The noncommutative space of light-like worldlines

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Abstract

The noncommutative space of light-like worldlines that is covariant under the light-like (or null-plane) \(\kappa\)-deformation of the (3+1) Poincaré group is fully constructed as the quantization of the corresponding Poisson homogeneous space of null geodesics. This new noncommutative space of geodesics is five-dimensional, and turns out to be defined as a quadratic algebra that can be mapped to a non-central extension of the direct sum of two Heisenberg–Weyl algebras whose noncommutative parameter is just the Planck scale parameter \(\kappa^{-1}\). Moreover, it is shown that the usual time-like \(\kappa\)-deformation of the Poincaré group does not allow the construction of the Poisson homogeneous space of light-like worldlines. Therefore, the most natural choice in order to model the propagation of massless particles on a quantum Minkowski spacetime seems to be provided by the light-like \(\kappa\)-deformation.

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1 Introduction

Noncommutative spacetimes have been proposed in different approaches to Quantum Gravity as useful algebraic tools in order to describe the minimal length scenarios and the spacetime fuzziness that is believed to arise as a consequence of quantum properties of spacetime at the Planck scale (see, for instance, [1–9] and references therein). In this context, the covariance properties of such “quantum” spacetimes becomes a relevant issue, which can be solved in the case of the noncommutative analogues of spacetimes that in the classical setting can be obtained as homogeneous spaces of kinematical groups, as it is the case of Minkowski and (Anti-)de Sitter spacetimes. In fact, these quantum homogeneous spacetimes can be defined as covariant objects under a quantum kinematical group of symmetries, and they can be explicitly obtained as quantizations of classical Poisson homogeneous spaces. Moreover, the latter are covariant under a Poisson–Lie group, which is just the semiclassical counterpart of the corresponding quantum group [10, 11].

In this setting, a noncommutative (Minkowski or (A)dS) spacetime is mathematically defined as a comodule algebra under the action of the corresponding quantum group of transformations. In this way, the latter provides the deformed analogue of the classical relativistic symmetries for the quantum spacetime, and allows the connection with the Deformed Special Relativity (DSR) approach to quantum gravity phenomenology in which the quantum deformation parameter is assumed to be related to the Planck scale [12–18]. So far, the most studied noncommutative spacetime model arising in this framework is the so-called $\kappa$-Minkowski spacetime (see [19–30] and references therein), which is covariant under the (time-like) quantum deformation known as the $\kappa$-Poincaré algebra [19, 31–35]. Also, its non-vanishing cosmological constant counterpart, the $\kappa$-(A)dS noncommutative spacetime, has also been recently constructed [27] from the corresponding $\kappa$-(A)dS quantum group introduced in [36].

However, there exists another class of very relevant homogeneous spaces for kinematical groups whose noncommutative analogues had not been considered until recently despite their physical significance: the homogeneous spaces of geodesics on the above-mentioned Lorentzian spacetimes with constant curvature. In fact, the construction of “quantum geodesics” on a given noncommutative spacetime is the essential problem underlying the definition of “quantum observers”. The explicit construction of the noncommutative space of time-like worldlines associated to the $\kappa$-Minkowski spacetime has been recently given in [25, 37], and the fuzziness of events has also been analysed in this quantum worldline setting [38].

The aim of this paper is the explicit construction of the noncommutative space of light-like geodesics associated to the $\kappa$-deformation. Indeed, the study of the propagation of photons on a quantum spacetime is one of the main theoretical problems in quantum gravity phenomenology (see [39, 40]) and we hope that this new model could shed some light in this context. Moreover, we will show that the construction of this new noncommutative space is possible only if the light-like (or null-plane) $\kappa$-deformation of the Poincaré algebra [41–43] is considered. As we will show in detail, this comes from the fact that the usual time-like $\kappa$-Poincaré deformation is not mathematically compatible with the construction of a Poisson homogenous space of light-like geodesics through a canonical projection from the corresponding Poisson–Lie group. This result seems to indicate that the time-like and light-like $\kappa$-deformations could be considered separately in order to model the propagation of, respectively, time-like and light-like quantum particles.

The structure of the paper is as follows. In the next section we provide a general discussion about the conditions that a given classical $r$-matrix (and, therefore, its associated quantum de-
formation) has to present in order to allow a canonical construction of the Poisson homogeneous space whose quantization will provide the noncommutative space under consideration. Section 3 provides the basics of the Poincaré group and its homogeneous spaces that will be needed in the rest of the paper, including the definition and properties of the so-called “null-plane” basis, which will be essential in the rest of the paper. In section 4, the specific case of the homogenous space $L$ of Minkowskian light-like geodesics will be analysed in detail from the viewpoint of the coisotropy condition described in section 2, thus arriving at the conclusion that the light-like $\kappa$-deformation is the only one that can be considered for the construction of the Poisson homogeneous space of light-like worldlines. The detailed construction of this space $L$ is provided in section 5, where its quantization is performed, thus giving rise to the noncommutative space of light-like worldlines, whose algebraic properties are analysed in detail. A final section containing several remarks and open problems close the paper.

2 From quantum deformations to noncommutative spaces

In general, the construction of a noncommutative analogue (endowed with quantum group invariance) of a given classical homogeneous space $G/H$ can be achieved through the following systematic and constructive procedure (see [10, 11] and references therein):

- Select a given (coboundary) Poisson–Lie structure $\Pi$ on the Lie group $G$ with Lie algebra $\mathfrak{g}$, which will be associated to a unique $r$-matrix on $\mathfrak{g} \wedge \mathfrak{g}$. The Poisson bivector $\Pi$ is explicitly given by means of the Sklyanin bracket

$$\{f, g\} = r^{ij} (X^L_i f X^L_j g - X^R_i f X^R_j g), \quad f, g \in C(G),$$

such that $X^L_i$ and $X^R_j$ are left- and right-invariant vector fields on $G$. The quantization (as a Hopf algebra [10, 11, 44, 45]) of the Poisson–Lie group $(G, \Pi)$ is the corresponding quantum group.

