Geometry of the logarithmic Hodge moduli space

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with an Appendix joint with Siqing Zhang

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Funding information
National Science Foundation, Grant/Award Numbers: 1901975, 2200492; Simons Fellowship in Mathematics, Grant/Award Number: 672936

Abstract
We show the smoothness over the affine line of the Hodge moduli space of logarithmic $t$-connections of coprime rank and degree on a smooth projective curve with geometrically integral fibers over an arbitrary Noetherian base. When the base is a field, we also prove that the Hodge moduli space is geometrically integral. Along the way, we prove the same results for the corresponding moduli spaces of logarithmic Higgs bundles and of logarithmic connections. We use smoothness to derive specialization isomorphisms on the étale cohomology rings of these moduli spaces; this includes the special case when the base is of mixed characteristic. In the special case where the base is a separably closed field of positive characteristic, we show that these isomorphisms are filtered isomorphisms for the perverse filtrations associated with the corresponding Hitchin-type morphisms.

MSC 2020
14D20 (primary), 14D23, 14F20 (secondary)
1  INTRODUCTION

Let $C$ be a compact Riemann surface. The nonabelian Hodge theorem (NAHT) of Simpson, Corlette, and others (see [37] and references therein), yields a canonical homeomorphism between three different moduli spaces of objects on the curve, namely, the moduli spaces of: semistable Higgs bundles of fixed rank $n$ and degree $d = 0$ (Higgs moduli space); algebraic flat connections of rank $n$ (de Rham moduli space); representations of the fundamental group of the curve into the general linear group $GL_n$ (Betti moduli space).

In particular, the cohomology rings of these moduli spaces are canonically isomorphic, a fact that we may call Cohomological NAHT. Following a suggestion by Deligne, Simpson has introduced the moduli space of semistable $t$-connections (Hodge moduli space) on the curve, which interpolates between the moduli space of Higgs bundles (set $t = 0$) and the one of algebraic flat connections (set $t = 1$). The Hodge moduli space is topologically trivial over the affine line (corresponding to the parameter $t$) and one can view this triviality as an incarnation of the NAHT.

The NAHT on a curve over the complex numbers has no direct analog for curves over fields of positive characteristic. In this context, there is a Frobenius-twisted NAHT (cf. [10, 19, 33]), but this does not identify moduli spaces. Absent such an NAHT, one can ask whether one still has an isomorphism between the cohomology rings of the Higgs and de Rham moduli spaces. Under natural conditions of coprimality involving rank, degree, and characteristic, the smoothness of the Hodge moduli space and the isomorphism on cohomology rings have been addressed in [13].

In this paper, we study logarithmic $t$-connections on a curve, that is, $t$-connections with at most simple poles along a fixed effective reduced divisor on the curve. We focus on the case of coprime rank $n$ and degree $d$.

We provide an explicit treatment of the deformation theory of $t$-connections on curves leading to the smoothness of the Hodge/Higgs/de Rham moduli spaces. As an application, we construct in the case of coprime rank and degree a canonical isomorphisms between the cohomology rings of these three moduli spaces.

There seems to be no complete study in the literature concerning the smoothness of the moduli of logarithmic $t$-connections, one that constructs modules of obstructions, obstructions classes, and provides an explicit criterion for smoothness. The rest of this paragraph is devoted to summarize the literature we are aware of on the subject. A classical reference for logarithmic Higgs bundles is [31, §6] where the dimension of tangent spaces is computed and smoothness in the coprime case can be derived indirectly, by use of the BNR correspondence (see [11, §2.1], e.g.). The papers [6, 28] have treatments of deformation theory of Hitchin pairs including descriptions of tangent spaces and obstructions, which are used in [28, Cor. 7.9] to prove the smoothness of the moduli space of stable (logarithmic) Higgs bundles. Yokogawa [40] deals with modules of parabolic Higgs bundles and proves that it is integral, normal, and smooth at parabolic stable points; for special parabolic data, that is, trivial filtration and weight adapted to the degree, this result specializes to our picture for logarithmic Higgs bundles. The paper [8] has a treatment of parabolic Higgs bundles when the rank is 2, and shows that the moduli space is simply connected [8, Thm. 4.2]. For a treatment of deformation theory of Hitchin pairs over the complex numbers using dgl language, see [29, Thm. 5.3]. A classical reference for logarithmic connections over the complex numbers is [32], where the tangent space of the moduli problem is described as the hypercohomology of a complex, but where there is no treatment of obstructions. In the case of logarithmic connections with parabolic structure over the curve $\mathbb{P}^1$, the smoothness and irreducibility of the moduli space for generic stability parameters was shown in [22, Thm. 1.1]. There is a treatment of deformation theory of parabolic logarithmic connections in [23], where also the irreducibility of the moduli...
of parabolic logarithmic connections is proven [23, Prop. 5.2, Prop. 5.3, Prop. 5.4]. Sun [39, §5.3] discusses the deformation theory in the more general setup of \(\Lambda\)-modules on Deligne–Mumford stacks, describing a tangent-obstruction theory in this context, but not addressing smoothness nor integrality questions. The article [3] discusses deformation theory and proves smoothness of the moduli space of modules for a Lie algebroid on a smooth projective curve over \(\mathbb{C}\) in the coprime case. This includes the moduli of \(t\)-connections as a special case.

Let us introduce the setup of this paper. Fix a family \(C_B/B\) of smooth projective geometrically integral curves over a Noetherian base scheme \(B\), which could be, for example, a field, or a discrete valuation ring (DVR) of equal or of mixed characteristic. Let \(D_B/B\) be a relative strictly effective reduced divisor on the family of curves. We fix the rank \(n\) and the degree \(d\) of \(t\)-connections (the rank and degree of the underlying vector bundles), and we assume that they are coprime \(g.c.d.(n,d) = 1\).

In this paper, we study the Hodge moduli space \(M_{\text{Hodge} \; \text{ss}} C_B \rightarrow \mathbb{A}^1_B\) of semistable logarithmic \(t\)-connections on \(C_B/B\) of coprime rank \(n\) and degree \(d\), with simple poles along \(D_B/B\). Our first main result is the following.

**Theorem 1.1.** Assume that \(n\) and \(d\) are coprime and the fibers of the divisor \(D\) are nonempty. The structural morphism (cf. Notation 2.2) \(\tau_B : M_{\text{Hodge} \; \text{ss}} C_B \rightarrow \mathbb{A}^1_B\) is smooth. For each \(a \in \mathbb{A}^1_B\), the fiber \((M_{\text{Hodge} \; \text{ss}} C_B)_a\) is geometrically integral of dimension \(n^2(2g - 2 + \deg(D_a)) + 1\). The same is true for the fibers over the points \(b \in B\) of the morphisms \(\nu_{\text{Higgs}, B} \; (2.5), \nu_{\text{de Rham}, B} \; (2.7)\), and (here add +1 to the dimension) \(\nu_{\text{Hodge}, B} \; (2.3)\).

The proof of Theorem 1.1 consists of first studying the deformation theory of \(t\)-connections and then proving that it is unobstructed. The vanishing of the obstruction class is proved using a degeneration argument from de Rham to Higgs inside of Hodge, involving a vanishing theorem that makes essential use of the nonemptiness of the divisor of poles \(D_B/B\) on the geometric fibers of \(C_B/B\). In fact, in Proposition 5.6, we show that under the same nonemptiness assumption, the Hodge stack of semistable objects is smooth over \(\mathbb{A}^1_B\) without imposing any conditions on degree and rank. In the non-coprime case, while the stack is smooth, the moduli space is usually singular; see Remark 5.8.

In §5.3, we complement Theorem 1.1 by proving a similar smoothness assertion in the case without poles, under necessary and thus natural numerical conditions.

We offer two applications (Theorems 3.6 and 3.8) of the smoothness result (Theorem 1.1) that relate to each other the cohomology rings of the Hodge, Higgs, and de Rham moduli spaces. The paper [13] proves a weaker version of Theorem 3.6 in the case without poles, when \(B\) is a field of positive characteristic and the rank and degree are subject to necessary, thus natural, conditions.

Theorems 3.6 and 3.8 could be viewed as the cohomological shadow of a currently nonexisting logarithmic NAHT in arbitrary, even mixed characteristic. Even in the case of curves over the complex numbers, it is not clear to us how the moduli space of logarithmic \(t\)-connections would fit into the context of the parabolic NAHT of Simpson and Mochizuki; see [35] and [30, Cor. 1.5].

The first application Theorem 3.6 (see the companion diagram (12)) is for the case when \(B = \text{Spec}(k)\) is the spectrum of a separably closed field. It shows that in the coprime case with poles, the natural restriction morphisms on cohomology rings (decorations omitted) \(H^*(M_{\text{Higgs}}) \rightarrow H^*(M_{\text{de Rham}})\) are isomorphisms. In fact, it shows that the specialization morphism relating \(H^*(M_{\text{Higgs}})\) and \(H^*(M_{\text{de Rham}})\) satisfies the following.
(1) It is defined; a priori such a morphism does not exist due to lack of properness of the morphism \( \tau_k : \text{MHodge} \to \mathbb{A}^1_k \). We circumvent the lack of properness by means of a suitable completion of the morphism \( \tau_k \).

(2) It is an isomorphism.

(3) If, furthermore, the field has positive characteristic, then all these isomorphisms are filtered isomorphisms for the perverse filtrations associated with the various Hitchin-type morphisms in the picture (see Section 2.4).

The second application Theorem 3.8 (see the companion diagram (16)) is for the coprime case with poles when \( B = \text{Spec}(R) \) is the spectrum of a discrete valuation ring \( R \). In this case, we have nine moduli spaces: Hodge/Higgs/de Rham over the geometric closed point, over the geometric generic point and over the DVR. Their cohomology groups are related by restriction maps (denoted by the letters \( \rho \) and \( r \)). We prove the following.

(a) All these restriction maps are isomorphisms.

(b) The resulting collection of specialization maps are defined and are isomorphisms. Again, we need to circumvent the lack of properness of various structural morphisms by means of suitable compactifications. For technical reasons, if the DVR \( R \) is of mixed characteristic \((0, p > 0)\), we assume that \( p > n \), that is, that the rank is smaller than \( p \).

(c) If, furthermore, the DVR \( R \) has equal positive characteristic, then all these isomorphisms are filtered isomorphisms for the perverse filtrations associated with the various Hitchin-type morphisms.

In both applications, we use compactification methods from [12], suitably generalized in [14]. To this end, we need to recall in §6.1 the construction of suitably good compactifications of the relevant moduli spaces given in [14]. Since such a compactification has not been constructed in the case of Hodge moduli spaces in [14], we provide one here.

Finally, in the Appendix jointly written with S. Zhang, we provide a construction of the Hodge–Hitchin morphism correcting minor inaccuracies in the literature.

2 | MODULI STACKS/SPACES WITH POLES

2.1 | Notation and setup

We work over a Noetherian scheme \( B \). Let \( \pi : C \to B \) be a smooth proper morphism of schemes with geometrically integral fibers of dimension 1. We refer to \( C \to B \) simply as a curve over \( B \), and we denote it by \( C_B \). Let \( D \ll C \) be a relative Cartier divisor such that every geometric \( B \)-fiber of \( D \) is nonempty and reduced.

We fix once and for all two integers \( n > 0 \)-the rank- and \( d \in \mathbb{Z} \)-the degree-. We assume that \( n \) and \( d \) are coprime.

For every morphism of schemes \( S \to B \), we denote the corresponding fiber product \( C \times_B S \to S \), simply by \( \pi_S : C_S \to S \). For example, if \( \bar{b} \to b \to B \) is a geometric point over a Zariski point \( b \) of \( B \), then \( C_b \) denotes the fiber over \( b \) and \( C_B \) denotes the corresponding geometric fiber.

We will often work over the base \( \mathbb{A}^1_B = \text{Spec}_B(\mathcal{O}_B[\tau]) \), equipped with the action of the multiplicative group scheme \( \mathbb{G}_{m,B} \) that assigns \( \tau \) weight 1. For any \( \mathbb{A}^1_B \)-scheme \( S \), we shall denote by \( t_S \) the global section of the structure sheaf \( \mathcal{O}_S \) obtained by pulling back \( \tau \).
If $\text{Spec}(A) \to \mathbb{A}^1_B$ is a morphism from an affine scheme, we might abuse notation and write $\pi_A : C_A \to \text{Spec}(A)$ and $t_A \in A$ as replacement of the notation described above.

If $S \to B$ is a morphism of schemes and $G$ is an object over $B$ (a scheme over $B$, an $\mathcal{O}_B$-module, an $\mathcal{O}_X$-module with $X$ a scheme over $B$, etc.), then $G_S$ denotes the pulled-back object via the morphism $S \to B$.

## 2.2 The Hodge moduli stack/space

### Definition 2.1.
We denote by $\mathcal{M}^{\text{ss}}_{\text{Hodge}}_{C_B} \to \mathbb{A}^1_B$ the moduli stack of (slope) semistable rank $n$ and degree $d$ logarithmic $t$-connections with poles along $D$. As a pseudofunctor, it sends an $\mathbb{A}^1_S$-scheme $S \to \mathbb{A}^1_S$ to the groupoid of pairs $(F, \nabla)$, where

(a) $F$ is a vector bundle of rank $n$ on $C_S$ such that its restriction to each geometric fiber of the morphism $C_S \to S$ has degree $d$.

(b) $\nabla : F \to F \otimes_{\mathcal{O}_{C_S}} \omega_{C_S/S}(D_S)$ is a logarithmic $t_S$-connection with (at most simple) poles allowed at the pullback $D_S$ of $D$.

(c) The restriction of the pair $(F, \nabla)$ to each geometric fiber $C_s$ of the morphism $C_S \to S$ is a semistable $t_s$-connection.

Since $n$ and $d$ are coprime, every semistable pair $(F, \nabla)$ is, in fact, stable.

When dealing with $t$-connections, one uses the sheaf of rings on $C_B \times_B \mathbb{A}^1_B$ given by the Rees degeneration with respect to the order filtration of the enveloping algebra of differential operators associated with the Lie algebroid of relative vector fields on $C \to B$ (see [36, $\tau$-connections, pg. 87] in characteristic 0 and [24, §2.2] for the enveloping algebra over a general base).

The degeneration takes the following shape: at the section $0_B$, it is the push-forward of the algebra of functions on the total space of the relative cotangent bundle $\omega_{C_B/B}$, whose modules are Higgs bundles on $C_B$; at the section $1_B$, it is the sheaf of crystalline differential operators, whose modules are flat connections on $C_B/B$.

In this paper, we deal with poles, so that instead, we use the Rees degeneration of the universal enveloping algebra of differential operators associated with the Lie algebroid of relative vector fields $(\omega_{C_B/B}(D))'$ on $C_B \to B$ vanishing at the poles $D$.

### Notation 2.2.
We denote by $\tau_B : \mathcal{M}^{\text{ss}}_{\text{Hodge}}_{C_B} \to \mathbb{A}^1_B$ the corresponding quasi-projective moduli space of rank $r$ and degree $d$ semistable logarithmic $t$-connections constructed over a base $B$ a $\mathbb{C}$-scheme of finite type in [36] by using geometric invariant theory, and more recently over a general Noetherian base $B$ in [25].

The natural $\mathbb{A}^1_B$-morphism $\mathcal{M}^{\text{ss}}_{\text{Hodge}}_{C_B} \to \mathcal{M}^{\text{ss}}_{\text{Hodge}}_{C_B}$ exhibits $\mathcal{M}^{\text{ss}}_{\text{Hodge}}_{C_B}$ as an adequate moduli space (as in [2]). In §5.1, by using the coprimality of rank and degree, we show that this is a good moduli space (as in [1]).

### Notation 2.3.
We denote the natural morphism obtained by composing with the projection onto $B$ by $\nu_{\text{Hodge}, B} : \mathcal{M}^{\text{ss}}_{\text{Hodge}}_{C_B} \to B$.

Both the stack $\mathcal{M}^{\text{ss}}_{\text{Hodge}}_{C_B}$ and the scheme $\mathcal{M}^{\text{ss}}_{\text{Hodge}}_{C_B}$ are of finite type over $\mathbb{A}^1_B$. The fact that the stack $\mathcal{M}^{\text{ss}}_{\text{Hodge}}_{C_B}$ is locally of finite type follows from the geometric invariant theory (GIT)
setup (or by, e.g., [21, Prop 2.2.2]). On the other hand, \( \mathcal{M}_{\text{Hodge}}^{ss} \) is quasi-projective over \( \mathbb{A}^1_B \), and so, it follows that the stack \( \mathcal{M}_{\text{Hodge}}^{ss} \) with moduli space \( \mathcal{M}_{\text{Hodge}}^{ss} \) is also quasi-compact.

The group scheme \( \mathbb{G}_m, B \) acts on \( \mathcal{M}_{\text{Hodge}}^{ss} \) by scaling the universal logarithmic \( t \)-connection; this induces an action of \( \mathbb{G}_m, B \) on the moduli space \( \mathcal{M}_{\text{Hodge}}^{ss} \). Both morphisms \( \mathcal{M}_{\text{Hodge}}^{ss} \to \mathbb{A}^1_B \) and \( \mathcal{M}_{\text{Hodge}}^{ss} \to \mathbb{A}^1_B \) are \( \mathbb{G}_m, B \)-equivariant.

### 2.3 The Higgs and de Rham moduli stacks/spaces

**Definition 2.4.** The Higgs moduli stack \( \mathcal{M}_{\text{Higgs}}^{ss} \) is defined by the following Cartesian diagram:

\[
\begin{array}{ccc}
\mathcal{M}_{\text{Higgs}}^{ss} & \xleftarrow{\sim} & \mathcal{M}_{\text{Hodge}}^{ss} \\
\downarrow & & \downarrow \\
B(=0_B) & \xrightarrow{t_s=0} & \mathbb{A}^1_B.
\end{array}
\]

For every \( B \)-scheme \( S \to B \), it classifies pairs \((F, \nabla)\) with \( F \) a vector bundle of rank \( r \) and degree \( d \) on \( C_S \) and \( \nabla \) a logarithmic Higgs field \( \nabla : F \to F \otimes_{\varphi_{CS}} \omega_{CS/S}(D_S) \) with poles at \( D_S \).

