Extremal theory of locally sparse multigraphs

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Abstract

An \((n, s, q)\)-graph is an \(n\)-vertex multigraph where every set of \(s\) vertices spans at most \(q\) edges. In this paper, we determine the maximum product of the edge multiplicities in \((n, s, q)\)-graphs if the congruence class of \(q\) modulo \(\binom{s}{2}\) is in a certain interval of length about \(3s/2\). The smallest case that falls outside this range is \((s, q) = (4, 15)\), and here the answer is \(a n^2 + o(n^2)\) where \(a\) is transcendental assuming Schanuel’s conjecture. This could indicate the difficulty of solving the problem in full generality. Many of our results can be seen as extending work by Bondy-Tuza [2] and Füredi-Kündgen [8] about sums of edge multiplicities to the product setting.

We also prove a variety of other extremal results for \((n, s, q)\)-graphs, including product-stability theorems. These results are of additional interest because they can be used to enumerate and to prove logical 0-1 laws for \((n, s, q)\)-graphs. Our work therefore extends many classical enumerative results in extremal graph theory beginning with the Erdős-Kleinman-Rothschild theorem [6] to multigraphs.

1 Introduction

Given a set \(X\) and a positive integer \(t\), let \(\binom{X}{t} = \{Y \subseteq X : |Y| = t\}\). A multigraph is a pair \((V, w)\), where \(V\) is a set of vertices and \(w : \binom{V}{2} \to \mathbb{N} = \{0, 1, 2, \ldots\}\).

Definition 1. Given integers \(s \geq 2\) and \(q \geq 0\), a multigraph \((V, w)\) is an \((s, q)\)-graph if for every \(X \in \binom{V}{s}\) we have \(\sum_{xy \in \binom{X}{2}} w(xy) \leq q\). An \((n, s, q)\)-graph is an \((s, q)\)-graph with \(n\) vertices, and \(F(n, s, q)\) is the set of \((n, s, q)\)-graphs with vertex set \([n] := \{1, \ldots, n\}\).

The goal of this paper is to investigate extremal, structural, and enumeration problems for \((n, s, q)\)-graphs for a large class of pairs \((s, q)\).

Definition 2. Given a multigraph \(G = (V, w)\), define

\[
S(G) = \sum_{xy \in \binom{V}{2}} w(xy) \quad \text{and} \quad P(G) = \prod_{xy \in \binom{V}{2}} w(xy),
\]

\[
\text{ex}_\Sigma(n, s, q) = \max \{S(G) : G \in F(n, s, q)\} \quad \text{and} \quad \text{ex}_\Pi(n, s, q) = \max \{P(G) : G \in F(n, s, q)\}.
\]

An \((n, s, q)\)-graph \(G\) is sum-extremal (product-extremal) if \(S(G) = \text{ex}_\Sigma(n, s, q)\) \((P(G) = \text{ex}_\Pi(n, s, q))\). Let \(S(n, s, q)\) \((P(n, s, q))\) be the set of all sum-extremal (product-extremal) \((n, s, q)\)-graphs with vertex set \([n]\).

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In [2], Bondy and Tuza determine the structure of multigraphs in $S(n,s,q)$ when $n$ is large compared to $s$ and $q \equiv 0,-1 \pmod{(1/2)}$ and when $s = 3$. In [3], Füredi and Kündgen (among other things) determine the asymptotic value of $\text{ex}_\Sigma(n,s,q)$ for all $s,q$ with a $O(n)$ error term, and the exact value is determined for many cases. Other special cases of these questions have appeared in [13]. A natural next step from the investigation of extremal problems for $(n,s,q)$-graphs is to consider questions of structure and enumeration. The question of enumeration for $(n,s,q)$-graphs was first addressed in [14], where it was shown the problem is closely related extremal results for $(n,s,q)$-graphs investigated in [2,9].

Definition 3. Given integers $s \geq 2$ and $q \geq (s+2)/2$, define the asymptotic product density and the asymptotic sum density, respectively, as the following limits (which both exist):

$$\text{ex}_\Pi(s,q) = \lim_{n \to \infty} \left( \text{ex}_\Pi(n,s,q) \right)^{1/(s+2)}$$

and

$$\text{ex}_\Sigma(s,q) = \lim_{n \to \infty} \frac{\text{ex}_\Sigma(n,s,q)}{(n+2)^{s+2}}.$$

In [14], the current authors showed $\text{ex}_\Pi(s,q)$ exists for all $s \geq 2$ and $q \geq 0$ and proved the following enumeration theorem for $(n,s,q)$-graphs in terms of $\text{ex}_\Pi(s,q + (s+2)/2)$.

Theorem 1. ([14]) Suppose $s \geq 2$ and $q \geq 0$ are integers. If $\text{ex}_\Pi(s,q + (s+2)/2) > 1$, then

$$\text{ex}_\Pi\left(s,q + \left(\frac{s}{2}\right)\right)^{1/(s+2)} \leq |F(n,s,q)| \leq \text{ex}_\Pi\left(s,q + \left(\frac{s}{2}\right)\right)^{(1+o(1))\left(\frac{s}{2}\right)},$$

and if $\text{ex}_\Pi(s,q + (s+2)/2) \leq 1$, then $|F(n,s,q)| \leq 2^{o(n^2)}$.

This result was used in [14] along with a computation of $\text{ex}_\Pi(4,15)$ to give an enumeration of $F(n,4,9)$. This case was of particular interest because it turned out that $|F(n,4,9)| = a n^{3+o(n^2)}$, where $a$ is transcendental under the assumption of Schanuel’s conjecture. In this paper, we continue this line of investigations by proving enumeration results for further cases of $s$ and $q$, and in some cases proving approximate structure theorems (the particular special case $(s,q) = (3,4)$ was recently studied in [7]). This generalizes many classical theorems about enumeration in extremal graph theory (beginning with the Erdős-Kleitman-Rothschild theorem [6]) to the multigraph setting. All of these results rely on computing $\text{ex}_\Pi(n,s,q)$, characterizing the elements in $P(n,s,q)$, and proving corresponding product-stability theorems, and this is the main content of this paper. Questions about $\text{ex}_\Pi(n,s,q)$ and $P(n,s,q)$ may also be of independent interest, as they are natural “product versions” of the questions about extremal sums for $(n,s,q)$-graphs investigated in [2,9].

2 Main Results

Given a multigraph $G = (V,w)$ and $xy \in (V)_2$, we will refer to $w(xy)$ as the multiplicity of $xy$. The multiplicity of $G$ is $\mu(G) = \max\{w(xy) : xy \in (V)_2\}$. Our first main result, Theorem 2 below, gives us information about the asymptotic properties of elements in $F(n,s,q)$, in the case when $\text{ex}_\Pi(s,q + (s+2)/2) > 1$. Suppose $G = (V,w)$ and $G' = (V,w')$ are multigraphs. We say that $G$ is a submultigraph of $G'$ if $V = V'$ and for each $xy \in (V)_2$, $w(xy) \leq w'(xy)$. Define $G^+ = (V,w^+)$ where for each $xy \in (V)_2$, $w^+(xy) = w(xy) + 1$. Observe that if $G \in F(n,s,q)$, then $G^+ \in F(n,s,q + (s+2)/2)$.

Definition 4. Suppose $\epsilon > 0$ and $n,s,q$ are integers satisfying $n \geq 1$, $s \geq 2$, and $q \geq 0$. Set

$$E(n,s,q,\epsilon) = \left\{G \in F(n,s,q) : P(G^+ > \text{ex}_\Pi\left(s,q + \frac{s}{2}\right)^{(1-\epsilon)\left(\frac{s}{2}\right)}\right\}.$$

Then set $E(n,s,q,\epsilon) = \{G \in F(n,s,q) : G$ is a submultigraph of some $G' \in E(n,s,q,\epsilon)\}$.
Theorem 2. Suppose $s \geq 2$ and $q \geq 0$ are integers satisfying $\exp(s,q + \binom{s}{2}) > 1$. Then for all $\epsilon > 0$, there is $\beta > 0$ such that for all sufficiently large $n$, the following holds.

$$\frac{|F(n,s,q) \setminus E(n,s,q,\epsilon)|}{|F(n,s,q)|} \leq 2^{-\beta n^2}. \quad (1)$$

Theorem 2 will be proved in Section 4 using a consequence of a version of the hypergraph container theorem for multigraphs from [14]. Our next results investigate ex$(s,q)$ for various values of $(s,q)$. Observe that if $q < \binom{s}{2}$, then for any $n \geq s$, every $(n,s,q)$-graph $G$ must contain an edge of multiplicity 0, and therefore $P(G) = 0$. Consequently, ex$(n,s,q) = 0$ and $P(n,s,q) = F(n,s,q)$, for all $n \geq s$. For this reason we restrict our attention to the cases where $s \geq 2$ and $q \geq \binom{s}{2}$. Suppose $G = (V,w)$ and $G' = (V',w')$. Then $G = (V,w)$ and $G' = (V',w')$ are isomorphic, denoted $G \cong G'$, if there is a bijection $f : V \rightarrow V'$ such that for all $xy \in \binom{V}{2}$, $w(xy) = w'(f(x)f(y))$. If $V = V'$, set $\Delta(G,G') = \{xy \in \binom{V}{2} : w(xy) \neq w'(xy)\}$. Given $\delta > 0$, $G$ and $G'$ are $\delta$-close if $|\Delta(G,G')| \leq \delta n^2$, otherwise they are $\delta$-far. If $X \subseteq V$, $G[X]$ denotes the multigraph $(X,w |_{\binom{X}{2}})$. Suppose that $q = b \pmod{\binom{s}{2}}$. Our results fall into three cases depending on the value of $b$.

2.1 The case $0 \leq b \leq s - 2$

Definition 5. Given $n \geq s \geq 1$ and $a \geq 1$, let $\mathbb{U}_{a}(n)$ be the set of multigraphs $G = ([n],w)$ such that there is a partition $A_0, A_1, \ldots, A_{\left\lfloor \frac{n}{a} \right\rfloor}$ of $[n]$ for which the following holds.

- For each $1 \leq i \leq \left\lfloor \frac{n}{s} \right\rfloor$, $|A_i| = s$, and $|A_0| = n - s\left\lfloor \frac{n}{s} \right\rfloor$.
- For each $0 \leq i \leq \left\lfloor \frac{n}{s} \right\rfloor$, and $G[A_i]$ is a star with $|A_i| - 1$ edges of multiplicity $a + 1$ and all other edges of multiplicity $a$.
- For all $xy \notin \bigcup \binom{A_i}{2}$, $w(xy) = a$.

Let $\mathbb{U}_{a}(n)$ be the unique element of $\mathbb{U}_{1,a}(n)$, i.e. $\mathbb{U}_{a}(n) = ([n],w)$ where $w(xy) = a$ for all $xy \in \binom{[n]}{2}$.

Theorem 3. Suppose $n,s,q,a$ are integers satisfying $n \geq s \geq 2$, $a \geq 1$, and $q = a\binom{s}{2} + b$ for some $0 \leq b \leq s - 2$.

- (Extremal) Then $a\binom{n}{2} \leq \exp(n,s,q) \leq a\binom{n}{2}(a + 1)/a^{\left\lfloor \frac{n}{a} \right\rfloor}$ and thus $\exp(s,q) = a$. Further,
  - (a) If $b = 0$, then $P(n,s,q) = \{\mathbb{U}_a(n)\}$ and $\exp(n,s,q) = a\binom{n}{2}$.
  - (b) If $b = s - 2$, then $\mathbb{U}_{a-1,0}(n) \subseteq P(n,s,q)$ and $\exp(n,s,q) = a\binom{n}{2}\left(\frac{a+1}{a}\right)^{\left\lfloor \frac{a-2}{a}n \right\rfloor}$. Also, $P(n,3,q) = \mathbb{U}_{2,0}(n)$.
- (Stability) For all $\delta > 0$, there is $\epsilon > 0$ and $M$ such that for all $n > M$ and $G \in F(n,s,q)$, if $P(G) > \exp(n,s,q)^{1-\epsilon}$, then $G$ is $\delta$-close to $\mathbb{U}_a(n)$.

One interesting phenomenon discovered in [2] is that $S(n,3,3a + 1)$ has many non-isomorphic multigraphs when $a \geq 1$ and $n$ is large. In contrast to this, Theorem 3 shows that all the multigraphs in $P(n,3,3a + 1) = \mathbb{U}_{2,0}(n)$ are isomorphic.
2.2 The case \( b = \binom{s}{2} - t \) for some \( 1 \leq t \leq \frac{s}{2} \)

Call a partition \( U_1, \ldots, U_k \) of a finite set \( X \) an **equipartition** if \( ||U_i| - |U_j|| \leq 1 \) for all \( i \neq j \).

Recall the Turán graph, \( T_s(n) \), is the complete \( s \)-partite graph with \( n \) vertices, whose parts form an equipartition of its vertex set.

**Definition 6.** Given integers \( a \geq 2 \) and \( n \geq s \geq 1 \), define \( \mathbb{T}_{s,a}(n) \) to be the set of multigraphs \( G = ([n], w) \) with the following property. There is an equipartition \( U_1, \ldots, U_s \) of \([n] \) such that

\[
    w(xy) = \begin{cases} 
    a - 1 & \text{if } xy \in \binom{U_i}{2} \text{ for some } i \in [s], \\
    a & \text{if } (x, y) \in U_i \times U_j \text{ for some } i \neq j \in [s].
    \end{cases}
\]

We think of elements of \( \mathbb{T}_{s,a}(n) \) as multigraph analogues of Turán graphs. Let \( t_s(n) \) be the number of edges in \( T_s(n) \).

