Glassy states of aging social networks

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Abstract

Individuals often develop reluctance to change their social relations, called "secondary homebody", even though their interactions with their environment evolve with time. Some memory effect is loosely present deforcing changes. In other words, in presence of memory, relations do not change easily. In order to investigate some history or memory effect on social networks, we introduce a temporal kernel function into the Heider conventional balance theory, allowing for the "quality" of past relations to contribute to the evolution of the system. This memory effect is shown to lead to the emergence of aged networks, thereby perfectly describing and the more so measuring the aging process of links ("social relations"). It is shown that such a memory does not change the dynamical attractors of the system, but does prolong the time necessary to reach the "balanced states". The general trend goes toward obtaining either global ("paradise" or "bipolar") or local ("jammed") balanced states, but is profoundly affected by aged relations. The resistance of elder links against changes decelerates the evolution of the system and traps it into so named glassy states. In contrast to balance configurations which live on stable states, such long lived glassy states can survive in unstable states.

Introduction

"Yesterdays’ friend (enemy) rarely become tomorrows’ enemy (friend)."

Tension reduction is a predominant principle that contributes to the formation of human interactions \cite{1}. This principle acts as a self-organizing process; it indicates that social communications are established based on the tendency towards balanced states \cite{2,3}. Interesting questions that follow concern what "parameters" have a pivotal role in the social network dynamics. An appropriate answer seems to lie in the history of relationships. The ability of human beings to remember sequences of events (sometimes unconsciously) over the time brings about social concepts, such as commitment and allegiance that lead to the formation of cultural communities, sects, alliances, and political groups \cite{4,5,6,7}. In psychological terms, the more potent the commitment is, the more probable the relations will remain unchanged over time. There are persons, such as family members, friends (or enemies), even business partners, with
In terms of social networks dynamics, links can carry various information such as type, age and strength of relations. This figure illustrates a network with two types of relations (for instance friendship and animosity) which are denoted by solid and dash lines respectively. Here colors display the gradient of age from young (red) to old (blue); the weight (strength) of links are represented by the line thickness. Increasing the age or weight of links can lead to decreasing the tendency towards modifying relationships.

whom most people have no inclination to modify the nature of their relationships. This resistance to change can be explained by the history of relations and their importance which can be referred to two distinct parameters "age of links" and "weight of links", respectively. Generally, depending on age or weight, breaking up or modifying the links among individuals gradually can become difficult.

In this study, the main intention is to explore the effect of relationships history following the hypothesis that newly formed relations have more chance to change than older ones; thus old relations are more resistant. Practically, we aim at mimicking the strength of relationship links through a model considering a combination of emotional intensity and life time duration of relations, but keeping these "parameters" as independent of each other as proposed by Granovetter \[9\]. This leads to links which contain both aspects, - as depicted on Fig. 1 through different thicknesses of lines (for the strengths) and different colors (for the ages).

Therefore, in quantitative terms and in view of modeling the dynamics of social networks, it is useful to develop some study in order to comprehend how history (or memory) has global consequences on the evolution of a social system. Thanks to technological advances, allowing collection and analysis of empirical data about human interactions on social networks, it is nowadays easily revealed that social activities act as sequences of correlated events in which each event or decision depends on previous ones and on the "node environment". In this context, much effort has been already devoted to describe the evolution of in- and out-cluster relationships over time \[10\]-\[14\]. These endeavors have led to uncover the heterogeneous dynamics to be a consequence of memory, e.g. in cellphone users \[15\], face to face contacts \[16\], pattern of rumor spreading process \[17\]-\[18\], individuals’ web browsing pattern \[19\], Boolean networks \[20\] or optimized strategy in Prisoner’s Dilemma games \[21\]-\[22\].

