Exact tail asymptotics for a discrete-time preemptive priority queue

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Abstract

In this paper, we consider a discrete-time preemptive priority queue with different service rates for two classes of customers, one with high-priority and the other with low-priority. This model corresponds to the classical preemptive priority queueing system with two classes of independent Poisson customers and a single exponential server. Due to the possibility of customers’ arriving and departing at the same time in a discrete-time queue, the model considered in this paper is more complicated. In this model, we focus on the characterization of exact tail asymptotics for the joint stationary distribution of the queue length of the two classes of customers, for the two boundary distributions and for the two marginal distributions, respectively. By using generating functions and kernel method, we get an explicit expression of exact tail asymptotics along the low-priority queue direction, as well as along the high-priority queue direction.

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1 Introduction

Preemptive priority queueing system is meaningful in real life, for which the cases that take place in the emergency room or in a bank could be good examples. Miller [16] provided a special structure on the rate matrix, and suggested an efficient computational scheme for M/M/1 priority queue. Since then, the priority queueing systems have attracted a lot of interest. See, for example, Gail, Hantler and Taylor [7, 8], Kao and Narayanan [11], Takine [17], Alfa [1], Isotupa and Stanford [10], Alfa, Liu and He [2], Drekic and Woolford [4], Zhao et al. [19] and so on.

The stationary distribution is very important for a queueing model to characterize its performance, but in most cases, it is difficult to obtain the explicit expression of the stationary distribution. This motivates the study of exact tail asymptotic behaviours of the stationary distribution. Actually, the property of exact tail asymptotics is important since it often leads to the performance bounds, approximations and properties of the queueing model. In recent years, many new results on exact tail asymptotics for the continuous-time models have been established. For example, Li and Zhao [12] obtained the tail asymptotic results for the classical continuous-time preemptive priority queueing model. However, in contrast to the extensive study of tail asymptotics for the continuous-time queueing systems, there has been little systematic investigation on exact tail asymptotics for discrete-time queueing systems. The main reason for this is that, in a discrete-time queueing system, all queueing activities (e.g. customers’ arrival and departure) could occur simultaneously, which results in the complexity and difficulty of the analysis. Considering the importance of the discrete-time queueing systems in both theory and applications, it is interesting and important to study tail asymptotics for those models. Recently, Xue and Alfa [18] discussed a discrete-time priority BMAP/PH/1 queue, and obtained exact tail asymptotics in the marginal distribution for the low-priority queue. Inspired by above, in this paper we will extend the model studied in Li

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and Zhao [12] to the discrete-time one, i.e., the discrete-time preemptive priority queue with single server and two types of customers, and study the tail asymptotics for this model. We will carry out analysis through the Kernel method, and the details about Kernel method could be found in Fayolle, Iasnogorodski and Malyshev [5], Li and Zhao [12, 13, 14], Li, Javad and Zhao [15]. Considering the reality, asymmetric service rates are allowed in our model.

The rest of the paper is organized as follows. Section 2 provides the model description and the fundamental form. Discussions about the fundamental form and the kernel equation are presented in Section 3, then expressions of the generating functions are obtained. In Section 4, the singularity analysis and the Tauberian-like theorem used to determine tail asymptotics are provided. Sections 5 and 6 detail the proofs of exact tail asymptotics along both queue directions.

2 Model description and fundamental form

In this paper we focus on the discrete-time preemptive priority queue with two classes of customers, one with high-priority and the other with low-priority. Both kinds of customers arrive independently according to two Bernoulli processes with probabilities $p$ and $q$, and their service times follow geometrically distributions with parameters $\mu_h$ and $\mu_l$, respectively. In each of the two classes, the service rule is first in, first out (FIFO). With the preemptive rule, the service of a low-priority customer is interrupted upon the arrival of a high-priority customer. The interrupted low-priority customer will stay at the head of the waiting line for restarting its service immediately after the last high-priority customer in the system completes its service. All the processes are mutually independent. Let $Q_1(n)$ and $Q_2(n)$ be the number of high- and low-priority customers in the system at the time slot division point $n$, including the one being served, respectively, then we develop a discrete-time Markov chain $\{Q_1(n), Q_2(n)\}$. Without loss of generality, we assume that $p + q + \mu_h + \mu_l = 1$. Denote $\rho_h = \frac{p}{\mu_h}$, $\rho_l = \frac{q}{\mu_l}$ and $\bar{x} = 1 - x$ for any real number $x \in [0, 1]$, the system must be stable if $\rho = \rho_h + \rho_l < 1$, which also implies $p < \bar{p}$ and $q < \bar{q}$. Under this condition, let $\pi_{ij}$ be the joint stationary distribution for the number of high- and low-priority customers, respectively, in the system. We also use the following convention: for two functions $f(n)$ and $g(n)$ of nonnegative integers, $f(n) \sim g(n)$ means that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$.

Differing from the continues-time queueing system, all queueing activities such as the customers’ potential arrival and departure could occur at the same time in a discrete-time queueing system, so for mathematical clarity, we consider an early arrival system (EAS) (see [9]) in this paper. Fig.1 explicates the occurrence order of the potential arrival and departure.

For a discrete-time Markov chain, the transition probabilities and the balance equations can be easily determined from the transition diagram in Fig.2.
The transition probabilities are given by the following:

\[ p_{1,0} = p^{(0)}_{1,0} = p^{(1)}_{1,0} = p^{(2)}_{1,0} = pq\bar{q}\mu_h, \]
\[ p_{1,1} = p^{(0)}_{1,1} = p^{(1)}_{1,1} = p^{(2)}_{1,1} = pq\mu_h, \]
\[ p_{0,1} = p^{(1)}_{0,1} = pq\mu_h + \bar{p}q\bar{\mu}_h, \]
\[ p_{0,0} = p^{(2)}_{0,0} = p\bar{q}h + \bar{p}q\mu_h, \]
\[ p_{-1,1} = p^{(1)}_{-1,1} = \bar{p}q\mu_h, \]
\[ p_{-1,0} = p^{(1)}_{-1,0} = \bar{p}q\mu_h, \]
\[ p_{0,0} = p^{(1)}_{0,0} = \bar{p}q\mu_h + p\bar{q}h, \]
\[ p_{0,0} = \bar{p}q + p\bar{q}h + \bar{p}q\mu, \]
\[ p_{0,0} = \bar{p}q\mu + pqh + \bar{p}q\mu, \]
\[ p_{0,1} = \bar{p}q\mu. \]

