A NOTE ON DIASTATICENTROPY AND BALANCED METRICS

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Abstract. We give an upper bound $\text{Ent}_d(\Omega, g) < \lambda$ of the diastatic entropy $\text{Ent}_d(\Omega, g)$ (defined by the author in [18]) of a complex bounded domain $(\Omega, g)$ in terms of the balanced condition (in Donaldson terminology) of the Kähler metric $\lambda g$. When $(\Omega, g)$ is a homogeneous bounded domain we show that the converse holds true, namely if $\text{Ent}_d(\Omega, g) < 1$ then $g$ is balanced. Moreover, we explicit compute $\text{Ent}_d(\Omega, g)$ in terms of Piatetskii-Shapiro constants.

1. Introduction and statements of the main results

Let $(M, g)$ be a real analytic Kähler manifold with associated Kähler form $\omega$. Fix a local coordinate system $z = (z_1, \ldots, z_n)$ on a neighbourhood $U$ of a point $p \in M$ and let $\phi : U \to \mathbb{R}$ be a real analytic Kähler potential (i.e. $\omega|_U = \frac{i}{2} \partial \bar{\partial} \phi$). By shrinking $U$, we can assume that the potential $\phi(z)$ can be analytically continued to $U \times U$. Denote this extension by $\phi(z, \bar{w})$. The Calabi’s diastasis function $D : U \times U \to \mathbb{R}$ (E. Calabi [4]) is defined by

$$D(z, w) = D_w(z) = \phi(z, \bar{z}) + \phi(w, \bar{w}) + \phi(z, \bar{w}) + \phi(w, \bar{z}).$$

Assume that $\omega$ has global diastasis function $D_p : M \to \mathbb{R}$ centered at $p$. The diastatic entropy at $p$ is defined as

$$\text{Ent}_d(M, g)(p) = \min \left\{ c > 0 \mid \int_M e^{-cD_p} \frac{\omega^n}{n!} < \infty \right\}. \quad (1)$$

The concept of diastatic entropy was defined by the author in [18] (following the ideas developed in [15]) where he obtained upper and lower bound for the first eigenvalue of a real analytic Kähler manifold.

In this paper we study the link between the diastatic entropy $\text{Ent}_d(\Omega, g)$ and the balanced condition of the metric $g$. S. Donaldson in [8], in order to obtain a link between the constant scalar curvature condition on a Kähler metric $g$ and the Chow stability of an its polarization $L$, gave the definition of a balanced metric on a compact manifold. Later, this definition has been generalized, by C. Arezzo and A. Loi in [3] (see also [2]), to the noncompact case.

In this paper we are interested on complex domains $\Omega \subset \mathbb{C}^n$ (connected open subset of $\mathbb{C}^n$) endowed with a real analytic Kähler metric $g$. Let $\omega$ be the Kähler
form associated to \( g \). Assume that \( \omega \) admits a global Kähler potential \( \phi : \Omega \to \mathbb{R} \).

We can define the weighted Hilbert space \( \mathcal{H}_\phi \) of square integrable holomorphic functions on \( \Omega \) with weight \( e^{-\phi} \)

\[
\mathcal{H}_\phi = \left\{ s \in \text{Hol}(\Omega) : \int_\Omega e^{-\phi} |s|^2 \frac{\omega^n}{n!} < \infty \right\}.
\]

(2)

If \( \mathcal{H}_\phi \neq \{0\} \) we can pick an orthonormal basis \( \{ s_j \}_{j=1,...,N} \leq \infty \),

\[
\int_\Omega e^{-\phi} s_j s_k \frac{\omega^n}{n!} = \delta_{jk}
\]

and define its reproducing kernel by

\[
K_\phi (z, \overline{w}) = \sum_{j=0}^{N} s_j (z) \overline{s_j (w)}.
\]

(3)

The so called \( \varepsilon \)-function is defined by

\[
\varepsilon_g (z) = e^{-\phi(z)} K_\phi (z, \overline{z}),
\]

(4)

as suggested by the notation \( \varepsilon_g \) depends only on the metric \( g \) and not on the choice of the Kähler potential \( \phi \) (see, for example, [14, Lemma 1] for a proof).

**Definition 1.** The metric \( g \) is **balanced** if and only if the function \( \varepsilon_g \) is a positive constant.

In the literature the constancy of \( \varepsilon_g \) was study first by J. H. Rawnsley in [22] under the name of \( \eta \)-function, later renamed as \( \varepsilon \)-function in [5]. It also appears under the name of distortion function for the study of Abelian varieties by J. R. Kempf [13] and S. Ji [12], and for complex projective varieties by S. Zhang [24]. (See also [16] and references therein).

