TIME ANALYTICITY FOR THE HEAT EQUATION AND NAVIER-STOKES EQUATIONS

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Abstract. We prove the analyticity in time for solutions of two parabolic equations in the whole space, without any decaying or vanishing conditions. One of them involves solutions to the heat equation of exponential growth of order 2 on $M$. Here $M$ is $\mathbb{R}^d$ or a complete noncompact manifold with Ricci curvature bounded from below by a constant. An implication is a sharp solvability condition for the Cauchy problem of the backward heat equation, which is a well known ill-posed problem. Another implication is a sharp criteria for time analyticity of solutions down to the initial time. The other pertains bounded mild solutions of the incompressible Navier-Stokes equations in the whole space.

There are many long established analyticity results for the Navier-Stokes equations. See for example [Ka] and [FT] for spatial and time analyticity in certain integral sense, [CN] for pointwise space-time analyticity of 3 dimensional solutions to the Cauchy problem, and also the pointwise time analyticity results of [Ma] and [Gi] under zero boundary condition. Our result seems to be the first general pointwise time analyticity result for the Cauchy problem for all dimensions, whose proof involves only real variable method. The proof involves a method of algebraically manipulating the integral kernel, which appears applicable to other evolution equations.

1. Introduction

In the study of heat and other parabolic equations, one often hopes to prove that solutions are real analytic in space and time. While the spatial analyticity is usually true for generic solutions, the time analyticity is harder to prove and is false in general. For example, it is not difficult to construct a solution of the heat equation in a finite space-time cylinder in the Euclidean setting, which is not time analytic in a sequence of moments. On the other hand, under extra assumptions, many time-analyticity results for the heat equation can be found in the literature. See, for example, [Wi]. Moreover, if one imposes zero boundary conditions on the lateral boundary of a smooth cylindrical domain, then certain solutions of the heat, Navier-Stokes, and many other parabolic equations are analytic in time. See, for example, [Ma], [Ko], [Gi], and [EMZ]. One can also consider solutions in certain $L_p$ spaces with $p \in (1, \infty)$. In this setting, by using complexification argument the time analyticity with values in an $L_2$-based Gevrey class of periodic functions was proved for the Navier-Stokes equations in [FT]. See also [Pr] for an extension to a large class of dissipative equations in the periodic setting.

In a related development, there have been renewed interest in the study of global solutions of the heat equation on the Euclidean and manifold setting. One example is the study of ancient solutions of the heat equation, i.e., solutions that exist in the whole space and in all negative time. Let $M = \mathbb{R}^d$ or a complete noncompact Riemannian manifold.
with nonnegative Ricci curvature. In [SZ], it was found that sublinear ancient solutions are constants. Later in [LZ], it was shown that sublinear ancient solutions are constants in time. Colding and Minicozzi [CM1] then obtained a sharp dimension bound of this space. See also the papers [Ca1] and [CM2] for applications to the study of mean curvature flows, and [Hu] in the graph case. In a recent paper [Z1], it was observed that ancient solutions on the above manifold with exponential growth in the space variable are analytic in time. One application of this result is a necessary and sufficient condition on the solvability of the backward heat equation in this class of solutions, which is well known to be ill-posed in general. Backward heat equations have been studied by many authors, see, for example, [Mi] and [Yo]; and treated in many books. See, for instance, [LL]. They have been applied in such diverse fields as control theory, stochastic analysis, Ricci flows etc. There does not appear to be a necessary and sufficient solvability criteria, except when the manifold is a bounded domain for which semigroup theory gives an abstract criteria. See [CJ, Theorem 9].

One goal of the current paper is to show that the result in [Z1] can be extended to solutions with exponential growth of order 2 (see 2.1), which is a sharp condition. Another goal is to prove the time analyticity for all bounded mild solutions of the Navier-Stokes equations in the whole space. One implication is that bounded mild solutions of the Navier-Stokes equations are analytic in space-time, which yields the unique continuation property of such solutions. This result may also have applications in the study of possible singularity whose blow up limit can be such a solution. We note that the space analyticity in more general setting has been proven in [Ka, GK, GS, MS, GPS, DL, BBT, Gu, CKV, Xu], to name a few.

In a subsequent work, we will study the corresponding problems in the half space.

We will present and prove the results for the heat equation and the Navier-Stokes equations in Sections 2 and 3 respectively.

