An extension of the Eshelby conjecture to domains of general shape *

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Abstract

A simply connected inclusion which induces a uniform field on the inclusion for probing by a uniform loading is only an ellipse or an ellipsoid, as known as the Eshelby conjecture. We extend the Eshelby conjecture to domains of general shape for the anti-plane elasticity. In particular, we show that for each \( N \in \mathbb{N} \), an inclusion induces a uniform field on the inclusion for a harmonic polynomial loading of degree \( N \) if and only if the inclusion is a domain of negative order \( N \), which is a simply connected bounded domain whose exterior conformal mapping is a Laurent series of a finite negative degree \( N \).

Key words. Eshelby conjecture; Anti-plane elasticity; Polynomial loading; Faber polynomial

1 Introduction and main results

We consider the elastic field perturbation resulting from an inclusion in a homogeneous background for the anti-plane elasticity. More precisely, we consider the following two-dimensional conductivity problem:

\[
\begin{cases}
\nabla \cdot \left( \sigma \chi(\Omega) + I_2 \chi(\mathbb{R}^2 \setminus \overline{\Omega}) \right) \nabla u = 0 & \text{in } \mathbb{R}^2, \\
u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty,
\end{cases}
\]

(1.1)

where \( \Omega \) is a simply connected bounded domain occupied by a homogeneous material with the conductivity \( \sigma \), which is possibly anisotropic, and \( H \) is an entire harmonic function. The symbol \( \chi \) indicates the characteristic function and \( I_2 \) is the 2-by-2 identity matrix. The solution \( u \) should satisfy the transmission condition

\[
u \cdot \nabla u \bigg|_+ = \nu \cdot \sigma \nabla u \bigg|_- \quad \text{on } \partial \Omega.
\]

(1.2)

Here, \( \nu \) is the outward unit normal vector on \( \partial \Omega \) and the symbols + and − indicate the limit from the exterior and interior of \( \Omega \), respectively. An inclusion with a different material parameter from that of the background induces the field perturbation in the exterior and interior of the

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inclusion. The resulting perturbation depends on the inclusion’s shape as well as its material parameter so that certain shapes admit extremal properties. In 1957, Eshelby found that a uniform loading induces a uniform strain inside an ellipsoid which is embedded in an infinite elastic medium [9]. Then, he conjectured: ‘Among closed surfaces, the ellipsoid alone has this convenient property...’ [10]. This Eshelby uniformity conjecture has been proved by Ru and Schiavone [25] (for the conductivity problem) and Sendeckyj [26] for two dimensions, and by Kang and Milton [15] and Liu [20] for three dimensions. The aim of this paper is to extend the Eshelby conjecture to domains of general shape in two dimensions.

Let us state some results on the Eshelby conjecture and related interface problems. The problem of computing stress distribution for an elastic elliptic inclusion in an isotropic matrix was treated by Donnell [5] using elliptic coordinates. Mindlin and Cooper [23] extended this method to the thermoelastic problem. Hardiman [13] treated the elliptic inclusion problem using the complex variables method, and he noticed that a uniform loading induces a uniform strain within an elliptic inclusion. After the Eshelby conjecture was posed, it has been investigated based on various methods. To prove the Eshelby conjecture in two dimensions, Ru and Schiavone [25] and Sendeckyj [26] used the complex analytic function theory. Kang and Milton [15] provided an alternative proof by using the hodographic transformation. For the three dimensions, non-ellipsoidal simply connected inclusions of various shape were shown not to satisfy the Eshelby uniformity property. Rodin [24] considered polyhedral inclusions and Markenscoff [22] inclusions with a planar piece on its boundary. Lubada and Markenscoff [21] showed that a similar consideration holds for nonconvex inclusions and for inclusions bounded by polynomial surfaces of degree higher than two, or by segments of two or more different surfaces. To prove the Eshelby conjecture in three dimensions, Kang and Milton [15] and Liu [20] used the properties of the Newtonian potential. It has been also shown that multiply connected domains can have the uniformity property [4, 16, 19, 20].

In the present paper we consider the simply connected inclusions of non-elliptical shape. It turns out that the Eshelby conjecture can be generalized to domains of general shapes in which the applied loading is now harmonic polynomials of finite degree; see Theorem 1.1 for the details. As far as we know, there has been no report on the extension of the Eshelby uniformity to domains of general shape. Our analysis is based on the series expansions of the boundary integral operators by using the Faber polynomials, recently derived by Jung and Lim in [14].

Let us introduce some terminology before stating the main results. For notational convenience we identify \( x = (x_1, x_2) \) in \( \mathbb{R}^2 \) with \( z = x_1 + i x_2 \) in \( \mathbb{C} \). The symbols \( \text{Re} \) and \( \text{Im} \) indicate the real and imaginary parts of complex numbers. From the Riemann mapping theorem, there exists uniquely a conformal mapping, say \( \Phi \), from \( \mathbb{C} \setminus \mathbb{D} \) onto \( \mathbb{C} \setminus \Omega \) satisfying \( \gamma := \Phi'(\infty) > 0 \). Here, \( \mathbb{D} \) is the unit disk centered at the origin. The mapping \( \Phi \) admits the series expansion

\[
\Phi(w) = \gamma w + \mu_0 + \frac{\mu_1}{w} + \frac{\mu_2}{w^2} + \cdots
\]

with complex coefficients \( \mu_k \)'s. If \( \mu_N \neq 0 \) and \( \mu_{N+1} = \mu_{N+2} = \cdots = 0 \) for some \( N \in \mathbb{N} \), we call \( \Omega \) a domain of negative order \( N \). For the sake of simplicity we call a disk (as well as an ellipse) a domain of negative order 1.

**Definition 1** (Infinite polynomial associated with \( \Phi \)). *We define a formal infinite polynomial

\[
\mathcal{F}(z) = \sum_{k=2}^{\infty} \frac{\mu_k}{\gamma^k} F_k(z),
\]
where \( F_k(z) \) is the so-called Faber polynomial associated with \( \Omega \). For each \( k \), \( F_k(z) \) is a monomial of degree \( k \) that is uniquely defined by \( \mu_j \gamma^j \) for \( j = 0, 1, \ldots, k-1 \); see section 3.1 for its definition and properties. For \( \Omega \) a domain of negative order \( N \), \( \mathfrak{F}(z) \) is a polynomial of degree \( N \).

We set
\[
\tau_1(t) = \frac{\mu_1}{\gamma} + 2t, \quad \tau_2(t) = \frac{-\mu_1}{\gamma} + 2t, \quad (1.3)
\]
and
\[
\tau(t) = (1 - 2t) \begin{bmatrix} \text{Re} \{\tau_1(t)\} & -\text{Im} \{\tau_2(t)\} \\ \text{Im} \{\tau_1(t)\} & \text{Re} \{\tau_2(t)\} \end{bmatrix}.
\]

The following is the main result of this paper.

**Theorem 1.1.** Assume that \( \Omega \) is a simply connected bounded domain in \( \mathbb{R}^2 \) enclosed by a piecewise \( C^{1,\alpha} \) Jordan curve for some \( \alpha \in (0,1) \) possibly with a finite number of corner points which are not inward or outward cusps. We let \( \Omega \) have the constant conductivity \( \sigma \), which is possibly anisotropic. For the solution \( u \) to the transmission problem (1.1) the following holds:

(a) For any \( N \geq 1 \), \( \Omega \) is a domain of negative order \( N \) if and only if \( u \) has a uniform strain in \( \Omega \) for a harmonic polynomial \( H \) of degree \( N \).

