The tunneling picture of electron-positron pair creation

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The common tunneling picture of electron-positron pair creation in a strong electric field is generalized to pair creation in combined crossed electric and magnetic fields. This new picture is formulated in a gauge invariant and Lorentz invariant manner for quasistatic fields. It may be used to infer qualitative features of the pair creation process. In particular, it allows for an intuitive interpretation of how the presence of a magnetic field modifies and in particular cases even enhances pair creation. The creation of electrons and positrons from the vacuum may be assisted by an energetic photon, which can also be incorporated into this picture of pair creation.

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1. Introduction and motivation

One of the most intriguing predictions of quantum electrodynamics (QED) is certainly the possible breakdown of the vacuum in the presence of ultra-strong electromagnetic fields into pairs of electrons and positrons. Since its first prediction by Sauter and others [1–4] pair creation has been studied in many papers, see [5, 6] for recent reviews. The Schwinger critical field strength of $E_S = 1.3 \times 10^{18}$ V/m, where spontaneous pair creation is expected to set in, can not be reached even by the strongest lasers available today. However, pair creation may be assisted by additional fields or particles. Current research covers among others pair creation in spatially and temporally oscillating electric fields [7–10], pair creation induced by the interaction of strong pulsed laser fields with relativistic electron beams or a nuclear Coulomb field (Bethe-Heitler process) [11–14] and pair creation induced by additional photons in presence of an external field [15–18]. Furthermore, different aspects of pair creation like the effect of magnetic fields [19, 20], the dynamics, real-time evolution and pair distributions with non-trivial field configurations in one effective dimension [21–25], and effective mass signatures in multiphoton pair creation [26] are investigated at present. Current interest in the old topic of pair creation is prompted by recent advances in laser technology [27, 28] aiming for laser intensities exceeding $10^{22}$ W/cm$^2$ (corresponding to electric field strengths of about $10^{18}$ V/m) and by experimental proposals for quantum simulators [29] that may allow to study pair creation via quantum simulation. The ultimate quest for higher and higher laser intensities for studying quantum electrodynamical effects such as pair creation may be limited, however, just by the onset of pair creation [30].

At field strengths below the Schwinger limit $E_S$, pair creation via ultra-strong electric fields may be interpreted as a tunneling effect [31] similar to tunnel ionization from bound states via strong electric fields [32, 33] using the method of imaginary times [34] (see also recent applications [35–37]). This method uses classical trajectories with imaginary times to approximate the exponential suppression for the transition amplitude of interest. In spite of conceptual difficulties [38, 39], the tunneling picture of ionization was recently extended into the relativistic domain where the laser’s magnetic field component can no longer be neglected [40, 41]. The purpose of this contribution is to establish a similar picture for pair creation in an electromagnetic field, including the magnetic field to full extend. The influence of an additional quantized photon is incorporated and special care is taken with respect to gauge and Lorentz invariance. The tunneling picture presented here, is established in the quasistatic limit, where the electromagnetic field is assumed to be constant and uniform during the pair creation process. Furthermore, spin effects are not taken into account.

The manuscript is organized as follows. In Sec. 2 we describe the electromagnetic field configuration for pair creation, lay down the theoretical framework, and introduce all necessary notations. Our main results are presented in Sec. 3 where the tunneling picture for electron-positron pair creation in quasistatic crossed electric and magnetic fields is developed. Three different cases need to be distinguished, depending on the electric field amplitude being larger, equal or smaller than the magnetic field amplitude. The effect of a quantized photon is also discussed for all these cases. Properties of the maximum probability trajectories, stemming from the imaginary-time method, are investigated in Sec. 4. Finally, we conclude in Sec. 5. Further details of the calculations have been deferred into two appendices.

2. Theoretical framework

2.1. Geometric setup and notation

Natural units will be used in this work, i. e., $c = \hbar = 1$. The setup of the considered pair production process in an electromagnetic field of an ultra-strong laser is depicted in Fig. 1. The electric field with amplitude $E$ points in the $x$ direction and the magnetic field with amplitude $B$ in the $y$ direction. This configuration corresponds to the quasistatic limit of either a plane wave field or two counterpropagating laser fields. In case of a plane wave field, its wave vector will be directed along the positive $z$ direction and the amplitude of the electric and
magnetic field will be equal. The superposition of two counterpropagating laser fields can lead to electric and magnetic fields of different magnitudes. The wave vector of a possibly assisting high-energy photon is parallel to the $z$ direction, i.e., it may be positive or negative. In the following sections, we will utilize a semiclassical description of pair creation based on classical trajectories. For an economical description of these trajectories, the following notation will be used. The electron’s kinetic four-momentum is written as

$$ p^\mu = (p_0, p_x, p_y, p_z), $$

and likewise its canonical momentum $P^\mu$. The kinetic and canonical momentum of the positron are denoted by $q^\mu$ and $Q^\mu$, respectively. The wave vector for the photon $k^\mu$ is written, according to the geometry used in this work, as

$$ k^\mu = (k_0, 0, 0, k_z) $$

with $k_0 = |k_z|$. As the motion of the particles can be reduced to a one-dimensional description along the $x$ direction, we denote the electron’s and positron’s $x$ coordinate by $x^-$ and $x^+$ (not to be confused with light cone coordinates). Variables and their values at the point of pair production $x_\nu$ will be sub-indexed by “$\nu$”, thus $p^\mu_\nu = p^\mu(x_\nu)$ or $x^\mu_\nu$ being the $x$ component of the electron’s position at $x_\nu$. Variables and their values at the point, where the electron or the positron leave the imaginary trajectory will be sub-indexed by “e” (for exit), thus $p_{\nu e} = p_e(x^e)$ being the kinetic momentum of the electron in $x$ direction at its point of exit.

The equations of motion for the canonical momenta of the electron and the positron with charge $\pm e$ are given by

$$ \frac{d}{dt}P^\mu = -e \frac{\partial x^\mu}{\partial t} A_\nu, $$

$$ \frac{d}{dt}Q^\mu = e \frac{\partial x^\nu}{\partial t} A_\mu, $$

where $A_\mu$ denotes the electromagnetic field’s four-potential.

The electron’s and the positron’s kinetic and canonical momenta are connected via

$$ p^\mu(x) = P^\mu(x) + eA^\mu(x), $$

$$ q^\mu(x) = Q^\mu(x) - eA^\mu(x). $$

### 2.2. Matrix elements for pair creation

The mathematical handle for pair production is given by its so-called matrix element or transition amplitude from an initial vacuum state (possibly including a photon) to a final state with an electron and positron (see [42] for a thorough treatment). For an external electromagnetic field only, this is given by

$$ M_{\bar{\mu}} = \left< 1, p, q, \text{out}\left| 0, \text{in}\right. \right> . $$

Both, the initial in-state and the final out-state are defined in the Furry picture [43] and refer to a common time (see also Appx. A or [42]). The asymptotic four-momenta of the electron and positron are indicated by $p$ and $q$.

In case of an additional quantized photon field, which may assist the process, the matrix element will read in first order perturbation theory

$$ M_{\bar{\mu}} = i \int dx \left< 1, p, q, \text{out}\right| \hat{H}_{\text{int}}(x) \left| 1, p, q, \text{in}\right> . $$

The four-momentum of the quantized photon is labeled by $k$ and $\hat{H}_{\text{int}}$ designates the QED interaction vertex. Using semiclassical methods, the exponential part of both matrix elements can be evaluated approximately. Here, semiclassical refers to that only classical trajectories (although possibly imaginary) connecting the in- and the out-state, are taken into account in the path-integral picture. As shown in Appx. A (see Eqs. (A32) and (A41)), this approximation yields

$$ M_{\bar{\mu}} \sim \exp \left[ -\text{Im}\left( W_k + W_p + W_q \right) \right] . $$

$W_p$ and $W_q$ are the gauge-dependent classical actions of the electron and positron with asymptotic momentum $p$ and $q$ in the external field. Likewise, $W_k$ gives the classical action of the quantized photon with momentum $k$. Note, that both transitions (with or without an additional high-energy photon) may coexist. The limit $k \to 0$ leads to a vanishing amplitude in Eq. (5b) due to prefactors, but the exponential approaches the same value as the exponential in Eq. (5a). Therefore, in case of no additional photon, $W_k$ in Eq. (6) is set to zero. The various actions in (6) are not the same as the commonly used actions $S(x', x)$, which connect a position eigenstate with another position eigenstate. Rather, they are Legendre transforms thereof connecting a position eigenstate with a momentum eigenstate ($x'$ being implicitly defined by the canonical momenta, see Appx. A),

$$ W_p(x) = S_p(x', x) - P \cdot x', $$

$$ W_q(x) = S_q(x', x) - Q \cdot x', $$

$$ W_k(x) = S_k(x, x') + K \cdot x'. $$

The classical trajectories will consist of a path of the incoming photon toward the point of pair creation $x_\nu$, where
the photon converts into an electron and positron, and two outgoing paths of the created particles from $x_s$ onward. If there is no photon, the pair is created out of the vacuum at $x_s$. The exponent of Eq. (6) is therefore the imaginary part of the action, acquired by a photon coming from the past and propagating with momentum $k$ to $x_s$, and the two actions, acquired by the electron and positron propagating from $x_s$ to the future with momenta $p$ and $q$. Although the classical actions are gauge dependent, the square modulus of Eq. (6) is not, as the boundary terms at $x_s$ cancel each other and the boundary terms at $\pm\infty$ result in unimportant phases. Due to the Lorentz invariance of the actions, Eq. (6) is also invariant under Lorentz transformations.