Then, the balance equations are

\[ (1 - p^{(0)}_{0,0})\pi_{0,0} = p^{(1)}_{-1,0}\pi_{1,0} + p^{(2)}_{0,-1}\pi_{0,1}, \tag{2.1} \]
\[ (1 - p^{(1)}_{0,0})\pi_{i,0} = p^{(1)}_{1,0}\pi_{i-1,0} + p^{(2)}_{-1,0}\pi_{i+1,0}, \quad i \geq 1, \tag{2.2} \]
\[ (1 - p^{(2)}_{0,0})\pi_{0,j} = p^{(2)}_{1,0}\pi_{0,j-1} + p^{(2)}_{0,-1}\pi_{0,j+1} + p_{-1,1}\pi_{1,j-1} + p_{1,0}\pi_{1,j}, \quad j \geq 1, \tag{2.3} \]
\[ (1 - p^{(0)}_{0,0})\pi_{i,j} = p_{1,0}\pi_{i-1,j} + p_{-1,0}\pi_{i+1,j} + p_{-1,1}\pi_{i+1,j-1} + p_{1,0}\pi_{i,j-1} + p_{1,1}\pi_{i-1,j-1}, \quad i \geq 1, j \geq 1. \tag{2.4} \]
As a specific case of random walks in the quarter plane, we define the following generating functions of the stationary distributions:

\[ \varphi_j(x) = \sum_{i=0}^{\infty} \pi_{i,j} x^i, \quad j \geq 0, \]

\[ \psi_i(y) = \sum_{j=0}^{\infty} \pi_{i,j} y^j, \quad i \geq 0, \]

\[ P(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_{i,j} x^i y^j = \sum_{i=0}^{\infty} \psi_i(y) x^i = \sum_{j=0}^{\infty} \varphi_j(x) y^j. \]

It is clear that \( \varphi_0(x) = P(x, 0) = P_1(x) \) and \( \psi_0(y) = P(0, y) = P_2(y) \). Fayolle, Iasnogorodski and Malyshev establish a functional equation in terms of the unknown generating functions, which is often referred to as the fundamental form. Using a similar argument in our case, we give the fundamental form as follows, which connects the bivariate unknown function \( P(x, y) \) and two univariate unknown functions \( P_1(x) \) and \( P_2(y) \), i.e.,

\[ H(x, y)P(x, y) = H_1(x, y)P_1(x) + H_2(x, y)P_2(y) + H_0(x, y)\pi_{0,0}, \quad (2.5) \]

where

\[ H(x, y) = -h(x, y), \]

\[ H_1(x, y) = -h(x, y) + h_1(x, y)y, \]

\[ H_2(x, y) = -h(x, y) + h_2(x, y)x, \]

\[ H_0(x, y) = h_0(x, y)xy + h(x, y) - h_1(x, y)y - h_2(x, y)x, \]

and

\[ h(x, y) = xy \left( \sum_{i=-1}^{1} \sum_{j=-1}^{1} p_{i,j} x^i y^j - 1 \right) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y), \]

\[ h_1(x, y) = x \left( \sum_{i=-1}^{1} \sum_{j=-1}^{1} p_{i,j}^{(1)} x^i y^j - 1 \right) = a_1(y)x^2 + b_1(y)x + c_1(y), \]

\[ h_2(x, y) = y \left( \sum_{i=-1}^{1} \sum_{j=-1}^{1} p_{i,j}^{(2)} x^i y^j - 1 \right) = a_2(y)x + b_2(y), \]

\[ h_0(x, y) = \left( \sum_{i=-1}^{1} \sum_{j=-1}^{1} p_{i,j}^{(0)} x^i y^j - 1 \right) = a_0(y)x + b_0(y). \]

After some simple calculations, we have

\[ H(x, y) = -y \left\{ p\tilde{\mu}_h(qy + \tilde{q})x^2 + [(p\mu_h + \tilde{p}\mu_h)(qy + \tilde{q}) - 1]x + \tilde{p}\mu_h(qy + \tilde{q}) \right\}, \quad (2.6) \]

\[ H_1(x, y) = 0, \quad (2.7) \]

\[ H_2(x, y) = \tilde{p}(qy + \tilde{q})[(\mu_h - \mu_1)xy - \mu_hy + \mu_1x], \quad (2.8) \]

and

\[ H_0(x, y) = \tilde{p}\mu_1(y - 1)x. \quad (2.9) \]

So the fundamental form \( (2.5) \) can be simplified as

\[ H(x, y)P(x, y) = H_2(x, y)\psi_0(y) + H_0(x, y)\pi_{0,0}. \quad (2.10) \]
3 Kernel equation and generating functions

In this section, the Kernel equation

\[ H(x, y) = 0 \]  

is considered at first.

By (2.6), we have that

\[ H(x, y) = -yK(x, y), \]

where \( K(x, y) \) is called key kernel with

\[ K(x, y) = p\bar{\mu}_h(qy + \bar{q})x^2 + [(p\mu_h + p\bar{\mu}_h)(qy + \bar{q}) - 1]x + \bar{\mu}_h(qy + \bar{q}). \]

For each fixed \( y \), we consider \( K(x, y) \) as a quadratic polynomial of \( x \) and rewrite it as

\[ K(x, y) = a(y)x^2 + b(y)x + c(y), \]  

where

\[ a(y) = \frac{\hat{a}(y)}{y} = p\bar{\mu}_h(qy + \bar{q}), \quad b(y) = \frac{\hat{b}(y)}{y} = (p\mu_h + p\bar{\mu}_h)(qy + \bar{q}) - 1 \quad \text{and} \quad c(y) = \frac{\hat{c}(y)}{y} = \bar{\mu}_h(qy + \bar{q}). \]

Let \( \Delta(y) \) be the determinant of \( K(x, y) = 0 \), i.e.,

\[ \Delta(y) = b(y)^2 - 4a(y)c(y) = (p\mu_h - p\bar{\mu}_h)^2(qy + \bar{q})^2 - 2(p\mu_h + p\bar{\mu}_h)(qy + \bar{q}) + 1. \]  

Hence, the two solutions to (3.2) are given by

\[ x_0(y) = 1 - \frac{(p\mu_h + p\bar{\mu}_h)(qy + \bar{q}) - \sqrt{\Delta(y)}}{2p\bar{\mu}_h(qy + \bar{q})}, \]

and

\[ x_1(y) = 1 - \frac{(p\mu_h + p\bar{\mu}_h)(qy + \bar{q}) + \sqrt{\Delta(y)}}{2p\bar{\mu}_h(qy + \bar{q})}. \]

We call \( y \) as a branch point if it satisfies \( \Delta(y) = 0 \). Here it is easy to get the two branch points which are given by

\[ y_0 = \frac{p\mu_h + p\bar{\mu}_h - 2\sqrt{p\mu_h p\bar{\mu}_h}}{(p\mu_h - p\bar{\mu}_h)^2 q} - \frac{\bar{q}}{q}, \]

and

\[ y_1 = \frac{p\mu_h + p\bar{\mu}_h + 2\sqrt{p\mu_h p\bar{\mu}_h}}{(p\mu_h - p\bar{\mu}_h)^2 q} - \frac{\bar{q}}{q}. \]

When \( y = 0 \), we have

\[ x_0 = x_0(0) = \frac{r_0}{w}, \]

\[ x_1 = x_1(0) = \frac{1}{r_0}, \]

where

\[ r_0 = \frac{1}{x_1(0)}, \quad w = \frac{p\bar{\mu}_h}{p\mu_h}. \]

The following lemma presents the properties of the branch points and the two branches.
Lemma 3.1 (i) Suppose that $y_b$ is the root of $b(y) = 0$, then $1 < y_0 < y_b < y_1$.

