DISCRETIZATION OF C*-ALGEBRAS

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Abstract. We investigate how a C*-algebra could consist of functions on a noncommutative set: a discretization of a C*-algebra \( A \) is a \( * \)-homomorphism \( A \rightarrow M \) that factors through the canonical inclusion \( C(X) \subseteq \ell^\infty(X) \) when restricted to a commutative C*-subalgebra. Any C*-algebra admits an injective but nonfunctorial discretization, as well as a possibly noninjective functorial discretization, where \( M \) is a C*-algebra. Any subhomogenous C*-algebra admits an injective functorial discretization, where \( M \) is a W*-algebra. However, any functorial discretization, where \( M \) is an AW*-algebra, must trivialize \( A = B(H) \) for any infinite-dimensional Hilbert space \( H \).

1. Introduction

In operator algebra it is common practice to regard C*-algebras as noncommutative analogues of topological spaces, and to regard W*-algebras as noncommutative analogues of measurable spaces. What would it mean to make precise how a C*-algebra is a ‘noncommutative ring of continuous functions’? Several natural approaches to this question cannot faithfully represent examples as simple as matrix algebras \( M_n(\mathbb{C}) \) \[ [35, 7, 36, 4] \]. Such obstructions suggest more carefully considering what ‘noncommutative sets’ in the foundations of noncommutative geometry should be, before attempting to topologize them.

This article explores the idea of embedding the C*-algebra in an appropriate noncommutative algebra of ‘bounded functions on the noncommutative set underlying its spectrum’, just like any topological space embeds in a discrete one. More precisely, consider the case of a commutative C*-algebra \( A \). A representation of \( A \) as operating on a Hilbert space \( H \) is equivalent to a \( * \)-homomorphism \( A \rightarrow B(H) \). Similarly, representing \( A \) as continuous complex-valued functions on a compact Hausdorff space \( X \) can equivalently be viewed as a \( * \)-homomorphism \( A \rightarrow \ell^\infty(X) \) to the algebra of bounded functions on the set \( X \). More generally, representing \( A \) as (discrete) functions on a set \( X \) can equivalently be viewed as a \( * \)-homomorphism to the algebra \( \mathbb{C}^X \) of all functions on \( X \).

In the spirit of noncommutative geometry, we thus seek a category \( \mathbf{A} \) of \( * \)-algebras to play the role of the dual to the category of ‘noncommutative sets’. This category should contain the commutative algebras \( \ell^\infty(X) \) (or \( \mathbb{C}^X \)) as a full subcategory, dual to the category of sets. In keeping with the programme of taking commutative subalgebras seriously \[ [13, 35, 7, 36, 6, 19, 16, 20, 17] \], we posit that a representation of a C*-algebra as an algebra of functions on a noncommutative set should be an algebra homomorphism \( \phi: A \rightarrow M \) for some \( M \) in \( \mathbf{A} \), whose restriction

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to every commutative C*-subalgebra $C \simeq C(X)$ of $A$ factors through the natural inclusion $C(X) \subseteq \ell^\infty(X)$ via a morphism $\ell^\infty(X) \to M$ in $A$. We call such a map $\phi$ a discretization of $A$.

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & M \\
\downarrow & & \downarrow \\
C(X) & \subseteq & \ell^\infty(X)
\end{array}
$$

Section 2 makes this definition precise, relative to a parameterizing category $A$ that can then remain unspecified. This approach to terminology gives most flexibility in investigating the open problem of finding a suitable noncommutative extension of the functor $C(X) \mapsto \ell^\infty(X)$ before us. We show that every C*-algebra admits a discretization into a C*-algebra $M$ that is injective but nonfunctorial. We also show that there is a universal candidate for a functorial discretization into the category of C*-algebras, but it remains open whether this functorial discretization is injective for every C*-algebra.

In Section 3 we show that a sizeable class of C*-algebras that are ‘close to being commutative’ does indeed have injective functorial discretizations, namely the subhomogeneous algebras: subalgebras of $M_n(C)$ for some commutative C*-algebra $C$. The discretization is achieved by profinite completion, suggesting that profinite completion for subhomogeneous algebras is a noncommutative substitute for the ‘underlying set functor’ that sends a compact Hausdorff space to its underlying discrete space.

On the other hand, in Section 4 we show that no subcategory of W*-algebras, or even AW*-algebras, can be dual to noncommutative sets in the sense of injectively discretizing every C*-algebra. In particular, every functor from C*-algebras to AW*-algebras taking each C*-algebra to a discretization must trivialize $A = B(H)$ for any infinite-dimensional Hilbert space $H$. A number of related examples and obstructions are discussed, including separable algebras $A$ for which the same trivialization occurs. Viewing *-homomorphisms out of a C*-algebra as representing it by functions on a noncommutative set dates back at least to Akemann [1] and Giles and Kummer [14], who took the representation to be the canonical homomorphism $A \to A^{**}$ into the bidual. They noted [2, p. 10] that their theory was not functorial. Our obstructions amplify this observation by suggesting that W*-algebras indeed cannot play the role of ‘noncommutative $\ell^\infty(X)$-algebras’ for C*-algebras as large as $B(H)$.

The article concludes with a discussion in Section 5 of the implications of our obstructions, with an eye toward future work on the problem of finding injective functorial discretizations of all C*-algebras.

2. Discretization

We assume throughout this article that all rings, algebras, and subalgebras are unital, and that all homomorphisms preserve units. Write Spec$(C)$ for the Gelfand spectrum of a commutative C*-algebra $C$. Write Cstar for the category of C*-algebras with *-homomorphisms and Wstar for the subcategory of W*-algebras with normal *-homomorphisms.

Recall that a pro-C*-algebra [31, 32] is a topological *-algebra that is a directed (or “inverse”) limit in the category of topological *-algebras of a system of
C*-algebras. Pro-C*-algebras with continuous ∗-homomorphisms form a category proCstar. The algebra C(X) of all complex-valued functions on a set X equipped with its topology of pointwise convergence is a pro-C*-algebra, as it is the directed limit of the finite-dimensional C*-algebras C^S for all finite subsets S ⊆ X.

Lemma 2.1. The functors X ↦→ ℓ∞(X) and X ↦→ C(X) are contravariant equivalences between the category of sets and full subcategories of Wstar and proCstar.

Proof. The proof for the functor ℓ∞ can be found in [40, Section 6.1]. We sketch an argument that covers both functors.

It is rather clear that each of the above assignments forms a contravariant functor into the specified category. It only remains to show that each is naturally bijective on Hom-sets. Fix x ∈ X. Let ev_x : C(X) → C denote the continuous ∗-homomorphism given by evaluation at x, whose restriction to ℓ∞(X) is normal. The maps X → proCstar(C^X, C) and X → Wstar(ℓ∞(X), C), given in each case by x ↦→ ev_x, are both bijections; this follows by verifying that the kernel of either kind of morphism C^X → C or ℓ∞(X) → C is generated as an ideal by a characteristic function χ_S, which entails that S = X \ {x} for some x ∈ X.

Now the argument that the functors in question are bijective on Hom-sets is purely formal, and can be proved by essentially the same argument as the one given in the algebraic context in [21, Theorem 4.7].

The previous lemma leads naturally to the following notion, in keeping with the programme of taking commutative subalgebras seriously. As mentioned in the introduction, the definition is made relative to a category A of complex algebras that is a candidate to contain ‘algebras of bounded functions on noncommutative sets.’

