Barbero-Immirzi Value from Experiment

Leonid Perlov,
Department of Physics, University of Massachusetts, Boston, USA
leonid.perlov@umb.edu

August 12, 2021

Abstract

We consider General Relativity as a limit case of the Scalar-Tensor theory with Barbero-Immirzi field when the field tends to a constant. We use Shapiro time delay experimental value of $1/w(x) = (2.1 \pm 2.3) \times 10^{-5}$ provided by the Cassini spacecraft to find the present Barbero-Immirzi parameter value.

This is a short note on obtaining Barbero-Immirzi value from the experimental data by using the Parametric Post Newtonian framework [1]. Currently there are only two types of theories that still agree with experiment: General Relativity (GR) and the Scalar-Tensor theory [2,3]. The Barbero-Immirzi parameter plays a major role in Loop Quantum Gravity and in Ashtekar’s GR formulation [4–6]. This paper determines Barbero-Immirzi parameter’s value $\beta$ directly from experimental data. The previous attempts to restrain the Barbero-Immirzi parameter can be found in [7]–[16]. Barbero-Immirzi parameter appears in the Holst action [17] that is used in Loop Quantum Gravity and in Ashtekar’s GR formulation as a coefficient of the topological term that vanishes due to the first Bianchi identity:

$$S = \frac{1}{16\pi G} \int d^4x \, e e^I_I e^J_J (R^{IJ}_{\mu\nu} - \frac{\beta}{2} \epsilon^{IJ}_{KL} R^{KL}_{\mu\nu}).$$

Thus, it is impossible to obtain its value directly from the classical GR theory. Formally Barbero-Immirzi parameter can be real or complex valued. In this paper we consider GR as a limit of the Scalar-Tensor theory with Barbero-Immirzi field (BI) and non-zero torsion. When torsion tensor tends to zero, BI field tends to a constant, as it follows from [18], also see details in Appendix B. The Barbero-Immirzi field theory was introduced and studied in [18–21], and [22]. Its action obtained from (1) when $\beta(x)$ is a field is as follows [18]:

$$S = \frac{1}{16\pi G} \int \sqrt{-g} \, d^4x \left[ R - \frac{3}{2} \left( \frac{1}{1 + \beta^2(x)} \right) \partial_\mu \beta(x) \partial^\mu \beta(x) \right],$$

where $R$ is a torsion-free Riemann scalar.

The Scalar-Tensor theories are rare types of theories besides GR that are still in agreement with the experimental data [1][23]. The Scalar-Tensor theory action written in
"Einstein frame" contains a scalar field $\phi(x)$ and a coupling function $w(\phi(x)) \ [2, 24]$:

$$S = \frac{1}{16\pi G} \int \sqrt{-\tilde{g}} \ d^4x \left[ \tilde{R} - \frac{3 + 2w(\phi(x))}{2\phi^2(x)} \partial_{\mu} \phi(x) \partial^{\mu} \phi(x) \right], \tag{3}$$

where $\tilde{R}$ is a Riemann curvature after the conformal transformation $g_{\mu\nu} = \tilde{g}_{\mu\nu}/\phi(x)$.

In the limit, when $w(\phi(x))$ is constant and tends to infinity, the theory becomes pure GR $[2, 24]$: $\phi(x) = \text{const} + O(\frac{1}{w})$, $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi GT_{\mu\nu} + O(\frac{1}{w})$

Moreover, one can use the scaling $g_{\mu\nu} = \tilde{g}_{\mu\nu}/\phi^2(x)$ instead of $g_{\mu\nu} = \tilde{g}_{\mu\nu}/\phi(x)$ (see Appendix A for details) to present it in the form:

$$S = \frac{1}{16\pi G} \int \sqrt{-\tilde{g}} \ d^4x \left[ \tilde{R}_{\beta(x)} - \frac{3}{2} \left( \frac{1}{1 + \beta^2(x)} + \frac{1}{\beta^3(x)} \right) \partial_{\mu} \beta(x) \partial^{\mu} \beta(x) \right]. \tag{4}$$

We then perform a conformal transformation $g_{\mu\nu} = \tilde{g}_{\mu\nu}/\beta(x)$ in (2) to obtain:

$$S = \frac{1}{16\pi G} \int \sqrt{-\tilde{g}} \ d^4x \left[ \frac{\tilde{R}}{\beta(x)} - \frac{3}{2} \left( \frac{1}{1 + \beta^2(x)} + \frac{1}{\beta^3(x)} \right) \partial_{\mu} \beta(x) \partial^{\mu} \beta(x) \right]. \tag{5}$$

