Anomalous spacings of the CMB temperature angular power spectrum

Md Ishaque Khan and Rajib Saha
Indian Institute of Science Education and Research (IISER) Bhopal, Madhya Pradesh, India, 462066

We propose a novel technique to analyse spacings of the observed CMB temperature angular power spectrum (APS) motivated by the behaviour of level spacings in the presence or absence of correlations between eigenvalues of a random matrix. We use data from WMAP 9 year ILC and 2018 Planck maps (Commander, NILC and SMICA) and estimate minimum, maximum, average spacings and ratios of maximum to minimum spacings between consecutive multipoles of APS measures \( C_{\ell} \) and \( \frac{\ell(\ell+1)}{2\pi} C_{\ell} \). Anomalous multipole ranges given by \( \ell_{\text{max}} \) and \( \ell_{\text{min}} \) exist for multipoles \( \ell \in [2,31] \) of this work. Sans parity distinctions, average spacing of 8 \( \leq \ell \leq 31 \) for \( C_{\ell} \)'s, and maximum and average spacings of \( \ell_{\text{min}} \in [8,12] \) of \( \frac{\ell(\ell+1)}{2\pi} C_{\ell} \)'s are anomalously low for all maps, with most anomalously low maximum spacing at 99.93% C.L. for \( \frac{\ell(\ell+1)}{2\pi} C_{\ell} \) (Planck NILC). With parity distinctions, even multipoles indicate anomalously low maximum and average spacings relative to odd multipoles, most consistently for \( \ell_{\text{max}} \in [6,30] \) of \( C_{\ell} \)'s of all maps, the most outstanding being the even multipole average spacing of 2 \( \leq \ell \leq 27 \) for \( \frac{\ell(\ell+1)}{2\pi} C_{\ell} \) (WMAP), at 99.60% C.L. For spacing ratios, multipole ranges are similar to those for anomalous spacings, and odd multipole spacing ratios are mostly anomalous, with odd multipole spacing ratio of 2 \( \leq \ell \leq 7 \) being consistently low for all maps.

Overall, we infer an unusual deficit of correlations between APS measures and deviations from isotropy due to consistently low, parity-biased anomalous spacings, which can be argued to have arisen primordially. The physical origin of these findings may be probed in future work.

I. INTRODUCTION

The investigation of anomalous signatures in the observed Cosmic Microwave Background (CMB) data in the range of low multipoles is important as these correspond to large scales on the CMB sky, which potentially encode information about inflation [1], and hence any signatures that might have arisen primordially, or even some local effects like foregrounds. Because the theory of the \( \Lambda CDM \) model, also known as the Standard Model of Cosmology, does not incorporate the concept of foregrounds in the observed CMB, hence it is also imperative to compare theoretical simulations with observed data of cleaned maps, to see if anomalous signatures may correspond to some remnants of foregrounds even after CMB maps have been purged of the same [2]. This is further necessary because the CMB is the rudimentary source for estimating cosmological parameters that constrain the Standard Model and shape our understanding of the early developments of the universe [3, 4]. For, if these parameters are not determined accurately, we may lack in our understanding of primordial effects or foreground residuals, that may then manifest as CMB anomalies.

In this work, we propose a novel method, that has not been explored in existing literature, to detect anomalies in the CMB. The method investigates the spacings between the CMB angular power spectrum (APS) measures, namely \( C_{\ell} \)'s and \( \frac{\ell(\ell+1)}{2\pi} C_{\ell} \)'s, which must be probed for any unexpected or unusual, and hence, anomalous forms of correlations between them. This makes it interesting to study an analogue of ‘level spacings’ in the form of absolute differences of the consecutive multipole APS measures of low multipoles (\( \ell \in [2,31] \) in this work), given by,

\[
\begin{align*}
    s_A &= |C_{\ell_1} - C_{\ell_2}|, \\
    s_B &= \left| \frac{\ell_1(\ell_1+1)}{2\pi} C_{\ell_1} - \frac{\ell_2(\ell_2+1)}{2\pi} C_{\ell_2} \right|, 
\end{align*}
\]

(1)

where, \( C_{\ell} \)'s are the unbiased APS estimators defined as [5],

\[
C_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2, \quad \ell \geq 2 \tag{2}
\]

and, the \( a_{\ell m} \)'s are coefficients of the expansion,

\[
\Delta T(\hat{n}) = \sum_{\ell} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{n}), \tag{3}
\]

\( \Delta T(\hat{n}) \)'s being the temperature anisotropies relative to the uniform mean temperature of nearly 2.726K [6] of the CMB, where \( \hat{n} \) denotes a direction in the sky.

Because, \( C_{\ell} \)'s arise from uncorrelated Gaussian random \( a_{\ell m} \)'s (2), these should also be uncorrelated, and so should \( \frac{\ell(\ell+1)}{2\pi} C_{\ell} \)'s, which differ only by a multiplicative factor. Therefore, low spacings (level clustering) should be favoured because of Poisson statistics [7]. On the other hand, if the observed \( C_{\ell} \)'s were specially correlated, the Wigner-Dyson statistics [8] (favouring level repulsion) would effectively describe these spacings, and preclude the possibility of consecutive \( C_{\ell} \)'s and \( \frac{\ell(\ell+1)}{2\pi} C_{\ell} \)'s being arbitrarily close. Thus, by comparing spacings from observed data with theoretical simulations, we seek to discover any unexpected signatures of the presence or absence of correlations of consecutive multipole APS measures. In order to investigate such anomalous behaviour, we employ four estimators (section II).
As aforementioned, Poisson statistics (describing a possibly integrable underlying system) and Wigner-Dyson statistics (describing quantum systems with classically chaotic or non-integrable counterparts) can be represented as in equation (4) for a spacing \( s \) between two consecutive eigenvalues of the associated random matrix, with a probability distribution \( p(s) \) \cite{9}.

\[
p(s) = e^{-s} \quad \text{(Poisson statistics)},
\]

\[
p(s) = a_\alpha s^\alpha e^{-bs^2} \quad \text{(Wigner-Dyson statistics), (4)}
\]

(where, \( \alpha > 0 \)), which implies that for classically chaotic systems, very low spacings may not be favoured whereas for classically integrable systems, very low spacings are highly favoured. These behaviours are called level repulsion and level clustering, and are seen for special correlations between eigenvalues or a lack thereof, respectively.

CMB \( C_\ell \)'s of pure (\( \Lambda CDM \)) CMB maps should lack correlations and hence a spacing distribution must obey Poisson statistics, whereas, an addition of foregrounds is expected to enhance correlations that should cause the spacing distribution to obey Wigner-Dyson statistics. To corroborate this, we have plotted the spacing distribution of \(|C_8 - C_9|\) (this choice is not special, and similar plots were also seen to occur for all other multipole spacings of both \( s_A, s_B \) of (1)). We have used 5000 simulations of the theoretical APS (best fitted to Planck 2018 data) as pure CMB maps to which small amounts of three foreground maps (synchrotron, dust and free-free emission of Planck Commander from its Legacy archive \cite{10}) were added to illustrate the change in the spacing distribution from Poisson to Wigner-Dyson statistics in figure 1.

