Limit theorems for a class of stationary increments Lévy driven moving averages

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Abstract

In this paper we present some new limit theorems for power variation of $k$th order increments of stationary increments Lévy driven moving averages. In the infill asymptotic setting, where the sampling frequency converges to zero while the time span remains fixed, the asymptotic theory gives very surprising results, which (partially) have no counterpart in the theory of discrete moving averages. More specifically, we will show that the first order limit theorems and the mode of convergence strongly depend on the interplay between the given order of the increments, the considered power $p > 0$, the Blumenthal–Getoor index $\beta \in (0, 2)$ of the driving pure jump Lévy process $L$ and the behaviour of the kernel function $g$ at 0 determined by the power $\alpha$. First order asymptotic theory essentially comprises three cases: stable convergence towards a certain infinitely divisible distribution, an ergodic type limit theorem and convergence in probability towards an integrated random process. We also prove the second order limit theorem connected to the ergodic type result. When the driving Lévy process $L$ is a symmetric $\beta$-stable process we obtain two different limits: a central limit theorem and convergence in distribution towards a $(1 - \alpha)\beta$-stable totally right skewed random variable.

Key words: Power variation, limit theorems, moving averages, fractional processes, stable convergence, high frequency data.

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1 Introduction and main results

In the recent years there has been an increasing interest in limit theory for power variations of stochastic processes. Power variation functionals and related statistics play a major role in analyzing the fine properties of the underlying model, in stochastic integration concepts

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and statistical inference. In the last decade asymptotic theory for power variations of various classes of stochastic processes has been intensively investigated in the literature. We refer e.g. to [6, 24, 25, 31] for limit theory for power variations of Itô semimartingales, to [4, 5, 17, 21, 30] for the asymptotic results in the framework of fractional Brownian motion and related processes, and to [15, 16, 39] for investigations of power variation of the Rosenblatt process.

In this paper we study the power variation for a class of stationary increments Lévy driven moving averages. More specifically, we consider an infinitely divisible process with stationary increments \( (X_t)_{t \geq 0} \), defined on a probability space \((\Omega, \mathcal{F}, P)\), given as

\[
X_t = \int_{-\infty}^t \left\{ g(t - s) - g_0(-s) \right\} dL_s.
\]  

(1.1)

Here \( L = (L_t)_{t \in \mathbb{R}} \) is a symmetric Lévy process on \( \mathbb{R} \) with \( L_0 = 0 \) and without Gaussian component. Furthermore, \( g, g_0 : \mathbb{R} \to \mathbb{R} \) are deterministic functions vanishing on \(( -\infty, 0)\). In the further discussion we will need the notion of Blumenthal–Getoor index of \( L \), which is defined via

\[
\beta := \inf \left\{ r \geq 0 : \int_{-1}^1 |x|^r \nu(dx) < \infty \right\} \in [0, 2],
\]  

(1.2)

where \( \nu \) denotes the Lévy measure of \( L \). When \( g_0 = 0 \), the process \( X \) is a moving average, and in this case \( X \) is a stationary process. If \( g(s) = g_0(s) = s^\alpha \), \( X \) is a so called fractional Lévy process. In particular, when \( L \) is a \( \beta \)-stable Lévy motion with \( \beta \in (0, 2) \), \( X \) is called a linear fractional stable motion and it is self-similar with index \( H = \alpha + 1/\beta \); see e.g. [31] (since in this case the stability index and the Blumenthal–Getoor index coincide, they are both denoted by \( \beta \)).

Probabilistic analysis of stationary increments Lévy driven moving averages such as semimartingale property, fine scale structure and integration concepts, have been investigated in several papers. We refer to the work of [7, 8, 9, 10, 27] among many others. However, only few results on the power variations of such processes are presently available. Exceptions to this are [8, Theorem 5.1] and [19, Theorem 2]; see Remark 3.2 for a closer discussion of a result from [8, Theorem 5.1]. These two results are concerned with certain power variations of fractional Lévy process and have some overlap with our Theorem 1.1(ii) for the linear fractional stable motion, but we apply different proofs. The aim of this paper is to derive a rather complete picture of the first order asymptotic theory for power variation of the process \( X \), and, in some cases, the associated second order limit theory. We will see that the type of convergence and the limiting random variables/distributions are quite surprising and novel in the literature. Apart from pure probabilistic interest, limit theory for power variations of stationary increments Lévy driven moving averages give rise to a variety of statistical methods. In particular, the theoretical results can be applied to identify and estimate the parameters \( \alpha \) and \( \beta \) of the model (cf. Section 3.2). We refer to e.g. [8, 15, 16, 20] for related statistical procedures. Furthermore, the asymptotic results provide a first step towards limit theory for power variation of stochastic processes, which contain \( X \) as a building block. In this context let us mention stochastic integrals with respect to \( X \) and Lévy semi-stationary processes, which have been introduced in [3].
To describe our main results we need to introduce some notation and a set of assumptions. In this work we consider the $k$th order increments $\Delta_{i,k}^n X$ of $X$, $k \in \mathbb{N}$, that are defined by

$$\Delta_{i,k}^n X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-j)/n}, \quad i \geq k.$$ 

For instance, we have that $\Delta_{i,1}^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$ and $\Delta_{i,2}^n X = X_{\frac{i}{n}} - 2X_{\frac{i-1}{n}} + X_{\frac{i-2}{n}}$. Our main functional is the power variation computed on the basis of $k$th order filters:

$$V(p;k)_n := \sum_{i=k}^{n} |\Delta_{i,k}^n X|^p, \quad p > 0. \tag{1.3}$$

Now, we introduce the following set of assumptions on $g$ and $\nu$:

**Assumption (A):** The function $g : \mathbb{R} \to \mathbb{R}$ satisfies $g \in C^k((0, \infty))$ and

$$g(t) \sim c_0 t^\alpha \quad \text{as } t \downarrow 0 \quad \text{for some } \alpha > 0 \text{ and } c_0 \neq 0, \tag{1.4}$$

where $g(t) \sim f(t)$ as $t \downarrow 0$ means that $\lim_{t \downarrow 0} g(t)/f(t) = 1$. For some $\theta \in (0,2]$, $\limsup_{t \to \infty} \nu(x \mid |x| \geq t)^\theta < \infty$ and $g \circ \nu$ is a bounded function in $L^\theta(\mathbb{R}_+)$. Finally, there exists a $\delta > 0$ such that $|g^{(k)}(t)| \leq K t^{\alpha-k}$ for all $t \in (0, \delta)$, $|g'|$ and $|g^{(k)}|$ are in $L^\theta((0, \infty))$ and decreasing on $(\delta, \infty)$.

**Assumption (A-log):** In addition to (A) suppose that $\int_0^{\infty} |g^{(k)}(s)|^\theta \log(1/|g^{(k)}(s)|) \, ds < \infty$.

Assumption (A) ensures in particular that the process $X$ is well-defined, cf. Section 4. When $L$ is a $\beta$-stable Lévy process, we always choose $\theta = \beta$ in assumption (A). Before we introduce the main results, we need some more notation. Let $h_k : \mathbb{R} \to \mathbb{R}$ be given by

$$h_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x-j)^\alpha, \quad x \in \mathbb{R}, \tag{1.5}$$

where $y_+ = \max\{y, 0\}$ for all $y \in \mathbb{R}$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by $(L_t)_{t \geq 0}$, $(T_m)_{m \geq 1}$ be a sequence of $\mathbb{F}$-stopping times that exhausts the jumps of $(L_t)_{t \geq 0}$. That is, $\{T_m(\omega) : m \geq 1\} \cap \mathbb{R}_+$ = $\{t \geq 0 : \Delta L_t(\omega) \neq 0\}$ and $T_m(\omega) \neq T_n(\omega)$ for all $m \neq n$ with $T_m(\omega) < \infty$. Let $(U_m)_{m \geq 1}$ be independent and uniform $[0,1]$-distributed random variables, defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space, which are independent of $\mathcal{F}$.

The following two theorems summarize the first and second order limit theory for the power variation $V(p;k)_n$. We would like to emphasize part (i) of Theorem 1.1 and part (i) of Theorem 1.2 which are quite unusual probabilistic results. We refer to [1], [33] and to Section 3 for the definition of $\mathcal{F}$-stable convergence in law which will be denoted $\overset{\mathcal{F}}{-\Rightarrow}$.

**Theorem 1.1 (First order asymptotics).** Suppose (A) is satisfied and assume that the Blumenthal–Getoor index satisfies $\beta < 2$. We obtain the following three cases:
we have the following two cases:

(i) Suppose that \((A\text{-log})\) holds if \(\theta = 1\). If \(\alpha < k - 1/p\) and \(p > \beta\) we obtain the \(\mathcal{F}\)-stable convergence

\[
n^{\alpha p}V(p; k) n^{\frac{p}{2}} \sum_{m: T_m \in [0,1]} |\Delta T_m|^p V_m \quad \text{where} \quad V_m = \sum_{l=0}^{\infty} |h_k(l + U_m)|^p. \tag{1.6}
\]

(ii) Suppose that \(L\) is a symmetric \(\beta\)-stable \(\text{Lévy process}\) with scale parameter \(\sigma > 0\), i.e. the characteristic function of \(L_1\) is given by \(\mathbb{E}[\exp(\mathrm{i}uL_1)] = \exp(-\sigma^{|u|^\beta})\). If \(\alpha < k - 1/\beta\) and \(p < \beta\) then it holds that

\[
n^{-1+\rho(p+1/\beta)}V(p; k) n \xrightarrow{\mathbb{P}} m_p \tag{1.7}
\]

where \(m_p = |c_0|^p \sigma^p (\int_\mathbb{R} |h_k(x)|^\beta dx)^p/\beta \mathbb{E}[|Z|^p]\) and \(Z\) is a symmetric \(\beta\)-stable random variable with scale parameter \(1\).

(iii) Suppose that \(p \geq 1\). If \(p = \theta\) suppose in addition that \((A\text{-log})\) holds. For all \(\alpha > k - 1/(\beta \vee p)\) we deduce that

\[
n^{-1+\rho k}V(p; k) n \xrightarrow{\mathbb{P}} \int_0^1 |F_u|^p du \tag{1.8}
\]

where \((F_u)_{u \in \mathbb{R}}\) is a measurable process satisfying

\[
F_u = \int_{-\infty}^u g^{(k)}(u - s) dL_s \quad \text{a.s. for all} \quad u \in \mathbb{R} \quad \text{and} \quad \int_0^1 |F_u|^p du < \infty \quad \text{a.s.}
\]

We remark that, except the critical cases where \(p = \beta, \alpha = k - 1/p\) and \(\alpha = k - 1/\beta\), Theorem 1.1 covers all possible choices of \(\alpha > 0, \beta \in [0, 2)\) and \(p \geq 1\). We also note that the limiting random variable in (1.1) is infinitely divisible, see Section 3.1.3 for more details. In addition, we note that there is no convergence in probability in (1.1) due to the fact that the random variables \(V_m, m \geq 1\), are independent of \(L\) and the properties of stable convergence. To be used in the next theorem we recall that a totally right skewed \(\rho\)-stable random variable \(S\) with \(\rho > 1\), mean zero and scale parameter \(\sigma > 0\) has characteristic function given by

\[
\mathbb{E}[e^{\mathrm{i}\theta S}] = \exp \left( -\sigma^{|\theta|^\rho} (1 - \text{sign}(\theta) \tan(\pi \rho/2)) \right), \quad \theta \in \mathbb{R}.
\]

For part (ii) of Theorem 1.1 which we will refer to as the ergodic case, we also show the second order asymptotic results under the additional condition \(p < \beta/2\). We remark that for \(k = 1\) we are automatically in the regime of Theorem 1.2.

**Theorem 1.2** (Second order asymptotics). Suppose that assumption \((A)\) is satisfied and \(L\) is a symmetric \(\beta\)-stable \(\text{Lévy process}\) with scale parameter \(\sigma > 0\). Let \(f : [0, \infty) \to \mathbb{R}\) be given by \(f(t) = g(t)t^{-\alpha}\) for \(t > 0\) and \(f(0) = c_0\). Fix \(k \geq 1\) and assume that \(f\) is \(k\)-times continuously right differentiable at 0 and \(|g^{(k)}(t)| \leq K t^{\alpha-k}\) for all \(t > 0\). For all \(p < \beta/2\) we have the following two cases:
(i) Suppose that \( \alpha \in (k - 2/\beta, k - 1/\beta) \). If \( \beta < 1/2 \) assume in addition that \( \alpha > k - \frac{1}{\beta(1-\beta)} \). Then it holds that

\[
n^{1-\frac{1}{\alpha}} \left( n^{-1+p(\alpha+1/\beta)} V(p; k) n - m_p \right) \xrightarrow{d} S,
\]

where \( S \) is a totally right skewed \((k - \alpha)\beta\)-stable random variable with mean zero and scale parameter \( \tilde{\sigma} \), which is defined in Remark 3.1(i).

(ii) If \( \alpha \in (0, k - 2/\beta) \) we deduce that

\[
\sqrt{n} \left( n^{-1+p(\alpha+1/\beta)} V(p; k) n - m_p \right) \xrightarrow{d} \mathcal{N}(0, \eta^2),
\]

where the quantity \( \eta^2 \in (0, \infty) \) is defined via

\[
\eta^2 = \lim_{n \to \infty} \left( \text{var} \left( \left| n^{\alpha+1/\beta} \Delta_n^{\alpha+1/\beta} X \right|^p \right) + 2 \sum_{l=1}^{n-k} \text{cov} \left( \left| n^{\alpha+1/\beta} \Delta_n^{\alpha+1/\beta} X \right|^p, \left| n^{\alpha+1/\beta} \Delta_n^{\alpha+1/\beta} X \right|^p \right) \right).
\]

This paper is structured as follows. The basic ideas and methodology of the proofs are demonstrated in Section 2. Section 3 presents some remarks about the nature and applicability of the main results. Section 4 introduces some preliminaries. We state the proof of Theorem 1.1 in Section 5, while the proof of Theorem 1.2 is demonstrated in Section 6.

2 Basic ideas and methodology

2.1 First order asymptotics

In this section we explain the basic intuition and the methodology of the proofs of Theorem 1.1. For simplicity of exposition we only consider the case \( g_0 = 0, k = 1 \) and we set \( \Delta_n X := \Delta_n X, h := h_1 \) and \( V(p)_n := V(p; 1)_n \).

In order to uncover the path properties of the process \( X \) we perform a formal differentiation with respect to time. Since \( g(0) = 0 \) we obtain a formal representation

\[
dX_t = g(0) dL_t + \left( \int_{-\infty}^{t} g'(t-s) dL_s \right) dt = F_t dt.
\]

We remark that the path \((F_t(\omega))_{t \in [0,1]}\) is not necessarily bounded under assumption (A). However, under conditions of Theorem 1.1(iii), the process \( X \) is differentiable almost everywhere and \( X' = F \in L^p([0,1]) \); see Lemma 5.3 for a detailed exposition. Thus, under conditions of Theorem 1.1(iii), an application of the mean value theorem gives an intuitive proof of (1.1):

\[
\mathbb{P} \lim_{n \to \infty} n^{-1+\rho} V(p)_n = \mathbb{P} \lim_{n \to \infty} \frac{1}{n^p} \sum_{i=1}^{n} |F_{n_i}|^p = \int_0^1 |F_u|^p du,
\]
where \( \xi_i^n \in ((i - 1)/n, i/n) \); we refer to Lemma 5.4 for a formal argument. This gives a sketch of the proof of the asymptotic result at (1.1).

Now, we turn our attention to the small scale behaviour of the stationary increments Lévy driven moving averages \( X \). Recall that under conditions of Theorem 1.1(ii), \( \alpha < 1 - 1/\beta \) and thus \( g' \) has an explosive behaviour at 0. Hence, we intuitively deduce the following approximation for the increments of \( X \) for a small \( \Delta > 0 \):

\[
X_{t+\Delta} - X_t = \int_{\mathbb{R}} \{ g(t + \Delta - s) - g(t - s) \} \, dL_s
\]

\[
\approx \int_{t+\Delta}^{t+\Delta-\varepsilon} \{ g(t + \Delta - s) - g(t - s) \} \, dL_s
\]

\[
\approx c_0 \int_{t+\Delta}^{t+\Delta-\varepsilon} \{ (t + \Delta - s)^\alpha_+ - (t - s)^\alpha_+ \} \, dL_s
\]

\[
\approx c_0 \int_{\mathbb{R}} \{ (t + \Delta - s)^\alpha_+ - (t - s)^\alpha_+ \} \, dL_s = \tilde{X}_{t+\Delta} - \tilde{X}_t,
\]

where

\[
\tilde{X}_t := c_0 \int_{\mathbb{R}} \{ (t - s)^\alpha_+ - (-s)^\alpha_+ \} \, dL_s,
\]

and \( \varepsilon > 0 \) is an arbitrary small real number with \( \varepsilon \gg \Delta \). In the classical terminology \( \tilde{X} \) is called the tangent process of \( X \). In the framework of Theorem 1.1(ii) the process \( \tilde{X} \) is a symmetric fractional \( \beta \)-stable motion. We recall that \( (\tilde{X}_t)_{t \geq 0} \) has stationary increments, symmetric \( \beta \)-stable marginals, Hölder index \( \alpha \) (cf. [38, Theorem 3.4]) and it is self-similar with index \( H = \alpha + 1/\beta \in (1/2, 1) \), i.e.

\[
(\tilde{X}_{at})_{t \geq 0} \overset{d}{=} a^H (\tilde{X}_t)_{t \geq 0} \quad \text{for any } a \in \mathbb{R}_+.
\]

Furthermore, the symmetric fractional \( \beta \)-stable noise \( (\tilde{X}_t - \tilde{X}_{t-1})_{t \geq 1} \) is mixing; see e.g. [14]. Thus, using Birkhoff’s ergodic theorem we conclude that

\[
n^{-1+\alpha(1+1/\beta)} V(p)_n = \frac{1}{n} \sum_{i=1}^{n} |n^H \Delta_i^n X|^p
\]

\[
\approx \frac{1}{n} \sum_{i=1}^{n} |n^H \Delta_i^n \tilde{X}|^p
\]

\[
\overset{d}{=} \frac{1}{n} \sum_{i=1}^{n} |\tilde{X}_i - \tilde{X}_{i-1}|^p \overset{p}{\rightarrow} \mathbb{E}[|\tilde{X}_1 - \tilde{X}_0|^p].
\]

This method sketches the proof of the convergence at (1.1).

**Remark 2.1.** We recall that \( L \) is assumed to be a symmetric \( \beta \)-stable Lévy process in Theorems 1.1(ii) and 1.2. We conjecture that this assumption can be relaxed following the discussion of tangent processes at (2.1). Indeed, the small jumps of the driving Lévy process \( L \) are dominating in the asymptotic results of Theorems 1.1(ii) and 1.2. Thus, it
seems to suffice when small jumps of $L$ are in the domain of attraction of a symmetric $eta$-stable Lévy process, e.g. the Lévy measure of the process $L$ satisfies the decomposition
\[ \nu(dx) = \left( \text{const} \cdot |x|^{-1-\beta} + \varphi(x) \right) dx, \]
where the function $\varphi$ satisfies the conditions $\int_{\mathbb{R}} (1 \wedge x^2) \varphi(x) dx < \infty$ and $\varphi(x) = o(|x|^{-1-\beta})$ as $x \to 0$. Such processes include for instance tempered or truncated symmetric $\beta$-stable Lévy motions. We believe that the statement of Theorem 1.1(ii) remains valid for the above form of the Lévy density. Theorem 1.2 would probably require a stronger condition on the function $\varphi$. \hfill \Box

At this stage we need to better understand the fine scale behaviour of the process $X$ in order to describe the intuition behind the non-standard result of Theorem 1.1(i). For the ease of exposition we will discuss the following simple framework: Assume that the driving motion $L$ jumps only once at random time $T$, which has a density on the interval $(0,1)$ (note that $L$ is not a Lévy process in this case). That is, $L$ has the representation
\[ L_t = \mathbb{1}_{(-\infty,T)}(t) \Delta L_T. \]
Now, we will describe the asymptotic distribution of the scaled increments $n^\alpha \Delta^n X$. Let $j_n$ be a random index satisfying $T \in [(j_n - 1)/n, j_n/n)$. Similarly to the approximation at (2.1), we obtain that
\[ \Delta^n_i X \approx A^n_i := c_0 \int_0^\infty \left\{ \left( \frac{i}{n} - s \right)^\alpha_+ - \left( \frac{i-1}{n} - s \right)^\alpha_+ \right\} dL_s. \]
Since $T \in [(j_n - 1)/n, j_n/n)$ is the only jump time of $L$, we observe that $A^n_i = 0$ for all $i < j_n$. More precisely, we deduce that
\[ \Delta^n_{j_n+l} X \approx c_0 \Delta L_T \left( \left( \frac{j_n + l}{n} - T \right)^\alpha_+ - \left( \frac{j_n + l - 1}{n} - T \right)^\alpha_+ \right), \quad l \geq 0. \]
Now, we use the following result, which is essentially due to Tukey [10] (see also [18] and Lemma 5.1 below): Let $Z$ be a random variable with an absolutely continuous distribution and let $\{x\} := x - [x] \in [0,1)$ denote the fractional part of $x \in \mathbb{R}$. Then it holds that
\[ \{nZ\} \overset{L^s}{\to} U \sim \mathcal{U}([0,1]), \]
where $U$ is defined on the extended probability space and $U$ is independent of $Z$. Since $j_n - nT = 1 - \{nT\}$ and $1 - U \sim \mathcal{U}([0,1])$, we conclude the stable convergence in law
\[ n^\alpha \Delta^n_{j_n+l} X \overset{L^s}{\to} c_0 \Delta L_T \left( (l+U)^\alpha_+ - (l-1+U)^\alpha_+ \right), \quad l \geq 0. \]
Thus, in this setting we obtain the result of (1.1) as follows:
\[ n^{\alpha p} V(X,p)_n \approx \sum_{i=j_n}^n |n^\alpha A^n_i|^p \overset{L^s}{\to} c_0 \Delta L_T^p \sum_{l=0}^{\infty} |(l+U)^\alpha_+ - (l-1+U)^\alpha_+|^p, \]
which gives an intuitive proof of Theorem 1.1(i). A formal proof of the stable convergence at (1.1) for a general Lévy motion $L$ requires a decomposition of the driving jump measure associated with $L$ into big and small jumps, and a certain time separation between the big jumps.
2.2 Weak limit theorems

In this section we highlight the basic methodology behind the proof of Theorem 1.2. For the sake of exposition, we will rather consider the power variation $V(\tilde{X}, p; k)_n$ of the tangent process $\tilde{X}$ defined at (2.1) driven by a symmetric $\beta$-stable Lévy motion $L$.

Weak limit theory for statistics of discrete moving averages has been a subject of a deep investigation during the last thirty years. In a functional framework a variety of different limit distributions may appear. They include Brownian motion, $m$th order Hermite processes, stable Lévy processes with various stability indexes and fractional Brownian motion. We refer to the papers [2, 22, 23, 28, 36, 37] for an overview.

Let us start with the treatment of Theorem 1.2(i). By self-similarity of the symmetric fractional $\beta$-stable motion $\tilde{X}$, we conclude that

$$n^{-\frac{1}{(k-\alpha)\beta}} \left( n^{-\frac{1}{(k-\alpha)\beta}} V(\tilde{X}, p; k)_n - m_p \right) \xrightarrow{d} n^{-\frac{1}{(k-\alpha)\beta}} \sum_{i=k}^n H \left( \Delta_{i,k} \tilde{X} \right), \quad (2.3)$$

where

$$\Delta_{i,k} \tilde{X} := \sum_{j=0}^k (-1)^j \binom{k}{j} \tilde{X}_{i-j} \quad \text{and} \quad H(x) := |x|^p - m_p.$$ 

In this framework the most important ingredient is the Appell rank of the function $H$ (cf. [2]). We recall that for a general function $H : \mathbb{R} \to \mathbb{R}$ with $E[H(\Delta_{k,k} \tilde{X})] = 0$ the Appell rank of $H$ is defined via

$$m^* := \min_{m \geq 1} \{ H^{(m)}(0) \neq 0 \} \quad \text{with} \quad H_\infty(x) := E[H(\Delta_{k,k} \tilde{X} + x)].$$

Notice that in the setting $H(x) = |x|^p - m_p$ we have that $m^* = 2$, since $\tilde{X}$ has symmetric distribution and the function $x \mapsto |x|^p$ is even. It turns out that Appell rank $m^*$ together with the parameter $\alpha$ and the tail behaviour of the noise $L_1$ determines the limiting behaviour of the statistic on the right hand side of (2.2). The weak limit theory for $m^* = 1, 2, 3$ in the context of discrete moving average processes has been investigated in [22, 23, 36, 37] among others. We remark however that for $m^* \geq 2$ the authors only consider bounded functions $H$ and existence of second moments of the noise process. Both assumptions are obviously not satisfied in our setting since $E[L_1^2] = \infty$.