(b) [Isotropic case] Assume that \( \Omega \) is a domain of any finite negative order with isotropic conductivity \( \sigma = \sigma I_2 \), \( 0 < \sigma \neq 1 < \infty \). If \( u \) has a uniform strain \( \nabla u = (e_1, e_2) \) in \( \Omega \) for a harmonic polynomial \( H \), then it holds that
\[
H(x) = \text{const.} + c_1 \text{Re} \{z + \tau_1(\lambda) \mathfrak{F}(z)\} + c_2 \text{Im} \{z - \tau_2(\lambda) \mathfrak{F}(z)\}
\]
with
\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \tau(\lambda)^{-1} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad \lambda = \frac{\sigma + 1}{2(\sigma - 1)}.
\]
Conversely, for such \( H \) defined with any real constants \( c_1, c_2 \) not simultaneously zero, the corresponding solution \( u \) has the uniform strain \( \nabla u = (e_1, e_2) \) in \( \Omega \).

(c) [Anisotropic case] Assume that \( \Omega \) is a domain of any finite negative order with anisotropic conductivity \( \sigma \) such that \( I_2 - \sigma \) is either positive or negative definite. If \( u \) has a uniform strain \( \nabla u = (e_1, e_2) \) in \( \Omega \) for a harmonic polynomial \( H \), then it holds that
\[
H(x) = \text{const.} + f_1 x_1 + f_2 x_2 + c_1 \text{Re} \{z + \tau_1(-1/2) \mathfrak{F}(z)\} + c_2 \text{Im} \{z - \tau_2(-1/2) \mathfrak{F}(z)\}
\]
with
\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \tau(-1/2)^{-1} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \sigma \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.
\]

Conversely, for such \( H \) defined with any real constants \( c_1, c_2 \) not simultaneously zero, the corresponding solution \( u \) has the uniform strain \( \nabla u = (e_1, e_2) \) in \( \Omega \).
It is worth to remark that we can regard Theorem 1.1 as an extension of the strong Eshelby conjecture following the terminology of [15].

Let us discuss the invertibility of the matrices $\tau(\lambda)$ and $\tau(-1/2)$. Popularly known as the Bieberbach conjecture (see [3]), it holds that

$$|\mu_1| < \gamma,$$

which can be proved by using the area theorem (see [12])

$$\sum_{k=1}^{\infty} k|\mu_k|^2 < \gamma^2.$$

It follows from (1.5) that

$$\det \tau(t) = (1 - 2t)^2 \left(-|\mu_1/\gamma|^2 + 4t^2\right)^{-1} \neq 0$$

for all $t \in (-\infty, -1/2] \cup (1/2, \infty)$. Hence, $\tau(\lambda)$ and $\tau(-1/2)$ are invertible. Figures 1.1–1.3 illustrate the potential function $u$ for $\Omega$ a domain of finite negative order and $H$ a harmonic polynomial given as in Theorem 1.1 (b) and (c). In all examples, $u$ has a uniform strain inside $\Omega$ and the potential difference in $u$ between the neighboring level curves is $1/2$.

We also have the following theorem for the isotropic conductivity case.

**Theorem 1.2.** We assume the same regularity for $\Omega$ as in Theorem 1.1 and $\sigma = \sigma I_2$, $0 < \sigma \neq 1 < \infty$. For the solution $u$ to the transmission problem (1.1) the following holds:

(a) For any $N \geq 1$, $\Omega$ is a domain of any finite negative order $N$ if and only if the function $$(\lambda + \frac{1}{2})(u - H) - D_{\partial \Omega}[(u - H)|_{\partial \Omega}]$$ with $\lambda = (\sigma + 1)/(2(\sigma - 1))$ is a harmonic polynomial of degree $N$ in $\Omega$ for a first degree polynomial $H$.

(b) If $\Omega$ is a domain of finite negative order $N \geq 2$ and $H$ has a degree smaller than $N$, then $u$ cannot be a polynomial of any finite degree.

The rest of the paper is organized as follows. Section 2 is devoted to the boundary integral formulation for the transmission problem. In section 3, we review the definition and properties of the Faber polynomials and provide series expansions for the boundary integral operators. In section 4, we derive relations for the density function associated with $u$ and $H$. The main results are proved in section 5. We finish with conclusion in section 6.

## 2 Boundary integral formulation for the transmission problem

We formulate the transmission problem (1.1) in terms of the boundary integral operators as follows.

For a Lipschitz domain $\Omega$, the single and double layer potential for a density $\varphi \in L^2(\partial \Omega)$ is defined as

$$S_{\partial \Omega}[\varphi](x) = \int_{\partial \Omega} \Gamma(x - y)\varphi(y) d\sigma(y), \quad x \in \mathbb{R}^2,$$

$$D_{\partial \Omega}[\varphi](x) = \int_{\partial \Omega} \frac{\partial}{\partial \nu_y} \Gamma(x - y)\varphi(y) d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial \Omega,$$
Figure 1.1: Level curves of $u$ for the domain $\Omega$ (thicker closed curve) of negative order $N = 1, 2, 3, 4$ with $\sigma = 0.2$, where the coefficients $\mu_j$ of $\Omega$ for $j \leq N$ is given by $\gamma = 1$, $\mu_1 = 0.1 + 0.1i$, $\mu_2 = 0.1 + 0.1i$, $\mu_3 = -0.1i$, $\mu_4 = 0.05$. The harmonic polynomial $H$ is given by Theorem 1.1 (b) with $(c_1, c_2) = (1, 0)$. For all four examples, $\nabla u = (1.5695, -0.1121)$ inside $\Omega$.

where $\Gamma$ is the fundamental solution to the Laplacian, i.e., $\Gamma(x) = \frac{1}{2\pi} \ln |x|$ and $\nu_y$ denotes the outward unit normal vector on $\partial \Omega$. We also define the so-called Neumann-Poincaré operator as

$$K^\ast_{\partial \Omega}[\varphi](x) = \text{p.v.} \int_{\partial \Omega} \frac{\langle x - y, \nu_x \rangle}{|x - y|^2} \varphi(y) d\sigma(y).$$

Here, p.v. stands for the Cauchy principal value. We identify $\mathbb{R}^2$ with $\mathbb{C}$ as stated before and set

$$S_{\partial \Omega}[\varphi](z) := S_{\partial \Omega}[\varphi](x) \quad \text{for} \quad x = (x_1, x_2), \; z = x_1 + ix_2.$$

Likewise, we define $D_{\partial \Omega}[\varphi](z)$ and $K^\ast_{\partial \Omega}[\varphi](z)$. On the interface $\partial \Omega$, the single layer potential satisfies the jump relations

$$S_{\partial \Omega}[\varphi]^\pm = S_{\partial \Omega}[\varphi]^\pm,$$

$$\frac{\partial}{\partial \nu} S_{\partial \Omega}[\varphi]^\pm = (\pm \frac{1}{2} I + K^\ast_{\partial \Omega})[\varphi].$$

**Isotropic case.** For the case $\sigma = \sigma I_2$, $0 < \sigma \neq 1 < \infty$, $u$ can be expressed as

$$u(x) = H(x) + S_{\partial \Omega}[\varphi](x), \quad x \in \mathbb{R}^2,$$

(2.1)
Figure 1.2: Level curves of $u$ for the domain $\Omega$ (thicker closed curve) of negative order $N = 4$ with $\sigma = 0.2$, where $\gamma = 1$, $\mu_1 = 0.1i \ast s$, $\mu_2 = -0.05i \ast s$, $\mu_3 = 0$, $\mu_4 = 0.15 \ast s$ for various scale factor $s$. The harmonic polynomial $H$ is given by Theorem 1.1 (b) with $(c_1, c_2) = (0, 1)$. As the domain $\Omega$ more resembles a disk, the corresponding $H$ has smaller coefficients for the components of orders $\geq 2$.

