FUNCTIONAL CALCULUS FOR $C_0$-GROUPS USING TYPE AND COTYPE

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Abstract. We study the functional calculus properties of generators of $C_0$-groups under type and cotype assumptions on the underlying Banach space. In particular, we show the following. Let $-iA$ generate a $C_0$-group on a Banach space $X$ with type $p \in [1,2]$ and cotype $q \in [2,\infty)$. Then $f(A) : (X,\mathcal{D}(A))_{\frac{1}{p} - \frac{1}{q}} \to X$ is bounded for each bounded holomorphic function $f$ on a sufficiently large strip. As a corollary of this result, for sectorial operators we quantify the gap between bounded imaginary powers and a bounded $\mathcal{H}\infty$-calculus in terms of the type and cotype of the underlying Banach space.

For cosine functions we obtain similar results as for $C_0$-groups. We extend our theorems to $R$-bounded operator-valued calculi, and we give an application to the theory of rational approximation of $C_0$-groups.

1. Introduction

Let $-iA$ generate a $C_0$-group $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space $X$, and set

\begin{equation}
\theta(U) := \inf \left\{ \theta \geq 0 \mid \exists M \geq 1 : \|U(s)\| \leq Me^{\theta|s|} \text{ for all } s \in \mathbb{R} \right\}.
\end{equation}

For each $\omega > 0$ let

\begin{equation}
\text{St}_\omega := \{ z \in \mathbb{C} \mid |\text{Im}(z)| > \omega \},
\end{equation}

and let $\mathcal{H}\infty(\text{St}_\omega)$ be the space of bounded holomorphic functions on $\text{St}_\omega$ with the supremum norm. There is a natural definition of $f(A)$ as an unbounded operator on $X$ for each $\omega > \theta(U)$ and each $f \in \mathcal{H}\infty(\text{St}_\omega)$ (see Section 2A). It was shown by Boyadzhiev and deLaubenfels in [9] that, if $X$ is a Hilbert space, then there exists a constant $C \geq 0$ such that $f(A) \in \mathcal{L}(X)$ with

\begin{equation}
\|f(A)\|_{\mathcal{L}(X)} \leq C \|f\|_{\mathcal{H}\infty(\text{St}_\omega)}
\end{equation}

for each $\omega > \theta(U)$ and $f \in \mathcal{H}\infty(\text{St}_\omega)$. One says that $A$ has a bounded $\mathcal{H}\infty$-calculus. It is useful to know that an operator has a bounded $\mathcal{H}\infty$-calculus. The theory of $\mathcal{H}\infty$-calculus has applications to questions of maximal regularity (see [2],[35],[38]) and played a crucial role in the solution to the Kato square root problem in [4],[6].

Also, functional calculus bounds can be used to determine convergence rates in the numerical approximation theory for solutions to evolution equations (see [10],[17],[21]).

One can obtain nontrivial functional calculus results for $C_0$-groups even when $X$ is not a Hilbert space. For example, it was shown in [25] that if $-iA$ generates a...
$C_0$-group $(U(s))_{s \in \mathbb{R}}$ on a UMD space $X$, then $A$ has a bounded $\mathcal{H}^\infty_t(\text{St}_\omega)$-calculus for all $\omega > \theta(U)$. Here $\mathcal{H}^\infty_t(\text{St}_\omega)$ consists of all $f \in \mathcal{H}^\infty(\text{St}_\omega)$ such that
\[
\|f\|_{\mathcal{H}^\infty_t(\text{St}_\omega)} := \sup_{z \in \text{St}_\omega} |f(z)| + (1 + |z|)|f'(z)| < \infty.
\]
In [20] a similar statement was obtained on general Banach spaces, where one restricts the calculus to real interpolation spaces between $X$ and the domain of $A$.

However, there are several drawbacks to functional calculus theory for function spaces which are strictly contained in the class of $\mathcal{H}^\infty$-functions. For example, in applications one might be interested in functions $f$ which are not contained in a smaller function space such as $\mathcal{H}^\infty_t(\text{St}_\omega)$. But even when dealing with a function $f$ which is known to yield a bounded operator $f(A)$, it can be of interest to know that (1.3) holds, instead of an estimate for $\|f(A)\|_{L(X)}$ with respect to a larger function norm. This is the case for numerical approximation schemes for solutions to evolution equations, where one can often improve convergence rates for operators with a bounded $\mathcal{H}^\infty$-calculus (see [17][20]).

Moreover, a bounded $\mathcal{H}^\infty$-calculus yields square function estimates that arise frequently in harmonic analysis (see [22,36,10]) and that are of use in the theory of stochastic evolution equations [50, 51]. Also, a bounded $\mathcal{H}^\infty$-calculus can be bootstrapped to yield large classes of $R$-bounded operators, and the notion of $R$-boundedness has various applications in the theory of evolution equations (see [38]). It seems that such connections are not available in full generality for subspaces of the class of $\mathcal{H}^\infty$-functions (if $A$ is self-adjoint then more can be said; see for example [5]).

Finally, it seems somewhat unsatisfactory that there is such a rough division between functional calculus properties on Hilbert spaces and on UMD spaces, in the sense that one goes from a bounded $\mathcal{H}^\infty$-calculus on Hilbert spaces to a bounded $\mathcal{H}^\infty_1$-calculus on UMD spaces. One might expect a finer division of functional calculus theorems within the class of UMD spaces, and on $L^p$-spaces one might hope that functional calculus properties improve as $p$ tends to 2. The latter is indeed the case for symmetric contraction semigroups on $L^p$-spaces, cf. [12][14].

In the present article we aim to address the issues above. We study $\mathcal{H}^\infty$-calculus for generators of $C_0$-groups in terms of the type and cotype of the underlying Banach space (see Definition 2.3). Our main result, proved as Theorem 4.1, is as follows. (For the real interpolation space $D_A(\frac{1}{p} - \frac{1}{q}, 1)$ see [17].)

**Theorem 1.1.** Let $-iA$ generate a $C_0$-group $(U(s))_{s \in \mathbb{R}} \subseteq L(X)$ on a Banach space $X$ with type $p \in [1, 2]$ and cotype $q \in [2, \infty)$. Let $\omega > \theta(U)$. Then there exists a constant $C > 0$ such that $D_A(\frac{1}{p} - \frac{1}{q}, 1) \subseteq D(f(A))$ and
\[
\|f(A)x\|_X \leq C \|f\|_{\mathcal{H}^\infty_t(\text{St}_\omega)} \|x\|_{D_A(\frac{1}{p} - \frac{1}{q}, 1)}
\]
for all $f \in \mathcal{H}^\infty(\text{St}_\omega)$ and $x \in D_A(\frac{1}{p} - \frac{1}{q}, 1)$.

Under additional geometric assumptions we improve Theorem 1.1. Indeed, in Theorem 4.3 we show that, if $X$ is isomorphic to a complemented subspace of a $p$-convex and $q$-concave Banach lattice for $p \in [1, 2]$ and $q \in [2, \infty)$, then for each $\lambda > \omega > \theta(U)$ there exists a constant $C > 0$ such that $D((\lambda + iA)^{-\frac{1}{\alpha} + \frac{1}{q}}) \subseteq D(f(A))$ and
\[
\|f(A)x\|_X \leq C \|f\|_{\mathcal{H}^\infty_t(\text{St}_\omega)} \|(\lambda + iA)^{-\frac{1}{\alpha} + \frac{1}{q}}x\|_X
\]
for all $f \in \mathcal{H}^\infty(\text{St}_\omega)$ and $x \in D((\lambda + iA)^{-\frac{1}{\alpha} + \frac{1}{q}})$. 

One might say that Theorem 1.1 shows that each generator $-iA$ of a $C_0$-group on a Banach space $X$ with type $p$ and cotype $q$ has a bounded $H^\infty$-calculus from $D_A(^1_p - ^1_q, 1)$ to $X$, and $A$ has a bounded $H^\infty$-calculus from $D((\lambda + iA)^{-\frac{1}{\theta}} + \frac{1}{2})$ to $X$ under additional assumptions on $X$. Since $\frac{1}{p} - \frac{1}{q} = 0$ if and only if $X$ is isomorphic to an $L^2$-space, (1.5) recovers as a special case (1.3), the main result of [9].

The interpolation spaces $D_A(\theta, 1)$ and the fractional domains $D((\lambda + iA)^{-\theta})$ increase as $\theta$ tends to zero. Therefore the statements in this paper show that many functional calculus properties depend in a quantitative manner on how close the geometry of the underlying space is to that of a Hilbert space. Since each UMD space has non-trivial type and finite cotype, Theorem 1.1 provides a scale of functional calculus results on UMD spaces.

For $(\Omega, \mu)$ a measure space and $p \in [1, \infty)$, the space $L^p(\Omega, \mu)$ is a $p$-convex and $p$-concave Banach lattice with type $\min(p, 2)$ and cotype $\max(p, 2)$. Hence our results show that the functional calculus properties of group generators on $L^p$-spaces improve as $p$ tends to 2. It should be noted that (1.5) holds without any assumptions on the Banach space if $\frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}$ (see Remark 4.2). Therefore most of the results in Sections 4 and 5 are only of interest on $L^p$-spaces when $p \in (1, \infty)$.

In Proposition 4.1 we deduce from Theorem 1.1 that each $f \in H^\infty(\text{St}_\omega)$ with polynomial decay of order $\alpha > \frac{1}{p} - \frac{1}{q}$ at infinity satisfies $f(A) \in L(X)$. Under the assumptions of (1.3) the case $\alpha = \frac{1}{p} - \frac{1}{q}$ is attained.

It was shown by McIntosh in [43] that, on Hilbert spaces, sectorial operators with bounded imaginary powers have a bounded sectorial $H^\infty$-calculus. It is also known that on general Banach spaces (even on $L^p$-spaces) an operator with bounded imaginary powers need not have a bounded $H^\infty$-calculus. In Theorem 5.1 we quantify the gap between bounded imaginary powers and a bounded $H^\infty$-calculus, by showing that each sectorial operator $A$ with bounded imaginary powers has a bounded sectorial $H^\infty$-calculus from $D_{\log(A)}(^1_p - ^1_q, 1)$ to $X$ if the underlying Banach space $X$ has type $p \in [1, 2]$ and cotype $q \in [2, \infty)$. As in the classical case of a bounded $H^\infty$-calculus on $X$, one obtains from this an unconditionality result and square function estimates.

For generators of cosine functions we derive similar results as for $C_0$-groups. In particular, in Theorem 5.3 we show that each generator $-A$ of a cosine function on a Banach space $X$ with type $p$ and cotype $q$ has a bounded parabola-type $H^\infty$-calculus from $D_A(^1_p - ^1_q, 1)$ to $X$.

We extend Theorem 1.1 and (1.5) to operator-valued functional calculi. Then in Theorem 6.2 we show that $A$ in fact has an $R$-bounded $H^\infty$-calculus from $D_A(^1_p - ^1_q, 1)$ to $X$ if $X$ additionally has property $(\alpha)$. In Theorem 6.3 we obtain an $R$-bounded version of (1.5). These results are sharp, as Example 6.5 shows.

To indicate the use of Theorem 1.1 we give an application to the study of numerical approximation methods for the solutions to evolution equations. In Proposition 7.3 we consider a method of rational approximation proposed in [34, 44], and for exponentially stable groups on $L^p$-spaces we improve the rates of convergence which were obtained in [17]. In Corollary 7.4 we show that many rational approximation methods of exponentially stable $C_0$-groups converge strongly on $D_A(^1_p - ^1_q, 1)$ if the underlying space $X$ has type $p \in [1, 2]$ and cotype $q \in [2, \infty)$.

To prove Theorem 1.1 we use transference principles going back to [8, 11, 13]. The transference technique has been applied to functional calculus theory in [14, 30]...
and \[25, 26, 28\]. In \[29\] an interpolation version of the transference principle for unbounded groups from \[25\] was established. This transference principle was then combined with a theorem about Fourier multipliers on vector-valued Besov spaces. In the present paper we also obtain a transference principle involving vector-valued Besov spaces, but we combine it with a theorem about Fourier multipliers between distinct Besov spaces. For (1.5) we use statements about Fourier multipliers from Besov spaces, but we combine it with a theorem about Fourier multipliers between unbounded groups from \[25\] was established. This transference principle was then and \[25, 26, 28, 29\]. In \[29\] an interpolation version of the transfer ence principle for \[28\] for \(C_0\)-semigroups, but this matter will not be explored in the present article.

This paper is organized as follows. In Section 2 we discuss some of the basics of functional calculus theory, vector-valued Besov spaces and the notions of type and cotype. We then state two Fourier multiplier results which are vital for that which follows. In Section 3 we establish two new transference principles for \(C_0\)-groups. These are used in Section 4 to prove Theorem 1.1 and (1.5), as well as several corollaries for \(C_0\)-groups. In Section 6 we obtain results for sectorial operators and generators of cosine functions, and in Section 7 we extend the results from previous sections to \(R\)-bounded operator-valued calculi. Finally, in Section 7 we give an application to the theory of rational approximation schemes.

1.1. Notation and terminology. We write \(\mathbb{N} := \{1,2,3,\ldots\}\) for the natural numbers and \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\). We write \(\mathbb{C}_+\) for the open right half-plane of complex numbers \(z \in \mathbb{C}\) with \(\text{Re}(z) > 0\), and \(\mathbb{C}_- := \mathbb{C} \setminus \mathbb{C}_+\).

We denote nonzero Banach spaces over the complex numbers by \(X\). The space of \(X\)-valued tempered distributions is \(\mathcal{S}'(\mathbb{R}; X)\). We write \(L^p(\mathbb{R}; X)\) for the Bochner space of \(p\)-integrable \(X\)-valued functions on \(\mathbb{R}\). We often write \(|||f|||_p = |||\|f\||_p\|\|_p\) for \(n \in \mathbb{N}_0\) we let \(W^{n,p}(\mathbb{R}; X)\) be the Sobolev space of \(n\) times weakly differentiable \(f \in L^p(\mathbb{R}; X)\) such that \(f^{(n)} \in L^p(\mathbb{R}; X)\). The Hölder conjugate of \(p\) is denoted by \(p'\) and is defined by \(1 = \frac{1}{p} + \frac{1}{p'}\).