2. The heat equation

Let $M$ be a $d$ dimensional, complete, noncompact Riemannian manifold, $Ric$ be the Ricci curvature and $0$ be a reference point on $M$, $d(x, y)$ be the geodesic distance of $x, y \in M$. We will use $B(x, r)$ to denote the geodesic ball of radius $r$ centered at $x$ and $|B(x, r)|$ to denote the volume. Given a point $(t, x)$ as vertex, the standard parabolic cylinder of size $r$ is

$$Q_r(t, x) = \{(s, y) \mid d(x, y) < r, s \in [t - r^2, t]\}.$$

Theorem 2.1. Let $M$ be a complete, $d$ dimensional, noncompact Riemannian manifold such that the Ricci curvature satisfies $\text{Ric} \geq -(d - 1)K_0$ for a nonnegative constant $K_0$.

Let $u$ be a smooth solution of the heat equation $\partial_t u - \Delta u = 0$ on $[-2, 0] \times M$ of exponential growth of order 2, namely

$$|u(t, x)| \leq A_1 e^{A_2 d^2(x, 0)}, \quad \forall (t, x) \in [-2, 0] \times M,$$

where $A_1$ and $A_2$ are positive constants. Then $u = u(t, x)$ is analytic in $t \in [-1, 0]$ with radius $r > 0$ depending only on $d$, $K_0$, and $A_2$. Moreover, we have

$$u(t, x) = \sum_{j=0}^{\infty} a_j(x) t^j j!$$
with \( \Delta a_j(x) = a_{j+1}(x) \), and

\[
|a_j(x)| \leq A_1 A_3^{j+1} j^j e^{2A_2 d^2(x,0)}, \quad j = 0, 1, 2, \ldots,
\]

where \( A_3 \) is a positive constants depending only on \( d, K_0, \) and \( A_2 \).

**Proof.** Since the equation is linear, without loss of generality, we may assume that \( A_1 = 1 \). It suffices to prove the result for the space time point \((0,x)\). Let us recall a well-known parabolic mean value inequality which can be found, for instance, in [Li, Theorem 14.7]. Suppose \( v \) is a positive subsolution to the heat equation on \([0,T] \times M\) and \( 0 \) is a point on \( M \). Let \( T_1, T_2 \in [0,T] \) with \( T_1 < T_2 \), \( R > 0 \), \( p > 0 \), and \( \delta, \eta \in (0,1) \). Then there exist positive constants \( C_1 \) and \( C_2 \), depending only on \( p \) and \( d \), such that

\[
\sup_{[T_1,T_2] \times B(0,(1-\delta)R)} v^p \leq C_1 \frac{\bar{V}(2R)}{|B(0,R)|} (R \sqrt{K_0} + 1) \exp(C_2 \sqrt{K_0} (T_2 - T_1)) \times \left( \frac{1}{\delta R} + \frac{1}{\eta T_1} \right)^{d+2} \int_{(1-\delta)T_1}^{T_2} \int_{B(0,R)} v^p(t,x) \, dx \, dt.
\]

Here \( \bar{V}(R) \) is the volume of geodesic balls of radius \( R \) in the simply connected space form with constant sectional curvature \(-K_0\).

Let \( u \) be the given solution to the heat equation, so that \( u^2 \) is a subsolution. Given \( x_0 \in M \) and a positive integer \( k \), with a translation of time, the above mean value inequality with \( T_1 = -1/k \), \( T_2 = 0 \), \( R = 1/\sqrt{k} \), \( \eta = \delta = 1/2 \) implies that

\[
\sup_{Q_{1/(2\sqrt{k})(0,x_0)}} u^2 \leq \frac{C_1 k^{d+2}}{|B(x_0,1/\sqrt{k})|} \int_{Q_{1/(2\sqrt{k})(0,x_0)}} u^2(t,x) \, dx \, dt \leq \frac{C_2 k^{3d/2+2}}{|B(x_0,1)|} \int_{Q_{1/(2\sqrt{k})(0,x_0)}} u^2(t,x) \, dx \, dt,
\]

where we have used the Bishop-Gromov volume comparison theorem. Note that the above mean value inequality is a local one since the size of the cubes is less than one. Hence the constants \( C_1 \) and \( C_2 \) are independent of \( k \). Since \( \partial_t^k u \) is also a solution to the heat equation, it follows that

\[
(2.2) \quad \sup_{Q_{1/(2\sqrt{k})(0,x_0)}} (\partial_t^k u)^2 \leq \frac{C_2 k^{3d/2+2}}{|B(x_0,1)|} \int_{Q_{1/(2\sqrt{k})(0,x_0)}} (\partial_t^k u)^2(t,x) \, dx \, dt.
\]

Next we will bound the right-hand side.

