Characterization of input-output group decoupling for linear descriptor systems using controllability subspaces

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Abstract. In this paper we study the problem of the characterization of group decoupled system for linear descriptor systems with index one using controllable subspaces. From the standard decomposition form, the linear descriptor systems can be separated into slow and fast subsystems. In this form, using controllable subspaces, we give some results for the characterization of input-output group decoupled for slow and fast subsystems, respectively. Furthermore, the characterization of the input-output group decoupling yields the normal forms for the input-output group decoupling of linear descriptor system with index one. This normal form can be used in order to define weakly coupled linear systems. All the results that have been obtained in this study can be used to determine the formulation of algorithms in solving the problem of linear quadratic optimal control that contains systems that are consider as weakly coupled subsystems.

1. Introduction

The decoupling problem in multivariable systems theory has been actively investigated in many literatures. Up to now some problems remain unsolved. The input-output decoupling problem is of great theoretical and practical importance since it reduces a multi input-multi output system to a set of single-input single-output systems. Decoupling problem for classical linear systems was first introduced by Morgan [1], which investigated the synthesis of decoupling by state feedback control using a state space approach. Then Falb and Wolovich [2] can be solved the problem posed by Morgan. Furthermore, Morse and Wonham [3] and Wonham and Morse [4] presented a condition that is more general for decoupling problem via geometric approach based on the concept of controllability subspace.

The first paper on the problem of the classical decoupling of linear descriptor systems was recently reported by Cristodoulou and Paraskevopoulos [5]. Its results have the drawback in that the decoupling controller involves not only a proportional but also a derivative term. Furthermore, Komboulis [6] developed a generalization of the decoupling problem into the problem of decoupling blocks for linear descriptor system through the regular static state feedback using algebraic approach. Then, Liu, et al [7] proposed the problem of input-output block decoupling using state feedback for time-varying singular systems. Block decoupling problems have also been discussed by Vaviadis and Karcanias [8] for singular systems through state feedback and input transformation using matrix fraction description approach (MFD). The necessary and sufficient condition of input-output group decoupling for regular linear descriptor system with index one was investigated in Arman, et al [9]. Other results are given equivalent formulation for the input-output group decoupling for linear descriptor systems with index one.
Based on studies conducted by many authors, the main focus for the problem of input-output decoupling is to use state feedback law through an algebraic approach. They derived the analytic expression and structural properties of the transfer matrix function for the closed-loop system. For example, Wonham & Morse [4] and Morse & Wonham [3] studied only the decoupling problem for the classical linear system using geometric approach. They derived more general conditions for decoupling problem by the geometric approach based on the concept of controllability subspace of classical linear systems. While further discussion regarding the problem of input-output decoupling for linear descriptor systems using a geometric approach is still very limited.

In general, the coupled system is very difficult to control. It is known that not all coupled system can always be converted into decoupled system. Therefore, we need to design a control law that a coupled system may become a decoupled system in the sense that every input controls only one output and every output is controlled by only one input. Consequently, a decoupled system can be considered as consisting of a set of independent single-variable systems.

In this paper, following Arman et al. [10], we study the problem of characterization of group decoupling for linear descriptor systems of index at most one using controllability subspace. By using the standard decomposition form, we derived the normal form of input-output group decoupled for slow and fast subsystems. This normal form is also used in order to define weakly coupled subsystems. In the future, the results that have been obtained in this study can be used to determine the formulation of algorithms in solving the problem of linear quadratic optimal control that contains systems that are consider as weakly coupled subsystems.

2. Preliminaries

In this section, some basic theories are related to the properties of linear descriptor systems. Consider the following linear descriptor systems in continuous-time described by

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t),
\end{align*}
\]

where \(x \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\), and \(y \in \mathbb{R}^p\) are the state vector, the input vector, and the output vector of the system, respectively; \(E, A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), \(C \in \mathbb{R}^{p \times n}\) are constant coefficient matrices and we assume \(E\) is singular. It is well-known that the existence and uniqueness of solution (1) are guaranteed if system \((E,A)\) is regular, i.e., \(\det(sE - A) \neq 0\) for some \(s \in \mathbb{C}\).

