On the pricing of currency options under variance gamma process

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Abstract

The pricing of currency options is largely dependent on the dynamic relationship between a pair of currencies. Typically, the pricing of options with payoffs dependent on multi-assets becomes tricky for reasons such as the non-Gaussian distribution of financial variable and non-linear macroeconomic relations between these markets. We study the options based on the currency pair US dollar and Indian rupee (USD-INR) and test several pricing formulas to evaluate the performance under different volatility regimes. We show the performance of the variance gamma and the symmetric variance gamma models during different volatility periods as well as for different moneyness, in comparison to the modified Black-Scholes model. In all cases, variance gamma model outperforms Black-Scholes. This can be attributed to the control of kurtosis and skewness of the distribution that is possible using the variance gamma model. Our findings support the superiority of variance gamma process of currency option pricing in better risk management strategies.

Keywords: Currency options, Variance gamma process, Risk management.

JEL Codes: F31, G15, G32

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Introduction

Currency options are an important instrument for traders to hedge against possible fluctuations in exchange rates and at the same time have high gain potentials while taking on limited risks. Since the introduction of option trading on USD-INR in NSE there has been an exponential growth in the volume of contracts traded from a monthly average of 139,296 contracts in 2010 to 54,078,805 in 2019. Thus it is imperative we have a robust model for pricing these contracts. In this paper we have carried out an empirical study to see the performance of variance gamma and the symmetric variance gamma model in pricing of USD-INR during different volatility periods as well as for different moneyness. The results were then compared with that obtained from the modified Black-Scholes. In all cases Variance gamma model was found to out-perform Black-Scholes. This can be attributed to the control of kurtosis and skewness of the distribution that is possible using the variance gamma model. Section II provide gives a brief overview of the different models we have considered and section III explores the empirical and statistical results from these models.
Methodology

1. **Black-Scholes Garman–Kohlhagen Model:**

The modified Black-Scholes Model was proposed by Garman and Kohlhagen in 1983 to take into account the difference in interest rates between two countries while valuing currency options. It is a pure diffusion process and does not involve any jump component. The modified Black-Scholes formula is:

\[
C_{BS} = S_0 N(d_1) e^{-r_f T} - K N(d_2) e^{-r_d T}
\]  
(1)

Where \(d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r_d - r_f + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}\) and \(d_2 = d_1 - \sigma \sqrt{T}\).

\(N(.)\) represents the cumulative normal distribution, \(S_0\) is the current exchange rate, \(r_f\) is the foreign interest rate, \(r_d\) is the domestic risk free interest rate and \(T\) is the time to expiration of the contract.

2. **Variance Gamma Model:**

The variance gamma model was proposed by Carr and Madan in 1992. Unlike most models in the literature, variance is a pure jump process with no continuous Martingale component [1]. Variance gamma process \((X(t; \sigma, \nu, \theta))\) is basically geometric Brownian motion evaluated at random time having a gamma distribution i.e.
\[ X(t; \sigma, \nu, \theta) = b(T_t; \theta, \sigma) \]  

(2)

Where \( b(T_t; \theta, \sigma) \) represents a Brownian motion having drift \( \theta \) and variance \( \sigma \) and \( T_t \) is the value of a Gamma process \( \gamma(t; \mu, \nu) \) at time \( t \) with mean \( \mu = 1 \) and variance \( \nu \). Thus the variance Gamma process involves two steps 1) picking a value of \( T_t \) 2) evaluating the Brownian motion \( b(T_t; \theta, \sigma) \). The conditional probability distribution of \( X \) is given by:

\[
f(X|T_t = g) = \frac{1}{\sigma \sqrt{2\pi g}} \exp \left( \frac{x-\theta g}{2\sigma^2 g} \right)
\]  

(3)

Thus the unconditional probability distribution is given by integrating over all possible values of \( T_t \) i.e.

\[
f_X(t)(X) = \int_0^\infty f(X|T_t) * \gamma(t; \mu, \nu) \]

