Spinors and the reference point\[1\] of quasilocal energy

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**Abstract**

This paper investigates the relationship between the quasilocal energy of Brown and York and certain spinorial expressions for gravitational energy constructed from the Witten-Nester integral. A key feature of the Brown-York method for defining quasilocal energy is that it allows for the freedom to assign the reference point of the energy. When possible, it is perhaps most natural to reference the energy against flat space, i.e. assign flat-space the zero value of energy. It is demonstrated that the Witten-Nester integral when evaluated on solution spinors to the Sen-Witten equation (obeying appropriate boundary conditions) is essentially the Brown-York quasilocal energy with a reference point determined by the Sen-Witten spinors. For the case of round spheres in the Schwarzschild geometry, these spinors determine the flat-space reference point. A similar viewpoint is proposed for the Schwarzschild-case quasilocal energy of Dougan and Mason.

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1 Introduction

The “local” concept of energy in general relativity is quasilocal or, in other words, associated with closed two-surfaces in spacetime. Recently, many attempts to define quasilocal energy with spinor methods have relied on the so-called “Witten-Nester two-form” (famous from the Witten-style proof \[1\] of the positive energy theorem in the asymptotically-flat scenario). When the two-form is integrated over a generic two-surface $B$, the resulting integral may be viewed as a functional of $B$ spinor fields. The goal is then to determine which are the “correct” spinors to be inserted into this functional such that it serves as the quasilocal energy. Further, the assumption is that the correct spinors arise as solutions to some supplementary equation (which may or may not be the Sen-Witten equation\[1, 2\] used in the asymptotically-flat scenario).

Though this approach for defining quasilocal energy has been the focus of considerable research, several aspects of such spinor constructions have remained unclear. For instance, the approach seems to make essential use of a spinor with one index. This appears to imply that spinors are somehow essential for defining quasilocal energy, in contrast to the asymptotically-flat scenario where the ADM energy \[3\] may be defined without spinors. Another problem in the quasilocal context is the absence of a rigorous interpretation of the Witten-Nester integral as a boundary value of the gravitational Hamiltonian. Yet another issue is whether or not the supplementary spinor equation should be independent of the spanning three-slice $\Sigma$, so that, as is often claimed, the quasilocal energy is independent of the $\Sigma$ chosen to span $B$.\[5\] This issue deserves some further comment. It has been argued that, in fact, the fundamental concept of quasilocal energy must depend on the spanning three-slice, since it is the spanning slice which determines the fleet of observers at $B$ (defining an equivalence class of spanning three-slices).\[6, 7, 8\] A total energy contained within the two-surface is associated with each fleet. Therefore, in the spinorial constructions it seems that one should use a supplementary spinor equation which does depend on the spanning three-slice (at least in the sense that on $B$ one may choose the timelike

\footnote{Whether or not it is possible to find a truly satisfactory Hamiltonian for a spatially bounded slice $\Sigma$ is a subtle issue in its own right. Following Ref. \[6\], this paper assumes that the correct Hamiltonian for a bounded region is the one which is “read off” from the canonical form of the gravitational action appropriate for a spatially bounded spacetime region.}
vector associated with the solution spinors as a boundary condition). From an alternative viewpoint, if one employs an “intrinsic” supplementary equation which is solved on the unbounded two-surface $B$, then one should allow the solution spinors to select a preferred equivalence class of spanning slices.

Independent of spinor methods, Brown and York have determined what geometric entity plays the role of quasilocal energy in general relativity. Their analysis yields an energy surface density which is defined on the bounding two-surface of a generic spacelike slice. The quasilocal energy, the integral over the two-surface of the energy surface density, possesses the following salient properties: (i) it can be derived from a first-principles approach from the gravitational action via the Hamilton-Jacobi method, (ii) it is the value of the on-shell Hamiltonian (defined from the canonical action) corresponding to the choice on $B$ of a unit lapse function and zero shift vector, and (iii) it has been shown to play the role of thermodynamical internal energy in the context of black-hole thermodynamics. Furthermore, (iv) the Brown-York quasilocal energy surface density transforms under switches of the $\Sigma$ slice spanning $B$ (generalized boosts) in a manner which is in full accord with the equivalence principle. Therefore, it is of interest to establish a connection between the Brown-York quasilocal energy and the Witten-Nester form. Indeed, such a connection supports the use of the Witten-Nester form (which has been employed so spectacularly in asymptopia) in the quasilocal context, and, furthermore, provides one with a non-spinor vantage point from which the various supplementary spinor equations and the issues mentioned in the previous paragraph can be examined. The supplementary equations studied in some detail here are the Sen-Witten equation and the Dougan-Mason equation.

A key feature of the Brown-York construction (which may be regarded as property (v)) is that it allows one the freedom to assign the reference point of the quasilocal energy. When possible, it is perhaps most natural to reference the energy against flat space, i.e. assign flat space the zero value of energy. This paper establishes the general relationship between the Brown-York quasilocal energy and the common “spinorial definition of quasilocal energy” constructed from the Witten-Nestor two-form. It is shown that spinors may always be chosen so that the spinorial definition is the “unreferenced” Brown-York quasilocal energy. However, when the Witten-Nester expression is evaluated on solution spinors to the Sen-Witten equation (obeying appropriate boundary conditions), an implicit reference point for the energy
is set. The origin of the reference point is the Sen-Witten equation. In solving the elliptic Sen-Witten equation on $\Sigma$, one encounters the seemingly desirable type of boundary-value problem in which the timelike vector on $B$ associated with the solutions spinors may be fixed as a boundary condition. However, the solution spinors to the Sen-Witten equation are altered when the spanning slice $\Sigma$ is perturbed or “wiggled” in the ambient spacetime $M$, even if the $\Sigma$ timelike normal $u^\mu$ is held fixed on $B$. Clearly then, an energy expression built with Sen-Witten spinors is certainly not quasilocal in the truest sense, since such an expression does not depend solely of the fleet of (Eulerian) observers at $B$\footnote{I thank J. Samuel for bringing this point to my attention.}. Nevertheless, as shown later, the Sen-Witten energy expression can always be written in a form which is essentially identical to the that of the Brown-York QLE. Moreover, for the case of round spheres embedded in the preferred time slices of the Schwarzschild geometry, the Sen-Witten equation determines the flat-space reference point. Finally, the general results are also used to further elucidate the remarkable features of the (Schwarzschild-case) quasilocal energy of Dougan and Mason. It is shown that this energy is also equivalent to the Brown-York quasilocal energy (for an distinguished spanning slice) and that a flat-space reference, arising from the chosen supplementary equation, is also implicit in the Dougan-Mason definition.

The organization of this work is as follows. A preliminary section fixes all of the relevant geometric and spinorial conventions. In addition, this section collects the basic Brown-York results necessary for the following discussion. The next section, devoted to spinorial definitions of quasilocal energy, examines the general relationship between the Brown-York quasilocal energy and the Witten-Nester form. This section also applies the general results to the specific Schwarzschild case for both the Sen-Witten and Dougan-Mason equations. For the convenience of the reader the appendix uses the Newman-Penrose method to present the general solutions to the Schwarzschild-case Sen-Witten and Dougan-Mason equations.