**Theorem 4.** Let \( s, q, a, t \) be integers satisfying \( a \geq 2 \), \( q = a\binom{s}{2} - t \) and either

(a) \( s \geq 2 \) and \( t = 1 \) or
(b) \( s \geq 4 \) and \( 2 \leq t \leq \frac{s}{2} \).

- (Extremal) Then for all \( n \geq s \), \( \mathbb{T}_{s-t,a}(n) \subseteq \mathcal{P}(n, s, q) \), \( \exp_{\Pi}(n, s, q) = (a - 1)^{\binom{s}{2}} (\frac{a}{a-1})^{t_{s-t}(n)} \), and
  \( \exp_{\Pi}(s, q) = (a - 1)^{\binom{s}{2}} (\frac{a}{a-1})^{\frac{s-t-1}{2}} \). If (a) holds and \( n \geq s \) or (b) holds and \( n \) is sufficiently large, then \( \mathcal{P}(n, s, q) = \mathbb{T}_{s-t,a}(n) \).
- (Stability) For all \( \delta > 0 \), there is \( M \) and \( \epsilon \) such that for all \( n > M \) and \( G \in \mathcal{F}(n, s, q) \), if \( P(G) > \exp_{\Pi}(n, s, q)^{1-\epsilon} \) then \( G \) is \( \delta \)-close to an element of \( \mathbb{T}_{s-t,a}(n) \).

2.3 The case \( (s, q) = (4, 9) \)

The case \( (s, q) = (4, 9) \) is the first pair where \( s \geq 2 \) and \( q \geq \binom{s}{2} \) which is not covered by Theorems 3 and 4 and is closely related to an old question in extremal combinatorics. Let \( \exp(n, \{C_3, C_4\}) \) denote the maximum number of edges in a graph on \( n \) vertices which contains no \( C_3 \) or \( C_4 \) as a non-induced subgraph.

**Theorem 5.** \( \exp_{\Pi}(n, 4, 9) = 2^{\exp(n, \{C_3, C_4\})} \) for all \( n \geq 4 \).

It is known that

\[
    \left( \frac{1}{2\sqrt{2}} + o(1) \right) n^{3/2} < \exp(n, \{C_3, C_4\}) < \left( \frac{1}{2} + o(1) \right) n^{3/2}
\]

and an old conjecture of Erdős and Simonovits [4] states that the lower bound is correct.

The next case not covered here is \( (s, q) = (4, 15) \) and it was shown in [14] that \( \exp_{\Pi}(n, 4, 15) = 2^{\gamma n^2 + o(n^2)} \) where \( \gamma \) is transcendental and \( 2^\gamma \) is also transcendental if we assume Schanuel’s conjecture from number theory. Many other cases were conjectured in [14] to have transcendental behaviour like the case \( (4, 15) \). This suggests that determining \( \exp_{\Pi}(s, q) \) for all pairs \((s, q)\) will be a hard problem.
2.4 Enumeration and structure of most \((n,s,q)\)-graphs

Combining the extremal results of Theorems 3 and 4 with Theorem 1 we obtain Theorem 6 below, which enumerates \(F(n,s,q)\) for many cases of \((s,q)\).

**Theorem 6.** Let \(s, q, a, b\) be integers satisfying \(s \geq 2\), \(a \geq 0\), and \(q = a\binom{s}{2} + b\).

(i) If \(0 \leq b \leq s - 2\), then \(|F(n,s,q)| = (a + 1)\binom{s}{2}2^o(n^2)\).

(ii) If \(b = \binom{s}{2} - t\) where \(2 \leq t \leq \frac{s}{2}\), then \(|F(n,s,q)| = (a + 1)\binom{s}{2}\left(\frac{a+2}{a+1}\right)^{t}n^{o(n^2)}\).

(iii) \(|F(n,4,3)| = 2^\Theta(n^{3/2})\).

In our last main result, Theorem 7 below, we combine the stability results of Theorems 3 and 4 with Theorem 2 to prove approximate structure theorems for many \((s,q)\). Given \(\delta > 0\) and a set \(E(n) \subseteq F(n,s,q)\), let \(E^\delta(n)\) be the set of \(G \in F(n,s,q)\) such that \(G\) is \(\delta\)-close to some \(G' \in E(n)\).

**Definition 7.** Suppose \(n, a, s\) are integers such that \(n, s \geq 1\).

(i) If \(a \geq 1\), set \(U_a(n) = \{G = ([n], w) : G\) is a submultigraph of some \(G' \in U_a(n)\}\}.

(ii) If \(a \geq 2\), set \(T_{s,a}(n) = \{G = ([n], w) : G\) is a submultigraph of some \(G' \in T_{s,a}(n)\}\}.

Observe that in each case, \(U_a(n) \subseteq U_a(n)\) and \(T_{s,a}(n) \subseteq T_{s,a}(n)\).

**Theorem 7.** Suppose \(s, q, a, t, b\) are integers such that \(n \geq s \geq 2\), and \(E(n)\) is a set of multigraphs such that one of the following holds.

(i) \(a \geq 0, q = a\binom{s}{2} + b\) for some \(0 \leq b \leq s - 2\), and \(E(n) = U_a(n)\).

(ii) \(a \geq 1, q = a\binom{s}{2} - t\) for some \(1 \leq t \leq \frac{s}{2}\), and \(E(n) = T_{s-t,a}(n)\).

Then for all \(\delta > 0\) there exists \(\beta > 0\) such that for all sufficiently large \(n\),

\[
\frac{|F(n,s,q) \setminus E^\delta(n)|}{|F(n,s,q)|} \leq 2^{-\beta(n)}.
\]

(2)

3 Proof of Theorems 6 and 7

In this section we prove Theorems 6 and 7 assuming Theorems 2, 3, and 4.

**Proof of Theorem 6.** Suppose first that case (i) holds. By Theorem 3 (Extremal),

\[
\text{ex}_\Pi\left(s, q + \left(\frac{s}{2}\right)\right) = \text{ex}_\Pi\left(s, (a + 1)\left(\frac{s}{2}\right) + b\right) = a + 1.
\]

If \(a = 0\), then \(\text{ex}_\Pi(s, q + \binom{s}{2}) = 1\), so Theorem 4 implies \(|F(n,s,q)| = 2^{o(n^2)} = (a + 1)\binom{s}{2}2^{o(n^2)}\). If \(a \geq 1\), then \(\text{ex}_\Pi(s, q + \binom{s}{2}) = a + 1 > 1\), so Theorem 4 implies

\[
|F(n,s,q)| = (a + 1)\binom{s}{2}^{1+o(n^2)} = (a + 1)\binom{s}{2}2^{o(n^2)}.
\]

Suppose now that case (ii) holds. So \(q = a\binom{s}{2} + \binom{s}{2} - t = (a + 1)\binom{s}{2} - t\). By Theorem 4 (Extremal),

\[
\text{ex}_\Pi\left(s, q + \left(\frac{s}{2}\right)\right) = \text{ex}_\Pi\left(s, (a + 2)\left(\frac{s}{2}\right) - t\right) = (a + 1)\left(\frac{a + 2}{a + 1}\right)^{\frac{t}{s+t}}.
\]
Since \(a \geq 0\), this shows \(\exp(n, s, q + \binom{s}{2}) > 1\), so Theorem \([1]\) implies

\[
|F(n, s, q)| = \left( a + 1 \right) \left( \frac{a + 2}{a + 1} \right)^{\frac{1}{2}} (\binom{n}{2})^{1+o(n^2)} = (a + 1) \left( \frac{a + 2}{a + 1} \right)^{1+o(n^2)}.
\]

For (iii) first observe that any subgraph of a graph of girth at least 5 is a \((4,3)\)-graph, and since \(\exp(n, \{C_3, C_4\}) \geq c_1n^{3/2}\) for some constant \(c_1 > 0\) (see [4]) we obtain the lower bound. For the upper bound, observe that in a \((4,3)\)-graph, there is at most one pair with multiplicity at least two and the set of pairs with multiplicity one forms a graph with no \(C_4\). By the Kleitman-Winston theorem [12], the number of ways to choose the pairs of multiplicity one is at most \(2^{c_2n^{3/2}}\) for some constant \(c_2 > 0\) and this gives the upper bound.

**Proof of Theorem 7** Fix \(\delta > 0\). Observe that if case (i) holds (respectively, case (ii)), then \((s, q + \binom{s}{2})\) satisfies the hypotheses of Theorem \([3]\) (respectively, Theorem \([4]\)). Let

\[
\mathbb{E}(n) = \begin{cases} 
\mathbb{U}_{a+1}(n) & \text{in case (i)} \\
\mathbb{T}_{s-t,a+1}(n) & \text{in case (ii)} 
\end{cases}
\]

By Theorem \([3]\) (Stability) in case (i) and Theorem \([4]\) (Stability) in case (ii), there is \(\epsilon > 0\) so that for sufficiently large \(n\), if \(G^+ \in F(n, s, q + \binom{s}{2})\) satisfies \(P(G^+) > \exp(n, s, q + \binom{s}{2})^{1-\epsilon}\), then \(G^+\) is \(\delta\)-close to some \(G' \in \mathbb{E}(n)\). Note that \(G' \in \mathbb{E}(n)\) implies there is \(H \in E(n)\) such that \(H^+ = G\). Combining this our choice of \(\epsilon\), we obtain the following. For all sufficiently large \(n\) and \(G \in F(n, s, q)\),

\[
\text{if } P(G^+) > \exp(n, s, q + \binom{s}{2})^{1-\epsilon}, \text{ then } G^+ \text{ is } \delta\text{-close to } H^+, \text{ for some } H \in E(n). \tag{3}
\]

By Theorem \([3]\) (Extremal) in case (i) and Theorem \([4]\) (Extremal) in case (ii), we must have that \(\exp(n, s, q + \binom{s}{2}) > 1\). So Theorem \([2]\) implies there is \(\beta > 0\) such that for all sufficiently large \(n\) the following holds.

\[
\frac{|F(n, s, q) \setminus E(n, s, q, \epsilon)|}{|F(n, s, q)|} \leq 2^{-\beta n^2}.
\]

So to show \([2]\), it suffices to show that for sufficiently large \(n\), \(E(n, s, q, \epsilon) \subseteq E^\delta(n)\). Fix \(n\) sufficiently large and suppose \(G = ([n], w^{G'}) \in E(n, s, q, \epsilon)\). By definition, this means there is \(G' \in F(n, s, q)\) such that \(P(G^+) > \exp(n, s, q + \binom{s}{2})^{1-\epsilon}\) and \(G\) is a submultigraph of \(G'\). By \([3]\), \(G^+\) is \(\delta\)-close to \(H^+\), for some \(H \in E(n)\). Define \(H' = ([n], w^{H'})\) such that \(w^{H'}(xy) = w^G(xy)\) if \(xy \in \binom{n}{2}\) \(\setminus \Delta(G', H)\), and \(w^{H'}(xy) = 0\) if \(xy \in \Delta(G', H)\). We claim \(H'\) is a submultigraph of \(H\). Fix \(xy \in \binom{n}{2}\). We want to show \(w^{H'}(xy) \leq w^H(xy)\). If \(xy \in \Delta(G', H)\), then \(w^{H'}(xy) = 0 \leq w^H(xy)\) is immediate. If \(xy \notin \Delta(G', H)\), then \(w^{H'}(xy) = w^{G'}(xy) \leq w^G(xy) = w^H(xy)\), where the inequality is because \(G\) is a submultigraph of \(G'\) and the last equality is because \(xy \notin \Delta(G', H)\). Thus \(H'\) is a submultigraph of \(H \in E(n)\), which implies \(H'\) is also in \(E(n)\). By definition of \(H'\), \(\Delta(G, H') \subseteq \Delta(G', H) = \Delta(G^+, H^+)\). Since \(G^+\) and \(H^+\) are \(\delta\)-close, this implies \(|\Delta(G, H')| \leq \delta n^2\), and \(G \in E^\delta(n)\).

**4 Proof of Theorem 2**

In this section we prove Theorem \([2]\). We will use Theorem \([8]\) below, which is a version of the hypergraph containers theorem of \([1]\)\([15]\) for multigraphs. Theorem \([8]\) was proved in \([14]\).
Definition 8. Suppose $s \geq 2$ and $q \geq 0$ are integers. Set

$$H(s, q) = \{G = ([s], w) : \mu(G) \leq q \text{ and } S(G) > q\}, \quad \text{and} \quad g(s, q) = |H(s, q)|.$$ 

If $G = (V, w)$ is a multigraph, let $H(G, s, q) = \{X \in \binom{V}{s} : G[X] \cong G' \text{ for some } G' \in H(s, q)\}$.

Theorem 8. For every $0 < \delta < 1$ and integers $s \geq 2$, $q \geq 0$, there is a constant $c = c(s, q, \delta) > 0$ such that the following holds. For all sufficiently large $n$, there is $G$ a collection of multigraphs of multiplicity at most $q$ and with vertex set $[n]$ such that

(i) for every $J \in F(n, s, q)$, there is $G \in \mathcal{G}$ such that $J$ is a submultigraph of $G$,

(ii) for every $G \in \mathcal{G}$, $|H(G, s, q)| \leq \delta(n)$, and

(iii) $\log |\mathcal{G}| \leq cn^{2 - \frac{1}{8s}} \log n$.

