Phase transitions in non-reciprocal active systems

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Crowds, flocks of sheep and robotic swarms are examples of active systems composed of agents that decide how to move according to the state of their neighbours [1–5]. The interaction between two such agents is not necessarily subject to physical constraints such as Newton’s third law — it can be visibly non-reciprocal, as we demonstrate using programmable robots. While non-reciprocal active media are known to exhibit exotic responses and wave phenomena [6–11], the nature of the phase transitions between their many-body phases remains elusive. Here, we show that microscopic non-reciprocity can persist at large scales and give rise to unique many-body phases and transitions controlled by singularities called exceptional points [12]. We illustrate this mechanism within a paradigmatic example: a binary fluid of self-propelled particles with non-reciprocal interactions. Our analysis reveals generic features of non-reciprocal matter ranging from exceptional-point enforced pattern formation to active time-(quasi)crystals. Besides active materials [2] and collective robotics [5, 13–15], our work sheds light on phase transitions in other non-reciprocal systems ranging from networks of neurons [16–18] to ecological predator–prey models [19, 20].

The interaction between agents with opposite objectives typically leads to frustration. While often overlooked in existing models of collective behavior, non-reciprocity is a common feature of active systems that arise in social sciences, biology and physics [21–27]. It can originate from synthetic physical interactions [28], biological reasons such as a limited vision cone [24, 26], leader-follower relationships [21, 23] or programmable robotic interactions [8, 15]. Here, we explore the consequences of non-reciprocal interactions for large-scale collective behavior by considering a minimal modification of the Vicsek model [29], an archetypal example of active matter. In this model, each agent $m$ moves on a plane at constant speed $v_0$ in a direction determined by an angle $\theta_m(t)$ that evolves according to the equation

$$\partial_t \theta_m = \sum_n J_{mn} \sin(\theta_n - \theta_m) + \eta(t) \tag{1}$$

where $J_{mn}$ is a coupling constant and $\eta(t)$ is a random noise with strength $\eta$. When $J_{mn}$ is positive (negative), the agent $m$ tries to align (anti-align) with the agent $n$. A common simplification consists in assuming that all agents interact with their neighbors reciprocally (i.e., $J_{mn} = J_{nm}$) and with the same strength (i.e., $J_{mn} = J$). When $J/\eta$ exceeds a critical value, an iconic non-equilibrium phase transition to flocking behavior occurs [29, 30]. Instead of moving in random directions, the agents spontaneously align and move together as a flock.

Within the framework of Eq. (1), non-reciprocity naturally arises when the agents belong to two different species $A$ and $B$, shown as blue and red robots in Fig. 1a. Were the interactions restricted to be reciprocal, two such agents could only align (if $J_{AB} = J_{BA} > 0$) or anti-align (if $J_{AB} = J_{BA} < 0$). Richer dynamical behaviors can occur when the agents do not want the same thing, i.e. $J_{AB} \neq J_{BA}$. For instance, when $J_{AB} = -J_{BA}$, agent A tries to align with agent B while agent B tries to anti-align with agent A. As the two agents are never satisfied, they continuously move in circles with the hope of reaching a better situation. This non-reciprocal behavior persists even when the self-propulsion speed $v_0$ is set to zero. In this case, Eq. (1) reduces to a non-reciprocal generalization of the XY model describing planar magnets [31].

We investigate this situation experimentally using programmable robots in lieu of non-reciprocal spins, see Fig. 1a and SI Movie. Each robot measures its absolute orientation using a magnetometer. This information is then communicated to the other robots through a wireless network. Based on their own measurements and the data they receive, the robots periodically update their orientations using motorized wheels. Figure 1b and the corresponding video in SI show the motion of two robots of different species A (blue) and B (red), with $J_{AB} = -J_{BA}$. The non-reciprocal robots start rotating in the same direction (set by initial conditions) with an approximately constant angular speed, see Fig. 1c. Unlike spin precession, the resulting dynamics cannot be described by a Hamiltonian because the couplings $J_{AB} = -J_{BA}$ are not symmetric. If self-propulsion were switched off, the robots would move in circular orbits. However, there is an important caveat: the time-dependent state of two agents (robots) is not stable at long times if any amount of reciprocal interactions is reinstated — the agents eventually align or anti-align (see SI). The effect of non-reciprocal interactions is apparently washed out at long times.

Can many-body effects stabilize these time-dependent states generated by non-reciprocal interactions? To an-

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swer this question, we performed molecular dynamics simulations of a binary mixture of many self-propelled agents described by Eq. (1), where $J_{mn}$ now depends on whether the agents $m$ and $n$ belong to species $A$ or $B$ (e.g., the blue and red robots in Fig. 1). We choose intra-species interactions that always promote alignment (i.e., $J_{AA} > 0$ and $J_{BB} > 0$). Depending on the values of $J_{AB}$, $J_{BA}$ and $\eta$, we observe several collective behaviors represented as snapshots in Fig. 2a-d (see also SI Movie). We observe (a) a disordered regime where the agents move in random directions (b) a flocking regime where both populations move in the same direction [29, 30], (c) an antiflocking regime where populations A and B move in opposite directions [32–34] and (d) a chiral collective behavior where the non-reciprocal agents move along trajectories approaching circles as their number increases.

Our numerical results hint at the existence of time-dependent many-body phases that are stable in the continuum limit. To study their nature and stability, we construct hydrodynamic equations for the densities $\rho_A$, $\rho_B$ and the velocity fields $\vec{v}_A$, $\vec{v}_B$ (strictly speaking, their polarizations) of a binary active fluid with non-reciprocal interactions, see Methods for full expressions and SI for a microscopic derivation. In the mean-field approximation, where all quantities are assumed to be uniform, the equations of motion reduce to the dynamical system

$$\partial_t \begin{pmatrix} \vec{v}_A \\ \vec{v}_B \end{pmatrix} = \begin{pmatrix} \alpha_A[\vec{v}_A, \vec{v}_B] \\ j_{BA} \alpha_B[\vec{v}_A, \vec{v}_B] \end{pmatrix} \begin{pmatrix} \vec{v}_A \\ \vec{v}_B \end{pmatrix}$$

(2)

where $j_{AB}$ and $j_{BA}$ are rescaled inter-species interactions while $\alpha_A$ and $\alpha_B$ are nonlinear functions of the velocities, see SI. Both the densities and the intra-species interactions are set to unity. We stress that the matrix in Eq. (2) is not symmetric when the inter-species interactions are non-reciprocal (i.e., $j_{AB} \neq j_{BA}$). The non-linearity in Eq. (2) allows the system to reach non-equilibrium steady-states.

Figure 2e-g shows the corresponding mean-field phase diagram as a function of the reciprocal and the non-reciprocal parts of the inter-species interactions $j_{\perp} = [j_{AB} \pm j_{BA}]/2$ respectively. This phase diagram exhibits a disordered phase (in gray), a flocking phase (in blue), an antiflocking phase (in red), and a chiral phase (in purple). In the SI, we prove that these phases are linearly stable against velocity fluctuations over large ranges of parameters. There is no direct transition between flocking and antiflocking phases, except when the non-reciprocal interactions $j_{\perp} = 0$ vanish; instead, another phase always appears in between. In addition to the chiral phase, we find a swap phase (green region in Fig. 2f,g) where $\vec{v}_A$ and $\vec{v}_B$ oscillate along a fixed direction and a phase where both swap and chiral motions coexist (dark green). We note in Fig. 2g the presence of a tetracritical point where the (anti)flocking, chiral, swap and mixed chiral/swap phases meet, whose origin is discussed in the SI. The non-reciprocity of the microscopic model, manifested by antisymmetric couplings in the hydrodynamic theory of Eq. (2), is responsible for the onset of the chiral phase through a peculiar phase transition mechanism unique to non-equilibrium systems.

The transition from the flocking to the chiral phase can be understood in terms of the transverse velocity fluctuations (i.e., the ones perpendicular to the (anti)flocking direction), see Fig. 2h. The crucial point is that an antisymmetric perturbation (for which the velocities $v_A$ and $v_B$ rotate in opposite directions, see $\delta V^\perp_{\perp}$ in Fig. 2h) does not generate a purely antisymmetric response in the presence of non-reciprocity – the restoring force on $v_A$ is not equal in magnitude to the force on $v_B$. Instead, a symmetric response is also generated that consists in a rotation of $v_A$ and $v_B$ in the same direction. By contrast, a symmetric perturbation does not lead to any restoring force (irrespective of non-reciprocity), because it corresponds to the Goldstone mode of broken rotational symmetry. As the reciprocal coupling $j_+ \rightarrow 1$ becomes smaller and smaller compared to the non-reciprocal coupling $j_-$ (see Fig. 2i), it reaches a critical value where all transverse perturbations lead to a symmetric (solid-body) rotation of $v_A$ and $v_B$ with a fixed angle between them. This heralds the onset of the chiral phase.

To elucidate the mathematical origin of the transition mechanism, we linearize Eq. (2) by separating the velocities into a steady-state component and the fluctuations around it. In momentum space, we obtain the linear equation

$$\partial_t \begin{pmatrix} \delta \vec{v}_A \\ \delta \vec{v}_B \end{pmatrix} = \begin{pmatrix} L_\parallel(k) & 0 \\ 0 & L_\perp(k) \end{pmatrix} \begin{pmatrix} \delta \vec{v}_A \\ \delta \vec{v}_B \end{pmatrix}$$

(3)

where we have decomposed the velocity fluctuations $\delta \vec{V} = (\delta \vec{v}_A, \delta \vec{v}_B)$ into transverse and longitudinal components $\delta \vec{V}_\perp$ and $\delta \vec{V}_\parallel$. To lowest order in wavevector $k$, these components are decoupled [1, 35]. The phase transition from the flocking to the chiral phase described above occurs when the two eigenmodes of the linear operator $L_{\parallel}(k)$ describing the relaxation of transverse fluctuations become collinear. This corresponds to a so-called exceptional point [12] of $L_{\parallel}(k)$. This coalescence of the modes of a many-body system defines a class of dynamical phase transitions that we dub exceptional transitions.

We contrast the exceptional transitions to the standard mechanism of phase transition that underlies, for instance, the transition from flocking to disordered phases (which is formally equivalent to a ferromagnet/paramagnet transition). This mechanism can also lead to time-dependent phases: Fig. 3a illustrates the flocking/swap transition which is controlled by longitudinal fluctuations. These fluctuations are less and less damped at $k = 0$, until at the critical point they are completely suppressed: this marks the onset of the transition. The growth rate $\sigma_{\parallel}(k = 0)$ of these fluctuations simply changes sign from negative to positive at the transition. By contrast, the exceptional flocking/chiral transition involves the coalescence of two eigenmodes of $L_{\perp}(k)$, see Fig. 3b. As in the standard mechanism, there is an eigenmode of $L_{\perp}(k)$ corresponding to fluctuations which are
damped away from the transition. However, its growth rate $\sigma_\perp(0)$ cannot simply change sign, because of the existence of the Goldstone mode of broken rotational invariance (see SI). Instead, the damped mode coalesces with the Goldstone mode at the exceptional point [36, 37], and this triggers the phase transition. This transition is similar to PT-unbroken/PT-broken transitions in non-Hermitian quantum mechanics [11] (see SI). There is excellent agreement between the phase boundaries predicted analytically from the exceptional points (red lines in Fig. 2g) and the numerical phase diagram up to the tetracritical points (marked by black dots) where the prediction loses any relevance (dashed red lines).

Both the chiral and swap phases exhibit a spontaneous breaking of time translation symmetry, in a way similar to time crystals [38–43]. The swap phase spontaneously breaks an additional symmetry: rotational invariance. The chiral phase, instead, spontaneously breaks parity symmetry under $(x, y) \mapsto (x, -y)$, as the direction of rotation is chosen at random. This direction is the same for both populations, and the angle between the populations continuously interpolates between the flocking and anti-flocking phases. This phase is reminiscent of many-body chiral states observed in spinor Bose gases [44, 45], as well as chiral active fluids [46–50] where parity is however explicitly broken by the drive. The mixed chiral/swap phase also spontaneously breaks time translation symmetry, but it exhibits quasiperiodic dynamics reminiscent of time quasicrystals [51, 52]. This is revealed by the frequency spectrum of $v_A(t)$, $v_B(t)$ in Fig. 3c. In the time-independent flocking phase, the only frequency present is $f = 0$. The chiral phase corresponds to a solid-body rotation of the velocities. In this phase, we observe a single curve, which gives the rotation frequency. In the swap phase, the motion is not harmonic: we observe multiple branches, that correspond to a fundamental frequency and some of its harmonics. In contrast, the mixed chiral/swap phase exhibits two branches with two fundamental frequencies (which are unrelated), as well as the harmonics of both frequencies. As a consequence, the motion is quasiperiodic instead of periodic.

The existence of exceptional points in the phase diagram is not merely a mathematical curiosity: besides controlling the topology of the phase diagram, it leads to pattern formation (i.e., instabilities at finite wavelength) near the exceptional phase boundaries (see Fig. 3b and Fig. 4). To understand why, we present a simplified model of the linear excitations on top of the (anti)flocking steady-state, whose time evolution we assume is ruled by the equation

$$\partial_t \delta \vec{V} = [L_{\text{EP}} + M(\vec{V}_\text{ss} \cdot \nabla)] \delta \vec{V}$$

(4)

This general model captures two essential ingredients of non-reciprocal hydrodynamics beyond our specific system. First, the excitation spectrum at the transition exhibits an exceptional point in the static limit ($k \to 0$ and $\omega \to 0$), which we capture with a singular matrix $L_{\text{EP}}$. This situation alone is unique to non-equilibrium systems and generically leads to anomalous fluctuations of the velocities [20, 36, 37, 53]. Here, we add a second crucial ingredient common in fluids: the steady-state exhibits a flow with constant velocity $\vec{v}_\text{ss}$, as observed in the (anti)flocking phases. Hence, we add a linearized convective term of the form $M(\vec{v}_\text{ss} \cdot \nabla)$ to describe the excitations at finite wavelength, where $M$ is a matrix mixing the two species $A$ and $B$. In momentum space, we are left with an exceptional point perturbed with a term of the form $ik_f M$ where $k_f$ is the wavevector parallel to the steady-state velocity $\vec{v}_\text{ss}$.

Since the eigenvalues of a perturbed singular matrix typically diverge as a square root of the perturbation, we expect a complex growth rate $\sigma_\pm \approx \pm \sqrt{k_f}$ for long wavelength modes (see Fig. 4c and SI). Hence, exceptional transitions must be accompanied by pattern forming instabilities in systems where convective terms are important, provided that a current exists in the steady-state. In Fig. 4a, we show how the phase diagram of Fig. 2g is modified in the flocking and antiflocking phases when these instabilities are taken into account (we also show the regions of stability for the chiral phase). Around the critical lines marking the mean-field phase transition, we find regions (indicated by bright red and blue) where pattern formation occurs. In Fig. 4b, we show how the growth rate of the perturbations depends on the wavevector $k$ (see also Fig. 4c, middle panel). The growth rate vanishes at large wavelength and exhibits a positive maximum $\sigma^*$ at a finite wavevector $k^*$. Hence, the perturbations are amplified until non-linearities stop their growth. Close to the onset of the pattern-forming instability, $k^*$ controls the characteristic wavelength of the patterns.

Our work highlights the relevance of the framework of non-Hermitian hydrodynamics for active matter and collective robotics, and it suggests strong analogies between these fields and open quantum systems [36, 37, 54]. It also sheds light on phase transitions in other contexts where non-reciprocity can occur, including non-reciprocal generalizations of synchronization [55–59], ecological predator-prey models [19, 20] and non-reciprocal neural networks [17, 18, 60–62]. For instance, neurons with non-reciprocal interactions can perform computations using their collective dynamics rather than energy minimization, unlike Hamiltonian neural networks that can only find static solutions.
Fig. 1. Robots with non-reciprocal interactions. (a) A collection of agents represented by robots separated in two populations $A$ (in blue) and $B$ (in red) interact through a Vicsek-like alignment rule. The interactions between members of the same population, represented by the coupling constants $J_{AA}$ and $J_{BB}$, are reciprocal. The interactions between members of different populations, represented by the coupling constants $J_{AB}$ and $J_{BA}$, are not necessarily reciprocal. (b) A non-reciprocal interaction leads to dynamical frustration: the two agents are never satisfied by their current state and evolve. We demonstrate this behavior with two programmable robots (GoPiGo3, Dexter Industries), see SI Movie and Methods. (c) The angles $\theta_A(t)$ and $\theta_B(t)$ made by the robots with a fixed direction (as measured by the robots through a magnetometer) are plotted as a function of time. The two angles increase approximately linearly, which corresponds to approximately uniform rotations.
Fig. 2. Mean-field phase diagram and many-body exceptional points. (a-d) Snapshots of molecular dynamics simulations of the particle-based model in different phases (see also SI Movies). (e-g) Slices of the phase diagram for different values of the noise strength $\eta$. The red (resp. black) lines correspond to the analytical phase transition lines from the (anti)flocking phase to the chiral (resp. disordered) phase. In (g), the analytical prediction for the (anti)flocking/chiral transition is valid only up to tetracritical points marked by black dots, after which a (anti)flocking/swap transition occurs. (h) Schematic showing how non-reciprocal interactions lead to an exceptional point (EP). Symmetric perturbations of the velocities lead to no restoring forces because the global rotation is a Goldstone mode. In a non-reciprocal system, the restoring force (black arrow) on $\vec{v}_A$ is different from the one on $\vec{v}_B$. This is a combination of an antisymmetric forces plus a global rotation (purple arrow). When the imbalance is big enough, at the EP, the antisymmetric perturbation entirely converts to a (symmetric) global rotation. (i) Zoom on the phase diagram (g) [rotated by 90°]. With non-reciprocal interactions ($j^- \neq 0$), the linear eigenmodes of the velocities fluctuations (black arrows) are not orthogonal, and coalesce at the phase boundaries of an intermediate chiral phase that mediates the transition between flocking and antiflocking. These modes are represented in the basis of symmetric solid-body rotations $\delta V^s_\perp$ (the Goldstone mode of broken rotation symmetry) and antisymmetric clapping motion $\delta V^a_\perp$. The phase diagrams are determined by solving Eq. (2) numerically from random initial conditions, with $\rho_A = \rho_B = 1$, $j_{AA} = j_{BB} = 1$, and (e) $\eta/\eta_c = 1.5$, (f) $\eta/\eta_c = 0.99$, (g) $\eta/\eta_c = 0.5$. 

