Induction of quantum group representations

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Abstract

In this paper we will define a generalized procedure of induction of quantum group representations both from quantum and from coisotropic subgroups proving also their main properties. We will then show that such a procedure realizes quantum group representations on generalized quantum bundles.

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1 Introduction

Among the various methods to construct representations of a given Lie group, the induction procedure plays with no doubt a central role. On one hand it gives a strong insight in the representation theory of a large class of non semisimple Lie groups. For example it solves completely the semidirect product case, linked as it is to the orbit method for nilpotent and solvable groups. On the other hand it gives foundations to the geometric theory of representations of semi simple Lie groups, from Borel-Weil-Bott theorem on (see for example [21]). Induction of quantum group representations was developed in [18] and, later, in [13]. Recently a very detailed work [16] summarizes results for the compact case. One of the problems with induced quantum group representations is connected with the extreme rarity of quantum subgroups. For example in [2], [17], [20] induced representations for some non semisimple quantum groups are constructed. The last two paper, however, does not fit well the Parshall-Wang setting because they do not start with a well defined
quantum subgroup. To overcome this problem the theory of coisotropic quantum subgroups introduced in [11] seems useful. In this paper we generalize induced quantum group representations to coisotropic quantum subgroups introduced in [11] and show how the corresponding corepresentation space of the function algebra can be interpreted as the space of sections of an associated vector bundle to a principal coalgebra quantum bundle, this last concept being recently introduced by Brzeziński in [5] and [8]. We will not deal with the measure-theoretic concepts needed to develop correctly all unitariness aspects, which are the subject of a forthcoming paper.

2 Quantum coisotropic subgroups

In this section we will briefly summarize some basic of the theory of subgroups of quantum groups, as developed in [4] and [11] (to which we refer for motivations). A more detailed account of the problems we’ll deal with can also be found in [12], while for the general theory of quantum embeddable homogeneous spaces we refer to [14].

Definition 2.1 Given a Hopf algebra $A_q$ we will call quantum subgroup of $A_q$ any pair $(B_q, \pi)$ such that $B_q$ is a Hopf algebra and $\pi : A_q \to B_q$ is a Hopf algebra epimorphism. We will call quantum coisotropic left (resp. right) subgroup any pair $(C_q, \sigma)$ such that $C_q$ is a coalgebra and a left (resp. right) $A_q$-module and $\sigma : A_q \to C_q$ is a surjective linear map which is also a coalgebra and a left (resp. right) $A_q$-module homomorphism.

As usual in the classical case subgroups can be identified through the kernel of the restriction epimorphism. Thus quantum subgroups are in 1:1 correspondence with Hopf ideals in $A_q$ while left (right) coisotropic subgroups are in 1:1 correspondence with bilateral coideals which are also left (right) ideals.

The weaker hypothesis defining coisotropic quantum subgroup is not descending from a need of generality for its own sake. It is rather quite naturally imposed by two strictly connected aspects. First of all, quantum subgroups in the semiclassical limit correspond to Poisson-Lie subgroups, a fairly rare object. Secondly, while not any quantum embeddable homogeneous space comes out of a quotient by a quantum subgroup, a 'big' family of those can be obtained as a quotient by a coisotropic subgroup. All known families of embeddable homogeneous spaces are of this kind. Furthermore it has been proven by Etingof and Kazhdan that any quotient by a classical coisotropic subgroup of a Poisson-Lie subgroup can be quantized. The same statement does not hold for more general Poisson homogeneous spaces.

More precisely we have:
Definition 2.2 Let $A_q$ be a Hopf algebra. A right (resp. left) quantum embeddable homogeneous space of $A_q$ is a subalgebra which is also a right (resp. left) coideal.

Proposition 2.3 Let $(C, \sigma)$ be a right (resp. left) coisotropic quantum subgroup of $A_q$. Then

$$B_C = \{ f \in A_q | (\sigma \otimes id) \Delta f = \sigma(1) \otimes f \}$$

is a quantum right embeddable homogeneous space and, respectively

$$B^C = \{ f \in A_q | (id \otimes \sigma) \Delta f = f \otimes \sigma(1) \}$$

is a quantum left embeddable homogeneous space. Conversely let $B_q$ be a right (resp. left) embeddable quantum homogeneous space; then the right (resp. left) ideal generated by $\{ b - \varepsilon(b)1 \}$ is a bilateral coideal in $A_q$ and identifies a right (resp. left) coisotropic subgroup.

The annoying necessity of dealing with both left and right coisotropic subgroups (which becomes rather involved adding considerations both on left and right corepresentations) disappear when considering the unitary side of the theory, which forces precise choices.

