A MSWF root-MUSIC based on Pseudo-noise resampling technique

M. Johnny¹,²,³ and M. R. Aref²
¹Department of electrical Engineering, Islamic Azad University, Science and Research Branch, Tehran, Iran
²Department of electrical Engineering, Sharif University of Technology, Tehran, Iran
³Correspondence
M. Johnny, Department of electrical Engineering, Islamic Azad University, Science and Research Branch, Tehran, Iran.
Email: Maryam.Johnny@gmail.com

This paper uses the shift-invariance property of uniform linear array in root-MUSIC estimator for obtaining signal and noise subspaces by applying multistage Wiener filter (MSWF) procedure. Also, the MSWF root-MUSIC based on the pseudo-noise resampling process for estimating the direction of arrival (DOA) of signals is proposed. By this process, a root estimator bank and a corresponding DOA estimator bank are constructed. Then, a hypothesis test is applied to the DOA estimator bank to detect the normal DOA estimators from abnormal DOA estimators called outliers. By averaging the corresponding root estimators of normal DOA estimators, the final DOAs can be determined more accurately. When all the DOA estimators fail to pass the hypothesis test, the criterion based on the Gaussian weight average of the root estimator bank is introduced. By applying this criterion, better outlier-free performance of MSWF root-MUSIC can be obtained. Simulations show that our method can improve the DOA estimations, especially in small sample sizes and low signal-to-noise ratios.

Introduction: Direction of arrival (DOA) estimation is the main topic in array signal processing for localizing radiating sources and it has a lot of applications in radar and wireless communications [1–4]. Subspace based methods such as multiple signal classification (MUSIC)[4], root-MUSIC [5], beamspace root-MUSIC[6], and estimation signal parameter via a rotational invariant technique (ESPRIT) [7] are based on eigenvalue decomposition (EVD) or singular value decomposition (SVD) on array output covariance matrix (AOCM). To reduce the complexity, real valued estimators including unitary root-MUSIC [8], unitary root-MUSIC with second forward/backward averaging [9] and unitary ESPRIT were proposed. These methods use a real-valued EVD/SVD computation on either the real part of AOCM (R-AOCM) or the imaginary part of AOCM (I-AOCM), but they still require O(p³) flops where p denotes the number of sensors. When massive arrays are used [10], p can be a very large number and this term of high complexity is unacceptable. Also, these subspace methods suffer considerable performance degradation when the signal-to-noise ratio or the number of snapshots is small [11, 12], which is known as a threshold effect. Numerous authors have attempted to lower this effect by pseudo-noise resampling (PR) technique [11–13]. To accurately estimate the DOA of signals with reduced computational burden and reduced SNR threshold, we propose a MSWF root-MUSIC based on PR process. Our method uses a PR process to construct a root estimator bank and a corresponding DOA estimator bank. Then, a hypothesis test [12, 13][14] is applied to the whole DOA estimator bank to detect the normal DOA estimators from abnormal DOA estimators. If there is no retained DOA estimator, that is, no normal DOA estimators, the final DOAs can be determined more accurately. When all the DOA estimators fail to pass the hypothesis test, we propose a new criterion for determining the final DOAs. This criterion uses the argument of the Gaussian weight average of the root estimator bank.

Problem formulation: Assume q narrowband far-field sources are impinging on a ULA with p isotropic sensors. The signals and noises are assumed stationary and uncorrelated random processes. Also, the noises are spatially and temporally white. The output of sensors is as follows:

\[ \mathbf{x}(t) = \mathbf{A}(\theta)\mathbf{s}(t) + \mathbf{n}(t), \quad t = 1, \ldots, N, \]  