We let \(\mathcal{M}(\mathbb{R})\) denote the space of complex Borel measures on \(\mathbb{R}\) with the total variation norm. For \(\omega \geq 0\) we let \(\mathcal{M}_\omega(\mathbb{R})\) be the convolution algebra of \(\mu \in \mathcal{M}(\mathbb{R})\) of the form \(\mu(ds) = e^{-|\omega s|} \nu(ds)\) for some \(\nu \in \mathcal{M}(\mathbb{R})\), with \(||\mu||_{\mathcal{M}(\mathbb{R})} := ||e^{i\omega s}||_{\mathcal{M}(\mathbb{R})}\).

For \(\Omega \neq \emptyset\) open in \(\mathbb{C}\), we denote by \(\mathcal{H}^\infty(\Omega)\) the space of bounded holomorphic functions \(f : \Omega \to \mathbb{C}\), a Banach algebra with the norm \(||f||_{\mathcal{H}^\infty(\Omega)} := \sup_{z \in \Omega} |f(z)|\) \((f \in \mathcal{H}^\infty(\Omega))\).

The class of \(X\)-valued rapidly decreasing smooth functions on \(\mathbb{R}\) is \(S(\mathbb{R}; X)\), and the space of \(X\)-valued tempered distributions is \(\mathcal{S}'(\mathbb{R}; X)\). The Fourier transform of \(\Phi \in \mathcal{S}'(\mathbb{R}; X)\) is \(\mathcal{F}\Phi\). If \(\mu \in \mathcal{M}_\omega(\mathbb{R})\) for \(\omega > 0\) then \(\mathcal{F}\mu \in \mathcal{H}^\infty(\text{St}_\omega)\) is given by \(\mathcal{F}\mu(z) := \int_{\mathbb{R}} e^{-izs} \mu(ds)\) \((z \in \text{St}_\omega)\).

An interpolation couple is a pair \((X, Y)\) of Banach spaces which are embedded continuously in a Hausdorff topological vector space \(Z\). The real interpolation
space of $(X,Y)$ with parameters $\theta \in [0,1]$ and $q \in [1,\infty]$ is denoted by $(X,Y)_{\theta,q}$. If $T : X + Y \to X + Y$ restricts to a bounded operator on $X$ and $Y$ then
\begin{equation}
\|T\|_{L((X,Y)_{\theta,q})} \leq \|T\|_{L(X)}^{1-\theta} \|T\|_{L(Y)}^\theta
\end{equation}
for all $\theta \in (0,1)$ and $q \in [1,\infty]$. We mainly consider real interpolation spaces for the interpolation couple $(X,D(A))$, where $A$ is a closed operator on $X$. We write
\begin{equation}
D_A(\theta,q) := (X,D(A))_{\theta,q}
\end{equation}
and $\|x\|_{\theta,q} := \|x\|_{D_A(\theta,q)}$ ($x \in D_A(\theta,q)$).

For an operator $B$ on $X$ and a continuously embedded space $Y \hookrightarrow X$, the part of $B$ in $Y$ is the operator $B_Y$ on $Y$ that satisfies $B_Y y = By$ for $y \in D(B_Y) := \{z \in D(B) \cap Y \mid Bz \in Y\}$. We write $B_{\theta,q} := B_{D_A(\theta,q)}$ for $\theta \in [0,1]$ and $q \in [1,\infty]$.

2. Functional calculus and Fourier multipliers

In this section we present the background on functional calculus and Fourier multipliers which will be needed for the rest of the article.

2.1. Functional calculus. We assume that the reader is familiar with the basics of the theory of $C_0$-groups from [19]. For more on the functional calculus for generators of $C_0$-groups see [20, Chapter 4].

An operator $A$ on a Banach space $X$ is a strip-type operator of height $\omega_0 \geq 0$ if $\sigma(A) \subseteq \text{St}_{\omega_0}$, where $\text{St}_0 := \mathbb{R}$, and $\sup_{\lambda \in \mathbb{C} \cap \text{St}_\omega} \|R(\lambda,A)\| < \infty$ for all $\omega > \omega_0$. For $\omega > 0$ set
\[ \mathcal{E}(\text{St}_\omega) := \{ g \in \mathcal{H}^\infty(\text{St}_\omega) \mid g(z) \in O(|z|^{-\alpha}) \text{ for some } \alpha > 1 \text{ as } |\text{Re}(z)| \to \infty \} . \]

The strip-type functional calculus for a strip-type operator $A$ of height $\omega_0$ is defined as follows. First, operators $f(A) \in \mathcal{L}(X)$ are associated with $f \in \mathcal{E}(\text{St}_\omega)$ for $\omega > \omega_0$:
\begin{equation}
f(A) := \frac{1}{2\pi i} \int_{\partial \text{St}_\omega} f(z) R(z,A) \, dz .
\end{equation}

Here $\partial \text{St}_\omega$ is the positively oriented boundary of $\text{St}_\omega$ for $\omega' \in (\omega_0,\omega)$. This procedure is independent of the choice of $\omega'$ by Cauchy’s theorem, and yields an algebra homomorphism $\mathcal{E}(\text{St}_\omega) \to \mathcal{L}(X)$, $f \mapsto f(A)$. The definition of $f(A)$ is extended to a larger class of functions by regularization:
\begin{equation}
f(A) := \epsilon(A)^{-1} (\epsilon f)(A)
\end{equation}
if there exists an $e \in \mathcal{E}(\text{St}_\omega)$ with $\epsilon(A)$ injective and $\epsilon f \in \mathcal{E}(\text{St}_\omega)$. Then $f(A)$ is a closed unbounded operator on $X$, and the definition of $f(A)$ is independent of the choice of the regularizer $\epsilon$. Each $f \in \mathcal{H}^\infty(\text{St}_\omega)$ is regularizable by the function $z \mapsto (\lambda - z)^{-\alpha}$ for $|\text{Im}(\lambda)| > \omega$.

Let $-iA$ generate a $C_0$-group $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$. Then $A$ is a strip-type operator of height $\theta(U)$, with $\theta(U)$ as in (1.1). Let $M \geq 1$ and $\omega \geq 0$ be such that $\|U(s)\| \leq Me^{\omega|s|}$ for all $s \in \mathbb{R}$, and for $\mu \in \mathcal{M}_X(\mathbb{R})$ set
\begin{equation}
U_\mu x := \int_{\mathbb{R}} U(s)x \, d\mu(s) \quad (x \in X).
\end{equation}
The mapping $\mu \mapsto U_\mu$ is an algebra homomorphism $\mathcal{M}_X(\mathbb{R}) \to \mathcal{L}(X)$ called the Hille-Phillips calculus, and the following lemma from [21, Lemma 2.2] shows that this calculus is consistent with the strip-type calculus for $A$. 

Lemma 2.1. Let $-iA$ generate a $C_0$-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space $X$, and let $\omega > \alpha > \theta(U)$. Then each $f \in \mathcal{E}(\text{St}_\omega)$ satisfies $f = \mathcal{F}\mu$ and $f(A) = U_\mu \in \mathcal{L}(X)$ for some $\mu \in \mathcal{M}_\omega(\mathbb{R})$. Conversely, if $\mu \in \mathcal{M}_\omega(\mathbb{R})$ then $f := \mathcal{F}\mu \in \mathcal{H}^\infty(\text{St}_\omega)$ is such that $f(A) = U_\mu \in \mathcal{L}(X)$.

In fact, it is observed in Remark 4.8 that the first statement in Lemma 2.1 holds for all $f \in \mathcal{H}^\infty(\text{St}_\omega)$ such that $f(z) = O(|z|^{-1/2})$ as $|\text{Re}(z)| \to \infty$.

A fundamental result in functional calculus theory is the Convergence Lemma. The following is a version of this lemma adapted to our setting.

Lemma 2.2 (Convergence Lemma). Let $A$ be a densely defined strip-type operator of height $\omega_0 \geq 0$ on a Banach space $X$. Let $Y$ be a Banach space continuously embedded in $X$ such that $D(A^2) \subseteq Y$ is dense. Let $\omega > \omega_0$ and let $(f_j)_{j \in J} \subseteq \mathcal{H}^\infty(\text{St}_\omega)$ be a net satisfying the following conditions:

- $\sup_{j \in J} \|f_j\|_{\mathcal{H}^\infty(\text{St}_\omega)} < \infty$;
- $f(z) := \lim_j f_j(z)$ exists for all $z \in \text{St}_\omega$;
- $\sup_{j \in J} \|f_j(A)\|_{\mathcal{L}(Y,X)} < \infty$.

Then $f \in \mathcal{H}^\infty(\text{St}_\omega)$, $f(A) \in \mathcal{L}(Y,X)$, $f_j(A)x \to f(A)x$ for all $x \in Y$ and

$$\|f(A)\|_{\mathcal{L}(Y,X)} \leq \limsup_{j \in J} \|f_j(A)\|_{\mathcal{L}(Y,X)}.$$  

Proof. The proof is similar to the proofs of [23 Proposition 5.1.7] and [7 Theorem 3.1]. Vitali’s theorem for nets from [3, Theorem 2.1] implies that $f(z) = O(|z|^{-1/2})$ as $|\text{Re}(z)| \to \infty$.

Applying the dominated convergence theorem to (2.1) yields

$$f_j(A)x = \left(\frac{f_j(\cdot)}{(i\lambda - \cdot)^2}\right)(A)(i\lambda - A)^2x \to \left(\frac{f(\cdot)}{(i\lambda - \cdot)^2}\right)(A)(i\lambda - A)^2x = f(A)x$$

and

$$\|f(A)x\|_X \leq \limsup_j \|f_j(A)\|_{\mathcal{L}(Y,X)} \|x\|_Y$$

for all $x \in D(A^2)$. The required statements now follow since $D(A^2) \subseteq Y$ is dense and $f(A)$ is a closed operator on $X$.  

2.2. Fourier multipliers and Banach space geometry. In this section we treat Fourier multiplier operators under geometric assumptions on the underlying space. For more on the prerequisite notions from Banach space geometry, as well as for proofs of some of the statements below, see e.g. [15, 31, 32, 41]. Recall that a standard complex Gaussian random variable is a random variable $\gamma$ on a probability space $(\Omega, \mathbb{P})$ such that $\gamma = \gamma_e + i\gamma_i$ for independent standard real Gaussian random variables $\gamma_e, \gamma_i$ on $\Omega$. A Gaussian sequence is a sequence of independent standard complex Gaussian random variables.

Definition 2.3. Let $X$ be a Banach space, $(\gamma_k)_{k \in \mathbb{N}}$ a Gaussian sequence on a probability space $(\Omega, \mathbb{P})$ and $p \in [1, 2]$, $q \in [2, \infty]$.

- $X$ has (Gaussian) type $p$ if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$,

$$\left(\mathbb{E}\left[\sum_{k=1}^n \gamma_k x_k\right]^2\right)^{1/2} \leq C\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}.$$  

(2.4)
\( \bullet \) \( X \) has (Gaussian) cotype \( q \) if there exists a constant \( C \geq 0 \) such that for all \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in X \),

\[
(\sum_{k=1}^{n} \|x_k\|^{q})^{1/q} \leq C \left( \mathbb{E} \left[ \sum_{k=1}^{n} \gamma_k x_k \right]^{2} \right)^{1/2},
\]

with the obvious modification for \( q = \infty \).

The minimal constants \( C \) in (2.4) and (2.5) are called the Gaussian type \( p \) constant and the Gaussian cotype \( q \) constant and will be denoted by \( \tau_{p,X} \) and \( c_{q,X} \). We say that \( X \) has nontrivial type if \( X \) has type \( p \in (1, 2] \) and that \( X \) has finite cotype if \( X \) has cotype \( q \in [2, \infty) \). By the Kahane-Khintchine inequalities, one may replace the exponent 2 in (2.4) and (2.5) by any \( r \in [1, \infty) \). This does not change the properties of type and cotype, only the minimal constants in (2.4) and (2.5).

It is common to replace the Gaussian sequence in Definition 2.3 by a Rademacher sequence, i.e. a sequence \( \{r_k\}_{k \in \mathbb{N}} \) of independent complex random variables on a probability space \( (\Omega, \mathcal{F}) \) that are uniformly distributed on \( \{z \in \mathbb{C} \mid |z| = 1\} \). This does not change the class of spaces under consideration, only the minimal constants in (2.4) and (2.5). We choose to work with Gaussian sequences because the constants \( \tau_{p,X} \) and \( c_{q,X} \) occur in Proposition 2.4.

Every Banach space \( X \) has type \( p = 1 \) and cotype \( q = \infty \), with \( \tau_{1,X} = c_{\infty,X} = 1 \). If \( X \) has type \( p \) and cotype \( q \) then it has type \( r \) with \( \tau_{r,X} \leq \tau_{p,X} \) for all \( r \in [1, p] \) and cotype \( s \) with \( c_{s,X} \leq c_{q,X} \) for all \( s \in [q, \infty] \). A Banach space \( X \) has type \( p = 2 \) and cotype \( q = 2 \) if and only if \( X \) is isomorphic to a Hilbert space, by Kwapień’s result [39]. A Banach space \( X \) with nontrivial type has finite cotype. Each UMD space has nontrivial type coming from the Kahane-Khintchine inequalities.

Let \( \psi \in \mathcal{C}^{\infty}(\mathbb{R}) \) be such that \( \psi \geq 0 \), \( \text{supp}(\psi) \subseteq [1/2, 2] \) and \( \sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1 \) for all \( s \in (0, \infty) \). For \( k \in \mathbb{N} \) and \( s \in \mathbb{R} \) let \( \varphi_k(s) := \psi(2^{-k}|s|) \) and \( \varphi_0(s) := 1 - \sum_{k=1}^{\infty} \varphi_k(s) \). Let \( X \) be a Banach space and let \( p, q \in [1, \infty] \) and \( r \in \mathbb{R} \). The (inhomogeneous) Besov space \( B^{r}_{p,q}(\mathbb{R}; X) \) is the space of all \( f \in \mathcal{S}'(\mathbb{R}; X) \) such that

\[
\|f\|_{B^{r}_{p,q}(\mathbb{R}; X)} := \left\| \left( 2^{kr}\|\mathcal{F}^{-1}[\varphi_k * f]\|_{L^p(\mathbb{R}; X)} \right)_{k \in \mathbb{N}_0} \right\|_{\ell^q} < \infty,
\]

endowed with the norm \( \|\cdot\|_{B^{r}_{p,q}(\mathbb{R}; X)} \). Then \( B^{r}_{p,q}(\mathbb{R}; X) \) is a Banach space, \( \mathcal{S}(\mathbb{R}; X) \subseteq B^{r}_{p,q}(\mathbb{R}; X) \) is dense if \( p, q < \infty \), and a different choice of \( \psi \) yields an equivalent norm on \( B^{r}_{p,q}(\mathbb{R}; X) \). More details on vector-valued Besov spaces can be found in [149].

