On the Scale Uncertainties in the $B \to X_s \gamma$ Decay

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Abstract

We analyze the theoretical uncertainties in $Br(B \to X_s \gamma)$ due to the choice of the high energy matching scale $\mu_W = \mathcal{O}(M_W)$ and the scale $\mu_t$ at which the running top quark mass is defined: $m_t(\mu_t)$. To this end we have repeated the calculation of the initial conditions confirming the final results of Adel and Yao and Greub and Hurth and generalizing them to include the dependences on $\mu_t$ and $\mu_W$ with $\mu_t \neq \mu_W$. In the leading order the $\mu_W$ and $\mu_t$ uncertainties in $Br(B \to X_s \gamma)$ turn out to be $\pm 13\%$ and $\pm 3\%$ respectively. We show analytically how these uncertainties are reduced after including next-to-leading QCD corrections. They amount to $\pm 1.1\%$ and $\pm 0.4\%$ respectively. Reanalyzing the uncertainties due to the scale $\mu_b = \mathcal{O}(m_b)$ we find that after the inclusion of NLO effects they amount to $\pm 4.3\%$ which is a factor 2/3 smaller than claimed in the literature. Including the uncertainties due to input parameters as well as the non-perturbative $1/m_b^2$ and $1/m_c^2$ corrections we find $Br(B \to X_s \gamma) = (3.60 \pm 0.33) \times 10^{-4}$ where the error is dominated by uncertainties in the input parameters. This should be compared with $(3.28 \pm 0.33) \times 10^{-4}$ found by Chetyrkin et al. where the error is shared evenly between the scale and parametric uncertainties.

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1. The inclusive $B \to X_s \gamma$ decay has been subject of considerable experimental and theoretical interest during the last ten years. Experimentally its branching ratio is found by the CLEO collaboration to be \cite{1}

$$Br(B \to X_s \gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4},$$  \hspace{1cm} (1)

and a very recent preliminary result from the ALEPH collaboration reads \cite{2}

$$Br(B \to X_s \gamma) = (3.38 \pm 0.74 \pm 0.85) \times 10^{-4}.\hspace{1cm} (2)$$

In (1) and (2) the first error is statistical and the second is systematic. On the other hand the complete NLO analysis gives \cite{3}

$$Br(B \to X_s \gamma) = (3.28 \pm 0.22 \text{ (scale)} \pm 0.25 \text{ (par)}) \times 10^{-4} = (3.28 \pm 0.33) \times 10^{-4}.\hspace{1cm} (3)$$

where the first error results from the scale uncertainty (see below) and the second error from the uncertainties in the input parameters. A similar result has been obtained in \cite{4}. The NLO analyses presented in \cite{3,4} reduced by a factor of 3-4 the $\mu_b$-uncertainties present in the leading order, where $\mu_b = O(m_b)$ is the scale at which the relevant decay matrix element is evaluated. This reduction of the $\mu_b$ uncertainty is very welcome because in the forthcoming years much more precise measurements of $Br(B \to X_s \gamma)$ are expected from the upgraded CLEO detector, as well as from the B-factories at SLAC and KEK. This is also the reason why continuing efforts are being made to estimate non-perturbative corrections to the $B \to X_s \gamma$ decay with higher precision \cite{5,6,7,8,9,10} as well. It appears that these latter corrections amount only to a few percent and constitute a rather small theoretical uncertainty.

2. In this letter we have repeated the numerical analysis of \cite{3} to find that the remaining scale uncertainties are by roughly a factor 1.5-2.0 smaller than quoted by these authors and in \cite{4}. This includes also two additional theoretical uncertainties which have not been addressed in the literature. They are related to the choice of the high energy matching scale $\mu_W = O(M_W)$ and the scale $\mu_t = O(m_t)$ at which the running top quark mass is defined: $\mu_t(\mu_t)$. These two scales enter the analysis of $B \to X_s \gamma$ in the process of calculating the initial conditions for the renormalization group running of the Wilson coefficients $C_7$ and $C_8$ of the operators

$$Q_7 = \frac{e}{8\pi^2} m_b \bar{s}_a \sigma^{\mu\nu} (1 + \gamma_5) b_\alpha F_{\mu\nu} \hspace{1cm} Q_8 = \frac{g_s}{8\pi^2} m_b \bar{s}_a \sigma^{\mu\nu} (1 + \gamma_5) T_{\alpha\beta} b_\beta G_{\mu\nu}^a.\hspace{1cm} (4)$$