Figure 1.3: The domain $\Omega$ is given as in Figure 1.2 with $s = 1/4$. The harmonic polynomial $H$ is given as in Theorem 1.1 (c) such that $f_1 + c_1 = 0$ and $f_2 + c_2 = 1$. 
where
\[ \varphi = (\lambda I - K^*_\partial \Omega)^{-1} [\nu \cdot \nabla H] \quad \text{with} \quad \lambda = \frac{\sigma + 1}{2(\sigma - 1)}. \]  

(2.2)
The boundary integral operator \( \lambda I - K^*_\partial \Omega \) is invertible on \( L^2_0(\partial \Omega) \) for \( |\lambda| \geq 1/2 \) as shown in [7, 17, 28]. We recommend the reader to see [18, 1, 2] and references therein for more properties of the NP operator.

**Anisotropic case.** We now assume \( \sigma = A \) for a positive definite symmetric matrix \( A \) such that \( I_2 - A \) is either positive or negative definite. We set \( B \) to be the matrix satisfying \( B^2 = A^{-1} \) and define the single layer potential associated with \( A \) as \[ \tilde{S}_{\partial \Omega}[\tilde{\varphi}](x) = \frac{1}{2\pi} \int_{\partial \Omega} \ln |B(x - y)| \tilde{\varphi}(y) \frac{1}{\sqrt{\det A}} \, d\sigma(y), \quad x \in \mathbb{R}^2, \]
for \( \tilde{\varphi} \in L^2(\partial \Omega) \). It is well known that the solution \( u \) admits the boundary integral expression (see [8])
\[ u(x) = \begin{cases} \tilde{S}_{\partial \Omega}[\tilde{\varphi}](x) & \text{in } \Omega, \\ H(x) + S_{\partial \Omega}[\varphi](x) & \text{in } \mathbb{R}^2 \setminus \Omega, \end{cases} \]  
(2.3)
where the density functions \( (\tilde{\varphi}, \varphi) \in L^2(\partial \Omega) \times L^2_0(\partial \Omega) \) satisfy the transmission condition
\[ \begin{cases} \tilde{S}_{\partial \Omega}[\tilde{\varphi}] - S_{\partial \Omega}[\varphi] = H & \text{on } \partial \Omega, \\ \nu \cdot A \nabla \tilde{S}_{\partial \Omega}[\tilde{\varphi}] - \nu \cdot \nabla S_{\partial \Omega}[\varphi] \Big|^- = \nu \cdot \nabla H & \text{on } \partial \Omega. \end{cases} \]  
(2.4)

3 Series expansions of boundary integral operators

We normalize the exterior conformal mapping as \( \Psi(w) := \Phi(\gamma^{-1}w), \quad |w| \geq \gamma. \) It is straightforward to see that \( \Psi \) conformally maps \( \{ w \in \mathbb{C} : |w| > \gamma \} \) onto \( \mathbb{C} \setminus \overline{\Omega} \) and that it admits the expansion \( \Psi(w) = w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots \)  
(3.1)
with
\[ a_k = \mu_k \gamma^k, \quad k \in \mathbb{N} \cup \{0\}. \]  
(3.2)

**3.1 The Faber polynomials**

The concept of the Faber polynomials was first introduced by G. Faber in [11] and has been one of the essential elements in geometric function theory; see [6] for further details. The Faber polynomials \( \{F_m(z)\} \) associated with \( \Psi \) are defined by the generating function relation
\[ \frac{w \Psi'(w)}{\Psi(w) - z} = \sum_{m=0}^{\infty} \frac{F_m(z)}{w^m}, \quad z \in \Omega, \quad |w| > \gamma. \]

It implies that for \( \tilde{z} = \Psi(w) \in \mathbb{C} \setminus \overline{\Omega}, \quad z \in \Omega, \)
\[ \log(\tilde{z} - z) = \log w - \sum_{m=1}^{\infty} \frac{1}{m} F_m(z) w^{-m} \quad (\text{modulo } 2\pi i) \]  
(3.3)
with a suitably chosen complex argument.

Each $F_m$ is an $m$-th order monic polynomial which is uniquely determined by the coefficients $a_0, a_1, \cdots, a_{m-1}$. For example, the first three polynomials are

$$F_0(z) = 1, \quad F_1(z) = z - a_0, \quad F_2(z) = z^2 - 2a_0z + (a_0^2 - 2a_1).$$

The Faber polynomials form a basis for analytic functions in $\Omega$ as shown in [27].

In terms of the variable $w$, the Faber polynomial admits the expansion

$$F_m(\Psi(w)) = w^m + \sum_{k=1}^{\infty} c_{m,k} w^{-k}, \quad m = 1, 2, \ldots.$$  \hspace{1cm} (3.4)

Here, $c_{m,k}$’s are called the Grunsky coefficients. It is well known the Grunsky identity

$$kc_{m,k} = mc_{k,m}.$$  \hspace{1cm} (3.5)

**Lemma 3.1.** Let $\Omega$ be a domain of finite negative order $N$, i.e.,

$$\Psi(w) = w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots + \frac{a_N}{w^N} \quad \text{with } a_N \neq 0.$$  \hspace{1cm} (3.6)

Then, for each $m \in \mathbb{N}$ it hold that

$$c_{m,Nm} = (a_N)^m \neq 0,$$

$$c_{m,k} = c_{k,m} = 0 \quad \text{for } k \geq Nm + 1.$$

**Proof.** Since $F_m(z)$ is a monomial of order $m$ (the highest order term is $z^m$), one can easily prove the lemma by plugging (3.6) into (3.4). \hfill \Box

### 3.2 Orthogonal coordinates associated with the exterior conformal mapping

We set $\rho_0 = \ln \gamma$ and define the curvilinear orthogonal coordinates $(\rho, \theta) \in [\rho_0, \infty) \times [0, 2\pi)$ for $z \in \mathbb{C} \setminus \Omega$ via the relation

$$z = \Psi(e^{\rho+i\theta}).$$

One can easily see that the scale factors $h_{\rho} := |\frac{\partial z}{\partial \rho}|$ and $h_{\theta} := |\frac{\partial z}{\partial \theta}|$ coincide with each other. We denote

$$h = h_{\rho} = h_{\theta}.$$  \hspace{1cm} (3.7)