For integers \( j = 1, 2, \ldots, k \), consider the domains:

\[
\Omega_j^1 = \{(t,x) \mid d(x, x_0) < j/\sqrt{k}, \ t \in [-j/k, 0]\},
\]

\[
\Omega_j^2 = \{(t,x) \mid d(x, x_0) < (j+0.5)/\sqrt{k}, \ t \in [-\frac{(j+0.5)}{k}, 0]\}.
\]

Then it is clear that \( \Omega_j^1 \subset \Omega_j^2 \subset \Omega_{j+1}^1 \).

Denote by \( \psi_j^{(1)} \) a standard Lipschitz cutoff function supported in

\[
\{(t,x) \mid d(x, x_0) < (j+0.5)/\sqrt{k}, \ t \in [\frac{-(j+0.5)}{k}, (j+0.5)/k]\}.
\]
such that \( \psi_j^{(1)} = 1 \) in \( \Omega_j^1 \) and \( |\nabla \psi_j^{(1)}|^2 + |\partial_t \psi_j^{(1)}| \leq Ck. \)

Since \( u \) is a smooth solution to the heat equation, we deduce, by writing \( \psi = \psi_j^{(1)} \), that

\[
\int_{\Omega_j^2} (u_t)^2 \psi^2 \, dx \, dt = \int_{\Omega_j^2} u_t \Delta \psi^2 \, dx \, dt
\]

\[
= -\int_{\Omega_j^2} ((\nabla u)_t \nabla u) \psi^2 \, dx \, dt - \int_{\Omega_j^2} u_t \nabla u \psi^2 \, dx \, dt
\]

\[
= -\frac{1}{2} \int_{\Omega_j^2} |\nabla u|^2_x \psi^2 \, dx \, dt - 2 \int_{\Omega_j^2} u_t \nabla u \nabla \psi \, dx \, dt
\]

\[
\leq \frac{1}{2} \int_{\Omega_j^2} |\nabla u|^2 \psi^2 \, dx \, dt + \frac{1}{2} \int_{\Omega_j^2} (u_t)^2 \psi^2 \, dx \, dt + 2 \int_{\Omega_j^2} |\nabla u|^2 |\nabla \psi|^2 \, dx \, dt.
\]

Therefore, (2.3)

\[
\int_{\Omega_j^2} (u_t)^2 \psi^2 \, dx \, dt \leq Ck \int_{\Omega_j^2} |\nabla u|^2 \, dx \, dt.
\]

Denote by \( \psi_j^{(2)} \) a standard Lipschitz cutoff function supported in

\[
\{(t, x) \mid d(x,x_0) < (j + 1)/\sqrt{k}, \quad t \in ((j + 1)/k, (j + 1)/k)\}
\]

such that \( \psi_j^{(2)} = 1 \) in \( \Omega_j^2 \) and

\[
|\nabla \psi_j^{(2)}|^2 + |\partial_t \psi_j^{(2)}| \leq Ck.
\]

Using \( (\psi_j^{(2)})^2 u \) as a test function in the heat equation, the standard Caccioppoli inequality (energy estimate) between the cubes \( \Omega_j^2 \) and \( \Omega_j^{1+1} \) shows that (2.4)

\[
\int_{\Omega_j^2} |\nabla u|^2 \, dx \, dt \leq Ck \int_{\Omega_j^{1+1}} u^2 \, dx \, dt.
\]

A combination of (2.3) and (2.4) gives us (2.5)

\[
\int_{\Omega_j^1} (u_t)^2 \, dx \, dt \leq C_0 k^2 \int_{\Omega_j^{1+1}} u^2 \, dx \, dt,
\]

where \( C_0 \) is a universal constant.