Due to the parity inversion property of spherical harmonics, i.e., \( \hat{n} \rightarrow -\hat{n}, \ Y_{\ell m}(-\hat{n}) = (-1)^{\ell}\ Y_{\ell m}(\hat{n}) \Rightarrow a_{\ell m} \rightarrow (-1)^{\ell}a_{\ell m} \), the temperature anisotropy field can be reconstructed as a sum of a symmetric and an anti-symmetric function of even and odd parity and the power for a multipole range can be rewritten as a sum of contributions from the even and odd multipole APS. Hence, we also study nearest neighbour spacings of separate sets of even and odd multipole APS measures in addition to an analysis of spacings of the APS measures without a parity distinction, so as to find any parity preference of anomalous signatures of spacings, which can in principle indicate deviations from the assumed isotropy of the universe on large scales.

In the past, CMB data analysis unveiled many anomalies in the observed patterns of WMAP \cite{11} and Planck \cite{12} satellite missions. This has necessitated the study of anomalies in both WMAP and Planck maps to make discrepancies more discernible if any such exist. Such anomalies include the north-south power asymmetry \cite{13}, which is relatively insignificant for all scales from Planck \cite{14}, the anomalous cold spot \cite{15–17}, high degree of octupole-quadrupole alignment \cite{18–20} that gets enhanced on removing the frequency dependent kinetic Doppler quadrupole \cite{21}, quadrupole power deficit \cite{22, 23} and planarity of the octupole \cite{24}, the power excess for lower odd multipoles \cite{25}, generalised as ‘parity asymmetry’ \cite{26–28}, and the like. Some of the latter ones concerning the quadrupole and octupole have been somewhat mitigated using a hypothetical foreground reduction \cite{29}.

Our study here focuses on an unprecedented analysis in an effort to discover unusual signatures of correlations or their deficit between consecutive multipole APS as manifested by anomalous spacings of the APS. Our paper is organised as follows: Section II specifies the four estimators used for our study. Section III describes our way of testing consecutive multipole spacings with simulations. We report anomalies in section IV, for all multipole and even/odd multipole spacings followed by anomalous ratios of the maximum to minimum spacings. And, in section V, we summarise these results, deduce any general trends in anomalies and opine on apparent discrepancies.

![FIG. 1. Distribution of the \(|C_8 - C_9|\) spacing of the pure CMB map without any foreground addition (in purple) and with addition of foreground maps (in green, light blue and ochre): \(10^{-4}\) times for each of the synchrotron and free-free emission maps and \(10^{-3}\) times the thermal dust map. Each of the foreground maps were added after appropriate conversions of antenna to thermodynamic temperature. The purple histogram resembles a Poisson distribution whereas the other coloured histograms (with added foregrounds) are seen to obey some appropriate Wigner-Dyson statistics (4). Similar behaviours are also seen to be exhibited for all other multipole spacings of the APS as in (1). It can be shown that the lower are the multiplicative factors in orders of 10 below \(10^{-4}, 10^{-3}\) respectively, the more coincident are all histograms with the one for pure CMB.](image)
II. ESTIMATORS

In this article, we present a new idea of investigating CMB temperature anomalies. We propose a novel set of estimators for this purpose, based on the APS of the observed CMB temperature maps. Such estimators have not been spoken of in existing literature but can be understood in terms of the absolute differences or spacings between any two consecutive $C_\ell$'s and $i(\ell+1)/2\pi C_\ell$'s. There is a need for such estimators since it is not possibly meaningful to compare all such spacings from data individually with theoretical simulations.

For a given range of multipoles $[\ell_{\text{min}}, \ell_{\text{max}} + 1]$, the spacings between consecutive $C_\ell$'s and $i(\ell+1)/2\pi C_\ell$'s were estimated. Say, such consecutive multipole spacings are $[\Delta_1, \Delta_2, ...]$, then four estimators to detect anomalies are

\[
\begin{align*}
\text{max}_i &= \max[\Delta_1, \Delta_2, ...], \\
\text{min}_i &= \min[\Delta_1, \Delta_2, ...], \\
\text{avg}_i &= \frac{\Delta_1 + \Delta_2 + ...}{N}, \\
r_i &= \frac{\text{max}_i}{\text{min}_i},
\end{align*}
\]

which stand for the maximum, minimum, average of a set of spacings and the ratio of the maximum to minimum spacings, respectively. Here, $i = a, o, e$, where $a, o, e$ stand for all, odd and even multipoles, and $N$ = number of spacings for a given range of $\ell$'s.

These estimators have been proposed here as exhaustive probes of anomalous spacings, because the lower and upper limits ($\ell_{\text{min}}, \ell_{\text{max}}$ respectively) of the multipole range have also been varied appropriately for all the analyses. Being extreme statistical measures, these estimators (excluding the average measure) represent rare occurrences of certain values of the nearest neighbour spacings, yet along with the average or mean spacing and the ratio of the maximum to minimum of a multipole range, these should characterise anomalous behaviours of spacings in an easily quantifiable way. For reference, we have included the plots of the two natural APS measures, $C_\ell$’s and $i(\ell+1)/2\pi C_\ell$’s, with multipoles in figure 2.

III. METHODOLOGY

We have estimated discrepancies relative to the theoretical APS (Planck 2018 best fit) with the help of the HEALPix [30] package, by considering $10^4$ simulations of the theoretical temperature APS to account for statistical fluctuations while studying the four estimators (section II) of the spacings between consecutive $C_\ell$’s and $i(\ell+1)/2\pi C_\ell$’s.

Observed data maps used here for probing anomalous spacings are full-sky cleaned maps of WMAP 9 year ILC and 2018 release full mission Planck Commander (COMM), NILC and SMICA maps from latest sources, i.e., [31] and [10], respectively. These have been downgraded with the help of HEALPix [30] software facilities to a HEALPix $n_{\text{side}} = 16, n_{\text{max}} = 32$ (hence, an appropriate pixel_window) with no beam smoothing (fwhm_arcmn = 0.0d0) and the simulations are obtained using the same resolution. With all data maps, the theoretical power spectrum, and simulations thereof on an equal footing, we have proceeded with the analysis. We have excluded multipoles $\ell = 0, 1$ as these correspond respectively to the monopole of uniform CMB temperature ($\approx 2.726 K$) [6], and the dipole which arises due to our peculiar motion relative to the CMB rest frame [32].

We have taken the minimum ($\text{min}_i$), maximum ($\text{max}_i$), average ($\text{avg}_i$) spacing and the ratio ($r_i$) of the maximum to the minimum spacings of the $C_\ell$’s and $i(\ell+1)/2\pi C_\ell$’s for consecutive multipoles firstly without any distinction between odd or even ones and later separately for odd/even multipoles to study anomalies that arise upon such parity based distinctions. Here, a spacing means the absolute value of the difference between two consecutive $C_\ell$’s and $i(\ell+1)/2\pi C_\ell$’s. For example, for the 6 multipoles in the range $[2, 7]$ for $C_\ell$’s, we have 5 spacings with no parity based distinction namely, $|C_2 - C_3|, |C_3 - C_4|, ..., |C_6 - C_7|$. Whereas for even and odd multipoles taken distinctly, in the same range, we have 2 spacings each for even and odd multipoles namely, $|C_2 - C_4|, |C_4 - C_6|$, and $|C_3 - C_5|, |C_5 - C_7|$ respectively. Spacings for consecutive $i(\ell+1)/2\pi C_\ell$’s are found in a similar fashion.

