Abstract
The time-harmonic Maxwell equations at high wavenumber $k$ are discretized by edge elements of degree $p$ on a mesh of width $h$. For the case of a ball as the computational domain and exact, transparent boundary conditions, we show quasi-optimality of the Galerkin method under the $k$-explicit scale resolution condition that (a) $kh/p$ is sufficient small and (b) $p/\ln k$ is bounded from below.

Keywords Maxwell equations · Time-harmonic · High-frequency · Wavenumber explicit · $hp$-FEM quasi-optimality

1 Introduction
High-frequency electromagnetic scattering problems are often modeled by the time-harmonic Maxwell equations (2.1), and the high-frequency case is characterized by a large wavenumber $k > 0$. The solution is then highly oscillatory, and its numerical resolution requires fine meshes. Besides this natural condition on the discretization, a second, more subtle issue arises in the high-frequency regime, namely the difficulty of
Galerkin discretizations to control dispersion errors. That is, in fixed-order methods the discrepancy between the best approximation from the discrete space and the Galerkin error widens as the wavenumber $k$ increases. It is the purpose of the present paper to show for a model problem that high-order methods are able to control these dispersion errors and can lead to quasi-optimality for a fixed (but sufficiently large) number of degrees of freedom per wavelength.

For the related, simpler case of high-frequency acoustic scattering, which is modeled by the Helmholtz equation, substantial progress in the understanding of the dispersive properties of low-order and high-order methods has been made in the last decades. We mention the dispersion analyses on regular grids for fixed-order Galerkin methods [7,27–29], the works [2,5,6] for high-order methods and [4] for a non-conforming discretization and refer the reader to [21,41] for a more detailed discussion. These analyses on regular grids give strong arguments for the numerical observation that high-order discretizations are much better suited to control dispersion errors than low-order methods. For general meshes, a rigorous argument in favor of high-order (conforming and non-conforming) methods is put forward in the works [21,34,38,40,41], where stability and convergence analyses that are explicit in the mesh size $h$, the approximation order $p$, and the wavenumber $k$ are provided for several classes of Helmholtz problems. The underlying principles in these works are not restricted to FEM discretizations; indeed, [31] applies these techniques in a Helmholtz BEM context.

The numerical analysis focusing on the dispersive properties of high-order methods for the time-harmonic Maxwell equations is to date significantly less developed. An analysis on regular grids that is explicit in the polynomial degree $p$ is available in [3]. A convergence analysis for a Maxwell problem on general grids that is explicit in the mesh size $h$, the polynomial degree $p$, and the wavenumber $k$ is the aim of the present article. To fix ideas, we consider as a model problem the time-harmonic Maxwell equations (2.1) in full space $\mathbb{R}^3$. Since a (high-order)-finite element method (FEM) is our goal, we study the equivalent reformulation of the full space problem as a problem in the unit ball $\Omega = B_1(0)$ complemented with transparent boundary conditions on $\Gamma = \partial \Omega$ [cf. (2.6)]. As we study conforming Galerkin discretizations, the starting point for the discretization is the variational formulation (2.27). For this model problem, our main result is Theorem 4.17, which establishes quasi-optimality of the Galerkin method based on Nédélec type I elements of degree $p$ under the scale resolution conditions

$$kh/p \leq c_1 \quad \text{and} \quad p \geq c_2 \ln k; \quad (1.1)$$

here $c_2 > 0$ may be chosen arbitrarily and $c_1 > 0$ is sufficiently small but independent of $h$, $k$, and $p$.

We focus on a conforming Galerkin discretization, which will require the scale resolution condition (1.1) to ensure existence of the discrete solution. It is worth pointing out that alternatives to conforming Galerkin methods have been proposed in the literature. Without attempting completeness and restricting ourselves to approaches based on higher-order polynomials, we mention stabilized methods for Helmholtz [22,23,25,54] and Maxwell [24,32] problems; hybridizable methods [15]; least-squares-type methods [8,16] and discontinuous Petrov Galerkin methods, [20,50]. In convex domains or domains with a smooth boundary, $H^1$-conforming discretiza-
tions for Maxwell problems can be employed instead of $H^{(\text{curl})}$-conforming ones; see [49], [48] for a $k$-explicit theory.

We close this introduction by emphasizing that, as in the case of the Helmholtz equation, the techniques employed in the present work are not restricted to the model problem under consideration here; in the forthcoming [42], we apply our techniques to Maxwell’s equations complemented with impedance boundary conditions. Finally, a general note on notation is warranted: as we aim at a $k$-explicit theory, we indicate constants that (possibly) depend on the wavenumber $k$ by a subscript $k$.

1.1 Road Map: Setting

Our $k$-explicit convergence analysis of high-order FEM for Maxwell’s equations requires a variety of tools including compactness arguments, $k$-explicit regularity based on decomposing the solution into parts with finite regularity and analytic parts as developed for the Helmholtz equation, and commuting diagram operators that are explicit in the polynomial degree $p$. It may therefore be useful to provide an outline of the key steps.

The reformulation of the original full space problem (2.1) as the problem (2.6) in a bounded domain $\Omega \subset \mathbb{R}^3$ uses transparent boundary conditions, which are expressed in terms of the capacity operator $T_k$ [see Sect. 2.2 and (5.7) for its explicit series representation in the case of the unit ball $\Omega = B_1(0)$]. The pertinent sesquilinear form that we consider in this work is then

$$A_k(u, v) = (\text{curl } u, \text{curl } v) - k^2(u, v) - ik(T_k u_T, v_T)_{\Gamma}.$$ 

Here, $(\cdot, \cdot)$ is the $L^2(\Omega)$ inner product and $(\cdot, \cdot)_\Gamma$ the $L^2(\Gamma)$ inner product with $\Gamma = \partial \Omega$. The corresponding norms are denoted by $\| \cdot \|_\Omega$ and $\| \cdot \|_\Gamma$ and we write $\| \cdot \|$ short for $\| \cdot \|_\Omega$ if the domain $\Omega$ is clear from the context. The subscript $T$ indicates that the tangential component of the trace is considered. For $\Omega = B_1(0)$, our analysis will be explicit in the wavenumber $k$ and we therefore focus on the case $\Omega = B_1(0)$ in this introduction.

1.2 Road Map: The Maxwell Aspect

Let us first discuss the key issues that are specific to discretizations of Maxwell’s equations; in the following Sect. 1.3, we will focus on the additional difficulties arising from making the error analysis explicit in $k$. The arguments that we highlight in the current Sect. 1.2 are essentially those of [9,13,26,44] and [43, Sec. 7.2].

To understand the Galerkin error for Maxwell’s equations, it is imperative to decompose the various fields in gradient fields and solenoidal fields, both in $\Omega$ and on the surface $\Gamma$. The tangential field $u_T$ is decomposed as a gradient part $u^\nabla$ and a (surface) divergence-free part $u^\text{curl}$. The decomposition $u_T = u^\nabla + u^\text{curl}$ leads to the decomposition of the sesquilinear form $A_k$ as [cf. (4.3)]

$$A_k(u, v) = (\text{curl } u, \text{curl } v) - ik\left(T_k u^\text{curl}, v^\text{curl}\right)_\Gamma - ik\left(k^2(u, v) + i k\left(T_k u^\nabla, v^\nabla\right)_\Gamma\right).$$
By \cite[Thm. 5.3.6]{47}, we have for $\Omega = B_1(0)$ sign properties of the expressions $i k(T_k u^{\text{curl}}_\omega, u^{\text{curl}}_\omega)^\tau$ and $(\mathbf{u}, \mathbf{u})$. Furthermore, the curl part $u^{\text{curl}}_\omega$ of the tangential trace $u_T$ vanishes for gradient fields $u = \nabla \varphi, \varphi \in H^1(\Omega)$. Collecting these observations, we have:

(I) $\Re \left( (\text{curl} \mathbf{u}, \text{curl} \mathbf{u}) - i k(T_k u^{\text{curl}}_\omega, u^{\text{curl}}_\omega)^\tau \right) \geq \| \text{curl} \mathbf{u} \|^2 \forall \mathbf{u} \in \mathbf{X}$, where $\mathbf{X} := \mathbf{H}(\Omega, \text{curl})$, \cite[cf. (4.3) in conjunction with Remark 2.3]{47};

(II) $\Re \left( \langle \nabla \varphi, \nabla \varphi \rangle \right) \geq (k \| \nabla \varphi \|^2 \forall \varphi \in H^1(\Omega), \text{cf. (4.20)}$; \cite[Thm. 5.3.6, Lemma 5.2]{47};

(III) $A_k(\mathbf{u}, \nabla \varphi) = -\langle (\mathbf{u}, \nabla \varphi) \rangle \forall \varphi \in H^1(\Omega), \mathbf{u} \in \mathbf{H}(\Omega, \text{curl})$, \cite[Thm. 5.3.6, Lemma 5.2]{47};

Assuming continuity of $A_k$ and $(\cdot, \cdot)$ with respect to the norm $\| \cdot \|_{\text{curl}, \Omega, k}$ [defined in (1.2)], this analysis shows that quasi-optimality of the Galerkin method can be achieved provided one can ensure

\[
2 \sup_{\mathbf{v}_h \in \mathbf{X}_h \setminus \{0\}} \frac{\Re \langle (\mathbf{e}_h, \mathbf{v}_h) \rangle}{\| \mathbf{v}_h \|_{\text{curl}, \Omega, k} \| \mathbf{e}_h \|_{\text{curl}, \Omega, k}} < 1.
\]

It is tempting to treat this term by a duality argument. However, the duality argument cannot be applied directly since the map $\mathbf{X} : \mathbf{v} \mapsto (\cdot, \mathbf{v}) \in \mathbf{X}'$ is not necessarily compact. In the numerical analysis of Maxwell’s equations, this lack of compactness is addressed by suitable “continuous” and “discrete” Helmholtz decompositions, thereby exploiting that $\mathbf{v}_h$ is from the discrete space $\mathbf{X}_h$. Specifically, we decompose $\mathbf{v}_h \in \mathbf{X}_h$ in two ways (“continuous Helmholtz decomposition” and “discrete Helmholtz decomposition”) into a divergence-free part and a gradient part:

\[
v_h = \Pi_{\text{curl}, \omega}^* \mathbf{v}_h + \Pi_{\text{div}, \omega}^\tau \mathbf{v}_h
\]

(with “continuous” $\Pi_{\text{curl}, \omega}^* \mathbf{v}_h \in \mathbf{X}, \Pi_{\text{div}, \omega}^\tau \mathbf{v}_h \in \mathbf{X} \cap \nabla H^1(\Omega)$; see (IV));

\[
v_h = \Pi_{\text{curl}, \omega}^* \mathbf{v}_h + \Pi_{\text{div}, \omega}^\tau \mathbf{v}_h
\]

(with “discrete” $\Pi_{\text{curl}, \omega}^* \mathbf{v}_h \in \mathbf{X}_h, \Pi_{\text{div}, \omega}^\tau \mathbf{v}_h \in \mathbf{X}_h \cap \nabla H^1(\Omega)$; see (V)).
decompositions (1.6), (1.7)

\[
(e_h, v_h) = (e_h, \Pi^\text{curl,*} v_h) + (e_h, \Pi^\text{curl,*}_h v_h - \Pi^\text{curl,*}_h v_h) =: T_2 + T_3. \tag{1.8}
\]

The convergence analysis based on this decomposition then relies on a) the fact that the term \( T_2 = (e_h, \Pi^\text{curl,*} v_h) \) can be estimated with a duality argument and b) that \( \Pi^\text{curl,*}_h v_h - \Pi^\text{curl,*}_h v_h \) is shown to be small.

The continuous and discrete Helmholtz decompositions (1.6), (1.7) are defined as follows:

(IV) (decomposition in gradient part and divergence-free part) The gradient part \( \Pi^\nabla,* \in \nabla H^1(\Omega) \) is defined by the “orthogonality” condition

\[
\left( (\nabla \psi, \Pi^\nabla,* v) \right) = \left( (\nabla \psi, v) \right) \quad \forall \psi \in H^1(\Omega),
\]

which is well posed by (II). We set \( \Pi^\text{curl,*} := I - \Pi^\nabla,* \) and denote its range by \( V_0^* \). We note that the operators \( \Pi^\nabla,* \) and \( \Pi^\text{curl} \) effect a stable decomposition of the direct sum \( X = V_0^* \oplus \nabla H^1(\Omega) \). The above-mentioned duality argument for \( T_2 \) relies on the compactness of \( X \ni v \mapsto (\cdot, \Pi^\text{curl,*} v) \in X' \), which is shown in Lemma 4.12 and ultimately relies on the embedding \( V_0^* \subset H^1(\Omega) \) (Lemma B.1).

(V) (decomposition of discrete functions in gradient part and discrete divergence-free part) Let \( S_h \subset H^1(\Omega) \) be defined by the requirement that the following (discrete) exact sequence property holds:

\[
S_h \xrightarrow{\nabla} X_h \xrightarrow{\text{curl}} \text{curl} X_h \tag{1.9}
\]

[cf. (3.8) for the specific example of \( hp\)-FEM]. We define the discrete version \( \Pi^\nabla,*_h : X \rightarrow \nabla S_h \) of \( \Pi^\nabla,* \) by the “orthogonality” condition

\[
\left( (\nabla \psi, \Pi^\nabla,*_h v) \right) = \left( (\nabla \psi, v) \right) \quad \forall \psi \in S_h
\]

and set \( \Pi^\text{curl,*}_h := I - \Pi^\nabla,*_h \).

While the term \( T_2 \) in (1.8) is treated by a duality argument, control of the term \( T_3 \) in (1.8) relies on the existence of a projector \( \Pi^E_h \) (and a companion operator \( \Pi^F_h \)) with a commuting diagram property:

(VI) (commuting diagram projector) Define \( V_0^* := \{ v \in V_0^* | \text{curl} v \in \text{curl} X_h \} \). We require the existence of an operator \( \Pi^E_h : V_0^* + X_h \rightarrow X_h \) with the commutating diagram property

\[
\text{curl} \quad \Pi^E_h \quad \text{curl} \quad \Pi^E_h = \Pi^F_h \text{ curl}.
\]
(c) $\Pi_h^E$ has some approximation properties in $L^2(\Omega)$:

$$k\|v - \Pi_h^E v\| \leq \eta_{6}^{\text{alg}}\|v\|_{\text{curl},k} \quad \forall v \in V_{0,h}^*, \quad (1.10)$$

where the parameter $\eta_{6}^{\text{alg}}$ quantifies certain approximation properties of $X_h$ (e.g., in terms of the mesh size $h$ and polynomial degree $p$).

**Remark 1.1** In the case of $hp$-FEM, the operators $\Pi_h^E$ and $\Pi_h^F$ will be constructed in an element-by-element fashion (cf. Definition 8.1) from the operators $\hat{\Pi}_p^{\text{curl},c}$ and $\hat{\Pi}_p^{\text{div},c}$ (cf. Theorem 8.3) that are defined on the reference tetrahedron $\hat{K}$. In the $hp$-FEM setting, the quantity $\eta_{6}^{\text{alg}}$ in (1.10) is estimated via Lemma 8.6, (iii) by $\eta_{6}^{\text{alg}} \lesssim kh/p$; see (4.72).

**Remark 1.2** Various approximation properties $\eta_{\ell}$ will appear in our analysis, which depend on the subspace $X_h$. In the context of $hp$-finite elements, these quantities $\eta_{\ell}$ will depend on the mesh width $h$, the polynomial order $p$ of approximation, and the regularity of the functions involved. Given that we focus on high-order FEM with the potential of exponential convergence, we employ the following notational convention: If some $\eta_{\ell}$ is (generically) algebraically small in $p$, we employ the superscript “alg” while we use the superscript “exp” if the quantity is exponentially small.

The use of the properties of $\Pi_h^E$ required in (VI) becomes apparent if we observe the following arguments for estimating $T_3$:

(i) The definition of $\Pi_h^{\text{curl},*}$ and $\Pi_h^{\text{curl},*}$ implies the “orthogonality”

$$\left(\nabla \tilde{\psi}_h, \left(\Pi_h^{\text{curl},*} - \Pi_h^{\text{curl},*}\right) v_h\right) = 0 \quad \forall \tilde{\psi}_h \in S_h. \quad (1.11)$$

(ii) From $\text{curl } \Pi_h^{\text{curl},*} = \text{curl } \Pi_h^{\text{curl},*} = \text{curl on } X_h$ by (1.6), (1.7), we get for any $v_h \in X_h$

$$\text{curl } \left(\Pi_h^{\text{curl},*} v_h - \Pi_h^E \Pi_h^{\text{curl},*} v_h\right) = \text{curl } v_h - \Pi_h^F \text{ curl } v_h \quad (VIIb)$$

$$= \text{curl } v_h - \text{curl } \Pi_h^F v_h \quad (VIIa) = \text{curl } (v_h - v_h) = 0. \quad (1.12)$$

(iii) The exact sequence property and the observation (1.12) imply that $\Pi_h^{\text{curl},*} v_h - \Pi_h^E \Pi_h^{\text{curl},*} v_h$ is the gradient of an element of $S_h$, i.e., $\Pi_h^{\text{curl},*} v_h - \Pi_h^E \Pi_h^{\text{curl},*} v_h = \nabla \psi_h$ for some $\psi_h \in S_h$.

(iv) Combining (II), (iii), (1.11) yields

$A \lesssim B$ is shorthand for $A \leq CB$ for some $C > 0$ that is independent of the wavenumber $k$, the mesh size $h$, the polynomial degree $p$, as well as functions appearing in $A$ and $B$. 

$\text{ Springer}$

$\text{ Springer}$
\[
\begin{align*}
k^2 \left\| \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h \right\|^2 \\
\leq \Re \left( \left( \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h, \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h \right) \right) \\
\leq \Re \left( \left( \left( I - \Pi^E \right) \Pi^{\text{curl},*} \mathbf{v}_h, \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h \right) \right). \tag{1.11}
\end{align*}
\]

(v) The continuity of \((\cdot, \cdot)\) (cf. (4.14), Proposition 5.12, Corollary 5.13) and the property \(\text{curl} \left( (I - \Pi^E) \Pi^{\text{curl},*} \mathbf{v}_h \right) = 0 = \text{curl} \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h\) (as a consequence of the above calculation) give

\[
\left\| \left( I - \Pi^E \right) \Pi^{\text{curl},*} \mathbf{v}_h \right\|_{\text{curl}, \Omega, k} = k \left\| \left( I - \Pi^E \right) \Pi^{\text{curl},*} \mathbf{v}_h \right\|
\]

so that we may continue the estimate (1.13):

\[
k^2 \left\| \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h \right\|^2 \leq C_{\text{cont}, k} \left\| \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h \right\|_{\text{curl}, \Omega, k} \left\| \left( I - \Pi^E \right) \Pi^{\text{curl},*} \mathbf{v}_h \right\|_{\text{curl}, \Omega, k}
\]

\[
= C_{\text{cont}, k} \left( k \left\| \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h \right\| \right) \left( k \left\| \left( I - \Pi^E \right) \Pi^{\text{curl},*} \mathbf{v}_h \right\| \right).
\]

Here, the constant \(C_{\text{cont}, k}\) could depend on \(k\).

(vi) The final step in treating \(T_3\) uses the continuity of \((\cdot, \cdot)\), the above steps, and the stability of the map \(\mathbf{v}_h \mapsto \Pi^{\text{curl},*} \mathbf{v}_h\):

\[
|T_3| = \left| \left( \mathbf{e}_h, \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h \right) \right|
\]

\[
\leq C_{\text{cont}, k} \left\| \mathbf{e}_h \right\|_{\text{curl}, \Omega, k} \left\| \left( \Pi^{\text{curl},*} - \Pi_h^{\text{curl},*} \right) \mathbf{v}_h \right\|_{\text{curl}, \Omega, k}
\]

\[
\leq C_k \left\| \mathbf{e}_h \right\|_{\text{curl}, \Omega, k} \left( k \left\| \left( I - \Pi^E \right) \Pi^{\text{curl},*} \mathbf{v}_h \right\| \right)
\]

\[
\leq C_k \eta_6^{\text{alg}} \left\| \mathbf{e}_h \right\|_{\text{curl}, \Omega, k} \left\| \Pi^{\text{curl},*} \mathbf{v}_h \right\|_{\text{curl}, \Omega, k} \leq C_k \eta_6^{\text{alg}} \left\| \mathbf{e}_h \right\|_{\text{curl}, \Omega, k} \left\| \mathbf{v}_h \right\|_{\text{curl}, \Omega, k}.
\]

Here, the constant \(C_k\) may depend on \(k\) (and is, of course, different in each occurrence). Recalling our starting point (1.5), we discover that the approximation space \(X_h\) and the operator \(\Pi^E_h\) should be such that \(\eta_6^{\text{alg}}\) can be made sufficiently small (see (4.72)).

A few more comments concerning the above procedure are in order:

Remark 1.3 (a) The basic estimate (1.3), (1.4) is formulated in such a way that one is led to study \((\mathbf{e}_h, \mathbf{v}_h)\) with \(\mathbf{v}_h \in X_h\) in the \textit{discrete} space \(X_h\). This seemingly innocuous choice has far reaching ramifications. First, one has that \(\text{curl} \Pi^{\text{curl},*} \mathbf{v}_h = \text{curl} \mathbf{v}_h = \text{curl} \Pi_h^{\text{curl},*} \mathbf{v}_h\), which allows one to replace the stronger \(\| \cdot \|_{\text{curl}, \Omega, k}\) norm by the weaker \(L^2\)-norm in the estimates of Step (v):
Second, the commuting diagram property of $\Pi_h^E$ and the (discrete) exact sequence property (1.9) are responsible for the “orthogonality” (1.11) (cf. Steps (i)—(iii)).

(b) The $L^2$-approximation properties of $\Pi_h^E$ stipulated in (Vic) can be met because of the special structure of the space $\mathbf{V}_{0,h}^*$; first, as we discovered in (IV), functions from $\mathbf{V}_{0,h}^*$ are in fact in $H^1(\Omega)$. Second, for functions $v \in \mathbf{V}_{0,h}^*$ one has that $\text{curl} \; v \in \text{curl} \; \mathbf{X}_h$ is a discrete object. For the specific case of Nédélec type I elements of degree $p$, an operator $\Pi_h^E$ is constructed on the reference tetrahedron in Theorem 8.3 (called $\widehat{\Pi}_p^{\text{curl},c}$ there) that exploits these properties and leads to the quantitative estimate $\eta_{\text{alg}}^3 = O(h k/p)$. We flag at this point that, while the space $\mathbf{V}_{0,h}^*$ is a space of divergence-free functions, the operator $\widehat{\Pi}_p^{\text{curl},c}$ is additionally defined for (elementwise) smooth (actually, elementwise $H^1(\text{curl})$) functions. This property will be needed in Sect. 1.3 to argue the benefits of high-order methods. \hfill \Box

1.3 Road Map: $k$-explicit Estimates

The argument outlined above does not take into account how the wavenumber $k$ enters the estimates, which occurs in various places, for example, in the continuity of $A_k$ and $\langle (\cdot, \cdot) \rangle$, the stability of the map $\Pi^{\text{curl},*}$, and the regularity properties of the solution $z$ of the dual problem $A_k(\cdot, z) = \langle (\cdot, \Pi^{\text{curl},*} v_h) \rangle$. Indeed, care is required as we only have the $k$-dependent continuity bounds (cf. Corollary 5.13)

$$ |\langle v, w \rangle| + |A_k(v, w)| \leq C k^3 \| v \|_{\text{curl}, \Omega, k} \| w \|_{\text{curl}, \Omega, k}. \quad (1.14) $$

1.3.1 Continuity of $A_k$, $\langle (\cdot, \cdot) \rangle$ and Treatment of $T_1$

The fundamental ingredient for $k$-explicit bounds that are useful for the analysis of high-order FEM is the ability to decompose functions $u \in X$ into “high-frequency” parts $H_\Omega u$ and “low-frequency” parts $L_\Omega u$. An overarching theme of the present work is that the high-frequency component $H_\Omega u$ leads to estimates uniform in $k$ in the expected Sobolev norms; the low-frequency component $L_\Omega u$ involves $k$-dependencies, but is smooth (even analytic), which can be exploited by high-order approximation spaces. We note that such decompositions $u = H_\Omega u + L_\Omega u$ of functions entail corresponding decompositions of sesquilinear forms such as $A_k$ and $\langle (\cdot, \cdot) \rangle$.

The frequency-splitting operators $L_\Omega$ and $H_\Omega$ are motivated by an analysis of the $k$-dependence of the continuity constants of $A_k$ and $\langle (\cdot, \cdot) \rangle$, e.g., in the bound $|A_k(u, v)| \leq C_{\text{cont}, k} \| u \|_{\text{curl}, \Omega, k} \| v \|_{\text{curl}, \Omega, k}$. One discovers that it is the capacity operator $T_k$ that introduces a $k$-dependence in $C_{\text{cont}, k}$. Inspection of the series expansion of $T_k$ in (5.7) (see in particular Lemma 5.3, which gives sharp bounds for the symbol of the operator $T_k$) shows that the $k$-dependence is due to the low-frequency parts of $u_T$.\hfill \Box
Having identified these components as the culprits for unfavorable $k$-dependencies, we introduce in Definition 4.2 the low-frequency operator $L_\Omega : \textbf{X} \rightarrow \textbf{X}$ and the high-frequency operator $H_\Omega = I - L_\Omega$ that have for $\Omega = B_1(0)$ the following properties:

(VII) (stability) $\|L_\Omega \textbf{u}\|_{\text{curl}, \Omega, k} \leq \|\textbf{u}\|_{\text{curl}, \Omega, k}$ and $\|H_\Omega \textbf{u}\|_{\text{curl}, \Omega, k} \leq 2\|\textbf{u}\|_{\text{curl}, \Omega, k}$ [cf. (5.28)].

(VIII) (smoothness) $L_\Omega \textbf{u}$ is analytic. Specifically, there are $C, \alpha, \gamma > 0$ independent of $k$ and $\textbf{u}$ such that $L_\Omega \textbf{u} \in \mathcal{A}(Ck^\alpha \|\textbf{u}\|_{\text{curl}, \Omega, k}, \gamma, \Omega)$ with the analyticity class $\mathcal{A}$ given by Definition 2.5 [cf. Theorem 5.9].

(IX) $(k$-uniform continuity at the expense of a compact perturbation) For some $C > 0$ independent of $k$ [cf. Proposition 5.12 and Lemma 4.6 in conjunction with Corollary 5.13]:

$$
|\left< (H_\Omega \textbf{u}, \textbf{v}) \right>| + |\left< (\textbf{v}, H_\Omega \textbf{u}) \right>| \leq C \|\textbf{u}\|_{\text{curl}, \Omega, k} \|\textbf{v}\|_{\text{curl}, \Omega, k}, \quad (1.15)
$$

$$
|A_k (H_\Omega \textbf{u}, \textbf{v})| + |A_k (\textbf{v}, H_\Omega \textbf{u})| \leq C \|\textbf{u}\|_{\text{curl}, \Omega, k} \|\textbf{v}\|_{\text{curl}, \Omega, k}. \quad (1.16)
$$

The refined continuity properties of $A_k$ and $\langle \cdot, \cdot \rangle$ given in (IX) allow us to estimate the terms $T_1$ in the basic error estimate (1.3), (1.4) explicitly in $k$. Abbreviating $\textbf{v} := \textbf{u} - \textbf{w}_h$ and decomposing $\textbf{v}^{\text{low}} := L_\Omega \textbf{v}$ and $\textbf{v}^{\text{high}} := H_\Omega \textbf{v}$, we write

$$
T_1 = \text{Re} \left[ A_k (\textbf{e}_h, \textbf{v}) + 2(\langle \textbf{e}_h, \textbf{v} \rangle) \right]
$$

$$
= \text{Re} \left[ A_k (\textbf{e}_h, H_\Omega \textbf{v}) + 2(\langle \textbf{e}_h, H_\Omega \textbf{v} \rangle) \right] + \text{Re} \left[ A_k (\textbf{e}_h, L_\Omega \textbf{v}) + 2(\langle \textbf{e}_h, L_\Omega \textbf{v} \rangle) \right]
$$

$$
= \text{Re} \left( A_k (\textbf{e}_h, H_\Omega \textbf{v}) + 2(\langle \textbf{e}_h, H_\Omega \textbf{v} \rangle) \right) + \text{Re}(\text{curl} \textbf{e}_h, \text{curl} L_\Omega \textbf{v})
$$

$$
=: T_{1.1} + T_{1.2}
$$

$$
+ \text{Re} \left[ -i k (T_k \text{curl} \textbf{e}_h, (L_\Omega \textbf{v})^{\text{curl}}) \right] + (\langle \textbf{e}_h, L_\Omega \textbf{v} \rangle)
$$

$$
=: T_{1.3}(\textbf{e}_h, L_\Omega \textbf{v})
$$

The sesquilinear forms in $T_{1.1}$ and $T_{1.2}$ have good continuity properties [cf. (IX) and (VII), respectively] and can be estimated with $k$-independent constants. The term $T_{1.3}$ is amenable to a treatment by a duality argument: Let $\psi \in \textbf{X}$ solve $A_k (\cdot, \psi) = T_{1.3} (\cdot, L_\Omega \textbf{v})$. By Galerkin orthogonality satisfied by $\textbf{e}_h$ and the stability estimate (1.14), one arrives at

$$
\left| T_{1.3} (L_\Omega \textbf{e}_h, \textbf{u} - \textbf{w}_h) \right| = |A_k (\textbf{e}_h, \psi)|
$$

$$
\leq C k^3 \|\textbf{e}_h\|_{\text{curl}, \Omega, k} \inf_{\psi_h \in \textbf{X}_h} \|\psi - \psi_h\|_{\text{curl}, \Omega, k}. \quad (1.17)
$$

Since $L_\Omega (\textbf{u} - \textbf{w}_h)$ is an analytic function by (VIII) and the geometry $\Gamma = \partial B_1(0)$ is analytic so is the dual solution $\psi$. As discussed in Proposition 7.5, [cf. (4.39)] and in Sect. 4.3 [cf. (4.59)], one has the following analytic regularity assertion:

(X) Given $\textbf{r} \in \textbf{X}$, the solutions $\psi_1, \psi_2 \in \textbf{X}$ of $A_k (\cdot, \psi_1) = T_{1.3} (\cdot, L_\Omega \textbf{r})$ and $A_k (\cdot, \psi_2) = \langle \cdot, L_\Omega \textbf{r} \rangle$ are analytic in $\overline{\Omega}$ with $\psi_1, \psi_2 \in \mathcal{A}(Ck^\alpha \|\textbf{r}\|_{\text{curl}, \Omega, k}, \gamma)$ for some $C, \gamma, \alpha \geq 0$ independent of $k$ and $\textbf{r}$. The analyticity classes $\mathcal{A}$ are introduced in Definition 2.5.
Since, by (X), the solution \( \psi \) in (1.17) is analytic, exponential approximation properties of \( hp \)-FEM spaces will be able to offset the algebraic factor \( k^3 \) in (1.17). Indeed, we will show in Lemma 8.5, (ii) for Nédélec elements of degree \( p \) that the infimum in (1.17) decays exponentially in \( p \) (provided that \( kh/p \) is sufficiently small).

1.3.2 Treatment of \( T^\text{p} \): The \( k \)-explicit Duality Argument for \( \Pi^\text{curl}*v_h \)

The analysis of \( T_2 = \langle (e_h, \Pi^\text{curl}*v_h) \rangle \) and \( T_3 = \langle (e_h, \Pi^\text{curl}*v_h - \Pi^\text{curl}*v_h) \rangle \) appearing in (1.8) requires us to make the decompositions \( v_h = \Pi^\text{curl}*v_h + \Pi^\text{curl}*v_h = \Pi^\text{curl}*v_h + \Pi^\text{curl}*v_h \) in a more careful, \( k \)-dependent way. The stability property (IX) implies \( \|\Pi^\text{curl}*v\|_{\text{curl}, k} \leq C \|v\|_{\text{curl}, k} \) with \( C > 0 \) independent of \( k \) so that

\[
\|\Pi^\text{curl}*H_\Omega v\|_{\text{curl}, k} \leq C \|v\|_{\text{curl}, k},
\]

again with \( C > 0 \) independent of \( k \) [cf. also (4.23b) and Corollary 5.13]. These favorable estimates for \( H_\Omega v \) instead of \( v \) directly suggest that we should study, for \( v_h \in X_h \), the following decompositions instead of (1.6) (1.7):

\[
\begin{align*}
v_h &= \Pi^\text{comp}*v_h + \Pi^\text{curl}*H_\Omega v_h & \Pi^\text{comp}* := L_\Omega + \Pi^\text{curl}*H_\Omega, \quad (1.19) \\
v_h &= \Pi^\text{comp}_h*v_h + \Pi^\text{curl}*H_\Omega v_h & \Pi^\text{comp}_h := L_\Omega + \Pi^\text{curl}*H_\Omega. \quad (1.20)
\end{align*}
\]

Consequently, we replace the expression \( T_2 \) by \( \tilde{T}_2 = \langle (e_h, \Pi^\text{comp}*v_h) \rangle \) and the term \( T_3 \) by \( \tilde{T}_3 = \langle (e_h, \Pi^\text{comp}*v_h - \Pi^\text{comp}*v_h) \rangle \). The duality argument for \( \tilde{T}_2 = \langle (e_h, \Pi^\text{comp}*v_h) \rangle = \langle (e_h, L_\Omega v_h) \rangle \) is split into two duality arguments. For the first term, one observes again that \( L_\Omega v_h \) is analytic and so will be the appropriate dual solution by (X), which in turn means that exponential approximability of \( hp \)-FEM space can be brought to bear. For the second term, the duality argument requires much more care since \( \Pi^\text{curl}*H_\Omega v_h \) has only finite regularity. We have (cf. Proposition 7.2):

(XI) The solution \( \psi \) of \( A_k(\cdot, \psi) = \langle (\cdot, \Pi^\text{curl}*H_\Omega v_h) \rangle \) can be decomposed as \( \psi = \psi_{H^2} + \psi_A \) with \( k^2 \|\psi_{H^2}\|_{H^2(\Omega)} \leq C \|H_\Omega v_h\|_{\text{curl}, k} \leq C \|v_h\|_{\text{curl}, k} \) and \( \psi_A \in A(Ck^\alpha \|H_\Omega v_h\|_{\text{curl}, k}, \gamma, \Omega) \) for some \( C, \gamma, \alpha \geq 0 \) independent of \( k \) (cf. Definition 2.5 for the definition of the analyticity class \( A \)).

The decomposition of (XI) into a part \( \psi_{H^2} \) with finite regularity in conjunction with \( k \)-uniform control of the second derivatives and an analytic part \( \psi_A \) is shown in Sect. 7.2; it relies on a solution formula based on Green’s function for the Helmholtz equation and the decomposition is then inferred from the one developed in [40].
1.3.3 Treatment of $\tilde{T}_3$: Estimating $(\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h$

For the final term, $\tilde{T}_3 = \left( (e_h, (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h) \right)$, a new type of duality argument appears. We start by writing

$$
\tilde{T}_3 = \left( (e_h, (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h) \right) = \left( (e_h, L_\Omega (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h) \right) + \left( (e_h, H_\Omega (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h) \right)
$$

Exploiting the analyticity of $L_\Omega (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h$, the first term, $\tilde{T}_{3,1}$ can be treated by a duality argument as in Sect. 1.3.1. For the second term, $\tilde{T}_{3,2}$, we use (IX) to estimate

$$
|\tilde{T}_{3,2}| = \left| \left( (e_h, H_\Omega (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h) \right) \right| \leq C \| e_h \|_{\text{curl}, \Omega}, k \left( (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h \right)_{\text{curl}, \Omega}, k
$$

where in the last step, we used $\text{curl} ((\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h) = 0$. The term $k \left( (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h \right)$ is estimated by

$$
\left( k \left( (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h \right) \right)^2 \leq \left( \text{Re} \left( ((\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h, (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h) \right) \right) \leq \left( \text{Re} \left( ((I - \Pi_h^E) \Pi^\text{comp,*} \, v_h, (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h) \right) \right)
$$

From (1.15) in (IX), we get

$$
|\tilde{T}_{4,1}| \leq C \left( k \left( (\Pi^\text{comp,*} - \Pi_h^\text{comp,*}) \, v_h \right) \right) \left( k \left( (I - \Pi_h^E) \Pi^\text{comp,*} \, v_h \right) \right)
$$

We remark that the above argument glossed over a minor point: In view of the modified definition of the decomposition (1.19), (1.20), we have to require that the operator $\Pi_h^E$ be additionally defined on the space of smooth functions (in (VI), the operator $\Pi_h^E$ is only defined on $V^*_0, h + X_h$) and satisfies some appropriate stability properties. The term $\| (I - \Pi_h^E) \Pi^\text{comp,*} \, v_h \|$ can be estimated as follows in view of the definition (1.19):
\[
\left\| (I - \Pi_h^E) \Pi_{\text{comp}}^* \mathbf{v}_h \right\| \\
\leq \left\| (I - \Pi_h^E) L_\Omega \mathbf{v}_h \right\| + \left\| (I - \Pi_h^E) \Pi_{\text{curl}}^* H_\Omega \mathbf{v}_h \right\| =: \tilde{T}_{5,1} + \tilde{T}_{5,2}.
\]

For Nédélec elements of degree \( p \), Theorem 8.3 provides an operator \( \Pi_h^E \) (its restriction to the reference element \( \tilde{K} \) is denoted there \( \tilde{\Pi}_p^{\text{curl},c} \)) that is also defined on (elementwise) smooth functions and has good polynomial approximation properties. In particular, by the analyticity of \( L_\Omega \mathbf{v}_h \), the term \( \tilde{T}_{5,1} \) is exponentially small in the polynomial degree \( p \) for Nédélec elements. The term \( \tilde{T}_{5,2} \) can be controlled by the assumption (VIc) and the stability bound (1.18) as

\[
\tilde{T}_{5,2} \overset{(\text{VIc})}{\leq} \eta_6^{\text{alg}} \| \Pi_{\text{curl}}^* H_\Omega \mathbf{v}_h \|_{\text{curl},\Omega,k} \overset{(1.18)}{\leq} \eta_6^{\text{alg}} C \| \mathbf{v}_h \|_{\text{curl},\Omega,k}.
\]

The term \( \tilde{T}_{4,2} \) requires a duality argument that exploits the orthogonality property (1.11). Specifically, the dual problem is to find \( \psi \in H^1(\Omega) \) such that

\[
((\nabla \psi, \nabla \tilde{\psi})) = ((L_\Omega (I - \Pi_h^E) \Pi_{\text{comp}}^* \mathbf{v}_h, \nabla \tilde{\psi})) \quad \forall \tilde{\psi} \in H^1(\Omega). \quad (1.21)
\]

Solvability is ensured by (II). The analyticity of \( L_\Omega (I - \Pi_h^E) \Pi_{\text{comp}}^* \mathbf{v}_h \) and \( \partial \Omega \) gives that \( \psi \) is analytic; we have by Proposition 7.4 [problem (1.21) is of Type 2 discussed in Sect. 7.1]:

(XII) The solution \( \psi \) of the problem (1.21) belongs to an analyticity class \( \psi \in A(C k^\alpha \| \mathbf{v}_h \|_{\text{curl},\Omega,k}, \gamma, \Omega) \) for some \( C, \alpha, \gamma \geq 0 \) independent of \( k \).

We obtain, noting that \( (\Pi_{\text{comp}}^* - \Pi_h^{\text{comp},*}) \mathbf{v}_h \) satisfies the same orthogonality condition (1.11) as the difference \( (\Pi_{\text{curl}}^* - \Pi_h^{\text{curl},*}) \mathbf{v}_h \),

\[
T_{4,2} = ((\nabla \psi, (\Pi_{\text{comp}}^* - \Pi_h^{\text{comp},*}) \mathbf{v}_h)) \\
\overset{(1.11)}{=} \inf_{\psi_h \in S_h} \left( (\nabla (\psi - \psi_h), (\Pi_{\text{comp}}^* - \Pi_h^{\text{comp},*}) \mathbf{v}_h) \right) \\
\overset{(1.14)}{\leq} C k^3 \inf_{\psi_h \in S_h} \| \nabla (\psi - \psi_h) \|_{\text{curl},\Omega,k} \| (\Pi_{\text{comp}}^* - \Pi_h^{\text{comp},*}) \mathbf{v}_h \|_{\text{curl},\Omega,k} \\
= C k^3 \inf_{\psi_h \in S_h} \left( k \| \nabla (\psi - \psi_h) \| \right) \| (\Pi_{\text{comp}}^* - \Pi_h^{\text{comp},*}) \mathbf{v}_h \|;
\]

a more detailed argument can be found in the proof of Proposition 6.1.

In summary, the main result of the present work is quasi-optimality of \( H(\Omega, \text{curl}) \)-conforming discretizations: In Theorem 4.15, we present a fairly abstract convergence result (which is not fully explicit in \( k \)). In Theorem 4.17 we consider high-order Nédélec elements and the specific case of the unit ball \( B_1(0) \) and show quasi-optimality of the Galerkin discretization under the scale resolution condition (1.1).
2 Maxwell’s Equations

In Sects. 2.1 and 2.2 we introduce the strong form of the Maxwell problem first in the full space \( \mathbb{R}^3 \) and then in an equivalent way on a bounded domain. At this stage, we are vague concerning the precise function spaces and mapping properties of trace operators. The variational formulation of the problem in a bounded domain is given in Sect. 2.4, where also the appropriate function spaces are introduced.

2.1 Maxwell’s Equations in the Full Space \( \mathbb{R}^3 \)

We consider the solution of the Maxwell equations in the full space \( \mathbb{R}^3 \) with Silver–Müller radiation conditions at infinity. The angular frequency is denoted by \( \omega \), the electric permittivity by \( \varepsilon \), and the magnetic permeability by \( \mu \). We formulate the problem in terms of the wavenumber \( k = \omega \sqrt{\varepsilon \mu} \), the scaled magnetic field \( \tilde{H} = \sqrt{\varepsilon / \mu} H \), and the scaled electric charge density \( \tilde{j} = \sqrt{\mu / \varepsilon} j \): Find the electric field \( E \) and the scaled magnetic field \( \tilde{H} \) such that

\[
\text{curl } E - i k \tilde{H} = 0 \quad \text{and} \quad \text{curl } \tilde{H} + i k E = \tilde{j} \quad \text{in } \mathbb{R}^3,
\]

\[
\left| E - \frac{\tilde{H} \times x}{r} \right| \leq \frac{c}{r^2} \quad \text{and} \quad \left| E \times \frac{x}{r} + \tilde{H} \right| \leq \frac{c}{r^2} \quad \text{for } r = \|x\| \to \infty
\]

(2.1)

is satisfied in a weak sense. Throughout the paper, we assume that the data \( \tilde{j} \) satisfies the following Assumption 2.1a, which is sufficient to prove quasi-optimality of the Galerkin discretization (cf. Theorems 4.15, 4.17), while further assumptions on \( \tilde{j} \) are needed to obtain convergence rates (cf. Corollary 4.18).

Assumption 2.1 (\( \Omega, \Gamma, \text{right-hand side } \tilde{j} \)) a) The scaled electric charge density \( \tilde{j} \) is a compactly supported distribution (functional on the space \( \mathcal{H}_{\text{loc}}(\text{curl}, \mathbb{R}^3) \) defined in Sect. 2.3) in the sense that there exists a bounded, smooth Lipschitz domain \( \Omega \subset \mathbb{R}^3 \) with simply connected boundary \( \Gamma := \partial \Omega \) that satisfies \( \text{supp } \tilde{j} \subset \Omega \). We denote by \( \mathbf{n} \) the unit normal vector on the boundary \( \Gamma \) oriented such that it points into the unbounded exterior \( \Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega} \).

b) The wavenumber \( k \) is considered as a real parameter in the range\(^2\)

\[
k \geq 1.
\]

(2.2)

2.2 Reformulation on a Bounded Domain

Assumption 2.1 allows us to formulate problem (2.1) in an equivalent way as a transmission problem. For this we have to introduce in (2.3) the trace operators \( \Pi_T \) and \( \gamma_T \), which map sufficiently smooth functions \( u \) in \( \overline{\Omega} \) to tangential fields on the surface

\(^2\) The condition \( k \geq 1 \) can be replaced by \( k \geq k_0 > 0 \). Our estimates remain valid for all choices of \( k_0 > 0 \). The constants in the estimates are uniform for all \( k \geq k_0 \), while they depend continuously on \( k_0 \) and, possibly, become large as \( k_0 \to 0 \). We use (2.2) simply to reduce technicalities.
\( \Gamma \) while the trace operators \( \Pi_T^+ \) and \( \gamma_T^+ \) denote the corresponding traces for function \( \mathbf{u}^+ \) in the exterior domain \( \Omega^+ \):

\[
\begin{align*}
\Pi_T : \mathbf{u} &\mapsto \mathbf{n} \times (\mathbf{u}|_{\Gamma} \times \mathbf{n}) , \\
\Pi_T^+ : \mathbf{u}^+ &\mapsto \mathbf{n} \times (\mathbf{u}^+|_{\Gamma} \times \mathbf{n}) , \\
\gamma_T : \mathbf{u} &\mapsto \mathbf{u}|_{\Gamma} \times \mathbf{n} , \\
\gamma_T^+ : \mathbf{u}^+ &\mapsto \mathbf{u}^+|_{\Gamma} \times \mathbf{n} .
\end{align*}
\tag{2.3}
\]

This allows us to define the jumps for sufficiently smooth functions \( \mathbf{w} \) in the interior and \( \mathbf{w}^+ \) in the exterior domain:

\[
[(\mathbf{w}, \mathbf{w}^+)]_{0, \Gamma} := \gamma_T \mathbf{w} - \gamma_T^+ \mathbf{w}^+ , \quad [(\mathbf{w}, \mathbf{w}^+)]_{1, \Gamma} := \gamma_T \text{curl} \mathbf{w} - \gamma_T^+ \text{curl} \mathbf{w}^+ .
\tag{2.4}
\]

With this notation, problem (2.1) takes the form: Find \( \mathbf{E}, \mathbf{E}^+, \tilde{\mathbf{H}}, \tilde{\mathbf{H}}^+ \) such that

\[
\begin{align*}
\text{curl} \mathbf{E} - i k \tilde{\mathbf{H}} & = 0 , & \text{curl} \tilde{\mathbf{H}} + i k \mathbf{E} & = \tilde{j} & \text{in} \ \Omega , & \tag{2.5a} \\
\text{curl} \mathbf{E}^+ - i k \tilde{\mathbf{H}}^+ & = 0 , & \text{curl} \tilde{\mathbf{H}}^+ + i k \mathbf{E}^+ & = 0 & \text{in} \ \Omega^+ , & \tag{2.5b} \\
[(\mathbf{E}, \mathbf{E}^+)]_{0, \Gamma} & = 0 , & [(\mathbf{E}, \mathbf{E}^+)]_{1, \Gamma} & = \mathbf{0} , & \tag{2.5c} \\
\left| \mathbf{E}^+ - \tilde{\mathbf{H}}^+ \times \frac{\mathbf{x}}{r} \right| & \leq \frac{c}{r^2} , & \left| \mathbf{E}^+ \times \frac{\mathbf{x}}{r} + \tilde{\mathbf{H}}^+ \right| & \leq \frac{c}{r^2} & \text{for} \ r = \|\mathbf{x}\| \to \infty . & \tag{2.5d}
\end{align*}
\]

The key role for formulating this problem as an equation on the bounded domain \( \Omega \) is played by the capacity operator \( T_k \). This operator associates with \( \mathbf{g}_T \in \mathbf{H}^{-1/2}_\text{curl} (\Gamma) \) the value of \( \gamma_T^+ \tilde{\mathbf{H}}^+ \) on \( \Gamma \) where the pair \( \mathbf{E}^+, \tilde{\mathbf{H}}^+ \) solves the homogeneous Maxwell problem in the exterior domain \( \Omega^+ \) with Silver–Müller radiation conditions at \( \infty \) [i.e., (2.5b), (2.5d)] together with Dirichlet boundary conditions \( \gamma_T^+ \mathbf{E}^+ = \mathbf{g}_T \times \mathbf{n} \). That is, \( T_k \mathbf{g}_T := \gamma_T^+ \tilde{\mathbf{H}}^+ \).

**Remark 2.2** From [47, Lemma 5.4.3, Thm. 5.4.6]\(^3\) we conclude that the exterior homogeneous Maxwell equation with given Dirichlet data \( \mathbf{g} \in \mathbf{H}^{-1/2}_\text{div} (\Gamma) \), i.e., \( \gamma_T^+ \mathbf{E}^+ = \mathbf{g} \) on \( \Gamma \), for the electric field has a weak solution \( \mathbf{E}^+ \in \mathbf{H}_{\text{loc}}(\text{curl}, \Omega^+) \), which is unique and satisfies

\[
\| \mathbf{E}^+ \|_{\text{curl}, B_R(0) \cap \Omega^+, 1} \leq C_{R, \Omega} \| \mathbf{g} \|_{\mathbf{H}^{-1/2}_\text{div} (\Gamma)} ,
\]

where \( B_R(0) \) is a ball with radius \( R \) centered at \( 0 \) such that \( \overline{\Omega} \subset B_R(0) \) and \( C_{R, \Omega} \) is a constant that depends only on \( \Omega \) and \( R \). This implies that the capacity operator

\[
T_k : \mathbf{H}^{-1/2}_\text{curl} (\Gamma) \to \mathbf{H}^{-1/2}_\text{div} (\Gamma)
\]

is continuous.

The Maxwell equations on the bounded domain are given by

\[
\begin{align*}
\text{curl} \mathbf{E} - i k \tilde{\mathbf{H}} & = 0 , & \text{curl} \tilde{\mathbf{H}} + i k \mathbf{E} & = \tilde{j} & \text{in} \ \Omega , & \tag{2.5a} \\
\gamma_T \text{curl} \mathbf{E} - i k T_k \Pi_T \mathbf{E} & = 0 & \text{on} \ \Gamma . & \tag{2.5d}
\end{align*}
\]

\(^3\) The function spaces appearing in these statements will be introduced in Sect. 2.3.
Eliminating $\tilde{H}$ from these equations, we arrive at the Maxwell equations for the electric field on a bounded domain $\Omega$

\[
\begin{align*}
\text{curl curl } E - k^2 E &= i k \tilde{j} \quad \text{in } \Omega, \\
\gamma_T \text{curl } E - i k T_k \Pi_T E &= 0 \quad \text{on } \Gamma.
\end{align*}
\]  

(2.6)

### 2.3 Sobolev Spaces in $\mathcal{O}$ and on $\Gamma$

We introduce the pertinent function spaces.

#### 2.3.1 Sobolev Spaces in $\mathcal{O}$

By $H^s(\Omega)$ we denote the usual Sobolev spaces of index $s \geq 0$ with norm $\| \cdot \|_{H^s(\Omega)}$. The closure of $C_0^\infty(\Omega)$ functions with respect to $\| \cdot \|_{H^s(\Omega)}$ is denoted by $H^s_0(\Omega)$. For $s \geq 0$, the dual space of $H^s_0(\Omega)$ is denoted by $H^{-s}(\Omega)$. If the functions are vector-valued, we indicate this by writing $H^s(\Omega)$, $H^s_0(\Omega)$.