2 Preliminaries
2.1 Conventions

Topology and index conventions  Consider a generic spatially-bounded spacetime region $M$ which is topologically the Cartesian product of an orientable three-manifold $\Sigma$ with spatial boundary $\partial\Sigma = B$ and a closed connected segment of the real line $I$. One need not assume that the boundary $B$ of the three-space $\Sigma$ is simply connected. The spacetime $M$ is foliated by a time function $t : M \to I$. The level hypersurfaces of this function or slices are the leaves of the foliation, and the slices labeled by the initial and final endpoints of $I$ are denoted respectively by $t'$ and $t''$. The product of $B$ with $I$ is an element of the three-boundary of $M$ and is denoted by $\mathcal{T}$. This element is often referred to as the three-boundary, even though technically the three-boundary of $M$ consists of $\mathcal{T}$, $t'$, and $t''$. One need not consider $\mathcal{T}$ to be a physical boundary. Rather, one may imagine $\mathcal{T}$ to be just some locus of points separating $M$ off from an ambient spacetime $\mathcal{U}$ (the universe).

Lowercase Greek letters $\mu, \nu, \sigma, \cdots$ are spacetime $M$ indices. Lowercase latin letters $i, j, k, \cdots$ from the middle of the alphabet are spatial $\Sigma$ indices, while lowercase latin letters $a, b, c, \cdots$ from the first part of the alphabet are spatial $B$ indices. There is no need to consider $\mathcal{T}$ indices. Orthonormal (or when appropriate pseudo-orthonormal) labels and indices for each space are represented by the same letters with hats. For example, $\hat{\mu}$ is a spacetime tetrad index and $\hat{a}$ is a $B$ orthonormal index.

Spin conventions  The $(-, +, +, +)$ metric-signature convention is used throughout this discussion. Since this is a somewhat uncommon choice when spinor methods are employed, a few basic conventions are listed now. Using a normalized spin dyad $\{e_A | A = 1, 2\}$ ($e_1^A = o^A, e_2^A = \imath^A$, and $\varepsilon_{AB} o^A \imath^B = 1$ where $\varepsilon_{AB}$ denotes skewsymmetric inner product), one can construct the following spin-tensor fields:

$$
\sigma_0^{AA'} = -\frac{i}{\sqrt{2}} (o^A \bar{\sigma}^{A'} + \imath^A \bar{\imath}^{A'}) \\
\sigma_1^{AA'} = -\frac{i}{\sqrt{2}} (o^A \bar{\tau}^{A'} + \imath^A \sigma^{A'}) \\
\sigma_2^{AA'} = -\frac{i}{\sqrt{2}} (-i o^A \bar{\imath}^{A'} + i \imath^A \sigma^{A'})
$$

(1)
\[ \sigma_3^{AA'} = -\frac{i}{\sqrt{2}} \left( o^A o^{A'} - t^A t^{A'} \right) . \]

The hat on the \( \hat{\rho} \) index of the \( \sigma^{\hat{\rho} AA'} \) is present because these indices are to be identified with a spacetime tetrad. The \( \hat{\rho} \) index is only a label and not a tensorial index. Given an orthonormal spacetime cotetrad \( \{ e^\hat{\rho} \} \) on \( M \) and the set of \( \sigma^{\hat{\rho} AA'} \) tensors above, one can construct a spacetime soldering form (or \( SL(2,\mathbb{C}) \) soldering form) on \( M \) with the definition

\[ e^{AA'}_\mu := \sigma^{\hat{\rho} AA'} e^\hat{\rho}_\mu , \] (2)

Note that the soldering form is imaginary \( \bar{e}^{AA'}_\mu = -e^{AA'}_\mu \) and that the following incredibly useful identities involving the soldering form and its inverse hold:

\[ e^{AA'}_\mu e_{BB'}^\mu = \varepsilon_B^A \varepsilon_{B'}^{A'} , \]
\[ e^{AA'}_\mu e_{AA'}^{\nu} = g_\nu^\nu , \]
\[ e^{AA'}_\mu e_{BA'}^\nu = \frac{1}{2} g_{\mu\nu} \varepsilon_B^A - \frac{i}{2} \epsilon_{\mu\nu\sigma\rho} e^{AA'}^\sigma e_{BA'}^\rho , \]
\[ e^{AA'}_\mu e_{AB'}^\nu = \frac{1}{2} g_{\mu\nu} \varepsilon_{B'}^{A'} + \frac{i}{2} \epsilon_{\mu\nu\sigma\rho} e^{AA'}^\sigma e_{AB'}^\rho . \] (3)

These may be verified by explicit calculation using the full expressions for the \( e_{AA'}^\mu \) and the formulas \( o_A t^A = 1, o_A o^A = 0, etc. \)

### 2.2 Brown-York quasilocal energy

**General considerations** The Brown-York notion of quasilocal energy (often \( QLE \) hereafter) stems from a Hamilton-Jacobi analysis of a gravitational action functional suitable for the bounded region \( M \). Their analysis in Ref. [6] leads to an energy surface density \( \varepsilon \), which is defined on \( B \) and is associated with the energy of the \( \Sigma \) gravitational and matter fields contained within \( B \). The energy surface density has the form

\[ \varepsilon = \varepsilon_1 - \varepsilon_0 = \frac{1}{\kappa} \left( k - k_0 \right) , \] (4)

where \( \kappa = 8\pi \) (in units with \( G = c = 1 \)) and \( k = \kappa \varepsilon_1 \) is the trace \( \sigma_{ab} k^{ab} \) of the extrinsic curvature of \( B \) as embedded in \( \Sigma \) which has three-metric \( h_{ij} \).
The reference term $k_0(\sigma_{ab})$ is an arbitrary quantity depending on the intrinsic geometry of $B$ (the $B$ metric is $\sigma_{ab}$) and its presence arises from the freedom to add boundary terms to the gravitational action which depend solely on the fixed boundary data. Usually, $k_0$ is assumed to be the flat-space reference, the trace of the extrinsic curvature of a two-surface which is isometric to the surface $B$ but which is embedded in $R^3$ rather than $\Sigma$. In general such a construction is not possible. Therefore, for certain two-surfaces it is not possible to define a flat-space density $\varepsilon_0$, however, the unreferenced $\varepsilon_1$ is always well-defined. The final discussion section of this work addresses this point more fully. The definition (4) ensures that the total quasilocal energy

$$E = \int_B d^2x \sqrt{\sigma} \varepsilon$$

(5)

is a functional on the $\Sigma$ gravitational phase space. It proves useful to make the following natural split of the quasilocal energy: $E = E_1 - E_0$, where

$$E_1 = \int_B d^2x \sqrt{\sigma} \varepsilon_1 ,$$

(6)

and is referred to hereafter as the unreferenced QLE. The reference term (or subtraction term) is

$$E_0 = \int_B d^2x \sqrt{\sigma} \varepsilon_0 .$$

(7)

This work considers $E_0$ to be a suitable reference point for the QLE whenever $k_0$ represents the trace of the extrinsic curvature of a two surface which is isometric to $B$ but is embedded in some other three-geometry (preferably, but not necessarily, $R^3$). Of course, this general definition of the reference term is not fully satisfactory, because there has been no mention of the initial-value constraints. Some commentary on the role of the constraints in this definition is found below.

Finally, the later analysis of the Dougan-Mason QLE demands that one consider a normal momentum density.\[10, 7\] Introduce the $B$ density

$$(j_1)_+ = -\frac{1}{\kappa} \sigma_{ij} K^{ij} .$$

(8)

where $K_{ij}$ represents the extrinsic curvature of $\Sigma$ as embedded in $M$. (The “$1$” notation has been employed in anticipation of allowing the freedom to consider a reference-term contribution to this normal momentum density, in
which case \( j_- = (j_1)_- - (j_0)_- \). In a Hamilton-Jacobi analysis of the gravitational action, \( j_- \) arises (apart from a factor of \( \sqrt{\sigma} \)) as minus the variation of the classical action with respect to a unit stretch of a “radial” lapse function \( \alpha \) associated with a \((1 + 2)\) decomposition of the metric on either \( t'' \) or \( t' \). This is similar to the origin of \( \varepsilon \), which is (apart from a factor of \( \sqrt{\sigma} \)) minus the variation of the classical action with respect to a unit stretch of a temporal lapse function \( N \) associated with a \((2 + 1)\) decomposition of the \( T \) metric. Ref. [6] and a forthcoming paper [7] devoted to “boosted” quasilocal stress-energy-momentum examine how \( (j_1)_- \) and \( \varepsilon_1 \) behave under switches of the three-slice spanning \( B \).

