Geodesics and Geodesic Deviation in Static Charged Black Holes

Ragab M. Gad
Mathematics Department, Faculty of Science, Minia University, 61915 El-Minia, EGYPT.

Abstract
The radial motion along null geodesics in static charged black hole space-times, in particular, the Reissner-Nordström and stringy charged black holes are studied. We analyzed the properties of the effective potential. The circular photon orbits in these space-times are investigated. We found that the radius of circular photon orbits in both charged black holes are different and differ from that given in Schwarzschild space-time. We studied the physical effects of the gravitational field between two test particles in stringy charged black hole and compared the results with that given in Schwarzschild and Reissner-Nordström black holes.

PACS numbers: 04.50.+h, 12.10.-g
Keywords: Geodesics and geodesic deviation; circular photon orbits; static charged black hole.

1 Introduction
The well known static, spherically symmetric black hole solutions in vacuum of Einstein’s general relativity are given by the charged Reissner-Nordström and uncharged Schwarzschild solutions. In the non-vacuum case of Einstein’s general relativity, several black hole solutions are known [1]- [3]. One of them is the stringy charged black hole discovered by Garfinkle, Horowitz and Strominger [3].

The study of timelike and null geodesics, the paths of freely moving particles and photons, is the key to understand the physical importance of a given space-time.

We wish to investigate in this paper the properties of the stringy charged black hole by studying its geodesic structure, that is, from the motion of photons. We compare the results by the aforementioned solutions in vacuum.

1Email Address: ragab2gad@hotmail.com
The space-time under consideration is almost identical to Schwarzschild space-time. The only differences are

1. The gravitational energy, using Møller’s prescription \[4\], depends on the mass parameter \(M\) and on the charge \(Q\) (see Gad \[5\]), while in the Schwarzschild black hole is given only by the mass parameter \(M\) (see Xulu \[6\]).

2. The areas of the spheres of constant \(r\) and \(t\) depend on the charge \(Q\).

The first aim of this paper is to sustain the second difference by analyzing the effective potential of radial motion along null geodesics.

Compared to the Reissner-Nordström black hole, the stringy charged black hole exhibits several different properties \[7\]- \[10\]. For example, first, this solution has only one horizon at \(r = 2M\) and not two as is the case for the Reissner-Nordström\(^2\). Second, this solution has singularities at \(r = \alpha\), \(r = 0\) whereas the Reissner-Nordström has a singularity at \(r = 0\). The extremal solution occurs at \(Q^2 = 4M^2e^{2\Phi_0}\) rather than at \(E^2 = M^2\) as for Reissner-Nordström solution.

It is therefore worthwhile to investigate other properties of the stringy charged black hole to see how this differs from the vacuum solutions.

The equation of geodesic deviation gives the relative accelerations between free test particles falling in a gravitational field and is a cornerstone to the understanding of the physical effects of the gravitational field and the geometry of space-time \[9\]. Geodesics deviations in vacuum space-times, namely, Schwarzschild and Reissner-Nordström space-times, are rigorously studied in \[11\]- \[17\]. Some studies on geodesic deviation can be found in \[18\]- \[20\]. Ghosh and Kar \[21\] investigated geodesic motion and geodesic deviation in warped space-times with a a time dependent extra dimension. We do follow here the same approach in \[16\], \[17\] to obtain the relative accelerations of nearby test particles in stringy charged black hole. This is the second aim of this paper.

The structure of the paper is as follows: In the next section, we investigate the geodesic curves and the circular photon orbits in the stringy charged black hole. In section 3, we study the photon trajectories in Reissner-Nordström by analyzing the properties of effective potential. Section 4 concerns with the tidal forces between nearby test particles in stringy charged black hole. In section 5, we investigate the behavior of geodesic deviation vector along a timelike geodesic near the singularities \(r = 0\) and \(r = \alpha\), by considering a

\(^2\)The Reissner-Nordström has two horizons given by the quadratic equation \(r^2 - 2Mr + e^2 = 0\).
special case. Finally, in section 6, brief summary of results and concluding remarks are presented.

