Constructing and Counting Number Fields

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Abstract

In this paper we give a survey of recent methods for the asymptotic and exact enumeration of number fields with given Galois group of the Galois closure. In particular, the case of fields of degree up to 4 is now almost completely solved, both in theory and in practice. The same methods also allow construction of the corresponding complete tables of number fields with discriminant up to a given bound.

2000 Mathematics Subject Classification: 11R16, 11R29, 11R45, 11Y40.

Keywords and Phrases: Discriminants, Number field tables, Kummer theory.

1. Introduction

Let \( K \) be a number field considered as a fixed base field, \( \overline{K} \) an algebraic closure of \( K \), and \( G \) a transitive permutation group on \( n \) letters. We consider the set \( \mathcal{F}_{K,n}(G) \) of all extensions \( L/K \) of degree \( n \) with \( L \subset \overline{K} \) such that the Galois group of the Galois closure \( \overline{L} \) of \( L/K \) viewed as a permutation group on the set of embeddings of \( L \) into \( \overline{L} \) is permutation isomorphic to \( G \) (i.e., \( n/m(G) \) times the number of extensions up to \( K \)-isomorphism, where \( m(G) \) is the number of \( K \)-automorphisms of \( L \)). We write

\[
N_{K,n}(G, X) = |\{ L \in \mathcal{F}_{n}(G), \ |N(d(L/K))| \leq X \}|,
\]

where \( d(L/K) \) denotes the relative ideal discriminant and \( N \) the absolute norm.

The aim of this paper is to give a survey of new methods, results, and conjectures on asymptotic and exact values of this quantity. It is usually easy to generalize the results to the case where the behavior of a finite number of places of \( K \) in the extension \( L/K \) is specified, for example if \( K = \mathbb{Q} \) when the signature \( (R_1, R_2) \) of \( L \) is specified, with \( R_1 + 2R_2 = n \).

Remarks.

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1. It is often possible to give additional main terms and rather good error terms instead of asymptotic formulas. However, even in very simple cases such as \( G = S_3 \), this is not at all easy.

2. The methods which lead to exact values of \( N_{K,n}(G, X) \) always lead to algorithms for computing the corresponding tables, evidently only when \( N_{K,n}(G, X) \) is not too large in comparison to computer memory, see for example [8] and [10].

General conjectures on the subject have been made by several authors, for example in [3]. The most precise are due to G. Malle (see [24], [25]). We need the following definition.

**Definition 1.1.** For any element \( g \in S_n \) different from the identity, define the index \( \text{ind}(g) \) of \( g \) by the formula \( \text{ind}(g) = n - |\text{orbits of } g| \). We define the index \( i(G) \) of a transitive subgroup \( G \) of \( S_n \) by the formula

\[
i(G) = \min_{g \in G, \, g \neq 1} \text{ind}(g).
\]

**Examples.**

1. The index of a transposition is equal to 1, and this is the lowest possible index for a nonidentity element. Thus \( i(S_n) = 1 \).

2. If \( G \) is an Abelian group, and if \( \ell \) is the smallest prime divisor of \( |G| \), then \( i(G) = |G|(1 - 1/\ell) \).

**Conjecture 1.2.** For each number field \( K \) and transitive group \( G \) on \( n \) letters as above, there exist a strictly positive integer \( b_K(G) \) and a strictly positive constant \( c_K(G) \) such that

\[
N_{K,n}(G, X) \sim c_K(G) X^{1/i(G)} (\log X)^{b_K(G) - 1}.
\]

In [25], Malle gives a precise conjectural value for the constant \( b_K(G) \) which is too complicated to be given here.

**Remarks.**

1. This conjecture is completely out of reach since it implies the truth of the inverse Galois problem for number fields.

2. If true, this conjecture implies that for any composite \( n \), the proportion of \( S_n \)-extensions of \( K \) of degree \( n \) among all degree \( n \) extensions is strictly less than 1 (but strictly positive), contrary to the case of polynomials.

The following results give support to the conjecture (see [2], [9], [18], [19], [20], [21], [22], [23], [28], [30]).