For characterising the extent to which a spacing may be anomalous in a quantitative way, we define the following probabilities:

\[
\begin{align*}
\text{s-value} & : P^s(x), \\
\text{p-value} & : P^p(x) = 1 - P^s(x),
\end{align*}
\]

where,

\[
x = (\text{min}_i, \text{max}_i, \text{avg}_i, r_i),
\]

\[
i = a, o, e,
\]

FIG. 2. $C_\ell$ and $i(\ell+1)/2\pi C_\ell$ versus $\ell$, the analysed APS : The plot compares the theoretical best fit to Planck (purple) with COMM (green), NILC (light blue), SMICA (ochre) and WMAP (yellow) maps used here, all equivalently with zero smoothing but with a pixel window application appropriate for $n_{\text{side}} = 16$. The monopole and dipole have been excluded in this figure. The vertical axis is in log-scale.
and $a, o, e$ as before, stand for all (no parity based distinction), odd and even multipoles respectively.

$P^t(x)$ (hereafter referred to as the ‘s-value’) is the probability of the theoretical spectrum having a value greater than a data map for the four entities, that are, the minimum, maximum, average spacing and the ratio of the maximum to minimum spacings, respectively, denoted by $x$. This is calculated by counting the number of simulations having these entities greater than the observed map and dividing the number by the total number of simulations, that being $10^4$ throughout this paper (except for the plots in figures 1 and 9 which were made using 5000 simulations).

$P^b(x)$ (referred to as the ‘p-value’) is similarly defined as the probability of simulations having a value of the quantity $x$ less than that from observational data. Probability plots for $P^b$ are in log-scale along the vertical axis. Plots for $P^t$ are not in log-scale. All probability plots mention 0.05, 0.95 (in dotted red lines) corresponding to the 5%–95% confidence range of hypothesis testing methodology [33–36]. Therefore all probabilities which feature below 5% or above 95% have been reported here. A considerable discourse against usage of such confidence intervals is given by [37] or that against the often assumed complementarity of the s- and p-values is presented in the work of [38].

As may be intuitively obvious from figure 2, the maximum and minimum spacings for $C_\ell$’s and $(\ell+1)C_\ell$’s may lie in any of the lower or higher multipole ranges and hence it becomes necessary to scan the multipole ranges by changing not only the upper bound of the multipole range ($\ell_{\text{max}}$, whilst keeping the lower bound fixed), but also the lower bound ($\ell_{\text{min}}$, whilst keeping the upper bound fixed). Also considering not just the minimum and maximum spacings, but also computing the average value of a set of spacings for a given $\ell_{\text{max}}$ or $\ell_{\text{min}}$ for the same reason, is needed.

We have, therefore, calculated the maximum, minimum and average spacings by varying $\ell_{\text{max}}$’s and $\ell_{\text{min}}$’s. So, for example, multipoles in the range [2, 7] will correspond to an $\ell_{\text{max}} = 6$ where the $\ell_{\text{min}}$ is fixed at $\ell_{\text{min}} = 2$. A similar scheme holds for the $\ell_{\text{min}}$’s which have been varied keeping $\ell_{\text{max}}$ fixed at $\ell_{\text{max}} = 30$. $\ell = 0, 1$ have already been excluded for reasons aforementioned. Also having a maximum upper limit of $\ell_{\text{max}} = 30$, implies that we are looking for anomalies in the range [2, 31]. This invariably means that $\ell = 32$ is excluded from our analysis. This is because, although including $\ell = 32$ would not have affected our results for all spacings (no parity distinctions), but, when we scan for anomalies after taking odd and even multipole spacings distinctly, we should have an equal number of odd and even spacings for any chosen range of multipoles.

Hence with this forethought, $\ell = 32$ is excluded not only for the analysis with parity based distinction but also when all multipoles are treated without distinction, so that comparisons of results between the two analyses later on seem legitimate. This is ensured further by choosing only even numbers for $\ell_{\text{min}}$’s and $\ell_{\text{max}}$’s. Thus, $\ell_{\text{max}}$’s have been varied from 6 → 30 (by fixing $\ell_{\text{min}} = 2$) in steps of $\Delta \ell = 2$ for all spacings and $\ell_{\text{min}}$’s have been varied from 2 → 26 (having fixed $\ell_{\text{max}} = 30$) in steps of $\Delta \ell = 2$ for all spacings. Similar variations of $\ell_{\text{min}}$ and $\ell_{\text{max}}$ have been made for odd and even multipole spacings when those are considered individually, but those are varied in steps of $\Delta \ell = 4$, for, that being the most sensible choice, keeps at least two spacings of odd and even multipoles for comparison when maximum and minimum among the set of spacings for a given $\ell_{\text{max}}$ or $\ell_{\text{min}}$ are computed. Thus, there are 13 choices each for $\ell_{\text{min}}, \ell_{\text{max}}$ when all multipole spacings are considered, and 7 choices each, when even and odd multipole sets are considered separately. Of these, $\ell_{\text{min}} = 2$ and $\ell_{\text{max}} = 30$ correspond to the same range, i.e., $2 \leq \ell \leq 31$. Hence, in all, there are 25 and 13 independent ranges of multipoles, respectively, for all and even/odd multipole sets.

IV. RESULTS

A. Spacings

We present the $\ell_{\text{min}}$ or $\ell_{\text{max}}$ ranges of the four CMB temperature anisotropy data sources which are anomalous along with their respective probabilities $P^t$ and/or $P^b$.

1. All multipoles in [2, 31]

For the entire set of multipoles in [2, 31], the anomalous spacings are presented here.

1. $C_\ell$’s: When considering all spacings, anomalies arise when $\ell_{\text{min}}$ is varied for all four maps. For a variation of $\ell_{\text{max}}$ however, only WMAP shows anomalies (See figure 3). The most anomalous case for COM is for $\ell_{\text{min}} = 8$ where the s-value of the average spacing is 99.1%, and for the maximum spacing, it is for $\ell_{\text{min}} = 16, 22$ and the higher s-value is 98.46% for $\ell_{\text{min}} = 22$. For NILC, $\ell_{\text{min}} = 8, 12, 20, 22$ are anomalous, with the highest s-value being 99.20% for $\ell_{\text{min}} = 8$ for the maximum spacing. Others are $\ell_{\text{min}} = 8, 12, 18, 20, 22$ for avg$_a$, the highest among them being $\ell_{\text{min}} = 20$ at 98.40%. For SMICA, the maximum spacing is anomalous for $\ell_{\text{min}} = 8, 12, 20$, with its highest s-value for $\ell_{\text{min}} = 8$ at 99.15%, also, for its average spacing, anomalous s-values are for $\ell_{\text{min}} = 8, 12, 20, 22$, but is highest for $\ell_{\text{min}} = 8$, again, at 98.94%. For WMAP, the minimum spacing turns up as being anomalously low for $\ell_{\text{min}} = 18, 20, 22, 24$, $\ell_{\text{max}} = 24, 26, 28$, a trend that is unprecedented; the highest s-value for this case being, 97.46% for $\ell_{\text{max}} = 24$. SMICA exhibits a maximal s-value of 99.02% for $\ell_{\text{min}} = 8$ for max$_a$, other ranges...
are $\ell_{\text{min}} = 10, 20$. For $\text{avg}_a$, SMICA shows up $\ell_{\text{min}} = 8$ as most anomalous again, with s-value 99.24%, other ranges are $\ell_{\text{min}} = 10, 20, 22$.

