Family of coherence measures and duality between quantum coherence and path distinguishability

Chunhe Xiong, 1,* Asutosh Kumar, 2,† and Junde Wu 1,‡

1 School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, PR China
2 P.G. Department of Physics, Gaya College, Magadh University, Rampur, Gaya 823001, India

Coherence measures and their operational interpretations lay the cornerstone of coherence theory. In this paper, we introduce a class of coherence measures with $\alpha$-affinity, say $\alpha$-affinity of coherence for $\alpha \in (0,1)$. Furthermore, we obtain the analytic formulae for these coherence measures and study their corresponding convex roof extension. We provide an operational interpretation for $1/2$-affinity of coherence by showing that it is equal to the error probability to discrimination a set of pure states with the least square measurement. Employing this relationship we regain the optimal measurement for equiprobable quantum state discrimination. Moreover, we compare these coherence quantifiers, and establish a complementarity relation between $1/2$-affinity of coherence and path distinguishability.

I. INTRODUCTION

Quantum coherence [1] is one of the fundamental features in quantum mechanics, characterizing the wave-like property for all objects. It is also a necessary condition for entanglement and other quantum correlations which manifests its core position in quantum information theory [2, 3]. As a key quantum resource, coherence may lead to an operational advantage over classical physics, and its important role in quantum algorithms has been investigated [4–7]. Hence, for a given quantum state, it is important to ask the amount of coherence it has and if the quantifier of coherence has any operational meaning? In Ref. [1], authors have established a resource theory of coherence which is a rigorous framework to quantify coherence. In this theory, coherence characterizes superposition of a quantum state relative to a fixed orthogonal basis and thereafter a lot of work has been done to enrich this theory [8–12]. This framework places certain important constraints on the measures of coherence, and different coherence measures may reflect different physical aspects of the quantum system [13–16]. Like other resource theories, the resource theory of coherence is composed of “free states” and “free operations”.

Let $\mathcal{H}$ be a finite dimensional Hilbert space with an orthogonal basis $\{|i\rangle\}_{i=1}^{d}$. Density matrices that are diagonal in this basis are free states and we call them incoherent states as they do not possess any coherence. We label this set of incoherent quantum states by $\mathcal{I}$. That is,

$$\mathcal{I} = \{\sigma | \sigma = \sum_{i=1}^{d} \lambda_{i} |i\rangle \langle i| \}. \quad (1)$$

Free operations in coherence theory are the completely positive and trace preserving (CPTP) maps which admit an incoherent Kraus representation. That is, there always exists a set of Kraus operators $\{K_{i}\}$ such that

$$\frac{K_{i} \sigma K_{i}^{\dagger}}{\text{Tr} K_{i} \sigma K_{i}^{\dagger}} \in \mathcal{I}, \quad (2)$$

for each $i$ and any incoherent state $\sigma$. These operations are also called incoherent operations and we label these operations by $\Phi$.

Analogous to the quantification of entanglement [17–20], any measure of coherence $C$ should satisfy the following axioms [1]:

(C1) Nonnegativity or faithfulness: $C(\rho) \geq 0$ with equality if and only if $\rho$ is incoherent.

(C2) Monotonicity: $C$ does not increase under the action of an incoherent operation, i.e., $C(\Phi(\rho)) \leq C(\rho)$ for any incoherent operation $\Phi$.

(C3) Strong monotonicity: $C$ does not increase on average under selective incoherent operations, i.e., $\sum_{i} p_i C(\sigma_i) \leq C(\rho)$ with probabilities $p_i = \text{Tr} K_i \rho K_i^{\dagger}$, post-measurement states $\sigma_i = p_i^{-1} K_i \rho K_i^{\dagger}$, and incoherent operators $K_i$.

(C4) Convexity: Nonincreasing under mixing of quantum states, i.e., $\sum_{i} p_i C(\rho_i) \geq C(\sum_{i} p_i \rho_i)$ for any set of states $\{\rho_i\}$ and $p_i \geq 0$ with $\sum_{i} p_i = 1$.

Conditions (C1) and (C2) highlight the role of free states and free operations in the coherence theory, i.e., the free states have zero coherence and the free operations cannot increase coherence of any state. (C3) and (C4) are two constraints imposed on coherence measures. Like in entanglement theory, a coherence quantifier which satisfies nonnegativity and (strong) monotonicity is called (strong) coherence monotone. Furthermore, if it also satisfies convexity, we call it convex (strong) coherence monotone.

It is worth noting that authors in Ref. [21] have provided a simple and interesting condition to replace (C3) and (C4) with the additivity of coherence for block-diagonal states,

$$C(p \rho \oplus (1-p) \sigma) = p C(\rho) + (1-p) C(\sigma), \quad (3)$$

for any $p \in [0, 1]$, $\rho \in \mathcal{E}(\mathcal{H}_1)$, $\sigma \in \mathcal{E}(\mathcal{H}_2)$ and $p \rho \oplus (1-p) \sigma \in \mathcal{E}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where $\mathcal{E}(\mathcal{H})$ denotes the set of density matrices on $\mathcal{H}$.

They proved that conditions (C1), (C2) and (3) are equivalent to conditions (C1) through (C4). This is surprising because (3) is operation-independent equality, whereas strong monotonicity and convexity are operation-dependent inequalities. In general, it is relatively easy to check whether a coherence quantifier satisfies (3) than (C3) and (C4).

In this paper we introduce a class of coherence measures, and attempt to answer the question posed in the beginning, by...
linking this coherence measure to ambiguous quantum state discrimination (QSD). QSD, as a fundamental problem in quantum mechanics, has been studied extensively [22–29]. It is not only an important problem of theoretical research, but also plays a key role in quantum communication and quantum cryptography [30–34].

We briefly review the ambiguous QSD. Suppose there are two persons, Alice and Bob. Alice chooses a state $\rho_i$ from a set of states $\{\rho_i\}_{i=1}^N$ with probability $\eta_i$ and sends it to Bob. Now Bob’s job is to determine which state he has received, as accurately as possible. To do this, Bob performs a positive-operator valued measure (POVM) on each $\rho_i$ and declares that the state is $\rho_j$ when the measurement outcome reads $j$. The POVM is a set of positive operators $\{M_i\}$ satisfying $\sum_i M_i = I$. As the probability to get the result $j$ with state $\rho_i$ is $p_{ji} = \text{Tr}(M_j \rho_i)$, the corresponding maximal success probability is

$$P^{\text{opt}}_S(\{\rho_i, \eta_i\}) = \max \sum_i \eta_i \text{Tr}(M_i \rho_i),$$

(4)

where the maximization is done over all POVMs. For $N = 2$ case, the analytic formula of $P^{\text{opt}}_S$ and the optimal measurement are known. However, no solution about optimal probability and measurement is known for general $N > 2$ case.