**Notation 2.5.** We denote by \( v_{\text{Higgs}, B} : \mathcal{M}_{\text{Higgs}}^{ss} \to B \) the quasi-projective moduli space of semistable logarithmic Higgs bundles with poles along \( D \) constructed using GIT as recalled in Notation 2.2.

Since the formation of good moduli spaces is compatible with arbitrary base change, by Lemma 5.3, the canonical morphism for moduli spaces below is an isomorphism (see also [25, Thm. 1.1]):

\[
\mathcal{M}_{\text{Higgs}}^{ss} \xrightarrow{\sim} (\mathcal{M}_{\text{Hodge}}^{ss})_{0_s}.
\]

**Definition 2.6.** The de Rham stack \( \mathcal{M}_{\text{de Rham}}^{ss} \) is defined by the following Cartesian diagram:

\[
\begin{array}{ccc}
\mathcal{M}_{\text{de Rham}}^{ss} & \xleftarrow{\sim} & \mathcal{M}_{\text{Hodge}}^{ss} \\
\downarrow & & \downarrow \\
B(=1_B) & \xrightarrow{t_s=1} & \mathbb{A}^1_B.
\end{array}
\]

It classifies pairs \((F, \nabla)\) with \( \nabla \) a logarithmic connection with poles at \( D \).

**Notation 2.7.** We denote by \( v_{\text{de Rham}, B} : \mathcal{M}_{\text{de Rham}}^{ss} \to B \) the corresponding quasi-projective moduli space of semistable logarithmic connections with poles along \( D \) constructed using GIT.
When restricted over the open $G_{m,B} \subseteq \mathbb{A}^1_B$, both the Hodge moduli stack and space are fiber products over $G_{m,B}$ over $B$ of the de Rham moduli: a $t$-connection with $t$ invertible $(F, V)$ can be rescaled to a connection $(F, \frac{1}{t} V)$. This trivialization is $G_{m,B}$-equivariant:

$$\left( \text{MHodge}^{ss}_{C_S} \right)_{G_{m,B}} \xrightarrow{\sim} \text{Mde Rham}^{ss}_{C_S} \times_B G_{m,B}, \quad (F, V) \mapsto ((F, \frac{1}{t} V), t).$$

(2)

In view of this triviality over $G_{m,B}$, one can show directly that there is an isomorphism

$$\text{Mde Rham}^{ss}_{C_S} \xrightarrow{\sim} (\text{MHodge}^{ss}_{C_S})_{1_B}.$$  

(3)

This isomorphism (3) holds without having to assume the coprimality of rank and degree. At present, we ignore if, absent the coprimality assumption, the same is true for the Higgs moduli space.

### 2.4 Hitchin-type morphisms and perverse filtrations

In this section, we use [25, esp. §4,5] as a reference, but we employ a notation closer to the one in [14, esp. §2.2].

**Notation 2.8.** The Higgs moduli space comes equipped with the Hitchin $B$-morphism:

$$h_{\text{Higgs},B} : MHiggs^{ss}_{C_S} \longrightarrow A(C_B),$$

(4)

with target the vector bundle on $B$ with fibers $A(C_B) := \Phi_{j=1}^n H^0(C_B, (\omega_{C_B/b}(D_B))^\otimes j)$, and which assigns to a Higgs bundle its characteristic polynomial.

**Notation 2.9.** If $B$ has characteristic $p > 0$, then there is the Hodge–Hitchin $B$-morphism (see §A)

$$h_{\text{Hodge},B} : MHodge^{ss}_{C_S} \longrightarrow A(C_B^{(p)}) \times_B \mathbb{A}^1_B,$$

(5)

where $C_B^{(p)} = B \times_{B,f_{B}} C_B$ (absolute Frobenius $f_{B} : B \to B$) is the Frobenius twist of $C_B$ relative to $B$, and the morphism assigns to a $t$-connection with poles the $p$th root of the characteristic polynomial of its $p$-curvature.

Both the Hitchin morphism $h_{\text{Higgs},B}$ (4) and the Hodge–Hitchin morphism $h_{\text{Hodge},B}$ (5) are proper by [25, Thm. 5.9].

**Notation 2.10.** By restricting the Hodge–Hitchin morphism to connections with poles (i.e., $t = 1$), we obtain the de Rham–Hitchin $B$-morphism (a.k.a. the $p$-Hitchin morphism; it is clearly proper):

$$h_{\text{de Rham},B} : M\text{de Rham}^{ss}_{C_S} \longrightarrow A(C_B^{(p)}).$$

(6)
By restricting the Hodge–Hitchin morphism to logarithmic Higgs bundles, we obtain the Hitchin morphism to $A(C_B)$ post-composed with the relative Frobenius $B$-morphism $Fr_{A(C_B)}$; see [14, Lemma 4.3], which, while stated and proved in the case without poles, can be proved in the same way in the logarithmic case.

Each of these Hitchin-type morphisms induces an increasing filtration, called the perverse (Leray) filtration (cf. [12, §2.1]) on the respective $\mathbb{Q}_\ell$-adic cohomology rings (with some decorations omitted)

$$(H^*(MHiggs), P_{Higgs}), (H^*(MHodge), P_{Hodge}), (H^*(M de Rham), P_{de Rham}).$$

Since the relative Frobenius morphism $Fr_{A(C_B)}$ is a universal homeomorphism, the perverse filtrations on the cohomology ring $H^*(MHiggs)$ associated with the Hitchin morphism and with the Hodge–Hitchin morphism restricted to the Higgs moduli space coincide.

### 3 SPECIALIZATION MORPHISMS

The goal of this section is to remind the reader of the notions of specialization morphism and of its filtered counterparts given in [12], so that the two applications of Theorem 1.1 we give in this paper, namely, Theorems 3.6 and 3.8, can be stated.

The typical setup for specialization morphisms is the one of a morphism to a DVR. In Theorem 3.6, the DVR in question is the Henselization of the local ring at the origin of the affine line over an algebraically closed field $k$, and the morphism to it is the restriction of the structural morphism $\tau_k$ of the Hodge moduli space in Notation 2.2. The specialization morphism then relates the cohomology rings of the Higgs and de Rham moduli space, with the one of the Hodge moduli space acting as an intermediary.

In Theorem 3.8, the DVR is arbitrary, and the morphism is the morphism $\tau_B$ in Notation 2.2. The specialization morphism then relates the cohomology rings of the Hodge moduli spaces over the geometric closed and generic points.

#### 3.1 (Filtered) Specialization morphisms

A reference for this section is [12]. We will freely employ the associated formalism of nearby/vanishing cycles [16, XIII]. Specialization morphisms appear in the statements of Theorems 3.6 and 3.8.

Let $(A, a, \overline{a})$ be the spectrum of strictly Henselian DVR, together with its closed and geometric point $i_a = i : a \to A$, open point $\alpha$, and a choice of geometric generic point $j_{\overline{a}} = j : \overline{a} \to A$ induced by a separable closure of $k(\alpha)$. Fix a prime $\ell$ that is invertible in the residue field of $a$.

Let $Y$ be a scheme and let $\nu_Y = \nu : Y \to A$ be a separated morphism of finite type. Let $D^b_c(Y)$ be the $\mathbb{Q}_\ell$-constructible derived category on $Y$. We have the distinguished triangle of functors $(i^*[−1], \psi_\nu[−1], \phi_\nu) : D^b_c(Y) \to D^b_c(Y_a)$, where $i : Y_a \to X$ is the closed embedding of the special fiber, and $\psi_\nu$ and $\phi_\nu$ are the nearby and vanishing cycle functors, with values supported both on $Y_a$ and on $a$ depending on the context. In our notation, $\psi_\nu[−1]$ and $\phi_\nu$ are $t$-exact. We have the base change morphism $i^*\nu_\ast \to \nu_a i^\ast$. 


Let $G \in D^b_c(Y)$. We have the natural morphisms in cohomology (cf. [15, II-6 p. 23]): (we omit pullback notation on $G$)

$$H^*(Y_a, G) \xrightarrow{r_a} H^*(Y = Y_A, G) \xrightarrow{r_\pi} H^*(Y_\pi, G),$$

where we employ the letter $r$ to denote the pullback/restriction in cohomology via the evident morphisms. If $G = \overline{Q}_\ell \cdot Y$, then these are morphisms of cohomology rings.

The reference [12, §1–3] works over the complex numbers with the classical topology. As it is pointed out in the introduction to [12] and in [14, §5], [12, §1–3] remains valid, with only calligraphic changes, in our setup over a DVR.

**Definition 3.1** [12, Defn. 3.1.3]. We say that the specialization morphism $s_{p\nu}(G)$ is defined if the base change morphism $b_{c!\nu}^!$ in [12, Def. 3.1.3, based on diagram (42)] is an isomorphism. In this case, the pullback morphism $r_a$ (8) is invertible, in which case, we define the specialization morphism:

$$s_{p\nu} := s_{p\nu}(G) = r_\pi \circ r_a^{-1} : H^*(Y_a, G) \longrightarrow H^*(Y_\pi, G).$$

If $\nu$ is proper, then the specialization morphism is defined by proper base change. If $\nu$ is not proper, then the specialization morphism can fail to be defined: for example, when $\nu : Y := A \setminus \{a\} \to A$.

For what follows, we refer to [14, §5.2 (rectified perverse $t$-structure over a DVR)]. In particular, we have O. Gabber’s rectified perverse $t$-structure on the $\overline{Q}_\ell$-constructible derived category on $Y$. One way to think of it is to view it, in first approximation, as gluing perverse sheaves on $Y_a$ to perverse sheaves on $Y_\pi$ shifted by $[1]$. We thus have the notion of the perverse filtration $P$ on the cohomology $H^*(Y, G)$ of a $\overline{Q}_\ell$-constructible complex $G$ on $Y$.

Let $f : X \to Y$ be a separated morphism of finite type. Let $\nu_X := \nu_Y \circ f : X \to A$. There is the notion of perverse Leray filtration $P^f$ relative to the morphism $f$ on the cohomology of a $\overline{Q}_\ell$-constructible complex $F$ on $X$, which is defined to be the perverse filtration on the cohomology of the derived direct image $f_*F$ on $Y$, that is, $(H^*(X, F), P^f) := (H^*(Y, f_*F), P)$. When dealing with the cohomology ring of $X$, for convenience, we number the perverse Leray filtration so that 1 lands in the 0th graded subquotient.

Let $F$ be a $\overline{Q}_\ell$-constructible complex on $X$. de Cataldo [12, Def. 3.3.3, based on diagram (55)] defines the notion of filtered specialization morphism for $F$ on $X$ and for the composition $X \to Y \to A$:

$$s_{p\nu} : (H^*(X_a, F), P^{f_a}) \to (H^*(X_\pi, F), P^{f_\pi})$$

relative to the perverse Leray filtrations $P^{f_a}$ and $P^{f_\pi}$. Our notation here differs slightly from [12], and we emphasize that we are considering the specialization morphism for the morphism $\nu_Y : Y \to A$ for the derived direct image complex $f_*F$ on $Y$, filtered by the perverse $t$-structure on $Y$. The special case $f = \text{Id}_Y$ gives the notion of filtered specialization morphism for $G$ on $Y$ for the morphism $Y \to A$; see [12, (49)].

**Definition 3.2** cf. [12, Defn. 3.2.3]. We say that the filtered specialization morphism is defined if the two sequences, labeled by the integers, of base change arrows on the left-hand-side column of [12, (55)] are invertible. In this case, we obtain the filtered morphism (10).
Remark 3.3. If the filtered specialization morphism is defined, then so is the specialization morphism, which is then the morphism underlying the filtered version. In the special case when the morphisms of type $\delta$ are isomorphisms, so that $r_\delta$ is a filtered isomorphism, then we have $P^{f/\delta}(1) \sim P^f \rightarrow P^{f/\tau}(1)$.

### 3.2 Specialization for the Hodge moduli space over a field

The purpose of this section is to introduce and discuss the commutative diagram (12), which we need to state (and to prove) Theorems 3.6.

Context 3.4. Let $B = Spec(k)$ be a separably closed field and let $C_k$ be our curve. Consider the Hodge moduli space $\tau_k : MHodges_{ss}^{C_k} \rightarrow A^1_k$, together with its fibers $M\text{Higgs}_{ss}^{C_k}$ over $0_k$, and $M\text{deRham}_{ss}^{C_k}$ over $1_k$.

Notation 3.5. We denote by the same symbol $\tau_k$ the morphism obtained by base changing $\tau_k$ via the morphism $Spec(\overline{O}_k, \mathbb{A}^1_k) \rightarrow \mathbb{A}^1_k$, where $\overline{O}_k$ is the strict Henselization of the local ring at the origin $0_k \in A^1_k$.

Let $\overline{\infty} \rightarrow \infty \in A^1_k$ be a fixed geometric generic point of the affine line $A^1_k$ induced by a choice of a separable closure of $k(\infty)$. We have the nearby/vanishing-cycle functors $\psi_{\tau_k}$ and $\phi_{\tau_k}$ for this new morphism $\tau_k$.

Because of the product structure of the Hodge moduli space over $G_{m,k} \subset A^1_k$, we have canonical isomorphisms:

$$H^*(0_k, \psi_{\tau_k} \tau_k* \overline{\sigma}_{\tau_k}) \sim H^*((M\text{Hodge}_{ss}^{C_k})_{\overline{\infty}}) \sim H^*(M\text{deRham}_{ss}^{C_k}),$$

(11)

where the first equality is the classical and general fact that the cohomology of the nearby cycle functor applied to the derived direct image via $\tau_k$ agrees with the cohomology of the geometric generic fiber; the second identification is due to the aforementioned product structure, in view of the natural morphism $\overline{\infty} \rightarrow k \rightarrow 1_k$.

We have a commutative diagram, where the arrows are the morphisms induced by restriction/pullback:

$$\begin{align*}
H^*(M\text{Higgs}_{ss}^{C_k}) & \xleftarrow{\overline{\rho}_{\tau_k}} H^*(M\text{Hodge}_{ss}^{C_k}) \xrightarrow{\overline{\rho}_{\tau_k}} H^*(M\text{deRham}_{ss}^{C_k}) \\
\downarrow & \downarrow \downarrow \\
H^*((M\text{Hodge}_{ss}^{C_k})_{0_k}) & \xleftarrow{\overline{\rho}_{\tau_k}} H^*((M\text{Hodge}_{ss}^{C_k})_{\overline{\sigma}_{0_k}A^1_k}) \xrightarrow{\overline{\rho}_{\tau_k}} H^*((M\text{Hodge}_{ss}^{C_k})_{\overline{\infty}}). \\
\end{align*}$$

(12)

The specialization morphism $sp_{\tau_k}$ associated with $\tau_k$ is defined if the morphism $r_{0_k}$ is an isomorphism so that we can set, by using the identification (11), special to our situation:

$$sp_{\tau_k} := r_{\overline{\infty}} \circ r_{0_k}^{-1} : H^*(M\text{Higgs}_{ss}^{C_k}) \rightarrow H^*(M\text{deRham}_{ss}^{C_k}).$$

(13)

The morphism $\tau_k$ is not proper, so that it is $a priori$ unclear that the specialization morphism is defined. de Cataldo and Zhang [13, Thm. 3.5] show that the restriction morphisms $\rho_{0_k}$ and $\rho_{1_k}$ in (12) are isomorphisms in the case of $\text{char}(k) > 0$, without poles, also under some suitable coprimality conditions. The same proof works in the case with poles; see the proof of Theorem 3.6. On
the other hand, de Cataldo and Zhang [13] do not address explicitly the existence and properties of the (filtered) specialization morphism; Theorem 3.6 puts a remedy to these omissions.

**Theorem 3.6.** Let \( B = k \) be a separably closed field and let \( C_k \) be our curve. Assume that the rank \( n \) and the degree \( d \) are coprime, and the fibers of the divisor \( D \) are nonempty. Then, all the morphisms in (12) are isomorphisms of cohomology rings, the specialization morphism \( sp_{\tau_k} \) (13) is defined, it is an isomorphism of cohomology rings, and we have an identification

\[
sp_{\tau_k} = \rho_{1_k} \circ \rho_{0_k}^{-1}.
\]

If, in addition, \( \text{char}(k) > 0 \), then all the morphisms in (12) and (13) are filtered isomorphisms for the respective perverse filtrations as in §3.1.

### 3.3 Specialization for the Hodge moduli space over a DVR

The purpose of this section is to introduce and discuss the commutative diagram (16), which we need to state (and to prove) Theorem 3.8.

**Context 3.7.** Let \((B, s, \overline{\eta})\) be the spectrum of a strictly Henselian DVR with closed geometric point \( s \in B \) and a choice of a geometric generic point \( \overline{\eta} \rightarrow \eta \in B \).

The morphisms (2.3), (2.5), and (2.7) of type \( v_B : M? \to B \) give rise to possible specialization morphisms that we denote by \( sp_{v_B} \). As usual, each one is defined if and when the associated pullback/restriction morphism, denoted as \( r_s \), is an isomorphism, so that we can set \( sp_{v_B} := r_s^{-1} : H^*(M?_s) \to H^*(M?_{\overline{\eta}}) \). There are three potential versions of such specialization morphisms of type \( sp_{v_B} \): the Hodge, the Higgs, and the de Rham version:

\[
sp_{v_{\text{Hodge},B}}, sp_{v_{\text{Higgs},B}}, sp_{v_{\text{de Rham},B}}.
\]

Moreover, in the Hodge case, according to §3.2, esp. (12), we have the possible specialization morphisms \( sp_{\tau_s} \) and \( sp_{\tau_{\overline{\eta}}} \) associated with the structural morphisms \( \tau_s : M\text{Hodge}^{ss}_{C_s} \to A^1_s \) and \( \tau_{\overline{\eta}} : M\text{Hodge}^{ss}_{C_{\overline{\eta}}} \to A^1_{\overline{\eta}} \).