2. $\frac{\ell(\ell+1)}{2\pi} C_{\ell}$’s: In this case too, anomalies arise mainly for variations of $\ell_{\text{min}}$ for all four maps, save NILC, which also shows anomalies for certain $\ell_{\text{max}}$’s (See figure 4). For COMM, there is an entire range of values of $\ell_{\text{min}} = 6 \rightarrow 16$, $\ell_{\text{min}} = 6 \rightarrow 14, 20$ which is anomalous for the maximum and average spacings, respectively. The most anomalous is $\ell_{\text{min}} = 8$ again, with s-values of 99.89% and 98.99% for $\text{max}_e, \text{avg}_a$, respectively. NILC has an anomalous range of $\ell_{\text{max}} = 14 \rightarrow 18$ for the minimum spacing, with $\ell_{\text{max}} = 14$ being most anomalous with an s-value of 96.66%. $\ell_{\text{min}} = 8 \rightarrow 16$ is anomalous for the maximum spacing with highest s-value of 99.93% for $\ell_{\text{min}} = 8$ for NLC. $\ell_{\text{min}} = 6 \rightarrow 14, 18 \rightarrow 22$ are anomalous for average spacings from NILC with maximum s-value for $\ell_{\text{min}} = 8$, which is 99.61%. For SMICA the anomalous ranges are $\ell_{\text{min}} = 8 \rightarrow 12$ for $\text{max}_a$ and $\ell_{\text{min}} = 6 \rightarrow 12$ for $\text{avg}_a$. The most probable is $\ell_{\text{min}} = 8$ again for both kinds of spacings, with s-values of 99.19%, 99.44% for $\text{max}_a, \text{avg}_a$ respectively. For WMAP, again the most probable is $\ell_{\text{min}} = 8$ for both maximum and average spacings. The s-values of WMAP for these are 99.51% and 99.44% respectively. Other anomalous ranges are $\ell_{\text{min}} = 10 \rightarrow 14$ for $\text{max}_a$, $\ell_{\text{min}} = 10, 12, 20$ for $\text{avg}_a$.

2. Odd and even multipoles in $[2, 31]$

On treating $C_{\ell}$’s and $\frac{\ell(\ell+1)}{2\pi} C_{\ell}$’s differently based on parity of the multipoles, some novel anomalies come up. For example, the minimum spacing for a certain $\ell_{\text{max}}$ or $\ell_{\text{min}}$ shows anomalous behaviour in some ranges with a much greater s-values relative to the case when all spacings regardless of parity were being treated without distinction. Also many a time, $C_{\ell}$’s or $\frac{\ell(\ell+1)}{2\pi} C_{\ell}$’s of a certain parity may only show up in anomalous ranges, whereas those of a different parity remain quiescent.

1. $C_{\ell}$’s: Mostly the maximum and average spacings in even multipoles (i.e., $\text{max}_a$ and $\text{avg}_e$) show anomalous behaviour for all four maps, but at times, the minimum spacing among odd multipoles ($\text{min}_o$) also shows some anomalies (see figure 5). Some extremely consistent ranges of multipoles show up for all four maps, for $\text{max}_e, \text{avg}_e$, as given by $\ell_{\text{min}} = 2$ and $\ell_{\text{max}} = 10 \rightarrow 30$.

COMM shows an anomaly for $\ell_{\text{max}} = 10$, for $\text{min}_o$, with an s-value of 97.55%. For $\text{max}_e, \text{avg}_e$, the entire range of $\ell_{\text{max}}$’s from $\ell_{\text{max}} = 6 \rightarrow 30$ is anomalous with highest s-values of 97.66% for $\ell_{\text{max}} = 10 \rightarrow 30$ for $\text{max}_e$ and 99.37% for $\ell_{\text{max}} = 30$ or $\ell_{\text{min}} = 2$ for $\text{avg}_e$.  

---

**FIG. 3.** Probabilities $P^t$ for $\text{min}_o, \text{max}_a, \text{avg}_a$ (in purple, green and light blue, respectively) for $C_{\ell}$’s versus $\ell_{\text{max}}$ or $\ell_{\text{min}}$ as mentioned in parentheses () beside the name of the data map. Red dotted-dashed lines indicate 5% (double dots) and 95% (single dots).

**FIG. 4.** Probabilities $P^t$ for $\text{min}_o, \text{max}_a, \text{avg}_a$ (in purple, green and light blue, respectively) for $\frac{\ell(\ell+1)}{2\pi} C_{\ell}$’s versus $\ell_{\text{max}}$ or $\ell_{\text{min}}$ as mentioned in parentheses () beside the name of the data map. Red dotted-dashed lines indicate 5% (double dots) and 95% (single dots).
For NILC, the entire range of $\ell_{\max} = 6 \rightarrow 30$ is anomalous for maximum and average spacings among the even multipoles reaching the maximum $s$-values of 95.80% for $\ell_{\max} = 10 \rightarrow 30$ for $\max_e$, and 98.93% for $\avg_e$ for $\ell_{\max} = 26, 30$. Again, only for $\ell_{\min} = 18, 22$, $\avg_e$ is anomalous for NILC, with a higher $s$-value of 97.33% for $\ell_{\min} = 22$.

SMICA shows anomalous $\min_o$, most anomalously for $\ell_{\min} = 26$ at 98.69%. Other anomalous range are $\ell_{\min} = 2, 6, 10 \rightarrow 22$. Again, the entire range of $\ell_{\max} = 6 \rightarrow 30$ is anomalous for both $\max_e$ and $\avg_e$, with maximal $s$-value of 96.68% for $\max_e$ for $\ell_{\max} = 10 \rightarrow 30$, and that of 99.29% for $\ell_{\max} = 30$ for $\avg_e$. Again, only for $\ell_{\min} = 18, 22$, $\avg_e$ is anomalous for SMICA, with a higher $s$-value of 96.91% for $\ell_{\min} = 22$.

For WMAP, $\ell_{\max} = 14, 18$ are anomalous for $\min_o$ with $\ell_{\max} = 14$ being most anomalous with a $s$-value of 98.11%. Again, $\ell_{\max} = 6 \rightarrow 30$ are all anomalous for $\max_e$ and $\avg_e$ for WMAP, with a maximal $s$-value of 96.61% for $\ell_{\max} = 10 \rightarrow 30$ for $\max_e$, and that of 98.93% for $\ell_{\max} = 26$ for $\avg_e$. 

