The averaging problem on the past null cone in inhomogeneous dust cosmologies

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Abstract
Cosmological models typically neglect the complicated nature of the spacetime manifold at small scales in order to hypothesize idealized general relativistic solutions for describing the average dynamics of the Universe. Although these solutions are remarkably successful in accounting for data, they introduce a number of puzzles in cosmology, and their foundational assumptions are therefore important to test. In this paper, we go beyond the usual assumptions in cosmology and propose a formalism for averaging the local general relativistic spacetime on an observer’s past null cone: we formulate average properties of light fronts as they propagate from a cosmological emitter to an observer. The energy-momentum tensor is composed of an irrotational dust source and a cosmological constant—the same components as in the \( \Lambda \)CDM model for late cosmic times—but the metric solution is not a priori constrained to be locally homogeneous or isotropic. This generally makes the large-scale dynamics depart from that of a simple Friedmann–Lemaître–Robertson–Walker solution through ‘backreaction’ effects. Our formalism quantifies such departures through a fully covariant system of area-averaged equations on the light fronts propagating towards an observer, which can be directly applied to analytical and numerical inves-

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tigations of cosmic observables. For this purpose, we formulate light front averages of observable quantities, including the effective angular diameter distance and the cosmological redshift drift and we also discuss the backreaction effects for these observables.

**Keywords**  Relativistic cosmology · Spacetime foliations · Light propagation · Cosmological backreaction · Dark Universe

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1 Introduction

Observational cosmology primarily relies on information that propagates along our past null cone; cf. Ref. [108]. Observational data is most commonly interpreted within the Friedmann–Lemaître–Robertson–Walker (FLRW) class of spacetime metrics. Perturbation theory at this background cosmology is then employed to refine the description of the propagation of null signals along the past null cone in an ‘almost FLRW’ cosmology that takes inhomogeneities into account (see, e.g., Refs. [12, 143, 144, 158]). The ΛCDM model (Cold Dark Matter including a positive cosmological constant Λ) is a particular general relativistic FLRW metric model, with ordinary matter, radiation, dark matter (CDM), and dark energy (modelled with Λ) as energy-momentum sources. The ΛCDM paradigm has proven successful in a largely consistent interpretation of observational data; however, with a number of ‘tensions’ remaining [6, 31, 45, 94, 128, 137]. Importantly, the success of the ΛCDM framework comes at the price of introducing dark components to the cosmological energy budget [3].

The specification of a realistic inhomogeneous spacetime metric for a lumpy model of the Universe and the description of realistic local properties for the propagation of null signals is an involved task. Cosmological observations involve the study of many sources and the propagation of null signals over large spatial distances. We
might thus aim at formulating average dynamical equations—suitable for obtaining effective laws for the propagation of null signals in a cosmological context—where exact knowledge of the local spacetime metric might not be required. Averaging of Einstein’s field equations in general relativity has been formulated in the context of a 3-dimensional volume-averaging of scalar-valued variables\(^1\) [24, 25]. In this approach, the spacetime metric is not specified and no approximation is employed to restrict inhomogeneities. Instead, one concentrates on integral properties of local spacetime variables such as the rest mass density, the expansion rate, the shear rate and the spatial Ricci curvature scalar. The resulting volume-averaged Einstein’s field equations can be viewed as balance equations for effective macroscopic spacetime variables. The system of equations governing the macroscopic variables is in general not closed and additional constraints must be imposed in order to solve the system. (A comprehensive discussion of the closure problem in \(N\) dimensions, in particular emphasizing topological constraints, may be found in Ref. [22].)

In such average Universe descriptions, the structure of the underlying local inhomogeneous distributions of the matter content and the spatial geometry manifests itself globally through correction terms to Friedmann’s equations. Such terms will be non-vanishing in general—also in cosmologies with a notion of statistical spatial homogeneity and isotropy—and represent the deviance of the ‘monopole state’ of a model universe with structure, i.e., an appropriate lowest-order and spatially uniform solution, from the strictly spatially homogeneous and isotropic FLRW solutions. The monopole state can, nevertheless, be mapped to an effective FLRW cosmology by interpreting the correction terms as effective source terms. For instance, volume-averaged fluctuation terms have been discussed as geometrical candidates of kinematical dark energy and dark matter sources; e.g. Refs. [14, 15, 23, 26–28, 34, 37, 37, 38, 40, 91, 105, 110, 126, 132, 135, 140, 154, 155, 159, 163–166].

Much work has been done in describing the past null cones of observers in various inhomogeneous and anisotropic model universes (e.g. Refs. [5, 16, 44, 46, 57, 82, 93, 98–100, 111, 113, 122, 124, 131, 138, 146, 162]), including coordinate-independent considerations within gauge invariant perturbation theory [21, 167, 168], and the distortion of the average distance—redshift relation relative to that of a reference FLRW cosmology in specific spatially inhomogeneous model universes [1, 13, 20, 68, 71, 72, 101, 123, 148]. General considerations on the propagation of null signals in statistically spatially homogeneous and isotropic cosmologies with slowly evolving structure have been carried out [133, 134]. Such considerations allow for relating leading-order observable quantities to volume-averaged variables on spacelike 3-surfaces under the assumption that null signals sample the Universe fairly in volume. Covariant null cone averaging formalisms suitable for averaging on light fronts have been developed in Refs. [70, 76], but have mainly been employed for the purpose of constraining errors in the determination of FLRW background parameters in a \(\Lambda\)CDM cosmology [7–9]. Formalism that can be used for model-independent analysis of standardizable objects and cosmic drift effects, while making no assumptions about the form of the cosmological spacetime metric, has been presented in Refs. [87, 89, 90, 156]—see also

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\(^1\) Averaging over tensor-valued variables introduces ambiguities in the volume-averaging procedure (see Refs. [26, 29, 54, 58, 127] and references therein).
Refs. [39, 83, 106, 107, 157] for other interesting formalisms which could be suitable for model-independent data analysis.

In this paper, we employ an averaging operation similar to that introduced in Refs. [24, 25], but with the averaging domain adapted to an observer’s past null cone. More precisely, we shall propagate 2-dimensional inhomogeneous light fronts on which we define area-averaged physical variables (of, e.g., matter density and optical scalars), and invoke Einstein’s field equations to obtain propagation equations for the same area-averaged variables. We do this without specifying the spacetime metric. We shall characterize average cosmological distance relations in terms of null cone averages over light fronts. We shall constrain the local setting to that of an irrotational dust cosmology. This approximation comes with limitations: on scales where gravitationally bound structures are forming, the irrotationality assumption for matter breaks down. At such scales, velocity dispersion arising from the internal motion effects within the structures must generally also be included through an effective anisotropic pressure term that violates the dust approximation. In the early history of the Universe, where radiation constitutes a significant fraction of the total energy density, pressure must also be included for an accurate description. The framework developed here is thus applicable for scales above the largest gravitationally virialized structures and for cosmic times well after the epoch of radiation and matter equality. In order to include modelling of smaller length scales and earlier cosmic times, the framework must be generalized in terms of the assumed matter content, which may be done by following similar approaches as in Ref. [36], see also Ref. [35]. We furthermore consider the idealized case where caustics in the geodesic null congruence constituting an observer’s past null cone can be ignored. This is consistent with the large-scale modelling of the matter content of the Universe described above, where we neglect the physics of strong gravitational lenses. For treatments including caustics, see, e.g., Refs. [56, 62].

The general framework investigated in this paper offers the possibility of employing non-perturbative and background-free approximations to describe the effect of inhomogeneities on cosmological measurements.

**Notation and conventions** We use units in which \( c = G/c^2 = 1 \). Greek letters \( \mu, \nu, \ldots \) label spacetime indices in a general basis. Summation over repeated indices is understood. The signature of the spacetime metric, \( g_{\mu\nu} \), is taken to be \((-+++)[120].\)

## 2 Local spacetime setting

Here we describe the assumptions made for the local cosmological spacetime. In Sect. 2.1 we describe the assumed matter content in the form of an irrotational dust source. In Sect. 2.2 we describe the observer’s past null cone and in Sect. 2.3 we introduce the foliation of the past null cone into light fronts that will be used throughout this paper. In Sects. 2.4 and 2.5 we derive propagation equations and constraint equations, respectively, for capturing the dynamics of relevant variables for the cosmological spacetime and null signals travelling along an observer’s past null cone. Finally, in

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\(^2\) The same averaging operation might be formulated in the notation of Refs. [70, 76].
Sect. 2.6 we consider photon number conservation and effective cosmological distance measures.

2.1 Field equations and matter content

We consider a cosmological spacetime which is dynamically described through Einstein’s field equations \[50, 51,\]

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + \Lambda g_{\mu \nu} = 8\pi T_{\mu \nu},
\]

(1)

where \(R_{\mu \nu}\) is the 4-Ricci curvature of the spacetime, \(R\) is its trace, and \(\Lambda\) denotes the cosmological constant. We concentrate on a matter-dominated epoch of the cosmological evolution and restrict the energy-momentum tensor \(T_{\mu \nu}\) to that of an irrotational continuum of dust,

\[
T_{\mu \nu} = \varrho u_\mu u_\nu; \quad \omega_{\mu \nu} := h_{[\alpha} h_{\nu]}^\beta \nabla_\alpha u_\beta = 0,
\]

(2)

where \(u^\mu\) is the future-directed 4-velocity of the matter congruence, \(\varrho\) is the matter’s rest mass density, \(\nabla_\mu\) is the covariant derivative, and square brackets denote antisymmetrization in the indices involved. We assume that the irrotational continuum of dust is transparent to propagating null signals. From Eq. (2) and the conservation of energy-momentum it follows that \(u^\mu\) is geodesic and can be expressed as the gradient of a scalar-valued variable,

\[
u^\nu \nabla_\nu u_\mu = 0; \quad u_\mu = -\partial_\mu \tau,
\]

(3)

where \(\tau\) can be interpreted as a proper time function along each individual matter world line, since \(u^\alpha \partial_\alpha \tau = 1\). The covariant \((1 + 3)\)-decomposition of the covariant derivative of the 4-velocity of the matter congruence thus results in (cf. Refs. \[49, 53, 59, 60, 64, 65, 67, 139, 160,\])

\[
\nabla_\mu u_\nu = \Theta_{\mu \nu} = \frac{1}{3} \theta h_{\mu \nu} + \sigma_{\mu \nu},
\]

(4)

where \(\theta\) is the isotropic expansion rate describing the volume expansion of the matter congruence, and \(\sigma_{\mu \nu}\) is the volume shear rate describing the volume-preserving deformation of the congruence. The covariant derivative of the 4-velocity is equal to its own projection \(\Theta_{\mu \nu} := h_{\mu}^\alpha h_{\nu}^\beta \nabla_\alpha u_\beta\) because of the vanishing of the 4-acceleration of \(u^\mu\). The spatial projection tensor

\[
h_{\mu}^\nu := \delta_{\mu}^\nu + u_\mu u^\nu,
\]

(5)

is orthogonal to the irrotational matter 4-velocity field and plays the role of the metric tensor on the spacelike 3-surfaces orthogonal to \(u^\mu\).
2.2 Past null cone

We now assume the presence of a non-gravitating, geodesic and irrotational null congruence pervading the cosmological spacetime such that its future-directed 4-momentum $k^\mu$ obeys [147]

$$k_\mu k^\mu = 0; \quad k^\nu \nabla_\nu k^\mu = 0; \quad k_{[\mu} \nabla_\nu k_{\alpha]} = 0. \quad (6)$$

This implies that $k^\mu$ is proportional to the gradient of a scalar-valued variable,

$$k_\mu = \eta \partial_\mu V; \quad k^\rho \partial_\rho V = 0; \quad k^\rho \partial_\rho \eta = 0, \quad (7)$$

where the conditions $\eta > 0$ and $n^\rho \partial_\rho V < 0$ (with $n^\mu$ a future-directed timelike vector field) ensure the future-directed nature of $k^\mu$. The function $V$ defines the direction of the 4-momentum $k^\mu$ and $\eta$ defines its normalization. The preservation of $V$ along each individual ray of the geodesic null congruence follows from the vanishing norm of $k^\mu$, and the preservation of $\eta$ follows from the affine geodesic equation. Spacetime domains singled out by $\{V = \text{constant}\}$ are 3-dimensional null surfaces.