For \( X \) a Banach space and \( m \in L^\infty(\mathbb{R}; L(X)) \), the Fourier multiplier operator \( T_m : S(\mathbb{R}; X) \to S'(\mathbb{R}; X) \) with symbol \( m \) is given by

\[
T_m(f) := \mathcal{F}^{-1}(m \cdot \mathcal{F} f) \quad (f \in S(\mathbb{R}; X)).
\]

For each \( \mu \in \mathcal{M}(\mathbb{R}) \),

\[
L_\mu(f) := \mu * f \quad (f \in S(\mathbb{R}; X))
\]

defines a Fourier multiplier operator with symbol \( \mathcal{F} \mu \in L^\infty(\mathbb{R}) \). We say that \( X \) is a UMD space if the function \( 1_{[0, \infty)} - 1_{(-\infty, 0)} \) is the symbol of a bounded Fourier multiplier on \( L^2(\mathbb{R}; X) \).
Let $X$ and $Y$ be Banach spaces and let $(r_k)_{k \in \mathbb{N}}$ be a Rademacher sequence on a probability space $(\Omega, \mathbb{P})$. A collection $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is \textit{R-bounded} if there exists a constant $C \geq 0$ such that, for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $T_1, \ldots, T_n \in \mathcal{T}$,

\begin{equation}
(2.7) \quad \left( \mathbb{E} \left[ \sum_{k=1}^{n} r_k T_k x_k \right]^2 \right)^{1/2} \leq C \left( \mathbb{E} \left[ \sum_{k=1}^{n} r_k x_k \right]^2 \right)^{1/2}.
\end{equation}

The minimal constant $C$ in (2.7) is denoted by $R(\mathcal{T})$. If we want to specify the underlying spaces $X$ and $Y$ then we write $R_{X,Y}(\mathcal{T}) = R(\mathcal{T})$, and we write $R_X(\mathcal{T}) = R(\mathcal{T})$ if $\mathcal{T} \subseteq \mathcal{L}(X)$.

By the Kahane contraction principle, each uniformly bounded collection $\mathcal{T} \subseteq \mathbb{C}$ is $R$-bounded as a subset of $\mathcal{L}(X)$, with $R_X(\mathcal{T})$ equal to the uniform bound of $\mathcal{T}$.

The following result was obtained in [37, Theorem 1.1].

\textbf{Proposition 2.4.} Let $X$ be a Banach space with type $p \in [1, 2]$ and cotype $q \in [2, \infty]$. Let $r \in \mathbb{R}$, $s \in [1, \infty]$ and $m \in L^\infty(\mathbb{R}; \mathcal{L}(X))$ be such that \{m(s) | s \in \mathbb{R}\} \in \mathcal{L}(X)$ is $R$-bounded. Then $T_m \in \mathcal{L}(B_{p,s}^{r+1/q}(\mathbb{R}; X), B_{p,s}^{r}(\mathbb{R}; X))$ and

\begin{equation}
\|T_m\|_{\mathcal{L}(B_{p,s}^{r+1/q}(\mathbb{R}; X), B_{p,s}^{r}(\mathbb{R}; X))} \leq 4^{\frac{p-1}{q}} \tau_{p,X} c_{q,X} R_X(\{m(s) | s \in \mathbb{R}\}).
\end{equation}

\textbf{Corollary 2.5.} Let $X$ be a Banach space with type $p \in [1, 2]$ and cotype $q \in [2, \infty]$, and let $\mu \in \mathcal{M}(\mathbb{R})$. Then $L_\mu \in \mathcal{L}(B_{p,1}^{1/p-1/q}(\mathbb{R}; X), L^q(\mathbb{R}; X))$ and

\begin{equation}
\|L_\mu\|_{\mathcal{L}(B_{p,1}^{1/p-1/q}(\mathbb{R}; X), L^q(\mathbb{R}; X))} \leq 4^{\frac{p-1}{q}} \tau_{p,X} c_{q,X} \|\mathcal{F}\mu\|_{L^\infty(\mathbb{R})}.
\end{equation}

\textbf{Proof.} By the Kahane contraction principle, $R_X(\{\mathcal{F}\mu(s) | s \in \mathbb{R}\}) = \|\mathcal{F}\mu\|_{L^\infty(\mathbb{R})}$. Since $B_{p,1}^q(\mathbb{R}; X)$ is contractively embedded in $L^q(\mathbb{R}; X)$, the result follows by applying Proposition 2.4 to $L_\mu = T_{\mathcal{F}\mu}$. \hfill \Box

\textbf{Remark 2.6.} If in Corollary 2.5 one assumes additionally that $X$ is a UMD space, then $L_\mu \in \mathcal{L}(B_{p,1}^{1/p-1/q}(\mathbb{R}; X), L^q(\mathbb{R}; X))$ and

\begin{equation}
(2.8) \quad \|L_\mu\|_{\mathcal{L}(B_{p,1}^{1/p-1/q}(\mathbb{R}; X), L^q(\mathbb{R}; X))} \leq C \tau_{p,X} c_{q,X} \|\mathcal{F}\mu\|_{L^\infty(\mathbb{R})}
\end{equation}

for each $\mu \in \mathcal{M}(\mathbb{R})$ and some $C \geq 0$ independent of $\mu$. This follows from Proposition 2.4 and the embedding $B_{p,q}^q(\mathbb{R}; X) \subseteq L^q(\mathbb{R}; X)$ from [52, Proposition 3.1].

We assume that the reader is familiar with the basics of Banach lattices from [11].

\textbf{Definition 2.7.} Let $X$ be a Banach lattice and $p, q \in [1, \infty]$.

\begin{itemize}
\item $X$ is $p$-\textit{convex} if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$,

\begin{equation}
\left( \mathbb{E} \left[ \sum_{k=1}^{n} |x_k|^p \right]^1 \right)^{1/p} \leq C \left( \mathbb{E} \left[ \sum_{k=1}^{n} \|x_k\|^p \right] \right)^{1/p},
\end{equation}

with the obvious modification for $p = \infty$.

\item $X$ is $q$-\textit{concave} if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$,

\begin{equation}
\left( \mathbb{E} \left[ \sum_{k=1}^{n} \|x_k\|^q \right]^1 \right)^{1/q} \leq C \left( \mathbb{E} \left[ \sum_{k=1}^{n} |x_k|^q \right] \right)^{1/q},
\end{equation}

with the obvious modification for $q = \infty$.
\end{itemize}
Every Banach lattice $X$ is 1-convex and $\infty$-concave. If $X$ is $p$-convex and $q$-concave then it is $r$-convex and $s$-concave for all $r \in [1,p]$ and $s \in [q,\infty]$. By Proposition 1.3, if $X$ is $q$-concave then it has cotype $\max(q,2)$, and if $X$ is $p$-convex and $q$-concave for some $q < \infty$ then $X$ has type $\min(p,2)$. For $(\Omega,\mu)$ a measure space and $r \in [1,\infty]$, $L^r(\Omega,\mu)$ is an $r$-convex and $r$-concave Banach lattice.

A subspace $X_0 \subseteq Y$ of a Banach space $Y$ is said to be complemented if there exists a projection $P \in \mathcal{L}(Y)$ with $P(Y) = X_0$.

**Proposition 2.8.** Let $X$ be isomorphic to a complemented subspace of a $p$-convex and $q$-concave Banach lattice, for $p \in [1,2]$ and $q \in [2,\infty)$. Then there exists a constant $C \geq 0$ such that $T_m \in \mathcal{L}(L^p(R;X),L^q(R;X))$ with

$$
\|T_m\|_{\mathcal{L}(L^p(R;X),L^q(R;X))} \leq C R_X \{\|s\|^\frac{1}{p} \frac{1}{q} m(s) \mid s \in R\}
$$

for each $m \in L^\infty(R;\mathcal{L}(X))$ such that $R_X \{\|s\|^\frac{1}{p} \frac{1}{q} m(s) \mid s \in R\} < \infty$.

**Proof.** Let $X_0$ be a subspace of a $p$-convex and $q$-concave Banach lattice $Y$, $S : X \to X_0$ an isomorphism and $P \in \mathcal{L}(Y)$ a projection with $P(Y) = X_0$. Let $m \in L^\infty(R;\mathcal{L}(X))$ be such that $R_X \{\|s\|^\frac{1}{p} \frac{1}{q} m(s) \mid s \in R\} < \infty$, and let $m_0 \in L^\infty([\Omega,\mathcal{L}(Y))]$ be given by $m_0(s) := Sm(s)S^{-1}P \in \mathcal{L}(Y)$ for $s \in R$. Then

$$
R_Y \{\|s\|^\frac{1}{p} \frac{1}{q} m_0(s) \mid s \in R\} \leq \|S\| ||S^{-1}|| \|P\| \|R_X \{\|s\|^\frac{1}{p} \frac{1}{q} m(s) \mid s \in R\}\|.
$$

It follows from \cite{25} Theorem 3.21 that there exists a constant $C \geq 0$ independent of $m$ such that $T_{m_0} \in \mathcal{L}(L^p(R;Y),L^q(R;Y))$ with

$$
\|T_{m_0}\|_{\mathcal{L}(L^p(R;Y),L^q(R;Y))} \leq C R_Y \{\|s\|^\frac{1}{p} \frac{1}{q} m_0(s) \mid s \in R\}.
$$

This concludes the proof since $S^{-1}T_{m_0}S = T_m$. \hfill $\square$

### 3. Transference principles

In this section we establish two new transference principles for $C_0$-groups, both based on the transference principles for unbounded groups from \cite{25} and \cite{20}.

For $\omega \geq 0$ and $\mu \in \mathcal{M}_\omega(R)$ let $\mu_\omega \in \mathcal{M}(R)$ be given by

$$
\mu_\omega(ds) := \cosh(\omega s) \mu(ds),
$$

and note that

$$
\mathcal{F}\mu_\omega(s) = \frac{\mathcal{F}\mu(s + i\omega) + \mathcal{F}\mu(s - i\omega)}{2}
$$

for all $s \in R$.

**Proposition 3.1.** Let $\omega > \omega_0 \geq 0$, $p \in [1,2]$ and $q \in [2,\infty)$. Then there exists a constant $C \geq 0$ such that the following holds. Let $X$ be a Banach space with type $p$ and cotype $q$, and let $-\omega A$ generate a $C_0$-group $(U(s))_{s \in R} \subseteq \mathcal{L}(X)$ such that $\|U(s)\|_{\mathcal{L}(X)} \leq M \cosh(\omega_0 s)$ for all $s \in R$ and some $M \geq 1$. Then

$$
\left\| \int_R U(s)x \mu(ds) \right\|_X \leq C \tau_{p,x,c,q,x} M^2 ||\mathcal{F}\mu_\omega||_{L^\infty(R)} \|x\|_{1/p-1/q,1}
$$

for all $\mu \in \mathcal{M}_\omega(R)$ and all $x \in D_A(\frac{1}{p} - \frac{1}{q},1)$.
\textbf{Proof.} Since \(D_A(0,1) = \{0\}\) we may assume that \(\frac{1}{p} - \frac{1}{q} \in (0,1)\) (for \(\frac{1}{p} - \frac{1}{q} = 0\) a stronger result is obtained in Proposition 3.3 below). Let

\begin{equation}
\psi(s) := \frac{1}{\cosh(2\omega s)} \quad \text{and} \quad \varphi(s) := \frac{\sqrt{8\omega}}{\pi} \frac{\cosh(\omega s)}{\cosh(2\omega s)}
\end{equation}

for \(s \in \mathbb{R}\). Define \(\iota : X \to L^p(\mathbb{R} ; X)\) by

\begin{equation}
\iota x(s) := \psi(-s)U(-s)x \quad (x \in X, s \in \mathbb{R})
\end{equation}

and \(P : L^q(\mathbb{R} ; X) \to X\) by

\begin{equation}
Pf := \int_{\mathbb{R}} \varphi(s)U(s)f(s)\,ds \quad (f \in L^q(\mathbb{R} ; X)).
\end{equation}

Then \(\iota\) is bounded and

\begin{equation}
\|\iota\|_{L(X,L^p(\mathbb{R} ; X))} \leq M \|\psi(\cdot)\cosh(\omega \cdot)\|_p.
\end{equation}

By H"older’s inequality, \(P\) is bounded and

\begin{equation}
\|P\|_{L(L^q(\mathbb{R} ; X) ; X)} \leq M \|\varphi(\cdot)\cosh(\omega \cdot)\|_{q'}.
\end{equation}

Let \(x \in D(A)\). Then \(\iota x \in C^1(\mathbb{R} ; X)\) and

\(\iota x)'(s) = -\psi'(-s)U(-s)x + i\psi(-s)U(-s)Ax = -2\omega \frac{\tanh(2\omega s)}{\cosh(2\omega s)}U(-s)x + i\frac{1}{\cosh(2\omega s)}U(-s)Ax\)

for all \(s \in \mathbb{R}\). Hence \((\iota x)\)' \(\in L^p(\mathbb{R} ; X)\) and

\begin{equation}
\|(\iota x)'\|_p \leq 2\omega M \|\tanh\|_{L^\infty(\mathbb{R})} \left(\frac{\cosh(\omega \cdot)}{\cosh(2\omega \cdot)}\right) \|x\|_X + M \left(\frac{\cosh(\omega \cdot)}{\cosh(2\omega \cdot)}\right) \|Ax\|_X.
\end{equation}

