TRIANGULATIONS AND VOLUME FORM ON MODULI SPACES OF FLAT SURFACES

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Abstract

In this paper, we study the moduli spaces of flat surfaces with cone singularities verifying the following property: there exists a union of disjoint geodesic trees on the surface such that the complement is a translation surface. Those spaces can be viewed as deformations of the moduli spaces of translation surfaces in the space of flat surfaces. We prove that such spaces are quotients of flat complex affine manifolds by a group acting properly discontinuously, and preserving a parallel volume form. Translation surfaces can be considered as a special case of flat surfaces with erasing forest, in this case, it turns out that our volume form coincides with the usual volume form (which are defined via the period mapping) up to a multiplicative constant. We also prove similar results for the moduli space of flat metric structures on the \( n \)-punctured sphere with prescribed cone angles up to homothety. When all the angles are smaller than \( 2\pi \), it is known (cf. [Th]) that this moduli space is a complex hyperbolic orbifold. In this particular case, we prove that our volume form induces a volume form which is equal to the complex hyperbolic volume form up to a multiplicative constant.

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# Introduction

While the moduli space of (half)-translation surfaces has been studied for a rather long time, little is known about the topology of its strata, we only know that each stratum contains finitely many (in fact, at most three) connected components thank to recent works of Kontsevich-Zorich [KZ], and Lanneau [L]. To investigate the topology of the strata, it is often useful to study their deformations. In this paper, we study some deformations of the moduli space of translation surfaces in the space of flat surfaces with conical singularities. For this purpose, we deform a translation surface by replacing a single singular point by a cluster of singular points with fixed cone angles, the angles are chosen so that this replacement can be carried out in a neighborhood of the former singular point without changing the geometric structure on its complement. For technical reasons, we also assume that the new singular points are the vertices of a geodesic tree which will be called an erasing tree, the union...
of such trees is called an erasing forest.

On a flat surface with erasing forest, vector fields which are invariant by parallel transport can be defined on the complement of the erasing forest, we will call such fields parallel vector fields.

As all the trees in an erasing forest shrink to points, a flat surface with erasing forest becomes a usual translation surface, therefore, the moduli space of surface with erasing forest can be viewed as a deformation of the moduli space of translation surfaces. Note that, by definition, a translation surface is a particular flat surface with erasing forest where all the erasing trees are points.

On a flat surface of genus zero, a geodesic tree whose vertex set is the set of singular points is automatically an erasing tree. Since such a tree always exists, the study of flat surface with erasing forest provides us with a common treatment for translation surfaces and flat surface of genus zero.

Before getting into formal definitions and precise statements, let us resume the main results of this paper in the following

**Theorem** The moduli space of flat surfaces with erasing forest and unitary parallel vector field (here, the erasing forest consists of trees which are isomorphic to those of some fixed family of topological trees, the cone angles at the vertices of the erasing forest are fixed) is a quotient of a flat complex affine manifold by a group acting properly discontinuously, preserving a parallel volume form \( \mu_{\mathcal{T}} \).

In the case of translation surfaces, up to a multiplicative constant, the volume form \( \mu_{\mathcal{T}} \) agrees with the usual one, which is defined by the period mapping.

For the case of flat surfaces of genus zero, the volume form \( \mu_{\mathcal{T}} \) induces a volume form \( \hat{\mu}_{\mathcal{T}}^1 \) on the moduli space of flat surfaces of genus zero of unit area with fixed cone angles. If all the cone angles are less than \( 2\pi \), then \( \hat{\mu}_{\mathcal{T}}^1 \) agrees with the complex hyperbolic volume form defined by Thurston up to a multiplicative constant.

In a forthcoming paper, we will prove that the volume of the moduli space of flat surfaces with erasing forest with respect to \( \mu_{\mathcal{T}} \), normalized by some energy functions involving the area of the surface and the length of the trees in the erasing forest, is finite. Using this result, one can recover the classical results of Masur-Veech, and Thurston on the finiteness of the volume of the moduli space of translation surface, and of the moduli space of flat surface of genus zero.

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2 Background and statement of main results

2.1 Flat surfaces and their moduli spaces

Let \( \Sigma \) be an oriented, connected, closed surface, and \( Z \) a finite subset of \( \Sigma \). A flat surface structure on \( \Sigma \) with conical singularities at \( Z \) is an Euclidean metric structure on \( \Sigma \setminus Z \) so that for every \( s \) in \( Z \), a neighborhood of \( s \) is isometric to an Euclidean cone. Note that if the cone angle at a point in \( Z \) is \( 2\pi \) then this point is actually a regular point. Throughout this paper, we will call a flat surface with conical singularities a flat surface.

Translation surfaces are flat surfaces such that the holonomy of any closed curve, which does not contain any singular point, is a translation of \( \mathbb{R}^2 \). On such surfaces, given a direction \( \theta \) in \( S^1 \), there exists a foliation \( \mathcal{F}_\theta \) of the surface by parallel lines in direction \( \theta \).

The moduli spaces of flat surfaces have been studied by numerous authors [BG], [EM], [Th], [V2]. For the case of polyhedral flat surfaces, let \( \alpha_1, \ldots, \alpha_n \) be \( n \) positive real numbers such that \( \alpha_1 + \cdots + \alpha_n = 2\pi(n - 2) \). Let \( C(\alpha_1, \ldots, \alpha_n) \) denote the space of all flat surface structures of unit area on the sphere \( S^2 \) having exactly \( n \) singular points with cone angles \( \alpha_1, \ldots, \alpha_n \). The following result is proved in [Th].

**Theorem (Thurston)** Assume that all the angles \( \alpha_i, \ i = 1, \ldots, n \) are smaller than \( 2\pi \), then \( C(\alpha_1, \ldots, \alpha_n) \) is a complex hyperbolic orbifold of dimension \( n - 3 \), whose volume is finite.

A necessary condition for a flat surface to be a translation surface is that the cone angle at every singular point must belong to \( 2\pi \mathbb{N} \). As a consequence, there are no translation surface structures on the 2-sphere.

A translation surface together with a foliation by parallel lines can be identified to a pair \((M, \omega)\), where \( M \) is a closed connected Riemann surface, and \( \omega \) is a holomorphic 1-form on \( M \). By this identification, a zero of order \( k \) of the holomorphic 1-form corresponds to a singular point with cone angle \( 2\pi(k + 1) \).

Given positive integers \( g, k_1, \ldots, k_n \) such that \( k_1 + \cdots + k_n = 2g - 2 \), let \( \mathcal{H}(k_1, \ldots, k_n) \) denote the set of all pairs \((M, \omega)\), where \( M \) is a Riemann surface of genus \( g \), and \( \omega \) is a holomorphic 1-form on \( M \) which has exactly \( n \) zeros with orders \( k_1, \ldots, k_n \). We denote by \( \mathcal{H}_1(k_1, \ldots, k_n) \) the subset of \( \mathcal{H}(k_1, \ldots, k_n) \) which corresponds to the set of translation surfaces of area one. It is well known (cf. [EO], [K], [V1], [Z]) that \( \mathcal{H}(k_1, \ldots, k_n) \) is a complex algebraic orbifold of dimension \( 2g + n - 1 \), and there exists a volume form \( \mu_0 \) on \( \mathcal{H}(k_1, \ldots, k_n) \) which is induced by the period mapping \( \Phi \). The map \( \Phi \) is defined locally by the integrals of \( \omega \) over a family of curves representing a basis of \( H_1(M, Z(\omega), \mathbb{Z}) \), where \( Z(\omega) \) is the set of zeros of \( \omega \). This map sends a neighborhood of \((M, \omega)\) in \( \mathcal{H}(k_1, \ldots, k_n) \) to an open subset of \( \mathbb{C}^{2g+n-1} \). The volume form \( \mu_0 \) is the pull-back under \( \Phi \) of the Lebesgue measure on \( \mathbb{C}^{2g+n-1} \).
2.2 Erasing forest

Given a flat surface \( \Sigma \), a tree in \( \Sigma \) is the image of an embedding from a topological tree into \( \Sigma \). We consider an isolated point as a special tree which has only one vertex and no edges. A forest in \( \Sigma \) is a union of disjoint trees in \( \Sigma \). A tree in \( \Sigma \) is said to be geodesic if each of its edges is a geodesic segment in \( \Sigma \). A forest is said to be geodesic if it is a union of geodesic trees.

Definition 2.1 (Erasing forest) Let \( \Sigma \) be a compact connected flat surface without boundary. An erasing forest on \( \Sigma \) is a geodesic forest whose vertex set contains all the singular points of \( \Sigma \) such that, if \( c \) is a closed curve in \( \Sigma \) which does not intersect any tree in the forest, then the holonomy of \( c \) is a translation of \( \mathbb{R}^2 \).

Remark:

- If an erasing forest contains only one tree, we will call this tree an erasing tree.

- Since points are trees, a translation surface has an obvious erasing forest which is the union of all singular points.

- If \( \Sigma \) is a flat surface homeomorphic to the 2-sphere, and \( A \) is a geodesic tree on \( \Sigma \) which connects all the singular points, then \( A \) is automatically an erasing tree, since \( \Sigma \setminus A \) is a topological disk. Throughout this paper, we will call a flat surface of genus zero, a spherical flat surface. On any flat surface, there always exists a geodesic trees whose vertex set is exactly the set of singular points (cf. Proposition 9.1), therefore, spherical flat surfaces can be considered, not in a unique way, as special flat surfaces with an erasing tree.

Given a flat surface \( \Sigma \) with an erasing forest \( \hat{A} \), we call a non vanishing vector filed defined on the complement of \( \hat{A} \) a parallel vector field if its integral lines are parallel lines in the local charts of the Euclidean metric structure.

2.3 Main results

We fix two integers \( g \geq 0, n > 0 \), such that \( 2g + n - 2 > 0 \), and positive real numbers \( \alpha_1, \ldots, \alpha_n \) verifying \( \alpha_1 + \cdots + \alpha_n = 2\pi(2g + n - 2) \). In what follows, \( S_g \) will be a fixed compact, oriented, connected flat surface of genus \( g \), without boundary. We also assume that there exists a geodesic erasing forest \( \hat{A} = \bigcup_{i=1}^{m} A_i \) on \( S_g \), where each \( A_i \) is a geodesic tree. Let \( p_1, \ldots, p_n \) denote the vertices of the trees in \( \hat{A} \), and assume that the cone angle at \( p_i \) is \( \alpha_i \). Let \( \mathcal{V} \) denote the set \( \{p_1, \ldots, p_n\} \). Recall that, by definition, all the singular points of \( S_g \) are contained in \( \mathcal{V} \), but some of the points \( p_i \) may be regular.

Let \( \text{Homeo}^+ (S_g, \hat{A}) \) denote the group of orientation preserving homeomorphisms of \( S_g \) which fix the points in \( \mathcal{V} \), and preserve the forest \( \hat{A} \). Let \( \text{Homeo}_0^+ (S_g, \hat{A}) \) be the normal subgroup of \( \text{Homeo}^+ (S_g, \hat{A}) \) consisting of all elements which can be connected to \( \text{Id}_{S_g} \) by an isotopy fixing the points in \( \mathcal{V} \).

Definition 2.2 (Mapping class group preserving a forest) The mapping class group of the pair \( (S_g, \hat{A}) \) is the quotient group
\[ \Gamma(S_g, \hat{A}) = \text{Homeo}^+(S_g, \hat{A})/\text{Homeo}_0^+(S_g, \hat{A}). \]

We start by defining the space of flat metric structures having an erasing forest isomorphic to \( \hat{A} \) together with a marking, i.e. the Teichmüller space of flat surfaces with erasing forest, we then identify the moduli space of flat surfaces with erasing forest with the quotient of this Teichmüller space under the action of the group \( \Gamma(S_g, \hat{A}) \).

Let \( \alpha \) denote the set \( \{\alpha_1, \ldots, \alpha_n\} \), and let \( \tilde{T}^{et}(\hat{A}, \alpha)^* \) denote the collection of all pairs \((\Sigma, \phi)\), where \( \Sigma \) is an oriented flat surface of genus \( g \), and \( \phi : S_g \to \Sigma \) is an orientation preserving homeomorphism verifying

(i) \( \phi \) maps the set \( \{p_1, \ldots, p_n\} \) bijectively to the set of singularities of \( \Sigma \) such that the cone angle at the point \( \phi(p_i) \) is \( \alpha_i \), \( i = 1, \ldots, n \).

(ii) The image of \( \hat{A} \) under \( \phi \) is an erasing forest on \( \Sigma \).

We define an equivalence relation on \( \tilde{T}^{et}(\hat{A}, \alpha)^* \) as follows: two pairs \((\Sigma_1, \phi_1)\) and \((\Sigma_2, \phi_2)\) are equivalent if there exists an isometry \( h : \Sigma_1 \to \Sigma_2 \) such that the homeomorphism \( \phi_2^{-1} \circ h \circ \phi_1 \) is an element of \( \text{Homeo}_0^+(S_g, \hat{A}) \). The equivalence class of a pair \((\Sigma, \phi)\) is denoted by \([\Sigma, \phi]\). Let \( T^{et}(\hat{A}, \alpha)^* \) be the set of equivalence classes of this relation.

Clearly, the group \( \Gamma(S_g, \hat{A}) \) acts on \( T^{et}(\hat{A}, \alpha)^* \), the quotient space \( T^{et}(\hat{A}, \alpha)^*/\Gamma(S_g, \hat{A}) \) will be called the moduli space of flat surfaces with marked erasing forest and denoted by \( M^{et}(\hat{A}, \alpha)^* \). We denote by \( T^{et}_1(\hat{A}, \alpha)^* \) the set of equivalence classes \([\Sigma, \phi]\), where \( \Sigma \) is a flat surface of area one, and by \( M^{et}_1(\hat{A}, \alpha)^* \) the quotient space \( T^{et}_1(\hat{A}, \alpha)^*/\Gamma(S_g, \hat{A}) \).

**Definition 2.3** The Teichmüller space of flat surfaces with marked erasing forest and parallel vector field is the set of equivalence classes of triples \((\Sigma, \phi, \xi)\), where \((\Sigma, \phi)\) is a pair in \( T^{et}(\hat{A}, \alpha)^* \), and \( \xi \) is a unitary parallel vector field on \( \Sigma \setminus \phi(\hat{A}) \), and the equivalence relation is defined as follows: \((\Sigma_1, \phi_1, \xi_1)\) and \((\Sigma_2, \phi_2, \xi_2)\) are equivalent if there exists an isometry \( h : \Sigma_1 \to \Sigma_2 \) such that

- \( h_{*}\xi_1 = \xi_2 \),
- \( \phi_2^{-1} \circ h \circ \phi_1 \in \text{Homeo}_0^+(S_g, \hat{A}) \).

We denote this space by \( T^{et}(\hat{A}, \alpha) \), the equivalence class of a triple \((\Sigma, \phi, \xi)\) will be denoted by \([\Sigma, \phi, \xi]\).

The moduli space of flat surfaces with marked erasing forest and parallel vector field is the quotient space \( T^{et}(\hat{A}, \alpha)/\Gamma(S_g, \hat{A}) \), we denote it by \( M^{et}(\hat{A}, \alpha) \).

A point in \( M^{et}(\hat{A}, \alpha) \) is then a triple \((\Sigma, \hat{A}, \xi)\), where \( \Sigma \) is a flat surface having exactly \( n \) singularities, with cone angles \( \alpha_1, \ldots, \alpha_n \), \( \hat{A} \) is an erasing forest on \( \Sigma \) isomorphic to \( \hat{A} \), and \( \xi \) is a unitary parallel vector field on \( \Sigma \setminus \hat{A} \). Two topological trees are isomorphic if there exists a continuous mapping from one to the other which induces bijections on the two sets of vertices, and the two set of edges.
Note that, by definition, we have a numbering on the set of vertices of \( \hat{A} \), and hence, a numbering on the set of edges of \( \hat{A} \).

Recall that a flat complex affine manifold is a manifold with an atlas whose transition maps are complex affine transformations. We can now state the main results of this paper.

**Theorem 2.4** The space \( \mathcal{T}^{et}(\hat{A}, \alpha) \) is a flat complex affine manifold of dimension

- \( 2g + n - 1 \) if \( \alpha_i \in 2\pi \mathbb{N} \) for every \( i \in \{1, \ldots, n\} \).
- \( 2g + n - 2 \) otherwise.

The group \( \Gamma(S_g, \hat{A}) \) acts properly discontinuously on \( \mathcal{T}^{et}(\hat{A}, \alpha) \), and preserves a parallel volume form which will be denoted by \( \mu_{TV} \).

In the case where \( \hat{A} \) is a union of points, which implies that \( S_g \) is a translation surface, the space \( \mathcal{M}^{et}(\hat{A}, \alpha) \) can be identified to a stratum \( \mathcal{H}(k_1, \ldots, k_n) \) for some appropriate integers \( k_1, \ldots, k_n \). Recall that, on \( \mathcal{H}(k_1, \ldots, k_n) \), we have a volume form \( \mu_0 \) which is defined by the period mapping. We have

**Proposition 2.5** On each connected component of \( \mathcal{H}(k_1, \ldots, k_n) \), there exits a constant \( \lambda \) such that \( \mu_{TV} = \lambda \mu_0 \).

Let us now focus on the case \( g = 0 \), in this case, we identify \( S_g \) to the standard sphere \( S^2 \), and fix \( n \) points \( p_1, \ldots, p_n \) on \( S^2 \) with \( n \geq 3 \). Fix \( n \) positive real numbers \( \alpha = (\alpha_1, \ldots, \alpha_n) \) such that \( \alpha_1 + \cdots + \alpha_n = 2\pi(n - 2) \). The Teichmüller space of spherical flat surfaces having \( n \) singularities with cone angles \( \alpha_1, \ldots, \alpha_n \) is the set of equivalence classes of pairs \((\Sigma, \phi)\), where

- \( \Sigma \) is a spherical flat surface having \( n \) singularities with cone angles \( \alpha_1, \ldots, \alpha_n \).
- \( \phi \) is a homeomorphism from \( S^2 \) to \( \Sigma \), which sends \( \{p_1, \ldots, p_n\} \) onto the set of singularities of \( \Sigma \) such that the cone angle at \( \phi(p_i) \) is \( \alpha_i \).
- The equivalence class of \((\Sigma, \phi)\) is the set of all pairs \((\Sigma, \phi')\), where \( \phi' \) is a homeomorphism isotopic to \( \phi \) by an isotopy which is constant on the set \( \{p_1, \ldots, p_n\} \).