As a special case of null surfaces we consider those associated with the past null cones of an observer. We consider the case where the timelike world line, $\gamma$, of this observer belongs to the matter congruence. For each point $P$ of $\gamma$, we consider incoming geodesic null rays of all spatial directions orthogonal to $u^\mu$ as initial conditions for the direction of the 4-momentum $k^\mu$. We extend this two-parameter family of null directions away from $P$ via the geodesic equation to construct the past null cone $C^-(P)$ at $P$. From $u^\rho \partial_\rho V < 0$, we have that the scalar $V$ uniquely labels the one-parameter family of past null cones $C^-(P)$ along $\gamma$. Since $\eta$ is a constant over each null cone, we have that $\eta$ is a function of $V$, and can be absorbed into a redefinition of $V$. We thus set $\eta = 1$ without loss of generality. The normalization of the 4-momentum $k^\mu$ might for instance be chosen by fixing the energy function,

$$E := -u^\mu k_\mu = -u^\rho \partial_\rho V, \quad (8)$$

along $\gamma$. This choice of normalization specifies the frequency of monochromatic null signals measured at the observer’s telescope. We might for instance consider the normalization $-u^\mu k_\mu |_{\gamma} = \text{constant}$ that corresponds to the situation where the central observer performs measurements at the same radiation frequency at different instances of proper time.

The cosmological redshift, $z$, of a luminous astrophysical source comoving with the matter congruence and located on the observer’s past null cone, as measured at the point of observation $P$ (“here and now”), is given by (cf. Ref. [59])

$$(1 + z) := \frac{-u^\mu k_\mu}{-u^\mu k_\mu(P)} = \frac{E}{E(P)}, \quad (9)$$

---

3 We might in principle choose any world line to initialize a past null cone, but we shall often be interested in observers comoving with the matter in the cosmological spacetime.
where \( E(P) \) denotes the energy of the incoming null ray as measured by the observer at the point of observation \( P \). The redshift function is independent of the normalization procedure chosen for \( k^\mu \).

We may decompose the 4-momentum \( k^\mu \) associated with the incoming geodesic null congruence in terms of the 4-velocity \( u^\mu \) of the matter congruence and a spatial unit vector \( e^\mu \) orthogonal to \( u^\mu \) according to

\[
k^\mu = E(u^\mu - e^\mu),
\]

where \( e^\mu \), as evaluated by the observer at \( P \), is multi-valued and takes values in the two-parameter family of all possible spatial directions of pointings of the observer’s telescope. For later reference, we introduce an auxiliary future-directed null 4-momentum,

\[
l^\mu = E(u^\mu + e^\mu).
\]

The null 4-momentum \( l^\mu \) points in the opposite spatial direction as \( k^\mu \), when viewed in the rest frame of the matter congruence. Note that \( l^\mu \) is not necessarily a geodesic null congruence (cf. Footnote 5 for further properties). We refer to geodesic null rays generated by \( k^\mu \) as incoming null rays and null rays generated by \( l^\mu \) as outgoing null rays. The field \( l^\mu \) does not incorporate any new physical information about the cosmological spacetime, but will be convenient for formulating constraint equations in the covariant \((1 + 1 + 2)\)-decomposition of the cosmological spacetime considered later in this analysis (see Sect. 2.5).

### 2.3 Foliation of the past null cone and light fronts

In order to describe the evolution along a fixed past null cone \( C^-(P) : \{ V = \text{constant} \} \), we consider its foliation into a union of spatial 2-surfaces that we refer to as light fronts. One way to construct light fronts is to consider the intersection of the past null cone with a family of 3-surfaces (timelike, spacelike or lightlike). An obvious choice of an intersecting foliation is that defined from the irrotational 4-velocity \( u^\mu \) of the matter congruence, so that the fronts of intersection are simultaneously orthogonal to the null congruence and to the matter congruence.

The resulting light fronts—also referred to as screen spaces, cf. Refs. [141, 161]—are constant-level surfaces of \( \tau \) and \( V \). The local screen basis propagated along the geodesics null rays is usually referred to as Sachs basis [97, 129, 141]. In this setting, the full observer’s past null cone can be considered to be the union of 2-dimensional light fronts with varying values of \( \tau \).\(^4\) We define the projection tensor onto the light fronts:

\[
p_{\mu}^{\quad v} := \delta_{\mu}^{\quad v} + u_{\mu}u^{v} - e_{\mu}e^{v}.
\]

\(^4\) Another option is to consider level surfaces of constant affine parameter \( \lambda \) of the geodesic null congruence defined from the propagation requirement \( k^\rho \partial_\rho \lambda = 1 \). In order to uniquely define \( \lambda \) as a spacetime function, we must specify initial conditions. We might, for instance, require setting \( \lambda|_y = 0 \). From this it follows immediately that \( u^\rho \partial_\rho \lambda|_y = 0 \), and the gradient of \( \lambda \) is thus spacelike in the vicinity of \( y \), and \( \lambda = \) constant-level surfaces define timelike cylinders in the same vicinity. Far away from the vertex, the use of \( \lambda \) as a meaningful foliation scalar must be carefully re-assessed.
We have $p^\mu{}^v u_v = p^\mu{}^ve_v = p^\mu{}^vk_v = p^\mu{}^vl_v = 0$, $p^\mu{}^\alpha{}^v = p^\mu{}^v$, and $p^\mu{}^\mu = 2$. The tensor $p^\mu{}^\nu$ is thus orthogonal to the space spanned by $k^\mu$ and $u^\mu$ while it acts as the metric tensor for tensorial fields intrinsic to the screen space. The surface density on the light fronts is given by

$$
\epsilon_{\mu\nu} := u^\alpha p^\mu{}^\beta p^\nu{}^\epsilon \epsilon_{\alpha\beta\gamma\delta} = \epsilon_{[\mu\nu]},
$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the spacetime permutation tensor, equal to $\sqrt{-g}$ for even and $-\sqrt{-g}$ for odd permutations of $0, 1, 2, 3$, and $g$ is the determinant of the cosmological spacetime metric. The surface density has the following properties:

$$
e_{\mu\nu} u^\nu = e_{\mu\nu} e^\nu = e_{\mu\nu} k^\nu = e_{\mu\nu} l^\nu = 0, e_{\mu\nu} e^{\alpha\beta} = 2! p_{[\mu}{}^\alpha p_{\nu]}{}^\beta$$

and $e_{\mu\nu} e_{\mu\nu} = 2$. The area expansion tensor associated with the light fronts can be decomposed as [97, 141]

$$
\hat{\Theta}_{\mu\nu} := p^\mu{}^\alpha p^\nu{}^\beta \nabla_\alpha k_\beta = \frac{1}{2} \hat{\theta} p_{\mu\nu} + \hat{\sigma}_{\mu\nu},
$$

where $\hat{\theta} := p^\mu{}^\nu \nabla_\mu k_\nu$, is the dimensionless area expansion rate of a geodesic null ray bundle and $\hat{\sigma}_{\mu\nu} := p^\mu{}^\alpha p^\nu{}^\beta \nabla_\alpha k_\beta - \frac{1}{2} \hat{\theta} p_{\mu\nu}$ is the dimensionless area-preserving deformation or ‘shearing’ rate of the same geodesic null ray bundle. The anti-symmetric part of the deformation vanishes by construction via the requirement of irrotationality; see the last condition in Eq. (6). We define an analogous area expansion tensor for the auxiliary null 4-momentum $l^\mu$ by

$$
\tilde{\Theta}_{\mu\nu} := p^\mu{}^\alpha p^\nu{}^\beta \nabla_\alpha l_\beta = \frac{1}{2} \tilde{\theta} p_{\mu\nu} + \tilde{\sigma}_{\mu\nu},
$$

where we have labelled the kinematic variables associated with $k^\mu$ by a hat and the kinematic variables associated with $l^\mu$ by a tilde. 5

It shall furthermore be useful to also define the analogous expansion tensor and kinematic variables associated with $e^\mu$ (labelled by a bar):

$$
\bar{\Theta}_{\mu\nu} = p^\mu{}^\alpha p^\nu{}^\beta \nabla_\alpha e_\beta = \frac{1}{2} \bar{\theta} p_{\mu\nu} + \bar{\sigma}_{\mu\nu}.
$$

The trace component of $\bar{\Theta}_{\mu\nu}$ is given by $\bar{\theta} := p^\mu{}^\nu \nabla_\mu e_\nu = \nabla_\mu e^\mu$, which follows from the orthonormality of $u^\mu$ and $e^\mu$ and the geodesic nature of $u^\mu$. The symmetric tracefree part of $\bar{\Theta}_{\mu\nu}$ is $\bar{\sigma}_{\mu\nu} := p^\mu{}^\alpha p^\nu{}^\beta \nabla_\alpha e_\beta - \frac{1}{2} \bar{\theta} p_{\mu\nu}$. The anti-symmetric component $p_{[\mu}{}^\alpha p_{\nu]}{}^\beta \nabla_\alpha e_\beta$ vanishes due to the requirement of irrotationality for both $u^\mu$ and $k^\mu$.

The evolution of the multi-valued spatial direction vector $e^\mu$ along the geodesic null congruence generated by $k^\mu$ can be formulated as [87]

$$
E^{-1} k^\rho \nabla_\rho e^\mu = -E^{-1} \alpha k^\mu - \left( \frac{1}{3} \theta h^\mu{}^\alpha + \sigma^\mu{}^\alpha \right) e^\nu,
$$

5 As a result of the fact that $l^\mu$ does not in general generate a geodesic null congruence, it does not in general satisfy the condition (6). For the same reason, $l^\mu$ is not necessarily hypersurface-forming. However, it is irrotational through its definition as a linear combination of irrotational vector fields, Eq. (10b), after projection onto the screen space normal to $k^\mu$ and $u^\mu$: $p^\mu{}^\alpha p^\nu{}^\beta \nabla_{[\alpha l_{\beta]} = 0$. 

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where we have made use of one of the two area-adapted longitudinal expansion rate variables for the matter congruence,

\[
\alpha := \frac{1}{3} \theta - (p^{\mu \nu} \sigma_{\mu \nu}), \quad \beta := \frac{1}{3} \theta + \frac{1}{2} \left( p^{\mu \nu} \sigma_{\mu \nu} \right),
\]

which were introduced in Ref. [151]. In the present irrotational dust cosmology setup the variable \(\alpha\) determines the logarithmic rate of change of the energy \(E\) along the geodesic null rays, see Sec. 2.4, and replaces the Hubble parameter in FLRW cosmology in the general cosmographic representation of the luminosity distance; see Refs. [41, 87, 156]. Using Eq. (14), we can write the ‘acceleration’ vector of \(e^\mu\) in the following way:

\[
h_{\nu}^{\mu} e^{\alpha} \nabla_{\alpha} e^{\nu} = \kappa^{\mu} + p_{\nu}^{\alpha} e^{\rho} \sigma_{\rho \nu}^{\alpha},
\]

where \(\kappa^{\mu} := p_{\nu}^{\mu} u^{\rho} \nabla_{\rho} e^{\nu}\) is the position drift describing the angular drift of the source on the observer’s sky as seen in a non-rotating (Fermi-propagated) reference frame [106]. The acceleration vector (16) involves the volume shear rate and the angular drift of the source on the observer’s sky and can thus be thought of as a measure of the violation of isotropy around the observer’s position. The acceleration vector (16) and \(\kappa^{\mu}\) are both constructed such that they are tangential to the 2-dimensional screen space.

The dimensionless kinematic variables \(\hat{\theta}, \hat{\sigma}_{\mu \nu}, \tilde{\theta}\) and \(\tilde{\sigma}_{\mu \nu}\) associated with the null 4-momenta \(k^{\mu}\) and \(l^{\mu}\) can be expressed algebraically in terms of the kinematic variables of \(u^{\mu}\) and \(e^{\mu}\), and the energy \(E\) of the incoming null rays as follows:

\[
\begin{align*}
\frac{\hat{\theta}}{E} &= \frac{2}{3} \beta - \tilde{\theta}, \\
\frac{\hat{\sigma}_{\mu \nu}}{E} &= p_{(\mu}^{\alpha} p_{\nu)}^{\beta} \sigma^{\alpha \beta} - \sigma_{\mu \nu}, \quad (17a) \\
\frac{\tilde{\theta}}{E} &= \frac{2}{3} \beta + \tilde{\theta}, \\
\frac{\tilde{\sigma}_{\mu \nu}}{E} &= p_{(\mu}^{\alpha} p_{\nu)}^{\beta} \sigma^{\alpha \beta} + \sigma_{\mu \nu}; \quad (17b)
\end{align*}
\]

this result is obtained from the covariant decompositions (10a) and (10b), and the definitions of the kinematic variables introduced in Eqs. (4), (13a), (13b) and (13c). Angular brackets single out the symmetric and tracefree part of the tensor in the involved indices.

The null kinematic variables \(\hat{\theta}, \hat{\sigma}_{\mu \nu}, \tilde{\theta}\) and \(\tilde{\sigma}_{\mu \nu}\) inherit the kinematics of \(u^{\mu}\) and \(e^{\mu}\) in the screen space. The longitudinal component of the shear rate, \((p^{\mu \nu} \sigma_{\mu \nu})\), enters in the expansion rates \(\hat{\theta}\) and \(\tilde{\theta}\) via \(\beta\). In general, the kinematic variables of \(k^{\mu}\) and \(l^{\mu}\) differ. This difference can be assigned to the focusing of the null congruences towards the observer, and is for instance present for radially propagating geodesic null congruences in FLRW cosmology.

