A Lower Bound Estimate of Life Span of Solutions to Stochastic 3D Navier-Stokes Equations with Convolution Type Noise

October 20, 2021

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Abstract

In this paper we investigate the stochastic 3D Navier-Stokes equations perturbed by linear multiplicative Gaussian noise of convolution type by transformation to random PDEs. We are not interested in the regularity of the initial data. We focus on obtaining bounds from below for the life span associated with regular initial data. The key point of the proof is the fixed point argument.

Key words: stochastic PDEs, Navier-Stokes, random PDEs, life span, fixed point

1 Introduction

Consider the following stochastic 3D Navier–Stokes equation

\[
\begin{aligned}
&du + (u \cdot \nabla u - \Delta u)dt = \sum_{i=1}^{n}(B_i(u) + \lambda_i u)d\beta_i(t) - \nabla pdt, \\
&\text{div } u = 0, \\
&u \mid_{t=0} = u_0,
\end{aligned}
\]

(1)
on the whole space \(\mathbb{R}^3\), where \(\beta_i(t), i = 1, \ldots, n\) are one dimensional independent Brownian motions on a given probability space \((\Omega, \mathcal{F}, \mathbb{P})\), \(\lambda_i, i = 1, \ldots, n\), are non-zero constants and \(B_i, i = 1, \ldots, n\) are the convolution operators such that

\[
B_i(u)(\xi) = \int_{\mathbb{R}^3} h_i(\xi - \xi)u(\xi)d\xi = (h_i * u)(\xi), \quad \xi \in \mathbb{R}^3,
\]
where \( h_i \in L^1(\mathbb{R}^3) \), \( i = 1, 2, \ldots, n \), and \( \Delta \) is the (weak) Laplacian on \( (L^2(\mathbb{R}^3))^3 \). The vorticity form of this system has been investigated in [2] by Barbu and Röckner, where the authors prove the existence and uniqueness in \( (L^p(\mathbb{R}^3))^3, \frac{3}{2} < p < 2 \), of a global mild solution to random vorticity equations associated to stochastic 3D Navier–Stokes equations for sufficiently small initial vorticity. In their paper the smallness of the initial values depend on the whole Brownian path, hence the solutions obtained are not adapted. In the paper [8] by Röckner, R. C. Zhu and X. C. Zhu, the authors prove that the solution satisfies the vorticity equation with the stochastic integration being understood in the sense of the integration of controlled rough paths. In the further paper [6], the authors generalize this result to gradient-type noise in 2 or 3 dimensions by a different type of transformation which is adapted to their noise. Then they also obtain the existence of a solution adapted to the Brownian filtration up to some stopping time.

We do not assume the initial values are small. Instead we only assume initial data are smooth. In other words, the initial data are in Sobolev spaces \( H^N \) for any \( N \in \mathbb{N} \). We focus on the life span of the \( \dot{H}^{\frac{3}{2}+\gamma} \)-mild solution on the fixed path (see Definition 4.1 and Theorem 4.2 for the definitions of the mild solution and life span).

**Main result of the paper (Theorem 4.2):** we obtain a lower bound estimate (7) of the life span.

For the deterministic classical 3D Navier–Stokes equations, by Fujita–Kato’s fixed point procedure, global wellposedness results have been obtained in certain scaling invariant spaces (one also calls them critical spaces). For general initial values, the solution may blow up (in the sense of strong solutions) after some time. For some spaces which are above the critical spaces, a fixed point procedure (Picard’s contraction principle) can also be applied to study the blow up time of the solution. We refer to Chapter 15 in the book [4] for such results for the Sobolev spaces \( \dot{H}^s \) and \( L^p \), for \( s > \frac{1}{2} \) and \( p \geq 3 \). Also in Poulan’s paper [7], the author introduces the notion of the minimal blow up Navier–Stokes solutions. The authors also show that the set of such solutions is not only nonempty but also compact in a certain sense. Based on this, a lower bound estimate of the maximal time up to which the solution remains regular for initial values in the space \( \dot{H}^s, \frac{1}{2} < s < \frac{3}{2} \), is obtained in Proposition 1.1 of a later paper [3] by J. -Y. Chemin and I. Gallagher. Note that if we find some time \( T \) and a solution in the space \( L^q([0, T]; \dot{H}^s) \), for \( \frac{1}{2} < s < \frac{3}{2} \), then by Sobolev’s embedding theorem we know that the solution up to time \( T \) satisfies Serrin’s condition, i.e.

\[
    u \in L^q([0, T]; L^p(\mathbb{R}^3)) \quad \text{for some} \quad p, q \quad \text{which satisfy} \quad \frac{2}{q} + \frac{3}{p} \leq 1.
\]

In other words, it is a strong solution up to time \( T \). For deterministic anisotropic 3D Navier–Stokes equations, in [5], we also study the maximal time up to which the solutions remain regular.

The idea of the proof of Theorem 4.2 is that we first apply the transformation in [2] to transform the equation to a random PDE and write the solution in the form of mild solutions. Then we apply Littlewood-Paley theory, thanks to the commutativity of the convolution (this is the reason why we have to limit our noise to convolution type multiplicative noise), our convolution operator \( B_i \) and transformation operator \( \Gamma \) operate on any Besov space. Afterwards, due to the contraction property of the semigroup \( e^{t\Delta} \), we obtain the estimates necessary for the fixed point argument. This method has also been used to calculate a
lower bound of the time $T$ up to which the regular solution exists for deterministic 3D Navier–Stokes equations.

