A Barth-Lefschetz theorem for submanifolds
of a product of projective spaces

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Abstract

Let $X$ be a complex submanifold of dimension $d$ of $\mathbb{P}^m \times \mathbb{P}^n$ ($m \geq n \geq 2$) and denote by $\alpha: \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \to \text{Pic}(X)$ the restriction map of Picard groups, by $N_{X|\mathbb{P}^m \times \mathbb{P}^n}$ the normal bundle of $X$ in $\mathbb{P}^m \times \mathbb{P}^n$. Set $t := \max\{\dim \pi_1(X), \dim \pi_2(X)\}$, where $\pi_1$ and $\pi_2$ are the two projections of $\mathbb{P}^m \times \mathbb{P}^n$. We prove a Barth-Lefschetz type result as follows: Theorem. If $d \geq \frac{m+n+1}{2}$ then $X$ is algebraically simply connected, the map $\alpha$ is injective and $\text{Coker}(\alpha)$ is torsion-free. Moreover $\alpha$ is an isomorphism if $d \geq \frac{m+n+2}{2}$, or if $d = \frac{m+n+1}{2}$ and $N_{X|\mathbb{P}^m \times \mathbb{P}^n}$ is decomposable. These bounds are optimal. The main technical ingredients in the proof are: the Kodaira-Le Potier vanishing theorem in the generalized form of Sommese ([18], [19]), the join construction and an algebraisation result of Faltings concerning small codimension subvarieties in $\mathbb{P}^n$ (see [9]).

Introduction

It is well known that if $X$ is a submanifold of the complex projective space $\mathbb{P}^n$ ($n \geq 3$) of dimension $d > \frac{n}{2}$ then a topological result of Lefschetz type, due to Barth and Larsen (see [16], [6]), asserts that the canonical restriction maps $H^i(\mathbb{P}^n, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ are isomorphisms for $i \leq 2d-n$, and injective with torsion-free cokernel, for $i = 2d-n+1$. As a consequence, the restriction map $\alpha: \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \to \text{Pic}(X)$ is an isomorphism if $d \geq \frac{n+2}{2}$, and injective with torsion-free cokernel if $n = 2d - 1$.

This topological result has been generalized by Sommese to the case when the ambient space $\mathbb{P}^n$ is replaced by any projective rational homogeneous space $M$ (see [21]). For example, if $M = \mathbb{P}^m \times \mathbb{P}^n$ (with $m \geq n \geq 2$) then Sommese’s topological result implies that the canonical restriction map

$$\alpha: \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \to \text{Pic}(X)$$

is injective with torsion-free cokernel for every submanifold $X$ of $\mathbb{P}^m \times \mathbb{P}^n$ of dimension $d \geq \frac{m+n+1}{2}$, and an isomorphism if $d \geq \frac{m+n+2}{2}$.

The aim of this paper is to prove (in a geometric way) an improved version of Sommese’s result concerning the Picard group of the small-codimensional submanifolds $X$ of $\mathbb{P}^m \times \mathbb{P}^n$ of dimension $d$. To state the main result, set

$$t := \max\{\dim \pi_1(X), \dim \pi_2(X)\}, \quad (1)$$

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where \( \pi_1 \) and \( \pi_2 \) are the two canonical projections of \( \mathbb{P}^m \times \mathbb{P}^n \).

**Main Theorem.** Let \( X \) be a complex submanifold of dimension \( d \) of \( \mathbb{P}^m \times \mathbb{P}^n \) (with \( m \geq n \geq 2 \)), and denote by \( \alpha \colon \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \to \text{Pic}(X) \) the restriction map of Picard groups, by \( N_X|_{\mathbb{P}^m \times \mathbb{P}^n} \) the normal bundle of \( X \) in \( \mathbb{P}^m \times \mathbb{P}^n \), and by \( t \) the integer defined by \cite{1}. Then the following statements hold true:

i) If \( d \geq \frac{m+n+t+1}{2} \) then \( X \) is algebraically simply connected (and in particular, \( H^1(\mathcal{O}_X) = 0 \)), the map \( \alpha \) is injective and \( \text{Coker}(\alpha) \) is torsion-free.

ii) If \( d \geq \frac{m+n+t+2}{2} \) then \( \alpha \) is an isomorphism.

iii) If \( d = \frac{m+n+t+1}{2} \) and \( N_X|_{\mathbb{P}^m \times \mathbb{P}^n} \cong E_1 \oplus E_2 \), with \( E_1 \) and \( E_2 \) vector bundles of rank \( \geq 1 \), then \( \alpha \) is an isomorphism.

iv) If \( d = \frac{m+n+t}{2} \) and \( N_X|_{\mathbb{P}^m \times \mathbb{P}^n} \cong E_1 \oplus E_2 \), with \( E_1 \) and \( E_2 \) vector bundles of rank \( \geq 1 \), then \( H^1(\mathcal{O}_X) = 0 \). If instead \( N_X|_{\mathbb{P}^m \times \mathbb{P}^n} \cong E_1 \oplus E_2 \), with \( E_1 \) and \( E_2 \) vector bundles of rank \( \geq 2 \), then \( \alpha \) is injective and \( \text{rank } \text{NS}(X) = 2 \), where \( \text{NS}(X) \) denotes the Néron-Severi group of \( X \).

We show by examples that the bounds given in Main Theorem, i) and ii) are optimal. As far as i) is concerned, take for instance \( m = 2s \) even and \( s \leq n \leq m = 2s \). For every \( s \geq 2 \) there exist elliptic scrolls \( Y \) of dimension \( s \) in \( \mathbb{P}^m = \mathbb{P}^{2s} \) (well known if \( s = 2 \) and \cite{5} for every \( s \geq 3 \)). Set \( X = Y \times \mathbb{P}^n \). Then \( d = s+n, t = n \) and \( d = \frac{m+n+t}{2} \). Since \( H^1(\mathcal{O}_X) = H^1(\mathcal{O}_Y) \neq 0 \), \( X \) is not algebraically simply connected, and \( \text{Coker}(\alpha) \) cannot be an isomorphism because \( \text{rank } \text{Pic}(X) = 3 \). Notice also that by iii) the normal bundle \( N_X|_{\mathbb{P}^m \times \mathbb{P}^n} = q^*(N_Y|_{\mathbb{P}^m}) \) (where \( q \colon X = Y \times \mathbb{P}^n \to Y \) is the canonical projection) is indecomposable (with \( q \colon X = Y \times \mathbb{P}^n \to Y \) the canonical projection). The indecomposability of \( N_Y|_{\mathbb{P}^m} \) was also previously proved in \cite{2}. To produce an example showing that the bound in ii) is also optimal, take \( m = 2s+1 \) with \( s \geq 2 \) and \( n \) such that \( s+1 \leq n \leq m = 2s+1 \). Let \( Y \) be the image of the Segre embedding \( \mathbb{P}^s \times \mathbb{P}^1 \to \mathbb{P}^m = \mathbb{P}^{2s+1} \), and set \( X = Y \times \mathbb{P}^n \). Then \( d = s+n+1 \) and \( t = n \) are such that \( d = \frac{m+n+t+1}{2} \). However, the map \( \alpha \) cannot be an isomorphism because \( \text{rank } \text{Pic}(X) = 3 \). Notice also that by iii) the normal bundle \( N_X|_{\mathbb{P}^m \times \mathbb{P}^n} = q^*(N_Y|_{\mathbb{P}^m}) \) is indecomposable (with \( q \colon X = Y \times \mathbb{P}^n \to Y \) the canonical projection). The indecomposability of \( N_Y|_{\mathbb{P}^m} \) was also previously proved in \cite{2}.

The proof of part i) makes use of the join construction to reduce the problem to an open subset of a small-codimensional subvariety of \( \mathbb{P}^{m+n+1} \) and then to apply a result of Faltings (see \cite{3}). This is done in section 2. The proof of parts ii)–iv) (which is inspired from \cite{2}) makes systematic use of Kodaira-Le Potier vanishing theorem in the generalized form given by Sommese (see \cite{18}) and is contained in section 1.

We want to mention the following interesting recent result of Arrondo and Caravantes \cite{1} which is related to our Main Theorem (although our approach is completely different from theirs):

**Theorem (Arrondo-Caravantes)** Let \( X \) be a complex submanifold of \( \mathbb{P}^m \times \mathbb{P}^n \) of dimension \( d \geq m+1 \) such that \( \pi_i(X) = \mathbb{P}^m \) for \( i = 1, 2 \). Then the restriction map \( \alpha \colon \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \to \text{Pic}(X) \) is injective and \( \text{Pic}(X) \) is a free abelian group of rank two.

In the result of Arrondo and Caravantes the codimension of \( X \) is relatively larger than in our Main Theorem, but it does not give any information on the torsion of \( \text{Coker}(\alpha) \).
All varieties considered throughout are defined over the field \( \mathbb{C} \) of complex numbers. By a manifold we mean a nonsingular irreducible complex algebraic variety. The terminology and the notation used are standard, unless otherwise specified.

