Abstract

In a certain class of differential-difference equations for dissipative systems, we show that hyperbolic tangent model is the only the nonlinear system of equations which can admit some particular solutions of the Toda lattice. We give one parameter family of exact solutions, which include as special cases the Toda lattice solutions as well as the Whitham’s solutions in the Newell’s model. Our solutions can be used to describe temporal-spatial density patterns observed in the optimal velocity model for traffic flow.
1 Introduction

This paper concerns analytic description of temporal-spatial density patterns generated in a class of discrete nonlinear systems of dynamical equations:

\[ G[\dot{x}_n(t), \ddot{x}_n(t), \dot{x}_n(t+\tau), \ddot{x}_n(t+\tau)] = V[\Delta x_n(t)], \quad (n = 1, 2, \ldots, N), \quad (1.1) \]

where the r.h.s. depends only on the forward difference between the discrete indices \( \Delta x_n(t) = x_{n-1}(t) - x_n(t) \). It would be interesting if we could find exact solutions for such dissipative systems.

It is the purpose of this paper to show that there is actually a nonlinear system of differential-difference equations for which we may find exact solutions. To show that, we shall construct nonlinear equations from given candidates for exact solutions instead of trying to solve a given system of \( G[\cdots] \) and \( V(\Delta x) \) exactly. This is analogous with Toda’s way of finding his lattice model\[1\][2]. A clue which led him to the discovery\[1\] is the observation that a specific combination of logarithm of theta functions obeys a simple system of equations of motion for an exponentially interacting lattice of particles. The particular solutions, referred to as Toda solutions hereafter, correspond to a periodic wave\[1\] as well as a solitary wave\[2\]. These are used here to describe temporal-spatial density patterns, and to construct nonlinear systems of equations.

Requiring \( V(\Delta x) \) to be single-valued, we show that only the first-order differential-difference equations\[1\][3],

\[ \dot{x}_n(t + \tau) = V[\Delta x_n(t)], \quad (1.2) \]

with a hyperbolic tangent (tanh) OV function \( V(\Delta x) \) can admit the Toda solutions\[2\](See (2.13)).

The first-order equations (1.2) and its truncated form of the second-order differential equations with “friction terms”\[3\][4]

\[ \tau \ddot{x}_n(t) + \dot{x}_n(t) = V[\Delta x_n(t)]. \quad (1.3) \]

have been extensively discussed in the optimal velocity (OV) model\[4\][5][6][7] to describe the formation of density pattern in a congested flow of traffic; they may be also relevant to the density patterns in granular materials\[8\]. To be concrete, we employ the OV model terminology: the basic coordinate \( x_n \) and the forward difference \( \Delta x_n \) correspond to the position of \( n \)th car and its headway, respectively. The latter is the distance between the car and the preceding \((n - 1)\)th car. The idea postulating eqs. (1.2) is that a driver adjusts the car’s velocity \( \dot{x}_n(t) \) according to the observed headway \( \Delta x_n \). The delay time \( \tau \) is the time lag for the driver and car to reach the optimal velocity \( V(\Delta x) \) when the traffic flow is changed. In the OV model, spontaneously appears a density pattern with regions of

\[ \text{See ref.}[3] \text{ for review articles.} \]

\[ \text{Although the functional form of the Toda solutions here is the same as the original one in the Toda lattice, the dispersion relations of the two systems are obviously different. We use the term of “Toda solutions” up to the dispersion relations.} \]

\[ \text{See, for example, ref.}[8]. \]
high density where cars move slowly, and low-density regions where the velocity of cars is high. When these two regions occur alternatively in a lane, a spatial-temporal pattern is generated, which may be viewed as a density wave. What we show here is that the pattern can be described by the Toda solutions of the nonlinear equations (2.15).

We also show that the our tanh model can admit one parameter family of exact solutions which contains a width parameter $\delta$. The Toda solution for a “cnoidal wave” consists of sharp pulses with a fixed width $\delta_T$. In contrast, our solutions describe pulses with arbitrary width. Unless $\delta = \delta_T$, the solutions do not obey the Toda equation\(^4\). The solution for $\delta = l\delta_T$ with an integer $l$ may be interpreted to describe a “bound state” of solitons. It is certainly intriguing that such solutions exist for dissipative systems. We have confirmed stability of the solutions via a computer simulation.

