Some spectral properties of chain graphs

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Abstract

A graph is called a chain graph if it is bipartite and the neighborhoods of the vertices in each color class form a chain with respect to inclusion. Alazemi, Andelić and Simić conjectured that no chain graph shares a non-zero (adjacency) eigenvalue with its vertex-deleted subgraphs. We disprove this conjecture. However, we show that the assertion holds for subgraphs obtained by deleting vertices of maximum degrees in either of color classes. We also give a simple proof for the fact that chain graphs have no eigenvalue in the interval (0, 1/2).

Keywords: Chain graph, Adjacency Matrix, Eigenvalue, Downer vertex

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1 Introduction

A graph is called a chain graph (or double nested graph [3]) if it is bipartite and the neighborhoods of the vertices in each color class form a chain with respect to inclusion. Chain graphs appear in different contexts and so several characterizations of them can be found in the literature. Here we mention a few: a graph $G$ is a chain graph if and only if it satisfies one of the following properties:

- every vertex $v_i$ of $G$ can be assigned a real number $a_i$ for which there exists a positive real
number $R$ such that $|a_i| < R$ for all $i$ and two vertices $v_i, v_j$ are adjacent if and only if $|a_i - a_j| \geq R$ (due to this property chain graphs are also called difference graphs) [8];

- $G$ is a bipartite graph and every induced subgraph with no isolated vertices has a dominating vertex on each color class, that is, a vertex adjacent to all the vertices of the other color class [8];

- $G$ is $(2K_2, C_5, C_3)$-free;

- $G$ is $2K_2$-free and bipartite;

- $G$ is $P_5$-free and bipartite.

Note that the last three characterizations follow easily from the second one.

In terms of graph eigenvalues, (connected) chain graphs have a remarkable feature. They are characterized as graphs whose largest eigenvalue is maximum among the connected bipartite graphs with the same number of vertices and edges ([3, 4]). Another family with similar properties as chain graphs are threshold graphs which are the graphs such that the neighborhoods of their vertices form a single chain with respect to inclusion. They have the largest maximum eigenvalue among the graphs with prescribed number of vertices and edges (see [7, Remarks 8.1.9]). In fact, any threshold graph can be obtained from a chain graph $G$ by replacing one color class of $G$ by a clique, and all other edges unchanged. For more information see [5, 9].

Alazemi, Andelić and Simić [1] conjectured that no chain graph shares a non-zero (adjacency) eigenvalue with its vertex-deleted subgraphs. We disprove this conjecture. However, we show that the assertion holds for subgraphs obtained by deleting vertices of maximum degrees in either of color classes. They [1] also proved that chain graphs have no eigenvalue in the interval $(0, 1/2)$. We give a simple proof for this result.

2 Preliminaries

The graphs we consider are all simple and undirected. For a graph $G$, we denote by $V(G)$ the vertex set of $G$. For two vertices $u, v$, by $u \sim v$ we mean that $u$ and $v$ are adjacent. If $V(G) = \{v_1, \ldots, v_n\}$, then the adjacency matrix of $G$ is an $n \times n$ matrix $A(G)$ whose $(i, j)$-entry is 1 if $v_i \sim v_j$ and 0 otherwise. By eigenvalues of $G$ we mean those of $A(G)$. The multiplicity of an eigenvalue $\lambda$ of $G$ is denoted by $\text{mult}(\lambda, G)$. For a vertex $v$ of $G$, let $N(v)$ denote the neighborhood of $v$, i.e. the set of all vertices of $G$ adjacent to $v$. Two vertices $u$ and $v$ of $G$ are called duplicate if $N(u) = N(v)$. For $v \in V(G)$, we use the notation $G - v$ to mean the subgraph of $G$ induced by $V(G) \setminus \{v\}$.

