A novel recurrence-transience transition and Tracy-Widom growth in a cellular automaton with quenched noise

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Abstract – We study the growing patterns formed by a deterministic cellular automaton, the rotor-router model, in the presence of quenched noise. By the detailed study of two cases, we show that: a) the boundary of the pattern displays KPZ fluctuations with a Tracy-Widom distribution, b) as one increases the amount of randomness, the rotor-router path undergoes a transition from a recurrent to a transient walk. This transition is analysed here for the first time, and it is shown that it falls in the 3D anisotropic directed percolation universality class.

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Introduction. – Growing surfaces in two dimensions which break time-reversal but not rotational or translational symmetry are known to fall in the Kardar-Parisi-Zhang (KPZ) universality class [1,2]. This means that the exponents governing the long-time and long-distance behaviour of single-point fluctuations and two-point correlations are the same for the models within this class. However, specific models belonging to this class have been shown to have a deeper connection, wherein the asymptotic distribution of the fluctuating quantities has been shown to be the same, as described below [3–5]. Examples of problems shown to have this connection are the TASEP on an infinite lattice, anisotropic directed percolation in 3D [6,7], the Bernoulli matching problem [8–10], random growth models [11], and the largest eigenvalue problem in random matrix theory [12,13]. In these problems, one is typically concerned with the distribution of fluctuations of a random variable $h(t)$ (which could be the height profile in a random growth model), and remarkably, in all these cases this distribution has the form

$$P(h,t) = At^{2/3}U_{\beta}(Bh/t^{2/3}),$$

where $U_{\beta}$ is the Tracy-Widom distribution with $\beta = 1$, 2 or 4 [14–17] ($A$ and $B$ are model-dependent constants). The growing evidence for Tracy-Widom distributions occurring in seemingly unrelated non-equilibrium phenomena points to a new underlying universality that is yet to be fully understood. There has been an explosion of interest in Tracy-Widom distributions amongst theoretical and experimental physicists [18–20], mathematicians [21], and biologists [22].

Many systems in nature are believed to spontaneously reach a critical state, a phenomenon known as self-organized criticality (SOC) [23]. Examples of simple models which show self-organized criticality are the sandpile model [24,25] and the rotor-router model [26]. In recent years, some of these models have been shown to exhibit intricate growing patterns which show proportionate growth [27,28], where all parts of the pattern, whether big or small, grow at the same rate, as happens in biological systems such a mammalian bodies, but without a preplanned program for growth or co-ordination. However, in the case of the sandpile model, noise in the initial condition eventually blurs the intricate structure of the noiseless pattern [29]. In this paper we study the patterns formed in the rotor-router model, which was introduced as the Eulerian Walkers model in the physics literature [26,30,31]. The intricately structured patterns observed in this model, when a pattern is grown on periodic backgrounds, have been studied previously [32]. Meanwhile, numerical studies on completely random backgrounds have shown that the boundary of the cluster of visited sites displays KPZ fluctuations [33–35]. In this letter, we study the effect of a controllable amount of initial (quenched) noise on the pattern, and show that the pattern is not destroyed if noise is below some threshold. In particular, we provide an exact mapping from the patterns formed by a rotor-router walker on a lattice with quenched disorder to the anisotropic directed percolation model, a height model in the Tracy-Widom class, thereby showing that the Tracy-Widom universality extends to deterministic cellular automata with quenched randomness.
The properties of the rotor-router are intimately related to the properties of the random walk on the corresponding lattice [36–39]. We study the rotor-router model in 2D, the critical dimension at which random walks change between recurrence and transience [40]. We show for the first time that the 2D rotor-router exhibits a transition between transience and recurrence. The classification of irreducible Markov chains as recurrent or transient is an important and fundamental problem [41]. Recently, there has been a surge of interest in the recurrent and transient properties of de-randomized, deterministic versions of Markov chains, of which the rotor-router model is one example [42–50]. We show that a recurrence-transience transition can be produced by tuning the amount of quenched disorder, and that this transition falls in the well-known 3D anisotropic directed percolation universality class. We provide both analytic arguments and extensive numerical evidence to support our claims.