Definition 2.2. Let A denote a category of C-algebras containing the algebras ℓ∞(X) for any set X with their normal ∗-homomorphisms. Given a C*-algebra A, a (bounded) A-discretization is a homomorphism φ : A → M whose restriction to each commutative C*-subalgebra C of A factors through the natural inclusion C → ℓ∞(Spec(C)) via a morphism φ_C : ℓ∞(Spec(C)) → M in A.

We call a discretization φ faithful when it is injective and all φ_C can be chosen injective. We call φ compatible if the morphisms φ_C can be chosen such that φ_C factors through φ_D via the induced morphism ℓ∞(Spec(C)) → ℓ∞(Spec(D)) for commutative C*-subalgebras C ⊆ D ⊆ A.

When A is Cstar or Wstar above, we will speak of C*- or W*-discretizations instead of A-discretizations.

Proposition 2.3. Every C*-algebra has a faithful C*-discretization.

Proof. Write L for the functor C ↦→ ℓ∞(Spec(C)). Given a finite family S = {C_1, ..., C_n} of commutative C*-subalgebras of A, write A_S for the colimit in Cstar of the diagram whose objects are A, the C_i, and the L(C_i), along with
the inclusions of each $C_i$ into both $A$ and $L(C_i)$. This can be constructed up to isomorphism as an iterated amalgamated free product:

$$A_S \simeq \cdots ((A \ast_{C_1} L(C_1)) \ast_{C_2} L(C_2)) \cdots \ast_{C_n} L(C_n).$$

Thus the natural maps from $A$ and the $L(C_i)$ into $A_S$ are all embeddings; see [8, Theorem 3.1] or [29, Theorem 4.2].

The finite families $S$ of commutative $C^*$-subalgebras of $A$ form a directed set under inclusion. Consider the directed colimit $M = \colim S A_S$. By construction the mediating map $\phi: A \to M$ is a $C^*$-discretization. For finite subfamilies $S \subseteq T$ of commutative $C^*$-subalgebras of $A$, the induced map $A_S \to A_T$ is injective because $A_T$ is formed from $A_S$ by iterated pushouts. Thus the natural maps $A_S \to M$ are injective [37, Theorem 1], from which it follows that $\phi$ is faithful. □

The discretization $\phi: A \to M$ constructed in the proof above is not compatible: for commutative $C^*$-subalgebras $C \subseteq D \subseteq A$, the algebra $M$ is obtained by gluing together distinct copies of $L(C)$ and $L(D)$ without regard to the natural inclusion $L(C) \to L(D)$. In Theorem 2.5 below we modify the construction to ensure compatibility, with the caveat that we no longer know that the discretization is even injective. This universally constructed $C^*$-discretization will in fact satisfy the following natural condition.

**Definition 2.4.** Let $A$ be a category as in Definition 2.2. A functorial $A$-discretization is a functor $F: \text{Cstar} \to A$ together with natural homomorphisms $\eta_A: A \to F(A)$ such that $\eta_C$ for each commutative $C^*$-algebra $C$ turns into the natural inclusion $C \to \ell\infty(\text{Spec}(C))$ by a natural isomorphism $F(C) \simeq \ell\infty(\text{Spec}(C))$.

A functorial discretization automatically gives compatible discretizations $A \to F(A)$ for every $C^*$-algebra $A$: writing $i_C: C \to A$ for the inclusion of a commutative $C^*$-subalgebra gives the following commutative diagram.

$$\begin{array}{ccc}
A & \xrightarrow{\eta_A} & F(A) \\
i_C & \downarrow & \downarrow F(i_C) \\
C & \xrightarrow{\eta_C} & F(C) \simeq \ell\infty(\text{Spec}(C))
\end{array}$$

Compatibility follows by applying $F$ to successive inclusions $C \subseteq D \subseteq A$.

Write $\text{cCstar}$ for the full subcategory of $\text{Cstar}$ of commutative $C^*$-algebras. Write $\mathcal{C}(A)$ for the small subcategory of $\text{cCstar}$ consisting of the commutative $C^*$-subalgebras of a $C^*$-algebra $A$ with their inclusion morphisms; we also view this as a partially ordered set.

**Theorem 2.5.** The functor $F: \text{Cstar} \to \text{Cstar}$ given by

$$F(A) = \colim_{C \in \mathcal{C}(A)} A \ast_C \ell\infty(\text{Spec}(C))$$

equipped with the naturally induced $*$-homomorphisms $\eta_A: A \to F(A)$ is a functorial $C^*$-discretization. For each $C^*$-algebra $A$, the $C^*$-discretization $A \to F(A)$ is universal among all compatible $C^*$-discretizations of $A$. Thus $F$ is universal among all functorial $C^*$-discretizations.

**Proof.** We follow the idea of [35, Theorem 2.15] but with arrows reversed. Write $L = \ell\infty \circ \text{Spec}: \text{cCstar} \to \text{Cstar}$. The assignment $A \mapsto \mathcal{C}(A)$ is a functor to the
category of small categories. Given a C*-algebra $A$, the assignment $C \mapsto A_C \mathcal{L}(C)$ is functorial $\mathcal{C}(A) \to \text{Cstar}$. So $F(A) = \text{colim}_{C \in \mathcal{C}(A)} A_C \mathcal{L}(C)$ defines a functor $F: \text{Cstar} \to \text{Cstar}$. Moreover, the induced $*$-homomorphisms $\eta_A: A \to F(A)$ are natural by construction. Finally, if $A$ is commutative so that $A \in \mathcal{C}(A)$, then one naturally has an isomorphism $F(A) \simeq \ell^\infty(\text{Spec}(A))$ that turns $\eta_A$ into the inclusion $A \to \ell^\infty(\text{Spec}(A))$. Thus $F$ is a functorial C*-discretization.

To verify universality of $\eta_A$, fix a compatible C*-discretization $\phi: A \to M$. Each $C \in \mathcal{C}(A)$ then makes the following outer square commute.

The morphisms $\phi$ and $\phi_C$ factor uniquely through the pushout $A_C \mathcal{L}(C)$. Compatibility of the $\phi_C$ means that these uniquely determined morphisms form a cocone from the diagram of the $A_C \mathcal{L}(C)$ to $M$. Thus we obtain a $*$-homomorphism $F(A) = \text{colim}_{C \in \mathcal{C}(A)} A_C \mathcal{L}(C) \to M$ through which $\phi$ factors uniquely, as desired.

Finally, if $(F', \eta')$ is any functorial C*-discretization, then by the local universality of the previous paragraph the natural morphisms $\eta'_A: A \to F'(A)$ factor uniquely through $\eta_A: A \to F(A)$, from which it readily follows that $F'$ factors through a unique natural transformation $F \Rightarrow F'$ whose composite with $\eta$ is $\eta'$. □

Whereas the ‘incompatible’ discretization of Proposition 2.3 is faithful, it is not clear whether the natural C*-discretizations $A \to F(A)$ of the last theorem are faithful or even injective. Abstract nonsense alone does not answer this question.

**Question 2.6.** Is the universal functorial C*-discretization $\eta_A: A \to F(A)$ of Theorem 2.5 injective or faithful for every C*-algebra $A$? Equivalently, does every C*-algebra have an injective or faithful compatible C*-discretization?

**Remark 2.7.** The definitions and results above carefully used the Gelfand spectrum $\text{Spec}(C)$ of a commutative C*-algebra $C$. Henceforth we loosen notation, and write $C = C(X)$ for an arbitrary commutative C*-algebra, and $C \simeq C(X)$ for an arbitrary commutative C*-subalgebra of a C*-algebra $A$.