1 A general result on submanifolds in \( \mathbb{P}^m \times \mathbb{P}^n \)

Let \( X \) be a closed irreducible subvariety of dimension \( d \geq 1 \) of \( \mathbb{P}^m \times \mathbb{P}^n \), with \( m \geq n \geq 2 \). Denote by \( \pi_1: \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m \) and \( \pi_2: \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n \) the canonical projections and by \( p_1: X \to p_1(X) \) and \( p_2: X \to p_2(X) \) the restrictions \( \pi_1|X \) and \( \pi_2|X \). Throughout this paper we shall assume that \( p_1(X) \) and \( p_2(X) \) are both positive dimensional. A closed irreducible subvariety \( X \) of \( \mathbb{P}^m \times \mathbb{P}^n \) satisfying this property will be called positive. It is well known (and easy to see) that \( X \) is positive if and only if \( X \) intersects every hypersurface of \( \mathbb{P}^m \times \mathbb{P}^n \). Set

\[
\mathcal{O}_X(a, b) := \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a, b)|X = p_1^*(\mathcal{O}_{p_1(X)}(a)) \otimes p_2^*(\mathcal{O}_{p_2(X)}(b)), \forall a, b \in \mathbb{Z}. \tag{2}
\]

Since the cotangent bundle of \( \mathbb{P}^m \times \mathbb{P}^n \) is given by \( \Omega^1_{\mathbb{P}^m \times \mathbb{P}^n} = \pi_1^*(\Omega^1_{\mathbb{P}^m}) \oplus \pi_2^*(\Omega^1_{\mathbb{P}^n}) \), we get

\[
\Omega^1_{\mathbb{P}^m \times \mathbb{P}^n}|X = p_1^*(\Omega^1_{\mathbb{P}^m}|p_1(X)) \oplus p_2^*(\Omega^1_{\mathbb{P}^n}|p_2(X)). \tag{3}
\]

Then a lot of information about the embedding \( X \subseteq \mathbb{P}^m \times \mathbb{P}^n \) is contained in the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & \rightarrow & N^\vee_X|_{\mathbb{P}^m \times \mathbb{P}^n} & \rightarrow & p_1^*(\Omega^1_{\mathbb{P}^m}|p_1(X)) \oplus p_2^*(\Omega^1_{\mathbb{P}^n}|p_2(X)) & \rightarrow & \Omega^1_X & \rightarrow & 0 \\
\vert & & \downarrow \text{id} & & \downarrow \beta' & & \downarrow \beta & & \downarrow \varepsilon \\
0 & \rightarrow & N^\vee_X|_{\mathbb{P}^m \times \mathbb{P}^n} & \rightarrow & p_1^*(\mathcal{O}_{p_1(X)}(-1)^{\oplus m+1}) \oplus p_2^*(\mathcal{O}_{p_2(X)}(-1)^{\oplus n+1}) & \rightarrow & F & \rightarrow & 0 \\
\vert & & \downarrow \gamma & & \vert & & \vert & & \vert \\
0 & & 0 & & p_1^*(\mathcal{O}_{p_1(X)}) \oplus p_2^*(\mathcal{O}_{p_2(X)}) & \rightarrow & \mathcal{O}^{\oplus 2}_X & \rightarrow & 0 \\
\end{array}
\]

in which \( N^\vee_X|_{\mathbb{P}^m \times \mathbb{P}^n} \) is the conormal bundle of \( X \) in \( \mathbb{P}^m \times \mathbb{P}^n \), the first row is the canonical exact sequence of cotangent bundles of \( X \) in \( \mathbb{P}^m \times \mathbb{P}^n \) (taking into account of (3)), the middle column is the direct sum of the restricted Euler sequences of \( \mathbb{P}^m \) and of \( \mathbb{P}^n \), and \( F := \text{Coker}(\beta') \).

We first prove the following general result:
Lemma 1.1 Let \( f: X \to Y \) be a proper surjective morphism from a projective manifold \( X \) of dimension \( d \geq 2 \) onto a projective variety \( Y \) of dimension \( e \) with \( d > e \geq 2 \). Then \( f \) has no fiber isomorphic to \( \mathbb{P}^{d-1} \).

Proof. Assume that there exists \( y \in Y \) such that the fiber \( F = f^{-1}(y) \) is isomorphic to \( \mathbb{P}^{d-1} \). Then \( F \) is an effective divisor on \( X \). Since \( \text{Pic}(\mathbb{P}^{d-1}) = \mathbb{Z}[\mathcal{O}_{\mathbb{P}^{d-1}}(1)] \), it follows that the conormal line bundle \( N_{F/X}^{\vee} \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^{d-1}}(s) \) for some \( s \in \mathbb{Z} \).

We claim that \( s > 0 \). Indeed, since \( \text{Pic}(\mathbb{P}^{d-1}) \cong \mathbb{Z} \), it is sufficient to show that we can find an irreducible curve \( C \subseteq F \) such that \( \deg_C(N_{F/X}^{\vee} | C) = (N_{F/X}^{\vee} | C) > 0 \). To produce such a curve (following an idea of P. Ionescu) we fix a projective embedding \( X \hookrightarrow \mathbb{P}^N \), and let \( H_1, \ldots, H_{d-2} \) be \( d-2 \) general hyperplanes of \( \mathbb{P}^N \), and set \( X' := X \cap H_1 \cap \cdots \cap H_{d-2} \) and \( C := F \cap H_1 \cap \cdots \cap H_{d-2} \). By Bertini, \( X' \) is a smooth projective surface and \( C \) is a smooth irreducible curve on \( X' \). By construction, the morphism \( f' := f|X': X' \to f(X') \) is generically finite and \( f'(C) = y \); then by a well known elementary fact in the theory of surfaces, \( (C^2)_{X'} < 0 \) (for instance this fact is an easy consequence of Hodge index theorem). On the other hand, since \( C \) is the proper intersection of \( F \) with \( H_1 \cap \cdots \cap H_{d-2} \) we infer that \( N_{F/X}^{\vee} | X' \cong N_C^{\vee} \). But since \( (C^2)_{X'} = \deg_C(N_C^{\vee}) = \deg_C(N_{F/X}^{\vee}) \), we get \( \deg_C(N_{F/X}^{\vee} | C) > 0 \), as claimed.

Now, by a generalization of a contractibility result of Castelnuovo-Kodaira (see [4]) the divisor \( F = \mathbb{P}^{d-1} \) of \( X \) of conormal bundle \( \mathcal{O}_{\mathbb{P}^{d-1}}(s) \) with \( s > 0 \) can be blown down to a normal point, i.e. there exists a birational morphism \( \varphi: X \to V \), with \( V \) a normal projective variety such that \( \varphi(F) = v \) is a point and \( \varphi|X \setminus F \) defines a biregular isomorphism \( X \setminus F \cong V \setminus \{v\} \). Then by a well known elementary fact, there is a unique morphism \( g: V \to Y \) such that \( g \circ \varphi = f \). In particular, the fiber \( g^{-1}(g(v)) \) is reduced to the point \( v \), which contradicts the theorem on the dimension of fibers because by hypothesis \( \dim(V) = d > e = \dim(Y) \).

Corollary 1.2 Let \( X \) be a submanifold of dimension \( d \geq m + 1 \) of \( \mathbb{P}^m \times \mathbb{P}^n \), with \( m \geq n \geq 2 \). If \( d = m + 1 \) assume moreover that \( \dim p_i(X) \geq 2 \) for \( i = 1, 2 \). Then all the fibers of \( p_i: X \to p_i(X), i = 1, 2 \), have dimension \( \leq d - 2 \).

Proof. The assertion is trivial if \( d \geq m + 2 \) because all the fibers of \( p_i \) \((i = 1, 2)\) are of dimension \( \leq m \). Assume therefore \( d = m + 1 \); if there exists a fiber \( F \) of \( p_i \) of dimension \( m \) (with \( i = 1 \) or \( i = 2 \)), then necessarily \( F \cong \mathbb{P}^m \). In this case the corollary follows from Lemma 1.1.

Now we turn to our general situation (under the hypotheses from the beginning). By [2] we have

\[
p_1^*(\mathcal{O}_{p_1(X)}(-1)^{\oplus m+1}) \cong \mathcal{O}_X(-1,0)^{\oplus m+1} \quad \text{and} \quad p_2^*(\mathcal{O}_{p_2(X)}(-1)^{\oplus n+1}) \cong \mathcal{O}_X(0,-1)^{\oplus n+1}.
\]

Moreover,

\[
H^0(p_1^*(\mathcal{O}_{p_1(X)}(-1)^{\oplus m+1}) \oplus p_2^*(\mathcal{O}_{p_2(X)}(-1)^{\oplus n+1})) = 0.
\]

This follows because \( p_1(X) \) and \( p_2(X) \) are positive dimensional, whence by the above isomorphisms \( \mathcal{O}_X(1,0)^{\oplus m+1} \) and \( \mathcal{O}_X(0,1)^{\oplus n+1} \) are direct sums of \((d-1)\)-ample line bundles (in the sense of Sommese [20]). Thus the cohomology of the second row of the above diagram yields the exact sequence

\[
0 \to H^0(F) \to H^1(N_{X|\mathbb{P}^m \times \mathbb{P}^n}^{\vee}) \to H^1(\mathcal{O}_X(-1,0)^{\oplus m+1} \oplus \mathcal{O}_X(0,-1)^{\oplus n+1}) \to H^1(F) \to H^2(N_{X|\mathbb{P}^m \times \mathbb{P}^n}^{\vee}) \to H^2(\mathcal{O}_X(-1,0)^{\oplus m+1} \oplus \mathcal{O}_X(0,-1)^{\oplus n+1}).
\]
On the other hand, the last column yields the cohomology exact sequence

$$0 \rightarrow H^0(\Omega^1_X) \rightarrow H^0(F) \rightarrow H^0(\Omega^2_X) \rightarrow H^1(\Omega^1_X) \rightarrow H^1(F) \rightarrow H^1(\Omega^2_X).$$  \hfill (5)

**Lemma 1.3** Under the above hypotheses, assume moreover that the projections $p_i: X \rightarrow p_i(X)$ have all fibers of dimension $\leq d - 2$ for $i = 1, 2$, e.g. if $d \geq m + 2$, or if $d = m + 1$ and $\dim p_i(X) \geq 2$ for $i = 1, 2$, (by Corollary 1.2 above). Then $H^1(\mathcal{O}_X(-1, 0)^{\oplus m + 1} \oplus \mathcal{O}_X(0, -1)^{\oplus n + 1}) = 0$.