Throughout this paper Latin indices run from 0 to 3; Greek indices run from 1 to 3. Units are such that $G = 1 = c$, ($G$ is the gravitational constant; $c$ is the velocity of light).

## 2 The equations of geodesic motion

In recent years there is considerable interest in obtaining black hole solutions in string theory and investigating their properties (see references [3], [7] and [22]). There are two metrics in this theory, which are called the sigma-model (or string) metric and Einstein metric. For uncharged static, spherically symmetric black hole, the solution in the low energy is the same as the Schwarzschild solution. This is, however, not the case when the black hole is charged.

We are interested to investigate the geodesic curves of a static spherically symmetric charged black hole. The line element representing this space-time is given by Garfinkle et al [3].

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + (1 - \frac{\alpha}{r})r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$  

where

$$\alpha = \frac{Q^2}{2M} e^{\Phi_0},$$

(2.1)

$M$ and $Q$ are, respectively, mass and charge parameters; $\Phi_0$ is the asymptotic value of dilaton field.

The equations and constraint for geodesics are given as

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0,$$

$$g_{ab} \dot{x}^a \dot{x}^b = \varepsilon,$$

where a superposed dot stands for a derivative with respect to the affine parameter $\tau$ associated to the geodesic, $x^a$ are the coordinates of a space-time point on the geodesic and $\varepsilon = -1$ or 0, for timelike or null geodesics, respectively.

The geodesic equations for the line element (2.1) are given by

$$\ddot{t} + \frac{2M}{r(r - 2M)} \dot{t}\dot{r} = 0,$$

(2.3)
\[ \ddot{r} - \frac{M}{r(r-2M)} r^2 + \frac{(r-2M)(\alpha - 2r)}{2r} \dot{\theta}^2 + \frac{(r-2M)(\alpha - 2r)}{2r} \sin^2 \theta \dot{\phi}^2 + \]
\[ \frac{M(r-2M)}{r^3} \dot{t}^2 = 0, \quad (2.4) \]
\[ \ddot{\theta} + \frac{(\alpha - 2r)}{r(\alpha - r)} \dot{r} \dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0, \quad (2.5) \]
\[ \ddot{\phi} + \frac{(\alpha - 2r)}{r(\alpha - r)} \dot{r} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0, \quad (2.6) \]

The constraint of timelike or null geodesics for the line element (2.1) is given by
\[ - (1 - 2M/r) \dot{t}^2 + (1 - \frac{2M}{r})^{-1} r^2 + (1 - \frac{\alpha}{r}) r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = \varepsilon. \quad (2.7) \]

Let a geodesic \( \gamma \) be given by \( \gamma(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau)) \). We suppose without loss of generality that \( \gamma(\tau_0) = (t(\tau_0), r(\tau_0), \pi/2, \phi(\tau_0)) \) and for all \( \tau : \theta = \pi/2 \) (equatorial orbits) that is the particle has at start, and continues to have, zero momentum in the \( \theta \)-direction; thus \( \dot{\theta} = 0 \). Consequently, the equation of geodesics (2.3) and (2.6) have the straightforward first integrals

\[ \dot{t} = c_1 \left( \frac{r}{r-2M} \right), \quad (2.8) \]
\[ \dot{\phi} = \frac{c_2}{r(r-\alpha)}, \quad (2.9) \]

The integration constants \( c_1 \) and \( c_2 \) can be found, if we know the initial conditions \( \gamma(\tau_0) \) and \( \frac{d\gamma}{d\tau}(\tau_0) \) for some \( \tau_0 \in \mathbb{R} \). In the case of Schwarzschild and Reissner-Nordström space-times, Clarke [23] and also Wald [24] demonstrate that \( c_1 \) represents the total energy, \( E \), per unit rest mass of a particle measured by a static observer, and \( c_2 \) represents the angular momentum, \( L \), per unit mass of a particle (see also [11]). We recognize the constant \( \varepsilon \) to represent the rest energy per unit mass for massive particles (timelike curve, \( \varepsilon = -1 \)) or the rest energy for massless particles (null curves, \( \varepsilon = 0 \)), traveling along the given geodesic [11, 12].