(a) disordered
(b) flocking
(c) antiflocking
(d) chiral

(e) high noise $\eta > \eta_c$
(f) critical noise $\eta = \eta_c$
(g) low noise $\eta < \eta_c$

\[ j^- = \frac{[j_{AB} - j_{BA}]}{2} \]

\[ j^+ = \frac{[j_{AB} + j_{BA}]}{2} \]
(a) Schematic plot of the growth rate of longitudinal perturbations $\sigma_\parallel(k)$ at the swap transition. The transition from (anti)flocking to the swap phase originates from the longitudinal channel through a standard mechanism: a massive (i.e., damped) mode become massless (undamped). The transition occurs when the growth rate $\sigma_\parallel(k = 0)$ becomes positive. This is the same mechanism at play in the ferromagnet/paramagnet transition in an Ising magnet (and in the flocking/disorder transition). (b) Schematic plot of the growth rate of transverse perturbations $\sigma_\perp(k)$ at the chiral transition. The exceptional transition from (anti)flocking to the chiral phase originates from a mechanism unique to non-equilibrium system, through the coalescence of a massive (i.e., damped) mode with the Goldstone mode of broken rotational invariance. (c) Plot of the frequencies present in the steady-state solution as a function of $j_+$, for $j_- = -0.6$. In the chiral phase, a single frequency is present in the spectrum (at each point), which corresponds to the solid-body rotation. In the swap phase, a single frequency accompanied by harmonics are present. In contrast, in the mixed chiral/swap phase, two independent frequencies are present (with their harmonics). These frequencies are unrelated (they are not harmonics of each other), leading to a quasiperiodic phase similar to a time quasicrystal. Inset: for $j_- = -0.25$, a direct transition between flocking and chiral phases is observed. In all cases, the frequency changes continuously at the transitions. We have used the same parameters as in figure 2(g).
Fig. 4. Exceptional-point enforced pattern formation. The combination of the singular operators at an exceptional point (where phase transitions occur) with convective terms gives rise to pattern formation near the transition lines. This singularity-enforced pattern-forming instability occurs whenever the convective terms lift the defectiveness of the linearized operator. (a) Numerical phase diagram including the linear stability analysis of the (anti)flocking and chiral phases. (b) Normalized growth rate $\sigma(k)/\sigma^*$ as a function of wavevector, for different values of $j_+$ at fixed $j_-$ [along the dashed line in (a)]. A maximum growth rate $\sigma^*$ (computed independently for each $j_+$) is found at finite wavevector $k^*$. The value of $k^*$ gives an estimation of the wavelength of the pattern near the onset. (c) Mechanism of the singularity-enforced instability. The presence of an exceptional point, combined with a linearized convective term, leads to a growth rate of the form $\sigma_\pm \simeq \pm \sqrt{\epsilon k}$ at the transition. Near the transition, this implies the existence of finite-momentum instabilities (see also Fig. 3b). We have used the same parameters as in figure 2 with $\eta/\epsilon = 0.5$, $v_\lambda^0 = 0.06$ and $v_B^0 = 0.01$. 
METHODS

EXPERIMENTAL DEMONSTRATION WITH PROGRAMMABLE ROBOTS

We demonstrate the effect of non-reciprocal interactions using programmable robots evolving according to a modified version of Eq. (1). The main differences are that (i) the evolution is discrete in time, (ii) the term \( \sin(\theta_n - \theta_m) \) is replaced by \( \text{sign} \sin(\theta_n - \theta_m) \) and (iii) we do not add artificial noise. Hence, Eq. (1) is replaced by

\[
\theta_m(t + T_{\text{mv}}) = \sum_n [J_{mn} T_{\text{mv}}] \text{sign} \sin(\theta_n(t) - \theta_m(t))
\]

In practice, additional differences such as delays and noises are also present due to imperfections in the implementation. This motivates the lack of artificial noise. We use two programmable robots (GoPiGo3, Dexter Industries). Each robot is connected to a magnetic sensor (Bosh BNO055 packaged in Dexter Industries IMU Sensor) as a compass to measure its absolute orientation. The magnetic sensor is attached to the body of the robot at a distance from the motors to reduce electromagnetic interferences. Each robot communicates its respective orientation to the other via Wi-Fi every \( T_{\text{ms}} = 0.1 \text{s} \). The communication is implemented through a central server, which allows to easily record the angles of each robot as a function of time (see Fig. 1c). Every \( T_{\text{mv}} = 0.5 \text{s} \), each robots computes the left hand side of the modified Eq. (1) described above, and actuates its two motors with opposite angular velocities for a given time to perform a rotation of \( \pm \theta_0 \) where \( \theta_0 = 15^\circ \), depending on the result of the computation. The change in angle is not instantaneous, because the rotation speed of the motors cannot be arbitrarily large. (Orientation measurements and communication are not instantaneous either, but they are much faster.) Hence, we have chosen to make new decisions only at discrete times. Performing rotations with very small angles (lower than \( \sim 4^\circ \)) is not effective; this is compensated by the replacement of \( \sin(\theta_n - \theta_m) \) by \( \text{sign} \sin(\theta_n - \theta_m) \) to avoid the presence of arbitrarily small angle increments. The robots and the server are implemented in Python, using the GoPiGo3 Python package (version 1.2.0) to control the robots, the DI_Sensors package (version 1.0.0) to access the magnetometer data and ZeroMQ (version 4.3.2) as a messaging library.

MOLECULAR DYNAMICS SIMULATIONS

We perform simple molecular dynamics simulations of a moderately large number of active agents following Eq. (1) in order to visually demonstrate the disordered, flocking, antiflocking, and chiral behaviours, as shown in Fig. 2a-d and SI Movie.

We simulate \( N \) agents following Eq. (1) discretized using the Euler–Maruyama scheme with a timestep \( \delta t \), with a ratio \( N_A/N_B \) between populations A and B, in a \( L \times L \) box with periodic boundary conditions, for a duration \( T_{\text{sim}} \). We choose the couplings in Eq. (1) to be \( J_{mn} = J_s(m)nH||r_i - r_j|| - R_0 \) where \( s(n) \) represents the species of particle \( n \) (A or B) and \( H \) is the Heaviside function. We set \( N = 512, N_A/N_B = 1, v_0 = 0.5, R_0 = 2, L = 8, T_{\text{sim}}/\delta t = 8000 \) with \( \delta t = 0.01 \). Figure 2a-d and SI Movie show simulations exhibiting (a) disordered, (b) flocking, (c) antiflocking and (d) chiral behaviours. For these simulations, the noise is (a) \( \eta = 200 \times 10^{-2} \) and (b,c,d) \( \eta = 2 \times 10^{-2} \) and the coupling matrices \((J_{AA}, J_{AB}, J_{BA}, J_{BB})\) are (a) \( 1 \times 10^{-4} \times (1,1;1,1) \); (b) \( (1,1;1,1) \); (c) \( (1,1,-1,-1) \); (d) \( 0.39 \times (1,-0.25;0.25,1) \).

HYDRODYNAMIC THEORY

We have derived hydrodynamic equations for the densities \( \rho_n(t,r) \) and polarizations \( \vec{P}_n(t,r) \) (or equivalently the velocities \( \vec{v}_n(t,r) \); in the main text, we denote the polarization fields by \( \vec{v}_n(t,r) \) for simplicity) of an arbitrary number of populations from Eq. (1). Our derivation, presented in the SI, follows the methods described in Refs. [1, 23, 63–67]. The set of hydrodynamic equations obtained generalize the Toner-Tu equations [30, 68] to several populations with non-reciprocal interactions, and are the basis of the analysis in the main text.

Our results for two populations also generalize the situation considered in Ref. [23], which considers aligners A (standard Vicsek-like self-propelling particles) and dissenters B that do not align at all with anyone (neither A or B), but with which the population A aligns. With our notations, this corresponds to \( J_{AA}, J_{AB} > 0 \) but \( J_{BB} = J_{BA} = 0 \).

Several methods of deriving continuum hydrodynamic equations from microscopics were applied to active matter, going from (i) approaches based on the Fokker–Planck (Smoluchowski) equation for the hydrodynamic variables [1, 63, 66], to (ii) kinetic theory approaches based on the Boltzmann equation [64, 65, 69], or (iii) directly from the Chapman-Kolmogorov equation [70] (in increasing order of complexity). Although coarse-graining microscopic models provides
invaluable qualitative insights on the behaviour of the system, even current state-of-the-art coarse-graining procedures only provide a qualitative agreement, at best semi-quantitative, with the microscopic starting point [67, 71]. With this in mind, we use the easiest coarse-graining method (i) along with several simplifying approximations (see SI). This procedure has the benefit of simplicity and allows to highlight the key features of a non-reciprocal multi-component fluid. However, the correspondence between the resulting hydrodynamic equations and the microscopic model is only qualitative, in the sense that the values of the coefficients might be inaccurate.

Here, we write the hydrodynamic equations for two populations $A$ and $B$. We refer to the SI for the general case of an arbitrary number of populations and its derivation from the microscopic equations.

The continuity conservation equation simply reads

$$\partial_t \rho_a + v_a^0 \text{div}(\vec{P}_a) = 0$$  \hspace{1cm} (6)

for $a = A, B$. The equation of motion for the polarizations reads

$$\partial_t \vec{P}_A = \left[ j_{AA} \rho_A - \frac{1}{2\eta} \| j_{AB} \rho_B + j_{AB} \vec{P}_B \|^2 \right] \vec{P}_A + j_{AB} \rho_A \vec{P}_B - \frac{v_a^0}{2} \vec{\nabla} \rho_A + D_A \vec{\nabla}^2 \vec{P}_A + \lambda_{AA} \left[ \frac{5}{2} \vec{\nabla} (\vec{P}_A \cdot \vec{\nabla} \vec{P}_A) - 3 (\vec{P}_A \cdot \vec{\nabla}) \vec{P}_A - 5 \vec{P}_A \text{div}(\vec{P}_A) \right]$$

\hspace{1cm} + \lambda_{AB} \left[ (\vec{P}_B \cdot \vec{\nabla}) \vec{P}_A - 2 (\vec{P}_A \cdot \vec{\nabla}) \vec{P}_B - 2 \vec{P}_B \text{div}(\vec{P}_A) + (\vec{P}_B \cdot \vec{\nabla}) \vec{P}_A^* + 2 (\vec{P}_A^* \cdot \vec{\nabla}) \vec{P}_B^* + 2 \vec{P}_B^* \text{div}(\vec{P}_A) \right] \hspace{1cm} (7)

The equation for $B$ is obtained by exchanging the indices. In this equation, the notation $U^*$ denotes the 2D vector $U = (U_x, U_y)$ rotated by 90 in the clockwise direction, namely $U^* = (U_y, -U_x)$. The polarizations denoted by $\vec{P}_A$ and $\vec{P}_B$ here and in the SI are called $\vec{v}_A$ and $\vec{v}_B$ in the main text. The hydrodynamic parameters in Eq. (7) are related to the microscopic parameters, see SI.

SUPPLEMENTARY INFORMATION

I. MICROSCOPIC AND HYDRODYNAMIC DESCRIPTIONS OF NON-RECIPROCAL MULTI-COMPONENTS ACTIVE FLUIDS

In this section, we describe a microscopic model of active self-propelled particles inspired by the Vicsek model [29], in which several populations of aligning self-propelled particles interact. The coupling between individuals belonging to different populations is not necessarily reciprocal. This model is defined by Eq. (S1). Using the methods described in references [23, 63–66] and summarized in the reviews [1, 67], we perform a coarse-graining of this microscopic model to obtain a set of hydrodynamic equations generalizing the Toner-Tu equations [30, 68], which is the basis of the analysis in the main text. The main results are Eq. (S42) and Eq. (S51), which are respectively the hydrodynamic equations for the densities of the active particles and for their polarization fields. These equations describe an arbitrary number of species. In the main text, we focus on the case of two species described by Eq. (S62). In this SI, we denote the polarization field $\vec{P}_a$, which is called $\vec{v}_a$ in the main text.

Several methods of deriving continuum hydrodynamic equations from microscopics were applied to active matter, going from (i) approaches based on the Fokker–Planck (Smoluchowski) equation for the hydrodynamic variables [1, 63, 66], to (ii) kinetic theory approaches based on the Boltzmann equation [64, 65, 69], or (iii) directly from the Chapman–Kolmogorov equation [70] (in increasing order of complexity). Although coarse-graining microscopic models provides invaluable qualitative insights on the behaviour of the system, even current state-of-the-art coarse-graining procedures only provide a qualitative agreement, at best semi-quantitative, with the microscopic starting point [67, 71]. With this in mind, we use the easiest coarse-graining method (i) along with several simplifying approximations (see SI). This procedure has the benefit of simplicity and allows to highlight the key features of a non-reciprocal multi-component fluid. However, the correspondence between the resulting hydrodynamic equations and the microscopic model is only qualitative, in the sense that the values of the coefficients might be inaccurate.

A. Microscopic particle-based model

Let us consider $N_{\text{pop}}$ populations $a = 1, \ldots, N_{\text{pop}}$ of $N_a$ active particles moving in a plane. Each particle is described by a position $r_i^a$ and an angle $\theta_i^a$, with $i = 1, \ldots, N_a$. The dynamics of the population is described by the
set of equations

\[ \dot{r}_i^a(t) = v_0^a \tilde{n}[\theta_i^a(t)] \]  
\[ \dot{\theta}_i^a(t) = \eta_i^a(t) + \sum_{b} \sum_{j=1}^{N_a} J_{ij}^{ab} \sin[\theta_j^b(t) - \theta_i^a(t)] \]

where we have defined

\[ \tilde{n}(\theta) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \]

and where \( \eta_i^a(t) \) are Gaussian white noises with \( \langle \eta_i^a(t) \rangle = 0 \) and

\[ \langle \eta_i^a(t) \eta_{j}^{b}(t') \rangle = 2 \eta_0 \delta_{ij} \delta^a b \delta(t - t'). \]

We set

\[ J_{ij}^{ab} = J^{ab} H(R_0 - \|r_i - r_j\|). \]

where \( H \) is the Heaviside step function. In the derivation of the hydrodynamic model, we will simplify the analysis by replacing the Heaviside functions by Dirac distributions.

**B. Coarse-graining of the microscopic model to hydrodynamic equations**

It will be convenient to write the equations of motion of section 1A in a slightly more general form as

\[ r_i^a = A_i^a(r_i^a, \theta_i^a) + \sum_{b} \sum_{j=1}^{N_a} B_{ij}^{ab}(r_i^a, \theta_i^a, r_j^b, \theta_j^b) \]  
\[ \dot{\theta}_i^a = A_{\theta i}^a(r_i^a, \theta_i^a) + \sum_{b} \sum_{j=1}^{N_a} \dot{B}_{ij}^{ab}(r_i^a, \theta_i^a, r_j^b, \theta_j^b) + \eta_i^a(t). \]

For our equations (S1), we have

\[ A_i^a(r, \theta) = v_0^a n(\theta) \quad \text{and} \quad A_{\theta i}^a(r, \theta) = 0 \]

while

\[ B_{ij}^{ab}(r, \theta, r', \theta') = 0 \quad \text{and} \quad \dot{B}_{ij}^{ab}(r, \theta, r', \theta') = J^{ab} H(R_0 - \|r_i - r_j\|) \sin(\theta' - \theta). \]

In order to obtain hydrodynamic equations, we first define the (stochastic) single-particle distributions

\[ c_i^a(r, \theta, t) = \frac{1}{N_a} \sum_{i=1}^{N_a} \delta(r - r_i^a(t)) \delta(\theta - \theta_i^a(t)). \]

They are the sum of the individual densities

\[ c_i^a(r, \theta, t) = \delta(r - r_i^a(t)) \delta(\theta - \theta_i^a(t)). \]

We follow the procedure of Ref. [63] to obtain a Langevin equation for this quantity. To do so, let us first compute the time derivative of the individual densities

\[ \frac{\partial}{\partial t}[c_i^a(r, \theta, t)] = \frac{\partial}{\partial t}[\delta(r - r_i^a(t))] \delta(\theta - \theta_i^a(t)) + \delta(r - r_i^a(t)) \frac{\partial}{\partial t}[\delta(\theta - \theta_i^a(t))] \]

so using Itô lemma,

\[ \frac{\partial}{\partial t}[c_i^a(r, \theta, t)] = [-\nabla_r \delta(r - r_i^a(t)) \cdot \dot{r}_i^a(t)] \delta(\theta - \theta_i^a(t)) \]

\[ + \delta(r - r_i^a(t)) [-\nabla_{\theta} \delta(\theta - \theta_i^a(t)) \cdot \dot{\theta}_i^a(t) + \eta_i^a(t) \nabla_{\theta} \delta(\theta - \theta_i^a(t))] \].
There is no diffusive term in the position equation, because there is no noise in the corresponding equation of motion \((S1a)\).

