The magnetic Rayleigh-Taylor instability around astrophysical black holes

D. B. Papadopoulos\textsuperscript{1}, I. Contopoulos\textsuperscript{2,3}\textsuperscript{*}

\textsuperscript{1} Physics Department, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece
\textsuperscript{2} Research Center for Astronomy and Applied Mathematics, Academy of Athens, Athens 11527, Greece
\textsuperscript{3} National Research Nuclear University (MEPhI), Moscow 115409, Russia

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ABSTRACT

We investigate the development of the magnetic Rayleigh-Taylor instability at the inner edge of an astrophysical disk around a spinning central black hole. We solve the equations of general relativity that govern small amplitude oscillations of a discontinuous interface in a Keplerian disk threaded by an ordered magnetic field, and we derive a stability criterion that depends on the central black hole spin and the accumulated magnetic field. We also compare our results with the results of GR MHD simulations of black hole accretion flows that reach a magnetically arrested state (MAD). We found that the instability growth timescales that correspond to the simulation parameters are comparable to the corresponding timescales for free-fall accretion from the ISCO onto the black hole. We thus propose that the Rayleigh-Taylor instability disrupts the accumulation of magnetic flux onto the black hole horizon as the disk reaches a MAD state.

Key words: accretion, accretion discs – black hole physics – relativistic processes

1 INTRODUCTION

Magnetic fields are believed to play a fundamental role in powering energetic astrophysical sources such as active galactic nuclei, X-ray binaries and gamma-ray bursts. Extensive theoretical research over the past four decades has most convincingly shown that magnetic fields contribute to the extraction of rotational energy from spinning astrophysical black holes as proposed forty years ago by Blandford & Znajek (1977). The fundamental parameter that determines the efficiency of this process is the amount of magnetic flux $\Phi_{\text{BH}}$ that threads the black hole horizon. It is well known that magnetized accretion may bring magnetic flux toward the black hole, but when matter finally crosses the horizon, the magnetic field decouples from the matter and leaves the black hole at light-crossing times, unless there is some external medium preventing it from escaping to infinity. In astrophysical black holes, the role of this external medium is played by the surrounding accretion disk.

This configuration may be naively described as the ‘heavy’ disk material holding the ‘light’ magnetic field from escaping ‘buoyantly’, as in water over oil in vertical gravitational equilibrium. We are thus very much interested in studying the development of the magnetic Rayleigh-Taylor (hereafter RT) instability around the inner edge of the accretion disk when a large scale vertical magnetic field is present inside it. This will help us understand what limits the maximum amount of magnetic flux that threads the black hole horizon which, as we said above, is the fundamental parameter that characterizes the efficiency of the Blandford-Znajek process.

In a previous work (Contopoulos et al. 2016, hereafter Paper I), we investigated the magnetic RT instability in a non-rotating equatorial disk of plasma at the position of the innermost stable circular orbit (hereafter ISCO) around a slowly rotating black hole. We obtained very low limits for the maximum flux that can be stably held inside the ISCO. We found that a disk around a Schwarzschild black hole is unstable, and that black hole rotation slightly stabilizes the system. On the contrary, in their simulations of magnetized black hole accretion, Tchekhovskoy et al. (2012) observed only a mild dependence of the accumulated dimensionless magnetic flux on the black hole spin. In Paper I, we speculated that this may be due to our neglect of rotation and/or the dynamics of accretion. We decided to extend our analysis and consider a Keplerian isothermal incompressible equatorial disk around a Kerr black hole. The matter distribution extends practically down to the ISCO, inside which free-fall accretion abruptly reduces the matter density $\rho$ (e.g. Penna et al. 2010). There is also a vertical magnetic field $B$ that accumulates inside it. We will thus treat the ISCO as a discontinuous interface along which we will investigate the development of the magnetic RT instability. In § 2 we establish the general theoretical problem, and apply it to the study of a Kerr black hole surrounded by a Keplerian disk. In § 3 we obtain the main equation that yields the stability criterion and the instability growth timescale as a function of the accumulated magnetic field.

\textsuperscript{*} E-mail: icontop@academyofathens.gr

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and the spin of the central black hole. Finally, in § 4 we discuss the astrophysical implications of our results.

2 SMALL PERTURBATION ANALYSIS

2.1 General relativistic MHD in 3+1 formalism

As in Paper I, we follow here the 3+1 (space+time) formalism of general relativistic magnetohydrodynamics (GRMHD) developed by Thorne & Macdonald (1982). In this paper we will work in geometrical units in which $c = G = 1$. We introduce spatial magnetic and electric fields $\mathbf{B}$ and $\mathbf{E}$ respectively measured by fiducial observers with 4-velocity $U^\mu$. In that formalism, Maxwell’s equations $F_{\mu\nu} = 4\pi J^\mu$, $F_{[\mu\nu]} = 0$, and $J^\mu = 0$ yield

$$\begin{align*}
\nabla \cdot B &= 0 \\
\nabla \cdot E &= 4\pi \rho_c \\
\nabla \times E &= \frac{4}{c} \frac{\partial B}{\partial t} \\
\nabla \times B &= \frac{4}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} J
\end{align*}$$

This yields

$$\begin{align*}
D_t E + \nabla \cdot \tilde{E} &= \frac{\partial}{\partial t} \frac{\tilde{\rho}}{\alpha} + \nabla \cdot \left( \frac{\tilde{\gamma}}{\alpha} \frac{\tilde{\rho}}{\alpha} \right) + \frac{1}{\alpha} \tilde{\nabla} \times (\alpha \tilde{B}) - 4\pi \tilde{J} \\
D_t B + \nabla \cdot \tilde{B} &= \frac{\partial}{\partial t} \frac{\tilde{\sigma}}{\alpha} + \nabla \cdot \left( \frac{\tilde{\gamma}}{\alpha} \frac{\tilde{\sigma}}{\alpha} \right) - \frac{1}{\alpha} \tilde{\nabla} \times (\alpha \tilde{E})
\end{align*}$$

Here, $D_t M^\mu = M^\mu \partial_t U^\mu - U^\mu a_\mu M^\nu$ is the Fermi derivative, $\theta$ and $\sigma$ are the expansion and shear of the spacetime metric respectively, and $\rho_c$ is the electric charge density in the rest frame of the fluid. In the Kerr spacetime with 4-velocity given by eq. (23), the evolution of the magnetized fluid is characterized by the divergence of the total stress-energy tensor $T^\mu_\nu = T_{\text{matter}}^\mu_\nu + T_{\text{EM}}^\mu_\nu$, namely

$$T^\mu_\nu = 0$$

2.2 Kerr spacetime

In Boyer-Lindquist coordinates the Kerr metric reads

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2$$

For simplicity, in what follows we will consider only an isothermal fluid with $\sigma^{\theta\phi} = 0$.