The energy space for the electric field is given by

\[
X := H(\Omega, \text{curl}) := \left\{ u \in L^2(\Omega) \mid \text{curl } u \in L^2(\Omega) \right\}
\]  

(2.7)

and equipped with the indexed scalar product and norm

\[
(f, g)_{\text{curl}, \Omega, k} := (\text{curl } f, \text{curl } g) + k^2 (f, g),
\]

\[
\|f\|_{\text{curl}, \Omega, k} := (f, f)_{\text{curl}, \Omega, k}^{1/2},
\]

where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$-scalar product

(2.8)

The corresponding norm is denoted by $\| \cdot \| = (\cdot, \cdot)^{1/2}$. To indicate the domain $\Omega$, we write sometimes more explicit $\| \cdot \|_\Omega$ instead of $\| \cdot \|$. Here, $(\cdot, \cdot)$ is the Euclidean scalar product in $\mathbb{C}^3$ (with complex conjugation in the second argument). The dual space of $X$ (i.e., the space of continuous linear functionals on $X$) is denoted by $X'$ and the anti-dual space (i.e., the space of the continuous anti-linear functionals on $X$) by $X^\times$. We also introduce the space

\[
H(\Omega, \text{div}) := \left\{ u \in L^2(\Omega) \mid \text{div } u \in L^2(\Omega) \right\}.
\]  

(2.9)

For unbounded domains $D \subset \mathbb{R}^3$, we denote $H_{\text{loc}}(D, \text{curl})$ the space of all distributions $f$ with the property that $\varphi f \in H(D, \text{curl})$ for all smooth, compactly supported functions $\varphi \in C_0^\infty(\mathbb{R}^3)$. 

\[\text{Springer}\]
2.3.2 Sobolev Spaces on $\Gamma$

The Sobolev spaces on the boundary $\Gamma$ are denoted by $H^s(\Gamma)$ for scalar-valued functions and by $H^s(\Gamma)$ for vector-valued functions. The range of differentiability $s$ depends on the smoothness of $\Gamma$. To avoid such technicalities, we assume throughout the paper that the boundary $\Gamma$ is sufficiently smooth so that the Sobolev spaces $H^s(\Gamma)$ are well defined. A formal definition may be found in [33]; however, below and throughout this work, we will use the characterization in terms of expansions via eigenfunctions of the Laplace–Beltrami operator. We will need the space $L^2_T(\Gamma)$ of tangential vector fields given by

\[
L^2_T(\Gamma) := \{ v \in L^2(\Gamma) \mid \langle n, v \rangle = 0 \text{ on } \Gamma \}.
\] (2.10)

For a sufficiently smooth scalar-valued function $u$ and vector-valued function $v$ on $\Gamma$, the constant (along the normal direction) extensions into a sufficiently small three-dimensional neighborhood $U$ of $\Gamma$ are denoted by $u^*$ and $v^*$. The surface gradient $\nabla \Gamma$, the tangential curl $\nabla \Gamma \times n$, and the surface divergence $\text{div} \Gamma$ are defined by (cf., e.g., [47], [10])

\[
\nabla \Gamma u := (\nabla^* u)|_\Gamma, \quad \nabla \Gamma \times n := \nabla \Gamma u \times n, \quad \text{and} \quad \text{div} \Gamma v = (\text{div}^* v)|_\Gamma.
\] (2.11)

The scalar counterpart of the tangential curl is the surface curl

\[
\text{curl} \Gamma v := \langle (\text{curl}^* v)|_\Gamma, n \rangle \quad \text{on } \Gamma.
\] (2.12)

The composition of the surface divergence and surface gradient leads to the scalar Laplace–Beltrami operator

\[
\Delta \Gamma u = \text{div} \Gamma \nabla \Gamma u.
\] (2.13)

From [47, (2.5.197)] we have the relation

\[
\text{div} \Gamma (v \times n) = \text{curl} \Gamma v.
\] (2.14)

The operator $\Delta \Gamma$ is self-adjoint with respect to the $L^2(\Gamma)$ scalar product $\langle \cdot, \cdot \rangle_\Gamma$ and positive semidefinite. It admits a countable sequence of eigenfunctions in $L^2(\Gamma)$ denoted by $Y^m_{\ell}$ such that

\[
- \Delta \Gamma Y^m_{\ell} = \lambda_{\ell} Y^m_{\ell} \quad \text{for } \ell = 0, 1, \ldots \text{ and } m \in \iota_{\ell}.
\] (2.15)

We choose the normalization such that $\langle Y^m_{\ell}, Y^{m'}_{\ell'} \rangle_\Gamma = \delta_{m,m'} \delta_{\ell,\ell'}$ with Kronecker’s $\delta_{m,\ell}$. Here, $\iota_{\ell}$ is a finite index set whose cardinality equals the multiplicity of the eigenvalue $\lambda_{\ell}$, and we always assume that the eigenvalues $\lambda_{\ell}$ are distinct and ordered increasingly. We have $\lambda_0 = 0$, and for $\ell \geq 1$, they are real and positive and accumulate at infinity. By Assumption 2.1 the surface $\Gamma$ is simply connected so that $\lambda_0 = 0$ is
a simple eigenvalue. The $Y^m_\ell$ form an orthonormal basis of $L^2(\Gamma)$ and from [47, Sec. 5.4] we know that any distribution $w$, defined on the surface $\Gamma$, can formally be expanded with respect to the basis $Y^m_\ell$ as

$$w = \sum_{\ell=0}^{\infty} \sum_{m \in \iota_\ell} w^m_\ell Y^m_\ell.$$ 

The space $H^s(\Gamma)$ can be characterized by

$$H^s(\Gamma) = \left\{ w \in (C^\infty(\Gamma))' \mid \|w\|^2_{H^s(\Gamma)} := \sum_{\ell=0}^{\infty} \left( (\delta_{\ell, 0} + \lambda_\ell)^s \sum_{m \in \iota_\ell} |w^m_\ell|^2 \right) < \infty \right\}. \quad (2.16)$$

The norm on $H^s(\Gamma)$ is given by $\|\cdot\|_{H^s(\Gamma)}$. Next, we define spaces of vector-valued functions. By [47, Sec. 5.4.1], every function $v^T \in L^2_T(\Gamma)$ can be written as

$$v^T = \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} \left( v^m_\ell \overrightarrow{\text{curl}}_\Gamma Y^m_\ell + V^m_\ell \nabla Y^m_\ell \right), \quad (2.17)$$

where the coefficients satisfy $\sum_{\ell=1}^{\infty} \lambda_\ell \sum_{m \in \iota_\ell} \left( |v^m_\ell|^2 + |V^m_\ell|^2 \right) < \infty$. We set

$$\|v^T\|^2_{H^s_T(\Gamma)} := \sum_{\ell=1}^{\infty} \lambda_\ell^{s+1} \sum_{m \in \iota_\ell} \left( |v^m_\ell|^2 + |V^m_\ell|^2 \right). \quad (2.18)$$

A tangential vector field $v^T$ can be decomposed into a surface gradient and a surface curl part as $v^T = v^{\text{curl}} + v^\nabla$, where [cf. (2.17)]

$$v^{\nabla} := \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} V^m_\ell \nabla Y^m_\ell, \quad (2.19a)$$

$$v^{\text{curl}} := \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} v^m_\ell \overrightarrow{\text{curl}}_\Gamma Y^m_\ell. \quad (2.19b)$$

**Remark 2.3** For gradient fields $\nabla \phi$, we have $(\Pi_T \nabla \phi)^{\text{curl}} = 0$ and $(\Pi_T \nabla \phi)^\nabla = \nabla_\Gamma \phi$. \hfill \Box

The decomposition (2.19) allows us to express the operators $\text{curl}_\Gamma$ and $\text{div}_\Gamma$ and the corresponding norms in terms of the Fourier coefficients: for a tangential field $v^T$ of the form (2.17), the surface divergence and surface gradient are defined (formally) as in [47, (5.4.18)–(5.4.21)] by

$$\text{div}_\Gamma v^T = \sum_{\ell=1}^{\infty} \lambda_\ell \sum_{m \in \iota_\ell} V^m_\ell Y^m_\ell \quad \text{and} \quad \text{curl}_\Gamma v^T = \sum_{\ell=1}^{\infty} \lambda_\ell \sum_{m \in \iota_\ell} v^m_\ell Y^m_\ell. \quad (2.20)$$
The $H^s(\Gamma)$ norm [cf. (2.16)] of $\text{curl}_\Gamma (\cdot)$ and $\text{div}_\Gamma (\cdot)$ can accordingly be expressed in terms of the Fourier expansions:

$$
\| \text{curl}_\Gamma v_T \|_{H^s(\Gamma)}^2 = \sum_{\ell=1}^{\infty} \lambda_\ell^{s+2} \sum_{m \in \ell} |v^m_\ell|^2 \quad \text{and} \quad (2.21a)
$$

$$
\| \text{div}_\Gamma v_T \|_{H^s(\Gamma)}^2 = \sum_{\ell=1}^{\infty} \lambda_\ell^{s+2} \sum_{m \in \ell} |V^m_\ell|^2. \quad (2.21b)
$$

We define

$$
\| v_T \|_{-1/2,\text{curl}\Gamma}^2 = \sum_{\ell=1}^{\infty} \lambda_\ell^{1/2} \sum_{m \in \ell} \left( (1 + \lambda_\ell) |v^m_\ell|^2 + |V^m_\ell|^2 \right), \quad (2.22a)
$$

$$
\| v_T \|_{-1/2,\text{div}\Gamma}^2 = \sum_{\ell=1}^{\infty} \lambda_\ell^{1/2} \sum_{m \in \ell} \left( |v^m_\ell|^2 + (1 + \lambda_\ell) |V^m_\ell|^2 \right). \quad (2.22b)
$$

The spaces $H^{-1/2}_\text{curl}(\Gamma)$ and $H^{-1/2}_\text{div}(\Gamma)$ allow for orthogonal decompositions on the surface $\Gamma$. From [47, (5.4.20), (5.4.21)] we conclude that

$$
v_T \in H^{-1/2}_\text{div}(\Gamma) \iff v_T \text{ is of the form (2.17) and } \| v_T \|_{-1/2,\text{div}\Gamma} < \infty, $$

$$v_T \in H^{-1/2}_\text{curl}(\Gamma) \iff v_T \text{ is of the form (2.17) and } \| v_T \|_{-1/2,\text{curl}\Gamma} < \infty. $$

The system $\{ \nabla_\Gamma Y^m_\ell, \overrightarrow{\text{curl}_\Gamma Y^m_\ell} \}$ forms an orthogonal basis in $L^2_T(\Gamma)$ (cf. [47, paragraph after (5.12)]) so that

$$
\left( v^\nabla, v^\text{curl} \right)_{L^2_T(\Gamma)} = 0 \quad \forall v \in L^2_T(\Gamma). \quad (2.23)
$$

The following theorem shows that $H^{-1/2}(\text{div}_\Gamma, \Gamma)$ and $H^{-1/2}(\text{curl}_\Gamma, \Gamma)$ are the correct spaces to define continuous trace operators.

**Theorem 2.4** The trace mappings

$$
\Pi_T : X \rightarrow H^{-1/2}_\text{curl}(\Gamma), \quad \gamma_T : X \rightarrow H^{-1/2}_\text{div}(\Gamma)
$$

are continuous and surjective. Moreover, there exist continuous divergence-free liftings $\mathcal{E}_\text{curl} : H^{-1/2}_\text{curl}(\Gamma) \rightarrow X$ and $\mathcal{E}_\text{div} : H^{-1/2}_\text{div}(\Gamma) \rightarrow X$ for these trace spaces.

For a proof, we refer to [14], [47, Thm. 5.4.2]. For a vector field $u \in X$, we will employ frequently the notation $u_T := \Pi_T u$. The continuity constant of $\Pi_T$ is

$$
C_\Gamma := \sup_{v \in X \setminus \{0\}} \frac{\| \Pi_T v \|_{-1/2,\text{curl}\Gamma}}{\| v \|_{\text{curl},\Omega,1}}. \quad (2.24)
$$

The spaces $H^{-1/2}_\text{curl}(\Gamma)$ and $H^{-1/2}_\text{div}(\Gamma)$ are in duality with respect to $L^2_T(\Gamma)$, i.e., $(\cdot, \cdot)_T$ extends continuously to the duality pairing $H^{-1/2}_\text{curl}(\Gamma) \times H^{-1/2}_\text{div}(\Gamma)$ and, cf. [47, Lemmas 5.3.1, 5.4.1], there holds for all $u_T \in H^{-1/2}_\text{curl}(\Gamma)$ and all $v_T \in H^{-1/2}_\text{curl}(\Gamma)$
\[(u_T, v_T) \leq \|u_T\|_{-1/2, \text{curl}_T} \|v_T\|_{-1/2, \text{div}_T} \]  

\[\text{(2.25)}\]

### 2.3.3 The Analyticity Classes \(\mathcal{A}\)

We introduce classes of analytic functions whose growth of the derivatives (as the order of differentiation grows) is controlled explicitly in terms of the wavenumber \(k\). For smooth tensor-valued functions \(u = (u_i)_{i \in I}\) on an open \(\omega \subset \mathbb{R}^d\), where \(I\) is a suitable finite index set and using the usual multi-index conventions \(\alpha = (\alpha_i)_{s=1}^d\), we set \(|\alpha| = \alpha_1 + \cdots + \alpha_d\), abbreviate \(D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}\), and introduce

\[|\nabla^n u(x)|^2 = \sum_{\alpha \in \mathbb{N}_0^d} \sum_{|\alpha| = n} \frac{n!}{\alpha_1! \cdots \alpha_d!} |D^\alpha u_i(x)|^2.\]  

\[\text{(2.26)}\]

We then define:

**Definition 2.5** For an open set \(\omega \subset \mathbb{R}^d\), constants \(C_1, \gamma_1 > 0\), and wavenumber \(k \geq 1\) (cf. (2.2)), we set

\[\mathcal{A}(C_1, \gamma_1, \omega) := \left\{u \in L^2(\omega) \mid \|\nabla^n u\|_{L^2(\omega)} \leq C_1 \gamma_1^n \max\{n + 1, k\}^n \forall n \in \mathbb{N}_0\right\},\]

\[\mathcal{A}^\infty(C_1, \gamma_1, \omega) := \left\{u \in L^\infty(\omega) \mid \|\nabla^n u\|_{L^\infty(\omega)} \leq C_1 \gamma_1^n \forall n \in \mathbb{N}_0\right\}.\]

For the unit sphere \(\Gamma\) in \(\mathbb{R}^3\), constants \(C_1, \gamma_1\), and the wavenumber \(k \geq 1\), we set

\[\mathcal{A}(C_1, \gamma_1, \Gamma) := \left\{f \in L^2(\Gamma) \mid \|\nabla^n f\|_{L^2(\Gamma)} \leq C_1 \gamma_1^n \max\{n + 1, k\}^n \forall n \in \mathbb{N}_0\right\},\]

where \(\nabla_{\Gamma}\) denotes the surface gradient as in (2.11), and the application of \(\nabla_{\Gamma}\) to \(f\) is defined componentwise.

Membership in the analyticity class \(\mathcal{A}\) is invariant under analytic changes of variables and multiplication by analytic functions:

**Lemma 2.6** Let \(d \in \mathbb{N}\). Let \(\omega_1, \omega_2 \subset \mathbb{R}^d\) be bounded, open sets. Let \(g : \omega_1 \to \omega_2\) be a bijection and analytic on the closure \(\overline{\omega_1}\), i.e., there are constants \(C_g, C_{g, \text{inv}}, \gamma_g\) such that

\[g \in \mathcal{A}^\infty(C_g, \gamma_g, \omega_1) \quad \text{and} \quad \|(g')^{-1}\|_{L^\infty(\omega_1)} \leq C_{g, \text{inv}}.\]

Let \(f\) be analytic on the closure \(\overline{\omega_2}\), i.e., \(f \in \mathcal{A}^\infty(C_f, \gamma_f, \omega_2)\) for some \(C_f, \gamma_f\). Let \(u \in \mathcal{A}(C_u, \gamma_u, \omega_2)\) for some \(C_u, \gamma_u\). Then there are constants \(C', \gamma' > 0\) depending solely on \(C_g, \gamma_g, C_{g, \text{inv}}, \gamma_u, \gamma_f,\) and \(d\), such that \(\tilde{u} := f \cdot (u \circ g)\) satisfies \(\tilde{u} \in \mathcal{A}(C'C_fC_u, \gamma', \omega_1)\).

**Proof** The case \(d = 2\) is proved in [35, Lemma 4.3.1]. Inspection of the proof shows, as was already observed in [40, Lemma C.1], that it generalizes to arbitrary dimension \(d \in \mathbb{N}\).
2.4 Variational Formulation of the Electric Maxwell Equations

We formulate (2.6) as a variational problem. We introduce the sesquilinear forms $a_k$, $b_k$, $A_k : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ by

$$a_k (u, v) := (\text{curl} \ u, \text{curl} \ v) - k^2 (u, v), \quad b_k (u_T, v_T) := (T_k u_T, v_T)_\Gamma, \quad A_k (\cdot, \cdot) := a_k (\cdot, \cdot) - i k b_k (\Pi_T \cdot, \Pi_T \cdot).$$

(2.27a)

Then, the weak form of the electric Maxwell equations on a bounded domain $\Omega \subset \mathbb{R}^3$ with transparent boundary conditions reads:

Given $F \in \mathbb{X}^\times$ find $E \in \mathbb{X}$ such that $A_k (E, v) = F (v) \ \forall v \in \mathbb{X}$. (2.27b)

Note that the strong formulation (2.6) corresponds to the choice $F (v) = (i k \bar{j}, v)$ in (2.27b).

Theorem 2.7  Let Assumption 2.1 be satisfied. Let $A_k$ have the form (2.27a). Then, for every $F \in \mathbb{X}^\times$, problem (2.27b) has a unique solution.

Proof  Let $B_R(0)$ denote a ball centered at the origin with sufficiently large radius $R$ such that $\Omega \subset B_R(0)$. We consider the electric Maxwell equation in $B_R(0)$ of the form: Find $E_R \in \mathbb{H}(B_R(0), \text{curl})$ such that

$$(\text{curl} \ E_R, \text{curl} \ v)_{L^2(B_R(0))} - k^2 (E_R, v)_{L^2(B_R(0))}$$

$$- i k (T_{k,R} E_R, v_T)_{L^2(\partial B_R(0))} = F_R (v) \ \forall v \in \mathbb{H}(B_R(0), \text{curl}),$$

(2.28)

where $T_{k,R}$ is the capacity operator for the exterior domain $\mathbb{R}^3 \setminus B_R(0)$ and $F_R$ is the extension of $F$ by zero, i.e., $F_R (v) := F(v|_\Omega)$. In [47, Lemma 5.4.4 (with $\Gamma = \emptyset$ therein)] an ansatz $E_R = u_R + \nabla p_R$ is employed, where $u_R$ and $p_R$ are the solutions of a variational saddle point problem. In [47, Thm. 5.4.6], an inf-sup condition is proved for this saddle point problem which implies the well-posedness of (2.28). The construction implies that $E := E_R|_\Omega$ then satisfies (2.27b). On the other hand, every solution $E$ of (2.27b) can be extended to a solution of (2.28) by employing the well-posedness of the exterior Dirichlet problem, [47, Thm. 5.4.6]. Since (2.28) has a unique solution, also the solution of (2.27b) is unique.

$\square$

3 Discretization

3.1 Abstract Galerkin Discretization

Let $\mathbb{X}_h \subset \mathbb{X}$ denote a finite-dimensional subspace. The Galerkin discretization of (2.27) reads: Find $E_h \in \mathbb{X}_h$ such that

$$A_k (E_h, v_h) = F (v_h) \ \forall v_h \in \mathbb{X}_h.$$
For the error analysis, we will impose an assumption (Assumption 4.14) on the space $X_h$ by requiring the existence of a suitable projection onto the space $X_h$. Also for the error analysis, we will make use of a space $S_h$ such that the following exact sequence property holds:

$$
S_h \xrightarrow{\nabla} X_h \xrightarrow{\text{curl}} \text{curl } X_h.
$$

In the next section, we will introduce the Nédélec space $\mathcal{N}_h^1(T_h)$; for the choice $X_h = \mathcal{N}_h^1(T_h)$, we will perform the error analysis explicitly in the wavenumber $k$, the mesh width $h$, and the polynomial degree $p$.

### 3.2 Curl-Conforming $hp$-finite Element Spaces

The classical example of curl-conforming FE spaces are the Nédélec elements, [46]. We restrict our attention here to so-called type I elements (sometimes also referred to as the Nédélec–Raviart–Thomas element) on tetrahedra. These spaces are based on a regular (no hanging nodes), shape-regular triangulation $T_h$ of $\Omega \subset \mathbb{R}^3$. That is, $T_h$ satisfies:

(i) The (open) elements $K \in T_h$ cover $\Omega$, i.e., $\overline{\Omega} = \bigcup_{K \in T_h} \overline{K}$.

(ii) Associated with each element $K$ is the element map, a $C^1$-diffeomorphism $F_K : \widehat{K} \to \overline{K}$. The set $\widehat{K} \subset \mathbb{R}^3$ is the reference tetrahedron.

(iii) Denoting $h_K = \text{diam } K$, there holds, with some shape-regularity constant $\gamma$,

$$
h_K^{-1} \| F_K' \|_{L^\infty(\widehat{K})} + h_K \| (F_K')^{-1} \|_{L^\infty(\widehat{K})} \leq \gamma.
$$

The following assumption requires that the element maps $F_K$ can be written as compositions of affine scalings with a smooth mappings that can be controlled uniformly in $h$. Specifically, we adopt the setting of [40, Sec. 5] and assume that the element maps $F_K$ of the regular, $\gamma$-shape-regular triangulation $T_h$ satisfy the following additional requirements:

**Assumption 3.1** (normalizable regular triangulation) Each element map $F_K$ can be written as $F_K = R_K \circ A_K$, where $A_K$ is an affine map and the maps $R_K$ and $A_K$ satisfy for constants $C_{\text{affine}}, C_{\text{metric}}, \gamma > 0$ independent of $K$:

$$
\| A_K' \|_{L^\infty(\widehat{K})} \leq C_{\text{affine}} h_K, \quad \|(A_K')^{-1} \|_{L^\infty(\widehat{K})} \leq C_{\text{affine}} h_K^{-1}, \\
\| (R_K')^{-1} \|_{L^\infty(\widehat{K})} \leq C_{\text{metric}}, \quad \| \nabla^n R_K \|_{L^\infty(\widehat{K})} \leq C_{\text{metric}} \gamma^n n! \quad \forall n \in \mathbb{N}_0.
$$

Here, $\widetilde{K} = A_K(\widehat{K})$ and $h_K > 0$ is the element diameter.
Remark 3.2 A prime example of meshes that satisfy Assumption 3.1 is patchwise structured meshes as described, for example, in [40, Ex. 5.1] or [35, Sec. 3.3.2]. These meshes are obtained by first fixing a macrotriangulation of \( \Omega \); the actual triangulation is then obtained as images of affine triangulations of the reference element. \( \square \)

On the reference tetrahedron \( \hat{K} \), we introduce the classical Nédélec type I and Raviart–Thomas elements of degree \( p \geq 0 \) (see, e.g., [43]):

\[
\begin{align*}
\mathcal{P}_p(\hat{K}) & := \text{span}\{ x^\alpha | |\alpha| \leq p \}, \\
\mathcal{N}^I_p(\hat{K}) & := \{ p(x) + x \times q(x) | p, q \in (\mathcal{P}_p(\hat{K}))^3 \}, \\
\mathcal{RT}_p(\hat{K}) & := \{ p(x) + xq(x) | p \in (\mathcal{P}_p(\hat{K}))^3, q \in \mathcal{P}_p(\hat{K}) \}.
\end{align*}
\]

The spaces \( S_{p+1}(T_h) \), \( \mathcal{N}^I_p(T_h) \), \( \mathcal{RT}_p(T_h) \), and \( Z_p(T_h) \) are then defined as in [43, (3.76), (3.77)] by transforming covariantly the space \( \mathcal{N}^I_p(\hat{K}) \) and with the aid of the Piola transform the space \( \mathcal{RT}_p(\hat{K}) \):

\[
\begin{align*}
S_{p+1}(T_h) & := \{ u \in H^1(\Omega) | u|_K \circ F_K \in \mathcal{P}_{p+1}(\hat{K}) \}, \\
\mathcal{N}^I_p(T_h) & := \{ u \in H(\Omega, \text{curl}) | (F_K')^T u|_K \circ F_K \in \mathcal{N}^I_p(\hat{K}) \}, \\
\mathcal{RT}_p(T_h) & := \{ u \in H(\Omega, \text{div}) | (\det F_K')(F_K')^{-1} u|_K \circ F_K \in \mathcal{RT}_p(\hat{K}) \}, \\
Z_p(T_h) & := \{ u \in L^2(\Omega) | u|_K \circ F_K \in \mathcal{P}_p(\hat{K}) \}.
\end{align*}
\]

A key property of these spaces is that we have the following exact sequence, [43]:

\[
\mathbb{R} \longrightarrow S_{p+1}(T_h) \overset{\nabla}{\longrightarrow} \mathcal{N}^I_p(T_h) \overset{\text{curl}}{\longrightarrow} \mathcal{RT}_p(T_h) \overset{\text{div}}{\longrightarrow} Z_p(T_h).
\]

4 Stability and Error Analysis

4.1 The Basic Error Estimate

4.1.1 Preliminaries

The basic error estimates for curl-conforming Galerkin discretization involve some \( k \)-dependent sesquilinear forms and corresponding \( k \)-dependent norms which, in turn, are based on Helmholtz decompositions on the surface \( \Gamma \). We start this section with these preliminaries. For the proof of the basic error estimate (Theorem 4.13), we introduce the sesquilinear form \( ((\cdot, \cdot)) : X \times X \to \mathbb{C} \) by

\[
((u, v)) := k^2 (u, v) + ik b_k (u^\nabla, v^\nabla).
\]

We need some definiteness assumptions for the sesquilinear form \( b_k (\cdot, \cdot) \). Throughout the paper, we will assume:

\[ \heartsuit \] Springer
Assumption 4.1 The sesquilinear form $b_k : X \times X \to \mathbb{C}$ of (2.27a) satisfies

\[
\begin{align*}
\text{Im } b_k(u^\nabla, u^\nabla) & \leq 0 \quad \text{and} \quad \text{Im } b_k(u^{\text{curl}}, u^{\text{curl}}) \geq 0 \quad \forall u \in X, \\
\text{Re } b_k(v_T, v_T) & > 0 \\
& \forall v \in X \setminus \{0\}
\end{align*}
\] (4.2a)

and

\[
b_k\left(u^\nabla, v^{\text{curl}}\right) = b_k\left(u^{\text{curl}}, v^\nabla\right) = 0 \quad \forall u, v \in X.
\] (4.2b)

For $\Omega$ being the unit ball, the statements in Assumption 4.1 are proved in [47, Sec. 5.3.2]; see also Remark 5.4. In particular, Assumption 4.1 implies:

\[
\begin{align*}
A_k(u, v) &= (\text{curl } u, \text{curl } v) - i k b_k\left(u^{\text{curl}}, v^{\text{curl}}\right) - (u, v), \\
A_k(u, \nabla \varphi) &= -((u, \nabla \varphi)) \quad \forall u \in X, \quad \varphi \in H^1(\Omega).
\end{align*}
\] (4.3, 4.4)

The stability and convergence analysis of the Galerkin discretization (3.1) involve a) some frequency splittings on the surface $\Gamma$ and in the domain $\Omega$ as well as b), a suitable Helmholtz decomposition for the space $X$. These splittings will be defined next, while their analysis (for the case of the unit ball) is postponed to Sect. 5.

Definition 4.2 (frequency splittings) Let $\lambda > 1$ be a parameter. For a tangential field with an expansion of the form (2.17), the low-frequency operator $L_\Gamma$ and high-frequency operator $H_\Gamma$ are given by

\[
L_\Gamma v_T := \sum_{1 \leq \ell \leq \lambda k} \sum_{m \in \iota \ell} \left( v^m_T Y_{m\ell} + V^m_{\ell} \nabla Y_{m\ell} \right) \quad \text{and} \quad H_\Gamma := I - L_\Gamma.
\]

The mapping $L_\Omega : X \to X$ is the solution operator of the minimization problem:

\[
\|L_\Omega u\|_{\text{curl}, \Omega, k} = \min_{v \in X} \|v\|_{\text{curl}, \Omega, k} \quad \text{subject to } \Pi_T v = L_\Gamma u_T.
\] (4.5)

Set $H_\Omega := I - L_\Omega$. We introduce the notation

\[
C^L_\Omega(k) := \sup_{u \in X \setminus \{0\}} \frac{\|L_\Omega u\|_{\text{curl}, \Omega, k}}{\|u\|_{\text{curl}, \Omega, k}} \quad \text{and} \quad C^H_\Omega(k) := \sup_{u \in X \setminus \{0\}} \frac{\|H_\Omega u\|_{\text{curl}, \Omega, k}}{\|u\|_{\text{curl}, \Omega, k}}.
\] (4.6)

Remark 4.3 Since

\[
X_0 := \{ w \in X \mid \Pi_T w = 0 \}
\] (4.7)

is a Hilbert space with respect to $\| \cdot \|_{\text{curl}, \Omega, k}$, the operator $L_\Omega : X \to X$ is well defined, bounded, and linear (see also [52] and [53, Lemma 3.3]). The function $L_\Omega u$
can be characterized equivalently to (4.5) as the solution of the following variational problem: Find \( L_\Omega u \in X \) with \( \Pi_T L_\Omega u = L_\Gamma u_T \) such that
\[
(\text{curl } L_\Omega u, \text{curl } w) + k^2 (L_\Omega u, w) = 0 \quad \forall w \in X_0. \tag{4.8}
\]
The strong formulation of (4.8) is thus
\[
\begin{align*}
\text{curl } \text{curl } L_\Omega u + k^2 L_\Omega u &= 0 \quad \text{in } \Omega, \quad \tag{4.9a} \\
\Pi_T L_\Omega u &= L_\Gamma u_T \quad \text{on } \partial\Omega. \quad \tag{4.9b}
\end{align*}
\]
By applying the divergence operator to (4.9a), we get
\[
\text{div } L_\Omega u = 0 \quad \text{in } \Omega. \quad \tag{4.9c}
\]

Remark 4.4 For the special case of a ball \( \Omega = B_1(0) \), we will derive in Sect. 5.3 \( k \)-independent estimates for the continuity constants \( C^{L,\Omega}_k \) and \( C^{H,\Omega}_k \). In the general case, one can show estimates of the form \( C^{L,\Omega}_k \leq \tilde{C} k \) and \( C^{H,\Omega}_k \leq 1 + \tilde{C} k \) for some \( \tilde{C} > 0 \) independent of \( k \) by the following argument based on the \( (k) \)-independent lifting operator \( \mathcal{E}_{\text{curl}} : H^{-1/2}(\Gamma) \to H(\Omega, \text{curl}) \) provided by Theorem 2.4: The ansatz \( L_\Omega u = U - U_0 \) with \( U = \mathcal{E}_{\text{curl}} L_\Gamma u_T \) leads to the equation
\[
\text{curl } \text{curl } U_0 + k^2 U_0 = \text{curl } \text{curl } U + k^2 U \quad \text{in } \Omega, \quad \Pi_T U_0 = 0 \quad \text{on } \partial\Omega.
\]
Hence,
\[
\|U_0\|_{\text{curl},\Omega,k} \leq \|U\|_{\text{curl},\Omega,k} \leq \tilde{C} k \|L_\Gamma u_T\|_{-1/2,\text{curl},\Gamma} \leq C \|u_T\|_{-1/2,\text{curl},\Gamma}
\]
\[
\leq CC \tilde{C} k \|u\|_{\text{curl},\Omega,1} \leq C C \tilde{C} k \|u\|_{\text{curl},\Omega,k}.
\]
From this we get \( \|L_\Omega u\|_{\text{curl},\Omega,k} \leq \tilde{C} k \|u\|_{\text{curl},\Omega,k} \), i.e., \( C^{L,\Omega}_k \leq \tilde{C} k \). The triangle inequality provides \( C^{H,\Omega}_k \leq 1 + C^{L,\Omega}_k \).

The operators \( L_\Gamma \) and \( L_\Omega \) map into low-frequency modes that correspond to smooth functions (since the eigenfunctions \( Y_{m,\ell} \) are smooth by the smoothness of \( \Gamma \) ) and, hence, can be approximated well by \( hp \)-finite elements. We also use the operators \( L_\Gamma \) and \( H_\Gamma \) to define the high- and low-frequency parts of the sesquilinear form \( b_k \).

Definition 4.5 The low- and high-frequency parts of the capacity operator \( T_k \) and the sesquilinear form \( b_k \) are given by
\[
\begin{align*}
T_k^{\text{low}} &= T_k L_\Gamma, \quad &T_k^{\text{high}} &= T_k H_\Gamma, \quad \tag{4.11a}
\end{align*}
\]
The continuity constants of the high-frequency parts of $b_k$ are given by
\begin{align}
C_{b,k}^{\nabla, \text{high}} := k \sup_{u,v \in X \backslash \{0\}} \frac{\max \left\{ |b_k (u^\nabla, (H_\Omega v)^\nabla)|, |b_k ((H_\Omega u)^\nabla, v^\nabla)| \right\}}{\|u\|_{\text{curl}, \Omega, k} \|v\|_{\text{curl}, \Omega, k}}, \quad (4.12a) \\
C_{b,k}^{\text{curl, high}} := k \sup_{u,v \in X \backslash \{0\}} \frac{\max \left\{ |b_k (u^{\text{curl}}, (H_\Omega v)^{\text{curl}})| \right\}}{\|u\|_{\text{curl}, \Omega, k} \|v\|_{\text{curl}, \Omega, k}}. \quad (4.12b)
\end{align}

**Lemma 4.6** The capacity operator $T_k : H^{1/2}_{\text{curl}}(\Gamma) \to H^{-1/2}_{\text{div}}(\Gamma)$ is continuous with continuity constant
\begin{equation}
C_{\text{DN}, k} := \|T_k\|_{H^{1/2}_{\text{curl}}(\Gamma) \to H^{-1/2}_{\text{div}}(\Gamma)} < \infty. \quad (4.13)
\end{equation}

The sesquilinear form $A_k : X \times X \to \mathbb{C}$ is continuous. For all $u, v \in X$, there holds
\begin{align}
\max \{|A_k (u, v)|, |(u, v)|\} \leq C_{\text{cont}, k} \|u\|_{\text{curl}, \Omega, 1} \|v\|_{\text{curl}, \Omega, 1}, \quad (4.14) \\
\max \{|(u, H_\Omega v)|, |(H_\Omega u, v)|\} \leq C_{b,k}^{\text{high}} \|u\|_{\text{curl}, \Omega, k} \|v\|_{\text{curl}, \Omega, k}, \quad (4.15) \\
\max \{|A_k (H_\Omega u, v)|, |A_k (u, H_\Omega v)|\} \leq C_{b,k}^{\text{high}} \|u\|_{\text{curl}, \Omega, k} \|v\|_{\text{curl}, \Omega, k}, \quad (4.16)
\end{align}

where $C_{\text{cont}, k} := k^2 + C_T^2 C_{\text{DN}, k} k$, $C_{b,k}^{\text{high}} := C_k^{H, \Omega} + C_{b,k}^{\nabla, \text{high}}$, and $C_{\text{cont}, k}^{\text{high}} := C_k^{H, \Omega} + C_{b,k}^{\nabla, \text{high}}$.

**Proof** The continuity of $T_k : H^{-1/2}_{\text{curl}}(\Gamma) \to H^{1/2}_{\text{div}}(\Gamma)$ is asserted in Remark 2.2. For the sesquilinear form $A_k$, we employ
\begin{equation}
|A_k (u, v)| \leq \|u\|_{\text{curl}, \Omega, k} \|v\|_{\text{curl}, \Omega, k} + k \|b_k (u^T, v^T)\|. \quad (4.25)
\end{equation}

For the last term, we obtain
\begin{align}
k \|b_k (u^T, v^T)\| = k \|(T_k u^T, v^T)\| \leq k \|T_k u^T\|_{-1/2, \text{div}^r} \|v^T\|_{-1/2, \text{curl}^r} \leq C_{\text{DN}, k} k \|u^T\|_{-1/2, \text{curl}^r} \|v^T\|_{-1/2, \text{curl}^r} \leq C_T^2 C_{\text{DN}, k} k \|u\|_{\text{curl}, \Omega, 1} \|v\|_{\text{curl}, \Omega, 1}. \quad (4.24)
\end{align}

For the continuity bound of the sesquilinear form $(\cdot, \cdot)$, we obtain similarly as before
\begin{align}
|(u, v)| \leq k^2 \|u\|^2 \|v\|^2 + C_{\text{DN}, k} k \|u^\nabla\|_{-1/2, \text{curl}^r} \|v^\nabla\|_{-1/2, \text{curl}^r} \leq k^2 \|u\|^2 \|v\|^2 + C_{\text{DN}, k} k \|u^T\|_{-1/2, \text{curl}^r} \|v^T\|_{-1/2, \text{curl}^r} \leq C_{\text{cont}, k} \|u\|_{\text{curl}, \Omega, 1} \|v\|_{\text{curl}, \Omega, 1}. \quad (4.22)
\end{align}
For the high-frequency estimate of \((\cdot, \cdot)\), we employ

\[
|\langle u, H_\Omega v \rangle| \leq (k \|u\|) (k \|H_\Omega v\|) + k |b_k (u^\nabla, (H_\Omega v)^\nabla)|
\]

\[
\leq (k \|u\|) C_k^{H,\Omega} \|v\|_{\text{curl},\Omega,k} + C_{b,k}^{\nabla,\text{high}} \|u\|_{\text{curl},\Omega,k} \|v\|_{\text{curl},\Omega,k}
\]

\[
\leq \left( C_k^{H,\Omega} + C_{b,k}^{\nabla,\text{high}} \right) \|u\|_{\text{curl},\Omega,k} \|v\|_{\text{curl},\Omega,k}.
\]

The estimates with interchanged arguments follow along the same lines. The bound (4.16) follows similarly. \(\square\)

Next, we introduce frequency-dependent Helmholtz decompositions for the space \(X\). Let \(V \subset H^1(\Omega)\) be a closed subspace (the choice \(V = H^1(\Omega)\) is allowed). Note that this implies \(\nabla V \subset X\). Consider the problems:

Given \(w \in X\), find \(\Pi_V^\nabla w \in \nabla V\) s.t. \(\left( \langle \Pi_V^\nabla w, \xi \rangle \right) = \langle (w, \xi) \rangle \quad \forall \xi \in \nabla V\). (4.18)

Given \(w \in X\), find \(\Pi_V^{\nabla,*} w \in \nabla V\) s.t. \(\left( \langle \xi, \Pi_V^{\nabla,*} w \rangle \right) = \langle (\xi, w) \rangle \quad \forall \xi \in \nabla V\). (4.19)

**Lemma 4.7** Let assumption (4.2a) be satisfied. Let \(V \subset H^1(\Omega)\) be a closed subspace. Then, problems (4.18) and (4.19) are both uniquely solvable. Thus, the operators \(\Pi_V^\nabla\) and \(\Pi_V^{\nabla,*}\) are well defined.

**Proof** The definiteness of \(\text{Im } b_k (\cdot \nabla, (\cdot)^\nabla)\) [cf. (4.2a)] yields, for all \(\xi \in H^1(\Omega)\),

\[
\text{Re} \left( \langle \nabla \xi, \nabla \xi \rangle \right) = (k \|\nabla \xi\|)^2 - k \text{Im } b_k \left( \langle \nabla \xi, (\nabla \xi)^\nabla \rangle \right) \geq (k \|\nabla \xi\|)^2. \quad (4.20)
\]

From (4.14) we furthermore get \(|\langle (w, \nabla \xi) \rangle| \leq C_{\text{cont},k} \|w\|_{\text{curl},\Omega,1} \|\nabla \xi\|_{\text{curl},\Omega,1} = C_{\text{cont},k} \|w\|_{\text{curl},\Omega,1} (k \|\nabla \xi\|)\), which shows the well-posedness of \(\Pi_V^\nabla\). The well-posedness of \(\Pi_V^{\nabla,*}\) is shown analogously. \(\square\)

For \(V = S_h\), we abbreviate \(\Pi^\nabla_{S_h}\) by \(\Pi^\nabla_h\) and \(\Pi^{\nabla,*}_{S_h}\) by \(\Pi^{\nabla,*}_h\) while for \(V = H^1(\Omega)\) we use the shorthands \(\Pi^\nabla\) for \(\Pi^\nabla_{H^1(\Omega)}\) and \(\Pi^{\nabla,*}\) for \(\Pi^{\nabla,*}_{H^1(\Omega)}\).

A central role in the analysis is played by the spaces (cf. [47, p.220])

\[
V_0 := \left\{ u \in X \mid \langle (u, \nabla \xi) \rangle = 0 \quad \forall \xi \in H^1(\Omega) \right\}
\]

\[
\overset{(4.4)}{=} \left\{ u \in X \mid A_k(u, \nabla \xi) = 0 \quad \forall \xi \in H^1(\Omega) \right\}, \quad (4.21a)
\]

\[
V^*_0 := \left\{ u \in X \mid \langle (\nabla \xi, u) \rangle = 0 \quad \forall \xi \in H^1(\Omega) \right\}
\]

\[
\overset{(4.4)}{=} \left\{ u \in X \mid A_k(\nabla \xi, u) = 0 \quad \forall \xi \in H^1(\Omega) \right\}. \quad (4.21b)
\]

These spaces of divergence-free functions are the ranges of the operators \(\Pi^\text{curl}\) and \(\Pi^\text{curl,*}\) given by

\[
\Pi^\text{curl} := I - \Pi^\nabla, \quad \Pi^\text{curl,*} := I - \Pi^{\nabla,*}. \quad (4.22)
\]
Lemma 4.8 Let Assumption 4.1 be satisfied. Then there holds for all \( \mathbf{v} \in \mathbf{X} \)

\[
\begin{align*}
\| \Pi^{\nabla,*} H_\Omega \mathbf{v} \|_{\text{curl}, \Omega, k} & \leq C_{b,k} \| \mathbf{v} \|_{\text{curl}, \Omega, k}, \quad (4.23a) \\
\| \Pi^{\text{curl,*}} H_\Omega \mathbf{v} \|_{\text{curl}, \Omega, k} & \leq \left( C_{k} H_\Omega + C_{b,k}^{\text{high}} \right) \| \mathbf{v} \|_{\text{curl}, \Omega, k}. \quad (4.23b)
\end{align*}
\]

Proof We employ (4.20) and curl \( \Pi^{\nabla,*} H_\Omega \mathbf{v} = 0 \) to obtain

\[
\left( k \| \Pi^{\nabla,*} H_\Omega \mathbf{v} \| \right)^2 \leq \text{Re} \left( \left( \Pi^{\nabla,*} H_\Omega \mathbf{v}, \Pi^{\nabla,*} H_\Omega \mathbf{v} \right) \right) = \text{Re} \left( \left( \Pi^{\nabla,*} H_\Omega \mathbf{v}, H_\Omega \mathbf{v} \right) \right)
\]

\[
\overset{(4.15)}{\leq} C_{b,k} \| \Pi^{\nabla,*} H_\Omega \mathbf{v} \| \| \mathbf{v} \|_{\text{curl}, \Omega, k},
\]

so that (4.23a) follows. Estimate (4.23b) is obtained from (4.23a) and the triangle inequality using \( \Pi^{\text{curl,*}} = I - \Pi^{\nabla,*} \):

\[
\| \Pi^{\text{curl,*}} H_\Omega \mathbf{v} \|_{\text{curl}, \Omega, k} \leq \| H_\Omega \mathbf{v} \|_{\text{curl}, \Omega, k} + \| \Pi^{\nabla,*} H_\Omega \mathbf{v} \|_{\text{curl}, \Omega, k} \overset{(4.6), (4.23a)}{=} \left( C_{k} H_\Omega + C_{b,k}^{\text{high}} \right) \| \mathbf{v} \|_{\text{curl}, \Omega, k}.
\]

It is finally convenient to introduce the discrete counterparts of these operators:

\[
\Pi^\nabla_h := I - \Pi^\nabla \quad \Pi^{\nabla,*}_h := I - \Pi^{\nabla,*}. \quad (4.24)
\]

The operators \( \Pi^\nabla \) and \( \Pi^{\text{curl}} \) (analogously \( \Pi^{\nabla,*} \) and \( \Pi^{\text{curl,*}} \)) can be used to define a Helmholtz decomposition of \( \mathbf{u} \in \mathbf{X} \) into a gradient part and a divergence-free part. Since favorable stability properties of \( \Pi^\nabla \) (and thus also of \( \Pi^{\text{curl}} \)) will only be available for high-frequency functions, the decomposition (4.25) below involves additionally the frequency-splitting operators \( H_\Omega \) and \( L_\Omega \).

Definition 4.9 (Helmholtz decompositions) For \( \mathbf{u}, \mathbf{v} \in \mathbf{X} \) we set

\[
\mathbf{u} = \Pi^{\text{comp}} \mathbf{u} + \Pi^\nabla H_\Omega \mathbf{u} \quad \text{with} \quad \Pi^{\text{comp}} := L_\Omega + \Pi^{\text{curl}} H_\Omega. \quad (4.25a)
\]

The adjoint splitting is based on the operator \( \Pi^{\nabla,*} \) and given by

\[
\mathbf{v} = \Pi^{\text{comp,*}} \mathbf{v} + \Pi^{\nabla,*} H_\Omega \mathbf{v} \quad \text{with} \quad \Pi^{\text{comp,*}} := L_\Omega + \Pi^{\text{curl,*}} H_\Omega. \quad (4.25b)
\]

The discrete counterparts of these splittings are

\[
\mathbf{u} = \Pi^\text{comp}_h \mathbf{u} + \Pi^\nabla_h H_\Omega \mathbf{u} \quad \text{with} \quad \Pi^\text{comp}_h := L_\Omega + \Pi^{\text{curl}_h} H_\Omega, \quad (4.26a)
\]

\[
\mathbf{v} = \Pi^{\text{comp,*}_h} \mathbf{v} + \Pi^{\nabla,*}_h H_\Omega \mathbf{v} \quad \text{with} \quad \Pi^{\text{comp,*}_h} := L_\Omega + \Pi^{\text{curl,*}_h} H_\Omega. \quad (4.26b)
\]
The next lemma characterizes the spaces $V_0$ and $V_0^*$ in terms of the capacity operator $T_k$:

**Lemma 4.10** Let Assumption 4.1 be satisfied. Then: $u \in V_0$ if and only if $u \in X$ satisfies

$$\text{div } u = 0 \text{ in } L^2(\Omega) \quad \wedge \quad i k \langle u, n \rangle + \text{div}_Γ T_k Π_T u = 0 \text{ in } H^{-1/2}(Γ). \quad (4.27)$$

Furthermore, $v \in V_0^*$ if and only if $v \in X$ satisfies

$$\text{div } v = 0 \text{ in } L^2(\Omega) \quad \wedge \quad i k \langle v, n \rangle - \text{div}_Γ T_{-k} Π_T v = 0 \text{ in } H^{-1/2}(Γ). \quad (4.28)$$

**Proof** We only show the equivalence (4.28), since (4.27) follows by the same reasoning. Integration by parts applied to the condition $(\langle \nabla ξ, v \rangle) = 0$ yields, for all $ξ \in H^1(\Omega)$,

$$0 = \langle \nabla ξ, v \rangle = k^2 \langle \nabla ξ, v \rangle + i k_b (\langle \nabla ξ \rangle, v) Γ = -k^2 \langle ξ, \text{div } v \rangle + k^2 \langle ξ, \langle v, n \rangle \rangle Γ + i k \langle T_k (\nabla ξ), v Γ \rangle = -k^2 \langle ξ, \text{div } v \rangle + i k \langle ξ, \langle v, n \rangle \rangle Γ - \text{div}_Γ T_k^* (T_k v) Γ,$$

where $T_k^*$ is the adjoint of $T_k$ given by

$$\langle T_k φ, ψ \rangle Γ = \langle φ, T_k^* ψ \rangle Γ \quad \forall φ, ψ ∈ H^{-1/2}_{\text{curl}}(Γ). \quad (4.29)$$

Since $4(i k T_k)^* = -i k T_k^* = -i k T_{-k}$, this is equivalent to (4.28). □

**Corollary 4.11** Let the right-hand side in (2.27b) be defined by $F(v) = (i k ˜j, v)$ for some $˜j \in H(\Omega, \text{div})$ with div $˜j = 0$ and $˜j \cdot n = 0$ on $Γ$. Then the solution $E$ satisfies $E \in V_0$.

**Proof** The conditions div $˜j = 0$ and $˜j \cdot n = 0$ imply $(˜j, \nabla p) = 0$ for all $p \in H^1(\Omega)$. Hence, $A_k(E, \nabla p) = F(\nabla p) = 0$ for all $p \in H^1(\Omega)$. By (4.21a) we get $E ∈ V_0$. □

Next, we will prove that the spaces $V_0$ and $V_0^*$ are subspaces of $H^1(\Omega)$. For the special case of $Γ$ being the unit sphere, the constants in the norm equivalences can be determined explicitly—these details can be found in Lemma B.1.

**Lemma 4.12** Let Assumption 4.1 be valid. Let $V_0, V_0^*$ be defined as in (4.21). Then,

$$V_0 \cup V_0^* \subset H^1(\Omega). \quad (4.30)$$

---

4 This follows by representing $T_k$ by trace operators and boundary/volume potentials for the electric Maxwell equation as, e.g., explained in [11], and by applying the rules for computing the adjoint of composite operators.
There exists a constant $C_{\Omega,k} > 0$ such that

$$
\|u\|_{H^1(\Omega)} \leq C_{\Omega,k} \|u\|_{\text{curl},\Omega,1}, \quad \forall u \in V_0 \cup V_0^*.
$$

(4.31)

Moreover, the mappings $X \ni u \mapsto ((\Pi_{\text{curl}}^u, \cdot)) \in X'$ and $X \ni v \mapsto \left((\cdot, \Pi_{\text{curl}}^v)\right) \in X'$ are compact.

**Proof** Let $u \in V_0$. The function $T_k \Pi_T u$ is computed by first solving the exterior problem (cf. Remark 2.2)

\[
\begin{aligned}
\text{curl } \text{curl } u^+ - k^2 u^+ &= 0 & \text{in } \Omega^+,
\\ \left[(u, u^+)\right]_{0, \Gamma} &= 0 & \text{on } \Gamma,
\end{aligned}
\]

(4.32)

with Silver–Müller radiation conditions and then setting $T_k \Pi_T u = \frac{1}{i k} \gamma_T^+ \text{curl } u^+$. Since the tangential components of $u$ and $u^+$ coincide on $\Gamma$, the function $U : \mathbb{R}^3 \to \mathbb{C}$ defined by $U|_{\Omega} = u$ and $U|_{\Omega^+} = u^+$ (and $\Gamma$ considered as a set of measure zero) is in $H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$. Then, for all $v \in C_0^\infty(\mathbb{R}^3)$ we have

\[
\begin{aligned}
\left(\text{curl } U, \text{curl } v\right)_{L^2(\mathbb{R}^3 \setminus \Gamma)} - k^2 (U, v)_{L^2(\mathbb{R}^3)} &= a_k (u, v) + (\text{curl } u^+, \text{curl } v)_{L^2(\Omega^+)} - k^2 (u^+, v)_{L^2(\Omega^+)} \\
&= a_k (u, v) - i k (T_k U_T, v_T)_{\Gamma} = a_k (u, v) - i k b_k (u_T, v_T).
\end{aligned}
\]

(4.33)

If we test with gradients $v = \nabla \varphi$ for $\varphi \in C_0^\infty(\mathbb{R}^3)$, we obtain

\[
\begin{aligned}
\left(\text{curl } U, \nabla \varphi\right)_{L^2(\mathbb{R}^3 \setminus \Gamma)} - k^2 (U, \nabla \varphi)_{L^2(\mathbb{R}^3)} &= -k^2 (U, \nabla \varphi)_{L^2(\mathbb{R}^3)} \\
&= k^2 (\text{div } U, \varphi)_{L^2(\mathbb{R}^3)} \\
\left(\text{curl } U, \nabla \varphi\right)_{L^2(\mathbb{R}^3 \setminus \Gamma)} - k^2 (U, \nabla \varphi)_{L^2(\mathbb{R}^3)} &= a_k (u, \varphi) - i k b_k (u_T, (\nabla \varphi)_T) \\
&\quad \overset{\text{Remark } 2.3}{=} (u, \nabla \varphi).
\end{aligned}
\]

(4.34)

Since $u \in V_0$ implies $((u, \nabla \varphi) = 0$, the combination of the previous two equations leads to $\text{div } U = 0$ in $\mathbb{R}^3$. Hence,

\[
U \in H_{\text{loc}}(\mathbb{R}^3, \text{div}) \cap H_{\text{loc}}(\mathbb{R}^3, \text{curl}).
\]

(4.34)

Let $B_R(0)$ denote the ball with radius $0 < R < \infty$ and centered at 0 such that $\overline{\Omega} \subset B_R(0)$. Next, we show $U \in H^1(B_R(0))$. From (4.34) we conclude that $v := \chi U \in H_{\text{loc}}(\mathbb{R}^3, \text{div}) \cap H_{\text{loc}}(\mathbb{R}^3, \text{curl})$ for any smooth cutoff function $\chi$; in particular, we choose $\chi$ such that $\chi|_{B_R(0)} = 1$. The Fourier transform $\hat{v}$ then satisfies $\langle \xi, \hat{v} \rangle \in L^2(\mathbb{R}^3)$ as well as $\xi \times \hat{v} \in L^2(\mathbb{R}^3)$. From $|\xi|^2 |\hat{v}(\xi)|^2 = |\xi|^2 + |\xi \times \hat{v}|^2$ we
infer $\nabla v \in L^2(\mathbb{R}^3)$ and, in turn, $\nabla U \in L^2(B_R(0))$. Since (4.34) directly implies $U \in L^2(B_R(0))$, we have proved $U \in H^1(B_R(0))$ with

$$\|u\|_{H^1(\Omega)} \leq \|U\|_{H^1(B_R(0))} \leq C_R \left( \|U\|_{\text{curl}, B_R(0), 1} + \|\text{div} \ U\|_{L^2(B_R(0))} \right).$$

We already know that $\text{div} \ U = 0$ in $\mathbb{R}^3$ so that

$$\|u\|_{H^1(\Omega)} \leq C_R \left( \|u\|_{\text{curl}, \Omega, 1} + \|u^+\|_{\text{curl}, \Omega+\cap B_R(0), 1} \right). \quad (4.35)$$

An inspection of the proof of [47, Thm. 5.4.6] implies that

$$\|u^+\|_{\text{curl}, \Omega+\cap B_R(0), 1} \leq C_k \|\nabla^+ u^+\|_{H^{1/2}(\Gamma)} \quad (4.32) \Rightarrow \quad C_k \|\nabla^+ u\|_{H^{-1/2}(\Gamma)} \quad \text{Thm. 2.4}.$$

The combination with (4.35) leads to (4.31) for $u \in V_0$ with a constant $C_{\Omega,k}$, possibly depending on $\Omega$ and $k$. The inclusion $V^*_0 \subset H^1(\Omega)$ in (4.30) and (4.31) for $u \in V^*_0$ follows by the same reasoning.

