I discuss some of the forces acting on vortices in charged superfluids, paying particular attention to the way that the Berry and Aharonov-Casher phases combine to reflect the classical magnetohydrodynamics.

1 Introduction

The motion of vortices is the only significant source of dissipation in a superconductor, making it a topic of some technological importance. Much of the modern work on vortex dynamics relates to many-vortex interaction effects so it is rather surprising to find that there is still considerable debate about the forces acting on an individual vortex. This debate was sparked by the claim in papers by Ao, Thouless and Niu\cite{1,2} that there exists a universal, exact expression for the transverse Magnus force on a vortex. Arising from topological effects, the magnitude of this Magnus force should not be affected by impurities or quasiparticles except in that they change the value of the superfluid density.

The Ao-Thouless-Niu papers stimulated a number of authors to re-examine the problem\cite{3,4,5,6,7,8,9} with conclusions ranging between support for their viewpoint, through the suggestion that spectral flow, yet another topological effect, partly cancels the Magnus force, to claims that there are no topological effects at all. It has also led to an ingenious experiment aimed at a direct measurement of the force\cite{10}.

Given the interests of the audience at this conference, I will focus on only one aspect of this problem — the effect, if any, electromagnetic interactions have on the topological Magnus force. In the next section I will review the connection between the topological phase and the Magnus force for neutral superfluids, and then show how the phase accounting is modified by Aharonov-Casher phases due to the motion of the magnetic flux which is tied to the vortices. In the third section I will show how the various phase cancellations reflect a purely classical re-routing of the momentum flux from the charged superfluid to the background lattice. A final discussion section will, for completeness, briefly review how spectral flow induces relaxation effects that need to be included to complete the picture.

2 Berry and Aharonov-Casher phases
2.1 Neutral Fluids

Variation of the action

\[ S = \int dt d^3x \left\{ -\frac{i}{2}(\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*) + \frac{1}{2m} |\nabla \varphi|^2 + \frac{\lambda}{2}(|\varphi|^2 - \rho_0)^2 \right\} \]

(1)
gives rise to a non-linear Schrödinger equation

\[ i \partial_t \varphi = -\frac{1}{2m} \nabla^2 \varphi + \lambda(|\varphi|^2 - \rho_0) \varphi \]

(2)
describing fluctuations of the scalar field \( \varphi \) about a stationary configuration with \(|\varphi|^2 = \rho_0\). This equation is called the Gross-Pitaevskii (GP) equation and is often used as a simple model for the motion of a Bose condensate. Long ago Madelung\[^{12}\] showed that the real and imaginary parts of the Schrödinger equation can be interpreted as the equations of mass and momentum conservation for a simple fluid, so it is no surprise that vortex solutions to the GP equation are consistent with Kelvin’s circulation theorem from elementary fluid dynamics — i.e the vortices are advected along with the flow. For example, in three dimensions, there exists a family of smoke-ring-like vortex solutions to the GP equation which move along their symmetry axis without change of radius\[^{13}\]. Any attempt to force a vortex to move with respect to the flow requires the application of a transverse lift force. In the superconductivity literature this is usually referred to as the Magnus force\[^{14}\].

We can simultaneously obscure and illuminate the existence of this force by some cosmetic rewriting of the action. We begin setting \( \varphi = \sqrt{\rho} e^{i\theta} \) to get

\[ S = \int dt d^3x \left\{ \rho \partial_t \theta + \frac{\rho_0}{2m} (\nabla \theta)^2 + \frac{\lambda}{2} \left( \rho - \rho_0 \right)^2 + \ldots \right\} \]

(3)

(where the dots indicate a presently uninteresting term depending on the gradient of \( \rho \)). Next we integrate over \( \rho \) to find (again ignoring the gradient of \( \rho \) terms)

\[ S = \int dt d^3x \left\{ \rho_0 \partial_t \theta + \frac{\rho_0}{2m} (\nabla \theta)^2 - \frac{1}{2\lambda} \left( \partial_t \theta + \frac{1}{2m} \nabla v_s^2 \right)^2 \right\} \]

(4)

where \( v_s = \frac{1}{m} \nabla \theta \) is the superfluid velocity.