$\tilde{\gamma}^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu$ is the projection tensor, and $\alpha$ is the lapse function. Latin indices take values 1, 2, 3 and Greek indices 0, 1, 2, 3. Vectors and tensors with tilde are purely spatial. For an ideal fluid with rest energy density $\rho$, 3-velocity $\mathbf{v}$, and pressure $p$ we have

$$\begin{align*}
\Gamma &= (1 - v^2)^{-1/2}, \\
\n\tilde{\gamma} &= (\rho + p)\nabla\tilde{v}, \\
\tilde{\nabla} &= (\rho + p)\nabla\tilde{v} + \tilde{\nabla} \tilde{v}
\end{align*}$$

Moreover, we assume conservation of mass (or equivalently baryon number) in the fluid, namely

$$\rho \tilde{u}_\nu = 0$$

(Chandrasekhar 1961). In Appendix A we show that, in 3+1 formalism, eq. (6) can be rewritten as

$$D_t (\Gamma \tilde{\rho}) + \Gamma \tilde{\rho} \theta + \tilde{\nabla} \cdot (\Gamma \tilde{\rho} \tilde{v}) = 0$$

We further assume an equation of state $p = \rho \tilde{v}^2$ from which one can deduce the ‘speed of sound’ $c_s = (dp/d\rho)^{1/2}$. For simplicity, in what follows we will consider only an isothermal fluid with

$$\frac{\rho}{\tilde{v}} = c_s^2 = \text{const.}$$

Finally, we assume ideal MHD conditions, namely

$$\tilde{E} = -\tilde{\nabla} \times \tilde{B}$$

As in Paper I, we investigate the development of the magnetic RT instability in the astrophysical context of a thin accretion disk, thus we restrict our analysis to the equatorial plane ($\theta = \pi/2$). This time the disk is not stationary, but is in Keplerian rotation around the central black hole.
2.3 Perturbed equations

We consider only small perturbations of physical quantities $f$ as

$$f(t, \tilde{r}) = f(\tilde{r}) + \delta f(t, \tilde{r})$$

where the perturbations in the equatorial plane $\theta = \pi/2$ are of the form

$$\delta f(t, r, \frac{\pi}{2}, \phi) = \delta f(r) e^{\iota (\iota t + \omega_0 t)}$$

and keep only linear terms of the perturbations. In this case, in the Cowling approximation of a fixed Kerr spacetime, the zeroth order MHD equations are:

$$\nabla \cdot \mathbf{E} = 4\pi \rho, \quad \nabla \times \mathbf{H} = 0$$

$$D_{\tilde{r}} \delta \tilde{E} = \tilde{\nabla} \times \tilde{B} \times \tilde{B} - \tilde{B} \times \tilde{B} \times \tilde{a} - 4\pi \tilde{\mathbf{J}}$$

$$D_{\tilde{r}} \delta \tilde{B} - \tilde{\nabla} \times \tilde{E} + \tilde{E} \times \tilde{a}$$

$$D_{\tilde{a}} \rho + \nabla \cdot (\mathbf{J} + \tilde{J} \times \tilde{B}) = 0$$

$$D_{\tilde{a}} \mathbf{E} + \nabla \times \tilde{S} + 2\tilde{S} \cdot \tilde{a} + W^\alpha \sigma_\alpha = -\tilde{J} \cdot \tilde{E}$$

$$D_{\tilde{a}} \delta \tilde{S} + \nabla \cdot \tilde{S} + \kappa \nabla \cdot \tilde{W} + \tilde{W} \cdot \tilde{a} = (\rho_0 \tilde{E} + \tilde{J} \times \tilde{B})$$

The first order MHD equations are:

$$\nabla \cdot \delta \mathbf{E} = 4\pi \delta \rho$$

$$\nabla \cdot \delta \mathbf{B} = 0$$

$$D_{\tilde{r}} \delta \tilde{E} = \tilde{\nabla} \times \delta \tilde{B} + \tilde{a} \times \delta \tilde{B} + \mathbf{E} \times \delta \tilde{E}$$

$$D_{\tilde{a}} \delta \tilde{B} - \tilde{\nabla} \times \delta \tilde{E} + \delta \tilde{E} \times \tilde{a}$$

$$D_{\tilde{a}} \delta \mathbf{E} + \nabla \times \delta \tilde{S} + \delta \tilde{S} \cdot \tilde{a} + \tilde{\mathbf{W}} \cdot \tilde{a} = -\delta \tilde{J} \cdot \tilde{E} + \tilde{\mathbf{J}} \times \delta \tilde{E}$$

The perturbed eqns. (5) become

$$\delta \tilde{v}^\mu = 2\tilde{v} \tilde{\partial} \tilde{v}^\mu, \quad \delta \mathbf{E}^2 = 2\tilde{v} \tilde{\partial} \tilde{v} (1 - \tilde{v}^2)^{-2}$$

$$\delta \mathbf{E} = \delta \mathbf{E}^\mu (\rho + \tilde{\omega}^2) + \Gamma^\mu (\rho + \tilde{\omega}^2) + \delta \rho \delta \tilde{v}^\mu$$

$$\delta \tilde{S} = [(\delta \rho + \tilde{\omega}) \tilde{\mathbf{I}}^2 + (\rho + \tilde{\omega}) \tilde{\mathbf{I}}^2 \delta \tilde{v}] \delta \tilde{v}$$

$$\delta \tilde{W} = [(\delta \rho + \tilde{\omega}) \tilde{\mathbf{I}}^2 + (\rho + \tilde{\omega}) \tilde{\mathbf{I}}^2] \delta \tilde{v} \otimes \tilde{v}$$

$$+ (\rho + \tilde{\omega}) \tilde{\mathbf{I}}^2 \delta \tilde{v} \otimes \tilde{v} \otimes \delta \tilde{v} + \tilde{\gamma} \nabla \tilde{v}$$

and the ideal MHD condition (eq. 9) yields

$$\delta \tilde{E} = -\tilde{\partial} \tilde{B} - \tilde{\nabla} \times \delta \mathbf{B}$$

2.4 Keplerian Disc

For our further study we will assume a simple Keplerian flow configuration in the equatorial plane, namely

$$\tilde{v} = (0, 0, u^\phi), \quad \tilde{\mathbf{J}} = (0, 0, 2f)$$

where, because of symmetry,

$$f = (0, 0, f) \quad \text{and} \quad \tilde{B} = (0, \tilde{B}^\phi, 0).$$

We have set $u^\varphi = 0$, but allow for nonzero $\delta u^\varphi$ equatorial velocity perturbations. We do not consider off-plane perturbations\(^1\) and set $\delta u^\varphi = 0$. Furthermore, we will assume for simplicity that $\rho_0 = 0$, $\delta \rho = 0$, and that the fluid is incompressible such that $\nabla \cdot \delta \tilde{v} = 0$ (zeroth order), $\nabla \cdot \delta \tilde{\mathbf{E}} = 0$ (first order).

The latter yields

$$\delta \rho^\alpha = -\frac{1}{\iota \tilde{\gamma}} \frac{2}{\iota \tilde{\gamma}} \delta \tilde{u}^\nu \delta \tilde{u}^\alpha$$

The perturbed eq. (7) (see Appendix A)

$$D_{\tilde{a}} \delta \mathbf{E} + \tilde{\partial} \delta \tilde{v}^\alpha - \tilde{\partial} \tilde{v}^\alpha \delta \tilde{E} + \tilde{\partial} \tilde{v}^\alpha \delta \tilde{B}$$

We now re-write eq. (25) as

$$\nabla \cdot \delta \tilde{W} = -D_{\tilde{a}} \delta \tilde{S} - \tilde{\nabla} \cdot \delta \tilde{E} - \delta \tilde{E} \cdot \tilde{a} - \tilde{\mathbf{E}} \times \delta \tilde{B}$$

We split $\delta \tilde{W} = \delta \tilde{W}^0 + \tilde{\gamma} \delta \mathbf{p}$, where $\delta \tilde{W}^0$ has contravariant components

$$\delta \tilde{W}^0 = f_1 u^\nu u^\phi + f_2 (u^\nu u^\phi + u^\nu u^\phi)$$

with

$$f_1 = (1 + \tilde{c}_1^2) (\Gamma^\nu \delta \rho + \rho \tilde{\omega} \tilde{\mathbf{I}}^\nu),$$

$$f_2 = (1 + c_1^2) \tilde{\mathbf{I}}^\nu$$

Recall that

$$\nabla \cdot \delta \tilde{W} = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\nu} \left[ \sqrt{\gamma} \tilde{W}^\nu \right] + \Gamma^\nu_{\alpha \beta} \tilde{W}^\alpha \delta \tilde{W}^\beta$$

where $\gamma = \det(\gamma_{ij}) = \Sigma^2 / \alpha^2$ is the determinant of the radial 3-metric of the Kerr space time. Next, eqs. (33) & (35) give

$$\gamma^{\nu} \delta \rho_{\nu} = -f_1 \gamma \delta u^0 \phi - f_2 \gamma \delta u^0 \phi$$

and

$$\gamma^{\nu \phi} \delta \rho_{\nu \phi} = -\gamma f_1 \gamma \delta u^0 \phi - f_2 \gamma \delta u^0 \phi$$

and

$$\gamma^{\nu \phi} \delta \rho_{\nu \phi} = -\gamma f_1 \gamma \delta u^0 \phi - f_2 \gamma \delta u^0 \phi$$

