Invariant star-products on $S^2$ and the canonical trace

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Abstract

In the literature there are two different ways of describing an invariant star product on $S^2$. We show that the products are actually the same. We also calculate the canonical trace and use the Fedosov-Nest-Tsygan index theorem to obtain the characteristic class of this product.

1 Introduction

The progress in understanding aspects of noncommutative geometry has been immense during the past ten years. The formal theory of deformation quantization \cite{1,2} has led to many beautiful discoveries and some use has also been found in the realm of physics \cite{3,4,5}, i.e. low energy effective theories for strings etc.

The most famous deformation quantization is the one which gives the Moyal product. All other deformations can be identified locally with this deformation, see for instance \cite{6}.

The important interplay between topology and geometry in the subject of deformation quantization is made explicit in the theorem by Fedosov, Nest and Tsygan \cite{7,8}. This theorem connects the canonical trace to a topological index of the noncommutative manifold.

This paper has two parts. In the first we investigate the two invariant star-products on $S^2$ \cite{9,10} and show that they are actually the same. In the last section we consider the canonical trace. We construct a local isomorphism connecting our star-product on the sphere and the Moyal product. We use this isomorphism to calculate the canonical trace of the product on $S^2$. This gives us the opportunity to calculate the characteristic class for the star-product on $S^2$, using the theorem by Fedosov, Nest and Tsygan.

2 The Star Product

There are two descriptions of invariant star-products on $S^2$ present in the literature today. These are presented in the papers \cite{9,10}. The main point of this
section will be to prove that the products described are actually the same.

We begin by investigating the star product on the sphere which is defined in \cite{9}, and is given by,

\[ f \star g = fg + \sum_{n=1}^{\infty} C_n \left( \frac{\hbar}{r} \right) J^{a_1 b_1} \cdots J^{a_n b_n} \partial_{a_1} \cdots \partial_{a_n} f \partial_{b_1} \cdots \partial_{b_n} g, \]

where

\[ C_n \left( \frac{\hbar}{r} \right) = \frac{\left( \frac{\hbar}{r} \right)^n}{n!(1 - \frac{1}{r})(1 - 2\frac{\hbar}{r}) \cdots (1 - (n-1)\frac{\hbar}{r})}, \]

and

\[ J^{ab} = r^2 \delta_{ab} - x_a x_b + i r \epsilon_{abc} x_c. \]

The star product is defined on \( \mathbb{R}^3 \setminus \{0\} \), but can be restricted to two-spheres centered at the origin since \( f(r^2) \star g(x) = g(x) \star f(r^2) = f(r^2)g(x) \) \cite{9}, and it is rotation invariant since \( J^{ab} \) is a covariant 2-tensor.

Now, let us turn to the product of the type that can be found in \cite{10}. Left invariant vector fields on \( \text{SU}(2) \) correspond to elements of the Lie algebra \( \mathfrak{su}(2) \).

If \( X \in \mathfrak{su}(2) \) we get a vector field acting on functions,

\[ X(f(a)) = \frac{d}{dt} f(a \exp(tX))|_{t=0}, \]

\( f \in C^\infty(\text{SU}(2)) \) and \( a \in \text{SU}(2) \). If \( Z \in \mathfrak{sl}(2, \mathbb{C}) \) then,

\[ Z(f) = X(f) + iY(f), \quad X, Y \in \mathfrak{su}(2). \]

Take \( L_a \in \mathfrak{su}(2) \) so that \([L_a, L_b] = -\epsilon_{abc} L_c\), and define \( L_\pm = L_1 \pm iL_2 \). The same symbols are used for the Lie algebra elements and the corresponding vector fields.

We have now described how the operators act on functions on \( \text{SU}(2) \), but we want to define operators acting on functions on \( S^2 \). We use the fact that \( S^2 = \text{SU}(2)/\text{U}(1) \), and let our \( \text{U}(1) \) be the subgroup generated by \( L_3 \). To lift functions on \( S^2 \) to \( \text{SU}(2) \) we use functions on \( \text{SU}(2) \) satisfying, \( f(a) = f(ah) : \forall a \in \text{SU}(2), \forall h \in \text{U}(1) \). For two such functions \( f \) and \( g \) it can be shown that \( L^n f L^n g \) again has the same property, but \( L_\pm f \) by itself would not be defined as a function on \( S^2 \). This implies that,

\[ f \circ g = fg + \sum_{n=1}^{\infty} C_n \left( \frac{\hbar}{r} \right) L_n f L_n g, \]

is a well defined product for functions on \( S^2 \), where we use the same symbols for functions on \( S^2 \) and their lifts.