The nonlinear system\(^4\) of eqs.(1.2) was first discussed by Newell\(^4\). He gave exact solutions for $\tau = 0$ in the model with $V_N(\Delta x) = b_1[1 - \exp\{-b_2(\Delta x - b_3)\}]$. Some years ago, Whitham\(^5\) obtained Toda-like exact solutions of the Newell’s model for $\tau \neq 0$. He found that a relation between parameters which specify a wave propagation is crucial for the existence of the exact solutions. In our construction, this relation, which we call the Whitham relation, is necessary for the OV function to be single-valued. We will see that the Whitham’s solutions belong to our family of solutions: they have a width $\delta = \delta_W = 1/2 - \delta_T/2$, and our tanh OV function reduces to the Newell’s function in this case.

This paper is organized as follows. The next section describes construction of the model equations by using the Toda solution for a “cnoidal wave”. In section 3, we present an ansatz for one parameter family of exact solutions as an extension of the Toda solutions, and discuss the stability of the exact solutions. We obtain a solitary wave solution in the large modulus limit in section 4. Summary and a few remarks on the exact solutions are given in the final section. Appendix lists some formulae used in this paper.

2 Construction of the tanh model

2.1 Toda solution for periodic wave

In a simulation of the OV model, we may generate a stable pattern of a congested flow. There, the pulses with almost the same maximal values are observed in the velocity as well as the headway of a car. Here we consider the density pattern analytically by using the Toda solution corresponding to a “cnoidal wave”\(^4\). For the position of the $n$th car, it is given by

$$x_n(t) = A \ln \frac{\vartheta_0(\nu t - \frac{n}{\lambda})}{\vartheta_0(\nu t - \frac{n+1}{\lambda})} + Ct - nh,$$

\(^4\)Some analytic solutions were found for the system \(^1\) with piecewise linear OV functions\(^9\)\(^10\), or in an asymptotic method\(^11\) to investigate the long time behavior in the vicinity of a critical point.
where \( A, \lambda, \nu, C, h \) are constants. \( \vartheta_0(v) \) is the theta function (See (A.4) in Appendix).

The headway is then given by

\[
\Delta x_n(t) = A \left[ \ln \vartheta_0 \left( \frac{\nu t - \frac{n}{\lambda}}{\lambda} \right) - \ln \vartheta_0 \left( \frac{\nu t - \frac{n-1}{\lambda}}{\lambda} \right) \right] + h. \tag{2.2}
\]

To write equations given below in compact forms, we introduce the notations:

\[
v = \nu t - \frac{n}{\lambda}, \quad v_0 = \frac{1}{\lambda}, \quad s_0 = \text{sn}(2Kv_0), \quad c_0 = \text{cn}(2Kv_0), \quad d_0 = \text{dn}(2Kv_0), \tag{2.3}
\]

where \( \text{sn} \), \( \text{cn} \) and \( \text{dn} \) are the Jacobian elliptic functions, \( K = K(k) \) is the complete elliptic integrals of the first kind. All the Jacobian elliptic functions used in this paper are assumed to have a modulus \( k \).

Using formula (A.1) in Appendix which relates the theta functions with the elliptic functions, one obtains

\[
k^2 s_0^2 \text{sn}^2(2Kv) = 1 - e^X, \tag{2.4}
\]

where

\[
X = \frac{\Delta x_n - \beta}{A}, \quad \beta = h + 2A \ln \frac{\vartheta_0(v_0)}{\vartheta_0(0)}. \tag{2.5}
\]

Therefore, the elliptic functions, \( \text{sn}(2Kv) \), \( \text{cn}(2Kv) \) and \( \text{dn}(2Kv) \), are expressed in terms of \( e^X \). The velocity and acceleration of the \( n \)th car are given by

\[
\dot{x}_n(t) = 2AK\nu [Z(2Kv) - Z(2K(v-v_0))] + C,
\]

\[
\ddot{x}_n(t) = 4A(K\nu)^2 \left[ \text{dn}^2(2Kv) - \text{dn}^2(2K(v-v_0)) \right], \tag{2.6}
\]

where \( Z(u) \) is the Jacobian zeta function. The formulæ (A.2), (A.6) and (A.7) in Appendix lead to

\[
\dot{x}_n(t) = \gamma - (2AK\nu k^2 s_0)\text{sn}(2Kv) \left[ \frac{c_0 d_0 \cdot \text{sn}(2Kv) - s_0 \cdot \text{cn}(2Kv)\text{dn}(2Kv)}{1 - k^2 s_0^2 \cdot \text{sn}^2(2Kv)} \right],
\]

\[
\gamma = 2AK\nu Z(2Kv_0) + C, \tag{2.7}
\]

and

\[
\ddot{x}_n(t) = 4A(K\nu)^2 \left[ \text{dn}^2(2Kv) - \left\{ \frac{d_0 \cdot \text{dn}(2Kv) + k^2 s_0 c_0 \cdot \text{sn}(2Kv)\text{cn}(2Kv)}{1 - k^2 s_0^2 \cdot \text{sn}^2(2Kv)} \right\}^2 \right]. \tag{2.8}
\]