If $\Omega$ is a piecewise $C^{1,\alpha}$ domain without inward or outward cusps, then one can show (see [14])

$$h(\rho_0, \theta), \quad \frac{1}{h(\rho_0, \theta)} \in L^1([0, 2\pi]).$$

For notational simplicity we set $v(\rho, \theta) = v(\Psi(e^{\rho+i\theta}))$ for a function $v$. One can easily see that the exterior normal derivative of $v(\rho, \theta)$ is

$$\frac{\partial v}{\partial \nu}(z) = \frac{1}{h} \frac{\partial}{\partial \rho} v(\Psi(e^{\rho+i\theta})) \bigg|_{\rho \to \rho_0^+}. \hspace{1cm} (3.7)$$
We denote $\langle \cdot, \cdot \rangle$ the inner-product in $L^2(\partial \Omega, h)$, which is the weighted $L^2$ space with the weight $h$. In other words, for functions $p, q$ on $\partial \Omega$ satisfying $\int_{\partial \Omega} |p|^2 h d\sigma$, $\int_{\partial \Omega} |q|^2 h d\sigma < \infty$ we set

$$\langle p, q \rangle = \frac{1}{2\pi} \int_{\partial \Omega} p(z) \overline{q(z)} h(z) d\sigma(z)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} p(\rho_0, \theta) \overline{q(\rho_0, \theta)} (h(\rho_0, \theta))^2 d\theta.$$

Here, we used the fact that 

$$d\sigma(z) = h(\rho_0, \theta) d\theta.$$

(3.8)

### 3.3 Series expansions for the boundary integral operators

We define the density functions as

$$\psi_m(z) := \psi_m(\Psi(e^{\rho_0+i\theta})) = \frac{e^{im\theta}}{h(\rho_0, \theta)}, \quad m \in \mathbb{Z}.$$

One can easily see that they form an orthonormal basis in $L^2(\partial \Omega, h)$. In particular,

$$\langle \psi_m, \psi_n \rangle = \delta_{m,n} \quad \text{for all } m, n \in \mathbb{Z}.$$

From (3.3), the real logarithm function satisfies

$$\ln|\tilde{z} - z| = \ln|w| - \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \left[ F_m(z) w^{-m} + \overline{F_m(z)} \overline{w^{-m}} \right].$$

(3.9)

The following series expansions for the boundary integral operators were derived in [14] by using (3.4) and the series expansion for the complex logarithm.

**Lemma 3.2** ([14]). Assume that $\Omega$ is a simply connected bounded domain in $\mathbb{R}^2$ enclosed by a piecewise $C^{1, \alpha}$ Jordan curve, possibly with a finite number of corner points which are not inward or outward cusps.

(a) We have (for $m = 0$)

$$S_{\partial \Omega}[^0]\psi_0](z) = \begin{cases} \ln \gamma & \text{if } z \in \Omega, \\ \ln |w| & \text{if } z \in \mathbb{C} \setminus \Omega. \end{cases} \quad (3.10)$$

For $m = 1, 2, \ldots$, we have

$$S_{\partial \Omega}[^m]\psi_m](z) = \begin{cases} -\frac{1}{2m^2}\frac{F_m(z)}{\gamma^m} & \text{if } z \in \overline{\Omega}, \\ -\frac{1}{2m^2} \left( \sum_{k=1}^{\infty} c_{m,k} e^{-k(\rho+i\theta)} + \gamma^{2m} e^{m(-\rho+i\theta)} \right) & \text{if } z \in \mathbb{C} \setminus \overline{\Omega}, \end{cases} \quad (3.11)$$

$$S_{\partial \Omega}[^{-m}]\psi_{-m}](z) = \begin{cases} -\frac{1}{2m^2}\frac{F_m(z)}{\gamma^m} & \text{if } z \in \overline{\Omega}, \\ -\frac{1}{2m^2} \left( \sum_{k=1}^{\infty} c_{m,k} e^{-k(\rho-i\theta)} + \gamma^{2m} e^{m(-\rho-i\theta)} \right) & \text{if } z \in \mathbb{C} \setminus \overline{\Omega}. \end{cases} \quad (3.12)$$

The series converges uniformly for all $(\rho, \theta)$ such that $\rho \geq \rho_1 > \rho_0$. 

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(b) We have (for \( m = 0 \))
\[
\mathcal{K}_{\partial \Omega}^*[\psi_0] = \frac{1}{2} \psi_0.
\] (3.13)

For all \( m, k \in \mathbb{N} \), it holds that
\[
\langle \mathcal{K}_{\partial \Omega}^*[\psi_m], \psi_k \rangle = 0,
\] (3.14)
\[
\langle \mathcal{K}_{\partial \Omega}^*[\psi_{-m}], \psi_k \rangle = 0,
\] (3.15)
\[
\langle \mathcal{K}_{\partial \Omega}^*[\psi_{-m}], \psi_{-k} \rangle = \frac{1}{2} \frac{k}{m} \sigma_{m,k} \gamma^m \gamma^{-k},
\]
\[
\langle \mathcal{K}_{\partial \Omega}^*[\psi_m], \psi_{-k} \rangle = \frac{1}{2} \frac{k}{m} \sigma_{m,k} \gamma^m \gamma^{-k},
\]
\[
\langle \mathcal{K}_{\partial \Omega}^*[\psi_1], \psi_{-N} \rangle \neq 0 \quad \text{for } N \neq 1.
\]

If \( \partial \Omega \) is \( C^{1,\alpha} \), then it further holds
\[
\mathcal{K}_{\partial \Omega}^*[\psi_m] = \frac{1}{2} \sum_{k=1}^{\infty} \frac{k}{m} \sigma_{m,k} \gamma^m \gamma^{-k} \psi_{-k},
\]
\[
\mathcal{K}_{\partial \Omega}^*[\psi_{-m}] = \frac{1}{2} \sum_{k=1}^{\infty} \frac{k}{m} \sigma_{m,k} \gamma^m \gamma^{-k} \psi_k,
\]
\[
\text{where the infinite series converges in the Sobolev space } H^{-1/2}(\partial \Omega).
\]

The following lemma is essential for characterizing the domain of finite negative order.

**Lemma 3.3.** We assume the same regularity for \( \Omega \) as in Lemma 3.2 and let \( N \) be an arbitrary natural number. Then, \( \Omega \) is a domain of finite negative order \( N \) if and only if
\[
\langle \mathcal{K}_{\partial \Omega}^*[\psi_1], \psi_{-k} \rangle = 0 \quad \text{for all } k \geq N + 1
\]
and
\[
\langle \mathcal{K}_{\partial \Omega}^*[\psi_1], \psi_{-N} \rangle \neq 0 \quad \text{for } N \neq 1.
\]

**Proof.** Thanks to (3.5) and (3.15), we prove the proposition. \( \square \)

The following relation is also useful in proving the main theorems:
\[
\frac{1}{2} (\nu_1 + i \nu_2) = \left( \frac{1}{2} I - \mathcal{K}_{\partial \Omega}^* \right) [\gamma \psi_1].
\] (3.16)

Indeed, from (3.11) with \( m = 1 \) it holds that
\[
S_{\partial \Omega}[\psi_1](z) = -\frac{1}{2\gamma} F_1(z) = -\frac{1}{2\gamma} (x_1 + ix_2 - a_0).
\]

By taking the interior normal derivative, we have
\[
\left( -\frac{1}{2} I + \mathcal{K}_{\partial \Omega}^* \right) [\psi_1] = -\frac{1}{2\gamma} (1, i) \cdot \nu = -\frac{1}{2\gamma} (\nu_1 + i \nu_2).
\]

This implies (3.16).
4 Density relations

In this section we derive the relations between the density functions associated with $u$ and $H$ assuming that $H$ is a harmonic polynomial of degree $\tilde{N}$ and $u$ a harmonic polynomial of degree $\tilde{M}$ inside $\Omega$ ($\tilde{M} = 1$ for the anisotropic case). We assume the same regularity for $\Omega$ as in Lemma 3.2.