We denote this Teichmüller space by \( \mathcal{T}(S^2, \underline{\alpha})^* \). The equivalence class of a pair \((\Sigma, \phi)\) in \( \mathcal{T}(S^2, \underline{\alpha})^* \) is denoted by \([[(\Sigma, \phi)]\). Let \( \mathcal{T}_1(S^2, \underline{\alpha})^* \) denote the subset of \( \mathcal{T}(S^2, \underline{\alpha})^* \) consisting of surfaces of unit area.

Let \( \Gamma(0; n) \) denote the modular group of the punctured sphere \( S^2 \setminus \{p_1, \ldots, p_n\} \). Let \( \mathcal{M}(S^2, \underline{\alpha})^* \), and \( \mathcal{M}_1(S^2, \underline{\alpha})^* \) denote the quotients \( \mathcal{T}(S^2, \underline{\alpha})^*/\Gamma(0; n) \), and \( \mathcal{T}_1(S^2, \underline{\alpha})^*/\Gamma(0; n) \) respectively.

Let \( \mathcal{T}(S^2, \underline{\alpha}) \) denote the product space \( \mathcal{T}(S^2, \underline{\alpha})^* \times S^1 \). We extend the action of \( \Gamma(0; n) \) onto \( \mathcal{T}(S^2, \underline{\alpha}) \) by assuming that \( \Gamma(0; n) \) acts trivially on the \( S^1 \) factor of \( \mathcal{T}(S^2, \underline{\alpha}) \). Let \( \mathcal{M}(S^2, \underline{\alpha}) \), and \( \mathcal{M}_1(S^2, \underline{\alpha}) \) denote the spaces \( \mathcal{M}(S^2, \underline{\alpha})^* \times S^1 \), and \( \mathcal{M}_1(S^2, \underline{\alpha})^* \times S^1 \) respectively. First, we have
Proposition 2.6 $\mathcal{T}(S^2, \alpha)$ can be endowed with a flat complex affine manifold structure of dimension $n - 2$, on which $\Gamma(0; n)$ acts properly discontinuously.

By Proposition 9.1, we know that, on any flat surface of genus zero, there always exists an erasing tree whose vertex set is the set of singularities. Therefore, we can identify a neighborhood of a point $([\Sigma, \phi], e^{i\theta})$ in $\mathcal{T}(S^2, \alpha)$ to a neighborhood of a point $([\Sigma, \phi, \xi])$ in $\mathcal{E}(\hat{A}, \alpha)$, where $\hat{A}$ contains only one tree. By this identification, we get a volume form on a neighborhood of $([\Sigma, \phi, \xi])$ in $\mathcal{E}(\hat{A}, \alpha)$, which is induced by the volume form $\mu_T$, depending on the choice of the erasing tree. Our main result in this case is the following

Theorem 2.7 Let $A_1$ and $A_2$ be two erasing trees on $\Sigma$, and let $\mu_{A_1}, \mu_{A_2}$ denote the volume forms corresponding to $A_1, A_2$ respectively which are defined on a neighborhood of $([\Sigma, \phi], e^{i\theta})$ in $\mathcal{T}(S^2, \alpha)$, then we have

$$\mu_{A_1} = \mu_{A_2}.$$  

Consequently, we get a well-defined volume form $\mu_T$ on $\mathcal{T}(S^2, \alpha)$ which is invariant under the action of $\Gamma(0; n)$.

The volume form $\mu_T$ induces naturally a volume form $\hat{\mu}_T^1$ on $\mathcal{M}_1(S^2, \alpha)^*$. In the case where $0 < \alpha_i < 2\pi$, for $i = 1, \ldots, n$, according to the Thurston’s result, $\mathcal{M}_1(S^2, \alpha)^*$ can be equipped with a complex hyperbolic metric structure, hence, we have a volume form $\mu_{\text{Hyp}}$ induced by this metric on $\mathcal{M}_1(S^2, \alpha)^*$. In this situation, we have

Proposition 2.8 There exists a constant $\lambda$ such that $\hat{\mu}_T^1 = \lambda \mu_{\text{Hyp}}$.

This paper is organized as follows: from Section 3 to Section 7 we prove Theorem 2.4, the proof of Proposition 2.5 is given in Section 8, and the proof of Proposition 2.6 in Section 9, Section 10 is devoted to the proof of Theorem 2.7, and finally the proof of Proposition 2.8 is given in Section 11.

3 Admissible triangulation

Let $[(\Sigma, \phi, \xi)]$ be a point in $\mathcal{T}^e(\hat{A}, \alpha)$. Following the method of Thurston in [Th1], we will construct local charts of $\mathcal{T}_T(\bar{\alpha}; \bar{\beta})$ about $[(\Sigma, \phi, \xi)]$ using geodesic triangulations of $\Sigma$. In preparation for this construction, we first define

Definition 3.1 (Admissible triangulation) Let $(\Sigma, \phi)$ be a pair in $\mathcal{T}^e(\hat{A}, \alpha)^*$, an admissible triangulation of $(\Sigma, \phi)$ is a triangulation $T$ of $\Sigma$ such that:

- The set of vertices of $T$ is the set $V = \phi(\{p_1, \ldots, p_n\})$.
- Every edge of $T$ is a geodesic segment.
- The erasing forest $\phi(\hat{A})$ is contained in the 1-skeleton of $T$. 

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The aim of this section is to show the existence and uniqueness up to isotopy of such triangulations on $\Sigma$. First, we have

**Proposition 3.2 (Existence of admissible triangulations)** There always exists an admissible triangulation $T$ for any pair $(\Sigma, \phi)$ in $\mathcal{T}^{ad}(\hat{A}, \alpha)^*$.

**Proof:** If we cut the surface $\Sigma$ along the edges in the forest $\phi(\hat{A})$, then we will obtain a flat surface with piece-wise geodesic boundary. Therefore, the proposition is a particular case of the following lemma

**Lemma 3.3** Let $\hat{\Sigma}$ be a flat surface with piece-wise geodesic boundary if the boundary is not empty, and $\hat{V}$ be a finite subset of $\hat{\Sigma}$ which contains all the singular points. Then there exists a geodesic triangulation of $\hat{\Sigma}$ whose vertex set is $\hat{V}$.

This is a well-known fact, for a proof, the reader can see for example [BS], [V2], or [EM].

Next, we have

**Proposition 3.4 (Uniqueness of admissible triangulations up to isotopy)** Let $T_1$ and $T_2$ be two admissible triangulations of $(\Sigma, \phi)$. If there exists a homeomorphism $h$ in $\text{Homeo}^+(S_g, \hat{A})$ such that $h(T_1) = T_2$ then $T_1 = T_2$.

**Proof:** This proposition is a direct consequence of Lemma 3.5 below.

**Lemma 3.5** Let $\Sigma$ be a flat surface without boundary. Let $V = \{x_1, \ldots, x_n\}$ be a finite subset of $\Sigma$ such that $\Sigma \setminus V$ contains only regular points, and suppose that $\chi(\Sigma \setminus V) < 0$. Let $\gamma$ and $\gamma'$ be two simple geodesic arcs of $\Sigma$ having the same endpoints in $V$ (the two endpoints may coincide). Assume that $\gamma$ and $\gamma'$ are homotopic with fixed endpoints relative to $V$, then we have $\gamma \equiv \gamma'$.

**Proof:** We first observe that there exist no Euclidean structures on a closed disk such that its boundary is the union of two geodesic segments. This is just a consequence of the Gauss-Bonnet Theorem.

Since $\chi(\Sigma \setminus V) < 0$, the universal covering of $\Sigma \setminus V$ is the open disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The flat metric structure on $\Sigma \setminus V$ give rise to a flat metric structure on $\Delta$ (which is not complete). Now, let $\tilde{\gamma}$ be a lift of $\gamma$ in $\Delta$ whose endpoints are contained in the boundary of $\Delta$ . By lifting the homotopy from $\gamma$ to $\gamma'$, we get a lift $\tilde{\gamma}'$ of $\gamma'$ which has the same endpoints as $\tilde{\gamma}$. Note that by assumption, $\tilde{\gamma}$ and $\tilde{\gamma}'$ are two geodesic in $\Delta$.

The two curves $\tilde{\gamma}$ and $\tilde{\gamma}'$ may have intersections, but in any case, we can find (at least) an open disk $D$ which is bounded by two arcs, one is a subsegment of $\tilde{\gamma}$, the other is a subsegment of $\tilde{\gamma}'$. Consequently, the open disk $D$ is isometric to the interior of an Euclidian disk which is bounded by two geodesic segments. Since such a disk cannot exist, we have $\tilde{\gamma} \equiv \tilde{\gamma}'$, and the lemma follows.
4 Flat complex affine structure

In this section, we prove that the space $T_{et}(\hat{A}, \alpha)$ is a flat complex affine manifold and compute its dimension. Let $T_R(S_g, \hat{A})$ denote the set of all equivalence classes of triangulations (not necessarily geodesic) of $S_g$ which contain the forest $\hat{A}$, and whose vertex set is $V$, two triangulations are equivalent if they are isotopic relative to $V$. Let $T$ be an element of $T_R(S_g, \hat{A})$, we consider $T$ as a particular triangulation of $S_g$, and denote by $U_T$ the subset of $T_{et}(\hat{A}, \alpha)$ consisting of triples $[(\Sigma, \phi, \xi)]$ where $\phi$ is a homeomorphism which maps $T$ onto an admissible triangulation of $\Sigma$. Proposition 3.2 implies that the family $\{U_T: T \in T_R(S_g, \hat{A})\}$ covers the space $T_{et}(\hat{A}, \alpha)$. We will define coordinate charts on $U_T$ for each $T$ in $T_R(S_g, \hat{A})$.

4.1 Definition of local charts

Given an equivalence class of triangulations $T$ in $T_R(S_g, \hat{A})$, let $[(\Sigma, \phi, \xi)]$ be a point in $U_T$. By definition, we can assume that $T = \phi(T)$ is an admissible triangulation of $\Sigma$. By Proposition 3.4, we know that $T$ is unique.

Slitting open $\Sigma$ along the erasing forest $\hat{A} = \phi(\hat{A})$, we obtain a translation surface with piece-wise geodesic boundary which will be denoted by $\hat{\Sigma}$. The triangulation $T$ of $\Sigma$ induces a geodesic triangulation $\hat{T}$ of $\hat{\Sigma}$. Let $N_1$ be the number of edges of $\hat{T}$, and $N_2$ be the number of triangles of $\hat{T}$. By computing the Euler characteristic of $\Sigma$, we see that:

$$N_1 = 3(2g + m - 2) + 4(n - m) \text{ and } N_2 = 2(2g + m - 2) + 2(n - m).$$

We construct a map from $U_T$ to $\mathbb{C}^{N_1}$ as follows: choose an orientation for every edge of $\hat{T}$, for each triangle $\Delta$ in $\hat{T}$, there exists an isometric embedding of this triangle into $\mathbb{R}^2$ such that the vector field $\xi$ is mapped to the constant vertical vector field $(0, 1)$ on the image of $\Delta$. By this embedding, each oriented side of the triangle $\Delta$ is mapped into a vector in $\mathbb{R}^2 \simeq \mathbb{C}$. As a consequence, we can associate to every oriented edge $e$ of $\hat{T}$ a complex number $z(e)$. Note that, even though each edge $e$ in the interior of $\Sigma$ belongs to two distinct triangles, the complex number $z(e)$ is well defined since the vector field $\xi$ is parallel and normalized. The procedure above defines a map from $U_T$ into $\mathbb{C}^{N_1}$, we denote this map by $\Psi_T$.

First, we have the following important observations:

**Lemma 4.1**

i) Let $e_i, e_j, e_k$ be three edges of $\hat{T}$ which bound a triangle. Then we have

$$\pm z(e_i) \pm z(e_j) \pm z(e_k) = 0,$$

where the signs are determined by the orientation of $e_i, e_j$ and $e_k$.

ii) If $e_1, \ldots, e_k$ are the $k$ edges of $\hat{T}$ which bound an open disk in $\hat{\Sigma}$, then we have
\[ \pm z(e_1) \pm \cdots \pm z(e_k) = 0, \tag{2} \]

where, again, the signs are determined by the orientations of the edges.

**Proof:** Assertion \( i \) is straightforward. Assertion \( ii \) follows from \( i \). Namely, let \( D \) denote the disk bounded by \( e_1, \ldots, e_k \). The disk \( D \) is divided into triangles by the triangulation \( \hat{T} \). By \( i \), three sides of a triangle verify (1). Note that every edge of \( \hat{T} \) inside \( D \) belongs to two distinct triangles. If, for each triangle, we choose the orientation of its boundary coherently with the orientation of the surface, and write the corresponding equation according to this orientation, then, by taking the sum over all the triangles inside \( D \), we get (2). \( \square \)

**Lemma 4.2** Let \((e, \bar{e})\) be a pair of edges in the boundary of \( \hat{\Sigma} \) which corresponds to an edge of a tree \( A_j \) in \( \hat{A} \). Suppose that \( e \) and \( \bar{e} \) are oriented following an orientation of \( \hat{\Sigma} \), then we have

\[ z(\bar{e}) = -e^\theta z(e) \tag{3} \]

where the number \( \theta \) is determined up to sign by the angles \( \alpha \), and the tree \( A_j \).

**Proof:** Let \( c \) be a path in \( \hat{\Sigma} \) joining the midpoint of \( e \) to the midpoint of \( \bar{e} \) such that \( \text{int}(c) \cap \partial \hat{\Sigma} = \emptyset \). Let \( \hat{c} \) denote the edge of \( A_j \) which corresponds to the pair \((e, \bar{e})\), and \( q \) denote the mid-point of \( \hat{c} \). Then \( c \) corresponds to a closed curve \( \gamma \) in \( \Sigma \) which intersects \( \hat{A} \) only at \( q \) transversely. Observe that (3) is verified when \( \theta \) is the rotation angle of the holonomy of \( \gamma \). Note that this angle is independent of the choice the base-point of \( \gamma \).

To simplify the notations, we denote by \( \text{orth}(\gamma) \) the linear part of the holonomy of \( \gamma \), which is a rotation. We need to show that the angle \( \theta \) is determined up to sign by the tree \( A_j \) and the angles \( \alpha_1, \ldots, \alpha_n \).

Since \( A_j \) is a tree, \( A_j \setminus \text{int}(\hat{c}) \) has two connected components, let \( A'_j \) denote one them, which is a sub-tree of \( A_j \). Choose an orientation of \( \gamma \), and let \( q_1 \) and \( q_2 \) be two points in \( \gamma \) close to \( q \) so that \( q \) is between \( q_1 \) and \( q_2 \). Let \( s_1, s_2 \) denote the two sub-arcs of \( \gamma \) with endpoints \( q_1, q_2 \), where \( s_2 \) is the sub-arc containing \( q \). We can then find a simple arc \( s_3 \) with endpoints \( q_1, q_2 \) such that \( s_2 \cup s_3 \) is the boundary of a disk \( D \) which contains the tree \( A'_j \), and no other singular points of \( \Sigma \) except the vertices of \( A'_j \) are contained in \( D \). Note that we must have

\[ s_3 \cap \hat{A} = \emptyset. \]

As a consequence, we see that \( \gamma \) is (freely) homotopic to the curve \( \gamma_1 \cdot \gamma_2 \), where \( \gamma_1 = \gamma \), and \( \gamma_2 = s_2 \cup s_3 \). Hence, the rotation angle of \( \text{orth}(\gamma) \) is equal to the sum of the rotation angles of \( \text{orth}(\gamma_1) \), and \( \text{orth}(\gamma_2) \). By definition of erasing forest, we know that \( \text{orth}(\gamma_1) = \text{Id} \), meanwhile, the rotation angle of \( \text{orth}(\gamma_2) \) is equal to the sum of all the angles at the singular points contained in the disk \( D \) modulo \( 2\pi \). Therefore, we have
\[ \theta = \sum_{\phi(p_i) \in A'_j} \alpha_i \mod 2\pi. \]

Observe that, the angle \( \theta \) is only determined up to sign since it depends on the orientation of \( \gamma \), and on the choice of \( A'_j \). \( \square \)

Let \( S_T \) denote the linear equation system consisting of \( N_2 \) equations of type (1) corresponding to the triangles of \( T \), and \( n - m \) equations of type (3) corresponding to the edges of \( \hat{A} \). From what we have seen, the vector \( \Psi_T([([\Sigma, \phi, \xi])] \) is a solution of the system \( S_T \). Let \( V_T \) denote the subspace of \( \mathbb{C}^{N_1} \) consisting of solutions of the system \( S_T \). We have

**Lemma 4.3** \( \Psi_T(\mathcal{U}_T) \) is an open subset of \( V_T \).

**Proof:** The fact that \( \Psi_T(\mathcal{U}_T) \) is contained in \( V_T \) is a direct consequence of Lemma 4.1, and Lemma 4.2.

Now, let \( Z \) be the image of \( [([\Sigma, \phi, \xi])] \) by \( \Psi_T \), and let \( Z' = (z'_1, \ldots, z'_N_1) \) be a vector in a neighborhood of \( Z \) in \( V_T \). Using the triangulation \( T \) of \( \Sigma \), we construct a flat surface from \( Z' \) as follows:

1. Construct an Euclidean triangle from \( z'_i, z'_j, z'_k \) if \( z'_i, z'_j, z'_k \) verify an equation of type (1).
2. Identify two sides of two distinct triangles if they correspond to the same complex number \( z'_i \).
3. Identify the edges corresponding to \( z_i \) and \( z_j \) if \( z_i \) and \( z_j \) satisfy an equation of type (3).

Clearly by this construction we obtain a translation surface \( \Sigma' \) homeomorphic to \( \Sigma \). The surface \( \Sigma' \) has \( n \) singular points of cone angles \( \alpha_1, \ldots, \alpha_n \) in the interior, there exists an erasing forest \( \hat{A}' \) on \( \Sigma' \). Moreover, we also get a triangulation \( T' \) of \( \Sigma' \) by geodesic segments. Each triangle in \( T' \) corresponds to a triangle in \( \mathbb{R}^2 \) specified by three complex numbers which are coordinates of \( Z' \), hence we get a normalized parallel vector field \( \xi' \) on \( \Sigma' \setminus \hat{A}' \) which is defined by the constant vertical vector field \((0, 1)\) on the Euclidean plan \( \mathbb{R}^2 \).