Recall from Lemma 2.1 that sets may also be encoded algebraically through the algebra of discrete (possibly unbounded) functions as $X \mapsto \mathbb{C}^X$. The rest of the paper will also discuss ‘unbounded’ discretizations.

**Definition 2.8.** Let $\mathbf{A}$ denote a category of C-algebras containing the algebras $\mathbb{C}^X$ for any set $X$ with the $*$-homomorphisms that are continuous with respect to the topology of pointwise convergence. Given a C*-algebra $A$, an **unbounded $\mathbf{A}$-discretization** is a homomorphism $\phi: A \to M$ whose restriction to each commutative C*-subalgebra $C \simeq C(X)$ of $A$ factors through the inclusion $C(X) \to \mathbb{C}^X$ via a morphism $\phi_C: \mathbb{C}^X \to M$ in $\mathbf{A}$. 
Define injective, faithful, and functorial unbounded discretizations analogous to the bounded case. For $A = \text{proCstar}$ we refer to unbounded pro-$C^*$-discretizations.

3. Functorial discretizations through profinite completion

For a compact Hausdorff space $X$, the natural inclusion $C(X) \to \ell^\infty(X)$ is a $W^*$-discretization of the corresponding commutative $C^*$-algebra. Also, if $A$ is a finite-dimensional $C^*$-algebra, then the identity map $A \to A$ is a $W^*$-discretization. This section provides a common generalization of these two examples: Theorems 3.3 and 3.5 below show that the profinite completion of a $C^*$-algebra is a functorial discretization that is faithful for a large class of algebras.

For a $C^*$-algebra $A$, let $\mathcal{F}(A)$ denote the family of closed ideals $I$ of $A$ for which $A/I$ is finite-dimensional. Then $\mathcal{F}(A)$ is closed under finite intersections, as is readily verified by embedding $A/(I \cap J) \to A/I \oplus A/J$ for ideals $I, J \in \mathcal{F}(A)$. Thus the finite-dimensional $C^*$-algebras $A/I$ for $I \in \mathcal{F}(A)$ form an inversely directed system. We may take the directed limit of this system either in the category $\text{Cstar}$ to obtain a $C^*$-algebra, or in the category of topological algebras to obtain a pro-$C^*$-algebra. We denote these two directed limits by

$$P_b(A) = \lim_{I \in \mathcal{F}(A)} A/I \quad \text{computed in } \text{Cstar},$$
$$P_u(A) = \lim_{I \in \mathcal{F}(A)} A/I \quad \text{computed in } \text{proCstar}.$$ 

Given a $*$-homomorphism $f: A \to B$ and $J \in \mathcal{F}(B)$, the induced embedding $A/f^{-1}(J) \to B/J$ ensures that $f^{-1}(J) \in \mathcal{F}(A)$. Universality provides a composite $*$-homomorphism

$$P_b(A) = \lim_{I \in \mathcal{F}(A)} A/I \to \lim_{J \in \mathcal{F}(B)} A/f^{-1}(J) \to \lim_{J \in \mathcal{F}(B)} B/J = P_b(B)$$

making the assignments $P_b$ and $P_u$ functorial.

Notice that the diagram over which the limit $P_b(A)$ is computed consists of $W^*$-algebras with normal $*$-homomorphisms. The subcategory $\text{Wstar}$ of $\text{Cstar}$ is closed under limits since the forgetful functor $\text{Wstar} \to \text{Cstar}$ is right adjoint to the universal enveloping $W^*$-algebra functor $\mathcal{U}$. Thus $P_b(A)$ is a $W^*$-algebra, and for $f: A \to B$ in $\text{Cstar}$ the induced morphism $P_b(f): P_b(A) \to P_b(B)$ is a normal $*$-homomorphism. Thus $P_b$ is a functor $\text{Cstar} \to \text{Wstar}$.

Each of the two functors $P_b$ and $P_u$ is a kind of profinite completion $[13]$. Definition 3.1. We call $P_b: \text{Cstar} \to \text{Wstar}$ the bounded profinite completion, and $P_u: \text{Cstar} \to \text{proCstar}$ the unbounded profinite completion.

Let $b(P) \subseteq P$ denote the set of bounded elements of a pro-$C^*$-algebra $P$: those elements whose spectrum forms a bounded subset of $\mathbb{C}$. This is a $C^*$-algebra that lies densely in $P$ $[31$, Proposition 1.11].

Proposition 3.2. If $A$ is a $C^*$-algebra, then $P_b(A) \simeq b(P_u(A))$: the $W^*$-algebra $P_b(A)$ is $*$-isomorphic to the algebra of bounded elements of the pro-$C^*$-algebra $P_u(A)$.

Proof. Suppose that a $C^*$-algebra $B$ forms a cone over the diagram of finite-dimensional algebras $A/I$ for $I \in \mathcal{F}(A)$. Then $B$ also forms a cone over this diagram in the category $\text{proCstar}$, and this cone factors uniquely through a morphism $B \to P_u(A)$. But the image of this morphism lands in the $C^*$-algebra $b(P_u(A))$ $[31$, Corollary 1.13]. Thus $b(P_u(A))$ satisfies the universal property of
Theorem 3.3. Bounded profinite completion is a functorial \( W^* \)-discretization. Unbounded profinite completion is an unbounded functorial pro-\( C^* \)-discretization.

Proof. For a commutative \( C^* \)-algebra \( C = C(X) \), each \( I \in \mathcal{F}(C) \) is of the form \( I = I_S = \{ f \in C \mid f(S) = 0 \} \) for some finite subset \( S \subseteq X \). The surjection \( C \twoheadrightarrow C/I \simeq C(S) \) is Gelfand dual to the inclusion \( S \hookrightarrow X \). Thus

\[
P_b(C(X)) = \lim_{S \subseteq X} C(S) \simeq \ell^\infty(X),
\]

\[
P_u(C(X)) = \lim_{S \subseteq X} C(S) \simeq \mathbb{C}^X,
\]

and under these isomorphisms the natural map \( \eta_C : C \to P_b(C) \subseteq P_u(C) \) corresponds to the natural inclusion \( C(X) \hookrightarrow \ell^\infty(X) \subseteq \mathbb{C}^X \).

It remains to verify that these functors behave as expected on morphisms. Fix a \( * \)-homomorphism \( f : B = C(Y) \to C = C(X) \), which is Gelfand dual to a continuous function \( \hat{f} : X \to Y \). For any finite set \( S \subseteq X \), the restriction of \( \hat{f} \) to \( S \to \hat{f}(S) \) is Gelfand dual to \( C(\hat{f}(S)) \simeq B/f^{-1}(I_S) \to C/I_S \simeq C(S) \). Taking the directed limit in \( \textbf{Wstar} \) over finite subsets \( S \subseteq X \), we see that the induced map \( P_b(f) : P_b(B) \to P_b(C) \) corresponds to \( \ell^\infty(\hat{f}) \) under the isomorphisms \( P_b(B) \simeq \ell^\infty(Y) \) and \( P_b(C) \simeq \ell^\infty(X) \). This completes the proof for \( P_b \); the analogous argument in \( \textbf{proCstar} \) also holds for \( P_u \).

Example 3.4. Let \( A = M_n(C(X)) \) for a compact Hausdorff space \( X \). Then \( P_b(A) = M_n(\ell^\infty(X)) \) and \( P_u(A) = M_n(\mathbb{C}^X) \).