Remark 1. (Structure of chain graphs) As it was observed in [3], the color classes of any chain graph $G$ can be partitioned into $k$ non-empty cells $U_1, \ldots, U_k$ and $V_1, \ldots, V_k$ such that

$$N(u) = V_1 \cup \cdots \cup V_{k-i+1} \text{ for any } u \in U_i, \ 1 \leq i \leq k.$$
Remark 2. (Sum rule) Let $x$ be an eigenvector for eigenvalue $\lambda$ of a graph $G$. Then the entries of $x$ satisfy the following equalities:

$$\lambda x(v) = \sum_{u: u \sim v} x(u), \text{ for all } v \in V(G).$$

(1)

From this it is seen that if $\lambda \neq 0$ and $N(v) = N(v')$, then $x(v) = x(v')$. In particular if $G$ is a chain graph, in the notations of Remark 1, $x$ is constant on each $U_i$ and on each $V_i$ for $i = 1, \ldots, k$.

We will make use of the interlacing property of graph eigenvalues which we recall below (see [6, Theorem 2.5.1]).

**Lemma 3.** Let $G$ be a graph of order $n$, $H$ be an induced subgraph of $G$ of order $m$, $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$ be the eigenvalues of $G$ and $H$, respectively. Then

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \text{ for } i = 1, \ldots, m.$$  

In particular, if $m = n - 1$, then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$  

From the case of equality in interlacing (see [6, Theorem 2.5.1]) the following can be deduced.

**Lemma 4.** If in Lemma 3 we have $\lambda_i = \mu_i$ or $\mu_i = \lambda_{n-m+i}$ for some $1 \leq i \leq m$, then $A(H)$ has an eigenvector $x$ for $\mu_i$, such that $\begin{pmatrix} 0 \\ x \end{pmatrix}$, with the 0 vector corresponding to $V(G) \setminus V(H)$, is an eigenvector of $A(G)$ for the eigenvalue $\mu_i$.

3 Eigenvectors and downer vertices

For a graph $G$ and an eigenvalue $\lambda$ of $G$, a vertex $v$ is called downer if $\text{mult}(\lambda, G - v) = \text{mult}(\lambda, G) - 1$. In [2] it was shown that all the non-zero eigenvalues of chain graphs are simple (this also readily follows from (the proof of) Theorem 7 below). As the subgraphs of any chain graph are also chain graphs, if $\lambda$ is an eigenvalue of a chain graph $G$, then removal of any vertex from $G$ does not increase the multiplicity of $\lambda$, i.e. $\text{mult}(\lambda, G - v) \leq \text{mult}(\lambda, G - v) = 1$. A question raises on the precise value of $\text{mult}(\lambda, G - v)$: is it always 0? This was actually conjectured in [1].

**Conjecture 5.** ([1]) In any chain graph, every vertex is downer with respect to every non-zero eigenvalue.
The conjecture is equivalent to say that for any chain graph $G$ and any $v \in V(G)$, $G - v$ shares no non-zero eigenvalue with $G$.

We disprove Conjecture 5 in this section. Indeed, Theorems 8 and 9 below show that there are infinitely many counterexamples for this conjecture. In spite of that, a weak version of the conjecture is true: in Theorem 7 it will be shown that for non-zero eigenvalues the vertices with maximum degrees in each color class of a chain graph are downer.

**Remark 6.** For a vertex $v$ being downer or not depends on the component corresponding to $v$ in the eigenvectors of $\lambda$. Let $W$ be the eigenspace corresponding to $\lambda$. If for all $x \in W$, we have $x(v) = 0$, then $v$ cannot be a downer vertex as for any $x \in W$, the vector $x'$ obtained by eliminating the the component corresponding to $v$, is an eigenvector of $\lambda$ for $G - v$, so we have

$$\text{mult}(\lambda, G - v) \geq \dim \{x': x \in W\} = \dim W = \text{mult}(\lambda, G).$$

From this and Lemma 4 it follows that, in the case that $\text{mult}(G, \lambda) = 1$, there exists an eigenvector $x$ for $\lambda$ with $x(v) = 0$ if and only if $v$ is not a downer vertex for $\lambda$.

**Theorem 7.** Let $G$ be a chain graph. Then the vertices having maximum degrees in each color class of $G$ are downer for any non-zero eigenvalue.

