Blind calibration for compressed sensing: State evolution and an online algorithm

Marylou Gabrié∗ 1, Jean Barbier2, Florent Krzakala1, Lenka Zdeborová3

1Laboratoire de Physique de l’Ecole normale supérieure, ENS, Université PSL, CNRS, Sorbonne Université, Université de Paris, F-75005 Paris, France
2International Center for Theoretical Physics, Trieste, Italy
3Institut de Physique Théorique, CEA, CNRS, Université Paris-Saclay

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Abstract

Compressed sensing, allows to acquire compressible signals with a small number of measurements. In applications, a hardware implementation often requires a calibration as the sensing process is not perfectly known. Blind calibration, that is performing at the same time calibration and compressed sensing is thus particularly appealing. A potential approach was suggested by Schülke and collaborators in [1, 2], using approximate message passing (AMP) for blind calibration (cal-AMP). Here, the algorithm is extended from the already proposed offline case to the online case, where the calibration is refined step by step as new measured samples are received. Furthermore, we show that the performance of both the offline and the online algorithms can be theoretically studied via the State Evolution (SE) formalism. Through numerical simulations, the efficiency of cal-AMP and the consistency of the theoretical predictions are confirmed.

1 Introduction

The efficient acquisition of sparse signals has been made possible by Compressed Sensing (CS) [3]. Using random projections, underdetermined linear system can be inverted, provided that the signal to recover is sparse enough and that the mixing matrix has certain properties. This technique has now many applications, in medical imaging [4, 5] for instance, where short acquisition times are preferable, or in imaging devices where measurements are costly [6]. For practical applications, a correct calibration of potential hardware problems is a central issue: performing at the same time calibration and compressed sensing when the training signals are sparse but unknown, which we refer to as blind calibration, is thus particularly appealing. Here we consider formally the case where the exact measurement process is known up to a set of variables, that need to be calibrated. In the event that complementary pairs of measurement-signal are available a priori, a supervised procedure of calibration can be imagined. We assume it is not the case and focus on blind calibration.

Gribonval and co-authors discussed in a series of articles [7, 8] a decalibration example where unknown gains multiply each sensor. More precisely, they considered for each signal $x \in \mathbb{R}^N$, an observation $y \in \mathbb{R}^M$ produced as

$$y = \mathcal{S}Wx$$

with $W \in \mathbb{R}^{M \times N}$ a known measurement matrix and $\mathcal{S}$ an unknown square diagonal matrix of calibration variables. The problem examined is to reconstruct both the corresponding signals $x$ and the calibration

∗Corresponding author: marylou.gabrie@ens.fr

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matrix $S$, provided several observations $y$. In [7, 8], the authors showed that this question can be exactly expressed as a convex optimization problem, and thus be solved using off-the-shelf algorithms. In subsequent papers [1, 2], Schülke and collaborators used instead an approximate message passing (AMP) approach, famously introduced in compressed sensing by [9], to the case of a Bayesian blind calibration. In this calibration-AMP approach (cal-AMP), both the calibration variables on the sensors and the elements of the signals are reconstructed simultaneously. The algorithm is not restricted to the case where the distortion of the measurements is a multiplication by a gain and is applicable in other settings. It relies on the extension of the AMP resolution technics to the Generalized Linear Model (GLM), for which linear mixing is followed by a generic probabilistic sensing process, corresponding to the Generalized-AMP (GAMP) algorithm [10]. Authors of [1, 2] demonstrated their approach through empirical numerical simulations, and found that considering relatively few measured samples at a time could already allow for a good calibration.

An ensuing question is whether the calibration could be performed online, that is when different observations are received successively instead of being treated at once. In learning applications, it is sometimes advantageous for speed concerns to only treat a subset of training examples at the time. Sometimes also, the size of the current data sets may exceed the available memory. Methods implementing a step-by-step learning, as the data arrives, are referred to as online, streaming or mini-batch learning, as opposed to offline or batch learning. For instance, in deep learning, Stochastic Gradient Descent is the most popular training algorithm [11]. From the theoretical point of view, the restriction to the fully online case, where a single data point is used at a time, offers interesting possibilities of analysis, as demonstrated in different machine learning problems by [12, 13, 14]. Here we will consider the Bayesian online learning of the calibration variables.

In the present paper, we revisit and extend cal-AMP with the following contributions

• We re-derive the message passing equations from a more general formulation than [1, 2]. This strategy allows us to theoretically analyze the algorithm through a State Evolution, as first done for regular AMP in [9]. This analysis was not straightforward and remained an open problem for [2]. These contributions are presented in Section 2.

• In Section 2.4, we write the free energy, or equivalently the mutual information, of the problem thanks to the general formulation and the natural connection to committee machine [15]. We conjecture our results to be rigorously exact and discuss the missing ingredient to turn our formula into a full theorem.

• Along the lines of [16], we also consider the online version of the problem in Section 3. We propose an online cal-AMP algorithm allowing for the Bayesian adjustment of the calibration as new observations arrive. The corresponding State Evolution analysis is presented as well.

• We validate numerically the performance of the algorithms and their consistency with the theoretical analyses, on the example of gain calibration, in Section 4. In particular, the algorithm remains fast and remarkably efficient going from the offline to the online setting.

Finally, we wish to clarify notations: vector variables are underlined as $\underline{x}$, matrix variables are doubly underlined as $\underline{W}$ and the symbol $\propto$ omits the normalization factor of probability distributions.
2 Cal-AMP revisited

2.1 Model of calibrated generalized linear estimation

We start by precising the model. We have \( P \) of observations \( y_{(k)} \in \mathbb{R}^M \) is gathered in a matrix \( Y \in \mathbb{R}^{M \times P} \), which were generated by the following teacher model

\[
x_0 \sim p_{x_0}(x_0)
\]

\[
\forall k = 1 \cdots P, \quad z_{0,(k)} \sim p_{z_0}(z_{0,(k)}) = \prod_{i=1}^{N} p_{x_0}(x_{0,i,(k)})
\]

\[
\forall k \quad y_{(k)} \sim p_{\text{out},0}(y_{(k)} | Wz_{0,(k)}, \xi_0) = \prod_{\mu=1}^{M} p_{\text{out},0}(y_{(k),\mu} | Wz_{0,(k)}, s_{0,\mu}).
\]

The \( z_{0,(k)} \in \mathbb{R}^N \) are \( P \) unknown signals, gathered in a matrix \( \Xi \in \mathbb{R}^{N \times P} \), linearly mixed by a known weight (or measurement) matrix \( W \) and pushed through a noisy channel denoted \( p_{\text{out},0} \). The channel is not perfectly known and depends on the realization of calibration variables \( \xi_0 \in \mathbb{R}^M \). We are interested in the estimation of the unknown signals and calibration variables. The priors and the channel of the teacher model are not necessarily available and we assume the following, a priori different, student model,

\[
p(x, s | Y, W) = \frac{1}{Z(Y, W)} p_x(x)p_s(s)p_{\text{out}}(Y | W X, s)
\]

\[
= \frac{1}{Z(Y, W)} \prod_{i=1}^{N} p_x(x_i) \prod_{\mu=1}^{M} p_s(x_{\mu})p_{\text{out}}(y_{\mu} | w_{\mu}^T x_{\mu}, s_{\mu}),
\]

with \( x_i \in \mathbb{R}^P \), and \( y_{\mu} \in \mathbb{R}^P \). The corresponding factor graph is drawn on Figure 1b. Note that the distribution could be further factorized over the index \( k \) of the \( P \) observations. The corresponding message passing was derived in \([12]\). Here we instead restrict to the level of factorization above and derive the AMP algorithm on vector variables in \( \mathbb{R}^P \). Thus cal-AMP can be seen as a special case of GAMP on vector variables as will be made clear in the next section. This point of view is key to obtain here for the first time the State Evolution analysis of the calibration problem.

2.2 Cal-AMP as a special case of GAMP

The idea of our derivation is to see calibration-AMP (cal-AMP) as a special case of Generalized Approximate Message Passing (GAMP) \([10]\), where (a) the algorithm acts simultaneously on the vector variables and (b) the channel includes a marginalization over the calibration variables. For this first variation, the AMP algorithm on vector variables was already derived by \([13]\), to treat committee machines (two-layer networks \([17]\)) without calibration. We shall recall here the main steps of the derivation, which will be useful in the following to understand how calibration variables can be taken into account, as well as the derivation of the State Evolution.

2.2.1 AMP for reconstruction of multiple samples

The systematic procedure to write AMP for a given joint probability distribution consists in first writing BP on the factor graph, second project the messages on a parametrized family of functions to obtain the corresponding relaxed-BP and third close the equations on a reduced set of parameters by keeping only leading terms in the thermodynamic limit.

For the generic GLM, without the calibration variable, the posterior measure we are interested in is

\[
p(X | Y, W) = \frac{1}{Z(Y, W)} \prod_{i=1}^{N} p(x_i) \prod_{\mu=1}^{M} p_{\text{out}}(y_{\mu} | w_{\mu}^T x_{\mu} / \sqrt{N}), \quad x_i \in \mathbb{R}^P, \quad y_{\mu} \in \mathbb{R}^P.
\]
where the known entries of matrix $W$ are drawn i.i.d from a standard normal distribution (the scaling in $1/\sqrt{N}$ is from now on made explicit). The corresponding factor graph is given on Figure 1a. We are considering the simultaneous reconstruction of $P$ signals $x_{\theta, (k)} \in \mathbb{R}^N$ and therefore write the message passing on the variables $x_i \in \mathbb{R}^P$. The major difference with the fully factorized version on scalar variables of [1,2] is that we will consider covariance matrices between variables coming from the $P$ observations instead of assuming complete independence.

**Belief propagation (BP)** We start with BP on the factor graph of Figure 1a. For all pairs of index $i - \mu$, we define the update equations of messages function

$$
\tilde{m}_\mu^{(t)}(x_i) = \frac{1}{Z_{\mu \rightarrow i}} \int \prod_{i' \neq i} \text{d}x_{i'} m_{i' \rightarrow \mu}(x_{i'}) \prod_{i' \neq i} \tilde{m}_\mu^{(t)}(x_{i'}) \quad (8)
$$

$$
m_{i \rightarrow \mu}^{(t+1)}(x_i) = \frac{1}{Z_{i \rightarrow \mu}} p_x(x_i) \prod_{\mu' \neq \mu} \tilde{m}_{\mu' \rightarrow i}^{(t)}(x_i) \quad (9)
$$

where $Z_{\mu \rightarrow i}$ and $Z_{i \rightarrow \mu}$ are normalization function that allow to interpret messages as probability distributions. To improve readability, we drop the time indices in the following derivation, and only specify them in the final algorithm.