We summarize the discussion above via the natural commutative diagram of restrictions/pullbacks and specializations, all of which are morphisms of cohomology rings: 

\[
\begin{array}{ccc}
H^*(M\text{Hodge}^{ss}_{C_s}) & \xrightarrow{\rho_{1s}} & H^*(M\text{Hodge}^{ss}_{C_s}) \\
sp_{\tau_s} & \Downarrow r_s & \Downarrow r_s \\
H^*(M\text{Hodge}^{ss}_{C_{\overline{\eta}}}) & \xrightarrow{\rho_{1\overline{\eta}}} & H^*(M\text{Hodge}^{ss}_{C_{\overline{\eta}}}) \\
sp_{\tau_{\overline{\eta}}} & \Downarrow r_{\overline{\eta}} & \Downarrow r_{\overline{\eta}} \\
H^*(M\text{Hodge}^{ss}_{C_{\eta}}) & \xrightarrow{\rho_{1\eta}} & H^*(M\text{Hodge}^{ss}_{C_{\eta}}) \\
sp_{\tau_{\eta}} & \Downarrow r_{\eta} & \Downarrow r_{\eta} \\
H^*(M\text{Hodge}^{ss}_{C_{\overline{\eta}}}) & \xrightarrow{sp_{v_{\text{Hodge},B}}} & H^*(M\text{Hodge}^{ss}_{C_{\overline{\eta}}}) \\
sp_{v_{\text{Hodge},B}} & \Downarrow sp_{\tau_{\overline{\eta}}} & \Downarrow sp_{\tau_{\overline{\eta}}} \\
H^*(M\text{Hodge}^{ss}_{C_{\eta}}) & \xrightarrow{sp_{v_{\text{Higgs},B}}} & H^*(M\text{Hodge}^{ss}_{C_{\eta}}) \\
sp_{v_{\text{Higgs},B}} & \Downarrow sp_{\tau_{\eta}} & \Downarrow sp_{\tau_{\eta}} \\
H^*(M\text{Hodge}^{ss}_{C_{\overline{\eta}}}) & \xrightarrow{sp_{v_{\text{de Rham},B}}} & H^*(M\text{Hodge}^{ss}_{C_{\overline{\eta}}}) \\
sp_{v_{\text{de Rham},B}} & \Downarrow sp_{\tau_{\overline{\eta}}} & \Downarrow sp_{\tau_{\overline{\eta}}} \\
\end{array}
\]

(16)
where we have omitted indicating the possible specialization arrow $s_{pv}^{Hodge,B}$ in the central column for graphical reasons, and the specialization arrows are labeled by a “?” because at this stage, we do not know whether they are defined.

de Cataldo and Zhang [13, Prop. 3.3. (ii)] prove that the filtered specialization morphism $s_{pv}^{Higgs,B}$ exists and is an isomorphism in the case without poles under suitable coprimality conditions. The same principle of proof applies here. de Cataldo and Zhang [13] do not address the similar question in the de Rham case, nor in the Hodge case.

Theorem 3.8 puts a remedy to these omissions. We show that the specialization morphisms of type $s_{pv}^{U_B}$ (15) exist and are isomorphisms and that, moreover, they are compatible with the specialization morphisms of type $s_{pt}s$ and $s_{pt}\eta$. For technical reasons, if $B$ is of mixed characteristic $(0, p > 0)$, then we need to assume that $p > n$. In the case of equal positive characteristic $p$, we prove, without the need to assume $p > n$, that this system of specialization morphisms of types $s_{pt}s$ and $s_{pv}^{U_B}$ is also compatible with the perverse filtrations.

**Theorem 3.8.** Let $(B, s, \eta)$ be a strictly Henselian DVR. Assume that the rank $n$ and degree $d$ are coprime, and the fibers of the divisor $D$ are nonempty. If $B$ has mixed characteristic $(0, p > 0)$, then, in addition, we assume that $p > n$.

The specialization morphisms $s_{pv}^{Hodge,B}, s_{pv}^{Higgs,B}, s_{pv}^{deRham,B}$ are defined, are isomorphisms, and they are compatible with the specialization morphisms $s_{pt}s$ and $s_{pt}\eta$ of Theorem 3.6, that is, we have the natural commutative diagram of restrictions/pullbacks and specializations, all of which are isomorphisms of cohomology rings: $(s_{pv}^{Hodge,B}$ is omitted for graphical reasons)

\[
\begin{array}{ccc}
H^*(M_{Higgs}^{ss}_{C_s}) & \leftarrow & H^*(M_{Hodge}^{ss}_{C_s}) \\
\simeq & & \simeq \\
\simeq r_s & & \simeq r_s \\
\simeq r_s & & \simeq r_s \\
H^*(M_{Higgs}^{ss}_{C_\eta}) & \leftarrow & H^*(M_{Hodge}^{ss}_{C_\eta}) \\
\simeq & & \simeq \\
\simeq r_\eta & & \simeq r_\eta \\
\simeq r_\eta & & \simeq r_\eta \\
H^*(M_{Higgs}^{ss}_{C_\eta}) & \leftarrow & H^*(M_{Hodge}^{ss}_{C_\eta}) \\
\simeq & & \simeq \\
\simeq s_{pv}^{deRham,B} & & \simeq s_{pv}^{deRham,B} \\
\simeq & & \simeq \\
\simeq & & \simeq \\
\simeq & & \simeq \\
\end{array}
\]

(17)

In particular, we have the identity:

\[s_{pt}\eta \circ s_{pv}^{Higgs,B} = s_{pv}^{deRham,B} \circ s_{pt}s.\]

The vertical morphisms on the left-hand-side Higgs column in (17) are filtered isomorphisms for the respective perverse filtrations as in §3.1.

If $B$ has equal characteristic $p > 0$, then all the morphisms in (17) are filtered isomorphisms for the respective perverse filtrations as in §3.1.
4 | DEFORMATION THEORY OF $t$-CONNECTIONS

In this section, we construct an obstruction module and an obstruction class to lifting $t$-connections for a square-zero thickening. We also prove some compatibilities of the obstruction module with base change. These results are then used in Section 5.2 to prove the smoothness assertion in Theorem 1.1.

4.1 | Čech cohomology and base change

Let $A$ be a Noetherian ring and let $Spec(A) \to \mathbb{A}^1_B$ be a morphism. If $M$ is an $A$-module, with associated $\mathcal{O}_{Spec(A)}$-module $\mathcal{M}$, and $\mathcal{E}$ is an $\mathcal{O}_{\mathbb{C}_A}$-module, then we denote $\mathcal{E} \otimes_{\mathcal{O}_{\mathbb{C}_A}} \pi_A^* \mathcal{M}$ simply by $\mathcal{E} \otimes_A M$.

Let $\mathcal{E}$ be a coherent $\mathcal{O}_{\mathbb{C}_A}$-module. Let $U = (U_i)_{i=1}^m$ be a finite affine open cover of the curve $C_A$.

Let us briefly recall the definition of the corresponding Čech complex $(\mathcal{E}^*, (\mathcal{E}, \delta))$ attached to the cover. We set $U_{i_0, i_2, \ldots, i_l} = U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_l}$, and $E_{i_0, i_2, \ldots, i_l} := \mathcal{E}(U_{i_0, i_2, \ldots, i_l})$. The (alternating) Čech complex has $l$th term given by $\mathcal{E}^l(U, \mathcal{E}) := \prod_{i_0 < i_1 < \ldots < i_l} \mathcal{E}_{i_0, i_1, \ldots, i_l}$. The differential $\delta^l$ is given by:

$$(\delta^l(c))_{i_0 < \ldots < i_{l+1}} := \sum_{j=0}^{l+1} (-1)^{l+1} c_{i_0 < i_1 < \ldots, i_{j-1} < \hat{i}_j < i_{j+1} < \ldots < i_{l+1}}.$$

The following facts are standard, except possibly for part (d), where the morphism is $\mathcal{O}_A$-linear, but not $\mathcal{O}_{\mathbb{C}_A}$-linear.

**Lemma 4.1.** With notation as above, the following hold:

(a) The $l$th cohomology group $\check{H}^l(U, \mathcal{E})$ of the Čech complex computes the sheaf cohomology $H^l(\mathcal{E}) = H^l(C_A, \mathcal{E})$. The Čech cohomology groups $\check{H}^l(U, \mathcal{E})$ vanish for $l \geq 2$.

(b) Suppose that $\mathcal{E}$ is $A$-flat. Then, for any $A$-module $M$, the natural morphism:

$$H^1(\mathcal{E}) \otimes_A M \cong \check{H}^1(U, \mathcal{E}) \otimes_A M \longrightarrow \check{H}^1(U, \mathcal{E} \otimes_A M) \cong H^1(\mathcal{E} \otimes_A M)$$

is an isomorphism.

(c) Suppose that $\mathcal{E}$ is $A$-flat. Let $S$ be an $A$-algebra, inducing a morphism $\sigma : Spec(S) \to Spec(A)$. Then, the natural morphism $H^1(\mathcal{E}) \otimes_A S \to H^1(\sigma^* \mathcal{E})$ is an isomorphism.

(d) With notation as in part (c), assume that $G$ is another $A$-flat coherent sheaf on $C_A$ equipped with a $A$-linear morphism of abelian sheaves $\varphi : \mathcal{E} \to G$. Then, the following induced diagram is commutative:

$$
\begin{array}{ccc}
H^1(\mathcal{E}) \otimes_A S & \xrightarrow{H^1(\varphi) \otimes_A S} & H^1(G) \otimes_A S \\
\downarrow \cong & & \downarrow \cong \\
H^1((\sigma_C)^* \mathcal{E}) & \xrightarrow{H^1((\sigma_C)^* \varphi)} & H^1((\sigma_C)^* G).
\end{array}
$$

**Proof.**

(a) The higher cohomology of the coherent sheaf $\mathcal{E}$ over each of the affine subsets $U_{i_0, \ldots, i_l}$ vanishes. Therefore, the Čech to sheaf cohomology spectral sequence (cf. [38, Tag 03O])
for $\mathcal{E}$ is $E_2$-degenerate, thus yielding a canonical identification $\tilde{H}^1(\mathcal{U}, \mathcal{E}) \cong H^1(\mathcal{E})$. The vanishing of $\tilde{H}^l(\mathcal{U}, \mathcal{E}) \cong H^l(\mathcal{E})$ for $l \geq 2$ follows from the theorem on formal functions, and the fact that the fibers of the morphism $C_A \to A$ have dimension 1 (cf. [38, Tag 02V7]).

(b) By virtue of the $A$-flatness of $\mathcal{E}$, it follows that all of the terms of the Čech complex $\check{C}^*(\mathcal{U}, \mathcal{E})$ are flat $A$-modules (cf. [38, Tag 01U4]). Consider the truncation $\tau_{\leq 1}(\check{C}^*(\mathcal{U}, \mathcal{E}))$, given by the two-term complex:

$$\tau_{\leq 1}(\check{C}^*(\mathcal{U}, \mathcal{E})) = \left[ \check{C}^0(\mathcal{U}, \mathcal{E}) \overset{\delta^0}{\to} \ker(\delta^1) \right].$$

This truncation is quasi-isomorphic to $\check{C}^*(\mathcal{U}, \mathcal{E})$, via the given inclusion, because the cohomology in degree $\geq 2$ vanishes by (a). Moreover, since all the terms $\check{C}^l(\mathcal{U}, \mathcal{E})$ are flat $A$-modules, it follows that for any $A$-module $M$, we have $\tau_{\leq 1}(\check{C}^*(\mathcal{U}, \mathcal{E}) \otimes_A M) \cong \tau_{\leq 1}(\check{C}^*(\mathcal{U}, \mathcal{E})) \otimes_A M$. Note that the Čech complex of $\mathcal{E} \otimes_A M$ coincides with $\check{C}^*(\mathcal{U}, \mathcal{E}) \otimes_A M$. By using this fact, we see that $\tau_{\leq 1}(\check{C}^*(\mathcal{U}, \mathcal{E} \otimes_A M)) \cong \tau_{\leq 1}(\check{C}^*(\mathcal{U}, \mathcal{E})) \otimes_A M$. Therefore, it follows that

$$\check{H}^1(\mathcal{E} \otimes_A M) \cong H^1(\tau_{\leq 1}(\check{C}^*(\mathcal{U}, \mathcal{E} \otimes_A M))) \cong H^1(\tau_{\leq 1}(\check{C}^*(\mathcal{U}, \mathcal{E})) \otimes_A M) \cong \ker(\delta^1) \otimes_A M.$$

On the other hand, since, irrespective of flatness, the operation $(-) \otimes_A M$ commute with taking cokernels, we have that:

$$\text{coker} \left[ \check{C}^0(\mathcal{U}, \mathcal{E}) \otimes_A M \overset{\delta^0 \otimes A \text{id}_M}{\longrightarrow} \ker(\delta^1) \otimes_A M \right] \cong \text{coker} \left[ \check{C}^0(\mathcal{U}, \mathcal{E}) \overset{\delta^0}{\longrightarrow} \ker(\delta^1) \otimes_A M \right] \cong H^1(\tau_{\leq 1}(\check{C}^*(\mathcal{U}, \mathcal{E}))) \otimes_A M \cong \check{H}^1(\mathcal{E}) \otimes_A M,$$

as predicated.

(c) This follows immediately from part (b) by setting $M = S$.

(d) By restricting $\varphi : \mathcal{E} \to \mathcal{G}$ to each $U_{i_0 \ldots i_l}$, we see that $\varphi$ induces an $A$-linear morphism of Čech complexes $\tilde{\varphi} : \check{C}^*(\mathcal{U}, \mathcal{E}) \to \check{C}^*(\mathcal{U}, \mathcal{G})$. It is readily seen that the morphism $\tilde{\varphi} \otimes_A \text{id}_S$ is the morphism corresponding to $\varphi \otimes A \text{id}_S$ for the corresponding Čech complexes of $\sigma^*_C \mathcal{E}$ and $\sigma^*_C (\mathcal{G})$ (notice that here we are using that $\varphi$ is $A$-linear to form the tensor product!). The commutativity of the diagram follows, by using the identifications provided by the previous part (c).

□

Fix a morphism $x_A : \text{Spec}(A) \to \mathcal{M} \text{Hodge}^{ss}_{CB}$ over $\mathbb{A}^1_{B'}$. This amounts to a pair $(\mathcal{F}, \nabla)$ consisting of a vector bundle $\mathcal{F}$ on $C_A$ and a logarithmic $t_A$-connection $\nabla$. 

Notation 4.2. We denote by:

\[ \varphi_{x_A} : \text{End}(F) \longrightarrow \text{End}(F) \otimes_{\mathcal{O}_{C_A}} \omega_{C_A/A}(D_A) \]  

the \( \mathcal{O}_A \)-linear morphism that sends a local section \( \theta \) in \( \text{End}(F) \) to the commutator \( \nabla \circ \theta - \theta \circ \nabla \).

Definition 4.3. The module of obstructions \( Q_{x_A} \) is defined to be the \( A \)-module cokernel of the following \( A \)-linear morphism in sheaf cohomology:

\[ Q_{x_A} := \text{coker} \left( H^1(\text{End}(F)) \longrightarrow H^1(\text{End}(F) \otimes_{\mathcal{O}_{C_A}} \omega_{C_A/A}(D_A)) \right). \]

Note that since \( C_A \rightarrow A \) is proper, \( Q_{x_A} \) is a finitely generated \( A \)-module.

4.2 Obstruction classes

Context 4.5. Let \( A \) be a local Artin algebra. Fix a morphism \( \text{Spec}(S) \rightarrow \text{Spec}(A) \). We shall denote by \( x_S = (\sigma^*_C(F), \sigma^*_C(V)) \) the \( t_S \)-connection on \( C_S \) obtained by pulling back \( x_A \) via \( \sigma \) (using \( A \)-linearity as in [36, Lemma 2.7]). More concretely, \( \sigma^*_C(V) \) is the unique connection that satisfies \( \sigma^*_C(V)(\sigma^*_C(s)) = \sigma^*_C(V(s)) \) for every local pullback section \( \sigma^*_C(s) \) of the sheaf \( \sigma^*_C(F) \) (this uniquely determines the morphism on all sections by the \( t_S \)-Leibniz rule).

Corollary 4.4. There is a natural isomorphism of \( S \)-modules \( Q_{x_S} \cong Q_{x_A} \otimes_A S \) of \( S \)-modules, that is, the formation of the obstruction module commutes with base change.

Proof. This follows from Lemma 4.1(d), the definition of \( Q_{x_A} \), and the fact that the formation of cokernels commutes with tensor products. \( \square \)

Remark 4.6. Let \( \tilde{M} \) be an \( \tilde{A} \)-module. We have the \( A \)-module \( M := \tilde{M} / \tilde{I} \tilde{M} \). Since \( \tilde{I}^2 = 0 \), there is no conflict with the notation we have chosen for \( I \). We have that \( I = \tilde{I} \otimes_{\tilde{A}} A \). Irrespective of \( \tilde{I} \) squaring to zero, we have that: \( M = \tilde{M} \otimes_{\tilde{A}} A \); if \( \tilde{M} \) is \( \tilde{A} \)-flat, then the natural surjective \( \tilde{A} \)-morphism \( \tilde{I} \otimes_{\tilde{A}} \tilde{M} \rightarrow \tilde{I} \tilde{M} \) is an isomorphism. We have a canonical isomorphism of \( A \)-modules: \( \tilde{I} \otimes_{\tilde{A}} \tilde{M} = I \otimes_A M \); if in addition \( \tilde{M} \) is \( \tilde{A} \)-flat, then these two \( A \)-modules are also \( A \)-isomorphic to \( \tilde{I} \tilde{M} \). We also have the analogous relations for \( \mathcal{O}_A \) and \( \mathcal{O}_{\tilde{A}} \)-modules respectively on \( C_A \) and \( C_{\tilde{A}} \), respectively. For example, we have that if \( \tilde{E} \) is a locally free \( \mathcal{O}_{C_{\tilde{A}}} \)-module on \( C_{\tilde{A}} \), with restriction \( E \) to \( C_A \), then we have a
canonical isomorphism of $O_{C_A}$-modules:

$$\tilde{E} \otimes_{O_{C_A}} \tilde{T} = E \otimes_{O_{C_A}} I.$$ (19)

Choose a compatible morphism $Spec(\tilde{A}) \to \mathbb{A}^1_B$, so that $\iota : Spec(A) \hookrightarrow Spec(\tilde{A})$ is a morphism over $\mathbb{A}^1_B$. This gives a well-defined lift $\iota_{A}$ of $t_{A}$ over $\mathbb{A}^1_B$. We thus have a commutative diagram of solid arrows:

$$
\begin{array}{ccc}
Spec(A) & \xrightarrow{x_A} & M\text{Hodge}^{ss}_{C_A} \\
\downarrow & & \downarrow \\
Spec(\tilde{A}) & \xrightarrow{\iota_{\tilde{A}}} & \mathbb{A}^1_B.
\end{array}
$$

We are interested in finding lifts as in the dotted arrow. This amounts to finding a $t_{\tilde{A}}$-connection $y_{\tilde{A}} = (\tilde{F}, \tilde{V})$ over $C_{\tilde{A}}$ such that the pullback $\iota^*(y_{\tilde{A}})$ is isomorphic to $x_{A}$.