Here $e$ and $g_s$ denote the electromagnetic and strong coupling constants respectively. These initial conditions have been calculated at NLO in \cite{11} and have been recently confirmed in \cite{12}. These NLO corrections are necessary to remove the renormalization scheme dependence present in the
renormalization group evolution from $\mu_W = O(M_W)$ down to $\mu_b = O(m_b)$. From our point of view the additional reason for performing these rather tedious calculations is the reduction of the uncertainties related to the choices of $\mu_W$ and $\mu_t$. These uncertainties have not been discussed in [3, 4, 11, 12].

To this end we have repeated the calculation of the initial condition for the by far dominant Wilson coefficient $C_7$ confirming the final result in [11, 12] and generalizing it to include the dependences on $\mu_t$ and $\mu_W$ with $\mu_t \neq \mu_W$. In [11] and [12] $\mu_W = \mu_t = \mu_{Wz}$ have been used. The technical details of our calculation which differs in certain aspects from the previous ones will be presented elsewhere [13]. Here we discuss first the issue of the $\mu_W$ and $\mu_t$ uncertainties and their reduction after the inclusion of NLO corrections. Subsequently we discuss the $\mu_b$ uncertainties and we present our estimate of $Br(B \to X_s \gamma)$ in the Standard Model.

3. In the leading logarithmic approximation one has

$$\frac{Br(B \to X_s \gamma)}{Br(B \to X_c e \bar{\nu}_e)} = \frac{|V_{ts}^* V_{tb}|^2}{|V_{cb}|^2} \frac{6 \alpha}{\pi f(z)} |C_7^{(0)\text{eff}}(\mu_b)|^2,$$

where

$$f(z) = 1 - 8z + 8z^3 - z^4 - 12z^2 \ln z \quad \text{with} \quad z = \frac{m_{c,pole}^2}{m_{b,pole}^2}$$

is the phase space factor in $Br(B \to X_c e \bar{\nu}_e)$ and $\alpha = e^2/4\pi$.

The effective renormalization scheme independent coefficient $C_7^{(0)\text{eff}}(\mu_b)$ introduced in [6] is given by

$$C_7^{(0)\text{eff}}(\mu_b) = \eta^{14} C_7^{(0)}(\mu_W) + \frac{8}{3} \left( \eta^{14} - \eta^{23} \right) C_8^{(0)}(\mu_W) + \sum_{i=1}^{8} h_i \eta^{a_i},$$

where

$$\eta = \frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)},$$

$$C_7^{(0)}(\mu_W) = \frac{3x_t^3 - 2x_t^2}{4(x_t - 1)^4} \ln x_t + \frac{-8x_t^3 - 5x_t^2 + 7x_t}{24(x_t - 1)^3},$$

$$C_8^{(0)}(\mu_W) = \frac{-3x_t^2}{4(x_t - 1)^4} \ln x_t + \frac{-x_t^3 + 5x_t^2 + 2x_t}{8(x_t - 1)^3}$$

with

$$x_t = \frac{m_t^2(\mu_t)}{M_W^2}.$$
Table 1: Magic Numbers.

| $i$ | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| $a_i$ | 14/23 | 16/23 | 6/23  | -12/23| 0.4086| -0.4230| -0.8994| 0.1456 |
| $h_i$ | 2.2996| -1.0880| -3/7  | -1/14 | -0.6494| -0.0380| -0.0185| -0.0057 |
| $e_i$ | 8461194/816831 | -516/2217 | 0     | 0     | -1.9043| -0.1008| 0.1216  | 0.0183  |
| $f_i$ | -17.3023| 8.5027 | 4.5508| 0.7519| 2.0040| 0.7476  | -0.5385 | 0.0914  |
| $g_i$ | 14.8088| -10.8090| -0.8740| 0.4218| -2.9347| 0.3971  | 0.1600  | 0.0225  |
| $l_i$ | 0.5784| -0.3921| -0.1429| 0.0476| -0.1275| 0.0317  | 0.0078  | -0.0031 |