The key to proving Theorem 1.2(i) are several projection techniques that are described in details in Section 6.1 (cf. Eq. (6.1)). In particular, they show the decomposition

$$n^{-\frac{1}{(k-\alpha)\beta}} \sum_{i=k}^n H \left( \Delta_{i,k} \tilde{X} \right) = n^{-\frac{1}{(k-\alpha)\beta}} \sum_{i=k}^n Z_i + o_p(1),$$

where $(Z_i)_{i \geq k}$ is a certain sequence of i.i.d. random variables. Thus, by [34, Theorem 1.8.1], it is sufficient to determine the tail behaviour of $Z_1$. That is, we prove the convergence

$$\lim_{x \to \infty} x^{(k-\alpha)\beta} \mathbb{P}(Z_1 > x) = \gamma \quad \text{and} \quad \lim_{x \to \infty} x^{(k-\alpha)\beta} \mathbb{P}(Z_1 < -x) = 0.$$
for a constant $\gamma \in (0, \infty)$, which completes the proof of Theorem 1.2(i) (cf. Section 6.3). At this stage we remark that Theorem 1.2(i) is similar in spirit to the results of [37]. Indeed, we apply a similar proof strategy to show the weak convergence. However, strong modifications due to unboundedness of $H$, triangular nature of summands in (1), stochastic integrals instead of sums, and the different set of conditions are required.

In order to describe the main ideas behind the proof of Theorem 1.2(ii) we again observe the identity in distribution
\[
\sqrt{n} \left( n^{-1+p(\alpha+1/\beta)} V(\tilde{X}, p; k) n - m_p \right) \xrightarrow{d} \frac{1}{\sqrt{n}} \sum_{i=k}^{n} H \left( \Delta_{i,k} \tilde{X} \right) =: \frac{1}{\sqrt{n}} S_n.
\]
In the first step the term $S_n$ is approximated by the quantity $S_{n,m}$, which is a sum of $m$-dependent identically distributed random variables. This approximation is obtained by a proper cut off in the integration region of the integral $\Delta_{i,k} \tilde{X}$. In the second step we will show that
\[
\frac{1}{\sqrt{n}} S_{n,m} \xrightarrow{d} N(0, \eta_m^2) \quad \text{as } n \to \infty \quad \text{and} \quad \lim_{m \to \infty} \eta_m^2 = \eta^2.
\]
Hence, the proof of Theorem 1.2(ii) is complete if we show the convergence
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \left( n^{-1} \mathbb{E}[(S_n - S_{n,m})^2] \right) = 0,
\]
which is the main part of the proof. It again relies on rather complex projection techniques similar to the proof of Theorem 1.2(i) (cf. Eq. (6.1)).

**Remark 2.2.** The symmetry condition on the Lévy process $L$ is assumed for sake of assumption simplification. Most asymptotic results of this paper would not change if we dropped this condition. However, the Appell rank of the function $H(x) = |x|^p - m_p$ might be 1 when $L$ is not symmetric and this does change the result of Theorem 1.2(i). We conjecture that the limiting distribution becomes $\beta$-stable in this framework (see e.g. [28] for the discrete moving average setting). However, we dispense with the exact exposition of this case.

## 3 Some remarks and applications

### 3.1 General comments

In this section we discuss the set of assumptions and the various statements of the main theoretical results. We start by commenting on the set of conditions introduced in assumption (A). First of all, assumption (A) ensures that the process $X$ introduced at (1) is well-defined (cf. Section 1). More importantly, the conditions of assumption (A) guarantee that the quantity
\[
\int_{-\infty}^{t-\varepsilon} g^{(k)}(t-s) dL_s, \quad \varepsilon > 0,
\]
Remark 3.1. (Definition of $\tilde{\sigma}$) In order to introduce the constant $\tilde{\sigma}$ appearing in Theorem 1.2(i) we set

$$
\kappa = \frac{k^{1/(k-\alpha)}}{k-\alpha} \int_0^\infty \Phi(y) y^{-1-1/(k-\alpha)} dy,
$$

where $\Phi(y) := \mathbb{E}[|\Delta_{k,k} \tilde{X} + y|^p - |\Delta_{k,k} \tilde{X}|^p], y \in \mathbb{R}, \tilde{X}_t$ is a linear fractional stable motion defined in (2.1) with $c_0 = 1$ and $L$ being a standard symmetric $\beta$-stable Lévy process, and $k_\alpha = \alpha(\alpha-1)(\alpha-2) \cdots (\alpha-k+1)$. In addition, set

$$
\tau_\rho = \frac{\rho - 1}{\Gamma(2-\rho) |\cos(\pi \rho/2)|}, \quad \text{for all } \rho \in (1,2),
$$

(3.1)

is well-defined (this does not hold true for $\varepsilon = 0$). The latter is crucial for the proof of Theorem 1.1(i). We recall that the condition $p \geq 1$ is imposed in Theorem 1.1(iii). We think that this condition might not be necessary, but the results of [13] applied in our proofs require $p \geq 1$. Finally, we note that assumption (A-log) will be used only for the case $\theta = 1$ (resp. $\theta = p$) in part (i) (resp. part (iii)) of Theorem 1.1.

The conditions $\alpha \in (0,k-1/p)$ and $p > \beta$ of Theorem 1.1(i) seem to be sharp. Indeed, Taylor expansion implies that $|h_k(x)| \leq K|x|^{\alpha-k}$ for large $x$. Consequently, we obtain from (1.1) that

$$
\sup_{m \geq 1} V_m < \infty
$$

when $\alpha \in (0,k-1/p)$. On the other hand $\sum_{m,T_m \in [0,1]} |\Delta L_{T_m}|^p < \infty$ for $p > \beta$, which follows from the definition of the Blumenthal–Getoor index at [1]. Notice that under assumption $\alpha \in (0,k-1/2)$ the case $p = 2$, which corresponds to quadratic variation, always falls under Theorem 1.1(i).

We remark that the distribution of the limiting variable in (1.1) does not depend on the chosen sequence $(T_m)_{m \geq 1}$ of stopping times which exhausts the jump times of $L$. Furthermore, the limiting random variable $Z$ in (1.1) is infinitely divisible with Lévy measure $(\nu \otimes \eta) \circ ((y,v) \mapsto |c_0 y|^p v)^{-1}$, where $\eta$ denotes the law of $V_1$. In fact, $Z$ has characteristic function given by

$$
\mathbb{E}[\exp(i\theta Z)] = \exp \left( \int_{\mathbb{R}_0 \times \mathbb{R}} (e^{i\theta|c_0 y|^p v} - 1) \nu(dy) \eta(dv) \right).
$$

To show this, let $\Lambda$ be the Poisson random measure given by $\Lambda = \sum_{m=1}^\infty \delta_{(T_m,\Delta L_{T_m})}$ on $[0,1] \times \mathbb{R}_0$ which has intensity measure $\lambda \otimes \nu$. Here $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and $\lambda$ denotes the Lebesgue measure on $[0,1]$. Set $\Theta = \sum_{m=1}^\infty \delta_{(T_m,\Delta L_{T_m},V_m)}$. Then $\Theta$ is a Poisson random measure with intensity measure $\lambda \otimes \nu \otimes \eta$, due to [35, Theorem 36]. Thus, the above claim follows from the stochastic integral representation

$$
Z = \int_{[0,1] \times \mathbb{R}_0 \times \mathbb{R}} (|c_0 y|^p v) \Theta(ds,dy,dv).
$$

As for Theorem 1.1(iii), we remark that for values of $\alpha$ close to $k-1/p$ or $k-1/\beta$, the function $g^{(k)}$ explodes at 0. This leads to unboundedness of the path $(F_t(\omega))_{t \in [0,1]}$. Nevertheless, the limiting random variable in (1.1) is still finite.

Remark 3.1. We set $\alpha = (k-1/p)/k$ and $\sigma = \kappa^{1/(k-\alpha)}$ in Theorem 1.2(i) we set

$$
\kappa = \frac{k^{1/(k-\alpha)}}{k-\alpha} \int_0^\infty \Phi(y) y^{-1-1/(k-\alpha)} dy,
$$

where $\Phi(y) := \mathbb{E}[|\Delta_{k,k} \tilde{X} + y|^p - |\Delta_{k,k} \tilde{X}|^p], y \in \mathbb{R}, \tilde{X}_t$ is a linear fractional stable motion defined in (2.1) with $c_0 = 1$ and $L$ being a standard symmetric $\beta$-stable Lévy process, and $k_\alpha = \alpha(\alpha-1)(\alpha-2) \cdots (\alpha-k+1)$. In addition, set

$$
\tau_\rho = \frac{\rho - 1}{\Gamma(2-\rho) |\cos(\pi \rho/2)|}, \quad \text{for all } \rho \in (1,2),
$$

(3.1)
where $\Gamma$ denotes the gamma function. Then, the scale parameter $\tilde{\sigma}$ is defined via

$$\tilde{\sigma} = |c_0|^p \sigma^p \left( \frac{\tau \beta}{\tau(1-\alpha) \beta} \right)^{\frac{1}{1-\alpha}} \kappa.$$ 

The function $\Phi(y)$ can be computed explicitly, see (6.2). This representation shows, in particular, that $\Phi(y) > 0$ for all $y > 0$, and hence the limiting variable $S$ in Theorem 1.2(i) is not degenerate, because $\tilde{\sigma} > 0$.

**Remark 3.2.** Theorem 5.1 of [8] studies the first order asymptotic of the power variation of some fractional fields $(X_t)_{t \in \mathbb{R}^d}$. In the case $d = 1$, they consider fractional Lévy processes $(X_t)_{t \in \mathbb{R}}$ of the form

$$X_t = \int_{\mathbb{R}} \left\{ |t - s|^{H-1/2} - |s|^{H-1/2} \right\} dL_s$$

(3.2)

where $L$ is a truncated $\beta$-stable Lévy process. This setting is close to fit into the framework of the present paper (1) with $\alpha = H - 1/2$ except for the fact that the stochastic integral (3.2) is over the whole real line. However, the proof of Theorem (1.1)(i) still holds for $X$ in (3.2) with the obvious modifications of $h_k$ and $V_m$ in (1) and (1.1), respectively. Notice also that [8] considers the power variation along the subsequence $2^n$, which corresponds to dyadic partitions, and their setting includes second order increments $(k = 2)$. For $p < \beta$, Theorem 5.1 of [8] claims that $2^{a_n p} V(p; 2^n) \to C$ almost surely, where $C$ is a positive constant. However, this contradicts Theorem (1.1)(i) together with the remark following it, namely that, convergence in probability can not take place under the conditions of Theorem (1.1)(i), not even trough a subsequence. It seems that the last three lines of the proof of [8, Theorem 5.1] are erroneous, since the derived estimates are not uniform in the parameters which are required for the stated conclusion to hold, see [8, p. 372].

### 3.2 Statistical applications

The asymptotic theory of this paper has a variety of potential applications in statistics. We have seen in Section 2 that in the framework of a symmetric fractional $\beta$-stable motion $(\tilde{X}_t)_{t \geq 0}$ the parameter $H = \alpha + 1/\beta \in (1/2, 1)$ is the self-similarity index of the process $\tilde{X}$ while $\alpha > 0$ determines the Hölder continuity index of $\tilde{X}$.

Having understood the role of the parameters $\alpha > 0$ and $H = \alpha + 1/\beta \in (1/2, 1)$ from the modelling perspective, it is obviously important to investigate estimation methods for these parameters when the underlying process is given by $(X_t)_{t \geq 0}$. We start with a direct estimation procedure that identifies the convergence rates in Theorem (1.1)(i)-(iii). We apply these convergence results only for $k = 1$. Since we have assumed that $\alpha > 0$ and $H = \alpha + 1/\beta \in (1/2, 1)$, it must hold that $\beta \in (1, 2)$. Notice also that the condition $p > 1$ is required in Theorem (1.1)(i) when $k = 1$. Now, we define the statistic

$$S_{\alpha, \beta}(n, p) := -\frac{\log V(p)_n}{\log n} \quad \text{with} \quad V(p)_n = V(p; 1)_n.$$ 

Assume that the underlying Lévy motion $L$ is symmetric $\beta$-stable, in which case Theorems (1.1)(i)-(iii) are all applicable. Then, under assumptions of Theorem (1.1) for $p > 1$ it
holds that

\[ S_{α,β}(n,p) \xrightarrow{p} S_{α,β}(p) := \left\{ \begin{array}{ll}
\alpha p : & \alpha < 1 - 1/p \text{ and } p > \beta \\
pH - 1 : & \alpha < 1 - 1/β \text{ and } p < \beta \\
p - 1 : & \alpha > 1 - 1/\max(p,β) 
\end{array} \right. \]  

(3.3)

for any fixed \( p \in (1,2) \). Indeed, the result of Theorem 1.1(i) implies that

\[ \frac{αp \log n + \log V(p)}{\log n} \xrightarrow{L} 0 \Rightarrow \frac{αp \log n + \log V(p)}{\log n} \xrightarrow{P} 0, \]

which explains the first line in (3.2). Similarly, Theorem 1.1(ii) and (iii) imply the other convergence results of (3.2). At this stage we remark that the limit \( S_{α,β}(p) \) is a piecewise linear function in \( p \in (1,2) \) with different slopes. Indeed, it suffices to only consider \( p \in (1,2) \) to uncover all three slopes. Now, it is natural to consider the \( L^2 \)-distance between the observed scale function \( S_{α,β}(n,p) \) and the theoretical \( S_{α,β}(p) \):

\[ (\hat{α}_n, \hat{β}_n) := \arg\min_{α > 0, \ α+1/β \in (1/2,1)} \int_1^2 (S_{α,β}(n,p) - S_{α,β}(p))^2 dp. \]  

(3.4)

In practice the integral in (3.2) needs to be discretised. This approach is somewhat similar to the estimation method proposed in [20].

If we are interested in the estimation of the self-similarity parameter \( H = α + 1/β ∈ (1/2,1) \), then there is an alternative estimator based on a ratio statistic. Recalling that \( β ∈ (1,2) \), we deduce for any \( p ∈ (0,1] \)

\[ R(n,p) := \frac{\sum_{i=2}^{n} |X_{i/n} - X_{i-2/n}|^p}{\sum_{i=1}^{n} |X_{i/n} - X_{i-1/n}|^p} \xrightarrow{P} 2^pH \]

by a direct application of Theorem 1.1(ii). Thus, we immediately conclude that

\[ \hat{H}_n := \frac{\log R(n,p)}{p \log 2} \xrightarrow{P} H. \]

This type of idea is rather standard in the framework of fractional Brownian motion with Hurst parameter \( H \). Theorem 1.2(i) suggests that the statistic \( \hat{H}_n \) has convergence rate \( n^{1-1/(1-α)β} \) when \( p ∈ (0,1/2] \). By Theorem 1.2(ii) this convergence rate can be improved to \( \sqrt{n} \) when the first order increments are replaced by \( k \)th order increments, \( k ≥ 2 \), in the definition of the statistic \( R(n,p) \).

4 Preliminaries

Throughout the following sections all positive constants will be denoted by \( K \), although they may change from line to line. Also the notation might change from subsection to subsection, but the meaning will be clear from the context. Throughout all the next sections we assume, without loss of generality, that \( c_0 = δ = σ = 1 \). Recall that \( g(t) = g_0(t) = 0 \) for all \( t < 0 \) by assumption.
For a sequence of random variables \((Y_n)_{n \in \mathbb{N}}\) defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we write \(Y_n \xrightarrow{L^p} Y\) if \(Y_n\) converges \(\mathcal{F}\)-stably in law to \(Y\). That is, \(Y\) is a random variable defined on an extension of \((\Omega, \mathcal{F}, \mathbb{P})\) such that for all \(\mathcal{F}\)-measurable random variables \(U\) we have the joint convergence in law \((Y_n, U) \xrightarrow{d} (Y, U)\).

In particular, \(Y_n \xrightarrow{L^p} Y\) implies \(Y_n \xrightarrow{d} Y\). For \(A \in \mathcal{F}\) we will say that \(Y_n \xrightarrow{L^p} Y\) on \(A\), if \(Y_n \xrightarrow{L^p} Y\) under \(\mathbb{P}|A\), where \(\mathbb{P}|A\) denotes the conditionally probability measure \(B \mapsto \mathbb{P}(B \cap A)/\mathbb{P}(A)\), when \(\mathbb{P}(A) > 0\). We refer to the work [1, 33] for a detailed exposition of stable convergence. In addition, \(\xrightarrow{P}\) will denote convergence in probability. We will write \(V(Y_n, p; k) = \sum_{n=1}^{\infty} \Delta_{n,k} Y\) when we want to stress that the power variation is built from a process \(Y\). On the other hand, when \(k\) and \(p\) are fixed we will sometimes write \(V(Y_n) = V(Y, p; k)\) to simplify the notation.

First of all, it follows from [32, Theorem 7] that the process \(X\) introduced in (1) is well-defined if and only if for all \(t \geq 0\),

\[
\int_{-t}^{\infty} \int_{\mathbb{R}} \left( |f_t(s)| x \right)^2 \nu(dx) ds < \infty,
\]

(4.5)

where \(f_t(s) = g(t + s) - g_0(s)\). By adding and subtracting \(g\) to \(f_t\) it follows by assumption (A) and the mean value theorem that \(f_t \in L^\theta(\mathbb{R}+)\) and \(f_t\) is bounded. For all \(\varepsilon > 0\), assumption (A) implies that

\[
\int_{\mathbb{R}} \left( |y| \right)^2 \nu(dx) \leq K \left( 1_{\{|y| \leq 1\}} |y|^{\theta} + 1_{\{|y| > 1\}} |y|^{\beta + \varepsilon} \right),
\]

which shows (4) since \(f_t \in L^\theta(\mathbb{R}_+)\) is bounded.

Now, for all \(n, i \in \mathbb{N}\), we set

\[
\begin{align*}
g_{i,n}(x) &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} g((i - j)/n - x), \\
h_{i,n}(x) &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} ((i - j)/n - x)^\alpha_+, \\
g_{n}(x) &= n^\alpha g(x/n), \quad x \in \mathbb{R}.
\end{align*}
\]

(4.6)

(4.7)

In addition, for each function \(\phi: \mathbb{R} \to \mathbb{R}\) define \(D^k \phi: \mathbb{R} \to \mathbb{R}\) by

\[
D^k \phi(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \phi(x - j), \quad x \in \mathbb{R}.
\]

(4.8)

In this notation the function \(h_k\), defined in [10], is given by \(h_k = D^k \phi\) with \(\phi: x \mapsto x^\alpha_+\).
Lemma 4.1. Assume that $g$ satisfies condition (A). Then we obtain the following estimates

\begin{align}
|g_{i,n}(x)| &\leq K(i/n - x)^\alpha, \quad x \in [(i-k)/n, i/n], \\
|g_{i,n}(x)| &\leq Kn^{-k}((i-k)/n - x)^{\alpha-k}, \quad x \in (i/n - 1, (i-k)/n), \\
|g_{i,n}(x)| &\leq Kn^{-k}\left(1_{[(i-k)/n-1,i/n-1]}(x) + g^{(k)}((i-k)/n - x)1_{(-\infty,(i-k)/n-1)}(x)\right),
\end{align}

(4.9)

(4.10)

(4.11)

$x \in (-\infty, i/n - 1].$

The same estimates trivially hold for the function $h_{i,n}.$

Proof. The inequality (4.1) follows directly from condition (i) of (A). The second inequality (4.1) is a straightforward consequence of Taylor expansion of order $k$ and the condition $|g^{(k)}(t)| \leq Kt^{\alpha-k}$ for $t \in (0, 1)$. The third inequality (4.1) follows again through Taylor expansion and the fact that the function $g^{(k)}$ is decreasing on $(1, \infty).$

5 Proof of Theorem 1.1

In this section we will prove the assertions of Theorem 1.1.

5.1 Proof of Theorem 1.1(i)

The proof of Theorem 1.1(i) is divided into the following three steps. In Step (i) we show Theorem 1.1(i) for the compound Poisson case, which stands for the treatment of big jumps of $L$. Step (ii) consists of an approximating lemma, which proves that the small jumps of $L$ are asymptotically negligible. Step (iii) combines the previous results to obtain the general theorem.

Before proceeding with the proof we will need the following preliminary lemma. Let $\{x\} := x - \lfloor x \rfloor \in [0, 1)$ denote the fractional part of $x \in \mathbb{R}$. The lemma below seems to be essentially known (cf. [18, 40]), however, we have not been able to find this particular formulation. Therefore it is stated below for completeness.

Lemma 5.1. For $d \geq 1$ let $V = (V_1, \ldots, V_d)$ be an absolutely continuous random vector in $\mathbb{R}^d$ with a density $v: \mathbb{R}^d \to \mathbb{R}_+$. Suppose that there exists an open convex set $A \subseteq \mathbb{R}^d$ such that $v$ is continuous differentiable on $A$ and vanish outside $A$. Then, as $n \to \infty$,

\[
\left(\{nV_1\}, \ldots, \{nV_d\}\right) \overset{d}{\longrightarrow} U = (U_1, \ldots, U_d)
\]

where $U_1, \ldots, U_d$ are independent $\mathcal{U}([0, 1])$-distributed random variables which are independent of $\mathcal{F}$.

Proof. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^k$ let $\{x\} = (\{x_1\}, \ldots, \{x_d\})$ be the fractional parts of its components. Let $f: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a $C^1$-function, which vanishes outside some closed
such that for all \( j \)

\[
D_{\rho} := \left| \int_{\mathbb{R}^d} f(x, \{x/\rho\}) v(x) \, dx - \int_{\mathbb{R}^k} \left( \int_{[0,1]^d} f(x, u) \, du \right) v(x) \, dx \right| \leq K_{\rho}. \tag{5.1}
\]

Indeed, by (5.1) used for \( \rho = 1/n \) we obtain that

\[
\mathbb{E}[f(V, \{nV\})] \rightarrow \mathbb{E}[f(V, U)] \quad \text{as } n \to \infty,
\]

with \( U = (U_1, \ldots, U_d) \) given in the lemma. Moreover, due to \textbf{[11] Proposition 2(D')}, (5.1) implies the stable convergence \( \{nV\} \xrightarrow{\mathcal{L}} U \) as \( n \to \infty \), and the proof is complete. Thus, it only remains to prove the inequality (5.1). At this stage we use a similar technique as in \textbf{[13] Lemma 6.1}. Define \( \phi(x, u) := f(x, u)v(x) \). Then it holds by substitution that

\[
\int_{\mathbb{R}^d} f(x, \{x/\rho\}) v(x) \, dx = \sum_{j \in \mathbb{Z}^d} \int_{(0,1]^d} \rho^d \phi(\rho j + \rho u, u) \, du
\]

and

\[
\int_{\mathbb{R}^d} \left( \int_{[0,1]^d} f(x, u) \, du \right) v(x) \, dx = \sum_{j \in \mathbb{Z}^d} \int_{(0,1]^d} \left( \int_{(\rho j, \rho(j+1)]} \phi(x, u) \, dx \right) \, du.
\]

Hence, we conclude that

\[
D_{\rho} \leq \sum_{j \in \mathbb{Z}^d} \int_{(0,1]^d} \left| \int_{(\rho j, \rho(j+1)]} \phi(x, u) \, dx - \rho^d \phi(\rho j + \rho u, u) \right| \, du
\]

\[
\leq \sum_{j \in \mathbb{Z}^d} \int_{(0,1]^d} \left( \int_{(\rho j, \rho(j+1)]} \phi(x, u) - \phi(\rho j + \rho u, u) \right) \, dx \, du.
\]

By mean value theorem there exists a positive constant \( K \) and a compact set \( B \subseteq \mathbb{R}^d \times \mathbb{R}^d \) such that for all \( j \in \mathbb{Z}^d, \ x \in (\rho j, \rho(j+1)] \) and \( u \in (0,1]^d \) we have

\[
\left| \phi(x, u) - \phi(\rho j + \rho u, u) \right| \leq K_{\rho} \mathbb{1}_{B}(x, u).
\]

Thus, \( D_{\rho} \leq K_{\rho} \int_{(0,1]^d} \int_{\mathbb{R}^d} \mathbb{1}_{B}(x, u) \, dx \, du \), which shows (5.1).

\[ \square \]

\textbf{Step (i): The compound Poisson case.} Let \( L = (L_t)_{t \in \mathbb{R}} \) be a compound Poisson process and let \( 0 \leq T_1 < T_2 < \ldots \) denote the jump times of the Lévy process \( (L_t)_{t \geq 0} \) chosen in increasing order. Consider a fixed \( \varepsilon > 0 \) and let \( n \in \mathbb{N} \) satisfy \( \varepsilon n > 4k \). We define

\[
\Omega_{\varepsilon} := \left\{ \omega \in \Omega : \text{for all } j \geq 1 \text{ with } T_j(\omega) \in [0,1] \text{ we have } |T_{j+1}(\omega) - T_j(\omega)| > \varepsilon/2 \right\}
\]

and \( \Delta L_s(\omega) = 0 \) for all \( s \in [-\varepsilon, \varepsilon] \cup [1 - \varepsilon, 1] \).

Notice that \( \mathbb{P}(\Omega_{\varepsilon}) \uparrow 1 \) as \( \varepsilon \downarrow 0 \). Now, we decompose for \( i = k, \ldots, n \)

\[
\Delta_{i,k} X = M_{i,n,\varepsilon} + R_{i,n,\varepsilon},
\]
where
\[ M_{i,n,\varepsilon} = \int_{\frac{i}{n}-\varepsilon}^{\frac{i}{n}} g_{n}(s) \, dL_s, \quad R_{i,n,\varepsilon} = \int_{-\infty}^{\frac{i}{n}-\varepsilon} g_{n}(s) \, dL_s, \]
and the function \( g_{n} \) is introduced in (1.1). The term \( M_{i,n,\varepsilon} \) represents the dominating quantity, while \( R_{i,n,\varepsilon} \) turns out to be negligible.