(1)
where
\[ A(\theta) = [a(\theta_1), \ldots, a(\theta_q)]. \]
(2)
is the steering matrix,
\[ s = [s_1(t), \ldots, s_q(t)]^T, \]
(3)
is the source signal vector.
\[ a(\theta_q) = [1, e^{i \theta_q}, \ldots, e^{i q \theta_q}]^T. \]
(4)
is the steering vector due to qth source where \( \cdot^T \) is the transpose operator, \( \lambda \) is the carrier wavelength and \( d \) is the inter-element spacing. Given \( N \) i.i.d. snapshots, \( x(1), \ldots, x(N) \), the sample covariance matrix is obtained as follows:
\[ \hat{R}_s = \frac{1}{N} \sum_{t=1}^{N} x(t)x(t)^H. \]
(6)
By eigenvalue decomposition of \( \hat{R}_s \), we have:
\[ \hat{R}_s = \hat{P}_s \hat{A}_s \hat{P}_s^H + \hat{P}_n \hat{A}_n \hat{P}_n^H, \]
(7)
where the columns of \( p \times q \) and \( p \times (p-q) \) matrices \( \hat{P}_s \) and \( \hat{P}_n \) contain the signal and noise eigenvectors.

**MSWF root-MUSIC:** As shown in [15], given a reference signal \( d_0(t) \), the MSWF can partition the observation data \( x(t) \) step by step to provide \( q \) desired signals \( d_i(t) \) and their orthogonal components \( \hat{x}(t) \) which at stage \( i, \)
\[ d_i(t) = h_i^H \hat{x}_{i-1}(t), \]
(8)
\[ \hat{x}(t) = x_{i-1}(t) - |h_i|d_i(t) = (1 - |h_i|)x_{i-1}(t) = B_i x_{i-1}(t). \]
(9)
In above formulas, \( h_i \) is the matched filter and \( B_i = 1 - |h_i|^2 \) is the blocking matrix. The matched filter \( h_i \) can be obtained by the normalization of cross-correlation between \( x_{i-1}(t) \) and \( d_{i-1}(t) \) as follows:
\[ h_i = \frac{E[x_{i-1}(t)d_{i-1}^*(t)]}{||E[x_{i-1}(t)d_{i-1}^*(t)]||}, \]
(10)
where \( ||.|| \) indicates the vector norm. In [16], the authors use MSWF for subspace decomposition of a linear array consisting of \( p \) sensors. They use the output of the first sensor as a reference signal \( d_0(t) \) and the output of \( p-1 \) remaining sensors as the observation data vector \( x_0(t) \).
\[ d_0(t) = x_1(t) = s^T(t)l + n_1(t), \]
(11)
\[ x_0(t) = [x_2(t), \ldots, x_p(t)] = A(\theta) x(t) + \hat{n}^T(t), \]
(12)
where \( l = [1, 1, \ldots, 1]^T \). By partitioning \( x(t) \) similar to that of MSWF procedure, \( p-1 \) desired signals and their orthogonal components can be attained. So, we have \( p-1 \) matched filters \( h_i, i = 1, \ldots, p-1 \) computed from the equation 10. By using the shift-invariance property of the Krylov subspace, the authors have demonstrated that for \( q \) uncorrelated narrowband signals impinging upon the array with \( p \) sensors
\[ \mathbf{S}^{(0)} = \text{span}\{h_1, h_2, \ldots, h_p\}, \]
(13)
\[ \mathbf{N}^{(p-1-q)} = \text{span}\{h_{q+1}, \ldots, h_{p-1}\}, \]
(14)
are the signal and noise subspaces respectively. So we can express the root-MUSIC polynomial as follows:
\[ F_{\text{root-MUSIC}} = a_i^H(z) \mathbf{E}_x \mathbf{E}_n^H a_i(z), \]
(15)
where \( a_i \) is obtained by removing the first row of \( a(\theta) \) and \( \mathbf{E}_x = [h_{q+1}, \ldots, h_p]. \)
(16)
\[ z_j = e^{j 2 \pi \lambda d \sin(\theta_j)}/d \]
denotes the roots of (15). After selecting \( p \) roots lying inside the unit circle, we can determine the DOA of signals from the \( q \) roots that are closest to the unit circle as follows:
\[ \hat{\theta}_j = \sin^{-1} \frac{\arg(z_j)}{2 \pi d}, i = 1, \ldots, q. \]
(17)
Since the calculation of matched filter \( h_i(t) \) requires \( p \) complex multiplications and \( p-1 \) additions in each snapshot, by assuming \( N \) snapshots we need \( O(pN) \) flops. So the total number of computations for obtaining \( \mathbf{E}_x \) is about \( O(p^2N) \) flops in contrast to EVD method that needs \( O(p^3N) + O(p^2) \) flops.