Proof. Write \( C = C(X) \), and recall that every closed ideal \( J \subseteq M_n(C) \) is of the form \( M_n(I) \) for some closed ideal \( I \subseteq C \) \cite{27} Corollary 17.8. Such an ideal \( J \) has finite codimension in \( A \) if and only if \( I \) has finite codimension in \( C \). Thus

\[
P_b(A) = \lim_{I \in \mathcal{F}(A)} A/J = \lim_{I \in \mathcal{F}(C)} M_n(C)/M_n(I)
\]

\[
\simeq \lim_{I \in \mathcal{F}(C)} M_n(C/I) \simeq M_n(\ell^\infty(X))
\]

and similarly \( P_u(A) \simeq M_n(\mathbb{C}^X) \).

Let us emphasize that, even though the profinite completion functors yield discretizations of all \( C^* \)-algebras, there are many \( C^* \)-algebras \( A \) for which \( P_b(A) = P_u(A) = 0 \) is trivial. Indeed, if \( A \) is any \( C^* \)-algebra with no finite-dimensional representations, then by construction of the profinite completions we necessarily have \( P_b(A) = P_u(A) = 0 \). Example include: the algebra \( B(H) \) of bounded operators on an infinite-dimensional Hilbert space \( H \); the CCR algebra \( \mathcal{O}_n \); the Calkin algebra \( B(H)/K(H) \); and the (separable) Cuntz algebra \( \mathcal{O}_n \) generated by \( n \geq 2 \) isometries \( [11] \). Thus it is interesting to see which algebras have injective or faithful discretizations to their profinite completion. This is addressed in the next theorem.

Recall that a \( C^* \)-algebra \( A \) is \textit{residually finite-dimensional} when it has a faithful family of finite-dimensional representations. Similarly, \( A \) is \textit{subhomogeneous} when
there is an integer $n \geq 1$ such that every irreducible representation of $A$ has dimension at most $n$; this is equivalent \cite[Proposition IV.1.4.3]{[13]} to $A$ being isomorphic to a C*-subalgebra of $M_k(C)$ for a commutative C*-algebra $C$ and an integer $k \geq 1$.

For a point $x$ in a set $X$, we let $\delta_x = \chi(x) \in \ell^\infty(X) \subseteq \mathbb{C}^X$ denote the indicator function of the singleton $\{x\}$.

**Theorem 3.5.** For a C*-algebra $A$, the functorial discretizations $P_b$ and $P_u$ are:

(i) injective if and only if $A$ is residually finite-dimensional;
(ii) faithful if $A$ is subhomogeneous.

*Proof.* (i) If $A$ is residually finite-dimensional, every nonzero $a \in A$ allows $I_a \in \mathcal{F}(A)$ with $a \notin I_a$ (meaning that $a$ has nonzero image in $A/I_a$). Thus $a$ is not in the kernel of $\eta_A: A \to \varinjlim_{I \in \mathcal{F}(A)} A/I = P_b(A) \subseteq P_u(A)$. Hence $\eta_A$ is injective. (See also \cite[Lemma 1.10]{[13]}.) The converse follows directly from the definition.

(ii) Consider a commutative C*-subalgebra $C(X) \subseteq A$, and $x \in X$. Because the homomorphisms $\ell^\infty(X) \simeq P_b(C(X)) \to P_b(A)$ and $\mathbb{C}^X \simeq P_u(C(X)) \to P_u(A)$ are respectively normal and continuous, it suffices to show that $\delta_x \in \ell^\infty(X) \subseteq \mathbb{C}^X$ is not in their kernel. Indeed, the kernel $I$ of either morphism is an ideal generated by a characteristic function $\chi_S$ for some $S \subseteq X$, so that $I$ contains exactly those $\delta_x$ with $x \in S$. Hence if all $\delta_x \notin I$, then $S = \emptyset$ and therefore $I = 0$.

Evaluation at $x$ is a pure state on $C(X)$, which extends \cite[II.6.3.2]{[13]} to a pure state $\rho_x$ on $A$. Because $A$ is subhomogeneous, the GNS construction applied to $\rho_x$ yields a finite-dimensional representation $\pi: A \to B(\mathbb{C}^n) \simeq M_n(\mathbb{C})$ for some integer $n \geq 1$, with cyclic vector $v_x \in \mathbb{C}^n$. Let $I \in \mathcal{F}(A)$ denote the kernel of $\pi$. The induced *-homomorphism $\psi: \ell^\infty(X) \to A/I \to M_n(\mathbb{C})$ has image isomorphic to $C(S)$ for some finite subset $S \subseteq X$; in fact, this set $S$ is characterized as those pure states on $C(X)$ that are induced by vector states of the representation $\pi$. Now $\pi(f)v_x = f(x)v_x$ for $f \in C(X)$ by construction of $\pi$. Thus $x \in S$, so that $\delta_x$ is not in the kernel of $\psi$. It follows that $\delta_x$ has nonzero image in each of the limit algebras $P_b(A)$ and $P_u(A)$, as desired. \hfill \Box

**Remark 3.6.** For C*-algebras $A$ that are residually finite-dimensional but not subhomogeneous, the natural map $A \to P_b(A)$ is technically an injective discretization, but it does not satisfy all desiderata for an ‘algebra of bounded functions on the noncommutative underlying set’ of $A$. Consider the C*-sum $A = \bigoplus_{k=1}^\infty M_k(\mathbb{C})$. Let $I_n \subseteq A$ denote the kernel of the projection $A \to M_1(\mathbb{C}) \oplus \cdots \oplus M_n(\mathbb{C})$ onto the first $n$ components. By an argument similar to that in \cite[Lemma 7.5]{[23]}, the kernel of any finite-dimensional representation of $A$ must contain some $I_n$. It follows that the $I_n$ form a cofinal chain in $\mathcal{F}(A)$, so that the profinite completion

$$A \to P_b(A) \simeq \varinjlim_{n \to \infty} A/I_n \simeq A$$

is an isomorphism. But this is far from the behavior one would expect when comparing to the commutative example $C = \bigoplus_{k=1}^\infty \mathbb{C} \simeq \ell^\infty(\mathbb{N}) \simeq C(\beta\mathbb{N})$; the profinite completion $C \to P_b(C)$ corresponds under this isomorphism to the embedding $C \simeq C(\beta\mathbb{N}) \to \ell^\infty(\beta\mathbb{N})$, indicating that $C$ is ‘far below’ $P_b(C)$ as a subalgebra.

Almost all faithful discretizations of C*-algebras we know are supplied by Theorem 3.5 above. We conclude this section by describing another significant example of a faithful compatible discretization that is not of this form.
Example 3.7. For an infinite-dimensional Hilbert space $H$, consider the C*-subalgebra $A = \mathbb{C} \oplus K(H)$ of $B(H)$ generated by the identity and the compact operators. The embedding $A \hookrightarrow B(H)$ is a faithful compatible W*-discretization.

Proof. Any commuting set of self-adjoint compact operators on $H$ has an orthonormal basis of simultaneous eigenvectors, so the same remains true for commuting sets of self-adjoint operators in $A$. Let $C \simeq C(X) \subseteq A$ be a commutative C*-subalgebra. For $x \in X$ let $p_x \in B(H)$ denote the projection onto the simultaneous eigenspace $\{ v \in H \mid f \cdot v = f(x)v \text{ for all } f \in C \}$. Now each $p_x \neq 0$ and $\sum p_x = 1$ in $B(H)$. It follows that the W*-subalgebra $W_C$ generated by the $p_x$ is isomorphic to $\ell^\infty(X)$, and the fact that $fp_x = p_xf = f(x) \cdot p_x$ for all $f \in C$ guarantees that the natural inclusion $C \subseteq W_C$ corresponds under this isomorphism to the natural inclusion $C(X) \subseteq \ell^\infty(X)$. Thus the discretization is faithful.