Using equations (2.8), (2.9) and \( \theta = \pi/2 \) in the constraint equation (2.7), we have

\[ \dot{r}^2 = E^2 - (1 - \frac{2M}{r}) \left[ - \varepsilon + \frac{L^2}{r(r-\alpha)} \right]. \quad (2.10) \]

Notice that we have not used equation (2.4) for the following reason: If we substitute \( \dot{r} \) and its derivative with respect to \( \tau \) from the constraint equation
and using equations (2.8) and (2.9) in equation (2.4), then it is satisfied identically.

Equation (2.10) can be written as

\[ \dot{r}^2 = E^2 - V^2(r), \]  

(2.11)

where \( V(r) \) is the "effective potential" defined by

\[ V^2(r) = (1 - \frac{2M}{r}) \left[ \frac{L^2}{r(r - \alpha)} \right], \]  

(2.12)

Equation (2.11) is in the form of the equation of a one-dimensional problem for a particle in a potential field \( V(r) \).

Since the left side of equation (2.10) is positive or zero, the energy \( E \) of the trajectory must not be less than the potential \( V \). So for an orbit of a given \( E \), the radial range is restricted to those radii for which \( V \) is smaller than \( E \).

In the following we investigate the circular photon orbits in the stringy charged black hole by analyzing the properties of the effective potential.

In the case of the photon trajectories, putting \( \varepsilon = 0 \), the effective potential (2.12) takes the form

\[ V^2(r) = (1 - \frac{2M}{r}) \left[ \frac{L^2}{r(r - \alpha)} \right], \]  

(2.13)

Differentiating equation (2.11) with respect to \( \tau \) gives

\[ 2 \left( \frac{dr}{d\tau} \right) \left( \frac{d^2r}{d\tau^2} \right) = - \frac{dV^2(r)}{dr} \frac{dr}{d\tau}, \]

or

\[ \frac{d^2r}{d\tau^2} = -\frac{1}{2} \frac{d}{dr}(V^2(r)). \]  

(2.14)

It is clear from equation (2.14) that a circular orbit \( (r = \text{constant}) \) is possible only at a minimum or maximum of \( V^2(r) \). We can quantitatively evaluate

\[ 0 = \frac{d}{dr} \left[ (1 - \frac{2M}{r}) \left[ \frac{L^2}{r(r - \alpha)} \right] \right], \]

and get

\[ r = \frac{1}{4} \left( (6M + \alpha) \pm \sqrt{(6M + \alpha)^2 - 32M\alpha} \right). \]  

(2.15)

Since \( \alpha \) must be less than \( r \), to keep the line element (2.1) to be in Lorentzian metric, then for \( \alpha = 2M \) and \( \alpha > 2M \) no circular is possible. From the
relation (2.2), \( \alpha \) is smaller than \( M \) (\( \alpha < M \)), therefore the larger of two roots given by equation (2.15) locates the maximum of the potential-energy curve \( V^2(r) \) defined by equation (2.13). Consequently, the unstable circular orbit is always at the same radius \( r = \frac{1}{4}(6M + \alpha + \sqrt{(6M + \alpha)^2 - 32M\alpha}) \), regardless of \( L \). For the smaller root no circular is possible.

There are many possible ways to compare the effective potential for a massless particle in stringy charged black hole with that in Schwarzschild black hole. Here we assume \( \alpha < M \) and fix the value of angular momentum \( L \); the values of \( r \) and \( \alpha \) are calculated to obtain the plot of the effective potential. The choice of \( \alpha << M \) allows the two paths to be the same. (This comparison is shown in figures (1) - (5).)

We notice that, when \( \alpha = 0 \), that is the Schwarzschild case, the radius is \( r = 3M \) which is the same radius obtained by Schutz [13] in the Schwarzschild black hole.

3 Reissner-Nordström metric

A well-known simple solution of the Einstein-Maxwell equation is the Reissner-Nordström solution. This solution represents a non-rotating charged black hole. Discussions of the basic properties of this solution can be found in many places including works by Chandrasekhar [14], and Hawking and Ellis [15]. The metric is defined on a four-dimensional manifold and its typically written in the form

\[
 ds^2 = -(1 - \frac{2m}{r} + \frac{e^2}{r^2})dt^2 + (1 - \frac{2m}{r} + \frac{e^2}{r^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{3.1}
\]

where \( m \) represents the gravitational mass and \( e \) the electric charge of the body.