We may eliminate the kinematic variables of the matter congruence in Eqs. (17a) and (17b) to obtain the useful relations:

\[
\tilde{\theta} = \hat{\theta} + 2E\tilde{\theta}, \quad \tilde{\sigma}_{\mu \nu} = \hat{\sigma}_{\mu \nu} + 2E\sigma_{\mu \nu}. \quad (18)
\]
The function $\theta$ quantifies the departure of the 2-dimensional screen space from a minimal surface area within the spacelike 3-surfaces orthogonal to $u^\mu$. When $k^\mu$ is a generator of a past null cone, which is narrowing towards the singularity at the vertex of the observer, we expect $\theta$ to be dominantly negative. 7

### 2.4 Propagation equations along the past null cone

We now formulate the evolution equations along the observer’s past null cone $C^-(P) : \{ V = \text{constant} \}$ for the variables of our main interest.

The propagation equations for the optical scalars associated with the incoming geodesic null congruence $k^\mu$ can be derived from light front projections of the Ricci identity $2\nabla_{[\mu} \nabla_{\nu]} k^\alpha = - R_{\mu\nu\beta\alpha} k^\beta$; cf. Refs. [161, pp. 222–223] and [97]. Evolving the energy-rescaled variables of Eq. (17a), we obtain:

$$
E^{-1} k^\rho \partial_\rho E = - \alpha E; \quad (19a)
$$

$$
E^{-1} k^\rho \partial_\rho \left( \frac{\hat{\theta}}{E} \right) = - \frac{1}{2} \left( \frac{\hat{\theta}}{E} - 2\alpha \right) \left( \frac{\hat{\theta}}{E} \right) - 2 \left( \frac{\hat{\sigma}}{E} \right)^2 - 8\pi \varrho; \quad (19b)
$$

$$
E^{-1} k^\rho \partial_\rho \left( \frac{\hat{\sigma}}{E} \right)^2 = - 2 \left( \frac{\hat{\sigma}}{E} - \alpha \right) \left( \frac{\hat{\sigma}}{E} \right) \left( \frac{\hat{\mu}^{\alpha \beta}}{E} \right) \left( p_{(\alpha} \rho_{p_{\beta} V]} \right)^{\beta} E_{\alpha \beta} + \epsilon_{(\alpha} \rho_{p_{\beta} V]} \hat{H}_{\alpha \beta}, \quad (19c)
$$

where $\hat{\sigma}^2 := \frac{1}{2} \hat{\sigma}_{\mu\nu} \hat{\sigma}^{\mu\nu}$ is the (squared) optical shear scalar. In Eq. (19c), the propagation of the energy-rescaled optical shear scalar is sourced by the incoming radiative Weyl curvature eigenfield defined by

$$
p_{(\mu} \rho_{p_{\nu} V]} \beta E_{\alpha \beta} + \epsilon_{(\mu} \rho_{p_{\nu} V]} \beta H_{\alpha \beta}, \quad (20)
$$

where $E_{\alpha \beta} := C_{\mu \nu \rho \sigma} u^\mu h_{\alpha}^\nu u^\rho h_{\beta}^\sigma$ and $H_{\alpha \beta} := - \frac{1}{2} \epsilon_{\rho \sigma \gamma \delta} C_{\mu \nu \gamma \delta} u^\rho h_{\alpha}^\gamma u^\mu h_{\beta}^\delta$ are the electric and magnetic Weyl curvature tensors with respect to the matter frame.

The propagation equations for the scalar-valued matter variables read: 8

$$
E^{-1} k^\rho \partial_\rho \theta = - \frac{1}{3} \theta^2 - 2\sigma^2 - 4\pi \varrho + \Lambda - \theta'; \quad (21a)
$$

$$
E^{-1} k^\rho \partial_\rho \varrho = - \theta \varrho - \varrho'; \quad (21b)
$$

---

6 It is a well-known result from calculus of variations in Riemannian geometry that surfaces minimizing the area measure locally have zero trace of the extrinsic curvature scalar of the embedding.

7 Since causal lines can only leave a past null cone (and not enter), we indeed expect a negative contribution to the overall expansion rate of the screen space from the drift of the screen space boundaries relative to the matter congruence: the screen space is sampling the cross section of fewer fluid elements as the vertex of the past null cone is approached. However, local differential expansion of the dust matter congruence and the spatial fluctuations in the rest mass density $\varrho$ could potentially compensate this tendency locally.

8 With Eq. (10a), when acting on scalar-valued variables, the operator identity $E^{-1} k^\rho \partial_\rho = u^\rho \partial_\rho - e^\rho \partial_\rho$ applies.

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\begin{equation}
E^{-1}k^\rho \partial_\rho \sigma^2 = -\frac{4}{3} \theta \sigma^2 - \sigma_\mu^\nu \sigma_\nu^\alpha \sigma_\alpha^\mu - (\sigma^{\mu \nu} E_{\mu \nu}) - (\sigma^2)'; \tag{21c}
\end{equation}

\begin{equation}
E^{-1}k^\rho \partial_\rho (p^{\mu \nu} \sigma_\mu^\nu) = -\frac{2}{3} \theta (p^{\mu \nu} \sigma_\mu^\nu) - p^{\mu \nu} \sigma_\mu^\alpha \sigma_\nu^\alpha \\
+ \frac{4}{3} \sigma^2 - (p^{\mu \nu} E_{\mu \nu}) - 2\sigma_{\mu \nu} e^\mu_k \kappa^\nu - (p^{\mu \nu} \sigma_\mu^\nu)', \tag{21d}
\end{equation}

where the operator \( \prime := e^\mu_\nu \nabla_\mu \) denotes the spatial derivative along \( e^\mu_\mu \), and \( (p^{\mu \nu} E_{\mu \nu}) \) is the longitudinal projection of the electric Weyl curvature tensor, which accounts for tidal effects in the evolution equations.

We can recast Eqs. (21a) and (21d) into the following evolution equations for the area-adapted longitudinal expansion rate variables \( \alpha \) and \( \beta \):

\begin{equation}
E^{-1}k^\rho \partial_\rho \alpha = -\alpha^2 - \alpha' + (p^{\mu \nu} E_{\mu \nu}) - \frac{4\pi}{3} \varrho + \frac{1}{3} \Lambda - (e^\mu_\nu \sigma_\mu^\alpha)(e^\nu_\sigma \sigma^\beta) p^{\alpha \beta} \\
+ 2\sigma_{\mu \nu} e^\mu_k \kappa^\nu, \tag{22a}
\end{equation}

\begin{equation}
E^{-1}k^\rho \partial_\rho \beta = -\frac{1}{9} (\alpha + 2\beta)(4\beta - \alpha) - \beta' - \frac{1}{2} (p^{\mu \nu} E_{\mu \nu}) - \frac{4\pi}{3} \varrho + \frac{1}{3} \Lambda \\
- \frac{1}{2} (e^\mu_\nu \sigma_\mu^\alpha)(e^\nu_\sigma \sigma^\beta) p^{\alpha \beta} - \frac{1}{2} \sigma_{\mu \alpha} \sigma_{\nu \beta} p^{\mu \nu} p^{\alpha \beta} - \sigma_{\mu \nu} e^\mu_k \kappa^\nu. \tag{22b}
\end{equation}

We may, in addition, formulate the evolution equation for the cosmological redshift, defined in Eq. (9). Using Eq. (19a), we have (cf. Ref. [53, p. 147f])

\begin{equation}
k^\rho \partial_\rho z = - E(P)\alpha (1 + z)^2. \tag{23}
\end{equation}

The cosmological redshift is a monotonic function along the geodesic null curves only when the area-adapted longitudinal expansion rate variable \( \alpha \) does not change sign. Thus, as pointed out by Räsänen [133], cosmological redshift can in general not be used as a parameter along the geodesic null congruence when scales of collapsing structures are considered; see Sect. 4 for a discussion on valid parametrizations of light fronts. For the same reason, cosmological redshift cannot in general be thought of as a cosmological time variable, since, typically, \( u^\rho \partial_\rho z \) will change sign along the world lines of the matter congruence. The proper time function \( \tau \) for the matter congruence, on the other hand, is a monotonic function along the geodesic null congruence,

\begin{equation}
k^\rho \partial_\rho \tau = -k^\mu u_\mu = E, \tag{24}
\end{equation}

since \( E \) is a positive-valued variable. We remark that level surfaces \( \{ \lambda = \text{constant} \} \)—with \( \lambda \) defined as an affine parameter along the geodesic null congruence satisfying \( k^\rho \partial_\rho \lambda = 1 \)—do not in general coincide with spacelike 3-surfaces \( \{ \tau = \text{constant} \} \). \( E \) is in general an inhomogeneous function on the screen space, which renders \( \lambda \) an invalid label for the light fronts \( \{ V = \text{constant}, \tau = \text{constant} \} \).
2.5 Constraint equations on the light fronts

The null kinematic variables given in Eqs. (17a) and (17b) are subject to constraints from the embedding of the 2-dimensional light fronts into the cosmological spacetime; cf. Refs. [150], [161, p. 258] and [149, Sec. 16.2]. When expressed in terms of the covariantly \((1+1+2)\)-decomposed variables, the Gauss embedding constraint reads:

\[
0 = K - (p^{\mu\nu} E_{\mu\nu}) - \frac{8\pi}{3} \varrho - \frac{1}{3} \Lambda + \frac{1}{4} \frac{\hat{\theta}}{E} \bar{\bar{\theta}} - \frac{1}{2} \frac{\bar{\bar{\sigma}}^{\mu\nu}}{E} \bar{\bar{\sigma}}^{\mu\nu}, \tag{25}
\]

while the two Codazzi embedding constraints are given by

\[
0 = p^{\mu\alpha} p^{\nu\beta} D_v \left( \frac{\tilde{\sigma}_{\alpha\beta}}{E} \right) - \frac{1}{2} p^{\mu\nu} D_v \left( \frac{\hat{\theta}}{E} \right) + \left( \frac{\hat{\sigma}^{\mu\nu}}{E} - \frac{1}{2} \frac{\hat{\theta}}{E} p^{\mu\nu} \right) \frac{D_v E}{E}; \tag{26a}
\]

\[
0 = p^{\mu\alpha} p^{\nu\beta} D_v \left( \frac{\tilde{\sigma}_{\alpha\beta}}{E} \right) - \frac{1}{2} p^{\mu\nu} D_v \left( \frac{\hat{\theta}}{E} \right) + \left( \frac{\tilde{\sigma}^{\mu\nu}}{E} - \frac{1}{2} \frac{\tilde{\theta}}{E} p^{\mu\nu} \right) \frac{D_v E}{E}, \tag{26b}
\]

respectively. In Eq. (25) the extrinsic curvature of the embedding has been formulated using the area expansion tensors associated with \(k^\mu\) and \(l^\mu\). The scalar \(K := \frac{1}{2} (2)R\) is the Gaussian curvature of the screen space, and \((2)R\) is the Ricci scalar of the same screen space. The sectional curvature of the light fronts is given by the projection

\[
\frac{1}{2} R_{\mu\nu\alpha\beta} p^{\mu\alpha} p^{\nu\beta} = (p^{\mu\nu} E_{\mu\nu}) + \frac{8\pi}{3} \varrho + \frac{1}{3} \Lambda, \tag{27}
\]

where \(R_{\mu\nu\alpha\beta}\) is the Riemann curvature tensor for the cosmological spacetime.

2.6 Photon conservation, distance measures, and redshift drift

In the present analysis we consider light rays propagating along the observer’s past null cone \(\overline{C}^{-}(P) : \{V = \text{constant}\}\) to be associated with freely streaming test particles which are—apart from the point of emission and the point of observation at the vertex \(P\) of the past null cone – non-interacting with the matter content of the cosmological spacetime. In this setting we are entitled to assume the conservation of the current density,

\[
J^\mu = n k^\mu, \tag{28}
\]

where \(n\) is the number density of light front continuum elements (photons) with respect to the area enclosed by the screen space. The conservation of \(J^\mu\) leads to

\[
\nabla_\mu J^\mu = 0 \iff k^\mu \partial_\mu n = -\hat{\theta} n, \tag{29}
\]
i.e. the number of radiation elements per area element is conserved. This conservation law is fundamental for the definition of cosmological distance measures, see, e.g., Refs. [53, 143], and is an important assumption in order to obtain the relativistically corrected luminosity distances.