**Relaxed BP (r-BP)** The second step of the derivation is to develop messages keeping only terms up to order $O(1/N)$ as we take the thermodynamic limit $N \rightarrow +\infty$ (at fixed $\alpha = M/N$). At this order, we will find that it is consistent to consider the messages to be approximately Gaussian, i.e. characterized by their means and co-variances. Thus we define

$$
\bar{x}_{i \rightarrow \mu} = \int \text{d}x \ m_{i \rightarrow \mu}(x) \quad (10)
$$

$$
\underline{C}_{i \rightarrow \mu} = \int \text{d}x \ x x^T m_{i \rightarrow \mu}(x) \quad (11)
$$

[Figure 1: (a) Factor graph of the Generalized Linear Model (GLM) on vector variables corresponding to the joint distribution (7). (b) Factor graph for the GLM on vector variables with not perfectly known channel including calibration variables, corresponding to the joint distribution (6).]
and
\[ \omega_{\mu \rightarrow i} = \sum_{i' \neq i} \frac{W_{\mu i'}}{\sqrt{N}} \xi_{i' \rightarrow \mu}, \]  
\[ V_{\mu \rightarrow i} = \sum_{i' \neq i} \frac{W_{\mu i'}^2}{N} C_{i' \rightarrow \mu}, \]  
where \( \omega_{\mu \rightarrow i} \) and \( V_{\mu \rightarrow i} \) are related to the intermediate variable \( \tilde{z}_\mu = w^T_X \).

**Expansion of \( \tilde{m}_{\mu \rightarrow i} \)** - We defined the Fourier transform \( \tilde{p}_{out} \) of \( p_{out}(y_{\mu} | \tilde{z}_\mu) \) with respect to its argument \( \tilde{z}_\mu = w^T_X \).

\[ \tilde{p}_{out}(y_{\mu}, \xi_{\mu}) = \int d\tilde{z}_\mu \tilde{p}_{out}(y_{\mu} | \tilde{z}_\mu) e^{-i\xi^T \tilde{z}_\mu}. \]  

Using reciprocally the Fourier representation of \( p_{out}(y_{\mu} | \tilde{z}_\mu) \),

\[ p_{out}(y_{\mu} | \tilde{z}_\mu) = \frac{1}{(2\pi M)^M} \int d\xi_{\mu} \tilde{p}_{out}(y_{\mu} | \xi_{\mu}) e^{i\xi^T \tilde{z}_\mu}, \]  
we decouple the integrals over the different \( \xi_{i'} \) in [8].

\[ \tilde{m}_{\mu \rightarrow i}(\xi_{i}) \propto \int d\tilde{z}_\mu \tilde{p}_{out}(y_{\mu} | \xi_{\mu}) e^{i\tilde{W}_{\mu}^T \tilde{z}_\mu} \prod_{i' \neq i} \int d\xi_{i'} m_{i' \rightarrow \mu}(\xi_{i'}) e^{i\tilde{W}_{i'}^T \xi_{i'}}, \]  
\[ \propto \int d\xi_{\mu} \tilde{p}_{out}(y_{\mu} | \xi_{\mu}) e^{i\xi^T \tilde{W}_{\mu} \xi_{\mu} + \tilde{w}_{\mu \rightarrow i} \xi_{\mu} - \frac{1}{2} \xi^T \tilde{V}_{\mu \rightarrow i} \xi}, \]  
where developing the exponentials of the product in [8] allows to express the integrals over the \( \xi_{i'} \) as a function of the definitions [12], [13], before re-exponentiating to obtain the final result [17]. Now reversing the Fourier transform and performing the integral over \( \xi \) we can further rewrite

\[ \tilde{m}_{\mu \rightarrow i}(\xi_{i}) \propto \int d\tilde{z}_\mu p_{out}(y_{\mu} | \tilde{z}_\mu) e^{-\frac{1}{2} \left( \tilde{z}_\mu - \tilde{w}_{\mu \rightarrow i} \right)^T \tilde{V}_{\mu \rightarrow i} \left( \tilde{z}_\mu - \tilde{w}_{\mu \rightarrow i} \right)} \]  
\[ \times \int d\xi_{\mu} p_{out}(\tilde{z}_\mu; \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}) e^{\left( \tilde{z}_\mu - \tilde{w}_{\mu \rightarrow i} \right)^T \tilde{V}_{\mu \rightarrow i}^{-1} \left( \tilde{z}_\mu - \tilde{w}_{\mu \rightarrow i} \right) - \frac{1}{2} \tilde{z}_\mu^T \tilde{V}_{\mu \rightarrow i}^{-1} \tilde{z}_\mu}, \]  
where we are led to introduce the output update functions,

\[ p_{out}(\tilde{z}_\mu; \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}) = p_{out}(y_{\mu} | \tilde{z}_\mu) \mathcal{N}(\tilde{z}_\mu; \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}), \]  
\[ Z_{out}(y_{\mu}, \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}) = \int d\tilde{z}_\mu p_{out}(y_{\mu} | \tilde{z}_\mu) \mathcal{N}(\tilde{z}_\mu; \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}), \]  
\[ g_{out}(y_{\mu}, \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}) = \frac{1}{Z_{out}} \frac{\partial Z_{out}}{\partial \omega} \quad \text{and} \quad \frac{\partial g_{out}}{\partial \omega}, \]  
where \( \mathcal{N}(\tilde{z}_\mu; \omega, V) \) is the multivariate Gaussian distribution of mean \( \omega \) and covariance \( V \). Further expanding the exponential in [19] up to order \( O(1/N) \) leads to the Gaussian parametrization

\[ \tilde{m}_{\mu \rightarrow i}(\xi_{i}) \propto 1 + \frac{W_{\mu i}}{\sqrt{N}} g_{out} \xi_{i} + \frac{W_{\mu i}^2}{2N} \xi_{i}^T (g_{out} g_{out}^T + \frac{\partial g_{out}}{\partial \omega}) \xi_{i}, \]
\[ \propto e^{\tilde{B}_{\mu \rightarrow i}^T \tilde{z}_{\mu} - \frac{1}{2} \tilde{z}_{\mu}^T \tilde{V}_{\mu \rightarrow i} \tilde{z}_{\mu}}, \]  
where \( \tilde{B}_{\mu \rightarrow i} = \sum_{i' \neq i} \frac{W_{\mu i'}}{\sqrt{N}} C_{i' \rightarrow \mu} \) and \( \tilde{V}_{\mu \rightarrow i} = \sum_{i' \neq i} \frac{W_{\mu i'}^2}{N} \).
with
\[ B_{\mu \rightarrow i} = \frac{W_{\mu i}}{\sqrt{N}} g_{\text{out}}(y_{\mu}, \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}) \] (25)
\[ A_{\mu \rightarrow i} = -\frac{W_{\mu i}^2}{N} \partial_y g_{\text{out}}(y_{\mu}, \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}). \] (26)

**Consistency with** \( m_{i \rightarrow \mu} \) - Inserting the Gaussian approximation of \( \tilde{m}_{\mu \rightarrow i} \) in the definition of \( m_{i \rightarrow \mu} \), we get the parametrization
\[ m_{i \rightarrow \mu}(x_i) \propto p_x(x_i) \prod_{\mu' \neq \mu} c_{\mu' \rightarrow i}^T (x_i - \bar{x}) \bar{\sigma}^{-1} (x_i - \bar{x}) \] (27)

with
\[ \bar{\Lambda}_{i \rightarrow \mu} = \bar{a}_{i \rightarrow \mu} \left( \sum_{\mu' \neq \mu} B_{\mu' \rightarrow i} \right) \] (28)
\[ \bar{a}_{i \rightarrow \mu} = \left( \sum_{\mu' \neq \mu} A_{\mu' \rightarrow i} \right)^{-1}. \] (29)

**Closing the equations** - Ensuring the consistency with the definitions (10)-(11) of mean and covariance of \( m_{i \rightarrow \mu} \) we finally close our set of equations by defining the input update functions,
\[ Z^x = \int dx p_x(x) e^{-\frac{1}{2} (x - \bar{x})^T \bar{\sigma}^{-1} (x - \bar{x})} \] (30)
\[ f^x_{1}(\Lambda, \sigma) = \frac{1}{Z^x} \int dx x p_x(x) e^{-\frac{1}{2} (x - \bar{x})^T \bar{\sigma}^{-1} (x - \bar{x})} \] (31)
\[ f^x_{2}(\Lambda, \sigma) = \frac{1}{Z^x} \int dx x^T p_x(x) e^{-\frac{1}{2} (x - \bar{x})^T \bar{\sigma}^{-1} (x - \bar{x})} - f^x_{1}(\Lambda, \sigma) f^x_{1}(\Lambda, \bar{\sigma})^T, \] (32)
so that
\[ \hat{x}_{i \rightarrow \mu} = f^x_{1}(\bar{\Lambda}_{i \rightarrow \mu}, \bar{a}_{i \rightarrow \mu}) \] (33)
\[ C^x_{i \rightarrow \mu} = f^x_{2}(\bar{\Lambda}_{i \rightarrow \mu}, \bar{a}_{i \rightarrow \mu}). \] (34)

The closed set of equations (12), (13), (25), (26), (28), (29), (33) and (34), with restored time indices, defines the r-BP algorithm. At convergence of the iterations, we obtain the approximated marginals
\[ m_i(x_i) = \frac{1}{Z_i} p_x(x_i) e^{-\frac{1}{2} (x_i - \bar{x}_i)^T \bar{\sigma}_i^{-1} (x_i - \bar{x}_i)} \] (35)

with
\[ \bar{\Lambda}_i = \bar{a}_i \left( \sum_{\mu=1}^{M} B_{\mu \rightarrow i} \right) \] (36)
\[ \bar{a}_i = \left( \sum_{\mu} A_{\mu \rightarrow i} \right)^{-1}. \] (37)
While BP requires to follow iterations over \( M \times N \) message distributions over vectors in \( \mathbb{R}^P \), r-BP only requires to track \( O(M \times N \times P) \) variables, which is a great simplification. Nonetheless, r-BP is scarcely used as such as the computational cost can be readily reduced with little more approximation. In the thermodynamic limit, the messages are closely related to the marginals as the contribution of the missing message between \( \mu \) and \( \nu \) is to a certain extent negligible. Careful book keeping of the order of contributions leads to a set of closed equations on parameters of the marginals, i.e. \( O(N) \) variables, corresponding to the GAMP algorithm.

