Morrey-Sobolev Spaces on Metric Measure Spaces

Yufeng Lu, Dachun Yang* and Wen Yuan

Abstract In this article, the authors introduce the Newton-Morrey-Sobolev space on a metric measure space $(\mathcal{X}, d, \mu)$. The embedding of the Newton-Morrey-Sobolev space into the Hölder space is obtained if $\mathcal{X}$ supports a weak Poincaré inequality and the measure $\mu$ is doubling and satisfies a lower bounded condition. Moreover, in the Ahlfors $Q$-regular case, a Rellich-Kondrachov type embedding theorem is also obtained. Using the Hajłasz gradient, the authors also introduce the Hajłasz-Morrey-Sobolev spaces, and prove that the Newton-Morrey-Sobolev space coincides with the Hajłasz-Morrey-Sobolev space when $\mu$ is doubling and $\mathcal{X}$ supports a weak Poincaré inequality. In particular, on the Euclidean space $\mathbb{R}^n$, the authors obtain the coincidence among the Newton-Morrey-Sobolev space, the Hajłasz-Morrey-Sobolev space and the classical Morrey-Sobolev space. Finally, when $(\mathcal{X}, d)$ is geometrically doubling and $\mu$ a non-negative Radon measure, the boundedness of some modified (fractional) maximal operators on modified Morrey spaces is presented; as an application, when $\mu$ is doubling and satisfies some measure decay property, the authors further obtain the boundedness of some (fractional) maximal operators on Morrey spaces, Newton-Morrey-Sobolev spaces and Hajłasz-Morrey-Sobolev spaces.

1 Introduction

In 1996, via introducing the notion of Hajłasz gradients, Hajłasz [13] obtained an equivalent characterization of the classical Sobolev space on $\mathbb{R}^n$, which becomes an effective way to define Sobolev spaces on metric spaces. From then on, several different approaches to introduce Sobolev spaces on metric measure spaces were developed; see, for example, [26, 11, 37, 16, 14, 22, 41, 27].

Throughout the paper, $(\mathcal{X}, d, \mu)$ denotes a metric measure space with a non-trivial Borel regular measure $\mu$, which is finite on bounded sets and positive on open sets. Let $f$ be a measurable function on $\mathcal{X}$. Recall that a non-negative function $g$ on $\mathcal{X}$ is called

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*Corresponding author.
a Hajłasz gradient of $f$ if there exists a set $E \subset X$ such that $\mu(E) = 0$ and, for all $x, y \in X \setminus E$,

$$|f(x) - f(y)| \leq d(x, y)[g(x) + g(y)].$$

The Hajłasz-Sobolev space $M^{1,p}(X)$ with $p \in [1, \infty]$ is then defined to be the space of all measurable functions $f \in L^p(X)$ which have Hajłasz gradients $g \in L^p(X)$. The norm of this space is defined by

$$\|f\|_{M^{1,p}(X)} := \|f\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all Hajłasz gradients $g$ of $f$. It was proved in [13] that, when $X = \mathbb{R}^n$ and $p \in (1, \infty]$, $M^{1,p}(\mathbb{R}^n)$ coincides with the classical Sobolev space $W^{1,p}(\mathbb{R}^n)$.

Over a decade ago, based on the notions of upper gradients and weak upper gradients, Shanmugalingam [37, 38] introduced another type of Sobolev spaces on metric measure spaces, which are called Newtonian spaces or Newton-Sobolev spaces. These spaces were also proved to coincide with the Hajłasz-Sobolev spaces if $X$ supports some Poincaré inequality and the measure is doubling. Now we recall their definitions.

Recall that we call $\gamma$ a curve if it is a continuous mapping from an interval into $X$. A curve $\gamma$ is said to be rectifiable if its length is finite. All rectifiable curve can be arc-length parameterized. Without loss of generality, we may assume that all curves appearing in this article are always treated as arc-length parameterized.

Let $p \in [1, \infty)$ and $\Gamma$ be a family of non-constant rectifiable curves on $X$. Recall that the admissible class $F(\Gamma)$ for $\Gamma$ is defined by

$$F(\Gamma) := \left\{ \rho \in [0, \infty] : \rho \text{ is Borel measurable and } \int_{\gamma} \rho(s) \, ds \geq 1 \text{ for all } \gamma \in \Gamma \right\}.$$ \hspace{1cm} (1.1)

If $\Gamma$ contains a constant curve, then $F(\Gamma) = \emptyset$. The $p$-modulus of $\Gamma$ is then defined by

$$\text{Mod}_p(\Gamma) := \inf_{\rho \in F(\Gamma)} \|\rho\|^p_{L^p(X)},$$

where the infimum is taken over all admissible functions $\rho$ in $F(\Gamma)$. We let the infimum over the empty set always be infinity. Let $f$ be a measurable function on $X$. A non-negative function $g$ is called an upper gradient of $f$ if, for any curve $\gamma \in \Gamma_{\text{rect}},$

$$|f \circ \gamma(0) - f \circ \gamma(l(\gamma))| \leq \int_{\gamma} g(s) \, ds,$$

where $\Gamma_{\text{rect}}$ is the class of all non-constant rectifiable curves in $X$. Moreover, if the inequality (1.2) holds for all the curves except for a family of curves of $p$-modulus zero, then we call $g$ a $p$-weak upper gradient of $f$. The notion of $p$-weak upper gradient was introduced by Heinonen and Koskela in [15]; see also [19] and [37, 38].

For all $p \in [1, \infty)$, denote by the symbol $\widetilde{N}^{1,p}(X)$ the space of all measurable functions $f \in L^p(X)$ which have $p$-weak upper gradients $g \in L^p(X)$ and, for all $f \in \widetilde{N}^{1,p}(X)$, let

$$\|f\|_{\widetilde{N}^{1,p}(X)} := \|f\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all $p$-weak upper gradients $g$ of $f$. It was proved in [13] that, when $X = \mathbb{R}^n$ and $p \in (1, \infty]$, $\widetilde{N}^{1,p}(\mathbb{R}^n)$ coincides with the classical Sobolev space $W^{1,p}(\mathbb{R}^n)$. 


where the infimum is taken over all $p$-weak upper gradients $g$ of $f$. The *Newton-Sobolev space* $N^{1,p}(\mathcal{X})$ is then defined to be the quotient space $N^{1,p}(\mathcal{X}) := \widetilde{N}^{1,p}(\mathcal{X})/\sim$ with the norm $\| \cdot \|_{N^{1,p}(\mathcal{X})} := \| \cdot \|_{\widetilde{N}^{1,p}(\mathcal{X})}$, where $\sim$ is an equivalence relation defined by setting, for all $f_1, f_2 \in \widetilde{N}^{1,p}(\mathcal{X})$, $f_1 \sim f_2$ if $\| f_1 - f_2 \|_{\widetilde{N}^{1,p}(\mathcal{X})} = 0$. It was proved in [37, Theorem 4.9] that the Newton-Sobolev space coincides with the Hajlasz-Sobolev space if $(X, \mu)$ supports some Poincaré inequality and the measure $\mu$ is doubling. We refer the reader to [37, 19, 7, 12, 6] for more properties about these spaces.

Recently, there were some attempts to study Newtonian type spaces in more general settings. Durand-Cartagena in [10] introduced and studied the Newtonian space $N^{1,\infty}(\mathcal{X})$ in the limit case $p = \infty$. Tuominen [40] considered Newtonian type spaces associated with Orlicz spaces by replacing the Lebesgue norm in the definition of $N^{1,p}(\mathcal{X})$ with Orlicz norms. Using Lorentz spaces instead of Lebesgue spaces, Costea and Miranda [9] introduced Newtonian type spaces related to Lorentz spaces. Malý [29, 30] studied the Newtonian type spaces associated with a general *quasi-Banach function lattice* $X$, namely, a quasi-Banach function space $X$ satisfying that, if $f \in X$ and $|g| \leq |f|$ almost everywhere, then $g \in X$ and $\|g\|_X \leq \|f\|_X$.

Let $0 < p \leq q \leq \infty$. Recall that the *Morrey space* $M^q_p(\mathcal{X})$ (see [33]) is defined to be the space of all measurable functions $f$ on $\mathcal{X}$ such that

$$\|f\|_{M^q_p(\mathcal{X})} := \sup_{B \subset \mathcal{X}} [\mu(B)]^{1/q-1/p} \left[ \int_B |f(x)|^p \, d\mu(x) \right]^{1/p} < \infty,$$

where the supremum is taken over all balls in $\mathcal{X}$. In recent years, Morrey spaces and the Morrey versions of many classical function spaces such as Hardy spaces and Besov spaces, namely, the spaces defined via replacing Lebesgue norms by Morrey norms in their norms, attract more and more attentions and have proved useful in the study of partial differential equations and harmonic analysis; see, for example, [1, 2, 3, 4, 34, 32, 28, 31, 43] and their references.

The main purpose of this article is to develop a theory of Newtonian type spaces based on Morrey spaces, namely, Newton-Morrey-Sobolev spaces, as well as the Hajlasz-Morrey-Sobolev spaces on metric measure spaces.

We begin with the following generalized modulus based on Morrey spaces.

**Definition 1.1.** Let $1 \leq p \leq q < \infty$ and $\Gamma$ be a collection of rectifiable curves. The *Morrey-modulus* of $\Gamma$ is defined by

$$\text{Mod}^q_p(\Gamma) := \inf_{\rho \in F(\Gamma)} \|\rho\|_{M^q_p(\mathcal{X})}^p,$$

where $F(\Gamma)$ is defined as in (1.1).

**Definition 1.2.** Let $f$ be a measurable function and $g$ a non-negative Borel measurable function. If the inequality (1.2) holds true for all non-constant rectifiable curves in $\mathcal{X}$ except a family of curves of Morrey-modulus zero, then $g$ is called a *Morrey-$p$-weak upper gradient* of $f$. 
Via these Mod$_p^q$-weak upper gradients, the Newton-Morrey-Sobolev space is introduced as follows.

**Definition 1.3.** Let $1 \leq p \leq q < \infty$. The space $\widetilde{NM}^q_p(\mathcal{X})$ is defined to be the set of all $\mu$-measurable functions $f$ such that $\|f\|_{\widetilde{NM}^q_p(\mathcal{X})} < \infty$, where

$$\|f\|_{\widetilde{NM}^q_p(\mathcal{X})} := \|f\|_{M^q_p(\mathcal{X})} + \inf \|g\|_{M^q_p(\mathcal{X})}$$

with the infimum being taken over all Mod$_p^q$-weak upper gradients $g$ of $f$. The *Newton-Morrey-Sobolev space* $NM^q_p(\mathcal{X})$ is then defined as the quotient space

$$\widetilde{NM}^q_p(\mathcal{X})/ \left\{ f \in \widetilde{NM}^q_p(\mathcal{X}) : \|f\|_{\widetilde{NM}^q_p(\mathcal{X})} = 0 \right\}$$

with

$$\|f\|_{NM^q_p(\mathcal{X})} := \|f\|_{\widetilde{NM}^q_p(\mathcal{X})}.$$

It is easy to see that $\| \cdot \|_{NM^q_p(\mathcal{X})}$ is a norm. Moreover, when $p = q$, the space $NM^q_q(\mathcal{X})$ is just the Newton-Sobolev space $N^{1,p}(\mathcal{X})$ introduced by Shanmugalingam [37]. We also remark that, since Morrey spaces are Banach function lattices, these Newton-Morrey-Sobolev spaces are special cases of the Newtonian type spaces associated with quasi-Banach function lattices considered by Malý [29, 30].

This article is organized as follows. In Section 2, we show that the Newton-Morrey-Sobolev space is non-trivial by proving that the set of Lipschitz functions with bounded support is contained in the Newton-Morrey-Sobolev space $NM^q_p(\mathcal{X})$ (see Theorem 2.4 below), but not dense in some examples (see Remark 2.5 below), which is different from the Newton-Sobolev space. Moreover, in Remark 4.8 below, we even show that the set of Lipschitz functions is not dense in $NM^q_p(\mathcal{R}^n)$ when $1 < p < q < \infty$.