Define an orientation preserving homeomorphism \( f : \Sigma \to \Sigma' \) as follows: \( f \) maps each edge of \( T \) onto the corresponding edge of \( T' \) (i.e. the edge of \( T \) that corresponds to the same coordinate), and the restriction \( f \) on each triangle of \( T \) is a linear transformation of \( \mathbb{R}^2 \). Let \( \phi' \) denote the map \( f \circ \phi \), it follows that the triple \( ([\Sigma', \phi', \xi']) \) represents a point of \( U_T \) close to \( [([\Sigma, \phi, \xi])] \). By construction, it is clear that \( Z' = \Psi_T([([\Sigma', \phi', \xi'])]) \). Hence, we deduce that \( \Psi_T(\mathcal{U}_T) \) is an open set of \( V_T \). \( \square \)

### 4.2 Injectivity of \( \Psi_T \)

**Lemma 4.4** The map \( \Psi_T \) is injective.

**Proof:** Let \( [([\Sigma_1, \phi_1, \xi_1])] \) and \( [([\Sigma_2, \phi_2, \xi_2])] \) be two points in \( \mathcal{U}_T \) such that

\[ \Psi_T([([\Sigma_1, \phi_1, \xi_1])]) = \Psi_T([([\Sigma_2, \phi_2, \xi_2])]). \]
By definition, we can assume that $T_1 = \phi_1(\mathcal{T})$ and $T_2 = \phi_2(\mathcal{T})$ are admissible triangulations of $(\Sigma_1, \phi_1)$ and $(\Sigma_2, \phi_2)$ respectively.

Now, the hypothesis $\Psi_T(\{(\Sigma_1, \phi_1, \xi_1)\}) = \Psi_T(\{(\Sigma_2, \phi_2, \xi_2)\})$ implies that there exists an isometry $h : \Sigma_1 \to \Sigma_2$, which maps each triangle of $T_1$ onto a triangle of $T_2$, and also $\xi_1$ to $\xi_2$. It follows that the homeomorphism $\phi_2^{-1} \circ h \circ \phi_1 : S_g \to S_g$ fixes all the points in $\mathcal{V}$, and preserves each triangles of $\mathcal{T}$. We deduce that the map $\phi_2^{-1} \circ h \circ \phi_1$ is isotopic to the identity of $S_g$ by an isotopy fixing all the points in $\mathcal{V}$. Therefore, by definition, we have $[(\Sigma_1, \phi_1, \xi_1)] = [(\Sigma_2, \phi_2, \xi_2)]$. 

4.3 Computation of dimensions

Lemma 4.5 $\dim_C V_T = \begin{cases} 2g + n - 1, & \text{if } \alpha_i \in 2\pi\mathbb{N}, \forall i = 1, \ldots, n; \\ 2g + n - 2 & \text{otherwise.} \end{cases}$

Proof: Recall that $V_T$ is the subspace of $\mathbb{C}^{N_1}$ consisting of solutions of the system $S_T$. Since the system $S_T$ contains $N_2 + n - m$ equations, we have

$$\dim V_T \geq N_1 - (N_2 + n - m) = 2g + n - 2. \quad (4)$$

Let $a_1, \bar{a}_1, \ldots, a_{n-m}, \bar{a}_{n-m}$ denote the edges of $\hat{T}$ which are contained in the boundary of $\hat{\Sigma}$ so that the pair $(a_i, \bar{a}_i)$ corresponds to an edge of $\hat{A}$. Choose a family of edges $\{b_1, \ldots, b_{2g+m-1}\}$ in $\hat{T}$ such that $\text{int}(\hat{\Sigma}) \setminus \bigcup_{j=1}^{2g+m-1} b_j$ is an open disk.

From Lemma 4.1 ii), we deduce that if $e$ is any edge of $\hat{T}$ which does not belong to the set

$$\{a_1, \bar{a}_1, \ldots, a_{n-m}, \bar{a}_{n-m}, b_1, \ldots, b_{2g+m-1}\},$$

then $z(e)$ can be written as a linear combination of

$$z(a_1), z(\bar{a}_1), \ldots, z(a_{s_1+s_2+s_3}), z(\bar{a}_{n-m}), z(b_1), \ldots, z(b_{2g+m+n-1})$$

with coefficients in $\{-1, 0, 1\}$. From Lemma 4.2, we know that $z(\bar{a}_i) = -e^{i\theta_i}z(a_i)$, where $\theta_i$ is determined by $\alpha$ and $\hat{A}$. Thus, the number $z(e)$ is a linear function of

$$(z(a_1), \ldots, z(a_{n-m}), z(b_1), \ldots, z(b_{2g+m-1}))$$

with coefficients determined by $\alpha$ and $\hat{A}$. We deduce that

$$\dim V_T \leq 2g + n - 1. \quad (5)$$

Suppose that the edges $a_1, \bar{a}_1, \ldots, a_{n-m}, \bar{a}_{n-m}$ are oriented coherently with the orientation of $\partial \hat{\Sigma}$, apply (2) to the disk $D = \text{int}(\hat{\Sigma}) \setminus \bigcup_{j=1}^{2g+m-1} b_j$, we get

$$\sum_{i=1}^{n-m} (z(a_i) + z(\bar{a}_i)) = \sum_{i=1}^{n-m} (1 - e^{i\theta_i})z(a_i) = 0 \quad (6)$$
The numbers $z(b_j)$, $j = 1, \ldots, 2g + m - 1$, do not appear in the equation (6) because each of the edges $b_j$ belongs to two distinct triangles. Here, we have two issues:

- Case 1: there exists $i \in \{1, \ldots, n\}$ such that $\alpha_i \notin 2\pi \mathbb{N}$. The equation (6) is then non-trivial, which means that the vector $(z(a_1), \ldots, z(a_{n-m}), z(b_1), \ldots, z(b_{2g+m-1}))$ belongs to a hyperplane of $\mathbb{C}^{2g+n-1}$. Therefore we have

$$\dim V_T \leq 2g + n - 2. \quad (7)$$

From (4) and (7), we conclude that $\dim \mathbb{C} V_T = 2g + n - 2$.

- Case 2: $\alpha_i \in 2\pi \mathbb{N}$ for every $i$ in $\{1, \ldots, n\}$. In this case, the equation (6) is trivial. However, this also means that the sum of all equations in the system $S_T$, with appropriate choices of signs, is the trivial equation $0 = 0$. This implies $\text{rk}(S_T) \leq N_2 + (n - m) - 1$. Hence

$$\dim V_T \geq N_1 - (N_2 + n - m - 1) = 2g + n - 1. \quad (8)$$

From (5) and (8), we conclude that $\dim V_T = 2g + n - 1$.

The lemma is then proved. \qed

4.4 Coordinate change

Let $T_1, T_2$ be two equivalence classes of triangulations in $\mathcal{T} \mathcal{R}(S_g, \hat{A})$. Suppose that $U_{T_1} \cap U_{T_2} \neq \emptyset$, and let $[(\Sigma, \phi, \xi)]$ be a point in $U_{T_1} \cap U_{T_2} \neq \emptyset$. Let $T_1, T_2$ be the admissible triangulations of $\Sigma$ corresponding to $T_1$ and $T_2$ respectively. As usual, we denote by $\Psi_{T_1}, \Psi_{T_2}$ the local charts on $U_{T_1}$ and $U_{T_2}$ respectively. We have:

**Lemma 4.6** There exists an invertible complex linear map

$$L : \mathbb{C}^{N_1} \rightarrow \mathbb{C}^{N_1}$$

such that $\Psi_{T_2}([(\Sigma', \phi', \xi')]) = L \circ \Psi_{T_1}([(\Sigma', \phi', \xi')])$, for every $[(\Sigma', \phi', \xi')]$ in a neighborhood of $[(\Sigma, \phi, \xi)]$.

**Proof:** Let $e$ be an edge of $T_2$. Let $\{\Delta_i, i \in I\}$ be the set of triangles in $T_1$ which are crossed by $e$, that is $\Delta_i \cap \text{int}(e) \neq \emptyset, \forall i \in I$. Using the developing map, we can construct a polygon $P_e$ in $\mathbb{R}^2$ by gluing successively isometric copies of $\Delta_i$’s ($i \in I$) so that $e$ corresponds to a diagonal $\tilde{e}$ inside $P_e$. Note that a triangle $\Delta_i$ may have several copies in the polygon $P_e$. We will call $P_e$ the developing polygon of $e$ with respect to $T_1$.

By this construction, we also get a map $\varphi : P_e \rightarrow \Sigma$ which is locally isometric, such that $\varphi(\tilde{e}) = e$. The inverse images of the edges of $T_1$ under $\varphi$ give rise to a triangulation of $P_e$ by diagonals. Note
that by construction, all of the diagonals of this triangulation intersect the diagonal $\tilde{e}$.

Since the map $\varphi$ sends segments in the boundary of $P_e$ onto edges of $T_1$, it follows that the complex numbers associated to the edge $e$ can be written as linear function of the complex numbers associated to some edges of $T_1$. The coefficients of these linear functions are unchanged if we replace $[(\Sigma, \phi, \xi)]$ by another point $[(\Sigma', \phi', \xi')]$ nearby in $U_{T_1} \cap U_{T_2}$, and this argument is reciprocal between $T_1$ and $T_2$. We deduce that the coordinate change between $\Psi_{T_1}$ and $\Psi_{T_2}$, in a neighborhood of $[(\Sigma, \phi, \xi)]$, is a complex linear transformation of $\mathbb{C}^{N_1}$ which sends $V_{T_1}$ onto $V_{T_2}$. The lemma is then proved. \hfill $\Box$

5 Action of Mapping Class Group

In this section, we will prove that the action of $\Gamma(S_g, \hat{A})$ on $\mathcal{T}^*(\hat{A}, \alpha)$ is properly discontinuous. For this purpose, we first define a map $\Xi$ from $\mathcal{T}^*(\hat{A}, \alpha)$ into $\mathcal{T}(g, n)$ as follows: the image of a point $[(\Sigma, \phi, \xi)]$ in $\mathcal{T}^*(\hat{A}, \alpha)$ under $\Xi$ is the point in $\mathcal{T}(g, n)$ represented by the pair $(\Sigma, \phi)$, where $\Sigma$ is now considered as a Riemann surface with $n$ marked points $\{\phi(p_1), \ldots, \phi(p_n)\}$.

Observe that the group $\Gamma(S_g, \hat{A})$ is naturally embedded into the mapping class group $\Gamma(g, n)$ of the punctured surface $S_g \setminus \{p_1, \ldots, p_n\}$, since by definition, we have

$$\text{Homeo}_0^+ (S_g, \hat{A}) = \text{Homeo}_0^+(S_g, \hat{A}) \cap \text{Homeo}_0^+(S_g, \{p_1, \ldots, p_n\}),$$

where $\text{Homeo}_0^+(S_g, \{p_1, \ldots, p_n\})$ is the set of orientation preserving homeomorphisms of $S_g$ which are isotopic to $\text{Id}_{S_g}$ relative to $\{p_1, \ldots, p_n\}$. Clearly, the map $\Xi$ is equivariant with respect to the actions of $\Gamma(S_g, \hat{A})$ on $\mathcal{T}^*(\hat{A}, \alpha)$, and on $\mathcal{T}(g, n)$. It is well known that the group $\Gamma(g, n)$ acts properly discontinuously on $\mathcal{T}(g, n)$, therefore, it suffices to prove

**Proposition 5.1** The map $\Xi : \mathcal{T}^*(\hat{A}, \alpha) \rightarrow \mathcal{T}(g, n)$ is continuous.

**Proof:** Let $[(\Sigma, \phi, \xi)]$ be a point in $\mathcal{T}^*(\hat{A}, \alpha)$, and $\{(\Sigma_k, \phi_k, \xi_k)\}, \ k \in \mathbb{N}$ be a sequence in $\mathcal{T}^*(\hat{A}, \alpha)$ converging to $[(\Sigma, \phi, \xi)]$. The point $[(\Sigma, \phi, \xi)]$ belongs to a set $\mathcal{U}_T$, where $T$ is an equivalence class in $\mathcal{T}(S_g, \hat{A})$. Without loss of generality, we can assume that the sequence $\{(\Sigma_k, \phi_k, \xi_k)\}, \ k \in \mathbb{N}$ is also contained in $\mathcal{U}_T$. Let $\Psi_T$ be the local chart of $\mathcal{T}^*(\hat{A}, \alpha)$ which is defined on $\mathcal{U}_T$. Put $Z = \Psi_T([(\Sigma, \phi, \xi)])$, and $Z_k = \Psi_T([(\Sigma_k, \phi_k, \xi_k)])$, by assumption we have $Z_k \xrightarrow{k \to \infty} Z$ in $\mathbb{C}^{N_1}$.

Let $T$ be the admissible triangulation of $\Sigma$ corresponding to $\mathcal{T}$. Recall that, by the definition of $\Psi_T$, for every point $[(\Sigma', \phi', \xi')]$ in $\mathcal{U}_T$, we can write $\phi' = f \circ \phi$, where $f : \Sigma \rightarrow \Sigma'$ is a homeomorphism verifying

- $f$ sends each edge of $T$ onto an edge of an admissible triangulation $T'$ of $\Sigma'$,
- the restriction of $f'$ into the a triangle of $T$ is a linear transformation of $\mathbb{R}^2$.

Therefore, for every $k \in \mathbb{N}$, we can assume that $\phi_k = f_k \circ \phi$, where $f_k : \Sigma \rightarrow \Sigma_k$ is a homeomorphism with the properties above. It is clear that, as $Z_k$ tends to $Z$, the restriction of $f_k$ on each triangle of $T$
tends to identity, which implies \( \lim_{k \to \infty} K(f_k) = 1 \), where \( K(f_k) \) is the dilatation of \( f_k \). By the definition of Teichmüller distance \( d_{\text{Teich}} \), it follows that

\[
\lim_{k \to \infty} d_{\text{Teich}}([\Sigma, \phi]), ([\Sigma_k, \phi_k]) = 0,
\]

and the proposition follows. \( \square \)

Thank to a result of Troyanov [Tr1], more can be said about the map \( \Xi \). Observe that the map \( \Xi \) can be defined on \( T^{\text{et}}(\tilde{A}, \omega)^* \), and we have

**Proposition 5.2** The restriction of \( \Xi \) to \( T_{1}^{\text{et}}(\tilde{A}, \omega)^* \) is injective.

**Proof:** Let \( [(\Sigma_1, \phi_1)] \) and \( [(\Sigma_2, \phi_2)] \) be two points in \( T_{1}^{\text{et}}(\tilde{A}, \omega)^* \) such that \( (\Sigma_1 \setminus \{ \phi_1(p_1), \ldots, \phi_1(p_n) \}, \phi_1) \) and \( (\Sigma_2 \setminus \{ \phi_2(p_1), \ldots, \phi_2(p_n) \}, \phi_2) \) belong to the same equivalence class in \( T(g, n) \), we have to prove that \( [(\Sigma_1, \phi_1)] = [(\Sigma_2, \phi_2)] \) in \( T^{\text{et}}(\tilde{A}, \omega)^* \).

By assumption, there exists a conformal homeomorphism \( h : \Sigma_1 \longrightarrow \Sigma_2 \) such that \( \phi_2^{-1} \circ h \circ \phi_1 \) is isotopic to \( \text{Id}_{S^2} \) by an isotopy which is identity on the set \( \{ p_1, \ldots, p_n \} \). First, we prove that \( h \) is also an isometry between the two flat surfaces \( \Sigma_1 \) and \( \Sigma_2 \).

Consider \( \Sigma_1 \) and \( \Sigma_2 \) as Riemann surfaces equipped with conformal flat metrics \( f_1 \) and \( f_2 \) respectively. Let \( \text{div}_1 \) denote the formal sum \( \sum_{k=1}^{n} s_k \phi_1(p_k) \), where \( s_k = \frac{\alpha_k}{2\pi} - 1 \), we say that the metric \( f_i \) represents the divisor \( \text{div}_i \), \( i = 1, 2 \).

Since \( h \) is a conformal homeomorphism, it follows that \( h^* f_2 \) is also a conformal flat metric on \( \Sigma_1 \). By assumption, we have \( h(\text{div}_1) = \text{div}_2 \), and we deduce that \( h^* f_2 \) represents \( \text{div}_1 \) too. Now, from Proposition 2 [Tr1], there exists \( \lambda > 0 \) such that \( f_1 = \lambda h^* f_2 \). Since we have \( \text{Area}_{f_1}(\Sigma_1) = \text{Area}_{f_2}(\Sigma_2) = 1 \), it follows that \( \lambda = 1 \). Therefore we have \( f_1 = h^* f_2 \), in other words, \( h \) is an isometry from the flat surface \( \Sigma_1 \) onto the flat surface \( \Sigma_2 \).

All we need to prove now is that \( \phi_2^{-1} \circ h \circ \phi_1 \) preserves the forest \( \tilde{A} \). Since \( h \) is an isometry of flat surfaces, \( h(\phi_1(\tilde{A})) \) is a union of geodesic trees whose vertices are singular points of \( \Sigma_2 \). Let \( a \) be an edge of a tree in \( \tilde{A} \), then \( \phi_1(a) \) is a geodesic segment on \( \Sigma_1 \), hence \( h(\phi_1(a)) \) is a geodesic segment of \( \Sigma_2 \). By definition, \( \phi_2(a) \) is also a geodesic segment of \( \Sigma_2 \) which has the same endpoints as \( h(\phi_1(a)) \).