In this paper we use $C$ to denote the constant which can be different from line to line. And we use the notation ‘$A \leq B$’ to mean that there is some constant $C$ such that $A \leq CB$.

2 Function Spaces on $\mathbb{R}^3$

Denote by $\mathcal{S}(\mathbb{R}^3)$ the Schwartz space and $\mathcal{S}'(\mathbb{R}^3)$ its dual space.

On $\mathbb{R}^3$, we recall the **non-homogeneous Sobolev spaces**:

$$H^s(\mathbb{R}^3) := \left\{ u \in \mathcal{S}'(\mathbb{R}^3); \| u \|_{H^s(\mathbb{R}^3)} := \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi < \infty \right\},$$

where $s \in \mathbb{R}$, and

$$\hat{u}(\xi) = \mathcal{F}u(\xi) := \int_{\mathbb{R}^3} u(x) e^{-ix\cdot\xi} \, dx,$$

denotes the Fourier transform of $u$ on $\mathbb{R}^3$. Then $H^s(\mathbb{R}^3)$ is a Hilbert space with $H^{-s}(\mathbb{R}^3)$ as its dual space.

On $\mathbb{R}^3$, we recall the **homogeneous Sobolev spaces**:

$$\dot{H}^s(\mathbb{R}^3) := \left\{ u \in \mathcal{S}'(\mathbb{R}^3), \dot{u} \in L^1_{\text{loc}}(\mathbb{R}^3); \| u \|_{\dot{H}^s(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi < \infty \right\},$$

where $\dot{u}$ denotes the Fourier transform of $u$.

3 Transform to Random PDEs

We use the same transform as in [2], but instead of vorticity equation, we apply it to the original equation. For $t \geq 0$, we consider the transformation

$$u(t) = \Gamma(t) y(t),$$

where $\Gamma(t) : (L^2(\mathbb{R}^3))^3 \rightarrow (L^2(\mathbb{R}^3))^3$ is the linear continuous operator defined by the equations

$$d\Gamma(t) = \sum_{i=1}^n (B_i + \lambda_i I) \Gamma(t) d\beta_i(t), \quad t \geq 0,$$

and $\Gamma(0) = I$. In other word, $\Gamma(t)$ is defined in the sense that, for every $z_0 \in (L^2(\mathbb{R}^3))^3$, the continuous $(\mathcal{F})$-adapted $(L^2(\mathbb{R}^3))^3$-valued process $z(t) := \Gamma(t) z_0, \quad t \geq 0$, solves the following SDE on $(L^2(\mathbb{R}^3))^3$,

$$dz(t) = \sum_{i=1}^n \tilde{B}_i z(t) d\beta_i(t), \quad z(0) = z_0.$$

Similar as in [2], we also set

$$\tilde{B}_i = B_i + \lambda_i I, \quad i = 1, ..., n,$$

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where \( I \) is the identity operator. Then we can also write \( \Gamma \) as the exponential form:

\[
\Gamma(t) = \prod_{i=1}^{N} \exp \left( \beta_i(t) \tilde{B}_i - \frac{t}{2} \tilde{B}_i^2 \right), \quad t \geq 0.
\]

Moreover, we immediately have all of \( \Gamma(t) \), \( \Gamma^{-1}(t) \) and \( \tilde{B}_i \) commute with (weak) derivatives. Note that

\[
\Gamma^{-1}(t) = \prod_{i=1}^{N} \exp \left( -\beta_i(t) \tilde{B}_i + \frac{t}{2} \tilde{B}_i^2 \right), \quad t \geq 0.
\]

Moreover, \( \tilde{B}_i \tilde{B}_j = \tilde{B}_j \tilde{B}_i \).

**Remark 3.1.** It is obvious that the operator \( B_i \), \( \Gamma(t) \) and \( \Gamma^{-1}(t) \) can be defined (as a continuous operator) in any \( L^p(\mathbb{R}^3)^3 \) for any \( p \geq 1 \) since the convolution with an \( L^1 \) function makes sense in any \( L^p \) space with the \( L^1 \) norm as the uniform bound of the operator (Young’s inequality). Moreover, we have the following lemma as is proved in \([2]\).

**Lemma 3.2.** We have

\[
\| \Gamma(t)z \|_{L^q} + \| \Gamma^{-1}(t)z \|_{L^q} \leq C \| z \|_{L^q}, \quad t \in [0, \infty), \quad \forall z \in L^q(\mathbb{R}^3), \quad \forall q \in [1, \infty),
\]

and

\[
\| \nabla(\Gamma(t)z) \|_{L^q} \leq \| \Gamma(t) \|_{L^q(L^q,L^q)} \| \nabla z \|_{L^q}, \quad \text{for all} \ z \text{ which satisfies} \ z, \nabla z \in L^q(\mathbb{R}^3).
\]

**Proof.** See Lemma 2.1 of \([2]\). The above lemma also holds when \( q = \infty \). Since the proof is a direct result of Young’s inequality

\[
\| h_i \ast u \|_{L^q} \leq \| h \|_{L^1} \| u \|_{L^q},
\]

which also holds as \( q = \infty \), and the second inequality is due to the fact that \( \Gamma \) commutes with derivatives.

