( \omega_J \) is the omitted index notation,

\[
\omega_J = \pm \prod_{a \notin J} \omega^a,
\]

with signs specified by the condition that

\[
\omega_I \wedge \omega_J = \omega_J \wedge \omega_I.
\]

As an element of \( \mathcal{I} \), the \((n + 1)\)-form \( \Upsilon \) is only defined up to scaling and addition of multiples of \( \theta_\emptyset \) and \( d\theta_\emptyset \). However, by the pointwise Lefschetz decomposition from symplectic linear algebra\(^8\), there is a unique multiple \( \gamma \wedge d\theta_\emptyset \) so that \( \Upsilon + \gamma \wedge d\theta_\emptyset \) is primitive,

\[
(\Upsilon + \gamma \wedge d\theta_\emptyset) \wedge d\theta_\emptyset \equiv 0 \quad (\text{mod } \theta_\emptyset).
\]

So without loss of generality, assume that

\[
\Upsilon \wedge d\theta_\emptyset \equiv 0 \quad (\text{mod } \theta_\emptyset).
\]

With this assumption, the representative \( \Upsilon \) in \( \mathcal{I} \) is uniquely defined up to scaling and multiples of \( \theta_\emptyset \). Furthermore, the primitivity condition guarantees that \( A_{I,J} = A_{J,I} \) in equation (4.2).

\(^8\)See, for example, [2], Proposition 1.1.
Monge-Ampère systems model the solutions of Monge-Ampère equations, as follows. Let $M_{-1} = J^1(\mathbb{R}^{n+1}, \mathbb{R})$ and fix the coframing

$$\theta_\emptyset = du - p_\alpha dx^\alpha, \quad \omega_a = dx^a, \quad \pi_\alpha = dp_\alpha.$$  

Corresponding to equation (4.1), define the $(n+1)$-form

$$\Upsilon = \sum_{|I|=|J|} A_{I,J}(x^a, u, p_\alpha) \pi_I \wedge \omega_{(J)}.$$  

The Monge-Ampère system $(M_{-1}, \{\theta_\emptyset, \Upsilon\})$ has a natural independence condition, defined by the forms $\omega_a$. Then any solution manifold is locally the 1-jet graph of a function $u(x^a)$, and the condition that $\Upsilon$ vanish forces $u$ to be a solution to (4.1).

Conversely, any Monge-Ampère system $(M_{-1}, \{\theta_\emptyset, \Upsilon\})$ is locally modelled by Monge-Ampère differential equations: By the Pfaff Normal Form Theorem, there are local coordinates $x^a, u, p_\alpha$, and a nonzero function $\lambda$ on $M_{-1}$ so that

$$\theta_\emptyset = \lambda(du - p_\alpha dx^a).$$  

The $dx^a$'s determine an independence condition, albeit not contact invariantly. Fix a local coframing

$$\theta_\emptyset = du - p_\alpha dx^a, \quad \omega_a = dx^a, \quad \pi_\alpha = dp_\alpha.$$  

There are functions $A_{I,J}$ so that

$$\Upsilon = \sum_{|I|=|J|} A_{I,J}(x^a, u, p_\alpha) \pi_I \wedge \omega_{(J)}.$$  

Consider a solution manifold $\Sigma$ of $M_{-1}$. Since

$$\theta_\emptyset|_\Sigma \quad \text{and} \quad dx^0 \wedge \ldots \wedge dx^n|_\Sigma \neq 0,$$

$\Sigma$ is locally the graph of functions $u(x^a)$ and $p_\alpha(x^a)$ so that

$$p_\alpha = \frac{\partial u}{\partial x^a}$$

and thus

$$dp_\alpha = \frac{\partial^2 u}{\partial x^a \partial x^b} dx^b.$$  

The condition that $\Upsilon$ vanishes when restricted to $\Sigma$ is equivalent to the condition that $u(x^a)$ solves the equation

$$\sum_{|I|=|J|} A_{I,J} H_{I,J} = 0.$$  

The class of Monge-Ampère systems is manifestly preserved by EDS equivalences, making clear that the class of Monge-Ampère equations is preserved under changes of variables. By contrast, the class of linear differential equations is not preserved by changes of variables. Indeed, a generic change of coordinates takes a linear equation to a non-linear equation.
4.2. Linear Type Systems and Their Symbol  

Generic Monge-Ampère equations do not have constant symbol on $M$. For example, the equation

$$F = \sum_{a \neq b} (p_{aa} p_{bb} - p_{ab}^2) = 0$$

is Monge-Ampère, and

$$dF = \sum_{a \neq b} (p_{aa} dp_{bb} + p_{bb} dp_{aa} - 2p_{ab} dp_{ab}).$$

Consequently, $F$ has any possible symbol, depending on a suitable choice of $(x^a, u, p_a, p_{ab}) \in M$ and it generally does not make sense to ask what the symbol is at a point $x$ of $M_{-1}$, because the answer can depend on the choice of a point in the fiber $(M)_x$.

