On varieties of groups generated by wreath products of abelian groups

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To Marine Mikaelian on her birthday

Abstract. Generalizing results of Higman and Houghton on varieties generated by wreath products of finite cycles, we prove that the (direct or cartesian) wreath product of arbitrary abelian groups $A$ and $B$ generates the product variety $\text{var}(A) \cdot \text{var}(B)$ if and only if one of the groups $A$ and $B$ is not of finite exponent, or if $A$ and $B$ are of finite exponents $m$ and $n$ respectively and for all primes $p$ dividing both $m$ and $n$, the factors $B[p^k]/B[p^{k-1}]$ are infinite, where $B[s] = \langle b \in B \mid b^s = 1 \rangle$ and where $p^k$ is the highest power of $p$ dividing $n$.

Introduction

The problem, whether the standard wreath product $A \wr B$ of abelian groups $A$ and $B$ generates the product variety $\text{var}(A) \cdot \text{var}(B)$, is solved by Higman for the case when $A = C_p$ and $B = C_n$ are finite cycles of orders $p$ and $n$, where $p$ is a prime [H59], and by Houghton for the case of arbitrary finite cycles $A = C_m$ and $B = C_n$. Namely, equality $\text{var}(C_m \wr C_n) = \text{var}(C_m) \cdot \text{var}(C_n) = \mathfrak{A}_m \cdot \mathfrak{A}_n$ holds if and only if $m$ and $n$ are coprime. As we are informed by Professor C. Houghton, his result never was published. However this theorem is frequently cited in the literature and can be found, say, in [N68]: this is not only a well-known result of independent interest, but also an argument frequently used in other constructions of the theory of varieties of groups: descriptions of lattices of subvarieties of certain product varieties, basis ranks of varieties (see for example [N68]).

The aim of this paper is to generalize Houghton’s result for the case of arbitrary abelian groups $A$ and $B$. Namely, for arbitrary abelian groups $A$ and $B$ the (direct or cartesian) wreath product of groups $A$ and $B$ generates the product variety $\text{var}(A) \cdot \text{var}(B)$ if and only if at least one of the groups $A$ and $B$ is not of finite exponent, or if $A$ and $B$ are of finite exponents $m$ and $n$ respectively and for all primes $p$ dividing both $m$ and $n$, the factors $B[p^k]/B[p^{k-1}]$ are infinite, where $B[s] = \langle b \in B \mid b^s = 1 \rangle$ and where $p^k$ is the highest power of $p$ dividing $n$ (Theorem 6.1). As we will see below these factors $B[p^k]/B[p^{k-1}]$ have very “understandable” structure and our criterion is easily applicable in concrete situations.

The structure of infinitely generated abelian groups of non-finite exponent is complicated and at first sight such a generalization may demand techniques very different from those of critical groups, of Cross varieties or of finite nilpotent $p$-groups. The
main idea that enables us to deal with the case of infinitely generated groups is the following main dichotomy: each abelian group is either of finite exponent and, thus, is a direct sum of (possibly infinitely many) copies of some finitely many cycles of prime power orders, or is a discriminating group (see definitions and notations below). The point is that if the “active” group $B$ is a discriminating group, then the cartesian or direct wreath product of $A$ and $B$ always generates (and discriminates) the variety $\text{var}(A) \cdot \text{var}(B)$ and we can restrict ourselves to the first case of the dichotomy. In this case if $p$ is a prime dividing $n$, then the $p$-primary component $B_p$ of $B$ is simply a direct sum of some (possibly infinitely many) copies of cycles $C_p, C_p^2, \ldots, C_p^{\infty}$, and our condition $|B[p^k]/B[p^{k−1}]| = \infty$ simply means that the direct decomposition of $B_p$ contains infinitely many summands isomorphic to the cycle $C_p^\infty$.

Since the proof of Theorem 6.1 consists of consideration of several cases and subcases, we have divided it into parts which occupy Sections 2–5 and each one of them is presented as an independent result (Theorems 2.5, 3.3, 4.5, 5.5 closing corresponding sections).

For arbitrary groups $A$ and $B$ the cartesian wreath product $A \wr B$ and direct wreath product $A \text{wr} B$ generate the very same variety of groups. We build our construction for the case of cartesian wreath products, bearing in mind, that our proofs are also true for direct wreath products of groups. Only in a few cases do we consider direct wreath products for some specific details of the proofs.

For general information on the theory of groups we refer to [R96, KM96]. Following [N68, N64] we denote by $A \wr B$ the cartesian wreath product of groups $A$ and $B$ and by $A \text{wr} B$ the direct wreath product of these groups. The base groups of the cartesian and direct wreath products of $A$ and $B$ will be denoted by $A^B$ and by $A^{(B)}$ respectively. Detail information on wreath products can be found in [N68, M95, N64, KM96]. For general information on varieties of groups we refer to the book of Hanna Neumann [N68]. We reserve notations $\mathfrak{A}$, $\mathfrak{A}_n$, $\mathfrak{N}_c$ and $\mathfrak{B}_c$ for varieties of all abelian groups, of all abelian groups of exponent dividing $n$, of all nilpotent groups of class at most $c$, and of all groups of exponents dividing $e$ respectively. For a set $\mathfrak{X}$ of groups we denote by $\text{var} \mathfrak{X}$, as usual, the variety generated by $\mathfrak{X}$. Information on the notion of discriminating group can be found in [BNN64, N68]. See also the articles of Bryce [B70, B76] and of Kovács and Newman [KN94] for results of more general nature related to the material of this paper. We write abelian groups additively, all other groups will be written multiplicatively. Background information on abelian groups used in this paper can be found in [F70, R96, KM96].

I am extremely grateful to Professor Alexander Yurievich Ol’shanskii, who introduced me to varieties of groups and guided and encouraged me in all parts of my work during my post-graduate study at the Lomonosov Moscow State University, where most of this investigation was done.

1. Wreath products and operations $Q, S, C$

As usual, for a given set $\mathfrak{X}$ of groups we denote by $Q\mathfrak{X}$, $S\mathfrak{X}$ and $C\mathfrak{X}$, the sets of all homomorphic images, subgroups and cartesian products of groups of $\mathfrak{X}$ respectively. According to Birkhoff’s Theorem [B35, N68], for the given set $\mathfrak{X}$ of groups the variety $\text{var} \mathfrak{X}$ can be realized as: $\text{var} \mathfrak{X} = Q\mathfrak{X} \cdot S\mathfrak{X} \cdot C\mathfrak{X}$.

For given $\mathfrak{X}$ and $\mathfrak{Y}$ denote $\mathfrak{X} \text{wr} \mathfrak{Y} = \{X \text{wr} Y \mid X \in \mathfrak{X}, Y \in \mathfrak{Y}\}$ and $\mathfrak{X} \text{wr} \mathfrak{Y} = \{X \text{wr} Y \mid X \in \mathfrak{X}, Y \in \mathfrak{Y}\}$. Since the product variety $\text{var} \mathfrak{X} \cdot \text{var} \mathfrak{Y}$ consists of all
extensions of groups $X^* \in \var \mathcal{X}$ by groups $Y^* \in \var \mathcal{Y}$ and, since by Kaloujnine and Krasner Theorem [KK52] each extension of such a type can be embedded into the appropriate wreath product $X^* \wr W^Y$, we get that the set $\mathcal{X} \wr \mathcal{Y}$ generates the variety $\var(\mathcal{X} \cdot \var \mathcal{Y})$ if and only if for each pair $X^* \in \mathcal{QSC} \mathcal{X}$ and $Y^* \in \mathcal{QSC} \mathcal{Y}$ variety $\var(\mathcal{X} \wr \mathcal{Y})$ contains $X^* \wr Y^*$.