The first term is a total derivative and therefore does not contribute to the classical equation of motion for \( \theta \). If we temporarily suppress the \( v_s^2 \) part of the third term, variation of the action gives rise to a wave equation which describes sound waves propagating in the fluid. We should not, however, use this truncated equation for the vortex dynamics. Wave-equation dynamics for an angular variable like \( \theta \) appears in the relativistic form of the \( X - Y \) model and it is well known that in such a model a vortex loop will rapidly shrink and disappear\[^{15}\]. Throwing away the higher order terms in \( v_s \), although common in the superconductivity literature\[^{14}\], is therefore not safe when one is interested in vortex motion. This because omitting these terms destroys galilean invariance\[^{17}\].

It is not safe to discard the first term \( \rho_0 \partial_t \theta \) either\[^{16}\]. If one wishes to use \( S \) to compute the partition function via an imaginary time path integral over configurations with periodicity \( \beta = 1/kT \), the first term will give rise to a topological phase.\[^{18}\]

This is done, for example, in the otherwise excellent textbook by Popov\[^{18}\].
Although the field $\phi$ must return to its original value at $\tau = \beta$, the angle $\theta$ will in general have wound through $2\pi N$ where $N$ is an integer. In two dimensions the order parameter phase associated with a vortex contains a part $\theta(r, t) = \text{Arg}(r - r_L(t))$ therefore moving the vortex in a closed path will contribute a phase factor equal to

$$\exp\{2\pi i \rho_0 (\text{Area})\}$$

where (Area) is the area enclosed by the vortex trajectory. This is exactly the same phase factor that would be generated by a charged particle with interaction action $\oint e \dot{\mathbf{x}} \cdot A_\mu \, dt$ moving in a uniform magnetic field. It must therefore signal the existence a transverse “Lorentz” force of magnitude $2\pi \rho_0 V_{\text{vortex}}$. Taking into account that the circulation around the vortex is $\kappa = 2\pi/m$ this is exactly the expected magnitude of the classical Magnus force $\rho_0 m \kappa V_{\text{vortex}}$.

This connection between this topological “Berry” phase and the Magnus force in a superfluid was pointed out in Ref.1. It became the motivation for a more general theorem on the existence of transverse forces on vortices.

2.2 Charged Fluids

What happens to the vortex-induced Berry phase when the fluid is composed of charged particles? To find out it is simplest to perform a duality map which raises the status of the vortices from mere defects in the order parameter to the central objects of discussion. There are many references for such maps, but since the notation and inspiration for this section comes directly from a paper written by a member of the audience I will recommend that for an introduction.

For geometric simplicity I will restrict myself to two dimensions. In the following roman subscripts such as $a$ will run over the two space dimensions while greek subscripts such as $\alpha$ will run over both euclidean time ($\alpha = 0$) and space dimensions.

When we couple our scalar field to a gauge field the Lagrange density becomes

$$L = \frac{1}{2}(\phi^* (\partial_\mu - ie A_\mu) \phi - \phi (\partial_\mu + ie A_\mu) \phi^*) + \frac{1}{2m} (\partial_a - ie A_a) \phi|^2 + \frac{\lambda}{2}(|\phi|^2 - \rho_0)^2 + \frac{1}{4} F^2. \quad (6)$$

Set $\phi = f e^{i \theta}$ to write this as

$$L = if^2 (\partial_\mu \theta - e A_\mu) + \frac{1}{2m} (\partial_a f^2)^2 + \frac{f^2}{2m} (\partial_a \theta - e A_a)^2 + \frac{\lambda}{2} (f^2 - \rho_0)^2 + \frac{1}{4} F^2. \quad (7)$$