The velocity and acceleration at a time \( t + \tau \) are calculated to be

\[
\dot{x}_n(t + \tau) = 2AK\nu [Z(2K(v + v_1)) - Z(2K(v + v_2))] + C
\]

\[
\ddot{x}_n(t + \tau) = 4A(K\nu)^2 \left[ \text{dn}^2(2K(v + v_1)) - \text{dn}^2(2K(v + v_2)) \right]. \tag{2.9}
\]
where \( v_1 = \nu \tau, \ v_2 = v_1 - v_0 \). These are again written by the elliptic functions,

\[
\begin{align*}
\dot{x}_n(t + \tau) &= \gamma - (2AK\nu k^2 s_0) \left[ \frac{c_1 d_1 \cdot \text{sn}(2Kv) + s_1 \cdot \text{cn}(2Kv)\text{dn}(2Kv)}{1 - k^2 s_1^2 \cdot \text{sn}^2(2Kv)} \right] \\
&\times \left[ \frac{c_2 d_2 \cdot \text{sn}(2Kv) + s_2 \cdot \text{cn}(2Kv)\text{dn}(2Kv)}{1 - k^2 s_2^2 \cdot \text{sn}^2(2Kv)} \right],
\end{align*}
\]

(2.10)

and

\[
\begin{align*}
\ddot{x}_n(t + \tau) &= 4A(K\nu)^2 \left[ \frac{\left\{ d_1 \cdot \text{dn}(2Kv) - k^2 s_1 c_1 \cdot \text{sn}(2Kv)\text{cn}(2Kv) \right\}^2}{1 - k^2 s_1^2 \cdot \text{sn}^2(2Kv)} \right] \\
&\quad - \left\{ \frac{d_2 \cdot \text{dn}(2Kv) - k^2 s_2 c_2 \cdot \text{sn}(2Kv)\text{cn}(2Kv)}{1 - k^2 s_2^2 \cdot \text{sn}^2(2Kv)} \right\}^2,
\end{align*}
\]

(2.11)

where

\[
s_i = \text{sn}(2Kv_i), \quad c_i = \text{cn}(2Kv_i), \quad d_i = \text{dn}(2Kv_i) \quad (i = 1, 2).
\]

(2.12)

Eqs. (2.7), (2.8), (2.10), (2.11) are used to construct the dynamical equations.

### 2.2 Uniqueness of the model

Here under appropriate assumptions we show that only the \( \dot{x}_n(t + \tau) \) is the fundamental element to be incorporated into the dynamical equations.

Let us rename the velocity and acceleration variables as,

\((y_1, y_2, y_3, y_4) = (\dot{x}_n(t), \ddot{x}_n(t), \dot{x}_n(t + \tau), \ddot{x}_n(t + \tau))\). In general, \( y \)'s are all double-valued functions of \( \Delta x_n \) because of the presence of terms proportional to \( \text{sn}(2Kv) \cdot \text{cn}(2Kv) \). One may impose some cancellation conditions which make their coefficients to vanish. We first examine if it is possible to construct single-valued dynamical equations without imposing such conditions. One might find a polynomial \( F = \sum_i f_i (y_i)^{n_i} \) in four variables, where \( f \)'s are chosen in such a way that \( F \) becomes a single-valued function of \( \Delta x_n \), \( F = V(\Delta x_n) \). It gives a system of single-valued equations, but takes of the form: \( F(y_1, y_2, y_3, y_4; \Delta x_n) = V(\Delta x_n) \), where the l.h.s. necessarily depends on \( \Delta x_n \) via the non-vanishing coefficients, \( f \)'s. The class of these models which have the Toda solution may not be out of our interest, but does not belong to the system (1.1).