- The high energy scale $\mu_W = \mathcal{O}(M_W)$ at which the full theory is matched with the effective five-quark theory. In LO this scale enters only $\eta$, $C_7^{(0)}(\mu_W)$ and $C_8^{(0)}(\mu_W)$, usually denoted by $C_7^{(0)}(M_W)$ and $C_8^{(0)}(M_W)$, serve in LO as initial conditions to the renormalization group evolution from $\mu_W$ down to $\mu_b$. As seen explicitly in (9) and (10) they do not depend on $\mu_W$.

- The scale $\mu_t = \mathcal{O}(m_t)$ at which the running top quark mass is defined. In LO it enters only $x_t$ in (11).

It should be stressed that $\mu_W$ and $\mu_t$ do not have to be equal. Initially when the top quark and the W-boson are integrated out, it is convenient in the process of matching to keep $\mu_t = \mu_W$. Yet one has always the freedom to redefine the top quark mass and to work with $m_t(\mu_t)$ where $\mu_t \neq \mu_W$. It is evident from the formulae above that in LO the variations of $\mu_b$, $\mu_W$ and $\mu_t$ remain uncompensated which results in potential theoretical uncertainties in the predicted branching ratio.

In the context of phenomenological analyses of $B \to X_s\gamma$, only the uncertainty due to $\mu_b$ has been discussed [5, 6, 3, 4]. It is the purpose of this letter to analyze the uncertainties due to $\mu_W$ and $\mu_t$ and to reanalyze the $\mu_b$-uncertainty.

It is customary to estimate the uncertainties due to $\mu_b$ by varying it in the range $m_b/2 \leq \mu_b \leq 2m_b$. Similarly one can vary $\mu_W$ and $\mu_t$ in the ranges $M_W/2 \leq \mu_W \leq 2M_W$ and $m_t/2 \leq \mu_t \leq 2m_t$ respectively. Specifically in our numerical analysis of $\mu_W$ and $\mu_t$ uncertainties we will consider the ranges

$$40 \text{ GeV} \leq \mu_W \leq 160 \text{ GeV} \quad 80 \text{ GeV} \leq \mu_t \leq 320 \text{ GeV}$$

setting $\mu_b = m_b \equiv m_{b,\text{pole}} = 4.8 \text{ GeV}$. 

4
In the LO analysis we use

\[ \alpha_s(\mu) = \frac{\alpha_s(M_Z)}{v(\mu)} \quad v(\mu) = 1 - \beta_0 \frac{\alpha_s(M_Z)}{2\pi} \ln \left( \frac{M_Z}{\mu} \right) \]  

(13)

with \( \alpha_s(M_Z) = 0.118 \) and

\[ \overline{m}_t(\mu_t) = \overline{m}_t(m_t) \left[ \frac{\alpha_s(\mu_t)}{\alpha_s(m_t)} \right]^{\frac{1}{30}}. \]  

(14)

We are using the parameters \( \alpha_s^{(5)} \) and \( \overline{m}^{(5)} \) defined in the effective theory with five flavours throughout in this work, hence \( \beta_0 = 23/3 \). We set \( \overline{m}_t(m_t) = 168 \) GeV and \( m_t \equiv m_{t,\text{pole}} = 176 \) GeV.

Varying \( \mu_W \) and \( \mu_t \) in the ranges (12) we find the following uncertainties in the branching ratio:

\[ \Delta Br(B \to X_s \gamma) = \begin{cases} 
\pm 13\% \ (\mu_W) \\
\pm 3\% \ (\mu_t) 
\end{cases} \]  

(15)

to be compared with the \( \pm 22\% \) uncertainty due to the variation of the scale \( \mu_b \) [3, 4]. The fact that the \( \mu_W \)-uncertainty is smaller than the \( \mu_b \) uncertainty is entirely due to \( \alpha_s(\mu_W) < \alpha_s(\mu_b) \).