We will also use the following two results appearing in [14].

Lemma 1 (Lemma 1 of [14]). Fix integers $s \geq 2$ and $q \geq 0$. For all $0 < \nu < 1$, there is $0 < \delta < 1$ such that for all sufficiently large $n$, the following holds. If $G = ([n], w)$ satisfies $\mu(G) \leq q$ and $|H(G, s, q)| \leq \delta(n)$, then $G$ is $\nu$-close to some $G'$ in $F(n, s, q)$.

Proposition 1 (Proposition 2 in [14]). For all $n \geq s \geq 2$ and $q \geq 0$, $\exp\pi(s, q)$ exists and $\exp\pi(n, s, q) \geq \exp\pi(s, q)(2)$. If $q \geq \left(\frac{4}{5}\right)^2$, then $\exp\pi(s, q) \geq 1$.

Proof of Theorem 2. Fix $\epsilon > 0$ and set $\nu = (\epsilon \log(\exp\pi(s, q + \left(\frac{4}{5}\right)^2)))/(8\log(q + 1))$. Choose $\delta > 0$ according to Lemma 1 so that the following holds for all sufficiently large $n$.

Any $G = ([n], w)$ with $\mu(G) \leq q$ and $|H(G, s, q)| \leq \delta(n)$ is $\nu$-close to some $G'$ in $F(n, s, q)$. (4)

Fix $n$ sufficiently large. Apply Theorem 8 to obtain a constant $c$ and a collection $\mathcal{G}$ of multigraphs of multiplicity at most $q$ and with vertex set $[n]$ satisfying (i)-(iii) of Theorem 8. Suppose that $H = ([n], w^H) \in F(n, s, q) \setminus E(n, s, q, \epsilon)$. By (i), there is $G = ([n], w^G) \in \mathcal{G}$ such that $H$ is a submultigraph of $G$ and $|H(G, s, q)| \leq \delta(n)$. We claim that $P(G^{+}) \leq \exp\pi(n, s, q + \left(\frac{4}{5}\right)^2)^{1-\epsilon/2}$. Suppose towards a contradiction this is not the case, so $P(G^{+}) > \exp\pi(n, s, q + \left(\frac{4}{5}\right)^2)^{1-\epsilon/2}$. By (4), $|H(G, s, q)| \leq \delta(n)$ implies there is $G' = ([n], w^G') \in F(n, s, q)$ which is $\nu$-close to $G$. Define $H' = ([n], w^{H'})$ by setting $w^{H'}(xy) = w^H(xy)$ for all $xy \in ([n]^2) \setminus \Delta(G, G')$ and $w^{H'}(xy) = 0$ for all $xy \in \Delta(G, G')$. By construction and because $H'$ is a submultigraph of $G'$, we have that $H$ is also a submultigraph of $G'$. Observe

$$P(G^{+}) = P(G^{+})\left(\prod_{xy \in \Delta(H, H')} \frac{w^{G'}(xy) + 1}{w^{G'}(xy) + 1}\right) \geq P(G^{+})(q + 1)^{-|\Delta(G, G')|},$$

where the inequality is because $1 \leq w^{G'}(xy) + 1, w^{G}(xy) + 1 \leq q + 1$ implies $\frac{w^{G'}(xy) + 1}{w^{G'}(xy) + 1} \geq \frac{1}{q + 1}$. Combining this with the fact that $G$ and $G'$ are $\nu$-close, the definition of $\nu$, and our assumption that $P(G^{+}) \geq \exp\pi(n, s, q + \left(\frac{4}{5}\right)^2)^{1-\epsilon/2}$, we have that $P(G^{+})$ is at least the following.

$$P(G^{+})(q + 1)^{-\epsilon n^2} = P(G^{+})\exp\pi\left(s, q + \left(\frac{s}{2}\right)^{1-\epsilon/2}\right) \geq \exp\pi\left(s, q + \left(\frac{s}{2}\right)^{1-\epsilon/2}\right)^{-\epsilon n^2/8}.$$
Since \( \text{ex}_\Pi(n,s,q + (s\choose 2))^{1/\left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right)} \geq \text{ex}_\Pi(s,q + (s\choose 2)) \) (see Proposition \[\square\]), we obtain that the right hand side is at least

\[
\text{ex}_\Pi\left(n,s,q + \left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right)\right)^{1-\epsilon/2} \text{ex}_\Pi\left(n,s,q + \left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right)^{-\epsilon n^2/(8\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right))} \right) \geq \text{ex}_\Pi\left(n,s,q + \left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right)\right)^{1-\epsilon},
\]

where the inequality is because \( n \) large implies \( cn^2/(8\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right)) \leq \epsilon/2 \). But now \( H \) is a submultigraph of \( G' \) and \( P(G') \geq \text{ex}_\Pi(n,s,q + (s\choose 2))^{1-\epsilon} \), contradicting that \( H \in F(n,s,q) \setminus E(n,s,q,\epsilon) \). Therefore, every element of \( F(n,s,q) \setminus E(n,s,q,\epsilon) \) can be constructed as follows.

- Choose some \( G \in \mathcal{G} \) with \( P(G') \leq \text{ex}_\Pi(n,s,q + (s\choose 2))^{1-\epsilon/2} \). There are at most \( cn^2/\pi \log n \) choices.

Since \( n \) is large and \( \text{ex}_\Pi(s,q + (s\choose 2)) > 1 \), we may assume \( cn^2/\pi \log n \leq \text{ex}_\Pi(s,q + (s\choose 2)^s/4 \). 

- Choose a submultigraph of \( G \). There are at most \( P(G') \leq \text{ex}_\Pi(n,s,q + (s\choose 2))^{1-\epsilon/2} \) choices.

This shows

\[
|F(n,s,q) \setminus E(n,s,q,\epsilon)| \leq \text{ex}_\Pi\left(s,q + \left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right)^{1-\epsilon/2}
\text{ex}_\Pi\left(n,s,q + \left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right)^{-\epsilon n^2/(8\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right))} \right) \leq \text{ex}_\Pi\left(s,q + \left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right)^{1-\epsilon/2}}
\end{align}

where the second inequality is because \( \text{ex}_\Pi(n,s,q + (s\choose 2)) \geq \text{ex}_\Pi(s,q + (s\choose 2))^{1/2} \). By Theorem \[\square\]

\[
\frac{|F(n,s,q) \setminus E(n,s,q,\epsilon)|}{|F(n,s,q)|} \leq \text{ex}_\Pi\left(s,q + \left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right)^{-\epsilon n^2/(8\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right))} \right).
\]

Setting \( \beta = \frac{4}{\pi} \log_2(\text{ex}_\Pi(s,q + (s\choose 2))) \) finishes the proof (note \( \beta > 0 \) since \( \text{ex}_\Pi(s,q + (s\choose 2)) > 1 \)). \( \square \)

5 Extremal Results

In this section we prove the extremal statements in Theorems \[\square\] and \[\square\]. We begin with some preliminaries. Suppose \( s \geq 2 \) and \( q \geq \left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right) \). It was shown in [9] that \( \text{ex}_\Sigma(s,q) \) exists, and the AM-GM inequality implies that

\[
\text{ex}_\Pi(s,q) = \lim_{n \to \infty} \text{ex}_\Pi(n,s,q)^{1/\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right)} \leq \lim_{n \to \infty} \frac{\text{ex}_\Sigma(n,s,q)}{\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right)} = \text{ex}_\Sigma(s,q) .
\] (5)

The following lemma is an integer version of the AM-GM inequality.

**Lemma 2.** If \( \ell \geq 2, k \in [\ell] \) and \( a, x_1, \ldots, x_\ell \) are positive integers such that \( \sum_{i=1}^\ell x_i \leq a\ell - k \), then \( \prod_{i=1}^\ell x_i \leq a^\ell - k(a-1)^k \). Moreover, equality holds if and only if exactly \( k \) of the \( x_i \) are equal to \( a-1 \) and the rest are equal to \( a \).

**Proof.** If there are \( x_i \) and \( x_j \) with \( x_i < x_j - 1 \), then replacing \( x_i \) with \( x_i + 1 \) and replacing \( x_j \) with \( x_j - 1 \) increases the product and keeps the sum unchanged. So no two of the \( x_i \)'s differ by more than one when the product is maximized. \( \square \)

**Corollary 1.** Let \( n \geq s \geq 2, a \geq 2 \), and \( (a-1)\left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right) \leq q < a\left(\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right) \). Suppose \( G \in \mathcal{S}(n,s,q) \) has all edge multiplicities in \( [a,a-1] \) and contains exactly \( k \) edges of multiplicity \( a-1 \). Then for all other \( G' \in F(n,s,q) \), \( G' \in \mathcal{P}(n,s,q) \) if and only if \( G' \) has \( k \) edges of multiplicity \( a-1 \) and all other edges of multiplicity \( a \). Consequently, \( G \in \mathcal{P}(n,s,q) \leq \mathcal{S}(n,s,q) \).
Proof. Fix $G$ so that the hypotheses hold. Then $S(G) = a(n/2) - k$ and $P(G) = a(n/2) - k(a - 1)^{k}$. Let $G' = ([n], w)$ be another element of $F(n, s, q)$. Since $G \in S(n, s, q)$, we have

$$S(G') \leq S(G) = a\left(\frac{n}{2}\right) - k.$$ 

By Lemma 2 with $\ell = \binom{n}{2}$, $P(G') \leq a\left(\frac{n}{2}\right) - k(a - 1)^{k}$ with equality if and only if $\{w(xy) : xy \in \binom{[n]}{2}\}$ consists of $k$ elements equal to $a - 1$ and the rest equal to $a$. This shows $G' \in P(n, s, q)$ if and only if $G'$ has $k$ edges of multiplicity $a - 1$ and the rest of multiplicity $a$. Consequently, $G \in P(n, s, q)$. To show $P(n, s, q) \subseteq S(n, s, q)$, let $G' \in P(n, s, q)$. Then by what we have shown, $S(G') = a\left(\frac{n}{2}\right) - k = S(G)$, so $G \in S(n, s, q)$ implies $G' \in S(n, s, q)$. 

The following is a consequence of Theorem 5.2 in [2] (case $b = 0$) and Theorems 8 and 9 in [9] (cases $0 < b \leq s - 2$).

**Theorem 9 (Bondy-Tuza [2], Füredi-Kündgen [9]).** Let $n \geq s \geq 2$, $a \geq 1$, $0 \leq b \leq s - 2$, and $q = a(n/2) + b$. Then

$$\text{ex}_S(n, s, q) \leq a\left(\frac{n}{2}\right) + \left\lfloor \frac{b}{b + 1}\right\rfloor,$$

with equality holding when $b = s - 2$ and when $b = 0$.

**Proof of Theorem 9 (Extremal).** Since $U_a(n) \in F(n, s, q)$, $a(n/2) \leq \text{ex}_P(n, s, q)$. On the other hand, let $G \in F(n, s, q)$. Then $S(G) \leq a\left(\frac{n}{2}\right) + \left\lfloor \frac{b}{b + 1}\right\rfloor$ by Lemma 2. This along with Lemma 2 implies that $P(G) \leq a\left(\frac{n}{2}\right) \left(\frac{a + 1}{a}\right)^{\frac{b}{b + 1}n}$. Thus $a\left(\frac{n}{2}\right) \leq \text{ex}_P(n, s, q) \leq a\left(\frac{n}{2}\right) \left(\frac{a + 1}{a}\right)^{\frac{b}{b + 1}n}$, which implies $\text{ex}_P(n, s, q) = a$.

Case (a): If $b = 0$, then Theorem 9 implies $U_a(n) \in S(n, s, q)$. Because $U_a(n)$ has all edge multiplicities in $\{a\}$, Corollary 1 implies $U_a(n) \in P(n, s, q)$ and moreover, every other element of $P(n, s, q)$ has all edges of multiplicity $a$. In other words, $\{U_a(n)\} = P(n, s, q)$, so $\text{ex}_P(n, s, q) = a\left(\frac{n}{2}\right)$.

Case (b): If $b = s - 2$, then it is straightforward to check $U_{s-1,a}(n) \subseteq F(n, s, q)$. Since $S(G) = a\left(\frac{n}{2}\right) + \left\lfloor \frac{b}{b + 1}\right\rfloor$ for all $G \in U_{s-1,a}(n)$, Theorem 10 implies $U_{s-1,a}(n) \subseteq S(n, s, q)$. Because every element in $U_{s-1,a}(n)$ has all edge multiplicities in $\{a + 1, a\}$, Corollary 1 implies $U_{s-1,a}(n) \subseteq P(n, s, q)$ and every $G' \in P(n, s, q)$ contains exactly $\left\lfloor \frac{s-2}{s-1}\right\rfloor$ edges of multiplicity $a + 1$, and all others of multiplicity $a$. Thus $\text{ex}_P(n, s, q) = a\left(\frac{n}{2}\right) \left(\frac{s+1}{a}\right)$. Suppose $s = 3$, $b = 1$, and $G' = ([n], w) \in P(n, s, q)$. If there are $x, y \neq z \in [n]$ such that $w(xy) = w(xz) = a + 1$, then because $G'$ contains only edges of multiplicity $a + 1$ and $a$, $S(\{x, y, z\}) \geq 2(a + 1) + a = 3a + 2 > q$, a contradiction. Thus the edges of multiplicity $a + 1$ form a matching of size $\left\lceil \frac{n}{2}\right\rceil$ in $G'$, so $G' \in U_{s-1,a}(n)$. This shows $U_{s-1,a}(n) = P(n, s, q)$.