The following proposition is key to the proof of the smoothness Theorem 1.1. While it is probably standard, we could not locate a reference in the literature.

**Proposition 4.7.** With notation as above, there exists a well-defined element $ob_{x_A} \in Q_{x_A} \otimes_A I$ such that $ob_{x_A} = 0$ if and only if a lift $y_{\tilde{A}}$ of $x_{A}$ exists. In particular, such a lift $y_{\tilde{A}}$ always exists if $Q_{x_A} = 0$.

In order to prove the proposition, we will make use of the following consequence of the nilpotent version of Nakayama’s lemma.

**Lemma 4.8.** Let $Spec(R)$ be an affine scheme, and let $J$ be a nilpotent ideal in $R$. Let $M$ be a locally-free $R$-module. Then, any trivialization of the vector bundle $M/JM$ on $Spec(R/J)$ lifts to a trivialization of $M$.

**Proof.** A trivialization $\overline{\psi} : (R/J)^{\oplus n} \sim M/JM$ amounts to a choice of $n$ independent elements $\overline{m}_1, \overline{m}_2, ..., \overline{m}_n \in M/JM$. For each $i$, fix the choice of a lift $m_i \in M$ mapping to $\overline{m}_i$ under the surjection $M \twoheadrightarrow M/JM$. We claim that the corresponding morphism $\psi : R^{\oplus n} \oplus m_i \rightarrow M$ is an isomorphism, thus concluding the proof of the lemma.

Surjectivity follows by the nilpotent version of Nakayama’s lemma [38, Tag 00DV (11)]. Next, we prove injectivity. Since $M$ is a projective $R$-module, the surjective morphism $R^{\oplus n} \twoheadrightarrow M$ splits as $R^{\oplus n} \cong M \oplus \ker(\psi)$. Since the reduction modulo $J \overline{\psi}$ of $\psi$ is injective, we must then have that $\ker(\psi) / J \ker(\psi) = 0$. It follows by [38, Tag 00DV (9)] that $\ker(\psi) = 0$, as desired. \[\square\]

**Proof of Proposition 4.7.** The closed embedding $\iota : C_A \rightarrow C_{\tilde{A}}$ induces a homeomorphism on the underlying topological spaces. In particular, an open cover $U'$ of $C_A$ induces an evident open cover $U'_{\tilde{A}}$ of $C_{\tilde{A}}$, compatibly with restrictions, that is, $U'_{\tilde{A}}$ restricts to $U'_{A}$. Fix a finite affine open cover $U = (U_{A_{i_1, ..., i_l}})_{i_1=1}^{\mu_i}$ of $C_A$ on which the restriction of $F$ is trivializable. We employ the notation $U_{A_{i_1, ..., i_l}} = U_{A_{i_1}} \cap ... \cap U_{A_{i_l}}$, and similarly for $U_{A_{\tilde{i}_1, ..., \tilde{i}_l}}$. By Lemma 4.1, the Čech cohomology with respect to these covers computes the corresponding sheaf cohomology groups. We do not make a distinction between Čech and sheaf cohomology, and we use the results in Lemma 4.1 freely without further mention.
By standard deformation theory for vector bundles [20, Thm. 7.1], the obstruction to lifting the vector bundle $\mathcal{F}$ from $C_A$ to a vector bundle $\tilde{\mathcal{F}}$ on $\tilde{C}_A$ lives in the second cohomology group $H^2(C_A, \text{End}(\mathcal{F}))$. This group is $\{0\}$ because $C_A$ is a curve over the affine Spec$(A)$ (cf. Lemma 4.1(a)).

Choose a locally free lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to $C_A$. We use the notation $\mathcal{F}_i := \mathcal{F}|_{U_{Ai}}$ and $\nabla_i := \nabla|_{U_{Ai}}$, and analogously for restrictions to multiple intersection. Similarly, we set $\tilde{\mathcal{F}}_i := \tilde{\mathcal{F}}|_{\tilde{U}_{Ai}}$.

Choose trivializations of the $\mathcal{F}_i$ on the $U_{Ai}$. We then notation $\mathcal{F}_i := \mathcal{F}_i|_{U_{Ai}}$, and $\nabla_i := \nabla_i|_{U_{Ai}}$, and analogously for restrictions to multiple intersection. Similarly, we set $\tilde{\mathcal{F}}_i := \tilde{\mathcal{F}}_i|_{\tilde{U}_{Ai}}$.

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in $\tilde{\mathcal{C}}^0(\mathcal{U}, \mathcal{E}nd(F) \otimes \omega_{\mathcal{C}_A/A}(D_A)) \otimes_A I$. By construction, we have:

$$c' = c + \delta^0(h),$$

so that $[c'] = [c] \in \tilde{H}^1(\mathcal{U}, \mathcal{E}nd(F) \otimes \omega_{\mathcal{C}_A/A}(D_A)) \otimes_A I$.

**N.B. I:** Conversely, for any 0-chain $h = (h_i)$, we can define $\tilde{\nabla}'_i := \tilde{\nabla}_i + h_i$, and then, we end up with cocohomologous cycles $c' = c + \delta^0(h)$.

Let $\mathcal{V}$ be a finite refinement of the given trivializing cover $\mathcal{U}$. Let $\tau, \tau' : \mathcal{I}_\mathcal{V} \to \mathcal{I}_\mathcal{U}$ be any two refinement maps on the indexing sets of the covers, so that $V_x \subseteq U_{\tau(x)} \cap U_{\tau'(x)}$. We denote by the same symbol the induced chain homotopic morphisms of cochain complexes $\tau, \tau' : \tilde{\mathcal{C}}(\mathcal{U}, -) \to \tilde{\mathcal{C}}(\mathcal{V}, -)$ (cf. [38, Tag 09UY]). A choice of lifts $\tilde{\nabla}_i$ gives rise to the cocycle $c$ for $\mathcal{U}$ as above. The two choices of lifts $\tilde{\nabla}_i| V_x$ and $\tilde{\nabla}'_i| V_x$ give rise to corresponding cocycles for $\mathcal{V}$ denoted as $\gamma$ and $\gamma'$. By construction, we see that $\tau(c) = \gamma$ and $\tau'(c) = \gamma'$, so that the formation of the cohomology class $[c]$, for a given lift $\tilde{\mathcal{F}}$, is compatible with finite refinements of covers. The usual argument involving common refinements tells us that the formation of the class $[c]$ depends only on the choice of lift $\tilde{\mathcal{F}}$. We are thus left with showing that given a trivializing cover, the obstruction class is independent of the choice of lift $\tilde{\mathcal{F}}$.

We choose a second locally free lift $\tilde{\mathcal{F}}'$ to $C_{\tilde{\mathcal{A}}}$ of the vector bundle $\mathcal{F}$ on $C_{\tilde{\mathcal{A}}}$. Since the restrictions $\mathcal{F}_i, \tilde{\mathcal{F}}_i, \tilde{\mathcal{F}}'_i$ are trivializable on their corresponding affine schemes, we can choose isomorphisms $\psi_i : \tilde{\mathcal{F}}_i \to \tilde{\mathcal{F}}'_i$ that restrict to the identity on $\mathcal{F}_i$. For each pair of indexes $i, j$, the isomorphism $\psi_{i,j}^{-1}|U_{\tilde{\mathcal{A}}_{i,j}} \circ \psi_{i,j}|U_{\tilde{\mathcal{A}}_{i,j}} : \tilde{\mathcal{F}}_{i,j} \to \tilde{\mathcal{F}}'_{i,j}$ is a lift of the identity, so that it is of the form $\text{Id} + B_{i,j}$ for a unique element $B_{i,j} \in \text{End}(\tilde{\mathcal{F}}_{i,j}) \otimes_A \tilde{\mathcal{I}} \cong \text{End}(\mathcal{F}_{i,j}) \otimes_A I$. We can view $B = (B_{i,j})$ as a cochain in $\tilde{\mathcal{C}}^1(\mathcal{U}_\mathcal{A}, \mathcal{E}nd(F) \otimes A I)$. By direct computation, we see that $B$ is a cocycle. We can use $\psi_i$ to define lifts $\tilde{\nabla}'_i = \psi_i \circ \tilde{\nabla}_i \circ \psi_{i}^{-1}$ on $\tilde{\mathcal{F}}'_i$, so that they fit into the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{\mathcal{F}}_i & \xrightarrow{\tilde{\nabla}_i} & \tilde{\mathcal{F}}_i \otimes \omega_{\mathcal{C}_A/A}(D_A) \\
\downarrow{\psi_i} & & \downarrow{\psi_i} \\
\tilde{\mathcal{F}}'_i & \xrightarrow{\tilde{\nabla}'_i} & \tilde{\mathcal{F}}'_i \otimes \omega_{\mathcal{C}_A/A}(D_A).
\end{array}
\]

The new Čech cocycle $c^1 = (c^1_{i,j})$ for this choice of $\tilde{\mathcal{F}}'$ and $\tilde{\nabla}'_i$ is given by:

$$c^1_{i,j} = \tilde{\nabla}'_i|_{U_{\tilde{\mathcal{A}}_{i,j}}} - \tilde{\nabla}_j|_{U_{\tilde{\mathcal{A}}_{i,j}}} = (\psi_i \circ \tilde{\nabla}_i \circ \psi_i^{-1})|_{U_{\tilde{\mathcal{A}}_{i,j}}} - (\psi_j \circ \tilde{\nabla}_j \circ \psi_j^{-1})|_{U_{\tilde{\mathcal{A}}_{i,j}}}.$$

We know that $c^1 = (c^1_{i,j})$ is actually a cocycle dwelling in the submodule $\Gamma(\mathcal{E}nd(F_{i,j}) \otimes \omega_{\mathcal{C}_A/A}(D_A)) \otimes_A I$. Since $\psi_j$ restricts to the identity on $\mathcal{F}_j$, we see that applying $\psi_{j}^{-1}|_{U_{\mathcal{A}_{i,j}}} \circ (-) \circ \psi_j|_{U_{\mathcal{A}_{i,j}}}$ does not affect the cocycle in $\Gamma(\mathcal{E}nd(F_{i,j}) \otimes \omega_{\mathcal{C}_A/A}(D_A)) \otimes_A I$. We can thus rewrite:

$$c^1_{i,j} = \psi^{-1}_{j}|_{U_{\tilde{\mathcal{A}}_{i,j}}} \circ c^1_{i,j} \circ \psi_{j}|_{U_{\tilde{\mathcal{A}}_{i,j}}} = (\psi_{j}^{-1} \circ \psi_i \circ \tilde{\nabla}_j \circ \psi_j^{-1} \circ \psi_j)|_{U_{\tilde{\mathcal{A}}_{i,j}}} - \tilde{\nabla}_j|_{U_{\tilde{\mathcal{A}}_{i,j}}}.$$

This can be rewritten as

$$c^1_{i,j} = (1 + B_{i,j}) \circ \tilde{\nabla}_i \circ (1 - B_{i,j})|_{U_{\tilde{\mathcal{A}}_{i,j}}} - \tilde{\nabla}_j|_{U_{\tilde{\mathcal{A}}_{i,j}}}. $$
By expanding, and using that \( \text{End}(F_{i,j}) \otimes A I \) and \( \tilde{I}^2 = 0 \), we get: (we omit denoting the restrictions to \( U_{A,i,j} \))

\[
c^1_{i,j} = -\tilde{\nabla}_{i} \circ B_{i,j} + B_{i,j} \circ \tilde{\nabla}_{i} + \tilde{\nabla}_{j} = -\tilde{\nabla}_{i} \circ B_{i,j} + B_{i,j} \circ \tilde{\nabla}_{i} + c_{i,j}.
\]

Now, since \( B_{i,j} \) lies in the submodule \( \text{End}(F_{i,j}) \otimes A I \) and \( \tilde{\nabla}_{i} \) is a lift of \( \nabla_{i} \), the commutator can be rewritten as: (omitting restrictions again, and recalling (18))

\[
\tilde{\nabla}_{i} \circ B_{i,j} - B_{i,j} \circ \tilde{\nabla}_{i} = \nabla_{i} \circ B_{i,j} - B_{i,j} \circ \nabla_{i} = (\tilde{C}^1(\varphi_{x_A})(B))_{i,j}.
\]

In conclusion, the new cocycle \( c^1 \) can be expressed as

\[
c^1 = -\tilde{C}^1(\varphi_{x_A})(B) + c.
\] (21)

Hence, it differs from the cocycle \( c \) by the image of a cocycle in \( \tilde{C}^1(U_A, \text{End}(\mathcal{F})) \otimes A I \), and so, given Definition 4.3 of the obstruction \( A \)-module \( Q_{x_A} \), it yields the same element \( \text{ob}_{x_A} \) in \( Q_{x_A} \otimes A I \). We have established the sought-after independence on the locally free lift \( \tilde{\mathcal{F}} \) of \( \mathcal{F} \).

**N.B. II:** In the last argument, the cocycle \( B = (B_{i,j}) \) depends on the choice of isomorphisms \( \psi_i \) up to the coboundary of a 0-chain. Indeed, we can always change the isomorphisms \( \psi_i \) by precomposing by an automorphism of \( \tilde{\mathcal{F}}|_{\tilde{U}_{i}} \), which will be of the form \( \text{Id} + M_i \) for some cochain \( M = (M_i) \) in \( \tilde{C}^0(U, \text{End}(\mathcal{F}) \otimes A I) \). It follows from the computations above that the new cocycle obtained by changing the \( \psi_i \) in this way will be of the form \( B + \delta^0(M) \). We conclude that the corresponding cohomology class \( \overline{B} \) of \( B \) in \( \tilde{H}^1(U, \text{End}(\mathcal{F})) \otimes A I \) is well defined.

Conversely, by standard deformation theory of vector bundles [20, Thm. 7.1], every such cohomology class \( \overline{B} \) arises this way from a choice of a locally free lift of \( \mathcal{F} \) to \( C_{A} \). This establishes a canonical bijection between isomorphism classes of lifts \( \tilde{\mathcal{F}} \) of \( \mathcal{F} \) and cohomology classes in \( \tilde{H}^1(U, \text{End}(\mathcal{F})) \otimes A I \).

Hence, for any given cocycle \( B \) in \( \tilde{C}^1(U, \text{End}(\mathcal{F}) \otimes A I) \), we can find a given lift \( \tilde{\mathcal{F}}^1 \) and isomorphisms \( \psi_i \) such that the corresponding cocycle is cohomologous to \( B \). By further changing the given \( \psi_i \) by a 0-cocycle \( M_i \), as described above, we can moreover assume that the corresponding cocycle in \( \tilde{C}^1(U_A, (\text{End}(\mathcal{F}) \otimes A I) \) is \( B \) on the nose. Hence, for any given cocycle \( B \), we find lifts as described above so that the new obstruction cocycle is \( c^1 = -\tilde{C}^1(\varphi_{x_A})(B) + c \), as in the computation above (Equation 21).

We conclude by showing that \( \text{ob}_{x_A} = 0 \) if and only if there exists a lift of the \( t_{A} \)-connection to \( C_{A} \).

First, suppose that \( \text{ob}_{x_A} = 0 \). Choose a suitably finite trivializing cover \( U \) and some lifts \( \tilde{\nabla}_i \) and \( \tilde{\mathcal{F}} \). The corresponding cocycle \( c = (c_{i,j}) \) satisfies \( [c] = 0 \), and so, it is cohomologous to an element in the image of \( \tilde{C}^1(\varphi_{x_A}) \). We can thus find \( h = (h_i) \in \tilde{C}^0(U, \text{End}(\mathcal{F}) \otimes \omega_{C_{A}/A}(D_A) \otimes A I) \) and \( B = (B_{i,j}) \in \tilde{Z}^1(U, \text{End}(\mathcal{F}) \otimes A I) \), such that \( c = \delta^0(h) + \tilde{C}^1(\varphi_{x_A})(B) \). Replacing the lifts with \( \tilde{\nabla}_i := \nabla_i - h_i \) instead, as in N.B. I, we can assume that the cocycle \( c \) is of the form \( c = \tilde{C}^1(\varphi_{x_A})(B) \). Now, by N.B.2, we can choose another lift \( \tilde{\mathcal{F}}^1 \) and isomorphisms \( \psi_i \) that correspond to the cocycle \( B \). As in N.B. II, this yields new choices of lifts such that the corresponding new cocycle \( c^1 = c - \tilde{C}^1(\varphi_{x_A})(B) = 0 \) vanishes. Hence, we can assume without loss of generality that \( c \) is identically 0. Since by definition, we have \( c_{i,j} = \tilde{\nabla}_i|_{\tilde{U}_{i,j}} - \tilde{\nabla}_j|_{\tilde{U}_{i,j}} \), this means that the \( \tilde{\nabla}_i \) agree on the intersections \( U_{A,i,j} \), and so, they glue to give a \( t_{A} \)-connection \( (\tilde{\mathcal{F}}, \tilde{\nabla}) \) that lifts \( (\mathcal{F}, \nabla) \).
Conversely, suppose that there exists a lift \( (\tilde{F}, \tilde{\nabla}) \) of the \( t_A \)-connection to \( C_A \). Then we can use \( \tilde{F} \) as the lift of the vector bundle, choose any trivializing cover, and set \( \tilde{\nabla}_i := \tilde{\nabla}|_{\tilde{U}_i} \) in the construction of a cocycle \( c = (c_{i,j}) \) representing \( \text{ob}_{\mathcal{X}} \). Since the \( \tilde{\nabla}_i \) agree on the intersections, we have \( c_{i,j} = 0 \), so that \( \text{ob}_{\mathcal{X}} = 0 \). \( \square \)

### 4.3 Relative tangent space

For any algebraic stack \( \mathcal{M} \) and any geometric point \( x : \text{Spec}(k) \to \mathcal{M} \), the tangent space \( T_{\mathcal{M},x} \) is defined to be the set of isomorphism classes of pairs \((y, \psi)\), where \( y : \text{Spec}(k[\varepsilon]/(\varepsilon^2)) \to \mathcal{M} \) is a \( k[\varepsilon]/(\varepsilon^2) \)-point of \( \mathcal{M} \) and \( \psi \) is an isomorphism \( y|_{\text{Spec}(k[\varepsilon]/(\varepsilon))} \sim \rightarrow x \). The tangent space \( T_{\mathcal{M},x} \) acquires a canonical structure of a \( k \)-vector space.