Still this uncertainty is rather disturbing as it introduces an error of approximately \( \pm 0.40 \cdot 10^{-4} \) in the branching ratio. The smallness of the \( \mu_t \)-uncertainty is related to the weak \( x_t \) dependence of \( C_7^{(0)}(\mu_W) \) and \( C_8^{(0)}(\mu_W) \) which in the range of interest can be well approximated by

\[ C_7^{(0)}(\mu_W) = -0.122 \ x_t^{0.30} \quad C_8^{(0)}(\mu_W) = -0.072 \ x_t^{0.19} \]  

(16)

Thus even if \( 161 \) GeV \( \leq \overline{m}_t(\mu_t) \leq 178 \) GeV for \( \mu_t \) in (12), the \( \mu_t \) uncertainty in \( Br(B \to X_s \gamma) \) is small. This should be contrasted with \( B_s \to \mu \bar{\mu} \), \( K_L \to \pi^0 \nu \bar{\nu} \) and \( B^0 - \bar{B}^0 \) mixing, where \( \mu_t \) uncertainties in LO have been found [14, 15] to be \( \pm 13\% \), \( \pm 10\% \) and \( \pm 9\% \) respectively.

4. We will next investigate how much the uncertainties in (15) are reduced after including NLO corrections.

The formula (5) modifies after the inclusion of NLO corrections as follows [3]:

\[ \frac{Br(B \to X_s \gamma)}{Br(B \to X_c e \nu_e)} = \frac{|V_{ts}^* V_{tb}|^2}{|V_{cb}|^2} \frac{6\alpha}{\pi f(z)} F \left( \frac{1}{2} + A \right), \]  

(17)

where

\[ F = \frac{1}{\kappa(z)} \left( \frac{m_b(\mu = m_b)}{m_{b,\text{pole}}} \right)^2 = \frac{1}{\kappa(z)} \left( 1 - \frac{8 \alpha_s(m_b)}{3\pi} \right), \]  

(18)

with \( \kappa(z) \) being the QCD correction to the semileptonic decay [16] and given to a good approximation by [17]

\[ \kappa(z) = 1 - \frac{2\alpha_s(\mu_b)}{3\pi} \left[ \left( \frac{\pi^2}{4} - \frac{31}{4}\right)(1 - \sqrt{z})^2 + \frac{3}{2} \right]. \]  

(19)
An exact analytic formula for $\kappa(z)$ can be found in [13]. Here $\bar{\mu}_b = O(m_b)$ is a scale in the calculation of QCD corrections to the semi-leptonic rate which is generally different from the one used in the $b \to s \gamma$ transition. In this respect we differ from Greub et al. [4] who set $\bar{\mu}_b = \mu_b$. In [3] the choice $\bar{\mu}_b = m_b$ has been made. We will return to this point below.

Next

$$D = C_{7}^{(0)}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} \left\{ C_{7}^{(1)}(\mu_b) + \sum_{i=1}^{8} C_{i}^{(0)}(\mu_b) \left[ r_i + \frac{\gamma_{i7}^{(0)}(\mu_b)}{2} \ln \frac{m_b^2}{\mu_b^2} \right] \right\}$$

(20)

where $C_{7}^{(1)}(\mu_b)$ is the NLO correction to the effective Wilson coefficient of $Q_7$:

$$C_{7}^{\text{eff}}(\mu_b) = C_{7}^{(0)}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_{7}^{(1)}(\mu_b).$$

(21)

