A Collatz-type conjecture on the set of rational numbers

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October 19, 2010

Abstract

Define \( \theta(x) = (x - 1)/3 \) if \( x \geq 1 \), and \( \theta(x) = 2x/(1 - x) \) if \( x < 1 \).
We conjecture that the orbit of every positive rational number ends in 0. In particular, there does not exist any positive rational fixed point for a map in the semigroup \( \Omega \) generated by the maps \( 3x + 1 \) and \( x/(x + 2) \). In this paper, we prove that the asymptotic density of the set of elements in \( \Omega \) that have rational fixed points is zero.

Introduction. Let \( \theta \) be the following function on \( \mathbb{Q}_+ \), the set of nonnegative rational numbers:

\[
\theta(x) = \begin{cases} 
(x - 1)/3 & x \geq 1 \\
2x/(1 - x) & x < 1 
\end{cases}
\]

We make the following conjecture:

Conjecture 1. For every \( x \in \mathbb{Q}_+ \), there exists \( n \geq 1 \) so that \( \theta^n(x) = 0 \).

Equivalently, the conjecture states, that the orbit of 1 under the action of the semigroup generated by \( r(x) = 3x + 1 \) and \( s(x) = x/(x + 2) \) is the entire set of positive rational numbers. In particular, there should not exist a function \( t = r^{\beta_1} \circ s^{\alpha_1} \ldots r^{\beta_k} \circ s^{\alpha_k}, k \geq 1 \), that has a positive rational fixed point, i.e.
a rational number \( x > 0 \) so that \( t(x) = x \). Here the composition of functions is denoted by \( \circ \) and the composition of a function \( u \) with itself \( n \) times is denoted by \( u^n \).

The more general problem is to find all pairs of real linear fractional maps with the property that the semigroup generated by the pair has an orbit that contains all of the rational numbers in some interval. A well-known example of such a pair is the pair of \( f(x) = x + 1 \) and \( g(x) = x/(x + 1) \). The orbit of 1 under the action of \( \langle f, g \rangle \), the semigroup generated by \( f \) and \( g \), is the set of all positive rational numbers. This can be seen by noticing that the map

\[
\phi(x) = \begin{cases} 
  x - 1 & \text{if } x \geq 1 \\
  x/(1 - x) & \text{if } x < 1
\end{cases}
\]

has the following property: Let \( \{p_i/q_i\}_{i=1}^\infty \) be the \( \phi \)-orbit of a given \( x = p_1/q_1 \) so that \( p_i, q_i \geq 0 \) and \( (p_i, q_i) = 1 \). Then \( p_i + q_i \) is non-increasing along the orbit, which implies that the orbit ends in zero.

Another way to see that the \( \langle f, g \rangle \)-orbit of 1 is \( \mathbb{Q}_+ \) is by recalling that the matrices

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]

provide a multiplicative basis for the matrices in \( SL_2(\mathbb{Z}) \) with nonnegative entries \(^2\). There is a natural homomorphism from invertible \( 2 \times 2 \) real matrices under matrix multiplication to real linear fractional transformations under composition:

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto t(x) = \frac{ax + b}{cx + d}.
\]

The matrices in \(^1\) correspond to \( f(x) = x + 1 \) and \( g(x) = x/(x + 1) \). Now, let \( b, d \in \mathbb{N} \) be coprime. We show that \( b/d \) belongs to the orbit of 1 under the action of \( \langle f, g \rangle \). Choose \( a, c \in \mathbb{N} \) so that \( ad - bc = 1 \). Since the matrix \( A = [a, b; c, d] \) belongs to the semigroup generated by the matrices in \(^1\), we conclude that \( (ax + b)/(cx + d) \) belongs to \( \langle f, g \rangle \). In particular, \( b/d \) belongs to the orbit of zero, hence it belongs to the orbit of 1 under the action of \( \langle f, g \rangle \).

If \( a, b, c, d \in \mathbb{Z} \) and \( t(x) \) has a rational fixed point, then \( A \) has integer eigenvalues. The converse is partially true: if \( A \) has integer eigenvalues, then either \( t(x) \) has a rational fixed point or \( c = 0 \), \( b \neq 0 \), and \( a \neq d \). Let

\[
R = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.
\]
The following conjecture is weaker than Conjecture 1.

**Conjecture 2.** The only matrices in the semigroup generated by $R$ and $S$ that have integer eigenvalues are $R^n$ and $S^n$, $n \geq 0$.

Let $\Lambda = \langle R, S \rangle$ denote the semigroup generated by the matrices $R$ and $S$, and $\Lambda_{k,M}$ denote the set of matrices of the form $f = R^{\beta_1}S^{\alpha_1} \cdots R^{\beta_k}S^{\alpha_k}$, where $0 < \alpha_i \leq M$ for $i < k$ and $0 < \beta_i \leq M$ for $i > 1$. Finally, let $\Omega_{k,M}$ denote the subset of $\Lambda_{k,M}$ consisting of matrices that have integer eigenvalues. In this paper, we prove the following theorem.