Nonetheless, there is a suitable class of Monge-Ampère systems that do have a well defined notion of symbol.

**Definition 4.2.** A Monge-Ampère system $(M_{-1}, I_{-1})$ in $n + 1$ variables is of linear type if it has an independence condition $J$, locally spanned by $\theta_0$ and 1-forms $\omega^0, \ldots, \omega^n$, so that

1. $J$ is Lagrangian with respect to $d\theta_0$,

$$d\theta_0 \equiv 0 \pmod{J}.$$

2. For any 1-form $\alpha \in J$,

$$\alpha \wedge \Upsilon \equiv 0 \pmod{\theta_0, \omega^0 \wedge \ldots \wedge \omega^n}.$$ 

The ideal $J$ is called a compatible independence condition for $(M_{-1}, I_{-1})$.

Note that linear-type Monge-Ampère systems do not necessarily arise from linear differential equations. The nomenclature is in line with linear Pfaffian systems, where ‘linear’ refers to linearity of $\Upsilon$ in the complement of the independence condition. Indeed, by condition 2, there are 1-forms $\eta_0$ and a function $L$ so that

$$\Upsilon \equiv \eta_0 \wedge \omega(\alpha) - L \omega(\alpha) \pmod{\theta_0}.$$ 

(4.3)

By condition 1 and the maximal non-integrability of $\theta_0$, the forms $\theta_0$ and $\omega^a$ can be completed to a coframing of $M_{-1}$ by forms $\pi_a$ so that

$$d\theta_0 \equiv -\pi_a \wedge \omega^a \pmod{\theta_0}. $$

It follows from equation (4.3) and the assumed primitivity of $\Upsilon$ that

$$0 \equiv d\theta_0 \wedge \Upsilon \equiv (\eta_0 \wedge \pi_0) \wedge \omega(\alpha) \pmod{\theta_0}. $$
By an application of Cartan’s lemma, there is a symmetric-matrix valued function \((h_{ab})\) so that
\[
\eta_a \equiv h_{ab} \pi_b \pmod{\theta_\emptyset, \omega_a},
\]
and thus,
\[
\Upsilon \equiv h_{ab} \pi_b \wedge \omega_a \pmod{\theta_\emptyset}.
\]
The function \((h_{ab})\) is the unadapted symbol of the linear-type Monge-Ampère system.

In a story analogous to that described for parabolic systems, one can define a \(G\)-structure of coframes adapted to the ideal \(\mathcal{I}_{-1}\), where the structure group consists of matrices of the form
\[
\begin{pmatrix}
k_\emptyset & 0 & 0 \\
* & B & 0 \\
* & tB^{-1}S & tB^{-1}
\end{pmatrix}
\]
for \(k_\emptyset \in \mathbb{R}^\times, B \in \text{GL}(W)\), and \(S \in S^2 W\).

It is not difficult to check that the function \(h_{ab}\), lifted to this \(G\)-structure, varies by the rule
\[
(h_{ab}(g \cdot u)) = \det(B)B^c_d h_{cd}B^d_b
\]
Consequently, there are choices of adapted coframes that diagonalize \(h_{ab}\). In the parabolic case, where the unadapted symbol \(h_{ab}\) is positive semi-definite everywhere, these reduced coframings take the form
\[
\theta_\emptyset, \omega^a, \pi_a
\]
so that
\[
d\theta_\emptyset \equiv -\pi_a \wedge \omega^a \pmod{\theta_\emptyset}
\]
and
\[
\Upsilon \equiv \pi_i \wedge \omega_i \pmod{\theta_\emptyset}.
\]

Observe that for such a coframing, \(\omega^0\) is specified (up to scaling and addition of multiples of \(\theta_\emptyset\)) by the condition that
\[
\omega^0 \wedge \Upsilon \equiv 0 \pmod{\theta_\emptyset}.
\]
In fact, the ideal \(\{\theta_\emptyset, \omega^0\}\) defines the characteristic hyper-plane distribution when restricted to solution manifolds. This provides a useful characterization of parabolic Monge-Ampère systems.

**Definition 4.3.** A Monge-Ampère system of linear type is parabolic if its compatible independence condition contains a 1-form \(\omega^0\) so that \(\theta_\emptyset \wedge \omega^0\) is non-vanishing and so that
\[
\omega^0 \wedge \Upsilon \equiv 0 \pmod{\theta_\emptyset}.
\]
In this case \(\omega^0\) is called the characteristic co-vector.
Equivalently, a parabolic Monge-Ampère system is a Monge-Ampère system
with a compatible coframing
\( \theta, \pi^a, \omega^a \)
so that
\[
d\theta = -\pi^a \wedge \omega^a \quad \text{mod} \ \theta,
\]
and
\[
\Upsilon = \pi^a \wedge \omega^a(1) - L \omega^a \quad \text{mod} \ \theta.
\]

We now have two systems that model the solutions of a parabolic Monge-Ampère
equation: the Monge-Ampère system \((M_{-1}, I_{-1})\) and the second order system \((M, I)\).
The first prolongation of \((M_{-1}, I_{-1})\) along its independence condition, denoted \((M^{(0)}, I^{(0)})\),
is naturally EDS equivalent to \((M, I)\).