The following lemmas, however, show that, to see whether $\var(\mathcal{X} \wr \mathcal{Y}) = \var(\mathcal{X} \cdot \var \mathcal{Y})$, for purposes of the current paper we have to check just one of six conditions assumed, namely, whether for abelian sets $\mathcal{X}$ and $\mathcal{Y}$ of groups the variety $\var(\mathcal{X} \wr \mathcal{Y})$ contains wreath products $X \wr Y$ for every $X \in \mathcal{X}$ and $Y^* \in \mathcal{Y}$.

**Lemma 1.1.** For arbitrary sets $\mathcal{X}$ and $\mathcal{Y}$ of groups and arbitrary groups $X^*$ and $Y^*$, where $X^* \in \mathcal{QX}$, $X^* \in \mathcal{SX}$ or $X^* \in \mathcal{CX}$ and where $Y \in \mathcal{Y}$, the group $X^* \wr Y^*$ belongs to variety $\var(\mathcal{X} \wr \mathcal{Y})$.

**Proof.** If $X^*$ is the homomorphic image of some $X \in \mathcal{X}$ under a homomorphism $f$, then for arbitrary $Y \in \mathcal{Y}$ the group $X^* \wr Y^*$ is the homomorphic image of the group $X \wr Y$ under the homomorphism $f_W: y \varphi \mapsto y \varphi_f$, where $y \in Y$, $\varphi \in X^Y$ and where $\varphi_f \in (X^*)^Y$ is set as: $\varphi_f(g) = f(\varphi(g))$ for each $g \in Y$ [N68, 22.11].

If $X^*$ is the subgroup of some $X \in \mathcal{X}$, then, clearly, $X^* \wr Y^*$ is the subgroup of $X \wr Y$ [N68, 22.12].

If $X^*$ is cartesian product of some groups $X_i \in \mathcal{X}$, $i \in I$, then we can define an embedding of $X^* \wr Y$ into the cartesian product $W^* = \prod_{i \in I}(X_i \wr Y)$ by the following rule: $y \varphi \rightarrow \theta \in W^*$, where $y \in Y$, $\varphi \in (X^*)^Y$ and $\theta$ it defined as $\theta(i) = y \varphi_i$ with $\varphi_i \in X_i^Y$, $\varphi_i(g) = [\varphi(g)](i)$, $g \in Y$, $i \in I$. \hfill \Box

**Lemma 1.2.** For arbitrary sets $\mathcal{X}$ and $\mathcal{Y}$ of groups and arbitrary groups $X$ and $Y^*$, where $X \in \mathcal{X}$ and where $Y^* \in \mathcal{SY}$, the group $X \wr Y^*$ belongs to variety $\var(\mathcal{X} \wr \mathcal{Y})$.

Moreover, if $\mathcal{X}$ is a set of abelian groups, then for each $Y^* \in \mathcal{SY}$ the group $X \wr Y^*$ also belongs to $\var(\mathcal{X} \wr \mathcal{Y})$.

**Proof.** The first statement of the lemma is obvious (see [N68, 22.13]).

Since the cartesian and direct wreath products $H \wr G$ and $H \wr G$ of arbitrary groups $H$ and $G$ generate the same variety [N68, 22.31, 22.32], it is sufficient to show that, if $Y^*$ is a homomorphic image of some $Y \in \mathcal{Y}$ under some homomorphism $h$, then the direct wreath product $X \wr Y^*$ is the homomorphic image of $X \wr Y$ under some homomorphism $h_W$, provided that, $X$ is abelian. $h_W$ is defined by its values $h_W(y)$ and $h_W(\varphi_{x,y})$ over the following set of elements generating $X \wr Y$:

$$\{ y \in Y, \varphi_{x,y} \in X^Y \mid \varphi_{x,y}(y) = x \text{ and } \varphi_{x,y}(g) = 1 \text{ for } g \in Y \setminus \{y\}, \ x \in X \}.$$ 

Namely:

$$h_W: y \mapsto h(y) \quad \text{and} \quad h_W: \varphi_{x,y} \mapsto \varphi_{x,h(y)} \in X^{(Y^*)}.$$ 

See also [S65, B63]. \hfill \Box

In particular, if each of $\mathcal{X}$ and $\mathcal{Y}$ consist of one group only, it follows from the previous two lemmas that:

**Lemma 1.3.** For arbitrary groups $A$ and $B$, if $A^* \cong A/N$ ($N$ is any normal subgroup of $A$), $A^* \leq A$ or $A^* = \prod_{i \in I} A$ ($I$ is any index set), then $A^* \wr B \in \var (A \wr B)$.

On the other hand, if $B^* \cong B/K$ ($A$ is abelian and $K$ is any normal subgroup of $B$) or if $B^* \leq B$, then $A \wr B^* \in \var (A \wr B)$. 

Corollary 1.4. If $A$ and $B$ are abelian groups, then $\text{var}(A \text{Wr} B) = \text{var}(A) \cdot \text{var}(B)$ if and only if $\text{var}(A \text{Wr} B)$ contains the wreath product $A \text{Wr} (\prod_{i \in I} B)$ for every index set $I$.

Remark 1.5. As we will see in Section 7, an even stronger result of independent interest can be proved: $\text{var}(A \text{Wr} B) = \text{var}(A) \cdot \text{var}(B)$ holds for abelian groups $A$ and $B$ if and only if $\text{var}(A \text{Wr} B)$ contains the wreath product $A \text{Wr} (B \oplus B)$ of $A$ and of the direct sum of two copies of $B$ (see Theorem 7.1).

2. Discriminating sets of groups
and the case of abelian groups of non-finite exponents

Let us begin by considering the case of wreath products of abelian groups $A$ and $B$, where at least one of these groups is not of finite exponent. As we will see, this situation can be described via properties of discriminating sets of groups.

Definition 2.1 (see [BNNN64]). The set $\mathfrak{D}$ of groups is said to be discriminating, if for arbitrary finite word set $V$ with the property that, for each $w \in V$ there exists a homomorphism $\delta_w$ of a free group $F_n$ into some group of $\mathfrak{D}$, such that $\delta_w(w) \neq 1$, there exist a group $D \in \mathfrak{D}$ and a single homomorphism $\delta$ of $F_n$ into $D$, such that $\delta(w) \neq 1$ for all $w \in V$.

A discriminating set of groups $\mathfrak{D}$ can be described by the following property: every finite set of identities $\{ w \equiv 1 \mid w \in V \}$ that can be separately falsified in some groups $\{ D_w \in \mathfrak{D} \mid w \in V \}$ can also be simultaneously falsified in a group $D = D_V \in \mathfrak{D}$ for certain choice of values $d_1, d_2, \ldots, d_n \in D$. Every discriminating set $\mathfrak{D}$ discriminates the variety $\text{var} \mathfrak{D}$ generated by $\mathfrak{D}$, that is, $\mathfrak{D} \subseteq \text{var} \mathfrak{D}$ and for every finite set $V$ of words in, say, $n$ variables, none of which is identically 1 in $\text{var} \mathfrak{D}$ there is a group $D \in \mathfrak{D}$ and elements $d_1, d_2, \ldots, d_n \in D$ such that for all $w \in V$ $w(d_1, d_2, \ldots, d_n) \neq 1$ holds [BNNN64]. Discriminating set $\mathfrak{D}$ always generates $\text{var} \mathfrak{D}$. If a discriminating set consists of one group $D$ we term discriminating group $D$.

Lemma 2.2 (see [BNNN64]). If $\mathfrak{D}$ discriminates the variety $\mathfrak{U} = \text{var} \mathfrak{D}$ and for the given set $\mathfrak{D}_1$ the relations $\mathfrak{D} \subseteq \mathfrak{Q}\mathfrak{S}\mathfrak{D}_1$ and $\mathfrak{D}_1 \subseteq \mathfrak{U}$ hold, then $\mathfrak{D}_1$ also discriminates $\mathfrak{U}$.