The dominating term: We claim that on \( \Omega_\varepsilon \) and as \( n \to \infty \),
\[ n^{\alpha_p} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^p \xrightarrow{\varepsilon \to 0} Z \quad \text{where} \quad Z = \sum_{m:T_m \in (0,1]} |\Delta L_{T_m}|^p V_m, \tag{5.3} \]
where \( V_m, m \geq 1 \), are defined in (1.1). To show (5.1) let \( i_m = i_m(\omega, n) \) denote the random index such that \( T_m \in ((i_m - 1)/n, i_m/n] \). The following representation will be crucial: On \( \Omega_\varepsilon \) we have that
\[ n^{\alpha_p} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^p = V_{n,\varepsilon} \quad \text{with} \quad V_{n,\varepsilon} = n^{\alpha_p} \sum_{m:T_m \in (0,1]} |\Delta L_{T_m}|^p \left( \sum_{l=0}^{[\varepsilon n]+v_m} |g_{i_m+l,n}(T_m)|^p \right) \tag{5.4} \]
for some random indexes \( v_m = v_m(\omega, n, \varepsilon) \in \{-2, -1, 0\} \) which are measurable with respect to \( T_m \). Indeed, on \( \Omega_\varepsilon \) and for each \( i = k, \ldots, n \), \( L \) has at most one jump in \((i/n - \varepsilon/2, i/n]\). For each \( m \in \mathbb{N} \) with \( T_m \in (0,1] \) we have \( T_m \in (i/n - \varepsilon, i/n] \) if and only if \( i_m \leq i < n(T_m + \varepsilon) \) (recall that \( \varepsilon n > 4k \)). Thus,
\[ \sum_{i \in \{k, \ldots, n\}: T_m \in (i/n - \varepsilon, i/n]} |M_{i,n,\varepsilon}|^p = |\Delta L_{T_m}|^p \left( \sum_{l=0}^{[\varepsilon n]+v_m} |g_{i_m+l,n}(T_m)|^p \right) \tag{5.5} \]
for some \( T_m \)-measurable random variable \( v_m \in \{-2, -1, 0\} \). Thus, by summing (5.1) over all \( m \in \mathbb{N} \) with \( T_m \in (0,1] \), (5.1) follows. In the following we will show that
\[ V_{n,\varepsilon} \xrightarrow{\varepsilon \to 0} Z \quad \text{as} \quad n \to \infty. \]

For \( d \geq 1 \) it is well-known that the random vector \((T_1, \ldots, T_d)\) is absolutely continuous with a \( C^1 \)-density on the open convex set \( A := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 < x_1 < x_2 < \cdots < x_d\} \), which is vanishing outside \( A \). Thus, by Lemma 5.1 we have
\[ (\{nT_m\})_{m \leq d} \xrightarrow{\varepsilon \to 0} (U_m)_{m \leq d} \quad \text{as} \quad n \to \infty \tag{5.6} \]
where \((U_i)_{i \in \mathbb{N}}\) are i.i.d. \( U((0,1])\)-distributed random variables. By (1) we may write \( g(x) = x^d f(x) \) where \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f(x) \to 1 \) as \( x \to 0 \). By definition of \( i_m \) we have that
\[\{nT_m\} = nT_m - (i_m - 1)\] and therefore for all \(l = 0, 1, 2, \ldots\) and \(j = 0, \ldots, k,\)

\[n^\alpha g\left(\frac{l + i_m - j}{n} - T_m\right) = n^\alpha \left(\frac{l + i_m - j}{n} - T_m\right)^\alpha f\left(\frac{l + i_m - j}{n} - T_m\right)\]

\[= (l - j + (i_m - nT_m))^\alpha f\left(\frac{l - j}{n} + n^{-1}(i_m - nT_m)\right)\]

\[= (l - j + 1 - \{nT_m\})^\alpha f\left(\frac{l - j}{n} + n^{-1}(1 - \{nT_m\})\right).\]

By \((5.1), (U_m)_{m \leq d} \overset{d}{=} (1 - U_m)_{m \leq d}\) and \(f(x) \to 1\) as \(x \downarrow 0\) we obtain that

\[\left\{n^\alpha g\left(\frac{l + i_m - j}{n} - T_m\right)\right\}_{l,m \leq d} \overset{\mathcal{L}-\mathbb{S}}{\longrightarrow} \left\{(l - j + U_m)^\alpha\right\}_{l,m \leq d}\quad\text{as } n \to \infty.\quad (5.7)\]

Eq. \((5.1)\) implies that

\[\left\{n^\alpha g_{i_m+l,n}(T_m)\right\}_{l,m \leq d} \overset{\mathcal{L}-\mathbb{S}}{\longrightarrow} \left\{h_k(l + U_m)\right\}_{l,m \leq d},\quad (5.8)\]

with \(h_k\) being defined at \((1)\). Due to the \(\mathcal{F}\)-stable convergence in \((5.1)\) we obtain by the continuous mapping theorem that for each fixed \(d \geq 1\) and as \(n \to \infty,\)

\[V_{n,\varepsilon,d} := n^{\alpha p} \sum_{m: m \leq d, T_m \in [0,1]} |\Delta L_{T_m}|^p \left(\sum_{l=0}^{[\varepsilon d] + v_m} |g_{i_m+l,n}(T_m)|^p\right)^\frac{1}{p}\]

\[\overset{\mathcal{L}-\mathbb{S}}{\longrightarrow} Z_d = \sum_{m: m \leq d, T_m \in [0,1]} |\Delta L_{T_m}|^p \left(\sum_{l=0}^{[\varepsilon d] + v_m} |h_k(l + U_m)|^p\right)^\frac{1}{p}.\]

Moreover, for \(\omega \in \Omega\) we have as \(d \to \infty,\)

\[Z_d(\omega) \uparrow Z(\omega).\]

Recall that \(|h_k(x)| \leq K(x - k)^{\alpha - k}\) for \(x > k + 1\), which implies that \(Z < \infty\) a.s. since \(p(\alpha - k) < -1\). For all \(l \in \mathbb{N}\) with \(k \leq l \leq n\), we have

\[n^{\alpha p} |g_{i_m+l,n}(T_m)|^p \leq K |l - k|^{(\alpha - k)p},\quad (5.9)\]

due to \((4.1)\) of Lemma \(4.1\). For all \(d \geq 0\) set \(C_d = \sum_{m > d; T_m \in [0,1]} |\Delta L_{T_m}|^p\) and note that \(C_d \to 0\) a.s. as \(d \to \infty\) since \(L\) is a compound Poisson process. By \((5.1)\) we have

\[|V_{n,\varepsilon} - V_{n,\varepsilon,d}| \leq K \left(C_d + C_0 \sum_{l=[\varepsilon d] + 1}^{\infty} |l - k|^{p(\alpha - k)}\right) \to 0\quad\text{as } d \to \infty\]

since \(p(\alpha - k) < -1\). Due to the fact that \(n^{\alpha p} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^p = V_{n,\varepsilon}\) a.s. on \(\Omega_\varepsilon\) and \(V_{n,\varepsilon} \overset{\mathcal{L}-\mathbb{S}}{\longrightarrow} Z\), it follows that \(n^{\alpha p} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^p \overset{\mathcal{L}-\mathbb{S}}{\longrightarrow} Z\) on \(\Omega_\varepsilon\), since \(\Omega_\varepsilon \in \mathcal{F}\). This proves \((5.1)\).
The rest term: In the following we will show that
\[
n^{\alpha p} \sum_{i=k}^{n} |R_{i,n,\varepsilon}|^p \xrightarrow{p} 0 \quad \text{as } n \to \infty. \tag{5.10}
\]

The fact that the random variables in (5.1) are usually not integrable makes the proof of (5.1) considerably more complicated. Similarly to (4.1) of Lemma 4.1 we have that
\[
n^k |g_{i,n}(s)| \mathbb{1}_{\{s \leq i/n - \varepsilon\}} \leq K \left( \mathbb{1}_{\{s \in [-1,1]\}} + \mathbb{1}_{\{s < -1\}} |g^{(k)}(-s)| \right) =: \psi(s)
\]
where \( K = K_\varepsilon \). We will use the function \( \psi \) several times in the proof of (5.1), which will be divided into the two special cases \( \theta \in (0,1) \) and \( \theta \in (1,2) \).

Suppose first that \( \theta \in (0,1] \). To show (5.1) it suffices to prove that
\[
\sup_{n \in \mathbb{N}, i \in \{k,\ldots,n\}} n^k |R_{i,n,\varepsilon}| < \infty \quad \text{a.s.} \tag{5.11}
\]
since \( \alpha < k - 1/p \). To show (5.11) we will first prove that
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \left( |\psi(s)x| \wedge 1 \right) \nu(dx) ds < \infty. \tag{5.12}
\]
Choose \( \tilde{K} \) such that \( \psi(x) \leq \tilde{K} \) for all \( x \in \mathbb{R} \). For \( u \in [-\tilde{K},\tilde{K}] \) we have that
\[
\int_{\mathbb{R}} \left( |ux| \wedge 1 \right) \nu(dx) \leq \tilde{K} \int_{1}^{\infty} (|ux| \wedge 1) x^{-1-\theta} dx \\
\leq \begin{cases} 
K|u|^\theta & \theta \in (0,1) \\
K|u|^\theta \log(1/u) & \theta = 1,
\end{cases} \tag{5.13}
\]
where we have used that \( \theta \leq 1 \). By (5.11) applied to \( u = \psi(s) \) and assumption (A) it follows that (5.11) is satisfied. Since \( L \) is a symmetric compound Poisson process we can find a Poisson random measure \( \mu \) with compensator \( \lambda \otimes \nu \) such that for all \( -\infty < u < t < \infty \), \( L_t - L_u = \int_{[u,t] \times \mathbb{R}} x \mu(ds,dx) \). Due to [26, Theorem 10.15], (5.11) ensures the existence of the stochastic integral \( \int_{\mathbb{R} \times \mathbb{R}} |\psi(s)x| \mu(ds,dx) \). Moreover, \( \int_{\mathbb{R} \times \mathbb{R}} |\psi(s)x| \mu(ds,dx) \) can be regarded as an \( \omega \) by \( \omega \) integral with respect to the measure \( \mu_\omega \). Now, we have that
\[
|n^k R_{i,n,\varepsilon}| \leq \int_{(-\infty,i/n-\varepsilon] \times \mathbb{R}} n^k g_{i,n}(s)x \mu(ds,dx) \leq \int_{\mathbb{R} \times \mathbb{R}} |\psi(s)x| \mu(ds,dx) < \infty, \tag{5.14}
\]
which shows (5.11), since the right-hand side of (5.11) does not depend on \( i \) and \( n \).

Suppose that \( \theta \in (1,2] \). Similarly as before it suffices to show that
\[
\sup_{n \in \mathbb{N}, i \in \{k,\ldots,n\}} \frac{n^k |R_{i,n,\varepsilon}|}{(\log n)^{1/q}} < \infty \quad \text{a.s.} \tag{5.15}
\]
where \( q > 1 \) denotes the conjugated number to \( \theta > 1 \) determined by \( 1/\theta + 1/q = 1 \). In the following we will show (5.11) using the majorizing measure techniques developed in [29].
In fact, our arguments are closely related to their Section 4.2. Set \( T = \{(i,n): n \geq k, i = k, \ldots, n\} \). For \((i,n) \in T\) we have

\[
\frac{n^k|\theta_{i,n}|}{(\log n)^{1/q}} = \left| \int_{\mathbb{R}} \zeta_{i,n}(s) dL_s \right|, \quad \zeta_{i,n}(s) := \frac{n^k}{(\log n)^{1/q}} g_{i,n}(s) \mathbb{1}_{\{s \leq i/n - \varepsilon\}}.
\]

For \( t = (i,n) \in T \) we will sometimes write \( \zeta_t(s) \) for \( \zeta_{i,n}(s) \). Let \( \tau: T \times T \to \mathbb{R}_+ \) denote the metric given by

\[
\tau((i,n),(j,m)) = \begin{cases} 
\log(n - k + 1)^{-1/q} + \log(m - k + 1)^{-1/q} & (i,n) \neq (j,l) \\
0 & (i,n) = (j,l).
\end{cases}
\]

Moreover, let \( m \) be the probability measure on \( T \) given by \( m(\{(i,n)\}) = Kn^{-3} \) for a suitable constant \( K > 0 \). Set \( B_r(t,r) = \{s \in T: \tau(s,t) \leq r\} \) for \( t \in T, r > 0, D = \sup\{\tau(s,t): s,t \in T\} \) and

\[
I_q(m,\tau;D) = \sup_{t \in T} \int_0^D \left( \log \frac{1}{m(B_r(t,r))} \right)^{1/q} dr.
\]

In the following we will show that \( m \) is a so-called majorizing measure, which means that \( I_q(m,\tau,D) < \infty \). For \( r < (\log(n - k + 1))^{-1/q} \) we have \( B_r((i,n),r) = \{(i,n)\} \). Therefore, \( m(B_r((i,n),r)) = Kn^{-3} \) and

\[
\int_{0}^{(\log(n-k+1))^{-1/q}} \left( \log \frac{1}{m(B_r((i,n),r))} \right)^{1/q} dr = \int_{0}^{(\log(n-k+1))^{-1/q}} \left( 3 \log n + \log K \right)^{1/q} dr.
\]

(5.16)

For all \( r \geq (\log(n - k + 1))^{-1/q}, (k,k) \in B_r((i,n),r) \) and hence \( m(B_r((i,n),r)) \geq m(\{(k,k)\}) = K(n - k + 1)^{-3} \). Therefore,

\[
\int_{(\log(n-k+1))^{-1/q}}^{D} \left( \log \frac{1}{m(B_r((i,n),r))} \right)^{1/q} dr 
\leq \int_{(\log(n-k+1))^{-1/q}}^{D} \left( 3 \log(k + 1) + \log K \right)^{1/q} dr.
\]

By (5.16) it follows that \( I_q(m,\tau,D) < \infty \). For \((i,n) \neq (j,l)\) we have that

\[
\frac{|\zeta_{i,n}(s) - \zeta_{j,l}(s)|}{\tau((i,n),(j,l))} \leq n^k|g_{i,n}(s)|\mathbb{1}_{\{s \leq i/n - \varepsilon\}} + t^k|g_{j,l}(s)|\mathbb{1}_{\{s \leq j/l - \varepsilon\}} \leq K\psi(s).
\]

(5.18)

Fix \( t_0 \in T \) and consider the following Lipschitz type norm of \( \zeta \),

\[
\|\zeta\|_\tau(s) = D^{-1}|\zeta_{t_0}(s)| + \sup_{t_1, t_2 \in T; \tau(t_1, t_2) \neq 0} \frac{|\zeta_{t_1}(s) - \zeta_{t_2}(s)|}{\tau(t_1, t_2)}.
\]

By (5.1) it follows that \( \|\zeta\|_\tau(s) \leq K\psi(s) \) and hence

\[
\int_{\mathbb{R}} \|\zeta\|_\tau^q(s) ds \leq K \left( 2 + \int_{1}^{\infty} |g^{(k)}(s)|^q ds \right) < \infty.
\]

(5.19)
By [29, Theorem 3.1, Eq. (3.11)] together with \( I_q(m, \tau, D) < \infty \) and (5.1) we deduce (5.1), which completes the proof of (5.1).

**End of the proof:** Recall the decomposition \( \Delta_{i,n}^n X = M_{i,n,\varepsilon} + R_{i,n,\varepsilon} \) in (5.1). Eq. (5.1), (5.1) and an application of Minkowski inequality yield that

\[
n^{op} V(p; k)_n \xrightarrow{L^\infty} Z \quad \text{on } \Omega_\varepsilon \text{ as } n \to \infty.
\]

Since \( P(\Omega_\varepsilon) \uparrow 1 \) as \( \varepsilon \downarrow 0 \), (5.1) implies that

\[
n^{op} V(p; k)_n \xrightarrow{L^\infty} Z.
\]

We have now completed the proof for a particular choice of stopping times \((T_{m})_{m \geq 1}\). However, the result remains valid for any choice of \( \mathcal{F} \)-stopping times, since the distribution of \( Z \) is invariant with respect to reordering of stopping times. \( \square \)

**Step (ii): An approximation.** To prove Theorem 1.1(i) in the general case we need the following approximation result. Consider a general symmetric Lévy process \( L = (L_t)_{t \in \mathbb{R}} \) as in Theorem 1.1(i) and let \( N \) be the corresponding Poisson random measure \( N(A) := \# \left\{ t : (t, \Delta L_t) \in A \right\} \) for all measurable \( A \subseteq \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \). By our assumptions (in particular, by symmetry), the process \( X(j) \) given by

\[
X_t(j) = \int_{(-\infty,t] \times [-\frac{1}{j}, \frac{1}{j}]} \left\{ (g(t - s) - g_0(-s))x \right\} N(ds, dx)
\]

is well-defined. The following estimate on the processes \( X(j) \) will be crucial.

**Lemma 5.2.** Suppose that \( \alpha < k - 1/p \) and \( \beta < p \). Then

\[
\lim_{j \to \infty} \limsup_{n \to \infty} P(n^{op} V(X(j))_n > \varepsilon) = 0 \quad \text{for all } \varepsilon > 0.
\]

**Proof.** By Markov’s inequality and the stationary increments of \( X(j) \) we have that

\[
P(n^{op} V(X(j))_n > \varepsilon) \leq \varepsilon^{-1} n^{op} \sum_{i=k}^{n} E[|\Delta_{i,k}^n X(j)|^p] \leq \varepsilon^{-1} n^{op+1} E[|\Delta_{k,k}^n X(j)|^p].
\]

Hence, it is enough to show that

\[
\lim_{j \to \infty} \limsup_{n \to \infty} E[|Y_{n,j}|^p] = 0 \quad \text{with } Y_{n,j} := n^{\alpha+1/p} \Delta_{k,k}^n X(j).
\]

To show (5.1) it suffices to prove that

\[
\lim_{j \to \infty} \limsup_{n \to \infty} \xi_{n,j} = 0 \quad \text{where } \xi_{n,j} = \int_{|x| \leq 1/j} \chi_n(x) \nu(dx) \quad \text{and}
\]

\[
\chi_n(x) = \int_{-\infty}^{k/n} \left( |n^{\alpha+1/p} g_{k,n}(s)x|^p 1_{\{|n^{\alpha+1/p} g_{k,n}(s)x| \geq 1\}}
\right.
\]

\[
+ |n^{\alpha+1/p} g_{k,n}(s)x|^2 1_{\{|n^{\alpha+1/p} g_{k,n}(s)x| \leq 1\}}
\]
which follows from the representation
\[ Y_{n,j} = \int_{[\infty,k/n] \times [-\frac{1}{2}, \frac{1}{2}]} \left( n^{\alpha+1/p} g_{k,n}(s)x \right) N(ds, dx), \]
and by [32, Theorem 3.3 and the remarks above it]. Suppose for the moment that there exists a finite constant \( K > 0 \) such that
\[ \chi_n(x) \leq K(|x|^{p} + x^2) \quad \text{for all } x \in [-1, 1]. \]
(5.23)

Then,
\[ \limsup_{j \to \infty} \left\{ \limsup_{n \to \infty} \xi_{n,j} \right\} \leq K \limsup_{j \to \infty} \int_{|x| \leq 1/j} (|x|^p + x^2) \nu(dx) = 0 \]
since \( p > \beta \). Hence it suffices to show the estimate (5.1), which we will do in the following.

Let \( \Phi_p : \mathbb{R} \to \mathbb{R}_+ \) denote the function \( \Phi_p(y) = |y|^2 \mathbb{1}_{|y| \leq 1} + |y|^p \mathbb{1}_{|y| > 1} \). We split \( \chi_n \)
into the following three terms which need different treatments
\[ \chi_n(x) = \int_{-k/n}^{k/n} \Phi_p\left(n^{\alpha+1/p} g_{k,n}(s)x\right) ds + \int_{-1}^{-k/n} \Phi_p\left(n^{\alpha+1/p} g_{k,n}(s)x\right) ds \]
\[ + \int_{-\infty}^{-1} \Phi_p\left(n^{\alpha+1/p} g_{k,n}(s)x\right) ds \]
\[ =: I_{1,n}(x) + I_{2,n}(x) + I_{3,n}(x). \]

**Estimation of \( I_{1,n} \):** By (4.1) of Lemma 4.1 we have that
\[ |g_{k,n}(s)| \leq K (k/n - s)^\alpha, \quad s \in [-k/n, k/n]. \]
(5.24)

Since \( \Phi_p \) is increasing on \( \mathbb{R}_+ \), (5.1) implies that
\[ I_{1,n}(x) \leq K \int_{0}^{2k/n} \Phi_p\left(x^{\alpha+1/p}s^\alpha\right) ds. \]
(5.25)

By basic calculus it follows that
\[ \int_{0}^{2k/n} |x^{\alpha+1/p}s^\alpha|^2 \mathbb{1}_{\{|x^{\alpha+1/p}s^\alpha| \leq 1\}} ds \]
\[ \leq K \left( \mathbb{1}_{\{|x| \leq (2k)^{-\alpha-n-1/p}\}} x^{2n^{2/p-1}} + \mathbb{1}_{\{|x| > (2k)^{-\alpha-n-1/p}\}} |x|^{-1/\alpha} n^{-1-1/(\alpha p)} \right) \]
\[ \leq K (|x|^p + x^2). \]
(5.26)

Moreover,
\[ \int_{0}^{2k/n} |x^{\alpha+1/p}s^\alpha|^p \mathbb{1}_{\{|x^{\alpha+1/p}s^\alpha| > 1\}} ds \leq \int_{0}^{2k/n} |x^{\alpha+1/p}s^\alpha|^p ds \leq K |x|^p. \]
(5.27)

Combining (5.1), (5.1) and (5.1) show the estimate \( I_{1,n}(x) \leq K (|x|^p + x^2). \)
**Estimation of \( I_{2,n} \):** By (4.1) of Lemma [4.1] it holds that
\[
|g_{k,n}(s)| \leq Kn^{-k}|s|^{\alpha-k}, \quad s \in (-1, -k/n).
\] (5.28)
Again, due to the fact that \( \Phi_p \) is increasing on \( \mathbb{R}_+ \), (5.1) implies that
\[
I_{2,n}(x) \leq K \int_{k/n}^{1} \Phi_p(xn^{\alpha+1/p-k}s^{\alpha-k}) \, ds.
\] (5.29)
For \( \alpha \neq k - 1/2 \) we have
\[
\int_{k/n}^{1} |xn^{\alpha+1/p-k}s^{\alpha-k}|^2 1_{\{|xn^{\alpha+1/p-k}s^{\alpha-k}| \leq 1\}} \, ds
\leq K \left( x^2 n^{2(\alpha+1/p-k)} + 1_{\{|x| \leq n^{-1/p_k-(\alpha-k)}\}} |x|^2 n^{2/p-1}ight.
\phantom{=} + 1_{\{|x| > n^{-1/p_k-(\alpha-k)}\}} |x|^{1/(k-\alpha)} n^{1/(p(k-\alpha)-1)}
\leq K \left( x^2 + |x|^p \right),
\] (5.30)
where we have used that \( \alpha < k - 1/p \). For \( \alpha = k - 1/2 \) we have
\[
\int_{k/n}^{1} |xn^{\alpha+1/p-k}s^{\alpha-k}|^p 1_{\{|xn^{\alpha+1/p-k}s^{\alpha-k}| \leq 1\}} \, ds
\leq x^2 n^{2(\alpha+1/p-k)} \int_{k/n}^{1} s^{-1} \, ds = x^2 n^{2(\alpha+1/p-k)} \log(n/k) \leq K x^2,
\] (5.31)
where we again have used \( \alpha < k - 1/p \) in the last inequality. Moreover,
\[
\int_{k/n}^{1} |xn^{\alpha+1/p-k}s^{\alpha-k}|^p 1_{\{|xn^{\alpha+1/p-k}s^{\alpha-k}| > 1\}} \, ds
\leq K |x|^p n^{p(\alpha+1/p-k)} \left( 1 + (1/n)^{p(\alpha-k)+1} \right) \leq K |x|^p.
\] (5.32)
By (5.1), (5.11), (5.1) and (5.1) we obtain the estimate \( I_{2,n}(x) \leq K(|x|^p + x^2) \).

**Estimation of \( I_{3,n} \):** For \( s < -1 \) we have that \( |g_{k,n}(s)| \leq Kn^{-k}|g^{(k)}(-k/n-s)| \), by (4.1) of Lemma [4.1] and hence
\[
I_{3,n}(x) \leq K \int_{1}^{\infty} \Phi_p(n^{\alpha+1/p-k}g^{(k)}(s)) \, ds.
\] (5.33)
We have that
\[
\int_{1}^{\infty} |xn^{\alpha+1/p-k}g^{(k)}(s)|^2 1_{\{|xn^{\alpha+1/p-k}g^{(k)}(s)| \leq 1\}} \, ds \leq x^2 n^{2(\alpha+1/p-k)} \int_{1}^{\infty} |g^{(k)}(s)|^2 \, ds.
\] (5.34)
Since \( |g^{(k)}| \) is decreasing on \((1, \infty)\) and \( g^{(k)} \in L^\theta((1, \infty)) \) for some \( \theta \leq 2 \), the integral on the right-hand side of (5.1) is finite. For \( x \in [-1, 1] \) we have
\[
\int_{1}^{\infty} |xn^{\alpha+1/p-k}g^{(k)}(s)|^p 1_{\{|xn^{\alpha+1/p-k}g^{(k)}(s)| > 1\}} \, ds
\leq |x|^p n^{p(\alpha+1/p-k)} \int_{1}^{\infty} |g^{(k)}(s)|^p 1_{\{|g^{(k)}(s)| > 1\}} \, ds.
\] (5.35)
From our assumptions it follows that the integral in (5.1) is finite. By (5.1), (5.1) and (5.1) we have that \( I_{3,n}(x) \leq K(|x|^p + x^2) \) for all \( x \in [-1, 1] \), which completes the proof of (5.1) and therefore also the proof of the lemma.

