Higher Dimensional Lattice Walks: Connecting Combinatorial and Analytic Behavior

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Abstract

We consider the enumeration of walks on the non-negative lattice $\mathbb{N}^d$, with steps defined by a set $S \subset \{-1,0,1\}^d \setminus \{0\}$. Previous work in this area has established asymptotics for the number of walks in certain families of models by applying the techniques of analytic combinatorics in several variables (ACSV), where one encodes the generating function of a lattice path model as the diagonal of a multivariate rational function. Melczer and Mishna obtained asymptotics when the set of steps $S$ is symmetric over every axis; in this setting one can always apply the methods of ACSV to a multivariate rational function whose set of singularities is a smooth manifold (the simplest case). Here we go further, providing asymptotics for models with generating functions that must be encoded by multivariate rational functions with non-smooth singular sets. In the process, our analysis connects past work to deeper structural results in the theory of analytic combinatorics in several variables. One application is a closed form for asymptotics of models defined by step sets which are symmetric over all but one axis. As a special case, we apply our results when $d = 2$ to give a rigorous proof of asymptotics conjectured by Bostan and Kauers; asymptotics for walks returning to boundary axes and the origin are also given.

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1 Introduction

Much modern research in enumeration concerns links between analytic function behaviour and combinatorial models. When one has sufficient information about a generating function — for instance, the locations and types of its singularities closest to the origin in the complex plane — the theory of analytic combinatorics in one variable gives, almost automatically in many cases, the asymptotics of the related sequence (see the compendium text of Flajolet and Sedgewick [17] for further details).

When a generating function is given in terms of an expansion of a multivariate power series, however, much less is known. Over the last two decades, several authors have been working towards the development of a theory of analytic combinatorics in several variables. This refers to methods for deriving asymptotics of a sequence

$$b_n = [z^n]F(z) = [z_1^{n\cdot r_1} \cdots z_d^{n\cdot r_d}] F(z_1, \ldots, z_d) := a_{n\cdot r_1, \ldots, n\cdot r_d}$$

for some fixed vector $r \in \mathbb{Z}^d$ and multi-dimensional sequence $(a_i)_{i \in \mathbb{Z}^d}$ such that

$$F(z) := \sum_{i \in \mathbb{Z}^d} a_i z^i$$

is a meromorphic function analytic in a specified domain. The theory, as it has developed, generally consists of two stages: first, one must find the contributing singularities of $F(z)$, which are the singularities of $F(z)$
where local behaviour of the function dictates its coefficients’ asymptotics. Once these contributing points have been found, which is often the most difficult step of the analysis, one must then calculate the asymptotic contribution of each point and sum the results.

In 2002, Pemantle and Wilson [28] derived asymptotics for the power series expansion of a rational function $F(z) = G(z)/H(z)$ admitting a finite number of contributing points, at which the variety of singularities is a complex manifold. Two years later [27], they extended this analysis to allow contributing points where the variety of singularities is the transverse union of complex manifolds. Modern approaches incorporate techniques from differential and algebraic geometry, topology, and singularity theory; the interested reader is referred to the text of Pemantle and Wilson [29] or, for a more elementary introduction, the thesis of Melczer [23].

1.1 Lattice walks in restricted regions

Enumerating lattice walks restricted to certain regions is a classical topic in combinatorics, tracing its roots back hundreds of years to work on what is now known as the ballot problem. In modern times, a large area of work on this topic centers around using the so-called kernel method to express generating functions of large classes of models as positive series extractions or diagonals of multivariate rational functions, proving that the generating functions are (or are not) D-finite; that is, determining whether they satisfy a linear differential equation with polynomial coefficients. Although there are a finite number of such models, the combinatorics of walks with steps in $\{\pm 1, 0\}^2$ which are restricted to a quarter-plane has been the subject of intense study in recent years [16, 9, 19, 26, 10, 2, 3, 31, 22, 24, 6, 11, 1, 8, 14]. The models considered in this paper are higher-dimensional generalizations of this two-dimensional setting.

A lattice path model is determined by a finite set of steps $S \subset \mathbb{Z}^d$ and a region $R \subset \mathbb{Z}^d$ to which the walks of the model are restricted. A model whose set of allowed steps is symmetric over every axis is called highly symmetric. Melczer and Mishna [24] combined the kernel method with techniques from analytic combinatorics in several variables to give asymptotics for the number of integer lattice walks restricted to an orthant (that is, $R = \mathbb{N}^d$) when the set of steps $S$ is a subset of $\{\pm 1, 0\}^d \setminus \{0\}$ and is highly symmetric. Their main result is the following.

**Theorem 1** (Melczer and Mishna [24, Theorem 3.4]). Let $S \subset \{-1, 0, 1\}^d \setminus \{0\}$ be a set of steps which is symmetric over every axis and moves forwards and backwards in each coordinate. Then the number $s_n$ of walks of length $n$ taking steps in $S$, beginning at the origin, and never leaving the positive orthant has asymptotic expansion

$$s_n = \left(\sum_{k=1}^{d} s_k^{(1)} \cdots s_k^{(d)}\right)^{-1/2} \pi^{-d/2} |S|^{d/2} \cdot n^{-d/2} \cdot |S|^n + O\left(n^{-d/2-1} \cdot |S|^n\right),$$

where $s_k^{(k)}$ denotes the number of steps in $S$ which have $k^{th}$ coordinate $1$.

In order to get this result, Melczer and Mishna used the kernel method to derive the expression

$$F(t) = \Delta \left(\frac{G(z,t)}{H(z,t)}\right) = \Delta \left(\frac{1+z_1 \cdots (1+z_d)}{1-t(z_1 \cdots z_d)}S(z)\right)$$

for the generating function $F(t)$ counting the number of walks of a given length defined by the model, where $\Delta$ is the diagonal operator defined in Section 3.2 below. After verifying certain technical conditions, Theorem 1 then follows from the main result of the original paper of Pemantle and Wilson [28]. The symmetry condition on the step set $S$ was chosen by Melczer and Mishna precisely because it leads to a smooth singular variety, the set of singular points of $G/H$, defined by the zero set of $H$.

Although not originally considered by Melczer and Mishna, Theorem 1 can be easily extended [23, Chapter 7] to handle positively weighted step sets with weights that are symmetric over every axis. Generalizations of this work have used multivariate singularity analysis to enumerate weighted walks [13] and walks with step coordinates having absolute value greater than one [7].
1.2 Our contributions

Building on the conference paper of Melczer and Wilson [25], in this work we generalize the results of Melczer and Mishna by giving asymptotics of lattice path models, restricted to the positive orthant, whose set of allowable steps is symmetric over all but one axis. The formulae we obtain give explicit and fairly simple descriptions of the exponential rate, leading order and leading coefficient in terms of the basic data of the walk step set $S$. The drift of a walk, the vector sum of all the steps in $S$, plays a crucial role in asymptotics; owing to symmetry, only one coordinate of the drift may be non-zero and we refer to a walk as negative, zero, or positive drift depending on the sign of this coordinate.

Additionally, we show how the arguments of Melczer and Mishna’s analysis fit into larger structure theorems about the singularities of multivariate rational functions and their asymptotic expansions. In the negative and zero drift cases, substantial extra work is needed because the expected leading order coefficient for generic problems of this dimension turns out to vanish. Unfortunately, due to this vanishing and other degeneracies in integrals which must be asymptotically approximated, we are currently unable to determine asymptotics for the general zero drift case. Such models will be the subject of future study.

Furthermore, we provide the first rigorous proofs of the guessed asymptotics of Bostan and Kauers [2] on 2D walks restricted to the non-negative quadrant, completing the outline in Melczer and Wilson [25]. Our analysis uncovers and explains periodicity of the asymptotic coefficients in the negative drift case, which was not noted in [2], and we give asymptotics for the number of walks returning to the boundary axes and the origin. Around the same time as the conference paper of Melczer and Wilson [25], Bostan et al. [8] rigorously gave annihilating differential equations for the generating functions of these lattice path models. Using these differential equations they were able to prove some of the guessed asymptotics of Bostan and Kauers [2], however due to issues related to the decidability of asymptotics for coefficients of D-finite functions they were unable to prove all asymptotics. We discuss the difficulties they faced, and how our results fit into this context, in Section 7.2. An accompanying Maple worksheet verifying our calculations can be found online\(^1\).

1.3 Organization

We begin in Section 2 by discussing our main results and some illustrative examples. Section 3 then gives an overview of the kernel method applied to lattice path enumeration, and shows how it can be used to derive expressions for lattice path generating functions which are amenable to the techniques of analytic combinatorics in several variables. Unlike the previous work of Melczer and Mishna — where complete symmetry of the step sets under consideration simplified the required manipulations of the kernel method — care must be taken here when manipulating diagonal and positive sub-series extractions in iterated Laurent series rings. Section 4 details the general methods of analytic combinatorics in several variables, and outlines how the asymptotic analysis will proceed; in Theorem 15 we give an explicit description of the contributing singularities for the models under consideration. We derive asymptotics using this characterization in Section 5. This work divides naturally into three cases: the positive, negative and zero drift, listed in increasing order of difficulty (as mentioned above, we do not treat the general zero drift case here, see Section 7.4 for more information). Detailed computations needed are collected in Appendix A. Section 6 proves asymptotics of 2D walks restricted to the non-negative quadrant. Section 7 discusses extensions and directions for future research.

A summary of results is displayed in Table 1.

2 Main Results and Examples

In order to simplify equations, we use the following notation:

\[
\zeta_i = z_i^{-1}; \quad z = (z_1, \ldots, z_d); \quad \mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d; \\
z_i^i = z_1^{i_1} \cdots z_d^{i_d}; \quad \zeta_k := (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_d).
\]

\(^1\)https://github.com/smelczer/HigherDimensionalLatticeWalks
We consider walks in dimension $d$ defined by a (finite) set $S \subset \{\pm 1, 0\}^d \setminus \{0\}$ of weighted steps, where $\mathbf{i} \in S$ is given real weight $w_{\mathbf{i}} \geq 0$, such that

- there exists some step forwards and some step backwards in each direction:
  
  For all $j = 1, \ldots, d$ there exist $\mathbf{i} \in S$ with $\mathbf{i}_j = 1$ and $w_{\mathbf{i}} \neq 0$;
  
  For all $j = 1, \ldots, d$ there exist $\mathbf{i} \in S$ with $\mathbf{i}_j = -1$ and $w_{\mathbf{i}} \neq 0$.

- the weighting $w_{\mathbf{i}}$ is symmetric over all axes except one;
- each walk is confined to the non-negative orthant $\mathbb{N}^d$.

The (weighted) inventory of $S$ is the Laurent polynomial

$$S(z) = \sum_{\mathbf{i} \in S} w_{\mathbf{i}} z^\mathbf{i}.$$

We may assume without loss of generality that the axis of non-symmetry is $z_d$. In other words, our step set $S$ is such that for each $j$ with $1 \leq j \leq d-1$,

$$S(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_d) = S(z),$$

so we may write

$$S(z) = z_d A(z_d) + Q(z_d) + z_d B(z_d)$$

for Laurent polynomials $A, B,$ and $Q$ which are symmetric in their variables. Note that $S(1) = |S|$ is the size of the stepset when each step has weight 1.

Also important is the drift $B(1) - A(1)$ of a walk, the weight of steps in the positive $z_d$ direction minus the weight of steps in the negative $z_d$ direction. For our models the sign of the drift will correspond to different asymptotic regimes; in general the relationship between drift and asymptotics is more nuanced [13, Section 6.4].

### 2.1 Positive drift models

For $1 \leq k \leq d-1$, define $b_k$ to be the total weight of the steps moving forwards (or backwards) in the $k$th coordinate,

$$b_k = \sum_{\mathbf{i} \in S, \mathbf{i}_k = 1} w_{\mathbf{i}} = \sum_{\mathbf{i} \in S, \mathbf{i}_k = -1} w_{\mathbf{i}}.$$

Our main asymptotic result for positive drift models is the following.

**Theorem 2** (Positive Drift Asymptotics). Let $S$ be a step set which is symmetric over all but one axis and takes a step forwards and backwards in each coordinate. If $S$ has positive drift, then then number of walks of length $n$ which never leave the non-negative orthant satisfies

$$s_n = S(1)^n \cdot n^{-\frac{(d-1)}{2}} \cdot \left(1 - \frac{A(1)}{B(1)}\right) \left(\frac{S(1)}{\pi}\right)^{\frac{d-1}{2}} \frac{1}{\sqrt{b_1 \cdots b_{d-1}}} \left(1 + O(n^{-1})\right).$$
Note that the result is trivial to apply to any given model, and is general enough to handle families of models in varying dimension.

**Example 3.** Consider the step set where \( B(\mathbf{z}_d) = \prod_{j<d}(z_j + \pi_j) \), \( Q(\mathbf{z}_d) = 0 \), and \( A(\mathbf{z}_d) = 1 \). Then

\[
s_n = (1 + 2^{d-1})^n \cdot n^{-(d-1)/2} \cdot \left[ \frac{2^{d-1} - 1}{(2d\pi)^{d/2}} \right] (1 + O(n^{-1})) .
\]

Theorem 2 is proven in Section 5.1.