- Coboundary Lie bialgebras $(\mathfrak{g}, \delta)$ are the tangent counterparts of coboundary Poisson–Lie groups $(G, \Pi)$, where the Lie bialgebra cocommutator map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is straightforwardly obtained from the $r$-matrix in the form

$$\delta(X) = [X \otimes 1 + 1 \otimes X, r], \quad \forall X \in \mathfrak{g}. \quad (2)$$

- A Poisson homogeneous space $(G/H, \pi)$ for a Poisson–Lie group $(G, \Pi)$ is the classical homogeneous space $G/H$ endowed with a Poisson structure $\pi$ which is covariant under the action of the Poisson–Lie group $(G, \Pi)$. The essential problem now is the explicit construction of the Poisson structure $\pi$, whose quantization will give rise to the noncommutative space with quantum group covariance we are looking for. In this respect, a strong mathematical constraint arises: such an explicit construction of $\pi$ is guaranteed only if the so-called coisotropy condition [46–48] for the cocommutator $\delta$ with respect to the isotropy subalgebra $\mathfrak{h}$ of $H$ holds, namely

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}. \quad (3)$$

This condition can be interpreted as a strong compatibility constraint between the specific isotropy subgroup $H$ (and, therefore, between the chosen homogeneous space) and the specific Poisson–Lie structure for $G$ given by $\Pi$, which is in one to one correspondence with the cocommutator $\delta$. In the case that (3) holds, we have a so-called coisotropic
Poisson homogeneous space and the Poisson structure $\pi$ is straightforwardly obtained by canonical projection from the Poisson–Lie structure $\Pi$ on $G$, which can always be explicitly constructed in terms of the Sklyanin bracket (1). Moreover, a particular case of coisotropy condition (3) is the so-called Poisson-subgroup condition, that holds when the Lie subalgebra $\mathfrak{h}$ is also a sub-Lie bialgebra:

$$\delta (\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h},$$  \hspace{1cm} (4)

which implies that $(H,\Pi|_H)$ is a Poisson–Lie subgroup of $(G,\Pi)$, i.e. $H$ is a Poisson submanifold of $G$.

• Noncommutative spaces (which can be thought of as the generating objects of quantum homogeneous spaces [49]) can then be obtained as the quantization (as a comodule algebra under the quantum group co-action) of coisotropic Poisson homogeneous spaces. In this quantum setting, the coisotropy condition (3) ensures that the commutation relations that define the noncommutative space at the first-order in all the quantum coordinates generate a Lie subalgebra which is just the annihilator $\mathfrak{h}^\perp$ of $\mathfrak{h}$ on the dual Lie algebra $\mathfrak{g}^*$ (see [29, 47, 50] for details).

As we will show in section 4, the most common $\kappa$-Poincaré deformation (namely, the time-like $\kappa$-Poincaré algebra [19, 31–35] ) does not fulfil the coisotropy condition for the homogeneous space of null-geodesics on Minkowski spacetime, and therefore the previous construction will be precluded. In contradistinction, we will show that the coisotropy condition is fulfilled by the light-like $\kappa$-Poincaré algebra [24, 41–43] (which is also known in the literature as the “null-plane” quantum Poincaré algebra), which therefore becomes the natural candidate as the quantum symmetry of the non-commutative space of null geodesics, which will be explicitly constructed in section 5.

3 The (3+1) Poincaré group and its homogeneous spaces

Let us consider the Poincaré Lie algebra $\mathfrak{g} = \mathfrak{iso}(3,1) \equiv \mathfrak{so}(3,1) \times \mathbb{R}^4$, which generates the (3+1)D Poincaré group $G = \text{ISO}(3,1)$. In the usual kinematical basis $\{P_0, P_a, K_a, J_a\}$ ($a = 1, 2, 3$) of generators of time translation, space translations, boosts and rotations, respectively, the commutation rules for $\mathfrak{g}$ read

\[
\begin{align*}
[J_a, J_b] &= \epsilon_{abc} J_c, & [J_a, P_b] &= \epsilon_{abc} P_c, & [J_a, K_b] &= \epsilon_{abc} K_c, & [J_a, P_0] &= 0, \\
[K_a, P_0] &= P_a, & [K_a, P_b] &= \delta_{ab} P_0, & [K_a, K_b] &= -\epsilon_{abc} J_c, & [P_\mu, P_\nu] &= 0, \\
\end{align*}
\]

where the speed of light is set $c = 1$. From now on sum over repeated indices will be assumed, unless otherwise stated. In the above kinematical basis, $a, b, c = 1, 2, 3$, and $\mu, \nu = 0, 1, 2, 3$. We shall denote 3-vectors by $\mathbf{v} = (v^1, v^2, v^3)$ and 4-vectors by $v = (v^0, \mathbf{v}) = (v^0, v^1, v^2, v^3)$.

A faithful representation $\rho : \mathfrak{g} \to \text{End}(\mathbb{R}^5)$ for a generic element $X \in \mathfrak{g}$ is given by

\[
\rho(X) = x^\mu \rho(P_\mu) + \xi^a \rho(K_a) + \theta^a \rho(J_a) = \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
x^0 & 0 & \xi^1 & \xi^2 & \xi^3 \\
x^1 & \xi^1 & 0 & -\theta^3 & \theta^2 \\
x^2 & \xi^2 & \theta^3 & 0 & -\theta^1 \\
x^3 & \xi^3 & -\theta^2 & \theta^1 & 0
\end{pmatrix}, \hspace{1cm} (6)
\]
and the corresponding exponential map provides a 5D representation the Poincaré group $G$.