(ii) For $-1 \leq y \leq 1$, we have $x_0(y) < x_1(y)$ and $0 < x_0(y) \leq 1$. When $y = 0$, $0 < x_0 < 1 < x_1$.

Proof: (i) One easily have that $b(y) = (pyh + \tilde{p}h)(qy + \tilde{q}) - 1 < 0$ for $y \leq 1$. This implies that $y_b > 1$, since $b(y)$ is a linear function. Moreover, $a(1) + b(1) + c(1) = 1$, so we can easily get

$$\Delta(1) = [a(1) - c(1)]^2 > 0.$$  

However,

$$\Delta(y_b) = -4a(y_b)c(y_b) < 0.$$  

Considering that $\Delta(\infty) > 0$ and $y_0 < y_1$, we can conclude $1 < y_0 < y_b < y_1$.

(ii) It is obvious that $x_0(y) < x_1(y)$. To prove $x_0(y) > 0$ for $y \in [-1, 1]$, we need to show

$$1 - (pyh + \tilde{p}h)(qy + \tilde{q}) > \sqrt{\Delta(y)},$$  

which is equivalent to

$$4pyh\tilde{p}h(qy + \tilde{q})^2 > 0.$$  

On the other hand, to prove $x_0(y) \leq 1$ for $-1 \leq y \leq 1$, we only need to show

$$1 - (pyh + \tilde{p}h)(qy + \tilde{q}) - \sqrt{\Delta(y)} \leq 2\tilde{p}h(qy + \tilde{q})$$  

which is equivalent to

$$(qy + \tilde{q})q(y - 1) \leq 0.$$  

Since $x_1(0) > 1$, the proof is completed. \hfill \square

Remark 3.1 Both the two branches $x_0(y)$ and $x_1(y)$ are analytic in the cut plane $C_y \setminus [y_0, y_1]$.

Next, we determine the generating functions $\varphi_j(x)$ and $\psi_0(y)$. Based on the balance equations, $\varphi_j(x)$ are recursively determined in the following lemma.

Lemma 3.2

$$\varphi_0(x) = \frac{\pi_{0,0}}{1 - r_0x},$$  

(3.11)

and

$$\varphi_j(x) = \frac{a_j}{x - x_1} - \frac{q\varphi_{j-1}(x)(x + x_0)}{\tilde{q}(x - x_1)} - \frac{q(\rho x + \tilde{p})(\muhx + \muh)\varphi_{j-1}(x) - \varphi_{j-1}(x_0)}{p\tilde{p}h(x - x_1)} x - x_0, \quad j = 1, 2, \ldots$$  

(3.12)

where

$$\pi_{0,0} = \frac{1 - \rho}{\tilde{p}q},$$  

(3.13)

and

$$a_j = \frac{[\tilde{p}\tilde{q}(\mu_l - \mu_h) - p\tilde{p}q\mu_l]\pi_{0,j} - \tilde{p}q\mu_l\pi_{0,j+1} + \tilde{p}q(\mu_l - \mu_h)\pi_{0,j+1} - q(\tilde{p}\tilde{p}h + p\muh)\varphi_{j-1}(x_0)}{p\tilde{p}h}.$$  

Proof: It follows from (2.2) that

$$\varphi_0(x) = \frac{p\tilde{p}h(\pi_{0,0} + \pi_{1,0}x) - [1 - (p\muh + \tilde{p}q)\bar{q}]\pi_{0,0}x}{p\tilde{p}h(x - x_0)(x - x_1)},$$  

(3.14)

where $x_0$ and $x_1$ are given by (3.8) and (3.9), respectively. By Lemma 3.1 and the fact that $\varphi_0(x)$ is analytic inside the unit circle, we get that $x_0 = \frac{\pi_0}{w}$ is also a zero of the numerator of the function on the right side of (3.14). By some calculations, we obtain

$$\varphi_0(x) = \frac{\pi_{0,0}}{1 - r_0x}.$$
Now we determine $\pi_{0,0}$ from (2.10). For convenience, let

$$\tilde{G}(x) = \frac{\mu_1 x}{\mu_h - (\mu_h - \mu)x}.$$ 

Then, the coefficient of $\psi_0(y)$ is zero in (2.10). Furthermore, $x \to 1$ implies $\tilde{G}(x) \to 1$. By L’Hospital’s rule, we can obtain the detailed expression of $\pi_{0,0}$.

Similarly, using the same argument as above, we can also get $\varphi_j(x)$ from (2.3) and (2.4). We omit the proof here.

At the end of this section, we determine the generating function $\psi_0(y)$.

**Lemma 3.3** Let

$$F(y) = (\bar{p} + \mu_h - 2\bar{p}\mu)y^2 + [(\bar{p} + \mu_h - 2\bar{p}\mu)\bar{q} + 2\bar{p}\mu_1 - 1]y + 2\bar{p}\mu_1,$$

$$T(y) = F(y) - y\sqrt{\Delta(y)}$$

and

$$T^*(y) = F(y) + y\sqrt{\Delta(y)},$$

where $\Delta(y)$ is defined in (3.3). Then, for $y \in [-1, 1]$, we have

$$\psi_0(y) = a\frac{T^*(y)}{(qy + q)(1 - \eta_1 y)} + b\frac{T^*(y)}{(qy + q)(1 - \eta_2 y)},$$

where

$$a = \frac{\pi_{0,0} \eta_1}{2\bar{p}\mu \eta_1 - \eta_2},$$

$$b = \frac{\pi_{0,0} \eta_2}{2\bar{p}\mu \eta_2 - \eta_1}.$$ 

$$\eta_1 = \frac{(1 - \mu_h \bar{q} - \bar{p}\mu_1 - \bar{p}\mu_1) + \sqrt{(1 - \mu_h \bar{q} - \bar{p}\mu_1 - \bar{p}\mu_1)^2 + 4\bar{p}\mu_1(\mu_h - \bar{p}\mu_1)\bar{q}}} {2\bar{p}\mu_1},$$

$$\eta_2 = \frac{(1 - \mu_h \bar{q} - \bar{p}\mu_1 - \bar{p}\mu_1) - \sqrt{(1 - \mu_h \bar{q} - \bar{p}\mu_1 - \bar{p}\mu_1)^2 + 4\bar{p}\mu_1(\mu_h - \bar{p}\mu_1)\bar{q}}} {2\bar{p}\mu_1}.$$ 