3 Quantum induction

In this section we define induced corepresentations from coisotropic quantum subgroups and prove their main properties. Let us begin recalling some definitions. Let $A$ be a Hopf algebra and let $V$ be a vector space (we’re considering a fixed base field of zero characteristic). $V$ is said a right corepresentation for $A$ if there exists a linear map

$$\psi_R : V \rightarrow V \otimes A \quad (3.1)$$

such that:

$$(id \otimes \varepsilon) \circ \psi_R = id_V \quad (3.2)$$

$$(id \otimes \Delta) \circ \psi_R = (\psi_R \otimes id_A) \circ \psi_R \quad (3.3)$$

We will say that $V$ is a left corepresentation for $A$ if there exists a linear map

$$\psi_L : V \rightarrow A \otimes V \quad (3.4)$$

such that:

$$(\varepsilon \otimes id) \circ \psi_L = id_V \quad (3.5)$$

$$(\Delta \otimes id) \circ \psi_L = (id_A \otimes \psi_L) \circ \psi_L \quad (3.6)$$

Let us remark explicitly that above definitions are well posed for any coalgebra $A$. 

Let us consider now fixed a quantum group \( A_q = A_q(G) \) and a coisotropic quantum subgroup \((C_q, \pi) = (A_q(K), \pi)\) (when we do not specify coisotropic subgroups to be left or right we mean the result is valid in both cases).

**Proposition 3.1** The map \( R = (id \otimes \pi) \circ \Delta_G : A_q(G) \rightarrow A_q(G) \otimes A_q(K) \) defines a right corepresentation of \( A_q(K) \) on \( A_q(G) \). Similarly the map \( L = (\pi \otimes id) \circ \Delta_G \) defines a left corepresentation.

**Proof:**

We will verify conditions 3.2 and 3.3 for \( R \). The proof for \( L \) proceeds along the same lines. Let us start with 3.3:

\[
Rf = \sum_{(f)} f_1 \otimes \pi(f_2)
\]

\[
(id \otimes \Delta_K)(Rf) = \sum_{(f)} f_1 \otimes (\Delta_K \circ \pi)(f_2) = \sum_{(f)} f_1 \otimes \pi(f_2) \otimes \pi(f_3)
\]

where we have used coassociativity of \( \Delta_G \) together with the fact that \( \pi \) is a coalgebra morphism. On the other side

\[
((R \otimes id) \circ R)f = \sum_{(f)} R(f_1) \otimes \pi(f_2) = \sum_{(f)} (id \otimes \pi)(f_1_1 \otimes f_1_2) \otimes \pi(f_2) =
\]

\[
= \sum_{(f)} f_1 \otimes \pi(f_2) \otimes \pi(f_3)
\]

again using coassociativity; this proves 3.3. For what concerns 3.2 we have:

\[
(id \otimes \epsilon_K)(Rf) = \sum_{(f)} f_1 \otimes (\epsilon_K \circ \pi)(f_2) =
\]

\[
= \sum_{(f)} f_1 \otimes \epsilon_G(f_2) = ((id \otimes \epsilon_G) \circ \Delta)f = f
\]

The proof for \( L \) goes along in the same way.

**Lemma 3.2** The following identities hold:

\[
(\Delta_G \otimes id) \circ R = (id \otimes R) \circ \Delta_G \quad (3.7)
\]

\[
(id \otimes \Delta_G) \circ L = (L \otimes id) \circ \Delta_G \quad (3.8)
\]

**Proof**

Let us prove the first, the second is proven in the same way.

\[
(\Delta_G \otimes id)(Rf) = \sum_{(f)} \Delta_G(f_1) \otimes \pi(f_2) = \sum_{(f)} f_1 \otimes f_2 \otimes \pi(f_3) =
\]

\[
(id \otimes R)(\Delta_G f) = \sum_{(f)} f_1 \otimes R(f_2) = \sum_{(f)} f_1 \otimes f_2 \otimes \pi(f_3)
\]

Straightforward calculations allow to prove also the following claim:
Lemma 3.3 Let \((A_q(K), \pi)\) be a left coisotropic subgroup; then we the multiplication properties:

\[
L(fg) = \Delta_G(f)L(g); \quad R(fg) = \Delta_f \cdot R(g) \quad \forall f, g \in \mathcal{F}_q(G)
\]  

(3.9)

where \(\cdot\) denotes the action of \(\mathcal{F}_q(G) \otimes \mathcal{F}_q(G)\) on \(\mathcal{F}_q(G) \otimes A_q(K)\) given by \((f \otimes g) \cdot (h \otimes c) = fh \otimes g \cdot c\). Similarly if \((A_q(K), \pi)\) is a right coisotropic subgroup we have:

\[
L(fg) = L(f)\Delta_G(g); \quad R(fg) = R(f)\Delta_G(g) \quad \forall f, g \in \mathcal{F}_q(G)
\]  

(3.10)

Let us start now from a right corepresentation \(\rho_R\) and a left corepresentation \(\rho_L\) of the coalgebra \(A_q(K)\) on the vector space \(V\). Define:

\[
\text{ind}_K^G(\rho_L) = \{F \in A_q(G) \otimes V \mid (R \otimes \text{id})(F) = (\text{id} \otimes \rho_L)(F)\}
\]  

(3.11)

\[
\text{ind}_K^G(\rho_R) = \{F \in V \otimes A_q(G) \mid (\text{id} \otimes L)F = (\rho_R \otimes \text{id})(F)\}
\]  

(3.12)

Both these spaces are kernels of certain linear operators and thus are closed vector subspaces of \(A_q(G) \otimes V\) and \(V \otimes A_q(G)\).