The equation of geodesics and the constraint of geodesic for the line element (3.1) are given as

\[
 \ddot{t} + \frac{2(-m + \frac{e^2}{r})}{r^2 - 2mr + e^2}r\dot{t} = 0, \tag{3.2}
\]

\[
 \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0, \tag{3.3}
\]

\[
 \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot \theta \dot{\theta}\dot{\phi} = 0, \tag{3.4}
\]

\[
 - (1 - \frac{2m}{r} + \frac{e^2}{r^2})\dot{t}^2 + (1 - \frac{2m}{r} + \frac{e^2}{r^2})^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = \varepsilon. \tag{3.5}
\]
We will assume, as section 2, that the orbit is in the $\theta = \frac{\pi}{2}$ plane. Equation (3.3) shows that if $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$ initially, then $\ddot{\theta} = 0$ and the orbit remains in this plane.

Equations (3.2) and (3.4) can be integrated directly, giving

$$\dot{t} = \frac{c_3 r^2}{r^2 - 2mr + e^2},$$  \hspace{1cm} (3.6)$$

$$\dot{\phi} = \frac{c_4}{r^2},$$  \hspace{1cm} (3.7)$$

where the integrating constant $c_3$ represents the energy, $\bar{E}$, (at $r \to \infty$) of a test particle and $c_4$ the angular momentum, $\bar{L}$.

Substituting (3.6) and (3.7) in equation (3.5) and using the condition $\dot{\theta} = 0$, we get

$$\dot{r}^2 = \bar{E}^2 - (1 - \frac{2m}{r} + \frac{e^2}{r^2})[-\varepsilon + \frac{\bar{L}^2}{r^2}].$$  \hspace{1cm} (3.8)$$

Equation (3.8) can be written as

$$\dot{r}^2 = \bar{E}^2 - \bar{V}^2(r),$$  \hspace{1cm} (3.9)$$

where $\bar{V}(r)$ is the "effective potential" defined by

$$\bar{V}^2(r) = (1 - \frac{2m}{r} + \frac{e^2}{r^2})[-\varepsilon + \frac{\bar{L}^2}{r^2}].$$  \hspace{1cm} (3.10)$$

In this paper we restrict our tension to the photon trajectories, by putting $\varepsilon = 0$ in equation (3.10), the effective potential becomes

$$\bar{V}^2(r) = (1 - \frac{2m}{r} + \frac{e^2}{r^2})[\frac{\bar{L}^2}{r^2}].$$  \hspace{1cm} (3.11)$$

Differentiating (3.9) with respect to $\tau$, as in the previous section, we get

$$\frac{d^2 r}{d\tau^2} = -\frac{1}{2} \frac{d}{dr}(\bar{V}^2(r)).$$  \hspace{1cm} (3.12)$$

This equation shows that a circular orbit ($r = \text{const.}$) is possible only at a minimum or maximum of $\bar{V}^2(r)$. We can quantitative by evaluating

$$0 = \frac{d}{dr}[(1 - \frac{2m}{r} + \frac{e^2}{r^2})[\frac{\bar{L}^2}{r^2}]$$

which gives

$$r = \frac{3m}{2} \left[1 \pm \sqrt{1 - \frac{8e^2}{9m^2}} \right].$$  \hspace{1cm} (3.13)$$
This equation shows that the two radii are identical for \( \frac{e}{m} = \frac{3\sqrt{2}}{4} \) (in this case a minimum radius for photon \( r_{\text{MIN}} = \frac{3m}{2} \)) and do not exist at all for \( \frac{e}{m} > \frac{3\sqrt{2}}{4} \). This indicates a qualitative change in the shape of the curve of \( V^2 \) for small \( \frac{e}{m} \). For \( \frac{e}{m} < \frac{3\sqrt{2}}{4} \) the larger of the two roots given by equation (3.13) locates the minimum of the potential-energy curve \( V^2 \) defined by equation (3.11), while the smaller root locates the maximum of the potential-energy curve. Therefore, the circular orbit of the larger radius will be stable in contrast to circular orbit of the smaller radius which will be unstable.

