CLASS-PRESERVING AUTOMORPHISMS
OF UNIVERSAL HYPERLINEAR GROUPS

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Abstract. We show that the group of class-preserving automorphisms of a
universal hyperlinear group has index 2 inside the group of all automorphisms.

1. Introduction

Suppose that \( G \) is a group. An automorphism \( \alpha \) of \( G \) is class-preserving if
for every \( g \in G \) the elements \( g \) and \( \alpha(g) \) of \( G \) belong to the same conjugacy
class. It is easily observed that class-preserving automorphisms of \( G \) form a normal
subgroup of the group of automorphisms of \( G \). It is shown in [13] that if \( \mathcal{U} \) is a
nonprincipal ultrafilter over \( \mathbb{N} \), then every automorphism of the ultraproduct \( \prod_{\mathcal{U}} S_n \)
of the finite symmetric groups endowed with the normalized Hamming metric (as
defined in [14, Section 2.4]) is class-preserving. The groups \( \prod_{\mathcal{U}} S_n \) are sometimes
called universal sofic groups, since they contain any countable discrete sofic group
as a subgroup, see [14, Section 3].

In this note we consider universal hyperlinear groups, i.e. ultraproducts \( \prod_{\mathcal{U}} U_n \)
of the finite rank unitary groups endowed with the normalized Hilbert-Schmidt
metric, as defined in [14, Section 2.4]. We show that for any nonprincipal ultrafilter \( \mathcal{U} \) over \( \mathbb{N} \) the group of class-preserving automorphism of \( \prod_{\mathcal{U}} U_n \) has index 2 in
the automorphism group of \( \prod_{\mathcal{U}} U_n \). The same statement holds when one replaces
\( \prod_{\mathcal{U}} U_n \) with the unitary group \( U(\mathcal{R}^{\mathcal{U}}) \) of the ultrapower \( \mathcal{R}^{\mathcal{U}} \) of the separable hyperfinite \( \Pi_1 \) factor (an introduction to \( \Pi_1 \) factors and a definition of the hyperfinite \( \Pi_1 \)
factor can be found in [2, Section III.1]). It is worth observing that by [4, Proposition 2.4.6]
if the Continuum Hypothesis holds, then \( U(\mathcal{R}^{\mathcal{U}}) \) and \( \prod_{\mathcal{U}} U_n \) have outer
automorphisms (and in fact \( 2^{\aleph_1} \) many of them).

The rest of this note is divided into two sections: In Section 2 we recall a fact
about automorphisms of countably saturated \( \Pi_1 \) factors whose proof is essentially
contained in [13]; In Section 3 we present a proof of the main theorem, based
on Theorem 2 from [6]. In the following \( \mathcal{U} \) will always be assumed to be a fixed
nonprincipal ultrafilter over \( \mathbb{N} \). The separable hyperfinite \( \Pi_1 \) factor will be denoted
as customary by \( \mathcal{R} \). The set of natural numbers \( \mathbb{N} \) will be assumed not to contain
0, and a positive real number will be assumed to be strictly greater than zero. A
natural number \( n \) will be regarded as a finite ordinal and hence identified with the
set \( \{0, 1, \ldots, n-1\} \) of its predecessors.

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2. AUTOMORPHISMS OF COUNTABLY SATURATED II$_1$ FACTORS

In this section we make use of terminology and results from the logic for metric structures. (An introduction to this subject can be found in [1].) In particular we consider II$_1$ factors as structures in the language of tracial von Neumann algebras, as described in [8, Section 2.3.2].

Suppose that $M$ is a II$_1$ factor, $\beta$ is an automorphism of $M$, and $a$ is a normal element in the unit ball of $M$. Approximating $a$ by normal elements with finite spectrum, it is easy to see that there is a sequence $(u_n)_{n\in\mathbb{N}}$ of unitary elements of $M$ such that

$$\lim_{n \to +\infty} \|\beta(a) - u_n au_n^*\|_2 = 0$$

where $\|\cdot\|_2$ is the Hilbert-Schmidt norm associated with the unique trace of $M$. Thus the formula (with parameters)

$$\psi(z) \equiv \|\beta(a) - zaz^*\|_2 + \|zz^* - 1\|_2 + \|zz^* - 1\|_2$$

is approximately realized in $M$, i.e. for every positive real number $\varepsilon$ there is an element $u$ in the unit ball of $M$ such that $\psi(u) < \varepsilon$. If moreover $M$ is countably saturated as in [8, Section 4.4], then the formula $\psi$ is actually realized in $M$, i.e. there is a (necessarily unitary) element $u$ of $M$ such that $\psi(u) = 0$ and hence

$$uau^* = \beta(a).$$

This concludes the proof of Proposition 1.

**Proposition 1.** Suppose that $M$ is a countably saturated II$_1$ factor, and $\beta$ is an automorphism of $M$. If $a$ is a normal element of $M$, then $a$ and $\beta(a)$ are conjugate by a unitary element of $M$.