Proposition 3.4 \(\Delta_G \otimes \text{id}\) defines a left corepresentation of \(A_q(G)\) on \(\text{ind}_K^G(\rho_L)\) and \(\text{id} \otimes \Delta_G\) defines a right corepresentation of the same space on \(\text{ind}_K^G(\rho_R)\).

Proof

Let us prove the first claim. To begin with we need to prove that \((\Delta_G \otimes \text{id})(F) \in A_q(G) \otimes \text{ind}_K^G(\rho_L)\). Linearity of all the operations involved implies that we can limit ourselves to the case \(F = f \otimes v\). Then what we have to prove is equivalent to

\[
(id \otimes R \otimes \text{id})(\Delta_G \otimes \text{id})(f \otimes v) = (id \otimes \text{id} \otimes \rho_L)(\Delta_G \otimes \text{id})(f \otimes v)
\]

Using formula 3.7 in lemma 3.1 the first term is equal to

\[
((id \otimes R) \circ \Delta_G) \otimes (id\otimes id)(f \otimes v) = ((\Delta_G \otimes id) \circ R) \otimes (id \otimes id)(f \otimes v) = (\Delta_G \otimes id \otimes id)(R \otimes id)(f \otimes v)
\]

which, applying the fact that \(f \otimes v\) is in \(\text{ind}_K^G(\rho_L)\), is equal to

\[
(\Delta_G \otimes id \otimes id)(id \otimes \rho_L)(f \otimes v) = (id \otimes id \otimes \rho_L)(\Delta_G \otimes id)(f \otimes v)
\]

Now the corepresentation condition \(3.3\) is equivalent to coassociativity of \(\Delta_G\) and the condition \(3.2\) is equivalent to properties of the counity \(\varepsilon_G\). The second claim follows similarly from 3.8. ■

Definition 3.5 Given a right corepresentation \(\rho_R\) of the quantum coisotropic subgroup \((A_q(K), \pi)\) the corresponding corepresentation \(\text{ind}_K^G(\rho_R)\) of \(A_q(G)\) is called induced representation.
Remark In case $\rho_L$ is a one-dimensional corepresentation the induced representation is given by
\[
\text{ind}_K^G(\rho_L) = \{ f \in A_q(G) | (R \otimes id) f = (id \otimes \rho_L) f \}
\]
with coaction $\Delta_G$. Similarly if $\rho_R$ is one-dimensional
\[
\text{ind}_K^G(\rho_R) = \{ f \in A_q(G) | (id \otimes L) f = (\rho_R \otimes id) f \}
\]
In both cases they can be seen as subrepresentation of the regular representation (left or right) $(A_q(G), \Delta_G)$. In analogy with classical terminology we will call such representations \textit{monomial}.

As usual in representation theory we will often identify the space of the representation with the representation itself, being clear which is the representation map.

\textbf{Proposition 3.6} \hspace{1em} If $\rho_R$ and $\rho'_R$ are equivalent right corepresentations of $A_q(K)$ then $\text{ind}_K^G(\rho_R)$ and $\text{ind}_K^G(\rho'_R)$ are equivalent right corepresentations. An obvious statement is valid for the case of left corepresentations.

\textbf{Proof}

Let $\rho_R : V \to A_q(K) \otimes V$ and $\rho'_R : W \to A_q(K) \otimes W$ be equivalent. Then there exists a vector space isomorphism $F : V \to W$ such that $\rho'_R \circ F = (id \otimes F) \circ \rho_R$. Define then the isomorphism
\[
\tilde{F} = F \otimes id : V \otimes A_q(G) \to W \otimes A_q(G)
\]
Let us prove that $\tilde{F}(\text{ind}_K^G(\rho_R)) \subset \text{ind}_K^G(\rho'_R)$. Let $v \otimes f \in \text{ind}_K^G(\rho_R)$. We want to prove that $F(v) \otimes f$ verifies
\[
F(v) \otimes L(f) = \rho'_R(F(v)) \otimes f
\]
We have that
\[
\rho'_R(F(v)) \otimes f = (id \otimes F) \circ \rho_R(v) \otimes f = (id \otimes F \otimes id)(\rho_R(v) \otimes f) = (F \otimes id \otimes id)(v \otimes L(f)) = F(v) \otimes L(f)
\]
Let us prove now that the vector space isomorphism $\tilde{F}$ intertwines corepresentations, i.e.
\[
(id \otimes \Delta_G) \circ \tilde{F} = (\tilde{F} \otimes id)(id \otimes \Delta_G)
\]
We have:
\[
(id \otimes \Delta_G) \circ \tilde{F}(v \otimes f) = (id \otimes \Delta_G)(F(v) \otimes f) = F(v) \otimes \Delta_G f = (\tilde{F} \otimes id)(v \otimes \Delta_G f) = (\tilde{F} \otimes id)(id \otimes \Delta_G)(v \otimes f)
\]
$\blacksquare$
Another major property is the behaviour of the induction procedure with respect to direct sum. Let us recall that if \( \rho_1 \) and \( \rho_2 \) are right corepresentations of the coalgebra \( C \) on the vector spaces \( V \) and \( W \) respectively, then we define:

\[
\rho_1 \oplus \rho_2 : V \bigoplus W \to C \otimes (V \bigoplus W)
\]