(4.1)

\[
= \int_0^\infty \frac{1}{\sigma \sqrt{2\pi g}} \exp \left( \frac{x-\theta g}{2\sigma^2 g} \right) \frac{t}{\nu^\nu \Gamma(\frac{\nu}{t})} \text{d}g
\]  

(4.2)

The kurtosis of the distribution is controlled by the factor \( \nu \) and the skewness is captured by the factor \( \theta \). If we assume the stock prices to follow a Variance Gamma process we have

\[
S(t) = S(0) \exp(X(t; \sigma, \nu, \theta) + \omega t + rt)
\]  

(5)
Where $r$ is the risk free interest rate and $\omega$ is correction term required to make the overall measure martingale and is given by

$$\omega = \frac{t}{v} \log \left( 1 - \theta v - \frac{\sigma^2 v}{2} \right)$$

(6)

The distribution of stock price is thus dependent on the realization of the random variable $T_t$ which then is given by a log-normal distribution. For the unconditional probability distribution we have to integrate as done earlier in (4) which then results in the following expression for the log returns $\left( z = \ln \left( \frac{S(t)}{S(0)} \right) \right)$:

$$\frac{2\exp(\frac{\theta x}{\sigma^2})}{\sqrt{\frac{v}{2} \pi \Gamma(\frac{t}{v})}} \cdot \left( \frac{x^2}{2\sigma^2 + \theta^2} \right)^{\frac{t-1}{2v^4}} \cdot K_{\frac{t}{v-2}} \left( \frac{x^2}{\sigma^2} \right) \sqrt{x^2 \left( \frac{2\sigma^2}{v} + \theta^2 \right)}$$

(7)

Where $K$ is the modified Bessel function of the second order and $x$ is given by

$$x = z - mt - \frac{t}{v} \log \left( 1 - \theta v - \frac{\sigma^2 v}{2} \right)$$

(8)

Once the parameters of the Variance gamma are selected i.e. $\nu$, $\theta$ and $\sigma$ we can price an European option as follows:

$$C_{VG} = e^{-rT} E(S(0) \exp(X(t; \sigma, \nu, \theta) + \omega t + rt))$$

(9)
Upon evaluating (9.1) one obtains the following closed form solution for the pricing of European currency option[2]:

$$C_{VG}(t) = S(t)e^{-rfT}\psi\left(d\sqrt{\frac{1-c_1}{\nu}},(\alpha + s)\sqrt{\frac{\nu}{1-c_1}},\frac{T}{\nu}\right) -$$

$$Ke^{-rfT}\psi\left(d\sqrt{\frac{1-c_2}{\nu}},as\sqrt{\frac{\nu}{1-c_1}},\frac{T}{\nu}\right)$$

(10)

Where $d = \frac{1}{s}\left[\ln\left(\frac{S(t)}{K}\right) + (r_d - r_f)T + \frac{T}{\nu}\ln\left(\frac{1-c_1}{1-c_2}\right)\right]$.

$c_1 = \nu(\alpha + s)^2/2$

$c_2 = \nu\alpha^2/2$

$\alpha = -\theta s/\sigma^2$

$s = \frac{\sigma}{\sqrt{1 + \left(\frac{\theta}{\sigma}\right)^2\nu}}$

The function $\psi$ is given in the appendix and is expressed in terms of modified Bessel function of second order and hypergeometric function.
Empirical Evidence