The following is a consequence of Theorem 5.2 of [2].

**Theorem 10 (Bondy-Tuza [2]).** Suppose $n \geq s \geq 2$, $a \geq 1$, and $q = a\left(\frac{n}{2}\right) - 1$. Then

$$\text{ex}_S(n, s, q) = (a - 1)\left(\frac{n}{2}\right) + t_{s-1}(n).$$

**Proof of Theorem 10 (Extremal).** Since $T_{s-1,a}(n) \subseteq F(n, s, q)$ and for all $G \in T_{s-1,a}(n)$, $S(G) = (a - 1)\left(\frac{n}{2}\right) + t_{s-1}(n)$, Theorem 10 implies that $T_{s-1,a}(n) \subseteq S(n, s, q)$. Therefore Corollary 1 implies $T_{s-1,a}(n) \subseteq P(n, s, q)$ and each $G \in P(n, s, q)$ has $t_{s-1}(n)$ edges of multiplicity $a$ and the
for all 

Suppose

Consequently, ex

Turán’s theorem, G′ = T_{s-1}(n) and thus G ∈ T_{s−1,a}(n). So we have shown, P(n, s, q) = T_{s−1,a}(n).

Therefore, ex_{Π}(n, s, q) = (a − 1)(s2−1)/2 + (a−1)(s−1)−1

To prove Theorem (b) (Extremal), we will need the following theorem, as well as a few lemmas.

**Theorem 11. [Dirac 3, Bondy-Tuza 2]** Let n ≥ s ≥ 4, a ≥ 1, and q = a(s−1)/2 − t for some 2 ≤ t ≤ s2−1. Then ex_{Π}(n, s, q) = ex_{Π}(n, s', q') where s' = s − t + 1 and q' = a(s−1)/2 − 1.

**Proof.** Let n ≥ s ≥ 4 and 2 ≤ t ≤ s/2. In 3, Dirac proved that ex_{Π}(n, s, (s−1)/2) = t_{s−1}(n). This along with Lemma 5.1 in 2 implies that for all a ≥ 1,

ex_{Π}(n, s, a(s−1)/2−t) = ex_{Π}(n, s, s−t)/2−t) + (a−1)(n−t) + (a−1)(n−t) = ex_{Π}(n, s', a(s−1)/2−1),

where the last equality is by Theorem 10 applied to s' and a(s−1)/2 − 1.

**Lemma 3.** If s, q, a, t are integers satisfying case (b) of Theorem 2 and s' = s − t + 1, q' = a(s−1)/2 − 1, then for all n ≥ s, T_{s−1,a}(n) ⊆ P(n, s, q) and ex_{Π}(n, s, q) = ex_{Π}(n, s', q').

**Proof.** Set s' = s − t + 1 and q' = a(s−1)/2 − 1, and fix n ≥ s. Fix G ∈ P(n, s, q). It is straightforward to check that G ∈ P(n, s, q). By Theorem 10, ex_{Π}(n, s', q') = ex_{Π}(n, s, q). Since S(G) = (a−1)(s−1)/2 + t_{s−1}(n), by Theorem 10 applied to s' and q', we have that S(G) = ex_{Π}(n, s', q') = ex_{Π}(n, s, q).

This shows G ∈ S(n, s, q). By Corollary 1 since G has all edge multiplicities in {a, a−1}, G ∈ P(n, s, q), so P(G) = ex_{Π}(n, s, q). Since G ∈ T_{s−1,a}(n) and T_{s−1,a}(n) ⊆ P(n, s', q') by Theorem 4 (Extremal), P(G) = ex_{Π}(n, s, q). Thus ex_{Π}(n, s, q) = P(G) = ex_{Π}(n, s', q').

We now fix some notation. Given n ∈ N, z ∈ [n], Y ⊆ [n], and G = ([n], w), set

S(Y) = \sum_{xy \in Y} w(xy), \quad S_z(Y) = \sum_{y \in Y} w(yz), \quad P(Y) = \prod_{xy \in Y} w(xy), \quad \text{and} \quad P_z(Y) = \prod_{y \in Y} w(yz)

If X ⊆ [n] is disjoint from Y, set P(X, Y) = \prod_{x \in X \times Y} w(xy).

**Claim 1.** Suppose s, q, a, t are integers satisfying the hypotheses of case (b) of Theorem 2. Then for all n ≥ 2s and s − t + 1 ≤ y ≤ s − 1,

ex_{Π}(n, y, s, q) ≤ ex_{Π}(n, s, q)(a−1)^{y−1}(a−1)^{y−2}(a−1)^{y−2}(a−1)^{y−2}.

**Proof.** Set s' = s − t + 1 and q' = a(s−1)/2 − 1. Fix n ≥ s and s' ≤ y ≤ s − 1. Choose some H = ([n−y], w) ∈ T_{s−1,a}(n−y) and let U_1, …, U_{s−1} be the partition of [n−y] corresponding to H. Observe that there is some i such that |U_i| ≥ \frac{n−y}{s−1}. Without loss of generality, assume |U_1| ≥ \frac{n−y}{s−1}.

Assign the elements of Y' := [n] \setminus [n−y] to the U_i in as even a way as possible, to obtain an equipartition U'_1, …, U'_{s−1} of [n] extending U_1, …, U_{s−1}. Observe that because s' ≤ |Y'| ≤ s − 1 and s' − 1 = s − t ≥ s/2, for each i, |U'_i| \setminus U_i| ∈ {1, 2}, and there is at least one i such that |U'_i \setminus U_i| = 1. Since |U_1| ≥ \frac{n−y}{s−1}, by redistributing Y' if necessary, we may assume that |U'_1 \setminus U_1| = 1.

Define a new multigraph H' = ([n], w') so that w'(xy) = a − 1 if xy \in \binom{Y'}{2} for some i ∈ [s'−1] and w'(xy) = a if (x, y) \in U'_i \times U'_j for some i ≠ j. Note that by construction H' ∈ T_{s−1,a}(n) and
$H'[n - y] = H$. By Lemma 3 since $n - y \geq s$, $H \in \mathbb{T}_{\alpha}(n - y)$ and $H' \in \mathbb{T}_{\alpha}(n)$ imply $H \in \mathcal{P}(n - y, s, q)$ and $H' \in \mathcal{P}(n, s, q)$. These facts imply the following.

$$\exp(n, s, q) = \exp(H') = \exp(H)\exp(Y')\exp(Y', [n - y]) = \exp(n - y, s, q)\exp(Y', [n - y]).$$

(6)

By definition of $H'$, if $|U_i^1 \setminus U_i^t| = 2$, then for all $z \in U_i, P_z(Y) = a^{y_2}(a - 1)^2$ and if $|U_i^1 \setminus U_i| = 1$, then for all $z \in U_i, P_z(Y') = a^{y_1}(a - 1)$. Since $|U_i^1 \setminus U_i| = 1$, this implies

$$P(Y', [n - y]) \geq \left(a^{y_2}(a - 1)^2 \right)^{n - y}|U_i^1| \left(a^{y_1}(a - 1) \right)^{|U_i|^1 = \left(a^{y_2}(a - 1)^2 \right)^{n - y} \left(\frac{a}{a - 1} \right)^{|U_i|}.$$  (7)

By construction, $P(Y') \geq (a - 1)^{\frac{y}{s - t}}$. Combining this with (6), (7), and the fact that $|U_i| \geq \frac{n - y}{s - t}$, we obtain

$$\exp(n, s, q) \geq \exp(n - y, s, q) |(a - 1)^{\frac{y}{s - t}} \left(a^{y_2}(a - 1)^2 \right)^{n - y} \left(\frac{a}{a - 1} \right)^{|U_i|}.$$  (8)

Rearranging this yields $\exp(n - y, s, q) \leq \exp(n, s, q) |(a - 1)^{\frac{y}{s - t}} \left(a^{y_2}(a - 1)^2 \right)^{n - y} \left(\frac{a}{a - 1} \right)^{|U_i|}$.

Lemma 4. Let $n \geq s \geq 4, a \geq 2$, and $q = a^{(s)} - t$ for some $2 \leq t \leq \frac{s}{2}$. Suppose $G \in F(n, s, q)$ and $Y \in \binom{[n]}{s-t+1}$ satisfies $S(Y) \geq a^{(s-t+1)}$. Then there is $Y \subseteq Y' \subseteq [n]$ such that $s - t + 1 \leq |Y'| \leq s - 1$ and for all $z \in [n] \setminus Y'$, $S_z(Y') \leq a|Y'| - 2$, and consequently, $P_z(Y') \leq a|Y'| - 2$.  (8)

Proof. Suppose towards a contradiction that $Y \in \binom{[n]}{s-t+1}$ satisfies $S(Y) \geq a^{(s-t+1)}$ but for all $Y \subseteq Y' \subseteq [n]$ such that $s - t + 1 \leq |Y'| \leq s - 1$, there is $z \in [n] \setminus Y'$ with $S_z(Y') > a|Y'| - 2$. Apply this fact with $Y' = Y$ to choose $z_1 \in [n] \setminus Y$ such that $S_{z_1}(Y) > a|Y| - 2$. Then inductively define a sequence $z_2, \ldots, z_{t-1}$ so that for each $1 \leq i \leq t - 2, S_{z_1}(Y \cup \{z_1, \ldots, z_i\}) \geq a(s - t + i - 1)$ (to define $z_{t+1}$, apply the fact with $Y' = Y \cup \{z_1, \ldots, z_i\}$). Then $|Y' \cup \{z_1, \ldots, z_{t-1}\}| = s$ and

$$S(Y \cup \{z_1, \ldots, z_{t-1}\}) \geq S(Y) + S_{z_1}(Y) + S_{z_2}(Y \cup \{z_1\}) + \ldots + S_{z_{t-1}}(Y \cup \{z_1, \ldots, z_{t-1}\})$$

$$\geq a \left(\frac{s - t + 1}{2} \right) + a(s - t + 1 - 1) + \ldots + a(s - 1 - 1)$$

$$= a \left(\frac{s}{2} \right) - (t - 1) > a \left(\frac{s}{2} \right) - t,$$

contradicting that $G \in F(n, s, q)$. Therefore there is $Y \subseteq Y' \subseteq [n]$ such that $s - t + 1 \leq |Y'| \leq s - 1$ and for all $z \in [n] \setminus Y'$, $S_z(Y') \leq a|Y'| - 2$. By Lemma 2, this implies $P_z(Y') \leq a|Y'| - 2$.  (8)

Lemma 5. Suppose $s, q, a, t$ are integers satisfying the hypotheses of case (b) of Theorem 4. Then there are constants $C > 1$ and $0 < \alpha < 1$ such that for all $n \geq 1$ the following holds. Suppose $G \in F(n, s, q)$ and $k(G)$ is the maximal number of pairwise disjoint elements of $Y \in \binom{[n]}{s-t+1}$: $S(G[Y]) \geq a^{(s-t+1)}$. Then

$$P(G) \leq \exp(n, s, q).$$  (8)

Proof. Set $\alpha = a^{\frac{t}{s-t+1}}$. Choose $C \geq \frac{a^{(s-t+1)}}{s-t+1} \alpha^{k(G)}$ sufficiently large so that $\exp(n, s, q) \leq C \alpha^{n^2}$ holds for all $1 \leq n \leq s^3$. We proceed by induction on $n$. If $1 \leq n \leq s^3$ and $G \in F(n, s, q)$, then (8) is clearly true of $k(G) = 0$. If $k(G) \geq 1$, then by choice of $C$ and since $k(G) \leq n$ and $\alpha < 1$,

$$P(G) \leq \exp(n, s, q) \leq C \alpha^{n^2} \leq C \alpha^{k(G)} \leq C^{k(G)} \alpha^{k(G)n} \exp(n, s, q).$$
Now let \( n > s^3 \) and suppose by induction \( \mathcal{F} \) holds for all \( G' \in F(n', s, q) \) where \( 1 \leq n' < n \). If \( G \in F(n, s, q) \), then \( \mathcal{F} \) is clearly true if \( k(G) = 0 \). If \( k(G) > 0 \), let \( Y_1, \ldots, Y_k \) be a maximal set of pairwise disjoint elements in \( \{ Y \in \binom{[n]}{s-t+1} : S(G[Y]) \geq a(s-t+1) \} \). Apply Lemma \( \mathcal{L} \) to find \( Y' \) such that \( Y_1 \subseteq Y' \subseteq [n] \), \( s-t+1 \leq |Y'| \leq s-1 \), and for all \( z \in [n] \setminus Y' \), \( P_z(Y') \leq a^{3s'-2}(a-1)^2 \). Let \( |Y'| = y \). Then note

\[
P(Y', [n] \setminus Y') = \prod_{z \in [n] \setminus Y'} P_z(Y') \leq \left(a^{y-2}(a-1)^2\right)^{n-y}.
\] (9)

Observe that \( G([n] \setminus Y') \) is isomorphic to some \( H \in F(n-y, s, q) \). Since \( Y' \) can intersect at most \( t-2 \) other \( Y_i \), and since \( Y_1, \ldots, Y_k \) was maximal, we must have \( k(H) + 1 \leq k(G) \leq k(H) + t - 1 \). By our induction hypothesis,

\[
P([n] \setminus Y') = P(H) \leq C^{k(H)} \alpha^{k(H)(n-y)} \exp(n - y, s, q).
\] (10)