**Proof:** We get the expression of the generating function $P(x, y)$ directly from the fundamental form (2.10), i.e.,

$$P(x,y) = \frac{H_2(x, y)\psi_0(y) + H_0(x, y)\pi_{0,0}}{-yK(x, y)}$$

$$= \frac{\bar{p}(qy + q)[(\mu_h - \mu_1)xy - \mu_1y + \mu_1x]\psi_0(y) + \bar{p}\mu_1(y - 1)x\pi_{0,0}}{-y(x - x_0(y))(x - x_1(y))},$$

where $x_0(y)$ and $x_1(y)$ are given by (3.4) and (3.5), respectively. For $-1 \leq y \leq 1$, $P(x_0(y), y)$ is analytic and nonzero, which implies that

$$H_2(x_0(y), y)\psi_0(y) + H_0(x_0(y), y)\pi_{0,0} = 0.$$ 

Therefore, the equations (3.5) and (3.20) lead to

$$\psi_0(y) = \frac{\bar{q}\mu_1(1 - y)x_0(y)\pi_{0,0}}{(qy + q)\{(\mu_h - \mu_1)y + \mu_1x_0(y) - \mu_1y\}}$$

$$= \frac{2\bar{p}\mu_1(1 - y)\pi_{0,0}T^*(y)}{T(y)T^*(y)},$$

since $x_0(y)x_1(y) = 1/w$. It is easy to confirm that $T(1) = 0$ and $T^*(-\frac{2}{y}) = 0$. Furthermore, we need to identify other zeros of $T(y)T^*(y)$, since they are the candidate poles of $\psi_0(y)$. We have

$$T(y)T^*(y) = F(y)^2 - y^2\Delta(y)$$

$$= 4\bar{p}(qy + q)(y - 1)[(\mu_h - \bar{p}\mu_1)\bar{q}y^2 + \mu_1(1 - \mu_h \bar{q} - \bar{p}\mu_1 - \bar{p}\mu_1)y - \bar{p}\mu_1^2]$$

$$= 4\bar{p}(qy + q)(y - 1)f(y).$$
where
\[ f(y) = (\mu_h - \bar{\mu}_l)\bar{\mu}_l y^2 + \mu_l (1 - \mu_h \bar{q} - \bar{\mu}_l q - \bar{\mu}_l y) - \bar{\mu}_l \bar{q} y. \]  \hfill (3.22)

Obviously, \( \eta_2 < \eta_1 \) and \( \eta_1, \eta_2 \) are two non-unit zeros of the denominator of the generating function \( \psi_0(y) \).

From Lemma 3.1 we know that \( 1 < y_0 < y_1 \), so \( T^*(y) \) is analytic in \([-1, 1]\). By substituting
\[ T(y)T^*(y) = 4\bar{\mu}_l \bar{\mu}_l \bar{q} y (1-y)(1-\eta_1 y)(1-\eta_2 y) \]
into \( \psi_0(y) \), and then using partial fractions, we can get that \( \text{[10]} \) holds for \( -1 \leq y \leq 1 \), which completes the proof of the lemma.

Below, we will study the exact tail asymptotics for this queueing system. For convenience and simplification, we only consider the case that \( \mu_l \leq \mu_h \). In fact, this condition is not critical since we can follow the same procedure and method used in this paper to study the case that \( \mu_l > \mu_h \).

4 Analysis of singularities

The analysis of exact tail asymptotics along the low-priority queue direction in the stationary distribution \( \pi_{i,j} \) relies on the analysis of the singularities of the generating function \( \psi_0(y) \), which is the focus of this section. Then according to the detailed asymptotic property at the dominant singularity of the generating function, asymptotics of the coefficients of the generating functions will be obtained by using Tauberian-like theorem. Before giving the Tauberian-like theorem, we first introduce the definition of \( \Delta^- \) domain in [6].

**Definition 4.1** For given numbers \( \epsilon > 0 \) and \( \phi \) with \( 0 < \phi < \frac{\pi}{2} \), the open domain \( \Delta(\phi, \epsilon) \) is defined by
\[ \Delta(\phi, \epsilon) = \{ z \in \mathbb{C} : |z| < 1 + \epsilon, \arg(z) > 1 \} \]

A domain is a \( \Delta^- \) domain at 1 if it is a \( \Delta(\phi, \epsilon) \) for some \( \epsilon > 0 \) and \( 0 < \phi < \frac{\pi}{2} \). For a complex number \( \xi \neq 0 \), a \( \Delta^- \) domain at \( \xi \) is defined as the image \( \xi \cdot \Delta(\phi, \epsilon) \) of a \( \Delta^- \) domain \( \Delta(\phi, \epsilon) \) at 1 under the mapping \( z \to \xi z \). A function is called \( \Delta^- \) analytic if it is analytic in some \( \Delta^- \) domain.

**Remark 4.1** When we apply the Tauberian-like theorem for obtaining the exact asymptotics, we often need to consider the region \( \Delta(\phi, \epsilon) \). The region \( \Delta(\phi, \epsilon) \) is an dented disk. In the sequel, without otherwise stated, the limit of a \( \Delta^- \) analytic function is always taken in \( \Delta(\phi, \epsilon) \).

The following Tauberian-like theorem is from Bender [3], which can also be found in Flajolet and Sedgewick [6].

**Theorem 4.1** (Tauberian-like theorem for single singularity) Suppose \( A(z) = \sum_{n \geq 0} a_n z^n \) is an analytic at zero with \( R \) the radius of convergence. Suppose that \( R \) is a singularity of \( A(z) \) on the circle of convergence such that \( A(z) \) can be continued to a \( \Delta^- \) domain at \( R \). If for a real number \( \alpha \notin \{0, -1, -2, \ldots\} \),
\[ \lim_{z \to R} (1 - z/R)^\alpha A(z) = g, \]
where \( g \) is a non-zero constant. Then
\[ a_n \sim \frac{g}{\Gamma(\alpha)} n^{\alpha-1} R^{-n}, \]
where \( \Gamma(\alpha) \) is the value of Gamma function at \( \alpha \).

To apply the above Tauberian-like theorem, we only need to focus on the singularity with modulus greater than 1. For the generating function \( \psi_0(y) \), the following Key lemma specifies when the dominant singularity is a pole or not a pole.