Let us now consider an arbitrary function \((r, \theta) \mapsto f(r, \theta)\). We indeed have

\[
\int \text{d}r \text{d} \theta \ c^a_i(r, \theta, t) f(r, \theta). \tag{S12}
\]

Hence,

\[
\frac{\text{d}}{\text{d}t} [f(r^a_i(t), \theta^a_i(t))] = \int \text{d}r \text{d} \theta \ \frac{\partial}{\partial t} c^a_i(r, \theta, t) f(r, \theta) \tag{S13}
\]

so we also have

\[
\frac{\text{d}}{\text{d}t} [f(r^a_i(t), \theta^a_i(t))] = \int \text{d}r \text{d} \theta [-(\nabla_r \delta)(r - r^a_i(t)) \cdot \dot{r}^a_i(t)] \delta(\theta - \theta^a_i(t)) \\
+ \delta(r - r^a_i(t)) \cdot [-((\nabla_\theta \delta)(\theta - \theta^a_i(t)) \cdot \dot{\theta}^a_i(t)] f(r, \theta) \tag{S14}
\]

By integration by part and replacing the Dirac distributions with \(c^a_i(r, \theta, t)\), we obtain

\[
\frac{\text{d}}{\text{d}t} [f(r^a_i(t), \theta^a_i(t))] = \int \text{d}r \text{d} \theta \left[ (\nabla_r f) \cdot \dot{r}^a_i(t) + (\nabla_\theta f)(r, \theta) \cdot \dot{\theta}^a_i(t) + \eta(\nabla^2 f)(r, \theta) \right] c^a_i(r, \theta, t). \tag{S15}
\]

Replacing the time derivatives with their values given by the equations of motion \((S5)\) yields

\[
\frac{\text{d}}{\text{d}t} [f(r^a_i(t), \theta^a_i(t))] = \int \text{d}r \text{d} \theta (\nabla_r f) \cdot \left( A^a_r(r^a_i, \theta^a_i) + \sum_{b} \sum_{j=1}^{N_b} B^{ab}_r(r^a_i, \theta^a_i, r^b_j, \theta^b_j) \right) c^a_i(r, \theta, t) \\
+ (\nabla_\theta f)(r, \theta) \cdot \left( A^a_\theta(r^a_i, \theta^a_i) + \sum_{b} \sum_{j=1}^{N_b} B^{ab}_\theta(r^a_i, \theta^a_i, r^b_j, \theta^b_j) + \eta^a_i(t) \right) c^a_i(r, \theta, t) \tag{S16}
\]

After integration by parts,

\[
\frac{\text{d}}{\text{d}t} [f(r^a_i(t), \theta^a_i(t))] = \int \text{d}r \text{d} \theta \ f(r, \theta) \left( -\nabla_r \cdot \left( A^a_r(r^a_i, \theta^a_i) + \sum_{b} \sum_{j=1}^{N_b} B^{ab}_r(r^a_i, \theta^a_i, r^b_j, \theta^b_j) \right) c^a_i(r, \theta, t) \right) \\
- \nabla_\theta \cdot \left( A^a_\theta(r^a_i, \theta^a_i) + \sum_{b} \sum_{j=1}^{N_b} B^{ab}_\theta(r^a_i, \theta^a_i, r^b_j, \theta^b_j) \right) c^a_i(r, \theta, t) \tag{S17}
\]

Comparing with \((S13)\), we obtain

\[
\frac{\partial}{\partial t} c^a_i(r, \theta, t) = \left( -\nabla_r \cdot \left( A^a_r(r^a_i, \theta^a_i) + \sum_{b} \sum_{j=1}^{N_b} B^{ab}_r(r^a_i, \theta^a_i, r^b_j, \theta^b_j) \right) c^a_i(r, \theta, t) \right) \\
- \nabla_\theta \cdot \left( A^a_\theta(r^a_i, \theta^a_i) + \sum_{b} \sum_{j=1}^{N_b} B^{ab}_\theta(r^a_i, \theta^a_i, r^b_j, \theta^b_j) \right) c^a_i(r, \theta, t) \tag{S18}
\]
Summing over \( i = 1, \ldots, N_0 \) and replacing \( r^a_i \to r \) and \( \theta^a_i \to \theta \) as allowed by the Dirac distributions gives

\[
\frac{\partial}{\partial t} [c^a(r, \theta, t)] = -\nabla_r \cdot \left( \left( A^a_0(r, \theta) + \sum_b \sum_{j=1}^{N_0} B^{ab}_0(r, \theta, r_j^b, \theta_j^b) \right) c^a(r, \theta, t) \right) -\nabla_\theta \cdot \left( A^a_0(r, \theta) + \sum_b \sum_{j=1}^{N_0} B^{ab}_0(r, \theta, r_j^b, \theta_j^b) \right) c^a(r, \theta, t) -\nabla_\theta \cdot \left[ \sum_i \eta^a_i(t) c^a_i(r, \theta, t) \right] + \eta \nabla_\theta^2 c^a(r, \theta, t). \tag{S19}
\]

Using again that for an arbitrary function \( f \)

\[
\int dr'd\theta' f(r', \theta', z)c^a_j(r', \theta') = f(r_j^0, \theta_j^0, z) \tag{S20}
\]

\[
\frac{\partial}{\partial t} [c^a(r, \theta, t)] = -\nabla_r \cdot \left( \left( A^a_0(r, \theta) + \sum_b \int dr'd\theta' B^{ab}_0(r, \theta, r', \theta') c^b(r', \theta'), t) \right) c^a(r, \theta, t) \right) -\nabla_\theta \cdot \left( A^a_0(r, \theta) + \sum_b \int dr'd\theta' B^{ab}_0(r, \theta, r', \theta') c^b(r', \theta'), t) \right) c^a(r, \theta, t) -\nabla_\theta \cdot \left[ \sum_i \eta^a_i(t) c^a_i(r, \theta, t) \right] + \eta \nabla_\theta^2 c^a(r, \theta, t). \tag{S21}
\]

The random contribution can then be handled to obtain a Markovian stochastic equation of motion following Ref. [63]. This derivation suggests that noise \( \eta(t) \) entering the equation of motion \( \frac{\partial}{\partial t} [c^a(r, \theta, t)] = (\text{deterministic part}) + \eta(t) \) (i) \( \eta(t) \) is multiplicative in the density and (ii) has a correlation function of the form \( \langle \eta(t, r) \eta(0, 0) \rangle \propto \delta(t) \nabla^2 \delta(r) \), similar to fluids at thermal equilibrium (like model A in Ref. [72, 73]). In the Toner-Tu model [30, 68, 74], this correlation function is usually assumed to be (i) not multiplicative in the density and (ii) with a correlation function of the form \( \langle \eta(t, r) \eta(0, 0) \rangle \propto \delta(t) \delta(r) \) (without Laplacian); in Ref. [73], it is argued that this form is chosen because of the lack of linear momentum conservation. We refer to Refs. [1, 71, 76] for discussions.

Its noise-averaged version (where we use the same symbols for simplicity) is obtained by removing the noise and reads

\[
\frac{\partial}{\partial t} [c^a(r, \theta, t)] = -\nabla_r \cdot \left( \left( A^a_0(r, \theta) + \sum_b \int dr'd\theta' B^{ab}_0(r, \theta, r', \theta') c^b(r', \theta'), t) \right) c^a(r, \theta, t) \right) -\nabla_\theta \cdot \left( A^a_0(r, \theta) + \sum_b \int dr'd\theta' B^{ab}_0(r, \theta, r', \theta') c^b(r', \theta'), t) \right) c^a(r, \theta, t) -\nabla_\theta \cdot \left[ \sum_i \eta^a_i(t) c^a_i(r, \theta, t) \right] + \eta \nabla_\theta^2 c^a(r, \theta, t). \tag{S22}
\]

We now replace the A and B’s with equations (S6) and (S7) to get

\[
\frac{\partial}{\partial t} [c^a(r, \theta, t)] = -\nabla_r \cdot \left[ v_0^a n(\theta) c^a(r, \theta, t) \right] -\nabla_\theta \cdot \left[ \sum_b \int dr'd\theta' J^{ab} H(R_0 - ||r - r'||) \sin(\theta' - \theta) c^b(r, \theta, t) c^b(r', \theta', t) \right] + \eta \nabla_\theta^2 c^a(r, \theta, t). \tag{S23}
\]

that can be reorganized as

\[
(\partial_t + v_0^a n(\theta) \cdot \nabla_r) c^a(r, \theta, t) = \eta \nabla_\theta^2 c^a(r, \theta, t) -\sum_b J^{ab} \nabla_\theta \cdot \left[ \int dr'd\theta' H(R_0 - ||r - r'||) \sin(\theta' - \theta) c^a(r, \theta, t) c^b(r', \theta', t) \right]. \tag{S24}
\]
To simplify this equation, we replace $H(R_0 - |r - r'|)$ by $2\pi R_0^2 \delta(r - r')$, see e.g. Refs. [1, 23] (the $2\pi$ is here to simplify notations later), so that we obtain

$$\left( \partial_t + v_0^a n(\theta) \cdot \nabla_r \right) c^a(r, \theta, t) = \eta \nabla^2 c^a(r, \theta, t) - \sum_b 2\pi R_0^2 J^{ab} \nabla_\theta \left[ \int d\theta' \sin(\theta' - \theta) c^a(r, \theta, t) c^b(r, \theta', t) \right]. \tag{S25}$$

Let us now define the angular moments

$$f^a_n(r, t) = \int d\theta \, e^{i n \theta} c^a(r, \theta, t). \tag{S26}$$

so that

$$c^a(r, \theta, t) = \frac{1}{2\pi} \sum_n e^{-i n \theta} f^a_n(r, t) \tag{S27}$$

Note that by reality

$$f^a_{-n}(r, t) = \overline{f^a_n(r, t)} \tag{S28}$$

where the overline represents complex conjugation.

We also define

$$\partial_0 = \partial - i \partial_y \quad \text{and} \quad \partial_0 = \partial + i \partial_y. \tag{S29}$$

Then

$$n(\theta) \cdot \nabla_r = \cos(\theta) \partial_0 + \sin(\theta) \partial_y = \frac{1}{2} \left[ e^{-i \theta} \partial_0 + e^{i \theta} \partial_0 \right]. \tag{S30}$$

Using the expansion (S27) into the equation (S25), we get

$$\sum_n e^{-i n \theta} \partial_t f^a_n(r, t) + \frac{v_0^a}{2} \sum_n e^{-i n \theta} \partial_0 f^a_n(r, t) + \frac{v_0^a}{2} \sum_n e^{-i n \theta} \partial_y f^a_n(r, t) = \eta \sum_n \nabla^2 \epsilon^{-i n \theta} f^a_n(r, t) \tag{S31}$$

i.e.

$$\sum_n e^{-i n \theta} \partial_t f^a_n(r, t) + \frac{v_0^a}{2} \sum_n e^{-i(n+1) \theta} \partial_y f^a_n(r, t) + \frac{v_0^a}{2} \sum_n e^{-i(n-1) \theta} \partial_z f^a_n(r, t) = \eta \sum_n (-n)^2 e^{-i n \theta} f^a_n(r, t) \tag{S32}$$

After reindexation,

$$\sum_n e^{-i n \theta} \partial_t f^a_n(r, t) + \frac{v_0^a}{2} \sum_n e^{-i n \theta} \partial_z f^a_{n+1}(r, t) + \frac{v_0^a}{2} \sum_n e^{-i n \theta} \partial_y f^a_{n-1}(r, t) = \eta \sum_n (-n)^2 e^{-i n \theta} f^a_n(r, t) \tag{S33}$$

Integrating over $\theta'$ gives $\delta_{n',0}$ that remove the corresponding sum, and after applying the last derivative we obtain

$$\sum_n e^{-i n \theta} \partial_t f^a_n(r, t) + \frac{v_0^a}{2} \sum_n e^{-i n \theta} \partial_z f^a_{n+1}(r, t) + \frac{v_0^a}{2} \sum_n e^{-i n \theta} \partial_y f^a_{n-1}(r, t) = \eta \sum_n (-n)^2 e^{-i n \theta} f^a_n(r, t) \tag{S34}$$
Finally, division by $\sum_n e^{-in\theta}$ produces

$$\partial_t f_n^a(r,t) + \frac{v_0^a}{2} \partial_x f_{n-1}^a(r,t) + \frac{v_0^a}{2} \partial_x f_{n+1}^a(r,t) = \eta (-in)^2 f_n^a(r,t)$$

$$- \sum_b R_0^{ab} \frac{1}{2i} (\overline{f}_{n-1}^a (r,t) f_1^b(r,t) - f_n^a (r,t) f_1^b(r,t))$$

(S35) and

$$\partial_t f_n^a(r,t) + \frac{v_0^a}{2} \left[ \partial_x f_{n-1}^a(r,t) + \partial_x f_{n+1}^a(r,t) \right] = -\eta^2 f_n^a(r,t)$$

$$+ \sum_b \frac{R_0^{ab}}{2} \left[ f_{n-1}^a(r,t) f_1^b(r,t) - f_{n+1}^a(r,t) f_1^b(r,t) \right]$$

(S36)

Hence, using the expansion (S27) into the equation (S25) finally yields

$$\partial_t f_n^a + \frac{v_0^a}{2} \left( \partial_x f_{n-1}^a + \partial_x f_{n+1}^a \right) = -n^2 \eta f_n^a + \frac{1}{2} \sum_b \left[ f_{n-1}^a f_1^b - f_{n+1}^a f_1^b \right].$$

(S37)

For $n = 0, 1, 2$ we get

$$\partial_t f_0^a + \frac{v_0^a}{2} \left( \partial_x f_0^a + \partial_x f_1^a \right) = 0$$

(S38a)

$$\partial_t f_1^a + \frac{v_0^a}{2} \left( \partial_x f_0^a + \partial_x f_2^a \right) = -\eta f_1^a + \frac{1}{2} \sum_b \left[ f_0^a f_1^b - f_2^a f_1^b \right]$$

(S38b)

$$\partial_t f_2^a + \frac{v_0^a}{2} \left( \partial_x f_1^a + \partial_x f_3^a \right) = -4\eta f_2^a + \sum_b \left[ f_2^a f_1^b - f_3^a f_1^b \right]$$

(S38c)

Following [1, 64, 65, 67], we close the hierarchy of moment equations by considering the last equation with the assumptions $f_3^a = 0$ and $\partial_t f_2^a = 0$, giving

$$f_2^a = \frac{1}{4\eta} \left[ -\frac{v_0^a}{2} \left( \partial_x f_1^a \right) + \sum_b \left[ f_0^a f_1^b f_1^b \right] \right].$$

(S39)

The replacement (S39) used in equation (S38b) gives

$$\partial_t f_1^a + \frac{v_0^a}{2} \partial_x f_0^a - \frac{(v_0^a)^2}{16\eta} \partial_x \partial_x f_1^a + \frac{1}{8\eta} \sum_b J_{ab} R_0^{ac} \left[ f_0^a f_1^b - f_2^a f_1^b \right]$$

(S40)

We identify the density $\rho^a$ and polarization $\vec{P}^a = (P_x^a, P_y^a)^T$ as

$$f_0^a = \rho^a \quad \text{and} \quad f_1^a = P_x^a - iP_y^a,$$

(S41)

We briefly note that we have used $\vec{v}_a$ in the main text.