FIG. 5. Probabilities $P^t$ for $\min_o$, $\max_o$, $\avg_o$, $\min_e$, $\max_e$, $\avg_e$ (in purple, green, light blue, ochre, yellow and dark blue) for $C_\ell$’s versus $\ell_{\max}$ or $\ell_{\min}$ as mentioned in parentheses () beside the name of the data map. Red dotted-dashed lines indicate 5% (double dots) and 95% (single dots).

FIG. 6. Probabilities $P^t$ for $\min_o$, $\max_o$, $\avg_o$, $\min_e$, $\max_e$, $\avg_e$ (in purple, green, light blue, ochre, yellow and dark blue) for $\ell(\ell+1)C_\ell$’s versus $\ell_{\max}$ or $\ell_{\min}$ as mentioned in parentheses () beside the name of the data map. Red dotted-dashed lines indicate 5% (double dots) and 95% (single dots).
2. $\frac{\ell(\ell+1)}{2\pi} C_\ell$'s: For COMM, there are no anomalies. For plots, see figure 6. For $\ell_{\text{max}}$ variations, only average spacings of even multipoles reveal anomalous ranges, which are $\ell_{\text{max}} = 22 \rightarrow 30$ for NILC, $\ell_{\text{max}} = 18 \rightarrow 30$ for SMICA, and $\ell_{\text{max}} = 14 \rightarrow 30$ for WMAP. Their maximum s-values are 97.04%, 97.59% and 99.60% for $\ell_{\text{max}} = 26$ for the three maps, respectively.

For $\ell_{\text{min}}$ variations, $\ell_{\text{min}} = 6, 10, 14$ and $\ell_{\text{min}} = 2, 6, 10$ are anomalous for $\text{max}_{e}$ and $\text{avg}_{e}$ for NILC, with maximum s-values of 98.43% for $\ell_{\text{min}} = 10$ and 97.06% for $\ell_{\text{min}} = 6$, respectively. For SMICA, $\ell_{\text{min}} = 10$ shows an anomaly with an s-value of 96.31% for $\text{max}_{e}$. For $\text{max}_{e}$, $\ell_{\text{min}} = 6, 10$ are anomalous with maximal s-value of 97.52% for $\ell_{\text{min}} = 6$, whereas for $\text{avg}_{e}$ for SMICA, $\ell_{\text{min}} = 2, 6$ are anomalous each with s-values of 96.46%. For WMAP, $\ell_{\text{min}} = 6$ for $\text{max}_{e}$ is anomalous with an s-value of 96.77%. Also, $\ell_{\text{min}} = 2, 6, 10$ are anomalous for $\text{avg}_{e}$ for WMAP, with a maximal s-value of 98.63% for $\ell_{\text{min}} = 6$.

B. Spacing ratios

In addition to checking solely for anomalous spacings, we also checked if the ratio ($r_i$) of the maximum to the minimum spacing (as defined in (5)) of a given range of multipoles is anomalous. The s-value plots of the most probably anomalous results are presented in figure 7, which show that SMICA is not problematic for spacing ratios sans parity distinction. Although the other three maps show anomalous ratios, yet COMM and WMAP most recurrently show anomalies with high s-values.

1. All multipoles in [2, 31]

1. $C_\ell$'s: For COMM, $\ell_{\text{min}} = 2, 8, 12 \rightarrow 18, 22$ are anomalous for $\text{rat}_{o}$, but the highest s-value is 98.51%. For NILC, $\ell_{\text{max}} = 12$ is anomalous for $\text{rat}_{o}$ at 95.41%. For WMAP, $\ell_{\text{min}} = 4, 6, 24$ have anomalously high $\text{rat}_{o}$, with maximally anomalous $\ell_{\text{min}} = 4$ at an s-value of 3.98%. Also for WMAP, $\ell_{\text{max}} = 22$ has anomalously low $\text{rat}_{o}$, at an s-value of 96.66%.

2. $\frac{\ell(\ell+1)}{2\pi} C_\ell$'s: For NILC, $\ell_{\text{max}} = 14, 16$ are anomalous with high $\text{rat}_{o}$, and $\ell_{\text{max}} = 14$ has the lower s-value of 3.61%. No anomalous ratios exist for other maps.

2. Odd and even multipoles in [2, 31]

1. $C_\ell$'s: For COMM, $\ell_{\text{max}} = 6, 10$ are anomalous for $\text{rat}_{o}$, with the higher s-value for $\ell_{\text{max}} = 6$, at 98.63%; $\ell_{\text{max}} = 26, 30$ are anomalous for $\text{rat}_{e}$, with the higher s-value for $\ell_{\text{max}} = 30$ at 98.24%. For NILC, $\ell_{\text{max}} = 6$ is anomalous for $\text{rat}_{o}, \ell_{\text{max}} = 22$ is anomalous for $\text{rat}_{e}$ at s-values of 98.29%, 97.30% respectively. SMICA exhibits anomalously high $\text{rat}_{o}$ in ranges of $\ell_{\text{min}} = 2 \rightarrow 22$, $\ell_{\text{max}} = 26$, with the most anomalous (lowest s-value) being for $\ell_{\text{min}} = 6$, at 1.47%. SMICA also exhibits an anomalously low $\text{rat}_{o}$ for $\ell_{\text{max}} = 6$ at 99.16%. SMICA has anomalously low $\text{rat}_{e}$'s for $\ell_{\text{max}} = 18, 22$, with the higher s-value for $\ell_{\text{max}} = 18$ at 96.90%. WMAP shows an anomalously low $\text{rat}_{o}$ for $\ell_{\text{max}} = 6$, at 95.08%, and an anomalously high $\text{rat}_{o}$ for $\ell_{\text{max}} = 14$ at an s-value of 3.3%. $\ell_{\text{max}} = 14, 18, 22, 30$ for WMAP have anomalously low $\text{rat}_{e}$'s with the highest s-value for $\ell_{\text{max}} = 14$ at 98.28%.

2. $\frac{\ell(\ell+1)}{2\pi} C_\ell$'s: Only NILC shows an anomalously low $\text{rat}_{o}$ for $\ell_{\text{min}} = 22$ at 95.29% and WMAP shows the same for $\ell_{\text{min}} = 26$ at 96.49%.

FIG. 7. Anomalous ratios of maximum to minimum spacings: Probabilities $P_\pi(r_i)$, $i = a, o$ versus $\ell_{\text{max}}$ or $\ell_{\text{min}}$ as mentioned in parentheses ( ). Top left: COMM, WMAP for $C_\ell$'s in purple and green, respectively. Top right: NILC, WMAP for $C_\ell$'s in purple and green, respectively; NILC for $\frac{\ell(\ell+1)}{2\pi} C_\ell$'s in light blue. Bottom left: SMICA ($r_o$) for $C_\ell$'s in purple; NILC ($r_o$), WMAP ($r_o$) for $\frac{\ell(\ell+1)}{2\pi} C_\ell$'s in green and light blue, respectively. Bottom right: COMM ($r_o$), NILC ($r_o$), SMICA ($r_o$), WMAP ($r_o$), COMM ($r_e$), NILC ($r_e$), SMICA ($r_e$), WMAP ($r_e$) for $C_\ell$'s in purple, green, light blue, ochre, yellow, dark blue, red, and black, respectively. Red dotted-dashed lines indicate 5% (double dots) and 95% (single dots).
TABLE I. \( \geq 3\sigma \) anomalies for all spacings (with \( \geq 3.25\sigma \) results highlighted)

| Data Map | \( \ell_{\text{min}} \) or \( \ell_{\text{max}} \) | Estimator | s-value \((P^t)\) |
|----------|-----------------|-----------|-----------------|
| COMM     | \( \ell_{\text{min}} = 8 \) \( \ell_{\text{max}} \) | maxa      | 99.89%          |
| NILC     | \( \ell_{\text{min}} = 8 \) \( \ell_{\text{max}} \) | maxa      | 99.93%          |
| NILC     | \( \ell_{\text{min}} = 10 \) \( \ell_{\text{max}} \) | maxa      | 99.73%          |