4 Geodesic Deviation

In this section, we use the tidal forces between free test particles falling in a gravitational field to investigate different properties between the stringy charged and vacuum black holes.

Consider a sphere of two non-interacting particles falling freely towards the center of the Earth. Each particle moves on a straight line, but nearer the Earth fall faster because the gravitational attraction is stronger. This means that the sphere does not remain a sphere but is distorted into an ellipsoid with the same volume. The same effect occurs in a body falling towards a spherical object in general relativity, but if the object is a black hole the effect becomes infinite as the singularity is reached. Jacobi vector fields provide the connection between the behavior of nearby particles and curvature, via the equation of geodesic deviation (Jacobi equation)

\[
\frac{D^2 \eta^a}{D\tau^2} + R^a_{\ bcd} v^b v^c \eta^d = 0, \tag{4.1}
\]

where \( v^a \) are the components of the tangent vector to geodesic and \( \eta^a \) are the components of the connecting vector between two neighboring geodesics.

In order to investigate in detail the behavior of Jacobi fields we consider a congruence of timelike geodesics (path of particles) with timelike unit tangent vector \( v \) \( (g(v, v) = -1) \). We define at some point \( q \) on the geodesic \( \gamma(\tau) \) dual bases \( e^0_0, e^1_1, e^2_2, e^3_3 \) and \( e^0_0, e^1_1, e^2_3, e^3_2 \) of the tangent space \( T_q M \) and dual tangent \( T^*_q M \) respectively in the following way [16] : We choose \( e^a_0 \) to be \( v^a \) and \( e^a_1, e^a_2, e^a_3 \) as unit spacelike vectors, orthogonal to each other and to \( v^a \). If we parallelly propagate the basis along the timelike geodesic \( \gamma(\tau) \) (that is, \( \frac{D}{D\tau} e^a_\alpha = 0, \ \alpha = 1, 2, 3 \)), \( e^a_0 \) will remain equal \( v^a \), and \( e^a_1, e^a_2, e^a_3 \) will remain to orthogonal to \( v^a \) (see [13] p. 80). The frame \( e^0_0, e^a_1, e^a_2, e^a_3 \) is called "parallel transported" (PT) frame. The orthogonal connecting vector, \( \eta^a \), between two neighboring timelike geodesics may be expressed as \( \eta^a = \eta^a e^\alpha_\alpha (\eta^0 = e^0_\alpha \eta^\alpha = 0) \).
The geodesic deviation vector \( \eta^a \) satisfy the following equation

\[
\frac{D \eta^a}{D\tau} = \eta^a_{\alpha} v^\alpha, \tag{4.2}
\]

\[
\frac{D^2 \eta^a}{D\tau^2} + \tilde{R}^a_{bde} e_c^b v^d e_c^d \eta^e = 0, \tag{4.3}
\]

where \( \eta^a \) are the space-like components of the orthogonal connecting vector \( \eta^a \) connecting two neighboring particles in free fall; \( \eta^0 = 0 \). The tilde denotes components in the PT frame and the components of the Riemann tensor \( \tilde{R}^a_{bde} \) are given by

\[
\tilde{R}^a_{bde} = e^a_c e^f_b e^g_d \Gamma^c_{fg}. \tag{4.4}
\]

From (2.1) the frame \( e^a_b \) in Reissner-Nordström metric is given by:

\[
e^0_0 = (1 - \frac{2M}{r})^{-\frac{1}{2}} (0, 0, 0, 1),
\]

\[
e^1_1 = (1 - \frac{2M}{r})^{\frac{1}{2}} (1, 0, 0, 0),
\]

\[
e^2_2 = (1 - \frac{2M}{r})^{-\frac{1}{2}} (0, 1, 0, 0),
\]

\[
e^3_3 = (1 - \frac{2M}{r})^{-\frac{1}{2}} \frac{r}{\sin \theta} (0, 0, 1, 0). \tag{4.5}
\]