The observer area distance $d_A$, henceforth referred to as the effective angular diameter distance, for a luminous astrophysical source is defined by [53, Eq. (6.27)]

$$d^2_A := \frac{\delta A}{\delta \Omega},$$

(30)

where $\delta A$ is the physical area covered by the source at its own position in directions perpendicular to the spatial propagation direction of the emanating geodesic null rays, and $\delta \Omega$ is the solid angle the source subponds at the position $P$ of the observer (with $0 < \delta \Omega < 4\pi$). The evolution of the effective angular diameter distance along the geodesic null congruence $k^\mu$ is given by (cf. Ref. [53, Eq. (6.19)])

$$E^{-1}k^\rho \partial_\rho d_A = \frac{1}{2} \frac{\dot{\theta}}{E} d_A; \quad E^{-1}k^\rho \partial_\rho d^{-2}_A = - \frac{\dot{\theta}}{E} d^{-2}_A.$$  

(31)

This, when combined with Eq. (19b), yields the focusing equation (cf. Ref. [129, Eq. (44)]):

$$E^{-1}k^\rho \partial_\rho (E^{-1}k^\sigma \partial_\sigma d_A) = \left( \frac{1}{2} \alpha \frac{\dot{\theta}}{E} - \left( \frac{\dot{\sigma}}{E} \right)^2 - 4\pi \varrho \right) d_A.$$  

(32)

Using the conservation law (29) and Etherington’s reciprocity theorem [59, 69, 157], the luminosity distance between a source and the observer at the vertex point $P$,

$$d_L := \sqrt{\frac{L^{\text{bol}}}{4\pi I^{\text{bol}}}},$$  

(33)

can be written in terms of the effective angular diameter distance as follows:

$$d_L = (1 + z)^2 d_A;$$  

(34)

$L^{\text{bol}}$ is the bolometric luminosity of the source, and $I^{\text{bol}}$ is the bolometric flux of energy as measured by the observer.

Let us finally consider the drift of cosmological redshift of a given source in time as measured by an observer at the vertex point $P$. Let the source and the observer be comoving with the irrotational 4-velocity field $u^\mu$ of the matter congruence and let the

---

9 In generic cosmological spacetimes the observer area distance for a luminous astrophysical source does not necessarily coincide with the linear size based angular diameter distance for the same source due to potential Weyl curvature induced shearing along the observer’s past null cone of an incoming geodesic null ray bundle. Only in an exact FLRW cosmology do their conceptions become identical as a consequence of the prevailing spatial isotropy, and thus vanishing Weyl curvature; cf. Refs. [149, Eq. (25,27)] and [59, Sec. 4.5.2].
incoming geodesic null rays connecting them be generated by $k^\mu$. The spacetime point of observation $P$ and the spacetime point of the source $S$ are shifted along the matter congruence at their respective locations, creating a change in cosmological redshift which we measure in the observer’s proper time $\tau_0$ [88]:

$$\frac{dz}{d\tau_0} = u^\rho \partial_\rho z |_P + \frac{d\tau}{d\tau_0} u^\rho \partial_\rho z |_S = (1 + z) \alpha |_P - (1 + z) \frac{E'}{E} |_P - \alpha |_S + \frac{E'}{E} |_S,$$

(35)

where $d\tau/d\tau_0 = 1/(1 + z)$ is the Jacobian of the change of the proper time measure between the source and observer world lines from the bijection induced by the geodesic null congruence. The second equality can be obtained by substituting $u^\mu = k^\mu / E + e^\mu$ and using Eq. (19a). We see that the cosmological redshift drift (35) depends on the area-adapted longitudinal expansion rate variable, $\alpha$, of the matter congruence (replacing the Hubble parameter in the analogous FLRW expression for redshift drift) and on the spatial gradient of the photon energy, $E$, along the spatial direction of the incoming null ray.

3 Area-averaging over light fronts

We now consider the evolution of area-averaged physical quantities along the observer’s past null cone. In Sect. 3.1 we define the area-averaging operation on the light fronts and construct evolution equations for the area of the light fronts. In Sect. 3.2 we consider conservation laws and the Gauss–Bonnet theorem as constraints for the average dynamics of the light fronts. In Sect. 3.3 we derive evolution equations for area-averaged cosmological redshift and distance measures, and in Sect. 3.4 we check the relations derived against the FLRW limit.

3.1 Area-averaging operation and evolution equations for area-averaged variables

We shall now define the covariant area-averaging operation on the light fronts. The area-averaging operation defined is invariant under coordinate transformations (though we shall define useful area-adapted coordinates on the 2-surfaces of integration), and it can be formulated in terms of covariant integrals in the cosmological spacetime, as detailed in Ref. [76].

Consider an orientable and simply connected compact 2-dimensional domain $\mathcal{F}$ in the light front which lies within a 2-surface of intersection between the observer’s past null cone $C^-(P)$ and a spacelike 3-surface $S(\tau)$, the latter labelled by the observer’s corresponding value of proper time: $\mathcal{F}(\tau) \subseteq (C^-(P) \cap S)(\tau)$. The total domain $(C^-(P) \cap S)(\tau)$ is assumed to be topologically closed (as, e.g., the all-sky last scattering surface of the Cosmic Microwave Background radiation). For the general spatially inhomogeneous case, this assumption corresponds to a choice of closed space form, e.g., a spherical topology in the simplest case. We shall allow $\mathcal{F}(\tau)$ to be any compact
2-dimensional subdomain of \((C^- (P) \cap S)(\tau)\) with boundaries propagated along \(k^\mu\) such that we track a system of constant number of radiation elements.

For the present purpose, it is convenient to use local spacetime coordinates 
\(x^\mu = (\tau, V, x^A)\), where \(x^A\), with indices \(A = 2, 3\), are local coordinates adapted to the screen spaces such that they are constant along the geodesic null rays: \(k^\rho \partial_\rho x^A = 0\). Gasperini, Veneziano and their collaborators introduced these coordinates as ‘Geodesic Light cone Coordinates (GLC)’ with \(\tau\) and \(V\) exchanged; see Refs. [73, 76] and Appendix A for details, and [chap.3 and references therein] [63] for the relation to George Ellis’ observational ‘Perturbed Light cone Gauge’ (PLG) coordinates. In GLC coordinates the components of the cosmological spacetime metric and its inverse read:

\[
g_{\mu\nu} = \begin{pmatrix} 0 & 1/E & 0 \\ 1/E & (1/E^2) + U^2 & U_B \\ 0 & U_A & p_{AB} \end{pmatrix}; \quad g^{\mu\nu} = \begin{pmatrix} -1 & E & -E U^B \\ E & 0 & 0 \\ -E U_A & 0 & p^{AB} \end{pmatrix},
\]

and the screen space-forming vectors and associated one-forms read (see Fig. 1):

\[
k_\mu = (0, 1, 0, 0); \quad \quad k^\mu = (E, 0, 0, 0);
\]

\[
l_\mu = (-2E, -1, 0, 0); \quad \quad l^\mu = (E, -2E^2, 2E^2 U^A);
\]

\[
u_\mu = (-1, 0, 0, 0); \quad \quad u^\mu = (1, -E, E U^A);
\]

\[
e_\mu = (-1, -1/E, 0, 0); \quad \quad e^\mu = (0, -E, E U^A),
\]

with the coordinate-independent projections \(k_\mu k^\mu = l_\mu l^\mu = u_\mu u^\mu = u^\mu e_\mu = e_\mu e^\mu = 0\), \(u_\mu u^\mu = -1\), \(e_\mu e^\mu = 1\), \(k_\mu u^\mu = k^\mu u_\mu = l_\mu u^\mu = l^\mu u_\mu = k_\mu e^\mu = k^\mu e_\mu = -E\), \(l_\mu e^\mu = l^\mu e_\mu = E\), \(k_\mu l^\mu = k^\mu l_\mu = -2E^2\). The 2-dimensional tensor \(p_{AB}\) is the induced metric on the screen space and \(p^{AB}\) is its inverse, while the 2-dimensional vector \(U^A = u^\rho \partial_\rho (x^A)/E\) defines the drift of the area-adapted coordinates \(x^A\) along the matter congruence, and \(U^2 := p_{AB} U^A U^B\).

We define the area of a 2-surface within a screen space by

\[
A_\mathcal{F}(\tau) = |\mathcal{F}(\tau)| = \int_\mathcal{F} \sqrt{p(\tau, x^A)} \, d^2x,
\]

where \(p(\tau, x^A) := \det(p_{AB})\) is the Riemannian area element of the 2-surface. We define the area-average of a general scalar-valued function \(F(\tau, x^A)\) as follows:

\[
\langle F \rangle_\mathcal{F}(\tau) := \frac{1}{A_\mathcal{F}(\tau)} \int_\mathcal{F} F(\tau, x^A) \sqrt{p(\tau, x^A)} \, d^2x.
\]

\(^{10}\) In general, local coordinates preserved along \(k^\mu\) are not preserved along \(u^\mu\), since \(k^\rho \partial_\rho x^A = 0\) implies that \(k^\rho \nabla_\rho (u^\mu \partial_\mu x^A) = \xi_k (u^\rho) \partial_\rho x^A\), where the operator \(\xi_k\) is the Lie derivative along the geodesic null rays associated with \(k^\mu\). Thus, if the change of \(u^\mu\) along the geodesic null rays has components tangential to the screen space (which happens generically in spatially inhomogeneous cosmologies), then \(u^\rho \partial_\rho x^A = 0\) cannot be satisfied globally on the past null cone.
The 4-vector fields \(k, l, u\) and \(e\) (right column of (37)) at a line of intersection \(\{V = \text{constant}, \tau = \text{constant}\}\) of the past null cone \(C^{-}(P)\), embedded into the cosmological spacetime with GLC coordinates \(\{\tau, V, x^{A=2}\}\), and a single spacelike 3-surface \(S(\tau)\). The third dimension of the family of spacelike 3-surfaces \(\{\tau = \text{constant}\}\), as parametrized by \(x^{A=3}\), is suppressed. The wave vector \(k\) is tangent to the past null cone, while the wave vector \(l\) represents an auxiliary null-like field pointing in the opposite spatial direction of \(k\) if seen in the restframe of the 4-velocity \(u\), which is orthogonal to the spacelike 3-surfaces (containing the vector \(e\) as a tangent). \(u\) is also, together with the wave vector \(k\), orthogonal to the 2-dimensional light fronts \(\{V = \text{constant}, \tau = \text{constant}\}\) (blue dashed line). Recall the vector relations (10a) and (10b), and see related figures in Ref. [73]. (Past null cone in this figure inspired by Fig. 12 of Ref. [129]; spacetime foliation in this figure: Pierre Mourier (priv. comm.))

It shall also be convenient to define area-averaging of products of scalar-valued functions \(F\) and \(G\) as evaluated at different points along the geodesic null rays (parametrized in terms of the observer’s proper time function \(\tau\)):

\[
\langle F(\tau, x^{A})G(\tau’, x^{A}) \rangle_{\mathcal{F}} := \frac{1}{A_{\mathcal{F}}(\tau)} \int_{\mathcal{F}} F(\tau, x^{A})G(\tau’, x^{A}) \sqrt{p(\tau, x^{A})} \, d^{2}x, \tag{40}
\]

where the evaluation of the area element is at the unprimed coordinate time \(\tau\).

The commutation rule for the operations of area-averaging and evolution along the observer’s past null cone is

\[
\partial_{\tau} \langle F \rangle_{\mathcal{F}} - \langle \partial_{\tau} F \rangle_{\mathcal{F}} = \text{Cov}_{\mathcal{F}} \left( \hat{\theta} E, F \right), \tag{41}
\]

where \(\partial_{\tau} := E^{-1}k^{\rho} \partial_{\rho}\) is the directional derivative along the null congruence as measured in units of \(\tau\) (cf. Eq. (24) for the Jacobian, \(k^{\rho} \partial_{\rho} \tau = E\)), and where, for scalar-valued functions \(X\) and \(Y\):

\[
\text{Cov}_{\mathcal{F}}(X, Y) := \langle XY \rangle_{\mathcal{F}} - \langle X \rangle_{\mathcal{F}} \langle Y \rangle_{\mathcal{F}}. \tag{42}
\]
The local evolution of a scalar-valued function per unit proper time is naturally measured along the geodesic null rays of $k^\mu$, since the boundaries of the spatial domain as well as the past null cone scalar $V$ are comoving with $k^\mu$. In deriving Eq. (41), the Jacobi identity,

$$E^{-1}k^\rho \partial_\rho \sqrt{p(\tau, x^A)} = \partial_\tau \sqrt{p(\tau, x^A)} = \frac{\dot{\theta}}{E} \sqrt{p(\tau, x^A)},$$  \hspace{1cm} (43)

has been used; cf. Refs. [53, Eq. (6.19)] and [143, Eq. (3.14)]. We define the domain-dependent dimensionless area scale factor for the light fronts:

$$a_{\mathcal{F}}(\tau) := \left( \frac{A_{\mathcal{F}}(\tau)}{A_{\mathcal{F}_i}} \right)^{1/2},$$  \hspace{1cm} (44)

where the reference area $A_{\mathcal{F}_i} := A_{\mathcal{F}}(\tau_i)$ is evaluated at a reference light front domain with time label $\tau_i$. Since the domain $\mathcal{F}(\tau)$ is defined as having no flow of geodesic null rays across its boundary, it fails in general to preserve the number of matter fluid elements. Thus, $a_{\mathcal{F}}(\tau)$ cannot be interpreted as an average scaling of the coordinate distance between individual world lines of the matter congruence, and so does not reduce to the FLRW scale factor in the case of spatial homogeneity and isotropy on the spacelike 3-surfaces orthogonal to the matter congruence. We formulate the evolution along the observer’s past null cone $\mathcal{C}^-(P) : \{V = \text{constant}\}$ of the area scale factor as a first part of a theorem.