**The rotor-router model.** We consider the rotor-router model on a two-dimensional square lattice. Each site on the lattice is equipped with a single arrow that points to one of the neighbouring sites (see fig. 1). The path of a walker on this lattice is directed by the arrows, and in turn changes their orientation: the rule is that a walker, when it visits a site, first rotates the arrow attached to the site by 90° counter-clockwise, and then walks to the next site along the new direction of the arrow. Starting from a configuration where every arrow points to the right, a walker inserted at the origin would walk straight along the y-axis to infinity, changing all arrows along the y-axis to point up. Such configurations, on which the walker visits each site only finitely many times, are called transient backgrounds. On arrow configurations where this does not happen, the walker visits every site on the lattice infinitely many times before it reaches infinity [47].

A simple example of a recurrent configuration is the periodic configuration constructed by tiling the lattice with the 2 × 2 unit cell (←↓→↑), also written as (0 3 4 3) with the notation 0 3 4 3. Given V(i, j, M) and the initial arrow configuration ρ(i, j), the final arrow configuration is given by ρf = [(ρi + V)/4]. For the rotor-router model, V(i, j, M) function obeys the Laplace-like equation

$$\mathcal{L}[V, ρ](x, y) ≡ \nabla^2 V(x, y) + \sum f(x', y', x, y) = 0,$$

where the summation is over the neighbours of site (x, y). The function f(p, ρf)(x', y', x, y) can take the value 0 or 1, and it counts the number of times the arrow at site (x', y') points towards site (x, y) when one rotates it from ρi(x', y') to ρf(x', y').

**Arrow configurations with quenched randomness.** We construct random arrow configurations by replacing some of the unit cells in the periodic configuration above by a certain fraction of “defect cells”. The fraction of defects controls the amount of randomness in the pattern. We first study the perturbation of the periodic initial condition defined above, such that a fraction p of all arrows on odd sites are turned counterclockwise once (this we call the “type-I” background). The probabilities of different unit cells are then

$$\text{Prob}([0 0 3]) = (1 – p)^2, \quad \text{Prob}([1 3]) = p(1 – p),$$
$$\text{Prob}([0 0 3]) = p(1 – p), \quad \text{Prob}([0 0 3]) = p^2,$$

with $p$ varying between 0 to 1. $p = 0$ is the unperturbed background, while at $p = 1$ the initial condition is a perfect tiling of the lattice by the unit cell (0 3 4), which is a
transient background, with a particle inserted at the origin moving in a straight line along the positive y-axis and leaving the lattice. The randomness in the initial condition has a local effect in patches 1, 3 and 4, but creates terrace-like structures in patch 2, as seen in fig. 2. We shall focus on the effect in patch 2.

We now show that the structure in patch 2 can be mapped to an exactly solvable height model. Equation (3), along with the fact that the final arrow configuration should be loopless [31], and the condition $V(0,0,M) = M$ can be used to generate a solution $V(i,j,M)$ for the Visit function, starting from an initial guess $V^0(i,j)$. (See the Supplementary Information Supplementarymaterial.pdf (SI) for details for the algorithm. A similar algorithm was earlier used to study rotor-router aggregation [51].) We start from the initial guess $V^0(i,j)$, eq. (2), which is the correct Visit function for $p = 0$, and determine the effect of adding randomness to the pattern. We prove that (for details and proof, see SI) the true visit function $V(i,j,M)$ can be written as $V_0(i,j,M) + h(i,j,M)$. We call $h(i,j,M)$ the “height field”, and it can be determined from the positions of the perturbed unit cells, by a few simple rules:

1) A single perturbed unit cell creates a terrace inside which the $h$ is increased by 4 compared to the $h$ outside the cone. Thus two nested terraces will successively increase $h$ by 4 when crossing from the region outside both the terraces to the region inside both terraces. (See fig. 3.)

2) Two intersecting “V”-shaped terraces merge to create a single terrace shaped like the letter “W”.

3) A perturbed unit cell lying on the boundary of a terrace does not have any effect.