**Proposition 1**

\[ f \circ g = f \star g, \quad \forall f, g \in C^\infty(S^2). \]

**Proof.** First we note that, at the unit element, left and right invariant vector fields agree, and the right invariant vector fields project down to \( S^2 \). For \( L_a \) the projections to \( S^2 \) embedded in \( \mathbb{R}^3 \) are the well known \( J_a = \epsilon_{abc} x_b \partial_c \). This
means that at the point corresponding to the orbit of the unit, which we call
the “north pole”, where \( x_1 = x_2 = 0 \) and \( x_3 = r \), we can use,

\[
L_+ = x_3 \partial_+ + x_- \partial_3, \quad L_- = x_3 \partial_- + x_+ \partial_3,
\]
where \( x_- = x_2 - ix_1 \), \( x_+ = x_2 + ix_1 \),

\[
\partial_+ = -\partial_{x_2} + i\partial_{x_1}, \quad \partial_- = -\partial_{x_2} - i\partial_{x_1}.
\] (7)

It is enough to show that the star products (1) and (6) are the same at one point
since they are both spherically symmetric. To first order the products agree at
the “north pole” since there

\[
J_{ab} \partial_a f \partial_b g = L_- f L_+ g = L_+^a L_-^b \partial_a f \partial_b g,
\] (8)

where we have written the \( L_\pm \) operators in cartesian coordinates as
\( L_\pm^a = L_\pm^a \partial_a \).

This means that \( J_{ab} = L_+^a L_-^b \) and with our choice of coordinates (7) it is easy
to see using \( \partial_+ x_- = \partial_- x_+ = 0 \), that at the “north pole” we have,

\[
L_\pm^n f = L_\pm^1 \ldots L_\pm^n \partial_a \ldots \partial_{a_n} f = r^n L_\pm^n f.
\] (9)

This means that,

\[
L_+^a L_-^b g = L_+^a \ldots L_+^1 \ldots L_+^a \partial_a \ldots \partial_{a_n} f \partial_{b_1} \ldots \partial_{b_n} g
= J^{a_1b_1} \ldots J^{a_nb_n} \partial_{a_1} \ldots \partial_{a_n} f \partial_{b_1} \ldots \partial_{b_n} g
= r^{2n} \partial_+^n f \partial_-^n g,
\] (10)

and when put into the definitions (1) and (6), the proposition follows. □

3 The Canonical Trace and the Characteristic
class θ

In this section we will calculate the canonical trace and use the Fedosov-Nest-
Tsygan index theorem [7, 8], to calculate the characteristic class of the invariant
star-product on \( S^2 \). The theorem identifies the canonical trace of the identity
to a topological index of the noncommutative manifold. Before we give the
formulation of the theorem, we will go through the main definitions of the
objects involved.

Given a manifold \( M \) and a star-product on it, one can define a map \( f \to Tr(f) \) s.t. \( Tr(f \ast g) = Tr(g \ast f) \). This map is cyclic and is called a trace. A
distinguished such trace is the canonical trace defined by the canonical trace
density of the star-product.

The canonical trace of a star-product may be defined via the canonical trace of
the Moyal product \( \ast_m \)[11],

\[
Tr_{can}(f) = \int f \mu_m
\] (11)

where \( \mu_m \) is the formal trace density of the Moyal product as stated in sec.
3.2. For any other star-product you just replace the canonical trace density of
the Moyal product by that of the new product. To find the new trace density
one does as follows. Given two star-products, the Moyal product \( \ast_m \) and some
other product \( \ast \), they may always be identified on some neighborhood [6]. The
two densities can be related due to the equivalence of the products on the neighborhood. Let $F$ denote the equivalence operator of the two star-products on the neighborhood and call the neighbourhood $U$. One may then calculate the trace densities, on this neighborhood, via the equivalence operator as follows,

$$\int_U F f \mu_{\text{can}} = \int_U f \mu_m$$

(12)

where $\mu_{\text{can}}$ and $\mu_m$ are the trace densities of the two star-products. Now to calculate the canonical trace globally for some product other than the Moyal product we would have to connect the trace densities on an atlas of charts. This may be done since the densities agree on the overlaps, which make the identification a well defined procedure. This can however be quite cumbersome. Due to the symmetries of the product on $S^2$ it will be shown that the information at one point will be enough for our calculation.