The components of \( \eta^a \) can be written as follows

\[\eta^a = (\eta^1, \eta^2, \eta^3) = (\eta^r, \eta^\theta, \eta^\phi).\]

Using (4.4), (4.5), \( v^a = e^a_0 \) and the components of Riemann tensor for the metric (2.1) (see appendix), in (4.3), we get

\[
\frac{D^2 \eta^r}{D\tau^2} = \frac{2M}{r^2} \eta^r,
\]

\[
\frac{D^2 \eta^\theta}{D\tau^2} = \frac{M(\alpha - 2r)}{2r^3(r-\alpha)} \eta^\theta,
\]

\[
\frac{D^2 \eta^\phi}{D\tau^2} = \frac{M(\alpha - 2r)}{2r^3(r-\alpha)} \eta^\phi. \tag{4.6}
\]

In order to write equation (4.6) in terms of ordinary derivative, we must evaluate the second covariant derivative \( \frac{D^2}{D\tau^2} \). Using \( e^a_0 = v^a \), equation (4.2) takes the form

\[
\frac{D \eta^a}{D\tau} = \frac{D \eta^a}{d\tau} + \hat{\Gamma}^a_{ab} \eta^b v^a, \tag{4.7}
\]

where \( \hat{\Gamma}^a_{ab} = e^c_a e^f_b e^g_d \Gamma^c_{fg}. \)

Differentiating (4.7) covariantly and using the Christoffel components of metric (2.1) (see appendix), we can write (4.6) in the form
\[
\frac{d^2 \eta^r}{d\tau^2} = \frac{2M}{r^3} \eta^r, \quad (4.8)
\]
\[
\frac{d^2 \eta^\theta}{d\tau^2} = \frac{M(\alpha - 2r)}{2r^3(r - \alpha)} \eta^\theta, \quad (4.9)
\]
\[
\frac{d^2 \eta^\phi}{d\tau^2} = \frac{M(\alpha - 2r)}{2r^3(r - \alpha)} \eta^\phi. \quad (4.10)
\]

Equation (4.8) indicates that tidal force in radial direction will stretch an observer falling in this fluid. To keep the line element (2.1) to be in Lorentzian metric, \( \alpha \) should be less than \( r \). Therefore equations (4.9) and (4.10) indicate a pressure or compression in the transverse directions.

5 Solution of the equations (4.8)-(4.10)

To solve the equations (4.8)-(4.10), we consider the special case when freely falling particles have zero angular momentum \( (L = 0) \). From equation (2.10) we obtain the relation between the radial coordinate, \( r \), and the affine parameter, \( \tau \), in the following form

\[
\frac{dr}{d\tau} = -\sqrt{E^2 - 1} + \frac{2M}{r}. \quad (5.11)
\]

Using this relation in equations (4.8)-(4.10), we consider, without loss of generality, the case when \( E^2 = 1 \), we obtain

\[
\frac{d^2 \eta^r}{dr^2} - \frac{1}{r} \frac{d\eta^r}{dr} = \frac{1}{r^2} \eta^r, \quad (5.12)
\]
\[
\frac{d^2 \eta^\theta}{dr^2} - \frac{1}{2r} \frac{d\eta^\theta}{dr} = -\frac{(2r - \alpha)}{4r^2(r - \alpha)} \eta^\theta, \quad (5.13)
\]
\[
\frac{d^2 \eta^\phi}{dr^2} - \frac{1}{r} \frac{d\eta^\phi}{dr} = -\frac{(2r - \alpha)}{4r^2(r - \alpha)} \eta^\phi. \quad (5.14)
\]

Solving the above equations, using MAPLE, we obtain the space-like components of the geodesic deviation vector in the following form

\[
\eta^r = \frac{C_1}{\sqrt{r}} + C_2 r^2, \quad (5.15)
\]
\[
\eta^\theta = C_3 F_1 \left( \frac{1}{2} + b, b, -\frac{1}{2} + 2b, \frac{r}{\alpha} \right) r^b(r - \alpha) + \]