As a manifold, \(M^{(0)} \subset \text{Gr}_{n+1}(TM_{-1})\) is the subset of \((n + 1)\)-planes \(E\) that
satisfy the independence condition and the condition \(I_{-1}|_E = 0\). The projection of the
Grassmanian bundle restricts to a submersion
\[
\pi^{(0)}: M^{(0)} \longrightarrow M_{-1},
\]
making \(M^{(0)}\) into an affine-space bundle. The independence condition guarantees that
elements \(E\) in the fiber \(M^{(0)}_x\) are parameterized by numbers \(p_{ab}\) so that
\[
E = \{\theta, \pi_a - p_{ab} \omega^b\}^\perp_x.
\]
Then \(I_{-1}|_E = 0\) ensures that
\[
p_{ab} = p_{ba}, \quad p_{ii} = L.
\]

Given the forms
\[
\theta_a = \pi_a - p_{ab} \omega^b \quad \text{mod} \ \theta
\]
on \(M^{(0)}\), the prolonged ideal of \(I_{-1}\) is defined to be the differential ideal
\[
I^{(0)} = \{\theta, \pi_a\}.
\]

By construction, \(\pi^{(0)}\) is an EDS morphism and the maximal solution manifolds
of \(M^{(0)}\) are in bijection with the maximal solution manifolds of \(M_{-1}\). Furthermore,
the local (extended) coframing
\[
\theta, \pi_a, \omega^a, d p_{ab}
\]
exhibits \(M^{(0)}\) as a parabolic system.

The EDS equivalence \(f\) from \(M\) to \(M^{(0)}\) is defined as follows. At each point
\(x \in M\), consider the \((n + 2)\)-codimension plane
\[
\bar{E}_x = \{\theta, \theta_a\}^\perp_x \subset T_x M.
\]
Its projection to \( M_{-1} \), given by
\[
E_x = d\pi^{(0)}(\tilde{E}_x),
\]
is an \( n + 1 \) dimensional integral element of \( \mathcal{I} \) that satisfies the independence condition. Since \( E_x \) is an element of \( M^{(0)} \), we may define \( f \) by the rule
\[
x \mapsto -E_x.
\]
It follows from the structure equations of \( M \) both that \( f \) is an EDS equivalence.

Given a 2nd order exterior differential system, an obvious question to ask is whether it has a Monge-Ampère deprolongation like the one just described. In the parabolic case, the Monge-Ampère invariants answer this question completely:

**Theorem 4.4.** A parabolic system \((M, \mathcal{I})\) has a deprolongation to a parabolic Monge-Ampère system if and only if its Monge-Ampère invariants vanish identically.

**Proof.** This has been proved for systems in \( n + 1 = 2 \) and \( n + 1 = 3 \) variables by Bryant & Griffiths [4] and Clelland [6] respectively, so I will assume that \( n \geq 3 \).

The forward implication is clear, for if \( M \) has a Monge-Ampère deprolongation, then the coframing \((4.5)\) of its “reprolongation” provides a coframing of \( M \) whose Monge-Ampère invariants vanish. Indeed,
\[
d\theta_i = d\pi_i - p_{ia} d\omega^a - \pi_{ia} \wedge \omega^a = -\pi_{ia} \wedge \omega^a \quad (\text{mod } \theta_\emptyset, \Lambda^2 \mathcal{J})
\]
since both \( \pi_i \) and \( \omega^a \) are pullbacks from \( M_{-1} \).

To see the reverse implication, consider a parabolic system whose Monge-Ampère invariants vanish. After reduction to the \( G_{MA} \)-structure \( B_{MA} \), the structure equations
\[
d\Theta = -\beta \wedge \Theta - \Pi \wedge \Omega \quad (\text{mod } \theta_\emptyset, \Lambda^2 \mathcal{J})
\]
\[
d\Omega = -(\kappa_\emptyset - t\beta) \wedge \Omega \quad (\text{mod } \theta_\emptyset, \Lambda^2 \mathcal{J})
\]
hold. (This is where the assumption that \( n \geq 3 \) is used.)