### 2.2 Negative drift models

Dominant asymptotics in the negative drift case are given by adding the asymptotic contributions of a finite collection of points. Let \( \rho = \sqrt{\frac{A(1)}{B(1)}} \), and for each \( 1 \leq k \leq d - 1 \) define

\[
b_k(\mathbf{z}_k) := [z_k]S(\mathbf{z}) = [z_k^{-1}]S(\mathbf{z}).
\]

Furthermore, define

\[
C_\rho := \frac{S(1, \rho) \rho}{2 \pi^{d/2} A(1)(1 - 1/\rho)^2} \cdot \sqrt{\frac{S(1, \rho)^d}{\rho b_1(1, \rho) \cdots b_{d-1}(1, \rho) \cdot B(1)}}
\]

and let \( C_{-\rho} \) be the constant obtained by replacing \( \rho \) by \( -\rho \) in \( C_\rho \) (the term in the square-root will always be real and positive, so there is no ambiguity).

**Theorem 4** (Negative Drift Asymptotics). Let \( S \) be a negative drift step set which is symmetric over all but one axis and takes a step forwards and backwards in each coordinate. If \( Q(\mathbf{z}_d) \neq 0 \) (i.e., if there are steps in \( S \) with \( z_d \) coordinate 0) then the number of walks of length \( n \) which never leave the non-negative orthant satisfies

\[
s_n = S(1, \rho)^n \cdot n^{-d/2 - 1} \cdot C_\rho \left( 1 + O(n^{-1}) \right).
\]

If \( Q(\mathbf{z}_d) = 0 \) then the number of walks of length \( n \) which never leave the non-negative orthant satisfies

\[
s_n = n^{-d/2 - 1} \cdot \left[ S(1, \rho)^n \cdot C_\rho + S(1, -\rho)^n \cdot C_{-\rho} \right] \left( 1 + O(n^{-1}) \right).
\]

Again, this result can be immediately applied to families of models.

**Example 5.** Consider the step set where \( A(\mathbf{z}_d) = \prod_{j<d}(z_j + \pi_j) \), \( Q(\mathbf{z}_d) = 0 \), and \( B(\mathbf{z}_d) = 1 \). Then \( \rho = 2^{d-1} \) and

\[
C_\rho = \frac{2^{2d-3/2}}{\pi^{d/2} (2(d-1)/2 - 1)}.
\]

\[
C_{-\rho} = \frac{2^{2d-3/2}}{\pi^{d/2} (2(d-1)/2 + 1)}.
\]

So

\[
s_n = \left( 2^{(d+1)/2} \right)^n \cdot n^{-d/2 - 1} \cdot \frac{2^{2d-3/2}}{\pi^{d/2} (2d-1)^2} \cdot c_n \left( 1 + O(n^{-1}) \right),
\]

where

\[
c_n = \begin{cases} 
2^d + 2 & : n \text{ is even} \\
2^{(d+3)/2} & : n \text{ is odd}
\end{cases}
\]

Theorem 2 is proven in Section 5.2.
3 Lattice Path Generating Functions

We now show how to derive several useful expressions for the generating functions of the lattice path models we consider. We closely follow the kernel method as outlined in Bousquet-Mélou and Mishna [10] and Melczer and Mishna [24].

3.1 A generating function expression via the kernel method

To apply the kernel method we introduce the symmetry group of the walk.

**Definition 6.** For $1 \leq j \leq d - 1$ define the map $\sigma_j : \mathbb{C}^d \to \mathbb{C}^d$ by

$$\sigma_j(z_1, \ldots, z_d) = (z_1, \ldots, z_{j-1}, \bar{z}_j, z_{j+1}, \ldots, z_d),$$

and the map $\gamma : \mathbb{C}^d \to \mathbb{C}^d$ by

$$\gamma(z_1, \ldots, z_d) = \left(z_1, \ldots, z_{d-1}, \frac{A(z_d)}{B(z_d)}\right).$$

We can view these maps as acting on Laurent polynomials $f \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_d, \bar{z}_d]$ through

$$\sigma \cdot f(z) := f(\sigma(z_1, \ldots, z_d))$$

and further view them as acting on elements $\sum_{n \geq 0} f_n(z) t^n \in \mathbb{C}[z_1, \bar{z}_1, \ldots, z_d, \bar{z}_d][[t]]$ by

$$\sigma \cdot \sum_{n \geq 0} f_n(z) t^n := \sum_{n \geq 0} (\sigma \cdot f_n(z)) t^n = \sum_{n \geq 0} f_n(\sigma(z)) t^n.$$ 

Finally, we let $\mathcal{G}$ be the group of birational transformations generated by $\sigma_1, \ldots, \sigma_{d-1}$ and $\gamma$.

**Remark 7.** Since $\mathcal{S}$ is symmetric over all but one axis we have, for each $j = 1, \ldots, d - 1,$

$$\sigma_j\left(A(z_d)\right) = A(z_d) \quad \sigma_j\left(B(z_d)\right) = B(z_d)$$

which, together with the fact that $\gamma$ fixes $\mathcal{S}(z)$, implies that $\mathcal{S}(z)$ is fixed by $\mathcal{G}$. Furthermore, these equalities show that the generators of $\mathcal{G}$, which are involutions, commute, meaning $\mathcal{G}$ is the finite group of order $2^d$ defined by

$$\mathcal{G} := \left\{ \sigma_1 \cdot \cdots \cdot \sigma_{d-1} : j_1, \ldots, j_d \in \{0, 1\} \right\}.$$ 

The group $\mathcal{G}$ is the direct sum of $d$ cyclic groups of order 2.

Let $F(z, t)$ be the multivariate generating function

$$F(z, t) = \sum_{i \in \mathbb{N}^d} a_{i,n} z^i t^n,$$

where $a_{i,n}$ counts the number of weighted walks of length $n$ using the steps in $\mathcal{S}$, beginning at the origin, ending at $i \in \mathbb{N}^d$, and never leaving the non-negative orthant in $\mathbb{Z}^d$. Describing a walk of length $n$ ending at $i \in \mathbb{N}^d$ recursively as a walk of length $n - 1$ followed by a single step, one can show (see Melczer and Mishna [24]) that the generating function satisfies a functional equation of the form

$$(1 - tS(z)) z^1 F(z, t) = z^1 + \sum_{k=1}^d L_k(z_k^1, t), \quad L_k(z_k^1, t) \in \mathbb{Q}[z_k^1][[t]]. \quad (2)$$
In particular, note that each $L_k(z_k, t)$ is independent of the variable $z_k$.

When manipulating the formal expressions which arise in our application of the kernel method, we may encounter rational functions in the variables $z_1, \ldots, z_d$ which, in addition to not being analytic at the origin, are not Laurent polynomials in these variables. Thus, we make use of the iterated Laurent series ring $\mathcal{R} = \mathbb{Q}((z_1)) \cdots ((z_d))[t]$; unless otherwise stated all computations below are assumed to take place in the ring $\mathcal{R}$, which contains both $\mathbb{Q}[z_1, \ldots, z_d, z_\sigma][t]$ and $\mathbb{Q}[[z, t]]$. Note that every rational function in $\mathbb{Q}(z)$ has an expansion in $\mathcal{R}$. For further details on iterated Laurent series, including their uses in combinatorics and a classification of which formal series are iterated Laurent series, the reader is referred to the PhD thesis of Xin [32]. We define the positive sub-series extraction operator $[z^{\geq 0}]: \mathcal{R} \to \mathbb{Q}[[z, t]]$ by

$$[z^{\geq 0}] \sum_{n \geq 0} \left( \sum_{i \in \mathbb{Z}^d} a_{i,n} z^i \right) t^n := \sum_{n \geq 0} \left( \sum_{i \in \mathbb{N}^d} a_{i,n} z^i \right) t^n.$$  

This setup leads to the following result, typical of the kernel method.

**Theorem 8.** If $S$ is symmetric over all but one axis, then the multivariate generating function $F(z, t)$ tracking endpoint and length satisfies

$$F(z, t) = [z^{\geq 0}] \sum_{\sigma \in \mathcal{G}} \text{sgn}(\sigma) \sigma(z_1 \ldots z_d) / (z_1 \cdots z_d)(1 - t S(z)), \quad (3)$$

where

$$\text{sgn} \left( \sigma_1^{j_1} \cdots \sigma_{d-1}^{j_{d-1}} \gamma j_d \right) = (-1)^{j_1 + \cdots + j_d}.$$  

The generating function $F(z, t)$, and thus the specialized generating function $F(1, t)$ which counts the walks of a given length ending anywhere, are $D$-finite.

**Note:** The order of the iterated Laurent fields that define $\mathcal{R}$ is important. If one works in an iterated Laurent field where $z_d$ is not the last variable before $t$, Equation (3) may not hold.

**Proof.** We begin by examining the expression $\sigma(z_1 \ldots z_d) F(\sigma(z), t)$ for some fixed $\sigma = \sigma_1^{j_1} \cdots \sigma_{d-1}^{j_{d-1}} \gamma j_d \in \mathcal{G}$. When $j_d = 1$, then every term in the expansion of $\sigma(z_1 \ldots z_d) F(\sigma(z), t)$ in the ring $\mathcal{R}$ will have negative power of $z_d$ (due to the order of the variables used when defining $\mathcal{R}$). Otherwise, if $j_d = 0$ and there is some $k \in \{1, \ldots, d-1\}$ such that $j_k = 1$ then every term in the expansion of $\sigma(z_1 \ldots z_d) F(\sigma(z), t)$ in the ring $\mathcal{R}$ will have negative power of $z_k$. Thus, we see $[z^{\geq 0}] \sigma(z_1 \ldots z_d) F(\sigma(z), t) = 0$ for $\sigma \in \mathcal{G}$ unless $\sigma$ is the identity element. This implies

$$[z^{\geq 0}] \sum_{\sigma \in \mathcal{G}} \text{sgn}(\sigma) \sigma(z_1 \ldots z_d) F(\sigma(z), t) = \sum_{\sigma \in \mathcal{G}} \text{sgn}(\sigma) \left( [z^{\geq 0}] \sigma(z_1 \ldots z_d) F(\sigma(z), t) \right) = (z_1 \cdots z_d) F(z, t).$$

By definition, for all $\sigma \in \mathcal{G}$ and $\tau \in \{\sigma_1, \ldots, \sigma_{d-1}, \gamma\}$,

$$\text{sgn}(\tau \sigma) = - \text{sgn}(\sigma).$$

As $S(z)$ is fixed by the elements of $\mathcal{G}$, to prove Equation (3) from Equation (2) it is sufficient to show that for each $k = 1, \ldots, d$,

$$\sum_{\sigma \in \mathcal{G}} \text{sgn}(\sigma) \sigma(z_1 \ldots z_d) \left( \sigma \cdot L_k(z_k, t) \right) = 0.$$  

Fix $k$ and write $\mathcal{G}$ as the disjoint union $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$, where

$$\mathcal{G}_0 = \left\{ \sigma_1^{j_1} \cdots \sigma_{d-1}^{j_{d-1}} \gamma j_d : j_1, \ldots, j_d \in \{0, 1\}, j_k = 0 \right\}$$  

$$\mathcal{G}_1 = \left\{ \sigma_1^{j_1} \cdots \sigma_{d-1}^{j_{d-1}} \gamma j_d : j_1, \ldots, j_d \in \{0, 1\}, j_k = 1 \right\}.$$
Then for all \( g \in \mathcal{G}_1, (\sigma_k g) \cdot L_k(z_{\vec{k}}, t) = g \cdot L_k(z_{\vec{k}}, t), \) and therefore
\[
\sum_{\sigma \in \mathcal{G}} \text{sgn}(\sigma)\sigma(z_1 \ldots z_d) (\sigma \cdot L_k(z_{\vec{k}}, t)) = \sum_{\sigma \in \mathcal{G}_0} \text{sgn}(\sigma)\sigma(z_1 \ldots z_d) (\sigma \cdot L_k(z_{\vec{k}}, t))
+ \sum_{\sigma \in \mathcal{G}_1} \text{sgn}(\sigma)\sigma(z_1 \ldots z_d) (\sigma \cdot L_k(z_{\vec{k}}, t))
= \sum_{\sigma \in \mathcal{G}_0} (\text{sgn}(\sigma) - \text{sgn}(\sigma))\sigma(z_1 \ldots z_d) (\sigma \cdot L_k(z_{\vec{k}}, t))
= 0,
\]
as desired. The results on D-finiteness follow from a classical result of Lipschitz [21] which states (in an equivalent form) that the class of D-finite functions is closed under positive sub-series extraction.

The next result determines an explicit expression for the generating function under consideration.