Compatibility for commutative C*-subalgebras $C \subseteq D \subseteq A$ is readily established from the simple observation that a simultaneous eigenspace for $D$ restricts to a simultaneous eigenspace for $C$. □

The example above is a faithful compatible W*-discretization for which we do not know of any extension to an unbounded discretization.

4. Obstructions to discretizations with many projections

Can the bounded faithful functorial W*-discretization for subhomogeneous C*-algebras of Theorem 3.5 be extended to general C*-algebras through some method other than profinite completion? Perhaps surprisingly, we prove in this section that the answer is no: any W*-discretization of the algebra $B(H)$ for an infinite-dimensional Hilbert space $H$ is necessarily zero. In fact, the obstruction is even more serious: if we replace the category of W*-algebras (‘noncommutative measurable spaces’) with the category of AW*-algebras [22, 5] (‘noncommutative complete Boolean algebras’ [20]), the obstruction persists.

The next definition is crucial to our obstructions, and relies on the following notions from measure theory. An atom of a measure space $(X, \mu)$ is a measurable subset $U \subseteq X$ with $\mu(U) > 0$, such that $\mu(V) < \mu(U)$ implies $\mu(V) = 0$ for any measurable subset $V \subseteq U$. An atom of a regular Borel measure on a locally compact Hausdorff space is necessarily a singleton [24, 2.IV]. A measure is diffuse if it has no atoms. We will say that a positive linear functional $\psi: C(X) \to \mathbb{C}$ of a commutative C*-algebra, given by $\psi(f) = \int f \, d\mu$ for a regular Borel measure $\mu$ on $X$, is diffuse when $\mu$ is diffuse.

Definition 4.1. Let $A$ be a C*-algebra. A pair of commutative C*-subalgebras $C$ and $D$ is relatively diffuse when every extension of a pure state of $D$ to a state of $A$ restricts to a diffuse state on $C$.

Example 4.2. Consider the separable Hilbert space $H = L^2[0,1]$, and the C*-algebra $A = B(H)$. Write $D$ for the discrete maximal abelian W*-subalgebra generated by the projections $q_n$ onto the Fourier basis vectors $e_n = \exp(2\pi i n \cdot -)$ for $n \in \mathbb{Z}$, and $C$ for the continuous maximal abelian W*-subalgebra $L^\infty[0,1]$. Then $C$ and $D$ are relatively diffuse.
Proof. There is a canonical conditional expectation $E: A \to D$ that sends $f \in A$ to its diagonal part $\sum q_n f q_n$. For $f \in C$ then $E(f) = \int_0^1 f(t) \, dt$ because
\[
\langle f e_n, e_n \rangle = \int_0^1 f(t) \cdot e^{2\pi i nt} \cdot e^{-2\pi i nt} \, dt = \int_0^1 f(t) \, dt.
\]
Because $\psi$ is a pure state of $D$ now $\psi = \psi \circ E$ by the solution of the Kadison-Singer problem [28]. Hence $\psi(f) = \psi(E(f)) = \psi(\int_0^1 f(t) \, dt) = \int_0^1 f(t) \, dt$. \hfill $\Box$

**Example 4.3.** For $H = L^2[0, 1]$, consider any separable $C^*$-subalgebra $C \subseteq L^\infty[0, 1] \subseteq B(H)$ for which the state $f \mapsto \int_0^1 f(t) \, dt$ is diffuse (such as $C = C[0, 1]$). Then there is a separable $C^*$-subalgebra $A \subseteq B(H)$ containing $C$ and a commutative $C^*$-subalgebra $D$ generated by projections, with $C$ and $D$ relatively diffuse.

**Proof.** Let $e_n$ and $E$ be as in Example 4.2. Because $C$ is separable, we can fix a sequence $\{f_i\}_{i=1}^\infty$ of elements whose linear span is dense in $C$. For each $f_i$ and for each integer $j \geq 1$, the positive solution to the paving conjecture [28] ensures that there is a finite set of projections $p_k = p_k^{(i,j)}$ in the discrete maximal abelian subalgebra of $B(H)$ relative to the Fourier basis $e_n$ with $\sum p_k = 1$ and $\|p_k(f_i - E(f_i))p_k\| \leq 1/j$. Let $D$ be the commutative $C^*$-subalgebra of $B(H)$ generated by the $p_k^{(i,j)}$ for all $i, j$, and $k$. Let $A$ be the $C^*$-subalgebra of $B(H)$ generated by $C$ and $D$; as both $C$ and $D$ are countably generated, the same is true of $A$, whence $A$ is separable. An argument familiar in the literature on the Kadison-Singer problem (as in [38] p310) shows that any extension of a pure state $\psi_0$ on $D$ to a state $\psi$ on $A$ satisfies $\psi(f) = \psi_0(E(f))$ for all $f \in C$. The same computation as in Example 4.2 shows that $\psi(f) = \int_0^1 f(t) \, dt$, which is diffuse on $C$ by hypothesis. \hfill $\Box$

**Remark 4.4.** It is possible to modify Examples 4.2 and 4.3 so that the conclusions can be reached without using the full force of Kadison-Singer. In either case, identify the algebra $C = C(\mathbb{T})$ of continuous functions on the unit circle with the subalgebra $\{ f \mid f(0) = f(1) \} \subseteq C[0, 1] \subseteq B(H)$. The algebra of Fourier polynomials—or more generally, the Wiener algebra $A(\mathbb{T})$—is a dense subalgebra of $C$ and lies in the algebra $M_0 \subseteq B(H)$ of operators that are $l_1$-bounded in the sense of Tanbay [35] with respect to the Fourier basis $\{ e_n \mid n \in \mathbb{Z} \}$. Thus $C$ lies in the norm closure $M_0$, and it was shown in [35] (without the full force of Kadison-Singer) that every element of $M$ is compressible (that is, the operator $f - E(f)$ satisfies paving with respect to the basis $e_n$ for any $f \in M$). The computations in either example given above may now proceed in the same manner.

The relatively diffuse subalgebras $C$ and $D$ in the examples above had pure states of $D$ inducing a unique diffuse state on $C$. We thank the referee for the following example which allows for possibly non-unique extensions.

**Example 4.5.** Let $A$ and $D$ be as in Example 4.2 but consider the commutative $C^*$-subalgebra of $A$ generated by the bilateral shift $e_n \mapsto e_{n+1}$, and let $C$ be its bicommutant. Then $C$ and $D$ are relatively diffuse.

**Proof.** Write $C_0$ for the $C^*$-subalgebra generated by the shift $u: H \to H$; its Gelfand spectrum is the unit circle $\mathbb{T} = \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \}$ [15], Problem 84]. Let $f_n \in C(\mathbb{T})$ be a decreasing sequence converging to the characteristic function $\delta_\lambda = \chi_\lambda(\cdot)$ of some $\lambda \in \mathbb{T}$. Then, since the bounded sequence $(f_n)$ converges pointwise to $\delta_\lambda$, the sequence $(f_n(u))$ in $C_0$ converges strongly to the projection $\delta_\lambda(u)$ in
C. But \( \lim_n (f_n(u), e_0) = (\delta_\lambda(e_0), e_0) \) vanishes because \( u \) has no eigenvectors. Hence \( \| E(f_n(u)) \| = \| (f_n(u), e_0), e_0 \|_H \rightarrow 0 \). Thus a state \( \psi \) of \( A \) that is pure on \( D \) satisfies \( \psi(f_n) = \psi(E(f_n)) \rightarrow 0 \), and is therefore diffuse on \( C \).