From the above results we are tempted to turn to the possibility of having single-valued \( y \)' by imposing cancellation conditions on the parameters, \( (s_m, c_m, d_m) \) \( (m = 0, 1, 2) \). They are functions of \( (\nu \tau, \lambda) \) and \( k \). As discussed in the next section, the periodic boundary condition \( x_{N+1} = x_1 \) requires that \( \lambda \) is order of the total number of the discrete elements (cars in the OV model), \( N \). Therefore, it is not unreasonable to assume, say \( \lambda > 2 \). This excludes \( c_0 = 0 \). It is also obvious that \( s_0 \neq 0 \), otherwise \( \Delta x_n(t) \equiv \) constant. In the same manner it can be shown that the cancellation conditions necessary for \( \ddot{x}_n(t + \tau) \) to be

\(^5\)This observation may apply to non-polynomial combinations of \( y \)' which are single-valued functions of \( \Delta x_n \).
single-valued are not acceptable. Thus, \( \dot{x}_n(t), \ddot{x}_n(t), \dddot{x}_n(t + \tau) \) cannot be single-valued, and are excluded from entering into the desired equations.

One is then only left with the cancellation condition to make \( \dot{x}_n(t + \tau) \) single-valued\
\[
s_1c_2d_2 + s_2c_1d_1 = 0. \tag{2.13}
\]
Solution for the condition, \( v_2 = -v_1 \), is essentially unique and acceptable. It leads to the Whitham relation\[3\]
\[
2\nu\lambda\tau = 1. \tag{2.14}
\]
One finds that \( \dot{x}_n(t + \tau) \) subject to the Whitham relation becomes only the variable which may be used to construct dynamical equations. For the first-order differential-difference equations, one obtains
\[
\dot{x}_n(t + \tau) = V(\Delta x_n)
\]
\[
= \xi + \eta \tanh \left( \frac{\Delta x_n - \rho}{2A} \right), \tag{2.15}
\]
where \( \xi, \eta \) and \( \rho \) are some constants determined by the parameters in the Toda solution (2.1). The \( \tanh \) OV function has been extensively used in computer simulations\[4][5\] for the second-order differential equations \( \ddot{x}_n = a[V(\Delta x_n) - \dot{x}_n] \). In the traffic flow application, we have shown here the system of first-order differential-difference equations with the \( \tanh \) OV function is only the model which admits the Toda solution.

The following remarks are in order.
(1) The nonlinear systems of equations as
\[
\ddot{x}_n(t) = a(V[\Delta x_n(t)] - \dot{x}_n(t)), \tag{2.16}
\]
\[
\dot{x}_n(t + \tau) = a(V[\Delta x_n(t)] - \dot{x}_n(t)), \tag{2.17}
\]
cannot admit the Toda solutions for single-valued OV functions. In other words, the OV functions become necessarily double-valued in order for the systems (2.16) to have the Toda solutions. Each OV function may take two different values for a fixed headway, depending whether the car is accelerating or decelerating. Although it would not be unrealistic to have the double-valued optimal velocities, the simplicity of the OV model is lost. As pointed out in the above discussion, a way out of the appearance of the double-valued functions is to consider the equations like \( F(y_i; \Delta x_n) = V[\Delta x_n] \). For example, if the parameter \( a \) in (2.16) is allowed to have an appropriate \( \Delta x_n \) dependence, the OV function \( V \) in the modified system of equations may become single-valued. Another possibility is to introduce the backward difference \( \Delta x_{n+1} \) as well as the forward difference \( \Delta x_n \). The OV function becomes then a single-valued function of two variables. We shall not consider here such a class of the models\[12\], though it will be intriguing.

(2) The Whitham relation (2.14) is found to be crucial for the existence of the exact
\footnote{For single-valued \( \dot{x}_n(t + \tau), \dot{x}_n(t) \) may be a double-valued function of \( \Delta x_n \) upon the formation of a congested flow.}
solutions. The relation is interpreted as follows. In a density wave propagation, the
headway may be written in the form
\[ \Delta x_n(t) = f\left(\nu t - \frac{n}{\lambda}\right) = f(v), \]
\[ \Delta x_{n-1}(t) = f\left(\nu(t + T) - \frac{n}{\lambda}\right) = f(v + \nu T), \quad T = \frac{1}{\nu \lambda}, \tag{2.18} \]
which implies that the \( n \)th headway has the same time dependence as that of the \((n - 1)\)th one apart from a time lag \( T \). The Whitham relation (2.14) then takes the form
\[ T = 2 \tau. \tag{2.19} \]
Imposing this relation, one obtains a differential-difference equation for a single universal function \( f(v) \)
\[ \nu \frac{d}{dv} f(v) = V [f(v + \nu(T - \tau))] - V [f(v - \nu \tau)] \]
\[ = V [f(v + \nu \tau)] - V [f(v - \nu \tau)]. \tag{2.20} \]
Note that the argument of the left in (2.20) is midway between those on the right. It suggests\(^5\) that a lattice model interpretation may be possible for the present system once a temporal-spatial pattern is generated.