Since a real harmonic polynomial is the real part of a complex polynomial and the Faber polynomial $F_m(z)$ is a monomial for each $m \geq 0$, we have $H(x) = \text{Re} \left\{ \sum_{m=0}^{\tilde{N}} \alpha_m F_m(z) \right\}$ for some complex coefficients $\alpha_m$’s ($\alpha_{\tilde{N}} \neq 0$). From (3.11) and (3.12) it holds that

$$S_{\partial \Omega} [ -m \gamma^m \psi_m ] (z) = \frac{1}{2} F_m(z),$$

$$S_{\partial \Omega} [ -m \gamma^m \psi_{m-} ] (z) = \frac{1}{2} F_m(z) \quad \text{on } \overline{\Omega}. \quad (4.1)$$

Hence, we have

$$H(x) = \frac{1}{2} \sum_{m=0}^{\tilde{N}} \left( \alpha_m F_m(z) + \overline{\alpha_m F_m(z)} \right) \quad \text{in } \mathbb{C}$$

$$= S_{\partial \Omega} [ -\psi ] (x) + \frac{1}{2} (\alpha_0 + \overline{\alpha_0}) \quad \text{on } \overline{\Omega} \quad (4.2)$$

with

$$\psi = \sum_{m=1}^{\tilde{N}} \left( \alpha_m m \gamma^m \psi_m + \overline{\alpha_m m \gamma^m \psi_{m-}} \right).$$

It then follows from the jump formula of the single layer potential that

$$\frac{\partial H}{\partial \nu} = \left( \frac{1}{2} I - \mathcal{K}_{\partial \Omega}^{*} \right) [\psi] \quad \text{on } \partial \Omega. \quad (4.3)$$

Similarly, we have

$$u(x) = \frac{1}{2} \sum_{m=0}^{\tilde{M}} \left( \tilde{\alpha}_m F_m(z) + \overline{\tilde{\alpha}_m F_m(z)} \right) \quad \text{on } \overline{\Omega}$$

$$= S_{\partial \Omega} [ -\tilde{\psi} ] + \frac{1}{2} (\tilde{\alpha}_0 + \overline{\tilde{\alpha}_0}) \quad \text{on } \overline{\Omega} \quad (4.4)$$

with

$$\tilde{\psi} = \sum_{m=1}^{\tilde{M}} \left( \tilde{\alpha}_m m \gamma^m \psi_m + \overline{\tilde{\alpha}_m m \gamma^m \psi_{m-}} \right)$$

(4.5)

and some complex coefficients $\tilde{\alpha}_m$’s satisfying $\tilde{\alpha}_{\tilde{M}} \neq 0$. The normal derivative of $u$ satisfies

$$\left. \frac{\partial u}{\partial \nu} \right|_{-} = \left( \frac{1}{2} I - \mathcal{K}_{\partial \Omega}^{*} \right) [\tilde{\psi}] \quad \text{on } \partial \Omega.
Lemma 4.1. Assume that the harmonic polynomial $H$ has the degree $\tilde{N}$ and that $u$ is a harmonic polynomial of degree $\tilde{M}$ inside $\Omega$ ($\tilde{M} = 0, 1$ for the anisotropic case). We set $\psi$ and $\tilde{\psi}$ as (4.2) and (4.5). We also set $\varphi$ to be the density function on $\partial \Omega$ satisfying (2.1) for the isotropic case or (2.3) for the anisotropic case. Then, we have

$$\varphi = \psi - \tilde{\psi}. \quad (4.6)$$

Furthermore, the following holds:

(a) If $\sigma$ is isotropic, then we have

$$K^*_\partial \Omega [\tilde{\psi}] = \lambda \tilde{\psi} - (\lambda - \frac{1}{2}) \psi. \quad (4.7)$$

(b) If $\sigma$ is anisotropic and $\nabla u$ is constant in $\Omega$, then $\psi$ satisfies

$$K^*_{\partial \Omega} \left[ (e - f) \gamma \psi_1 + (e - f) \gamma \psi_{-1} \right] = -\frac{1}{2} (e + f) \gamma \psi_1 - \frac{1}{2} (e + f) \gamma \psi_{-1} + \psi \quad (4.8)$$

with $e = e_1 + i e_2$ and $f = f_1 + i f_2$ such that

$$\nabla u = (e_1, e_2), \quad \sigma \nabla u = (f_1, f_2) \quad \text{in } \Omega.$$

Proof. Since $S_{\partial \Omega} [\varphi] = u(x) - H(x)$ for $x \in \mathbb{C} \setminus \Omega$, even when $\sigma$ is anisotropic, we have

$$S_{\partial \Omega} [\varphi] = S_{\partial \Omega} [-\tilde{\psi}] - S_{\partial \Omega} [-\psi] + \text{const.} \quad \text{in } \Omega.$$

Indeed, the equality holds for $x \in \partial \Omega$ from (4.1) and (4.4). Since the both sides are harmonic in $\Omega$ and continuous on $\overline{\Omega}$, the equality holds in $\Omega$ as well. By taking the interior normal derivative we have

$$\left( -\frac{1}{2} I + K^*_\partial \Omega \right) [\varphi] = \left( -\frac{1}{2} I + K^*_\partial \Omega \right) [\psi - \tilde{\psi}] \quad \text{on } \partial \Omega.$$

Since $-\frac{1}{2} I + K^*_\partial \Omega$ is invertible on $L^2(\partial \Omega)$ and $\varphi, \psi - \tilde{\psi}$ are in $L^2(\partial \Omega)$, one can deduce (4.6).

If $\sigma$ is isotropic, the relations (2.2) and (4.3) imply that

$$(\lambda I - K^*_\partial \Omega)[\varphi] = \frac{\partial H}{\partial \nu} = \left( \frac{1}{2} I - K^*_\partial \Omega \right) [\psi].$$

From (4.6), it is straightforward to obtain (4.7).

Let us now assume $\sigma$ to be anisotropic and $\nabla u = (e_1, e_2)$ in $\Omega$ for some real constants $e_1, e_2$. Then, $\tilde{M} = 1$ and

$$u(x_1, x_2) = \text{const.} + \frac{1}{2} \bar{e} z + \frac{1}{2} e \bar{z}$$

so that

$$\tilde{\alpha}_1 = \bar{e} \quad (4.9)$$

and

$$\tilde{\psi} = e \gamma \psi_1 + e \gamma \psi_{-1}. \quad (4.10)$$
The definition of $f$ and (3.16) imply
\[
\nu \cdot \sigma \nabla u \left|_{\nu} \right. = \nu_1 f_1 + \nu_2 f_2 \\
= \frac{1}{2}(\nu_1 + i\nu_2)\tilde{f} + \frac{1}{2}(\nu_1 - i\nu_2)f \\
= \left(\frac{1}{2}I - K^*_{\partial \Omega}\right) [\tilde{f} \gamma \psi_1 + f \gamma \psi_{-1}].
\] (4.11)

On the other hand, we have from (4.3), (4.6) and the transmission condition on $\partial \Omega$ that
\[
\nu \cdot \sigma \nabla u \left|_{\nu} \right. = \frac{\partial H}{\partial \nu} + \frac{\partial}{\partial \nu} S_{\partial \Omega}[\varphi] \left|_{\nu} \right. = -K^*_{\partial \Omega} [\tilde{\psi}] + \psi - \frac{1}{2} \tilde{\psi}.
\]

By use of (4.10) and (4.11) we deduce
\[
\left(\frac{1}{2}I - K^*_{\partial \Omega}\right) [\tilde{f} \gamma \psi_1 + f \gamma \psi_{-1}] + \left(\frac{1}{2}I + K^*_{\partial \Omega}\right) [\tilde{\varphi} \gamma \psi_1 + \varphi \gamma \psi_{-1}] = \psi,
\] (4.12)
and this implies (4.8).