From Lemma 3.2 we can view \( \Gamma \) as a linear continuous operator from \( (L^p(\mathbb{R}^3))^3 \) to \( (L^p(\mathbb{R}^3))^3 \) for \( 1 \leq p \leq \infty \). Moreover, there is a common upper bound of \( \| \Gamma(t) \|_{L(L^p,L^p)} \) which does not depend on \( p \), but only depends on \( h_i, \lambda_i, t \) and, of course, the path \( \omega \).

The next lemma tells us for any \( s \in \mathbb{R} \), and \( h \in L^1 \), the convolution with \( h \) is a continuous operator map from any Sobolev space \( H^s \) to itself. Therefore, we can also extend the definition of operator \( B_i \), \( \Gamma(t) \) and \( \Gamma^{-1}(t) \) to continuous operators from any Sobolev space \( H^s \) to itself.

**Lemma 3.3.** For any \( h \in L^1 \) and \( s \in \mathbb{R} \), the convolution with \( h \) is a continuous operator mapping from any Sobolev space \( H^s \) to itself.

**Proof.** It suffices to prove that for any \( s > 0 \), and \( a \in H^s \), we have \( a \ast h \in H^s \).
\[
\|a \ast h\|_{H^s}^2 = \int_{\mathbb{R}^3} |\mathcal{F}(a \ast h)(\xi)|^2(1 + |\xi|^2)^s d\xi \\
= \int_{\mathbb{R}^3} |\hat{a}(\xi)|^2 |\hat{h}(\xi)|^2(1 + |\xi|^2)^s d\xi \\
\leq \|a\|_{H^s}^2 \|\hat{h}\|_{L^s}^2 \\
\leq \|a\|_{H^s}^2 \|h\|_{L^1}^2.
\]

Therefore, we have proved the result.

Thus we transform (1) to the following random PDEs of \( y \):

\[
dy + \Gamma^{-1}(\Gamma y \cdot \nabla \Gamma y)dt - \Delta y dt = -\Gamma^{-1}\nabla pdt, \\
y(0) = u_0.
\] (3)

Note that

\[
\nabla p = \nabla(-\Delta)^{-1} \text{div} \div(G \Gamma y \otimes \Gamma y) = \sum_{1 \leq i,j \leq 3} \nabla(-\Delta)^{-1}(\hat{\partial}_i \partial_j(G \Gamma y))^j).
\]

Let \( Q \) be the following bilinear operator from \((L^2(\mathbb{R}^3))^3 \times (L^2(\mathbb{R}^3))^3 \) to \((S'(\mathbb{R}^3))^3 \):

\[
Q(x, y) := \text{div}(x \otimes y) + \sum_{1 \leq i,j \leq 3} \nabla(-\Delta)^{-1}(t)(\hat{\partial}_i \partial_j(x^i y^j)) = Q(y, x).
\]

Thus we can rewrite the equation in the following form of mild solution

\[
y(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \Gamma^{-1}(\omega)Q(\Gamma(\omega) y(s), \Gamma(\omega) y(s))ds.
\] (4)

For simplicity from now on in this section we skip the dimension notation \( \mathbb{R}^3 \) if it is 3 dimension and there is no confusion.

**Remark 3.4.**

1. \( \Delta \) is an operator mapping from tempered distribution space \( S' \) to \( S' \).

2. We will show that \( Q, \Gamma Q \) and \( \Gamma^{-1} Q \) are well defined and continuous from \((L^p)^3 \times (L^p)^3 \) to (at least) the Sobolev space \((H^{-2})^3 \subset (S')^3 \) for any \( 2 \leq p < \infty \).

   **Case of \( 2 < p < \infty \).**
   It immediately follows from the \( L^p \) boundedness of the Riesz transform.

   **Case of \( p = 2 \).**
   Recall that the inverse of the (minus) Laplacian \((\Delta)^{-1} \) can be defined by Fourier multipliers: for any \( u \) which is in the range of the \( \Delta \),

   \[
   (-\Delta)^{-1} u = \mathcal{F}^{-1}(\xi \partial (|\xi|^2 u(\xi))).
   \]
Claim that for any \( f \in L^1 \), \( \partial_i \partial_j f \) is in the domain of \( (-\Delta)^{-1} \) (i.e. the range of the \( \Delta \)). Indeed since \( f \in L^1 \), we have \( f \in L^\infty \), and \( \xi_i \xi_j |\xi|^{-2} \hat{f}(\xi) \in L^\infty \subset S' \). Therefore, the term of Fourier inverse transform \( F^{-1}(\xi_i \xi_j |\xi|^{-2} \hat{f}(\xi)) \) is at least meaningful in \( S' \).

