Boundary feedback stabilization of a critical nonlinear JMGT equation with Neumann-undissipated part of the boundary

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Abstract

Boundary feedback stabilization of a critical, nonlinear Jordan–Moore–Gibson–Thompson (JMGT) equation is considered. JMGT arises in modeling of acoustic waves involved in medical/engineering treatments like lithotripsy, thermotherapy, sonochemistry, or any other procedures using High Intensity Focused Ultrasound (HIFU). It is a well-established and recently widely studied model for nonlinear acoustics (NLA): a third–order (in time) semilinear Partial Differential Equation (PDE) with the distinctive feature of predicting the propagation of ultrasound waves at finite speed due to heat phenomenon know as second sound which leads to the hyperbolic character of heat propagation. In practice, the JMGT dynamics is largely used for modeling the evolution of the acoustic velocity and, most importantly, the acoustic pressure as sound waves propagate through certain media. In this work, critical refers to (usual) case where media–damping effects are non–existent or non–measurable and therefore cannot be relied upon for stabilization purposes.

In this paper the issue of boundary stabilizability of originally unstable (JMGT) equation is resolved. Motivated by modeling aspects in HIFU technology, boundary feedback is supported only on a portion of the boundary, while the remaining part of the boundary is left free (available to control actions). Since the boundary conditions imposed on the "free" part of the boundary fail to satisfy Lopatinski condition (unlike Dirichlet boundary conditions), the analysis of uniform stabilization from the boundary becomes very subtle and requires careful geometric considerations.

keywords: Nonlinear acoustics, second sound, third–order in time, heat-conduction, boundary stabilization, degenerate viscoelasticity.

1 Introduction

1.1 PDE Model and an Overview

Let \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) denote a bounded domain with sufficiently smooth boundary \( \Gamma := \partial \Omega \) within which a sound wave propagates. In HIFU technology, as well as in other contexts, one is interested
in tracking – and often controlling – the evolution of the acoustic pressure $u = u(t,x)$ ($t \in \mathbb{R}_+, x \in \Omega$) caused by such wave propagation. In media on which heat propagates hyperbolically (which is the case of most biological tissues), the evolution of the acoustic pressure can be assumed to obey the semilinear JMGT-equation which is given by the third order in time abstract evolution equation

$$\tau u_{ttt} + (\alpha - 2ku)u_{tt} - c^2 \Delta u - (\delta + \tau c^2)\Delta u_t = 2ku_t^2, \quad (1.1)$$

where $c, \delta, k > 0$ are constants representing the speed and diffusivity of sound and a nonlinearity parameter, respectively, while the function $\alpha : \overline{\Omega} \to \mathbb{R}^+$ accounts for natural frictional damping provided by the media. The parameter $\tau > 0$ – also media–dependent – accounts for thermal relaxation and essentially transfers the hyperbolicity of the heat to the acoustic wave.

The semilinear equation (1.1) can be viewed as a singular perturbation and, to some extent, a refinement of the classical quasilinear Westervelt’s equation

$$(\alpha - 2ku)u_{tt} - c^2 \Delta u - \delta \Delta u_t = 2ku_t^2, \quad (1.2)$$

obtained by setting $\tau = 0$. Physically, the main difference between (1.1) and (1.2) is that the latter predicts that these waves propagate at a finite speed. From the modeling point of view this results from using Maxwell-Cattaneo Law – rather than Fourier’s Law – to model a heat flux for acoustic heat waves. The parameter $\tau > 0$ corresponds to time relaxation. For more about the physical interpretation of (1.1), its derivation and overall discussion see [6, 19, 10, 11, 42, 13, 20]. This also includes an analysis of asymptotic behavior of solutions when the parameter of relaxation tends to zero [26, 27, 4, 3].

The issues of wellposedness and stability of solutions under homogeneous Dirichlet and Neumann boundary data were first addressed for both nonlinear and linearized ($k = 0$) dynamics around 2010 with the works of I. Lasiecka, R. Triggiani and B. Kaltenbacher [23, 24, 36, 25]. For the analysis of the long–time dynamics of (1.1) for both linear and nonlinear cases, the function

$$\gamma : \overline{\Omega} \to \mathbb{R}, \quad \gamma(x) \equiv \alpha(x) - \frac{\tau c^2}{b} \quad (1.3)$$

plays a central role. In fact, the existence of a positive constant $\gamma_0$ such that $\gamma(x) \geq \gamma_0 > 0$ a.e. in $\Omega$ ensures both that linear dynamics is uniformly exponentially stable and that stable nonlinear flows can be constructed via “barriers” [25] method. A natural question, in light of the above results, is concerning the other profiles of $\gamma$. It is known that if $\gamma < 0$ one may have chaotic solutions [14] and if $\gamma \equiv 0$ then the energy is conserved [24, 23]. This raises the interesting question of what mechanisms could be employed to ensure stability of the dynamics when $\gamma$ degenerates, i.e., $\gamma(x) \geq 0$.

From a practical point of view, the quantity $\gamma(x)$ is interpreted as the viscoelasticity of the material point $x \in \Omega$ and, in particular in the medical field, is not expected to be known for all points of $\Omega$. By making the physically relevant assumption that $\gamma \in L^\infty(\Omega)$, $\gamma(x) \geq 0$ a.e. in $\Omega$ (allowing the critical case $\gamma \equiv 0$, or the case where measurements can only be made at isolated points of the domain), we ask ourselves whether a non–invasive (boundary) action can drive the acoustic pressure to zero at large times regardless of the particular knowledge of $\gamma$ (as long as it is nonnegative). This question, besides being of independent interest in stability theory, is critical in
ensuring global wellposedness of nonlinear solutions. Otherwise the nonlinearity may cause "blow" up of solutions [12].

It has been recently shown that viscoelastic effects produces, in some cases, the asymptotic decay of the energy, cf. e.g. [33, 34, 15, 18, 17, 16]. In this work we concentrate on a physically attractive boundary stabilization – where control action can be applied on the boundary, hence easily accessible to external manipulations. Of particular interest is a configuration arising in the ultrasound technology where an acoustic medium is excited on one part of the boundary, while the remaining part of the boundary is subject to absorbing boundary conditions. This control model was introduced in [22, 21] in the case of Westervelt-Kuznetsov equation and later pursued in [9] for MGT equation. This corresponds to the following boundary conditions

\[ \lambda \partial_{\nu} u + \kappa_0(x) u = 0 \text{ on } \Sigma_0 \quad \partial_{\nu} u + \kappa_1(x) u_t = 0 \text{ on } \Sigma_1 \]

(1.4)

with \( \Gamma_0, \Gamma_1 \subset \Gamma \) relatively open, \( \Gamma_0 \neq \emptyset, \overline{\Gamma_0} \cup \overline{\Gamma_1} = \Gamma, \Gamma_0 \cap \Gamma_1 = \emptyset, \lambda > 0, \kappa_0 \in L^\infty(\Gamma_0) \) and \( \kappa_1 \in L^\infty(\Gamma_1) \), \( \kappa_1(x) \geq \kappa_1 > 0, \kappa_0 > 0 \) a.e.

Notice that the boundary condition (1.4)1 – where there is no dissipation –, do not satisfy strong Lopatinski condition, a fact that leads to new challenges at the level of proving controllability or stabilization even for a wave equation in dimension higher than one. The technical (mathematical) reason is that the presence of tangential boundary derivatives cannot be handled by standard flow multipliers methods. In fact, past contributions to the subject include [5, 6] where linear dynamics is considered in the case \( \lambda = 0 \) and \( \kappa_0 \equiv \kappa_1 \equiv 1 \) in (1.4). Thus, the uncontrolled part of the boundary is subject to Dirichlet boundary conditions where Lopatinski condition is satisfied and tangential derivatives (appearing in the applications of flux multipliers) vanish altogether on \( \Gamma_0 \). In [5] star–shaped boundary condition is assumed on \( \Gamma_1 \). This restriction has been removed in [7] by resorting to a microlocal analysis argument.

The present paper addresses the challenging case of a nonlinear, critical JMGT dynamics subject to Neuman/Robin boundary conditions on the undissipated part of the boundary \( \Gamma_0 \) (\( \lambda = 1 \)). The model is important in the context of HIFU control theory where boundary open loop strategic control is activated precisely on this “free” part of the boundary. On the other hand, in such case one encounters a well recognized PDE predicament: seeking stabilization for a hyperbolic dynamics when Lopatinski condition [39, 40] fails on undissipated portion of the boundary. This leads to major difficulties when applying flux multipliers or geometric optics in order to carry the analysis of uniform stability. Clearly, one expects some restrictions on the geometry of the boundary to cooperate. In the case of Dirichlet boundary conditions, star–shaped condition suffices. Instead, for the Neumann case (non–Lopatinski), it turns out that star–shaped along with some convexity is a sufficient condition. Precise formulation of the corresponding results will be given in the next section. In addition to new geometric constructs, nonlinearity in the model forces considering stability properties at higher topological levels with a restricted “smallness” condition imposed on the initial data. The key point here is that this would make the model “close” to linear. In order to contend with this limitation, the results presented require smallness conditions imposed only at the low energy level, while higher derivatives can remain large. We will be able to achieve this goal through boundary dissipation and small initial data imposed only at the lowest energy level. So the model and the resulting acoustic
waves remain genuinely nonlinear.

For other relatively recent references related to regularity questions for linear MGT equation, an interested reader may be referred to: [8, 38, 44]

2 Main Results and Discussion.

We consider the system comprised of (1.1), boundary conditions (1.4) and initial conditions

\[ u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1, \quad u_{tt}(0, \cdot) = u_2 \]  

(2.1)

with regularity to be specified in what follows.

Here and throughout the paper, by \( L^2(\Omega) \) and \( L^2(\Gamma) \) we denote the sets of measurable (in the Lebesgue and Hausdorff senses, respectively) functions whose squares are integrable on \( \Omega \) and \( \Gamma \) respectively equipped with the norms induced by the inner products

\[ (u, v) = \int_\Omega uv \, d\Omega \quad \text{and} \quad (u, v)_\Gamma = \int_\Gamma uv \, d\Gamma. \]

and denoted respectively by \( \| \cdot \|_2 \) and \( \| \cdot \|_\Gamma \). The remaining \( L^p(\Omega) \)–spaces \((1 \leq p \leq \infty)\) will also have their norms denoted by \( \| \cdot \|_p \). Additionally, by \( H^s(\Omega) \) we denote the \((L^2–based)\) Sobolev space of order \( s \) and define the particular spaces \( H^1_\Gamma(\Omega) \) and \( H^2_\Gamma(\Omega) \) as

\[ H^1_\Gamma(\Omega) = \{ u \in H^1(\Omega); u|_\Gamma = 0 \} \quad \text{and} \quad H^2_\Gamma(\Omega) = H^2(\Omega) \cap H^1_\Gamma(\Omega) \]

in order to avoid confusion with the standard \( H^1_0(\Omega) \).

2.1 Functional Analytic Setting.

Let \( A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega) \) be the operator defined as

\[ A\xi = -\Delta \xi, \quad \mathcal{D}(A) = \{ \xi \in H^2(\Omega); \partial_\nu \xi|_{\Gamma_1} = 0, [\partial_\nu \xi + \kappa_0 \xi]|_{\Gamma_0} = 0 \}. \]  

(2.2)

In this setting, \( A \) is a positive, self–adjoint operator with compact resolvent and for \( \kappa_0 > 0 \), \( \mathcal{D}(A^{1/2}) = H^1(\Omega) \) with the – equivalent to \( H^1(\Omega) \) – topology of \( \mathcal{D}(A^{1/2}) \) given by

\[ \| u \|^2_{\mathcal{D}(A^{1/2})} := \| \nabla u \|^2_2 + \int_{\Gamma_0} \kappa_0 |u|^2 \, d\Gamma_0. \]

In addition, with some abuse of notation we (also) denote by \( A : L^2(\Omega) \to [\mathcal{D}(A)]' \) the extension (by duality) of the operator \( A \).

Let us introduce the phase space \( \mathbb{H} \) given by

\[ \mathbb{H} := \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega) \sim H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega). \]  

(2.3)

Next, we rewrite (1.1) along with (1.4) and (2.1) as a first–order abstract system on \( \mathbb{H} \). For this, we introduce the classic boundary \( \to \) interior harmonic extension for the Neumann data on \( \Gamma_1 \) as
follows: for \( \varphi \in L^2(\Gamma_1) \), let \( \psi := N(\varphi) \), be the unique solution of the elliptic problem

\[
\begin{align*}
\Delta \psi &= 0 & \text{in } \Omega \\
\partial_\nu \psi &= \varphi|_{\Gamma_1} & \text{on } \Gamma_1 \\
\partial_\nu \psi + \kappa_0 \psi &= 0 & \text{on } \Gamma_0.
\end{align*}
\]  \tag{2.4}

From elliptic theory, it follows that that \( N \in \mathcal{L}(H^s(\Gamma_1), H^{s+3/2}(\Omega)) \) (\( s \in \mathbb{R} \)) and

\[
N^* A \xi = \begin{cases} 
\xi & \text{on } \Gamma_1 \\
0 & \text{on } \Gamma_0.
\end{cases}
\]  \tag{2.5}

for all \( \xi \in \mathcal{D}(A) \), where \( N^* \) represents the adjoint of \( N \) when it is considered as an operator from \( L^2(\Gamma_1) \) to \( L^2(\Omega) \) \([31]\).

Thus, the \( u \)-problem can be written (via duality on \( [\mathcal{D}(A)]' \)) as

\[
\tau u_{tt} + \alpha u_t + \kappa_1 N^* A u_t + c^2 AN(\kappa_1 N^* A u_t) + b AN(\kappa_1 N^* A u_{tt}) = u_t^2 + uu_{tt}
\]  \tag{2.6}

where we have taken \( k = 1/2 \) without any loss of generality.

Next, we introduce the operator \( A : \mathcal{D}(A) \subset \mathbb{H} \rightarrow \mathbb{H} \) with the action:

\[
A \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} := \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\frac{c^2}{\tau} A & -\frac{c^2}{\tau} AN(\kappa_1 N^* A) - b \frac{\tau}{\alpha} A & -b \frac{\tau}{\alpha} AN(\kappa_1 N^* A) - \alpha \frac{\tau}{\alpha} I \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}
\]  \tag{2.7}

and domain (with \( \bar{\xi} \equiv (\xi_1, \xi_2, \xi_3) \))

\[
\mathcal{D}(A) = \left\{ \bar{\xi} \in \mathbb{H} ; \ \xi_3 \in \mathcal{D}(A^{1/2}) , \ \xi_i + N(\kappa_1 N^* A \xi_{i+1}) \in \mathcal{D}(A) , \text{ for } i = 1, 2 \right\}
\]

\[
= \left\{ \bar{\xi} \in [H^2(\Omega)]^2 \times H^1(\Omega) ; \ [\partial_\nu \xi_1 + \kappa_0 \xi_1]_{\Gamma_0} = [\partial_\nu \xi_2 + \kappa_0 \xi_2]_{\Gamma_0} = 0 \right. \\
& \left. \quad [\partial_\nu \xi_1 + \kappa_1 \xi_3]_{\Gamma_1} = [\partial_\nu \xi_2 + \kappa_1 \xi_3]_{\Gamma_1} = 0 \right\}
\]  \tag{2.8}

where the second characterization follows from elliptic regularity. This gives

\[
\mathcal{D}(A) \subset H^2(\Omega) \times H^2(\Omega) \times H^1(\Omega)
\]

with a proper, but not dense injection.