The data we have considered is the USD-INR option price for call options obtained from the National Stock Exchange-NSE for the duration of Nov 1\textsuperscript{st} 2010 to Sep 28\textsuperscript{th} 2012. We have considered contracts having volume greater than 100 in order to avoid taking into consideration illiquid contracts which may not be priced appropriately. We have a total of 7312 contracts which averages to approximately 80 contracts a week. To further understand the performance of the models in periods of high volatility and low volatility we have divided our timeline into two periods Nov 1\textsuperscript{st} 2010 to July 28\textsuperscript{th} 2011 as low volatility period and from July 28\textsuperscript{th} 2011 to Sep 28\textsuperscript{th} 2012 as high volatility period as shown in Fig.1. The option data is then further divided based on the moneyness as at the money (ATM; S/K>0.95 and S/K<1.05) in the money (ITM; S/K>1.05) and out of the money (S/K<0.95) for each period as shown in Table 1.
### TABLE 1

| Moneyness | Low Volatility Period | High Volatility Period |
|-----------|-----------------------|------------------------|
| ITM       | 18                    | 675                    |
| ATM       | 2565                  | 5909                   |
| OTM       | 48                    | 648                    |
| **Total** | **2631**              | **7232**               |

1. **Historical data parameters:**

In table II we have shown the parameters for each distribution which fits with the historical data for the two time periods we have considered. For the case of Black-Scholes volatility ($\sigma$) was calculated by taking the variance of the historical data. For the case of Variance Gamma and symmetric Variance Gamma was calculated by making an initial approximation using the moment functions (11.1)-(11.4). These values were then used as initial guesses to fit the historical data with (7) using Nelder-Mead optimization algorithm.

\[
\mathbb{E}(X_t) = c + \theta \quad (11.1)
\]

\[
\text{Var}(X_t) = \sigma^2 \quad (11.2)
\]

\[
\mathbb{E}(X_t - \mathbb{E}(X_t))^3 = 3\sigma^2 \theta \nu \quad (11.3)
\]

\[
\mathbb{E}(X_t - \mathbb{E}(X_t))^4 = 3\sigma^4(1 + \nu) \quad (11.4)
\]
The volatility($\sigma$) for the low volatility time period was found to be 0.054 for Black-Scholes and 0.0544 and 0.0545 for the case of symmetric Variance Gamma and Variance Gamma respectively. For high volatility period it was found to be 0.1039 for the case of Black-Scholes and 0.1044 for both the symmetric Variance Gamma and Variance Gamma models. The much higher value of $\sigma$ obtained for all models in the high volatility period compared to the low volatility period is consistent with the time period division. The kurtosis of Variance Gamma and symmetric Variance Gamma given by $\kappa = 3(1 + \nu)$ gives a value of 3.606 and 3.379 in the case of symmetric VG for low and high volatile periods respectively whereas for VG it was observed to be 3.633 and 3.249 for low and high volatile periods respectively. Skewness of the distribution specified by $\theta$ indicates a positive skewness for the low volatile period however for the high volatile period it is found to have a negative skewness.

2. **Risk Neutral parameters:**

| Parameters | Black-Scholes | Variance Gamma |
|------------|---------------|----------------|
| $\sigma$ (HV) | 0.1039 | 0.1044 |
| (LV) | (0.054) | (0.0545) |
| $\theta$ (HV) | - | -0.00118 |
| (LV) | | (0.00682) |
| $\nu$ (HV) | - | 0.211 |
| (LV) | | (0.083) |

Above Table shows the distribution parameters for the three models i.e. Black Scholes, Symmetric Variance Gamma and Variance Gamma. These values have been obtained by calibrating the model distribution with the daily log returns of the USD-INR exchange rate in the interval Dec 30th 2010 to April 30th 2012.
In Table III is shown the mean weekly risk neutral parameters of the different models and their standard deviation. The weekly parameters have been calculated by minimizing the log-likelihood function $\sum_{i=0}^{M} |\log(C_{Model_i}(\theta_1, \theta_2, \theta_3, \ldots)) - \log(C_{Market_i})|$, where $C_{Model}(\theta_1, \theta_2, \theta_3, \ldots)$ is the option price calculated from the respective model and $\theta_1, \theta_2, \theta_3, \ldots$ are the various model parameters for which $f$ is minimum. The optimization algorithm used is Nelder-Mead and for the initial guess we used the parameters obtained from the historical values shown in Table II.