We have

\[
\Delta F^{-1}(\xi_i \xi_j |\xi|^{-2} \hat{f}(\xi)) = \partial_i \partial_j f
\]

and

\[
(-\Delta)^{-1}(\partial_i \partial_j f) = -F^{-1}(\xi_i \xi_j |\xi|^{-2} \hat{f}(\xi)).
\]

Moreover, since \( \xi_i \xi_j |\xi|^{-2} \hat{f}(\xi) \in L^\infty \),

\[
\|F^{-1}(\xi_i \xi_j |\xi|^{-2} \hat{f}(\xi))\|_{L^2}^2 = \int_{\mathbb{R}^3} |\xi_i \xi_j |\xi|^{-2} \hat{f}(\xi)|^2 (1 + |\xi|^2)^{-2} d\xi \leq \|f\|_{L^1} < \infty.
\]

Therefore, \( Q \) maps \((L^p)^3 \times (L^p)^3 \) continuously to \((H^{-2})^3 \). Therefore, by Lemma 3.3 we know that \( \Gamma Q \) and \( \Gamma^{-1} Q \) also map \((L^p)^3 \times (L^p)^3 \) continuously to \((H^{-2})^3 \).

3. Therefore, by Sobolev embedding \( Q \) and \( \Gamma Q \) can be well defined as continuous maps from Sobolev space \((H^s)^3 \times (H^s)^3 \) for \( 0 < s < \frac{3}{2} \) to \((H^{-2})^3 \).

4. From our definition of \( Q(x,y) \), we immediately have \( \text{div} \, Q(x,y) = 0 \)

5. Let \( X \) be \( L^p \), \( p \geq 2 \) or any Sobolev space \( H^s \) for \( 0 < s < \frac{3}{2} \). We have shown that \( Q \) is a continuous map from \( X^3 \times X^3 \) to \((H^{-2})^3 \), thus if \( u \) and \( v \) are \( \mathcal{L}([0,t])/\mathcal{B}(X^3) \)-measurable, \( Q(u,v) \) is \( \mathcal{L}([0,t])/\mathcal{B}((H^{-2})^3) \)-measurable, where \( \mathcal{L}([0,t]) \) is the \( \sigma \)-algebra of the Lebesgue measurable sets in the interval \([0,t] \).

Since \( X \) is dense in \((H^{-2})^3 \), and both \( X \) and \((H^{-2})^3 \) are Banach spaces, we have \( X \in \mathcal{B}((H^{-2})^3) \) and \( \mathcal{B}(X) = \mathcal{B}((H^{-2})^3) \cap X \).

Therefore, \( Q(u,v)1_{Q(u,v) \in X} \) is also \( \mathcal{L}([0,t])/\mathcal{B}(X) \)-measurable.

6. Due to 5, and the fact that \( \Gamma \) maps from any Sobolev or Lebesgue space to itself continuously, (which we would prove later in the Remark 4.7, ) the integral in (4) is meaningful in any Sobolev space if we can show the integration of their corresponding Sobolev norms are finite on the interval \([0,T] \).

Lemma 3.5. Assume that \( \lambda_i, h_i \) satisfy

\[
|\lambda_i| > (\sqrt{12} + 3)\|h_i\|_{L^1}, \quad \forall i = 1,2,...,N.
\]

Let

\[
\eta(t) = \|\Gamma(t)\|_{L(L^2,L^2)}^2 \|\Gamma^{-1}(t)\|_{L(L^2,L^2)}, \quad t \geq 0,
\]

where for \( q \in (1, \infty), \| \cdot \|_{L(L^q,L^q)} \) is the norm of the space \( L(L^q,L^q) \) of linear continuous operators on \( L^q \). Then we have

\[
\sup_{t \geq 0} \eta(t) < \infty, \mathbb{P}-\text{a.e.}
\]
Proof. The proof is the same to Remark 1.2 of [2] if we note that the following still holds by calculation directly,

$$\eta(t) \leq \prod_{i=1}^{N} \exp\left(3|\beta_i(t)(\|h_i\|_{L^1} + |\lambda_i|) - t\alpha_i)\right), \ t \in [0, \infty),$$

where $\alpha_i := \frac{1}{2}\lambda_i^2 - \frac{3}{2}(\|h_i\|_{L^1}^2 + 2|\lambda_i| \|h_i\|_{L^1})$ is strictly positive followed by (5).

\[ \square \]

4 The Main Theorem

Define the function space $Z_T^\gamma$ to be functions from $[0, T] \times \mathbb{R}^3$ to $\mathbb{R}^d (d \geq 1)$ with the corresponding norm

$$\|u\|_{Z_T^\gamma}^2 := \|u\|_{L^2([0, T]; H^{\frac{1}{2}+\gamma})}^2 + \int_0^T \|\nabla u(t)\|_{H^{\frac{1}{2}+\gamma}}^2 dt$$

finite.

The completeness of $Z_T^\gamma$ when $0 < \gamma < 1$ is proved in the Appendix A.

Definition 4.1 ($\dot{H}^{\frac{1}{2}+\gamma}$-mild solution on path $\omega$). Fix $0 < \gamma < 1$ and the path $\omega$. Let $y(t, x)$ be a function from $[0, T] \times \mathbb{R}^3$ to $\mathbb{R}$. We say that $y$ is a $\dot{H}^{\frac{1}{2}+\gamma}$-mild solution on path $\omega$ to (3) in the interval $[0, T]$ if

1. $y \in Z_T^\gamma$;
2. (4) holds for any $0 \leq t \leq T$.

