Evolution of stationary flows of a viscous incompressible fluid in a plane diffuser

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Abstract. For the classical formulation of the Jeffery-Hamel problem of the stationary flows of a viscous incompressible fluid in a plane diffuser the solutions are found. The bifurcation of the basic single-mode flow into flows containing sectors of flow and outflow (multi-mode flows) is found. The second bifurcation point (the bifurcation of three mode flow into four and five modes flows) is analyzed as well as the third one. The values of Reynolds number corresponding to the bifurcation points are calculated. The dependence of first bifurcation value vs. the half-expansion angle is presented.

1. Statement of the problem

Let us consider the flat diffuser with the angle between the walls is equal to $2\beta \ (0 < \beta < \pi)$ where a fluid with dynamic viscosity $\nu$ and density $\rho$ flows. Fluid region in the polar coordinates $(r, \theta)$ has the form $|\theta| < \beta$, $r > 0$. The classical formulation of this Jeffery-Hamel problem \[1, \ 2\] demand a constant-power source $Q < 0$ at $r = 0$ (the flow has a singularity). On the walls we have no-slip conditions.

The velocity field for a self-similar solution is radial \[1, \ 3\]

$$v_r = -\frac{Q}{r} V(\theta), \quad v_\theta = 0. \quad (1)$$

The incompressibility condition for an arbitrary smooth function $V(\theta)$ are automatically guaranteeing. For the components of two tensors — the stress and the strain rate — we have

$$v_{rr} = -v_{\theta\theta} = \frac{Q}{r^2} V(\theta), \quad v_{r\theta} = -\frac{Q}{2r^2} V'(\theta);$$

$$\sigma_{rr,\theta\theta} = -p \pm \frac{2\rho Q^2}{r^4 Re} V(\theta), \quad \sigma_{r\theta} = -\frac{\rho Q^2}{r^4 Re} V'(\theta). \quad (2)$$

These components are depends from unknown functions (the pressure $p$ and velocity $V(\theta)$). Now substituting expressions (2) into the two Navier-Stokes equations we can obtain

$$V'' + 4V - Re V^2 = C, \quad C = \text{const}; \quad (3)$$

$$p = \frac{\rho Q^2}{2r^2 Re} (C - 4V). \quad (4)$$
Also we have no-slip conditions (boundary condition) and the important integral condition, which is following from the constant rate of outflow:

$$ V(\pm \beta) = 0, \quad \int_{-\beta}^{\beta} V(\theta) d\theta = 1. \quad (5) $$

The parameters of flow, such as (1), (2), (4), etc., may be calculated after determination from formulas (3) and (5) the unknown function $V(\theta)$ (velocity profile) and constant $C = V''(\pm \beta)$ for arbitrary values $Re < 0$ and $0 < \beta \leq \pi$.

To simplify computational difficulties, usually, the integral condition (5) is ignored. We would not do this, because this approach is strongly distort the classical formulation of Jeffery-Hamel problem.

Also, the non zero velocity value on the axis of diffuser is assumed. As it will be clear below, this assumption leads to ignoring the unsymmetric solutions.

Another widespread approach based on the first integral of formula (3). One can get the set of three transcendental equations with elliptic functions and the unknown integration constants. The solution of this problem leads to sufficient computational difficulties.

Here we introduce another approach to the solution of this problem. We will get a solution for the nonlinear boundary value problem given by formulas (3) and (5) with the help of special developed numerical-analytical method [5]. It is based on the modified Newton algorithm and provide the solution of boundary value problem with high accuracy.

Let us introducing the convenient change of variables: normalized velocity profile $y(x)$ and unknown parameters $g$ and $l$:

$$ y(x) = 2\beta V(\theta), \quad x = \frac{1}{2} \left( \frac{\theta}{\beta} + 1 \right), \quad 0 \leq x \leq 1, $$

$$ a = 4\beta, \quad b = 2\beta Re, \quad \lambda = 8\beta^3 C, \quad \gamma = y'(0) \quad (6) $$

Now we can obtain the target boundary value problem

$$ y'' + a^2 y - b y^2 = \lambda, \quad y(0) = y(1) = 0, \quad y'(0) = \gamma, $$

$$ z' = y - 1, \quad z(0) = z(1) = 0. \quad (7) $$

Here the known parameters $a$ and $b$ characterized the diffuser and the Reynolds number. The problem is to obtain the unknown parameters $\gamma$ and $\lambda$. Then we can find the velocity profile $y(x)$.