In Section 3, the embedding of the Newton-Morrey-Sobolev space into the Hölder space is obtained when $\mathcal{X}$ supports a weak Poincaré inequality, the measure $\mu$ is doubling and satisfies a lower bounded condition (see Theorem 3.1 below). Moreover, if the space $\mathcal{X}$ is Ahlfors $Q$-regular and supports a weak Poincaré inequality, via proving the boundedness of some fractional integrals on Morrey spaces, we also obtain a Rellich-Kondrachov type embedding theorem of the Newton-Morrey-Sobolev space (see Theorem 3.6 below). Both embedding properties on Newton-Morrey-Sobolev spaces generalize the corresponding results for Newton-Sobolev spaces obtained by Shanmugalingam in [37, Theorems 5.1 and 5.2].

In Section 4, using the Hajłasz gradient, we introduce the Hajłasz-Morrey-Sobolev space on metric measure spaces and show that, when $\mathcal{X}$ supports a weak Poincaré inequality and the measure $\mu$ is doubling, the Newton-Morrey-Sobolev space coincides with the Hajłasz-Morrey-Sobolev space (see Theorem 4.6 below). This generalizes the result on the relation between Newton-Sobolev spaces and Hajłasz-Sobolev spaces obtained by Shanmugalingam in [37, Theorem 4.9]. In particular, when $\mathcal{X} = \mathcal{R}^n$ and $1 < p \leq q < \infty$, both the Newton-Morrey-Sobolev space $NM^q_p(\mathcal{R}^n)$ and the Hajłasz-Morrey-Sobolev space $HM^q_p(\mathcal{R}^n)$ are proved to coincide with the classical Morrey-Sobolev space on $\mathcal{R}^n$ (see Theorem 4.7 below).
Finally, Section 5 is devoted to the boundedness of some fractional maximal operators on Morrey and Morrey-Sobolev spaces. We first show, in Subsection 5.1, the boundedness of some modified maximal operators on modified Morrey spaces over geometrically doubling metric measure spaces (see Theorem 5.8 below). As an application, the boundedness of related fractional maximal operators on modified Morrey spaces is obtained (see Proposition 5.10 below). As further applications, in Subsection 5.2, we show the boundedness of (fractional) maximal operators on Hajłasz-Morrey-Sobolev spaces when $\mathcal{X}$ is a doubling metric measure space satisfying the relative 1-annular decay property and the measure lower bound condition (see Theorem 5.13 below). If $\mathcal{X}$ supports a weak Poincaré-inequality, and the measure is doubling and satisfies the measure lower bound condition, then the boundedness of discrete (fractional) maximal operators on Newton-Morrey-Sobolev spaces is also obtained (see Theorem 5.14 below). All these conclusions generalize the corresponding known results on Newton-Sobolev spaces and Hajłasz-Sobolev spaces by Heikkinen et al. in [17, 18].

At the end of this section, we make some conventions on notation. Throughout the paper, we denote by $C$ a **positive constant** which is independent of the main parameters, but it may vary from line to line. The symbols $A \lesssim B$ and $A \gtrsim B$ means $A \leq CB$ and $A \geq CB$, respectively, where $C$ is a positive constant. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$. If $E$ is a subset of $\mathcal{X}$, we denote by $\chi_E$ its **characteristic function**.

## 2 Some basic properties

In this section, we consider some basic properties of Newton-Morrey-Sobolev spaces including their completeness and non-triviality. Throughout this section, we only assume that $\mu$ is a non-trivial Borel regular measure.

Recall that the Newton-Morrey-Sobolev space is a special case of the Newtonian spaces based on quasi-Banach function lattice $X$ introduced in [29]. The following result is a special case of [29, Theorem 7.1].

**Theorem 2.1.** For all $1 \leq p \leq q < \infty$, the space $NM^q_p(\mathcal{X})$ is a Banach space.

The next lemma is usually called the **truncation lemma**, which shows how a $\text{Mod}^q_p$-weak upper gradient behaves when multiplying a characteristic function. Its proof is similar to those of [9, Lemmas 4.6 and 4.7], the details being omitted.

**Lemma 2.2.** Let $f \in NM^q_p(\mathcal{X})$ and $g_1, g_2 \in \mathcal{M}^q_p(\mathcal{X})$ be two $\text{Mod}^q_p$-weak upper gradients of $f$.

(i) If $f$ is a constant on a closed set $E$, then $g := g_1\chi_{\mathcal{X}\setminus E}$ is also a $\text{Mod}^q_p$-weak upper gradient of $f$.

(ii) If $E$ is closed in $\mathcal{X}$, then

$$h := g_1\chi_E + g_2\chi_{\mathcal{X}\setminus E}$$

is also a $\text{Mod}^q_p$-weak upper gradient of $f$.

We also need the following conclusion.
Proposition 2.3. Let \( 1 \leq p \leq q < \infty \). For any set \( E \subset \mathcal{X} \) with finite measure, \( \| \chi_E \|_{M_p^q(\mathcal{X})} \) is bounded by a positive constant multiple of \( [\mu(E)]^{1/q} \) with the positive constant independent of \( E \).

Proof. Notice that

\[
\| \chi_E \|_{M_p^q(\mathcal{X})} = \sup_{B \subset \mathcal{X}} \left( \frac{\mu(B \cap E)}{\mu(B)} \right)^{1/p} \left( \mu(B) \right)^{1/q}.
\]

If \( \mu(B) \geq \mu(E)/2 \), then by \( p \leq q \), we have

\[
\left( \frac{\mu(B \cap E)}{\mu(B)} \right)^{1/p} \leq \left( \frac{\mu(B)}{\mu(E)} \right)^{1/q} \leq [\mu(E)]^{1/q}.
\]

If \( \mu(B) \leq \mu(E)/2 \), then

\[
\left( \frac{\mu(B \cap E)}{\mu(B)} \right)^{1/p} \leq [\mu(E)]^{1/q}.
\]

This finishes the proof of Proposition 2.3. \( \square \)

Recall that \( NM_p^q(\mathcal{X}) = N^{1,p}(\mathcal{X}) \), which is a non-trivial space, namely, the space \( N^{1,p}(\mathcal{X}) \) contains more than just the zero function and might be a proper subspace of \( L^p(\mathcal{X}) \) if \( \mathcal{X} \) has enough rectifiable paths (see [37]). The following conclusion shows that, even when \( q > p \geq 1 \), \( NM_p^q(\mathcal{X}) \) is also a non-trivial space. In what follows, \( \text{Lip}_b(\mathcal{X}) \) denotes the set of all Lipschitz functions on \( \mathcal{X} \) with bounded support.

Theorem 2.4. Let \( 1 \leq p \leq q < \infty \). Then,

\[
\text{Lip}_b(\mathcal{X}) \subset NM_p^q(\mathcal{X}) \subset N^{1,p}_{\text{loc}}(\mathcal{X}),
\]

where \( N^{1,p}_{\text{loc}}(\mathcal{X}) \) denotes the collection of functions which belong to \( N^{1,p}(B) \) for any ball \( B \subset \mathcal{X} \).

Proof. To show the first embedding, let \( B \) be a ball in \( \mathcal{X} \). By Proposition 2.3, we know that \( \chi_B \in M_p^q(\mathcal{X}) \) and

\[
\| \chi_B \|_{M_p^q(\mathcal{X})} \lesssim [\mu(B)]^{1/q} < \infty.
\]

Recall, by our conventions on notation at the end of Section 1, that the symbol \( \lesssim \) means that the implicit positive constant here is independent of \( B \).

Now let \( f \in \text{Lip}_b(\mathcal{X}) \) with \( \text{supp}(f) \subset B \) and \( L \) be the Lipschitz constant of \( f \), which means that, for all \( x, y \in \mathcal{X} \),

\[
|f(x) - f(y)| \leq Ld(x, y).
\]

Since \( f \in \text{Lip}_b(\mathcal{X}) \), we know that there exists a positive constant \( M_0 \) such that \( |f| \leq M_0 \chi_B \). Hence, by (1.3), we see that \( f \in M_p^q(\mathcal{X}) \) and

\[
\| f \|_{M_p^q(\mathcal{X})} \lesssim M_0 [\mu(B)]^{1/q}.
\]
On the other hand, notice that, for all rectifiable curves \( \gamma \), it holds true that
\[
|f \circ \gamma(\ell(0)) - f \circ \gamma(\ell(\gamma))| \leq L d(\gamma(\ell(0)), \gamma(\ell(\gamma))) \leq \int_{\gamma} L \, ds.
\]
Hence \( L \) is an upper gradient of \( f \). Then, by Lemma 2.2, \( L \chi_{2B} \) is a \( \text{Mod}^p_{\mu} \)-weak upper gradient of \( f \), which further implies that \( f \in \text{NM}^p_\mu(X) \) and
\[
\|f\|_{\text{NM}^p_\mu(X)} \lesssim (M_0 + L) \mu(B)^{1/q}.
\]
Thus, \( \text{Lip}_b(X) \subset \text{NM}^p_\mu(X) \).

The second embedding follows directly from definitions, together with [29, Corollary 5.7]. Indeed, let \( f \in \text{NM}^p_\mu(X) \). Then, by [29, Definition 2.4 and Corollary 5.7], we know that
\[
\|f\|_{\text{NM}^p_\mu(X)} = \|f\|_{\mathcal{M}^p_\mu(X)} + \inf \|h\|_{\mathcal{M}^p_\mu(X)} < \infty,
\]
where the infimum is taken over all the upper gradients of \( f \). From this, we deduce that \( f \) has an upper gradient \( h \in \mathcal{M}^p_\mu(X) \) and hence \( h \in L^p(E) \) for any ball \( E \subset X \). Since it is obvious that \( f \in L^p(E) \), by [29, Definition 2.4 and Corollary 5.7] again, we obtain that \( f \in N^{1,p}(E) \), which, together with the arbitrariness of \( E \subset X \) and the definition of \( N^{1,p}_{\text{loc}}(X) \), implies that \( f \in N^{1,p}_{\text{loc}}(X) \) and hence completes the proof of Theorem 2.4.

**Remark 2.5.** We point out that \( \text{Lip}_b(X) \) might not be dense in \( \text{NM}^p_\mu(X) \) when \( p < q \). Indeed, even the set of Lipschitz functions, \( \text{Lip}(X) \), might not be dense in \( \text{NM}^p_\mu(X) \) when \( p < q \). This behavior of \( \text{NM}^p_\mu(X) \) (non-density of Lipschitz functions) is different from the Newton-Sobolev space \( N^{1,p}(X) = \text{NM}^p_\mu(X) \), since \( \text{Lip}(X) \) is dense in \( N^{1,p}_{\text{loc}}(X) \) (see [37, Theorem 4.1]). A counterexample in the Euclidean setting is given in Remark 4.8 below.

### 3 Sobolev embeddings

Let \( \alpha \in (0,1] \) and \( C^{0,\alpha}(X) \) denote the \( \alpha \)-Hölder space on \( X \), namely, the space of all functions \( f \) satisfying that, for all \( x, y \in X \),
\[
|f(x) - f(y)| \leq C[d(x,y)]^\alpha,
\]
where \( C \) is a positive constant independent of \( x \) and \( y \).

It is well known that, when \( X = \mathbb{R}^n \), the following Sobolev embeddings hold true:
\[
W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{np/(n-p)}(\mathbb{R}^n) \quad \text{if} \quad p < n,
\]
and
\[
W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-n/p}(\mathbb{R}^n) \quad \text{if} \quad p > n,
\]
where the symbol \( \hookrightarrow \) means continuous embedding. The generalizations of (3.1) and (3.2) to the Newton-Sobolev space and the Hajlasz-Sobolev space on metric measure spaces were
obtained in [37] and [15, 16], respectively. This section is devoted to the corresponding Sobolev embedding theorems for Newton-Morrey-Sobolev spaces.