**The Schwarzschild example** Following Ref. [6], one may specialize the preceding results to the simple yet illustrative case of the Schwarzschild geometry. Work with the line element in standard coordinates [13],

\[
\text{ds}^2 = -N^2 \text{dt}^2 + \alpha^2 \text{dr}^2 + r^2 \left( \text{d} \theta^2 + \sin^2 \theta \text{d} \phi^2 \right),
\]  

(9)

where the lapse function is

\[
N = \alpha^{-1} = \sqrt{1 - \frac{2M}{r}}.
\]  

(10)

The mass of the hole is (“block-Roman”) \( M \). For a constant time slice, the two-surface \( B \) is the round sphere specified by \( r = r_+ > 2M \). For Schwarzschild the quasilocal energy density is

\[
\varepsilon = \varepsilon_1 - \varepsilon_0 = \frac{2}{\kappa} \left( -\frac{1}{\alpha \, r} + \frac{1}{r} \right) \bigg|_{r=r_+}.
\]  

(11)

Notice that the reference term

\[
E_0 = \int_B d^2x \sqrt{\sigma} \left( -\frac{2}{\kappa \, r_+} \right)
\]  

(12)

ensures that the full quasilocal energy (13),

\[
E = r_+ \left[ 1 - \left( 1 - \frac{2M}{r_+} \right)^{1/2} \right],
\]  

(13)
is referenced against flat-space. In the limit $r_+ \to \infty$ the quasilocal energy $E$ becomes the Schwarzschild mass $M$. The momentum density $(j_1)_-\text{vanishes identically for the preferred } t = \text{constant \ Schwarzschild \ slices. Nevertheless, for future purposes it is instructive to retain } (j_1)_- \text{ in the formalism.}$

3 Spinorial quasilocal energy

This purpose of this section is twofold: (i) to investigate the general relationship between the Witten-Nester two-form and the Brown-York QLE density and (ii) to examine the general relationship for the specific case of the Schwarzschild geometry. From the results of this section one can also infer the relationship between the Witten-Nester form and the Brown-York quasilocal momentum densities $j_a$ of Ref. [6]. However, though there is some commentary concerning the relationship with the $j_a$, the focus of the discussion rests on the link with the energy surface density $\varepsilon$.

The sought-for link is made in the following way. First, $\varepsilon_1 \sqrt{\sigma} d^2x$ and $(j_1)_- \sqrt{\sigma} d^2x$ are expressed as Sparling two-forms. Once this is achieved, it is quite straightforward to derive a formula which relates the Witten-Nester two-form directly to these expressions. This formula (number (27) below) has the following form:

$$E_s = \int_B d^2x \sqrt{\sigma} \left( \gamma \varepsilon_1 - v \gamma (j_1)_- \right) - \mathcal{E},$$

where $E_s$ is the “spinorial definition of quasilocal energy” built from the Witten-Nester form and associated with a timelike vector $\bar{u}^\mu$ normal to $B$. At $B$ the vector $\bar{u}^\mu$ is related to the future-pointing $\Sigma$ normal $u^\mu$ by $\bar{u}^\mu = \gamma u^\mu + v \gamma n^\mu$, where $n^\mu$ is the normal of $B$ in $\Sigma$ and $v$ is a point-dependent boost velocity which defines a local relativistic factor $\gamma = (1 - v^2)^{-1/2}$. Also, $\mathcal{E}$ is an “anomalous term.” Note that $E_s$ is spinor-dependent, in the sense that $E_s$ depends on how $\bar{u}^\mu$ is broken down into constituent spinors. Therefore, the anomalous term $\mathcal{E}$ is also spinor-dependent in the same fashion. It is shown that one can easily set $v = 0$ and $\mathcal{E} = 0$ by “hand picking” the spinors (to be inserted into $E_s$) in an obvious way. However, when the energy expression is built from spinors which obey the Sen-Witten equation (and natural boundary conditions), though $v$ again vanishes the anomalous term $\mathcal{E}$ serves as a reference point of the energy. Hence, $E_s$ with Sen-Witten spinors has the same form as the full quasilocal energy (5).
After this general discussion is completed, the following two central results are given. First, for the Schwarzschild geometry the anomalous term $E$ in the expression (14) serves as the correct flat-space subtraction $E_0$ from (12) when the spinors inserted into $E_s$ obey the Sen-Witten equation. Second, though the analysis is more subtle, precisely the same type of interpretation can made for the Dougan-Mason Schwarzschild-case QLE.

### 3.1 General considerations

**Witten-Nester two-form** With the $(-,+,+)$ metric-signature convention of this work the *Witten-Nester two-form* is written as

$$ F[\bar{\eta}, \xi] = -\frac{2}{\kappa} \bar{\eta}_{A'} D\xi_A \wedge e^{AA'}, $$  \hspace{1cm} (15)

where $D$ is the spinorial exterior derivative. Its action on general spin tensors may be inferred from its action on spin covectors (“cospinor-valued scalars”): $D\xi_A = d\xi_A - \xi_B A^B_A$, where $A^A_{B\mu}$ are the (unprimed) spin connection coefficients which specify the $SL(2,C)$ connection. The $SL(2,C)$ spin connection is compatible with the soldering form in the following sense.

$$ D_{\mu} e^{AA'}_{\lambda} = e^{AA'}_{\lambda} \Gamma^\sigma_{\lambda\mu}. $$  \hspace{1cm} (16)

The identity (16) allows one to show that $D e^{AA'} = 0$ which implies that the imaginary part of $F[\bar{\xi}, \xi]$ is a pure divergence, and, hence, the integral $\int_B F[\bar{\xi}, \xi]$ is a real quantity.

Most spinorial definitions of QLE employ the integrated Witten-Nester form in a construction of the following type [5]:

$$ P^{AA'} = -\frac{2}{\kappa} \int_B \bar{\lambda}^{A'}_A D\lambda^A_A \wedge e^{AA'}. $$  \hspace{1cm} (17)

(Often only the real part of $P^{AA'}$ is considered. Since the concern here is only with the energy, the above formula suffices.) The set $\{\lambda^A_A | A = 1, 2\}$ comprises a basis for the solution space corresponding to some supplementary differential equation $O\lambda^A_A = 0$ ($O$ is some operator linear in $D_{\mu}$). Various combinations of the $P^{AA'}$ are then offered as the value of the energy or momentum associated with the gravitational and matter fields contained within $B$. In particular, the spinorial expression for the energy is
\[ E_s = \frac{1}{\sqrt{2}} \left( P_{11} + P_{22} \right). \] (18)

At this point, since the spinors which are to be inserted into \( P_{\Delta \Delta} \) are not known, the \( E_s \) notion of QLE is not at all well-defined. Notice that this definition is most natural when the timelike vector associated with the solution spinors is of unit magnitude on the two-surface \( B \).