**Proof.** In the notations of Remark 1, the vertices in $U_1$ and $V_1$ have the maximum degree in color classes of $G$. We show that the vertices of $U_1$ and $V_1$ are downer with respect to any non-zero eigenvalue $\lambda$ of $G$. We may assume that $G$ has no isolated vertices. Let $u_1 \in U_1$, so $N(u_1) = V_1 \cup \cdots \cup V_k$. Let $x$ be any eigenvector for $\lambda$. We claim that $x(u_1) \neq 0$, from which the result follows. For a contradiction, assume that $x(u_1) = 0$. So, $x$ is zero on the whole $U_1$. For any $v \in V_k$, $N(v) = U_1$, so by the sum rule, $x(v) = 0$. Hence for any $u_2 \in U_2$,

$$0 = \lambda x(u_1) = \sum_{v \in N(u_1)} x(v) = \sum_{v \in V_1 \cup \cdots \cup V_k} x(v) = \sum_{v \in V_1 \cup \cdots \cup V_{k-1}} x(v) = \sum_{v \in N(u_2)} x(v) = \lambda x(u_2).$$

It follows that $x$ is zero on $U_2$ as well. For any $v \in V_{k-1}$, $N(v) = U_1 \cup U_2$, so again by the sum rule, $x(v) = 0$. Hence for any $u_3 \in U_3$,

$$0 = \lambda x(u_1) = \sum_{v \in V_1 \cup \cdots \cup V_k} x(v) = \sum_{v \in V_1 \cup \cdots \cup V_{k-2}} x(v) = \sum_{v \in N(u_3)} x(v) = \lambda x(u_3).$$

It follows that $x$ is zero on $U_3$, too. Continuing this argument, it follows that $x = 0$, a contradiction. \[\square\]

A chain graph for which $|U_1| = \cdots = |U_k| = |V_1| = \cdots = |V_k| = 1$ is called a *half graph*, where we denote it by $H(k)$. As we will see in what follows, specific half graphs provide counterexamples to Conjecture 5. Let

$$(a_1, \ldots, a_6) := (1, 0, -1, -1, 0, 1).$$
Let 

\[ x := (x_1, \ldots, x_k) \] where \( x_i = a_s \) if \( i \equiv s \pmod{6} \).

In the next theorem, we show that the vector \((x x)\) (each \(x\) corresponds to a color class) is an eigenvector of a non-zero eigenvalue of \(H(k)\) for some \(k\). In view of Remark 6, this disproves Conjecture 5.

**Theorem 8.** In any half graph \(H(k)\), the vector \((x x)\) is an eigenvector for eigenvalue 1 if \(k \equiv 1 \pmod{6}\) and it is an eigenvector for eigenvalue \(-1\) if \(k \equiv 4 \pmod{6}\).

**Proof.** From Table 1, we observe that for \(1 \leq s \leq 6\),

\[
\sum_{i=1}^{5-s} a_i = -a_s \quad \text{and} \quad \sum_{i=1}^{2-s} a_i = a_s,
\]

where we consider \(5-s\) and \(2-s\) modulo 6 as elements of \(\{1, \ldots, 6\}\).

| \(s\) | \(a_s\) | \(5-s\) | \(\sum_{i=1}^{5-s} a_i\) | \(2-s\) | \(\sum_{i=1}^{2-s} a_i\) |
|-----|-----|-----|----------------|-----|----------------|
| 1   | 1   | 4   | -1            | 1   | 1              |
| 2   | 0   | 3   | 0             | 6   | 0              |
| 3   | -1  | 2   | 1             | 5   | -1             |
| 4   | -1  | 1   | 1             | 4   | -1             |
| 5   | 0   | 6   | 0             | 3   | 0              |
| 6   | 1   | 5   | -1            | 2   | 1              |

Table 1: The values of \(\sum_{i=1}^{5-s} a_i\) and \(\sum_{i=1}^{2-s} a_i\).