**Lemma 4.1** (i) If \( F(y_0) \neq 0 \), then \( 1 < 1/\eta_1 < y_0 \).
(ii) If \( \rho = \rho_h + \rho_l < 1 \), then \( T'(1) < 0 \).

Proof: (i) It is easy to get that \( f(0) = -\bar{\rho} \mu_h^2 < 0 \) and \( f(1) = \mu_h \rho - 1 < 0 \). From (3.19), (3.20), (3.22), we can also confirm that \( 1/\eta_2 < 0 \) and \( 1/\eta_1 > 1 \). If \( F(y_0) \neq 0 \), then \( f(y_0) > 0 \) follows from \( T(y_0)T'(y_0) = F(y)^2 > 0 \). Since the branch point \( y_0 > 1 \), therefore, \( 1 < 1/\eta_1 < y_0 \).

(ii) Based on

\[
\frac{dT(y)}{dy} = \frac{dF(y)}{dy} - \sqrt{\Delta(y)} - \frac{y}{2\sqrt{\Delta(y)}} \frac{d\Delta(y)}{dy},
\]

we calculate \( T'(1) = -2\bar{\eta}\mu_l(1-\rho)/(1-\rho_h) < 0 \), since \( \rho = \rho_h + \rho_l < 1 \). \( \square \)

Lemma 4.2 (Key lemma) For the property of \( 1/\eta_1 \), there are three cases:

(i) If \( F(y_0) > 0 \), then \( 1 < 1/\eta_1 < y_0 \) and \( 1/\eta_1 \) is a zero of \( T(y) \), but not \( T^*(y) \). Hence \( 1/\eta_1 \) is the dominant singularity of \( \psi_0(y) \), which is a pole.

(ii) If \( F(y_0) = 0 \), then \( 1 < 1/\eta_1 = y_0 \) and \( 1/\eta_1 = y_0 \) is a zero of both \( T(y) \) and \( T^*(y) \). Hence \( 1/\eta_1 = y_0 \) is the dominant singularity of \( \psi_0(y) \), which is a branch point.

(iii) If \( F(y_0) < 0 \), then \( 1 < 1/\eta_1 < y_0 \) and \( 1/\eta_1 \) is a zero of \( T^*(y) \), but not \( T(y) \). Hence \( y_0 \) is the dominant singularity of \( \psi_0(y) \), which is a branch point.

Proof: We prove the lemma based on the property of \( F(y) \). Since \( \bar{\rho} + \mu_h - 2\bar{\rho}\mu_l > 0 \), \( F(0) = 2\bar{\rho}\mu_l > 0 \) and \( F(1) = \mu_h - \bar{\rho} > 0 \), there are only three cases for the quadratic function \( F(y) \):

(a) If the roots of \( F(y) = 0 \) are in \((0,1)\), then \( T(y) > 0 \) for \( y \in (-\infty,0) \) implies \( T^*(\frac{1}{\eta_2}) = 0 \). But from Lemma 4.1, we know \( 1 < 1/\eta_1 < y_0 \), so \( T^*(\frac{1}{\eta_2}) > 0 \), which would yield \( T(\frac{1}{\eta_2}) = 0 \).

(b) If the roots of \( F(y) = 0 \) are in \((1,\infty)\), then \( T(y) > 0 \) for \( y \in (-\infty,0) \) implies \( T^*(\frac{1}{\eta_2}) = 0 \). If \( F(y_0) > 0 \), from Lemma 4.1, we have \( 1 < 1/\eta_1 < y_0 \), which would yield \( T(\frac{1}{\eta_1}) = 0 \) since \( T(1) = 0 \), \( T'(1) < 0 \) and \( T(y_0) > 0 \). If \( F(y_0) = 0 \), it implies \( f(y_0) = 0 \), then \( 1 < y_0 = 1/\eta_1 \) and \( T(\frac{1}{\eta_1}) = T^*(\frac{1}{\eta_1}) = 0 \). If \( F(y_0) < 0 \), then \( T(y_0) < 0 \). From \( 1 < 1/\eta_1 < y_0 \) and \( T^*(1) > 0 \), we can easily show that \( T^*(\frac{1}{\eta_1}) = 0 \).

(c) If the roots of \( F(y) = 0 \) are in \((-\infty,0)\), then \( T^*(y) > 0 \) for \( y \in (0,y_0) \). From Lemma 4.1, we have \( 1 < 1/\eta_1 < y_0 \) leading to \( T^*(\frac{1}{\eta_1}) > 0 \), which implies \( T(\frac{1}{\eta_1}) = 0 \). Since \( T(-\infty) > 0 \) follows from \( F(-\infty) > 0 \) and we also notice that \( T(1) = 0 \), \( T^*(-\frac{1}{\eta_1}) = 0 \), then we obtain \( T(\frac{1}{\eta_2}) \neq 0 \) but \( T^*(\frac{1}{\eta_2}) = 0 \).

Finally, we summarize the above cases as the Key lemma. \( \square \)

Remark 4.2 In our model, it is obvious to see that \( \psi_0(y) \) must be analytic on the whole complex plane except \( [y_0,y_1] \cup \{ \frac{1}{\eta_1} \} \), so the Tauberian-like theorem will be applied directly in the next section.

Remark 4.3 In the next section, we will see that the three cases in the Key lemma correspond to the three types of exact tail asymptotics along the low priority queue direction: (i) exact geometric; (ii) geometric with a prefactor \( n^{-\frac{1}{2}} \); (iii) geometric with a prefactor \( n^{-\frac{1}{2}} \).

5 Exact tail asymptotics for the low-priority queue

In this section, we provide exact tail asymptotic properties in the joint stationary distribution as well as in the marginal distribution. Firstly, we study the tail asymptotics for the boundary probabilities \( \pi_{0,j} \), and then based on the mathematical induction, we obtain the exact tail asymptotic characterization in the joint probabilities \( \pi_{i,j} \) for any fixed \( i \geq 1 \).

5.1 Exact tail asymptotics of boundary probabilities

Stationary probabilities \( \pi_{0,j} \) for \( j \geq 0 \) are referred to as the boundary probabilities. In this subsection, we apply properties obtained in the previous section to characterize the asymptotic behavior of \( P_2(y) \), i.e., \( \psi_0(y) \). The exact tail asymptotics of boundary probabilities is a direct consequence of the Tauberian-like theorem and the asymptotic behavior of \( \psi_0(y) \).
Define
\[ C_{l,1} = \frac{2aF\left(\frac{1}{y}\right)}{q\eta_1^{-1} + q}, \]  
(5.1)

\[ C_{l,2} = \frac{a(\bar{p} - \mu_b)qy_0 \sqrt{y_0(y_1 - y_0)}}{(qy_0 + \bar{q})\sqrt{\pi}}, \]  
(5.2)

\[ C_{l,3} = \left[ \frac{a}{(qy_0 + \bar{q})(1 - \eta_1 y_0)} + \frac{b}{(qy_0 + \bar{q})(1 - \eta_2 y_0)} \right] \frac{(\mu_b - \bar{p})q}{2\sqrt{\pi}} \sqrt{y_0(y_1 - y_0)}, \]  
(5.3)

where \( a, b, \eta_1, \eta_2, y_0, y_1 \) and \( F(\cdot) \) are given by (3.17), (3.18), (3.19), (3.6), (3.7), (3.15), respectively.