**Proof.** The hypothesis implies that $\mathcal{O}_X(1, 0)^{\oplus m + 1} \cong p_1^*(\mathcal{O}_{p_1(X)}(1))^{\oplus m + 1}$ and $\mathcal{O}_X(0, 1)^{\oplus n + 1} \cong p_2^*(\mathcal{O}_{p_2(X)}(1))^{\oplus n + 1}$ are both $(d - 2)$-ample vector bundles which are direct sums of line bundles. Then the conclusion follows from Kodaira vanishing theorem in the generalized form of Sommese, see [19], page 96, Corollary 5.20.

**Corollary 1.4** Under the hypotheses of Lemma 1.3 one has $h^1(N^\vee_X|p_m \times \mathbb{P}^n) = h^0(F)$ and $h^1(F) \leq h^2(N^\vee_X|p_m \times \mathbb{P}^n)$.  

**Proof.** The corollary follows from the exact sequence (4) and from Lemma 1.3.

**Corollary 1.5** Under the hypotheses of Lemma 1.3 one has $h^1(\Omega^1_{p_m \times \mathbb{P}^n}|X) = 2$.

**Proof.** The corollary follows from the cohomology sequence of the first column of the above diagram and from Lemma 1.3, taking into account of the isomorphism (3).

Now we analyze the exact sequence (5). The pull backs of the Euler exact sequences

$$0 \rightarrow p_1^*(\Omega^1_{p_m}|p_1(X)) \rightarrow p_1^*(\mathcal{O}_{p_1(X)}(-1)^{\oplus m + 1}) = \mathcal{O}_X(-1, 0)^{\oplus m + 1} \xrightarrow{\gamma_1} \mathcal{O}_X \rightarrow 0,$$

$$0 \rightarrow p_2^*(\Omega^1_{p_m}|p_2(X)) \rightarrow p_2^*(\mathcal{O}_{p_2(X)}(-1)^{\oplus n + 1}) = \mathcal{O}_X(0, -1)^{\oplus n + 1} \xrightarrow{\gamma_2} \mathcal{O}_X \rightarrow 0$$

do not split, because $p_1(X)$ and $p_2(X)$ are positive dimensional. This means that $H^0(\gamma_i) = 0$ for $i = 1, 2$. Since the first vertical column of the above diagram is the direct sum of these exact sequences, it follows that the map

$$H^0(\gamma) = H^0(\gamma_1) \oplus H^0(\gamma_2)$$

is also zero. Thus from the cohomology sequence of the first column we infer that the map

$$\delta_1: H^0(\Omega^2_X) \rightarrow H^1(p_1^*(\Omega^1_{p_m}|p_1(X)) \oplus p_2^*(\Omega^1_{p_m}|p_2(X)))$$

is injective.

On the other hand, in the commutative square

$$\begin{array}{ccc}
H^0(\Omega^2_X) & \xrightarrow{\text{id}} & H^0(\Omega^2_X) \\
\delta_1 \downarrow & & \delta_2 \\
H^1(p_1^*(\Omega^1_{p_m}|p_1(X)) \oplus p_2^*(\Omega^1_{p_m}|p_2(X))) & \rightarrow & H^1(\Omega^1_X)
\end{array}$$

the bottom horizontal map is not zero. Indeed, by hypothesis $p_1(X)$ and $p_2(X)$ are both positive dimensional. Since $\dim p_1(X) > 0$ the map $H^1(\Omega^1_{p_m}) \rightarrow H^1(\Omega^1_X)$ is non-zero (the image of the class of $\mathcal{O}_{p_m}(1)$ is not zero in $H^1(\Omega^1_X)$). Since this map is the composition

$$H^1(\Omega^1_{p_m}) \rightarrow H^1(p_1^*(\Omega^1_{p_m}|p_1(X))) \rightarrow H^1(\Omega^1_X),$$
it follows that the second map cannot be zero.

Now, since \( \delta_1 \) is injective we infer that \( \delta_2 \neq 0 \). Thus (5) yields the exact sequences

\[
0 \to H^0(\Omega^1_X) \to H^0(F) \text{ and } 0 \to V \to H^1(\Omega^1_X) \to H^1(F) \to H^1(\mathcal{O}_X^{\oplus 2}),
\]

in which \( V \) is a \( \mathbb{C} \)-vector space of dimension 2 if \( \delta_2 \) is injective (i.e. if \( H^0(\varepsilon) = 0 \)), and 1 otherwise. In particular,

\[
h^1(\Omega^1_X) \leq \begin{cases} 
2 + h^1(F), & \text{if } H^0(\varepsilon) = 0 \\
1 + h^1(F), & \text{if } H^0(\varepsilon) \neq 0.
\end{cases}
\]

(7)

Moreover, in both cases we have equality if \( H^1(\mathcal{O}_X) = 0 \).

Putting everything together and using Corollary 1.4 we get:

**Theorem 1.6** Under the hypotheses of the beginning assume moreover that both projections \( p_i : X \to p_i(X), i = 1, 2 \), have fibers all of dimension \( \leq d - 2 \). (This is always the case if \( d \geq m + 2 \), or by Corollary 1.3 above, if \( d = m + 1 \) and \( \dim p_i(X) \geq 2 \) for \( i = 1, 2 \).) Then the following statements hold true:

i) \( h^0(\Omega^1_X) \leq h^1(N^\vee_X|P^m \times P^n) \).

ii) \( \text{rank } \text{NS}(X) \leq \begin{cases} 2 + h^2(N^\vee_X|P^m \times P^n), & \text{if } H^0(\varepsilon) = 0. \\
1 + h^2(N^\vee_X|P^m \times P^n), & \text{if } H^0(\varepsilon) \neq 0.
\end{cases} \)

where \( \text{NS}(X) \) is the Néron-Severi group of \( X \).

iii) If \( d \geq m + 1 \) and \( H^1(N^\vee_X|P^m \times P^n) = 0 \) then \( H^1(\mathcal{O}_X) = 0 \) and \( 2 \leq \text{rank } \text{NS}(X) \leq 2 + h^2(N^\vee_X|P^m \times P^n) \).

**Proof.** i) follows from Corollary 1.4 and from (6). ii) follows from the well known inequality \( \text{rank } \text{NS}(X) \leq h^1(\Omega^1_X) \) (valid in characteristic zero, see [13], Exercise 1.8, page 367, if \( X \) is a surface, and [2], the claim in the proof of Theorem 2.1, in general), and from (7). The assertion about \( H^1(\Omega^1_X) \) in iii) follows from i), using the Hodge symmetry \( H^0(\Omega^1_X) \cong H^1(\mathcal{O}_X) \) (via Serre’s GAGA). The last part of iii) follows from Lemma 2.2 and from ii) because \( H^0(\varepsilon) = 0 \) if \( H^1(N^\vee_X|P^m \times P^n) = 0 \), by Corollary 1.4. \( \square \)

**Remark 1.7** Take \( m = n \geq 2 \) and \( X = \Delta \cong P^n \) the diagonal of \( P^n \times P^n \). Then \( N^\vee_X|P^m \times P^n = \Omega^1_{P^n} \), whence \( H^1(N^\vee_X|P^m \times P^n) = H^1(\Omega^1_{P^n}) \cong \mathbb{C} \). So by Theorem 1.6 i), \( H^0(F) \cong \mathbb{C} \). In this case, \( H^0(\varepsilon) \neq 0 \), whence \( 1 = \text{rank } \text{Num}(X) \leq 1 + h^1(F) \). Moreover, \( H^2(N^\vee_X|P^m \times P^n) = H^2(\mathcal{O}^\vee_{P^n}) = 0 \), whence \( h^1(F) = 0 \) by Corollary 1.4. Thus the above inequality becomes equality. In this case we also have \( H^0(\Omega^1_X) = 0 \) and \( H^1(N^\vee_X|P^m \times P^n) \neq 0 \). In particular, the second possibility in (7) really occurs.

**Corollary 1.8** Assume that \( N_X|P^m \times P^n \) is ample and \( d \geq m + 1 \). Then the irregularity of \( X \) is zero, and \( \text{rank } \text{Pic}(X) = 2 \).