**Sparling two-forms** The intimate link between the differential Sparling forms and notions of gravitational energy is well-known [14]. Refs. [10, 13] have established the connection between the unreferenced Brown-York quasilocal energy-momentum surface densities and the tetrad versions of the Sparling forms. It is appropriate to briefly recall this correspondence. First, introduce a spacetime cotetrad \( \hat{e}_{\rho \mu} \) and its associated connection coefficients \( \Gamma_{\hat{\rho}\hat{\sigma}\hat{\mu}}[e] \). One need only consider the Sparling two-forms

\[ \sigma_{\hat{\rho}} \equiv -\frac{1}{2} \epsilon_{\hat{\rho}\hat{\sigma}\hat{\tau}} \Gamma_{\hat{\sigma}\hat{\tau}} \wedge \hat{e}_{\hat{\mu}}, \] (19)

which are completely determined by the cotetrad. Clearly, these two-forms do not transform homogeneously under tetrad transformations. However, the boundary structure of \( B \) as embedded in \( \Sigma \subset M \) provides a natural (almost unique) gauge. Assume that the time leg \( e_{\perp} \equiv e_{\hat{0}} \) of the tetrad is tied to the hypersurface normal \( u \), and, further, that the third space leg \( e_\tau \equiv e_3 \) is tied to the normal \( n \) of \( B \) in \( \Sigma \). Such a tetrad is said to be radial time-gauge or RT-gauge. Let \( s \) denote the inclusion mapping of the two-surface \( B \) in spacetime, \( s: B \rightarrow M \). As demonstrated in [13], one can easily verify that the pullbacks to \( B \) of \( \sigma_\perp \) and \( \sigma_\tau \) are the following:

\[-\frac{1}{\kappa}s^*(\sigma_\perp) = \varepsilon_1 \sqrt{\sigma} d^2x \]
\[ \frac{1}{\kappa}s^*(\sigma_\tau) = (j_\perp)_- \sqrt{\sigma} d^2x, \] (20)

Likewise, the pullbacks of \( 1/\kappa \sigma_\hat{a} \) (\( \hat{a} = \hat{1}, \hat{2} \)) are the orthonormal components of the momentum densities \( (j_\perp)_a \) from Ref. [7].
It simplifies matters somewhat - but seems in no way necessary - to work instead with the complex Sparling two-forms

\[ \sigma(+)_{\hat{\rho}} \equiv -\epsilon_{\hat{\rho}\hat{\sigma}+\hat{\mu}} \Gamma(+)_{\hat{\sigma}} \wedge e_{\hat{\mu}}, \]  

which are built from the self-dual connection forms

\[ \Gamma(+)_{\hat{\rho}\hat{\sigma}} = \frac{1}{2} \left( \Gamma_{\hat{\rho}\hat{\sigma}} - \frac{i}{2} \epsilon_{\hat{\rho}\hat{\sigma}+\hat{\mu}} \Gamma_{\hat{\tau}\hat{\mu}} \right). \]  

(22)

Notably, subject to the RT-gauge choice the situation is just as it was before in (21)

\[ -\frac{1}{\kappa} s^* \left( \sigma(+) \right) = \varepsilon_1 \sqrt{\sigma} d^2 x \]

\[ \frac{1}{\kappa} s^* \left( \sigma(+) \right) = (j_1)_- \sqrt{\sigma} d^2 x. \]

One can quickly discern the relationship between the Sparling two-forms and the expression for \( P^{AA'} \) given in (17). Let \( e_A = e_1^A \) and \( \iota_A = e_2^A \) be the normalized spin dyad which corresponds to the RT-gauge tetrad. One has the following formula between the RT-gauge self-dual connection coefficients and the spin connection coefficients with respect to the normalized dyad [10, 11]:

\[ \Gamma(+)_{\hat{\rho}\hat{\sigma}} = e_{AA'} \hat{\rho} \epsilon^{BA'}_{AA'} A_{B\mu}. \]

(23)

Next, define the vector-field \( \nu_{AA'} = -i \lambda_{AA'} \lambda_{AA'} e^{AA'}_{\mu} \). Evidently, \( \nu_{AA'} \) and \( \nu_{BB'} \) are real future-pointing null vectors, while \( \nu_{CC'} \) and its complex conjugate \( \nu_{CC'} \) are spacelike complex null vectors. Starting from the expression

\[ -\frac{1}{\kappa} \nu_{AA'} \sigma(+) \hat{\rho} \]  

with (23) inserted into (21), one obtains the formula

\[ -\frac{1}{\kappa} \nu_{AA'} \sigma(+) \hat{\rho} = \frac{2}{\kappa} \lambda_{AA'} \lambda_{BB'} A_{A} A_{B} \wedge e^{AA'}, \]  

(24)

where along the way one must appeal to the identities (3) and realized that with respect to a normalized dyad the connection coefficients \( A_{AB\mu} = A_{(AB)\mu} \) are symmetric in their spin indices. This leads directly to the result

\[ P^{AA'} = -\frac{1}{\kappa} \int_B \nu_{AA'} \sigma(+) \hat{\rho} - \frac{2}{\kappa} \int_B \lambda_{AA'} d\lambda_{AA'} \wedge e^{AA'}. \]  

(25)
This is a fundamental formula for the considerations of this paper, because it establishes a connection between the Witten-Nester two-form and the Brown-York quasilocal surface densities.

To evaluate $P_{A\bar{A}'}$ on general spinors it helps to cast the anomalous term in (25) in a more convenient form. Consider the complex null vector $m^a = \frac{1}{\sqrt{2}}(e_1^a + i e_2^a)$ associated with the normalized spin frame and its complex conjugate $\bar{m}^a$ which are tangent to $B$ and normalized so that $m^a \bar{m}_a = 1$.

On $B$ the action of the exterior derivative may be written as $d\, f = m[j] \bar{m} + \bar{m}[j] m$, where as one-forms $m = m_a \, dx^a$ and $\bar{m} = \bar{m}_a \, dx^a$. Now the expression (25) for $P_{A\bar{A}'}$ may be written as

$$P_{A\bar{A}'} = -\frac{1}{\kappa} \epsilon_{A'}^A \sigma^{(+)} - \frac{2}{\kappa} \int_B d^2 x \sqrt{\sigma} \left( \bar{\lambda}^A_{A'} m \left[ \lambda^A_{12} \right] - \bar{\lambda}^A_{A'} \bar{m} \left[ \lambda^A_{12} \right] \right),$$

where the $B$ volume form can also be expressed as $\sqrt{\sigma} \, d^2 x = i m \wedge \bar{m}$. The identity $m \wedge e^{AA'} = - o^A \bar{v}^{A'} \sqrt{\sigma} \, d^2 x$ and its complex conjugate help in obtaining this last result.