**Theorem 1.3.** We will say that the above conjecture is true in the weak sense if there exists \( c_K(G) > 0 \) such that for all \( \varepsilon > 0 \) we have

\[
c_K(G) \cdot X^{1/i(G)} < N_{K,n}(G, X) < X^{1/i(G) + \varepsilon}.
\]

1. (Mäki, Wright). The conjecture is true for all Abelian groups \( G \).

2. (Davenport-Heilbronn, Datskovsky-Wright). The conjecture is true for \( n = 3 \) and \( G = S_3 \).
3. (Cohen-Diaz-Olivier). The conjecture is true for $n = 4$ and $G = D_4$.
4. (Bhargava, Yukie). The conjecture is true for $n = 4$ and $G = S_4$, in the weak sense if $K \neq \mathbb{Q}$.
5. (Klüners-Malle). The conjecture is true in the weak sense for all nilpotent groups.
6. (Kable-Yukie). The conjecture is true in the weak sense for $n = 5$ and $G = S_5$.

The methods used to prove these results are quite diverse. In the case of Abelian groups $G$, one could think that class field theory gives all the answers so nothing much would need to be done. This is not at all the case, and in fact Kummer theory is usually more useful. In addition, Kummer theory allows us more generally to study solvable groups. We will look at this method in detail.

Apart from Kummer theory and class field theory, the other methods have a different origin and come from the classification of orders of degree $n$, interpreted through suitable classes of forms. This can be done at a very clever but still elementary level when the base field is $\mathbb{Q}$, and includes the remarkable achievement of M. Bhargava in 2001 for quartic orders. Over arbitrary $K$, one needs to use and develop the theory of prehomogeneous vector spaces, initiated at the end of the 1960’s by Sato and Shintani (see for example [26] and [27]), and used since with great success by Datskovsky-Wright, and more recently by Wright-Yukie (see [29]), Yukie and Kable-Yukie.

2. Kummer theory

This method applies only to Abelian, or more generally solvable extensions.

2.1. Why not class field theory?

It is first important to explain why class field theory, which is supposed to be a complete theory of Abelian extensions, does not give an answer to counting questions. Let us take the very simplest example of quadratic extensions, thus with $G = C_2$. A trivial class-field theoretic argument gives the exact formula

$$N_{K,2}(C_2, X) = -1 + \sum_{\mathcal{N}(\mathfrak{a}) \leq X} 2^{\text{rk}(\mathcal{O}^+_1(K))} M_K\left(\frac{X}{\mathcal{N}(\mathfrak{a})}\right),$$

where $\mathfrak{a}$ runs over all integral ideals of $K$ of norm less than or equal to $X$, $\mathcal{O}^+_1(K)$ denotes the narrow ray class group modulo $\mathfrak{a}$, $\text{rk}(G)$ denotes the 2-rank of an Abelian group $G$, and $M_K(n)$ is the generalization to number fields of the summatory function $M(n)$ of the Möbius function.

This formula is completely explicit, the quantities $\mathcal{O}^+_1(K)$ and the function $M_K(n)$ are algorithmically computable with reasonable efficiency, so we can compute $N_{K,2}(C_2, X)$ for reasonably small values of $X$ in this way. Unfortunately, this formula has two important drawbacks.

The first and essential one is that, if we want to deduce from it asymptotic information on $N_{K,2}(C_2, X)$, we need to control $\text{rk}(\mathcal{O}^+_1(K))$, which can be done,
although rather painfully, but we also need to control $M_K(n)$, which cannot be done (recall for instance that the Riemann Hypothesis can be formulated in terms of this function).

The second drawback is that, even for exact computation it is rather inefficient, compared to the formula that we obtain from Kummer theory. Thus, even though Kummer theory is used in a crucial way for the constructions needed in the proofs of class field theory, it must not be discarded once this is done since the formula that it gives are much more useful, at least in our context.