Now introduce a pair of Hubbard-Stratovitch fields $C_a$ and also set $C_0 = f^2$ to get

$$L \rightarrow L' = i C_\mu (\partial_\mu \theta - e A_\mu) + \frac{1}{8m} (\partial_a C_0)^2 C_0 + \frac{m}{2C_0} C_a^2 + \frac{\lambda}{2} (C_0 - \rho)^2 + \frac{1}{4} F^2. \quad (8)$$

Next we isolate the vortex part of the phase $\theta$ by writing $\theta = \bar{\theta} + \eta$ where $\bar{\theta} = \sum_i \text{Arg}(r - r_i(t))$ is the singular part of the phase due to vortices at $r_i$, while $\eta$ is the remaining non-singular part. Integration over $\eta$ enforces the mass conservation equation $\partial_\tau C_0 + \partial_a C_a = 0$. This is automatically satisfied if we write

$$C_\mu = \epsilon_{\mu\nu\sigma} \partial_\nu b_\sigma. \quad (9)$$
Regarding $b_\sigma$ as a gauge potential then suggests renaming

$$C_\alpha \rightarrow \tilde{E}_\alpha \quad C_0 = \tilde{H},$$

(10)
a set of pseudo electric and magnetic fields. After defining

$$K_\mu = \epsilon_{\mu\nu\sigma} \partial_\nu \partial_\sigma \tilde{\theta}$$

(11)
to be the vortex 3-current density we find that the Lagrange den sity takes the form

$$L' = ib_\mu (K_\mu - eF_\mu^\ast) + \frac{1}{2\rho_0} |\tilde{E}|^2 + \frac{1}{2}(\tilde{H} - \rho_0)^2 + \frac{1}{4} F^2 + \ldots$$

(12)

where the dots represent the same higher order gradients of the density that were omitted in the earlier section. Assuming $\rho \approx \rho_0$, we can write the Lagrange density as

$$L' = ib_\mu (K_\mu - eF_\mu^\ast) + \frac{1}{2\rho_0} |\tilde{E}|^2 + \frac{1}{2}(\tilde{H} - \rho_0)^2 + \frac{1}{4} F^2$$

(13)

where $F_\mu^\ast = \frac{1}{2} \epsilon_{\mu\nu\sigma} F^{\nu\sigma}$ is the dual of the real electromagnetic flux.

We see that the vortex current is coupled to a gauge field $\tilde{E}$ that apparently mediates coulomb interactions between vortices but actually accounts for the attractive and repulsive forces due to the kinetic energy of the fluid, and to a magnetic field $\tilde{H}$ which has an equilibrium value $\rho_0$ even when the vortex is at rest. Motion with respect to this background field gives rise to a Lorentz force which is just the Magnus force in this pseudo-field language. We also see that the source for the fields is not the bare vortex current but the vortex current less the flux of the genuine magnetic field that accompanies the vortex core. As in any Abrikosov flux tube solution, this magnetic flux completely screens the vortex current and appears to eliminate both the “coulomb” interaction between the vortices and the Magnus force. The coulomb screening is a genuine effect – the dual of the Anderson-Higgs mechanism. Is the Magnus force banished as well? The answer is no! We have omitted an important term in Eq.3. This is the interaction of the electromagnetic field with the background ions. Consideration of this is necessary if we are not to have infinite coulomb energy. Including an $L_{\text{ions}} = i e \rho_{\text{ion}} A_0$ of the standard form and writing the ionic 3-current (a static charge of course) as the curl of a field $b_{\mu}^{\text{ion}}$ the final form of the vortex field interaction becomes

$$ib_\mu (K_\mu - eF_\mu^\ast) + ie b_{\mu}^{\text{ion}} F_\mu^\ast + \frac{1}{2\rho_0} |\tilde{E}|^2 + \frac{1}{2}(\tilde{H} - \rho_0)^2 + \frac{1}{4} F^2.$$  

(14)

When the fluid is at rest with respect to the background charges away from the vortices (which it has to be unless one is within a penetration depth of the boundary of the system) the two terms prortional to $F_\mu^\ast$ cancel and the Magnus force is restored.