As a suboptimal choice, least square measurement (LSM) is an alternative to discriminate quantum states [35–41]. In comparison to the optimal measurement, the LSM has several nice properties. First, its construction is relatively simple as it can be determined directly from the given ensemble. Second, it is very close to the optimal measurement when the states to be distinguished are almost orthogonal [37, 42]. The construction of LSM is as follows.

Given an ensemble $\{\rho_i, \eta_i\}_{i=1}^N$ and denoting $\rho_{\text{out}} = \sum_i \eta_i \rho_i$, the least square measurements are [43]

$$M_i^{\text{lsm}} = \eta_i \rho_{\text{out}}^{-1/2} \rho_i \rho_{\text{out}}^{-1/2}, \quad i = 1, 2, ..., N.$$

(5)

As a result, the minimal error probability of this measurement is

$$P^{\text{lsm}}_E(\{\rho_i, \eta_i\}) = 1 - \sum_i \eta_i \text{Tr}(M_i^{\text{lsm}} \rho_i).$$

(6)

The paper is structured as follows. In section II we introduce $\alpha$-affinity of coherence. We reveal the connection between the $1/2$-affinity of coherence and QSD with least square measurement in section III. Furthermore, we deal with quantum state discrimination with coherence theory in section IV and section V. Besides, we establish a duality between $1/2$-affinity of coherence and path distinguishability in section VI, and finally conclude in section VI with a summary and outlook.

II. QUANTIFYING COHERENCE WITH AFFINITY

A. $\alpha$-affinity and $\alpha$-affinity of distance

Distances in state space are good candidates for quantifying quantum correlations. In this subsection, we introduce a distance using which we can establish a bona fide measure to quantify coherence. In classical statistical theory [44], affinity is defined as

$$A(f, g) = \sum_x \sqrt{f(x)g(x)},$$

where $f$ and $g$ are discrete probability distributions. Classical affinity quantifies the closeness of two probability distributions. Borrowing the notion from classical statistical theory, Luo and Zhang [45] have introduced quantum affinity as follows. Let $\mathcal{H}$ be a $d$-dimensional Hilbert space and $\mathcal{E}(\mathcal{H})$ be the set of density matrix on $\mathcal{H}$. For any $\rho, \sigma \in \mathcal{E}(\mathcal{H})$, quantum affinity is defined as

$$A(\rho, \sigma) := \text{Tr}(\sqrt{\rho} \sqrt{\sigma}).$$

(7)

Quantum affinity, similar to fidelity [46], describes how close two quantum states are. We drop the adjective “quantum” in the rest of this paper unless there is any ambiguity.

The notion of affinity has been extended to $\alpha$-affinity ($0 < \alpha < 1$), and is defined as

$$A^{(\alpha)}(\rho, \sigma) := \text{Tr} \rho^{\frac{1}{2}(1-\alpha)}.$$

(8)

For each $\alpha \in (0, 1)$, $A^{(\alpha)}(\rho, \sigma)$ satisfies the following properties: (1) Boundness. $A^{(\alpha)}(\rho, \sigma) \in [0, 1]$ with $A^{(\alpha)}(\rho, \sigma) = 1$ if and only if $\rho = \sigma$. (2) Monotonicity. $A^{(\alpha)}(\rho, \sigma) \leq A^{(\alpha)}(\Phi(\rho), \Phi(\sigma))$ for any CPTP map $\Phi$. (3) Joint concavity. If $\rho_i, \sigma_i \in \mathcal{E}(\mathcal{H})$ and $p_i > 0$, $\sum_i p_i = 1$, then $A^{(\alpha)}(\sum_i p_i \rho_i, \sum_i p_i \sigma_i) \geq \sum_i p_i A^{(\alpha)}(\rho_i, \sigma_i)$. The proof of property (1) is given in Appendix B. See Ref. [47] for the property (2), and property (3) is the result of Lieb’s concavity theorem [48].

It is well-known that $\alpha$-affinity plays an important role in quantum hypothesis testing. For the two state discrimination with many identical copies, one has [49, 50]

$$- \lim_{N \to \infty} \frac{1}{N} P^{\text{opt}}_{E,N}(\{\rho_i^{\otimes N}, \eta_i\}_{i=1}^N) = - \inf_{\alpha \in (0, 1)} \{\ln(\text{Tr} \rho_1^{\frac{1}{2}(1-\alpha)})\}.$$

This limit defines a function of $\alpha$-affinity, and

$$Q(\rho, \sigma) := \min_{\alpha \in (0, 1)} A^{(\alpha)}(\rho, \sigma),$$

(8)

is nonlogarithmic version of quantum Chernoff bound (QCB) [50].

Moreover, we can see that $\alpha$-affinity is related to $\alpha$-relative Rényi entropy [47]

$$S_{\alpha,z}(\rho||\sigma) = \frac{1}{\alpha - 1} \ln F_{\alpha,z}(\rho||\sigma),$$

where

$$F_{\alpha,z}(\rho||\sigma) := \text{Tr}(\rho^{\frac{1}{1-\alpha}} \sigma^\frac{\alpha}{1-\alpha} \rho^z),$$

(9)

and

$$A^{(\alpha)}(\rho, \sigma) = F_{\alpha,1}(\rho, \sigma).$$

(10)
Note that the family of $\alpha$-z-relative Rényi entropies includes relative entropy $S$ and max-relative entropy $S_{\text{max}}$ [47]

$$S = \lim_{a\to 1} S_{a,\alpha}, \quad S_{\text{max}} = \lim_{\alpha\to \infty} S_{a,\alpha}.$$ 

It’s worth noting that several coherence measures like relative entropy [1], geometric coherence [9] and max-relative entropy [16] are related to $\alpha$-z-relative Rényi entropy. In the next subsection, we introduce yet another measure of coherence, namely $\alpha$-affinity of coherence which is related to $\alpha$-z-relative Rényi entropy.

Based on $\alpha$-affinity, we introduce $\alpha$-affinity of distance as

$$d_a^{\alpha}(\rho, \sigma) := 1 - |A^{\alpha}(\rho, \sigma)|^{1/\alpha},$$

(11)

where $\rho, \sigma \in \mathcal{E}(\mathcal{H})$. Obviously, $\alpha$-affinity of distance satisfies the following properties.

(P1) $d_a^{\alpha}(\rho, \sigma) \geq 0$ with equality if and only if $\rho = \sigma$.

(P2) $d_a^{\alpha}$ is contractive under CPTP maps.