This gives us some insight into the trace part, but there is more information needed to formulate the theorem. There are topological quantities involved in the theorem, since it connects the canonical trace to certain cohomology classes of the manifold $M$. These are the three characteristic classes, $\theta(M)$, $c_1(M)$ and $\text{Todd}(M)$. The first one, for the symplectic case, belongs to $\omega/2\pi \hbar + H^2(M)[[\hbar]]$ and classifies star-products up to equivalence. The other two classes are combined into $\hat{A}(M) = e^{-c_1(M)/2\hbar} \text{Todd}(M)$. Here $c_1(M)$ is the first Chern class of the manifold $M$, with the bundle structure given by the complex structure induced by the symplectic form $\omega$. For $S^2$ we have $\hat{A}(S^2) = 1$.

Now, let us formulate the index theorem that we will investigate, the Fedosov-Nest-Tsygan index theorem for a compact symplectic manifold $M$.

**Theorem 1** $Tr_{\text{can}}(1) = \int_M e^\theta \hat{A}(M)$.

We will calculate the characteristic class $\theta$ of our product by comparing the trace with the topological index. First we will find the equivalence operator needed to calculate the trace.

### 3.1 Transforming the Star Product

Two star-products $\star_1$ and $\star_2$ defined on the same manifold $M$ are called equivalent if there exists an operator $F = \sum (\frac{\hbar}{2})^k D_k$, where $D_k$ are differential operators, such that,

$$f \star_2 g = F^{-1}(F f \star_1 F g), \quad \forall f, g \in C^\infty(M).$$

(13)

In our case we want to look at an operator that does such a transformation around the “north pole”. We transform the product in (6), here called $\star_1$, to a star product, $\star_2$, which we call the polarized Moyal product and is given by the following expression,

$$f \star_2 g = fg + \sum_{n=1}^{\infty} \frac{(\frac{\hbar}{2})^n}{n!} r^{2n} \partial_n f \partial_n^2 g.$$  

(14)

This can be transformed into the real Moyal product by doing one further trivial transformation using the operator $F_2 = \exp(\frac{\hbar}{2\pi} \partial_- \partial_+)$.
\( \mathcal{F} \) may be taken to be U(1) invariant, a proof of which may be found in the appendix. This and the fact that it must transform the original star product correctly give us some restrictions on how \( \mathcal{F} \) can be written. A general differential operator could be written,

\[
\mathcal{F} = \sum c_{ijkl} x^i_+ x^j_+ \partial^k \partial^l_+,
\]

where the \( c_{ijkl} \) are formal numbers, that is elements of \( \mathbb{C}[[\hbar]] \). The U(1) invariance gives us immediately the restriction,

\[
i + l \neq j + k \Rightarrow c_{ijkl} = 0.
\]

The total \( \mathcal{F} \) will have two types of parameters. Some will contribute at the “north pole” and some will not. To see this take equation (17) and perform all derivatives. At the “north pole” all terms for which the derivatives have not annihilated precisely enough \( x^+ \) or \( x^- \) will be zero. Below we will concentrate at coefficients that contribute to \( \mathcal{F} \) at the north pole, i.e. those for which the derivatives annihilated precisely enough \( x^+ \) or \( x^- \) to make the contribution an \( \hbar \)-dependent constant. The set of parameters, that are ignored here, might be needed to extend our \( \mathcal{F} \) to a neighborhood of this point, but these values will not be needed in the calculation of the trace. The value of \( \mathcal{F} \) at the “north pole” is invariant under the change of these parameters.

At the north pole the relation,

\[
\mathcal{F}(f \ast_2 g) = \mathcal{F} f \ast_1 \mathcal{F} g,
\]

reduces to a form where one sees that the \( c_{ijkl} \) coefficients with both \( i \neq 0 \) and \( j \neq 0 \) do not contribute to the result in that specific point. Of the remaining coefficients we claim that only those of the form \( c_{0k0k} \) or \( c_{k0k0} \) can differ from zero. This can be seen by putting in suitable functions \( f \) and \( g \) on both sides and deriving contradictions. For instance, assume that \( c_{00kk} \) differs from zero then choose \( f = x^k_+ \) and \( g = x^k_- \) and derive a contradiction. Then assume we have a nonzero coefficient of the type \( c_{0abc} \) and use two choices of functions to derive a contradiction. Take first \( f = x^b_+ x^c_-, \ g = x^a_+ \) and then \( f = x^a_+ g = x^a_- \), and similarly for the remaining possibility \( c_{a0bc} \neq 0 \). All this means that we can write,