We give an alternative proof for the Eshelby conjecture by using Lemma 3.3 and Lemma 4.1 as follows.

**Corollary 4.2** (The Eshelby conjecture). Assume that $\Omega$ is a simply connected bounded domain in $\mathbb{R}^2$ enclosed by a piecewise $C^{1, \alpha}$ Jordan curve for some $\alpha \in (0, 1)$ possibly with a finite number of corner points which are not inward or outward cusps. For any $\sigma$, either isotropic or anisotropic, $\Omega$ is an ellipse if and only if the solution $u$ to (1.1) has a uniform strain in $\Omega$ for a uniform loading $H$.

**Proof.** We only prove that $\Omega$ is an ellipse if $u$ has a uniform strain in $\Omega$ for a uniform loading $H$. From the assumption
\[
\bar{M} = 0, 1 \quad \text{and} \quad \bar{N} = 1.
\]
It follows from Lemma 4.1 (b) that
\[
K^*_{\partial \Omega} \left( [e - f] \gamma \psi_1 + (e - f) \gamma \psi_{-1} \right) \\
= -\frac{1}{2} (e + f) \gamma \psi_1 - \frac{1}{2} (e + f) \gamma \psi_{-1} + \alpha_1 \gamma \psi_1 + \bar{\alpha}_1 \gamma \psi_{-1}.
\] (4.13)

We have
\[
e \neq 0 \quad \text{and, hence,} \quad \bar{M} = 1.
\]
Indeed, if $e = 0$, then $f = 0$ from the definition of $f$. It implies $\alpha_1 = 0$ form (4.13). This contradicts the assumption that $\bar{N} = 1$ (which implies $\alpha_1 \neq 0$). Now, from the assumption that $I_2 - \sigma$ is either positive or negative definite we deduce $e \neq f$.

Note that the right-hand side of (4.13) belongs to the linear space spanned by $\{\psi_1, \psi_{-1}\}$. By taking the inner-product with $\psi_{-k}$ for both sides of (4.13) and applying Lemma 3.2 (b), we observe that
\[
(e - f) \gamma \langle K^*_{\partial \Omega} [\psi_1], \psi_{-k} \rangle = 0 \quad \text{for all} \ k \geq 2.
\]
Thanks to Lemma 3.3, $\Omega$ is an ellipse. So we prove the corollary.
5  Proof of the main results

5.1  Proof of Theorem 1.1

We prove (a) separately for the isotropic and anisotropic cases, where (b) and (c) are also proved in the meantime.

**Isotropic case.** We first prove the ‘if’ direction by assuming that $H$ is a harmonic polynomial of degree $N$ and $u$ is a first order polynomial in $\Omega$. In other words,

$$\tilde{\bar{N}} = N, \quad \tilde{\bar{M}} = 1.$$  

From (4.7), we have

$$\mathcal{K}_{\partial \Omega}^* \left[ \bar{\alpha}_1 \gamma \psi_1 + \bar{\alpha}_1 \gamma \psi_{-1} \right]$$

$$= \lambda \left( \bar{\alpha}_1 \gamma \psi_1 + \bar{\alpha}_1 \gamma \psi_{-1} \right) - (\lambda - \frac{1}{2}) \sum_{m=1}^{N} (\alpha_m m \gamma^m \psi_m + \overline{\alpha_m} m \gamma^m \psi_{-m}).$$

By taking the inner-product for both sides with $\psi_{-m}$ and applying Lemma 3.2 (b), we obtain

$$\tilde{\bar{\alpha}}_1 \gamma \langle \mathcal{K}_{\partial \Omega}^* [\psi_1], \psi_{-m} \rangle = \begin{cases} \lambda \tilde{\overline{\alpha}}_1 - (\lambda - \frac{1}{2}) \tilde{\overline{\alpha}}_1 \gamma & \text{for } m = 1, \\ - (\lambda - \frac{1}{2}) \alpha_m m \gamma^m & \text{for } m = 2, \cdots, N, \\ 0 & \text{for } m \geq N + 1. \end{cases}$$

On the other hand, from (3.2), (3.5) and (3.15) we have

$$\tilde{\bar{\alpha}}_1 \gamma \langle \mathcal{K}_{\partial \Omega}^* [\psi_1], \psi_{-m} \rangle = \tilde{\bar{\alpha}}_1 \gamma \frac{m!}{m} \frac{c_1 m}{\gamma^{m+1}} = \tilde{\bar{\alpha}}_1 \frac{m!}{m} \mu_m, \quad m = 1, 2, \cdots.$$  

Hence it holds that

$$\mu_N \neq 0, \quad \mu_m = 0 \quad \text{for } m \geq N + 1,$$

and $\Omega$ is a domain of negative order $N$. Moreover, we have the linear algebraic relations

$$\frac{\mu_1}{\gamma} \tilde{\overline{\alpha}}_1 - 2 \lambda \tilde{\overline{\alpha}}_1 = (1 - 2 \lambda) \tilde{\overline{\alpha}}_1,$$

$$\frac{\mu_m}{\gamma^m} \tilde{\overline{\alpha}}_1 = (1 - 2 \lambda) \tilde{\overline{\alpha}}_1, \quad m = 2, \cdots, N,$$

which are equivalent to

$$\tilde{\overline{\alpha}}_1 = \frac{(1 - 2 \lambda)}{\left| \frac{\mu_1}{\gamma} \right|^2 - 4 \lambda^2} \left[ \frac{\overline{\mu_1}}{\gamma} \tilde{\overline{\alpha}}_1 + 2 \lambda \tilde{\alpha}_1 \right]$$

$$= \frac{(1 - 2 \lambda)}{\left| \frac{\mu_1}{\gamma} \right|^2 - 4 \lambda^2} \left[ \left( \frac{\overline{\mu_1}}{\gamma} + 2 \lambda \right) \text{Re} \{ \alpha_1 \} + i \left( - \frac{\overline{\mu_1}}{\gamma} + 2 \lambda \right) \text{Im} \{ \alpha_1 \} \right]$$

$$= (1 - 2 \lambda) \left( \frac{1}{\tau_1(\lambda)} \text{Re} \{ \alpha_1 \} + i \tau_2(\lambda) \text{Im} \{ \alpha_1 \} \right)$$  

(5.1)
and
\[
\alpha_m = \frac{1}{1 - 2\lambda \gamma^m} \overline{\mu_m \alpha_1} = \left[ \tau_1(\lambda) \text{Re}\{\alpha_1\} - i\tau_2(\lambda) \text{Im}\{\alpha_1\} \right] \frac{\mu_m}{\gamma^m}, \quad m = 2, \ldots, N, \quad (5.2)
\]
where \(\tau_1(\lambda)\) and \(\tau_2(\lambda)\) are given by (1.3). Setting
\[
\alpha_1 = c_1 - ic_2, \quad (5.3)
\]
we have
\[
H(z) = \text{Re} \left\{ \alpha_1 F_1(z) + \sum_{m=2}^{N} \alpha_m F_m(z) \right\}
= \text{Re} \left\{ c_1 F_1(z) - ic_2 F_1(z) + \left[ \tau_1(\lambda)c_1 + i\tau_2(\lambda)c_2 \right] \overline{\psi}(z) \right\}
= \text{const.} + c_1 \text{Re} \left\{ z + \tau_1(\lambda) \overline{\psi}(z) \right\} + c_2 \text{Im} \left\{ z - \tau_2(\lambda) \overline{\psi}(z) \right\}.
\]

Here we used the fact that \(\mu_m = 0\) for \(m \geq N + 1\) and \(F_1(z) = z - a_0\).