The terraces constructed using these rules can be used to determine $h(i,j)$, and hence $V(i,j)$, for all $p$. The rules given above are in fact the same as those used to create terraces of wetted sites in 3D anisotropic directed percolation (ADP) [6,7], which can also be exactly mapped to the Bernoulli matching problem and the longest increasing subsequence problem [5]. The exact solution for the height field is known in this case, and thus we can write down that $h(i,j) = 4h_{BM}(\frac{i-j}{2}, \frac{i+j}{2})$ (as the patch is aligned with the $j$-axis as its diagonal, and the terrace boundaries have a width of 2 lattice units), where $h_{BM}$ is the height field in the Bernoulli matching problem. Hence [15],

$$h(i,j) = \frac{4}{j-i}$$

$$= \begin{cases} 
j - i, & \text{if } \frac{j}{i} < \frac{1-p}{1+p}, \\
i + j, & \text{if } \frac{j}{i} > \frac{1-p}{1+p}, \\
\sqrt[p]{p(j^2 - i^2) - pij} - \frac{ij}{1 - p} + A(p,i,j)\xi_{GUE}, & \text{otherwise}
\end{cases}$$

(5)

where $\xi_{GUE}$ is a random variable distributed according to the Tracy-Widom distribution corresponding to the Gaussian Unitary Ensemble, also known as the Tracy-Widom type-2 distribution, and

$$A(p,i,j) = \left(\frac{p(j^2 - i^2)}{21/3(1-p)}\right)^{1/6} \left(1 + p - 2\sqrt{\frac{p}{(j^2 - i^2)^2}}\right)^{2/3}.$$
Fig. 5: (Colour online) (a) Left: the final pattern formed by a walker on a type-II background with $p=0.05$. The pattern shows terraces in three patches. Colour code as in fig. 1. The walker keeps returning back to the origin for this value of $p$. (b) Right: for $p=0.45$, this figure shows only the sites visited by the walker, coloured by the number of visits to each site, before the walker reaches the edge of the lattice. Note that most sites are not visited, a sign of the transient nature of the walk. This shows that there is a transition from a recurrent walk to a transient walk below $p=0.45$.

To compare with simulations, we define the shifted Tracy-Widom type-2 distribution $TW_2(\xi_s, m, s)$ as the distribution followed by the variable $\xi_s = m + s_{GUE}$. Thus,

$$TW_2(\xi_s, m, s) = U_{\beta=2} \left( \frac{\xi_s - m}{s} \right), \quad (6)$$

where $U_{\beta=2}$ is the Tracy-Widom distribution with $\beta = 2$. Figure 4 shows that the measured distribution of height fluctuations in a type-I pattern (within patch 2) agrees extremely well with the prediction in eq. (5). Equation (5) also shows that the background in eq. (4) is transient only at $p=1$. Thus, a transition from recurrence to transience happens at $p=1$, and it falls in the universality class of the wetting transition seen in 3D anisotropic directed percolation [6].

A background with a transition at $p < 1$. We next study the recurrence-transience transition in detail for a case that cannot be exactly mapped to a solvable model. Consider the pattern formed by the walker on the random arrow configuration constructed out of $2 \times 2$ unit cells with the following probabilities ("type II"):

$$\text{Prob} \left[ (0, 3) \right] = (1-p)^2, \quad \text{Prob} \left[ (1, 2) \right] = p(1-p), \quad \text{Prob} \left[ (3, 3) \right] = p(1-p), \quad \text{Prob} \left[ (2, 2) \right] = p^2. \quad (7)$$

For $p=0$, this is the unperturbed background whose visit function is given by eq. (2), while for $p=1$ it is transient. Figure 5(a) shows that the defects create terraces in three patches of the unperturbed background. Figure 5(b) shows that for $p=0.45$, on the other hand, the path of the walker visits only a small fraction of the sites before leaving the lattice. In particular, it visits the origin only $O(1)$ times even on a large lattice (see fig. 7). The recurrence-transience transition happens at $p_c \approx 0.4$. The rules for constructing the terraces can be determined, and are not the same as for the previous case, and cannot be exactly mapped to a solvable model (see SI). But the terraces and the corresponding height field $h(i, j) = V(i, j) - V^0(i, j)$ can still be constructed given the positions of the defect cells. The height increases by 4 when crossing from the outside of a terrace to the inside. It can be seen that the height fluctuations follow a Tracy-Widom distribution in this case as well (see fig. 6), although the data in the tails is not good enough to distinguish between the cases $\beta = 1$ and 4.