Since \( \mu(G) \leq q \) and \( y \leq s-1 \), and by our choice of \( C \), \( P(Y') \leq q^{\left(\frac{y}{2}\right)} \leq C \). Combining this with \( \mathcal{F} \), (10) and the fact that \( \mu(H) \leq \mu(G) \) we obtain that

\[
P(G) = P([n] \setminus Y') P(Y', [n] \setminus Y') P(Y') \leq C^{k(H)} \alpha^{k(H)(n-y)} \exp(n - y, s, q) \left(a^{y-2}(a-1)^2\right)^{n-y} C
\]

\[
= C^{k(H)+1} \alpha^{k(H)(n-y)} \exp(n - y, s, q) \left(a^{y-2}(a-1)^2\right)^{n-y}.
\]

Plugging in the upper bound for \( \exp(n - y, s, q) \) from Claim \( \mathcal{C} \) yields that \( P(G) \) is at most

\[
C^{k(H)+1} \alpha^{k(H)(n-y)} \exp(n, s, q)(a-1)^{-\left(\frac{y}{2}\right)} \left(\frac{a-1}{a}\right)^{\frac{n-y}{s-t}} \leq C^{k(H)+1} \alpha^{k(H)(n-y)+2t(n-y)} \exp(n, s, q),
\] (11)

where the last inequality is because \( (a-1)^{-\left(\frac{y}{2}\right)} < 1 \) and by definition of \( \alpha \), \( (\frac{a-1}{a})^{1/(s-t)} = \alpha^{2t} \). We claim that the following holds.

\[
k(H)(n-y) + 2t(n-y) \geq (k(H) + t - 1)n.
\] (12)

Rearranging this, we see \( \mathcal{D} \) is equivalent to \( yk(H) \leq tn + n - 2ty \). Since \( 2 \leq t \leq s/2 \) and \( y \leq s-1 \), \( tn + n - 2ty \geq 3n - s(s-1) \), so it suffices to show \( yk(H) \leq 3n - s(s-1) \). By definition, \( k(H) \leq \frac{n-t}{s-t+1} \) so \( yk(H) \leq \frac{y(n-y)}{s-t+1} \). Combining this with the facts that \( s-t+1 \leq y \leq s-1 \) and \( s/2 < s-t+1 \) yields

\[
yk(H) \leq \frac{(s-1)(n-(s-t+1))}{s-t+1} = n\left(\frac{s-1}{s-t+1}\right) - s + 1 < 2n\left(\frac{s-1}{s}\right) - s + 1.
\]

Thus it suffices to check \( 2n\left(\frac{s-1}{s}\right) - s + 1 \leq 3n - s(s-1) \). This is equivalent to \( (s-1)^2 \leq n(\frac{s+1}{s}) \), which holds because \( n \geq s^3 \). This finishes the verification of \( \mathcal{D} \). Combining \( \mathcal{C}, \mathcal{D}, \mathcal{F} \), and the fact that \( k(H) + 1 \leq k(G) \leq k(H) + t - 1 \) yields

\[
P(G) \leq C^{k(H)+1} \alpha^{k(H)(t-1)n} \exp(n, s, q) \leq C^{k(G)} \alpha^{k(G)n} \exp(n, s, q).
\]
Lemma 6. Let \( P \) be the product of \( \Pi \) and \( \delta > n \), where the second inequality is by assumption on \( n \).

Proof. Fix \( P \) in the definition of \( s' \). Rearranging exponents yields \( \Pi \geq n \), where the second inequality is because \( \exp \Pi \). Assume \( \Pi \geq n \). We have left to show that \( P(n, s, q) \subseteq \Pi \nu_{-1} \). Note \( \exp \Pi(n, s, q) = \exp \Pi(n, s', q') \) implies \( P(n, s, q) \cap F(n, s', q') \subseteq \Pi \nu_{-1} \), where the equality is by Theorem 4(a) (Extremal). So it suffices to show \( P(n, s, q) \subseteq F(n, s', q') \). Suppose towards a contradiction that there exists \( G = ([n], w) \in P(n, s, q) \setminus F(n, s', q') \). Then in the notation of Lemma 5, \( k(G) \geq 1 \). Combining this with Lemma 5, we have

\[
\delta \geq n, \quad \text{where the second inequality is because } n \text{ is large, } \alpha < 1, \text{ and } k(G) \geq 1. \quad \text{But now } P(G) < \exp \Pi(n, s, q),
\]

which contradicts that \( G \in P(n, s, q) \). \( \square \)

6 Stability

In this section we prove the product-stability results for Theorems 3 and 4(a). We will use the fact that for any \((s, q)\)-graph \( G, \mu(G) \leq q \). If \( G = (V, w) \) and \( a \in \mathbb{N} \), let \( E_a(G) = \{ xy \in \binom{V}{2} : w(xy) = a \} \) and \( e_a(G) = |E_a(G)| \). In the following notation, \( p \) stands for “plus” and \( m \) stands for “minus.”

\[
p_p(G) = |\{ xy \in \binom{V}{2} : w(xy) > a \}| \quad \text{and} \quad m_a(G) = |\{ xy \in \binom{V}{2} : w(xy) < a \}|.
\]

Lemma 6. Let \( s \geq 2, q \geq \binom{s}{2} \) and \( a > 0 \). Suppose there exist \( 0 < \alpha < 1 \) and \( C > 1 \) such that for all \( n \geq s \), every \( G \in P(n, s, q) \) satisfies

\[
P(G) \leq \exp \Pi(n, s, q) q^{Cn \alpha_{pa}(G)}.
\]

Then for all \( \delta > 0 \) there are \( \epsilon, M > 0 \) such that for all \( n > M \) the following holds. If \( G \in P(n, s, q) \) and \( P(G) \geq \exp \Pi(n, s, q)^{1 - \epsilon} \) then \( p_a(G) \leq \delta n^2 \).

Proof. Fix \( \delta > 0 \). Choose \( \epsilon > 0 \) so that \( \frac{2 \epsilon \log q}{\log(1/\alpha)} = \delta \). Choose \( M \geq s \) sufficiently large so that \( n \geq M \) implies \((\epsilon n^2 + Cn) \log q \leq 2 \epsilon \log n^2 \). Let \( n > M \) and \( G \in P(n, s, q) \) be such that \( P(G) \geq \exp \Pi(n, s, q)^{1 - \epsilon} \). Our assumptions imply

\[
\exp \Pi(n, s, q)^{1 - \epsilon} \leq P(G) \leq \exp \Pi(n, s, q) q^{Cn \alpha_{pa}(G)}.
\]

Rearranging \( \exp \Pi(n, s, q)^{1 - \epsilon} \leq \exp \Pi(n, s, q) q^{Cn \alpha_{pa}(G)} \) yields

\[
\left( \frac{\epsilon}{n} \right)^{p_a(G)} \leq \exp \Pi(n, s, q) q^{Cn} \leq q^{\epsilon n^2 + Cn},
\]

where the second inequality is because \( \exp \Pi(n, s, q) \leq q^{n^2} \). Taking logs of both sides, we obtain

\[
p_a(G) \log(1/\alpha) \leq (\epsilon n^2 + Cn) \log q \leq 2 \epsilon n^2 \log q,
\]

where the second inequality is by assumption on \( n \). Dividing both sides by \( \log(1/\alpha) \) and applying the definition of \( \epsilon \) yields \( p_a(G) \leq \frac{2 \epsilon n^2 \log q}{\log(1/\alpha)} = \delta n^2 \). \( \square \)
We now prove the key lemma for this section.

**Lemma 7.** Let \(s, q, b, a\) be integers satisfying \(s \geq 2\) and either

(i) \(a \geq 1, 0 \leq b \leq s - 2\), and \(q = a^{(s)}(2) + b\) or

(ii) \(a \geq 2, b = 0\), and \(q = a^{(s)}(2) - 1\).

Then there exist \(0 < \alpha < 1\) and \(C > 1\) such that for all \(n \geq s\) and all \(G \in F(n, s, q)\),

\[
P(G) \leq \exp(n, s, q) q^{C_n \alpha p_a(G)}.\]  

(13)

**Proof.** We prove this by induction on \(s \geq 2\), and for each fixed \(s\), by induction on \(n\). Let \(s \geq 2\) and \(q, b, a\) be as in (i) or (ii) above. Set

\[
\xi = \begin{cases} 
0 & \text{if case (i) holds.} \\
1 & \text{if case (ii) holds.} 
\end{cases}
\]

Suppose first \(s = 2\). Set \(\alpha = 1/2\) and \(C = 2\). Since \(G\) is an \((n, 2, a - \xi)\)-graph, \(p_a(G) = 0\). Therefore for all \(n \geq 2\),

\[
P(G) \leq \exp(n, s, q) \leq \exp(n, s, q) q^{C_n} = \exp(n, s, q) q^{C_n \alpha p_a(G)}.
\]

Assume now \(s > 2\). Let \(\mathcal{I}\) be the set of \((s', q', b') \in \mathbb{N}^3\) such that \(2 \leq s' < s\) and \(s', q', b', a\) satisfy (i) or (ii). Observe that \(\mathcal{I}\) is finite. Suppose by induction on \(s\) that \((s', q', b') \in \mathcal{I}\) implies there are \(0 < \alpha(s', q', b') < 1\) and \(C(s', q', b') > 1\) such that for all \(n' \geq s'\) and \(G' \in F(n', s', q')\),

\[
P(G) \leq \exp(n, s', q') q^{C(s', q', b') n \alpha(s', q', b') p_a(G')}. 
\]

Set

\[
\alpha = \max \left( \left\{ q^{-1}, \frac{a - 1}{a} \right\} \cup \left\{ \alpha(s', q', b') : (s', q', b') \in \mathcal{I} \right\} \right).
\]

Observe \(0 < \alpha < 1\). Choose \(C \geq \binom{s - 1}{2}\) sufficiently large so that for all \(n \leq s\)

\[
q^{(n)}(2) \leq q^{C_n (a - \xi)}(2) \left( \frac{a}{a - \xi} \right)^{\frac{1}{2} s - 1(n)} \alpha^{(n)}(2),
\]

(14)

and so that for all \((s', q', b') \in \mathcal{I}\), \(C(s', q', b') \leq C/2\) and \(\frac{a + 1}{a}^{(s-3)/(s-2)} \leq q^{C/2}\). Given \(G \in F(n, s, q)\), set

\[
\Theta(G) = \left\{ Y \subseteq \left[ \begin{array}{c} n \\ s - 1 \end{array} \right] : S(Y) \geq a \left( \frac{s - 1}{2} \right) + (1 - \xi) b \right\},
\]

and let \(A(n, s, q) = \{ G \in F(n, s, q) : \Theta(G) \neq \emptyset \}\). We show the following holds for all \(n \geq 1\) and \(G \in F(n, s, q)\) by induction on \(n\).

\[
P(G) \leq q^{C_n (a - \xi)}(2) \left( \frac{a}{a - \xi} \right)^{\frac{1}{2} s - 1(n)} \alpha^{p_a(G)}. 
\]

(15)

This will finish the proof since \(\left( \frac{a - \xi}{2} \right) \left( \frac{a}{a - \xi} \right) \left( \frac{1}{2} s - 1(n) \right) \leq \exp(n, s, q)\) (by Theorem 3 (Extremal) for case (i) and Theorem 4(a) (Extremal) for case (ii)). If \(n \leq s\) and \(G \in F(n, s, q)\), then (15) holds because of (14) and the fact that \(P(G) \leq q^{(n)}(2)\). So assume \(n > s\), and suppose by induction that
holds for all \( s \leq n' < n \) and \( G' \in F(n', s, q) \). Let \( G = ([n], w) \in F(n, s, q) \). Suppose first that \( G \in A(n, s, q) \). Choose \( Y \in \Theta(G) \) and set \( R = [n] \setminus Y \). Given \( z \in R \), note that

\[
\begin{align*}
a \left( \frac{s - 1}{2} \right) + (1 - \xi)b + S_2(Y) & \leq S(Y) + S_2(Y) = S(Y \cup \{z\}) \leq a \left( \frac{s}{2} \right) + (1 - \xi)b - \xi,
\end{align*}
\]

and therefore \( S_2(Y) \leq a(s - 1) - \xi \). Then for all \( z \in R \), Lemma 2 implies \( P_2(Y) \leq a^{s-2}(a - \xi) \), with equality only if \( \{ w(yz) : y \in Y \} \) consists of \( s - 1 - \xi \) elements equal to \( a \) and \( \xi \) elements equal to \( a - 1 \). Let \( R_1 = \{ z \in R : \exists y \in Y, w(yz) > a \} \) and \( R_2 = R \setminus R_1 \). Then \( z \in R_1 \) implies \( P_2(Y) < a^{s-2}(a - \xi) \), so \( P_2(Y) \leq a^{s-2}(a - \xi) - 1 \). Let \( k = |R_1| \). Observe that \( G[R] \) is isomorphic to an element of \( F(n', s, q) \), where \( n' = n - |R| \geq 1 \). By induction (on \( n \)) and these observations we have that the following holds, where \( p_a(R) = p_a(G[R]) \).