**Theorem 1.a** (Effective evolution equations for light fronts) The evolution along the observer’s past null cone $\mathcal{C}^-(P) : \{V = \text{constant}\}$ of the area scale factor is governed by the screen space area expansion rate:

$$\mathcal{H}_\mathcal{F} := \frac{\partial_\tau a_{\mathcal{F}}(\tau)}{a_{\mathcal{F}}(\tau)} = \frac{1}{2} \left\langle \frac{\dot{\theta}}{E} \right\rangle_{\mathcal{F}},$$  \hspace{1cm} (45)

where we defined the 2-surface area expansion functional $\mathcal{H}_\mathcal{F}$. The second derivative of the area scale factor yields the ‘screen space area acceleration law’:

$$\frac{\partial^2 a_{\mathcal{F}}(\tau)}{a_{\mathcal{F}}(\tau)} = -4\pi \langle \rho \rangle_{\mathcal{F}} + Q^k_{\mathcal{F}} + S^k_{\mathcal{F}},$$  \hspace{1cm} (46)

where we have used the optical evolution equations (19a) and (19b). The function

$$Q^k_{\mathcal{F}} := \frac{1}{4} \text{Cov}_{\mathcal{F}} \left( \frac{\dot{\theta}}{E}, \frac{\dot{\theta}}{E} \right) - \left\langle \frac{\dot{\theta}^2}{E^2} \right\rangle_{\mathcal{F}},$$  \hspace{1cm} (47)

is a ‘screen space kinematic backreaction’ term arising from local spatial inhomogeneity and anisotropy of the incoming geodesic null ray bundle associated with $k^\mu$. The function
\[ S^k_F := \frac{1}{2} \left\langle \hat{\theta} \hat{E} \alpha \right\rangle_F = \frac{1}{2} \left\langle \hat{\theta} \frac{k^\rho \partial_\rho \left( \frac{1}{E} \right)}{E} \right\rangle_F = -\frac{1}{2} \left\langle \hat{\theta} \frac{\partial_\tau E}{E} \right\rangle_F \] (48)

arises from the re-parametrization of the incoming geodesic null congruence and measures the failure of \( \tau \) to be an affine parameter along this null congruence.

**Remarks to Theorem 1.a**

Removing the term (48) is possible by averaging over \( \{ \lambda = \text{constant} \} \) light fronts instead of \( \{ \tau = \text{constant} \} \) light fronts. We stick to the \( \{ \tau = \text{constant} \} \) foliation of the observer’s past null cone in the present paper. The term \( S^k_F \) vanishes for energy functions \( E = k^\rho \partial_\rho \tau \), cf. Eq. (24), that are constant along the integral curves of the geodesic null congruence, i.e., when the geodesic null rays are not subject to cosmological redshift. The function \( 1/E \) plays the role of an inhomogeneous lapse function along the geodesic null congruence, cf. Eq. (36) and Appendix A.11

We formulate the area-average of the Gauss embedding constraint (25) as a second part of this theorem.

**Theorem 1.b** (Effective energy constraint on light fronts) Averaging the Gauss embedding constraint (25) over the compact domain \( F \) of the light fronts yields the ‘screen space area expansion law’:

\[
\mathcal{H}_F + \mathcal{H}_F \langle \theta \rangle_F = \frac{8\pi}{3} \langle \varrho \rangle_F + \frac{1}{3} \Lambda - \langle K \rangle_F - Q_{kl}^F, \tag{49}
\]

where

\[
Q_{kl}^F := \frac{1}{4} \text{Cov}_F \left( \frac{\hat{\theta}}{E}, \frac{\hat{\theta}}{E} \right) - \frac{1}{2} \left\langle \frac{\hat{\sigma}_{\mu\nu}}{E} \frac{\hat{\sigma}^{\mu\nu}}{E} \right\rangle_F - \left\langle p_{\mu\nu} E_{\mu\nu} \right\rangle_F \tag{50}
\]

is a second backreaction term on the screen space, which may be substituted by the linearly transformed backreaction variable, using the relations (18):

\[
Q_{ke}^F := Q_{kl}^F - Q_k^F = \frac{1}{2} \text{Cov}_F \left( \frac{\hat{\theta}}{E}, \vartheta \right) - 2 \left\langle \frac{\hat{\sigma}_{\mu\nu}}{E} \tilde{\sigma}^{\mu\nu} \right\rangle_F - \left\langle p_{\mu\nu} E_{\mu\nu} \right\rangle_F \tag{51}
\]

**Remarks to Theorem 1.b**

Notice that the new backreaction term (50) contains products of the kinematic variables associated with both the incoming and outgoing null congruences, \( k^\mu \) and \( l^\mu \), contrary to the backreaction term (47).12 Moreover, in both backreaction terms, (50) and (51),

11 For a discussion of the degeneracy at the null cone of the standard ADM (Arnowitt, Deser and Misner) \( 3 + 1 \) slicing formalism [4] and its generalization for lightlike foliations, exemplified for double null foliations, see Refs. [18, 95].

12 We remark that this backreaction term appears to bear a relationship to the Hawking–Hayward energies as defined in Refs. [85, 86]; see Refs. [10, 152, 153] for recent work.
tidal effects enter through the screen space projection of the electric Weyl curvature tensor. This is contrary to the 3-dimensional volume-averaging operation of Refs. [24, 25], where Weyl curvature only enters implicitly in the large-scale volume evolution.

Since Eq. (49) must be the integral of Eq. (46), we take the derivative of Eq. (49) with respect to $\tau$ and re-insert Eqs. (49) and (46) to obtain the following integrability condition on the light fronts that we formulate as a third part of theorem 1.

**Theorem 1.c** (Integrability condition) A necessary condition of integrability for Eq. (46) to yield Eq. (49) is given by the integral constraint:

$$
\partial_\tau Q^k_F + 4\mathcal{H}_F Q^k_F + \partial_\tau Q^{ke}_F + 2\mathcal{H}_F Q^{ke}_F + \partial_\tau \langle K \rangle_F + 2\mathcal{H}_F \langle K \rangle_F
$$

$$
+ \partial_\tau \left( \mathcal{H}_F \langle \tilde{\theta} \rangle_F \right) + 2\mathcal{H}_F \left( \mathcal{H}_F \langle \tilde{\theta} \rangle_F \right) + 2\mathcal{H}_F \left( S^k_F - \frac{1}{3} \Lambda \right)
$$

$$
= \frac{8\pi}{3} \left( \partial_\tau \langle \varrho \rangle_F + 5\mathcal{H}_F \langle \varrho \rangle_F \right),
$$

(52a)

where it follows from averaging of the local law for rest mass flow Eq. (21b) that

$$
\partial_\tau \langle \varrho \rangle_F + \langle \theta \rangle_F \langle \varrho \rangle_F = -\langle \varrho' \rangle_F - \text{Cov}_F (\theta, \varrho) + \text{Cov}_F \left( \frac{\tilde{\theta}}{E}, \varrho \right).
$$

(52b)

The integrability condition in Eq. (52a) can be written in compact form:

$$
\frac{1}{a^4_F} \partial_\tau \left[ Q^k_F a^4_F \right] + \frac{1}{a^2_F} \partial_\tau \left[ \left( Q^{ke}_F + S^k_F - \frac{1}{3} \Lambda + \langle K \rangle_F + \mathcal{H}_F \langle \tilde{\theta} \rangle_F \right) a^2_F \right]
$$

$$
= \frac{8\pi}{3} \frac{1}{a^5_F} \partial_\tau \left[ \langle \varrho \rangle_F a^5_F \right] + \partial_\tau S^k_F.
$$

(52c)

**Remarks to Theorem 1.c**

Looking at Eq. (52c), we appreciate that separate conservation of screen space kinematic backreaction implies the following scaling behaviour $Q^k_F \propto a^{-4}_F$. Furthermore, separate conservation of the area-averaged Gaussian curvature implies $\langle K \rangle_F \propto a^{-2}_F$, being inversely proportional to the square of the area scale factor; see Sect. 3.2 below for a discussion on the Gauss–Bonnet theorem and area-averaged Gaussian curvature. (Compare also Ref. [22] for a discussion of the averaged equations in a $(2+1)$-spacetime.) The scaling behaviours derived from requiring separate conservation laws of the cosmological variables in Eq. (52c) can in some cases provide some guidance, but will in some cases also be misleading. In general we expect non-trivial couplings between the various macroscopic variables.

**Discussion of Theorem 1**

The area-averaged equations (45), (46), (49) and (52) appear more involved than the corresponding evolution equations for the volume of fluid-orthogonal spacelike 3-surfaces in an irrotational dust universe given in Ref. [24]; cf. Ref. [25] for the corresponding formulation including non-zero pressure in the fluid description. This is not
surprising, since the consideration of generic cosmological dynamics and corresponding observations along the observer’s past null cone introduces additional variables. In particular, in a covariant $(1+1+2)$-decomposition we see the effects induced by the Weyl curvature. Even though the incoming geodesic null rays in the present description are non-gravitating test particles—and thus do not collectively act as a source in Einstein’s field equations—they enter in the description of the observer’s past null cone and in the evolution along this past null cone of quantities to be measured.

At this stage, the screen space area acceleration law (46) of theorem 1 for the area scale factor $a_F(\tau)$ in 1.a is subject to the area-averaged constraint equation (49) in 1.b. These equations involve three backreaction terms, $Q^k_F$, $Q^{kl}_F$ and $Q^{ke}_F$, where the latter two are trivially related to the first, $Q^{kl}_F - Q^{ke}_F = Q^k_F$. The integrability condition (52) in 1.c serves as a balance equation among the various sources and backreaction terms, and is not an independent equation. In addition, these equations involve the further variables $\langle \theta \rangle_F$, the area-averaged rest mass density $\langle \rho \rangle_F$, the area-averaged Gaussian curvature scalar $\langle K \rangle_F$, and the non-affine term $S^k_F$ (48). This forms a set of two independent equations for seven unknown functions, in contrast to the 3-dimensional dust case in Ref. [24], where the counting yields three independent equations (those that correspond to the two above, but also the continuity equation for the averaged fluid density) for four unknown functions. However, the present case is to be formally compared with the 3-dimensional case in Ref. [25], where the counting also yields two independent equations for seven unknown functions (that can be reduced by one via an equation of state). Unlike the dust case [24], but similar to the more involved case [25], the evolution equations for $\partial_\tau \langle \theta \rangle_F$ and $\partial_\tau \langle \rho \rangle_F$, which can be derived from Eqs. (21a) and (21b), i.e. $\partial_\tau \langle H_F \langle \theta \rangle_F \rangle + 2H_F \langle H_F \langle \theta \rangle_F \rangle = \ldots$, $\partial_\tau \langle \rho \rangle_F + 5H_F \langle \rho \rangle_F = \ldots$, do not immediately serve to constrain the system further due to failure of the light front domain to be comoving with the matter congruence, and due to the appearance of further backreaction terms.

As detailed in the next section, invoking the Gauss–Bonnet theorem has the potential to add a further constraint to the non-closed system of area-averaged equations.

### 3.2 Average conservation laws and integral-geometric measures

Conservation laws can play an important role in the closure of averaged evolution equations. In this section we investigate the possibility of defining globally conserved quantities over the light fronts. The local photon number conservation law (29) leads to the area-averaged equivalent:

$$\partial_\tau \langle n \rangle_F + 2H_F \langle n \rangle_F = 0 \iff \langle n \rangle_F = \frac{\langle n \rangle_F}{a_F^2(\tau)}, \quad (53)$$

In the case when we consider a global area-average over a total light front domain, a 2-surface $\Sigma := (C^-(P) \cap S)(\tau)$ which we assume to be compact and topologically closed, we may invoke the Gauss–Bonnet theorem for the area-averaged Gaussian

---

13 We still refer to the dust case here; a generalization including fluid pressure with general considerations on lapse and shift can be found in [121].
curvature scalar \( K \):

\[
\langle K \rangle \Sigma = \langle K \rangle_{\Sigma_i} A_{\Sigma_i} = 2\pi \chi,
\]

where \( \chi = 2 - 2g \) is the Euler characteristic of \( \Sigma \) with genus \( g \). Thus, in this case the area-averaged Gaussian curvature scalar obeys its own separate conservation equation and decouples from the integrability condition (52a). For light fronts \( \Sigma \) of spherical topology we have \( \chi = 2 \), and so \( \langle K \rangle_{\Sigma_i} A_{\Sigma_i} = 4\pi \). (Level 2-surfaces of a past null cone must always have spherical topology, unless caustics change the topology by destroying the bijection between the surfaces [56, 62, 152].)