### 2.3. Kinetic considerations at the point of pair production

At the point of pair production $x'_s$, the classical energy-momentum conservation

$$p^\mu_s + q^\mu_s = k^\mu$$  \hspace{1cm} (8)

must be satisfied for the trajectory being classical. This cannot happen on real classical trajectories, but on imaginary ones. Squaring both sides of $k^\mu - p^\mu_s = q^\mu_s$ leads to

$$kp_s = 0 = k_0p_{0,s} - k_qp_{z,s},$$  \hspace{1cm} (9)

yielding with $k_0 = |k_z|$

$$p^2_{0,s} = p^2_{z,s}$$  \hspace{1cm} (10)

and likewise for $q^2_s$. Hence, using the relativistic dispersion relation and squaring $p_s$ and $q_s$, gives

$$m^2 = -p^2_{x,s} - p^2_{y,s} = -q^2_{x,s} - q^2_{y,s}.$$  \hspace{1cm} (11)

Due to the fact that $p_s$ and $q_s$ are constants of motion in the aforementioned setup, they have to be real to be consistent with a real asymptotic momentum. Therefore, $p_{x,s}$ and $q_{x,s}$ must be purely imaginary

$$p_{x,s} = -q_{x,s} = \pm i\sqrt{m^2 + p^2_z} = \pm im_s,$$  \hspace{1cm} (12)

$$m_s = \sqrt{m^2 + p^2_z}.$$  \hspace{1cm} (13)

Starting from $x_s$, both trajectories of the electron and the positron need to be followed until the exit points in order to calculate the imaginary part of the action along these trajectories. The trajectories will be called “imaginary”, as long as the momenta $p_s$ and $q_s$ are still imaginary. Hence the exit points are given by the condition that $p_x$ and $q_x$ become zero, and consequently, real

$$p_{x,e} = 0 = q_{x,e}.$$  \hspace{1cm} (14)

### 3. Tunneling picture for the constant field approximation

The tunneling picture, presented in this work, is based on the assumption that the external electromagnetic field can be treated as constant and uniform during the imaginary part of the trajectory. In Sec. 3.1 this assumption will be discussed in more detail and the kinetic equations, necessary for determining the pair creation probability, will be derived. Based on these findings an intuitive picture of pair creation will be derived in Sec. 3.2 and discussed in detail in the remaining subsections.

### 3.1. Kinetics within the constant field approximation

Expanding the electromagnetic field in the region of interest just up to linear terms in $x^\mu$ leads to a constant and uniform electric and magnetic field. For the case of an external field due to two counterpropagating lasers, the approximation is justified if the space time region of the imaginary dynamics, which is $|x'_s - x^\mu_s|$, is small compared to the space time region on which the electromagnetic field varies. In case of an external plane wave field, this approximation is also valid, if the imaginary trajectory is such, that both the electron and the positron move closely to the plane wave’s phase and therefore see the same field everywhere. Hence applying the constant field approximation requires to check, after calculating the trajectories, that along these, the electromagnetic field under consideration can be really treated as constant. In general, the electromagnetic field is evaluated at complex spacetimes. Using the constant field approximation, the vector potential for the electromagnetic field may be written as

$$A_\mu(t,z) = (0, Et - Bz, 0, 0).$$  \hspace{1cm} (15)

Here, $E$ and $B$ correspond to the value of the electric and magnetic field at the point where the electromagnetic field is expanded. For crossed or counterpropagating lasers, it might be possible that $|E| \neq |B|$. Thus allowing for arbitrary values of $E$ and $B$, there are three different cases $|E| > |B|$, $|E| = |B|$, and $|E| < |B|$ [44]. The corresponding Lorentz invariant quantity

$$\mathcal{F} \equiv \sqrt{|E^2 - B^2|}$$  \hspace{1cm} (16)

will also be used in this work.

Integrating the equations of motion (3) with constant crossed fields of the geometry in Fig. 1 yields

$$p_0(x^-) = p_{0,e} - eE(x^- - x^e),$$  \hspace{1cm} (17a)

$$p_x(x^-) = p_{x,e} - eB(x^- - x^e),$$  \hspace{1cm} (17b)

$$q_0(x^+) = q_{0,e} + eE(x^+ - x^e),$$  \hspace{1cm} (17c)

$$q_x(x^+) = q_{x,e} + eB(x^+ - x^e).$$  \hspace{1cm} (17d)

Thus, both the kinetic energy and the $z$ momentum depend only linearly on the position in $x$ direction, simplifying the calculation considerably. In contrast to Schwinger pair creation
with an electric field only, the electron and the positron will be accelerated along the z direction in the presence of a magnetic field, see Eqs. (17b) and (17d). Furthermore, the magnetic field guides both particles in the same direction while the electric field accelerates the particles into opposite directions. The \( x \) dependence for \( p_x \) and \( q_x \) is given implicitly by the dispersion relation \( p_x^2 = m^2 \)

\[
p_x^2(x^-) = p_0^2(x^-) - \left( m_0^2 + p_z^2(x^-) \right), \quad (18a)
\]

\[
q_x^2(x^-) = q_0^2(x^-) - \left( m_0^2 + q_z^2(x^-) \right). \quad (18b)
\]

Using Eq. (17) and the relation from Eq. (9), the length of the imaginary trajectory from \( x_s \) to \( x_e \) along the \( x \) direction for both the electron and the positron is determined by

\[
x_e^+ - x_s^- = \frac{k_0 p_{0,e} - k_z p_{z,e}}{e(Ek_0 - Bk_z)}, \quad (19a)
\]

\[
x_e^+ - x_s^- = \frac{k_0 q_{0,e} - k_z q_{z,e}}{e(Ek_0 - Bk_z)}. \quad (19b)
\]

Plugging Eq. (17) into Eq. (8) gives the relations

\[
eE(x_e^+ - x_s^-) = p_{0,e} + q_{0,e} - k_0, \quad (20a)
\]

\[
eB(x_e^+ - x_s^-) = p_{z,e} + q_{z,e} - k_z. \quad (20b)
\]

Multiplying (20a) with \( B \), (20b) with \( E \), and subtracting both yields

\[
B(p_{0,e} + q_{0,e} - k_0) = E(p_{z,e} + q_{z,e} - k_z). \quad (21)
\]

This equation gives the relation between \( p_x^0, q_x^0 \) and \( k^\mu \) as they are not independent of each other.

As a consequence of Eqs. (17) and (18), the kinetic momenta of the particles depend only on the \( x \) direction of space. Furthermore, it turns out that the imaginary part of the exponent is solely determined by the momenta in \( x \) direction [45] via

\[
W = \int_{x_s}^{x_e} p_x \, dx + \int_{x_s}^{x_e} q_x \, dx = W^+ + W^-, \quad (22)
\]

although the particles’ (imaginary) dynamics can be three-dimensional. The imaginary part is made up from a part due to the electron moving between \( x_s \) and \( x_e^+ \), where \( p_x \) is imaginary, and a part due to the positron moving between \( x_e^+ \) and \( x_s^- \), where \( q_x \) is imaginary. The photon’s action cancels with the electron’s and the positron’s actions along the \( z \) and \( t \) directions due to energy and momentum conservation. The photon will, however, influence the whole dynamics and in this way the momenta \( p_x \) and \( q_x \). The integrals for the exponent Eq. (22) can be calculated analytically using Eq. (18) for the momenta \( p_x \) and \( q_x \) and (19) for the integration limits. Note that \( W^+ \) and \( W^- \) are not independent of each other due to momentum conservation at the point of pair creation, see Eq. (8).