(3.13)

\[
\rho_1 \oplus \rho_2 = (\text{id} \otimes \iota_V) \circ \rho_1 \circ \rho_V + (\text{id} \otimes \iota_W) \circ \rho_2 \circ \rho_W
\]

where \( \iota_V : V \to V \bigoplus W \) and \( \iota_W : W \to V \bigoplus W \) are the natural immersion and \( \rho_V : V \bigoplus W \to V \) and \( \rho_W : V \bigoplus W \to W \) are the natural projections.

**Proposition 3.7** Let \( (\rho_k, V_k), k \in \mathbb{N} \) be right \( A_q(K) \)-corepresentations of a given coisotropic quantum subgroup \( (A_q(K), \pi) \) of \( A_q(G) \). Then we have an equivalence of right corepresentations

\[
\text{ind}^G_K(\bigoplus \rho_k) \equiv \bigoplus \text{ind}^G_K(\rho_k)
\]

In particular if \( \text{ind}^G_K(\rho) \) is irreducible then \( \rho \) is irreducible.

**Proof**

We will prove the proposition for a finite sum. Due to associativity of the direct sum it is sufficient to prove the theorem for \( k = 1, 2 \). We will begin proving that \( \rho_V \otimes \text{id} + \rho_W \otimes \text{id} \) is an isomorphism of vector spaces of \( \text{ind}^G_K(\rho_1 \oplus \rho_2) \) in \( \text{ind}^G_K(\rho_1 \oplus \rho_2) \). Due to the fact that the above map is a linear isomorphism of \((V \bigoplus W) \otimes A_q(G) \) in \((V \otimes A_q(G)) \bigoplus (W \otimes A_q(G)) \) it is sufficient to prove that \( (\rho_V \otimes \text{id})F \in \text{ind}^G_K(\rho_1) \) and \( (\rho_W \otimes \text{id})F \in \text{ind}^G_K(\rho_2) \). We have:

\[
(id \otimes L)(\rho_V \otimes \text{id})F = (\rho_V \otimes \text{id} \otimes \text{id})(\rho_V \otimes \text{id}) = (\rho_V \otimes \text{id} \otimes \text{id})(\rho_V \otimes \rho_2)(\rho_1) = (\rho_V \otimes \rho_2)(\rho_1) = \]

\[
= [\rho_V \otimes \rho_2](\rho_1)(\rho_2) = (\rho_2 \otimes \rho_1)(\rho_1)F
\]

and similarly for \( (\rho_W \otimes \text{id})F \). We can then identify the vector spaces writing for every \( f \in \text{ind}^G_K(\rho_1 \oplus \rho_2) \), \( f = f_1 + f_2 \) with \( f_1 \in \text{ind}^G_K(\rho_1) \) and \( f_2 \in \text{ind}^G_K(\rho_2) \). Let us prove then the intertwining property:

\[
(id \otimes \Delta_G)f = (id \otimes \Delta_G)(f_1 + f_2) = (id \otimes \Delta_G)f_1 + (id \otimes \Delta_G)f_2
\]

which is the sum of the induced corepresentations.

Another interesting properties of the classical induction procedure that we want to mimic is the so called double induction. At this purpose let us remark that a coisotropic quantum subgroup cannot have a quantum subgroup but only coisotropic quantum subgroups. Let us consider then the case in which \( A_q(K) \) is a quantum subgroup of \( A_q(G) \), coisotropic or not, and \( A_q(H) \) is a coisotropic quantum subgroup of \( A_q(K) \).
Proposition 3.8 Let $\rho_R$ be a right corepresentation of $A_q(H)$ on $V$. Then there is an equivalence of right $A_q(G)$-corepresentations between
\[
\text{ind}^G_H(\rho_R) \equiv \text{ind}^G_K(\text{ind}^K_H(\rho_R))
\]
The same property holds for left corepresentations.

Proof

Let us denote with $\pi_{KH}$ and $\pi_{GK}$ respectively the maps defining $H$ as a quantum subgroup of $K$ and $K$ as a quantum subgroup of $G$ and let $\pi_{GH} = \pi_{KH} \circ \pi_{GH}$ defining $H$ as a quantum subgroup of $G$. Let us denote with $L_{GH}$, $L_{KH}$ and $L_{GK}$ the corresponding left corepresentations granted by [3,4]. Let us remark that $L_{GH} = (\pi_{KH} \otimes id) \circ L_{GK}$. The required isomorphism between representation spaces is given by:
\[
id \otimes L_{GH} : \text{ind}^G_H(\rho) \rightarrow \text{ind}^G_K(\text{ind}^K_H(\rho))
\]