Recall that a space \( X \) is said to support a weak \((1, p)\)-Poincaré inequality if there exist positive constants \( C \) and \( \tau \geq 1 \) such that, for all open balls \( B \) in \( X \) and all pairs of functions \( f \) and \( \rho \) defined on \( \tau B \), whenever \( \rho \) is an upper gradient of \( f \) in \( \tau B \) and \( f \) is integrable on \( B \), then

\[
(3.3) \quad \frac{1}{\mu(B)} \int_B |f(x) - f_B| \, d\mu(x) \leq C \operatorname{diam}(B) \left\{ \frac{1}{\mu(\tau B)} \int_{\tau B} [\rho(x)]^p \, d\mu(x) \right\}^{1/p},
\]

where above and in what follows, \( f_B \) denotes the integral mean of \( f \) on \( B \), namely,

\[
(3.4) \quad f_B = \frac{1}{\mu(B)} \int_B f(y) \, d\mu(y),
\]

diam(\( B \)) the diameter of \( B \) and \( \tau B \) the ball with the same center as \( B \) but \( \tau \) times the radius of \( B \). In particular, if \( \tau = 1 \), then we say that \( X \) supports a \((1, p)\)-Poincaré inequality.

It is well known that the Euclidean space supports a \((1, p)\)-Poincaré inequality. For more information on Poincaré inequalities, we refer the reader to [20, 21, 16] and their references.

A measure \( \mu \) on \( X \) is said to be doubling if there exists a positive constant \( C \) such that, for all balls \( B \) in \( X \), it holds true that

\[
\mu(2B) \leq C \mu(B).
\]

As a generalization of (3.2) to Newton-Morrey-Sobolev spaces, we have the following conclusion.

**Theorem 3.1.** Let \( 1 \leq p \leq q < \infty \) and \( Q \in (0, q) \). Assume that \((X, d, \mu)\) is a metric measure space, with doubling measure \( \mu \), and supports a weak \((1, p)\)-Poincaré inequality. If there exists a positive constant \( C \) such that \( \mu(B(x, r)) \geq Cr^Q \) for all \( x \in X \) and \( 0 < r < 2 \operatorname{diam}(X) \), then

\[
NM^p_q(X) \hookrightarrow C^{0, 1-Q/q}(X).
\]

**Proof.** By the same reason as that stated in the proof of [37, Theorem 5.1], we only need to show that, if \( f \in NM^p_q(X) \) and \( x, y \) are Lebesgue points of \( f \), then

\[
|f(x) - f(y)| \lesssim [d(x, y)]^{1-Q/q} \|f\|_{NM^p_q(X)}.
\]

To this end, let \( B_1 := B(x, d(x, y)), B_{-1} := B(y, d(x, y)) \) and, for all \( i > 1 \),

\[
B_i = \frac{1}{2} B_{i-1} \quad \text{and} \quad B_{-i} = \frac{1}{2} B_{-i+1}.
\]

Let \( B_0 := B(x, 2d(x, y)) \). Since \( x, y \) are Lebesgue points, it follows that

\[
|f(x) - f(y)| \leq \sum_{i \in \mathbb{Z}} |f_{B_i} - f_{B_{i+1}}|.
\]

Let \( \rho \) be an upper gradient of \( f \) such that

\[
\|f\|_{M^p_q(X)} + \|\rho\|_{M^p_q(X)} \lesssim \|f\|_{NM^p_q(X)}.
\]
Let $r_i$ be the radius of the ball $B_i$. Then, by this, (1.3), the doubling condition of $\mu$ and
the weak $(1,p)$-Poincaré inequality, together with $\mu(\tau B_i) \gtrsim r_i^Q$, we see that, when $i \in \mathbb{N}$,

\begin{equation}
|f_{B_i} - f_{B_{i+1}}| \lesssim \frac{1}{\mu(B_i)} \int_{B_i} |f_{B_i} - f(x)| \, d\mu(x)
\lesssim \text{diam}(B_i) \left\{ \frac{1}{\mu(\tau B_i)} \int_{\tau B_i} [\rho(x)]^p \, d\mu(x) \right\}^{1/p}
\lesssim r_i \|\mu(\tau B_i)|^{-1/q} \|\rho\|_{M^p_q(X)} \lesssim r_i^{1-Q/q} \|\rho\|_{M^p_q(X)}
\lesssim 2^{-i(1-Q/q)} [d(x,y)]^{1-Q/q} \|f\|_{NM^p_q(X)}. \tag{3.5}
\end{equation}

Similarly, for all $i \leq -2$, we also have

\[ |f_{B_i} - f_{B_{i+1}}| \lesssim 2^{i(1-Q/q)} [d(x,y)]^{1-Q/q} \|f\|_{NM^p_q(X)}. \]

On the other hand, by the Hölder inequality and the doubling condition of $\mu$, we see that

\[ |f_{B_{-1}} - f_{B_0}| \lesssim \frac{1}{\mu(B_{-1})} \int_{B_{-1}} |f_{B_0} - f(z)| \, d\mu(z) \lesssim \frac{1}{\mu(B_0)} \int_{B_0} |f_{B_0} - f(x)| \, d\mu(x) \]

and then, similar to (3.5), we further conclude that

\[ |f_{B_{-1}} - f_{B_0}| \lesssim [d(x,y)]^{1-Q/q} \|f\|_{NM^p_q(X)}. \]

Meanwhile, by the same method as above, we also find that

\[ |f_{B_0} - f_{B_1}| \lesssim [d(x,y)]^{1-Q/q} \|f\|_{NM^p_q(X)}. \]

Thus, combining the above estimates, by $Q \in (0, q)$, we see that

\[ |f(x) - f(y)| \lesssim [d(x,y)]^{1-Q/q} \sum_{i \in \mathbb{Z}} 2^{-|i|(1-Q/q)} \|f\|_{NM^p_q(X)} \]

which completes the proof of Theorem 3.1.

\[ \square \]

**Remark 3.2.** Theorem 3.1 generalizes [37, Theorem 5.1] by taking $p = q$.

Next we give a Rellich-Kondrachov type embedding theorem for $NM^p_q(X)$ when $p$ is small, which can be seen as a generalization of (3.1). We begin with the following notion of the Ahlfors $Q$-regular measure spaces; see, for example, [19].

**Definition 3.3.** Let $Q \in (0, \infty)$. A metric measure space $\mathcal{X}$ is said to be Ahlfors $Q$-regular (or $Q$-regular), if there exists a constant $C \geq 1$ such that, for any $x \in \mathcal{X}$ and any $r \in (0, 2\text{diam}(\mathcal{X}))$,

\[ \frac{1}{C} r^Q \leq \mu(B(x,r)) \leq C r^Q. \]
Let $L^1_{\text{loc}}(\mathcal{X})$ be the collection of all locally integrable functions on $\mathcal{X}$. The Hardy-Littlewood maximal operator $M$ is defined by setting, for all $f \in L^1_{\text{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$,

$$Mf(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),$$

where the supremum is taken over all balls $B$ in $\mathcal{X}$ containing $x$. The following statement shows that the operator $M$ is bounded on Morrey spaces. For its proof, we refer the reader to [5] for example.

Lemma 3.4. Let $(\mathcal{X}, d, \mu)$ be a metric space with doubling measure $\mu$ and $1 < p \leq q \leq \infty$. Then there exists a positive constant $C$ such that, for all $f \in \mathcal{M}_p^q(\mathcal{X})$,

$$\|Mf\|_{\mathcal{M}_p^q(\mathcal{X})} \leq C\|f\|_{\mathcal{M}_p^q(\mathcal{X})}.$$

We also need the following boundedness of fractional integral operators on Morrey spaces.

Proposition 3.5. Let $\mathcal{X}$ be Ahlfors $Q$-regular with $Q \in (0, \infty)$, $1 < p \leq q < \infty$ and $\alpha > 0$ such that $q < Q/\alpha$. Then, the fractional integral $I_\alpha$ is bounded from $\mathcal{M}_p^q(\mathcal{X})$ to $\mathcal{M}_{p^*}^{q^*}(\mathcal{X})$, where $p^* := \frac{pq}{q-qp}$, $q^* := \frac{pq}{q-pq}$ and $I_\alpha$ is defined by setting, for all $f \in \mathcal{M}_p^q(\mathcal{X})$ and $x \in \mathcal{X}$,

$$I_\alpha(f)(x) := \int_{\mathcal{X}} \frac{f(y)}{|d(x,y)|^{Q-\alpha}} \, d\mu(y).$$

Proof. Without loss of generality, we may assume that $f \in \mathcal{M}_p^q(\mathcal{X})$ is non-negative. For any $x \in \mathcal{X}$, fix $\delta > 0$ and write

$$I_\alpha(f)(x) = \int_{B(x,\delta)} \frac{f(y)}{|d(x,y)|^{Q-\alpha}} \, d\mu(y) + \int_{\mathcal{X}\setminus B(x,\delta)} \frac{f(y)}{|d(x,y)|^{Q-\alpha}} \, d\mu(y)
= b_\delta^{(\alpha)}(x) + g_\delta^{(\alpha)}(x).$$

By the Hölder inequality, (1.3) and the Ahlfors $Q$-regular property of $\mathcal{X}$, together with $q < Q/\alpha$, we see that

$$g_\delta^{(\alpha)}(x) \leq \sum_{j=0}^\infty \int_{B(x,2j+1\delta) \setminus B(x,2j\delta)} \frac{f(y)}{|d(x,y)|^{Q-\alpha}} \, d\mu(y)
\leq \sum_{j=0}^\infty (2^{j+1}\delta)^{\alpha-Q} \left[ \mu \left( B \left( x, 2^{j+1}\delta \right) \right) \right]^{1-1/p} \left\{ \int_{B(x,2j+1\delta)} |f(y)|^p \, d\mu(y) \right\}^{1/p}
\leq \sum_{j=0}^\infty (2^{j+1}\delta)^{\alpha-Q} \left[ \mu \left( B \left( x, 2^{j+1}\delta \right) \right) \right]^{1-1/q} \|f\|_{\mathcal{M}_p^q(\mathcal{X})}
\leq \sum_{j=0}^\infty (2^{j+1}\delta)^{\alpha-Q} (2^{j}\delta)^{(1-1/q)Q} \|f\|_{\mathcal{M}_p^q(\mathcal{X})} \leq \delta^{\alpha-Q/q} \|f\|_{\mathcal{M}_p^q(\mathcal{X})}.$$
For $b_\delta$, let $A_j := B_j \setminus B_{j+1} := B(x, 2^{-j}\delta) \setminus B(x, 2^{-j-1}\delta)$ for all $j \in \mathbb{N} \cup \{0\}$. Then, by the Ahlfors $Q$-regular property of $\mathcal{X}$, together with $\alpha > 0$, we see that, for all $x \in \mathcal{X}$,

$$b_\delta^{(\alpha)}(x) = \sum_{j \in \mathbb{Z}_+} \int_{A_j} \frac{f(y)}{d(x,y)^q} \, d\mu(y) \approx \sum_{j \in \mathbb{Z}_+} (2^{-j}\delta)^{\alpha - Q} \int_{B_j} f(y) \, d\mu(y) \lesssim \delta^{\alpha} \sum_{j \in \mathbb{Z}_+} 2^{-j\alpha} \frac{1}{\mu(B_j)} \int_{B_j} f(y) \, d\mu(y) \lesssim \delta^{\alpha} M(f)(x).$$

Combining (3.7) and (3.8), we have

$$I_\alpha(f)(x) \lesssim \delta^{\alpha} M(f)(x) + \delta^{\alpha - Q/q} \|f\|_{\mathcal{M}_p^q(\mathcal{X})}.$$

Now let $\delta := \|f\|^q_{\mathcal{M}_p^q(\mathcal{X})}[M(f)(x)]^{-q/Q}$. Then for any $x \in \mathcal{X}$,

$$I_\alpha(f)(x) \lesssim \|f\|^{q\alpha/q}_{\mathcal{M}_p^q(\mathcal{X})} [M(f)(x)]^{1 - \alpha q/Q},$$

which, together with Lemma 3.4, further implies that

$$\|I_\alpha(f)\|_{\mathcal{M}_p^{q\alpha/q}(\mathcal{X})} \lesssim \|f\|^{q\alpha/q}_{\mathcal{M}_p^q(\mathcal{X})} \|M(f)[1 - \alpha q/Q]\|_{\mathcal{M}_p^{q\alpha/q}(\mathcal{X})} \approx \|f\|^{q\alpha/q}_{\mathcal{M}_p^q(\mathcal{X})} \|M(f)[1 - \alpha q/Q]\|_{\mathcal{M}_p^{q\alpha/q}(\mathcal{X})} \lesssim \|f\|_{\mathcal{M}_p^q(\mathcal{X})}.$$

This finishes the proof of Proposition 3.5.