### 3.2. Graphical interpretation of the enhanced tunneling picture

Pair creation in strong electric fields is commonly interpreted as tunneling from a negative-energy state in the Dirac sea to a positive-energy state [31]. In other words the creation of an electron positron pair is described by a transition from a one-particle state to another particle state. This picture can, however, be translated into a picture where tunneling starts under the tunneling barrier and the final state after tunneling is a classical two-particle state, see Fig. 2. Although, both interpretations are equivalent, a pair creation picture that involves two particles may be more intuitive. In the following a pair creation picture will be introduced that goes beyond the standard tunneling picture of [31] by incorporating also an external magnetic field and possibly a high-energy photon. This picture is inspired by the tunneling picture for atomic ionization.

The standard tunneling picture for atomic ionization gives an intuitive measure for the suppression of ionization by means of an area that is determined by the potential function and the particle’s energy. If the potential is increased, for example, this area gets larger, thus indicating a reduced tunneling probability. In the same way, the enhanced tunneling picture of pair creation presented in this work will give a measure for the suppression of pair production by means of an area. The exact exponents are given by the integrals in Eq. (22), which can be pictured by areas given by the integrals of

\[
\text{Im} \ p_x(x) = \sqrt{m_0^2 + p_x^2(x) - p_0^2(x)}, \quad (23a)
\]

\[
\text{Im} \ q_x(x) = \sqrt{m_0^2 + q_x^2(x) - q_0^2(x)}. \quad (23b)
\]

along the \( x \) direction. Plotting these quantities directly would yield the exact areas, but unfortunately, does not give any intuitive account on how the pair production changes, if, for example, the electric or magnetic field amplitude or the energy of the additional photon changes. However, it will be sufficient to use quantities that are monotonically correlated to the momenta in Eq. (23). These quantities will be called \( \tilde{p}_x \) and \( \tilde{q}_x \) and will increase or decrease if \( p_x \) and \( q_x \) increase or decrease. This monotonic correlation assures that the areas spanned by \( \tilde{p}_x \) and \( \tilde{q}_x \) along the \( x \) direction also increase or decrease if the exponents in Eq. (22) increase or decrease. Although the
choice for $\tilde{p}_x$ and $\tilde{q}_x$ is not unique, we defined them as
\begin{align}
\text{Im} \tilde{p}_x &= \left| \sqrt{m^2 + p_x^2 + p_y^2 + p_z^2 - \sqrt{m^2 + p_x^2 + p_z^2}} \right| \\
&= |p_0 - \tilde{p}_0| = \tilde{p}_0 - p_0, \quad (24a) \\
\text{Im} \tilde{q}_x &= \left| \sqrt{m^2 + q_x^2 + q_y^2 + q_z^2 - \sqrt{m^2 + q_x^2 + q_z^2}} \right| \\
&= |q_0 - \tilde{q}_0| = \tilde{q}_0 - q_0, \quad (24b)
\end{align}

with the pseudo energies $\tilde{p}_0$ and $\tilde{q}_0$ defined as
\begin{align}
\tilde{p}_0(x) &= \sqrt{m^2 + p_x^2 + p_z^2(x)} = \sqrt{m^2 + p_z^2(x)}, \quad (25a) \\
\tilde{q}_0(x) &= \sqrt{m^2 + q_x^2 + q_z^2(x)} = \sqrt{m^2 + q_z^2(x)}. \quad (25b)
\end{align}

The areas that are spanned by $\tilde{p}_x$ and $\tilde{q}_x$ along the $x$ direction are the areas enclosed by the curves of the kinetic energy and the pseudo energy; both amenable for an intuitive interpretation. Furthermore, $\tilde{p}_x$ and $\tilde{q}_x$ are zero if $p_x$ and $q_x$ are zero, which corresponds to the exit points and hence the exit points are automatically given by the points of intersection between the kinetic and pseudo energy curves. Despite the fact, that this approximation to the area of the exponents is not exact, it is sufficient to derive qualitative results and gives more intuition as we will show in the following.

The now introduced quantities can be presented graphically as a function of the $x$ coordinate giving a visual representation of the tunneling picture of pair creation as shown exemplarily in Fig. 3. The upper black solid line represents the zero energy reference. Lines for the electron are drawn in blue, whereas lines for the positron are drawn in red. The bold solid colored lines represent the particles’ pseudo energies in Eq. (25) as a function of the particles’ real $x$ coordinate and the imaginary $z$ coordinate. The motion into the imaginary $z$ direction results from a real $z$ momentum due to the presence of a magnetic field or the initial momentum transfer due to the additional photon during an imaginary time interval. Note, however, that there is no motion in $z$ direction in real space because tunneling is instantaneous in real time [40]. Because the semiclassical tunneling trajectories do not represent realistic particle paths, the exit coordinate, that is where the particles’ momenta become real, is not necessarily real, i.e., $\text{Im} \, z^* \neq 0$. This may be related to that each semiclassical trajectory corresponds to a delocalized quantum state with a well-defined momentum, which is given by the exit momentum of the semiclassical trajectory. A real motion into $z$ direction would require an inhomogeneous electromagnetic field similarly to the near-threshold-tunneling regime of relativistic tunnel ionization [41].

The particles’ pseudo energies in Eq. (25) as well as their kinetic energies in Eqs. (17a) and (17c) can be expressed as function of the $x$ coordinate only and are represented in Fig. 3 by the solid and dashed lines, respectively. The solid colored lines originate for both the electron and the positron from the initial point for pair creation $x_s$ and end at their respective exit points $x_{e}$ and $x_{e}'$, where they intersect the dashed lines. Hence, by following, for example, the blue dashed and solid lines, the change in kinetic energy and pseudo energy can be traced along the $x$ direction of the imaginary trajectory from the point of pair creation until the exit for the electron; and similarly for the positron by following the red lines. Finally, a blue shaded area for the electron and a red shaded area for the positron are shown, which are enclosed by the dashed and solid colored lines. These areas can be related to exponents for pair production.

After introducing the different constituents of the enhanced tunneling picture, their physical interpretation can be explained now. At the point of pair creation $x_s$, the sum of the kinetic energies of the electron and the positron must be equal to the initial energy already existing in the system. This is either zero or $k_0$ in case of an additional photon. From $x_s$, both the electron and the positron follow their imaginary trajectories until their respective exits. The trajectories will be imaginary as long as $p_s$ and $q_s$ are imaginary or equivalently as long as the pseudo energies $\tilde{p}_0$ and $\tilde{q}_0$ are larger than their respective kinetic energies $p_0$ and $q_0$. In the graphs this can be seen by the solid colored lines lying below the dashed colored lines. At the exits, $p_s = 0$ and $q_s = 0$, both lines intersect as both types of energy (pseudo and kinetic) become equal. Note that the kinetic energies go below their minimum allowed energy behind the exit points towards $x_s$, where $p_s$ and $q_s$ get imaginary. It is just this behavior, that allows the fulfillment of Eq. (8), i.e., energy-momentum conversation at $x_s$. Furthermore, the differ-
ence between the pseudo energy and kinetic energy is taken as an approximate measure for the imaginary part of the particles' momenta, see Eq. (24). The exponent for pair production Eq. (22) is given by the integral over $p_x$, and $q_z$ along the $x$ direction from the point of pair creation $x$, to both the exit points of the electron and positron. Hence, the absolute value of these exponents can be inferred approximately by the shaded areas in the graphs. Thus, the shaded blue area corresponds to $W^x$ and the shaded red area corresponds to $W^z$. In the following three subsections the qualitative behavior of these shaded areas, that is the pair production exponent, will be discussed depending on different factors like the ratio of electric and magnetic field strength or the impact of an additional photon.

### 3.3. Case $|E| > |B|$ 

Figure 4 shows some common cases, where the electric field is stronger in magnitude than the magnetic field. It can be readily seen that for these cases, pair production is always possible due to the existence of the exit points, given by the intersection of the dashed and solid lines. For $|E| > |B|$, this intersection is always possible, because the pseudo energy line (solid) never falls steeper than the kinetic energy line (dashed) and hence, they need to intersect somewhere.