**Lemma 9.** For the group \( \mathcal{G} \),
\[
\sum_{\sigma \in \mathcal{G}} \text{sgn}(\sigma)\sigma(z_1 \ldots z_d) = (z_1 - \bar{z}_1) \cdots (z_{d-1} - \bar{z}_{d-1}) \left( z_d - \frac{A(z_d)}{B(z_d)} \right).
\]
Consequently,
\[
F(z, t) = [z^{>0}] R(z, t)
\]
where
\[
R(z, t) = \frac{(1 - z_1^{-2}) \cdots (1 - z_{d-1}^{-2}) (B(z_d) - z_d^{-2} A(z_d))}{B(z_d) (1 - tS(z))}.
\] (4)

**Proof.** The first statement follows directly from the definition of \( \mathcal{G} \) and the sign operator (formally it can be proven by induction). The second statement comes from combining Lemma 9 with (3).

### 3.2 A diagonal representation

Next, we turn back to the sequence counting the total number of walks of a given length (regardless of endpoint). The generating function of this sequence is simply \( F(1, t) \), since specializing each \( z_j \) variable to 1 sums over its possible values.

We may translate the positive sub-series extraction given by Equation (3) into an expression for \( F(1, t) \) using the diagonal operator \( \Delta : \mathcal{R} \to \mathbb{Q}[[t]] \) defined by
\[
\Delta \left( \sum_{n \geq 0} \left( \sum_{i \in \mathbb{Z}^d} a_{1,n} z_i^i \right) t^n \right) := \sum_{n \geq 0} a_{n,\ldots,n} t^n.
\]

Our asymptotic results will follow from an analysis of a diagonal expression for \( F(1, t) \). Establishing this diagonal expression is more complicated than in Melczer and Mishna [24], because we must consider expressions whose coefficients in \( t \) are not Laurent polynomials. In the completely symmetric case \( A = B \) in (4), and cancellation leaves only \( 1 - tS(z) \) in the denominator.

The following technical lemma is elementary (see also Melczer and Mishna [24, Proposition 2.6]).

**Lemma 10.** Let \( P(z, t) \in \mathbb{Q}[z_1, \bar{z}_1, \ldots, z_d, \bar{z}_d][[t]] \subset \mathcal{R} \). Then
\[
([z^{>0}]P(z, t)) \bigg|_{z_1 = 1, \ldots, z_d = 1} = \Delta \left( \frac{P(\bar{z}_1, \ldots, \bar{z}_d, z_1 \cdots z_d \cdot t)}{(1 - z_1) \cdots (1 - z_d)} \right),
\] (5)
where the diagonal on the right hand side is taken as an expansion in \( \mathcal{R} \), as usual.
The proof follows from the definition of the diagonal after writing out the geometric series and expansion of $P$ on the right hand side.

We would like to use Lemma 10 directly, but unfortunately $R$ in (4) does not lie in the correct ring and the substitutions indicated (replacing $z_i$ by $\tau_i$) are not formally justified. This problem can be circumvented through a tedious but elementary generating function argument, taking into account the precise structure of the rational function under consideration, giving Proposition 11. Ultimately, we use a different diagonal expression for the generating function more suitable to an asymptotic analysis, so we omit the proof.

**Proposition 11.** Let $\mathcal{S}$ be a weighted step set satisfying the conditions above. Then the generating function counting the number of walks of a given length in the lattice path model defined by $\mathcal{S}$ satisfies

$$F(1,t) = \Delta \left( \frac{(1 + z_1) \cdots (1 + z_{d-1}) (B(z_d) - z_d^2 A(z_d))}{(1 - z_d)B(z_d) (1 - t z_1 \cdots z_d S(z_1, \ldots, z_{d-1}, \tau_d))} \right).$$

The generating function expression in Proposition 11 presents a challenge for the integral manipulations necessary to compute asymptotics, as one can only easily deform domains of integration where the integrand is analytic; the factor $B(z_d)$ present in the denominator can give strange surfaces of singularities. Instead we use the following alternative expression, which is a power series in $t$ with Laurent polynomial coefficients in the other variables.

**Theorem 12.** Let $\mathcal{S}$ be a weighted step set satisfying the conditions above. Then the generating function counting the number of walks of a given length in the lattice path model defined by $\mathcal{S}$ satisfies

$$F(1,t) = \Delta \left( \frac{G(z,t)}{H(z,t)} \right),$$

where

$$G(z,t) = (1 + z_1) \cdots (1 + z_{d-1}) \left( 1 - t z_1 \cdots z_d (Q(z_d) + 2z_d A(z_d)) \right),$$

$$H(z,t) = (1 - z_d) \left( 1 - t z_1 \cdots z_d S(z) \right) \left( 1 - t z_1 \cdots z_d (Q(z_d) + z_d A(z_d)) \right),$$

and

$$S(z) = S(z_d, \tau_d) = \tau_d B(z_d) + Q(z_d) + z_d A(z_d).$$

**Proof.** Expanding the expression in (4) we obtain

$$(1 - z_d^2) \cdots (1 - z_d^{d-1}) \cdot \left( 1 - z_d^2 A(z_d) / B(z_d) \right) \sum_{n \geq 0} t^n \left( z_d A(z_d) + Q(z_d) + z_d B(z_d) \right)^n.$$ 

As the sub-expression

$$(1 - z_d^2) \cdots (1 - z_d^{d-1}) \cdot \left( -z_d^2 A(z_d) / B(z_d) \right) \sum_{n \geq 0} t^n \left( z_d A(z_d) + Q(z_d) \right)^n$$

contains no positive powers of $z_d$, we can subtract it from $R(z)$ and extract the positive part of

$$\frac{(1 - z_d^2) \cdots (1 - z_d^{d-1}) \left( 1 - z_d A(z_d) / B(z_d) \right)}{1 - tS(z)} + \frac{(1 - z_d^2) \cdots (1 - z_d^{d-1}) \left( z_d^2 A(z_d) / B(z_d) \right)}{1 - t (z_d A(z_d) + Q(z_d))}.$$ 

This final rational function simplifies to

$$\frac{(1 - z_d^2) \cdots (1 - z_d^{d-1}) \left( 1 - t (z_d A(z_d) + Q(z_d)) \right)}{(1 - t (z_d A(z_d) + Q(z_d)) + z_d B(z_d)) \left( 1 - t (z_d A(z_d) + Q(z_d)) \right)},$$

and we can now apply Lemma 10. □
Note that the power series expansion of $1/H(z, t)$ has all non-negative coefficients, which will simplify the arguments below. In the special case where $S$ is symmetric over all axes, we obtain an expression different from that in [24]: by forcing positivity on our series coefficients we have lost some symmetry and less cancellation occurs. For example, the generating function of the model with all possible steps in 2 dimensions is the diagonal of

$$
\frac{(1 + x)(1 + y)}{1 - t((1 + x + y + x^2 + y^2 + x^2y + xy^2 + x^2y^2))}
$$

using the expression in Proposition 11, which coincides with that in [24], but the diagonal of

$$
\frac{(1 + x)(1 - 2t(y + y^2 + x^2y + xy^2 + x^2y^2))}{(1 - y)(1 - t(1 + x + y + x^2 + y^2 + x^2y + xy^2 + x^2y^2))(1 - t((y + y^2 + x^2y + xy^2 + x^2y^2)))}
$$

using the expression in Theorem 12.

### 3.3 Why these methods do not extend to smaller groups

We end this section with a justification of why we only consider models missing one symmetry (instead of two, three, etc.). Indeed, as the following theorem shows, in any dimension $d \geq 2$ there exists a model that is missing two symmetries and admits a generating function which is non D-finite. As the diagonal of a multivariate rational function must be D-finite [12, 21], this shows that it is impossible to determine the asymptotics of all models missing two symmetries uniformly through multivariate rational diagonals and analytic combinatorics in several variables.

**Theorem 13.** There exists a sequence of step sets $S_2, S_3, \ldots$ with $S_d$ defining a step set of dimension $d$ that is symmetric over all but two axes, such that the generating function of each model is non D-finite.

**Proof.** If a counting sequence $(c_n)_{n \geq 0}$ has asymptotics of the form $c_n \sim K \cdot \rho^n \cdot n^\alpha$ for constants $K, \rho, \alpha \in \mathbb{R}$ and its generating function is D-finite, then $\rho$ is algebraic and $\alpha$ is rational (see Theorem 3 of Bostan, Raschel, and Salvy [5]).

When $d = 2$, consider the set of steps

$$S_2 = \{(-1, -1), (0, -1), (0, 1), (1, 0), (-1, 0)\}.$$

Bostan, Raschel, and Salvy [5] show that $(\epsilon_n)_{n \geq 0}$, the number of walks on these steps staying the first quadrant which begin and end at the origin, has dominant asymptotics

$$\epsilon_n \sim K_{\epsilon} \cdot \rho_{\epsilon}^n \cdot n^{\alpha_{\epsilon}},$$

where $\alpha$ is an irrational number (approximately 2.757466) equal to $-1 - \pi/\arccos(-c)$, with $c$ an algebraic number satisfying $c^3 - c^2 + (3/4)c - (1/8) = 0$. Work of Dura[15] implies that—since $S_2$ has a negative vector sum in both coordinates, which is known as ‘negative drift’—the sequence counting the total number of steps has dominant asymptotics

$$s_n^{(2)} \sim K_2 \cdot \rho_2^n \cdot n^{\alpha_2},$$

where $\alpha_2 = \alpha_{\epsilon}$ and is thus irrational.

For $d \geq 3$ let $S_d = S_2 \times \{\pm 1\}^{d-2}$. Every walk of length $n$ on the steps $S_d$ is constructed uniquely from a walk of length $n$ on the steps $S_2$ in the non-negative quadrant and $d - 2$ independent walks of length $n$ on the steps $(-1, 1)$ restricted to the non-negative integers (this is a simple version of the Hadamard decomposition of walks studied in Bostan et. al. [6]). Thus, the number of walks of length $n$ taking steps in $S_d$ restricted to the $d$-dimensional non-negative orthant is

$$s_n^{(d)} = s_n^{(2)} \cdot c_n^{d-2},$$

\footnote{That article takes a probabilistic view of exit times for random walks to leave certain cones; see its Example 7 for the case of two dimensional random walks in a quadrant.}
where \( c_n \) is the number of Dyck paths which do not have to end at 0. It is a classical result of enumerative combinatorics that \( c_n = \binom{n}{\lceil n/2 \rceil} \) with dominant asymptotics of the form

\[
c_n \sim \sqrt{2/\pi} \cdot 2^n \cdot n^{-1/2},
\]
which implies

\[
s_n^{(d)} \sim K_d \cdot (\rho_d)^n \cdot n^{\alpha_d},
\]
with \( \alpha_d = \alpha_2 - d/2 + 1 \notin \mathbb{Q} \).

It would be of great interest to find ‘simple’ diagonal expressions involving more general multivariate meromorphic functions for walk models with non-D-finite generating functions. Such multivariate functions could not be D-finite, in the sense that the vector space of all partial derivatives over \( \mathbb{Q}(z) \) would need to be infinite dimensional.

We now move on to the analysis of the expressions obtained using the methods of analytic combinatorics in several variables. We use the methods developed by Pemantle and Wilson [29] for asymptotics controlled by points where the zero set of \( H(z, t) \) is locally a manifold or a union of finitely many transversely intersecting manifolds.

## 4 Contributing Singularities

Suppose \( Q(z, t) \) is a rational function analytic at the origin. As in the univariate case, a multivariate singularity analysis starts from the Cauchy integral formula, which implies

\[
b_n := [(z_1 \cdots z_d t)^n]Q(z, t) = \frac{1}{(2\pi i)^d+1} \int_{\mathcal{C}} Q(z) \cdot \frac{dz\,dt}{(z_1 \cdots z_d t)^{n+1}}
\]

for any \( n \in \mathbb{N} \) and \( \mathcal{C} \) a product of circles sufficiently close to the origin. If \( \mathcal{D} \) is the domain of convergence of the power series of \( Q(z, t) \) at the origin, and \( \mathcal{V} \) is the set of singularities of \( Q(z, t) \), then any singularity on the boundary \( \partial \mathcal{D} \) of \( \mathcal{D} \) is called minimal. When \( P(z, t) \) is a polynomial we say that \( (w, s) \) is a minimal zero of \( P(z, t) \) if \( P(w, s) = 0 \) and \( P(y, r) \neq 0 \) whenever

\[
|w_j| \leq |y_j|, \quad j = 1, \ldots, d, \quad |r| \leq |s|
\]

and one of the inequalities is strict. Note that a minimal point of \( Q \) is a minimal zero of its denominator, and vice-versa.

As \( Q(z, t) \) is rational, \( b_n \) grows at most exponentially and standard integral bounds imply

\[
\limsup_{n \to \infty} |b_n|^{1/n} \leq |w_1 \cdots w_d s|^{-1} \quad (8)
\]

for any minimal point \( (w, s) \in \partial \mathcal{D} \cap \mathcal{V} \). In the simplest cases, one hopes to identify a finite set of minimal points achieving the optimal bound in Equation (8). When such a set exists, and the local geometry of the algebraic set \( \mathcal{V} \) is sufficiently nice, asymptotics can then be determined.