The first order abstract version of the \( u \)-problem is thus given by

\[
\begin{align*}
\Phi_t &= A \Phi + \mathcal{F}(\Phi) \\
\Phi(0) &= \Phi_0 = (u_0, u_1, u_2) \top,
\end{align*}
\]  \tag{2.9}

in the variable \( \Phi = (u, u_t, u_{tt}) \top \) with \( A \) defined in (2.7) and \( \mathcal{F}(\Phi) \top \equiv (0, 0, \tau^{-1}(u_t^2 + uu_{tt})) \).

In order to treat nonlinear problem one needs to consider “smoother” solutions than generated by the topology of \( \mathbb{H} \). This leads to the following construction of the second phase space denoted
by \( \mathbb{H}_1 \), which is “thighter” than \( \mathbb{H} \) but strictly larger than \( \mathcal{D}(A) \). The new phase space \( \mathbb{H}_1 \) is defined below

\[
\mathbb{H}_1 = \{ \xi \in \mathbb{H}; \Delta \xi_1 \in L^2(\Omega); [\lambda \partial_{\nu} \xi_1 + \kappa_0 \xi_1]_{\Gamma_0} = 0; [\partial_{\nu} \xi_1 + \kappa_1 \xi_2]_{\Gamma_1} = 0 \} \tag{2.10}
\]

and endowed with the norm

\[
\| \xi \|_{\mathbb{H}_1}^2 = \| \xi \|_{\mathbb{H}}^2 + \| \Delta \xi_1 \|_2^2 + \| \xi_1 \|_{H^{1/2}(\Gamma_0)}^2 + \| \xi_2 \|_{H^{1/2}(\Gamma_1)}^2
\]

or equivalently

\[
\| \xi \|_{\mathbb{H}_1}^2 = \| \xi \|_{\mathbb{H}}^2 + \| \Delta \xi_1 \|_2^2 + \| \partial_{\nu} \xi_1 \|_{H^{1/2}(\Gamma)}^2
\]

Note that the boundary conditions in the definition of the space \( \mathbb{H}_1 \) are well defined due to the property: \( \Delta \xi_1 \in L^2(\Omega) \) and \( \xi_1 \in H^1(\Omega) \) then \( \partial_{\nu} \xi_1 \in H^{-1/2}(\Gamma) \) – the latter allowing to define the boundary conditions as a distribution. We also note that since \( \xi_1, \xi_2 \in H^1(\Omega) \) we have \( \xi_i|_{\Gamma} \in H^{1/2}(\Gamma) \) (\( i = 1, 2 \)) and therefore \( \partial_{\nu} \xi_1 \in H^{1/2}(\Gamma) \). This along with the elliptic regularity implies:

\[
\mathbb{H}_1 \subset H^2(\Omega) \times H^1(\Omega) \times L^2(\Omega)
\]

with a proper but not dense injection. We shall show that the operator \( A \) also generates a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) on \( \mathbb{H}_1 \). Notice that the nonlinear term is invariant under \( \mathbb{H}_1 \) topology in dimensions up to 3.

### 2.2 Main Results

Our first preliminary result is stated below.

**Theorem 2.1.** The operator \( A \) generates a \( C_0 \)-semigroup \( \{S(t)\}_{t \geq 0} \) on \( \mathbb{H} \). Moreover, the family \( T(t) := S(t)|_{\mathbb{H}_1}, t \geq 0, \) is also a \( C_0 \)-semigroup with generator \( A \) and its realization on \( \mathbb{H}_1 \).

The second result deals with an exponential stability of the semigroups on the phase space \( \mathbb{H} \) and \( \mathbb{H}_1 \). For this, one needs to introduce the following geometric condition.

**Assumption 1.** The boundary \( \Gamma_0 \) is star-shaped and convex. This is to say: there exists \( x_0 \in \mathbb{R}^n \) such that \((x - x_0) \cdot \nu(x) \leq 0 \) for all \( x \in \Gamma_0 \) where \( \nu(x) \) is the outwards normal vector to the boundary at \( x \). In addition, there exists a convex level set function which defines \( \Gamma_0 \). See [32].

**Theorem 2.2 (Two level uniform stability).** Let Assumption 1 on \( \Gamma_0 \) be in force and let \( \gamma(x) \geq 0 \). Then \( \textbf{(i)} \) the semigroup \( \{S(t)\}_{t \geq 0} \) generated by \( A \) in \( \mathbb{H} \) is uniformly exponentially stable with decay rate \( \omega_0 > 0 \) and \( \textbf{(ii)} \) the semigroup \( \{T(t)\}_{t \geq 0} \) generated by \( A \) in \( \mathbb{H}_1 \) is uniformly exponentially stable with decay rate \( \omega_1 > 0 \), where \( \omega_1 < \omega_0 \).

Once linear wellposedness and uniform stability of the linear \( (k = 0) \) problem are established with respect to the appropriate topologies, our next task is to prove generation of nonlinear semigroup on \( \mathbb{H}_1 \). To accomplish this, initial data need to be assumed sufficiently small. How small? This is an important question as argued in [4]. We will be able to show that some smallness will be only imposed at the lowest level of regularity, while higher derivatives can be large. As a consequence, in
the following theorem we show existence of $H_1$–valued solutions given $H_1$ initial data which are small in $H$ only. The proof, given in Section 5, relies on estimates derived via interpolation inequalities which allows to demonstrate certain “invariance” of a $H$-small ball under the nonlinear dynamics in $H_1$.

We start specifying the notion of solution for the semilinear problem (1.1) supplemented with (1.4) and (2.1). We denote the initial data here by $\Phi_0 = (u_0, u_1, u_2)^\top$. Given $T > 0$, we say that

$$\Phi(t) = (u(t), u_t(t), u_{tt}(t))$$

is a **mild solution** for the system (1.1), (1.4) and (2.1) provided $\Phi \in C([0, T], H_1)$ and

$$\Phi(t) = T(t)\Phi_0 + \int_0^t T(t - \tau)\mathcal{F}(\Phi)(\tau)d\tau, \quad (2.11)$$

Before stating the theorem, we denote by $H^\rho$ (for $\rho > 0$) the set

$$H^\rho := \{\Phi \in H_1; \|\Phi\|_H < \rho\}.$$

**Theorem 2.3 (Global Solutions).** Let Assumption 1 on $\Gamma_0$ be in force. Then, there exists $\rho > 0$ sufficiently small such that, given any $\Phi_0 \in H^\rho$ the formula (2.11) defines a continuous $H_1$–valued mild solution for the system (1.1), (1.4) and (2.1). Moreover, for such $\rho > 0$, there exists $R = R(\|\Phi_0\|_{H_1})$ such that all trajectories starting in $B_{H^\rho}(0, R)$ remain in $B_{H^\rho}(0, R_1)$ for all $t \geq 0, R_1 > R$.

Once global solutions are shown to exist, we take on the issue of asymptotic (in time) stability. The final result is positive, as we expected, and holds uniformly (w.r.t $\gamma$) as long as $\gamma \in L^\infty(\Omega)$ and $\gamma(x) > 0$ a.e. in $\Omega$.

**Theorem 2.4 (Nonlinear Uniform Stability).** Let Assumption 1 on $\Gamma_0$ be in force and assume $\gamma \in L^\infty(\Omega)$ and $\gamma(x) > 0$. Then, there exists $\rho > 0$ sufficiently small and $M(\rho), \omega > 0$ such that if $\Phi_0 \in H^\rho$ then

$$\|\Phi(t)\|_{H_1} \leq M(\rho)e^{-\omega t}\|\Phi_0\|_{H_1}, \quad t \geq 0 \quad (2.12)$$

where $\Phi$ is the mild solution given by Theorem 2.3.

We notice that repeating the statement there exists $\rho > 0$ such that if $\Phi_0 \in H^\rho$ in the Theorem above is not redundant. In fact, one might need to require even smaller initial data to yield exponential decay. This again highlights the advantage of requiring smallness only in $H$. For more details, see Section 6.

**2.3 Discussion**

The main novelty of this paper is that we study stabilizability of a nonlinear critical JMGT equation with Neumann–Robin undissipated portion of the boundary. Should the problem be subcritical (i.e. $\gamma(x) > \gamma_0 > 0$ for $x \in \Omega$, the difficulty created by the failure of Lopatinski condition would not enter

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1The $H^\rho$-ball centered at the origin and with radius $R$. 

---
the picture. Simply because there will be no need to propagate stability from the boundary into the interior. As already mentioned before, linear dynamics with absorbing boundary conditions on \( \Gamma_1 \) and zero Dirichlet data on \( \Gamma_0 \) subject to star–shaped conditions has been considered in \([5, 7]\). Mathematical difficulties in propagating stability through the undissipated part of the boundary are not present in this case. In order to cope with the difficulties we shall employ geometric constructs developed earlier in \([32]\). These allow to construct suitable–non–radial–vector fields which result from tangential bending of radial and star–shaped ones. These newly constructed fields propagate the needed estimates through un–dissipated part of the boundary.

In order to treat the nonlinear problem, the approach used in the past (for subcritical case) was to use the so called “barrier’s method” based on contradiction argument. However, this presents several technical difficulties in the present scenario, even at the level of low frequencies (lower order terms). Hence, in this paper, we exploit another technique which, to the best of our knowledge, is new and makes a strong use of the fact that we only require initial data to be small in \( H \). One of the advantages of such construction (for JMGT) was already exploited by the authors in \([4]\) in allowing extension by density in the nonlinear environment. In this paper we discovered that it also allows to:

\(\text{a) prove global existence and exponential stability by the representation of the solution and two-level stability of linear flows. Here, the smallness interplay comes to the picture through a nonlinear propagation of the estimate of the type}

\[
\|F(\Phi)\|_{H_1} \leq C_1(\|\Phi\|_{H})C_2(\|\Phi\|_{H_1},.) \tag{2.13}
\]

\(\text{where the size of } C_1(\|\Phi\|_{H}) \text{ can be controlled by } \rho. \text{ See Theorem 2.3.}
\]

\(\text{b) obtain, to some extent, a continuity property of the decay rate with respect to the } H \text{–size of the initial data and the decay rate of the linear flow, } \omega_1. \text{ In general, we prove that if } \varepsilon \text{ is the } H \text{–size of the initial data and } \omega(\varepsilon) \text{ is the corresponding decay rate, then there exists } \underline{\omega}(\varepsilon) \text{ such that } \omega(\varepsilon) \geq \underline{\omega}(\varepsilon) \text{ and } \underline{\omega}(\varepsilon) \rightarrow \omega_1 \text{ as } \varepsilon \rightarrow 0^+.
\]

The rest of this paper is devoted to the proofs.

3 Linear Semigroups – Proof of Theorem 2.1

We notice that \( H \) has its topology induced by the inner product

\[
\left(\begin{array}{c}
\xi_1 \\
\xi_2 \\
\xi_3
\end{array}\right) , \left(\begin{array}{c}
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{array}\right)_{H} = (A^{1/2}\xi_1, A^{1/2}\varphi_1) + \frac{b}{\tau}(A^{1/2}\xi_2, A^{1/2}\varphi_2) + (\xi_3, \varphi_3), \tag{3.1}
\]

\(\text{for all } (\xi_1, \xi_2, \xi_3)^\top, (\varphi_1, \varphi_2, \varphi_3)^\top \in H.
\)

\(\text{We first show that } A : D(A) \subset H \rightarrow H \text{ generates a } C_0 \text{ semigroup on } H. \text{ This part of the argument follows essentially } [7] \text{ with some rather straightforward modifications. We shall outline the main details – as these are needed for the proof of generation on the higher level of } H_1 \text{ topology.} \)
For notational convenience and future use, we introduce the following change of variables $bz = bu_t + c^2u$ which reduces the problem to a PDE–abstract ODE coupled system. The change from the coordinates $(u, u_t, u_{tt})$ to $(u, z, z_t)$ is described through the isomorphism $M \in \mathcal{L}(\mathbb{H})$ given by (see [36])

$$M = \begin{bmatrix} 1 & 0 & 0 \\ c^2 / b & 1 & 0 \\ 0 & c^2 / b & 1 \end{bmatrix}. $$

The next lemma makes the above topological statement precise.

**Lemma 3.1.** Assume that the compatibility conditions

$$\lambda \partial_\nu u_0 + \kappa_0 u_0 = 0 \text{ on } \Gamma_0, \quad \partial_\nu u_0 + \kappa_1 u_0 = 0 \text{ on } \Gamma_1 $$

hold. Then $\Phi \in C^1(0,T;\mathbb{H}) \cap C(0,T;\mathcal{D}(A))$ is a strong solution of

$$\begin{cases} \\
\Phi_t = A\Phi \\
\Phi(0) = \Phi_0 \end{cases}$$

if, and only if, $\Psi = M\Phi \in C^1(0,T;\mathbb{H}) \cap C(0,T;\mathcal{D}(A))$ is a strong solution for

$$\begin{cases} \\
\Psi_t = \mathbb{A}\Psi \\
\Psi(0) = \Psi_0 = M\Phi_0 \end{cases}$$

where $\mathbb{A} = MAM^{-1}$ with

$$\mathcal{D}(\mathbb{A}) = \left\{ \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \in [H^2(\Omega)]^2 \times \mathcal{D}(A^{1/2}) \mid [\lambda \partial_\nu \xi_2 + \kappa_0 \xi_2]_{\Gamma_0} = 0, [\partial_\nu \xi_2 + \kappa_1 \xi_3]_{\Gamma_1} = 0 \right\}$$

**Proof.** We only check the matching of the boundary conditions. Assume that $\Psi = (u, z, z_t) \in C^1(0,T;\mathbb{H}) \cap C(0,T;\mathcal{D}(A))$ is a strong solution for (3.4). Let

$$\Upsilon(t) := (\lambda \partial_\nu u(t) + \kappa_0 u(t))|_{\Gamma_0}, \ t \geq 0$$

and notice that $b\Upsilon_t + c^2\Upsilon = 0$ for all $t$. This along with the compatibility condition (3.2)$_1$ ($\Upsilon(0) = 0$) implies that $\Upsilon \equiv 0$. The same argument *mutatis mutandis* recovers the boundary condition for $u$ on $\Gamma_1$. \hfill \Box

For convenience, we explicitly write a formula for the new operator $\mathbb{A} = MAM^{-1}$. We have

$$\mathbb{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} -\frac{c^2}{b} I & I & 0 \\ 0 & 0 & I \\ -\gamma \frac{c^4}{\tau b^2} I & \gamma \frac{c^2}{\tau b} I - \frac{b}{\tau} A & -\gamma \frac{1}{\tau} I - \frac{b}{\tau} AN(\kappa_1 N^* A) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

(3.6)
where $\gamma = \alpha - \frac{\tau c^2}{b} \in L^\infty(\Omega)$.