Since \( \partial_t^k u \) is a solution, we can replace \( u \) in (2.5) with \( \partial_t^k u \) to deduce, after induction:

(2.6)

\[
\int_{\Omega_k^1} (\partial_t^k u)^2 \, dx \, dt \leq C_0 k^{2k} \int_{\Omega_k^1} u^2 \, dx \, dt.
\]

Note that \( \Omega_k^1 = Q_{1/\sqrt{k}}(0, x_0) \) and \( \Omega_k^1 = [-1, 0] \times B(x_0, \sqrt{k}) \). Substituting (2.6) into (2.2), we find that

\[
(\partial_t^k u)^2(0, x_0) \leq \frac{C_0 k^{3d/2+2}}{|B(x_0, 1)|} C_0(1) k^{2k} \int_{[-1,0] \times B(x_0, \sqrt{k})} u^2 \, dx \, dt.
\]
This implies, by the growth condition \((2.1)\) and the Bishop-Gromov volume comparison theorem, that
\[
|\partial_t^k u(0, x_0)| \leq A_3^{k+1} k^k e^{2A_2 d^2(x_0,0)}
\]
for all integers \(k \geq 1\). Here \(A_3\) is a positive constant depending only on \(A_2, K_0,\) and \(d\) .
We remark that the volume of the ball in the denominator is cancelled since
\[
|B(x_0, \sqrt{k})| \leq e^{c\sqrt{k}} |B(x_0, 1)|.
\]

Fixing a number \(R \geq 1\), for \(x \in B(0, R)\), choose a positive integer \(j\) and \(t \in [-\delta, 0]\) for some small \(\delta > 0\). Taylor’s theorem implies that
\[
(2.8) \quad u(t,x) - \sum_{i=0}^{j-1} \partial_t^i u(0,x) \frac{t^i}{i!} = \partial_j^t u(s,x),
\]
where \(s = s(x,t,j) \in [t,0]\). By \((2.7)\), for sufficiently small \(\delta > 0\), the right-hand side of \((2.8)\) converges to 0 uniformly for \(x \in B(0, R)\) as \(j \to \infty\). Hence
\[
u(t,x) = \sum_{j=0}^{\infty} \partial_j^t u(0,x) \frac{t^j}{j!},
\]
i.e., \(u\) is analytic in \(t\) with radius \(\delta\). Writing \(a_j = a_j(x) = \partial_j^t u(0,x)\). By \((2.7)\) again, we have
\[
\partial_t u(t,x) = \sum_{j=0}^{\infty} a_{j+1}(x) \frac{t^j}{j!} \quad \text{and} \quad \Delta u(t,x) = \sum_{j=0}^{\infty} \Delta a_j(x) \frac{t^j}{j!},
\]
where both series converge uniformly for \((t,x) \in [-\delta,0] \times B(0,R)\) for any fixed \(R > 0\).
Since \(u\) is a solution of the heat equation, this implies
\[
\Delta a_j(x) = a_{j+1}(x)
\]
with
\[
|a_j(x)| \leq A_3^{j+1} j^j e^{2A_2 d^2(x,0)}.
\]
Here \(A_3\) a positive constant depending only on \(A_2, K_0,\) and \(d\). This completes the proof of the theorem. \(\square\)

An immediate application is the following:

**Corollary 1.** Let \(M\) be as in the theorem. Then the Cauchy problem for the backward heat equation
\[
(2.9) \begin{cases} 
\partial_t u + \Delta u = 0, \\
u(0,x) = a(x)
\end{cases}
\]
has a smooth solution of exponential growth of order 2 in \((0, \delta) \times M\) for some \(\delta > 0\) if and only if
\[
(2.10) \quad |\Delta^j a(x)| \leq A_3^{j+1} j^j e^{A_4 d^2(x,0)}, \quad j = 0, 1, 2, \ldots,
\]
where \(A_3\) and \(A_4\) are some positive constants.
Proof. Suppose \((2.9)\) has a smooth solution of exponential growth of order 2, say \(u = u(t, x)\). Then \(u(x, -t)\) is a solution of the heat equation with exponential growth of order 2. By the theorem

\[ u(x, -t) = \sum_{j=0}^{\infty} a_j(x) \frac{(-t)^j}{j!}. \]

Then \((2.10)\) follows from the theorem since \(\Delta_j a(x) = a_j(x)\) in the theorem.