We now specialize our arguments to the rational function \( Q(z, t) = G(z, t)/H(z, t) \) defined by Theorem 12; note that \( G \) and \( H \) are co-prime, so the singularities of \( Q \) are the zeroes \( \mathcal{V} = \mathcal{V}(H) \) of the polynomial \( H \). Because of the nice form of \( H \), we are able to characterize its minimal zeroes achieving the best bound in Equation (8), which is typically the hardest step of any asymptotic analysis. We make use of the following result.

**Lemma 14.** Suppose \( \mathcal{P} \subset \mathbb{Z}^d \) is a finite set not contained in a hyperplane of \( \mathbb{R}^d \), and \( a_i > 0 \) are positive constants for each \( i \in \mathcal{P} \). Then every critical point of

\[
P(z) = \sum_{i \in \mathcal{P}} a_i z^i
\]

on \( (\mathbb{R}_{>0})^d \) is a global minimum and \( P \) admits at most one critical point on this domain. Furthermore, such a global minimum exists if and only if \( \mathcal{P} \) is not contained in a halfspace containing the origin.
Proof. This result follows from the strict convexity of the Laplace transform $P(e^{x_1}, \ldots, e^{x_d})$; for details see Garbit and Raschel [18, Lemma 7].

In order to reason about minimal zeroes of $H(z, t)$, we define the factors
\[
H_1 := 1 - tz_1 \cdots z_d S(z) = 1 - tz_1 \cdots z_{d-1} \left( z_d^2 A(z_d) + z_d Q(z_d) + B(z_d) \right)
\]
\[
H_2 := 1 - tz_1 \cdots z_d \left( Q(z_d) + z_d A(z_d) \right)
\]
\[
H_3 := 1 - z_d,
\]
and set $V_j = V(H_j)$.

Under our assumptions on $S$ the conditions of Lemma 14 are satisfied by $S(z)$, giving the following.

**Proposition 15.** The unique minimal zero of $H(z, t)$ with positive coordinates which minimizes $|z_1 \cdots z_d t|^{-1}$ is
\[
p_1 := \left(1, 1, \ldots, 1, \sqrt{B(1)/A(1)}, \frac{\sqrt{A(1)/B(1)}}{2\sqrt{A(1)B(1)} + Q(1)} \right)
\]
if the drift is negative
\[
p_2 := \left(1, 1, \ldots, 1, \frac{1}{S(1)} \right)
\]
otherwise.

Proof. As $|z_1 \cdots z_d t|^{-1}$ decreases as $(z, t)$ moves away from the origin, any such minimizer must be a zero of $H_1$ or $H_2$. Since $S(z)$ has non-negative coefficients, any zero of $H_1$ with positive coordinates is a minimal zero as $t = (z_1 \cdots z_d S(z))^{-1}$ increases as one of the $z_j$ decrease and the others are constant. Furthermore, on $V_1 \cap (\mathbb{R}_{>0})^d$
\[
|z_1 \cdots z_d t|^{-1} = S(z),
\]
which by Lemma 14 has a unique minimum corresponding to a unique critical point. The system
\[
S_{z_1}(z) = \cdots = S_{z_d}(z) = 0
\]
can be reduced to
\[
(1 - z_1^2) \cdot |z_1^{-1}| S(z) = \cdots = (1 - z_{d-1}^2) \cdot |z_{d-1}^{-1}| S(z) = B(z_d) - z_d^2 A(z_d) = 0,
\]
which has only the solution $(1, \sqrt{B(1)/A(1)})$ with positive coordinates, as $S$ has all non-negative coefficients.

If the drift is non-positive, $B(1) \leq A(1)$ and $p_1$ is a minimal zero of the product $H_1(z, t)H_3(z, t)$. Otherwise, any minimal zero of $H_1(z, t)H_3(z, t)$ which minimizes $|z_1 \cdots z_d t|^{-1}$ must lie on $V_1 \cap V_3$, where
\[
|z_1 \cdots z_d t|^{-1} = S(z_d, 1),
\]
and Lemma 14 implies the minimizer is $p_2$.

Finally, if $(z, t) \in V_2 \cap (\mathbb{R}_{>0})^d$ then
\[
t = \frac{1}{z_1 \cdots z_d (Q(z_d) + z_d A(z_d))} > \frac{1}{z_1 \cdots z_d S(z, t)}
\]
since $z_1 \cdots z_d B(z) > 0$. But this implies $(z, t)$ is not a minimal zero of $H$, as there exists $(z, s) \in V_1$ with $0 < s < t$.

In order to perform a local singularity analysis we will need to describe $V$ near points of interest. In our case, the singular set $V$ is the union of smooth manifolds $V_1$, $V_2$, and $V_3$ (for each $i$, the gradient of $H_i$ never vanishes when $H_i = 0$). Furthermore, we show in the proof of Theorem 16 that any minimal singularity will not lie on $V_2$, so that any minimal singularity is either in $V_1$ alone, $V_3$ alone, or the intersection $V_1 \cap V_3$. As the gradients of $H_1$ and $H_3$ are linearly independent at any common zero, we say $V_1$ and $V_3$ are transverse.
In this setting, the stratum of minimal \( w \in \mathcal{V} \) is the intersection of the \( \mathcal{V}_j \) containing \( w \). Minimal \( w \in \mathcal{V} \) with non-zero coordinates is called a minimal critical point if it is a critical point (in the differential geometry sense) of the map \( \phi(z) = \log(z_1 \cdots z_d) \) from the stratum of \( w \) to \( \mathbb{C} \). Algebraically, this means that the gradient of \( \phi(z) = \log(z_1 \cdots z_d) \) at \( z = w \) can be written as a linear combination of the gradients of the \( H_j \) polynomials defining the strata of \( w \). Critical points are those where the Cauchy integral can be locally manipulated into a so-called Fourier-Laplace integral, where saddle-point methods can be applied to obtain asymptotics.

General definitions of critical and contributing points, where local function behaviour dictates coefficient asymptotics, can be found in [29]. In particular, Proposition 10.3.6 of [29] gives an explicit characterization of contributing points: in our setting, a singularity of \( Q(z, t) \) is a contributing point if it is a minimal critical point which minimizes \( |z_1 \cdots z_d t|^{-1} \) on \( \partial D \) (the exponential order of the asymptotic contribution of that point is maximum).

**Theorem 16.** When the drift is positive, there are at most \( 2^{d-1} \) contributing points. The point \( p_2 \) is one, and the others are the points \( (w, 1, t) \) where

\[
\begin{align*}
  w &\in \{\pm 1\}^{d-1}, \\
  t &= \frac{1}{w_1 \cdots w_{d-1} \cdot S(w, 1)}, \text{ and } |t| = \frac{1}{S(1, 1)}.
\end{align*}
\]

When the drift is negative, there are at most \( 2^d+1 \) contributing points. The point \( p_1 \) is one, and the others are the points \( (w, w_d, t) \) where

\[
\begin{align*}
  w &\in \{\pm 1\}^{d-1}, \\
  w_d &= \nu \sqrt{\frac{B(w)}{A(w)}}, \\
  |w_d| &= \frac{\sqrt{B(1)}}{\sqrt{A(1)}}, \text{ and } |t| = \frac{\sqrt{A(1)}}{\sqrt{B(1)}S(1, \sqrt{A(1)}/B(1))},
\end{align*}
\]

with \( \nu \) a fourth root of unity (note that in order to satisfy the condition on \( |t| \) it is necessary that \( B(w)/A(w) > 0 \), so the square root can be taken unambiguously).

When the drift is zero, there are at most \( 2^d \) contributing points. The point \( p_1 = p_2 \) is one, and the others are the points \( (w, w_d, t) \) where

\[
\begin{align*}
  (w, w_d) &\in \{\pm 1\}^d, \\
  t &= \frac{1}{w_1 \cdots w_{d-1}S(w, w_d)}, \text{ and } |t| = \frac{1}{S(1, 1)}.
\end{align*}
\]

**Proof.** As the power series expansion of \( 1/H(z, t) \) has non-negative coefficients, every minimal point has the same coordinate-wise modulus as an element of \( \mathcal{V} \) with positive coordinates (an element of \( \partial D \) is the limit of a sequence in \( \mathcal{D} \) which makes the power series of \( 1/H \) approach infinity, but as the power series coefficients are non-negative the series only gets larger when each coordinate is replaced by its modulus).

Thus, we search for points in \( \mathcal{V} \) with the same coordinate-wise modulus as \( p_1 \) and \( p_2 \). First, we note that no point in \( \mathcal{V}_2 \) is minimal as its \( t \) variable will be smaller than required. On \( \mathcal{V}_1 \), we seek points \( (w, s) \) such that

\[
|S(w, s)| = S(p_1) \quad \text{or} \quad |S(w, s)| = S(p_2).
\]

Since \( z_1 \cdots z_d S(z, t) \) is a polynomial with non-negative coefficients, the triangle inequality implies the only way this can happen is if every monomial of \( S(z, t) \) has the same argument when evaluated at \( w \). Our assumptions on \( S \) imply that \( w_1, \ldots, w_{d-1} \) must be real (and thus \( \pm 1 \)) so the points in the statement of Theorem 16 are the only potential minimizers of \( |z_1 \cdots z_d t|^{-1} \), and are minimal points.

Computing the gradient of \( H_1(z) = 1 - t z_1 \cdots z_d S(z) \) shows that a point \( (z, t) \) with stratum \( \mathcal{V}_1 \) is critical if and only if \( z \) satisfies

\[
S_{z_1}(z) = \cdots = S_{z_d}(z) = 0,
\]

while a point with stratum \( \mathcal{V}_1 \cap \mathcal{V}_3 \) is critical if and only if

\[
S_{z_1}(z) = \cdots = S_{z_d}(z) = 0, \quad z_d = 1.
\]

These equations are satisfied by the stated points, therefore we have found the set of contributing points. \( \Box \)
5 Asymptotic Expansions

The results of Pemantle and Wilson [29] apply broadly to compute asymptotics when contributing points are known. Now that the set of contributing points is characterized by Theorem 16 it is a straightforward (though computationally intensive) matter to compute asymptotics, which we do for each of the cases in Theorem 16. Some of the technical and laborious proofs which are not of theoretical interest are given in Appendix A.

We make an exponential change of variables to convert complex contour integrals to integrals over \( \mathbb{R}^d \). To this end we introduce some basic notation.

**Definition 17.** For \( p \in \mathbb{C}^d \), define \( E \) on \( [-\pi, \pi]^d \) by

\[
E_p(\theta) = (p_1 e^{i\theta_1}, \ldots, p_d e^{i\theta_d})
\]

For fixed \( p \), every function \( f(z) \) of \( z \) yields a corresponding function \( \tilde{f}(\theta) := f \circ E_p \) of \( \theta \) under this change of variable.

For \( 1 \leq j \leq d \), we use the usual notation \( \partial_j \) for the partial differential operator \( (\partial/\partial \theta_j) \) and define the functions \( B_1(z_1), \ldots, B_{d-1}(z_{d-1}) \) by stipulating that \( B_k \) is the unique Laurent polynomial such that

\[
\overline{S}(z) = (z_k + \bar{z}_k)B_k(z_k) + Q_k(z_k)
\]

for some Laurent polynomial \( Q_k \). For notational convenience we set \( B_d(z_d) := B(z_d) \).

**Remark 18.** We note that for each \( j \neq k \) with \( j, k < d \), each \( B_j \) is symmetric in \( z_k \) and \( \bar{z}_k \), and does not involve any power of \( z_j \). Also \( A, B, Q \) (and \( S \) and \( \overline{S} \)) are symmetric in \( z_k \) and \( \bar{z}_k \).

We also need the following quantities.

**Definition 19.** For each \( j < d \), define Laurent polynomials \( A_j', B_j', A_j'', B_j'' \) by

\[
A_j(z_d) = (z_j + \bar{z}_j)A_j'(z_j) + A_j''(z_j)
\]

\[
B_j(z_d) = (z_j + \bar{z}_j)B_j'(z_j) + B_j''(z_j).
\]

Finally, for a differentiable function \( f(z) \) we define

\[
\nabla \log f(z) := (z_1 \partial_1 f, \ldots, z_d \partial_d f).
\]

We now have all the necessary tools to compute asymptotics, beginning with the positive drift case.

5.1 The Positive Drift Models

In the positive drift case, when \( A(1) < B(1) \), Theorem 16 implies that we are dealing with contributing points on the stratum \( V_1 \cap V_3 \). The next result follows from Theorem 10.3.4 of Pemantle and Wilson [29], where \( e_j \) is the \( j \)th standard basis vector (with a 1 in its \( j \)th entry and zeroes elsewhere).