Relatively diffuse pairs of commutative \( C^* \)-subalgebras are inherited along \(*\)-homomorphisms, as follows.

**Lemma 4.6.** Let \( \phi: A \rightarrow B \) be a morphism in \( \text{Cstar} \). If two commutative \( C^* \)-subalgebras \( C, D \subseteq A \) are relatively diffuse, then so are \( \phi(C), \phi(D) \subseteq B \).

**Proof.** Fix a pure state \( \psi_0 \) on \( \phi(D) \), and let \( \psi \) be any extension to a state on \( B \). Then \( \psi \circ \phi \) is a state on \( A \) that extends \( \psi_0 \circ \phi \) from \( D \); observe that the latter is a pure state on \( D \) as it is a composition of a \(*\)-homomorphism with a pure state. By hypothesis, the restriction of \( \psi \circ \phi \) to \( C \) is diffuse. As the restriction of \( \phi \) to \( C \rightarrow \phi(C) \) is Gelfand dual to the inclusion \( \text{Spec}(\phi(C)) \rightarrow \text{Spec}(C) \) of a closed subspace, the measure on \( \text{Spec}(\phi(C)) \) corresponding to \( \psi|_{\phi(C)} \) is the restriction of the measure on \( \text{Spec}(C) \) corresponding to \( \psi_0|_C \), which is diffuse. It follows that the restriction of \( \psi \) to \( C' \) is diffuse.

The major result below and its many corollaries will refer to commutative diagrams of the following kind, where \( A \) is a \( C^* \)-algebra with relatively diffuse commutative \( C^* \)-subalgebras \( C \simeq C(X) \) and \( D \simeq C(Y) \).

\[
\begin{array}{ccc}
C \simeq C(X) & \xrightarrow{\phi} & \ell^\infty(X) \\
\downarrow & & \downarrow \phi_C \\
A & \xrightarrow{\phi} & M \\
\downarrow & & \downarrow \phi_D \\
D \simeq C(Y) & \xrightarrow{\phi} & \ell^\infty(Y)
\end{array}
\]

(4.1)

**Theorem 4.7.** If a \( C^* \)-algebra \( A \) has relatively diffuse commutative \( C^* \)-subalgebras \( C \simeq C(X) \) and \( D \simeq C(Y) \), and if there is a \( C^* \)-algebra \( M \) with \(*\)-homomorphisms \( \phi, \phi_C \) and \( \phi_D \) making the diagram (4.1) commute, then for any \( x \in X \) and \( y \in Y \):

\[
\phi_C(\delta_x)\phi_D(\delta_y) = 0.
\]

**Proof.** Let \( x \in X \) and \( y \in Y \), and write \( p = \phi_C(\delta_x) \) and \( q = \phi_D(\delta_y) \). Fix any state \( \sigma \) on the \( C^* \)-algebra \( qBq \), and let \( \psi \) denote the state on \( A \) given by \( \psi(a) = \sigma(q\phi(a)q) \). For \( g \in D \), observe \( \psi(g) = \sigma(\phi_D(\delta_y)g) = \sigma(\phi_D(g(\delta_y))) = g(\delta_y)\sigma(q) = g(y) \), so that \( \psi \) restricts to a pure state on \( D \). By hypothesis, the restriction of \( \psi \) to \( C \) is of the form \( f \mapsto \int f \, d\mu \) for some diffuse Radon measure \( \mu \) on \( X \). Thus for each integer \( n \geq 1 \) we may find an open neighborhood \( U_n \) of \( x \) with \( \mu(U_n) \leq \frac{1}{n} \). Urysohn’s lemma provides a continuous function \( f_n: X \rightarrow [0, 1] \) that vanishes on \( X \setminus U_n \) and satisfies \( f_n(x) = 1 \). Since \( \delta_x \leq f_n \) in \( \ell^\infty(X) \) we have \( p = \phi_C(\delta_x) \leq \phi_C(f_n) \). Positivity of \( b \mapsto \sigma(qbp) \) yields

\[
\sigma(qpq) \leq \sigma(q\phi_C(f_n)q) = \psi(f_n) = \int f_n \, d\mu \leq \mu(U_n) \leq \frac{1}{n}.
\]

As \( n \rightarrow \infty \) we find that \( \sigma(qpq) = 0 \) for all states \( \sigma \) on \( B \), making \( qpq = 0 \). It follows that \( \|qp\|^2 = \|qpq\| = 0 \) and thus \( pq = (qp)^* = 0 \). 

\( \square \)
Write $\text{AWstar}$ for the category of $\text{AW}^*$-algebras with $*$-homomorphisms whose restriction to the projection lattices preserve arbitrary least upper bounds. We call $\text{AWstar}$-discretizations $\text{AW}^*$-discretizations.

**Corollary 4.8.** If a $\text{C}^*$-algebra $A$ has two relatively diffuse commutative $\text{C}^*$-subalgebras, then any $\text{AW}^*$-discretization $\phi: A \to M$ satisfies $M = 0$. Consequently, every functorial $\text{AW}^*$-discretization $F: \text{Cstar} \to \text{AWstar}$ has $F(A) = 0$ for such $A$.

**Proof.** Let $C \simeq C(X)$ and $D \simeq C(Y)$ be the relatively diffuse commutative $\text{C}^*$-subalgebras, and let $\phi_C: \ell^\infty(X) \to M$ and $\phi_D: \ell^\infty(Y) \to M$ be the discretizing morphisms as in Definition 2.2 yielding a commuting diagram (4.1). For $x \in X$ and $y \in Y$, set $p_x = \phi_C(\delta_x)$ and $q_y = \phi_D(\delta_y)$. As $\sum \delta_x = 1_C$ and $\sum \delta_y = 1_D$ (in the sense of least upper bounds of orthogonal projections), and as $\phi_C$ and $\phi_D$ are morphisms in $\text{AWstar}$, we have $\sum p_x = 1 = \sum q_y$ in $M$. By Theorem 4.7, each $p_x$ is orthogonal to all of the $q_y$, so that $p_x$ is orthogonal to $\sum q_y = 1 \in M$. Therefore $p_x = 0$ for all $x \in X$, whence $1 = \sum p_x = 0$ in $M$ and $M = 0$. □

**Example 4.9.** If there is a morphism $B(H) \to A$ in $\text{Cstar}$ for some infinite-dimensional Hilbert space, then $A$ has no nontrivial $\text{AW}^*$-discretization.

**Proof.** First note that $H$ as above is unitarily isomorphic to $L^2[0, 1] \otimes H$, so $a \mapsto a \otimes 1$ is a $*$-homomorphism $B(L^2[0, 1]) \to B(L^2[0, 1]) \otimes B(H) \simeq B(H)$. Example 4.2 along with Lemma 4.6 show that $A$ contains a relatively diffuse commutative $\text{C}^*$-subalgebra, so that Corollary 4.8 applies. □

In particular, by the last example the Calkin algebra $A = B(H)/K(H)$ has no nontrivial $\text{AW}^*$-discretization for $H = L^2[0, 1]$.

Theorem 4.7 has the following consequence for purely ring-theoretic discretizations, with much tamer conclusion than those of Corollaries 4.8 or 4.11.