Theorem 4.2. Fix $0 < \gamma < 1$. Assume that (5) holds. Given any (fixed) vector field $u_0$ such that $u_0 \in \dot{H}^N$ for any positive integer $N$, then for $\mathbb{P}$-a.e. path $\omega$, a positive time $T(u_0, \omega)$ exists such that a unique $\dot{H}^{\frac{1}{2}+\gamma}$-mild solution on path $\omega$ to (3) exists in the interval $[0, T(u_0, \omega)]$. Let $T^*(u_0, \omega)$ be the supremum of such $T(u_0, \omega)$. In this case we call $T^*(u_0, \cdot)$ the life span in $\dot{H}^{\frac{1}{2}+\gamma}$ associated with the regular initial value $u_0$. (Note that although $u_0$ is not random, the life span depends on the path). Then we have $\mathbb{P}$-a.e. path,

$$T^*(u_0, \omega) \geq T_*(u_0, \omega) := c_{\gamma}\left(\sup_{t \geq 0} \eta(t)\right)^{-1} \|u_0\|_{\dot{H}^{\frac{1}{2}+\gamma}}^{-\frac{2}{\gamma}},$$

and the strict positive number $c_{\gamma}$ depends only on $\gamma$.

We put the proof in the Section 4.2.

Remark 4.3.

1. In particular, in (7), if there is no noise, we do not need any transformation, thus both $\Gamma$ and $\Gamma^{-1}$ are identity and $\eta_1 = 1$ for any $t$, then we obtain the result which is consistent to the deterministic cases.

2. Similar to [2], the solution we obtain is not adapted.
The proof of the main theorem relies heavily on the fixed point theorem. There are two key steps in the following proof: one is to show that \( \Gamma \) and \( \Gamma^{-1} \) could extend as operators in Besov spaces (see the next section for the definition of the Besov spaces), the proof of which relies on the commutative property of \( \Gamma \) and Littlewood-Paley operators (we will introduce the tool of Littlewood-Paley theory in the next section). In other word, it relies on the commutative property of convolutions since Littlewood-Paley operators are actually convolution operators. This is the reason why we need the noise to be also convolution types. Another step is that just like what we do in deterministic equations, we write the solution in the form of mild solution and use the contraction property of the semigroup \( e^{t\Delta} \) in order to obtain the estimates we need for the fixed point theorem.

4.1 Littlewood-Paley Theory

Let us first recall the (homogeneous) Littlewood-Paley decomposition in the book [1]. We will give a brief introduction of Littlewood-Paley theory, the details of which could be found in the book [1]. For \( a \in S' \), as usual, denote \( \mathcal{F}a \) and \( \hat{a} \) the Fourier transform of the distribution \( a \).

**Definition 4.4** (the Space \( S'_h \), see Definition 1.26 of [1]). Let \( S'_h \) be the space of tempered distributions \( u \) such that

\[
\lim_{\lambda \to \infty} \| \theta(\lambda D)u \|_{L^\infty} = 0
\]

for any \( \theta \in C_c^\infty \), where \( \theta(\lambda D)u \) is the Fourier multiplier defined as follows

\[
\theta(\lambda D)u := \mathcal{F}^{-1}(\theta(\lambda \cdot)\hat{u}).
\]

The next remark comes from Remark 1.27 of [1] and the examples afterwards.

**Remark 4.5.**

1. Whether or not a tempered distribution \( u \) belongs to \( S'_h \) depends only on low frequencies. \( u \) belongs to \( S'_h \) if and only if one can find some smooth compactly supported function \( \theta \) such that \( \theta(0) \neq 0 \) and (8) holds.

2. Directly by the definition we immediately know that the space \( S'_h \) contains all the tempered distributions whose Fourier transforms are locally integrable around 0. In particular, all the Sobolev spaces (homogeneous and non-homogeneous) are subsets of \( S'_h \).

For \( a \in S'_h \), we set

\[
\hat{\Delta}_ka = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\hat{a}),
\]

where \( \varphi(\tau) \) is a smooth function value in \([0, 1]\) such that

\[
\text{Supp } \varphi \subset \left\{ \tau \in \mathbb{R}; \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{k \in \mathbb{Z}} \varphi(2^{-k}\tau) = 1.
\]
Then we have
\[ \forall \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1. \]
Moreover, we have the following equality for \( a \in \mathcal{S}'_h \),
\[ a = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j a. \tag{10} \]
For \( s \in \mathbb{R} \) and \((p, r) \in [1, \infty]^2\), define the homogeneous Besov spaces \( \dot{B}^s_{p,r} \) which consists of those distributions in \( \mathcal{S}'_h \) such that
\[ \| u \|_{\dot{B}^s_{p,r}} := \left( \sum_{j \in \mathbb{Z}} 2^{js} \| \hat{\Delta}_j u \|_{L^p} \right)^\frac{1}{r} < \infty. \]
From the definition we immediately know the norm of \( \dot{H}^s \) coincides with \( \dot{B}^s_{2,2} \).