B. Quantifying coherence

Quantification of entanglement from the geometric point of view began in [18, 19]. Authors in these two papers put forward the scheme to quantify entanglement with the minimal distance between a given quantum state and all the separable states with relative entropy and Bures distance. Later, Luo and Zhang [45] studied the quantification of entanglement using Hellinger distance. Bures distance and Hellinger distance with $d_a^{\alpha}$ are related to $\alpha$-affinity of distance over all incoherent states, namely

$$\text{relative entropy} \quad \text{as the following properties.}$$

An advantage of $C_a^{\alpha}$ over geometric coherence, $C_g(\rho) := 1 - \max_{\sigma \in \mathcal{I}} (\text{Tr}(\sqrt{\sqrt{\rho} \sigma \sqrt{\sigma}}))^2$ [29], is that it is relatively easy to compute. Let $\sigma = \sum_i \mu_i |i\rangle \langle i|$ be an incoherent state. Then

$$A^{\alpha}(\rho) \equiv \max_{\sigma \in \mathcal{I}} \text{Tr}(\rho^\alpha \sigma^{1-\alpha})$$

$$= \max_{\mu_i} \left( \sum_i \mu_i \left| \langle i | \rho^\alpha | i \rangle \right| \right)^{1-\alpha}$$

$$\leq \max_{\mu_i} \left( \sum_i \mu_i \right)^{1-\alpha} \left( \sum_i \langle i | \rho^\alpha | i \rangle \right)^{1/\alpha}$$

$$= \left( \sum_i \langle i | \rho^\alpha | i \rangle \right)^{\alpha/\alpha},$$

(13)

where the inequality follows from the Hőlder’s inequality:

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \left( \sum_{i=1}^n |y_i|^q \right)^{1/q}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Here $p = \frac{1}{\alpha - 1} > 1$ and $q = \frac{1}{\alpha} > 1$. Inequality (13) gives an upper bound on $A^{\alpha}(\rho)$. This suggests that we can choose suitable $\{\mu_i\}$’s such that above inequality becomes an equality. As a result, we obtain the analytic expression for $C_a^{\alpha}$ as,

$$C_a^{\alpha}(\rho) = 1 - \sum_i \langle i | \rho^\alpha | i \rangle^{1/\alpha},$$

(14)

and the closest incoherent state which minimizes $C_a^{\alpha}(\rho)$ is

$$\sigma_\rho = \sum_i \frac{|i\rangle \langle i|}{\sqrt{\sum_j |j\rangle \langle j| \rho^{1/\alpha}}} |i\rangle \langle i|.$$ 

(15)

With (P1), (P2) and (3), we have the following theorem.

Theorem 1. $\alpha$-affinity of coherence is a coherence measure.

Proof. First, it is obvious that $C_a^{\alpha}(\rho) \geq 0$. Since $d_a^{\alpha}(\rho, \sigma) = 0$ iff $\rho = \sigma$, one has $C_a^{\alpha}(\rho) = 0$ if and only if $\rho \in \mathcal{I}$. In addition, since $d_a^{\alpha}(\rho, \sigma)$ obeys monotonicity under CPTP maps, we have $C_a^{\alpha}(\rho) \geq C_a^{\alpha}(\Phi(\rho))$ for any incoherent operation $\Phi$. Now, instead of (C3) and (C4), we prove that $C_a^{\alpha}$ satisfies additivity of coherence for block-diagonal states. We have

$$C_a^{\alpha}(\rho \oplus (1 - p)\sigma)$$

$$= 1 - \sum_i \langle i | (\rho \oplus (1 - p)\sigma)^\alpha | i \rangle^{1/\alpha}$$

$$= 1 - \sum_i \langle i | (\rho)^\alpha \oplus [(1 - p)\sigma]^\alpha | i \rangle^{1/\alpha}$$

$$= p(1 - \sum_i \langle i | (\rho)^\alpha | i \rangle^{1/\alpha}) + (1 - p)(1 - \sum_i \langle i | \sigma^\alpha | i \rangle^{1/\alpha})$$

$$= pC_a^{\alpha}(\rho) + (1 - p)C_a^{\alpha}(\sigma),$$

Thus, $C_a^{\alpha}$ is a coherence measure for each $\alpha \in (0, 1)$. □

Similarly, we define quantum Chernoff bound of coherence, $C_{qcb}(\rho)$, and affinity of coherence, $\tilde{C}_a(\rho)$, respectively as

$$C_{qcb}(\rho) := \min_{\sigma \in \mathcal{I}} (1 - Q^{1/\alpha}(\rho, \sigma))$$

$$= 1 - \max_{\alpha \in (0, 1)} \min_{\sigma \in \mathcal{I}} (\text{Tr}(\rho^\alpha \sigma^{1-\alpha}))^{1/\alpha}$$

$$= \max_{\alpha \in (0, 1)} C_a^{\alpha}(\rho),$$

(16)

and

$$\tilde{C}_a(\rho) := \min_{\sigma \in \mathcal{I}} (1 - A(\rho, \sigma)) = 1 - \max_{\sigma \in \mathcal{I}} \text{Tr}(\sqrt{\rho} \sqrt{\sigma})$$

$$= 1 - \sqrt{\sum_i \langle i | \sqrt{\rho} | i \rangle^2},$$

(17)

and the closest incoherent state is again $\sigma_\rho$ in Eq. (15).

As $C_a^{\alpha}$ is a coherence measure for each $\alpha \in (0, 1)$, $C_{qcb}(\rho)$ is also a coherence measure. Also, we can show that $\tilde{C}_a$ is a convex weak coherence monotone. Following
the same lines of the proof of Theorem 1, \( \tilde{C}_a \) satisfies (C1) and (C2). Moreover, convexity of \( \tilde{C}_a \) can be derived from the joint concavity of \( A(\rho, \sigma) \). However, \( \tilde{C}_a \) does not satisfy strong monotonicity.

Let \( \rho_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) and \( \rho_2 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \).

Then \( \tilde{C}_a(\rho_1) = 1 - \sqrt{\frac{2}{3}} \), \( \tilde{C}_a(\rho_2) = 1 - \sqrt{\frac{1}{3}} \), and

\[
\tilde{C}_a \left( \frac{1}{2} \rho_1 \otimes \frac{1}{2} \rho_2 \right) = 1 - \sqrt{\frac{5}{12}} \neq \frac{1}{2} (\tilde{C}_a(\rho_1) + \tilde{C}_a(\rho_2)).
\]

In conclusion, \( \tilde{C}_a \) is a convex weak coherence monotone.