**Approximate message passing** Given the scaling of the weights in \( O(1/\sqrt{N}) \) it is possible to further simplify the algorithm in the thermodynamic limit. We define parameters \( \tilde{w}_\mu, V_\mu \) and \( \tilde{z}_\mu, C^x_\mu \), likewise \( \tilde{\lambda}_\mu \) and \( \tilde{\sigma}_\mu \) defined above and consider their relations to the original \( \tilde{\lambda}_\mu \rightarrow \mu, \tilde{\sigma}_\mu \rightarrow \mu, \tilde{V}_\mu \rightarrow \mu, \tilde{z}_\mu \rightarrow \mu \) and \( C^x_\mu \rightarrow \mu \). As a result we obtain the vectorized AMP for the GLM presented in Algorithm [1] and [5].

Note that, similarly to GAMP, relaxing the Gaussian assumption on the weight matrix entries to any distribution with finite second moment yields the same algorithm using the Central Limit Theorem. Furthermore, the algorithmic procedure should also generalize to a wider class of random matrices allowing for correlations between entries: the ensemble of orthogonally invariant matrices. In the singular value decomposition of such weight matrices \( W = U S V^\top \in \mathbb{R}^{M \times N} \) the orthogonal basis matrices \( U \) and \( V \) are drawn uniformly at random from respectively \( O(M) \) and \( O(N) \), while the diagonal matrix of singular values \( S \) has an arbitrary spectrum. For the GLM, without calibration variables, the signal is recovered in such cases by the (Generalized) Vector-Approximate Message Passing (G-VAMP) algorithm [18, 19], inspired from prior works in statistical physics [20, 21, 22, 23, 24] and statistical inference [25, 26].

### 2.2.2 Treatment of calibration variables

**Heuristic derivation** To include calibration variables, we need to consider the factor graph on Figure [1b] and augment the belief propagation with a set of messages related to the \( s_\mu \). Without going back through the entire derivation the final algorithm can easily be deduced. The posterior distribution on \( \hat{x}_\mu \) in the presence of the calibration variable \( s \) is a special case of the GLM on vector variables examined above with the effective channel

\[
p_{\text{out}}(y_\mu | w_\mu^\top X) = \int ds_\mu p_{\text{out}}^s(y_\mu | w_\mu^\top X, s_\mu) p_s(s_\mu).
\]

(46)

Thus to reconstruct \( X \) one can directly use Algorithm [1] with output functions [20, 22] that will include a marginalization over \( s_\mu \):

\[
Z_{\text{out}}(y_\mu, w_\mu, V_\mu) = \int d\hat{z}_\mu \int ds_\mu p_{\text{out}}^s(y_\mu | \hat{z}_\mu, s_\mu) p_s(s_\mu) N(\hat{z}_\mu; w_\mu, V_\mu).
\]

(47)

The estimate of a variable constructed by (sum-product) AMP is always the mean of the approximate posterior marginal distribution. For the calibration variable, the AMP posterior is already displayed in the above \( Z_{\text{out}} \),

\[
m^*_\mu(s_\mu) = \frac{1}{Z_{\text{out}}} \int d\hat{z}_\mu p_{\text{out}}^s(y_\mu | \hat{z}_\mu, s_\mu) p_s(s_\mu) N(\hat{z}_\mu; w_\mu, V_\mu).
\]

(48)

So that at convergence of Algorithm [1] we can compute the estimate and uncertainty on the calibration variable

\[
\hat{s}_\mu = f^1_\mu(y_\mu, w_\mu, V_\mu) = \frac{1}{Z_{\text{out}}} \int ds_\mu s_\mu \int d\hat{z}_\mu p_{\text{out}}^s(y_\mu | \hat{z}_\mu, s_\mu) p_s(s_\mu) N(\hat{z}_\mu; w_\mu, V_\mu)
\]

(49)

\[
C^s_\mu = f^2_\mu(y_\mu, w_\mu, V_\mu) = \frac{1}{Z_{\text{out}}} \int ds_\mu s_\mu^2 \int d\hat{z}_\mu p_{\text{out}}^s(y_\mu | \hat{z}_\mu, s_\mu) p_s(s_\mu) N(\hat{z}_\mu; w_\mu, V_\mu) - \hat{s}_\mu^2.
\]

(50)
Algorithm 1 Generalized Approximate Message Passing (GAMP) for vectors

**Input:** matrix $Y \in \mathbb{R}^{M \times P}$ and matrix $W \in \mathbb{R}^{M \times N}$

**Initialize:** $\hat{x}_i, C_{x_i}, \forall i$ and $g_{\text{out}_\mu}, \partial g_{\text{out}_\mu}, \forall \mu$

repeat

1) Estimate mean and variance of $z_\mu$ given current $\hat{x}_i$

$$V_\mu(t) = \sum_{i=1}^{N} \frac{W_{\mu i}^2}{N} C_{x_i}^{(t)}$$

$$\omega_\mu(t) = \sum_{i=1}^{N} \frac{W_{\mu i} z_i^{(t)}}{\sqrt{N}} - \sum_{i=1}^{N} \left( \frac{W_{\mu i}^2}{N} g_{\text{out}_\mu}^{(t)} - 1 \right) C_{x_i}^{(t)} \sigma_i^{(t)} g_{\text{out}_\mu}^{(t-1)}$$

2) Estimate mean and variance of the gap between optimal $z_\mu$ and $\omega_\mu$ given $y_\mu$

$$\partial g_{\text{out}_\mu}^{(t)} = \partial g_{\text{out}_\mu}(y_\mu, \omega_\mu^{(t)}, V_\mu^{(t)})$$

$$g_{\text{out}_\mu}^{(t)} = g_{\text{out}}(y_\mu, \omega_\mu^{(t)}, V_\mu^{(t)})$$

3) Estimate mean and variance of $x_i$ given current optimal $z_\mu$

$$\sigma_i^{(t)} = \left( - \sum_{\mu=1}^{M} \frac{W_{\mu i}^2}{N} \partial g_{\text{out}_\mu}^{(t)} \right)^{-1}$$

$$\lambda_i^{(t)} = \hat{x}_i^{(t)} + \sigma_i^{(t)} \left( \sum_{\mu=1}^{M} \frac{W_{\mu i}}{\sqrt{N}} g_{\text{out}_\mu}^{(t)} \right)$$

4) Estimate of mean and variance of $x_i$ augmented of the information about the prior

$$C_{x_i}^{(t+1)} = f_x^{(t)}(\lambda_i^{(t)}, \sigma_i^{(t)})$$

$$\hat{x}_i^{(t+1)} = f_x^{(t)}(\lambda_i^{(t)}, \sigma_i^{(t)})$$

until convergence

**Relation to original cal-AMP derivation** In [1, 2], the cal-AMP algorithm was derived from the belief propagation equations on the scalar variables of the fully factorized distribution (over $N, M$ and $P$). It is equivalent to Algorithm 1 if covariance matrices $V_\mu, \partial g_{\text{out}_\mu}, \sigma_i, C_{x_i}$ are assumed to be diagonal. However, we recall that BP is only exact on a tree, where the incoming message at each node are truly independent. On the dense factor graph of the GLM on scalar variables, this is approximately exact in the thermodynamic limit due to the random mixing, and small scaling, of the weight matrix $W$. Here, by considering the reconstruction of $P$ samples at the same time sharing given realizations of the calibration variable $s$, as well as the measurement matrix $W$, additional correlations may arise. Writing the message passing on the vector variables in $\mathbb{R}^P$ allows not to neglect them.

2.3 State Evolution for cal-AMP

In the large size limit, where $N \to +\infty$ at fixed $\alpha = M/N$, the performance of the AMP algorithm can be characterized by a set of simpler equations corresponding to a quenched average over the disorder (here the realizations of $\lambda_\mu, s_0, Y$ and $W$), referred to as State Evolution (SE) [9]. Remarkably, the SE equations are equivalent to the saddle point equations of the replica free energy associated with the problem under the Replica Symmetric (RS) ansatz [27]. In [19], the teacher-student matched setting of the GLM on vectors is examined through the replica approach and the Bayes optimal State Evolution equations are obtained
through this second strategy. In the following we present the alternative derivation of the State Evolution equations from the message passing and without assuming a priori matched teacher and student. To this end, our starting point will be the r-BP equations. Finally, we also introduce new State Evolution equations following the reconstruction of calibration variables.