In order to introduce the influence of memory on social networks, we have followed the Heider balance theory \[1\] that has provided a fundamental platform for tackling many sociological problems ranging from social systems \[23\]-\[26\] to political ones \[27\], and e.g. online networks \[28\]-\[32\] (see \[33\] and references therein). In this theory, relations between agents are considered as positive or negative links: the positive sign for a link indicate for instance; friendship \[28\]-\[29\], attractiveness \[30\], profit \[31\], tolerance \[23\], whereas a negative sign would indicate the opposite. This theory proposes a model based on triadic configurations in which relations evolve in order to reduce the number of unbalanced triads and in view of attaining "minimum tension states". Assigning a potential energy to these systems allows a more quantitative view of the landscape of the network’s dynamic over time \[34\].

Let us denote by $s_{ij}$, the link sign, corresponding to a feeling of friendship or of animosity between the nodes $i$ and $j$. Let the social network potential energy be calculated by summing all products of links in
all triads, normalized by the total number of triangles. From this energy point of view, minimum tension states can be distributed between global minima, namely "balanced states", and local minima, named "jammed states" \[34-36\]. In presence of memory, agent relations (friendships and animosities) build up and can become vigorous over some long time. This age gain for links decreases the probability of relationship changes \[37\]. Gallos et al. \[38\] explored the effect of nodes’ attributes (e.g. age, genders) on the formation of specific configurations in real social networks. They observed that younger individuals have less triadic closure in their relations (friends of friends tie), while raising age increases this propensity. In the spirit of scaling ideas, based on the nature of memory features in real systems \[39-41\], we mimic this aging process through a kernel function which emphasizes that the strength of the relationships increases like a power law, from past to present. In so doing, we generalize the order of the derivative in the conventional continuous equation of balance theory \[42-45\] in physics to a non-integer order. The results presented here below reveal the emergence of some novel states, called "glassy states" that refer to conditions in which, for a reasonably long time interval, there is no propensity to reduce the amount of stress within the system in favor of equilibrium.

It can be informative to compare the notions of jammed and glassy states. Although both of these states are not global minimum states, they have distinct characteristic that make them distinguishable. According to the tension reduction condition, complex networks tend to move toward lower energy levels and reach the global minimum states (paradise of bipolar). During this evolution a system may be trapped in some local minimum states, namely jammed states, for a while. In such a situation there is no possibility to reduce tension (energy) within a system by altering links. It has been shown that jammed states are rare events in networks dynamics and occur with negative energies. In contrast, in glassy states agents resist to modify the quality of their relationships in order to reduce the stress which preserves the system in unstable states. Glassy states are established by a combination of aged and young links. Here, aged links refer to those that have not changed for a long time, while young links change quickly without resistance. Although there exist some links in glassy states, if changed, can guide the system toward lower energy states, - thanks to their age the probability of change is low. It will be shown that the energy of glassy states can be negative or even positive; due to the memory intensity, they can be the only final states of the networks dynamical evolution.

It will be further explained that the present study considers an endogenous condition for a link switch. Exogenous shocks, or threshold of awareness \[46\], are outside the present study. So we imply a difference between simultaneous or sequential updating \[47\]. Glassy states, in condensed matter, are indeed subjects to frustration considerations and specific phase transition patterns.

The evolving network

The dynamics of social networks is a sort of self-organized process in which relationships are modified based upon agents’ benefits. These modifications occur at a microscopic level; they impose a collective behavior that guides the network along particular structural evolution paths. In this context, Heider \[1\] proposed a "balance theory" based on triadic relations among two persons and their "attitude" towards an issue (as the third node) where connections are rearranged to become stable \[48\]. According to this description, relations among individuals can be categorized into two classes based on balanced and unbalanced triadic relations. Tension, in a unbalanced triad, stimulates agents to modify their current relations to reach some balance. Later, by putting the concept of "attitude" in the background and substituting an individual as the third node, Cartwright et al. \[49\] generalized Heider’s approach in terms of "sign graphs", in which the links among members can be \(\pm 1\) (without any weight). Intuitively, a positive value denotes friendship, corporation, or tolerance, to name but a few, while a negative value depicts animosity, rivalry or intolerance in social groups \[28-31\]. According to the balance theory, a triadic relation among any three agents is balanced (unbalanced) only if the product of signs assigned to the links is positive (negative). Obviously, a network is "balanced" if all triangles are balanced.
Study on the dynamics of such sign networks, from the viewpoint of balance theory, can be traced back to Antal et al. [35]. They proposed a model based on the evolution of links for fully connected networks, namely Constrained Triadic Dynamics (CTD), which could describe how an initially imbalanced society attains a balanced state [35, 50].