**Form of the spinorial QLE** Adopt (18) as the spinorial notion of QLE, and, further, assume that the set $\{\lambda^A_1, \lambda^A_2\}$ is subject to the requirement that the associated timelike vector field has the form $\bar{u}^\mu = \gamma u^\mu + v \gamma n^\mu$ on $B$. Subject to this assumption, the expression for $E_s$ is

$$E_s = \int_B d^2 x \sqrt{\sigma} (\gamma \varepsilon_i - v \gamma (j_i),)$$

$$- \frac{\sqrt{2}}{\kappa} \int_B d^2 x \sqrt{\sigma} \left( \bar{\lambda}^A_{12} m \left[ \lambda^A_{12} \right] - \bar{\lambda}^A_{12} \bar{m} \left[ \lambda^A_{12} \right] + \bar{\lambda}^A_{12} m \left[ \lambda^A_{21} \right] - \bar{\lambda}^A_{12} \bar{m} \left[ \lambda^A_{21} \right] \right).$$

The integrand of the first term can also be expressed as $1/\kappa \bar{k}$, where $\bar{k}$ is the extrinsic curvature of $B$ as embedded in a different spanning slice $\Sigma$ defined by $\bar{u}^\mu$. (Actually, $\bar{u}^\mu$ defines an equivalence class of slices, since the extension of $\bar{u}^\mu$ off $B$ is not determined.) Therefore, the spinorial definitions of QLE based on a construction of the type considered here can be viewed as essentially the Brown-York notion of unreferenced QLE plus a contribution from an anomalous term. All of ambiguity associated with how the vector $\bar{u}^\mu$ is broken down into constituent spinors resides in this anomalous term. Note that in the anomalous term the components of the $\lambda^A_A$ are with respect
to the spin dyad associated with the RT-gauge tetrad determined by the embedding of $B$ in $\Sigma$. It is easy to write the anomalous term in terms of $\lambda_A^A$ components with respect to the dyad determined by the $\bar{\Sigma}$ RT-gauge tetrad (and in fact in terms of these components $\mathcal{E}$ has exactly the same form, since $\bar{w}^\mu$ is normal to $B$). Therefore, the need to consider the $(j_1)_-$ term can be done away with. However, here the slices are “chosen first” and considered primary so the $(j_1)_-$ term is retained.

Relation between $E_s$ and $E_I$ Suppose that rather than being determined by a supplementary equation, the spinors $\lambda_A^A$ are constructed from the normalized spin dyad (which has been tailored to the $\Sigma$ slicing) in the following trivial way. Take $\lambda_1^A = 1/\sqrt{2}(o_A + e^{i\psi}l_A)$ and $\lambda_2^A = 1/\sqrt{2}(o_A - e^{i\psi}l_A)$ where $\psi$ is some fixed angle (constant on $B$). The null vectors associated with these spinors are respectively $1/\sqrt{2}(u + \cos \psi e_1 + \sin \psi e_2)$ and $1/\sqrt{2}(u - \cos \psi e_1 - \sin \psi e_2)$. Notice that by construction (i) the associated timelike vector is the hypersurface normal $u^\mu$ and that (ii) the components $\lambda_1^A$ and $\lambda_2^A$ of these spinors with respect to the normalized dyad $\{o^A, l^A\}$ are constants on $B$. Therefore, with this “hand-picked” selection, the spinorial QLE (18) is

$$E_s = E_I = \int_B d^2x \sqrt{\sigma} \varepsilon_1.$$  \hfill (28)

Sen-Witten spinors The expression (27) takes a particularly interesting form when evaluated on solution spinors to the Sen-Witten equation,

$$\Sigma D^{AA'} \lambda_{A'}^A = 0.$$  \hfill (29)

Here the Sen-Witten derivative is defined by $\Sigma D_{AA'} = h_{\mu\nu} e_{AA',\mu} D_{\nu}$, where the spacetime expression for the $\Sigma$ metric is $h_{\mu\nu} = g_{\mu\nu} + u_{\mu} u_{\nu}$. With the operation of $\Sigma D_{\mu} = h_{\mu\nu} D_{\nu}$ restricted to unprimed (or primed) spinors, $\Sigma D_{\mu}$ is the derivative operator specified by the $\Sigma$ Sen connection which has well-known properties. Assume that the boundary conditions placed on the solution spinors ensure that their associated timelike vector

$$t^\mu = \frac{i}{\sqrt{2}} \left( \lambda_1^A \bar{\lambda}_{A'}^A + \lambda_2^A \bar{\lambda}_{A'}^A \right) e^{AA'\mu}$$  \hfill (30)
is the hypersurface normal $u^\mu$ on the two-surface $B$ (further, take the solution set to comprise a normalized dyad on $B$). Since on $B$ one has $-i o_A \tilde{\partial}_A t^{AA'} = -i \iota_A \tilde{\partial}_A t^{AA'} = 0$, the expression (27) can be written as

$$E_{SW} = \int_B d^2x \sqrt{\sigma} \varepsilon_1$$

$$- \frac{1}{\sqrt{2} \kappa} \int_B d^2x \sqrt{\sigma} \left( \check{\lambda}_1 m [\lambda_2] - \lambda_2 m [\check{\lambda}_1] + \check{\lambda}_2 m [\lambda_2] - \lambda_2 m [\check{\lambda}_2] \right)$$

$$- \frac{1}{\sqrt{2} \kappa} \int_B d^2x \sqrt{\sigma} \left( \check{\lambda}_1 \check{m} [\lambda_2] - \lambda_2 \check{m} [\check{\lambda}_1] + \check{\lambda}_2 \check{m} [\lambda_2] - \lambda_2 \check{m} [\check{\lambda}_2] \right)$$

Inserting full form of the Sen-Witten equation (62) into (31) and appealing to the imposed boundary condition, one finds that

$$E_{SW} = \int_B d^2x \sqrt{\sigma} \varepsilon_1$$

$$- \frac{1}{\kappa} \int_B d^2x \sqrt{\sigma} \sqrt{2} \left[ \psi^{-1} (\mu + \Delta \log \psi) + \psi^{-1} (\rho - D \log \psi) \right] .$$

The conformal factor $\psi \equiv (-t_\mu w^\mu)^{1/2}$ is unity on $B$. Also in this relation, $\rho$ and $\mu$ are spin coefficients and $D$ and $\Delta$ are respectively the derivative operators in the directions $-i o^A \tilde{\partial}_A$ and $-i u^A \tilde{\iota}_A$. (These are defined in the appendix equations (60) and (61) for the Schwarzschild case, but these formulas are fully general.) Note that since $B$ is a spacelike two-surface, $\mu$ and $\rho$ are real [11] and that the derivatives $D$ and $\Delta$ appear only in a combination $D - \Delta = \sqrt{2} n$. Further, $k = \sqrt{2} (\mu + \rho)$. Therefore,

$$k_0 = \sqrt{2} \left[ \psi^{-1} (\mu + \Delta \log \psi) + \psi^{-1} (\rho - D \log \psi) \right] , (33)$$

is the trace of the extrinsic curvature of the two-surface $B$ as embedded in the conformally transformed geometry which has three-metric $\psi^2 h_{ij}$. Notice that this is an isometric embedding, because the conformal factor is unity on $B$. Therefore, $k_0$ may be interpreted as a reference-point contribution to the energy, and, hence, (32) has the form of the full QLE expression (3). Since $\psi = 1$ on $B$, another expression for the energy is

$$E_{SW} = \frac{2}{\kappa} \int_B d^2x \sqrt{\sigma} n [\psi] , (34)$$
which is reminiscent of the conformal expression for the ADM energy. Of course, one is really interested in on-shell expressions for the energy in which case the pair \((h_{ij}, K^{ij})\) obey the initial-value constraints. The expression above is most natural when the conformal geometry \(\hat{h}_{ij} = \psi^2 h_{ij}\) can be augmented by \(\hat{K}^{ij}\) such that the pair \((\hat{h}_{ij}, \hat{K}^{ij})\) also obey the initial-value constraints. In this case, \(E_{SW}\) is the energy difference between two instantaneous “states” of the gravitational field.