Given the following theorem shows that the equivalence of a regular linear descriptor system.

**Theorem 1. Duan [11].** Given the linear descriptor systems (1) with \(E,A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), \(C \in \mathbb{R}^{p \times n}\) and \((E, A)\) is regular, then there exist two nonsingular matrices \(Q\) and \(P\) such that

\[
(E, A, B, C) \xrightarrow{(Q,P)} (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})
\]

with \(\tilde{E} = QEP = \text{diag}(I_{n_1}, N)\); \(\tilde{A} = QAP = \text{diag}(A_1, I_{n_2})\); \(\tilde{B} = QB = [B_1 \ B_2]^T\) and \(\tilde{C} = CP = [C_1 \ C_2]\), where \(n_1 + n_2 = n\), and the involved partitions are compatible. Furthermore, the matrix \(N \in \mathbb{R}^{n_2 \times n_2}\) is nilpotent.

Based on Theorem 1 for regular linear descriptor system (1), there exist two nonsingular matrices \(Q\) and \(P\) such that the systems (1) are a restricted system equivalent by the following systems:

\[
\begin{align*}
\dot{x}_1(t) &= A_1x_1(t) + B_1u(t) \\
y_1(t) &= C_1x_1(t), \\
N\dot{x}_2(t) &= x_2(t) + B_2u(t) \\
y_2(t) &= C_2x_2(t),
\end{align*}
\]

where the output systems is

\[
y(t) = y_1(t) + y_2(t) = C_1x_1(t) + C_2x_2(t)
\]
In this form, subsystem (2) and (3) are called the slow and fast subsystems, respectively. This form usually called the standard decomposition form of linear descriptor systems (1). Furthermore, if the nilpotent matrix \( N \) in (3) has index \( h \) (i.e. \( N^h = 0 \) and \( N^{h-1} \neq 0 \)), then the system (1) is called a system with index \( h \). If \( N = 0 \) in the fast subsystem (3), then the original system (1) is called a regular linear descriptor system of index at most one.

**Lemma 1. Dai [12] & Gertsner et al [13].** Let \( E, A \in \mathbb{R}^{n \times n} \). Then the following statements are equivalent:

(i) \( (E, A) \) is regular and of index at most one.
(ii) \( \text{rank } [E \ A S_w(E)] = n \), where \( S_w(E) \) denotes a matrix with orthogonal columns spanning the kernel of matrix \( E \).
(iii) \( \text{deg}(\text{det}(sE - A)) = \text{rank}(E) \).

It is well known that the solution of regular linear descriptor systems (1) take the form

\[
x(t) = P \begin{bmatrix} I \\ 0 \end{bmatrix} x_1(t) + P \begin{bmatrix} 0 \\ I \end{bmatrix} x_2(t)
\]

where

\[
x_1(t) = e^{At} x_1(0) + \int_0^t e^{A(t-\tau)} B_1 u(\tau) \, d\tau
\]

\[
x_2(t) = -\sum_{i=0}^{h-1} N^{\frac{\tau}{h}} B_2 u(i)(t)
\]

It is obvious that the solution \( x(t) \) will not contain derivatives of the input function \( u \) if and only if \( h \leq 1 \). In that case, the solution \( x(t) \) is called impulse-free. In general, the solution \( x(t) \) involves derivatives of the order \( h - 1 \) of the forcing input function \( u \) if the system (1) has index \( h \).

Suppose that the systems (1) are a regular linear descriptor systems with index one. Given \( u_1, u_2, \ldots, u_m \) are input elements and \( y_1, y_2, \ldots, y_p \) are output elements of the linear descriptor systems (1). Then the relationship between input and output of systems can be presented in the following definition.