We shall describe the tangent spaces of the fibers of \( \mathcal{M}_{\text{Hodge}^{ss} C_B} \to \mathbb{A}^1_B \). Fix a geometric point \( a : \text{Spec}(k) \to \mathbb{A}^1_B \), and choose a geometric point \( x : \text{Spec}(k) \to (\mathcal{M}_{\text{Hodge}^{ss} C_B})_a \) of the fiber \((\mathcal{M}_{\text{Hodge}^{ss} C_B})_a \). The point \( x \) represents a pair \((F, \nabla)\) of a vector bundle and a logarithmic \( t_a \)-connection. In Subsection 4.1, we made use of the following complex of sheaves of \( k \)-vector spaces to define the obstruction module

\[
C^*(x) := \left[ \text{End}(F) \xrightarrow{\varphi_x} \text{End}(F) \otimes_{O_{C_A}} \omega_{C_A/A}(D_A) \right].
\]

Here, by convention, we place the left term \( \text{End}(F) \) in cohomological degree 0. We shall denote by \( H^i(C^*(x)) := H^i(C_A^*, C^*(x)) \) denote the \( i \)th hypercohomology of the complex. By the hypercohomology spectral sequence, we have a natural identification \( H^2(C^*(x)) \cong Q_x \). The spectral sequence also identifies \( H^0(C^*(x)) \) with the \( k \)-vector space \( \text{End}(x) \) consisting of endomorphisms of the vector bundle \( F \) that commute with the logarithmic \( t_a \)-connection \( \nabla \). The argument in [32, Thm. 4.2] generalizes without change to the setting of logarithmic \( t_a \)-connections to show that there is a natural identification of \( k \)-vector spaces \( H^1(C^*(x)) \cong T_{(\mathcal{M}_{\text{Hodge}^{ss} C_B})_a,x} \) (see also [39, §5] for a treatment in the generality of \( \Lambda \)-modules). Using these identifications, we give a dimension formula for the tangent space \( T_{(\mathcal{M}_{\text{Hodge}^{ss} C_B})_a,x} \).

**Corollary 4.9.** Fix \( a \in \mathbb{A}^1_B \). For any geometric point \( x \in (\mathcal{M}_{\text{Hodge}^{ss} C_B})_a \) of the fiber, the dimension of the tangent space \( T_{(\mathcal{M}_{\text{Hodge}^{ss} C_B})_a,x} \) of \( x \) in \((\mathcal{M}_{\text{Hodge}^{ss} C_B})_a \) is given by

\[
\dim(T_{(\mathcal{M}_{\text{Hodge}^{ss} C_B})_a,x}) = n^2(2g - 2 + \deg(D_a)) + \dim(\text{End}(x)) + \dim(Q_x).
\]

In particular, if the rank \( n \) and degree \( d \) are coprime, then the dimension of the tangent space of the fiber is given by

\[
\dim(T_{(\mathcal{M}_{\text{Hodge}^{ss} C_B})_a,x}) = n^2(2g - 2 + \deg(D_a)) + 1 + \dim(Q_x).
\]

**Proof.** The point \( x \) represents a logarithmic \( t_a \)-connection \((F, \nabla)\). By the hypercohomology spectral sequence and Riemann–Roch, we get the following formula for the Euler characteristic;

\[
\chi(C^*(x)) = \chi(\text{End}(F)) + \chi\left( \text{End}(F) \otimes \Omega^1_{C_A/A}(D_A) \right) = -n^2(2g - 2 + \deg(D_a)).
\]
Since by definition $\chi(C^*(x)) = H^0(C^*(x)) - H^1(C^*(x)) + H^2(C^*(x))$, we get
\[
\dim(H^1(C^*(x))) = n^2(2g - 2 + \deg(D_a)) + \dim(H^0(C^*(x))) + \dim(H^0(C^*(x))).
\]

Using the natural identifications $H^0(C^*(x)) \cong \text{End}(x)$, $H^1(C^*(x)) \cong T_{(\mathcal{M}Hodge_{ss}^i(\mathbb{C}_B))_{a,x}}$ and $H^2(C^*(x)) \cong Q_x$ yields the desired formula
\[
\dim\left(T_{(\mathcal{M}Hodge_{ss}^i(\mathbb{C}_B))_{a,x}}\right) = n^2(2g - 2 + \deg(D_a)) + \dim(\text{End}(x)) + \dim(Q_x).
\]

In the special case when $n$ and $d$ are coprime, then the space of endomorphisms $\text{End}(x)$ is one dimensional, consisting of the scalar endomorphisms of $F$ (cf. the proof of Lemma 5.1). Hence, we can set $\dim(\text{End}(x)) = 1$. □

5 | SMOOTHNESS AND IRREDUCIBILITY OF THE MODULI SPACE

5.1 | Reduction to the smoothness of the stack

There is a central copy of $\mathbb{G}_m$ in the automorphisms of every point of $\mathcal{M}Hodge_{ss}^i(\mathbb{C}_B)$, because multiplication by constants commutes with any logarithmic $t$-connection. Therefore, we can form the $\mathbb{G}_m$-rigidification $(\mathcal{M}Hodge_{ss}^i)^{rig}$, as in [4, Appendix A]. By the proof of [4, Thm. A.1], there is a smooth cover $U \to (\mathcal{M}Hodge_{ss}^i)^{rig}$ by a scheme $U$ and a Cartesian diagram

\[
\begin{array}{ccc}
B(\mathbb{G}_m,U) & \longrightarrow & \mathcal{M}Hodge_{ss}^i \\
\downarrow & & \downarrow \\
U & \longrightarrow & (\mathcal{M}Hodge_{ss}^i)^{rig}
\end{array}
\]

Since the left vertical arrow $B(\mathbb{G}_m,U) \to U$ is a smooth good moduli space morphism, and being a good moduli space morphism can be checked étale locally on the target, it follows that the rigidification morphism $\mathcal{M}Hodge_{ss}^i_{\mathbb{C}_B} \to (\mathcal{M}Hodge_{ss}^i)^{rig}_{\mathbb{C}_B}$ is a smooth good moduli space morphism. In particular, since being Noetherian can be checked smooth locally, it also follows that $(\mathcal{M}Hodge_{ss}^i)^{rig}_{\mathbb{C}_B}$ is Noetherian.

Lemma 5.1. Assume that the rank $n$ and degree $d$ are coprime. Then $(\mathcal{M}Hodge_{ss}^i)^{rig}_{\mathbb{C}_B}$ is an algebraic space.

Proof. We need to show that inertia is trivial [38, Tag 04SZ]. This means that for every $T$-point $p^{rig} : T \to (\mathcal{M}Hodge_{ss}^i)^{rig}_{\mathbb{C}_B}$, we need to show that the group algebraic space of automorphisms $\text{Aut}(p^{rig}) \to T$ is trivial. Since $(\mathcal{M}Hodge_{ss}^i)^{rig}_{\mathbb{C}_B}$ is locally Noetherian, we can without loss of generality take $T$ to be Noetherian. By Lemma 5.2 below, it suffices to show that the fibers of $\text{Aut}(p^{rig})$ over any geometric point of $T$ are trivial, and therefore, we can assume without loss of generality that $T = \text{Spec}(k)$ for an algebraically closed field $k$. Then, $p^{rig}$ is a $k$-point coming from a point $p \in \mathcal{M}Hodge_{ss}^i_{\mathbb{C}_B}$. The group automorphisms of $p^{rig}$ is just the quotient of the group automorphisms $\text{Aut}(p)$ by the central $\mathbb{G}_m$. Therefore, it suffices to show that
the group scheme of automorphisms of any point \( p \in \mathcal{M}_{H^0}^{ss} \) is equal to the constant scalars \( \mathbb{G}_m \).

The automorphisms of a pair \( \mathcal{X} = (\mathcal{F}, \nabla) \in \mathcal{M}_{H^0}^{ss}(k) \) consist of the automorphisms of the vector bundle \( \mathcal{F} \) that commute with the logarithmic \( t \)-connection \( \nabla : \mathcal{F} \to \mathcal{F} \otimes \omega_{\mathcal{L}}(D) \). We have a closed immersion of algebraic groups \( \mathbb{G}_m \subset \text{Aut}(\mathcal{X}) \). Since \( \mathcal{X} \) is stable (which is the same as semistable because \( n \) and \( d \) are coprime), the usual argument (cf. [36, pg. 90]) shows that \( \mathbb{G}_m \hookrightarrow \text{Aut}(\mathcal{X}) \) induces a bijection at the level of \( k \)-points, and so, \( \mathbb{G}_m \) must be the reduced subgroup scheme of \( \text{Aut}(\mathcal{X}) \). To show equality of schemes, it suffices to show that the scheme of automorphisms \( \text{Aut}(\mathcal{X}) \) is smooth over \( k \), which would follow if we can prove that the Lie algebra of the group scheme of automorphisms is one-dimensional. But standard deformation theory shows that the Lie algebra consists of endomorphisms of \( \mathcal{F} \) that commute with \( \nabla \). By the same argument, this just consists of the one-dimensional space of constant endomorphisms, as desired. □

**Lemma 5.2.** Let \( T \) be a Noetherian scheme, and let \( G \) be an group algebraic space of finite type over \( T \). Suppose that for all geometric points \( \bar{t} \in T \), the fiber \( G_{\bar{t}} \) is the trivial group scheme over \( \bar{t} \). Then, \( G \) is the trivial group scheme over \( T \).

**Proof.** Let \( e : T \to G \) denote the identity section. We know that for all geometric points \( \bar{t} \in T \), the restriction \( e_{\bar{t}} : \bar{t} \to G_{\bar{t}} \) is an isomorphism. Since the property of being an isomorphism can be checked flat locally, this actually implies that for all points \( t \in T \), we have that \( e_t \) is an isomorphism. We want to conclude that \( e \) is an isomorphism. Since this statement is étale local on \( G \), after choosing an étale atlas \( X \to G \), we just need to show that the monomorphism \( e_X : T \times_G X \to X \) is an isomorphism. Consider the following commutative diagram of schemes.

\[
\begin{array}{ccc}
T \times_G X & \xrightarrow{e_X} & X \\
\downarrow & & \downarrow \\
T & &
\end{array}
\]

By assumption, for every point \( t \in T \), the restriction \( (e_X)_t : t \times_x X_t \to X_t \) is an isomorphism. Note that \( T \times_X X \to T \) is étale, and hence flat. By the fiberwise criterion for flatness [38, Tag 05VK], we conclude that \( e_X \) is flat. So, \( e_X \) is a flat monomorphism of finite type, and hence an open immersion. We also know that \( e_X \) is surjective, because it is an isomorphism over every point of \( T \). Therefore, \( e_X \) is an isomorphism, as desired. □

**Lemma 5.3.** Assume that the rank \( n \) and degree \( d \) are coprime. There is an isomorphism \( \mathcal{M}_{H^0}^{ss} \cong (\mathcal{M}_{H^0}^{ss})^{rig} \). In particular, the morphism \( \mathcal{M}_{H^0}^{ss} \to \mathcal{M}_{H^0}^{ss} \) is a smooth good moduli space morphism.

**Proof.** By the universal property of good moduli spaces, we have a canonical morphism \( \psi : (\mathcal{M}_{H^0}^{ss})^{rig} \to \mathcal{M}_{H^0}^{ss} \) (in this case, we could have also used the universal property of rigidifications). By applying [1, Prop. 4.5] to the good moduli space morphism \( f : \mathcal{M}_{H^0}^{ss} \to (\mathcal{M}_{H^0}^{ss})^{rig} \), we see that we have a natural isomorphism of functors \( \psi*(\mathcal{O}) = \psi_*f_*\mathcal{O}(\mathcal{M}_{H^0}^{ss}) = \mathcal{O}(\mathcal{M}_{H^0}^{ss}) \).
by the pushforward property in the definition of adequate moduli space morphism [2, Defn. 5.1.1 (2)]. It follows that \( \mathcal{M}_{\text{Hodge}}^{ss} \rightarrow (\mathcal{M}_{\text{Hodge}}^{ss})^{rig} \cong \mathcal{M}_{\text{Hodge}}^{ss} \) is a smooth good moduli space morphism.

**Remark 5.4.** Since good moduli space morphisms commute with arbitrary base change, it follows from Lemma 5.3 that the formation of the good moduli space \( \mathcal{M}_{\text{Hodge}}^{ss} \) commutes with arbitrary base change over \( \mathbb{A}^1_B \). This also follows from [25, Thm. 1.1]. Note that, a priori, the formation of the GIT quotient in arbitrary characteristic is only known to commute with flat base change.

**Corollary 5.5.** If the rank \( n \) and degree \( d \) are coprime, the moduli space \( \mathcal{M}_{\text{Hodge}}^{ss} \) is smooth over \( \mathbb{A}^1_B \) if and only if the stack \( \mathcal{M}_{\text{Hodge}}^{ss} \) is smooth over \( \mathbb{A}^1_B \).

**Proof.** This is immediate from Lemma 5.3 and the fact that property of being a smooth morphisms can be checked smooth locally. \( \square \)

### 5.2 Proof of the smoothness and dimension assertions in Theorem 1.1

In view of Corollary 5.5, in order to prove the smoothness assertion in Theorem 1.1, it suffices to show that the stack \( \mathcal{M}_{\text{Hodge}}^{ss} \) is smooth over \( \mathbb{A}^1_B \). Our proof of smoothness of the stack \( \mathcal{M}_{\text{Hodge}}^{ss} \) applies even when the rank and degree are not coprime.

**Proposition 5.6.** Without coprimeness assumptions on the rank \( n \) and degree \( d \), the morphism of stacks \( \mathcal{M}_{\text{Hodge}}^{ss} \rightarrow \mathbb{A}^1_B \) is smooth.

**Proof.** We use the lifting criterion for smoothness [38, Tag 0DP0] for the finite type morphism \( \mathcal{M}_{\text{Hodge}}^{ss} \rightarrow \mathbb{A}^1_B \). Since both the target and the source are locally Noetherian, by [38, Tag 02HT], it suffices to show the existence of lifting for square-zero thickenings of local Artin algebras.

Let \( A \) be a local Artin \( k \)-algebra with maximal ideal \( m \) and residue field \( k \). Fix a morphism \( Spec(A) \rightarrow \mathcal{M}_{\text{Hodge}}^{ss} \), inducing a composition \( Spec(A) \rightarrow \mathbb{A}^1_B \), defining a function \( t_A \in A \). Choose a square-zero thickening \( \widetilde{A} \rightarrow A \) with defining ideal \( \widetilde{I} \) (cf. §4.2). Let \( Spec(\widetilde{A}) \rightarrow \mathbb{A}^1_B \) be a choice of a morphism so that \( Spec(A) \hookrightarrow Spec(\widetilde{A}) \) is a morphism over \( \mathbb{A}^1_B \). We need to find a lifting as in the dotted arrow below.

\[
\begin{array}{ccc}
Spec(A) & \longrightarrow & \mathcal{M}_{\text{Hodge}}^{ss} \\
\downarrow & & \downarrow \\
Spec(\widetilde{A}) & \rightarrow & \mathbb{A}^1_B.
\end{array}
\]

The family \( Spec(A) \rightarrow \mathcal{M}_{\text{Hodge}}^{ss} \) is represented by a pair \( x_A = (F, V) \), that is, a logarithmic \( t_A \)-connection on \( C_A \). By Proposition 4.7, in order to show the existence of a lift for this family, it suffices to prove that \( Q_{x_A} = 0 \).

By Nakayama’s lemma, if we show \( Q_{x_A} \otimes_A A/m = 0 \), then we have \( Q_{x_A} = 0 \). Using the compatibility of the obstruction module with base change (Corollary 4.4), we see that \( Q_{x_A} \otimes_A A/m \cong Q_{x_A/m} \), where \( x_A/m \) is obtained by pulling back \( x_A \) to \( C_A/m \). Therefore,
without loss of generality, we can assume that \( A = k \). In particular, \( x_A = x_k \) is a \( k \)-point of the stack \( \mathcal{MHodge}^S_{\mathbb{C}_k} \), and \( Q_{x_A} = Q_{x_k} \) is a \( k \)-vector space. We also assume, without loss of generality, that \( B = \text{Spec}(k) \), and that \( k \) is algebraically closed.

Recall that there is a lift of the \( \mathbb{G}_m \)-action on \( \mathbb{A}_k^1 \) to the stack \( \mathcal{MHodge}^S_{\mathbb{C}_k} \), given by scaling the universal logarithmic \( t \)-connection. Starting with our point \( x_k \), we consider the morphism \( \mathbb{G}_m \rightarrow \mathcal{MHodge}^S_{\mathbb{C}_k} \) induced by the action \( y \mapsto y \cdot x_k \). In this family, the vector bundle \( \mathcal{L} \) remains constant, and we scale the \( t \)-connection \( \nabla \). This can be completed to a \( \mathbb{G}_m \)-equivariant morphism \( \mathbb{A}_k^1 \rightarrow \mathcal{MHodge}_{\mathbb{C}_k}^S \) to the stack \( \mathcal{MHodge}_{\mathbb{C}_k}^S \) of all \( t \)-connections, with no semistability condition. The image of \( 0 \in \mathbb{A}_k^1 \) is given by the pair \((F, 0)\) consisting of the vector bundle \( F \) and the zero Higgs-field. Using the argument for the “semistable reduction” theorem in [24, Thm. 5.1], we modify \( \mathbb{A}_k^1 \rightarrow \mathcal{MHodge}_{\mathbb{C}_k}^S \) to a \( \mathbb{G}_m \)-equivariant semistable family \( x_{\mathbb{A}_k^1} : \mathbb{A}_k^1 \rightarrow \mathcal{MHodge}^S_{\mathbb{C}_k} \). Consider the corresponding \( \mathbb{G}_m \)-equivariant \( k[z] \)-module of finite type \( N := \mathcal{O}_{x_{\mathbb{A}_k^1}} \), where \( z \) is the coordinate of \( \mathbb{A}_k^1 \). By construction, the fiber over \( 1 \in \mathbb{A}_k^1 \) is the \( k[z]/(z - 1) \)-module \( N_1 = \mathcal{O}_{x_k} \) that we are interested in. To show \( N_1 = 0 \), it suffices to show that \( N_0 = 0 \) for the fiber at 0.