In Fig. 4a, Schwinger pair creation is considered with an electric field only and without an additional photon. Furthermore, the most probable case with zero momentum at the exit is taken. The pseudo energies $p_0$ and $q_0$ (the solid colored lines) are constant, because $p_x$ and $q_z$ do not change, when there is no magnetic field, and $p_y$ as well as $q_y$ are constants of motion. At $x$, both kinetic energies of the electron and the positron are zero in sum, as there is no additional energy without a photon. The Lorentz boosted version of Fig. 4a, seen in a reference frame boosted along the $z$ direction, is shown in Fig. 4b. Due to the boost, the electric field gets larger and hence the dashed colored lines get steeper. The stronger electric field alone would increase the tunneling probability. But due to the boost, also a magnetic field is experienced now by the particles, which transfers momentum from the $x$ direction to the $z$ direction. This momentum transfer reduces the acceleration into the electric field direction and makes pair creation less probable. In the tunneling picture the momentum transfer is represented by the bent pseudo energies $p_0$ and $q_0$; see Fig. 4b. Furthermore, the energy of the created particles is increased, which is consistent with the boost in $z$ direction. The growth of the relativistic mass is seen by the bending of the pseudo energy lines. Both effects, the increase of the electric field and the increase of the relativistic energy, cancel each other exactly as the transition amplitude is Lorentz invariant. This invariance is also represented in Figs. 4a and 4b, since the size of the shaded areas remains almost constant under the Lorentz boost, although the areas are a qualitative measure for the tunneling probability. Note also that the $x$ coordinate of the exit points $x^e$ and $x^p$ does not change as it is not affected by a boost along the $z$ direction.

In Fig. 4c, an additional photon is incorporated into the setting of $E$ and $B$ in Fig. 4b. Here, two very important things can be seen immediately. First, the additional energy of $k_0$ at $x$. Both, the electron and the positron, can now share this energy, which is $k_0/2$ per particle for maximum pair production probability. For this reason, the dashed colored lines are shifted down by $k_0/2$. This decreases the shaded areas and therefore increases the pair production probability. Secondly, the additional photon does not only transfer its kinetic energy, but it also transfers its momentum along the $z$ direction. This leads to an additional $z$ momentum of $k_0/2$ for both the electron and the positron, increasing their pseudo energy, or equivalently, their relativistic mass at $x$. Without a magnetic field, this extra...
energy due to the $z$ momentum remains until the exit and must be supported by the electric field, ultimately decreasing the pair production probability due to a longer “time” until the exit. To summarize, the additional photon transfers energy, which enhances the probability, but it also transfers momentum, which degrades the probability. In total, this results always in an enhanced probability. The degradation due to the momentum transfer along the $z$ direction can be reduced by tuning the magnetic field in such a way that it decelerates the particles along the $z$ direction, and thus, making them “lighter” at the exits. The appropriate tuning of the magnetic field will be discussed in more detail in subsection 4. From Eq. (20) it can be shown that the direction, in which the electron and the positron are accelerated along the $x$ direction is determined only by the orientation of the electric field

$$\text{sgn}(x^e_x - x^e_z) = \text{sgn} E.$$ \hspace{1cm} (26)

This is represented in the tunneling picture by the slope of the kinetic energy lines being positive or negative, forcing the particles in the one or the other direction.

Calculating the tunneling probability via the integral in Eq. (22) gives for the electron

$$W^e = \frac{(E_{p_0,e} - E_{p_z,e})^2}{2F^3} \left( \arccos \Gamma - \Gamma \sqrt{1 - \Gamma^2} \right)$$ \hspace{1cm} (27)

with

$$\Gamma = 1 - e^{-\frac{E^2 - B^2}{E_{p_0,e} - E_{p_z,e}}(x^e_x - x^e_z)}.$$ \hspace{1cm} (28)

The quantity $W^+$ has the same analytical structure, but the momenta and the position of the electron need to be replaced by that of the positron. For the standard Schwinger case, that is just an electric field without an additional photon and with zero momentum at the exit, $W^+$ in Eq. (27) reduces to $i\pi m^2/4E$. Adding $W^+$ and multiplying by two, which corresponds to the square modulus of the matrix element, gives exactly the Schwinger exponent $\pi m^2/E$.

3.4. Case $|E| = |B|$ 

This case is somewhat special, because it connects the previous and the next case in a singular manner. It corresponds to an external plane wave field under the constant crossed field approximation. Figures 5a and 5b show the tunneling picture for a quantized photon of energy $k_0 = m/2$ and $k_0 = m/100$. In general, the picture is quite similar to that of Fig. 4c, but here a quantized photon is mandatory for pair production. Due to the magnetic field being as strong as the electric field, the energy transferred to the particles in $z$ momentum by the magnetic field is as strong as the kinetic energy transfered to the particles by the electric field. In this sense, the electric field without an additional photon is not strong enough to let the particles leave the imaginary trajectories, as the magnetic field makes them more and more heavier. This can be seen by comparing in Fig. 5a and Fig. 5b how the colored dashed and solid lines approach each other. In the limit of $k$ going to zero, the kinetic and pseudo energy lines become tangent and will not intersect, leading to an infinite exponent and thus no pair production. Depending on the orientation of $E$ and $B$, which is connected with the orientation of $k_L$, for a plane wave ($E \times B$ points into the same direction as $k_L$), the quantized photon has to be directed...
anti-parallel to the external field wave vector. In the given geometry, this means that
\[ \text{sgn} \, k_x = - \text{sgn} \,(EB) \quad (29) \]
has to be satisfied. Otherwise there will be no pair production, as the quantized photon and external field will be parallel which can be thought of as one plane wave field, which is known to produce no pairs. If the additional photon propagates into the opposite direction, then the pseudo energy lines falls down such that it do not cross the kinetic energy lines, in contrast to the case in Fig. 5a. This is because the photon momentum points into the direction in which as the magnetic field accelerates the particles, leading to no intersection and hence infinite suppression of pair creation. As in the previous case \(|E| > |B|\), the direction of movement for the electron and positron along \(x\) is given by Eq. (26).

The imaginary part of the exponent for the electron now reads
\[ W^- = i \frac{2}{3} \left[ \frac{2e(Ep_{0,e} - Bp_{z,e})(x_e^x - x_e^z)}{|2e(Ep_{0,e} - Bp_{z,e})|} \right]^{1/2}. \quad (30) \]
For comparison, the same result will be derived in Appx. B by a different approach. Instead of the constant field approximation, the action integral \(W\) will be evaluated perturbatively in terms of the classical nonlinearity parameter \(\xi = eE/m\omega_L \gg 1\) for the special case of an external plane wave field with the characteristic frequency \(\omega_L\).

### 3.5. Case \(|E| < |B|\)

For this case there is a minimal energy, the quantized electron must have in order to produce a pair. Boosting into a reference frame, where the electric field vanishes, it is clear that no work will be done by the electromagnetic field. Hence, the whole energy in this frame for the pair must be carried by the quantized photon. In the case of no electric field and both electron and positron carrying only their rest mass after creation, the quantized photon has to carry an energy of exactly \(2m\), as shown in Fig. 5c. Both kinetic energy lines are horizontal because of the electric field being zero and their sum is equal to the initial photon energy of \(2m\). If this initial energy would be smaller than \(2m\), then the pseudo energy lines would not hit the kinetic energy lines, instead, they would stay below them. This minimum energy condition is also given by Eq. (21), which yields for \(E = 0\)
\[ k_0 = p_{0,e} + q_{0,e}. \quad (31) \]
The effect of a non-zero electric field may also be deduced from Fig. 5c. A non-zero electric field tilts the kinetic energy lines to the one or the other direction and consequently the exit points will be closer to or farther from \(x_s\) and hence pair production will be enhanced or suppressed.

Interpreting Fig. 5c in terms of imaginary trajectories, it can be understood in the following way. At the point of pair production \(x_e\), the momentum along the \(x\) direction for the electron and the positron has the value \(\pm im\). The momentum along the \(z\) direction, transferred by the photon, is \(m\). Following the imaginary trajectories from \(x_s\) on, the magnetic field rotates the momentum along the \(z\) direction onto the \(x\) direction and cancels exactly the initial imaginary momentum. Thus the particles become real and pair production is possible. If the initial energy of the photon is smaller, then the magnetic field can not turn the particles into real ones. If there is also an electric field, which reduces \(p_x\) and \(q_z\) in the same direction as the magnetic field, then the particles can leave the imaginary trajectory sooner, making pair production more probable. Pair production will be reduced or even completely cut off, if the electric field works in the opposite direction and, therefore, the energy carried by the photon does not suffice anymore.

In contrast to the previous two cases, the direction of tunneling is now determined by the orientation of the \(B\) field and the quantized photon
\[ \text{sgn} \,(x_e^x - x_e^z) = - \text{sgn} \,(Bk_e). \quad (32) \]
If the electric field works in the tunneling direction, then the probability will be enhanced; otherwise it will be reduced. For \(|E| < |B|\) the exponent for the electron computes to
\[ W^- = i \frac{(Ep_{0,e} - Bp_{z,e})^2}{2|E|^3} \left( \arccosh \Gamma - \gamma \sqrt{\Gamma^2 - 1} \right). \quad (33) \]

### 4. Maximum probability trajectories

Each tunneling trajectory is characterized by its final momenta of the created electron and positron and has a specific pair production probability, which depends on an exponential term with exponent \(\text{Im} \, W = \text{Im} \, W^+ + \text{Im} \, W^-\) given in Eqs. (27), (30) and (33) as well as on a prefactor. The maximum probability trajectories are defined as the trajectories, where the exponential \(\text{Im} \, W\) is minimized for given electromagnetic field strengths \(E\) and \(B\) and a given photon energy \(k_0\). In general, the most probable trajectories are always the most symmetric ones. For the kinetic momenta at the exit this means
\[ p_{x,e} = q_{x,e} = 0, \quad (34a) \]
\[ p_{y,e} = q_{y,e} = 0, \quad (34b) \]
\[ p_{z,e} = q_{z,e}, \quad (34c) \]
where (34a) follows from the definition of the exit point and (34b) is required by minimizing the kinetic energy of the created particles.