**Proof.** Since \( N_X|P^m \times P^n \) is ample from Le Potier vanishing theorem and \( d \geq m + 1 \) it follows that \( H^i(N^\vee_X|P^m \times P^n) = 0 \) for \( i \leq 2 \). Then the conclusion follows from Theorem 1.6. \( \square \)

In the sequel we shall be interested in the submanifolds of \( P^m \times P^n \) of dimension \( d \) with \( d \geq \frac{m+n+t+1}{2} \). We shall need the following lemma:
Lemma 1.9 Let $X$ be a submanifold of $\mathbb{P}^m \times \mathbb{P}^n$ of dimension $d$ with $m \geq n \geq 2$.

i) Assume that $d \geq \frac{m+n+t+1}{2}$. Then $d \geq m+1$ and if $d = m+1$ then $\dim p_i(X) \geq 2$, $i = 1, 2$.

ii) Assume that $d = \frac{m+n+t}{2}$. Then $d \geq m$. If $d = m$ then $m \geq 2n$ and $X = X_1 \times \mathbb{P}^n$, with $X_1$ a submanifold of $\mathbb{P}^m$ of dimension $m-n$, with $m-n \geq n \geq 2$. If $d \geq m+1$ and $\dim p_i(X) = 1$ for some $i \in \{1, 2\}$ then $X$ is isomorphic to $\mathbb{P}^m \times X_2$, with $X_2$ a smooth curve in $\mathbb{P}^n$, with $n = 2$.

Proof. i) Assume $d \leq m$, i.e. $d = m-r$, with $r \geq 0$. The hypothesis implies $m+n+t+1 \leq 2d = 2m-2r$, whence $d-t = m-r-t \geq n+r+1$. Let $F_1$ be a general fiber of $p_1 : X \rightarrow p_1(X)$. Since $t \geq \dim p_1(X)$, by the theorem on dimension of fibers we get

$$\dim(F_1) = d - \dim p_1(X) \geq d - t \geq n + r + 1 \geq n + 1.$$ 

However this is impossible because $F_1 \cong p_2(F_1) \subseteq \mathbb{P}^n$. This proves that $d \geq m+1$.

Now we prove the last part of i). The only case in which we can have $\dim p_1(X) = 1$ for some $i \in \{1, 2\}$ is when $d = m+1$. Thus

$$d = m+1 \geq \frac{m+n+t+1}{2}.$$ (8)

If $m = n$ the inequality (8) implies $t = 1$. On the other hand, since $X \subseteq p_1(X) \times p_2(X)$, it follows that $d \leq 2$, and in particular, by the above inequality we get $m = n = 1$, which contradicts the hypothesis that $m \geq n \geq 2$. If instead $m > n$ then $\dim(F_1) \leq n$, whence $\dim p_1(X) = m+1 - \dim(F_1) \geq m+1 - n \geq 2$. Assume now that $\dim p_2(X) = 1$; then all fibers of $p_2$ are $m$-dimensional, whence all of them are isomorphic to $\mathbb{P}^m$ (since they are contained in $\mathbb{P}^m \times p$, with $p \in \mathbb{P}^n$). It follows that $X = p_2^{-1}(p_2(X)) = \mathbb{P}^m \times p_2(X)$ and in particular, $t = m$. Finally, since $m+1 > \frac{m+n+t+1}{2} = m + \frac{n+1}{2}$ we get $n = 1$, which again contradicts the hypotheses. This proves i).

ii) Assume first that $d \leq m$, i.e. $d = m-r$, with $r \geq 0$. Since $t \geq \dim p_1(X)$ and $F_1 \cong p_2(F_1) \subseteq \mathbb{P}^n$, the equality $d = \frac{m+n+t}{2}$ and the theorem on dimension of fibers yield

$$n + r = d - t \leq d - \dim p_1(X) = \dim(F_1) \leq n,$$

where (as above) $F_1$ is a general fiber of $p_1 : X \rightarrow p_1(X)$. It follows that $r = 0$, i.e. $d = m$, $t = \dim p_1(X)$ and $F_1 = \mathbb{P}^n$. Hence all fibers of $p_1$ are isomorphic to $\mathbb{P}^n$, i.e. $X = X_1 \times \mathbb{P}^n$, with $X_1 = p_1(X)$ a submanifold of $\mathbb{P}^m$ of dimension $m-n$. Moreover, since $t = \dim p_1(X) = d - n = m - n$ and $\dim p_2(X) = n$ it follows that $m - n \geq n$, i.e. $m \geq 2n$. In particular, every fiber of the projections $p_1$ and $p_2$ is of dimension $\leq d-2$.

Assume now $d \geq m+1$. Then as above we can have $\dim p_i(X) = 1$ for some $i \in \{1, 2\}$ only if $d = m+1$. Thus $m+1 = \frac{m+n+t}{2}$ yields $m = n + t - 2$. If $m = n$ then $t = 2$, and therefore (say) $\dim p_1(X) = 2$ and $\dim p_2(X) = 1$. Using $X \subseteq p_1(X) \times p_2(X)$ this immediately yields $d \leq 3$ and $X$ is the hypersurface $\mathbb{P}^2 \times p_2(X) \subseteq \mathbb{P}^2 \times \mathbb{P}^2$. If $m > n$ then only $X_2 := p_2(X)$ can be a curve, and in this case $X = \mathbb{P}^m \times X_2$. Since in this case $t = m$ it follows $m = n + m - 2$, i.e. $n = 2$ and $m \geq 3$. □

Now we come back to the above commutative diagram with exact rows and columns. Then from the second row of this diagram it follows that $N_{X|\mathbb{P}^m \times \mathbb{P}^n}$ is a quotient of

$$p_1^*(O_{p_1(X)(1)}^{{\oplus m+1}}) \oplus p_2^*O_{p_2(X)}(1)^{{\oplus n+1}} = O_X(1,0)^{{\oplus m+1}} \oplus O_X(0,1)^{{\oplus n+1}}.$$ (9)
Clearly $t \leq m$, where $t$ is defined by formula (11) of the introduction. Moreover, it is
easy to see that the fibers of the morphisms $p_1: X \to p_1(X)$ and $p_2: X \to p_2(X)$ are all of
dimension $\leq t$. Indeed, if for example $F$ is a fiber of $p_1$ then $F \cong p_2(F)$ (via $p_2$), whence\ndim($F$) = dim $p_2(F) \leq$ dim $p_2(X) \leq t$.

It follows that the vector bundle (11) is $t$-ample, whence its quotient $N_{X|\mathbb{P}^m \times \mathbb{P}^n}$ is a $t$-
ample vector bundle of rank $m + n - d$. Then using Le Potier-Sommese vanishing theorem
(see [19], page 96, Corollary (5.20)) we get:

\[ H^i(N_{X|\mathbb{P}^m \times \mathbb{P}^n}^\vee) = 0, \quad \text{for } i \leq d - (m + n - d) - t = 2d - m - n - t. \]

In particular,

\[ H^1(N_{X|\mathbb{P}^m \times \mathbb{P}^n}^\vee) = 0 \quad \text{if } d \geq \frac{m + n + t + 1}{2} \quad \text{and} \]
\[ H^2(N_{X|\mathbb{P}^m \times \mathbb{P}^n}^\vee) = 0 \quad \text{if } d \geq \frac{m + n + t + 2}{2}. \]

Now we are ready to prove the following:

**Theorem 1.10** Let $X$ be a submanifold of $\mathbb{P}^m \times \mathbb{P}^n$ of dimension $d$ with $m \geq n \geq 2$.

Then the following statements hold true:

i) If $d \geq \frac{m + n + t + 1}{2}$ then $H^1(\mathcal{O}_X) = 0$, and if $d \geq \frac{m + n + t + 2}{2}$ then rank Pic($X$) = 2.

ii) If $d = \frac{m + n + t + 1}{2}$ and $N_{X|\mathbb{P}^m \times \mathbb{P}^n} \cong E_1 \oplus E_2$, with $E_1$ and $E_2$ vector bundles of rank

$\geq 1$, then rank Pic($X$) = 2.

iii) If $d = \frac{m + n + t + 2}{2}$ and $N_{X|\mathbb{P}^m \times \mathbb{P}^n} \cong E_1 \oplus E_2$, with $E_1$ and $E_2$ vector bundles of rank $\geq 1$,

then $H^1(\mathcal{O}_X) = 0$. If moreover $N_{X|\mathbb{P}^m \times \mathbb{P}^n} \cong E_1 \oplus E_2$, with $E_1$ and $E_2$ vector bundles

of rank $\geq 2$, then rank Pic($X$) = 2.