**Lemma 5.7.** Suppose \( B = \text{Spec}(k) \) for an algebraically closed field \( k \). Let \( x_0 \) be a \( k \)-point of \( \mathcal{MHodge}^S_{\mathbb{C}_k} \) in the 0-fiber over \( \mathbb{A}_k^1 \), represented by a semistable logarithmic Higgs bundle \( x_0 = (\mathcal{L}, \nabla) \). Then, \( Q_{x_0} = 0 \).

**Proof.** By a degeneration and semicontinuity argument, it suffices to prove \( Q_{x_0} = 0 \) for closed points of the stack. In other words, we can assume that \( x_0 \) represents a polystable logarithmic Higgs bundle. We need to prove the vanishing of the cokernel \( Q_{x_0} \) of the morphism (cf. Definition 4.3):

\[
H^1(\mathcal{E}nd(F)) \xrightarrow{H^1(\varphi_{x_0})} H^1(\mathcal{E}nd(F) \otimes_{\mathcal{O}_C} \omega_C/k(D_k)).
\]

Under our assumptions, the commutator \( \varphi_{x_0} \) is \( \mathcal{O}_C \)-linear. Therefore, we can consider the dual twisted morphism \( \varphi_{x_0}^\vee \otimes_{\mathcal{O}_C} \text{id}_{\omega_C/k} : \mathcal{E}nd(F)^\vee(-D) \rightarrow \mathcal{E}nd(F)^\vee \otimes_{\mathcal{O}_C} \omega_C/k \). Under the identifications provided by Serre duality, the morphism \( H^1(\varphi_{x_0}) \) is identified with the dual

\[
H^0(\varphi_{x_0}^\vee \otimes_{\mathcal{O}_C} \text{id}_{\omega_C/k})^\vee : H^0(\mathcal{E}nd(F)^\vee \otimes_{\mathcal{O}_C} \omega_C/k)^\vee \rightarrow H^0(\mathcal{E}nd(F)^\vee(-D))^\vee.
\]

Therefore, the \( k \)-vector space \( Q_{x_0} \) is canonically isomorphic to the dual \( \mathcal{K}^\vee \) of the following kernel:

\[
\mathcal{K} := \ker \left[ H^0(\mathcal{E}nd(F)^\vee(-D)) \xrightarrow{H^0(\varphi_{x_0}^\vee \otimes_{\mathcal{O}_C} \text{id}_{\omega_C/k})} H^0(\mathcal{E}nd(F)^\vee \otimes_{\mathcal{O}_C} \omega_C/k) \right].
\]

We want to show that \( \mathcal{K} \) vanishes. Note that there is a transposition isomorphism \( \tau : \mathcal{E}nd(F) \rightarrow \mathcal{E}nd(F)^\vee \) given by the swap (transposition of matrices):

\[
\tau : \mathcal{E}nd(F) = F^\vee \otimes_{\text{swap}} F \rightarrow F \otimes F^\vee \rightarrow \mathcal{E}nd(F)^\vee.
\]
This also induces identifications \( \text{End}(F)^{\vee} \otimes_{\mathcal{O}_C/k} \omega_{C/k} \cong \text{End}(F) \otimes_{\mathcal{O}_C/k} \omega_{C/k} \) and \( \text{End}(F)^{\vee}(-D) \cong \text{End}(F)(-D) \). Consider the diagram of \( \mathcal{O}_C \)-modules:

\[
\begin{array}{ccc}
\text{End}(F)^{\vee}(-D) & \xrightarrow{\varphi_{x_0} \otimes_{\mathcal{O}_C} \text{id}_{\mathcal{O}_C(-D)}} & \text{End}(F)^{\vee} \otimes_{\mathcal{O}_C} \omega_{C/k} \\
\tau \otimes_{\mathcal{O}_C} \text{id}_{\mathcal{O}_C(-D)} & & \tau \otimes_{\mathcal{O}_C} \text{id}_{\mathcal{O}_C(-D)} \\
\text{End}(F)(-D) & \xrightarrow{\varphi_{x_0} \otimes_{\mathcal{O}_C} \text{id}_{\mathcal{O}_C(-D)}} & \text{End}(F) \otimes_{\mathcal{O}_C} \omega_{C/k}.
\end{array}
\]

The diagram is commutative by the linear algebra fact that the dual of the commutator morphism of matrices is identified with the commutator morphism itself under transposition.

From the commutativity of the diagram, we see that \( \mathcal{K} \) is identified with the following kernel \( \mathcal{K}_D \):

\[\mathcal{K}_D := \ker \left[ H^0(\text{End}(F)(-D)) \xrightarrow{H^0(\varphi_{x_0} \otimes_{\mathcal{O}_C} \text{id}_{\mathcal{O}_C(-D)})} H^0(\text{End}(F) \otimes_{\mathcal{O}_C} \omega_{C/k}) \right].\]

The inclusion \( \mathcal{O}_C(-D) \hookrightarrow \mathcal{O}_C \) induces an inclusion of vector spaces \( \mathcal{K}_D \subseteq \mathcal{G} \), where \( \mathcal{G} := \ker(H^0(\varphi_{x_0})) \) is the subset global of endomorphisms of \( F \) that commute with the Higgs field \( \nabla \). Since \( k \) is algebraically closed and \( (F, \nabla) \) is polystable, we know that \( \mathcal{G} \) consists of a direct sum of “constant” matrix endomorphisms in \( M_{n_i \times n_i}(k) \) of \( F \) that act on each isotypic component of \( F \) consisting of a direct sum of \( n_i \) isomorphic stable logarithmic Higgs bundles (cf. the proof of Lemma 5.3). Notice that \( \mathcal{K}_D \subseteq \mathcal{G} \) is the subset of endomorphisms in \( \mathcal{G} \) that vanish on the divisor \( D \). But any nonzero “constant” matrix in \( \mathcal{G} \setminus \{0\} \) is nowhere vanishing. Since \( D \) is nonempty, we conclude that \( \mathcal{K}_D = 0 \), as desired.

**Remark 5.8.** If we have \( 2g - 2 + \deg(D) \geq 2 \), then the strictly semistable points of the moduli space of logarithmic Higgs bundles are singular points (the same holds for the moduli space of logarithmic connections). Therefore, under the assumption \( 2g - 2 + \deg(D) \geq 2 \), the moduli space is singular in the non-coprime case, even though we know that the stack of semistable objects is smooth by Lemma 5.6.

**Corollary 5.9.** Suppose that \( n \) and \( d \) are coprime. For any point \( a \in \mathbb{A}^1_B \), the fiber \( (\text{M} \text{Hodge}^{ss}_{C_B})_a \) of the moduli space is equidimensional of dimension \( n^2(2g - 2 + \deg(D_a)) + 1 \).

**Proof.** In view of the smoothness of \( (\text{M} \text{Hodge}^{ss}_{C_B})_a \), it suffices to prove that for every closed geometric point \( x \in (\text{M} \text{Hodge}^{ss}_{C_B})_a \), the dimension of the tangent space \( T_{(\text{M} \text{Hodge}^{ss}_{C_B})_a,x} \) of \( x \) in \( (\text{M} \text{Hodge}^{ss}_{C_B})_a \), is equal to \( n^2(2g - 2 + \deg(D_a)) + 1 \). Choose a lift \( \bar{x} \) of \( x \) in the stack \( (\text{M} \text{Hodge}^{ss}_{C_B})_a \). By Lemma 5.3, it follows that the morphism \( (\text{M} \text{Hodge}^{ss}_{C_B})_a \to (\text{M} \text{Hodge}^{ss}_{C_B})_a \) is a \( \mathbb{G}_m \)-gerbe. In other words, étale locally on \( (\text{M} \text{Hodge}^{ss}_{C_B})_a \), the fibers of \( (\text{M} \text{Hodge}^{ss}_{C_B})_a \to (\text{M} \text{Hodge}^{ss}_{C_B})_a \) are isomorphic to the classifying stack \( B_0 \mathbb{G}_m \). This implies, by the definition of tangent space, that \( (\text{M} \text{Hodge}^{ss}_{C_B})_a \to (\text{M} \text{Hodge}^{ss}_{C_B})_a \) induces an isomorphism of tangent spaces \( T_{(\text{M} \text{Hodge}^{ss}_{C_B})_a,\bar{x}} \sim T_{(\text{M} \text{Hodge}^{ss}_{C_B})_a,\bar{x}} \). By Corollary 4.9, we have

\[
\dim(T_{(\text{M} \text{Hodge}^{ss}_{C_B})_a,x}) = \dim(T_{(\text{M} \text{Hodge}^{ss}_{C_B})_a,\bar{x}}) = n^2(2g - 2 + \deg(D_a)) + 1 + \dim(Q_{\bar{x}}).
\]
By the vanishing of the obstruction module $Q_X$ proven in Proposition 5.6, it follows that $\dim(T_{(\mathcal{M}_{\text{Hodge}}^{ss})_A^x}) = n^2(2g - 2 + \deg(D_d)) + 1$, as desired. □

5.3 Smoothness in the coprime case without poles

In this subsection, we consider flat connections without poles. By [7, Prop. 3.1], a vector bundle on a curve over an algebraically closed field admits a flat connection if and only if each of its indecomposable summands has degree not invertible in the field.

In particular, in characteristic zero, the de Rham moduli stack is empty unless the degree is zero. In degree 0, the de Rham moduli space of semistable flat connections is singular, and similarly for the Higgs and Hodge moduli spaces.

In positive characteristic $p$, in the case when the degree $d = d'p$ is a multiple of $p$ and rank and degree are coprime $(n, d'p) = 1$, the smoothness has been proven in [13, Prop. 3.1] under the assumption that the base $B$ is reduced and Noetherian. The proof given in loc. cit. is ad hoc and based on earlier related smoothness results.

The methods in the proof of Theorem 5.2 can be modified to prove the smoothness of the Hodge moduli space $\mathcal{M}_{\text{Hodge}}^{ss} \to \mathbb{A}_B^1$ when: $D$ is the empty divisor, $n$ and $d$ are coprime, and the integer $d$ maps to zero in all residue fields of points of the not necessarily reduced but Noetherian $B$. Note that when $B$ is connected, these conditions can be met only when $B$ has positive characteristic, say, $p$, so that we are then in the aforementioned case where $(n, d = d'p) = 1$ with $B$ Noetherian.

Here, we give a sketch of the proof of the smoothness assertion made above. For this remark, we need to keep track of the rank $n$, and so, we use the notation $\mathcal{M}_{\text{Hodge}}^{ss}_{n,C_B}$ for the moduli space.

We need to prove that the obstruction $ob_x$ vanishes for any given geometric point $x : \text{Spec}(k) \to \mathcal{M}_{\text{Hodge}}^{ss}_{n,C_B}$. We can assume that $B = \text{Spec}(k)$. If we write $x = (F, \nabla)$, then the determinant connection $\det(x) := (\det(F), \det(\nabla))$ is an element in $\mathcal{M}_{\text{Hodge}}^{1, C_k}$. There is a commutative diagram of morphisms induced by the trace $\text{tr} : \mathcal{E}nd(F) \to \mathcal{O}_C$:

\[
\begin{array}{ccc}
H^1(\mathcal{E}nd(F)) & \xrightarrow{H^1(\varphi_x)} & H^1(\mathcal{E}nd(F) \otimes_{\mathcal{O}_C} \omega_{C/k}) \\
\downarrow H^1(\text{tr}) & & \downarrow H^1(\text{tr} \otimes \text{id}_{\omega_{C/k}}) \\
H^1(\mathcal{O}_C) & \xrightarrow{H^1(\varphi_{\det(x)})} & H^1(\omega_{C/k})
\end{array}
\]

which induces a trace map $Q_x \to Q_{\det(x)}$ on the cokernels. It can be checked directly from the construction that this maps sends $ob_x$ to $ob_{\det(x)}$. Since $d$ is assumed to be divisible by the characteristic of $k$, it follows that every line bundle of degree $d$ admits a $t$-connection. The Hodge stack $\mathcal{M}_{\text{Hodge}}^{1, C_k}$ is isomorphic to a smooth affine bundle with fibers $H^0(C, \omega_k)$ over the Picard stack. In particular, $\mathcal{M}_{\text{Hodge}}^{1, C_k}$ is smooth, and so, $ob_{\det(x)} = 0$. This shows that $ob_x$ lies in the kernel of the trace morphism $Q_x \to Q_{\det(x)}$, and so, it lies in the trace-zero part of this module. One verifies that this latter is the trace-zero obstruction module $Q_x^0$ formed using trace-zero endomorphisms $\mathcal{E}nd^0(F)$. Hence, it suffices to show that the trace-zero obstruction module $Q_x^0$ vanishes. The same degeneration argument as in the proof of Theorem 1.1 above shows that we can take $x$ to be a Higgs bundle. Since the divisor $D$ is empty, the obstruction module $Q_x$ is dual to the space $\mathcal{K}$ of endomorphism of the Higgs bundle. This consists of the constant scalar matrices $k$, because $n$ and $d$ are coprime. The trace-zero obstruction module $Q_x^0$ will be dual to the space of trace-zero endomorphisms. In other words, the trace-zero module $Q_x^0$ is isomorphic to the dual of the
kernel of the trace on scalar matrices \( k \xrightarrow{n(-)} k \). Since \( n \) is coprime to \( d = d'p \), it is also coprime to \( \text{char}(k) \), and hence, this kernel is 0.

### 5.4 Proof of the integrality assertion in Theorem 1.1

**Proposition 5.10.** All of the fibers of \( M_{\text{Hodge}}^{ss}_{C_k} \to \mathbb{A}^1_B \) are smooth and geometrically integral.

**Context 5.11.** In order to show the proposition, we can assume without loss of generality that \( B = \text{Spec}(k) \) is a field. We shall assume this for the rest of this section.

We start by proving the proposition for the 0-fiber \( M_{\text{Higgs}}^{ss}_{C_k} \). This is the moduli space of logarithmic Higgs bundles with poles at \( D \). We use the Hitchin fibration \( M_{\text{Higgs}}^{ss}_{C_k} \to A(C) \), where \( A(C) \) denotes the Hitchin base \( A(C) = \bigoplus_{i=1}^n H^0((\omega_C/k(D))^\otimes i) \), viewed as an affine space over \( k \).

**Definition 5.12 (Spectral curve).** Let \( W = \text{Spec}(\text{Sym}(\omega_C/k(D)^\vee)) \) be the total space of the line bundle \( \omega_{C/k}(D) \), with projection \( \pi_W : W \to C \). There is the tautological section \( x : \mathcal{O}_W \to \pi_W^*(\omega_{C/k}(D)) \). For any morphism \( \text{Spec}(k) \to A(C) \) corresponding to a tuple of sections \( (\sigma_i \in H^0(\omega_{C/k}(D))^\otimes i)_{i=1}^n \), we define the spectral curve \( C(\sigma_i) \subset W \) to be the vanishing locus of the section:

\[
x^n + \pi_W^*(\sigma_1)x^{n-1} + \cdots + \pi_W^*(\sigma_{n-1})x + \pi_W^*(\sigma_n) \in H^0(\pi_W^*(\omega_{C/k}(D))^\otimes n).
\]

**Lemma 5.13.** Suppose that \( k = \overline{k} \).

1. The spectral curve assigned to the generic point of \( A(C) \) is singular if and only if \( g = 0, n > 1 \) and \( \deg(D) = 1 \).
2. The generic spectral curve is reducible if and only if \( g = 0, n > 1 \) and \( \deg(D) = 2 \).

**Proof.** When \( g = 0, n > 1 \), and \( \deg(D) = 1 \), the Hitchin base consists of a single point corresponding to the 0 section. The unique spectral curve is then an \( n \)th infinitesimal thickening of \( C \), and therefore, it is singular and irreducible.

We are left with the following remaining cases.

(A) \( n = 1 \).

(B) \( (\omega_{C/k}(D))^\otimes n \) is very ample on \( C \).

(C) \( g = 0, n > 1 \), and \( \deg(D) = 2 \).

(D) \( g = 1, n = 2 \), and \( \deg(D) = 1 \).

Since smoothness is an open condition, it suffices to show that there exists a single spectral curve that is smooth to conclude smoothness. The same holds for (geometric) integrality.

The first case (A) is clear, because then every spectral curve is isomorphic to \( C \). On the other hand, (B) follows from an application of Bertini’s theorem ([38, Tag 0FD6]+ [38, Tag 0G4F]).

In case (C), we have that \( H^0(\omega_{k/k}(D))^\otimes i) = H^0(\mathcal{O}_{k/k}) = k \). For generic choice of constants \( \sigma_i \in k \), the polynomial \( x^n + \sigma_1 x^{n-1} + \cdots + \sigma_n \) in \( k[x] \) splits into distinct linear factors, and then, the corresponding spectral curve will be a disjoint union of \( n \) copies of \( C \). Hence, it will be smooth and reducible.
Let us assume (D) with char\( k \neq 2 \). For a section \( \sigma_2 \in H^0(\omega_C/k(D))^\otimes 2 = H^0(\mathcal{O}_C(2D)) \), we consider the spectral curve \( f : C_{\sigma_2} \to C \) defined by \( x^n + \pi_W^*(\omega_{\sigma_2}) \) inside \( W \). Let \( c \) be a closed point of \( C \). Choose a uniformizer \( t \) for the completion of the local ring \( \mathcal{O}_{C,p} \), and a trivialization of the stalk of \( \omega_C/k(D) \) at \( p \). Using these choices, we can write the formal fiber of \( f : C_{\sigma_2} \to C \) at \( p \) as \( \text{Spec}(k[[t]][x]/(x^2 + \sigma_2(t))) \). By the Jacobi criterion for smoothness, \( C_{\sigma_2} \) will be smooth at the points lying over \( t \) if the following three polynomials in \( k[x] \) do not have a common root

\[
\begin{cases} 
    x^2 + \sigma_n(0) \\
    2x \\
    \partial_t(\sigma_n)(0). 
\end{cases}
\]

Since \( 2 \) is coprime to the characteristic of \( k \), the second equation forces \( x \) to be 0. Therefore, the points lying over \( p \) will be smooth if one of \( \sigma_2(0) \) or \( \partial_t(\sigma_2)(0) \) does not vanish. This is true as long we choose a section \( \sigma_2 \in H^0(\mathcal{O}_C(2D)) \) whose vanishing locus consists of two distinct points of \( C \), which is always possible.