Generalizing the formula (21) of [3] to include $\mu_t$ and $\mu_W$ dependences we find

$$C_{7}^{(1)}(\mu_b) = \eta_{32}^{2n} C_{7}^{(0)}(\mu_W) + \frac{8}{3} \left( \frac{\eta_{32}^{2n} - \eta_{32}^{2n}}{\eta_{32}^{2n}} \right) C_{8}^{(1)}(\mu_W)$$

$$+ \left( \frac{297664}{14283} \eta_{32}^{16} - \frac{7164416}{357075} \eta_{32}^{16} + \frac{256868}{14283} \eta_{32}^{16} - \frac{6698884}{357075} \eta_{32}^{16} \right) C_{8}^{(0)}(\mu_W)$$

$$+ \left( \eta_{32}^{2n} - \eta_{32}^{2n} \right) C_{7}^{(0)}(\mu_W)$$

$$+ \sum_{i=1}^{8} \left( e_i \eta E(x_i) + f_i + g_i \eta + \eta \left[ \frac{2}{3} e_i + 6 l_i \right] \ln \frac{\mu_W^2}{M_W^2} \right) \eta^a,$$

(22)

where in the $\overline{\text{MS}}$ scheme

$$C_{7}^{(1)}(\mu_W) = C_{7}^{(1)}(M_W) + 8 x_i \frac{\partial C_{7}^{(0)}(\mu_W)}{\partial x_i} \ln \frac{\mu^2}{M_W^2} + \left( \frac{16}{3} C_{7}^{(0)}(\mu_W) - \frac{16}{9} C_{8}^{(0)}(\mu_W) + \frac{\gamma_{27}^{(0)}(\mu_W)}{2} \right) \ln \frac{\mu^2}{M_W^2}$$

(23)

$$C_{8}^{(1)}(\mu_W) = C_{8}^{(1)}(M_W) + 8 x_i \frac{\partial C_{8}^{(0)}(\mu_W)}{\partial x_i} \ln \frac{\mu^2}{M_W^2} + \left( \frac{14}{3} C_{8}^{(0)}(\mu_W) + \frac{\gamma_{28}^{(0)}(\mu_W)}{2} \right) \ln \frac{\mu^2}{M_W^2}$$

(24)

Here $(x = x_i)$

$$C_{7}^{(1)}(M_W) = -16 x^4 - 122 x^3 + 80 x^2 - 8 x \frac{\text{Li}_2 \left( 1 - \frac{1}{x} \right)}{9(x - 1)^4} + \frac{6 x^4 + 46 x^3 - 28 x^2}{3(x - 1)^5} \ln^2 x$$

$$+ \frac{-102 x^5 - 588 x^4 - 2262 x^3 + 3244 x^2 - 1364 x + 208}{81(x - 1)^5} \ln x$$

$$+ \frac{1646 x^4 + 12205 x^3 - 10740 x^2 + 2509 x - 436}{486(x - 1)^4}$$

(25)

$$C_{8}^{(1)}(M_W) = -4 x^4 + 40 x^3 + 41 x^2 + x \frac{\text{Li}_2 \left( 1 - \frac{1}{x} \right)}{6(x - 1)^4} + \frac{-17 x^3 - 31 x^2}{2(x - 1)^5} \ln^2 x$$

\[1\] We would like to thank the authors of [13] for pointing out the missing logarithmic term in the original version of this work. See also the discussion after equation (22).
\[ + \frac{-210x^5 + 1086x^4 + 4893x^3 + 2857x^2 - 1994x + 280}{216(x-1)^5} \ln x \]
\[ + \frac{737x^4 - 14102x^3 - 28209x^2 + 610x - 508}{1296(x-1)^4} \]

and

\[ E(x) = \frac{x(18 - 11x - x^2)}{12(1-x)^3} + \frac{x^2(15 - 16x + 4x^2)}{6(1-x)^4} \ln x - \frac{2}{3} \ln x. \]  

(27)

The formulae for \( C^{(1)\text{eff}}_{7,8}(M_W) \) given above and presented in [3] are obtained from the results in [11, 12] by using the general formulae for the effective coefficient functions [6]. The formula for \( C^{(1)\text{eff}}_{7}(M_W) \) has been confirmed by us [13]. The numbers \( e_i-l_i \) are given in Table 1. We have confirmed these numbers as well as the numerical coefficients in (22) using the anomalous dimension matrices in [3]. Next the \( \eta \) in (8) should now be calculated using the NLO expression

\[ \alpha_s(\mu) = \alpha_s(M_Z) \left[ 1 - \frac{\beta_1}{4\pi} \frac{\alpha_s(M_Z) \ln v(\mu)}{v(\mu)} \right], \]  

(28)

where \( v(\mu) \) is given in [13] and \( \beta_1 = \frac{116}{3} \).