In the case of a light front with boundary, we may invoke the Gauss–Bonnet theorem with boundary term, now considering a compact domain \( F \subset \Sigma \):

\[
\langle K \rangle_F A_F + \int_{\partial F} \kappa d\ell = 2\pi \chi,
\]

where \( \kappa \) is the extrinsic curvature of the 1-dimensional boundary \( \partial F \) of \( F \), and \( d\ell \) is the line element for this boundary.

We notice from the above considerations the natural appearance of integral-geometric measures of the light fronts, which can be viewed as generalizations of the Minkowski functionals of convex sets in Euclidian space; cf. Ref. [145] and references therein: the 2-surface area, \( A \) (content), the length of the circumference, \( L := \int_{\partial F} d\ell \) (shape), and the Euler characteristic, \( \chi \) (connectivity). As the area of a light front evolves, the shape of the light front also evolves, while its Euler characteristic is preserved according to our assumption of caustic-free evolution along the observer’s past null cone, irrespective of the choice of compact submanifold of the total light front domain \( (C^-(P) \cap S)(\tau) \).

The evolution of the area-averaged Gaussian curvature scalar \( \langle K \rangle_F \) is in general determined by the evolution of the area of the 2-surface \( A_F(\tau) \) and the boundary term \( \int_{\partial F} \kappa d\ell \). Whereas the evolution of the area is given by the area-averaged expansion rate scalar of the geodesic null congruence, implicitly given by Eq. (45), the evolution of the boundary term involves shear degrees of freedom as well as Weyl curvature degrees of freedom.

Let the boundary at each 2-dimensional screen space be determined by the level surfaces of value \( B_0 \) of a spacetime function \( B \), such that

\[
n^\mu := \frac{p^{\mu\nu} \partial_\nu B}{\mathcal{N}}; \quad \mathcal{N} := \sqrt{p^{\mu\nu} \partial_\mu B \partial_\nu B}; \quad k^\rho \partial_\rho B = 0,
\]

defines the outward directed normal to the boundary.\(^{14}\) The last condition ensures that the boundaries are constant along the integral curves of the geodesic null congruence generating the observer’s past null cone. The normalized tangent vector of the curve

\(^{14}\) For an analogous definition of domain boundaries in the 3 + 1 slicing formalism, see Ref. [75].
in the light front plane, $q^\mu$, is determined by:

$$q^\mu n_\mu = 0; \quad q^\mu u_\mu = 0; \quad q^\mu e_\mu = 0,$$

and the induced metric on the boundary is given by

$$q_\mu q^\nu = p_\mu \nu - n_\mu n^\nu = \delta_\mu \nu + u_\mu u^\nu - e_\mu e^\nu - n_\mu n^\nu,$$

such that $\{u^\mu, e^\mu, n^\mu, q^\mu\}$ constitute an orthonormal vector basis for the cosmological spacetime. The extrinsic curvature associated with the embedding of the boundary in the 2-dimensional screen space can now be expressed as follows:

$$\kappa \equiv n^\nu q^\mu \nabla_\mu q_\nu.$$  

From the normalization requirement, $n_\mu k^\nu \nabla_\nu n^\mu = 0$, and the propagation rule, $k^\rho \partial_\rho B = 0$, we have

$$k^\rho \partial_\rho N = -\frac{1}{2} \hat{\theta} - n_\mu n^\nu \hat{\sigma}_{\mu \nu}.$$

The evolution of the normal $n^\mu$ from one screen space to the next is given by

$$k^\nu \nabla_\nu n^\mu = -q^\mu q^\nu n^\nu \hat{\sigma}_{\alpha \nu} - k^\mu n_\alpha e^\nu \sigma_{\alpha \nu},$$

which follows from the definition of $n^\mu$ in Eq. (55). Similarly, we may compute the evolution of $q^\mu$ along the geodesic null congruence:

$$k^\nu \nabla_\nu q^\mu = n_\mu q^\alpha n^\nu \hat{\sigma}_{\alpha \nu} - k_\mu q^\alpha e^\nu \sigma_{\alpha \nu},$$

which follows from the orthogonality requirements (56). The evolution of the boundary term is then given by

$$\partial_\tau \int_{\partial F} \kappa \, d\ell = \int_{\partial F} \frac{\nabla_\mu (\kappa N k^\mu)}{N E} \, d\ell = \int_{\partial F} \left( E^{-1} k^\rho \partial_\rho \kappa + \frac{\kappa q^\mu q^\nu \nabla_\mu k_\nu}{E} \right) \, d\ell,$$

where Eqs. (56) and (59) have been used, and where

$$q^\mu q^\nu \nabla_\mu k_\nu = \frac{1}{2} \hat{\theta} + q^\mu q^\nu \hat{\sigma}_{\mu \nu}$$

is the projected expansion tensor along the tangent of the curve, describing the expansion of the length of the boundary.

An important observation is that the intermediate equality in Eq. (62) shows that the boundary term is conserved, if $(\kappa N k^\mu)$ is a conserved current in the vicinity of the boundary, corresponding to $\kappa$ being a conserved density on the boundary domain.
A further insight is obtained when considering the evolution of the extrinsic curvature of the boundary:

\[
k^\rho \partial_\rho \kappa = - \left( \frac{1}{2} \hat{\theta} + q^\mu q^v \hat{\sigma}_{\mu v} \right) (\kappa - n^\alpha e^\beta \sigma_{\alpha \beta}) - n^\mu q^v \hat{\sigma}_{\mu v} \left( q^\alpha e^\beta \sigma_{\alpha \beta} - \frac{q^\beta \partial_\beta E}{E} \right) + q^\alpha \nabla_\alpha (n^\mu q^v \hat{\sigma}_{\mu v}) - E \left( E_{\mu \nu} e^\mu n^v + \epsilon_{\mu \nu \beta} H_{\alpha \beta} n^\mu q^\nu q^\alpha \right) + q^\alpha \nabla_\alpha (n^\mu q^v \hat{\sigma}_{\mu v}) - E \left( E_{\mu \nu} e^\mu n^v + \epsilon_{\mu \nu \beta} H_{\alpha \beta} n^\mu q^\nu q^\alpha \right).
\]

(64)

In deriving Eq. (64), the evolution equations (60) and (61) have been invoked and it has been used that

\[
n^\mu k^\nu q^\alpha q^\beta R_{\mu \alpha \nu \beta} = n^\mu k^\nu q^\alpha q^\beta C_{\mu \alpha \nu \beta} = -E \left( E_{\mu \nu} e^\mu n^v + \epsilon_{\mu \nu \beta} H_{\alpha \beta} n^\mu q^\nu q^\alpha \right),
\]

(65)

where the first equality follows from the form of the energy-momentum tensor (2), and the second equality follows from the decomposition of the Weyl curvature tensor into its electric and magnetic parts with respect to the matter frame [112].

It can be seen from Eq. (64) that vanishing on the boundary of each of the projected Weyl curvature terms of Eq. (65), the off-diagonal components of the projected volume shear rate of the matter congruence, \( n^\alpha e^\beta \sigma_{\alpha \beta} \), and the incoming optical shear, \( n^\mu q^v \hat{\sigma}_{\mu v} \), results in conservation of the boundary term. In this special case, we have from Eq. (54b) that \( \langle K \rangle_{\mathcal{F}} \propto 1/A_\mathcal{F} \), with the constant of proportionality depending on the initial boundary chosen. This is a scaling behaviour of the area-averaged Gaussian curvature scalar equivalent to that of the boundary-free case. Given the strict constraints which must be imposed in order to mimic the boundary-free case, the scaling law \( \langle K \rangle_{\mathcal{F}} \propto 1/A_\mathcal{F} \) is likely not satisfied in realistic cosmological spacetimes.

### 3.3 Area-averaged observables

We shall now perform the area-average over observable quantities for an observer viewing multiple objects on the same 2-dimensional screen space. Using Eqs. (9) and (23), the evolution of the area-average of the logarithmic cosmological redshift along the observer’s past null cone reads:

\[
\partial_\tau \langle \ln(1+z) \rangle_{\mathcal{F}} = - \langle \alpha \rangle_{\mathcal{F}} + \text{Cov}_{\mathcal{F}} \left( \frac{\hat{\theta}}{E}, \ln(1+z) \right).
\]

(66)

Unlike the local cosmological redshift function, which, in general, cannot be thought of as a time variable along the world lines of the matter congruence, the area-averaged logarithmic cosmological redshift can be used as a time label for the 2-dimensional light fronts as long as the right-hand side of Eq. (66) does not change sign.

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15 Any restriction imposed on the volume shear rate of the matter congruence leads to constraints on the Weyl curvature and on the spatial gradient of the expansion rate through the geodesic deviation equation and constraint equations for the relevant congruence [55]. For example, \( n^\mu q^\nu \hat{\sigma}_{\mu \nu} = 0 \Rightarrow k^\alpha n^\mu q^\beta k^\nu C_{\alpha \mu \beta \nu} = 0 \) in the present set-up. By imposing such restrictions it must be checked whether there exist non-trivial cosmological spacetimes fulfilling such conditions for the volume shear rate.
We can, furthermore, compute the evolution along the observer’s past null cone of the area-average of the logarithm of the effective angular diameter distance, which, by Eq. (31), reads:\(^{16}\)

\[ \partial_\tau \langle \ln(d_A) \rangle_\mathcal{F} = \mathcal{H}_\mathcal{F} + \int_\tau^T \mathcal{M}(\tau, \tau') \, d\tau', \]

(67)

where \( \mathcal{M}(\tau, \tau') := \frac{1}{2} \text{Cov}_\mathcal{F} \left( \frac{\hat{\theta}(\tau, x^A)}{E(\tau, x^A)}, \frac{\hat{\theta}(\tau', x^A)}{E(\tau', x^A)} \right) \)

(68)

defines the memory function for the observer’s past null cone—cf. the definition (40)—memorizing the spatial inhomogeneities and auto-correlation properties in the expansion rate and in the energy function of the incoming geodesic null rays encountered along the path from the source to the observer. We have used that

\[ \ln(d_A) = \frac{1}{2} \int^\lambda \hat{\theta}(\lambda', x^A) \, d\lambda' = \frac{1}{2} \int^\tau \frac{\hat{\theta}(\tau', x^A)}{E(\tau', x^A)} \, d\tau', \]

(69)

where integration is along each geodesic null ray labelled by comoving coordinates \( x^A \), together with Eqs. (31) and (24). The memory of shear and Weyl curvature along the observer’s past null cone is encoded in the shape of the boundary \( \partial \mathcal{F} \) of the compact light front domain \( \mathcal{F} \), according to Eqs. (62) and (63).

The time-range of dependence is expected to be determined by the size of typical matter structures in the cosmological spacetime. Differentiating Eq. (67) and using Leibniz’ rule leads to the area-averaged logarithmic focusing equation:

\[ \partial^2_\tau \langle \ln(d_A) \rangle_\mathcal{F} = \partial_\tau \mathcal{H}_\mathcal{F} + \mathcal{M}(\tau, \tau) + \int_\tau^T \partial_\tau \mathcal{M}(\tau, \tau') \, d\tau', \]

(70a)

with

\[ \partial_\tau \mathcal{H}_\mathcal{F} = -\mathcal{H}^2_\mathcal{F} - 4\pi \langle \theta \rangle_\mathcal{F} + Q^k_\mathcal{F} + S^k_\mathcal{F}. \]

(70b)

The area-averaged logarithmic version of the reciprocity relation (34) reads:

\[ \langle \ln(d_L) \rangle_\mathcal{F} = 2 \langle \ln(1 + z) \rangle_\mathcal{F} + \langle \ln(d_A) \rangle_\mathcal{F}, \]

(71)

whose evolution along the observer’s past null cone can be determined through relations (66) and (67). There is a sense in which the logarithmic variables \( \ln(1 + z) \), \( \ln(d_A) \) and \( \ln(d_L) \) are natural for discussing area-averaged null signal propagation, as the logarithmic transformation simplifies evolution equations and preserves the local form of the reciprocity theorem. We note that the distance modulus is defined as a linear function of \( \ln(d_L) \) by \( \mu := (5/\ln(10)) \ln(d_L/10 \text{ parsecs}). \)

\(^{16}\) We omit the normalization of the dimensionfull quantities \( d_A \) and \( d_L \) inside the logarithm; the corresponding equations hold for any choice of normalization.
We now consider area-averaging of the cosmological redshift drift relation (35) over sources sprinkled continuously and uniformly in volume over the 2-dimensional screen space:

\[
\left\langle \frac{dz}{d\tau} \right\rangle_F = (1 + z) \frac{\alpha}{p} - (1 + z) \frac{E'}{E} \left|_p \right. - \langle \alpha \rangle_F - \left\langle \frac{E'}{E} \right\rangle_F. \tag{72}
\]

Spatially inhomogeneous and anisotropic contributions enter the cosmological redshift drift, also when area-averaged over many sources. In particular, systematic effects are expected to be introduced through the terms evaluated at the point of observation \( P \), while non-cancelling effects from structure along the individual null rays might also play a role; see Refs. [88, 89] for detailed investigations of systematic effects entering the cosmological redshift drift signal.