Next, we prove that the mapping $X \ni u \mapsto \left( (\Pi \text{curl} \ u, \cdot) \right) \in X^\times$ is compact. The $L^2(\Omega)$ part of this mapping is compact since $\Pi \text{curl} \ u \in V_0 \subset H^1(\Omega) \hookrightarrow L^2(\Omega)$. Hence, it remains to prove the compactness of

$$X \ni u \mapsto \left( T_k (\Pi \text{curl} \ u)^\nabla, (\cdot)^\nabla \right)_\Gamma \in X^\times. \quad (4.36)$$

We set $u_0 := \Pi \text{curl} \ u$ and write $\Pi_T u_0 := u_0^\text{curl} + u_0^\nabla$ according to (2.19). For an element $v \in X$, we decompose $v_T = v^\text{curl} + \nabla_T \varphi$ with $\varphi \in H^{1/2}(\Gamma)/\mathbb{R}$; the mapping $X \ni v \mapsto \varphi \in H^{1/2}(\Gamma)/\mathbb{R}$ is continuous. Then

$$\left( T_k u_0^\nabla, v^\nabla \right)_\Gamma \overset{(4.2b)}{=} \left( T_k u_0, v^\nabla \right)_\Gamma \overset{(4.27)}{=} i k \left( (u_0 \cdot n), \varphi \right)_\Gamma.$$

Since $u_0 \in V_0 \subset H^1(\Omega)$, we have $(u_0 \cdot n) \in H^{1/2}(\Gamma)$. Hence, we arrive at

$$\left| \left( T_k (\Pi \text{curl} \ u)^\nabla, (v)^\nabla \right)_\Gamma \right| = \left| i k \left( (u_0 \cdot n), \varphi \right)_\Gamma \right| \leq k \|(u_0 \cdot n)\|_{H^{1/2}(\Gamma)} \|\varphi\|_{H^{-1/2}(\Gamma)}.$$}

Since $\varphi \in H^{1/2}(\Gamma)^\text{comp} \hookrightarrow H^{-1/2}(\Gamma)$, the compactness of the mapping (4.36) follows.

The compactness of the mapping $X \ni v \mapsto \left( (\cdot)^\nabla, T_k^*(\Pi \text{curl}^* v)^\nabla \right)_\Gamma \in X'$ follows analogously. \qed
4.1.2 Abstract Error Estimate

We have collected all ingredients to prove the quasi-optimal error estimate for the Galerkin solution in the following Theorem 4.13. It is the “Maxwell generalization” of the Galerkin convergence theory for sesquilinear forms satisfying a Gårding inequality, going back to [44]; various generalizations of this technique can be found in [9,26]. We follow [43, Sec. 7.2]. For \( w \in X \setminus \{0\} \), we introduce the quantity

\[
\delta_k(w) := \sup_{v \in X \setminus \{0\}} \left( \frac{2 \text{Re} ((w, v))}{\|w\|_{\text{curl}, \Omega} \|v\|_{\text{curl}, \Omega}} \right). \tag{4.37}
\]

We need an adjoint approximation property \( \tilde{\eta}_1^{\text{exp}} \) defined via the following dual problem: For given \( w, h \in X \), find \( \hat{N}(w, h) \in X \) such that

\[
A_k(v, \hat{N}(w, h)) = ((v, w)) - ikb_k \left( v^{\text{curl}}, h^{\text{curl}} \right) \quad \forall v \in X. \tag{4.38}
\]

In (7.15) we will present an explicit solution formula for this problem, which directly implies existence of a solution. The operator \( \hat{N}_1^A : X \to X \) then is given by \( \hat{N}_1^A(w) := \tilde{N}(L_{\Omega} w, L_{\Omega} w) \), i.e.,

\[
A_k(v, \hat{N}_1^A w) = ((v, L_{\Omega} w)) - ikb_k \left( v^{\text{curl}}, (L_{\Omega} w)^{\text{curl}} \right) \quad \forall v \in X. \tag{4.39}
\]

The adjoint approximation property \( \tilde{\eta}_1^{\text{exp}} \) is defined by

\[
\tilde{\eta}_1^{\text{exp}} := \tilde{\eta}_1^{\text{exp}}(X_h) := \sup_{w \in X \setminus \{0\}} \inf_{\tilde{z}_h \in X_h} \frac{\|\hat{N}_1^A w - \tilde{z}_h\|_{\text{curl}, \Omega}}{\|w\|_{\text{curl}, \Omega}}. \tag{4.40}
\]

**Theorem 4.13** Let Assumption 4.1 hold. Let \( E \in X \) and \( E_h \in X_h \) satisfy

\[
A_k(E - E_h, v_h) = 0 \quad \forall v_h \in X_h. \tag{4.41}
\]

Assume that \( \delta_k(e_h) < 1 \) for \( e_h := E - E_h \). Then, \( e_h \) satisfies, for all \( w_h \in X_h \), the quasi-optimal error estimate

\[
\|e_h\|_{\text{curl}, \Omega} \leq \frac{C_k^I + \delta_k(e_h)}{1 - \delta_k(e_h)} \|E - w_h\|_{\text{curl}, \Omega} \tag{4.42}
\]

with

\[
C_k^I := 1 + C_{b,k}^{\text{high}} + C_{b,k}^{\text{curl, high}} + C_{\text{cont}, k} \tilde{\eta}_1^{\text{exp}}. \tag{4.43}
\]
Proof The assumed sign conditions of $T_k$ [cf. (4.2a)] imply

$$\|\mathbf{e}_h\|_{\text{curl,}\Omega,k}^2 \leq (\text{curl } \mathbf{e}_h, \text{curl } \mathbf{e}_h) + k^2 (\mathbf{e}_h, \mathbf{e}_h) - k \text{ Im } b_k \left( \mathbf{e}_h^\nabla, \mathbf{e}_h^\nabla \right) + k \text{ Im } b_k \left( \mathbf{e}_h^{\text{curl}}, \mathbf{e}_h^{\text{curl}} \right) = \text{Re } A_k (\mathbf{e}_h, \mathbf{e}_h) + 2 \text{ Re } ((\mathbf{e}_h, \mathbf{E} - \mathbf{w}_h)).$$

We employ Galerkin orthogonality for the first term to obtain for any $\mathbf{w}_h \in X_h$

$$\|\mathbf{e}_h\|_{\text{curl,}\Omega,k}^2 \leq \text{Re } A_k (\mathbf{e}_h, \mathbf{E} - \mathbf{w}_h) + 2 \text{ Re } ((\mathbf{e}_h, \mathbf{E} - \mathbf{w}_h))$$

$$+ \delta_k (\mathbf{e}_h) \|\mathbf{e}_h\|_{\text{curl,}\Omega,k} \|\mathbf{E} - \mathbf{w}_h\|_{\text{curl,}\Omega,k} \leq \|\mathbf{e}_h\|_{\text{curl,}\Omega,k} + \|\mathbf{E} - \mathbf{w}_h\|_{\text{curl,}\Omega,k}.$$

We write $A_k$ in the form (4.3) so that

$$(1 - \delta_k (\mathbf{e}_h)) \|\mathbf{e}_h\|_{\text{curl,}\Omega,k}^2 \leq |(\text{curl } \mathbf{e}_h, \text{curl } (\mathbf{E} - \mathbf{w}_h))|$$

$$+ \text{Re } \left\{ (\mathbf{e}_h, \mathbf{E} - \mathbf{w}_h) - i k b_k \left( \mathbf{e}_h^{\text{curl}}, (\mathbf{E} - \mathbf{w}_h)^{\text{curl}} \right) \right\}$$

$$+ \delta_k (\mathbf{e}_h) \|\mathbf{e}_h\|_{\text{curl,}\Omega,k} \|\mathbf{E} - \mathbf{w}_h\|_{\text{curl,}\Omega,k}.$$  (4.44)

The sesquilinear forms in braces $\{\ldots\}$ will be seen to allow for good continuity constants when applied to high-frequency functions, while these constants grow with $k$ when being applied to low-frequency functions. For a function $\mathbf{v} \in X$, we therefore introduce the splitting into a high- and a low-frequency part $\mathbf{v} = \mathbf{v}^{\text{high}} + \mathbf{v}^{\text{low}} := H_{\Omega} \mathbf{v} + L_{\Omega} \mathbf{v}$ and get by using (4.39)

$$(\mathbf{e}_h, \mathbf{v}) - i k b_k \left( \mathbf{e}_h^{\text{curl}}, \mathbf{v}^{\text{curl}} \right)$$

$$= (\mathbf{e}_h, \mathbf{v}^{\text{high}}) - i k b_k \left( \mathbf{e}_h^{\text{curl}}, (\mathbf{v}^{\text{high}})^{\text{curl}} \right) + (\mathbf{e}_h, \mathbf{v}^{\text{low}}) - i k b_k \left( \mathbf{e}_h^{\text{curl}}, (\mathbf{v}^{\text{low}})^{\text{curl}} \right)$$

$$= (\mathbf{e}_h, \mathbf{v}^{\text{high}}) - i k b_k \left( \mathbf{e}_h^{\text{curl}}, (\mathbf{v}^{\text{high}})^{\text{curl}} \right) + A_k (\mathbf{e}_h, \mathbf{N}^A \mathbf{v}).$$  (4.45)

We employ the continuity estimate of (4.12a) to get

$$\left| (\mathbf{e}_h, \mathbf{v}^{\text{high}}) \right| \leq (k \|\mathbf{e}_h\|) (k \|\mathbf{v}^{\text{high}}\|) + k \left| \mathbf{e}_h^{\nabla}, (\mathbf{v}^{\text{high}})^{\nabla} \right|$$

$$\leq \|\mathbf{e}_h\|_{\text{curl,}\Omega,k} \|H_{\Omega} \mathbf{v}\|_{\text{curl,}\Omega,k} + C_{b,k}^{\text{high}} \|\mathbf{e}_h\|_{\text{curl,}\Omega,k} \|\mathbf{v}\|_{\text{curl,}\Omega,k}$$

$$\leq C_{b,k}^{\text{high}} \|\mathbf{e}_h\|_{\text{curl,}\Omega,k} \|\mathbf{v}\|_{\text{curl,}\Omega,k}.$$  (4.45)

For the second term in (4.45), we use (4.12b) and obtain in a similar fashion

$$|k b_k \left( \mathbf{e}_h^{\text{curl}}, (\mathbf{v}^{\text{high}})^{\text{curl}} \right)| \leq C_{b,k}^{\text{curl,high}} \|\mathbf{e}_h\|_{\text{curl,}\Omega,k} \|\mathbf{v}\|_{\text{curl,}\Omega,k}.$$
For the last term in (4.45), the combination of Galerkin orthogonality, the continuity estimate (4.14), and the definition of $\tilde{\eta}_1^\text{exp}$ in (4.40) gives

$$\left| A_k(e_h, \mathcal{N}_k^A v) \right| = \inf_{\mathbf{w}_h \in X_h} \left| A_k(e_h, \mathcal{N}_k^A v - \mathbf{w}_h) \right| \leq \tilde{\eta}_1^\text{exp} C_{\text{cont}, k} \| e_h \|_{\text{curl}, \Omega, k} \| v \|_{\text{curl}, \Omega, k}.$$ 

Thus,

$$\left| (e_h, v) - i k b_k (e_h^\text{curl}, v^\text{curl}) \right| \leq \left( C_{b, k}^{\text{high}} + C_{b, k}^{\text{curl, high}} + \tilde{\eta}_1^\text{exp} C_{\text{cont}, k} \right) \| e_h \|_{\text{curl}, \Omega, k} \| v \|_{\text{curl}, \Omega, k}.$$

This allows us to continue the error estimation in (4.44) resulting in

$$(1 - \delta_k (e_h)) \| e_h \|_{\text{curl}, \Omega, k} \leq \left( 1 + C_{b, k}^{\text{high}} + C_{b, k}^{\text{curl, high}} + \delta_k (e_h) + \tilde{\eta}_1^\text{exp} C_{\text{cont}, k} \right) \| E - w_h \|_{\text{curl}, \Omega, k}.$$

Theorem 4.13 implies that quasi-optimality of the Galerkin method is ensured if $\delta_k (e_h) < 1$. As will be shown in Theorem 4.15, this condition also implies existence and uniqueness of the Galerkin approximation $E_h$. In the following, we will focus on estimating $\delta_k (e_h)$, heavily exploiting the Galerkin orthogonality (4.41). For the case $\Omega = B_1(0)$, we will derive $k$-explicit estimates for the constants in (4.42) in Corollary 5.13. In this case, the constants $C_{b, k}^{\text{high}}, C_{b, k}^{\text{curl, high}}$ are independent of $k$; $C_{\text{cont}, k} = O(k^3)$ grows algebraically in $k$, which can be offset by controlling $\tilde{\eta}_1^\text{exp}$.

### 4.2 Splittings of $v_h$ to Estimate $\delta(e_h)$

It remains to estimate $\delta(e_h)$ in (4.37). In this section, we will introduce some frequency-dependent Helmholtz decompositions for a splitting of the term $((e_h, v_h))$. For $v \in X$, we introduce two decompositions according to Definition 4.9. Let $v^{\text{low}} := L_\Omega v$ and $v^{\text{high}} := H_\Omega v$. Then,

$$v = \Pi_h^{\text{comp, *}} v + \Pi_h^{\text{\nabla, *}} v^{\text{high}} \quad \text{with} \quad \Pi_h^{\text{comp, *}} \text{ as in (4.26)},$$

$$v = \Pi_h^{\text{comp, *}} v + \Pi_h^{\nabla, *} v^{\text{high}} \quad \text{with} \quad \Pi_h^{\text{comp, *}} \text{ as in (4.25b)}.$$ 

An important point to note is that for $v_h \in X_h$ we have $\Pi_h^{\text{comp, *}} v_h \in X_h$ and, for any $v \in X$, we have $\Pi_h^{\nabla, *} v^{\text{high}} \in \nabla S_h \subset X_h$. However, $\Pi_h^{\text{comp}} v_h$ and $\Pi_h^{\nabla} v^{\text{high}}$ are only in $X$ and $\nabla H^1(\Omega)$. From $\text{curl} \left( \Pi_h^{\nabla, *} v^{\text{high}}_h \right) = 0$ and Galerkin orthogonality, we conclude that

$$\left( (e_h, \Pi_h^{\nabla, *} v^{\text{high}}_h) \right) \overset{(4.3), \text{Rem. 2.3}}{=} -A_k(e_h, \Pi_h^{\nabla, *} v^{\text{high}}_h) = 0 \quad (4.47)$$
since $\Pi_h^\nabla v_h^{\text{high}} \in \nabla S_h \subset X_h$. We employ the splitting
\[ v_h = \Pi_{\text{comp}}^* v_h + (\Pi_h^\text{comp} - \Pi_h^\circ \Pi_{\text{comp}}^*) v_h + \Pi_h^\nabla v_h^{\text{high}} \]
and arrive via (4.47) at our main splitting
\[ \langle (e_h, v_h) \rangle = \langle (e_h, (\Pi_h^\text{comp} - \Pi_h^\circ \Pi_{\text{comp}}^*) v_h) \rangle + \langle (e_h, \Pi_h^\circ \Pi_{\text{comp}}^* v_h) \rangle + \langle (e_h, v_h^{\text{low}}) \rangle + \langle (e_h, v_h^{\text{high}}) \rangle. \]  

(4.48a)

(4.48b)

(4.48c)

4.3 Adjoint Approximation Properties

The error analysis involves solution operators for some adjoint problems, and we now introduce corresponding approximation measures that quantify how well these adjoint solutions can be approximated from the Galerkin space $X_h$ and its companion space $S_h$. One of these approximation measures involves the existence of a projector $\Pi_h^E$ that will also be introduced in this section.

Recall the definition of $V_0^*$ in (4.21b). We set
\[ V_{0,h}^* := \{ v \in V_0^* | \text{curl} v \in \text{curl} X_h \}. \]  

(4.49)

The following assumption stipulates the existence of a projector $\Pi_h^E : V_{0,h}^* + \text{Range}(L_\Omega) + X_h \rightarrow X_h$.

Assumption 4.14 There exists a linear operator $\Pi_h^E : V_{0,h}^* + \text{Range}(L_\Omega) + X_h \rightarrow X_h$ with the following properties:

(a) $\Pi_h^E$ is a projection, i.e., the restriction $\Pi_h^E |_{X_h}$ is the identity on $X_h$.

(b) There exists a companion operator $\Pi_h^F : \text{curl} X_h \rightarrow \text{curl} X_h$ with the property $\text{curl} \Pi_h^E = \Pi_h^F |_{\text{curl}}$.

Now we formulate the adjoint problems along with their solution operators: we introduce the solution operators $N_2^*, N_3^A$ for the following adjoint problems:

\[ A_k (w, N_2 r) = \langle (w, r) \rangle \quad \forall w \in X, \forall r \in V_0^*. \]  

(4.50a)

\[ A_k (w, N_3^A r) = \langle (w, L_\Omega r) \rangle \quad \forall w \in X, \forall r \in X, \]  

(4.50b)

i.e.,
\[ N_2 r := \widehat{N}(r, 0) \quad \text{and} \quad N_3^A r := N_2 (L_\Omega r) = \widehat{N}(L_\Omega r, 0). \]
The solution operator $\mathcal{N}^A_4 : X \to H^1(\Omega)/\mathbb{R}$ is related to some Poisson problem and given by

$$-A_k(\nabla \mathcal{N}^A_4 r, \nabla \xi) \overset{(4.4)}{=} \left(\nabla \mathcal{N}^A_4 r, \nabla \xi\right) = \langle (L_\Omega r, \nabla \xi) \rangle \quad \forall \xi \in H^1(\Omega). \ (4.50c)$$

We introduce the adjoint approximation properties $^5$

$$\eta_2^\text{alg} := \eta_2^\text{alg}(X_h) := \sup_{v_0 \in V_h^0} \inf_{w_h \in X_h} \frac{\|N_2 v_0 - w_h\|_{\text{curl}, \Omega, k}}{\|v_0\|_{\text{curl}, \Omega, k}}, \quad \eta_3^\text{exp} := \eta_3^\text{exp}(X_h) := \sup_{r \in X_h} \inf_{w_h \in X_h} \frac{\|N_2^k r - w_h\|_{\text{curl}, \Omega, k}}{\|r\|_{\text{curl}, \Omega, k}},$$

$$\eta_4^\text{exp} := \eta_4^\text{exp}(S_h) := \sup_{r \in X_h} \inf_{v_h \in S_h} \frac{\|\nabla (N_2^k r - v_h)\|}{\|r\|_{\text{curl}, \Omega, 1}}, \quad \eta_5^\text{exp} := \eta_5^\text{exp}(X_h) := \sup_{r \in X_h} \inf_{w_h \in X_h} \frac{\|L_\Omega r - w_h\|_{\text{curl}, \Omega, k}}{\|r\|_{\text{curl}, \Omega, k}},$$

$$\eta_6^\text{alg} := \eta_6^\text{alg}(X_h, \Pi_h^E) := \sup_{v_h \in \Pi_h^E} \frac{\|\nabla r\|_{H^1(\Omega)}}{\|v_h\|_{\text{curl}, \Omega, k}}, \quad \eta_7^\text{exp} := \eta_7^\text{exp}(X_h, \Pi_h^E) := \sup_{r \in X_h} \frac{\|L_\Omega r - \Pi_h^E L_\Omega r\|}{\|r\|_{\text{curl}, \Omega, k}}. \quad \ (4.56)$$

In Sect. 6 we will derive the following estimates for the terms in (4.48). Let $r := \Pi_h^{\text{comp},*} v_h - \Pi_h^{\text{comp},*} v_h$. Then

$$\left| \left( e_h, r^{\text{high}} \right) \right|_{\text{curl}, \Omega, k} \overset{\text{Prop. } 6.1}{\leq} C^{\text{high}}_{b,k} C_{r,k} \|e_h\|_{\text{curl}, \Omega, k}, \quad \|e_h\|_{\text{curl}, \Omega, k} \|v_h\|_{\text{curl}, \Omega, k} \overset{\text{Prop. } 6.2}{\leq} C^{\text{curl},*}_{b,k} \left( C^{\#}_{b,k} + C^{\text{curl},\text{high}}_{b,k} + C_{\text{cont},k} \eta_5^\text{exp} \right) \eta_2^\text{alg},$$

$$\left| \left( e_h, \Pi_h^E r \right) \right| + \left| \left( e_h, L_\Omega v_h \right) \right| \overset{\text{Prop. } 6.3}{\leq} C_{\text{cont},k} \eta_3^\text{exp} (1 + C_{r,k}) \|e_h\|_{\text{curl}, \Omega, k}, \quad \|v_h\|_{\text{curl}, \Omega, k}.$$

We insert these in (4.48) and combine with (4.37) to obtain

$$\delta_k(e_h) \leq \delta_k^I := 2 \left( C^{\text{high}}_{b,k} C_{r,k} + C^{\#}_{b,k} \left( C^{\#}_{b,k} + C^{\text{curl},\text{high}}_{b,k} + C_{\text{cont},k} \eta_5^\text{exp} \right) \eta_2^\text{alg} + C_{\text{cont},k} \eta_3^\text{exp} \left( 1 + C_{r,k} \right) \right). \ (4.57)$$

---

$^5$ We write $\hat{\eta}_\ell$ for an approximation property which involves a *solution operator* and $\eta_\ell$ for a “pure” approximation property for a given space/set of functions.
Theorem 4.15 Let Assumption 2.1 be satisfied, and let $E$ be the solution of Maxwell's equations (2.27b). Assume that $\delta_k^I$ in (4.57) is smaller than 1. Then the discrete problem (3.1) has a unique solution $E_h$, which satisfies the quasi-optimal error estimate

$$
\|e_h\|_{\text{curl},\Omega,k} \leq \frac{C_k^I + \delta_k^I}{1 - \delta_k^I} \inf_{w_h \in X_h} \|E - w_h\|_{\text{curl},\Omega,k}.
$$

(4.58)

Proof The proof uses the same arguments as the proof of [31, Thm. 3.9]. Assuming that a solution exists, the quasi-optimal error estimate (4.58) follows from (4.42) and the assumption $\delta_k^I < 1$. Next, we will prove uniqueness of problem (3.1). We show that if $E_h$ solves

$$
A_k (E_h, v_h) = 0 \quad \forall v_h \in X_h,
$$

then $E_h = 0$. This is the Galerkin discretization of the continuous problem: Find $E \in X$ such that $A_k (E, v) = 0$ for all $v \in X$. Theorem 2.7 implies that $E = 0$ is the unique solution. Hence, $e_h = E - E_h = -E_h$ satisfies the error estimate

$$
\|E_h\|_{\text{curl},\Omega,k} = \|e_h\|_{\text{curl},\Omega,k} \leq \frac{C_k^I + \delta_k^I}{1 - \delta_k^I} \inf_{w_h \in X_h} \|E - w_h\|_{\text{curl},\Omega,k} = 0
$$

since $E = 0$. Hence, $E_h = 0$. Since problem (3.1) is finite-dimensional with as many equations as unknowns, uniqueness implies existence. \qed

4.4 $k$-explicit $hp$-FEM

In this section, we show that the choice $(X_h, S_h) := \left( \mathcal{N}_p^I(T_h), S_{p+1}(T) \right)$ for properly chosen mesh size $h$ and $k$-dependent polynomial degree $p \geq 1$ leads to a $k$-independent quasi-optimality constant in (4.42). We adopt the setting described in Sect. 3.2. That is, we let $T_h$ be a mesh satisfying the assumptions of Sect. 3.2 and Assumption 3.1. The operators $\Pi_h^E$ and $\Pi_h^F$, whose existence is required in Sects. 3.1–8, may be chosen to be $\Pi_p^{\text{curl},c}$ and $\Pi_p^{\text{div},c}$ of Theorem 8.3.

4.4.1 Applications to the Case $\Omega = B_1(0)$

The adjoint approximation properties $\eta_{\text{alg}}^\ell, \eta_{\text{exp}}^\ell$ involve solution operators whose regularity is investigated in Sects. 5.3 and 7 for the unit ball $\Omega = B_1(0)$. In particular, we show in Proposition 7.2 that the solution operator $\mathcal{N}_2$ allows for a stable additive splitting $\mathcal{N}_2 = \mathcal{N}_2^{\text{rough}} + \mathcal{N}_2^{A}$, where $\mathcal{N}_2^{A}$ maps into some analyticity class and $\mathcal{N}_2^{\text{rough}} : V_0^* \rightarrow H^2(\Omega)$ satisfies the estimate $\|\mathcal{N}_2^{\text{rough}} v_0\|_{H^2(\Omega)} \leq C_{\text{rough}} k \|v_0\|_{\text{curl},\Omega,k}$. In Theorem 5.9 and Propositions 7.2, 7.3, 7.4, 7.5, we show that all other solution operators map into some analyticity class; more precisely, for all $r \in X$ and $v_0 \in V_0^*$, it

$\square$ Springer
holds with $\alpha_1 = 3$, $\alpha_2 = 3$, $\alpha_3 = 3$, $\alpha_4 = 5/2$, $\alpha_5 = 3/2$

\[ \mathcal{N}_j^A \mathbf{r} \in A \left( C_A, j k^{\alpha_j} \| \mathbf{r} \|_{\text{curl}, \Omega, k}, \gamma A, j, \Omega \right), \quad j = 1, 3, \quad (4.59a) \]

\[ \mathcal{N}_2^A \mathbf{v}_0 \in A \left( C_A, 2k^{\alpha_2} \| \mathbf{v}_0 \|_{\text{curl}, \Omega, k}, \gamma A, 2, \Omega \right), \quad (4.59b) \]

\[ \nabla \mathcal{N}_4^A \mathbf{r} \in A \left( C_A, 4k^{\alpha_4} \| \mathbf{r} \|_{\text{curl}, 1}, \gamma A, 4, \Omega \right), \quad (4.59c) \]

\[ L_{\Omega} \mathbf{r} \in A \left( C_A, 5k^{\alpha_5} \| \mathbf{r} \|_{\text{curl}, 1}, \gamma A, 5, \Omega \right). \quad (4.59d) \]

This allows us to estimate those adjoint approximations that involve solution operators by simpler approximation properties that we introduce next:

\[ \eta_1^\text{exp} (\gamma, \mathbf{X}_h) := \sup_{z \in A(1, \gamma, \Omega)} \inf_{w_h \in \mathbf{X}_h} \| z - w_h \|_{\text{curl}, \Omega, k}, \quad (4.60) \]

\[ \eta_2^\text{alg} (\mathbf{X}_h) := \sup_{z \in H^2(\Omega)} \inf_{w_h \in \mathbf{X}_h} \frac{\| z - w_h \|_{\text{curl}, \Omega, k}}{\| z \|_{H^2(\Omega)}}, \quad (4.61) \]

\[ \eta_3^\text{exp} (\gamma, \mathbf{S}_h) := \sup_{\nabla z \in A(1, \gamma, \Omega)} \inf_{w_h \in \mathbf{S}_h} \| \nabla (z - w_h) \|. \quad (4.62) \]

\[ \eta_7^\text{exp} (\gamma, \mathbf{X}_h) := k \sup_{z \in A(1, \gamma, \Omega)} \left\| z - \Pi_h^E z \right\|. \quad (4.63) \]

We obtain

\[ \eta_1^\text{exp} \leq C A, 1 k^{\alpha_1} \eta_1^\text{exp} (\gamma A, 1, \mathbf{X}_h), \quad (4.64a) \]

\[ \eta_2^\text{alg} \leq C_{\text{rough}} \eta_2^\text{alg} (\mathbf{X}_h) + C A, 2 k^{\alpha_2} \eta_1^\text{exp} (\gamma A, 2, \mathbf{X}_h), \quad (4.64b) \]

\[ \eta_3^\text{exp} \leq C A, 3 k^{\alpha_3} \eta_1^\text{exp} (\gamma A, 3, \mathbf{X}_h), \quad \eta_4^\text{exp} \leq C A, 4 k^{\alpha_4} \eta_4^\text{exp} (\gamma A, 4, \mathbf{S}_h), \quad (4.64c) \]

\[ \eta_5^\text{exp} \leq C A, 5 k^{\alpha_5} \eta_1^\text{exp} (\gamma A, 5, \mathbf{X}_h), \quad \eta_7^\text{exp} \leq C A, 5 k^{\alpha_5} \eta_7^\text{exp} (\gamma A, 5, \mathbf{X}_h). \quad (4.64d) \]

**Corollary 4.16** Let $\Omega = B_1(0)$. Recall the definition of $\alpha_{\ell}$ before (4.59). Define

\[ \eta_{1, \ast}^\text{exp} := \max_{j \in \{1, 2, 3, 5\}} C A, j \eta_1^\text{exp} (\gamma A, j, \mathbf{X}_h), \quad \eta_{2, \ast}^\text{alg} := C_{\text{rough}} \eta_2^\text{alg} (\mathbf{X}_h), \quad (4.65a) \]

\[ \eta_{4, \ast}^\text{exp} := C A, 4 \eta_4^\text{exp} (\gamma A, 4, \mathbf{S}_h), \quad \eta_{7, \ast}^\text{exp} := C A, 5 \eta_7^\text{exp} (\gamma A, 5, \mathbf{X}_h). \quad (4.65b) \]

For $0 < \tau \leq 1$ sufficiently small but independent of $k$, and any $0 < \varepsilon_{\ell} \leq \tau$, $\ell \in \{1, 2, 4, 6, 7\}$, select the mesh size $h$ and the polynomial degree $p$ for the $hp$-finite element space $\mathbf{X}_h$ such that $\mathbf{X}_h$ and its companion space $\mathbf{S}_h$ [cf. (3, 2)] satisfy Assumption 4.14 and the following approximation properties:

\[ k^{\alpha_3 + 3} \eta_{1, \ast}^\text{exp} \leq \varepsilon_1, \quad \eta_{2, \ast}^\text{alg} \leq \varepsilon_2, \quad k^{\alpha_4 + 1} \eta_{4, \ast}^\text{exp} \leq \varepsilon_4, \quad \eta_6^\text{alg} \leq \varepsilon_6, \quad k^{\alpha_5} \eta_{7, \ast}^\text{exp} \leq \varepsilon_7. \quad (4.66) \]

---

For the last relation, we have estimated $\| \cdot \|_{\text{curl}, \Omega, 1} \leq \| \cdot \|_{\text{curl}, \Omega, k}$ in (5.30) (using (2.2)) to simplify technicalities.
Then, the quantity $\delta_k^1$ in (4.57), (4.58) can be estimated by $\delta_k^1 < 1/2$, and the discrete problem (3.1) has a unique solution $E_h$, which satisfies the quasi-optimal error estimate

\[
\|e_h\|_{\text{curl},\Omega,k} \leq C \inf_{w_h \in X_h} \|E - w_h\|_{\text{curl},\Omega,k}
\] (4.67)

for a constant $C$ independent of $k$.

Proof We estimate $\delta_k^1$ of (4.57) termwise by using (4.64), (4.65), and the values of $\alpha_j$. From Corollary 5.13, we get that the constants $C_{\#,k}, C_{\#\#,k}$ in (6.1) and (6.14) are actually bounded uniformly in $k$. Hence,

\[
Cr_{r,k} \leq C(\epsilon_6 + \epsilon_7)
\]

for a constant $C$ independent of $k$. Again from Corollary 5.13 and (4.64), we get

\[
\delta_k^1 \leq C \left( \epsilon_6 + \epsilon_7 + \left( 1 + k^{\alpha_5 + 3} \tilde{\eta}_{1,*}^{\text{exp}} \right) \left( \tilde{\eta}_{2,*}^{\text{alg}} + k^{\alpha_2} \tilde{\eta}_{1,*}^{\text{exp}} \right) + k^{\alpha_3 + 3} \tilde{\eta}_{1,*}^{\text{exp}} \left( 1 + \epsilon_6 + \epsilon_7 \right) \right)
\]

for a constant $C$ independent of $k$. We use $\alpha_3 + 3 \geq \max \{ \alpha_1 + 3, \alpha_5 + 3, \alpha_2 \}$ and the conditions in (4.66) along with $\epsilon_k \leq \tau \leq 1$ to obtain

\[
\delta_k^1 \leq C \left( \epsilon_1 + \epsilon_2 + \epsilon_6 + \epsilon_7 \right) \leq \tilde{C} \tau
\]

for a constant $\tilde{C}$ independent of $k$. Hence, the condition $0 < \tau < 1/(2\tilde{C})$ implies $\delta_k^1 < 1/2$, and existence and uniqueness of the discrete solution follow from Theorem 4.15.

To prove that the quasi-optimality constant $C$ in (4.67) is independent of $k$, we use (4.58) so that it remains to prove that $C_k^1$ in (4.58) [cf. (4.43)] is bounded independently of $k$. This, in turn, is a direct consequence of Corollary 5.13 and

\[
C_{\text{cont},k} \tilde{\eta}_{1,*}^{\text{exp}} \leq C k^{\alpha_1 + 3} \tilde{\eta}_{1,*}^{\text{exp}} \leq C k^{\alpha_3 + 3} \tilde{\eta}_{1,*}^{\text{exp}} \leq C \epsilon_1 \leq C \tau \leq C
\]

independent of $k$.

\[\square\]

4.4.2 hp-FEM: Results

Theorem 4.17 Let $\Omega = B_1(0)$ be the unit ball, and let $E$ denote the exact solution of (2.27b). Let the mesh $T_h$ satisfy Assumption 3.1 and set $h := \max_{K \in T} h_K$. Let $S_h = S_{p+1}(T_h)$ and $X_h = N_{p}^p(T_h)$. Fix $c_2 > 0$. Then there exist constants $c_1, C > 0$ depending solely on $R$, $c_2$, and the constants $C_{\text{affine}}, C_{\text{metric}}, \gamma$ of Assumption 3.1 such the following holds: If $k \geq 1$, $p \geq 1$, $h > 0$ satisfy

\[
\frac{kh}{p} \leq c_1 \quad \text{and} \quad p \geq c_2 \ln k,
\] (4.68)
then the Galerkin approximation \( E_h \in X_h \) [cf. (3.1)] exists and satisfies

\[
\| E - E_h \|_{\Omega, \text{curl}, k} \leq C \inf_{w_h \in X_h} \| E - w_h \|_{\Omega, \text{curl}, k} .
\] (4.69)

**Proof** The proof consists in checking the conditions of Corollary 4.16. The infima in \( \eta^\text{alg}_2, \eta^\text{exp}_j, j \in \{1, 4\} \), are estimated with the aid the specific approximation operator \( \Pi^\text{curl}, s \) analyzed in Lemma 8.5:

\[
\eta^\text{alg}_2 \leq \sup_{z \in H^2(\Omega)} \| z \| \| z - \Pi^\text{curl}, s z \|_{\text{curl}, \Omega, k} \lesssim \left( \frac{h}{p} + \frac{h^2}{p^2} k \right)^k \left( \frac{h}{p} + \left( \frac{kh}{p} \right)^2 \right)^k.
\] (4.70)

The terms \( \eta^\text{exp}_j, j \in \{1, 4\} \), involve the approximation of analytic functions: The term \( \eta^\text{exp}_1 \) is an approximation from \( X_h = N^1_p(\mathcal{T}_h) \) and estimated with Lemma 8.5, (ii); the term \( \eta^\text{exp}_4 \) contains an approximation from \( S_h = S_{p+1}(\mathcal{T}_h) \) and is taken from the proof of [40, Thm. 5.5]:

\[
\sum_{j \in \{1, 4\}} \eta^\text{exp}_j \lesssim \left( \frac{h}{h + \sigma} \right)^p + k \left( \frac{kh}{\sigma p} \right)^p + k \left\{ \left( \frac{h}{h + \sigma} \right)^{p+1} + \left( \frac{kh}{\sigma p} \right)^{p+1} \right\}.
\] (4.71)

The terms \( \eta^\text{alg}_6, \eta^\text{exp}_7 \) involve the operator \( \Pi^\text{curl}, c \). These are estimated in Lemma 8.6. Specifically, \( \eta^\text{alg}_6 \) is controlled with Lemma 8.6, (iii), and \( \eta^\text{exp}_7 \) is controlled with Lemma 8.6, (ii) after the observation (4.59) that \( L_\Omega v \) is in an analyticity class:

\[
\eta^\text{alg}_6 \lesssim \frac{hk}{p}, \quad \eta^\text{exp}_7 \lesssim k \left( \left( \frac{h}{h + \sigma} \right)^{p+1} + \left( \frac{kh}{\sigma p} \right)^{p+1} \right).
\] (4.72)

Selecting \( c_1 \) sufficiently small and using Lemma 8.7 allows us to conclude the proof. 

\[ \square \]

**Corollary 4.18** Let Assumption 2.1 be satisfied, and let \( F \) in \( (2.27b) \) be of the form \( F(v) = (i k \tilde{j}. v) \) for some \( \tilde{j} \in \{ u \in L^2(\Omega) \mid \text{div} u = 0 \land (u|_{\Gamma} \cdot n) = 0 \} \). Let the assumptions of Theorem 4.17 be satisfied. Then under the scale resolution condition (4.68), the Galerkin approximation \( E_h \in X_h \) [cf. (3.1)] exists and satisfies

\[
\| E - E_h \|_{\Omega, \text{curl}, k} \leq C \frac{kh}{p} \| \tilde{J} \|_{L^2(\Omega)} .
\] (4.74)
**Proof** Under the assumption of this corollary, the solution $E$ is the restriction of the electric field of the full space problem (2.1) (with right-hand side defined as the extension of $\tilde{j}$ to $\mathbb{R}^3$ by zero). In Sect. 7.1, we will derive the solution formula (7.15) for an adjoint Maxwell problem that can easily be adapted to the original Maxwell problem and to our assumption on the data $\tilde{j}$. We obtain

$$E(\mathbf{x}) = i k \int_{\Omega} g_k(\|\mathbf{x} - \mathbf{y}\|) \tilde{j}(\mathbf{y}) \, d\mathbf{y} \quad \forall \mathbf{x} \in \Omega,$$

where $g_k$ is the fundamental solution of the Helmholtz equation (7.9). By [40, Lemma 3.5] there exist constants $c, C > 0$ independent of $k$ and $\tilde{j}$ such that, for every $\mu > 1$, there exists a $\mu$- and $k$-dependent splitting $E = E_{H^2} + E_A$ with

$$\|\nabla^m E_{H^2}\|_{L^2(\Omega)} \leq C \left(1 + \frac{1}{\mu^2 - 1}\right)(\mu k)^{m-1}\|\tilde{j}\|_{L^2(\Omega)} \quad \forall m \in \{0, 1, 2\}, \quad (4.75a)$$

$$\|\nabla^n E_A\|_{L^2(\Omega)} \leq C \mu (\gamma \mu k)^n \|\tilde{j}\|_{L^2(\Omega)} \quad \forall n \in \mathbb{N}_0. \quad (4.75b)$$

As in (4.70) and (4.71) we obtain $C, \sigma > 0$ independent of $k, h, p,$ and $\tilde{j}$ such that

$$\left\| E_{H^2} - \Pi_p^{\text{curl}, s} E_{H^2} \right\|_{\text{curl}, \Omega, k} \leq C \frac{k h}{p} \|\tilde{j}\|_{L^2(\Omega)},$$

$$\left\| E_A - \Pi_p^{\text{curl}, s} E_A \right\|_{\text{curl}, \Omega, k} \leq C \left[ \left( \frac{h}{h + \sigma} \right)^p + k \left( \frac{k h}{\sigma p} \right)^{p+1} + \left( \frac{k h}{\sigma p} \right)^{p+1} \right] \|\tilde{j}\|_{L^2(\Omega)}.$$
with radius 1 centered at the origin. These estimates will be derived in Sect. 6 and
applied to the different terms of the splitting of \( \langle e_h, v_h \rangle \) in Sects. 6.1–6.3.

We also analyze in the present section the operator \( L_\Omega \) and show that it maps into
an analyticity class. The fact that we consider \( \Omega = B_1(0) \) here implies the \textit{a priori}
bound \( \| L_\Omega v \|_{\text{curl}, \Omega, k} \leq \| v \|_{\text{curl}, \Omega, k} \) which, in turn, leads to the quantitative assertion
\( L_\Omega v \in A(Ck^{3/2} \| v \|_{\text{curl}, \Omega, 1, \gamma, \Omega}) \) in Theorem 5.9.

5.1 The Capacity Operator \( T_k \) on the Sphere

We restrict to the case that \( \Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega} \), where \( \Omega = B_1(0) \) is the open unit ball
with boundary \( \Gamma \). Let \( T_k : H^{-1/2}(\Gamma) \to H^{-1/2}_{\text{div}}(\Gamma) \) be the capacity operator that was
introduced in the paragraph before Remark 2.2.

In the case of the sphere, the eigenfunctions of the negative Laplace–Beltrami
operator “\(-\Delta_{\Gamma}\)” are given by the spherical harmonics \( Y^m_\ell \) (cf. [47, Sec. 2.4.1]) with
eigenvalues \( \lambda_\ell = \ell (\ell + 1) \). In this case, the index set \( \iota_\ell \) in (2.15) is given by
\[
\iota_\ell = \{-\ell, -\ell + 1, \ldots, \ell - 1, \ell\}. \tag{5.1}
\]

We introduce the decomposition of \( E_T \) according to (2.19) (cf. [47, (5.3.87)])
\[
E_T = E_{\text{curl}} + E^\nabla, \tag{5.2a}
\]
where
\[
E_{\text{curl}} := \sum_{\ell=1}^\infty \sum_{m \in \iota_\ell} u^m_\ell T^m_\ell \quad \text{and} \quad E^\nabla = \nabla_{\Gamma} p \quad \text{with} \quad p := \sum_{\ell=1}^\infty \sum_{m \in \iota_\ell} U^m_\ell Y^m_\ell \tag{5.2b}
\]
and the vectorial spherical harmonics \( T^m_\ell := \overrightarrow{\text{curl} \Gamma} Y^m_\ell \) (cf. [47, (2.4.152), (2.4.173)]). This implies \( \text{div}_{\Gamma} E_{\text{curl}} = 0 \).

Remark 5.1 For the expansion of a tangential field, e.g., \( E_T \), the summation starts with
\( \ell = 1 \) since \( T_0 = \nabla_{\Gamma} Y^0_0 = 0 \), i.e.,
\[
E_T = \sum_{\ell=1}^\infty \sum_{m \in \iota_\ell} \left( u^m_\ell T^m_\ell + U^m_\ell \nabla Y^m_\ell \right). \tag{5.3}
\]

Lemma 5.2 Let \( E_T \in H^{-1/2}_{\text{curl}}(\Gamma) \) be decomposed as in (5.2). Then (cf. Assumption 4.1)
\[
\text{div}_{\Gamma} E_{\text{curl}} = 0 \quad \text{and} \quad \left( E_{\text{curl}}, E^\nabla \right)_{\Gamma} = b_k \left( E_{\text{curl}}, E^\nabla \right) = b_k \left( E^\nabla, E_{\text{curl}} \right). \tag{5.4b}
\]

Furthermore, we have the definiteness relations: For all \( E \in X \), there holds
\[
\text{Im} \ b_k \left( E_{\text{curl}}, E_{\text{curl}} \right) \geq 0 \quad \text{and} \quad \text{Im} \ b_k \left( E^\nabla, E^\nabla \right) \leq 0. \tag{5.5}
\]
**Proof** Equation (5.4a) follows from \[47, (5.3.87)\) and (5.3.91). Integration by parts and using \(\text{div}_E \mathbf{E}^{\text{curl}} = 0\) shows the first equality in (5.4b). The second equality in (5.4b) is a consequence of \(\text{div}_E \mathbf{E}^{\text{curl}} = 0\) and \[47, (5.3.109)\]. The last term is zero since \(T_k \mathbf{E}^\nabla\) is a linear combination of \(\nabla \gamma Y^m_m\) (cf. \[47, (5.3.87)\) and (5.3.88)) and \((\nabla \gamma Y^m_m, \mathbf{E}^{\text{curl}})_\Gamma = - (Y^m_m, \text{div}_E \mathbf{E}^{\text{curl}})_\Gamma = 0\).

The first inequality in (5.5) follows from \[47, (5.3.107)\], and the second one is a consequence of \[47, (5.3.106)\]. □

Any tangential vector field \(\mathbf{u}_T \in H^{-1/2}_\text{curl}(\gamma)\) can be expanded in terms of surface gradients of spherical harmonics \(Y^m_m\) and vectorial spherical harmonics \(T^m_m\) via

\[
\mathbf{u}_T = \mathbf{u}^{\text{curl}} + \mathbf{u}^\nabla
\]  

(5.6)

with

\[
\mathbf{u}^{\text{curl}} := \sum_{\ell=1}^\infty \sum_{m \in \iota \ell} u^{m \ell}_T T^m_m \quad \text{and} \quad \mathbf{u}^\nabla := \nabla_p \quad \text{with} \quad p := \sum_{\ell=1}^\infty \sum_{m \in \iota \ell} U^m_m Y^m_m.
\]

The application of \(T_k\) to \(\mathbf{u}_T\) has the explicit form (cf. \[47, (5.3.88)\])

\[
T_k \mathbf{u}_T = \sum_{\ell=1}^\infty \frac{z_\ell(k) + 1}{k} \sum_{m \in \iota \ell} u^{m \ell}_T T^m_m + \sum_{\ell=1}^\infty \frac{i k}{z_\ell(k) + 1} \sum_{m \in \iota \ell} U^m_m \nabla \gamma Y^m_m
\]  

(5.7)

with

\[
z_\ell(r) := r h_\ell^{(1)}(r) = - p_\ell(r^{-2}) + i q_\ell(r^{-2}),
\]  

(5.8)

and the spherical Hankel functions \(h_\ell^{(1)}\), and \(p_\ell, q_\ell\) are polynomials of degree \(\ell\) with real coefficients (cf. \[47, (2.6.19)–(2.6.22)\]).

**Lemma 5.3** Let \(\lambda_0 > 1\) arbitrary but fixed. Then there exists \(C_0\) depending only on \(\lambda_0\) such that for any \(\lambda \geq \lambda_0\):

\[
\frac{k}{|z_n(k) + 1|} \leq \begin{cases} 
2\sqrt{2} \frac{k}{n} & \forall n \in \mathbb{N}_0, \\
2\sqrt{2} \frac{k}{\lambda_0 + 1} \frac{k}{(n + 1)} & n > \lambda k^2, \\
C_0 \frac{k}{n + 1} & n \geq \lambda k,
\end{cases}
\]  

(5.9)

\[
\frac{|z_n(k) + 1|}{k} \leq 1 + \frac{n}{k},
\]  

(5.10)

Estimate (5.10) follows from \[47, (5.3.95)\]. The proof of (5.9) is rather technical and postponed to Appendix A.

**Remark 5.4** Equations (5.5), (5.7), and (5.8) imply that Assumption 4.1 is satisfied for the sphere. □
5.2 Analysis of Frequency Splittings $L_{\Gamma}, H_{\Gamma}$ on the Surface of the Sphere

5.2.1 Analyticity of $L_{\Gamma}$

Lemma 5.5 Let $\Omega = B_1(0)$, and let the frequency filter $L_{\Gamma}$ be given by Definition 4.2 with a cutoff parameter $\lambda \geq \lambda_0 > 1$. Then:

(i) There exists a fixed tubular neighborhood $U_{\Gamma}$ of $\Gamma$ and constants $C_2, \gamma_2$ independent of $k$ (but dependent on $\Gamma, \lambda$) such that for each $u \in H(\text{curl}, \Omega)$ there is an extension $U \in A(C_2 k^{3/2} \|u\|_{\text{curl},\Omega,1}, \gamma_2, U_{\Gamma})$ of $L_{\Gamma} u_T$ to $U_{\Gamma}$.

(ii) The function $L_{\Gamma} u_T$ belongs to the class $A(C_1 k^{3/2} \|u\|_{\text{curl},\Omega,1}, \gamma_1, \Gamma)$ for constants $C_1, \gamma_1$ independent of $k$ and $u$. In particular, $\|L_{\Gamma} u_T\|_{H^{1/2}(\Gamma)} \leq C' k^2 \|u\|_{\text{curl},\Omega,1}$.

Proof Before proving Lemma 5.5, we mention that the algebraic growth rates with respect to $k$ are likely suboptimal. However, sharper estimates would require more technicalities. We start by noting that the analyticity of $\Gamma$ provides that the eigenfunctions $Y_m^\ell$ of the Laplace–Beltrami operator have analytic extensions $\tilde{Y}_m^\ell$ to a tubular neighborhood $U_{\Gamma}$ of $\Gamma$. A quantitative bound in terms of the eigenvalue $\lambda_\ell$ is given in [37, Lemma C.1]

$$\|\nabla^n \tilde{Y}_\ell\|_{L^2(U_{\Gamma})} \leq C_S \max\{\sqrt{\lambda_\ell}, n\}^n \gamma_S^n \forall n \in \mathbb{N}_0 \quad (5.11)$$

for some $C_S, \gamma_S$ depending solely on $\Gamma$. We recall specifically that the eigenvalues $\lambda_\ell$ of the Laplace–Beltrami operator on the sphere are $\lambda_\ell = \ell(\ell + 1)$.

Proof of (i): Let $u_T$ be a tangential field on the sphere represented as [cf. (5.3)]

$$u_T = \sum_{\ell=1}^{\infty} \sum_{m \in \iota_\ell} (u_\ell^m U_\ell^m \nabla_{\Gamma} Y_\ell^m) \quad , \quad T_\ell^m = \text{curl}_\Gamma Y_\ell^m = \nabla_{\Gamma} Y_\ell^m \times n^*.$$

With the extension $n^*$ of the normal vector $n$ that is constant in normal direction, we may define the extension $U$ of $L_{\Gamma} u_T$ as

$$U = \sum_{1 \leq \ell \leq \lambda k} \sum_{m \in \iota_\ell} u_\ell^m \nabla \tilde{Y}_\ell^m \times n^* + U_\ell^m \nabla \tilde{Y}_\ell^m.$$

By the analyticity of $n^*$, we get from Lemma 2.6 and (5.11) that, for some $C', \tilde{\gamma} > 0$ depending solely on $\Gamma$, there holds for all $n \in \mathbb{N}_0$

$$\|\nabla^n (\nabla \tilde{Y}_\ell^m \times n^*)\|_{L^2(U_{\Gamma})} + \|\nabla^n (\nabla \tilde{Y}_\ell^m)\|_{L^2(U_{\Gamma})} \leq C' \sqrt{\lambda_\ell} \tilde{\gamma}^n \max\{\sqrt{\lambda_\ell}, n\}^n \quad . \quad (5.12)$$

Taking $\ell \leq \lambda k$ into account (and using $\lambda_\ell = \ell(\ell + 1)$) allows us to estimate $U$ by
\[ \| \nabla^n U \|_{L^2(\mathcal{U}_T)} \leq \sum_{1 \leq \ell \leq \lambda k} \sum_{m \in \mathcal{I}} \left( |u^m_\ell| \right) \| \nabla^n (\nabla \tilde{Y}_\ell^m \times n^\ast) \|_{L^2(\mathcal{U}_T)} + |U^m_\ell| \| \nabla^n+1 \tilde{Y}_\ell^m \|_{L^2(\mathcal{U}_T)} \]

\[ \lesssim \tilde{Y}^n \sum_{1 \leq \ell \leq \lambda k} \max \left\{ \sqrt{\lambda_\ell}, n \right\}^{n/4} (|u^m_\ell| + |U^m_\ell|) \]

\[ \lesssim \tilde{Y}^n \left( \sum_{1 \leq \ell \leq \lambda k} \max \left\{ \sqrt{\lambda_\ell}, n \right\}^{2n/4} \right)^{1/2} \left( \sum_{1 \leq \ell \leq \lambda k} \lambda_\ell^{1/2} (|u^m_\ell| + |U^m_\ell|)^2 \right)^{1/2} \]

\[ \lesssim \tilde{Y}^n (\lambda k + 1)^{3/2} \max \{\lambda k + 1, n\} \| L \Gamma u_T \|_{-1/2, \text{curl} \Gamma} \cdot \] (5.13)

Since

\[ \| L \Gamma u_T \|_{-1/2, \text{curl} \Gamma} \leq \| u_T \|_{-1/2, \text{curl} \Gamma} \lesssim \| u \|_{\text{curl}, \Omega, 1}, \] (5.14)

the proof of (i) is complete.