**Proposed method:** In this section, the PR-based MSWF root-MUSIC is presented. First, we use the MSWF root-MUSIC estimator to obtain \( q \) DOA estimates as final DOAs, or else the PR technique with two new strategies is used.

**Pseudo-noise resampling technique:** The idea of this technique is to perturb the original measured data matrix \( \mathbf{X} = [x(t_1), \ldots, x(t_N)] \) by means of artificially generated pseudo random noise as follows:
\[ \mathbf{Y}_i = \mathbf{X} + \mathbf{Z}_i, \]
(18)
where \( \mathbf{Y}_i = [y_i(t_1), \ldots, y_i(t_N)] \) is the \( p \times N \) resampled data matrix, \( i = 1, \ldots, F. \) \( \mathbf{Z}_i \) is the matrix of independent zero mean circular pseudo-noise obtained by a Gaussian random generator such that \( E(\mathbf{Z}) = 0, E(\mathbf{Z}\mathbf{Z}^H) = (\sigma^2 I_N) \mathbf{1}, E(\mathbf{Z}\mathbf{Z}^H) = 0. \) To maintain an acceptable signal-to-noise ratio (SNR) in the resampled data, the variance of pseudo-noise should be approximately the same as the variance of the original noise \( \sigma_n^2. \) For estimating \( \hat{\sigma}_n^2, \) we use a diagonal matrix \( \mathbf{R}_n. \) This matrix is constructed from the variations of desired signals of the MSWF \( \mathbf{R}_n^q, (i = q + 1, \ldots, p) \) after the \( p \)-th stage as follows:
\[ \mathbf{R}_n = \text{diag}(\sigma_{n_{q+1}}^2, \ldots, \sigma_{n_p}^2), \]
(19)
\[ \sigma_{n_i}^2 = E[d_i(t)d_i^*(t)], i = q + 1, \ldots, p. \]
(20)
So
\[ \hat{\sigma}_n^2 = \frac{1}{p-q} tr(\mathbf{R}_n). \]
(21)
As in [16], these \( d_i(t), (i = q + 1, \ldots, p) \) are uncorrelated with each other,
\[ E[d_i(t)d_j^*(t)] = 0 (i \neq j), \]
(22)
and their variances equal the noise variance.
For each resampling run, the MSWF root-MUSIC is done to obtain a root estimator $\hat{Z}$ and the corresponding DOA estimator $\hat{\theta}$ each contains $q$ roots and $q$ DOA estimates, respectively. In some resampling runs, the original noise $N = [n(t_1), \ldots, n(t_N)]$ is permuted in a favourable way by pseudo-noise $\xi_t$ for the exploited DOA estimator. By applying a hypothesis test $H$, we can select these successfully resampled estimators in such runs and improve the DOA estimation performance. This hypothesis is defined as follows:

$$H: \text{All the } q \text{ DOA estimates in a DOA estimator are localized in } \hat{\Theta},$$

$$\hat{\Theta} = [\hat{\theta}_1^{\text{left}}, \ldots, \hat{\theta}_1^{\text{right}}] \cup \ldots \cup [\hat{\theta}_q^{\text{left}}, \hat{\theta}_q^{\text{right}}],$$

(23)

where $\hat{\theta}_i^{\text{left}}, i = 1, \ldots, q$, are the highest peaks of the conventional beamformer output $[11, 13]$ and $\hat{\theta}_i^{\text{right}}$ are the left and right boundaries of the $q$th subinterval. These boundaries are chosen as angular distances between the maximum of the $q$th peak and the left/right neighbor point with 3 dB drop, respectively.