Eqs. (24) & (17) yield (see Appendix B)

$$\frac{\alpha + \iota m \omega \iota \delta u^\alpha}{\alpha} = -2\gamma \delta f_1 (u^\nu u^\phi + f_1 (u^\nu u^\phi + u^\nu u^\phi))$$

\(^1\) Off-plane perturbations along an initially vertical magnetic field may be related to the well studied magnetorotational instability (MRI; Balbus & Hawley 1991). Here, we are interested only in the development of the Rayleigh-Taylor instability triggered by equatorial motions.

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Notice that because of eqs. (26), (30) and (34)

\[ \mathbf{\nabla} \cdot \delta \mathbf{S} = \mathbf{\nabla} \cdot (f_1 \delta \mathbf{S}) + \mathbf{\nabla} \cdot (f_2 \delta \mathbf{S}) = \mathbf{\nabla} \cdot f_1 \delta \mathbf{S} + \delta \mathbf{S} \cdot \mathbf{\nabla} f_2 \]

Eqs. (37), (38) & (39) simplify considerably:

\[ \delta p_{r_a} = -\frac{T_{e_a}}{a} [(n + im \omega + im \alpha \delta) f_2 \delta \mathbf{v}_t - \gamma_{\mathbf{v}_t} \Gamma_a \delta \mathbf{v}_t - \gamma_{\mathbf{v}_t} \Gamma_a \delta \mathbf{v}_t] - \frac{T_{e_a}}{a} \Xi_{\mathbf{\Omega} \delta a} \delta \mathbf{v}_t + \alpha a \omega_{a} \nu_{a} \delta \mathbf{v}_t + \frac{\gamma_{\mathbf{v}_t}}{\sqrt{\gamma}} \delta \mathbf{v}_t \mathbf{\nabla} f_1 \delta \mathbf{S} \]

\[ \delta p_{r_\phi} = -\frac{T_{e_a}}{a} [(n + im \omega + im \alpha \delta) f_2 \delta \mathbf{v}_t + f_3 \delta \mathbf{v}_t] - \frac{T_{e_a}}{a} \Xi_{\mathbf{\Omega} \delta a} \delta \mathbf{v}_t - \gamma_{\mathbf{v}_t} \nu_{a} \delta \mathbf{v}_t f_2 \delta \mathbf{v}_t - \gamma_{\mathbf{v}_t} \nu_{a} \delta \mathbf{v}_t f_2 \delta \mathbf{v}_t - \omega_{a} \nu_{a} \alpha \delta \mathbf{v}_t f_2 \delta \mathbf{v}_t - \frac{\gamma_{\mathbf{v}_t}}{\sqrt{\gamma}} \delta \mathbf{v}_t \mathbf{\nabla} f_1 \delta \mathbf{S} \]

where

\[ \delta \mathbf{v} = \mathbf{v}_t / \mathbf{\nabla} f_1 \]

\[ f_1 = \nu_{a} f_1, \quad f_2 = (1 + c_1^2 \frac{\nu_{a}}{\sqrt{\gamma}}) \left[ \frac{\nu_{a}}{\sqrt{\gamma}} \mathbf{\nabla} f_1 \right] + \frac{\gamma_{\mathbf{v}_t}}{\sqrt{\gamma}} \delta \mathbf{v}_t \mathbf{\nabla} f_1 \delta \mathbf{S} \]

We define here the angular velocity of the flow as

\[ \Omega = \omega + \alpha \delta \mathbf{v}_t \]

From eq. (32) and (31), we obtain (see Appendix A)

\[ (n + \mathbf{\Omega} \delta \mathbf{v}_t) \mathbf{\nabla} \mathbf{v}_t = -1 \frac{\mathbf{\nabla} \cdot \mathbf{\nabla} \mathbf{v}_t}{\mathbf{\nabla} \cdot \mathbf{v}_t} \mathbf{\nabla} \mathbf{\nabla} \mathbf{v}_t \mathbf{\nabla} \mathbf{\nabla} \mathbf{v}_t \]

\[ \frac{\mathbf{\nabla} \cdot \mathbf{\nabla} \mathbf{v}_t}{\mathbf{\nabla} \cdot \mathbf{\nabla} \mathbf{v}_t} \mathbf{\nabla} \mathbf{\nabla} \mathbf{v}_t \mathbf{\nabla} \mathbf{\nabla} \mathbf{v}_t \]

Furthermore, eq. (42) with the aid of eqs. (43), (45) and (17) becomes

\[ \delta p = n_1 + i n_2 \]

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\[ \frac{\mathbf{\nabla} \cdot \mathbf{\nabla} \mathbf{v}_t}{\mathbf{\nabla} \cdot \mathbf{\nabla} \mathbf{v}_t} \mathbf{\nabla} \mathbf{\nabla} \mathbf{v}_t \mathbf{\nabla} \mathbf{\nabla} \mathbf{v}_t \]

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where \( n_1 \) and \( n_2 \) are complex expressions that can be found in the Appendix C. Eq. (41) with the aid of eq. (43) becomes

\[ \delta p_{r_a} = A_1 + i A_2 \]

where again \( A_1 \) and \( A_2 \) can be found in the Appendix C.

Taking the \( r \)-derivative of eq. (46) and comparing it with eq. (47), we obtain two independent equations in the complex plane, namely

\[ N_{1_r} = A_1 \text{ and } N_{2_r} = A_2 \]
We will assume that our physical quantities $\rho$, $p$, and $B$ are constant inside and outside $r_{\text{isco}}$ (at least in its vicinity), but change discontinuously across $r_{\text{isco}}$. The radial velocity perturbations $\delta v^r$ and the total pressure $(\rho + B^2/8\pi)$ are continuous across the interface between the two fluids, but $\mu$, $B^r$, and $(\delta v^r)_r$, in general are not.

We make here a further simplifying assumption that the interface lies far from the black hole horizon, namely that $M/r_{\text{isco}} \ll 1$ (this is obviously not valid for fast rotating black holes where the ISCO approaches the horizon). Under that approximation, we expand in powers of $M/r$ and keep terms only up to $1/r^3$. Eq. (49) now takes the form

$$\frac{d^2 w}{dr^2} + \left\{ \frac{M(1 + c^2)}{2r^2} + \frac{5M^2(1 + c^2)}{4r^3} \right\} \frac{dw}{dr} + \left\{ \frac{m^2}{r^2} - \frac{m^2 M(1 + c^2)}{2r^3} \frac{2u_1^2}{u_1^2} \right\} w = 0 \quad (52)$$

inside and outside the discontinuous interface, where $u_1^2 \equiv B^2/(4\pi \rho)$. Eq. (52) admits two independent solutions $w_1(r), w_2(r)$ that apply inside and outside the interface respectively (see Appendix D). Notice that we haven’t made here the assumption of negligibly small magnetic field as we did in Paper I.