### 3.4 FLRW limit

It is worth considering the FLRW limit of the area-averaged equations (45), (46), (49) and (52). We consider the FLRW spacetime metric written in hyper-spherical local coordinates \( \{t, r, \theta, \phi\} \) with line element,

\[
ds^2 = -dt^2 + a(t)^2 \left( dr^2 + S_k^2(r) d\Omega^2 \right), \quad S_k(r) := \sqrt{-k-1} \sinh(\sqrt{-kr}), \tag{73}
\]

where \( d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2 \) is the solid angular element on the unit sphere, and \( a(t)^2 S_k^2(r) \) is the adapted area measure on a sphere of proper radius \( a(t) r \) contained in a spacelike 3-surface labelled by \( t \). The dimensionless scale factor \( a(t) \) is the conformal scaling of the static metric, and thus describes the temporal dependence of the spatial volume in the comoving frame; we employ the usual convention that \( a(t_0) = 1 \), where \( t_0 \) is the proper time elapsed since the big bang singularity at the present epoch (“here and now”). The spatial coordinates \( \{r, \theta, \phi\} \) are comoving with a central observer, where \( \{\theta, \phi\} \) are angular coordinates describing directions on the observer’s sky, and \( r \) is a radial coordinate with dimension of length from which we can get the proper geodesic distance \( a(t) r \) away from the observer at an instant of proper time \( t \). The curvature of the 3-dimensional spatial sections \( k \) has dimensions of inverse length scale squared, and the spatially flat FLRW model is obtained when \( k \to 0 \). The observer’s time variable \( t \) is synchronous with the proper time function \( \tau \) associated with the irrotational 4-velocity \( u^\mu \) of the matter congruence, of uniform rest mass density \( \varrho(t) \). The incoming geodesic null rays which constitute the central observer’s past null cone are per construction of the local coordinates propagating along paths with \( \{\theta = \text{constant}, \phi = \text{constant}\} \). We thus have in this reference frame:

\[
u^\mu = \delta_t^\mu, \quad e^\mu = \frac{1}{a(t)} \delta_r^\mu, \quad k^\mu = E \left( \delta_t^\mu - \frac{1}{a(t)} \delta_r^\mu \right). \tag{74}
\]

The kinematic variables associated with \( u^\mu \) and \( e^\mu \) and the energy function \( E \) of the geodesic null congruence—obtained from the connection associated with the line element (73) and the differential equation (19a) for \( E \)—are:
\begin{equation}
\theta = 3 \frac{\dot{a}(t)}{a(t)}, \quad \sigma_{\mu\nu} = 0, \quad \bar{\theta} = 2 \frac{\partial_r S_k(r)}{S_k(r)}, \quad \bar{\sigma}_{\mu\nu} = 0, \quad E = \frac{\mathcal{E}(L)}{a(t)},
\end{equation}

where \( \cdot \) := \partial_t, and \( L := \int^t dt'/a(t') + r \) is constant over each past null cone of the central observer. The function \( \mathcal{E}(L) \) is an initial condition for the geodesic null congruence and might be chosen in accordance with the observational frequency of interest; cf. Sect. 2.2. The null scalar variable of the geodesic null congruence is given by

\begin{equation}
V(L) = - \int^L \mathcal{E}(L') dL',
\end{equation}

which can be verified by computing its gradient and recovering \( k^\mu \).

The kinematic variables associated with \( k^\mu \) and \( l^\mu \) follow from plugging the expressions (75) into the relations (17a) and (17b), yielding:

\begin{align}
\dot{\theta} &= \frac{2}{3} \theta - \bar{\theta} = 2 \frac{\dot{a}(t)}{a(t)} - 2 \frac{\partial_r S_k(r)}{S_k(r)}, \quad \dot{\sigma}_{\mu\nu} = 0, \\
\bar{\theta} &= \frac{2}{3} \theta + \bar{\theta} = 2 \frac{\dot{a}(t)}{a(t)} + 2 \frac{\partial_r S_k(r)}{S_k(r)}, \quad \bar{\sigma}_{\mu\nu} = 0.
\end{align}

The contribution \( \pm 2 \partial_r (S_k(r))/S_k(r)/a(t) \) to the area expansion rates accounts for the change of the area measure as the proper radius of the 2-dimensional screen space is decreased or increased, respectively, when propagating the screen towards or away from the vertex point \( P \) of the observer’s past null cone along the incoming/outgoing null rays. This contribution becomes singular at the vertex point \( P \) at coordinate value \( r = 0 \), where the angular measure of the 2-dimensional screen space tends to zero.

We now consider the area-averaged equations (46, 49), in the limit of the FLRW cosmologies. In this limit, the backreaction terms \( Q^k_{\mu\nu} \) and \( Q^{kl}_{\mu\nu} \) and the right-hand side of Eq. (52b) vanish. The Ricci scalar of the embedded 2-surfaces is

\begin{equation}
(2)^R = \frac{(3)^R}{3} + \frac{1}{2} \bar{\theta}^2; \quad (3)^R = \frac{6k}{a^2(t)},
\end{equation}

where \( (3)^R \) is the 3-Ricci curvature scalar associated with the canonical FLRW space-like 3-surfaces \( S(t) \). Using these results, along with Eqs. (45) and (77a), on both sides of the area-averaged Gauss embedding constraint (49) gives:

\begin{equation}
\left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{4} \bar{\theta}^2 - \frac{\dot{a}}{a} \bar{\theta} = \frac{8\pi}{3} \varrho + \frac{1}{3} \Lambda - \frac{k}{a^2} - \frac{\dot{a}}{a} \bar{\theta} + \frac{1}{4} \bar{\theta}^2,
\end{equation}

where it has been used that all area-averaged variables are constant over the 2-dimensional screen space. The terms involving \( \bar{\theta} \) cancel, and we arrive at the first of Friedmann’s equations:

\begin{equation}
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} \varrho + \frac{1}{3} \Lambda - \frac{k}{a^2}.
\end{equation}
The Raychaudhuri equation can be recovered by noticing that

\[
E^{-1} k^\rho \partial_\rho \left( \frac{\dot{\theta}}{E} \right) = (u^\rho - e^\rho) \partial_\rho \left( 2 \frac{\dot{a}}{a} \right) = 2 \frac{\ddot{a}}{a} - 2 \left( \frac{\dot{a}}{a} \right)^2 \frac{1}{2} \tilde{\theta} + \frac{1}{a} \tilde{\theta}^2,
\]

(80a)

where \( \tilde{\dot{\theta}} = -\frac{\dot{a}}{a} \tilde{\theta}, \quad \tilde{\theta}' = -\frac{2k}{a} - \frac{a^2}{2}. \quad \text{(80b)} \)

follows from the definition of \( \tilde{\theta} \), Eq. (75). This gives for the acceleration equation (46):

\[
\frac{\partial^2 a_x(\tau)}{a_x(\tau)} = \frac{\ddot{a}}{a} - \frac{1}{2} \frac{\dot{a}}{a} \tilde{\theta} - \frac{k}{a^2}.
\]

(81)

We also have from Eq. (46) that

\[
\frac{\partial^2 a_x(\tau)}{a_x(\tau)} = -4\pi \rho + \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{2} \tilde{\theta} \frac{\ddot{a}}{a} = -\frac{4\pi}{3} \rho + \frac{1}{3} \Lambda - \frac{k}{a^2} - \frac{1}{2} \frac{\dot{\theta}}{a}.
\]

(82)

where it has been used that

\[
\dot{\theta} E^{-1} k^\rho \partial_\rho E = -2 \left( \frac{\dot{a}}{a} \right)^2 + \tilde{\theta} \frac{\ddot{a}}{a}.
\]

(83)

Finally, equating Eqs. (81) and (82), we arrive at Friedmann’s acceleration law:

\[
\frac{\ddot{a}}{a} = \frac{4\pi}{3} \rho + \frac{1}{3} \Lambda.
\]

(84)

It is illustrative to derive the scaling of the Gaussian curvature scalar of the FLRW screen spaces:

\[
K = \frac{1}{2} \frac{(2)}{R} = \frac{1}{a^2(t) S_k^2(r)} \left[ k S_k^2(r) + \left( \partial_r S_k(r) \right)^2 \right] = \frac{1}{a^2(t) S_k^2(r)}.
\]

(85)

This expression is manifestly positive when both \( a(t) \) and \( S_k(r) \) are non-zero. The first equality in Eq. (85) follows from Eqs. (78) and (77a), while the second follows from the definition of \( S_k(r) \) and hyperbolic identities. The FLRW expression for the Gaussian curvature scalar (85) is nothing but the inverse area measure of the light fronts. The integral of Eq. (85) over the full light front yields the area of the unit sphere, \( 4\pi \), consistent with the Gauss–Bonnet theorem for closed surfaces (54a). For light front subdomains, the integral of \( K \) is simply the constant area covered by the subdomain of the unit sphere.
4 Discussion

This paper offers an area-averaging formalism for describing the evolution of dynamical variables and observable quantities along the observer’s past null cone in the setting of generic irrotational dust cosmologies with a cosmological constant. This formalism applies to all kinds of null signals, e.g., those carried by photons, but also a future gravitational wave sky could be examined using this formalism. The formalism is based on the assumption that the cosmological dynamics progresses sufficiently smoothly so that no caustics will form on the past null cone of the observer.

The area-averaged macroscopic system of equations on 2-dimensional light fronts derived in Sect. 3.1 involves two independent backreaction terms. The system of light front averaged equations comprises a larger set of global variables than the analogous volume-averaged macroscopic system of equations adapted to the spacelike 3-surfaces given in Ref. [24]. This is to be expected, as additional dynamical variables are introduced when the dynamics of the null cone generating congruence is included. Since the area-averaging operation is adapted to the observer’s past null cone, there is a flow of matter world lines across the averaging domain, which further complicates the light front averaged equations. We note that caution is required when invoking simplifying assumptions. A radical example would be to get rid of all of the shear variables of the problem by simply setting the area shear rate of the geodesic null congruence and the volume shear rate of the matter congruence to zero. However, such approximations at the local level of the cosmological spacetime turn out to be extremely restrictive; see Ref. [55] and references therein. If, for instance, the volume shear rate of the matter congruence is required to be zero, $\sigma_{\mu\nu} = 0$, then—since its vorticity is already required to be zero—this implies that the dust cosmology considered must be of the FLRW class [52]. Vanishing of the area shear rate of the geodesic null congruence, on the other hand, implies that the evolution of this null congruence is affected only by the matter (hence, Ricci curvature) it encounters, and that this null congruence must be a principal null direction of the Weyl curvature tensor; this fact might be viewed as a generalization of the Goldberg–Sachs theorem for vacuum spacetimes [55, 81].

The generality of the presented formalism leads, by construction, to a set of balance equations that do not form a closed set. Quantification of the level of backreaction is intimately related to the implementation of closure conditions for the screen-space averaged variables. Concrete inhomogeneous models at the level of area-averaged variables can be studied in order to arrive at appropriate statistical descriptions for the propagation of geodesic null ray bundles in spatially inhomogeneous dust cosmologies. One might for instance employ exact solutions such as Lemaître–Tolman–Bondi solutions or the more general classes of Szekeres solutions. They can be employed to design ‘Swiss cheese models’, within which light propagation has already been studied in great detail; see [104, 109] for recent investigations into observational backreaction effects in Swiss cheese models. Generic structure formation models, based on perturbative assumptions and not restricted by symmetries, are a natural next step of investigation. In the present setup of the irrotational matter congruence, appropriate models have been constructed that are based on relativistic perturbation

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17 The Weyl curvature has no influence on the properties of the geodesic null congruence in this case.
theory combined with exact averages, e.g. [110], and the relativistic generalization of Lagrangian perturbation theory, recently reviewed in [32]. Ongoing work investigates a non-perturbative generalization of these schemes, controlled by the Szekeres class of solutions and that contain the most general Szekeres solutions as the exact body [43].