\[
P(G) = P(R)P(Y) \prod_{z \in R_1} P_2(Y) \prod_{z \in R_2} P_2(Y)
\leq q^{C(n-s+1)}(a - \xi) \left( a \left( \frac{a - \xi}{2} \right) \right) t_{s-1}(n-1) \alpha_{p_a(R)} q \left( \frac{s-1}{2} \right) \left( a^{s-2}(a - \xi) - 1 \right)^k \left( a^{s-2}(a - \xi) \right)^{n-s+1-k},
\]

where the second inequality is because \( \binom{s-1}{2} \leq C \). Since \( \alpha \geq \left( \frac{a^{s-2}(a - \xi) - 1}{a^{s-2}(a - \xi)} \right)^{1/(s-2)} \), this is at most

\[
q^{C(n-s+2)}(a - \xi) \left( a \left( \frac{a - \xi}{2} \right) \right) t_{s-1}(n-1) \alpha_{p_a(R) + k(s-1)} \left( a^{s-2}(a - \xi) \right)^{n-s+1}.
\]

Because \( C(n-s+2) \leq Cn - \binom{s-1}{2} \) and \( q^{-1} \leq \alpha \), we have \( q^{C(n-s+2)} \leq q^{Cn} \alpha^{\binom{s-1}{2}} \). Combing this with the fact that \( p_a(G) \leq p_a(R) + k(s-1) + \binom{s-1}{2} \) implies that (16) is at most

\[
q^{Cn}(a - \xi) \left( a \left( \frac{a - \xi}{2} \right) \right) t_{s-1}(n-1) \alpha_{p_a(R) + k(s-1) + \binom{s-1}{2}} \left( a^{s-2}(a - \xi) \right)^{n-s+1}
= q^{Cn}(a - \xi)^{\binom{s-1}{2} + (s-1)(n-s+1)} \left( a \left( \frac{a - \xi}{2} \right) \right) t_{s-1}(n-1+s-2)(n-s+1) \alpha_{p_a(R) + k(s-1) + \binom{s-1}{2}}
\leq q^{Cn}(a - \xi)^{\binom{s-1}{2}} \left( a \left( \frac{a - \xi}{2} \right) \right) t_{s-1}(n) \alpha_{p_a(G)}.
\]

We now have that \( P(G) \leq q^{Cn}(a - \xi)^{\binom{s-1}{2}} \left( a \left( \frac{a - \xi}{2} \right) \right) t_{s-1}(n) \alpha_{p_a(G)} \), as desired. Assume now \( G \notin A(n, s, q) \). Then for all \( Y \in \binom{[n]}{s} \), \( S(Y) \leq \binom{s-1}{2} + (1 - \xi)b - 1 \). Thus \( G \) is an \( (n, s', q') \)-graph where \( s' = s - 1 \) and \( q' = a \left( \binom{s-1}{2} \right) + (1 - \xi)b - 1 \). Suppose \( a = 1 \), \( \xi = 0 \), and \( b = 0 \). Then \( q' = \binom{s-1}{2} \) and any \( (n, s', q') \)-graph must contain an edge of multiplicity 0. This implies \( P(G) = 0 \) and (15) holds. We have the following three cases remaining, where \( b' = \max\{b - 1, 0\} \).

1. \( \xi = 0 \), \( b = 0 \), and \( a \geq 2 \). In this case \( q' = \binom{s-1}{2} - 1 \) and \( b' = 0 \).
2. \( \xi = 1 \), \( b = 0 \), and \( a \geq 2 \). In this case \( q' = \binom{s-1}{2} - 1 \) and \( b' = 0 \).
3. \( \xi = 0 \), \( 1 \leq b \leq s - 2 \), and \( a \geq 1 \). In this case \( q' = \binom{s-1}{2} + b' \) and \( 0 \leq b' \leq s' - 2 \).
It is clear that in all three of these cases, \((s', q', b') \in \mathcal{I}\), so by our induction hypothesis (on \(s\)), there are \(\alpha' = \alpha(s', q', b') \leq \alpha\) and \(C' = C(s', q', b)\) such that

\[ P(G) \leq \exp_\Pi(n, s', q')(q')^{C'n} (\alpha')^{p_0(G)} \leq \exp_\Pi(n, s', q') q'^{C'n} \alpha^{p_0(G)}, \tag{17} \]

where the inequality is because \(q' \leq q\) and \(\alpha' \leq \alpha\). By Theorem \([4a]\) (Extremal) if cases 1 or 2 hold, and by Theorem \([3]\) (Extremal) if case 3 holds, we have the following.

\[ \exp_\Pi(n, s', q') \leq (a - \xi)\left(\frac{a}{a - \xi}\right)^{t_s(n)} (a + 1) \left(\frac{b'}{b' + 1}\right)^{t_s(n)} \leq (a - \xi)\left(\frac{a}{a - \xi}\right)^{t_s(n)} (a + 1) \left(\frac{b'}{b' + 1}\right)^{t_s(n)} \frac{a - 3}{a - 2} n, \]

where the last inequality is because \(t_s(n) \leq t_s(n)\) and \(\frac{b'}{b' + 1} n \leq \frac{b'}{b' + 1} n \leq \frac{a - 3}{a - 2} n\). By choice of \(C, C' = (\frac{a + 1}{a})^{\frac{b - 3}{b - 2}} \leq q^{Cn/2}\). Thus \(\exp_\Pi(n, s', q') \leq (a - \xi)\left(\frac{a}{a - \xi}\right)^{t_s(n)} q'^{C'n} \alpha^{p_0(G)}\). Combining this with \((17)\) implies

\[ P(G) \leq (a - \xi)\left(\frac{a}{a - \xi}\right)^{t_s(n)} q'^{C'n} \alpha^{p_0(G)} \leq (a - \xi)\left(\frac{a}{a - \xi}\right)^{t_s(n)} q'^{C'n} \alpha^{p_0(G)} , \]

where the last inequality is because \(C' \leq C/2\). Thus \((15)\) holds.

**Proof of Theorem \([3]\) (Stability).** Let \(s \geq 2, a \geq 1, \) and \(q = a(\delta) + b\) for some \(0 \leq b \leq s - 2\). Fix \(\delta > 0\). Given \(G \in F(n, s, q)\), let \(p_G = p_a(G)\) and \(m_G = m_a(G)\). Note that if \(G \in F(n, s, q)\), then

\[ |\Delta(G, \cup_{s}(n))| = m_G + p_G. \]

Suppose first \(a = 1\), so \(m_G = 0\). Combining Lemma \([7]\) with Lemma \([6]\) implies there are \(\epsilon_1\) and \(M_1\) such that if \(n > M_1\) and \(G \in F(n, s, q)\) satisfies \(P(G) \geq \exp_\Pi(n, s, q)^{1-\epsilon_1}\), then \(|\Delta(G, \cup_{s}(n))| = p_G \leq \delta n^2\). Assume now \(a > 1\). Combining Lemma \([7]\) with Lemma \([6]\) implies there are \(\epsilon_1\) and \(M_1\) such that if \(n > M_1\) and \(G \in F(n, s, q)\) satisfies \(P(G) \geq \exp_\Pi(n, s, q)^{1-\epsilon_1}\), then \(p_G \leq \delta n^2\), where

\[ \delta' = \min\left\{\frac{\delta}{2}, \frac{\delta \log(a/(a - 1))}{4 \log q}\right\}. \]

Set \(\epsilon = \min\{\epsilon_1, \frac{\delta \log(a/(a - 1))}{4 \log q}\}\). Suppose \(n > M_1\) and \(G \in F(n, s, q)\) satisfies \(P(G) \geq \exp_\Pi(n, s, q)^{1-\epsilon}\).

Our assumptions imply \(p_G \leq \delta n^2 \leq \delta n^2/2\). Observe that by definition of \(p_G\) and \(m_G\),

\[ P(G) \leq a(\delta)^{m_G} a(1 - 1)^m q^{p_G} = a(\delta)^{m_G} q^{p_G}. \tag{18} \]

By Theorem \([3a]\) (Extremal), \(\exp_\Pi(n, s, q) \geq a(\delta)^{1-\epsilon}\). Therefore \(P(G) \geq \exp_\Pi(n, s, q)^{1-\epsilon} \geq a(\delta)^{(1-\epsilon)}\).

Combining this with \((18)\) yields

\[ a(\delta)^{(1-\epsilon)} \leq a(\delta)^{m_G} q^{p_G}. \]

Rearranging this, we obtain

\[ \left(\frac{a}{a - 1}\right)^{m_G} \leq a(\delta)^{p_G} \leq q^{p_G} \leq q^{m_G-q_G}. \]

Taking logs, dividing by \(\log(a/(a - 1))\), and applying our assumptions on \(p_G\) and \(\epsilon\) yields

\[ m_G \leq \frac{\epsilon n^2 \log q}{\log(a/(a - 1))} + \frac{p_G \log q}{\log(a/(a - 1))} \leq \frac{\delta n^2}{4} + \frac{\delta n^2}{4} = \frac{\delta n^2}{2}. \]

Combining this with the fact that \(p_G \leq \frac{\delta n^2}{2}\) we have that \(|\Delta(G, \cup_{s}(n))| \leq \delta n^2\). 

The following classical result gives structural information about \(n\)-vertex \(K_s\)-free graphs with close to \(t_{s-1}(n)\) edges.
Theorem 12 (Erdős-Simonovits [5,16]). For all $\delta > 0$ and $s \geq 2$, there is an $\epsilon > 0$ such that every $K_s$-free graph with $n$ vertices and $t_{s-1}(n) - \epsilon n^2$ edges can be transformed into $T_{s-1}(n)$ by adding and removing at most $\delta n^2$ edges.

Proof of Theorem 4(a) (Stability). Let $s \geq 2$, $a \geq 2$, and $q = a\binom{s}{2} - 1$. Fix $\delta > 0$. Given $G \in F(n,s,q)$, let $p_G = p_s(G)$, $m_G = m_{a-1}(G)$. Choose $M_0$ and $\mu$ such that $\mu < \delta/2$ and so that Theorem 12 implies that any $K_s$-free graph with $n \geq M_0$ vertices and at least $(1 - \mu)t_{s-1}(n)$ edges can be made into $T_{s-1}(n)$ by adding or removing at most $\frac{\delta n^2}{3}$ edges. Set

$$A = \begin{cases} \frac{2}{a-1} & \text{if } a = 2 \\ \frac{2}{a} & \text{if } a > 2 \end{cases}$$

Combining Lemma 6 with Lemma 5 implies there are $\epsilon_1, M_1$ so that if $n > M_1$ and $G \in F(n,s,q)$ satisfies $p_G \geq \exp(n,s,q)^{1-\epsilon}$, then $p_G \leq \delta n^2$, where

$$\delta' = \min \left\{ \frac{\delta}{3}, \frac{\mu \log(a/(a-1))}{2 \log q}, \delta \log A \right\}.$$ 

(19)

Let

$$\epsilon = \min \left\{ \epsilon_1, \frac{\delta \log A}{6 \log q}, \frac{\mu \log(a/(a-1))}{2 \log q} \right\} \quad \text{and} \quad M = \max\{M_0, M_1\}.$$  

Suppose now that $n > M$ and $G \in F(n,s,q)$ satisfies $P(G) \geq \exp(n,s,q)^{1-\epsilon}$. By assumption, $p_G \leq \delta' n^2 \leq \frac{\delta n^2}{3}$. We now bound $m_G$. Note that if $a = 2$ and $P(G) \neq 0$, then $m_G = 0$. If $a > 2$, observe that by definition of $p_G$ and $m_G$,

$$P(G) \leq q^{p_G(a-2)m_G} a^{e_s(G)}(a-1)^{e_{s-1}(G)} \leq q^{p_G(a-2)^{m_G}} a^{e_s(G)}(a-1)^{\binom{s}{2}-e_s(G)},$$

(20)

where the last inequality is because $e_{a-1}(G) + m_G \leq \binom{n}{2} - e_s(G)$. Note that Turán’s theorem and the fact that $G$ is an $(n,s,q)$-graph implies that $e_s(G) \leq t_{s-1}(n)$, so

$$a^{e_s(G)}(a-1)^{\binom{s}{2}-e_s(G)} \leq a^{t_{s-1}(n)}(a-1)^{\binom{s}{2}-t_{s-1}(n)} = \exp(n,s,q),$$

where the last equality is from Theorem 4(a) (Extremal). Combining this with (20) yields

$$\exp(n,s,q)^{1-\epsilon} \leq P(G) \leq q^{p_G(a-2)^{m_G}} \exp(n,s,q).$$

Rearranging $\exp(n,s,q)^{1-\epsilon} \leq q^{p_G(a-2)^{m_G}} \exp(n,s,q)$ and using that $\exp(n,s,q) \leq q^{n^2}$, we obtain

$$A^{m_G} = \left( \frac{a-1}{a-2} \right)^{m_G} \leq q^{p_G} \exp(n,s,q)^{\epsilon} \leq q^{p_G + \epsilon n^2}.$$ 

Taking logs, dividing by $\log A$, and applying our assumptions on $p_G$ and $\epsilon$ we obtain $m_G \leq \frac{\delta n^2}{3}$. Using (20) and $a^{t_{s-1}(n)}(a-1)^{\binom{s}{2}-t_{s-1}(n)} = \exp(n,s,q)$, we have

$$\exp(n,s,q)^{1-\epsilon} \leq P(G) \leq q^{p_G} a^{e_s(G)}(a-1)^{\binom{s}{2}-e_s(G)} = q^{p_G} \exp(n,s,q) \left( \frac{a}{a-1} \right)^{e_s(G)-t_{s-1}(n)}.$$ 