2.2. Quadratic extensions

As an example, let us see how to treat quadratic extensions using Kummer theory instead of class field theory. Of course in this case Kummer theory is trivial since it tells us that quadratic extensions of $K$ are parameterized by $K^*/K^{*2}$ minus the unit class. This is not explicit enough. By writing for any $\alpha \in K^*$, $\alpha \mathbb{Z}_K = a q^2$ with $a$ an integral squarefree ideal, it is clear that $K^*/K^{*2}$ is in one-to-one correspondence with pairs $(a, u)$, where $a$ are integral squarefree ideals whose ideal class is a square, and $u$ is an element of the so-called Selmer group of $K$, i.e., the group of elements $u \in K^*$ such that $u \mathbb{Z}_K = q^2$ for some ideal $q$, divided by $K^{*2}$. We can then introduce the Dirichlet series $\Phi_{K,2}(C_2, s) = \sum N(L/K)/s$, where the sum is over quadratic extensions $L/K$ in $\overline{K}$. A number of not completely trivial combinatorial and number-theoretic computations (see [9]) lead to the explicit formula

$$\Phi_{K,2}(C_2, s) = -1 + \frac{2^{-r_2}}{\zeta_K(2s)} \sum_{\epsilon \mid 2} \frac{N(2/\epsilon)}{N(2/\epsilon)^s} \sum_{\chi} L_K(\chi, s),$$

where $\chi$ runs over all quadratic characters of the ray class group $Cl_{c_2}(K)$ and $L_K(\chi, s)$ is the ordinary Dirichlet-Hecke $L$-function attached to $\chi$.

There are two crucial things to note in this formula. First of all, the sum on $\epsilon$ is only on integral ideals dividing 2, so is finite and very small. Thus, $\Phi_{K,2}(C_2, s)$ is a finite linear combination of Euler products, and can directly be used much more efficiently than the formula coming from class field theory to compute $N_{Q,2}(C_2, X)$ exactly. For example (but this of course does not need the above machinery) we obtain $N_{Q,2}(C_2, 10^{25}) = 6079271018540266286517795$.

Second, since $L_K(\chi, s)$ extends to a meromorphic function in the whole complex plane with no pole if $\chi$ is not a trivial character, the polar part of $\Phi_{K,2}(C_2, s)$, which is the only thing that we need for an asymptotic formula, comes only from the contributions of the trivial characters, in which case $L_K(\chi, s)$ is equal to $\zeta_K(s)$ times a finite number of Euler factors. We thus obtain

$$N_{K,2}(C_2, X) \sim \frac{1}{2} \frac{\zeta_K(1)}{\zeta_K(2)} X,$$

where $\zeta_K(1)$ is a convenient abuse of notation for the residue of $\zeta_K(s)$ at $s = 1$. Apparently this simple result was first obtained by Datskovsky-Wright in [18], although their proof is different.
2.3. General finite Abelian extensions

The same method can in principle be applied to any finite Abelian group $G$. I say “in principle”, because in practice several problems arise. For the base field $K = \mathbb{Q}$, a complete and explicit solution was given by Mäki in [23]. For a general base field, a solution has been given by Wright in [28], but the problem with his solution is that the constant $c_K(G)$, although given as a product of local contributions, cannot be computed explicitly without a considerable amount of additional work. It is always a finite linear combination of Euler products.

In joint work with F. Diaz y Diaz and M. Olivier, using Kummer theory in a manner analogous but much more sophisticated than the case of quadratic extensions, we have computed completely explicitly the constants $c_K(G)$ for $G = C_\ell$ the cyclic group of prime order $\ell$, for $G = C_4$ and for $G = V_4 = C_2 \times C_2$. Although our papers are perhaps slightly too discursive, to give an idea the total number of pages for these three results exceeds 100. We refer to [7], [13], [11], [15], [16] for the detailed proofs, and to [12] and [14] for surveys and tables of results. We mention here the simplest one, for $G = V_4$. We have

$$N_{K,4}(V_4, X) \sim c_K(V_4) X^{1/2} \log^2 X \quad \text{with}$$

$$c_K(V_4) = \frac{1}{48 \cdot 4 \zeta(1)} \prod_p \left( 1 + \frac{3}{N_p} \right) \left( 1 - \frac{1}{N_p} \right)^3$$

$$\prod_{p \mid 22K} \left( 1 + \frac{4}{N_p} + \frac{2}{N_p^2} + \frac{1}{N_p^3} - \frac{(1 - 1/N_p^2)e(p) + (1 + 1/N_p^2)}{N_p e(p) + 1} \right).$$

Of course, the main difficulty is to compute correctly the local factor at 2.