Equation (S38a) gives

$$\partial_t \rho^a + v_0^a \text{div} (\vec{P}^a) = 0.$$  

(S42)

Removing spatial derivative terms $\partial_x$ and $\partial_z$ for now, equation (S40) yields

$$\partial_t \left( \begin{array}{c} P_x^a \\ P_y^a \end{array} \right) = -\eta \left( \begin{array}{c} P_x^a \\ P_y^a \end{array} \right) + \frac{1}{2} \sum_b J_{ab} R_0^{ac} \left( \begin{array}{c} P_x^b \\ P_y^b \end{array} \right)$$

$$- \frac{R_0^{ab}}{8\eta} \sum_{b,c} J_{ab} J_{ac} \left( \begin{array}{c} P_x^a P_x^b P_x^c + P_y^a P_y^b P_x^c - P_y^a P_y^c P_x^c + P_y^a P_y^b P_x^c + P_y^a P_y^b P_y^c + P_y^a P_y^c P_y^c \right) + \mathcal{O}(\nabla)$$

(S43)
Let us define the notation \((\bar{x}^\ast)_\mu = \epsilon_{\mu\nu}(\bar{x})_\nu\) and \(\epsilon_{\mu\nu}\) is the Levi-Civita symbol so we can rewrite the sum in the last term in Eq. (S43) as \(\bar{P}^a(\bar{P}^b, \bar{P}^c) + \bar{P}^{ab*}(\bar{P}^{ba*}, \bar{P}^{ac})\) where \(\langle \cdot, \cdot \rangle\) is the standard Euclidean scalar product and write

\[
\partial_t \bar{P}^a = -\eta \bar{P}^a + \sum_b \frac{J_{ab} R_{k}^2}{2} \rho^b \bar{P}^b - \frac{R_0^4}{8\eta} \sum_{b,c} J_{ab} J_{ac} \left[ \bar{P}^{a}(\bar{P}^b, \bar{P}^c) + \bar{P}^{ab*}(\bar{P}^{ba*}, \bar{P}^{ac}) \right] + O(\nabla). \tag{S44}
\]

As \(\langle \bar{P}^a, \bar{Q} \rangle = -\langle \bar{P}, \bar{Q}^* \rangle\), the term \(\langle \bar{P}^{ba*}, \bar{P}^c \rangle\) is antisymmetric in the exchange \(b \leftrightarrow c\) while \(J_{ab} J_{ac}\) is symmetric, so after the sum is applied this term is removed and we get

\[
\partial_t \bar{P}^a = -\eta \bar{P}^a + \sum_b \frac{J_{ab} R_{k}^2}{2} \rho^b \bar{P}^b - \frac{R_0^4}{8\eta} \sum_{b,c} \bar{P}^a \langle J_{ab} J_{ac}, \bar{P}^c \rangle + O(\nabla). \tag{S45}
\]

We can look at the gradient contributions to \(\partial_t f^a_0\), namely

\[
-\frac{v_0^a}{2} \partial_t f^a_0 + \frac{(v_0^a)^2}{16\eta} (\partial_2 \partial_t f^a_1) + \sum_b \frac{J_{ab} R_{k}^2 v_0^b}{16\eta} \left[ f_1^a (\partial_2 f^a_1) - 2 \partial_1 (f^a_1 f^b_1) \right]. \tag{S46}
\]

The simple terms are

\[
(\partial_2 \partial_t f^a_1) \to \nabla^2 \bar{P}^a, \quad (\partial_1 \partial_t f^a_1) \to \nabla^2 \bar{P}^a. \tag{S47}
\]

The third one is a mess, but the following should hold

\[
\bar{f}_1^a (\partial_2 f^a_1) \to (\bar{P}^b : \nabla \bar{P}^a) + (\bar{P}^{ba*} : \nabla \bar{P}^{ac}) \tag{S49}
\]

\[
\partial_2 (f^a_1 f^b_1) \to (\bar{P}^b : \nabla \bar{P}^a) + \bar{P}^b \text{div}(\bar{P}^a) - (\bar{P}^{ab*} : \nabla \bar{P}^{ac}) \tag{S50}
\]

where the grad are written in letters for clarity, but will soon be replaced by \(\nabla\).

Hence, the hydrodynamic equation finally reads

\[
\partial_t \bar{P}^a = -\eta \bar{P}^a + \sum_b \frac{J_{ab} \rho^b}{2} \bar{P}^b - \frac{1}{2\eta} \sum_{b,c} \bar{P}^a \langle j_{ab} \bar{P}^b, j_{ac} \bar{P}^c \rangle - \frac{v_0^a}{2} \nabla \rho^a + D_a \nabla^2 \bar{P}^a + \sum_b \lambda_{ab} \left[ (\bar{P}^b : \nabla) \bar{P}^a + (\bar{P}^{ba*} : \nabla) \bar{P}^{ac} - 2 \left[ (\bar{P}^{ab*} : \nabla) \bar{P}^{ba*} - (\bar{P}^{ab*} : \nabla) \bar{P}^{ac} \right] \right]. \tag{S51}
\]

where we have defined

\[
J_{ab} = \frac{R_0^2}{2} J_{ab}, \quad D_a = \frac{(v_0^a)^2}{16\eta}, \quad \lambda_{ab} = \frac{v_0^a v_0^b}{8\eta}. \tag{S52}
\]

In the stability analysis of section III, we will set \(\lambda_{ab} = v_0^a v_0^b\). We refer the reader to Ref. [1] for a discussion on this point in the case of a single population.

We also note that the relations between the numerous terms in Eq. (S51) are merely a consequence of the particular derivation used here. In generic non-reciprocal binary fluids, these terms might have unrelated coefficients. Nonetheless, we will focus on Eq. (S62) for simplicity.

In some cases (such as when there is only one kind of active particles), Eq. (S51) will be simplified by using the following identities for two vectors fields \(u\) and \(v\)

\[
(u^* \cdot \nabla) v^* + (v^* \cdot \nabla) u^* = \text{grad}(u \cdot v) - u \text{div}(v) - v \text{div}(u) \tag{S53a}
\]

\[
u^* \text{div}(v^*) + v^* \text{div}(u^*) = \text{grad}(u \cdot v) - (u \cdot \nabla)v - (v \cdot \nabla)u. \tag{S53b}
\]

They can also be written into a more symmetric but less useful way as

\[
\text{grad}(u \cdot v) = \langle u^* \cdot \nabla \rangle v^* + \langle v^* \cdot \nabla \rangle u^* + u \text{div}(v) + v \text{div}(u) \tag{S54a}
\]

\[
\text{grad}(u \cdot v) = \langle u \cdot \nabla \rangle v + \langle v \cdot \nabla \rangle u + u^* \text{div}(v^*) + v^* \text{div}(u^*). \tag{S54b}
\]
1. Hydrodynamic equations for a single population

In the paragraph, we first specialize equation (S51) to the case of a single population to recover the standard Toner-Tu equations.

Starting from (S51), we use $\bar{P}^* = \bar{P} = 0$ and the identity

$$
(\bar{P}^* \cdot \nabla)\bar{P}^* + 2 \left( (\bar{P}^* \cdot \nabla)\bar{P}^* + \bar{P}^* (\nabla \cdot \bar{P}^*) \right) = 5\nabla(\bar{P}^2/2) - 3\bar{P}(\nabla \bar{P}) - 2(\bar{P}^* \cdot \nabla)\bar{P}.
$$

(S55)

obtained from (S53) to get

$$
\partial_t \bar{P} + \lambda_1 (\bar{P} \cdot \nabla)\bar{P} + \lambda_2 \bar{P} \text{div}(\bar{P}) + \lambda_3 \nabla(\bar{P}^2) = - \left[ \alpha(\rho) + \beta \parallel \bar{P} \parallel^2 \right] \bar{P} - \frac{v_0}{2} \nabla \rho + D \nabla^2 \bar{P}
$$

(S56)

where

$$
\alpha(\rho) = \eta - j \rho \quad \beta = \frac{j^2}{2\eta} \quad D = \frac{(v_0)^2}{16\eta} \quad \lambda_0 = \frac{j v_0}{8\eta} \quad j = \frac{JR^2}{2}
$$

(S57)

and $\lambda_1 = 3\lambda_0$, $\lambda_2 = 5\lambda_0$, $\lambda_3 = -5/2\lambda_0$.

2. Hydrodynamic equations for two populations

We now specialize to the case where there are only two populations $a, b, c = A, B$ (here the capital letters $A$ and $B$ refer to the two populations and are not abstract indices), which is the situation analyzed in the main text.

A special case of this situation was derived and analyzed in Ref. [23], with which our results agree.

We set $a = A$ for simplicity, and remove all spatial derivatives. In this case, the hideous sum in equation (S43) reads

$$
J_{AA} J_{AA} \left( P_y^A P_y^A + P_x^A P_x^A + P_y^A P_x^A \right) + 2 J_{AB} J_{AB} \left( P_y^A P_y^B + P_x^A P_y^B + P_y^A P_x^B \right) + J_{AB} J_{AB} \left( P_y^B P_y^B + P_x^B P_y^B + P_y^B P_x^B \right)
$$

(S58)

Factoring out the polarization, one recognizes

$$
\left[ J_{AA} J_{AA} \parallel \bar{P}^A \parallel^2 + 2 J_{AB} J_{AB} (\bar{P}^A, \bar{P}^B) + J_{AB} J_{AB} \parallel \bar{P}^B \parallel^2 \right] \bar{P}^A = \parallel J_{AA} B^A + J_{AB} B^B \parallel^2 \bar{P}^A
$$

(S59)

i.e. (all quantities are real)

$$
\left[ \parallel J_{AA} \bar{P}^A \parallel^2 + 2 \langle J_{AA} B^A, J_{AB} \bar{P}^B \rangle + \parallel J_{AB} B^B \parallel^2 \right] \bar{P}^A = \parallel J_{AA} B^A + J_{AB} B^B \parallel^2 \bar{P}^A
$$

(S60)

Using again $j_{ab} = (R_0^2/2) J_{ab}$, we obtain

$$
\partial_t \bar{P}^A = -\eta \bar{P}^A + \eta j_{AA} B^A \bar{P}^A + j_{AB} B^B \bar{P}^A - \frac{1}{2\eta} \parallel j_{AA} \bar{P}^A + j_{AB} B^B \parallel^2 \bar{P}^A + O(\nabla)
$$

(S61)

and a similar equation for $\bar{P}^B$ is obtained by permuting the indices.

Including the gradient terms, (S51) becomes for $a = A$

$$
\partial_t \bar{P}^A = \left[ j_{AA} B^A - \eta - \frac{1}{2\eta} \parallel j_{AA} \bar{P}^A + j_{AB} B^B \parallel^2 \right] \bar{P}^A + j_{AB} B^A \bar{P}^B - \frac{v_0}{2\eta} \nabla B^A + D A \nabla^2 \bar{P}^A
$$

(S62)

$$
+ \lambda_{AA} \left[ 5/2 \nabla(\bar{P}^A \cdot \bar{P}^A) - 3(\bar{P}^A \cdot \nabla)\bar{P}^A - 5\bar{P}^A \text{div}(\bar{P}^A) \right]
$$

$$
+ \lambda_{AB} \left[ (\bar{P}^B \cdot \nabla)\bar{P}^A - 2(\bar{P}^A \cdot \nabla)\bar{P}^B - 2\bar{P}^B \text{div}(\bar{P}^A) + (\bar{P}^B \cdot \nabla)\bar{P}^A + 2(\bar{P}^A \cdot \nabla)\bar{P}^B + 2\bar{P}^B \text{div}(\bar{P}^A) \right]
$$

where we already have used equation (S55) to simplify the $AA$ terms, and where we have defined

$$
j_{ab} = \frac{R_0^2}{2} J_{ab} \quad D_a = \frac{(v_0)^2}{16\eta} \quad \lambda_{ab} = \frac{v_0 J_{ab}}{8\eta}
$$

(S63)

In the stability analysis of section III, we will set $\lambda_{ab} = v_0^a v_0^b$, see discussion above.

The equation for $a = B$ is obtained in the same way.
II. MEAN-FIELD PHASE DIAGRAM IN THE STEADY STATE

In this section, to grasp the influence of nonreciprocal interaction to the many-body state, we perform a mean-field approximation to the hydrodynamic theory derived in Sec. I, thereby neglecting the gradient terms in Eq. (S62):

$$\partial_t \begin{pmatrix} \tilde{\rho}_A \\ \tilde{\rho}_B \end{pmatrix} = -\hat{W}[\tilde{\rho}_A, \tilde{\rho}_B] \begin{pmatrix} \tilde{\rho}_A \\ \tilde{\rho}_B \end{pmatrix}$$  \hspace{1cm} (S64)

where

$$\hat{W}[\tilde{\rho}_A, \tilde{\rho}_B] = \begin{pmatrix} W_{AA}[\tilde{\rho}_A, \tilde{\rho}_B] & W_{AB} \\ W_{BA}[\tilde{\rho}_A, \tilde{\rho}_B] & W_{BB}[\tilde{\rho}_A, \tilde{\rho}_B] \end{pmatrix} = \begin{pmatrix} \eta - j_{AB} \rho^A + \frac{1}{2\eta} ||\tilde{Q}^A(t)||^2 & -j_{AB} \rho^A \\ -j_{AB} \rho^B & \eta - j_{BB} \rho^B + \frac{1}{2\eta} ||\tilde{Q}^B(t)||^2 \end{pmatrix}$$  \hspace{1cm} (S65)

and

$$\tilde{Q}^A(t) = j_{AA} \tilde{\rho}_A(t) + j_{AB} \tilde{\rho}_B(t),$$  \hspace{1cm} (S66a)

$$\tilde{Q}^B(t) = j_{BA} \tilde{\rho}_A(t) + j_{BB} \tilde{\rho}_B(t).$$  \hspace{1cm} (S66b)

The matrix $\hat{W}$ is in general non-Hermitian, i.e. $\hat{W} \neq \hat{W}^\dagger$. We are especially interested in cases where the nonreciprocal interaction is strong such that the inter-species couplings have opposite signs ($j_{AB} j_{BA} < 0$). In such situation, there are no configuration that can make both species satisfied. This situation shares conceptual similarities with the geometrical frustration present in systems ranging from (spin) glasses [77–82] to liquid crystals and colloidal systems [83, 84], which occurs when the interactions between different entities, such as spins or atoms, have competing effects (like for three spins with antiferromagnetic couplings on the vertices on a triangle). The dynamical frustration present here has a different origin: instead of coming from multiple competing interactions, it arises from each individual non-reciprocal interaction. It gives rise to an time-dependent phase where the direction of flocking continuously changes over time, see Fig. S1a. We call this the "chiral phase".

In this section, we first give an analytical argument that illustrates how a spontaneous symmetry breaking from a flocking or anti-flocking phase (similar to the antiferromagnetic phase) to the chiral phase may occur by increasing the non-reciprocity of the coupling strength. We show that this phase transition is marked by a so-called exceptional points [12] of the matrix $\hat{W}$, which are the points where two of the eigenvectors of $\hat{W}$ coalesce, and discuss its relation to PT symmetry breaking. This mechanism unique to out-of-equilibrium systems originates from the non-Hermitian structure of the matrix $\hat{W}$ that controls the dynamics. As such, it is expected to be a generic feature of nonreciprocal interaction. Using the relation between the phase transition and the exceptional points of $\hat{W}$, we determine the phase boundaries in terms of the microscopic coupling strength $j_{ab}$ and compare them with numerics. We also show how chiral phase interpolates the flocking and the antiflocking phase (see Fig. S1b).

In addition, we also find from direct numerical simulations of the mean-field equation (Eq. (S64)) that another time-dependent phase appears in the phase diagram, which we call the "swap phase", see Fig. S1c. The swap phase exhibits a time oscillation in the amplitude of the macroscopic polarization (in contrast to the chiral phase exhibiting oscillations in their direction of the orientation), which is again triggered by the dynamical frustration. Further, we find an interesting regime where these two oscillations coexist with different frequencies, which its time dependence of the polarization field, as a result, becomes quasiperiodic, see Figs. 2e-g and 3d in the main text. Discussions on the origin of these phases from the point of view of fluctuation modes are provided in Sec. III.

We note that the mean-field approximation employed in this section assumes that the system reaches a uniform state. This is not always true; we discuss finite momentum instabilities and pattern formation [85] in Sec. III.

A. Emergence of the chiral phase by the PT symmetry breaking

Before attempting to directly solve the full nonlinear equation (S64), here we provide an argument based on the non-Hermiticity and the symmetry of the matrix $\hat{W}$, that explains how a spontaneous breaking of time translation symmetry may emerge in this system. In particular, we show that, in addition to the uniform flocking and the antiflocking phase, the chiral phase, where the direction of the orientation continuously changes over time, can emerge as a steady state solution as a result of the non-Hermitian nature of the matrix $\hat{W}$ (see also Fig. S1). We also discuss its relation to the spontaneous PT symmetry breaking, often discussed in the context of open quantum mechanics [11, 86–88]. Since the details of the $\tilde{\rho}_A(t), \tilde{\rho}_B(t)$ dependence on the matrix $\hat{W}$ is essentially irrelevant to this discussion, we expect the emergence of the chiral phase to be a generic feature of non-reciprocally interacting active fluids.
1. Flocking and antiflocking phase

Let us first look for the conventional, time-independent solutions, by assuming that the polarization eventually converges to a constant, i.e., $\vec{P}_A(t) = \vec{P}_B(t) = \text{const}$. Although, at a glance, it seems possible for the relative angle between $\vec{P}_A$ and $\vec{P}_B$ to take any value, we shall see in the following that it is only possible to be parallel or antiparallel to each other, which we call the flocking and antiflocking phase, respectively.