**Proposition 20.** Let \( \Gamma \) be the square matrix whose first 2 rows are \( \nabla \log H_1(p) \) and \( \nabla \log H_3(p) \), and whose last \( d - 1 \) rows are \( p_j e_j \) for \( j = 1, \ldots, d - 1 \). Furthermore, define

\[
g(\theta) := \log \left( \frac{1}{(p_1 \cdots p_d) e^{i(\theta_1 + \cdots + \theta_d)} \overline{S}(p_1 e^{i\theta_1}, \ldots, p_d e^{i\theta_d})} \right).
\]

Then

\[
[t^{n_1}] \Delta Q(z, t) = p^{n_1} \cdot n^{-(d-1)/2} \left( \frac{(2\pi)^{-(d-1)/2} (d + 1)^{-(d-1)/2}}{\det \Gamma \cdot \sqrt{\det g''(0)}} \right) + O(n^{-1}),
\]

where \( g''(0) \) denotes the Hessian of \( g(\theta) \) at the origin.
Proof. As $p_2$ is the only contributing point where $G(z, t)$ does not vanish, Theorem 10.3.4 of Pemantle and Wilson [29] gives the above asymptotic result and the bound of $O(n^{-1})$ on the lower order terms.

Applying Proposition 20 in our situation gives asymptotics in the positive drift case.

Proof of Theorem 2. Theorem 16 implies that the point $p = (1, 1, 1/S(1, 1))$ is the unique contributing point at which $G(z, t)$ does not vanish. At this point, one can calculate that

$$
\Gamma = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & -1 & 0 \\
-1 & -1 & -1 & \ldots & -1 & -r & -1 \\
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 
\end{pmatrix}
$$

for real number $r < 1$, and

$$
\frac{G(1, 1/[S])}{H_2(1, 1/[S])} = 2^{d-1} \frac{(B(1) - A(1)) S(1)}{S(1) B(1)} = 2^{d-1} \left( 1 - \frac{A(1)}{B(1)} \right).
$$

The Chain Rule and Lemma 27 of Appendix A imply that $g''(0)$ is the diagonal matrix

$$
g''(0) = \begin{pmatrix}
\frac{2B_1(1)}{(d+1)S(1)} & 0 & 0 & \ldots & 0 \\
0 & \frac{2B_2(1)}{(d+1)S(1)} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \frac{2B_{d-1}(1)}{(d+1)S(1)} & 0 \\
0 & 0 & \ldots & 0 & \frac{2B_{d-1}(1)}{(d+1)S(1)}
\end{pmatrix},
$$

so that Proposition 20 gives

$$
s_n = S(1)^n n^{-1/2} (2\pi)^{-\frac{d+1}{2}} (d+1)^{-\frac{(d-1)}{2}} \frac{2^{d-1} \left( 1 - \frac{A(1)}{B(1)} \right)}{\sqrt{(d+1)^{-\frac{(d-1)}{2}} 2^{d-1} S(1)^{-\frac{(d-1)}{2}} (b_1 \ldots b_{d-1})}} + O(S(1)^n n^{-3/2}),
$$

which simplifies to (1).

5.2 The Negative Drift Models

In the negative drift case, when $A(1) < B(1)$, Theorem 16 implies that we are dealing with contributing points on the stratum $V_1$ where $V$ itself is locally a manifold. This simplifies computations, but an added difficulty is that the numerator vanishes to at least first order at every critical point. We now state a general theorem which allows one to calculate the asymptotics of smooth points where the numerator of the rational function under consideration vanishes, coming from Raichev and Wilson [30] (note that for us the dimension $d$ is one less than the number of variables of $F$).
Theorem 21. Fix natural number \( N > 0 \) and recall the above notation from this section. In a neighborhood in \( \mathcal{V} \) of a smooth critical point \( p \) on \( \mathcal{V}_1 \), write \( t = h(z) \). Define \( u, \tilde{g} \) and \( \tilde{g} \) by

\[
\begin{align*}
    u(z) &= -\frac{G(z, h(z))}{h(z)(\partial H/\partial t)(z, h(z))} \\
    \tilde{g}(\theta) &= \log \frac{\tilde{h}(\theta)}{h(0)} + i \sum_{j=1}^{d} \theta_j \\
    \tilde{g}(\theta) &= \tilde{g}(\theta) - \frac{1}{2} \theta^T\partial H/\partial t(0) \theta.
\end{align*}
\]

Supposing that the Hessian determinant \( \det \tilde{g}''(0) \neq 0 \), define

\[
\Psi_{n,N}^{(p)} := p^{-n} \cdot n^{-d/2} \cdot (2\pi)^{-d/2} (\det \tilde{g}''(0))^{-1/2} \sum_{k=0}^{N-1} n^{-k} L_k(\tilde{u}, \tilde{g}),
\]

where

\[
L_k(\tilde{u}, \tilde{g}) = \sum_{l=0}^{2k} \frac{\mathcal{H}^{k+l}(\tilde{u} \tilde{g}^l)(0)}{(-1)^k 2^{k+l} (k + l)!}
\]

with \( \mathcal{H} \) the differential operator

\[
\mathcal{H} = -\sum_{1 \leq a,b \leq d} (\tilde{g}''(0))^{-1} a_\alpha b_\beta \partial_\alpha \partial_\beta.
\]

Then, as \( n \to \infty \),

\[
[t^{n} \Delta Q(z, t)] = \sum_{p \in W} \Psi_{n,N}^{(p)} + O(n^{-N}).
\]

Lemma 28 of Appendix A shows that the term \( L_1(\tilde{u}, \tilde{g}) \), which will determine dominant asymptotics for negative drift models, simplifies considerably for the functions we consider. In the setting of this section we have

\[
\tilde{u}(\theta) = \frac{(1 + p_1 e^{i \theta}) \cdots (1 + p_{d-1} e^{i \theta_{d-1}}) \left( 1 - p_d e^{2i \theta_d} A(p_d)/B(p_d) \right)}{1 - p_d e^{i \theta_d}}
\]

\[
\tilde{g}(\theta) = \log S(p) - \log S(p_1 e^{i \theta_1}, \ldots, p_{d-1} e^{i \theta_{d-1}}),
\]

and Lemma 27 of Appendix A implies that \( \tilde{g}''(0) \) is the diagonal matrix

\[
\tilde{g}''(0) = \begin{pmatrix}
\frac{2p_1 B_1(p_1)}{S(p)} & 0 & 0 & \cdots & 0 \\
0 & \frac{2p_2 B_2(p_2)}{S(p)} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{2p_{d-1} B_{d-1}(p_{d-1})}{S(p)} & 0 \\
0 & 0 & \cdots & 0 & \frac{2B_d(p_d)}{p_d S(p)}
\end{pmatrix}.
\]

Extensive calculations, given by Proposition 29 and Proposition 30 in Appendix A, then give the following.

Proposition 22. Let \( S \) be a step set which is symmetric over all but one axis and takes a step forwards and backwards in each coordinate, and let \( W \) be the set of contributing points given by Theorem 16. If \( S \) has negative drift, then the number of walks of length \( n \) which never leave the non-negative orthant satisfies

\[
s_n = \sum_{p \in W} \left[ (p_1 \cdot \cdot \cdot p_d p_t)^{-n} \left( n^{-d/2-1} K_p C_p + O(n^{-1}) \right) \right],
\]
where

\[
K_p = 2^{-d} \pi^{-d/2} S(p)^{d/2} \left( p_1 \cdots p_{d-1} \cdot B_1(p) \cdots B_{d-1}(p) \right) B(p)/p_d)^{-1/2}
\]

\[
C_p = \frac{S(p)}{1 - p_d} \prod_{j<d} (1 + p_j) \left[ \frac{1}{A(p)p_d(1-p_d)} + \sum_{j=1}^{d-1} \frac{1 - p_j}{2p_jB_j(p)} \left( \frac{A'_j(p)}{A(p)} - \frac{B'_j(p)}{B(p)} \right) \right].
\]

Examining the set of contributing points given by Theorem 16 implies only those whose first \( d - 1 \) coordinates are 1 contribute to dominant asymptotics (otherwise they have a −1 coordinate and \( C_p \) is zero). Furthermore, if \( c = \sqrt{B(1)/A(1)} \) and \( |S(1, c \nu)| = S(1, c) \) then \( \nu \) must be 1 if \( Q \neq 0 \) and must be either 1 or −1 if \( Q = 0 \). Putting everything together gives Theorem 4. Note that \( p \) is the reciprocal of \( c \), so \( S \) is replaced by \( S \) in the theorem statement, and the radicand appearing in \( C_{-\rho} \) is positive as \( S(1, -\rho) \) and each \( b_j(1, -\rho) \) are negative when \( Q = 0 \).

6 Applications to 2D Models Restricted to a Quadrant

The study of two dimensional lattice walks restricted to the non-negative quadrant has been a very active area of interest in several sub-areas of combinatorics (see, for instance, the citations in our Introduction above) and has applications to several branches of applied mathematics, including queuing theory and the study of linear polymers. The seminal work of Mishna and Bousquet-Mélou [10] gave a uniform approach to several enumerative questions, including the nature of a model's generating function (algebraic, D-finite, etc.) and the determination of exact or asymptotic counting formulas. In particular, they used the orbit sum method (in a manner similar to Section 3) to prove that the generating functions corresponding to 22 of the 79 non-equivalent two dimensional models were D-finite (they conjectured that one additional model was D-finite and that the rest were not). Around the same time, Bostan and Kauers [2] used computer algebra approaches to guess differential equations satisfied by the generating functions of 23 models (the 22 proven D-finite by Mishna and Bousquet-Mélou and the one they conjectured to be D-finite) which they then used to guess dominant asymptotics for these models (see Table 2). They later proved, using another computer algebra approach, that the 23rd conjectured D-finite model of Mishna and Bousquet-Mélou is in fact algebraic (and thus also D-finite).

Furthermore, 5 of the remaining 56 models were proven to admit non D-finite generating functions by Melczer and Mishna [22] and strong evidence for non D-finiteness of the final 51 generating functions has been provided by several sources [20, 5].

We now look at the application of the general formulas developed in Section 5 to proving the guessed asymptotics in Table 2.

6.1 The Highly Symmetric Models

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{highly_symmetric_models.png}
\end{array}
\]

Four of the models in Table 2 have step sets which are symmetric over every axis. This means that their asymptotics follow directly from the work of Melczer and Mishna [24] (see Theorem 1 above). The asymptotic order is \( |S|^n n^{-1} \) in each case.

6.2 Positive Drift Models Missing One Symmetry

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{positive_drift_models.png}
\end{array}
\]

There are six models whose step sets are missing one symmetry and have positive drift; one can directly apply Theorem 2 to prove the asymptotics listed. The asymptotic order is \( |S|^n n^{-1/2} \) in each case.

---

3 This was later proven using non-computer based arguments by Bostan, Kurkova, and Raschel [4] and Bousquet-Mélou [11].
Asymptotics

| $S$ | Asymptotics | $S$ | Asymptotics | $S$ | Asymptotics |
|-----|-------------|-----|-------------|-----|-------------|
| ![ ] | $\frac{1}{\pi} \cdot \frac{4^n}{n^2}$ | ![ ] | $\frac{\sqrt{3}}{2\pi} \cdot \frac{3^n}{\sqrt{n}}$ | ![ ] | $\frac{1}{\pi} \cdot \frac{4^n}{n^2}$ |
| ![ ] | $\frac{\sqrt{3}}{\pi} \cdot \frac{6^n}{n^2}$ | ![ ] | $\frac{\sqrt{5}}{3\pi} \cdot \frac{5^n}{\sqrt{n}}$ | ![ ] | $\frac{\sqrt{3}}{\pi} \cdot \frac{6^n}{n^2}$ |
| ![ ] | $\frac{2\sqrt{2}}{3\pi} \cdot \frac{3^n}{n^{3/2}}$ | ![ ] | $\frac{2\sqrt{2}}{3\pi} \cdot \frac{6^n}{\sqrt{n}}$ | ![ ] | $\frac{\sqrt{3}}{\pi} \cdot \frac{6^n}{n^{3/2}}$ |
| ![ ] | $\frac{\sqrt{6}}{\pi} \cdot \frac{3^n}{n^{3/2}}$ | ![ ] | $\frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{\sqrt{n}}$ | ![ ] | $\frac{\sqrt{3}}{\pi} \cdot \frac{6^n}{n^{3/2}}$ |
| ![ ] | $\frac{4\sqrt{3}}{3\pi} \cdot \frac{4^n}{n^{2/3}}$ | ![ ] | $\frac{\sqrt{3}}{2\pi} \cdot \frac{3^n}{\sqrt{n}}$ | ![ ] | $\frac{\sqrt{3}}{\pi} \cdot \frac{6^n}{n^{3/2}}$ |

Table 2: Asymptotics for the 23 D-finite models of Bostan and Kauers.

$A_n = \begin{cases} 24\sqrt{2} : n \text{ even} \\ 32 : n \text{ odd} \end{cases}$, $B_n = \begin{cases} 12\sqrt{3} : n \text{ even} \\ 18 : n \text{ odd} \end{cases}$, $C_n = \begin{cases} 12\sqrt{30} : n \text{ even} \\ 144/\sqrt{5} : n \text{ odd} \end{cases}$

6.3 Negative Drift Models Missing One Symmetry

There are six models whose step sets are missing one symmetry and have negative drift; one can apply Theorem 4 to prove the asymptotics listed (we note that the original table of guessed asymptotics by Bostan and Kauers [2] has small errors in the constants for the first three of these models). The asymptotic order is $(\sqrt{B(1)/A(1)})^n n^{-2}$ in each case.