For any physical quantity $f$ discontinuous across $r_{\text{isco}}$, we now define

$$D(f) \equiv f_{\text{in}} - f_{\text{out}}, \quad \mathcal{D}(f) \equiv f_{\text{in}} + f_{\text{out}}, \quad (53)$$

where $f_{\text{in}} \equiv f(r_{\text{isco}} - \epsilon)$ and $f_{\text{out}} \equiv f(r_{\text{isco}} + \epsilon)$. The two independent solutions $w_1(r), w_2(r)$ are then inserted in the full eq. (49) which, at the discontinuous interface yields

$$\frac{A}{\rho}(1 + \frac{\tilde{v}^2}{1 - \tilde{v}^2} f_2 + \frac{B^2}{4\pi})[w_{\text{in}}] - \frac{A}{2\rho^2}(1 + \frac{c^2}{1 - \tilde{v}^2}) \mathcal{D}[\rho w_{\text{in}}] = -\rho \mathcal{D}(\frac{1}{1 + \frac{c^2}{1 - \tilde{v}^2}} w_{\text{in}}) = -\frac{\alpha G_0(r)}{\Delta} \frac{m^2(\alpha \tilde{v}^2)}{m + m^2 \Omega^2} \left[ \frac{1 + c^2}{1 - \tilde{v}^2} \right] \mathcal{D}[\rho w] - \frac{\alpha^2 \tilde{v}^2 m^2 \alpha \tilde{v}^2}{m + m^2 \Omega^2} \mathcal{D}[\rho w] - \frac{m^2}{m^2 + m^2 \alpha \tilde{v}^2} \frac{\tilde{v}^2}{\Delta} \times \left\{ 1 - \frac{c^2}{1 - \tilde{v}^2} \right\} \mathcal{D}[\rho w] - \frac{3}{8\pi} \mathcal{D}[B^2 w] - \left\{ \frac{1 + c^2}{1 - \tilde{v}^2} \right\} \frac{\alpha G_0(r)}{\Delta} \frac{m^2 \Omega(\alpha \tilde{v}^2)}{m + m^2 \Omega^2} \mathcal{D}[\rho w]$$

Inserting eqs. (114) in eq. (54), we end up with an equation of the form

$$R = -\frac{m^2}{m^2 + m^2 \alpha \tilde{v}^2} [L_1 + \frac{m^2 L_1}{m^2 + m^2 \Omega^2}] - \frac{m^2}{m^2 + m^2 \Omega^2} [L_2 + L_3] \quad (55)$$

where the expressions for $R$, $L_1$, $L_1$, $L_2$, and $L_3$ can be found in Appendix E. Next, we write eq. (55) as

$$n^4 + n^2 m^2 \omega^2 + \Omega^2 \frac{L_1 + L_2 + L_3}{R \mathcal{L}_1} + \frac{L_3}{R \mathcal{L}_1} + \frac{L_2}{R \mathcal{L}_1} + \frac{\omega^2 L_2}{R \mathcal{L}_1} = 0 \quad (56)$$

This is the main equation of our analysis. Its roots

$$n^2_{1,2} = \frac{m^2}{2} \omega^2 + \Omega^2 + \frac{L_1 + L_2 + L_3}{R \mathcal{L}_1} + \frac{L_2}{R \mathcal{L}_1} + \frac{\omega^2 L_2}{R \mathcal{L}_1} \pm \left( \frac{L_1 + L_2 + L_3}{R \mathcal{L}_1} + \frac{\omega^2 L_2}{R \mathcal{L}_1} \right)^{1/2} \quad (57)$$

characterize the time evolution of our perturbations according to eq. (17). Whenever either one of $n^2_1, n^2_2$ is found to be positive, the system will be unstable to the development of the magnetic RT instability. Stability requires that both roots are negative.

At this point we would like to notice that, in our present work, we expanded eq. (49) up to third order in terms of $M/r$, whereas previously, in our Paper I, we expanded in terms of $a/M$. In the limit of no black hole rotation, the two approaches yield small differences in the denominator $R$ of eq. (57).

Let us here study the stability of a simple configuration at $r = r_{\text{isco}}$. As we have already said, the density drops inside the ISCO (e.g. Penna et al. 2010) because of a corresponding increase in the accretion velocity, from a value $v_{\text{isco}}$, just outside the ISCO to $v_{\text{ISCO}}$ just inside the ISCO. Thus, mass conservation across the ISCO requires that

$$\rho_{\text{ISCO}} = \rho, \quad \rho_{\text{ISCO}} = \rho \left( \frac{v_{\text{ISCO}}}{v_{\text{isco}}} \right) > \rho. \quad (58)$$

Let us now assume that there is a significant uniform vertical magnetic field $B$ accumulated inside the ISCO, and no magnetic field outside, i.e.

$$B_{\text{ISCO}}^r = B, \quad B_{\text{ISCO}}^r = 0 \quad (59)$$

Pressure balance across the interface requires that $\rho_{\text{ISCO}} c_s^2 + B^2/(8\pi) = \rho_{\text{ISCO}} c_s^2$, and therefore

$$c_s^2 = \frac{p}{\rho} = \frac{B^2}{8\pi \rho} \left( \frac{v_{\text{ISCO}}}{v_{\text{isco}}} \right) \left( \frac{v_{\text{ISCO}}}{v_{\text{isco}}} \right)^{-1} \quad (60)$$

We can now calculate the stability regime in which both roots of eq. (56) are negative. Since we are interested in the global flow disruption by the RT instability, we first consider the mode $m = 1$ (higher $m$ modes are also considered in Table 1). We also take $v_{\text{ISCO}} = 3v_{\text{isco}}$. The second root is found to be negative for all magnetic field values, the first root is negative only for small values of the magnetic field, and becomes positive at some finite value of $B$. In Fig. 1 we plot with a continuous red line the maximum value of $B/\sqrt{4\pi \rho}$ that is stable to the development of the magnetic RT instability as a function of the dimensionless black hole spin parameter $a/M$. We observe that, similarly to the case of no disk rotation studied in Paper I, a non-rotating non-accreting (Schwarzschild) black hole cannot stably hold any finite amount of magnetic field. As we noted, our present stability analysis differs slightly from that in Paper I. For black hole spins beyond about $a/M \gtrsim 0.8$, the ISCO approaches the black hole horizon, and the approximation used to obtain the red line (namely that $M/r_{\text{isco}} \ll 1$) breaks down.

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4 ASTROPHYSICAL IMPLICATIONS

The idealized conditions of a Keplerian disk with a discontinuous interface at the ISCO are rather different from the actual conditions in a real astrophysical accretion flow. If the plasma is ideal, magnetic flux is carried along by the flow in such a way that it conserves the flux to mass ratio $B/\sigma$, where $\sigma$ is the surface density in the disk. As accretion proceeds through the ISCO and the density drops, $B$ also drops, thus the conditions discussed in the previous section do not develop at the ISCO.

Such conditions (namely a drop in density with an increase in the accumulated magnetic field) develop instead on the black hole horizon where our formalism does not apply ($M/r$ is of order unity). We will, thus, adopt a discontinuous configuration at the ISCO and compare our conclusions with the results of GR MHD numerical simulations. As an example, figure 4a of Tchekhovskoy et al. (2012) shows the maximum dimensionless magnetic flux $\Phi_{BH} \equiv \Phi_{BH}/M^{1/2} \dot{r}_h$ accumulated on the black hole horizon as a function of the dimensionless black hole spin (for prograde flow rotation). In their notation, $\Phi_{BH}$ is the actual accumulated magnetic flux, $M = 2\pi\rho\nu_i$, is the mass accretion rate, $\dot{r}_h$ is the disk thickness, and $\nu_i$ is the accretion velocity. In order to connect $\Phi_{BH}$ to $B/\sqrt{4\pi\rho}$ at the ISCO, we set $\Phi_{BH} \equiv \Phi_{ISCO} \equiv \pi r_{ISCO}^4 B$. We also calculate $M$ at the ISCO. Thus we obtain

$$\frac{M}{\sqrt{4\pi\rho}} \approx \frac{\dot{r}_h}{\pi} \left( \frac{r_{ISCO}}{r_h} \right)^{-3/4} \left( \frac{h}{2\sigma} \right)^{1/2}. \quad (61)$$

The ratio $h/r$ around the inner edge of the disk is not known. We can roughly estimate it from figure 3a of Tchekhovskoy et al. (2012) as $h/2r \sim 0.2$. The simulation points resulting from eq. (61) (blue points in Fig. 1) lie above the RT instability limit (red line) for black hole spins below about $0.9M$, thus, according to our present analysis, they must be unstable to the development of the RT instability.