### TABLE 3

| Model          | Mean    | Standard Deviation | Maximum  | Minimum  |
|----------------|---------|--------------------|----------|----------|
| Black-Scholes: | $\sigma$| 0.08275            | 0.0216   | 0.1233   |
| Variance       | $\nu$  | 0.099              | 0.06     | 0.3471   |
|                | $\theta$| 0.0026             | 0.0048   | 0.0188   |
|                | $\sigma$| 0.116              | 0.00517  | 0.8651   |

The weekly risk neutral parameters obtained are tabulated and shown in table 3 along with their statistical distribution.

**Out of Sample Performance:**

In order to test the out of sample performance of the different models we have used one week on implied model parameters to predict the option price for the
next week. This is in line with the previous works of Madan (1998). Below shown is the statistics of the percentage error of the two models.

**TABLE 4**

| Model                  | Mean | Standard Deviation | Maximum | Minimum |
|------------------------|------|--------------------|---------|---------|
| Black-Scholes          | 0.244| 0.0786             | 0.511   | 0.11    |
| LV period              | 0.191| 0.052              | 0.301   | 0.11    |
| HV period              | 0.272| 0.0707             | 0.511   | 0.158   |
| Variance Gamma         | 0.05848| 0.03041           | 0.156   | 0.0012  |
| LV period              | 0.0446| 0.0334             | 0.149   | 0.016   |
| HV period              | 0.068 | 0.024              | 0.156   | 0.034   |

In table 4 above is shown the mean absolute relative error i.e. \( \text{MAPE} = \frac{\sum_{i=1}^{n}(C_{Market_i} - C_{Model_i})}{n} \). It is seen that overall mean of MAPE of Variance Gamma model with 0.0584 is much lower than that of Black Scholes’s 0.244. This is evidence to the robustness of the VG model compared to Black-Scholes. In order to understand the performance of the two models in different cases such as for based on the maturity of the contract as well as based on moneyness we have further compiled its performance in table 5 and 6. In table 5 is shown performance of the two models based on the maturity period.

**TABLE 5**

[11]
| Parameter                        | Mean  | Standard Deviation | Maximum | Minimum |
|---------------------------------|-------|--------------------|---------|---------|
| Black-Scholes <30days           | 0.123 | 0.076              | 0.307   | 0.0012  |
| Black-Scholes >30days and <60days | 0.081 | 0.053              | 0.2876  | 0.023   |
| Black-Scholes >60days           | 0.038 | 0.041              | 0.238   | 0.015   |
| VG Short Term <30 days          | 0.035 | 0.023              | 0.125   | 0.0015  |
| VG Med Term >30 and <60 days    | 0.015 | 0.031              | 0.301   | 0.0012  |
| VG Long Term >60                | 0.06  | 0.063              | 0.623   | 0.0018  |

**TABLE 6**
| Model Type       | ITM   | ATM   | OTM   | OTM   |
|------------------|-------|-------|-------|-------|
| Black-Scholes    | 0.0034| 0.0039| 0.017 | 0.0001|
| ITM              |       |       |       |       |
| Black-Scholes    | 0.223 | 0.066 | 0.45  | 0.1014|
| ATM              |       |       |       |       |
| Black-Scholes    | 0.014 | 0.020 | 0.097 | 0.015 |
| OTM              |       |       |       |       |
| VG Short Term    | 0.0118| 0.021 | 0.121 | 0.003 |
| ITM              |       |       |       |       |
| VG Med Term      | 0.0458| 0.0271| 0.116 | 0.012 |
| ATM              |       |       |       |       |
| VG Long Term     | 0.011 | 0.021 | 0.1215| 0.0013|
| OTM              |       |       |       |       |

**Conclusions**

In this paper we have carried out an empirical study to see the performance of variance gamma and the symmetric variance gamma model in pricing of USD-INR during different volatility periods as well as for different moneyness. The results were then compared with that obtained from the modified Black-Scholes. In all cases Variance gamma model was found to out-perform Black-Scholes. This can be attributed to the control of kurtosis and skewness of the distribution that is possible using the variance gamma model.
References

1. Agrawal, A. and K. Tandon (1994). Anomalies or Illusions? Evidence from Stock Markets in Eighteen Countries. Journal of International Money and Finance 13, 83-106.

