Solution of the $\kappa$-deformed Dirac equation with scalar, vector and tensor interactions in the context of pseudospin and spin symmetries

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(Dated: April 18, 2017)

The deformed Dirac equation invariant under the $\kappa$-Poincaré-Hopf quantum algebra in the context of minimum, vector and scalar couplings under spin and pseudospin symmetric limits is considered. Expressions for the energy eigenvalues and wave functions are determined for both symmetry limits. We verify that the energies and wave functions of the particle are modified by the deformation parameter.

PACS numbers: 03.65.Ge, 03.65.Pm, 11.30.Cp, 71.70.Di

I. INTRODUCTION

Quantum deformations based on the $\kappa$-Poincaré-Hopf algebra constitute an important branch of research that enables us to address problems in condensed matter and high energy physics through field equations. These field equations were first presented in Ref. [1], where a new real quantum Poincaré algebra with standard real structure, obtained by contraction of $U_q(O(3,2))$. The resulting algebra of this contraction is a standard real Hopf algebra and depends on a dimension-full parameter $\kappa$ instead of $q$. Since then, the algebraic structure of the $\kappa$-deformed Poincaré algebra has been investigated intensively and have become a theoretical field of increasing interest [2 21]. Through the field equations from the $\kappa$-Poincare algebra ($\kappa$-Dirac equation [22 24]), we can study the physical implications of the quantum deformation parameter $\kappa$ in relativistic and nonrelativistic quantum systems. In this context, we highlight the study of relativistic Landau levels [3], the Aharonov-Bohm effect taking into account spin effects [17], the Dirac oscillator [25 26] and the integer quantum Hall effect [27].

When we want to study the relativistic quantum dynamics of particles with spin, we must obviously consider the presence of external fields, which include the vector and scalar fields. The inclusion of vector and scalar potentials in the Dirac equation reveals interesting properties of symmetries in nuclear theory. The first contributions in this subject revealed the existence of $SU(2)$ symmetries, which are known in the literature as pseudospin and spin symmetries [23 28]. Some investigations have been made in this scenario in order to give a meaning to these symmetries. However, it was only in a work by Ginocchio, that pseudospin symmetry was revealed. He verified that pseudospin symmetry in nuclei could arise from nucleons moving in a relativistic mean field, which has an attractive scalar and repulsive vector potential nearly equal in magnitude [30] (for a more detailed description see Ref. [31]). Spin and pseudo-spin symmetries in the Dirac equation have been studied under different aspects in recent years (see Refs. [22 28]). Some studies have been developed taking into account the spin and pseudospin symmetry limits to study relativistic dynamics of physical systems interacting with a class of potentials [14 11].

The present work is proposed to investigate the $\kappa$-deformed Dirac equation derived in Refs. [23] in the context of minimum, vector and scalar couplings under spin and pseudospin symmetric limits.

The structure of the paper is as follows. In Sec. II, we present the $\kappa$-deformed Dirac equation with couplings from which we derive the $\kappa$-deformed Pauli-Dirac equation, by using the usual procedure that consists of squaring the $\kappa$-deformed Dirac equation. In Sec. III we consider the equation of Pauli and establish the spin and pseudospin symmetries limits. As an application, we consider the particle interacting with an uniform magnetic field in the $z$-direction in two different physical situations: (i) particle interacting with a harmonic oscillator and (ii) particle interacting with a linear potential. We obtain expressions for the energy eigenvalues and wave functions in both limits. In Sec. IV we present our comments and conclusions.

II. THE $\kappa$-DEFORMED DIRAC EQUATION WITH COUPLINGS

We begin with the deformed Dirac equation invariant under the $\kappa$-deformed Poincaré quantum algebra [22 23]

$$\left\{ (\gamma_0 p_0 - \gamma_i p_i) + \frac{1}{2} \varepsilon \left[ \gamma_0 \left( p_0^2 - p_i p_i \right) - M p_0 \right] \right\} \psi = M \psi. \tag{1}$$

The interactions can be performed through the following prescriptions [22]

$$p_i \rightarrow p_i - e A_i, \quad E \rightarrow E - \nu (r), \quad M \rightarrow M + w (r). \tag{2\-4}$$