Let us also introduce the so called null-plane basis for the Poincaré algebra [51], which was the one used in the construction of the “null-plane” quantum Poincaré algebra [41–43], since this basis will be essential for our construction. We consider the null-plane $N^*_n$ orthogonal to the light-like vector $n = (\frac{1}{3}, 0, 0, \frac{1}{3})$ in the classical Minkowski spacetime with Cartesian coordinates $x$ and we define

$$x^+ = x^0 + x^3, \quad x^- = \frac{1}{2}(x^0 - x^3) = \tau.$$  \hspace{1cm} (7)

A point $x \in N^*_n$ is labelled by the coordinates $(x^+, x^1, x^2)$, while $x^- = \tau$ plays the role of a “time” or evolution parameter, and the chosen null-plane is associated with the boost generator $K_3$. It can be checked that under the action of the boost transformation generated by the $K_3$ uniparametric subgroup, the initial null-plane $N^0_n (x^- = 0)$ is invariant and the transverse coordinates $(x^1, x^2)$ remain unchanged, while $\exp(\xi^3 \rho(K_3))$ maps $x^+$ into $e^{\xi^3} x^+$. The generators of the Poincaré algebra (5) can be sorted into three different classes according to the adjoint action of $K_3$ onto them, namely

$$[K_3, X] = \gamma X, \quad X \in \mathfrak{g},$$  \hspace{1cm} (8)

where the parameter $\gamma$ is called the “goodness” of the generator $X$ [51]. Such property allows us to introduce a null-plane Poincaré algebra basis associated with the coordinates (7), which is consistent with all the results already presented in [41–43]. Explicitly, the sign of the rotation generators has to be changed

$$L_a = -J_a,$$  \hspace{1cm} (9)

and the so-called “null-plane basis” \{\(P_\pm, P_1, E_i, F_i, K_3, L_3\) \((i=1, 2)\) is defined as

\[
\begin{align*}
\gamma = +1: & \quad P_+ = \frac{1}{2}(P_0 + P_3), \quad E_1 = \frac{1}{2}(K_1 + L_2), \quad E_2 = \frac{1}{2}(K_2 - L_1), \\
\gamma = 0: & \quad K_3, \quad L_3, \quad P_1, \quad P_2, \\
\gamma = -1: & \quad P_- = P_0 - P_3, \quad F_1 = K_1 - L_2, \quad F_2 = K_2 + L_1.
\end{align*}
\]

(10)

From the commutation relations (5) we obtain that [41, 43]

\[
\begin{align*}
[L_3, E_i] &= -\epsilon_{ij3}E_j, \quad [L_3, F_i] = -\epsilon_{ij3}F_j, \quad [L_3, P_i] = -\epsilon_{ij3}P_j, \quad [L_3, P_\pm] = 0, \\
[K_3, E_i] &= E_i, \quad [K_3, F_i] = -F_i, \quad [K_3, P_i] = 0, \quad [K_3, P_\pm] = \pm P_\pm, \\
[E_i, P_j] &= \delta_{ij}P_+, \quad [E_i, P_+] = 0, \quad [E_i, P_-] = P_i, \quad [E_1, E_2] = 0, \\
[F_i, P_j] &= \delta_{ij}P_-, \quad [F_i, P_+] = P_i, \quad [F_1, P_-] = 0, \quad [F_1, F_2] = 0, \\
[K_3, L_3] &= 0, \quad [P_\alpha, P_\beta] = 0, \quad [E_i, F_j] = \delta_{ij}K_3 + \epsilon_{ij3}L_3,
\end{align*}
\]

(11)

where the indices in the null-plane basis are $i,j = 1, 2$ and $\alpha, \beta = \pm, 1, 2$.

From the kinematical representation (6) we obtain the corresponding matrices for the null-plane basis generators (9) and (10), and the generic element $X \in \mathfrak{g}$ reads [42]

$$\rho(X) = x^+ \rho(P_+) + x^- \rho(P_-) + x^i \rho(P_i) + e^i \rho(E_i) + f^i \rho(F_i) + \xi^3 \rho(K_3) + \phi^3 \rho(L_3)$$

\[
= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}x^+ + x^- & 0 & \frac{1}{2}e^1 + f_1 & \frac{1}{2}e^2 + f_2 & \xi^3 \\
x^1 & \frac{1}{2}e^1 + f_1 & 0 & e^3 & -\frac{1}{2}e^1 + f_1 \\
x^2 & \frac{1}{2}e^2 + f_2 & -e^3 & 0 & -\frac{1}{2}e^2 + f_2 \\
\frac{1}{2}x^+ - x^- & \xi^3 & \frac{1}{2}e^1 - f_1 & \frac{1}{2}e^2 - f_2 & 0
\end{pmatrix}.
\]

(12)
From the representation (12) it can be directly checked that the seven generators with \( \gamma = +1 \) and \( \gamma = 0 \) span the isotropy subgroup of the initial null-plane \( N^0_n \), keeping \( x^- = 0 \), while the three remaining ones with \( \gamma = -1 \) act non-trivially on \( N^0_n \). In particular, the transformations generated by \( F_i \) rotate \( N^0_n \), and \( \exp(\gamma \rho(P_-)) \) transforms \( N^0_n \) into \( N^0_n \). Hence the latter generators, which span an abelian subgroup, determine the dynamical evolution of \( N^0_n \) with time \( x^- = \tau \).

It can also be shown that the five generators \( \{P_+, E_i, K_3, L_3\} \) span the isotropy subgroup of the light-like geodesic determined by \( n = (\frac{1}{2}, 0, 0, \frac{1}{2}) \) which under the \( 5 \times 5 \) matrix representation (12) is associated to the 5-vector 
\[
\ell = \left(1, \frac{1}{2}, 0, 0, \frac{1}{2}\right).
\]
This means that 
\[
\exp \left(x^+ \rho(P_+)\right) \exp \left(e^1 \rho(E_1)\right) \exp \left(e^2 \rho(E_2)\right) \exp \left(\xi^3 \rho(K_3)\right) \exp \left(\phi^3 \rho(L_3)\right) \ell^T = \left(1, \frac{1}{2} e^{\xi^3} + \frac{1}{2} x^+, 0, 0, \frac{1}{2} e^{\phi^3} + \frac{1}{2} x^+\right)^T,
\]
where \( T \) denotes transpose. The transformations generated by the five remaining generators \( \{P_-, P_i, F_i\} \) can be used to transform \( \ell \) (13) into any other light-like worldline, and they generate a subgroup which can be thought of as a central extension through \( P_- \) of the 4D abelian group. Therefore, the space of light-like worldlines will have dimension 5.