By assumption, there exists an isotopy relative to \( \{ p_1, \ldots, p_n \} \) from \( h \circ \phi_1 \) to \( \phi_2 \), it follows that \( \phi_2(a) \) and \( h(\phi_1(a)) \) are homotopic with fixed endpoints in \( \Sigma_2 \) relative to \( \{ \phi_2(p_1), \ldots, \phi_2(p_n) \} \). Now, from Lemma 3.5, we have \( h(\phi_1(a)) = \phi_2(a) \). Since this is true for every edges in \( \tilde{A} \), we conclude that \( h \circ \phi_1(\tilde{A}) = \phi_2(\tilde{A}) \), or equivalently, \( \phi_2^{-1} \circ h \circ \phi_1(\tilde{A}) = \tilde{A} \). It follows immediately that \( \phi_2^{-1} \circ h \circ \phi_1 \in \text{Homeo}_0(S_g, \tilde{A}) \), in other words, \( (\Sigma_1, \phi_1) \) and \( (\Sigma_2, \phi_2) \) are equivalent in \( T_1^{\text{et}}(\tilde{A}, \omega)^* \). \( \square \)

Since \( T^{\text{et}}(\tilde{A}, \omega) \) is a \( C^\ast \)-bundle over \( T_1^{\text{et}}(\tilde{A}, \omega)^* \), we can then consider \( T^{\text{et}}(\tilde{A}, \omega) \) as a \( C^\ast \)-bundle over the subset \( \Xi(T_1^{\text{et}}(\tilde{A}, \omega)^*) \) of \( T(g, n) \).
6 Changes of triangulations

Let \([\Sigma, \phi]\) be an element of the space \(T^{et}(\hat{A}, \alpha)^{\star}\), we have seen that an admissible triangulation of \((\Sigma, \phi)\) (cf. Definition 3.1) allows us to construct a local chart for \(T^{et}(\hat{A}, \alpha)\). In this section, we are interested in relations between geodesic triangulations of \(\Sigma\). More precisely, we want to answer the question: How to go from an admissible triangulation to another one. This will play a crucial role in our construction of the volume form on \(T^{et}(\hat{A}, \alpha)\).

Let us start with a simple example: let \(ABCD\) be a convex quadrilateral in \(\mathbb{R}^2\), there are only two ways to triangulate \(ABCD\): one by adding the diagonal \(AC\), and the other by adding the diagonal \(BD\).

This example suggests

**Definition 6.1 (Elementary Move and Related Triangulations)** Let \(\Sigma\) be a flat surface with geodesic boundary. Let \(T\) be a triangulation of \(\Sigma\) by geodesic segments whose set of vertices contains the set of singularities of \(\Sigma\). An elementary move of \(T\) is a transformation as follows: take two adjacent triangles of \(T\) which form a convex quadrilateral, replace the common side of the two triangles by the other diagonal of the quadrilateral (if these two triangles have more than one common side, just take one of them). After such a move, we obtain evidently another geodesic triangulation of \(\Sigma\) with the same set of vertices as \(T\).

Let \(T_1, T_2\) be two geodesic triangulations of \(\Sigma\) whose sets of vertices coincide. We say that \(T_1\) and \(T_2\) are related if there exists a sequence of elementary moves which transform \(T_1\) into \(T_2\).

The main result of this section is the following theorem

**Theorem 6.2** Let \(\Sigma\) be a flat surface with piece-wise geodesic boundary. Let \(V\) be a finite subset of \(\Sigma\) which contains all the singularities. Suppose that \(\Sigma\) satisfies the following condition

\[(Q)\quad \text{for every closed curve } c \subset \text{int}(\Sigma) \setminus V, \text{ we have } \text{orth}(c) \in \{\pm \text{Id}\},\]

where \(\text{orth}(c)\) is the orthogonal part of the holonomy of \(c\). Let \(T_1, T_2\) be two geodesic triangulations of \(\Sigma\) such that the set of vertices of \(T_i\) is \(V, i = 1, 2\), then \(T_1\) and \(T_2\) are related.
Remark: The changes of triangulations by elementary moves, which are also called flips, are already studied in the context of general flat surfaces (not necessarily translation surfaces). In the general situation, Theorem 6.2 is already known, it results from the fact that any geodesic triangulation whose vertex set contains all the singularities can be transformed by flips into a special one, called Delaunay triangulation, which is unique up to some flips (see [BS] for further detail).

For the case at hand, we have an elementary proof of this fact which is based on an observation on polygons, and uses some basic properties of translation and half-translation surfaces, this proof is given in Appendices, Section A.

7 Volume form

In this section, we define a volume form on the manifold $T^* (\hat{A}, (\hat{\alpha}))$. Recall that, if $L : E \to F$ is a linear map between (real) vector spaces which is surjective, then given a volume form $\mu_E$ on $E$, and a volume form $\mu_F$ on $F$, one can define a volume form $\mu$ on $\ker(L)$ as follows: let $\iota$ be the embedding of $\ker(L)$ into $E$, the volume form $\mu$ on $\ker(L)$ is defined to be $\iota^* \tilde{\mu}$, where $\tilde{\mu}$ is any element of $\Lambda^{(\dim E - \dim F)} E^*$ such that:

$$\mu_E = \tilde{\mu} \wedge L^* \mu_F.$$

7.1 Definitions

Let $\mathcal{T}$ be a triangulation of $S_g$, which represents an equivalence class in $\mathcal{T}R(S_g, \hat{\Delta})$. Cut open $S_g$ along the edges of the forest $\hat{A}$, and denote the new surface by $\hat{S}_g$. Let $\hat{T}$ denote the triangulation of $\hat{S}_g$ which is induced by $\mathcal{T}$. As usual, let $N_1, N_2$ denote the number of edges, and the number of triangles in $\hat{T}$ respectively. Let $\Psi_T : UT \to C^{N_1}$ be the local chart associated to $\mathcal{T}$. Recall that $\Psi_T(UT)$ is an open subset of the solution space $V_T$ of a system $S_T$, which consists of $N_2$ equations of type (1) and $(n - m)$ equations of type (3).

Let $a_1, \ldots, a_{N_2+(n-m)}$ denote the vectors of $(C^{N_1})^*$ which correspond to the equations of the system $S_T$. A vector $a_i$ is said to be normalized if each of its coordinates is either 0, or a complex number of module 1. Consider the complex linear map $A_T : C^{N_1} \to C^{N_2+(n-m)}$, which is defined in the canonical basis of $C^{N_1}$ and $C^{N_2+(n-m)}$ by the matrix

$$A_T = \begin{pmatrix} a_1 \\ \vdots \\ a_{N_2+(n-m)} \end{pmatrix}.$$ 

We have $V_T = \ker A_T$. The map $A_T$ is said to be normalized if each row of its matrix in the canonical basis is normalized. We have two cases:

- **Case 1:** there exist $i \in \{1, \ldots, n\}$ such that $\alpha_i \notin 2\pi N$. In this case, we have seen that (cf. Lemma 4.5) $\dim V_T = 2g + n - 2$, hence, $\text{rk}(A_T) = \text{rk}(S_T) = N_2 + (n - m)$. Let $\lambda_{2N_1}$ et $\lambda_{2(N_2+(n-m))}$...
denote the Lebesgue measures on $\mathbb{C}^{N_1} \simeq \mathbb{R}^{2N_1}$ and $\mathbb{C}^{N_2+(n-m)} \simeq \mathbb{R}^{2(N_2+(n-m))}$ respectively. Since $A_T$ is surjective, $\lambda_{2N_1}$ and $\lambda_{2N_2}$ induce a volume form $\nu_T$ on $V_T$ via the following exact sequence:

$$0 \rightarrow V_T \leftarrow \mathbb{C}^{N_1} \xrightarrow{A_T} \mathbb{C}^{N_2+(n-m)} \rightarrow 0$$

(9)

- **Case 2:** for every $i \in \{1, \ldots, n\}$, $\alpha_i \in 2\pi \mathbb{N}$. In this case, $\dim V_T = 2g + n - 1$, and $\text{rk}(A_T) = \text{rk}(S_T) = N_2 + (n - m) - 1$. If the vectors $a_1, \ldots, a_{N_2+(n-m)}$ are normalized, and if their signs are chosen suitably, we have $a_1 + \cdots + a_{N_2} = 0$. Thus, without loss of generality, we can assume that $\text{Im}A_T = W$, where $W$ is the complex hyperplane of $\mathbb{C}^{N_2+(n-m)}$ defined by

$$W = \{(z_1, \ldots, z_{N_2+(n-m)}) \in \mathbb{C}^{N_2+(n-m)} \mid z_1 + \cdots + z_{N_2+(n-m)} = 0\}.$$

Let $\tilde{\lambda}_{2(N_2+(n-m)-1)}$ denote the volume form of $W$ which is induced by the Lebesgue measure of $\mathbb{C}^{N_2+(n-m)}$. The volume forms $\lambda_{2N_1}$ and $\tilde{\lambda}_{2(N_2+(n-m)-1)}$ induce a volume form $\nu_T$ on $V_T^*$ via the exact sequence:

$$0 \rightarrow V_T \leftarrow \mathbb{C}^{N_1} \xrightarrow{A_T} W \rightarrow 0$$

(10)

In other words, in this case, $\nu_T$ is defined by the torsion of the following exact sequence:

$$0 \rightarrow V_T \leftarrow \mathbb{C}^{N_1} \xrightarrow{A_T} \mathbb{C}^{N_2+(n-m)} \xrightarrow{s} \mathbb{C} \rightarrow 0$$

(11)

where $s((z_1, \ldots, z_{N_2+(n-m)})) = z_1 + \cdots + z_{N_2+(n-m)}$, together with the Lebesgue measures on $\mathbb{C}^{N_1}, \mathbb{C}^{N_2+(n-m)}$, and $\mathbb{C}$.

In both cases, let $\mu_T$ denote the volume form $\Psi^*_T \nu_T$ which is defined on $U_T$.

### 7.2 Invariance by coordinate changes

Let $T_1$, and $T_2$ be two triangulations of $S_g$ which represent two different equivalence classes in $\mathcal{TR}(S_g, \hat{A})$. Assume that $U_{T_1} \cap U_{T_2} \neq \varnothing$. Then we have

**Lemma 7.1** $\mu_{T_1} = \mu_{T_2}$ on $U_{T_1} \cap U_{T_2}$.

**Proof:** Let $[(\Sigma, \phi, \xi)]$ be a point in $U_{T_1} \cap U_{T_2}$, and let $T_1$, $T_2$ be the admissible triangulations of $(\Sigma, \phi)$ corresponding to $T_1$ and $T_2$ respectively. By Theorem 6.2, we can assume that $T_2$ is obtained from $T_1$ by only one elementary move. Let $Z_i = \Psi_{T_i}([(\Sigma, \phi, \xi)])$, it is clear that the coordinates of $Z_2$ are linear functions of $Z_1$ with integer coefficients and vice versa. We deduce that coordinates change between $\Psi_{T_1}$, and $\Psi_{T_2}$ is realized by (complex) linear transformation $F$ of $\mathbb{C}^{N_1}$, that is to say we have the following commutative diagram:
\[
\begin{align*}
0 \rightarrow V_{T_1} &\leftrightarrow \mathbb{C}^{N_1} \xrightarrow{A_{T_1}} X \rightarrow 0 \\
\downarrow H &\quad \downarrow F \\
0 \rightarrow V_{T_2} &\leftrightarrow \mathbb{C}^{N_1} \xrightarrow{A_{T_2}} X \rightarrow 0
\end{align*}
\]

where \(X\) is either \(\mathbb{C}^{N_2+(n-m)}\), or \(W\), and \(H\) is the coordinates change between \(\Psi_{T_1}\), and \(\Psi_{T_2}\). Note that the map \(F\) is written in the canonical basis of \(\mathbb{C}^{N_1}\) as an integer matrix, and so is its inverse. As a matter of fact, we can arrange so that

\[
A_{T_1} = \begin{pmatrix}
-1 & 1 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & 1 & \cdots & 0 \\
0 & * & * & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & * & * & * & * & \cdots & *
\end{pmatrix};
A_{T_2} = \begin{pmatrix}
-1 & 0 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 0 & 0 & 1 & \cdots & 0 \\
0 & * & * & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & * & * & * & * & \cdots & *
\end{pmatrix},
\]

and

\[
F = \begin{pmatrix}
1 & -1 & 0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]

in the canonical basis of \(\mathbb{C}^{N_1}\) and \(\mathbb{C}^{N_2+(n-m)}\), where for \(j \geq 3\), the \(j\)-th rows of \(A_{T_1}\), and \(A_{T_2}\) are the same. Note that the first two rows of \(A_{T_1}\), and \(A_{T_2}\) correspond to two triangles in the quadrilateral where the elementary move occurs. Since \(|\det F| = 1\), it follows that \(\nu_{T_1} = H^*\nu_{T_2}\), and the lemma follows.

\[\square\]

### 7.3 Invariance by action of \(\Gamma(S_g, \hat{A})\)

Lemma 7.1 implies that the volume forms \(\{\mu_T : T \in \mathcal{T}\mathcal{R}(S_g, \hat{A})\}\) give a well defined volume form, which will be denoted by \(\mu_{T_1}\), on \(\mathcal{T}^{eq}(\hat{A}, \alpha)\). To complete the proof of Theorem 2.4, we need the following

\[\text{Lemma 7.2} \quad \text{The volume form } \mu_{T_1} \text{ is invariant by the action of } \Gamma(S_g, \hat{A}).\]

\[\text{Proof:} \quad \text{This lemma follows readily from the arguments of Lemma 7.1.} \quad \square\]

The proof of Theorem 2.4 is now complete.

\[\square\]
8 Proof of Proposition 2.5

Let \((M, \omega)\) be a pair in \(\mathcal{H}(k_1, \ldots, k_n)\), and let \(\Sigma\) denote the induced translation surface. Let \(x_1, \ldots, x_n\) denote the singularities of \(\Sigma\) so that the cone angle at \(x_i\) is \(2\pi(k_i + 1)\). The vertical geodesic flow determined by \(\omega\) is induced by a unitary parallel vector field \(\xi\) on \(\Sigma \setminus \{x_1, \ldots, x_n\}\). The pair \((M, \omega)\) in \(\mathcal{H}(k_1, \ldots, k_n)\) is then identified to the point \((\Sigma, \{x_1, \ldots, x_n\}, \xi)\) in \(\mathcal{M}^{ct}(\mathcal{A}, \mathcal{Q})\), where \(\mathcal{A}\) is the union of \(n\) points.

Let \(T\) be a geodesic triangulation of \(\Sigma\) whose vertex set is \(\{x_1, \ldots, x_n\}\). Note that, in this case, any geodesic triangulation whose set of vertices coincides with the set of singularities is admissible. We call a set of \(2g + n - 1\) edges of \(T\) such that the complement of the union of those edges is a topological open disk a family of primitive edges of \(T\). Remark that such a family always exists since it corresponds to the complement of a maximal tree, i.e. a tree which contains all the vertices, in the dual graph of \(T\). Let \(\{b_1, \ldots, b_{2g+n-1}\}\) be a family of primitive edges of \(T\). Observe that \(\{b_1, \ldots, b_{2g+n-1}\}\) is a basis of the group \(H_1(\Sigma, \{x_1, \ldots, x_n\}, \mathbb{Z})\).

Let \(\phi : S_g \to \Sigma\) be a homeomorphism which maps \(p_i\) to \(x_i\), \(i = 1, \ldots, n\), and let \(T\) denote triangulation \(\phi^{-1}(T)\) of \(S_g\) whose vertex set is \(\{p_1, \ldots, p_n\}\). Let \(\Psi_T\) be the local chart associated to \(T\). As usual, let \(S_T\) denote the system of linear equations associated to \(T\), \(V_T\) denote the space of solutions of \(S_T\), and \(A_T\) denote the normalized linear map associated to \(S_T\). We can assume that

\[
\text{Im} A_T = W = \{(z_1, \ldots, z_{N_2}) \in \mathbb{C}^{N_2} | z_1 + \cdots + z_{N_2} = 0\}.
\]

Recall that in this case, \(\dim \mathbb{C} V_T = 2g + n - 1\). Under \(\Psi_T\), a neighborhood of \((\Sigma, \{x_1, \ldots, x_n\}, \xi)\) in \(\mathcal{M}^{ct}(\mathcal{A}, \mathcal{Q})\) is identified to an open subset of \(V_T\). There exists a neighborhood \(U\) of \((\Sigma, \{x_1, \ldots, x_n\}, \xi)\) such that, for any point \((\Sigma', \{x'_1, \ldots, x'_n\}, \xi')\) in \(U\), there exists a homeomorphism \(f_{\Sigma'} : \Sigma \to \Sigma'\) such that \(f_{\Sigma'}(T) = T'\) is an admissible triangulation of \(\Sigma'\). Let \(b'_1\) denote \(f_{\Sigma'}(b_i)\), \(i = 1, \ldots, 2g + n - 1\), then the segments \(\{b'_1, \ldots, b'_{2g+n-1}\}\) form a basis of the group \(H_1(\Sigma', \{x'_1, \ldots, x'_n\}, \mathbb{Z})\). Hence, we can define a local chart of \(\mathcal{H}(k_1, \ldots, k_n)\) by the following period mapping

\[
\Phi : U \to \mathbb{C}^{2g+n-1}
\]

\[
(\Sigma', \{x'_1, \ldots, x'_n\}, \xi') \xrightarrow{\sim} (M', \omega') \mapsto (\int_{b'_1} \omega', \ldots, \int_{b'_{2g+n-1}} \omega')
\]

By the construction of \(\Psi_T\), we can assume that, if \(\Psi_T(\Sigma', \{x'_1, \ldots, x'_n\}, \xi') = (z_1, \ldots, z_{N_1})\), then the complex numbers \(z_1, \ldots, z_{2g+n-1}\) are associated to the edges \(b'_1, \ldots, b'_{2g+n-1}\). It follows that the map \(\Psi_T \circ \Phi^{-1} : \Phi(U) \subset \mathbb{C}^{2g+n-1} \to \mathbb{C}^{N_1}\) is a restriction of a linear map from \(\mathbb{C}^{2g+n-1}\) into \(\mathbb{C}^{N_1}\) which is injective, hence, \(\Psi_T \circ \Phi^{-1}\) is a restriction to \(\Phi(U)\) of an isomorphism from \(\mathbb{C}^{2g+n-1}\) onto \(V_T\).