### 3.2 The Schwarzschild example

Assume that \(E_s\) is to be evaluated for a round sphere \(B\) in the Schwarzschild geometry determined by \(r = r_+ > 2M\). It is convenient to set

\[
m = \frac{e^{-i\phi}}{\sqrt{2}r_+} \left( \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right) = -\frac{P}{\sqrt{2} r_+} \frac{\partial}{\partial \zeta},
\]

where the stereographic coordinate \(\zeta = e^{i\phi} \cot \theta/2\) and \(P = 1 + \zeta \bar{\zeta}\). Therefore, for the round sphere \(B\) the formula (27) can be written as

\[
E_s = \int_B d^2x \sqrt{\sigma} (\gamma \varepsilon_i - v \gamma (j_i)) \]

\[+ \frac{1}{\kappa} \int_B d^2x \sqrt{\sigma} \frac{P}{r_+} \left( \lambda_1^1 \frac{\partial \lambda_2^1}{\partial \zeta} - \lambda_2^1 \frac{\partial \lambda_1^1}{\partial \zeta} + \lambda_2^2 \frac{\partial \lambda_1^2}{\partial \zeta} - \lambda_1^2 \frac{\partial \lambda_2^2}{\partial \zeta} \right).
\]

The remainder of this section evaluates this expression on both the Sen-Witten and Dougan-Mason spinors.

**Sen-Witten spinors** Assume that the spinors \(\lambda_A^1\) and \(\lambda_A^2\) are solutions to the Sen-Witten equation. The equation (23) must be solved on the Schwarzschild background subject to required boundary conditions on \(B\): (i) \(\{\lambda_A^1, \lambda_A^2\}\) comprise a normalized spin dyad and (ii) that the associated timelike vector field is the \(\Sigma\) normal \(u^\mu\). Appendix B.2 establishes that the following two spinors are solutions to the Sen-Witten equation and obey these boundary conditions:

\[
\lambda_A^1 = \frac{X}{X_+} \frac{i}{\sqrt{P}} \left( \bar{\zeta} \iota_A + o_A \right)
\]
\[ \lambda_A^2 = \frac{X}{X_+} \frac{i}{\sqrt{\mathcal{P}}} (-\iota_A + \zeta o_A), \]  

where \( X = \alpha^{-2} + 2\alpha^{-1} + 1 \) and \( X_+ = X(r_+) \). One has the desired normalization

\[ \varepsilon^{AB} \lambda_A^1 \lambda_B^2 = \left( \frac{X}{X_+} \right)^2, \]  

and the associated timelike vector is

\[ t^\mu = \left( \frac{X}{X_+} \right)^2 u^\mu. \]

Hence, the conformal factor \( \psi = X/X_+ \). At \( r = r_+ \) the general considerations above establish that

\[ E_{SW} = E_1 - \mathcal{E}, \]  

where \( E_1 \) is the specific Schwarzschild expression

\[ E_1 = \int_B d^2 x \sqrt{\sigma} \left( -\frac{2}{\kappa \alpha_+ r_+} \right) \]  

with \( \alpha_+ = \alpha(r_+) \). Furthermore, with the components from (37) inserted into (36), direct calculation shows that

\[ \mathcal{E} = E_\theta = \int_B d^2 x \sqrt{\sigma} \left( -\frac{2}{\kappa r_+} \right), \]  

which is the correct flat-space reference term. Notice that the exact flat-space density \( \varepsilon_\theta \) is recovered. One can also verify directly that \( \psi^2 h_{ij} \) has vanishing curvature. Since any spinors which obey the demanded boundary conditions on \( B \) yield \( E_1 \) as the leading term in the expression for \( E_s \), it seems that in the Schwarzschild context the Sen-Witten equation is responsible for the flat-space reference of the QLE.
**Dougan-Mason spinors**  Now assume that the spinors are solutions to the (holomorphic-case) **Dougan-Mason equation**, 

$$ \delta \lambda^A_A = 0, \quad (43) $$

where $\delta \equiv m^\mu D_\mu$. The quasilocal energy $E$ from (13) is associated with the fleet of observers at $B$ who are instantaneously at rest in the spanning $\Sigma$ slice (their world lines are the integral curves of $u$). However, notice that the $\delta$ operator is insensitive to local boosts of the $B$ timelike normal, and, therefore, the Dougan-Mason equation does not depend on the $\Sigma$ slice spanning $B$. In general one would not expect to find a set of solution spinors for (43) which provide the $\Sigma$ normal $u$ as their associated timelike vector. Rather, one should, perhaps, turn the situation around and let (43) **determine** a preferred spanning slice $\Sigma$. In the Schwarzschild context it is possible to achieve a satisfactory interpretation of Dougan-Mason energy from this perspective.

**Appendix B.3** presents the general solution to (43) for the case of round spheres in Schwarzschild. One may choose 

$$ \begin{align*}
\lambda^1_A &= i \left( \frac{\alpha}{p} \right)^{1/2} \left( -\alpha^{-1} \zeta o_A + i_A \right), \\
\lambda^2_A &= i \left( \frac{\alpha}{p} \right)^{1/2} \left( \alpha^{-1} o_A + \bar{\zeta} i_A \right)
\end{align*} $$

(44)

as the the set of solution spinors. This set is normalized dyad, 

$$ \varepsilon^{AB} \lambda^A_A \lambda^B_B = 1, \quad (45) $$

however, the associated timelike vector,

$$ \bar{u}^\mu = \left( \frac{1 + \alpha^2}{2\alpha} \right) u^\mu + \left( \frac{1 - \alpha^2}{2\alpha} \right) n^\mu, \quad (46) $$

is not the hypersurface normal of a $t = \text{constant}$ slice $\Sigma$. However, it has the boost-form

$$ \bar{u}^\mu = \gamma u^\mu + v \gamma n^\mu, \quad (47) $$

where $v = (1 - \alpha^2)(1 + \alpha^2)^{-1}$. For each value of $r_+$ one just lets $\bar{u}^\mu$ **select** a new spanning three-slice $\Sigma$. 

17
Inserting the Dougan-Mason spinors in the spinorial QLE expression (36), one finds

\[ E_{DM} = \int_B d^2x \sqrt{\sigma} (\gamma_+ \varepsilon_1 - v_+ \gamma_+(j_1)_-) - E_0 , \]  

(48)

where \( v_+ = v(r_+) \) and \( E_0 \) has exactly the same form as before in (12) (as may be verified by direct calculation). The above expression can also be written as

\[ E_{DM} = \int_B d^2x \sqrt{\sigma} \bar{\varepsilon}_1 - \bar{E}_0 , \]  

(49)

where \( \bar{\varepsilon}_1 = 1/\kappa \bar{k} \) with \( \bar{k} \) representing the trace of the extrinsic curvature of \( B \) as embedded in the selected spanning slice \( \bar{\Sigma} \). The first term is the Brown-York unreferenced QLE associated the fleet of \( \Sigma \) observers at \( B \). Further, \( E_0 \) is the correct flat-space reference term. It is the metric data of \( B \) which is crucial when determining the reference term (\( B \) must be isometrically embedded in \( R^3 \) in order to obtain \( \bar{\varepsilon}_0 = 1/\kappa \bar{k}_0 \)). Of course the induced metric on \( B \) is the same whether \( B \) is viewed as embedded in \( \Sigma \) or \( \bar{\Sigma} \). Hence, the flat-space reference terms \( \bar{E}_0 \) and \( E_0 \) should agree. Of course, for Schwarzschild the \( (j_1)_- \) term vanishes, and one can verify that (with the chosen metric-signature convention) the expression (48) enjoys the remarkable feature of giving the value of \( M \) for any value of \( r_+ \).