Finally, we recall that, as a vector space, the Poincaré algebra \( \mathfrak{g} \) can be decomposed in the following ways:
\[
\mathfrak{g} = \mathfrak{t}_{st} \oplus \mathfrak{b}_{st}, \quad t_{st} = \text{span}\{P_0, P\}, \quad \mathfrak{b}_{st} = \text{span}\{K, J\} = \mathfrak{so}(3, 1),
\]
\[
\mathfrak{g} = \mathfrak{t}_{tl} \oplus \mathfrak{b}_{tl}, \quad t_{tl} = \text{span}\{\mathbf{P}, K\}, \quad \mathfrak{b}_{tl} = \text{span}\{P_0, J\} = \mathbb{R} \oplus \mathfrak{so}(3),
\]
\[
\mathfrak{g} = \mathfrak{t}_{sl} \oplus \mathfrak{b}_{sl}, \quad t_{sl} = \text{span}\{P_0, P_i, K_3, J_i\}, \quad \mathfrak{b}_{sl} = \text{span}\{P_3, K_i, J_3\} = \mathbb{R} \oplus \mathfrak{so}(2, 1),
\]
\[
\mathfrak{g} = \mathfrak{t}_{ll} \oplus \mathfrak{b}_{ll}, \quad t_{ll} = \text{span}\{P_-, P_i, F_i\}, \quad \mathfrak{b}_{ll} = \text{span}\{P_+, E_i, K_3, L_3\},
\]
where here \( i = 1, 2 \) and ‘st’, ‘tl’, ‘sl’ and ‘ll’ means, in this order, spacetime, time-like, space-like and light-like. From these decompositions the corresponding homogeneous spaces of points and lines can be constructed as the left coset spaces \( G/H \) between the Poincaré group \( G \) and the isotropy subgroup \( H \) with Lie algebra \( \mathfrak{h} \) (15), namely

- The \( (3+1) \)D Minkowski spacetime is obtained as the homogeneous space \( \mathcal{M} = G/H_{st} \), where the isotropy subgroup \( H_{st} \) is just the Lorentz group.
- The 6D space of time-like lines is obtained as the homogeneous space \( \mathcal{W}_{tl} = G/H_{tl} \).
- The 6D space of space-like lines would be given by \( \mathcal{W}_{sl} = G/H_{sl} \).
- And, finally, the 5D space of light-like lines can be constructed as \( \mathcal{L} = G/H_{ll} \).

The aim of this paper is the construction of a noncommutative version of \( \mathcal{L} \) which is covariant under a quantum Poincaré group.

## 4 \( \kappa \)-Poincaré deformations and light-like worldlines

We recall that all Lie bialgebra structures for the \( (3+1) \)D Poincaré algebra \( \mathfrak{g} \) are coboundary ones [52–54]. Therefore any quantum deformation is determined by an underlying classical \( r \)-matrix which provides the cocommutator \( \delta \) in the form (2).
Among all the possible inequivalent $r$-matrices for the Poincaré algebra (see [52] for a classification), the most studied cases are the so-called $\kappa$-Poincaré $r$-matrices, which in covariant notation can be written as (see [24, 52])

$$r = a^\mu M_{\mu\nu} \wedge P_\nu,$$

(16)

where $a^\mu$ are the components of a Minkowskian four-vector $a$ and $M_{\mu\nu}$ denote the generators of the Lorentz group with the following identifications:

$M_{0\nu} \equiv K_\nu, \quad M_{12} = J_3, \quad M_{23} = J_1, \quad M_{31} = J_2.$

(17)

In this way, three neat classes of $\kappa$-deformations arise:

- **The time-like deformation:** if we consider the time-like vector $a = (1/\kappa, 0, 0, 0)$, we obtain

$$r_{\text{time}} = \frac{1}{\kappa} (M_{0\nu} \wedge P_\nu) = \frac{1}{\kappa} (K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3).$$

(18)

- **The space-like deformation,** generated by the space-like vector $a = (0, 0, 0, -1/\kappa)$:

$$r_{\text{space}} = -\frac{1}{\kappa} (M_{3\nu} \wedge P_\nu) = \frac{1}{\kappa} (K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1).$$

(19)

- **The light-like deformation:** obtained from the light-like vector $a = (1/\kappa, 0, 0, -1/\kappa)$:

$$r_{\text{light}} = \frac{1}{\kappa} (M_{0\nu} \wedge P_\nu - M_{3\nu} \wedge P_\nu)$$

$$= \frac{1}{\kappa} (K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1).$$

(20)

The first $r$-matrix, $r_{\text{time}}$, is just the $r$-matrix underlying the well-known $\kappa$-Poincaré algebra [19, 31–35], whose time-like nature is not frequently emphasized in the literature. In this case, the deformation parameter $\kappa$ has dimensions of a time$^{-1}$ (recall that $c = 1$). The second one, $r_{\text{space}}$, determines the $q$-Poincaré algebra obtained in [55] (c.f. Type 1. (a) with $q = e^z$ and $z = 1/\kappa$), having $\kappa$ dimensions of a length$^{-1}$. Both $r_{\text{time}}$ and $r_{\text{space}}$ provide quasitriangular (or standard) deformations of the Poincaré algebra since they are solutions of the modified classical Yang–Baxter equation with non-vanishing Schouten bracket. In fact, both $r$-matrices/deformations can be obtained through coboundary Lie bialgebra contractions from the Drinfel’d–Jimbo bialgebra for $\mathfrak{so}(5)$ (see [29] and references therein).