We are ready to prove Theorem 2.1. This will be done by first showing that $A$ generates a $C_0$-semigroup on $\mathbb{H}$, from which the semigroup generated by $A$ can be recovered via $M$. The semigroup on $\mathbb{H}_1$ will then be obtained by a restriction argument. The details are below.

We write $A = A_d + P$ where

$$P := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -\gamma \frac{c^4}{\tau b^2} & \gamma \frac{c^2}{\tau b} & 1 - \frac{1}{\tau}\end{bmatrix} \in \mathcal{L}(\mathbb{H})$$

is bounded in $\mathbb{H}$ and

$$A_d \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} := \begin{bmatrix} -\frac{c^2}{b} I & 0 & 0 \\ 0 & 0 & I \\ 0 & -\frac{b}{\tau} A & -I - \frac{b}{\tau} A N(\kappa_1 N^* A) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix},$$

where $\mathcal{D}(A_d) := \mathcal{D}(A)$. It then suffices to prove generation of $A_d$ on $\mathbb{H}$, see [37, p. 76] and this will be done by verifying the hypothesis of Lumer Philips Theorem: dissipativity and maximality.

For dissipativity we consider $(\xi_1, \xi_2, \xi_3)^\top \in \mathcal{D}(A)$ and compute via (3.1)

$$\begin{pmatrix} A_d & \xi_2 \\ \xi_3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = -\frac{c^2}{b} \|A^{1/2} \xi_1\|_{L^2(\Omega)}^2 + \frac{b}{\tau} (A^{1/2} \xi_3, A^{1/2} \xi_2)$$

$$- \|\xi_3\|_{L^2(\Omega)}^2 - \frac{b}{\tau} (A^{1/2} \xi_2, A^{1/2} \xi_3) - \frac{b}{\tau} \|\sqrt{\kappa_1} \xi_3\|_{L^2(\Gamma_1)}^2$$

$$= -\frac{c^2}{b} \|A^{1/2} \xi_1\|_{L^2(\Omega)}^2 - \|\xi_3\|_{L^2(\Omega)}^2 - \frac{b}{\tau} \|\sqrt{\kappa_1} \xi_3\|_{L^2(\Gamma_1)}^2 \leq 0,$$

hence, $A_d$ is dissipative in $\mathbb{H}$.

For maximality in $\mathbb{H}$, given any $L = (f, g, h) \in \mathbb{H}$ we need to find $\Psi = (\xi_1, \xi_2, \xi_3)^\top \in \mathcal{D}(A)$ such that $(s - A_d)\Psi = L$, for some $s > 0$. This is equivalent to solving

$$\begin{cases} s\xi_1 + \frac{c^2}{b} \xi_1 = f, \\ s\xi_2 - \xi_3 = g, \\ s\xi_3 + \xi_3 + bA(\xi_2 + N(\kappa_1 N^* A_3)) = h, \end{cases}$$

which readily implies

$$\xi_1 = \frac{b}{bs + c^2} f \in \mathcal{D}(A^{1/2}).$$

Moreover, since $A^{-1} \in \mathcal{L}(L^2(\Omega))$ a combination of the second and third equations above yields

$$K_s \xi_3 = sA^{-1} h - bg$$

(3.9)
where \( K_s : L^2(\Omega) \to L^2(\Omega) \) acts on an element \( \xi \in L^2(\Omega) \) as

\[
K_s \xi = [(s^2 + s)A^{-1} + b(I + sN(\kappa_1N^*A))] \xi.
\]

We now notice the restriction \( K_s|_{\mathcal{D}(A^{1/2})} \) is strictly positive. Indeed it follows by (2.5) that, given \( \xi \in \mathcal{D}(A^{1/2}) \) we have

\[
(K_s \xi, \xi)_{\mathcal{D}(A^{1/2})} = (s^2 + s)\|\xi\|^2_2 + b\|A^{1/2} \xi\|^2_2 + bs(A^{1/2}N(\kappa_1N^*A)\xi, A^{1/2}\xi)
\]

\[
= (s^2 + s)\|\xi\|^2_2 + b\|A^{1/2} \xi\|^2_2 + bs\|\sqrt{\kappa_1}N^*A\xi\|_1^2 > 0.
\]

Therefore \( K_s^{-1}_{\mathcal{D}(A^{1/2})} \in \mathcal{L}(\mathcal{D}(A^{1/2})) \) and since \( sA^{-1}h - bg \in \mathcal{D}(A^{1/2}) \) we have that

\[
\xi_3 := K_s^{-1}(sA^{-1}h - bg) \in \mathcal{D}(A^{1/2})
\]

is the solution of (3.9). Finally,

\[
\xi_2 = s^{-1}(\xi_3 + g) \in \mathcal{D}(A^{1/2}).
\]

For the final step to conclude membership of \((\xi_1, \xi_2, \xi_3)\) in \(\mathcal{D}(\mathcal{A})\) we look at the abstract version of the description of \(\mathcal{D}(\mathcal{A})\):

\[
\mathcal{D}(\mathcal{A}) = \{ (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{H}; \quad \xi_2 + N(\kappa_1N^*A\xi_3) \in \mathcal{D}(A) \}
\]

whereby one only needs to check that \(\xi_2 + N(\kappa_1N^*A\xi_3) \in \mathcal{D}(A)\) since the regularity for the triple \((\xi_1, \xi_2, \xi_3)^\top\) to belong to \(\mathbb{H}\) was already established. The desired regularity will follow from (3.8), which implies

\[
bs(\xi_2 + N(\kappa_1N^*A\xi_3)) = -(s^2 + s)A^{-1}\xi_3 + sA^{-1}h \in \mathcal{D}(A),
\]

since \(\xi_3, h \in L^2(\Omega)\).

This proves that \(\mathcal{A}_d\) is maximal dissipative, therefore generates a \(C_0\)-semigroup of contractions due to Lummer Phillips Theorem and, since \(P\) is bounded, \(\mathcal{A} = \mathcal{A}_d + P\) generates a \(C_0\)-semigroup on \(\mathbb{H}\).

For generation in \(\mathbb{H}_1\) we use an argument inspired by the one presented in ([36], p. 26) with the needed modifications. Since we already know that \(\mathcal{A}\) generates a \(C_0\) semigroup \(\{S(t)\}_{t \geq 0}\) on a larger space \(\mathbb{H}\), we only show that

\[
\{T(t)\}_{t \geq 0} := \{S(t)|_{\mathbb{H}_1}\}_{t \geq 0}
\]

is also a semigroup and that its infinitesimal generator is \(\mathcal{A}\) when considered as an operator in \(\mathbb{H}_1\).

This entails the proof of two things: \(\{T(t)\}_{t \geq 0}\) satisfies the semigroup property – which follows from the fact that the problem is autonomous – and invariance: \(T(t)(\mathbb{H}_1) \subset \mathbb{H}_1\) for all \(t \geq 0\).

If \(\Phi_0 = (u_0, u_1, u_2)^\top \in \mathbb{H}_1\) then \(\partial_t u_0 + k_1u_1 = 0\) on \(\Gamma_1\). We then need to show that this condition is invariant under the dynamics and, in addition, the regularity \(\Delta u \in C ([0, T); L^2(\Omega))\) holds true. This along with the boundary conditions and regularity of elliptic problems will prove that \(u \in C ([0, T); H^2(\Omega))\). In order to show that \(\Delta u \in C ([0, T); L^2(\Omega))\), we appeal to the change of variables \(bz_t = bu_t + c^2u\). By the variation of parameters formula we have

\[
u(t) = e^{-\frac{c^2}{b}t} u_0 + \int_0^t e^{-\frac{c^2}{b}(t-s)} z(\sigma) d\sigma
\]
and since $\Delta u_0 \in L^2(\Omega)$ ($\Phi_0 \in H^1$), it suffices to verify that

$$\int_0^t e^{-\frac{\Delta^2}{2}(t-\sigma)} \Delta z(\sigma) d\sigma \in L^2(\Omega), \ \forall t \geq 0.$$ 

To this end we recall that $(z, z_t) \in C([0, T); H^1(\Omega) \times L^2(\Omega))$. Writing the linear solution of (1.1) (with $k = 0$) in the $z$-variable yields

$$\int_0^t e^{-\frac{\Delta^2}{2}(t-\sigma)} \Delta z(\sigma) d\sigma = \frac{\tau}{b} \int_0^t e^{-\frac{\Delta^2}{2}(t+\sigma)} [z_{tt}(\sigma) + \gamma u_{tt}(\sigma)] d\sigma$$

$$= \frac{\tau}{b} \left[ z_t(t) + \gamma u_t(t) - e^{-\frac{\Delta^2}{2}t}[z_t(0) + \gamma u_1] + \frac{\epsilon^2}{b^2} \int_0^t e^{-\frac{\Delta^2}{2}(t-\sigma)} [z_t(\sigma) + \gamma u_t(\sigma)] d\sigma \right]$$

as needed. Now, in order to show that the boundary conditions are also invariant under the dynamics, we write (for continuous $D(A)$-valued solutions):

$$\partial_\nu u(t) + k_1 u_t(t) = e^{-\frac{\Delta^2}{2}t} \partial_\nu u_0 + \int_0^t e^{-\frac{\Delta^2}{2}b^{-1}(t-\sigma)} \partial_\nu z(\sigma) d\sigma$$

$$+ k_1 \left( -\frac{\epsilon^2}{b} e^{-\frac{\Delta^2}{2}t} u_0 + e^{-\frac{\Delta^2}{2}t} z(0) + \int_0^t e^{-\frac{\Delta^2}{2}t}(t-\sigma) z_t(\sigma) d\sigma \right)$$

$$= e^{-\frac{\Delta^2}{2}t} \partial_\nu u_0 + \int_0^t e^{-\frac{\Delta^2}{2}b^{-1}(t-\sigma)} \partial_\nu z(\sigma) d\sigma$$

$$+ k_1 \left[ -\frac{\epsilon^2}{b} e^{-\frac{\Delta^2}{2}t} u_0 + e^{-\frac{\Delta^2}{2}t} \left( u_1 + \frac{\epsilon^2}{b} u_0 \right) + \int_0^t e^{-\frac{\Delta^2}{2}(t-\sigma)} z_t(\sigma) d\sigma \right]$$

$$= e^{-\frac{\Delta^2}{2}t} [\partial_\nu u_0 + k_1 u_1] + \int_0^t e^{-\frac{\Delta^2}{2}b^{-1}(t-\sigma)} [\partial_\nu z(\sigma) + k_1 z_t(\sigma)] d\sigma = 0,$$

where the conclusion follows from the fact that the initial conditions for $u$ satisfy the absorbing boundary conditions and the variable $z$ satisfies the absorbing boundary conditions along the trajectory. This completes the proof of Theorem 2.1.

4 Exponential decays – Proof of Theorem 2.2

In this section we work with the linearized version of (1.1) – i.e., we take $k = 0$ – in the $z$-variable. Moreover, since $\tau$ is fixed, we lose no generality by setting $\tau = 1$, therefore this is assumed for the rest of the paper. Recall that the change of variables $z = u_t + \frac{\epsilon^2}{b} u$ transforms (1.2) into

$$z_{tt} + bA(z_t + N(\kappa_1 N^* A) z) = -\gamma u_{tt} + f.$$  \hfill (4.1)

We assume smooth initial conditions and a $u$–independent forcing term $f \in C^1(\mathbb{R}_+, L^2(\Omega))$. This ensures the existence and uniqueness of classical solutions which are, in addition, continuously dependent on the initial data. We can eventually extend the results that follow to semigroup solutions by the virtue of density.
The energy for a solution $\Phi = (u, u_t, u_{tt}) \in D(A)$ will be computed in two levels. We define the **lower energy** functional $E(t) = E_0(t) + E_1(t)$ where

$$E_1(t) := \frac{b}{2} \|A^{1/2} z\|_2^2 + \frac{1}{2} \|z_t\|_2^2 + \frac{c^2}{2b} \|\gamma^{1/2} u_t\|_2^2$$

(4.2)

$$E_0(t) := \frac{1}{2} \|\alpha^{1/2} u_t\|_2^2 + \frac{c^2}{2} \|A^{1/2} u\|_2^2$$

(4.3)

and the **higher energy** functional $E(t) = E(t) + E_2(t)$ where

$$E_2(t) = \frac{b}{2} \|\Delta u\|_2^2$$

(4.4)

where $\alpha$ and $\gamma$ may depend on $x$. We also note that

$$\|A^{1/2} u\|_2^2 = \|\nabla u\|_2^2 + \|\sqrt{\kappa_0} u\|_{1_0}^2$$

. By Poincare- Wirtenberg inequality we obtain that

$$\|A^{1/2} u\|_2 \sim \|u\|_{H^1(\Omega)}$$

It is standard to see that $E(t) \sim \|\Phi(t)\|_{2_\Omega}^2$, see [5] for details. The following lemma follows from classic elliptic theory and provides the estimate which is necessary for justifying our choice of higher energy functional. As a remark, here and hereafter we use the notation $a \lesssim b$ to say that $a \leq Cb$ where $C$ is a constant possibly depending on the physical parameters of the model ($\tau, c, b > 0$) but independent of space, time and $\gamma \in L^\infty(\Omega)$.

**Lemma 4.1.** Let $\Omega$ be a smooth domain and consider a function $u : \Omega \to \mathbb{R}$ such that $\Delta u \in L^2(\Omega)$ and $\partial u|_{\partial \Omega} \in H^{1/2}(\Gamma)$, then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \lesssim \|\Delta u\|_2 + \|\partial u\|_{H^{1/2}(\Gamma)}.$$  

**4.1 Propagation of boundary dissipation – Flow multipliers**

We begin with energy identity for $E_1$. Since the problem is linear we work with smooth solutions (in the domain of the generator) which can be extended by density to the phase space solutions.