Under the assumption stated above, the matrix becomes time-independent, i.e.

$$0 = \tilde{W}_0 \left( \begin{array}{c} \vec{P}_A \\ \vec{P}_B \end{array} \right) = \begin{pmatrix} W_{AA}^{0} & W_{AB}^{0} \\ W_{BA}^{0} & W_{BB}^{0} \end{pmatrix} \begin{pmatrix} \vec{P}_A \\ \vec{P}_B \end{pmatrix}$$

$$= \begin{pmatrix} \eta - j_{AA} \rho^A + \frac{1}{2} \| \vec{Q}_A \|^2 & -j_{AB} \rho^A \\ -j_{BA} \rho^B & \eta - j_{BB} \rho^B + \frac{1}{2} \| \vec{Q}_B \|^2 \end{pmatrix} \begin{pmatrix} \vec{P}_A \\ \vec{P}_B \end{pmatrix}, \quad (S67)$$

with

$$\vec{Q}_A = j_{AA} \vec{P}_A + j_{AB} \vec{P}_B, \quad (S68a)$$

$$\vec{Q}_B = j_{BA} \vec{P}_A + j_{BB} \vec{P}_B. \quad (S68b)$$

Diagonalizing this matrix $\tilde{W}_0$ in Eq. (S67) gives

$$0 = \begin{pmatrix} \Gamma_- & 0 \\ 0 & \Gamma_+ \end{pmatrix} \begin{pmatrix} \vec{P}_{-} \\ \vec{P}_{+} \end{pmatrix}. \quad (S69)$$

Here, the eigenvalues $\Gamma_{\pm}$ roughly correspond to the decay rate of the eigenmodes, where their explicit expressions are given by

$$\Gamma_{\pm} = \frac{1}{2} [W_{AA}^{0} + W_{BB}^{0} \pm \sqrt{\Lambda_0}], \quad (S70)$$

$$\Lambda_0 = (W_{AA}^{0} - W_{BB}^{0})^2 + 4W_{AB}^{0}W_{BA}^{0} \quad (S71)$$
with the corresponding eigenmodes,

\[ \mathbf{u}_- = \left( \frac{\sqrt{\Lambda_0} - (W_0^{AA} - W_0^{BB})}{2 W_0^{BA}}, -W_0^{AB} \right), \quad (S72a) \]

\[ \mathbf{u}_+ = \left( \frac{\sqrt{\Lambda_0} - (W_0^{AA} - W_0^{BB})}{2 W_0^{AB}}, \frac{-W_0^{AB}}{\sqrt{\Lambda_0} - (W_0^{AA} - W_0^{BB})} \right). \quad (S72b) \]

The order parameter of the flocking phase is transformed accordingly into

\[ \left( \frac{\tilde{P}_0^-}{\tilde{P}_0^+} \right) = \hat{U}_0 \left( \frac{\tilde{P}_0^A}{\tilde{P}_0^B} \right) = \frac{1}{\det \hat{U}_0^{-1}} \left( \frac{\sqrt{\Lambda_0} - (W_0^{AA} - W_0^{BB})}{2 W_0^{BA}}, -W_0^{AB} \right) \left( \frac{\tilde{P}_0^A}{\tilde{P}_0^B} \right), \quad (S73) \]

where \( \hat{U}_0^{-1} = (\mathbf{u}_-, \mathbf{u}_+) \), or inversely,

\[ \left( \frac{\tilde{P}_0^A}{\tilde{P}_0^B} \right) = \left( \begin{array}{cc} [U_0^{-1}]_A^- & [U_0^{-1}]_A^+ \\ [U_0^{-1}]_B^- & [U_0^{-1}]_B^+ \end{array} \right) \left( \frac{\tilde{P}_0^-}{\tilde{P}_0^+} \right) = \left( \begin{array}{cc} \frac{\sqrt{\Lambda_0} - (W_0^{AA} - W_0^{BB})}{2 W_0^{BA}} & \frac{W_0^{AB}}{-W_0^{BA}} \\ \frac{-W_0^{AB}}{-W_0^{BA}} & \frac{\sqrt{\Lambda_0} - (W_0^{AA} - W_0^{BB})}{2 W_0^{BA}} \end{array} \right) \left( \begin{array}{c} \tilde{P}_0^- \\ \tilde{P}_0^+ \end{array} \right). \quad (S74) \]

It can be shown from Eq. (S69) that in addition to a trivial solution \( \tilde{P}_0^- = \tilde{P}_0^+ = 0 \) that corresponds to a disordered phase, nontrivial solutions with \( (\tilde{P}_0^-, \tilde{P}_0^+)^T \neq 0 \) can always be classified into two types [36]: solutions that satisfy \( (\tilde{P}_0^- \neq 0, \tilde{P}_0^+ = 0) \) and \( (\tilde{P}_0^- = 0, \tilde{P}_0^+ \neq 0) \), which we call "-" and "+" solutions, respectively. This is readily seen as follows. Let us assume that \( \tilde{P}_0^{-(+)} \neq 0 \). Then, it is necessary for the eigenvalue \( \Gamma^{-(+)} \) to vanish in order to satisfy the first (second) line of Eq. (S69). In such case, since the eigenvalue of "+(−)" is finite \( \Gamma^{−(+)} \neq 0 \) as long as \( \Gamma \neq \Gamma^{−(+)} \), \( \tilde{P}_0^{+(−)} \) necessarily vanishes because of the second (first) line of Eq. (S69). Thus, \( \tilde{P}_0^- \) and \( \tilde{P}_0^+ \) cannot be nonzero simultaneously, letting us classify the solutions into two types.

The above property has a direct consequence that the polarization field of \( A \) and \( B \) agents can only be either parallel or antiparallel in the uniform steady state, which we call the flocking and antiflocking phase, respectively. For example, for the "-" solution, polarization fields are given by

\[ \tilde{P}_0^A = [U_0^{-1}]_A^- \tilde{P}_0^- = \frac{\sqrt{\Lambda_0} - (W_0^{AA} - W_0^{BB})}{2} \tilde{P}_0^-, \quad (S75) \]

\[ \tilde{P}_0^B = [U_0^{-1}]_B^- \tilde{P}_0^- = -W_0^{BA} \tilde{P}_0^-, \quad (S76) \]

explicitly showing that \( \tilde{P}_0^A \) and \( \tilde{P}_0^B \) are either parallel or antiparallel to each other, depending on the relative sign between \([U_0^{-1}]_A^-\) and \([U_0^{-1}]_B^-\).

As mentioned earlier, the eigenvalues \( \Lambda \) roughly corresponds to the decay rate of the corresponding modes. The condition \( \Gamma^{−(+)} = 0 \) for the "-(+)" solution can be regarded as the defining property of a steady state. Assuming \( \Lambda_0 > 0 \) (that assures \( \Lambda \) to be real), the "-(+)" solution is likely to be (un)stable since \( 0 < \Gamma^{−} \). \( \Gamma^{−} < 0 \), implying a positive (negative) decay rate of the "-(+)" mode, where we have used the relation \( \Gamma^{−} < \Gamma^{−} \). This strongly suggest that the "-" solution is the solution that would be realized. Indeed, as shown in Sec. III B 2 from a stability analysis, it can be proven that "+" solution is always unstable, limiting the possible stable solution to the "-" solution.

It is important to emphasize that, for the flocking or antiflocking phase to be realized, it is necessary for \( \Lambda_0 \) to be positive since it requires \( \sqrt{\Lambda_0} \) to be real such that the state can satisfy the condition \( \Gamma^{−} = 0 \) (\( \Gamma^{−} = 0 \)).\( "-(+)" \) solution. This condition is assured to be satisfied when the sign of the inter-species coupling is the same, i.e., \( j_{ABjBA} > 0 \) (which is equivalent to \( W_0^{AB} W_0^{BA} > 0 \)), which includes the reciprocal case, since the first term of Eq. (S71), \( W_0^{AA} - W_0^{BB} \), is non-negative.

However, when the inter-species couplings have opposite sign (i.e. \( j_{ABjBA} < 0 \) or \( W_0^{AB} W_0^{BA} < 0 \)), \( \Lambda_0 \) may become negative, hence the eigenvalues can turn imaginary implying the existence of a phase transition to a time-oscillating phase. As we show in the following section, the system indeed exhibits a phase transition to an exotic phase where the direction of the orientation continuously oscillates in time, which we call the chiral phase. The phase transition is driven by the non-Hermitian nature of the matrix \( \hat{W} \), which can be seen from the observation that the phase transition point, \( \lambda_0 = 0 \) is the point where the two eigenvectors \( \mathbf{u}_± \) (see Eq. (S72)) coalesce; This is the so-called exceptional point [12]. We will further show in Sec. II A 3 that this transition can be regarded as a spontaneous PT symmetry breaking discussed in the field of non-Hermitian quantum mechanics.
2. Chiral phase

Below, we look for solutions with an oscillating polarization field, described by the ansatz,

\[ \vec{B}^A(t) = R^A \left( \frac{\cos(\Omega t + \phi^A)}{\sin(\Omega t + \phi^A)} \right), \quad \text{(S77a)} \]

\[ \vec{B}^B(t) = R^B \left( \frac{\cos(\Omega t + \phi^B)}{\sin(\Omega t + \phi^B)} \right), \quad \text{(S77b)} \]

with \( \Omega \) being the frequency of the oscillation and \( R^A, R^B > 0 \) the amplitude of the polarization fields of \( A \) and \( B \) species, respectively. This solution exhibits a "chiral" motion, in the sense that the direction of the orientation continuously evolves in time (while the amplitude of the polarization remains fixed) implying a collective chiral motion of agents, which is exactly what is observed in our microscopic Vicsek model simulation (Fig. 2d in the main text and Supplemental video).

We note that, for solutions of the form (S77), the \( O(2) \) symmetry of the mean-field system assures that \( \vec{W} = \vec{W}_0 \) does not depend on time. This can be directly checked from the observation that the amplitudes of the vectors \( \vec{Q}^A(t), \vec{Q}^B(t) \) given by Eq. (S66) which shows that the magnitude of the nonlinearity in \( W_0^{AA} \) and \( W_0^{BB} \) are time-independent.

We show below that the ansatz (S77) satisfies,

\[ \Omega = \Omega_{\pm} = \pm \frac{1}{2} \sqrt{|\Lambda_0|}, \quad \text{(S78)} \]

\[ \Delta \phi^{AB} = \phi^A - \phi^B = \begin{cases} \arccos \left( \sqrt{\frac{W_0^{AA}}{W_0^{BB} + W_0^{AA}}} \right) = \arccos \left( \sqrt{1 - \frac{|\Lambda_0|}{|\Lambda_0|}} \right) & (W_0W_0^{AB} < 0) \\ \arccos \left( -\sqrt{\frac{W_0^{AA}}{W_0^{BB} + W_0^{AA}}} \right) = \arccos \left( -\sqrt{1 - \frac{|\Lambda_0|}{|\Lambda_0|}} \right) & (W_0W_0^{AB} > 0) \end{cases} \quad \text{(S79)} \]

with

\[ W_0^{AA} = -W_0^{BB} \equiv W_0, \quad R^B = \sqrt{\frac{W_0^{BA}}{W_0^{AB}}} R^A, \quad \text{(S80)} \]

and importantly, \( \Lambda_0 < 0 \). The last condition is satisfied only if \( W_0^{AB}W_0^{BA} < 0 \), i.e. nonreciprocal coupling with opposite signs.

Two comments are in order. Firstly, the fact that we find two solutions, \( \Omega = \Omega_+ > 0 \) and \( \Omega = \Omega_- < 0 \), indicates the occurrence of a spontaneous chiral \( (Z_2) \) symmetry breaking to a left and right-handed phase, respectively. Secondly, the exceptional point \( \Lambda_0 = 0 \) with \( W_0^{AB} > 0(< 0) \) is a continuous transition point to the (anti)flocking phase, with \( \Delta \phi^{AB} = 0(\neq \pi) \). Noting that \( W_0 \) may switch its sign inside the chiral phase, this implies that the chiral flocking phase lies in between the flocking and antiflocking phase (See Fig. S1b.), which is indeed the case (see also Figs. 2f and g).

The mean-field equation (S64) we wish to solve with the ansatz (S77) takes the form

\[ -\Omega RA \sin(\Omega t + \phi^A) = -W_0^{AA} R^A \cos(\Omega t + \phi^A) - W_0^{AB} R^B \cos(\Omega t + \phi^B), \quad \text{(S81)} \]

\[ \Omega RA \cos(\Omega t + \phi^A) = -W_0^{AA} R^A \sin(\Omega t + \phi^A) - W_0^{AB} R^B \sin(\Omega t + \phi^B), \quad \text{(S82)} \]

\[ -\Omega RB \sin(\Omega t + \phi^B) = -W_0^{BA} R^A \cos(\Omega t + \phi^A) - W_0^{BB} R^B \cos(\Omega t + \phi^B), \quad \text{(S83)} \]

\[ \Omega RB \cos(\Omega t + \phi^B) = -W_0^{BA} R^A \sin(\Omega t + \phi^A) - W_0^{BB} R^B \sin(\Omega t + \phi^B). \quad \text{(S84)} \]

Note that Eqs. (S81) and (S82) (Eqs. (S83) and (S84)) are equivalent. These can be factorized as

\[ \hat{R}^a \cos(\Omega t + \bar{\phi} a) = 0, \quad \text{(S85)} \]

\[ \hat{R}^a \sin(\Omega t + \bar{\phi} a) = 0, \quad \text{(S86)} \]

where \( a = A, B \),

\[ (\hat{R}^A)^2 = (-W_0^{AA} R^A \cos \phi^A - W_0^{AB} R^B \cos \phi^B + \Omega A \sin \phi^A)^2 + (W_0^{AA} R^A \sin \phi^A + W_0^{AB} R^B \sin \phi^B + \Omega A \cos \phi^A)^2 \]

\[ (\hat{R}^B)^2 = (-W_0^{BA} R^A \cos \phi^A - W_0^{BB} R^B \cos \phi^B + \Omega B \sin \phi^B)^2 + (W_0^{BA} R^A \sin \phi^A + W_0^{BB} R^B \sin \phi^B + \Omega B \cos \phi^B)^2 \]

and \( \bar{\phi}^A, \bar{\phi}^B \) are real constant numbers determined from the parameters \( W_0^{ab}, R^a, \phi^a \). Equations (S85) and (S86) are satisfied at arbitrary \( t \) when \( \hat{R}^A = \hat{R}^B = 0 \).
Let us first determine $\Omega$ by solving $\tilde{R}A = 0$. This gives,

$$\Omega = \frac{R^B}{R^A} W_0^{AB} \sin \Delta \phi^{AB} \pm i(W_0^{AA} + W_0^{AB} \frac{R^B}{R^A} \cos \Delta \phi^{AB}).$$  \hspace{1cm} (S87)

Since we require the frequency $\Omega$ to be real, we demand the imaginary part of Eq. (S87) to vanish,

$$\Delta \phi^{AB} = \arccos \left[ -\frac{W_0^{AA}}{W_0^{AB}} \frac{R^A}{R^B} \right].$$  \hspace{1cm} (S88)

Plugging this back into Eq. (S87), we get

$$\Omega = \Omega \pm = \frac{R^B}{R^A} W_0^{AB} \sin \Delta \phi^{AB} = \pm \frac{R^B}{R^A} W_0^{AB} \sqrt{1 - \left(\frac{R^A W_0^{AA}}{R^B W_0^{AB}}\right)^2}. \hspace{1cm} (S89)$$

We can similarly compute for $\Omega$ and $\Delta \phi^{AB}$ by solving $\tilde{R}B = 0$, where we get

$$\Delta \phi^{AB} = \arccos \left[ -\frac{W_0^{BB}}{W_0^{BA}} \frac{R^A}{R^B} \right], \hspace{1cm} (S90)$$

$$\Omega = \Omega \pm = -\frac{R^A}{R^B} W_0^{BA} \sin \Delta \phi^{AB} = \mp \frac{R^A}{R^B} W_0^{BA} \sqrt{1 - \left(\frac{R^B W_0^{BB}}{R^A W_0^{BA}}\right)^2}. \hspace{1cm} (S91)$$

The solution sets given by Eqs. (S88), (S89) and by Eqs. (S90), (S91) should be identical. Noting that $W_0^{AB} W_0^{BA} < 0$ and thus $\Omega_\pm$ in Eqs. (S89) and (S91) have the same sign, we get the relation

$$\frac{W_0^{AA} R^A}{W_0^{AB} R^B} = \frac{W_0^{BB} R^B}{W_0^{BA} R^A}, \hspace{1cm} (S92)$$

$$\frac{R^B}{R^A} W_0^{AB} \sqrt{1 - \left(\frac{R^A W_0^{AA}}{R^B W_0^{AB}}\right)^2} = -\frac{R^A}{R^B} W_0^{BA} \sqrt{1 - \left(\frac{R^B W_0^{BB}}{R^A W_0^{BA}}\right)^2}. \hspace{1cm} (S93)$$

Solving the above yields,

$$W_0^{AA} = -W_0^{BB} \equiv W_0, \hspace{1cm} (S94)$$

$$R^A = \sqrt{-\frac{W_0^{AB}}{W_0^{BA}}} R^B = \sqrt{\frac{W_0^{AB}}{W_0^{BA}}} R^B, \hspace{1cm} (S95)$$

giving,

$$\Omega_\pm = \pm \sqrt{W_0^{AB} W_0^{BA}} - W_0^2 = \pm \frac{V_0}{2}, \hspace{1cm} (S96)$$

and

$$\Delta \phi^{AB} = \arccos \left[ \sqrt{\frac{W_0^2}{W_0^{AB} W_0^{BA}}} \right], \hspace{1cm} (S97)$$

for $W_0 W_0^{AB} < 0$, and for $W_0 W_0^{AB} > 0,$

$$\Delta \phi^{AB} = \arccos \left[ -\sqrt{\frac{W_0^2}{W_0^{AB} W_0^{BA}}} \right]. \hspace{1cm} (S98)$$

Hence, we arrive at the relations (S77)-(S79).
In this subsection, we make a remark that our system is PT symmetric (which we define below) and the (anti)flocking-to-chiral phase transition can be regarded as an example of a spontaneous PT symmetry breaking often discussed in the field of non-Hermitian quantum mechanics [11, 86–88]. In the following, we consider the operations executed by the operators $P$ and $T$, which are the generalized parity and time reversal operator, respectively. Here, $P$ is defined to be a general Hermitian and unitary operator and $T$ is a general operator expressible as $K$ times a unitary matrix (where $K$ is a complex conjugate parameter) that satisfies

$$P^2 = T^2 = 1, \quad [P, T] = 0,$$

as in conventional parity and time reversal operators.