\[
\frac{d}{dt} X_{ki}(t) \equiv \sum_{j=1}^{n} X_{kj}(t)X_{ji}(t).
\]  

(1)

Having this CTD model in hands, to investigate the memory effects on the evolution of the system, we have considered the following assumption: although changes in a relationship (link) would guide the system toward a minimum tension state, agents have less tendency to alter the nature of their connections, due to some memory of pertinent relations. This approach results in the concept of aged networks, where the history of links is associated with their ages. In previous studies, there was no discrepancy between age (duration) and strength of links in terms of weighted networks [9, 51–53].

Many studies, e.g. [54–57], have considered fractional integrals or derivatives as a generalization of ordinary differential-integral operators to non-integer ones, in view of describing the effects of past events on the present one. Such a fractional calculus approach can also be taken for the conventional balance differential equation in order to explore these effects on the social interactions.

\[
\frac{c}{t_0} D_\alpha^\alpha X_{ki}(t) \equiv \sum_{j=1}^{n} X_{kj}(t)X_{ji}(t).
\]  

(2)

The left hand side of Eq. (2) is the Caputo [58] fractional differential operator of order \( \alpha \), where \( 0 < \alpha < 1 \). This \( \alpha \) "the fractional order of derivation" denotes the significance of the memory in the interaction mechanisms: i.e., \( \alpha = 1 \) refers to the balance theory master equation with no memory [43–45].

Eq. (2) can be rewritten in the form of its equivalent Volterra integral [59, 60],

\[
X_{ki} = X_{ki0} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} dt' (t-t')^{(\alpha-1)} \left[ \sum_{l=1}^{n} X_{kl}(t')X_{li}(t') \right].
\]  

(3)

This functional form with a scale invariance time dependent kernel makes it possible to consider historical effects. Indeed, due to the "non-local" (more exactly, "time lag dependent") nature of these operators, past events play a "non-Markovian" role on the system dynamics. This mathematical technique practically means that those past events which are "further away" from the present time, are those which less likely contribute to the current state, - unless those incidents hold high levels of importance.

To solve the integral of Eq. (3) numerically, the predictor-corrector algorithm is employed [60, 62]. In this way, the product rectangle rule [60] is used, in which the time domain is divided in an equispaced grid \( t_j = t_0 + hj \) with equal space \( h \). The right hand side of Eq. (3) is approximated on this grid, such that Eq. (3) becomes

\[
X_{ki} = X_{ki0} + h^\alpha \sum_{j=0}^{n-1} b_{n-j-1} \sum_{l=1}^{n} \langle X_{kl}X_{li} \rangle_j .
\]  

(4)

In so doing, the \( b_n = \frac{(n+1)^\alpha - (n)^\alpha}{\Gamma(\alpha+1)} \) coefficients are the essential terms which indicate and control the role of the past events in the model.

For the simulation part of this study, a fully connected network including \( N \) nodes where all agents are acquainted with each other is first considered. Here a link between two agents \( i \) and \( j \) is represented by \( s_{ij} \) with initial value \(+1\) (\(-1\)) which denotes friendship (animosity). We may start from a fully antagonistic network, where a link between any two agents has \(-1\) for initial value. In order to check the evolution of the network at each time step, any link must fulfill two conditions. The first condition: a link is selected randomly; switching the sign of the link is made permissible, if only the total number of
unbalance triangles is reduced. This description is accordance with the reduction of the total energy of the system [34],

\[ U = -\frac{1}{3} \sum_{i,j,k} s_{ij}s_{jk}s_{ki}, \]  

(5)

where the sum is over all triadic relations. \( U \) can accept values from 1 (antagonistic configuration) to \(-1\) (balance configuration).