\[
\mathcal{F} = \mathcal{F}_+ + \mathcal{F}_- + \text{irrelevant part},
\]

where \( \mathcal{F}_+ \) only contains \( c_{0k0k} \) coefficients and \( \mathcal{F}_- \) only contains \( c_{k0k0} \) coefficients. Observe that this only holds at the “north pole”. Now \( \mathcal{F}_+ \) can be rewritten as,

\[
\mathcal{F}_+ = F_+(\frac{\hbar}{r}, -\frac{x_+}{2} \partial_+) = \sum_{k=0}^{\infty} \left(\frac{\hbar}{r}\right)^k (-1)^k P^+_k \left( -\frac{x_+}{2} \partial_+ \right),
\]

where \( P^+_k \) are polynomials. \( \mathcal{F}_+ \) satisfies the relation,

\[
\partial^n_+ F_+ \left( \frac{\hbar}{r}, -\frac{x_+}{2} \partial_+ \right) = F_+ \left( \frac{\hbar}{r}, n + \frac{x_+}{2} \partial_+ \right) \partial^n_+.
\]

We have a corresponding expression for \( \mathcal{F}_- \), just exchange + with − indices everywhere.
We must take \( P_0^+(z) + P_0^-(z) = 1 \), and \( P_n^+(0) + P_n^-(0) = 0 \), \( \forall n : n > 0 \), to satisfy the necessary conditions that \( \mathcal{F}(f) \) is equal to \( f \) to zeroth order in \( \hbar/r \) and that \( \mathcal{F}(1) = 1 \). This implies that at the “north pole” \( \mathcal{F}^{-1} = 1 \). Furthermore we have that \( \partial_x^n \mathcal{F} = \partial_x^n f \). This means that for the complete expression we get,

\[
\mathcal{F}^{-1}(\mathcal{F} f \star_1 \mathcal{F} g) = fg + \sum_{n=1}^{\infty} C_n \left( \frac{\hbar}{r} \right) r^{2n} \partial_x^n f \partial_x^n g,
\]

(21)

where \( \mathcal{F} = \mathcal{F}_+ + \mathcal{F}_- \). Now we see that if we find a function \( F \) such that

\[
F(\frac{\hbar}{r}, n) = (1 - \frac{\hbar}{r})(1 - 2\frac{\hbar}{r}) \cdots (1 - (n - 1)\frac{\hbar}{r}), \quad n \in \mathbb{Z}^+, \quad (22)
\]

we will have the wanted product, \( \star_2 \). This \( F \) will have the same form as \( F_+ \) but we have \( P_k = P_k^+ + P_k^- \). We can now formulate the following,

**Proposition 2** The recursion relation \( P_k(z) = (z - 1)P_{k-1}(z - 1) + P_k(z - 1) \), with \( P_0(z) = 1 \) and \( P_k(0) = 0 \), \( \forall n : n > 0 \) uniquely defines \( F \) so that it satisfies the relation (22).

**Proof.** The function is uniquely specified due to the uniqueness of polynomials specified at enough points. This can be seen by looking at the problem combinatorially. Each polynomial \( P_k(z) \) is specified at enough points to make it unique. The proof of the recursion relation is by induction. Restrict the variable to \( z = n \in \mathbb{Z}^+ \). We first note that the theorem is valid when \( n = 1 \) which immediately follows from the recursion relation and the specified values of the polynomials. Now assume that the theorem is valid for all \( n \leq m \). We then look at \( z = m + 1 \) and use \( P_0(z) = 1 \) to write,

\[
F(\frac{\hbar}{r}, m + 1) = 1 + \sum_{k=1}^{\infty} (\frac{\hbar}{r})^k (-1)^k P_k(m + 1) =
\]

\[
= 1 + \sum_{k=1}^{\infty} (\frac{\hbar}{r})^k (-1)^k (mP_{k-1}(m) + P_k(m))
\]

\[
= F(\frac{\hbar}{r}, m)(1 - m\frac{\hbar}{r}) = (1 - \frac{\hbar}{r})(1 - 2\frac{\hbar}{r}) \cdots (1 - m\frac{\hbar}{r}).
\]

(23)

This induction step implies that the relation holds for all \( n \in \mathbb{Z}^+ \), since we know it to be valid for \( n = 1 \).