Moreover, for this choice of \( \sigma_2 \), the spectral curve \( C_{\sigma_2} \) is integral, even if the characteristic is \( 2 \). Indeed, the spectral curve \( C_{\sigma_2} \to C \) is flat over \( C \), because it is a relative global complete intersection over \( C \) [38, Tag 00SW]. Therefore, it suffices to check integrality of the generic fiber. So, we think of \( \sigma_2 \) as an element of the ring of functions \( k(C) \), and we want to show that \( \text{Spec}(k[x]/(x^2 - \sigma_2)) \) is integral. This is true because \( \sigma_2 \) is not a square in \( k(C) \) (\( \sigma_2 \) has simple zeroes by construction). Therefore, the generic spectral curve is integral in case (D) regardless of characteristic.

We are left to show the smoothness in case (D) with char\( k = 2 \). Choose a nonzero \( \sigma_1 \in H^0(\omega_C/k(D)) = H^0(\mathcal{O}_C(D)) \), and choose \( \sigma_2 \in H^0(\omega_C/k(D)^\otimes 2) = H^0(\mathcal{O}_C(2D)) \) linearly independent to \( (\sigma_1)^2 \). Notice that \( \sigma_1 \) has only one zero at \( D \), and it is a simple zero. On the other hand, \( \sigma_2 \) does not vanish at \( D \), since the linear system spanned by \( (\sigma_1)^2 \) and \( \sigma_2 \) is base-point free. Using that the characteristic is \( 2 \), the local Jacobi criterion in this case tells us that the spectral curve is smooth at a point 0 with uniformizer \( t \) whenever the following polynomials in \( k[x] \) do not have a common zero:

\[
\begin{cases} 
    x^2 + x\sigma_1(0) + \sigma_2(0) \\
    \sigma_1(0) \\
    x\partial_t(\partial_t)(0) + \partial_t(\sigma_2)(0). 
\end{cases}
\]

Since \( \sigma_1 \) only vanishes at \( D \), the second equation forces the point 0 to be \( D \in C(k) \). The morphism \( C \to \mathbb{P}^1_k \) corresponding to the linear series spanned by \( (\sigma_1)^2 \) and \( \sigma_2 \) is ramified at \( D \), and therefore, we have \( \partial_t(\partial_t)(0) = 0 \). Since \( D \) is a simple zero of \( \sigma_1 \), we have \( \partial_t(\sigma_1)(0) \neq 0 \), and so, the vanishing of the third equation would imply \( x = 0 \). Going back to the first equation, we see that the vanishing of all three equations forces 0 to be the point \( D \) and \( \sigma_2(0) = 0 \), which is not true because \( \sigma_2 \) does not vanish at \( D \). 

\[ \square \]

Lemma 5.14. \( M_{\text{Higgs}}^{ss} \) is smooth and geometrically connected. It is empty when \( n > 1 \), \( g = 0 \) and \( \deg(D) \leq 2 \).

\textbf{Proof.} We have already shown smoothness in Theorem 1.1, we just need to prove geometric connectedness. For this, we can replace \( k = \overline{k} \). We start by dealing with the cases
when the generic spectral curve is smooth and irreducible, as characterized in Lemma 5.13. Note that $M_{\text{Higgs}}^{ss}_{k} \to A(C)$ is flat by miracle flatness [38, Tag 00R4], because it is a morphism between integral $k$-smooth schemes (Theorem 1.1) with equidimensional fibers of the same dimension [9, Cor. 8.2]. By flatness, it suffices to show that the generic fiber of the Hitchin fibration is irreducible. The generic fiber of the Hitchin morphism will be a connected component of fixed degree of the Picard scheme associated to the smooth and irreducible generic spectral curve [5, Prop. 3.6], [34, §5]. Therefore, it is connected, as desired.

We are left with the case when $n > 1$, $g = 0$, and $\deg(D) \leq 2$. Then there are no stable Higgs bundles under our coprime assumption $(n,d) = 1$. Therefore, the Higgs moduli space is empty in that case, and hence vacuously irreducible. □

Now we are ready for the proof of the more general proposition.

Proof of Proposition 5.10. We have already seen in Theorem 1.1 that all fibers are smooth; we only need to prove that they are geometrically connected. Without loss of generality, we replace $B$ with $\text{Spec}(k)$ for $k = \bar{k}$. We already know that the 0-fiber $M_{\text{Higgs}}^{ss}_{\bar{k}}$ is geometrically connected. Using this and the “semistable reduction” theorem in [24, Thm. 5.1]), we see that the total space $M_{\text{Hodge}}^{ss}_{\bar{k}}$ is connected. Since $M_{\text{Hodge}}^{ss}_{\bar{k}}$ is smooth, this means that $M_{\text{Hodge}}^{ss}_{\bar{k}}$ is integral. Therefore, the generic fiber of $M_{\text{Hodge}}^{ss}_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ is integral. Since $M_{\text{Hodge}}^{ss}_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ is a constant family away from 0, and $k = \bar{k}$, this means that all the fibers away from 0 are also geometrically irreducible. □

6 | PROOF OF THE COHOMOLOGICAL THEOREMS 3.6 AND 3.8

Theorems 3.6 and 3.8 are concerned with specialization morphisms in the context of moduli spaces of $t$-connections with poles and coprime rank and degree.

If the morphism to a DVR is not proper, as it is the case for the moduli spaces above, then the desirable specialization morphisms may fail to be defined. The paper [12] studies this problem and provides criteria for the existence of specialization morphisms. These criteria are often based on the existence of a suitable completion of the morphism to the DVR, where one leverages the existence of the specialization morphism after the completion to deduce the existence before the completion.

In this section, we recall some of the techniques, rooted in [12] and [14], and employed in [13] to study specialization morphisms for moduli spaces of $t$-connections without poles under suitable coprimality assumptions; we recall the constructions of the completions of moduli spaces used in this study; we observe that these techniques apply to the case of poles; we finally prove Theorems 3.6 and 3.8.

6.1 | Completion of Hodge, Higgs, and de Rham moduli spaces

In the remainder of this paper, we need suitable completions of the structural morphisms $\upsilon_{\text{Hodge},B}$ (2.3), $\upsilon_{\text{Higgs},B}$ (2.5), and $\upsilon_{\text{de Rham},B}$ (2.7) to the Noetherian base $B$ and of the structural morphism $\tau_B$ (2.2) to $\mathbb{A}^1_B$. 
de Cataldo and Zhang [14, §2.4] develop a general compactification technique and apply it to Hodge, Higgs, and de Rham moduli spaces without poles. This can also be applied to the moduli spaces of $t$-connections with poles appearing in this paper, as soon as we have the properness of the Hodge–Hitchin morphism, which we do by [25, Thm. 5.2]. These techniques give us all the desired completions, except for the morphism $u_{\text{Hodge}, B} (2.3)$. We complete the morphism $u_{\text{Hodge}, B} (2.3)$ by means of a simple additional construction, akin to the completion of $\mathbb{A}^1_B$ given by $\mathbb{P}^1_B$.

Let us summarize the construction of all these completions, and list the properties relevant to the proof of Theorems 3.6 and 3.8.

**Context 6.1.** For the compactification results in this section, we do not assume that the rank and degree are coprime, or that the fibers of the divisor $D$ of poles are nonempty.

**Notation 6.2.** In what follows, we omit many decorations, and the moduli spaces in question may be with or without poles.

We have the following commutative diagram with Cartesian square of $\mathbb{G}_m$-equivariant morphisms (see [14, (48)]):

\[
\begin{array}{ccc}
M & \xrightarrow{\tau} & \mathbb{A}^2_t \\
\downarrow & & \downarrow \\
\mathbb{A}^2_{x,y} & \xrightarrow{\tau'} & \mathbb{A}^1_t, \\
\downarrow & & (x,y) \rightarrow t = xy \\
\mathbb{A}^1_x & & (x, \lambda x) \rightarrow t = \lambda x \\
\end{array}
\] (22)

where the $\mathbb{G}_m$ action on $\mathbb{A}^2_{x,y}$ is defined by setting $\lambda(x, y) := (x, \lambda y)$, the $\mathbb{G}_m$-action on $\mathbb{A}^1_t$ is the usual dilation $\lambda \cdot t = \lambda t$, and the $\mathbb{G}_m$ action on $\mathbb{A}^1_x$ is trivial.

The completions of $u_{\text{Hodge}, B} (2.3)$, $u_{\text{Higgs}, B}(2.5)$, and of the structural morphism $\tau_B (2.2)$ to $\mathbb{A}^1_B$ are obtained as follows. We refer to [14, Theorem 2.13 and (48) (resp. Theorem 2.14 and (49), if we wish to incorporate the Hitchin-type morphisms)] for more details. Note that these theorems follow from the generalization [14, Theorem 2.7] of a well-known compactification technique of Simpson’s, generalized in [14, Theorem 2.7].

Recall that the nilpotent cone $N\text{Higgs}$ is the fiber of the proper Hitchin morphism $h : M\text{Higgs} \rightarrow A$ over the origin $o_A$ of the Hitchin base $A$.

**Definition 6.3.** We define $M^* \subset M$ as the open complement of the union of all nilpotent cones in the preimage $M_{x=0}$ of the $x$-axis inside $\mathbb{A}^2_{x,y}$.

**Definition 6.4.** We define the following $\mathbb{A}^1_x$-schemes obtained by taking quotients by the $\mathbb{G}_m$-action:

- $\text{MHodge} := (M^*)/\mathbb{G}_m$ (proper over $\mathbb{A}^1_x$, but not over $B$);
- $\text{MHiggs} := ((M^*)_{x=0})/\mathbb{G}_m$ (proper over $B$);
- $\text{Mde Rham} := ((M^*)_{x=1})/\mathbb{G}_m$ (proper over $B$);
- $\delta\text{MHiggs} := ((M^*)_{x=0, y=0})/\mathbb{G}_m \equiv'' (M^*)_{x=1, y=0}/\mathbb{G}_m = \delta\text{Mde Rham}$.
Note that: $\partial M_{\text{de Rham}} = \partial M_{\text{Higgs}} = (M_{\text{Higgs}} \setminus N_{\text{Higgs}}) / G_m$ (proper over $B$).

The resulting proper morphism $\tau : M_{\text{Hodge}} \to \mathbb{A}^1_B$ is $G_m$-equivariant for the natural $G_m$-action on $\mathbb{A}^1_B$ given by $t \cdot x = tx$. After restriction over $G_m \subseteq \mathbb{A}^1_B$, we have $G_m$-equivariant isomorphisms:

$$(\partial M_{\text{Hodge}}, M_{\text{Hodge}}, M_{\text{Hodge}})_{G_m} \simeq (\partial M_{\text{de Rham}}, M_{\text{de Rham}}, M_{\text{de Rham}}) \times G_m. \quad (23)$$

In particular, we have natural isomorphisms: $\partial M_{\text{Hodge}} = (M^*)_v = 0 / G_m \simeq \partial M_{\text{Higgs}} \times \mathbb{A}^1 = \partial M_{\text{de Rham}} \times \mathbb{A}^1$ (proper over $\mathbb{A}^1_B$, but not over $B$).

**Notation 6.5.** If in place of (22) (i.e., [14, (48)]), we consider its version [14, (49)], augmented by the Hodge–Hitchin morphism, and we obtain that the completions above factor through suitable completions of the Hodge–Hitchin, Hitchin, and de Rham–Hitchin morphisms, with suitably completed targets. We thus have the following three completions of morphisms (cf. [14, Theorems 2.19, 2.18, and 2.14, respectively]).

- A completion of the morphism $v_{\text{Higgs}, B} : M_{\text{Higgs}} \to A(C_B) \to B$ to a morphism:

$$v_{\text{Higgs}} : M_{\text{Higgs}} \xrightarrow{h_{\text{Higgs}}} (A(C^B)) \to B. \quad (24)$$

- (If $B$ has positive equicharacteristic) A completion of the morphism $v_{\text{de Rham}, B} : M_{\text{de Rham}} \to A(C_B) \to B$ to a morphism:

$$v_{\text{de Rham}} : M_{\text{de Rham}} \xrightarrow{h_{\text{de Rham}}} A(C_B) \to B. \quad (25)$$

- (If $B$ has positive equicharacteristic) A completion of the morphism $\tau_B : M_{\text{Hodge}} \to A(C^{(B)}_B) \times_B \mathbb{A}^1_B \to \mathbb{A}^1_B$ to a morphism:

$$\tau_B : M_{\text{Hodge}} \xrightarrow{h_{\text{Hodge}}} A(C^{(B)}_B) \times_B \mathbb{A}^1_B \to \mathbb{A}^1_B. \quad (26)$$

The compositum morphism $\overline{v_{\text{Hodge}}} : \overline{M_{\text{Hodge}}} \to \mathbb{A}^1_B \to B$ is not proper as soon as the intermediate morphism to $\mathbb{A}^1_B$ is surjective (e.g., in the case of coprime rank and degree and nonempty divisor of poles $D$; or in the case of degree 0 and empty $D$) and therefore does not yield the desired completion.

Next, we construct such a completion.

**Definition 6.6.** Let $\overline{M_{\text{Hodge}}}$ denote the scheme over $\mathbb{P}^1_B$ obtained by gluing $M_{\text{Hodge}}$ to $M_{\text{de Rham}} \times \mathbb{A}^1_B$ along their open subsets over $G_m$ by using the isomorphism (23) and using the same prescription that yields $\mathbb{P}^1$ from two copies of $\mathbb{A}^1$. In particular, we obtain a proper morphism $\overline{\tau} : \overline{M_{\text{Hodge}}} \to \mathbb{P}^1_B$. 
If $B$ is of positive equicharacteristic, then we get a canonical factorization:

$$
\begin{array}{c}
\text{\overline{v}_{\text{Hodge}}} : \text{M} \to \text{M} \to \tilde{\text{r}} \\
\text{\text{\overline{h}}}_{\text{Hodge}} \to \text{A}(C_{B}^{(R)}) \times_{B} \mathbb{P}_{B}^{1} \to \text{proj} \to \mathbb{P}_{B}^{1} \to B.
\end{array}
$$

(27)

The boundary $\partial \overline{\text{M}}_{\text{Hodge}} = \partial' \cup \partial''$, complement of $\text{M}_{\text{Hodge}}$, is made up of two relative to $B$ hypersurfaces where $\partial''$ is the preimage of $\infty_{B}$ via the morphism to $\mathbb{P}_{B}^{1}$ and $\partial'$ is the closure of $\partial_{\text{M}_{\text{Hodge}}}$.

We have two charts $\overline{\text{M}}_{\text{Hodge}} = M^{*}/G_{m}$ and $\overline{\text{M}}_{\text{de} \text{Rham}} \times \mathbb{A}^{1} = ((M^{*})_{x=1}/G_{m}) \times \mathbb{A}^{1}$, and in each of these two charts, before taking the quotient by $G_{m}$, the hypersurfaces $\partial'$ and $\partial''$ are given by relative Cartier divisors.

The key observation here is that, when the Hodge moduli space is smooth, for example, in our case with poles and coprime rank and degree (cf. Theorem 1.1), these Cartier divisors form a simple normal crossing divisor over $B$.

### 6.2 Vanishing of vanishing cycles

We return to our assumption that we are working with poles and that rank and degree are coprime. For this section, we assume that $B$ is a DVR. We denote by $\phi$ the vanishing cycle functors associated with morphisms to $B$. We denote by $\mathbb{Q}_{\ell}$ the $\mathbb{Q}_{\ell}$-adic constant sheaf of rank 1 on a scheme $X$; we drop the space decoration if it is clear in the context.

Let us start by proving the following complement to SGA.

**Lemma 6.7.** Let $X$ be a Noetherian regular scheme. Let $v : X \to B$ be a smooth morphism from a Noetherian regular scheme to a DVR. Let $a : D \to X$ be a closed embedding with $D$ a divisor in simple normal crossings relative to $v$; in particular, the irreducible components of $D$ are smooth over $B$. Assume that the prime $\ell$ is invertible in $X$ and in $B$. Let $b : X^{0} := X \setminus D \to X$ be the open immersion. Then, we have

$$
\phi b_{*}b^{*}(\overline{\mathbb{Q}}_{\ell})_{X} = 0.
$$

(28)

**Proof.** We offer two proofs. The first one consists of applying Beilinson’s theorem, to the effect that the vanishing cycle functor $\phi$ commutes with the Verdier duality functor $\mathbb{D}$ up to a Tate shift (cf. [27, Cor. 0.2]), that is, there is an isomorphism of functors $\mathbb{D}X, \phi \simeq \phi \mathbb{D}X(-1)$. In fact, since $X$ is smooth over $B$, the constant sheaf on $X$ is self-dual up to shifts, so that the conclusion follows from the fact that one knows that $\phi b_{1}b^{*}(\overline{\mathbb{Q}}_{\ell})_{X} = 0$ (cf. [16, XIII, LM. 2.1.11, p. 105]) by an application of Beilinson’s theorem together with the standard $\mathbb{D}b_{1}b^{*} = b_{1}b^{*}\mathbb{D}$.

For the second one, we argue as follows. Let $D = \bigcup_{i \in I} D_{i}$ be the decomposition into irreducible components. For $J \subseteq I$, let $D_{J} := \bigcap_{i \in J} D_{i}$. By considering the distinguished triangle of functors $(a_{*}, a^{!}, \text{id}, b_{*}, b^{*})$, since the smoothness of $X/B$ implies that $\phi \overline{\mathbb{Q}}_{\ell} = 0$, it is enough to show that $\phi a_{*}\overline{\mathbb{Q}}_{\ell} = 0$. By the absolute purity conjecture proved by Gabber [18], we have that if $a_{E} : E \to X$ is the closed immersion of a pure codimension $e$ regular subscheme in $X$, then
\[ a^!_E(\overline{Q}_\ell)_X = (\overline{Q}_\ell)_E[-2c](-c), \] so that \( \phi a^!_E(\overline{Q}_\ell)_X = 0 \). We thus have \( \phi a^!_{D,J}(\overline{Q}_\ell)_{D,J} = 0 \), for every \( J \subset I \).