Recall that \( \mathcal{J} = \{\theta_\emptyset, \theta_a, \omega^a\} \) is a Frobenius distribution and consider a small enough neighborhood in \( M \) so that the space of leaves \( M_{-1} \) is a manifold, which will be the space of the deprolongation. Let
\[
\pi^{(0)}: M \longrightarrow M_{-1}
\]
denote the submersion of \( M \) onto its leaf space.

Consider the \((n + 1)\)-form
\[
\Upsilon_0 = \theta_i \wedge \omega_{(i)}
\]
defined on \( B_{MA} \) and the ideal
\[
\tilde{\mathcal{I}} = \{\theta_\emptyset, \Upsilon_0\}.
\]
From Equation 4.6 it follows that
\[
d(\theta_i \wedge \omega(i)) \equiv -((n-2)(\kappa_\emptyset - \beta_{\tau\nu}) - \beta_0^0) \wedge \theta_i \wedge \omega(i) \quad \text{(mod } \theta_\emptyset, \Lambda^{n+2} J) \tag{4.7}
\]
\[
eq -((n-2)(\kappa_\emptyset - \beta_{\tau\nu}) - \beta_0^0) \wedge \theta_i \wedge \omega(i) \quad \text{(mod } \theta_\emptyset, \ d\theta_\emptyset),
\]
where the second equation follows from \(\Lambda^{n+2} J \subset \{\theta_\emptyset, \ d\theta_\emptyset\}\). From Equations (4.6) and (4.7) it follows that \(\tilde{I}\) is invariant\(^9\) in each fiber of \(B_{MA}\), so is the pullback of an ideal on \(M\), also denoted \(\tilde{I}\).

Furthermore, the Cartan system of \(\tilde{I}\) is \(J\), so there exists a 1-form \(\tilde{\theta}\) and an \((n+1)\)-form \(\Upsilon\) on \(M-1\) so that the ideal
\[
\mathcal{I}_{-1} = \{\tilde{\theta}, \Upsilon\}
\]
pulls back by \(\pi^{(0)}\) to generate \(\tilde{I}\). The pair \((M-1, \mathcal{I}_{-1})\) is a parabolic type Monge-Ampère system: There is a non-zero function \(\lambda\) on \(M\) so that
\[
(\pi^{(0)})^* \theta = \lambda \theta_\emptyset,
\]
so if \(\theta \wedge (d\theta)^{n+1}\) vanished somewhere, then its pullback to \(M\) would also vanish, contradicting the structure equations of \(\theta_\emptyset\) for a parabolic system. Similarly, \(\Upsilon \neq 0 \pmod{\tilde{I}}\), or else the pullback equation \(\Upsilon_0 \equiv 0 \pmod{\theta_\emptyset, \ d\theta_\emptyset}\) would hold.

To see that \(M_{-1}\) is of linear parabolic type, observe that (4.6) also implies that \(J\) is the Cartan system of the ideal
\[
\tilde{J} = \{\theta_\emptyset, \omega^0\},
\]
so \(\tilde{J}\) pushes down to an ideal \(J\) on \(M_{-1}\) for which \((\pi^{(0)})^* J = \tilde{J}\). This ideal is necessarily generated locally by \(n+2\) independent 1-forms, including \(\theta\), so it defines an independence condition. The compatibility of this independence condition follows from the injectivity of \((\pi^{(0)})^*\) on forms. For example, for \(\alpha \in J\), the equation
\[
(\pi^{(0)})^* (d\theta) \equiv \lambda d\theta_\emptyset \equiv 0 \pmod{\tilde{J}}
\]
implies
\[
d\theta \equiv 0 \pmod{J}.
\]

Finally, the ideal
\[
\{\theta_\emptyset, \omega^0\}
\]
also has Cartan system \(J\), and the corresponding ideal on \(M_{-1}\) provides the characteristic form for \((M_{-1}, \mathcal{I}_{-1})\).

The map \(f\) defined prior to the proposition gives an EDS equivalence between \(M\) and the first prolongation of \(M_{-1}\), so the theorem is proved. \(\square\)

**Example 4.5.** It is not difficult to check from the coframing given in Example 1.6 that Monge-Ampère invariants of the mean curvature flow vanish.

\(^9\)The Lie derivative of any element of \(\tilde{I}\) in any fiber direction is again in \(\tilde{I}\). This follows from Cartan’s formula for the Lie derivative.
5 The Goursat Invariant

5.1. Definition Continuing on the equivalence problem, assume that the primary Monge-Ampère invariants of a parabolic system $M$ vanish. Then there are functions $a_{ij}$ and $a_i$ on $B_1$ so that

$$\xi_0^i \equiv a_{ij}\omega^j + a_i\omega^0 \quad (\text{mod } \theta_\varnothing, \theta_a).$$