3 Ansatz for one parameter family of solutions

The “cnoidal wave” described by the Toda solution (2.1) consists of periodic pulses. There is a parameter, \( \lambda \), which adjusts the period of the pulses. In the Toda solution, however, the width of each pulse is fixed. In this section, we present one parameter family of solutions which consist of pulses with arbitrary width.

In the context of the OV model, the motivation for constructing such solutions is explained as follows: In a computer simulation, the density pattern is generated under suitable conditions. It is described by the alternative appearance of the low-density (high-velocity) regions and the high-density (low-velocity) regions in a traffic flow. A congested region contains a bunch of cars. For the Toda solution, take a circuit of length \( L \), and impose the periodic boundary condition, \( x_{N+1} = x_1 \). Then, the parameter \( \lambda \) is fixed to be \( \lambda = N/m \), where \( m \) is the number of the congested regions in the circuit. The solution (2.1) with \( A > 0 \) (\( A < 0 \)) describes a car which spends most of its time in congested (free) regions. Thus, the distance between a pair of a kink and an anti-kink in \( \dot{x}_n \) in the Toda solution is much shorter than that observed in the simulation; the pair looks like a sharp pulse. It is desirable therefore to explore analytic solutions which consist of “trapezoidal pulses” with larger width.

Our ansatz for one parameter family of solutions is given by
\[ x_n(t) = A \ \ln \frac{\varphi_0 \left(\nu t - \frac{n}{\lambda} - \frac{1}{\lambda} + \delta\right)}{\varphi_0 \left(\nu t - \frac{n}{\lambda} - \frac{1}{\lambda} - \delta\right)} + Ct - nh, \tag{3.1} \]
where $\delta$ is related to the size of a congested or free region. Without loss of generality, the range of $\delta$ is restricted as $0 \leq \delta < 1/2$. One may observe that $\delta = \delta_T = 1/(2\lambda)$ corresponds to the Toda solution. Let us emphasize that the ansatz otherwise does not solve the Toda equation\[1\].

Let us see that this ansatz does solve the tanh model (2.15). The ansatz (3.1) slightly modifies the expression for the headway,

$$k^2\text{sn}^2(2Kv) = \frac{e^{\hat{X}} - 1}{e^{\hat{X}} \cdot \text{sn}^22K(\delta - \frac{1}{2}) - \text{sn}^22K(\delta + \frac{1}{2})},$$

where

$$\hat{X} = \frac{\Delta x_n - \hat{\beta}}{A}, \quad \hat{\beta} = h + 2A \ln \frac{\vartheta_0(\delta + \frac{1}{2\lambda})}{\vartheta_0(\delta - \frac{1}{2\lambda})}. (3.3)$$

Let us introduce the following notations:

$$P = \text{sn}^2(2K\delta) - \text{sn}^2(2K(\delta - \frac{1}{2\lambda})),
Q = -\text{sn}^2(2K\delta) + \text{sn}^2(2K(\delta + \frac{1}{2\lambda})),
\text{s}_\delta = \text{sn}(2K\delta), \quad \text{c}_\delta = \text{cn}(2K\delta), \quad \text{d}_\delta = \text{dn}(2K\delta). (3.4)$$

For $\tau$ which satisfies the relation (2.14),

$$\dot{x}_n(t + \tau) = 4AK\nu Z(2K\delta) - s_\delta c_\delta d_\delta \frac{k^2\text{sn}^2(2Kv)}{1 - k^2s_\delta^2 \cdot \text{sn}^2(2Kv)} + C. (3.5)$$

Taking the same step as before, we obtain the following OV function for the present case,

$$V(\Delta x) = 4AK\nu Z(2K\delta) + C + (4AK\nu \cdot s_\delta c_\delta d_\delta) \frac{e^{\hat{X}} - 1}{P \cdot e^{\hat{X}} + Q}. (3.6)$$

It may be rewritten in a more familiar form:

$$V(\Delta x) = C + 2AK\nu \left[2Z(2K\delta) + s_\delta c_\delta d_\delta \left(\frac{1}{P} - \frac{1}{Q}\right)\right] + 2AK\nu s_\delta c_\delta d_\delta \left(\frac{1}{P} + \frac{1}{Q}\right) \tanh\left(\Delta x - \hat{\beta} \frac{1}{2A} - \frac{1}{2} \ln Q \frac{P}{P}\right). (3.7)$$

This completes the proof that the ansatz (3.1) solves (2.15).