C. Coherence for pure states and single-qubit states

In this subsection, we evaluate \( \alpha \)-affinity of coherence for pure states and single-qubit states. For any pure state \( |\psi\rangle \),

\[
C_{\alpha}^{(\alpha)}(|\psi\rangle) = 1 - \sum_i |\langle i | \psi \rangle|^2 / \alpha,
\]

is a non-increasing function of \( \alpha \), and \( C_{qcb}(|\psi\rangle) = \max_{\alpha \in (0, 1]} C_{\alpha}^{(\alpha)}(|\psi\rangle) \). We have \( C_{qcb}(|\psi\rangle) \to 1 \) when \( \alpha \to 0 \). This is very interesting observation that all coherent pure states are almost the maximally coherent states in the sense of QCB of coherence.

For a single-qubit state \( \rho = \frac{1}{2} (I + \sum_i c_i \sigma_i) \) with \( \sigma_i \) (\( i = 1, 2, 3 \)) being Pauli matrices, the eigenvalues are \( \lambda_{1,2} = (1 \mp |c|)/2 \) and

\[
\rho^{\alpha} = \left( \begin{array}{cc} \frac{\lambda_2}{2} & \frac{c_3 (\lambda_2 - \lambda_1)}{2 |c|} \\ \frac{c_3 (\lambda_2 - \lambda_1)}{2 |c|} & \frac{\lambda_1}{2} \end{array} \right) + \frac{1}{2} c_2 \sigma_3 + \frac{1}{2} c_3 \sigma_1.
\]

Therefore, the corresponding \( \alpha \)-affinity of coherence is

\[
C_{\alpha}^{(\alpha)}(\rho) = 1 - (A + B)^{1/\alpha} - (A - B)^{1/\alpha},
\]

where

\[
A = \frac{(1 - |c|)^\alpha + (1 + |c|)^\alpha}{2}, \quad \text{and} \quad B = \frac{c_3 ((1 + |c|)^\alpha - (1 - |c|)^\alpha)}{2 |c|}.
\]

D. Convex roof of \( \alpha \)-affinity of coherence

A measure of coherence defined for all pure states can be extended to mixed quantum states (similar to entanglement theory [54]). The convex roof extension of \( \alpha \)-affinity of coherence for a mixed state \( \rho \) can be defined as

\[
C_{\alpha}^{(\alpha)}(\rho) := \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_{\alpha}^{(\alpha)}(|\psi_i\rangle),
\]

where the minimization is performed over all pure state decompositions of \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \). Then, we have the following theorem.

**Theorem 2.** \( C_{\alpha}^{(\alpha)}(\rho) \) is a coherence measure for each \( \alpha \in (0, 1) \).

**Proof.** See Appendix A.

**Remark 2.1.** In fact, since each coherence quantifier \( C \) for pure states satisfies (C1) and (C3), its convex roof is also a coherence measure. The proof follows the same lines as in Theorem 2.

III. 1/2-AFFINITY OF COHERENCE AND LEAST SQUARE MEASUREMENT

Spehner and Orszag [55] first revealed the connection between quantum correlation (Hellinger distance based quantum discord) and QSD with least square measurement. In coherence theory, there is a very close relationship between geometric coherence and QSD. Authors in Ref. [56] have recently shown that geometric coherence of a pure state is equal to the minimum error probability to discriminate a set of linearly independent pure states \( \{ |\psi_i\rangle, \eta_i \}_{i=1}^d \) with von Neumann measurement, where \( |\psi_i\rangle = \eta_i^{-1/2} \sqrt{\rho} |i\rangle \), \( \eta_i = \rho_{ii} \) and \( d = \text{rank}(\rho) \). Since the optimal measurement is not easy to find, we consider the least square measurement for \( \{ |\psi_i\rangle, \eta_i \}_{i=1}^d \).

For \( \{ |\psi_i\rangle, \eta_i \}_{i=1}^d \), there are two cases. If \( \eta_i \neq 0 \) for \( i = 1, \ldots, d \), then the ensemble contains \( d \) states. Since \( \sum_i \eta_i |\psi_i\rangle \langle \psi_i| = \rho \), the least square measurement is

\[
M_l^{\text{ism}} = \eta_i \rho^{-1/2} |\psi_i\rangle \langle \psi_i| \rho^{-1/2} = |i\rangle \langle i|, \quad (21)
\]

where \( \rho^{-1/2} := \sum_i \lambda^{-1/2} |a_i\rangle \langle a_i| \) if \( \rho = \sum_i \lambda_i |a_i\rangle \langle a_i| \) is the spectral decomposition. Thus, \( \sum_i M_l^{\text{ism}} = I \) and the successful probability to discriminate the ensemble \( \{ |\psi_i\rangle, \eta_i \}_{i=1}^d \)

is

\[
P_{\text{disc}}^{\text{ism}}(\{ |\psi_i\rangle, \eta_i \}_{i=1}^d) = \sum_i \eta_i \text{Tr}(M_l^{\text{ism}} |\psi_i\rangle \langle \psi_i|) = \sum_i |\langle i| \sqrt{\rho} |i\rangle|^2 = |A^{(1/2)}(\rho)|^2, \quad (22)
\]

If \( \eta_i = 0 \) for some \( i = i_1, i_2, \ldots, i_s \), then the ensemble \( \{ |\psi_i\rangle, \eta_i \}_{i=1}^d \) reduces to \( \{ |\psi_i\rangle, \eta_i \}_{i\neq i_s} \). In fact, as \( \eta_i = |\langle i| \sqrt{\rho} |i\rangle|^2 \), \( \eta_i = 0 \) implies \( |\psi_i\rangle \) is a zero vector. If \( S \) is the subspace spanned by \( \{ |\psi_{i'}\rangle \}_{i' \neq i_s} \), then

\[
M_l^{\text{ism}} = \eta_i \rho^{-1/2} |\psi_{i'}\rangle \langle \psi_{i'}| \rho^{-1/2} = |i'\rangle \langle i'|, \quad (23)
\]
for all $i'$, and $\sum_{i'=1}^{d-\alpha} M_i^{lsm} = I_S$. Moreover, the successful probability to discriminate the ensemble $\{ |ψ_i' \rangle, η_i \}_{i'=1}^{d-\alpha}$ with $\{ M_i^{lsm} \}$ is

$$P_S^{lsm}(\{ |ψ_i' \rangle, η_i \}_{i'=1}^{d-\alpha}) = \sum_{i'=1}^{d-\alpha} Tr(M_i^{lsm} |ψ_i' \rangle \langle ψ_i' |)$$

$$= \sum_{i'=1}^{d-\alpha} \langle i' | \sqrt{ρ} | i' \rangle^2$$

$$= \sum_{i'=1}^{d-\alpha} \langle i | \sqrt{ρ} | i \rangle^2$$

$$= [A^{(1/2)}(ρ)]^2.$$

In other words, the corresponding error probability to discriminate linearly independent pure states $\{ |ψ_i \rangle, η_i \}_{i=1}^d$ is

$$P_{E}^{lsm}(\{ |ψ_i \rangle, η_i \}_{i=1}^d) = 1 - P_S^{lsm}(\{ |ψ_i \rangle, η_i \}_{i=1}^d) = C^{(1/2)}(ρ).$$

Thus, we have the following theorem.