Now (3.6) implies that \(\iota x \in W^{1,p}(\mathbb{R};X)\), with

\begin{equation}
\|\iota x\|_{1,p} \leq M(2\omega \|\tanh\|_{L^\infty(\mathbb{R})} + 1) \left(\frac{\cosh(\omega \cdot)}{\cosh(2\omega \cdot)}\right) \|x\|_{D(A)}.
\end{equation}

Hence \(\iota : D(A) \to W^{1,p}(\mathbb{R};X)\) is bounded and

\begin{equation}
\|\iota\|_{L(D(A),W^{1,p}(\mathbb{R};X))} \leq M(2\omega \|\tanh\|_{L^\infty(\mathbb{R})} + 1) \left(\frac{\cosh(\omega \cdot)}{\cosh(2\omega \cdot)}\right) \|x\|_{D(A)}.
\end{equation}

By equation (5.9) in \([1]\),

\[B_{p,1}^{1/p-1/q}(\mathbb{R};X) = (L^p(\mathbb{R};X),W^{1,p}(\mathbb{R};X))_{\frac{1}{p} - \frac{1}{q},1}\]

with equivalent norms. Moreover, it follows from a direct sum argument that the constant in the norm equivalence does not depend on \(X\). Hence (1.6), (3.6) and (3.8) imply that \(\iota : D_A(\frac{1}{p} - \frac{1}{q},1) \to B_{p,1}^{1/p-1/q}(\mathbb{R};X)\) is bounded with

\begin{equation}
\|\iota\|_{L(D_A(\frac{1}{p} - \frac{1}{q},1),B_{p,1}^{1/p-1/q}(\mathbb{R};X))} \leq C_1 M,
\end{equation}

for some constant \(C_1 \geq 0\) independent of \(A\) and \(X\).
It is shown in [25, Theorem 3.2] that \( \varphi \ast \psi(s) = \frac{1}{\cosh(s)} \) for all \( s \in \mathbb{R} \). Let \( U_\mu \in \mathcal{L}(X) \) be as in [25, Proposition 3.1]. Then the abstract transference principle from [25, Section 2] yields the commutative diagram

\[
\begin{array}{ccc}
B_{p,1}^{1/p-1/q}(\mathbb{R}; X) & \xrightarrow{L_{\mu_\omega}} & L^q(\mathbb{R}; X) \\
\downarrow & & \downarrow P \\
D_A(\frac{1}{q}, 1) & \xrightarrow{U_\mu} & X
\end{array}
\]

of bounded maps. Finally, estimate the norms of \( P \) and \( \iota \) using \([\text{3.17}]\) and \([\text{3.30}]\) and apply Corollary \([\text{2.25}]\) to \( L_{\mu_\omega} \).

\[\Box\]

**Remark 3.2.** If \( X \) is a UMD space then one can replace the space \( D_A(\frac{1}{q}, 1) \) by the larger \( D_A(\frac{1}{q} - \frac{1}{p}, 1) \). This follows in the exact same way as Proposition 3.1 except that one uses \([\text{2.28}]\) instead of Corollary \([\text{2.25}]\) at the very end of the proof.

For Banach lattices we establish another transference principle. Recall that each Banach space \( X \) with type \( p = 2 \) and cotype \( q = 2 \) is isomorphic to an \( L^2 \)-space, by [39]. Hence the following proposition deals with the case \( p = q = 2 \) in Proposition 3.1.

**Proposition 3.3.** Let \( \omega > \omega_0 \geq 0 \), \( p \in [1, 2] \) and \( q \in [2, \infty) \). Let \( X \) be isomorphic to a complemented subspace of a \( p \)-convex and \( q \)-concave Banach lattice. Then there exists a constant \( C \geq 0 \) such that the following holds. Let \( -iA \) generate a \( C_0 \)-group \( (U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X) \) such that \( \|U(s)\|_{\mathcal{L}(X)} \leq M \cosh(\omega_0 s) \) for all \( s \in \mathbb{R} \) and some \( M \geq 1 \). Then

\[
\left\| \int_{\mathbb{R}} U(s) x \mu(ds) \right\|_X \leq CM^2 \sup_{s \in \mathbb{R}} \left( |s|^{\frac{1}{p} - \frac{1}{q}} |F\mu_\omega(s)| \right) \|x\|_X
\]

for all \( \mu \in \mathcal{M}_\omega(\mathbb{R}) \) and all \( x \in X \).

**Proof.** For \( \mu \in \mathcal{M}_\omega(\mathbb{R}) \) let \( U_\mu \in \mathcal{L}(X) \) be as in \([\text{2.28}]\). Let \( \iota \) and \( P \) be as in \([\text{3.1}]\) and \([\text{3.3}]\), with \( \psi \) and \( \varphi \) as in \([\text{3.3}]\). As in the proof of Proposition 3.1 one can factorize \( U_\mu \) as \( U_\mu = P \circ L_{\mu_\omega} \circ \iota \). Hence \([\text{3.10}]\), \([\text{3.17}]\) and Proposition 3.3 conclude the proof.

\[\Box\]

**4. Results for \( C_0 \)-groups**

In this section we obtain functional calculus results for generators of \( C_0 \)-groups.

**4.1. The main result for \( C_0 \)-groups.** We now prove our main functional calculus result for \( C_0 \)-groups, already stated in the Introduction as Theorem 1.1.

**Theorem 4.1.** Let \( -iA \) generate a \( C_0 \)-group \( (U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X) \) on a Banach space \( X \) with type \( p \in [1, 2] \) and cotype \( q \in [2, \infty) \). Let \( \omega > \theta(U) \). Then there exists a constant \( C \geq 0 \) such that \( D_A(\frac{1}{q} - \frac{1}{p}, 1) \subseteq D(f(A)) \) and

\[
\|f(A)\|_{\mathcal{L}(D_A(\frac{1}{q} - \frac{1}{p}, 1), X)} \leq C \|f\|_{\mathcal{H}^\infty(\mathcal{S}_\omega)}
\]

for all \( f \in \mathcal{H}^\infty(\mathcal{S}_\omega) \).
Proof. First consider $f \in \mathcal{E}(\text{St}_\omega)$ and let $\alpha \in (\theta(U), \omega)$. By Lemma 2.1 there exists a $\mu \in \mathcal{M}_\alpha(\mathbb{R})$ with $f = \mathcal{F}\mu$ and $f(A) = U_\mu$. By Proposition 5.1

$$\|f(A)x\|_X \leq C \|\mathcal{F}\mu\|_{L^\infty(\mathbb{R})} \|x\|_{\frac{1}{\beta} - \frac{1}{\beta_1}}$$

for all $x \in D_A(1/p - 1/q, 1)$ and some constant $C \geq 0$ independent of $f$ and $x$. This implies (4.1) since $\|\mathcal{F}\mu\|_{L^\infty(\mathbb{R})} \leq \|f\|_{\mathcal{H}^\infty(\text{St}_\omega)}$, as follows from (3.2).

For general $f \in \mathcal{H}^\infty(\text{St}_\omega)$, define $\tau_k(z) := -k^2(i(k - z)^{-2}$ for $k \in \mathbb{N}$ with $k > \omega$ and $z \in \text{St}_\omega$. Then $f\tau_k \in \mathcal{E}(\text{St}_\omega)$ for all $k$,

$$\sup_k \|f\tau_k\|_{\mathcal{H}^\infty(\text{St}_\omega)} \leq \|f\|_{\mathcal{H}^\infty(\text{St}_\omega)} \sup_k \|\tau_k\|_{\mathcal{H}^\infty(\text{St}_\omega)} < \infty,$$

and $(f\tau_k)(z) \to f(z)$ as $k \to \infty$, for all $z \in \text{St}_\omega$. By what we have already shown,

$$\|f\tau_k(A)x\|_X \leq C \|f\tau_k\|_{\mathcal{H}^\infty(\text{St}_\omega)} \|x\|_{\frac{1}{\beta} - \frac{1}{\beta_1}} \leq C' \|f\|_{\mathcal{H}^\infty(\text{St}_\omega)} \|x\|_{\frac{1}{\beta} - \frac{1}{\beta_1}}$$

for $C' := C \sup_k \|\tau_k\|_{\mathcal{H}^\infty(\text{St}_\omega)}$. Since $D(A^2) \subseteq D_A(\frac{1}{p} - \frac{1}{q}, 1)$ is dense and since $\lim \sup_k \|\tau_k\|_{\mathcal{H}^\infty(\text{St}_\omega)} = 1$, Lemma 5.2 yields $f(A) \in \mathcal{L}(D_A(\frac{1}{p} - \frac{1}{q}, 1), X)$ with

$$\|f(A)\|_{\mathcal{L}(D_A(\frac{1}{p} - \frac{1}{q}, 1), X)} \leq C \|f\|_{\mathcal{H}^\infty(\text{St}_\omega)},$$

which concludes the proof. \qed

Remark 4.2. If in Theorem 4.1 one assumes in addition that $X$ is a UMD space, then $D_A(\frac{1}{p} - \frac{1}{q}, 1)$ may be replaced by $D_A(\frac{1}{p} - \frac{1}{q}, p)$. This follows by using Remark 3.2 instead of Proposition 3.1 in the proof.

Under additional assumptions one can obtain stronger results. Note that (1.2) improves (4.1), since $D_A(\theta, 1) \subseteq D((\lambda + iA)^\theta)$ for each generator group $A, \lambda \in \mathbb{R}$ sufficiently large and $\theta \in [0, 1)$ by [12] Proposition 4.1.7.

Theorem 4.3. Let $-iA$ generate a $C_0$-group $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space $X$. Suppose that $X$ is isomorphic to a complemented subspace of a $p$-convex and $q$-concave Banach lattice, for $p \in [1, 2]$ and $q \in [2, \infty)$. Let $\lambda > \omega > \theta(U)$. Then

$$\|f(A)\|_{\mathcal{L}(D((\lambda + iA)^\theta), X)} \leq C \|f\|_{\mathcal{H}^\infty(\text{St}_\omega)}$$

for all $f \in \mathcal{H}^\infty(\text{St}_\omega)$.

Proof. Write $\theta := \frac{1}{p} - \frac{1}{q}$. For $f \in \mathcal{E}(\text{St}_\omega)$ and $\alpha \in (\theta(U), \omega)$ there exists a $\mu \in \mathcal{M}_\alpha(\mathbb{R})$ with $f = \mathcal{F}\mu$ and $f(A) = U_\mu$, by Lemma 2.1. Let $x \in D((\lambda + iA)^\theta)$. By [23] Corollary 3.3.6,

$$f(A)x = f(A)(\lambda + iA)^{-\theta}(\lambda + iA)^\theta x = U_\mu U_\nu(\lambda + iA)^\theta x = U_{\mu \ast \nu}(\lambda + iA)^\theta x,$$

where $\nu \in \mathcal{M}_\omega(\mathbb{R})$ is given by $\nu(ds) := \frac{1}{1(\theta)}1_{[0, \infty)}(s)s^{-\theta - 1}e^{-\lambda s}ds$. Since $\mathcal{F}(\mu \ast \nu)(s) = f(s)(\lambda + is)^{-\theta}$ for $s \in \mathbb{R}$, [82] yields

$$\sup_{s \in \mathbb{R}} |s|^{\theta} |\mathcal{F}(\mu \ast \nu)_\alpha(s)| \leq C_1 \|f\|_{\mathcal{H}^\infty(\text{St}_\omega)}$$

for some constant $C_1 \geq 0$. Now Proposition 3.3 yields a constant $C_2 \geq 0$ such that

$$\|f(A)x\|_X \leq C_2 \sup_{s \in \mathbb{R}} |s|^{\theta} |\mathcal{F}(\mu \ast \nu)_\alpha(s)||((\lambda + iA)^\theta x||X$$

$$\leq C_1 C_2 \|f\|_{\mathcal{H}^\infty(\text{St}_\omega)} \|((\lambda + iA)^\theta x||X,$$
which proves (4.2) for \( f \in \mathcal{E}(\text{St}_\omega) \). Proceed as in the proof of Theorem 4.1 to obtain (4.2) for general \( f \in \mathcal{H}_{\infty}(\text{St}_\omega) \). \( \square \)

**Remark 4.4.** It follows from the proofs of Theorems 4.1 and 4.3 that the constants in (4.1) and (4.2) depend on \( A \) only through the norm \( \| U(s) \|_{L(X)} \) of \( U(s) \) for all \( s \in \mathbb{R} \). In (4.1) the constant \( C \) depends on the underlying space \( X \) only through the Gaussian type \( p \) constant and the Gaussian cotype \( q \) constant of \( X \). In particular, the dependence of \( C \) on these constants is as in Proposition 2.4. In (4.2) the constant depends on the underlying space only through the constant in (2.9). Note from the proof of Proposition 2.8 that, if (2.9) holds with constant \( C \geq 0 \) on \( Y \) and if \( X \) is complemented in \( Y \) by a projection \( P \), then (2.9) holds on \( X \) with constant \( C \| P \|_{L(Y)} \). This fact will be used in Theorem 6.3.

4.2. **More results for \( C_0 \)-groups.** Here we derive some additional results for group generators from Theorems 4.1 and 4.3.

**Proposition 4.5.** Let \(-iA\) generate a \( C_0 \)-group \( (U(s))_{s \in \mathbb{R}} \subseteq L(X) \) on a Banach space \( X \) with type \( p \in [1, 2] \) and cotype \( q \in [2, \infty) \). Let \( \omega > \theta(U) \) and \( \alpha, \lambda \in \mathbb{C} \) with \( \text{Re}(\alpha) > \frac{1}{p} - \frac{1}{q} \) and \( \text{Re}(\lambda) > \omega \). Then there exists a constant \( C \geq 0 \) such that

\[
\| f(A)(\lambda + iA)^{-\alpha} \|_{L(X)} \leq C \| f \|_{\mathcal{H}_{\infty}(\text{St}_\omega)}
\]

for all \( f \in \mathcal{H}_{\infty}(\text{St}_\omega) \).