We now prove the 'only if' direction. Let us assume that \(\Omega\) is a domain of negative order \(N\) and that the corresponding exterior conformal mapping is
\[
\Phi(w) = \gamma w + \mu_1 w + \frac{\mu_2}{w^2} + \cdots + \frac{\mu_N}{w^N}.
\]
In other words,
\[
\Psi(w) = w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots + \frac{a_N}{w^N}.
\]
Since \(\Omega\) is a smooth domain for such a case, we have from Lemma 3.2 (b)
\[
K_{\partial\Omega}^{*}[\psi_1] = \frac{1}{2\gamma} \sum_{m=1}^{N} \mu_m m \psi_{-m}. \quad (5.4)
\]
Choose any nonzero complex number for \(\alpha_1\). We then set \(\tilde{\alpha}_1, \alpha_2, \ldots, \alpha_N\) as (5.1) and (5.2) and define \(H\) and \(\psi\) as (4.1) and (4.2) with \(\alpha_0 = 0\). Then, it holds that
\[
(1 - 2\lambda) \sum_{m=1}^{N} \overline{\alpha_m} m \gamma^m \psi_{-m} = \tilde{\alpha}_1 \sum_{m=1}^{N} \mu_m m \psi_{-m} - 2\lambda \overline{\tilde{\alpha}_1} \gamma \psi_{-1}.
\]
In other words,
\[
K_{\partial\Omega}^{*}[\tilde{\alpha}_1 \gamma \psi_1] = (\frac{1}{2} - \lambda) \sum_{m=1}^{N} \overline{\alpha_m} m \gamma^m \psi_{-m} + \lambda \overline{\tilde{\alpha}_1} \gamma \psi_{-1}
\]
and, by taking the complex conjugate,
\[
K_{\partial\Omega}^{*}[\overline{\tilde{\alpha}_1} \gamma \psi_{-1}] = (\frac{1}{2} - \lambda) \sum_{m=1}^{N} \alpha_m m \gamma^m \psi_m + \lambda \overline{\tilde{\alpha}_1} \gamma \psi_1.
\]
Therefore we have
\[ K^*_{\partial \Omega}[\bar{\alpha}_1 \gamma \psi_1 + \bar{\alpha}_1 \gamma \psi_{-1}] = (\frac{1}{2} - \lambda) \psi + \lambda \bar{\alpha}_1 \gamma \psi_1 + \lambda \bar{\alpha}_1 \gamma \psi_{-1}. \]

By using the fact that \( \frac{\partial H}{\partial \nu} = (\frac{1}{2} I - K^*_{\partial \Omega})[\psi] \) one can easily show
\[ (\lambda I - K^*_{\partial \Omega})[\varphi] = \frac{\partial H}{\partial \nu} \]
for
\[ \varphi := \psi - \bar{\alpha}_1 \gamma \psi_1 - \bar{\alpha}_1 \gamma \psi_{-1}. \]

The solution to the transmission problem (1.1) is then
\[ u(x) = H(x) + S_{\partial \Omega}[\varphi] = S_{\partial \Omega}[\psi] + S_{\partial \Omega}[\psi - \bar{\alpha}_1 \gamma \psi_1 - \bar{\alpha}_1 \gamma \psi_{-1}] \]
\[ = \frac{1}{2} \left( \bar{\alpha}_1 F_1(z) + \bar{\alpha}_1 F_1(\bar{z}) \right) \]
\[ = \text{const.} + \text{Re}\{\bar{\alpha}_1\}x_1 - \text{Im}\{\bar{\alpha}_1\}x_2 \quad \text{in } \Omega. \]

Hence we complete the proof.

**Anisotropic case.** We first prove the ‘if’ direction by assuming that \( H \) is a harmonic polynomial of degree \( N \) and \( u \) is a first order polynomial in \( \Omega \). In other words,
\[ \bar{N} = N, \quad \bar{M} = 1. \]

As discussed in the proof of Corollary 4.2 we have \( e \neq f \). From (4.8) and Lemma 3.2 (b), it holds that \( K^*_{\partial \Omega}[\psi_1] \in \text{span}(\psi_{-1}, \cdots, \psi_{-N}) \) and \( \Omega \) is a domain of negative order \( N \) thanks to Lemma 3.3.

We remind the reader that equation (4.8) can be written as
\[ K^*_{\partial \Omega}[ar{\psi}^{\text{aniso}}] = \lambda^{\text{aniso}} \bar{\psi}^{\text{aniso}} - \left( \lambda^{\text{aniso}} - \frac{1}{2} \right) \psi^{\text{aniso}} \]
with
\[ \lambda^{\text{aniso}} := -\frac{1}{2}, \]
\[ \psi^{\text{aniso}} := \psi - f \gamma \psi_1 - f \gamma \psi_{-1}, \]
\[ \bar{\psi}^{\text{aniso}} := (e - f) \gamma \psi_1 + (e - f) \gamma \psi_{-1}. \]

In view of the definition of \( \bar{\psi}^{\text{aniso}} \) and the fact that
\[ \psi^{\text{aniso}} = (\alpha_1 - f) \gamma \psi_1 + (\bar{\alpha}_1 - f) \gamma \psi_{-1} + \sum_{m=2}^{N} (\alpha_m \gamma^m \psi_m + \bar{\alpha}_m \gamma^m \psi_{-m}), \]
we can interpret \( \alpha_1 - \overline{f} \) and \( e - \overline{f} \) as \( \alpha_1 \) and \( \overline{\alpha}_1 \) in the isotropic case, respectively. By following the same computation as in the isotropic case, one arrives at the relations (which correspond to (5.1) and (5.2))

\[ e - \overline{f} = (1 - 2\lambda^{\text{aniso}}) \left[ \tau_1 \left( \lambda^{\text{aniso}} \right) \text{Re} \{ \alpha_1 - \overline{f} \} + i \tau_2 \left( \lambda^{\text{aniso}} \right) \text{Im} \{ \alpha_1 - \overline{f} \} \right], \quad (5.5) \]

\[ \alpha_m = \left[ \tau_1 \left( \lambda^{\text{aniso}} \right) \text{Re} \{ \alpha_1 - \overline{f} \} - i \tau_2 \left( \lambda^{\text{aniso}} \right) \text{Im} \{ \alpha_1 - \overline{f} \} \right] \frac{m_m}{\gamma_m}, \quad m = 2, \ldots, N. \quad (5.6) \]

Letting

\[ \alpha_1 - \overline{f} = c_1 - ic_2, \]

one can derive

\[ (I_2 - \sigma) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \text{Re} \{ e - f \} \\ \text{Im} \{ e - f \} \end{bmatrix} = 2 \begin{bmatrix} \text{Re} \{ \tau_1(-1/2) \} - \text{Im} \{ \tau_2(-1/2) \} \\ \text{Im} \{ \tau_1(-1/2) \} + \text{Re} \{ \tau_2(-1/2) \} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \]

and

\[ H(z) - \text{const.} = -S_{\partial \Omega}[\psi](z) = -S_{\partial \Omega}[\overline{f} \gamma \psi_1 + f \gamma \psi_{-1}] - S_{\partial \Omega}[\psi^{\text{aniso}}](z) \]

\[ = f_1 x_1 + f_2 x_2 + c_1 \text{Re} \{ z + \tau_1(-1/2) \gamma \psi(z) \} + c_2 \text{Im} \{ z - \tau_2(-1/2) \gamma \psi(z) \}. \]

We now prove the 'only if' direction. Let us assume that \( \Omega \) is a domain of negative order \( N \). Similar to the isotropic case, we will construct \( H \) with which the corresponding solution \( u \) has a uniform strain in \( \Omega \).