### 2.3.1 State Evolution derivation in mismatched prior and channel setting

**Definition of the overlaps** The important quantities to follow the dynamic of iterations and fixed points of AMP are the overlaps. Here, they are the $P \times P$ matrices

$$q_i = \frac{1}{N} \sum_{i=1}^{N} \xi_{i} \xi_{i}^T, \quad m_i = \frac{1}{N} \sum_{i=1}^{N} \xi_{i} x_{0,i}^T, \quad q_{2i} = \frac{1}{N} \sum_{i=1}^{N} x_{0,i} x_{0,i}^T. \quad (51)$$

**Output parameters** Under independent statistics of the entries of $W$ and under the assumption of independent incoming messages, the variable $w_{\mu \rightarrow i}$ defined in (12) is a sum of independent variables and follows a Gaussian distribution by the Central Limit Theorem. Its first and second moments are

$$E_{W}[w_{\mu \rightarrow i}] = \frac{1}{\sqrt{N}} \sum_{i' \neq i} E_{W}[W_{\mu,i}^2] \xi_{i'} \rightarrow \mu = 0, \quad (52)$$

$$E_{W}[w_{\mu \rightarrow i}^2] = \frac{1}{N} \sum_{i' \neq i} E_{W}[W_{\mu,i}^2] \xi_{i'} \rightarrow \mu \xi_{i'} \rightarrow \mu = \frac{1}{N} \sum_{i=1}^{N} \xi_{i} \xi_{i}^T + O(1/N) \quad (53)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \xi_{i} \xi_{i}^T - \partial_{A} f_{\chi}^T B_{\mu \rightarrow i} \xi_{i} \rightarrow \mu - (\partial_{A} f_{\chi}^T B_{\mu \rightarrow i} \xi_{i} \rightarrow \mu)^T + O(1/N) \quad (54)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \xi_{i} \xi_{i}^T + O(1/\sqrt{N}) \quad (55)$$

where we used the fact that $B_{\mu \rightarrow i}$ defined in (25) is of order $O(1/\sqrt{N})$. Similarly, the variable $z_{\mu \rightarrow i} = \sum_{i' \neq i} W_{\mu,i'} x_{i'}$ is Gaussian with first and second moments

$$E_{W}[z_{\mu \rightarrow i}] = \frac{1}{\sqrt{N}} \sum_{i' \neq i} E_{W}[W_{\mu,i}] x_{0,i'} = 0, \quad (56)$$

$$E_{W}[z_{\mu \rightarrow i}^2] = \frac{1}{N} \sum_{i=1}^{N} x_{0,i} x_{0,i}^T + O(1/\sqrt{N}). \quad (57)$$

Furthermore, their covariance is

$$E_{W}[z_{\mu \rightarrow i} z_{\mu \rightarrow i}^T] = \frac{1}{N} \sum_{i' \neq i} E_{W}[W_{\mu,i}^2] x_{0,i'} x_{0,i'}^T = \frac{1}{N} \sum_{i=1}^{N} x_{0,i} x_{0,i}^T + O(1/N) \quad (58)$$

$$= \frac{1}{N} \sum_{i=1}^{N} x_{0,i} x_{0,i}^T - x_{0,i} \partial_{A} f_{\chi}^T B_{\mu \rightarrow i} + O(1/N) \quad (59)$$

$$= \frac{1}{N} \sum_{i=1}^{N} x_{0,i} x_{0,i}^T + O(1/\sqrt{N}). \quad (60)$$
Hence we find that for all $\mu$-s and all $i$-s, $\omega_{\mu \rightarrow i}$ and $z_{\mu \rightarrow i}$ are approximately jointly Gaussian in the thermodynamic limit following a unique distribution $\mathcal{N}\left(z_{\mu \rightarrow i}, \omega_{\mu \rightarrow i}; 0, Q\right)$ with the block covariance matrix

$$Q = \begin{bmatrix} q & 0 \\ m & m^T \end{bmatrix}.$$  \quad (61)

For the variance message $V_{\mu \rightarrow i}$, defined in (13), we have

$$E_W \left[V_{\mu \rightarrow i} \right] = \sum_{i' \neq i} E_W \left[W_{\mu i}^2 / N \right] C_{\mu i' \rightarrow \mu} = \sum_{i=1}^N \frac{1}{N} C_{\mu i \rightarrow \mu} + O \left(1/N\right) \quad (62)$$

$$= \sum_{i=1}^N \frac{1}{N} C_{\mu i \rightarrow \mu} + O \left(1/\sqrt{N}\right), \quad (63)$$

where using the developments of $\lambda_{i \rightarrow \mu}$ and $\sigma_{i \rightarrow \mu}$ along with the scaling of $B_{\mu i \rightarrow i}$ in $O(1/\sqrt{N})$ we replaced

$$C_{\mu i \rightarrow \mu} = f_x \left(\lambda_{i \rightarrow \mu}, \sigma_{i \rightarrow \mu}\right) = f_x \left(\lambda_{i}, \sigma_{i}\right) - \partial_{\lambda_{i}} f_x \sigma_{i} B_{\mu i \rightarrow i} = f_x \left(\lambda_{i}, \sigma_{i}\right) + O \left(1/\sqrt{N}\right). \quad (64)$$

Furthermore, we can check that

$$\lim_{N \to +\infty} E_W \left[V_{\mu \rightarrow i}^2 - E_W \left[V_{\mu \rightarrow i} \right]^2 \right] = 0, \quad (65)$$

meaning that all $V_{\mu \rightarrow i}$ concentrate on their identical mean in the thermodynamic limit, which we note

$$V = \sum_{i=1}^N \frac{1}{N} C_{\mu i \rightarrow \mu}. \quad (66)$$

Input parameters Here we use the re-parametrization trick to express $y_{\mu}$ as a function $g_0(\cdot)$ taking the calibration variable $s_{\mu}$ and a noise $\xi_{\mu} \sim p_\epsilon(\xi_{\mu})$ as inputs: $y_{\mu} = g_0 \left(\frac{w_{\mu}^T X_0}{\sqrt{N}}, s_{0, \mu}, \xi_{\mu}\right)$. Following (25) and (35),

$$\sigma_{\mu i}^{-1} \lambda_{i} = \sum_{\mu=1}^M \frac{W_{\mu i}}{\sqrt{N}} g_{out} \left(y_{\mu}, \omega_{\mu \rightarrow i}; V_{\mu \rightarrow i}\right) \quad (67)$$

$$= \sum_{\mu=1}^M \frac{W_{\mu i}}{\sqrt{N}} g_{out} \left(g_0 \left(\sum_{i' \neq i} \frac{W_{\mu i'}}{\sqrt{N}} \omega_{0, i'}, \omega_{\mu i}, \xi_{\mu}\right), \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}\right) \quad (68)$$

$$= \sum_{\mu=1}^M \frac{W_{\mu i}}{\sqrt{N}} g_{out} \left(g_0 \left(\sum_{i' \neq i} \frac{W_{\mu i'}}{\sqrt{N}} \omega_{0, i'}, s_{\mu}, \xi_{\mu}\right), \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}\right) + \sum_{\mu=1}^M \frac{W_{\mu i}^2}{N} \partial_{s_\mu} g_{out} \left(g_0 \left(\omega_{\mu \rightarrow i}, s_{\mu}, \xi_{\mu}\right), \omega_{\mu \rightarrow i}, V_{\mu \rightarrow i}\right) \omega_0, i. \quad (69)$$

The first term is again a sum of independent random variables, given the $W_{\mu i}$ are i.i.d. with zero mean, of which the messages of type $\mu \rightarrow i$ are assumed independent. The second term has non-zero mean and can be
shown to concentrate. Finally recalling that all \( V_\rightarrow \) also concentrate on \( V \) we obtain the distribution
\[
\bar{\sigma}^{-1} \Delta_i \sim \mathcal{N} \left( \bar{\sigma}^{-1} \Delta_i; \alpha \bar{\mu}_\rightarrow x_{0,i}, \sqrt{\alpha \bar{q}} \right)
\] (70)

with
\[
\hat{q} = \int d\xi \, p_\rightarrow(\xi) ds_0 \, p_{s_0}(s_0) \int d\omega \, d\tilde{z} \, \mathcal{N}(\tilde{z}, \omega; 0, Q) g_{out}(g_0(\tilde{z}, s_0, \xi), \omega, V) \times
\]
\[
\bar{g}_{out}(g_0(\tilde{z}, s_0, \xi), \omega, V)^T,
\]
\[
\hat{m} = \int d\xi \, p_\rightarrow(\xi) ds_0 \, p_{s_0}(s_0) \int d\omega \, d\tilde{z} \, \mathcal{N}(\tilde{z}, \omega; 0, Q) \bar{\partial}_\rightarrow g_{out}(g_0(\tilde{z}, s_0, \xi), \omega, V).
\] (72)

For the inverse variance \( \bar{\sigma}^{-1} \) one can check again that it concentrates on its mean
\[
\bar{\sigma}^{-1} = \sum_{\mu=1}^{M} \frac{W_\mu^2}{N} \bar{\partial}_\rightarrow g_{out}(y_\rightarrow; \omega_{\rightarrow \rightarrow}, V) \sim \alpha \chi^2,
\] (73)
\[
\chi = -\int d\xi \, p_\rightarrow(\xi) ds_0 \, p_{s_0}(s_0) \int d\xi \, d\tilde{z} \, \mathcal{N}(\tilde{z}, \omega; 0, Q) \bar{\partial}_\rightarrow g_{out}(g_0(\tilde{z}, s_0, \xi), \omega, V).
\] (74)

**Closing the equations** These statistics of the input parameters must ensure that consistently
\[
\tilde{V} = \frac{1}{N} \sum_{i=1}^{N} \bar{G}_i = \mathbb{E}_{\bar{\lambda}, \bar{\sigma}} \left[ f^x(\bar{\lambda}, \bar{\sigma}) \right],
\] (75)
\[
\tilde{q} = \frac{1}{N} \sum_{i=1}^{N} \bar{\xi}_i \bar{\chi}_i^T = \mathbb{E}_{\bar{\lambda}, \bar{\sigma}} \left[ f^x(\bar{\lambda}, \bar{\sigma}) f^x(\bar{\lambda}, \bar{\sigma})^T \right],
\] (76)
\[
\tilde{m} = \frac{1}{N} \sum_{i=1}^{N} \bar{\xi}_i \bar{\xi}_0,i^T = \mathbb{E}_{\bar{\lambda}, \bar{\sigma}} \left[ f^x(\bar{\lambda}, \bar{\sigma}) \bar{\xi}_0,i^T \right],
\] (77)

which gives upon making the computation of the expectations
\[
\tilde{V} = \int d\xi_0 \, p_{x_0}(\xi_0) \int d\xi \, f^x_\rightarrow \left( (\alpha \bar{\chi})^{-1} \left( \sqrt{\alpha \bar{q}} + \alpha \bar{m} x_0 \right) ; (\alpha \bar{\chi})^{-1} \right),
\] (78)
\[
\tilde{m} = \int d\xi_0 \, p_{x_0}(\xi_0) \int d\xi \, f^x_\rightarrow \left( (\alpha \bar{\chi})^{-1} \left( \sqrt{\alpha \bar{q}} + \alpha \bar{m} x_0 \right) ; (\alpha \bar{\chi})^{-1} \right) \bar{\xi}_0^T,
\] (79)
\[
\tilde{q} = \int d\xi_0 \, p_{x_0}(\xi_0) \int d\xi \, f^x_\rightarrow \left( (\alpha \bar{\chi})^{-1} \left( \sqrt{\alpha \bar{q}} + \alpha \bar{m} x_0 \right) ; (\alpha \bar{\chi})^{-1} \right) \times
\]
\[
f^x_\rightarrow \left( (\alpha \bar{\chi})^{-1} \left( \sqrt{\alpha \bar{q}} + \alpha \bar{m} x_0 \right) ; (\alpha \bar{\chi})^{-1} \right)^T.
\] (80)

The State Evolution analysis of the GLM on the vector variables finally consists in iterating alternatively the equations (71), (72), (74), and the equations (78), (79), (80) until convergence.