The constants \( r_i \) resulting from the calculations of NLO corrections to decay matrix elements [4] are collected in [3] where also explicit formulae for \( C^{(0)\text{eff}}_i(\mu_b) \) with \( i = 1 \sim 6, 8 \) and the values of \( \gamma^{(0)\text{eff}}_7 \) can be found. It should be stressed that the basis of the operators with \( i = 1 \sim 6 \) used in [3] differs from the standard basis used in the literature [23]. For the discussion below it will be useful to have [24]

\[ \gamma^{(0)\text{eff}}_{27} = \frac{416}{81}, \quad \gamma^{(0)\text{eff}}_{28} = \frac{70}{27} \]  

(29)

which enter (23) and (24) respectively.

Finally the term \( A \) in (17) originates from the bremsstrahlung corrections and the necessary virtual corrections needed for the cancellation of the infrared divergences. These have been calculated in [20, 21] and are also considered in [3, 4] in the context of the full analysis. The explicit formula for \( A \), which we use in our numerical analysis, can be found in equation (32) of [3].

Setting \( \mu_W = \mu_t = \mu_{Wt} \), replacing \( \gamma^{(0)\text{eff}}_{27} \) by its value in the NDR scheme \( \gamma^{(0)\text{NDR}}_{27} = 464/81 \) and adding all \( \mu_i \) dependent terms in (28) we recover the \( \mu_{Wt} \) dependence of \( C^{(1)\text{NDR}}_7(\mu_{Wt}) \) found in [12]. Similarly replacing \( \gamma^{(0)\text{eff}}_{28} \) by \( \gamma^{(0)\text{NDR}}_{28} = 76/27 \) in (24) we recover the \( \mu_{Wt} \) dependence of \( C^{(1)\text{NDR}}_8(\mu_{Wt}) \) given in [12]. For \( \mu_W = \mu_t = M_W \) the formulae above reduce to the ones given in [3].

\footnote{In the replacement version of [3] several quantities entering the formula for \( A \) have been corrected. In this paper the updated values are used. We thank M. Neubert [22], P. Gambino and M. Misiak for informing us about these modifications. Accordingly the numerical results of this work are changed slightly.}
5. Before entering the numerical analysis let us demonstrate analytically that the $\mu_t$ and $\mu_W$ dependences present in $C_{7}^{(0)\text{eff}}(\mu_b)$ are indeed cancelled at $\mathcal{O}(\alpha_s)$ by the explicit scale dependent terms in (23). The scale dependent terms in (24) do not enter this cancellation at this order in $\alpha_s$ in $B \to X_s\gamma$. On the other hand they are responsible for the cancellation of the scale dependences in $C_{8}^{(0)\text{eff}}(\mu_b)$ relevant for the $b \to s$ gluon transition.

Expanding the three terms in (7) in $\alpha_s$ and keeping the leading logarithms we find:

$$\eta_{16}^{23} C_{7}^{(0)}(\mu_W) = \left(1 + \frac{\alpha_s}{4\pi} \frac{16}{3} \ln \frac{\mu_b^2}{\mu_W^2} \right) C_{7}^{(0)}(\mu_W)$$  \hspace{1cm} (30)

$$\frac{8}{3} \left( \eta_{14}^{23} - \eta_{16}^{23} \right) C_{8}^{(0)}(\mu_W) = -\frac{\alpha_s}{4\pi} \frac{16}{9} \ln \frac{\mu_b^2}{\mu_W^2} C_{8}^{(0)}(\mu_W)$$  \hspace{1cm} (31)