**Proof.** The case \( p = q = 2 \) follows from Theorem 4.3. If \( \frac{1}{p} - \frac{1}{q} \in (0, 1) \), then, by [12] Propositions 1.1.4 and 4.1.7, \( D((\lambda + iA)^{\alpha}) \subseteq D_A((\frac{1}{p} - \frac{1}{q}, 1) \) continuously. Hence the proof is concluded by appealing to Theorem 4.1. \( \square \)

One may equivalently formulate Proposition 4.5 as a statement about boundedness on \( X \) of the calculus for functions with sufficient decay:

**Proposition 4.6.** Let \(-iA\) generate a \( C_0 \)-group \( (U(s))_{s \in \mathbb{R}} \subseteq L(X) \) on a Banach space \( X \) with type \( p \in [1, 2] \) and cotype \( q \in [2, \infty) \). Let \( \omega > \theta(U) \) and \( \alpha, \lambda \in \mathbb{C} \) with \( \text{Re}(\alpha) > \frac{1}{p} - \frac{1}{q} \) and \( \text{Re}(\lambda) > \omega \). Then there exists a constant \( C \geq 0 \) such that the following holds. Let \( f \in \mathcal{H}_{\infty}(\text{St}_\omega) \) be such that \( f(z) \in O(|z|^{-\alpha}) \) as \( |\text{Re}(z)| \to \infty \). Then \( f(A) \in L(X) \) with

\[
(4.3) \quad \| f(A) \|_{L(X)} \leq C \sup_{z \in \text{St}_\omega} |\lambda + iz|^{-\alpha} |f(z)|.
\]

**Proof.** Apply Proposition 4.4 to \((\lambda + i)^{\alpha} f(\cdot) \in \mathcal{H}_{\infty}(\text{St}_\omega) \). \( \square \)

In Propositions 4.5 and 4.6 one may let \( \alpha = \frac{1}{p} - \frac{1}{q} \) if \( X \) is isomorphic to a complemented subspace of a \( p \)-convex and \( q \)-concave Banach lattice, as follows from Theorem 4.3.

**Corollary 4.7.** Let \( \omega > 0 \) and let \( f \in \mathcal{H}_{\infty}(\text{St}_\omega) \) be such that \( f(z) \in O(|z|^{-1/2}) \) as \( |\text{Re}(z)| \to \infty \). Then \( f = F\mu \), where \( \mu \in \mathcal{M}_\omega(\mathbb{R}) \) for all \( \omega' \in [0, \omega) \). If in addition \( f \) is bounded and holomorphic on \( \{ z \in \mathbb{C} \mid \text{Im}(z) > -w \} \) then \( \text{supp}(\mu) \subseteq [0, \infty) \).

**Proof.** For \( \omega' \in [0, \omega) \), let \( A := \frac{d}{dt} \) with maximal domain on \( X := L^1(\mathbb{R}, e^{\omega'\lambda}|dt) \). Then \(-iA\) generates the left translation group \((U(s))_{s \in \mathbb{R}} \subseteq L(X) \), and \( \theta(U) = \omega' \). By Proposition 4.6 \( f(A) \in L(X) \). Now [24] Proposition 2.3 implies that \( f = F\mu \) for some \( \mu \in \mathcal{M}_{\omega'}(\mathbb{R}) \). By uniqueness of the Fourier transform, \( \mu \) is independent
of the choice of \( \omega' \in [0, \omega) \). The final statement follows from an application of Liouville’s theorem.

\[ \square \]

**Remark 4.8.** It follows from Corollary 4.7 that the conclusion of Theorem 4.3 holds without any assumptions on the Banach space \( X \) if \( \frac{1}{p} - \frac{1}{q} \geq \frac{1}{2} \). Indeed, let \(-iA\) generate a \( C_0\)-group \((U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)\) on a general Banach space \( X \), and let \( \lambda > \omega > \Theta(U) \) and \( f \in \mathcal{H}^\infty(St_\omega) \). By Corollary 4.7 and Lemma 2.1 \( f(A)(\lambda + iA)^{-1/2} = (f(\cdot)(\lambda + i\cdot)^{-1/2})(A) \in \mathcal{L}(X) \) and hence \( f(A) \in \mathcal{L}(D((\lambda + iA)^{1/2}), X) \).

Also note that Corollary 4.7 extends \[25\] Lemma 2.4 from \( \alpha > 1/2 \) to \( \alpha = 1/2 \).

**Remark 4.9.** Let \(-iA\) generate a \( C_0\)-group \((U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)\) on a UMD space \( X \). In \[25\] it is shown that \( A \) has a bounded \( \mathcal{H}^\infty(St_\omega)\)-calculus for all \( \omega > \Theta(U) \), where \( \mathcal{H}^\infty(St_\omega) \) is defined by \[1.4\]. Since \( X \) has type \( p \in (1, 2] \) and cotype \( q \in [2, \infty) \), one can compare our results with those in \[24\]. We note that our results do not imply those in \[25\], nor does \[25\] imply the results in this article. Indeed, let \( \omega > \Theta(U) \). Then \( f(z) := (\lambda + iz)^{-\alpha} \) defines an element of \( \mathcal{H}^\infty(St_\omega) \) for all \( \alpha > 0 \) and \( \lambda > \omega \), but Proposition 1.6 does not apply to \( f \) if \( \alpha \in (0, \frac{1}{p} - \frac{1}{q}) \). Also, the function \( f \in \mathcal{H}^\infty(St_\omega) \) given by \( f(z) := e^{-iz}(\lambda + iz)^{-\alpha} \) is not an element of \( \mathcal{H}^\infty(St_\omega) \) if \( \alpha \in (\frac{1}{p} - \frac{1}{q}, 1) \) but decays with order \( \alpha > \frac{1}{p} - \frac{1}{q} \) at infinity.

Although both examples it is clear that \( f(A) \in \mathcal{L}(X) \), the difference between estimating \( \|f(A)\|_{\mathcal{L}(X)} \) by \[1.4\] or using \[1.3\] is relevant for numerical approximation methods, as is shown in Section 7.

We now obtain a version of Theorem 4.1 for other interpolation spaces.

**Proposition 4.10.** Let \(-iA\) generate a \( C_0\)-group \((U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)\) on a Banach space \( X \) with type \( p \in [1, 2] \) and cotype \( q \in [2, \infty) \). Let \( \omega > \Theta(U) \), \( r \in (0, 1 - \frac{1}{p} + \frac{1}{q}) \) and \( u \in [1, \infty] \). Then there exists a constant \( C \geq 0 \) such that

\[ \|f(A)x\|_{r,u} \leq C \|f\|_{\mathcal{H}^\infty(St_\omega)} \|x\|_{r+\frac{1}{p} - \frac{1}{q}, u} \]

for all \( f \in \mathcal{H}^\infty(St_\omega) \) and \( x \in D_A(r + \frac{1}{p} - \frac{1}{q}, u) \).

**Proof.** Let \( f \in \mathcal{H}^\infty(St_\omega) \) and first consider the case where \( u = 1 \). Then the part \(-iA_{r,1}\) of \(-iA\) in \( D_A(r, 1) \) generates the \( C_0\)-group \((U(s)|_{D_A(r, 1)})_{s \in \mathbb{R}} \subseteq \mathcal{L}(D_A(r, 1)) \) which satisfies \( \theta(U|_{D_A(r, 1)}) \leq \Theta(U) \), by \[29\] Lemma 2.2. Hence Theorem 1.1 yields

\[ (4.4) \quad \|f(A_{r,1})x\|_{r,1} \leq C \|f\|_{\mathcal{H}^\infty(St_\omega)} \|x\|_{(D_A(r, 1), D(A_{r,1}))\frac{1}{p} - \frac{1}{q}, 1} \]

for all \( x \in (D_A(r, 1), D(A_{r,1}))\frac{1}{p} - \frac{1}{q}, 1 \). Moreover, it follows from \[12\] Proposition 3.1.5 that \( D_A(r, 1) = (X, D(A^2))\frac{1}{p} - \frac{1}{q}, 1 \) and \( D(A_{r,1}) = (X, D(A^2))\frac{1}{p} - \frac{1}{q}, 1 \) with equivalent norms. Hence, by the Reiteration Theorem (see \[12\] Theorem 1.3.5) and again by \[12\] Proposition 3.1.5,

\[ (D_A(r, 1), D(A_{r,1}))\frac{1}{p} - \frac{1}{q}, 1 = ((X, D(A^2))\frac{1}{p} - \frac{1}{q}, 1, (X, D(A^2))\frac{1}{p} - \frac{1}{q}, 1)\frac{1}{p} - \frac{1}{q}, 1 = (X, D(A^2))\frac{1}{p} - \frac{1}{q}, 1 = D_A(r + \frac{1}{p} - \frac{1}{q}, 1). \]

Combine this with \[4.4\], using that \( f(A_{r,1})x = f(A)x \) for all \( x \in D(f(A_{r,1})) \) by \[29\] Lemma 2.2, to conclude the proof in the case where \( u = 1 \).

For general \( u \in [1, \infty] \), the Reiteration Theorem yields

\[ D_A(r, u) = (D_A(\theta_1, 1), D(\theta_2, 1))_{\theta, u} \]
and

\[ D_A(\tau + \frac{1}{p} - \frac{1}{q}, u) = (D_A(\theta_1 + \frac{1}{p} - \frac{1}{q}, 1), D_A(\theta_2 + \frac{1}{p} - \frac{1}{q}, 1))_{\theta, u} \]

for certain \( \theta_1, \theta_2 \in (0, 1 - \frac{1}{p} + \frac{1}{q}) \) and \( \theta \in (0, 1) \). Now apply (1.9) to what we have already shown to conclude the proof. \( \square \)

**Remark 4.11.** For \( u < \infty \) Proposition 4.10 can be proved directly, without using Theorem 4.1. To do so, adapt Proposition 3.1 to the setting of Proposition 4.10, using instead of Corollary 2.5 the more general Proposition 2.4, and then proceed as in the proof of Theorem 4.1.

Similarly, Theorem 4.3 extends to other fractional domains.

**Proposition 4.12.** Let \(-iA\) generate a \( C_0\)-group \((U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)\) on a complemented subspace \( X \) of a \( p\)-convex and \( q\)-concave Banach lattice, where \( p \in [1, 2] \) and \( q \in [2, \infty) \). Let \( \lambda > \omega > \theta(U) \). Then there exists a constant \( C \geq 0 \) such that

\[ \| (\lambda + iA)^\alpha f(A)x \|_X \leq C \| f \|_{H^\infty(S_{\lambda_\omega})} \| (\lambda + iA)^{\alpha + \frac{1}{p} - \frac{1}{q}}x \|_X \]

for all \( \alpha \in \mathbb{C} \), \( f \in H^\infty(S_{\lambda_\omega}) \) and \( x \in D((\lambda + iA)^{\alpha + \frac{1}{p} - \frac{1}{q}}) \).

**Proof.** Apply Theorem 4.3 to \((\lambda + iA)^\alpha x \in D((\lambda + iA)^{\frac{1}{p} - \frac{1}{q}})\) for each \( x \in D((\lambda + iA)^{\alpha + \frac{1}{p} - \frac{1}{q}}) \). \( \square \)

### 5. Results for sectorial operators and cosine functions

In this section we derive from Theorem 4.1 some results for sectorial operators and generators of cosine functions. By Remark 4.12 the results in this section can be improved on UMD spaces, and by Remark 4.13 the statements hold on general Banach spaces for \( \frac{1}{p} - \frac{1}{q} \geq \frac{1}{2} \).

#### 5.1. Sectorial operators.

For \( \varphi \in (0, \pi) \) let \( S_{\varphi} := \{ z \in \mathbb{C} \mid |\arg(z)| < \varphi \} \). An operator \( A \) on a Banach space \( X \) is said to be a sectorial operator of angle \( \varphi \) if \( \sigma(A) \subseteq \mathbb{C}_- \) and \( \sup \{ |zR(z, A)| \mid \psi \in \mathbb{C} \setminus S_{\varphi} \} < \infty \) for all \( \psi \in (\varphi, \pi) \).

For sectorial operators one can construct a functional calculus in a similar manner as for strip-type operators. Define \( f(A) \in \mathcal{L}(X) \) via a Cauchy-type integral for

\[ f \in H^\infty(S_{\varphi}) := \left\{ g \in H^\infty(S_{\varphi}) \mid \exists C, \delta > 0 : |g(z)| \leq C \frac{|z|^\delta}{|1 + z|^\beta} \right\}, \]

set \( 1(A) := I_X \) and \((1 + \cdot)^{-1}(A) := (1 + A)^{-1} \), and then extend linearly and regularize as in (2.2). For details see [23 Chapter 2]. If \( A \) is an injective sectorial operator of angle \( \varphi \in (0, \pi) \), then \( \log(A) \) is defined via the sectorial calculus for \( A \), as is \( f(A) \) for all \( \psi \in (\varphi, \pi) \) and \( f \in H^\infty(S_{\varphi}) \). A sectorial operator \( A \) of angle \( \varphi \in (0, \pi) \) has bounded imaginary powers if \( A \) is injective and if \(-i \log(A)\) generates a \( C_0\)-group \((U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)\). Then \( U(s) = A^{-is} \) for all \( s \in \mathbb{R} \), and \( A \) is sectorial of angle \( \theta_A := \theta(U) \) if \( \theta(U) \in [0, \pi) \), by [16, Theorem 2]. We write \( A \in \text{BIP}(X) \).

If \( A \in \text{BIP}(X) \) and if there exists an \( \omega \in [0, \pi) \) such that \( \{ e^{-\omega |s|}A^{-is} \mid s \in \mathbb{R} \} \subseteq \mathcal{L}(X) \) is \( R \)-bounded, then \( A \) has a bounded \( H^\infty(S_{\varphi}) \) calculus for all \( \psi \in (\omega, \pi) \). Moreover, if \( X \) has property \((\alpha)\) as below, then a bounded \( H^\infty(S_{\varphi}) \)-calculus for \( A \) implies that \( \{ e^{-\omega |s|}A^{-is} \mid s \in \mathbb{R} \} \subseteq \mathcal{L}(X) \) is \( R \)-bounded for \( \omega > \psi \). See [30 Theorem 7.5]. Hence the following result is only of use if \( A \) does not have \( R \)-bounded imaginary powers.
Theorem 5.1. Let $A \in \text{BIP}(X)$ with $\theta_A < \pi$, where $X$ is a Banach space with type $p \in [1,2]$ and cotype $q \in [2,\infty)$. Let $\psi \in (\theta_A,\pi)$. Then there exists a constant $C \geq 0$ such that $D_{\log(A)} \left( \frac{1}{p} - \frac{1}{q}, 1 \right) \subseteq D(f(A))$ and

$$
\|f(A)\|_{L(D_{\log(A)} \left( \frac{1}{p} - \frac{1}{q}, 1 \right), X)} \leq C \|f\|_{\mathcal{H}^{\infty}(S_\psi)}
$$

for all $f \in \mathcal{H}^{\infty}(S_\psi)$.