It is important to realize that the instability does not manifest itself instantaneously, but grows with an exponential e-folding time $t_{\text{inst}} \equiv 1/n_1$, where $n_1$ is the positive first root of eq. (57) in that region. During that time, accretion proceeds and brings the accumulated magnetic field towards the horizon on a free-fall timescale of the order of $t_f \equiv \sqrt{r_{ISCO}^3 - r_{h}^3}/(3GM)$, where $r_h$ is the radius of the black hole horizon (see Table 1; the timescale calculation is Newtonian). For the magnetic field parameters that correspond to the blue (simulation) points above the red line we found that the instability growth timescales are close to the (classical) free-fall accretion times from the ISCO onto the black hole horizon. This implies that the RT instability has enough time to begin manifesting itself.

The latter result is rather interesting. Tchekhovskoy et al. (2012) conclude that, in steady state, the black hole is saturated with magnetic flux, and the magnetic field is so strong that it obstructs the accretion and leads to a magnetically-arrested disk (MAD). It is very interesting that the maximum accumulated dimensionless magnetic flux is found to be roughly equal to its equipartition value independent of the black hole spin (for prograde flow rotation). Our present results suggest that the accumulation of the magnetic field may also be limited by the RT instability. In other words, the process of accretion and magnetic flux accumulation on the black hole horizon is probably disrupted both by the strong magnetic field and the RT instability.

In summary, our investigation of the magnetic Rayleigh-Taylor instability in this series of two papers showed that the amount of magnetic flux that can be stably accumulated inside the ISCO of a Keplerian accretion disk around a black hole is small for a slowly spinning black hole, and increases for higher black hole spins. We also found that, for black hole spins $a < 0.9M$ for which our present analysis is valid, the disk reaches a magnetically arrested state (MAD) and the accretion flow is disrupted at about the same time that the magnetic flux accumulation onto the black hole horizon is disrupted by the Rayleigh-Taylor instability.

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APPENDIX A

We derive here the law of conservation of mass (or baryon number) in 3+1 formalism. According to our notation,

\[ u^a = \frac{dx^a_{\text{fluid}}}{dt_{\text{fluid}}} \]

is the fluid 4-velocity

\[ U^a = \frac{dx^a_{\text{ZAMO}}}{dr_{\text{ZAMO}}} \]

is the ZAMO 4-velocity

\[ \tilde{v}^i = \frac{dx^i_{\text{fluid w.r. to ZAMO}}}{dr_{\text{ZAMO}}} \]

is the fluid 3-velocity w.r to ZAMO

\[ \alpha = \frac{dr_{\text{ZAMO}}}{dt_{\text{fluid}}} \]

\[ \tilde{u}^i = \frac{dr}{dt_{\text{fluid}}} = \frac{dr_{\text{ZAMO}}}{dr_{\text{ZAMO}}} = \frac{r}{\alpha} \]

where \( dr_{\text{ZAMO}}/dt_{\text{fluid}} = \Gamma \) is the fluid Lorentz factor w.r. to ZAMO observers. Obviously,

\[ dx^i_{\text{fluid}} = dx^i_{\text{ZAMO}} + dx^i_{\text{fluid w.r. to ZAMO}} \]

thus

\[ u^i = (\tilde{v}^i + U^i) \left( \frac{dr_{\text{ZAMO}}}{dr_{\text{ZAMO}}} \right) = (\tilde{v}^i + U^i) \Gamma . \]

According to Chandrasekhar (1969), the mass conservation becomes

\[ \rho (u^a)_\mu (\rho u^\mu)_\nu = (\rho u^\mu)(\rho u^\mu) = 0 , \]

or equivalently

\[ [(\rho \Gamma) / \alpha] + U'(\alpha(\rho \Gamma) / \alpha)) + \rho \Gamma (U')_\nu + \rho(\rho \Gamma / \alpha) , \]

\[ = aD_i (\rho \Gamma) + \rho \Gamma \theta + \tilde{v} \cdot (\rho \Gamma \tilde{v}) = 0 \]

(we remind the reader that the expansion \( \theta = \tilde{v} \cdot \tilde{U} \) is equal to zero in Kerr space time). We perturb Eq. (65) and find

\[ D_i (\tilde{v} \cdot \rho \Gamma) + \tilde{v} \cdot (\rho \Gamma \tilde{v}) = 0 \]

or

\[ D_i (\tilde{v} \cdot \rho \Gamma) = - \tilde{v} \cdot \rho D_i \Gamma + \mu D_i \delta \Gamma = 0 \]

where \( \delta \Gamma = \nu / \alpha \Gamma \)

\[ \tilde{v} \cdot (\rho \theta) = \tilde{v} \cdot (\rho \theta) + \tilde{v} \cdot (\rho \theta) = \frac{\rho \theta^2}{2} \]

\[ = \frac{\rho \theta^2}{2} \cdot \tilde{v} \delta \theta^2 \]

Taking into account that \( \Gamma = (1 - \tilde{v} \cdot \tilde{v})^{-1/2} \) and that the Fermi derivative includes \( \delta \) and \( \phi \) derivatives, we find immediately that \( D_i \delta = 0 \). Furthermore,

\[ \delta \Gamma = \frac{1}{2} \cdot (\rho \theta^2)^{-1/2} = \frac{1}{2} \cdot (\rho \theta^2)^{-1/2} \cdot \tilde{v} \cdot \tilde{v} = \frac{1}{2} \cdot (\rho \theta^2)^{-1/2} \cdot \tilde{v} \cdot \tilde{v} \]

and

\[ D_i \delta \tilde{v} = \frac{1}{2} \cdot (\rho \theta^2)^{-1/2} \cdot \tilde{v} \delta \theta^2 \]

Inserting all values in eq. (67), we find

\[ D_i \delta \rho + \tilde{v} \cdot \chi = \frac{1}{2} \cdot (\rho \theta^2)^{-1/2} \cdot \tilde{v} \delta \theta^2 \]

Notice that since \( \tilde{v} \) has only a \( \phi \)-component, and \( \delta \theta \) does not depend on \( \phi, \tilde{v}, \tilde{v} \cdot \tilde{v} \). We have also assumed for simplicity that \( \tilde{v} \cdot \tilde{v} = 0 \) (eq. 30). Thus, eq. (71) becomes

\[ D_i \delta \rho + \tilde{v} \cdot \chi = \frac{1}{2} \cdot (\rho \theta^2)^{-1/2} \cdot \tilde{v} \delta \theta^2 \]

For our further calculations we use mathematical formulas from Chandrasekhar (1969), De Villiers & Hawley (2003) and compute

\[ \delta \Gamma \rightarrow \frac{1}{2} \cdot (\rho \theta^2)^{-1/2} \cdot \tilde{v} \delta \theta^2 \]

\[ = \frac{1}{2} \cdot (\rho \theta^2)^{-1/2} \cdot \tilde{v} \delta \theta^2 \]

\[ \tilde{v} \cdot \chi \delta \theta^2 = \frac{1}{2} \cdot (\rho \theta^2)^{-1/2} \cdot \tilde{v} \delta \theta^2 \]

In eq. (72), the term \( \tilde{v} \cdot \tilde{u} \) becomes

\[ \tilde{v} \cdot \tilde{u} = \frac{\nu}{\alpha} \cdot (n + im \omega) \delta \theta^2 \]

\[ \tilde{v} \cdot \tilde{u} = \frac{\nu}{\alpha} \cdot (n + im \omega) \delta \theta^2 \]

\[ = \nu \cdot (n + im \omega) \delta \theta^2 \]

\[ \tilde{v} \cdot \tilde{u} = \frac{\nu}{\alpha} \cdot (n + im \omega) \delta \theta^2 \]

Eventually, from eqs. (72)-(76) we have

\[ (n + im \omega) \delta \theta = \frac{\nu}{\alpha} \cdot (n + im \omega) \delta \theta^2 \]

\[ \frac{\nu}{\alpha} \cdot (n + im \omega) \delta \theta^2 \]

which is eq. (45) in the text.