Now we can prove the following:

Lemma 2.3. If the group $B$ is not of finite exponent, then the cartesian wreath product $A \text{Wr} B$ generates the variety $\text{var}(A) \cdot \text{var}(B) = \text{var}(A) \cdot \mathfrak{A}$.

Proof. Assume, firstly, that $B$ contains an element $c$ of infinite order. Then $B$ contains an infinite cycle $C = \langle c \rangle$ which is a discriminating group for the variety $\mathfrak{A}$ [BNNN64]. According to Lemma 2.2, $B$ also discriminates $\mathfrak{A}$. Therefore for an arbitrary (and not only abelian) group $A$ the wreath product $A \text{Wr} B$ discriminates $\text{var}(A) \cdot \mathfrak{A}$ because in this situation the direct wreath product $A \text{wr} B$ discriminates $\text{var}(A) \cdot \mathfrak{A}$ [BNNN64]: $A \text{wr} B \subseteq S(A \text{Wr} B)$.

Assume now that $B$ is not of finite exponent but that $B$ contains no element of infinite order. Then there exists a sequence of elements
\begin{equation}
(2.1) \quad c_1, c_2, \ldots \in B
\end{equation}
such that for arbitrary $l \in \mathbb{N}$ there is such a $c_{i(l)}$ whose exp $c_{i(l)} \geq l$. Let
\begin{equation}
(2.2) \quad w_1(x_1, \ldots, x_d), \ldots, w_k(x_1, \ldots, x_d)
\end{equation}
be a finite set of words none of which is an identically 1 for all abelian groups (we can assume all these words to contain the same variables \(x_1, \ldots, x_d\) because the number of these words is finite). Since the infinite cycle \(C = \langle c \rangle\) discriminates \(\mathfrak{A}\), there exist elements \(c^{i_1}, \ldots, c^{i_d} \in C\) \((i_1, \ldots, i_d \in \mathbb{Z})\) such that

\[
w_1(c^{i_1}, \ldots, c^{i_d}) \neq 1, \ldots, w_k(c^{i_1}, \ldots, c^{i_d}) \neq 1. \tag{2.3}
\]

Now let us consider all these values (2.3) together with all values of all subwords of words (2.2) over elements \(c^{i_1}, \ldots, c^{i_d}\). Since in this way we will get only finitely many values, that is, only finitely many elements \(c^j \in C\), we can choose a power \(c^{j_0}\), such that the absolute value of \(j_0\) is greater than that of all \(j\)’s obtained. Now take such a \(c_{i(j_0)}\) in (2.1) that \(\exp c_{i(j_0)}\) is greater than \(|j_0|\). Since, clearly, all values (2.3) remain unchanged if they are calculated not in \(C\) but in \(\langle c_{i(j_0)} \rangle\) (that is, modulo \(\exp c_{i(j_0)}\)), we get that cycles

\[
\langle c_1 \rangle, \langle c_2 \rangle, \ldots \leq B
\]
form a discriminating set for the variety \(\mathfrak{A}\). And since \(\{\langle c_1 \rangle, \langle c_1 \rangle, \ldots\} \subseteq SB\), the group \(B\) discriminates \(\mathfrak{A}\) according to Lemma 2.2. \(\square\)

An analog of this lemma for the “passive” group \(A\) is also true:

Lemma 2.4. If the group \(A\) is not of finite exponent, then cartesian wreath product \(A \text{ Wr } B\) generates the variety \(\var(A) \cdot \var(B) = \mathfrak{A} \cdot \mathfrak{A}_n\).

Proof. Taking into account Lemma 2.3 we assume, without loss of generality, that \(B\) is a group of finite exponent \(n\). Since the infinite cycle \(C\) belongs to \(\var(A) = \mathfrak{A}\), it sufficient, according to Lemma 1.3, to prove that

\[
\var(C \text{ Wr } C_n) = \var(A) \cdot \var(B) = \mathfrak{A} \cdot \mathfrak{A}_n,
\]
where \(C_n\) is a finite cycle of order \(n\). We can choose infinitely many cycles

\[
C_{p_1}, C_{p_2}, \ldots \in \mathbb{Q}(C)
\]
of orders \(p_1, p_2, \ldots\) all coprime to \(n\) and to each other. Thus \(\var(C \text{ Wr } C_n)\) contains each one of the wreath products \(C_{p_1} \text{ Wr } C_n, C_{p_2} \text{ Wr } C_n, \ldots\) and, thus, varieties \(\mathfrak{A}_{p_1} \cdot \mathfrak{A}_n, \mathfrak{A}_{p_2} \cdot \mathfrak{A}_n\) generated by these wreath products according to the result of Houghton \([\text{N68}]\). It remains to use the fact that

\[
(\mathfrak{A}_{p_1} \cdot \mathfrak{A}_n) \cup (\mathfrak{A}_{p_2} \cdot \mathfrak{A}_n) \cup \cdots = (\mathfrak{A}_{p_1} \cup \mathfrak{A}_{p_2} \cup \cdots) \cdot \mathfrak{A}_n = \mathfrak{A} \cdot \mathfrak{A}_n.
\]

\(\square\)

We collect the information of Lemmas 2.3 and 2.4 below:

Theorem 2.5. If at least one of the groups \(A\) and \(B\) is not of finite exponent, then cartesian or direct wreath product of groups \(A\) and \(B\) generates the variety \(\var(A) \cdot \var(B)\).

3. The case of finitely generated abelian groups

Assume \(A\) and \(B\) to be arbitrary finitely generated abelian groups. If in a direct decomposition of \(A\) or \(B\) an infinite cycle is present, then we apply the construction of Section 2. So what we have to deal with are merely finite abelian groups \(A\) and \(B\).
LEMMA 3.1. For finite abelian groups $A$ and $B$ of exponents $m$ and $n$ respectively the wreath product of $A$ and $B$ generates the variety $\text{var}(A) \cdot \text{var}(B) = \mathcal{A}_m \cdot \mathcal{A}_n$ if and only if the exponents $m$ and $n$ are coprime.

REMARK 3.2. So, as we see, the results of Higman [H59] and Houghton [N68] remain true not only for finite cycles, but also for arbitrary finite groups. On the other hand, as the familiar example $\text{var}(C_p \text{Wr} \prod_{i=1}^{\infty} C_p) = \mathcal{A}_p \cdot \mathcal{A}_p$ shows ($p$ is a prime), the criterion of Higman and Houghton has no direct analog for the case of infinite groups, even for the case of infinite groups of finite exponent.

PROOF OF LEMMA 3.1. If $\text{var}(A) \cdot \text{var}(B) = \mathcal{A}_m \cdot \mathcal{A}_n = \text{var}(A \text{Wr} B)$, then $\mathcal{A}_m \cdot \mathcal{A}_n$ is a Cross variety [N68] and $m$ and $n$ are coprime according to result of Šmelkin on product varieties of group generated by a finite group [S65].

On the other hand, if the condition of the lemma is satisfied, we can choose a cyclic subgroup $\langle a_m \rangle$ of order $m$ in $A$ and a cyclic subgroup $\langle b_n \rangle$ of order $n$ in $B$. Now according to Lemma 1.3 $\text{var}(A \text{Wr} B) = \text{var}(\langle a_m \rangle \text{Wr} \langle b_n \rangle)$ and the latter is equal $\mathcal{A}_m \cdot \mathcal{A}_n$ according to the result of Houghton.