Proof. By [23, Theorem 4.2.4], $(g \circ \log)(A) = g(\log(A))$ for all $g \in \mathcal{H}^{\infty}(S_\psi)$. Since $g \mapsto g \circ \log$ is an isometric isomorphism $\mathcal{H}^{\infty}(S_\psi) \to \mathcal{H}^{\infty}(S_\psi)$, Theorem 4.1 concludes the proof.

Remark 5.2. It was shown by Dore in [15] that each sectorial operator $A$ with dense range on a general Banach space $X$ has a bounded $\mathcal{H}^{\infty}$-calculus on the real interpolation spaces $(X, D(A) \cap \text{ran}(A))_{\theta,r}$ for all $\theta \in (0,1)$ and $q \in [1,\infty]$. We note that this statement implies neither Proposition 4.10 nor Theorem 5.1. Indeed, after rotation Proposition 4.10 deals with functions on strips around the imaginary axis, whereas for unbounded group generators [16] only applies to $f \in \mathcal{H}^{\infty}(S_\psi)$ for $\psi > \frac{\pi}{2}$.

Moreover, for all $\theta \in (0,1)$ and $q \in [1,\infty]$ it holds that $(X, D(A) \cap \text{ran}(A))_{\theta,r} \subseteq D(\log(A))$. Since in general $D(\log(A)) \nsubseteq D_{\log(A)} \left( \frac{1}{p} - \frac{1}{q}, 1 \right)$, Theorem 5.1 does not follow from [16].

It follows from Proposition 4.10 that $f(A) : D_{\log(A)} \left( r + \frac{1}{p} - \frac{1}{q}, u \right) \to D_{\log(A)} \left( r, u \right)$ is bounded for each $r \in (0, 1 - \frac{1}{p} + \frac{1}{q})$ and each $u \in [1,\infty]$, and

$$
\|f(A)\|_{L(D_{\log(A)} \left( r + \frac{1}{p} - \frac{1}{q}, u \right), D_{\log(A)} \left( r, u \right))} \leq C \|f\|_{\mathcal{H}^{\infty}(S_\psi)}.
$$

In the same manner one can deduce from Proposition 4.6 that each $f \in \mathcal{H}^{\infty}(S_\psi)$ such that $f(z) \in O((2\pi - i \log(z))^{-\alpha})$ as $z \to 0$ or $z \to \infty$ for some $\alpha > \frac{1}{p} - \frac{1}{q}$ satisfies $f(A) \in \mathcal{L}(X)$. If $X$ is isomorphic to a complemented subspace of a $p$-convex and $q$-concave Banach lattice then one may let $\alpha = \frac{1}{p} - \frac{1}{q}$.

From Theorem 5.1 one obtains unconditionality of the functional calculus and square function estimates in the same manner as in [28, Theorem 12.2].

Corollary 5.3. Let $A \in \text{BIP}(X)$ with $\theta_A < \pi$, where $X$ is a Banach space with type $p \in [1,2]$ and cotype $q \in [2,\infty)$. Let $\psi \in (\theta_A,\pi)$, $f \in \mathcal{H}^{\infty}(S_\psi)$, and let $(r_k)_{k \in \mathbb{Z}}$ be a Rademacher sequence on a probability space $(\Omega, \mathcal{F})$. Then the following assertions hold:

- $\sup \{ \| \sum_{k=-n}^{n} \varepsilon_k f(2^k t A) \|_{L(D_{\log(A)} \left( \frac{1}{p} - \frac{1}{q}, 1 \right), X)} \mid n \in \mathbb{N}, t > 0, |\varepsilon_k| = 1 \} < \infty$;
- there exists a constant $C \geq 0$ such that

$$
\sup_{t > 0} \left\| \sum_{k=-\infty}^{\infty} r_k f(2^k t A) x \right\|_{L^2(\Omega; X)} \leq C \left\| f \right\|_{D_{\log(A)} \left( \frac{1}{p} - \frac{1}{q}, 1 \right)}
$$

for all $x \in D_{\log(A)} \left( \frac{1}{p} - \frac{1}{q}, 1 \right)$, and

$$
\sup_{t > 0} \left\| \sum_{k=-\infty}^{\infty} r_k f(2^k t A)^* x^* \right\|_{L^2(\Omega; D_{\log(A)} \left( \frac{1}{p} - \frac{1}{q}, 1 \right)^*)} \leq C \left\| x^* \right\|_{X^*}
$$

for all $x^* \in X^*$.
5.2. Cosine functions. For $\omega \geq 0$ let $\Pi_{\omega} := \{ z^2 \mid z \in \text{St}_\omega \}$. An operator $A$ on a Banach space $X$ is of parabola-type $\omega$ if $\sigma(A) \subseteq \Pi_{\omega}$ and if for all $\omega' > \omega$ there exists a $M_{\omega'} \geq 0$ such that

$$
\| R(\lambda, A) \| \leq \frac{M_{\omega'}}{\sqrt{\lambda} \left| \text{Im}(\sqrt{\lambda}) - \omega' \right|} \quad (\lambda \in \mathbb{C} \setminus \Pi_{\omega'}).$$

For operators of parabola-type $\omega \geq 0$ there is a natural functional calculus, constructed similarly as the strip-type and sectorial functional calculi, and $f(A)$ is defined as an unbounded operator for all $f \in \mathcal{H}^\infty(\Pi_{\omega'})$, $\omega' > \omega$. For details see [24].

A cosine function $\text{Cos} : \mathbb{R} \rightarrow \mathcal{L}(X)$ on a Banach space $X$ is a strongly continuous mapping such that $\text{Cos}(0) = I$ and

$$
\text{Cos}(t + s) + \text{Cos}(t - s) = 2 \text{Cos}(t) \text{Cos}(s) \quad (s, t \in \mathbb{R}).
$$

Then

$$
\theta(\text{Cos}) := \inf \{ \omega \geq 0 \mid \exists M \geq 0 : \| \text{Cos}(t) \| \leq M e^{\omega |t|} \text{ for all } t \in \mathbb{R} \} < \infty.
$$

The generator of a cosine function is the unique operator $-A$ on $X$ that satisfies

$$
\lambda R(\lambda^2, -A) = \int_0^\infty e^{-\lambda t} \text{Cos}(t) \, dt \quad (\lambda > \theta(\text{Cos})).
$$

Then $A$ is an operator of parabola-type $\theta(\text{Cos})$.

We now prove a version of Theorem [4.4] for generators of cosine functions.

**Theorem 5.4.** Let $-A$ generate a cosine function $(\text{Cos}(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space $X$ with type $p \in [1, 2]$ and cotype $q \in [2, \infty)$. Let $\omega > \theta(\text{Cos})$. Then there exists a constant $C \geq 0$ such that $D_A((\frac{1}{p} - \frac{1}{q}), 1) \subseteq D(f(A))$ and

$$
\| f(A) \|_{\mathcal{L}(D_A((\frac{1}{p} - \frac{1}{q}), 1), X)} \leq C \| f \|_{\mathcal{H}^\infty(\Pi_{\omega})}
$$

for all $f \in \mathcal{H}^\infty(\Pi_{\omega}).$

**Proof.** The proof follows the same lines as that of [24 Proposition 5.5]. It suffices to assume that $\theta := \frac{1}{p} - \frac{1}{q} \in (0, 1)$. By [37 Theorem 2] there is a unique subspace $V \subseteq X$ such that $D(A) \subseteq V$ and such that $-iA$ generates a $C_0$-group $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(V \times X)$ on $V \times X$, where

$$
A := i \begin{bmatrix}
0 & 1_V \\
-A & 0
\end{bmatrix}
$$

with domain $D(A) := D(A) \times V$. Moreover, by [24 Theorem 6.2], $\theta(\text{Cos}) = \theta(U)$. Hence Theorem [4.4] yields a constant $C \geq 0$ such that $g(A) \in \mathcal{L}(D_A(\theta, 1), V \times X)$ with

$$
\| g(A) \|_{\mathcal{L}(D_A(\theta, 1), V \times X)} \leq C \| g \|_{\mathcal{H}^\infty(\Pi_{\omega})}
$$

for all $g \in \mathcal{H}^\infty(\Pi_{\omega})$.

Let $f \in \mathcal{H}^\infty(\Pi_{\omega})$. Then $[z \mapsto g(z) := f(z^2)] \in \mathcal{H}^\infty(\text{St}_\omega)$ and $\| g \|_{\mathcal{H}^\infty(\text{St}_\omega)} = \| f \|_{\mathcal{H}^\infty(\Pi_{\omega})}$. Moreover, it is straightforward to see that

$$
\| f(A_V) \| \leq C \frac{1}{\sqrt{\lambda}} \left( \left| \text{Im}(\sqrt{\lambda}) - \omega_0 \right| \right) \quad (\lambda \in \mathbb{C} \setminus \Pi_{\omega_0}).
$$

Now, $\mathcal{A}^2 := \begin{bmatrix} A_V & 0 \\ 0 & A \end{bmatrix}$.
with $D(A^2) = D(A^1) \times D(A)$. By Proposition 3.1.4,
\[ D(A) \times V \in K_{1/2}(V \times X, D(A^1) \times D(A^1)) \cap J_{1/2}(V \times X, D(A^1) \times D(A^1)), \]
where $K_{1/2}$ and $J_{1/2}$ are as in Definition 1.3.1. Hence
\[ V \in K_{1/2}(X, D(A)) \cap J_{1/2}(X, D(A)). \]

Now Theorem 1.3.5 yields
\[ D_A(\theta, 1) = (V \times X, D(A) \times V)_{\theta, 1} = (V, D(A))_{\theta, 1} \times (X, V)_{\theta, 1} \]
\[ = D_A \left( \frac{1+\theta}{2}, 1 \right) \times D_A \left( \frac{\theta}{2}, 1 \right). \]
Combining this with Proposition 5.4 as in the proof of Proposition 5.4, using the complex interpolation method (2.2) extends this functional calculus to all bounded holomorphic
\[ \|f(A)\|_{\mathcal{L}(D_A(\theta, 1), V \times X)} \leq C \|g\|_{H^\infty(S_{\omega})} \leq C \|f\|_{H^\infty(I_{\omega})}, \]
as required.

From Proposition 3.10 one deduces in a similar manner that, under the assumptions of Proposition 5.4 and for all $r \in (0, 1 - \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right))$ and $u \in [1, \infty]$, there exists a constant $C \geq 0$ such that
\[ \|f(A)x\|_{r,u} \leq C \|f\|_{H^\infty(I_{\omega})} \|x\|_{r+\frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right), u} \]
for all $f \in H^\infty(I_{\omega})$ and $x \in D_A(r + \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right), u)$. We leave the formulation of the obvious analogue of Proposition 4.10 for cosine functions to the reader.

**Remark 5.5.** Let $-A$ generate a cosine function $(\text{Cos}(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$, where $X$ is isomorphic to a complemented subspace of a $p$-convex and $q$-concave UMD Banach lattice, for $p \in (1, 2]$ and $q \in [2, \infty)$. Let $\lambda > \omega > \theta(\text{Cos})$. Then there exists a constant $C \geq 0$ such that
\[ \|f(A)x\|_{X} \leq C \|f\|_{H^\infty(I_{\omega})} \|x\|_{\lambda^2 + A} \left( \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \right) \]
for all $f \in H^\infty(I_{\omega})$ and $x \in D((\lambda^2 + A)^{\frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)})$. This follows from Theorem 4.3 as in the proof of Proposition 5.4 using the complex interpolation method and Theorem 6.6.9 and Theorem 5.5.

It should be noted that generators of cosine functions on UMD spaces have a bounded sectorial $H^\infty$-calculus, by Theorem 5.5. Hence on UMD spaces Theorem 5.4 is only of use when it does not suffice to obtain an estimate for $\|f(A)x\|_{X}$ with respect to the supremum norm of $f$ on a sector. The latter is e.g. the case if $f(z) = g(z^2)$ for $g \in H^\infty(S_{\omega})$ which is unbounded on any double sector $S_{\psi} \cup -S_{\psi}$ for $\psi \in (0, \frac{\pi}{2})$, such as $g(z) = e^{-iz}$.

6. Operator-valued functional calculus

In this section we extend our main functional calculus theorems to $R$-bounded operator-valued calculi. Since many of the ideas and proofs are similar to those in the sections before, we leave some details to the reader.

Let $A$ be a strip-type operator of height $\omega_0 \geq 0$ on a Banach space $X$. Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be the algebra of bounded operators that commute with $A$, and let $\omega > \omega_0$. For a bounded holomorphic $f : St_{\omega} \to \mathcal{A}$ such that $\|f(z)\|_{\mathcal{L}(X)} \leq C|z|^{-\alpha}$ for all $z \in St_{\omega}$ and certain $C \geq 0$ and $\alpha > 1$, define $f(A)$ as in (2.2). Regularization as in (2.2) extends this functional calculus to all bounded holomorphic $f : St_{\omega} \to \mathcal{A}$. 

For $\omega \geq 0$ let $\mathcal{M}_\omega(\mathbb{R}; A)$ consist of the $A$-valued Borel measures $\mu$ on $\mathbb{R}$ such that $e^{\omega |s|}\mu(ds)$ has bounded variation. If $-iA$ generates a $C_0$-group $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$ then one can define $U_\mu \in \mathcal{L}(X)$ as in \cite{23} for all $\omega > \theta(U)$ and $\mu \in \mathcal{M}_\omega(\mathbb{R}; A)$. Versions of Lemmas 2.1 and 2.2 hold for this operator-valued calculus, with the same proofs.