\( \square \)

**Step (iii): The general case.** In the following we will prove Theorem 1.1(i) in the general case by combining the above Steps (i) and (ii).

**Proof of Theorem 1.1(i).** Let \((T_m)_{m \geq 1}\) be a sequence of \( \mathbb{F}\)-stopping times that exhausts the jumps of \((L_t)_{t \geq 0}\). For each \( j \in \mathbb{N} \) let \( \hat{L}(j) \) be the Lévy process given by

\[
\hat{L}_t(j) - \hat{L}_u(j) = \sum_{u \in [s,t]} \Delta L_u \mathbb{1}_{\{\Delta L_u > \frac{1}{j}\}}, \quad s < t,
\]

and set

\[
\hat{X}_t(j) = \int_{-\infty}^t \left( g(t-s) - g_0(-s) \right) d\hat{L}_s(j).
\]

Moreover, set

\[
T_{m,j} = \begin{cases} T_m & \text{if } |\Delta L_{T_m}| > \frac{1}{j} \\ \infty & \text{else,} \end{cases}
\]

and note that \((T_{m,j})_{m \geq 1}\) is a sequence of \( \mathbb{F}\)-stopping times that exhausts the jumps of \((\hat{L}_t(j))_{t \geq 0}\). Since \( \hat{L}(j) \) is a compound Poisson process, Step (i) shows that

\[
n^{op}V(\hat{X}(j))_n \overset{L_s}{\longrightarrow} Z_j := \sum_{m: T_{m,j} \in [0,1]} |\Delta \hat{L}_{T_{m,j}}(j)|^p V_m \quad \text{as } n \to \infty, \quad (5.36)
\]

where \( V_m, m \geq 1, \) are defined in (1.1). By definition of \( T_{m,j} \) and monotone convergence we have as \( j \to \infty, \)

\[
Z_j = \sum_{m: T_m \in [0,1]} |\Delta L_T|^p V_m \mathbb{1}_{\{|\Delta L_T| > \frac{1}{j}\}} \overset{\text{a.s.}}{\longrightarrow} \sum_{m: T_m \in [0,1]} |\Delta L_T|^p V_m =: Z. \quad (5.37)
\]

Suppose first that \( p \geq 1 \) and decompose

\[
(n^{op}V(X)_n)^{1/p} = (n^{op}V(\hat{X}(j))_n)^{1/p} + \left( (n^{op}V(X)_n)^{1/p} - (n^{op}V(\hat{X}(j))_n)^{1/p} \right)
\]

\[
= Y_{n,j} + U_{n,j}.
\]

Eq. (5.1) and (5.1) show

\[
Y_{n,j} \overset{L_s}{\longrightarrow} Z_j^{1/p} \quad \text{and} \quad Z_j^{1/p} \overset{p}{\longrightarrow} \overset{j \to \infty}{\longrightarrow} Z_j^{1/p}. \quad (5.38)
\]

Note that \( X - \hat{X}(j) = X(j) \), where \( X(j) \) is defined in (5.1). For all \( \varepsilon > 0 \) we have by Minkowski’s inequality

\[
\lim_{j \to \infty} \sup_{n \to \infty} \mathbb{P}\left( |U_{n,j}| > \varepsilon \right) \leq \lim_{j \to \infty} \sup_{n \to \infty} \mathbb{P}\left( n^{op}V(X(j))_n > \varepsilon^p \right) = 0, \quad (5.39)
\]

where the last equality follows by Lemma 5.2. By a standard argument, see e.g. [12, Theorem 3.2], (5.1) and (5.1) implies that \( (n^{op}V(X)_n)^{1/p} \overset{L_s}{\longrightarrow} Z^{1/p} \) which completes the proof of Theorem 1.1(i) when \( p \geq 1 \). For \( p < 1 \), Theorem 1.1(i) follows by (5.1), (5.1), the inequality \( |V(X)_n - V(\hat{X}(j))_n| \leq V(X(j))_n \) and [12, Theorem 3.2]. \( \square \)
5.2 Proof of Theorem 1.1(ii)

Suppose that $\alpha < k - 1/\beta$, $p < \beta$ and $L$ is a symmetric $\beta$-stable Lévy process. In the proof of Theorem 1.1(ii) we will use the following notation: For all $n \geq 1$, $r \geq 0$ set

$$
\phi^n_r(s) = D^n g_n(r - s), \quad \phi^\infty_r(u) = h_k(r - u),
$$

where $g_n$ and $D^k$ are defined at (4.1) and (4.2), and the function $h_k$ is defined in (4.3). For all $n \in \mathbb{N} \cup \{\infty\}$ and $t \geq 0$

$$
Y^n_t = \int_{-\infty}^t \phi^n_r(s) dL_s.
$$

By self-similarity of $L$ of index $1/\beta$ we have for all $n \in \mathbb{N}$,

$$
\{n^{\alpha+1/\beta} \Delta^n_{t,k} X : i = k, \ldots, n\} \overset{d}{=} \{Y^n_t : i = k, \ldots, n\},
$$

where $\overset{d}{=}$ means equality in distribution. For $\alpha < 1 - 1/\beta$, $Y^n$ is the $k$-order increments of a linear fractional stable motion. For $\alpha \geq 1 - 1/\beta$ the linear fractional stable motion is not well-defined, but $Y^n$ is well-defined since the function $h_k$ is locally bounded and satisfies $|h_k(x)| \leq K x^{\alpha-k}$ for all $x \geq k + 1$, which implies that $h_k \in L^\beta(\mathbb{R})$. We are now ready to prove Theorem 1.1(ii), which is done by approximate $Y^n_t$ by $Y_t^\infty$ and applying the ergodic properties of $Y_t^\infty$.

To show that $Y^n_k \to Y^\infty_k$ in $L^p$ as $n \to \infty$ we recall that for $\phi : s \mapsto s^{\alpha}$ we have $D^k \phi = h_k \in L^\beta(\mathbb{R})$. For $s \in \mathbb{R}$ let $\psi_n(s) = g_n(s) - s^{\alpha}$. Since $p < \beta$,

$$
\mathbb{E}[|Y^n_k - Y^\infty_k|^p] = K \left( \int_0^\infty |D^k \psi_n(s)|^{p/\beta} ds \right)^{p/\beta}.
$$

To show that the right-hand side of (5.2) converge to zero we note that

$$
\int_{n+k}^\infty |D^k \psi_n(s)|^{p/\beta} ds \leq K n^{\beta(\alpha-k)} \int_{n+k}^\infty |g^{(k)}((s - k)/n)|^{\beta} ds
$$

$$
= K n^{\beta(\alpha-k)+1} \int_1^\infty |g^{(k)}(s)|^{\beta} ds \to 0 \quad \text{as } n \to \infty,
$$

which implies that

$$
\int_{n+k}^\infty |D^k \psi_n(s)|^{p/\beta} ds \leq K \left( \int_{n+k}^\infty |D^k g_n(s)|^{p/\beta} ds + \int_{n+k}^\infty |D^k \phi(s)|^{p/\beta} ds \right) \to 0 \quad \text{as } n \to \infty.
$$

By (4.1) of Lemma 4.1 it holds that

$$
|D^k g_n(s)| \leq K (s - k)^{\alpha-k}
$$

for $s \in (k + 1, n)$. Therefore, for $s \in (0, n]$ we have

$$
|D^k \psi_n(s)| \leq K \left( \mathbb{1}_{s \leq k+1} + \mathbb{1}_{s > k+1} (s - k)^{\alpha-k} \right),
$$
where the function on the right-hand side of (5.2) is in $L^\beta(\mathbb{R}_+)$. For fixed $s \geq 0$, $\psi_n(s) \to 0$ as $n \to \infty$ by assumption (1), and hence $D^k\psi_n(s) \to 0$ as $n \to \infty$. By (5.2) and the dominated convergence theorem this shows that

$$
\int_0^n |D^k\psi_n(s)|^\beta \, ds \to 0.
$$

By (5.2), (5.2) and (5.2) we have

$$
E[|Y^n_k - Y^\infty_k|^p] \to 0 \quad \text{as} \quad n \to \infty,
$$

which implies that

$$
E \left[ \frac{1}{n} \sum_{i=k}^n |Y^n_i - Y^\infty_i|^p \right] = \frac{1}{n} \sum_{i=k}^n E[|Y^n_i - Y^\infty_i|^p] \leq E[|Y^n - Y^\infty|^p] \to 0
$$

as $n \to \infty$. Moreover, $(Y^\infty_t)_{t \in \mathbb{R}}$ is mixing since it is a symmetric stable moving average, see e.g. [14]. This implies, in particular, that the discrete time stationary sequence $\{Y_j\}_{j \in \mathbb{Z}}$ is mixing and hence ergodic. According to Birkhoff’s ergodic theorem (cf. [26, Theorem 10.6])

$$
\frac{1}{n} \sum_{i=k}^n |Y^\infty_i|^p \overset{a.s.}{\to} E[|Y_k|^p] =: m_p \in (0, \infty) \quad \text{as} \quad n \to \infty.
$$

We note that $m_p$ defined at (5.2) coincide with the definition in Theorem 1.1(ii), cf. [34, Property 1.2.17 and 3.2.2]. By (5.2), Minkowski’s inequality and (5.2), we deduce

$$
\frac{1}{n} \sum_{i=k}^n |Y^n_i|^p \overset{P}{\to} m_p \quad \text{as} \quad n \to \infty.
$$

By (5.2) it shows that

$$
n^{-1+p(\alpha+1/\beta)} V(X)_n = \frac{1}{n} \sum_{i=k}^n |n^{\alpha+1/\beta}\Delta_{i,k}X|^p \overset{d}{=} \frac{1}{n} \sum_{i=k}^n |Y^n_i|^p \overset{P}{\to} m_p
$$

as $n \to \infty$. This completes the proof of Theorem 1.1(ii).

5.3 Proof of Theorem 1.1(iii)

We will derive Theorem 1.1(iii) from the two lemmas below. For $k \in \mathbb{N}$ and $p \in [1, \infty)$ let $W^{k,p}$ denote the Wiener space of functions $\zeta : [0,1] \to \mathbb{R}$ which are $k$-times differentiable with $\zeta^{(k)} \in L^p([0,1])$ where $\zeta^{(k)}(t) = \partial^k \zeta(t)/\partial t^k$ $\lambda$-a.s. First we will show that, under the conditions in Theorem 1.1(iii), $X \in W^{k,p}$ almost surely.

**Lemma 5.3.** Suppose that $p \neq 0$, $p \geq 1$ and (A) holds. If $\alpha > k - 1/(p \lor \beta)$ then

$$
X \in W^{k,p} \quad \text{a.s.} \quad \text{and} \quad \frac{\partial^k}{\partial t^k} X_t = \int_t^u g^{(k)}(t-s) \, dL_s \quad \lambda \otimes \mathbb{P}\text{-a.s.}
$$

Eq. (5.3) remains valid for $p = \theta$ if, in addition, (A-log) holds.
Proof. We will not need the assumption (1) on $g$ in the proof. For notation simplicity we only consider the case $k = 1$, since the general case follows by similar arguments. To prove (5.3) it is sufficient to show that the three conditions (5.3), (5.4) and (5.6) from [13, Theorem 5.1] are satisfied (this result uses the condition $p \geq 1$). In fact, the representation (5.3) of $(\partial / \partial t) X_t$ follows by the equation below (5.10) in [13]. In our setting the function $\dot{\sigma}$ defined in [13, Eq. (5.5)] is constant and hence (5.3), (5.4) and (5.6) in [13] simplifies to

$$\int_{\mathbb{R}} \nu\left( \frac{1}{\|g'\|_{L^p([s,1+s])}}, \infty \right) ds < \infty,$$

(5.51)

$$\int_{0}^{\infty} \int_{\mathbb{R}} \left( |xg'(s)|^2 \wedge 1 \right) \nu(dx) ds < \infty,$$

(5.52)

$$\int_{0}^{1} \int_{\mathbb{R}} |g'(t+s)|^p \left( \int_{r/|g'(t+s)|}^{1/\|g'(t+s)\|_{L^p([s,1+s])}} x^p \nu(dx) \right) ds dt < \infty$$

(5.53)

for all $r > 0$. When the lower bound in the inner integral in (5.3) exceed the upper bound the integral is set to zero. Since $\alpha > 1 - 1/\beta$ we may choose $\varepsilon > 0$ such that $(\alpha - 1)(\beta + \varepsilon) > -1$. To show (5.3) we use the estimates

$$\|g'\|_{L^p([s,1+s])} \leq K \left( 1_{\{s \in [-1,1]\}} + 1_{\{s > 1\}} |g'(s)| \right), \quad s \in \mathbb{R},$$

and

$$\nu((u, \infty)) \leq \begin{cases} Ku^{-\theta} & u \geq 1 \\ Ku^{-\beta-\varepsilon} & u \in (0,1], \end{cases}$$

which both follow from assumption (A). Hence, we deduce that

$$\int_{\mathbb{R}} \nu\left( \frac{1}{\|g'\|_{L^p([s,1+s])}}, \infty \right) ds$$

$$\leq \int_{-1}^{1} \nu\left( \frac{1}{K}, \infty \right) ds + \int_{1}^{\infty} \nu\left( \frac{1}{|g'(s)|}, \infty \right) ds$$

$$\leq 2\nu\left( \frac{1}{K}, \infty \right) + K \int_{1}^{\infty} \left( |g'(s)|^\theta 1_{\{|g'(s)| \leq 1\}} + |g'(s)|^{\beta+\varepsilon} 1_{\{|g'(s)| > 1\}} \right) ds < \infty$$

which shows (5.3) (recall that $|g'|$ is decreasing on $(1, \infty)$). To show (5.3) we will use the following two estimates:

$$\int_{0}^{1} \left( |x|^{\alpha-1} |x|^2 \wedge 1 \right) dx \leq \begin{cases} K \left( 1_{\{|x| \leq 1\}} |x|^{1/(1-\alpha)} + 1_{\{|x| > 1\}} \right) & \alpha < 1/2 \\ K \left( 1_{\{|x| \leq 1\}} x^2 \log(1/x) + 1_{\{|x| > 1\}} \right) & \alpha = 1/2 \\ K \left( 1_{\{|x| \leq 1\}} x^2 + 1_{\{|x| > 1\}} \right) & \alpha > 1/2, \end{cases}$$

(5.54)

and

$$\int_{\{|x| > 1\}} \left( |xg'(s)|^2 \wedge 1 \right) \nu(dx) \leq K \int_{1}^{\infty} \left( |xg'(s)|^2 \wedge 1 \right) x^{1-\theta} dx \leq K |g'(s)|^\theta.$$
For \( \alpha < 1/2 \) we have

\[
\int_0^\infty \int_\mathbb{R} \left( |s|^\alpha - 1 \right) \nu(dx) \, ds \\
\leq K \left\{ \int_\mathbb{R} \left( |s|^\alpha - 1 \right) \nu(dx) + \int_1^\infty \int_\{ |x| \leq 1 \} \left( |s|^\alpha - 1 \right) \nu(dx) \, ds \\
+ \int_1^\infty \int_\{ |x| > 1 \} \left( |s|^\alpha - 1 \right) \nu(dx) \, ds \right\}
\]

\[
\leq K \left\{ \int_\mathbb{R} \left( 1_{\{ |x| \leq 1 \}} |x|^{1/(1-\alpha)} + 1_{\{ |x| > 1 \}} \right) \nu(dx) \\
+ \int_1^\infty |s|^2 \nu(dx) \left( \int_\{ |x| \leq 1 \} x^2 \nu(dx) \right) + \int_1^\infty |s|^\theta \nu(dx) \right\} < \infty,
\]

where the first inequality follows by assumption (A), the second inequality follows by \((5.3)\) and \((5.3)\), and the last inequality is due to the fact that \( 1/(1-\alpha) > \beta \) and \( g' \in L^\theta((1, \infty)) \cap L^2((1, \infty)) \). This shows \((5.3)\). The two remaining cases \( \alpha = 1/2 \) and \( \alpha > 1/2 \) follow similarly.

Now, we will prove that \((5.3)\) holds. Since \( |g'| \) is decreasing on \((1, \infty)\) we have for all \( t \in [0, 1] \) that

\[
\int_1^\infty |g'(t + s)|^p \left( \int_{r/|g'(t+s)|}^{1/|g'(t+s)|} x^p \nu(dx) \right) ds \\
\leq \int_1^\infty |g'(s)|^p \left( \int_{r/|g'(t+s)|}^{1/|g'(t+s)|} x^p \nu(dx) \right) ds \\
\leq \frac{K}{p - \theta} \int_1^\infty |g'(s)|^\theta ds < \infty,
\]

and for \( p < \theta \), \((5.3)\) is less than or equal to

\[
\frac{K}{p - \theta} \int_1^\infty |g'(s)|^\theta ds < \infty,
\]

and for \( p > \theta \), \((5.3)\) is less than or equal to

\[
\frac{K r^{p-\theta}}{\theta - p} \int_1^\infty |g'(s)|^p |g'(s) + 1|^{\theta - p} ds \leq \frac{K r^{p-\theta}}{\theta - p} \int_1^\infty |g'(s)|^\theta ds < \infty,
\]

where the first inequality is due to the fact that \( |g'| \) is decreasing on \((1, \infty)\). Hence we have shown that

\[
\int_0^1 \int_1^\infty |g'(t + s)|^p \left( \int_{r/|g'(t+s)|}^{1/|g'(t+s)|} x^p \nu(dx) \right) ds dt < \infty \quad (5.57)
\]

for \( p \neq \theta \). Suppose that \( p > \beta \). For \( t \in [0, 1] \) and \( s \in [-1, 1] \) we have

\[
\int_{r/|g'(t+s)|}^{1/|g'|} x^p \nu(dx) \leq \int_{r/|g'|}^{1/|g'|} x^p \nu(dx) + \int_{r/|g'(t+s)|}^1 x^p \nu(dx) \leq K \left( |g'|^{\theta - p} + 1 \right)
\]
and hence
\[
\int_0^1 \int_{-1}^1 |g'(t + s)|^p \left( \int_{r/[g'(t+s)]}^{1/\|g'\|_{L^p([s,1+s])}} \nu(x) \, dx \right) \, ds \, dt \\
\leq K \left( \int_{-1}^1 \|g'\|^\|_{L^p([s,1+s])} \, ds + \int_{-1}^1 \|g'\|_{L^p([s,1+s])}^\| \, ds \right) < \infty.
\]
(5.59)

Suppose that \( p \leq \beta \). For \( t \in [0, 1] \) and \( s \in [-1,1] \) we have
\[
\int_{r/[g'(t+s)]}^{1/\|g'\|_{L^p([s,1+s])}} \nu(x) \, dx \leq K \left( \|g'\|^\|_{L^p([s,1+s])} + |g'(t + s)|^{\beta + \varepsilon - p} \right)
\]
and hence
\[
\int_0^1 \int_{-1}^1 |g'(t + s)|^p \left( \int_{r/[g'(t+s)]}^{1/\|g'\|_{L^p([s,1+s])}} \nu(x) \, dx \right) \, ds \, dt \\
\leq K \left( \int_{-1}^1 \|g'\|^\|_{L^p([s,1+s])} \, ds + \int_{-1}^1 \|g'\|_{L^{\beta + \varepsilon}([s,1+s])}^\| \, ds \right) < \infty
\]
(5.61)
since \((\alpha - 1)(\beta + \varepsilon) > -1\). Thus, (5.3) follows by (5.3), (5.3)–(5.3) and (5.3)–(5.3).

For \( p = \theta \) the above proof remains valid except for (5.3), where we need the additional assumption (A-log). This completes the proof. \( \square \)

**Lemma 5.4.** For all \( \zeta \in W^{k,p} \) we have as \( n \to \infty \),
\[
n^{-1+k}V(\zeta, p; k)_n \to \int_0^1 |\zeta^{(k)}(s)|^p \, ds.
\]
(5.62)

**Proof.** First we will assume that \( \zeta \in C^{k+1}(\mathbb{R}) \) and afterwards we will prove the lemma by approximation. Successive applications of Taylor’s theorem gives
\[
\Delta_{n,k} \zeta = \zeta^{(k)} \left( \frac{i - k}{n} \right) \frac{1}{n^k} + a_{i,n}, \quad n \in \mathbb{N}, \ k \leq i \leq n
\]
where \( a_{i,n} \in \mathbb{R} \) satisfies
\[
|a_{i,n}| \leq Kn^{-k-1}, \quad n \in \mathbb{N}, \ k \leq i \leq n.
\]
By Minkowski’s inequality,
\[
\left| \left( n^{kp-1}V(\zeta) \right)^{1/p} - \left( \sum_{j=k}^n \zeta^{(k)} \left( \frac{i - k}{n} \right) \frac{1}{n^k} \right)^{1/p} \right|
\leq \left( \sum_{j=k}^n |a_{i,n}|^p \right)^{1/p} \leq Kn^{-1-1/p} \to 0.
\]

By continuity of \( \zeta^{(k)} \) we have
\[
n^{kp-1} \sum_{i=k}^n \zeta^{(k)} \left( \frac{i - k}{n} \right) \frac{1}{n^k} \to \int_0^1 |\zeta^{(k)}(s)|^p \, ds
\]
as \( n \to \infty \), which shows (5.4).

The statement of the lemma for a general \( \zeta \in W^{k,p} \) follows by approximating \( \zeta \) through a sequence of \( C^{k+1}([\mathbb{R}]) \)-functions and Minkowski’s inequality. This completes the proof. \( \square \)

The Lemmas 5.3 and 5.4 yield Theorem 1.1(iii). \( \square \)

6 Proof of Theorem 1.2

Throughout this section we suppose that the assumptions stated in Theorem 1.2 hold, which in particular means that \( p < \beta/2 \). Suppose in addition that \( \alpha \in (0, k - 1/\beta) \) which will be satisfied in both (i) and (ii) of Theorem 1.2, and note that this condition is equivalent to \( (\alpha - k)\beta > -1 \). Without loss of generality we will assume that the symmetric \( \beta \)-stable Lévy process \( L \) has scale parameter \( \sigma = 1 \) and (A) holds with \( \delta = c_0 = 1 \). In the following subsection we will consider some notation and decompositions to be used in the proof of Theorem 1.2.

6.1 Notation and outline of the proof

In addition to the notation introduced in Subsection 5.2 we define the following truncated version of \( Y^n_r \) in (5.2) by

\[
Y^{n,m}_r = \int_{r-m}^{r} \phi^n_r(s) \, dL_s, \quad n \in \mathbb{N} \cup \{\infty\}, m, r \geq 0,
\]

where the function \( \phi^n_r \) has been introduced in (5.2). For \( n, m \in \mathbb{N} \) we set

\[
S_n = \sum_{r=k}^{n} \left( |Y^n_r|^p - \mathbb{E}[|Y^n_r|^p] \right) \quad \text{and} \quad S_{n,m} = \sum_{r=k}^{n} \left( |Y^{n,m}_r|^p - \mathbb{E}[|Y^{n,m}_r|^p] \right).
\]

By (5.2) we have that

\[
n^{p(\alpha+1/\beta)} V(p; k)_n \overset{d}{=} S_n + (n - k + 1)\mathbb{E}[|Y^n_1|^p], \quad (6.1)
\]

and hence when proving Theorem 1.2 we may instead analyse the right-hand side of (6.1). For all \( n \in \mathbb{N} \cup \{\infty\}, j \geq 1 \) and \( m \geq 0 \) we also set

\[
\rho^n_j = \|\phi^n_j\|_{L^p([0,1])}, \quad \rho^{n,m}_j = \|\phi^n_j\|_{L^p([j-m,j]\setminus[0,1])},
\]

\[
U^{n,j}_r = \int_{r}^{r+1} \phi^n_j(u) \, dL_u. \quad (6.2)
\]

For all \( r \in \mathbb{R} \) we consider the following \( \sigma \)-algebras

\[
\mathcal{G}_r = \sigma(L_s - L_u : s, u \leq r) \quad \text{and} \quad \mathcal{G}^1_r = \sigma(L_s - L_u : r \leq s, u \leq r + 1).
\]
We note that \((G^1_r)_{r \geq 0}\) is not a filtration. Let \(W\) denote a symmetric \(\beta\)-stable random variable with scale parameter \(\rho \in (0, \infty)\) and \(\Phi_\rho : \mathbb{R} \to \mathbb{R}\) be defined by

\[\Phi_\rho(x) = \mathbb{E}[|W + x|^p] - \mathbb{E}[|W|^p], \quad x \in \mathbb{R}.\]

(6.3)

For all \(n \geq 1, m, r \geq 0\) let

\[V^{n,m}_r = |Y^{n,m}_r|^p - |Y^{n,m}_r|^p - \mathbb{E}[|Y^{n,m}_r|^p - |Y^{n,m}_r|^p],\]

\[\zeta^{n,m}_{r,j} = \mathbb{E}[V^{n,m}_r | G_{r-j+1}] - \mathbb{E}[V^{n,m}_r | G_{r-j}] - \mathbb{E}[V^{n,m}_r | G^1_{r-j}],\]

(6.4)

\[R^{n,m}_r = \sum_{j=1}^\infty \zeta^{n,m}_{r,j} \quad \text{and} \quad Q^{n,m}_r = \sum_{j=1}^\infty \mathbb{E}[V^{n,m}_r | G^1_{r-j}].\]

(6.5)

According to Remark 6.1 below the two series \(R^{n,m}_r\) and \(Q^{n,m}_r\) converge with probability one, and the following decomposition of \(S_n - S_{n,m}\) holds with probability one

\[S_n - S_{n,m} = \sum_{r=k}^n R^{n,m}_r + \sum_{r=k}^n Q^{n,m}_r.\]

(6.6)

Decompositions of the type (6.1) have been successfully used in theory of discrete time moving averages, see e.g. Ho and Hsing [22], and will also play a crucial role in the proof of Theorem 1.2. Indeed, for the proof of Theorem 1.2(i) we will choose \(m = 0\) in (6.1) and since \(S_{n,0} = 0\) we have the following decomposition of \(S_n:\)

\[S_n = \sum_{r=k}^n R^{n,0}_r + \sum_{r=k}^n (Q^{n,0}_r - Z_r) + \sum_{r=k}^n Z_r,\]

(6.7)

where

\[Z_r = \sum_{j=1}^\infty \left\{ \Phi_{\rho_j}^{\infty}(U_r^{j,\infty}) - \mathbb{E}[\Phi_{\rho_j}^{\infty}(U_r^{j,\infty})] \right\}.