Now we have the following Rellich-Kondrachov type embedding result, which generalizes [37, Theorem 5.2] by taking $p = q$, $\alpha = 1$ and $\mathcal{X}$ being bounded.

**Theorem 3.6.** Let $1 \leq r < p \leq q < \infty$. Let $1 < p/r \leq q/r < Q/\alpha < \infty$, $\alpha \in (0,r) \cap (0,Q)$ and $\mathcal{X}$ be an Ahlfors $Q$-regular metric measure space supporting a weak $(1,r)$-Poincaré inequality. Then there exists a positive constant $C$ such that, for all functions $f \in NM_p^q(\mathcal{X})$, upper gradients $\rho$ of $f$ and $R \in (0,\infty)$,

$$\|f - f_{B(\cdot, R)}\|_{\mathcal{M}_p^{Q^-}\mathcal{X}} \lesssim CR^{1 - \alpha/r} \|\rho\|_{\mathcal{M}_p^{q\alpha/q}(\mathcal{X})},$$

where $Q^* := \frac{Qr}{q \alpha - q}.$

**Proof.** Let $f \in NM_p^q(\mathcal{X})$ and $\rho$ be an upper gradient of $f$. For any Lebesgue point $x$ for $f$, we write $B_0 := B(x, R)$ and $B_i := B(x, 2^{-i}R)$ for all $i \in \mathbb{N}$. Since an Ahlfors $Q$-regular space is doubling, by the weak $(1,r)$-Poincaré inequality (namely, the inequality (3.3) with $p$ replaced by $r$) and $r > \alpha$, we see that

$$|f(x) - f_{B(x, R)}| \leq \sum_{i=0}^{\infty} |f_{B_i} - f_{B_{i+1}}| \lesssim \sum_{i=0}^{\infty} \frac{1}{\mu(B_i)} \int_{B_i} |f(z) - f_{B_i}| \, d\mu(z).$$
which completes the proof of Theorem 3.6.

holds true for the case \( r > \alpha \) which has its own interest. However, it is not clear whether the above conclusion still applies to \( y \in [37, \text{Theorem } 5.2] \).

Applying Proposition 3.5, together with \( 1 < p/r < Q < \alpha \), we conclude that

\[
\|f - f_{B(.R)}\|_{M^{q,p}_{\alpha/r}(\mathcal{X})} \lesssim R^{1-\alpha/r} \left\| \{I_{a}(\rho^p)\}^{1/p}_{\alpha/r} \right\|_{M^{q,p}_{\alpha/r}(\mathcal{X})} \approx R^{1-\alpha/r} \left\| \{I_{a}(\rho^p)\}^{1/p}_{\alpha/r} \right\|_{M^{q,p}_{\alpha/r}(\mathcal{X})} \lesssim R^{1-\alpha/r} \left\| \rho \right\|_{M^{q,p}_{\alpha/r}(\mathcal{X})},
\]

which completes the proof of Theorem 3.6. \( \square \)

**Remark 3.7.** (i) Let \( 1 < r < p < \infty, 1 < p/r < Q < \infty \), and \( \mathcal{X} \) be an Ahlfors \( Q \)-regular metric measure space supporting a weak \( (1,r) \)-Poincaré inequality. Then, by Theorem 3.6 with \( \alpha = 1 \), we see that there exists a positive constant \( C \) such that, for all functions \( f \in N^{1,p}(\mathcal{X}) \), upper gradients \( \rho \) of \( f \) and \( R \in (0, \infty) \),

\[
\|f - f_{B(.R)}\|_{L^{Q,p}_{\alpha/r}(\mathcal{X})} \leq CR^{1-1/r} \|\rho\|_{L^p(\mathcal{X})},
\]

which has its own interest. However, it is not clear whether the above conclusion still holds true for the case \( r = 1 \) or not, since, we had to use Theorem 3.6 with \( \alpha = 1 \) and, to this end, we need \( r > \alpha = 1 \).

(ii) We also remark that Theorem 3.6 generalizes the classical result for Newton-Sobolev spaces in [37, Theorem 5.2]. Indeed, if we further assume that \( \mathcal{X} \) is bounded, then we know that \( f_{\mathcal{X}} = f_{B(x,\text{diam}(\mathcal{X}))} \) for almost all \( x \in \mathcal{X} \). Thus, it follows, from (i), that, under the same assumptions on \( Q, r, p \) as in (i), there exists a positive constant \( C \) such that, for all functions \( f \in N^{1,p}(\mathcal{X}) \) and upper gradients \( \rho \) of \( f \),

\[
\|f - f_{\mathcal{X}}\|_{L^{Q,p}_{\alpha/r}(\mathcal{X})} \leq C[\text{diam}(\mathcal{X})]^{1-1/r} \|\rho\|_{L^p(\mathcal{X})},
\]

which is just [37, Theorem 5.2].

(iii) The condition on the weak \( (1,r) \)-Poincaré inequality in Theorem 3.6 can be replaced by the weak \( (1,1) \)-Poincaré inequality, due to the Hölder inequality.

## 4 Hajlasz-Morrey-Sobolev spaces

In this section, we introduce Morrey-Sobolev spaces associated with Hajlasz gradients and consider the relation between the Hajlasz-Morrey-Sobolev space and the Newton-Morrey-Sobolev space.
Definition 4.1. Let $0 < p \leq q \leq \infty$. The Hajłasz-Morrey-Sobolev space $HM^q_p(\mathcal{X})$ is defined to be the space of all measurable functions $f$ that have a Hajłasz gradient $h \in \mathcal{M}^q_p(\mathcal{X})$. The norm of $f \in HM^q_p(\mathcal{X})$ is defined as

$$
\|f\|_{HM^q_p(\mathcal{X})} := \|f\|_{\mathcal{M}^q_p(\mathcal{X})} + \inf \|h\|_{\mathcal{M}^q_p(\mathcal{X})},
$$

where the infimum is taken over all Hajłasz gradients $h$ of $f$.

We remark that $HM^q_p(\mathcal{X})$ when $p = q$ is just the Hajłasz-Sobolev space $M^{1,p}(\mathcal{X})$ of [13]. Moreover, $\|f\|_{HM^q_p(\mathcal{X})} = 0$ if and only if $f = 0$ almost everywhere.

To consider the relation between the Hajłasz-Morrey-Sobolev space and the Newton-Morrey-Sobolev space, we need the following technical lemma, which is a special case of [29, Lemma 5.6].

Lemma 4.2. Let $1 \leq p \leq q < \infty$ and $g$ be a Mod$^q_p$-weak upper gradient of $f$. Then, for any $\varepsilon \in (0, \infty)$, there exists a function $g_\varepsilon$, which is an upper gradient of $f$, such that $\|g_\varepsilon - g\|_{\mathcal{M}^q_p(\mathcal{X})} \leq \varepsilon$ and $g_\varepsilon \geq g$ everywhere on $\mathcal{X}$.

Applying Lemma 4.2, we obtain the following conclusion.

Theorem 4.3. Let $1 < p \leq q < \infty$. If $\mathcal{X}$ supports a weak $(1,p)$-Poincaré inequality and the measure $\mu$ is doubling, then

$$
NM^q_p(\mathcal{X}) \hookrightarrow HM^q_p(\mathcal{X}).
$$

Proof. Let $f \in NM^q_p(\mathcal{X})$. By [25, Theorem 1.0.1], we see that $\mathcal{X}$ supports a weak $(1,r)$-Poincaré inequality for some $r \in (1,p)$. By Lemma 4.2, there exists an upper gradient $g$ of $u$ such that

$$
\|f\|_{\mathcal{M}^q_p(\mathcal{X})} + \|g\|_{\mathcal{M}^q_p(\mathcal{X})} \lesssim \|f\|_{NM^q_p(\mathcal{X})}.
$$

Since $\mathcal{X}$ supports a weak $(1,r)$-Poincaré inequality for some $r \in (1,p)$, by [16, Theorem 3.2], we know that there exists a set $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that, for all $x, y \in \mathcal{X} \setminus E$,

$$
|f(x) - f(y)| \lesssim d(x,y) \left\{ [M(g^r)(x)]^{1/r} + [M(g^r)(y)]^{1/r} \right\}.
$$

Hence a positive constant multiple of $h := [M(g^r)]^{1/r}$ is a Hajłasz gradient of $f$. Then, by Lemma 3.4, we know that

$$
\|f\|_{HM^q_p(\mathcal{X})} \lesssim \|f\|_{\mathcal{M}^q_p(\mathcal{X})} + \|h\|_{\mathcal{M}^q_p(\mathcal{X})} \lesssim \|f\|_{\mathcal{M}^q_p(\mathcal{X})} + \|g\|_{\mathcal{M}^q_p(\mathcal{X})} \lesssim \|f\|_{NM^q_p(\mathcal{X})},
$$

which completes the proof of Theorem 4.3.

Remark 4.4. When $p = q$, under the same assumptions as in Theorem 4.3, it was proved by Shanmugalingam in [37, Theorem 4.9] that $NM^q_p(\mathcal{X}) = HM^q_p(\mathcal{X})$ with equivalent norms.

Next we turn to consider the inverse embedding of Theorem 4.3.
\textbf{Theorem 4.5.} Let $1 \leq p \leq q < \infty$. Then,

$$HM^q_p(\mathcal{X}) \hookrightarrow NM^q_p(\mathcal{X}).$$

\textit{Proof.} Let $f \in HM^q_p(\mathcal{X})$. Then, there exists a Hajlasz gradient $h \in \mathcal{M}^q_p(\mathcal{X})$ of $f$ such that

$$|f(x) - f(y)| \leq d(x,y)[h(x) + h(y)], \quad x, y \in \mathcal{X}\setminus E,$$

for some $E$ of measure 0, and $\|f\|_{\mathcal{M}^q_p(\mathcal{X})} + \|h\|_{\mathcal{M}^q_p(\mathcal{X})} \lesssim \|f\|_{HM^q_p(\mathcal{X})}$. It was proved in [24, Theorem 1.1] that, if $f, g \in L^1_{\text{loc}}(\mathcal{X})$ and $g$ is a Hajlasz gradient of $f$, then there exist $\tilde{f}$ and $\tilde{g}$ such that $\tilde{f} = f$ and $\tilde{g} = g$ almost everywhere, and $8\tilde{g}$ is an upper gradient of $\tilde{f}$. Since $\tilde{f} = f$ in $HM^q_p(\mathcal{X})$, we identify $f$ and $\tilde{f}$. In this sense, $8\tilde{h}$ is an upper gradient of $f$. Therefore,

$$\|f\|_{NM^q_p(\mathcal{X})} \lesssim \|f\|_{\mathcal{M}^q_p(\mathcal{X})} + \|\tilde{h}\|_{\mathcal{M}^q_p(\mathcal{X})} \sim \|f\|_{\mathcal{M}^q_p(\mathcal{X})} + \|h\|_{\mathcal{M}^q_p(\mathcal{X})} \lesssim \|f\|_{HM^q_p(\mathcal{X})},$$

which completes the proof of Theorem 4.5. \hfill \Box

Combining Theorems 4.3 and 4.6, we have the following conclusion.

\textbf{Theorem 4.6.} Let $1 < p \leq q < \infty$. If $\mathcal{X}$ supports a weak $(1,p)$-Poincaré inequality and the measure $\mu$ is doubling, then $NM^q_p(\mathcal{X}) = HM^q_p(\mathcal{X})$ with equivalent norms.

Next we consider the relations among the Hajlasz-Morrey-Sobolev space, the Newton-Morrey-Sobolev space and the classical Morrey-Sobolev space on $\mathbb{R}^n$. Let $1 \leq p \leq q < \infty$. Recall that the classical \textit{Morrey-Sobolev space} $WM^q_p(\mathbb{R}^n)$ is defined by

$$WM^q_p(\mathbb{R}^n) := \left\{ f \in \mathcal{M}^q_p(\mathbb{R}^n) : \|\nabla f\|_{\mathcal{M}^q_p(\mathbb{R}^n)} < \infty \right\},$$

where $\nabla f$ denotes the \textit{weak derivative} of $f$. The \textit{norm} of $f \in WM^q_p(\mathbb{R}^n)$ is given by

$$\|f\|_{WM^q_p(\mathbb{R}^n)} := \|f\|_{\mathcal{M}^q_p(\mathbb{R}^n)} + \|\nabla f\|_{\mathcal{M}^q_p(\mathbb{R}^n)}.$$}

Observe that $WM^q_p(\mathbb{R}^n)$ is just the Sobolev space $W^{1,p}(\mathbb{R}^n)$.