The system is said to be PT symmetric if we can find a $PT$ operator that commutes with the matrix $\hat{W}_0$ that controls the dynamics:

$$[PT, \hat{W}_0] = 0. \quad \text{(S100)}$$

The PT symmetry of a PT symmetric system is said to be unbroken if any eigenstate of the matrix $\hat{W}_0$ is simultaneously an eigenstate of the $\hat{PT}$ operator. Otherwise, the PT symmetry is said to be spontaneously broken [87].

We argue below that (1) our system is PT symmetric, (2) the (anti)flocking phase is in a PT unbroken phase, and (3) the chiral phase is in PT broken phase, therefore the (anti)flocking-to-chiral phase transition is an example of a spontaneous PT symmetry breaking.

Our system is PT symmetric since we can find the operators $P$ and $T$ that satisfies Eq. (S100). To see this explicitly, following Ref. [88], we express the matrix $\hat{W}_0$ and the operators $P$ and $T$ in terms of Pauli matrices,

$$\hat{W}_0 = w_0^1 1 + w_0 \cdot \sigma \quad \text{(S101)}$$

$$P = p^0 1 + p \cdot \sigma, \quad \text{(S102)}$$

$$T = K \sigma_3 (t^0 1 + t \cdot \sigma), \quad \text{(S103)}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)^T$ is a vector composed of Pauli matrices. From their definitions introduced above, the vectors $p$ and $t$ need to be real with

$$p^0 = t^0 = 0, p \cdot p = t \cdot t = 1, p \cdot t = 0. \quad \text{(S104)}$$

Further decomposing the vector $w_0$ into real and imaginary part as $w_0 = w_0^R + iw_0^I$, we find [88]

$$PT\hat{W}_0(PT)^{-1} - \hat{W}_0 = 2\sigma \cdot F - 2i\sigma \cdot G \quad \text{(S105)}$$

where

$$F = (w_0^R \cdot p)p + (w_0^R \cdot t)t - w_0^R, \quad \text{(S106)}$$

$$G = (w_0^I \cdot p)p + (w_0^I \cdot t)t. \quad \text{(S107)}$$

The existence of $p$ and $t$ that makes the right-hand side of Eq. (S105) vanish means that the system is PT symmetric.

We can indeed find such $p$ and $t$ for our system, by using the property that $\hat{W}_0$ is a real matrix that restrict $w_0^0$ to be real and the real vectors $w_0^R, w_0^I$ to take the form $w_0^R = (w_0^R, 0, w_0^R)^T$ and $w_0^I = (0, w_0^I, 0)^T$. Thus,

$$w_0^R \cdot w_0^I = 0. \quad \text{(S108)}$$

By choosing

$$p = \frac{w_0^R}{|w_0^R|}, \quad \text{(S109)}$$

since $p \cdot t = 0$, we get $w_0^R \cdot t = 0$. Plugging these into Eq. (S106) yields $F = 0$. Further, since $p \propto w_0^R$, Eq. (S108) gives

$$p \cdot w_0^I = 0, \quad \text{(S110)}$$

leading to $G = (w_0^I \cdot t)t$. Since $w_0^I$ and $t$ are in a plane orthogonal to $p$, we can always find $t$ that are also orthogonal to $w_0^I$. Choosing such a vector $t$, i.e.,

$$t = \frac{w_0^R \times w_0^I}{|w_0^R \times w_0^I|}, \quad \text{(S111)}$$
we get \( \mathbf{G} = 0 \) and therefore the right-hand side of Eq. (S105) vanishes. Thus, our system is PT symmetric.

Now we argue that the flocking and antiflocking phases are in a PT unbroken phase while the chiral phase is in a PT broken phase. In a PT symmetric system, the eigenstates \( \mathbf{u}_\pm \), defined as states that satisfies
\[
\hat{W}_0 \mathbf{u}_\pm = \Gamma_\pm \mathbf{u}_\pm, \tag{S112}
\]
also satisfies
\[
\hat{W}_0 (\mathcal{PT} \mathbf{u}_\pm) = \Gamma_\pm^* (\mathcal{PT} \mathbf{u}_\pm), \tag{S113}
\]
showing that \( \mathcal{PT} \mathbf{u}_\pm \) is also an eigenstate of this system. Here, we have operated \( \mathcal{PT} \) from the left and used Eq. (S100).

In a PT unbroken phase, since \( \mathcal{PT} \mathbf{u}_\pm \propto \mathbf{u}_\pm \) and thus satisfies
\[
\hat{W}_0 \mathbf{u}_\pm = \Gamma_\pm \mathbf{u}_\pm, \tag{S114}
\]
the eigenvalues of PT unbroken phase is real (\( \Gamma_\pm = \Gamma_\pm^* \)). On the other hand, in a PT broken phase, \( \mathcal{PT} \mathbf{u}_{\mp(-)} \) is a distinct vector from \( \mathbf{u}_{\mp(-)} \). Thus, the eigenstate with the eigenvalue \( \Gamma_{\mp(-)}^* \) is a different state from that with \( \Gamma_{\mp(-)}^* \).

Since there are at most two eigenvalues in our two component system,
\[
\Gamma_+ = \Gamma_-^*, \tag{S115}
\]
should be true; the two eigenvalues should be complex conjugate pairs of one another.

Recalling that, while the flocking and antiflocking phase have real eigenvalues, the chiral phase has eigenvalues that are complex conjugate to each other resulting in an oscillation in time, we can conclude from the above argument that the former is in a PT unbroken phase while the latter is in a PT broken phase, marking the phase transition point as a PT symmetry breaking point.

B. Mean-field phase diagram

So far, we have observed how the flocking/antiflocking phase may be destabilized into a chiral phase, without paying too much attention to the concrete form of the matrix \( \hat{W}_0 \). Here, we directly compute analytically the mean-field equation (S64) to show that the flocking, antiflocking, and the chiral phase predicted from the above analysis indeed arise. We determine the phase diagram in terms of the microscopic coupling strengths \( j_{ab}(a, b = A, B) \).

Below, we parameterize
\[
j_\pm = \frac{1}{2} (j_{AB} \pm j_{BA}), \tag{S116}
\]
for our convenience, where \( j_+ \) and \( j_- \) characterize the reciprocal and non-reciprocal component of the coupling, respectively.

1. Ordered-to-disordered phase transition

Firstly, we analyze the ordered-to-disordered phase transition point within mean-field approximation. Starting from the ordered phase (i.e. flocking and antiflocking phase), the order parameter \( \vec{P}_0 \) approaches zero as moving towards the phase boundary. Thus, the ordered-to-disordered phase transition point should satisfy,
\[
\det \hat{W}_0 (\vec{P}_0^A \to 0, \vec{P}_0^B \to 0) = \Gamma_- (\vec{P}_0^A \to 0, \vec{P}_0^B \to 0) \Gamma_+ (\vec{P}_0^A \to 0, \vec{P}_0^B \to 0) = 0. \tag{S117}
\]
This can be solved analytically with the result (For simplicity, we assume below \( \rho^A = \rho^B = \rho \)),
\[
j_+^* = \pm \frac{\sqrt{\eta^2 - \eta (j_{AA} + j_{BB}) \rho + (j_{AA} j_{BB} + j_+^2) \rho^2}}{\rho}. \tag{S118}
\]

Note that, as discussed earlier, the ordered phase should be described as the stable "\(-\)" solution and not the unstable "\(+\)" solution. While the "\(-\)" solution satisfies \( \Gamma_- = 0, \Gamma_+ > 0 \), the "\(+\)" solution satisfies \( \Gamma_- < 0, \Gamma_+ = 0 \). Thus, the sign of the average \( (\Gamma_+ + \Gamma_-)/2 \) indicates which type the obtained solution is. Since
\[
\Gamma_\pm (\vec{P}_0^A \to 0, \vec{P}_0^B \to 0) = -\frac{1}{2} [(j_{AA} + j_{BB}) \rho - 2\eta \pm \sqrt{\Lambda_0}], \tag{S119}
\]
the average of the two eigenvalues are given by
\[
\frac{\Gamma_+ + \Gamma_-}{2} = -\frac{1}{2}[(j_{AA} + j_{BB})\rho - 2\eta]. \tag{S120}
\]
Thus, Eq. (S118) is valid only when
\[
\eta > (j_{AA} + j_{BB})\rho \tag{S121}
\]
is satisfied such that it describes the destabilization towards the stable "−" solution. We have drawn this ordered-to-disordered phase boundary in Fig. 2e in the main text (black line), giving an excellent agreement with the numerical result.

2. Exceptional point

We next determine the exceptional point that marks the phase transition point from the (anti)flocking to a chiral phase for a given parameter set \((\rho^A, \rho^B, j_{AA}, j_{BB})\). At the transition point, the following relations are satisfied:
\[
\Gamma_- = W_0 - \frac{\sqrt{\Lambda_0}}{2} = 0, \tag{S122}
\]
\[
\Lambda_0 = (\Delta W_0)^2 + 4W_0^{AB}W_0^{BA} = 0, \tag{S123}
\]
\[
R^A = \left|\frac{\sqrt{\Lambda_0} - \Delta W_0}{2W_0^{BA}}\right| R^B, \tag{S124}
\]
where \(R^a = \|\vec{P}_0^a\| (> 0)\),
\[
W_0 = \frac{W_0^{AA} + W_0^{BB}}{2} = -\frac{1}{2}[(j_{AA} + j_{BB})\rho - 2\eta - \frac{1}{2\eta}(\|\vec{Q}_0^A\|^2 + \|\vec{Q}_0^B\|^2)], \tag{S125}
\]
\[
\Delta W_0 = \frac{W_0^{AA} - W_0^{BB}}{2} = -\frac{1}{2}[(j_{AA} - j_{BB})\rho - 2\eta - \frac{1}{2\eta}(\|\vec{Q}_0^A\|^2 - \|\vec{Q}_0^B\|^2)]. \tag{S126}
\]

with
\[
\|\vec{Q}_0^A\|^2 = \|j_{AA}\vec{P}_0^A + j_{AB}\vec{P}_0^B\|^2, \tag{S127}
\]
\[
\|\vec{Q}_0^B\|^2 = \|j_{BA}\vec{P}_0^A + j_{BB}\vec{P}_0^B\|^2. \tag{S128}
\]

Here, Eq. (S122) is the steady state condition, Eq. (S123) is condition for the exceptional point, and Eq. (S124) is derived from Eqs. (S75) and (S76). Eq. (S123) shows that non-reciprocal interaction with \(j_{AB}j_{BA}, W_0^{AB}W_0^{BA} < 0\) is necessary for the chiral flocking phase to appear. Below, we assume without loss of generality that the interaction are nonreciprocal with opposite sign, i.e. \(j_{AB}j_{BA}, W_0^{AB}W_0^{BA} < 0\).

From Eqs. (S122) and (S123),
\[
\Delta W_0 = \pm 2\sqrt{|W_0^{AB}W_0^{BA}|}, \tag{S129}
\]
Plugging this into Eq. (S124), we get
\[
R_B = \sqrt{\left|\frac{W_0^{BA}}{W_0^{AB}}\right| R_A} = \sqrt{\left|\frac{j_{BAP}}{j_{BPA}}\right| R_A}. \tag{S130}
\]
For the flocking phase,
\[
\|\vec{Q}_0^A\|^2 = (j_{AA}R_A + j_{AB}R_B)^2, \tag{S131}
\]
\[
\|\vec{Q}_0^B\|^2 = (j_{BA}R_A + j_{BB}R_B)^2, \tag{S132}
\]
while for the antiflocking phase,
\[
\|\vec{Q}_0^A\|^2 = (j_{AA}R_A - j_{AB}R_B)^2, \tag{S133}
\]
\[
\|\vec{Q}_0^B\|^2 = (j_{BA}R_A - j_{BB}R_B)^2. \tag{S134}
\]
We solve the coupled equations

$$W_0 = 0,$$  \hspace{1cm} (S135)

Eq. (S129), and Eq. (S130) for $R_A, R_B$ and the critical value $j_{EP}^2$ for a given $j_-$.

We can solve the above equations analytically in the case $j_{AA} = j_{BB} = j$, as we perform in the following. We only consider here the flocking-to-chiral phase transition point, as the antiflocking-chiral transition point can be computed in a similar way. Using Eq. (S130) to eliminate $R_B$, Eq. (S129) yields the relation

$$R_A = \frac{4nj_{AB}\sqrt{\rho}}{2jj_{AB}(j_{AB} - j_{BA})/j_{AB} + j^2(j_{AB} + j_{BA}) - j_{AB}j_{BA}(j_{AB} + j_{BA})},$$  \hspace{1cm} (S136)

We substitute this to Eq. (S135) to get,

$$(j^2 + j_{AB}j_{BA})\left[j(j_{AB} + j_{BA}) + \sqrt{-j_{AB}j_{BA}(j_{AB} - j_{BA})}\right]\rho = \left[(j^2 - j_{AB}j_{BA})(j_{AB} + j_{BA}) + 2j\sqrt{-j_{AB}j_{BA}(j_{AB} - j_{BA})}\right]\eta.$$

This can be rewritten in terms of $j_{\pm}$ introduced in Eq. (S116),

$$(j^2 + j_{+}^2 - j_{-}^2)(jj_{+} + j - \sqrt{j_{+}^2 - j_{-}^2})\rho = (2jj_{-} - \sqrt{j_{+}^2 - j_{-}^2} + j_{+})(jj_{+} + j - \sqrt{j_{+}^2 - j_{-}^2})\eta,$$

which can be organized into a cubic equation in terms of $j_{+}^2$ as,

$$aj_{+}^6 + bj_{+}^4 + cj_{+}^2 + d = 0$$

with

$$a = (\eta + j\rho)^2 + j^2\rho^2,$$

$$b = 2j^2\rho^2 - 2\eta j^2(j_{-}^2 - j_{+}^2) - 8\eta jj_{-}^2 - 3jj_{-}^2\rho^2,$$

$$c = \eta^2 j^4 + 6\eta^2 j^2j_{-}^2 + \eta^2 j_{-}^4 - 2\eta^2 j^3\rho - 4\eta j\rho j_{-}^2 + 10\eta jj_{-}^2\rho + j^6\rho^2 - j^4j_{-}^2\rho^2 - 3jj_{-}^4\rho^2 + 3j_{-}^6\rho^2,$$

$$d = j^4[-4\eta^2 j^2(1 + j\rho) - j^4\rho^2 - 4\eta jj_{-}^2 + j^4\rho^4 + 2j^2\rho^2j_{-}^2 - \rho^2j_{-}^4].$$

This can be solved exactly using Cardano’s formula. Its real solution is plotted as a red line in Fig. 2g in the main text, giving an excellent agreement with the (anti)flocking/chiral phase transition lines obtained numerically.

### III. EXCITATION SPECTRUM AND STABILITY ANALYSIS

In this section, we provide a linear analysis on the fluctuations around the mean-field solution obtained in Sec. II to study the excitations properties as well as the stability of the phases described there as well as the nature of the phase transitions between them. We confirm that there is a wide range of parameters where the flocking, antiflocking, and chiral phases are stable, as summarized in the phase diagram in Fig. 4a in the main text.

We show from this analysis that the chiral phase emerges by the coalescence of the collective eigenmodes in the transverse channel, which is a fundamentally different mechanism from the conventional phase transition scenarios [37]. We further show that the emergence of the swap phase can be understood as the instability of the flocking/antiflocking phase against the global longitudinal fluctuations. Based on these results, we argue that it leads the appearance of chiral-swap mixed phase exhibiting quasiperiodic oscillation in time, and explain the occurrence of tetracritical points with reduced codimension in the phase diagram. We also describe how the combination of exceptional points and the convective terms enforces the occurrence of the pattern-forming instabilities at the (anti)flocking-to-chiral phase transition.