In terms of topological phases, what is happening is that the Berry phase from the superfluid is being supplemented by two Aharonov-Casher phases arising from moving the magnetic flux line round the charges in the system. One Aharonov-Casher phase comes from moving the flux through the mobile fluid charges and one
from moving it through the static background ion charges. Coulomb interactions force the net charge inside any macroscopic region to be zero so the two Aharonov-Casher phases must cancel.

The results of this section differ from those in some recent papers. Although in the end they obtain the same total force on the vortex, the authors of these papers claim that inclusion of electromagnetic interactions causes the topological phase terms in the effective action to cancel. This cancellation originates in the form they chose for the interaction with the background charge. I have chosen to represent it as \( ie\rho_{\text{ion}}A_0 \) whereas these papers argue that gauge invariance requires us to write this interaction term as a gauge covariant derivative \( ie\rho_{\text{ion}}(A_0 - \partial_0\theta) \), where \( \theta \) is the phase of the electron order parameter. Then, because \( \rho_{\text{ion}} = \rho_0 \) the terms in \( \partial_0\theta \) cancel. I would argue that since the background ion current is conserved separately from the electron current, the conventional form is gauge invariant without any additional term. Similar opinions have been expressed by Zhu, Gaitan and Volovik. In the next section I will argue that my choice of interaction is the correct one by relating the phase cancellations to the actual mechanical forces acting on the system.

3 Kutta-Joukowski Theorem for Charged Fluids

The original example exhibiting the Aharonov-Chasher phase was constructed to be similar to the Bohm-Aharonov effect in that there was a quantum effect even though the classical force acting on the moving object vanished. However, just as the Bohm-Aharonov phase in a region with non-vanishing magnetic field becomes path dependent and signals the existence of the Lorentz force, so the Aharonov-Casher phase in a region of non-vanishing charge density becomes path dependent and signals that moving flux through a charge distribution requires application of transverse force — the reaction force to the Lorentz force the charges feel in the moving magnetic field. (More accurately the charges feel the electric field arising from the Lorentz-transformed magnetic field). We can therefore avoid all controversy over cancelling phases and gauge invariance by focussing on the equivalent classical forces experienced by the fluid and the background ions.

3.1 Neutral Fluids

To understand the forces on the vortex, or equivalently the momentum flux into and out of the vortex core, we may as well look at the forces on a hypothetical solid body occupying the same region as the core. We will begin by reviewing the traditional derivation of the lift force on an aerofoil with circulation \( \kappa \) which is being held stationary in electrically neutral, two-dimensional, incompressible potential flow.

This force is most easily obtained from the momentum flux tensor.

\[
T_{ij} = \rho m v_i v_j + g_{ij} P. \tag{15}
\]

Since we are interested in steady, irrotational motion with constant density we may use Bernoulli’s theorem, \( P + \frac{1}{2}\rho m|v|^2 = \text{const.} \), to substitute \(-\frac{1}{2}\rho m|v|^2\) in place...
of $P$. (the constant will not affect the momentum flow). With this substitution $T_{ij}$ becomes a traceless symmetric tensor

$$T_{ij} = \rho m(v_iv_j - \frac{1}{2}g_{ij}|v|^2). \tag{16}$$

In two dimensions it is convenient to use complex coordinates $z = x + iy$ where the metric tensor becomes $g_{zz} = g_{zz} = 0$, $g_{xx} = \frac{1}{2}$ and $g_{yy} = g_{yy} = 2$ etc. In these coordinates $v_z = \frac{1}{2}(v_x - iv_y)$ and

$$T \equiv T_{zz} = \frac{1}{4}(T_{xx} - T_{yy} - 2iT_{xy}) = \rho m(v_z)^2. \tag{17}$$

$T$ is the only component of $T_{ij}$ we will need to consider. $T_{zz}$ is simply $T$ while $T_{xx} = T_{yy}$ because $T_{ij}$ is traceless.