After suitable scaling we show that the first two sums on the right-hand side of (6.1) are negligible, see (6.3). To analyse the third sum we note the random variables \(\{Z_r : r \geq 1\}\) are independent and identically distributed, which follows from their definition. Hence, to complete the proof of Theorem 1.2(i), it is enough to show that the common law of \(\{Z_r : r \geq 1\}\) belong to the domain of attraction of an \((k - \alpha)\beta\)-stable random variable, which is done in (6.3).

The main part of the proof of Theorem 1.2(ii) consists in showing that

\[\lim_{m \to \infty} \limsup_{n \to \infty} \left( n^{-1}\mathbb{E}[(S_n - S_{n,m})^2] \right) = 0,\]

(6.8)

see (6.4). We prove (6.1) by estimating each of the two sums on the right-hand side of (6.1) separately. We note that for a fixed \(m \geq 1\) the sequences \(\{S_{n,m} : n \geq 1\}\) are \(m\)-dependent, which means that for all \(k \geq 1\) the random variables \(\{S_{1,m}, \ldots, S_{k,m}\}\) are independent of \(\{S_{n,m} : n \geq k + 1 + m\}\). Hence, using a standard result for \(m\)-dependent sequences one can deduce a central limit theorem for the sequences \(\{S_{n,m} : n \geq 1\}\) and by using (6.1) transfer this result to \(S_n\), which will prove Theorem 1.2(ii). In the next subsection we present some estimates which play a key role in the proof of Theorem 1.2.
6.2 Preliminary estimates

The assumption \(|g^{(k)}(x)| \leq K x^{\alpha-k}\) for all \(x > 0\) implies that
\[
\|\phi_j^\alpha\|_{L^\beta([0,1])} \leq K_j^{\alpha-k}
\]
for some finite constant \(K\) which do not depend on \(j \in \mathbb{N}, n \in \mathbb{N} \cup \{\infty\}\). The estimate \((6.2)\) will be used repeatedly throughout the proof. In the following we will collect some estimates on the functions \(\Phi_\rho\) defined in \((6.1)\) which will be used various places in the proofs. We first observe the identity
\[
|x|^p = a_p^{-1} \int_{\mathbb{R}} (1 - \exp(iux))|u|^{-1-p}du \quad \text{for } p \in (0,1),
\]
with \(a_p = \int_{\mathbb{R}} (1 - \exp(iux))|u|^{-1-p}du \in \mathbb{R}_+\), which can be shown by substitution \(y = ux\).

Secondly, for any deterministic function \(\varphi : \mathbb{R} \to \mathbb{R}\) satisfying \(\varphi \in L^\beta(\mathbb{R})\), it holds that
\[
E \left[ \exp \left( iu \int_{\mathbb{R}} \varphi(s) dL_s \right) \right] = \exp \left( -|u|^\beta \int_{\mathbb{R}} |\varphi(s)|^\beta ds \right).
\]
Applying the identities \((6.2)\) and \((6.2)\), we obtain the representation
\[
\Phi_\rho(x) = a_p^{-1} \int_{\mathbb{R}} (1 - \cos(ux))e^{-|x|^\beta|u|^{-1-p}} du.
\]
From \((6.2)\), we deduce that \(\Phi_\rho \in C^3(\mathbb{R})\) and it holds that
\[
\Phi'_\rho(x) = a_p^{-1} \int_{\mathbb{R}} \sin(ux)|u|^{-p}e^{-|x|^\beta|u|^\beta} du
\]
\[
\Phi''_\rho(x) = a_p^{-1} \int_{\mathbb{R}} \cos(ux)|u|^{-1}e^{-|x|^\beta|u|^\beta} du
\]
\[
\Phi'''_\rho(x) = -a_p^{-1} \int_{\mathbb{R}} \sin(ux)|u|^{2-p}e^{-|x|^\beta|u|^\beta} du
\]
In the following we let \(\varepsilon > 0\) be a fixed number. The identities at \((6.2)\) imply that for \(v = 1, 2, 3\) there exists a finite constant \(K_\varepsilon\) such that for all \(\rho \geq \varepsilon\) and all \(x \in \mathbb{R}\)
\[
|\Phi^{(v)}_\rho(x)| \leq K_\varepsilon.
\]
By \((6.2)\) we also deduce the following estimate by several applications of the mean value theorem
\[
|\Phi_\rho(x) - \Phi_\rho(y)| \leq K_\varepsilon \left( |x| \wedge 1 + |y| \wedge 1 \right) |x - y| |1_{\{|x-y|\leq 1\}} + |x - y|^p 1_{\{|x-y|>1\}}
\]
which holds for all \(\rho \geq \varepsilon\) and all \(x, y \in \mathbb{R}\). Eq. \((6.2)\) used on \(y = 0\) yields that
\[
|\Phi_\rho(x)| \leq K_\varepsilon (|x|^p \wedge |x|^2).
\]
In particular, it implies that
\[
|\Phi_\rho(x)| \leq K_\varepsilon |x|^l \quad \text{for all } l \in (p, \beta).
\]
Moreover, for all \(r \in [p,2]\) and \(\rho_1, \rho_2 \geq \varepsilon\) we deduce by \((6.2)\) that
\[
|\Phi_{\rho_1}(x) - \Phi_{\rho_2}(x)| \leq K_\varepsilon \rho_1^\beta - \rho_2^\beta \cdot |x|^r \quad \text{for all } x \in \mathbb{R}.
\]
Remark 6.1. In the following we will show that the three series \( R^{n,m}_r, Q^{n,m}_r \) and \( Z_r \) defined in (6.1) converge almost surely, and the identity (6.1) holds almost surely. To show the above claim we will first prove that for all \( n \geq 1 \) and \( m \geq 0 \) the two series

\[
(a) : \sum_{j=1}^{\infty} \mathbb{E}[V^{n,m}_r | G^1_{r-j}] \quad \quad (b) : \sum_{j=m+1}^{\infty} \left( \Phi_{\rho_j}^{\infty}(U^{\infty}_{j,r}) - \mathbb{E}[\Phi_{\rho_j}^{\infty}(U^{\infty}_{j,r})] \right)
\]

(6.19)

converge absolutely with probability one. For \( j \geq 1 \) we have that

\[
\mathbb{E}[V^{n,m}_r | G^1_{r-j}] = \Phi_{\rho_j}^{n}(U^{n}_{r,r-j}) - \Phi_{\rho_j}^{n,m}(U^{n}_{r,r-j}) \mathbb{1}_{\{j \leq m\}}
\]

\[
- \mathbb{E}\left[ \left( \Phi_{\rho_j}^{n}(U^{n}_{r,r-j}) - \Phi_{\rho_j}^{n,m}(U^{n}_{r,r-j}) \mathbb{1}_{\{j \leq m\}} \right) \right].
\]

(6.20)

We have that \( \rho_j^n \to \|\phi^n\|_{L^\beta(\mathbb{R})} > 0 \) as \( j \to \infty \), and hence \( \{\rho_j^n : j \geq N\} \) is bounded away from zero for \( N \) large enough. For all \( j > \max\{m, N\} \) and all \( \gamma \in (p, \beta) \) we have

\[
\mathbb{E}\left[ \left| \mathbb{E}[V^{n,m}_r | G^1_{r-j}] \right| \right] \leq 2\mathbb{E}\left[ \left| \Phi_{\rho_j}^{n}(U^{n}_{r,r-j}) \right| \right] \leq K\mathbb{E}[|U^{n}_{r,r-j}|^\gamma]
\]

\[
\leq K\|\phi^n\|_{L^\beta([0,1])}^\gamma \leq K_j^{(\alpha-k)\gamma},
\]

where the first inequality follows by (6.1), the second inequality follows by (6.2) and the last inequality follows by (6.2). By choosing \( \gamma \) close enough to \( \beta \) and using the assumption \( (\alpha-k)\beta < -1 \), it follows that the series (a) in (6.1) converges absolutely almost surely. A similar application of (6.2) and (6.2) also shows that the series (b) in (6.1) converge absolutely almost surely. Next we note that \( V^{n,m}_r = \mathbb{E}[V^{n,m}_r | G_r] \) and \( \mathbb{E}[V^{n,m}_r | G_{r-j}] \to \mathbb{E}[V^{n,m}_r] = 0 \) almost surely as \( j \to \infty \). The latter claim follows from Kolmogorov’s 0-1 law and the backward martingale convergence theorem. From these two properties we deduce that \( V^{n,m}_r \) has the following telescoping sum representation

\[
V^{n,m}_r = \sum_{j=1}^{\infty} \left( \mathbb{E}[V^{n,m}_r | G_{r-j+1}] - \mathbb{E}[V^{n,m}_r | G_{r-j}] \right),
\]

(6.21)

where sum converge almost surely. Convergence of the three series in (6.1) and (6.1) show the claim in the remark together with the observation that \( S_n = S_{n,m} = \sum_{r=k}^{\infty} V^{n,m}_r \).

The following estimates will play a key role in the proof of Theorem 1.2.

Proposition 6.2. Suppose that the conditions of Theorem 1.2 hold, and hence in particular \( p < \beta/2 \) and \( \alpha < k - 1/\beta \). For all \( \epsilon > 0 \) there exists a finite constant \( K \) such that for all \( n \geq 1 \) and \( m \geq 0 \) we have the following estimates:

\[
\mathbb{E}\left[ \left( \sum_{r=k}^{n} R^{n,m}_r \right)^2 \right] \leq K \left[ n \left( (m+1)^{(\alpha-k)+1} \log^2 (m+1) + (m+1)^2 (\alpha-k)+3 \right) + n^2 (\alpha-k)+4 + \log(n) \right].
\]

(6.22)
If in addition \( \alpha < k - 2/\beta \) then the estimate (6.2) holds:

\[
\mathbb{E} \left[ \left( \sum_{r=k}^{n} Q_r^{n,m} \right)^2 \right] \leq K \left( n^{(\alpha-k)\beta+3+\varepsilon} + n(m + 1)^{(\alpha-k)\beta+2+\varepsilon} + 1 \right).
\]

(6.23)

On the other hand, if \( \alpha > k - 2/\beta \) then the following estimate holds:

\[
\mathbb{E} \left[ \left| \sum_{r=k}^{n} (Q_r^{n,0} - Z_r) \right| \right] \leq K \left( n^{(\alpha-k)\beta+2+\varepsilon} + n^{1-\beta+\varepsilon} \right).
\]

(6.24)

The proof of Proposition 6.2 is carried out in Subsections 6.5 and 6.6. We will also need the following inequality.

**Lemma 6.3.** Assume that the conditions of Theorem 1.2 hold. Then there exists a finite constant \( K \) such that for all \( j, n \geq 1 \) we have

\[
\left| \| \phi_j^n \|_{L^\beta([0,1])}^{\beta} - \| \phi_j^\infty \|_{L^\beta([0,1])}^{\beta} \right| \leq K \begin{cases} 
\frac{n^{-1}}{n^{(\alpha-k)\beta+1}} & \text{when } \alpha \in (0, k - 2/\beta) \\
n^{(\alpha-k)\beta+1} & \text{when } \alpha \in (k - 2/\beta, k - 1/\beta) 
\end{cases}
\]

where the functions \( \phi_j^n \) and \( \phi_j^\infty \) has been introduced at (5.2).

The proof of Lemma 6.3 is postponed to Subsection 6.7. We are now ready to show Theorem 1.2(i).

### 6.3 Proof of Theorem 1.2(i)

To prove Theorem 1.2(i) we will first state and prove the following lemma:

**Lemma 6.4.** There exists \( \delta > 0 \) and a finite constant \( K > 0 \) such that for all \( \varepsilon \in (0, \delta), \rho > \delta, \kappa, \tau \in L^\beta([0,1]) \) with \( \| \kappa \|_{L^\beta([0,1])}, \| \tau \|_{L^\beta([0,1])} \leq 1 \) the following inequality holds

\[
\left\| \Phi_\rho \left( \int_0^1 \kappa(s) \, dL_s \right) - \Phi_\rho \left( \int_0^1 \tau(s) \, dL_s \right) \right\|_{L^\beta([0,1])} \leq K \left( \| \kappa - \tau \|_{L^\beta([0,1])} + \| \kappa - \tau \|_{L^\beta([0,1])}^{\delta/\beta} \right).
\]

(6.25)

Moreover,

\[
\left\| \Phi_\rho \left( \int_0^1 \kappa(s) \, dL_s \right) - \Phi_\rho \left( \int_0^1 \tau(s) \, dL_s \right) \right\|_{L^1} \leq K \left\{ \begin{array}{ll}
\left( \| \kappa \|_{L^\beta([0,1])}^{\beta-1-\varepsilon} + \| \tau \|_{L^\beta([0,1])}^{\beta-1-\varepsilon} \right) & \| \kappa - \tau \|_{L^\beta([0,1])} + \| \kappa - \tau \|_{L^\beta([0,1])}^{\beta} \beta > 1 \\
\| \kappa - \tau \|_{L^\beta([0,1])}^{\beta-\varepsilon} & \beta \leq 1.
\end{array} \right.
\]

(6.26)

To prove Lemma 6.4 we will among others use the following simple estimates.

**Lemma 6.5.** Let \( W \) be a symmetric \( \beta \)-stable random variable with scale parameter \( \rho \).
(i) Let $\gamma < \beta$. For all $\rho \leq 1$ we have that
\[
\mathbb{E}[|W|^\gamma \mathbb{1}_{\{|W| \geq 1\}}] \leq K \rho^\beta.
\]

(ii) Let $\gamma > \beta$. For all $\rho \leq 1$ we have that
\[
\mathbb{E}[(|W| \wedge 1)^\gamma] \leq K \rho^\beta. \tag{6.27}
\]

Proof of Lemma 6.5. Let $\eta$ be the density of a standard symmetric $\beta$-stable random variable. According to [41, Theorem 1.1] we have that $\eta(x) \leq K(1 + |x|)^{-1-\beta}$, $x \in \mathbb{R}$. To prove (i) we use substitution to get
\[
\mathbb{E}[|W|^\gamma \mathbb{1}_{\{|W| \geq 1\}}] = \int_{\mathbb{R}} |x|^\gamma \mathbb{1}_{\{|x| \geq 1\}} \eta(x) \, dx \leq K \rho^{-1} \int_{\mathbb{R}} |x|^\gamma \mathbb{1}_{\{|x| \geq 1\}} |\rho^{-1} x|^{-1-\beta} \, dx \leq K \rho^\beta,
\]
where we use that $\gamma < \beta$ in the last inequality. To show (ii) we note that the assumption $\gamma > \beta$ implies that
\[
\mathbb{E}[|W|^\gamma \mathbb{1}_{\{|W| \leq 1\}}] = \int_{\mathbb{R}} |x|^\gamma \mathbb{1}_{\{|x| \leq 1\}} \eta(x) \, dx \leq K \rho^{-1} \int_{\mathbb{R}} |x|^\gamma \mathbb{1}_{\{|x| \leq 1\}} |\rho^{-1} x|^{-1-\beta} \, dx \leq K \rho^\beta. \tag{6.28}
\]
Moreover, if $W_0$ denotes symmetric $\beta$-stable random variable with scale parameter 1 then
\[
\mathbb{E}[\mathbb{1}_{\{|W| \geq 1\}}] = \mathbb{P}(|W_0| \geq \rho^{-1}) \leq K \rho^\beta,
\]
which together with (6.3) completes the proof of Lemma 6.5.

Proof of Lemma 6.4. For notation simplicity set $U = \int_0^1 \kappa(s) \, dL_s$, $V = \int_0^1 \tau(s) \, dL_s$ and $r_\varepsilon = (k - \alpha)\beta + \varepsilon$ for all $\varepsilon > 0$. To prove (6.4) fix $\delta > 0$ and let $\rho \geq \delta$. According to (6.2) and Minkowski inequality we have that
\[
\left\| \Phi_\rho(U) - \Phi_\rho(V) \right\|_{L^\infty} \leq K \left( \left\| U - V \right\|_{L^\infty} + \left\| U - V \right\|_{L^\infty} \right), \tag{6.29}
\]
where $K = K_\delta$ is a finite constant only depending on $\delta$. To estimate the second term on the right-hand side of (6.3) we note that $p\beta(k - \alpha) < 2p < \beta$ by our assumptions, and hence for all $\varepsilon > 0$ small enough we have that $pr_\varepsilon < \beta$. Therefore, according to Lemma 6.5(i), we have
\[
\left\| U - V \right\|_{L^\infty} \leq K \| \kappa - \tau \|_{L^\beta([0,1])}^{\frac{1}{\beta + \varepsilon}} = K \| \kappa - \tau \|_{L^\beta([0,1])}^{\frac{1}{\beta + \varepsilon}}.
\]
To estimate the first term on the right-hand side of (6.3) we assume first that $k - \alpha \geq 1$ which implies that $r_\varepsilon > \beta$ for all $\varepsilon > 0$, and hence by Lemma 6.5(i)
\[
\left\| U - V \right\|_{L^\infty} \leq K \| \kappa - \tau \|_{L^\beta([0,1])}^{\frac{1}{\beta + \varepsilon}} = K \| \kappa - \tau \|_{L^\beta([0,1])}^{\frac{1}{\beta + \varepsilon}}.
\]
On the other hand, if \( k - \alpha < 1 \) then \( r < \beta \) for all \( \varepsilon > 0 \) close enough to 0 which implies that
\[
\left\| U - V |\{U - V| < 1\} \right\|_{L^{r\varepsilon}} \leq \| U - V \|_{L^{r\varepsilon}} \leq K \| \kappa - \tau \|_{L^{\beta}(\{0,1\})},
\]
and completes the proof of (6.4).

To prove (6.3) we are applying (6.2) to get
\[
\left\| \Phi^\rho(U) - \Phi^\rho(V) \right\|_{L^1} \leq K \left( \left\| \left( |U| \land 1 + |V| \land 1 \right) |U - V|1_{\{U - V| < 1\}} \right\|_{L^1} + \left\| |U - V|^p1_{\{U - V| > 1\}} \right\|_{L^1} \right) . (6.30)
\]
By using that \( p < \beta \) we have by Lemma 6.5(i)
\[
\left\| |U - V|^p1_{\{U - V| > 1\}} \right\|_{L^1} \leq K \| \kappa - \tau \|_{L^{\beta}(\{0,1\})}^\beta .
\]
Suppose first that \( \beta > 1 \). To estimate the first term in (6.3) we let \( r \in (1, \beta) \) and \( q = r/(r - 1) \) denote the conjugated number to \( r \). By Hölder’s inequality we have
\[
\left\| \left( |U| \land 1 + |V| \land 1 \right) |U - V|1_{\{U - V| < 1\}} \right\|_{L^1} \leq \left( \| U \|_{L^q} + \| V \|_{L^q} \right) \left\| |U - V|1_{\{U - V| < 1\}} \right\|_{L^{r\varepsilon}} \leq K \left( \| \kappa \|_{L^{\beta}(\{0,1\})}^{\beta/q} + \| \tau \|_{L^{\beta}(\{0,1\})}^{\beta/q} \right) \| \kappa - \tau \|_{L^{\beta}(\{0,1\})}, (6.31)
\]
where we have used Lemma 6.5(ii) and \( r < \beta < q \) in the second inequality. By (6.3) we obtain (6.4) by choosing \( r \) close enough to \( \beta \). For \( \beta \leq 1 \) and all \( \varepsilon > 0 \) the first term in (6.3) is less than or equal to
\[
2\mathbb{E}[|U - V|1_{\{U - V| \leq 1\}}] \leq 2\mathbb{E}[|U - V|^{1+\varepsilon}1_{\{U - V| \leq 1\}}]^{1/(1+\varepsilon)} \leq K \| \kappa - \tau \|_{L^{\beta}(\{0,1\})}^{\beta/(1+\varepsilon)}
\]
where we have used Lemma 6.5(ii) in the last inequality. Hence choosing \( \varepsilon \) small enough yields (6.4).

To prove Theorem 1.2(i) we use (6.1) to obtain the decomposition
\[
n^{-1/(k-\alpha)\beta} \left( n^{-1+p(\alpha+1/\beta)} V(p; k)n - m_p \right) \overset{d}{=} n^{1/(\alpha-k)\beta} S_n + n^{-1/(\alpha-k)\beta} \left( \frac{n-k+1}{n} \mathbb{E}[|Y_1|^p] - m_p \right) . (6.32)
\]
First we will prove that
\[
n^{-1/(\alpha-k)\beta} S_n \overset{d}{\rightarrow} S \quad \text{as } n \rightarrow \infty , (6.33)
\]
where the random variable \( S \) is defined in Theorem 1.2(i). Afterwards we show that the second term on the right-hand side of (6.3) converges to zero. To show (6.3) we will use
the decomposition (6.1), which shows that it suffices to prove that

\[
\frac{1}{n} \sum_{r=0}^{n} R_{r,0}^{n} \xrightarrow{\Pr} 0, \quad \frac{1}{n} \sum_{r=0}^{n} (Q_{r,0}^{n} - Z_{r}) \xrightarrow{\Pr} 0, \quad (6.34)
\]

as \( n \to \infty \). According to (6.2) of Proposition 6.2 we have that

\[
\mathbb{E} \left[ \left( \frac{1}{n} \sum_{r=0}^{n} R_{r,0}^{n} \right)^{2} \right] \leq K \left( n^{\frac{2}{(k-\alpha)\beta} + 1} + n^{2 \frac{1}{(\alpha-k)\beta} + (\alpha-k)\beta + 2} + n^{\frac{2}{(\alpha-k)\beta}} \log(n) \right) \to 0
\]

as \( n \to \infty \), where we have used the inequality \( 2 < x + 1/x \) for all \( x > 1 \) and the fact that \( (k-\alpha)\beta > 1 \) by assumption. Furthermore, for all \( \varepsilon > 0 \) we have according to (6.2) of Proposition 6.2 and the assumption \( \alpha > k - 2/\beta \) that as \( n \to \infty \)

\[
\mathbb{E} \left[ \left| \frac{1}{n} \sum_{r=0}^{n} (Q_{r,0}^{n} - Z_{r}) \right| \right] \leq K \left( n^{\frac{1}{(k-\alpha)\beta} + (\alpha-k)\beta + 2 + \varepsilon} + n^{\frac{1}{(\alpha-k)\beta} - (\alpha-k)\beta + 1 + \varepsilon} \right) \to 0 \quad (6.35)
\]

for all \( \varepsilon \) close enough to zero. The first term on the right-hand side of (6.3) converge to zero by the inequality \( 2 < x + 1/x \) for all \( x > 1 \) and the assumption \( (k-\alpha)\beta > 1 \). Convergence of the second term on the right-hand side of (6.3) to zero is equivalent to \( \alpha > k - 2/\beta \) for \( \beta \geq 1/2 \) and by explicit assumption for \( \beta < 1/2 \).