We can also give the function \( F \) in closed form,

\[
F(\frac{\hbar}{r}, z) = (\frac{\hbar}{r})^{z-1} \frac{\Gamma(\frac{z}{r})}{\Gamma(\frac{z}{r} - (z - 1))}
\]

(24)

as can be seen by a simple calculation where one rewrites the Gamma functions to obtain (22). It can also be shown to fulfill the form given in (19).

### 3.2 The Canonical Trace Density

We now take a look at the trace density and calculate it for our star product. Assume that we have a 2-dimensional symplectic manifold \( M \) with symplectic
2-form $\omega$. Given this data one can define the canonical trace density on the Moyal product to be,

$$\mu_m = \frac{\omega}{2\pi\hbar},$$

(25)

where,

$$\int_M \omega = 2\pi r.$$  

(26)

The canonical trace density for any other star product on our manifold can now be calculated by using the pullback of the canonical trace density of the Moyal product using the operator $\mathcal{F}$. The canonical trace densities of the Moyal product and the polarized Moyal products are the same, so we could use our operator instead of the one that transforms to the real Moyal product. According to theorem [11], any two trace densities of the same star product can only differ by a multiple of a formal number. That is, if $\mu_1$ and $\mu_2$ are two different trace densities of the same star product, $\mu_1 = c(\hbar)\mu_2$.

**Definition 1** A star product is called strongly closed if $\omega$ is a trace density.

This together with the above given observations imply that for a strongly closed star product $\mu_{\text{can}} = c(\hbar)\mu_m$. We also state the following theorem which can be found in [11].

**Theorem 2** If a Lie group $G$ acts transitively on $M$ by symplectomorphism in such a way that the corresponding shifts in the algebra are automorphisms i.e. if $G$ is a symmetry group of the star product, then the star product is strongly closed.

In our case since $\star_1$ is spherically symmetric and SO(3) is a symmetry group of our star product we know that it is strongly closed. So the canonical trace density will differ everywhere from the canonical Moyal trace density by a formal number $c(\hbar)$.

To completely define the canonical trace density of our product, $\star_1$, it would therefore be enough to find the value of $c(\hbar)$. Due to the spherical symmetry this can be done by looking at one point, which gives us the relation

$$c(\hbar) \int_U \mathcal{F}\delta_0\mu_m = \int_U \delta_0\mu_m$$

(27)

$$c(\hbar) = \left( \frac{\int_U \delta_0\mu_m}{\int_U F\delta_0\mu_m} \right) = F\left(\frac{\hbar}{r}, -1\right)^{-1}$$

(28)

c(\hbar) can now be calculated using the following proposition.

**Proposition 3** $F(h/r, -1) = (1 + h/r)^{-1}, \quad h/r \neq -1$

*Proof.* From the recursion relation it immediately follows that $\forall k: P_k(-1) = 1$ when this is put into the definition of $F(h/r, -1)$ we get the Taylor expansion of $(1 + h/r)^{-1}$, which converges to the function except when $h/r = -1$ and this
proves the theorem. □

This result means that,

\[
\text{Tr}_{\text{can}}(1) = \left(1 + \frac{\hbar}{r}\right) \int \mu_m = 1 + \frac{r}{\hbar}.
\]  

(29)

**Proposition 4** The characteristic class of the invariant star-product on \(S^2\) is \(\theta = \omega/2\pi\hbar + c_1(S^2)/2\).

**Proof.** We know that \(\theta(S^2) = \omega/2\pi\hbar + \rho\) with \(\rho \in H^2(S^2)\) \([3]\), and that \(\hat{A}(S^2) = 1\) as stated in the introduction to this section. For compact two dimensional manifolds with a complex structure we know that the integral of \(c_1\) equals the Euler character. If we choose a complex structure \(J\), as in \([8]\), such that \(\omega(Jx, y)\) is positive definite, one has \(\chi = 2\). This information is all that is needed for our calculation.