\textbf{Theorem 4.7.} Let $1 < p \leq q < \infty$. Then,

$$WM^q_p(\mathbb{R}^n) = NM^q_p(\mathbb{R}^n) = HM^q_p(\mathbb{R}^n)$$

with equivalent norms.

\textit{Proof.} Observe that the conclusion of Theorem 4.7 when $1 < p = q < \infty$ is just [13, Theorem 1]. Thus, in what follows of this proof, we always assume that $1 < p < q < \infty$.

By Theorem 4.6, it suffices to prove that $WM^q_p(\mathbb{R}^n) \hookrightarrow HM^q_p(\mathbb{R}^n)$ and

$$NM^q_p(\mathbb{R}^n) \hookrightarrow WM^q_p(\mathbb{R}^n).$$
We first show that $WM_p^q(\mathbb{R}^n) \hookrightarrow HM_p^q(\mathbb{R}^n)$. Let $f \in WM_p^q(\mathbb{R}^n)$. By the definition of $WM_p^q(\mathbb{R}^n)$, we see that $|\nabla f| \in L^p(Q)$ for all cubes $Q$ in $\mathbb{R}^n$ and then, following the argument as in [13, p. 404], we know that, for all Lebesgue points $x, y \in \mathbb{R}^n$ of $f$,

$$|f(x) - f(y)| \lesssim |x - y| \left[ M(|\nabla f|)(x) + M(|\nabla f|)(y) \right].$$

Hence a positive constant multiple of $M(|\nabla f|)$ is a Hajlasz gradient of $f$. Moreover, by Definition 4.1 and Lemma 3.4, we further see that

$$\|f\|_{HM_p^q(\mathbb{R}^n)} \leq \|f\|_{\mathcal{M}_p^q(\mathbb{R}^n)} + \|M(|\nabla f|)\|_{\mathcal{M}_p^q(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}_p^q(\mathbb{R}^n)} + \||\nabla f||\mathcal{M}_p^q(\mathbb{R}^n) \approx \|f\|_{WM_p^q(\mathbb{R}^n)}.$$

This shows that $WM_p^q(\mathbb{R}^n) \subset HM_p^q(\mathbb{R}^n)$.

Next we prove $NM_p^q(\mathbb{R}^n) \subset WM_p^q(\mathbb{R}^n)$. Let $f \in NM_p^q(\mathbb{R}^n)$. Then, by Definition 1.3, there exists $g \in \mathcal{M}_p^q(\mathbb{R}^n)$ such that $g$ is a weak upper gradient of $f$. Moreover, for any ball $B \subset \mathbb{R}^n$, we have $g \in L^q(B)$, which implies that $f \in N^{1,q}(B)$. By [6, Theorem A.2], we know that, for $i \in \{1, \ldots, n\}$ and almost every $x \in B$, $\frac{\partial f}{\partial x_i}(x)$ exists and $|\frac{\partial f}{\partial x_i}(x)|$ is controlled by $g(x)$. Since $B \subset \mathbb{R}^n$ is arbitrary, it follows that, for almost every $x \in \mathbb{R}^n$,

$$\left| \frac{\partial f}{\partial x_i}(x) \right| \leq g(x),$$

which, together with $g \in \mathcal{M}_p^q(\mathbb{R}^n)$, implies that $\frac{\partial f}{\partial x_i} \in \mathcal{M}_p^q(\mathbb{R}^n)$. Furthermore, we have $|\nabla f| \in \mathcal{M}_p^q(\mathbb{R}^n)$, from which, together with $f \in \mathcal{M}_p^q(\mathbb{R}^n)$, we deduce that $f \in WM_p^q(\mathbb{R}^n)$. This finishes the proof of Theorem 4.7.

We remark that Theorem 4.7 when $p = q$ goes back to the equivalence between Sobolev spaces and Hajlasz-Sobolev spaces on $\mathbb{R}^n$ obtained in [13].

**Remark 4.8.** (i) We remark that, for all $1 < p < q < \infty$, the set $C^1(\mathbb{R}^n)$ of functions having continuous derivatives up to order 1 is not dense in $NM_p^q(\mathbb{R}^n)$. To see this, by Theorem 4.7, we only need to consider $WM_p^q(\mathbb{R}^n)$. For simplicity, we only consider the case $n = 1$. Let $\phi \in C^\infty_c(\mathbb{R})$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $(-1, 1)$, and $\phi \equiv 0$ on $(-2, 2)^c$. Write $g(x) := |x|^{1-1/q}\phi(x)$ for all $x \in \mathbb{R}$. Then, the function

$$g'(x) := \begin{cases} (1 - 1/q)x^{-1/q}\phi(x) + x^{1-1/q}\phi'(x), & x \in (0, \infty); \\ 0, & x = 0; \\ -(1 - 1/q)(-x)^{-1/q}\phi(x) + (-x)^{1-1/q}\phi'(x), & x \in (-\infty, 0) \end{cases}$$

is a weak derivative of $g$. Since it is known that $|x|^\alpha \chi_{(-2, 2)}(x) \in \mathcal{M}_p^q(\mathbb{R})$ if and only if $\alpha \geq -1/q$, we then see that $g \in WM_p^q(\mathbb{R})$.

Now we apply an approach from [44, pp. 587-588] to show that $g$ can not be approximated by $C^1(\mathbb{R})$ functions in $WM_p^q(\mathbb{R})$. Indeed, it suffices to prove that $g'$ can not be approximated by continuous functions in $\mathcal{M}_p^q(\mathbb{R})$. To see this, for any continuous function
\( h \), write \( N := \sup_{x \in (-1,1)} |h(x)|^p < \infty \). Notice that \( \phi \equiv 1 \) on \((-1,1)\). We then see that, for all \( R \in (0,1) \),

\[
\int_{-R}^{R} |g'(x) - h(x)|^p \, dx \geq 2^{-p} \int_{-R}^{R} |g'(x)|^p \, dx - 2NR.
\]

Notice that

\[
\int_{-R}^{R} |g'(x)|^p \, dx \geq (1 - 1/q)^p \int_{0}^{R} x^{-p/q} \, dx = \frac{(1 - 1/q)^p}{1 - p/q} R^{1-p/q}.
\]

We know that

\[
\int_{-R}^{R} |g'(x) - h(x)|^p \, dx \geq 2^{-p} \left( 1 - 1/q \right)^p R^{1-p/q} - 2NR
\]

\[
= R^{1-p/q} \left[ 2^{-p} \left( 1 - 1/q \right)^p - 2NR^{p/q} \right].
\]

Hence, taking \( R \) small enough such that

\[
2^{-p} \left( 1 - 1/q \right)^p - 2NR^{p/q} \geq 2^{-p-1} \frac{(1 - 1/q)^p}{1 - p/q},
\]

we then see that

\[
\|g' - h\|_{\mathcal{M}_p^q(\mathbb{R})} \gtrsim \frac{R^{p/q-1}}{1 - p/q} \int_{-R}^{R} |g'(x) - h(x)|^p \, dx
\]

\[
\gtrsim 2^{-p-1} \frac{(1 - 1/q)^p}{1 - p/q} > 0.
\]

This implies the above claim.

(ii) We point out that the key property we used in (i) is the locally boundedness of continuous functions, which ensures that the number \( N \) is finite. If we replace continuous functions \( h \) by any locally bounded functions, then the subsequent argument remains true. From this observation, together with the well-known fact that any Lipschitz function \( f \) on \( \mathbb{R}^n \) is differentiable almost everywhere and the absolute value \( |\partial_i f| \) of its weak derivative \( \partial_i f \) is dominated by its Lipschitz constant \( L_f \) almost everywhere, we deduce that \( g \) cannot be approximated by any Lipschitz function \( f \) in the norm of \( WM_p^q(\mathbb{R}^n) \). Therefore, the set of Lipschitz functions is not dense in \( NM_p^q(\mathbb{R}^n) \) when \( 1 < p < q < \infty \).

## 5 Boundedness of (fractional) maximal operators

This section is devoted to the boundedness of (fractional) maximal operators on Morrey type spaces over metric measure spaces.

In Subsection 5.1, for a geometrically doubling metric measure space \((\mathcal{X}, \rho, \mu)\) in the sense of Hytönen [23], we show, in Theorem 5.8 below, that the modified maximal operator
$M_0^{(β)}$ (see (5.1) below) is bounded on the modified Morrey space $\mathcal{M}_p^{q,(k)}(\mathcal{X})$, which, when $(\mathcal{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, \mu)$ with $\mu$ being a Radon measure satisfying the polynomial growth condition (also called the non-doubling measure), was introduced by Sawano and Tanaka [36]. As an application, the boundedness of the fractional maximal operator $M_α^{(β)}$ on this space is also obtained in Proposition 5.10 below.

In Subsection 5.2, if $\mu$ is a doubling measure, as applications of Theorem 5.8 and Proposition 5.10, we show the boundedness of the fractional maximal operator $M_α$ on $\mathcal{X}$ (see Corollary 5.11 below), from which, we further deduce, in Corollary 5.12 below, the boundedness of the discrete fractional maximal function $\tilde{M}_α^*$ is bounded on $NM_0^p(\mathcal{X})$ (see Theorem 5.13 below). Finally, we prove that, if $\mu$ is doubling and satisfies (5.9), and $\mathcal{X}$ supports a weak $(1, p)$-Poincaré inequality, then the discrete fractional maximal function $\tilde{M}_α^*$ is bounded on $NM_0^p(\mathcal{X})$ (see Theorem 5.14 below).

5.1 Maximal operators on $\mathcal{M}_p^{q,(k)}(\mathcal{X})$

In 2010, Hytönen [23] introduced the notion of geometrically doubling metric measure spaces which include both spaces of homogeneous type and the Euclidean spaces with non-doubling measures satisfying the polynomial growth condition as special cases; see also the monograph [42] for some recent developments of this subject.

Now we recall the following notion of the geometrically doubling from [23], which is also known as metrically doubling (see, for example, [19, p. 81]).

Definition 5.1. A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling, if there exists $N_0 \in \mathbb{N}$ such that any given ball contains no more than $N_0$ points at distance exceeding half its radius.

From the geometrically doubling property, we deduce the following conclusion, which is used later on.

Proposition 5.2. Let $(\mathcal{X}, d)$ be a geometrically doubling metric space. Then, for any ball $B(x, r) \subset \mathcal{X}$, with $x \in \mathcal{X}$ and $r \in (0, \infty)$, and any $n_1 \geq n_2 > 1$, there exist $r_0 \in (0, \infty)$ and $\tilde{N}$ balls $\{B(x_i, r_0)\}_{i=1}^{\tilde{N}}$ such that $n_1 B(x, r_0) \subset n_2 B(x, r)$ for all $i \in \{1, \ldots, \tilde{N}\}$ and

$$B(x, r) \subset \bigcup_{i=1}^{\tilde{N}} B(x_i, r_0),$$

where $\tilde{N} \in \mathbb{N}$ depends only on $n_1$, $n_2$ and the constant $N_0$ in Definition 5.1.