The second condition: a competition occurs between the tendency of a link to reduce energy and the insistence on maintaining the past relation due to the memory effect. This competitive process identifies whether the selected link (which should fulfill the condition of the first step) tends to switch into the opposite sign. In this respect, a random number is generated from a normal distribution over \((0, 1)\): when this number is less than \( A_{ij}^{\alpha - 1} \), a switch into the opposite sign \((\pm 1)\) is made for the selected link \((s_{ij})\). The magnitude of \( A_{ij} \) is equivalent to somewhat measured by the age of the link and controls the lapse of time (“resting time”) during which agents do not alter their relations. Here, \( \alpha \) exhibits the rate of memory effects on relations and its value changes over \((0, 1)\). Obviously, the lower the magnitude of \( A_{ij} \) or the higher the value of \( \alpha \) is, the more probable a link will change sign.

In the aging process, for any \( N \) (the number of nodes in the system) steps, the links which do not change sign, their age \((A_{ij})\) increases by one unit. According to this dynamical evolution, a system in various paths towards minimum tension states remains unchanged within some time intervals. When the system remains so static for time intervals of the order of the number of links (i.e., in order to give the same probability for each link to be chosen), we call those states glassy states.

**Results and discussions**

The dynamics of a network, in Heider balance theory, is a sort of Markovian process in which there is no evidence of the past incidents (memory) in link rearrangements, whence the evolution of relationships is entirely stochastic. Thus, in such a basic theory, when a person decides to modify a connection, apart from what happened in the past, she or he only checks the quality of a specific relation at the present moment. In terms of social networks, links carry various information such as type, age and strength of links. Age denotes the duration of a relations and strength of a link is described by its weight. Fig. 1 displays type, age and weight of links through solid (dashed) lines, colored lines and thick (thin) lines respectively for 9 nodes which are part of a larger network. Since age of links indicates the effect of past relationships on current decisions, it can be interpreted that this parameter (age) is responsible for the memory of relationships. According to recent studies [10–13], memory imposes a scaling (increasing power law) behavior onto a system in such a way that the older the relations get, the more significant role they perform for the network destiny. Converting the conventional continuous time equation of balance theory [63,64] to the fractional form, Eq. (1), appropriately allows us to include and describe a memory effect over time. Thus, the interactions among individuals become time dependent and the system experiences a non-Markovian process in an endogenous way. Notice that considering the fractional space only rescales the evolution time without any impact on the phase space and the dynamics of the system. However, changes are slowing down; in other words, time intervals between changes appear to become longer; the system finally attains a balanced state later than if it is a memory-less system.

In the investigation of networks dynamics based on the strict CTD model, Antal et al. [35] demonstrated that systems may follow various paths before obtaining the network (global) minimum tension, in a finite time interval. These systems mostly move towards final balanced states (either paradise or bipolar), where all triangles are balanced. A ”paradise state” refers to a system in which all members are friendly with each other, while a ”bipolar state” refers to a system which consists of two main clusters, in which the members of each group are friendly with each other, but not friendly with any member of
Figure 2. The maximum time interval during which a network resists against any change (long lived states). The left panel depicts this time interval for networks with 21 nodes in various paths (10000 realizations). The right panel illustrates that the maximum time intervals follow a Poisson distribution.

the other group. Antal et al. [35] also showed that there are a few "rare" states named as "jammed states" (local minima) in which a system is divided into several (≠ 2) communities.

According to the present model, i.e., when introducing memory effects into the CTD model, one even describes the formation of aged (and aging) networks. Hence, over the course of time, depending on the value of $\alpha$ which indicates the significance of relations, links can gain age with probability $A_{ij}^{(\alpha-1)}$. It can be concluded that the larger $\alpha$ is, the less relevant are effects of past events on the individual's decision. For $\alpha = 1$, the memory effect completely disappears and the current model is reduced to the CTD model. In the presence of memory, even though the least tension principle is enforced, the network, before obtaining either a global or a local equilibrium, can be trapped into intermediate states during several time intervals. This practically means that people do not forget their long lasting friendships or animosities readily; in other words, the system can resist toward changes over time. Fig. 2 illustrates the most prolonged periods in various paths that the system remains unchanged before reaching its balanced state, in the case of a network with 21 nodes for $\alpha = 0.7$, after 10000 realizations. These time intervals are found to follow a Poisson distribution function $p(x) = \frac{1}{k!} e^{-\lambda} \lambda^k$, where $\lambda$ controls the expected frequency with which an event can occur. Such a Poisson distribution demonstrates that long time intervals become probable; this deviation from the normal distribution leads to the emergence of inhomogeneity in the time intervals during which the system remains unchanged. Here, it is found that $\lambda$ has values 83.640, 13.157 for $\alpha = 0.5$ and 0.7 respectively. For larger values of $\alpha$, $\lambda$ tends to zero. In such a situation, as $\alpha$ decreases, the time intervals in which the system shows no inclination towards changes increases.