**Definition 1. Suparwanto [14].** Let the regular linear descriptor systems of the form (1). The output \( y_i \) is not controlled by the input \( u_j \) or equivalently, the input \( u_j \) does not control the input \( y_i \), if we have, for all \( x_0 \in \mathbb{R}^n \) and all admissible inputs \( u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_m \),

\[
y_i(t; x_0, u_1, \ldots, u_{j-1}, v, u_{j+1}, \ldots, u_m) = y_i(t; x_0, u_1, \ldots, u_{j-1}, w, u_{j+1}, \ldots, u_m)
\]

for all \( t \in [0, T] \) and all admissible input \( v, w \).

Suppose that the input vector \( u \in \mathbb{R}^m \) and the output vector \( y \in \mathbb{R}^p \) can be partitioned into \( q \) sub-vectors, i.e.

\[
u = [u_1, u_2, \ldots, u_q]^T ; \ y = [y_1, y_2, \ldots, y_q]^T
\]

where \( u_i \in \mathbb{R}^{l_i}, \ l_1 + l_2 + \cdots + l_q = m \) and \( y_i \in \mathbb{R}^{k_i}, \ k_1 + k_2 + \cdots + k_q = p \), with \( i = 1, 2, \ldots, q \). Therefore, we have the output systems \( y_i(t) = C_i x(t), \ i = 1, 2, \ldots, q \) and the input

\[
u(t) = \sum_{i=1}^{q} G_i u_i(t),
\]

where \( C_i \in \mathbb{R}^{k_i \times n} \) and \( G_i \in \mathbb{R}^{m \times l_i} \) are determined by

\[
C = [C_1, C_2, \ldots, C_q]^T \quad \text{and} \quad G_i = [0, \ldots, l_i, \ldots, 0]^T, \ \text{for } i = 1, 2, \ldots, q.
\]

Note that \( I_i \in \mathbb{R}^{l_i \times l_i} \) is the identity matrix. In this part, if we defined matrix \( B_i \in \mathbb{R}^{n \times l_i} \) by \( B_i = B G_i, \ i = 1, 2, \cdots, q \), then the linear descriptor systems (1) can be stated as

\[
E \dot{x}(t) = A x(t) + \sum_{i=1}^{q} B_i u_i(t).
\]
Furthermore, from this result the linear descriptor systems (1) can be rewritten completely as
\[
\begin{align*}
E\dot{x}(t) &= Ax(t) + \sum_{i=1}^{q} B_i u_i(t) \\
y_i(t) &= C_i x(t), \quad i = 1, 2, \ldots, q,
\end{align*}
\] (10)
where \( u_i \) and \( y_i \) are group of inputs and group of outputs, respectively, with
\[
u_i = [u_{i1}, u_{i2}, \ldots, u_{il}]^T; \quad y_i = [y_{i1}, y_{i2}, \ldots, y_{ik}], \quad \text{for} \ i = 1, 2, \ldots, q.
\]

Thus we can say that a regular linear descriptor system (10) is said to be input-output group decoupling if for given a set of inputs \( \{u_1, u_2, \ldots, u_q\} \), then the group of inputs \( u_i \) only controls the group of outputs \( y_i \) and does not control other output \( y_j \) for \( j \neq i \). This can be stated in the following definition.

**Definition 2 Suparwanto [14].** A regular linear descriptor system (10) is said to be input-output group decoupling if the following statements are true:
(i) The group \( y_i \) of output is not controlled by the group \( u_j \) of input, for \( i \neq j \) with \( i, j = 1, 2, \ldots, q \).
(ii) The group \( y_i \) of output is controlled by the group \( u_i \) of input for \( i = 1, 2, \ldots, q \).

Based on the standard decomposition form, the linear descriptor systems can be decomposed into slow and fast subsystems. Note that the slow subsystem (2) is the classical linear systems. This systems have the controllability subspace
\[
\mathcal{R}_1 = \sum_{k=0}^{n-1} \text{Im} A_1^k B_1,
\]
and the fast subsystems (3) have the controllability subspace
\[
\mathcal{R}_2 = \sum_{k=1}^{n-1} \text{Im} A_2^k B_2.
\]
This controllability subspaces can be used to characterize the problem of input-output group decoupling for linear descriptor systems with index one.