**Proof.** Assume first that $d \geq \frac{m + n + t + 1}{2}$. Then by Lemma [19], i) we have $d \geq m + 1$ and
dim $p_i(X) \geq 2$ for $i = 1, 2$. Then by Serre’s GAGA and Hodge symmetry, $h^0(\Omega^1_X) = h^1(\mathcal{O}_X)$. Moreover, NS($X$) = Pic($X$) if $X$ is a regular variety. Then i) follows from (10)
and from Theorem [1,6] i) and iii).

ii) By hypothesis $N_{X|\mathbb{P}^m \times \mathbb{P}^n} = E_1 \oplus E_2$, with $E_1$ and $E_2$ vector bundles of rank $\leq$

rank($N_{X|\mathbb{P}^m \times \mathbb{P}^n}$) = 1 = $m + n - d - 1$. Since $N_{X|\mathbb{P}^m \times \mathbb{P}^n}$ is $t$-ample, $E_1$ and $E_2$ are also $t$-ample. Thus by Le Potier vanishing theorem in the generalized form given by Sommese
(see [19], page 96, Corollary (5.20)), we have

\[ H^2(N_{X|\mathbb{P}^m \times \mathbb{P}^n}^\vee) \cong H^2(E_1^\vee) \oplus H^2(E_2^\vee) = 0, \]

because in this case $d - \text{rank}(E_i) - t \geq d - (m + n - d - 1) - t = 2d - (m + n + t - 1) = (m + n + t + 1) - (m + n + t - 1) = 2$, for $i = 1, 2$. Then by Theorem [1,6] iii), rank Pic($X$) = 2.

iii) By Lemma [1,9] ii) we may assume that both projections $p_i: X \to p_i(X)$, $i = 1, 2$, have fibers all of dimension $\leq d - 2$. Indeed, if $d = m$ by Lemma [1,9] ii) we have

$X = X_1 \times \mathbb{P}^n$ with dim($X_1$) = $m - n \geq n \geq 2$. If instead $d \geq m + 1$ then we can have
dim $p_i(X) = 1$ for some $i \in \{1, 2\}$ only if $X = \mathbb{P}^m \times X_2$, with $X_2$ a smooth curve in $\mathbb{P}^n$,
with $n = 2$, in which case $X$ is a hypersurface in $\mathbb{P}^m \times \mathbb{P}^n$. However this situation is ruled
out by the hypotheses which imply codim$_{\mathbb{P}^m \times \mathbb{P}^n}(X) \geq 2$. 

\[ \text{L. Bădescu and F. Repetto} \]
If \( d = \frac{m+n+t}{2} \) and \( N_{X|\mathbb{P}^m \times \mathbb{P}^n} = E_1 \oplus E_2 \), with \( E_1 \) and \( E_2 \) vector bundles of rank \( \leq \text{rank}(N_{X|\mathbb{P}^m \times \mathbb{P}^n}) - 1 = m + n - d - 1 \), then as above,

\[
H^1(N_{X|\mathbb{P}^m \times \mathbb{P}^n}' ) \cong H^1(E_1') \oplus H^1(E_2') = 0,
\]

by Le Potier-Sommese vanishing theorem, because \( d = \frac{m+n+t}{2} \) implies \( d - \text{rank}(E_i) - t \geq 1 \), for \( i = 1, 2 \). Then the statement follows from Theorem 1.6 i).

If instead \( N_{X|\mathbb{P}^m \times \mathbb{P}^n} = E_1 \oplus E_2 \), with \( E_1 \) and \( E_2 \) vector bundles of rank \( \geq 2 \), i.e. of rank \( \leq \text{rank}(N_{X|\mathbb{P}^m \times \mathbb{P}^n}) - 2 = m + n - d - 2 \). Then as in the first part of iii), by Le Potier-Sommese vanishing theorem we have

\[
H^2(N_{X|\mathbb{P}^m \times \mathbb{P}^n}' ) \cong H^2(E_1') \oplus H^2(E_2') = 0,
\]

because \( d = \frac{m+n+t}{2} \) implies \( d - \text{rank}(E_i) - t \geq 2 \), for \( i = 1, 2 \). Then the statement follows from Theorem 1.6 ii). \( \square \)

2 Torsion-freeness of \( \text{Coker}(\alpha) \)

In this section we shall prove the following:

**Theorem 2.1** Let \( X \) be a submanifold of dimension \( d \) of \( \mathbb{P}^m \times \mathbb{P}^n \) (with \( m \geq n \geq 2 \)). If \( d \geq \frac{m+n+1}{2} \) then \( X \) is algebraically simply connected, the restriction map \( \alpha: \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \to \text{Pic}(X) \) is injective and \( \text{Coker}(\alpha) \) is torsion-free, where \( t \) is defined by formula (1) of the introduction.

We start with the following simple observation:

**Lemma 2.2** Let \( X \) be a positive closed irreducible subvariety of \( \mathbb{P}^m \times \mathbb{P}^n \) (\( m \geq n \geq 1 \)) such that at least one of the morphisms \( p_i: X \to p_i(X), i = 1, 2 \) has a positive-dimensional fiber, e.g. if \( d > n \). Then the restriction map \( \alpha: \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \to \text{Pic}(X) \) is injective.

**Proof.** Assume for instance that \( p_1 \) has a positive-dimensional fiber \( F \); by hypotheses we also have \( \text{dim}(F) > 0 \). Let \( (a, b) \in \mathbb{Z} \times \mathbb{Z} \) such that \( \mathcal{O}_X(a, b) \cong \mathcal{O}_F \). Then this isomorphism implies \( \mathcal{O}_X(a, b)|F \cong \mathcal{O}_F \), and since \( \mathcal{O}_X(a, b)|F \cong \mathcal{O}_F(a, b) \cong \mathcal{O}_F(b) \), we get \( b = 0 \) because \( \text{dim}(F) > 0 \) and the restriction map \( p_2|F: F \to p_2(F) \) is an isomorphism. Thus \( \mathcal{O}_X(a, b) = p_1^*(\mathcal{O}_{p_1(X)}(a)) \). Finally, from \( p_1^*(\mathcal{O}_{p_1(X)}(a)) \cong \mathcal{O}_X \) and \( \text{dim}(p_1(X)) > 0 \) we get \( a = 0 \) because if \( a \neq 0 \) one of the line bundles \( p_1^*(\mathcal{O}_{p_1(X)}(a)) \) or \( p_1^*(\mathcal{O}_{p_1(X)}(-a)) \) is \((d-1)\)-ample, while \( \mathcal{O}_X \) is not. \( \square \)

**Remark 2.3** In Lemma 2.2 the hypothesis that one of \( p_1 \) or \( p_2 \) has a positive-dimensional fiber is essential; indeed, if we take \( m = n \) and \( X \) the diagonal of \( \mathbb{P}^m \times \mathbb{P}^m \) then \( \text{Pic}(X) \cong \mathbb{Z} \), while \( \text{Pic}(\mathbb{P}^m \times \mathbb{P}^m) = \mathbb{Z} \times \mathbb{Z} \).

**Corollary 2.4** If \( X \) is a closed irreducible subvariety of \( \mathbb{P}^m \times \mathbb{P}^n \) of dimension \( d \geq \frac{m+n+t}{2} \) (with \( m \geq n \geq 2 \) and \( t \) given by formula (1)) then the restriction map \( \alpha: \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \to \text{Pic}(X) \) is injective.
Proof. We shall show that the hypotheses of Lemma 2.2 are fulfilled. First we observe that the hypotheses that $d \geq \frac{m+n+1}{2}$ and $m \geq n \geq 2$ imply $d > n$. Therefore by the theorem on the dimension of fibers we get $\dim p_1(X) > 0$ and the morphism $p_2 : X \to p_2(X)$ has all fibers positive-dimensional. Thus it remains to show that $\dim p_2(X) > 0$. Assuming $\dim(p_2(X)) = 0$, i.e. $X \subseteq \mathbb{P}^m \times \{p\} \cong \mathbb{P}^m$, with $p \in \mathbb{P}^n$, then $X \cong p_1(X)$, and in particular, $t = d \leq m$. Then the hypothesis that $d \geq \frac{m+n+1}{2}$ yields $d \geq m + n$, a contradiction. \hfill $\Box$

Now, to prove the non-trivial part of Theorem 2.1 we need some preparation. Let $X$ be a submanifold of $\mathbb{P}^m \times \mathbb{P}^n$ (with $m \geq n \geq 1$). Let us recall the join construction of $X$. Note that this construction has been already used in algebraic geometry in various circumstances, e.g. by Lascu and Scott in [17] to determine the behaviour of Chern classes undergoing a blowing up, by Deligne in [8] to simplify Fulton-Hansen connectedness theorem [10], and by the first named author in [5] to prove Lefschetz-type results for proper intersections. In the projective space $\mathbb{P}^{m+n+1} := \text{Proj}(k[x_0, \ldots, x_m, y_0, \ldots, y_n])$ consider the disjoint linear subspaces

$$L_1 := \left\{ [x_0, \ldots, x_m, y_0, \ldots, y_n] \in \mathbb{P}^{m+n+1} | x_0 = \cdots = x_m = 0 \right\},$$

$$L_2 := \left\{ [x_0, \ldots, x_m, y_0, \ldots, y_n] \in \mathbb{P}^{m+n+1} | y_0 = \cdots = y_n = 0 \right\},$$

and set $U := \mathbb{P}^{m+n+1} \setminus (L_1 \sqcup L_2)$. Consider also the rational map

$$\pi : \mathbb{P}^{m+n+1} \dashrightarrow \mathbb{P}^m \times \mathbb{P}^n$$

defined by

$$\pi([x_0, \ldots, x_m, y_0, \ldots, y_n]) := ([x_0, \ldots, x_m], [y_0, \ldots, y_n]).$$

Then $\pi$ is defined precisely on $U$ and it is the projection of a locally trivial $\mathbb{G}_m$-bundle (in the Zariski topology), where $\mathbb{G}_m$ is the multiplicative group of $k$. Observe also that the rational map $\pi_{L_1} := \pi_1 \circ \pi : \mathbb{P}^{m+n+1} \dashrightarrow \mathbb{P}^m$ (resp. $\pi_{L_2} := \pi_2 \circ \pi : \mathbb{P}^{m+n+1} \dashrightarrow \mathbb{P}^n$) is nothing but the linear projection of $\mathbb{P}^{m+n+1}$ of center $L_1$ (resp. the linear projection of $\mathbb{P}^{m+n+1}$ of center $L_2$). In particular, $\pi_{L_2}|L_1$ defines an isomorphism $L_1 \cong \mathbb{P}^n$ and $\pi_{L_1}|L_2$ an isomorphism $L_2 \cong \mathbb{P}^m$. Moreover,

$$\pi^*(\pi_1^*(\mathcal{O}_{\mathbb{P}^m}(1))) = \mathcal{O}_U(1) \quad \text{and} \quad \pi^*(\pi_2^*(\mathcal{O}_{\mathbb{P}^n}(1))) = \mathcal{O}_U(1).$$