In this case,

$$0 = d^2\theta_i = d(-\kappa_i \wedge \theta_\varnothing - \beta_i^j \wedge \theta_j - \pi_{ij} \wedge \omega^j - \pi_{i0} \wedge \omega^0 - \xi_0^i \wedge \theta_0 - \xi_0^j \wedge \theta_j) \equiv (-\beta_i^j \wedge \xi_0^j - \gamma_0^i \wedge \omega^j - d\xi_0^i + \beta_0^i \wedge \xi_0^j) \wedge \theta_0 \quad (\text{mod } \theta_\varnothing, \theta_a, \omega^0, \Omega_{ab}^3).$$

gives, after an application of Cartan’s lemma,

$$d\xi_0^i \equiv \beta_0^0 \wedge \xi_0^i - \beta_i^j \wedge \xi_0^j - \gamma_0^i \wedge \omega^j \quad (\text{mod } \theta_\varnothing, \theta_a, \omega^0, \Omega_{ab}^2).$$

Plugging in for $\xi_0^i$, one finds that

$$da_{ij} \equiv (\beta_0^0 + \kappa_0)a_{ij} - \beta_i^k a_{kj} - a_{ik}\beta_j^k - \gamma_0^i \quad (\text{mod } \Omega_{ab}^1),$$

and thus that the functions $a_{ij}$ vary in the fiber of $B_1$ by the rule

$$(a_{ij}(g \cdot u)) = B_{0}^0 k_{\varnothing} (B' \cdot 1 (a_{ij}(u)) / (B')^{-1} - D_{ij}^0 \quad (5.1)$$

for all $u \in B_1$ and $g \in G_1$. Since $D_{ij}^0$ is symmetric, there are choices of coframe absorbing the symmetric part of $a_{ij}$. These adapted coframes form a $G_2$-structure on $M$, where $G_2$ consists of matrices as in (2.1) for which

$$S_{0a} = 0 \quad \text{and} \quad D_{ij}^0 = 0.$$

The pseudo-connection forms $\gamma_0^i$ reduce to semi-basic torsion forms on $B_2$.

Upon restricting to $B_2$, two relative invariants become well defined: 1) the remaining anti-symmetric component of $(a_{ij})$ is the primary Goursat invariant, which measures integrability of the characteristic distribution by vanishing for a parabolic system if and only if it can be put in evolutionary form (see Theorem 5.2 for proof.) and 2) the classical Goursat invariant.

The classical Goursat invariant is given by the function $a$ on $B_2$ so that

$$\sum_{i=1}^{n} \pi_{ii} \equiv a\theta_0 \quad (\text{mod } \theta_\varnothing, \theta_i, \omega^a).$$

---

$^10$This group is unrelated to the exceptional simple group $G_2$. 

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Taking the derivative of both sides shows that
\[
((\kappa_\emptyset - 2\beta_{tr} + \beta_0^0)a - da) \wedge \rho_0 \equiv 0 \quad \text{(mod } \theta_\emptyset, \theta^a, \omega^a, \Omega^2_{sb}) ,
\]
so that \(a\) is a relative invariant that scales by the rule
\[
(a(g \cdot u)) = \frac{B_0^0 F_{\emptyset}}{b^2} a(u).
\]
It follows that for a point \(x \in M\), either \(a\) is identically zero in the fiber of \(B_2\) over \(x\) or it is nonzero in each fiber over a neighborhood of \(x\). In the latter case, there a reduction to the bundle of coframes for which \(a = 1\), which reduces \(B_2\) to a \(G_3\)-structure, where \(G_3\) consists of matrices in \(G_2\) for which
\[
B_0^0 = b^2 / \kappa_\emptyset .
\]
Furthermore, \(a\) is non-zero at \(x \in M\) if and only if the subprincipal symbol is non-degenerate. Indeed, fix a parabolic coframing
\[
\theta_\emptyset, \theta^a, \omega^a, \pi_{ab}
\]
compatible with the \(G_3\)-structure on \(M\) near \(x\) and an EDS embedding of \(M\) into \(J^2(\mathbb{R}^{n+1}, \mathbb{R})\) so that the canonical coframing agrees with the parabolic coframing at \(x\),
\[
(\theta_\emptyset)_x = (\hat{\theta}_\emptyset)_x, \quad (\theta_a)_x = (\hat{\theta}_a)_x, \quad (\omega^a)_x = (\hat{\omega}^a)_x, \quad (\pi_{ab})_x = (\hat{\pi}_{ab})_x.
\]
Then at \(x\),
\[
(\pi_{ij})_x \equiv (\theta_0)_x \quad \text{(mod } \theta_\emptyset, \theta^a, \omega^a)
\]
if and only if
\[
0 = dF_x \equiv \frac{\partial F}{\partial p_{ab}}(x) dp_{ab} - \frac{\partial F}{\partial p_0}(x) dp_0 \equiv \theta_0 \wedge (\pi_{ij} - (\theta_0)_x) \quad \text{(mod } \theta_\emptyset, \theta^a, \omega^a) ,
\]
where \(F\) is a local function on \(J^2(\mathbb{R}^{n+1}, \mathbb{R})\) cutting out \(M\).