Proof. Let $n_1$ and $n_2$ be as in Proposition 5.2, and

$$k := \left\lfloor \log_2 \left( \frac{n_1 + 1}{n_2 - 1} \right) \right\rfloor + 1,$$
where \([t]\) denotes the maximal integer not more than \(t \in \mathbb{R}\). We claim that, for any \(y \in \mathcal{X}\) and ball \(B(x, r) \subset \mathcal{X}\) with \(x \in \mathcal{X}\) and \(r \in (0, \infty)\), if \(B(y, \frac{r}{2^k}) \cap B(x, r) \neq \emptyset\), then \(n_1 B(y, \frac{r}{2^k}) \subset n_2 B(x, r)\). Indeed, by choosing \(z \in B(y, \frac{r}{2^k}) \cap B(x, r)\) and observing that \(k > \log_2 \frac{n_1 + 1}{n_2 - 1}\), we have

\[
d(x, y) \leq d(x, z) + d(z, y) < \left(1 + \frac{1}{2^k}\right) r < \left(n_2 - \frac{n_1}{2^k}\right) r.
\]

Thus, for all \(w \in n_1 B(y, \frac{r}{2^k})\),

\[
d(w, x) \leq d(w, y) + d(y, x) < \frac{n_1 r}{2^k} + \left(n_2 - \frac{n_1}{2^k}\right) r = n_2 r,
\]

which shows the above claim. Then, by repeating the proof that (1) implies (2) in [23, Lemma 2.3], we obtain the desired conclusion, which completes the proof of Proposition 5.2.

Now we recall the definition of the modified Morrey space, which, when \((\mathcal{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, \mu)\) with \(\mu\) being a Radon measure satisfying the polynomial growth condition, was originally introduced by Sawano and Tanaka [36].

**Definition 5.3.** Let \(k \in (0, \infty)\), \(1 \leq p \leq q < \infty\) and \(\mathcal{X}\) be a metric measure space. The modified Morrey space \(\mathcal{M}^{q,(k)}_p(\mathcal{X})\) is defined as

\[
\mathcal{M}^{q,(k)}_p(\mathcal{X}) := \left\{ f \in L^p_{\text{loc}}(\mathcal{X}) : \|f\|_{\mathcal{M}^{q,(k)}_p(\mathcal{X})} < \infty \right\},
\]

where

\[
\|f\|_{\mathcal{M}^{q,(k)}_p(\mathcal{X})} := \sup_{B(x, r) \subset \mathcal{X}} \left[ \mu(B(x, kr)) \right]^{1/q-1/p} \left[ \int_{B(x, r)} |f(y)|^p d\mu(y) \right]^{1/p},
\]

where the supremum is taken over all balls \(B(x, r)\), with \(x \in \mathcal{X}\) and \(r \in (0, \infty)\), of \(\mathcal{X}\).

Recall that a geometrically doubling metric measure space \((\mathcal{X}, d, \mu)\) means that \((\mathcal{X}, d)\) is geometrically doubling and \(\mu\) is a non-negative Radon measure on \((\mathcal{X}, d)\).

**Proposition 5.4.** Let \((\mathcal{X}, d, \mu)\) be a geometrically doubling metric measure space and \(1 \leq p \leq q < \infty\). Then, the space \(\mathcal{M}^{q,(k)}_p(\mathcal{X})\) is independent of the choice of \(k \in (1, \infty)\).

**Proof.** Let \(k_1, k_2 \in (1, \infty)\). We need to show that \(\mathcal{M}^{q,(k_1)}_p(\mathcal{X})\) and \(\mathcal{M}^{q,(k_2)}_p(\mathcal{X})\) coincide with equivalent norms. To this end, without loss of generality, we may assume that \(k_1 < k_2\). By Definition 5.3, we easily find that \(\mathcal{M}^{q,(k_1)}_p(\mathcal{X}) \subset \mathcal{M}^{q,(k_2)}_p(\mathcal{X})\). Thus, we still need to show the inverse embedding. Let \(B\) be a ball in \(\mathcal{X}\). By Proposition 5.2, there exist \(\tilde{N}\) balls \(\{B_i\}_{i=1}^{\tilde{N}}\) with the same radius such that, for all \(i \in \{1, \ldots, \tilde{N}\}\), \(k_2 B_i \subset k_1 B\) and \(B \subset \bigcup_{i=1}^{\tilde{N}} B_i\), where \(\tilde{N}\) depends only on \(k_1, k_2\) and \(N_0\) in Definition 5.1. By these, we see that

\[
[\mu(k_1 B)]^{1/q-1/p} \left[ \int_B |f(x)|^p d\mu(x) \right]^{1/p} \leq \sum_{i=1}^{\tilde{N}} [\mu(k_1 B)]^{1/q-1/p} \left[ \int_{B_i} |f(x)|^p d\mu(x) \right]^{1/p}
\]

and

\[
\|f\|_{\mathcal{M}^{q,(k_1)}_p(\mathcal{X})} \leq \|f\|_{\mathcal{M}^{q,(k_2)}_p(\mathcal{X})}
\]

for all \(f \in \mathcal{M}^{q,(k_1)}_p(\mathcal{X})\). Therefore, it follows that \(\mathcal{M}^{q,(k_1)}_p(\mathcal{X}) \subset \mathcal{M}^{q,(k_2)}_p(\mathcal{X})\). Hence, \(\mathcal{M}^{q,(k)}_p(\mathcal{X})\) is independent of the choice of \(k \in (1, \infty)\).
A Proof. Let $B$ be a maximal subset in $A$. By the arbitrariness of $B$ and Definition 5.3, we conclude that

$$
\|f\|_{\mathcal{M}^{q(k_1)}_p(\mathcal{X})} \leq \tilde{N}\|f\|_{\mathcal{M}^{q(k_2)}_p(\mathcal{X})},
$$

which further implies that $\mathcal{M}^{q(k_2)}_p(\mathcal{X}) \subset \mathcal{M}^{q(k_1)}_p(\mathcal{X})$ and hence completes the proof of Proposition 5.4.

Recall that, for $\alpha \in [0,1]$ and $\beta \in [1,\infty)$, the modified fractional maximal operator $M^{(\beta)}_\alpha$ is defined by setting, for all $f \in L^1_{\text{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$,

$$(5.1) \quad M^{(\beta)}_\alpha f(x) := \sup_{r>0} [\mu(B(x,\beta r))]^{\alpha-1} \int_{B(x,r)} |f(y)| \, d\mu(y).$$

In particular, we write $M_\alpha := M^{(1)}_\alpha$.

The maximal operator $M_\beta^{(\beta)}$ where $\beta \in (1,\infty)$ is bounded on the modified Morrey spaces. To prove this, we need the following technical lemma.

**Lemma 5.5.** Let $\beta \in (1,\infty)$ and $(\mathcal{X},d)$ be a geometrically doubling metric space. Suppose that $\mathcal{B} := \{B(x,\lambda \delta)\}_{x \in \Lambda} \cup \{B(x,\lambda \tau)\}_{x \in \Lambda}$ such that $\sup_{x \in \Lambda} \lambda \delta < \infty$. Then, there exist $J_\beta \in \mathbb{N}$, depending only on $\beta$ and $N_0$ in Definition 5.1, and sub-families of balls of $\mathcal{B}$, $\mathcal{B}_i := \{B(x,\lambda \tau)\}_{x \in \Lambda}$ with $i \in \{1,\ldots,J_\beta\}$, such that

(i) for each $i \in \{1,\ldots,J_\beta\}$, $\mathcal{B}_i$ consists of disjoint balls;

(ii) for any $\lambda \in \Lambda$, there exists $\lambda' \in \bigcup_{i=1}^{J_\beta} \Lambda_i$ such that $B(x,\lambda \delta) \subset B(x,\lambda' \delta \lambda')$.

**Proof.** Let $R := \sup_{x \in \Lambda} \lambda \delta$ and, for all $j \in \mathbb{Z}_+$,

$$(5.2) \quad A_j := \{B(x,\lambda \delta) : \lambda \in \Lambda, R\sqrt{\beta^{-j-1}} < \lambda \delta \leq R\sqrt{\beta^{-j}}\}.$$

Here, we need $\beta \in (1,\infty)$ and, otherwise, $A_j = \emptyset$ for all $j \in \mathbb{Z}_+$. Let $D_0 \subset A_0$ be a maximal subset in $A_0$ such that, for any two distinct balls $B(x,\lambda \delta)$ and $B(x,\lambda' \delta \lambda')$ in $D_0$, it holds true that $d(x,\lambda \delta) > R(\sqrt{\beta} - 1)$. By such a choice, we know that, for any ball $B(x,\lambda \delta) \in A_0$, there exists $B(x,\lambda' \delta \lambda') \in D_0$ such that $d(x,\lambda \delta) \leq R(\sqrt{\beta} - 1)$. Furthermore, from $r_{\lambda} \leq R$ and $\beta r_{\lambda'} > R\sqrt{\beta}$, it follows that

$$B(x,\lambda \delta) \subset B(x,\lambda' \delta \lambda').$$

Let $E_0$ be the collection of balls $B(x,\lambda \delta)$ which belong to $\mathcal{B}$ and satisfy that, for some $B(x,\lambda' \delta \lambda') \in D_0$, $B(x,\lambda \delta) \subset B(x,\lambda' \delta \lambda')$. Obviously, $A_0 \subset E_0$. 
Let $m \geq 1$. We now define $D_m$ and $E_m$ recursively. Suppose that $D_j$ and $E_j$ for $j \in \{0, \ldots, m-1\}$ has already been defined. Let $D_m \subset A_m \setminus \bigcup_{j=0}^{m-1} E_j$ be a maximal subset satisfying that, for all distinct balls $B(x_\lambda, r_\lambda)$ and $B(x_{\lambda'}, r_{\lambda'})$ in $D_m$, it holds true that

$$d(x_\lambda, x_{\lambda'}) > R \sqrt{\beta^{-j}} (\sqrt{\beta} - 1).$$

Let $E_m$ be the collection of balls $B(x_\lambda, r_\lambda)$ which belong to $B$ and satisfy that, for some $B(x_{\lambda'}, r_{\lambda'}) \in D_m$, $B(x_\lambda, r_\lambda) \subset B(x_{\lambda'}, \beta r_{\lambda'})$. Notice that, for any ball $B \in A_m$, we have either $B \in E_j$ for some $j \in \{0, \ldots, m-1\}$ or $B \in A_m \setminus \bigcup_{j=0}^{m-1} E_j$. In the first case, we can find a ball $B(x_{\lambda'}, r_{\lambda'}) \in D_j$ such that $B \subset B(x_{\lambda'}, \beta r_{\lambda'})$. In the second case, we can find $B(x_{\lambda'}, r_{\lambda'}) \in D_m$ such that $B \subset B(x_{\lambda'}, \beta r_{\lambda'})$.

Due to the geometrically doubling condition, we can partition each $D_j$ into disjoint sub-families, $D_{j,1}, \ldots, D_{j,L_\beta}$, where $L_\beta$ is a positive constant depending only on $\beta$ and the geometrically doubling constant $N_0$, since, for any $j \in \mathbb{Z}_+$ and any $B := B(x_\lambda, r_\lambda) \in D_j$, there are at most $L_\beta$ balls in $D_j$ intersect $B$. Indeed, let $F_j$ be the collection of balls $B' := B(x_{\lambda'}, r_{\lambda'})$ which belong to $D_j$ and intersect $B$. Let $y \in B'$ and $z \in B \cap B'$. Then

$$d(y, x_\lambda) \leq d(y, x_{\lambda'}) + d(x_{\lambda'}, z) + d(z, x_\lambda) < 2r_{\lambda'} + r_\lambda \leq 3R \sqrt{\beta^{-j}}.$$ 

Thus, $B' \subset B(x_\lambda, 3R \sqrt{\beta^{-j}})$. On the other hand, by the choice of $D_j$, we know that

$$d(x_\lambda, x_{\lambda'}) > R \sqrt{\beta^{-j}} (\sqrt{\beta} - 1)$$

and hence

$$B \left( x_\lambda, \sqrt{\beta - 1}^{-\frac{r_\lambda}{2}} \right) \cap B \left( x_{\lambda'}, \sqrt{\beta - 1}^{-\frac{r_{\lambda'}}{2}} \right) = \emptyset.$$ 

By the geometrically doubling property of $\mathcal{X}$ and [23, Lemma 2.3], we see that there exists a constant $L_\beta$, depending on $N_0$ and $\beta$, such that $F_j$ has no more than $L_\beta$ balls.