In the following we will show the last statement of (6.3). Since \( (Z_{r})_{r \geq k} \) are i.i.d. with mean zero it is enough to show that

\[
\lim_{x \to \infty} x^{(k-\alpha)\beta} \mathbb{P}(Z > x) = \gamma \quad \text{and} \quad \lim_{x \to \infty} x^{(k-\alpha)\beta} \mathbb{P}(Z < -x) = 0 \quad (6.36)
\]

with \( Z := Z_{k} \), cf. [34, Theorem 1.8.1]. The constant \( \gamma \) is defined in (6.3) below. To show (6.3) let us define the function \( \Phi : \mathbb{R} \to \mathbb{R}_{+} \) via

\[
\Phi(x) := \sum_{j=1}^{\infty} \Phi_{j}^{\infty}(\phi_{j}^{\infty}(0)x).
\]

Note that (6.2) implies that \( \Phi_{j}^{\infty}(x) \geq 0 \) and hence \( \Phi \) is positive. Note that \( \rho_{j}^{\infty} \to \rho_{\infty} := \|h_{k}\|_{L^{\beta}(\mathbb{R})} > 0 \) which implies that \( (\rho_{j}^{\infty})_{j \geq 1} \) is bounded away from 0, and hence by (6.2) and for \( l \in (p, \beta) \) with \( (\alpha-k)l < -1 \) we have

\[
|\Phi(x)| \leq K|x|^{\sum_{j=1}^{\infty} \phi_{j}^{\infty}(0)^{l}} \leq K|x|^{\sum_{j=1}^{\infty} j^{l(\alpha-k)}} < \infty, \quad (6.37)
\]

which shows that \( \Phi \) is well-defined. Eq. (6.3) shows moreover that \( \mathbb{E}[\Phi(L_{k+1} - L_{k})] < \infty \), and hence we can define a random variable \( Q \) via

\[
Q = \Phi(L_{k+1} - L_{k}) - \mathbb{E}[\Phi(L_{k+1} - L_{k})] = \sum_{j=1}^{\infty} \left( \Phi_{j}^{\infty}(\phi_{j}^{\infty}(0)(L_{k+1} - L_{k})) - \mathbb{E}[\Phi_{j}^{\infty}(\phi_{j}^{\infty}(0)(L_{k+1} - L_{k}))] \right),
\]
where the last sum converges absolutely almost surely. Since $Q \geq -\mathbb{E}[\Phi(L_{k+1} - L_k)]$, we have that
\[
\lim_{x \to -\infty} x^{(k-\alpha)\beta} \mathbb{P}(Q < -x) = 0.
\] (6.38)

By the substitution $t = (x/u)^{1/(k-\alpha)}$ we have that
\[
x^{1/(\alpha - k)} \Phi(x) = x^{1/(\alpha - k)} \int_0^\infty \Phi_{\rho \infty}(\phi_{\alpha \infty}(0) x) \, dt
\]
\[
= (k - \alpha)^{-1} \int_0^\infty \Phi_{\rho \infty}(\phi_{\alpha \infty}((x/u)^{1/(k-\alpha)}) (0) x) u^{-1 + 1/(\alpha - k)} \, du
\]
\[
\to (k - \alpha)^{-1} \int_0^\infty \Phi_{\rho \infty}(k_\alpha u) u^{-1 + 1/(\alpha - k)} \, du =: \kappa \quad \text{as } x \to \infty, \quad (6.39)
\]
where $k_\alpha = \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)$. Here we have used that $(\rho_{\alpha \infty})_{\alpha \geq 1}$ are bounded away from zero together with the estimate $\Phi_{\rho \infty}$ on $\Phi_{\rho \infty}$ and Lebesgue’s dominated convergence theorem. Note that the constant $\kappa$ defined in (6.3) coincides with the $\kappa$ defined in Remark 3.1. The connection between the tail behaviour of a symmetric $\rho$-stable random variable $S_\rho$, $\rho \in (1, 2)$, and its scale parameter $\bar{\sigma}$ is given via
\[
\mathbb{P}(S_\rho > x) \sim \tau_\rho \bar{\sigma}^\rho x^{-\rho} / 2 \quad \text{as } x \to \infty,
\]
where the function $\tau_\rho$ has been defined in (3.1) (see e.g. [34, Eq. (1.2.10)]). Hence,
\[
\mathbb{P}(|L_{k+1} - L_k| > x) \sim \tau_\beta x^{-\beta} \quad \text{as } x \to \infty, \quad \text{and by (6.3) we readily deduce that as } x \to \infty
\]
\[
\mathbb{P}(Q > x) \sim \gamma x^{(k-\alpha)\beta} \quad \text{with } \gamma = \tau_\beta k_\alpha^{(k-\alpha)\beta}. \quad (6.40)
\]

Next we will show that for some $r > (k - \alpha)\beta$ we have
\[
\mathbb{P}(|Z - Q| > x) \leq K x^{-r} \quad \text{for all } x \geq 1, \quad (6.41)
\]
which implies (6.3), cf. (6.3) and (6.3). To show (6.3) it is sufficient to find $r > (k - \alpha)\beta$ such that
\[
\mathbb{E}[|Z - Q|^r] < \infty
\]
by Markov’s inequality. Furthermore, by Minkowski inequality and the definitions of $Q$ and $Z$ it suffices to show that
\[
\sum_{j=1}^{\infty} \left\| \Phi_{\rho_j \infty}(U_j, k) - \Phi_{\rho_j \infty}(\phi_{\rho_j \infty}(0)(L_{k+1} - L_k)) \right\|_{L^r} < \infty \quad (6.42)
\]
(recall that $(k - \alpha)\beta > 1$). To show (6.3) we note that for all $x \in [0, 1]$ and $j \in \mathbb{N}$ there exists $\theta_{j, x} \in [j, j + x]$ such that
\[
|\phi_{\rho_j \infty}(x) - \phi_{\rho_j \infty}(0)| = |h_k(j + x) - h_k(j)| \leq |h_k'(\theta_{j, x})| \leq K_j^{\alpha - k - 1}. \quad (6.43)
\]
Choose $\delta > 0$ according to Lemma 6.4 and let $r_\varepsilon = (k - \alpha)\beta + \varepsilon$ for all $\varepsilon \in (0, \delta)$. By Lemma 6.4 and (6.3) we have that
\[
\left\| \Phi_{\rho_j \infty}(U_{j, k}) - \Phi_{\rho_j \infty}(\phi_{\rho_j \infty}(0)(L_{k+1} - L_k)) \right\|_{L^{r_\varepsilon}} \leq K \left( \left\| \phi_{\rho_j \infty} - \phi_{\rho_j \infty}(0) \right\|_{L^3([0, 1])} + \left\| \phi_{\rho_j \infty} - \phi_{\rho_j \infty}(0) \right\|_{L^{3/(\alpha - 1 + \beta)}([0, 1])} \right) \leq K \left( j^{\alpha - k - 1} + j^{3/(\alpha - 1 + \beta)} \right). \quad (6.44)
\]
Our assumption \( \alpha < k - 1/\beta \) implies that \( \alpha - k < 0 \). Furthermore, since
\[
\frac{\alpha - k - 1}{k - \alpha + \varepsilon / \beta} \to -1 - 1/(k - \alpha) < -1 \quad \text{as } \varepsilon \to 0,
\]
we may, according to (6.3), choose \( \varepsilon > 0 \) such that (6.3) holds for \( r = r_\varepsilon \) which satisfies the condition \( r > (k - \alpha)\beta \). This completes the proof of (6.3) and hence also of (6.3).

To complete the proof of Theorem 1.2(ii) we show that the second term in (6.3) converges to zero. For this purpose it is enough to show that
\[
n^{-1} \frac{1}{(k - \alpha)\beta} \left( \mathbb{E}[|Y_1^n|^p] - m_p \right) \to 0 \quad \text{as } n \to \infty, \tag{6.45}
\]
since \( 1 - \frac{1}{(k - \alpha)\beta} < 1 \). Recall that \( m_p = \|h_k\|_{L^{p}(\mathbb{R})}^p \mathbb{E}[|Z|^p] \), where \( Z \) is a standard symmetric \( \beta \)-stable random variable and \( \|h_k\|_{L^{p}(\mathbb{R})} = \|\phi_1^N\|_{L^{p}(\mathbb{R})} \). By Lemma 6.3 we have that
\[
\left| \|\phi_1^N\|_{L^{p}(\mathbb{R})}^{\beta} - \|\phi_2^N\|_{L^{p}(\mathbb{R})}^{\beta} \right| \leq Kn^{(\alpha - k)\beta + 1} \to 0, \tag{6.46}
\]
where the convergence to zero is due to the fact that \( (k - \alpha)\beta > 1 \) under our assumptions. Since the function \( x \mapsto x^{p/\beta} \) is continuously differentiable on \((0, \infty)\) and \( \|h_k\|_{L^{p}(\mathbb{R})} > 0 \), it follows by the mean value theorem that
\[
\left| \|\phi_1^N\|_{L^{p}(\mathbb{R})}^{\beta} - \|h_k\|_{L^{p}(\mathbb{R})}^{\beta} \right| \leq K\left( \|\phi_1^N\|_{L^{p}(\mathbb{R})}^{\beta} - \|h_k\|_{L^{p}(\mathbb{R})}^{\beta} \right),
\]
which together with (6.3) and the definition of \( Y_1^n \) in (5.2) shows that
\[
n^{-1} \frac{1}{(k - \alpha)\beta} \left( \mathbb{E}[|Y_1^n|^p] - m_p \right) = n^{-1} \frac{1}{(k - \alpha)\beta} \mathbb{E}[|Z|^p] \left| \|\phi_1^N\|_{L^{p}(\mathbb{R})}^{\beta} - \|h_k\|_{L^{p}(\mathbb{R})}^{\beta} \right|
\leq Kn^{-1} \frac{1}{(k - \alpha)\beta} \mathbb{E}[|Z|^p] \left| \|\phi_1^N\|_{L^{p}(\mathbb{R})}^{\beta} - \|h_k\|_{L^{p}(\mathbb{R})}^{\beta} \right|
\leq Kn^{-1} \frac{1}{(k - \alpha)\beta + 1} \to 0.
\]
By (6.3) and the assumption \( (k - \alpha)\beta > 1 \) we obtain (6.3), and the proof of Theorem 1.2(ii) is complete. \( \square \)

### 6.4 Proof of Theorem 1.2(ii)

To prove Theorem 1.2(ii) we start by noticing that
\[
\sqrt{n} \left( n^{-1 + p(\alpha + 1/\beta)} V(p; k)n - m_p \right) \overset{d}{=} \frac{1}{\sqrt{n}} S_n + \sqrt{n} \left( \frac{n - k + 1}{n} \mathbb{E}[|Y_1^n|^p] - m_p \right) \tag{6.48}
\]
due to (6.1). First we will show that
\[
\frac{1}{\sqrt{n}} S_n \overset{d}{\to} \mathcal{N}(0, \eta^2) \quad \text{as } n \to \infty, \tag{6.49}
\]
where \( \eta^2 \in (0, \infty) \) is given in Theorem 1.2(ii). Afterwards we will show that the second term on the right-hand side of (6.4) converges to zero, which will complete the proof.
of Theorem 1.2(ii). To prove (6.3) it is according to a standard result (see e.g. [12, Theorem 3.2]) enough to show the following (i)–(iii):

(i): We have that
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \left( n^{-1} \mathbb{E}[(S_n - S_{n,m})^2] \right) = 0. \quad (6.50)
\]

(ii): For all \( m \geq 1 \) there exists \( \eta_m^2 \in [0, \infty) \) such that
\[
\frac{1}{\sqrt{n}} S_{n,m} \overset{d}{\to} \mathcal{N}(0, \eta_m^2) \quad \text{as } n \to \infty. \quad (6.51)
\]

(iii): We have that
\[
\eta_m^2 \to \eta^2 \quad \text{as } m \to \infty.
\]

To prove (i) we use Proposition 6.2 and the assumption \( \alpha < k - 2/\beta \) to obtain that
\[
\frac{1}{n} \mathbb{E}\left[ \left( \sum_{r=k}^{n} R_{r,m}^{n} \right)^2 \right] \leq K \left( (m + 1)^{(\alpha - k)\beta/4 + 1/2} + n^{2(\alpha - k)\beta + 3} + n^{-1} \log n \right), \quad (6.52)
\]
\[
\frac{1}{n} \mathbb{E}\left[ \left( \sum_{r=k}^{n} Q_{r,m}^{n} \right)^2 \right] \leq K \left( n^{(\alpha - k)\beta + 2 + \varepsilon} + (m + 1)^{(\alpha - k)\beta + 2 + \varepsilon} + n^{-1} \right). \quad (6.53)
\]

Thus, by the decomposition (6.1) of \( S_n - S_{n,m} \), (6.4), (6.4) and the assumption \( \alpha < k - 2/\beta \) we deduce (6.4), which completes the proof of (i).

To prove (ii) we note that for fixed \( n,m \geq 1 \), \( \{ |Y_{i,m}^n|^p : i = k, \ldots, n \} \) is a stationary \( m \)-dependent sequence, and hence
\[
n^{-1} \text{var}(S_{n,m}) = n^{-1}(n-k)\theta_0^{n,m} + 2n^{-1} \sum_{i=1}^{m} (n-k-i)\theta_i^{n,m} \quad (6.54)
\]
where we set \( \theta_i^{n,m} = \text{cov}(Y_k^{n,m}, |Y_{k+i}^n|^p) \) for all \( n \in \mathbb{N} \cup \{ \infty \} \), \( m, i \geq 1 \). By the symmetrisation inequality we have for all \( u > 0 \) that
\[
\mathbb{P}(|Y_i^{n,m} - Y_i^{\infty,m}| > u) \leq 2\mathbb{P}(|Y_i^n - Y_i^{\infty}| > u),
\]
where the quantities \( Y_i^n \) and \( Y_i^{\infty} \) have been introduced in (5.2). By the equivalence of moments of stable random variables we have for all \( q < \beta \) that
\[
\mathbb{E}[|Y_i^{n,m} - Y_i^{\infty,m}|^q] \leq K_q \mathbb{E}[|Y_i^n - Y_i^{\infty}|^q] \to 0 \quad (6.55)
\]
as \( n \to \infty \), where the convergence to zero follows by (5.2). Since \( p < \beta/2 \), (6.4) implies that \( \theta_i^{n,m} \to \theta_i^{\infty,m} \) as \( n \to \infty \), and by (6.4) we deduce that
\[
n^{-1} \text{var}(S_{n,m}) \to \theta_0^{\infty,m} + 2 \sum_{i=1}^{m} \theta_i^{\infty,m} =: \eta_m^2 \quad \text{as } n \to \infty. \quad (6.56)
\]

By (6.4), (6.4), and since for all \( n \geq 1 \), the sequences \( \{ |Y_i^{n,m}|^p : i = k, \ldots, n \} \) are \( m \)-dependent, the convergence (6.4) follows by the main theorem of [11], and the proof of (ii) is complete.
The proof of (iii) follows by the same arguments as in [22, p. 1650]. Indeed, for all \( m, j \geq 1 \) we have by the triangle inequality that
\[
|\eta_m| - |\eta_j| = \lim_{n \to \infty} \left( n^{-1/2} \left\| S_{n,m} \right\|_{L^2} - \left\| S_{n,j} \right\|_{L^2} \right) \leq \limsup_{n \to \infty} \left( n^{-1/2} \left\| S_{n,m} - S_{n,j} \right\|_{L^2} \right) \\
\leq \limsup_{n \to \infty} \left( n^{-1/2} \left\| S_{n,m} - S_n \right\|_{L^2} \right) + \limsup_{n \to \infty} \left( n^{-1/2} \left\| S_n - S_{n,j} \right\|_{L^2} \right),
\]
which according to (6.4) shows that \((\eta_m)_{m \geq 1}\) is a Cauchy sequence in \( \mathbb{R}_+ \). Hence, \( (\eta^2_{m})_{m \geq 1}\) is convergent.

To show that the second term on the right-hand side of (6.4) converges to zero it suffices to prove that
\[
\sqrt{n} \left( \mathbb{E}[|Y_1^n|^p] - m_p \right) \to 0 \quad \text{as } n \to \infty. \tag{6.57}
\]
By Lemma [6.3] we have that
\[
\left| \left\| \phi_{n}^{p} \right\|_{L^\beta(\mathbb{R})}^{\beta} - \left\| \phi_{1}^{\infty} \right\|_{L^\beta(\mathbb{R})}^{\beta} \right| \leq Kn^{-1} \to 0. \tag{6.58}
\]
Since the function \( x \mapsto x^{p/\beta} \) is continuously differentiable on \((0, \infty)\) and \( \left\| \phi_{1}^{\infty} \right\|_{L^\beta(\mathbb{R})}^{\beta} > 0 \) it follows by the mean-value theorem that
\[
\left| \left\| \phi_{n}^{p} \right\|_{L^\beta(\mathbb{R})}^{\beta} - \left\| \phi_{1}^{\infty} \right\|_{L^\beta(\mathbb{R})}^{\beta} \right| \leq K \left| \left\| \phi_{n}^{\infty} \right\|_{L^\beta(\mathbb{R})} - \left\| h_{k} \right\|_{L^\beta(\mathbb{R})} \right|.
\]
Together with (6.4) and the definition of \( Y_1^n \) in (5.2) it shows that
\[
\sqrt{n} \left| \mathbb{E}[|Y_1^n|^p] - m_p \right| = \sqrt{n} \mathbb{E}[|Z|^p] \left| \left\| \phi_{n}^{p} \right\|_{L^\beta(\mathbb{R})}^{\beta} - \left\| \phi_{1}^{\infty} \right\|_{L^\beta(\mathbb{R})}^{\beta} \right| \\
\leq Kn^{-1/2} \to 0 \tag{6.59}
\]
as \( n \to \infty \). Eq. (6.4) shows (6.4) and completes the proof of Theorem 1.2(ii). \( \square \)

### 6.5 An estimate

This subsection is devoted to proving the following lemma, which is used in the proof of (6.2) of Proposition 6.2.

**Lemma 6.6.** Let \( \zeta_{r,j}^{n,m} \) be defined in (6.1). Then there exists a finite constant \( K \) such that for all \( n \geq 1 \), \( r = k, \ldots, n \), \( m \geq 0 \) and \( j \geq 1 \) we have
\[
\mathbb{E}[|\zeta_{r,j}^{n,m}|^2] \leq K \begin{cases} 
(m + 1)^{(\alpha-k)\beta+1}j^{(\alpha-k)\beta} & \text{if } j = 1, \ldots, m \\
 j^{2(\alpha-k)\beta+1} & \text{if } j > m.
\end{cases}
\]

To show Lemma 6.6 we will use the following telescoping sum decomposition of \( \zeta_{r,j}^{n,m} \):
\[
\zeta_{r,j}^{n,m} = \sum_{l=j}^{\infty} \eta_{r,j,l}^{n,m}, \quad \eta_{r,j,l}^{n,m} := \mathbb{E}[\zeta_{r,j}^{n,m}|\mathcal{G}_{r-j} \cup \mathcal{G}_{r-l}] - \mathbb{E}[\zeta_{r,j}^{n,m}|\mathcal{G}_{r-j} \cup \mathcal{G}_{r-l-1}]. \tag{6.60}
\]
The series \((6.5)\) converges almost surely and the representation follows from the fact that \(\lim_{n \to \infty} \mathbb{E}[\ell_{r,j}^n | \mathcal{G}_{r,j}^1 \lor \mathcal{G}_{r,l}] = \mathbb{E}[\ell_{r,j}^n | \mathcal{G}_{r,j}^1] = 0\) almost surely, similarly to the argument used in Remark 6.1. The next lemma gives a moment estimate for \(\ell_{r,j,l}^n\).

**Lemma 6.7.** Let \(\ell_{r,j,l}^n\) be defined in \((6.5)\) and suppose that \(\beta < \gamma < \beta/p\). Then there exists \(N \geq 1\) such that for all \(n \geq N\), \(r = k, \ldots, n\), \(j \geq 1\) and \(m \geq 0\) we have that

\[
\mathbb{E}[|\ell_{r,j,l}^n|^\gamma] \leq K \left\{ \begin{array}{ll}
 j^{(\alpha-k)\beta} l(\alpha-k) & l \geq m \\
 (m+1)^{(\alpha-k)\beta+1} j^{(\alpha-k)\beta} l & l = j, \ldots, m-1.
\end{array} \right.
\] (6.61)

To prove Lemma 6.7 we use the following estimate on \(\Phi_{\rho}\) defined in \((6.1)\).

**Lemma 6.8.** There exists a finite constant \(K\) such that for all \(\rho \in [\varepsilon, \varepsilon^{-1}]\), all \(x, y, z \geq 0\) and all \(a \in \mathbb{R}\) we have that

\[
\int_0^z \int_0^x \int_0^y |\Phi_{\rho}''(a + u_1 + u_2 + u_3)| \, du_1 \, du_2 \, du_3 
\leq K \left( (x \wedge 1)(y \wedge 1)(z \mathbb{1}_{\{z \leq 1\}} + z^p \mathbb{1}_{\{z > 1\}}) \right). \tag{6.62}
\]

and

\[
\int_0^z \int_0^x \int_0^y |\Phi_{\rho}''(a + u_1 + u_2)| \, du_1 \, du_2 
\leq K \left( (x \wedge 1)(y \mathbb{1}_{\{y \leq 1\}} + y^p \mathbb{1}_{\{y > 1\}}) \right). \tag{6.63}
\]

**Proof of Lemma 6.8** Throughout the proof \(K\) will denote a finite constant only depending on \(\beta, \varepsilon\) and \(p\), but might change from line to line. First we will show that for all \(v = 1, 2, 3\), all \(a \in \mathbb{R}\) and all \(z > 0\) we have that

\[
\int_0^z \Phi_{\rho}^{(v)}(a + u) \, du \leq K(1 \wedge |z|^{p-v}), \tag{6.64}
\]

where \(\Phi_{\rho}^{(v)}\) denotes the \(v\)-th derivative of \(\Phi_{\rho}\). To this aim we first show that for \(v = 1, 2, 3\) we have that

\[
|\Phi_{\rho}^{(v)}(x)| \leq K(1 \wedge |x|^{p-v}) \quad \text{for all } x \in \mathbb{R}, \tag{6.65}
\]

which, in particular, yields that

\[
|\Phi_{\rho}^{(v)}(x)| \leq K(1 \wedge |x|^{p-1}) \quad \text{for all } x \in \mathbb{R}. \tag{6.66}
\]

For all \(u > 0\) we let

\[
q(u) = u^{v-1-p} e^{-\rho u} \quad \text{and} \quad \psi(u) = u^{v-1-p}(e^{-\rho u} - e^{-\rho u^3}).
\]

By recalling \((6.2)\) we have by the triangle inequality that

\[
|\Phi_{\rho}^{(v)}(x)| \leq 2e^{v-1}(\int_0^\infty \cos(xu)\psi(u) \, du) + \int_0^\infty \cos(xu)q(u) \, du \right). \tag{6.67}
\]
To estimate the second integral on the right-hand side of (6.5) we note that \( q(u) \rho^{\beta(v-p)}/\Gamma(v-p) \) is the density of a gamma distribution with shape parameter \( v-p \) and rate parameter \( \rho^\beta \). Hence using the expression for the characteristic function for the gamma distribution we get for all \( x \neq 0 \) that

\[
\left| \int_0^\infty \cos(xu)q(u)\,du \right| \leq \left| \int_0^\infty e^{ixu}q(u)\,du \right| = \frac{\Gamma(v-p)}{\rho^{\beta(v-p)}} \left| (1 - ix\rho^{-\beta})^{p-v} \right| = \frac{\Gamma(v-p)}{\rho^{\beta(v-p)}} \left( 1 + x^2 \rho^{-2\beta} \right)^{\frac{p-v}{2}} \leq \Gamma(v-p)|x|^{p-v}.
\]

To estimate the first integral on the right-hand side of (6.5) we set

\[
K(\zeta(u)) = \Gamma(\beta)\zeta(0). \quad \text{Hence using the expression for the characteristic function for the gamma distribution}
\]

we get for all \( x \neq 0 \) that \( \psi(u) = u^{v-1-p}\zeta(u) \). For all \( j = 0, 1, 2, 3 \) we obtain the estimates

\[
|\zeta^{(j)}(u)| \leq \begin{cases} Ku^{\beta \lambda - 1 - j} & u \in (0, 1), \\ Ku^2e^{-\epsilon u^{\beta \lambda}} & u \geq 1, \end{cases}
\]

which implies that

\[
|\psi^{(j)}(u)| \leq \begin{cases} Ku^{\beta \lambda - 1 - 1 - p} & u \in (0, 1), \\ Ku^3e^{-\epsilon u^{\beta \lambda}} & u \geq 1. \end{cases}
\]

Since \( p < \beta \wedge 1 \) by assumption, we deduce by (6.3) used on \( j = 0 \) and \( j = 1 \) that \( \lim_{u \to \infty} \psi(u) = \lim_{u \to 0} \psi(u) = \lim_{u \to \infty} \psi'(u) = 0 \). Hence, by integration by parts, we have for all \( x > 0 \) that

\[
\left| \int_0^\infty \cos(xu)\psi(u)\,du \right| = \begin{cases} x^{-v} \int_0^\infty \cos(xu)\psi^{(v)}(u)\,du & v \text{ even} \\ x^{-v} \int_0^\infty \sin(xu)\psi^{(v)}(u)\,du & v \text{ odd} \end{cases}
\]

\[
\leq x^{-v} \int_0^\infty \left| \psi^{(v)}(u) \right|\,du \leq Kx^{-v},
\]

where the last inequality follows from (6.3) used on \( j = v \). The estimates (6.5), (6.5) and (6.5) imply (6.5).

