A NOTE ON SIMPLICIAL DIMENSION SHIFTING

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Abstract. We discuss a simplicial dimension shift which associates to each $n$-manifold an $n-1$-manifold. As a corollary we show that an invariant which was recently proposed by Ooguri and by Crane and Yetter for the construction of 4-dimensional quantum field theories out of 3-dimensional theories is trivial.

Ooguri [1] proposed recently a method for obtaining 4-manifold invariants and studying quantum gravity, with a construction inspired statistical mechanics. This procedure was further formalized by Crane and Yetter [2]. Their method is associating a 3-manifold with boundary to every 4-simplex $\Delta^4$. Given 4-manifold $M$ with a triangulation $\tau$ and an invariant for 3-manifolds with boundary, they evaluate the invariant on the 3-manifolds with boundary coming from each 4-simplex, they match boundary conditions according to the triangulation and then they compute a partition function. They mention that the proof of the invariance of the result to the choice of the triangulation is long and technical.

In fact the matching and summation of boundary conditions is equivalent to the glueing together of the corresponding portions of the boundary and then computing the invariant of the 3-manifold without boundary obtained this way. With this observation the choice of the 3-manifold invariant becomes irrelevant. One obtains a topological construction, which associates to each triangulated 4-manifold a 3-manifold, which we call the dimension shift of $M$. The aim of this note is to prove that the dimension shift of a 4-manifold is essentially trivial, and that for a singular triangulated 4-manifold the dimension shift yields essentially the direct sum of the 3-singularities of $M$. In particular, the invariants of Ooguri, Crane and Yetter are always 1.

We were led independently to the same construction in a different context [3]. We showed an equivalence between systems of $n$-modules over operator algebras and simplicial invariants for $n + 1$-manifolds which have the glueing property. Working with subfactors, or 2-modules, an asymptotic construction produces naturally some 3-modules. The 4-manifold invariants associated to these 3-modules are

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obtained from the 3-manifold invariants associated to the 2-modules by means of dimension shifting. Because of the result which follows we used this construction only to show that the axioms for 4-dimensional invariants are nonempty.

The idea of dimension shifting is the following. Let $\Delta^4$ denote the 4-simplex. The boundary $\partial \Delta^4$ of $\Delta^4$ consists of 5 copies of the 3-simplex $\Delta^3$, and is topologically equivalent to the 3-sphere $S^3$. Consider the 1-skeleton $\sigma_1$ of $\partial \Delta^4$ and the dual 1-skeleton $\hat{\sigma}_1$ of $\partial \Delta^4$. Let $T$ be a small tubular neighborhood of $\hat{\sigma}_1$ in $S^3$.

Let now $M^4$ be a 4-manifold with a triangulation $\tau$. To each 4-simplex $\Delta^4 \in \tau$ associate the 3-manifold with boundary $T$ as above. When two 3-faces $\Delta^3$ and $\Delta^3'$ of the 4-simplices $\Delta^4$ and $\Delta^4'$ in $\tau$ are glued, glue the corresponding components $\partial T \cap \Delta^3$ and $\partial T' \cap \Delta'^3$ of the boundary of $T$ and respectively $T'$.

This way one obtains a 3-manifold without boundary denoted by $\text{sh}_{\text{int}} M$. In another version of this construction one glues the exterior $S^3 - T$ of $T$ the same way as above, to obtain a 3-manifold $\text{sh}_{\text{ext}} M$. Note that $S^3 - T$ is a tubular neighborhood of the 1-skeleton $\sigma_1$.

**Theorem 0.1.** Let $M^4$ be a connected compact 4-manifold without boundary, with a triangulation $\tau$ having $n_k$ simplices in each dimension $k = 0, \ldots, 4$. Then we have

$$\text{sh}_{\text{int}} M = (10n_4 - n_3 + 1)(S^1 \times S^2)$$

and

$$\text{sh}_{\text{ext}} M = (n_1 - n_0 + 1)(S^1 \times S^2).$$

Here $mM$ denotes the connected sum of $m$ copies of the manifold $M$.

