Primordial magnetic seed field amplification by gravitational waves: comment on gr-qc/0503006

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Abstract
We consider the amplification of cosmological magnetic fields by gravitational waves as it was recently presented in [1]. That study confined to infinitely conductive environments, arguing that on spatially flat Friedmann backgrounds the gravito-magnetic interaction proceeds always as if the universe were a perfect conductor. We explain why this claim is not correct and then re-examine the Maxwell-Weyl coupling at the limit of ideal magnetohydrodynamics. We find that the scales of the main results of [1] were not properly assessed and that the incorrect scale assessment has compromised both the physical and the numerical results of the paper. This comment aims to clarify these issues on the one hand, while on the other it takes a closer look at the gauge-invariance and the nonlinearity of [1].

1 Introduction

The interaction between electromagnetic fields and gravitational waves and the possible energy transfer between the Weyl and the Maxwell fields have a long research history. A mechanism for the amplification of large-scale magnetic fields by gravity waves of similar size soon after inflation was recently proposed in [2, 3]. In the poorly conductive environment of early reheating the analysis indicated a resonant magnetic amplification proportional to the square of the field’s scale (see [3] for details). This meant that Weyl-curvature distortions could provide a very efficient early-universe dynamo of superhorizon-sized magnetic fields. For example, fields with a current comoving scale of approximately 10 kpc and a strength of $\sim 10^{-34}$ G, like those produced in [4], could be amplified by many orders of magnitude by the end of reheating.

The same gravito-magnetic interaction has been applied to infinitely conductive cosmologies in [1]. Central to that study is the claim that on spatially flat Friedmann-Robertson-Walker (FRW) backgrounds the Maxwell-Weyl coupling proceeds always as if the universe were a perfect conductor. We argue that this is not the case and explain why the above mentioned paper arrived at the opposite result. In addition, we find serious problems in the scale assessments of [1]. These have prevented the authors from recognising the main effect of the interaction, namely the large-scale superadiabatic amplification of the $B$-field, and compromised their results. With the present comment we draw attention to these issues by correcting the mathematics, where necessary, and by clarifying the physics. We also re-examine and question the gauge-invariance and the nonlinearity of the formalism proposed in [1].
2 On the electric curl and the role of conductivity

Section III.B of [1] argues that the Maxwell-Weyl coupling proceeds unaffected by the conductivity of the medium and as if the universe were a perfect conductor. This follows the claim that the gravitationally induced, second order, electric field \( E_a \) is curl-free if the zero order FRW model has flat spatial sections. This is not the case, however, because

\[
\text{curl}E_a = -\Theta \text{curl}E_a + \mathcal{R}_{ab} \dot{B}^b - \text{curl} \mathcal{J}_a - D^2 B_a ,
\]

at second order. Here \( \Theta \) is the volume expansion, \( \mathcal{R}_{ab} \) is the first-order 3-Ricci tensor, \( \dot{B}_a \) is the first order magnetic field, \( \mathcal{J}_a \) is the spatial current and \( D^2 = D_a D^a \) is the 3-dimensional Laplacian operator. Therefore, even when \( \text{curl}E_a \) initially vanishes, there are sources in the right-hand side of (1) that will generally lead to a nonzero electric curl. Switching the latter off is not a consistent constraint. Note that the current term depends on the conductivity of the medium, while the 3-curvature distortions and the magnetic field fluctuations are caused by the presence of gravity-wave perturbations. These set the field lines into motion causing magnetic fluctuations that produce an electric component. The latter has \( \text{curl}E_a \neq 0 \) because of (1). This means that the gravito-magnetic interaction does not always proceed as if the conductivity of the universe were infinite. In [1], the authors arrived at the opposite conclusion because they considered the second time-derivative of \( \text{curl}E_a \) instead of the first, namely (see Eq. (29) in [1])

\[
\text{curl}E_a \cdot = \frac{7}{3} \Theta \text{curl}E_a \cdot - D^2 \text{curl}E_a + \left[ \frac{7}{3} \Theta^2 + \frac{1}{3} (\mu - 9p) + \frac{5}{3} \Lambda \right] \text{curl}E_a = \text{curl}K_a .
\]