**Lemma 4.6.** For \( 0 < \gamma < 1 \),
\[ \| u \otimes u \|_{\dot{B}_{2,1}^{2\gamma - \frac{1}{2}}} \lesssim \| u \|_{\dot{H}_{2,1}^{1 + \gamma}}^2. \]

**Proof.** See Corollary 2.55 of [1]. \( \Box \)

**Remark 4.7.**
1. By definition \( \hat{\Delta}_j \) commutates with \( \Gamma(t) \) and \( \Gamma^{-1}(t) \). That is, for \( u \in L^p \), we have
\[ \Gamma \hat{\Delta}_j u = \hat{\Delta}_j \Gamma u \text{ and } \Gamma^{-1} \hat{\Delta}_j u = \hat{\Delta}_j \Gamma^{-1} u. \tag{11} \]
2. From (10), we know for any \( p, L^p \) is dense in \( \mathcal{S}'_h \), thus we can extend \( \Gamma(t) \) and \( \Gamma^{-1}(t) \) continuously and uniquely to an operator from \( \mathcal{S}'_h \) to \( \mathcal{S}'_h \).
3. We claim for \( u \in \dot{B}^s_{p,r} \),
\[ \Gamma u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j \Gamma u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u, \tag{12} \]
and
\[ \| \Gamma(t) \|_{L(\dot{B}^s_{p,r}, \dot{B}^s_{p,r})} \lesssim \| \Gamma(t) \|_{L(L^p, L^p)}, \tag{13} \]
where \( L(X, X) \) denote the operator norm from \( X \) to \( X \).
Indeed, first we note that (11) still holds for \( u \in \dot{B}^s_{p,r} \), since by the way of expansion, we have
\[ \Gamma u := \sum_{k \in \mathbb{Z}} \Gamma \hat{\Delta}_k u, \]
and

\[ \Gamma \hat{\Delta} j u = \sum_{k \in \mathbb{Z}} \Gamma \hat{\Delta} k \hat{\Delta} j u \]
\[ = \sum_{k \in \mathbb{Z}} \Gamma \hat{\Delta} j \hat{\Delta} k u \]
\[ = \sum_{k \in \mathbb{Z}} \hat{\Delta} j \Gamma \hat{\Delta} k u \]
\[ = \hat{\Delta} j \Gamma u, \]

where the third inequality is due to (11) in \( L^p \). Thus we finish the proof of (12). For the proof of (13), immediately follows by the definition of Besov norms we have

\[ \| \Gamma u \|_{B_{p,r}^{s}} = \left( \sum_{j \in \mathbb{Z}} 2^{jr} \| \hat{\Delta} j \Gamma u \|_{L^p}^r \right)^{\frac{1}{r}} \]
\[ = \left( \sum_{j \in \mathbb{Z}} 2^{jr} \| \hat{\Delta} j u \|_{L^p}^r \right)^{\frac{1}{r}} \]
\[ \leq \left( \sum_{j \in \mathbb{Z}} 2^{jr} \| \Gamma \|_{L(L^p,L^p)} \| \hat{\Delta} j u \|_{L^p} \right)^{\frac{1}{r}} \]
\[ \leq \| \Gamma \|_{L(L^p,L^p)} \| u \|_{B_{p,r}^{s}}. \]

Roughly speaking, the above preparation is to show how ‘good’ the operators \( \Gamma \) and \( \Gamma^{-1} \) are. They could be expanded to any Besov (hence Sobolev) space and could commute with derivatives and Littlewood-Paley operators. After the preparation, we now show the following lemma, which is the crucial step of the proof of the main theorem.

4.2 Proof of Main Theorem

Define

\[ F(y)(t) = - \int_{0}^{t} e^{(t-s)\Delta} \Gamma^{-1} Q(\Gamma y, \Gamma y) ds. \]

Lemma 4.8. There exists some constant \( C \), which depends on the path \( \omega \), such that

\[ \| F(y) \|_{\mathcal{Z}_1^2} \leq CT \| y \|_{\mathcal{Z}_1^2}. \]

Proof. The key point of the proof is based on the contraction property of the semigroup \( e^{t\Delta} \). By definition

\[ \| F(y)(t) \|_{\dot{H}^{\frac{1}{2}+\gamma}} \leq \int_{0}^{t} \| e^{(t-s)\Delta} \Gamma^{-1} Q(\Gamma y(s), \Gamma y(s)) \|_{\dot{H}^{\frac{1}{2}+\gamma}} ds \]
\[ \leq \int_{0}^{t} \| e^{(t-s)\Delta} \Gamma^{-1} (\Gamma y(s) \otimes \Gamma y(s)) \|_{\dot{H}^{\frac{1}{2}+\gamma}} ds, \]
where the second inequality is due to the reason that $Q(y, y)$ is the sum of first order derivatives of $y \otimes y$. Moreover, by Lemma 2.4 of [1], there exists some constant $c$,

\[
\| \Delta_j \left( e^{(t-s)} \Delta \Gamma^{-1} (y \otimes y) \right) \|_{L^2} \\
\leq e^{-c(t-s)^{2j}} \| \Delta_j \left( \Gamma^{-1} (y \otimes y) \right) \|_{L^2} \\
\leq e^{-c(t-s)^{2j}} \| \Gamma^{-1}(s) \|_{L(L^2, L^2)} 2^{-j(2\gamma - \frac{1}{2})} d_j \| y \otimes y \|_{\dot{B}_{2,1}^{\gamma}} \tag{14}
\]