APPENDIX B

We derive here eq. (39). In the text we re-write eq. (24) as follows:

\[ D_i \delta \rho = -2a \tilde{v} \cdot \tilde{a} - \tilde{v} \cdot \delta \tilde{v} \cdot \delta \tilde{v} - \tilde{a} \cdot \delta \tilde{v} - \delta \tilde{v} \cdot \tilde{E} - \delta \tilde{E} \cdot \tilde{E} , \]

where the Fermi derivative of the scalar \( \delta \rho \) is

\[ D_i \delta \rho = \frac{\nu}{\alpha} \cdot (n + im \omega) \delta \theta^2 \]

\[ = \frac{1}{\alpha} \cdot [n + im \omega] \delta \theta^2 \]

\[ = \frac{1}{\alpha} \cdot [n + im \omega] \delta \theta^2 \]
Eq. (79) is written as
\[
\frac{1}{\alpha} (n + i m \omega) \delta \varepsilon = -2 \gamma_{i j} \delta S^i - \nabla \cdot \delta S - \sigma_{i j} \delta W_{B j}^i - \gamma_{i j} \delta E_{j}^i
\]
(80)
The third eqs. (25), with the aid of eqs. (34), gives
\[
\delta S = f_j \delta v + f_2 \delta \psi
\]
(81)
where eqs. (34) reduce to
\[
f_1 = (\delta \rho + \delta p) \delta \Gamma^2 + (\rho + p) \alpha \delta \Gamma^2
\]
\[
f_2 = (\rho + p) \delta \Gamma^2 = \frac{1}{1 - \upsilon^2} (\rho + p) \delta \Gamma^2
\]
(82)
for an isothermal fluid with \( \upsilon^2 = p/\rho \). Furthermore, under our assumption that the fluid is incompressible (namely \( \nabla \cdot \delta \psi = 0 \) and \( \nabla \cdot \delta v = 0 \)), we obtain
\[
\nabla \cdot \delta S = \nabla \cdot (f_j \delta v) + \nabla \cdot (f_2 \delta \psi)
\]
\[
f_j = f_j \delta v + \delta v \cdot \nabla f_j + f_2 \delta \psi + \delta \psi \cdot \nabla f_2
\]
\[
= \delta \psi \cdot \nabla f_j + \delta \psi \cdot \nabla f_2
\]
(83)
and thus eq. (80) reads
\[
\frac{1}{\alpha} (n + i m \omega) \delta \varepsilon = -2 \gamma_{i j} \delta S^i - \gamma_{i j} \delta \psi \partial \phi [f_j \delta v] + \gamma_{i j} \delta \psi [f_i \delta v] - \gamma_{i j} \delta E_{j}^i - \gamma_{i j} \delta E_{j}^i
\]
(84)
On the equatorial plane, there exist only one non-zero component of the acceleration \( (\delta^\phi) \) and one component of the shear tensor \( (\sigma_{\alpha \beta}) \). Also, we assume that the current \( J^x \) has only one non-zero component \( (J^y) \), and similarly for the velocity \( \psi^x \) \( (\psi^y) \). Eq. (84), now reads:
\[
\frac{1}{\alpha} (n + i m \omega) \delta \varepsilon = -2 \gamma_{i j} \delta S^i - \gamma_{i j} \delta \psi \partial \phi [f_j \delta v] + \gamma_{i j} \delta \psi [f_i \delta v] - 2 \sigma_{\alpha \beta} \delta \Gamma^\beta - \gamma_{i j} \delta E_{j}^i - \gamma_{i j} \delta E_{j}^i
\]
(85)
where from eq. (81) with \( i = r \) we obtain the \( \delta S^r \) and from the forth of eqs. (25) we obtain
\[
\delta \Gamma^\beta = f_i \delta \psi^i + f_2 (\delta \psi^\psi + \delta \psi^\psi)
\]
(86)
From eqs. (85), (81) and (86) we find
\[
\frac{1}{\alpha} (n + i m \omega) \delta \varepsilon = -2 \gamma_{i j} \delta S^i - \gamma_{i j} \delta \psi \partial \phi [f_j \delta v] + \gamma_{i j} \delta \psi [f_i \delta v] - 2 \sigma_{\alpha \beta} \delta \Gamma^\beta - \gamma_{i j} \delta E_{j}^i - \gamma_{i j} \delta E_{j}^i
\]
(87)
Further, we will compute the term \(-2 \gamma_{i j} (\alpha f_2 \delta \psi^i) \delta \psi^r \) in eq. (87) using the zero-order eq. (87).
We consider small perturbations of the form (16) with (17) in this case we have
\[
(\delta \psi \partial \phi) \delta \psi^i = \upsilon_i \delta \psi^i + \nu \delta \psi^i
\]
\[
f_i = f_i \delta v + \frac{\partial f_i}{\partial \phi} \delta \psi = \nu \delta \psi^i
\]
\[
f_2 \delta = 0
\]
\[
\delta \psi_{y}^i = \upsilon_i \delta \psi
\]
(88)
and
\[
\nabla \cdot \delta \psi = 0 \Rightarrow -i m \delta \psi = \frac{1}{\rho} (\rho^2 \delta \psi)_{x} = \chi
\]
(89)
For our further computations we will use some of the zero-order eqs. (18) in the text. From the zeroth order MHD equations we keep only
\[
D \dot{S}_i + \dot{\psi} \cdot \dot{S} + \dot{\psi} \cdot \dot{\psi} + \dot{\psi} \cdot \dot{W} + \dot{W} \cdot \dot{\psi} = (\rho E + \dot{J} \times \dot{B})
\]
(90)
where
\[
\dot{S} = f_2 \delta \psi + \dot{W} = f_2 \delta \psi + \dot{\psi} + \dot{\psi}
\]
\[
\epsilon = \Gamma^2 (\rho + p \delta \psi) = \Gamma^2 (\rho + c_s^2)
\]
(91)
\[
D \dot{S} \delta = (S^\delta + \Gamma^\delta \Gamma^\psi \dot{S}^\psi) \epsilon = \frac{\partial}{\partial \sigma} [\sqrt{\gamma^y} f_2 \epsilon^y] + \Gamma^\delta \epsilon^y f_2 \epsilon^y + \gamma^y p^y}
\]
(92)
Furthermore, eqs. (90) with the aid of eqs. (91)-(92) read
\[
D \dot{S}^y + f_2 \sigma^y \gamma_{x y} \dot{S} + \epsilon = \frac{\partial}{\partial \sigma} [\sqrt{\gamma^y} f_2 \epsilon^y] + \gamma^y p^y}
\]
(93)
Taking the \( r \)-component of the last equation (93) we find
\[
D \dot{S}^r + f_2 \sigma^r \gamma_{x y} \dot{S} + \epsilon = \frac{\partial}{\partial \sigma} [\sqrt{\gamma^y} f_2 \epsilon^y] + \gamma^y p^y}
\]
(94)
We re-write eq. (94) as follows:
\[
D \dot{S}^r + f_2 \sigma^r \gamma_{x y} \dot{S} + \epsilon = \frac{\partial}{\partial \sigma} [\sqrt{\gamma^y} f_2 \epsilon^y] + \gamma^y p^y}
\]
(95)
Because of the form of \( \epsilon \) we find that
\[
p + \epsilon = P + \Gamma^2 (\rho + c_s^2) p = \rho \epsilon^y E + (J \times B)
\]
(96)
Substitution of eq. (96) into (95) we have
\[
\dot{a}^r f_2 = -\gamma^r p_r - D_r \dot{S}^r - f_2 \sigma^r \gamma_{x y} \dot{S}^- - \frac{\partial}{\partial \sigma} [\sqrt{\gamma^y} f_2 \epsilon^y] + \gamma^y p^y}
\]
(97)
Further, we will compute the term \(-2 \gamma_{i j} (\alpha f_2 \delta \psi^i) \delta \psi^r \) in eq. (87) using the zero-order eq. (87).
From eqs. (97) and (98) and the definition of the Fermi derivative \( D_r \) we find
\[
\dot{a}^r f_2 [1 + \gamma_{r r} \dot{u}^r] = \gamma^r p_r - \frac{\partial}{\partial \sigma} [n + i m \omega] f_2 \dot{u}^r
\]
(99)
Finally, the definition of the Fermi derivative \( D_r \) we find
\[
\dot{a}^r f_2 [1 + \gamma_{r r} \dot{u}^r] = \gamma^r p_r - \frac{\partial}{\partial \sigma} [n + i m \omega] f_2 \dot{u}^r
\]
(99)
MNras 000. 000-000 (00000)
In the text, we consider $\nu' = (0,0,\nu')$, and $\rho_s = 0$. In this case, eqs. (87) and (99) simplify considerably and read