Let $N_\beta \in \mathbb{N}$ satisfy

$$1 + 2 \sqrt{\beta^{-N_\beta}} < \sqrt{\beta}. \quad (5.3)$$

We claim that, if $j_1 \geq j_2 + N_\beta$, then, for any pair of balls, $(B_1, B_2) \in D_{j_1} \times D_{j_2}$, $B_1$ and $B_2$ do not intersect. To see this, assume that $B_1 \cap B_2 \neq \emptyset$ and $x \in B_1 \cap B_2$. Then, by (5.3), for any $y \in B_1$, we have

$$d(y, x_2) \leq d(y, x) + d(x, x_2) < 2r(B_1) + r(B_2) \leq 2R \sqrt{\beta^{-j_1}} + R \sqrt{\beta^{-j_2}} \leq R \sqrt{\beta^{-j_2}} (2 \sqrt{\beta^{-N_\beta}} + 1) \leq R \sqrt{\beta^{-j_2-1}} \beta < \beta r(B_2),$$

where $x_i$ and $r(B_i)$ denote the center and the radius of $B_i$, for $i \in \{1, 2\}$, respectively. Thus, $B_1 \subset \beta B_2$ and hence belongs to $E_{j_2}$, which contradicts to the definition of $D_{j_1}$, since $D_{j_1} \cap E_{j_2} = \emptyset$. Thus, the above claim holds true.

Therefore, if, for $i \in \{1, \ldots, N_\beta\}$ and $n \in \{1, \ldots, L_\beta\}$, let

$$B_{i,n} := \bigcup_{j=0}^{\infty} D_{N_\beta j + i, n}.$$
Then, \( \{B_{i,n} : i \in \{1, \ldots, N_\beta\}, n \in \{1, \ldots, L_\beta\}\} \) are the desired families, which completes the proof of Lemma 5.5.

The boundedness of the modified maximal operator on \( L^p(\mathcal{X}) \) could be deduced from the above lemma by borrowing some ideas used in the proof of [39, Section 3.1, Theorem 1]. We give some details as follows.

**Theorem 5.6.** Let \( \beta \in (1, \infty) \) and \( (\mathcal{X}, d, \mu) \) be a geometrically doubling metric measure space.

(i) Then, there exists a positive constant \( C \) such that, for all \( \lambda \in (0, \infty) \) and \( f \in L^1(\mathcal{X}) \),

\[
\mu \left( \left\{ x \in \mathcal{X} : M_0^{(\beta)}(x) > \lambda \right\} \right) \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{X})}.
\]

(ii) Let \( p \in (1, \infty] \). Then, there exists a positive constant \( C \) such that, for all \( f \in L^p(\mathcal{X}) \),

\[
\left\| M_0^{(\beta)} f \right\|_{L^p(\mathcal{X})} \leq C \|f\|_{L^p(\mathcal{X})}.
\]

**Proof.** The boundedness of \( M_0^{(\beta)} \) on \( L^\infty(\mathcal{X}) \) is obvious. Next we only prove (i), since (ii) can be deduced from (i) and the \( L^\infty(\mathcal{X}) \)-boundedness of \( M_0^{(\beta)} \) via interpolation.

Let 
\[
E_\lambda := \left\{ x \in \mathcal{X} : M_0^{(\beta)}(x) > \lambda \right\}.
\]

Then, by the definition of \( E_\lambda \), for any \( x \in E \), there exists a ball \( B_x \) such that

\[
\mu(E_\lambda) = \lim_{k \to \infty} \frac{1}{\mu(\beta B_x)} \int_{B_x} |f(y)| \, d\mu(y) > \lambda.
\]

For all \( k \in \mathbb{Z}_+ \), let \( \mathcal{B}^{(k)} \) be the collection of all balls \( B_x \) for \( x \in E \), whose radius \( r(B_x) \in (0, 2^k] \), and \( E_\lambda^{(k)} := \{ x \in E_\lambda : B_x \in \mathcal{B}^{(k)} \} \). Then, \( E_\lambda = \bigcup_{k \in \mathbb{Z}_+} E_\lambda^{(k)} \) and \( E_\lambda^{(k)} \subset E_\lambda^{(k+1)} \) for any \( k \in \mathbb{Z}_+ \).

For each \( k \in \mathbb{Z}_+ \), by Lemma 5.5, we can find \( J_\beta \in \mathbb{N} \), independent of \( k \), and sub-families \( \mathcal{B}_i^{(k)} \subset \mathcal{B}^{(k)} \), \( i \in \{1, \ldots, J_\beta\} \) such that

\[
\bigcup_{B \in \mathcal{B}^{(k)}} B \subset \bigcup_{i=1}^{J_\beta} \bigcup_{B \in \mathcal{B}_i^{(k)}} \beta B,
\]

where \( \beta B \) denotes the ball with the same center as \( B \) but \( \beta \) times the radius of \( B \). Thus, by the fact that \( E_\lambda^{(k)} \) increasingly converges to \( E_\lambda \) as \( k \to \infty \), and the disjointness of balls in \( \mathcal{B}_i^{(k)} \) over \( i \), we see that

\[
\mu(E_\lambda) = \lim_{k \to \infty} \mu \left( E_\lambda^{(k)} \right) \leq \lim_{k \to \infty} \mu \left( \bigcup_{B \in \mathcal{B}^{(k)}} B \right) \leq \lim_{k \to \infty} \mu \left( \bigcup_{i=1}^{L_\beta} \bigcup_{B \in \mathcal{B}_i^{(k)}} \beta B \right)
\]
\[ \lim_{k \to \infty} \sum_{B \in B_i^{(k)}} \mu(\beta B) \leq \lim_{k \to \infty} \sum_{B \in B_i^{(k)}} \frac{1}{X} \int_B |f(y)| \, d\mu(y) \lesssim \frac{L_\beta}{X} \|f\|_{L^1(\mathcal{X})}. \]

This finishes the proof of Theorem 5.6. \(\square\)

**Remark 5.7.**

(i) It is worth pointing out that Theorem 5.6 also holds true for the non-centered maximal operator, whose proof is similar, the details being omitted.

(ii) We should point out that Lemma 5.5 and Theorem 5.6 are generously provided to us by Professor Yoshihiro Sawano from Tokyo Metropolitan University of Japan.

(iii) Lemma 5.5 and Theorem 5.6 in the case \(\beta = 1\) are still unknown.

Then we have the following conclusion, which generalizes [36, Theorem 2.3], wherein the corresponding result on the non-doubling measure satisfying the polynomial growth condition on \(\mathbb{R}^n\) was obtained. The proof of Theorem 5.8 is similar to that of [36, Theorem 2.3], and one key tool used in the proof is the \(L^p(\mu)\)-boundedness in Theorem 5.6. For the sake of convenience, we give the details.

**Theorem 5.8.** Let \(\mathcal{X}\) be a geometrically doubling metric measure space, \(1 < p \leq q < \infty\), \(\beta \in (1, \infty)\) and \(k \in (1, \infty)\). Then, there exists a positive constant \(C\) such that, for all \(f \in M_{p,k}^{\beta}(\mathcal{X})\),

\[
\left\| M_0^{(\beta)} f \right\|_{M_{p,k}^{\beta}(\mathcal{X})} \leq C \| f \|_{M_{p,k}^{\beta}(\mathcal{X})}.
\]

**Proof.** By Proposition 5.4, it suffices to consider the case that \(k := \frac{2\beta}{\beta + 1} > 1\). Let \(f \in M_{p,k}^{\beta}(\mathcal{X})\) and \(B_0 \subset \mathcal{X}\) be a ball. Define \(\tilde{\beta} := \frac{\beta + 7}{\beta} > 1\), \(f_1 := f \chi_{\tilde{\beta}B_0}\) and \(f_2 := f - f_1\). Then, by Definition 5.3, together with \(1 < p \leq q < \infty\), we have

\[
\frac{1}{|\mu(\beta B_0)|^{1/p-1/q}} \left\{ \int_{B_0} \left[ M_0^{(\beta)} f_1(y) \right]^p d\mu(y) \right\}^{1/p} \leq \frac{1}{|\mu(\beta B_0)|^{1/p-1/q}} \left\{ \int_{\mathcal{X}} \left[ M_0^{(\beta)} f_1(y) \right]^p d\mu(y) \right\}^{1/p} \leq \| f \|_{M_{p,k}^{\beta}(\mathcal{X})},
\]

where we used the fact that \(M_0^{(\beta)}\) is bounded on \(L^p(\mu)\), for \(p \in (1, \infty)\) (see Theorem 5.6).

To estimate \(f_2\), observe that, if \(B \subset \mathcal{X}\) is a ball satisfying that \(B \cap B_0 \neq \emptyset\) and \(B \cap (\mathcal{X} \setminus \tilde{\beta}B_0) \neq \emptyset\), then the radius \(r_B > \frac{\tilde{\beta} - 1}{2} r_{B_0} = \frac{4}{\beta - 1} r_{B_0}\), where \(r_B\) and \(r_{B_0}\) denote, respectively, the radii of \(B\) and \(B_0\), and hence \(B_0 \subset \frac{\beta + 1}{2} B\). Therefore, we see that, for any \(x \in B_0\),

\[
M_0^{(\beta)} f_2(x) \leq \sup_{x \in B} \frac{1}{\mu(\beta B)} \int_B |f_2(y)| \, d\mu(y) \leq \sup_{B_0 \subset \frac{\beta + 1}{2} B} \frac{1}{\mu(\beta B)} \int_B |f(y)| \, d\mu(y)
\]
where

\[ \tilde{\beta} \]

which completes the proof of Theorem 5.8.

Proposition 5.10. Let \( \mathcal{X} \) be a geometrically doubling metric measure space, \( 1 < p \leq q < \infty \), \( \beta \in (1, \infty) \), \( \alpha \in (0, 1/q) \) and \( k \in (1, \infty) \). Then, there exists a positive constant \( C \) such that, for all \( f \in \mathcal{M}_p^{q,(k)}(\mathcal{X}) \),

\[
\| M^{(\beta)}_\alpha f \|_{\mathcal{M}^{\tilde{p},(k)}_p(\mathcal{X})} \leq C \| f \|_{\mathcal{M}^{q,(k)}_p(\mathcal{X})},
\]

where \( \tilde{p} := \frac{p}{1-\alpha q} \) and \( \tilde{q} := \frac{q}{1-\alpha q} \).

Proof. By Proposition 5.4, it suffices to consider the case that \( k = \beta \in (1, \infty) \).

For any ball \( B(x, r) \subset \mathcal{X} \), with \( x \in \mathcal{X} \) and \( r > 0 \), and \( f \in \mathcal{M}_p^{q,(k)}(\mathcal{X}) \), we know, by the Hölder inequality, (1.3) and (3.6), that

\[
[\mu(B(x, \beta r))]^{\alpha-1} \int_{B(x, r)} |f(y)| \, d\mu(y)
\]
Moreover, since \( \mu \) is doubling, we see that, for any \( x \in \mathcal{X} \),
\[
M_\alpha^{(\beta)} f(x) \leq \|f\|_{\tilde{M}_p^{(\beta)}(\mathcal{X})} \left[ M_0^{(\beta)} f(x) \right]^{1-aq},
\]
which, together with (5.1), implies that, for all \( x \in \mathcal{X} \),
\[
M_\alpha^{(\beta)} f(x) \leq \|f\|_{\tilde{M}_p^{(\beta)}(\mathcal{X})} \left[ M_0^{(\beta)} f(x) \right]^{1-aq}.
\]
Then, by Theorem 5.8, we see that
\[
\left\| M_\alpha^{(\beta)} f \right\|_{\tilde{M}_p^{(\beta)}(\mathcal{X})} \leq \|f\|_{\tilde{M}_p^{(\beta)}(\mathcal{X})} \left[ M_0^{(\beta)} f \right]^{1-aq} \leq \|f\|_{\tilde{M}_p^{(\beta)}(\mathcal{X})},
\]
which completes the proof of Proposition 5.10. \( \square \)

### 5.2 Fractional maximal operators on \( HM_p^q(\mathcal{X}) \) and \( NM_p^q(\mathcal{X}) \)

Recently, Heikkinen et al. [17, 18] studied the boundedness of some (fractional) maximal operators on the Newton-Sobolev space and the Hajlasz-Sobolev space over metric measure spaces. In this section, we consider the corresponding problem for Newton-Morrey-Sobolev spaces and Hajlasz-Morrey-Sobolev spaces.