**Proposition 4.2 (Energy Identity).** Let $T > 0$. If $\Psi = (u, z, z_t)$ is a weak solution of (4.1) then

$$E_1(T) + \int_t^T D_\Psi(s) ds = E_1(t) + \int_t^T \int_\Omega f z_t d\Omega ds$$

(4.5)

holds for $0 \leq t \leq T$, where $D_\Psi$ represents the interior/boundary damping and is given by

$$D_\Psi := b \int_{\Gamma_1} \kappa_1 z_t^2 d\Gamma_1 + \int_\Omega \gamma u_{tt}^2 d\Omega ds$$

(4.6)
Proof. The energy identity (4.5) is first derived for strong solutions and then extended by density to weak solutions. Consider the bilinear form \( \langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} \) be given by
\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix}
\begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{bmatrix}
:= b \left( A^{1/2} \xi_2, A^{1/2} \varphi_2 \right) + \left( \xi_3, \varphi_3 \right) + \frac{c^2}{b} \left( \gamma \left( \xi_2 - \frac{c^2}{b} \xi_1 \right), \varphi_2 - \frac{c^2}{b} \varphi_1 \right) \tag{4.7}
\]
which is continuous. Moreover, recalling that \( \Psi = (u, z, z_t) \) it follows that \( 2E_1(t) = \langle \Psi(t), \Psi(t) \rangle \). Therefore, with \( G = (0, 0, f)^T \) we have, after straightforward calculations (see [6] for details)
\[
2 \frac{dE_1(t)}{dt} = \left\langle \frac{d\Psi(t)}{dt}, \Psi(t) \right\rangle = \langle A\Psi(t) + G, \Psi(t) \rangle
\]
\[
= - \int_{\Omega} \gamma u_t^2 d\Omega - b \int_{\Gamma_1} \kappa_1 z_t^2 d\Gamma_1 + \int_{\Omega} f z_t d\Omega.
\]
Identity (4.5) then follows by an integration in time on \([t, T]\).

In the next step we reconstruct the integral of the full energy on a truncated time interval \((s, T-s)\) for \( 0 < s < T/2 \).

Proposition 4.3. Let \( T > 0 \). If \((u, z, z_t)\) is a classical solution of (3.4) then the inequality
\[
\int_s^{T-s} E_1(t) dt \lesssim \left[ E_1(s) + E_1(T-s) \right] + C_T \left[ \int_0^T D_\Psi(s) ds + \int_Q f^2 dQ + \text{lot}_\delta(z) \right]. \tag{4.8}
\]
holds for \( 0 < s < T/2 \). Here, \( \text{lot}_\delta(z) \) is a collection of lower order terms satisfying
\[
\text{lot}_\delta(z) \leq C_\delta \sup_{t \in [0,T]} \left\{ \| z(t) \|_{H^{1-\delta}(\Omega)}^2 + \| z_t(t) \|_{H^{-\delta}(\Omega)}^2 \right\},
\]
for some \( \delta > 0 \).

Proof. We study stability properties of both \( S(t) \) and \( T(t) \) assuming the general degenerated case for \( \gamma \), i.e., \( \gamma \in L^\infty(\Omega) \) and \( \gamma(x) \geq 0 \) a.e. in \( \Omega \). This includes a completely degenerate (critical) case when \( \gamma = 0 \) and the uncontrolled dynamics is unstable. With a feedback boundary control, we will be able to show that the semigroups can be stabilized – but under additional geometric conditions which however are stronger than the ones typically assumed in a boundary stabilization theory of hyperbolic dynamics. It is clear that if \( \Gamma_0 = \emptyset \) then the entire boundary \( \Gamma \) is dissipated and therefore stability results would hold true without any additional geometric restrictions. Assuming \( \Gamma_0 \neq \emptyset \), and \( \Gamma_0 \) is star-shaped (standard condition), classical stability methods (multipliers) do not work due to conflicting signs of radial vector fields on the boundary \( \Gamma_1 \) when acting on tangential derivatives. This fact has been recognized already in [32]. However, due to convexity of \( \Gamma_0 \) and Assumption 1 one obtains the following construction [32]: there exists a vector field \( h(x) = [h_1(x), \ldots, h_d(x)] \in C^2(\overline{\Omega}) \) such that
\[
h \cdot \nu = 0 \text{ on } \Gamma_0. \tag{4.9a}
\]
with $\nu$ being the unit outward normal, and that for some constant $\delta > 0$ and all vector $u(x) \in [L^2(\Omega)]^n$, we have
\[
\int_{\Omega} J(h)|u(x)|^2 d\Omega \geq \delta \int_{\Omega} |u(x)|^2 d\Omega,
\]
(4.9b)
where $J(h)$ represents the Jacobian matrix of $h$.

Remark 4.1. We note that the more general typical star–shaped condition $h \cdot \nu \leq 0$ on $\Gamma_0$ is not sufficient. This is due to the presence of tangential derivatives on uncontrolled part of the boundary which can not be “absorbed” via dissipation by the microlocal argument [30, 43, 29].

This all leads to the following local estimate
\[
\|\partial_\tau u\|_{\Sigma_0} \leq C\|u_t\|_{\Sigma_0} + C\|\partial_\nu u\|_{\Sigma_0} + lot_Q
\]
valid on solutions. Above, $lot_Q$ mean lower order terms on $Q = \Omega \times [0,T]$. By “bending” on the boundary $\Gamma_0$ the radial vector field allows to eliminate its contribution of tangential derivatives. See Remark 4.3.

Remark 4.2. Convexity of $\Gamma_0$ is only one sufficient condition. Several examples where the construction in (4.9) holds for other types of domains are given in [32].

As mentioned earlier, since we have dissipation only on a portion of the boundary, while the remaining part is subject to Neuman/Robin type of boundary conditions, standard multiplier theory with radial vector fields does no apply- due to sign inconsistency in the estimates of tangential boundary integrals. The introduction of the vector field $h \in C^2$ in (4.9) is then critical for the results in this paper.

We start with energetic calculations performed first on regular (strong) solutions. Let’s multiply equation (4.1) by $h \cdot \nabla z$ integrate by parts in $(s,T-s) \times \Omega$, for $s \in [0,T/2)$. This gives
\[
\frac{1}{2} \int_{s}^{T-s} \int_{\Gamma} \left( z^2_t - b|\nabla z|^2 \right) (h \cdot \nu) d\Gamma dt + b \int_{s}^{T-s} \int_{\Gamma} \partial_\nu z (h \cdot \nabla z) d\Gamma dt
\]
\[
= \int_{s}^{T-s} \int_{\Omega} \gamma u_{tt} (h \cdot \nabla z) d\Omega dt + \int_{s}^{T-s} \int_{\Omega} z_t (h \cdot \nabla z) d\Omega dt
\]
\[
+ \frac{b}{2} \int_{s}^{T-s} \int_{\Omega} J(h)|\nabla z|^2 d\Omega dt + \frac{1}{2} \int_{s}^{T-s} \int_{\Omega} \left( z^2_t - b|\nabla z|^2 \right) \text{div}(h) d\Omega dt
\]
\[
- \int_{s}^{T-s} \int_{\Omega} f(h \cdot \nabla z) d\Omega dt,
\]
(4.10)
where $J(h)$ is the Jacobian. Now notice that equipartition of kinetic and potential energy appears in the above identity via the term
\[
\int_{\Omega} \left( z^2_t - b|\nabla z|^2 \right) \text{div}(h) d\Omega,
\]
(4.11)
We next multiply (4.1) by $z\text{div}(h)$ and again integrate by parts to obtain the following identity

$$\frac{b}{2} \int_{s}^{T-s} \int_{\Gamma} \partial_{\nu} z z \text{div}(h) d\Gamma dt = \frac{1}{2} \int_{s}^{T-s} \int_{\Omega} \gamma u_{tt} z \text{div}(h) d\Omega dt$$

$$+ \frac{1}{2} \int_{s}^{T-s} \int_{\Omega} z_{t} z \text{div}(h) d\Omega dt + \frac{1}{2} \int_{s}^{T-s} \int_{\Omega} (b|\nabla z| - z_{t}^{2}) \text{div}(h) d\Omega dt$$

$$- \frac{b}{2} \int_{s}^{T-s} \int_{\Omega} z \nabla z \cdot \nabla (\text{div}(h)) d\Omega dt - \frac{1}{2} \int_{s}^{T-s} \int_{\Omega} f z \text{div}(h) d\Omega dt$$

(4.12)

where we have, as in (4.10), kept the boundary terms on the left–hand–side. Adding (4.10) and (4.12) we obtain

$$\frac{1}{2} \int_{s}^{T-s} \int_{\Gamma} (z_{t}^{2} - b|\nabla z|^{2}) (h \cdot \nu) d\Gamma dt + b \int_{s}^{T-s} \int_{\Omega} \partial_{\nu} z (h \cdot \nabla z) d\Omega dt + \frac{b}{2} \int_{s}^{T-s} \int_{\Omega} \partial_{\nu} z z \text{div}(h) d\Omega dt$$

$$= \int_{s}^{T-s} \int_{\Omega} \gamma u_{tt} (h \cdot \nabla z) d\Omega dt + \frac{1}{2} \int_{s}^{T-s} \int_{\Omega} \gamma u_{tt} z \text{div}(h) d\Omega dt + \frac{b}{2} \int_{s}^{T-s} \int_{\Omega} J(h)|\nabla z|^{2} d\Omega dt$$

$$+ \int_{s}^{T-s} \int_{\Omega} z_{t} (h \cdot \nabla z) d\Omega dt + \frac{1}{2} \int_{s}^{T-s} \int_{\Omega} z_{t} z \text{div}(h) d\Omega dt - \frac{b}{2} \int_{s}^{T-s} \int_{\Omega} z \nabla z \cdot \nabla (\text{div}(h)) d\Omega dt$$

$$- \int_{s}^{T-s} \int_{\Omega} f (h \cdot \nabla z) d\Omega dt - \frac{1}{2} \int_{s}^{T-s} \int_{\Omega} f z \text{div}(h) d\Omega dt,$$

and then the boundary terms can be written more compactly in terms of the interior terms as

$$\int_{s}^{T-s} B(\Gamma)(t) dt = \int_{s}^{T-s} \int_{\Omega} (\gamma u_{tt} - f) M_{h}(z) d\Omega dt + \int_{s}^{T-s} \int_{\Omega} z_{t} M_{h}(z) d\Omega dt$$

$$+ \frac{b}{2} \int_{s}^{T-s} \int_{\Omega} J(h)|\nabla z|^{2} d\Omega dt - \frac{b}{2} \int_{s}^{T-s} \int_{\Omega} z \nabla z \cdot \nabla (\text{div}(h)) d\Omega dt.$$  

(4.13)

by defining $M_{h}(z) := h \cdot \nabla z + \frac{1}{2} z \text{div}(h)$ and

$$B(\Gamma) := \frac{1}{2} \int_{\Gamma} (z_{t}^{2} - b|\nabla z|^{2}) (h \cdot \nu) d\Gamma + b \int_{\Gamma} \partial_{\nu} z M_{h}(z) d\Gamma.$$  

(4.14)

Now, the second part of geometrical condition (4.9) allow us to obtain an estimate for the potential $z$–energy. To see this, first notice that $M_{h}(z)$ is controlled by the potential energy, indeed,

$$\|M_{h}(z)\| \leq \sup_{x \in \Omega} (|h(x)| + \text{div}(h)(x)) \left(\|\nabla z\|_{2} + \frac{1}{2}\|z\|_{2}\right) \lesssim \|z\|_{H^{1}(\Omega)} \lesssim \|A^{1/2} z\|_{2},$$  

(4.15)

due to Robin boundary condition imposed on $\Gamma_{0}$ which allows to control $L_{2}$ norms by the gradient.

Moreover, the last term in (4.13) can be estimated as

$$\left| \int_{s}^{T-s} \int_{\Omega} z \nabla z \nabla (\text{div}(h)) d\Omega dt \right| \lesssim \varepsilon \int_{s}^{T-s} \int_{\Omega} \|\nabla z\|^{2} d\Omega dt + C_{\varepsilon} \|z\|^{2}_{L^{2}(s,T-s; L^{2}(\Omega))},$$  

(4.16)

for any $\varepsilon > 0$, due to Peter–Paul’s Inequality and boundedness of $D^{2} h$ in $\overline{\Omega}$. Similarly, for any $\varepsilon > 0$

$$\left| \int_{s}^{T-s} \int_{\Omega} \gamma u_{tt} A_{h}(z) d\Omega dt \right| \lesssim \varepsilon \int_{s}^{T-s} \int_{\Omega} \|\nabla z\|^{2} d\Omega dt + C_{\varepsilon} \int_{s}^{T-s} \int_{\Omega} \gamma |u_{tt}|^{2} d\Omega dt.$$


and finally
\[ \int_{\Omega} z_t M_h(z) d\Omega \bigg|_{s}^{T-s} \lesssim E_1(s) + E_1(T - s). \] (4.17)

Therefore, combining (4.15), (4.16), (4.17) and the second part of assumption (4.9) we obtain, for \( \varepsilon > 0 \) sufficiently small,
\[ \int_{s}^{T-s} \int_{\Omega} |\nabla z|^2 d\Omega dt \lesssim E_1(s) + E_1(T - s) + \int_{0}^{T} D_{\psi}(s) ds + \int_{s}^{T-s} B(\Gamma)(t) dt + \int_{Q} f^2 dQ + \|z\|^2_{L^2(s,T-s;L^2(\Omega))}. \] (4.18)

Now, given the definition of \( E_1 \), we need an estimate for the kinetic energy in order to continue. For that, we multiply (4.1) by \( z \) and again integrate by parts over \( (s,T-s) \times \Omega \) to obtain
\[ \int_{s}^{T-s} \int_{\Omega} [b|\nabla z|^2 - z_t^2] d\Omega dt + b \int_{s}^{T-s} \int_{\Gamma} z \partial_n d\Gamma dt = - \int_{s}^{T-s} \int_{\Omega} \gamma u_t z d\Omega dt - \int_{\Omega} z_t \partial_n d\Omega \bigg|_{s}^{T-s} + \int_{s}^{T-s} \int_{\Omega} f z d\Omega dt. \] (4.19)

Identity (4.19) implies the following upper estimate for the kinetic energy
\[ \int_{s}^{T-s} \int_{\Omega} |z_t|^2 d\Omega dt \lesssim [E_1(s) + E_1(T - s)] + \int_{s}^{T-s} \int_{\Omega} |\nabla z|^2 d\Omega dt + \int_{s}^{T-s} \int_{\Gamma} z \partial_n z d\Gamma dt + \int_{0}^{T} D_{\psi}(s) ds + \int_{Q} f^2 dQ + \|z\|^2_{L^2(s,T-s;L^2(\Omega))}. \] (4.20)
and then combining (4.18) and (4.20) [adding and subtracting the boundary term defining \( A^{1/2}z \) from the gradient], and noticing that the term \( \|\gamma^{1/2} u_t\| \) in \( E_1(t) \) can be obtained by the first two, we conclude
\[ \int_{s}^{T-s} E_1(t) dt \lesssim E_1(s) + E_1(T - s) + \int_{0}^{T} D_{\psi}(s) ds + \int_{s}^{T-s} B(\Gamma)(t) dt + \int_{Q} f^2 dQ + \|z\|^2_{L^2(s,T-s;L^2(\Omega))}. \] (4.21)

where
\[ B(\Gamma) := \frac{1}{2} \int_{\Gamma} \left( z_t^2 - b|\nabla z|^2 \right) (h \cdot \nu) d\Gamma + b \int_{\Gamma} \partial_n z M_h(z) d\Gamma + \int_{\Gamma} z \partial_n z d\Gamma + \kappa_0 \int_{\Gamma_0} |z|^2 d\Gamma_0 \] (4.22)

We notice that for all the computations carried out so far we only needed the second part of our geometric assumption, that is, it would work with any \( C^2 \)-vector field such that its Jacobian is strictly positive. Only now, in the analysis of the boundary term \( B(\Gamma) \) we will use assumption (4.9a) to prove the following lemma.