Let us present the information of Sections 2 and 3 in an “easy-to-use” form:

THEOREM 3.3. Let $A$ and $B$ be arbitrary finitely generated abelian groups with direct decompositions respectively

$$A = C \oplus \cdots \oplus C \oplus C_{p_1}^{n_1} \oplus \cdots \oplus C_{p_s}^{n_s},$$

and $B = C \oplus \cdots \oplus C \oplus C_{q_1}^{k_1} \oplus \cdots \oplus C_{q_d}^{k_d}$. Then the cartesian or direct wreath product of groups $A$ and $B$ generates the product variety $\text{var}(A) \cdot \text{var}(B)$ if and only if

1. at least one of the decompositions of $A$ and $B$ contains a non trivial infinite cycle, that is, $r_A \neq 0$ or $r_B \neq 0$,
2. or all cycles in the decompositions of $A$ and $B$ are finite, that is, $r_A = 0$, $r_B = 0$, and the sets of primes $\{p_1, \ldots, p_s\}$ and $\{q_1, \ldots, q_d\}$ have empty intersection.

4. The case of arbitrary abelian $p$-groups

4.1. Some notations. As we mentioned in Remark 3.2, in the case of infinitely generated abelian groups $A$ and $B$ of finite exponent no analogs of Lemma 3.1 and of Theorem 3.3 do exist. What a variety can be generated by wreath product, say, $C_p \text{Wr}(C_{p^2} \oplus \sum_{i=1}^{\infty} C_p)$ of groups of exponent $p$ and $p^2$? We clear this situation by means of a specially defined function $\lambda(A, B, t)$:

DEFINITION 4.1. For given abelian $p$-groups $A$ and $B$ of finite exponents and for given $t \in \mathbb{N}$ the value of the function

$$\lambda = \lambda(A, B, t),$$

is defined to be the maximum of the nilpotency classes of the $t$-generated groups of variety $\text{var}(A \text{Wr} B)$.

Firstly we have to observe that this definition is correct: $t$-generated groups of $\text{var}(A \text{Wr} B)$ belong to the variety generated by all $t$-generated subgroups of the group $A \text{Wr} B$ [N68, 16.31]. The number of non-isomorphic copies of mentioned $t$-generated subgroups is finite and every one of them is a finite $p$-group and, thus, nilpotent (in the
next subsection we will find concrete upper bounds for nilpotency classes of $t$-generated groups of var $(A \Wr B)$.

For the purposes of the rest of this paper for the given $p$-group $B$ of finite exponent $p^k$ let us denote: $k(B,p) = k$. If $B$ is not a $p$-group but still has finite exponent, then let us denote by $k(B,p)$ the largest $k$ for which $p^k$ divides $\exp B$. Further, for the given $B$ and positive integer $s$ let us denote $B[s] = \langle b \in B | b^s = 1 \rangle$. The factor groups

$$B[p^k]/B[p^{k-1}] = B[p^{k(B,p)}]/B[p^{k(B,p)-1}]$$

will play a key role in our construction. So let us clear what do they mean in our situation of $p$-groups. According to Prüfer’s Theorem the group $B$ is a direct sum of finite cycles which can be arranged as:

(4.1)

$$B = C_{p^{k_1}} \oplus C_{p^{k_2}} \oplus \cdots,$$

where $k_1 = k = k(B,p) \geq k_2 \geq \cdots$. From this decomposition it is clear that

(4.2)

$$B[p^k]/B[p^{k-1}] = \underbrace{C_p \oplus \cdots \oplus C_p}_{\mu \text{ times}},$$

where $\mu$ is such an ordinal that $k_1 = \ldots, k_\mu = k$ and $k_{\mu+1} < k$ (notation is correct for the set of all ordinals greater than $\mu$ can be well-ordered).

4.2. An upper bound for the function $\lambda(A,B,t)$. According to the remark following Definition 4.1, $\lambda(A,B,t)$ is bounded by the maximum of the nilpotency classes of $t$-generated subgroups of $A \Wr B$. Let $H$ be such a subgroup and $A_H$ be its intersection with the base group $A^B$ of $A \Wr B$. So $A_H$ is a group of the variety $\mathfrak{A}_{p^u}$, where $p^u = \exp A$. $A_H$ is normal in $H$ and $H/A_H$ is isomorphic to an, at most, $t$-generated subgroup $A_B$ of

$$(A \Wr B)/A^B \cong A$$

and, therefore, according to Kaloujnine and Krasner Theorem [KK51], we get, that $H$ is embeddable into the cartesian wreath product $A_H \Wr B_H$, where $A_H$ is of exponent $p^u$ dividing $p^u$ and where

$$B_H = C_{k'_1} \oplus C_{k'_2} \oplus \cdots \oplus C_{k'_\ell} \quad (k'_1 \geq k'_2 \geq \cdots \geq k'_\ell)$$

is certain subgroup of $B$. So $B_H$ is finite and $A_H \Wr B_H$ is a direct wreath product. Thus, applying the result of Liebeck [L62] (on the nilpotency class of the direct wreath product of finite abelian $p$-groups), we calculate the class $c$ of $A_H \Wr B_H$:

$$c = \sum_{i=1}^{t} (p^{k'_i} - 1) + (u' - 1)(p - 1)p^{k'_i-1} + 1.$$

And since $u' \leq u$, $k'_1 \leq k_1 = k$, $k'_2 \leq k_2$, \ldots, we have:

**Lemma 4.2.** For given abelian $p$-group $A$ of finite exponent $p^u$ and for the abelian $p$-group $B$ of form (4.1):

(4.3)

$$\lambda(A,B,t) \leq \sum_{i=1}^{t} (p^{k'_i} - 1) + (u - 1)(p - 1)p^{k(B,p)-1} + 1.$$
4.3. An example of a \( t \)-generated group in \( \mathfrak{A}_{p^u} \cdot \mathfrak{A}_{p^k} \), the general criterion for wreath products of abelian \( p \)-groups.

**Example 4.3.** The product variety \( \mathfrak{A}_{p^u} \cdot \mathfrak{A}_{p^k} \) contains the \( t \)-generated group \( T(p, t) = C_{p^u} \wr \sum_{i=1}^{t-1} C_{p^k} \). According to [L62] the nilpotency class of \( T(p, t) \) is equal to:

\[
\nu(p, t) = \sum_{i=1}^{t-1} (p^k - 1) + (u - 1)(p - 1)p^{k-1} + 1.
\]

**Lemma 4.4.** If \( |B[p^k]/B[p^{k-1}]| < \infty \), then for sufficiently large values of \( t \):

\[
\nu(p, t) > \lambda(A, B, t)
\]

and, thus, the group \( T(p, t) \) does not belong to the variety \( \text{var}(A \wr B) \).

On the other hand, if \( |B[p^k]/B[p^{k-1}]| = \infty \), then \( \text{var}(A \wr B) = \mathfrak{A}_{p^u} \cdot \mathfrak{A}_{p^k} \).

**Proof.** Assume \( B[p^k]/B[p^{k-1}] \) is finite and is of order \( p^\mu \), according to (4.2). If \( t \leq \mu \), then all powers of \( p \) on the right side of (4.3) are equal to \( p^k \). But when \( t \) becomes greater than \( \mu \), then the later summands in sum \( \sum_{i=1}^{t-1} (p^k - 1) \) become less or equal to \( p^{k-1} - 1 \). Thus for sufficiently large \( t \):

\[
\nu(p, t) - \lambda(A, B, t) > \sum_{i=1}^{t-1} (p^k - 1) - \left( \sum_{i=1}^{\mu} (p^k - 1) + \sum_{i=\mu+1}^{t-1} (p^{k-1} - 1) \right)
\]

\[
= \sum_{i=\mu+1}^{t-1} \left[ (p^k - 1) - (p^{k-1} - 1) \right] - (p^{k-1} - 1) = (t - 1 - \mu)(p^k - p^{k-1}) + 1 - p^{k-1}.
\]

So taking an arbitrary positive integer \( t_0 > (p^{k-1} - 1)/(p^k - p^{k-1}) + \mu + 1 \) we get that \( \nu(p, t_0) - \lambda(A, B, t_0) > 0 \).