By definition, \(\mu_0 = \Phi^* \lambda_2(2g+n-1)\), where \(\lambda_2(2g+n-1)\) is the Lebesgue measure of \(\mathbb{C}^{2g+n-1}\), and \(\mu_T = \Psi_T^* \nu_T\), where \(\nu_T\) is the volume form on \(V_T\) which is defined by the exact sequence (10). Clearly, on \(\mathbb{C}^{2g+n-1}\) we have

\[
(\Psi_T \circ \Phi^{-1})^* \nu_T = \lambda \lambda_2^{2g+n-1},
\]

where \(\lambda\) is a non-zero constant. This implies \(\mu_T = \lambda \mu_0\) on a neighborhood of \((\Sigma, \{x_1, \ldots, x_n\}, \xi)\). We
deduce that $\frac{\mu_{Tr}}{\mu_0}$ is locally constant, consequently, $\frac{\mu_{Tr}}{\mu_0}$ is constant on every connected component of $\mathcal{H}(k_1, \ldots, k_n)$. \hfill \Box

9 Flat complex affine structure on moduli space of flat surfaces of genus zero

9.1 Existence of erasing forest

To see that spherical flat surfaces are locally a special case of flat surfaces with erasing forest, let us prove the following

**Proposition 9.1** Let $\Sigma$ be compact flat surface without boundary. Let $\{p_1, \ldots, p_n\}$ denote the singularities of $\Sigma$, then there exists a geodesic tree whose vertex set is $\{p_1, \ldots, p_n\}$.

**Proof:** Let $C_1$ be a path of minimal length from $p_1$ to $p_2$. The path $C_1$ is a finite union of geodesic segments whose endpoints are singular points of $\Sigma$. Apart from $p_1$ and $p_2$, $C_1$ can contain other points in $\{p_1, \ldots, p_n\}$. Since $C_1$ is a path of minimal length, it has no self intersections. By renumbering the set of singular points if necessary, we can assume that $C_1$ is a path joining $p_1$ to $p_r$, and contains points $p_2, \ldots, p_{r-1}$. Note that for every point $p \in C_1$, the length of the path from $p_1$ to $p$ along $C_1$ is the distance $d(p_1, p)$ between them, where $d$ is the distance on $\Sigma$ which is induced by the flat metric.

If $r = n$, then we have obtained a geodesic tree whose vertices are $\{p_1, \ldots, p_n\}$. Assume that $r < n$, let $C_2$ be a path of minimal length from $p_1$ to $p_{r+1}$. If $C_1 \cap C_2 = \{p_1\}$, then we get a geodesic tree whose vertex set contains at least $r + 1$ points in $\{p_1, \ldots, p_n\}$. If this is not the case, let us prove that $C_2$ can not intersect $C_1$ transversely at a regular point.

Suppose that $p$ is a regular point where $C_2$ intersects $C_1$ transversely. Let $V$ be a neighborhood of $p$ such that $S_1 = V \cap C_1$ and $S_2 = V \cap C_2$ are two geodesic segments, and $p$ is the unique common point of $S_1$ and $S_2$. Let $C_1'$ be the paths from $p_1$ to $p$ along $C_1$ and $C_2'$ be the path from $p_1$ to $p$ along $C_2$, we have $\text{leng}(C_1') = \text{leng}(C_2') = d(p_1, p)$.

Let $q$ be a point in $S_2 \setminus C_2'$, and $r$ be a point in $S_1 \cap C_1'$. Let $\overline{pq}$ denote the sub-segment of $S_2$ whose endpoints are $p$ and $q$, and $\overline{pr}$ denote the sub-segments of $S_1$ whose endpoints are $p$ and $r$. We have
Proof of Proposition 2.6

Since the complement of a tree in the sphere $S^2$ is a topological disk, on a spherical flat surface, any geodesic tree whose vertex set is the set of singular points is automatically an erasing tree. Therefore, the arguments for flat surfaces with erasing forest can be applied in the case of spherical flat surfaces.

Let $T$ be the geodesic triangulations of $\Sigma$ consisting of pairs $([\Sigma, \phi], e^{i\theta})$, such that $\phi(T)$ is a geodesic triangulation of $\Sigma$.

Pick a tree $A$ in the 1-skeleton of $T$ whose vertex set is $\{p_1, \ldots, p_n\}$, for any $([\Sigma, \phi], e^{i\theta})$ in $U_T$, $\phi(A)$ is a geodesic erasing tree of $\Sigma$. Therefore, we can identify $U_T$ to an open subset in $\mathcal{T}(S^2, \alpha)$ consisting of pairs $([\Sigma, \phi], e^{i\theta})$, such that $\phi(T)$ is a geodesic triangulation of $\Sigma$.

Since in this case $\dim_{C} V_{T, A} = n - 2$, it follows that $\dim_{C} \mathcal{T}(S^2, \alpha) = n - 2$. It is worth noticing that $\Psi_{T, A}$ is only defined up to a rotation.

Given two triangulations $T_1, T_2$ representing distinct equivalence classes in $\mathcal{T}(S^2, \{p_1, \ldots, p_n\})$, let $([\Sigma, \phi], e^{i\theta})$ be a point in $U_{T_1} \cap U_{T_2}$, and let $T_1, T_2$ be the geodesic triangulations of $\Sigma$ corresponding to $T_1$ and $T_2$ respectively. Choose a tree $A_1$ (resp. $A_2$) in $T_1$ (resp. $T_2$) which connects all the points in $\{p_1, \ldots, p_n\}$, and let $\Psi_{T_1, A_1}$ and $\Psi_{T_2, A_2}$ be the two local charts of $\mathcal{T}(S^2, \alpha)$ corresponding.

Let $e$ be an edge of $T_2$ which is not contained in $T_1$, and $P_e$ be the developing polygon of $e$ with respect to $T_1$ (see Lemma 4.6). By construction, the complex number associated to the edge $e$ in the local chart $\Psi_{T_2, A_2}$ can be written as a linear function of complex numbers associated to edges of $T_1$, which correspond the segments in the boundary of $P_e$ in the local chart $\Psi_{T_1, A_1}$. Since the roles of $T_1$
and $T_2$ in this reasoning can be interchanged, we deduce that the coordinate change between $\Psi_{T_1,A_1}$ and $\Psi_{T_2,A_2}$ is a linear isomorphism of $\mathbb{C}^{4n-7}$ which sends $V_{T_1,A_1}$ onto $V_{T_2,A_2}$. Thus, we can conclude that $T(S^2, \alpha)$ is a flat complex affine manifold of dimension $n-2$.

We can define a map $\Xi : T(S^2, \alpha) \rightarrow T(0; n)$ by associating to each point $([\Sigma, \phi], e^{i\theta}) \in T(S^2, \alpha)$, the equivalence class of $(\Sigma, \phi)$ in $T(0; n)$, where $\Sigma$ is now considered as a Riemann surface. Using the same arguments as in Proposition 5.1, we can see that $\Xi$ is continuous. Clearly, the map $\Xi$ is equivariant with respect to the actions of $\Gamma(0; n)$ on $T(S^2, \alpha)$, and $T(0; n)$, it follows that the action of $\Gamma(0; n)$ on $T(S^2, \alpha)$ is properly discontinuous. The proof of Proposition 2.6 is now complete. \(\square\)

Remark: The map $\Xi$ can be defined on $T(S^2, \alpha)^*$. Using Proposition 2 [Tr1], we can show that the restriction of $\Xi$ onto $T_1(S^2, \alpha)^*$, the subspace of $T(S^2, \alpha)^*$ consisting of surfaces of unit area, is a bijection. Therefore, we can consider $T(S^2, \alpha)$ as a $\mathbb{C}^*$-bundle over $T(0; n)$.

### 10 Volume form on moduli space of flat surfaces of genus zero

In this section, we will give the proof of Theorem 2.7. Set $N_1 = 4n - 7$, $N_2 = 3n - 5$, and let $T$ be a triangulation of $S^2$ representing an equivalence class in $T\mathcal{R}(S^2, \{p_1, \ldots, p_n\})$. Let $A$ be a tree contained in $T$, which connects all the points in $\{p_1, \ldots, p_n\}$. Let $\Psi_{T,A}$ be the local chart associated to $(T, A)$, which is defined on the set $U_T$. Following the method in Section 7, we then get a volume form $\mu_{T,A}$ on $U_T$. Recall that there is a normalized linear map $A_{T,A} : \mathbb{C}^{N_1} \rightarrow \mathbb{C}^{N_2}$ associated to the local chart $\Psi_{T,A}$, and by definition, $\mu_{T,A} = \Psi_{T,A}^* \nu_{T,A}$, where $\nu_{T,A}$ is the volume form on $\ker A_{T,A}$ which is induced by the Lebesgue measures on $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ via the following exact sequence

$$0 \rightarrow \ker A_{T,A} \rightarrow \mathbb{C}^{N_1} \xrightarrow{A_{T,A}} \mathbb{C}^{N_2} \rightarrow 0 \quad (12)$$

The following proposition shows that the volume form $\mu_{T,A}$ does not depend on the choice of $A$.

**Proposition 10.1** Let $T$ be a triangulation representing an equivalence class in $T\mathcal{R}(S^2, \{p_1, \ldots, p_n\})$. Let $A_1, A_2$ be two trees contained in the 1-skeleton of $T$, each of which connects all the points in $\{p_1, \ldots, p_n\}$. Let $A_{T,A_1}$ and $A_{T,A_2}$ denote the linear maps from $\mathbb{C}^{N_1}$ onto $\mathbb{C}^{N_2}$ corresponding to $A_1$ and $A_2$ respectively. Let $\nu_i$, $i = 1, 2$ denote the volume form on $\ker A_{T,A_i}$ which is defined by the exact sequence (12). Let $H = \Psi_{T,A_2} \circ \Psi_{T,A_1}^{-1}$ be the coordinate change between $\Psi_{T,A_1}$ and $\Psi_{T,A_2}$, then we have $H^* \nu_2 = \nu_1$.

To show that the volume form $\mu_{T,A}$ actually does not depend on the choice of $T$, we need the following

**Theorem 10.2** Let $\Sigma$ be a spherical flat surface. If $T_1$ and $T_2$ are two geodesic triangulations of $\Sigma$ whose sets of vertices coincide, and contain the set of singularities of $\Sigma$, then $T_1$ and $T_2$ are related (i.e. one can be transformed into the other by elementary moves).

Theorem 2.7 follows directly from Proposition 10.1, and Theorem 10.2, since by Lemma 7.1 we know that, if $T_2$ is obtained from $T_1$ by an elementary move then the volume forms corresponding
to $T_1$ and $T_2$ coincide. Theorem 10.2 is of course a consequence of the fact that any geodesic triangulation of a spherical flat surface whose vertex set coincides with the set of singularities can be transformed into a Delaunay triangulation by elementary moves. In Appendices, Section B, we give a proof of Theorem 10.2 using similar ideas to the proof of Theorem 6.2. The remainder of this section is devoted to the proof of Proposition 10.1.

10.1 Cutting and gluing

Let us consider pairs $(\Sigma_0, T_0)$ where

- $\Sigma_0$ is a flat surface homeomorphic to a closed disk, with geodesic boundary, and having no singularities in the interior.

- $T_0$ is a triangulation of $\Sigma_0$ by geodesic segments whose vertex set is contained in the boundary of $\Sigma_0$.

- The edges of $T_0$ on the boundary of $\Sigma_0$ are paired up. Two edges in a pair have the same length.

We will call such a pair a well triangulated flat disk. Remark that a geodesic tree $A$ contained in $T$ which connects all the singular points of $\Sigma$ gives rise to a well triangulated flat disk, which is obtained by slitting open $\Sigma$ along $A$.

Let $\mathcal{T}, A_1, A_2$ be as in Proposition 10.1. Given a point $([\Sigma, \phi], e^{i\theta})$ in $U_T$, let $T$ be the geodesic triangulation of $\Sigma$ corresponding to $\mathcal{T}$, and $A_1, A_2$ be the geodesic trees corresponding to $A_1, A_2$ respectively. Let $\Sigma^1_0$ and $\Sigma^2_0$ denote the flat surface with geodesic boundary obtained by slitting open the surface $\Sigma$ along the trees $A_1$ and $A_2$ respectively. Let $T^1_0$ (resp. $T^2_0$) denote the geodesic triangulation of $\Sigma^1_0$ (resp. $\Sigma^2_0$) which is induced by $T$. By definition, $(\Sigma^1_0, T^1_0)$ and $(\Sigma^2_0, T^2_0)$ are well triangulated flat disks. Consider the following the following operation:

- Choose a pair of edges $(a, \bar{a})$ of $T_0$ in the boundary of $\Sigma_0$, and an edge $b$ in the interior of $\Sigma_0$ so that $a$ and $\bar{a}$ do not belong to the same connected component of $\Sigma_0 \setminus b$.

- Cut $\Sigma_0$ along $b$, then glue two the sub-disks by identifying $a$ to $\bar{a}$.

Clearly, by this operation, we get another pair $(\Sigma'_0, T'_0)$ with is also a well triangulated flat disk. We will call this operation the cutting-gluing operation. We have:

**Lemma 10.3** The pair $(\Sigma^2_0, T^2_0)$ can be obtained from $(\Sigma^1_0, T^1_0)$ by a sequence of cutting-gluing operations.

**Proof:** First, let us prove that $A_2$ is obtained from $A_1$ by replacing edges of $A_1$ by edges of $A_2$ one by one, successively. Indeed, let $e$ be an edge of $T$ which is contained in $A_2$, but not in $A_1$. Let $v_1$ and $v_2$ denote the endpoints of $e$, by assumption, there is a path $c$ in $A_1$ which joins $v_1$ to $v_2$. The union of $c$ and $e$ is then a cycle in $T$, therefore there exists an edge $e'$ in $c$, different from $e$, which does not belong to $A_2$. Replacing $e'$ by $e$, we get a new tree which connects all the singular points,
and contains one more common edge with $A_2$ than $A_1$. The claim follows by induction.

Now, we just need to observe that the operation of replacing $e'$ by $e$ corresponds to a cutting-gluing operation on the well triangulated flat disk corresponding to the tree $A_1$, and the lemma follows. □

### 10.2 Increased exact sequence

Given a well triangulated flat disk $(\Sigma_0, T_0)$ arising from a tree $A$ contained in the 1-skeleton of $T$ which connects all the singular points of $\Sigma$, we have a system of linear equations $S_0$ defined as in 4.1, and a normalized complex linear map $A_0 : \mathbb{C}^{N_1} \rightarrow \mathbb{C}^{N_2}$ corresponding. Let $a_1, \ldots, a_{N_2}$ denote the row vectors of $A_0$. We know that $\text{rk}(A_0) = N_2$. Now, choose an edge $e_0$ of $T_0$ which is contained inside $\Sigma_0$, and cut $\Sigma_0$ along $e_0$, we then get two flat disks $D_1, D_2$ with the geodesic triangulations $T_1, T_2$ respectively. Again, we can associate to each edge $e$ of $T_1$ and $T_2$ a complex number $z(e)$ to get a vector $\tilde{z}_0$ in $\mathbb{C}^{N_1+1}$. We also have a system of $N_2$ linear equations arising from the equations of $S_0$. We add to this system the following equation

$$z(e'_0) + z(e''_0) = 0$$

where $e'_0$ and $e''_0$ are the edges of $T_1$ and $T_2$ which arise from $e_0$ (with appropriate orientation). Let $\hat{S}_0$ denote the new system, and $\hat{A}_0 : \mathbb{C}^{N_1+1} \rightarrow \mathbb{C}^{N_2+1}$ denote the normalized linear map corresponding. We will call $\hat{A}_0$ the \textit{increased linear map} of $A_0$ associated to the splitting along $e_0$.

By construction, there exists injective linear maps $J_1 : \mathbb{C}^{N_1} \rightarrow \mathbb{C}^{N_1+1}$, $J_2 : \mathbb{C}^{N_2} \rightarrow \mathbb{C}^{N_2+1}$, and an isomorphism $I : \ker A_0 \rightarrow \ker \hat{A}_0$ so that the following diagram is commutative:

$$
\begin{array}{ccc}
0 & \rightarrow & \ker A_0 & \xleftarrow{\iota} & \mathbb{C}^{N_1} & \xrightarrow{A_0} & \mathbb{C}^{N_2} & \rightarrow & 0 \\
\downarrow{I} & & \downarrow{J_1} & & \downarrow{J_2} & & \\
0 & \rightarrow & \ker \hat{A}_0 & \xleftarrow{\iota} & \mathbb{C}^{N_1+1} & \xrightarrow{\hat{A}_0} & \mathbb{C}^{N_2+1} & \rightarrow & 0
\end{array}
$$

(13)

As usual, let $\lambda_{2k}$ denote the Lebesgue measure of $\mathbb{C}^k$, $\forall k \in \mathbb{N}$. Let $\nu$ denote the volume form on $\ker A_0$ which is induced by $\lambda_{2N_1}$ and $\lambda_{2N_2}$ via the upper exact sequence in (13), and $\tilde{\nu}$ is the volume form on $\ker \hat{A}_0$ which is induced by $\lambda_{2(N_1+1)}$ and $\lambda_{2(N_2+1)}$ via the lower exact sequence in (13). We can choose a numbering of the edges of $T_0, T_1, T_2$ so that

$$J_1((z_1, \ldots, z_{N_1})) = (z_1, \ldots, z_{N_1}, -z_1), \text{ and } J_2((z_1, \ldots, z_{N_2})) = (z_1, \ldots, z_{N_2}, 0).$$

Let $a_i$ (resp. $\hat{a}_i$), $i = 1, \ldots, N_2$ denote the $i$-th row of $A_0$ (resp. $\hat{A}_0$), we consider $a_i$ (resp. $\hat{a}_i$) as complex linear map from $\mathbb{C}^{N_1}$ (resp. $\mathbb{C}^{N_1+1}$) into $\mathbb{C}$. By assumption, we have $a_i = J_1^* \hat{a}_i$. Let $h : \mathbb{C}^{N_1+1} \rightarrow \mathbb{C}$ be the complex linear map corresponding to the last row of $\hat{A}_0$, by assumption, we have $h((z_1, \ldots, z_{N_1+1})) = z_1 + z_{N_1}$, and the following sequence

$$0 \rightarrow \mathbb{C}^{N_1} \xrightarrow{J_1} \mathbb{C}^{N_1+1} \xrightarrow{h} \mathbb{C} \rightarrow 0$$

(14)
is exact. Let $\lambda_{2N_1}'$ be the volume form induced by $\lambda_{2(N_1+1)}$ and $\lambda_2$ via the exact sequence (14). Let $\nu'$ be the volume form on ker $A_0$ which is induced by $\lambda_{2N_1}'$ and $\lambda_{2N_2}$ via the upper exact sequence in (13). We have

**Lemma 10.4** $\Gamma^*\hat{\nu}$ is equal to $\nu'$.

**Proof:** Let $\hat{\eta}$ be a $2(N_1 - N_2)$-real form on $C^{N_1+1}$ such that

$$\hat{\eta} \wedge (\text{Re}(\hat{a}_1) \wedge \text{Im}(\hat{a}_1)) \wedge \cdots \wedge (\text{Re}(\hat{a}_{N_2}) \wedge \text{Im}(\hat{a}_{N_2})) \wedge (\text{Re}(h) \wedge \text{Im}(h)) = \lambda_{2(N_1+1)}.$$

By definition, we have $\hat{\nu} = \hat{\iota}_* \hat{\eta}$, where $\hat{\iota} : \ker \hat{A}_0 \hookrightarrow C^{N_1+1}$ is the natural embedding. On the other hand, by definition, we have

$$\lambda_{2N_1}' = J_1^*(\hat{\eta} \wedge (\text{Re}(\hat{a}_1) \wedge \text{Im}(\hat{a}_1)) \wedge \cdots \wedge (\text{Re}(\hat{a}_{N_2}) \wedge \text{Im}(\hat{a}_{N_2})).$$

Therefore

$$\lambda_{2N_1}' = J_1^* \hat{\eta} \wedge (\text{Re}(a_1) \wedge \text{Im}(a_1)) \wedge \cdots \wedge (\text{Re}(a_{N_2}) \wedge \text{Im}(a_{N_2})), $$

and by definition, the volume form $\nu'$ is equal to $i^*(J_1^* \hat{\eta})$. Since $J_1 \circ i = i \circ \iota$, it follows $\nu' = \Gamma^*(i^* \hat{\eta}) = \Gamma^* \hat{\nu}$. □

**Corollary 10.5** We have $\Gamma^* \hat{\nu} = c_0 \nu$, where $c_0$ is a constant which does not depend on the choice of the edge $e_0$.