**Theorem 3.** If quantum state $ρ$ describes a quantum system in $d$-dimensional Hilbert space $H$ with $\{ |i \rangle \}_{i=1}^d$ being a reference basis, then the $α$-affinity of coherence is equal to the error probability to discriminate $\{ |ψ_i \rangle, η_i \}_{i=1}^d$ with least square measurement. That is,

$$C^{(1/2)}(ρ) = P_{E}^{lsm}(\{ |ψ_i \rangle, η_i \}_{i=1}^d),$$

where $η_i = ⟨ i | ρ | i ⟩$ and $|ψ_i⟩$ is $η_i^{-1/2} \sqrt{ρ} | i ⟩$.

**Remark 3.1.** If $ρ$ is an incoherent state, then $C^{(1/2)}(ρ) = 0$ which means that $\{ |ψ_i \rangle, η_i \}_{i=1}^d$ can be perfectly discriminated by the least square measurement. In other words, the LSM is actually the optimal measurement.

### IV. LEAST SQUARE MEASUREMENT AND OPTIMAL MEASUREMENT

#### A. QSD with LSM and 1/2-affinity of coherence

In this section, we review a connection between the least square measurement (as a suboptimal choice) and the optimal measurement in QSD protocol. Authors in Ref. [56] have linked quantum state discrimination to geometric coherence. Let us consider QSD of a set of pure states $\{ |ψ_i \rangle, η_i \}_{i=1}^d$. Denote a matrix $M$ with $M_{ij} = \sqrt{η_i} ⟨ ψ_i | ψ_j ⟩$, $1 \leq i, j \leq d$, that is,

$$M = \begin{pmatrix}
η_1 & \sqrt{η_2/η_1} ⟨ ψ_2 | ψ_1 ⟩ & \cdots & \sqrt{η_d/η_1} ⟨ ψ_d | ψ_1 ⟩ \\
\sqrt{η_2/η_1} ⟨ ψ_2 | ψ_1 ⟩ & η_2 & \cdots & \sqrt{η_d/η_2} ⟨ ψ_d | ψ_2 ⟩ \\
\vdots & \ddots & \ddots & \vdots \\
\sqrt{η_d/η_1} ⟨ ψ_d | ψ_1 ⟩ & \cdots & \sqrt{η_d/η_d} ⟨ ψ_d | ψ_d ⟩ & η_d
\end{pmatrix}.\tag{24}$$

Then, $M$ is a density matrix and we call it the QSD-state of $\{ |ψ_i \rangle, η_i \}_{i=1}^d$.

#### Theorem 4. [56] Let $H$ be a $d$-dimensional Hilbert space and $\{ |i \rangle \}_{i=1}^d$ be the computable basis, that is, $|i⟩ = (0, ..., 0, 1, 0, ..., 0)$, the $i$-th entry is 1 for each $i$. For $|ψ_i⟩ \in H$, the minimal error probability to discriminate the collection of linearly independent pure states $\{ |ψ_i⟩, η_i \}_{i=1}^d$ is equal to the geometric coherence of the corresponding QSD-state $ρ$, that is,

$$P_{E}^{opt}(\{ |ψ_i⟩, η_i \}_{i=1}^d) = C^{(1/2)}(M).$$

For 1/2-affinity and the least square measurement, there exists a similar relationship. If we denote the corresponding QSD-state by $M$, namely, $ν_i = M_{ii} = η_i$, $|φ_i⟩ = η_i^{-1/2} \sqrt{M} | i ⟩$ for each $i$, then $⟨ φ_i | φ_j ⟩ = (ν_i ν_j)^{-1/2} (i) M (j) = ⟨ ψ_i | ψ_j ⟩$, $1 \leq i, j \leq d$. With Lemma 8 in Ref. [56], there exists a unitary $V$ such that $|φ_i⟩ = V |ψ_i⟩$ for each $i$.

As a result, the least square measurement for $\{ |ψ_i⟩, η_i \}_{i=1}^d$ is

$$M_i = η_i ν_i^{-1/2} |ψ_i⟩ ν_i^{-1/2} |ψ_i⟩^\dagger, i = 1, ..., d,$$

with $ρ_{out} = \sum_i η_i |ψ_i⟩ ν_i^{-1/2} |ψ_i⟩$. Since $σ_{out} = η_i |ψ_i⟩ ⟨ ψ_i | V ρ V^\dagger$, the LSM for $\{ |φ_i⟩, η_i \}_{i=1}^d$ is

$$N_i = η_i σ_{out}^{-1/2} |φ_i⟩ ⟨ φ_i | σ_{out}^{-1/2} = V M_i V^\dagger.\tag{27}$$

In addition, one has

$$P_S^{lsm}(\{ |ψ_i⟩, η_i \}_{i=1}^d) = \sum_i η_i tr(M_i |ψ_i⟩ ⟨ ψ_i | M_i^\dagger)$$

$$= \sum_i η_i tr(N_i |φ_i⟩ ⟨ φ_i | N_i^\dagger)$$

$$= P_S^{lsm}(\{ |φ_i⟩, η_i \}_{i=1}^d) = C^{(1/2)}(M).$$

In conclusion, we have the following result.

**Theorem 5.** Let $H$ be a $d$-dimensional Hilbert space and $\{ |i⟩ \}_{i=1}^d$ be the computable basis, that is, $|i⟩ = (0, ..., 0, 1, 0, ..., 0)^\dagger$, the $i$-th entry is 1 for $i = 1, ..., d$. For $|ψ_i⟩ \in H, i = 1, ..., d$, the error probability to discriminate the collection of pure states $\{ |ψ_i⟩, η_i \}_{i=1}^d$ with least square measurement is equal to the geometric coherence of the corresponding QSD-state $M$, that is,

$$P_{E}^{opt}(\{ |ψ_i⟩, η_i \}_{i=1}^d) = C^{(1/2)}(M),$$

where the incoherent pure states are $\{ |i⟩ \}_{i=1}^d$.

#### B. Least square measurement and optimal measurement

First, we recall the following result.