### DOA estimation strategy:

After applying the MSWF root-MUSIC for each resampling run, a root estimator $\hat{Z}$ and the corresponding DOA estimator $\hat{\Theta}$ are obtained. So the $i$th estimators be

$$\hat{Z}_i = \{\hat{z}_i^{(1)}, \ldots, \hat{z}_i^{(q)}\}, \hat{\Theta}_i = \{\hat{\theta}_i^{(1)}, \ldots, \hat{\theta}_i^{(q)}\},$$

(24)

where $\hat{z}_i^{(j)}$ is the root corresponding to $\hat{\theta}_i^{(j)}$ and $\hat{\theta}_i^{(j)} \leq \hat{\theta}_i^{(j+1)} \leq \ldots \leq \hat{\theta}_i^{(q)}$. So after $F$ PR runs, two estimator banks are constructed as follows:

$$\beta_\ell = [\hat{Z}_1^{(\ell)}, \ldots, \hat{Z}_J^{(\ell)}], \beta_\ell = [\hat{\Theta}_1^{(\ell)}, \ldots, \hat{\Theta}_F^{(\ell)}],$$

(25)

By applying the reliability test $H$ to $\beta_\ell$ the following two subsets

$$\beta_{p,\ell} = (\hat{\Theta}_1^{(\ell)}, \ldots, \hat{\Theta}_J^{(\ell)}), \beta_{r,\ell} = (\hat{\Theta}_1^{(\ell)}, \ldots, \hat{\Theta}_F^{(\ell)})$$

(26)

are yielded where $\beta_{p,\ell}$ consists of $J_i$ DOA estimators that are passed by $H$ and $\beta_{r,\ell}$ contains the DOA estimators that are rejected by $H$. Also

$$\hat{\theta}_{p,\ell} = \{\hat{\theta}_1^{(1)}, \ldots, \hat{\theta}_J^{(1)}\}, \hat{\theta}_{r,\ell} = \{\hat{\theta}_1^{(1)}, \ldots, \hat{\theta}_F^{(1)}\}$$

(27)

are their corresponding root estimators. In (26) and (27),

$$\hat{\Theta}_J^{(\ell)} = [\hat{\theta}_1^{(1)}, \ldots, \hat{\theta}_J^{(1)}], \hat{\Theta}_F^{(\ell)} = [\hat{\theta}_1^{(1)}, \ldots, \hat{\theta}_F^{(1)}],$$

(28)

$$\hat{Z}_J^{(\ell)} = [\hat{z}_1^{(1)}, \ldots, \hat{z}_J^{(1)}], \hat{Z}_F^{(\ell)} = [\hat{z}_1^{(1)}, \ldots, \hat{z}_F^{(1)}]$$

(29)

represent the $i$th DOA estimator and corresponding root estimator accepted by $H$. If $0 < J_i < F$, we calculate the $q$th DOA by the argument of the following new criterion:

$$\hat{\theta}_q = \text{arg } \hat{z}_q$$

(30)

$$\hat{z}_q = \frac{1}{J_i} \sum_{j=1}^{J_i} \hat{z}_j^{(1)} e^{i \hat{\theta}_j^{(1)}} \quad (31)$$

If all the member of the DOA estimator bank in (25) do not satisfy the reliability test $J_i = 0$, we propose another criterion that exploits the Gaussian weight average (GWA) of root estimator bank. The steps of this strategy are as follows:

1) $\beta_{\ell,\ell}$ is divided into $q$ subsets, with each subset contains $F$ roots corresponding to a DOA. In this case, the $i$th subset is expressed as follows:

$$G_{i,\ell} = [\hat{z}_i^{(1)}, \ldots, \hat{z}_i^{(q)}], \quad i = 1, \ldots, q.$$  

(32)

2) The modulus of each $\hat{z}_i$ in (32) is calculated.

$$|G_{i,\ell}| = [\hat{z}_i^{(1)}, \ldots, \hat{z}_i^{(q)}], \quad i = 1, \ldots, q.$$  

(33)

3) The final root corresponding to $i$th DOA is computed by the Gaussian weight average of $G_i$ as follows:

$$Z_{i,\ell,\text{final}} = \sum_{j=1}^{q} e^{-\alpha(\hat{z}_j^{(1)} - \hat{z}_i^{(1)})^2} / \sum_{j=1}^{q} e^{-\alpha(\hat{z}_j^{(1)} - \hat{z}_i^{(1)})^2}, \quad 0 \leq \alpha < \infty.$$  

(34)

4) Substitute $Z_{i,\ell,\text{final}}$ into (17) to obtain the final DOA estimate.