Turning to the primary Goursat invariant, suppose that \(a_{ij}\) is identically zero. Then there is a matrix valued function \((N^b_a)\) and an anti-symmetric matrix valued function \((M^{ab})\) so that
\[
d\omega^0 \equiv -\kappa^0 \wedge \theta \emptyset + M^{ab} \theta_a \wedge \theta_b + N^b_a \theta_b \wedge \omega^a \quad \text{(mod } \omega^0) .
\]
Since
\[
0 = d^2 \omega^0 \equiv -N^b_a \pi_{bc} \wedge \omega^c \wedge \omega^a \quad \text{(mod } \theta_\emptyset, \theta_a, \omega^0) ,
\]
it must be true \((n > 2)\) that
\[
N^b_a = N_{tr} \delta^b_a
\]
for a function \(N_{tr}\) on \(M\). Replacing \(\omega^0\) with \(\omega^0 + N_{tr} \theta_\emptyset\) gives in a new 3-adapted coframing where furthermore
\[
d\omega^0 \equiv M^{ab} \theta_a \wedge \theta_b \quad \text{(mod } \theta_\emptyset, \omega^0) .
\]
The set of coframes with $N_{tr}$ absorbed is a $G_4$-structure $B_4$, where $G_4$ consists of the matrices in $G_3$ for which

$$\kappa^0 = 0.$$  

The psuedo-connection form $\kappa^0$ is semi-basic when restricted to $B_4$. Furthermore, with $N_{tr} = 0$,

$$0 = d^2 \omega^0 \equiv 2 M^{ab} \pi_{ai} \wedge \theta_b \wedge \omega^i - \kappa^0 \wedge \theta_i \wedge \omega^i \quad \text{(mod } \theta, \omega_0, \Lambda^2 I^1),$$

and it follows that

$$(M^{ab}) = 0.$$  

Finally, since

$$0 = d^2 \omega^0 \equiv \kappa^0 \wedge \theta_i \wedge \omega^i \quad \text{(mod } \theta, \omega^0),$$

it follows that

$$\kappa^0 \equiv 0 \quad \text{(mod } \theta, \omega^0),$$

and thus

$$d\omega^0 \equiv 0 \quad \text{(mod } \omega^0).$$

in any 4-adapted coframing.

To summarize, when the primary Goursat invariant vanishes, there is a reduction of coframes to a $G_4$-structure, on which the structure equations simplify to

$$d\theta_i \equiv -\pi_{ij} \wedge \omega^j \equiv \pi_{i0} \wedge \omega^0 \quad \text{(mod } \theta, \theta_i)$$

and

$$d\omega^0 \equiv 0 \quad \text{(mod } \omega^0).$$

5.2. Geometric interpretation of the Goursat Invariant  

It follows from the general theory of characteristics for linear Pfaffian systems (see [1], Chapter IV) that a parabolic system $M$ has a well defined characteristic ideal, which given in a coframing by

$$C = \{\theta, \theta_a, \omega\}.$$  

The forms $\eta_i$ and $\eta_a$ vanish on any solution manifold $\Sigma$ in $M$, so the restriction of $C$ to a $\Sigma$ defines a hyperplane distribution, the characteristic distribution.

For a generic parabolic system, $d\omega^0 \wedge \omega^0 \neq 0$, so the characteristic distribution is non-integrable on solution manifolds. By contrast, for a parabolic system corresponding to an evolutionary equation

$$p_0 = F(x^a, u, p_i, p_{ij}),$$

the level sets of the coordinate $x_0$ define a natural foliation of $M$ into ‘time’ slices. It follows from equation (5.2) that there is a 0-adapted coframing of $M$ for which $\omega^0$ is a multiple of $dx^0$. Thus, by explicit construction, the characteristic distribution of an evolution equations is Frobenius on all solution manifolds.

The following theorem proves the converse and makes a connection with the invariants just developed.

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Theorem 5.1. For a real analytic parabolic system \((M,I)\) in \(n+1 > 3\) variables, the following conditions are equivalent:

1. The primary Monge-Ampère invariants and the Goursat invariant \(\alpha_{ij}\) vanish identically.

2. \(M\) has a 3-adapted coframing so that
   \[ d\omega^0 \equiv 0 \pmod{\theta_\varnothing, \theta_a, \omega^0}. \]

3. \(M\) has a 4-adapted coframing so that
   \[ d\omega^0 \equiv 0 \pmod{\omega^0}. \]

4. \(M\) is locally equivalent to a parabolic equation in evolutionary form.

