2-Local derivations of real AW*-algebras are derivation

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Abstract
2-Local derivations on real matrix algebras over unital semi-prime Banach algebras are considered. Using the real analogue of the result that any 2-local derivation on the algebra $M_{2^n}(A)$ ($n \geq 2$) is a derivation, it is shown that any 2-local derivation on real AW*-algebra for which the enveloping algebra is (complex) AW*-algebra, is a derivation, where $A$ is a unital semi-prime Banach algebra with the inner derivation property.

Keywords Matrix algebra · Real AW*-algebra · Derivation · Inner derivation property · 2-local derivation

Mathematics Subject Classification 46L10, 46L37, 46L57, 47B47, 47C15

1 Introduction

Given an algebra $\mathcal{A}$, a linear operator $D : \mathcal{A} \to \mathcal{A}$ is called a derivation, if $D(xy) = D(x)y + xD(y)$, for all $x, y \in \mathcal{A}$. Each element $a \in \mathcal{A}$ implements a derivation $D_a$ on $\mathcal{A}$ defined as $D_a(x) = [a, x] = ax - xa, x \in \mathcal{A}$. Such derivations are said to be inner derivations. A map $\Delta : \mathcal{A} \to \mathcal{A}$ (not linear in general) is called a 2-local derivation, if for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

In the paper [1] P. Semrl introduced the notion of 2-local derivations and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional Hilbert space $H$. This was extended by P. Semrl and P. Šemrl in [2] to derivations on real algebras and to multilinear derivations on any real algebra.

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dimensional separable (complex) Hilbert space $H$. A similar description for the finite-dimensional case appeared later in [2]. In the papers [3–5] and [6] the authors extended the Semrl’s result for arbitrary finite, semi-finite and purely infinite von Neumann algebras, respectively. The real analogue of Semrl’s result is received in the paper [7], i.e. it is described 2-local derivations on the real $W^*$-algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable real Hilbert space $H$.

In the paper [8] the authors investigated 2-local derivations on matrix algebras over unital semi-prime Banach algebras. For a unital semi-prime Banach algebra $A$ with the inner derivation property it is proved that any 2-local derivation on the algebra $M_{2^n}(A)$, $n \geq 2$, is a derivation. They apply this result to $AW^*$-algebras and it is showed that any 2-local derivation on an arbitrary (complex) $AW^*$-algebra is a derivation. In the present paper step by step we will prove analogous result for real $AW^*$-algebras.

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2 Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space $H$. A weakly closed *-subalgebra $M$ containing the identity operator $1$ in $B(H)$ is called a $W^*$-algebra. A real *-subalgebra $R \subset B(H)$ is called a real $W^*$-algebra if it is closed in the weak operator topology, $1 \in R$ and $R \cap iR = \{0\}$. A real $W^*$-algebra $R$ is called a real factor if its center $Z(R)$ consists of the elements $\{\lambda 1, \lambda \in \mathbb{R}\}$. We say that a real $W^*$-algebra $R$ is of the type $I_{fin}, I_\infty, II_1,$ or $III_\lambda$, $(0 \leq \lambda \leq 1)$ if the enveloping $W^*$-algebra $R + iR$ has the corresponding type in the ordinary classification of $W^*$-algebras.

We say that an algebra $A$ has the inner derivation property if every derivation on $A$ is inner. Recall that an algebra $A$ is said to be semi-prime if $aAa = 0$ implies that $a = 0$.

In work [8] (Theorem 2.1) it is proved, that if $A$ is a unital semi-prime (complex) Banach algebra with the inner derivation property and $M_{2^n}(A)$ is the algebra of $2^n \times 2^n$-matrices over $A$, then any 2-local derivation on $M_{2^n}(A)$ is a derivation.

The proof of this theorem without changes to pass for real Banach algebras. Therefore we shall formulate this result in a real case.

**Theorem 1** Let $A$ be a unital semi-prime real Banach algebra with the inner derivation property and let $M_{2^n}(A)$ is the real algebra of $2^n \times 2^n$-matrices over $A$. Then any 2-local derivation on $M_{2^n}(A)$ is a derivation.

We apply Theorem 1 to the description of 2-local derivations on real $AW^*$-algebras.

3 2-Local derivations on real $AW^*$-algebras

Firstly, we shall remind some definitions and the facts from the theory of complex and real $AW^*$-algebras. Let $A$ be a real or complex *-algebra and let $S$ be a nonempty subset of $A$. Put
\[ R(S) = \{ x \in A \mid sx = 0 \text{ for all } s \in S \} \]

and call \( R(S) \) the right-annihilator of \( S \). Similarly

\[ L(S) = \{ x \in A \mid xs = 0 \text{ for all } s \in S \} \]

denotes the left-annihilator of \( S \). Following [12] we introduce the following notions.