In the above, which is said to hold at the second perturbative level, the pair \( \mu \) and \( p \) represents the density and the pressure of the matter, \( \Lambda \) is the cosmological constant and \( K_a \) is a gravito-magnetic source term. Arguing that \( \text{curl}K_a \) vanishes when the FRW background is spatially flat (see Eq. (30) in [1]), the paper claims that the electric field will remain curl-free if it was so initially. However, this a priori sets \( \text{curl}E_a \cdot = 0 \) in Eq. (2), although the latter is not necessarily the case because of expression (1).

By switching the electric curl off the authors have inadvertently confined their study to idealised perfectly conductive universes, bypassing all plasma effects and the implications of finite conductivity. Because of that [1] cannot follow the gravito-magnetic interaction in the poorly conductive stages of early reheating and therefore the comparison with [2, 3] was inappropriate.

3 On the scale assessment and the nature of the amplification

Restricting to perfectly conducting cosmological environments we have (see § II.C, IV.B in [1])

\[
\dot{B}_a + \frac{4}{3} \Theta B_a = \sigma_{ab} \ddot{B}^b \equiv I_a ,
\]

where \( B_a \) is the total magnetic field, \( \dot{B}_a \) is the original and \( \sigma_{ab} \) is the transverse shear component. Then, the gravitationally induced \( B \)-fields during the radiation era is

\[
B_R = \tilde{B}_0 \left( \frac{\dot{a}_0}{a} \right)^2 \left\{ 1 + \frac{2}{3} \left( \frac{\sigma}{H} \right)_0 \left[ \left( \frac{\dot{a}_0}{a} \right) - 1 \right] + \frac{5}{6} \left( \frac{\sigma}{H} \right)_0 \left[ \left( \frac{a}{\dot{a}_0} \right)^2 - 1 \right] \right\} ,
\]

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while for dust and late reheating we have

\[ B_{D/RH} = \tilde{B}_0 \left( \frac{a_0}{a} \right)^2 \left\{ 1 + \frac{2}{3} \left( \frac{\sigma}{H} \right)_0 \left[ \left( \frac{a_0}{a} \right)^{3/2} - 1 \right] + 2 \left( \frac{\sigma}{H} \right)_0 \left( \frac{a_0}{a} - 1 \right) \right\}, \quad (5) \]

(see Eqs. (46), (47) in [1]). Above \( a \) is the scale factor, \( H \) is the Hubble parameter and the zero suffix marks the onset of the gravito-magnetic interaction. These expressions were treated as infinite wavelength solutions (see § IV.B.1 in [1]), when in reality they cover all finite superhorizon scales.\(^1\) This incorrect scale assessment has led to the sequence of problems that we will discuss below. Here we note that the domains of (4) and (5) are essential for the additional reason that both results show a deviation from the standard \( a^{-2} \) magnetic decay-law and a superadiabatic amplification of the field (i.e. \( B = \) constant for radiation and \( B \propto a^{-1} \) for dust/reheating).

Returning to [1] we find (see Eq. (51) there) that the final solution for the gravitationally induced magnetic field, during both the radiation and the dust eras, is

\[ \frac{B}{\rho_\gamma^{1/2}} \simeq \left[ 1 + \frac{1}{10} \left( \frac{\lambda_B}{\lambda_H} \right)_0^2 \left( \frac{\sigma}{H} \right)_0 \right] \left( \frac{\tilde{B}}{\rho_\gamma^{1/2}} \right)_0 \Rightarrow B \simeq \tilde{B}_0 \left[ 1 + \frac{1}{10} \left( \frac{\lambda_B}{\lambda_H} \right)_0^2 \left( \frac{\sigma}{H} \right)_0 \right] \left( \frac{a_0}{a} \right)^2, \quad (8) \]