To prove it remark that elements of the second space are those $v \otimes g \otimes f$ in $V \otimes A_q(K) \otimes A_q(G)$ verifying
\[
1) \quad \rho_R(v) \otimes g \otimes f = \sum_{(g)} v \otimes \pi_{KH}(g(1)) \otimes g(2) \otimes f
\]
\[
2) \quad v \otimes \Delta_K(g) \otimes f = \sum_{(f)} v \otimes g \otimes \pi_{GK}(f(1)) \otimes f(2)
\]
Applying $id \otimes \varepsilon_K \otimes id \otimes id$ to this second equality gives
\[
v \otimes g \otimes f = \sum_{(f)} \varepsilon_K(g)v \otimes \pi_{GK}(f(1)) \otimes f(2)
\]
from which one can prove that $v \otimes g \otimes f \mapsto \varepsilon(g)v \otimes f$ is both a left and a right inverse of $id \otimes L_{GH}$. The fact that the actions are intertwined results from coassociativity of $\Delta_G$. ■

Let us prove now how the induction procedure interacts with automorphisms of the Hopf algebra structure. Let $A_q(G)$ be the quantum group and $(A_q(K), \pi)$ a coisotropic quantum subgroup. Let $\alpha \in \text{Aut } (A_q(G))$ be a Hopf-algebra automorphism. Then there exists one and only one coalgebra automorphism $\tilde{\alpha} : A_q(K) \rightarrow A_q(K)$ such that: $\tilde{\alpha} \circ \pi = \pi \circ \alpha$.

Proposition 3.9 In the above hypothesis we have the equivalence of representations
\[
\text{ind}^G_K((id \otimes \tilde{\alpha}) \circ \rho_R) = (id \otimes \alpha)(\text{ind}^G_K(\rho_R))
\]

Proof

The vector space of the corepresentation on the left is:
\[
\{F \in V \otimes A_q(G) | \quad (id \otimes L)F = (id \otimes \tilde{\alpha} \otimes id)(\rho_R \otimes id)F\}
\]
If $F$ belongs to $\text{ind}_K^G(\rho)$ then

$$(id \otimes L)(id \otimes \alpha)F = v \otimes L(\alpha(f)) = v \otimes ((\pi \otimes id)\Delta(\alpha(f))) =$$

$$= v \otimes ((\pi \otimes id)(\alpha \otimes \alpha(\Delta f))) = v \otimes ((\hat{\alpha} \otimes \hat{\alpha})(\pi \otimes id(\Delta f))) = (id \otimes \hat{\alpha} \otimes \hat{\alpha})(v \otimes L(f))$$

and this proves the isomorphism between the vector spaces carrying the representation. The intertwining property is a simple consequence of $\alpha$ being a coalgebra morphism. $lacksquare$

4 Geometric realization on homogeneous quantum bundles

The purpose of this section is to explicit the relations between induced corepresentations from coisotropic subgroups and embeddable quantum homogeneous spaces. We will split the two cases of quantum and coisotropic subgroup, although the first one is a special case of the second, to point out the differences. Much of what follows is both inspired from and intimately related to results in references [5], [7], [8], where the theory of quantum principal bundles with structure group given by a coalgebra has been developed. Those results are reinterpreted (and slightly generalized) here in the context of induced corepresentations. This reflects what happens in the classical case where there is a bijective correspondence between induced representations and homogeneous vector bundles (i.e. vector bundles associated to principal bundles, [21]). For a more detailed relation on the quantum bundle interpretation we refer to [16], which appeared while this paper was in preparation and which contains more details, although limited to the compact and subgroup case.

Let $\mathcal{F}_q(G)$ be a Hopf algebra with invertible antipode and let $(\mathcal{F}_q(K), \pi)$ be a quantum subgroup. Let $B_K$ be the corresponding quantum quotient space

$$B_K = \{ f \in \mathcal{F}_q(G) \mid (\pi \otimes id)\Delta(f) = 1 \otimes f \}$$

Let $\rho_R$ be a $\mathcal{F}_q(K)$ right corepresentation on $V$ and let $\mathcal{H}_R$ be the space of the induced $\mathcal{F}_q(G)$ corepresentation.

**Lemma 4.1** $\mathcal{H}_R$ is a left and right $B_K$-module, where the action is given by the linear extension of

$$b \cdot (f \otimes v) = (bf \otimes v)$$

$$(f \otimes v) \cdot b = fb \otimes v$$
Proof

Let \( v \otimes f \in \mathcal{H}_R \). Then

\[
L(v \otimes bf) = \sum_{(b)(f)} v \otimes \pi(b_{(1)}f_{(1)}) \otimes b_{(2)}f_{(2)} = \sum_{(b)(f)} v \otimes \varepsilon(b_{(1)})\pi(f_{(1)}) \otimes b_{(2)}\pi(f_{(2)}) = \\
= \sum_{(f)} v \otimes f_{(1)} \otimes bf_{(2)} = \sum_{(v)} v(0) \otimes v_{(1)} \otimes bf = b \cdot (\rho_R \otimes id)(v \otimes f)
\]

The same proof holds for right action (\( \pi \) is an algebra morphism).