### 3. Input-output group decoupling for slow subsystems using controllability subspaces

In this section, we give the structure of decoupling for slow subsystems based on the results obtained by Wonham and Morse [4], Morse and Wonham [3], Suparwanto [14], and Arman et al [10]. This construction results the structural matrix of the group decoupling system. However, in a practical system it is always desired that the decoupling system has a matrix representation in a simple form, therefore in this part we give theorems which present the matrix representation for the problem of input-output group decoupling for slow subsystems. The system with this form is called the normal form for the input-output group decoupling. In this form, the problem of input-output group decoupling can be seen as a problem of input-output decoupling of control systems that are mutually independent.

If condition (9) can be substituted into slow subsystem (2) and defined the matrix \( B_{1i} \in \mathbb{R}^{n_1 \times l_i} \) by
\[
B_{1i} = B_1 G_i, \quad i = 1, 2, \ldots, q,
\]
thus we obtain the system of the form
\[
\dot{x}_1(t) = A_1 x_1(t) + \sum_{i=1}^{q} B_{1i} u_i(t),
\] (11)

Next, we give the construction of controllability subspaces to obtain the structure decoupled of input-output group decoupling for slow subsystems. Let \( \mathcal{R}_{1i}, \ i = 1, 2, \ldots, q \) be a controllability subspace of system (11), then we have
\[
\mathcal{R}_{1i} = \sum_{k=0}^{n_1-1} \text{Im} A_1^k B_{1i}.
\]
The first step, we defined a subspace $\beta_0$ by $\beta_0 = \bigcap_{j=1}^{q} \mathbb{R}_{ij}^+$, where $\mathbb{R}_{ij}^+ = \sum_{i=1}^{q} \mathbb{R}_{1i}$. Therefore the subspace $\beta_0$ can be rewritten completely as

$$\beta_0 = \bigcap_{j=1}^{q} \sum_{i=1}^{q} \sum_{k=0}^{n_l-1} \text{Im} A_1^k B_{1i}.$$  \hspace{1cm} (12)

Furthermore, suppose that the subspaces $\beta_i$, $i = 1, 2, \cdots, q$ can be chosen such that $\mathbb{R}_{1i} = \beta_i (\beta_0 \cap \mathbb{R}_{1i})$. This relation can be written completely as

$$\sum_{k=0}^{n_l-1} \text{Im} A_1^k B_{1i} = \beta_i \bigoplus (\beta_0 \cap \sum_{k=0}^{n_l-1} \text{Im} A_1^k B_{1i}).$$ \hspace{1cm} (13)

The important results from the relationship of these subspaces can be presented in the following lemma.

**Lemma 2. Arman et al [10].** Assume that the slow subsystem (11) is controllable. Then the subspaces $\beta_i$, $i = 1, 2, \cdots, q$ satisfy

$$\mathbb{R}^{n_1} = \beta_0 \oplus \beta_1 \oplus \cdots \oplus \beta_q = \bigoplus_{q=0}^{q=0} \beta_i.$$ 

The detail proof of Lemma 2 can be seen in Arman et al [10].

In view of Lemma 2 we assume for the rest of this section that the slow subsystem (2) is controllable and suppose that $T_i$ be the projection onto $\beta_i$ along the subspaces $\sum_{j=0}^{j=0} \beta_j$. Based on this statement we have the following theorem:

**Theorem 2. Arman et al [10].** Given a decoupled subsystem (11) and let $i, j = 1, 2, \cdots, q$ with $i \neq j$. Then the following statements are satisfied:

(i) $T_i A_1 T_j = 0$ and $T_i A_1 = T_i A_1 T_i$.
(ii) $C_{1i} T_j = 0$ and $C_{1i} T_i = C_{1i}$.
(iii) $T_i B_{1j} = 0$ and $T_i B_{1j} = B_{1j}$.
(iv) $\sum_{k=0}^{n_l-1} \text{Im} T_i A_1^k T_i B_{1j} = \beta_i$.