5. The characteristic distribution is Frobenius on every solution manifold.

Proof. Conditions (1), (2), and (3) are equivalent by the work of the previous sections. In particular, the coframing given in condition (3) is necessarily 4-adapted. The discussion immediately prior to this theorem shows that (4) implies (5).

Proof of (5) \(\Rightarrow\) (2): Fix a local 0-adapted coframing \(\eta\) near \(x \in M\). Over \(x\), the set of integral elements \(E \in T_x M\) satisfying the independence condition is in bijection with symmetric traceless \((p_{abc}) \in (S^3 \mathbb{R}^{n+1})_0\) via the rule
\[
E = \{ \theta_\varnothing, \theta_a, \pi_{ab} - p_{abc} \omega^c \}_x.
\]
Since \((M,I)\) is involutive, the Cartan-Kähler theorem\(^{11}\) guarantees that there exists an integral manifold
\[
\iota : \Sigma \to M
\]
for which \(x \in \Sigma\) and
\[ T_x \Sigma = E. \]

At \(x\), the pullback of \(\omega^0\) to \(\Sigma\) satisfies
\[
(\iota^* d\omega^0)_x \equiv (U_{ij}^0 + U_{ik}^j p_{jkl} + U_{ij}^k p_{jkl} + a_{ij}) \omega^j \wedge \omega^i \pmod{\iota^\ast \omega^0}. \]

On the other hand, the characteristic foliation is integrable on \(\Sigma\), so
\[
(\iota^* d\omega^0)_x \equiv 0 \pmod{\iota^\ast \omega^0}, \]
and thus
\[
(U_{ij}^0 + U_{ik}^j p_{jkl} + U_{ij}^k p_{jkl} + a_{ij}) \omega^j \wedge \omega^i = 0.
\]

\(^{11}\)Here is where the assumption of real analyticity is needed. For reference, see [1] Chapter III.
Since the choice of integral element was arbitrary, this holds for any choice of \((p_{abc})\) symmetric and trace-free.

By choosing \(p_{jko} = p_{jkl} = 0\) and \(p_{j00} \neq 0\), it is clear that \((U_i)\) vanishes identically. (Recall that as a relative invariant, when \((U_i)\) vanishes anywhere in the fiber of \(B_0\) over \(x\), it vanishes in the whole fiber.) Next, setting \(p_{jko} = 1\) for each \(j \neq k\), one finds that \(U_j^j = U_k^k\), so the matrix the matrix \(U_i^i\) must be a multiple of the identity. Assume without loss of generality that \(\eta\) is adapted so that \(U_k^i\) vanishes identically.

Finally, setting \(p_{ijk} = 1\) for \(i \neq j \neq k\) demonstrates that \(U_{ij}^i = U_{ji}^j\) (and for cyclic permutations of \(i, j, k\)), so \(\eta\) can be adapted to the Goursat invariant. It is now clear that the \(a_{ij}\) must also vanish.

**Proof of (3) \(\Rightarrow\) (4):** Fix a 4-adapted framing near a point \(y \in M\). By rescaling \(\omega^0\) if necessary, there is a function \(x^0\) near \(y\) on \(M\) so that
\[
\omega^0 = dx^0.
\]

I construct jet coordinates near \(y\) that extend \(x^0\) and give a local EDS embedding of \(M\) into \(J^1(R^m+1, R)\), exhibiting \(M\) as locally equivalent to a parabolic equation in evolutionary form.

First note that
\[
\theta_\emptyset \wedge d\theta_\emptyset^{n+1} \wedge dx^0 = 0,
\]
so after possibly rescaling \(\theta_\emptyset\), there are functions \(x^i, u, p_a\) on \(M\) so that
\[
\theta_\emptyset = du - p_i dx^i - p_0 dx^0.
\]

Since
\[
d\theta_\emptyset \equiv - dp_i \wedge dx^i - dp_0 \wedge dx^0 \pmod{\theta_\emptyset}
\equiv - \theta_i \wedge \omega_i - \theta_0 \wedge dx^0 \pmod{\theta_\emptyset},
\]
there is a symplectic-matrix valued function, with block components of the form
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]
\(tAC, tBD\) symmetric,
\(tAD - tCB = I\)

so that
\[
\theta_a \equiv A^a_i dp_i + B_{ab} dx^b \pmod{\theta_\emptyset}
\]
\[
\omega^a \equiv C^{ab} dp_b + D_a^b dx^b \pmod{\theta_\emptyset}.
\]