(4) and (5) are equal when the second terms are equal and the scales are equal: $\phi^2(x) = \beta(x)$, since in one case we used $\phi^2(x)$ scaling, while in the other $\beta(x)$. Thus, we have obtained a system of two equations:

$$\phi^2(x) = \beta(x), \tag{6}$$

$$\frac{3}{2} \left( \frac{1}{1 + \beta^2(x)} + \frac{1}{\beta^3(x)} \right) = \frac{6 + w(\beta(x))}{\phi^4(x)}. \tag{7}$$

After substituting the first equation into the second and simplifying, we obtain the following equation for $\phi(x)$:

$$(12 + 2w(\phi(x))) \frac{\partial^7(x)}{\phi^6(x)} - 3\frac{\partial^6(x)}{\phi^6(x)} - 3\frac{\phi^4(x)}{\phi^4(x)} + (12 + 2w(\phi(x))) \phi^3(x) - 3 = 0. \tag{8}$$

When $w(x)$ tends to infinity, we receive in the limit equation by dividing by $12 + 2w(x)$

$$\lim_{w \to \infty} \left[ \frac{\partial^7(x)}{12 + 2w(\phi(x))} - 3\frac{\partial^6(x)}{(12 + 2w(\phi(x)))} + \frac{\phi^4(x)}{(12 + 2w(\phi(x)))} + \frac{3}{(12 + 2w(\phi(x)))} = 0 \right]. \tag{9}$$

or

$$\phi^7(x) + \phi^3(x) = 0. \tag{10}$$

Recalling from (6) that $\phi^2(x) = \beta(x)$, we rewrite (10) with $\beta(x)$

$$\beta^{3/2}(x)(\beta^2(x) + 1) = 0. \tag{11}$$
The asymptotic solutions are:

\[ \beta(x) = \pm i \quad \text{or} \quad \beta(x) = 0 \]  \hspace{1cm} (12)

By solving (7) numerically with \( w(\phi(x)) = 10^5/(2.1 \pm 2.3) \), we obtain the value, which is very close to the asymptotic one:

\[ \beta(x) = \pm (i + 1.27 \times 10^{-5}) \pm (0.1i + 0.06)10^{-5} \] \hspace{1cm} (13)

As it seen from (10), each of two roots of (7) is of a multiplicity two, while the zero root is of a multiplicity three. Of course, we disregard the zero root, since we divide by \( \phi(x) \) and \( \beta(x) \) in our original equation (6).

Let us now discuss the experimental data obtained for \( w(\phi(x)) \). The most recent experimental value of \( w(\phi(x)) \) is \( 10^5/(2.1 \pm 2.3) \). It was provided by Shapiro time delay data in 2003 from the Cassini spacecraft on its way to Saturn [1, 25, 26]. It’s worth mentioning that back in 1972 this value was much lower, equal to 6 [2, 23], reaching thousands at the end of 70th, and order of \( 10^5 \) today. The ongoing BepiColombo mission to Mercury, launched in 2018, will further increase Cassini’s result 4 folds in July 2022 when it will experience the first solar conjunction [27].

For the combined graph of the different experimental data we refer to [25]. The values provided by the most recent binary pulsars are not competitive with the solar system Shapiro time delay measurement provided by Cassini due to the near equality of the star masses suppressing dipole radiation [11].

The bigger \( w(\beta(x)) \) the closer is the Scalar-Tensor theory to GR. When \( w(\beta(x)) \) is constant, which is the original Brans-Dicke theory [2], the limit \( w \to \infty \) implies \( \beta(x) = \text{const} + O(\frac{1}{w}) \) [24]. As we see from (12) the limit of \( \beta = \beta(x) \to \text{const} \) in this case is \( \pm i \), which corresponds to the Ashtekar self-dual GR formalism. At the end we would like to address the solution uniqueness. Indeed, by looking at (2) we see that when \( \beta(x) \) is any constant, not necessarily \( \pm i \), the action becomes GR. However, if we want the theory to be in agreement with the experiment all the time, while going to a limit, then the limit value of \( \beta(x) \) cannot be a random constant, but necessary \( \pm i \). To conclude, today’s value of the Barbero-Immirzi parameter detected from experiment is \( \beta(x) = \pm (i + 1.27 \times 10^{-5}) \pm (0.1i + 0.06)10^{-5} \), with the expected BepiColombo improvement it will become \( \beta(x) = \pm (i + 2.95 \times 10^{-6}) \pm (0.0.5i + 0.16)10^{-6} \), and with further experiments it will be going closer and closer to Ashtekar’s limit value \( \pm i \).

