ON THE PRODUCTS IN THE FINITE GROUPS

V.V. Genk

Belarusian State Politechnical College
Tsentralnaya sq. 2, Molodechno, Minsk region, 222310, The Republic of Belarus
E-mail: gravitation@mail.admiral.ru

All possible products of all elements of an odd order finite group are considered. A set of all such products is called a $K$-set.

A hypothesis of $K$-set coincidence of any group of an odd order with its commutant is proposed and the hypothesis validity for groups with a commutant of a simple structure is shown.

Sequences of elements of a finite group can form “a good” in this or that sense, part of a group (see, for example, [2]-[7]), and a set of products of all commutators is a commutant of a group. A question arises: what information about a group does a set made of every possible products of all elements of the finite group taken in an unspecified order carry?

Let $G = \{1, a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$ be a finite group of an odd order $2n + 1$ (later on simply “an odd group”), where $a_i^{\pm 1}$ — the whole of different elements of the group.

Let’s form a set:

$$K = \{g_{\sigma(1)} \ldots g_{\sigma(2n)} \mid \sigma \in S_{2n}, \quad g_i \in G \setminus 1, \quad g_i \neq g_j \quad \text{for} \quad i \neq j\},$$

where $S_{2n}$ — a set of substitutions in $2n$ symbols. Let’s later on call a set $K$ as a $K$-set. This paper proposes a hypothesis on the structure of a $K$-set of odd groups and a special case of this hypothesis is proved.

**The basic hypothesis.**

A $K$-set of any odd group coincides with its commutant.

The following two suggestions take place (the first is evident, the second is proved by induction on word length out of a $K$-set).

**Suggestion 1.**

A $K$-set of an odd group is its invariant subset, containing the unit of a group and also together with any element and its inverse.
Suggestion 2.
A K-set of an odd group is contained in its commutant.

Just from these suggestions by the Feit-Thompson theorem such a consequence arises (indicated by prof. L.S.Kazarin):

Consequence.
A K-set of an odd group is solvable too, i.e. a repeat procedure of taking of a K-sets through the finite number of steps leads to the unit of a group.

When proving the following theorem commutator identities are systematically used
\[
[a, b] = [a, ab], \quad [a, b] = [b^{-1}, ab], \quad [a, b]^{-1} = [b, a],
\]
and an obvious lemma as well:

Lemma.
Not any other element of an odd group but the unit can be conjugated with its inverse.

Theorem.
The product of two commutators in an odd group belongs to its K-set.

Proof.
It’s sufficient to check that the inclusion takes place when one of the commutators is made of elements which themselves or their inverse enter the other commutator.

Sixteen variants of accurance of common elements or their inverse into commutators are possible with an accuracy to designation of the elements of a group:
\[
\begin{align*}
[a_1, a_2][a_1, a_3], & \quad [a_1, a_2][a_2, a_3], & \quad [a_1, a_2][a_3, a_2], \\
[a_1, a_2][a_3, a_1], & \quad [a_1, a_2][a_3, a_1^{-1}], & \quad [a_1, a_2][a_1^{-1}, a_3], \\
[a_1, a_2][a_2^{-1}, a_3], & \quad [a_1, a_2][a_3, a_2^{-1}], & \quad [a_1, a_2][a_2, a_1], \\
[a_1, a_2][a_1, a_2^{-1}], & \quad [a_1, a_2][a_2, a_1^{-1}], & \quad [a_1, a_2][a_2^{-1}, a_1^{-1}], \\
[a_1, a_2][a_1^{-1}, a_2^{-1}].
\end{align*}
\]
(2)

Let’s consider the first of them in more detail. The rest are analyzed similarly or reduced to already analyzed ones. It’s clear that the product of elements $a_1a_2 \notin \{a_1, a_2, a_1^{-1}, a_2^{-1}\}$.

Let
\[
a_1a_2 \in \{a_3, a_3^{-1}, a_4^\varepsilon\}, \quad \text{where } i > 3, \quad \varepsilon = \pm 1.
\]

1) If $a_1a_2 = a_4^\varepsilon$,
then by the second of the identities (1):
\[
[a_1, a_2][a_1, a_3] = [a_2^{-1}, a_4^\varepsilon][a_1, a_3] \in K.
\]

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2) Let \( a_1a_2 = a_3 \), then:

\[
[a_1, a_2][a_1, a_3] = [a_2^{-1}, a_3][a_1, a_3].
\]

Consider the product of elements \( a_2^{-1}a_3 \). As shown above

\[ a_2^{-1}a_3 \in \{a_1, a_1^{-1}, a_i^\xi\}, \quad i > 3. \]

a) For \( a_2^{-1}a_3 = a_i^\xi \) by the first of the identities (1):

\[ [a_2^{-1}, a_i^\xi][a_1, a_3] \in K. \]

b) If \( a_2^{-1}a_3 = a_1 \), then \( a_3a_1 = a_1a_2a_2^{-1}a_3 = a_1a_3 \) and

\[ [a_1, a_2][a_1, a_3] \in K. \]

c) It remains \( a_2^{-1}a_3 = a_1^{-1} \), from which \( a_1^{-1} = a_2^{-1}a_1a_2 \), i.e. \( a_1 \sim a_1^{-1} \) and this case is impossible due to the lemma.

3) Let, finally, \( a_1a_2 = a_3^{-1} \). Then

\[
[a_1, a_2][a_1, a_3] = [a_2^{-1}, a_3^{-1}][a_1, a_3].
\]

Considering the product

\[ a_2^{-1}a_3^{-1} \in \{a_1, a_1^{-1}, a_i^\xi\}, \quad i > 3 \]

as above one can make sure that in any case

\[ [a_1, a_2][a_1, a_3] \in K. \]

All another variants in (2) may be considered similarly. So the proof is completed.

Thus, the basic hypothesis is true for odd groups with a commutant elements of which reach the limit of products of not more than two commutators. What concerns noncommutative groups of an even order the question about a structure of \( K \)-sets of such groups remains open. An example of symmetric group \( S_3 \) shows that a \( K \)-set of an even group can disagree with its commutant.

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References

[1] Magnus W., Karrass A., Solitar D., “Combinatorial group theory”, N.Y., 1966.

[2] Chynihin S.A. – Reports of Belarusian Academy of Sciences, 1965, V.9, N10, p.641-642 (in russian).

[3] Keedwell A.D. Sequenceable groups: a survey. – London math. lect. note ser. 1980, N49, p.205-215.

[4] Carlitz L. A note on abelian groups. – Proc. Amer. Math. Soc., 1953, V.4, N6, p.937-938.

[5] Schenkman E. On the lower theorem for finite groups. – Pacif. J.Math. Suppl., 1955, V.5, N2, p.995-998.

[6] Olson John E. An addition theorem for finite abelian groups. – J.Number theory, 1977, V.9, N1, p.63-70.

[7] Iwahori Nagayos, Hattori Akira. On associative compositions in finite nilpotent groups. – Nagoya Math.J., 1954, V.7, June, 145-148.