We next give an interpretation of the transition in terms of the height field. Since the boundary of the pattern is given by $V^0(i, j, M) + h(i, j) = 0$, the transition point is simply the point at which the (positive) slope of the height

Fig. 6: (Colour online) The histogram for fluctuations about the average height for a type-II pattern with $p = 0.2$. Also shown is the best-fit shifted Tracy-Widom type-2 distribution, with the best-fit values $m = 197.32$ and $s = 22.08$. Since this is only the central part of the distribution, the fits for shifted Tracy-Widom distributions with $\beta = 1$ and 4 are equally good, but with different parameter values (not shown).

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field equals the (negative) slope of $V^0$, which is 2 (see eq. (2)). For type-II patterns, the height field increases fastest along the positive $x$-axis, and thus, the recurrence-transience transition at $p = p_c$ happens along a preferred direction, the positive $x$-axis. Above $p = p_c$ the height field calculated from the constructed terraces does not describe the path of the walker, as $V = V^0 + h$ has a positive slope and thus cannot be the solution of eq. (3). However, nothing singular happens to the height field itself at $p = p_c$. Even though an exact mapping to a solvable model cannot be constructed for type-II patterns, simulations show that the height field $h(i, j)$, and hence also the boundary of the pattern, shows fluctuations which increase as $(h)^{1/3}$, or as $R^{1/3}$ where $R$ is the distance from the origin (see fig. 7, inset).

Properties of the transition can be determined by looking at the height field. Below $p_c$, the probability that the pattern extends a distance $L$ is, for large $L$, a decaying function of the quantity $L/\xi(p)$, where $\xi(p)$ is a length scale that diverges as $(p-p_c)^{-\nu}$, defining the correlation length exponent $\nu$. We now calculate the exponent $\nu$. As mentioned earlier, the height field does not exhibit any discontinuity at $p_c$, only that its slope along the $x$-axis crosses the value $2 \equiv m_0$. Let the slope of the height field along the $x$-axis be $m = m_0 - \delta m$ for $p = p_c - \delta p$, where $\delta m \sim \delta p$, assuming nothing singular happens to the height field at $p = p_c$. At a distance $L$ from the origin, thus, $\langle V(L) \rangle = -L(\delta m)$. For this site to be part of the pattern, $V(L) > 0$. The variable $\langle V(L) \rangle$ fluctuates about its average with variance $L^{1/3}$. The probability that one observes a fluctuation of order $(V(L))$ is thus a decaying function of $(V(L))/L^{1/3}$, and hence of the variable $L(\delta m)^{3/2} \sim L(\delta p)^{3/2}$, giving $\nu = 3/2$.

The same correlation length exponents are found for the wetting transition in the 3D anisotropic directed percolation model [6], along with KPZ fluctuations of the height field. This strongly suggests that the recurrence-transience transition in rotor-router belongs to the 3D ADP class.

Conclusions and outlook. – We have shown that, for some types of quenched randomness, there is an exact mapping of the pattern formed by the rotor-router walker to the anisotropic directed percolation model. This allowed us to calculate the distribution of the number of visits to a site, which follows a Tracy-Widom type-2 distribution. We also found that for another case where an exact mapping cannot be performed, a height representation exists below $p_c$, and the pattern also shows KPZ fluctuations. The properties of the recurrence-transience transition can be described by the wetting transition in 3D ADP. The properties of the transient walk above $p_c$ are yet to be elucidated.

Although the KPZ property can only be proved in certain cases, we have shown that it holds for a wider range of recurrent backgrounds. For the cases when the recurrence-to-transience transition happens along a preferred direction, and the height field in that direction shows KPZ fluctuations, the analysis performed for type-II patterns can be generalized, and thus the transition should be in the ADP universality class. Also of interest is the effect of quenched randomness in the initial configuration on the intricately structured patterns created by a large number of walkers starting at the origin on periodic transient backgrounds [32]. Similar patterns are also seen when the initial conditions of linearly growing sandpiles [29] are perturbed [52]. It would be interesting to extend the KPZ analysis to this case.

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