Chirotepes of Random Points in Space are Realizable on a Small Integer Grid

Jean Cardinal\textsuperscript{*} Ruy Fabila-Monroy\textsuperscript{†} Carlos Hidalgo-Toscano\textsuperscript{†}

Abstract

We prove that with high probability, a uniform sample of \( n \) points in a convex domain in \( \mathbb{R}^d \) can be rounded to points on a grid of step size proportional to \( 1/\sqrt[3]{n^{d+1}+\varepsilon} \) without changing the underlying chirotope (oriented matroid). Therefore, chirotope of random point sets can be encoded with \( O(n \log n) \) bits. This is in stark contrast to the worst case, where the grid may be forced to have step size \( 1/2^{\Omega(n)} \) even for \( d=2 \).

This result is a high-dimensional generalization of previous results on order types of random planar point sets due to Fabila-Monroy and Huemer (2017) and Devillers, Duchon, Glisse, and Goaoc (2018).

1 Introduction

Chirotope, order types, and oriented matroids.

Many interesting properties of planar point sets in general position are captured by the combinatorial abstraction consisting of the orientation – clockwise or counterclockwise – of every triple of points. This generalizes naturally to \( d \)-dimensional point sets. Consider a set \( S \subset \mathbb{R}^d \) of \( n \) points in general position (no \( d+1 \) on a hyperplane). Let us denote by \( \Lambda(S,k) \) the set of all ordered \( k \)-tuples of distinct points of \( S \). With every ordered \( (d+1) \)-tuple \( P = (p_1, p_2, \ldots, p_{d+1}) \in \Lambda(S, d+1) \) of points we can associate a binary value \( \chi(P) \) indicating the orientation of the corresponding simplex. This can be expressed as the sign of a determinant:

\[
\chi(P) = \text{sgn} \left| \begin{array}{cccc}
p_{1,1} & p_{1,2} & \ldots & p_{1,d} \\
p_{2,1} & p_{2,2} & \ldots & p_{2,d} \\
\vdots & \vdots & \ddots & \vdots \\
p_{d+1,1} & p_{d+1,2} & \ldots & p_{d+1,d}
\end{array} \right|
\]

The map \( \chi \) is referred to as the chirotope of the point set \( S \). The values of \( \chi(P) \) obey the chirotope axioms, in particular the Grassmann-Pfucker relations, and completely characterize the rank-\((d+1)\) oriented matroid defined by the point set. Note that not all maps satisfying the chirotope axioms are chirotope of point sets. The **Topological Representation Theorem**, however, ensures that they always have a representation as a collection of pseudohyperplanes. For \( d=2 \), such a representation is a pseudoline arrangement. We refer the reader to the classical text from Björner, Las Vergnas, Sturmfels, White, and Ziegler [4] for more background on oriented matroids and chirotope.

We say that two sets of points have the same order type whenever there exists a bijection between them such that the chirotope is preserved. By extracting the purely combinatorial features of a set of points, oriented matroids and order types are useful tools in discrete and computational geometry. In computational geometry, it is the case for instance that the chirotope of a point set is sufficient to compute its convex hull, and therefore provides a simple query-based computation model for this task, in a way that is reminiscent to comparison-based sorting. Knuth explored such a model and several generalizations in his book **Axioms and Hulls** [19]. As for discrete geometry, Eppstein’s recent book on forbidden configurations [9] contains a thorough, unified treatment of major results in geometry of planar point sets through the lens of monotone properties of chirotopes. The **Order Type Database** from Aichholzer et al. contains all equivalence classes of chirotopes realized by sets of up to 10 points in the plane [3].

Algebraic universality and bit complexity. It is not clear, however, how much information is contained in a chirotope. For \( d=2 \), it is known that a chirotope can be encoded using \( O(n^2) \) bits [14, 25, 11, 12]. Recently, it has even been shown that chirotopes induced by sets of \( n \) points in \( \mathbb{R}^d \) could be stored using \( O(n^d \log \log n)^2/\log n) \) bits, in such a way that the orientation of every \((d+1)\)-tuple can be recovered in \( O(\log n/\log \log n) \) time on a word RAM [6].

These representations, however, are distinct from the natural representation of a set of points by \( d \)-tuples of coordinates. The reason is that such a representation can have exponential bitsize: for every \( n \) there exists a chirotope of a set of \( n \) points in the plane, every geometric realization of which requires coordinates that are doubly exponential in \( n \) [15]. This is only one consequence of a more general phenomenon, known as algebraic universality of rank-three oriented matroids, and
proved by Mnëv [21, 22] and Richter-Gebert [23]. In a nutshell, it states that for any semialgebraic set $A$, there exists an oriented matroid whose realization space is stably equivalent, in particular homotopic, to $A$. Algebraic universality holds for other discrete geometric structures such as unit disk graphs [20, 16], 4-dimensional polytopes [24], simplicial polytopes [1], and $d$-dimensional Delaunay triangulations [2].