The kinetic momenta \(p_{z,e}\) and \(q_{z,e}\) depend on the ratio \(\beta = B/E\) as well on the momentum of the photon \(k_0\). Figure 6a illustrates the \(\beta\)-dependence of \(p_{z,e}\) and \(q_{z,e}\) for various \(k_0\). In the left part of Fig. 6a \(|E| > |B|\), while in the right part \(|E| < |B|\). At the center and at the outer borders, \(E\) and \(B\) have the same magnitude. For the Schwinger case with \(B = 0\) and \(k_0 = 0\) the final momentum of the maximum probability trajectories is zero as shown in the figure. Varying \(\beta\) while keeping \(k_0 = 0\) yields non-zero momenta \(p_{z,e}\) and \(q_{z,e}\), see solid black line in Fig. 6a. The momentum \(p_{z,e}\) corresponds exactly to the
For a given photon momentum \( k_0 = 0 \) pair creation may be maximized by varying \( \beta \) and minimizing the imaginary part of the exponent for the plane wave case (black solid line) and for the optimal case (gray solid line). At \( k_0 = k_0^\ast \), the ratio \( \beta \) becomes one, corresponding to the plane wave case. The exponent for the plane wave case is also given for reference by the thin black line. For \( k_0 < k_0^\ast \), a tuned magnetic field can therefore enhance the pair production probability. While for values greater than \( k_0 = k_0^\ast \), \( \beta = 1 \) is always optimal for a given fixed maximum field strength.

\( \beta \) boosted momentum of the zero momentum of \( B = 0 \). In the limit \( |\beta| \rightarrow 1 \) the momenta \( p_{c,e} \) and \( q_{c,e} \) diverge, which is related to the fact that pair production is not allowed in this regime without a photon.

In the presence of a photon with momentum the created electron and positron share the photons momentum at \( x_s \) if \( \beta = 0 \) and keep it until the exit because there is no magnetic field and, therefore, \( p_{c,e} = q_{c,e} = k_0 / 2 \) as also shown in Fig. 6a. Going to the left of \( \beta = 0 \), shows that \( p_{c,e} \) and \( q_{c,e} \) diverge also for \( k_0 > 0 \), because the electric and the magnetic fields work into different directions. Going to the right of \( \beta = 0 \), they will work into the same region and now pair production is also possible in the region \( |\beta| > 1 \), i.e., where \( |B| > |E| \). For \( k_0 < 2m \), the lines for the momenta \( p_{c,e} \) and \( q_{c,e} \) end before \( 1/\beta = 0 \) because the photon’s energy is not enough to create pairs without an electric field. The line for \( k_0 = 2m \) ends exactly at \( 1/\beta = 0 \) with \( p_{c,e} = 0 \). This corresponds to the case discussed in Fig. 5c. Going further to the right of \( 1/\beta = 0 \), the electric and magnetic field will work into opposite directions and in this way rendering pair production impossible.

For a given photon momentum \( k_0 \) pair creation may be maximized by varying \( \beta \) and minimizing the imaginary part of the exponent for the maximum probable trajectories. The corresponding exponent is shown in Fig. 6b. The blue dashed line indicates the optimal ratio \( \beta \). At \( k_0 = 0 \) it is optimal to have only an electric field. For \( k_0 > 0 \) also the optimal \( \beta \) is non-zero, meaning that a magnetic field will enhance pair production. If the photon momentum is larger than the critical momentum

\[
k_0^\ast = \sqrt{4/5} m \approx 0.89 m
\]

then the optimal \( \beta \) is one, which corresponds to a plane wave. This means, that for a fixed maximum field strength, the plane wave field is always optimum for photon energies larger than \( k_0^\ast \) in the setup treated in this work. The value (35) may be calculated by taking the derivative of Eq. (27) with respect to the magnetic field. This derivative is evaluated for the limit \( B \rightarrow E \) and then set to zero, which gives an implicit equation for \( k_0^\ast \) that can be solved analytically. Note, that \( k_0^\ast \) does not depend on the electromagnetic field magnitude \( E \). The imaginary values of the exponent \( W \) is also shown in Fig. 6b for the plane wave case (black solid line) and for the optimal case (gray solid line). The exponent for the optimum tuned case starts at \( k_0 = 0 \) with \( \text{Im} W = 0 \) (Schwinger case) and then decreases with increasing \( k_0 \) until it coincides with the value of the exponent for the plane wave case at \( k_0 = k_0^\ast \).

5. Conclusion

We introduced an intuitive tunneling picture for pair creation, which also incorporates effects due to a magnetic field and an additional high-energy photon. This picture is Lorentz invariant and does not depend on a particular gauge and is valid for homogeneous constant electromagnetic fields. This constant field approximation is valid, e.g., for the long wave limit of pair creation in counter propagating laser fields.

The enhanced tunneling picture can be drawn due to the quasi one-dimensional nature of the used setup in the quasistatic limit. Various features of pair creation can be inferred qualitatively from the introduced tunneling picture. For example, an additional photon lowers the potential barrier due to the photons energy but also increases the particle’s relativistic mass due to the photon’s additional momentum. Due to this increased relativistic mass the electron and positron stay longer.
(in terms of imaginary time) under the barrier until they gained enough energy to become real. An additional magnetic field, however, will also change the momentum under the barrier and in this way the relativistic mass. Depending on the magnetic field’s direction and magnitude it may counteract the increase of the relativistic mass due to the photons momentum and, therefore, increase the pair production probability.

The tunneling picture has been devised on the basis of a semiclassical approximation using classical trajectories. This approach also allowed us to calculate the exponents of the transition amplitudes. The determined exponents converge to their known values for the typical Schwinger case [3] with an electric field only and for the case of the external laser field being plane wave with an incoming photon [46] in the limit \( \xi \ll 1 \).

Other geometries as well as time and space varying fields may also be treated in the tunneling regime by imaginary trajectories, but their respective picture will be inherently multidimensional and hard to visualize. Nevertheless, simple trends due to time and space variation may be interpreted with the current picture and are subject to future work.

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**A. From quantum field theory to classical actions**

This paragraph recapitulates, how to derive the semiclassical approximation for matrix elements, which is usually symbolically written as

\[
M_{fi} \sim \exp (-\text{Im} S) .
\]

This form in general is quite ambiguous, because the definition of the action \( S \) itself due to its gauge dependence is not unique and furthermore it also depends on the type of initial and final states which can be eigenstates of position, momentum or of a specific Hamiltonian (to name the most common).

Following the treatment in [42], the mode Hamiltonian \( \mathcal{H}_c \) of the matter part (e.g., electrons and positrons), describing the free modes of these particles is given by

\[
\mathcal{H}_c = \alpha (-i \nabla + \beta m) .
\]

The quantization leads to the following field Hamiltonian for the free quantized matter field

\[
\hat{\mathcal{H}}_c = \int \text{d}x \hat{\psi}^\dagger(x) \mathcal{H}_c \hat{\psi}(x)
= \int \text{d}x \hat{\psi}(x) (-i \gamma \nabla + m) \hat{\psi}(x) .
\]

The same procedure applied to the free quantized radiation field leads to its corresponding field Hamiltonian \( \hat{\mathcal{H}}_\gamma \) which explicit form is not of interest here. The interaction between both fields is given by

\[
\hat{\mathcal{H}}_{\text{int,full}} = e \int \text{d}x \hat{\tilde{\psi}}(x) \gamma_{\mu} \hat{\psi}(x) \hat{A}^\mu_{\text{full}}(x) .
\]