**Proof of (ii):** An application of the multiplicative trace inequality would allow us to infer from (i) the assertion \( L \Gamma u_T \in \mathcal{A}(C_1 k^2 \| u \|_{\text{curl}, \Omega, 1}, \gamma_1, \Gamma) \) for suitable \( C_1, \gamma_1 \). The sharper statement follows by repeating the arguments of (i) starting with the assertion of [37, Lemma C.1] that

\[ \| \nabla^n Y_\ell^m \|_{L^2(\Gamma)} \leq C S Y^n \max \{\sqrt{\lambda_\ell}, n\} \forall n \in \mathbb{N}_0. \] (5.15)

\[ \square \]

### 5.2.2 Estimates for High- and Low-Frequency Parts of the Capacity Operator

In this section, we derive continuity estimates for the sesquilinear form \( b_k \). The \( k \)-dependence is different for the low- and high-frequency parts of the tangential fields and for the summands in the splitting \( u_T = u^\text{curl} + u^\nabla \). In Proposition 5.7 we derive such estimates for the tangential fields while these estimates are lifted to the space \( \mathbf{X} \) and some subspaces thereof in Proposition 5.8.

**Remark 5.6** If \( \Gamma = \partial B_1(0) \), then there holds for all \( u_T, v_T \in H^{-1/2}_\text{curl} (\Gamma) \)

\[ b^\text{high}_k (u_T, v_T) \overset{\text{Def.} 4.5}{=} (T_k u_T, H_T v_T)_{\Gamma} \overset{\text{Def.} 4.2}{=} (H_T T_k u_T, v_T)_{\Gamma} \overset{\text{(5.7)}}{=} (T_k H_T u_T, v_T)_{\Gamma} \overset{\text{Def.} 4.5}{=} \left( I_{k}^\text{high}_T u_T, v_T \right)_{\Gamma}. \]

Analogous relations hold for the low-frequency part \( b^\text{low}_k \). \[ \square \]

**Proposition 5.7** With the frequency filters \( L \Gamma, H_T \) of Definition 4.2 given by a cutoff parameter \( \lambda \geq \lambda_0 > 1 \), the sesquilinear form \( b_k \) can be written for \( u_T, v_T \in H^{-1/2}_\text{curl} (\Gamma) \) as

\[ b_k (u_T, v_T) = b_k (u^\text{curl}, v^\text{curl}) + b^\text{high}_k (u^\nabla, v^\nabla) + b^\text{low}_k (u^\nabla, v^\nabla). \] (5.16)
There is $C_b > 0$ depending solely on $\lambda_0$ such that the following holds:

$$
|b_k(u_{\text{curl}}, v_{\text{curl}})| \leq C_b \left( \frac{1}{k} \left\| \text{curl}_T u_T \right\|_{H^{-1/2}(\Gamma)} \left\| \text{curl}_T v_T \right\|_{H^{-1/2}(\Gamma)} + \left( 1 + \lambda \right) \left\| \text{curl}_T L T u_T \right\|_{H^{-1}(\Gamma)} \left\| \text{curl}_T L T v_T \right\|_{H^{-1}(\Gamma)} \right),
$$

$$
|b_k^{\text{high}}(u_{\text{curl}}, v_{\text{curl}})| \leq C_b \frac{1}{k} \left\| \text{curl}_T u_T \right\|_{H^{-1/2}(\Gamma)} \left\| \text{curl}_T v_T \right\|_{H^{-1/2}(\Gamma)},
$$

$$
|b_k^{\text{low}}(u_{\nabla}, v_{\nabla})| \leq C_b \lambda^\rho k^{\theta+1} \left\| \text{div}_T L T u_T \right\|_{H^{-1-\rho/2}(\Gamma)} \left\| \text{div}_T L T v_T \right\|_{H^{-1-\rho/2}(\Gamma)}
$$

(5.17)

for $0 \leq \rho \leq 2$. If $\text{div}_T u_T \in H^{\rho_1}(\Gamma)$ and $\text{div}_T v_T \in H^{\rho_2}(\Gamma)$ for some $\rho_1 + \rho_2 + 3 \geq 0$ we have

$$
|b_k^{\text{high}}(u_{\nabla}, v_{\nabla})| \leq C_b \frac{k}{(\lambda k)^{\rho_1+\rho_2+3}} \left\| \text{div}_T u_T \right\|_{H^{\rho_1}(\Gamma)} \left\| \text{div}_T v_T \right\|_{H^{\rho_2}(\Gamma)}. \quad (5.18)
$$

**Proof** The equality (5.16) follows from Lemma 5.2.

Using the orthogonality relations of $T^m_\ell$ and $\nabla T^m_\ell$, the representations in [47, Sec. 5.3.2] give us

$$
|b_k(u_{\text{curl}}, v_{\text{curl}})| = \left| \sum_{\ell=1}^{\infty} \left( -\frac{z_\ell(k)+1}{k} \right) \sum_{m=-\ell}^{\ell} u^m_\ell v^m_\ell (T^m_\ell \cdot T^m_\ell) \right|
$$

\[\text{[47, (2.4.155)]} \leq \left| \sum_{\ell=1}^{\infty} (\ell+1) \left( -\frac{z_\ell(k)+1}{k} \right) \sum_{m=-\ell}^{\ell} u^m_\ell v^m_\ell \right|

\leq \sum_{\ell=1}^{\infty} (\ell+1) \left( 1 + \frac{\ell}{k} \right) \sum_{m=\ell}^{\ell+m} u^m_\ell v^m_\ell

\leq 2 \left( \frac{1}{k} \left\| \text{curl}_T u_T \right\|_{H^{-1/2}(\Gamma)} \right) \left\| \text{curl}_T v_T \right\|_{H^{-1/2}(\Gamma)}

+ \left( 1 + \lambda \right) \left\| \text{curl}_T L T u_T \right\|_{H^{-1}(\Gamma)} \left\| \text{curl}_T L T v_T \right\|_{H^{-1}(\Gamma)}.
$$

This leads to the first estimate in (5.17). In a similar way, we obtain for the high-frequency part

$$
|b_k^{\text{high}}(u_{\text{curl}}, v_{\text{curl}})| \leq \sum_{\ell \geq \lambda k} (\ell+1) \left( 1 + \frac{\ell}{k} \right) \sum_{m=\ell}^{\ell+m} u^m_\ell v^m_\ell

\leq \frac{2}{k} \sum_{\ell \geq \lambda k} \ell^2 (\ell+1) \sum_{m=\ell}^{\ell+m} \left| u^m_\ell \right| \left| v^m_\ell \right|

\leq \frac{2}{k} \left\| \text{curl}_T H T u_T \right\|_{H^{-1/2}(\Gamma)} \left\| \text{curl}_T H T v_T \right\|_{H^{-1/2}(\Gamma)}.
$$

(2.21)
producing the second estimate in (5.17). For the third one and (5.18), we obtain

\[ b_k^{\text{low}}(u \nabla, v \nabla) = i \sum_{1 \leq \ell \leq \lambda k} \ell (\ell + 1) \sum_{m \in \ell} \left( \frac{k}{z_k (k) + 1} U^m_k \nabla^m_k \right), \]

\[ b_k^{\text{high}}(u \nabla, v \nabla) = i \sum_{\ell > \lambda k} \ell (\ell + 1) \sum_{m \in \ell} \left( \frac{k}{z_k (k) + 1} U^m_k \nabla^m_k \right). \]

Using (5.3) and (5.9), we get for any \( 0 \leq \rho \leq 2 \)

\[
\left| b_k^{\text{low}}(u \nabla, v \nabla) \right| \leq 2 \sqrt{2k} \sum_{1 \leq \ell \leq \lambda k} \ell (\ell + 1) \sum_{m \in \ell} \left| U^m_k \right| \left| \nabla^m_k \right| \\
\leq 4 \sqrt{2k} (\lambda k)^\rho \sum_{1 \leq \ell \leq \lambda k} \ell^{2-\rho} \sum_{m \in \ell} \left| U^m_k \right| \left| \nabla^m_k \right| \\
\leq 16 \sqrt{2} \lambda^\rho k^{\rho+1} \sum_{1 \leq \ell \leq \lambda k} \ell^2 (\ell + 1)^{-\rho} \sum_{m \in \ell} \left| U^m_k \right| \left| \nabla^m_k \right| \\
\overset{(2.21)}{\leq} 16 \sqrt{2} \lambda^\rho k^{\rho+1} \| \text{div} \Gamma L \Gamma u_T \|_{H^{-1-\rho/2}(\Gamma)} \| \text{div} \Gamma L \Gamma v_T \|_{H^{-1-\rho/2}(\Gamma)}. \]

For \( \rho_1 + \rho_2 + 3 \geq 0 \), we get from (5.9)

\[
\left| b_k^{\text{high}}(u \nabla, v \nabla) \right| \leq C_0 k \sum_{\ell > \lambda k} \ell \sum_{m \in \ell} \left| U^m_k \right| \left| \nabla^m_k \right| \\
\leq \frac{C_0 k}{(\lambda k)^{\rho_1+\rho_2+3}} \sum_{\ell > \lambda k} \ell^{4+\rho_1+\rho_2} \sum_{m \in \ell} \left| U^m_k \right| \left| \nabla^m_k \right| \\
\overset{(2.21)}{\leq} C \frac{k}{(\lambda k)^{\rho_1+\rho_2+3}} \| \text{div} \Gamma u_T \|_{H^{\rho_1}(\Gamma)} \| \text{div} \Gamma v_T \|_{H^{\rho_2}(\Gamma)}. \]

\[ \Box \]

**Proposition 5.8** There is a constant \( C'_b > 0 \) depending solely on \( \lambda_0 \) such that the following holds:

Let \( u, v \in X \). Then:

\[ \left| b_k(u \nabla, v \nabla) \right| \leq C'_b k^2 \| u \|_{\text{curl}, \Omega, 1} \| v \|_{\text{curl}, \Omega, 1}. \] (5.19)

Let \( u_0 \in V_0 \) and \( v \in X \). Then:

\[ \left| k b_k^{\text{high}}(u_0 \nabla^1, v \nabla) \right| \leq C'_b \frac{k}{\lambda} \| u_0 \|_{\text{curl}, \Omega, 1} \| v \|_{\text{curl}, \Omega, 1}, \] (5.20a)

\[ \left| k b_k^{\text{low}}(u_0 \nabla, v \nabla) \right| \leq C'_b \lambda k^3 \| \text{div} \Gamma L \Gamma u_0, T \|_{H^{-3/2}(\Gamma)} \| \text{div} \Gamma L \Gamma v_T \|_{H^{-3/2}(\Gamma)}. \] (5.20b)
Let \( u \in X \) and \( v_0 \in V_0^* \). Then:

\[
\begin{aligned}
|k b_k^\text{high} \left( u^\nabla, v_0^\nabla \right) | & \leq C_b \frac{k}{\lambda} \| u \|_{\text{curl}, \Omega, 1} \| v_0 \|_{\text{curl}, \Omega, 1} , \\
|k b_k^\text{low} \left( u^\nabla, v_0^\nabla \right) | & \leq C_b \lambda k^3 \| \text{div}_\Gamma L \Gamma \mathbf{u}_T \|_{H^{-3/2}(\Gamma)} \| \text{div}_\Gamma L \Gamma \mathbf{v}_T \|_{H^{-3/2}(\Gamma)} .
\end{aligned}
\] (5.21a)

Let \( u_0 \in V_0, v_0 \in V_0^* \) and \( p, q \in H^1(\Omega) \). Then:

\[
\begin{aligned}
|k b_k^\text{high} \left( u_0^\nabla, (\nabla p)^\nabla \right) | & \leq C_b \frac{e}{\lambda} \| u_0 \|_{\text{curl}, \Omega, 1} \| \nabla p \|_{\text{curl}, \Omega, k} , \\
|k b_k^\text{high} \left( (\nabla p)^\nabla, v_0^\nabla \right) | & \leq C_b \frac{e}{\lambda} \| \nabla p \|_{\text{curl}, \Omega, k} \| v_0 \|_{\text{curl}, \Omega, 1} , \\
|k b_k^\text{high} \left( (\nabla p)^\nabla, (\nabla q)^\nabla \right) | & \leq C_b \frac{e}{\lambda} \| \nabla p \|_{\text{curl}, \Omega, k} \| \nabla q \|_{\text{curl}, \Omega, k} .
\end{aligned}
\] (5.21b)

Let \( u_0 \in V_0 \) and \( v_0 \in V_0^* \). Then:

\[
\begin{aligned}
|k b_k^\text{high} \left( u_0^\nabla, v_0^\nabla \right) | & \leq C_b \frac{e}{\lambda^2} \| u_0 \|_{\text{curl}, \Omega, 1} \| v_0 \|_{\text{curl}, \Omega, 1} , \\
|k b_k^\text{low} \left( u_0^\nabla, v_0^\nabla \right) | & \leq C_b \lambda k^3 \| u_0 \|_{\text{curl}, \Omega, 1} \| v_0 \|_{\text{curl}, \Omega, 1} .
\end{aligned}
\] (5.21c)

Let \( u, v \in H^1(\Omega) \) and \( w \in X \). Then:

\[
\begin{aligned}
|k b_k^\text{high} \left( u^\nabla, v^\nabla \right) | & \leq C_b \frac{e}{\lambda^2} \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)} , \\
|k b_k^\text{high} \left( w^\nabla, v^\nabla \right) | & \leq C_b \frac{e}{\lambda} \| w \|_{\text{curl}, \Omega, 1} \| v \|_{H^1(\Omega)} .
\end{aligned}
\] (5.21d)

Proof Proof of (5.19): We combine the last estimate in (5.17) (for \( \rho = 1 \) and \( \lambda = \lambda_0 \)) with (5.18) (for \( \rho_1 = \rho_2 = -3/2 \) and \( \lambda = \lambda_0 \)) and obtain

\[
\begin{aligned}
| b_k \left( u^\nabla, v^\nabla \right) | & \leq | b_k^\text{low} \left( u^\nabla, v^\nabla \right) | + | b_k^\text{high} \left( u^\nabla, v^\nabla \right) | \\
& \leq C_b \left( \lambda_0 k^2 \| \text{div}_\Gamma L \Gamma \mathbf{u}_T \|_{H^{-3/2}(\Gamma)} \| \text{div}_\Gamma L \Gamma \mathbf{v}_T \|_{H^{-3/2}(\Gamma)} \right) \\
& + k \| \text{div}_\Gamma \mathbf{u}_T \|_{H^{-3/2}(\Gamma)} \| \text{div}_\Gamma \mathbf{v}_T \|_{H^{-3/2}(\Gamma)} \\
& \leq C (1 + \lambda_0 k) k \| \text{div}_\Gamma \mathbf{u}_T \|_{H^{-3/2}(\Gamma)} \| \text{div}_\Gamma \mathbf{v}_T \|_{H^{-3/2}(\Gamma)} \\
& \leq C (1 + \lambda_0 k) k \| \mathbf{u}_T \|_{H^{-1/2}(\Gamma)} \| \mathbf{v}_T \|_{H^{-1/2}(\Gamma)} \\
& \leq C (1 + \lambda_0 k) \| \mathbf{u} \|_{\text{curl}, \Omega, 1} \| \mathbf{v} \|_{\text{curl}, \Omega, 1} .
\end{aligned}
\] (5.25)
Proof of (5.20), (5.21), (5.24b): Let \( u \in H^1(\Omega) \) and \( v \in X \). Choose \( \rho_1 = -1/2 \) and \( \rho_2 = -3/2 \) in (5.18) to obtain

\[
|k b_k^{\text{high}} \left( (u, v) \right)| \leq C_b \frac{k}{\lambda} \| \text{div}_\Gamma u_T \|_{H^{-1/2}(\Gamma)} \| \text{div}_\Gamma v_T \|_{H^{-3/2}(\Gamma)}
\leq C \frac{k}{\lambda} \| u_T \|_{H^{1/2}(\Gamma)} \| v_T \|_{H^{-1/2}(\Gamma)}
\leq C \frac{k}{\lambda} \| u \|_{H^1(\Omega)} \| v \|_{\text{curl}, \Omega, 1}.
\] (5.26)

This shows (up to interchanging the roles of \( u \) and \( v \)) the estimate (5.24b). Since \( V_0 \subset H^1(\Omega) \), we may apply estimate (5.26) to \( u \in V_0 \) and \( v \in X \). Lemma B.1 implies the estimate \( \| u \|_{H^1(\Omega)} \leq \| u \|_{\text{curl}, \Omega, 1} \) so that (5.20a) follows. For the low-frequency part, we get from (5.17) for \( \rho = 1 \) the estimate

\[
|k b_k^{\text{low}} \left( (u, v) \right)| \leq C_b \lambda k^3 \| \text{div}_\Gamma L_{\Gamma} u_{0,T} \|_{H^{-3/2}(\Gamma)} \| \text{div}_\Gamma L_{\Gamma} v_T \|_{H^{-3/2}(\Gamma)},
\]

which is (5.20b). For \( u \in X \) and \( v_0 \in V_0^* \), estimates (5.21) follow by the same arguments and interchanging the roles of \( u \) and \( v \).

Proof of (5.22): For \( u_0 \in V_0 \) and \( p \in H^1(\Omega) \), we employ (5.20a) with \( v = \nabla p \) and \( \text{curl} \, \nabla p = 0 \) so that

\[
|k b_k^{\text{high}} \left( (u_0, (\nabla (p)) \right)| \leq C \frac{k}{\lambda} \| u_0 \|_{\text{curl}, \Omega, 1} \| \nabla p \|_{\text{curl}, \Omega, 1} = C \frac{k}{\lambda} \| u_0 \|_{\text{curl}, \Omega, 1} (k \| \nabla p \|)
\leq C \frac{k}{\lambda} \| u_0 \|_{\text{curl}, \Omega, 1} \| \nabla p \|_{\text{curl}, \Omega, k},
\]

which shows (5.22a). The proof of (5.22b) is just a repetition of the previous arguments while the proof of (5.22c) uses (5.18) with \( \rho_1 = \rho_2 = -3/2 \):

\[
|k b_k^{\text{high}} \left( (\nabla (p)), (\nabla q) \right)| \leq C_b k^2 \| \text{div}_\Gamma (\nabla (p))_T \|_{H^{-3/2}(\Gamma)} \| \text{div}_\Gamma (\nabla q)_T \|_{H^{-3/2}(\Gamma)}
\leq C k^2 \| \nabla p \|_{\text{curl}, \Omega, 1} \| \nabla q \|_{\text{curl}, \Omega, 1}
= C k^2 \| \nabla p \| \| \nabla q \| = C \| \nabla p \|_{\text{curl}, \Omega, k} \| \nabla q \|_{\text{curl}, \Omega, k},
\]

where the second step uses the same arguments as in (5.25).

Proof of (5.23), (5.24a): For any \( u, v \in H^1(\Omega) \), we may choose \( \rho_1 = \rho_2 = -1/2 \) in (5.18) to obtain

\[
|k b_k^{\text{high}} \left( (u, v) \right)| \leq C_b \frac{k}{\lambda^2} \| \text{div}_\Gamma u_T \|_{H^{-1/2}(\Gamma)} \| \text{div}_\Gamma v_T \|_{H^{-1/2}(\Gamma)}
\leq C \frac{k}{\lambda^2} \| u_T \|_{H^{1/2}(\Gamma)} \| v_T \|_{H^{1/2}(\Gamma)} \leq C \frac{k}{\lambda^2} \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)}.
\] (5.27)
This proves (5.24a). If we assume additionally \( u_0 \in V_0 \) and \( v_0 \in V_0^* \), we can appeal to Lemma B.1 to get
\[
|k b_{\text{high}}(u_0, v_0)| \leq \frac{C}{\lambda^2} \| u_0 \|_{\text{curl}, \Omega, 1} \| v_0 \|_{\text{curl}, \Omega, 1} ,
\]
i.e., (5.23a). For (5.23b), we employ the last equation in (5.17) for \( \rho = 1 \) and proceed as in the proof of (5.23a).

\[ \Box \]

5.3 Analysis of Frequency Splittings \( L_\Omega, H_\Omega \) for the Case \( \Omega = B_1(0) \)

The operator \( L_\Omega \) is defined in Definition 4.2 as the minimum norm extension of \( L_\Gamma \) with respect to the norm \( \| \cdot \|_{\text{curl}, \Omega, k} \). From Lemma C.1 we have the following stability estimate for the case \( \Omega = B_1(0) \):
\[
\| L_\Omega u \|_{\text{curl}, \Omega, k} \leq \| u \|_{\text{curl}, \Omega, k} .
\]
(5.28a)

By the triangle inequality, we infer that also \( H_\Omega \) is stable:
\[
\| H_\Omega u \|_{\text{curl}, \Omega, k} \leq 2 \| u \|_{\text{curl}, \Omega, k} .
\]
(5.28b)

**Theorem 5.9** Let \( \Omega = B_1(0) \). Then the low-frequency part \( L_\Omega u \) satisfies
\[
\| L_\Omega u \|_{\text{curl}, \Omega, k} \leq \| u \|_{\text{curl}, \Omega, k} \quad \text{and} \quad \text{div} L_\Omega u = 0.
\]
(5.29)

Furthermore, \( L_\Omega u \in A(C_{A,5} C'' u, \gamma'_{A,5}, \Omega) \) with
\[
C'' u = k^{3/2} \| u \|_{\text{curl}, \Omega, 1} .
\]
(5.30)

The constants \( C_{A,5}, \gamma'_{A,5} \) are independent of \( k \) and \( u \) but depend on the choice of the cutoff parameter \( \lambda \). Furthermore, there exists a tubular neighborhood \( U_\Gamma \) of \( \Gamma \) such that \( L_\Omega u \) is analytic on \( \Omega \cup U_\Gamma \) with \( L_\Omega u \in A(C'_{A,5} C'' u, \gamma''_{A,5}, \Omega \cup U_\Gamma) \).

**Proof** 1. Step (interior regularity) Using the vector identity
\[
\text{curl curl} = -\Delta + \nabla \text{div} ,
\]
(5.31)

we infer from (4.9) that \( -\Delta L_\Omega u + k^2 L_\Omega u = 0 \) in \( \Omega \). Interior regularity in the form [35, Prop. 5.5.1] then gives \( L_\Omega u \in A(C_{R,k}^{-1} \| L_\Omega u \|_{\text{curl}, \Omega, k}, \gamma_R, B_R) \) for any ball \( B_R \subset \Omega \), where the constants \( C_{R}, \gamma_R \) are independent of \( k \) and \( u \) (but depend on \( R \)). Noting (5.28) shows the desired analyticity assertion for the interior of \( \Omega \).

2. Step (smoothness up to the boundary and \( H^1 \)-estimates) Let the tubular neighborhood \( U_\Gamma \) of \( \Gamma \) and the extension \( U \in A(C k^{3/2} \| u \|_{\text{curl}, \Omega, 1}, \gamma_2, U_\Gamma) \) of \( L_\Gamma u_\Gamma \) be given by Lemma 5.5 and write \( L_\Omega u = U + \tilde{u} \). By the triangle inequality, we have
\[
k \| \tilde{u} \|_{L^2(U_\Gamma)} \leq \| L_\Omega u \|_{\text{curl}, \Omega, k} + \| U \|_{\text{curl}, U_\Gamma, k} \leq C k^{3/2} \| u \|_{\text{curl}, \Omega, 1} ,
\]
(5.32a)
The third condition is not essential but leads to a significant simplification as the ensuing (5.37) effects a decoupling of the elliptic system (5.38) into three scalar problems at 0.

For notational convenience, we will henceforth denote $\hat{U}$ by $U$.

The analyticity of $U$, Lemma D.1, and a simple induction argument (to deal with the presence of the lower-order term $k^2 \hat{\tilde{u}}$) shows that $\hat{\tilde{u}}$ is in $C^{\infty}(\Omega \cap \Omega)$. By suitably localizing, Lemma D.1, (i) gives for a suitable subset $\hat{\mathcal{U}} \subset \mathcal{U}$

$$\|\tilde{u}\|_{H^1(\hat{\mathcal{U}})} \leq C \left[ \|\tilde{u}\|_{L^2(\mathcal{U})} + \|\text{curl}\, \tilde{u}\|_{L^2(\Omega \cap \hat{\mathcal{U}})} + \|\text{div}\, \tilde{u}\|_{L^2(\Omega \cap \hat{\mathcal{U}})} \right]$$

(5.32), (5.33b) \leq C k^{5/2} \|u\|_{\text{curl}, \Omega, 1}. \tag{5.35}

For notational convenience, we will henceforth denote $\mathcal{U}$ by $\mathcal{U}$. 3. Step (analytic regularity of $\hat{u}$) Quantitative bounds for higher derivatives of $\hat{u}$ are obtained by locally flattening the boundary. By the analyticity of $\Gamma$ and the compactness of $\hat{\mathcal{U}}$, there are $R_0, C_X, \gamma_X > 0$ such that for each $x_0 \in \Gamma$ we can find a parameterization $\chi_{x_0} \in A(C_X, \gamma_X, B^{+}_{R_0}(0))$ with the following properties:

1. $\chi_{x_0}(0) = x_0$ and, for $B^{+}_{R_0} := \{ \hat{x} \in B^{+}_{R_0}(0) | \hat{x}_3 > 0 \}$ and $\hat{R}_{R_0} := \{ \hat{x} \in B^{+}_{R_0}(0) | x_3 = 0 \}$, we have $V_{x_0} := \chi_{x_0}(B^{+}_{R_0}) \subset \Omega$ as well as $\chi_{x_0}(\hat{R}_{R_0}) \subset \Gamma$.
2. For $\hat{x} \in \hat{R}_{R_0}$ the vectors $t_{x_0} := \partial_i \chi_{x_0}(\hat{x})$, $i \in \{1, 2\}$, span the tangent plane of $\Gamma$ at $\chi_{x_0}(\hat{x})$ and $n(x) := -\partial_3 \chi_{x_0}(\hat{x})$ is the outward normal vector.
3. The Jacobian $D\chi_{x_0}(0) \in \mathbb{R}^{3 \times 3}$ is orthogonal, i.e., $(D\chi_{x_0}(0))^T(D\chi_{x_0}(0)) = I$.

The transformation of the system (5.33) on $V_{x_0}$ to the half-ball $B^{+}_{R_0}$ is effected with a covariant transformation of the dependent variable $\tilde{u}$ by setting $\tilde{u}^{\text{cov}} := (D\chi_{x_0})^T\tilde{u} \circ \chi_{x_0}$. We recall the formula (see, e.g., [43, Cor. 3.58])

$$\frac{1}{\det(D\chi_{x_0})} (D\chi_{x_0}) \text{curl } w^{\text{cov}} = (\text{curl } w) \circ \chi_{x_0}$$

and introduce the two pointwise symmetric positive definite matrices

$$A := \frac{(D\chi_{x_0})^T(D\chi_{x_0})}{\det(D\chi_{x_0})}, \quad B := \left(\det(D\chi_{x_0})\right)^{-1}(D\chi_{x_0})^{-T}; \tag{5.36}$$

The third condition is not essential but leads to a significant simplification as the ensuing (5.37) effects a decoupling of the elliptic system (5.38) into three scalar problems at 0.
note that $\mathbf{A}, \mathbf{B} \in \mathcal{A}^\infty(C', \gamma', B_{R_0}^+)$ for some constants $C', \gamma'$ that depend solely on $\Gamma$ (which is fixed in our case $\Gamma := \partial B_1(0)$). We also note that, since $D\chi_0(0)$ is assumed to be orthogonal, we have

$$\mathbf{A}(0) = \mathbf{B}(0) = \mathbf{I} \in \mathbb{R}^{3 \times 3}. \quad (5.37)$$

From (5.33a) we obtain for all $\mathbf{V} \in C_0^\infty(B_{R_0}^+)$

$$\int_{B_{R_0}^+} \left( \frac{1}{\det D\chi_0} \left( (D\chi_0) \text{curl} \, \tilde{\mathbf{u}}^{\text{cov}}, (D\chi_0) \text{curl} \, \mathbf{V} \right) + (\det D\chi_0) k^2 \left( \tilde{\mathbf{u}}^{\text{cov}}, (D\chi_0)^{-1} (D\chi_0)^{-T} \mathbf{V} \right) \right) = \int_{B_{R_0}^+} \left( \hat{\mathbf{f}}, \mathbf{V} \right)$$

with $\hat{\mathbf{f}} := \det(D\chi_0)(D\chi_0)^{-1}f \circ \chi_0$. The strong form of this equation is

$$\text{curl} \left( \mathbf{A} \text{curl} \, \tilde{\mathbf{u}}^{\text{cov}} \right) + k^2 \mathbf{B}\tilde{\mathbf{u}}^{\text{cov}} = \hat{\mathbf{f}} \quad \text{in} \ B_{R_0}^+. \quad (5.38a)$$

The transformation of the divergence condition (5.33b) to $B_{R_0}^+$ is:

$$\text{div} \left( \mathbf{B}\tilde{\mathbf{u}}^{\text{cov}} \right) = \hat{G} := \det(D\chi_0) G \circ \chi_0 \quad \text{in} \ B_{R_0}^+. \quad (5.38b)$$

The covariant transformation leaves the homogeneous tangential trace (5.33c) invariant:

$$\Pi_T \tilde{\mathbf{u}}^{\text{cov}} = 0 \quad \text{on} \ \hat{\Gamma}_{R_0}. \quad (5.38c)$$

We rewrite the equations (5.38) in the form (D.13). To that end, we note that the solution $\tilde{\mathbf{u}}^{\text{cov}}$ is smooth (up to the boundary $\hat{\Gamma}_{R_0}$) by Step 2 so that the manipulations are admissible; we also note $\mathbf{A}(0) = \mathbf{B}(0) = \mathbf{I}$ by (5.37). Adding the gradient of equation (5.38b) to equation (5.38a) and taking the trace of (5.38b) on $\hat{\Gamma}_{R_0}$ as well as taking note of (5.38c) (to obtain both (5.39b) and (5.39c)) give a system of the following form:

$$- \sum_{\alpha, \beta, j=1}^3 \partial\alpha \left( A_{\alpha\beta}^{ij} \partial\beta \tilde{u}_j^{\text{cov}} \right) + \sum_{j, \beta=1}^3 B_{\beta j}^{ij} \partial\beta \tilde{u}_j^{\text{cov}} + \sum_{j=1}^3 \left( C_{ij}^{\alpha} + k^2 B_{ij} \right) \tilde{u}_j^{\text{cov}} = \hat{f}_i + \partial\beta \hat{G}, \quad \text{on} \ B_{R_0}^+, \quad i = 1, 2, 3, \quad (5.39a)$$

$$\tilde{u}_i^{\text{cov}} = 0 \quad \text{on} \ \hat{\Gamma}_{R_0}, \quad i = 1, 2, \quad (5.39b)$$

$$\partial_3 \tilde{u}_3^{\text{cov}} = \hat{G} - \left( \sum_{i=1}^3 \partial_3 B_{i3} \right) \tilde{u}_3^{\text{cov}} - \sum_{i=1}^2 B_{i3} \partial_3 \tilde{u}_3^{\text{cov}} - \sum_{j=1}^2 B_{3j} \partial_3 \tilde{u}_j^{\text{cov}} - (B_{33} - 1) \partial_3 \tilde{u}_3^{\text{cov}} \quad \text{on} \ \hat{\Gamma}_{R_0}. \quad (5.39c)$$
The tensors \((A_{ij})^{\alpha\beta}, B^{ij}_{\beta},\) and \((C^{ij})_{i,j}\) are analytic on \(B^+_R\) and, with constants \(C'', \gamma''\), depending solely on \(\Gamma\) (being fixed in our case by \(\Gamma := \delta B_1(0)\)), we have \((A_{ij})^{\alpha\beta}, B^{ij}_{\beta},(C^{ij})_{i,j} \in A^\infty(C'', \gamma'', B^+_R)\). Additionally, we have the structural property [cf. (5.31) and (5.37)]

\[
A_{ij}^{\alpha\beta}(0) = \delta_{ij} \delta_{\alpha\beta}, \quad \forall i, j, \alpha, \beta, \\
B_{3j}(0) = B_{3j}(0) = 0 = B_{33}(0) - 1 \quad \text{for} \ j \in \{1, 2\}. \tag{5.40}
\]

Lemma 2.6 and (5.34) imply, for suitable constants \(C, \gamma_3\),

\[
\hat{f} \in A(C \varepsilon k^{1/2} \|u\|_{\text{curl}, \Omega, 1}, \gamma_3, B^+_R, C f), \quad \hat{G} \in A(C \varepsilon k^{5/2} \|u\|_{\text{curl}, \Omega, 1}, \gamma_3, B^+_R, C_G), \quad k \|u^{\text{cov}}\|_{L^2(B^+_R)} + \|u^{\text{cov}}\|_{H^1(B^+_R)} \leq C \varepsilon k^{5/2} \|u\|_{\text{curl}, \Omega, 1}. \tag{5.41}
\]

Dividing (5.39) by \(k^2\) makes Theorem D.5 applicable with \(\varepsilon = k^{-1}\) and the constants \(C_f, C_G, C_C\) there of the form \(C_f = O(k^{3/2} \|u\|_{\text{curl}, \Omega, 1}), C_G = O(k^{3/2} \|u\|_{\text{curl}, \Omega, 1}), C_C = O(1)\). Theorem D.5 yields a \(R > 0\) and constants \(C, \gamma\) such that for \(B^+_R := \{\hat{x} \in B_R(0) | \hat{x}_3 > 0\}\) we have \(u^{\text{cov}} \in A(C u, \gamma, B^+_R)\), where

\[
C_u = k^{3/2} \|u\|_{\text{curl}, \Omega, 1}.
\]

Transforming back using again Lemma 2.6 gives for \(V_n = \chi_n(B^+_R)\) the analytic regularity assertion \(\tilde{u} \in A(C C_u, \gamma, V_n)\) for suitable constants \(C, \gamma\). A covering argument completes the estimate of \(\tilde{u}\) on \(U_\Gamma\).

The normal trace \(\langle L_\Omega v, n \rangle\) is also analytic. We have:

**Lemma 5.10** Let \(\Omega = B_1(0)\). There is a tubular neighborhood \(U_\Gamma\) of \(\Gamma\), and there are constants \(C A, \gamma, \gamma A, \Gamma, C' A, \gamma', C' A, \gamma, C', \gamma' A, \Gamma, b > 0\) depending only on the choice of cutoff parameter \(\lambda\) such that for any \(v \in H(\text{curl}, \Omega)\) the normal trace \(\langle L_\Omega v, n \rangle\) on \(\Gamma\) satisfies the following:

(i) \(g_1 := \langle L_\Omega v, n \rangle\) has an analytic extension \(g_1^*\) to \(U_\Gamma\), which satisfies \(g_1^* \in A(C A, \Gamma k^{3/2} \|v\|_{\text{curl}, \Omega, 1}, \gamma A, \Gamma, U_\Gamma)\).

(ii) \(\langle L_\Omega v, n \rangle \in A(C A, \Gamma k^2 \|v\|_{\text{curl}, \Omega, 1}, \gamma A, \Gamma, \Gamma)\).

(iii) The expansion coefficients \(\kappa^m_\ell\) of

\[
\langle L_\Omega v, n \rangle = \sum_{\ell=1}^{\infty} \sum_{m \in \ell} \kappa^m_\ell Y^m_\ell 
\]

satisfy

\[
|\kappa^m_\ell| \leq C' A, \Gamma \|v\|_{\text{curl}, \Omega, 1} \left\{ \begin{array}{ll}
k^{1/2} & \text{if } \ell \leq \gamma' A, \Gamma k \\
k^2 e^{-b \ell} & \text{if } \ell > \gamma' A, \Gamma k.
\end{array} \right. \tag{5.44}
\]
\[ \sum_{\ell \leq k'_{A,r}} \sum_{m \in \ell} |\kappa^m_{\ell}| \leq C_{A',r} k^{1/2} \|v\|_{\text{curl}, \Omega, 1}, \]  
(5.45)

\[ \sum_{\ell > k'_{A,r}} \sum_{m \in \ell} |\kappa^m_{\ell}| (\ell + 1)^{\alpha} \leq C''_{A',r} k^2 (\gamma''_{A,r})^{\alpha+1} (\alpha + 1)^{\alpha+1} \|v\|_{\text{curl}, \Omega, 1} \]  
(5.46)

for all \( \alpha \geq 0 \).

**Proof** Proof of (i): From Theorem 5.9 we infer for suitable \( C, \gamma \) that \( L_\Omega v \) is in fact analytic on \( \Omega \cup U_\Gamma \) and satisfies there

\[ L_\Omega v \in A(Ck^{3/2} \|v\|_{\text{curl}, \Omega, 1}, \gamma, \Omega \cup U_\Gamma). \]  
(5.47)

The extension \( g_1^* \) of \( g_1 = (L_\Omega v, n) \) into \( U_\Gamma \) is taken as \( g_1^* := \langle L_\Omega v, n^* \rangle \) where \( n^* (x) := x/\|x\| \) is the extension of the normal vector \( n \) to \( U_\Gamma \). By the analyticity of \( n^* \) and (5.47), we may apply Lemma 2.6 to get with suitable constants \( \tilde{C}, \tilde{\gamma} \) independent of \( k \) and \( v \),

\[ g_1^* \in A \left( \tilde{C}k^{3/2} \|v\|_{\text{curl}, \Omega, 1}, \tilde{\gamma}, U_\Gamma \right). \]  
(5.48)

Proof of (ii): Since for smooth \( w \) we have the pointwise bound \( |\nabla_{\Gamma} w| \leq |(\nabla w)|_{\Gamma}| \), we get from a multiplicative trace inequality (see, e.g., [36, Thm. A.2])

\[ \|\nabla^n g_1\|_{\Gamma} \leq C \left( \|\nabla^n g_1^*\|_{L^2(U_\Gamma)} \|\nabla^n g_1^*\|_{H^1(U_\Gamma)} \right)^{1/2} \quad \forall n \in \mathbb{N}_0 \]

so that \( g_1 \in A \left( C_1 k^2 \|v\|_{\text{curl}, \Omega, 1}, \gamma_1, \Gamma \right) \) for suitable \( C_1, \gamma_1 \); this is the second statement.

Proof of (iii): 1. Step: By [47, (2.5.212)], the Laplace–Beltrami operator can be expressed in terms of differential operators in ambient space: \( \Delta u = \Delta_{\Gamma} u + 2H\partial_n u + \partial_n^2 u \), where \( H \equiv 1 \) is the mean curvature of the unit sphere. Applying this to \( u = g_1^* \) implies for some \( C, \gamma_2 > 0 \) independent of \( k \) and \( j \) again with the trace inequality

\[ \|\Delta_{\Gamma}^j g_1\|_{L^2(\Gamma)} \leq Ck^2 \|v\|_{\text{curl}, \Omega, 1} \gamma_2^{2j} \max\{k, 2j\}^{2j}, \]  
(5.49)

2. Step: Recall that by (5.1) we have card \( \iota_\ell = 2\ell + 1 \) and that, by (5.29), we have

\[ \|L_\Omega v\|_{H^{(\iota_\ell)}(\Omega)} = \|L_\Omega v\|_{L^2(\Omega)} \leq k^{-1} \|L_\Omega v\|_{\text{curl}, \Omega, k} \leq k^{-1} \|v\|_{\text{curl}, \Omega, k}. \]

By orthonormality of the \( Y_\ell^m \), the expansion coefficients \( \kappa^m_{\ell} \) are given by \( \kappa^m_{\ell} = \langle g_1, Y_\ell^m \rangle_{\Gamma} \). We estimate the low-order coefficients \( (\ell \leq k'_{A,r}) \) by

\[ |\kappa^m_{\ell}| \leq \sum_{\ell \leq k'_{A,r}} \sum_{m \in \ell} |\kappa^m_{\ell}| \leq \left( \sum_{\ell \leq k'_{A,r}} \sum_{m \in \ell} |\kappa^m_{\ell}|^2 \kappa_{\ell}^{-1/2} \right)^{1/2} \left( \sum_{\ell \leq k'_{A,r}} \sum_{m \in \ell} \kappa_{\ell}^{1/2} \right)^{1/2} \]

\[ \lesssim \|\langle L_\Omega v, n \rangle\|_{H^{-1/2}(\Gamma)} k^{3/2} \]

\[ \lesssim k^{3/2} \|L_\Omega v\|_{H^{(\iota_\ell)}(\Omega)} \lesssim k^{1/2} \|v\|_{\text{curl}, \Omega, k}, \]  
(5.50)
which implies the first estimate in (5.44) and (5.45).

3. Step: The minimum of \( x \mapsto x^x \) is attained at \( 1/e \) with value \( e^{-1/e} < 1 \). Hence, there are \( q \in (0, 1) \) and \( \delta > 0 \) such that

\[
x^x \leq q < 1 \quad \forall x \in [1/e - \delta, 1/e + \delta].
\]

(5.51)

4. Step: For \( \gamma_3 := \max\{1, 2\gamma_2/\delta\} > 0 \), the following implication holds for \( k \geq 1 \):

\[
\ell \geq \gamma_3 k \implies j := \left\lfloor \frac{\ell}{2\gamma_2} \right\rfloor \quad \text{satisfies} \quad j \geq k \quad \text{and} \quad \frac{2j\gamma_2}{\ell} \in [1/e - \delta, 1/e + \delta].
\]

(5.52)

5. Step: Given \( \ell \geq \gamma_3 k \) we select \( j \) as in (5.52). Using the orthonormality of the \( Y_m^\ell \) with the eigenvalues \( \lambda_\ell = \ell(\ell + 1) \) of \(-\Delta r\), we compute

\[
|k_m^\ell| = \lambda_\ell^{-j} \left| \left( g_1, (-\Delta r)^j Y_m^\ell \right)_r \right| = \lambda_\ell^{-j} \left| \left( (-\Delta r)^j g_1, Y_m^\ell \right)_r \right|
\]

\[
\leq CK^2 \|v\|_{\text{curl}, \Omega, 1}^2 \gamma_2^j (\ell(\ell + 1))^{-j} \max\{k, 2j\}^{2j}
\]

\[
\leq CK^2 \|v\|_{\text{curl}, \Omega, 1} \gamma_2^j (\ell(\ell + 1))^{-j} (2j)^{2j} \leq CK^2 \|v\|_{\text{curl}, \Omega, 1} \left( (2j\gamma_2/\ell)^{2j}/\ell \right)^{\ell/\gamma_2}
\]

\[
\leq CK^2 \|v\|_{\text{curl}, \Omega, 1} q^{\ell/\gamma_2}.
\]

(5.49), (5.51)

This and (5.50) show the bound (5.44) for \( k_m^\ell \).

6. Step: We show (5.46). We start with the observation

\[
\sup_{x > 0} x^\alpha e^{-x} \leq \alpha^\alpha e^{-\alpha} \quad \forall \alpha \geq 0.
\]

(5.53)

Then,

\[
\sum_{\ell \geq k_0^r} \sum_{m \in \ell^\ell} |k_m^\ell| (\ell + 1)^\alpha \leq C'_{A, r} k^2 \|v\|_{\text{curl}, \Omega, 1} \sum_{\ell > k_0^r} (\ell + 1)^\alpha (2\ell + 1)e^{-b\ell}
\]

\[
\leq C'_{A, r} k^2 \|v\|_{\text{curl}, \Omega, 1} \sum_{\ell=1}^{\infty} (\ell + 1)^{\alpha+1} e^{-b(\ell+1)}.
\]

Upon writing

\[
(\ell + 1)^{\alpha+1} e^{-b(\ell+1)} = \left( \frac{(\ell + 1)b}{2} \right)^{\alpha+1} e^{-b(\ell+1)/2} \left( \frac{2}{b} \right)^{\alpha+1} e^{-b(\ell+1)/2}
\]

\[
\leq (\alpha + 1)^{\alpha+1} \left( \frac{2}{b} \right)^{\alpha+1} e^{-b(\ell+1)/2},
\]

we see that the infinite sum can be controlled in the desired fashion. □
5.4 Helmholtz Decomposition

The stability properties of the operators \( \Pi^\text{comp} \), \( \Pi^\nabla \), \( \Pi^\text{comp}_h \), \( \Pi^\nabla_h \) and the splittings induced by them in Definition 4.9 are characterized in Lemma 4.8 in terms of the constants \( C^\text{high}_{b,k} \), \( C^H_{\Omega,k} \), \( C^\nabla_{b,k} \). For the case of the unit ball \( B_1(0) \), we show in Lemma 5.11 that these constants can be bounded uniformly in \( k \). We furthermore track the dependence of these constants on the cutoff parameter \( \lambda > 1 \) that enters the definition of \( L_\Omega \) and \( H_\Omega \) (cf. Definition 4.2). We track the \( \lambda \)-dependence with the aid of the norm

\[
\| u \|_{\text{curl},\Omega,k,\lambda} := \left( k^2 \| u \|^2 + \frac{1}{\lambda^2} \| \text{curl} \ u \|^2 \right)^{1/2}.
\]

Lemma 5.11 (Stability of the splitting) Let \( \Omega = B_1(0) \) and \( \lambda \geq \lambda_0 > 1 \). Then there exists \( C > 0 \) depending solely on \( \lambda_0 \) such that the decompositions of \( u, v \in X \) as

\[
\begin{align*}
\text{curl} \ u &= \Pi^\text{comp} u + \Pi^\nabla H_\Omega u = (u^\text{low} + \Pi^\text{curl} u^\text{high}) + \Pi^\nabla u^\text{high}, \\
\text{curl} \ v &= \Pi^\text{comp}_* v + \Pi^\nabla_* H_\Omega v = (v^\text{low} + \Pi^\text{curl}_* v^\text{high}) + \Pi^\nabla_* v^\text{high},
\end{align*}
\]

with \( u^\text{low} := L_\Omega u, u^\text{high} := H_\Omega u, v^\text{low} := L_\Omega v, \) and \( v^\text{high} := H_\Omega v \) satisfy:

\[
\begin{align*}
\sum_{0}^{k} \Pi^\text{curl} u^\text{high} \leq C \| u^\text{high} \|_{\text{curl},\Omega,k,\lambda}, \quad \| \text{curl} \left( \Pi^\text{curl} u^\text{high} \right) \| \leq 2 \| u^\text{high} \|_{\text{curl},\Omega,k}, \quad (5.55a) \\
\| \Pi^\nabla u^\text{high} \|_{H^1(\Omega)} \leq C \| u^\text{high} \|_{\text{curl},\Omega,k}. \quad (5.55d)
\end{align*}
\]

Analogous estimates hold for \( \Pi^\text{curl}_*, v^\text{high} \) and \( \Pi^\nabla_* v^\text{high} \).

Proof For \( u \in X \), choose \( p \in H^1(\Omega)/\mathbb{R} \) such that \( \Pi^\nabla u^\text{high} = \nabla p \), and set \( u_0 := \Pi^\text{curl} u^\text{high} \in V_0 \). A direct consequence is the relation (5.55c): \( \text{curl} \left( \Pi^\nabla u^\text{high} \right) = 0 \).

For the remaining estimates, we first collect some simple facts about this splitting.

1. Step: The definition of the space \( V_0^* \) implies \( 0 = (\nabla p, v_0) = (u^\text{high} - u_0, v_0) \) for all \( v_0 \in V_0^* \).

2. Step: In Lemma B.1, we prove for the unit ball

\[
\| u_0 \|_{H^1(\Omega)} \leq \| u_0 \|_{\text{curl},\Omega,1}. \quad (5.56)
\]

Together with (5.28), we obtain (5.55d).

3. Step: \( \text{curl} \ \nabla p = 0 \) implies

\[
\text{curl} \ u^\text{high} = \text{curl} u_0. \quad (5.57)
\]

The combination with (5.28) leads to the estimate (5.55b). Note that (5.28) also implies the second estimate in (5.55a).
4. Step: Since \( \mathbf{u}_0 \in \mathbf{V}_0 \), the definition \((4.21)\) implies

\[
\text{Re} \left( \langle \mathbf{u}_0, \mathbf{u}_0 \rangle \right) = \text{Re} \left( \left( \mathbf{u}_0, \mathbf{u}^{\text{high}} \right) \right) - \text{Re} \left( \langle \mathbf{u}_0, \nabla p \rangle \right) \overset{(4.21a)}{=} \text{Re} \left( \left( \mathbf{u}_0, \mathbf{u}^{\text{high}} \right) \right). \tag{5.58}
\]

5. Step: The weighted \( L^2 (\Omega) \)-norm of \( \mathbf{u}_0 \) can be estimated via

\[
k^2 \| \mathbf{u}_0 \|^2 \\
\leq k^2 \| \mathbf{u}_0 \|^2 - \text{Im} \ k b_k \left( \mathbf{u}_0^\nabla, \mathbf{u}_0^\nabla \right) = \text{Re} \left( \langle \mathbf{u}_0, \mathbf{u}_0 \rangle \right) \overset{(5.58)}{=} \text{Re} \left( \left( \mathbf{u}_0, \mathbf{u}^{\text{high}} \right) \right) \tag{5.59}
= \text{Re} \left[ k^2 \left( \mathbf{u}_0, \mathbf{u}^{\text{high}} \right) + i k \left\{ b_k^{\text{low}} \left( \mathbf{u}_0^\nabla, \left( \mathbf{u}^{\text{high}} \right)^\nabla \right) + b_k^{\text{high}} \left( \mathbf{u}_0^\nabla, \left( \mathbf{u}^{\text{high}} \right)^\nabla \right) \right\} \right]
\leq \frac{1}{2} (k \| \mathbf{u}_0 \|^2) + \frac{1}{2} \left( k \| \mathbf{u}^{\text{high}} \| \right)^2
+ k \left| b_k^{\text{low}} \left( \mathbf{u}_0^\nabla, \left( \mathbf{u}^{\text{high}} \right)^\nabla \right) \right| + k \left| b_k^{\text{high}} \left( \mathbf{u}_0^\nabla, \left( \mathbf{u}^{\text{high}} \right)^\nabla \right) \right|. \tag{5.60}
\]

From \((4.10)\), we conclude that \( \left( \mathbf{u}^{\text{high}} \right)^\nabla = \sum_{\ell > \lambda} \sum_{m \in \ell} U_m^\ell \nabla \gamma \gamma^\ell \), and it follows from the definition of \( b_k^{\text{low}} \) in \((4.11)\) that \( b_k^{\text{low}} \left( \mathbf{u}_0^\nabla, \left( \mathbf{u}^{\text{high}} \right)^\nabla \right) = 0 \). Next, we estimate the last term in \((5.60)\). Our decomposition \( \mathbf{u}^{\text{high}} = \mathbf{u}_0 + \nabla p \) leads to

\[
k \left| b_k^{\text{high}} \left( \mathbf{u}_0^\nabla, \left( \mathbf{u}^{\text{high}} \right)^\nabla \right) \right| \leq k \left| b_k^{\text{high}} \left( \mathbf{u}_0^\nabla, \mathbf{u}_0^\nabla \right) \right| + k \left| b_k^{\text{high}} \left( \mathbf{u}_0^\nabla, \nabla p \right) \right|. \tag{5.61}
\]

The first term can be estimated by using \((5.24a)\):

\[
k \left| b_k^{\text{high}} \left( \mathbf{u}_0^\nabla, \mathbf{u}_0^\nabla \right) \right| \overset{(5.24a)}{=} \frac{C_b'}{\lambda^2} \| \mathbf{u}_0 \|^2 \overset{(5.60)}{=} \frac{C_b'}{\lambda^2} \| \mathbf{u}_0 \|^2 \overset{(5.56)}{=} \frac{C_b'}{\lambda^2} \| \mathbf{u}_0 \|^2 \overset{(\text{curl})}{=} \frac{C_b'}{\lambda^2} \left( \| \mathbf{u}_0 \|^2 + \| \text{curl} \ \mathbf{u}^{\text{high}} \|^2 \right). \tag{5.62}
\]

For the second term of the right-hand side of \((5.61)\), we assume that \( p \in C^\infty(\overline{\Omega}) \) since the result for general \( p \in H^1(\Omega) \) follows by a density argument. We obtain

\[
k \left| b_k^{\text{high}} \left( \mathbf{u}_0^\nabla, \nabla p \right) \right| \overset{(5.24b)}{=} \frac{C_b'}{\lambda} k \| \nabla p \| \| \mathbf{u}_0 \| \overset{(\text{curl})}{=} \frac{C_b'}{\lambda} k \| \nabla p \| \| \mathbf{u}_0 \| \| \text{curl} \ \mathbf{u}^{\text{high}} \| \overset{(5.56)}{=} \frac{C_b'}{\lambda} \left( k \| \mathbf{u}^{\text{high}} \| + k \| \mathbf{u}_0 \| \right) \left( \| \mathbf{u}_0 \| + \| \text{curl} \ \mathbf{u}^{\text{high}} \| \right). \tag{5.63}
\]
Inserting (5.62), (5.63) into (5.61) and employing Cauchy–Schwarz inequalities with \( \eta > 0 \) lead to

\[
\|b_{k}^{\text{high}}(u_{\nabla 0}, (u^{\text{high}})^{\nabla})\| \leq C_{b}^{\prime}\left(\left(\frac{3}{2\lambda^{2}k^{2}} + \frac{1}{\lambda k} + \eta \right) (k \|u_{0}\|)^{2} + \left(k \|u^{\text{high}}\|\right)^{2} + \left(\frac{3 + \eta^{-1}}{2}\right) \left(\frac{\|\text{curl } u^{\text{high}}\|}{\lambda}\right)^{2}\right).
\]

We combine this estimate with (5.59) and absorb the first term on the right-hand side of (5.60) into the left-hand side of (5.59) to obtain (using \( \lambda k > 1 \))

\[
\frac{k^{2}}{2} \|u_{0}\|^{2} \leq C_{1}^{\prime}k^{2} \|u^{\text{high}}\|^2 + \frac{C_{2}}{\lambda^{2}} \|\text{curl } u^{\text{high}}\|^{2}
\]

with \( C_{1} := \left(\frac{1}{2} + C_{b}^{\prime}\right), \quad C_{2} := C_{b}^{\prime}\left(\frac{3 + 4C_{b}^{\prime}}{2}\right) \). (5.64)

We first consider the case \( k \geq \max\{1, \frac{20C_{b}^{\prime}}{\lambda}\} \) and choose \( \eta = \frac{1}{4C_{b}^{\prime}} \). This leads to

\[
\frac{k^{2}}{4} \|u_{0}\|^{2} \leq C_{1}k^{2} \|u^{\text{high}}\|^2 + \frac{C_{2}}{\lambda^{2}} \|\text{curl } u^{\text{high}}\|^{2}
\]

This yields the first estimate for the first term in (5.55a) (for the considered range of \( k \)). For \( 1 \leq k \leq \max\{1, \frac{20C_{b}^{\prime}}{\lambda}\} \), we estimate the term \( k \|b_{k}^{\text{high}}(u_{\nabla 0}, (u^{\text{high}})^{\nabla})\| \) in (5.60) by using (5.20a) and \( \eta > 0 \)

\[
k \|b_{k}^{\text{high}}(u_{\nabla 0}, (u^{\text{high}})^{\nabla})\| \leq C_{b}^{\prime}\frac{k}{\lambda} \|u_{0}\|_{\text{curl},\Omega,1} \|u^{\text{high}}\|_{\text{curl},\Omega,1}
\]

\[
\leq C_{b}^{\prime}\frac{k}{2\lambda} \left(\eta \|u_{0}\|_{\text{curl},\Omega,1} + \frac{1}{\eta} \|u^{\text{high}}\|_{\text{curl},\Omega,1}^{2}\right)
\]

\[
\leq C_{b}^{\prime}\frac{k}{2\lambda} \left(\eta \|u_{0}\|^{2} + \left(\eta + \frac{1}{\eta}\right) \|u^{\text{high}}\|_{\text{curl},\Omega,1}^{2}\right).\quad (5.57)
\]

Combining this estimate with (5.59) and taking into account \( b_{k}^{\text{low}}(u_{\nabla 0}, (u^{\text{high}})^{\nabla}) = 0 \) lead to

\[
k^{2} \|u_{0}\|^{2} \leq \frac{1}{2} (k \|u_{0}\|)^{2} + \frac{1}{2} \left(k \|u^{\text{high}}\|\right)^{2} + C_{b}^{\prime}\frac{k}{2\lambda} \left(\eta \|u_{0}\|^{2} + \left(\eta + \frac{1}{\eta}\right) \|u^{\text{high}}\|_{\text{curl},\Omega,1}^{2}\right).
\]
Recall $\lambda k \geq \lambda > 1$. The choice $\eta = \frac{1}{2C_b}$ leads to

$$k^2 \|u_0\|^2 \leq \left( \frac{1}{2} + \frac{1}{4\lambda k} \right) (k \|u_0\|^2) + \left( \frac{1}{2} + \frac{\eta}{4\lambda k} \right) \left( \frac{1}{\lambda k} \right) k^2 \|u^\text{high}\|_{\text{curl},\Omega,1}^2 \leq \frac{3}{4} (k \|u_0\|^2) + \left( \frac{1}{2} + \left( \frac{1}{4} + \left( C'_b \right)^2 \right) \right) k^2 \|u^\text{high}\|_{\text{curl},\Omega,1}^2.$$  

The first term on the right-hand side can be absorbed into the left-hand side. Since $k \leq \frac{20C'_b}{\lambda}$, we get

$$k^2 \|u_0\|^2 \leq C \|u^\text{high}\|_{\text{curl},\Omega,k,\lambda}^2 \leq C \|u\|_{\text{curl},\Omega,k}^2.$$  

The $L^2$ estimate for $\nabla p$ follows by a triangle inequality:

$$k \|\nabla p\| \leq k \left( \|u^\text{high}\| + \|u_0\| \right) \leq C_3 \left( k \|u^\text{high}\| + \lambda^{-1} \|\nabla u^\text{high}\| \right) \leq C_3' \|u^\text{high}\|_{\text{curl},\Omega,k,\lambda} \leq C_3'' \|u\|_{\text{curl},\Omega,k}.$$  

The estimates for $\|\Pi_\nabla \cdot v^\text{high}\| + k \|\Pi_\nabla \cdot v^\text{high}\|_{\text{curl},\Omega,1}$ are derived by repeating the arguments above. □

By similar techniques we will prove next that if one argument in $(\cdot, \cdot)$ has only high-frequency components then we get $k$-independent continuity estimates (cf. also (4.15) for the general case):

**Proposition 5.12** Let $\Omega = B_1(0)$ and $\lambda \geq \lambda_0 > 1$. Then there exists $\tilde{C}_b > 0$ depending solely on $\lambda_0$ such that for all $u, v \in X$

$$|((H\Omega u, v))| + |((H\Omega, H\Omega v))| \leq \tilde{C}_b \|u\|_{\text{curl},\Omega,k,\lambda} \|v\|_{\text{curl},\Omega,k,\lambda}, \quad (5.65)$$  

$$|((u, v))| \leq C_{\text{cont},k} \|u\|_{\text{curl},\Omega,1} \|v\|_{\text{curl},\Omega,1}, \quad (5.66)$$

where $C_{\text{cont},k} \leq \tilde{C}_b k^3$.