On the other hand, suppose \((2.10)\) holds. Then it is easy to check that

\[ u(t, x) = \sum_{j=0}^{\infty} \Delta_j a(x) \frac{t^j}{j!} \]

is a smooth solution of the heat equation for \(t \in [-\delta, 0]\) with \(\delta\) sufficiently small. Indeed, the bounds \((2.10)\) guarantee that the above series and the series

\[ \sum_{j=0}^{\infty} \Delta^{j+1} a(x) \frac{t^j}{j!} \quad \text{and} \quad \sum_{j=0}^{\infty} \Delta^j a(x) \frac{\partial t^j}{j!} \]

all converge absolutely and uniformly in \([-\delta, 0] \times B(0, R)\) for any fixed \(R > 0\). Hence \(\partial_t u - \Delta u = 0\). Moreover \(u\) has exponential growth of order 2 since

\[ |u(t, x)| \leq \sum_{j=0}^{\infty} |\Delta^j a(x)| \frac{|t|^j}{j!} \leq A_3 e^{A_4 d^2(x, 0)} \sum_{j=0}^{\infty} \frac{(A_3 j |t|^j)}{j!} \leq A_3 e^{A_4 d^2(x, 0)} \]

provided that \(t \in [-\delta, 0]\) with \(\delta\) sufficiently small. Thus \(u(x, -t)\) is a solution to the Cauchy problem of the backward heat equation \((2.9)\) of exponential growth of order 2. \(\square\)

Remark 2.1. For the conclusion of the theorem to hold, some growth condition for the solution is necessary. Tychonov’s non-uniqueness example can be modified as follows. Let \(v = v(t, x)\) be Tychonov’s solution of the heat equation in \((\mathbb{R}, \mathbb{R}^d)\), which is 0 when \(t \leq 0\) but nontrivial for \(t > 0\). Then \(u \equiv v(x, t + 1)\) is a nontrivial ancient solution. It is clearly not analytic in time. Note that \(|u(t, x)|\) grows faster than \(e^{c|x|^2}\) for any \(c > 0\), but for any \(\varepsilon > 0\), \(|u(x, t)|\) is bounded by \(Ce^{c|x|^2 + \varepsilon}\) for some positive constants \(c\) and \(C\). This implies that our growth condition is sharp.

Remark 2.2. If \(M = \mathbb{R}^d\), it is well known that the solution in the theorem is also analytic in space variables. In fact, in this case for the time analyticity the Laplace operator can be replaced with a uniformly elliptic operator in divergence form \(D_i(a_{ij} D_j)\), where \(a_{ij}\) are measurable functions depending only on \(x\). For general manifolds, the space analyticity requires certain bounds on curvature and its derivatives.

Remark 2.3. The time analyticity and backward solvability results are still valid if the manifold \(M\) is replaced with a bounded Lipschitz domain \(D\) and the solutions are required to satisfy the Dirichlet or Neumann boundary condition. To justify, we can modify the proof of the theorem by replacing the geodesic balls with the fixed domain \(D\) intersected with balls. The proof goes through for the following reasons. The first is that the mean value inequality with the geodesic balls replaced with \(D\) intersected with balls still holds for \(\partial_D u\) which satisfies the same boundary condition as \(u\) itself. The second is that the boundary terms in the integration by parts procedure all vanish when the geodesic balls are replaced with \(D\) intersected with balls.
All these results are also true with the appearance of a time independent inhomogeneous term, which is smooth and has at most exponential growth of order 2, on the right-hand side of the heat equation. This may have applications in control theory.

Next we present a further application of the main result in the section: time analyticity of solutions of the heat equation down to the initial time 0. Recall the following classical example from [Kow]. Given the initial value \( u_0(x) = 1/(1 + x^2) \) in \( \mathbb{R} \), the solution to the Cauchy problem of the heat equation in \([0, \infty) \times \mathbb{R}\) is not analytic in time at \( t = 0 \). In the same paper, analyticity of solution down to \( t = 0 \) is linked to the extension property of the initial value to an entire function in the complex plane with certain growth condition. See also [Wi] Corollary 3.1b for a relatively modern approach. We mention that under the growth condition, time analyticity down to time 0 also implies solvability of the backward heat equation, at least in short time. However it seems that the classical authors were mainly concerned with the concept of analytic extension in space time without realizing that under the growth condition, solutions of the backward heat equation are analytic in time automatically. The following result, which is an immediate consequence of the above corollary, provides a necessary and sufficient condition for solutions to be analytic up to time 0 without dependence on spatial analyticity, which is not available in the general setting.