Using the concept develop so far, it is possible to discuss the problem of characterization for structure of input-output group decoupling of slow subsystem (11). Theorem 2 presented a matrix representation for the input-output group decoupling problem of slow subsystems (11). However, considering that in a practical system, it is always desirable that the decoupling system has a matrix representation in a simple form. Therefore, in this section the construction of the matrix representation for the input-output group decoupling for slow subsystem (11) is given. For this reason, we develop such a representation now.

Let $r_i = \text{rank } T_i = \text{dim } \beta_i$, for $i = 0, 1, \cdots, q$. According to the theory of matrices, there exist two matrices $V_i \in \mathbb{R}^{n_1 \times r_i}$ and $W_i \in \mathbb{R}^{r_i \times n_1}$ such that $T_i = V_i W_i$ and $\text{rank } V_i = \text{rank } W_i = r_i$. Furthermore, we define the matrices:

$$\tilde{A}_{1i} = W_i A_1 V_i ; \quad \tilde{B}_{1i} = W_i B_{1i} ; \quad \tilde{C}_{1i} = C_{1i} V_i , \quad i = 1, 2, \cdots, q$$

$$\tilde{A}_{01i} = W_0 A_1 V_i , \quad i = 0, 1, \cdots, q ; \quad \tilde{B}_{01i} = W_0 B_{1i} , \quad i = 1, 2, \cdots, q.$$ 

Next, we give the theorem which is very usefull in determining the normal form for input-output group decoupling of slow subsystems (11).

**Theorem 3. Arman et al [10].** Given a decoupled subsystem (11) and let $i, j = 1, 2, \cdots, q$ with $i \neq j$. Then the following statements are satisfied:

(i) $W_i A_1 = \tilde{A}_{1i} W_i$.
(ii) $C_{1i} = \tilde{C}_{1i} W_i$.
(iii) $W_i B_{1j} = 0$.
(iv) The system $(\tilde{A}_{1i}, \tilde{B}_{1i})$ is controllable.
The completely proof of this theorem can be seen in Arman et al [10].

At this point, we could immediately draw a conclusion regarding the structure of a decoupled system. Therefore, we introduce the new state variable by \( \tilde{x}_1 = Wx_1 \). Let \( \tilde{x}_{1i} = W_i x_1 \in \mathbb{R}^{r_i} \). We have, for \( i = 0, 1, \ldots, q \),

\[
\dot{x}_{1i} = W_i A_i x_1 + \sum_{j=1}^{q} W_i B_{ij} u_j. \tag{14}
\]

From (14), for \( i = 0 \), we obtain

\[
\dot{x}_{10}(t) = \sum_{l=0}^{i} \tilde{A}_{01l} \tilde{x}_{1l} + \sum_{j=1}^{q} \tilde{B}_{01j} u_j(t).
\]

and for \( i = 1, 2, \ldots, q \) we have

\[
\dot{x}_{1i}(t) = \tilde{A}_{11i} \dot{x}_{11}(t) + \tilde{B}_{11i} u_i(t).
\]

While the output of systems (11) for \( i = 1, 2, \ldots, q \) are given by

\[
y_{1i}(t) = \tilde{C}_{11} \ddot{x}_{11}(t).
\]

So far we have developed a convenient matrix representation for decoupled system. Hence the transformed system can be written as

\[
\dot{x}_3(t) = A_3 \ddot{x}_3(t) + B_3 u(t). \tag{15}
\]

\[
y_1(t) = C_1 \ddot{x}_1(t). \tag{16}
\]