**Proof** For $u, v \in X$, write $u^\text{high} := H\Omega u, v^\text{high} := H\Omega v$. Choose $p, q \in H^1(\Omega)$ such that $\Pi_\nabla u^\text{high} = \nabla p$ and $\Pi_\nabla \cdot v^\text{high} = \nabla q$. Set $u_0 = u^\text{high} - \nabla p$ and $v_0 = v^\text{high} - \nabla q$. Since $\Pi_T H\Omega = H_T \Pi_T$ (cf. (4.10)) we have

$$\left| (\Pi_\nabla u^\text{high}, v^\text{high}) \right| \leq \left( k \|u^\text{high}\| \right) (k \|v\|) + k^3 \|b^\text{high}\| \left( (u^\text{high})^\nabla, v^\nabla \right).$$

For the boundary term, we get
Combining (5.68) and (5.69) leads to
\[ \| b_k^{\text{high}} \left( (u^{\text{high}})^\nabla, v^\nabla \right) \| \leq \left| b_k^{\text{high}} \left( u_0^\nabla, v_0^\nabla \right) \right| + \left| b_k^{\text{high}} \left( (\nabla p)^\nabla, v_0^\nabla \right) \right| + \left| b_k^{\text{high}} \left( (\nabla q)^\nabla, (\nabla q)^\nabla \right) \right| \]

\[ \leq C_b' \| u_0 \|_{\text{curl}, \Omega, 1} \| v_0 \|_{\text{curl}, \Omega, 1} + \frac{C_b'}{\lambda} (k \| \nabla p \|) \| v_0 \|_{\text{curl}, \Omega, 1} \]

\[ \leq C_b' \| u \|_{\text{curl}, \Omega, k, \lambda} \| v \|_{\text{curl}, \Omega, k, \lambda} \]  

The estimate for \((u, v^{\text{high}})\) follows from the same arguments.

It remains to prove estimate (5.66). We choose \( \lambda = \lambda_0 = O(1) \) in all splittings and start with

\[ \| (L_{\Omega} u, v) \| \leq k^2 \| (L_{\Omega} u, v) \| + \left| b_k \left( (L_{\Omega} u)^\nabla, v^\nabla \right) \right| \]

\[ \leq (k \| L_{\Omega} u \|)(k \| v \|) + \left| b_k^{\text{low}} \left( u^\nabla, v^\nabla \right) \right| . \]

We employ (5.17) with \( \rho = +1 \) to obtain

\[ \left| b_k^{\text{low}} \left( u^\nabla, v^\nabla \right) \right| \leq C_k k^2 \| \text{div}_{\Gamma} L_{\Gamma} u_T \|_{H^{-1/2}(\Gamma)} \| \text{div}_{\Gamma} L_{\Gamma} v_T \|_{H^{-1/2}(\Gamma)} \]

\[ \leq C k^2 \| L_{\Gamma} u_T \|_{H^{-1/2}(\Gamma)} \| L_{\Gamma} v_T \|_{H^{-1/2}(\Gamma)} \]  

\[ \leq C k^2 \| u \|_{\text{curl}, \Omega, 1} \| v \|_{\text{curl}, \Omega, 1} . \]  

Combining (5.68) and (5.69) leads to

\[ \left| b_k \left( u^\nabla, v^\nabla \right) \right| \leq C \| u \|_{\text{curl}, \Omega, k, \lambda} \| v \|_{\text{curl}, \Omega, k, \lambda} + C k^3 \| u \|_{\text{curl}, \Omega, 1} \| v \|_{\text{curl}, \Omega, 1} \]

\[ \leq C k^3 \| u \|_{\text{curl}, \Omega, 1} \| v \|_{\text{curl}, \Omega, 1} . \]

Taking into account the \( L^2(\Omega) \) part in \((\cdot, \cdot)\) results in the estimate (5.66). \( \square \)

**Corollary 5.13** For \( \Omega = B_1(0) \), the constants in (4.13), (4.14), (4.12), (4.6), and (4.15) can be estimated by

\[ C_{\text{DN}, k} \leq C k^2, \quad C_{\text{cont}, k} \leq \tilde{C}_b k^3, \quad C_{b,k}^{\nabla, \text{high}} \leq \tilde{C}_b, \quad C_{b,k}^{\text{curl, high}} \leq C b C_{\Gamma}^2, \quad C_{k,\Omega}^{H, \text{high}} \leq 2, \quad C_{b,k}^{\text{high}} \leq 2 + \tilde{C}_b \]  

with \( k \)-independent constants \( C, C_b \) (cf. Proposition 5.7), \( \tilde{C}_b \) (cf. Proposition 5.12), and \( C_{\Gamma} \).
Proof The estimate of \( C_{\text{DN},k} \) follows by combining (5.17) and (5.25). Proposition 5.12 implies the bound for \( C_{\text{cont},k} \). Estimate (5.68) implies the estimate of \( C^{\nabla\text{high},b,k} \) as in (4.12a). For \( C^{\text{curl}\text{high},b,k} \) we use (5.17) to obtain

\[
\begin{align*}
  k \left| b_k \left( u^{\text{curl}}, (v^{\text{high}})^{\text{curl}} \right) \right| &= k \left| b_k^{\text{high}} \left( u^{\text{curl}}, v^{\text{curl}} \right) \right| \\
  &\leq C_b \| \nabla \Upsilon \|_{H^{-1/2}(\Gamma)} \| \nabla \Upsilon \|_{H^{-1/2}(\Gamma)} \leq C_b C_2 \| u \|_{\text{curl},\Omega,1} \| v \|_{\text{curl},\Omega,1}
\end{align*}
\]

so that the estimate for \( C^{\text{curl}\text{high},b,k} \) is shown. Finally, \( C_{H,\Omega}^k \leq C \) is proved in (5.28b) and the estimate of \( C_{\text{high}} = C_{H,\Omega}^k + C^{\nabla\text{high},b,k} \) follows by combining the previous estimates.

\( \square \)

6 Estimating the Terms in the Splitting (4.48) of \( (e_h, v_h) \)

6.1 Estimating \( \left( \left( e_h, \left( (\Pi^\text{comp,*}_h - \Pi^\text{comp,*} \right) v_h \right)^{\text{high}} \right) \right) \) in (4.48a)

In this section, we will prove the following Proposition 6.1. Recall the definition of \( \tilde{\eta}_4, \tilde{\eta}_6, \tilde{\eta}_7 \) in (4.53), (4.55), (4.56), which involve the operator \( \Pi^E_h \), which is required to satisfy Assumption 4.14.

Proposition 6.1 Let \( e_h = E - E_h \) denote the Galerkin error. For \( v_h \in X_h \) let \( \Pi^\text{comp,*}_h, \Pi^\text{comp,*} \) be given by Definition 4.9. Let Assumption 4.14 hold. Then

\[
\left| \left( \left( e_h, \left( (\Pi^\text{comp,*}_h - \Pi^\text{comp,*} \right) v_h \right)^{\text{high}} \right) \right| \leq C_{b,k}^{\text{high}} C_{r,k} \| e_h \|_{\text{curl},\Omega,k} \| v_h \|_{\text{curl},\Omega,k}
\]

with

\[
C_{r,k} := \left( C_{b,k}^{\text{high}} + \frac{C_{\text{cont},k}}{k^2} \tilde{\eta}_4 \right) \left( \tilde{\eta}_7^{\exp} + C_{\text#,k} \eta_6^{\text{alg}} \right) \quad \text{and} \quad C_{\text#,k} := \left( C_{H,\Omega}^k + C^{\nabla\text{high},b,k} \right) C_{\Omega,k}.
\]

The constant \( C_{b,k}^{\text{high}} \) is as in (4.15), \( C_{\text{cont},k} \) as in (4.14), and \( C_{\Omega,k} \) as in (4.31).

For the case \( \Omega = B_1(0) \), we have \( C_{\text{cont},k} \leq C k^3 \) while \( C_{b,k}^{\text{high}}, C_{r,k}, C_{\Omega,k} \text{ and } C_{\text#,k} \) are bounded independently of \( k \).

Proof From (4.46) we conclude

\[
\begin{align*}
  \text{curl } \Pi^\text{comp,*}_h v_h &= \text{curl } \Pi^\text{comp,*} v_h = \text{curl } v_h \\
  \text{and } \text{curl } \Pi^\text{curl,*}_h H_{\Omega} v_h &= \text{curl } \Pi^\text{curl,*} H_{\Omega} v_h = \text{curl } H_{\Omega} v_h \\
  \forall v_h \in X_h.
\end{align*}
\]
Let \( r := (\Pi^\text{comp.*} - \Pi^\text{comp.*}_h) v_h \) and let \( q := (I - \Pi^E_h) \Pi^\text{comp.*} v_h \). First we prove some curl-free properties. We have

\[
\text{curl} \left( \Pi^E_h \Pi^\text{comp.*} - \Pi^\text{comp.*}_h \right) v_h
\]

(4.46), Ass. 4.14

\[
= \text{curl} \left( \Pi^E_h - I \right) L_\Omega v_h + \left( \Pi^E_h \text{curl} \Pi^\text{curl,*} - \text{curl} \Pi^\text{curl,*}_h \right) H_\Omega v_h
\]

= \text{curl} \left( \Pi^E_h - I \right) L_\Omega v_h + \Pi^E_h \text{curl} H_\Omega v_h - \text{curl} H_\Omega v_h

= \text{curl} \left( \Pi^E_h - I \right) L_\Omega v_h + \text{curl} \left( \Pi^E_h - I \right) H_\Omega v_h = \text{curl} \left( \Pi^E_h - I \right) v_h

Ass. 4.14(a)

\[
= \text{curl} (v_h - v_h) = 0,
\]

(6.3)

and also

\[
\text{curl} \ r \overset{(6.2)}{=} 0,
\]

(6.4)

\[
\text{curl} \ q = \text{curl} \left( \Pi^\text{comp.*} - \Pi^\text{comp.*}_h \Pi^\text{comp.*} \right) v_h \overset{(6.2)}{=} \text{curl} \left( \Pi^\text{comp.*}_h - \Pi^\text{comp.*}_h \Pi^\text{comp.*} \right) v_h \overset{(6.3)}{=} 0.
\]

(6.5)

We start our estimate with a continuity bound for the sesquilinear form \((\cdot, H_\Omega \cdot)\) and employ (4.15) to get

\[
\left| \left(\left( e_h, r^{\text{high}} \right) \right) \right| \leq C_{b,k}^{\text{high}} \| e_h \|_{\text{curl},k} \| r \|_{\text{curl},k} \overset{(6.4)}{=} C_{b,k}^{\text{high}} \| e_h \|_{\text{curl},k} (k \| r \|) .
\]

(6.6)

The coercivity of \((\cdot, \cdot)\) in the form (4.20) leads to

\[
(k \| r \|)^2 \leq \text{Re} (\langle r, r \rangle) = \text{Re} (\langle q, r \rangle) + \text{Re} (\langle (\Pi^E_h \Pi^\text{comp.*} - \Pi^\text{comp.*}_h) v_h, r \rangle) .
\]

(6.7)

We use the definition of \( \Pi^{\nabla,*}, \Pi^{\text{curl,*}}, \Pi^{\text{comp.*}} \) and its discrete versions as in (4.19) and Definition 4.9 to get

\[
(\langle w_h, r \rangle) = \left( \left( w_h, \left( \Pi^{\text{curl,*}} - \Pi^{\text{curl,*}_h} \right) H_\Omega v_h \right) \right) = 0 \ \forall w_h \in \nabla S_h .
\]

(6.8)

As a consequence of equation (6.3) and the exact sequence property (3.2) we get \((\Pi^E_h \Pi^\text{comp.*} - \Pi^\text{comp.*}_h) v_h = \nabla \psi_h \) for some \( \psi_h \in S_h \). The combination of this with (6.8) for \( w_h = \nabla \psi_h \) implies that the last term in (6.7) vanishes. Hence,

\[
(k \| r \|)^2 \leq \text{Re} (\langle q, r \rangle) = \text{Re} (\langle H_\Omega q, r \rangle) + \text{Re} (\langle L_\Omega q, r \rangle) .
\]

(6.9)

For the high-frequency part on the right-hand side, we employ (4.15) and obtain

\[
\text{Re} (\langle H_\Omega q, r \rangle) \leq C_{b,k}^{\text{high}} \| q \|_{\text{curl},k} \| r \|_{\text{curl},k} \overset{(6.4),(6.5)}{=} C_{b,k}^{\text{high}} (k \| q \|) (k \| r \|) .
\]

(6.10)
The term \( \|q\| \) can be estimated using the definition of \( \Pi^{\text{comp.}*} \) (cf. Definition 4.9)

\[
k \|q\| \leq k \left\| \left( I - \Pi^E_h \right) L_\Omega v_h \right\| + k \left\| \left( I - \Pi^E_h \right) \Pi^{\text{curl.}*} H_\Omega v_h \right\|
\]

\[
\leq \tilde{\eta}_7 \exp \|v_h\|_{\text{curl},k} + \eta_6 \|\Pi^{\text{curl.}*} H_\Omega v_h\|_{H^1(\Omega)}
\]

\[
\leq \tilde{\eta}_7 \exp \|v_h\|_{\text{curl},k} + C_{\text{alg},k} \eta_6 \|\Pi^{\text{curl.}*} H_\Omega v_h\|_{\text{curl},1}
\]

Lem.4.12

\[
\leq \left( \tilde{\eta}_7 \exp + C_{\text{alg},k} \eta_6 \right) \|v_h\|_{\text{curl},k}.
\]

(6.11)

To estimate the low-frequency part in (6.9), we observe that

\[ \zeta := \Pi \nabla L_\Omega q \quad \text{[cf. (4.50c)]} \]

satisfies

\[ \langle \zeta, \xi \rangle = \langle L_\Omega q, \xi \rangle \quad \forall \xi \in \nabla H^1(\Omega). \]

By choosing \( \xi = r \), we can use a Galerkin orthogonality in the form (6.8) to obtain for any \( w_h \in \nabla S_h \)

\[
\text{Re} \left( \langle L_\Omega q, r \rangle \right) = \text{Re} \left( \langle \zeta, r \rangle \right) = \text{Re} \left( \langle \zeta - w_h, r \rangle \right) \leq C_{\text{cont.,k}} \|r\|_{\text{curl,1}} \|\zeta - w_h\|_{\text{curl,1}}.
\]

The last factor can be estimated by using (4.53), (6.11), and the definition of \( \zeta \):

\[
\inf_{v_h \in S_h} \|\nabla \left( N^A_q v_h \right)\|_{\text{curl,1}} = \inf_{v_h \in S_h} \|\nabla \left( N^A_q v_h \right)\| \leq \tilde{\eta}_4 \exp \|q\|_{\text{curl,1}}
\]

\[
\equiv \tilde{\eta}_4 \exp \|q\| \leq \frac{\tilde{\eta}_4 \exp}{k} \left( \tilde{\eta}_7 \exp + C_{\#.,k} \eta_6 \right) \|v_h\|_{\text{curl},k}.
\]

(6.12)

Finally, we combine this estimate with (6.9), (6.10), (6.11) to bound the last factor in (6.6) by

\[
k \|r\| \leq C_{r,k} \|v_h\|_{\text{curl},k}.
\]

(6.13)

We insert (6.13) into (6.6) and arrive at the assertion. The bounds for the constants are stated in Corollary 5.13.

\[\Box\]

6.2 Estimating \((e_h, \Pi^{\text{curl.}*} v^\text{high}_h)\) in (4.48c)

In this section, we investigate the second term of the right-hand side in (4.48c). Recall the definition of the adjoint solution operators (4.50) and the corresponding adjoint approximation properties (4.51)–(4.56).

Proposition 6.2 Let \( e_h = E - E_h \) denote the Galerkin error with splitting of \( v_h \in X_h \) as in (4.46). Let Assumption 4.14 be satisfied. Then
\[
\left| \left( \left( e_h, \Pi^{\text{curl},*} v^{\text{high}}_h \right) \right) \right| \\
\leq C_{##,k} \left( C_{##,k}^{\text{high}} + C^{\text{curl,high}}_{b,k} + C_{\text{cont},k} \eta^2_{\text{exp}} \right) \| e_h \|_{\text{curl},\Omega,k} \| v_h \|_{\text{curl},\Omega,k} 
\]  
(6.14)

with \( C_{##,k} := C^{H,\Omega}_k + C^{\text{high}}_{b,k} \). For \( \Omega = B_1(0) \), it holds \( C_{\text{cont},k} \leq Ck^3 \) while all other constants are bounded independently of \( k \).

**Proof** Let \( s := \Pi^{\text{curl},*} v^{\text{high}}_h \in V_0^* \). We consider the adjoint problem [cf. (4.50a)] with solution operator \( N_2 \) and set \( z := N_2 s \). Galerkin orthogonality with arbitrary \( z_h \in X_h \) gives

\[
\left( (e_h, s) \right) = A_k(e_h, z) = A_k(e_h, z - z_h) \\
= A_k(e_h, H_\Omega(z - z_h)) + A_k(e_h, L_\Omega(z - z_h)).
\]  
(6.15)

For the first term, we obtain

\[
|A_k(e_h, H_\Omega(z - z_h))| \leq \| \text{curl } e_h \| \| \text{curl } (H_\Omega(z - z_h)) \| + |(e_h, H_\Omega(z - z_h))| \\
+ \left| k b_k \left( e^{\text{curl}}_h, (H_\Omega(z - z_h))^{\text{curl}} \right) \right|.
\]

The three terms on the right-hand side can be estimated by using the constants in (4.12), (4.6), (4.15):

\[
\| \text{curl } (H_\Omega(z - z_h)) \| \leq \| H_\Omega(z - z_h) \|_{\text{curl},\Omega,k} \leq C^{H,\Omega}_k \| z - z_h \|_{\text{curl},\Omega,k}, \\
\left| k b_k \left( e^{\text{curl}}_h, (H_\Omega(z - z_h))^{\text{curl}} \right) \right| \leq C^{\text{curl,high}}_{b,k} \| e_h \|_{\text{curl},\Omega,k} \| z - z_h \|_{\text{curl},\Omega,k}, \\
| (e_h, H_\Omega(z - z_h)) | \leq C^{\text{high}}_{b,k} \| e_h \|_{\text{curl},\Omega,k} \| z - z_h \|_{\text{curl},\Omega,k}.
\]

This leads to

\[
|A_k(e_h, H_\Omega(z - z_h))| \leq \left( C_{##,k} + C^{\text{curl,high}}_{b,k} \right) \| e_h \|_{\text{curl},\Omega,k} \| z - z_h \|_{\text{curl},\Omega,k}.
\]

For the second term in (6.15), we obtain for arbitrary \( \tilde{z}_h \in X_h \)

\[
|A_k(e_h, L_\Omega(z - z_h))| \leq |A_k(e_h, L_\Omega(z - z_h) - \tilde{z}_h)| \\
\overset{(4.14)}{\leq} C_{\text{cont},k} \| e_h \|_{\text{curl},\Omega,1} \| L_\Omega(z - z_h) - \tilde{z}_h \|_{\text{curl},\Omega,1}. 
\]  
(6.16)

This leads to the estimate

\[
|(e_h, s)| \leq \left( C_{##,k} + C^{\text{curl,high}}_{b,k} \right) \| e_h \|_{\text{curl},\Omega,k} \| z - z_h \|_{\text{curl},\Omega,k} \\
+ C_{\text{cont},k} \| e_h \|_{\text{curl},\Omega,1} \| L_\Omega(z - z_h) - \tilde{z}_h \|_{\text{curl},\Omega,1}.
\]  
(6.17)
With the definition of the adjoint approximation properties (cf. Sect. 4.3), we get

$$\inf_{z_h \in X_h} \| z - z_h \|_{\operatorname{curl}, \Omega, k} \leq \hat{\eta}_2 \| \mathcal{J}^{\operatorname{curl}, \star} \| \Pi_{\operatorname{curl}, \star} v_h \|_{\operatorname{curl}, \Omega, k},$$

$$\inf_{z_h} \inf_{\tilde{z}_h} \| L_\Omega (z - z_h) - \tilde{z}_h \|_{\operatorname{curl}, \Omega, k} \leq \hat{\eta}_5 \inf_{z_h} \| z - z_h \|_{\operatorname{curl}, \Omega, k} \leq \hat{\eta}_2 \hat{\eta}_5 \| \mathcal{J}^{\operatorname{curl}, \star} \| \Pi_{\operatorname{curl}, \star} v_h \|_{\operatorname{curl}, \Omega, k}.$$ \hspace{1cm} (6.18)

The combination of these estimates with Lemma 4.8 leads to (6.14). The estimates of the constants for the case $\Omega = B_1(0)$ are stated in Corollary 5.13.

### 6.3 Estimating $(\langle e_h, L_\Omega (\mathcal{J}^{\operatorname{comp}, \star} v_h - \mathcal{J}^{\operatorname{comp}, \star} v_h) \rangle)$ and $(\langle e_h, L_\Omega v_h \rangle)$ in (4.48)

Next, we investigate the terms (4.48b) and the first term of (4.48c).

**Proposition 6.3** Let $e_h = E - E_h$ denote the Galerkin error with splitting of $v_h \in X_h$ as in (4.46) and let Assumption 4.14 be satisfied. Then:

$$|\langle e_h, L_\Omega r \rangle| + |\langle e_h, L_\Omega v_h \rangle| \leq C_{\text{cont}, k} \hat{\eta}_3 \exp \left(1 + C_{r, k}\right) \| e_h \|_{\operatorname{curl}, \Omega, k} \| v_h \|_{\operatorname{curl}, \Omega, k} \| r \| \leq C_{r, k} \| v_h \|_{\operatorname{curl}, \Omega, k}.$$ \hspace{1cm} (6.19)

with $r := \Pi_{\operatorname{comp}, \star} v_h - \Pi_{\operatorname{comp}, \star} v_h$ and $C_{r, k}$ as in (6.1).

**Proof** Recall the definition of the solution operator $\mathcal{N}^A_3$ from (4.50b) satisfying, for given $s \in X$,

$$A_k (w, \mathcal{N}^A_3 s) = \langle w, L_\Omega s \rangle \quad \forall w \in X.$$  

For the first term in (6.19), we get in a similar fashion as in (6.16)

$$|\langle e_h, L_\Omega s \rangle| = \inf_{z_h \in X_h} \left| A_k \left( e_h, \mathcal{N}^A_3 s - z_h \right) \right| \leq C_{\text{cont}, k} \| e_h \|_{\operatorname{curl}, \Omega, 1} \inf_{z_h \in X_h} \left\| \mathcal{N}^A_3 s - z_h \right\|_{\operatorname{curl}, \Omega, 1} \leq C_{\text{cont}, k} \hat{\eta}_3 \exp \| e_h \|_{\operatorname{curl}, \Omega, k} \| s \|_{\operatorname{curl}, \Omega, k}.$$  

This leads directly to the estimate of the second term in (6.19) by choosing $s = v_h$. For the choice $s = r$, we combine (6.4) with (6.13) to get $\| r \|_{\operatorname{curl}, \Omega, k} = k \| r \| \leq C_{r, k} \| v_h \|_{\operatorname{curl}, \Omega, k}$. \hspace{1cm} $\square$
7 Analysis of the Dual Problems

For the stability and convergence analysis, we have introduced various adjoint approximation properties in Sect. 4.3. In this section, we analyze the regularity of the adjoint solutions in Sect. 7.2 based on a solution formula which we will derive in Sect. 7.1. The quantitative convergence rates require approximation operators for \(hp\)-finite element spaces that will be presented in Sect. 8.3.

7.1 Solution Formulae

In this section, we will develop a regularity theory to estimate the solutions of the dual problems which have been introduced in Sect. 4.3. They belong to one of the following two types.

**Type 1:**

Given \(v \in H(\Omega, \text{div}), \, g, \, h \in X\) find \(z \in X\) such that

\[
A_k (w, z) = k^2 (w, v) + i k b_k \left( w^\nabla, g^\nabla \right) - i k b_k \left( w^{\text{curl}}, h^{\text{curl}} \right) \quad \forall w \in X. \tag{7.1}
\]

This is problem (4.50a) with \(v := g := r\) and \(h := 0\), problem (4.50b) with \(v := g := L_\Omega r\) and \(h := 0\), and problem (4.39) with \(v = h = g := L_\Omega w\).

**Type 2:**

Given \(r \in X\) find \(Z \in H^1(\Omega)/\mathbb{R}\) s.t. \((\nabla Z, \nabla \xi) = (L_\Omega r, \nabla \xi) \forall \xi \in H^1(\Omega). \tag{7.2}\)

This is problem (4.50c).

7.1.1 Solution Formula for Problems of Type 1

Integration by parts in the sesquilinear form \(A_k (\cdot, \cdot)\) gives

\[
A_k (w, z) = (\text{curl } w, \text{curl } z) - k^2 (w, z) - i k \left( T_k w_T, z_T \right)_\Gamma
\]

\[
= \left( w, \text{curl } z - k^2 z \right) - \left( \gamma_T w, \Pi_T \text{curl } z \right)_\Gamma + \left( w_T, i k T_{-k} z_T \right)_\Gamma
\]

\[
= \left( w, \text{curl } z - k^2 z \right) + \left( w_T, \gamma_T \text{curl } z \right)_\Gamma + \left( w_T, i k T_{-k} z_T \right)_\Gamma. \tag{7.3}
\]

In a similar way, we can express the right-hand side in (7.1) by

\[
r.h.s. = k^2 (w, v) + i k \left[ \left( T_k w^\nabla, g^\nabla \right)_\Gamma - \left( T_k w^{\text{curl}}, h^{\text{curl}} \right)_\Gamma \right]
\]

\[
= k^2 (w, v) + \left( w^\nabla, (i k T_k)^* g^\nabla \right)_\Gamma - \left( w^{\text{curl}}, (i k T_k)^* h^{\text{curl}} \right)_\Gamma
\]

\[
= k^2 (w, v) + \left( w_T, - i k T_{-k} (g^\nabla - h^{\text{curl}}) \right)_\Gamma. \tag{7.4}
\]
The right-hand sides in (7.3) and (7.4) must be equal, which leads to

\[
\text{curl curl } z - k^2 z = k^2 v \quad \text{in } \Omega,
\]
\[
\gamma_T \text{curl } z + i k T_{-k} z_T = -i k T_{-k} \left( g^\nabla - h^{\text{curl}} \right) \quad \text{on } \Gamma.
\] (7.5)

In the next step, we eliminate the capacity operator \( T_{-k} \) by considering a full space problem with transmission condition. Note that for any given \( q_T \in H^{-1/2}_\text{curl} (\Gamma) \) the adjoint capacity operator \( T_{-k} q_T \) is computed by first solving the exterior problem (cf. also [47, (5.2.43)] for the radiation condition)

\[
- i k z^+ + \text{curl } \tilde{H} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},
\]
\[
i k \tilde{H} + \text{curl } z^+ = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},
\]
\[
\gamma_T^+ z^+ = q_T \times n \quad \text{on } \Gamma,
\]
\[
z^+(x) \leq c/r \quad \text{as } r = \|x\| \to \infty
\] (7.6)

so that \( \gamma_T^+ \tilde{H} = T_{-k} q_T \). In the following, we always choose \( q_T = \Pi_T z \) in (7.6) with \( z \) being the solution of (7.1). From the third equation in (7.6) we obtain \( \left( z, z^+ \right)_{0,\Gamma} = 0 \) and from the second equation in (7.6)

\[
\gamma_T^+ \text{curl } z^+ = - i k \gamma_T^+ \tilde{H} = - i k T_{-k} z_T.
\] (7.7)

Hence,

\[
\left( z, z^+ \right)_{1,\Gamma} \overset{(2.4)}{=} \gamma_T \text{curl } z - \gamma_T^+ \text{curl } z^+ \overset{(7.5), (7.7)}{=} - i k T_{-k} \left( g^\nabla - h^{\text{curl}} \right).
\]

Let \( v_{\text{zero}} \) be the extension of \( v \) to the full space by 0 and define \( Z \in H_{\text{loc}}(\mathbb{R}^3, \text{curl}) \) by \( Z|_{\Omega} = z \) and \( Z|_{\partial \Omega} = z^+ \). The combination with (7.5) leads to (see [47, (5.2.22)]) for the radiation condition

\[
\text{curl curl } Z - k^2 Z = k^2 v_{\text{zero}} \quad \text{in } \mathbb{R}^3 \setminus \Gamma,
\]
\[
\left( [z, z^+] \right)_{0,\Gamma} = 0,
\]
\[
\left( [z, z^+] \right)_{1,\Gamma} = - i k T_{-k} \left( g^\nabla - h^{\text{curl}} \right),
\]
\[
\partial_r z^+(x) + i k z^+(x) \leq c/r^2, \quad \text{as } r = \|x\| \to \infty.
\] (7.8)

We first construct a particular solution for the corresponding full space problem by ignoring the transmission conditions. Then we adjust this solution to satisfy the transmission condition.

For this purpose, we need the fundamental solution for the electric part of the Maxwell problem in the full space:

\[
\text{curl curl } G_k - k^2 G_k = \delta I \quad \text{in } \mathbb{R}^3,
\]
\[
\partial_r G_k(x) - i k G_k(x) \leq c/r^2 \quad \text{as } r = \|x\| \to \infty.
\]
We eliminate in [47, (5.2.1)] the magnetic field to get the equations
\[
\begin{align*}
\text{curl} \text{ curl} \ E - k^2 E &= \delta \mathbf{I} \quad \text{in} \ \mathbb{R}^3, \\
|\partial_r E (\mathbf{x}) - i k E (\mathbf{x})| &\leq c/r^2 \quad \text{as} \ r = \|\mathbf{x}\| \to \infty.
\end{align*}
\]
Hence, the fundamental solution is obtained by dividing the one in [47, (5.2.8)] by \((i \omega \mu)\) to obtain
\[
G_k (\mathbf{x}) = g_k (\|\mathbf{x}\|) I + \frac{1}{k^2} \nabla \nabla^\top g_k (\|\mathbf{x}\|) \quad \text{with} \quad g_k (r) := \frac{e^{ikr}}{4\pi r}.
\]
(7.9)
The second term in the sum is understood as a distribution, i.e., the convolution with a function \(f \in C_\infty (\mathbb{R}^3, \mathbb{C}^3)\) is defined by
\[
(G_k * f) (\mathbf{x}) = \int_{\mathbb{R}^3} g_k (\|\mathbf{x} - \mathbf{y}\|) f (\mathbf{y}) \, d\mathbf{y} + \frac{1}{k^2} \nabla \int_{\mathbb{R}^3} g_k (\|\mathbf{x} - \mathbf{y}\|) \text{div} f (\mathbf{y}) \, d\mathbf{y}.
\]
(7.10)
From (7.10) we conclude that
\[
z_1 = k^2 \int_{\Omega} g_{-k} (\|\cdot - \mathbf{y}\|) v (\mathbf{y}) \, d\mathbf{y} + \frac{1}{k^2} \nabla \int_{\mathbb{R}^3} g_{-k} (\|\mathbf{x} - \mathbf{y}\|) \text{div} v_{\text{zero}} (\mathbf{y}) \, d\mathbf{y}
\]
solves the differential equation [first line in (7.8)] in \(\mathbb{R}^3 \setminus \Gamma\) and the radiation condition. The function \(v_{\text{zero}}\) has a jump across \(\Gamma\), and it is easy to verify that the distributional divergence is given by
\[
(\text{div}_{\mathbb{R}^3} v_{\text{zero}}) (\psi) = \int_{\Omega} (\text{div} v) \psi - \int_{\Gamma} \langle v, \mathbf{n} \rangle \psi \quad \forall \psi \in C_\infty (\mathbb{R}^3).
\]
Hence,
\[
z_1 = k^2 \mathcal{N}_{-k}^{\text{Hh}} (v) + \nabla \mathcal{N}_{-k}^{\text{Hh}} (\text{div} v) - \nabla \mathcal{S}_{-k}^{\text{Hh}} ((v, \mathbf{n})) =: z_{1,1} + z_{1,2} + z_{1,3}
\]
with the acoustic single-layer potential
\[
\mathcal{S}_{k}^{\text{Hh}} \phi := \int_{\Gamma} g_k (\|\cdot - \mathbf{y}\|) \phi (\mathbf{y}) \, d\Gamma_y
\]
(7.11)
and the acoustic Newton potential
\[
\mathcal{N}_{k}^{\text{Hh}} w := \int_{\Omega} g_k (\|\cdot - \mathbf{y}\|) w (\mathbf{y}) \, d\Gamma_y.
\]
(7.12)
We assumed \(v \in H(\Omega, \text{div})\). Well-known mapping properties of \(\mathcal{S}_{k}^{\text{Hh}}\) and \(\mathcal{N}_{k}^{\text{Hh}}\) (cf. [51]) imply that
\[
z_{1,1} \in H^2_{\text{loc}} (\mathbb{R}^3) \quad \text{so that} \quad [z_{1,1}]_{0, \Gamma} = 0 \quad \text{and} \quad [z_{1,1}]_{1, \Gamma} = 0.
\]
By the same reasoning, we know that \( z_{1,2} \in H^1_{\text{loc}}(\mathbb{R}^3) \) and also \( \text{curl} \, z_{1,2} = 0 \). Hence, 
\[
[z_{1,2}]_{0, \Gamma} = [z_{1,2}]_{1, \Gamma} = 0.
\]
Since \( \langle v, n \rangle \in H^{-1/2}(\Gamma) \), we know that \( S^H_{k}((v, n)) \in H^1_{\text{loc}}(\mathbb{R}^3) \) and \( \text{curl} \, \nabla S^H_{k}((v, n)) = 0 \) so that 
\[
[z_{1,3}]_{1, \Gamma} = 0.
\]
In all, we have obtained 
\[
[z_{1}]_{0, \Gamma} = 0 \quad \text{and} \quad [z_{1}]_{1, \Gamma} = 0. \tag{7.13}
\]
To obtain the full solution, we introduce the single-layer operator for the Maxwell problem (cf. \cite[3.11]{12}) by
\[
S^M_{k\omega} (\phi) = S^H_{k} (\phi) + \frac{1}{k^2} \nabla S^H_{k} (\text{div} \Gamma \phi). \tag{7.14}
\]
From \cite[(5.5.29)]{47}, we get that
\[
[z_{2}]_{1, \Gamma} = -\phi.
\]
The combination of this, the third equation in (7.8), and (7.13) shows that
\[
z_2 := S^M_{-k} \left( i k T_{-k} \left( g^\nabla - h^{\text{curl}} \right) \right)
\]
satisfies \( \text{curl} \, \text{curl} \, z_2 - k^2 z_2 = 0 \) in \( \mathbb{R}^3 \setminus \Gamma \), the transmission condition (second and third equation in (7.8)), and the Silver–Müller radiation conditions for the dual problem. Next, we give a formula for the full solution of (7.8)
\[
Z = k^2 \int_\Omega g_{-k} (|| \cdot - y ||) v(y) \, dy + \nabla \int_\Omega g_{-k} (|| \cdot - y ||) (\text{div} \, v) (y) \, dy
\]
\[
- \nabla \int_\Gamma g_{-k} (|| \cdot - y ||) \langle v, n \rangle (y) \, dy
\]
\[
+ (i k) \int_\Gamma g_{-k} (|| \cdot - y ||) T_{-k} \left( g^\nabla - h^{\text{curl}} \right) (y) \, d\Gamma_y
\]
\[
- \frac{1}{i k} \nabla \int_\Gamma g_{-k} (|| \cdot - y ||) \text{div} \Gamma \, T_{-k} g^\nabla (y) \, d\Gamma_y, \tag{7.15}
\]
where we used \( \text{div} \Gamma \, T_{-k} h^{\text{curl}} = 0 \) (cf. \cite[2.20]{20}).

**Theorem 7.1**  
(i) For \( v \in V^*_0 \), \( g = v \), and \( h = 0 \), the solution of (7.1) is given by
\[
z = k^2 \int_\Omega g_{-k} (|| \cdot - y ||) v(y) \, dy + i k \int_\Gamma g_{-k} (|| \cdot - y ||) T_{-k} v^\nabla (y) \, d\Gamma_y. \tag{7.16}
\]
(ii) For \( v = 0 \), formula (7.15) simplifies to a combined layer potential
\[
\square \text{ Springer } \square
\[
\begin{align*}
z = & \ i k \int_{\Gamma} g_{-k} (\| \cdot - y \|) T_{-k} \left( g^\nabla - h^{\text{curl}} \right)(y) \, d\Gamma_y \\
& - \frac{1}{i k} \nabla \int_{\Gamma} g_{-k} (\| \cdot - y \|) \, \text{div} \, T_{-k} g^\nabla (y) \, d\Gamma_y.
\end{align*}
\]  
(7.17)

**Proof** The assumptions \( h = 0, \ g = v \in V_0^* \) and the properties (4.28) allow us to simplify (7.15) to obtain (7.16). Formula (7.17) follows by setting \( v = 0 \) in (7.15). \( \square \)

### 7.1.2 Solution Formula for Type 2 Problems in the Unit Ball \( B_1(0) \)

Let \( \Omega = B_1(0) \) be the unit ball. The problem of Type 2 [cf. (7.2)] is a Poisson-type problem. Integration by parts leads to its strong formulation. We recall \( \text{div} \, L_{\Omega} r = 0 \) by (4.9c) so that

\[
- \Delta Z = 0 \quad \text{in} \ \Omega, \\
\frac{\partial Z}{\partial n} - \frac{i}{k} \text{div} \, T_k \nabla_G Z = \langle L_{\Omega} r, n \rangle - \frac{i}{k} \text{div} \, T_k^{\text{low}} r_T \quad \text{on} \ \Gamma.
\]  
(7.18)

To analyze this problem, we introduce the Dirichlet-to-Neumann operator \( T_{\Delta} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \) that maps \( g \in H^{1/2}(\Gamma) \) to \( \partial_n u \), where \( u \) is the (weak) solution of

\[
\Delta u = 0 \quad \text{in} \ \Omega, \quad u = g \quad \text{on} \ \Gamma.
\]

This allows us to formulate (7.18) as follows (with \( L_{\Gamma} \) as in Definition 4.2)

\[
- \Delta Z = 0 \quad \text{in} \ \Omega, \\
T_{\Delta} Z - \frac{i}{k} \text{div} \, T_k \nabla_G Z = \langle L_{\Omega} r, n \rangle - \frac{i}{k} \text{div} \, T_k^{\text{low}} r_T \quad \text{on} \ \Gamma.
\]  
(7.19)

We employ expansions of \( \langle L_{\Omega} r, n \rangle \) and \( r_T \) in the forms

\[
\langle L_{\Omega} r, n \rangle = \sum_{\ell=0}^{\infty} \sum_{m \in \iota_{\ell}} \kappa_{m,\ell}^m y_{m,\ell}^m \quad \text{and} \quad r_T = \sum_{\ell=1}^{\infty} \sum_{m \in \iota_{\ell}} \left( r_{m,\ell}^m T_{m,\ell}^\nabla + R_{m,\ell}^m \nabla_G Y_{m,\ell}^m \right)
\]  
(7.20)

so that the right-hand side in the second equation of (7.19) is

\[
\langle L_{\Omega} r, n \rangle - \frac{i}{k} \text{div} \, T_k^{\text{low}} r_T = \sum_{\ell=0}^{\infty} \sum_{m \in \iota_{\ell}} \left( \kappa_{m,\ell}^m Y_{m,\ell}^m - \frac{i}{k} \text{div} \, T_k^{\text{low}} \left( r_{m,\ell}^m T_{m,\ell}^\nabla + R_{m,\ell}^m \nabla_G Y_{m,\ell}^m \right) \right)
\]

\[
[47,(2.4.173),(5.3.93)] \sum_{\ell>\lambda k} \sum_{m \in \iota_{\ell}} \kappa_{m,\ell}^m Y_{m,\ell}^m + \sum_{\ell \leq \lambda k} \sum_{m \in \iota_{\ell}} \left( \kappa_{m,\ell}^m - \frac{\ell (\ell + 1)}{z_{\ell}(k) + 1} R_{m,\ell}^m \right) Y_{m,\ell}^m.
\]  
(7.21)

Note that \( Y_0^0 = \frac{1}{\sqrt{4\pi}} \) is constant. Hence, \( \kappa_0^0 = \langle L_{\Omega} r, n \rangle, Y_0^0 \rangle = \left( \text{div} \, L_{\Omega} r, \frac{1}{\sqrt{4\pi}} \right) \Omega = 0 \). Hence, the summation index for the second sum in (7.21) can be restricted to \( 1 \leq \ell \leq \lambda k \). The representation (7.21) motivates the ansatz for the trace of \( Z \)
\[ Z|_\Gamma = \sum_{\ell=0}^{\infty} \sum_{m \in \mathbb{I}_\ell} Z^m_{\ell} Y^m_{\ell}. \]

The left-hand side of the second equation of (7.19) becomes
\[
T_{\Delta} Z - \frac{i}{k} \text{div}_\Gamma T_k \nabla Z = \sum_{\ell=0}^{\infty} \sum_{m \in \mathbb{I}_\ell} Z^m_{\ell} \left( T_{\Delta} Y^m_{\ell} - \frac{i}{k} \text{div}_\Gamma T_k \nabla Y^m_{\ell} \right)
\]
\[ \equiv \sum_{\ell=1}^{\infty} \ell \left( 1 - \frac{\ell + 1}{z_\ell(k) + 1} \right) \sum_{m \in \mathbb{I}_\ell} Z^m_{\ell} Y^m_{\ell}. \] (7.22)

The right-hand sides in (7.21) and (7.22) must be equal. Thus,
\[
Z^m_{\ell} = \begin{cases} 
\varphi^m_{\ell} := \frac{1}{\ell} \left( \frac{z_\ell(k) + 1}{z_\ell(k) - \ell} \right) k^m_{\ell} - \frac{\ell + 1}{z_\ell(k) - \ell} R^m_{\ell} \leq \lambda k, \\
\Phi^m_{\ell} := \frac{1}{\ell} \left( \frac{z_\ell(k) + 1}{z_\ell(k) - \ell} \right) k^m_{\ell} \quad \ell > \lambda k.
\end{cases} \] (7.23)

Hence, the solution \( Z \) of (7.19) is the solution of the following Laplace equation with non-homogeneous Dirichlet boundary conditions
\[
-\Delta Z = 0 \quad \text{in } \Omega, \\
Z = g_D \quad \text{on } \Gamma,
\]
with \( g_D := \sum_{\ell \leq \lambda k} \sum_{m \in \mathbb{I}_\ell} \varphi^m_{\ell} Y^m_{\ell} + \sum_{\ell > \lambda k} \sum_{m \in \mathbb{I}_\ell} \Phi^m_{\ell} Y^m_{\ell}. \) (7.24)

### 7.2 Regularity of the Dual Problems for the Unit Ball \( \Omega = B_1(0) \)

#### 7.2.1 The High-Frequency Case

We study the regularity of the solution of (4.50a) for the right-hand side \( r \leftarrow v_0 \in \mathbb{V}_0^* \). Recall the definition of \( \nabla^p \) in (2.26).

**Proposition 7.2** Let \( \Omega = B_1(0) \) be the unit ball. Let \( v_0 \in \mathbb{V}_0^* \) and \( z = \mathcal{N}_2 v_0 \) with \( \mathcal{N}_2 \) given by (4.50a). Then there exists a \( k \)-dependent splitting \( \mathcal{N}_2 v_0 = \mathcal{N}_2^{\text{rough}} v_0 + \mathcal{N}_2^{\text{A}} v_0 \) such that
\[
\| \mathcal{N}_2^{\text{rough}} v_0 \|_{H^2(\Omega)} \leq C_{\text{rough}} k \| v_0 \|_{\text{curl}, \Omega, k}, \\
\| \nabla^p \mathcal{N}_2^{\text{A}} v_0 \| \leq C_{A,2} \gamma_{A,2}^p (\lambda + 1) \| v_0 \|_{\text{curl}, \Omega, k} \forall p \in \mathbb{N}_0,
\]
where \( C_{\text{rough}}, C_{A,2}, \gamma_{A,2} > 0 \) are constants independent of \( p, k, \) and \( v_0 \).

**Proof** The solution of the dual problem (4.50a) is given (cf. (7.16), (4.2b)) by
\[
z = (-i k z_1 + z_2) i k \quad \text{with}
\]
\[
z_1 := \int_\Omega g_{-k} \| \cdot - y \| v_0(y) \, dy \quad \text{and} \quad z_2 := \int_\Gamma g_{-k} \| \cdot - y \| T_{-k} v_0^\wedge(y) \, d\Gamma_y.
\]
From the decomposition lemma in [40, Lemma 3.5], we get a $k$-dependent additive splitting $z_1 := z_1^{\text{rough}} + z_1^A$ such that

$$\begin{align*}
\| \nabla^m z_1^{\text{rough}} \| &\leq C k^{m-2} \| v_0 \| \quad \forall m \in \{0, 1, 2\}, \\
\| \nabla^p z_1^A \| &\leq C k^{p-1} \| v_0 \| \quad \forall p \in \mathbb{N}_0
\end{align*}$$

(7.26)

for a constant $C$ independent of $k$ and $v_0$. For $z_2$, we employ the splitting

$$v_0^{\nabla, \text{low}} := L^\ell (v_0^{\nabla}) \quad \text{and} \quad v_0^{\nabla, \text{high}} := H^\ell (v_0^{\nabla})$$

and define $z_2^{\text{low}} := S_{k}^h (T_{-k} v_0^{\nabla, \text{low}})$ and $z_2^{\text{high}} := z_2 - z_2^{\text{low}}$. From [37, Lemma 3.4, Thm. 5.3], we get a splitting $z_2^{\text{high}} = z_2^{\text{rough}} + z_2^A$ such that, for $w := T_{-k} v_0^{\nabla, \text{high}}$,

$$\begin{align*}
\| \nabla^m z_2^{\text{rough}} \| &\leq C \| w \|_{H^2 (\Gamma)} \quad \forall m \in \{0, 1, 2\}, \\
\| \nabla^p z_2^A \| &\leq \tilde{C} \gamma^p \max \{ p + 1, k \} \| w \|_{H^{3/2} (\Gamma)} \quad \forall p \in \mathbb{N}_0.
\end{align*}$$

(7.27)

Here the constants $C, \tilde{C}, \gamma$ are independent of $p, k$, and $w$. This motivates the definition of the operator $N_2^{\text{rough}} : V_0^\ast \to H^2 (\Omega)$ by

$$N_2^{\text{rough}} v_0 := z_1^{\text{rough}} + z_2^{\text{rough}}.$$  

(7.28)

To estimate the norms of $w$ in (7.27), we employ the third estimate in (5.9) in Lemma 5.3 for $s \leq 3/2$ (we also use that (5.8) gives $z_\ell (-k) = \tilde{z}_\ell (k)$ and $\left| \frac{\tilde{k}}{\tilde{k} (-k + 1)} \right| = \left| \frac{k}{\tilde{k} (k) + 1} \right|$): From the definition of $w$ and (5.6), (5.7), we conclude that $w$ has the representation

$$v_0^{\nabla} = \sum_{\ell = 1}^{\infty} \sum_{m \in \ell^t} V_{\ell}^m \nabla^m Y_{\ell}^m$$

for some coefficients $V_{\ell}^m$. Hence,

$$\begin{align*}
\| w \|_{H^s (\Gamma)}^2 &\leq \sum_{\ell > k} \sum_{m \in \ell^t} \left( \frac{k}{z_\ell (k) + 1} \right)^{s+1} \left| \frac{k}{z_\ell (k) + 1} \right|^2 \left| V_{\ell}^m \right|^2 \\
&\leq C k^2 \sum_{\ell > k} \sum_{m \in \ell^t} (\ell + 1)^{2s-3} (\ell (\ell + 1))^{3/2} \left| V_{\ell}^m \right|^2 \\
&\leq C k^{2s-1} \sum_{\ell > k} \sum_{m \in \ell^t} (\ell (\ell + 1))^{3/2} \left| V_{\ell}^m \right|^2 \\
&\leq C k^{2s-1} \left\| \text{div}_{\Gamma} v_0^{\nabla} \right\|_{H^{-1/2} (\Gamma)}^2 \leq C k^{2s-1} \left\| v_0^{\nabla} \right\|_{H^{1/2} (\Gamma)}^2 \\
&\leq C k^{2s-1} \left\| v_0 \right\|_{H^s (\Omega)}^2 \leq C k^{2s-1} \left\| v_0 \right\|_{\text{curl}, \Omega, 1}^2.
\end{align*}$$

(7.29)
We set \( s = 1/2 \) in (7.29) to derive

\[
\|w\|_{H^{1/2}_T(\Gamma)} \leq C \|v_0\|_{\text{curl}, \Omega, k}.
\] (7.30)

The combination of the first lines in (7.26) and (7.27) with (7.30) leads to the first estimate in (7.25).