Throughout this section, we always assume that the measure \( \mu \) is doubling. We call a measure is doubling if there exists a constant \( C_0 \in [1, \infty) \) such that, for all \( x \in \mathcal{X} \) and \( r > 0 \),
\[
\mu(B(x, 2r)) \leq C_0 \mu(B(x, r)) \quad \text{(doubling property)}.
\]

Recall that it is well known that any space of homogeneous type is also geometrically doubling (see [8, pp. 66-67]). Moreover, since \( \mu \) is doubling, we see that, for any \( \beta \in (1, \infty) \) and \( \alpha \in [0, 1] \), there exists a positive constant \( C \), depending on \( \beta \) and \( \alpha \), such that, for all \( f \in L_{loc}^{1}(\mathcal{X}) \) and \( x \in \mathcal{X} \), \( M_\alpha f(x) \leq C M_\alpha^{(\beta)} f(x) \). From these facts, Theorem 5.8 and Proposition 5.10, we immediately deduce the following conclusion.

**Corollary 5.11.** Let \( 1 < p \leq q < \infty \) and \( \alpha \in [0, 1/q) \). Then, there exists a positive constant \( C \), depending on \( \alpha \), \( p \) and \( q \), such that, for all \( f \in M_p^q(\mathcal{X}) \),
\[
\left\| M_\alpha f \right\|_{\tilde{M}_p^{(\beta)}(\mathcal{X})} \leq C \|f\|_{M_p^q(\mathcal{X})},
\]
where \( \tilde{p} := \frac{p}{\frac{1}{1-aq}} \) and \( \tilde{q} := \frac{q}{\frac{1}{1-aq}} \).
As an application of Corollary 5.11, we obtain the following boundedness of (fractional) maximal operators \( \tilde{M}_\alpha \) on modified Morrey spaces.

Recall that a measure \( \mu \) is said to satisfy the measure lower bound condition, if there exists a positive constant \( C \) such that, for any \( x \in X \) and \( r \in (0, \infty) \),

\[
\mu(B(x, r)) \geq Cr^Q
\]

for some \( Q \in (0, \infty) \).

Recently, if \( \mu \) satisfies (5.9), Heikkinen et al. [18] established the boundedness from \( L^p(X) \) to \( L^s(X) \) for \( p \in (1, Q) \) and \( s := \frac{Qp}{Q-\alpha p} \) of the following modified (fractional) maximal function \( \tilde{M}_\alpha \), defined by setting, for any \( \alpha \in [0, 1] \), \( f \in L^1_{\text{loc}}(X) \) and \( x \in X \),

\[
(5.10) \quad \tilde{M}_\alpha f(x) := \sup_{r>0} \frac{r^\alpha}{\mu(B(x, r))} \int_{B(x, r)} |f(y)| \, d\mu(y).
\]

It is easy to see that, in the present setting, there exists a positive constant \( C \), depending on \( \alpha \) and \( Q \), such that, for all \( f \in L^1_{\text{loc}}(X) \) and \( x \in X \), \( \tilde{M}_\alpha f(x) \leq CM_\alpha f(x) \), which, together with Corollary 5.11, implies the following conclusion.

**Corollary 5.12.** Let \( 1 < p \leq q < \infty \) and \( \alpha \in [0, 1/q) \). Assume that \( \mu \) satisfies (5.9). Then, there exists a positive constant \( C \), depending on \( \alpha \), \( p \) and \( q \), such that, for all \( f \in M^q_p(X) \),

\[
(5.11) \quad \left\| \tilde{M}_\alpha f \right\|_{M^q_p(X)} \leq C \left\| f \right\|_{M^q_p(X)},
\]

where \( \tilde{p} := \frac{p}{1-\alpha q} \) and \( \tilde{q} := \frac{q}{1-\alpha q} \).

Recall that \( X \) is said to satisfy the relative 1-annular decay property, if there exists a positive constant \( C \) such that, for all \( x \in X \), \( R \in (0, \infty) \) and \( h \in (0, R) \),

\[
(5.12) \quad \mu(B \cap [B(x, R) \setminus B(x, R-h)]) \leq C \frac{h}{r_B} \mu(B)
\]

for all balls \( B \) with radius \( r_B < 3R \); see, for example, [18, (2.5)].

Now we turn to the boundedness of the (fractional) maximal operator \( \tilde{M}_\alpha \) on Hajłasz-Morrey-Sobolev spaces.

**Theorem 5.13.** Assume that \( \mu \) satisfies (5.9) and \( X \) has the relative 1-annular decay property (5.12). Let \( 1 < p \leq q < \infty \) and \( \alpha \in [0, Q/q) \). Then, for any \( f \in HM^q_p(X) \), \( \tilde{M}_\alpha f \in HM^{p_\ast}_p(X) \), where \( p_\ast := \frac{Qp}{Q-aq} \) and \( q_\ast := \frac{Qq}{Q-aq} \). Moreover, there exists a positive constant \( C \), depending only on the doubling constant, \( Q \), \( p \), \( q \) and \( \alpha \), such that, for all \( f \in HM^q_p(X) \),

\[
\left\| \tilde{M}_\alpha f \right\|_{HM^{p_\ast}_p(X)} \leq C \left\| f \right\|_{HM^q_p(X)}.
\]
Proof. The proof is similar to that of [18, Theorem 4.5]. We present some details. Let \( f \in HM_p^q(\mathcal{X}) \) and \( g \in M_p^q(\mathcal{X}) \) be a Hajlasz gradient of \( f \) such that \( \|g\|_{M_p^q(\mathcal{X})} \lesssim \|f\|_{HM_p^q(\mathcal{X})} \). It is easy to see that \( g \) is also a Hajlasz gradient of \( |f| \). Let \( r \in (1,p) \) and define

\[
\tilde{g} := \left[ \widetilde{M}_{\alpha r}(g^r) \right]^{1/r}.
\]

By an argument similar to that used in the proof of [18, Theorem 4.5], we know that \( \tilde{g} \) is a Hajlasz gradient of \( \widetilde{M}_\alpha(|f|) \), as well as \( \widetilde{M}_\alpha f \), since \( \widetilde{M}_\alpha(|f|) = \widetilde{M}_\alpha f \). Moreover, by \( p/r > 1 \), (1.3) and (5.11), we see that

\[
\|g\|_{M_p^q(\mathcal{X})} \lesssim \|g\|_{M_p^q(\mathcal{X})}^{1/r} \lesssim \|g\|_{M_p^q(\mathcal{X})} \approx \|g\|_{M_p^q(\mathcal{X})}.
\]

Combining (5.13) and Definition 4.1, we obtain the desired conclusion and then complete the proof of Theorem 5.13.

We point out that Theorem 5.13 when \( p = q \) goes back to [18, Theorem 4.5].

Now we recall the discrete (fractional) maximal operator \( M_\alpha^* \) introduced in [17, Section 5]. Let \( \{B(x_i,r)\}_{i \in \mathbb{N}} \) be a ball covering of \( \mathcal{X} \) such that \( \{B(x_i,r)\}_{i \in \mathbb{N}} \) are of finite overlap. Since \( \mathcal{X} \) is doubling, the overlap number \( N \) depends only on the doubling constant and is independent of \( r \). Let \( \{\varphi_i\}_{i \in \mathbb{N}} \) be a partition of unity related to \( \{B(x_i,r)\}_{i \in \mathbb{N}} \) such that \( 0 \leq \varphi_i \leq 1 \), \( \varphi_i = 0 \) on \( \mathcal{X} \setminus B(x_i,6r) \), \( \varphi_i \geq v \) on \( B(x_i,3r) \) and \( \varphi_i \) is Lipschitz function with Lipschitz constant \( L/r \), where \( L \in (0,\infty) \) and \( v \in (0,1] \) are constants depending only on the doubling constant, and \( \sum_{i \in \mathbb{N}} \varphi_i \equiv 1 \). The discrete convolution of \( u \in L^1_{\text{loc}}(\mathcal{X}) \) at the scale \( 3r \) is defined by setting, for all \( x \in \mathcal{X} \),

\[
u_r(x) := \sum_{i \in \mathbb{N}} \varphi_i(x) u_{B(x_i,3r)},
\]

where \( u_{B(x_i,3r)} \) denotes the integral mean of \( u \) on \( B(x_i,3r) \) (see (3.4)). Now, let \( \{r_j\}_{j \in \mathbb{N}} \) be a sequence of the positive rational numbers, and \( \{B(x_{ij},r_j)\}_{i \in \mathbb{N}} \) for each \( j \) is a ball covering of \( \mathcal{X} \) as above. Then, the discrete (fractional) maximal function \( M_\alpha^* u \) of \( u \) is defined by setting, for all \( x \in \mathcal{X} \),

\[
M_\alpha^* u(x) := \sup_{j \in \mathbb{N}} r_j^{\alpha} |u| r_j(x).
\]

Similar to the proof of [17, Theorem 6.3], we obtain the following result on the boundedness of \( M_\alpha^* \) on Newton-Morrey-Sobolev spaces.

**Theorem 5.14.** Let \( \mu \) satisfy (5.9), \( 1 < p \leq q < \infty \) and \( \alpha \in [0,Q/q) \). Assume that \( \mathcal{X} \) is complete and supports a weak \((1,p)\)-Poincaré inequality. Then, for any \( f \in NM_p^q(\mathcal{X}) \), it holds true that \( M_\alpha^* f \in NM_p^q(\mathcal{X}) \) with \( p^* := Qp/(Q - \alpha q) \) and \( q^* := Qq/(Q - \alpha q) \). Moreover, there exists a positive constant \( C \), independent of \( f \), such that

\[
\|M_\alpha^* f\|_{NM_p^{q^*}(\mathcal{X})} \leq C \|f\|_{NM_p^q(\mathcal{X})}.
\]
Proof. Let \( f \in NM^q_p(\mathcal{X}) \) and \( g \in M^q_p(\mathcal{X}) \) be a Mod\(_p\)-weak upper gradient of \( f \) such that

\[
\|g\|_{\mathcal{M}^q_p(\mathcal{X})} \leq 2\|f\|_{NM^q_p(\mathcal{X})}.
\]

By [17, Lemma 5.1] and (5.11), we have

\[
\|M^*_\alpha f\|_{\mathcal{M}^p_q(\mathcal{X})} \lesssim \|f\|_{\mathcal{M}^p_q(\mathcal{X})}.
\]

By the same reason as that used in the proof [17, Theorem 6.3], observing that the pointwise Lipschitz constant of a function is also an upper gradient of that function, we see that a positive constant multiple of \( (M^*_\alpha \theta g^{\theta})^{1/\theta} \) is a Mod\(_p\)-weak upper gradient of \( M^*_\alpha f \), where \( \theta \) lies in \((1, p)\) such that the weak \((1, \theta)\)-Poincaré inequality is supported by \( \mathcal{X} \). By \( g^{\theta} \in M^{q/\theta}_p(\mathcal{X}) \) and \( p/\theta > 1 \), together with [17, Lemma 5.1] and (5.11), we know that

\[
\|\left[ M^*_\alpha \theta (g^{\theta}) \right]^{1/\theta}\|_{\mathcal{M}^p_q(\mathcal{X})} \lesssim \|g\|_{\mathcal{M}^q_p(\mathcal{X})}.
\]

Combining (5.14), (5.15) and (5.16), we obtain

\[
\|M^*_\alpha f\|_{NM^q_p(\mathcal{X})} \lesssim \|M^*_\alpha f\|_{\mathcal{M}^p_q(\mathcal{X})} + \|\left[ M^*_\alpha \theta (g^{\theta}) \right]^{1/\theta}\|_{\mathcal{M}^p_q(\mathcal{X})} \lesssim \|f\|_{\mathcal{M}^p_q(\mathcal{X})} + \|g\|_{\mathcal{M}^q_p(\mathcal{X})} \lesssim \|f\|_{NM^q_p(\mathcal{X})},
\]

which completes the proof of Theorem 5.14.

\[ \square \]

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Yufeng Lu and Dachun Yang (Corresponding author)
School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People’s Republic of China
E-mails: yufeng.lu@mail.bnu.edu.cn (Y. Lu)
          dcyang@bnu.edu.cn (D. Yang)

Wen Yuan
School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People’s Republic of China and
Mathematisches Institut, Friedrich-Schiller-Universität Jena, Jena 07743, Germany
E-mail: wenyuan@bnu.edu.cn