For $\omega > 0$ and $T \subseteq \mathcal{L}(X)$ let $RH^\infty(St_\omega; T)$ be the collection of bounded holomorphic functions $f : St_\omega \to T$ such that $\{f(z) \mid z \in St_\omega\} \subseteq \mathcal{L}(X)$ is $R$-bounded. If $T$ is an algebra then $RH^\infty(St_\omega; T)$ is a Banach algebra with the norm

$$\|f\|_{RH^\infty(St_\omega; T)} := RX(\{f(z) \mid z \in St_\omega\}) \quad (f \in RH^\infty(St_\omega; T)).$$

The following result generalizes Theorem 4.1 since each $f \in \mathcal{H}^\infty(St_\omega)$ defines an element $\hat{f} \in RH^\infty(St_\omega; A)$ by $f(z) = f(z)I_X$ for $z \in St_\omega$, and $\|f\|_{RH^\infty(St_\omega; A)} = \|f\|_{\mathcal{H}^\infty(St_\omega)}$.

**Proposition 6.1.** Let $-iA$ generate a $C_0$-group $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space $X$ with type $p \in [1, 2]$ and cotype $q \in [2, \infty)$, and let $\omega > \theta(U)$. Then there exists a constant $C \geq 0$ such that $D_A(\frac{1}{p} - \frac{1}{q}, 1) \subseteq D(f(A))$ and

$$\|f(A)\|_{\mathcal{L}(D(\frac{1}{p} - \frac{1}{q}, 1))} \leq C \|f\|_{RH^\infty(St_\omega; A)}$$

for all $f \in RH^\infty(St_\omega; A)$.

**Proof.** First extend Proposition 3.1 to all $\mu \in \mathcal{M}_\omega(\mathbb{R}; A)$ for which $RX(\{F_\mu(s) \mid s \in \mathbb{R}\}) < \infty$. To this end, note that the abstract transference principle in \cite{20} Section 2] extends to $A$-valued measures, and appeal to Proposition 2.4 instead of Corollary 2.5. Then proceed as in the proof of Theorem 4.1. $\square$

The other results in the previous sections can be extended to statements about operator-valued calculi in a similar manner. In particular, if in Proposition 6.1 $X$ is isomorphic to a complemented subspace of a $p$-convex and $q$-concave Banach lattice, then for each $\lambda > \omega$ there exists a constant $C \geq 0$ such that $D((\lambda + iA)^{\frac{1}{p} - \frac{1}{q}}) \subseteq D(f(A))$ and

$$\|f(A)\|_{\mathcal{L}(D((\lambda + iA)^{\frac{1}{p} - \frac{1}{q}}))} \leq C \|f\|_{RH^\infty(St_\omega; A)}$$

for all $f \in RH^\infty(St_\omega; A)$. Remark 4.4 applies to Proposition 6.1 and 6.1.

Let $(r_j)_{j \in \mathbb{N}}$ and $(r'_j)_{j \in \mathbb{N}}$ be mutually independent Rademacher sequences on a probability space $(\Omega, \mathcal{F})$. We say that a Banach space $X$ has property $(\alpha)$ if there exists a constant $C \geq 0$ such that, for all $m \in \mathbb{N}$, $\{x_{j,k}\}_{j,k=1}^m \subseteq X$ and $\{\alpha_{j,k}\}_{j,k=1}^m \subseteq \mathbb{C}$ with $|\alpha_{j,k}| \leq 1$ for all $j, k \in \{1, \ldots, m\}$,

$$\left(EE\left[\sum_{j,k=1}^m r_k r'_j \alpha_{j,k} x_{j,k}\right]^2\right)^{1/2} \leq C\left(EE\left[\sum_{j,k=1}^m r_k r'_j x_{j,k}\right]^2\right)^{1/2}.$$

Each Banach lattice with finite cotype has property $(\alpha)$, and if $X$ has property $(\alpha)$ then so do the closed subspaces of $X$ and $L^p(\mu, X)$ for each $\sigma$-finite measure space $(\Omega, \mu)$ and each $p \in [1, \infty)$.

For $T$ equal to the unit ball of $\mathcal{C} \subseteq \mathcal{L}(X)$, the following theorem implies that the $\mathcal{H}^\infty(St_\omega)$-calculus for $A$ is $R$-bounded from $D_A(\frac{1}{p} - \frac{1}{q}, 1)$ to $X$. That is, $\{f(A) \mid \|f\|_{\mathcal{H}^\infty(St_\omega)} \leq 1\} \subseteq \mathcal{L}(D_A(\frac{1}{p} - \frac{1}{q}, 1), X)$ is $R$-bounded.
Theorem 6.2. Let $-iA$ generate a $C_0$-group $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space $X$ with property $(\alpha)$, type $p \in [1, 2]$ and cotype $q \in [2, \infty]$. Let $\omega > \theta(U)$. Then there exists a constant $C \geq 0$ such that

$$R_{D(A)}(\frac{1}{p} - \frac{1}{q}, 1, X) \left( \{ f(A) \mid f \in \mathcal{R} \mathcal{H}^{\infty}(\text{St}_\omega; \mathcal{A} \cap T) \} \right) \leq CR_X(T)$$

for each $R$-bounded $T \subseteq \mathcal{L}(X)$.

Proof. The proof is similar to that of Theorem 12.8. Write $\theta := \frac{1}{p} - \frac{1}{q}$ and let $(r_k)_{k \in \mathbb{N}}$ be a Rademacher sequence on $[0, 1]$. Fix $n \in \mathbb{N}$ and let $\text{Rad}^n(X) := \{ \sum_{k=1}^n r_k x_k \mid (x_k)_{k=1}^n \subseteq X \} \subseteq L^2([0, 1]; X)$. Let $\tilde{A}$ be the operator on $\text{Rad}^n(X)$ with domain

$$D(\tilde{A}) = \left\{ \sum_{k=1}^n r_k x_k \in \text{Rad}^n(X) \mid (x_n)_{k=1}^n \subseteq D(A) \right\}$$

such that $\tilde{A}(\sum_{k=1}^n r_k x_k) := \sum_{k=1}^n r_k A x_k$ for $\sum_{k=1}^n r_k x_k \in D(\tilde{A})$. Then $-i\tilde{A}$ generates the $C_0$-group $(\tilde{U}(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(\text{Rad}^n(X))$ given by

$$\tilde{U}(s) \left( \sum_{k=1}^n r_k x_k \right) = \sum_{k=1}^n r_k \tilde{U}(s) x_k$$

for $s \in \mathbb{R}$ and $\sum_{k=1}^n r_k x_k \in \text{Rad}^n(X)$. Note that $\| \tilde{U}(s) \|_{\mathcal{L}(\text{Rad}^n(X))} = \| U(s) \|_{\mathcal{L}(X)}$ for all $s \in \mathbb{R}$. Moreover, $\text{Rad}^n(X) \subseteq L^2([0, 1]; X)$ has type $p$ and cotype $q$ with $\tau_p, \text{Rad}^n(X) \leq C_p \tau_p, X$ and $c_q, \text{Rad}^n(X) \leq C_q c_q, X$ for constants $C_p, C_q \geq 0$ depending only on $p$ and $q$ that come from the Kahane-Khintchine inequalities. By Proposition 6.1 and Remark 4.1 there exists a constant $C_1 \geq 0$ independent of $n$ such that

$$\| f(\tilde{A}) \|_{\mathcal{L}(D^{(\theta, 1, \text{Rad}^n(X))})} \leq C_1 \| f \|_{\mathcal{R} \mathcal{H}^{\infty}(\text{St}_\omega; \tilde{A})}$$

for all $f \in \mathcal{R} \mathcal{H}^{\infty}(\text{St}_\omega; \tilde{A})$, where $\tilde{A} \subseteq \mathcal{L}(\text{Rad}^n(X))$ is the algebra of operators commuting with $\tilde{A}$.

Let $T \subseteq \mathcal{L}(X)$ be $R$-bounded and let $f_1, \ldots, f_n \in \mathcal{R} \mathcal{H}^{\infty}(\text{St}_\omega; \mathcal{A} \cap T)$. Define

$$f(z) \left( \sum_{k=1}^n r_k x_k \right) = \sum_{k=1}^n r_k f_k(z) x_k$$

for $z \in \text{St}_\omega$ and $\sum_{k=1}^n r_k x_k \in \text{Rad}(X)$. We will now show that the range of $f$ is $R$-bounded in $\mathcal{L}(\text{Rad}(X))$, from which it will follow in particular that $f : \text{St}_\omega \rightarrow \tilde{A}$ is well-defined.

Let $(r'_j)_{j \in \mathbb{N}}$ be a Rademacher sequence on $[0, 1]$, independent of $(r_k)_{k \in \mathbb{N}}$. Let $m \in \mathbb{N}$, $(z_j)_{j=1}^m \subseteq \text{St}_\omega$ and $(y_j)_{j=1}^m \subseteq \text{Rad}^n(X)$. Write $y_j = \sum_{k=1}^n r'_k x'_k$ for $j \in \{ 1, \ldots, m \}$ and $(x'_k)_{k=1}^n \subseteq X$. Then Lemma 4.11 and Remark 4.10 yield a constant $C_2 \geq 0$ depending only on $X$ such that

$$\left\| \sum_{j=1}^m r'_j f(z_j) y_j \right\|_{L^2([0, 1]; \text{Rad}^n(X))}^2 \leq C_2^2 R_X(T)^2 \int_0^1 \int_0^1 \left\| \sum_{j=1}^m \sum_{k=1}^n r'_j(v) r_k(u) f(z_j) x'_k \right\|_X^2 \, du \, dv$$

for all $f \in \mathcal{R} \mathcal{H}^{\infty}(\text{St}_\omega; \mathcal{A} \cap T)$.
Hence \( f \in RH^{\infty}(ST;\tilde{\mathcal{A}}) \) with \( \|f\|_{RH^{\infty}(ST;\tilde{\mathcal{A}})} \leq C_2 R_X(T) \). Combining this with (6.3) yields

(6.4) \[ \|f(\tilde{\mathcal{A}})\|_{L(D(\theta,1),Rad^m(X))} \leq C_1 C_2 R_X(T). \]

Note that \( \|\sum_{k=1}^{n} r_kx_k\|_{Rad^m(D(\mathcal{A}))} \leq \|\sum_{k=1}^{n} r_kx_k\|_{D(\mathcal{A})} \leq 2\|\sum_{k=1}^{n} r_kx_k\|_{Rad^m(D(\mathcal{A}))} \)
for all \( \sum_{k=1}^{n} r_kx_k \in D(\mathcal{A}) \), hence \( D(\mathcal{A},\theta,1) = Rad^m(D(\mathcal{A},\theta,1)) \) with

(6.5) \[ \|\sum_{k=1}^{n} r_kx_k\|_{Rad^m(D(\mathcal{A},\theta,1))} \leq \|\sum_{k=1}^{n} r_kx_k\|_{D(\mathcal{A},\theta,1)} \leq 2\|\sum_{k=1}^{n} r_kx_k\|_{Rad^m(D(\mathcal{A},\theta,1))} \]

for all \( \sum_{k=1}^{n} r_kx_k \in D(\mathcal{A},\theta,1) \). Also, it is straightforward to check (by regularization) that \( f(\mathcal{A})(\sum_{k=1}^{n} r_kx_k) = \sum_{k=1}^{n} r_kf(\mathcal{A})x_k \) for all \( \sum_{k=1}^{n} r_kx_k \in D(f(\mathcal{A})). \) Hence (6.4) and (6.5) yield

\[
\left\| \sum_{k=1}^{n} r_kf(\mathcal{A})x_k \right\|_{L^2([0,1];X)} = \left\| f(\mathcal{A}) \left( \sum_{k=1}^{n} r_kx_k \right) \right\|_{Rad^m(X)} \\
\leq C_1 C_2 R_X(T) \left\| \sum_{k=1}^{n} r_kx_k \right\|_{D(\mathcal{A},\theta,1)} \\
\leq CR_X(T) \left\| \sum_{k=1}^{n} r_kx_k \right\|_{L^2([0,1];D(\mathcal{A},\theta,1))}
\]

for all \( x_1,\ldots,x_n \in D(\mathcal{A},\theta,1) \), where \( C \geq 0 \) is independent of \( T \subseteq L(X), n \in \mathbb{N}, f_1,\ldots,f_n \in RH^{\infty}(ST;\mathcal{A} \cap T) \) and \( x_1,\ldots,x_n \in D(\mathcal{A},\theta,1) \). This concludes the proof.

In the same manner we deduce an \( R \)-bounded version of Theorem 4.3 under the extra assumption of \( p \)-convexity for some \( p > 1 \). By [41, Corollary 1.f.9] the latter is equivalent to the assumption of nontrivial type. Recall that any closed subspace of a Banach lattice with finite cotype has property \( (\alpha) \).

**Theorem 6.3.** Let \( X \) be isomorphic to a complemented subspace of a \( p \)-convex and \( q \)-concave Banach lattice, for \( p \in (1,2] \) and \( q \in [2,\infty) \). Let \(-i\mathcal{A}\) generate a \( C_0 \)-group \( (U(s))_{s \in \mathbb{R}} \subseteq L(X) \), and let \( \lambda > \omega > \theta(U) \). Then there exists a constant \( C \geq 0 \) such that

\[
R_{D((\lambda+i\mathcal{A})^{\frac{1}{p}-\frac{1}{q}}),X}(\{f(\mathcal{A}) \mid f \in RH^{\infty}(ST;\mathcal{A} \cap T)\}) \leq CR_X(T)
\]

for each \( R \)-bounded \( T \subseteq L(X) \).

**Proof.** We use notation as in the proof of Theorem 6.2. It suffices to show that there exists a constant \( C \geq 0 \) independent of \( n \in \mathbb{N} \) such that

(6.6) \[ \|f(\tilde{\mathcal{A}})\|_{L(D((\lambda+i\tilde{\mathcal{A}})^{\frac{1}{p}-\frac{1}{q}}),Rad^m(X))} \leq C \|f\|_{RH^{\infty}(ST;\tilde{\mathcal{A}})} \]

for all \( f \in RH^{\infty}(ST;\tilde{\mathcal{A}}) \). Indeed, once this has been established, the rest of the proof is identical to that of Theorem 6.2. To obtain (6.6) apply (6.1) to \( \tilde{\mathcal{A}} \) on \( Rad^m(X) \). To see that the constant \( C \) that one gets from this can be chosen to be independent of \( n \), it suffices by Remark 4.4 to show that \( Rad^m(X) \) is complemented in \( L^2([0,1];X) \) by a projection \( P_n \in L(L^2([0,1];X)) \) with \( \|P_n\|_{L(L^2([0,1];X))} \leq C' \) for some \( C' \geq 0 \) independent of \( n \). The latter in turn follows from the fact that
X has nontrivial type and from Pisier’s characterization in \[45\] of the K-convex Banach spaces as the spaces with nontrivial type.