Studies of exact scaling solutions of the area-averaged system could also provide more insight, similar to those employed in the $3 + 1$ slicing formalism [140], and they can be explored to provide closure conditions for the area-averaged equations. The scaling behaviours suggested by the integrability condition (52c) at first glance are those where the relevant variables are uncoupled, but we expect from Ref. [140] that generic scaling solutions dynamically couple the area-averaged variables. In this context, topological closure conditions might be relevant: an insight from the Gauss–Bonnet theorem applied to light fronts shows that there is no such coupling of the scalar curvature of light fronts to other dynamical variables for all-sky averages, unlike in the 3-dimensional case: the area-averaged scalar parts of Einstein’s field equations on the light fronts simplify through the Gauss–Bonnet theorem when the total (all-sky) light front is considered, or when the curvature of the embedding of the boundary is associated with a conserved current, as detailed in Sect. 3.2. Integral-geometric properties analogous to the Minkowski functionals in Euclidian spaces appear naturally when all-sky averaging over light fronts is performed. Hence, it is for the case of all-sky averages, where we could expect simplifications after evaluating global contributions on topologically closed light fronts. We do not in general expect the light front backreaction terms to vanish globally. However, in cases where those terms are covariant divergences, a closed space would erase them, as in the case of flat light fronts (corresponding insights have been developed in Newtonian cosmology [33]).

Another arena of application of the formalism is that of general relativistic numerical simulations. Cosmological simulations have already been investigated in the context of backreaction and observables; cf. [2, 11, 47, 48, 78, 79, 117] for recent works. In these works, cosmological backreaction was generally found to be small at large scales where observables were also found to agree well with the FLRW prediction, whereas on smaller scales Newtonian simulations of structures combined with relativistic ray tracing could account for distortions of the FLRW background observations. The FLRW metric has thus proven itself robust towards the initial perturbations employed within these simulated universe scenarios. This robustness might seem surprising given the highly non-linear structures which are allowed to develop in these codes. However, this may be related to the fact that a foliation that inherits the properties of the longitudinal gauge of standard perturbation theory is present throughout the cosmic evolution of these simulations despite the presence of non-linear structures [42, 80]. Furthermore, the global architecture remains that of Newtonian cosmology of a flat 3-torus topology, which forces the global model to evolve according to the assumed background cosmology. These are conjectures that should be examined in future works, together with the question of which type of initial perturbations and topological constraints may cause the FLRW solution to be globally unstable as a

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18 See the references therein, also to earlier work on other relativistic perturbation theories, the history of Lagrangian perturbation theory, and e.g. [37] for their application as a closure condition for the $(3 + 1)$ averaged system.
model for the average evolution, an instability that generically occurs in a dynamical system analysis of the averaged $3 + 1$ system [140], pointing to negative average curvature on large scales [30]. The formalism developed in this paper sheds light on the conditions that must be satisfied for backreaction to be present in the context of observables, and may lead to interesting developments on simulated spacetime scenarios with backreaction.

Turning to observables, the evolution equations for the area-averaged effective angular diameter distance derived in Sect. 3.3 are partly expressed in terms of the area-averaged light front variables introduced in Sect. 3.1, but involve additional terms expressed through the memory function $\mathcal{M}(\tau, \tau')$, for which one needs to prescribe a model to close the area-averaged macroscopic system of equations. Backreaction in the area-averaged effective angular diameter distance might arise due to ‘direct backreaction’ through the memory function or $Q^k_F$, or through ‘indirect backreaction’ causing both $\mathcal{H}_F$ and $\langle \varrho \rangle_F$ to evolve differently than in the Friedmannian case. The evolutions along the observer’s past null cone of the area-averaged cosmological redshift and of the area-averaged cosmological redshift drift both induce additional terms which must be dealt with—for detailed investigations of the expression for cosmological redshift drift in a generic cosmological spacetime, and the effect of regional spatial inhomogeneities and anisotropies on the cosmological redshift drift signal, see Refs. [88–90, 106].

We emphasize that care must be taken when using the area-averaged relations to interpret cosmological data sets. Some challenges for interpreting the theoretically given area-averaged observable quantities include (i) identifying spacelike 3-surfaces of constant proper time on which the area-averaging operation is formulated; (ii) relating the effective area-averaging operation employed when observing many emitters over the sky to the volume-averaging operation. Emitters might not sample the area of the emission screen space fairly, and null signals will interact with the matter of the spacetime; (iii) incorporating the properties of the observer’s local vicinity relative to that of an observer with a volume-unbiased view of the Universe. Here follows a discussion of these challenges.

(i) Relative age measurements [96] might serve to distinguish spacelike 3-surfaces of constant proper time, although the errors associated with current relative age measurements are of the order of 20% [136]. The foliation of the observer’s 3-dimensional past null cone is not unique, and the choice to foliate it in terms of a family of spacelike 3-surfaces $S: \{\tau = \text{constant} \}$ of constant proper time with respect to the irrotational matter congruence could be modified, which would change the dynamical system of equations accordingly. It has been proposed to consider level surfaces of constant cosmological redshift $z$ as an observationally relevant foliation of the observer’s past null cone [70]. Regions of constant cosmological redshift are identifiable to a high precision, and are therefore observationally preferred. However, when $z$ is not a monotonic function along null paths, this destroys the otherwise observationally intuitive notion of $z$ as a foliation scalar. In situations where the effective fluid matter model is describing scales well above that of collapsing structures, then the redshift function might be re-established as a parametrization of light fronts. It must be assessed in
the physical situation at hand whether the cosmological redshift function is indeed a hypersurface-forming scalar. In the general case, where the cosmological redshift function is not hypersurface-forming, we might, nevertheless, average over regions of constant cosmological redshift, without these regions being well-defined as causally ordered and non-intersecting submanifolds.

(ii) Luminous astrophysical sources emitting electromagnetic radiation or gravitational wave signals are in general not uniformly distributed in volume over the Universe. Such sources are concentrated in over-densities of the matter distribution, and in this regard an observer should be observing a mass-biased picture of the Universe. On the other hand, after being emitted from a source, the null signals, which a given observer receives, tend to propagate in empty space (since signals propagating in regions with matter are likely to be scattered or absorbed by other particles, and, therefore, simply do not arrive at the observer’s position). Such subtleties must ideally be taken into account in an observationally matched averaging procedure. However, correctly accounting for such biases would be involved and the correction procedures would depend on the type of cosmological probe. As a first approximation we might simply assume an ideal situation where sources and null signals are probing the volume-averaged cosmological spacetime.

(iii) All cosmological measurements are dependent on the properties that hold in the immediate vicinity of the small segment of a single world line from which we observe the Universe. Whereas cosmological observations are directly or indirectly averaged over the emitter positions when performing statistical analysis with many cosmological data points, the same does not apply to the observer’s position. Some cosmological measurements might be sensitive to the observer’s position, whereas others might be subject to less bias.

The area-averaging formalism provided in this paper may be combined with appropriate statistical assumptions in order to formulate an average dynamical theory for dynamics and observables on our past null cone. Synthetic multi-variate cosmological data on the joint distribution of luminous astrophysical sources and spatial geometry inside and on a modelled observer’s past null cone may be generated by simulation and analyzed by means of methods from inductive statistical inference; cf. the textbook introductions, Refs. [66, 77, 119]. This approach allows for the possibility of factoring in all information that is relevant to dealing systematically with the uncertainty pertaining to a quantitative problem at hand. In this way (and when linked to exact cosmological dust solutions to begin with), a set of interval estimates for area-averaged observables such as, e.g., the cosmological distance measures may be calculated for specific configurations and then compared to standard FLRW-based values for these quantities. The amount of available observational data to be integrated in this process is steadily increasing. On the technical side, a long term project of this kind of cosmological data analysis may be informed by and validated through supervised solution algorithms developed in contemporary machine learning such as neural networks; cf., e.g., the textbook introduction Ref. [125], and examples on applications in Refs. [74, 130].
Our analytical results can be profitably employed for numerical simulations of null cones. As mentioned in the discussion above, it is important to examine the regime of global stability of the FLRW solution for predicting observables by means of numerical relativity. Relativistic effects in light propagation have been computed in linearized weak-field approximations of general relativity, while using (N-body or hydrodynamic) Newtonian simulations as input to generate the matter distribution of the universe model [17, 19]. Light rays have also been tracked fully relativistically through idealized but non-trivial inhomogeneous and anisotropic model universes with backreaction [102, 103, 109] and within post-Newtonian model universes [84, 142]. Weak-field relativistic N-body simulations, which allow to treat certain perturbative modes non-linearly, have been employed in order to model both the matter distribution and the null cone of an observer [1]. Null cones have been generated in general relativistic simulations codes; however, only tracking the evolution in the linear regime of density contrasts [79]. See also [47] for a study comparing observables in Newtonian N-body codes and general relativistic hydrodynamic simulations of structure formation, while employing relativistic ray-tracing in both cases. Finally, ray tracing codes for the general relativistic and non-linear universe simulations in Refs. [116, 117] have recently been developed [114] with exciting applications to relativistic modelling of cosmic observables [118].

The framework developed in this paper is complementary to model-independent cosmographic frameworks first considered in detail by [61, 108] and more recently by [41, 87, 89, 90], with applications to numerical simulations in [92, 115]. While the cosmography developed in these works is powerful for extracting cosmological information from data without the assumption of a cosmological metric or a dynamical theory, and primarily apply to measurements made at low redshifts, the framework considered in this paper considers the dynamical evolution and theoretical average of light fronts as they propagate towards the observer from astrophysical sources that may be distant. The cosmographic frameworks in [87, 90] are furthermore concerned with the angular dependence of observables over the sky of the observer, while the framework developed here is concerned with the average over (a patch of) the sky of the observer, which is applicable in general for finite-area screen spaces with or without boundary. Future investigations may consider a link to the analyses of galaxy catalog data on shells around the observer or Cosmic Microwave Background radiation data that are often investigated globally on topological all-sky support manifolds with the help of integral-geometric measures like the Minkowski functionals.

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Data availability  Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

A Light front-adapted spacetime metric

We shall consider the embedding of the light fronts into the cosmological spacetime, and write the metric tensor in local coordinates that are adapted to the screen space, $x^\mu = (\tau, V, x^A)$. We require that the local coordinates $x^A$, with $A = 2, 3$, satisfy the propagation law $k^\rho \partial_\rho x^A = 0$. The propagation laws for $\tau$ and $V$ are also fixed through $k^\rho \partial_\rho \tau = E$ and $k^\rho \partial_\rho V = 0$ (see Eqs. (24) and (7)), and the full adapted system of local coordinates $x^\mu = (\tau, V, x^A)$ is thus specified throughout the past null cone domain, once initial conditions for the local coordinates are fixed at a screen space. In these local coordinates, we have that

$$k_\mu = (0, 1, 0, 0); \quad u_\mu = (-1, 0, 0, 0); \quad \text{(A.1)}$$

$$k^\mu = (E, 0, 0, 0); \quad u^\mu = (1, -E, EU^A), \quad \text{(A.2)}$$

where $U^A := u^\rho \partial_\rho (x^A)/E$ defines the drift of the screen space coordinates in the matter frame. The one-form components in (A.1) follow directly from the definitions $k_\mu := \partial_\mu V$ and $u_\mu := -\partial_\mu \tau$. The vector components in (A.2) follow from the definition of the energy function $E := -k_\mu u_\mu$ and the transport rules (shift vectors) $k^\rho \partial_\rho (x^A) = 0$ and $u^\rho \partial_\rho (x^A) = EU^A$. We may now write the projection tensor onto the light fronts (11) in the adapted local coordinate system $x^\mu = (\tau, V, x^A)$:

$$p_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & U^2 & U_B \\ 0 & U_A & p_{AB} \end{pmatrix}; \quad p^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p^{AB} \end{pmatrix}, \quad \text{(A.3)}$$

where $U_A := p_{AB} U^B$ and $U^2 := p_{AB} U^A U^B$, and where the area-adapted screen space metric $p_{AB}$ has inverse $p^{AB}$. The tensor components in (A.3) follow from the orthogonality conditions $p_{\mu\nu} k^{\nu} = p_{\mu\nu} u^{\nu} = 0$ and $p^{\mu\nu} k^{\nu} = p^{\mu\nu} u^{\nu} = 0$, respectively. Note that in general the values of the components $p_{11}, p_{1A}$ and $p_{A1}$ are non-zero, which comes from generally non-zero values for $u^1$ and $u^A$ in (A.2).

Using the definitions (10a) and (11), we might formulate the metric tensor for the cosmological spacetime as $g_{\mu\nu} = k_\mu k_\nu/E^2 - (k_\mu u_\nu + u_\mu k_\nu)/E + p_{\mu\nu}$, and insert Eqs. (A.1), (A.2), and (A.3) in this formulation to obtain

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1/E \\ 1/E & (1/E^2)+U^2 \\ 0 & U_A \end{pmatrix}; \quad g^{\mu\nu} = \begin{pmatrix} -1 & E & -EU^B \\ E & 0 & 0 \\ -EU_A & 0 & p^{AB} \end{pmatrix}. \quad \text{(A.4)}$$
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