By adding \(p_i x^i\) to \(u\), any \(x^i\) can be exchanged with the corresponding \(p_i\) if necessary, so assume without loss of generality that \(A\) is invertible and write
\[
A = \begin{pmatrix}
a & t\nu \\
v & A'
\end{pmatrix} \in \begin{pmatrix}
\mathbb{R}^\times & \mathbb{R}^n \\
\mathbb{R}^n & \text{GL(}\mathbb{R}^n\text{)}
\end{pmatrix}.
\]
It is clear from $\omega^0 = dx^0$ that
\[ C^{0a} = 0, \quad D_i^0 = 0, \quad \text{and} \quad D_0^0 = 1, \]
so write
\[ C = \begin{pmatrix} 0 & 0 \\ u & C' \end{pmatrix} \in \left( \mathbb{R}^\times \times \mathbb{R}^n \oplus \mathfrak{gl}(\mathbb{R}^n) \right). \]

Since $tAC$ is symmetric, $C'(A')^{-1}$ is too, and after the parabolic change of coframing
\[ \omega^i \mapsto (C'(A')^{-1})^{ij} \omega_j, \]
we may assume without loss of generality that $C' = 0$. In fact, once $C' = 0$, it follows from the symmetry of $tAC$ that $C = 0$ and then that
\[ D = tA^{-1}. \]

Since it remains true that $\omega^0 = dx^0$, it must also be true that $v = 0$ and $a = 1$ in $A$. Let
\[ p_{ab} = -(A^{-1})^c_b B_{cb}, \]
which is symmetric in the indices $a, b$ because $tBD = tB'A^{-1}$ is symmetric. Then
\[ \theta_a = (A^{-1})^c_b dx^c \quad (mod \theta_a) \]
and from
\[ d\theta_a = -(A^{-1})^c_b dx^c \quad (mod \theta_a), \quad (5.3) \]
it follows that
\[ \pi_{ab} = A_c^{a} dp_{cd} A_d^b \quad (mod \theta_a, \theta_a, \omega^a). \]

In particular, the exterior derivatives of the functions $x^a, u, p_a, p_{ab}$ generate the differential forms on $M$ and satisfy a single relation
\[ F(x^a, u, p_a, p_{ab}) = 0. \]

The functions $x^a, u, p_a, p_{ab}$ define a local EDS embedding of $M$ into $J^2(\mathbb{R}^{n+1}, \mathbb{R})$, and the image of $M$ is cut out by $F$.

$F$ defines an evolutionary equation equivalent to $M$. First, note that the forms $\pi_{ab}$ are independent of the rest of the coframe forms for any parabolic coframing of $M$,
\[ \pi_{00} \wedge \pi_{01} \wedge \ldots \wedge \pi_{mn} \neq 0 \quad (mod \theta_a, \theta_a, \omega^a, \omega^a). \]

So, from the symbol relation of $F$,
\[ 0 = dF \equiv \frac{\partial F}{\partial p_{ab}} dp_{ab} \equiv (A^{-1})^0_a \frac{\partial F}{\partial p_{ab}} (tA^{-1})^c_a dx_c \quad (mod \theta_a, \theta_a, \omega^a, \omega^a). \]
it follows that
\[(A^{-1})^0 \frac{\partial F}{\partial p_{ab}} (tA^{-1})^0 = 0\]
for \(c = 0, \ldots, n\). For \(c = 0\), this simplifies to
\[(A^{-1})^0 \frac{\partial F}{\partial p_{00}} (tA^{-1})^0 = 0,\]
so \(\frac{\partial F}{\partial p_{00}} = 0\). When \(c = i = 1, \ldots, n\),
\[(A^{-1})^0 \frac{\partial F}{\partial p_{ij}} (tA^{-1})^0 = 0,\]
and since \(A'\) is invertible, it follows that \(\frac{\partial F}{\partial p_{ij}} = 0\). We conclude that

\[F = F(x^a, u, p_i, p_{ij}).\]

Finally, since there is only one relation between the coframe forms of \(M\), there is a nonzero function \(\lambda\) so that

\[\lambda(\pi_{ii} - \theta_0) \equiv dF \equiv \frac{\partial F}{\partial p_{ab}} dp_{ab} + \frac{\partial F}{\partial p_0} dp_0 \quad (\text{mod } \theta, \theta_i, \omega^a) \quad (5.4)\]

For 4-adapted coframings, a stronger version of Equation 5.3 holds:

\[d\theta_i \equiv -(A^c_i dp_{cd} A^d_i) \wedge \omega \quad (\text{mod } \theta, \theta_i)\]

and so

\[\pi_{ij} \equiv A^c_i dp_{cd} A^d_i \quad (\text{mod } \theta, \theta_k, \omega) \quad k = 1, \ldots, n.\]

Then Equation 5.4 simplifies to

\[-\lambda \theta_0 \equiv \frac{\partial F}{\partial p_0} dp_0 \quad (\text{mod } \theta, \theta_i, \omega^a, \pi_{ij}).\]

It follows immediately that \(\frac{\partial F}{\partial p_0} \neq 0\), so by the implicit function theorem, \(F\) is equivalent to an evolutionary equation

\[p_0 = G(x^a, u, p_i, p_{ij}).\]

\[\square\]

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