Let us now recall that linear maps \( f, g : \mathcal{F}_q(K) \to \mathcal{F}_q(G) \) can be multiplied according to convolution:

\[
(f \ast g)(k) = \sum_{(k)} f(k_{(1)})g(k_{(2)})
\]

If \( f : \mathcal{F}_q(K) \to \mathcal{F}_q(G) \) is its convolution inverse, if it exists, is a map \( f^{-1} : \mathcal{F}_q(K) \to \mathcal{F}_q(G) \) such that:

\[
\sum_{(k)} f(k_{(1)})f^{-1}(k_{(2)}) = \varepsilon(k)1 = \sum_{(k)} f^{-1}(k_{(1)})f(k_{(2)})
\]

Definition 4.2 A quantum subgroup is said to have a left (resp. right) section if there exists a linear, convolution invertible, map \( \phi : \mathcal{F}_q(K) \to \mathcal{F}_q(G) \) such that:

i) \( \phi(1) = 1 \);

ii) \( (\pi \otimes id)\Delta_G \circ \phi = (1 \otimes \phi) \circ \Delta_K \) (or, respectively ii') \( (id \otimes \pi)\Delta_G \circ \phi = (\phi \otimes 1) \circ \Delta_K \).

If, furthermore, \( \phi \) is an algebra morphism then \( (\mathcal{F}_q(K), \pi) \) is said to be trivializable.

The second condition is an intertwining condition between the corepresentation of \( \mathcal{F}_q(K) \) on itself and its corepresentation on \( \mathcal{F}_q(G) \).

Proposition 4.3 Let us suppose that \( (\mathcal{F}_q(K), \pi) \) has a section \( \phi \). Then:

i) \( \mathcal{F}_q(G) \) is isomorphic to \( \mathcal{F}_q(K) \otimes B_K \) as a vector space;

ii) \( \mathcal{H}_R \) is isomorphic to \( V \otimes B_K \) as a \( B_K \)-module (both left and right).

Proof

Let us consider, first of all, the following lemma, which can be proved with usual Hopf algebra techniques.

Lemma 4.4 The convolution inverse of the section \( \phi \) verifies

\[
(\pi \otimes id)\Delta_G \phi^{-1} = (S \otimes \phi^{-1})\tau_{1,2}\Delta_K
\]

where \( \tau_{1,2} \) is the map interchanging terms in tensor product.
Now let us consider \( f \in \mathcal{F}_q(G) \). Then
\[
f = \sum_{(f)} \varepsilon(f(1)) f(2) = \sum_{(f)} \varepsilon(\pi(f(1))) f(2) = \sum_{(f)} \phi(\pi(f(1))) \phi^{-1}(\pi(f(2))) f(3)
\]

Let us now prove that \( \sum_{(f)} \phi^{-1}(\pi(f(1))) f(2) \) belongs to \( B_K \).
\[
(\pi \otimes id) \Delta(\sum_{(f)} \phi^{-1}(\pi(f(1))) f(2)) = \sum_{(f)} \pi(\phi^{-1}(\pi(f(1)))) f(2(1)) \otimes \phi^{-1}(\pi(f(1))) f(2(2))) =
\]
\[
= \sum_{(f)} S_K(\pi(f(2))) \pi(f(3)) \otimes \phi^{-1}(\pi(f(1))) f(4)
\]
\[
= \sum_{(f)} 1 \otimes \varepsilon(f(2)) \phi^{-1}(\pi(f(1))) f(3) =
\]
\[
= 1 \otimes \sum_{(f)} \phi^{-1}(\pi(f(1))) f(2)
\]
This proves that the map \( A_{\phi} : \mathcal{F}_q(K) \otimes B_K \to \mathcal{F}_q(G) \) linearly extending \( k \otimes b \mapsto \phi(k)b \) is surjective and its right inverse is given by
\[
A^{-1}_{\phi} : f \mapsto \sum_{(f)} \pi(f(1)) \otimes \phi^{-1}(\pi(f(2))) f(3)
\]
Let us prove that \( A^{-1}_{\phi} \) is also a left inverse:
\[
A^{-1}_{\phi}(\phi(k)b) = \sum_{(b)(\phi(k))} \pi(\phi(k)(1)b(1)) \otimes \phi^{-1}(\pi(\phi(k)(2)b(2))) \phi(k)(3)b(3) =
\]
using the fact that \( ii) \) of definition 4.1 implies that \( (\pi \otimes \pi \otimes id)(\Delta_G \otimes id) \Delta_G \phi \) equals \( (id \otimes id \otimes \phi)(\Delta_K \otimes id) \Delta_K \) we have
\[
= \sum_{(\phi(k))} \pi(\phi(k)(1)) \otimes \phi^{-1}(\pi(\phi(k)(2))) \phi(k)(3) b = \sum_{(k)} k(1) \otimes \phi^{-1}(k(2)) \phi(k(3)) b
\]
\[
= \sum_{(k)} \varepsilon(k(2)) k(1) \otimes b = k \otimes b
\]
which proves the claim.