\[
\frac{(n + im\omega)\delta e}{\alpha} = -2\gamma r_{\alpha \beta} f_2 \delta u' - \tilde{\nabla} f_1 - \delta \tilde{\nabla} \cdot \tilde{\nabla} f_2
- 2\psi_\alpha [f_2 u' \delta u'] - \gamma_{\phi\phi} f_2^2 \delta E^\phi
\]

Thus, we re-write eqs. (101) as

\[
\frac{(n + im\omega)\delta e}{\alpha} = -2\gamma r_{\alpha \beta} f_2 \delta u' - \frac{1}{\alpha} \left[ \Gamma_{\phi \phi} + \omega \Gamma_{\psi \phi} + \alpha \Gamma_{\psi \phi} \right] f_2 u' - \psi_\phi [f_2 u' \delta u'] - \gamma_{\phi\phi} f_2^2 \delta E^\phi
\]

We substitute eq. (101) into eq. (100) and find

\[
\frac{(n + im\omega)\delta e}{\alpha} = -2\gamma r_{\alpha \beta} f_2 \delta u' - \frac{1}{\alpha} \left[ \Gamma_{\phi \phi} + \omega \Gamma_{\psi \phi} + \alpha \Gamma_{\psi \phi} \right] f_2 u' - \frac{\psi_\phi [f_2 u' \delta u']}{\sqrt{\gamma}} - \frac{\gamma_{\phi\phi} f_2^2 \delta E^\phi}{\sqrt{\gamma}}
\]

Because of the relation $\tilde{\nabla} \cdot \tilde{\nabla} f_1 = \nu f_{1,\phi}$ and $\delta \tilde{\nabla} \cdot \tilde{\nabla} f_2 = \delta \nu f_{2,\phi}$, eq. (102) reads

\[
\frac{(n + im\omega)\delta e}{\alpha} = -2\gamma r_{\alpha \beta} f_2 \delta u' - \frac{1}{\alpha} \left[ \Gamma_{\phi \phi} + \omega \Gamma_{\psi \phi} + \alpha \Gamma_{\psi \phi} \right] f_2 u' - \frac{\psi_\phi [f_2 u' \delta u']}{\sqrt{\gamma}} - \gamma_{\phi\phi} f_2^2 \delta E^\phi
\]

Furthermore, we re-write eqs. (82) as

\[
f_1 = \left( \partial \phi + \partial \rho \right) \Gamma^2 + (\rho + p) \partial \Gamma^2
= \left( 1 + \frac{c^2}{r^2} \right) \delta \rho + 2 \frac{f_2}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \delta \nu\ \rho
\]

\[
f_2 = (\rho + p) \Gamma^2 = \left( 1 + \frac{c^2}{r^2} \right) \delta \rho + 2 \frac{f_2}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \delta \nu\ \rho
\]

and because of the form of the perturbations we find

\[
f_{2,\phi} = 0
\]

\[
f_{1,\phi} = \left( 1 + \frac{c^2}{r^2} \right) \delta \rho \phi + 2 \frac{f_2}{1 - \frac{1}{\alpha} \sqrt{\gamma}} (\nu \delta \nu \phi)
= im \left( 1 + \frac{c^2}{r^2} \right) \delta \rho + 2 \frac{f_2}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \delta \nu \phi
\]

\[
f_{2,\phi} = (\rho + p) \Gamma^2 + (\rho + p) \left[ \Gamma^2 \right] \phi
= \frac{\rho_s + p_s}{1 - \frac{1}{\alpha} \sqrt{\gamma}} + f_2 \frac{\left( \frac{c^2}{r^2} \right) \phi}{1 - \frac{1}{\alpha} \sqrt{\gamma}}
\]

From eqs. (103), (105) and (106) we find

\[
\frac{(n + im\omega)\delta e}{\alpha} = -2\gamma r_{\alpha \beta} f_2 \delta u' - \frac{1}{\alpha} \left[ \Gamma_{\phi \phi} + \omega \Gamma_{\psi \phi} + \alpha \Gamma_{\psi \phi} \right] f_2 u' - \frac{\psi_\phi [f_2 u' \delta u']}{\sqrt{\gamma}} - \gamma_{\phi\phi} f_2^2 \delta E^\phi
\]

Finally, for an isothermal equation of state (eq. 8), we re-write eq. (107) as

\[
\frac{(n + im\omega)\delta e}{\alpha} = -\frac{\alpha(n - im\omega)}{n^2 + m^2 \omega^2} \left[ -2\nu^2 \left( \frac{f_2}{\alpha} \right) \delta u' \nu_\alpha \Gamma_{\phi \phi} + \alpha \nu^2 \Gamma_{\phi \phi} \right]
+ \delta u' \left( \frac{\nu_s}{1 - \frac{1}{\alpha} \sqrt{\gamma}} + 2 \frac{c^2}{r^2} \frac{\left( \frac{c^2}{r^2} \right)}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \right) + \text{im} f_{1,\phi} u'
- \frac{2}{\sqrt{\gamma}} \nu_\alpha B_{\phi\phi} \delta u' + \gamma_{\phi\phi} f_2^2 \delta E^\phi
\]

which is equivalent to eq. (39).