Recall that by [8, Proposition 4.11] an ultraproduct of a sequence of tracial von Neumann algebras with respect to a nonprincipal ultrafilter is countably saturated.

Proposition 1 can also be proved using methods and results from [15]. In fact it is not difficult to see that the same proof as in [15] shows that Theorem 3.1 and Corollary 3.6 from [15] hold not only for ultrapowers of II$_1$ factors, but also more generally for any countably saturated II$_1$ factor. If now $a$ is a normal element of $M$, and $A$ is the C*-subalgebra of $M$ generated by $a$, then $A$ is abelian and, in particular, nuclear. It therefore follows from the described generalization of [15, Corollary 3.6] that the restriction of $\beta$ to $A$ coincides with the restriction of some inner automorphism of $M$. In particular $a$ and $\beta(a)$ are conjugate by a unitary element of $M$.

3. AUTOMORPHISMS OF U(R$^d$)

The main result of this section is Theorem 1. Recall that an automorphism $\alpha$ of a group $G$ is class-preserving if for every $x \in G$ the elements $x$ and $\alpha(x)$ of $G$ are conjugate. The class-preserving automorphisms of $G$ form a normal subgroup of the automorphisms group of $G$.

**Theorem 1.** The group of class-preserving automorphisms of $U(R^d)$ has index 2 in the automorphisms group of $U(R^d)$. Moreover if $\alpha$ is any automorphism of $U(R^d)$ then $\alpha$ is an isometry with respect to the distance induced by the Hilbert-Schmidt norm on $R^d$. The same statement holds for automorphisms of the ultraproduct $\prod U_n$ of the sequence of unitary groups endowed with the normalized Hilbert-Schmidt metric.
Identify $\mathcal{R}$ with the von Neumann algebra tensor product of infinitely many copies of $M_2$. If $a \in M_n$ define $\overline{a}$ the element of $M_2$ obtained replacing every entry of $a$ with the corresponding complex conjugate. It is immediate to verify that the function $a \mapsto \overline{a}$ is a conjugate linear automorphism of $M_2$. Moreover the function $a_0 \otimes \cdots \otimes a_{n-1} \mapsto \overline{a}_0 \cdot \cdots \cdot \overline{a}_{n-1}$ induces a conjugate linear automorphism of $\mathcal{R}$. Passing to the ultrapower one obtains a conjugate linear automorphism $\gamma$ of $\mathcal{R}^U$ preserving the Hilbert-Schmidt norm. The restriction of $\gamma$ to $U(\mathcal{R}^U)$ defines an automorphism of $U(\mathcal{R}^U)$ that is not class-preserving but it is an isometry with respect to the distance induced by the Hilbert-Schmidt norm on $\mathcal{R}^U$. We will show in the following that if $\alpha$ is any automorphism of $U(\mathcal{R}^U)$, then either $\alpha$ or $\gamma \circ \alpha$ is class-preserving. This in particular will show that $\alpha$ is an isometry with respect to the distance induced by the Hilbert-Schmidt norm on $\mathcal{R}^U$.

Suppose that $\alpha$ is an automorphism of $U(\mathcal{R}^U)$. Recall that by cornerstone results of Dixmier-Lance [5] and McDuff [12] an ultrapower of a II$_1$ factor is a II$_1$ factor (this also follows from the fact that the class of II$_1$ factors is axiomatizable, see [1] Proposition 3.4). Thus $\mathcal{R}^U$ is a II$_1$ factor, which is moreover countably saturated by [8] Proposition 4.11. By Theorem 2 from [6] together with the result of Broise from [3] that the unitary group of a II$_1$ factor does not have characters, there is a linear or conjugate linear *-isomorphism $\beta : \mathcal{R}^U \to \mathcal{R}^U$ whose restriction to $U(\mathcal{R}^U)$ is $\alpha$. If $\beta$ is a linear *-isomorphism then by Proposition 1 for every $u \in U(\mathcal{R}^U)$ there is $v \in U(\mathcal{R}^U)$ such that $vuv^* = \beta(u) = \alpha(u)$.

and hence $\alpha$ is class-preserving. If $\beta$ is a conjugate linear *-isomorphism, then $\gamma \circ \beta$ is a linear *-isomorphism whose restriction to $U(\mathcal{R}^U)$ is $\gamma \circ \alpha$. Reasoning as before one concludes that $\gamma \circ \alpha$ is class-preserving. This concludes the proof Theorem 1.

In order to prove the analogous statement for $\prod_n U_n$ observe that $\prod_n U_n$ can be identified with the unitary group of $\prod_n M_n$ by [10] Proposition 2.1, see also [11] Exercise II.9.6. One can thus run the same argument where $\mathcal{R}^U$ is replaced by $\prod_n M_n$ which is again a countably saturated II$_1$ factor by [8] Proposition 4.11.

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