**Theorem 1.** For any fixed $k \geq 2$, the asymptotic density of $\Omega_{k,M}$ in $\Lambda_{k,M}$ is zero, i.e.

$$
\lim_{M \to \infty} \frac{|\Omega_{k,M}|}{|\Lambda_{k,M}|} = 0 .
$$

We note that $\Lambda$ is a free semigroup, i.e. every element in $\Lambda$ can be written in a unique way as a word in $R$ and $S$. On the contrary suppose there are distinct words $f$ and $g$ so that $f = g$ as matrices. We can assume, without loss of generality that $f = Rf'$ and $g = Sg'$, where $f'$ and $g'$ are words in $R$ and $S$ (possibly the empty word). But then the image of $f$ as a map from $\mathbb{R}^2_+$ to $\mathbb{R}^2_+$ is included in the region $\{(x,y) : x \geq y\}$, while the image of $g$ is included in the region $\{(x,y) : x \leq y\}$, which implies that $f$ and $g$ cannot equal each other, and so $\Lambda$ is a free semigroup. In particular, $|\Lambda_{k,M}| = (M+1)^2M^{2k-2}$.

In general, the problem of finding a matrix with integer eigenvalues in a semigroup of matrices might be undecidable, i.e. there might not exist an algorithm that can determine if the semigroup generated by two given matrices contains a matrix with integer eigenvalues. See [1] for some examples of undecidable problems on semigroups generated by two matrices. On the other hand, the probability of an integer matrix having integer eigenvalues is zero [4]; more precisely, for any $\epsilon > 0$, the probability that an $n \times n$ matrix with integer entries bounded in absolute value by $k$ has an integer eigenvalue is less than $Ck^{\epsilon-1}$, where $C$ depends on $\epsilon$ and $n$.

It is also worth mentioning that the orbit of 1 under the action of the semigroup generated by the maps $3x + 1$ and $x/(x+2)$ is indeed dense in the set of $[0, \infty)$. More generally, the orbit of every $x > 0$ under the action of the semigroup generated by $ax + 1$ and $x/(x+b)$ is dense, if $a, b > 1$; see [3] for a complete list of pairs of real linear fractional transformations that generate a semigroup with dense orbits.
A trace formula. We now find closed-form formulas for the entries of the matrix \( f = R^{\beta_1} S^{\alpha_1} \ldots R^{\beta_k} S^{\alpha_k} \). Fix \( k \geq 1 \), and let \( P_k \) denote the set of subsets of \( \{1, 2, \ldots, k\} \). For \( 1 \leq i, j \leq k \), let

\[
\sigma_{ij}^k = \begin{cases} 
-1 & j = i, i + 1 \\
+1 & \text{otherwise}
\end{cases}
\]

For \( A, B \in P_k \), let

\[
\sigma^k(A, B) = 2^{-k + \sum_{i \in A} \alpha_i + \sum_{j \in B} \beta_j} \prod_{i \notin A, j \notin B} \sigma_{ij}^k .
\]

Next, we define:

\[
U_{00}^k = \sum_{k \notin A, 1 \notin B} \sigma^k(A, B),
\]

(3)

\[
U_{10}^k = \sum_{k \in A, 1 \notin B} \sigma^k(A, B),
\]

(4)

\[
U_{01}^k = \sum_{k \notin A, 1 \in B} \sigma^k(A, B),
\]

(5)

\[
U_{11}^k = \sum_{k \in A, 1 \in B} \sigma^k(A, B).
\]

(6)

**Proposition 2.** For \( f = R^{\beta_1} S^{\alpha_1} \ldots R^{\beta_k} S^{\alpha_k} \), the four entries of the matrix \( f \) are given by

\[
f_{11} = U_{00}^k + U_{01}^k - U_{10}^k + U_{11}^k,
\]

(7)

\[
f_{12} = U_{11}^k - U_{10}^k,
\]

(8)

\[
f_{21} = 2U_{10}^k - 2U_{00}^k,
\]

(9)

\[
f_{22} = 2U_{10}^k.
\]

(10)

In particular, the trace of \( f \) is given by

\[
\text{tr}(f) = \sum_{A, B \in P_k} \sigma^k(A, B).
\]

(11)

**Proof.** Proof is by induction on \( k \). Suppose the proposition is true for \( f \), and let \( g = f R^{\beta} S^{\alpha} \). We prove that \( g_{22} = 2U_{00}^{k+1} \), the proof for other entries is similar and is omitted. By the matrix multiplication, we have

\[
g_{22} = \frac{1}{2} f_{21} (3^{\beta} 2^\alpha - 2^\alpha) + f_{22} 2^\alpha .
\]
Thus, using the inductive hypothesis, we need to show that

\[ 2U_{10}^{k+1} = -U_{10}^k 3^\beta 2^\alpha + U_{00}^k 2^\alpha + U_{10}^k 3^\beta 2^\alpha + U_{10}^k 2^\alpha. \]  \tag{12}