**Lemma 4.4 (Key Lemma).** The boundary term \( B(\Gamma) \) satisfies the following estimate
\[ \int_{s}^{T-s} B(\Gamma)(t) dt \lesssim E_1(s) + \varepsilon \int_{s}^{T-s} \|\nabla z\|^2_{L^2} dt \]
\[ + C_T \int_{0}^{T} D_{\psi}(s) ds + C_T \|f\|^2_{H^{-1/2+\varepsilon}(Q)} + C_T \text{lot}_\delta(z), \] (4.23)
where \( \text{lot}_\delta(z) \) has the properties stated in Proposition (4.3).
Proof. We need to estimate all the terms in (4.22). We immediately notice that the first boundary term contains $|z_t|^2 - |\nabla z|^2$ evaluated on the boundary. However, on $\Gamma_0$ we have no information on either tangential derivative – which provides contribution unbounded with respect to the energy level. And it is at this point where we will be using orthogonality of the constructed vector field $h$ with respect to normal to the boundary direction. The details are given below.

We start with the last term in (4.22). Notice that

$$
\int_{s}^{T-s} \int_{\Gamma} z \partial_{\nu} z d\Gamma d\tau = \int_{s}^{T-s} \int_{\Gamma_0} z \left( -\frac{\kappa_0(x)}{\lambda} \right) d\Gamma_0 d\tau + \int_{s}^{T-s} \int_{\Gamma_1} z (-\kappa_1(x) z_t) d\Gamma_1 d\tau
$$

$$
= -\frac{1}{\lambda} \int_{s}^{T-s} \int_{\Gamma_0} \kappa_0 z^2 d\Gamma_0 - \frac{1}{2} \int_{\Gamma_1} \kappa_1(x) z^2 (T-s) d\Gamma_1 + \frac{1}{2} \int_{\Gamma_1} \kappa_1(x) z^2 (s) d\Gamma_1
$$

$$
\leq \frac{1}{2} \int_{\Gamma_1} \kappa_1(x) z^2 (s) d\Gamma_1 \lesssim \|A^{1/2} z(s)\|^2 \lesssim E_1(s),
$$
due to trace inequality. Next, we notice that

$$
\int_{s}^{T-s} \int_{\Gamma} \left( z_t^2 - b|\nabla z|^2 \right) (h \cdot \nu) d\Gamma d\tau = \int_{s}^{T-s} \left( \int_{\Gamma_0} + \int_{\Gamma_1} \right) \left( z_t^2 - b|\nabla z|^2 \right) (h \cdot \nu) d\Gamma d\tau
$$

$$
\lesssim \int_{0}^{T} D_\Psi(s) ds - b \int_{s}^{T-s} \int_{\Gamma_1} |\nabla z|^2 (h \cdot \nu) d\Gamma d\tau,
$$
due to $h \cdot \nu = 0$ on $\Gamma_0$ and the definition of the damping term (4.6).

Remark 4.3. Notice that assuming only $h \cdot \nu \leq 0$, would allow to dispense with the term

$$
\int_{s}^{T-s} \int_{\Gamma_0} z_t^2 (h \cdot \nu) \leq 0.
$$

However, the gradient term

$$
\int_{s}^{T-s} \int_{\Gamma_0} |\nabla z|^2 (h \cdot \nu) = \int_{s}^{T-s} \int_{\Gamma_0} (|\partial_{\nu} z|^2 + |\partial_t z|^2) (h \cdot \nu)
$$

where $\partial_t$ indicates derivative in the tangential direction, poses difficulties. Boundary condition on $\Gamma_0$ provide good estimate for the first part. However, for the second no estimate is available unless $z_t|_{\Gamma_0}$ is under control, which is given through the dissipation or $h \cdot \nu = 0$.

Back to (4.25), for $\Gamma_1$ a tangential–trace estimate is available. In fact, using an adaptation of Lemma 2.1 in [28], which was obtained via microlocal analysis of the homogeneous case, and accounting for our non–homogeneity $-\gamma u_t + f$ in (4.1) we obtain

$$
\int_{s}^{T-s} \int_{\Gamma_1} |\partial_t z|^2 d\Gamma_1 d\tau \leq C_T \int_{0}^{T} \int_{\Gamma_1} (|\partial_{\nu} z|^2 + z_t^2) d\Gamma_1 d\tau + C_T \left[ \|\gamma u_t + f\|^2_{H^{-1/2+\delta}(Q)} + lot_\delta(z) \right]
$$

$$
\lesssim C_T \int_{0}^{T} D_\Psi(s) ds + C_T \|f\|^2_{H^{-1/2+\delta}(Q)} + C_T lot_\delta(z),
$$
(4.26)
with $\text{lot}_\delta(z)$ complying with the condition stated in Proposition 4.3. With this we then improve estimate (4.25) as follows
\[
\int_s^{T-s} \int_\Gamma (z_t^2 - b|\nabla z|^2) (h \cdot \nu) d\Gamma dt \lesssim \int_0^T D_\Psi(s) ds + \int_s^{T-s} \int_{\Gamma_1} |\partial_\tau z|^2 d\Gamma dt
\]
\[
\lesssim C_T \int_0^T D_\Psi(s) ds + C_T \|f\|_{H^{-1/2+\delta}(Q)}^2 + C_T \text{lot}_\delta(z),
\]
(4.27) due to (4.26). Finally, we tackle the more involved term. We notice that,
\[
\int_s^{T-s} \int_\Gamma \partial_\nu z M_h(z) d\Gamma dt = \int_s^{T-s} \int_\Gamma \partial_\nu z \left( h \cdot \nabla z + \frac{1}{2} z \text{div}(h) \right) d\Gamma dt
\]
\[
\lesssim \int_s^{T-s} \int_\Gamma \partial_\nu z (h \cdot \nabla z) d\Gamma dt + E_1(s),
\]
(4.28) (4.29) where we have used the fact that the second integral in (4.28) is exactly the one in (4.24) up to an uniformly bounded term. For the first integral in (4.28), we use the identity
\[
\partial_\nu (h \cdot \nabla z) = |\partial_\nu z|^2 (h \cdot \nu) + \partial_\nu z \partial_\tau z (h \cdot \tau)
\]
which is obtained by writing the coordinates of the vector $\nabla z$ in the basis $\{\tau, \nu\}$. This allows us to write, recalling the damping terms (4.6) and the tangential trace inequality (4.26):
\[
\int_s^{T-s} \int_\Gamma \partial_\nu z (h \cdot \nabla z) d\Gamma dt \lesssim \int_s^{T-s} \int_{\Gamma_0} \partial_\nu z \partial_\tau z (h \cdot \tau)
\]
\[
+ C_T \left[ \int_0^T D_\Psi(s) ds + \|f\|_{H^{-1/2+\delta}(Q)}^2 + \text{lot}_\delta(z) \right]
\]
(4.30) and we now invoke Sobolev Embedding’s Theory. Recall that
\[
\lambda \partial_\nu z = -\kappa_0 z \in H^{3/2}(\Gamma_0) \hookrightarrow H^{\delta_1}(\Gamma_0)
\]
($\delta_1 \leq 3/2$) since $z \in H^2(\Omega)$ if $\Psi$ is a classic solution. On the other hand,
\[
\partial_\tau z \in H^{1/2}(\Gamma_0) \hookrightarrow H^{\delta_2}(\Gamma_0)
\]
($\delta_2 \leq 1/2$). Taking any $\delta_1 = \delta \in (0,1/2]$ and $\delta_2 = -\delta_1 \in [-1/2,0)$ we have, by duality pairing along with continuity of the operator $\partial_\tau|_{\Gamma_0} : H^{3/2-\delta}(\Omega) \to H^{-\delta}(\Gamma_0)$
\[
\int_s^{T-s} \int_{\Gamma_0} \partial_\nu z \partial_\tau z (h \cdot \tau) d\Gamma_0 dt \lesssim \int_s^{T-s} \left[ C_\varepsilon \|\partial_\nu z\|_{H^\delta(\Gamma_0)}^2 + \varepsilon \|\partial_\tau z\|_{H^{-\delta}(\Gamma_0)}^2 \right] dt
\]
\[
\lesssim \int_s^{T-s} \left[ C_\varepsilon \|z\|_{H^\delta(\Gamma_0)}^2 + \varepsilon \|z\|_{H^{1/2-\delta}(\Omega)}^2 \right] dt
\]
\[
\lesssim C_\varepsilon \int_s^{T-s} \|z\|_{H^{1/2+\delta}(\Omega)}^2 dt + \varepsilon \int_s^{T-s} \|\nabla z\|_{L^2(\Omega)}^2 dt
\]
\[
\lesssim \varepsilon \int_s^{T-s} \|\nabla z\|_{L^2(\Omega)}^2 dt + C_{T,\varepsilon} \|z\|_{H^{1/2+\delta}(\Omega)}^2, \quad (4.31)
\]
finishing the proof of Lemma 4.4.
Finally, Lemma 4.4 yields Proposition 4.3 after taking $\varepsilon$ small enough.

Our next result aims at improving Lemma 4.3 by absorbing $\text{lot}_\delta(z)$ by the damping. For the linear problem the compactness uniqueness argument used for achieving it is stated below.

**Proposition 4.5.** For $T > 0$ there exists a constant $C_T > 0$ such that the following inequality holds:

$$ \text{lot}_\delta(z) \leq C_T \int_0^T D_\Psi(s) ds $$

(4.32)

**Proof.** As pointed out in the statement of Proposition (4.3), we have

$$ \text{lot}_\delta(z) \leq C_\delta \sup_{t \in [0,T]} \left\{ \|z\|^2_{H^{1-\delta}(\Omega)} + \|z_t\|^2_{H^{-\delta}(\Omega)} \right\} $$

for $\delta \in (0,1/2)$. Then we prove Proposition (4.5) as a corollary of the following Lemma

**Lemma 4.6.** There exists a constant $C_T$ such that

$$ \| (z, z_t) \|^2_{L^2(0,T;H^{1-\delta}(\Omega) \times H^{-\delta}(\Omega))} \leq C_T \int_0^T D_\Psi(s) ds $$

(4.33)

**Proof.** The proof is based on compactness-uniqueness argument. Compactness follows from compactness of Sobolev’s embeddings implicated in the definition of of lower order terms with respect to the finite energy space for variables $(z, z_t)$ which are $H^1(\Omega) \times L_2(\Omega)$. Uniqueness, instead follows from the overdetermination of the wave equation with overdetermined Neuman-Dirichlet data on the boundary $\Gamma_1$. Using the notation of [41], let $X = H^1(\Omega)$, $B = H^{1-\delta}(\Omega)$ and $Y = H^{-\delta}(\Omega)$. Then it follows from [35, Theorem 16.1] that the injection of $X$ in $B$ is compact. Moreover, since $\delta \in (0,1/2)$, [35, Theorem 12.4] allows us to write

$$ Y = H^{-\delta}(\Omega) = [L^2(\Omega), H^{-1}(\Omega)]_\delta, $$

and then the injection of $B$ in $Y$ is continuous (even dense). Introduce the space $\Lambda$ as

$$ \Lambda \equiv \{ v \in L^2(0,T;X); \dot{v} \in L^2(0,T;Y) \} $$

equipped with the norm

$$ \|v\|_\Lambda = \|v\|_{L^2(0,T;X)} + \|\dot{v}\|_{L^2(0,T;Y)}. $$

Then it follows from [41] that the injection of $W$ into $L^2(0,T;B)$ is compact. We are then ready for proving (4.33)

By contradiction, suppose that there exists a sequence of initial data $\{u_{0n}, u_{1n}, u_{2n}\}$ with corresponding $E^n_1(0)$ energy uniformly (in $n$) bounded generating a sequence $\{u_n, \dot{u}_n, \ddot{u}_n\}$ of solutions of problem (2.9) with related sequence $\{z_n = \frac{c^2}{b}u_n + \dot{u}_n, \dot{z}_n = \frac{c^2}{b}\ddot{u}_n + \dddot{u}_n\}$ solutions of problem 3.4 such that

$$ \begin{cases} 
\|z_n\|^2_{L^2(0,T;H^{1-\delta}(\Omega))} + \|\dot{z}_n\|^2_{L^2(0,T;H^{-\delta}(\Omega))} \equiv 1 \\
\frac{c^2}{b} \int_0^T \int_{\Omega} \gamma(\dddot{u}_n)^2 dQ + \int_0^T \int_{\Gamma_1} \kappa_1(\dot{z}_n)^2 d\Sigma_1 \to 0, \text{ as } n \to +\infty.
\end{cases} $$

(4.34)

(4.34b)
From identity (4.5) (with \( f = 0 \)) we see that the uniform boundedness \( E^n_1(0) \) implies uniform boundedness of \( E^n_1(t), \ t \in [0, T] \). Therefore, one might choose a (non–relabeled) subsequence satisfying

\[
z_n \to \text{some } \zeta, \ \text{weak* in } L^\infty(0, T; H^1(\Omega)) \tag{4.35a}
\]
\[
\dot{z}_n \to \text{some } \zeta_1, \ \text{weak* in } L^\infty(0, T; L^2(\Omega)) \hookrightarrow L^2(0, T; H^{-\delta}(\Omega)); \tag{4.35b}
\]
\[
\gamma^{1/2} \dot{u}_n \to \text{some } \eta, \ \text{weak* in } L^\infty(0, T; L^2(\Omega)); \tag{4.35c}
\]

It easily follows from distributional calculus that \( \dot{\zeta} = \zeta_1 \) and, in the limit, the functions \( \zeta \) and \( \eta \) satisfy the equation

\[
\begin{cases}
\dot{\zeta} = b\Delta \zeta - \gamma^{1/2} \dot{\eta} & \text{in } Q \tag{4.36a} \\
\gamma^{1/2} \dot{\zeta} = \frac{c^2}{b} \eta + \dot{\eta} & \text{in } Q \tag{4.36b} \\
\left[ \frac{\partial \zeta}{\partial \nu} + \kappa_1 \dot{\zeta} \right]_{\Sigma_1} = 0; \quad \left[ \frac{\partial \zeta}{\partial \nu} + \kappa_0 \dot{\zeta} \right]_{\Sigma_0} = 0. & \text{on } \Sigma \tag{4.36c}
\end{cases}
\]

plus respective initial data.