Any body-force $f_i$ acting on the fluid is given by the divergence of the momentum flux tensor

$$f_i = g^{kl}\partial_k T_{li}. \tag{18}$$

or, in our coordinates,

$$f_z = g^{zz}\partial_z T_{zz} + g^{z\bar{z}}\partial_{\bar{z}} T_{zz} = 2\partial_z T. \tag{19}$$

From this we see that, in a steady flow, the momentum flux $P$ out of a region $\Omega$ must be given by

$$\dot{P}_z = \frac{1}{2i}\int f_z d\bar{z}dz = \frac{1}{i}\int \partial_z T d\bar{z}dz = \frac{1}{i}\oint_{\partial \Omega} T dz. \tag{20}$$

When there is no body-force, $T$ is analytic and the integral will be independent of the choice of contour $\partial \Omega$.

To apply this result to our aerofoil we should take $\partial \Omega$ to be its boundary. Then $\dot{P}_z$ becomes minus the integral of the pressure over the boundary of the body, i.e the force exerted on the fluid by the aerofoil. Evaluating the integral is not immediately possible because the velocity $V_z$ will be a complicated function of the shape of the body. We can, however, exploit the contour independence of the integral and evaluate the integral over a contour encircling the aerofoil at large distance where the flow field takes the form

$$v_z = U_z + \frac{\kappa}{4\pi i z} + O\left(\frac{1}{z^2}\right). \tag{21}$$

To confirm that this flow has the correct circulation we compute

$$\oint v \cdot d\mathbf{r} = \oint v_z dz + \oint v_{\bar{z}} d\bar{z} = \kappa. \tag{22}$$

Substituting in Eq.21 we find that

$$\dot{P}_z = -i\rho m \kappa U_z. \tag{23}$$
In conventional coordinates the reaction force on the body $F = -\dot{P}$ is

$$
F_x = \rho m k U_y, \\
F_y = -\rho m k U_x.
$$

(24)

The body therefore exerts a transverse lift force on the fluid proportional to the product of the circulation with the asymptotic velocity. This is the Kutta-Joukowski theorem.

### 3.2 Charged Fluids

How is this result modified when the neutral fluid is replaced by one composed of charged particles? We wish to model a charged superfluid where the superfluid velocity is obtained from the order parameter phase, $\theta$, via the equation

$$
v = \frac{1}{m} (\nabla \theta - eA).
$$

(25)

The flow is therefore no longer irrotational but instead has vorticity

$$
\omega = \nabla \times v = -\frac{e}{m} \nabla \times A = -\frac{e}{m} B.
$$

(26)

The relation $eB + m\omega = 0$ gives rise to the Meissner effect: using the Maxwell equation $\nabla \times H = J$ with $J = \rho e v$ we find

$$
\nabla^2 B + \mu_0 \rho e^2 \frac{B}{m} = 0.
$$

(27)

Eq.27 implies that both the magnetic flux and the vorticity decay exponentially away from a flux-creating disturbance. A vortex defect is such a disturbance. The $2\pi$ phase winding produces a solenoid-like current near the vortex core. This current creates a magnetic field which in turn induces a local vorticity that tends to screen that of the defect over the penetration length $\lambda_0 = \sqrt{m/(\rho e^2 \mu_0)}$. The circulation about the vortex is therefore path-dependent and tends to zero as the integration path becomes further from the vortex core.

From the fact that the induced vorticity exactly screens the circulation round the bare vortex we can find the net flux

$$
\Phi = \int B_3 \, dx \, dy = -\frac{m}{e} \int \omega \, dx \, dy = \frac{m}{e} \frac{2\pi}{m} = \frac{2\pi}{e}.
$$

(28)

For the Cooper pairs of a BCS superfluid $m = 2\times$ (mass of electron) and $e = 2\times$ (charge of electron), so this flux quantum is half that appearing in the quantum Hall effect.