2. Amihud, Y., and H. Mendelson (1980). Dealership Market: Market-making with Inventory. Journal of Financial Economics 8, 31-53.

3. Amihud, Y., and H. Mendelson (1986). Asset Pricing and the Bid-ask Spread. Journal of Financial Economics 17, 223-249.

4. Amihud, Y., and H. Mendelson (2006). Stock and Bond Liquidity and its Effect of Prices and Financial Policies, Financial Markets and Portfolio Management 20, 19-32.

5. Ariel, R. A. (1987). A Monthly Effect in Stock Returns. Journal of Financial Economics 18, 161-174.

6. Benston, G., and R. Hagerman (1974). Determinants of Bid-ask Spreads in the Over-the-counter Market. Journal of Financial Economics 1, 353-364.

7. Brennan, M.J., and A. Subrahmanyam (1996). Market Microstructure and Asset Pricing: On the Compensation for Illiquidity in Stock Returns. Journal of Financial Economics 41, 441-464.

8. Brunnermeier, Markus, and L.H. Pedersen (2009). Market Liquidity and Funding Liquidity. Review of Financial Studies 22, 2201-2238.

9. Cadsby, Charles B., and M. Ratner (1992). Turn-of-month and Pre-holiday Effect on Stock Returns: Some International Evidence. Journal of Banking and Finance 16, 497-509.

10. Campbell, J. Y., S.J. Grossman, and J. Wang (1993). Trading Volume and Serial Correlation in Stock Returns. The Quarterly Journal of Economics 108, 905-939.

11. Chalmers, J.M.R., and G.B. Kadlec (1998). An Empirical Examination of the Amortized Spread. Journal of Financial Economics 48, 159-188.
13. Chan, K., and Y. Chung (1993). Intra-day Relationship among Index Arbitrage, Spot, and Futures Volatility, and Spot Market Volume: A Transaction Data Set. Journal of Banking and Finance 17, 663-687.
14. Chordia, T., R. Roll, and A. Subrahmanyam (2001). Market Liquidity and Trading Activity. Journal of Finance 56, 501-530.
15. Copeland, T. E., and D. Galai (1983). Information Effects on the Bid-ask Spread. Journal of Finance 38, 1457-1469.
16. Coopu, S.K., J.C. Goth, and W.E. Avera (1985). Liquidity, Exchange Listing and Common Stock Performance. Journal of Economics and Business 37, 19-33.
17. Datar, V.T, N.Y. Naik, and R. Radcliffe (1998). Liquidity and Stock Returns: An Alternative Test Journal of Financial Markets 1, 205-219.
18. Demsetz, H. (1968). The Cost of Transacting. The Quarterly Journal of Economics 82, 33-53.
19. Easley, D., S. Hvidjaer, and M. O’Hara (1999). Is Information Risk a Determinant of Asset Returns? Working paper, Cornell University, USA.
20. Eleswarapu, V.R. (1997). Cost of Transacting and Expected Returns in the NASDAQ Market. Journal of Finance 52, 2113-2127.
21. Stoll, H. R. (2000). Friction. Journal of Finance 54, 1479-1514.
22. Tinic, S. M. (1972). The Economics of Liquidity Services. The Quarterly Journal of Economics 86, 79-93.
23. Vives, X. (1995a). Short-term Investment and the Information Efficiency of the Market. Review of Financial Studies 8, 3-40.
24. Vives, X. (1995b). The Speed of Information Revelation in a Financial Market Mechanism. Journal of Economic Theory 67, 178-204.
25. Ward, Ronald W. (1974). Market Liquidity in the FCOJ Futures Market. American Journal of Agricultural Economics 56, 150-154.
26. World Federation of Exchanges Annual Statistics Report: Derivative Markets (2011). http://www.world-exchanges.org/statistics/annual.