To estimate \( \|w\|_{H^{-3/2}_T(\Gamma)} \), we employ (7.29) with \( s = -3/2 \) to obtain

\[
\|w\|_{H^{-3/2}_T(\Gamma)} \leq Ck^{-2} \|v_0\|_{\text{curl}, \Omega, 1}.
\] (7.31)

Taking into account the second estimate in (7.27) results in

\[
\|\nabla^p z_2^A\| \leq C \gamma^p \max \{ p + 1, k \}^{p-1} \|v_0\|_{\text{curl}, \Omega, 1}.
\]

The term \( z_2^\text{low} \) is defined as the acoustic single-layer potential applied to the function \( T - k v_0 \). The analysis of such a term will be carried out in Sect. 7.2.2, and it follows from (7.40) (where the function \( c \) corresponds to \( z_2^\text{low} \)) that

\[
z_2^\text{low} \in A \left( Ck^2 \|v_0\|_{\text{curl}, \Omega, 1} \cdot \gamma, \Omega \right),
\] (7.32)

where \( C \) and \( \gamma \) are positive constants independent of \( k \) and \( v_0 \). The combination of the second estimates in (7.26), (7.27) with (7.31) and (7.32) leads to the second estimate in (7.25).

7.2.2 The Low-Frequency Cases

First, we study the regularity of the solution operator \( N_3^A \) as in (4.50b), which is of Type 1 with \( r = g = L_\Omega v \) and \( h = 0 \). Since \( L_\Omega v \) is, in general, not in \( V_0^* \), we have to employ the solution formula (7.15), where the second summand can be dropped due to \( \text{div} L_\Omega v = 0 \) [cf. (4.9c)]. We set

\[
a := N_{Hh}^k (L_\Omega v), \quad b := S_{Hh}^k (\langle L_\Omega v, n \rangle), \quad c := S_{Hh}^k (T - k (L_\Omega v)^\nabla), \quad d := S_{Hh}^k (\text{div}^\Gamma T - k (L_\Omega v)^\nabla)
\] (7.33)

so that

\[
z := N_3^A v = k^2 a - \nabla b + ik c + \frac{i}{k} \nabla d.
\] (7.34)

**Proposition 7.3** Let \( \Omega = B_1(0) \) be the unit ball. There exist positive constants \( C_{A,3} \) and \( \gamma_{A,3} \) independent of \( k \) such that for any \( v \in X \)

\[
N_3^A v \in A \left( C_{A,3} k^3 \|v\|_{\text{curl}, \Omega, k} \cdot \gamma_{A,3}, \Omega \right).
\]
Proof We determine the analyticity classes for the functions in the splitting (7.34), distinguishing between the terms related to the acoustic Newton potential $N_{Hh}^{-k}$ and the acoustic single-layer operator.

(a) Newton potential We start by writing a function $q = N_{Hh}^{-k}(g)$ as a solution of a transmission problem: Let

$$-\Delta q - k^2 q = g_{\text{zero}} \quad \text{in} \quad \mathbb{R}^3 \setminus \Gamma$$

with $g_{\text{zero}} := \begin{cases} g & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \mathbb{R}^3 \setminus \Omega. \end{cases}$

$$[q]_{\Gamma} = \left[ \frac{\partial q}{\partial n} \right]_{\Gamma} = 0,$$

$$\left[ \frac{\partial q}{\partial r} + i k q \right] = o \left( \|x\|^{-1} \right) \quad \text{as} \quad \|x\| \to \infty.$$ 

Next, we will determine the class of analyticity for the function $q$ by using the results in [37]. For this, we have to investigate the analyticity class of $g = L_{\Omega} v$. From Theorem 5.9 we conclude with $C_1, \gamma_1$ independent of $k$ and $v$

$$L_{\Omega} v \in \mathcal{A} \left( C_{v,1}, \gamma_1, \Omega \right) \quad \text{with} \quad C_{v,1} := C_1 k^{3/2} \|v\|_{\text{curl}, \Omega, k}.$$ 

This allows us to use [37, Thm. B.4] to deduce the analyticity class for $N_{Hh}^{-k} (L_{\Omega} v)$. Introducing the $k$-weighted $H^1$-norm $\|v\|_{H^1, \Omega}$ := $(\| \nabla v \|_{\Omega}^2 + k^2 \| v \|_{\Omega}^2)^{1/2}$, we obtain

$$N_{Hh}^{-k} (L_{\Omega} v) \in \mathcal{A} (C_{v,2}, \gamma_3, \Omega),$$

with

$$C_{v,2} := C_3 \left( k^{-2} C_{v,1} + k^{-1} \left\| N_{Hh}^{-k} (L_{\Omega} v) \right\|_{H^1, B_R(0)} \right);$$

here, $B_R(0)$ is an (arbitrarily chosen) ball containing $\overline{\Omega}$. From [40, Lemma 3.5], we get $\left\| N_{Hh}^{-k} (L_{\Omega} v) \right\|_{H^1, B_R(0)} \leq C \|L_{\Omega} v\|$ so that

$$C_{v,2} \leq CC_3 \left( k^{-2} C_{v,1} + k^{-1} \|L_{\Omega} v\| \right)$$

$$\leq CC_3 \left( C_1 k^{-1/2} \|v\|_{\text{curl}, \Omega, k} + k^{-2} \|v\|_{\text{curl}, \Omega, k} \right)$$

$$\leq C_4 k^{-1/2} \|v\|_{\text{curl}, \Omega, k}.$$ 

Hence,

$$k^2 a \in \mathcal{A} \left( C_4 k^{3/2} \|v\|_{\text{curl}, \Omega, k}, \gamma_4, \Omega \right).$$ 

(b) Single-layer potential We write a function $q = S_{Hh}^{-k}(g)$ as a solution of a transmission problem: Let $\gamma_0$ denote the standard, one-sided trace operator for $\Gamma$ from the interior and $\gamma_0^+$ the one from the exterior. The one-sided normal trace (from
the interior) is denoted by $\gamma_1 := \partial/\partial n$ and by $\gamma_1^+$ from the exterior. The respective jumps are $[u]_\Gamma = \gamma_0^+ u - \gamma_0 u$ and $[u]_{n,\Gamma} = \gamma_1^+ u - \gamma_1 u$. The well-known jump relations for the single-layer potential yield for the potential $q = S^H_{\perp k} (g)$

$$- \Delta q - k^2 q = 0 \quad \text{in } \mathbb{R}^3 \backslash \Gamma,$$

$$[q]_\Gamma = 0 \quad \text{and} \quad [q]_{n,\Gamma} = -g,$$

$$\left| \frac{\partial q}{\partial r} + i k q \right| = o \left( \|x\|^{-1} \right) \quad \text{as} \quad \|x\| \to \infty.$$

The essential part of the regularity estimates are those near the boundary/interface $\Gamma$, where the analyticity of the jump $g$ and the geometry $\Gamma$ come into play. We follow the standard procedure of locally flattening $\Gamma$ so that [35, Thm. 5.5.4] becomes applicable.

In view of (7.33), we have to analyze the transmission problem for 3 different choices of $g$:

$$g \in \{g_1, g_2, g_3\} \quad \text{with} \quad g_1 := \langle L_\Omega v, n \rangle, \quad g_2 := \text{div}_\Gamma T_{-k} (L_\Omega v) \nabla, \quad g_3 := T_{-k} (L_\Omega v) \nabla. \quad (7.37)$$

1. Step (analyticity classes of $g$): In the following $\mathcal{U}_\Gamma$ is a sufficiently small neighborhood of $\Gamma$ whose size depends solely on $\Gamma$. Lemma 5.10 directly implies the existence of an extension $g_1^*$ of $g_1$ into $\mathcal{U}_\Gamma$

$$g_1^* \in \mathcal{A} \left( \tilde{C} k^{3/2} \|v\|_{\text{curl}, \Omega, k}, \tilde{\gamma}, \mathcal{U}_\Gamma \right). \quad (7.38)$$

To define extensions of $g_2, g_3$, we repeat the arguments of Lemma 5.5. From the expansion

$$\Pi_T v := \sum_{\ell=1}^{\infty} \sum_{m \in \ell} \left( V^m_\ell Y^m_\ell + V^m_\ell \nabla_\Gamma Y^m_\ell \right),$$

we get

$$g_3 = T_{-k} (L_\Omega v) \nabla \quad (5.7) = \sum_{1 \leq \ell \leq \lambda_k} \frac{ik}{z_\ell(k) + 1} \sum_{m \in \ell} V^m_\ell \nabla_\Gamma Y^m_\ell,$$

$$g_2 = \text{div}_\Gamma T_{-k} (L_\Omega v) \nabla \quad (2.20) = \sum_{1 \leq \ell \leq \lambda_k} \frac{i k \ell (\ell + 1)}{z_\ell(k) + 1} \sum_{m \in \ell} V^m_\ell Y^m_\ell.$$

Recall the analytic extension $\tilde{Y}^m_\ell$ of $Y^m_\ell$ with the property (5.11) from the proof of Lemma 5.5. We define the analytic extensions of $g_2, g_3$ by

$$g_2^* := \sum_{1 \leq \ell \leq \lambda_k} \frac{ik \ell (\ell + 1)}{z_\ell(k) + 1} \sum_{m \in \ell} V^m_\ell \tilde{Y}^m_\ell$$

and
\[ g_3^* := \sum_{1 \leq \ell \leq \lambda k} \left( \frac{i k}{z_{\ell}(k) + 1} \right) \sum_{m \in \ell} V_{\ell}^m \nabla \tilde{\nabla}_m. \]

We obtain by using Cauchy–Schwarz inequalities
\[
\| \nabla^n g_2^* \|_{L^2(\Omega)} \leq \sum_{1 \leq \ell \leq \lambda k} \left| \frac{i k \ell (\ell + 1)}{z_{\ell}(k) + 1} \right| \sum_{m \in \ell} V_{\ell}^m \| \nabla^n \tilde{\nabla}_m \|_{L^2(\Omega)}
\]
\[
\leq C k^{7/2} \gamma^n \max \{ k, n \}^n \| v_T \|_{-1/2, \text{curl}_r},
\]
\[
\| \nabla^n g_3^* \|_{L^2(\Omega)} \leq \sum_{1 \leq \ell \leq \lambda k} \left| \frac{i k}{z_{\ell}(k) + 1} \right| \sum_{m \in \ell} V_{\ell}^m \| \nabla^{n+1} \tilde{\nabla}_m \|_{L^2(\Omega)}
\]
\[
\leq \tilde{C} \gamma^{n+1} k \max \{ k, n + 1 \}^n \sum_{1 \leq \ell \leq \lambda k} \sqrt{\ell (\ell + 1)} \sum_{m \in \ell} | V_{\ell}^m |
\]
\[
\leq \tilde{C} \gamma^n k^{5/2} \max \{ k, n + 1 \}^n \| v_T \|_{-1/2, \text{curl}_r}.
\]

We combine this with Theorem 2.4 and have proved that the extensions \( g_1^*, g_2^*, g_3^* \) belong to the analyticity classes
\[
g_1^* \in A \left( C_1 k^{3/2} \| v \|_{\text{curl}, \Omega, k}, \gamma_1, \mathcal{U}_r \right), \quad g_2^* \in A \left( C_2 k^{7/2} \| v \|_{\text{curl}, \Omega, 1}, \gamma_2, \mathcal{U}_r \right), \quad g_3^* \in A \left( C_3 k^{5/2} \| v \|_{\text{curl}, \Omega, 1}, \gamma_3, \mathcal{U}_r \right),
\]
where \( C_j, \gamma_j \) are independent of \( k \) and \( v \).

2. Step (a priori bounds for potential \( q \)): For an (arbitrary) fixed ball \( B_R(0) \) with \( \Omega \subset B_R(0) \) [37, Lemma 3.4, Thm. 5.3] imply
\[
\| S_{-k}^H (g_3) \|_{H^{3/2+\ell}(\mathcal{U}_r)} \leq C \sum_{\ell=0}^1 k^{2-\ell} \| g_3 \|_{H^{3/2+\ell}(\mathcal{U}_r)} \quad \text{and}
\]
\[
\| S_{-k}^H (g_i) \|_{H^{3/2+\ell}(\mathcal{U}_r)} \leq C \sum_{\ell=0}^1 k^{2-\ell} \| g_i \|_{H^{3/2+\ell}(\mathcal{U}_r)}, \quad i = 1, 2.
\]

The \( \| \cdot \|_{H^{3/2+\ell}(\mathcal{U}_r)} \) norm of \( g_i \) can be estimated for \( \ell = 0, 1 \) as follows:

**Estimating \( g_1 \):**
\[
\| (L_{\Omega} v, n) \|_{H^{3/2+\ell}(\mathcal{U}_r)} \leq C \| \langle L_{\Omega} v, n \rangle \|_{H^{1/2}(\mathcal{U}_r)} \leq C \| L_{\Omega} v \|_{H(\Omega, \text{div})}
\]
\[
\overset{(4.9c)}{=} C \| L_{\Omega} v \| \leq \frac{C}{k} \| L_{\Omega} v \|_{\text{curl}, \Omega, k} \leq C k^{-1} \| v \|_{\text{curl}, \Omega, k}.
\]

**Estimating \( g_2 \):**
\[
\| g_2 \|_{H^{3/2+n}(\mathcal{U}_r)}^2 \overset{(2.16)}{=} \sum_{1 \leq \ell \leq \lambda k} (\ell (\ell + 1))^{-3/2+n} \left| \frac{i k \ell (\ell + 1)}{z_{\ell}(k) + 1} \right|^2 \sum_{m \in \ell} | V_{\ell}^m |^2
\]
\[
\overset{\circ \text{Springer}}{\leq} \text{...}
\]
\[
\leq Ck^2 \sum_{1 \leq \ell \leq \lambda k} (\ell (\ell + 1))^{1/2 + n} \sum_{m \in \ell} |V_m^\ell|^2
\]

\[
\leq Ck^{2+2n} \|v\|_\text{curl,}\Omega,1^2 \leq Ck^{2+2n} \|v\|_\text{curl,}\Omega,1^2.
\]

**Estimating \(g_3\):**

\[
\|g_3\|_{H^{-3/2+n}(\Gamma')}^2 \leq \sum_{1 \leq \ell \leq \lambda k} (\ell (\ell + 1))^{-1/2 + n} \left| \frac{i\kappa}{z_\ell(k)} + 1 \right|^2 \sum_{m \in \ell} |V_m^\ell|^2
\]

\[
\leq Ck^{2+2n} \sum_{1 \leq \ell \leq \lambda k} (\ell (\ell + 1))^{-1/2} \sum_{m \in \ell} |V_m^\ell|^2 \leq Ck^{2+2n} \|v\|_\text{curl,}\Omega,1^2.
\]

The combination with (7.39) leads to

\[
\left\| S_{-k}^{Hh} (g_1) \right\|_{H, B_R(0)} \leq Ck \|v\|_\text{curl,}\Omega,k,
\]

\[
\left\| S_{-k}^{Hh} (g_2) \right\|_{H, B_R(0)} \leq Ck^3 \|v\|_\text{curl,}\Omega,1,
\]

\[
\left\| S_{-k}^{Hh} (g_3) \right\|_{H, B_R(0)} \leq Ck^3 \|v\|_\text{curl,}\Omega,1.
\]

**3. Step (analyticity of potential \(q\)):** The above steps and [35, Thm. 5.5.4] give

\[
\nabla b \in A \left( Ck^{3/2} \|v\|_\text{curl,}\Omega,k, \gamma, \Omega \right), \ k c \in A \left( Ck^3 \|v\|_\text{curl,}\Omega,1, \gamma, \Omega \right), \ 
\frac{1}{k} \nabla d \in A \left( Ck^{5/2} \|v\|_\text{curl,}\Omega,1 \cdot \gamma, \Omega \right).
\]  

(7.40)

From the decomposition (7.34) and the assertions (7.36), (7.40), we conclude

\[
\mathcal{N}_3^A v \in A \left( C_{A,3} k^3 \|v\|_\text{curl,}\Omega,k, \gamma_{A,3}, \Omega \right)
\]

for constants \(C_{A,3}, \gamma_{A,3}\) independent of \(k\) and \(v\). \(\square\)

Next, we analyze the regularity of the solution operator \(\mathcal{N}_4^A\) of (4.50c).

**Proposition 7.4** Let \(\Omega = B_1(0)\). There exist positive constants \(C_{A,4}\) and \(\gamma_{A,4}\) depending only on \(\Gamma\) and the cutoff parameter \(\lambda\) such that for any \(r \in X\)

\[
\nabla \mathcal{N}_4^A r \in A \left( C_{A,4} k^{5/2} \|r\|_\text{curl,}\Omega,1 \cdot \gamma_{A,4}, \Omega \right).
\]

**Proof** We first analyze \(g_D\) of (7.24) (in Steps 1–3) and subsequently the solution \(Z\) of (7.24) in Step 4. As in the proof of Proposition 7.3, we let \(U_\Gamma\) be a sufficiently small neighborhood of \(\Gamma\).

**1. Step (analyticity class of \(g_D^*\)):** With the analytic extensions \(\tilde{Y}_m^\ell\) of the eigenfunctions \(Y_m^\ell\) (cf. (5.11)), we extend \(g_D\) to \(U_\Gamma\) by

\[
\square \ Springer \quad \square \ Springer
\]
\[
\begin{align*}
g_D^n \&= \sum_{\ell = 1}^{\infty} \sum_{m \in \iota_{\ell}} Z_{\ell}^m \tilde{Y}_{\ell}^m \\
&\ \overset{(7.23)}{=} \sum_{\ell = 1}^{\infty} \sum_{m \in \iota_{\ell}} \frac{1}{\ell} \left( \frac{z_{\ell}(k) + 1}{z_{\ell}(k) - \ell} \right) \kappa_{\ell, n}^{m} \tilde{Y}_{\ell}^m = \sum_{\ell \leq \lambda k} \sum_{m \in \iota_{\ell}} \frac{\ell + 1}{z_{\ell}(k) - \ell} R_{\ell}^m \tilde{Y}_{\ell}^m, 
\end{align*}
\]

where \( \kappa_{\ell, n}^{m} \) and \( R_{\ell}^m \) are given by (7.20). We note that the coefficients \( \kappa_{\ell, n}^{m} \) are controlled by Lemma 5.10. For the coefficients \( R_{\ell}^m \), we estimate

\[
\sum_{\ell \leq \lambda k} \sum_{m \in \iota_{\ell}} \ell |R_{\ell}^m| \lesssim k^{3/2} \left( \sum_{\ell \leq \lambda k} \sum_{m \in \iota_{\ell}} \ell |R_{\ell}^m|^2 \right)^{1/2} \overset{(2.22a)}{\lesssim} k^{3/2} \|r_T\|_{-1/2, \text{curl}_r}.
\]

**2. Step (symbol estimates):** We have

\[
|z_{\ell}(k) + 1|^2 = (1 + \text{Re}(z_{\ell}(k)))^2 + (\text{Im} z_{\ell}(k))^2.
\]

From [47, (2.6.23)] it follows \(-\ell \leq 1 + \text{Re}(z_{\ell}(k)) \leq 0\). This implies the generous estimate \(\text{Re}(z_{\ell}(k)) \leq 0\) so that \(-\ell + \text{Re}(z_{\ell}(k)) \leq 1 + \text{Re}(z_{\ell}(k)) \leq 0\) and, in turn, \(|1 + \text{Re}(z_{\ell}(k))| \leq |-\ell + \text{Re}(z_{\ell}(K))|\). Thus,

\[
\left| \frac{1}{\ell} \left( \frac{z_{\ell}(k) + 1}{z_{\ell}(k) - \ell} \right) \right| \leq \frac{1}{\ell} \leq \frac{1}{\ell + 1}
\]

and also

\[
\left| \frac{\ell}{z_{\ell}(k) - \ell} \right| \leq \frac{\ell}{|z_{\ell}(k) + 1|} \leq C \left\{ \begin{array}{l}
\ell, \quad \ell \leq \lambda k, \\
1, \quad \ell \geq \lambda k.
\end{array} \right.
\]

**3. Step (analyticity classes of \( g_D^n \)):** We claim: there are \( C, \gamma' > 0 \) independent of \( k \) and \( r \) such that

\[
g_D^n \in \mathcal{A}(Ck^{3/2}\|r\|_{\text{curl},\Omega, 1}, \gamma', \mathcal{U}_\Gamma).
\]

Using (5.11) and the symbol estimates of Step 2, we estimate with the abbreviation \( \lambda_{\ell} = \ell(\ell + 1) \) and \( \gamma'_{A,\Gamma} \) of Lemma 5.10

\[
\begin{align*}
\| \nabla^n g_D^n \|_{L^2(\mathcal{U}_\Gamma)} &\lesssim \gamma^n \left\{ \sum_{\ell \leq \lambda_{\ell, A, \Gamma}} \sum_{m \in \iota_{\ell}} \kappa_{\ell, n}^{m} \right\} \max \left\{ \sqrt{\lambda_{\ell}}, n \right\}^n + \sum_{\ell > \lambda'_{\ell, A, \Gamma}} \sum_{m \in \iota_{\ell}} \frac{1}{\ell + 1} |\kappa_{\ell, n}^{m}| \max \left\{ \sqrt{\lambda_{\ell}}, n \right\}^n \\
&\quad + \sum_{\ell \leq \lambda k} \sum_{m \in \iota_{\ell}} \ell |R_{\ell}^m| \max \left\{ \sqrt{\lambda_{\ell}}, n \right\}^n \right\} =: \gamma^n \left\{ \ldots \right\}.
\end{align*}
\]
We estimate the expression \{\cdot\} in curly braces further with Lemma 5.10:

\[
\left\{ \cdots \right\} \lesssim k^{3/2} \| \nabla \times \Omega, 1 \tilde{\gamma}^n \max\{k, n\}^n + \sum_{\ell > k\gamma^r} \sum_{m \in \ell} \frac{1}{\ell + 1} |k_m^n| \left[ \gamma^{\ell/2} + n^n \right]
\]

\[
\lesssim k^{3/2} \| \nabla \times \Omega, 1 \tilde{\gamma}^n \max\{k, n\}^n + k^{-1} \sum_{\ell > k\gamma^r} \sum_{m \in \ell} |k_m^n| \left[ \gamma^{\ell/2} + n^n \right]
\]

\[
\lesssim k^{3/2} \| \nabla \times \Omega, 1 \tilde{\gamma}^n \max\{k, n\}^n + k \tilde{\gamma}^n \alpha^n \| \nabla \times \Omega, 1.
\]

for suitable $\tilde{\gamma} > 0$; in the last step, we employed (5.46) once with $\alpha = n$ and once with $\alpha = 0$. This shows (7.41). We also note

\[
\| g_D \|_{H^{1/2}((\Gamma))} \lesssim \| g_D^\ast \|_{H^1(\mathcal{U})} \overset{(7.41)}{\lesssim} k^{5/2} \| \nabla \times \Omega, 1 \|.
\]

3. Step (interior regularity): Given $r \in X$, the function $Z = \mathcal{N}_A^r r$ solves (7.24). First, interior regularity as derived in [35, Prop. 5.5.1] gives for all $n \in \mathbb{N}_0$

\[
\| \nabla^{n+1} Z \|_{L^2(\Omega \setminus \mathcal{U}_r)} \leq C \gamma^n (n + 1)^n \| Z \|_{H^1(\Omega)} \leq C \gamma^n (n + 1)^n \| g_D \|_{H^{1/2}((\Gamma))}.
\]

This is the desired bound away from $\Gamma$ in view of (7.43).

4. Step: For the behavior of $Z$ near $\Gamma$, we write $Z = Z^0 - g_D^\ast$. Near $\Gamma$, the function $Z^0$ satisfies

\[
- \Delta Z^0 = - \Delta g_D^\ast \quad \text{in } \mathcal{U}_r \quad \text{and } Z^0|_{\Gamma} = 0.
\]

From (7.41) we get $\Delta g_D^\ast \in \mathcal{A}(Ck^{7/2}, \gamma, \mathcal{U}_r)$ for suitably adjusted constants $C, \gamma > 0$. Also we have

\[
\| \nabla Z^0 \|_{L^2(\mathcal{U}_r)} \leq \| \nabla Z \| + \| \nabla g_D^\ast \|_{L^2(\mathcal{U}_r)} \lesssim \| g_D \|_{H^{1/2}((\Gamma))} + Ck^{5/2} \| \nabla \times \Omega, 1
\]

\[
\lesssim k^{5/2} \| \nabla \times \Omega, 1 \|.
\]

One concludes with Theorem E.2 (and suitable localization as well as flattening of the boundary) that $Z^0$ in (7.45) satisfies, again with adjusted constants $C, \gamma$,

\[
\nabla Z^0 \in \mathcal{A} \left( Ck^{5/2} \| \nabla \times \Omega, 1 \cdot \gamma, \mathcal{U}_r \right).
\]

This in turn implies $\nabla Z|_{\mathcal{U}_r} \in \mathcal{A} \left( Ck^{5/2} \| \nabla \times \Omega, 1 \cdot \gamma, \mathcal{U}_r \right)$.

\begin{proposition}
Let $\Omega = B_1(0)$. There exist positive constants $C_{\mathcal{A}, 1}$ and $\gamma_{\mathcal{A}, 1}$ depending only on $\Gamma$ and the cutoff parameter $\lambda$ such that for any $v \in X$

\[
\mathcal{N}_A^1 v \in \mathcal{A} \left( C_{\mathcal{A}, 1} k^3 \| v \|_{H^1(\Omega \setminus \mathcal{U}_r)}, \Omega \right).
\]
\end{proposition}
Proof Given \( v \in X \), the solution \( z := N_A^1 v \) can be split into

\[
   z = N_A^3 v + \tilde{z}
\]

with the solution \( \tilde{z} \in X \) of

\[
   A_k (w, \tilde{z}) = -i k b_k \left( w^{\text{curl}}, (L_{\Omega} v)^{\text{curl}} \right) \quad \forall w \in X.
\]

From (7.15) we get the following representation of the solution

\[
   \tilde{z} = -i k \int_{\Gamma} g_{-k} (\| \cdot - y \|) T_{-k} \left( (L_{\Omega} v)^{\text{curl}} \right)(y) \, dy.
\]

Fourier expansion of \((L_{\Omega} v)^{\text{curl}}\) leads to [cf. (5.7)]

\[
   \mu := T_{-k} (L_{\Omega} v)^{\text{curl}} = \sum_{1 \leq \ell \leq \lambda k} \left( \frac{z_{\ell}(k) + 1}{i k} \right) \sum_{m \in \iota_{\ell}} v_{m_{\ell}} T_{\ell_{m}}^m.
\]

An extension of \( \mu^* \) is given by

\[
   \mu^* := \sum_{1 \leq \ell \leq \lambda k} \left( \frac{z_{\ell}(k) + 1}{i k} \right) \sum_{m \in \iota_{\ell}} v_{m_{\ell}} \tilde{T}_{\ell_{m}}^m,
\]

where \( \tilde{T}_{\ell_{m}}^m := \nabla \tilde{Y}_{\ell_{m}} \times n^* \) with \( n^*(x) := x / \| x \| \) and \( \tilde{Y}_{\ell_{m}}^m \) as in (5.11). Now we proceed as in the proof of Proposition 7.3. First, we derive the estimates

\[
   \| \mu \|^2_{H^3_T(\Gamma)} \leq C \| \nabla_T \|^2_{H^{1/2}_{\text{curl}}(\Gamma)} \leq C \| v \|^2_{\text{curl}, \Omega, 1}
\]

and

\[
   \| \nabla^n \mu^* \|_{L^2(\Omega_T)} \leq C \max \{ k, n + 1 \}^{n+1} \tilde{c}^n \| v \|_{\text{curl}, \Omega, 1}.
\]
The application of $\mathcal{S}_{h-k}$ to $\mu$ can then be estimated by

$$\|k\mathcal{S}^{h-k}_{-k}(T_{-k}(L_{\Omega}\mathbf{v})_{\text{curl}})\|_{\mathcal{H},\Omega} \leq C k^3 \|\mu\|_{H^{-3/2}(\Gamma)} \leq C k^3 \|\mathbf{v}\|_{\text{curl},\Omega,1}$$

and $k\mathcal{S}^{h-k}_{-k}(T_{-k}(L_{\Omega}\mathbf{v})_{\text{curl}}) \in \mathcal{A}(C k^2 \|\mathbf{v}\|_{\text{curl},\Omega,k,\gamma,\Omega})$ follows. The combination with Proposition 7.3 leads to the assertion. $\square$

8 Approximation Operators for $S_{p+1}(\mathcal{T}_h)$ and $A^1_p(\mathcal{T}_h)$

The relevant $hp$-finite element spaces have been introduced in Sect. 3.2. A key property of these spaces is that both lines in the diagram in Fig. 1 are exact sequences, [26,43,46]. In particular, therefore, (3.2) is satisfied for the pair $(S_h, X_h) = (S_{p+1}(\mathcal{T}_h), A^1_p(\mathcal{T}_h))$. The operators $\Pi_F^E$ and $\Pi_F^C$ of Assumption 4.14 are constructed to satisfy the stronger “commuting diagram property” that make the diagram in Fig. 2 commute. In that case, the operator $\Pi_F^E$ is defined on the space $\prod_{K \in \mathcal{T}_h} H^1(K, \text{curl}) \cap X \supset \{ \mathbf{u} \in H^1(\Omega) \mid \text{curl} \mathbf{u} \in \text{curl} X_h \}$.

8.1 Optimal Simultaneous $hp$-approximation in $L^2$ and $H(\text{curl})$

We restrict our attention to approximation operators that are constructed element-by-element.

**Definition 8.1** (element-by-element construction of operators) An operator $\hat{\Pi}^\text{grad} : H^2(\hat{K}) \to P_{p+1}^+$ is said to admit an element-by-element construction if the operator $\Pi^\text{grad}$ that is defined elementwise on $H^1(\Omega) \cap \prod_{K \in \mathcal{T}_h} H^2(K)$ by $(\Pi^\text{grad} u)|_K := (\hat{\Pi}^\text{grad}(u \circ F_K)) \circ F_K^{-1}$ maps into the conforming subspace $S_{p+1}^+(\mathcal{T}_h) \subset H^1(\Omega)$.

An operator $\hat{\Pi}^\text{curl} : H^1(\hat{K}, \text{curl}) \to A^1_p(\hat{K})$ is said to admit an element-by-element construction if the operator $\Pi^\text{curl}$ that is defined elementwise on $H(\Omega, \text{curl}) \cap \prod_{K \in \mathcal{T}_h} H^1(K, \text{curl}) \cap \prod_{K \in \mathcal{T}_h} H^1(K, \text{div})$ maps into the conforming subspace $A^1_p(\mathcal{T}_h) \subset H^1(\Omega)$.

Fig. 1 Continuous and discrete exact sequences

| $\mathbb{R}$ | $H^1(\Omega)$ | $\nabla$ | $H(\Omega, \text{curl})$ | $\text{curl}$ | $H(\Omega, \text{div})$ | $\text{div}$ | $L^2(\Omega)$ |
|-------------|----------------|---------|------------------------|------------|------------------------|------------|----------------|
| $\uparrow^\iota$ | $\uparrow^\iota$ | $\uparrow^\iota$ | $\uparrow^\iota$ | $\uparrow^\iota$ | $\uparrow^\iota$ |

Fig. 2 Commuting diagram on reference element $\hat{K}$

| $\mathbb{R}$ | $H^2(\hat{K})$ | $\nabla$ | $H^1(\hat{K}, \text{curl})$ | $\text{curl}$ | $H^1(\hat{K}, \text{div})$ | $\text{div}$ | $H^1(\hat{K})$ |
|-------------|----------------|---------|------------------------|------------|------------------------|------------|----------------|
| $\downarrow$ | $\downarrow^{\hat{\Pi}^\text{grad},c}$ | $\downarrow^{\hat{\Pi}^\text{curl},c}$ | $\downarrow^{\hat{\Pi}^\text{div},c}$ | $\downarrow^{\hat{\Pi}^L^2}$ |

| $\mathbb{R}$ | $P_{p+1}(\hat{K})$ | $\nabla$ | $A^1_p(\hat{K})$ | $\text{curl}$ | $RT_p(\hat{K})$ | $\text{div}$ | $P_p(\hat{K})$ |
|-------------|----------------|---------|----------------|------------|----------------|------------|----------------|
Let \( \Pi_{K \in \mathcal{T}_h} \mathbf{H}^1(K, \text{curl}) \) by \((\Pi_{\text{curl}} u)|_K := (F'_K)^{-T} (\hat{\Pi}_{\text{curl}} (F'_K)^T u \circ F_K) \circ F_K^{-1}\) maps into the conforming subspace \( \mathcal{N}^I_p(\mathcal{T}_h) \subset \mathbf{H}(\Omega, \text{curl}) \).

An operator \( \hat{\Pi}_{\text{div}} : \mathbf{H}^1(\hat{K}, \text{div}) \to \mathbf{RT}_p(\hat{K}) \) is said to admit an element-by-element construction if the operator \( \Pi_{\text{div}} \) defined elementwise on \( \mathbf{H}(\Omega, \text{div}) \cap \Pi_{K \in \mathcal{T}_h} \mathbf{H}^1(K, \text{div}) \) by

\[
(\Pi_{\text{div}} u)|_K := (\det (F'_K))^{-1} F_K' (\hat{\Pi}_{\text{div}} (\det F'_K) (F'_K)^{-1} u \circ F_K) \circ F_K^{-1}
\]

maps into the conforming subspace \( \mathbf{RT}_p(\mathcal{T}_h) \subset \mathbf{H}(\Omega, \text{div}) \). Finally, any operator \( \hat{\Pi}_{L^2} : L^2(\hat{K}) \to \mathcal{P}_p(\hat{K}) \) leads to a globally defined \( L^2(\Omega) \)-conforming operator by the following element-by-element construction: \((\Pi_{L^2} u)|_K := (\hat{\Pi}_{L^2} (u \circ F_K)) \circ F_K^{-1}\).

As it is typical, we will construct such operators on the reference tetrahedron \( \hat{K} \) in such a way that the value of the operator restricted to a lower-dimensional entity (i.e., a vertex, an edge, or a face) is completely determined by the value of the function on that entity. For scalar functions, the operator \( \Pi_p \) of \([40, \text{ Def. 5.3, Thm. B.4}]\) is an example that we will build on; it can be viewed as a variant of the projection-based interpolation technique of \([18]\) that also underlies the construction of the operator \( \Pi^I_p \).

Important features of the construction of \( \Pi_p \) are: \( (\Pi_p u)(V) = u(V) \) for all vertices \( V \); it has the property that \( (\Pi_p u)|_e \) is the projection of \( u|_e \) onto a space of polynomials of degree \( p \) on each edge \( e \) under the constraint that \( \Pi_p u \) has already been fixed in the vertices; it has the property that \( (\Pi_p u)|_f \) is the (constrained) projection of \( u|_f \) onto a space of polynomials of degree \( p \) on each face \( f \) under the constraint that \( \Pi_p u \) has already been fixed on edges. We note that the fact that \( \Pi_p \) is a (constrained) projection on polynomial spaces for the edges and faces makes the definition independent of the parameterization of the edges and faces of the reference tetrahedron.

We need approximation operators for the approximation in the norm \( \| \cdot \|_{\text{curl}, \Omega, k} \). Such an operator can be defined in an element-by-element fashion on the reference tetrahedron:

**Lemma 8.2** Let \( s > 3/2 \). There exist operators \( \hat{\Pi}_{p}^{\text{curl}, s} : \mathbf{H}^s(\hat{K}) \to \mathcal{N}^I_p(\hat{K}) \) with the following properties:

(i) \( \hat{\Pi}_{p}^{\text{curl}, s} \) admits an element-by-element construction as in Definition 8.1.

(ii) For \( p \geq s - 1 \) we have

\[
(p + 1)\|u - \hat{\Pi}_{p}^{\text{curl}, s} u\|_{L^2(\hat{K})} + \|u - \hat{\Pi}_{p}^{\text{curl}, s} u\|_{\mathbf{H}^1(\hat{K})} \leq C p^{-(s-1)}|u|_{\mathbf{H}^s(\hat{K})}.
\]

(iii) Let \( u \) satisfy, for some \( C_u, \overline{v}, h > 0, \) and \( \kappa \geq 1, \)

\[
\|\nabla^n u\|_{L^2(\hat{K})} \leq C_u (\overline{v} h)^n \max\{n, \kappa\}^n \quad \forall n \in \mathbb{N}, \; n \geq 2.
\]

Assume furthermore

\[
h + \kappa h/p \leq \widetilde{C}.
\]
Then there exist constants $C, \sigma > 0$ depending solely on $\tilde{C}$ and $\overline{\gamma}$ such that

$$
\|u - \hat{\Pi}_p^{\text{curl}, s} u\|_{W^{2,\infty}(\hat{K})} \leq CC_u \left[ \left( \frac{h}{\sigma + h} \right)^{p+1} + \left( \frac{\kappa h}{\sigma p} \right)^{p+1} \right].
$$

(8.4)

**Proof** Let $\Pi_p : H^s(\hat{K}) \to \mathcal{P}_p, s > 3/2$, be the scalar polynomial approximation operator of [40, Def. 5.3, Thm. B.4]. A key property of $\Pi_p$ is that, as described above, one has that the restriction of $\Pi_p u$ to a vertex, edge, or face is completely determined by $u$ restricted to that entity. We write, e.g., for a face $f$: $\Pi_p (u|_f) := (\Pi_p u)|_f$. We define the operator $\hat{\Pi}_p^{\text{curl}, s} : H^s(\hat{K}) \to (\mathcal{P}_p)^3 \subset \mathcal{N}_p^{\text{curl}}(\hat{K})$ by componentwise application to $u = (u_i)_{i=1}^3$, i.e.,

$$
\hat{\Pi}_p^{\text{curl}, s} u := \left( \Pi_p u_i \right)_{i=1}^3.
$$

1. Step: We show that $\hat{\Pi}_p^{\text{curl}, s}$ admits an element-by-element construction. We show this by asserting that the tangential component $\Pi_T \hat{\Pi}_p^{\text{curl}, s} u$ depends solely on the tangential component $\Pi_T u$. Fix a face $f$ of $\hat{K}$ with normal $n$. Note that $n$ is constant on $f$. The tangential component of $\hat{\Pi}_p^{\text{curl}, s} u$ on $f$ is

$$
\left( \Pi_T \left( \Pi_p^{\text{curl}, s} u \right) \right)|_f = \left( \left( \Pi_p u_i \right)_{i=1}^3 \right)|_f - \left( \sum_{j=1}^3 n_j \Pi_p u_j \right) n|_f.
$$

Using that $(\Pi_p u)|_f$ is completely determined by the values of $u$ on $f$ and using that the normal vector $n$ is constant on $f$, we infer with the understanding that $\Pi_p$ acts componentwise on a vector-valued object

$$
\left( \Pi_T \left( \Pi_p^{\text{curl}, s} u \right) \right)|_f = \Pi_p (u|_f) - \Pi_p \left( (\n \cdot u|_f) n \right) = \Pi_p (u|_f) - \left( (\n \cdot u|_f) n \right)
$$

which is the desired claim.

2. Step: Estimate (8.1) then follows from [40, Thm. B.4].

3. Step: From [40, Lemma C.2], we conclude that (8.4) holds. \hfill \Box

### 8.2 Projection Operators with Commuting Diagram Property

The operator $\hat{\Pi}_p^{\text{curl}, s}$, which is obtained by an elementwise use of the operator $\hat{\Pi}_p^{\text{curl}, s}$ of Lemma 8.2 (cf. Definition 8.1 for the transformation rule), has $(p\text{-optimal})$ approximation properties in $\| \cdot \|_{\text{curl}, \Omega, k}$ as it has simultaneously $p\text{-optimal approximation
properties in $L^2$ and $H^1$. However, it is not a projection and does not have the commuting diagram property. We therefore present a second operator, $\hat{\Pi}^{\text{curl},c}$, in Theorem 8.3 with this property. The construction is given in [39] and similar to that in [18, 19]. We point out that the difference between Theorem 8.3 from [39] and the works [18, 19] is that, by assuming $H^2(\hat{K})$- and $H^1(\hat{K}, \text{curl})$-regularity, Theorem 8.3 features the optimal $p$-dependence, thus avoiding the factors of $\ln p$ present in [18, 19].

**Theorem 8.3** ([39]) There are linear projection operators $\hat{\Pi}^{\text{grad},c}_{p+1}$, $\hat{\Pi}^{\text{curl},c}_p$, $\hat{\Pi}^{\text{div},c}_p$, $\hat{\Pi}^{L^2}_p$ such that the following holds:

(i) The diagram in Fig. 2 commutes.

(ii) The operators $\hat{\Pi}^{\text{grad},c}_{p+1}$, $\hat{\Pi}^{\text{curl},c}_p$, $\hat{\Pi}^{\text{div},c}_p$, $\hat{\Pi}^{L^2}_p$ admit element-by-element constructions as in Definition 8.1. The global operators $\Pi^{\text{grad},c}_{p+1}$, $\Pi^{\text{curl},c}_p$, $\Pi^{\text{div},c}_p$, $\Pi^{L^2}_p$ obtained from the operators $\hat{\Pi}^{\text{grad},c}_{p+1}$, $\hat{\Pi}^{\text{curl},c}_p$, $\hat{\Pi}^{\text{div},c}_p$, $\hat{\Pi}^{L^2}_p$ by an element-by-element construction are also linear projection operators, and the diagram in Fig. 3 commutes.

(iii) For all $\varphi \in H^2(\hat{K})$, there holds

$$\|\varphi - \hat{\Pi}^{\text{grad},c}_{p+1} \varphi\|_{H^s(\hat{K})} \leq C_s p^{-1-(1-s)} \inf_{v \in \mathcal{P}_{p+1}(\hat{K})} \|\varphi - v\|_{H^2(\hat{K})}, \quad s \in [0, 1].$$

(iv) For all $u \in \mathbf{H}^1(\hat{K}, \text{curl})$, there holds

$$\|u - \hat{\Pi}^{\text{curl},c}_p u\|_{\mathbf{H}(\hat{K}, \text{curl})} \leq C p^{-1} \inf_{v \in \mathcal{N}_p(\hat{K})} \|u - v\|_{\mathbf{H}^1(\hat{K}, \text{curl})}.$$  

(v) For all $k \geq 1$ and all $u \in \mathbf{H}^k(\hat{K})$ with curl $u \in \mathcal{P}_p$, there holds

$$\|u - \hat{\Pi}^{\text{curl},c}_p u\|_{L^2(\hat{K})} \leq C_k p^{-k} \|u\|_{\mathbf{H}^k(\hat{K})}. \quad (8.5)$$

For $p \geq k-1$ the full norm $\|u\|_{\mathbf{H}^k(\hat{K})}$ can be replaced with the seminorm $|u|_{\mathbf{H}^k(\hat{K})}$.

### 8.3 hp-FEM Approximation

Our $hp$-FEM convergence result will be formulated for the specific class of meshes that were introduced in Sect. 3.2. For such meshes, we can formulate approximation results for both operators $\Pi^{\text{curl},s}_p$ and $\Pi^{\text{curl},c}_p$. In both cases, we will need to relate...
functions defined on $K$ to their pull-back to the reference tetrahedron $\hat{K}$. The appropriate transformations are described in Definition 3.1: For scalar functions, $\varphi$ defined on $K$ and vector-valued functions $\mathbf{u}$ defined on $K$, we let

$$\hat{\varphi} = \varphi \circ F_K, \quad \hat{\mathbf{u}} = (F'_K)^\top (\mathbf{u} \circ F_K).$$ \hfill (8.6)

**Lemma 8.4** Let the regular mesh $\mathcal{T}_h$ satisfy Assumption 3.1.

(i) With implied constants depending only on $C_{\text{affine}}, C_{\text{metric}}, \gamma$, there holds for all $K \in \mathcal{T}_h$

$$|\hat{\varphi}|_{H^j(\hat{K})} \lesssim h_K^{j-3/2} |\varphi|_{H^j(K)}, \quad j \in \{0, 1\}, \quad |\hat{\varphi}|_{H^2(\hat{K})} \lesssim h_K^{2-3/2} \|\varphi\|_{H^2(K)}, \quad (8.7)$$

$$\|\hat{\mathbf{u}}\|_{L^2(\hat{K})} \lesssim h_K^{1-3/2} \|\mathbf{u}\|_{L^2(K)}, \quad \|\text{curl} \hat{\mathbf{u}}\|_{L^2(\hat{K})} \lesssim h_K^{2-3/2} \|\text{curl} \mathbf{u}\|_{L^2(K)},$$

$$\|\hat{\mathbf{u}}\|_{H^2(\hat{K})} \lesssim h_K^{3-3/2} \|\mathbf{u}\|_{H^2(K)}. \quad (8.8)$$

(ii) Let $\overline{\gamma} > 0$. Then there exist $\gamma', C > 0$ depending only on $\overline{\gamma}$ and the constants of Assumption 3.1 such that the following two implications hold:

$$\|\nabla^n \varphi\|_{L^2(K)} \leq C_{\varphi} \overline{\gamma}^n \max \{n, k\}^n \quad \forall n \in \mathbb{N}_0 \quad \implies$$

$$\|\nabla^n \varphi\|_{L^2(\hat{K})} \leq \overline{C_{\varphi}} h_K^{n-3/2} (h_k \gamma')^n \max \{n, k\}^n \quad \forall n \in \mathbb{N}_0, (8.9)$$

$$\|\nabla^n \mathbf{u}\|_{L^2(K)} \leq C_{\mathbf{u}} \overline{\gamma}^n \max \{n, k\}^n \quad \forall n \in \mathbb{N}_0 \quad \implies$$

$$\|\nabla^n \hat{\mathbf{u}}\|_{L^2(\hat{K})} \leq Ch_K^{1-3/2} C_{\mathbf{u}} (h_k \gamma')^n \max \{n, k\}^n \quad \forall n \in \mathbb{N}_0. \quad (8.10)$$

**Proof** We will not show (8.7). For (8.8), the first and third estimate in (8.8) follow by inspection, the second equivalence follows from (cf., e.g., [43, Cor. 3.58])

$$F'_K \text{ curl} \hat{\mathbf{u}} = (\det F'_K) (\text{curl} \mathbf{u}) \circ F_K.$$

The implications (8.9), (8.10) are obtained by similar arguments. We will therefore focus on (8.10). Recalling that the element map $F_K$ has the form $F_K = R_K \circ A_K$, we introduce the function $\hat{\mathbf{u}} := (R'_K)^\top \mathbf{u} \circ R_K$, which is defined on $\hat{K} := A_K(\hat{K})$. Using [35, Lemma 4.3.1] (and noting as in the proof [40, Lemma C.1] that the original 2d arguments extend to 3d), we get the existence of $C, \, \overline{\gamma}'$, which depend solely on the constants of Assumption 3.1 and on $\overline{\gamma}$, such that

$$\|\nabla^n \hat{\mathbf{u}}\|_{L^2(\hat{K})} \leq C_{\mathbf{u}} (\overline{\gamma}')^n \max \{n, k\}^n \quad \forall n \in \mathbb{N}_0.$$ 

Next, we observe $\hat{\mathbf{u}} = (A'_K)^\top \mathbf{u} \circ A_K$. Using that $A_K$ is affine, it is easy to deduce

$$\|\nabla^n \hat{\mathbf{u}}\|_{L^2(\hat{K})} \leq Ch_K^{1-3/2} C_{\mathbf{u}} \max \{n, k\}^n (h_k \gamma')^n \quad \forall n \in \mathbb{N}_0,$$

which is the desired estimate. \hfill $\square$

**Lemma 8.5** Let $\mathcal{T}_h$ be a regular mesh satisfying Assumption 3.1. Assume $p \geq 1$. 

\[\text{208, Foundations of Computational Mathematics (2021) 21:125–241}\]
Proof of (i): From Lemma 8.2 with $s = 2$ we have on the reference tetrahedron
\[ p \| \hat{\mathbf{u}} - \Pi_p^{\text{curl}, s} \hat{\mathbf{u}} \|_{L^2(K)} + \| \hat{\mathbf{u}} - \Pi_p^{\text{curl}, s} \hat{\mathbf{u}} \|_{H^1(K)} \lesssim p^{-1} \| \mathbf{u} \|_{H^2(K)}. \]
Hence, using (8.8) we infer
\[ p h_K^{1-3/2} \| \mathbf{u} - \Pi_p^{\text{curl}, s} \mathbf{u} \|_{L^2(K)} + h_K^{2-3/2} \| \text{curl} (\mathbf{u} - \Pi_p^{\text{curl}, s} \mathbf{u}) \|_{L^2(K)} \lesssim \frac{h_K^{3-3/2}}{p} \| \mathbf{u} \|_{H^2(K)}. \]

Proof of (ii): We proceed as above. The transformation rules of Lemma 8.4 and Lemma 8.2 give
\[ \left\| \hat{\mathbf{u}} - \Pi_p^{\text{curl}, s} \hat{\mathbf{u}} \right\|_{W^{2,\infty}(\hat{K})} \leq C C_H h_K^{1-3/2} \left( \frac{h_K}{h_K + \sigma} \right)^{p+1} + \left( \frac{k h_K}{\sigma p} \right)^{p+1}. \]

Since the norm $\| \cdot \|_{W^{2,\infty}(\hat{K})}$ is stronger than $\| \cdot \|_{L^2(\hat{K})}$ and $\| \text{curl} \cdot \|_{H^1(\hat{K})}$, the result follows by transforming back to $K$ using Lemma 8.4.

For the operator $\Pi_p^{\text{curl}, c}$, we have the following approximation results:

Lemma 8.6 Let $\mathcal{T}_h$ be a regular mesh satisfying Assumption 3.1. Then for $p \geq 1$:
(i) $h_K^{-1} \| \mathbf{u} - \Pi_p^{\text{curl}, c} \mathbf{u} \|_{L^2(K)} + \| \text{curl} (\mathbf{u} - \Pi_p^{\text{curl}, c} \mathbf{u}) \|_{L^2(K)} \leq C h_K (p+1)^{-1} \| \mathbf{u} \|_{H^2(K)}$. (ii) Assume the hypotheses of Lemma 8.5, (ii). Then
\[ h_K^{-1} \| \mathbf{u} - \Pi_p^{\text{curl}, c} \mathbf{u} \|_{L^2(K)} + \| \text{curl} (\mathbf{u} - \Pi_p^{\text{curl}, c} \mathbf{u}) \|_{L^2(K)} \lesssim \left( \frac{h_K}{h_K + \sigma} \right)^{p} + \frac{k}{p} \left( \frac{k h_K}{\sigma p} \right)^{p}. \]
For $u \in H^1(K, \text{curl})$ with curl $\hat{u} \in (P_p(\widehat{K}))^3$, there holds

$$\|u - P^\text{curl,e}_p u\|_{L^2(K)} \leq C h K p^{-1} \|u\|_{H^1(K)}.$$  

**Proof** Proof of (i): Using Lemma 8.4, we get from Theorem 8.3 and $p \geq 1$

$$h_K^{-1} \|u - P^\text{curl,e}_p u\|_{L^2(K)} + \|\text{curl}(u - P^\text{curl,e}_p u)\|_{L^2(K)}$$

$$\sim h_K^{-2+3/2} \|\hat{u} - \hat{P}_p \text{curl,e}\hat{u}\|_{H(\widehat{K}, \text{curl})}$$

$$\lesssim p^{-1} h_K^{-2+3/2} \inf_{v \in P_p^3} \|\hat{u} - v\|_{H^1(\widehat{K}, \text{curl})} \lesssim p^{-1} h_K^{-2+3/2} \|\hat{u}\|_{H^1(\widehat{K})} \lesssim p^{-1} h_K \|u\|_{H^2(K)}.$$  

Proof of (ii): We start as above. The novel aspect is that $\inf_{v \in N_p^0(\widehat{K})} \|\hat{u} - v\|_{H^1(\widehat{K}, \text{curl})}$ can be estimated as in the proof of Lemma 8.5

$$h_K^{-1} \|u - P^\text{curl,e}_p u\|_{L^2(K)} + \|\text{curl}(u - P^\text{curl,e}_p u)\|_{L^2(K)}$$

$$\lesssim p^{-1} h_K^{-2+3/2} \inf_{v \in P_p^3} \|\hat{u} - v\|_{H^1(\widehat{K}, \text{curl})}$$

$$\lesssim CC_u \left( \left( \frac{h_K}{h_K + \sigma} \right)^p + \frac{k h_K}{\sigma p} \right).$$  

Proof of (iii): With Lemma 8.4 and Theorem 8.3, (v) we estimate

$$\|u - P^\text{curl,e}_p u\|_{L^2(K)} \sim h_K^{-1+3/2} \|\hat{u} - \hat{P}_p \text{curl,e}\hat{u}\|_{L^2(\widehat{K})} \lesssim h_K^{-1+3/2} p^{-1} \|\hat{u}\|_{H^1(\widehat{K})}$$

$$\lesssim h_K^{-1+3/2} p^{-1} h_K^{-2+3/2} \|u\|_{H^1(K)},$$

which completes the proof.  