As we are interested in a planar dynamics, i.e., when the third directions of the fields involved are zero, we choose
the following representation for the gamma matrices (8):

\[ \gamma_0 = \sigma_3, \]
\[ \alpha_1 = \gamma_0 \gamma_1 = \sigma_1, \]
\[ \alpha_2 = \gamma_0 \gamma_2 = 8 \sigma_2, \]

where the parameter \( s \), which has a value of twice the spin value, can be introduced to characterizing the two spin states, with \( s = +1 \) for spin "up" and \( s = -1 \) for spin "down". In the above representation, the \( \kappa \)-deformed Dirac including the interactions can be written as

\[
\begin{align*}
&[\alpha \cdot (p - eA) + \gamma_0 (M + w (r))] \psi - [E - \nu (r)] \psi \\
&+ \frac{\varepsilon}{2} \{ e \sigma \cdot [B \times p] \} + M \gamma_0 (\alpha \cdot p) \psi
\end{align*}
\]

In this configuration, Eq. (11) reads

\[
X + \frac{\varepsilon}{2} Y = 0,
\]

with

\[
X = \begin{bmatrix} \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} + ieB \frac{\partial \psi}{\partial \varphi} \\
\frac{1}{4} e^2 B^2 r^2 \psi + [M + \nu (r)]^2 \psi - [E - \nu (r)]^2 \psi
\end{bmatrix}
\]

and

\[
Y = \begin{bmatrix} \gamma_0 \frac{\partial^2 w (r)}{\partial r^2} + \frac{1}{r} \frac{\partial w (r)}{\partial r} \psi - \gamma_0 \frac{\partial w (r)}{\partial r} \frac{\partial \psi}{\partial r} \\
-\frac{1}{r} \frac{\partial \psi}{\partial r} - 2is \frac{1}{r} \frac{\partial w (r)}{\partial r} \frac{\partial \psi}{\partial \varphi}
\end{bmatrix}
\]

where the matrices (8)-(11) are now given in cylindrical coordinates, \( \gamma_\varphi = i \sigma_\varphi, \quad \gamma_\varphi = -is \sigma_\varphi, \) with (12).

In order to apply this equation to some physical system, we need to choose a representation for the vector potential \( A \) and the scalar potentials \( w (r) \) and \( \nu (r) \). For certain particular choices of these quantities, we can study the physical implications of quantum deformation on the properties of various physical systems of interest.

For the field configuration, we consider a constant magnetic field along the z-direction (in cylindrical coordinates), \( B = B \hat{z} \), which is obtained from the vector potential (in the Landau gauge) (14).

\[
A = \frac{Br}{2} \varphi.
\]
A. Particle interacting with a harmonic oscillator

Because of applications to various physical systems, we consider the potential of a harmonic oscillator, \( \nu (r) = ar^2 \), where \( a \) is a constant. By adopting solutions of the form

\[
\psi_\pm = \left( \frac{\sum_m f_+ (r) e^{im\varphi}}{i \sum_m f_- (r) e^{i(m+s)\varphi}} \right),
\]

we arrive at radial equations

\[
\frac{d^2 f_\pm (r)}{dr^2} + \left( \frac{1}{r} + \varepsilon ar \right) \frac{df_\pm (r)}{dr} - \frac{(m^2 \pm 1)}{r^2} f_\pm (r) - \left( \omega^2 \right)^2 r^2 f_\pm (r) + k^\pm f_\pm (r) = 0,
\]

where \( k^+ = E^2 - M^2 + (m + s) eB - \varepsilon (2a + 3sa M + E M B) \), \( k^- = E^2 - M^2 + eB (m + s) + eS B - \varepsilon (2a - 3sa (m + s) + E M B) \), \( (\omega^+)^2 = \frac{e^2 B^2}{4} + 2 (M + E) a - \varepsilon \omega E M B / 2 \), \( (\omega^-)^2 = \frac{e^2 B^2}{4} + 2 (E - M) a - \varepsilon \omega E M B / 2 \), and \( m^+ = m + s \), with \( s = -m \).