Assume now \( B[p^k]/B[p^{k-1}] \) is infinite, that is, \( \mu \) is an infinite ordinal in (4.2). Thus \( B \) contains a subgroup \( D \) isomorphic with an infinite direct power of cycle \( C_{p^k} \). \( D \) is a discriminating group in the variety \( \mathfrak{A}_{p^k} \) [BNNN64]. Therefore by Lemma 2.2 the group \( B \) itself is a discriminating group for the variety \( \mathfrak{A}_{p^k} \) and so \( \text{var}(A \wr B) = \mathfrak{A}_{p^u} \cdot \mathfrak{A}_{p^k} \).

Now let us summarise:

**Theorem 4.5.** Let \( A \) and \( B \) be arbitrary abelian \( p \)-groups. Then:

1. if at least one of the groups \( A \) and \( B \) is not of finite exponent, then \( \text{var}(A \wr B) = \text{var}(A) \cdot \text{var}(B) \);
2. if \( A \) and \( B \) are groups of finite exponents \( p^u \) and \( p^k \) respectively then \( \text{var}(A \wr B) = \text{var}(A) \cdot \text{var}(B) = \mathfrak{A}_{p^u} \cdot \mathfrak{A}_{p^k} \) if and only if in direct decomposition (4.1) of \( B \) infinitely many cycles of order \( p^k \) are present, that is, if the factor group \( B[p^k]/B[p^{k-1}] \) is infinite.

**Example 4.6.** Applying this result, we easily find an answer to the question asked at the beginning of this section: the group \( C_p \wr (C_p^2 \oplus \sum_{i=1}^\infty C_p) \) does not generate the variety \( \mathfrak{A}_p \cdot \mathfrak{A}_{p^2} \) because decomposition of the “active” group of this wreath product contains only one direct summand of order \( p^2 = p^k \).
5. The case of abelian groups of finite composite exponents

5.1. $p$-groups in variety var $(A \text{ Wr } B)$. Assume $\exp A = m$, $\exp B = n$ and denote for a given prime $p$ (not necessarily dividing both $m$ and $n$) by $A_p$ and $B_p$ the $p$-primary components of $A$ and $B$ respectively.

**Lemma 5.1.** Using this notation:

\begin{equation}
\var (A \text{ Wr } B) \cap \mathfrak{A}_{p^u} \cdot \mathfrak{A}_{p^k} = \var (A_p \text{ Wr } B_p),
\end{equation}

where $u = k(A, p)$, $k = k(B, p)$.

**Proof.** The right side of (5.1) lies in the left side. So it is sufficient to prove that every $p$-group $P$ of var $(A \text{ Wr } B)$ belongs to var $(A_p \text{ Wr } B_p)$. Moreover, since var $(A \text{ Wr } B)$ is a locally finite variety, we can assume $P$ to be a finite group [N68]. We omit the trivial case, when $p$ is coprime with $m$ or with $n$, that is, when $A_p = \{1\}$ or $B_p = \{1\}$. The variety var $(A \text{ Wr } B)$ is generated by the set $\{R_i | i \in I\}$ of all finite subgroups of $A \text{ Wr } B$. So there is a finite subset $\{R_1, \ldots, R_l\}$ of this set, such that $P \in \var (R_1, \ldots, R_l)$ [N68] and:

$P \in \text{ QSC}(R_1, \ldots, R_l)$.

That is, $P$ is a surjective image of a subgroup $R$ of a direct product $R_1 \times \ldots \times R_l$ under some homomorphism $\varphi : R \to P$:

$P = \varphi(R), \quad R \leq R_1 \times \ldots \times R_l$.

$P$ is a $p$-group and, thus, is an image of some Sylow $p$-subgroup $P^*_i$ of $R$. In turn $P^*_i$ is a sub-direct product of its projections $P^*_i$ on $R_i$, $i = 1, \ldots, l$. Denote by $P^*_{i,B}$ the intersection of $P^*_i$ with the base group $A^B$ of $A \text{ Wr } B$. Clearly, $P^*_{i,B}$ is normal in $P^*_i$ and the factor group $P^*_i / P^*_{i,B}$ is isomorphic to some subgroup $P^*_{i,A}$ of the factor group $(A \text{ Wr } B) / A^B \cong A$. Thus $P^*_i$ is isomorphically embeddable into the wreath product $P^*_{i,A} \text{ Wr } P^*_{i,B}$. The group $P^*_{i,B}$ is a $p$-subgroup of $B$ and, thus, lies in the $B_p$. $P^*_{i,A}$ is a $p$-subgroup of the base group $A^B$ and, thus, lies in cartesian power $(A_p)^B$ of $A_p$. So $P^*_{i,A} \in \var (A_p)$ and, according to Lemma 1.3,

$P^*_i \leq P^*_{i,A} \text{ Wr } P^*_{i,B} \in \var (A_p \text{ Wr } B_p)$.

☐

5.2. Wreath products of abelian groups of finite composite exponents. Assume $\exp A = m$, $\exp B = n$ as above and

$A = A_{p^s} \oplus \cdots \oplus A_{p^1}, \quad B = B_{q_1} \oplus \cdots \oplus B_{q_d}$

are direct decompositions of $A$ and $B$ as direct sums of finitely many primary components $A_{p^i}$, $i = 1, \ldots, s$ and $B_{q^j}$, $j = 1, \ldots, d$.

Let us begin with the case when the set $\{q_1, \ldots, q_d\}$ is a subset of $\{p_1, \ldots, p_s\}$. Assume $p_1 = q_1$ and denote for brevity $p = p_1 = q_1$. Assume further that $\exp A_p = p^s$ and $\exp B_p = p^k$.

**Lemma 5.2.** If $A$, $B$, $p$ are as above, then

$\var (A \text{ Wr } B) = \var (A) \cdot \var (B) = \mathfrak{A}_m \cdot \mathfrak{A}_n$

if and only if for each $p$ dividing $n$ the factor $B[p^k]/B[p^{k-1}]$, where $k = k(B, p)$, is infinite.
PROOF. If \( A \Wr B \) generates \( \mathfrak{A}_m \cdot \mathfrak{A}_n \), then according to Lemma 5.1:
\[
\var (A_p \Wr B_p) = \var (A \Wr B) \cap \mathfrak{A}_p = \mathfrak{A}_p \cdot \mathfrak{A}_p.
\]
Therefore the group \( B_p \) must satisfy the condition of Lemma 4.4:
\[
[B_p[p^{\ell(B_p,p)}]/B_p[p^{\ell(B_p,p)-1}]] = \infty.
\]
But, since \( B_p \) is the \( p \)-primary component of \( B \), we have:
\[
k(B_p,p) = k(B,p) = k \quad \text{and} \quad B_p[p^{\ell(B_p,p)}] = B_p[p^k] = B[p^k].
\]

Now assume, on the other hand, that for all \( q_1, \ldots, q_d \) all factors
\[
B[q_1^{k_1}]/B[q_1^{k_1-1}], \ldots, B[q_d^{k_d}]/B[q_d^{k_d-1}],
\]
where \( k_i = k(B_{q_i}, q_i) \) \( (i = 1, \ldots, d) \) are infinite. Since the cycle \( C_n \) is the direct sum of cycles \( C_{p_1}, \ldots, C_{p_d} \), we get that \( B \) contains the infinite direct power \( D \) of cycle \( C_n \). And since \( D \) discriminates \( \mathfrak{A}_n \), the group \( B \) also discriminates \( \mathfrak{A}_n \) and \( A \Wr B \) discriminates \( \mathfrak{A}_m \cdot \mathfrak{A}_n \). \( \blacksquare \)

Lemma 5.3. In the above notation the wreath product \( A \Wr B \) generates the variety \( \mathfrak{A}_m \cdot \mathfrak{A}_n \) if and only if the wreath product \( A \Wr B_1 \) generates the variety \( \mathfrak{A}_m \cdot \mathfrak{A}_n = \mathfrak{A}_m \cdot \var (B_1) \).