**Corollary 2.** Let \( M \) be as in the theorem. Then the Cauchy problem for the heat equation

\[
\begin{cases}
\partial_t u - \Delta u = 0, \\
u(0, x) = a(x)
\end{cases}
\tag{2.11}
\]

has a smooth solution of exponential growth of order 2, which is also analytic in time in \([0, \delta) \times M\) for some \( \delta > 0 \) with a radius of convergence independent of \( x \) if and only if

\[
|\Delta^j a(x)| \leq A_3^{j+1} A_4^j e^{A_4 d^2(x,0)}, \quad j = 0, 1, 2, \ldots,
\tag{2.12}
\]

where \( A_3 \) and \( A_4 \) are some positive constants.

**Proof.** In the proof, all solutions are of exponential growth of order 2.

Assuming (2.12), it is a standard result that problem (2.11) has a solution \( u = u(t, x) \) for some \( \delta > 0 \). By Corollary 1 the following backward problem also has a solution

\[
\begin{cases}
\partial_t v + \Delta v = 0, \\
v(0, x) = a(x)
\end{cases}
\tag{2.13}
\]

in \([0, \delta) \times M\). Define the function \( U = U(t, x) \) by

\[
U(t, x) = \begin{cases} u(t, x), & t \in [0, \delta), \\
v(-t, x), & t \in (-\delta, 0]. \end{cases}
\tag{2.14}
\]

It is straightforward to check that \( U \) is a solution of the heat equation in \((-\delta, \delta) \times M\). By the theorem, \( U \) and hence \( u \) is analytic at time 0. We mention that the time interval in the theorem is normalized to \([-2, 0]\) which can be changed to any finite interval.

On the other hand, suppose \( u \) is a solution of the problem (2.11), which is analytic in time at \( t = 0 \) with a radius of convergence independent of \( x \). Then, by definition, \( u \) has a power series expansion in a time interval \((-\delta, \delta)\), for some \( \delta > 0 \). Hence (2.12) holds following the proof of Corollary 1 the first half. \( \square \)
3. The Navier-Stokes equations

The main result of this section is the following theorem.

**Theorem 3.1.** Assume that \( u \) is a mild solution to the incompressible Navier-Stokes equations
\[
    u_t - \Delta u + u \cdot \nabla u + \nabla p = 0
\]
on \([0, 1] \times \mathbb{R}^d\) and
\[(3.1)\quad |u| \leq C_2 \quad \text{in} \quad [0, 1] \times \mathbb{R}^d.
\]
Then for any \( n \geq 1 \),
\[
    \sup_{t \in (0, 1)} t^n \| \partial_t^n u(t, \cdot) \|_{L_\infty(\mathbb{R}^d)} \leq N^{n+1} n^n
\]
for some sufficiently large constant \( N \geq 1 \). Consequently, \( u(t, x) \) is analytic in time for any \( t \in (0, 1) \).

The proof of the theorem relies on taking time derivative of the integral representation of the solution involving the Stokes kernel. One difficulty to overcome is that the time derivative of the Stokes kernel is not locally integrable in space time, let alone higher order derivatives. We will manipulate the kernel function algebraically to allow differentiation. This method seems to be applicable to other types of equations. We also mention that non-mild solutions of the Navier-Stokes equations need not be analytic in time, as given by Serrin’s example \( u = a(t) \nabla h(x) \) where \( h \) is a harmonic function and \( a = a(t) \) is an arbitrary smooth function.

The following lemma will be used frequently.

**Lemma 3.1.** For any \( n \geq 1 \), we have
\[
    \sum_{j=1}^{n-1} \binom{n}{j} j^{-2/3} (n-j)^{n-j-2/3} \leq C n^{-2/3},
\]
where \( C > 0 \) is a constant independent of \( n \).