Of course, we have a fixed angle $\beta$ of the diffuser in the range $0 < \beta \leq \pi/2$. It means, that parameter $a$ is fixed in the range $0 < a \leq 2\pi$.

With the help of the numerical-analytical method [5] we can find $\gamma(b)$ and $\lambda(b)$ (for fixed $a$ and initial value $b$). By substituting these parameters in Cauchy problem (7) we can integrating it by any standard numerical method and obtain the desired function $y$.

Then, for the other parameter $b$ (Reynolds number), we can use the old values of $\gamma$ and $\lambda$ and refine it by the numerical-analytical method [5].

For the case $Re = 0$, an analytical solution of problem (7) can be presented. For $b = 0$, we have

$$ y_0(x) = \frac{a}{2D} \cos \left( ax - \frac{a}{2} \right), \quad \gamma_0 = \frac{a^2}{2D} \sin \frac{a}{2}, \quad \lambda_0 = \frac{a^3}{2D} \cos \frac{a}{2}, $$

$$ D = \sin \frac{a}{2} - \frac{a}{2} \cos \frac{a}{2} \neq 0. \quad (8) $$

Now we can use this solution as initial point. Thus we realize the procedure of continuation by the parameter.
Figure 1. The curves $\gamma(b)$ and $\lambda(b)$ for $\beta = 10^\circ (\pi/18)$. Subscript 1 corresponding to the basic single-mode flow, subscript 2 and 3 to the double- and triple-mode flows respectively.

2. Numerical solution

Let us apply the described above procedure to Jeffery-Hamel problem. We use the solution (8) corresponding to the case $Re = 0$ as initial point. Using the special developed method for solution of boundary value problem we obtain the unknown $\gamma(b)$ and $\lambda(b)$. Then we make a small step by Reynolds number and take the values of $\gamma(b)$ and $\lambda(b)$ from the previous step as initial values. The method [5] allow us to refine new $\gamma(b)$ and $\lambda(b)$. Now we can substitute these new values to Cauchy problem (7) and by integrating it by any standard numerical method we can obtain the velocity profile $y(x)$.

To be more specific we will take a small angle $\beta = 10^\circ (\pi/18)$, because the diffusers with such angle is used as a part of hydromechanical and engineering mechanisms [6].

The result of such calculations is shown on the Figure 1. On it the curves $\gamma(b)$ and $\lambda(b)$ are depicted. The values of Reynolds number one can estimate by the parameter $b = 2\beta Re$ which is in the interval $0 \geq b \geq -25$.

2.1. The first point of bifurcation of one-mode flow

It is clear from the figure 1 that the single-mode flow has a bifurcation at the critical value $b_* \approx -18.8$ [7]. This type of flows exists only before bifurcation point. After bifurcation the other types of flows are appeared: two-mode and three-mode flows. While Reynolds number is increasing the jets near the walls transforms the single mode flow.

The main direction of fluid in diffuser is to flow out. The inflow jet near the wall transform the one mode flow into two-mode flow. The flow rate $Q$ for this two-mode flow is the same, as for one mode. The velocity profile for two-mode flows are asymmetrical. For these type of flows we have two symmetric curves $\gamma_{(2)}^{\pm}$ (see figure 1). This is because the inflow jet can appear near any wall.
Figure 2. Three-mode flows: 1 and 2 for $b = -21$ coexisting simultaneously; the curve 3 corresponds to $b = b^*$; the curve 4 corresponds to $b = b^*$; 5 is single-mode velocity profile before bifurcation.

Figure 3. Velocity profiles for $b = -10$: basic flow 1, two-mode flows 2 and 3, and three-mode flow 4.

If the inflow jets appear near the both walls simultaneously the three-mode flow is appeared. This type of flow have a critical value $b = b^* \approx -21.7$, when it disappear. As it shown on figure 1 at this value the curves $\gamma_3(b)$ and $\lambda_3(b)$ have vertical tangents and a turning point.

The examples of velocity profiles for the Reynolds number near the first bifurcation point is shown on figure 2. As it was mentioned above one can obtain it by substitute of certain $\gamma(b)$ and $\lambda(b)$ from the figure 1 to the boundary value problem (7). Now it is only Cauchy problem where the boundary conditions are satisfying automatically. The velocity profiles $y(x)$ is shown on figure 3.