Proof. If \( \var (A \Wr B) = \mathfrak{A}_m \cdot \mathfrak{A}_n \), then \( \var (A \Wr B_1) = \mathfrak{A}_m \cdot \mathfrak{A}_n \) according to Lemma 5.1 and to the first part of the proof of Lemma 5.2.

Assume, on the other hand, \( \mathfrak{A}_m \cdot \mathfrak{A}_n = \var (A \Wr B_1) \). Then \( B_1 \) contains the direct sum \( B_1^* \) of infinitely many copies of \( C_{q_{i_1}} \) for each \( i = 1, \ldots, s' \):
\[
B_1^* = \sum_{j=1}^{\infty} C_{q_{i_1}} \oplus \cdots \oplus \sum_{j=1}^{\infty} C_{q_{i_{s'}}} \leq B_1.
\]
The groups \( B_2 \) contains the cycle
\[
B_2^* = C_{q_{s_1}} \oplus \cdots \oplus C_{q_{s_d}}.
\]
Thus, according to Lemma 1.3, it is sufficient to prove that \( \mathfrak{A}_m \cdot \mathfrak{A}_n = \var (A \Wr B^*) \), where \( B^* = B_1^* \oplus B_2^* \).

\( \mathfrak{A}_m \cdot \mathfrak{A}_n \) is a locally finite variety generated by its critical groups [N68]. Let \( Q \) be such a critical group. \( Q \) is an extension of a normal subgroup \( H \in \mathfrak{A}_n \) by means of group \( G = Q/H \in \mathfrak{A}_n \). So \( G = G_1 \oplus G_2 \), where \( G_1 \in \mathfrak{A}_n \) and \( G_2 \in \mathfrak{A}_n \). Let \( M \) be the monolith of \( Q \) [N68]. \( M \) is a finite direct product of, say, \( r \) copies of some cycle \( C_p, p \in \{p_1, \ldots, p_s\} \). So \( M \) is an \( r \)-dimensional space over the finite field \( \mathbb{F}_p \) and the operation of conjugation of elements of \( M \) by elements of \( Q \) defines a linear representation of the group \( G \) degree \( r \) over the field \( \mathbb{F}_p \). Since \( \exp G_2 \) is coprime with \( p \) we think of this groups to be isomorphically embedded in \( Q \). Let us use the same
notion $G_2$ for that isomorphic copy in $Q$. Since $M$ is a minimal normal subgroup of $Q$, this representation of $G$ is irreducible. Let us apply the theorem of Clifford [CR62] to the representation of normal subgroup $G_2$ of $G$, that is, to $\mathbb{F}_p G_2$-module $M_{G_2}$. Our $\mathbb{F}_p G$-module $M$ is, thus, a direct sum of its submodules, each of which is homogenous regarding $G_2$: $M = M_1 \oplus \cdots \oplus M_l$. Since $H$ and $Q/H$ are both abelian, $M_1, \ldots, M_l$ are normal in $Q$. Thus $l = 1$ and $M = M_1$.

The representation of $G_2$ is faithful. For, if not, the direct sum $H \oplus K$ of $H$ and of non-trivial kernel $K$ would be normal in $Q$ and, so, $D$ would contain a “second monolith” of $Q$. Thus, as an abelian group with irreducible faithful representation, $G_2$ has to be a cycle. Since this cycle belongs to $\mathfrak{A}_{n_2}$, we get that $G_2$ is a subgroup of $B_2$. Further, since $G_1$ is a finite group in $\mathfrak{A}_{n_1}$, then $G_1$ is a subgroup of $B_2$. Therefore $G = G_1 \oplus G_2$ is a subgroup of $B_1 \oplus B_2$ and, thus, the extension $Q$ of $H$ by $G$ lies in $H \text{Wr} G$ and

$$H \text{Wr} G \leq H \text{Wr} B \in \var (A \text{Wr} B^*) .$$

□

Example 5.4. Replacing the wreath product of Example 4.6 by the following one:

$$W = C_p \text{Wr} (C_{p^2} \oplus \sum_{i=1}^{\infty} C_p \oplus \sum_{i \in I} C_q) ,$$

where $q$ is a prime different from $p$, we get that $W$ does not generate the variety $\mathfrak{A}_p \cdot \mathfrak{A}_{p^2 \cdot q}$ for any index set $I$, in spite of the fact that $C_p \text{Wr} \sum_{i \in I} C_q$ does generate the variety $\mathfrak{A}_p \cdot \mathfrak{A}_q$ for arbitrary non-empty index set $I$. On the other hand, replacing in $W$ the summand $C_{p^2}$ by $\sum_{i=1}^{\infty} C_p$, we will obtain a wreath product $C_p \text{Wr} \left[ \sum_{i=1}^{\infty} C_{p^2} \oplus \sum_{i=1}^{\infty} C_p \oplus \sum_{i \in I} C_q \right]$ generating the variety $\mathfrak{A}_p \cdot \mathfrak{A}_{p^2 \cdot q}$ for any non-empty $I$. Clearly, this wreath product will generate variety $\mathfrak{A}_p \cdot \mathfrak{A}_{p^2 \cdot q}$ even if we remove the summand $\sum_{i=1}^{\infty} C_p$ in the active group.

The information of this section can be collected as:

Theorem 5.5. Let $A$ and $B$ be arbitrary abelian groups of finite exponents $m$ and $n$ respectively and let $m = p_1^{n_1} \cdots p_s^{n_s}$ and $n = q_1^{k_1} \cdots q_{s'}^{k_{s'}}$, where $s' \leq s$ and where the prime divisors $p_i$ and $q_j$ are grouped such that $p_1 = q_1, \ldots, p_{s'} = q_{s'}$ and $p_1 \neq q_j$ for all $i = 1, \ldots, s$; $j = s'+1, \ldots, d$. Then $\var (A \text{Wr} B) = \var (A \text{wr} B) = \var (A) \cdot \var (B) = \mathfrak{A}_m \cdot \mathfrak{A}_n$ if and only if the factors $B[\{ q^{k(x,y)} \}] / B[\{ q^{k(x,y)} \}^{-1}]$ are infinite for all $q = q_1, \ldots, q_{s'}$.

6. The general case of arbitrary abelian groups and wreath products of groups “near” to abelian ones

6.1. The general criterion for arbitrary abelian groups. Statements of Theorems 2.5, 3.3, 4.5 and 5.5 are constituent parts of the following main criterion for cartesian and direct wreath products of arbitrary abelian groups:

Theorem 6.1 (Main Criterion). For arbitrary abelian groups $A$ and $B$ their cartesian wreath product $A \text{Wr} B$ (or direct wreath product $A \text{wr} B$) generates the product variety $\var (A) \cdot \var (B)$ if and only if

1. at least one of the groups $A$ and $B$ is not of finite exponent,

2. or if $A$ and $B$ are of finite exponents $m$ and $n$ respectively and for each prime $p$ dividing both $m$ and $n$ the factors $B[p^k] / B[p^k]^{-1}$ are infinite, where $B[s] = \langle b \in B \mid b^s = 1 \rangle$ and $p^k$ is the highest power of $p$ dividing $n$.