In (34), $\alpha$ is a controlling parameter. In each subset, that is, $|G_{i,\ell}|$, some roots are far from the unit circle and some roots are close to the unit circle. By selecting $\alpha \geq 0$ in (34), we can increase the effect of roots close to the unit circle and reduce the effect of roots that are far from the unit circle. Thus, all the DOA estimates determined by our method are always closest to the true DOAs. The effectiveness of our method will be further verified in simulation section.

### Simulation:

In this section, some simulations have been carried out to show the effectiveness of our method. In all simulations, we assume that the source localization sectors and the number of sources are known or estimated by [16–17]. The array is a ULA with 8 sensors spaced at $d = \lambda/2$. The sources are two far-field and uncorrelated narrowband Gaussian signals with the same SNR. The additive noise is assumed to be a white stationary Gaussian random process uncorrelated with signals. The vector $\Theta$ denotes the DOAs. In the first simulation, we organize two experimental settings as follows:

Setting 1 (see Figure 1): $N = 100$, $\Theta = [20, 25]$, $\text{SNR} = [-7]$  
Setting 2 (see Figure 2): $N \in [10 1000]$, $\Theta = [20, 25]$, $\text{SNR} = -2$ dB.

In these settings, we compare the MSWF root-MUSIC (MSWF-RM) with EVD root-MUSIC (EVD-RM), unitary root-MUSIC (U-RM), second forward/backward unitary root-MUSIC (SBF-U-RM) and real-valued root-MUSIC (RV-RM) in terms of root mean square error (RMSE). 1000 Monte Carlo simulations are carried out to evaluate the RMSE as follows:

$$\text{RMSE} = \sqrt{\frac{1}{1000q} \sum_{i=1}^{q} \sum_{j=1}^{1000} (\hat{\theta}_{i,j} - \theta_i)^2}.$$  

(35)

For RMSE performance comparisons, the unconditional Cramer–Rao lower bound (CRLB) [8] is plotted as a benchmark. Figures 1 and 2 show that the MSWF root-MUSIC outperforms EVD root-MUSIC and RV root-MUSIC. In the next simulation, we set $\sigma$ to one and vary the signal power from $-7$ to $7$ dB. The number of snapshots is $N = 100$. According to (23), we use the conventional beamformer to pre-estimate $\hat{\Theta}$ in each independent run. The performance of our method with $\alpha = 1$ is examined with a different number of the PR process. The RMSEs are shown in Figure 3. As it is obvious, a relatively larger $F$ can lead to better performance for our method, especially in low SNR. At high SNR, since all the DOA estimates are passed by the reliability test $H$, our proposed method is reduced to be the MSWF root-MUSIC one. Also, to illustrate the effectiveness of our GWA criterion in determining the final DOAs, we add another GWA based algorithm for comparison which is obtained by replacing the median average method [12, 13] in PR root-MUSIC with the proposed GWA. For all the PR algorithms, we set $F = 10$. As Figure 3 shows, the GWA criterion for PR root-MUSIC provides a considerable improvement than the median one at low SNRs. Let us now study the RMSE of the proposed method versus the number of snapshots. For this simulation, we fix the SNR to $-2$ dB. As shown in Figure 4, for $N < 100$, our method achieves better performance than PR root-MUSIC. Also, the effectiveness of using the GWA criterion instead of the median average method in PR root-MUSIC versus $N$ has been shown. In the last simulation, the comparison of our method with PR unitary ESPRIT versus SNRs and $N$ is made. Figures 3 and 4 show the results of this comparison.

As we see, our method can be considered as a good estimator for determining the DOAs of signals.