FIG. 8. Plots of p-values \( (P^t) \) versus \( \ell_{\text{min}} \) or \( \ell_{\text{max}} \) as indicated in parentheses (), for \( \geq 3\sigma \) anomalies for all and \( \geq 2.75\sigma \) for odd/even multipole spacings as mentioned. Left: COMM \( (\maxa) \) in purple, NILC \( (\maxa) \) in green. Right: WMAP \( (\avg) \) in purple. Dotted-dashed red lines (top to bottom) represent: 5\%\((\sim 2\sigma)\), 0.59\%\((\sim 2.75\sigma)\), 0.3\%\((\sim 3\sigma)\).

V. SUMMARY AND CONCLUSIONS

We have proposed a new technique for investigating anomalies in the CMB temperature APS for its nearest neighbour spacings inspired by the notion of level spacings of random matrices, and associated concepts of chaos (specially correlated eigenvalues) and integrability (uncorrelated eigenvalues), associated with Wigner-Dyson and Poisson statistics, respectively. We have analysed spacings between \( C_\ell \)'s and \( \tfrac{\ell(\ell+1)}{2\pi} C_\ell \)'s as analogous 'level spacings' with the help of four estimators (section II) and found that such anomalous spacings arise in various multipole ranges, across all four data maps, with a variation of the lower or upper limit of the multipole range. In this section, we will try to deliberate on some obvious trends.

To summarise section IV A, we further present the plots of p-values (in log-scale along the vertical axis), which are those of the probabilities \( P^t = 1 - P^t \). These plots give a better insight on the extent to which the observational data are deviant from the theoretical predictions, indicated by the lowermost dips in the plots of figure 8. These plots showcase only the highest s-values as presented in tables I and II.

For all multipole spacings, we realise that on restricting the anomalous ranges as those corresponding to or beyond 99.7\% \(( \geq 3\sigma \) confidence) as the s-value, (see table I), we can sieve out the maximally anomalous results. For all multipole spacings, the most anomalous \( \ell_{\text{min}} = 8 \), has a maximal probability for NILC \( (\tfrac{\ell(\ell+1)}{2\pi} C_\ell)'s \) of 99.93\% for its maximum spacing. This corresponds to a \( \approx 3.4\sigma \) deviant anomaly and is the highest (lowest) probability among all the s-(p-)values in all our analyses. Besides, for all other maps also, highest s-values exist for \( \ell_{\text{min}} = 8 \) for \( \avg \) for \( C_\ell \)'s and \( \ell_{\text{min}} = 8, 10, 12 \) for both \( \maxa, \avg \) for \( \tfrac{\ell(\ell+1)}{2\pi} C_\ell \)'s. So mainly, the maximum and average spacings turn out to be anomalously low for the complete set of multipoles with highest s-values (lowest p-values) for the aforementioned ranges. These anomalies at a confidence level \( \approx 3.25\sigma \) are highlighted in table I.

For odd/even multipole spacings, we present the most probable anomalies for \( \geq 2.75\sigma \) range (confidence range \( \geq 99.40\% \)) in table II. The general trend however, seems to hold for that of \( \maxa, \avg \) as being anomalously low for all maps for \( C_\ell \)'s, with highest s-values for \( \ell_{\text{max}} = 6 \rightarrow 30, \ell_{\text{min}} = 2 \) \((= \ell_{\text{max}} = 30)\). Further, for \( \tfrac{\ell(\ell+1)}{2\pi} C_\ell \)'s, three maps, namely, NILC, SMICA and WMAP exhibit quite consistently: anomalously low \( \maxa, \avg \) for \( \ell_{\text{min}} = 6 \) and \( \avg \) for \( \ell_{\text{min}} = 2, 6 \) and \( \ell_{\text{max}} = 22, 26, 30 \). It is also evident from figures 5 & 6, that, among odd multipole \( C_\ell \)'s and \( \tfrac{\ell(\ell+1)}{2\pi} C_\ell \)'s, \( \maxa, \avg, \min \), respective, are never anomalous.

Parity based distinction (section IV A 2) reveals that the even multipoles have a considerable tendency to keep their maximum and average spacings much smaller than what may be theoretically expected. Also a noticeable feature of the Planck maps (COMM, NILC, SMICA) is that without any parity based distinction, the minimum spacing does not usually show up as an anomaly. However, when the odd and even multipoles are considered separately, the minimum spacing for odd multipoles begins to show up as being anomalously lower than theoretical simulations (especially for a wide range of \( C_\ell \)'s for SMICA), which could be a possible cause of the high odd multipole

TABLE II. \( \geq 2.75\sigma \) anomalies for odd/even spacings

| Data Map | \( \ell_{\text{min}} \) or \( \ell_{\text{max}} \) | Estimator | s-value \((P^t)\) |
|----------|-----------------|-----------|-----------------|
| WMAP     | \( \ell_{\text{max}} = 22 \)              | avg       | 99.45%          |
| WMAP     | \( \ell_{\text{max}} = 26 \)              | avg       | 99.60%          |

TABLE III. Highly consistent ranges of anomalously low estimators exhibited by all four maps: COMM, NILC, SMICA & WMAP

| Spacings of \( \tfrac{\ell(\ell+1)}{2\pi} C_\ell \)'s | \( \ell_{\text{min}} \) or \( \ell_{\text{max}} \) | Estimator | s-value \((P^t)\) |
|-------------------------------------------------|-----------------|-----------|-----------------|
| \( \tfrac{\ell(\ell+1)}{2\pi} C_\ell \)'s       | \( \ell_{\text{min}} = 8 \) \( \ell_{\text{max}} = 10, 12 \) | maxa      | 99.39%          |
| \( \tfrac{\ell(\ell+1)}{2\pi} C_\ell \)'s       | \( \ell_{\text{min}} = 8 \) \( \ell_{\text{max}} = 10, 12 \) | avg       | 99.39%          |
| \( C_\ell \)'s                                  | \( \ell_{\text{max}} = 6 \rightarrow 30 \) | maxa      | 99.39%          |
| \( C_\ell \)'s                                  | \( \ell_{\text{max}} = 6 \rightarrow 30 \) | avg       | 99.39%          |
| \( C_\ell \)'s                                  | \( \ell_{\text{max}} = 6 \)                          | rat        | 99.39%          |
spacing ratio for SMICA (section IV B) in similar multipole ranges. On the contrary, the maximum of even spacings is almost always anomalously low and at relatively much higher deviation from the confidence interval as compared to odd multipole spacings.