In particular, for every closed irreducible subvariety $X$ of $\mathbb{P}^m \times \mathbb{P}^n$ one has

$$\pi_X^*(\mathcal{O}_X(1,0)) = \mathcal{O}_{U_X}(1) \quad \text{and} \quad \pi_X^*(\mathcal{O}_X(0,1)) = \mathcal{O}_{U_X}(1),$$

where $U_X := \pi^{-1}(X)$ and $\pi_X : U_X \to X$ the restriction of $\pi$. Since $\pi : U \to \mathbb{P}^m \times \mathbb{P}^n$ is a locally trivial $\mathbb{G}_m$-bundle, so is $\pi_X : U_X \to X$. In particular, $U_X$ is irreducible. Denote by $Y := U_X$ the closure of $U_X$ in $\mathbb{P}^{m+n+1}$, then

$$Y = U_X \sqcup Z,$$

with $Z = (Y \cap L_1) \sqcup (Y \cap L_2)$ (disjoint union). The following well known fact follows easily:

**Lemma 2.5** Let $X$ be a closed irreducible subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ (with $m \geq n \geq 2$) of dimension $d > 0$. In the above notation, one has $Y \cap L_1 \cong p_2(X)$ and $Y \cap L_2 \cong p_1(X)$. 
We also have the following well known lemma (which is very similar to Lemma 3 in [5]):

**Lemma 2.6** Under the above notation, let \( P_X := \mathbb{P}(\mathcal{O}_X(1,0) \oplus \mathcal{O}_X(0,1)) \) be the projective bundle associated to \( \mathcal{O}_X(1,0) \oplus \mathcal{O}_X(0,1) \), and denote by \( p_X : P_X \to X \) the canonical projection of \( P_X \). Then the variety \( U_X \) can be canonically embedded in \( P_X \) as an open dense subset such that the morphism \( \pi_X : U_X \to X \) extends to \( p_X : P_X \to X \), and the complement of \( U_X \) in \( P_X \) is the union of two irreducible effective divisors \( E_i' \) and \( E_2' \) with the property that \( p_X|E_i' \) defines an isomorphism between \( E_i' \) and \( X \), for \( i = 1,2 \). Moreover there is a canonical morphism \( h_X : P_X \to Y \) which is an isomorphism on \( U_X \), such that \( p_X = \pi_X \circ h_X \) and \( h_X(E_i') = Y \cap L_i, i = 1,2 \).

**Proof.** Let \( h : P \to \mathbb{P}^{m+n+1} \) be the blowing up of \( \mathbb{P}^{m+n+1} \) of center \( L_1 \sqcup L_2 \), and set \( E_i := h^{-1}(L_i), i = 1,2 \). Then it is well known (and easy to see) that \( P \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(1,0) \oplus \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(0,1)) \) and that \( E_1 \) and \( E_2 \) are disjoint sections of the canonical projection \( p : \mathbb{P}(\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(1,0) \oplus \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(0,1)) \to \mathbb{P}^m \times \mathbb{P}^n \).

By construction, \( h^{-1}(U) \cong U \) is the complement of \( E_1 \sqcup E_2 \). Set \( P_X := p^{-1}(X) \) and \( E_i' := E_i \cap P_X \) (scheme theoretic intersection), \( i = 1,2 \). Then it easy to check that \( P_X = p^{-1}(X) \) dominates \( Y \) and, together with \( E_1' \) and \( E_2' \), satisfies all the requirements of the lemma (see the proof of Lemma 3 in [5] for more details).

**Lemma 2.7** The map \( \pi_X : \text{Pic}(X) \to \text{Pic}(U_X) \) is surjective and \( \ker(\pi_X^*) \cong \mathbb{Z}[\mathcal{O}_X(1, -1)] \).

**Proof.** We first prove the following:

Claim 1. The equality \( \ker(\pi_X^*) \cong \mathbb{Z}[\mathcal{O}_X(1, -1)] \) holds if \( X = \mathbb{P}^m \times \mathbb{P}^n \).

Indeed, for every \( a,b \in \mathbb{Z} \) by [11] we have \( \pi^*(\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a,b)) = \mathcal{O}_U(a+b) \), whence \( \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a,b) \in \ker(\pi^*) \) if and only if \( \mathcal{O}_U(a+b) \cong \mathcal{O}_U \), i.e. if and only if \( a + b = 0 \) (since \( \text{codim}_{\mathbb{P}^{m+n+1}}L_i \geq 2 \) for \( i = 1,2 \), the restriction map \( \text{Pic}(\mathbb{P}^{m+n+1}) \to \text{Pic}(U) \) is an isomorphism).

We shall also need the following:

Claim 2. Under the notation of Lemma 2.6 one has \( \mathcal{O}_{P_X}(E_i' - E_2') \cong p_X^*(\mathcal{O}_X(a,-a)) \) for some \( a \in \mathbb{Z} \).

Indeed, according to the proof of Lemma 2.6 and the definition of \( E_i' \), it will be sufficient to show that \( \mathcal{O}_P(E_1 - E_2) \cong p^*(\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a,-a)) \) for some \( a \in \mathbb{Z} \). To prove this latter formula, since \( P \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(1,0) \oplus \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(0,1)) \) and since \( E_i \) is a section of the canonical projection \( p : P \to \mathbb{P}^m \times \mathbb{P}^n \), a well known formula for the Picard group yields

\[
\text{Pic}(P) \cong p^*(\text{Pic}(\mathbb{P}^m \times \mathbb{P}^n)) \oplus \mathbb{Z}[\mathcal{O}_P(E_i)], \quad i = 1,2.
\]

In particular, the subgroup \( p^*(\text{Pic}(\mathbb{P}^m \times \mathbb{P}^n)) \) of \( \text{Pic}(P) \) is identified with those line bundles \( L \) on \( P \) whose restriction to every fiber of \( p : P \to \mathbb{P}^m \times \mathbb{P}^n \) is trivial. Clearly, the restriction of \( L = \mathcal{O}_P(E_1 - E_2) \) to every fiber of \( p \) is trivial (every fiber of \( p \) is \( \mathbb{P}^1 \) and \( E_1 \) and \( E_2 \) are sections of \( p \)), whence \( \mathcal{O}_P(E_1 - E_2) \cong p^*(\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a,b)) \) for some \( a,b \in \mathbb{Z} \). Finally, since \( \mathcal{O}_P(E_1 - E_2)|P \setminus (E_1 \sqcup E_2) \cong \mathcal{O}_P|P \setminus (E_1 \sqcup E_2) \), it follows that \( \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a,b) \in \ker(\pi^*) \), whence by claim 1, \( b = -a \), which proves claim 2.
Now we can prove the equality \( \text{Ker}(\pi_X^\ast) = \mathbb{Z}[\mathcal{O}_X(1, -1)] \) in general. The inclusion \( \mathbb{Z}[\mathcal{O}_X(1, -1)] \subseteq \text{Ker}(\pi_X^\ast) \) follows from (11) as above. On the other hand, according to Lemma 2.6, consider the compactification \( p_X : P_X \to X \) of \( \pi_X : U_X \to X \), and let \( L \in \text{Ker}(\pi_X^\ast) \). Then \( p_X^\ast(L) \in \text{Pic}(P_X) \) is such that \( p_X^\ast(L)|_{U_X} \cong \mathcal{O}_{U_X} \). Since \( U_X = P_X \setminus (E'_1 \cup E'_2) \), with \( E'_1 \) and \( E'_2 \) effective irreducible divisors of \( P_X \), it follows that

\[
p_X^\ast(L) \cong \mathcal{O}_{P_X}(cE'_1 + dE'_2), \quad \text{with} \ c, d \in \mathbb{Z}.
\]

Combining claim 2 and (12), we get \( p_X^\ast(L \otimes \mathcal{O}_X(-ca, ca)) \cong \mathcal{O}_{P_X}((c + d)E'_2) \). Recalling that \( \text{Pic}(P_X) = p_X^\ast(\text{Pic}(X)) \oplus \mathbb{Z}[\mathcal{O}_{P_X}(E'_2)] \), we get \( L \otimes \mathcal{O}_X(-ca, ca) \cong \mathcal{O}_X \), or else, \( L \cong \mathcal{O}_X(ca, -ca) \subset \mathbb{Z}[\mathcal{O}_X(1, -1)] \). This proves the formula \( \text{Ker}(\pi_X^\ast) = \mathbb{Z}[\mathcal{O}_X(1, -1)] \) in general.