#### A. General considerations

We assume the existence of a homogeneous solution $(\rho_a(t), P_a(t))$ to the equation (S51), which therefore also satisfies the mean-field Eq. (S64). This equation is of the form

$$\partial_t \psi(t,r) = f(\psi(t,r), \nabla \psi(t,r), \ldots)$$  \hspace{1cm} (S144)
The linear stability and the fate of small excitations (e.g. waves) on top of the steady-state are ruled by the linearized equation of motion for small perturbations $\delta \psi = \psi - \psi_{ss}$ on top of a steady-state $\psi_{ss}$. At first order in $\delta \psi$, the perturbations are described by the equation

$$
\partial_t \delta \psi(t, r) = f(\psi(t, r), \nabla \psi(t, r), \ldots) - f(\psi_{ss}(t), \nabla \psi_{ss}(t) \equiv 0, \ldots) \simeq \hat{L}(t) \delta \psi(t, r) + O(\delta \psi^2) \tag{S145}
$$

where $\hat{L}(t)$ is a linear (differential) operator, that might depend on time through the steady-state $\psi_{ss}(t)$. As this operator is linear, we use the Fourier transform to block-diagonalize the differential operators in momentum space, where we are left with a family of linear equations of the form

$$
\partial_t \delta \psi(t, k) = \mathcal{L}(t, k) \delta \psi(t, k) \tag{S146}
$$

where $\mathcal{L}(t, k)$ are finite matrices. In terms of the perturbations $\delta P^a_\mu$ and $\delta \rho^a$ of the polarization fields and density fields,

$$
\begin{align*}
\partial_t \delta P^a_\mu &= [\mathcal{L}_{PP}]^{ab}_\mu \delta P^b_\nu + [\mathcal{L}_{P\rho}]^{ab}_\mu \delta \rho^b \\
\partial_t \delta \rho^a &= [\mathcal{L}_{\rho P}]^{ab} \delta P^b_\nu + [\mathcal{L}_{\rho \rho}]^{ab} \delta \rho^b. 
\end{align*} \tag{S147a,b}
$$

Hence, the matrix elements of $\mathcal{L}$ are

$$
[\mathcal{L}_{PP}]^{ab}_\mu = -D_\theta \delta_{0ab} \delta_{\mu\nu} + j_{ab} \delta^\rho \delta_{\mu\nu} - \frac{1}{2D_\theta} \sum_{c,d,\rho} j_{ac} j_{ad} P^c_\mu P^d_\rho \delta_{\mu\nu} - \frac{1}{D_\theta} \sum_{c} j_{ab} j_{ac} P^a_\mu P^c_\nu + D_\theta (-k^2) \delta_{ab} \delta_{\mu\nu} + \sum_{c} \lambda_{ac} \left[ -ik_{\mu} P^c_\rho \delta_{ab} \delta_{\mu\nu} + 3ik_{\nu} P^c_\rho \delta_{ab} - 3P^c_\mu ik_{\nu} \delta_{ab} \right] \tag{S148}
$$

and

$$
\begin{align*}
[\mathcal{L}_{P\rho}]^{ab}_\mu &= \left( \sum_{c} j_{ac} P^c_\rho \right) - \frac{v_{0}}{2} ik \\
[\mathcal{L}_{\rho P}]^{ab}_\mu &= -\delta_{0ab} v_{0} ik_{\mu} \\
[\mathcal{L}_{\rho \rho}]^{ab}_\mu &= 0. \tag{S149, 150, 151}
\end{align*}
$$

In the following, we neglect density fluctuations, for instance by assuming that the system is incompressible (see Refs. [89, 90] for an analysis of the consequences in the Toner-Tu model for a single population). Hence, we will only consider the stability of the polarization channel described by $\hat{L} = \mathcal{L}_{PP}$.

### B. Stability of the flocking/antiflocking phase

We start with the stability analysis on the flocking and the antiflocking phase. In these phases, the steady state solutions can be written as

$$
\vec{F}_0^a = P^a_0 \vec{e}_f. \tag{S152}
$$

for $a = A, B$, where $P^a_0$ are real numbers and $\vec{e}_f$ is a unit vector pointing at the flocking direction. Crucially, $\vec{e}_f$ is the same for $a = A$ and $a = B$. With this notation, $P^A_0 P^B_0 > 0$ ($< 0$) corresponds to the (anti)flocking phase. It is convenient to decompose the fluctuations into longitudinal ($f$) and transverse ($\perp$) components,

$$
\delta \vec{F}^a(r, t) = \vec{e}_f \delta F^a_\parallel(r, t) + \vec{e}_\perp \delta F^a_\perp(r, t) = P^a_0 \left[ \vec{e}_f \delta n^a(r, t) + \vec{e}_\perp \delta \phi^a(r, t) \right], \tag{S153}
$$

where $\vec{e}_\perp$ is a unit vector perpendicular to the flocking direction $\vec{e}_f$. In this way, $\delta n^a$ represents longitudinal fluctuations normalized by $P^a_0$, and $\delta \phi^a$ is the angle describing the deviation from the flocking direction.

The relevant part of the linearized version (S146) of the coupled Toner-Tu equations (S62) then reads, in Fourier space,

$$
\hat{L}(k) d\vec{y}(k) = \omega(k) d\vec{y}(k), \tag{S154}
$$
where \( \mathbf{k} = k_x \hat{e}_x + k_z \hat{e}_z \),

\[
d\mathbf{\bar{y}}(k) = \begin{pmatrix} \delta \phi^A(k) \\ \delta \phi^B(k) \\ \kappa^A(k) \\ \kappa^B(k) \end{pmatrix},
\]

and \( \omega(k) \) is a dispersion relation of a collective mode. We decompose the matrix \( \hat{L}(k) \) controlling the linear excitations of the system as the block matrix

\[
\hat{L}(k) = \begin{pmatrix} \hat{L}_{\perp \perp}(k) & \hat{L}_{\perp \parallel}(k) \\ \hat{L}_{\parallel \perp}(k) & \hat{L}_{\parallel \parallel}(k) \end{pmatrix},
\]

where the blocks correspond to the transverse and longitudinal channels (blocks on the diagonal) and their coupling and have the explicit form

\[
\hat{L}_{\perp \perp}(k) = -i \left( \begin{array}{cc} W_{0}^{AA} & \frac{W_{0}^{AB} P_{0}^{B}}{P_{0}^{A}} \\ \frac{W_{0}^{BA} P_{0}^{B}}{P_{0}^{A}} & W_{0}^{BB} \end{array} \right) + \tilde{\lambda}_{\perp \perp} k_{\hat{e}_x} - i \left( \begin{array}{cc} D_{0}^{A} & 0 \\ 0 & D_{0}^{B} \end{array} \right) k^{2},
\]

(\ref{S155})

\[
\hat{L}_{\perp \parallel}(k) = -\tilde{\lambda}_{\perp \parallel} k_{\hat{e}_z},
\]

(\ref{S157b})

\[
\hat{L}_{\parallel \perp}(k) = \tilde{\lambda}_{\parallel \perp} k_{\hat{e}_x},
\]

(\ref{S157c})

\[
\hat{L}_{\parallel \parallel}(k) = -i \left( \begin{array}{cc} W_{0}^{AA} & \frac{W_{0}^{AB} P_{0}^{B}}{P_{0}^{A}} \\ \frac{W_{0}^{BA} P_{0}^{B}}{P_{0}^{A}} & W_{0}^{BB} \end{array} \right) - i \left( \begin{array}{cc} j_{BA} P_{0}^{A} Q_{0}^{A} & j_{AB} P_{0}^{B} Q_{0}^{B} \\ j_{BA} P_{0}^{A} Q_{0}^{A} & j_{AB} P_{0}^{B} Q_{0}^{B} \end{array} \right) + \tilde{\lambda}_{\parallel \parallel} k_{\hat{e}_z} - i \left( \begin{array}{cc} D_{0}^{A} & 0 \\ 0 & D_{0}^{B} \end{array} \right) k^{2},
\]

(\ref{S157d})

where

\[
Q_{0}^{A} = j_{AA} P_{0}^{A} + j_{AB} P_{0}^{B},
\]

(\ref{S158a})

\[
Q_{0}^{B} = j_{BA} P_{0}^{A} + j_{BB} P_{0}^{B},
\]

(\ref{S158b})

and \( \tilde{\lambda}_{\perp \perp}, \tilde{\lambda}_{\perp \parallel}, \tilde{\lambda}_{\parallel \perp}, \) and \( \tilde{\lambda}_{\parallel \parallel} \) are 2×2 real matrices that originates from the convective terms, given by

\[
\tilde{\lambda}_{\perp \perp} = \tilde{\lambda}_{\parallel \parallel} = \begin{pmatrix} 3 P_{0}^{A} \lambda_{AA} + P_{0}^{B} \lambda_{AB} & 2 P_{0}^{B} \lambda_{AB} \\ 2 P_{0}^{A} \lambda_{BA} & 3 P_{0}^{A} \lambda_{BA} + 3 P_{0}^{B} \lambda_{BB} \end{pmatrix},
\]

(\ref{S159})

\[
\tilde{\lambda}_{\perp \parallel} = \tilde{\lambda}_{\parallel \perp} = \begin{pmatrix} 5 P_{0}^{A} \lambda_{AA} + 3 P_{0}^{B} \lambda_{AB} & 2 P_{0}^{B} \lambda_{AB} \\ 2 P_{0}^{A} \lambda_{BA} & 3 P_{0}^{A} \lambda_{BA} + 5 P_{0}^{B} \lambda_{BB} \end{pmatrix}.
\]

(\ref{S160})

We use

\[
D_{a} = \frac{(v_{0}^{a})^{2}}{16 \eta} \quad \lambda_{ab} = v_{0}^{a} v_{0}^{b}
\]

(\ref{S161})

in our analysis (see Sec. I for further discussion on this point).

1. Computation of the stability regions of the phase diagram

We determine the regions of stability of the (anti)flocking phases (Fig. 4a of the main text) by first solving numerically the mean-field dynamical system in Eq. (S64) to obtain the quantities \( R_{a} = \| \delta \mathbf{P}_{a} \| (a = A, B) \) and \( \Delta \phi^{AB} = \angle (\mathbf{P}_{A}, \mathbf{P}_{B}) \). This allows to identify the phase, and to obtain the matrix \( \mathbf{L}(k) \) using Eq. (S148). The eigenvalues \( \kappa_{a}(k) = \sigma_{a}(k) + i \omega_{a}(k) \) of this matrix give the growth rates \( \sigma_{a}(k) \) and the frequencies \( \omega_{a}(k) \) of the perturbations. Because of the existence of a Goldstone mode, the largest growth rate is pinned to \( \sigma = 0 \) at \( k = 0 \). To evaluate the stability of the phase, we determine whether \( k = 0 \) is a local maximum by computing the sign of the eigenvalues of the Hessian matrix of \( \sigma(k) \) at \( k = 0 \), which is obtained by discretizing the second derivatives in momentum space. The result is presented in Fig. 4b of the main text. We have verified manually (by directly computing the growth rates as a function of \( k \)) that for a large range of parameters, an instability is present only if \( k = 0 \) is not a local minimum. However, this hypothesis might fail in particular cases. This shortcoming will be addressed in future works by determining the stability regions directly from the full momentum dependent growth rates. The largest growth rates as a function of the wavevector in Fig. 4b are directly obtained by diagonalizing \( \mathbf{L}(k) \) at each point.
2. Goldstone’s theorem and destabilization towards the chiral phase

Since the flocking and antiflocking phase are spontaneous symmetry broken phases, Goldstone’s theorem assures that at least one gapless eigenmode (i.e. the Goldstone mode) exists [91]. This can be shown explicitly as follows. Using the steady state mean-field equation (S67), or

\[
W_{0}^{A A} = - \frac{W_{0}^{A B} P_{0}^{B}}{P_{0}^{A}},
\]

\[
W_{0}^{B B} = - \frac{W_{0}^{B A} P_{0}^{A}}{P_{0}^{B}},
\]

the transverse-transverse block of the dynamical matrix \( \hat{L} \) at \( k = 0 \) can be simplified to

\[
L_{\perp \perp}(k = 0) = -i \left( \begin{array}{cc}
W_{0}^{A A} & -W_{0}^{A A} \\
-W_{0}^{B B} & W_{0}^{B B}
\end{array} \right).
\]

Then, noting that the transverse and the longitudinal fluctuations decouple at \( k = 0 \) since \( L_{\perp 
(\text{k} = 0) = 0 } \), the vector

\[
d\vec{y}(k = 0) = \left( \begin{array}{c}
\delta \phi_{A}(0) \\
\delta \phi_{B}(0) \\
\delta n_{A}(0) \\
\delta n_{B}(0)
\end{array} \right) = \left( \begin{array}{c} 1 \\
1 \\
0 \\
0
\end{array} \right),
\]

which corresponds to the global in-phase rotation of the direction of the flocks, is a gapless mode:

\[
\omega(k = 0) = 0.
\]

We also find another eigenmode

\[
d\vec{y}(k = 0) = \left( \begin{array}{c}
\delta \phi_{A}(0) \\
\delta \phi_{B}(0) \\
\delta n_{A}(0) \\
\delta n_{B}(0)
\end{array} \right) = \left( \begin{array}{c} 1 \\
1 \\
-W_{0}^{B B}/W_{0}^{A A} \\
0
\end{array} \right),
\]

associated with a gapped spectrum

\[
\omega(k = 0) = -i(W_{0}^{A A} + W_{0}^{B B}) = i\sigma_{\perp \perp},
\]

where \( \sigma_{\perp \perp} \) is a growth rate of this mode. Since the "-" solution satisfies (note that \( \Lambda_{0} > 0 \) in the flocking and antiflocking phase)

\[
0 = \Gamma_{-} = \frac{1}{2}(-\sigma_{\perp \perp} - \sqrt{\Lambda_{0}}),
\]

we get

\[
\sigma_{\perp \perp} = -\sqrt{\Lambda_{0}} < 0,
\]

meaning that the "-" solution is stable, at least against the global transverse fluctuations involving a change in the relative flocking angle between the two species. In contrast, the "+" solution that satisfies,

\[
0 = \Gamma_{+} = \frac{1}{2}(-\sigma_{\perp \perp} + \sqrt{\Lambda_{0}}),
\]

gives a positive growth rate of the mode,

\[
\sigma_{\perp \perp} = \sqrt{\Lambda_{0}} > 0,
\]

thus is always unstable. This tells us that the only possible stable solution of the flocking and antiflocking phase is the "-" solution.

The damping gap \( \gamma_{\perp \perp} \equiv -\sigma_{\perp \perp} = \sqrt{\Lambda_{0}} \) of the "-" solution softens and the transverse mode destabilizes when \( \Lambda_{0} \) approaches zero. This point is nothing but an exceptional point of the dynamic matrix \( \hat{L}(k = 0) \) (which is simultaneously the exceptional point the matrix \( W_{0} \)), where the eigenvector given in Eq. (S167) coalesces with the Goldstone mode given by Eq. (S165). This mechanism by which the (anti)flocking phase gets destabilized into the chiral phase by the coalescence of a transverse fluctuations mode with the Goldstone mode is a mechanism unique to non-equilibrium systems where the dynamics is controlled by non-Hermitian matrices, and is in sharp contrast with the conventional phase transition mechanism where the gapped mode simply softens at the critical point but are still orthogonal to other eigenmodes [37].
3. Destabilization towards the swap phase and the emergence of the active time-quasicrystal

In this paragraph, we show that the longitudinal fluctuations can also get destabilized by the nonreciprocal interaction through a conventional phase transition mechanism, implying a phase transition to the swap phase.

The existence of two independent mechanisms of destabilization, with one in the transverse channel (that leads to the chiral phase) and one in the longitudinal channel (that leads to the swap phase), suggests the existence of a phase where both channels destabilize. As we show in Figs. 2f and 2g and in Fig. 3d of the main text, a mixed chiral-swap phase indeed exists, as a result of the simultaneous occurrence of these two instabilities.

We focus on the uniform longitudinal fluctuations in this channel, described by the linear operator

\[
\dot{\hat{X}}_\|^{j}_{\perp}(k=0) = -i \left( W_{0}^{AA} - W_{0}^{AB} \right) - i \left( \eta_{AA} W_{0}^{AA} P_{0}^{Q_{0}^{A}} - \frac{\eta_{BB}}{\rho_{BB}} P_{0}^{Q_{0}^{B}} \right),
\]

where we have used the steady state condition (S67) or (S162), (S163). The collective eigenmodes are given by

\[
\omega^{\pm}_{\|}(0) = -i \left[ \zeta_{\|} \pm \sqrt{\zeta_{\|}^2 - \eta_{\|}} \right],
\]

where

\[
\eta_{\|} = \frac{4}{\eta} \left[ j^{AA} W_{0}^{BB} P_{0}^{Q_{0}^{A}} + j^{BB} W_{0}^{AA} P_{0}^{Q_{0}^{B}} + W_{0}^{AA} W_{0}^{BB} (P_{0}^{Q_{0}^{A}} + P_{0}^{Q_{0}^{B}}) + \mu_{\|} P_{0}^{Q_{0}^{A}} P_{0}^{Q_{0}^{B}} \right],
\]

\[
\mu_{\|} = \frac{1}{\eta} \left[ j^{AA} j^{BB} - W_{0}^{AA} W_{0}^{BB} \right] = \frac{1}{\eta} \left[ j^{AA} j^{BB} - \left( \frac{\eta_{AA}}{\rho_{AA}} - \frac{\eta_{BB}}{\rho_{BB}} \right) j^{BB} - \frac{\eta_{BB}}{\rho_{BB}} \right] > 0,
\]

\[
\zeta_{\|} = W_{0}^{AA} + W_{0}^{BB} + \frac{1}{\eta} \left[ j^{AA} P_{0}^{Q_{0}^{A}} + j^{BB} P_{0}^{Q_{0}^{B}} \right].
\]

Equation (S174) shows that this mode is (un)stable when \( \eta_{\|} > 0(0) \), because the largest growth rate of this channel is given by

\[
\sigma_{\|} \equiv \max \left[ \text{Im}[\omega_{-}(k=0)], \text{Im}[\omega_{+}(k=0)] \right] = \sqrt{\zeta_{\|}^2 - \eta_{\|} - |\zeta_{\|}|} < 0(0).
\]

In the case where the inter-species coupling has the same sign \( j^{AB} j^{BA} > 0 \) (which includes the reciprocal case \( j^{AB} = j^{BA} \)), \( \eta_{\|} \) is always positive and thus stable. This is seen as follows. When \( j^{AB}, j^{BA} > 0(<0) \), the system is in a (anti)flocking phase at low enough noise strength, giving \( P_{0}^{A} P_{0}^{B} > 0(<0) \). Then, from Eqs. (S162) and (S163), we get \( W_{0}^{AA}, W_{0}^{BB} > 0 \). Similarly, from the definition of \( Q_{0}^{A} \) given by Eq. (S158), \( P_{0}^{A(B)} \) and \( Q_{0}^{A(B)} \) have the same signs irrespective of the sign of the inter-species coupling \( j^{AB}, j^{BA} \), i.e. \( P_{0}^{AA} Q_{0}^{B} > 0 \). Since all the terms contributing to \( \eta_{\|} \) in Eq. (S175) is thus positive as long as \( j^{AA} \) and \( j^{BB} \) has the same sign, the parameter \( \eta_{\|} \) is positive and thus stable (\( \sigma_{\|} < 0 \)).