Reasonably, since all four maps, namely, COMM, NILC, SMICA, and WMAP, have been obtained by using different foreground removal methods, the levels of foreground residuals in these maps are different [39–41]. These systematic differences are clearly visible if we subtract any two of these maps at low resolution. This can be a possible cause of minor differences in results obtained from differently cleaned maps [42].

However, as shown in table III, we find highly consistent (as for all four maps) anomalously low maximum spacing for $\ell \in [8, 31]$ with all multipoles of $C_\ell$’s and maximum and average spacings for $\ell \in [\ell_{\text{min}}, 31]$ ($\ell_{\text{min}} = 8, 10, 12$) with all multipoles of $\frac{\ell(\ell+1)}{2\pi} C_\ell$’s. With parity based distinction of multipoles, highly consistent anomalously low maximum and average spacings for $\ell \in [2, \ell_{\text{max}} + 1]$ ($\ell_{\text{max}} = 6, 10, 14, 18, 22, 26, 30$) with even multipoles of $C_\ell$’s are observed. These account for a probable level clustering or some other spacing distribution that favours low spacings, as opposed to a probable level repulsion for maximum and average spacings as seen from simulations of theoretical $C_\ell$’s (purple histograms in figure 9) for some underlying possibly anomalous integrable system. However, the actual observed data based APS measures ($C_\ell$’s and $\frac{\ell(\ell+1)}{2\pi} C_\ell$’s) may not be based on any such underlying analogous physical system, or the system could be a mixed system [43], or a transitory one between those of chaos and integrability [44, 45]. Because, as [46] say, the idea of quantum chaos is not well defined and go on to show for an exceptional case of integrable Hamiltonians that obey Wigner’s distribution of nearest neighbour eigenvalue spacings. Hence, the anomalies found could also be due to an exceptional case of our lack of understanding of chaotic systems vis a vis non-chaotic ones if not due to any unknown foreground based suppression of spacings.

Besides, as multipoles with distinct parities exhibit anomalous behaviour quite differently, in that, even multipole spacings are almost always anomalously low, this could hint at a primordial origin of these anomalies, such as that of an anisotropic Finsler spacetime model [47] with a correction term that lowers even multipole $C_\ell$’s, unless of course, a parity bias for low nearest neighbour spacings may be manifested in the foreground residuals or due to other systematics. However, such a model may not necessarily help explain some low spacings for odd multipoles as found in subsection IV A 2. A theoretical model that alters the correlations of primordial fluctuation modes [48], could also inspire some alternate work to shed light on the anomalies observed.

We must note that despite the fact that these results of anomalous behaviours of spacings and their ratios are specific to certain maps as aforementioned, yet, all the four maps show mostly consistent anomalous multipole ranges, especially as presented in table III. Coincidentally, these consistent results also majorly correspond to those with highest s-values of anomalously low spacings and spacing ratios. Because of the consistency of results for all four observed maps, as in table III, we can intuitively reason that these anomalies have a primordial origin (and are not due to foreground residuals), also because of the parity bias (unless foregrounds also show a parity bias). To lend support to this inference, we plot the spacing distributions of the anomalous estimators of table III in figure 9, for the pure (Planck 2018 best fit of $\Lambda$CDM) CMB maps and for pure CMB maps added with small amounts of 3 typical foreground maps, just as the earlier plots in figure 1. For sake of brevity, we have limited the first three anomalous estimators to $\ell_{\text{min}} = 8$, as this range is most anomalous, and the last three estimators are plotted for $\ell_{\text{max}} = 6$ only, because the lowest multipoles correspond to the scales that exited the horizon first. Decreasing the amounts (than those used for plotting) of foreground addition make the synchrotron, thermal dust and free-free emission added CMB histograms move towards the left and coincide with the pure CMB histogram, but these never move beyond the leftmost edges of the pure CMB histogram. This implies that the anomalously low estimators from Planck and WMAP data tend to lie on the leftmost tail ends of the pure CMB histogram, where the existence of foreground residuals will be highly unlikely. Thus, anomalously low estimators point towards a possible primordial origin of the anomalies. To understand the effect of thermal dust on the distribution of $r_{\text{tot}}$, we use a much greater number of bins and zoom in on a smaller range of $r_{\text{tot}}$. The probability distributions separate out showing that the effect of thermal dust is small but non-zero for very low $r_{\text{tot}}$ for $\ell_{\text{max}} = 6$. Hence unlike the other consistently anomalous estimators, $r_{\text{tot}}$ could be due to systematic errors or foreground residuals.

Although our knowledge of foregrounds is limited, and some other systematic errors could also be present in the maps studied, yet within the constraints of our understanding, the anomalies are highly unlikely to have arisen due to foreground residuals, as shown in figure 9. Even the addition of other known foregrounds (apart from the three shown) to pure CMB maps will enhance correlations between $C_\ell$’s leading to similar plots as in figure 9. The high consistency of the anomalies also indicates that it is most improbable for the four cleaned maps obtained from four independent cleaning methods to have the same residual uncertainties to manifest as anomalies of table III. Hence it can be concluded that the anomalies have originated primordially. In this vein, a study of the physical origin of these anomalies is essential to advance cosmology, but being beyond the scope of this paper, it may constitute a work in the future.
FIG. 9. Plots of distributions $p(s_{\text{est}})$ versus $s_{\text{est}}$ (where $s_{\text{est}} = \text{estimator}$) of the most consistently anomalous estimators as in Table III. Plots show that foreground residuals should favour anomalously high spacing estimators. Thus it is most likely that the anomalously low estimators have a primordial origin. Here, the distributions have been plotted using 5000 simulations of the pure ($\Lambda$CDM) CMB map, best fitted to Planck 2018 data. The different amounts of foreground maps have been chosen to illustrate their respective behaviours. Because, lower orders of foreground addition than those plotted above, make the histograms for foreground added maps almost indistinguishable from the pure CMB histogram, but those never move beyond its leftmost edges. Foreground maps used are synchrotron, thermal dust and free-free emission maps of Planck Commander, after appropriate antenna to thermodynamic temperature conversions.

ACKNOWLEDGEMENTS

We acknowledge the use of the publicly available HEALPix [30] software package (http://healpix.sourceforge.io). Our analyses are based on observations from Planck (http://www.esa.int/Planck), an ESA science mission with instruments and contributions directly funded by ESA Member States, NASA, and Canada. We acknowledge the use of the Legacy Archive for Microwave Background Data Analysis (LAMBDA), part of the High Energy Astrophysics Science Archive Center (HEASARC). HEASARC/LAMBDA is a service of the Astrophysics Science Division at the NASA Goddard Space Flight Center. We thank the anonymous reviewer and Scientific Editor for critical comments on an earlier version of this paper. MIK would like to thank Ujjal Purkayastha for immense help in guiding him through the basics of HEALPix relevant to this project.

DATA AVAILABILITY

Data generated in the analyses presented in this article can be obtained by interested individuals if the corresponding author is reasonably requested for the same.