\[
\leq e^{-c(t-s)^{2j}} \| \Gamma^{-1}(s) \|_{L(L^2, L^2)} 2^{-j(2\gamma - \frac{1}{2})} d_j \| y \|_{H^{\frac{1}{2} + \gamma}}^2 \\
\leq \eta_s e^{-c(t-s)^{2j}} 2^{-j(2\gamma - \frac{1}{2})} \| y(s) \|_{H^{\frac{1}{2} + \gamma}}^2 d_j,
\]

where $d_j$ is a sequence in $\ell^1$, the third inequality is due to Lemma 4.6 and the last inequality is due to the reason that from Remark 4.7, we know

\[
\| \Gamma(t) \|_{L(H^{\frac{1}{2} + \gamma}, H^{\frac{1}{2} + \gamma})} \leq \| \Gamma(t) \|_{L(L^2, L^2)}.
\]

Thus we have

\[
\| e^{(t-s)} \Delta \Gamma^{-1} (y \otimes y) \|_{H^{\frac{1}{2} + \gamma}} \leq \eta_s \| y(s) \|_{H^{\frac{1}{2} + \gamma}}^2 \sup_j [e^{-c(t-s)^{2j}} 2^{j(2\gamma -)}].
\]

Since there exists some constant $C(\gamma)$, such that

\[
e^{-c(t-s)^{2j}} \leq C(\gamma) [(t-s)^{2j}]^{-1 + \frac{3}{2}},
\]

we obtain

\[
\| e^{(t-s)} \Delta \Gamma^{-1} (y \otimes y) \|_{H^{\frac{1}{2} + \gamma}} \leq \eta_s \| y(s) \|_{H^{\frac{1}{2} + \gamma}}^2 (t-s)^{-1 + \frac{3}{2}}. \tag{15}
\]

Therefore,

\[
\| F(y)(t) \|_{L^\infty([0, T]; H^{\frac{1}{2} + \gamma})} \leq T^{\frac{3}{2}} \| y(s) \|_{L^\infty([0, T]; H^{\frac{1}{2} + \gamma})} \sup_{t \geq 0} \eta_t. \tag{16}
\]

On the other hand,

\[
\| \nabla F(y)(t) \|_{H^{\frac{1}{2} + \gamma}} \leq \int_0^t \| e^{(t-s)} \Delta \Gamma^{-1} (y \otimes y(s)) \|_{H^{\frac{1}{2} + \gamma}} ds. \tag{17}
\]

Following the same way that we obtain (15), by replacing $\gamma$ by $\gamma + 1$, we obtain

\[
\| e^{(t-s)} \Delta \Gamma^{-1} (y \otimes y(s)) \|_{H^{\frac{1}{2} + \gamma}} \leq \eta_s \| y(s) \|_{H^{\frac{1}{2} + \gamma}}^2 (t-s)^{-\frac{2}{2}}. \tag{18}
\]

Therefore, by combining (17) and (18), we deduce

\[
\int_0^T \| \nabla F(y)(t) \|_{H^{\frac{1}{2} + \gamma}}^2 dt \leq \int_0^T \left( \int_0^t \| y(s) \|_{H^{\frac{1}{2} + \gamma}}^2 (t-s)^{-\frac{2}{2}} ds \right)^2 dt \leq \sup_{t \geq 0} \eta_t^2 \int_0^T \left( \int_0^t \| y(s) \|_{H^{\frac{1}{2} + \gamma}}^2 ds \right)^2 dt \\
\leq \sup_{t \geq 0} \eta_t^2 \int_0^T t^{-1} dt \int_0^T \| y(t) \|_{H^{\frac{1}{2} + \gamma}} dt \\
\leq \sup_{t \geq 0} \eta_t^2 T^{\gamma} \int_0^T \| y(t) \|_{H^{\frac{1}{2} + \gamma}} dt.
\]
That is,
\[
\|\nabla F(y)(t)\|_{L^2([0,T];\dot{H}^{\frac{1}{2}+\gamma})} \leq T^\frac{\gamma}{2} \|\nabla y(s)\|_{L^2([0,T];\dot{H}^{\frac{1}{2}+\gamma})} \sup_{t \geq 0} \eta_t. \tag{19}
\]

The conclusion follows directly from (16) and (19).

**Lemma 4.9.** For any \( T > 0 \) and \( 0 < \gamma < 1 \),
\[
\|e^{t\Delta} u_0\|_{Z_T^\gamma} \leq \|u_0\|_{\dot{H}^{\frac{1}{2}+\gamma}}.
\]

**Proof.** The proof is trivial. We write a simple proof here for completeness.
Set \( v = e^{t\Delta} u_0 \), then \( v \) is the solution of the following heat equations
\[
\begin{cases}
\partial_t u = \Delta u \\
u \big|_{t=0} = u_0.
\end{cases}
\tag{20}
\]
Taking \( \dot{H}^{\frac{1}{2}+\gamma} \) inner product of both sides with \( u \) immediately yields the result.