since \( \rho_\gamma \propto a^{-4} \) is the radiation density.\(^2\) It is also argued (see § IV.B.2 after Eq. (51) in [1]) that the above spans all finite wavelengths irrespective of their size relative to the horizon. This cannot be correct because (8) means \( B \propto a^{-2} \) on all scales, while (4) and (5) give \( B = \) constant and \( B \propto a^{-1} \) on super-Hubble lengths. Consider, for example, a large-scale \( B \)-mode propagating through the radiation era and ignore any effects from reheating. Then, Eq. (4) gives

\[ B \simeq \tilde{B}_0 \left[ 1 + \left( \frac{a}{a_0} \right)_0^2 \left( \frac{\sigma}{H} \right)_0 \right] \left( \frac{a_0}{a} \right)^2 = \tilde{B}_0 \left[ 1 + \left( \frac{\lambda_B}{\lambda_H} \right)_0^2 \left( \frac{\sigma}{H} \right)_0 \right] \left( \frac{\lambda_H}{\lambda_B} \right)_0^2 \left( \frac{a_0}{a} \right)^2. \quad (9) \]

Well outside the horizon \( \lambda_H/\lambda_B \ll 1 \), which means that using (8) instead of (9) on those scales will grossly overestimate the amplification of the field. The two expressions agree only at horizon crossing when \( \lambda_B = \lambda_H \). Therefore, at best, Eq. (8) works for scales that have crossed the Hubble radius. However, the magnetic growth takes place earlier outside the horizon and results from the superadiabatic amplification of the field seen in (4) and (5). In [1] the reader

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1Assuming dust and setting \( \tau = t/t_0 \) the gravito-magnetic source term in the right-hand side of (4) reads

\[ I(\ell) = \left[ D_1 J_{5/2} \left( 2\ell \tau^{1/3}/a_0 H_0 \right) + D_2 Y_{5/2} \left( 2\ell \tau^{1/3}/a_0 H_0 \right) \right] \tau^{-5/2}, \quad (6) \]

where \( \ell \) the wavenumber of the gravitational wave (see Eq. (48) in [1]). Finite superhorizon scales have \( \ell \tau^{1/3}/a_0 H_0 \ll 1 \) and the series expansion of the Bessel functions (with the initial conditions of [1]) give

\[ I(\tau) = 2\sigma_0 \tilde{B}_0 \tau^{-5/3} - \sigma_0 \tilde{B}_0 \tau^{-10/3}. \quad (7) \]

Inserted into Eq. (4) the above leads to solution (5), which therefore applies to all finite super-Hubble lengths and not only to infinite wavelengths. Similarly one can show that (5) also covers all finite superhorizon lengths.

2Expression (8) was obtained after replacing \( \lambda_{GW} \) with \( \lambda_B \) in Eqs. (49) and (50) of [1]. However, this substitution was arbitrary because no relation between the two scales was previously established and the original magnetic field was given a zero wavenumber (see § IV.B in [1]). The latter means that \( \lambda_B \) is ill defined (i.e. \( \lambda_B \rightarrow \infty \)) and misleadingly suggests an infinitely strong induced \( B \)-field in (8). Note that assigning a nonzero wavenumber to \( \tilde{B} \) provides a useful relation between \( \lambda_B, \lambda_{GW} \) and \( \lambda_B \) (see Eq. (10) in [3]).
is unaware of the nature of the magnetic growth and where it occurs. The root of the problem was assigning (4) and (5) to infinite wavelengths. As a result, these solutions were sidestepped (see in particular § V after Eq. (60) in [1]), the suparadiabatic nature of the amplification was not recognised, the domain of (8) was incorrectly assessed and ultimately the physics of the gravito-magnetic interaction was misrepresented.