This restriction is lifted in the strong nonreciprocal case where the inter-species interaction have opposite signs \( j^{AB} j^{BA} < 0 \). In particular, \( W_{0}^{AA}, W_{0}^{BB}, P_{0}^{Q_{0}^{A}} \) may all become negative. Thus, as the nonreciprocal interaction

---

**Fig. S2. Growth rate for the longitudinal perturbations near the swap transition.** We have used \( j_{AA} = j_{BB} = 1, \rho_{A} = \rho_{B} = 1, j_{-} = 0.75, \eta = 0.5, v_{0}^{A} = 0.01 \) and \( v_{0}^{B} = 0.03 \). Here \( j_{\text{flock}}^{\text{swap}} \approx 0.399 \) is the flocking-to-swap phase transition point.
Fig. S3. Bifurcation diagrams. Unless otherwise specified, the parameters are the same as in Fig. 2g of the main text. An exceptional point (marked by a black star) separates the branches labelled (+) and (−). The tetracritical points are found in the panel c and Fig. 2g in the main text at \((j^\ast_-, j^\ast_+) \approx (\pm 0.38, \pm 0.07)\). Note that the x-axis scales are different in the four panels.

increases, \(\eta_{\parallel}\) gets smaller and smaller until it approaches zero, or equivalently, \(\sigma_{\parallel} = 0\). Such softening of the longitudinal mode implies a phase transition to a swap phase.

This scenario is confirmed in Fig. S2, which shows the growth rate \(\sigma_{\parallel}\) of the collective mode in this channel, in the vicinity of the phase transition point to the swap phase. As expected, the negative growth rate \(\sigma_{\parallel}\) in the flocking phase softens (i.e. \(\sigma_{\parallel} = \text{Im} \omega_{\perp}(k = 0)\) approaches zero) at the flocking-to-swap phase transition point \((j_+ = j^\ast_{\text{swap}})\).

4. Tetracritical point

Here we argue that, from the properties shown above, a tetracritical point, which is the point where the four phases (i.e. the (anti)flocking, chiral, swap, and the chiral-swap phase) meet at a single point, naturally emerges at least within the mean-field approximation. This is demonstrated in Fig. 2g in the main text, where the tetracritical points...
are marked by black dots.

The properties that we have shown so far can be summarized as follows:

1. In the (anti)flocking phase, there exists two types of solutions, namely, the "−" and "+" solutions, but the "+" solution is always unstable. On the other hand, as long as the system is away from the exceptional point $\Lambda_0 > 0$, the "−" solution is always stable against the global ($k = 0$) transverse channel.

2. Approaching the exceptional point $\Lambda_0 = 0$ (which is the point where the "−" and the "+" solution coalesces), however, the (anti)flocking phase destabilizes in the transverse channel, signaling the phase transition to the chiral phase.

3. Independently from the above mechanism, "−" solution can also destabilize in the global longitudinal channel, implying the phase transition into the swap phase.

4. In the parameter region between the chiral and swap phase, a mixed chiral-swap time quasicrystal phase emerges.

The properties 1 and 2 show that the (anti)flocking-to-chiral phase transition always occurs at the many-body exceptional point. As an example of such a situation, in Figs. S3a and b, we have plotted the amplitude of the polarization $|\psi| = (\tilde{P}_0^A, \tilde{P}_0^B)^T$ for both the stable and unstable solutions in the (anti)flocking phase with $j_− < j^*_−$. As seen in the figure, the stable "−" solution and the unstable "+" solution coalesce at the exceptional point, which marks the phase transition point from the (anti)flocking phase to the chiral phase.

On the other hand, property 3 shows that there are cases where the "−" solution of the (anti)flocking phase exhibits phase transition to the swap phase. The property that this phase transition is associated with the destabilization of the longitudinal channel implies that the "susceptibility" of the amplitude of the polarization (i.e. the sensitivity of the amplitude of the polarization against a parameter change) diverges. This is indeed seen in the region $j_− > j^*_−$ in Fig. S3d at the (anti)flocking-to-swap phase transition point, where the derivative of the amplitude $|\psi|$ in the stable branch of the "−" solution diverges, i.e. $|\partial |\psi||/\partial j_+ | \to \infty$ at the transition point. In this case, the exceptional point sits in the unstable branch of solutions.

At $j_− = j^*_−$, these two types of transition points (i.e. the exceptional point and the diverging susceptibility point) merge at $j_+ = j^*_+$ as shown in Fig. S3c. Since these are the signals of the transition to the chiral and swap phase, respectively, and keeping in mind that the chiral/swap time quasicrystal phase occurs in the region between the chiral and the swap phase (property 4), the point $(j^*_−, j^*_+)$ is nothing but the tetracritical point.

5. Exceptional point enforced pattern formation

The chiral and swap phases appear in mean-field theory as spatially uniform instabilities of the (anti)flocking phases. Now we show that the flocking and antiflocking phases generically exhibit a finite wavelength instability in the transverse fluctuation channel in the vicinity of the exceptional point (except in highly fine-tuned situations), implying the occurrence of pattern formation. This originates from the singular behavior of the mean-field operator at this point combined with the presence of convective terms. At the exceptional point $\Lambda_0 = 0$ of the matrix $\tilde{W}_0$,

$$0 = \Gamma_- = \Gamma_+ = \frac{1}{2}(W_0^{AA} + W_0^{BB}),$$

and hence, $W_0^{AA} = -W_0^{BB}(\equiv W_0)$. As a consequence, the block $\tilde{L}_{\perp \perp}(k)$ corresponding to transverse fluctuations in Eq. (S157) reduces to

$$\tilde{L}_{\perp \perp}(k) = -i\begin{pmatrix} W_0 & -W_0 \\ W_0 & -W_0 \end{pmatrix} + \tilde{\lambda}_\perp^\perp i k_\parallel - i\begin{pmatrix} D_0^A & 0 \\ 0 & D_0^B \end{pmatrix} k^2.$$

The matrix $\tilde{L}_{\perp \perp}(k)$ has an exceptional point at $k = 0$, where the two collective modes given by Eqs. (S165) and (S167) coalesce [37]. Below, we restrict ourselves to $k_\perp = 0$, where the transverse fluctuations decouple from the longitudinal mode such that the eigenvalues of $\tilde{L}_{\perp \perp}(k = k_\parallel \vec{e}_\parallel)$ describe the collective modes of the system in this channel (i.e. $\tilde{L}_{\parallel \parallel}(k = k_\parallel \vec{e}_\parallel) = \tilde{L}_{\parallel \parallel}(k = k_\parallel \vec{e}_\parallel) = 0$). The frequencies of the collective modes are

$$\omega_\parallel^\perp(k_\parallel \vec{e}_\parallel) = \pm \sqrt{iC_0 k_\parallel + O(k_\parallel^2) + \lambda_\parallel^\perp k_\parallel - iD_0 k_\parallel^2}.$$

Here,

$$C_0 = -2W_0(\lambda_2^\perp + \lambda_3^\perp) \quad \text{and} \quad D_0 = \frac{D_0^A + D_0^B}{2},$$
where \( \hat{L} \perp \perp \) is parameterized as

\[
\hat{L} \perp \perp = \lambda_0 \hat{\sigma}_0 + \lambda_1 \hat{\sigma}_1 + i\lambda_2 \hat{\sigma}_2 + \lambda_3 \hat{\sigma}_3,
\]

(S183)

where \( \hat{\sigma}_\alpha \) (\( \alpha = 0, 1, 2, 3 \)) are the Pauli matrices, and \( \lambda_\alpha \) are real numbers since \( \hat{L} \perp \perp \) is a real matrix.

The leading order contribution in respect to \( k_\parallel \) has a singular form \( \omega_\parallel \perp \perp \sim \pm \sqrt{\Delta C_0} k_\parallel \perp \perp \) as long as \( C_0 \neq 0 \). As a result, since the quantity inside the square root is purely imaginary, at least one of the two modes is inevitably unstable, irrespective of the sign of \( C_0 \). This shows that in the vicinity of a phase transition point from a (anti)flocking to the chiral phase, the uniform state is always destabilized by transverse fluctuations.

This originates from the exceptional point physics in the presence of convective terms. Typically, the eigenvalues in the vicinity of the exceptional point behave as \( \omega_\parallel \perp \perp \sim \pm \sqrt{\Delta} \), where \( \Delta \) is a characteristic distance from the exceptional point. In our situation, recalling that \( \hat{L} \perp \perp (k = 0) \) is at an exceptional point, the finite momentum \( k \) contribution to \( \hat{L} \perp \perp (k_\parallel \perp \perp) \) can be regarded as the contributions that makes \( \hat{L} \perp \perp (k_\parallel \perp \perp) \) deviate from that point. Because of the presence of the convective term \( \hat{L} \perp \perp k_\parallel \perp \perp \), the leading order is \( O(k_\parallel \perp \perp) \). As a result, \( \Delta \sim i k_\parallel \perp \perp \), leading to \( \omega_\parallel \perp \perp \sim \pm \sqrt{i} k_\parallel \perp \perp \), implying an instability leading to pattern formation. The uniform flocking phase may be stabilized by moving away from the exceptional point.

C. Floquet stability analysis of the chiral phase

Here we provide a stability analysis on the chiral phase. As the mean-field solution in this phase depends on time, the method used to analyze the stability of the (anti)flocking phases cannot be directly applied. Instead, we take advantage of the periodicity in time of the mean-field solution to analyze the stability of the (anti)flocking phases. As the mean-field solution in this phase depends on time, the method used to analyze the stability of the (anti)flocking phases cannot be directly applied. Instead, we take advantage of the periodicity in time of the mean-field solution to analyze the stability of the (anti)flocking phases. As the mean-field solution in this phase depends on time, the method used to analyze the stability of the (anti)flocking phases cannot be directly applied. Instead, we take advantage of the periodicity in time of the mean-field solution to analyze the stability of the (anti)flocking phases. As the mean-field solution in this phase depends on time, the method used to analyze the stability of the (anti)flocking phases cannot be directly applied. Instead, we take advantage of the periodicity in time of the mean-field solution to analyze the stability of the (anti)flocking phases. As the mean-field solution in this phase depends on time, the method used to analyze the stability of the (anti)flocking phases cannot be directly applied. Instead, we take advantage of the periodicity in time of the mean-field solution to analyze the stability of the (anti)flocking phases.
where $\vec{p}_\alpha(t, k) = V(t)\delta \vec{P}_\alpha(k) = p_\alpha(t + T, k)$ is periodic in time. The eigenvalues $\lambda_\alpha(k)$ of $U(T, k)$ are called Floquet multipliers (or characteristic multipliers), and they can be written as $\lambda_\alpha(k) = e^{s_\alpha(k)}$ where $s_\alpha(k)$ is called a Floquet exponent (the Floquet exponents are also the eigenvalues of $L^{\text{eff}}$, and they are only defined up to a phase). The Floquet exponent $s_\alpha(k) = \sigma_\alpha(k) + i\omega_\alpha(k)$ can be decomposed into real and imaginary parts, which correspond to the growth rate $\sigma_\alpha(k)$ of the corresponding fundamental solution and its oscillation quasi-frequency $\omega_\alpha(k)$ (only defined up to multiples of $2\pi/T$). A positive (negative) growth rate corresponds to an unstable (stable) solution. Equivalently, the solution is stable when the absolute value of the Floquet multiplier $|\lambda_\alpha|$ is smaller than the unity.

To determine the stability of the chiral phase, we first solve numerically the mean-field dynamical system in Eq. (S64) to obtain the time-independent quantities $R_\alpha = ||\vec{P}_\alpha||$ ($a = A, B$) and $\Delta \phi^{AB} = \angle(\vec{P}_A, \vec{P}_B)$. We obtain the time-dependent mean-field solution $\vec{P}_{a,\text{ss}}(t)$ as well as its period $T$ using Eqs. (S77) and (S89). This allows us to compute the time-dependent matrix $L(t)$ from Eq. (S148), where $\vec{P}_a$ is replaced by $\vec{P}_{a,\text{ss}}(t)$. We then use a discrete version of Eq. (S186) (where $\delta t$ is finite) to compute $U(T, k)$, which is diagonalized to determine the Floquet multipliers $\lambda_\alpha(k)$.

A direct inspection of the spectra shows that as in the time-independent case, we always have $|\lambda_\alpha(0)| \leq 1$ in the chiral phase (because in is the mean-field solution), with one marginal eigenvalue pinned at $|\lambda| = 1$. We focus on the multiplier $\lambda(0)$ with maximal absolute value, for which $|\lambda(0)| = 1$. We use the same procedure as for the (anti)flocking phases to estimate the stability of the phase from the sign of the eigenvalues of the Hessian matrix of the function $k \mapsto |\lambda(k)|$ at $k = 0$, and the same caveat applies. By carrying out this procedure, we find wide regions in parameter space where the chiral phase is stable, as shown in Fig. 4b of the main text.

**IV. ANALYTICAL SOLUTION FOR TWO AGENTS**

Let us consider two agents $A$ and $B$. In this section, we assume that they are always close enough to interact, and neglect the effect of noise. The evolution of their angles $\theta_A$ and $\theta_B$ is then described by the equations

\begin{align}
\partial_t \theta_A &= J_{AB} \sin(\theta_B - \theta_A) \\
\partial_t \theta_B &= J_{BA} \sin(\theta_A - \theta_B).
\end{align}

(S189) (S190)

It is convenient to define

\[ \bar{\theta} = \theta_A + \theta_B \quad \Delta \theta = \theta_A - \theta_B \quad J_{\pm} = J_{AB} \pm J_{BA} \]

(S191)

so that the dynamical system above is equivalent to

\begin{align}
\partial_t \bar{\theta} &= -J_- \sin(\Delta \theta) \\
\partial_t \Delta \theta &= -J_+ \sin(\Delta \theta).
\end{align}

(S192) (S193)

When the reciprocal interactions vanish, $J_+ = 0$, then $\Delta \theta(t) = \Delta \theta(0)$ is a constant (equal to its initial value), and the average angle $\bar{\theta}(t)$ increases linearly. This corresponds to a circular motion at a frequency $1/[J_- \sin(\Delta \theta(0))]$, whose characteristics are highly sensitive to the initial conditions. In a purely anti-reciprocal system (and in the absence of noise), this circular motion goes on forever. However, we shall see momentarily that in this very simple model, any amount of reciprocal interaction $J_+ \neq 0$ leads to the eventual suppression of the circular motion, on a time scale of order $1/J_+$.

Indeed, when the reciprocal part of the interaction $J_+$ is nonzero, $\Delta \theta$ relaxes to either 0 or $\pi$ depending on the sign of $J_+$. In terms of $g = \tan \Delta \theta(t)/2$, for which $g'(t) = ag(t)$, this is an exponential relaxation with a time scale $1/J_+$, namely

\[ \tan \left( \frac{\Delta \theta(t)}{2} \right) = \tan \left( \frac{\Delta \theta(0)}{2} \right) \exp \left( -J_+ t \right). \]

(S194)

The evolution of the average angle $\bar{\theta}(t)$ is slaved to the evolution of $\Delta \theta$, as

\[ \bar{\theta}(t) = \bar{\theta}(0) - J_- \int_0^t \sin(\Delta \theta(t')) dt'. \]

(S195)

As a consequence, $\bar{\theta}(t)$ becomes approximately constant when $\Delta \theta(t)$ is close to 0 or $\pi$. More precisely, we find

\[ \bar{\theta}(t) = \bar{\theta}(0) - 2\frac{J_-}{J_+} [\arccot (a \exp (-J_+ t)) - \arccot (a)]. \]

(S196)
where $a = \tan(\Delta \theta(0)/2)$. This explicit form is interesting as it exhibits an indeterminate form as $J_+ \to 0$, that is can however be resolved and yields a linear behavior in time consistent with the previous discussion,

$$\bar{\theta}(t) = \bar{\theta}(0) - 2J_- \frac{at}{1 + a^2} = \bar{\theta}(0) - J_- \sin[\Delta \theta(0)]t. \quad (S197)$$
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