To show (6.5) it suffices, cf. (6.5), to show the estimate

\[
\int_0^\infty (1 \wedge |a + u|^{p-1})\,du \leq K(\mathbb{1}_{\{z \leq 1\}}z + \mathbb{1}_{\{z > 1\}}z^p).
\]

It is important that the constant \( K \) does not depend on \( a \in \mathbb{R} \). To show (6.5) we may and do assume that \( z > 1 \) since the estimate (6.5) holds for \( z \leq 1 \) by dominating the integrand by 1. We split the integral in three parts

\[
\int_0^\infty (1 \wedge |a + u|^{p-1})\,du = \int_{(1-a,1-a) \cap [0,z]} 1\,du
\]

\[
+ \int_{(1-a,\infty) \cap [0,z]} (a + u)^{p-1}\,du + \int_{(-\infty,-a-1) \cap [0,z]} (a - u)^{p-1}\,du.
\]
Since \( p \in (0,1] \) we have by subadditivity that \( x^p - y^p \leq (x - y)^p \) for all \( 0 \leq y \leq x \). Hence

\[
\int_{(1-a,\infty)\cap[0,z]} (a + u)^{p-1} \, du = \mathbb{1}_{\{z \geq 1-a\}} \frac{1}{p} \begin{cases} (a+z)^p - a^p & a \geq 1 \\ (a+z)^p - 1 & a < 1. \end{cases}
\]

\[
\leq \mathbb{1}_{\{z \geq 1-a\}} \frac{1}{p} z^p,
\]

and

\[
\int_{(-\infty,-a-1)\cap[0,z]} (-a - u)^{p-1} \, du = \mathbb{1}_{\{z < -a-1\}} \frac{1}{p} \begin{cases} (-a)^p - 1 & -a < z \\ (-a)^p - (z + a)^p & z \leq -a - 1 \end{cases}
\]

\[
\leq \mathbb{1}_{\{z < -a-1\}} \frac{1}{p} z^p.
\]

Thus, by (6.5) we obtain for \( z \geq 1 \) that

\[
\int_0^z \left( 1 \wedge |a + u|^{p-1} \right) \, du \leq 2 + \frac{2}{p} z^p \leq 2 \left( 1 + \frac{1}{p} \right) z^p,
\]

which implies (6.5), and completes the proof of (6.5).

We will now deduce (6.8) from (6.5). For \( x \geq 1 \) we have that with \( \tilde{a} = a + x \)

\[
\left| \int_0^y \int_0^z \int_0^x \Phi''_{\rho}(a + u_1 + u_2 + u_3) \, du_1 \, du_2 \, du_3 \right| 
\]

\[
\leq \left| \int_0^z \int_0^y \int_0^x \Phi''_{\rho}(\tilde{a} + u_1 + u_2) \, du_1 \, du_2 \right| + \left| \int_0^y \int_0^z \int_0^x \Phi''_{\rho}(a + u_1 + u_2) \, du_1 \, du_2 \right|.
\]

For \( x < 1 \) there exists an \( \tilde{a} \in \mathbb{R} \) such that

\[
\int_0^x \int_0^y \int_0^z \Phi''_{\rho}(a + u_1 + u_2 + u_3) \, du_2 \, du_3 \, du_1 = x \int_0^y \int_0^z \Phi''_{\rho}(\tilde{a} + u_2 + u_3) \, du_2 \, du_3.
\]

Repeating this argument shows that for any \( \tilde{a} \in \mathbb{R} \) and \( v = 2, 3 \) we have for \( y \geq 1 \) that with \( \tilde{a} = \tilde{a} + y \)

\[
\int_0^y \int_0^z \Phi^{(v)}_{\rho}(\tilde{a} + u_2 + u_3) \, du_3 \, du_2 
\]

\[
\leq \left| \int_0^z \Phi^{(v-1)}_{\rho}(\tilde{a} + u_2) \, du_3 \right| + \left| \int_0^z \Phi^{(v-1)}_{\rho}(\tilde{a} + u_3) \, du_3 \right|,
\]

and for \( y < 1 \) there exists \( \tilde{a} \in \mathbb{R} \) such that

\[
\int_0^y \int_0^z \Phi^{(v)}_{\rho}(\tilde{a} + u_2 + u_3) \, du_3 \, du_2 \leq y \int_0^z \Phi^{(v)}_{\rho}(\tilde{a} + u_3) \, du_3.
\]

By collecting all the terms and using (6.5) we obtain (6.8). Eq. (6.8) follows by similar arguments. In this case, we use (6.5) with \( v = 2 \) and conclude the proof by using (6.3) as above. \qed
We are now ready to prove Lemma 6.7.

Proof of Lemma 6.7: For fixed $n, m, j, l$, $\{\vartheta_{r,j,l}^{n,m} : r \geq 1\}$ is a stationary sequence, and hence we may and do assume that $r = 1$. Furthermore, we may assume that $l \geq j \vee 2$, since the case $l = j = 1$ can be covered by choosing a new constant $K$. By definition of $\vartheta_{1,j,l}^{n,m}$ we obtain the representation

$$\vartheta_{1,j,l}^{n,m} = \mathbb{E}[V_{r}^{n,m} | \mathcal{G}_{1-j} \oplus \mathcal{G}_{1-l}] - \mathbb{E}[V_{r}^{n,m} | \mathcal{G}_{1-j} \cup \mathcal{G}_{1-l}] + \mathbb{E}[V_{r}^{n,m} | \mathcal{G}_{1-l}]. \quad (6.74)$$

Set $\rho_{j,l}^{n} = \|\phi_{1}^{n}\|_{L^{p}([1-l,1-j] \cup [j-2,1])}$. For large enough $N \geq 1$ there exists $\varepsilon > 0$ such that $\rho_{j,l}^{n} \geq \varepsilon$ for all $n \geq N, j \geq 1, l \geq j \vee 2$ (we have $\rho_{j,l}^{n} = 0$ for $l = 1$). Hence by (6.2) there exists a finite constant $K$ such that

$$\Phi_{\rho_{j,l}^{n}}(x) \leq K \quad \text{for all } n \geq N, j \geq 1, l \geq j \vee 2, x \in \mathbb{R}.$$

Let

$$A_{1}^{n} = \int_{-\infty}^{-l} \phi_{1}^{n}(s) \, dL_{s} \quad \text{and} \quad A_{1}^{n,m} = \int_{1-m}^{-l} \phi_{1}^{n}(s) \, dL_{s}$$

and $(\tilde{U}_{1-j}^{n}, \tilde{U}_{1-j}^{n})$ denote a random vector, which is independent of $L$, and which equals $(U_{1-l}^{n}, U_{1-1}^{n})$ in law (cf. definition (6.1)). Let moreover $\tilde{E}$ denote the expectation with respect to $(\tilde{U}_{1-l}^{n}, \tilde{U}_{1-1}^{n})$ only. For all $j = 1, \ldots, m$ and $l = j, \ldots, m - 1$ we deduce from (6.5) that

$$\vartheta_{1,j,l}^{n,m} = \tilde{E}\left[\Phi_{\rho_{j,l}^{n}}(A_{1}^{n} + U_{1-j}^{n} + U_{1-1-j}^{n}) - \Phi_{\rho_{j,l}^{n}}(A_{1}^{n} + \tilde{U}_{1-j}^{n} + U_{1-1-j}^{n})
- \Phi_{\rho_{j,l}^{n}}(A_{1}^{n} + U_{1-j}^{n} + \tilde{U}_{1-j}^{n}) + \Phi_{\rho_{j,l}^{n}}(A_{1}^{n} + \tilde{U}_{1-j}^{n} + \tilde{U}_{1-j}^{n})
- \Phi_{\rho_{j,l}^{n}}(A_{1}^{n,m} + U_{1-j}^{n} + U_{1-1-j}^{n}) - \Phi_{\rho_{j,l}^{n}}(A_{1}^{n,m} + U_{1-j}^{n} + U_{1-1-j}^{n})
- \Phi_{\rho_{j,l}^{n}}(A_{1}^{n,m} + U_{1-j}^{n} + \tilde{U}_{1-j}^{n}) + \Phi_{\rho_{j,l}^{n}}(A_{1}^{n,m} + \tilde{U}_{1-j}^{n} + \tilde{U}_{1-j}^{n})\right]\n= \tilde{E}\left[\int_{A_{1}^{n,m}}^{A_{1}^{n,m}} \int_{U_{1-j}^{n}}^{U_{1-j}^{n}} \int_{U_{1-j}^{n}}^{U_{1-j}^{n}} \Phi_{\rho_{j,l}^{n}}(u_{1} + u_{2} + u_{3}) \, du_{1} \, du_{2} \, du_{3}\right],$$

where $\int_{x}^{y}$ denotes $\int_{x}^{y} f$ if $x < y$. Hence, by substitution, there is a random variable $W_{j,l}^{n,m}$ such that

$$|\vartheta_{1,j,l}^{n,m}| \leq \tilde{E}\left[\int_{0}^{[A_{1}^{n} - A_{1}^{n,m}]} \int_{0}^{[\tilde{U}_{1-j}^{n} - U_{1-j}^{n}]} \int_{0}^{[\tilde{U}_{1-j}^{n} - U_{1-j}^{n}]} |\Phi_{\rho_{j,l}^{n}}(W_{j,l}^{n,m} + u_{1} + u_{2} + u_{3})| \, du_{1} \, du_{2} \, du_{3}\right]. \quad (6.75)$$

For $l > m$ we have that

$$\vartheta_{1,j,l}^{n,m} = \tilde{E}\left[\Phi_{\rho_{j,l}^{n}}(A_{1}^{n} + U_{1-j}^{n} + U_{1-1-j}^{n}) - \Phi_{\rho_{j,l}^{n}}(A_{1}^{n} + \tilde{U}_{1-j}^{n} + U_{1-1-j}^{n})
- \Phi_{\rho_{j,l}^{n}}(A_{1}^{n} + U_{1-j}^{n} + \tilde{U}_{1-j}^{n}) + \Phi_{\rho_{j,l}^{n}}(A_{1}^{n} + \tilde{U}_{1-j}^{n} + \tilde{U}_{1-j}^{n})\right]\n= \tilde{E}\left[\int_{\tilde{U}_{1-j}^{n}}^{\tilde{U}_{1-j}^{n}} \int_{\tilde{U}_{1-j}^{n}}^{\tilde{U}_{1-j}^{n}} \Phi_{\rho_{j,l}^{n}}(A_{1}^{n} + u_{2} + u_{3}) \, du_{1} \, du_{2}\right].$$
Let \( l = j, \ldots, m - 1 \). By (6.5) and (6.8) we have that

\[
\mathbb{E}[|\theta_{l,j,l}^{n,m}|^\gamma] \leq K \left( \mathbb{E}[|A_{l} - A_{l}^{m}|p\gamma 1_{\{|A_{l} - A_{l}^{m}| \geq 1\}}] + \mathbb{E}[|A_{l} - A_{l}^{m}|p\gamma 1_{\{|A_{l} - A_{l}^{m}| \leq 1\}}] \right)
\times \mathbb{E}[\mathbb{E}[((U_{1,1-j}^{n} - U_{1,1-j}^{m}) \wedge 1)^\gamma]]
\leq K ||\phi_{l}^{n}||_{L^\beta((1,1+m]j)} ||\phi_{l}^{n}||_{L^\beta((-\infty,1-m]j)} ||\phi_{l}^{n}||_{L^\beta((-1,-1-l]j)} \leq Km(\alpha-k)^{\beta+1}j(\alpha-k)^{\beta}l(\alpha-k)^{\beta}.
\]

We use Lemma 6.5(i) and (ii), \( p\gamma < \beta < \gamma \) and \( |x-y| \wedge 1 \leq |x| \wedge 1 + |y| \wedge 1 \). For \( l \geq m \) we have by (6.8) that

\[
\mathbb{E}[|\theta_{l,j,l}^{n,m}|^\gamma] \leq K \mathbb{E}[\mathbb{E}[((U_{1,1-j}^{n} - \tilde{U}_{l,j}^{n}) \wedge 1)^\gamma]]
\times \left( \mathbb{E}[\mathbb{E}[|U_{1,1-j}^{n} - \tilde{U}_{l,j}^{n}|p\gamma 1_{\{|U_{1,1-j}^{n} - \tilde{U}_{l,j}^{n}| \geq 1\}}]] + \mathbb{E}[\mathbb{E}[|U_{1,1-j}^{n} - \tilde{U}_{l,j}^{n}|p\gamma 1_{\{|U_{1,1-j}^{n} - \tilde{U}_{l,j}^{n}| \leq 1\}}]] \right)
\leq K ||\phi_{l}^{n}||_{L^\beta((1,1+j)j)} ||\phi_{l}^{n}||_{L^\beta((-1,1-j]j)} \leq K j^{(\alpha-k)^{\beta}l(\alpha-k)^{\beta}}
\]

again using Lemma 6.5(i) and (ii), \( p\gamma < \beta < \gamma \) and \( |x-y| \wedge 1 \leq |x| \wedge 1 + |y| \wedge 1 \). This completes the proof of (6.7). \( \square \)

We are now ready to prove Lemma 6.6.

\textbf{Proof of Lemma 6.6} We will use Lemma 6.7 for \( \gamma = 2 \) which satisfies \( \beta < \gamma < \beta/p \). Suppose that \( j = 1, \ldots, m \). By orthogonality of \( \{\theta_{l,j,l}^{n,m} : l = 1, 2, \ldots\} \) in \( L^2 \) we have that

\[
\mathbb{E}[|\zeta_{l,j}^{n,m}|^2] = \sum_{l=j}^{\infty} \mathbb{E}[|\theta_{l,j,l}^{n,m}|^2] = K \left( \sum_{l=j}^{m-1} m(\alpha-k)^{\beta+1}j(\alpha-k)^{\beta}l(\alpha-k)^{\beta} + \sum_{l=m}^{\infty} l(\alpha-k)^{\beta}j(\alpha-k)^{\beta} \right)
\leq K \left( (m+1)(\alpha-k)^{\beta+1}j(\alpha-k)^{\beta+1} + j(\alpha-k)^{\beta}(m+1)(\alpha-k)^{\beta+1} \right)
\leq K(m+1)(\alpha-k)^{\beta+1}j(\alpha-k)^{\beta}
\]

since \( 2(\alpha-k) < 1 < (\alpha-k) \). Similarly, for \( j > m \) we have that

\[
\mathbb{E}[|\zeta_{l,j}^{n,m}|^2] = \sum_{l=j}^{\infty} \mathbb{E}[|\zeta_{l,j}^{n,m}|^2] \leq K j^{(\alpha-k)^{\beta}} \sum_{l=j}^{\infty} l(\alpha-k)^{\beta} \leq K j^{2(\alpha-k)^{\beta}+1},
\]

which completes the proof. \( \square \)

\textbf{6.6 Proof of Proposition 6.2}

We will start with the proof of (6.2). By rearranging the terms using the substitution \( s = r - j \), we have

\[
\sum_{r=k}^{n} P_{r}^{n,m} = \sum_{s=-\infty}^{n-1} M_{s}^{n,m} \text{ with } M_{s}^{n,m} := \sum_{r=1}^{n} \zeta_{r,s}^{n,m}.
\]
Recalling the definition of $\zeta_{n,m}$ in (6.1), we note that $\mathbb{E}[\zeta_{n,m}^r | G_s] = 0$ for all $s$ and $r$, showing that that $\{M_{s,n,m} : s \in (-\infty, n) \cap \mathbb{Z}\}$ are martingale differences. By orthogonality we have that

$$
\mathbb{E}\left[\left(\sum_{r=k}^n R_{r,n,m}^r\right)^2\right] = \sum_{s=-\infty}^{n-1} \mathbb{E}[|M_{s,n,m}|^2]
$$

$$
\leq \sum_{s=-\infty}^{n-1} \left( \sum_{r=1}^n \mathbb{E}[|\zeta_{n,m}^{s,r}|^2]^{1/2} \right)^2 := A_{n,m}. \quad (6.76)
$$

We split $A_{n,m} = \sum_{s=1}^n + \sum_{s=-n}^0 + \sum_{s=-\infty}^{-n} = A_{n,m}' + A_{n,m}'' + A_{n,m}'''$. By the substitution $\tilde{s} = n - s$ and $\tilde{r} = r - s$ we obtain

$$
A_{n,m}' = \sum_{s=1}^n \left( \sum_{r=1}^s \mathbb{E}[|\zeta_{n,m}^{s-r,n-s,r}|^2]^{1/2} \right)^2.
$$

For $s = 1, \ldots, n$ we have (cf. Lemma 6.7)

$$
\sum_{r=k}^s \mathbb{E}[|\zeta_{n,m}^{s-r,n-s,r}|^2]^{1/2} \leq K \left( \sum_{r=k}^m r^{(a-k)\beta/2} + \sum_{r=m}^s r^{2(a-k)\beta+1} \right)
$$

$$
\leq K \left( m^{(a-k)\beta+1/2} \log(m) + m^{(a-k)\beta+3/2} \right)
$$

(6.77)

where we have used the assumption $(a-k)\beta < -1$ in the second inequality. Eq. (6.6) shows that

$$
A_{n,m}' \leq Kn \left( m^{(a-k)\beta+1} \log(m)^2 + m^{2(a-k)\beta+3} \right). \quad (6.78)
$$

The substitution $\tilde{s} = -s$ and $\tilde{r} = r - s$ together with Lemma 6.7 yields that

$$
A_{n,m}'' = \sum_{s=0}^n \left( \sum_{r=s+1}^{n+s} \mathbb{E}[|\zeta_{n,m}^{r,s+n,s}|^2]^{1/2} \right)^2 \leq K \sum_{s=0}^n \left( \sum_{r=s+1}^{n+s} r^{(a-k)\beta+1/2} \right)^2. \quad (6.79)
$$

For $\alpha < k - 2/\beta$ the inner sum on the right-hand side of (6.6) is summable. Thus, we deduce

$$
A_{n,m}'' \leq K \sum_{s=0}^n s^{2(a-k)\beta+3} \leq K \left( n^{2(a-k)\beta+1} + \log(n) \right). \quad (6.80)
$$

On the other hand, for $\alpha \geq k - 2/\beta$ we have by Jensen’s inequality that

$$
A_{n,m}'' \leq Kn \sum_{s=0}^n \left( \sum_{r=s+1}^{n+s} r^{2(a-k)\beta+1} \right) \leq Kn \sum_{s=0}^n s^{2(a-k)\beta+2} \leq Kn^{2(a-k)\beta+4}, \quad (6.81)
$$
where we have used the assumption \((\alpha - k)\beta < -1\) in the second inequality and the fact that \(\alpha > k - \frac{3}{2}\beta\) in the third inequality. Again by the substitution \(\tilde{s} = -s\) and \(r = r - s\) and Lemma 6.7 we have

\[
A''_{n,m} = \sum_{s=n}^{\infty} \left( \sum_{r=s+1}^{n+s} \mathbb{E}[|s_{r+s,r}|^2]^{1/2} \right)^2 \leq K \sum_{s=n}^{\infty} \left( \sum_{r=s+1}^{n+s} r^{(\alpha-k)\beta+1/2} \right)^2 \\
\leq K \sum_{s=n}^{\infty} \left( n s^{(\alpha-k)\beta+1/2} \right)^2 \leq K n^{2(\alpha-k)\beta+4},
\]

(6.82)

where we have used the assumption \((\alpha - k)\beta < -1\) in the last inequality. Combining the estimates (6.6)–(6.8) yields (6.2).

In the proof of (6.2) and (6.2) we will use the following decomposition

\[
\sum_{r=k}^{n} Q_{r,m} = \sum_{s=-\infty}^{n-1} \sum_{j=(k-s)/\beta+1}^{n-s} \mathbb{E}[V_{s+j}^{n,m} | G_s^1]
\]

(6.83)

which follows by the substitution \(s = r - j\). To prove (6.2) we assume that \(\alpha < k - 2/\beta\) and let \(\epsilon > 0\). By (6.2) we have for all \(p \leq \gamma < \beta/2\) that

\[
\mathbb{E}\left[ |\Phi_{\rho_j}^{n,m}(U_{s+j,s}) - \Phi_{\rho_j}^{n,m}(U_{s+j,s})|^2 \right] \leq \left| \rho_j^n \right|^{\beta} - \left| \rho_j^{n,m} \right|^{\beta} \mathbb{E}[|U_{s+j,s}|^{2\gamma}] \\
\leq K \left| \rho_j^n \right|^{\beta} - \left| \rho_j^{n,m} \right|^{\beta} j^{(\alpha-k)2\gamma} \leq K \left| \rho_j^n \right|^{\beta} - \left| \rho_j^{n,m} \right|^{\beta} j^{(\alpha-k)+2\epsilon},
\]

(6.84)

where the last inequality holds for \(\gamma\) close enough to \(\beta/2\). We have that

\[
\left| \rho_j^n \right|^{\beta} - \left| \rho_j^{n,m} \right|^{\beta} = \int_{(-\infty,s+j)[s,s+1]} |\phi_{s+j}^n(u)|^{\beta} du - \int_{(-s+j-m,s+j)[s,s+1]} |\phi_{s+j}^{n,m}(u)|^{\beta} du \\
\leq \int_{-\infty}^{-m} |\phi_{0}^n(u)|^{\beta} du \leq m^{(\alpha-k)\beta+1}.
\]

(6.85)

By recalling the identity (6.1) we have

\[
\left\| \mathbb{E}[V_{s+j}^{n,m} | G_s^1] \right\|_{L^2} \leq \left\| \Phi_{\rho_j}^{n,m}(U_{s+j,s}) - \Phi_{\rho_j}^{n,m}(U_{s+j,s}) \right\|_{L^2} \\
\leq K \left\{ \begin{array}{ll} m(\alpha-k)\beta+1 & j = 1, \ldots, m \\
 j^{(\alpha-k)\beta+2+\epsilon} & j > m \end{array} \right.
\]

(6.86)

where the last inequality follows from (6.6) and (6.6). By orthogonality in \(L^2\) of the inner
sums on the right-hand side of (6.6) we have that
\[
\mathbb{E}\left[\left(\sum_{r=k}^{n} Q_{r}^{n,m}\right)^{2}\right] = \sum_{s=-\infty}^{n-1} \mathbb{E}\left[\left(\sum_{j=(k-s)^{\vee}1}^{n-s} \mathbb{E}[V_{s+j}^{n,m}|G_{s}^{1}]\right)^{2}\right]
\]
\[
\leq \sum_{s=-\infty}^{n-1} \left( \sum_{j=(k-s)^{\vee}1}^{n-s} \left\| \mathbb{E}[V_{s+j}^{n,m}|G_{s}^{1}] \right\|_{L^2}^{2} \right)
\]
\[
= K \left[ \sum_{s=-\infty}^{k-1} \left( \sum_{j=k-s}^{n-s} \left\| \mathbb{E}[V_{s+j}^{n,m}|G_{s}^{1}] \right\|_{L^2}^{2} \right)^{2} + \sum_{s=k}^{n-1} \left( \sum_{j=1}^{n-s} \left\| \mathbb{E}[V_{s+j}^{n,m}|G_{s}^{1}] \right\|_{L^2}^{2} \right)^{2} \right]
\]
\[
=: K[A'_{n,m} + A''_{n,m}], \quad (6.87)
\]
By (6.6) we obtain the following estimate on \(A'_{n,m}\):
\[
A'_{n,m} \leq K \sum_{s=-\infty}^{k-1} \left( \sum_{j=k-s}^{n-s} j^{(\alpha-k)\beta/2+\varepsilon} \right)^{2}
\]
\[
= K \left( \sum_{s=-\infty}^{n-k} \left( \sum_{j=k-s}^{n+s} j^{(\alpha-k)\beta/2+\varepsilon} \right)^{2} + \sum_{s=n-k}^{\infty} \left( \sum_{j=k-s}^{n-s} j^{(\alpha-k)\beta/2+\varepsilon} \right)^{2} \right)
\]
\[
=: K(B_{n} + C_{n}), \quad (6.88)
\]
Since \((\alpha-k)\beta < -2\) we obtain the estimate
\[
B_{n} \leq K \sum_{s=-\infty}^{n} s^{(\alpha-k)\beta+2+2\varepsilon} \leq K(n^{(\alpha-k)\beta+3+2\varepsilon} + 1), \quad (6.89)
\]
and by using \((\alpha-k)\beta < -1\) we get
\[
C_{n} \leq K \sum_{s=n-k}^{\infty} n^{2}s^{(\alpha-k)\beta+2+2\varepsilon} \leq Kn^{(\alpha-k)\beta+3+2\varepsilon}. \quad (6.90)
\]
By the substitution \(\tilde{s} = n - s\) and (6.6) we have
\[
A''_{n,m} = \sum_{s=1}^{n-k-1} \left( \sum_{j=1}^{s} \left\| \mathbb{E}[V_{n+s+j}^{n,m}|G_{n+s}^{1}] \right\|_{L^2}^{2} \right)
\]
\[
\leq \sum_{s=1}^{n-1} \left( m^{(\alpha-k)\beta+1} \sum_{j=1}^{m} j^{(\alpha-k)\beta/2+\varepsilon} + \sum_{j=m+1}^{s} j^{(\alpha-k)\beta/2+\varepsilon} \right)^{2}
\]
\[
\leq n \left( m^{2(\alpha-k)\beta+1} + m^{(\alpha-k)\beta+2+2\varepsilon} \right) \leq nm^{(\alpha-k)\beta+2+2\varepsilon} \quad (6.91)
\]
where the last inequality follows by the assumption \((\alpha-k)\beta < -2\). The above estimates (6.6)–(6.6) yield (6.2).