By using solutions of the form

\[
f_\pm (\rho) = e^{-\frac{1}{2}(1+\kappa^\pm) \rho} \rho^{|m^\pm|} F_\pm (\rho), \quad \rho = \omega^2 r^2
\]

where \( \kappa^\pm = \varepsilon a / 2 \omega^\pm \), Eq. (19) becomes

\[
\rho \frac{d^2 F_\pm}{d\rho^2} + \left( 1 + |m^\pm| - \rho \right) \frac{dF_\pm}{d\rho} - \left[ \frac{1}{2} \left( 1 + |m^\pm| + \kappa^\pm \right) - \frac{k^\pm}{4\omega^\pm} \right] F_\pm = 0.
\]

Equation (20) is of the confluent hypergeometric equation type and its solution is given in terms of the Kummer functions. In this manner, the general solution for Eq. (19) is given by

\[
f_\pm (\rho) = c_1 e^{-\frac{1}{2}(1+\kappa^\pm) \rho} \rho^{|m^\pm|} M \left( \frac{1}{2} \left( 1 + |m^\pm| + \kappa^\pm \right) - \frac{k^\pm}{4\omega^\pm}, 1 + |m^\pm|, \rho \right)
\]

\[
+ c_2 e^{-\frac{1}{2}(1+\kappa^\pm) \rho} \rho^{-\frac{1}{2}|m^\pm|} M \left( \frac{1}{2} \left( 1 - |m^\pm| + \kappa^\pm \right) - \frac{k^\pm}{4\omega^\pm}, 1 - |m^\pm|, \rho \right),
\]

where \( M \) are the Kummer functions. In particular, when \( (1 + |m^\pm| + \kappa^\pm) / 2 - k^\pm / 4\omega^\pm = -n \), with \( n = 0, 1, 2, \ldots \), the function \( M \) becomes a polynomial in \( \rho \) of degree not exceeding \( n \). From this condition, we extract the energies for the spin and pseudospin symmetries limits, given respectively by

\[
E^2 - M^2 = 2 \sqrt{\frac{e^2 B^2}{4} + 2 (M + E) a - \frac{1}{2} \varepsilon \omega E M B} \times (2n + |m| + 1) - eB (m + s) + \varepsilon (3a + 3sa M + E M B),
\]

\[
E^2 - M^2 = 2 \sqrt{\frac{e^2 B^2}{4} + 2 (E - M) a + \frac{1}{2} \varepsilon \omega E M B} \times (2n + 1 + |m| + s) - eB (m + s) - \varepsilon (a - 3sa (m + s) + E M B).
\]

These energies are a relativistic generalization of the Landau levels in the context of quantum deformation. When \( a \) and \( \varepsilon \) are null, we obtain

\[
E^2 - M^2 = eB (2n + 1 + |m| - m - s),
\]

\[
E^2 - M^2 = eB (2n + 1 + |m| - m - s).
\]

which are the usual relativistic Landau levels with the inclusion of the element of spin.

B. Particle interacting with a linear potential

Let us consider the case where the particle interacts with a linear potential, \( ar \). In this case, we make \( w(r) = \nu (r) = ar \) (where \( a \) is a positive constant) in Eq. (14) to the limits of spin and pseudo-spin symmetries and proceed as before.

In the case of the spin symmetry limit, the resulting equation is given by

\[
\frac{d^2 f (r)}{dr^2} + \left( \frac{1}{r} + \varepsilon a \right) \frac{df (r)}{dr} - \frac{(m^2 + 1)}{r^2} f (r) - \omega^2 f (r) - \frac{\mu^+}{r} r f (r) - \frac{k^\pm}{\rho} f (r) + \frac{l^\pm}{\rho} f (r) = 0,
\]

with \( m^+ = m, \quad m^- = m + s, \quad \omega = eB / 2, \quad \mu^+ = 2 (E + M) a + 3a eB / 4, \quad \mu^- = 2 (E - M) a - 3a eB / 4, \quad k^+ = \varepsilon a (1 + 3a E M / 2), \quad k^- = \frac{1}{2} \varepsilon a (1 - 3a (m + s)), \quad l^+ = E^2 - M^2 + eB (m + s) - \varepsilon M E B \) and \( l^- = E^2 - M^2 + eB (m + s) + eS B (1 - \varepsilon M) \). In Eq. (27), the \((+,-)\) sign refer to spin and pseudo-spin symmetries, respectively. By performing the variable change, \( x = \sqrt{\omega} r \), Eq. (27) assumes the form

\[
\frac{d^2 f (x)}{dx^2} + \left[ 1 + \frac{1}{2} \kappa \right] \frac{df (x)}{dx} - \frac{m^2}{x^2} f (x) - x^2 f (x) - a^\pm_1 x f (x) - a^\pm_2 f (x) + \frac{l^\pm}{\omega} f (x) = 0,
\]