As usual, we can use our methods to compute very efficiently the $N$ function. For example, we obtain (see [4]):

$$N_{\mathbb{Q},3}(C_3, 10^{37}) = 501310370031289126,$$
$$N_{\mathbb{Q},4}(C_4, 10^{32}) = 1220521363354404,$$
$$N_{\mathbb{Q},4}(V_4, 10^{36}) = 22956815681347605884.$$

2.4. Dihedral $D_4$-extensions

We can also apply our method to solvable extensions. The case of quartic $D_4$-extensions, where $D_4$ is the dihedral group of order 8, is especially simple and pretty. Such an extension is imprimitive, i.e., is a quadratic extension of a quadratic extension. Conversely, imprimitive quartic extensions are either $D_4$-extensions, or Abelian with Galois group $C_4$ or $V_4$. These can easily be counted as explained above, and in any case will not contribute to the main term of the asymptotic formula, so they can be neglected (or subtracted for exact computations). Since we have treated completely the case of quadratic extensions, it is just a matter of showing
that we are allowed to sum over quadratic extensions of the base field to obtain the desired asymptotic formula (for the exact formula nothing needs to be proved), and this is not difficult. In this way, we obtain that \( N_{K,4}(D, X) \sim c_K(D, X) \) for an explicit constant \( c_K(D) \) (in fact we obtain an error term \( O(X^{3/4} + \varepsilon) \)). This result is new even for \( K = \mathbb{Q} \), although its proof not very difficult. In the case \( K = \mathbb{Q} \), we have for instance

\[
c_{\mathbb{Q}}(D_4) = \frac{6}{\pi^2} \sum_D \frac{2^{-r_2(D)} L\left(\frac{D}{D}, 1\right)}{D^2} L\left(\frac{D}{D}, 2\right) = 0.1046520224 \ldots ,
\]

where the sum is over fundamental discriminants \( D \), \( r_2(D) = r_2(\mathbb{Q}(\sqrt{D})) \), and \( L\left(\frac{D}{D}, 1\right) \) is the usual Dirichlet series for the character \( \chi \).

Remark. In the Abelian case, it is possible to compute the Euler products which occur to hundreds of decimal places if desired using almost standard zeta-product expansions, see for example [6]. Unfortunately, we do not know if it is possible to express \( c_{\mathbb{Q}}(D) \) as a finite linear combination of Euler products (or at least as a rapidly convergent infinite series of such), hence we have only been able to compute 9 or 10 decimal places of this constant. We do not see any practical way of computing 20 decimals, say.

Our method also allows us to compute \( N_{\mathbb{Q},4}(D, X) \) exactly. However, here a miracle occurs: when \( k \) is a quadratic field, in the formula that we have given above for \( \Phi_k(\mathbb{C}, s) \) all the quadratic characters \( \chi \) which we need are genus characters in the sense of Gauss, in other words there is a decomposition

\[
L_k(\chi, s) = L\left(\frac{D}{D}, s\right) L\left(\frac{D}{D}, 1\right)
\]

into a product of two suitable ordinary Dirichlet \( L \)-series. This gives a very fast method for computing \( N_{\mathbb{Q},4}(D, X) \), and in particular we have been able to compute \( N_{\mathbb{Q},4}(D, 10^{17}) = 10465196820067560.10465196820067560 \ldots \)

We can also count the number of extensions with a given signature. The method is completely similar, but here not all characters are genus characters. In fact, it is only necessary to add a single nongenus character to obtain all the necessary ones, but everything is completely explicit, and closely related to the rational quartic reciprocity law. I refer to [5] for details.

2.5. Other solvable extensions

We can also prove some partial results in the case where \( G = A_4 \) or \( G = S_4 \) (of course the results for \( S_4 \) are supereceded by Bhargava’s for \( K = \mathbb{Q} \), and by Yukie’s for general \( K \); still, the method is also useful for exact computations), see [17].