**Proof.** The proof for the case of the interior dimension shift is the following. Let $\Delta^4$ be a 4-simplex in $\tau$, let $T$ be its thickened dual 1-skeleton and let $\Delta^3$ be a 3-simplex in $\partial \Delta^4$. Then $T \cap \Delta^3$ is topologically a 3-ball, and its boundary consists of $\partial T \cap \Delta^3$ and of $T \cap \partial \Delta^3$, which is the union of 4 copies of a 2-disk $D^2$. When $\Delta^4$ is glued to another 4-simplex $\Delta'^4$ in $\tau$, and $\Delta^3$ is identified to its pair $\Delta'^3$ in $\partial \Delta'^4$, the balls $T \cap \Delta^3$ and $T' \cap \Delta'^3$ are glued together to yield a copy of $S^3$ which has a 3-ball hole $B^3$ carved out for each face $\Delta^2$ of $\Delta^3$. The boundary $S^2$ of the hole $B^3$ consists of a pair of disks $D^2$ and $D'^2$ as described above. The disk $D^2$ is then glued to a disk coming from the neighbor $\Delta'^3$ of $\Delta^3$ in $\partial \Delta^4$ opposite to $\Delta^3$.

This procedure yields a 3-sphere for every 3-simplex of $\tau$ and a handle $S^2 \times I$ for every 2-face $\Delta^2$ of each 4-simplex $\Delta^4$ of $\tau$, for a total of $10n_4$
handles. From these, \( n_3 - 1 \) handles are needed to connect the 3-spheres, and \( 10n_4 - n_3 + 1 \) handles remain.

In the case of the exterior shift, note that the exterior \( S^3 - T \) corresponding to a 4-simplex \( \Delta^4 \) is a tubular neighborhood of the 1-skeleton of \( \Delta^4 \). We decompose \( S^3 - T \) into balls \( B^3_v \) for each vertex \( v \) of \( \Delta^4 \) joined by bars \( b_e = D^2_e \times I \), with \( I \) the 1-simplex, obtained by thickening each 1-edge \( e \) of \( \Delta^4 \).

Let now \( v \) be a vertex of the triangulation \( \tau \) of \( M^4 \). The simplicial neighborhood, or star, of \( v \) is a cone over a 3-manifold \( M_v \) called the link of \( v \). The link \( M_v \) is obtained by joining the faces \( \Delta^3_i \) opposed to \( v \) in the 4-simplices \( \Delta^4_i \) which contain \( v \). For each vertex \( v \), each neighboring 4-simplex \( \Delta^4_i \) contributes to \( \text{sh}^{\text{ext}} \tau \) with a 3-ball \( B^3_{vi} \).

The crucial observation is that the ball \( B^3_{vi} \) is obtained by a homothety from the 3-simplex \( \Delta^3_i \) opposite to \( v \) in \( \Delta^4_i \), and the glueing of the 3-balls \( B^3_{vi} \) is carried over by this homothety from the glueing of the corresponding link simplices \( \Delta^3_i \). This yields a 3-manifold \( \tilde{M}_v \) isomorphic to \( M_v \) for each vertex \( v \). The manifold \( \tilde{M}_v \) has a hole for each edge \( e \) of \( \tau \), with boundary obtained by joining the disks \( D^2_{vei} \) which connect the balls \( B_{vi} \) to the bars \( b_{ei} \). By the same homotopy, the disks \( D^2_{vei} \) join to make a 2-manifold \( \tilde{M}_e \) which is homothetic to the link \( M_e \) of \( e \). The bars themselves glue to give copies of \( M_e \times I \).

For a manifold triangulation \( \tau \) each \( M_v \) is a 3-sphere and each \( M_e \) is a 2-sphere, and we obtain \( n_0 \) spheres \( S^3 \) and \( n_1 \) handles \( S^2 \times I \). A counting argument similar to the one done before ends the proof.

Remark 0.1. Analogous dimension shifts can be defined from an \( n \)-dimensional triangulated manifold \( M^n \) to an \( n-1 \)-manifold, using tubular neighborhoods of the \( k \)-skeleton of \( \Delta^n \). The proof above extends to this context and shows that the interior dimension shift carries essentially no information at all about \( M \). If \( M \) is a triangulated manifold with singularities, i.e. such that the links of \( M \) are not all spheres, then the exterior dimension shift describes the link singularities of \( M \). Thus, for the 4-dimensional case described above, the shifted invariant of a singular triangulated 4-manifold is the product of the 3-dimensional invariants of the links of its vertices. This is the result that we described in [3].

References

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