$$\sum_{i=1}^{8} h_i \eta_{8}^{a_i} = \frac{\alpha_s}{4\pi} \frac{23}{3} \ln \frac{\mu_b^2}{\mu_W^2} \sum_{i=1}^{8} h_i a_i = -\frac{208 \alpha_s}{81} \frac{\mu_b^2}{\mu_W^2}$$  \hspace{1cm} (32)

respectively. In (32) we have used $\sum h_i = 0$. Inserting these expansions into (24), we observe that the $\mu_W$ dependences in (30), (31) and (32) are precisely cancelled by the three explicit logarithms in (23) involving $\mu_W$, respectively. Similarly one can convince oneself that the $\mu_t$-dependence of $C_{7}^{(0)\text{eff}}(\mu_b)$ is cancelled at $\mathcal{O}(\alpha_s)$ by the $\ln \mu_t^2/M_W^2$ term in (23).

Interestingly the last logarithm in (22) does not contribute to any cancellation of the $\mu_W$ dependence at this order in $\alpha_s$ due to the relation $\sum_{i=1}^{8} \left( \frac{2}{3} e_i + 6 l_i \right) = 0$ which can be verified by using the Table 1.

Clearly there remain small $\mu_t$ and $\mu_W$ dependences in (17) which can only be reduced by going beyond the NLO approximation. They constitute the theoretical uncertainty which should be taken into account in estimating the error in the prediction for $Br(B \to X_s\gamma)$.

Using the well known two-loop generalization of (14) and varying $\mu_W$ and $\mu_t$ in the ranges (12) we find that the respective uncertainties in the branching ratio after the inclusion of NLO corrections are negligible:

$$\Delta Br(B \to X_s\gamma) = \begin{cases} 
\pm1.1\% & (\mu_W) \\
\pm0.4\% & (\mu_t)
\end{cases}$$  \hspace{1cm} (33)

6. We have next performed the NLO analysis of the $\mu_b$ dependence. Varying $\mu_b$ in the range $2.5\text{GeV} \leq \mu_b \leq 10\text{GeV}$ we find

$$\Delta Br(B \to X_s\gamma) = \pm4.3\% \hspace{1cm} (\mu_b)$$  \hspace{1cm} (34)

This reduction of the $\mu_b$-uncertainty by roughly a factor of five relative to $\pm22\%$ in LO is caused by the presence of the explicit logarithm $\ln m_b^2/\mu_b^2$ in (20). We note that our result in (34) differs
from the $\mu_b$-uncertainty of $\pm 6.6\%$ quoted in [3]. A discussion with the latter authors confirmed our result.

Next we would like to comment on the uncertainty due to variation of $\bar{\mu}_b$ in $\kappa(z)$ given in (19). In [4] the choice $\bar{\mu}_b = \mu_b$ has been made. Yet in our opinion such a treatment is not really correct, since the scale $\bar{\mu}_b$ in the semi-leptonic decay has nothing to do with the scale $\mu_b$ in the renormalization group evolution in the $B \to X_s \gamma$ decay. Varying $\bar{\mu}_b$ in the range $2.5 \text{ GeV} \leq \mu_b \leq 10 \text{ GeV}$ we find

$$\Delta Br(B \to X_s \gamma) = \pm 1.7\% \quad (\bar{\mu}_b)$$  \hfill (35)

Since the $\mu_b$ and $\bar{\mu}_b$ uncertainties are uncorrelated we can add them in quadrature finding $\pm 4.6\%$ for the total scale uncertainty due to $\mu_b$ and $\bar{\mu}_b$. This is smaller by roughly 30% than the case in which $\bar{\mu}_b = \mu_b$ is used. The addition of the uncertainties in $\mu_t$ and $\mu_W$ in (33) modifies this result slightly and the total scale uncertainty in $Br(B \to X_s \gamma)$ amounts then to

$$\Delta Br(B \to X_s \gamma) = \pm 4.8\% \quad (\text{scale})$$  \hfill (36)

which is roughly by a factor of 1.5 smaller than quoted in [3, 4].