The following theorem represents the first result of this paper.

**Theorem 1.** Let \( \Omega \subset \mathbb{C}^n \) be a complex domain and \( g \) a Kähler metric such that \( \lambda g \) is balanced for some \( \lambda > 0 \) then \( \text{Ent}_d (\Omega, g) (z) < \lambda \), for any \( z \in \Omega \).

It is then natural to ask when the converse of the previous theorem holds true. In the next theorem we show that this is the case when \( (\Omega, g) \) is assumed to be homogeneous.

**Theorem 2.** Let \( (\Omega, g) \) be a homogeneous bounded domain, then \( g \) is balanced if and only if \( \text{Ent}_d (\Omega, g) (z) < 1 \), at some point \( z \in \Omega \).

Finally, we explicitly compute the diastatic entropy of a homogeneous bounded domain in terms of the constants \( p_k, b_k, q_k \) and \( \gamma_k \) defined by the Piatetskii-Shapiro’s root structure (see (8) in Appendix below):
Theorem 3. Let \((\Omega, g)\) be a homogeneous bounded domain. Then the diastatic entropy of \((\Omega, g)\) is the positive constant given by

\[
\text{Ent}_d(\Omega, g)(z) = \max_{1 \leq k \leq r} \frac{1 + p_k + b_k + q_k/2}{\gamma_k}, \quad \forall z \in \Omega.
\]

Example 2. If \(g\) is the Bergman metric on \(\Omega\), then \(\gamma_k = 2 + p_k + q_k + b_k, k = 1, \ldots, r\) (see [10, Theorem 5.1] or [20, (2.19)]), so by Theorem 3

\[
\text{Ent}_d(\Omega, g)(z) = \max_{1 \leq k \leq r} \frac{1 + p_k + b_k + q_k/2}{2 + p_k + q_k + b_k}, \quad \forall z \in \Omega. \tag{5}
\]

In particular, when \((\Omega, g)\) is a bounded symmetric domain (i.e. is a homogeneous bounded domain and a symmetric space as Riemannian manifold), there exists integers \(a\) and \(b\) such that

\[
p_k = (k - 1) a, \quad q_k = (r - k) a, \quad b_k = b, \quad \gamma_k = (r - 1) a + b + 2.
\]

where \(r\) is the rank of \((\Omega, g)\). Therefore, denoted by \(\gamma\) the genus of \(\Omega\), by (5) we obtain

\[
\text{Ent}_d(\Omega, g)(z) = \max_{1 \leq k \leq r} \frac{1 + (k - 1) a + b + (r - k) a/2}{\gamma} = \frac{1 + (r - 1) a + b + \gamma - 1}{\gamma}, \quad \forall z \in \Omega.
\]

See also [19] for a similar formula relating the volume entropy with the invariants \(a\), \(b\), and \(r\). The table below summarizes the numerical invariants and the dimension of irreducible bounded symmetric domains according to its type (for a more detailed description of these invariants, see e.g. [1], [24]).

| Type | \(\Omega_{n,n,m}\) | \(\Omega_{II}[n]\) | \(\Omega_{III}[n]\) | \(\Omega_{IV}[n]\) | \(\Omega_V\) | \(\Omega_{VI}\) |
|------|----------------|----------------|----------------|----------------|------|------|
| \(d\) | \(nm\) | \(\frac{(n-1)n}{2}\) | \(\frac{(n+1)n}{2}\) | \(n\) | 16 | 27 |
| \(r\) | \(n\) | \(\left[\frac{n}{2}\right]\) | \(n\) | 2 | 2 | 3 |
| \(a\) | 2, if \(2 \leq n\) | 4 | 1 | \(n - 2\) | 6 | 8 |
| \(b\) | \(m - n\) | 0, if \(n\) even | 0 | 0 | 4 | 0 |
| \(\gamma\) | \(m + n\) | \(2n - 2\) | \(n + 1\) | \(n\) | 12 | 18 |

Figure 1. Numerical invariants of irreducible bounded symmetric domains.