**Reconstruction of the calibration variable** In parallel, one can follow the reconstruction of \( \bar{c} \) by introducing the scalar overlaps
\[
r = \frac{1}{M} \sum_{\mu=1}^{M} \bar{s}_\mu^2, \quad \nu = \frac{1}{M} \sum_{\mu=1}^{M} \bar{s}_\mu s_{0,\mu}, \quad r_0 = \frac{1}{M} \sum_{\mu=1}^{M} s_{0,\mu}^2.
\] (81)
Recalling the definition of the estimator \( \hat{s} \) \(^{(49)}\), and after following the steps of the above derivation, one can see that the calibration overlaps can be computed from the previously introduced SE variables,

\[
\begin{align*}
    r &= \int d\bar{\epsilon} p_{\epsilon}(\bar{\epsilon}) ds_0 p_{s_0}(s_0) \int d\omega' d\tilde{z}' \mathcal{N}(\tilde{z}', \omega'; 0, Q) \hat{s}(g_0 (\tilde{z}', s_0, \epsilon), \omega', V)^2, \\
    \nu &= \int d\bar{\epsilon} p_{\epsilon}(\bar{\epsilon}) ds_0 p_{s_0}(s_0) \int d\omega' d\tilde{z}' \mathcal{N}(\tilde{z}', \omega'; 0, Q) \hat{s}(g_0 (\tilde{z}', s_0, \epsilon), \omega', V) s_0.
\end{align*}
\]

(82) (83)

**Performance analysis** The mean squared error (MSE) on the reconstruction of \( \underline{X} \) by the AMP algorithm is then predicted by

\[
\text{MSE}(\underline{X}) = q - 2m + q_0,
\]

(84)

where the scalar values used here correspond to the (unique) value of the diagonal elements of the corresponding overlap matrices. This MSE can be computed throughout the iterations of State Evolution. Similarly, the MSE in the reconstruction of the calibration variable can be computed as

\[
\text{MSE}(\underline{s}) = r - 2\nu + r_0,
\]

(85)

throughout the iterations. Remarkably, the State Evolution MSEs follow precisely the MSE of the cal-AMP predictors along the iterations of the algorithm provided the procedures are initialized consistently. A random initialization of \( \hat{x}_i \) and \( \hat{s} \) in cal-AMP corresponds to an initialization of zero overlap \( m = 0, \nu = 0 \), with variance of the priors \( q = q_0, r = r_0 \), in the State Evolution.

### 2.3.2 Bayes optimal State Evolution

The SE equations can be greatly simplified in the Bayes optimal setting where the statistical model used by the student (priors \( p_x \) and \( p_s \), and channel \( p_{\text{out}} \)) is known to match the teacher. In this case, the true unknown signal \( \underline{X}_0 \) is in some sense statistically equivalent to the estimate \( \hat{X} \) coming from the posterior. More precisely one can prove the Nishimori identities \([28, 29, 30]\) (or \([31]\) for a concise demonstration and discussion) implying that

\[
\begin{align*}
    q &= m, \\
    V &= q_0 - m, \\
    \hat{q} &= \hat{\chi} \quad \text{and} \quad r = \nu.
\end{align*}
\]

(86)

As a result the State Evolution reduces to a set of three equations

\[
\begin{align*}
    r &= \int d\bar{\epsilon} p_{\epsilon}(\bar{\epsilon}) ds_0 p_{s_0}(s_0) \int d\omega' d\tilde{z}' \mathcal{N}(\tilde{z}', \omega'; 0, Q) \hat{s}(g_0 (\tilde{z}', s_0, \epsilon), \omega', q_0 - m)^2, \\
    \hat{m} &= \int d\bar{\epsilon} p_{\epsilon}(\bar{\epsilon}) ds_0 p_{s_0}(s_0) \int d\omega' d\tilde{z}' \mathcal{N}(\tilde{z}', \omega'; 0, Q) \hat{s}(g_0 (\tilde{z}', s_0, \epsilon), \omega', q_0 - m) \frac{1}{\sqrt{\alpha m} + \alpha \hat{m} \gamma_0} \gamma_0 \oplus (\alpha \hat{m})^{-1} \\gamma_0, \\
    \hat{\chi} &= \int d\bar{\epsilon} p_{\epsilon}(\bar{\epsilon}) ds_0 p_{s_0}(s_0) \int d\omega' d\tilde{z}' \mathcal{N}(\tilde{z}', \omega'; 0, Q) g_{\text{out}}(g_0 (\tilde{z}, s_0, \epsilon), \omega, q_0 - m) \times g_{\text{out}}(g_0 (\tilde{z}, s_0, \epsilon), \omega, q_0 - m),
\end{align*}
\]

(87) (88) (89)

with the block covariance matrix

\[
Q = \begin{bmatrix} q_0 & m \\ m^T & m \end{bmatrix}.
\]

(90)
2.4 Replica free energy

The approximate inference via message passing presented in the previous sections is related to the replica approximation of the free energy of the problem defined as

\[ F_N = - \ln p(Y) = - \ln \int dX \, ds \, p_{\text{out}}^N(Y|X, s) p_x(X) p_s(s). \tag{91} \]

More precisely, the asymptotic expected free energy density

\[ f = - \lim_{N \to \infty} \frac{1}{N} \int dY \, p(Y) \ln p(Y) \tag{92} \]

as approximated by the replica formalism \cite{22} can be directly derived from the formula for GAMP on vector variables presented in \cite{15} (there applied to committee machines). In our notations, in the Bayes optimal case,

\[-f = \max \left\{ \text{extr}_{\hat{m}, \hat{\omega}} \left[ - \frac{1}{2} \text{Tr}(\hat{m} \hat{\omega}) + I_x(\hat{m}) + I_z(\hat{m}) \right] \right\} \tag{93} \]

with

\[ I_x(\hat{m}) = \int_{\mathbb{R}^p} D\xi \int_{\mathbb{R}^p} d\hat{\xi}_0 \, p_x(\hat{\xi}_0) e^{-\frac{1}{2} \hat{\xi}_0 \hat{\xi}_0^\top + \xi^\top \hat{m} \hat{\omega} + \frac{1}{2} \xi^\top \hat{m} \hat{\omega} \xi}, \tag{94} \]

\[ I_z(\hat{m}) = \int_{\mathbb{R}^q} D\eta \int_{\mathbb{R}^q} d\hat{\eta}_0 \, p_s(\hat{\eta}_0) p_{\text{out}}^N(y|q - m) \frac{1}{2} \hat{\eta}_0 \hat{\eta}_0^\top + q \frac{1}{2} \xi; s_0) \times \log \left( \int_{\mathbb{R}^q} D\eta \, ds \, p_{\text{out}}^N(y|q - m) \frac{1}{2} \hat{\eta}_0^\top + \frac{1}{2} q \xi + s \right). \tag{95} \]

In the case of mismatched teacher and student, the formula can be generalized and the expected free energy density

\[ f = - \lim_{N \to \infty} \frac{1}{N} \int dX_0 \, ds_0 \, p_{\text{out}}(Y|X_0, s) p_{\text{in}}(X_0, s_0) \ln \left( \int dX \, ds \, p_{\text{out}}^N(Y|X, s) p_x(X) p_s(s) \right) \tag{96} \]

is now approximated as the extremum of a potential over all the overlap and auxiliary matrices

\[ q = \frac{1}{N} \sum_{i=1}^N x_{a,i}^\top x_{a,i}, \quad m = \frac{1}{N} \sum_{i=1}^N x_{0,i} x_{a,i}^\top, \quad q_{12} = \frac{1}{N} \sum_{i=1}^N x_{a,i} x_{b,i}. \tag{97} \]

We obtain

\[-f = \max \left\{ \text{extr}_{q, \hat{q}, \hat{m}, \hat{\omega}, \hat{\eta}} \phi(q, \hat{q}, \hat{m}, \hat{\omega}, \hat{\eta}) \right\}, \tag{98} \]

with

\[ \phi(q, \hat{q}, \hat{m}, \hat{\omega}, \hat{\eta}) = -\text{Tr}(\hat{m} \hat{\omega}) + \frac{1}{2} \text{Tr}(q \hat{\eta}) + \frac{1}{2} \text{Tr}(q \hat{\eta}) + I_x(\hat{q}, \hat{m}, \hat{\eta}) + \alpha I_x(q, m, q), \tag{99} \]

and

\[ I_x(\hat{q}, \hat{m}, \hat{\eta}) = \int D\xi \int d\hat{\xi}_0 \, p_x(\hat{\xi}_0) \log \left( \int d\xi \, p_x(\xi) e^{-\frac{1}{2} \xi \xi^\top + \frac{1}{2} \xi^\top \hat{m} \hat{\omega} + \frac{1}{2} \xi^\top \hat{m} \hat{\omega} \xi} \right), \tag{100} \]

\[ I_z(q, m, q_{12}) = \int d\eta \int D\eta_0 \, d\eta_0 \, p_s(\eta_0)p_{\text{out}}^N(y|q - m q_{12}) \frac{1}{2} \eta_0 \eta_0^\top + \eta q_{12} + \alpha q_{12} \]

\[ \times \log \left( \int D\eta \, ds \, p_{\text{out}}^N(y|q - q_{12}) \frac{1}{2} \hat{\eta}_0 \hat{\eta}_0^\top + q_{12} \hat{\eta}_0 + q_{12} \xi + q_{12} \xi \right). \tag{101} \]
Algorithm 2 Offline Gain Calibration State Evolution