It follows from the weak convergence that there exist \( M \) independent of \( n \) such that

\[
\| (z_n, \dot{z}_n) \|_{L^\infty(0, T; H^1(\Omega) \times H^{-\delta}(\Omega))} = \| \dot{z}_n \|_\Lambda \leq M, \tag{4.37}
\]
for all \( n \). Then, by compactness (of \( \Lambda \) in \( L^2(0, T; H^{1-\delta}(\Omega)) \)) there exists a subsequence, still indexed by \( n \), such that

\[
z_n \to \zeta \text{ strongly in } L^2(0, T; H^{1-\delta}(\Omega)). \tag{4.38}
\]

Next we show that \( \eta \) and \( \zeta \) are zero elements. Indeed, from (4.34b) we obtain that \( \gamma^{1/2} \dot{u}_n \to 0 \) in \( L^2(0, T; L^2(\Omega)) \) and \( \dot{z}_n|_{\Gamma_1} \to 0 \) in \( L^2(0, T; L^2(\Gamma_1)) \). This implies that \( \dot{\eta} = 0 \) and \( \dot{\zeta}|_{\Gamma_1} = 0 \). Indeed, the last claim follows from \( \gamma^{1/2} \dot{u}_n \to \dot{\eta} \) in \( H^{-1}(0, T; L^2(\Omega)) \) where by the uniqueness of the limit one must have \( \dot{\eta} \equiv 0 \). Similar argument applies to infer \( \dot{\zeta}|_{\Gamma_1} = 0 \).

Next, passing to the limit as \( n \to \infty \) yields the following over determined (on \( \Gamma_1 \)) problem:

\[
\begin{cases}
\ddot{\zeta} = b\Delta \zeta & \text{in } Q \tag{4.39a} \\
\gamma^{1/2} \ddot{\zeta} = \frac{c^2}{b} \eta \tag{4.39b} \\
\left[ \frac{\partial \zeta}{\partial \nu} \right]_{\Sigma_1} = 0; \quad \left[ \frac{\partial \zeta}{\partial \nu} + \kappa_0 \dot{\zeta} \right]_{\Sigma_0} = 0; \quad \dot{\zeta}|_{\Gamma_1} = 0 & \text{on } \Sigma \tag{4.39c}
\end{cases}
\]

plus respective initial data. The overdetermined \( \zeta \)-problem implies in particular with \( v \equiv \zeta_t \)

\[
\ddot{v} = b\Delta v
\]

with the overdetermined boundary conditions

\[
\frac{\partial v}{\partial \nu} \bigg|_{\Gamma_1} = 0; \quad v|_{\Gamma_1} = 0
\]
which yields overdetermination of boundary data on $\Gamma_1$ for the wave operator. Here we have used contradiction assumption (4.34b). This gives $v \equiv 0$, hence $\zeta_t \equiv 0$ and $\zeta_{tt} = 0$ distributionally. Using this information in (4.36a) yields

$$\Delta \zeta = 0; \quad \frac{\partial \zeta}{\partial \nu} \bigg|_{\Gamma_1} = 0; \quad \left[ \frac{\partial \zeta}{\partial \nu} + \kappa_0 \zeta \right]_{\Gamma_0} = 0.$$ 

Standard elliptic estimate, along with $\kappa_0 > 0$ gives $\zeta \equiv 0$ in $Q$.

Finally, weak* convergence of $\dot{z}_n$ in $L^\infty(0,T; L^2(\Omega))$ and the compacity of $L^2(\Omega)$ into $H^{-\delta}(\Omega)$ (see [35, Theorem 16.1 with $s = 0$ and $\varepsilon = \delta$]) we have $\dot{z}_n(t) \to \dot{\zeta}(t)$ strongly in $H^{-\delta}(\Omega)$ for a.e. $t \in [0,T]$ and this allow us to compute (due to Lebesgue dominated convergence theorem):

$$\lim_{n \to \infty} \| \dot{z}_n \|^2_{L^2(0,T; H^{-\delta}(\Omega))} = \lim_{n \to \infty} \int_0^T \| \dot{z}_n(t) \|^2_{H^{-\delta}(\Omega)} dt = \int_0^T \lim_{n \to \infty} \| \dot{z}_n(t) \|^2_{H^{-\delta}(\Omega)} dt = \| \dot{\zeta} \|^2_{L^2(0,T; H^{-\delta}(\Omega))} = 0$$

since $\dot{\zeta} \equiv 0$ in $Q$. Then, passing with the limit as $n \to \infty$ in (4.34a) we have

$$0 = \| \zeta \|^2_{L^2(0,T; H^{1-\delta}(\Omega))} = 1,$$

which is a contradiction. The Lemma is proved.

Lemma 4.6 implies in a straightforward way the result of the proposition 4.5.

We are ready to establish the exponential decay of the the energy functional $E_1$.

**Theorem 4.7.** Assume that $f = 0$. Hence, the energy functional $E_1$ is exponentially stable, i.e. there exists $T > 0$ and constants $M, \omega > 0$ such that

$$E_1(t) \leq Me^{-\omega t}E_1(0), \quad \text{for } t \geq 0.$$  \hfill (4.40)

**Proof.** For $f = 0$, identity (4.5) implies

$$\left( \int_0^s \! \! + \int_{T-s}^T \right) E_1(t) dt \leq 2sE_1(0).$$

Since $s < T/2$ can be taken arbitrarily small, we fix $s < 1/2$ in the above inequality. Then by dissipativity of $E_1$ (for $f = 0$) along with Propositions 4.3 and 4.5 we infer

$$\int_0^T E_1(t) dt \lesssim E_1(T) + C_T \int_0^T D_\Psi(s) ds.$$  \hfill (4.41)

On the other hand, using identity (4.5) (with $f = 0$) once more, we deduce

$$TE_1(T) \lesssim \int_0^T E_1(t) dt + C_T \int_0^T D_\Psi(s) ds.$$  \hfill (4.42)

Combining (4.41) and (4.42) we arrive at

$$(T - C)E_1(T) + \int_0^T E_1(t) dt \leq C_T \int_0^T D_\Psi(s) ds$$
for some $C > 0$. Choosing $T = 2C$ and replacing the “damping” term using identity (4.5) (with $f = 0$) we rewrite the above estimate as follows

$$E_1(T) + \int_0^T E_1(t) dt \lesssim C_T [E_1(0) - E_1(T)]$$

which implies

$$E_1(T) \leq \frac{C_T}{1 + C_T} E_1(0) = \mu E_1(0),$$

where $0 < \mu < 1$ does not depend on the solution. This implies (4.40) with $\omega = |\ln \mu|/T$ and $M = 1/\mu$.

The result of Theorem 4.7 is the key to establish the exponential stability of of the semigroup $S(t)$, generated by $A$ on $\mathbb{H}$.

### 4.2 Proof of Theorem 2.2

**Part (i).** Notice that the exponential decay for $E_1$ obtained in Theorem (4.7) implies exponential decay of the quantities $\|z\|_{D(A^{1/2})}, \|z_t\|_{L^2(\Omega)}$, and we will show that this implies exponential decay for the total energy $E(t)$, provided that the initial data $u_0$ is controlled with respect to the topology induced by $A^{1/2}$. For this, the only remaining quantity we need to show exponential decay is $\|u\|_{D(A^{1/2})}$ and this follows from the fact that $bu_t + c^2u = z$. Indeed, the variation of parameters formula implies that

$$u(t) = e^{-\frac{c^2}{b}t}u_0 + \int_0^t e^{-\frac{c^2}{b}(t-\tau)}z(\tau) d\tau,$$

then, computing the $D(A^{1/2})$–norm both sides we estimate

$$\|u(t)\|_{D(A^{1/2})} \leq e^{-\frac{c^2}{b}t}\|u_0\|_{D(A^{1/2})} + \int_0^t e^{-\frac{c^2}{b}(t-\tau)}\|z(\tau)\|_{D(A^{1/2})} d\tau$$

hence it follows from (4.40) that

$$\|u(t)\|_{D(A^{1/2})} \leq e^{-\frac{c^2}{b}t}\|u_0\|_{D(A^{1/2})} + M E_1(0) \int_0^t e^{-\frac{c^2}{b}(t-\tau)} - \omega \tau d\tau$$

$$\leq e^{-\frac{c^2}{b}t}E(0) + \frac{(c^2 - b\omega)(e^{-\omega t} - e^{-\frac{c^2}{b}t})}{\omega c^2} ME(0) \leq Me^{-\omega_1 t}E(0).$$

where we have made the benign assumption that $\frac{c^2}{b} > \omega$ from (4.40), as if $\omega \geq \frac{c^2}{b}$ we use formula (4.40) with $\omega_1 := \frac{c^2}{b} - \varepsilon$ so $\omega > \omega_1$ and $\frac{c^2}{b} > \omega_1$.

The proof is complete.

**Part (ii).** The first step towards showing $\mathbb{H}_1$–level stabilization is to derive energy estimate for the higher order energy functional $E_2$. We start with a basic multiplier identity.
\[
(\forall t)(\exists \tilde{c}) \quad \left( \partial_T f \int + \partial_T \rho \int 0 + (i) \mathcal{H} \right)^{\tilde{c}} + \\
\left( \partial_T \rho \| n \nabla \| \int + \tilde{c} \| n \nabla \| \right) \forall + (0) \mathcal{F} \gtrless \tilde{c} \| n \nabla \| \int \tilde{c}^2 + \tilde{c} \| (\mathcal{L}) n \nabla \| q
\]

That is, or as bounded above by the damping, it follows that for each \(0 < \tilde{c} \) there exists \( \tilde{c} \) with \( \tilde{c} \) and \( (i) \mathcal{F} \) are better in the sense that all others are bounded. Now, since all terms in \( (i) \mathcal{F} \) (or \( (i) \mathcal{H} \)) are better, we take the inner product of each term with \( n \nabla \) and integrate in time. By connecting it with one obtains that

\[
z \nabla q = (n \nabla + \tilde{c} q) \nabla
\]

We now derive the estimate for \( \tilde{c} \mathcal{F} \mathcal{H} \). We take the inner product of

\[
\partial_T \rho \| n f \| \int + \partial_T \rho \left[ \| n \nabla \| n \nabla + \tilde{c} \| q \| \right] \int + \partial_T \rho \| n f \| \int =
\]

\[
\partial_T \rho \| n f \| \int + \partial_T \rho \left[ \| n \nabla \| n \nabla + \tilde{c} \| q \| \right] \int
\]

We start by noticing that since \( F \) is classical, we have the following identity holds

\[
\partial_T f \int + \partial_T \rho \int 0 + (i) \mathcal{H} \int \forall + (0) \mathcal{F} \gtrless \tilde{c} \| n \nabla \| \int \tilde{c}^2 + \tilde{c} \| (\mathcal{L}) n \nabla \| q
\]
Then, taking $\varepsilon$ small and using (4.40) we have

$$b\|\Delta u(T)\|_2^2 + c^2 \int_0^T \|\Delta u\|_2^2 \lesssim \mathcal{E}(0) + \int_0^T E_1(\sigma) d\sigma + \int_Q f^2 dQ,$$

(4.48)

From (4.48) we have obtained that $\Delta u(t) \in L^2(\Omega)$. In addition $(u, u_t, u_{tt}) \in \mathbb{H}$ implies that $u(t) \in H^1(\Omega)$ and $u_t(t) \in H^1(\Omega)$. By a standard duality argument one this obtains that $\partial_\nu u(t) \in H^{-1/2}(\Gamma)$. We will be able to improve this regularity by appealing to $\mathbb{H}$ regularity already obtained in the previous section. On the other hand, by using invariance of boundary conditions we also have

$$\partial_\nu u(t)|_{\Gamma_0} = -\kappa_0 u(t) \in H^{1/2}(\Gamma_0) \quad \partial_\nu u(t)|_{\Gamma_1} = -\kappa_1 u(t) \in H^{1/2}(\Gamma_1).$$

By the definition of the norm in $\mathbb{H}_1$, the above implies that $(u, u_t, u_{tt}) \in \mathbb{H}_1$, as desired. Moreover we have a control of the norms:

$$\|(u, u_t, u_{tt})\|_{\mathbb{H}_1} \leq C \|(u, u_t, u_{tt})\|_{\mathbb{H}} + \|\Delta u(t)\|_2 + \|\sqrt{\kappa_0} u(t)\|_{H^{1/2}(\Gamma_0)} + \|\sqrt{\kappa_1} u(t)\|_{H^{1/2}(\Gamma_1)}$$

$$\leq C \left(\|(u, u_t, u_{tt})\|_{\mathbb{H}} + \|\Delta u(t)\|_2\right)$$

which proved the desired regularity in $\mathbb{H}_1$. We are ready to complete the proof.

Let $f = 0$. Adding $E(T) + \int_0^T E(\sigma) d\sigma$ to both sides of (4.48) we obtain,

$$\mathcal{E}(T) + \int_0^T \mathcal{E}(\sigma) d\sigma \lesssim \mathcal{E}(0) + E(T) + \int_0^T E(\sigma) d\sigma$$

$$\lesssim \mathcal{E}(0) + ME(0)e^{-\omega t} + ME(0) \int_0^t e^{-\omega \sigma} d\sigma$$

$$= \mathcal{E}(0) + ME(0)e^{-\omega t} - \omega^{-1} ME(0) \left[e^{-\omega t} - 1\right] < +\infty,$$

for all $t \geq 0$, for some $\omega, M > 0$. By making $T \to \infty$ we see that

$$\int_0^\infty \mathcal{E}(\sigma) d\sigma < +\infty,$$

and the result follows by Pazy–Datko’s Theorem [37].

5 Proof of Theorem 2.3 – Construction of Global $\mathbb{H}_1$-valued Solutions

Our goal now is to prove that fixed-point solutions can be constructed for the nonlinear problem in $\mathbb{H}_1$. To this end, fix $r > 0$ such that $\|\Phi_0\|_{\mathbb{H}_1} \leq r$ and let $X$ be the set defined as

$$X^\beta = \left\{ \Psi = \begin{bmatrix} w \\ w_t \\ w_{tt} \end{bmatrix} \in C([0, T]; \mathbb{H}_1); \sup_{t \in [0, T]} \|\Psi(t)\|_{\mathbb{H}_1} \lesssim r + 1 \quad \text{and} \quad \sup_{t \in [0, T]} \|\Psi(t)\|_{\mathbb{H}} < \beta \right\}$$

where $\beta > 0$ is for the time being a given positive number but we will take it to be sufficiently small later. Moreover, the condition $\sup_{t \in [0, T]} \|\Psi(t)\|_{\mathbb{H}_1} \lesssim r + 1$ simply means that solutions will exist in bounded sets of $C([0, T]; \mathbb{H}_1)$ with respect to $\mathbb{H}$ but this introduces no restriction on the size of the
Recall that \( H \equiv H \in \mathbb{H}_1 \). The number 1 could, then, be replaced by any other positive number. Let’s equip \( X^\beta \) it with the norm
\[
\|\Psi\|_{X^\beta}^2 := \sup_{t \in [0, T]} \|\Psi(t)\|_{\mathbb{H}_1}^2.
\]

We start with a regularity lemma.