Now we state the main result of this section.

**Theorem 5.1** For the discrete-time preemptive priority queue with two classes of customers satisfying \( \rho < 1 \), characterizations of the exact tail asymptotics in the joint stationary distribution along the low-priority queue direction are given below for \( i = 0 \) of high-priority customers:

(i) (Exact geometric decay) In the region defined by \( F(y_0) > 0 \),
\[ \pi_{0,j} \sim C_{l,1} \eta_1^j. \]  
(5.4)

(ii) (Geometric with a prefactor \( n^{-\frac{1}{2}} \)) In the region defined by \( F(y_0) = 0 \),
\[ \pi_{0,j} \sim C_{l,2} j^{-\frac{1}{2}} y_0^j. \]  
(5.5)

(iii) (Geometric with a prefactor \( n^{-\frac{1}{2}} \)) In the region defined by \( F(y_0) < 0 \),
\[ \pi_{0,j} \sim C_{l,3} j^{-\frac{1}{2}} y_0^j. \]  
(5.6)

**Proof:** (i) In this case, \( 1 < 1/\eta_1 < y_0 \). So, it follows from the Key lemma 4.2 that \( T^*(-\frac{\bar{q}}{q}) = 0, T^*(\frac{1}{y_1}) = 0 \) and \( T^*(\frac{1}{y_2}) = 0 \). Clearly, \( \frac{T^*(y)}{(qy + \bar{q})(1 - \eta_1 y_0)} \) is analytic in \( \Delta(\phi, \epsilon) = \{ y_1 : |y_1| < 1 + \epsilon, y_1 \neq 1, |\text{Arg}(y_1 - 1)| > \phi, \epsilon > 0, 0 < \phi < \pi/2 \} \).

On the other hand, we have
\[ \lim_{y_1 \to 1} (1 - y_1)\psi_0(y) = \lim_{y_1 \to 1} \frac{aT^*(y)}{qy + \bar{q}} = \frac{2aF\left(\frac{1}{y}\right)}{q\eta_1^{-1} + q} = C_{l,1}, \]

where \( C_{l,1} \) is given by (5.1).

Therefore, by Theorem 4.1
\[ \pi_{0,j} \sim C_{l,1} \eta_1^j. \]

(ii) In this case, \( 1 < 1/\eta_1 = y_0 \). According to the Key lemma 4.2 \( T^*(\frac{1}{y_1}) = T^*(\frac{1}{y_2}) = 0 \) and \( T^*(\frac{1}{y_2}) = 0 \). Then
\[ \lim_{y_1 \to 1} \sqrt{1 - y_1} \psi_0(y) = \lim_{y_1 \to 1} \left[ a \frac{T^*(y)}{(qy + \bar{q})\sqrt{1 - y_1}} + b \frac{T^*(y)\sqrt{1 - y_1}}{(qy + \bar{q})(1 - \eta_2 y_0)} \right] \]
\[ = \lim_{y_1 \to 1} \frac{aT^*(y)}{(qy + \bar{q})\sqrt{1 - y_1}} = \lim_{y_1 \to 1} \frac{a}{qy + \bar{q}} \cdot \frac{F(y) + y\sqrt{\Delta(y)}}{\sqrt{1 - y_1}} \]
\[ = \lim_{y_1 \to 1} \frac{a}{qy + \bar{q}} \left[ \sqrt{F(y)} \sqrt{1 - y_1} \sqrt{F(y)} + y\sqrt{\frac{y_0\Delta(y)}{y_0 - y}} \right]. \]
Rewrite $\Delta(y)$ as follows

$$\Delta(y) = [(p\mu_h - \bar{p}\bar{\mu}_h)q]^2(y - y_0)(y - y_1).$$

Since $F(\frac{1}{\eta}) = 0$, $F(y)$ is a polynomial of degree 1. Therefore,

$$\lim_{y \to 1} \sqrt{\frac{F(y)}{1 - y\eta}} \sqrt{F(y)} = 0,$$

which implies

$$\lim_{y \to 1} \sqrt{1 - y\eta} \psi_0(y) = \frac{a(\bar{p} - \mu_h)qy_0}{qy_0 + \bar{q}} \sqrt{y_0(y_1 - y_0)}.$$

Applying Theorem 4.1, we have

$$\pi_{0,j} \sim C_{t,2} j^{-\frac{1}{2}} y_0^{-j},$$

where $C_{t,2}$ is given by (5.2).

(iii) In this case, $1 < 1/\eta_1 < y_0$ and $T^*(\frac{1}{\eta_1}) = 0$. We have

$$\frac{d\psi_0(y)}{dy} = T^*(y) \frac{d}{dy} \left[ \frac{a}{(qy + \bar{q})(1 - y\eta_1)} + \frac{b}{(qy + \bar{q})(1 - y\eta_2)} \right] + \left[ \frac{a}{(qy + \bar{q})(1 - y\eta_1)} + \frac{b}{(qy + \bar{q})(1 - y\eta_2)} \right] \frac{dT^*(y)}{dy},$$

where

$$\frac{dT^*(y)}{dy} = \frac{dF(y)}{dy} + \sqrt{\Delta(y)} + \frac{y}{2\sqrt{\Delta(y)}} \frac{d\Delta(y)}{dy}.$$

Therefore,

$$\lim_{y \to y_0} \sqrt{1 - \frac{y}{y_0}} \frac{d\psi_0(y)}{dy} = \lim_{y \to y_0} \left[ \frac{a}{(qy + \bar{q})(1 - y\eta_1)} + \frac{b}{(qy + \bar{q})(1 - y\eta_2)} \right] \sqrt{1 - \frac{y}{y_0}} \frac{dT^*(y)}{dy}$$

$$= \lim_{y \to y_0} \left[ \frac{a}{(qy + \bar{q})(1 - y\eta_1)} + \frac{b}{(qy + \bar{q})(1 - y\eta_2)} \right] \frac{y}{2\sqrt{\Delta(y)}} \frac{d\Delta(y)}{dy}$$

$$= \left[ \frac{a}{(qy + \bar{q})(1 - y\eta_1)} + \frac{b}{(qy + \bar{q})(1 - y\eta_2)} \right] \frac{(\mu_h - \bar{p})q}{2\sqrt{y_0(y_1 - y_0)}}.$$

Finally,

$$\pi_{0,j} \sim C_{t,3} j^{-\frac{1}{2}} y_0^{-j},$$

where $C_{t,3}$ is given by (5.3).