**Corollary 4.10.** If a $\text{C}^*$-algebra $A$ has relatively diffuse $\text{C}^*$-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$, and if there is a commutative diagram of the form (4.1) where $M$ is a ring and $\phi, \phi_C, \phi_D$ are ring homomorphisms, then for every $x \in X$ and $y \in Y$:

$$\phi_C(\delta_x)\phi_D(\delta_y) = \phi_D(\delta_y)\phi_C(\delta_x) = 0.$$  

**Proof.** Invoking Theorem 4.7 in the case where

$$M_1 = (A \ast_{C(X)} \ell^\infty(X)) \ast_{C(Y)} \ell^\infty(Y)$$

is the colimit in $\text{Cstar}$ of the diagram (4.1) with $M$ deleted, we obtain that the images of $\delta_x$ and $\delta_y$ are orthogonal in $M_1$. Now let $R \ast_S T$ denote the amalgamated free product of rings (which coincides with the amalgamated free product of $\text{C}$-algebras when $S$ is a unital subalgebra of algebras $R$ and $T$), and let

$$M_0 = (A \ast_{\ell^\infty(X)} \ell^\infty(X)) \ast_{C(Y)} \ell^\infty(Y)$$

be the colimit in the category of rings of the diagram (4.1) with $M$ deleted. There is a natural map $M_0 \to M_1$ induced by the universal property of $M_0$. It is a folk result that this is an embedding [10, 33]. Thus the images of $\delta_x$ and $\delta_y$ in $M_0$ are already orthogonal. But the morphisms $\phi$, $\phi_C$, and $\phi_D$ of (4.1) factor universally through $M_0$, so the images of $\delta_x$ and $\delta_y$ in $M$ are orthogonal. □

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1See [20, Lemma 2.2] for further characterizations of these morphisms.
We conclude this section with an obstruction for unbounded discretizations into topological algebras. Write $\mathbf{TAlg}$ for the category of Hausdorff topological $\mathbb{C}$-algebras with continuous homomorphisms. Recall [39, Chapter 10] that a family $(a_i)_{i \in I}$ of elements in a Hausdorff topological ring $R$ is summable if the net $(a_j)$ indexed by finite subsets $J \subseteq I$ converges, where $a_J = \sum_{j \in J} a_j$; in that case we write $\sum a_i$ for the limit.

**Corollary 4.11.** Let $A$ be a $C^*$-algebra with relatively diffuse $C^*$-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$. Then every unbounded $\mathbf{TAlg}$-discretization of $A$ is zero. More precisely: if there is a commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\phi} & \mathbb{C}^X \\
\downarrow & & \downarrow \phi_C \\
A & \xrightarrow{\phi} & M \\
\downarrow & & \downarrow \phi_D \\
D & \xrightarrow{\phi} & \mathbb{C}^Y
\end{array}
$$

where $M$ is a Hausdorff topological ring, $\phi_C$ and $\phi_D$ are continuous homomorphisms, and $\phi$ is a homomorphism, then $M = 0$.

**Proof:** It suffices to prove the second, more general claim. Because the natural embedding $C(X) \to \mathbb{C}^X$ has image in the subring $\ell^{\infty}(X) \subseteq \mathbb{C}^X$ and similarly for $C(Y)$, we may apply Corollary [4.10] to conclude that the idempotents $p_x = \phi_C(\delta_x)$ and $q_y = \phi_D(\delta_y)$ satisfy $p_x q_y = 0$ for all $x \in X$ and $y \in Y$.

The orthogonal set of idempotents $\{\delta_x \mid x \in X\}$ is summable with $\sum \delta_x = 1$ in $\mathbb{C}^X$, so the family of images $(p_x)_{x \in X}$ under the continuous homomorphism $\phi_C$ is also summable in $M$ with $\sum p_x = 1$. Similarly, we have $(q_y)_{y \in Y}$ summable in $M$ with $\sum q_y = 1$.

Now consider the net $(p_I q_J)$ indexed by the directed set of all ‘rectangular’ subsets $I \times J \subseteq X \times Y$ with both $I \subseteq X$ and $J \subseteq Y$ finite. As both $(p_I)$ and $(q_J)$ converge to 1, we have $p_I q_J \to 1^2 = 1$. But each $p_I q_J = \sum_I \sum_J p_x q_y = 0$, so we have $1 = \lim p_I q_J = 0$. Thus $M = 0$. \qed

Just as in Example [4.9] if there is a morphism $B(H) \to A$ in $\mathbf{Cstar}$ with $H$ an infinite-dimensional Hilbert space, then every unbounded $\mathbf{TAlg}$-discretization of $A$ is trivial.

**Remark 4.12.** Similar to the $C^*$-discretization in Proposition [2.8], one could construct a pro-$C^*$-discretization by replacing the pushouts $A \ast C \ell^{\infty}(\text{Spec}(C))$ in $\mathbf{Cstar}$ with the pushouts $A \ast C \mathbb{C}^\text{Spec}(C)$ in $\text{proCstar}$. However, the previous corollary shows that this construction must trivialize for algebras $A$ that have relatively diffuse commutative $C^*$-subalgebras.

We close with one further example of a separable algebra having no injective $W^*$-discretizations. We only sketch its proof, as the complete argument would require us to modify several results above to account for possibly nonunital commutative subalgebras, a technicality that we have avoided for the sake of readability.

**Example 4.13.** Let $H = L^2[0,1]$ and $C = C[0,1] \subseteq L^\infty[0,1] \subseteq B(H)$. Then $A = C + K(H)$ is a separable $C^*$-algebra of type I for which every AW*-discretization and every unbounded $\mathbf{TAlg}$-discretizations has nonzero kernel. (It does, however,
have nonzero non-injective such discretizations that factor through the commutative C*-algebra $A/K(H).$

Proof. Let $e_n$ and $q_n$ be as in Example 4.2. Within $B(H)$, write $C_0(\mathbb{Z}) \simeq D \subseteq K(H)$ for the nonunital commutative C*-subalgebra generated by the $q_n$. If one alters Definition 4.1 to allow for possibly nonunital C*-subalgebras, then $C$ and $D$ are relatively diffuse. Indeed, each pure state $\psi_0$ on $D$ is supported on some projection $p = q_n$, and every extension of $\psi_0$ to a state $\psi$ on $A$ satisfies $\psi(f) = \psi(pfp) = (\int_0^1 f \, dt)\psi(p) = \int_0^1 f \, dt$ for all $f \in C[0,1]$. A suitable modification of Theorem 4.7 holds for such $C$ and $D$, with hardly a change to the proof.

Now if $\phi: A \to M$ is an AW*-discretization or an unbounded $\mathbf{TAlg}$-discretization, then we claim that $K(H) \subseteq \ker(\phi)$. Indeed, the same method of proof of Corollaries 4.8 and 4.11 shows that $D$ is contained in $\ker(\phi)$ (noting that $C$ is still a unital subalgebra), and $K(H)$ is the ideal generated by $D$. □

5. Conclusion

In contrast to the obstructions [35, 7, 4], based on the Kochen-Specker theorem [25] from quantum physics, the fact that profinite completion faithfully discretizes all finite-dimensional C*-algebras shows that the results in Section 4 are truly infinite-dimensional obstructions and are therefore independent of the Kochen-Specker theorem.

From the perspective of discretization as discussed in this paper, the search for a suitable candidate $A$ for a category of algebras dual to ‘noncommutative sets’ remains open. Having ruled out various candidates, we now briefly discuss the implications, including possible avenues to avoid these obstructions.

Within the category $\mathbf{Cstar}$, there remains the interesting open Question 2.6 of whether every C*-algebra has a functorial (or equivalently, compatible) C*-discretization that is injective or faithful. This question is addressed in recent work of Kornell [26] that takes a radically different approach: passing to a model of set theory in which every subset of $\mathbb{R}$ is measurable, so that the Axiom of Choice fails.