An additional complication is that Eq. (8) alone cannot provide the total amplification of a mode that went through different epochs (e.g. reheating and radiation), because each domain has different growth rates (compare (4) and (5)). Consider an inflationary $B$-mode that crosses the horizon in the radiation era (like that in § V of [1]). Assuming for simplicity that the growth is always strong, the residual field according to (8) is

$$B = \tilde{B}_0 \left( \frac{\sigma}{H} \right)_0 \left( \frac{\sigma}{H} \right)_{RH} \left( \frac{\lambda_B}{\lambda_H} \right)_0 \left( \frac{a_0}{a} \right)^2,$$

where $RH$ marks the end of reheating. The difference in the above is made by extra factor $(\sigma/H)_{RH}$. The latter is typically very small and this accounts for the fact that the magnetic growth rate during (late) reheating is considerably slower than that of the radiation era. Therefore, not appreciating the role of (4) and (5) has also compromised the numerical results of [1].

4 On the gauge-invariance and the nonlinearity

Studies of cosmological perturbations are known to suffer from the gauge problem. The aim of [1] is to provide a nonlinear treatment of the gravito-magnetic interaction free from gauge ambiguities. This meant integrating the magnetic induction equation (i.e. Eq. (3) above) with respect to $B_a$ (see § IV in [1]). However, $\tilde{B}_a$ is treated as the perturbation of $\tilde{B}_a$ (see § II.C in [1]). The latter has nonzero linear value and this makes $B_a$ a gauge-dependent vector at second order by known theorems [6]. In an attempt to circumvent the problem the auxiliary variable $\beta_a$, with $\beta_a \equiv \dot{B}_a + \Theta B_a/3$ was temporarily introduced (see § II.C in [1]). This has zero linear value and is therefore gauge-invariant at second order. Nevertheless, $\beta_a$ has been of little practical use because the authors still had to solve for the gauge-dependent vector $B_a$ to extract any meaningful information about the evolution of the field (see Eqs. (4), (5) and (8) here). This fact makes the analysis and the results of [1] gauge dependent.

The nonlinearity of [1] is built on a set of four spacetimes (see § II there). However, the new setting has not improved our understanding of the interaction because still only the gravito-magnetic effects of [2, 3] are accounted for and all other nonlinearities (including the magnetic effects on the shear) are excluded. There is no new information in the nonlinear equations of [1], relative to what is already encoded in the linear formulae of [2, 3]. This should not have

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3 If the authors had found a way of evaluating the magnetic growth by means of $\beta_a$ exclusively, their results would have been gauge-invariant. This was not possible however. Moreover, at the ideal magnetohydrodynamics (MHD) limit $\beta_a$ is all but redundant because $\beta_a \equiv \sigma_{ab}B^b$ (see § II.C, IV.B in [1] or Eq. (3) here).

4 Expressions (4) and (10), the latter with $\Lambda = 0$, of [1] are identical to Eqs. (2b) and (4) in [3]. Also, at the MHD limit, the magnetic wave equation of [3] (see Eq. (3) there) and expression (12) in [1] reduce to Eq. (3) here. Moreover, the constraints used in [1] (see § II.B.2 there) are the standard linear ones. These constraints ensure that the traceless tensors remain transverse at all times, a highly nontrivial issue for any truly nonlinear study.
happened and the reason it does is the selective setting of [1], which restricts the study and prevents it from going beyond the linear level.

5 Discussion

The Maxwell-Weyl coupling and the possible energy transfer between the two fields has a long research history. Recently this interaction was proposed as a very efficient (resonant) amplification mechanism of large-scale magnetic fields during the poorly conductive stages of early reheating [2, 3]. The same mechanism was studied at the ideal MHD limit in [1], claiming that the gravito-magnetic interaction proceeds always as if the universe were a perfect conductor when the background spatial geometry is Euclidean. We explained here why the aforementioned claim is not correct and how it restricts the generality of [1]. We also demonstrated that the aforementioned paper did not properly monitor the gravito-magnetic interaction. In particular, solutions which in [1] were assigned to infinite wavelengths only, were found to hold on all finite super-Hubble scales. Also solutions that work only inside the horizon were applied to all finite scales. As a result, the physical interpretation and the numerical results of [1] were seriously compromised. With this comment we have attempted to clarify these issues by correcting, where necessary, the mathematics of the analysis and by explaining the physics of the interaction. In the process we also considered and questioned the gauge-invariance and the nonlinearity of the formalism proposed in [1].

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