It is likely, however, that this worst-case exponential coordinate bitsize is irrelevant for “typical” point sets, or for point sets occurring in applications. It is therefore natural to wonder what is the required coordinate bitsize for chirotopes of random point sets. The model we will assume here is that of a uniform distribution on a fixed compact, convex domain.

**Related works.** The question of random order types has first been tackled by Bokowski, Richter-Gebert, and Schindler [5]. They attribute the question of estimating the probability of an order type to Goodman and Pollack, and investigate it under the assumption of a uniform distribution on the Grassmannian. They discuss the problem of finding an efficient random generator on the Grassmann manifold from a generator for the unit interval. They also perform experiments supporting the conjecture that the maximum probability is reached by the oriented matroid corresponding to the cyclic polytope.

Recently, Fabila-Monroy and Huemer [10] proved that with high probability, a uniform sample of points in the plane can be rounded to a $n^{3+\epsilon} \times n^{3+\epsilon}$ grid without altering its chirotope. Even more recently, Devillers, Duchon, Glisse, and Goaoc [7] investigated the number of bits that need to be read from the coordinates of random points to know their order type, and obtain the same result in a slightly more general setting. They also raise the question of whether uniform samples yield a vanishing fraction of order types. The difficulty of sampling order types uniformly was also discussed by Goaoc, Hubard, de Joannis de Verclos, Sereni, and Volec [13].

Another closely related line of work is that of random alignment and shape distribution of triangles [18, 17]. Probabilistic analyses supporting the idea that near-alignment of points occur naturally in random sets were applied in particular to alignments of quasars [8], and debunking pseudoscientific claims on mysterious alignments between archaeological sites in Great Britain [26]. It is known from these works, for instance, that for a uniform random sample of $n$ points in the unit square, the expected number of triples contained in a slab of width $\epsilon$ is proportional to $en^3$. Hence unless the width is chosen to be at most proportional to $n^{-3}$, the sample contains near-aligned points, whose orientation is likely to be flipped by a rounding procedure.

**Our contribution.** We generalize the result of Fabila-Monroy and Huemer [10] and Devillers et al. [7] to $d$-dimensional point sets. We prove that in a uniform sample of $n$ points in $\mathbb{R}^d$, points can be rounded to a $n^{d+1+\epsilon} \times \cdots \times n^{d+1+\epsilon}$ grid without altering their chirotope. We believe that the proof is simpler than the previous ones, even in the case $d = 2$.

**2 Chirotopes**

We begin with a simple observation on the structure of cells in an arrangement of $d+1$ hyperplanes in $\mathbb{R}^d$.

**Lemma 1** Let $P := \{p_0, \ldots, p_d\}$ be a set of $d+1$ points in general position in $\mathbb{R}^d$. Let $\mathcal{H}$ be the hyperplane arrangement generated by all the hyperplanes passing through $d$ points of $P$. Let

- $R_1, \ldots, R_{d+1}$ be the unbounded cells of $\mathcal{H}$ that do not contain a facet of $\text{conv}(P)$ in their boundary; and
- $S_1, \ldots, S_{d+1}$ be the unbounded cells of $\mathcal{H}$ that do contain a facet of $\text{conv}(P)$ in their boundary.

Then there is no hyperplane that simultaneously intersects all the $S_i$ or all the $R_i$.

**Proof.** By doing an affine transformation we may assume that $p_0 = 0$ and $p_i$ is the vector with 1 in its $i$-th
coordinate and 0 in all other coordinates. Note that the $R_i$ and $S_i$ are defined by

$$R_i := \{x \in \mathbb{R}^d : x_j < 0 \forall j \in [d] \setminus \{i\} \land \sum_{j \in [d]} x_j > 1\},$$

$$R_{d+1} := \{x \in \mathbb{R}^d : x_j < 0 \forall j \in [d]\}$$

and

$$S_i := \{x \in \mathbb{R}^d : x_i < 0 \land x_j > 0 \forall j \in [d] \setminus \{i\} \land \sum_{j \in [d]} x_j < 1\},$$

$$S_{d+1} := \{x \in \mathbb{R}^d : x_j > 0 \forall j \in [d] \land \sum_{j \in [d]} x_j > 1\}.$$

For $d = 2$, the observation is direct and illustrated on Figure 1. For $d > 2$, we proceed by induction and suppose that the result holds for $d - 1$. Consider a hyperplane $h$ intersecting the regions $R_1, R_2, \ldots, R_d$. In both cases, if $h$ intersects $R_{d+1}$, then it must intersect $R_{d+1} \cap h'$, where $h'$ is one of the hyperplanes of equation $x_j = 0$ for $j \in [d]$. The intersection of the whole arrangement with $h'$ yields a similar situation in dimension $d - 1$, for which the statement holds by induction. Therefore, $h$ cannot intersect $R_{d+1}$. The proof for the $S_i$ is similar.