**Theorem 6.** [42, 57] Let $\{ ρ_i, μ_i \}_{i=1}^m$ be an ensemble of $m$ states of a system in an $n$-dimensional Hilbert space $H$ ($m \leq n$), then

$$P_S^{opt}(\{ ρ_i, μ_i \}_{i=1}^m) \leq \sqrt{P_S^{lsm}(\{ ρ_i, μ_i \}_{i=1}^m)}.$$\tag{29}
As $P_{S}^{opt} = 1 - P_{S}^{opt}$ is the minimal error probability of QSD, the error probability with LSM is

$$P_{L}^{sm} = 1 - P_{S}^{sm} \leq 1 - (P_{S}^{opt})^2 \leq 2P_{E}^{opt}. \quad (30)$$

Therefore, if $P_{S}^{opt}$ is very close to 0, so is $P_{L}^{sm}$. In fact, LSM is very close to the optimal measurement for almost orthogonal states.

As the LSM to discriminate a set of pure states is actually a von Neumann measurement and the result of Theorem 4, one has

$$2C_g(\rho) \geq C_{1/2}(\rho) \geq C_{g}(\rho) \geq \tilde{C}_{\alpha}(\rho),$$

for any $\rho$. In addition, since $C_g(\rho) \leq \frac{C_{1}(\rho)}{d^2}$ for any $\rho > 0$ (that is, $\rho$ is invertible) [56], where $C_{1}(\rho)$ is $l_1$-norm of coherence defined as $C_{1}(\rho) := \sum_{i \neq j} | \langle i | \rho | j \rangle |$, we have that for any $\rho > 0$, the following inequality holds

$$\frac{2}{d-1}C_{1}(\rho) \geq 2C_g(\rho) \geq C_{1/2}(\rho) \geq C_{g}(\rho) \geq \tilde{C}_{\alpha}(\rho).$$

On the other hand, we consider the connection between least square measurement and optimal measurement through coherence.

In Ref.[58], Zhang et al. give an upper bound for geometric coherence as

$$C_g(\rho) \leq \min\{l_1, l_2\}, \quad (31)$$

where $l_1 = 1 - \max_i \{\rho_{ij}\}$ and $l_2 = 1 - \sum_i b_{i}^{2}$ with $b_{ij}$ being the $(i, j)$-th entry of $\sqrt{\rho}$. This is interesting to note that $l_2$ is actually equal to $C_{1/2}(\rho)$, and moreover, they also show that $l_2$ is tight for the maximally coherent mixed states given by

$$\rho_m = p |\psi_d\rangle \langle \psi_d| + \frac{1-p}{d} I_d, \quad (32)$$

where $0 < p < 1$, and $|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle$ is the maximally coherent state.

In other words, one has

$$C_g(\rho_m) = C_{1/2}(\rho_m). \quad (33)$$

Combining Theorem 4, Theorem 1 and (33), we recover the following result.

**Theorem 7.** [22, 28] For the equiprobable quantum state discrimination task $\{|\phi_i\rangle, 1/d\}_{i=1}^d$ with $\langle \phi_i | \phi_j \rangle = p$ for $i \neq j$, the least square measurement is optimal. Moreover, the maximum successful probability is

$$P_{S}^{opt} \{|\phi_i\rangle, 1/d\}_{i=1}^d = \left[ \frac{d-1}{d} \sqrt{1-p} + \frac{1}{d} \sqrt{1-p + dp} \right]^2. \quad \text{Proof.}$$

Note that the QSD-state of the above-mentioned task is $\rho_m$. As

$$P_{E}^{opt} \{|\phi_i\rangle, 1/d\}_{i=1}^d = C_g(\rho_m) = C_{1/2}(\rho_m) = P_{E}^{sm} \{|\phi_i\rangle, 1/d\}_{i=1}^d,$$

then the least square measurement is optimal. The first equality is the result of Theorem 4 and the fact that $\{|\phi_i\rangle\}$ is linearly independent. The last equality is due to Theorem 5. Using the result in Ref. [58],

$$C_g(\rho_m) = 1 - \left[ \frac{d-1}{d} \sqrt{1-p} + \frac{1}{d} \sqrt{1-p + dp} \right]^2,$$

the maximum successful probability is

$$P_{S}^{opt} \{|\phi_i\rangle, 1/d\}_{i=1}^d = \left[ \frac{d-1}{d} \sqrt{1-p} + \frac{1}{d} \sqrt{1-p + dp} \right]^2,$$

and the corresponding optimal measurement is

$$M_{i}^{opt} = \frac{1}{d} \rho_{\text{out}}^{-1/2} |\phi_i\rangle \langle \phi_i| \rho_{\text{out}}^{-1/2},$$

where $\rho_{\text{out}} = \frac{1}{d} \sum_i |\phi_i\rangle \langle \phi_i| \ (i = 1, \ldots, d). \quad \square$

**V. WHEN IS LSM OPTIMAL?**

Theorem 7 indicates that LSM is optimal for the equiprobable case. However, we find that this is not the only case as discussed below.

**A. Two pure states case**

Since we have the explicit expressions of geometric coherence and $1/2$-affinity of coherence for single-qubit states, we can derive the condition for LSM being optimal for an ensemble containing two pure states. Given an ensemble $\{|\psi_i\rangle, \eta_i\}_{i=1}^2$, the corresponding QSD-state is a single-qubit state $\rho = \sum_i \eta_i \sigma_i$. From Eq. (19), one has

$$[A^{1/2}(\rho)]^2 = \frac{1}{2} \left( 1 + \sqrt{1 - |c|^2} + \frac{c_3^2}{1 + \sqrt{1 - |c|^2}} \right).$$

On the other hand, with fidelity $F(\rho, \sigma) := \text{tr} \sqrt{\sigma \rho \sigma}$,

$$F(\rho) := \max_{\sigma \in I} F(\rho, \sigma) = \frac{1}{2} \left( 1 + \sqrt{1 - c_1^2 - c_2^2} \right).$$

The above expressions reduce to simpler forms when $\rho$ is a pure state ($|c| = \sqrt{c_1^2 + c_2^2 + c_3^2} = 1$). That is, $[A^{1/2}(\rho)]^2 = \frac{1}{2}(1 + c_3^2)$ and $F^2(\rho) = \frac{1}{2}(1 + |c_3|)$. Then, $A^{1/2}(\rho) = F(\rho)$ if and only if $c_3 = 0$ or $\pm 1$. The same can be shown true for mixed states with some tedious calculation. Hence, the least square measurement is optimal for two pure states case if and only if these states are orthogonal or have equal probabilities.