**Lemma 5.1.** For \( \Psi = (w, w_t, w_{tt})^T \) let the action \( \mathcal{F} \) on \( \Psi \) be given by
\[
\mathcal{F}(\Psi) = \frac{1}{\tau} \begin{bmatrix}
0 \\
0 \\
w_t^2 + ww_{tt}
\end{bmatrix}.
\]

Then the following assertions hold true:

(i) \( \mathcal{F} \) defines a continuous map \( \mathcal{F} : X^\beta \to C([0, T]; \mathbb{H}_1) \) and, in particular, for each \( t \) the inequality
\[
\| \mathcal{F}(\Psi(t)) \|_{\mathbb{H}_1} \leq \frac{C \beta}{\tau} \|\Psi(t)\|_{\mathbb{H}_1}, \quad \Psi \in X^\beta
\]
holds for some \( C > 0 \) fixed.

(ii) Stronger than continuity, the following estimate holds:
\[
\tau \| \mathcal{F}(\Phi) \|_{C([0, T]; \mathbb{H}_1)} \lesssim \beta^2 + \beta^{3/2} \sqrt{r + 1}.
\]

**Proof.** Recall that \( w_t \in H_{\mathbb{H}_1}^1(\Omega) \hookrightarrow L^4(\Omega) \) and then \( w_t^2 \in C([0, T]; L^2(\Omega)) \). Moreover, since \( H^2(\Omega) \hookrightarrow L^\infty(\Omega) \) it follows that \( ww_{tt} \in C([0, T]; L^2(\Omega)) \). For each \( t \), interpolation inequalities\(^{23}\) give
\[
\tau \| \mathcal{F}(\Phi)(t) \|_{\mathbb{H}_1} = \|w_t^2 + ww_{tt}\|_2 \leq \|w_t\|_4^2 + \|w\|_\infty \|w_{tt}\|_2
\]
\[
\lesssim \|A^{1/2}w_t\|_2^2 + \|A^{1/2}w\|_2^{1/2} \|w\|_H^2(\Omega) \|w_{tt}\|_2
\]
\[
\lesssim \|\Psi(t)\|_{\mathbb{H}_1} \|\Psi(t)\|_{\mathbb{H}_1} \lesssim \beta \|\Psi(t)\|_{\mathbb{H}_1}
\]
which yields (5.2) and, by taking the supremum on both sides, also (i) altogether. Moreover, returning to the intermediate estimate (5.4), we further notice
\[
\tau \| \mathcal{F}(\Phi)(t) \|_{\mathbb{H}_1} \lesssim \|A^{1/2}w_t\|_2^2 + \|A^{1/2}w\|_2^{1/2} \|w\|_H^2(\Omega) \|w_{tt}\|_2
\]
\[
\lesssim \left[ \sup_{t \in [0, T]} \|\Psi(t)\|_{\mathbb{H}_1}^2 \right] + \left[ \sup_{t \in [0, T]} \|\Psi(t)\|_{\mathbb{H}_1} \right]^{3/2} \left[ \sup_{t \in [0, T]} \|\Psi(t)\|_{\mathbb{H}_1} \right]^{1/2}
\]
\[
\lesssim \beta^2 + \beta^{3/2} \sqrt{\sup_{t \in [0, T]} \|\Psi(t)\|_{\mathbb{H}_1}},
\]
which yields (5.3) and completes the proof. \( \square \)

\(^{23}\) \( 2\|w\|_4 \lesssim \|w\|_2^{1/2} \|w\|_H^{1/2} \) for all \( w \in H^1(\Omega) \)
\( 3\|w\|_\infty \lesssim \|w\|_H^{1/2} \|w\|_H^{1/2} \) for all \( w \in H^2(\Omega) \)
The validity of the previous Lemma along with the fact that \( \mathcal{A} \) generates \( C_0 \)-semigroups \( T(t) \) and \( S(t) \) on \( \mathbb{H}_1 \) and \( \mathbb{H} \) respectively, guarantees that, for each \( \Psi \in X \) there exists a unique \( \Phi = (u, u_t, u_{tt})^T =: \Theta(\Psi) \in C([0, T]; \mathbb{H}_1) \) solution of (2.9) characterized as the variation of parameters formula with forcing term \( \mathcal{F}(\Psi) \) and initial condition \( \Phi_0 = (u_0, u_1, u_2) \in \mathbb{H}_1 \), i.e.,

\[
\Theta(\Psi)(t) = T(t)\Phi_0 + \int_0^t T(t - \sigma)\mathcal{F}(\Psi)(\sigma)d\sigma
\]  
(5.5)

and we note that the same formula is valid if we replace \( T(t) \) by \( S(t) \). Moreover, uniform exponential stability implies the existence of numbers \( \omega_0, \omega_1, M_0, M_1 > 0 \) such that

\[
\|T(t)\Phi_0\|_{\mathbb{H}_1} \leq M_1 e^{-\omega_1 t}\|\Phi_0\|_{\mathbb{H}_1} \quad \text{and} \quad \|T(t)\Phi_0\|_{\mathbb{H}} \leq M_0 e^{-\omega_0 t}\|\Phi_0\|_{\mathbb{H}}
\]  
(5.6)

for all \( t \geq 0 \). Among other properties, the exponential stability of the linear problem implies invariance of the map \( \Theta \) in \( X^\beta \), as we make precise below.

**Lemma 5.2.** Given \( \Phi_0 \in \mathbb{H}_1 \) such that \( \|\Phi_0\|_{\mathbb{H}_1} \leq C \). Then, there exist \( \beta > 0 \) and \( \rho_\beta > 0 \) with the property that if \( \|\Phi_0\|_{\mathbb{H}} < \rho_\beta \) then the map \( \Theta \) is \( X^\beta \)-invariant.

**Proof.** Proving this claim is equivalent to prove that there exists \( \beta > 0 \) for which \( \|\Theta(\Psi)(t)\|_{\mathbb{H}_1} \leq r + 1 \) and \( \|\Theta(\Psi)(t)\|_{\mathbb{H}} < \beta \) for all \( t \in [0, T) \) and each \( \Psi \in X^\beta \), provided \( \|\Phi_0\|_{\mathbb{H}} < \rho_\beta \), with \( \rho_\beta \) conveniently chosen. From (5.5) and (5.6) it follows, for each \( t \in [0, T) \),

\[
\|\Theta(\Psi)(t)\|_{\mathbb{H}_1} \leq \|T(t)\Phi_0\|_{\mathbb{H}_1} + \int_0^t \|T(t - \sigma)\mathcal{F}(\Psi)(\sigma)\|_{\mathbb{H}_1} d\sigma
\]

\[
\leq M_1 \left( \|\Phi_0\|_{\mathbb{H}_1} + \int_0^t e^{-\omega_1 (t - \sigma)}\|\mathcal{F}(\Psi)(\sigma)\|_{\mathbb{H}_1} d\sigma \right)
\]

\[
\lesssim M_1 \left[ \|\Phi_0\|_{\mathbb{H}_1} + \frac{C_{\omega_1}}{\tau} \sup_{t \in [0, T]} \tau\|\mathcal{F}(\Psi)(t)\|_{\mathbb{H}_1} \right]
\]

\[
\lesssim M_1 C + M_1 \tau^{-1} C_{\omega_1} \left( \beta^2 + \beta^{3/2} \sqrt{r + 1} \right) \lesssim r + 1,
\]  
(5.7)

provided \( M_1 C < 1/2(r + 1) \) and \( \beta \) is sufficiently small. Moreover, by Lemma 5.1 (and again (5.5) and (5.6))

\[
\|\Theta(\Psi)(t)\|_{\mathbb{H}} \leq \|T(t)\Phi_0\|_{\mathbb{H}} + \int_0^t \|T(t - \sigma)\mathcal{F}(\Psi)(\sigma)\|_{\mathbb{H}} d\sigma
\]

\[
\leq M_0 \left( \|\Phi_0\|_{\mathbb{H}} + \int_0^t e^{-\omega_0 (t - \sigma)}\|\mathcal{F}(\Psi)(\sigma)\|_{\mathbb{H}_1} d\sigma \right)
\]

\[
\lesssim M_0 \|\Phi_0\|_{\mathbb{H}} + \frac{M_0 C_{\omega_0}}{\tau} \sup_{t \in [0, T]} \tau\|\mathcal{F}(\Psi)(t)\|_{\mathbb{H}_1}
\]

\[
\lesssim \rho_\beta + \left( \beta^2 + \beta^{3/2} \sqrt{r + 1} \right) < \beta,
\]  
(5.8)

provided \( \beta \) and \( \rho_\beta < 1/2\beta \) are sufficiently small. \( \square \)

We are then ready to prove that for a (possibly smaller) \( \beta \), the map \( \Theta \) is a contraction.
Lemma 5.3. There exist $\beta > 0$ and $\rho_\beta > 0$ with the property that if $\|\Phi_0\|_\mathcal{H} < \rho_\beta$ then $\Theta$ is a contraction.

Proof. Let $\Psi_1, \Psi_2 \in X^\beta$, $\Psi_1 = (v, v_t, v_{tt})^T$ and $\Psi_2 = (w, w_t, w_{tt})^T$. The key point of this proof is to estimate $\|\mathcal{F}(\Psi_1) - \mathcal{F}(\Psi_2)\|_{C([0,T];\mathcal{H}_1)}$, which is where we start. First notice that, since the first two coordinates of both $\mathcal{F}(\Psi_1)$ and $\mathcal{F}(\Psi_2)$ are zero, we just care about the third one, whose difference, for each $t$, is given by

$$v_t^2 + vv_{tt} - w_t^2 - ww_{tt} = (v_t + w_t)(v_t - w_t) + (v - w)v_{tt} + w(v_{tt} - w_{tt}) = I_1(t) + I_2(t) + I_3(t).$$

(5.9)

Now we estimate the supremum of the $L^2$-norm of $I_1$. For this we notice that a combination of H"older's Inequality with the Sobolev Embedding $H^2_\Gamma(\Omega) \hookrightarrow L^4(\Omega)$ yields

$$\|I_1(t)\|_2 = \|(v_t + w_t)(v_t - w_t)\|_2 \leq (\|v_t\|_4 + \|w_t\|_4) \|v_t - w_t\|_4 \lesssim (\|\nabla v_t\|_2 + \|\nabla w_t\|_2) \|\nabla (v_t - w_t)\|_2 \lesssim \beta\|\Psi_1 - \Psi_2\|_{X^\beta},$$

for each $t$. Then $\sup_{t \in [0,T]} \|I_1(t)\|_2 \leq \beta\|\Psi_1 - \Psi_2\|_{X^\beta}$. Next, for estimating the supremum of the $L^2$-norm of $I_2$ we notice that the sobolev embedding $H^2_{\Gamma_1}(\Gamma) \hookrightarrow L^\infty(\Omega)$ yields

$$\|I_2(t)\|_2 = \|v_{tt}(v - w)\|_2 \leq \|v_{tt}\|_2 \|v - w\|_\infty \lesssim \|v_{tt}\|_2 \|\Delta (v - w)\|_2 \lesssim \beta\|\Psi_1 - \Psi_2\|_{X^\beta},$$

for each $t \in [0,T]$. Then $\sup_{t \in [0,T]} \|I_2(t)\|_2 \leq \beta\|\Psi_1 - \Psi_2\|_{X^\beta}$. Finally, for estimating the supremum of the $L^2$-norm of $I_3$ we will use the (further to the Sobolev embedding $H^2_{\Gamma_1}(\Omega) \hookrightarrow L^\infty(\Omega)$) the interpolation inequality $\|w\|_\infty \lesssim \|\nabla w\|_2^{1/2} \|\Delta w\|_2^{1/2}$ which holds for all $w \in H^2_{\Gamma_1}(\Omega)$. We have

$$\|I_3(t)\|_2 = \|w(v_{tt} - w_{tt})\|_2 \leq \|w\|_\infty \|v_{tt} - w_{tt}\|_2 \lesssim \|\nabla w\|_2^{1/2} \|\Delta w\|_2^{1/2} \|v_{tt} - w_{tt}\|_2 \lesssim \beta^{1/2} \sqrt{r + 1} \|\Psi_1 - \Psi_2\|_{X^\beta},$$

for each $t \in [0,T]$. Then $\sup_{t \in [0,T]} \|I_3(t)\|_2 \lesssim \beta^{1/2} \sqrt{r + 1} \|\Psi_1 - \Psi_2\|_{X^\beta}$.

Therefore, the proof for contractivity goes as follows:

$$\|\Theta(\Psi_1) - \Theta(\Psi_2)\|_{X^\beta} = \sup_{t \in [0,T]} \left\| \int_0^t T(t - \sigma) \left[ \mathcal{F}(\Psi_1) - \mathcal{F}(\Psi_2) \right] d\sigma \right\|_{\mathcal{H}_1} \lesssim \frac{C_\omega}{\tau} \sup_{t \in [0,T]} \|\mathcal{F}(\Psi_1)(t) - \mathcal{F}(\Psi)(t)\|_{\mathcal{H}_1} \lesssim \sup_{t \in [0,T]} (\|I_1(t)\|_2 + \|I_2(t)\|_2 + \|I_3(t)\|_2) \lesssim (2\beta + \beta^{1/2} \sqrt{r + 1}) \|\Psi_1 - \Psi_2\|_{X^\beta} = C_\beta \|\Psi_1 - \Psi_2\|_{X^\beta}$$

(5.10)

owing the property $C_\beta < 1$ to the smallness of $\beta$. □
Notice that exponential stability of the linear problem in $H$ and $H_1$ allows us to obtain the estimates (5.7), (5.8) and (5.10) with right hand side time-independent. This allows us to take $T = \infty$ is all of them and repeat the same construction to obtain a fixed–point of $\Theta$ defined in the whole $\mathbb{R}_+$.

This completes the proof of theorem 2.3 by taking $\rho = \rho_\beta$.

6 Proof of Theorem 2.4-Uniform Nonlinear Stability

In this section we show that one can easily show that the solution of the nonlinear problem decay exponentially to zero as $t \to \infty$ by taking advantage of three facts established in this paper.

(i) The fact that the solution is a fixed point of the map $\Theta$ defined in (5.5), and therefore can be implicitly represented as

$$\Phi(t) = T(t)\Phi_0 + \int_0^t T(t - \sigma)F(\Phi)(\sigma)d\sigma \quad (6.1)$$

(ii) The fact that our existence of global solution result requires smallness of initial data only in the a lower topology and the use of this along with interpolation inequalities allowed us to obtain the key estimate (5.2).

(ii) The fact that the semigroup $T(t)$ in (6.1) is uniformly exponentially stable in both $H$ and $H_1$.

The final result of this section is the following.

Theorem 6.1. There exists $\rho > 0$ such that the solution $\Phi$ constructed in (2.3) is such that

$$\|\Phi(t)\|_{H_1} \leq 2M_1e^{-\frac{\omega}{2}t}\|\Phi_0\|_{H_1} \quad (6.2)$$

for all $t \geq 0$, where $M_1, \omega_1$ are the constants involved in the uniform stability of the linear semigroup $T(t)$.