where

\[
\tilde{x}_1 = \begin{bmatrix} \tilde{x}_{10} \\ \vdots \\ \tilde{x}_{1q} \end{bmatrix} ; u = \begin{bmatrix} u_1 \\ \vdots \\ u_q \end{bmatrix} ; y_1 = \begin{bmatrix} y_{11} \\ \vdots \\ y_{1q} \end{bmatrix} ; A_1 = \begin{bmatrix} \tilde{A}_{010} & \tilde{A}_{011} & \tilde{A}_{012} & \cdots & \tilde{A}_{01q} \\ 0 & \tilde{A}_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \tilde{A}_{1q} \end{bmatrix} ; B_1 = \begin{bmatrix} \tilde{B}_{011} & \tilde{B}_{012} & \cdots & \tilde{B}_{01q} \\ \tilde{B}_{11} & 0 & \cdots & 0 \\ \vdots & \tilde{B}_{12} & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{B}_{1q} \end{bmatrix} \]

\[
C_1 = \begin{bmatrix} \tilde{C}_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \tilde{C}_{1q} \end{bmatrix}.\]

This representation exhibits the slow subsystems (11) as \( q \)-independent completely controllable systems. The system (15)-(16) is called the normal form for the input-output group decoupling of slow subsystems (11). This representation shows that subsystem (11) can be stated as an input-output group decoupling system consisting of independently controllable subsystem.

4. Input-output group decoupling for fast subsystems using controllability subspaces

In this section, we provide a decoupling structure for the problem of input-output group decoupling for fast subsystems (3). Using the same way as the case in slow subsystems, we can derived the results to obtain the normal form of the fast subsystem for input-output group decoupling of linear descriptor system with index one.

If condition (9) substituted into the fast subsystem (3) and defined

\[ B_{2i} = B_2 H_i, \text{ for } i = 1, 2, \ldots, q. \]

we obtain

\[
x_2(t) = -\sum_{i=1}^{q} B_{2i} u_i(t). \tag{17}
\]
Furthermore, in this part we shall construct the structure decoupled for input-output group decoupling of the system (17). As a first step, suppose that $\mathcal{R}_{2i}$ is the controllability subspace of system (17) for $i = 1,2,\ldots,q$. Then we have the controllability subspace

$$
\mathcal{R}_{2i} = \text{Im} \, B_{2i}, \text{ for } i = 1,2,\ldots,q.
$$

We defined a subspace $\gamma_0 = \cap_{j=1}^{q} \sum_{i=1}^{q} \sum_{k=0}^{n_{i}-1} \text{Im} \, B_{2i}$ and then choose the subspaces $\gamma_i$, $i = 1,2,\ldots,q$ such that $\text{Im} \, B_{2i} = \gamma_i \oplus (\gamma_0 \cap \text{Im} \, B_{2i})$.

From the relationship of these subspaces, we have the following lemma:

**Lemma 3. (Arman et.al, [10]).** Suppose that the fast subsystem (17) is controllable. Then we have

$$
\mathbb{R}^{n_2} = \bigoplus_{i=0}^{q} \gamma_i.
$$

In the same way as the case for slow subsystems. suppose that a fast Subsystem (17) is controllable and let $P_i$ be the projection onto $\gamma_i$ along the subspace $\sum_{j=0}^{q} \gamma_j$. The following theorem is given to present a matrix representation for the problem of input-output group decoupling for the fast subsystem (17).

**Theorem 4. Arman et al [10].** Given a decoupled subsystems (17) and let $i, j = 1,2,\ldots,q$ with $i \neq j$. Then the following statements are satisfied:

(i) $P_i B_{2j} = 0$ and $P_i B_{2i} = B_{2i}$,

(ii) $C_2 l P_i = 0$ and $C_2 l P_i = C_2 l$.

Next, we develop the structure of decoupled systems for fast subsystems. Therefore in this section we give the matrix representation of input-output group decoupling for the system (17). For this purpose, let $s_i = \text{rank} \, P_i = \text{dim} \, \gamma_i$, for $i = 0,1,\ldots,q$. Then there exist two matrices $L_i \in \mathbb{R}^{n_2 \times s_i}$ and $M_i \in \mathbb{R}^{s_i \times n_2}$ such that

$$
P_i = L_i M_i \text{ and } \text{rank} \, L_i = \text{rank} \, M_i = s_i.
$$

Using the concepts developed so far, we define the following matrices:

$$
\tilde{B}_{2i} = M_i B_{2i}, \quad \tilde{C}_{2i} = C_2 l L_i, \quad \tilde{B}_{02i} = M_0 B_{2i}, \text{ for } i = 1,2,\ldots,q.
$$

Then we give the following theorem to determine the structure of the input-output group decoupling for fast subsystem (17).