Finally, we stress that $r_{\text{light}}$ gives rise exactly to the null-plane quantum Poincaré algebra introduced in [41–43] (where $z = 1/\kappa$). This is a triangular (or nonstandard) deformation with vanishing Schouten bracket, which indeed, verifies

$$r_{\text{light}} = r_{\text{time}} + r_{\text{space}}.$$  

(21)

In Table 1 the three $r$-matrices are written in both the kinematical (5) and null-plane basis (11). As it can be easily appreciated, the latter is very well adapted to the light-like deformation, while the kinematical basis provides the simplest possible form for both the time-like and space-like deformations.
The classical $r$-matrices for the time-like, space-like and light-like $\kappa$-deformations of the (3+1)D Poincaré algebra in both the kinematical (5) and null-plane basis (11).

\begin{align*}
\bullet \ r_{\text{time}} &= \frac{1}{\kappa} (K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) \\
&= \frac{1}{\kappa} (K_3 \wedge P_+ + E_1 \wedge P_1 + E_2 \wedge P_2) - \frac{2}{\kappa} (K_3 \wedge P_- - F_1 \wedge P_1 - F_2 \wedge P_2) \\
\bullet \ r_{\text{space}} &= \frac{1}{\kappa} (K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1) \\
&= \frac{1}{\kappa} (K_3 \wedge P_+ + E_1 \wedge P_1 + E_2 \wedge P_2) + \frac{2}{\kappa} (K_3 \wedge P_- - F_1 \wedge P_1 - F_2 \wedge P_2) \\
\bullet \ r_{\text{light}} &= \frac{1}{\kappa} (K_3 \wedge P_0 + K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + J_1 \wedge P_2 - J_2 \wedge P_1) \\
&= \frac{2}{\kappa} (K_3 \wedge P_+ + E_1 \wedge P_1 + E_2 \wedge P_2)
\end{align*}

Now let us face the explicit construction of the noncommutative analogue of the homogeneous space of light-like worldlines $\mathcal{L} = G/H_{\parallel}$ where

$$
\mathfrak{h}_{\parallel} = \text{span}\{P_+, E_i, K_3, L_3\}. \tag{22}
$$

As it was explained in section 2, firstly we have to obtain a coisotropic Poisson homogeneous space $(\mathcal{L}, \pi)$ where $\pi$ is the canonical projection of a Poisson–Lie structure II on the Poincaré group generated by a given classical $r$-matrix, which has an associated cocommutator map $\delta$ determined by (2). But this construction is only possible if the coisotropy condition with respect to the isotropy subalgebra $\mathfrak{h}_{\parallel}$ (22) of the space of light-like geodesics is fulfilled, namely:

$$
\delta(\mathfrak{h}_{\parallel}) \subset \mathfrak{h}_{\parallel} \wedge \mathfrak{g}. \tag{23}
$$

A straightforward computation shows that the light-like $r$-matrix (20) does fulfill this condition, while both the time-like (18) and the space-like (19) $r$-matrices do not. Explicitly, the cocommutator obtained from the light-like $r$-matrix (20), via the relation (2), reads

\begin{align*}
\delta(X) &= 0, \quad X \in \{P_+, E_i, L_3\}, \\
\delta(Y) &= \frac{2}{\kappa} Y \wedge P_+, \quad Y \in \{P_-, P_i\}, \\
\delta(K_3) &= \frac{2}{\kappa} (K_3 \wedge P_+ + E_1 \wedge P_1 + E_2 \wedge P_2), \\
\delta(F_1) &= \frac{2}{\kappa} (F_1 \wedge P_+ + E_1 \wedge P_- + L_3 \wedge P_2), \\
\delta(F_2) &= \frac{2}{\kappa} (F_2 \wedge P_+ + E_2 \wedge P_- - L_3 \wedge P_1).
\end{align*} \tag{24}

Therefore (20) generates the unique $\kappa$-deformation that can provide a coisotropic Poisson homogeneous space of null geodesics, whose explicit construction will be presented in the following section, together with its quantization.

## 5 Construction of the noncommutative space

In order to construct the noncommutative space of light-like worldlines we will follow a similar approach to the one presented in [37] for the construction of its time-like analogue. We start by...
considering the following Poincaré group element obtained from the matrix realization (12):

\[
G_\mathcal{L} = \exp \left( y^- \rho(P_-) \right) \exp \left( f^1 \rho(F_1) \right) \exp \left( f^2 \rho(F_2) \right) \exp \left( y^1 \rho(P_1) \right) \exp \left( y^2 \rho(P_2) \right) H_{\Pi},
\]

(25)

where \( H_{\Pi} \) is the isotropy subgroup or stabilizer of the light-like line \( \ell \) (13), since the latter is taken as the origin of the 5D homogeneous space \( \mathcal{L} = G/H_{\Pi} \) (see (14)). Explicitly,

\[
H_{\Pi} = \exp \left( y^+ \rho(P_+) \right) \exp \left( e^1 \rho(E_1) \right) \exp \left( e^2 \rho(E_2) \right) \exp \left( \xi^3 \rho(K_3) \right) \exp \left( \phi^3 \rho(L_3) \right).
\]

(26)

We recall that under this parametrization the generators \( \{P_-, F_i, P_i\} \) become the generators of translations on the 5D classical coset space \( \mathcal{L} \), that is parametrized in terms of the \((y^-, f^i, y^i)\) coordinate functions. Note also that the order chosen for calculating the group element \( G_\mathcal{L} \) (25) is consistent with the coset structure induced by the isotropy subgroup of the null-plane, which is generated by \( \{P, P_+, E_i, K_3, L_3\} \).

From (25), a long but straightforward computation leads to the left- and right-invariant vector fields on \( G \), and afterwards the full Poisson–Lie group structure \( \Pi \) on the Poincaré group (1) for the \( r \)-matrix \( r_{\text{light}} \) (20). The non-vanishing Poisson brackets defining \( \Pi \) are the following ones:

\[
\begin{align*}
\{y^-, y^i\} &= -\frac{1}{\kappa} f^i y^-, & \{y^-, e^i\} &= -\frac{1}{\kappa} \left( 2 f^i + e^i ((f^1)^2 + (f^2)^2) \right), \\
\{y^-, \xi^3\} &= -\frac{1}{\kappa} ((f^1)^2 + (f^2)^2), & \{y^+, y^i\} &= -\frac{2}{\kappa} y^i, \\
\{y^+, e^i\} &= -\frac{2}{\kappa} (\exp \xi^3 - 1) e^i, & \{y^+, f^i\} &= -\frac{2}{\kappa} f^i, \\
\{y^+, \xi^3\} &= -\frac{2}{\kappa} (\exp \xi^3 - 1), & \{y^+, f^i\} &= -\frac{2}{\kappa} \delta_{ij} ((f^1)^2 + (f^2)^2), \\
\{y^1, e^i\} &= -\frac{2}{\kappa} (\exp \xi^3 - 1 - e^1 f^2), & \{y^1, e^2\} &= -\frac{2}{\kappa} e^1 f^2, \\
\{y^2, e^2\} &= -\frac{2}{\kappa} (\exp \xi^3 - 1 - e^1 f^1), & \{y^2, e^1\} &= -\frac{2}{\kappa} e^2 f^1, \\
\{y^i, \phi^3\} &= \frac{2}{\kappa} \epsilon_{ij3} f^j, & \{y^1, y^2\} &= \frac{2}{\kappa} (f^1 y^2 - f^2 y^1), \\
\{y^-, y^+\} &= \frac{1}{\kappa} (2 y^- - 2 (f^1 y^1 + f^2 y^2) - y^+ ((f^1)^2 + (f^2)^2)).
\end{align*}
\]

(27)

Now, the Poisson structure \( \pi \) among the five coordinates \((y^-, y^i, f^i)\) is obtained as the canonical projection of \( \Pi \) onto this set of coordinates, which indeed generate a Poisson subalgebra. From (27) it is immediate to check that the Poisson algebra generated by \((y^-, y^i, f^i)\) is quadratic homogeneous, and defines the Poisson homogeneous space \((\mathcal{L}, \pi)\) associated to the light-like \( \kappa \)-deformation. Note that this result is fully consistent with (24) since the linearization of \( \pi \) is zero, this means that at the r.h.s. of the cocommutators (24) there should not exist any term \( X \wedge Y \) with \( X, Y \in \{P_-, P_i, F_i\} \), and this is indeed the case.

Moreover, since the Poisson algebra \( \pi \) presents no ordering problems, it can be directly quantized and the algebra \( \mathcal{L}_\kappa \) so obtained defines the noncommutative space of light-like geodesics:

\[
\begin{align*}
[y^-, \hat{y}^i] &= -\frac{1}{\kappa} \hat{f}^i y^- , & [\hat{y}^1, \hat{y}^2] &= \frac{2}{\kappa} (\hat{f}^1 \hat{y}^2 - \hat{f}^2 \hat{y}^1), \\
[\hat{y}^i, \hat{f}^j] &= \frac{1}{\kappa} \delta_{ij} ((\hat{f}^1)^2 + (\hat{f}^2)^2), & [\hat{y}^-, \hat{f}^i] &= [\hat{f}^1, \hat{f}^2] = 0, & i, j = 1, 2.
\end{align*}
\]

(28)
We remark that the three quantum coordinates \((\hat{y}^-, \hat{f}^i)\) can be thought to define the quantum null-plane, which generates a 3-dimensional abelian subalgebra.

Let us analyse the quantum space of light-like geodesics \(\mathcal{L}_\kappa\) (28) in more detail. The three commutative quantum coordinates \((\hat{y}^-, \hat{f}^i)\) associated with the null-plane provide a (rational) Casimir operator \(\hat{C}\) for the whole \(\mathcal{L}_\kappa\) algebra, which is given by

\[
\hat{C} = \hat{y}^- \left( (\hat{f}^1)^2 + (\hat{f}^2)^2 \right)^{-1},
\]

provided that \(( (\hat{f}^1)^2 + (\hat{f}^2)^2 )^{-1} \) exists, and where we have made use of the expressions \([u, v^{-1}] = -v^{-1}[u, v]v^{-1}\), and the fact that \([\hat{y}^-, (\hat{f}^1)^2 + (\hat{f}^2)^2] = 0\).

The four quantum coordinates \((\hat{y}^i, \hat{f}^i)\) generate a subalgebra of \(\mathcal{L}_\kappa\) which resembles a pair of two Heisenberg–Weyl algebras, and the commutator \([\hat{y}^1, \hat{y}^2]\) in (28) gives rise to a kind of angular momentum operator. In fact, we can introduce a differential realization of \((\hat{y}^i, \hat{f}^i)\) as operators acting on the space of functions \(\Psi(f^1, f^2)\) in the form

\[
\hat{y}^i \Psi = \frac{1}{\kappa} ( (\hat{f}^1)^2 + (\hat{f}^2)^2 ) \frac{\partial \Psi}{\partial f^i}, \quad \hat{f}^i \Psi = f^i \Psi,
\]

which can be interpreted as an action on a classical 2D space with momentum-type coordinates \((f^1, f^2)\). Finally, the action of the remaining quantum coordinate \(\hat{y}^-\) on \(\Psi\) is straightforward by considering the Casimir operator \(\hat{C}\) (29):

\[
\hat{y}^- \Psi = \hat{C} ( (\hat{f}^1)^2 + (\hat{f}^2)^2 ) \Psi = C ( (\hat{f}^1)^2 + (\hat{f}^2)^2 ) \Psi,
\]

where \(C\) is the eigenvalue of the operator \(\hat{C}\) acting on \(\Psi\).

Surprisingly enough, the representation (30) suggests the following definition for two pairs of quantum canonical variables \((\hat{q}^i, \hat{p}^i)\) in terms of the quantum worldline coordinates \((\hat{y}^i, \hat{f}^i)\):

\[
\hat{q}^i := ( (\hat{f}^1)^2 + (\hat{f}^2)^2 )^{-1} \hat{y}^i, \quad \hat{p}^i := \hat{f}^i, \quad [\hat{q}^i, \hat{p}^j] = \frac{1}{\kappa} \delta_{ij}.
\]

Consequently, the representation (30) and the definition (32) lead to the usual realization

\[
\hat{q}^i \Psi = \frac{1}{\kappa} \frac{\partial \Psi}{\partial f^i}, \quad \hat{p}^i \Psi = f^i \Psi,
\]

on functions \(\Psi(f^1, f^2)\) defined on a classical momentum space, where now the quantum deformation parameter \(\kappa\) plays the same role as the Planck constant \(\hbar\) in ordinary quantum mechanics. Moreover, for the fifth quantum worldline coordinate \(\hat{y}^-\) we have that

\[
[\hat{y}^-, \hat{p}^i] = 0, \quad [\hat{y}^-, \hat{q}^i] = -\frac{2C}{\kappa} \hat{p}^i, \quad i = 1, 2,
\]

for a given realization characterized by \(C\). Therefore, we can say that the noncommutative space of light-like worldlines (28) can be mapped to a non-central extension, generated by \(\hat{y}^-, \) of the direct sum of two Heisenberg–Weyl algebras \((\hat{q}^i, \hat{p}^i)\). (As a side remark, note that light-ray noncommutativity arises in spinning light propagation [56].)