We now give a sufficient condition for two order tuples to have the same orientation after a perturbation.

**Lemma 2** Consider two ordered $(d + 1)$-tuples $P := (p_1, p_2, \ldots, p_{d+1})$ and $Q := (q_1, q_2, \ldots, q_{d+1})$ of points in $\mathbb{R}^d$. For every $1 \leq i \leq d + 1$, let $f_i$ be the hyperplane passing through the facet of conv($P$) opposite to $p_i$.

Suppose that for every $1 \leq i \leq d + 1$ the following two conditions hold.

1) $p_i$ and $q_i$ are on the same open halfspace bounded by $f_i$; and

2) the distances from $q_i$ to $f_i$ and from $p_i$ to $f_i$ are both larger than the distance from $q_j$ to $f_i$, for all $j \neq i$.

Then $P$ and $Q$ have the same orientation.

**Proof.** By 1) and 2), for every $1 \leq i \leq d + 1$ there exists a hyperplane $h_i$ parallel to $f_i$ that separates $p_i$ and $q_i$ from the other $q_j$ ($j \neq i$). Let $H$ be the hyperplane arrangement generated by the $h_i$. Note that for every $1 \leq i \leq d + 1$, $p_i$ and $q_i$ lie in the same cell of $H$. Since the hyperplanes $h_i$ are parallel to the facets of conv($P$), $H$ can be of one of two types, depending on whether the unique bounded cell of $H$ has the same or the opposite orientation as conv($P$), see Figure 2.

Let $C_1, \ldots, C_{d+1}$ be the cells of $H$ containing $p_i$ and $q_i$, respectively. Note that the $C_i$ are either the $R_i$ or the $S_i$ defined in Lemma 1. In both cases no hyperplane can intersect all the $C_i$ simultaneously. For every $1 \leq i \leq d + 1$, let $f_i'$ be the hyperplane passing through the $q_i$ different from $q_i$. Note, $f_i'$ intersects all the $C_j'$ with $j \neq i$. Thus, $f_i'$ does not intersect $C_i$, and $p_i$ and $q_i$ are on the same open halfspace defined by $f_i$. Therefore, $P$ and $Q$ have the same orientation.

We now prove our main result by showing that if $S$ is a random point set and $S'$ is obtained by rounding $S$ on a sufficiently dense grid, then the conditions of the lemma hold for every pair composed of a $d$-simplex in $S$ and its corresponding rounded version in $S'$.

**Theorem 3** Let $S$ be a uniform sample of $n$ points in the $d$-dimensional unit ball. Then for every $\epsilon > 0$, with probability at least $1 - O\left(\frac{1}{n^{d+1}}\right)$, the points of $S$ can be rounded to a grid of step size $1/(n^{d+1+\epsilon})$ without changing their chirotope.

**Proof.** Let $M := n^{d+1+\epsilon}$. Let $S'$ be the image of $S$ after rounding each point to its nearest neighbor on a grid of step size $1/M$. Consider a $(d + 1)$-tuple of points $P := (p_1, \ldots, p_{d+1})$ in $S$ and the corresponding $(d + 1)$-tuple of rounded points $Q := (q_1, \ldots, q_{d+1})$ in $S'$. As in Lemma 2 let $f_i$ be the hyperplane passing through the facet of conv($P$) opposite to $p_i$. We prove that the conditions of Lemma 2 hold with high probability. By definition, for any given $j$, the absolute difference between $p_j$ and $q_j$ is at most $\sqrt{d}/M$. Thus the conditions of Lemma 2 hold if the distance from $p_i$ to $f_i$ is at least $2\sqrt{d}/M$. Let $B_d$ be the unit $d$-dimensional ball. The $d - 1$-volume of the intersection of $B_d$ and the hyperplane containing $f_i$ is at most $\text{vol}(B_d - 1)$. Thus, the probability that for a given $1 \leq i \leq d + 1$ the distance from $p_i$ to $f_i$ is less or equal to $2\sqrt{d}/M$ is at most $1 - O\left(\frac{1}{n^{d+1}}\right)$. Therefore, the probability that for every $1 \leq i \leq d + 1$ the distance from $p_i$ to $f_i$ is less or equal to $2\sqrt{d}/M$ is at most $1 - O\left(\frac{1}{n^{d+1}}\right)$.
We apply the union bound over all such bad events. There are \((d + 1)\binom{n}{d+1}\) such events to consider. Thus, the probability that no \((d + 1)\)-tuple has a different orientation as the corresponding \((d + 1)\)-tuple in \(S'\) is at least

\[
1 - \binom{n}{d+1} \frac{(d+1)d^{3/2}}{\sqrt{\pi M}} = 1 - O\left(\frac{1}{n^e}\right).
\]

Note that we considered the uniform distribution on the unit ball for convenience. The same analysis holds for any fixed convex body, where the probability of a bad event happening depends on the discrepancies in the distributions of the projections in different directions.

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