Rearranging this we obtain

$$\left( \frac{a}{a-1} \right)^{t_{s-1}(n)-e_s(G)} \leq q^{p_G} \exp(n,s,q)^{\epsilon} \leq q^{p_G + \epsilon n^2}.$$
Taking logs, dividing by \( \log(a/(a-1)) \), and using the assumptions on \( p_G \) and \( \epsilon \) we obtain that

\[
t_{s-1}(n) - e_a(G) \leq \frac{p_G \log q}{\log(a/(a-1))} + \frac{en^2 \log q}{\log(a/(a-1))} \leq \frac{\mu n^2}{2} + \frac{\mu n^2}{2} = \mu n^2.
\]

Let \( H \) be the graph with vertex set \([n]\) and edge set \( E = E_a(G) \). Then \( H \) is \( K_s \)-free, and has \( e_a(G) \) many edges. Since \( t_{s-1}(n) - e_a(G) \leq \mu n^2 \), Theorem 12 implies that \( H \) is \( \frac{\delta}{2} \)-close to some \( H' = T_{s-1}(n) \). Define \( G' \in F(n,s,q) \) so that \( E_a(G') = E(H') \) and \( E_{a-1}(G') = \binom{n}{2} \setminus E_a(G') \). Then \( G' \in T_{s-1,a}(n) \) and

\[
\Delta(G,G') \subseteq (E_a(G) \Delta E_a(G')) \cup \bigcup_{i \notin \{a,a-1\}} E_i(G) = \Delta(H,H') \cup \bigcup_{i \notin \{a,a-1\}} E_i(G).
\]

This implies \(|\Delta(G,G')| \leq |\Delta(H,H')| + p_G + m_G \leq \frac{\delta}{3} n^2 + \frac{\delta}{3} n^2 + \frac{\delta}{3} n^2 = \delta n^2.\]

6.1 Proof of Theorem 4(b) (Stability)

In this subsection we prove Theorem 4(b) (Stability). We first prove two lemmas.

**Lemma 8.** Let \( s \geq 4, a \geq 2, \) and \( q = a \binom{s}{2} - t \) for some \( 2 \leq t \leq \frac{s}{2} \). For all \( \lambda > 0 \) there are \( M \) and \( \epsilon > 0 \) such that the following holds. Suppose \( n > M \) and \( G \in F(n,s,q) \) satisfies \( P(G) > \exp(\eta n^2) \). Then \( k(G) < \lambda n \), where \( k(G) \) is as defined in Lemma 5.

**Proof.** Fix \( \lambda > 0 \). Set \( \eta = a^{\frac{\lambda^2-1}{2}} - t \) and choose \( C \) and \( \alpha \) as in Lemma 5. Choose \( \epsilon > 0 \) so that \( \alpha \lambda^2 = \eta^{-\epsilon} \). By Theorem 4(b) (Extremal), \( \exp(n,s,q) = \eta^{\binom{n}{2} + \alpha n^2} \). Assume \( M \) sufficiently large so that for all \( n \geq M, \) (i) holds for all \( G \in F(n,s,q), \) \( \exp(n,s,q) < \eta^2, \) \( C\lambda n \leq \eta n^2, \) and \( C\alpha n < 1. \) Fix \( n \geq M \) and suppose towards a contradiction that \( G \in F(n,s,q) \) satisfies \( P(G) \geq \exp(\eta n^2) \) and \( k(G) \geq \lambda n. \) By Lemma 5 and the facts that \( C\alpha n < 1 \) and \( k(G) \geq 1, \) we obtain that

\[
P(G) \leq C^{k(G)\alpha^{nk(G)}} \exp(n,s,q) = (C\alpha^n)^{k(G)} \exp(n,s,q) \leq (C\alpha^n)^{\lambda n} \exp(n,s,q).
\]

By assumption on \( n \) and definition of \( \epsilon, \) \( (C\alpha^n)^{\binom{n}{2}} = C^{\lambda n^{\alpha} \lambda n^2} = C^{\lambda n^{\alpha} \eta^{-2\alpha n^2}} \leq \eta^{-\alpha n^2}. \) Thus

\[
P(G) \leq \eta^{-\alpha n^2} \exp(n,s,q) < \exp(n,s,q)^{1-\epsilon},
\]

where the last inequality is because by assumption, \( \exp(n,s,q) < \eta n^2. \) But this contradicts our assumption that \( P(G) > \exp(n,s,q)^{1-\epsilon}. \)

**Lemma 9.** Let \( s,q,m \geq 2 \) be integers. For all \( 0 < \delta < 1, \) there is \( 0 < \lambda < 1 \) and \( N \) such that \( n > N \) implies the following. If \( G = \binom{n}{2}, \) \( G \) has \( \mu(G) \leq m \) and \( \mathcal{H}(G,s,q) \) contains strictly less than \( [\lambda n] \) pairwise disjoint elements, then \( G \) is \( \delta \)-close to an element in \( F(n,s,q). \)

**Proof.** Fix \( 0 < \delta < 1. \) Observe we can view any multigraph \( G \) with \( \mu(G) \leq m \) as an edge-colored graph with colors in \( \{0, \ldots, m\}. \) By Theorem 1 there is \( \epsilon \) and \( M \) such that if \( n > M \) and \( G = \binom{n}{2}, \) \( G \) has \( \mu(G) \leq m \) and \( \mathcal{H}(G,s,q) \leq \epsilon \binom{n}{2}, \) then \( G \) is \( \delta \)-close to an element of \( F(n,s,q). \) Let \( \lambda := \epsilon/s \) and \( N = \max\{M, \frac{s}{\epsilon} \}. \) We claim this \( \lambda \) and \( N \) satisfy the desired conclusions. Suppose towards a contradiction that \( n > M \) and \( G = \binom{n}{2} \) has \( \mu(G) \leq m, \) \( \mathcal{H}(n,s,q) \) contains
strictly less than $[\lambda n]$ pairwise disjoint elements, but $G$ is $\delta$-far from every element in $F(n, s, q)$. Then $\mathcal{H}(G, s, q) > \epsilon(n)$ by choice of $M$ and $\lambda$. By our choice of $N$, $[\lambda n]s \leq (\lambda n + 1)s \leq n$. Then Proposition 11.6 in [11] and our assumptions imply $|\mathcal{H}(G, s, q)| \leq ([\lambda n] - 1)^{(n-1)}_{s-1})$. But now

$$|\mathcal{H}(G, s, q)| \leq ([\lambda n] - 1)\left(\frac{n-1}{s-1}\right) < \lambda n \left(\frac{s-1}{n-1}\right) = \left(\frac{\epsilon n}{s}\right) \left(\frac{n}{s}\right) = \epsilon\left(\frac{n}{s}\right),$$

a contradiction. \qed

**Proof of Theorem 4(b) (Stability).** Let $s \geq 4$, $a \geq 2$, and $q = a\left(\frac{s}{2}\right) - t$ for some $2 \leq t \leq \frac{s}{2}$. Fix $\delta > 0$. Let $s' = s - t + 1$ and $q' = a\left(\frac{s'}{2}\right) - 1$. Note Theorem 4 (Extremal) implies that for sufficiently large $n$, $\mathcal{P}(n, s, q) = \mathcal{T}_{s'-1, a}(n)$, $\mathcal{e}_n(n, s', q') = \mathcal{e}_n(n, s, q)$, and $\mathcal{e}_n(s', q') = \mathcal{e}_n(s, q) = \eta$, where $\eta = (a - 1)(\frac{a}{a-1})^{(s'-2)/(s'-1)}$.

Apply Theorem 4 (a) (Stability) for $(s', q')$ to obtain $\epsilon_0$. By replacing $\epsilon_0$ if necessary, assume $\epsilon_0 < 4\delta / \log n$. Set $\epsilon_1 = \epsilon_0 \log n / (8 \log q)$ and note $\epsilon_1 < \delta / 2$. Apply Lemma 9 to $\epsilon_1$ and $m = q$ to obtain $\lambda$ such that for large $n$ the following holds. If $G = ([n], w)$ has $\mu(G) \leq q$ and $\mathcal{H}(G, s', q')$ contains strictly less than $[\lambda n]$ pairwise disjoint elements, then $G$ is $\epsilon_1$-close to an element in $F(n, s', q')$. Finally, apply Lemma 8 for $s, q, t$ to $\lambda$ to obtain $\epsilon_2 > 0$.

Choose $M$ sufficiently large for the desired applications of Theorems 4 (a) (Stability) and 4 (b) (Extremal) and Lemmas 8 and 9. Set $\epsilon = \min\{\epsilon_2, \epsilon_0 / 2\}$. Suppose $n > M$ and $G \in F(n, s, q)$ satisfies $P(G) \geq \mathcal{e}_n(n, s, q)^{1-\epsilon}$. Then Lemma 8 and our choice of $\epsilon$ implies $k(G) < \lambda n$. Observe that by the definitions of $s', q'$,

$$\{Y \in \left([n] \atop s-t+1\right): S(Y) \geq a\left(s-t+1\right) / 2\} = \{Y \in \left([n] \atop s'\right): S(Y) \geq q' + 1\} = \mathcal{H}(G, s', q').$$

Thus $k(G) < \lambda n$ means $\mathcal{H}(G, s', q')$ contains strictly less than $[\lambda n]$ pairwise disjoint elements. Lemma 9 then implies $G$ is $\epsilon_1$-close to some $G' \in F(n, s', q')$. Combining this with the definition of $\epsilon_1$ yields

$$P(G') \geq P(G)q^{-|\Delta(G, G')|} \geq P(G)q^{-\epsilon_1 n^2} = P(G)\eta^{-\epsilon_1 n^2 / 8} \geq \mathcal{e}_n(n, s, q)^{1-\epsilon_1 n^2 / 8}. \quad (21)$$

By Proposition 1 $\mathcal{e}_n(n, s, q) \geq \mathcal{e}_n(n, q)^{n / 2} = \eta^{n / 2}$. Combining this with (21) and the definition of $\epsilon$ yields

$$P(G') \geq \mathcal{e}_n(n, s, q)^{1-\epsilon_1 n^2 / 8} \geq \mathcal{e}_n(n, s, q)^{1-\epsilon_1 n^2 / 8} \geq \mathcal{e}_n(n, s, q)^{1-\epsilon_1 n^2 / 8}. \quad (22)$$

Since $\mathcal{e}_n(n, s, q) = \mathcal{e}_n(n, s', q')$, (22) implies $P(G') \geq \mathcal{e}_n(n, s', q')^{1-\epsilon_1 n^2}$. So Theorem 4 (a) (Stability) implies $G'$ is $\delta / 2$-close to some $G'' \in \mathcal{T}_{s'-1, a}(n) = \mathcal{T}_{s-t, a}(n)$. Now we are done, since

$$|\Delta(G, G'')| \leq |\Delta(G, G')| + |\Delta(G', G'')| \leq \epsilon_1 n^2 + \delta n^2 / 2 \leq \delta n^2.$$

\qed

**7 Extremal Result for $(n, 4, 9)$-graphs**

In this section we prove Theorems 5. We first prove one of the inequalities needed for Theorem 5.

**Lemma 10.** For all $n \geq 4$, $2^{\mathcal{e}_n(n, \{C_5, C_4\})} \leq \mathcal{e}_n(n, 4, 9)$.

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For all Lemma 11, of multiplicity larger than 2, then $P$ or 2. Consequently, for all $xy \in G$ because $G(n, 3, 5)$ is $C_3$-free, so $ex(6, C_3)$ contains at most 3 elements equal to 2 and the rest equal to 1, so $S(X) \leq 9$. This shows $G' \in F(n, 4, 9)$. Thus $2|E| = 2^{ex(n, \{C_3, C_4\})} = P(G') \leq ex\Pi(n, 4, 9)$.

To prove the reverse inequality, our strategy will be to show that if $G \in F(n, 4, 9)$ has no edges of multiplicity larger than 2, then $P(G) \leq 2^{ex(n, \{C_3, C_4\})}$ (Theorem 13). We will then show that all product extremal $(4, 9)$-graphs have no edges of multiplicity larger than 2 (Theorem 14). Theorem 5 will then follow. We begin with a few definitions and lemmas.

**Definition 9.** Suppose $n \geq 1$. Set $F_{\leq 2}(n, 4, 9) = \{G \in F(n, 4, 9) : \mu(G) \leq 2\}$ and
$$D(n) = F_{\leq 2}(n, 4, 9) \cap F(n, 3, 5).$$

**Lemma 11.** For all $n \geq 4$, if $G = ([n], w) \in D(n)$, then $P(G) \leq 2^{ex(n, \{C_3, C_4\})}$.

**Proof.** If $P(G) = 0$ we are done, so assume $P(G) > 0$. Let $H = ([n], E)$ be the graph where $E = \{xy \in \binom{[n]}{2} : w(xy) = 2\}$. Since $P(G) > 0$ and $\mu(G) \leq 2$, $G$ contains all edges of multiplicity 1 or 2. Consequently, $P(G) = 2^{|E|}$. Since $G \in F(n, 3, 5)$, $H$ is $C_3$-free and since $G \in F(n, 4, 9)$, $H$ is $C_4$-free, so $|E| \leq ex(n, \{C_3, C_4\})$. This shows $P(G) = 2^{|E|} \leq 2^{ex(n, \{C_3, C_4\})}$.