**Definition 1** A \(*\)-algebra \( A \) is called a Baer *-algebra if for any nonempty \( S \subset A \), \( R(S) = gA \) for an appropriate projection \( g \).

Since \( L(S) = (R(S^*))^* = (hA)^* = Ah \) the definition is symmetric and can be given in terms of the left-annihilator and a suitable projection \( h \). Here \( S^* = \{ s^* \mid s \in S \} \).

In the particular case, where we consider only one point sets \( S = \{ x \}, \ x \in A \), we obtain the more general definition of a Rickart *-algebra. It is known that a Rickart *-algebra is a Baer *-algebra if and only if its projections form a complete lattice or every orthogonal family of projections has a supremum (i.e. a least upper bound).

Let \( A \) be a Banach *-algebra over the field \( \mathbb{C} \). The algebra \( A \) is called a C*-algebra, if \( \|aa^*\| = \|a\|^2 \) for any \( a \in A \). A real Banach *-algebra \( R \) is called a real C*-algebra, if \( \|aa^*\| = \|a\|^2 \) and an element \( \mathbf{1} + aa^* \) is invertible for any \( a \in R \). It is easy to see that \( R \) is a real C*-algebra if and only if a norm on \( R \) can be extended onto the complexification \( A = R + iR \) of the algebra \( R \) so that algebra \( A \) is a C*-algebra.

**Definition 2** A (complex) C*-algebra \( M \) which is a Baer *-algebra is called an AW*-algebra.

This definition is equivalent to the definition given by Kaplansky [9], namely a C*-algebra is an AW*-algebra if and only if it satisfies the following conditions:

(A) In the partially ordered set of projections, any set of orthogonal projections has a supremum;  
(B) Any maximal abelian *-subalgebra is generated by its projections.

Every W*-algebra is, of course, an AW*-algebra, however, the converse is not true as it was shown by Dixmier [10]. Given an AW*-algebra \( M \), its center is \( Z_M = \{ x \in M \mid xy = yx \text{ for all } y \in M \} \). An AW*-algebra is called an AW*-factor, if its center consists of complex multiples of the identity \( \mathbf{1} \), i.e. \( Z_M = \{ \lambda \mathbf{1} \mid \lambda \in \mathbb{C} \} \).

Now following Kaplansky [11, Appendix III] we introduce the main subject of the paper.

**Definition 3** A real C*-algebra which is a Baer *-ring is called a real AW*-algebra.

It is clear that any real AW*-algebra contains an identity \( \mathbf{1} \), and we say that a real AW*-algebra \( A \) is a real AW*-factor if its center consists of real multiples of \( \mathbf{1} \).

**Remark 1** 1. Unlike the complex case in the real case it is not possible to give a definition in terms of conditions (A) and (B) above, because in maximal abelian *-subalgebras skew-hermitian elements can not be generated by projection.
2. A slightly more general notion of real AW*-algebras was given also by Berberian [12, p. 26, Exercise 14A]. He defined a real AW*-algebra as a Banach *-algebra over the field of real numbers such that \( \|x^*x\| = \|x\|^2 \) for all \( x \in A \) and such that \( A \) is a Baer *-ring. In this case the field \( \mathbb{C} \) of complex numbers with the identical involution \( x^* = x \) becomes a real AW*-algebra, but it is not a real C*-algebra, because it is not a symmetric *-algebra, which means \( I + x^*x \) is invertible for any \( x \in A \).

Any real W*-algebra (real W*-factor) is a real AW*-algebra (resp. a real AW*-factor). But the converse is not true. Any complex AW*-algebra is a real AW*-algebra. Complex AW*-factors are not real AW*-factors, because their centers are complex multiples of \( I \).

For real C*-algebras and W*-algebras we know that their complexification are C*- and W*-algebras respectively. But in AW*-algebras case we have: there is a real AW*-algebra \( R \) such that the complex C*-algebra \( R + iR \) is not an AW*-algebra (see [13, Proposition 4.2.3]).

Now, we shall prove the main result of paper.

**Theorem 2** Let \( R \) be an arbitrary real AW*-algebra and suppose that its complexification \( M = R + iR \) is a (complex) AW*-algebra. Then any 2-local derivation \( \Delta \) on \( R \) is a derivation.

**Proof** Let us first note that any (complex) AW*-algebra is semi-prime, and it is clear that a real algebra \( A \) is semi-prime if and only if its complexification \( A + iA \) is semi-prime. Therefore, any real AW*-algebra is also semi-prime. It is also known [14, Theorem 2] that AW*-algebra has the inner derivation property.