**Lemma 8.7** Let $h_0, \sigma, c_2 > 0, \alpha \geq 0, k \geq 1$. Then for every $\epsilon > 0$ there is $c_1 > 0$ (depending only on $h_0, \sigma, c_2, \alpha, \epsilon$) such that for any $h \in (0, h_0]$ and $p \geq \max\{1, c_2 \ln k\}$ with $kh/p \leq c_1$ there holds

$$k^\alpha \left( \frac{h}{h + \sigma} \right)^p \leq \epsilon.$$  

**Proof** Without loss of generality, we restrict to $\epsilon \in (0, 1]$. The case $k \in [1, \epsilon]$ is easily seen, and we therefore focus on $k \geq \epsilon$. We note that $h \mapsto h/(h + \sigma)$ is monotone increasing and we consider the two cases, “$h$ small” and “$h$ large.” For the first case, we select $\bar{h} \in (0, h_0]$ such that

$$\alpha + c_2 \ln(\bar{h}/(\bar{h} + \sigma)) \leq \ln \epsilon \leq 0.$$  

We emphasize that $\bar{h}$ depends only on $\epsilon, \alpha, \sigma$, and $c_2$ but is independent of $k$. By the monotonicity of $h \mapsto h/(h + \sigma)$ and the constraint $p \geq c_2 \ln k$, we have for $0 < h \leq \bar{h}$

$$\alpha + c_2 \ln(\bar{h}/(\bar{h} + \sigma)) \leq \ln \epsilon \leq 0.$$  

$$\square$$
and in view of $\ln k \geq \ln e = 1$ the estimate

$$
\ln \left( k^\alpha \left( \frac{h}{\bar{h} + \sigma} \right)^p \right) \leq \ln \left( k^\alpha \left( \frac{\bar{h}}{\bar{h} + \sigma} \right)^{c_2 \ln k} \right) = \ln k \left[ \alpha + c_2 \ln \frac{\bar{h}}{\bar{h} + \sigma} \right] \leq \ln \varepsilon.
$$

For the second case, i.e., $h \in (\bar{h}, h_0]$, we fix $c_1$ such that

$$
\alpha + \frac{\bar{h}}{c_1} \ln \frac{h_0}{h_0 + \sigma} \leq \ln \varepsilon \leq 0.
$$

We note that $c_1$ depends only on $\varepsilon, \alpha, \sigma, h_0$, and $c_2$. (Recall that $\bar{h}$ depends only on $\varepsilon, \alpha, c_2$.) For $h \in (\bar{h}, h_0]$, we get from $p \geq kh/c_1 \geq k\bar{h}/c_1$ and the monotonicity of $h \mapsto h/(h + \sigma)$

$$
\ln \left( k^\alpha \left( \frac{h}{\bar{h} + \sigma} \right)^p \right) \leq \ln \left( k^\alpha \left( \frac{h_0}{h_0 + \sigma} \right)^{k\bar{h}/c_1} \right) = \alpha \ln k + \frac{k\bar{h}}{c_1} \ln \frac{h_0}{h_0 + \sigma}
$$

$$
\leq \ln \varepsilon.
$$

\[\square\]

### A Proof of Lemma 5.3

In this appendix, we prove Lemma 5.3. The first two estimates in (5.9) are proved in the following lemma.

**Lemma A.1** For any $\lambda > 1$, there holds

$$
\frac{k}{|z_n(k)| + 1} \leq \begin{cases} 
2\sqrt{2} k, & n \in \mathbb{N}_0, \\
2\sqrt{2} \left( \frac{2}{\lambda} + 1 \right) \frac{k}{(n + 1)}, & n > \lambda k^2.
\end{cases}
$$

**Proof** We follow the reasoning in [47, Thm. 2.6.1]. The coefficient $z_n(k)$ can be expressed as

$$
z_n(k) = -\frac{(m_n^2)^2}{m_n^2} + k \frac{i}{m_n^2}.
$$
where
\[
\mu = (2n + 1)^2 \quad \text{and} \quad \begin{cases}
m_n^2 = \sum_{m=0}^{n} \frac{\delta_m(\mu)}{k^{2m}}, \\
(m_n^2)' = \sum_{m=0}^{n} (m + 1) \frac{\delta_m(\mu)}{k^{2m}}, \\
\delta_m(\mu) = (2m)! \gamma_m(\mu), \\
\gamma_m(\mu) := \prod_{s=1}^{m} (\mu - (2s - 1)^2).
\end{cases}
\]

Define
\[
a_{m,n} := \delta_m((2n + 1)^2) = \frac{(2m)! (n + m)!}{(m!)^2 4^m (n - m)!}.
\]

With the function
\[
\rho_n(k) := \frac{\sum_{m=0}^{n} a_{m,n}}{k^{2m}}(m + 1) \frac{a_{m,n}}{k^{2m}},
\]

we estimate
\[
\frac{k}{|z_n(k) + 1|} = \left| \frac{km_n^2}{m_n^2 - (m_n^2)' + ki} \right| \leq \sqrt{2k} \sum_{m=0}^{n} \frac{a_{m,n}}{k^{2m}} \leq \sqrt{2k} \sum_{m=0}^{n} \frac{a_{m,n}}{k^{2m}} + \sum_{m=1}^{n} m \frac{a_{m,n}}{k^{2m}}
\]

\[
\leq 2\sqrt{2k} \rho_n(k) \leq 2\sqrt{2k} \left( \frac{k^2 + \beta}{k^2 + C_n \beta} \right).
\]

The ansatz (A.3) is equivalent to
\[
\left( k^2 + C_n \beta \right) \sum_{m=0}^{n} \frac{a_{m,n}}{k^{2m}} \leq \left( k^2 + \beta \right) \sum_{m=0}^{n} (m + 1) \frac{a_{m,n}}{k^{2m}},
\]

which, by multiplying out and rearranging terms, is equivalent to
\[
k^2 a_{0,n} + \sum_{m=0}^{n-1} (a_{m+1,n} + C_n \beta a_{m,n}) \frac{1}{k^{2m}} + \frac{C_n \beta a_{n,n}}{k^{2n}}
\]

\[
\leq k^2 a_{0,n} + \sum_{m=0}^{n-1} (m + 2) a_{m+1,n} + \beta (m + 1) a_{m,n} \frac{1}{k^{2m}} + \beta (n + 1) \frac{a_{n,n}}{k^{2n}}.
\]
Hence, we have to stipulate
\[
(a_{m+1,n} + C_n \beta a_{m,n}) \leq (m + 2) a_{m+1,n} + \beta (m + 1) a_{m,n}, \quad m = 0, \ldots, n - 1,
\]
\[C_n \leq n + 1.
\]
We select \(C_n := (n + 1)\) and insert this in the left-hand side of the first condition to obtain
\[
0 \leq (m + 1) a_{m+1,n} + \beta (m + 1 - (n + 1)) a_{m,n}.
\]
Inserting the definitions of \(a_{m,n}\) leads to
\[
0 \leq (m + 1) \frac{(2m + 2)! (n + m + 1)!}{((m + 1)!)^2 4^{m+1} (n - (m + 1))!} + \beta (m + 1 - (n + 1)) \frac{(2m)! (n + m)!}{(m!)^2 4^m (n - m)!},
\]
This is equivalent to
\[
\beta (n - m) \frac{(2m)! (n + m)!}{(m!)^2 4^m (n - m)!} \leq (m + 1) \frac{(2m + 2)! (n + m + 1)!}{((m + 1)!)^2 4^{m+1} (n - (m + 1))!},
\]
which in turn leads to the condition
\[
\beta \leq (m + 1) \frac{(2m + 1) (2m + 2) (n + m + 1)}{(m + 1)^2 4} = \left(m + \frac{1}{2}\right) (n + m + 1), \quad m = 0, \ldots, n - 1.
\]
We select \(\beta = \frac{n + 1}{2}\), which finally leads to
\[
\frac{k}{|z_n(k) + 1|} \leq 2 \sqrt{2} k \left(\frac{2k^2 + n + 1}{2k^2 + (n + 1)}\right) \leq \begin{cases} 2 \sqrt{2} k & \forall n \in \mathbb{N}_0 \\ 2 \sqrt{2} \left(\frac{2}{\lambda} + 1\right) \frac{k}{(n + 1)} & n + 1 > \lambda k^2. \end{cases}
\]

The proof of the third estimate in (5.9) is more technical and is the assertion of the next lemma.

Lemma A.2 For every \(\lambda_0 > 1\), there is \(C_0 > 0\) depending only on \(\lambda_0\) such that
\[
\frac{n + 1}{|z_n(k) + 1|} \leq C_0 \quad \forall n \geq \lambda_0 k.
\]

Proof Recall the definition of the function \(\rho_n\) in (A.1). We will prove
\[
(n + 1)\rho_n (k) \leq \tilde{C}_0 \quad \forall n \geq \lambda_0 k
\]
from which the statement follows in view of (A.2).
1. Step: We claim that $\rho_n$ is monotone increasing with respect to $k$. To see this, we compute

$$
\rho'_n(k) = -\left( \sum_{m=0}^{n} (m+1) \frac{a_{m,n}}{k^{2m}} \right) \left( \sum_{m=0}^{n} 2m \frac{a_{m,n}}{k^{2m+1}} \right) + \sum_{m=0}^{n} \frac{a_{m,n}}{k^{2m}} \sum_{m=0}^{n} (2m) (m+1) \frac{a_{m,n}}{k^{2m+1}} \left( \sum_{m=0}^{n} (m+1) \frac{a_{m,n}}{k^{2m}} \right)^2.
$$

Thus, it is sufficient to prove that the numerator (denoted by $d_n(k)$) is positive. We write

$$
d_n(k) = 2 \sum_{m=0}^{n} \sum_{\ell=0}^{n} \ell (\ell - m) \frac{a_{\ell,n}a_{m,n}}{k^{2m+2\ell+1}}.
$$

We now exploit the fact that the coefficients $a_{\ell,n}$ are nonnegative. The double sum on the right-hand side can be interpreted as a quadratic form. Note that we have, for vectors $x$ and matrices $B$,

$$
2x^T B x = x^T (B^T + B) x \geq 0
$$

if the vector $x$ has nonnegative entries and the symmetric part $(B^T + B)/2$ of the matrix $B$ has nonnegative entries. For $B_{\ell,m} := \ell (\ell - m)$, we compute

$$
B_{\ell,m} + B_{m,\ell} = (\ell - m)^2 \geq 0.
$$

2. Step: The monotonicity of $\rho_n$ shown in Step 1 implies for $n \geq \lambda_0 k$

$$
\rho_n(k) \leq \rho_n(n/\lambda_0) = \frac{\sum_{m=0}^{n} a_{m,n} \left( \frac{\lambda_0}{n} \right)^{2m}}{\sum_{m=0}^{n} (m+1) a_{m,n} \left( \frac{\lambda_0}{n} \right)^{2m}} =: \rho_n^1.
$$

We next show that the dominant contribution to the sums in (A.4) arises from few coefficients with index $m$ close to $n\sqrt{1 - \lambda_0^{-2}}$. To that end, we analyze the coefficients $a_{m,n}$ with Stirling’s formula in the form

$$
\sqrt{2\pi} \exp \left( \frac{1}{12} \right) \sqrt{n+1} \left( \frac{n}{e} \right)^n \geq n! \geq \sqrt{n+1} \left( \frac{n}{e} \right)^n.
$$
Upon setting \( C_1 := 2\pi \exp(1/6) \) and \( C_2 := (2\pi)^{-3/2} \exp(-1/4) \), we get

\[
a_{m,n} = \frac{(2m)! (n+m)!}{(m!)^2 4^m (n-m)!} \leq C_1 \frac{\sqrt{n+m+1}}{\sqrt{n-m+1}} \frac{\sqrt{2m+1}}{m+1} \frac{(n+m)^{n+m}}{(n-m)^{n-m} e^{2m}} \quad \text{(A.5)}
\]

\[
\leq 2C_1 \frac{(n+m)^{n+m}}{(n-m)^{n-m} e^{2m}}. \quad \text{(A.6)}
\]

\[
a_{m,n} \geq C_2 \frac{\sqrt{2m+1}}{\sqrt{n+m+1}} \frac{(n+m)^{n+m}}{(n-m)^{n-m} e^{2m}}. \quad \text{(A.7)}
\]

The dominant contribution of \( a_{m,n}(\lambda_0/n)^{2m} \) is

\[
b_{m,n} := \frac{(n+m)^{n+m}}{(n-m)^{n-m} e^{2m}} \left( \frac{\lambda_0^2}{n} \right)^{2m}.
\]

The maximum of \( m \mapsto b_{m,n} \) in the interval \([0, n] \subset \mathbb{R}\) is attained at \( \tilde{m} = n\mu_0 \) with \( \mu_0 = \sqrt{1 - \lambda_0^{-2}} \) and value

\[
\tilde{b}_n = c_{\mu_0}^n \quad \text{with} \quad c_{\mu_0} = \frac{(1 + \mu_0)^{1+\mu_0}}{(1 - \mu_0)^{1-\mu_0}} \left( \frac{\lambda_0}{e} \right)^{2\mu_0}.
\]

We also introduce the factor

\[
f_{m,n} := \frac{\sqrt{n+m+1}}{\sqrt{n-m+1}} \frac{\sqrt{2m+1}}{m+1},
\]

so as to be able to describe \( a_{m,n}(\lambda_0/n)^{2m} \sim f_{m,n}b_{m,n} \) uniformly in \( m, n \).

**Case 1** We consider the range

\[
0 \leq n \leq \max \left\{ \frac{2}{1 - \mu_0}, \frac{2}{\mu_0}, \frac{\lambda_0^2}{\mu_0 c_0}, c_5 \right\},
\]

where the parameter \( c_0 \) is given by Lemma A.3 (with \( \lambda = \lambda_0 \) there) and \( c_5 \) is defined in (A.14); both constants depend solely on \( \lambda_0 \). This is a finite set so

\[
\sup_{0 \leq n \leq \max \left\{ \frac{2}{1 - \mu_0}, \frac{2}{\mu_0}, \frac{\lambda_0^2}{\mu_0 c_0}, c_5 \right\} } (n+1) \rho_1^n =: \tilde{C}_1 < \infty
\]

depends solely on \( \lambda_0 \).

**Case 2** We assume

\[
n \geq \max \left\{ \frac{2}{1 - \mu_0}, \frac{2}{\mu_0}, \frac{\lambda_0^2}{\mu_0 c_0}, c_5 \right\}.
\]

(A.8)
We split the summations \( \sum_{m=0}^{n} \) in the representation of \( \rho_n^I \) (cf. (A.4)) as \( S_n^I + S_n^II \) with

\[
S_n^I := \sum_{n \delta_0 \leq m \leq n} \frac{a_{m,n}}{b_n} \left( \frac{\lambda_0}{n} \right)^{2m}, \quad S_n^II := \sum_{0 \leq m < n \delta_0} \frac{a_{m,n}}{b_n} \left( \frac{\lambda_0}{n} \right)^{2m},
\]

where

\[
\tilde{\delta}_0 := \mu^3_0. \tag{A.9}
\]

In view of

\[
\min \left\{ m + 1 : m \geq n \tilde{\delta}_0 \right\} \geq 1 + n \tilde{\delta}_0,
\]

we have

\[
\rho_n^I \leq \frac{S_n^I + S_n^II}{(1 + n \tilde{\delta}_0) S_n^I} =: \rho_n^II.
\]

In order to estimate the terms \( S_n^I, S_n^II \), we have to investigate the behavior of \( a_{m,n} \left( \frac{\lambda_0}{n} \right)^{2m} / b_n \) depending on the distance of \( m \) from \( \tilde{m} \). We write \( m = n \mu_0 (1 + \varepsilon) \) for some \( \varepsilon \in \mathbb{R} \) that satisfies \( 0 < \mu_0 (1 + \varepsilon) < 1 \). This gives

\[
C_2 f_{m,n} \left( \gamma_{\lambda_0}(\varepsilon) \right)^n \leq \frac{a_{m,n}}{b_n} \left( \frac{\lambda_0}{n} \right)^{2m} \leq 2C_1 \left( \gamma_{\lambda_0}(\varepsilon) \right)^n
\]

with \( \gamma_{\lambda_0}(\varepsilon) := \frac{(1 + \mu_0 (1 + \varepsilon))^{1+\mu_0(1+\varepsilon)} (1 - \mu_0)^{1-\mu_0} \left( \frac{\lambda_0}{\varepsilon} \right)^{2\mu_0 \varepsilon}}{(1 - \mu_0 (1 + \varepsilon))^{1-\mu_0(1+\varepsilon)} (1 + \mu_0)^{1+\mu_0} \left( \frac{\lambda_0}{\varepsilon} \right)^{2\mu_0}} \). \( \tag{A.10} \)

**Estimate of \( S_n^I \):** The dominant contribution in the numerator of \( \rho_n^II \) will be seen to be \( S_n^I \), for which we therefore need a lower bound. Our strategy is to estimate this sum by a single summand, namely the summand corresponding to an integer \( m \) close to \( \tilde{m} = n \mu_0 \). For \( m \in \{ \lfloor n \mu_0 \rfloor, \lceil n \mu_0 \rceil \} \), we have

\[
m - n \mu_0 = n \mu_0 \varepsilon_m \text{ with } \varepsilon_m \in \left\{ \frac{-n \mu_0 - \lfloor n \mu_0 \rfloor}{n \mu_0}, \frac{\lfloor n \mu_0 \rfloor - n \mu_0}{n \mu_0} \right\}.
\]

For these two values of \( m \) (in fact, we will only need the one with \( m \leq \mu_0 n \)), we have \( m = n \mu_0 (1 + \varepsilon_m) \) with \( |\varepsilon_m| \leq (n \mu_0)^{-1} \) and (cf. (A.8))

\[
\frac{\mu_0}{2} n \leq n \mu_0 - 1 \leq m \leq n \mu_0 + 1 \leq \frac{1 + \mu_0}{2} n, \tag{A.11}
\]

\[
\lambda_0^2 |\varepsilon_m| \leq \frac{\lambda_0^2}{n \mu_0} \leq c_0. \tag{A.12}
\]
The estimates (A.11), (A.12) make Lemma A.3 applicable, which gives
\[ 1 \geq \gamma_{\lambda_\theta}(\varepsilon_m) \geq 1 - c_2\lambda_0^2\varepsilon_m^2 \geq 1 - c_2 c_0 |\varepsilon_m| \geq 1 - \frac{c_6}{n} \text{ with } c_6 = \frac{c_2 c_0}{\mu_0}. \] (A.13)

The estimate (A.11) leads to two-sided bounds for \( f_{m,n} \):
\[
\begin{align*}
    f_{m,n} &\leq \frac{2n + 1}{n\mu_0/2} \frac{n \geq 1}{\sqrt{n(1 - \mu_0)/2}} \frac{6\sqrt{2}}{\mu_0\sqrt{1 - \mu_0}} n^{-1/2} =: c_7 n^{-1/2}, \\
    f_{m,n} &\geq \frac{\sqrt{n + \mu_0^n} \sqrt{\mu_0^n}}{\sqrt{n - \mu_0^n (1 + \mu_0)n}} =: c_8 n^{-1/2}.
\end{align*}
\]

Define \( c_5 > 0 \) such that, with \( c_0 \) given by Lemma A.3,
\[
n \geq c_5 \implies \left( 1 - \frac{c_6}{n} \right)^n \geq \frac{1}{2} e^{-c_6} \text{ and } \frac{\lambda_0^2}{n\mu_0} \leq c_0 \right) \] (A.14)

This leads to
\[
S_{II}^n \geq C_2 f_{m,n} \left( \gamma_{\lambda_\theta}(\varepsilon_m) \right)^n \geq C_2 c_8 n^{-1/2} \left( 1 - \frac{c_6}{n} \right)^n \geq \frac{1}{2} C_2 c_8 e^{-c_6} n^{-1/2}.
\] (A.15)

**Estimate of** \( S_{II}^n \). Let \( c_0 \in (0, 1) \) be the constant in Lemma A.3. (Note that we may assume, without loss of generally, \( c_0 < 1 \).) Upon writing \( m \in \{0, \ldots, \lfloor n \delta_0 \rfloor \} \) in the form \( m = \mu_0 n (1 + \varepsilon_m) \), we find in view of \( \delta_0 = \mu_0^3 \) that \( |\varepsilon_m| \geq \lambda_0^{-2} \). Hence, the monotonicity properties of the function \( \gamma_{\lambda_\theta} \) of Lemma A.3 imply
\[
\gamma_{\lambda_\theta}(\varepsilon_m) \leq 1 - \frac{c_2}{c_0} \lambda_0^{-2}.
\] (A.16)

We therefore get
\[
S_{II}^n = \sum_{0 \leq m \leq \lfloor \delta_0 n \rfloor} \frac{a_{m,n}}{b_n} \left( \frac{\lambda_0}{n} \right)^{2m} \leq 2C_1 \sum_{0 \leq m \leq \lfloor \delta_0 n \rfloor} \left( 1 - \frac{c_2}{c_0} \lambda_0^{-2} \right)^n \leq 2C_1 (n + 1) \left( 1 - \frac{c_2}{c_0} \lambda_0^{-2} \right)^n.
\] (A.17)

The combination of (A.17) and (A.15) shows \( S_{II}^n \leq C S_{II}^n \) for some constant \( C > 0 \) that depends solely on \( \lambda_0 \). This concludes the proof. \( \Box \)

**Lemma A.3** For \( \lambda > 1 \) and \( \mu := \sqrt{1 - \lambda^{-2}} \), introduce the function
\[
(-1, \mu^{-1} - 1) \ni \varepsilon \mapsto \gamma_\lambda(\varepsilon) := \frac{(1 + \mu (1 + \varepsilon))^{1+\mu(1+\varepsilon)} (1 - \mu)^{1-\mu} (\lambda \varepsilon)}{(1 - \mu (1 + \varepsilon))^{1-\mu(1+\varepsilon)} (1 + \mu)^{1+\mu}}.
\]
Let $\lambda_0 > 1$. Then there are constants $c_0$, $c_1$, $c_2 > 0$ depending solely on $\lambda_0$ such that the following holds for every $\lambda \geq \lambda_0$: For every $\varepsilon$ satisfying

$$|\varepsilon|\lambda^2 \leq c_0$$

the function $\gamma_\lambda$ satisfies

$$1 - c_1\lambda^2\varepsilon^2 \leq \gamma_\lambda(\varepsilon) \leq 1 - c_2\lambda^2\varepsilon^2. \quad (A.19)$$

Furthermore, the function $\gamma_\lambda$ is monotone increasing on $(-1, 0)$ and monotone decreasing on $(0, \mu^{-1} - 1)$. In particular, therefore,

$$0 < \gamma_\lambda(\varepsilon) \leq 1 - \frac{c_2}{c_0}\lambda^{-2} \quad \forall \varepsilon \in (-1, \mu^{-1} - 1) \setminus (-c_0\lambda^{-2}, c_0\lambda^{-2}). \quad (A.20)$$

**Proof** Define the function

$$g_\lambda(\varepsilon) := \ln \left( (1 - \mu^2(1 + \varepsilon)^2)\lambda^2 \right), \quad (A.21)$$

and observe

$$g'_\lambda(\varepsilon) = -2\mu^2 \frac{1 + \varepsilon}{1 - \mu^2(1 + \varepsilon)^2}, \quad g''_\lambda(\varepsilon) = -2\mu^2 \frac{1 + \mu^2(\varepsilon + 1)^2}{(1 - \mu^2(1 + \varepsilon)^2)^2}, \quad (A.22)$$

$$\gamma'_\lambda = \mu\gamma_\lambda g_\lambda, \quad \gamma''_\lambda = \mu\gamma_\lambda \left( \mu g^2_\lambda + g'_\lambda \right), \quad \gamma'''_\lambda = \mu\gamma_\lambda \left( \mu^2 g^3_\lambda + 3\mu g_\lambda g'_\lambda + g''_\lambda \right). \quad (A.23)$$

1. **Step:** (monotonicity properties of $\gamma_\lambda$) The function $\gamma_\lambda$ is defined in the interval $(-1, \mu^{-1} - 1)$.

   **Claim:** $\gamma_\lambda$ is strictly increasing on $(-1, 0)$, strictly decreasing on $(0, \mu^{-1} - 1)$ and thus has a proper maximum at $\varepsilon = 0$. To see these monotonicity properties, we note that $\gamma_\lambda \geq 0$ and that $g_\lambda(\varepsilon) > 0$ for $\varepsilon < 0$ and $g_\lambda(\varepsilon) < 0$ for $\varepsilon > 0$. We calculate

$$\gamma_\lambda(0) = 1, \quad \gamma'_\lambda(0) = 0, \quad \gamma''_\lambda(0) = -2(\lambda^2 - 1)^{3/2}\lambda^{-1}. \quad (A.24)$$

2. **Step:** Use $\mu = \sqrt{1 - \lambda^{-2}}$ to write

$$\left( 1 - \mu^2(1 + \varepsilon)^2 \right)\lambda^2 = 1 - (\lambda^2 - 1)(2\varepsilon + \varepsilon^2). \quad (A.25)$$

Fix $q \in (0, 1)$ and consider $\varepsilon$ satisfying

$$0 < \mu(1 + \varepsilon) < 1 \quad \text{and} \quad (\lambda^2 - 1)|2\varepsilon + \varepsilon^2| \leq q < 1. \quad (A.26)$$

From (A.25) and (A.26), we infer

$$(1 - q)\lambda^{-2} \leq 1 - \mu^2(1 + \varepsilon)^2 \leq (1 + q)\lambda^{-2}.$$
This, together with $0 < \mu (1 + \varepsilon) < 1$ and $\mu \in (0, 1)$ implies

$$
|g_\lambda(\varepsilon)| \leq \max\{ |\ln(1 - q)|, |\ln(1 + q)| \}, \quad |g_\lambda'(\varepsilon)| \leq \frac{2\lambda^2}{1 - q},
$$

$$
|g_\lambda''(\varepsilon)| \leq \frac{4\lambda^4}{(1 - q)^2}. \quad (A.27)
$$

Taylor’s theorem now implies for every $\varepsilon$ satisfying (A.26) the existence of an $\varepsilon'$ in the interval $(0, \varepsilon)$ such that

$$
\gamma_\lambda(\varepsilon) = \gamma_\lambda(0) + \gamma_\lambda'(0)\varepsilon + \frac{1}{2}\gamma_\lambda''(0)\varepsilon^2 + \frac{1}{3!}\gamma_\lambda'''(\varepsilon')\varepsilon^3 = 1 - 2(\lambda^2 - 1)^{3/2}\lambda^{-1}\varepsilon^2 + \frac{1}{3!}\gamma_\lambda'''(\varepsilon')\varepsilon^3. \quad (A.28)
$$

The remainder term $\gamma_\lambda'''(\varepsilon')$ is estimated using (A.27) as follows (note that $\gamma_\lambda \geq 0$ and has maximum 1):

$$
|\gamma_\lambda'''(\varepsilon')| \leq \max\{ |\ln(1 + q)|, |\ln(1 - q)| \}^3
$$

$$
+ 6\lambda^2 \max\{ |\ln(1 + q)|, |\ln(1 - q)| \}(1 - q)^{-1} + 4\lambda^4 (1 - q)^{-2}
$$

$$
\leq C_1 \lambda^4
$$

for a constant $C_1$ that depends solely on $\lambda_0 > 1$ and the chosen $q$. Finally, there are constants $C_2, C_3 > 0$ depending solely on $\lambda_0 > 1$ such that

$$
C_2 \lambda^2 \leq 2(\lambda^2 - 1)^{3/2}\lambda^{-1} \leq C_3 \lambda^2. \quad (A.29)
$$

We conclude for $\varepsilon$ satisfying (A.26)

$$
1 - C_2 \lambda^2 \varepsilon^2 - \frac{C_1}{3!} \lambda^4 \varepsilon^3 \leq \gamma_\lambda(\varepsilon) \leq 1 - C_3 \lambda^2 \varepsilon^2 + \frac{C_1}{3!} \lambda^4 \varepsilon^3.
$$

The two-sided bound (A.19) now follows if we assume (A.18) for $c_0$ sufficiently small so that the terms $\lambda^4 \varepsilon^3$ are small compared to the terms involving $\lambda^2 \varepsilon^2$. We note that the condition (A.18) for sufficiently small $c_0$ also implies (A.26). Finally, the estimate (A.20) is a consequence of (A.19) and the monotonicity properties of $\gamma_\lambda$. \quad \Box

**B Equivalence of $\| \cdot \|_{H^1(\Omega)}$ and $\| \cdot \|_{\text{curl}, \mathcal{Q}, 1}$ in $V_0$ and $V_0^*$**

The spaces $V_0$ and $V_0^*$ as in (4.21) involve the capacity operator (cf. Lemma 4.10). For the case that $\Gamma$ is the surface of the ball, they are subspaces of $H^1(\Omega)$ as shown in the following lemma. In contrast to Lemma 4.12, we obtain $k$-explicit bounds for the norm estimates.
Lemma B.1 Let $\Omega = B_1(0)$ and let $V_0$, $V_0^*$ be defined as in (4.21). Then, $V_0 \cup V_0^* \subset H^1(\Omega)$ and

$$\|u\|_{H^1(\Omega)} \leq \|u\|_{\text{curl}, \Omega, 1} \quad \forall u \in V_0 \cup V_0^*. \quad (B.1)$$

i.e., the constant $C_{\Omega, k}$ in Lemma 4.12 equals 1 for $\Omega = B_1(0)$.

Proof The inclusion $V_0 \cup V_0^* \subset H^1(\Omega)$ follows from Lemma 4.12, and it remains to prove the norm estimates. Let $u \in V_0$. Then, from [47, (2.5.151), Lemma 5.4.2] we have

$$\langle \nabla u, \nabla v \rangle = \langle \text{curl} u, \text{curl} v \rangle - \langle \text{div} u, \text{div} v \rangle - \langle \text{div} \Gamma u_T, \langle v, n \rangle \rangle - \langle \langle u, n \rangle, \langle v, n \rangle \rangle - \langle u_T, \langle v, n \rangle \rangle - \langle u, \langle v, n \rangle \rangle. \quad (4.27)$$

We choose $v = u$ and employ (4.27) to obtain after rearranging terms

$$\|\nabla u\|^2 = \|\text{curl} u\|^2 - 2 \text{Re} \langle \text{div} \Gamma u_T, \langle u, n \rangle \rangle - 2 \|\langle u, n \rangle\|^2 - \|\langle u_T, \langle v, n \rangle \rangle\|^2 \leq \|\text{curl} u\|^2 + \frac{2}{k} \text{Im} \langle \text{div} \Gamma T_k u_T, \text{div} \Gamma \langle u_T, \langle v, n \rangle \rangle \rangle - 2 \|\langle u, n \rangle\|^2 - \|\langle u_T, \langle v, n \rangle \rangle\|^2. \quad (B.2)$$

From [47, (5.3.91), (5.3.93)], we conclude that

$$\langle \text{div} \Gamma T_k u_T, \text{div} \Gamma \langle u_T, \langle v, n \rangle \rangle \rangle = \sum_{\ell=1}^{\infty} \sum_{m \in \ell} i \ell^2 (\ell + 1)^2 \frac{k}{z_{\ell}(k) + 1} |U^m_{\ell}|^2.$$

Since

$$\text{Im} \left( \frac{i}{z_{\ell}(k) + 1} \right) = \frac{\text{Im} \left( \frac{1}{z_{\ell}(k) + 1} \right)}{|z_{\ell}(k) + 1|^2} = \frac{\text{Re} \left( z_{\ell}(k) + 1 \right)}{|z_{\ell}(k) + 1|^2} \leq 0,$$

the second summand in (B.2) is non-positive so that $\|\nabla u\| \leq \|\text{curl} u\|$. This implies the first estimate in (B.1). The statement about $u \in V_0^*$ is simply a repetition of these arguments. \qed

C Vector Spherical Harmonics

For $x \in \mathbb{R}^3$, $r = \|x\|$, and $\hat{x} := x/r$, we introduce the vectorial spherical harmonics (VSH) as in [30, Thm. 2.46] (with a different scaling)

$$Y_{\ell}^m(\hat{x}) := \hat{x} Y_{\ell}^m(\hat{x}), \quad U_{\ell}^m(\hat{x}) := \nabla_{\hat{x}} Y_{\ell}^m(\hat{x}), \quad V_{\ell}^m(\hat{x}) := \nabla_{\hat{x}} Y_{\ell}^m(\hat{x}) \times \hat{x}.$$
From [30, Thm. 5.36] we conclude that any \( \mathbf{u} \in \mathbf{X} \) has an expansion of the form

\[
\mathbf{u}(r\hat{x}) = \sum_{\ell=0}^{\infty} \sum_{m \in \mathbb{N}} (u^m_{\ell}(r)Y^m_{\ell}(\hat{x}) + v^m_{\ell}(r)U^m_{\ell}(\hat{x}) + w^m_{\ell}(r)V^m_{\ell}(\hat{x})). \tag{C.1}
\]

We use the relations (cf. [30, p. 271])

\[
\text{curl}(u^m_{\ell}(r)Y^m_{\ell}(\hat{x})) = \frac{u^m_{\ell}(r)}{r} \mathbf{V}^m_{\ell}(\hat{x}), \quad \text{curl}(v^m_{\ell}(r)U^m_{\ell}(\hat{x})) = -\frac{1}{r} (r v^m_{\ell}(r))' \mathbf{V}^m_{\ell}(\hat{x}), \quad \text{curl}(w^m_{\ell}(r)V^m_{\ell}(\hat{x})) = \frac{1}{r} (r w^m_{\ell}(r))' \mathbf{U}^m_{\ell}(\hat{x}) + w^m_{\ell}(r) \frac{\ell (\ell + 1)}{r} \mathbf{Y}^m_{\ell}(\hat{x}),
\]

so that \( \text{curl} \ \mathbf{u} \) is given by

\[
\text{curl} \ \mathbf{u}(r\hat{x}) = \sum_{\ell=0}^{\infty} \sum_{m \in \mathbb{N}} \frac{1}{r} \left[ (u^m_{\ell}(r) - (rv^m_{\ell}(r))') \mathbf{V}^m_{\ell}(\hat{x}) + (rw^m_{\ell}(r))' \mathbf{U}^m_{\ell}(\hat{x}) + w^m_{\ell}(r) \frac{\ell (\ell + 1)}{r} \mathbf{Y}^m_{\ell}(\hat{x}) \right].
\]

Using the orthogonality relations of the vectorial spherical harmonics, we get

\[
\|\mathbf{u}\|^2 = \sum_{\ell=0}^{\infty} \sum_{m \in \mathbb{N}} \int_{\mathbb{R}} r^2 \left( |u^m_{\ell}(r)|^2 + \ell (\ell + 1) \left( |v^m_{\ell}(r)|^2 + |w^m_{\ell}(r)|^2 \right) \right) dr., \tag{C.2}
\]

\[
\|\text{curl} \ \mathbf{u}\|^2 = \sum_{m \in \mathbb{N}} \int_{\mathbb{R}} \ell (\ell + 1) \left( \sum_{\ell=0}^{\infty} |u^m_{\ell}(r) - (rv^m_{\ell}(r))'|^2 + |rw^m_{\ell}(r))'|^2 + \ell (\ell + 1) |w^m_{\ell}(r)|^2 \right) dr. \tag{C.3}
\]

For \( a > 0 \), we introduce the operators \( L^\text{VSH}_a : \mathbf{X} \to \mathbf{X} \) and \( H^\text{VSH}_a : \mathbf{X} \to \mathbf{X} \) for functions \( \mathbf{u} \) as in (C.1) by

\[
L^\text{VSH}_a \mathbf{u} = \sum_{\ell:0 \leq \ell \leq a} \sum_{m \in \mathbb{N}} (u^m_{\ell}(r)Y^m_{\ell}(\hat{x}) + v^m_{\ell}(r)U^m_{\ell}(\hat{x}) + w^m_{\ell}(r)V^m_{\ell}(\hat{x})), \tag{C.4a}
\]

\[
H^\text{VSH}_a \mathbf{u} = \mathbf{u} - L^\text{VSH}_a \mathbf{u}. \tag{C.4b}
\]

From (C.2), (C.3) we conclude the stability of the splitting

\[
\|L^\text{VSH}_a \mathbf{u}\| \leq \|\mathbf{u}\|, \quad \|\text{curl} \ L^\text{VSH}_a \mathbf{u}\| \leq \|\text{curl} \ \mathbf{u}\|, \tag{C.5}
\]

\[
\|H^\text{VSH}_a \mathbf{u}\| \leq \|\mathbf{u}\|, \quad \|\text{curl} \ H^\text{VSH}_a \mathbf{u}\| \leq \|\text{curl} \ \mathbf{u}\|.
\]

In addition, the splitting is orthogonal:

\[
\left(L^\text{VSH}_a \mathbf{u}, H^\text{VSH}_a \mathbf{u}\right) = \left(\text{curl} \ L^\text{VSH}_a \mathbf{u}, \text{curl} \ H^\text{VSH}_a \mathbf{u}\right) = 0.
\]

\footnote{There is a sign error in the second last relation on [30, p.271].}
Note that on the unit sphere, we have
\[
\Pi T Y_m^\ell (\hat{x}) = \hat{x} \times (Y_m^\ell (\hat{x}) \hat{x} \times \hat{x}) = Y_m^\ell (\hat{x})
\]
\[
\Pi T U_m^\ell (\hat{x}) = \hat{x} \times (\nabla \Gamma Y_m^\ell (\hat{x}) \hat{x} \times \hat{x}) = \nabla \Gamma Y_m^\ell (\hat{x}) = T_m^\ell
\]
where \(T_m^\ell\) is as in [47, (2.4.173)]. Hence, the application of the trace map \(\Pi T\) yields
\[
u_T = \Pi_T u = \sum_{\ell=0}^{\infty} \sum_{m \in \ell} (v_m^\ell \nabla \Gamma Y_m^\ell + w_m^\ell T_m^\ell),\tag{C.6}
\]
where \(v_m^\ell = v_m^\ell (1), w_m^\ell := w_m^\ell (1)\). A key observation for the case of the unit sphere is that for any \(u \in X\), the function \(L_{VSH}^{\lambda k} u\) satisfies \(\Pi_T L_{VSH}^{\lambda k} u = L_{\Gamma} \Pi_T u\), where \(L_{\Gamma}\) was introduced in Definition 4.2.

**Lemma C.1** Let \(\Omega = B_1(0)\) and \(L_{\Omega}\) be as in Definition 4.2. Then: \(\Pi_T L_{\Omega}^{VSH} = L_{\Gamma} \Pi_T\) and
\[
\|L_{\Omega} u\|_{\text{curl}, \Omega, k} \leq \|u\|_{\text{curl}, \Omega, k}\quad \forall u \in X.
\]
Furthermore, the stability constants in (4.6) satisfy \(C_{L, \Omega}^{k} \leq 1\) and \(C_{H, \Omega}^{k} \leq 2\).

**Proof** Since \(L_{\Omega}\) is the minimum norm extension (cf. Definition 4.2) the bound (C.5) lead to
\[
\|L_{\Omega} u\|_{\text{curl}, \Omega, k}^2 \leq \|L_{\lambda k}^{VSH} u\|_{\text{curl}, \Omega, k}^2 = k^2 \|L_{\lambda k}^{VSH} u\|^2 + \|\text{curl} L_{\lambda k}^{VSH} u\|^2 \leq k^2 \|u\|^2 + \|\text{curl} u\|^2
\]
\[
\leq \|u\|_{\text{curl}, \Omega, k}^2.
\]

\[
\square
\]

**D Analytic Regularity of Maxwell and Maxwell-Like Problems**

**D.1 Local Smoothness**

Consider for a bounded Lipschitz domain \(\omega \subset \mathbb{R}^3\)

\[
\text{curl} (A(x) \text{curl} u) = f \quad \text{in } \omega, \quad \text{div} (B(x) u) = g \quad \text{in } \omega, \quad \Pi_T u = 0 \quad \text{on } \partial \omega.
\]

We have smoothness of \(u\) under regularity assumptions on the right-hand sides:

**Lemma D.1** Let \(\partial \omega\) be a smooth bounded Lipschitz domain that is star-shaped with respect to a ball. Let \(A, B \in C^\infty (\overline{\omega})\) be pointwise symmetric positive definite. Then:
(i) If \( u \in H_0(\omega, \text{curl}) \) and \( \text{div}(Bu) \in L^2(\omega) \), then \( u \in H^1(\omega) \) with
\[
\|u\|_{H^1(\omega)} \leq C \left( \|\text{div}(Bu)\|_{L^2(\omega)} + \|\text{curl}u\|_{L^2(\omega)} \right).
\]

(ii) If \( u \in H_0(\omega, \text{curl}) \) satisfies (D.1) for some \( f \in H^s(\omega), g \in H^{s+1}(\omega) \), \( s \in \mathbb{N}_0 \), then \( u \in H^{s+2}(\omega) \) and
\[
\|u\|_{H^{s+2}(\omega)} \leq C_s \left[ \|f\|_{H^s(\omega)} + \|g\|_{H^{s+1}(\omega)} \right].
\]

**Proof** We use the right inverse \( R \text{curl} \) of the curl-operator and use its mapping properties due to [17] as formulated in [39, Lemma 6.4]; specifically, we employ \( R \text{curl} : H^s(\omega) \to H^{s+1}(\omega) \) for any \( s \in \mathbb{N}_0 \). We will also repeatedly use decompositions formulated in [39, Lemma 6.5], i.e., for \( s \in \mathbb{N}_0 \) and \( v \in H^s(\omega, \text{curl}) \) there is \( \varphi \in H^{s+1}(\omega) \) such that
\[
v = \nabla \varphi + R \text{curl}(\text{curl} v). \tag{D.2}
\]

**Proof of (i):** Using (D.2), we write
\[
u = \nabla \varphi + R \text{curl}(\text{curl} u). \tag{D.3}
\]
The mapping property \( R \text{curl} : L^2(\omega) \to H^1(\omega) \) implies \( R \text{curl}(\text{curl} u) \in H^1(\omega) \). Using \( \Pi_T u = 0 \), we infer \( \nabla_{\partial\omega} \varphi = -\Pi_T R \text{curl}(\text{curl} u) \in H^{1/2}(\partial\omega) \) so that, by the smoothness of \( \partial\omega \), we have \( g_D := \varphi|_{\partial\omega} \in H^{3/2}(\partial\omega) \). Multiplying (D.3) by \( B \) and applying the divergence reveal that \( \varphi \) solves
\[
g = \text{div}(Bu) = \text{div}(B\nabla \varphi) + \text{div}(BR \text{curl}(\text{curl} u)) \quad \text{in } \omega, \quad \varphi = g_D \quad \text{on } \partial\omega. \tag{D.4}
\]
This is a standard Poisson-type problem for \( \varphi \), and the smoothness of \( \partial\omega \) and \( B \) then implies \( \varphi \in H^2(\omega) \) with
\[
\|\varphi\|_{H^2(\omega)} \lesssim \|g - \text{div}(BR \text{curl}(\text{curl} u))\|_{L^2(\omega)} + \|g_D\|_{H^{3/2}(\partial\omega)}
\lesssim \|g\|_{L^2(\omega)} + \|\text{curl}u\|_{L^2(\omega)}. \tag{D.5}
\]

**Proof of (ii):** We set \( w := \text{curl} u \) and note
\[
\text{div } w = 0, \quad n \cdot w = n \cdot \text{curl} u = \text{curl}_{\partial\omega} \Pi_T u = 0. \tag{D.6}
\]

1. **Step:** From (D.2) we see that we can we write, for some \( \varphi \in H^1(\omega) \),
\[
Aw = \nabla \varphi + R \text{curl}(\text{curl}(Aw)) = \nabla \varphi + R \text{curl}(f). \tag{D.1a}
\]
Hence, \( w = A^{-1} (R \text{curl}(f) + \nabla \varphi) \), and we get from (D.6) that \( \varphi \) satisfies
\[
-\text{div} \left( A^{-1} \nabla \varphi \right) = \text{div} \left( A^{-1} R \text{curl}(f) \right) \quad \text{in } \omega.
\]
\[ n \cdot A^{-1} \nabla \varphi = -n \cdot A^{-1} R^{\text{curl}}(f) \text{ on } \partial \omega. \]  

(D.8)

The mapping properties of \( R^{\text{curl}} : H^s(\omega) \to H^{s+1}(\omega) \) give \( R^{\text{curl}}(f) \in H^{s+1}(\omega) \) so that the scalar shift theorem for Poisson-type problems gives in fact \( \varphi \in H^{s+2}(\omega) \) with \( \|\varphi\|_{H^{s+2}(\omega)} \leq C \|f\|_{H^{s}(\omega)} \). Inserting this regularity information in (D.7) provides \( w \in H^{s+1}(\omega) \) with

\[ \|w\|_{H^{s+1}(\omega)} \leq C \|f\|_{H^{s}(\omega)}. \]  

(D.9)

2. Step: From (i) we have \( u \in H^1(\omega) \) and from the first step we get \( \text{curl } u \in H^{s+1}(\omega) \). In particular, \( u \in H^1(\omega, \text{curl}) \). Hence, (D.2) allows us to write, for some \( \varphi \in H^2(\omega) \),

\[ u = \nabla \varphi + R^{\text{curl}}( \text{curl } u \big|_{H^{s+1}(\omega)}). \]  

(D.10)

3. Step: An equation for \( \varphi \) is obtained in two steps: using \( \Pi_T u = 0 \), we see again that \( \nabla \partial_\omega \varphi = -\Pi_T R^{\text{curl}}(\text{curl } u) \in H^{s+3/2}(\partial \omega) \), where we used the trace estimate and the mapping properties of \( R^{\text{curl}} \). We conclude \( g_D := \varphi|_{\partial \omega} \in H^{s+5/2}(\partial \omega) \). Multiplying (D.10) with \( B \) and applying the divergence operator reveal a Poisson-type problem for \( \varphi \):

\[ g = \text{div } Bu = \text{div } (B \nabla \varphi) + \text{div } (BR^{\text{curl}}(\text{curl } u)) \text{ in } \omega, \quad \varphi = g_D \text{ on } \partial \omega. \]  

(D.11)

Standard elliptic regularity and the smoothness of \( \partial \omega \) and \( B \) give \( \varphi \in H^{s+3}(\omega) \) with

\[ \|\varphi\|_{H^{s+3}(\omega)} \lesssim \|g - \text{div } (BR^{\text{curl}}(\text{curl } u))\|_{H^{s+1}(\omega)} + \|g_D\|_{H^{s+5/2}(\partial \omega)} \]
\[ \lesssim \|g\|_{H^{s+1}(\omega)} + \|\text{curl } u\|_{H^{s+1}(\omega)} \lesssim \|g\|_{H^{s+1}(\omega)} + \|f\|_{H^{s}(\omega)}. \]  

(D.12)

4. Step: Inserting the information (D.12) in (D.10) implies \( u \in H^{s+2}(\omega) \) together with the bound \( \|u\|_{H^{s+2}(\omega)} \lesssim \|g\|_{H^{s+1}(\omega)} + \|f\|_{H^{s}(\omega)} \).

\[ \square \]

D.2 Local Analytic Regularity

We show analytic regularity of solutions of elliptic systems of the form (D.13) on half-balls \( B^+_R := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x| < r, x_3 > 0\} \). We denote \( \Gamma_R := \{x \in B_R(0) \mid x_3 = 0\} \).

On \( B^+_R \) with \( R \leq 1 \) we consider smooth functions \( u \) that satisfy the following equations for some \( \varepsilon > 0 \):

\[ -\varepsilon^2 \sum_{\alpha, \beta, i=1}^{3} \partial_\alpha \left( A^{ij}_{\alpha \beta} \partial_\beta u_j \right) + \varepsilon \sum_{\beta, j=1}^{3} B^{ij}_\beta \partial_\beta u_j + \sum_{j=1}^{3} C^{ij} u_j = f_i, \quad i = 1, 2, 3, \]

(D.13a)
\( \mathbf{u}_1 = \mathbf{u}_2 = 0 \) on \( \Gamma_R \), \hspace{1cm} (D.13b)

\[
\partial_3 \mathbf{u}_3 = \epsilon^{-1} (G + b \mathbf{u}_3) + \sum_{j=1}^{3} d_j \partial_j \mathbf{u}_3 + \sum_{j=1}^{3} e_j \partial_3 \mathbf{u}_j \quad \text{on} \quad \Gamma_R.
\] \hspace{1cm} (D.13c)

We assume that the coefficients are analytic, i.e., (cf. Definition 2.5)

\[
(A_{ij}^{\alpha\beta})_{i,j,\alpha,\beta} \in \mathcal{A}^\infty(C_A, \gamma_A, B_R^+), \quad (B_{ij}^{\alpha})_{i,j,\alpha} \in \mathcal{A}^\infty(C_B, \gamma_B, B_R^+),
\]

\[
(C^{ij})_{i,j} \in \mathcal{A}^\infty(C_C, \gamma_C, B_R^+), \quad b \in \mathcal{A}^\infty(C_b, \gamma_b, B_R^+),
\]

\[
(d_j)_{j} \in \mathcal{A}^\infty(C_d, \gamma_d, B_R^+), \quad (e_j)_{j} \in \mathcal{A}^\infty(C_e, \gamma_e, B_R^+);
\] \hspace{1cm} (D.14a)

here, we have written, e.g., \((d_j)_{j}\) to emphasize that the objects are tensor-valued and the multi-index notation is understood as in (2.26). Concerning the tensor \(A_{ij}^{\alpha\beta}\) and the coefficients \(d_j, e_j\) we will furthermore make the following structural assumption:

\[
A_{ij}^{\alpha\beta}(0) = \delta_{ij} \delta_{\alpha\beta}, \quad d_j(0) = 0, \quad e_j(0) = 0.
\] \hspace{1cm} (D.15)

This structural assumption implies that the leading-order differential operator in (D.13) reduces to a block Laplace operator at the origin and that the boundary conditions for the third component \(u_3\) reduce to Neumann boundary conditions. In other words, the system decouples at the origin. Hence, for sufficiently small \(R\), we can reduce the regularity analysis of the system to that of scalar problems, and this is the avenue taken in the remainder of this appendix.

**Remark D.2** The structural assumption on \(A_{ij}^{\alpha\beta}\) implies the “very strong ellipticity”/Legendre condition for the leading-order differential operator (near the origin). No sign conditions are imposed on the coefficients \(B_{ij}^{\alpha}, C^{ij}, b_j, d_j\), which could even be complex. The condition \(\epsilon > 0\) can always be enforced by a scaling so that \textit{mutatis mutandis} the ensuing theory is also valid for complex \(\varepsilon\). \(\square\)

It is convenient to introduce \(\mathcal{E} \in (0, 1]\) by

\[
\mathcal{E}^{-1} := \frac{C_B}{\varepsilon} + \frac{\sqrt{C_C}}{\varepsilon} + \frac{C_b}{\varepsilon} + 1,
\] \hspace{1cm} (D.16)

which implies the estimates

\[
\frac{C_C}{\varepsilon^2} \leq \mathcal{E}^{-2}, \quad \frac{C_B}{\varepsilon} \leq \mathcal{E}^{-1}, \quad \frac{C_b}{\varepsilon} \leq \mathcal{E}^{-1}, \quad \frac{\mathcal{E}}{\varepsilon} \leq \frac{1}{C_B + \sqrt{C_C} + C_b}.
\] \hspace{1cm} (D.17)

We will make the following assumptions on the right-hand sides

\[
\| \nabla^p f \|_{L^2(B_R)} \leq C_f \gamma_f^p \max\{p/R, \mathcal{E}^{-1}\}^p \quad \forall p \in \mathbb{N}_0, \quad (D.18a)
\]

\[
\| \nabla^p G \|_{L^2(B_R)} \leq C_G \gamma_G^p \max\{p/R, \mathcal{E}^{-1}\}^p \quad \forall p \in \mathbb{N}_0. \quad (D.18b)
\]
Given the special role of the variable \( x_3 \), we will interchangeably use the notation \( x = (x, y) \) with \( x = (x_1, x_2) \) and \( y = x_3 \). Analytic regularity of the solution of (D.13) will be characterized in Theorem D.5 by the following seminorms:

\[
N'_{R,p,q}(u) = \frac{1}{[p+q]!} \sup_{R/2 \leq r < R} (R-r)^{p+q+2} \| \partial_y^{q+2} \partial_x^p v \|_{L^2(B^+_r)},
\]

\[ p \geq 0, \quad q \geq -2. \quad (D.19)\]

Our procedure to control \( N'_{R,p,q}(u) \) is the standard one by first controlling tangential derivatives and then using the differential equation to control normal derivatives. We follow [35, Sec. 5.5]. In the proofs, we implicitly assume that the solution \( u \in C^\infty(B^+_R) \). This could be proved by carefully arguing with the difference quotient method or, alternatively, by asserting the smoothness of the solution by a separate argument. (This is how we proceed in the present application of Theorem D.5.)