**Example 23.** Consider the model defined by step set $S = \{(0, 1), (-1, -1), (0, -1), (1, -1)\} = \{N, SE, S, SW\}$.

Here we have

$$G(z, t) = (1 + x)(1 - y(x^2 + 1) + x) / (1 - y)$$

$$H(z, t) = 1 - t(x + y^2 + xy^2 + x^2 y^2),$$

and two of the eight possible points described by Theorem 16 are contributing points: $p_1 = (1, 1/\sqrt{3}, 1/2)$ and $p_2 = (1, -1/\sqrt{3}, 1/2)$. Using Sage to implement Proposition 29, we can calculate the contribution at each contributing point to be

$$\Psi_n(p_1) = \frac{3\sqrt{3}(2 + \sqrt{3})}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2}$$

$$\Psi_n(p_2) = \frac{3\sqrt{3}(2 - \sqrt{3})}{\pi} \cdot \frac{(-2\sqrt{3})^n}{n^2}$$

so that the number of walks of length $n$ satisfies

$$s_n = \frac{(2\sqrt{3})^n}{n^2} \cdot \frac{3\sqrt{3}}{\pi} \left(\sqrt{3}(1 - (-1)^n) + 2(1 + (-1)^n) + O(n^{-1})\right)$$

$$= \begin{cases} \frac{(2\sqrt{3})^n}{n^2} \cdot \frac{12\sqrt{3}}{\pi} + O(n^{-1}) : n \text{ even} \\ \frac{(2\sqrt{3})^n}{n^2} \cdot \frac{12\sqrt{3}}{\pi} + O(n^{-1}) : n \text{ odd} \end{cases}$$
The first three models here each have two contributing points which determine the dominant asymptotics, giving a periodicity in the coefficients as seen in the above example.

6.4 The Exceptional (Zero Orbit Sum) Algebraic Cases

There are four models for which the orbit sum method fails to give an expression for the walk generating functions as the diagonals of rational functions (meaning the techniques of analytic combinatorics in several variables as described above cannot by directly used). Luckily, these four models are algebraic and explicit minimal polynomials for the generating functions are known: the first was found by Mishna [26], the next two by Bousquet-Mélou and Mishna [10] and the final model—known as Gessel’s walk—is treated in Bostan and Kauers [3]. It is effective to determine the asymptotics of a sequence from its generating function’s minimal polynomial (under gentle technical conditions), so the asymptotics for these cases follow rigorously through univariate methods (see Chapter VII.7 of Flajolet and Sedgewick [17] and [26, 3]). In fact, the multivariate generating function $F(x, y, t)$ for each model is algebraic, a stronger property.

6.5 The Remaining Three Models

There are three models not covered by the above cases. They do not exhibit any symmetries, but the orbit sum method still gives a diagonal expression which can be analyzed as above. Asymptotics of these models were previous given by Bousquet-Mélou and Mishna [10]: although the multivariate generating functions $F(x, y, t)$ of these models are transcendental, the first two models have algebraic generating functions $F(1, 1, t)$ counting walks ending anywhere. The final model does not have an algebraic specialization, but the coefficients of $F(x, y, t)$ are Gosper summable; see [10, Prop 11] for details.

6.5.1 Case 1: $\mathcal{S} = \{N, W, SE\}$

Applying the kernel method, we see that the counting generating function satisfies

$$F(t) = \Delta \left( \frac{(x^2 - y)(1 - xy)(x - y^2)}{(1 - x)(1 - y)(1 - xyt(\bar{y} + y\bar{x} + x))} \right).$$

Although this rational function has smooth and transverse multiple points which are minimal and critical, we cannot directly apply the asymptotic methods discussed above at the point $(1, 1, 1/3)$—which turns out to be a contributing singularity—as the gradient of $H_1$ at that point is parallel to the gradient of the function $\phi(x, y, t) = \log(xy)$ occurring in the Fourier-Laplace integral which must be analyzed to determine asymptotics. Luckily, we can write

$$x^2 - y = (x - 1)(x + 1) - (y - 1)$$

to decompose the rational function as

$$\frac{(x^2 - y)(1 - xy)(x - y^2)}{(1 - x)(1 - y)(1 - xyt(\bar{y} + y\bar{x} + x))} = \frac{(1 - xy)(x - y^2)(x + 1)}{(1 - y)(1 - xyt(\bar{y} + y\bar{x} + x))} - \frac{(1 - xy)(x - y^2)}{(1 - x)(1 - xyt(\bar{y} + y\bar{x} + x))}.$$

Both of the summands in this decomposition admit exactly three minimal critical points:

$$p_1 = (1, 1, 1/3) \quad p_2 = (\nu, \nu^2, \nu^2/3) \quad p_2 = (\nu^2, \nu, \nu/3),$$

19
where $\nu = e^{2\pi i/3}$. Each summand admits $p_1$ as a multiple point, and adds a contribution of

$$3^n \cdot n^{-3/2} \cdot \frac{3\sqrt{3}}{4\sqrt{\pi}} + O(3^n \cdot n^{-5/2})$$

to the asymptotics of the diagonal at that point. Note that the gradient of $H_1$ is still parallel to the gradient of $\phi(x, y, t)$, however when this occurs with only two factors in the denominator then the expected leading constant is simply divided by 2 (see Pemantle and Wilson [29, Cor 10.4.11]). Since the original rational function admits both $p_2$ and $p_3$ as smooth minimal critical points where its numerator vanishes, Theorem 21 implies that the contributions at both points are $O(3^n/n^2)$. Thus, the counting sequence for the number of walks on these steps satisfies

$$s_n = \frac{3^n}{n^{3/2}} \cdot \frac{3\sqrt{3}}{2\sqrt{\pi}} + O\left(\frac{3^n}{n^2}\right).$$

6.5.2 Case 2: $S = \{E, SE, W, NW\}$

Applying the kernel method, we see that the counting generating function satisfies

$$F(t) = \Delta \left(\frac{(x+1)(x^2+y^2)(x-y)(x+y)}{1-xyt(x+x\overline{y}+y\overline{x}+\overline{y})}\right).$$

This case turns out to be easy to analyze, since the denominator is smooth. There are two points which satisfy the critical point equations: $p_1 = (1, 1, 1/4)$ and $p_2 = (-1, 1, 1/4)$, both of which are minimal and smooth. As the numerator has a zero of order 2 at $p_1$ but order 3 at $p_2$, in fact only $p_1$ contributes to the dominant asymptotics. The Sage package of Raichev – implementing Theorem 21 – computes the contributions at both points and shows that the counting sequence for the number of walks on these steps satisfies

$$s_n = \frac{4^n}{n^2} \cdot \frac{8}{\pi} + O\left(\frac{4^n}{n^3}\right).$$

Weighted versions of this stepset were studied in detail by Courtiel et al. [13].

6.5.3 Case 3: $S = \{NW, SE, N, S, E, W\}$

Applying the kernel method, we see that the counting generating function satisfies

$$F(t) = \Delta \left(\frac{(x-y^2)(1-x\overline{y})(x^2-y)}{(1-x)(1-y)(1-txy(x+y+x\overline{y}+y\overline{x}+\overline{y}))}\right).$$

Here there is only one minimal critical point, the point $p = (1, 1, 1/6)$. Because of the form of the denominator, we know this must be a contributing point and that it is the only such point. Noticing that the numerator can be written as $xy(1-x^3)(x+1)+xy(-x^3y^2+x^4-x^3y+xy^3-x^3+xy^2+xy-y^2+2x-y-1)(1-y)$, one can decompose this multivariate function as the sum of two rational functions which can be analyzed analogously to Case 1 above. This argument shows that the counting sequence for the number of walks on these steps satisfies

$$s_n = \frac{6^n}{n^{3/2}} \cdot \frac{3\sqrt{3}}{2\sqrt{\pi}} + O\left(\frac{6^n}{n^{5/2}}\right).$$

7 Further Considerations

We end with some additional remarks and generalizations.
7.1 Weighted models

Although our results holds for models whose steps have positive real weights, we have not yet given an example with positive weights not equal to one. We do so now.

Example 24. Consider the model defined by the step set of cardinal directions \( S = \{(1,0), (-1,0), (0,1), (0,-1)\} \), where the south step \((0,-1)\) has weight \(a > 0\) and the north step \((0,1)\) has weight \(b\) (when \(a\) and \(b\) are integers we can think of having multiple copies of each step with different colours). Then

\[
A(x) = a \quad Q(x) = \pi + x \quad B(x) = b
\]

and

\[
s_n \sim \begin{cases} 
(2 + 2\sqrt{ab})^n \cdot n^{-2} \cdot \frac{2a^{1/4}(1+\sqrt{ab})^2}{\pi b^{1/4}(\sqrt{\pi}-\sqrt{\eta})^2} : b < a \\
(2 + 2a)^n \cdot n^{-1} \cdot \frac{2(1+a)}{\sqrt{a} \pi} : b = a \\
(2 + a + b)^n \cdot n^{-1/2} \cdot \frac{(a+b)\sqrt{2+a+b}}{2\sqrt{\pi}} : b > a
\end{cases}
\]

with the different cases corresponding to negative drift, zero drift, and positive drift.

7.2 Decidability of asymptotics

The techniques of analytic combinatorics in several variables are currently at the front line of research into computability questions in enumerative combinatorics. Given a univariate rational generating function, or an algebraic generating function encoded by its minimal polynomial and a sufficient number of initial terms, there are algorithms which take the function and return asymptotics of its power series coefficients at the origin. On the other hand, it is an open problem whether it is decidable to take a D-finite generating function encoded by an annihilating linear differential equation and initial conditions and determine asymptotics of its counting sequence. In slightly restricted settings (for instance, when the D-finite generating function has integer coefficients and positive radius of convergence) a careful singularity analysis allows one to determine a so-called asymptotic basis: a finite collection of terms \(\Delta_1, \ldots, \Delta_d\) with asymptotic expansions of the form

\[
\Delta_j = \rho_j^n n^\alpha (\log n)^\kappa_j \left(C_0^{(j)} + \frac{C_1^{(j)}}{n} + \cdots\right)
\]

which can be determined explicitly to any finite order, such that asymptotics of the coefficient sequence \(c_n\) is an \(\mathbb{R}\)–linear combination of the \(\Delta_j\),

\[
c_n \sim K_1 \Delta_1 + \cdots + K_r \Delta_r, \quad K_j \in \mathbb{R}.
\]

See Flajolet and Sedgewick [17, Sec VII. 9] for details.

In this way, decidability of asymptotics can be reduced to the determination of the connection coefficients \(K_j\). If, without loss of generality, asymptotics of \(\Delta_1\) dominate asymptotics of the other \(\Delta_j\) then \(\Delta_1\) typically determines (up to a scaling multiple) asymptotics of \(c_n\). However, if the constant \(K_1\) is zero then \(c_n\) can have drastically different asymptotic behaviour than \(\Delta_1\). Determining the coefficients \(K_j\) is known as the connection problem.

Because the class of multivariate rational diagonals contains the class of algebraic functions, and is contained in the class of D-finite functions, the techniques of analytic combinatorics in several variables offer tools to investigate the connection problem (see Melczer [23] for an in-depth look at this approach). For instance, Bostan et al. [8] give annihilating differential equations for each lattice path generating function in Table 2, even representing them in terms of explicit hypergeometric functions; however, they were not able to prove all asymptotics in that table, because of the connection problem. For instance, they show [8,
Conjecture 2] that the number of walks with step set \( S = \{(0, -1), (-1, 1), (1, 1)\} \) has dominant asymptotics of the form \( \frac{\sqrt{3}}{2\sqrt{\pi}} 3^k k^{-1/2} \) if and only if the integral

\[
I := \int_0^{1/3} \left\{ \frac{(1 - 3v)^{1/2}}{v^3(1 + v^2)^{1/2}} \left[ 1 + (1 - 10v^3) \cdot {}_2F_1 \left( \begin{array}{c} 3/4, 5/4 \\ 1 \end{array} \right) 64v^4 \right] \right. \\
\left. + 6v^3(3 - 8v + 14v^2) \cdot {}_2F_1 \left( \begin{array}{c} 5/4, 7/4 \\ 2 \end{array} \right) \frac{64v^4}{\frac{2}{v^3} + \frac{4}{v^2}} \right\} dv
\]

has the value \( I = 1 \) (see that paper for details on the notation used). Using the multivariate singularity analysis discussed above, we are able to circumvent these difficulties, and resolve the connection problem for these lattice path models. As an indirect corollary of our asymptotic results, we thus determine the values of certain complicated integral expressions involving hypergeometric functions.