It should be stressed that this pure theoretical uncertainty related to the truncation of the perturbative series should be distinguished from parametric uncertainties related to $\alpha_s$, the quark masses etc. discussed below.

In our numerical calculations we have included all corrections in the NLO approximation. To work consistently in this order, we have in particular expanded the various factors in (17) in $\alpha_s$ and discarded all NNLO terms of order $\alpha_s^2$ which resulted in the process of multiplication. This treatment is different from [3, 4], where the $\alpha_s$ corrections in (18) have not been expanded in the evaluation of (17) and therefore some higher order corrections have been kept. Different scenarios of partly incorporating higher order corrections by expanding or not expanding various factors in (17) affect the branching ratio by $\Delta Br(B \to X_s \gamma) \approx \pm 3.0\%$. This number indicates that indeed the scale uncertainty in (36) realistically estimates the magnitude of yet unknown higher order corrections.

The remaining uncertainties are due to the values of the various input parameters. In order to obtain the final result for the branching ratio we have used the same parameters as in [3]. They are given in Table 2. In addition we have included small $1/m_b^2$ corrections as in [3] and also a 3% enhancement [11] from $1/m_c^2$ corrections [12] which were not known at the time of the analysis [3]. The relative importance of various uncertainties is shown in Table 3. Comparing

\footnote{In contrast to a 3% suppression found originally in [3] (except for the second paper which actually discusses the exclusive channels), the $1/m_c^2$ corrections to the $B \to X_s \gamma$ decay have been shown to be positive in [11].}
this table with the first row in the corresponding table in [3] we observe that except for the scale uncertainties discussed above, our error analysis agrees well with the one presented in [3].

Table 2. Input parameter values and their uncertainties. The masses are given in GeV.

| Scales | $\alpha_s(M_Z)$ | $m_{t,pole}$ | $m_{c,pole}/m_{b,pole}$ | $m_{b,pole}$ | $\alpha_{em}^{-1}$ | $|V_{ts}V_{tb}|/V_{cb}$ | $Br(B \rightarrow X_{c}e\bar{\nu}_e)$ |
|--------|-----------------|-------------|--------------------------|-------------|-----------------|------------------------|-----------------------------|
| Central| 0.118           | 176         | 0.29                     | 4.8         | 130.3           | 0.976                  | 0.104                      |
| Error  | $\pm0.003$      | $\pm6.0$    | $\pm0.02$                | $\pm0.15$  | $\pm2.3$        | $\pm0.010$             | $\pm0.004$                 |

Table 3. Uncertainties in $Br(B \rightarrow X_{c}e\bar{\nu}_e)$ due to various sources.

Adding all the uncertainties in quadrature we find

$$Br(B \rightarrow X_{c}e\bar{\nu}_e) = (3.60 \pm 0.17 \text{ (scale)} \pm 0.28 \text{ (par)}) \times 10^{-4} = (3.60 \pm 0.33) \times 10^{-4} \quad (37)$$

where we show separately scale and parametric uncertainties.

Comparing this result with the one of [3] as given in (3) we observe that in spite of the smaller scale uncertainties in (37) our final result is compatible with the one of [3] and the one given in [1]. This is due to the parametric uncertainties which dominate the theoretical error at present. Once these parametric uncertainties will be reduced in the future the smallness of the scale uncertainties achieved through very involved QCD calculations, in particular in [3, 4, 21, 22, 11, 12, 13], can be better appreciated. This reduction of the theoretical error in the Standard Model prediction for $Br(B \rightarrow X_{c}\gamma)$ could turn out to be very important in the searches for new physics. To this end also a better understanding of non-perturbative corrections [8] beyond those considered here should be achieved.

The theoretical estimate in (37) is somewhat higher than the CLEO result in (1) and rather close to the ALEPH result in (2). In any case we conclude that within the remaining theoretical and in particular experimental uncertainties, the Standard Model value is compatible with experiment. It will be exciting to watch the improvements in the theoretical estimate and in the experimental value in the coming years.

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