It is easy to shown that any real AW*-algebra has also the inner derivation property, i.e. every derivation of real AW*-algebra is an inner. Indeed, let \( R \) be a real AW*-algebra and let \( D : R \to R \) be a derivation. \( D \) can be extended by the linearity to a derivation on \( M = R + iR \) as \( \overline{D}(x + iy) = D(x) + iD(y) \). Since \( \overline{D} \) is an inner there is an element \( z = a + ib \) \((a, b \in R)\) such that \( \overline{D}(x + iy) = [z, x + iy] \) for all \( x, y \in R \). Hence \( D(x) = \overline{D}(x) = [z, x] = [a, x] + i[b, x] \). Therefore from \( D(x) \in R \) we have \( b = 0 \), i.e. \( z = a \in R \). Thus \( D(x) = [a, x] \).

Now, let \( z \) be a central projection in \( R \). Then \( z \) is a central projection in \( M \). It is known that \( \overline{D}(z) = 0 \), and therefore \( D(z) = 0 \). Then it is easy to see that \( \Delta(z) = 0 \) for any 2-local derivation \( \Delta \) on \( R \). For \( x \in R \) we consider the elements \( x \) and \( zx \). Then there is a derivation \( D \) on \( R \) such that \( \Delta(zx) = D(zx) \) and \( \Delta(x) = D(x) \). Then we have

\[
\Delta(zx) = D(zx) = D(z)x + zD(x) = zD(x) = z\Delta(x).
\]

It means that every 2-local derivation \( \Delta \) maps \( zR \) into \( zR \) for each central projection \( z \in R \). Thus we may consider the restriction of \( \Delta \) onto \( eR \). By [13, Proposition 4.4.3] an arbitrary real AW*-algebra can be decomposed along a central projection into the direct sum of an abelian real AW*-algebra, and real AW*-algebras of type I_\( n \), \( n \geq 2 \), type I_\( \infty \), type II and type III. We will consider these cases separately.
Let $R$ be an abelian real $AW^*$-algebra. It is well-known that any derivation on an abelian (complex) $W^*$-algebra $R + iR$ is identically zero. Therefore, the derivation $D(x + iy) = D(x) + iD(y)$ on $R + iR$ is identically zero, where $D$ is a derivation on $R$. Hence $D$ is identically zero, i.e. any 2-local derivation on an abelian $AW^*$-algebra is also identically zero.

If $R$ is a real $AW^*$-algebra of type $I_n$, $n \geq 2$, with the center $Z(R)$, then it is isomorphic to the algebra $M_n(Z(R))$. By Lemma 2.3 [8] (as it is already told above, that the proof of theorem 2.1 and Lemmas from [8] without changes to pass for real Banach algebras) there exists a derivation $D$ on $R \equiv M_n(Z(R))$ such that $D = D$. So, $D$ is a derivation.

Let the real $AW^*$-algebra $R$ have one of the types $I_\infty$, $II$ or $III$. Then using the methods developed in [13, §§4.3-4.7] and similarly following the scheme of the proof of Lemmas 4.5 and 4.12 in [9], the algebra $R$ can be represented as a sum of mutually equivalent orthogonal projections $e_1, e_2, e_3, e_4$ from $R$. Then the map $x \to \sum_{i, j=1}^4 e_i x e_j$ defines an isomorphism between the algebra $R$ and the matrix algebra $M_4(Q)$, where $Q = e_{1,1} Re_{1,1}$. It is easy to see that $Q$ and $Q + iQ$ are real and complex $C^*$-algebras, respectively. Since $M = R + iR$ is $AW^*$-algebra by [12, Proposition 8 (iii), 23p.] $C^*$-algebra $eMe$ is also $AW^*$-algebra, where $e$ is an arbitrary projection in $M$. Then by [13, Proposition 4.3.1] a real $C^*$-algebra $Q = e_{1,1} Re_{1,1}$ is a real $AW^*$-algebra and its complexification $Q + iQ = e_{1,1} Me_{1,1}$ is also a (complex) $AW^*$-algebra. Therefore $Q$ is a unital semi-prime real Banach algebra with the inner derivation property. Hence Theorem 1 implies that any 2-local derivation on $R$ is a derivation. $\square$

**Remark 2** Everywhere in the work we considered a real $AW^*$-algebra with a (complex) $AW^*$-algebra of its complexification. Moreover, in the definition of real $C^*$-algebra the condition of convertibility of an element $I + xx^*$ (for all $x$) is required. It is equivalent to that a norm on real $C^*$-algebra can be extended onto its complexification so that it is a (complex) $C^*$-algebra. But in the [12, Exercise 14A] in definition of real $C^*$-algebra convertibility of $I + xx^*$ is not required. In this connection we shall formulate following questions.

**Question** let $R$ be a real Baer $*$-ring. Suppose that

(i) $R$ is a real Banach $*$-algebra with $\|xx^*\| = \|x\|^2$, for any $x \in R$, or

(ii) $R$ is a real $AW^*$-algebra (not necessary its complexification is a (complex) $AW^*$-algebra). Then

(1) is any derivation of $R$ is inner?

(2) is any 2-local derivation on $R$ is derivation?

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