The proof of this result relies heavily on the facts (i)–(ii) outlined above and a Grownwall type inequality. This inequality seems to have been originally introduced in [1], but here we are using [2, Corollary 1, p. 389]. We state the inequality here for convenience, but in a version which is suitable for our use in what follows. We invite the reader to consult [1, 2] and references therein for more details

Lemma 6.2 (Grownwall–Beesack Inequality). Let $u, f, g, h : \mathbb{R} \to \mathbb{R}$ measurable functions such that $fh, gh$ and $uh$ are integrable. If $u, f, g, h$ are nonnegative and

$$u(t) \leq f(t) + g(t) \int_0^t h(\sigma)u(\sigma)d\sigma \quad (6.3)$$

then

$$u(t) \leq f(t) + g(t) \int_0^t f(\sigma)h(\sigma)\exp\left\{\int_\sigma^t g(s)h(s)ds\right\}d\sigma \quad (6.4)$$
Proof of Theorem 6.1. We use the same constants as in (5.6), that is, we use that
\[ \|T(t)\|_{\mathcal{L}(\mathbb{H}_1)} \leq M_1 e^{-\omega_1 t} \] (6.5)
for all \( t \). Moreover, we know that the solution \( \Phi \) exists in some \( X^\beta \) for \( \beta > 0 \) small and that the whole argument of the proof for the existence of global solution would still be true if one decreased \( \beta \). Therefore, by possibly taking it smaller, we assume
\[ \beta = \beta(\tau) < \frac{\tau \omega_1}{2M_1 C} \] (6.6)
where \( \omega_1 = \omega_1(\tau) \) is the rate of exponential decay of the semigroup \( T(t) \) in \( \mathbb{H}_1 \) for a fixed \( \tau > 0 \). As in the proof of global wellposedness, we take \( \rho = \rho_0 \). We then compute, via (6.5) and (5.2)
\[
\|\Phi(t)\|_{\mathbb{H}_1} \leq \|T(t)\Phi_0\|_{\mathbb{H}_1} + \int_0^t \|T(t-\sigma)\mathcal{F}(\Phi(\sigma))\|_{\mathbb{H}_1} d\sigma
\]
\[
\leq M_1 e^{-\omega_1 t} \|\Phi_0\|_{\mathbb{H}_1} + \int_0^t M_1 e^{-\omega_1 (t-\sigma)} \|\mathcal{F}(\Phi(\sigma))\|_{\mathbb{H}_1} d\sigma
\]
\[
\leq M_1 e^{-\omega_1 t} \|\Phi_0\|_{\mathbb{H}_1} + \frac{M_1C\beta}{\tau} e^{-\omega_1 t} \int_0^t e^{\omega_1 \sigma} \|\Phi(\sigma)\|_{\mathbb{H}_1} d\sigma.
\]
We then apply the Grownwall–Beesack inequality with
\[
u(t) = \|\Phi(t)\|_{\mathbb{H}_1}, \quad f(t) = M_1 e^{-\omega_1 t} \|\Phi_0\|_{\mathbb{H}_1}, \quad g(t) = \frac{M_1C\beta}{\tau} e^{-\omega_1 t}, \quad h(t) = e^{\omega_1 t}
\]
to obtain
\[
\|\Phi(t)\|_{\mathbb{H}_1} \leq M_1 e^{-\omega_1 t} \|\Phi_0\|_{\mathbb{H}_1} + \frac{M_1^2C^2\beta}{\tau} \|\Phi_0\|_{\mathbb{H}_1} e^{-\omega_1 t} \int_0^t \exp \left\{ \frac{M_1C\beta}{\tau} (t - \sigma) \right\} d\sigma
\]
\[
= M_1 e^{-\omega_1 t} \|\Phi_0\|_{\mathbb{H}_1} + \frac{M_1^2C^2\beta}{\tau} \|\Phi_0\|_{\mathbb{H}_1} \exp \left\{ \frac{M_1C\beta}{\tau} (t - \omega_1) \right\} (1 - \exp \left\{ \frac{M_1C\beta}{\tau} t \right\})
\]
\[
\leq M_1 e^{-\omega_1 t} \|\Phi_0\|_{\mathbb{H}_1} + M_1 \|\Phi_0\|_{\mathbb{H}_1} \exp \left\{ \frac{M_1C\beta}{\tau} (t - \omega_1) \right\} \leq 2M_1 e^{-\frac{\omega_1}{2} t} \|\Phi_0\|_{\mathbb{H}_1},
\] (6.7)
and we observe that due to (6.6) we have
\[
\frac{M_1C\beta}{\tau} - \omega_1 < -\frac{\omega_1}{2} < 0.
\]
The proof is complete. \( \square \)

Corollary 6.3. With reference to Section 5, let \( \beta_0 \) be the largest number such that the map \( \Theta \) has a fixed point in \( X^{\beta_0} \) which is, moreover, uniformly exponentially stable as in Theorem 2.4. Let \( \omega : (0, \beta_0] \to \mathbb{R}_+ \) be the function that maps each \( \beta > 0 \) to the decay rate \( \omega(\beta) \). Then there exists another function \( \omega : (0, \beta_0] \to \mathbb{R}_+ \) such that \( \omega(\beta) \geq \omega(\beta) \) for all feasible \( \beta \) and
\[
\lim_{\beta \to 0} \omega(\beta) = \omega_1,
\] (6.8)
where \( \omega_1 \) is the decay rate of the linear semigroup \( T(t) \).
Proof of Corollary 6.3. The proof of Theorem 2.4 already provides a proof of Corollary 6.3. Indeed, it suffices to define $\omega : (0, \beta_0] \to \mathbb{R}_+$ by

$$
\omega(\beta) = \omega_1 - \frac{M_1 C \beta}{\tau} > 0.
$$

\[\square\]

References

[1] P. R. Beesack. *Gronwall inequalities*. Carleton University, Department of Mathematics, 1975.

[2] P. R. Beesack. On some Gronwall–type integral inequalities in n independent variables. *Journal of mathematical analysis and applications*, 100(2):393–408, 1984.

[3] M. Bongarti, S. Charoenphon, and I. Lasiecka. Singular thermal relaxation limit for the Moore–Gibson–Thompson equation arising in propagation of acoustic waves. *Banasia J., Bobrowski A., Lachowicz M., Tomilov Y. (eds) Semigroups of Operators – Theory and Applications, SOTA 2018, Springer Proceedings in Mathematics and Statistics*, 325:147–182, 2020. doi:10.1007/978-3-030-46079-2_9.

[4] M. Bongarti, S. Charoenphon, and I. Lasiecka. Vanishing relaxation time dynamics of the Jordan Moore-Gibson-Thompson equation arising in nonlinear acoustics. *Journal of Evolution Equations*, 21:3553–3584, 2021. doi:10.1007/s00028-020-00654-2.

[5] M. Bongarti and I. Lasiecka. Boundary stabilization of the linear MGT equation with feedback Neumann control. In B. Jadamba, A. A. Khan, S. Migórski, and M. Sama, editors, *Deterministic and Stochastic Optimal Control and Inverse Problems*, chapter 7, pages 150–168. CRC Press, 2021. URL: 10.1201/97810003050575.

[6] M. Bongarti, I. Lasiecka, and J. H. Rodrigues. Boundary stabilization of the linear MGT equation with partially absorbing boundary data and degenerate viscoelasticity. *Discrete & Continuous Dynamical Systems - S*, 2021, to appear.

[7] M. Bongarti, I. Lasiecka, and R. Triggiani. The SMGT equation from the boundary: regularity and stabilization. *Applicable Analysis*, 0(0):1–39, 2021. doi:10.1080/00036811.2021.1999420.

[8] F. Bucci and M. Eller. The Cauchy–Dirichlet problem for the Moore–Gibson–Thompson equation. *Comptes Rendus Mathématique*, 359(7):881–903, 2021. doi:10.5802/crmath.231.

[9] F. Bucci and I. Lasiecka. Feedback control of the acoustic pressure in ultrasonic wave propagation. *Optimization*, 68(10):1811–1854, 2019. doi:10.1080/02331934.2018.1504051.

[10] C. Cattaneo. A form of heat–conduction equations which eliminates the paradox of instantaneous propagation. *Comptes Rendus*, 247:431, 1958. URL: https://ci.nii.ac.jp/naid/10018112216/en/.
[11] C. Cattaneo. Sulla conduzione del calore. In A. Pignedoli, editor, Some Aspects of Diffusion Theory, pages 485–485. Springer Berlin Heidelberg, 2011. doi:10.1007/978-3-642-11051-1_5.

[12] W. Chen and A. Palmieri. A blow-up result for the semilinear Moore-Gibson-Thompson equation with nonlinearity of derivative type in the conservative case. Evolution Equations & Control Theory, 10(4), 2021.

[13] C. I. Christov and P. M. Jordan. Heat conduction paradox involving second-sound propagation in moving media. Physical Review Letters, 94(15):154301, 2005. doi:10.1103/PhysRevLett.94.154301.

[14] J. A. Conejero, C. Lizama, and F. Rodenas. Chaotic behaviour of the solutions of the Moore–Gibson–Thompson equation. Applied Mathematics & Information Sciences, 9(5):2233–2238, 2015. doi:10.12785/amis/090503.

[15] F. Dell’Oro, I. Lasiecka, and V. Pata. The Moore–Gibson–Thompson equation with memory in the critical case. Journal of Differential Equations, 261(7):4188–4222, 2016. doi:10.1016/j.jde.2016.06.025.

[16] F. Dell’Oro and V. Pata. On a fourth–order equation of Moore–Gibson–Thompson type. Milan Journal of Mathematics, 85(2):215–234, 2017. doi:10.1007/s00032-017-0270-0.

[17] F. Dell’Oro and V. Pata. On the Moore–Gibson–Thompson Equation and its relation to linear viscoelasticity. Applied Mathematics & Optimization, 76(3):641–655, 2017.

[18] F. Dell’Oro, I. Lasiecka, and V. Pata. A note on the Moore–Gibson–Thompson equation with memory of type II. Journal of Evolution Equations, 20:1251–1268, 2020. doi:10.1007/s00028-019-00554-0.

[19] F. Ekoue, A. F. Halloy, D. Gigon, G. Plantamp, and E. Zajdman. Maxwell-Cattaneo Regularization of Heat Equation. World Academy of Science, Engineering and Technology, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 7(5).

[20] B. Kaltenbacher. Mathematics of nonlinear acoustics. Evolution Equations and Control Theory, 4(4):447–491, 2015.

[21] B. Kaltenbacher and C. Christian. Avoiding degeneracy in the Westervelt equation by state constrained optimal control. Evolution Equations and Control Theory, 2(2):281–300, 2013.

[22] B. Kaltenbacher, C. Christian, and V. Slobodan. Boundary optimal control of the westervelt and kuznetsov equations. JMAA, 356(2):738–751, 2009.

[23] B. Kaltenbacher and I. Lasiecka. Exponential decay for low and higher energies in the third order linear Moore–Gibson–Thompson equation with variable viscosity. Palestine Journal of Mathematics, 1(1):1–10, 2012.
[24] B. Kaltenbacher, I. Lasiecka, and R. Marchand. Wellposedness and exponential decay rates for the Moore–Gibson–Thompson equation arising in high intensity ultrasound. *Control and Cybernetics*, 40(4):971–988, 2011.

[25] B. Kaltenbacher, I. Lasiecka, and M. Pospieszalska. Wellposedness and exponential decay of the energy in the nonlinear jmgt equation arising in high intensity ultrasound. *Math.Models Methods Appl.Sci*, 22(11):34 p, 2012.

[26] B. Kaltenbacher and V. Nikolic. On the Jordan–Moore–Gibson–Thompson equation: well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time. *Mathematical Models and Methods in Applied Sciences*, 29(13):2523–2556, 2019.

[27] B. Kaltenbacher and V. Nikolić. Vanishing relaxation time limit of the Jordan–Moore–Gibson–Thompson wave equation with Neumann and absorbing boundary conditions. *Pure and Applied Functional Analysis*, 5(1):1–26, 2020.

[28] I. Lasiecka and C. Lebiedzik. Asymptotic behaviour or nonlinear structural acoustic interactions with thermal effects on the interface. *Nonlinear Analysis*, 49(5):703–735, 2002. doi:10.1016/S0362-546X(01)00135-3.

[29] I. Lasiecka and D. Tataru. Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential Integral Equations*, 6(3):507–533, 1993. URL: https://projecteuclid.org:443/euclid.die/1370378427.

[30] I. Lasiecka and R. Triggiani. Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions. *Applied Mathematics and Optimization*, 25(2):189–224, 1992. doi:10.1007/BF01182480.

[31] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. Volume 1*,. Cambridge University Press, Cambridge, 2010.

[32] I. Lasiecka, R. Triggiani, and X. Zhang. Nonconservative wave equations with unobserved Neumann BC: global uniqueness and observability in one shot. *Contemporary Mathematics*, 268:227–326, 2000. doi:10.1007/978-0-387-35690-7_24.

[33] I. Lasiecka and X. Wang. Moore–Gibson–Thompson equation with memory, part II: General decay of energy. *Journal of Differential Equations*, 259(12):7610–7635, 2015. doi:10.1016/j.jde.2015.08.052.

[34] I. Lasiecka and X. Wang. Moore–Gibson–Thompson equation with memory, part I: exponential decay of energy. *Journal of Applied Mathematics and Physics*, 67(2):17, 2016. doi:10.1007/s00033-015-0597-8.

[35] J. L. Lions and E. Magenes. *Non-homogeneous Boundary Value Problems and Applications: vol I, Problemes aux Limites Non–homogenes et Applications*. Springer-Verlag, Berlin, 1972.
[36] R. Marchand, T. McDevitt, and R. Triggiani. An abstract semigroup approach to the third-order Moore–Gibson–Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability. *Mathematical Methods in the Applied Sciences*, 35(15):1896–1929, 2012. doi:10.1002/mma.1576.

[37] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Number 44 in Applied mathematical sciences. Springer, New York, NY, corr. 2. print edition, 1992.

[38] M. Pellicer and J. Solà-Morales. Optimal scalar products in the Moore–Gibson–Thompson equation. *Evolution Equations & Control Theory*, 8(1):203–220, 2019. doi:10.3934/eect.2019011.

[39] R. Sakamoto. *Hyperbolic Boundary Value Problems*. Cambridge University Press, Cambridgeshire, New York, 1st english edition, 1982.

[40] R. Sakamoto. *Hyperbolic Boundary Value problems*. Cambridge University Press, Cambridge, 2009.

[41] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Annali di Matematica Pura et Applicata*, 146(1):65–96, 1986. doi:10.1007/BF01762360.

[42] R. Spigler. More around Cattaneo equation to describe heat transfer processes. *Mathematical Methods in the Applied Sciences*, 43(9):5953–5962, 2020. doi:10.1002/mma.6336.

[43] D. Tataru. On the regularity of boundary traces for the wave equation. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 26(1):185–206, 1998.

[44] R. Triggiani. Sharp interior and boundary regularity of the SMGTJ–equation with Dirichlet or Neumann boundary control. In *Semigroups of Operators – Theory and Applications*, volume 325, pages 379–426. Springer, 2020. doi:10.1007/978-3-030-46079-2_22.