The Dougan-Mason equation (48) determines just the “right” slice such that for each \( r_+ \) the associated Brown-York QLE is \( \bar{E} = M \). It is of interest to examine these slices in greater detail.\footnote{For a single isolated sphere the Dougan-Mason spinors actually determine only an equivalence class of slices. In determining the new foliation of \( M \) below, one is solving the Dougan-Mason equation on each of a whole family of spheres in spacetime.} As a one-form the \( \bar{\Sigma} \) normal is

\[ - \bar{u}_\mu dx^\mu = \frac{\gamma}{\alpha} (dt - v \alpha^2 dr) . \]  

(50)

Defining the new coordinate

\[ \bar{t} = t - M \log \left( \alpha^4 - 1 \right) , \]  

(51)

one can rewrite this one-form as

\[ - \bar{u}_\mu dx^\mu = \bar{N} d\bar{t} , \]  

(52)
where $\bar{N} = \gamma N = \gamma/\alpha = 1/\bar{\alpha}$. The spatial slices selected by the Dougan-Mason equation (48) are level hypersurfaces of $\bar{t}$. Consider the coordinate transformation $(t, r, \theta, \phi) \rightarrow (t', r', \theta', \phi') = (\bar{t}, r, \theta, \phi)$, under which the line element becomes

$$ds^2 = -\bar{N}^2 d\bar{t}^2 + \left[\bar{\alpha} dr - v \bar{N} d\bar{t}\right]^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right).$$

(53)

Clearly, the induced positive definite metric on $\bar{\Sigma}$ is

$$\bar{h}_{ij} dx^i dx^j = \bar{\alpha}^2 dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right).$$

(54)

For this metric it is straightforward to calculate that

$$\bar{\varepsilon}_1 = \frac{1}{\kappa} \bar{k} = -\frac{2}{\kappa \bar{\alpha} r_+} = -\gamma_+ \frac{2}{\kappa \bar{\alpha} r_+},$$

(55)

which is in accord with the result (48). Note, however, that the barred slicing is not a “rest frame” in the sense that

$$(\bar{j}_1)_+ = v_+ \gamma_+ \frac{2}{\kappa \bar{\alpha} r_+}.$$

(56)

The relation between the momentum integral

$$\bar{P}_1)_+ \equiv \int_B d^2 x \sqrt{\sigma} (\bar{j}_1)_+$$

(57)

and the Dougan-Mason $P_{AA'}$ is not immediate, since none of the spacelike vectors associated with the solution spinors are $\bar{n}^\mu$, the spacelike normal of $B$ in $\bar{\Sigma}$.

4 Discussion

This paper has interpreted in a new light the role played by the Sen-Witten equation in the spinorial expression for energy. Arguably, the “job” of the Sen-Witten equation is to provide a definite reference point for the energy. It should be stressed that the energy expression (18) determined by Sen-Witten spinors is not offered as a substitute for the definition of Brown-York QLE given in (5), since the Sen-Witten energy expression does not depend solely on the fleet of Eulerian observers at $B$. However, the Sen-Witten expression for
the energy always has the same form as the Brown-York expression. Further, as long as $u^\mu$ at $B$ is preserved, perturbations of $\Sigma$ only affect the reference point of the Sen-Witten energy expression. For the case of round spheres embedded in the preferred time slices of the Schwarzschild geometry, the reference point determined by the Sen-Witten equation is flat-space. Clearly, the Sen-Witten equation does not determine the flat-space reference for a generic two-surface embedded in a spatial section of an arbitrary spacetime, because in general the spatial section is not conformally flat. It is of interest to determine under what criteria the Sen-Witten equation does determine the flat-space reference. Perhaps, by choosing a special spanning slice in some way, one could use the spinor machinery as an effective way to solve the embedding problem and compute the appropriate flat-space $\varepsilon_0 = 1/\kappa k_0$ (when it is possible to do so).

The Schwarzschild-case QLE of Dougan and Mason has also been viewed from a new perspective. Using the Sen-Witten equation, one “hand selects” the spanning slice for which the energy is to be evaluated. However, at least for the Schwarzschild-case, the Dougan-Mason equation selects a distinguished slice $\bar{\Sigma}$ which spans the two-sphere $B$ such that the associated Brown-York QLE is the Schwarzschild mass $M$. It is known that in general the Dougan-Mason construction breaks down for “exceptional” two-surfaces.\[12\] Perhaps, this is related to an inability to define a flat-space (or otherwise) reference point for the energy. The construction of the flat-space $\varepsilon_0 = 1/\kappa k_0$ previously described makes sense only when $B$ can be embedded isometrically in $\mathbb{R}^3$ (and if the embedding is in a certain sense unique). Riemannian manifolds with two-sphere topology and everywhere positive curvature may be globally immersed (an immersion allows self-intersection) in $\mathbb{R}^3$ (the immersion is unique up to translations and rotations).\[6,\] This may also be contrasted with the fact that in order for the (holomorphic-case) Dougan-Mason QLE to be non-negative, one must assume that the spin coefficient $\rho' = -\mu$, the convergence of the inward null normal to $B$, must be non-negative. This implies that the two-surface is “suitably” convex or, in other words, has no indentations.\[12\]
5 Acknowledgments

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A Supplementary spinor equations

This appendix presents the Schwarzschild-case solutions to the Sen-Witten and Dougan-Mason spinorial equations. The vehicle for examining these equations is the Newman-Penrose formalism. Since the Newman-Penrose formalism is usually employed with the $(+,−,−,−)$ metric-signature convention, it is helpful to cast some of the basic results in the $__(−,+,+,+)__$ convention adopted here.

A.1 Newman-Penrose formalism

Null tetrad A convenient null tetrad associated with the line element [4] is

\[
\begin{align*}
e_1 &= k = \frac{1}{\sqrt{2}} \left( \frac{1}{N} \frac{\partial}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial r} \right) \\
e_2 &= l = \frac{1}{\sqrt{2}} \left( \frac{1}{N} \frac{\partial}{\partial t} - \frac{1}{\alpha} \frac{\partial}{\partial r} \right) \\
e_3 &= m = e^{-i\phi} \left( \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right) \\
e_4 &= \bar{m} = e^{i\phi} \left( \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right)
\end{align*}
\]

(58)

where the convention is that barred Greek indices $\bar{\mu}$ represent null-tetrad indices and labels and run over $(1,2,3,4)$. This frame has been rigged to ensure that

\[
m = -\frac{P}{\sqrt{2}r} \frac{\partial}{\partial \zeta},
\]

(59)

as seen before in (35).
Associated spin coefficients Define the following derivative operators:

\[ D \equiv k^\mu D_\mu ; \quad \delta \equiv m^\mu D_\mu ; \quad \bar{\delta} \equiv \bar{m}^\mu D_\mu ; \quad \Delta \equiv l^\mu D_\mu . \quad (60) \]

Recalling that the \( SL(2, C) \) spin connection is compatible with the affine connection in the sense of (44), one can calculate the spin coefficients,

\[
\alpha = -\iota_A \delta o^A = \frac{1}{2} (\Gamma_{124} + \Gamma_{434}) = -\frac{\zeta}{2\sqrt{2} r} \\
\beta = -\iota_A \delta o^A = \frac{1}{2} (\Gamma_{123} + \Gamma_{433}) = \frac{\bar{\zeta}}{2\sqrt{2} r} \\
\mu = -\iota_A \delta l^A = \Gamma_{423} = -\frac{1}{\sqrt{2} \alpha r} \\
\rho = -o_A \delta o^A = \Gamma_{134} = -\frac{1}{\sqrt{2} \alpha r} \\
\gamma = -\iota_A \Delta o^A = \frac{1}{2} (\Gamma_{122} + \Gamma_{432}) = \frac{1}{2\sqrt{2} \alpha} \frac{d}{dr} \log N \\
\varepsilon = -\iota_A D o^A = \frac{1}{2} (\Gamma_{121} + \Gamma_{431}) = \frac{1}{2\sqrt{2} \alpha} \frac{d}{dr} \log N ,
\]

where the affine connection coefficients with respect to the null frame (58) are given by \( \Gamma_{\bar{\mu} \bar{\sigma} \bar{\rho}} = e_{\bar{\rho}}^\lambda e_\bar{\sigma}^\kappa \nabla_\kappa e_\bar{\lambda} \) and may calculated by any of a variety of methods. For the chosen null frame, all other spin coefficients vanish. One should beware that on the left-hand side of these relations \( \alpha \) denotes a spin coefficient, while on the right-hand side \( \alpha \) denotes the radial lapse function. Also, here \( \varepsilon \) is a spin coefficient and not the QLE density.