where we have defined the parameters \( \kappa = \varepsilon a / \sqrt{\omega}, \quad a^\pm_1 = \mu^\pm / \omega \sqrt{\omega}, \quad a^\pm_2 = k^\pm / \sqrt{\omega} \). Note that the choice \( w(r) = \nu (r) = ar \) induces a Coulomb-like interaction in the resulting eigenvalue equation. The origin of this Coulomb potential is due purely to the quantum deformation and boundary symmetries involved.
Equation (28) is of the Heun equation type, which is a homogeneous, linear, second-order, differential equation defined in the complex plane. This equation can be put into its canonical form using the solution

\[ f(x) = x^{m_{\pm}}e^{-\frac{x}{2}y(x)} \]

where \( y_+ \) satisfies the biconfluent Heun differential equation

\[ y'' + \left( \frac{\alpha_+ + 1}{x} - 2x - \beta_+ \right) y' + \left( \gamma_+ - \alpha_+ - 2 - \frac{1}{2x} \right) y = 0, \]

and \( \alpha_\pm = 2|m\pm|, \beta_\pm = a_L, \gamma_\pm = (\beta_\pm)^2/4 + l^2/\omega \) and \( \delta_\pm = \kappa/2 + 2a_L^2 \). Equation (30) has a regular singularity at \( x = 0 \), and an irregular singularity at \( \infty \) of rank 2. Usually, the solution of this equation is given in terms of two linearly independent solutions as

\[ y_+(x) = N(\alpha^+, \beta^+, \gamma^+, \delta^+; x) + x^{-\alpha^+}N(-\alpha^+, \beta^+, \gamma^+, \delta^+; x), \]

where (assuming that \( \alpha^+ \) is not a negative integer)

\[ N(\alpha^+, \beta^+, \gamma^+, \delta^+; x) = \sum_{q=0}^{\infty} A^+_q(\alpha^+, \beta^+, \gamma^+, \delta^+) x^q, \]

are the Heun functions. After the insertion of this solution into Eq. (33), we find (\( q \geq 0 \))

\[ A_0 = 1, \]

\[ A_1^\pm = \frac{1}{2} [\delta^\pm + \beta^\pm (1 + \alpha^\pm)], \]

\[ A_{q+2}^\pm = \left\{ (q + 1) \beta^\pm + \frac{1}{2} \delta^\pm + \beta^\pm (1 + \alpha^\pm) \right\} A_{q+1}^\pm - (q + 1) (q + 1 + \alpha^\pm) [\gamma^\pm - \alpha^\pm - 2 - 2q] A_q^\pm, \]

\[ (1 + \alpha^\pm)_q = \frac{\Gamma(q + \alpha^\pm + 1)}{\Gamma(\alpha^\pm + 1)}, \quad q = 0, 1, 2, 3, \ldots \]

From the recursion relation (35), the function \( N(\alpha^+, \beta^+, \gamma^+, \delta^+; x) \) becomes a polynomial of degree \( n \) if and only if the two following conditions are imposed:

\[ \gamma^\pm - \alpha^\pm - 2 = 2n, \quad n = 0, 1, 2, \ldots \]

\[ A_{n+1}^\pm = 0, \]

where \( n \) is a positive integer. In this case, the \((n+1)th\) coefficient in the series expansion is a polynomial of degree \( n \) in \( \delta^\pm \). When \( \delta^\pm \) is a root of this polynomial, the \( n+1 \)th and subsequent coefficients cancel and the series truncates, resulting in a polynomial form of degree \( n \) for \( N(\alpha^+, \beta^+, \gamma^+, \delta^+; x) \). From condition (37), we extract the energies at the spin and pseudo symmetries limit, given respectively by:

\[ E_{nm}^2 - M^2 = 2\omega (n + |m| + 1) - \frac{a^2}{\omega^2} (E_{nm} + M)^2 \]

\[ - \frac{3a^2}{2\omega} (E_{nm} + M) \varepsilon s + 2\omega [\varepsilon M s - (m + s)], \]

\[ E_{nm}^2 - M^2 = 2\omega (n + |m + s| + 1) - \frac{a^2}{\omega^2} (E_{nm} - M)^2 \]

\[ + \frac{3a^2}{2\omega} (E_{nm} - M) \varepsilon s - 2\omega [s (1 - \varepsilon M) + (m + s)]. \]