Choose any \((e_1, e_2) \neq (0,0)\) and set \( f \) and \( c \) such that (1.4). We then set \( \alpha_1 = f_1 - if_2 + c_1 - ic_2 \) and \( \alpha_2, \ldots, \alpha_N \) to satisfy (5.6) and define \( H \) and \( \psi \) as (4.1) and (4.2) with constant term zero. Then, it holds that

\[ K^*_\partial \Omega \left[ (e - f) \gamma \psi_1 + (e - f) \gamma \psi_{-1} \right] = -\frac{1}{2} (e + f) \gamma \psi_1 - \frac{1}{2} (e + f) \gamma \psi_{-1} + \psi \quad (5.7) \]

and

\[ H(x) = S_{\partial \Omega}[\psi](x) \quad \text{in} \ \Omega. \quad (5.8) \]

We define

\[ \tilde{u}(x) = \begin{cases} -S_{\partial \Omega} [\overline{\gamma} \psi_1 + e \gamma \psi_{-1}] (x) & \text{for} \ x \in \Omega, \\ H(x) + S_{\partial \Omega}[\varphi](x) & \text{for} \ x \in \mathbb{C} \setminus \Omega \end{cases} \quad (5.9) \]

with

\[ \varphi := \psi - (\overline{e} \gamma \psi_1 + e \gamma \psi_{-1}). \]

It is straightforward to see from (3.11) that

\[ \tilde{u}(x) = \text{const.} + e_1 x_1 + e_2 x_2 \quad \text{for} \ x \in \Omega, \]

and one can easily show that \( \tilde{u} \) satisfies the boundary transmission condition (1.2) due to (5.7) and (5.8). Hence, we complete the proof.

\[ \square \]
5.2 Proof of Theorem 1.2

Proof of (a). Let $H$ be an arbitrary first order polynomial, then it hold that

$$H(x) = \text{const.} + \mathcal{S}_\partial \lambda [\gamma \alpha \psi_1 - \gamma \bar{\alpha} \bar{\psi}_1] \quad \text{in } \Omega$$

for some constant $\alpha \neq 0$ and

$$\nu \cdot \nabla H = \left( \frac{1}{2} I - K^*_\partial \right) [\gamma \alpha \psi_1 + \gamma \bar{\alpha} \bar{\psi}_1]. \quad (5.10)$$

By use of (2.2) we have the relation

$$\mathcal{S}_\partial [\nu \cdot \nabla H] = \mathcal{S}_\partial (\lambda I - K^*_\partial) [\varphi] = \lambda S_\partial [\varphi] - S_\partial K^*_\partial [\varphi] = (\lambda + \frac{1}{2}) S_\partial [\varphi] - D_\partial [S_\partial [\varphi]] -$$

$$= (\lambda + \frac{1}{2}) (u - H) - D_\partial [(u - H)|\partial \Omega]$$

on $\partial \Omega$. (5.11)

Since both sides are harmonic in $\Omega$, we have

$$\mathcal{S}_\partial [\nu \cdot \nabla H] = (\lambda + \frac{1}{2}) (u - H) - D_\partial [(u - H)|\partial \Omega] \quad \text{in } \Omega. \quad (5.11)$$

($\Leftarrow$) If $(\lambda + \frac{1}{2}) (u - H) - D_\partial [(u - H)|\partial \Omega]$ is a harmonic polynomial of degree $N$ in $\Omega$, then so is $\mathcal{S}_\partial [\nu \cdot \nabla H]$. From the discussion at the beginning of section 4 we have

$$\nu \cdot \nabla H \in \text{span} (\psi_N, \psi_{N+1}, \ldots, \psi_{N-1}, \psi_N). \quad (5.12)$$

From (5.10) we obtain

$$K^*_\partial [\psi_1] \in \text{span} (\psi_N, \psi_{N+1}, \ldots, \psi_{N-1}, \psi_N). \quad (5.13)$$

Hence, $\Omega$ is a domain of negative order $N$ from Lemma 3.3

($\Rightarrow$) Assume that $\Omega$ is a domain of negative order $N$. Actually, we can show that $(\lambda + \frac{1}{2}) (u - H) - D_\partial [(u - H)|\partial \Omega]$ is a harmonic polynomial of degree $N$ in $\Omega$ for any uniform loading $H$. From the assumption on $\Omega$, it is smooth and (5.13) holds. For any uniform loading $H$, from (5.10) we have (5.12). From (5.11) we deduce that $(\lambda + \frac{1}{2}) (u - H) - D_\partial [(u - H)|\partial \Omega]$ is a harmonic polynomial of degree $N$ in $\Omega$.

Proof of (b). We will prove by contrapositive. Assume that $H$ is a harmonic polynomial of degree $\tilde{N}$ for some $\tilde{N} < N$ and $u$ is a harmonic polynomial of degree $\tilde{M}$ for some $\tilde{M} \in \mathbb{N}$. We have (4.7) from the discussion in section 4 with $\psi$ and $\bar{\psi}$ defined there.

We also assume that $\Omega$ is a domain of finite negative order $N \geq 2$. Then, it follows from Lemma 3.1 that

$$K^*_\partial [\bar{\psi}] = C \psi_{N\tilde{M}} + C' \psi_{N\tilde{M}} + R \quad \text{with some constants } C, C' \neq 0$$

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and \( R \in \text{span}\{\psi_k : |k| < N\tilde{M}\} \). On the other hand, from (4.2) and (4.5) we have
\[
\psi, \tilde{\psi} \in \text{span}\{\psi_k : |k| \leq \max(\tilde{N}, \tilde{M})\}
\]
and, thus,
\[
\lambda\tilde{\psi} - (\lambda - \frac{1}{2})\psi \in \text{span}\{\psi_k : |k| \leq \max(\tilde{N}, \tilde{M})\}.
\]
From (4.7), one can deduce
\[
N\tilde{M} \leq \max(\tilde{N}, \tilde{M}).
\]
It is not possible to have \( \tilde{M} \leq \tilde{N} \) owing to \( \tilde{N} < N \) and \( \tilde{M} \geq 1 \). Hence, we have \( \tilde{M} > \tilde{N} \). Then, \( N\tilde{M} \leq \tilde{M} \) so that \( N \leq 1 \). This contradicts the assumption \( N \geq 2 \).

6 Conclusion

In this paper we investigated the Eshelby uniformity principle for the anti-plane elasticity based on the series expansion of the boundary integral operators obtained in [14]. We extended the uniformity principle to domains of general shape with polynomial loadings.

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