**B. Multiple copy QSD with LSM**

We consider QSD protocol with multiple copies, as the error probability of a QSD task decreases when we have more copies of states.
For the $N$-copy case $|\psi_i\rangle_{i=1}^\otimes$, the $(i,j)$-th entry of the corresponding QSD-state is
\[ \rho_{ij}^{(N)} = \sqrt{\eta_i \eta_j} |\psi_i\rangle \langle \psi_j|^N \ (1 \leq i, j \leq d). \]

Let $N \to \infty$ and $\rho_{ij}^{(N)} \to 0$ for each $i \neq j$. Since $|\psi_i\rangle_{i=1}^\otimes$ is linearly independent for large $N$, the QSD-state $\rho^{(N)}$ is invertible. Then,
\[ C_{a}^{1/2}(\rho) \leq \frac{2}{d-1} C_{lsm}(\rho), \]
and the error probability to discriminate $|\psi_i\rangle_{i=1}^\otimes, \eta_i \rangle$ tends to zero. In other words, if we have enough copies of states, pure states $|\psi_i\rangle_{i=1}^\otimes, \eta_i \rangle$ can be almost perfectly distinguished by the LSM. In other words, we prove that LSM is asymptotically optimal for discrimination of pure states in the sense that the corresponding QSD-state $\rho \to \rho^{diag} = \sum_i \langle i | \rho | i \rangle |i\rangle$.\]

VI. DUALITY BETWEEN 1/2-AFFINITY OF COHERENCE AND PATH DISTINGUISHABILITY

Bera et al. [59] obtained a complementarity relation between $l_1$-norm of coherence and path distinguishability in the case of Yang’s $n$-slit experiment. Here, we establish the complementarity between geometric coherence and path distinguishability as follows.

Consider the case of $d$-slit quantum interference with pure quantons. In the Yang’s $n$-slit experiment, if the quanton passes through the $i$th slit or takes the $j$th path, then we denote $|i\rangle$ as the possible state. As a result, the state of the quanton can be represented with $d$ basis states $\{|1\rangle, \ldots, |d\rangle\}$ as
\[ |\Psi\rangle = c_1 |1\rangle + \ldots + c_d |d\rangle, \tag{34} \]
where $|i\rangle$ represents the $i$th slit and $c_i$ is the amplitude of taking the $i$th slit. To determine through which slit the quanton passes, one needs to perform a quantum measurement. According to quantum measurement theory, the quanton will interact with a detector state and the compound state is given by
\[ U(|\Psi\rangle |0_d\rangle) = \sum_i c_i |i\rangle |d_i\rangle, \tag{35} \]
where $|d_i\rangle$ are normalized but not necessarily orthogonal states of the detector.

To quantify the coherence of quanton, one considers the reduced density matrix of the quanton after tracing out the detector states,
\[ \rho_s = \sum_{i,j=1}^{d} c_i c_j \langle d_j | d_i \rangle |i\rangle \langle j|. \tag{36} \]

From Theorem 3, the 1/2-affinity of coherence is
\[ C_{a}^{1/2}(\rho_s) = 1 - P_{\text{ls}}^{\text{slm}}(|\psi_i\rangle_{i=1}^\otimes, \eta_i \rangle), \tag{37} \]
where $\eta_i = |c_i|^2, |\psi_i\rangle = \exp(\sqrt{-1} \theta_i) \eta_i^{-1/2} \sqrt{\rho_s} |i\rangle$ and $\theta_i$ is the argument of $c_i$.

Now, to know which path the quanton takes, one has to discriminate the detector states $|d_i\rangle, |c_i|^2 |d_i\rangle$. In other words, the path distinguishability is actually equivalent to the discrimination of the corresponding detector states.

Since $\langle \psi_i | \psi_j\rangle = \langle d_j | d_i \rangle = \langle d_i | d_j \rangle$, there exists a unitary matrix $V$ such that $|d_i\rangle = V |\psi_i\rangle$. Therefore, one has
\[ \rho_{out} = \sum_i |c_i|^2 |d_i\rangle \langle d_i| = V \sum_i |c_i|^2 |\psi_i\rangle \langle \psi_i| V^\dagger = V \rho_s V^\dagger, \]
and then the corresponding LSM for $|d_i\rangle, |c_i|^2$ is
\[ N_{ls}^{\text{slm}} = |c_i|^2 \rho_{out}^{-1/2} |d_i\rangle \langle d_i| \rho_{out}^{-1/2} = V |i\rangle \langle i| V^\dagger. \]

As a result, one has
\[ P_{\text{ls}}^{\text{slm}}(|d_i\rangle, |c_i|^2)_{i=1}^{d} = \sum_i |c_i|^2 \langle i| V^\dagger |d_i\rangle^2 = \sum_i |\langle i| \sqrt{\rho_s} |i\rangle|^2 = P_{\text{ls}}^{\text{slm}}(|\psi_i\rangle_{i=1}^\otimes, |c_i|^2)_{i=1}^{d}. \]

Even though it is not the optimal choice for quantum state discrimination, LSM is very close to the optimal one when the states to be distinguished are almost orthogonal, and its construction is also relatively simple. Moreover, the complementarity between coherence and path distinguishability holds just for linearly independent detector states [56, 59]. Therefore, if we define the optimal successful probability to discriminate the detector states with LSM as path distinguishability, $D_q := P_{\text{ls}}^{\text{slm}}(|d_i\rangle, |c_i|^2)_{i=1}^{d}$, and the 1/2-affinity of coherence as coherence, $C := C_{a}^{1/2}(\rho_s)$, we obtain the complementarity between 1/2-affinity of coherence and path distinguishability as
\[ C + D_q = 1. \tag{38} \]

Thus, the wave nature of the quanton can also be characterized by $C_{a}^{1/2}(\rho_s)$. If the quantum system is exposed to the environment, that is, the quanton state is a mixed state $\rho = \sum_{i,j} \rho_{ij} \langle i| \langle j|$, we can obtain a generalized complementarity. The composite system of the quanton and the path detector after the unitary interaction can be given as
\[ \rho_{sd} = \sum_{i,j} \rho_{ij} |i\rangle \langle j| \otimes |d_i\rangle \langle d_j|, \tag{39} \]
and the reduced density matrix of the quanton after tracing out the detector states is
\[ \rho_s = \sum_{i,j=1}^{d} \rho_{ij} \langle d_j | d_i \rangle |i\rangle \langle j|. \tag{40} \]

As every principal $2 \times 2$ submatrix in Eq. (40) is positive semidefinite [60, p.434], we have
\[ \sqrt{\rho_{ii} \rho_{jj}} - |\rho_{ij}| \geq 0 \ (1 \leq i, j \leq d). \tag{41} \]
Assuming that the corresponding ensemble to $\rho_n$ is $\{|\psi_i\rangle, \rho_{ii}\}$, we have

$$|\langle \psi_i | \psi_j \rangle| = \sqrt{|\langle \psi_i | \rho_{jj} | \psi_j \rangle|} = \sqrt{\frac{|\rho_{ij}|}{\sqrt{\rho_{ii}\rho_{jj}}}} |\langle d_i | d_j \rangle| \leq |\langle d_i | d_j \rangle|,$$

for each $i$ and $j$. In other words, a pair of states in $\{|d_i\rangle, \rho_{ii}\}_{i=1}^d$ is more difficult to distinguish than the corresponding pair in $\{|\psi_i\rangle, \rho_{ii}\}_{i=1}^d$. As a result of the above inequality, we have

$$1 - C_d^{(1/2)}(\rho_n) = P_{\text{LSM}}^d(\{|\psi_i\rangle, \rho_{ii}\}_{i=1}^d) \geq P_{\text{LSM}}^d(\{|d_i\rangle, \rho_{ii}\}_{i=1}^d) = D_q.$$