At this point we recall that the 6D noncommutative space of time-like lines constructed in [37] was shown to be generated by the direct sum of three Heisenberg–Weyl algebras, and this enabled the analysis presented in [38] of the fuzziness properties of events defined as the crossing
of light-like quantum geodesics. In the light-like case we are dealing with a 5D quantum space, whose fuzziness could be studied in a similar manner by taking into account the mapping (32). We plan to face this analysis in a forthcoming paper.

For the sake of completeness, a few words on the noncommutative Minkowski spacetime arising from the light-like $\kappa$-deformation and its relation with the previous construction seem to be appropriate. This space can be constructed in terms of the Minkowskian null-plane coordinates by considering the Poincaré group element obtained from the $5 \times 5$ matrix realization (12) in the form

$$G_M = \exp (x^+ \rho(P_+)) \exp (x^2 \rho(P_2)) \exp (x^- \rho(P_-)) H_{st},$$

(35)

where

$$H_{st} = \exp (e^1 \rho(E_1)) \exp (e^2 \rho(E_2)) \exp (f^1 \rho(F_1)) \exp (f^2 \rho(F_2)) \exp (\xi^3 \rho(K_3)) \exp (\phi^3 \rho(L_3))$$

(36)

is the Lorentz subgroup SO(3,1) that plays the role of the isotropy subgroup of the point $O = (1,0,0,0,0)$, which is taken as the origin of the (3+1)D Minkowski spacetime $\mathcal{M} = G/H_{st}$.

From (35), the left- and right-invariant vector fields on $G$ are obtained, and the Sklyanin bracket (1) can be computed in terms of the $r$-matrix $r_{\text{light}}$ (20). Afterwards, the Poisson structure $\pi$ among the four null-plane Minkowski coordinates $x = (x^\pm, x^i)$ is obtained as the canonical projection of the Sklyanin bracket onto these coordinates. The resulting Poisson structure is linear, and its quantization is straightforward leading to the noncommutative null-plane Minkowski spacetime given by

$$[\hat{x}^i, \hat{x}^+] = \frac{2}{\kappa} \hat{x}^i, \quad [\hat{x}^-, \hat{x}^+] = \frac{2}{\kappa} \hat{x}^-, \quad i = 1, 2.$$  

(37)

Notice that the classical spacetime coordinates $x = (x^\pm, x^i)$ in $G_M$ (35) can be expressed in terms of those in $G_L$ (25) as

$$x^+ = y^+, \quad x^i = y^i + y^+ f^i, \quad x^- = y^- + y^1 f^1 + y^2 f^2 + \frac{1}{2} y^+(f^1)^2 + (f^2)^2.$$  

(38)

Obviously, the linearization of (38) gives $(x^\pm, x^i) \equiv (y^\pm, y^i)$. Note also that if we compute the Poisson brackets for $(x^\pm, x^i)$ by starting from (27) we recover the linear Poisson structure for the Poisson spacetime (37), as it should be. Therefore, the nonlinear map (38) will also provide the starting point for the study of the link between the noncommutative spaces of points (37) and light-like geodesics (28) arising from the light-like $\kappa$-deformation.

6 Concluding remarks

The definition and explicit construction of the quantum analogue of geodesics on a noncommutative space is indeed an interesting problem from both the mathematical and physical viewpoints. In particular, the case of null geodesics on a noncommutative Minkowski spacetime provides a first toy model that will be (hopefully) useful in order to describe some expected features of the propagation of massless particles on quantum Minkowski spacetimes.

In this paper we have approached this problem by constructing explicitly the 5D noncommutative space of null geodesics arising from the light-like $\kappa$-deformation of the (3+1) Poincaré
group through the following procedure: firstly, the space of geodesics is constructed as a homogeneous space \( \mathcal{L} \) of the Poincaré group, secondly this space is endowed with a Poisson-noncommutative structure which is provided by the \( r \)-matrix defining the chosen deformation, and finally the quantization of this Poisson structure is performed, thus obtaining the noncommutative space. We emphasize that, within this framework, each geodesic is represented as a point in the homogeneous space \( \mathcal{L} \), and a similar thing happens within the noncommutative setting here presented.

We have focused on the particular class of Hopf algebra deformations of the Poincaré group given by the so-called \( \kappa \)-deformations, which can be naturally splitted into time-, space- and light-like cases. Remarkably enough, we have found that only the light-like (or null-plane) \( \kappa \)-deformation can be used to provide a noncommutative space of light-like worldlines. This could seem to be a natural fact, but it is important to emphasize that these three quantum deformations are different Hopf algebras, and this result points in the direction of considering the complete family of \( \kappa \)-deformations of the Poincaré algebra as the proper Planck-scale symmetry, which has to be specialized to particles with different masses, in the same manner that different particles carry different irreducible representations of the Poincaré group. To the best of our knowledge, this viewpoint has not been considered in the literature yet, and its consequences should be elaborated in more detail.

Also, this mathematical compatibility issue between a chosen quantum deformation and a specific homogeneous space in order to construct the noncommutative version of the latter is worth to be analysed. In particular, the coisotropy condition (3) of the chosen Lie bialgebra with respect to the isotropy subalgebra of the classical homogeneous space is the keystone for this analysis. Specifically, we can consider the three \( r \)-matrices given in Table 1, since each of them leads to a Lie bialgebra structure \((\mathfrak{g}, \delta)\) with commutator given by (2) which, in turn, allows us to analyse whether the coisotropy relation (3) is fulfilled with respect to the four isotropy subalgebras (15).