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2. Proofs of Theorem 1, 2 and 3

2.1. Proof of Theorem 1. Assume that $\lambda g$ is balanced, namely that the $\varepsilon$-function is constant. Consider the Kähler potential $\lambda \phi (z)$ of $\lambda g$, let $\lambda \phi (z,w)$ be its analytic continuation and let $K_{\lambda \phi} (z,w)$ be the reproducing kernel for the weighted Hilbert space $H_{\lambda \phi}$. We have

$$\varepsilon_{\lambda g} (z,w) = e^{-\lambda \phi (z,w)} K_{\lambda \phi} (z,w) = C > 0,$$

so $K_{\lambda \phi} (z,w)$ never vanish. Hence, fixed a point $z_0$, the function

$$\psi (z,w) = \frac{K_{\lambda \phi} (z,w) K_{\lambda \phi} (z_0, \overline{z}_0)}{K_{\lambda \phi} (z_0, \overline{z}_0)}$$

is well defined. Note that $\psi (z_0, w) = \psi (z, z_0) = 1$ and that

$$D_{z_0} (z) = \log \psi (z, \overline{z})$$

is the diastasis centered in $z_0$ associated to $g$. So the diastasis $D_{z_0}$ is globally defined and $\lambda D_{z_0}$ is a global Kähler potential for $\lambda g$. Consider the Hilbert space $H_{\lambda D_{z_0}}$ given by:

$$H_{\lambda D_{z_0}} = \left\{ f \in \text{Hol} (\Omega) : \int_{\Omega} e^{-\lambda D_{z_0}} |f|^2 \frac{\omega^n}{n!} < \infty \right\}.$$

Let $K_{\lambda D_{z_0}} (z,w)$ be its reproducing kernel, since the $\varepsilon$-function does not depend on the Kähler potential, by (6) we have

$$\varepsilon_{\lambda g} (z,w) = e^{-\lambda D_{z_0}} (z,w) K_{\lambda D_{z_0}} (z,w) = C$$

where $D_{z_0} (z,w)$ denote the analytic continuation of $D_{z_0} (z) = D_{z_0} (z, \overline{z})$. In particular

$$C = K_{\lambda D_{z_0}} (z,w) = K_{\lambda D_{z_0}} (z_0, \overline{z}_0) \in H_{\lambda D_{z_0}},$$

thus $H_{\lambda D_{z_0}}$ contains the constant functions and

$$\int_{\Omega} e^{-\lambda D_{z_0}} \frac{\omega^n}{n!} < \infty.$$

We conclude, by the very definition of the diastatic entropy, that $\text{Ent}_d (\Omega, g) (z_0) < \lambda$ for any $z_0 \in \Omega$. As wished.

2.2. Proof of Theorem 2. Recall that a homogeneous bounded domain $(\Omega, g)$ is a bounded domain of $\mathbb{C}^n$ with a Kähler metric $g$ such that the group $G = \text{Aut}(\Omega) \cap \text{Isom}(\Omega, g)$ act transitively on it, where $\text{Aut}(\Omega)$ denotes the group of biholomorphism of $\Omega$ and $\text{Isom}(\Omega, g)$ the group of isometries of $(\Omega, g)$. It is well-known that a such domain $\Omega$ is contractible and that $\omega = \frac{i}{2} \partial \overline{\partial} \phi$ for a globally defined Kähler potential $\phi$. Indeed $\Omega$ is pseudoconvex being biholomorphically equivalent to a Siegel domain (see, e.g. [11] for a proof) and so the existence of a global potential follow by a classical result of Hormander (see [6]) asserting that
the equation $\bar{\partial}u = f$ with $f$ $\bar{\partial}$-closed form has a global solution on pseudoconvex domains (see also [19] and the proof of Theorem 4 in [7], for an explicit construction of the potential $\phi$ following the ideas developed in [9]).

We have to prove that if $1 > \text{Ent}_d(\Omega, g)(z)$ then $g$ is balanced (the converse follows by Theorem 1). Assume that $1 > \text{Ent}_d(\Omega, g)(z)$ i.e.

$$\int_{\Omega} e^{Dz} \frac{\omega^n}{n!} < \infty,$$

then the Hilbert space $H_{Dz}$ contains the constant functions and the associated $\varepsilon$-function is strictly positive. By the homogeneity of $(\Omega, g)$ we conclude that $\varepsilon_g$ has to be a positive constant (see the proof of [14, Lemma 1]), that is $g$ is balanced. This conclude the proof of Theorem 2.