Fig. 3 illustrates such a situation: the energy of final states, Eq. (5), versus the mean value of positive and negative links for a 45 node network at different $\alpha$ values, with 10000 realizations. It can be seen that for $\alpha = 0.3$ (left top panel) for all realizations, the system is trapped in glassy states, before it reaches its balanced states. Interestingly, however balanced and jammed states always happen in negative energies; however, here some of glassy states occur at positive energies which imply an intrinsic instability within the system. In such cases, the system "tolerates" some high tension condition. However, no agent tends
Figure 3. Energy of final states (Glassy, Balance and Jammed states) versus mean of positive and negative links. We show these final states for networks with 45 nodes at different $\alpha$ (strength of memory) values. With increasing $\alpha$ values, the system becomes more flexible against changes and the probability of reaching balance and jammed states increases. The right bottom panel shows a sudden ”phase transition” in the percentage (density) of glassy states to balanced states about $\alpha \sim 0.6$.

Conclusions

The memory footprint is evident in most complex systems, where life exits, from biological to economy and social systems. Thanks to the presence of memory, when we have a friendship (animosity), this relationship can get vigorous over time and the propensity of change may diminish; in other words, breaking up of aged friendship (animosity) gets difficult. Here, introducing a kernel function into the
differential equation of balance theory, we provide a better insight toward considering the contribution of past events on the dynamics of a social system. We have mimicked the aging process through a kernel function which emphasizes that the disinclination of relationships to change, increases as a power law from past to present. This function leads to a fractional differential equation of order $\alpha$ where $\alpha - 1$ indicates the strength of the memory effect. In other words, at high $\alpha$, links have a high probability to change and the system is more flexible toward social tension reduction.

We have investigated the feasible paths and final states that a (fully connected) network can experience, using the Constrained Triadic Dynamics (CTD) model with memory. We have employed a probability of links’ disinclination to change as a $A_{ij}^{(\alpha-1)}$ function. We have found out that memory is a key factor indeed, which sometimes withstands against the quick evolution of the network and eventually preserves the system in unstable, but long lived states, namely glassy states. Under such circumstances, for various time intervals, the system has no tendency to evolve towards global or local minima. Evidence, depending on the value of the strength parameter $\alpha$, indicate that a system is enabled to reach a lower tension and energy, in non trivial path ways, visiting glassy states. In such situations, individuals become dormant or accustomed to the current state of the network. In contrast to jammed states, i.e. local minimum states, in which systems only experience negative energies, glassy states can occur in positive energy states, thereby imposing instability to and keeping stress in the system.

In conclusion, the natural concept of aging of agent based networks provides a powerful tool to investigate the dynamical behavior of memory prone social agents, but is also surely valid for other realistic complex systems. In real technological networks, several links have necessarily a temporary lifetime and fleeting; they can develop into new relations (or functions) due to some feedback, automatic or external control, or more often reinforcement learning by memory effect over time. Therefore the concept of aged links might be a suitable language to study a wide variety of temporal networks, e.g. ranging from trading between companies, friendly and hostile relationships, scientific collaborations, political influences through cooperation or competition, to tweeting news in social media, beside strictly psychological aspects.

Acknowledgment

We would like to express our gratitude to Prof. Krzysztof Kulakowski for reading the paper and for his constructive comments. The research of GRJ was supported by the Higher Education Support Program of OSF and the Central European University.

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