**Proof.** By the Stirling formula,
\[
    \sum_{j=1}^{n-1} \binom{n}{j} j^{-2/3} (n-j)^{n-j-2/3} \leq C n^{-2/3} \sum_{j=1}^{n-1} \frac{n^{7/6}}{j^{7/6} (n-j)^{7/6}}
\]
\[
    = C n^{-2/3} \sum_{j=1}^{n-1} \left( \frac{1}{j} + \frac{1}{n-j} \right)^{7/6}
\]
\[
    \leq C n^{-2/3} \sum_{j=1}^{n-1} \left( \frac{1}{j^{7/6}} + \frac{1}{(n-j)^{7/6}} \right) \leq C n^{-2/3}.
\]
The lemma is proved. \( \square \)

The next combinatorial lemma can be proved by using induction.
Lemma 3.2. Let $f$ and $g$ be two smooth functions on $\mathbb{R}$. For any integer $n \geq 1$, we have
\[
D^n(t^n f(t)g(t)) = \sum_{j=0}^{n} \binom{n}{j} D^j(t^j f(t)) D^{n-j}(t^{n-j} g(t)) - n \sum_{j=0}^{n-1} \binom{n-1}{j} D^j(t^j f(t)) D^{n-1-j}(t^{n-1-j} g(t)),
\]
where we denote $D = \partial_t$.

Proof. It follows from a straightforward computation by using
\[
D^n(t^n f(t)g(t)) = tD^n(t^{n-1} f(t)g(t)) + nD^{n-1}(t^{n-1} f(t)g(t))
\]
and the inductive assumption. \qed

Let $P$ be the Helmholtz (Leray–Hopf) projection in $\mathbb{R}^d$, and $E(t, x) = P \Gamma(t, x)$ be the Stokes-Oseen kernel, where
\[
\Gamma = (4\pi t)^{-d/2} e^{-|x|^2/(4t)}
\]
is the heat kernel. Recall that $E$ satisfies the homogeneous heat equation, the semigroup property, and
\[
E(t, x) = t^{-d/2}E(1, x/\sqrt{t}),
\]
where $E(1, \cdot)$ is a smooth function on $\mathbb{R}^d$ and decays like $C/|x|^d$ as $x \to \infty$. Moreover, $(\partial_t E)(1, x)$ and $(\nabla E)(1, x)$ decay like $C/|x|^{d+2}$ and $C/|x|^{d+1}$ respectively as $x \to \infty$. See, for instance, \cite{Sol}. Using these properties, we easily obtain
\[
\|\nabla E(t, \cdot)\|_{L^1} \leq C_0 t^{-1/2},
\]
and
\[
\|\partial_t^k E(t, \cdot)\|_{L^1} \leq C_0^{k+1} k^{k-2/3} t^{-k}, \quad \|\partial_t^k \nabla E(t, \cdot)\|_{L^1} \leq C_0^{k+1} k^{k-2/3} t^{-1-k/2}
\]
for any integer $k \geq 1$ and $t > 0$, where $C_0 \geq 1$ is a constant. It then follows from the Leibniz rule that
\[
\|\partial_t^k (t^k \nabla E(t, \cdot))\|_{L^1} \leq C_1^{k+1} k^{k-2/3} t^{-1/2}.
\]
Similarly, we have
\[
\|\partial_t^k (t^k \Gamma(t, \cdot))\|_{L^1} \leq C_1^{k+1} k^{k-2/3}.
\]

To prove Theorem 3.1, we first establish the following proposition.

Proposition 3.1. Under the conditions of Theorem 3.1, for any $n \geq 1$, we have
\[
\sup_{t \in (0, 1]} \|\partial_t^n (t^n u(t, \cdot))\|_{L^\infty(\mathbb{R}^d)} \leq N^{n-1/2} n^{n-2/3}
\]
for some sufficiently large constant $N \geq 1$ depending only on the dimension and $\|u\|_{L^\infty}$.

Proof. We shall prove the proposition inductively. As $u$ is a mild solution, we have
\[
u(t, x) = E(t, x) * u(0, x) - \int_0^t E(t - s, x) * \nabla (u \otimes u)(s, x) \, ds,
\]
where \( * \) denotes the spatial convolution. Then, by using integration by parts,
\[
\partial_t^n (t^n u(t, x)) = \partial_t^n (t^n E(t, x) * u(0, x)) - \partial_t^n \left( \int_0^t t^n \nabla E(t - s, x) * (u \otimes u)(s, x) \, ds \right) =: I_1 + I_2.
\]

By using (3.1) and (3.4),
\[
|I_1| \leq C_2C_1^{n+1} n^{n-2/3} \leq N^{n-2/3} n^{n-2/3}
\]
for sufficiently large \( N \).