Hence, we have the following complementarity relation between coherence and path distinguishability,

$$C + D_q \leq 1. \quad (42)$$

VII. CONCLUSION

In this paper, we have introduced a family of coherence measures, namely $\alpha$-affinity of coherence for $\alpha \in (0, 1)$. Moreover, we obtained the analytic formulae for these quantifiers and also studied their convex roof extension. In particular, we have offered an operational meaning for 1/2-affinity of coherence, by showing that this equals the error probability to discriminate a set of pure states with least square measurement. Based on the relationship between the LSM and the optimal measurement, we obtained the optimal measurement for the equiprobable quantum state discrimination. Furthermore, we obtained conditions for the LSM to be the optimal measurement for two pure states from the perspective of coherence theory. In addition, we also studied the multiple copy QSD and concluded that LSM is optimal in the asymptotical sense. At last, we established the complementary relationship between 1/2-affinity of coherence and path distinguishability.

Our results not only offer a class of bona fide coherence quantifiers, but also reveal a close link between the quantification of coherence and quantum state discrimination. However, the operational interpretation of general $\alpha$-affinity coherence needs further investigation.

Note. After completion of this work we have been informed by Hyukjoon Kwon that 1/2-affinity of coherence has been computed and proven to be a coherence measure independently in Refs. [61, 62] by different methods, yielding the same result.

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Appendix A: $C_{ca}^{(\alpha)}$ is a coherence measure

(C1) Nonnegativity: It is obvious that $C_{ca}^{(\alpha)}(\rho) \geq 0$. Since $d_n^{(\alpha)}(\rho, \sigma) = 0$ iff $\rho = \sigma$, one has $C_{ca}^{(\alpha)}(\rho) = 0$ if and only if there exist a pure state decomposition $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ with each $|\psi_i\rangle$ being an incoherent state, that is, $\rho \in \mathcal{I}$.

(C3) Strong monotonicity: Assume that $\rho_n = p_n^{-1} K_n \rho K_n^\dagger$ is the output state of an incoherent operation $\Phi$ with probability $p_n = \text{Tr}(K_n \rho K_n^\dagger)$ for each $n$, and $\rho$ admits the optimal pure state decomposition $\rho = \sum_j q_j |\psi_j\rangle \langle \psi_j|$. Then $\rho_n = \sum_j \frac{q_j p_n}{p_n} |\psi_j\rangle \langle \psi_j|$ is a pure state decomposition with $|\phi_n^\dagger\rangle = \frac{1}{\sqrt{p_n}} K_n^\dagger |\psi_j\rangle$ and $q_{j,n} = \text{Tr}(K_n |\psi_j\rangle \langle \psi_j| K_n^\dagger)$ for each $\rho_n$. Thus, one gets

$$\sum_n p_n C_{ca}^{(\alpha)}(\rho_n) \leq \sum_n p_n \sum_j \frac{q_j p_n}{p_n} C_{ca}^{(\alpha)}(|\phi_n^\dagger\rangle) \leq \sum_j q_j C_{ca}^{(\alpha)}(|\psi_j\rangle) = C_{ca}^{(\alpha)}(\rho).$$

The second inequality is due to the strong monotonicity of $C_{ca}^{(\alpha)}$ for $|\psi_j\rangle$, that is, $\sum_n q_{j,n} C_{ca}^{(\alpha)}(|\phi_n^\dagger\rangle) \leq C_{ca}^{(\alpha)}(|\psi_j\rangle)$.

(C4) Convexity: Suppose $\rho_n = \sum_j q_j^n |\psi_j^n\rangle \langle \psi_j^n|$ is the optimal pure state decomposition for each $\rho_n$. Then $\rho = \sum_i p_i \rho_i$ has a pure state decomposition $\sum_i \rho_i q_i^n |\psi_j^n\rangle \langle \psi_j^n|$, and

$$C_{ca}^{(\alpha)}(\rho) = C_{ca}^{(\alpha)} \left( \sum_i p_i \rho_i \rangle \langle \rho_i \right) \leq \sum_i p_i C_{ca}^{(\alpha)}(\rho_i) \leq \sum_i p_i C_{ca}^{(\alpha)}(\rho_i).$$

The monotonicity (C2) of $C_{ca}^{(\alpha)}$ can be obtained from strong monotonicity (C3) and convexity (C4).

Appendix B: $A^{(\alpha)}$ is bounded

Proposition 8. $0 \leq A^{(\alpha)}(\rho, \sigma) \leq 1$, with $A^{(\alpha)}(\rho, \sigma) = 1$ if and only if $\rho = \sigma$.

Proof. As $\rho^{1/2} \sigma^{1-\alpha} \rho^{1/2}$ is a positive matrix, one has

$$\text{Tr}(\rho^{\alpha} \sigma^{1-\alpha} \rho^{1/2}) = \text{Tr}(\rho^{1/2} \sigma^{1-\alpha} \rho^{1/2}) \geq 0.$$

The other part can be proved as in Ref. [63]. Let $\{|x\rangle\}_x$ be a basis of $\mathcal{H}$, then $M_x = \{|x\rangle \langle x|\}$ is an informationally-complete measurement. Denoting $\Phi(\rho) = \sum_x \langle x | \rho | x \rangle |x\rangle \langle x|$, we have from the monotonicity of $A^{(\alpha)}(\rho, \sigma)$ and Jensen’s inequality

$$A^{(\alpha)}(\rho, \sigma) \leq A^{(\alpha)}(\Phi(\rho), \Phi(\sigma)) \leq \sum_x \left( \frac{\langle x | \rho | x \rangle}{\langle x | \sigma | x \rangle} \right)^\alpha \langle x | \sigma | x \rangle \leq \left( \sum_x \langle x | \rho | x \rangle \right)^\alpha = 1.$$
As the equality holds iff $\langle x | \rho | x \rangle = \langle x | \sigma | x \rangle$ for any informationally-complete measurement, one has $A^{(\alpha)}(\rho, \sigma) = 1$ if and only if $\rho = \sigma$. □

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