### Conclusion:

A PR-based MSWF root-MUSIC algorithm has been derived for DOA estimation. Our method combines the MSWF
root-MUSIC algorithm and PR technique to form a root estimator bank and a corresponding DOA estimator bank. Unlike the conventional root-MUSIC, MSWF root-MUSIC does not involve estimating the sample covariance matrix and its EVD. So it is suitable for practical applications where massive arrays are used. Also, two strategies for determining the final DOAs in the PR technique are introduced. Since these strategies use the modulus information of the root estimator bank, we can determine the final DOA estimates more accurately. Simulation results verify the effectiveness of our proposed method.

© 2021 The Authors. Electronics Letters published by John Wiley & Sons Ltd on behalf of The Institution of Engineering and Technology

This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

Received: 27 April 2021 Accepted: 3 May 2021
doi: 10.1049/el.2021.12220

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References
1 Alexion, A., Haardt, M.: Smart antenna technologies for future wireless systems: trend and challenges. IEEE Magazin. Commun. 42, 90–97 (2004)
2 Fan, D., et al.: Angle domain signal processing-aided channel estimation for indoor 60-GHZ TDD/FDD massive MIMO system. IEEE Selected Areas. Commun. 35, 1948–1961 (2017)
3 Zhao, J., et al.: Angle domain hybrid precoding and channel tracking for millimeter wave massive MIMO systems. IEEE Trans. Wireless Commun 16, 6868–6880 (2017)
4 Barabell, A. J.: Multiple emitter location and signal parameter estimation. IEEE Trans. Antenna Propag. 34, 276–280 (1986)
5 Barabell, A. J.: Improving the resolution performance of eigen structure based direction finding algorithms. In: Proceedings of IEEE Conference on Acoustics, Speech and Signal Processing, Boston, MA, pp. 336–339 (1983)
6 Vasylyshyn, V.: DOA estimation using beamspace root-music based estimator bank. In: Proceedings of the IEEE Conference Antenna Theory. Kyiv, Ukraine, pp. 367–369 (2017)
7 Qian, C., Huang, L., So, H.C.: Computationally efficient ESPRIT algorithm for direction of arrival estimation based on Nystrom method. Signal Process 94, 74–80 (2014)
8 Pesavento, M., Gersman, A.B., Haardt, M.: Unitary root-MUSIC with a real-valued eigendecomposition: A theoretical and experimental performance study. IEEE Trans. Signal Process 48, 1306–1314 (2000)
9 Yan, F.G., et al.: Unitary direction of arrival estimation based on a second forward/backward averaging technique. IEEE Letters. Commun. 22, 554–557 (2018)
10 Pham, G.T., Loubaton, P., Vallet, P.: Performance analysis of spatial smoothing schemes in the context of large array. IEEE Trans. Signal Process 64, 160–172 (2016)
11 Gersman, A.B., Bohme, J.E.: A pseudo-noise resampling approach to direction of arrival estimation using estimator banks. In: Proceedings of the 9th IEEE SP Workshop on Statistical Signal Array Processing, Portland, OR, pp. 244–247 (1998)
12 Vasylyshyn, V.: Removing the outliers in root-MUSIC via pseudo-noise resampling and conventional beamformer. Signal Process. 93, 3423–3429 (2013)
13 Gersman, A.B., Haardt, M.: Improving the performance of unitary ESPRIT via pseudo-noise resampling. IEEE Trans. Signal Process. 47, 2305–2308 (1999)
14 Qian, C., Huang, L., So, H.C.: Improved unitary root-MUSIC for DOA estimation based on Pseudo-noise resampling. IEEE Let. Signal Process. 21, 140–144 (2014)
15 Goldstein, J.S., Reed, I.S., Scharf, L.L.: A multistage representation of the Wiener filter based on orthogonal projection. IEEE Trans. Inf. Theory 44, 2943–2959 (1998)
16 Huang, L., Wu, S., Li, X.: Reduced-rank MDL method for source enumeration in high-resolution array processing. IEEE Trans. Signal process. 55, 5658–5667 (2007)
17 Lu, Z., Zoubir, A.M.: Source enumeration in array processing using a two-step test. IEEE Trans. Signal process. 63, 2718–2727 (2015)