As before, we must not forget the charge of the background ions. Without a background charge $\rho_{\text{ion}} \approx \rho$ there would be a huge Coulomb field. Indeed any deviation from equality between the two charges will tend to zero over a Debye screening length so we will assume that $\rho_{\text{ion}} = \rho$ everywhere. Further any tendency of the superfluid to flow relative to this background charge will create a magnetic
field and the resulting Meissner effect will restrict the flow to within a penetration length $\lambda_0$ of the boundary of the system. To avoid this current inhomogeneity and to obtain a theorem analogous to that of Kutta-Joukowski we will therefore assume that the asymptotic fluid flow $U$ is equal to that of the rigid motion background ions. This means that we are working in the rest frame of the vortex which is itself moving at a steady velocity $-U$ with respect to both the asymptotic fluid and the background ion crystal. We also assume that our body $\Omega$ contains static charge with density equal to that of the surrounding fluid.

Despite the non-zero vorticity it is easy to see that the momentum flux tensor remains $T_{ij} = \rho m (v_i v_j - \frac{1}{2} g_{ij} |v|^2)$. The momentum supplied by the aerofoil is therefore still given by

$$\dot{P}_z = \frac{1}{i} \oint_{\partial\Omega} T \, dz,$$  \hspace{1cm} (29)

but now $T$ is no longer analytic. Indeed we have that

$$\partial_z T = \frac{i}{2} e \rho B v_z.$$  \hspace{1cm} (30)

We must therefore take $\partial\Omega$ as the boundary of the aerofoil. (From now on we write $B$ for $B_3$ since this is the only non-zero component of the field). Although we cannot use analyticity to send the integration contour off to infinity, we can still integrate by parts and find

$$\frac{1}{i} \int_{\partial\Omega} T \, dz = -\frac{1}{i} \int_{\mathbb{R}^2 \setminus \Omega} \partial_z T \, d\vec{z} \, dz$$  \hspace{1cm} (31)

$$= -\frac{e \rho}{2} \int_{\mathbb{R}^2 \setminus \Omega} B v_z \, d\vec{z} \, dz.$$  \hspace{1cm} (32)

This last expression is simply the Lorentz force on the fluid outside the aerofoil.

We now note that the Maxwell equation $\nabla \wedge H = J$ is in our complex coordinates

$$i \partial_z B = e \mu_0 \rho (v_z - U_z) \quad \text{in} \quad \mathbb{R}^2 \setminus \Omega$$

$$= -e \mu_0 \rho U_z \quad \text{in} \quad \Omega.$$  \hspace{1cm} (33)

So

$$\dot{P}_z = -\frac{i}{2 \mu_0} \int_{\mathbb{R}^2 \setminus \Omega} (B \partial_z B) \, d\vec{z} \, dz - \frac{e \rho}{2} \int_{\mathbb{R}^2 \setminus \Omega} B U_z \, d\vec{z} \, dz.$$  \hspace{1cm} (34)

The last term in Eq.34 is independent of the flow field outside the body but still depends on details of the system through the domain of integration. We will see that we can combine both terms to get an expression for the lift that is completely independent of any such details.

We observe that

$$-\frac{i}{2 \mu_0} \int_{\mathbb{R}^2 \setminus \Omega} (B \partial_z B) \, d\vec{z} \, dz = -\frac{i}{2 \mu_0} \int_{\mathbb{R}^2 \setminus \Omega} \partial_z \frac{1}{2} B^2 \, d\vec{z} \, dz$$

$$= -\frac{i}{2 \mu_0} \oint_{\partial\Omega} \left( \frac{1}{2} B^2 \right) \, d\vec{z}.$$  \hspace{1cm} (35)
\[
\begin{align*}
\dot{P}_z &= -\frac{i}{2\mu_0} \int_\Omega \partial_z \frac{1}{2} B^2 \, dz \, d\vec{r} \\
&= \frac{1}{2} \int_\Omega e \rho \, B U_z \, dz \, d\vec{r}.
\end{align*}
\]  

(35)

Putting the two parts together then gives

\[
\dot{P}_z = \frac{1}{2} e \rho \int_{\mathbb{R}^2} B U_z \, dz \, d\vec{r} = -ie \rho \Phi U_z
\]  

(36)

This expression for the lift force on the aerofoil depends only on the total flux and the asymptotic current. Taking into account the relationship between the neutral-fluid quantum of circulation and the flux in the Abrikosov vortex we see that the force on the aerofoil, or equivalently the momentum flux into or out of a vortex, is unchanged by the introduction of electromagnetic interactions.