It remains to prove that the map \( \pi_X^\ast \) is surjective. To check this, consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Pic}(X) & \xrightarrow{id} & \text{Pic}(X) \\
\downarrow p_X^\ast & & \downarrow \pi_X^\ast \\
\text{Pic}(P_X) & \xrightarrow{i_X^\ast} & \text{Pic}(U_X)
\end{array}
\]

in which \( i_X^\ast \) is surjective (since by hypothesis \( X \) is nonsingular, whence \( P_X \) is also nonsingular, as a \( \mathbb{P}^1 \)-bundle over \( X \)) and \( p_X^\ast \) is injective (because \( \text{Pic}(P_X) = p_X^\ast(\text{Pic}(X)) \oplus \mathbb{Z}[\mathcal{O}_{P_X}(E'_1)] \)).

Let \( L \in \text{Pic}(U_X) \) be an arbitrary line bundle on \( U_X \). Since \( X \) is nonsingular, so is \( P_X \). In particular, \( i_X^\ast \) is surjective; therefore there is a line bundle \( \mathcal{T} \in \text{Pic}(P_X) \) such that \( \mathcal{T}|_{U_X} \cong L \). Moreover, \( \mathcal{T} \cong p_X^\ast(M) \otimes \mathcal{O}_{P_X}(kE'_1) \), with \( k \in \mathbb{Z} \) and \( M \in \text{Pic}(X) \), whence \( p_X^\ast(M) \cong \mathcal{T} \otimes \mathcal{O}_{P_X}(-kE'_1) \). It follows that

\[
p_X^\ast(M)|_{U_X} \cong \mathcal{T} \otimes \mathcal{O}_{P_X}(-kE'_1)|_{U_X} \cong \mathcal{T}|_{U_X} \cong L
\]

(because \( U_X \) is the complement of \( E'_1 \cup E'_2 \) in \( P_X \)), and since \( p_X \circ i_X = \pi_X \), we infer that \( p_X^\ast(M)|_{U_X} = \pi_X^\ast(M) \), i.e. \( \pi_X^\ast \) is surjective. \( \square \)

Now we need the following:

**Definition 2.8** Let \( Y \) be a closed subvariety of an irreducible quasi-projective variety \( X \). According to Grothendieck (see [12], cf. also [13]) we say that the pair \((X, Y)\) satisfies the Grothendieck-Lefschetz condition \( \text{Lef}(X, Y) \) if for every open subset \( V \) of \( X \) containing \( Y \) the functor \( E \to E' = E|_Y \), defined on the category of vector bundles on \( V \) into the category of vector bundles on \( Y \) on the formal completion \( X/Y = V/Y \) of \( X \) along \( Y \), is fully faithful. Equivalently, for every vector bundle \( E \) on \( V \), the canonical map \( H^0(V, E) \to H^0(X/Y, E') \) is an isomorphism. On the other hand, according to Hironaka–Matsumura (see [14], or [13], or also [3], page 95) we say that \( Y \) is \( G3 \) in \( X \) if the canonical map \( \alpha_{X/Y} : K(X) \to K(X/Y) \), defined on the field of rational functions of \( X \) into the ring of formal-rational functions on \( X \) along \( Y \), is an isomorphism.

We shall use the following result of Hironaka–Matsumura (see [14], cf also [13]):
**Theorem 2.9 (Hironaka–Matsumura)** Let $Y$ be a connected positive dimensional subvariety of the projective space $\mathbb{P}^N$. Then $Y$ is $G3$ in $\mathbb{P}^N$.

**Corollary 2.10** Let $Y$ be a closed irreducible subvariety of $\mathbb{P}^N$ of dimension $s \geq 2$, and let $Z \subseteq Y$ be a closed subscheme. If $\text{codim}_Y(Z) \geq 2$ then $Y \setminus Z$ is $G3$ in $\mathbb{P}^N \setminus Z$.

**Proof.** If $Z = \emptyset$ then this is Theorem 2.9. Assume $Z \neq \emptyset$, and let $L = L_{N-s+1}$ be a general linear subspace of $\mathbb{P}^N$ of dimension $N-s+1$. Then $Z \cap L = \emptyset$ and $C := Y \cap L$ is a projective irreducible curve on $Y$ (by Bertini’s theorem).

By Theorem 2.9 $C$ is $G3$ in $\mathbb{P}^N$, i.e. the canonical map $\alpha_{\mathbb{P}^N,C} : K(\mathbb{P}^N) \to K(\mathbb{P}^N/C)$ is an isomorphism. Consider the commutative diagram:

\[
\begin{array}{ccc}
K(\mathbb{P}^N \setminus Z) & \xrightarrow{\alpha_{\mathbb{P}^N \setminus Z,Y \setminus Z}} & K((\mathbb{P}^N \setminus Z)/Y \setminus Z) \\
\cong & & \\
K(\mathbb{P}^N) & \xrightarrow{\alpha_{\mathbb{P}^N,C}} & K(\mathbb{P}^N/C)
\end{array}
\]

where $\varphi$ is the canonical restriction map. Note that since $\mathbb{P}^N \setminus Z$ is smooth and $Y \setminus Z$ is irreducible $K((\mathbb{P}^N \setminus Z)/Y \setminus Z)$ is a field by [13], cf. also [3], Corollary 9.10. Hence the map $\varphi$ is injective and consequently $\alpha_{\mathbb{P}^N \setminus Z,Y \setminus Z}$ is an isomorphism (because $\alpha_{\mathbb{P}^N,C}$ is so).

**Lemma 2.11** Under the hypotheses of Theorem 2.1 and the notation of Lemma 2.6 the Grothendieck-Lefschetz condition $\text{Lef}(U, U_X)$ holds.

**Proof.** We need the following two claims:

**Claim 1.** Let $U'$ be a smooth quasi-projective irreducible variety of dimension $\geq 2$, and let $W$ be a closed subvariety of $U'$. Assume that $W$ is $G3$ in $U'$ and that $W$ intersects every hypersurface of $U'$. Then $\text{Lef}(U', W)$ holds.

Claim 1 is a result of Hartshorne and Speiser (see [13], Proposition 2.1, page 200) in the case when $U'$ is a projective and nonsingular. Practically the same proof given in [3], page 113, (with minor changes) works in our situation as well (cf. also [3], page 113).

**Claim 2.** One has $\text{codim}_U(U \setminus V) \geq 2$ for every open subset $V$ of $U$ containing $U_X$.

Indeed claim 2 is equivalent to proving that $\dim(U \setminus V) \leq \dim(U) - 2 = m + n - 1$; since $U \setminus V$ is open in $\mathbb{P}^{m+n+1} \setminus V$, it is enough to show that $\dim(\mathbb{P}^{m+n+1} \setminus V) \leq m + n - 1$. Assume that there is an irreducible hypersurface $H$ of $\mathbb{P}^{m+n+1}$ such that $H \subseteq \mathbb{P}^{m+n+1} \setminus V$. Then $H \cap V = \emptyset$, whence $H \cap U_X = \emptyset$ (because $U_X \subseteq V$). This yields

\[H \cap U_X = H \cap Y \subseteq Y \setminus U_X,\]

and therefore $\dim(H \cap Y) \leq \dim(Y \setminus U_X) = t$, because $Y \setminus U_X \cong p_1(X) \cup p_2(X)$ by Lemma 2.5. Thus:

\[t \geq \dim(H \cap Y) \geq \dim(Y) - 1 = d.\]

Combining $t \geq d$ with the hypothesis $d \geq \frac{m+n+t+1}{2}$ one gets the absurd inequality $t \geq m + n + 1$. This proves claim 2.
Now using Corollary 2.10 and these two claims we can easily prove Lemma 2.11. In fact in Corollary 2.10 we take \( N = m + n + 1, Y = U_X \) and \( Z = Y \setminus U_X \cong p_1(X) \cup p_2(X) \) (by Lemma 2.5). Clearly, \( \text{codim}_Y(Z) \geq 2 \), whence by claim 1, \( U_X \) is \( G_3 \) in \( U \). By claim 2, \( \text{codim}_U(U \setminus V) \geq 2 \) for every open neighbourhood \( V \) of \( U_X \) in \( U \), i.e. \( U_X \) intersects every hypersurface of \( U \). Then the conclusion follows from claim 1, taking \( U' = U \) and \( W = U_X \). □

Now we come back to prove the non-trivial parts of Theorem 2.1, i.e. the fact that \( \text{Coker}(\alpha) \) is torsion-free and \( X \) is algebraically simply connected. The main technical ingredient is the following result of Faltings:

**Theorem 2.12 (Faltings [9], Corollary 5)** Let \( Y \) be a closed irreducible subvariety of \( \mathbb{P}^N \), and let \( Z \subseteq Y \) be a closed subscheme. If \( \dim(Y) \geq 1 + \frac{N + \dim Z}{2} \) (with the convention that \( \dim(Z) = -1 \) if \( Z = \emptyset \)) then every formal vector bundle \( E \) on the formal completion \( (\mathbb{P}^N \setminus Z) \times Y \setminus Z \) of \( \mathbb{P}^N \setminus Z \) along \( Y \setminus Z \) is algebraisable, i.e. there is an open subset \( U \) of \( \mathbb{P}^N \setminus Z \) containing \( Y \setminus Z \) and a vector bundle \( E \) on \( U \) such that the formal completion of \( E \) along \( Y \setminus Z \) is isomorphic to \( E \).

**Corollary 2.13** Let \( X \) be a closed irreducible subvariety of \( \mathbb{P}^m \times \mathbb{P}^n \) (with \( m \geq n \geq 2 \)) of dimension \( d := \dim(X) \geq \frac{m + n + t + 1}{2} \), with \( t \) defined by (1). Then, under the notation of Lemma 2.6, every formal vector bundle on \( U/U_X \) is algebraisable.