### D.2.1 Control of Tangential Derivatives

We introduce the following auxiliary notation suitable for controlling tangential derivatives (cf. [35, Sec. 5.5])

\[ [p] := \max\{1, p\}, \quad (D.20)\]

\[
M'_{R,p}(v) = \frac{1}{[p]!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \| \partial_x^p v \|_{L^2(B^+_r)},
\]

\[ p \geq 0. \quad (D.21)\]

\[
N'_{R,p}(v) = \begin{cases} 
\frac{1}{[p]!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \| \partial^2_x \partial_x^p v \|_{L^2(B^+_r)} & \text{if } p \geq 0 \\
\frac{1}{[p]!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \| \partial^2_x v \|_{L^2(B^+_r)} & \text{if } p = -2, -1,
\end{cases}
\]

\[ p \geq 0. \quad (D.22)\]

\[
H_{R,p}(v) := \frac{1}{[p-1]!} \sup_{R/2 \leq r < R} \left( R-r \right)^{p+1} \left\| \partial_x^p v \right\|_{L^2(B^+_r)} + \frac{R-r}{[p]} \left\| \partial_x^p \partial_y v \right\|_{L^2(B^+_r)}.
\]

\[ p \geq 0. \quad (D.23)\]

**Lemma D.3** There exists a universal constant \( C_I > 0 \) such that for \( f, G \) sufficiently smooth, there holds:

(i) Let \( u \) solve \( -\Delta u = f \) on \( B^+_R \) and \( u|_{\Gamma_R} = 0 \). Then:

\[
N'_{R,p}(u) \leq C_I \left[ M'_{R,p}(f) + N'_{R,p-1}(u) + N'_{R,p-2}(u) \right] \quad \forall p \geq 0. \quad (D.24)
\]

For \( p = 0 \), we have the sharper estimate \( N'_{R,0}(u) \leq C_I \left[ M'_{R,0}(f) + N'_{R,-1}(u) \right]. \)

(ii) Let \( u \) solve \( -\Delta u = f \) on \( B^+_R \) and \( \partial_y u|_{\Gamma_R} = G \). Then

\[
N'_{R,p}(u) \leq C_I \left[ M'_{R,p}(f) + H_{R,p}(G) + N'_{R,p-1}(u) + N'_{R,p-2}(u) \right] \quad \forall p \geq 0.
\]

\[ (D.25)\]
For $p = 0$, we have the sharper estimate $N'_{R,0}(u) \leq C_I \left[ M'_{R,0}(f) + H_{R,0}(G) + N'_{R,-1}(u) \right]$.  

**Proof** For the proof of (i), see [35, Lemma 5.5.15] or [45, Lemma 5.7.3']. Statement (ii) is essentially taken from [35, Lemma 5.5.23]. The special cases $p = 0$ follow from the general case and the first Poincaré inequality in the case (i) and the second Poincaré inequality in the case (ii). □

**Lemma D.4** Let $u$ satisfy (D.13) with coefficients and data satisfying (D.14), (D.15), and (D.18). Let $C_I$ be given by Lemma D.3. Let $R \leq 1$ be such that

$$3C_I \left( C_{A\gamma_A} + C_d\gamma_d + C_e\gamma_e \right) R \leq \frac{1}{2}. \quad \text{(D.26)}$$

Then there is $K > 1$ depending only on the constants appearing in (D.14) and on $\gamma_f, \gamma_G$ such that

$$N'_{R,p}(u) \leq C_u K^{p+2} \max \left\{ R/\varepsilon, p + 3 \right\}^{p+2}, \quad p \geq -1,$$

$$C_u = \min \{ 1, R/\varepsilon \} (1 + \varepsilon C_{A\gamma_A}) \varepsilon \| \nabla u \|_{L^2(B_R^+)}$$

$$+ \min \{ 1, R/\varepsilon \} (\varepsilon/\varepsilon)^2 \left[ C_f + C_c \| u \|_{L^2(B_R^+)} \right]$$

$$+ C_G (1 + \gamma_G) \min \{ 1, R/\varepsilon \} (\varepsilon/\varepsilon)$$

$$+ C_b (1 + \gamma_b R) \min \{ 1, R/\varepsilon \} (\varepsilon/\varepsilon) \| u \|_{L^2(B_R^+)}$$

$$+ C_b \min \{ 1, R/\varepsilon \} (\varepsilon/\varepsilon) \| \nabla u \|_{L^2(B_R^+)}$$

$$+ (C_d\gamma_d + C_e\gamma_e) \min \{ 1, R/\varepsilon \} (\varepsilon/\varepsilon) \| \nabla u \|_{L^2(B_R^+)}.$$

\text{(D.27)}

**Proof** We start with the observation

$$\min \{ 1, R/\varepsilon \} \max \{ 1, R/\varepsilon \} = R/\varepsilon. \quad \text{(D.28)}$$

The proof is by induction on $p$, and we will employ Lemma D.3. To that end, recall $A_{ij}^{ij}(0) = \delta_{ij}$ from (D.15). We write (D.13) as

$$-\Delta u_i = \varepsilon^{-2} f_i - \varepsilon^{-2} \sum_{j=1}^{3} C^{ij} u_j - \varepsilon^{-1} \sum_{\beta,j=1}^{3} \tilde{B}_i^{ij} \partial_\beta u_j + \sum_{\alpha,\beta,j=1}^{3} \left( A_{ij}^{ij} - A_{ij}^{ij}(0) \right) \partial_\alpha \partial_\beta u_j,$$

\text{(D.29a)}

$$u_1 = u_2 = 0 \quad \text{on } \Gamma,$$

\text{(D.29b)}

$$\partial_3 u_3 = \varepsilon^{-1} (G + b u_3) + \sum_{j=1}^{3} d_j \partial_j u_3 + \sum_{j=1}^{3} e_j \partial_3 u_j \quad \text{on } \Gamma,$$

\text{(D.29c)
where the coefficient
\[
\tilde{B}^{ij}_{\beta} := B^{ij}_{\beta} + \varepsilon \sum_{\alpha=1}^{3} \partial_\alpha \left( A^{ij}_{\alpha\beta} - A^{ij}_{\alpha\beta}(0) \right) = B^{ij}_{\beta} + \varepsilon \sum_{\alpha=1}^{3} \partial_\alpha A^{ij}_{\alpha\beta}
\]
is again an analytic function with \((\tilde{B}^{ij})_{i,j,\beta} \in A^\infty(C_{\tilde{B}}, \gamma_{\tilde{B}}, B_R^+)\) with \(C_{\tilde{B}} := C_B + C_A \gamma_A \varepsilon\) and \(\gamma_{\tilde{B}} := \gamma_B + 2\gamma_A\). (Note: \(C_B \gamma_B^p + \varepsilon C_A \gamma_A^{p+1}(p + 1) \leq C_B \gamma_B^p + \varepsilon C_A \gamma_A(2\gamma_A)^p\)). The system (D.29) is of the form analyzed in Lemma D.3. We therefore get
\[
N'_{R,p}(u) \leq C_{\varepsilon} \left[ \varepsilon^{-2} M_{R,p}'(f) + \sum_{i=1}^{3} M_{R,0}' \left( \varepsilon^{-2} \sum_{j} C^{ij} u_j + \varepsilon^{-1} \sum_{p,j} \tilde{B}^{ij}_{\beta} \partial_\beta u_j - \sum_{\alpha,\beta,j} \tilde{A}^{ij}_{\alpha\beta} \partial_\alpha \partial_\beta u_j \right) + \varepsilon^{-1} H_{R,p}(G) + \varepsilon^{-1} H_{R,p}(b u_3) + H_{R,p}(\sum_{j=1}^{3} d_j \partial_j u_3) + H_{R,p}(\sum_{j=1}^{3} e_j \partial_3 u_j) + N'_{R,p-1}(u) + N'_{R,p-2}(u) \right]. \quad \text{(D.30)}
\]

1. Step: For \(p = -1\), the assertion (D.27) follows directly from \(K \geq 1\), the definition of \(C_u\), and (D.28) since
\[
N'_{R,-1}(u) \leq R \|\nabla u\|_{L^2(B^+_R)} \quad \text{(D.28)} \equiv \min \{1, R/\mathcal{E}\} \|\nabla u\|_{L^2(B^+_R)} \max \{1, R/\mathcal{E}\}. \quad \text{(D.31)}
\]

2. Step: For \(p = 0\), we employ the sharpened versions of Lemma D.3, which leads to (D.30) for \(p = 0\) where the last term, \(N_{p-2}(u)\), is dropped. In view of (D.15), we have
\[
\| (d_j)\|_{L^\infty(B^+_R)} \leq C d \gamma d R, \quad \| (e_j)\|_{L^\infty(B^+_R)} \leq C e \gamma e R, \quad \| (\tilde{A}^{ij}_{\alpha\beta})_{i,j,\alpha,\beta}\|_{L^\infty(B^+_R)} \leq C A \gamma A R. \quad \text{(D.32)}
\]

We estimate with the sharpened version of Lemma D.3:
\[
N'_{R,0}(u) \leq C I \left[ \left\{ \varepsilon^{-2} M_{R,0}'(f) + \sum_{i=1}^{3} M_{R,0}' \left( \varepsilon^{-2} \sum_{j} C^{ij} u_j + \varepsilon^{-1} \sum_{p,j} \tilde{B}^{ij}_{\beta} \partial_\beta u_j - \sum_{\alpha,\beta,j} \tilde{A}^{ij}_{\alpha\beta} \partial_\alpha \partial_\beta u_j \right) + \varepsilon^{-1} H_{R,0}(G) + \varepsilon^{-1} H_{R,0}(b u_3) + H_{R,0}(\sum_{j=1}^{3} d_j \partial_j u_3) + H_{R,0}(\sum_{j=1}^{3} e_j \partial_3 u_j) + N'_{R,-1}(u) \right] \right]
\[
\leq 3C I \left[ (R/2)^2 \varepsilon^{-2} C_f + (R/2)^2 \varepsilon^{-2} C_C \|u\|_{L^2(B^+_R)} + C_B (R/2)^2 \varepsilon^{-1} \|\nabla u\|_{L^2(B^+_R)} \right.
\[
+ C_A \gamma A R N'_{R,0}(u) + C_G R/2 \varepsilon^{-1} + C_G \gamma G (R/2)^2 \varepsilon^{-1} \max \{1/R, \varepsilon^{-1}\}
\]
\[
+ C_b (1 + \gamma_B R) R/2 \varepsilon^{-1} \|u\|_{L^2(B^+_R)} + C_b (R/2)^2 \varepsilon^{-1} \|\nabla u\|_{L^2(B^+_R)} + 3C d \gamma d (R/2)^2 \|\nabla u\|_{L^2(B^+_R)} \right].
\]
\[\square \text{ Springer} \]
The condition (D.26) allows us to absorb the three terms \( C_{AY} R N'_{R,0}(u) \), \( C_d y_d R N'_{R,0}(u) \), and \( C_{e} y_e R N'_{R,0}(u) \) of the right-hand side in the left-hand side at the expense of a factor 2. We next use (D.28), the trivial estimates \( 1 \leq \max\{1, R/\varepsilon\} \), \( \min\{1, R/\varepsilon\} \leq \min\{1, R/\varepsilon\} \), the observation \( (\varepsilon/\varepsilon) C_B = (\varepsilon/\varepsilon) C_B + C_{AY} A \varepsilon \leq 1 + C_{AY} A \varepsilon \) (by (D.17)), and (D.31) to see that we have arrived at \( N'_{R,0}(u) \leq C C_u \max\{1, R/\varepsilon\}^2 \). Thus, by selecting \( K \) sufficiently large, we have shown the case \( p = 0 \).

3. Step: For \( p \geq 1 \), we proceed by induction, assuming that (D.27) is valid up to \( p - 1 \). Then

\[
N'_{R, p-q}(u) \leq C_u K^{p+2-q} \frac{\max\{p+3, R/\varepsilon\}^{p+2}}{p!}, \quad q = 1, \ldots, p+1.
\]

(D.33)

We need to estimate the terms in (D.30). To bound the terms \( M'_{R, p}(\sum_{j, \beta} \tilde{B}^{ij}_{\beta} \partial_{\beta} u) \) in terms of \( N'_{R, p-1}(u) \), it is useful to note the simple facts (cf. also [45, (5.7.19)])

\[
|\nabla \nabla^p u|^2 \leq |\nabla \nabla^{p-1} u|^2, \quad p \geq 1, \quad |\nabla \nabla^p u|^2 = |\nabla u|^2, \quad p = 0.
\]

(D.34)

To estimate these terms, we compute (cf. [35, Lemma 5.5.13] for similar calculations) with (D.32) for the third estimate:

\[
\varepsilon^{-2} M'_{R, p} \left( (\sum_j C^{ij} u_j) \right) \leq C_c \frac{\gamma_c}{2} \sum_{q=0}^{p} \left( \gamma_c R/\varepsilon \right)^q \left( R/\varepsilon \right)^{2} \frac{[p-q-2]!}{(p-q)!} N'_{R, p-q+2}(u),
\]

\[
\varepsilon^{-1} M'_{R, p} \left( (\sum_{j, \beta} \tilde{B}^{ij}_{\beta} \partial_{\beta} u_j) \right) \leq C_{\tilde{B}} \frac{1}{2} \sum_{q=0}^{p} \left( \gamma_{\tilde{B}} \right)^q \left( R/\varepsilon \right)^{2} \frac{[p-q-1]!}{(p-q)!} N'_{R, p-q+1}(u),
\]
\[
M'_{R,p} \left( \sum_{j, \alpha, \beta} \tilde{A}_{ij}^{\alpha \beta} \partial_\alpha \partial_\beta u_j \right)_{ij} \]
\[
\leq C_A Y_A R N'_{R,p}(u) + C_A \sum_{q=1}^{p} \left( \frac{\gamma_A R}{2} \right)^q N'_{R,p-q}(u),
\]
\[
\varepsilon^{-1} \frac{1}{[p-1]!} \sup_{R/2 \leq r < R} (R - r)^{p+1} \left\| \nabla_x^p (b u_3) \right\|_{L^2(B^+)} \leq \frac{C_b R[p]}{\varepsilon} \sum_{q=0}^{p} \left( \frac{\gamma_b R}{2} \right)^q \frac{[p - q - 2]!}{(p-q)!} N'_{R,p-q-2}(u_3),
\]
\[
\varepsilon^{-1} \frac{1}{[p]!} \sup_{R/2 \leq r < R} (R - r)^{p+2} \left\| \nabla_x^p \nabla (b u_3) \right\|_{L^2(B^+)} \leq \frac{C_b R(p+1)}{\varepsilon} \sum_{q=0}^{p+1} \left( \frac{\gamma_b R}{2} \right)^q \frac{[p - q - 1]!}{(p-q+1)!} N'_{R,p-q-1}(u_3),
\]
\[
\frac{1}{[p-1]!} \sup_{R/2 \leq r < R} (R - r)^{p+1} \left\| \nabla_x^3 \left( \sum_{j=1}^{3} d_j \partial_j u_3 \right) \right\|_{L^2(B^+)} \leq \frac{C_d \gamma_d R N'_{R,p-1}(u_3)}{\gamma_d} + C_d \sum_{q=1}^{p} \left( \frac{\gamma_d R}{2} \right)^q \frac{[p]}{[p-q]} N'_{R,p-q-1}(u_3),
\]
\[
\frac{1}{[p]!} \sup_{R/2 \leq r < R} (R - r)^{p+2} \left\| \nabla_x^3 \nabla \left( \sum_{j=1}^{3} d_j \partial_j u_3 \right) \right\|_{L^2(B^+)} \leq \frac{C_d R \gamma_d N'_{R,p}(u_3)}{\gamma_d} + C_d \sum_{q=1}^{p+1} \left( \frac{\gamma_d R}{2} \right)^q \frac{p+1}{[p-q+1]} N'_{R,p-q}(u_3),
\]
\[
\frac{1}{[p-1]!} \sup_{R/2 \leq r < R} (R - r)^{p+1} \left\| \nabla_x^3 \left( \sum_{j=1}^{3} e_j \partial_3 u_j \right) \right\|_{L^2(B^+)} \leq \frac{C_e \gamma_e R N'_{R,p-1}(u)}{\gamma_e} + C_e \sum_{q=1}^{p} \left( \frac{\gamma_e R}{2} \right)^q \frac{[p]}{[p-q]} N'_{R,p-q-1}(u),
\]
\[
\frac{1}{[p]!} \sup_{R/2 \leq r < R} (R - r)^{p+2} \left\| \nabla_x^3 \nabla \left( \sum_{j=1}^{3} e_j \partial_3 u_j \right) \right\|_{L^2(B^+)} \leq \frac{C_e R \gamma_e N'_{R,p}(u)}{\gamma_e} + C_e \sum_{q=1}^{p+1} \left( \frac{\gamma_e R}{2} \right)^q \frac{p+1}{[p-q+1]} N'_{R,p-q}(u).
\]

We choose

\[
K > \max\{1, \gamma_f/2, \gamma_G/2, \gamma_A/2, \gamma_B/2, \gamma_C/2, \gamma_b/2, \gamma_d/2, \gamma_e/2\}
\]
such that the expression in brackets \([\cdots]\) in (D.38) is smaller than \(1/(6C_f)\) (and, of course, such that the case \(p = 0\) is proved); note that \(E/\varepsilon\) is controlled in view of (D.17). The calculation in [35, p. 206, bottom] for \(M'_{R,p}(f)\) and similar calculations for \(H_{R,p}(G)\) give

\[
\varepsilon^{-2} M'_{R,p}(f) \leq C_f \min(1, R/E)^2 (E/\varepsilon)^2 K^{p+2} \frac{\max\{p + 3, R/E\}^{p+2}}{p!} \frac{1}{4} K^{-2} \left( \frac{\gamma_f}{2K} \right)^p,
\]

(D.35)

\[
\varepsilon^{-1} H_{R,p}(G) \leq C_G \min(1, R/E)^{(E/\varepsilon)} K^{p+2} \frac{\max\{p + 3, R/E\}^{p+2}}{p!} \times \left[ \frac{1}{2} K^{-2} \left( \frac{\gamma G}{2K} \right)^p + \frac{1}{2K} \left( \frac{\gamma G}{2K} \right)^{p+1} \right].
\]

(D.36)

We use the induction assumption (D.33) and (D.28) (to deal with the cases where \(N'_{R,-2}(u)\) is involved) to estimate

\[
R^2 \varepsilon^{-2} \frac{[p - q - 2]!}{(p - q)!} N'_{R,p-q-2}(u)
\]

\[
\leq (E/\varepsilon)^2 C_u K^{p+2-q-2} \frac{\max\{p + 3, R/E\}^{p+2}}{p!}, \quad q = 0, \ldots, p,
\]

\[
\varepsilon^{-1} \frac{[p - q - 1]!}{(p - q)!} N'_{R,p-q-1}(u)
\]

\[
\leq (E/\varepsilon) C_u K^{p+2-q-1} \frac{\max\{p + 3, R/E\}^{p+2}}{p!}, \quad q = 0, \ldots, p,
\]

\[
[p] R/\varepsilon \frac{[p - q - 2]!}{(p - q)!} N'_{R,p-q-2}(u)
\]

\[
\leq (E/\varepsilon) C_u K^{p+2-q-2} \frac{\max\{p + 3, R/E\}^{p+2}}{p!}, \quad q = 0, \ldots, p,
\]

\[
\frac{(p + 1) R}{\varepsilon} \frac{[p - q - 1]!}{(p - q + 1)!} N'_{R,p-q-1}(u)
\]

\[
\leq \frac{\varepsilon}{C_u} K^{p+2-q-1} \frac{\max\{p + 3, R/E\}^{p+2}}{p!}, \quad q = 0, \ldots, p + 1,
\]

\[
\frac{[p]}{[p - q]} N'_{R,p-q-1}(u)
\]

\[
\leq C_u K^{p+2-q-1} \frac{\max\{p + 3, R/E\}^{p+2}}{p!}, \quad q = 1, \ldots, p,
\]

\[
\frac{[p + 1]}{[p - q + 1]} N'_{R,p-q}(u)
\]

\[
\leq C_u K^{p+2-q} \frac{\max\{p + 3, R/E\}^{p+2}}{p!}, \quad q = 1, \ldots, p + 1,
\]
We note that (D.17) gives
\[ C_C(\mathcal{E}/\varepsilon)^2 \leq 1, \quad C_{\overline{B}}(\mathcal{E}/\varepsilon) \leq 1 + C_A \gamma \mathcal{E}, \quad C_b(\mathcal{E}/\varepsilon) \leq 1. \] (D.37)

Inserting all of the above in (D.30) yields\(^{10}\) together with the geometric series
\[ N'_{R, p}(u) \leq 3C_I (C_A \gamma^A R + C_d \gamma^d R + C_e \gamma^e R) N'_{R, p}(u) + C_u K^{p+2} \max\{p, 3, R/\mathcal{E}\}^{p+2} \]
\[ \times 3C_I \left[ \frac{1}{4K} \left( \frac{\gamma_f}{2K} \right)^p + \frac{1}{2} \gamma^{-2} \left( \frac{\gamma_G}{2K} \right)^p + \frac{1}{4K^2} \frac{1}{1 - (\gamma_r R/(2K))} \right] \]
\[ + K^{-1} \frac{1 + C_A \gamma^A \mathcal{E}}{1 - R \gamma^B/(2K)} + \frac{C_A \gamma^A R}{2K} \frac{1}{1 - \gamma^A R/(2K)} + \frac{1}{2K^2} \frac{1}{1 - \gamma^B R/(2K)} \]
\[ + \frac{1}{2K} \frac{1}{1 - \gamma^B R/(2K)} + (1 + K^{-1}) \frac{C_d \gamma^d R}{2K} \frac{1}{1 - \gamma^d R/(2K)} \]
\[ + (1 + K^{-1}) \frac{C_e \gamma^e R}{2K} \frac{1}{1 - \gamma^e R/(2K)} + K^{-1} + K^{-2} \]. (D.38)

By the choice of \( K \), the expression in brackets, \([ \cdots ]\), is smaller than \((6C_I)^{-1}\) and by (D.26) the expression \(3C_I (C_A \gamma^A R + C_d \gamma^d R + C_e \gamma^e R) \leq 1/2\). Hence, the induction step is completed. \(\square\)

D.2.2 Control of Normal Derivatives

Recall \( N'_{R, p, q} \) from (D.19) and define
\[ M'_{R, p, q}(v) := \frac{1}{[p + q]!} \sup_{R/2 < r < R} (R - r)^{p+q+2} \| \partial_y^q \nabla_x^p v \|_{L^2(B_r^+)} \]. (D.39)

**Theorem D.5** Let \( u \) satisfy (D.13) with coefficients and data satisfying (D.14), (D.15), and (D.18). Let \( R \leq 1 \) be such that, with the universal constant \( C_I \) given by Lemma D.3,
\[ (3C_I + 6) (C_A \gamma^A + C_d \gamma^d + C_e \gamma^e) R \leq \frac{1}{2}. \]
Then there are \( K_1, K_2 > 1 \) depending only on the constants appearing in (D.14) and on \( \gamma_f, \gamma_G \) such that with \( C_u \) given by (D.27)
\[ N'_{R, p, q}(u) \leq C_u K_1^{p+2} K_2^{q+2} \max\{R/\mathcal{E}, p + q + 3\}^{p+q+2} \]
\[ \forall p \geq 0, q \geq -2 \text{ with } (p, q) \neq (0, -2). \] (D.40)

**Proof** Let \( K \) be given by Lemma D.4. We select \( K_1 \geq K, K_2 \) such that
\[ K_2 > \max\{1, \gamma_f/2, \gamma_A/2, \gamma_B/2, \gamma_C/2\} \]
\(^{10}\) the factor 3 in \( 3C_I \) is due to the summation over \( i \) and likely suboptimal.
and such that the expression in brackets, \([\cdots]\), in \((D.46)\) is smaller than \(1/12\). This is indeed possible by first selecting \(K_1 \geq K\) sufficiently large (e.g., such that 

\[ C_A \gamma_A/(2K_1)/(1 - (\gamma_A/(2K_1))) < 1/12 \]

and then selecting \(K_2\) sufficiently large. 

The proof is by induction on \(q\). For \(q \in \{-2, -1, 0\}\) and all \(p \in \mathbb{N}_0\) (with the exception of the excluded case \((q, p) = (-2, 0)\)), the result follows directly from Lemma \(D.4\).

Let us assume that \((D.40)\) holds (for all \(p \in \mathbb{N}_0\)) up to \(q - 1\) for some \(q \geq 1\).

Starting point is the observation that for a smooth solution \(\tilde{\mathbf{u}}\)

\[ -\partial_y^2 \tilde{\mathbf{u}} = \Delta \tilde{\mathbf{u}} + \tilde{\mathbf{f}} \quad \text{on} \quad B_R^+, \quad \text{(D.41)} \]

we have by the definition of the seminorms \(N'_{R, r, s}, M'_{R, r, s}\) the estimate

\[ N'_{R, p, q}(\tilde{\mathbf{u}}) \leq 2 \left[ N'_{R, p+2, q-2}(\tilde{\mathbf{u}}) + M'_{p, q}(\tilde{\mathbf{f}}) \right], \quad p \geq 0, \quad q \geq 0. \quad \text{(D.42)} \]

The system \((D.29)\) is of the form \((D.41)\) with

\[ \tilde{\mathbf{f}} = \sum_{j=1}^{3} \tilde{A}_{33}^{ij} \partial_y^2 \mathbf{u}_j + \varepsilon^{-2} \tilde{\mathbf{f}} - \varepsilon^{-2} \sum_j C^{ij} \mathbf{u}_j - \varepsilon^{-1} \sum_{j, \beta} \tilde{B}_{ij}^\beta \partial_\beta \mathbf{u}_j + \sum_{j, \alpha, \beta \neq (3,3)} \tilde{A}_{ij}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{u}_j. \quad \text{(D.43)} \]

We estimate

\[
\varepsilon^{-2} M'_{R, p, q}(\tilde{\mathbf{f}}) \\
\leq \frac{C_f}{4} \min \{ 1, R/\varepsilon \} (\varepsilon/\varepsilon)^2 \left( \frac{2}{\gamma_f} \right)^{p+q} \max \{ p + q + 3, R/\varepsilon \}^{p+q+2} \frac{\max \{ p + q + 3, R/\varepsilon \}^{p+q+2}}{\| \mathbf{u} \|_{L^2(\Omega)}}.
\]

\[
\varepsilon^{-2} M'_{R, p, q} \left( \left( \sum_j C^{ij} \mathbf{u}_j \right)_i \right) \\
\leq C \varepsilon^{-2} \sum_{r=0}^{p} \sum_{s=0}^{q} \left( \frac{q}{r} \right) \left( \frac{q}{s} \right) \sup_{R/2 \leq \rho < R} \frac{\rho^{r+s}}{(p+q)!} \| \mathbf{u} \|_{L^2(\Omega)}.
\]

\[
\varepsilon^{-1} M'_{R, p, q} \left( \left( \sum_{j, \beta} \tilde{B}_{ij}^\beta \partial_\beta \mathbf{u}_j \right)_i \right) \\
\leq \frac{C}{2} \sum_{r=0}^{p} \sum_{s=0}^{q} \left( \frac{q}{r} \right) \left( \frac{q}{s} \right) \frac{\gamma_r^{r+s}}{(p+q)!} \sup_{R/2 \leq \rho < R} \frac{\rho^{r+s}}{(p+q)!} \| \mathbf{u} \|_{L^2(\Omega)}.
\]

\[
M'_{R, p, q} \left( \left( \sum_{j, \alpha, \beta \neq (3,3)} \tilde{A}_{ij}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{u}_j \right)_i \right) \\
\leq \left[ N'_{R, p-r, q-s-1}(\mathbf{u}) + N'_{R, p+1-r, q-s-2}(\mathbf{u}) \right],
\]

\[
M'_{R, p, q} \left( \left( \sum_{j, \alpha, \beta \neq (3,3)} \tilde{A}_{ij}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{u}_j \right)_i \right).
\]
\[
\frac{1}{2} \sum_{r=0}^{p} \sum_{s=0}^{q} \binom{p}{r} \binom{q}{s} \left( \frac{\gamma R}{2} \right)^{r+s} \frac{s!r!([p + q - r - s]!}{(p+q)!} N'_{R_p+2-r,q-s-2}(u) + N'_{R_p+1-r,q-s-1}(u),
\]
\[
M'_{R,p,q}(\sum_j \tilde{A}^j_{3,s} a^j_2 u_i),
\]
\[
\leq C_A \gamma \sum_{(r,s) \neq (0,0)} \binom{p}{r} \binom{q}{s} \left( \frac{\gamma R}{2} \right)^{r+s} \frac{s!r!([p + q - r - s]!}{(p+q)!} N'_{R_p+2-r,q-s-2}(u).
\]

We introduce the abbreviation
\[
m_{p,q} := \max\{p + q + 3, R/\mathcal{E}\}^{p+q+2},
\]
Then, for \( q' \leq q - 1, \) the induction hypothesis reads \([p' + q']!N'_{R,p',q}(u) \leq C_uK_1^{p'+2}K_2^{q'+2}m_{p',q}' . \]
We have
\[
\varepsilon^{-2}M'_{R,p,q}(f) \leq \left[ \min\{1, R/\mathcal{E}\}^2 \frac{C_f (\mathcal{E}/\varepsilon)^2}{4K_1^2K_2^2} \left( \frac{\gamma f}{2K_1} \right)^p \left( \frac{\gamma f}{K_2} \right)^q \right] \left[ K_1^{p+2}K_2^{q+2} \frac{m_{p,q}}{(p+q)!} \right].
\]
We recall the elementary estimates
\[
\binom{p}{r} r! \leq p^r, \quad 0 \leq r \leq p.
\]
Recalling (D.17) (cf. also (D.37)), we get from the induction hypothesis
\[
\varepsilon^{-2}M'_{R,p,q}(\sum_j C^{ij} u_j)
\]
\[
\leq C_u \frac{C_C (\mathcal{E}/\varepsilon)^2}{4} \sum_{r,s} \binom{p}{r} \binom{q}{s} \left( \frac{\gamma C}{2} \right)^{r+s} \frac{r!s!(R/\mathcal{E})^2}{[p + q]!} K_1^{p-r+2}K_2^{q-s} m_{p-r,q-s-2}
\]
\[
\leq C_u \frac{1}{4} \sum_{r,s} \left( \frac{\gamma C}{2} \right)^{r+s} K_1^{p-r+2}K_2^{q-s} \frac{m_{p,q}}{(p+q)!},
\]
\[
\varepsilon^{-1}M'_{R,p,q}(\sum_{j,\beta} \tilde{\partial}^j_{\beta} \tilde{B}^{ij}_\alpha u_j)
\]
\[
\leq C_u \frac{C_B (\mathcal{E}/\varepsilon)}{4} \sum_{r,s} \binom{p}{r} \binom{q}{s} \left( \frac{\gamma B}{2} \right)^{r+s} \frac{r!s!(R/\mathcal{E})}{[p + q]!} \left[ K_1^{p-r+2}K_2^{q-s+1} m_{p-r,q-s-1} + K_1^{p-r+3}K_2^{q-s} m_{p+1-r,q-s-2} \right]
\]
\[
\leq C_u (1 + C_A \gamma A \mathcal{E}) \sum_{r,s} \left( \frac{\gamma B}{2} \right)^{r+s} K_1^{p-r+2}K_2^{q-s+2} \frac{m_{p,q}}{(p+q)!} \left[ K_2^{-1} + K_1 K_2^{-2} \right],
\]
We consider, on the half-ball $B_R^+$, solutions $u$ of

$$- \text{div} (A(x) \nabla u) = f \quad \text{in} \quad B_R^+, \quad u|_{\Gamma_R} = 0. \quad (E.1)$$

### E Analytic Regularity for Poisson Problems

We consider, on the half-ball $B_R^+$, solutions $u$ of

$$- \text{div} (A(x) \nabla u) = f \quad \text{in} \quad B_R^+, \quad u|_{\Gamma_R} = 0. \quad (E.1)$$
Here, the matrix $A$ is pointwise symmetric positive definite and satisfies
\[
A \in \mathcal{A}^\infty(C_A, \gamma_A, B_R^+), \quad A \geq \lambda_{\min} > 0. \tag{E.2}
\]
The data $f$ are assumed to satisfy, for some $\varepsilon \in (0, 1]$
\[
\|\nabla^p f\|_{L^2(B_R^+)} \leq C_f \gamma_f^p \max\{p/R, \varepsilon^{-1}\}^p \quad \forall p \in \mathbb{N}_0. \tag{E.3}
\]
Note that this problem has been considered in [35, Lemma 5.5.15] where a recursion for the tangential derivatives, i.e., for the seminorm $N'_{R,p}(u)$, is derived. We use this result here to derive the following estimate.

**Lemma E.1** Assume (E.2) and (E.3) and $R \leq 1$. Then there exists $K > 0$ depending solely on $\lambda_{\min}, C_A, \gamma_A, \gamma_f$ such that a solution $u$ of (E.1) satisfies
\[
N'_{R,p}(u) \leq K^{p+2} \left[ C_f R^2 \max\{p+1, R/\varepsilon\}^p p! + (p+1)R\|\nabla u\|_{L^2(B_R^+)} \right] \quad \forall p \geq 0. \tag{E.4}
\]
Additionally, $N'_{R,-1}(u) \leq R/2\|\nabla u\|_{L^2(B_R^+)}$.

**Proof** The estimate $N'_{R,-1}(u) \leq R/2\|\nabla u\|_{L^2(B_R^+)}$ is a direct consequence of the definition. The case $p = 0$ follows directly from [35, Lemma 5.5.15]. The case $p = 1$ follows from an inspection of the arguments below. For $p \geq 2$, the proof is by induction on $p$, assuming that (E.4) holds for all $p' \leq p - 1$ for some $p \geq 2$. From [35, Lemma 5.5.15], we get
\[
\begin{align*}
N'_{R,p}(u) & \leq C_B \left[ C_f \left( \frac{R}{2} \right)^2 \left( \frac{\gamma_f}{2} \right)^p \frac{\max\{p, R/\varepsilon\}^p}{p!} + C_A(p+1) \left( \frac{\gamma_A R}{2} \right)^{p+1} \right] \tag{E.5}
\end{align*}
\]
\[
+ C_A \sum_{q=1}^{p} \frac{(p+1)!}{(p+1-q)!} \left( \frac{\gamma_A R}{2} \right)^q \frac{(p-q)!}{p!} N'_{R,p-q}(u)
\]
\[
+ N'_{R,p-1}(u) + N'_{R,p-2}(u).
\]
The induction hypothesis gives for $q = 1, \ldots, p$
\[
\begin{align*}
\frac{(p+1)!}{(p+1-q)!} \frac{(p-q)!}{p!} N'_{R,p-q}(u)
\end{align*}
\]
\[
\leq K^{p+2-q} \left[ C_f R^2 \max\{p+1, R/\varepsilon\}^p \frac{p!}{p!} + (p+1)R\|\nabla u\|_{L^2(B_R^+)} \right]
\]
\[
= K^{p+2-q} B_p,
\]
where $B_p$ abbreviates the expression in brackets, $[\cdots]$. Inserting the above and the induction hypothesis in (E.5) gives, assuming $\gamma_A R/(2K) < 1$,

$$
N'_{R,p}(u) 
\leq K^{p+2} C'_B B_p \left[ \frac{1}{4K^2} \left( \frac{\gamma_f}{2K} \right)^p + C_A K^{-1} \left( \frac{\gamma_A R}{2K} \right)^{p+1} 
+ C_A \frac{\gamma_A R}{2K} \frac{1}{1 - \gamma_A R/(2K)} + K^{-1} + K^{-2} \right].
$$

Selecting $K$ sufficiently large shows that the factor $C'_B[\cdots]$ can be made smaller than 1, which concludes the induction argument. □

The following theorem generalizes [35, Prop. 5.5.2] from homogeneous Dirichlet boundary condition on the whole boundary of $B_R^+$ to the condition $u|_{\Gamma_R} = 0$.

**Theorem E.2** Assume (E.2) and (E.3) and $R \leq 1$. Then there exist $K_1, K_2 \geq 1$ depending solely on $\lambda_{\min}, C_A, \gamma_A, \gamma_f$ such that a solution $u$ of (E.1) satisfies, for all $p \geq 0, q \geq -2$ with $(p,q) \neq (0,-2)$

$$
N'_{R,p,q}(u) \leq K_1^{p+2} K_2^{q+2} \left[ C_f R^2 \frac{\max(p + q + 3, R/\varepsilon)^{p+q}}{(p+q)!} + (p + q + 3) R \|\nabla u\|_{L^2(B_R^+)} \right].
$$

**Proof** We control the normal derivatives as in the proof of Theorem D.5. Inspection of the arguments leading to [35, (5.5.30)] shows that we have

$$
-\partial^2_\gamma u = \tilde{f} + \tilde{A} \nabla u + B : \nabla^2 u,
$$

where, for $C', \gamma > 0$ depending solely on $\lambda_{\min}, C_A, \gamma_A, \gamma_f$,

$$
\|\nabla^p \tilde{f}\|_{L^2(B_R^+)} \leq C' C_f \gamma^p \max(p/R, \varepsilon^{-1})^p \quad \forall p \in \mathbb{N}_0,
$$

$$
\tilde{A}, B \in A^\infty(C', \gamma, B_R^+), \quad B_{33} = 0.
$$

We abbreviate

$$
M_{p,q} := \left[ C_f R^2 \frac{\max(p + q + 3, R/\varepsilon)^{p+q}}{(p+q)!} + (p + q + 3) R \|\nabla u\|_{L^2(B_R^+)} \right]
= \quad M_{p,q}^{(1)} + M_{p,q}^{(2)}.
$$

The proof is by induction on $q$, the cases $q \in \{-2, -1, 0\}$ being shown in Lemma E.1 if we select $K_1 = K$ with $K$ given by Lemma E.1. Assume that (E.6) holds for all $q' \leq q - 1$ for some $q \geq 0$ and all $p$. From (E.7), we get

$$
N'_{R,p,q}(u) \leq M'_{R,p,q}(\tilde{f}) + M'_{R,p,q}(\tilde{A} \nabla u) + M'_{R,p,q}(B : \nabla^2 u),
$$

$$
M'_{R,p,q}(\tilde{f}) \leq \frac{C'}{4} \left( \frac{\gamma}{2} \right)^{p+q} M_{p,q},
$$

$$
\text{ Springer}
$$
where the estimate for \( \hat{M}'_{R,p,q}(\vec{f}) \) follows from a direct calculation. The terms \( \hat{M}'_{R,p,q}(\vec{A}\nabla u) \) and \( \hat{M}'_{R,p,q}(B : \nabla^2 u) \) are treated as in the proof of Theorem D.5. This is done in the following steps.

1. Step: We note that for \( p, \alpha \in \mathbb{N}_0 \) the function

\[
 r \mapsto \frac{(p - r + \alpha)!}{(p - r)!} = (p - r + 1) \cdots (p - r + \alpha) \quad \text{is monotone decreasing.}
\]

(E.13)

2. Step: The induction hypothesis and the monotonicity assertion (E.13) imply

\[
 \binom{p}{r} \binom{q}{s} \frac{s!r!(p + q - r - s)!}{(p + q)!} \left[ N'_{R,p-r+2,q-s-2}(u) + N'_{R,p+1-r,q-s-1}(u) \right] \\
 \leq K_1^{p-r+2} K_2^{q-s+2} M_{p,q} \left[ K_1^2 K_2^{-2} + K_1 K_2^{-1} \right].
\]

(E.14)

To see this, one write \( M_{p,q} = M_{p,q}^{(1)} + M_{p,q}^{(2)} \) as in (E.10). The terms involving \( M_{p,q}^{(1)} \) are treated exactly as in the treatment of \( \hat{M}'_{R,p,q}(\sum_{j,\alpha,\beta: (\alpha,\beta) \neq (3,3)} \vec{A}_{ij}^{(j)} \partial_\alpha \partial_\beta u) \) in the proof of Theorem D.5. The terms involving \( M_{p,q}^{(2)} \) are handled by noting that a twofold application of the monotonicity assertion (E.13) implies

\[
 \binom{p}{r} \binom{q}{s} s!r!(p + q - r - s)! \frac{1}{(p + q)!} \leq 1.
\]

(E.15)

3. Step: Analogous to the treatment of the term \( \hat{M}'_{R,p,q}(\sum_{j,\alpha,\beta: (\alpha,\beta) \neq (3,3)} \vec{A}_{ij}^{(j)} \partial_\alpha \partial_\beta u) \) in the proof of Theorem D.5 we get for \( M'_{R,p,q}(B : \nabla^2 u) \) by observing that the assumption \( B_{33} \equiv 0 \) leads to a sum of terms of the form (E.14)

\[
 M'_{R,p,q}(B : \nabla^2 u) \leq K_1^{p+2} K_2^{q+2} M_{p,q} \left[ K_1^2 K_2^{-2} + K_1 K_2^{-1} \right] C' \sum_{r,s} \left( \frac{\gamma R}{2 K_1} \right)^r \left( \frac{\gamma R}{2 K_2} \right)^s.
\]

(E.16)

4. Step: The induction hypothesis gives

\[
 \binom{p}{r} \binom{q}{s} s!r!(p + q - r - s - 1)! \frac{1}{(p + q)!} \left[ N'_{R,p-r,q-s-1}(u) + N'_{R,p+1-r,q-s-2}(u) \right] \\
 \leq 3 K_1^{p-r+2} K_2^{q-s+2} M_{p,q} \frac{1}{p + q + 3} \left[ K_1^{-1} + K_1 K_2^{-2} \right];
\]

(E.17)

in this estimate, for the contribution \( M_{p,q}^{(1)} \) of \( M_{p,q} \), the calculation is as in the treatment of \( \hat{M}'_{R,p,q}(\vec{B}_i^{(ij)} \partial_\beta u) \); for the contribution \( M_{p,q}^{(2)} \) of \( M_{p,q} \) one has to consider \( \sum_{j,\alpha,\beta} (\vec{B}_i^{(ij)} \partial_\beta u) \).
where the term \((p - r + q - s + 3)\) arises from \(M^{(2)}_{p-r,q-s}\). Since \(M^{(2)}_{p,q} = (p + q + 3)R\|\nabla u\|_{L^2(B^R_\mathcal{K})}\), the result (E.17) follows.

5. **Step:** As in the treatment of the terms \(M'_{R,p,q}(\tilde{B}^{ij}_\beta \partial_\beta u_j)\) in the proof of Theorem D.5, we get from (E.17) the bound

\[
M'_{R,p,q}(\tilde{A}\nabla u) \leq 3K_1^{p+2}K_2^{q+2}M_{p,q}\left[K_2^{-1} + K_1K_2^{-2}\right]\frac{C'}{p + q + 3} \sum_{r,s} \left(\frac{\gamma R}{2K_1}\right)^r \left(\frac{\gamma R}{2K_2}\right)^s.
\]

6. **step:** Selecting \(K_2\) sufficiently large depending solely on \(C'\) and \(\gamma\) completes the induction argument. \(\square\)

**References**

1. R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
2. M. Ainsworth. Discrete Dispersion Relation for *hp*-Finite Element Approximation at High Wave Number. *SIAM J. Numer. Anal.*, 42(2):553–575, 2004.
3. M. Ainsworth. Dispersive properties of high-order Nédélec/edge element approximation of the time-harmonic Maxwell equations. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 362(1816):471–491, 2004.
4. M. Ainsworth, P. Monk, and W. Muniz. Dispersive and dissipative properties of discontinuous Galerkin finite element methods for the second-order wave equation. *J. Sci. Comput.*, 27(1-3):5–40, 2006.
5. M. Ainsworth and H. A. Wajid. Dispersive and dissipative behavior of the spectral element method. *SIAM J. Numer. Anal.*, 47(5):3910–3937, 2009.
6. M. Ainsworth and H. A. Wajid. Optimally blended spectral-finite element scheme for wave propagation and nonstandard reduced integration. *SIAM J. Numer. Anal.*, 48(1):346–371, 2010.
7. I. Babuška and S. A. Sauter. Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers. *SIAM J. Numer. Anal.*, 34(6):2392–2423, 1997.
8. M. Bernkopf and J. Melenk. Analysis of the *hp*-version of a first order system least squares method for the helmholtz equation. In T. Apel, U. Langer, A. Meyer, and O. Steinbach, editors, *Advanced Finite Element Methods with Applications—Selected Papers from the 30th Chemnitz FEM Symposium 2017*, volume 128 of *Lecture Notes in Computational Science and Engineering*, pages 57–84. Springer Verlag, 2019.
9. A. Buffa. Remarks on the discretization of some noncoercive operator with applications to heterogeneous Maxwell equations. *SIAM J. Numer. Anal.*, 43(1):1–18, 2005.
10. A. Buffa, M. Costabel, and D. Sheen. On traces for \(H(\text{curl}, \Omega)\) in Lipschitz domains. *J. Math. Anal. Appl.*, 276(2):845–867, 2002.
11. A. Buffa and R. Hiptmair. Galerkin boundary element methods for electromagnetic scattering. In *Topics in computational wave propagation*, volume 31 of *Lect. Notes Comput. Sci. Eng.*, pages 83–124. Springer, Berlin, 2003.
12. A. Buffa, R. Hiptmair, T. von Petersdorff, and C. Schwab. Boundary element methods for Maxwell transmission problems in Lipschitz domains. *Numer. Math.*, 95(3):459–485, 2003.
13. S. Caorsi, P. Fernandes, and M. Raffetto. On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems. *SIAM J. Numer. Anal.*, 38(2):580–607, 2000.
14. M. Cessenat. *Mathematical methods in electromagnetism*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
15. H. Chen, P. Lu, and X. Xu. A hybridizable discontinuous Galerkin method for the Helmholtz equation with high wave number. *SIAM J. Numer. Anal.*, 51(4):2166–2188, 2013.

16. H. Chen and W. Qiu. A first order system least squares method for the Helmholtz equation. *J. Comput. Appl. Math.*, 309:145–162, 2017.

17. M. Costabel and A. McIntosh. On Bogovski˘ı and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. *Math. Z.*, 265(2):297–320, 2010.

18. L. Demkowicz. Polynomial exact sequences and projection-based interpolation with applications to Maxwell’s equations. In D. Boffi, F. Brezzi, L. Demkowicz, L. Durán, R. Falk, and M. Fortin, editors, *Mixed Finite Elements, Compatibility Conditions, and Applications*, volume 1939 of *Lectures Notes in Mathematics*. Springer Verlag, 2008.

19. L. Demkowicz and A. Buffa. $H^1$, $H(\text{curl})$ and $H(\text{div})$-conforming projection-based interpolation in three dimensions. Quasi-optimal $p$-interpolation estimates. *Comput. Methods Appl. Mech. Engrg.*, 194(2-5):267–296, 2005.

20. L. Demkowicz, J. Gopalakrishnan, I. Muga, and J. Zitelli. Wavenumber explicit analysis of a DPG method for the multidimensional Helmholtz equation. *Comput. Methods Appl. Mech. Eng.*, 213/216:126–138, 2012.

21. S. Esterhazy and J. M. Melenk. On stability of discretizations of the Helmholtz equation. In I. Graham, T. Hou, O. Lakkis, and R. Scheichl, editors, *Numerical Analysis of Multiscale Problems*, volume 83 of *Lect. Notes Comput. Sci. Eng.*, pages 285–324. Springer, Berlin, 2012.

22. X. Feng and H. Wu. Discontinuous Galerkin methods for the Helmholtz equation with large wave number. *SIAM J. Numer. Anal.*, 47(4):2872–2896, 2009.

23. X. Feng and H. Wu. $hp$-discontinuous Galerkin methods for the Helmholtz equation with large wave number. *Math. Comput.*, 80:1997–2024, 2011.

24. X. Feng and H. Wu. An absolutely stable discontinuous Galerkin method for the indefinite time-harmonic Maxwell equations with large wave number. *SIAM J. Numer. Anal.*, 52(5):2356–2380, 2014.

25. R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, 11:237–339, 2002.

26. F. Ihlenburg. *Finite Element Analysis of Acousting Scattering*. Springer, New York, 1998.

27. F. Ihlenburg and I. Babuška. Finite Element Solution to the Helmholtz Equation with High Wave Number. Part I: The h-version of the FEM. *Comput. Math. Appl.*, 39(9):9–37, 1995.

28. F. Ihlenburg and I. Babuška. Finite Element Solution to the Helmholtz Equation with High Wave Number. Part II: The h-p version of the FEM. *SIAM J. Num. Anal.*, 34(1):315–358, 1997.

29. A. Kirsch and F. Hettlich. *The mathematical theory of time-harmonic Maxwell’s equations*. Springer, Cham, 2015.

30. M. Löhndorf and J. M. Melenk. Wavenumber-explicit $hp$-BEM for high frequency scattering. *SIAM J. Numer. Anal.*, 49(6):2340–2363, 2011.

31. P. Lu, H. Chen, and W. Qiu. An absolutely stable $hp$-HDG method for the time-harmonic Maxwell equations with high wave number. *Math. Comput.*, 86(306):1553–1577, 2017.

32. W. McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge, Univ. Press, 2000.

33. J. M. Melenk. *On Generalized Finite Element Methods*. PhD thesis, University of Maryland at College Park, 1995.

34. J. M. Melenk. *hp-Finite Element Methods for Singular Perturbations*. Springer, Berlin, 2002.

35. J. M. Melenk. On approximation in meshless methods. In *Frontiers of numerical analysis*, pages 65–141. Springer, Berlin, 2005.

36. J. M. Melenk. Mapping properties of combined field Helmholtz boundary integral operators. *SIAM J. Math. Anal.*, 44(4):2599–2636, 2012.

37. J. M. Melenk, A. Parsania, and S. A. Sauter. General DG-methods for highly indefinite Helmholtz problems. *J. Sci. Comput.*, 57(3):536–581, 2013.

38. J. M. Melenk and C. Rojik. On commuting $p$-version projection-based interpolation on tetrahedra. *Math. Comput.*, 89:45–87, 2020.

39. J. M. Melenk and S. A. Sauter. Convergence Analysis for Finite Element Discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary condition. *Math. Comput.*, 79:1871–1914, 2010.

40. J. M. Melenk and S. A. Sauter. Wave-Number Explicit Convergence Analysis for Galerkin Discretizations of the Helmholtz Equation. *SIAM J. Numer. Anal.*, 49(3):1210–1243, 2011.
42. J. M. Melenk and S. A. Sauter. Wavenumber-explicit $hp$-FEM analysis for Maxwell’s equations with impedance boundary conditions. Technical report, in prep.
43. P. Monk. Finite element methods for Maxwell’s equations. Oxford University Press, New York, 2003.
44. P. Monk. A simple proof of convergence for an edge element discretization of Maxwell’s equations. In Computational electromagnetics (Kiel, 2001), volume 28 of Lect. Notes Comput. Sci. Eng., pages 127–141. Springer, Berlin, 2003.
45. C. Morrey. Multiple Integrals in the Calculus of Variations. Springer, Berlin, 1966.
46. J.-C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. Numer. Math., 35(3):315–341, 1980.
47. J. C. Nédélec. Acoustic and Electromagnetic Equations. Springer, New York, 2001.
48. S. Nicaise and J. Tomezyk. Convergence analysis of a $hp$-finite element approximation of the time-harmonic Maxwell equations with impedance boundary conditions in domains with a analytic boundary. Technical report, hal-ouvertes, Mar. 2019. https://hal.archives-ouvertes.fr/hal-02063271.
49. S. Nicaise and J. Tomezyk. The time-harmonic Maxwell equations with impedance boundary conditions in polyhedral domains. In U. Langer, D. Pauly, and S. Repin, editors, Maxwell’s Equations: Analysis and Numerics, Radon Series on Computational and Applied Mathematics 24, pages 285–340, Berlin, 2019. De Gruyter.
50. S. Petrides and L. F. Demkowicz. An adaptive DPG method for high frequency time-harmonic wave propagation problems. Comput. Math. Appl., 74(8):1999–2017, 2017.
51. S. A. Sauter and C. Schwab. Boundary Element Methods. Springer, Heidelberg, 2010.
52. I. Terrasse. Résolution mathématique et numérique des équations de Maxwell instationnaires par une méthode de potentiels retardés. PhD thesis, Ecole Polytechnique, Paris, 1993.
53. A. Veit. Convolution Quadrature for Time-Dependent Maxwell Equations. Master’s thesis, Institut für Mathematik, Universität Zürich, 2009.
54. L. Zhu and H. Wu. Preasymptotic error analysis of CIP-FEM and FEM for Helmholtz equation with high wave number. Part II: $hp$ version. SIAM J. Numer. Anal., 51(3):1828–1852, 2013.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.