**Proof:** Set $c_0 = \frac{\lambda_{2N_1}'}{\lambda_{2N_1}}$, then we have $\nu' = c_0 \nu$. By lemma 10.4, we deduce that $\Gamma^* \hat{\nu} = c_0 \nu$. Now, the splitting of $\Sigma_0$ along another edge of $T_0$ inside $\Sigma_0$ corresponds to a permutation of coordinates in $C^{N_1+1}$, hence the exact sequence (14) gives the same volume form $\lambda_{N_1}'$ on $C^{N_1}$, and the lemma follows. □

### 10.3 Proof of Proposition 10.1

By Lemma 10.3, it suffices to consider the case where $(\Sigma_0^2, T_0^2)$ is obtained from $(\Sigma_0^1, T_0^1)$ by only one cutting-gluing operation. We can then assume $(\Sigma_0^2, T_0^2)$ is obtained by cutting $\Sigma_0^1$ along an edge $e_1$ of $T_0^1$ inside $\Sigma_0^1$, and gluing a pair $(e, \bar{e})$ of edges of $T_0^1$ in $\partial \Sigma_0^1$ which gives raise to an edge $e_2$ of $T_0^2$. 
Let $\hat{A}_1, \hat{A}_2 : \mathbb{C}^{N_1+1} \to \mathbb{C}^{N_2+1}$ denote the linear maps corresponding to the splitting of $\Sigma_1^0$ and $\Sigma_2^0$ along $e_1$ and $e_2$ respectively. There exist the isomorphisms $\hat{F} : \mathbb{C}^{N_1+1} \to \mathbb{C}^{N_1+1}, \hat{G} : \mathbb{C}^{N_2+1} \to \mathbb{C}^{N_2+1}$ such that the following diagram is commutative

\[
\begin{array}{cccc}
0 & \to & \ker \hat{A}_1 & \overset{\iota_1}{\to} & \mathbb{C}^{N_1+1} & \overset{\hat{A}_1}{\to} & \mathbb{C}^{N_2+1} & \to & 0 \\
\downarrow \hat{H} & & \downarrow \hat{F} & & \downarrow \hat{G} & & & & \\
0 & \to & \ker \hat{A}_2 & \overset{\iota_2}{\to} & \mathbb{C}^{N_1+1} & \overset{\hat{A}_2}{\to} & \mathbb{C}^{N_2+1} & \to & 0
\end{array}
\]  

where $\hat{H}$ is the isomorphism induced by $\hat{F}$. Let $k$ be the number of edges of $T_1$ which are contained in one of the two disks obtained from the splitting of $\Sigma_0^1$ along $e_1$, and $\theta$ be the angle of the rotation $\text{orth}(\gamma(e, \bar{e}))$, where $\gamma(e, \bar{e})$ is a closed curve on $\Sigma$ corresponding to a curve in $\Sigma_0^1$ joining the midpoint of $e$ to the midpoint of $\bar{e}$. With appropriate numberings of the edges of $T_0^1$, and the edges of $T_0^2$, we have

\[
\hat{F} = \begin{pmatrix} e^{i\theta} \text{Id}_k & 0 \\ 0 & \text{Id}_{N_1+1-k} \end{pmatrix}.
\]

Consequently, $\hat{G}$ is a diagonal matrix in $\mathbb{M}_{N_2+1}(\mathbb{C})$ whose diagonal entries are either 1 or $e^{i\theta}$. Clearly, we have

\[
|\det \hat{F}| = |\det \hat{G}| = 1.
\]

It follows that $\hat{H}^* \hat{\nu}_2 = \hat{\nu}_1$, where $\hat{\nu}_i$ is the volume form on $\ker \hat{A}_i$ which is induced by the Lebesgue measures on $\mathbb{C}^{N_1+1}$, and $\mathbb{C}^{N_2+1}$.

Let $I_i : \ker A_i \to \ker \hat{A}_i$, $i = 1, 2$, denote the isomorphism in (13) corresponding to the splitting of $\Sigma_i$ along $e_i$. We have the following commutative diagram

\[
\begin{array}{cccc}
\ker A_1 & \overset{I_1}{\to} & \ker \hat{A}_1 \\
\downarrow \hat{H} & & \downarrow \hat{H} \\
\ker A_2 & \overset{I_2}{\to} & \ker \hat{A}_2 
\end{array}
\]  

By Corollary 10.5, we know that $\frac{I_1^* \hat{\nu}_1}{\nu_1} = \frac{I_2^* \hat{\nu}_2}{\nu_2}$, and the proposition follows. \qed

11 Comparison with complex hyperbolic volume form

11.1 Definitions

In this section, we assume that all the angles $\alpha_1, \ldots, \alpha_n$ are less than $2\pi$. Put $\kappa_i = 2\pi - \alpha_i$, $i = 1, \ldots, n$, we have $\kappa_1 + \cdots + \kappa_n = 4\pi$. Following Thurston [Th], we denote by $C(\kappa_1, \ldots, \kappa_n)$ the moduli space of spherical flat surface having $n$ singularities with cone angles $\alpha_1, \ldots, \alpha_n$, or equivalently, with curvatures $\kappa_1, \ldots, \kappa_n$, up to homothety. In [Th], Thurston proves that $C(\kappa_1, \ldots, \kappa_n)$ admits a complex
hyperbolic metric structure with finite volume, and the metric closure of $C(\kappa_1, \ldots, \kappa_n)$ has cone manifold structure.

The complex hyperbolic metric provides a volume form $\mu_{\text{Hyp}}$ on $C(\kappa_1, \ldots, \kappa_n)$. On the other hand, the volume form $\mu_T$ gives another volume form $\tilde{\mu}_T^1$ on $C(\kappa_1, \ldots, \kappa_n)$ which is defined as follows:

- First, we identify $C(\kappa_1, \ldots, \kappa_n)$ to the subset $\mathcal{M}_1(S^2, \omega)^*$ of all surfaces of area 1 in $\mathcal{M}(S^2, \omega)^*$. Let $f : \mathcal{M}(S^2, \omega) \rightarrow \mathbb{R}^+$ be the function which associates to a pair $(\Sigma, \theta)$ in $\mathcal{M}(S^2, \omega)$ the area of $\Sigma$. The space $\mathcal{M}_1(S^2, \omega)^*$ can be considered as the quotient of the locus $f^{-1}(\{1\})$ by the action of $\mathbb{S}^1$.

- By Proposition 2.6, we know that $\mathcal{M}(S^2, \omega)$ is a complex orbifold, let $J$ denote the complex structure of $\mathcal{M}(S^2, \omega)$. Let $\rho : f^{-1}(\{1\}) \rightarrow f^{-1}(\{1\})/\mathbb{S}^1 = \mathcal{M}_1(S^2, \omega)^*$ denote the natural projection. We define the volume form $\rho^* \tilde{\mu}_T^1$ on $\mathcal{M}_1(S^2, \omega)^*$ to be the one such that:

$$\rho^* \tilde{\mu}_T^1 \wedge df \wedge (df \circ J) = \mu_T$$

11.2 Local formulae

First, we recall the construction of local charts for $C(\kappa_1, \ldots, \kappa_n)$ as presented in [Th], and consequently the definition of $\mu_{\text{Hyp}}$. Given a surface $\Sigma$ in $\mathcal{M}_1(S^2, \omega)^*$, we consider $\Sigma$ as a point in $C(\kappa_1, \ldots, \kappa_n)$. Let $T$ be a triangulation of $\Sigma$ by geodesic segments whose set of vertices is the set of singular points.

Choose a singular point of $\Sigma$ and denote this point by $\kappa_1$. We will call all the edges of $T$ which contain $\kappa_1$ as an endpoint followers. Pick a tree $\tilde{T}$ in $T$ which connects all other singular points of $\Sigma$, and call the edges of this tree leaders. The remaining edges of $T$ are also called followers.

Using a developing map, one can associate to each of the leaders a complex number, there are $n - 2$ of them. Let $(z_1, \ldots, z_{n-2})$ denote those complex numbers. The same developing map also defines an associated complex number for each of the followers, but these numbers can be calculated from those associated to leaders by complex linear functions. Thus, the complex numbers associated to leaders determine a local coordinate system $\varphi : U \rightarrow \mathcal{M}(S^2, \omega)$ for $\mathcal{M}(S^2, \omega)$ in a neighborhood of $(\Sigma, 1)$, where $U$ is a neighborhood of $(z_1, \ldots, z_{n-2})$ in $\mathbb{C}^{n-2}$. Consequently, a neighborhood of $\Sigma$ in $C(\kappa_1, \ldots, \kappa_n)$ is then identified to an open set of $\mathbb{P}\mathbb{C}^{n-3}$ which contains $[z_1 : \cdots : z_{n-2}]$.

If we add to $\tilde{T}$ a follower which joins $x_{\text{last}}$ to the tree $\tilde{T}$, then we have an erasing tree $T$ on $\Sigma$. We can then construct a local chart $\Psi_{T,A}$ for $\mathcal{M}(S^2, \omega)$ from $T$ and $A$ associated to a complex linear surjective map $A_T : \mathbb{C}^{N_1} \rightarrow \mathbb{C}^{N_2}$ is determined by the tree $A$. In this local chart, $\mu_T$ is identified to the volume form on $\ker A_T$ which is induced by the exact sequence (12).

Now, observe that the map $\Psi_{T,A} \circ \varphi$ is the restriction to an open subset of $\mathbb{C}^{n-2}$ of an isomorphism from $\mathbb{C}^{n-2}$ to $\ker A_T$. Therefore, the following sequence is exact

$$0 \rightarrow \mathbb{C}^{n-2} \xrightarrow{\Psi_{T,A} \circ \varphi} \mathbb{C}^{N_1} \xrightarrow{A_T} \mathbb{C}^{N_2} \rightarrow 0.$$

Consequently, in the local chart $\varphi$, we have
\[ \mu_{\text{Tr}} = c\lambda_{2(n-2)}, \]
where \( \lambda_{2(n-2)} \) is the Lebesgue measure of \( \mathbb{C}^{n-2} \), and \( c \) is a constant.

In the local chart \( \varphi \), the area function \( f \) on \( \mathcal{M}(S^2, \alpha) \) is expressed as a Hermitian form \( H \). More precisely, if \( v \in \mathbb{C}^{n-2} \) is a vector such that \( \varphi(v) = (\Sigma, \theta) \in \mathcal{M}(S^2, \alpha) \) then \( f((\Sigma, \theta)) = \text{Area}(\Sigma) = \frac{i}{2} v H v \).
It is proven in [Th] that \( H \) is of signature \((1, n-3)\). Changing the basis and the sign of \( H \), we can assume that

\[
H = \begin{pmatrix} \text{Id}_{n-3} & 0 \\ 0 & -1 \end{pmatrix}
\]

Thus we can write \( f(z_1, \ldots, z_{n-2}) = |z_1|^2 + \cdots + |z_{n-3}|^2 - |z_{n-2}|^2 \). Note that by these changes, the vectors of \( \mathbb{C}^{n-2} \) representing surfaces in \( \mathcal{M}_1(S^2, \alpha)^* \) are contained in the set \( Q_1 = f^{-1}(\{-1\}) \), and we still have \( \mu_{\text{Tr}} = c_0\lambda_{2(n-2)} \) with \( c_0 \) a constant.

We use the symbol \( \langle \cdot, \cdot \rangle \) to denote the scalar product defined by Hermitian form \( H \). By definition \( f(Z) = \langle Z, Z \rangle \), \( \forall Z \in \mathbb{C}^{n-2} \). Let \( J \) be the natural complex structure of \( \mathbb{C}^{n-2} \), that is \( J(z_1, \ldots, z_{n-2}) = (iz_1, \ldots, iz_{n-2}) \), and \( \eta \) be the real symmetric form induced by \( \langle \cdot, \cdot \rangle \), that is

\[
\eta(X, Y) = \text{Re} \langle X, Y \rangle.
\]

Let \( Z \) be a vector in \( Q_1 \) which represents a surface in \( \mathcal{M}_1(S^2, \alpha)^* \). The tangent space of \( Q_1 / S^1 \) at the orbit \( S^1 \cdot Z \) is naturally identified to the orthogonal complement of \( Z \) with respect to \( \langle \cdot, \cdot \rangle \), we denote this space by \( Z^\perp \). The restriction of \( \langle \cdot, \cdot \rangle \) on \( Z^\perp \) is a definite positive Hermitian form, which determines the complex hyperbolic metric on \( \mathcal{M}_1(S^2, \alpha)^* = C(\kappa_1, \ldots, \kappa_n) \). We have

\[
df = (\bar{z}_1 dz_1 + \cdots + \bar{z}_{n-3} dz_{n-3} - \bar{z}_{n-2} dz_{n-2}) + (z_1 d\bar{z}_1 + \cdots + z_{n-3} d\bar{z}_{n-3} - z_{n-2} d\bar{z}_{n-2}),
\]

and

\[
df \circ J = i(\bar{z}_1 dz_1 + \cdots + \bar{z}_{n-3} dz_{n-3} - \bar{z}_{n-2} dz_{n-2}) - i(z_1 d\bar{z}_1 + \cdots + z_{n-3} d\bar{z}_{n-3} - z_{n-2} d\bar{z}_{n-2}).
\]

Note that both \( df \) and \( df \circ J \) are invariant by the action of \( S^1 \). Put

\[
U_k = (0, \ldots, 0, \bar{z}_{n-2}, 0, \ldots, \bar{z}_k), \quad V_k = J \cdot U_k = iU_k, \quad \text{for } k = 1, \ldots, n-3.
\]

One can easily check that \( \{U_1, V_1, \ldots, U_{n-3}, V_{n-3}\} \) span \( Z^\perp \) as a real vector space. We consider \( \{U_1, V_1, \ldots, U_{n-3}, V_{n-3}\} \) as a basis of the tangent space of \( \mathcal{M}_1(S^2, \alpha)^* \) at \( \varphi(Z) \). We know that the restriction of the symmetric form \( \eta \) on \( Z^\perp \) defines a Riemannian metric. Let \( U_k, V_k^* \) denote the \( \mathbb{R} \)-linear 1-forms dual to \( U_k \) and \( V_k \) respectively with respect to \( \eta \). We have

\[
U_k^* = \frac{1}{2}[(z_{n-2} d\bar{z}_k - z_k d\bar{z}_{n-2}) + (\bar{z}_{n-2} d\bar{z}_k - \bar{z}_k d\bar{z}_{n-2})],
\]

and
\[ V_k^* = -\frac{1}{2} [(z_{n-2}dz_k - z_kdz_{n-2}) - (\bar{z}_{n-2}d\bar{z}_k - \bar{z}_kd\bar{z}_{n-2})]. \]

We can consider \( \{U_1^*, V_1^*, \ldots, U_{n-3}^*, V_{n-3}^*\} \) as a basis of the cotangent space of \( \mathcal{M}_1(S^2, \Omega)^* \) at \( \varphi(Z) \). Let \( \rho \) be the projection from \( Q_1 \) to \( Q_1/S^1 \). By definition, the volume form \( \hat{\mu}_{\text{Tr}}^1 \) on \( Q_1/S^1 \) verifies

\[ \rho^* \hat{\mu}_{\text{Tr}}^1 \wedge df \wedge (df \circ \mathcal{J}) = d\lambda_{2(n-2)} = \left(\frac{\lambda}{2}\right)^{n-2}dz_1 \cdots dz_{n-2}d\bar{z}_{n-2} \]

Since \( df \) and \( df \circ \mathcal{J} \) are invariant by the action of \( S^1 \), the volume form \( \hat{\mu}_{\text{Tr}}^1 \) is well defined by this condition. We will express \( \hat{\mu}_{\text{Tr}}^1(S^1 \cdot Z) \) in terms of \( U_k^*, V_k^*, k = 1, \ldots, n-3. \)

**Lemma 11.1** We have

\[ \hat{\mu}_{\text{Tr}}^1(S^1 \cdot Z) = \frac{c_0}{|z_{n-2}|^{2(n-4)}} (U_1^* \wedge V_1^*) \wedge \cdots \wedge (U_{n-3}^* \wedge V_{n-3}^*), \]

where \( c_0 = \frac{\mu_{\text{Tr}}}{\lambda_{2(n-2)}}. \)