Both, \( \hat{\psi}(x) \) and \( \hat{A}^\mu_{\text{full}}(x) \), are the field operators of the quantized matter and radiation field in the Schrödinger picture, whence they only depend on \( x \). This theory describes the full QED. But, so far, it does not allow to solve even very simple problems and hence some approximations need to be done. The most fundamental approximation, as common for problems of QED in external fields, is the separation of the full electromagnetic field into a quantized radiation field \( \hat{A}^\mu(x) \) and a classical external field \( A^\mu_{\text{ext}}(x) \). Treating the external field as a classical field is motivated by the fact, that it should be very intense and quantum effects are negligible. Its interaction with the matter field is given by the following Hamiltonian

\[
\hat{\mathcal{H}}_{\text{int,ext}} = e \int \text{d}x \hat{\tilde{\psi}}(x) \gamma_{\mu} \hat{\psi}(x) A^\mu_{\text{ext}}(x) .
\]

Note the explicit time dependence of this Hamiltonian due to the possible time dependence of \( A^\mu_{\text{ext}}(x) \). A further important fact of the external field approximation is neglecting the influence of the matter field on the dynamics of the external field, also called back reaction. That way, \( A^\mu_{\text{ext}}(x) \) is assumed to behave like a classical free electromagnetic field, that doesn’t see the current

\[
\hat{\mathcal{J}}^\mu = e \hat{\tilde{\psi}} \gamma_{\mu} \hat{\psi}
\]

created by the matter field. Putting together \( \hat{\mathcal{H}}_c \) and \( \hat{\mathcal{H}}_{\text{int,ext}} \) gives the (possible time-dependent) quantized field Hamiltonian for the matter field in an external field

\[
\hat{\mathcal{H}}_{c,\text{ext}} = \hat{\mathcal{H}}_c + \hat{\mathcal{H}}_{\text{int,ext}}
= \int \text{d}x \hat{\tilde{\psi}}(x) \left( \gamma (-i \nabla - A_{\text{ext}}) + m + e A^0_{\text{ext}} \right) \hat{\psi}(x) .
\]

Thus the modes of the matter field in the external electromagnetic field are described by the following Hamiltonian, known as the Dirac Hamiltonian with an external field,

\[
\mathcal{H}_{c,\text{ext}} = \alpha (-i \nabla - A_{\text{ext}}) + \beta m + e A^0_{\text{ext}}
\]

and therefore take the influence of the external field into full account. The interaction with the quantized field is still treated perturbatively. For doing this, it is convenient to switch to the interaction picture, with the free part

\[
\hat{\mathcal{H}}_0 = \hat{\mathcal{H}}_\gamma + \hat{\mathcal{H}}_{c,\text{ext}}
\]

and the interaction part

\[
\hat{\mathcal{H}}_{\text{int}} = e \int \text{d}x \hat{\tilde{\psi}}(x) \gamma_{\mu} \hat{\psi}(x) \hat{A}^\mu(x) .
\]

This type of interaction picture, including the interaction with the external field in \( \hat{\mathcal{H}}_0 \), is also called the Furry picture.
Thus, as shown in Ref. [42], for all the transition elements of interest one only needs to know the following propagator elements

\[ G^{(+)}_{mn}(\xi) = \int dx\, dy\, (\xi(x)\chi_{+}(x)\chi_{-}(y)) G^{(+)}(y,x)\chi_{-}(y) \chi_{+}(x), \]

\[ G^{(-)}_{mn}(\xi) = G^{(+)}_{mn}(\xi)^{*}, \]

with \( m, n \) being the quantum numbers of the modes at \( t_{\text{out/in}} \) and \( \zeta, \kappa = \pm \) denoting positive or negative energy (i.e., particle or antiparticle). \( G^{(+)}(x_{\text{out}}, y_{\text{in}}) \) is the full propagator of Eq. (A8). These propagator elements represent the probability, that a positive/negative mode from \( t_{\text{in}} \) evolves into a positive/negative mode at \( t_{\text{out}} \). According Ref. [42], the vacuum stability up to 0th order in the quantized radiation field is given by the transition amplitude between \( |0, \text{in}\rangle \) and \( |0, \text{out}\rangle \)

\[ c_{\nu} = \langle 0, \text{out} | 0, \text{in} \rangle = \det G^{(+)}_{\nu} = \det G^{(-)}_{\nu}. \]  

Furthermore, the transition amplitude for producing one pair, that is evolving from an initial vacuum state \(|0, \text{in}\rangle \) to a pair state \(|n, m, \text{out}\rangle \) (electron/positron having quantum number \( n/m \)) yields

\[ \langle n, m, \text{out} | 0, \text{in} \rangle = \langle 0, \text{out} | a_{m}(\text{out}) b_{n}(\text{out}) | 0, \text{in} \rangle = \omega(|mn\rangle c_{\nu}, \]  

with

\[ \omega(|mn\rangle 0) = \left[ G^{-1}(+) G^{(-)} \right]_{mn} = -\left[ G^{(-)} G^{-1}(+) \right]_{mn}. \]

For the semiclassical case, where pair production is exponentially suppressed, it is possible to write

\[ c_{\nu} = 1 + O(e^{-}), \]

\[ G^{(+)}_{\nu} = e^{i\delta_{\nu}^{+}} e^{i\delta_{\nu}^{-}} + O(e^{-}), \]

\[ G^{(-)}_{\nu} = e^{i\delta_{\nu}^{+}} e^{i\delta_{\nu}^{-}} + O(e^{-}), \]

\[ G^{(+)}_{\nu} = e^{i\delta_{\nu}^{+}} e^{i\delta_{\nu}^{+}} + O(e^{-}), \]

\[ G^{(-)}_{\nu} = e^{i\delta_{\nu}^{+}} e^{i\delta_{\nu}^{+}} + O(e^{-}). \]

The notation \( O(e^{-}) \) stands for the exponentially small corrections. Physically this means, that the positive/negative modes at \( t_{\text{in}} \) mainly evolve into their corresponding positive/negative modes at \( t_{\text{out}} \) up to a phase \( \theta^{+}/\theta^{+} \). Only an exponentially small fraction of their norm goes into different modes, regardless whether they are positive or negative. Inverting \( G^{(+)}_{\nu} \) and \( G^{(-)}_{\nu} \) yields therefore

\[ G^{(+)}_{\nu}^{-1} = e^{-i\delta_{\nu}^{+}} e^{i\delta_{\nu}^{+}} + O(e^{-}), \]

\[ G^{(-)}_{\nu}^{-1} = e^{-i\delta_{\nu}^{+}} e^{i\delta_{\nu}^{+}} + O(e^{-}). \]

Plugging this into the first line of Eq. (A21), results in

\[ \omega(|mn\rangle 0) = e^{i\delta_{\nu}^{+}} G^{(+)} e^{i\delta_{\nu}^{+}} + O^{2}(e^{-}). \]

Thus, up to first order in the exponential suppression, the single pair production elements in Eq. (A21) are given by the matrix \( G^{(+)}_{\nu} \) or alternatively by \( G^{(-)}_{\nu} \). The matrix element \( G^{(+)}_{\nu} \) gives the overlap between the negative mode \( \varphi_{-} \), propagated from \( t_{\text{out}} \) to \( t_{\text{in}} \), with the positive mode \( +\varphi_{+} \) at \( t_{\text{in}} \). Explicitly, this reads

\[ G^{(+)}_{mn} = \int dx\, dx'\, \varphi_{m}^{+}(x') G(x_{\text{out}}, x_{\text{in}}) \varphi_{m}(x). \]
This means, that somewhere in between \( t_{\text{out}} \) and \( t_{\text{in}} \) a small part of \( -\varphi_n \) splits up and evolves into \( +\varphi_m \). Still the main part goes into \( -\varphi_i \) \((G(\cdot|\cdot))\). This is shown schematically in the upper part of Fig. 7. Also shown there is a length, the so called formation length, because typically the timescale on which the conversion from \( -\varphi_n \) to \( +\varphi_m \) happens, is finite. By finite we mean, that only exponentially small corrections occur if the formation length is increased further. In this sense, the formation length isn’t strictly defined, rather it gives some estimate for the timescale on which most of the conversion has already happened (see also Ref. [47] for the case of a constant electromagnetic field).