The pairs \((A, B)\) with conditions \(k + 1 \in A\) and \(1 \notin B\) in the definition of \(U_{10}^{k+1}\) (equation (11)) can be divided into four groups: group I: \(k \in A\) and \(k + 1 \in B\), group II: \(k \notin A\) but \(k + 1 \in B\), group III: \(k \in A\) but \(k + 1 \notin B\), and group IV: \(k \notin A\) and \(k + 1 \notin B\). Now, for \((A, B)\) in group I, we have:

\[ \prod_{i \notin A, j \in B} \sigma_{ij}^{k+1} = \prod_{i \notin A \cap P_k, j \in B \cap P_k} \sigma_{ij}^k \prod_{j \notin B \cap P_k} 1, \]  \tag{13}

since \(\sigma_{ij}^{k+1} = 1\) for \(i \notin A\) and \(j = k + 1\). It follows that for \((A, B)\) in group I, we have

\[ 2 \sum_{(A, B) \in I} \sigma_{k+1}^{(A, B)} = U_{10}^k 3^\beta 2^\alpha. \]  \tag{12}

Similarly one can prove that the other three terms in the right side of (12) are accounted for by the other three groups. \[ \square \]

Let \(\eta\) be the following map on \(P_k\). For \(B \in P_k\), let \(\eta(B) = \{ i - 1, i \in B\}\), where the indices are understood to be modulo \(k\), i.e. the index 0 is identified with the index \(k\). Also let \(\overline{B} = B \oplus \eta(B)\) denote the symmetric difference of \(B\) and \(\eta(B)\). Then another way to write formula (11) is

\[ \text{tr}(f) = 2^{-k} \sum_{B \in P_k} 3 \sum_{i \in \eta(B)} \prod_{j \in \overline{B}} (1 + 2^\alpha_j) \prod_{j \in B} (1 + 2^\alpha_j). \]  \tag{14}

Hence, we have the following upper and lower bounds for \(\text{tr}(f)/\det(f)\).

Corollary 3. For \(f = R^{B_1} S^{\alpha_1} \ldots R^{B_k} S^{\alpha_k}\), we have

\[ \det(f) \leq 2^k \text{tr}(f) \leq \prod_{i=1}^k (1 + 3^\beta_i)(1 + 2^\alpha_i). \]  \tag{15}

Now, we are ready to present the proof of Theorem 1.

**Proof of Theorem 1** Let \(\lambda\) and \(\mu\) be the eigenvalues of \(f = R^{B_1} S^{\alpha_1} \ldots R^{B_k} S^{\alpha_k}\)
so that $\lambda \leq \mu$. Both $\mu$ and $\lambda$ are positive, since $\text{tr}(f)$ and $\det(f)$ are positive. It follows from (15) that $\lambda \mu \leq 2^k(\lambda + \mu)$ and so

$$(\lambda - 2^k)(\mu - 2^k) \leq 4^k.$$ 

Suppose that $\lambda$ and $\mu$ are integers. The above inequality implies that either $\lambda \leq 2^k$ or $\mu \leq 4^k$. In the latter case $\det(f) \leq \mu^2 \leq (4^k + 2^k)^2$. Since there are only a finite number of matrices in $\Lambda_{k,M}$ that have their determinant bounded by $(4^k + 2^k)^2$, we can ignore this case in computing the asymptotic density. Thus, we suppose that $\lambda \leq 2^k$. On the other hand, again by (15), we have

$$\lambda > \frac{\det(f)}{\text{tr}(f)} \geq 2^k \prod_{i=1}^{k} \frac{2^{\alpha_i}3^{\beta_i}}{(2^{\alpha_i} + 1)(3^{\beta_i} + 1)}.$$ 

We can choose $N = N(k)$ large enough so that the right hand side of the inequality above is greater than $2^k - 1$ if $\alpha_i, \beta_i > N(k)$ for all $i = 1, \ldots, k$. It follows that if $\alpha_i, \beta_i > N(k)$ for all $i$, then $2^k - 1 < \lambda \leq 2^k$, and so $\lambda = 2^k$. This cannot occur, since otherwise $4^k + \det(f) = 2^k \text{tr}(f) \geq \det(f) + 3 \sum \beta_i \geq \det(f) + 3^k N(k)$, which is a contradiction. So we have proved that for $N(k)$ large enough, there is no $f$ with $\alpha_i, \beta_i > N(k)$ that has integer eigenvalues, and so

$$\frac{|\Omega_{k,M}|}{|\Lambda_{k,M}|} \leq 1 - \frac{(M - N(k))^{2k}}{(M + 1)^2 M^{2^k-2}} \to 0,$$

as $M \to \infty$.

Remark. A similar argument proves the following more general result. For integers $a, b \geq 2$ and $u, v \geq 1$, let $A = [a, u; 0, 1]$ and $B = [1, 0; v, b]$. Then for every $k \geq 1$, there exists $N(k)$ so that if $\alpha_i, \beta_i > N(k)$ for $i = 1, \ldots, k$, then the matrix $f = B^{\beta_1}A^{\alpha_1} \ldots B^{\beta_k}A^{\alpha_k}$ has no integer eigenvalues.

References

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