5.2 Exact tail asymptotics of joint probabilities

In the previous subsection, exact tail asymptotic properties in the boundary probabilities are obtained. In the current subsection, we provide details for the exact tail asymptotic characterizations in the joint probabilities $\pi_{i,j}$ for any fixed $i \geq 1$. 

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From the balance equations (2.1) to (2.4), we obtain

\[ b_2(y)\psi_0(y) + yc(y)\psi_1(y) = \tilde{p}\mu_i(1 - y)\pi_0, \]  

(5.7)

and

\[ a(y)\psi_{i-1}(y) + b(y)\psi_i(x) + c(y)\psi_{i+1}(y) = 0, \quad i \geq 1. \]  

(5.8)

Now, we are ready to give the main result of this subsection, which shows exact tail asymptotics for the joint probabilities.

**Theorem 5.2** For the discrete-time preemptive priority queue with two classes of customers satisfying \( \rho < 1 \), characterizations of the exact tail asymptotics in the joint stationary distribution along the low-priority queue direction are given below for \( i > 0 \) of high-priority customers:

(i) (Exact geometric decay) In the region defined by \( F(y_0) > 0 \),

\[ \pi_{i,j} \sim C_{i,1}\eta^j_1, \]  

(5.9)

where

\[ C_{i,1} = \frac{1 - (p\mu_h + \tilde{p}\mu_i)(q\eta_1^{-1} + \bar{q}) - \tilde{p}\mu_i(q + \bar{q}\eta_1)}{\tilde{p}\mu_i(q\eta_1^{-1} + \bar{q})} \left[ \frac{p\tilde{h}}{\tilde{p}\mu_i - \tilde{p}\mu_i(1 - \eta_1)} \right]^{i-1} C_{1,1}. \]

(ii) (Geometric with a prefactor \( n^{-\frac{\delta}{2}} \)) In the region defined by \( F(y_0) = 0 \),

\[ \pi_{i,j} \sim C_{i,2}n^{-\frac{\delta}{2}}\eta^j_0, \]  

(5.10)

where

\[ C_{i,2} = \frac{1 - (p\mu_h + \tilde{p}\mu_i)(q\eta_0 + \bar{q}) - \tilde{p}\mu_i(q + \bar{q}\eta_0^{-1})}{\tilde{p}\mu_i(q\eta_0 + \bar{q})} \left[ \frac{p\tilde{h}}{\tilde{p}\mu_i - \tilde{p}\mu_i(1 - \eta_0)} \right]^{i-1} C_{1,2}. \]

(iii) (Geometric with a prefactor \( n^{-\frac{\delta}{2}} \)) In the region defined by \( F(y_0) < 0 \),

\[ \pi_{i,j} \sim C_{i,3}n^{-\frac{\delta}{2}}\eta^j_0, \]  

(5.11)

where

\[ C_{i,3} = \left[ \frac{b_2(y_0)}{yc(y_0)} \right] + \left[ \frac{h_2(x_0(y_0), y_0)}{yc(y_0)} \right] \left[ \frac{2p\tilde{h}(q\eta_0 + \bar{q})}{1 - (p\mu_h + \tilde{p}\mu_i)(q\eta_0 + \bar{q})} \right]^{i-1} C_{1,3}. \]

**Proof:** (i) Noting that \( \lim_{y\eta_1 \rightarrow 1}(1 - y\eta_1)\psi_0(y) = C_{1,1} \), by the equations (5.7) and (5.8), we assume that for \( i \geq 1 \),

\[ \lim_{y\eta_1 \rightarrow 1}(1 - y\eta_1)\psi_i(y) = C_{1,1}. \]

Then

\[ b_2(\eta_1^{-1})C_{i,1} + \eta_1^{-1}c(\eta_1^{-1})C_{1,1} = 0, \]

and

\[ a(\eta_1^{-1})C_{i-1,1} + b(\eta_1^{-1})C_{i,1} + c(\eta_1^{-1})C_{i+1,1} = 0, \quad i \geq 1. \]

Since \( C_{i,1} \), \( i \geq 1 \) satisfies the above relations, it should take the form of

\[ C_{i,1} = A_1 \left( \frac{1}{x_1(\eta_1)} \right)^{i-1} + B_1 \left( \frac{1}{x_0(\eta_1)} \right)^{i-1}, \quad i \geq 1. \]
To determine the coefficients $A_1$ and $B_1$, we solve the following initial equations:

\[
\begin{align*}
    b_2(\eta_1^{-1})C_{i1} + \eta_1^{-1}c(\eta_1^{-1})(A_1 + B_1) &= 0, \\
ap(\eta_1^{-1})C_{i1} + b(\eta_1^{-1})(A_1 + B_1) + c(\eta_1^{-1}) \left[ A_1 \left( \frac{1}{x_1(\eta_1)\eta_1} \right) + B_1 \left( \frac{1}{x_0(\eta_1)\eta_1} \right) \right] &= 0.
\end{align*}
\] (5.12)

Let $y = \eta_1^{-1}$, then

\[H_2(x_0(\eta_1^{-1}), \eta_1^{-1}) = 0.\] (5.13)

By (5.10) and (5.13), one can easily get that

\[h_2(x_0(\eta_1^{-1}), \eta_1^{-1}) = \eta_1^{-1}a(\eta_1^{-1})x_0(\eta_1^{-1}) + b(\eta_1^{-1}) = 0.\] (5.14)

since \(h(x_0(y), y) = y[a(y)x_0(y)^2 + b(y)x_0(y) + c(y)] = 0\). So, by (5.12), (5.14) and the equation that \(a(y)x_0(y)x_1(y) = c(y)\), we get that

\[
\begin{align*}
    A_1 &= -\frac{\eta_1^{-1}b(x_0(\eta_1^{-1}))}{c(\eta_1^{-1})}C_{i1}, \\
    B_1 &= 0.
\end{align*}
\]

On the hand, we have

\[
\sqrt{T(\frac{1}{\eta_1})} = \eta_1 F(\frac{1}{\eta_1}),
\] (5.15)

since \(T(\frac{1}{\eta_1}) = 0\). Then, by above argument, we can get \(x_1(\eta_1^{-1})\), and finally (5.9) by Theorem 4.1.

(ii) The proof for this case is similar to the case (i). So, we omitted here.