[1] C. Cheng and Q.-G. Huang, Physics Letters B 738, 140 (2014).
[2] P. Bielewicz, K. M. Górski, and A. J. Banday, Monthly Notices of the Royal Astronomical Society 355, 1283 (2004), https://academic.oup.com/mnras/article-pdf/355/4/1283/9275233/355-4-1283.pdf.
[3] G. Hinshaw, D. Larson, E. Komatsu, D. N. Spergel, C. L. Bennett, J. Dunkley, M. R. Nolta, M. Halpern, R. S. Hill, N. Odegard, L. Page, K. M. Smith, J. L. Weiland, B. Gold, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, G. S. Tucker, E. Wollack, and E. L. Wright, The Astrophysical Journal Supplement Series 208, 19 (2013).
[4] Planck Collaboration, N. Aghanim, Y. Akrami, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini, A. J. Banday, R. B. Barreiro, N. Bartolo, S. Basak, R. Battye, K. Benabed, J. P. Bernard, M. Bersanelli, P. Bielewicz, J. J. Bock, J. R. Bond, J. Borrill, F. R. Bouchet, F. Boulanger, M. Bucher, C. Burigana, R. C. Butler, E. Calabrese, J. F. Cardoso, J. Carron, A. Challinor, H. C. Chiang, J. Chluba, L. P. L. Colombo, C. Combet, D. Contreras, B. P. Crill, F. Cuttaia, P. de Bernardis, G. de Zotti, J. Delabrouille, J. M. Delouis, E. Di Valentino, J. M. Diego, O. Doré, M. Douspis, A. Ducout, X. Dupac, S. Dusini, G. Efstathiou, F. Elsner, T. A. Enßlin, H. K. Eriksen, Y. Fantaye, M. Farhang, J. Fer
M. Cruz, E. Martínez-González, P. Vielva, and L. Cayón, Monthly Notices of the Royal Astronomical Society 356, 29 (2005), https://academic.oup.com/mnras/article-pdf/356/1/29/3591691/356-1-29.pdf.

M. Cruz, L. Cayon, E. Martínez-González, P. Vielva, and J. Jin, The Astrophysical Journal 655, 11 (2007).

M. Tegmark, A. de Oliveira-Costa, and A. Hamilton, Phys. Rev. D 68, 123523 (2003), arXiv:astro-ph/0302496.

A. de Oliveira-Costa, M. Tegmark, M. Zaldarriaga, and A. Hamilton, Phys. Rev. D 69, 063516 (2004), arXiv:astro-ph/0307282.

D. J. Schwarz, G. D. Starkman, D. Huterer, and C. J. Copi, Phys. Rev. Lett. 93, 221301 (2004), arXiv:astro-ph/0403535.

A. Notari and M. Quartin, Journal of Cosmology and Astroparticle Physics 2015 (06), 047.

C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, L. Page, D. N. Spergel, G. S. Tucker, E. Wollack, E. L. Wright, C. Barnes, M. R. Greason, R. S. Hill, E. Komatsu, M. R. Nolta, N. Odegard, H. V. Peiris, L. Verde, and J. L. Weiland, The Astrophysical Journal Supplement Series 148, 1 (2003).

E. Gaztañaga, J. Wagg, T. Multamäki, A. Montaña, and D. H. Hughes, Monthly Notices of the Royal Astronomical Society 346, 47 (2003), https://academic.oup.com/mnras/article-pdf/346/1/47/9375589/346-l-17.pdf.

A. de Oliveira-Costa, M. Tegmark, M. Zaldarriaga, and A. Hamilton, Phys. Rev. D 69, 063516 (2004).

K. Land and J. a. Magueijo, Phys. Rev. D 72, 101302 (2005).

J. Kim and P. Naselsky, The Astrophysical Journal 714, L265 (2010).

P. K. Aluri and P. Jain, Monthly Notices of the Royal Astronomical Society 419, 3378 (2012), http://oup.prod.sis.lan/mnras/article-pdf/419/4/3378/9507459/mnras0419-3378.pdf.

W. Zhao, Phys. Rev. D 89, 023010 (2014).

L. R. Abramo, L. Sodrê, and C. A. Wuenesco, Phys. Rev. D 74, 083515 (2006).

K. M. Górski, E. Hinov, A. J. Banday, B. D. Wandelt, F. K. Hansen, M. Reinecke, and M. Bartelmann, The Astrophysics Journal 622, 759 (2005), arXiv:astro-ph/0409513 [astro-ph].

National Aeronautics and Space Administration, WMAP 9 year ILC map, https://lambda.gsfc.nasa.gov/product/map/dr5/ILC_map.get.cfm (2012).

M. Bucher, Int. J. Mod. Phys. D24, 1530005 (2015), arXiv:1501.04288 [astro-ph.CO].

H. Blaker, Canadian Journal of Statistics 28, 783 (2000), https://onlinelibrary.wiley.com/doi/pdf/10.2307/3315916.

R. Fisher, Statistical methods for research workers (Edinburgh Oliver & Boyd, 1925).

C. Pernet, F1000Research 4, 10.12688/f1000research.6963.5 (2017).

J. Newman, Phil. Trans. Roy. Soc. Lond. A236, 333 (1937).

R. D. Morey, R. Hoekstra, J. N. Rouder, M. D. Lee, and E.-J. Wagenmakers, The fallacy of placing confidence in confidence intervals (2016).

M. Wood, arXiv e-prints , arXiv:0912.3878 (2009), arXiv:0912.3878 [stat.ME].

D. Larson, J. L. Weiland, G. Hinshaw, and C. L. Bennett, The Astrophysical Journal 801, 9 (2015).

P. Ade, N. Aghanim, C. Armitage-Caplan, M. Arnaud, M. Ashdown, F. Atrio-Barandela, J. Aumont, C. Bacci-
galupi, A. Banday, R. Barreiro, J. Bartlett, E. Battaner, K. Benabed, A. Benoît, A. Benoit-Lévy, J.-P. Bernard, M. Bersanelli, P. Bielewicz, J. Bobin, and A. Zonca, Astronomy and Astrophysics 571 (2013).

[41] H. K. Eriksen, A. J. Banday, K. M. Górski, and P. B. Lilje, The Astrophysics Journal 612, 633 (2004), arXiv:astro-ph/0403098 [astro-ph].

[42] A. Rassat, J. L. Starck, P. Paykari, F. Sureau, and J. Bobin, Journal of Cosmology and Astroparticle Physics 2014, 006 (2014), arXiv:1405.1844 [astro-ph.CO].

[43] S. Schierenberg, F. Bruckmann, and T. Wettig, Phys. Rev. E 85, 061130 (2012).

[44] M. Robnik, Journal of Physics A: Mathematical and General 20, L495 (1987).

[45] A. Y. Abul-Magd, Journal of Physics A: Mathematical and General 29, 1 (1996).

[46] A. Relaño, J. Dukelsky, J. M. G. Gómez, and J. Retamosa, Phys. Rev. E 70, 026208 (2004).

[47] Z. Chang, P. K. Rath, Y. Sang, D. Zhao, and Y. Zhou, Monthly Notices of the Royal Astronomical Society 479, 1327 (2018), https://academic.oup.com/mnras/article-pdf/479/1/1327/25444851/sty1689.pdf.

[48] B. Yu and T. Lu, Phys. Rev. D 79, 043015 (2009).