**Proof.** We apply Theorem 2.12 to \( Y = U_X \subset \mathbb{P}^N \) and \( Z = Y \setminus U_X \), with \( N = m + n + 1 \). By Lemma 2.5 \( Z \cong p_1(X) \cup p_2(X) \), whence \( \dim(Z) = t \). Then the hypothesis \( d \geq \frac{m + n + t + 1}{2} \) translates into \( \dim(Y) = d + 1 \geq 1 + \frac{N + \dim Z}{2} \). Then the conclusion of the corollary follows from Theorem 2.12. □

**Proof of Theorem 2.1.** In view of Corollary 2.13 it remains to prove that \( X \) is algebraically simply connected and that \( \text{Coker}(\alpha) \) is torsion-free. We first prove that \( \text{Coker}(\alpha) \) is torsion-free. Consider the following commutative diagram

```
0 \longrightarrow \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \overset{\alpha}{\longrightarrow} \text{Pic}(X) \longrightarrow \text{Coker}(\alpha) \longrightarrow 0
\downarrow \pi^* \quad \downarrow \pi^* \quad \downarrow \pi
0 \longrightarrow \text{Pic}(U) \overset{\beta}{\longrightarrow} \text{Pic}(U_X) \longrightarrow \text{Coker}(\beta) \longrightarrow 0
```

Now we come back to prove the non-trivial parts of Theorem 2.1, i.e. the fact that \( \text{Coker}(\alpha) \) is torsion-free and \( X \) is algebraically simply connected. The main technical ingredient is the following result of Faltings:
A Barth-Lefschetz theorem

in which:

i) the first two columns are exact by Lemma 2.7

ii) the map $\alpha$ is injective by Corollary 2.4 whence the middle row is exact,

iii) the third row is also exact (the injectivity of the map $\beta$ comes from the injectivity of $\alpha$ and from the fact that the first two columns are exact, taking into account that the top horizontal map is an isomorphism).

Since the top horizontal map is an isomorphism, from this diagram with exact rows and columns it follows that the map

$$\pi: \text{Coker}(\alpha) \to \text{Coker}(\beta)$$

is also an isomorphism. Thus the proof of the theorem reduces to proving the following:

(*) $\text{Coker}(\beta)$ is torsion-free.

To check (*) the crucial point is the following:

Claim. The canonical map $\gamma: \text{Pic}(U) \to \text{Pic}(U/U_X)$ is surjective.

To prove the claim let $\mathcal{L}$ be a line bundle on $U/U_X$. By Corollary 2.13 there exists an open subset $V$ of $U$ containing $U_X$ and a line bundle $L$ on $V$ such that the completion $\hat{L}$ of $L$ along $U_X$ is isomorphic to $\mathcal{L}$. Since $U$ is nonsingular, $L$ can be extended to a line bundle $L'$ on $U$ which still satisfies $\hat{L'} \cong \mathcal{L}$. This proves the claim. (Actually using claim 2 of the proof of Lemma 2.11 it follows easily that $\gamma$ is also injective, but we don’t need this fact here.)

Now we have the commutative diagram with natural arrows

\[
\begin{array}{ccc}
\text{Pic}(U) & \xrightarrow{\gamma} & \text{Pic}(U/U_X) \\
& \searrow^{\beta} & \text{Pic}(U_X) \\
& & \text{Pic}(U_X)
\end{array}
\]

By the above claim $\beta$ is surjective, so $\text{Coker}(\beta) = \text{Coker}(\text{Pic}(U/U_X) \to \text{Pic}(U_X))$. Thus (*) translates into:

(**) $\text{Coker}(\text{Pic}(U/U_X) \to \text{Pic}(U_X))$ is torsion-free.

But (** is a general well known fact (see [9], cf also [3], Proposition 10.10), see also Corollary 2.15 below.

We finally prove that $X$ is algebraically simply connected. This can be done in two different ways:

First proof of simply connectedness of $X$. We first claim that it is enough to prove that $U_X$ is algebraically simply connected. Indeed since $\pi_X: U_X \to X$ is a locally trivial $\mathbb{G}_m$-bundle, by [11], XIII, Example 4.4 and Proposition 4.1, there exists an exact sequence of algebraic fundamental groups associated to $\pi_X$

$$\pi_1^{\text{alg}}(U_X) \to \pi_1^{\text{alg}}(X) \to 1,$$
from which it follows that if $U_X$ is algebraically simply connected, so is $X$.

Now, from Lemma 2.11 and Corollary 2.13 it follows (in the terminology of [12]) that the effective Grothendieck-Lefschetz condition $\text{Leff}(U, U_X)$ holds. Since $U$ is smooth, we can therefore apply Theorem 3.10 of [12]. Exposé X, to deduce that the natural map $\pi_1^{\text{alg}}(U_X) \to \pi_1^{\text{alg}}(U)$ is an isomorphism. Finally, since the complement $L_1 \cup L_2$ of $U$ in $\mathbb{P}^{m+n+1}$ is of codimension $\geq 2$ it follows that $U$ is algebraically simply connected, and therefore $U_X$ is algebraically simply connected as well.

**Second proof of simply connectedness of $X$.** We shall show that the simply connectedness of $X$ is a consequence of a result of Debarre (see [7], Corollary 2.4). For this, according with Debarre’s result (loc. cit.) it is sufficient to show that $\dim(X) > \frac{m+n}{2}$, $\dim p_1(X) > \frac{n}{2}$ and $\dim p_2(X) > \frac{n}{2}$. The first inequality follows from the hypothesis that $d \geq \frac{m+n^2+1}{2}$, so it remains to check the last two inequalities. Assume first that $\dim p_2(X) \leq \frac{m}{2}$, and let $F_i$ be a general fiber of $p_i: X \to p_i(X)$, $i = 1, 2$. Then by the theorem of dimension of fibers,

$$d = \dim(F_2) + \dim p_2(X) \leq \dim(F_2) + \frac{n}{2} \leq t + \frac{n}{2},$$

or else, using the hypothesis, we get $m + n + t + 1 \leq 2d \leq 2t + n$. It follows that $t \geq m+1$, which is absurd because $m \geq n$.

Assume now that $\dim p_1(X) \leq \frac{m}{2}$. Then exactly as above we get $t \geq n+1$, and in particular, $t = \dim p_1(X)$. Thus $n + 1 \leq t \leq \frac{m}{2}$, i.e. $m \geq 2n + 2$. Moreover, since $X \subseteq p_1(X) \times p_2(X)$ we get $\dim(X) \leq \dim p_1(X) + \dim p_2(X)$. Thus $d \leq \frac{m}{2} + n$. Putting everything together we get

$$m + n + t + 1 \leq 2d \leq m + 2n,$$

or else, $n \geq t + 1$, which contradicts the previous inequality $t \geq n+1$.

This completes the second proof of the fact that $X$ is algebraically simply connected, and thereby (modulo two standard facts, namely Lemma 2.14 below and Lemma 2.11) the proof of Theorem 2.1.

**Lemma 2.14** Let $W$ be a closed subvariety of an irreducible algebraic variety $V$ over a field $k$ of characteristic zero. Then for every formal line bundle $\mathcal{L} \in \text{Pic}(V/W)$ such that $\mathcal{L}|W \cong M^e$ for some $M \in \text{Pic}(W)$ and some integer $e \geq 1$, there exists a formal line bundle $\mathcal{M} \in \text{Pic}(V/W)$ such that $\mathcal{L} \cong \mathcal{M}^e$ and $\mathcal{M}|W \cong M$. If the characteristic of $k$ is $p > 0$ then the same conclusion holds provided $e$ is prime to $p$.

The proof of this lemma is completely standard and works by induction using infinitesimal neighbourhoods (see [9], cf also [3], page 115, Proposition 10.10). Note that in loc cit. one assumes that $V$ is projective, but this fact is not really used in the proof. In fact the cohomology spaces occurring in the proof are $k$-vector spaces which might be infinite-dimensional. The only fact which is used is that the underlying additive group of a (possibly infinite-dimensional) vector space over a field of characteristic zero is torsion-free and (uniquely) divisible. An obvious consequence of Lemma 2.14 is the following:

**Corollary 2.15** The abelian group $\text{Coker}(\text{Pic}(V/W) \to \text{Pic}(W))$ is torsion-free if the characteristic of $k$ is zero, and has no $e$-torsion for every positive integer $e$ which is prime to the characteristic $p$ of $k$ if $p > 0$.
Remarks 2.16 i) Under the hypothesis of Theorem 2.1 if \( t = m \) (i.e. \( \dim(X) \geq \frac{2m+n+1}{2} \) or, equivalently, \( \text{codim}_{\mathbb{P}^m \times \mathbb{P}^n}(X) \leq \frac{n-1}{2} \)) the result is already known as a consequence of a more general theorem due to Sommese ([21]).

ii) If in Theorem 2.1 the characteristic of the ground field is \( p > 0 \), then the fact that \( X \) is algebraically simply connected still holds (with the same arguments). Moreover, the map \( \alpha \) is injective and \( \text{Coker}(\alpha) \) has no \( e \)-torsion for every positive integer \( e \) which is prime to \( p \).

iii) Both proofs of simply connectedness of \( X \) work even in the case when \( X \) is singular.

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