### 7.3 Walks returning to boundaries

The kernel method as presented here uses the multivariate generating function \( F(z, t) \) tracking walk length and endpoint to derive a rational diagonal expression for the univariate generating function \( F(1, t) \) counting the number of walks ending anywhere. Also of interest is the number of walks ending on one or more of the boundary hyperplanes in the first orthant; if \( V \subset \{1, \ldots, d\} \) then

\[
F(z, t) \bigg|_{z_j = 0, j \in V}^{z_j = 1, j \notin V}
\]

counts the number of walks returning to the intersection of the boundary hyperplanes \( \{z_j = 0\} \) for \( j \in V \). Lemma 10 can easily be generalized to the following.

**Lemma 25.** Let \( P(z, t) \in \mathbb{Q}[z_1, \overline{z}_1, \ldots, z_d, \overline{z}_d][[t]] \subset \mathcal{R} \). Then

\[
\left( [z^g] P(z, t) \right) \bigg|_{z_j = 0, j \in V}^{z_j = 1, j \notin V} = \Delta \left( \frac{P(\overline{z}_1, \ldots, \overline{z}_d, z_1 \cdots z_d \cdot t)}{(1 - z_1) \cdots (1 - z_d)} \prod_{j \in V} (1 - z_j) \right).
\]

Thus, following the arguments above, if \( Q(z, t) = G(z, t)/H(z, t) \) is the rational function given in Theorem 12 such that \( F(1, t) = (\Delta Q)(t) \) then

\[
F(z, t) \bigg|_{z_j = 0, j \in V}^{z_j = 1, j \notin V} = \Delta \left( Q(z, t) \cdot \prod_{j \in V} (1 - z_j) \right).
\]

This close link between the diagonal expressions for walks ending anywhere and walks ending on boundary hyperplanes allows us to reuse most of the work above to derive asymptotics for walks ending on boundary hyperplanes. In particular, if \( V \) does not contain \( d \) then the singular sets of both multivariate rational functions obtained are the same, so the contributing points calculated by Theorem 16 are still contributing. Analysis of asymptotics is easy for any fixed model, however the additional zeroes in the numerator of \( Q(z, t) \cdot \prod_{j \in V} (1 - z_j) \) at contributing points makes explicit expressions for generic models harder to calculate.

When \( V \) contains \( d \), then the factor of \( 1 - z_d \) in the numerator of \( Q(z, t) \) will cancel the new factor \( 1 - z_d \) in the numerator. In the negative drift and zero drift cases this has no bearing on the contributing singularities, and hence on the exponential growth of the number of walks returning to the hyperplane \( \{z_d = 0\} \). However in the positive drift case the contributing singularities will change and the exponential growth will be smaller for walks returning to the hyperplane \( \{z_d = 0\} \) than for general walks.
Using the Sage package of Raichev to compute asymptotic contributions, Table 3 gives asymptotics for the number of walks returning to one or both of the boundary axes on the 2D quadrant models analyzed above, where

\[ \delta_n = \begin{cases} 1 & : n \equiv 0 \mod 2 \\ 0 & : \text{otherwise} \end{cases} \quad \sigma_n = \begin{cases} 1 & : n \equiv 0 \mod 3 \\ 0 & : \text{otherwise} \end{cases} \quad \epsilon_n = \begin{cases} 1 & : n \equiv 0 \mod 4 \\ 0 & : \text{otherwise} \end{cases} \]

and

\[ \gamma_n = \begin{cases} 448\sqrt{2} & : n \equiv 0 \mod 4 \\ 640 & : n \equiv 1 \mod 4 \\ 416\sqrt{2} & : n \equiv 2 \mod 4 \\ 512 & : n \equiv 3 \mod 4 \end{cases} \]

help account for periodicities which appear, and the algebraic constants \(A, B,\) and \(C\) are given by

\[ A = (156 + 41\sqrt{6})\sqrt{23 - 3\sqrt{6}}, \quad B = (583 + 138\sqrt{6})\sqrt{23 - 3\sqrt{6}}, \quad C = (4571 + 1856\sqrt{6})\sqrt{23 - 3\sqrt{6}}. \]

This gives the first complete proof of conjectured asymptotics given by Bostan et al. [8].

### 7.4 Zero drift models

In the non-highly symmetric zero drift case, when \(A(1) = B(1)\) but \(A(z_1) \neq B(z_1)\), Theorem 16 implies that we can have contributions from the point \(p := p_1 = p_2\) on the stratum \(V_1 \cap V_3\), possibly with other points lying on locally smooth parts of \(V_1\). Note that the numerator vanishes to at least first order at every critical point, and that this case cannot occur for unweighted steps in dimension 2 (where every zero drift model is highly symmetric).

Since \(p\) is on the intersection of \(V_1\) and \(V_3\), and the numerator vanishes, we expect it to give an asymptotic contribution of \(C \cdot |S|^n \cdot n^{-d/2-1/2}\), while the other (locally smooth) contributing points have a contribution of \(O \left( |S|^n \cdot n^{-d/2-1} \right)\). Thus, if we can determine a second order contribution at \(p\) and show that it does not vanish, we will have found dominant asymptotics.

Asymptotic contributions of minimal critical points are determined by analyzing integrals of the form

\[ \int_{[-1,1]^r} A(\theta) e^{-n\phi(\theta)} d\theta \]

where \(r \in \mathbb{N}\), and \(A\) and \(\phi\) analytic functions from \([-1,1]^r\) to \(\mathbb{C}\) (see [29] for details). When the gradient of \(\phi\) vanishes in the interior of \([-1,1]^r\), and other technical conditions on \(A\) and \(\phi\) which are satisfied here hold, the asymptotic formulae in Theorem 21 follow. Unfortunately, in the non-highly symmetric zero drift case the gradient of \(\phi\) vanishes on the boundary of the domain of integration, meaning the relevant asymptotic constants are not the same as those in Theorem 21. In fact, general asymptotics for such a situation have not yet been worked out in the context of ACSV.

Furthermore, while non-vanishing of the second order contribution at \(p\) happens generically, there are models where vanishing does occur and finding dominant asymptotics requires a detailed analysis at several contributing singularities. Because of these added difficulties, including a need to extend the underlying analytic theory, a more nuanced study of the zero drift models will be the subject of future work.

### 7.5 Connecting analytic and combinatorial behaviour

As we have seen, the kernel method shows how nice combinatorial properties of a step set (like symmetry over axes) correspond to nice analytic properties of a multivariate rational function (like a singular set defined as the union of a small number of smooth manifolds) encoding the corresponding generating function. Furthermore, it is possible to turn this around: because diagonal sequences of multivariate rational functions with ‘simple’ geometry at contributing singularities can only capture a restricted set of asymptotic behaviour, certain step sets whose asymptotics are sufficiently complicated cannot have their generating functions encoded as the diagonals of ‘nice’ rational functions.
For instance, it was previously observed that the exponential growth of 2D quadrant walks ending anywhere and the exponential growth of walks ending at the origin was the same for negative drift models but different for positive drift models. The strong connections between the diagonal representations of the corresponding generating functions explains why this is the case.

The connection between analytic and combinatorial behaviour also helps explain patterns in asymptotics. For instance, it was previously observed that the exponential growth of 2D quadrant walks ending anywhere and the exponential growth of walks ending at the origin was the same for negative drift models but different for positive drift models. The strong connections between the diagonal representations of the corresponding generating functions explains why this is the case.

We believe that the tools of analytic combinatorics in several variables have much to offer the immensely popular area of lattice path enumeration, and hope that others will pick up and utilize the tools discussed here.

| $S$ | Return to $x$-axis | Return to $y$-axis | Return to origin |
|-----|---------------------|--------------------|------------------|
| $\mathcal{H}$ | $\frac{8}{n} \cdot \frac{4^n}{n^2}$ | $\frac{8}{n} \cdot \frac{4^n}{n^2}$ | $\delta_n \frac{32}{\pi^2} \cdot \frac{4^n}{n^2}$ |
| $\mathcal{L}$ | $\delta_n \frac{4}{n} \cdot \frac{4^n}{n^3}$ | $\delta_n \frac{4}{n} \cdot \frac{4^n}{n^3}$ | $\delta_n \frac{8}{n} \cdot \frac{4^n}{n^3}$ |
| $\mathcal{W}$ | $\frac{3\sqrt{3}}{\pi} \cdot \frac{6^n}{n^4}$ | $\delta_n \frac{2\sqrt{3}}{\pi} \cdot \frac{6^n}{n^4}$ | $\delta_n \frac{3\sqrt{3}}{\pi} \cdot \frac{6^n}{n^4}$ |
| $\mathcal{O}$ | $\frac{3\sqrt{3}}{4\sqrt{\pi}} \cdot \frac{9^n}{n^5}$ | $\delta_n 4\sqrt{3} \cdot \frac{(2\sqrt{3})^n}{n^5}$ | $\epsilon_n 16\sqrt{3} \cdot \frac{(2\sqrt{3})^n}{n^5}$ |
| $\mathcal{N}$ | $\frac{3\sqrt{3}}{5\sqrt{\pi} n^{3/2}} \cdot \frac{3^n}{n^5}$ | $\frac{\sqrt{3}}{\sqrt{n} n^{3/2}} \cdot \frac{3^n}{n^5}$ | $\delta_n \frac{12\sqrt{3}}{25} \cdot \frac{(2\sqrt{3})^n}{n^5}$ |
| $\mathcal{C}$ | $\frac{3\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{5^n}{n^7/2}$ | $\frac{3\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{5^n}{n^7/2}$ | $\delta_n \frac{76\sqrt{3}}{\pi} \cdot \frac{4^n}{n^7}$ |
| $\mathcal{D}$ | $\frac{3\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{6^n}{n^7/2}$ | $\frac{3\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{6^n}{n^7/2}$ | $\delta_n \frac{76\sqrt{3}}{\pi} \cdot \frac{4^n}{n^7}$ |
| $\mathcal{E}$ | $\frac{3\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{7^n}{n^7/2}$ | $\frac{3\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{7^n}{n^7/2}$ | $\delta_n \frac{76\sqrt{3}}{\pi} \cdot \frac{4^n}{n^7}$ |
| $\mathcal{F}$ | $\frac{3\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{8^n}{n^7/2}$ | $\frac{3\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{8^n}{n^7/2}$ | $\delta_n \frac{76\sqrt{3}}{\pi} \cdot \frac{4^n}{n^7}$ |

Table 3: Asymptotics of quadrant walks returning to the $x$-axis, the $y$-axis, and the origin, respectively.
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A Calculus Computations

General Results

Here we collect the results and proofs which are necessary to determine asymptotics but theoretically uninteresting. Our first two results will be useful for calculating derivatives.
Lemma 26. Let \( P, Q \) be smooth functions from \( I \subseteq \mathbb{R}^d \) to \( \mathbb{C} \), and suppose that \( 0 \) lies in the interior of \( I \) and \( Q(0) \neq 0 \). Let \( \partial : = \partial_{t_k} \) be a partial derivative operator such that \( (\partial P)(0) = 0 = (\partial Q)(0) \). Then \( \partial(P/Q)(0) = 0 \) and

\[
\partial^2(P/Q)(0) = \frac{Q(0)\partial^2 P(0) - P(0)\partial^2 Q(0)}{Q(0)^2}.
\]

When

\[
P(\theta) = (e^{i\theta} + e^{-i\theta}) P_k(\theta) + R_k(\theta) \]
\[
Q(\theta) = (e^{i\theta} + e^{-i\theta}) Q_k(\theta) + R_k(\theta)
\]

then \( \partial P(0) = 0 = \partial Q(0) \) and

\[
\partial^2(P/Q)(0) = \frac{-2P_k(0)Q + P(0)Q_k(0)}{Q(0)^2}.
\]

Proof. The first assertion follows from expanding the second derivative using the quotient rule, and applying the obvious simplification. The second follows from the first by direct substitution, since \( (\partial^2 P)(0) \) simplifies to \(-2P_k(0)\) and similarly for \( Q \). \( \square \)

Lemma 27. For a point \( p \in \mathbb{C}^d \), let

\[
\tilde{S}(\theta) := \frac{S(p_1 e^{i\theta_1}, \ldots, p_d e^{-i\theta_d})}{A(p_d)}.
\]

Then if \( p_d \in \{\pm 1\}^{d-1} \) and \( p_d = \pm \sqrt{B(p_d)/A(p_d)} \), we have for \( 1 \leq j \leq k \leq d \):

\[
\partial_j \tilde{S}(0) = 0
\]

and

\[
\partial_j \partial_k \tilde{S}(0) = \begin{cases} 
0 & \text{if } j \neq k; \\
-2B_j(p_d)/p_d & \text{if } j = k = d; \\
-2p_j B_j(p_j) & \text{if } j = k < d.
\end{cases}
\]

Furthermore, if \( j < d \) then

\[
\partial_d \partial_j^2 \tilde{S}(0) = -2i p_j \left( p_j A_j'(0) - p_d^{-1} B_j'(0) \right).
\]

If \( p_d' = i p_d \) is the imaginary number corresponding to \( p_d \), and \( S(1, 1) = S(p_d, p_d') \), then the values of all partial derivatives above equal the values of the derivatives calculated for \( (p_d, p_d') \), potentially up to sign.