**Theorem 5. Arman et al [10].** Given a decoupled subsystems (17) and let $i, j = 1,2,\ldots,q$ with $i \neq j$. Then the following statements are satisfied:

(i) $M_i B_{2j} = 0$,

(ii) $\tilde{C}_{2i} M_i = C_2 l$.

The completely proof of Theorem 5 can be seen in Arman et al [10]. Furthermore to obtain the structure of group decoupled for fast subsystem (17) as a first step, we define the new state variables by $\tilde{x}_2 = M x_2$. Let $\tilde{x}_{2i} = M_i \tilde{x}_2 \in \mathbb{R}^{s_i}$. Hence for $i = 0,1,\ldots,q$, we have

$$
\tilde{x}_{2i} = M_i x_2 = M_i \left(-\sum_{j=1}^{q} B_{2j} u_j\right) = -\sum_{j=1}^{q} (M_i B_{2j}) u_j.
$$

For $i = 0$ we have

$$
\tilde{x}_{20}(t) = -\sum_{j=1}^{q} (M_0 B_{2j}) u_j(t) = -\sum_{j=1}^{q} \tilde{B}_{02j} u_j(t).
$$

While for $i = 1,2,\ldots,q$ we can get

$$
\tilde{x}_{2i}(t) = -\sum_{j=1}^{q} (M_i B_{2j}) u_j(t) = -M_i B_{2i} u_i(t) = -\tilde{B}_{2i} u_i(t).
$$

Finally, the output of system are given by
\[ y_{2i}(t) = \hat{C}_2 \hat{x}_{2i}(t), \quad i = 1, 2, \ldots, q. \]

From this relation the transformed systems have the following form

\[
\begin{align*}
\dot{x}_2(t) &= -\mathbb{B}_2 u(t) \\
y_2(t) &= \mathbb{C}_2 \hat{x}_2(t)
\end{align*}
\]  

(18)

where

\[
\mathbb{B}_2 = \begin{bmatrix}
\bar{B}_{021} & \bar{B}_{022} & \ldots & \bar{B}_{02q} \\
\bar{B}_{21} & 0 & \ldots & 0 \\
0 & \bar{B}_{22} & \ddots & \vdots \\
0 & \vdots & \ddots & 0 \\
0 & \ldots & 0 & \bar{B}_{2q}
\end{bmatrix};

\mathbb{C}_2 = \begin{bmatrix}
\bar{C}_{21} & 0 & \ldots & 0 \\
0 & \bar{C}_{22} & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \bar{C}_{2q}
\end{bmatrix}.
\]

This representation shows that the system (17) can be stated as an input-output group decoupling consisting of \( q \)-controllable subsystems that are mutually independent. Furthermore, the system (18) - (19) is called the normal form for fast subsystems (17).

5. Linear quadratic control problem for input-output group decoupling of linear descriptor systems

In this section, we shall introduce that the normal form of input-output group decoupling that has been obtained in the previous section can be used to define weakly coupled system for regular linear descriptors systems and then we give the definition of weakly coupled systems for input-output group decoupling of linear descriptor system with index one. Since for the remaining part of this research we shall consider systems in the form of (15)-(16) for slow subsystems and (18)-(19) for fast subsystem. Suppose that a perturbation is given to input-output group decoupling by assuming that the decoupled systems is already transformed to the normal form (15)-(16) and (18)-(19). We consider

\[
\dot{x}_1(t) = \mathcal{A}_1 x_1(t) + \mathcal{B}_1 u(t)
\]  

(20)

\[
y_1(t) = \mathcal{C}_1 x_1(t).
\]