The real space propagator is approximated in the semiclassical approximation by

\[
G(x',x) \sim \exp [iS(x',x)] .
\]  

(A27)

Similarly, the propagator from a momentum eigenstate to another (in our notation taking the quantum numbers \( n = p \) and \( m = q \)) is approximated semiclassically as

\[
G(p',q) \sim \exp [iS(x',x) - iP\cdot x + iQ\cdot x] \quad \text{(A28)}
\]

with \( x' \) and \( x \) implicitly given by

\[
P = \left. \frac{\partial S}{\partial x'} \right|_{x = x'} \quad \text{and} \quad Q = \left. \frac{\partial S}{\partial x} \right|_{x = x} .
\]  

(A29)

The additional terms are due to the in- and out-modes being momentum and not position eigenstates, see also [48, 49]. Thus, the full propagator gets approximated only by its classical trajectory connecting the appropriate in- and out-modes instead of all trajectories in the path integral picture. One peculiarity of Eq. (A28) is the matrix structure of its left hand side corresponding to the full propagator. Its right hand side doesn’t have this matrix structure, as the exponential is a scalar. As shown in [50], the semiclassical treatment of the Dirac equation leads to a prefactor with matrix structure, which must not be necessarily diagonal in the 0th order of \( \hbar \). Therefore, a transition from a positive energy spinor to a negative energy spinor is possible. Nevertheless, the main contribution to the matrix elements stems from the exponential factor. In the classical picture, the transition from a negative energy eigenstate to a positive must happen on an imaginary trajectory because an electron with positive energy will always stay an electron with positive energy on a real trajectory. But if it goes over to an imaginary path and wraps around the root of its kinetic energy \( p_0 = \sqrt{m^2 + p^2} \), it is possible, that it continues on the other branch of the square root, that is \( p_0 = -\sqrt{m^2 + p^2} \) and therefore becomes a negative energy electron. This point on the imaginary part of the trajectory, where this transition occurs, might be defined by \( p_0 = 0 \) and will be denoted by \( x_s \). It should be mentioned, that it has no direct physical meaning, i.e. it cannot be measured. Nevertheless it can be related to the region of the formation length. Using this point as the point of pair production in the classical picture, the propagator in Eq. (A28) can be split into

\[
G(p_{\text{in}}, q_{\text{out}}) \sim e^{iS(x'_{\text{in}},x_{\text{in}}) - iP\cdot x'} e^{iS(x_{\text{out}},x_{\text{out}}) + iQ\cdot x},
\]  

(A30)

where the back part corresponds to a negative energy electron traveling back in time from \( t_{\text{out}} \) to \( x_{\text{in}} \), which can be also interpreted as a positron going from \( x_{\text{in}} \) to \( t_{\text{out}} \) (changing sign in \( q \) and charge). For the full matrix element in Eq. (A21), the front part of Eq. (A30) gets multiplied by \( G(\cdot|\cdot)^{-1} \). In our approximation this corresponds to a propagation of the positive energy electron back from \( t_{\text{in}} \) to \( t_{\text{out}} \) and one can therefore write

\[
G(\cdot|\cdot)^{-1} e^{iS(x'_{\text{in}},x_{\text{in}}) - iP\cdot x'} \sim e^{iS(x_{\text{out}},x_{\text{out}}) + iQ\cdot x} .
\]  

(A31)

Accordingly, the transition element in Eq. (A21) can now be written as

\[
\omega^{\pm}(mm|0) = \exp \left[ - \text{Im} (W_p + W_q) \right] .
\]  

(A32)

where \( W_p = S(x'_{\text{out}},x_{\text{in}}) - P\cdot x' \) and \( W_q = S(x_{\text{out}},x_{\text{in}}) - Q\cdot x \) correspond to the modified actions of the electron and positron along their classical imaginary trajectories. This result is schematically drawn in the lower part of Fig. 7 with \( G(\cdot|x_s) \sim e^{iW_p} \) and \( G(-|x_s) \sim e^{iW_q} \). If there are, for a specific field configuration, different imaginary paths possible for connecting the electron and positron in their specific out-states, their contribution must be added and might lead to interference phenomena.
2. **Pair production due to perturbative treatment of the quantized electromagnetic field**

For this case, the first order correction to the time evolution operator in Eq. (A16) is given by

\[
\hat{U}(t_{\text{out}}, t_{\text{in}}) \to -i \int_{t_{\text{in}}}^{t_{\text{out}}} \hat{H}_{\text{int}}(t) \, dt
\]

\[
\to -ie \int \hat{\phi}_f(x) \gamma^\mu \hat{\phi}_i(x) A^\mu_i(x) \, dx . \tag{A33}
\]

The field operators in their interaction representation can be expanded (as before) into the solutions of their corresponding mode Hamiltonians. For the matter field operator \( \hat{\phi}_i \), the decomposition Eq. (A14b) for the out-states is used again, as the main interested lies in pair creation and not pair annihilation. The field operator for the free quantized electromagnetic field is decomposed as

\[
\hat{A}^\mu(x) = \sum_{\lambda=0}^{3} \int \left[ \hat{c}_{\lambda k} f_{\lambda k}(x) + \hat{c}^*_{\lambda k} f^{+*}_{\lambda k}(x) \right] dk \tag{A34}
\]

and \( f_{\lambda k}(x) \) being the photon wave function proportional to \([51]\)

\[
f_{\lambda k}(x) \sim \exp[-i k x] = \exp [i W_k(x)] , \tag{A35}
\]

where \( W_k(x) = -k x \) usually is called its classical action. Care should be taken here, since this action does not correspond to an action of the type \( S(x', x) \) but rather to the modified action discussed before, where on one side the momentum \( k \) and on the other side the spacetime \( x \) is fixed.

Expressing also the solutions \( \hat{\psi}_n \) to the Dirac equation in their semiclassical approximation and setting the final quantum number \( n \) to momentum eigenstates \( (p \text{ for the electron and } q \text{ for the positron}) \), they read \( \hat{\psi}_n(x) \sim \exp[-i k x] \) and \( \hat{\psi}_n(x) \sim \exp[i W_k(x)] \) for the electron and the positron, respectively. The "\( +i \)" in the positrons exponent is due to its quantization as an antiparticle. Again, \( W_p \) and \( W_q \) are usually called the electrons and positrons classical actions, but the same distinction as mentioned before applies here. Note that in case of an external plane-wave field, these actions are given by the so called Volkov actions and the semiclassical exponent is exact.

By taking into account the first order of the interaction with the quantized field, pair production can now happen with the additional help of an incoming quantized photon. The in-state is defined as

\[
|0, 0, 1_k \rangle \equiv \hat{c}_k^+ |0, \text{in} \rangle \tag{A36}
\]

and the out-state as

\[
|1_p, 1_q, 0 \rangle \equiv \hat{a}_p^+(\text{out}) \hat{b}_q^+(\text{out}) |0, \text{out} \rangle . \tag{A37}
\]

The transition amplitude for this process to first order is given by

\[
M_{fi} = -ie \left\langle 1_p, 1_q, 0 | \right. \int \hat{\phi}_f(x) \gamma^\mu \hat{\phi}_i(x) A^\mu_i(x) \, dx \left| 0, 0, 1_k \right. \rangle . \tag{A38}
\]

Expanding the field operators according to Eq. (A14b) and (A34) and writing (approximating) the field mode solutions in terms of their modified classical action \( W \) yields

\[
M_{fi} \sim -ie c_v \int dx \, \exp \left[i \left( W_{\text{in}}(x) + W_{\text{out}}(x) + W_{\text{ph}}(x) \right) \right] . \tag{A39}
\]

Applying the method of stationary phase to this integral yields the following condition for a stationary phase

\[
0 = \frac{\partial}{\partial x^\alpha} \left| W_k + \frac{\partial}{\partial x^\alpha} \right|_{x_\alpha} = K^\mu(x_\alpha) - P^\mu(x_\alpha) - Q^\mu(x_\alpha) = k^\mu(x_\alpha) - p^\mu(x_\alpha) - q^\mu(x_\alpha) . \tag{A40}
\]

The last line follows from the previous one due to the fact, that the gauges at the same space-time point \( x_\alpha \) cancel each other exactly \([52]\). Although the modified actions \( W \) are gauge dependent, the point of stationary phase \( x_\alpha \) is given by a gauge independent and Lorentz invariant kinetic equation, which can be interpreted from a classical point of view. According to Eq. (A39), the matrix element is an interference from all space-time points \( x \) where the photon converts into electron and positron obeying the asymptotic momenta. The outcome of this interference, e.g. the matrix element, can be approximated by the point of stationary phase. Due to energy-momentum conservation \( p + q = k \) at this point, the laws of classical mechanics are satisfied. Therefore, the particle conversion stemming from the creation and annihilation operators in quantum field theory can be brought to a classical level in the semiclassical approximation.

The classical trajectory will now look like an incoming photon, that will convert at \( x_\alpha \) instantaneously into electron and positron while satisfying energy-momentum conservation. The electron and positron will travel further in the external field and approach their asymptotic momenta.