**Proof:** Set

- \( X_k = z_{n-2}dz_k - z_kdz_{n-2}, \bar{X}_k = \bar{z}_{n-2}d\bar{z}_k - \bar{z}_kd\bar{z}_{n-2}, k = 1, \ldots, n-3, \)
- \( X = \bar{z}_1dz_1 + \cdots + \bar{z}_{n-3}dz_{n-3} - \bar{z}_{n-2}d\bar{z}_{n-2}, \bar{X} = z_1d\bar{z}_1 + \cdots + z_{n-3}d\bar{z}_{n-3} - z_{n-2}d\bar{z}_{n-2}. \)

We have

- \( U_k^* \wedge V_k^* = -\frac{1}{4} (X_k + \bar{X}_k) \wedge (X_k - \bar{X}_k) = \frac{1}{2} X_k \wedge \bar{X}_k, k = 1, \ldots, n-3, \)
- \( df \wedge (df \circ \mathcal{J}) = 2tX \wedge \bar{X}. \)

Thus

\[ \begin{align*}
(U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^*) \wedge df \wedge (df \circ \mathcal{J}) &= -\left(\frac{1}{2}\right)^{n-4} (X_1 \wedge \bar{X}_1 \wedge \cdots \wedge X_{n-3} \wedge \bar{X}_{n-3}) \wedge X \wedge \bar{X} \\
&= -\left(\frac{1}{2}\right)^{n-4} (-1)^{\frac{(n-2)(n-3)}{2}} (X_1 \wedge \cdots \wedge X_{n-3} \wedge X) \wedge (\bar{X}_1 \wedge \cdots \wedge \bar{X}_{n-3} \wedge \bar{X}).
\end{align*} \]

Simple computations give

\[ X_1 \wedge \cdots \wedge X_{n-3} \wedge X = z_{n-2}^{n-4} |z_1|^2 \cdots |z_{n-3}|^2 |z_{n-2}|^2 dz_1 \cdots dz_{n-2} = -z_{n-2}^{n-4} dz_1 \cdots dz_{n-2}. \]

Similarly, \( \bar{X}_1 \wedge \cdots \wedge \bar{X}_{n-3} \wedge \bar{X} = -\bar{z}_{n-2}^{n-4} d\bar{z}_1 \cdots d\bar{z}_{n-2}. \) Therefore,

\[ \begin{align*}
(X_1 \wedge \cdots \wedge X_{n-3} \wedge X) \wedge (\bar{X}_1 \wedge \cdots \wedge \bar{X}_{n-3} \wedge \bar{X}) &= |z_{n-2}|^{2(n-4)} dz_1 \cdots dz_{n-2} d\bar{z}_1 \cdots d\bar{z}_{n-2} \\
&= 2^{n-2} |z_{n-2}|^{2(n-4)} d\lambda_{2(n-2)},
\end{align*} \]

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and we get

\[ U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^* \wedge df \wedge (df \circ \mathcal{J}) = 4|z_{n-2}|^{2(n-4)}d\lambda_{2(n-2)}. \]

By the definition of \( \hat{\mu}_1 \), we obtain

\[ \hat{\mu}_1^1(S^1 \cdot Z) = \frac{c_0}{4|z_{n-2}|^{2(n-4)}} U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^*. \]

\[ \square \]

**Remark:**

- Even though the 1-forms \( U_k^* \) and \( V_k^* \) are not invariant by the \( S^1 \)-action, the 2-form \( U_k^* \wedge V_k^* \) is. Hence, the \( 2(n-3) \)-form \( U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^* \) is invariant by the \( S^1 \) action, and we get a well defined volume form on \( Q_1/S^1 \).

- Let \( \mu_1^1 \) be the volume form on \( Q_1 \) verifying the following condition

\[ \mu_1^1 \wedge df = \mu_1. \]

The tangent vector to the \( S^1 \) orbit at a point \( Z \in \mathbb{C}^2 \) is given by \( iZ \), and we have

\[ df \circ \mathcal{J}(iZ) = -df(Z) = -(Z, Z) = 1. \]

Therefore, the volume form \( \hat{\mu}_1^1 \) can be considered as the push-forward of \( \mu_1^1 \) onto \( Q_1/S^1 \).

Now, we will proceed to compute the volume form defined by \( \eta \) on \( Z^1 \) in terms of \( U_k^*, V_k^* \). Let \((\eta_{ij})\) with \( i, j = 1, \ldots, 2(n-3) \) be the (real) matrix of \( \eta \) in the basis \( \{U_1, V_1, \ldots, U_{n-3}, V_{n-3}\} \). Since the volume form \( \mu_{\text{Hyp}} \) is defined by the metric \( \eta \), we have

\[ \mu_{\text{Hyp}}(S^1 \cdot Z) = \frac{1}{\sqrt{\det(\eta_{ij})}} U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^* \] \hspace{1cm} (18)

**Lemma 11.2** We have \( \det(\eta_{ij}) = |z_{n-2}|^{4(n-4)} \).

**Proof:** Since \( \eta \) is the real part of \( H \), the matrix \((\eta_{ij})\) is the real interpretation of the matrix \((H_{ij})\) of \( H \) in the complex basis \( \{U_1, \ldots, U_{n-3}\} \) of \( Z^1 \). This implies \( \det(\eta_{ij}) = |\det(H_{ij})|^2 \).

We have \( H_{ij} = \langle U_i, U_j \rangle = \left\{ \begin{array}{ll} -\bar{z}_i \bar{z}_j, & \text{if } i \neq j; \\ |z_{n-2}|^2 - |z_i|^2, & \text{if } i = j. \end{array} \right. \)

hence

\[ \det(H_{ij}) = |z_{n-2}|^{2(n-3)} \begin{vmatrix} 1 - |\varepsilon_1|^2 & -\bar{\varepsilon}_1 \varepsilon_2 & \cdots & -\bar{\varepsilon}_1 \varepsilon_{n-3} \\ -\bar{\varepsilon}_2 \varepsilon_1 & 1 - |\varepsilon_2|^2 & \cdots & -\bar{\varepsilon}_2 \varepsilon_{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ -\bar{\varepsilon}_{n-3} \varepsilon_1 & -\bar{\varepsilon}_{n-3} \varepsilon_2 & \cdots & 1 - |\varepsilon_{n-3}|^2 \end{vmatrix}, \]

where \( \varepsilon_k = z_k/z_{n-2}, \) \( k = 1, \ldots, n-3 \). Since we have
\[ 1 - |\varepsilon_1|^2 \quad -\bar{\varepsilon}_1 \varepsilon_2 \quad \ldots \quad -\bar{\varepsilon}_1 \varepsilon_{n-3} \\
-\bar{\varepsilon}_2 \varepsilon_1 \quad 1 - |\varepsilon_2|^2 \quad \ldots \quad -\bar{\varepsilon}_2 \varepsilon_{n-3} \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
-\bar{\varepsilon}_{n-3} \varepsilon_1 \quad -\bar{\varepsilon}_{n-3} \varepsilon_2 \quad \ldots \quad 1 - |\varepsilon_{n-3}|^2 \]

it follows that

\[ \det(H_{ij}) = |z_{n-2}|^{2(n-3)}(1 - (|\varepsilon_1|^2 + \cdots + |\varepsilon_{n-3}|^2)) \]

\[ = |z_{n-2}|^{2(n-4)}(|z_{n-2}|^2 - (|z_1|^2 + \cdots + |z_{n-3}|^2)) \]

\[ = |z_{n-2}|^{2(n-4)} \]

Consequently, we have \( \det(\eta_{ij}) = |\det(H_{ij})|^2 = |z_{n-2}|^{4(n-4)} \). The lemma is then proved. \( \square \)

11.3 Proof of Proposition 2.8

From Lemma 11.1, Lemma 11.2, and (18), we know that the quotient \( \frac{\mu_T}{\mu_{\text{Hyp}}} \) is a locally constant function on \( \mathcal{M}_1(S^2, \alpha)^* \). It remains to show that \( \mathcal{M}_1(S^2, \alpha)^* \) is connected, but this is known since, by Proposition 2 [Tr1], we can identify \( \mathcal{M}_1(S^2, \alpha)^* \) to modular space of \( n \)-punctured sphere, which is connected. \( \square \)

Appendices

A Proof of Theorem 6.2

Let \( n_1 \) be the cardinal of \( V \cap \partial \Sigma \), and \( n_2 \) be the cardinal of \( V \cap \text{int}(\Sigma) \). Observe that any triangulations of \( \Sigma \) whose vertex set is \( V \) has a fixed number \( N_e \) of edges. Let \( k, 0 \leq k \leq N_e \), be the number of common edges of \( T_1 \) and \( T_2 \). Since the boundary of \( \Sigma \) contains \( n_1 \) edges, we have \( k \geq n_1 \). Assume that \( n_1 \leq k < N_e \), we will proceed by induction.

First, let us prove the following technical lemma

**Lemma A.1** Let \( P \) be a polygon in \( \mathbb{R}^2 \) whose vertices are denoted by \( A_1, A_2, A_3, B_1, \ldots, B_l \). Let \( x : \mathbb{R}^2 \rightarrow \mathbb{R} \), and \( y : \mathbb{R}^2 \rightarrow \mathbb{R} \) denote the two coordinate functions of \( \mathbb{R}^2 \). Assume that the vertices of \( P \) verify the following conditions:

1. \((A_1, A_2, A_3)\) are ordered in the clockwise sense.
2. \( y(A_i) \geq 0, i = 1, 2, 3, y(A_1) < y(A_2), \) and \( y(A_2) \geq y(A_3) \).
. \( y(B_j) < 0, \ j = 1, \ldots, l; \)

. \( B_1, \ldots, B_l \) are ordered in the counter-clockwise sense.

. For all \( j \in \{1, \ldots, l\} \), the segment \( \overline{A_2B_j} \) is a diagonal of \( P \).

Let \( T \) denote the triangulation of \( P \) by the diagonals \( \overline{A_2B_1}, \ldots, \overline{A_2B_l} \). Let \( \{s_0, \ldots, s_k\} \) be a family of disjoint horizontal segments in \( P \) whose endpoints are contained the boundary of \( P \), where \( s_0 \) is a segment lying on the horizontal axis \( y = 0 \). Let \( r \) be the number of intersection points of the edges of \( T \) with the set \( \bigcup_{i=0}^{k} s_i \). Then there exists a sequence of elementary moves which transform \( T \) into a new triangulation \( T' \) whose edges intersect the set \( \bigcup_{i=0}^{k} s_i \) at at most \( r - 1 \) points.

**Proof:** Let \( j_0 \) be the smallest index such that \( y(B_{j_0}) = \min\{y(B_j) : j = 1, \ldots, l\} \), and consider the following algorithm:

1. If \( P \) is a quadrilateral, that is \( l = 1 \), then \( P \) must be convex since its two diagonals intersect. Apply an elementary move inside \( P \) and stop the algorithm.

2. If \( 1 < j_0 < l \), then consider the quadrilateral \( A_2B_{j_0-1}B_{j_0}B_{j_0+1} \). By the choice of \( j_0 \), this quadrilateral is convex, hence, we can apply an elementary move inside it, and the algorithm stops.

3. If \( j_0 = 1 \) and \( l \geq 2 \), then consider the quadrilateral \( A_2A_1B_1B_2 \). Observe that this quadrilateral is convex. Apply an elementary move inside it. By this move, we get a new triangulation of \( P \) which contains the triangle \( \Delta A_1B_1B_2 \). Cut off this triangle from \( P \). Replace \( P \) by the remaining sub-polygon and restart the algorithm.

4. If \( j_0 = l > 1 \), then consider the quadrilateral \( A_2A_3B_lB_{l-1} \). Since this quadrilateral is convex, we can apply an elementary move inside it, then cut off the triangle \( \Delta A_3B_lB_{l-1} \). Replace \( P \) by the remaining sub-polygon and restart the algorithm.

Observe that, at each step of the algorithm above, the number of intersection points of the set \( \bigcup_{i=0}^{k} s_i \) with the edges of the new triangulation cannot exceed the number of intersection points with those of the ancien one. Indeed, suppose that we are in the case 2., by the choice of \( j_0 \), we have
There exists a sequence of elementary moves which transform \( \text{Proposition A.2} \) into a new triangulation containing \( s_1 \), i.e. \( y \). Moreover, at the final step of the algorithm, \( i.e. \) case 1. or 2., we replace a diagonal intersecting \( s_0 \) by another one which does not intersect \( s_0 \). Therefore, by this algorithm, we get a new triangulation \( T' \) of \( P \) whose edges have strictly less intersection points with the set \( \cup_{i=0}^{k} s_i \) than those of \( T_4 \). 

Let \( a_1, \ldots, a_{N_1}, \) and \( b_1, \ldots, b_{N_2} \) denote the edges of \( T_1 \) and \( T_2 \) respectively. We can assume that \( a_i = b_i \), for \( i = 1, \ldots, k \). All we need to prove is the following

**Proposition A.2** There exists a sequence of elementary moves which transform \( T_1 \) into a new triangulation containing \( b_1, \ldots, b_k, \) and \( b_{k+1} \).

**Proof:** Since \( b_{k+1} \) is not an edge of \( T_1 \), it must intersect some edges of \( T_1 \). Let \( P \) be the developing polygon of \( b_{k+1} \) with respect to \( T_1 \), and \( \varphi : P \rightarrow \Sigma \) be the associated immersion. Let \( T_3 \) be the triangulation of \( P \) by diagonals which is induced by \( T_1 \), \( i.e. \) \( T_3 = \varphi^{-1}(T_1) \). By definition, each diagonal in \( T_3 \) is mapped by \( \varphi \) onto an edge of \( T_1 \) which intersects \( \text{int}(b_{k+1}) \). Let \( d \) be the diagonal of \( P \) such that \( \varphi(d) = b_{k+1} \). Observe that \( \text{int}(d) \) intersects all the diagonals which are edges of \( T_3 \).

Let \( m \) be the number of intersection points of \( b_{k+1} \) with the edges of \( T_1 \) excluding the two endpoints of \( b_{k+1} \). Note that \( b_{k+1} \) may intersect an edge of \( T_1 \) more than once. By construction, the polygon \( P \) is triangulated by \( m \) diagonals, hence it has \( m + 3 \) sides. We prove the proposition by induction.

- If \( m = 1 \), then \( P \) is a quadrilateral. The quadrilateral \( P \) must be convex because its two diagonals intersect. If \( P \) is mapped by \( \varphi \) to a single triangle of \( T_1 \), then there is a singular point of \( \Sigma \) with cone angle strictly less than \( \pi \), but this is impossible since we have assumed that \( \Sigma \) verifies Property (Q). Thus, we conclude that \( \varphi \) maps \( \text{int}(P) \) isometrically onto a domain consisting of two triangles in \( T_1 \). Clearly, by applying the elementary move inside \( \varphi(P) \), we obtain a new triangulation which contains \( b_{k+1} \).

- If \( m > 1 \), it is enough to show that there exists a sequence of elementary moves which transform \( T_1 \) into a new triangulation \( T'_1 \) containing \( b_1 = (a_1), \ldots, b_k = (a_k), \) such that \( b_{k+1} \) intersects the edges of \( T'_1 \) at most \( m - 1 \) times.

Equip the plane \( \mathbb{R}^2 \) with a system of Cartesian coordinates such that the diagonal \( d \subset P \) is a horizontal segment lying in the axis \( Ox \). Let \( x : \mathbb{R}^2 \rightarrow \mathbb{R}, \) and \( y : \mathbb{R}^2 \rightarrow \mathbb{R} \) denote the two coordinate functions. Let \( A_1, \ldots, A_r \) denote the vertices of \( P \) such that \( y(A_i) > 0, \) and \( B_1, \ldots, B_s \) denote the vertices of \( P \) such that \( y(B_j) < 0 \). Let \( A_0 \) and \( A_{r+1} \) denote the left and the right endpoints of \( d \) respectively. We set, by convention, \( B_0 = A_0, \) and \( B_{s+1} = A_{r+1} \). Since \( P \) has \( m + 3 \) vertices, we have \( r + s + 2 = m + 3 \). We can assume that \( r \geq s \) (if it is not the case, reverse the orientation of \( Oy \)). We denote the vertices of \( P \) so that \( A_0, \ldots, A_{r+1} \) are ordered in
the clockwise sense, and \( B_0, \ldots, B_{s+1} \) are ordered in the counter-clockwise sense.

Without loss of generality, we can assume that \( r \geq 2 \) because \( m > 1 \). Let \( i_0 \) be the smallest index such that \( y(A_{i_0}) = \max\{y(A_i) : i = 1, \ldots, r\} \), that is \( y(A_{i_0}) \geq y(A_i) \) \( \forall i = 1, \ldots, r \), and \( y(A_{i_0}) > y(A_i) \) if \( i < i_0 \). Consider the polygon \( P_1 \) which is the union of all triangles in \( T_3 \) having \( A_{i_0} \) as a vertex. There exist \( j_0 \) and \( l \) such that the vertices of \( P_1 \) are \( A_{i_0} - 1, A_{i_0}, A_{i_0} + 1 \) and \( B_{j_0}, \ldots, B_{j_0 + l} \), note that \( P_1 \) is triangulated by the diagonals \( A_{i_0}B_{j_0}, \ldots, A_{i_0}B_{j_0 + l} \). Let \( T_4 \) denote this triangulation of \( P_1 \).

By Lemma A.3 below, we know that the restriction of \( \varphi \) into \( \text{int}(P_1) \) is injective. Let \( Q_1 \) be the image of \( \text{int}(P_1) \) under \( \varphi \). Since \( b_1, \ldots, b_k, b_{k+1} \) are edges of the triangulation \( T_2 \), we have \( \text{int}(b_i) \cap \text{int}(b_{k+1}) = \varnothing \), \( \forall i = 1, \ldots, k \). Recall that \( b_1, \ldots, b_k \) are also edges of the triangulation \( T_1 \) of \( \Sigma \), from this we deduce that \( \text{int}(b_i) \cap Q_1 = \varnothing \), since if \( e \) is an edge of \( T_1 \) and \( \text{int}(e) \cap Q_1 \neq \varnothing \), then \( \text{int}(e) \cap \text{int}(b_{k+1}) \neq \varnothing \). This implies that an elementary move inside \( Q_1 \) does not affect the edges \( b_1, \ldots, b_k \).