(21)

and

\[
\dot{x}_2(t) = -\mathbb{B}_2 u(t)
\]  

(22)

\[
y_2(t) = \mathbb{C}_2 x_2(t).
\]

(23)

where

\[
\mathcal{A}_1 = \begin{bmatrix}
\bar{A}_{010} & \bar{A}_{011} & \bar{A}_{012} & \ldots & \bar{A}_{01q} \\
\bar{R}_{110} & \bar{A}_{11} & \bar{R}_{112} & \ldots & \bar{R}_{11q} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\bar{R}_{1q0} & \bar{R}_{1q1} & \ldots & \bar{R}_{1q(q-1)} & \bar{A}_{1q}
\end{bmatrix};
\]

\[
\mathcal{B}_1 = \begin{bmatrix}
\bar{B}_{011} & \bar{B}_{012} & \ldots & \bar{B}_{01q} \\
\bar{B}_{11} & \bar{S}_{112} & \ldots & \bar{S}_{11q} \\
\vdots & \ddots & \ddots & \ddots \\
\bar{S}_{1q1} & \ldots & \bar{S}_{1q(q-1)} & \bar{B}_{1q}
\end{bmatrix};
\]

\[
\mathbb{C}_2 = \begin{bmatrix}
\bar{T}_{110} & \bar{T}_{112} & \ldots & \bar{T}_{11q} \\
\vdots & \ddots & \ddots & \vdots \\
\bar{T}_{1q0} & \bar{T}_{1q1} & \ldots & \bar{T}_{1q(q-1)} \\
\end{bmatrix};
\]

\[
\bar{C}_2 = \begin{bmatrix}
\bar{C}_{11} & \bar{C}_{12} & \ldots & \bar{C}_{1q}
\end{bmatrix}.
\]
Thus, we derived the definition of weakly coupled for the input-output group decoupling for linear descriptor systems of index at most one.

**Definition 3.** System (20)-(23) is weakly coupled if and only if

\[
\max_{0 \leq i \leq q} \left\{ \|R_{1i}\|, \|S_{1i}\|, \|T_{1i}\|, \|S_{2i}\|, \|T_{2i}\| \right\} \leq \alpha
\]

where \(\alpha\) can be chosen sufficiently small.

Sufficiently small in Definition 3 means that the number \(\alpha\) is small enough so that the algorithm for solving the linear quadratic optimal control problem is convergent, that is the interaction between the subsystems is sufficiently weak, i.e., interaction between inputs \(u_i\) and outputs \(y_j\) is sufficiently weak in case \(i \neq j\). The number \(\alpha\) can be determined in term of the matrices of the decoupled systems, the final time \(T\) in the control problem and the matrices appearing in the cost functional. In the future research we propose an algorithm for solving linear quadratic control problems involving systems which can be viewed as in some sense weakly coupled subsystems for input-output group decoupling of linear descriptor systems with index one.

**6. Conclusion**

The problem of characterization for input-output group decoupling of linear descriptor systems has been studied using a geometric approach through controllable subspaces. Based on the standard decomposition form, the structure decoupled for linear descriptor system with index one has been derived. A regular linear descriptor system can be separated into slow and fast subsystems. Then through the controllability subspaces, we obtained the structure of the group decoupled system for slow and fast subsystems. This structure yields a normal form of input-output group decoupling for regular linear descriptor system of index at most one. In such form, the problem of input-output group decoupling can be considered as an input-output decoupled system which is controllable consists of independently controllable subsystems. This normal form can be used to define weakly coupled linear system for input-output group decoupling of linear descriptor system with index one. The results presented in this paper can be considered as a first step towards a more general theory for nonlinear control systems consisting of weakly coupled subsystems for input-output group decoupling of linear descriptor system with index one.

**Acknowledgments**

The first author is grateful for the support of the Directorate Research and Community Service (DRPM), Ministry of Research, Technology and Higher Education, Indonesia, with grant contract number: 056/SP2H/LT/DRPM/2018.

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