The following lemma gives us useful information about elements of $F(n, 4, 9) \setminus F(n, 3, 5)$.

**Lemma 12.** Suppose $n \geq 4$ and $G = ([n], w) \in F(n, 4, 9)$ satisfies $P(G) > 0$. If there is $X \in \binom{[n]}{3}$ such that $S(X) \geq 6$, then $P(X) \leq 2^3$ and $w(xy) = 1$ for all $x \in X$ and $y \in [n] \setminus X$. Consequently
$$P(G) = P(X)P([n] \setminus X) \leq 2^3 P([n] \setminus X).$$

**Proof.** Let $y \in [n] \setminus X$. Since $P(G) > 0$, every edge in $G$ has multiplicity at least 1, so $S_y(X) \geq 3$. Thus
$$3 + S(X) \leq S_y(X) + S(X) = S(X \cup \{y\}) \leq 9,$$
which implies $S(X) \leq 6$. By Lemma 2 this implies $P(X) \leq 2^3$. By assumption, $S(X) \geq 6$, so we have $6 + S_y(X) \leq S(X) + S_y(X) = S(X \cup \{y\}) \leq 9$, which implies $S_y(X) \leq 3$. Since every edge in $G$ has multiplicity at least 1 and $|X| = 3$, we must have $w(xy) = 1$ for all $x \in X$. Therefore $P(G) = P([n] \setminus X)P(X) \leq P([n] \setminus X)^2$.

**Fact 1.** For all $n \geq 4$ and $1 \leq i < n$, $ex(n, \{C_3, C_4\}) \geq ex(n - i, \{C_3, C_4\}) + i$.

**Proof.** Suppose $n \geq 4$ and $1 \leq i < n$. Fix $G = ([n - i], E)$ an extremal $\{C_3, C_4\}$-free graph. Let $G' = ([n], E')$ where $E' = E \cup \{(n - 1), n\}$, then $G'$ is $\{C_3, C_4\}$-free graph because $G = G'[n - i]$ is $\{C_3, C_4\}$-free and because the elements of $[n] \setminus [n - i]$ all have degree 1 in $G'$. Therefore $ex(n, \{C_3, C_4\}) \geq ex(n - i, \{C_3, C_4\}) + |E' \setminus E| = ex(n - i, \{C_3, C_4\}) + i$.

We now prove Theorem 13. We will use that $ex(4, \{C_3, C_4\}) = 3$, $ex(5, \{C_3, C_4\}) = 5$, and $ex(6, \{C_3, C_4\}) = 6$ (see 10).

**Theorem 13.** For all $n \geq 4$ and $G \in F_{\leq 2}(n, 4, 9)$, $P(G) \leq 2^{ex(n, \{C_3, C_4\})}$.
Proof. We proceed by induction on $n$. Assume first $4 \leq n \leq 6$ and $G \in F_{\leq 2}(n, 4, 9)$. If $P(G) = 0$ then we are done. If $G \in D(n)$, then we are done by Lemma 11. So assume $P(G) > 0$ and \( G \in F_{\leq 2}(n, 4, 9) \setminus D(n) \). By definition of $D(n)$ this means $G \notin F(n, 3, 5)$, so there is $X \in \binom{[n]}{3}$ such that $S(X) \geq 6$. By Lemma 12 this implies $P(G) \leq P([n] \setminus X)2^3 \leq 2^{(n^2 - 3) + 3}$, where the second inequality is because $\mu(G) \leq 2$. The explicit values for $ex(n, \{C_3, C_4\})$ show that for $n \in \{4, 5, 6\}$, $2^{(n^2 - 3) + 3} \leq 2^{ex(n, \{C_3, C_4\})}$.

Consequently, $P(G) \leq 2^{(n^2 - 3) + 3} \leq 2^{ex(n, \{C_3, C_4\})}$.

Suppose now $n \geq 7$ and assume by induction that for all $4 \leq n' < n$ and $G' \in F_{\leq 2}(n', 4, 9), P(G') \leq 2^{ex(n', 4,9)}$. Fix $G \in F_{\leq 2}(n, 4, 9)$. If $P(G) = 0$ then we are done. If $G \in D(n)$, then we are done by Lemma 11. So assume $P(G) > 0$ and $G \in F_{\leq 2}(n, 4, 9) \setminus D(n)$. By definition of $D(n)$ this means $G \notin F(n, 3, 5)$, so there is $X \in \binom{[n]}{3}$ such that $S(X) \geq 6$. By Lemma 12 this implies $P(G) \leq P([n] \setminus X)2^3$. Clearly there is $H \in F_{\leq 2}(n - 3, 4, 9)$ such that $G[[n] \setminus X] \cong H$. By our induction hypothesis applied to $H$, $P([n] \setminus X) = P(H) \leq 2^{ex(n - 3, \{C_3, C_4\})}$. Therefore

$$P(G) \leq P([n] \setminus X)2^3 \leq 2^{ex(n - 3, \{C_3, C_4\})+3} \leq 2^{ex(n, \{C_3, C_4\})},$$

where the last inequality is by Fact 11 with $i = 3$. \( \square \)

We will use the following lemma to prove Theorem 14. Observe for all $n \geq 2$, $ex_3(n, 4, 9) > 0$ implies that for all $G \in \mathcal{P}(n, 4, 9)$, every edge in $G$ has multiplicity at least 1. We will write $xyz$ to denote the three element set $\{x, y, z\}$.

Lemma 13. Suppose $n \geq 4$ and $G = ([n], w) \in \mathcal{P}(n, 4, 9)$ satisfies $\mu(G) \geq 3$. Then one of the following hold.

(i) There is $xyz \in \binom{[n]}{3}$ such that $\mu(G[[n] \setminus xyz]) \leq 2$ and $P(G) \leq 6 \cdot P([n] \setminus xyz)$.

(ii) There is $xy \in \binom{[n]}{2}$ such that $\mu(G[[n] \setminus xy]) \leq 2$ and $P(G) \leq 3 \cdot P([n] \setminus xy)$.

Proof. Suppose $n \geq 4$ and $G = ([n], w) \in \mathcal{P}(n, 4, 9)$ is such that $\mu(G) \geq 3$. Fix $xy \in \binom{[n]}{2}$ such that $w(xy) = \mu(G)$. We begin by proving some preliminaries about $G$ and $xy$. We first show $w(xy) = 3$. By assumption, $w(xy) \geq 3$. Suppose towards a contradiction $w(xy) \geq 4$. Choose some $u \neq v \in [n] \setminus xy$. Since every edge in $G$ has multiplicity at least 1, $5 + w(xy) \leq S(\{x, y, u, v\}) \leq 9$. This implies $w(xy) \leq 9 - 5 = 4$, and consequently $w(xy) = 4$. Combining this with the fact that every edge has multiplicity at least 1, we have

$$9 \leq 4 + w(uv) + w(uw) + w(vx) + w(yu) + w(yv) = S(\{x, y, u, v\}) \leq 9.$$  

Consequently, $w(uv) = w(uw) = w(vx) = w(yu) = w(yv) = 1$. Since this holds for all pairs $uv \in \binom{[n]}{2} \setminus xy$, we have shown $P(G) = w(xy) = 4$. Because $n \geq 4$, Fact 11 implies

$$2^{ex(n, \{C_3, C_4\})} \geq 2^{ex(4, \{C_3, C_4\})} = 2^3 > 4 = P(G).$$

Combining this with Lemma 10 shows $P(G) < 2^{ex(n, \{C_3, C_4\})} \leq ex_3(n, 4, 5)$, a contradiction. Thus $\mu(G) = w(xy) = 3$. We now show that for all $uv \in \binom{[n]}{2} \setminus xy$, $w(uv) \leq 2$. Fix $uv \in \binom{[n]}{2} \setminus xy$ and suppose towards a contradiction $w(uv) \geq 3$. Choose some $X \in \binom{[n]}{4}$ containing $\{x, y, u, v\}$. Because every edge in $G$ has multiplicity at least 1, we have that $S(X) \geq w(uv) + w(xy) + 4 \geq 10$, a contradiction. Thus $w(uv) \leq 2$ for all $uv \in \binom{[n]}{2} \setminus xy$. We now show that for all $z \in [n] \setminus xy$, at most one of $w(xz)$ or $w(yz)$ is equal to 2. Suppose towards a contradiction there is $z \in [n] \setminus xy$ such that $w(xz) = w(yz) = 2$. Note $S(xyz) \geq 7$. So for each $z' \in [n] \setminus xyz$, $S_{z'}(xyz) \leq 9 - S(xyz) = 9 - 7 = 2$.  

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But since every edge has multiplicity at least 1 this is impossible. Thus for all \( z \in [n] \setminus xy \), at most one of \( w(xz) \) or \( w(yz) \) is equal to 2.

We now prove either (i) or (ii) holds. Suppose there is \( z \in [n] \setminus xy \) such that one of \( w(xz) \) or \( w(yz) \) is equal to 2. Then by what we have shown, \( \{w(xy), w(xz), w(yz)\} = \{3, 1, 2\} \), and consequently \( P(xyz) = 6 \). By Lemma 12 since \( S(xyz) \geq 6 \), we have that

\[
P(G) = P(xyz)P([n] \setminus xyz) = 6 \cdot P([n] \setminus xy).
\]

By the preceding arguments, \( \mu(G([n] \setminus xyz])) \leq 2 \). Thus (i) holds. Suppose now that for all \( z \in [n] \setminus xy, w(xz) = w(yz) = 1 \). Then \( P(G) = w(xy)P([n] \setminus xy) = 3 \cdot P([n] \setminus xy) \). By the preceding arguments, \( \mu(G([n] \setminus xy)) \leq 2 \). Thus (ii) holds.

**Theorem 14.** For all \( n \geq 4 \), \( P(n, 4, 9) \subseteq F_{\leq 2}(n, 4, 9) \).

**Proof.** Fix \( n \geq 4 \) and \( G = ([n], w) \in P(n, 4, 9) \). Suppose towards a contradiction \( G \notin F_{\leq 2}(n, 4, 9) \). We show \( P(G) < 2^{ex(n, \{C_3,C_4\})} \), contradicting that \( G \) is product-extremal (since by Lemma 10, \( 2^{ex(n,\{C_3,C_4\})} \leq ex_H(n,4,9) \)).

Since \( G \notin F_{\leq 2}(n, 4, 9) \), either (i) or (ii) of Lemma 13 holds. If (i) holds, choose \( xyz \in \binom{[n]}{3} \) with \( \mu(G([n] \setminus xy)) \leq 2 \) and \( P(G) \leq 6 \cdot P([n] \setminus xy) \). Let \( H \in F_{\leq 2}(n-2, 4, 9) \) be such that \( G([n] \setminus xy) \cong H \). If \( n \in \{4, 5, 6\} \), then \( P(G) \leq 6 \cdot P(H) \leq 6 \cdot 2^{ex(n,\{C_3,C_4\})} \), where the second inequality is because \( \mu(H) \leq 2 \), and the strict inequality is from the exact values for \( ex(n, \{C_3, C_4\}) \) for \( n \in \{4, 5, 6\} \). If \( n \geq 7 \), then by Lemma 13 and because \( n-3 \geq 4 \), \( P(H) \leq 2^{ex(n-3,\{C_3,C_4\})} \). Therefore,

\[
P(G) \leq 6 \cdot P(H) \leq 6 \cdot 2^{ex(n-3,\{C_3,C_4\})} < 2^{ex(n-3,\{C_3,C_4\})+3} \leq 2^{ex(n,\{C_3,C_4\})},
\]

where the last inequality is by Fact 1. If (ii) holds, choose \( xy \in \binom{[n]}{2} \) with \( \mu(G([n] \setminus xy)) \leq 2 \) and \( P(G) \leq 3 \cdot P([n] \setminus xy) \). Let \( H \in F_{\leq 2}(n-2, 4, 9) \) be such that \( G([n] \setminus xy) \cong H \). If \( n \in \{4, 5\} \), then \( P(G) \leq 3 \cdot P(H) \leq 3 \cdot 2^{ex(n,\{C_3,C_4\})} \), where the second inequality is because \( \mu(H) \leq 2 \), and the strict inequality is from the exact values for \( ex(n, \{C_3, C_4\}) \) for \( n \in \{4, 5\} \). If \( n \geq 6 \), then \( n-2 \geq 4 \) and Lemma 13 imply \( P(H) \leq 2^{ex(n-2,\{C_3,C_4\})} \). Therefore,

\[
P(G) \leq 3 \cdot P([n] \setminus xy) \leq 3 \cdot 2^{ex(n-2,\{C_3,C_4\})} < 2^{ex(n-2,\{C_3,C_4\})+2} \leq 2^{ex(n,\{C_3,C_4\})},
\]

where the last inequality is by Fact 1.

**Proof of Theorem 5.** Fix \( n \geq 4 \) and \( G \in P(n, 4, 9) \). By Theorem 14 \( G \in F_{\leq 2}(n, 4, 9) \). By Theorem 13 this implies \( P(G) \leq 2^{ex(n,\{C_3,C_4\})} \). By Lemma 10 \( P(G) \geq 2^{ex(n,\{C_3,C_4\})} \). Consequently, \( P(G) = 2^{ex(n,\{C_3,C_4\})} = ex_H(n,4,9) \). 

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