To estimate \( I_2 \), we first note that
\[
\int_0^t t^n \nabla E(t - s, x) * (u \otimes u)(s, x) \, ds = \sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) \int_0^t (t - s)^k \nabla E(t - s, x) * \left( s^{n-k}(u \otimes u)(s, x) \right) \, ds.
\]
Therefore,
\[
I_2 = -\sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) \partial_t^n \int_0^t ((t - s)^k \nabla E(t - s, x)) * \left( s^{n-k}(u \otimes u)(s, x) \right) \, ds
\]
\[
= -\sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) \partial_t^{n-k} \int_0^t \partial_t^k ((t - s)^k \nabla E(t - s, x)) * \left( s^{n-k}(u \otimes u)(s, x) \right) \, ds
\]
\[
= -\sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) \partial_t^{n-k} \int_0^t \partial_s^k (s^k \nabla E(s, x)) * \left( (t - s)^{n-k}(u \otimes u)(t - s, x) \right) \, ds
\]
\[
= -\sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) \int_0^t \partial_s^k (s^k \nabla E(s, x)) * \partial_t^{n-k} (t - s)^{n-k}(u \otimes u)(t - s, x) \, ds,
\]
where in the third equality, we made a change of time variable \( s \rightarrow t - s \), which allows us to pass the time derivatives \( \partial_t^{n-k} \) through the integral without producing additional terms.

When \( n = 1 \), we compute
\[
|\partial_t (t(u \otimes u)(t, x))| = |\partial_t (tu) \otimes u(t, x) + u \otimes \partial_t (tu)(t, x) - u \otimes u(t, x)| \leq C \|u\|_{L_\infty} |\partial_t (t u)(t, x)| + C \|u\|_{L_\infty}^2,
\]
where \( C > 0 \) depends only on the dimension \( d \). In general, by the inductive assumption and Lemmas 3.2 and 3.1 we have for \( k = 1, \ldots, n - 1 \),
\[
(3.6) \quad |\partial_t^k (tk^k(u \otimes u)(t, x))| \leq N^{k-1/3} k^{k-2/3}
\]
and
\[
(3.7) \quad |\partial_t^n (t^n u \otimes u)(t, x))| \leq 2C_2 |\partial_t^n (t^n u(t, x))| + N^{n-3/4} n^{n-2/3}
\]
provided that $N$ is sufficiently large, depending only on the dimension $d$ and $\|u\|_{L_\infty}$. It then follows from Lemma 3.1, (3.2), (3.3), (3.7), and (3.6) that
\[
|I_2| \leq \int_0^t (t-s)^{-1/2} \left[ C_1^{n+1} n^{n-2/3} C_2^2 + C_0 (2C_2 \|\partial^p_u ((t-s)^n u(t-s, \cdot))\|_{L_\infty} \right. \\
+ N^{n-3/4} n^{n-2/3} + \sum_{k=1}^{n-1} \binom{n}{k} C_1^{k+1} n^{n-2/3} \cdot N^{n-1/3} (n-k)^{n-k-2/3} \left. \right] ds \\
\leq N^{n-2/3} n^{n-2/3} t^{1/2} + 2C_1 C_2 \int_0^t s^{-1/2} \|\partial^p_n (s^n u(s, \cdot))\|_{L_\infty} ds
\]
for sufficiently large $N$ depending on $C_1$ and $C_2$.

Combining the estimates of $I_1$ and $I_2$, we get (3.5) by applying the Gronwall inequality, and complete the proof of the proposition. □

Finally, we complete the proof of Theorem 3.1.

**Proof of Theorem 3.1** Note that
\[
\partial^p_t (t^k u) = n \partial^{p-1}_t (t^{k-1} u) + t \partial^p_t (t^{k-1} u).
\]
Taking $k = n$ and using (3.5), we obtain
\[
\sup_{t \in (0,1]} \|t \partial^p_t (t^{n-1} u(t, \cdot))\|_{L_\infty(\mathbb{R}^d)} \leq N^n (1 + 1/N)n^n.
\]
By induction,
\[
\sup_{t \in (0,1]} \|t^n \partial^p_t u(t, \cdot)\|_{L_\infty(\mathbb{R}^d)} \leq N^n (1 + 1/N)n^n = (N + 1)^n n^n.
\]
The theorem is proved. □

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