**Input:** matrix $Y \in \mathbb{R}^{M \times P}$ and matrix $W \in \mathbb{R}^{M \times N}$:

**Initialize:**
$t = 0$, $m^{(0)} = 0$, $V^{(0)} = q_0 = \rho$,
$\forall \mu = 1 \cdots N_{MC}$ $s_{0,\mu} \sim p_{s_0}(s_{0,\mu})$

**repeat**

1) Draw Monte Carlo samples for update of $\hat{m}^{(t+1)}$ and $r^{(t)}$
$\forall \mu = 1 \cdots N_{MC}$
$\tilde{z}_{\mu}, \omega_{\mu} \sim \mathcal{N}(\tilde{z}_{\mu}, \omega_{\mu}; 0, Q^{(t)})$
$\xi \sim \mathcal{N}(\xi)$
$y_{\mu} = g_0(\tilde{z}_{\mu}, s_{0,\mu}, \xi)$

with $Q^{(t)} = \begin{bmatrix} q_0 I_P & m^{(t)} I_P; m^{(t)} I_P \end{bmatrix} \in \mathbb{R}^{2P \times 2P}$

2) Compute integrands
2.2) Compute $\hat{s}_\mu(y_{\mu}, \tilde{z}_{\mu}, \omega_{\mu}, V^{(t)})$
2.3) Compute $g_{\text{out}}(y_{\mu}, \tilde{z}_{\mu}, \omega_{\mu}, V^{(t)})$

3) Update $r$ and $\hat{m}$

$$r^{(t)} = \frac{1}{N_{MC}} \sum_{\mu=1}^{N_{MC}} (\hat{s}_{\mu})^2, \quad \hat{m}^{(t)} = \frac{1}{N_{MC}} \sum_{\mu=1}^{N_{MC}} (g_{\text{out},\mu})^2$$

3) Update $m^{(t+1)}$ by numerical integration using $\{111\}$, and $V^{(t+1)} = q_0 - m^{(t+1)}$.
$t = t + 1$

**until** convergence

**Output:** time series $\{V^{(t)}, m^{(t)}, \hat{m}^{(t)} ; t = 1 \cdots t_{\text{max}}\}$

Writing the extremization conditions for the potential yields a set of self consistency equations on the overlap matrices. It can be shown to be equivalent to the SE fixed points, $\{88\}$-$\{89\}$ for the Bayes optimal case, and $\{71\}$-$\{80\}$ for the mismatched case. Thus the equivalence between the apparently unrelated replica method and message passing algorithms is recovered.

Finally, let us comment on the validity of the formula for the asymptotic free energy in the Bayes Optimal case. We conjecture that when one selects the maximum value between all the extrema in $\{93\}$, the replica prediction is asymptotically correct. A rigorous justification can be provided using the approach used for the committee machine in $\{15\}$, based on interpolation technics discussed in $\{33\}$-$\{34\}$. It is rather straightforward to repeat these steps for the present problem, which should lead to a complete proof of the conjecture $\{93\}$. The only caveat is that, as in the case of the committee machine the proof would unfortunately rely on a non-proven assumption discussed in detail in section 5.3 of $\{15\}$.

### 3 Online algorithm and analysis

In learning applications, the **offline** strategy, where all the training data is exploited altogether, is at times impossible because of limited memory. In such cases, data is partitioned into mini-batches of a few samples, or even considered one sample at the time. This approach is sometimes preferred even when memory is not an issue, as for instance in deep learning with Stochastic Gradient Descent (SGD) $\{11\}$. Here we will be interested in the Bayesian online learning of the calibration variables as the observations are received. The main concept behind Bayesian online learning algorithms, such as Assumed Density Filtering (ADF) $\{35\}$-$\{25\}$, is the update of a prior along the reception of the different data points. In the statistical physics literature,
Bayesian online learning algorithms were proposed and analyzed for simple neural networks \[36,37,38,39\] and compressed sensing [10].

In the present section, we propose an online algorithm, online cal-AMP, for the calibration problem of the GLM we examine. Our algorithm takes inspiration from the streaming version of AMP for the simple GLM of [16], which we review in the following section. We also derive the corresponding State Evolution analysis.

### 3.1 Streaming AMP for online reconstruction in the GLM

In [16], a mini-batch version of the GAMP algorithm is proposed. On the example of the GLM, one imagines receiving at each iteration \( k \) a subset \( y_{(k)} \) of the components of \( y \in \mathbb{R}^M \) generated via a channel

\[
y \sim p_{\text{out}}(y|Wx),
\]

(102)

to reconstruct \( x \in \mathbb{R}^N \). Bayes formula gives the posterior distribution over \( x \) after seeing \( k \) mini-batches

\[
p(x|y_{(k)}; \{y_{(k-1)}, \cdots, y_{(1)}\}) = \frac{p(y_{(k)}|x)p(x|\{y_{(k-1)}, \cdots, y_{(1)}\})}{\int dx \ p(y_{(k)}|x)p(x|\{y_{(k-1)}, \cdots, y_{(1)}\})}.
\]

(103)

The formula suggests the iterative scheme of using as a prior on \( x \) at iteration \( k \) the posterior obtained at iteration \( k - 1 \). This idea can be implemented in different approximate inference algorithms, as proposed by [41] using a variational method. In the regular version of GAMP [10] an effective factorized posterior is given at convergence by the input update functions (scalar equivalent of (31)-(32)):

\[
p(x|y, W) \approx \prod_{i=1}^{N} \frac{1}{Z_x(\lambda_i, \sigma_i)} p_x(x_i) e^{-\frac{(\lambda_i - z_i)^2}{2\sigma_i^2}}.
\]

(104)

Plugging this posterior approximation in the iterative scheme yields the mini-AMP algorithm using the converged values of \( \lambda_{(t)}, \sigma_{(t)} \) and \( \sigma_{(t)}, \sigma_{(t)} \) at each anterior mini-batch \( \ell < k \) to compute the prior

\[
p_x^{(k)}(x) = p(x|\{y_{(k-1)}, \cdots, y_{(1)}\}, W) \approx \prod_{i=1}^{N} \frac{1}{Z_{x,i}} p_x(x_i) e^{-\sum_{i=1}^{k-1}\frac{\lambda_{(i)} - z_i^2}{2\sigma_{(i)}^2}},
\]

(105)

where the \( Z_{x,i} \) normalize each marginal factor. Compared to a naive mean-field variational approximation of the posterior, AMP takes into account more correlations and is indeed found to perform better in experiments reported by [16]. Another advantage of the AMP based online inference is that it is amenable to theoretical analysis by a corresponding State Evolution. In the following we apply these ideas to the online learning of the calibration variables in the model we defined in Section 2.1.

### 3.2 Online cal-AMP

We consider here the analysis of the online reconstruction of the calibration variable \( s \) as the observations \( y_{(k)} \in \mathbb{R}^M \) are treated successively. We readily adapt the strategy of [16], by using the cal-AMP algorithm and at step \( k + 1 \), the approximate posterior on \( s \) at step \( k \) as an effective prior on \( s_{(k+1)} \).

**Algorithm** The AMP algorithm consists here in restarting cal-AMP at each new observation \( y_{(k)} \) (that is with \( P = 1 \) samples in the notation of Section 2), while updating the prior used for the calibration variable. From the definition of the approximate posterior [16], we obtain the recursion on the effective prior \( p_{s_{(k+1)}} \) on \( s_{(k+1)} \):

\[
p_{s_{(k+1)}}(s_{(k+1)}) = m_{s_{(k)}}(s_{(k+1)})
\]

(106)

\[
= \frac{1}{Z_{s_{(k)}}} \int dz_{(k),\mu} p_{\text{out}} \left( y_{(k),\mu}|z_{(k),\mu}, s_{(k)} \right) p_{s_{(k)}}(s_{(k)}|z_{(k),\mu}, \omega_{(k),\mu}, V_{(k),\mu}),
\]

(107)
where the output variables $\omega_{(k),\mu}$ and $V_{(k),\mu}$ correspond to the values at convergence (or at the last iteration $t_{\text{max}}$) of the cal-AMP algorithm at the previous step $(k)$. In the Section 4.1, we will examine the gain calibration problem and specify effective strategies to implement this recursion within the AMP algorithm.

**State Evolution** The streaming State Evolution analysis of the online cal-AMP is also adapted using the above recursion. Note that the effective prior at a given step $P$ depends on the output variables of the algorithm for all the previously seen samples. Expanding the recursion above we have:

$$p_{s_{\mu}}^{(P)}(s) = \frac{1}{Z_{\text{out}}^{(P-1)}} \int \prod_{k=1}^{P-1} \left( dz_{(k),\mu} \ p_{\text{out}}^{s_{\mu}}(y_{(k),\mu} \ | \ z_{(k),\mu}, s_{\mu}) \ N(z_{(k),\mu} ; \omega_{(k),\mu}, V_{(k),\mu}) \right) p_{s}(s_{\mu}) \tag{108}$$

with

$$Z_{\text{out}}^{(P)} = Z_{\text{out}}^{(P)} \left( \{ y_{(k),\mu}, \omega_{(k),\mu}, V_{(k),\mu} \}_{k=1}^{P} \right), \tag{109}$$

where again for each $k$ the output variables $\omega_{(k)}$ and $V_{(k)}$ are the converged values for the corresponding step $k$.

The dependence of the normalization of $Z_{\text{out}}^{(P)}$ on the output variables is reflected in the definitions at step $P$ of the output update function $g_{\text{out}}^{(P)} = \partial \log Z_{\text{out}}^{(P)}/\partial \omega_{(P)}$ and calibration update function $\hat{s}^{(P)} = \int ds \ s \ m^{*}(s)^{(P)}$. Therefore, the State Evolution involving these functions features an averaging on all the output variables $\omega_{(k)}$ and $V_{(k)}$ relative to the already processed samples.