Proof. For \( j < d \), applying \( \partial_j \) to

\[
\tilde{S}(\theta) = (p_j e^{i\theta_j} + p_j^{-1} e^{-i\theta_j}) B_j(\theta) + \hat{Q}_j(\theta)
\]

yields

\[
\partial_j \tilde{S}(\theta) = (i p_j e^{i\theta_j} - i p_j^{-1} e^{-i\theta_j}) B_j(\theta).
\]

Note that from this point, applying any \( \partial_k \) with \( k \neq j \) will not change the factor \( i p_j e^{i\theta_j} - i p_j^{-1} e^{-i\theta_j} \). Also, evaluating at zero gives \( i(p_j - p_j^{-1}) = 0 \).

Repeating this with higher powers of \( \partial_j \) gives a formula that is periodic in the exponent, with period 4. In particular when we evaluate at \( 0 \) we obtain

\[
\partial_j^n \tilde{S}(0) = \begin{cases} 
(-1)^{n/2} 2p_j B_j(0) & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}.
\end{cases}
\]

27
A similar computation with \( j = d \) yields

\[
\partial_d^n \tilde{S}(0) = \begin{cases} 
(-1)^{n/2} 2 \tilde{B}(0)/p_d & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}.
\end{cases}
\]

Finally, consider the third order derivative \( \partial_d \partial_j^2 \tilde{S} \). Writing

\[
\tilde{S}(\theta) = p_d e^{i \theta_d} A(\theta) + Q + p_d^{-1} e^{-i \theta_d} B(\theta)
\]

and differentiating using the formulae in Definition 19 yields the stated result.

The statement about \( p'_d \) follows from the same considerations, using the fact that if \( S(1, 1) = S(p'_d, p'_d) \) then \( B(p'_d, p'_d)/p'_d \) and \( A(p'_d, p'_d) p'_d \) have the same argument.

**Negative Drift Model Calculations**

We now show that the quantities appearing in Theorem 21 simplify for us. Since in our situation of interest we always have \( L_0(\tilde{u}, \tilde{g}) = \tilde{u}(0) = 0 \), we begin by considering the term corresponding to \( k = 1 \) in (10). For possible independent interest we show that some simplification is possible even in the general case.

**Lemma 28.** In the general smooth case

\[
L_1(\tilde{u}, \tilde{g}) = -\frac{1}{2} \left( \mathcal{H}(\tilde{u})(0) + \frac{\mathcal{H}^2(\tilde{u}\tilde{g})(0)}{4} + \frac{\tilde{u}\mathcal{H}^3(\tilde{g}^2)(0)}{48} \right).
\]

If \( \tilde{u} \) vanishes to order at least 1 at 0, then

\[
L_1(\tilde{u}, \tilde{g}) = -\frac{1}{2} \left( \mathcal{H}(\tilde{u})(0) + \frac{\mathcal{H}^2(\tilde{u}\tilde{g})(0)}{4} \right)
\]

and only terms involving third partial derivatives of \( \tilde{g} \) contribute to the \( \mathcal{H}^2 \) term.

If \( \tilde{u} \) vanishes to order at least 2 at 0 then

\[
L_1(\tilde{u}, \tilde{g}) = -\frac{1}{2} \mathcal{H}(\tilde{u})(0).
\]

If \( \tilde{u} \) vanishes to order at least 3 at 0 then

\[
L_1(\tilde{u}, \tilde{g}) = 0.
\]

**Proof.** First note that any partial derivative of order at most 2 of \( \tilde{g} \) is zero when evaluated at 0, by construction. Furthermore all derivatives of \( \tilde{g} \) of degree more than 2 yield the same result when evaluated at 0 as the corresponding derivative of \( \tilde{g} \), since the difference between the two functions is quadratic. Thus in each nonzero term in an expansion of \( L_1 \) we may replace \( \tilde{g} \) by \( \tilde{g} \).

Since \( \tilde{g} \) vanishes to order 3 at 0, the term involving \( \mathcal{H}^3 \) simplifies substantially, since in order to obtain a nonzero term all the 6th partial derivatives must be applied to \( \tilde{g}^2 \) and so \( \mathcal{H}^3(\tilde{u}\tilde{g}^2) \) simplifies to \( \hat{u}\mathcal{H}^3(\tilde{g}^2) \).

Similarly, \( \mathcal{H}^2(\tilde{u}\tilde{g}) \) simplifies, since each 4th order partial derivative, when applied to the product \( \hat{u}\tilde{g} \) and then evaluated at 0, only yields a nonzero result when at least 3 of the derivations are applied to \( \tilde{g} \). If \( \tilde{u} \) vanishes to order 2 then even these terms are zero. If \( \tilde{u} \) vanishes to order 1 then it is exactly the 3rd partials of \( \tilde{g} \) that can contribute.

This, combined with Theorem 21, directly gives the following.
Proposition 29. Let $S$ be a step set which is symmetric over all but one axis and takes a step forwards and backwards in each coordinate, and let $W$ be the set of contributing points determined by Theorem 16. If $S$ has negative drift, then the number of walks of length $n$ which never leaves the non-negative orthant satisfies

$$s_n = \sum_{p \in W} \Psi_n^{(p)}$$

for

$$\Psi_n^{(p)} = (p_1 \cdots p_d p_e)^{-n} \left[ n^{-d/2-1} K_p C_p + O(n^{-1}) \right],$$

where

$$K_p = 2^{-d} n^{-d/2} \mathcal{S}(p)^{d/2} \left( p_1 B_1(p_1) \cdots p_{d-1} B_{d-1}(p_{d-1}) B_d(p_d)/p_d \right)^{-1/2},$$

$$C_p = \frac{1}{2} \left( \mathcal{H}(\tilde{u})(0) + \frac{\mathcal{H}^2(\tilde{u}\bar{g})(0)}{4} \right)$$

for differential operator

$$\mathcal{H} = -\mathcal{S}(p)^2 \left( \frac{p_d}{B_d(p_d)} \partial_d^2 + \sum_{j<d} \frac{1}{p_j B_j(p_j)} \partial_j^2 \right)$$

and

$$\tilde{u}(\theta) = (1 + p_1 e^{i\theta_1}) \cdots (1 + p_{d-1} e^{i\theta_{d-1}}) \left( 1 - p_d^2 e^{2i\theta_d} \frac{A(p_d e^{i\theta_d})}{B(p_d e^{i\theta_d})} \right) (1 - p_d e^{i\theta_d})^{-1}.$$ 

We now show that the derivatives of $\tilde{g}$ and $\tilde{u}$ simplify substantially, giving Theorem 4.

Proposition 30. In the situation of Proposition 29, we have

$$C_p = \mathcal{S}(p)^{1-d} \left[ \frac{1}{A(p)p_d(1-p_d)} + \sum_{j=1}^{d-1} \frac{1-p_j}{2p_j B_j(p)} \left( \frac{A_j'(p)}{A(p)} - \frac{B_j'(p)}{B(p)} \right) \right].$$

Proof. Note that $\partial_k \tilde{g} = -\partial_k \tilde{S}/\tilde{S}$. This evaluates to zero at $0$. It follows from Lemma 26 that $\partial_k^n \tilde{g}$ evaluates at $0$ to $-\partial_k^n \tilde{S}(0)/\tilde{S}(0)$. Also, when we evaluate $\partial_d \partial_d^2 \tilde{g}$ at $0$, it simplifies to $\partial_d \partial_d^2 \tilde{S}(0)/\tilde{S}(0)$.

Now define

$$X := \prod_{j<d} (1 + p_j e^{i\theta_j})$$

$$Y := 1 - p_d^2 e^{2i\theta_d} \frac{A(p_d e^{i\theta_d})}{B(p_d e^{i\theta_d})}$$

$$Z := (1 - p_d e^{i\theta_d})^{-1}$$

so that

$$\tilde{u} = XYZ.$$

We first seek to compute

$$-\frac{1}{2} \mathcal{H}(\tilde{u})(0) = \mathcal{S}(p)^{1-d/2} \left( \frac{p_d}{B_d(p_d)} \partial_d^2 \tilde{u}(0) + \sum_{j<d} \frac{1}{p_j B_j(p_j)} \partial_j^2 \tilde{u}(0) \right).$$
When \( k < d \), we have \( \partial^2_{t} \tilde{u} = \partial^2_{t}(XYZ) \). Expanding via the product rule and evaluating at \( \theta = 0 \) we see that each term with \( Y \) as a factor vanishes because \( Y \) vanishes at \( \mathbf{p} \), and each term with \( \partial_{k} Y \) as a factor vanishes by Lemma 26. This leaves

\[
\partial^2_{t} \tilde{u} = X(\partial^2_{t} Y)Z
\]

which simplifies to

\[
\partial^2_{t} \tilde{u}(0) = -p_{d} \prod_{j<d}(1+p_{j}) \left( \frac{\partial^2_{t} Y}{A(p)} \right) \left( \frac{-2A^\prime(p)B(p) + 2A(p)B^\prime(p)}{B(p)^2} \right) \]

\[
= 2 \frac{B(p) \prod_{j<d}(1+p_{j})}{A(p)} \left[ \frac{A^\prime(p)B(p) - A(p)B^\prime(p)}{B(p)^2} \right] \]

\[
= \frac{2 \prod_{j<d}(1+p_{j})}{1-p_{d}} \left[ \frac{A^\prime(p)B(p) - A(p)B^\prime(p)}{B(p)^2} \right]
\]

by Lemma 26.

Now consider \( k = d \). Then since \( X \) is independent of \( \theta_{d} \), \( \partial^2_{t} \tilde{u} \) evaluates at \( \theta = 0 \) to \( X \left( (\partial^2_{t} Y)Z + 2(\partial_{d} Y)(\partial_{d} Z) \right) \). At this point we readily compute \( \partial_{d} Y = -2i \), \( \partial_{d} Z = p_{d}/(1-p_{d}) \), \( \partial^2_{t} Y = 4p_{d} \). Thus

\[
\partial^2_{t} \tilde{u}(0) = \frac{4 \prod_{j<d}(1+p_{j})}{1-p_{d}}.
\]

Thus

\[
\frac{1}{2} \mathcal{H}(\tilde{u})(0) = \frac{\mathcal{S}(p) \prod_{j<d}(1+p_{j})}{4} \left[ \frac{2}{1-p_{d}} \left( \sum_{j=1}^{d-1} \frac{A^\prime(p)B(p)}{A(p)} - \frac{B^\prime(p)}{B(p)} \right) + \frac{4}{A(p)p_{d}(1-p_{d})} \right].
\]

We now compute the term \( (1/8)\mathcal{H}^2(\tilde{u}\tilde{g})(0) \). The diagonal nature of the Hessian implies that \( \mathcal{H}^2 \) has the form \( \sum_{j,k} c_{j,k} \partial^2_{j} \partial^2_{k} \). Lemma 27 now allows further simplification, because it implies that each third partial derivative of the form \( \partial^3 \tilde{g} \) evaluates to zero. Thus, since \( \tilde{u} \) vanishes to order at least 1 at 0, when we expand \( \mathcal{H}^2(\tilde{u}\tilde{g}) \) fully the only nonzero terms remaining on evaluation at 0 are of the form \( \partial_{d} \tilde{u} \partial_{d} \partial_{j} \tilde{g} \). The coefficient of each such term is \( 4c_{j,d} \).

It is easily computed that

\[
\partial_{d} \tilde{u}(0) = X(0) \partial_{d} Y(0) Z(0) = -2i \frac{\prod_{j<d}(1+p_{j})}{1-p_{d}}.
\]

By Lemma 27,

\[
\partial_{d} \partial^2_{j} \tilde{g}(0) = \partial_{d} \partial^2_{j} \tilde{g}(0) \frac{2p_{d} \left( p_{d} \tilde{A}_{j}(0) - p_{d}^{-1} \tilde{B}_{j}(0) \right)}{\mathcal{S}(p)}.
\]

Thus,

\[
(1/8)\mathcal{H}^2(\tilde{u}\tilde{g})(0) = -\frac{\mathcal{S}(p) \prod_{j<d}(1+p_{j})}{2p_{d}(1-p_{d})A(p)} \sum_{j<d} \left( p_{d} \tilde{A}_{j}(0) - p_{d}^{-1} \tilde{B}_{j}(0) \right)
\]

and hence, using a little more algebraic simplification (particularly the defining relation for \( p_{d} \)), we obtain

\[
C_{p} = \frac{\mathcal{S}(p) \prod_{j<d}(1+p_{j})}{1-p_{d}} \left[ \frac{1}{A(p)p_{d}(1-p_{d})} + \sum_{j=1}^{d-1} \frac{1}{2p_{d}B_{j}(p)} \left( \frac{A^\prime(p)}{A(p)} - \frac{B^\prime(p)}{B(p)} \right) \right].
\]

\( \square \)