In the case of quartic \( A_4 \) and \( S_4 \)-extensions (or, for that matter, of cubic \( S_3 \)-extensions), we use the diagram involving the cubic resolvent (the quadratic one for \( S_3 \)-extensions), also called the Hasse diagram. We then have a situation which bears some analogies with the \( D_4 \) case. The differences are as follows. Instead of having to sum over quadratic extensions of the base field \( K \), we must sum over cubic extensions, cyclic for \( A_4 \) and noncyclic for \( S_4 \). As in the \( D_4 \)-case, we then have to consider quadratic extensions of these cubic fields, but generated by an element of
square norm. It is possible to go through the exact combinatorial and arithmetic computation of the corresponding Dirichlet series, the cubic field being fixed. This in particular uses some amusing local class field theory. As in the $D_4$ case, we then obtain the Dirichlet generating series for discriminants of $A_4$ (resp., $S_4$) extensions by summing the series over the corresponding cubic fields.

Unfortunately, we cannot obtain from this any asymptotic formula. The reason is different in the $A_4$ and the $S_4$ case. In the $A_4$ case, the rightmost singularity of the Dirichlet series is at $s = 1/2$. Unfortunately, this is simultaneously the main singularity of each individual Dirichlet series, and also that of the generating series for cyclic cubic fields. Thus, although the latter is well understood, it seems difficult (but not totally out of reach) to paste things together. On the other hand, we can do two things rigorously in this case. First, we can prove an asymptotic formula for $A_4$-extensions having a fixed cubic resolvent. Tables show that the formula is very accurate. Second, we can use our formula to compute $N_{K,4}(s_4, X)$ exactly. For instance, we have computed $N_{Q,4}(A_4, 10^{16}) = 218369252$. This computation is much slower than in the $D_4$-case, because we do not have the miracle of genus characters, and we must compute the class and unit group of all the cyclic cubic fields.

In the $S_4$ case, the situation is different. The main singularity of each individual Dirichlet series is still at $s = 1/2$ (because of the square norm condition), and the rightmost singularity of the generating series for noncyclic cubic fields is at $s = 1$, so the situation looks better (and analogous to the $D_4$ situation with $s$ replaced by $s/2$). Unfortunately, as already mentioned we know almost nothing about the generating series for noncyclic cubic fields, a fortiori with coefficients. So we cannot go further in the asymptotic analysis. As in the $A_4$ case, however, we can compute exactly either the number of $S_4$-extensions corresponding to a fixed cubic resolvent, or even $N_{K,4}(s_4, X)$ itself. The problem is that here we must compute class and unit groups of all noncyclic cubic fields of discriminant up to $X$, while cyclic cubic fields of discriminant up to $X$ are much rarer, of the order of $X^{1/2}$ instead. We have thus not been able to go very far and obtained for example $N_{Q,4}(S_4, 10^7) = 6541232$.

3. Prehomogeneous vector spaces

The other methods for studying $N_{K,n}(G, X)$ are two closely related methods: one is the use of generalizations of the Delone-Faddeev map, which applies when $K = \mathbb{Q}$. The other, which can be considered as a generalization of the first, is the use of the theory of prehomogeneous vector spaces, initiated by Sato and Shintani in the 1960’s.

3.1. Orders of small degree

We briefly give a sketch of the first method. We would first like to classify quadratic orders. It is well known that, through their discriminant, such orders are in one-to-one correspondence with the subset of nonsquare elements of $\mathbb{Z}$ congruent
to 0 or 1 modulo 4, on which $SL_1(\mathbb{Z})$ (the trivial group) acts. Thus, for fixed discriminant, the orbits are finite (in fact of cardinality 0 or 1). For maximal orders, we need to add local arithmetic conditions at each prime $p$, which are easy for $p > 2$, and slightly more complicated for $p = 2$.

We do the same for small higher degrees. For cubic orders, the classification is due to Davenport-Heilbronn (see [19], [20]). These orders are in one-to-one correspondence with a certain subset of $\text{Sym}^3(\mathbb{Z}^2)$, i.e., binary cubic forms, on which $SL_2(\mathbb{Z})$ acts. Since once again the difference in “dimensions” is $4 - 3 = 1$, for fixed discriminant the orbits are finite, at least generically. For maximal orders, we again need to add local arithmetic conditions at each prime $p$. These are easy to obtain for $p > 3$, but are a little more complicated for $p = 2$ and $p = 3$. An alternate way of explaining this is to say that a cubic order can be given by a nonmonic cubic equation, which is almost canonical if representatives are suitably chosen.