The proof of the second isomorphism, although more involved, is quite similar to the preceding one. Let us define first of all
\[
T_{\phi} : V \to \mathcal{H}_R; \quad v \mapsto \sum_{(v)} v(0) \otimes \phi(v(1))
\]
We have to prove that \( T_{\phi} \) really takes its values in \( \mathcal{H}_R \). This is true if and only if
\[
\sum_{(v)} \rho_R(v(0)) \otimes \phi(v(1)) = \sum_{(v)} v(0) \otimes (\pi \otimes id) \Delta_G(\phi(v(1)))
\]
which is a straightforward calculation. Let us define now $I_\phi : V \otimes B_K \to \mathcal{H}_R$ as the linear extension of $v \otimes b \to T_\phi(v) \cdot b$. We want to prove that $I_\phi$ is bijective, which we will do by showing that its bilateral inverse is the linear extension of $I_\phi(v \otimes f) = \sum_{(v)} v(0) \otimes \phi^{-1}(v(1)) f = v \otimes f$

First of all

$I_\phi(I_\phi(v \otimes b)) = \sum_{(v)} I_\phi(v(0) \otimes \phi(v(1))) b$

$= \sum_{(v)(b)(\phi(v(1)))} v(0) \otimes \phi^{-1}(\pi(\phi(v(1)))(b(1))) \phi(v(1))(2)b(2)$

$= \sum_{(v)(\phi(v(1)))} v(0) \otimes \phi^{-1}(\pi(\phi(v(1)))(1)) \phi(v(1))(2)b = \sum_{(v)} v(0) \otimes \phi^{-1}(v(1))(2)b = v \otimes b$

The fact that $I_\phi$ takes its values in $V \otimes B_K$ follows from the first part of the proof. Finally

$I_\phi(\sum_{(f)} v \otimes \phi^{-1}(\pi(f(1)))(f(2))) = I_\phi(\sum_{(v)} v(0) \otimes \phi^{-1}(v(1)) f) = \sum_{(v)} v(0) \otimes \phi(v(1))(1) \phi^{-1}(v(1)(2)) f = v \otimes f$

The second isomorphism of the proposition can be considered a morphism of $\mathcal{F}_q(G)$-comodules provided we put on $V \otimes B_K$ the right coaction. Explicitly it is given by:

$\sigma_R : V \otimes B_K \to V \otimes B_K \otimes \mathcal{F}_q(G)$

$\sigma_R(v \otimes b) = \sum_{(b)(\phi(v(2)))(v)} v(0) \phi^{-1}(v(0)(1)) \phi(v(1))(1) \otimes b(1) \otimes \phi(v(1))(2)b(2)$

$= \sum_{(v)(b)} v(0) \otimes b(1) \otimes \phi(v(1))b(2)$

Remark also that in the case of right sections we would have obtained a $B^K$-module homomorphism for $\mathcal{H}_L$ both on left and right side.

Let us now consider the more delicate situation in which we start from a right corepresentation $\rho_R$ of a left (resp. right) coisotropic quantum subgroup $(C, \pi)$ of $\mathcal{F}_q(G)$. Let $B_C$ be the corresponding embeddable quantum homogeneous space. Then one can multiply functions on the space of induced representation by functions on the homogeneous space only on one side.

**Lemma 4.5** If $(\mathcal{F}_q(K), \pi)$ is a left (resp. right) coisotropic quantum subgroup then $\mathcal{H}_R$ is a right (resp. left) sub-$B_C$-module (resp. sub $B^C$-module) of $V \otimes \mathcal{F}_q(G)$.
Let us consider the case of a right coisotropic subgroup.

\[ v \otimes (\pi \otimes \text{id}) \Delta (fb) = \sum_{(f)(b)} v \otimes \pi(f(1)b(1)) \otimes f(2)b(2) \]

\[ = \sum_{(f)(b)} v \otimes \pi(f(1)\varepsilon(b(2))b(1)) \otimes f(2) = \sum_{(v)} v(0) \otimes v(1) \otimes fb \]

\[ \blacksquare \]

Let us remark that the convolution product of linear maps \( f, g : C \to \mathcal{F}_q(G) \) is still well defined.

**Definition 4.6** A right (resp. left) coisotropic subgroup \((C, \pi)\) is said to have a section \( \phi \) if there exists a linear map \( \phi : C \to \mathcal{F}_q(G) \) convolution invertible and such that:

i) \( \phi(\pi(1)) = 1; \)

ii) \( \sum_c \pi(\phi(c)(1)u) \otimes \phi(c)(2) = \sum_v \pi(v(1)u) \otimes \phi(v(2))) \) for every \( c \in C, u \in \mathcal{F}_q(G) \) and \( v \in \pi^{-1}(c) \).

(resp. ii') \( \sum_c \phi(c)(1) \otimes \pi(u\phi(c)(2)) = \sum_v \phi(v(1)) \otimes \pi(u\phi(v(2))) \) for every \( c \in C, v \in \mathcal{F}_q(G) \) and \( v \in \pi^{-1}(c) \).

If, furthermore, \( \phi \) is a right (resp. left) \( \mathcal{F}_q(G) \)-module map then it is said to be a trivialization.