Note that the OV function is described by a tanh function only when $P/Q > 0$ or in the range, $1/(4\lambda) < \delta < 1/2 - 1/(4\lambda)$, outside of which it becomes a coth function;\footnote{One may take $\delta$ to be $-1/2 \leq \delta < 1/2$ because of the periodicity of the $\vartheta_0$-function. Furthermore, the difference between $x_n$ for $\delta = -\delta_0$ and that for $\delta = 1/2 - \delta_0$ can be absorbed into a constant shift of $t$.}
on the boundaries it is exponential functions. In particular, it should be noted that if one takes \( \delta = \delta_W = 1/2 - 1/(4\lambda) \) so that \( Q = 0 \), the solution \((3.1)\) reproduces the Whitham’s solution \([5]\), and the OV function in \((3.7)\) reduces to the one in the Newell model, \( V(\Delta x) = b_1[1 - \exp\{-b_2(\Delta x - b_3)\}] \).

Comparing the OV function in \((2.15)\) with that of \((3.7)\), we have

\[
\begin{align*}
\xi &= C + 2AK\nu \left[ 2Z(2K\delta) + s_\delta c_\delta d_\delta \cdot \left( \frac{1}{P} - \frac{1}{Q} \right) \right], \\
\eta &= 2AK\nu s_\delta c_\delta d_\delta \cdot \left( \frac{1}{P} + \frac{1}{Q} \right), \\
\rho &= h + 2A \ln \frac{\varphi_0(\delta + 1/(2\lambda))}{\varphi_0(\delta - 1/(2\lambda))} + A \ln \frac{Q}{P}.
\end{align*}
\tag{3.8}
\]

It is crucially important to see the stability of our “trapezoidal pulse” solutions. We have confirmed it via a numerical study.

Suppose \( N \) cars run on the lane with length \( L \) according to the equations of motion given with the time delay \( \tau \) and the OV function. The linear stability analysis tells us that above some critical value for \( \tau \), we find a range of density, or the average headway \( h = L/N \), for which a homogeneous flow is linearly unstable. What we have learned from numerical studies are: (1) once a congested flow is fully developed, a car repeats the behaviour of its preceding one with the time delay \( T \) and the entire pattern moves backward with some fixed velocity \( v_B \); (2) the trajectory of a car on the headway-velocity plane crosses the OV function at two points well outside the unstable region and the line connecting the two points has the slope \( \sim 1/T \) and passes through the inflection point. Obviously, the \( T \) and \( v_B \) are the outputs of the simulation.

In our analytical study reported in this paper, the set of parameters \((N, L, \tau, \xi, \rho, \eta)\) as well as \((T, v_B)\) for the global pattern are written with the parameters in our “trapezoidal pulse” solutions. This implies that there are nontrivial relations among the quantities stated above. Among others, the relation, \( T = 2\tau \), cannot be foreseen from numerical study and should be understood as a prediction of our analytical study.

We have performed a simulation to compare with our analytical results. To start a simulation we need to choose appropriate values for \( \tau \), \( N \) and \( L \). Since they are written with parameters in our “trapezoidal pulse” solutions, one needs to search a set of parameters in the solutions which give natural values for the three quantities as those for the OV model. The following conditions are important: (1) the value for \( \tau \) allows the linearly unstable region; (2) \( h \) is in the unstable region\([7]\). The simulation with the chosen values for \( \tau \), \( N \) and \( L \) gave us a hysteresis loop on the headway-velocity plane. This is to be compared with the loop obtained from our analytical solutions\([7]\). As clearly shown

\[ ^8 \text{Since we started from the solutions, logically we do not have any strong reason for the presence of appropriate set of parameters which allows an interpretation in the context of the OV model.} \]

\[ ^9 \text{Here are parameters for the simulation: } \tau = 0.582, N = \lambda = 10, h = 1.89; \text{ the OV function is } V(\Delta x) = \tanh 2 + \tanh(\Delta x - 2). \text{ The result is to be compared with eq.\([8]\) for } 2K \delta = 3, k^2 = 0.99999 \text{ and } C = 0.866. \text{ The DODAM package is used for solving differential equations.} \]