A.2 Sen-Witten equation

Transvecting the Sen-Witten equation \( \Sigma D^{AA'} \lambda_{A} = 0 \) with \( \bar{o}^{A'} \) and \( \bar{\iota}^{A'} \), one obtains the set

\[
\bar{\delta} \lambda_1 - \frac{1}{2} (D - \Delta) \lambda_2 + \lambda_1 \left( -\alpha + \frac{\pi}{2} - \frac{\nu}{2} \right) + \lambda_2 \left( \rho - \frac{\varepsilon}{2} + \frac{\gamma}{2} \right) = 0
\]
\[
\delta \lambda_2 + \frac{1}{2} (D - \Delta) \lambda_1 + \lambda_1 \left( -\mu - \frac{\varepsilon}{2} + \frac{\gamma}{2} \right) + \lambda_2 \left( \beta + \frac{\kappa}{2} - \frac{\tau}{2} \right) = 0.
\]  
(As mentioned, the coefficients \(\pi, \nu, \kappa,\) and \(\tau\) vanish for the Schwarzschild geometry. With the adopted conventions, the general expressions for these coefficients in terms of the spin dyad have the same forms as found in [11].)

For the case at hand these equations take the following specific form:

\[
P \frac{\partial \lambda_1}{\partial \zeta} + \frac{r}{\alpha} \frac{\partial \lambda_2}{\partial r} - \frac{\zeta \lambda_1}{2} + \frac{\lambda_2}{\alpha} = 0
\]

\[
P \frac{\partial \lambda_2}{\partial \zeta} - \frac{r}{\alpha} \frac{\partial \lambda_1}{\partial r} - \frac{\bar{\zeta} \lambda_2}{2} - \frac{\lambda_1}{\alpha} = 0,
\]

where \(\alpha\) is the radial lapse in these equations. The ansatz

\[
\lambda_1 = f(r) P^{-1/2} \left( a - b \bar{\zeta} \right)
\]

\[
\lambda_2 = g(r) P^{-1/2} (a \zeta + b)
\]

\((f, g\) are real functions and \(a, b \in \mathbb{C})\) reduces these equations to the following system of coupled ordinary differential equations:

\[
\frac{dg}{dr} + \frac{g}{r} - \frac{\alpha f}{r} = 0
\]

\[
\frac{df}{dr} + \frac{f}{r} - \frac{\alpha g}{r} = 0.
\]

Viewed as an initial-value problem with data specified at \(r = 2M\), the solution to this set of equations is

\footnote{This ansatz is most readily suggested if one works in the Geroch-Held-Penrose compacted spin coefficient formalism \([11]\) in the manner of Dougan in Ref. \([20]\).}
\[
\begin{pmatrix}
  f \\
g
\end{pmatrix} = \begin{pmatrix}
\frac{X}{2} + \frac{4M^2}{2Xr^2} & \frac{X}{2} - \frac{4M^2}{2Xr^2} \\
\frac{X}{2} - \frac{4M^2}{2Xr^2} & \frac{X}{2} + \frac{4M^2}{2Xr^2}
\end{pmatrix}
\begin{pmatrix}
f_0 \\
g_0
\end{pmatrix},
\]
where \(f_0 = f(2M), g_0 = g(2M),\) and \(X = \alpha^{-2} + 2\alpha^{-1} + 1.\) One should notice that

\[
\lim_{r \to 2M} X = 1; \quad \lim_{r \to \infty} X = 4,
\]
and that on the interval \([2M, \infty)\) \(X\) is monotonically increasing. Hence, one finds that the general solution to the Sen-Witten equation associated with the Schwarzschild geometry may be written as

\[
\lambda_1 = X P^{-1/2} \left( a - b \bar{\zeta} \right) \left( \frac{f_0 + g_0}{2} \right) + \frac{4M^2}{2Xr^2} P^{-1/2} \left( a - b \bar{\zeta} \right) \left( \frac{f_0 - g_0}{2} \right)
\]
\[
\lambda_2 = X P^{-1/2} \left( a \zeta + b \right) \left( \frac{f_0 + g_0}{2} \right) - \frac{4M^2}{2Xr^2} P^{-1/2} \left( a \zeta + b \right) \left( \frac{f_0 - g_0}{2} \right)
\]

One seeks two linearly independent solutions \(\{\lambda_A^A = 1, 2\}\) which have the appropriate behavior at the finite radius \(r_+ > 2M.\) To obtain the desired behavior, first define \(X_+ = X(r_+)\) and set \(f_0 = g_0 = X_+^{-1}.\) With this choice, from the set \(\lambda_A^A\) \(\lambda_A^1\) corresponds to the selection \(a = 0\) and \(b = i\) while \(\lambda_A^2\) is determined by \(a = i\) and \(b = 0.\)

### A.3 Dougan-Mason equation

Dougan has given a full treatment \([20]\) of the Dougan-Mason equation for the case of round spheres, and the method followed here has been inspired by this treatment. Transvection of \(\delta \lambda_A = 0\) by each of the members of the spin dyad yields two coupled equations

\[
m [\lambda_1] - \beta \lambda_1 + \sigma \lambda_2 = 0
\]
\[
m [\lambda_2] + \beta \lambda_2 - \mu \lambda_1 = 0,
\]

24
For the present Schwarzschild case the first equation is

\[
\frac{\partial}{\partial \zeta} \log \lambda_1 + \frac{\bar{\zeta}}{2P} = 0. \tag{70}
\]

Direct integration yields that

\[
\lambda_1 = P^{-1/2} \left( c + d \bar{\zeta} \right), \tag{71}
\]

where \( c(\bar{\zeta}, r) \) and \( d(\bar{\zeta}, r) \) are undetermined functions of \( \bar{\zeta} \) and \( r \) (or one may hold that \( r = r_{+} \)). Next, insert the result for \( \lambda_1 \) into the second equation of (69) to find (again, \( \alpha \) is the radial lapse below)

\[
\frac{\partial \lambda_2}{\partial \zeta} - \frac{\bar{\zeta} \lambda_2}{2P} - \frac{\left( c + d \bar{\zeta} \right)}{\alpha P^{3/2}} = 0. \tag{72}
\]

Integration of this equation gives

\[
\lambda_2 = \alpha^{-1} P^{-1/2} \left( c \zeta - d \right). \tag{73}
\]

Therefore, the general solution to the (Schwarzschild-case) Dougan-Mason equation (13) is

\[
\lambda_A = \alpha^{-1} P^{-1/2} \left( c \zeta - d \right) \sigma_A - P^{-1/2} \left( c + d \bar{\zeta} \right) \iota_A. \tag{74}
\]

One seeks a set linearly independent solutions which serves as a normalized spin dyad. From the set (14) one sees that \( \lambda_A^1 \) corresponds to the choice \( c = -i \sqrt{\alpha} \) and \( d = 0 \), while \( \lambda_A^2 \) corresponds to the choice \( d = -i \sqrt{\alpha} \) and \( c = 0 \).

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