The equations (39) and (40) cannot represent the spectrum for the system in question. The energy of a physical system must be a function involving all the parameters present in the equation of motion. As expected, from the condition (37) alone one cannot predict the energy of the system. In the case of the energies above, this is justified by the absence of the parameters \( a_L^2 \). Moreover, it also does not provide the energy spectrum for all values of \( n \). On the other hand, by analyzing more carefully the condition (38), we see that it admits a natural application of (33), so that it is a necessary and sufficient condition for the derivation of the energies of the particle. Let us consider the solution (33) up to second-order in \( x \) of the expansion,

\[ N(\alpha^+, \beta^+, \gamma^+, \delta^+; x) = \frac{A_0}{(1 + \alpha^+)_0} + \frac{A_1^\pm}{(1 + \alpha^\pm)_1} x^\pm + \frac{A_2^\pm}{(1 + \alpha^\pm)_2} x^2 + \cdots. \]

By using the relation of recurrence (35) and Eqs. (33)-(34), the coefficient above \( A_0^\pm \) can be determined. If we\( ^\) want to truncate solution (33) in \( x \), we must impose that \( A_0^\pm = 0 \) through the condition (33); when we truncate in \( x^2 \), we make \( A_2^\pm = 0 \), and so on. For each of these cases, we have an associated energy. Thus, for \( A_0^\pm = 0 \), it means that we are investigating the particular solution for \( n = 0 \). Then, from Eq. (34), we have

\[ \frac{1}{2} (\delta^\pm + \beta^\pm \tilde{m}^\pm) = 0, \]

where \( \tilde{m}^\pm = 1 + \alpha^\pm \). Solving (33) for \( E \), we find the energies corresponding to the spin and pseudo-spin symmetries. They can be written explicitly as

\[ E_{0m} + M = \frac{\varepsilon \omega}{2 (1 + 2|m|)} \left[ 3s (|m| + m) + \frac{3}{2} (s + 1) \right], \]

\[ E_{0m} + M = \frac{\varepsilon \omega}{2 (1 + 2|m + s|)} \times \left[ 3s (|m + s| + m + s) + \frac{3}{2} (s - 1) \right]. \]
These energies, after imposing $\epsilon = 0$, lead to the ground state, $E_{0m} = -M$ and $E_{0m} = M$. Analogously, for $A_2^\pm = 0$, we get $(n = 1)$

$$\frac{1}{2} \beta^\pm (\delta^\pm + \beta^\pm \tilde{m}^\pm) + \frac{1}{4} (\delta^\pm + \beta^\pm \tilde{m}^\pm)^2 - 2\tilde{m}^\pm = 0. \quad (45)$$

In this relation, we can observe that the only parameter that depends on the energy $E$ is the parameter $a_2^\pm$ (through the parameter $\beta^\pm$). So, we only need to solve it for $a_2^\pm$. The result is given by

$$E_{1m} + M = \pm \frac{\omega}{a} \sqrt{\frac{2\omega}{2|m| + 3}}$$

$$- \frac{\epsilon\omega}{2} \left[ \frac{(1 + |m|) [1 + 2 (1 + 3sm)]}{(1 + 2|m|)(3 + 2|m|)} + \frac{3}{2} \right], \quad (46)$$

for the spin symmetry limit, and

$$E_{1m} - M = \pm \frac{\omega}{a} \sqrt{\frac{2\omega}{3 + 2|m| + s}}$$

$$- \frac{\epsilon\omega}{2} \left[ \frac{3 (1 + |m| + s)}{(1 + 2|m|))(3 + 2|m| + s)) - \frac{3}{2} \right], \quad (47)$$

for the pseudo-spin symmetry limit. If $\epsilon = 0$ in Eqs. (46)-(47), we find

$$E_{1m} + M = \pm \frac{\omega}{a} \sqrt{\frac{2\omega}{3 + 2|m|}}, \quad (48)$$

$$E_{1m} - M = \pm \frac{\omega}{a} \sqrt{\frac{2\omega}{3 + 2|m| + s}}. \quad (49)$$

The energies obtained from condition (37) (Eqs. (39)-(41)) together with those obtained from (38) (Eqs. (43)-(44)) and (40)-(41) specify the energy eigenvalues for the system governed by equation (17). However, we can verify that these energies are connected to each other through the parameters $\omega$ and $a$, so that we have a constraint on the energies. In particular, if we solve Eqs. (46)-(47) for $\omega$ and replace them in (38)-(44), we will have expressions for the energies involving the quantities $\alpha^\pm, \beta^\pm, \gamma^\pm, \delta^\pm$, which contains the Coulomb potential coupling constant $a_2^\pm$, the mass of the particle $M$, the effective angular momentum $m^\pm$ and the frequency $\omega$.