To prove (6.2) we suppose that \(\alpha > k - 2/\beta\). We will again use the decomposition (6.6), which by the decomposition \(\sum_{s=-\infty}^{n-1} = \sum_{s=-\infty}^{k-1} + \sum_{s=k}^{n-1}\) gives
\[
\sum_{r=k}^{n} \left( Q_{r}^{n,0} - Z_{r} \right) = H_{n}^{(1)} - H_{n}^{(2)} + H_{n}^{(3)}, \quad (6.92)
\]
where
\[ H_n^{(1)} = \sum_{s=-\infty}^{-k+1} \sum_{j=k-s}^{n-s} \mathbb{E}[V_s^{n,0}|G_s], \quad H_n^{(2)} = \sum_{s=k}^{n} \sum_{j=n-s+1}^{\infty} \left\{ \Phi_{\rho_j}^{\infty}(U_{j,r}^\infty) - \mathbb{E}[\Phi_{\rho_j}^{\infty}(U_{j,r}^\infty)] \right\}, \]
\[ H_n^{(3)} = \sum_{s=k}^{n} \sum_{j=1}^{\infty} \left( \mathbb{E}[V_s^{n,0}|G_s] - \{ \Phi_{\rho_j}^{\infty}(U_{j,r}^\infty) - \mathbb{E}[\Phi_{\rho_j}^{\infty}(U_{j,r}^\infty)] \} \right). \]

In the following we will show that for all \( \varepsilon > 0 \) there exists a finite constant \( K \) such all \( i = 1, 2, 3 \) and \( n \geq 1 \) we have
\[ \mathbb{E}[|H_n^{(i)}|] \leq K \left( n^{(\alpha-k)\beta+2+\varepsilon} + n^{1-\beta+\varepsilon} \right), \tag{6.93} \]
which by (6.6) yields (6.2). To be used in the proof of (6.6) we recall that according to (6.1) we have
\[ \mathbb{E}[V_s^{n,0}|G_s] = \Phi_{\rho_j}^{\infty}(U_{s+j,s}^n) - \mathbb{E}[\Phi_{\rho_j}^{\infty}(U_{s+j,s}^n)]. \tag{6.94} \]

For all \( \gamma \in (p, \beta) \) such that \(-2 < (\alpha - k)\gamma < -1\) we have by (6.6) that
\[ \mathbb{E}[|H_n^{(1)}|] \leq 2 \sum_{s=-\infty}^{-k+1} \sum_{j=k-s}^{n-s} \mathbb{E}[|\Phi_{\rho_j}^{\infty}(U_{s+j,s}^n)|] \leq K \sum_{s=-\infty}^{-k+1} \sum_{j=k-s}^{n-s} \mathbb{E}[|U_{s+j,s}^n|^\gamma] \]
\[ \leq K \sum_{s=-k+1}^{\infty} \sum_{j=-k+1}^{n+s} j^{(\alpha-k)\gamma} \leq K \left( \sum_{s=-k+1}^{n} \sum_{j=-k+1}^{n+s} j^{(\alpha-k)\gamma} + \sum_{s=-k+1}^{\infty} \sum_{j=0}^{n+s} j^{(\alpha-k)\gamma} \right) \]
\[ \leq K \left( \sum_{s=-k+1}^{n} s^{(\alpha-k)\gamma+1} + \sum_{s=n+1}^{\infty} ns^{(\alpha-k)\gamma} \right) \leq Kn^{(\alpha-k)\gamma+2}, \]
where the second inequality follows by (6.2), the third inequality follows by (6.2), the fourth inequality follows by \((\alpha - k)\gamma < -1\), and the last inequality follows by \((\alpha - k)\gamma + 1 > -1\).

Similarly, we have for all \( \gamma \in (p, \beta) \) with \(-2 < (\alpha - k)\gamma < -1\) that
\[ \mathbb{E}[|H_n^{(2)}|] \leq 2 \sum_{s=k}^{n} \sum_{j=n-s+1}^{\infty} \mathbb{E}[|\Phi_{\rho_j}^{\infty}(U_{j+n-s,n-s}^\infty)|] \leq K \sum_{s=0}^{n-k} \sum_{j=s+1}^{\infty} \mathbb{E}[|U_{j+n-s,n-s}^\infty|^\gamma] \]
\[ \leq K \sum_{s=0}^{n-k} \sum_{j=s+1}^{\infty} j^{(\alpha-k)\gamma} \leq K \sum_{s=1}^{n-1} s^{\alpha-k)\gamma+1} \leq Kn^{(\alpha-k)\gamma+2}. \]

We will need more involved estimates on \( H_n^{(3)} \). To this end we start with the following simple inequality
\[ \mathbb{E}[|H_n^{(3)}|] \leq 2 \sum_{s=k}^{n-1} \sum_{j=1}^{n-s} \mathbb{E}[|\Phi_{\rho_j}^{\infty}(U_{s+j,s}^n) - \Phi_{\rho_j}^{\infty}(U_{s+j,s}^\infty)|] \]
\[ \leq 2n \sum_{j=1}^{n} \mathbb{E}[|\Phi_{\rho_j}^{\infty}(U_{j,0}^n) - \Phi_{\rho_j}^{\infty}(U_{j,0}^\infty)|]. \tag{6.95} \]
By adding and subtracting $\Phi_{\rho_j^n}(U_{j,0}^\infty)$ we get the decomposition
\[ \Phi_{\rho_j^n}(U_{j,0}^n) - \Phi_{\rho_j^n}(U_{j,0}^\infty) = C_j^n + D_j^n \]
where
\[ C_j^n = \Phi_{\rho_j^n}(U_{j,0}^n) - \Phi_{\rho_j^n}(U_{j,0}^\infty) \quad \text{and} \quad D_j^n = \Phi_{\rho_j^n}(U_{j,0}^\infty) - \Phi_{\rho_j^n}(U_{j,0}^n). \]
In the following we will show that for all $\varepsilon > 0$ we have that
\[ (a): \sum_{j=1}^n E[|C_j^n|] \leq K\left( n^{(\alpha-k)\beta + 1 + \varepsilon} + n^{-\beta + \varepsilon} \right) \quad \text{and} \quad (b): \sum_{j=1}^n E[|D_j^n|] \leq Kn^{(\alpha-k)\beta + 1}. \]

(6.96) To prove (6.6)(a) we note that $g_n(s) = n^\alpha g(s/n)$ and $g(s) = s^\alpha f(s)$ we have for $s \geq 0$ that
\[ \eta_n(s) := g_n(s) - s^\alpha = n^\alpha (s/n) \{ f(s/n) - f(0) \} = n^\alpha \psi_1(s/n) \psi_2(s/n) \]
where $\psi_1(s) = s^\alpha$ and $\psi_2(s) = f(s) - f(0)$ for $s \geq 0$. For all $s > k$ there exists, as a consequence of the mean-value theorem, a $\xi_s \in [s - k, s]$ such that
\[ (D^k \eta_n)(s) = \eta_n^{(k)}(\xi_s) = n^{\alpha-k} \sum_{l=0}^{k} \binom{k}{l} \psi_1^{(l)}(\xi_s/n) \psi_2^{(k-l)}(\xi_s/n). \]

Eq. (6.6) implies that
\[ |(D^k \eta_n)(s)| \leq K \left[ \sum_{l=0}^{k-1} n^{l-k} |\xi_s^{n|\alpha-l|} | + |\xi_s^{n|\alpha-k+1} n^{-1} |, \right. \]
where we have used that $\psi_1^{(l)}(t) = \alpha(\alpha-1) \cdots (\alpha-l+1) t^{\alpha-l}$ for $t > 0$, that $\psi_2^{(l)}$ is bounded on $(0, \infty)$ for $l = 1, \ldots, k$, and that $|\psi_2(t)| \leq K t$ for all $t > 0$. Since $\phi_j^n(s) - \phi_j^\infty(s) = D^k \eta_n(j - s)$ we obtain by (6.6) the estimate
\[ \| \phi_j^n - \phi_j^\infty \|_{L^\beta([0,1])} \leq K \sum_{l=0}^{k} a_{l,j,n} \quad \text{where} \quad a_{l,j,n} = n^{l-k} j^{\alpha-l} \quad \text{for} \quad l = 0, \ldots, k - 1, \quad \text{and} \quad a_{k,j,n} = n^{-1} j^{\alpha-k+1}. \]

We note that $a_{k-1,j,n} = a_{k,j,n}$, and for all $l = 0, \ldots, k - 1$ and $j = 1, \ldots, n$ we have $a_{l,j,n} = (n/j)^{l-k} j^{\alpha} \leq (n/j)^{k-1} n^{-k} j^{\alpha} = n^{-1} j^{\alpha-k+1}$, which by (6.6) shows that
\[ \| \phi_j^n - \phi_j^\infty \|_{L^\beta([0,1])} \leq Kn^{-1} j^{\alpha-k+1} \quad j = 1, \ldots, n. \]

First we suppose that $\beta > 1$. For all $\varepsilon > 0$ we have according to (6.4) of Lemma 6.4 (6.2) and (6.6) that
\[ \sum_{j=1}^n E[|C_j^n|] \leq K \sum_{j=1}^n \left( j^{(\alpha-k)(\beta-1-\varepsilon)}(n^{-1} j^{\alpha-k+1}) + (n^{-1} j^{\alpha-k+1})^{\beta} \right) \]
\[ = K \left( n^{-1} \sum_{j=1}^n j^{(\alpha-k)(\beta+1-\varepsilon) - \beta} + n^{-\beta} \sum_{j=1}^n j^{(\alpha-k+1)\beta} \right) \]
\[ \leq K \left( n^{-1+(\alpha-k)\beta+2-\varepsilon(\alpha-k)} + n^{-\beta+(\alpha-k)\beta+\beta+1} \right), \]

(6.101)
where we have used that \((\alpha - k)\beta > -2\) and \(\beta > 1\) in the second inequality. Eq. (6.6) shows (6.6)(a) by choosing \(\tilde{\epsilon}\) close enough to zero. On the other hand, suppose that \(\beta \leq 1\). For all \(\tilde{\epsilon} > 0\) we have according to (6.4) of Lemma 6.4 and (6.6) that
\[
\sum_{j=1}^{n} \mathbb{E}[|C_j^n|] \leq K \sum_{j=1}^{n} \left( n^{-1} j^{-(\alpha-k+1)} \beta - \tilde{\epsilon} \right) = Kn^{-\beta + \tilde{\epsilon}} \sum_{j=1}^{n} j^{-(\alpha-k+1)(\beta - \tilde{\epsilon})}.
\]
(6.102)

For \((\alpha - k + 1)\beta > -1\) and \(\tilde{\epsilon}\) chosen small enough, (6.6) implies that
\[
\sum_{j=1}^{n} \mathbb{E}[|C_j^n|] \leq Kn^{(\alpha-k)\beta + 1 + \tilde{\epsilon}},
\]
which shows (6.6)(a). When \((\alpha - k + 1)\beta \leq -1\) then the sum on the right-hand side of (6.6) converges and we obtain the estimate
\[
\sum_{j=1}^{n} \mathbb{E}[|C_j^n|] \leq Kn^{-\beta + \tilde{\epsilon}},
\]
which completes the proof of (6.6)(a).

To show (6.6)(b) we use Lemma 6.3 to obtain
\[
\left| \rho_j^n|^{\beta} - \rho_j^{\infty}|^{\beta} \right| \leq \left( \| \phi_j^n\|^{\beta}_{L^\alpha(\mathbb{R})} - \| \phi_j^{\infty}\|^{\beta}_{L^\alpha(\mathbb{R})} \right) \leq Kn^{(\alpha-k)\beta + 1}.
\]
(6.103)

For any \(\gamma \in (p, \beta)\) such that \((\alpha - k)\gamma < -1\) we have
\[
\mathbb{E}[\|D_j^n\|] = \mathbb{E}[\|\Phi_j^n(U_{j,0}) - \Phi_j^{\infty}(U_{j,0})\|] \leq K \left| \rho_j^n|^{\beta} - \rho_j^{\infty}|^{\beta} \right| \mathbb{E}[\|U_{j,0}\|^{\gamma}] \\
\leq K \left| \rho_j^n|^{\beta} - \rho_j^{\infty}|^{\beta} \right| \left( \| \phi_j^n\|^{\gamma}_{L^\beta([0,1])} \right) \leq Kn^{(\alpha-k)\beta + 1} j^{(\alpha-k)\gamma},
\]
where the first inequality follows by (6.2), the second inequality follows by (6.2) and the last inequality follows by (6.6). Since \((\alpha - k)\gamma < -1\), the estimate (6.6)(b) follows.

The estimates (6.6) and (6.6) yields that
\[
\mathbb{E}[\|H_n^{(3)}\|] \leq 2n \sum_{j=1}^{n} \left( \mathbb{E}[|C_j^n|] + \mathbb{E}[|D_j^n|] \right) \leq Kn^{(\alpha-k)\beta + 2} + n^{1-\beta + \tilde{\epsilon}},
\]
and completes the proof of the proposition. 

\[\Box\]

6.7 Proof of Lemma 6.3

We have that \(f(x) = g(x)x^{-\alpha}\) for \(x > 0\), \(f(0) = 1\) and \(f\) is assumed to be right differentiable at zero, and hence we may and do extend \(f\) to a differentiable function from \(\mathbb{R}\) which also will be denoted \(f\). We recall the notation from (5.2):
\[
\phi_j^n(s) = D^k g_n(j - s), \quad g_n(x) = n^{\alpha} g(x/n), \quad \phi_j^{\infty}(u) = h_k(j - u).
\]
By substitution we have that

$$
\| \phi_j^{n\alpha} \|_{L^\beta(\mathbb{R})}^\beta = \int_0^\infty |D^kg_n(x)|^\beta \, dx \quad \text{and} \quad \| \phi_j^{\infty \alpha} \|_{L^\beta(\mathbb{R})}^\beta = \int_0^\infty |h_k(x)|^\beta \, dx.
$$

From Lemma 4.1 and condition $\alpha < k - 1/\beta$ we obtain for all $n \geq 1$ that

$$
A_n := \int_n^\infty |h_k(x)|^\beta \, dx \leq K \int_n^\infty x^{(\alpha-k)\beta} \, dx \leq Kn^{(\alpha-k)\beta+1}. \tag{6.104}
$$

The same estimate holds for the quantity $\int_n^\infty |D^kg_n(x)|^\beta \, dx$. On the other hand, we have that

$$
B_n := \left| \int_0^k |D^kg_n(x)|^\beta \, dx - \int_0^k |h_k(x)|^\beta \, dx \right| \leq Kn^{-1}. \tag{6.105}
$$

This follows by the estimate $||x|^{\beta} - |y|^{\beta}| \leq K \max\{|x|^{\beta-1},|y|^{\beta-1}\}|x-y|$ for all $x, y > 0$, and that for all $x \in [0, k]$ we have by differentiability of $f$ at zero that

$$
|D^kg_n(x) - h_k(x)| \leq Kn^{-1}x^\alpha.
$$

Recalling that $g(x) = x^\alpha f(x)$ and using $k$th order Taylor expansion of $f$ at $x$, we deduce the following identity

$$
D^kg_n(x) = n^\alpha \sum_{j=0}^k (-1)^j \binom{k}{j} g((x-j)/n)
$$

$$
= \sum_{j=0}^k (-1)^j \binom{k}{j} (x-j)^\alpha + \sum_{l=0}^{k-1} \frac{f^{(l)}(x/n)}{l!} (-j/n)^l + \frac{f^{(k)}(\xi_{j,x})}{k!} (-j/n)^k
$$

$$
= \sum_{l=0}^{k-1} \frac{f^{(l)}(x/n)}{l!} \left( \sum_{j=0}^k (-1)^j \binom{k}{j} (-j/n)^l (x-j)^\alpha \right)
$$

$$
+ \left( \sum_{j=0}^k \frac{f^{(k)}(\xi_{j,x})}{k!} (-1)^j \binom{k}{j} (-j/n)^k (x-j)^\alpha \right),
$$

where $\xi_{j,x}$ is a certain intermediate point. Now, by rearranging terms we can find coefficients $\lambda_n^0, \cdots, \lambda_n^k : [k,n] \to \mathbb{R}$ and $\tilde{\lambda}_n^0, \cdots, \tilde{\lambda}_n^k : [k,n] \to \mathbb{R}$ (which are in fact bounded functions in $x$ uniformly in $n$) such that

$$
D^kg_n(x) = \sum_{l=0}^k \lambda_l^0(x)n^{-l} \left( \sum_{j=l}^k (-1)^j \binom{k}{j} j(j-1) \cdots (j-l+1)(x-j)^\alpha \right)
$$

$$
= \sum_{l=0}^k \tilde{\lambda}_l^0(x)n^{-l} \left( \sum_{j=l}^k (-1)^j \binom{k-l}{j-l} (x-j)^\alpha \right) =: \sum_{l=0}^k r_{l,n}(x).
$$
At this stage we remark that the term $r_{l,n}(x)$ involves $(k-l)$th order differences of the function $x^\alpha$ and $\lambda_n^\alpha(x) = \tilde{\lambda}_n^\alpha(x) = f(x/n)$. Now, observe that

$$C_n := \int_k^n \left| D^k g_n(x) \right|^\beta \left| h_k(x) \right|^\beta \, dx$$

$$\leq K \int_k^n \max\{\left| D^k g_n(x) \right|^\beta - 1, \left| h_k(x) \right|^\beta - 1\} \left| D^k g_n(x) - h_k(x) \right| \, dx.$$ 

Since $r_{0,n}(x) = f(x/n)h_k(x)$ and $f(0) = 1$, it holds that

$$\left| r_{0,n}(x) - h_k(x) \right| \leq K (x/n) |h_k(x)|.$$ 

We deduce that

$$\int_k^n \max\{\left| D^k g_n(x) \right|^\beta - 1, \left| h_k(x) \right|^\beta - 1\} \left| r_{0,n}(x) - h_k(x) \right| \, dx \leq K^{-1} \int_k^n x^{(\alpha-k)\beta+1} \, dx$$

$$\leq K \begin{cases} n^{-1} & \text{when } \alpha \in (0, k - 2/\beta) \\ n^{(\alpha-k)\beta+1} & \text{when } \alpha \in (k - 2/\beta, k - 1/\beta) \end{cases}$$ 

(6.106)

For $1 \leq l \leq k$, we readily obtain the approximation

$$\int_k^n \max\{\left| D^k g_n(x) \right|^\beta - 1, \left| h_k(x) \right|^\beta - 1\} \left| r_{l,n}(x) \right| \, dx \leq K n^{-1} \int_k^n x^{(\alpha-k)\beta+l} \, dx.$$ 

If $\alpha \in (k - 2/\beta, k - 1/\beta)$, then $(\alpha - k)\beta + l > -1$ and we have

$$\int_k^n x^{(\alpha-k)\beta+l} \, dx \leq K n^{(\alpha-k)\beta+l+1}.$$ 

(6.107)

When $\alpha \in (0, k - 2/\beta)$ it holds that

$$\int_k^n x^{(\alpha-k)\beta+l} \, dx \leq \begin{cases} K & (\alpha - k)\beta + l < -1 \\ K \log(n)n^{(\alpha-k)\beta+l+1} & (\alpha - k)\beta + l \geq -1 \end{cases}$$ 

(6.108)

By (6.7), (6.7) and (6.7) we conclude that

$$C_n \leq K \begin{cases} n^{-1} & \text{when } \alpha \in (0, k - 2/\beta) \\ n^{(\alpha-k)\beta+1} & \text{when } \alpha \in (k - 2/\beta, k - 1/\beta) \end{cases}$$

Since $(\alpha - k)\beta + 1 < -1$ if and only if $\alpha < k - 2/\beta$, the result readily follows from (6.7) and (6.7).

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References

[1] D.J. Aldous and G. K. Eagleson (1978). On mixing and stability of limit theorems. *Ann. Probab.* 6(2), 325–331.

[2] F. Avram and M.S. Taqqu (1987). Noncentral limit theorems and Appell polynomials. *Ann. Probab.* 15(2), 767–775.

[3] O.E. Barndorff-Nielsen, F.E. Benth, and A.E.D. Veraart (2013). Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. *Bernoulli* 19(3), 803–845.

[4] O.E. Barndorff-Nielsen, J.M. Corcuera and M. Podolskij (2009). Power variation for Gaussian processes with stationary increments. *Stochastic Process. Appl.* 119(6), 1845–1865.

[5] O.E. Barndorff-Nielsen, J.M. Corcuera, M. Podolskij and J.H.C. Woerner (2009). Bipower variation for Gaussian processes with stationary increments. *J. Appl. Probab.* 46(1), 132–150.

[6] O.E. Barndorff-Nielsen, S.E. Graversen, J. Jacod, M. Podolskij and N. Shephard (2005). A central limit theorem for realised power and bipower variations of continuous semimartingales. In: Kabanov, Yu., Liptser, R., Stoyanov, J. (eds.), *From Stochastic Calculus to Mathematical Finance. Festschrift in Honour of A.N. Shiryaev*, 33–68, Springer, Heidelberg.

[7] A. Basse-O’Connor and J. Rosiński (2016). On infinitely divisible semimartingales. *Probab. Theory Relat. Fields* 164(1-2), 133–163.

[8] A. Benassi, S. Cohen and J. Istas (2004). On roughness indices for fractional fields. *Bernoulli* 10(2), 357–373.

[9] C. Bender, A. Lindner and M. Schicks (2012). Finite variation of fractional Lévy processes. *J. Theoret. Probab.* 25(2), 595–612.

[10] C. Bender and T. Marquardt (2008). Stochastic calculus for convoluted Lévy processes. *Bernoulli* 14(2), 499–518.

[11] K.N. Berk (1973). A central limit theorem for $m$-dependent random variables with unbounded $m$. *Ann. Probab.* 1(2), 352–354.

[12] P. Billingsley (1999). *Convergence of Probability Measures (second edition).* Wiley Series in Probability and Statistics: Probability and Statistics.

[13] M. Braverman and G. Samorodnitsky (1998). Symmetric infinitely divisible processes with sample paths in Orlicz spaces and absolute continuity of infinitely divisible processes. *Stochastic Process. Appl.* 78(1), 1–26.

[14] S. Cambanis, C.D. Hardin, Jr., and A. Weron (1987). Ergodic properties of stationary stable processes. *Stochastic Process. Appl.* 24(1), 1–18.
[15] A. Chronopoulou, C.A. Tudor and F.G. Viens (2009). Variations and Hurst index estimation for a Rosenblatt process using longer filters. *Electron. J. Stat.* 3, 1393–1435.

[16] A. Chronopoulou, C.A. Tudor and F.G. Viens (2011). Self-similarity parameter estimation and reproduction property for non-Gaussian Hermite processes. *Communications on Stochastic Analysis* 5, 161–185.

[17] J.-F. Coeurjolly (2001). Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. *Stat. Inference Stoch. Process.* 4(2), 199–227.

[18] S. Delattre and J. Jacod (1997). A central limit theorem for normalized functions of the increments of a diffusion process, in the presence of round-off errors. *Bernoulli* 3(1), 1–28.

[19] S. Glaser (2015). A law of large numbers for the power variation of fractional Lévy processes. *Stoch. Anal. Appl.* 33(1), 1–20.

[20] D. Grahovac, N.N. Leonenko and M.S. Taqqu (2015). Scaling properties of the empirical structure function of linear fractional stable motion and estimation of its parameters. *J. Stat. Phys.* 158(1), 105–119.

[21] L. Guyon and J. Leon (1989): Convergence en loi des $H$-variations d’un processus gaussien stationnaire sur $\mathbb{R}$. *Ann. Inst. H. Poincaré Probab. Statist.* 25(3), 265–282.

[22] H.-C. Ho and T. Hsing (1997). Limit theorems for functionals of moving averages. *Ann. Probab.* 25(4), 1636–1669.

[23] T. Hsing (1999). On the asymptotic distributions of partial sums of functionals of infinite-variance moving averages. *Ann. Probab.* 27(3), 1579–1599.

[24] J. Jacod (2008). Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Process. Appl.* 118(4), 517–559.

[25] J. Jacod and P. Protter (2012). *Discretization of Processes.* Springer, Berlin.

[26] O. Kallenberg (2002). *Foundations of Modern Probability (second edition).* Springer-Verlag, New York.

[27] F.B. Knight (1992). *Foundations of the Prediction Process.* Oxford Science Publications, New York.

[28] H.L. Koul and D. Surgailis (2001). Asymptotics of empirical processes of long memory moving averages with infinite variance. *Stochastic Process. Appl.* 91(2), 309–336.

[29] M.B. Marcus and J. Rosiński (2005). Continuity and boundedness of infinitely divisible processes: a Poisson point process approach. *J. Theoret. Probab.* 18(1), 109–160.

[30] I. Nourdin and A. Réveillac (2009). Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case $H = 1/4$. *Ann. Probab.* 37(6), 2200–2230.
[31] M. Podolskij and M. Vetter (2010). Understanding limit theorems for semimartingales: a short survey. *Stat. Neerl.* 64(3), 329–351.

[32] B. Rajput and J. Rosiński (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Relat. Fields* 82(3), 451–487.

[33] A. Rényi (1963). On stable sequences of events. *Sankhyā Ser. A* 25, 293–302.

[34] G. Samorodnitsky and M.S. Taqqu (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance.* Chapman and Hall, New York.

[35] R. Serfozo (2009). *Basics of Applied Stochastic Processes.* Probability and its Applications (New York), Springer-Verlag, Berlin.

[36] D. Surgailis (2002). Stable limits of empirical processes of moving averages with infinite variance. *Stochastic Process. Appl.* 100(1–2), 255–274.

[37] D. Surgailis (2004). Stable limits of sums of bounded functions of long-memory moving averages with finite variance. *Bernoulli* 10(2), 327–355.

[38] K. Takashima (1989). Sample path properties of ergodic self-similar processes. *Osaka J. Math.* 26(1), 159–189.

[39] C.A. Tudor and F.G. Viens (2009). Variations and estimators for self-similarity parameters via Malliavin calculus. *Ann. Probab.* 37(6), 2093–2134.

[40] J.W. Tukey (1938). On the distribution of the fractional part of a statistical variable. *Rec. Math. [Mat. Sbornik] N.S.,* 4(46):3, 561–562.

[41] T. Watanabe (2007). Asymptotic estimates of multi-dimensional stable densities and their applications. *Trans. Amer. Math. Soc.* 359(6), 2851–2879.