---

The text above is a natural representation of the content, ensuring that all mathematical notations and equations are correctly formatted. The bibliography references and page numbers are not included in this representation.
\[ C_4 F_2 \left( \frac{3}{2} - b, 2 - b, \frac{5}{2} - 2b, \frac{r}{\alpha} \right) r^{\left( \frac{3}{2} - b \right)} (r - \alpha), \quad (5.16) \]

\[ \eta^\phi = C_5 F_1 \left( \frac{1}{2} + b, b, -\frac{1}{2} + 2b, \frac{r}{\alpha} \right) r^b (r - \alpha) + \]

\[ C_6 F_2 \left( \frac{3}{2} - b, 2 - b, \frac{5}{2} - 2b, \frac{r}{\alpha} \right) r^{\left( \frac{3}{2} - b \right)} (r - \alpha), \quad (5.17) \]

where \( C_1, C_2, C_3 \) and \( C_4 \) are constants of integration and \( b = \frac{3}{4} \pm \frac{\sqrt{5}}{4} \). \( F_1 \) and \( F_2 \) are hypergeometric functions.

The spacial solution given by (5.15)-(5.17) allows one, by a suitable choice of initial conditions and constants appearing in (4.8)-(4.10), to describe various physical situations connected with relative motion of freely falling particles.

At the singularity \( r = 0 \) the radial component \( \eta^r \) of geodesic deviation vector becomes infinite, while the transverse components \( \eta^\theta \) and \( \eta^\phi \) vanish. Therefore, the behavior of geodesic deviation vector in the space-time under consideration is the same as in the case of Schwarzschild space-time, when the singularity, \( r = 0 \), is approached. In fact the Schwarzschild singularity has time-like geodesic for which one component of the geodesic deviation vector becomes infinite while the other two vanish \[11\]. The singularity \( r = 0 \) is a strong curvature singularity \[3\], as the Schwarzschild singularity \[26\], for the two vanishing components of geodesic deviation vector go to zero faster than the remaining component becomes infinite. The misbehavior of these components could in concert so that it is not visible in the volume element magnitude.

At the singularity \( r = \alpha \), the component \( \eta^r \) has a finite value, while the other two components, \( \eta^\theta \) and \( \eta^\phi \), vanish. Then the volume element, defined by the three space-like components of geodesic deviation vector \[25\], \[26\], along a time-like geodesic vanishes as the geodesic approaches this singularity.

### 6 Conclusion

In this paper we have studied the circular photon orbits in charged black holes by analyzing the properties of effective potential. Considering the light-like geodesics, we classified and analyzed the different cases between the stringy charged black hole and the vacuum solutions. These differences arose from considering the orbits associated with stable and unstable circular orbits. In the context of the Schwarzschild geometry there is an unstable circular orbit which is always at the same radius, \( r = 3M \). In the Reissner-Nordström

\[^3\text{A singular point is called a strong curvature singularity if any object hitting it is crushed to zero volume}[26].\]
space-time there are two radii, the circular orbit of the larger radius will be stable while that of the smaller radius will be unstable. In the case of stringy charged black hole there is an unstable circular orbit as the Schwarzschild black hole, but the difference is that the radius in the case of stringy charged black hole depends on the charge $Q$.

Equations (4.8)-(4.10) provide the explicit expressions of the relative accelerations in a stringy charged black hole. Two comments are worth making about expression (4.8). First there is no divergence in the radial direction at $r = 2m$. Secondly, the tidal field at the horizon in radial direction is larger for smaller black hole. This is simply because

$$
d^2 \eta^r \sim \frac{2M}{r^3} \eta^r \sim \frac{1}{M^2} \eta^r \quad \text{at} \quad r \sim M.
$$

The radial component of the geodesic deviation vector field, equation (4.8), is the same as the component obtained in the case of Schwarzschild black hole and different from the component obtained in the case of Reissner-Norström. Consequently, in the case of stringy charged and Schwarzschild black holes, the tidal forces in the radial direction will stretch an observer falling in these black holes, while in the case of Reissner-Norström the radial component depend on the quantity $2e^2 - 2mr$ to indicate a tension or stretching in the radial direction. In the transverse directions the relative acceleration components (see equations (4.9) and (4.10)) indicate a compression in these directions. This property is similar as in the case of Schwarzschild black hole, while in the case of Reissner-Nordström depends on the quantity $e^2 - 2mr$ to indicate compression or tension in these directions.