Consider the intersection of \( P_1 \) and \( \varphi^{-1}(b_{k+1}) \). A priori, this set is a family of geodesic segments with endpoints in the boundary of \( P_1 \). Clearly, the segment \( s_0 = \overline{A_0A_{r+1}} \cap P_1 \) is contained in the set \( P_1 \cap \varphi^{-1}(b_{k+1}) \). Since \( \Sigma \) satisfies (Q), all the segments in this family are parallel, therefore, all of them are parallel to the horizontal axis. Let \( \delta \) be the number of intersection points of the set \( P_1 \cap \varphi^{-1}(b_{k+1}) \) and the edges of \( T_4 \).

Now, Lemma A.1 shows that there exists a sequence of elementary moves which transform \( T_4 \) into a new triangulation of \( P_1 \) whose edges intersect the set \( P_1 \cap \varphi^{-1}(b_{k+1}) \) at at most \( \delta - 1 \) points. It follows that there exists a sequence of elementary moves inside the domain \( Q_1 \subset \Sigma \) which transform \( T_1 \) into a new triangulation of \( \Sigma \) whose edges have at most \( m - 1 \) intersection points with \( b_{k+1} \). As we have seen, those elementary moves do not affect the edges \( b_1, \ldots, b_k \).

By induction, the proposition is then proved. \( \square \)

We need the following lemma to complete the proof of Proposition A.2.
Lemma A.3 With the same notations as in the proof of A.2, the restriction of \( \varphi \) to \( \text{int}(P_1) \) is an isometric embedding.

Proof: Since \( \varphi \) maps each triangle of \( T_3 \) onto a triangle of \( T_1 \), it is enough to show that the images by \( \varphi \) of the triangles of \( T_3 \) which are contained in \( P_1 \) are all distinct. Suppose that there exist two triangles \( \Delta_1 \) and \( \Delta_2 \) such that \( \varphi(\Delta_1) = \varphi(\Delta_2) \). Since \( \varphi \) is locally isometric, and by assumption, the orthogonal part of the holonomy of any closed curve in \( \text{int}(\Sigma \setminus \{p_1, \ldots, p_n\}) \) is either \( \text{Id} \) or \( -\text{Id} \), it follows that either \( \Delta_2 = \Delta_1 + v \), or \( \Delta_2 = -\Delta_1 + v \), where \( -\Delta_1 \) is the image of \( \Delta_1 \) by \( -\text{Id} \), and \( v \in \mathbb{R}^2 \).

Note that, by definition, the triangles \( \Delta_1 \) and \( \Delta_2 \) have a common vertex, which is \( A_{i_0} \).

- If \( \Delta_2 = \Delta_1 + v \), excluding the case \( \Delta_1 \equiv \Delta_2 \), we have two possible configurations. In these both cases, we see that the angle of \( P_1 \) at the point \( A_{i_0} \) is at least \( \pi \). But, by assumption, this is impossible since we have \( y(A_{i_0}) > y(A_{i_0-1}) \) and \( y(A_{i_0}) \geq y(A_{i_0+1}) \).

- If \( \Delta_2 = -\Delta_1 + v \), we have three possible configurations. In the case where \( \Delta_1 \) and \( \Delta_2 \) have only one common vertex, we see that the angle of \( P_1 \) at \( A_{i_0} \) must be greater than \( \pi \), which is, as we have seen above, impossible. In the other two cases, \( \Delta_1 \) and \( \Delta_2 \) are adjacent. As we have seen, this implies the existence of a singular point of \( \Sigma \) with cone angle strictly less than \( \pi \). This is again impossible.

The lemma is then proved. \( \square \)

B Proof of Theorem 10.2

Let \( x_1, \ldots, x_n \) denote the vertices of \( T_1 \) and \( T_2 \). By convention, we consider \( \{x_1, \ldots, x_n\} \) as the set of singular points of \( \Sigma \) even though some of them may be regular. In what follows, if \( T \) is a triangulation of \( \Sigma \) whose vertex set is \( \{x_1, \ldots, x_n\} \), we will call a tree contained in the 1-skeleton of \( T \) which connects all the vertices of \( T \) a maximal tree. Let \( A_i, \ i = 1, 2 \) be a maximal tree of \( T_i \). If \( A_1 \equiv A_2 \), then the theorem follows from Theorem 6.2. Thus, it is enough to prove the following
Proposition B.1 There exists a sequence of elementary moves which transforms $T_1$ into a triangulation containing $A_2$.

We start by the following lemma

Lemma B.2 If $c_1,\ldots,c_k$ are geodesic segments with endpoints in $\{x_1,\ldots,x_n\}$ such that $\text{int}(c_i) \cap \text{int}(c_j) = \emptyset$ if $i \neq j$, and $\text{int}(c_i) \cap A_1 = \emptyset$, $i = 1,\ldots,k$, then there exists a sequence of elementary moves which transforms $T_1$ into a new triangulation containing $A_1$, and all the segments $c_1,\ldots,c_k$.

Proof: Let $P_{c_1}$ be the developing polygon of $c_1$ with respect to $T$. Since $\text{int}(c_1) \cap A_1 = \emptyset$, the isometric immersion $\varphi_{c_1}: \text{int}(P_{c_1}) \to \Sigma$ is an embedding, therefore, by applying Theorem 6.2 to $P_{c_1}$, we deduce that $T$ can be transformed by elementary moves into a triangulation containing $A_1$ and $c_1$. We can then restart the procedure with $c_2$ in the place of $c_1$. Since $\text{int}(c_1) \cap \text{int}(c_2) = \emptyset$, the developing polygon of $c_2$ does not contain $c_1$ in the interior. Thus, the lemma follows by induction. $\square$

Now, let $a_1,\ldots,a_{n-1}$ denote the edges of the tree $A_1$, and $b_1,\ldots,b_{n-1}$ denote the edges of the tree $A_2$. We will proceed by induction. Suppose that $T_1$ contains already the $k$ edges $b_1,\ldots,b_k$ of $A_2$. We will show that $T_1$ can be transformed by a sequence of elementary moves into a new triangulation containing $b_1,\ldots,b_k$ and $b_{k+1}$. Let $m$ be the number of intersection points of $b_{k+1}$ with the tree $A_1$ excluding the endpoints of $b_{k+1}$. If $m = 0$, then Lemma B.2 allows us to get the conclusion. Therefore, if $m \geq 1$, all we need to show is the following

Lemma B.3 The triangulation $T_1$ can be transformed by elementary moves into a new triangulation $T'_1$ which contains a maximal tree $A'_1$, and the edges $b_1,\ldots,b_k$, such that the number of intersecting points of $b_{k+1}$ with $A'_1$, excluding the endpoints of $b_{k+1}$, is at most $m - 1$.

Proof: We can assume that the endpoints of $b_{k+1}$ are $x_1$ and $x_2$. We consider $b_{k+1}$ as a geodesic ray exiting from $x_1$. Let $y_1$ denote the first intersection point of $b_{k+1}$ with the tree $A_1$, which is contained in the interior of an edge $\overline{x_{j_1}y_{j_1+1}}$ of $A_1$. Let $\overline{x_{j_1}y_1}$ denote the subsegment of $b_{k+1}$ whose endpoints are $x_1$ and $y_1$. Without loss of generality, we can assume that $x_{j_1}$ is contained in the unique path along $A_1$ from $x_1$ to $x_{j_1+1}$.

Cutting open the surface $\Sigma$ along the tree $A_1$, we get a flat surface $\Sigma'$ with geodesic boundary homeomorphic to a close disk. By construction, we have a surjective map $\pi_{A_1}: \Sigma' \to \Sigma$ verifying the following properties

1. $\pi_{A_1} | \text{int}(\Sigma')$ is an isometry,
2. $\pi_{A_1}(\partial \Sigma') = A_1$.
3. There are $2(n-1)$ geodesic segments in the boundary of $\Sigma'$ such that the restriction of $\pi_{A_1}$ into each of which is an isometry.
4. For every edge $e$ in $A_1$, $\pi_{A_1}^{-1}(\text{int}(e))$ is the union of two open segments in the boundary of $\Sigma'$.
Let $s_1$ denote $\pi_{A_1}^{-1}(x_1 y_1)$, then $s_1$ is a geodesic segment with endpoints in $\partial \Sigma'$. Let $x'_i$ and $y'_i$ denote the endpoints of $s_1$ with $\pi_{A_1}(x'_i) = x_1$, and $\pi_{A_1}(y'_i) = y_1$. Let $x'_1, \ldots, x'_{2(n-1)}$ denote the points in $\pi_{A_1}^{-1}\{(x_1, \ldots, x_n)\}$ following an orientation of $\partial \Sigma'$. By choosing the suitable orientation, we can assume that the point $y'_1$ is between $x'_{j_1}$ and $x'_{j_1+1}$ where $\pi_{A_1}(x'_{j_1}) = x_{j_1}$, and $\pi_{A_1}(x'_{j_1+1}) = x_{j_1+1}$. For every $j$ in $\{1, \ldots, 2(n-1)\}$, we denote by $x'_j x'_{j+1}$ the segment in the boundary of $\Sigma'$ with endpoints $x'_j$ and $x'_{j+1}$, with the convention $x'_{2n-1} = x'_1$. Note that $\pi_{A_1}(x'_{j_1}, x'_{j_1+1})$ is an edge of $A_1$.

Let $c_0$ be a path in $\Sigma'$ joining $x'_1$ and $x'_{j_1+1}$ with minimal length. First, we prove

**Lemma B.4** We have $c_0 \cap s_1 = \{x'_1\}$.

**Proof:** Suppose that $c_0 \cap \text{int}(s_1) \neq \emptyset$, then let $y'_2$ denote the first intersection point of $c_0$ with $s_1 \setminus \{x'_1\}$. Let $c_1$ denote the path from $x'_1$ to $y'_2$ along $c_0$, and let $\overline{x'y'_2}$ denote the subsegment of $s_1$ with endpoints $x'_1$ and $y'_2$. By definition, we see that $c_1 \cup \overline{x'y'_2}$ is the boundary a flat disk with piecewise geodesic boundary $D$. Since the path $c_0$ is of minimal length, it follows that the interior angle between two consecutive segments of $c_1$ is at least $\pi$. Therefore, if the number of segments in $c_1$ is $l$, the boundary of $D$ contains then $l + 1$ geodesic segments, and the sum of all the interior angles is at least $(l-1)\pi$. But this is impossible by the Gauss-Bonnet Theorem, hence we conclude that $c_0 \cap (s_1 \setminus \{x'_1\}) = \emptyset$. □

Let $\overline{y'_1 x'_{j_1+1}}$ denote the subsegment of $x'_j x'_{j+1}$ with endpoints $y'_1$ and $x'_{j_1+1}$. From Lemma B.4, we see that $s_1 \cup \overline{y'_1 x'_{j_1+1}} \cup c_0$ is the boundary of a disk $D_0$ contained in $\Sigma'$. We have immediately the following

**Lemma B.5** Let $s$ be a geodesic ray that intersects the interior of $D_0$. If $s$ intersects $D_0$ by a point in the path $c_0$, then $s$ must exit $D_0$ by a point in $(s_1 \cup \overline{y'_1 x'_{j_1+1}}) \setminus \{x'_1, x'_{j_1+1}\}$.

**Proof:** If $s$ exits $D_0$ by another point in $c_0$, then we have a flat disk with piecewise geodesic boundary which violates the Gauss-Bonnet Theorem. □

Let $\hat{c}_0$ denote the image of $c_0$ under $\pi_{A_1}$. The path $\hat{c}_0$ is then a finite union of geodesic segments on $\Sigma$ with endpoints in the set $\{x_1, \ldots, x_n\}$. It is clear that $\hat{c}_0$ contains a path $\hat{c}_1$ joining $x_1$ to $x_{j_1+1}$. Let us prove the following

**Lemma B.6** The path $\hat{c}_1$ does not contain the segment $\overline{x_{j_1} x_{j_1+1}}$.

**Proof:** Suppose, on the contrary, that $\hat{c}_1$ contains $\overline{x_{j_1} x_{j_1+1}}$. This implies that $c_0$ contains a segment $\overline{x_{k'} x_{k'+1}}$, with $k' \neq j_1$, such that $\pi_{A_1}(x'_{k'} x_{k'+1}) = \pi_{A_1}(x'_1 x'_1) = x_{j_1} x_{j_1+1}$.

Let $y'_2$ denote the unique point in $\overline{x_{k'} x_{k'+1}}$ such that $\pi_{A_1}(y'_2) = \pi_{A_1}(y'_1) = y_1$. Observe that $\pi_{A_1}^{-1}(b_{k'+1})$ is a sequence of $(m+1)$ geodesic segments of $\Sigma'$ with endpoints in the boundary of $\Sigma'$, $s_1$ is the first one of this sequence, $y'_2$ is then one endpoint of the next segment in this sequence, which will be denoted by $s_2$.

By assumption, $y'_2$ is an intersection point of the segment $s_2$ and the disk $D_0$. Consider the segment $s_2$ as a geodesic ray exiting from $y'_2$. By Lemma B.5, the ray $s_2$ exits $D_0$ by a point $z'_2$ in
(s_1 \cup y_1^{x_j^{j_1+1}}) \setminus \{x_1', x_{j_1+1}'\}. Since the geodesic b_{k+1} is a simple, z_2' cannot be a point in s_1, hence z_2' must be a point in \text{int}(y_1^{x_j^{j_1+1}}).

Now, since the segments \overline{x_j^{j_1+1}} and \overline{x_k^{x_{k'+1}}} are identified by \pi_{A_1}, the point z_2' is identified to a point y_3' in \overline{x_k^{x_{k'+1}}}. Consequently, the argument above can be applied infinitely many times, which implies that \pi_{A_1}^{-1}(b_{k+1}) contains infinitely many segments, and we have a contradiction to the fact that \pi_{A_1}^{-1}(b_{k+1}) contains only m + 1 segments. \hfill \Box

Since A_1 is a tree, the set \{A_1 \setminus \text{int}(y_1^{x_j^{j_1+1}})\} has two connected components, the one containing x_1 will be denoted by C_1, the other one containing x_{j_1+1} will be denoted by C_2. From Lemma B.6, we know that the path \hat{c}_1, which joins x_1 to x_{j_1+1} does not contain \overline{x_j^{x_{j_1+1}}} Therefore the path \hat{c}_1 must contain a segment \hat{s}, with endpoints in \{x_1, \ldots, x_{b+1}\}, such that one of the two endpoints is in C_1, and the other is in C_2. Let s denote \pi_{A_1}^{-1}(\hat{s}). Evidently, \hat{s} is not an edge of A_1, hence s is a segment contained inside \Sigma'. It follows that \text{int}(\hat{s}) \cap A_1 = \emptyset. Let us prove

**Lemma B.7** \text{int}(\hat{s}) \cap \text{int}(b_i) = \emptyset, for every i = 1, \ldots, k.

**Proof:** Let b_i', i = 1, \ldots, k, denote \pi_{A_1}^{-1}(b_i). Since \text{int}(b_i) \cap A_1 = \emptyset, b_i' is a geodesic segment with endpoints in \partial \Sigma'. Suppose that \text{int}(\hat{s}) \cap \text{int}(b_i) \neq \emptyset, it follows that \text{int}(b_i') \cap \text{int}(s) \neq \emptyset. Let y_i' be the intersection point of \text{int}(b_i') and \text{int}(s). Recall that s is included in the path \hat{c}_0, we can then consider the segment b_i' as a ray which intersects D_0 by y_i'. By Lemma B.4, we know that b_i' must exit D_0 by a point z_i' which is contained in \text{int}(y_1^{x_j^{j_1+1}}), but it would imply that either \text{int}(b_i) \cap b_{k+1} \neq \emptyset, or \text{int}(b_i) \cap A_1 \neq \emptyset, which is impossible by assumption. The lemma is then proved. \hfill \Box

We can now finish the proof of Lemma B.3. Using Lemma B.2, we deduce that there exists a sequence of elementary moves which transforms T_1 into a new triangulation T_1' containing A_1, the edges b_1, \ldots, b_n, and the segment \hat{s}. By replacing \overline{x_j^{x_{j_1+1}}} by \hat{s}, we get a new maximal tree A_1'. Let us show that the number of intersection points of b_{k+1} with A_1', excluding the endpoints of b_{k+1}, is at most m – 1. We have

\text{Card}\{\text{int}(b_{k+1}) \cap A_1'\} = \text{Card}\{\text{int}(b_{k+1}) \cap A_1\} - \text{Card}\{\text{int}(b_{k+1}) \cap \text{int}(x_j^{x_{j_1+1}})\} + \text{Card}\{\text{int}(b_{k+1}) \cap \text{int}(\hat{s})\}

Let y be a point in \text{int}(b_{k+1}) \cap \text{int}(\hat{s}), and let y' = \pi_{A_1}^{-1}(y). Let b' be the segment in \pi_{A_1}^{-1}(b_{k+1}) which contains y', note that y' = b' \cap s. By Lemma B.5, and since \text{int}(b') \cap \text{int}(s) = \emptyset, it follows that b' contains a point z' in \overline{x_j^{x_{j_1+1}}} We deduce that there is a one-to-one mapping from \{\text{int}(b_{k+1}) \cap \text{int}(\hat{s})\} into \{\text{int}(b_{k+1}) \cap \text{int}(\overline{x_j^{x_{j_1+1}}})\}. Clearly, the point y_1 does not belong to the image of this map, therefore we have

\text{Card}\{\text{int}(b_{k+1}) \cap \text{int}(\overline{x_j^{x_{j_1+1}}})\} \geq \text{Card}\{\text{int}(b_{k+1}) \cap \text{int}(\hat{s})\} + 1.

It follows immediately that

\text{Card}\{\text{int}(b_{k+1}) \cap A_1'\} \leq \text{Card}\{\text{int}(b_{k+1}) \cap A_1\} - 1 = m - 1.
The proof of Lemma B.3 is now complete. □

From what we have seen, Proposition B.1, and hence Theorem 10.2, follow directly from Lemma B.3. □

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