The following fixed point theorem comes from Lemma 5.5 of [1]:

**Lemma 4.10.** Let \( E \) be a Banach space, \( B \) a continuous bilinear map from \( E \times E \) to \( E \), and \( \alpha \) a positive real number such that
\[
\alpha < \frac{1}{4\|B\|} \text{ with } \|B\| := \sup_{\|u\| \leq 1, \|v\| \leq 1} \|B(u,v)\|.
\]
For any \( a \) in the ball \( B(0,\alpha) \) (i.e., with center 0 and radius \( \alpha \)) in \( E \), a unique \( x \) then exists in \( B(0,2\alpha) \) such that
\[
x = a + B(x,x).
\]

**Proof of Theorem 4.2**
The result comes immediately by Lemma 4.8, Lemma 4.9 and Lemma 4.10.

**A the Completeness of the Normed Space \( Z_T^\gamma \)**

We prove the completeness of the space \( Z_T^\gamma \) when \( 0 < \gamma < 1 \).
Firstly, recall that the norm of \( Z_T^\gamma \) is:
\[
\|u\|_{Z_T^\gamma}^2 := \|u\|_{L^2([0,T];\dot{H}^{\frac{1}{2}+\gamma})}^2 + \int_0^T \|\nabla u(t)\|_{\dot{H}^{\frac{1}{2}+\gamma}}^2 dt.
\]

Let \( \{u_n\}_{n \geq 1} \) be a Cauchy sequence in \( Z_T^\gamma \), which means \( \{u_n\}_{n \geq 1} \) is a Cauchy sequence in \( L^2([0,T];\dot{H}^{\frac{1}{2}+\gamma}) \) and \( \{\nabla u_n\}_{n \geq 1} \) is a Cauchy sequence in \( L^2([0,T];\dot{H}^{\frac{1}{2}+\gamma}) \).

Our aim is to find some \( u \in Z_T^\gamma \), such that \( u_n \) converges to \( u \) in the norm of \( Z_T^\gamma \). By Prop 1.34 of [1], we know that when \( 0 < \gamma < 1 \), the homogeneous Sobolev space \( \dot{H}^{\frac{1}{2}+\gamma} \) is a Hilbert
space. Therefore, both $L^\infty([0, T]; \dot{H}^{\frac{1}{2}+\gamma})$ and $L^2([0, T]; \dot{H}^{\frac{1}{2}+\gamma})$ are complete. Therefore, there exists $v_1$ and $v_2$, such that

$$u_n \rightharpoonup v_1 \text{ in } L^\infty([0, T]; \dot{H}^{\frac{1}{2}+\gamma}),$$

and

$$\nabla u_n \rightharpoonup v_2 \text{ in } L^2([0, T]; \dot{H}^{\frac{1}{2}+\gamma}).$$

Now note that if we can prove that $\nabla v_1(t) = v_2(t)$ for a.e. $t \in [0, T]$, (of course the derivatives mean weak derivatives, which can be at least defined for those $v_1(t)$ which satisfy $v_1(t) \in \dot{H}^{\frac{1}{2}+\gamma}$ and the derivatives are at least in the space $S'$), then we immediately have

$$\nabla u_n \rightharpoonup \nabla v_1 \text{ in } L^2([0, T]; \dot{H}^{\frac{1}{2}+\gamma}),$$

hence $v_1 \in Z^\gamma_T$ and $u_n$ converges to $v_1$ in the norm of $Z^\gamma_T$.

**To prove: $\nabla v_1(t) = v_2(t)$ for a.e. $t \in [0, T]$:**

Since $L^p$ convergence implies convergence in probability, hence we can find a subsequence $n_k$, such that

$$u_{n_k}(t) \rightharpoonup v_1(t) \text{ in } \dot{H}^{\frac{1}{2}+\gamma} \text{ a.e.},$$

and

$$\nabla u_{n_k}(t) \rightharpoonup v_2(t) \text{ in } \dot{H}^{\frac{1}{2}+\gamma} \text{ a.e.}.$$ 

Therefore, there exists a subset $A$ of the interval $[0, T]$ which has the full Lebesgue measure $T$, such that for any $t \in A$,

$$u_{n_k}(t) \rightharpoonup v_1(t) \text{ in } \dot{H}^{\frac{1}{2}+\gamma},$$

and

$$\nabla u_{n_k}(t) \rightharpoonup v_2(t) \text{ in } \dot{H}^{\frac{1}{2}+\gamma}.$$ 

Then for any $\phi \in (C^\infty_c(\mathbb{R}^3))^3$, and $t \in A$, we have

$$\langle v_2(t), \phi \rangle = \lim_{k \to \infty} \langle \nabla u_{n_k}(t), \phi \rangle$$

$$= -\lim_{k \to \infty} \langle u_{n_k}(t), \text{div} \phi \rangle$$

$$= -\langle v_1(t), \text{div} \phi \rangle$$

$$= \langle \nabla v_1(t), \phi \rangle,$$

where same as before, derivatives mean weak derivatives, $\langle \cdot, \cdot \rangle$ is the duality bracket and the second and the fourth equalities are due to the definition of the weak derivatives. Therefore, we have shown $\nabla v_1(t) = v_2(t)$ for a.e. $t \in [0, T]$, which finishes our proof.

**Acknowledgments.** S. Liang is grateful for the financial support from Deutsche Forschungsgemeinschaft (DFG) through the program IRTG 2235. The author thanks Prof. Dr. Michael Röckner for pointing out the precise way to state Definition 4.1 and Theorem 4.2.
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