The conclusion follows by a simple devissage argument involving the application of the functors \( \phi a^! \) applied to the graded pieces of the stupid filtration on the acyclic complex providing a resolution of the constant sheaf on \( D \):

\[
0 \to (\overline{Q}_\ell)_{D,J} \to \bigoplus_{|J|=1} (\overline{Q}_\ell)_{D,J} \to \bigoplus_{|J|=2} (\overline{Q}_\ell)_{D,J} \to \ldots \to \bigoplus_{|J|=|I|} (\overline{Q}_\ell)_{D,J} \to 0.
\]

We have the closed and open immersions:

\[
\partial \text{MHodge} \xrightarrow{a} \text{MHodge} \xleftarrow{b} \text{MHodge}. \tag{29}
\]

In this paragraph, we work on the two charts before taking the quotient by \( \mathbb{G}_m \). We have morphisms as in (29). By the smoothness of \( M^* \to B \) (cf. Theorem 1.1), we know that \( \phi \overline{Q}_\ell = 0 \). The boundary is a simple normal crossing divisor on \( M^* \) relative to \( B \). By Lemma 6.7, we obtain the identity \( \phi b^*b^* \overline{Q}_\ell = 0 \). By the exactness of \( \phi \) applied to the distinguished triangle of functors \( (a^! a^!, \text{id}, b^* b^*) \), we see that \( \phi a^! a^! \overline{Q}_\ell = 0 \). These vanishing of vanishing cycle complexes occur on the two charts before taking the quotient by \( \mathbb{G}_m \). The purpose of the following lemma is to descend these three identities to the quotient by \( \mathbb{G}_m \), that is, to the two charts of MHodge.

**Lemma 6.8.** Let rank and degree be coprime and let us assume that we are in the situation with poles. We have \( \phi \overline{Q}_\ell = \phi a^! a^! \overline{Q}_\ell = \phi b^* b^* \overline{Q}_\ell = 0 \) on MHodge.

**Proof.** For the first \( \phi \overline{Q}_\ell = 0 \), we apply [12, Lemma 4.1.5]. The second \( \phi a^! a^! \overline{Q}_\ell = 0 \) would follow from \( \phi \overline{Q}_\ell = \phi b^* b^* \overline{Q}_\ell = 0 \) by applying \( \phi \) to the distinguished triangle \( (a^! a^!, \text{id}, b^* b^*) \). We are therefore left with proving \( \phi b^* b^* \overline{Q}_\ell = 0 \).

We use the notation of [12, (72)] freely, where the quotient morphism \( \pi = q \circ p : (M^*)_{x=1} \times \mathbb{A}^1 \to \text{MHodge} \) by \( \mathbb{G}_m \) is written as the composition of a quotient by a finite group (containing the stabilizers of the action) followed by a quotient by the free residual \( \mathbb{G}_m \)-action.

We shall show that the desired identity \( \phi b^* b^* \overline{Q}_\ell = 0 \) holds on the second chart \( \text{Mde Rham} \times \mathbb{A}^1 \), with quotient map \( \pi = q \circ p : (M^*)_{x=1} \times \mathbb{A}^1 \to \text{Mde Rham} \times \mathbb{A}^1 \). The same proof applies for the first chart.

We have the following chain of implications: (caution, the first identity is on the chart before taking the quotient, and the last is on the chart itself, i.e., after the application of \( \pi = q \circ p \))

\[
(\phi b^* b^* \overline{Q}_\ell = 0) \text{ (on } (M^*)_{x=1} \times \mathbb{A}^1) \Rightarrow
\]

\[
(p_* (\phi b^* b^* \overline{Q}_\ell) = 0) \Rightarrow (\phi p_* b^* b^* \overline{Q}_\ell = 0) \Rightarrow (\phi b^* p_* b^* \overline{Q}_\ell = 0) \Rightarrow
\]

\[
(\phi b^* b^* p_* \overline{Q}_\ell = 0) \Rightarrow (\phi b^* b^* \overline{Q}_\ell = 0) \Rightarrow (\phi b^* q^* \overline{Q}_\ell = 0) \Rightarrow
\]

\[
(\phi b^* q^* b^* \overline{Q}_\ell = 0) \Rightarrow (\phi q^* b^* b^* \overline{Q}_\ell = 0) \Rightarrow (q^* \phi b^* b^* \overline{Q}_\ell = 0) \Rightarrow
\]

\[
(\phi b^* b^* \overline{Q}_\ell = 0) \text{ (on } \text{Mde Rham} \times \mathbb{A}^1),
\]

where the first implication is a mere application of \( p_* \); the second is because \( p \) is finite, hence proper, so that \( p_* \phi = \phi p_* \); the third is by the commutativity of [12, (72)]; the fourth is because
\begin{align*}
b^* &= b^! \quad \text{for open immersions and we always have base change } p_*b^! = b^!p_*; \quad \text{the fifth is because } \\
\overline{Q}_c \text{ is a direct summand of } p_*\overline{Q}_c \quad \text{(cf. [14, Lemma 5.3] applied to } p); \quad \text{the sixth is simply because } \\
q^*\overline{Q}_c &= \overline{Q}_c; \quad \text{the seventh is by the commutativity of } [12, (72)]; \quad \text{the eight is because } q \text{ is smooth} \\
of relative dimension } 1 \quad \text{so that } q^! \text{ equals } q^*[2] \text{ and base change; the ninth is again by the} \\
smoothness of } q \text{ since then } \phi q^* = q^* \phi; \quad \text{and the tenth is because } q^* \text{ preserves stalks and } q \text{ is} \\
surjective. \quad \square
\end{align*}

### 6.3 Proof of Theorems 3.6 and 3.8

**Notation 6.9.** When we are working in positive equicharacteristic, there is a filtered version of the statements of Theorems 3.6 and 3.8. When we do not wish to repeat verbatim an argument that has been provided for the unfiltered version in order to prove the filtered version, we resort to locutions such as “(filtered) isomorphism.”

**Proof of Theorems 3.6.** By virtue of Lemma 6.8, the hypotheses of the unfiltered version of [12, Prop. 3.4.2.(A)] are met when applied to the completion $\tau_k$ (26) of the structural morphism $\tau_k$ of the Hodge moduli space. We deduce that the arrows on the bottom row of (12) are isomorphisms of cohomology rings, that the specialization morphism is defined, and that it is an isomorphism of cohomology rings. For the filtered version of the sought-after statement, we use the filtered version of [12, Prop. 3.4.2.(A)].

By applying the same method of proof of [13, Thm. 3.5], we see that we reach the desired conclusions for the top row of (12) (filtered and unfiltered version).

The left-hand-side vertical arrow in (12) is the identity, and hence, the sought-after properties are automatically valid.

The right-hand-side vertical arrow, being identified with the morphism associated with an extension of separably closed fields, is also a (filtered) isomorphism.

Every arrow in diagram (12), except for the middle vertical arrow, is a (filtered) isomorphism, forcing the middle vertical arrow to be one as well. \quad \square

**Proof of Theorem 3.8.** The goal is to prove that all the arrows in (17) exist and are (filtered) isomorphisms.

As a starting point, we use the commutative diagram of noncurved morphisms of cohomology rings (16). The noncurved arrows in the top and bottom row of (16) are (filtered) isomorphisms of cohomology rings by Theorem 3.6 applied to $\tau_\varepsilon$ and to $\tau_{\overline{\varepsilon}}$. We also have that the corresponding specialization morphisms on the top and bottom rows are defined and are (filtered) isomorphisms.

We use the completion $\overline{v}_{\text{Higgs}}$ (24) of the Higgs moduli spaces. Since we have proven smoothness of the morphism $v_{\text{Higgs}} : M_{\text{Higgs}}^{ss} \rightarrow B$, we can check the hypotheses of [12, Prop. 3.4.2.(A)] exactly in the same way as we did for Hodge in Lemma 6.8 (note that in this case, we do not need to consider the second chart). It follows that all the noncurved arrows in the left-hand-side Higgs column of (16) are filtered isomorphisms, and that the corresponding filtered specialization morphism is defined and is a filtered isomorphism. Here, we are using universal corepresentability $M_{\text{Higgs}}^{ss} \rightarrow M_{\text{Higgs}}^{ss}$ (Remark 5.4) to identify the special fiber with $M_{\text{Higgs}}^{ss}$. 

The arrows in the right-hand-side de Rham column of (16) are well-defined isomorphisms (filtered, when charB > 0) by the same argument using the completion \( v_{\text{de Rham}} \) (25) of de Rham moduli spaces with poles.

We use the completion \( v_{\text{Hodge}} \) (27) of the Hodge moduli space over \( B \). In view of Lemma 6.8, we can apply [12, Prop. 3.4.2.(A)] and deduce that the middle Hodge column is made up of (filtered) isomorphisms and that the (filtered) specialization morphism for this middle Hodge columns is defined and is a (filtered) isomorphisms.

We are now left with showing that the two horizontal arrows \( \rho_0B \) and \( \rho_1B \) in the middle row are (filtered) isomorphisms. This follows formally from the commutativity of the diagram of noncurved arrows (16), and the fact that all the remaining noncurved arrows have been proven to be (filtered) isomorphisms.

\[ \square \]

APPENDIX A: FACTORIZATION OF THE \( p \)-CURVATURE MORPHISM BY MARK ANDREA DE CATALDO, ANDRES FERNANDEZ HERRERO, AND SIQING ZHANG

For the definition of the Hodge–Hitchin morphism (5) in the case of connections without poles, see [26, Prop. 3.2], which works with a curve over a field. For a stronger result, which covers the case of curves — and of higher dimensions as well — over a Noetherian base, see [25, Cor. 5.7].

The proof of [26, Prop. 3.2] contains a minor inaccuracy, for it is stated that the stack of \( t \)-connections (without poles) is smooth over the base field, whereas even the open substack of semistables is not smooth. This purported smoothness is used in the proof of loc. cit.

Theorem 1.1 proves that the stack of semistable \( t \)-connections with poles is smooth over the Noetherian base \( B \). In this paper, we apply this smoothness result to the case when \( B \) is a field and when \( B \) is a DVR, both of which are reduced. Therefore, the proof given in [26, Prop. 3.2] works, with the trivial modification stemming from the fact that: while in the case with no poles loc. cit. uses, in the context of the elegant “Bost’s Trick,” the elementary identity \( \partial_x\partial_x\cdots\partial_x(p\text{ times}) = 0 \), in the case with poles, we can use the identity \( (x\partial_x)^{[p]} = x\partial_x \). In the end, while loc. cit. ends with a factor \( t \) in the case without poles, we end with a factor \( tx \) in the case with poles, and the logic to reach the desired conclusion, namely, the existence of the Hodge–Hitchin morphism for families of curves over a reduced Noetherian scheme, is the same.

In this appendix, we remove the assumption made above of semistability, as well as the assumption on the Noetherian \( B \) being reduced. We give a proof of the existence of the Hodge–Hitchin morphism (5) from the stack of \( t \)-connections, with or without poles for a family of curves over a Noetherian base \( B \). Moreover, we correct the minor inaccuracy in the proof of [26, Prop. 3.2].

The key step is to reduce to an auxiliary family of curves over a suitable complete and reduced ring, where then the Lazlo–Pauly logic is valid, without poles (factor \( t \), and with poles (factor \( tx \)). We now give the details of this key reduction step.

By Noetherian approximation (more precisely: choose a relative polarization and then use the fact that the stack of polarized smooth geometrically connected curves over \( F_p \) is locally of finite presentation over \( F_p \) [38, Tag 0DSS]+ [38, Tag 0E81]+[38, Tag 0DQ0], combined with [38, Tag 0CMX] applied to a colimit \( B = \text{colim} B_i \) as in [38, Tag 01ZA]) the curve \( C \to B \) fits into a Cartesian diagram:

\[
\begin{array}{ccc}
C & \longrightarrow & C' \\
\downarrow & & \downarrow \\
B & \longrightarrow & B',
\end{array}
\]
where $B'$ is of finite type over the prime field $\mathbb{F}_p$ and $C' \rightarrow B'$ is a smooth projective morphism with geometrically connected fibers of dimension 1. This means that the Hodge stacks fit into the following diagram:

$$
\begin{array}{ccc}
\mathcal{M}\text{Hodge}(C) & \longrightarrow & \mathcal{M}\text{Hodge}(C') \\
\downarrow & & \downarrow \\
B & \longrightarrow & B',
\end{array}
$$

Since the formation of the $p$-curvature morphism is compatible with base change in the base $B$, it suffices to show the desired factorization for $\mathcal{M}\text{Hodge}(C')$, and so, we can assume without loss of generality that $B$ is of finite type over $\mathbb{F}_p$.

Since the stack $\mathcal{M}\text{Hodge}(C)$ is locally of finite type over $B$ [21, Prop 2.2.2], it suffices to check the desired factorization for any family over an affine scheme $\text{Spec}(R)$ with $R$ of finite type over $\mathbb{F}_p$. Such a point $\text{Spec}(R) \rightarrow \mathcal{M}\text{Hodge}(C)$ corresponds to a function $t \in R$, a vector bundle $\mathcal{F}$ on $C \times R$, and a $t$-connection $\nabla$ on $\mathcal{F}$.

We can write $R = \mathbb{F}_p[t_1, t_2, \ldots, t_r]/I$ for some ideal $I$. We denote by $\hat{S}$ the completion of the polynomial ring $\mathbb{F}_p[t_1, t_2, \ldots, t_r]$ with respect to the ideal $I$. Since $\mathbb{F}_p[t_1, t_2, \ldots, t_r]$ is a reduced G-ring [38, Tag 07PX], the completion $\hat{S}$ is reduced ([38, Tag 0AH2] + [38, Tag 0C21]). Choose a $\text{Spec}(R)$-ample line bundle $\mathcal{L}$ on the family $C$. The deformation theory of smooth curves equipped with ample line bundles is unobstructed ([38, Tag 0AH2] + [38, Tag 0E84]). Similarly, the deformation theory of the vector bundle $\mathcal{F}$ has obstructions in the groups $H^2(C, I^j \otimes \mathcal{E}nd(\mathcal{F})) = 0$ [17, Thm. 8.5.3], and so, it is unobstructed as well. Therefore, we can get a compatible family of lifts of the triple $(C, \mathcal{F}, \mathcal{L})$ for every nilpotent thickening $\mathbb{F}_p[t_1, t_2, \ldots, t_r]/I^j$ as $j$ ranges over the positive integers. By Grothendieck’s existence and algebraization theorems ([38, Tag 089A] + [38, Tag 03O]), we can algebraize this formal tuple into families $(\tilde{C}, \tilde{\mathcal{F}}, \tilde{\mathcal{L}})$ over $\text{Spec}(\hat{S})$. Therefore, we get a Cartesian diagram of families of smooth curves:

$$
\begin{array}{ccc}
C & \longrightarrow & \tilde{C} \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \hookleftarrow & \text{Spec}(\hat{S}),
\end{array}
$$

and a vector bundle $\tilde{\mathcal{F}}$ on $\tilde{C}$ such that its restriction to $C$ recovers $\mathcal{F}$. Choose a lift $\tilde{t} \in \hat{S}$ of $t \in R$. In order to show the factorization of the $p$-curvature morphism as in [26, Prop. 3.2], we need to show that certain canonically defined sections of powers of the line bundle $\omega_{C/S}$ vanish. This can be done Zariski locally on $C$. Choose an affine open covering $U_i$ of $\tilde{C}$ that trivializes $\tilde{\mathcal{F}}$. We fix trivializations of $\tilde{\mathcal{F}}|_{U_i}$. We denote by $U_i$ the restriction to $C$, which yields an affine open covering with trivializations of the restriction $\mathcal{F}$. It suffices to show that the factorization of the $p$-curvature map on every $U_i$. The $t$-connection $\nabla$ on the trivial bundle $\mathcal{F}|_{U_i}$ can be written as $td_{U_i} + M$, where $d_{U_i} : O_{U_i} \rightarrow \Omega^1_{U_i/S}$ denotes the exterior derivative on $U_i$ and $M \in H^0(\omega_{U_i/R}^{\otimes 2})$ is a matrix of differentials. Choose a lift $\tilde{M} \in H^0(\omega_{\tilde{U}_i/S}^{\otimes 2})$ of $M$, and define $\tilde{\nabla}$ to be the $\tilde{t}$ connection $\tilde{t}d_{\tilde{U}_i} + \tilde{M}$ on the trivial bundle $\tilde{\mathcal{F}}|_{\tilde{U}_i}$. The $\tilde{t}$-connection $(\tilde{\mathcal{F}}|_{\tilde{U}_i}, \tilde{\nabla})$ on $\tilde{U}_i$ restricts to the $t$-connection $(\mathcal{F}|_{U_i}, \nabla)$ under the base change by $\text{Spec}(R) \hookrightarrow \text{Spec}(\hat{S})$. Since the formation of the $p$-curvature is compatible with such base change, it suffices to show the desired factorization for the $\tilde{t}$-connection $(\tilde{\mathcal{F}}|_{\tilde{U}_i}, \tilde{\nabla})$ on
Thus, we can work over the reduced ring $\tilde{S}$ and on affine open subsets of $\tilde{C}$ to prove the desired factorization. By passing to each irreducible component of $\text{Spec}(\tilde{S})$, we can furthermore assume that $\tilde{S}$ is an integral domain. Hence, we can use the local computation outlined in the proof of [26, Prop. 3.2], which assumes that the base ring is an integral domain. The calculation is carried out in the case without poles by using the vector field $\partial_x$. The case with poles is analogous, once we replace the vector field $\partial_x$ with $x\partial_x$. Note also that the case without poles implies directly the case with poles: the sections are trivial away from the poles, and hence, are trivial across the poles.

ACKNOWLEDGMENTS

We thank Roberto Fringuelli, Jochen Heinloth, Pengfei Huang, Georgios Kydonakis, Mirko Mauri, Hao Sun, Siqing Zhang, and Lutian Zhao for useful conversations. We would also like to thank the referee for carefully reading the manuscript. The first-named author has been partially supported by NSF grants 1901975 and DMS-2200492, and by a Simons Fellowship in Mathematics Award n. 672936.

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The Journal of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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