We do not know whether the assumption in Theorem \[63\] that X has nontrivial type is necessary.

From Theorems \[62\] and \[63\] one can deduce R-bounded versions of Theorems \[5,4\] and \[5,4\] in the obvious manner. Also, as a corollary of our results, for C₀-groups we improve Theorem \[33\] Theorem 6.1).

**Corollary 6.4.** Let \(-iA\) generate a C₀-group \((U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)\) on a Banach space X with property \((\alpha)\), type \(p \in [1, 2]\) and cotype \(q \in [2, \infty)\). Let \(\omega > \theta(U)\). Then \(\{e^{-\omega|s|}U(s) \mid s \in \mathbb{R}\} \subseteq \mathcal{L}(D(A(\frac{1}{p} - \frac{1}{q}, 1), X)\) is R-bounded.

If X is isomorphic to a complemented subspace of a p-convex and q-concave Banach lattice for \(p \in (1, 2)\) and \(q \in [2, \infty)\), then \(\{e^{-\omega|s|}U(s) \mid s \in \mathbb{R}\} \subseteq \mathcal{L}(D(\lambda + iA)^{\frac{1}{p} - \frac{1}{q}}, X)\) is R-bounded for each \(\lambda > 0\).

**Proof.** Let \(\omega' \in (\theta(U), \omega)\) and let \(T\) be the unit ball of \(\mathbb{C} \subseteq \mathcal{L}(X)\). Now apply Theorems \[62\] and \[63\] to \(\{e^{-\omega'|s|}|s|^{i\omega'} \mid s \in \mathbb{R}\} \subseteq RH^\infty(St_{\omega' \in A \cap T})\).

As the following example shows, Corollary 6.4 and Theorem 6.3 are sharp.

**Example 6.5.** Let \(p \in [1, \infty)\) and let \((U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)\) be the left translation group on \(X := L^p(\mathbb{R})\) with generator \(-iA\), where \(Af := if'\) for \(f \in D(A) = W^{1, p}(\mathbb{R})\). Then \(D((iA)^\alpha) = H^{\alpha,p}(\mathbb{R})\), where \(H^{\alpha,p}(\mathbb{R})\) is a Bessel-potential space. It is shown in \[33\] Example 6.2] that \(\{U(s) \mid s \in [-1, 1]\} \subseteq \mathcal{L}(H^{\alpha,p}(\mathbb{R}), L^p(\mathbb{R}))\) is not R-bounded for \(\alpha \in [0, \frac{1}{p} - \frac{1}{q}]\). Hence \(\{e^{-\omega|s|}U(s) \mid s \in \mathbb{R}\} \subseteq \mathcal{L}(D((iA)^\alpha), X)\) is not R-bounded for \(w \in \mathbb{R}\) and \(\alpha \in [0, \frac{1}{p} - \frac{1}{q}]\).

### 7. Rational approximation

In this section we give an application of the results in previous sections to the theory of rational approximation of C₀-groups. Note that the results in Section 6 can be used to replace the uniform bounds in this section by R-bounds.

Recall that a C₀-semigroup \((T(t))_{t\geq 0} \subseteq \mathcal{L}(X)\) on a Banach space X is **exponentially stable** if there exist \(M \geq 1\) and \(\omega > 0\) such that \(\|T(t)\|_{\mathcal{L}(X)} \leq Me^{-\omega t}\) for all \(t \geq 0\). We note that, if \(-A\) generates an exponentially stable C₀-semigroup \((T(t))_{t\geq 0} \subseteq \mathcal{L}(X)\) such that \(T(t) \in \mathcal{L}(X)\) is invertible for each \(t \geq 0\), then \(-A\) in fact generates the C₀-group \((U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)\), where \(U(s) := T(s)\) for \(s \geq 0\) and \(U(s) := T(-s)^{-1}\) for \(s < 0\). Then \(f(A)\) is defined as an unbounded operator for each \(f \in H^\infty(C_+)\) by a shifted version of the strip-type calculus from Section 2.1.

**Lemma 7.1.** Let \(-A\) generate an exponentially stable C₀-semigroup \((T(t))_{t\geq 0} \subseteq \mathcal{L}(X)\) with type \(p \in [1, 2]\) and cotype \(q \in [2, \infty)\). Suppose that \(T(t)\) is invertible for all \(t \geq 0\). Then there exists a constant \(C \geq 0\) such that

\[
\|f(A)\|_{\mathcal{L}(D(A(\frac{1}{p} - \frac{1}{q}, 1), X))} \leq C \|f\|_{H^\infty(C_+)}
\]

for all \(f \in H^\infty(C_+)\). For each \(\beta > \frac{1}{p} - \frac{1}{q}\) there exists a constant \(C' \geq 0\) such that

\[
\|f(A)A^{-\beta}\|_{\mathcal{L}(X)} \leq C' \|f\|_{H^\infty(C_+)}
\]

for all \(f \in H^\infty(C_+)\).
Proof. Let $f \in \mathcal{H}^\infty(\mathbb{C}_+)$ and apply Theorem 3.1 and Proposition 3.5 to the strip-type operator $-i(A - \omega)$ and the function $f(i \cdot + \omega) \in \mathcal{H}^\infty(St_\omega)$ for $\omega > 0$ large enough. Then use the composition rule

$$f(i \cdot + \omega)(-i(A - \omega)) = f(A),$$

which is straightforward to prove in the same manner as [23, Theorem 4.2.4]. □

Lemma 7.1 applies to the important question of the power-boundedness of the Cayley transform $(1 - A)(1 + A)^{-1}$ of $A$, which in turn is equivalent to the stability of the Crank-Nicholson approximation scheme associated with $A$.

Corollary 7.2. Let $-A$ generate an exponentially stable $C_0$-semigroup $(T(t))_{t \geq 0} \subseteq \mathcal{L}(X)$ on a Banach space $X$ with type $p \in [1, 2]$ and cotype $q \in [2, \infty)$. Suppose that $T(t)$ is invertible for all $t \geq 0$. Then

$$\sup_{n \in \mathbb{N}} \left\| (1 - A)^n(1 + A)^{-n} \right\|_{\mathcal{L}(D_A(\frac{1}{p} - \frac{1}{q}, 1), X)} < \infty.$$

For $n \in \mathbb{N}$ let $p_n$ and $q_n$ be the unique polynomials of degree $n$ and $n + 1$, respectively, such that $p_n(0) = q_n(0) = 1$ and such that

$$\left| \frac{p_n(z)}{q_n(z)} - e^z \right| \leq C |z|^{2n+2}$$

for all $z$ in a neighborhood of $0 \in \mathbb{C}$. Let $r_n := \frac{p_n}{q_n}$. Then $r_n$ is the $n$-th subdiagonal Padé approximation of the exponential function.

For $\frac{1}{p} - \frac{1}{q} \in [0, \frac{1}{2})$ the following proposition improves convergence rates obtained in [17, Theorem 4.1] for uniformly bounded $C_0$-semigroups on general Banach spaces. Note that, on a Banach space $X$ with type $p \in [1, 2]$ and cotype $q \in [2, \infty)$, for $\alpha > \frac{1}{p} - \frac{1}{q}$ we obtain strong convergence of $r_n(-tA)$ to $T(t)$ on $D(A^\alpha)$ with rate $C_{\alpha} \leq O(n^{-\alpha + \frac{1}{p} - \frac{1}{q}})$, locally uniformly in $t$.

Proposition 7.3. Let $-A$ generate an exponentially stable $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$ with type $p \in [1, 2]$ and cotype $q \in [2, \infty)$. Suppose that $T(t)$ is invertible for all $t \geq 0$. Let $\alpha > \frac{1}{p} - \frac{1}{q}$ and $a \in (0, \alpha - \frac{1}{p} + \frac{1}{q})$. Then there exists a constant $C \geq 0$ such that

$$\|r_n(-tA)x - T(t)x\|_X \leq C t^a (n + 1)^{-a} \|A^\alpha x\|_X$$

for all $t \in (0, \infty)$, all $n \in \mathbb{N}$ with $n > \frac{a}{2} - 1$ and all $x \in D(A^\alpha)$. Moreover, $(r_n(-tA))_{n \in \mathbb{N}}$ converges strongly on $D_A(\frac{1}{p} - \frac{1}{q}, 1)$ to $T(t)$, locally uniformly in $t$.

Proof. Let $t \in (0, \infty)$ and $n \in \mathbb{N}$ with $n \geq \frac{a}{2} - 1$. Set

$$f(z) := \frac{r_n(-tz) - e^{-tz}}{z^a} \quad (z \in \mathbb{C}_+).$$

Then Lemma 3.4 yields a constant $C' \geq 0$ such that

$$\left\| (r_n(-tA) - T(t))A^{-\alpha} \right\|_{\mathcal{L}(X)} = \left\| f(A)A^{-\alpha + a} \right\|_{\mathcal{L}(X)} \leq C' \| f \|_{\mathcal{H}^\infty(\mathbb{C}_+)}.$$

By [17, Lemmas 3.3 and 3.5],

$$\| f \|_{\mathcal{H}^\infty(\mathbb{C}_+)} = t^a \sup_{z \in \mathbb{C}_+} \frac{r_n(-tz) - e^{-tz}}{(tz)^a} \leq 2t^a(n + 1)^{-a}.$$
Therefore, with \( C := 2C' \),
\[
\| r_n(-tA)x - T(t)x \|_X \leq \| (r_n(-tA) - T(t))A^{-\alpha} \|_{\mathcal{L}(X)} \| A^\alpha x \|_X \\
\leq C t^\alpha (n + 1)^{-\alpha} \| A^\alpha x \|_X
\]
for all \( x \in D(A^\alpha) \), which proves the first statement.

Since \( \| r_n \|_{\mathcal{H}^\infty(\mathbb{C}_-)} \leq 1 \) for all \( n \in \mathbb{N} \) by [15], Lemma 7.1 yields that
\[
\{ r_n(-tA) - T(t) \mid n \in \mathbb{N}, t \geq 0 \} \subseteq \mathcal{L}(D_A(\frac{1}{p} - \frac{1}{q}, 1), X)
\]
is uniformly bounded. The proof is now concluded by what we have already shown and by the fact that \( D(A^2) \) is dense in \( D_A(\frac{1}{p} - \frac{1}{q}, 1) \).

The same method that was used in Proposition 7.3 to yield strong convergence on \( D_A(\frac{1}{p} - \frac{1}{q}, 1) \) also works for other rational approximation methods. Recall that a rational function \( r \in \mathcal{H}^\infty(\mathbb{C}_-) \) is said to be \( A \)-stable if \( \| r \|_{\mathcal{H}^\infty(\mathbb{C}_-)} \leq 1 \), and \( r \) is a rational approximation (of the exponential function) of order \( k \in \mathbb{N} \) if there exists a constant \( C \geq 0 \) such that \( |r(z) - e^z| \leq C|z|^{k+1} \) for all \( z \) in a complex neighborhood of 0.

**Corollary 7.4.** Let \( r \) be an \( A \)-stable rational approximation of order \( k \in \mathbb{N} \). Let \( -A \) generate an exponentially stable \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) on a Banach space \( X \) with type \( p \in [1, 2] \) and cotype \( q \in [2, \infty) \). Suppose that \( T(t) \) is invertible for all \( t \geq 0 \). Then \( (r(-\frac{1}{n}A)A)^n \) converges strongly on \( D_A(\frac{1}{p} - \frac{1}{q}, 1) \) to \( T(t) \), locally uniformly in \( t \geq 0 \).

**Proof.** Lemma 7.1 yields a constant \( C \geq 0 \) such that
\[
\| r(-\frac{1}{n}A)^n - T(t) \|_{\mathcal{L}(D_A(\frac{1}{p} - \frac{1}{q}, 1), X)} \leq C \| r(-\frac{1}{n})^n - e^{-t} \|_{\mathcal{H}^\infty(\mathbb{C}_-)} \\
\leq C(\| r^n \|_{\mathcal{H}^\infty(\mathbb{C}_-)} + 1) \leq 2C
\]
for all \( n \in \mathbb{N} \) and \( t \geq 0 \). Since, by [10] Theorem 3], \( r(-\frac{1}{n}A)^n \) converges locally uniformly in \( t \) to \( T(t) \) on \( D(A^{k+1}) \), the uniform boundedness of
\[
\{ r(-\frac{1}{n}A)^n - T(t) \mid t \geq 0, n \in \mathbb{N} \} \subseteq \mathcal{L}(D_A(\frac{1}{p} - \frac{1}{q}, 1), X)
\]
and the fact that \( D(A^{k+1}) \) is dense in \( D_A(\frac{1}{p} - \frac{1}{q}, 1) \) yield the desired statement.

**Remark 7.5.** If \( X \) is isomorphic to a complemented subspace of a \( p \)-convex and \( q \)-concave Banach lattice, for \( p \in [1, 2] \) and \( q \in [2, \infty) \), then the case \( \beta = \frac{1}{p} - \frac{1}{q} \) is attained in Lemma 7.1. Hence, in the setting of Corollary 7.2
\[
\sup_{n \in \mathbb{N}} \| (1-A)^n(1+A)^{-n-\beta} \|_{\mathcal{L}(X)} < \infty.
\]
Moreover, one obtains rate \( O(n^{-\alpha + \frac{1}{p} - \frac{1}{q}}) \) in Proposition 7.3 and strong convergence on \( D(A^{\frac{1}{p} - \frac{1}{q}}) \) in Corollary 7.4.

**Funding.** This work was supported by the Netherlands Organisation for Scientific Research (NWO) [grant number 613.000.908 “Applications of Transference Principles”].

**Acknowledgements.** The author thanks Mark Veraar for numerous helpful suggestions, and the referee for carefully checking the manuscript.
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