A positive answer to Question 2.6 would still not entail a candidate category of algebras dual to ‘noncommutative sets’. That would require isolating a suitable subcategory $A$ of $\mathbf{Cstar}$ containing the algebras $\ell^\infty(X)$ and their normal *-homomorphisms as a full subcategory (dual to ‘classical’ sets). One of the most notable feature of the algebras $\ell^\infty(X)$ and $\mathbb{C}^X$ is their abundance of projections. But using this structure as a guide makes Corollaries 4.8 and 4.11 particularly troubling. Suppose that $A$, $C(X)$, and $C(Y)$ are as in Theorem 4.7. Let $\phi: A \to M$ be the discretization of Proposition 2.3. On the one hand, that proposition demonstrates that $\ell^\infty(X)$ and $\ell^\infty(Y)$ embed faithfully into $M$. On the other hand, for all $x \in X$ and $y \in Y$, Theorem 4.7 implies that the images of $\delta_x \in \ell^\infty(X)$ and $\delta_y \in \ell^\infty(Y)$ are orthogonal in $M$. So it is not contradictory to faithfully embed both $\ell^\infty(X)$ and $\ell^\infty(Y)$ into a common discretization making all $\delta_x \delta_y$ vanish.

Thus Corollaries 4.8 and 4.11 merely indicate that globally ‘gluing’ projections via the structure of an AW*-algebra or via convergence of nets of finite sums is inadequate for discretization. This suggests exploring new structures imposing a suitable ‘global coherence’ on projections in noncommutative *-algebras beyond AW*-algebras or topological algebras. To speculate only about a single possibility:
the notion of contramodule formalizes ‘infinite summation’ operations that cannot be interpreted as convergence of sums in any topology.

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References

[1] C. A. Akemann. The general Stone-Weierstrass problem. J. Funct. Anal., 4:277–294, 1969.
[2] C. A. Akemann. A Gelland representation theory for $C^*$-algebras. Pacific J. Math., 39:1–11, 1971.
[3] J. Anderson. Extensions, restrictions, and representations of states on $C^*$-algebras. Trans. Amer. Math. Soc., 249(2):303–329, 1979.
[4] M. Ben-Zvi, A. Ma, and M. L. Reyes. A Kochen-Specker theorem for integer matrices and noncommutative spectrum functors. arXiv:1509.03618, 2015.
[5] S. K. Berberian. Baer $^*$-rings. Springer, 1972, second printing.
[6] B. van den Berg and C. Heunen. Noncommutativity as a colimit. Applied Categorical Structures, 20(4):393–414, 2012.
[7] B. van den Berg and C. Heunen. Extending obstructions to noncommutative functorial spectra. Theory and Applications of Categories, 29:No. 17, 457–474, 2014.
[8] B. E. Blackadar. Weak expectations and nuclear $C^*$-algebras. Indiana Univ. Math. J., 27(6):1021–1026, 1978.
[9] B. E. Blackadar. Operator Algebras: Theory of $C^*$-algebras and von Neumann algebras, volume 122 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 2006.
[10] D. P. Blecher and V. I. Paulsen. Explicit construction of universal operator algebras and applications to polynomial factorization. Proceedings of the American Mathematical Society, 112(3):839–850, 1991.
[11] J. Cuntz. Simple $C^*$-algebras generated by isometries. Communications in Mathematical Physics, 57:173–185, 1977.
[12] J. Dauns. Categorical W*-tensor product. Transactions of the American Mathematical Society, 166:439–456, 1972.
[13] R. El Harti, N. C. Phillips, and P. R. Pinto. Profinite pro-$C^*$-algebras and pro-$C^*$-algebras of profinite groups. Houston J. Math., 40(3):791–816, 2014.
[14] R. Giles and H. Kummer. A non-commutative generalization of topology. Indiana University Mathematics Journal, 21(1):91–102, 1971.
[15] P. Halmos. A Hilbert space problem book. Springer, 2nd edition, 1982.
[16] J. Hamhalter. Isomorphisms of ordered structures of abelian $C^*$-algebras. Journal of Mathematical Analysis and Applications, 383:391–399, 2011.
[17] J. Hamhalter. Dye’s theorem and Gleason’s theorem for $AW^*$-algebras. Journal of Mathematical Analysis and Applications, 422(2):1103–1115, 2015.
[18] C. Heunen. The many classical faces of quantum structures. arXiv:1412.2177, 2014.
[19] C. Heunen, N. P. Landsman, and B. Spitters. A topos for algebraic quantum theory. Communications in Mathematical Physics, 291:63–110, 2009.
[20] C. Heunen and M. L. Reyes. Active lattices determine $AW^*$-algebras. Journal of Mathematical Analysis and Applications, 416:289–313, 2014.
[21] M. C. Iovanov, Z. Mesyan, and M. L. Reyes. Infinite-dimensional diagonalization and semisimplicity. Israel J. Math., 215(2), 2016.
[22] I. Kaplansky. Projections in Banach algebras. Ann. of Math. (2), 53:235–249, 1951.
[23] I. Kaplansky. The structure of certain operator algebras. Trans. Amer. Math. Soc., 70:219–255, 1951.
[24] J. D. Knowles. On the existence of non-atomic measures. Mathematika, 14:62–67, 1967.
[25] S. Kochen and E. P. Specker. The problem of hidden variables in quantum mechanics. J. Math. Mech., 17:59–87, 1967.
[26] A. Kornell. $V^*$-algebras. arXiv:1502.01516, 2015.
[27] T.-Y. Lam. *Lectures on Modules and Rings*. Springer, 1999.
[28] A. Marcus, D. A. Spielman, and N. Srivastava. Interlacing families II: mixed characteristic polynomials and the Kadison-Singer problem. *Ann. Math.*, 182(1):327–350, 2015.
[29] G. K. Pedersen. Pullback and pushout constructions in C*-algebra theory. *Journal of Functional Analysis*, 167:243–344, 1999.
[30] D. Petz. *An invitation to the algebra of canonical commutation relations*. Leuven University Press, 1990.
[31] N. C. Phillips. Inverse limits of C*-algebras. *J. Operator Theory*, 19(1):159–195, 1988.
[32] N. C. Phillips. Inverse limits of C*-algebras and applications. In *Operator algebras and applications, Vol. I*, volume 135 of *London Math. Soc. Lecture Note Ser.*, pages 127–185. Cambridge Univ. Press, Cambridge, 1988.
[33] L. Positselski. Contramodules. [arXiv:1503.00991] 2015.
[34] C. Ramsey and M. L. Reyes. Amalgamated free products of C*-algebras are nondegenerate. forthcoming, 2016.
[35] M. L. Reyes. Obstructing extensions of the functor Spec to noncommutative rings. *Israel Journal of Mathematics*, 192(2):667–698, 2012.
[36] M. L. Reyes. Sheaves that fail to represent matrix rings. In *Ring theory and its applications*, volume 699 of *Contemp. Math.*, pages 285–297. American Mathematical Society, 2014.
[37] Z. Takeda. Inductive limit and infinite direct product of operator algebras. *Tohoku Math. J.* (2), 7:67–86, 1955.
[38] B. Tanbay. Pure state extensions and compressibility of the l1-algebra. *Proceedings of the American Mathematical Society*, 113(3):707–713, 1991.
[39] S. Warner. *Topological Rings*, volume 178 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1993.
[40] N. Weaver. *Mathematical Quantization*. Studies in Advanced Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2001.

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