For quartic orders, the classification is due to M. Bhargava in 2001, who showed in complete detail how to generalize the above. These orders are now in one-to-one correspondence with a certain subset of $\mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^3)$, i.e., pairs of ternary quadratic forms, on which $SL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$ acts. Once again the difference in “dimensions” is $2 \times 6 - (3 + 8) = 1$, so for fixed discriminant the orbits are finite, at least generically. For maximal orders, we again need to add local arithmetic conditions at each prime $p$, which Bhargava finds after some computation. An alternate way of explaining this is to say that a quartic order can be given by the intersection of two conics in the projective plane, the pencil of conics being almost canonical if representatives are suitably chosen.

For quintic orders, only part of the work has been done, by Bhargava and Kable-Yukie in 2002. These are in one-to-one correspondence with a certain subset of $\mathbb{Z}^4 \otimes \Lambda^2(\mathbb{Z}^5)$, i.e., quadruples of alternating forms in 5 variables, on which $SL_4(\mathbb{Z}) \times SL_5(\mathbb{Z})$ acts. Once again the difference in “dimensions” is $4 \times 10 - (15 + 24) = 1$, so for fixed discriminant the orbits are finite, at least generically. The computation of the local arithmetic conditions, as well as the justification for the process of point counting near the cusps of the fundamental domain has however not yet been completed.

Since prehomogeneous vector spaces have been completely classified, this theory does not seem to be able to apply to higher degree orders, at least directly.

### 3.2. Results

Using the above methods, and generalizations to arbitrary base fields, the following results have been obtained on the function $N_{K,n}(G, X)$ (many other deep and important results have also been obtained, but we fix our attention to this function). It is important to note that they seem out of reach using more classical methods such as Kummer theory or class field theory mentioned earlier.

**Theorem 3.1.** Let $K$ be a number field of signature $(r_1, r_2)$, and as above write $\zeta_K(1)$ for the residue of the Dedekind zeta function of $K$ at $s = 1$.

1. (Davenport-Heilbronn [19], [20]). We have $N_{Q,3}(S_3, X) \sim c_Q(S_3) X$ with

   $$c_Q(S_3) = \frac{1}{\zeta(3)}.$$
2. (Datskovsky-Wright [18]). We have $N_{K,3}(S_3, X) \sim c_K(S_3) X$ with

$$c_K(S_3) = \left(\frac{2}{3}\right)^{r_1-1} \left(\frac{1}{6}\right)^{r_2} \zeta_K(1) \zeta_K(3).$$

3. (Bhargava [1], [2]). We have $N_{Q,4}(S_4, X) \sim c_Q(S_4) X$ with

$$c_Q(S_4) = \frac{5}{6} \prod_p \left(1 + \frac{1}{p^2} - \frac{1}{p^3} - \frac{1}{p^4}\right).$$

4. (Yukie [30]). There exist two strictly positive constants $c_1(K)$ and $c_2(K)$ such that

$$c_1 X < N_{K,4}(S_4, X) < c_2 X \log^2(X).$$

Under some very plausible convergence assumptions we should have in fact $N_{K,4}(S_4, X) \sim c_K(S_4) X$ with

$$c_K(S_4) = 2 \left(\frac{5}{12}\right)^{r_1} \left(\frac{1}{24}\right)^{r_2} \prod_p \left(1 + \frac{1}{Np^2} - \frac{1}{Np^3} - \frac{1}{Np^4}\right).$$

5. (Kable-Yukie [21]). There exists a strictly positive constant $c_1$ such that for all $\varepsilon > 0$ we have

$$c_1 X < N_{Q,5}(S_5, X) < X^{1+\varepsilon}.$$

Remark. It should also be emphasized that, although the above methods give important and deep results on $N_{K,n}(G, X)$ for certain groups $G$, they shed almost no light on the possible analytic continuation of the corresponding Dirichlet series of which $N_{K,n}(G, X)$ is the counting function. For example, in the simplest case where $K = Q$, $n = 3$, and $G = S_3$, for which the result dates back to Davenport-Heilbronn, no one knows how to give an analytic continuation of the Dirichlet series $\sum_L |d(L)|^{-s}$ even to $\Re(s) = 1$ (the sum being over cubic fields in $Q$ and $d(L)$ being the absolute discriminant of $L$).

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