Using the results above we calculate,

\[
\int_{S^2} e^{\theta \hat{A}} = \frac{r}{\hbar} + \int_{S^2} \rho,
\]

(30)

so that we get, from the index theorem, the value \(\int \rho = 1\) for our product. Now the choice \(\rho = c_1(S^2)/2\) gives the desired result. Here one uses that \(H^2(S^2, \mathbb{R}) = \mathbb{R}\). Hence, the integral of \(\rho\) is sufficient to identify \(\rho\). □

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**Appendix**

We argue that we can choose the map between \(*_{1}\) and \(*_{2}\) to be \(U(1)\) invariant. To prove this we need some definitions.

Let the functions \(a, b, f \in C^{\infty}(U)\), be such that \(a\) is holomorphic and \(b\) antiholomorphic. A star-product will be called a star-product with separation of variables if for any open subset \(U \subset M\) of a symplectic manifold \(M\) it holds that \(a \star f = af\) and \(f \star b = fb\).

Furthermore let the star-product algebra \(\mathcal{A}\) be the algebra of functions written as formal series in \(\hbar\), given by the product \(\star\). Let \(C^k(\mathcal{A}, \mathcal{A})\) be the space of mappings from, the space of \(k\)-multi-differential operators on \(M, \mathcal{A} \otimes \ldots \otimes \mathcal{A}\), to \(\mathcal{A}\).

Between \(C^k(\mathcal{A}, \mathcal{A})\) and \(C^{k+1}(\mathcal{A}, \mathcal{A})\), define the Hochschild coboundary operator \(b\), such that,

\[
(bc)(u_0, \ldots, u_k) = u_0 \star c(u_1, \ldots, u_k) + \sum_{i=0}^{k-1} (-1)^{i+1} c(u_0, \ldots, u_i \star u_{i+1}, \ldots, u_k) \\
+ (-1)^{k+1} c(u_0, \ldots, u_{k-1}) \star u_k,
\]

(31)
for \(u_0, \ldots, u_k \in \mathcal{A}\) and \(c \in C^k(\mathcal{A}, \mathcal{A})\). The Hochschild cohomology groups are the cohomology groups of this complex.

Now consider the specific products of section 3. First note that deformations of Kähler forms \(\omega_i\) are then in 1-1 correspondence with certain star-products \(\star_i\) with separation of variables\[^{12}\]. Hence to each of the products there is a specific Kähler forms. Since \(\omega_{*2}\) belongs to the same cohomology class as \(\omega_{*1}\), on a contractible subset, we know, due to Mosers argument, that for some smooth family of 1-forms \(\beta_i \in \Omega^1(S^2)\) there exists a family of diffeomorphisms \(\phi_i\) such that for \(\omega_t = \omega_0 + d\beta_t\) one has \(\phi_t^* \omega_{*1} = \omega_{*t}\). This gives us \(\phi_t^* \omega_{*2} = \omega_{*1}\). Hence we may identify \(\omega_{*1}\) with \(\omega_{*2}\), using \(\phi_t\).

This will induce an isomorphism between the algebras \(\mathcal{A}_1\) and \(\mathcal{A}_2\), were the index indicates which star product it corresponds to, since there is a 1-1 correspondence between the deformation quantization and the deformations of Kähler forms.

Writing the isomorphism \(D_t\) one has \(f \star_1 g = D_t^{-1}(D_t(f) \star_1 D_t(g))\) which infinitesimally may be written as \(f \star_t g = f \star_1 g + c\alpha_t(f, g)\) where \(\alpha_t\) is a map from \(\mathcal{A} \otimes \mathcal{A}\) to \(\mathcal{A}\).

Now, since demanding \(\star_t\) to be associative is equivalent to demanding that \(\hat{b}\alpha_t = 0\), where \(\hat{b}\) is the Hochschild coboundary operator we know that we may write \(\alpha_t\) as \(\hat{b}c\) for some \(c: \mathcal{A} \to \mathcal{A}\), due to the vanishing of Hochschild cohomology groups on contractible subsets\[^{14}\]. Hence \(\alpha_t\) may be written,

\[
\alpha_t(f, g) = -c_t(f \star_1 g) + c_t(f) \star_1 g + f \star_1 c_t(g) \tag{32}
\]

We now know that there exists an isomorphism \(D_t\) and also its infinitesimal properties given by the Hochschild coboundary operator. Now we may average over \(c_t\)'s using the U(1)-action and produce a U(1) invariant isomorphism which may be integrated and thus extended to a non infinitesimal isomorphism. Hence we may choose \(\mathcal{F}\) to be U(1) invariant.

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