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9 Here are parameters for the simulation: \( \tau = 0.582, N = \lambda = 10, h = 1.89; \) the OV function is \( V(\Delta x) = \tanh 2 + \tanh(\Delta x - 2). \) The result is to be compared with eq.\([8]\) for \( 2K \delta = 3, k^2 = 0.99999 \) and \( C = 0.866. \) The DODAM package is used for solving differential equations.
in Fig. 1, they are actually found to be the same! In particular, the prediction from the solution, \( T = 2\tau \), is confirmed.

![Graph showing velocities from a simulation and “trapezoidal pulse” solution.](image)

\[ \text{Fig.} 1: \]

Velocities from a simulation and “trapezoidal pulse” solution; dots are from the simulation and the line denotes the solution.

Therefore we conclude that our “trapezoidal pulse” solutions are the exact stable solutions for the OV model, which describe the fully developed congested flows.

## 4 A solitary wave solution

We have discussed so far the periodic solution, and constructed the model with the corresponding exact solution. In the Toda lattice, it has been known that a solitary wave solution \( [3] \) is obtained from the periodic wave solution \( [4] \) by taking the large modulus limit \( k \to 1 \). We use the same procedure here. In the solution (3.4), the exponential of \( \Delta x_n \) is related to \( \text{dn}^2(2K\nu) \). It consists of an infinite number of pulses, each of which is expressed by square of the hyperbolic secant (See the decomposition formula \( (A.5) \) in Appendix). A solitary wave limit is defined by \( k \to 1 \) \((K \to \infty), \nu \to 0, \lambda \to \infty \) and \( \delta \to \infty \) with \( 2K\nu = \nu_R, 2K/\lambda = 1/\lambda_R \) and \( 2K\delta = \delta_R \) held fixed. In this limit, only one of the pulses survives to describe a solitary wave, while all the others disappear. Actually, since

\[
\lim_{k \to 1} \frac{\partial_0 ((v_R + \delta_R - v_{0R}/2)/2K)}{\partial_0 ((v_R - \delta_R - v_{0R}/2)/2K)} = \frac{\cosh(v_R + \delta_R - \frac{v_{0R}}{2})}{\cosh(v_R - \delta_R - \frac{v_{0R}}{2})} \tag{4.1}
\]

where \( v_R = \nu_R t - n/\lambda_R \) and \( v_{0R} = 1/\lambda_R \), \( x_n \) is given by

\[
x_n(t) = A \ln \frac{\cosh(v_R + \delta_R - \frac{v_{0R}}{2})}{\cosh(v_R - \delta_R - \frac{v_{0R}}{2})} + Ct - nh. \tag{4.2}
\]
Therefore, \( \dot{x}_n(t) \) consists of a pair of a kink and an anti-kink; \( \ddot{x}_n(t) \) has two pulses with different sign:

\[
\dot{x}_n(t) = A \nu_R \left[ \tanh \left( v_R + \delta_R - \frac{v_0 R}{2} \right) - \tanh \left( v_R - \delta_R - \frac{v_0 R}{2} \right) \right] + C,
\]

\[
\ddot{x}_n(t) = A \nu_R^2 \left[ \text{sech}^2 \left( v_R + \delta_R - \frac{v_0 R}{2} \right) - \text{sech}^2 \left( v_R - \delta_R - \frac{v_0 R}{2} \right) \right]. \tag{4.3}
\]

This implies that the kink and the anti-kink is separated by the distance \( 2\delta_R \). One can directly confirm that (4.2) and (4.3) provide another particular solution of (2.15).

5 Summary and outlook

We have given a method to construct the dynamical equations for many-body dissipative systems, which admit the Toda lattice solutions describing periodic as well as solitary density waves. The uniqueness of the tanh model of the first-order differential-difference equations is shown. It should be emphasized again that the Whitham relation plays an important role. It is remarkable that the model admits one parameter family of exact solutions which are not present in the exact solutions of the Toda lattice.

For the width parameter \( \delta = \delta_W = 1/2 - 1/(4\lambda) \), our solutions become the Whitham’s solutions, and the tanh OV function reduces to the Newell’s function. As described above, we do know there are ranges for parameters to make our solutions stable. However it is not obvious to us for the moment the parameters for the Whitham’s solutions fall into these ranges. So further study via computer simulation as analytical method will be needed to clarify this point.