APPENDIX C

We collect here complex expressions that are used in the main text of the paper.

\[
N_1 = \frac{n}{m} \frac{A}{r^4} \frac{1 + \frac{c^2}{r^2}}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \left[ \tilde{\nabla} \nu \left( r^2 \delta \nu' \right)_s - \frac{n}{m} \frac{A}{r^4} \frac{1 + \frac{c^2}{r^2}}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \left( r^2 \delta \nu' \right)_s - \frac{1}{4\pi} \text{Re}(B_{\phi\phi} B^\phi) + \frac{n}{m} \frac{A}{r^4} \alpha \nu^2 \left( \frac{1}{4\pi} \right) \text{Re}(iB_{\phi\phi} B^\phi)
\]

\[
N_2 = \frac{A}{2mr^4} \left( \frac{1 + \frac{c^2}{r^2}}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \left[ G_s(r) \left( r^2 \delta \nu' \right) - m(\omega + 2\alpha \nu^2) \left( r^2 \delta \nu' \right)_s \right] - \frac{A}{2mr^4} \left( \frac{1 + \frac{c^2}{r^2}}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \left[ G_s(r) \left( r^2 \delta \nu' \right) - m(\omega + 2\alpha \nu^2) \left( r^2 \delta \nu' \right)_s \right] + \frac{A}{m} \left( \frac{1 + \frac{c^2}{r^2}}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \left[ G_s(r) \left( r^2 \delta \nu' \right) - m(\omega + 2\alpha \nu^2) \left( r^2 \delta \nu' \right)_s \right] + \frac{A^2}{m} \left( \frac{1 + \frac{c^2}{r^2}}{1 - \frac{1}{\alpha} \sqrt{\gamma}} \left[ G_s(r) \left( r^2 \delta \nu' \right) - m(\omega + 2\alpha \nu^2) \left( r^2 \delta \nu' \right)_s \right] \right)
\]
\[\Lambda_1 = -\left(\frac{n}{\Delta N}\right)\left(f_1 + \frac{B_i^2}{4\pi} r^2 \delta \nu'\right)
+ G_2(r) \frac{1 + c_1^2}{\alpha N} \left(\alpha N^2 + \frac{n^2 + m^2 \Omega^2}{\Delta N}\right) \times \left[\alpha N r(r^2 \delta \nu') + \frac{\rho \delta \nu r}{2(1 - \bar{v}^2)} \left(r^2 \delta \nu'\right) - \frac{\rho \bar{v}^2}{2(1 - \bar{v}^2)} \left(r^2 \delta \nu'\right) + \frac{n}{4\pi \alpha N} \frac{B_i B_p}{n^2 + m^2 \Omega^2} \left(r^2 \delta \nu'\right)
+ \frac{1}{\alpha N} \left[1 - c_1^2 \right] \frac{1}{n^2 + m^2 \Omega^2} \times (\alpha N r(r^2 \delta \nu') + \frac{\rho \delta \nu r}{2(1 - \bar{v}^2)} \left(r^2 \delta \nu'\right) - \frac{\rho \bar{v}^2}{2(1 - \bar{v}^2)} \left(r^2 \delta \nu'\right) + \frac{2n a^2}{4\pi \alpha N} \frac{B_i B_p}{n^2 + m^2 \Omega^2} \left(\alpha N^2 + \frac{\rho \delta \nu r}{2(1 - \bar{v}^2)} \left(r^2 \delta \nu'\right) - \frac{\rho \bar{v}^2}{2(1 - \bar{v}^2)} \left(r^2 \delta \nu'\right) + \frac{n}{4\pi \alpha N} \frac{B_i B_p}{n^2 + m^2 \Omega^2} \left(r^2 \delta \nu'\right)\right) + \frac{n}{4\pi \alpha N} \frac{B_i B_p}{n^2 + m^2 \Omega^2} \left(r^2 \delta \nu'\right)\right)\]
where

\[ \xi = \sqrt{1 + 4m^2} \]

\[ H_1 = \text{HeunB}[\xi, k_2, k_1, k_4, \frac{k_5}{r}] \]

\[ H_2 = \text{HeunB}[\xi, k_2, k_1, k_4, \frac{k_5}{r}] \]

\[ k_2 = -\frac{\sqrt{10}}{5} \sqrt{\frac{7(1 + c_2^2) + 2u_4^2}{1 + c_2^2}} \]

\[ k_3 = 3 \]

\[ k_4 = 2 \frac{\sqrt{10}}{5} \frac{[7(1 + c_2^2) + 2u_4^2 + m^2(1 + c_2^2 + 4u_4^2)]}{\sqrt{[1 + c_2^2][7(1 + c_2^2) + 2u_4^2]}} \]

\[ k_5 = M \frac{\sqrt{10}}{4} \frac{[1 + c_2^2][7(1 + c_2^2) + 2u_4^2]}{1 + c_2^2 + u_4^2} \]  \hspace{1cm} (115)

Here, \( c_1 \) is an arbitrary constant, and \( c_2 = c_1 r \) \( |H_1 - H_2| = r_{\text{ISCO}} \) is chosen such as to guarantee the continuity of \( u(r) \) at the interface at \( r = r_{\text{ISCO}} \). \( u_4^2 \equiv B^2(A\Delta) \) and \( \text{HeunB}[\xi, k_2, k_1, k_4, k_5/r] \) is the Heun Biconfluent function which is the solution of the Heun Biconfluent equation (Ronveaux 1995).

**APPENDIX E**

Here we present the explicit expressions \( R, L_1, L_2, \bar{L}_1, \) and \( \bar{L}_2 \) of eq. (55). \( R \equiv R_1 + \bar{R}_1, \) where

\[ R_1 = \frac{1 + \tilde{c}^2}{(1 - \tilde{c}^2)^2} (1 + c_1^2) D(\rho) + D(\frac{B^2}{4\pi}) \]

\[ - \xi \frac{1 + \tilde{c}^2}{(1 - \tilde{c}^2)^2} (1 + c_1^2) \mathcal{P}(\rho) + \mathcal{P}(\frac{B^2}{4\pi}) \]

\[ - \frac{k_2 k_5}{r} \frac{1 + \tilde{c}^2}{(1 - \tilde{c}^2)^2} (1 + c_1^2) D(\rho) + D(\frac{B^2}{4\pi}) \]

\[ - \frac{k_2 k_5}{r} \frac{1 + \tilde{c}^2}{(1 - \tilde{c}^2)^2} (1 + c_1^2) \mathcal{P}(\rho) + \mathcal{P}(\frac{B^2}{4\pi}) \]

\[ \bar{R}_1 = \frac{\tilde{c}^2(1 + c_1^2)}{2(1 - \tilde{c}^2)^2} (\mathcal{D}(\rho) - \mathcal{E} \mathcal{P}(\rho)) \]

\[ + \frac{\tilde{c}^2(1 + c_1^2)}{2(1 - \tilde{c}^2)^2} (\frac{k_2 k_5}{r} |D(\rho)| \mathcal{P}(\rho)) \]

\[ - \frac{\tilde{c}^2(1 + c_1^2)}{2(1 - \tilde{c}^2)^2} (\frac{k_2 k_5}{r} |D(\rho)| - \mathcal{E} \mathcal{P}(\rho)) \]

\[ L_1 = \frac{2r^3 A^2 B}{A \Delta} [1 - 1 - \tilde{c}^2 + 2c_1^2 c_2^2 |D(\rho)| - \frac{3 \mathcal{D}(\frac{B^2}{8\pi})}{4 \pi}] \]

\[ L_1 = -\frac{2r^3 A^2 B}{A \Delta} [\frac{1 + c_1^2}{1 - \tilde{c}^2} |\alpha^\phi(2\omega + \alpha^\rho)| D(\rho)] \]

\[ L_2 = \frac{2r^3 A^2 B}{8\pi} [\mathcal{D}(\frac{B^2}{8\pi}) (2\alpha^\rho a), \]

\[ + a(-a^\rho A \frac{r^\phi}{r^\rho}) |\Omega + (\frac{r^2 \alpha^\phi}{\Delta}) |G_2(r)| \]

\[ L_2 = \frac{2r^3 A^2 B}{8\pi} [(\alpha c_1^2) D(\rho) + (\frac{2 \alpha G_2(r) r^\phi}{A \Delta}) [\frac{1 + c_1^2}{1 - \tilde{c}^2} |\alpha^\phi| D(\rho)] \]  \hspace{1cm} (116)

All of the above expressions are evaluated at the interface at \( r = r_{\text{ISCO}} \). We remind the reader that we have defined here \( \mathcal{D}(f) \equiv f_{\Delta} - f_{\Delta} \) and \( \mathcal{P}(f) \equiv f_{\Delta} + f_{\Delta} \) at the ISCO.