As the mentioned Theorems 2.5, 3.3, 4.5 and 5.5 show, this criterion is effective in the sense that in concrete cases the factors $B[p^k] / B[p^k]^{-1} \cong \sum_{i \in I} C_p$ (if need of their
consideration arises) have simple and “understandable” meaning: we have to consider them only in the case when \( A \) and \( B \) are of finite exponents and there is a prime \( p \) dividing that exponents; in this circumstances the condition \(|B[p^k]/B[p^{k-1}]| = \infty\) simply means that in the direct decomposition \( G_p = C_{p^{k_1}} \oplus C_{p^{k_2}} \oplus \cdots \) (where \( k_1 \geq k_2 \geq \cdots \)) of the \( p \)-primary component \( B_p \) of the group \( B \) infinitely many cycles (direct summands) \( C_{p^{k_1}} = C_{p^{k(B,p)}} \) are present.

An immediate consequence of Theorem 6.1 is the following:

**Corollary 6.2.** If for abelian groups \( A \) and \( B \) the wreath product \( A \wr B \) does not generate the product variety \( \text{var}(A) \cdot \text{var}(B) \), then the wreath product \( A \wr \sum_{i=1}^l B_i \) does not generate \( \text{var}(A) \cdot \text{var}(B) \) for any positive integer \( l \in \mathbb{N} \).

### 6.2. Parallels for nilpotent groups of class 2 and metabelian groups.

**Problems.** The following two examples show, that Theorem 6.1 does *not* have obvious generalizations even for cases of “small” groups of classes of groups “near” to abelian groups.

**Example 6.3.** Let \( \mathfrak{W} = \mathfrak{N}_2 \cap \mathfrak{W}_3 \) be the variety of all nilpotent groups of class at most 2 and exponent dividing 3. Then for no one group \( A \) generating \( \mathfrak{W} \) and cycle \( B = C_2 \cdot \text{var}(A \wr B) = \mathfrak{W} \cdot \mathfrak{A}_2 \). On the other hand \( \exp(A) = 3 \) is coprime with \( \exp(B) = 2 \). According to Lemma 1.3, it is sufficient to prove this for one group \( A \) generating \( \mathfrak{W} \). Let \( A = F_2(\mathfrak{W}) = \langle x_1, x_2 \mid [x_1, x_2, x_1] = [x_1, x_2, x_2] = x_1^3 = x_2^3 = 1 \rangle \) be the \( \mathfrak{W} \)-free group of rank 2 and let \( R \) be the extension of \( A \) by means of the group of operators generated by automorphisms \( \nu_1, \nu_2 \in \text{Aut}(A) \) defined as: \( \nu_1 : x_1 \mapsto x_1^{-1}, \nu_2 : x_2 \mapsto x_2; \nu_1 : x_1 \mapsto x_1, \nu_2 : x_2 \mapsto x_2^{-1} \). Clearly \( \langle \nu_1, \nu_2 \rangle \cong C_2 \oplus C_2 \in \mathfrak{A}_2 \).

As it is shown in \([B]\), \( R \) is a critical group. Every one of its proper factors, but not \( R \) itself, satisfies \([x_1, x_2], [x_3, x_4, x_5] \equiv 1 \). On the other hand the wreath product \( A \wr B \) satisfies this identity because its second commutator subgroup lies in the center.

**Example 6.4.** Let \( A \) and \( B \) be arbitrary finite groups generating varieties \( \mathfrak{A}_p \cdot \mathfrak{A}_q \) and \( \mathfrak{A}_r \) respectively, where \( p, q, r \) are arbitrary different primes. Then \( \text{var}(A \wr B) \neq (\mathfrak{A}_p \cdot \mathfrak{A}_q) \cdot \mathfrak{A}_r \), in spite of the fact that \( \exp(A) = p q \) is coprime with \( \exp(B) = r \). For, the product of three non-trivial varieties \( (\mathfrak{A}_p \cdot \mathfrak{A}_q) \cdot \mathfrak{A}_r = \mathfrak{A}_p \cdot \mathfrak{A}_q \cdot \mathfrak{A}_r \) cannot, by theorem of Smelkin [S65], be generated by a single finite group \( A \wr B \).

This examples, and the results of Section 1 proved not only for the wreath products of single groups but also for sets \( \mathfrak{X} \wr \mathfrak{Y} \) set the following problems of generalization of our main criterion (Theorem 6.1) in two possible directions:

**Problem 6.5.** Let \( A \) and \( B \) be arbitrary (nilpotent, metabelian, soluble) groups. Find a criterion under which \( \text{var}(A \wr B) = \text{var}(A) \cdot \text{var}(B) \).

**Problem 6.6.** Let \( \mathfrak{X} \) and \( \mathfrak{Y} \) be arbitrary sets of (abelian) groups. Find a criterion under which \( \text{var}(\mathfrak{X} \wr \mathfrak{Y}) = \text{var}(\mathfrak{X}) \cdot \text{var}(\mathfrak{Y}) \).

### 7. Wreath products and finite direct sums of abelian groups.

As we saw in Section 1, \( \text{var}(A \wr B) = \text{var}(A) \cdot \text{var}(B) \) if and only if \( \text{var}(A \wr B) \) contains wreath products \( A \wr (\prod_{i \in I} B) \) for every index set \( I \). It is a result of independent interest, that here instead of arbitrary set \( I \) we can take a two-element set, that is:
Theorem 7.1. For arbitrary abelian groups $A$ and $B$ equality \( \text{var}(A \text{Wr} B) = \text{var}(A) \cdot \text{var}(B) \) takes place if and only the variety \( \text{var}(A \text{Wr} B) \) contains the group \( A \text{Wr} (B \times B) \). The analogous statement is also true for direct wreath products of groups.

This theorem follows from a more general result:

Theorem 7.2. For arbitrary abelian groups $A$ and $B$ one and only one of the following two alternatives holds:

1. the variety \( \text{var}(A \text{Wr} B) \) is equal to the variety \( \text{var}(A) \cdot \text{var}(B) \) and, thus, to every subvariety \( \text{var}(A \text{Wr} (\prod_{i \in I} B)) \) of the latter for any index set \( I \);
2. the variety \( \text{var}(A \text{Wr} B) \) is a proper subvariety of variety \( \text{var}(A) \cdot \text{var}(B) \). Then for any positive integer \( s \in \mathbb{N} \) the variety \( \text{var}(A \text{Wr} (\prod_{i \in I} B)) \) is a proper subvariety of variety \( \text{var}(A) \cdot \text{var}(B) \) and, moreover, of the variety \( \text{var}(A \text{Wr} (\prod_{i \in I} B)) \) for any integer \( u > s \). On the other hand for any infinite index set \( I \): \( \text{var}(A \text{Wr} (\prod_{i \in I} B)) = \text{var}(A) \cdot \text{var}(B) \).

The first alternative holds for any abelian $A$ and $B$ apart from the following case: $A$ and $B$ are of finite exponents $m$ and $n$ respectively and for some prime $p$ dividing both $m$ and $n$ the corresponding factor $B[p^k]/B[p^{k-1}]$ is finite.

The analogous statement is also true for direct wreath products of groups.

Remark 7.3. Applying this theorem, having any wreath product $A \text{Wr} B$ which does not generate the variety $\text{var}(A) \cdot \text{var}(B)$, we get a countably infinite set of linearly ordered proper subvarieties of $\text{var}(A) \cdot \text{var}(B)$, namely, varieties generated by wreath products: \( \{A \text{Wr} (B \oplus \cdots \oplus B) | s \in \mathbb{N}\} \) (see examples below).