So for the first bifurcation point the analysis of flows transformation in plane diffuser is provided.
2.2. The second point of bifurcation of three-mode flow

Now we can establish that the one-, two- and three-mode flows are disappear while the Reynolds number is increasing.

But the new types of flows are arrived. Now three-, four-, and five-mode flows are possible [8], [9].

The role of the basic flow belongs to the new three-mode flow. This new flow has the other
Figure 6. The curves $\gamma(b)$ and $\lambda(b)$ for the third bifurcation point.

structure which is different from the three-mode flow from the first domain. Now it has an outflows near walls and inflow in the center of diffuser. This flow is transformed into four-, and five-mode flows at the second bifurcation point.

The transformation mechanism at the second point of bifurcation is similar to discussed above. For this point the bifurcation value is $b^{(2)\ast} \approx -75.4$.

The picture of second bifurcation in terms of $\gamma(b)$ and $\lambda(b)$ is shown on figure 4. From this figure it is clear that the structures of the second and first bifurcation domains is rather similar.

In the frame of procedure we can again substitute the certain values of $\gamma(b)$ and $\lambda(b)$ from the figure 4 to the boundary value problem (7). Thus we obtain the corresponding velocity profiles. Some examples are shown on figure 5.

Let us analyze the four-mode velocity profile. For this type of flow the velocity value of liquid is zero on the axis of diffuser. That means, that the attempt to solve the Jeffery-Hamel problem, based on proposal that velocity on the center of diffuser is nonzero, is wrong.

Now the second bifurcation point is found and new types of three-, four-, and five-mode flows are calculated. The examples of these new velocity profiles for $b = -60$ are presented on figure 5.

2.3. The third point of bifurcation of three-mode flow

With further increase of Reynolds number three-, four-, and five-mode flows are disappear now. The new types of flows are arrived. Now the new version of five-, so as six-, and seven-mode flows are possible [8], [9].

Now the role of the basic flow belongs to the new five-mode flow. This new flow has the other structure which is different from the five-mode flow from the previous domain. Now it has an outflows near walls and at the center of diffuser and inflow between. This flow is transformed into two six- and one seven-mode flows.

The transformation mechanism of bifurcation at this point is similar to the first and second ones. For this point the bifurcation value is $b^{(3)\ast} \approx -170.0$.

The picture of third bifurcation in terms of $\gamma(b)$ and $\lambda(b)$ is shown on figure 6. Structure of third bifurcation domain is rather similar to the first and second ones.
3. Evolution of first bifurcation point by the diffuser angle

It is interesting to find how the value of the first bifurcation point \( b_\ast (\beta) \) for the interval \( 0 < \beta < \pi/2 \) depends from the diffuser angle [6], [7]. By this curve one can obtain the boundary of the single-mode flow existence. It can be found with the help of the first integral of the first of Eqs. (7):

\[
-b_\ast = 6\xi K(\xi)L(\xi), \quad \beta = K(\xi)/\sqrt{2}, \quad 0 \leq \xi < \infty;
\]

\[
K(\xi) = \int_0^1 \frac{dq}{\sqrt{f(q,\xi)}}, \quad L(\xi) = \int_0^1 \frac{qdq}{\sqrt{f(q,\xi)}}, \quad f(q,\xi) = q(1-q)[2+\xi(1+q)].
\]

It is following from Eqs. (9) that for a very small angle between walls of the diffuser \( \beta = +0 \) the bifurcation value \( b_\ast \) is equal to \(-6\pi\). For the other side for the fluid flow through a flat slot when the angle is \( \beta = \pi/2 \) the bifurcation value is equal to zero. Figure 7 shows the curve \( b_\ast (\beta) \) vs the diffuser half-expansion angle \( 0 < \beta < \pi/2 \).

4. Conclusion

Using the special developed method for solution of boundary value problem [5] based on the modified Newton algorithm and the procedure of continuation by parameter we obtain the solution of Jeffery-Hamel problem. Multi-mode velocity profiles of flows in a flat diffuser are calculated. The analysis for the first, second and third bifurcation points for the flat diffuser are provided. It is found out that the structures and bifurcation mechanisms of all these domains are qualitatively similar to each other. The dependence of first bifurcation point value vs the half-expansion angle of the diffuser is found.

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Figure 7. Critical value \( b_\ast \) vs. the half-expansion angle \( \beta \) of the diffuser.
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