Note that, since \(\sum_{i=1}^{6} a_i = 0\), if \(1 \leq \ell \leq k\), \(1 \leq s \leq 6\) and \(\ell \equiv s \pmod{6}\), then

\[
\sum_{i=1}^{\ell} x_i = \sum_{i=1}^{s} a_i.
\]

Let \(\{u_1, \ldots, u_k\}\) and \(\{v_1, \ldots, v_k\}\) be the color classes of \(H(k)\). Let \(k = 6t + 4\). We show that \((x x)\) satisfies the sum rule with \(\lambda = -1\). By the symmetry, we only need to show this for \(u_i\)'s.

Let \(i = 6t' + s\) for some \(1 \leq s \leq 6\). Then \(n - i + 1 = 6(t - t') + 5 - s\).

\[
\sum_{j: v_j \sim u_i} x_j = \sum_{j=1}^{n-i+1} x_j = \sum_{j=1}^{5-s} a_j = -a_s = -x_i.
\]

Now, let \(k = 6t + 1\). We show that in this case \((x x)\) satisfies the sum rule with \(\lambda = 1\). Let \(i = 6t' + s\) for some \(1 \leq s \leq 6\). Then \(n - i + 1 = 6(t - t') + 2 - s\).

\[
\sum_{j: v_j \sim u_i} x_j = \sum_{j=1}^{n-i+1} x_j = \sum_{j=1}^{2-s} a_j = a_s = x_i.
\]

\[\Box\]
Now we give another class of counterexamples to Conjecture 5. For this, let
\[ \omega^2 + \omega - 1 = 0, \]
and
\[(b_1, \ldots, b_{10}) := (\omega, -1, 0, 1, -\omega, -\omega, 1, 0, -1, \omega).\]
Let
\[ x := (x_1, \ldots, x_k) \text{ where } x_i = b_s \text{ if } i \equiv s \pmod{10}. \]

**Theorem 9.** In any half graph \( H(k) \), the vector \((x \ x)\) is an eigenvector for eigenvalue \( \omega \) if \( k \equiv 7 \pmod{10} \) and it is an eigenvector for eigenvalue \( -\omega \) if \( k \equiv 2 \pmod{10} \).

**Proof.** From Table 2, we observe that for \( 1 \leq s \leq 10 \),
\[ \sum_{i=1}^{8-s} b_i = \omega b_s \text{ and } \sum_{i=1}^{3-s} b_i = -\omega b_s, \]
where we consider \( 8 - s \) and \( 3 - s \) modulo 10 as elements of \( \{1, \ldots, 10\} \).

| \( s \) | \( b_s \) | \( 8 - s \) | \( \sum_{i=1}^{8-s} b_i \) | \( 3 - s \) | \( \sum_{i=1}^{3-s} b_i \) |
|------|------|------|-------|------|-------|
| 1    | \( \omega \) | 7    | 1 - \( \omega \) | 2    | \( \omega - 1 \) |
| 2    | -1   | 6    | -\( \omega \) | 1    | \( \omega \) |
| 3    | 0    | 5    | 0     | 10   | 0     |
| 4    | 1    | 4    | \( \omega \) | 9    | -\( \omega \) |
| 5    | -\( \omega \) | 3    | \( \omega - 1 \) | 8    | 1 - \( \omega \) |
| 6    | -\( \omega \) | 2    | \( \omega - 1 \) | 7    | 1 - \( \omega \) |
| 7    | 1    | 1    | \( \omega \) | 6    | -\( \omega \) |
| 8    | 0    | 10   | 0     | 5    | 0     |
| 9    | -1   | 9    | -\( \omega \) | 4    | \( \omega \) |
| 10   | \( \omega \) | 8    | 1 - \( \omega \) | 3    | \( \omega - 1 \) |

Table 2: The values of \( \sum_{i=1}^{8-s} b_i \) and \( \sum_{i=1}^{3-s} b_i \)