Proof of Theorem 7.2. First let us consider the cases, when the statement of this theorem easily follows from one of the results already established. If one of the groups $A$ and $B$ is not of finite exponent, then the first alternative holds. Thus assume $\exp A = m$, $\exp B = n$. If the index set $I$ is infinite, then for any $B$ the direct sum $\sum_{i \in I} B$ always discriminates $\text{var}(B)$ and, thus, $A \text{Wr} (\sum_{i \in I} B)$ discriminates $\text{var}(A) \cdot \text{var}(B)$. So assume $I$ to be finite. If $\text{var}(A \text{Wr} (\sum_{i \in I} B)) \neq \text{var}(A) \cdot \text{var}(B)$, then by Theorem 6.1 there is a prime $p$ such that the corresponding factor $B[p^k]/B[p^{k-1}]$ is finite. Denote for brevity by $B_{s,p}$ the $p$-primary component of direct sum $\sum_{i=1}^{s} B$. If for given $r > s$ \( \text{var}(A \text{Wr} \sum_{i=1}^{s} B) = \text{var}(A \text{Wr} \sum_{i=1}^{r} B) \) holds, then following the proof of Lemma 5.1:

\[
(7.1) \quad \text{var}(A_p \text{Wr} B_{s,p}) = \text{var}(A_p \text{Wr} B_{r,p}).
\]

So to complete the proof it is sufficient to show that (7.1) leads to a contradiction to the fact that $B[p^k]/B[p^{k-1}]$ is finite. We can also omit the case when the group $B_{s,p}$ (and, therefore, the group $B_{r,p}$) is finite, for in such a case by Liebeck’s Theorem [L62] the groups $A_p \text{Wr} B_{s,p}$ and $A_p \text{Wr} B_{r,p}$ have different nilpotency classes.

For the rest of proof the concrete form of $B_p$ is essential. Since $B[p^k]/B[p^{k-1}]$ is finite, $B_p$ contains in its direct decomposition only finitely many, say $l_0$, summands $C_{p^k}$. It may turn out that $B[p^{k-1}]/B[p^{k-2}]$ is also finite and the number of summands $C_{p^{k-1}}$ is finite, say $l_1$. But since the group $B_p$ is infinite, there exists the first number $d$ such that the corresponding factor $B[p^{k-d+1}]/B[p^{k-d}]$ is finite, it consists of, say, $l_{d-1}$ summands $C_{p^{k-d+1}}$, and the factor $B[p^{k-d}]/B[p^{k-d-1}]$ is infinite. Thus $B_p$ can be
presented as:

\[ B_p = C_{p^k} \oplus \cdots \oplus C_{p^k} \oplus \cdots \oplus C_{p^{k-d+1}} \oplus \cdots \oplus C_{p^{k-d+1}} \]

(7.2)

\[ \oplus C_{p^k - d} \oplus \cdots \oplus C_{p^{k-d}} \oplus \cdots \oplus \hat{B}, \]

where \( \exp \hat{B} \leq p^{k-d-1} \) and where some of \( l_1, \ldots, l_{d-1} \) may be equal to 0. Let \( \lambda = \lambda(A_p, B_{s,p}, t) \) be the function defined in Section 4 and let \( \exp A_p = p^n \). It follows from the proof in Subsection 4.2 and from decomposition (7.2) that, for any \( t > s \cdot \sum_{i=0}^{d-1} l_i \) the function \( \lambda(A_p, B_{s,p}, t) \) is bounded by

\[ s \cdot \sum_{i=0}^{d-1} l_i (p^{k-i} - 1) + (t - s \cdot \sum_{i=0}^{d-1} l_i) \cdot (p^{k-d} - 1) + (u - 1)(p - 1)p^{k-1} + 1. \]

(7.3)

On the other hand for each \( t > r \cdot \sum_{i=0}^{d-1} l_i + 1 \) the variety \( \text{var}(A_p \wr B_{r,p}) \) contains the following \( t \)-generated group:

\[ T(r, t) = C_{p^s} \wr \left[ \sum_{i=1}^{r} C_{p^k} \oplus \cdots \oplus \sum_{i=1}^{r_{d-1}} C_{p^{k-d+1}} \oplus \sum_{i=1}^{r_{d-1}} C_{p^{k-d}}, \right] \]

where \( \omega(t) = t - r \cdot \sum_{i=0}^{d-1} l_i - 1 \). The group \( T(r, t) \) is nilpotent of class:

\[ \nu(p, r, t) = r \cdot \sum_{i=0}^{d-1} l_i (p^{k-i} - 1) + (t - r \cdot \sum_{i=0}^{d-1} l_i - 1) \cdot (p^{k-d} - 1) \]

\[ + (u - 1)(p - 1)p^{k-1} + 1. \]

It remains to verify that for sufficiently large integers \( t \) the value of \( \nu(p, r, t) \) is greater than that of \( \lambda(A_p, B_{s,p}, t) \). It is sufficient to make calculations for the value \( r = s + 1 \); the generality of the statement of Theorem 7.2 remains unaffected but our calculations become much shorter.

\[ \nu(p, s + 1, t) - \lambda(A_p, B_{s,p}, t) = \sum_{i=0}^{d-1} l_i (p^{k-i} - 1) \]

\[ + \left( t - (s + 1) \cdot \sum_{i=0}^{d-1} l_i - 1 - t + s \cdot \sum_{i=0}^{d-1} l_i \right) \cdot (p^{k-d} - 1) \]

\[ = \sum_{i=0}^{d-1} l_i p^{k-i} - \sum_{i=0}^{d-1} l_i - \left( \sum_{i=0}^{d-1} l_i + 1 \right) \cdot (p^{k-d} - 1) \]

\[ = \sum_{i=0}^{d-1} l_i (p^{k-i} - p^{k-d}) - p^{k-d} + 1 > 0. \]

\[ \square \]

Here are two examples concerning subvarieties generated by wreath products in the lattice of all subvarieties of product varieties of abelian groups.

**Example 7.4.** Since \( C_p \wr C_p \) does not generate variety \( \mathfrak{A}^2_p \), wreath products \( C_p \wr \sum_{i=1}^s C_p \quad (s = 1, 2, \ldots) \) generate infinitely many subvarieties of \( \mathfrak{A}^2_p \).
are able to locate them in the lattice of subvarieties of $\mathfrak{A}_m^2$ using its description due to Kovács and Newman [KN71]. For any proper subvariety $\mathfrak{Y}$ of $\mathfrak{A}_m^2$ there is a number $s \geq 1$ such that $\operatorname{var} (C_p \operatorname{Wr} \sum_{i=1}^s C_p) \subseteq \mathfrak{Y} \subseteq \operatorname{var} (C_p \operatorname{Wr} \sum_{i=1}^{s+1} C_p)$ or $\mathfrak{Y}$ lies in $\operatorname{var} (C_p \operatorname{Wr} C_p)$. Moreover, for any $s \geq 1$ there are exactly the following $p - 2$ subvarieties of $\mathfrak{A}_m^2$ “between” $\operatorname{var} (C_p \operatorname{Wr} \sum_{i=1}^s C_p)$ and $\operatorname{var} (C_p \operatorname{Wr} \sum_{i=1}^{s+1} C_p)$: $\mathfrak{A}_m^2 \cap \mathfrak{A}_{mp} \cap \mathfrak{A}_j$, $j = s(p - 1) + 2, \ldots, (s+1)(p - 1)$. And there are $2p - 3$ subvarieties of $\mathfrak{A}_m^2$ “between” $\operatorname{var} (C_p \operatorname{Wr} \{1\}) = \mathfrak{A}_m$ and $\operatorname{var} (C_p \operatorname{Wr} C_p)$ (see [KN71]).

Example 7.5. On the other hand for arbitrary coprime numbers $m$ and $n$ the variety $\mathfrak{A}_m \cdot \mathfrak{A}_n$ contains only finitely many subvarieties [N68]. According to Theorem 7.2 this fact already guarantees, that for an arbitrary pair $A \in \mathfrak{A}_m$, $B \in \mathfrak{A}_n$: $\operatorname{var} (A \operatorname{Wr} B) = \operatorname{var} (A) \cdot \operatorname{var} (B)$.

Closing the current work we would like to announce our recent papers [M00, Msub1, Msub2] as well as our common paper with Professor H. Heineken [HM00], where some related properties of wreath products and their verbal subgroups are considered.

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