The model considered here seems to be not exactly solvable, yet carries some nature to form patterns stably. This might suggest that there are important classes of “partly solvable models”, sharing stability of solutions with solitons in the exactly solvable models. They may serve to reveal the underlying universal nature of pattern formations. In this connection, it is a highly non-trivial task to investigate soliton picture in the present model. In the Toda lattice, solitons are stable and independent except the time interval during collisions, and multi-soliton states are essentially described by a succession of two-soliton collisions. Soliton picture associated with our model seems to be different from this. What we have observed in simulations is as follows: when two congested regions collide, they make a single large congested region; after a certain time, some number of such large congested regions move with the same velocity and the entire pattern on the lane does not change. So we expect that solitons in this model are stable against perturbations, but may not necessarily “solitary”: suppose that two solitons run in the same direction and one of them catches up the other, they would form one-soliton state, their “bound state”, whose size is given by a sum of those for two solitons. Our new solutions discussed in the previous section could be understood as “bound states”. Given such a picture, it would be very important and interesting to know how two solitons would behave in this model. In order to make the soliton picture clear, analysis of the two-soliton sector would therefore be crucial.
From our results, it is tempting to speculate that there are dissipative systems which acquire (at least, partial) integrability when temporal-spatial patterns are formed: there could be some crucial relations similar to the Whitham’s for other system to recover such properties.

Acknowledgements

We are grateful to K. Hasebe, A. Nakayama and Y. Sugiyama for their assistance on computer simulation: they kindly let us use their code for the delayed OV model. K. N thanks H. Hayakawa for informing him of ref.[5]. The hospitality of E-Laboratory of Nagoya University extended to K. N is also acknowledged.

A Appendix

Here we list useful formulae used in the text.

\[
\ln \frac{\vartheta_0(v + w)}{\vartheta_0(v)} - \ln \frac{\vartheta_0(v)}{\vartheta_0(v - w)} - 2\ln \frac{\vartheta_0(w)}{\vartheta_0(0)} = \ln [1 - k^2 \text{sn}^2(2Kv)\text{sn}^2(2Kw)], \tag{A.1}
\]

\[
Z(u) - Z(w) = Z(u - w) - k^2 \text{sn}u \text{sn}w \text{sn}(u - w), \tag{A.2}
\]

\[
Z(u + w) - Z(u - w) - 2Z(w) = -\frac{2k^2 \text{sn}w \text{cn}w \text{dn}w \text{sn}^2u}{1 - k^2 \text{sn}^2w \text{sn}^2u}, \tag{A.3}
\]

where \( K \) is the complete elliptic integral of the first kind. The theta function \( \vartheta_0(v) \) and the Jacobian zeta function \( Z(u) \) are defined by

\[
\vartheta_0(v) = q_0 \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2\pi v + 2^{4n-2}), \quad q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}) \tag{A.4}
\]

\[
Z(u) = \frac{d}{du} \ln \vartheta_0(u \frac{2K}{K}), \quad Z'(u) = \text{dn}^2 u - \frac{E}{K}, \tag{A.5}
\]

with \( q = e^{-\pi K' / K} \). \( K' \) is the complete elliptic integral of the first kind for the complementary modulus \( k' = \sqrt{1 - k^2} \), and \( E \) the complete elliptic integral of the second kind.

Other formulae are:

\[
\text{sn}(u - w) = \frac{\text{sn}u \text{cn}w \text{dn}w - \text{sn}w \text{cn}u \text{dn}u}{1 - k^2 \text{sn}^2u \text{sn}^2w}, \tag{A.6}
\]

\[
\text{dn}(u - w) = \frac{\text{dn}u \text{dn}w + k^2 \text{sn}u \text{cn}u \text{sn}w \text{cn}w}{1 - k^2 \text{sn}^2u \text{sn}^2w}, \tag{A.7}
\]

and

\[
\text{dn}^2(2Kv) - \frac{E}{K} = \left(\frac{\pi}{2K'}\right)^2 \sum_{l=-\infty}^{\infty} \text{sech}^2 \left( \frac{\pi K}{K'} (v - l) \right) - \frac{\pi}{2KK'}. \tag{A.8}
\]

\[\text{See ref.} \] for formulae of the elliptic functions.
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