What is happening is that in the neutral case momentum is flowing out of the vortex and increasing the net momentum of the fluid. In the charged case the moving vortex supplies exactly the same amount of momentum to the fluid, but the magnetic field acts as a clutch coupling the fluid to the background ions. Any relative motion creates a magnetic field that tends to inhibit it. The momentum is therefore transferred to the background ions. That is why the force the aerofoil exerts on the fluid is exactly equal to the Lorentz force the magnetic field exerts on the background ions as a result of their being dragged through the vortex’s flux tube. A similar mechanism occurs in the quantum Hall effect where motion of either a Laughlin quasiparticle or a skyrmion transfers momentum to the electrons, and they in turn hand it on to the magnetic field. An alternative way of thinking of the same physics is to follow Nozieres and Vinen and realise that in the charged fluid the circulation, and hence the Magnus force, is reduced as we look at contours at larger and larger distance from the vortex. The lost Magnus force is however exactly compensated by the Lorentz force on the fluid within the contours. This re-routing of the momentum exactly mirrors the phase cancellations described in section 2.

4 Discussion

We have treated the superfluid as if it were a Bose condensate of Cooper pairs and therefore modeled its motion by the GP equation. This model differs in several ways from BCS electrons in a real metal. Firstly the GP equation is galilean invariant. This, in my opinion, is not a problem. One often models electrons in a metal by taking a parabolic band dispersion curve \( \epsilon = k^2/2m^* \), and the resulting pseudo-galilean invariance provides a useful check on one’s computation.

Secondly some people object to modeling the dynamics by a first-order equation at all. For the neutral case this should not be an issue. It has been shown by various authors that this is the correct form of the low energy dynamics for a galilean invariant BCS system at \( T = 0 \). The quantity \( \varphi \) appearing in the GP equation is not, however, the BCS order parameter. It is instead the combination \( \varphi = \sqrt{\rho} e^{i\theta} \), where \( \theta \) is the phase of the order parameter and \( \rho \) is the fluid density.
You may have noticed that throughout this talk I have never referred to the scalar field $\phi$ as the “order parameter”. I have only referred its phase as the “phase of the order parameter”! These variables are those associated with the low-energy Goldstone mode. The magnitude of the BCS order parameter $|\Delta|$ is not associated with Goldstone-mode dynamics and so does not appear in the low-energy effective action.

Finally, and most importantly, BCS vortices have internal dynamics associated with quasiparticle core states. Volovik suggested that there should be a novel axial-anomaly driven spectral flow among these states in a moving vortex, and that this spectral flow would cancel most of the Magnus force. It turns out that once one takes the discreteness of the core states into account then one must include relaxation processes in order to see their effect. With relaxation included this turns out to be a previously known effect known as the Kopnin-Kravtsov (KK) force, a sort of transverse friction which exchanges momentum with the crystal lattice by scattering of core quasiparticles off lattice defects. How does this extra transverse force fit in with the claims of Ao et al. that the Magnus force is all there is? The theorem of Ao-Thouless and Niu states that if a vortex moves against the flow, then momentum enters the superfluid at exactly the rate given by the Magnus force. When we include the KK friction a vortex placed in a flow field will find itself moving at a velocity different from the asymptotic flow. Momentum must therefore enter the fluid at exactly the rate given by the theorem. In order to solve for the relative speed of the vortex and the fluid, though, we need another expression for the rate at which momentum is entering the system. The actual mechanism by which the momentum is being gained is that momentum is being transfered to the core states from the lattice by KK scattering, and from there flows out into the fluid in accord with Eq. 20. The rate of KK scattering depends on the relative velocity of the vortex and the crystal lattice, so equating these two expressions for the rate of momentum transfer determines the steady state vortex velocity in terms of the relative velocity of the lattice and the flow field.

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