2.3. Proof of Theorem 3

In order to prove Theorem 3 we need of the following result:

**Lemma 3.** ([14, Theorem 2] and [14, (12)]) Let $(\Omega, \lambda g)$ be a homogeneous bounded domain and let $p_k, b_k, q_k$ and $\gamma_k$ be the constants defined by [5] in Appendix. Then the metric $\lambda g$ is balanced if and only if

$$\lambda > \frac{1 + p_k + b_k + q_k/2}{\gamma_k}, \quad 1 \leq k \leq r.$$

Fix $z_0 \in \Omega$ and $\lambda > 0$. If $\text{Ent}_d(\Omega, \lambda g)(z_0) < 1$, then by Theorem 2 the metric $\lambda g$ is balanced. Hence by Theorem 1 $\text{Ent}_d(\Omega, \lambda g)(z) < 1$ for any $z \in \Omega$. Since (by (1)),

$$\text{Ent}_d(\Omega, \lambda g)(z) = \frac{\text{Ent}_d(\Omega, g)(z)}{\lambda},$$

we deduce that, if $\text{Ent}_d(\Omega, g)(z_0) < \lambda$ at a point $z_0$, then $\text{Ent}_d(\Omega, g)(z) < \lambda$ for any $z \in \Omega$. In other words we just prove that $\text{Ent}_d(\Omega, g)(z)$ is constant and that

$$\text{Ent}_d(\Omega, g)(z) = \min \{ \lambda > 0 \mid \lambda g \text{ is balanced} \}, \quad \forall z \in \Omega.$$

Therefore we conclude the proof by applying Lemma 3.

3. Appendix

In this section we give the definition of the constants $p_k, b_k, q_k$ and $\gamma_k$ associated to the Piatetskii-Shapiro’s root structure (see [21] for details).

Let $(\Omega \subset \mathbb{C}^n, g)$ be an homogenous bounded domain, and set $G = \text{Aut}(\Omega) \cap \text{Isom}(\Omega, g)$, where $\text{Aut}(\Omega)$ (resp. $\text{Isom}(\Omega, g)$) denotes the group of invertible holomorphic maps (resp. $g$-isometries) of $\Omega$. By [9, Theorem 2 (c)], there exists a connected split solvable Lie subgroup $S \subset G$ acting simply transitively on the domain $\Omega$. Fix a point $p_0 \in \Omega$, we have a diffeomorphism $\iota : S \to \Omega$ with $\iota(h) = h \cdot p_0$. By differentiation, we get a linear isomorphism between the Lie algebra $\mathfrak{s}$ of $S$ and
the tangent space $T_{p_0}M$, given by $X \mapsto X \cdot p_0$. Then the evaluation of the Kähler form $\omega$ on $\mathfrak{s}$ is given by

$$\omega(X \cdot p_0, Y \cdot p_0) = \beta([X, Y]) \quad X, Y \in \mathfrak{s}$$

with a certain linear form $\beta \in \mathfrak{s}^*$. Let $j: \mathfrak{s} \to \mathfrak{s}$ be the linear map defined in such a way that

$$(jX) \cdot p_0 = \sqrt{-1} (X \cdot p_0) \quad \text{for } X \in \mathfrak{s}.$$ 

We have $g(X \cdot p_0, Y \cdot p_0) = \beta([jX, Y])$ for $X, Y \in \mathfrak{s}$, so the right-hand side defines a positive inner product on $\mathfrak{s}$. Let $\mathfrak{a}$ be the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$ with respect to this inner product. Then $\mathfrak{a}$ is a commutative Cartan subalgebra of $\mathfrak{s}$. Given a linear form $\alpha$ on the Cartan algebra $\mathfrak{a}$, we denote by $\mathfrak{s}_\alpha$ the subspace

$$\mathfrak{s}_\alpha = \{ X \in \mathfrak{s} : [C, X] = \alpha(C) X, \forall C \in \mathfrak{a} \}$$

of $\mathfrak{s}$. We say that $\alpha$ is a root if $\mathfrak{s}_\alpha \neq \{0\}$ and $\alpha \neq 0$. Thanks to [21, Chapter 2, Section 3] or [23, Theorem 4.3], there exists a basis $\{ \alpha_1, \ldots, \alpha_r \}$, $(r := \dim \mathfrak{a})$ of $\mathfrak{a}^*$ such that every root is one of the following:

$$\alpha_k, \alpha_k/2, \quad (k = 1, \ldots, r), \quad (\alpha_l \pm \alpha_k)/2, \quad 1 \leq k < l \leq r.$$ 

For $k = 1, \ldots, r$ we define (see [23, Definition 4.7] and [20, (2.7)])

$$p_k := \sum_{i<k} \dim \mathfrak{s}_{(\alpha_k - \alpha_i)/2}, \quad q_k := \sum_{l<k} \dim \mathfrak{s}_{(\alpha_l - \alpha_k)/2}, \quad b_k := \frac{1}{2} \dim \mathfrak{s}_{\alpha_k/2}, \quad \gamma_k := 4 \beta([jA_k, A_k]).$$

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