Note that, since \( \sum_{i=1}^{10} b_i = 0 \), if \( 1 \leq \ell \leq k, 1 \leq s \leq 10 \) and \( \ell \equiv s \pmod{10} \), then
\[ \sum_{i=1}^{\ell} x_i = \sum_{i=1}^{s} b_i. \]
Let \( k = 10t + 7 \). We show that \((x \ x)\) satisfies the sum rule with \( \lambda = \omega \). Let \( i = 10t' + s \) for some \( 1 \leq s \leq 10 \). Then \( n - i + 1 = 10(t - t') + 8 - s \).
\[ \sum_{j: v_j \sim u_i} x_j = \sum_{j=0}^{n-i+1} x_j = \sum_{j=1}^{8-s} b_j = \omega b_s = \omega x_i. \]
Now, let $k = 10t+2$. Assume that $i = 10t' + s$ for some $1 \leq s \leq 10$. Then $n - i + 1 = 6(t-t') + 3 - s$.

$$\sum_{j : v_j \sim u_i} x_j = \sum_{j=1}^{n-i+1} x_j = \sum_{j=1}^{3-s} b_j = -\omega b_s = -\omega x_i.$$  

It follows that in this case $(x, x)$ satisfies the sum rule with $\lambda = -\omega$. $\square$

**Remark 10.** (i) Given $(x, x)$ as eigenvector of $H(k)$ for $\lambda \in \{\pm 1, \pm \omega\}$, then $(x, -x)$ is an eigenvector of $H(k)$ for $-\lambda$. This gives more eigenvalues of $H(k)$ with eigenvectors containing zero components. (ii) Let $x$ be an eigenvector for eigenvalue $\lambda$ of a graph $G$ with $x(v) = 0$ for some vertex $v$. If we add a new vertex $u$ duplicate to $v$ and add a zero component to $x$ corresponding to $u$, then the new vector is an eigenvector of $H$ for eigenvalue $\lambda$. So, we can extend any graph presented in Theorems 8 or 9 to construct infinitely many more counterexamples for Conjecture 5.

4 An eigenvalue-free interval

In [1], it was proved that chain graphs have no eigenvalues in the interval $(0, 1/2)$ (and hence no eigenvalue in the interval $(-1/2, 0)$, as the eigenvalues of bipartite graphs are symmetric with respect to zero). Here we give a simple proof for this result.

**Theorem 11.** ([1]) Chain graphs have no eigenvalue in the interval $(0, 1/2)$.

**Proof.** The proof goes by induction on the number of vertices. The assertion holds for bipartite graphs with at most 4 vertices (see [6, p. 17]). It suffices to consider connected graphs. So let $G$ be a connected chain graph with at least 5 vertices.

First assume that $G$ has a pair of duplicates $u, v$ and $H = G - v$. Let $\lambda_1 \geq \cdots \geq \lambda_\ell$ and $\mu_1 \geq \cdots \geq \mu_{\ell-1}$ be the eigenvalues of $G$ and $H$, respectively. Also suppose that $\mu_t > \mu_{t+1} = \cdots = \mu_{t+j} = 0 > \mu_{t+j+1}$ (with possibly $j = 0$). By the induction hypothesis, $\mu_t > 1/2$ (the equality is impossible). By interlacing, we have $\lambda_{t+1} \geq 0 = \lambda_{t+2} = \cdots = \lambda_{t+j} = 0 \geq \lambda_{t+j+1} \geq \mu_{t+j+1}$. Note that $\text{mult}(0, G) = \text{mult}(0, H) + 1 = j + 1$. This is possible only if both $\lambda_{t+1}$ and $\lambda_{t+j+1}$ are zero. On the other hand, again by interlacing, $\lambda_t \geq \mu_t > 1/2$. Hence $G$ has no eigenvalue in $(0, 1/2)$.

Now, suppose that $G$ has no pair of duplicates. It follows that $G$ is a half graph and $$A(G) = \begin{pmatrix} O & C \\ C^\top & O \end{pmatrix},$$ with $C + C^\top = J_n + I_n$ where $J_n$ is the all 1’s $n \times n$ matrix. We have that

$$(2C - I)(2C - I)^\top = 4CC^\top - 2C - 2C^\top + I = 4CC^\top - I - 2J.$$
This means that $4CC^\top - I = (2C - I)(2C - I)^\top + 2J$ is positive semidefinite and so the eigenvalues of $CC^\top$ are not smaller than $1/4$. It turns out that $G$ has no eigenvalue in the interval $(-1/2, 1/2)$. This completes the proof.

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\square
\]

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