**Lemma 4.7** A right coisotropic subgroup section verifies

\[ \sum_c \pi(\phi^{-1}(c)(1)u) \otimes \phi^{-1}(c)(2) = \sum_v \pi(S(v(2)u)\phi^{-1}(\pi(v(1)))) \]

where \( c \in C, v \in \pi^{-1}(c) \).

Let us remark that \( u = 1 \) yields again condition (ii) of 4.2.

**Proposition 4.8** Let us suppose that \((C, \pi)\) is a right coisotropic subgroup with a section \( \phi \). Then:

i) \( \mathcal{F}_q(G) \) is isomorphic to \( C \otimes B_C \) as a vector space;

ii) \( \mathcal{H}_R \) is isomorphic to \( V \otimes B_C \) as a left \( B_C \)-module.

Proof

Let \( f \in \mathcal{F}_q(G) \). Using lemma 4.7 one can prove like in 4.3 that \( \sum_{(f)} \phi^{-1}(\pi(f(1)))f(2) \) belongs to \( B_C \). The inverse isomorphisms \( A_\phi \) and \( A_{\phi^{-1}} \) are then realized exactly as in 4.7. Also the second part of the proof goes along exactly in the same way. \( \blacksquare \)
Remark that the $BC$-module isomorphism in 4.3 and 4.8 proves that $\mathcal{H}_R$ is a projective $BC$-module, which is a natural property to ask, as generalization of Swan’s theorem, to spaces of sections (see [13]).

The explicit isomorphism of 4.8 can be used to describe the corepresentation directly on this space. The coaction is given by:

$$\rho(b \otimes v) = \sum_{(b)(v)(v(1))} b(1)\phi(v(2))_1 \otimes b(2)\phi(v(2))_2 \phi^{-1}(v(1)) \otimes v(0)$$

$$= \sum_{(v)(b)} v(0) \otimes b(1) \otimes \phi(v(1)) b(2)$$

This allow a complete description of the induced corepresentation from the following data: the homogeneous space $BC$ and the section $\phi$. It is very useful in applications as the following examples show.

Let us give two examples in which one cannot avoid the use of coisotropic subgroups. We start with the non standard Euclidean quantum group $E_\kappa(2)$ as described in [3]. The coisotropic subgroup we will consider is the coalgebra $C$ with a denumerable family of group-like generators $c_p, p \in \mathbb{Z}$ and with restriction epimorphism

$$\pi(v^p) = c_p \quad \pi(a_1) = \pi(a_2) = 0$$

The corresponding quantum homogeneous space is the $\kappa$-plane

$$[a_1, a_2] = \kappa(a_1 - a_2)$$

As shown in [3] this quantum homogeneous space has a section

$$\phi : C \to E_\kappa(2); \quad c_p \mapsto v^p$$

Starting from one-dimensional irreducible corepresentations of this subgroup $\rho_n : 1 \to 1 \otimes c_n$ we obtain as corepresentation space the $\kappa$-plane with coaction

$$\text{ind}(\rho_n)(a_1) = (v + v^{-1})v^n \otimes a_1 - i(v - v^{-1})v^n \otimes a_2 + a_1v^n \otimes 1$$

$$\text{ind}(\rho_n)(a_2) = i(v - v^{-1})v^n \otimes a_1 + (v + v^{-1})v^n \otimes a_2 + a_2v^n \otimes 1$$

For $n = 0$ this is simply the regular corepresentation on the $\kappa$-plane. Using the duality pairing explicitly described in [3] it is easy to derive an algebra representation of the corresponding quantized enveloping algebra, which is never irreducible. The corresponding decomposition into irreducibles amounts to $E_\kappa(2)$-harmonic analysis on the $\kappa$ plane and has been carried through in [3] for the $n = 0$ case.

The second, similar, example is given by the standard Euclidean quantum group $E_q(2)$ with the family of right coisotropic subgroups $(C, \pi_\lambda)$, where $C$ is as before and the restriction epimorphism is given by

$$\pi(v^r n^s \bar{n}^k) = q^{2r(k+s)}\lambda^k c_r(c_{-1}; q^{-2})_k(c_1; q^2)_s$$
The corresponding quantum homogeneous spaces are called quantum hyperboloids and described in [1] and [11]; they are algebras in two generators, $z$ and $\bar{z}$, with relation $z\bar{z} = q^2 \bar{z}z + (1 - q^2)$. In [3] these subgroups are shown to admit sections

$$\phi_r : C \to E_q(2); \quad c_p \mapsto v^{p-r}$$

Starting from one-dimensional irreducible corepresentations $\rho_m : 1 \mapsto 1 \otimes c_m$ we obtain corepresentations on the quantum hyperboloids given by

$$\text{Ind}(\rho_m)(z) = v^{m-r+1} \otimes z + v^m n \otimes 1$$
$$\text{Ind}(\rho_m)(\bar{z}) = v^{m+r-1} \otimes \bar{z} + v^m \bar{n} \otimes 1$$

The decomposition into irreducibles for the $\infty$-dimensional corresponding $U_q(e(2))$-representation is dealt with, in the $r = 0$-case, in [3].

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