**Appendix A - Scalar-Tensor Action Scaling**

The original form of the Scalar-Tensor theory action is as follows [2][24][29]:

\[ S = \frac{1}{16\pi G} \int \sqrt{-g} \, d^4x \left[ \phi(x) R - \frac{w(\phi(x))}{\phi(x)} g^{\alpha\beta} \partial_\alpha \phi(x) \partial_\beta \phi(x) \right], \]  \hspace{1cm} (14)

Let us make the transform used in this paper: \( g_{\alpha\beta} = \tilde{g}_{\alpha\beta} / \phi^2(x) \). We have correspondingly \( g^{\alpha\beta} = \phi^2(x) \tilde{g}^{\alpha\beta} \), the volume will transforms as \( \sqrt{-g} = \sqrt{-\tilde{g}} / \phi^4(x) \), and the
Riemann scalar as \[29\]:

\[
R = \phi^2(x) \left( \tilde{R} + 6 \frac{\partial_\alpha \phi(x) \partial_\beta \phi(x)}{\phi^2(x)} - 6 \tilde{g}^{\alpha \beta} \partial_\alpha \phi(x) \partial_\beta \phi(x) \right) \tag{15}
\]

By substituting all these components into (14) and eliminating \(6 \frac{\partial_\alpha \phi(x)}{\phi^2(x)}\) by the Gauss theorem converting it to a surface integral \[29\] p. 740 we obtain:

\[
S = \frac{1}{16\pi G} \int \sqrt{-\tilde{g}} \, d^4x \left[ \frac{\tilde{R}}{\phi(x)} - 6 + \frac{w(\phi(x))}{\phi^3(x)} \tilde{g}^{\alpha \beta} \partial_\alpha \phi(x) \partial_\beta \phi(x) \right], \tag{16}
\]

Appendix B - BI Field is Constant when the Torsion Tensor is Zero

We repeat here the equations 8, 15-17 of \[18\] to demonstrate this point. The torsion tensor can be split into irreducible components:

\[
T_{\mu\nu\rho} = \frac{1}{3} (T_\nu g_{\mu\rho} - T_\rho g_{\mu\nu}) - \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} \mathbb{S}^\sigma + q_{\mu\nu\rho}, \tag{17}
\]

where \(T_\mu = T_\mu^\nu\), \(S_\mu = \epsilon_{\nu\rho\sigma} T_\nu^{\mu\rho\sigma}\), \(q_{\mu\nu\rho}\) - the antisymmetric tensor, such that \(q_{\nu\rho\nu} = 0 = \epsilon_{\mu\nu\rho\sigma} q_{\mu\nu\rho}\)

By varying the Holst action with Barbero Immirzi being a field, and using the fore-mentioned components of the torsion tensor, we obtain the system of equations:

\[
S_\mu = \frac{6}{1 + \beta^2(x)} \partial_\mu \beta, \tag{18}
\]

\[
T_\nu = \frac{3}{2} \frac{\beta(x)}{1 + \beta^2(x)} \partial_\nu \beta, \tag{19}
\]

\[
q_{\mu\nu\rho} = 0. \tag{20}
\]

It follows then from (17), (18), (19), and (20) that the BI field is constant if and only if the torsion tensor is zero.

Acknowledgments

Dedicated to the memory of my beloved father who was very supportive and much interested in the fate of this paper.

I am very grateful to Michael Bukatin for reviewing this note and for his supportive enthusiastic spirit.
References

[1] C. Will "Theory and experiment in gravitational physics", Cambridge University Press, Cambridge, (2018)

[2] C. Brans and R.H. Dicke, Phys. Rev. 124, 925 (1961)

[3] C. Brans "Jordan-Brans-Dicke-Theory", doi:10.4249/scholarpedia.31358 (2014)

[4] A. Ashtekar, Phys. Rev. Lett. 57, 2244, (1986).