(iii) In this case, \(\Delta(y_0) = 0\). So \(x_0(y_0) = x_1(y_0)\). We assume that

\[
\lim_{y \to y_0} \sqrt{1 - \frac{y}{y_0} \frac{d\psi_0}{dy}} = C_{i,3}.
\]

By using the same method as in the case (i), we get

\[b_2(y_0)C_{i,3} + y_0c(y_0)C_{i,3} = 0,
\]

and

\[a(y_0)C_{i-1,3} + b(y_0)C_{i,3} + c(y_0)C_{i+1,3} = 0, \quad i \geq 1.
\]

The solution is

\[C_{i+1,3} = (A_3 + B_3i) \left( \frac{1}{x_1(y_0)} \right)^i, \quad i \geq 1.
\]

After the similar argument as in the case (i), we can determine \(A_3\), \(B_3\) and \(x_1(y_0)\).

\[\square\]

### 5.3 Exact tail asymptotics for the marginal distribution

In this subsection, we provide details for the exact tail asymptotics of the marginal distribution \(\pi_j^{(l)} = \sum_i \pi_{i,j}\), which can be characterized by computing \(P(1, y)\).

Based on the fundamental form (2.10), we have

\[P(1, y) = \frac{\bar{p}_1(1)}{qy} \left[ (qy + \bar{q})P_2(y) - \bar{q}p_{00} \right],\]

which leads to

\[\pi_j^{(l)} = \bar{p}_1 \pi_{0,j}.\]

Hence, up to constant factors, the exact tail asymptotics for the marginal distribution \(\pi_j^{(l)}\) are same as those of the boundary probability \(\pi_{0,j}\).
6 Exact tail asymptotics for the high-priority queue

In this section, we characterize the exact tail asymptotics in the joint stationary distribution as well as the marginal distribution along the high-priority queue direction. The former, \( \pi_{i,j} \) for fixed \( j \geq 0 \), can be derived from the generating function of \( \varphi_j(x) \) given by (3.12), and the latter \( \sum_i \pi_{i,j} \) can be obtained from the fundamental form (2.10).

Lemma 6.1 For \( j \geq 0 \),

\[
\varphi_j(x) \sim C^j \pi_{0,0}(1-r_0 x)^{-j-1}, \quad \text{as} \quad r_0 x \to 1,
\]

where

\[
C = \frac{q \left[ p \mu \bar{r}_0 + (p \mu + \bar{p} \bar{u}) r_0 + p \bar{\mu} h \right]}{p \bar{\mu} h - p \bar{\mu} r_0^2}.
\]

Proof: If \( j = 0 \), then by (3.12), one can easily get that the lemma holds.

Next, we will prove the lemma for \( j \geq 1 \) by using the induction. By (3.12), we have

\[
\varphi_j(x) = \frac{a_j}{x - x_1} - \frac{q \varphi_{j-1}(x_0)(x + x_0)}{q(x - x_1)} \frac{q(p x + \bar{p})(\bar{\mu} x_1 + \mu h)}{p \bar{\mu} h (x - x_1)} \varphi_{j-1}(x) - \varphi_{j-1}(x_0),
\]

where \( a_j \) and \( \varphi_{j-1}(x_0) \) are constants depending on \( j \). It follows from Lemma 3.1 that for \( j = 1 \),

\[
\lim_{x \to x_1} \left( 1 - \frac{x}{x_1} \right)^{j+2} \varphi_1(x) = \frac{q(p x_1 + \bar{p})(\bar{\mu} x_1 + \mu h)}{p \bar{\mu} h (x_1 - x_0) x_1} \pi_{0,0} = C \pi_{0,0}.
\]

Substituting \( x_0 = \frac{a}{w} \), \( x_1 = \frac{1}{r_0} \) and \( w = \frac{\bar{p} \bar{u}}{\mu h} \) into the equation (6.3) and after some elementary manipulations, we get the expression of \( C \).

Now, we assume that (6.1) is true for \( j = k \), then by using the induction

\[
\lim_{x \to x_1} \left( 1 - \frac{x}{x_1} \right)^{j+2} \varphi_{j+1}(x) = -\frac{q(p x_1 + \bar{p})(\bar{\mu} x_1 + \mu h)}{p \bar{\mu} h (x_1 - x_0) x_1} \lim_{x \to x_1} \frac{\varphi_j(x)}{x - x_1} (x_1 - x)^{j+2} \frac{1}{x_1^{j+1}}
\]

\[
= C \lim_{x \to x_1} \varphi_j(x) \left( 1 - \frac{x}{x_1} \right)^{j+1}
\]

\[
= C^{j+1} \pi_{0,0}.
\]

This completes the lemma.

Theorem 6.1 For the discrete-time preemptive priority queue with two classes of customers satisfying \( \rho < 1 \), the exact tail asymptotics in the joint stationary distribution along the high-priority queue direction is characterized by: for a fixed number \( j \geq 0 \) of low-priority customers,

\[
\pi_{i,j} \sim \frac{C^j \pi_{0,0}}{j!} r_0^i,
\]

with \( r_0 \), \( \pi_{0,0} \) and \( C \) are given, respectively, by (3.10), (3.13) and (6.3).

Proof: It is obvious that for \( j \geq 0 \), \( \varphi_j(x) \) is analytic in the region \( \Delta(\phi, \epsilon) = \{ x : |r_0 x| < 1 + \epsilon, r_0 x \neq 1, |Arg(r_0 x - 1)| > \phi,\epsilon > 0, 0 < \phi < \pi/2, \} \). By Theorem 4.1 and Lemma 6.1 we have for fixed \( j \geq 0 \),

\[
\pi_{i,j} \sim \frac{C^j \pi_{0,0}}{\Gamma(j+1)} \left( \frac{1}{r_0} \right)^{-i} = \frac{C^j \pi_{0,0}}{j!} r_0^i.
\]

Finally, we present tail asymptotic property of the marginal distribution \( \pi^{(h)}_i = \sum_j \pi_{i,j} \). Taking \( y = 1 \) in the fundamental form (2.10), we have

\[
P(x,1) = \frac{\bar{p} \mu h \psi_0(1)}{\bar{p} \mu h - \bar{p} \mu h x}.
\]
On the other hand, by (3.21),

$$\psi_0(1) = \frac{1 - \rho}{\bar{p}}. \quad (6.5)$$

By (6.4) and (6.5),

$$P(x, 1) = \frac{\mu_h (1 - \rho)}{\bar{p} \mu_h - \bar{p} \mu_h x}. \quad (6.6)$$

Finally, from the assumption of this model, we have

$$\frac{\bar{p} \mu_h}{\bar{p} \mu_h} > 1. \quad (6.7)$$

Therefore, $\frac{\bar{p} \mu_h}{\bar{p} \mu_h}$ is the dominant singularity and

$$\pi_i^{(h)} \sim \left(1 - \rho \left(\frac{\bar{p} \mu_h}{\bar{p} \mu_h}\right)^i \right).$$

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