In the Bayes optimal setting, it is easy to see from the cal-AMP SE, that the online algorithm is characterized by the equations:

$$r_{(P)} = \int d\ell \ p_{\ell}(\ell) ds_{0} \ p_{s_{0}}(s_{0}) \int d\omega \ dz \ N(z, \omega; 0, Q_{(P)}) \tag{110}$$

$$m_{(P)} = \int dx_{0} \ p_{x_{0}}(x_{0}) \int d\ell \ f_{\ell}^{r_{\ell}} \left( (m_{(P)})^{-1} \left( \sqrt{\alpha m_{(P)}} + \alpha \hat{m}_{(P)} x_{0} \right) ; (m_{(P)})^{-1} \right) x_{0}, \tag{111}$$

$$\hat{m}_{(P)} = \int d\epsilon \ p_{\epsilon}(\epsilon) ds_{0} \ p_{s_{0}}(s_{0}) \int d\omega \ dz \ N(z, \omega; 0, Q_{(P)}) \tag{112}$$

$$m_{(P)} = \int dx_{0} \ p_{x_{0}}(x_{0}) \int \left( dz_{(k)} \ N(z_{(k)}, \omega_{(k)}; 0, Q_{(k)}) \right) g_{\text{out}}^{(P)}(g_{0}(z, s_{0}, \epsilon), \omega, q_{0} - m)^{2},$$

where the block covariance matrices $Q_{(k)}$, are given by (90) for each step $(k)$ using the corresponding fixed points $m_{(k)}$. In the following Section we will discuss how to implement this State Evolution in practice, focusing on the specific problem of gain calibration.

### 4 Numerical tests

#### 4.1 Gain calibration setting and update functions

As a test case, we consider the following problem of gain calibration [7, 8, 1, 2]. The input signal is known to be $p$-sparse and distributed according to a Gauss-Bernoulli distribution,

$$\forall i = 1 \cdots N, \ p_{x_{0}}(x_{0}) = \prod_{k=1}^{P} \left( p_{N}(x_{(k)}, i; 0, 1) + (1 - p) \delta(x_{(k), i}) \right). \tag{113}$$
Algorithm 3 Online Gain Calibration State Evolution

Input: matrix $Y \in \mathbb{R}^{M \times P}$ and matrix $W \in \mathbb{R}^{M \times N}$:

Initialize: $t = 0$; 
∀$\mu = 1 \ldots N_{MC}$

$s_\mu^0 \sim p_\sigma(s_\mu^0)$

$\Lambda_{0,\mu} = 0, \Sigma_{0,\mu} = 0$

for $k = 1 \ldots P$ do

Initialize: $t = 0, m_0^{(0)} = 0, V_0^{(0)} = q_0 = \rho$

repeat

1) Draw Monte Carlo samples for update of $\hat{m}$ (112) and $r$ (110)

∀$\mu = 1 \ldots N_{MC}$

$z_{\mu,k}, \omega_{\mu,k} \sim \mathcal{N}(z_{\mu,k}, \omega_{\mu,k}; 0, Q_{(k)}^{(t)})$

$\epsilon \sim p_\epsilon(\epsilon)$

$y_{\mu,k} = g^0(z_{\mu,k}, s_0^\mu, \epsilon)$

with $Q_{(k)}^{(t)} = \begin{bmatrix} q_0 & m_0^{(t)} \\ m_0^{(t)} & m_0^{(t)} \end{bmatrix}$

2) Following step (2) of Algorithm 4 with

- samples of previous steps \{ $y_{\mu,l}, z_{\mu,l}, \omega_{\mu,l}$ \} $l \leq k - 1$ at convergence

- current $y_{\mu,k}, z_{\mu,k}, \omega_{\mu,k}$

- current $V_{(k)}^{(t)}$

2.1) Update $\Lambda_{k,\mu}, \Sigma_{k,\mu}$

2.2) Compute $\hat{s}_\mu$

2.3) Compute $g_{\text{out},\mu,k}$

3) Update $r_{(k)}$ and $\hat{m}_{(k)}$

\[ r_{(k)}^{(t)} = \frac{1}{N_{MC}} \sum_{\mu=1}^{N_{MC}} \left( \hat{s}_\mu^{(t)} \right)^2 \]

\[ \hat{m}_{(k)}^{(t)} = \frac{1}{N_{MC}} \sum_{\mu=1}^{N_{MC}} \left( g_{\text{out},\mu,k}^{(t)} \right)^2 \]

4) Update $m_{(k)}^{(t+1)}$ by numerical integration using (111), and $V_{(k)}^{(t+1)} = q_0 - m_{(k)}^{(t+1)}$.

$t = t + 1$

until convergence

end for

Output: time series \{ $V_{(k)}^{(t)}, m_{(k)}^{(t)}, \hat{m}_{(k)}^{(t)} : t = 1 \ldots t_{\text{max}} \}^P_{k=1}$

Each component of the output includes a division by a calibration variable that is uniformly distributed in a positive interval $[a, b]$: \[ \forall \mu = 1 \ldots M, \ p_\sigma(s_\mu) = \mathbb{1}_{[a, b]}/(b - a), \ 0 < a < b, \quad (114) \]

\[ y_{\mu} = \frac{1}{s_\mu^0} (w_\mu^\top X_0 + \xi), \quad \xi \sim \sqrt{\Delta} \mathcal{N}(0, I_P). \quad (115) \]

We will consider both the offline reconstruction where the $P$ samples are exploited simultaneously and the online case.
Figure 2: Full covariance matrices and departure from the diagonal ansatz for Gaussian inputs $\rho = 1$ (top row) $P = 2$ (bottom row) $P = 3$. Left: Comparison of the MSEs achieved by cal-AMP for full unconstrained covariance matrices and diagonal covariance matrices for $N = 1000$. Middle: Average value at fixed points of the diagonal elements and the off-diagonal elements of $V$ when unconstrained as a function of the size of the problem $N$. Right: Average absolute value of the ratio between off-diagonal and diagonal elements of $V$ as a function of $N$.

Output functions For any value of $P$, the output functions have analytical expressions in this setting. The channel distribution and partition function are

$$p_{\text{out}}(y_{\mu}|z_{\mu}) = \int_a^b ds_{\mu} p_s(s_{\mu})(s_{\mu})^P \mathcal{N}(z_{\mu}; s_{\mu}y_{\mu}, \Delta),$$

$$Z_{\text{out}}(y, \omega, V) = \int d\bar{z} p_{\text{out}}(y|\bar{z}) = \int_a^b ds p_s(s)(s)^P \mathcal{N}(\bar{z}; sy, \Delta)$$

which give

$$g_{\text{out}}(y, \omega, V) = \frac{1}{Z_{\text{out}}} \partial_{\omega} Z_{\text{out}} = (V + \Delta I_P)^{-1}(\hat{s}(y, \omega, V)y - \omega),$$

$$\partial_{\omega} g_{\text{out}}(y, \omega, V) = C^s(y, \omega, V)(V + \Delta I_P)^{-1}y y^\top(V + \Delta I_P)^{-1} - (V + \Delta I_P)^{-1}.$$
These expressions involve the estimate and variance of the calibration variables under the approximate posterior which can be computed as

$$
\delta(y, \omega, V) = f_1^s(y, \omega, V) = \int_a^b ds \frac{ds}{P} P^{1/2} N(y, \Delta) = \frac{\mathcal{I}(P + 1, \nu, \delta, a, b)}{\mathcal{I}(P, \nu, \delta, a, b)},
$$

$$
C^s(y, \omega, V) = f_2^s(y, \omega, V) = \frac{\mathcal{I}(P + 2, \nu, \delta, a, b)}{\mathcal{I}(P, \nu, \delta, a, b)} - \delta(y, \omega, V)^2,
$$

where

$$
\mathcal{I}(P, \nu, \delta, a, b) = \int_a^b ds \frac{s^P e^{-(s-\nu)^2}}{P^{1/2} e^{-(s-\nu)^2}}.
$$

$$
\delta = (y^T (V + \Delta I_P)^{-1} y)^{-1},
$$

$$
\delta^{-1} \nu = y^T (V + \Delta I_P)^{-1} \omega,
$$

and \(\mathcal{I}(P, \nu, \delta, a, b)\) can be computed using gamma functions as explained in [1, 2].

**Input functions** For a Gauss-Bernoulli prior on the entries of \(X\) and assuming the AMP variances \(\sigma_{ik}\) are diagonal matrices, the input update functions can be written component-wise with scalar arguments:

$$
f_1^l(\lambda, \sigma) = \left( \frac{\lambda}{(1 + \sigma)^{3/2}} e^{-\frac{\lambda^2}{2(1 + \sigma)}} \right) / \left( \frac{\rho \left( \frac{e^{-\frac{\lambda^2}{2(1 + \sigma)}}}{(1 + \sigma)^{1/2}} + (1 - \rho) \frac{e^{-\frac{\lambda^2}{2\sigma}}}{\sigma^{1/2}} \right)} {\mathcal{I}(P, \nu, \delta, a, b)} \right),
$$

and

$$
f_2^l(\lambda, \sigma) = \left( \frac{\rho \left( \frac{e^{-\frac{\lambda^2}{2(1 + \sigma)}}}{(1 + \sigma)^{1/2}} + (1 - \rho) \frac{e^{-\frac{\lambda^2}{2\sigma}}}{\sigma^{1/2}} \right)} {\mathcal{I}(P, \nu, \delta, a, b)} \right) - f_{1,k}^l \cdot
$$

so that \(\mathcal{Z}_{i,k} = f_1^l(\lambda_{i,k}, \sigma_{i,k})\) and \(C_{i,k}^s = f_2^l((\lambda_{i,k}, \sigma_{i,k})\).
4.2 Offline tests

The offline AMP algorithm is directly given by Algorithm 1 replacing the values of the output and input functions presented in the previous paragraph. In our numerical tests, we would like to additionally impose that covariance matrices $V_p$, $\partial g_{\text{out}}$, and $\sigma$ are diagonal. This assumption lightens numerics so as to consider larger number of samples $P$.

Consequently, we first investigate how the algorithm behaves under a diagonal ansatz for the covariances compared to the full parametrization. On Figure 2, we show that the performance of the algorithm with diagonal covariance does not deteriorate in terms of signal reconstruction. Furthermore, we find that the off-diagonal elements tend to vanish as we increase the size of the problem, and conjecture that they are null in the thermodynamic limit. In the following, we adopt the diagonal ansatz.

Under the assumption of diagonal covariance matrices, the State Evolution equations (88), (89) involve $P \times P$ matrices proportional to the identity (given the $P$ examples are statistically equivalent). Therefore it is sufficient to consider the update of one diagonal element noted respectively $m$ and $\hat{m}$. While the update of $m$ only requires a two dimensional integral that can be performed numerically, we resort to a Monte Carlo for the update of $\hat{m}$. The procedure is described in Algorithm 2.