For a specific physical system described by Eq. (27), its corresponding energy spectrum are the modified Landau levels. In the absence of magnetic field, the energy eigenvalues are equivalent to those of a planar harmonic oscillator being corrected only by the parameter of quantum deformation. Because the spectrum of the system has the generalized form of the spectrum of a relativistic oscillator is more convenient to fix the frequency the frequency $\omega$ to give the energies corresponding to each value of $n$. It is an immediate calculation to solve the equations Eqs. (43)-(44) and (40)-(41) for $\omega$. For each specific frequency, $\omega_{0m}, \omega_{01}$, we have the following energies:

$$E_{0m}^2 - M^2 = 2\omega_{0m} (|m| + 1) - \frac{a^2}{\omega_{0m}^2} (E_{0m} + M)^2$$

$$- \frac{3a^2}{2\omega_{0m}} (E_{0m} + M) \epsilon s + 2\omega_{0m} [\epsilon Ms - (m + s)], \quad (50)$$

$$E_{0m}^2 - M^2 = 2\omega_{0m} (|m + s| + 1) - \frac{a^2}{\omega_{0m}^2} (E_{0m} - M)^2$$

$$+ \frac{3a^2}{2\omega_{0m}} (E_{0m} - M) \epsilon s - 2\omega_{0m} [s (1 - \epsilon M) + (m + s)], \quad (51)$$

and

$$E_{1m}^2 - M^2 = 2\omega_{1m} (|m| + 2) - \frac{a^2}{\omega_{1m}^2} (E_{1m} + M)^2$$

$$- \frac{3a^2}{2\omega_{1m}} (E_{1m} + M) \epsilon s + 2\omega_{1m} [\epsilon Ms - (m + s)], \quad (52)$$

$$E_{1m}^2 - M^2 = 2\omega_{1m} (|m + s| + 2) - \frac{a^2}{\omega_{1m}^2} (E_{1m} - M)^2$$

$$+ \frac{3a^2}{2\omega_{1m}} (E_{1m} - M) \epsilon s - 2\omega_{1m} [s (1 - \epsilon M) + (m + s)], \quad (53)$$

To determine the energies corresponding to $n = 2, 3, 4, \ldots$, we must make use of the above recipe. However, the polynomials of degree $n \geq 3$ resulting from condition (28), in general, only some roots are physically acceptable.

IV. CONCLUSIONS

We have studied the relativistic quantum dynamics of a spin-1/2 charged particle with minimal, vector and scalar couplings in the quantum deformed framework generated by the $\kappa$-Poincaré-Hopf algebra. The problem have been formulated using the $\kappa$-deformed Dirac equation in two dimensions. The $\kappa$-deformed Pauli equation was derived to study the dynamics of the system taking into account the spin and pseudospin symmetries limits. For the $\kappa$-deformed Dirac-Pauli equation obtained (Eq. (4)), we have argued that only particular choices of radial function $\nu (r)$ lead to exactly solvable differential equations. We have considered the case where the particle interacts with an uniform magnetic field, a planar harmonic oscillator and a linear potential. We have verified that the linear potential leads to a Coulomb-type term in the $\kappa$-deformed sector of the radial equation. The resulting equation obtained is a Heun-type differential equation. Analytical solutions for both spin and pseudospin symmetries limits enabled us to obtain expressions for the energy eigenvalues (through the use of the Eqs. (43) and (45)) and wave functions. Because of the limitations imposed by the condition (28), we have derived expressions for the energies corresponding only to $n = 0$ (Eqs. (54)-(55)) and $n = 1$ (Eqs. (56)-(57)). We have shown that
the presence of the spin element in the equation introduces a correction in the expressions for the bound state energy and wave functions.

ACKNOWLEDGMENTS

This work was supported by the Brazilian agencies CAPES, CNPq and FAPEMA.

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