It is straightforward to prove by direct computation that the three classical \( r \)-matrices defining the \( \kappa \)-deformations do satisfy the coisotropy condition for the construction of a Poisson homogeneous Minkowski spacetime, i.e. \( \delta(\mathfrak{h}_{\text{at}}) \subseteq \mathfrak{h}_{\text{at}} \wedge \mathfrak{g} \) in all the cases. This means that three noncommutative Minkowski spacetimes can be explicitly constructed from the \( \kappa \)-Poincaré deformations. In fact, all of them can be found in the previous literature, and their non-vanishing commutation relations in terms of the corresponding quantum spacetime coordinates \( \hat{x}^\mu \) (dual to \( P_\mu \)) are given by the linear relations (see [19, 29, 42], respectively)

\[
\begin{align*}
\mathcal{r}_{\text{time}} : & \quad [\hat{x}^a, \hat{x}^0] = \frac{1}{\kappa} \hat{x}^a, \quad a = 1, 2, 3, \\
\mathcal{r}_{\text{space}} : & \quad [\hat{x}^0, \hat{x}^3] = \frac{1}{\kappa} \hat{x}^0, \quad [\hat{x}^1, \hat{x}^3] = \frac{1}{\kappa} \hat{x}^1, \quad [\hat{x}^2, \hat{x}^3] = \frac{1}{\kappa} \hat{x}^2, \\
\mathcal{r}_{\text{light}} : & \quad [\hat{x}^i, \hat{x}^+ ] = \frac{2}{\kappa} \hat{x}^i, \quad [\hat{x}^-, \hat{x}^+ ] = \frac{2}{\kappa} \hat{x}^-, \quad i = 1, 2.
\end{align*}
\]

Therefore, they are noncommutative spacetimes of Lie-algebraic type and, moreover, all of them are isomorphic as Lie algebras.

Likewise, Table 2 shows that the three \( r \)-matrices enable the construction of noncommutative spaces of time- and space-like geodesics associated with isotropy subalgebras \( \mathfrak{h}_{\text{tl}} \) and \( \mathfrak{h}_{\text{sl}} \) (15), respectively. However, observe that the coisotropy condition (3) is trivial \( (\delta(\mathfrak{h}) = 0) \) for \( \mathcal{r}_{\text{time}} \) with \( \mathfrak{h}_{\text{tl}} \) and for \( \mathcal{r}_{\text{space}} \) with \( \mathfrak{h}_{\text{sl}} \), and hence the Poisson-subgroup condition (4) is satisfied in the cases when the type of deformation and the type of homogenous space fit. We recall that the time-like case has been studied at the Lie bialgebra level for \( \mathcal{r}_{\text{space}} \) and \( \mathcal{r}_{\text{time}} \) in [29] (c.f. classes A and C, respectively), and the structural differences between the two conditions \( \delta(\mathfrak{h}_{\text{tl}}) \subseteq \mathfrak{h}_{\text{tl}} \wedge \mathfrak{g} \)
Table 2: Coisotropy condition (3) for the three classical r-matrices in Table 1 with respect to the three different isotropy subalgebras (15) that define the time-like (tl), space-like (sl) and light-like (ll) homogeneous spaces of worldlines.

| Condition | $\mathfrak{h}_{tl}$ | $\mathfrak{h}_{sl}$ | $\mathfrak{h}_{ll}$ |
|-----------|---------------------|---------------------|---------------------|
| $r_{\text{time}}$ | $\delta(\mathfrak{h}_{tl}) = 0$ | $\delta(\mathfrak{h}_{sl}) \subset \mathfrak{h}_{sl} \wedge \mathfrak{g}$ | $\delta(\mathfrak{h}_{ll}) \not\subset \mathfrak{h}_{ll} \wedge \mathfrak{g}$ |
| $r_{\text{space}}$ | $\delta(\mathfrak{h}_{tl}) \subset \mathfrak{h}_{tl} \wedge \mathfrak{g}$ | $\delta(\mathfrak{h}_{sl}) = 0$ | $\delta(\mathfrak{h}_{ll}) \not\subset \mathfrak{h}_{ll} \wedge \mathfrak{g}$ |
| $r_{\text{light}}$ | $\delta(\mathfrak{h}_{tl}) \subset \mathfrak{h}_{ll} \wedge \mathfrak{g}$ | $\delta(\mathfrak{h}_{sl}) \subset \mathfrak{h}_{sl} \wedge \mathfrak{g}$ | $\delta(\mathfrak{h}_{ll}) \subset \mathfrak{h}_{ll} \wedge \mathfrak{g}$ |

and $\delta(\mathfrak{h}_{tl}) = 0$ have been shown. Finally, as we have discussed in section 4, the construction of noncommutative spaces of light-like lines is precluded for both the time-like and space-like deformations. Therefore, the only quantum $\kappa$-deformation which can provide a noncommutative counterpart of the four homogeneous spaces of points and lines is the light-like $\kappa$-Poincaré algebra.

To end with, we comment on two open problems that are worth to be worked out in the near future. Firstly, we recall that the complete classification of those (3+1)D quantum Poincaré and (A)dS algebras that present a quantum Lorentz subgroup has been recently obtained in [30], where it has been shown that all of them fulfill the coisotropy condition (3) for the Minkowski spacetime (thus with respect to $\mathfrak{h}_{st}$). We stress that $\kappa$-deformations are not included within this class of deformations and therefore the analysis of the coisotropy condition for the three spaces of geodesics (15) that are covariant under these novel quantum Poincaré groups deserves a further study. Secondly, we recall that the time-like $\kappa$-deformation of the (A)dS algebra (with nonvanishing cosmological constant $\Lambda$) was explicitly constructed in [36], and its associated noncommutative spacetime has also been presented in [27]. Therefore, the methodology here proposed can also be applied to the case of the homogeneous spaces of worldlines associated to this and to the previously mentioned quantum deformations of the (A)dS groups, thus providing a novel approach to the interplay between the cosmological constant and the Planck scale deformation parameter [57] based on the properties of quantum worldlines.

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