On the top row of Figure 3 and respectively of Figure 4 we report the performance of reconstruction by the cal-AMP algorithm of the signal $X$ and the calibration variable $s$, in the plane $\rho - \alpha$, for different values of $P$. These phase diagrams are similar to the ones reported in [1, 2] and will be compared with the online case described in the next Section. The performances are compared with the information theoretic threshold $\alpha_{\text{min}}$ corresponding to the minimal number of observations necessary to reconstruct the unknown signal and calibration variables,

$$M_{\text{min}} \times P = \rho N \times P + M_{\text{min}} \Rightarrow \alpha_{\text{min}} = \frac{M_{\text{min}}}{N} = \frac{\rho P}{P - 1},$$

(127)

and the sharp threshold of reconstruction of the fully calibrated Bayesian AMP $\alpha_{\text{rmCS}}$ [2].

On Figure 3a we check numerically that the derived State Evolution predicts the behavior of the AMP algorithm for gain calibration. The MSEs of the two procedures are indeed consistent along the iterations of the algorithm. On Figure 3b we also report an almost perfect agreement of the fixed points of SE and AMP in terms of the MSE on $X$ as we vary the number of samples $P$.

4.3 Online tests

The online algorithms include supplementary operations to update the effective prior on the calibration variable from one step to the next. In this setting the recursion is

$$p_{s_{\mu}}^{(k+1)}(s_\mu) = \frac{1}{Z_{\text{out}}^{(k+1)}} \int \mathcal{D}z_{(k),\mu} \mathcal{N}(y_{(k),\mu}; z_{(k),\mu}/s_\mu, \Delta) \mathcal{N}(z_{(k),\mu}; w_{(k),\mu}, V_{(k),\mu}) p_{s_\mu}^{(k)}(s_\mu),$$

(128)

so that

$$p_{s_{\mu}}^{(P)}(s_\mu) \propto s_\mu^P e^{-\frac{(s_{\mu} - s_{\mu})^2}{2C_{\mu}^{P}}}$$

with

$$\Sigma_{(P),\mu}^{-1} = \sum_{k=1}^{P} \delta_{(k),\mu}^{-1},$$

(130)

$$\Lambda_{(P),\mu} = \Sigma_{(P),\mu} \left( \sum_{k=1}^{P} \delta_{(k),\mu}^{(P)} \right),$$

yielding a posterior on the calibration variable at step $P$ with an identical form to the posterior of the offline algorithm (albeit with parameters computed differently). Therefore $\hat{s}_\mu$ and $C_\mu^P$ can still be computed analytically following [120] and [121]. We provide a pseudo-code for the online cal-AMP in Algorithm 4 and a pseudo-code for online SE in Algorithm 3.
On the bottom rows of Figure 4 and 5 we plot phase diagrams obtained with online cal-AMP. Compared to the offline diagrams we find that the reconstruction requires more samples to achieve comparable levels of accuracy. On Figure 3c and 3d we check the consistency of the cal-AMP and SE fixed points in terms of MSE on $X_0$.

Figure 4: Normalized cross correlation between cal-AMP estimate $\hat{X}$ and teacher signal $X_0$ for the gain calibration problem with calibration variables uniformly distributed in $[0.95, 1.05]$ and $N = 10^3$. Diagrams are plotted as a function of the measurement rate $\alpha = M/N$ and the sparsity level $\rho$ for the offline (top row) and online (bottom row) algorithms, for increasing number of available samples $P$. The blue line $\alpha_{CS}$ is the phase transition threshold for a perfectly calibrated channel. The pink line $\alpha_{min}$ marks the strict lower bound on the number of measurements necessary for reconstruction. The online cal-AMP requires more samples than the offline version to achieve comparable errors, nevertheless above the transition relatively low errors are already reached at $P = 10$. We note that the algorithms are sometimes unstable at low $\rho$, leading to unexpectedly high MSEs (top left corner of top right diagram).
Figure 5: Normalized cross correlation between cal-AMP estimate $\hat{s}$ and teacher signal $s_0$ for the gain calibration problem with calibration variables uniformly distributed in $[0.95, 1.05]$ and $N = 10^3$. Diagrams are plotted as a function of the measurement rate $\alpha = M/N$ and the sparsity level $\rho$ for the offline (top row) and online (bottom row) algorithms, for increasing number of available samples $P$. The pink line $\alpha_{\text{min}}$ marks the strict lower bound on the number of measurements necessary for reconstruction. Results are similar to the diagrams in terms of errors on the signal $\hat{X}$ of Figure 4.
Algorithm 4 Online Gain Calibration Approximate Message Passing

**Input:** matrix $Y \in \mathbb{R}^{M \times P}$ and matrix $W \in \mathbb{R}^{M \times N}$.

**Initialize:** $\Lambda_{0,\mu} = 0$, $\Sigma_{0,\mu} = 0$.

**for** $k = 1 \ldots P$ **do**

**Initialize:** $\hat{x}_{i,k}, C_{i,k}^x \ \forall i$ and $g_{out,\mu,k}, \partial_{\omega} g_{out,\mu,k} \ \forall \mu$

**repeat**

1) Estimate mean and variance of $\hat{z}_{(t)}$ given current $\hat{z}_{(t)}$

$$V_{\mu,k}^{(t)} = \sum_{i=1}^{N} W_{i,\mu}^2 C_{i,k}^x$$

$$\omega_{\mu,k}^{(t)} = \sum_{i=1}^{N} W_{i,\mu} \hat{x}_{i,k}^{(t)} - \sum_{i=1}^{N} W_{i,\mu}^2 C_{i,k}^x g_{out,\mu,k}^{(t-1)}$$

2) Exploit current $y_{(t)}$ and inherited $\Lambda_{k-1,\mu}, \Sigma_{k-1,\mu}$

2.1) Update recursion parameters given $y_{(t)}$

$$\delta_{\mu,k}^{(t)} = \frac{y_{\mu,k}^2}{(V_{\mu,k}^{(t)} + \Delta)}$$

$$\nu_{\mu,k}^{(t)} = \frac{\delta_{\mu,k} y_{\mu,k} \omega_{\mu,k}^{(t)}}{(V_{\mu,k}^{(t)} + \Delta)}$$

$$\Sigma_{k,\mu}^{(t)} = \Sigma_{k-1,\mu} + \delta_{\mu,k}^{(t)}$$

$$\Lambda_{k,\mu}^{(t)} = \Sigma_{k,\mu} \left( \Sigma_{k-1,\mu} \Lambda_{k-1,\mu} + \delta_{\mu,k}^{(t)} \nu_{\mu,k}^{(t)} \right)$$

2.2) Update estimates $\hat{s}$ and $C^s$

$$\hat{s}_{(t)}^{(t)} = \frac{I(k+1, \Lambda_{k,\mu}, \Sigma_{k,\mu}, a, b)}{I(k, \Lambda_{k,\mu}, \Sigma_{k,\mu}, a, b)}$$

$$C_{(t)}^{s} = \frac{I(k+2, \Lambda_{k,\mu}, \Sigma_{k,\mu}, a, b)}{I(k, \Lambda_{k,\mu}, \Sigma_{k,\mu}, a, b)}$$

2.3) Update $g_{out}$ and $\partial_{\omega} g_{out}$

$$\partial_{\omega} g_{out,\mu,k}^{(t)} = C_{\mu}^s \frac{y_{\mu,k}^2}{(V_{\mu,k}^{(t)} + \Delta)} - (V_{\mu,k}^{(t)} + \Delta)^{-1}$$

$$g_{out,\mu,k}^{(t)} = \frac{\delta_{\mu,k} y_{\mu,k} - \omega_{\mu,k}^{(t)}}{(V_{\mu,k}^{(t)} + \Delta)}$$

3) Estimate mean and variance of $\hat{z}$ given current optimal $z$

$$\sigma_{i,k}^{(t)} = \left( -\sum_{\mu=1}^{M} W_{i,\mu} \partial_{\omega} g_{out,\mu,k}^{(t)} \right)^{-1}$$

$$\lambda_{i,k}^{(t)} = \hat{x}_{i,k}^{(t)} + \sigma_{i,k}^{(t)} \left( \sum_{\mu=1}^{M} W_{\mu,\hat{z}_{(t)}^{out,\mu,k}} \right)$$

4) Estimate of mean and variance of $\hat{z}$ augmented of the information about the prior

$$C_{i,k}^{x} \left( t+1 \right) = f_{z}^{2} (\lambda_{i,k}^{(t)}, \sigma_{i,k}^{(t)})$$

$$\hat{x}_{i,k}^{(t+1)} = f_{z}^{1} (\lambda_{i,k}^{(t)}, \sigma_{i,k}^{(t)})$$

$t = t + 1$

**until** convergence

**end for**

**Output:** Estimates and variances $\hat{z}_{(t)}, C_{i,k}^{x}$, $\hat{s}_{(t)}, C_{(t)}^{s}$
5 Conclusion

In this paper we presented a theoretical characterization of the cal-AMP algorithm for blind calibration. The derivation of the corresponding State Evolution was presented from the message passing equations of a vectorial version of the GAMP algorithm. Furthermore, we extended the algorithm and its analysis to the online scenario, which allows to adapt the calibration upon the reception of new observations. The newly proposed analyses and algorithms were demonstrated through numerical experiments on the case of gain calibration. In the offline setting, cal-AMP achieves near perfect reconstruction close to the theoretical lower-bound. In the online setting, cal-AMP is also able to calibrate and reconstruct from a small number of samples.

For practical applications, the extension to the online case is of particular interest. It enables a continuous improvement of an hardware sensing channel, without requiring to keep past observations in memory. A natural extension will also be to consider the G-VAMP [18, 19] version of cal-AMP, which will enable to generalize the algorithm to measurement matrices with non-i.i.d. entries. From the theoretical point of view, note that the introduced SE analysis in the non-Bayes optimal setting should make it possible to characterize other reconstruction algorithms, as for instance algorithms proposed in [7, 8]. Also it will be interesting to confirm whether cal-AMP can also succeed in a multi-layer setting, similarly to the recent extension of GAMP to ML-AMP [43]. This generalization will open the way to message passing algorithms solutions in more complex calibration problems.

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