The condition \( p+q=k \) can not be satisfied in real space time, as it can be seen by squaring both sides of this equation, but in imaginary space time. This leads to an imaginary classical trajectory. Therefore, approximating Eq. (A39) semiclassically yields

\[
M_{fi} \sim -ie c_v e^{-i \int W_k + W_p + W_q} . \tag{A41}
\]

B. **Perturbative evaluation of the exponent for the plane-wave case**

For the plane wave four-potential we choose the following gauge (with \( \eta = k_L x \) being the phase of the laser, \( E \) the maximum field amplitude and \( \omega_L = k_{L,0} \) the characteristic frequency)

\[
A_{\mu}(\eta) = \begin{pmatrix} 0, E \frac{\omega_L}{\omega_L} f(\eta), 0, 0 \end{pmatrix} , \tag{B1}
\]

yielding for the electron the constants of motion (according to Eq. (3)) \( P_x = p_x \), and \( P_0 - P_z = p_0 - p_z \). The last
expression can be written in a Lorentz invariant way \( \Lambda_p = k_L p \).
Accordingly, the same is true for the positron. For the quantized photon, this notation can be used too \( \Lambda_k = k_L k \), yielding also a constant of motion.
To make use of these constants of motion, the transformation
\[
(t, x, y, z) \rightarrow (x', y', z') = (k_L t - z, x, y, z/k_L) \tag{B2}
\]
into a different coordinate system is applied \((x, y \text{ and } z \text{ are space-time coordinates and not four-vectors in this paragraph.})\). The total differential of the space-time dependent part of the action \( W \) in this coordinate system
\[
W(x, y, z, t) = W(x(x'), y(y'), z(z'), t(z', \eta)) \tag{B3}
\]
is given for the electron and analogously for the positron by
\[
dW(x', y', z', \eta) = -P_L dx' - P_y dy' + \omega_L (P_0 - P_z) dz' + P_0/\omega_L d\eta, \tag{B4}
\]
which can be written in the gauge of Eq. (B1) as
\[
dW(x', y', z', \eta) = -P_x dx' - P_y dy' + \Lambda_p dz' + \frac{P_0}{\omega_L} d\eta. \tag{B5}
\]
The first three terms are the constants of motion and can be integrated
\[
W(x', y', z', \eta) = -P_x x' - P_y y' + \Lambda_p z' - \int_0^{\eta} \frac{P_0}{\omega_L} \, d\varphi. \tag{B6}
\]
Similarly, the photon’s action can be written this way
\[
W(x, y, z, t) = k_x x + k_y y' - \Lambda_k z' - \int_0^{\eta} \frac{k_0}{\omega_L} \, d\varphi. \tag{B7}
\]
Allowing only for the classical trajectory, that is, four-momentum conservation at the point of pair creation, yields the following set of constraints
\[
\begin{align}
P_x + Q_x &= p_x + q_x = 0, \tag{B8a} \\
P_y + Q_y &= p_y + q_y = 0, \tag{B8b} \\
\Lambda_p + \Lambda_q &= \Lambda_k, \tag{B8c} \\
p_0(\eta^i) + q_0(\eta^i) &= k_0. \tag{B8d}
\end{align}
\]
The last line, Eq. (B8d), implicitly determines the phase \( \eta^i \) of the electromagnetic field, where the pair will be produced on the classical trajectory. This trajectory as well as \( \eta^i \) has to be imaginary. Again, this imaginary trajectory will not be the physical trajectory of the process. It is only the trajectory where the exponential part of the matrix element gets stationary and hence yields a good approximation for its value. The part of the exponent, which has an imaginary value, is given by the sum of the actions of electron, positron and photon
\[
\Sigma W = \frac{1}{\omega_L} \left( \int_0^{\eta_f} p_0 \, d\varphi + \int_0^{\eta_f} q_0 \, d\varphi - \int_0^{\eta_f} k_0 \, d\varphi \right). \tag{B9}
\]
In the general case, \( \eta^i \) and subsequently \( \text{Im} \Sigma W \) need to be calculated depending on the analytic form of the given electromagnetic field. Usually, this has to be done numerically.

In case of the classical nonlinearity parameter \( \xi = eE_0/m\omega_L \) being much larger than 1, the integral can be calculated perturbatively up to order \( \mathcal{O}(1/\xi) \) and gives the same results as a constant crossed field with \(|E| = |B|\) as shown in the following. Using the identity
\[
\Lambda_p = k_L p = \omega_L(p_0 - p_z) \tag{B10}
\]
it follows
\[
p_z = p_0 - \frac{\Lambda_p}{\omega_L}, \tag{B11}
\]
\[
p_z^2 = m^2 + p_y^2 + \frac{p_z^2}{\omega_L^2} \tag{B12}
\]
and from Eqs. (B8) and (4)
\[
p_s(\eta) \rightarrow p_s^* (\eta) = q_s^* (\eta). \tag{B14}
\]
Plugging Eqs. (B13) and (B14) into Eq. (B9) and using Eq. (B8c) gives
\[
\Sigma W = -\left( \frac{1}{2\Lambda_p} + \frac{1}{2\Lambda_q} \right) \int_0^{\eta_f} d\varphi \left[ m^2 + p_s^2(\varphi) \right]. \tag{B15}
\]
\( \eta_f \) is implicitly fixed by Eq. (B8d) leading to
\[
p_s(\eta_f) = \pm im. \tag{B16}
\]
with \( m \) being defined as in Eq. (12). Using the four-potential Eq. (B1) and rewriting in terms of the canonical momentum \( P_s \) (constant of motion) yields
\[
\pm \frac{1}{\xi} + \frac{\omega_L P_s}{eE_0} = f(\eta_f) \quad \text{with} \quad \xi = \frac{eE_0}{m\omega_L}. \tag{B17}
\]
Assuming \( \xi \) being large, \( \eta_f \) can be evaluated perturbatively in powers of \( 1/\xi \) as
\[
\eta_f = \eta_f^0 + \frac{1}{\xi} \eta_f^1 + \frac{1}{\xi^2} \eta_f^2 + \cdots \tag{B18}
\]
Solving Eq. (B17) up to first order in \( 1/\xi \) gives
\[
\eta_f^0 = f^{-1} \left( \frac{\omega_L P_s}{eE_0} \right), \tag{B19}
\]
\[
\eta_f^1 = \pm i f'(\eta_f^0). \tag{B20}
\]
Only the part \( \eta_f^0 \rightarrow \eta_f^0 + \eta_f^1/\xi \) of the integration contour for \( \Sigma W \) gives an imaginary contribution (in \( \mathcal{O}(1/\xi) \)) and therefore the integral needs to be calculated just along this path. The integrand of \( \Sigma W \) is rewritten as
\[
m^2 + p_s^2 = m^2 + m^2 \left[ 1 + \frac{P_s^2}{m^2} \right] = m^2 \left[ 1 + \left( \frac{P_s}{m} - \xi f(\varphi) \right)^2 \right] \tag{B21}
\]
and expanding \( f(\varphi) \) around \( \eta_f^0 \)
\[
f(\varphi) \sim f(\eta_f^0) + f' (\eta_f^0) (\varphi - \eta_f^0) + \cdots \tag{B22}
\]
yields
\[ m^2 + p_x^2 \sim m^2 \left[ 1 + \left[ \xi f'(\eta_0^0) (\phi - \eta_0^2) \right]^2 \right]. \] (B23)

Consequently, \( \Sigma W \) calculates to
\[
\sim m^2 \int_{\eta_0^0}^{\eta_0^0 + \omega_0 / \xi} \mathrm{d} \phi \left[ 1 + \left[ \xi f'(\eta_0^0) (\phi - \eta_0^2) \right]^2 \right]
= \pm \frac{2}{3} m^2 + O\left( \frac{1}{\xi} \right). \] (B24)

Taking the correct sign of \( \pm i \) for exponential suppression and
writing \( \xi \) explicitly gives the final result
\[
\Sigma W = W^- + W^+ = \frac{i m^2}{3} \frac{\omega_L}{\Lambda_p} eE(\eta_0^0) + \frac{i m^2}{3} \frac{\omega_L}{\Lambda_p} eE(\eta_0^0). \] (B25)

Taking only the electron part \( W^- \) and rewriting it as
\[
W^- = \frac{i m^2}{3} \omega_L (p_0 - p_x) eE(\eta_0^0) + \frac{i m^2}{3} \frac{1}{p_0 - p_x} eE(\eta_0^0), \] (B26)

it can be seen that this yields the same result as the constant field approximation Eq. (30) by using the identities \( B = -E \kappa_1 / k_0 \) and \( p_0^2 - p_x^2 = m^2 \). This is because the expansion only up to the order \( O(1/\xi) \) corresponds to expanding the external field up to its first derivative giving a constant crossed field.

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[51] As we do not consider spin effects in this paper, the given actions do not include any spin or polarization terms.

[52] Using usual quantization rules, the photon will have no gauge and therefore $K = k$. Furthermore $P = p - eA$ and $Q = q + eA$, which gives $P + Q - K = p + q - k$. 