[5] C. Rovelli. (2004) "Quantum Gravity", Cambridge University Press, Cambridge

[6] T. Thiemann, "Introduction to Modern Canonical Quantum General Relativity", (Cambridge University Press, 2007)

[7] A. Perez and C. Rovelli, Phys. Rev. D 73 (2006) 044013, e-Print: gr-qc/0505081 [gr-qc].

[8] L. Freidel, D. Minic, T. Takeuchi, Phys. Rev. D 72 (2005) 104002, e-Print: hep-th/0507253 [hep-th].

[9] M. Kazmierczak, Phys. Rev. D 79 (2009) 064029, e-Print: 0812.1298 [gr-qc].

[10] A. Torres-Gomez and K. Krasnov, Phys. Rev. D 79 (2009) 104014, e-Print: 0811.1998 [gr-qc].

[11] S. Mercuri and V. Taveras, Phys. Rev. D 80 (2009) 104007, e-Print: 0903.4407 [gr-qc].

[12] G. de Berredo-Peizoto, L. Freidel, I. L. Shapiro, C. A. de Souza, JCAP 06 (2012) 017, e-Print: 1201.5423 [gr-qc].

[13] A. Cisterna, C. Corral, S. del Pino, Eur. Phys. J. C 79 (2019) 5, 400, e-Print: 1809.02903 [gr-qc].

[14] S. Boudet, F. Bombacigno, G. Montani, M. Rinaldi, Phys. Rev. D 103 (2021) 8, 084034, e-Print: 2012.02700 [gr-qc].

[15] M. Shaposhnikov, A. Shkerin, I. Timiryasov, S. Zell, Phys. Rev. Lett. 126 (2021) 16, 161301, e-Print: 2008.11686 [hep-ph].

[16] M. Shaposhnikov, A. Shkerin, I. Timiryasov, S. Zell, JCAP 02 (2021) 008, e-Print: 2007.14978 [hep-ph].

[17] S. Holst. (1996) "Barbero’s Hamiltonian derived from a generalized Hilbert-Palatini action", Phys. Rev. D 53, 5966

[18] G Calcagni, S Mercuri, "Barbero–Immirzi field in canonical formalism of pure gravity", Phys. Rev. D 79, 084004 (2009)

[19] A. Torres-Gomez, K. Krasnov, "Implications of the Holst term in a f(R) theory with torsion", Phys. Rev. D 79, 104014 (2009)
[20] V. Taveras, N. Yunes, "The Barbero-Immirzi Parameter as a Scalar Field: K-Inflation from Loop Quantum Gravity?", Phys. Rev. D 78, 064070 (2008)

[21] F. Bombacigno, G. Montani, "Implications of the Holst term in a f(R) theory with torsion", Phys. Rev. D 99, 064016 (2019)

[22] O. Castillo-Felisola, C. Corral, S. Kovalenko, I. Schmidt, V. Lyubovitskij "Axions in gravity with torsion", Phys. Rev. D 91, 8, 085017 (2015)

[23] C. Will "Theoretical tools of experimental gravitation", Cambridge University Press, Cambridge, (1972)

[24] S. Weinberg, "Gravitation and cosmology principles and applications of the general theory of relativity", (Wiley 1972)

[25] P.C.C. Freire, N. Wex, G. Esposito-Farese, J.P.W. Verbiest, M. Bailes, B. A. Jacoby, M. Kramer, I. H. Stairs, J. Antoniadis, G.H. Janssen, "The relativistic pulsar–white dwarf binary PSR J1738+0333 – II. The most stringent test of scalar–tensor gravity", Monthly Notices of the Royal Astronomical Society, V 423, Issue 4, (2012)

[26] B. Bertotti, P Tortora, "A test of general relativity using radio links with the Cassini spacecraft", Nature volume 425, pages 374–376(2003)

[27] L. Imperi, Lucano Iass, "The determination of the post-Newtonian parameter $\gamma$ during the cruise phase of BepiColombo", Class Quant. Grav. Volume 34, Number 7, (2017)

[28] BarberoG.,J.Fernando, "Real Ashtekar variables for Lorentzian signature spacetimes", Phys. Rev. D 51, 5507 (1995)

[29] E. Poisson, C. W. Will "Gravity. Newtonian, Post-Newtonian, Relativistic", Cambridge University Press, Cambridge, (2014)