To describe various physical situations connected with relative motion of freely falling photons, in stringy charged black hole, we tried to solve equations (4.8)-(4.10) by considering a special case. We found that any object hitting the singularity $r = \alpha$ is crushed to zero volume.

Appendix

We use $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$ so that the non-vanishing Christoffel symbols of the second kind of the line element (2.1) are

$$
\begin{align*}
\Gamma^1_{11} &= \frac{M}{r^2(2M-r)}, & \Gamma^2_{12} &= \frac{\alpha-2r}{r(\alpha-r)}, \\
\Gamma^1_{22} &= \frac{2}{r(2M)(\alpha-2r)}, & \Gamma^3_{13} &= \frac{\alpha-2r}{2(r(\alpha-r))}, \\
\Gamma^1_{33} &= \frac{(r-2M)(\alpha-2r)}{2r^2} \sin^2 \theta, & \Gamma^2_{33} &= -\sin \theta \cos \theta, \\
\Gamma^0_{00} &= \frac{\alpha}{M(r-2M)}, & \Gamma^3_{23} &= \cot \theta, \\
\Gamma^0_{10} &= \frac{\alpha}{r(r-2M)}. &
\end{align*}
$$
The non-zero Riemann tensor are:

\[
R_{1212}^1 = 2M(r-a)(2r-a)+a^2(2M-r) \frac{4r^2(r-a)}{4r^2(a-r)}
\]

\[
R_{313}^1 = 2M(r-a)(2r-a)+a^2(2M-r) \frac{4r^2(a-r)}{4r^2(a-r)} \sin^2 \theta
\]

\[
R_{010}^1 = 2M(2M-r) \frac{r}{4r^2(a-r)}
\]

\[
R_{323}^2 = 8Mr(r-a)-a^2(r-2M) \frac{4r^2(r-a)}{4r^2(r-a)} \sin^2 \theta
\]

\[
R_{620}^2 = R_{030}^3 = \frac{M(a-2r)(2M-r)}{2r^4(r-a)}
\]

References

[1] J. D. Bekenstein, Ann. Phys. (N. Y.), 91, 75 (1975).

[2] E. Ayon-Beato and A. Garcia, Gen. Relat. Grav. 31, 629 (1999).

[3] D. Garfinkle, G. T. Horowitz and A. Strominger, Phys. Rev. D43, 3140 (1991); D45, 3888(E) (1992).

[4] C. Möller, Ann. Phys. (N. Y.), 4, 347 (1958).

[5] R. M. Gad, Astrophys. Space Sci. 295, 459 (2004).

[6] S. S. Xulu, Astrophys. Space Sci. 283, 23 (2003).

[7] J. Preskill, P. Schwarz, A. Shapere, S. Trivedi and F. Wilczek, Mod. Phys. Lett., 6, 2353 (1991).

[8] J. A. Harvey and A. Strominger, "Quantum aspects of black holes"; Preprint EFI-92-41, [hep-th/9209055](http://arxiv.org/abs/hep-th/9209055)

[9] F. A. E. Piran, Phys. Rev. 105, 1089 (1957).

[10] K. P. Tod, Proc. R. Soc. Lond. A388, 467 (1983).

[11] C. Misner, K. Thorne and J. Wheeler, (1973), "Gravitation", Freeman, San Francisco.

[12] R. Adler, M. Bazin and M. Schiffer, (1975), "Introduction to General Relativity", (McGraw-Hill, New York, 2nd ed.).

[13] B. F. Schutz, (1985), "A First Course in General Relativity" (Cambridge Uni. Press, Cambridge, London, New York, New Rochelle, Melbourne Sydney).
[14] S. Chandrasekhar, (1983), "The Mathematical Theory of Black Holes", (Oxford Uni. Press, Cambridge, England).

[15] S. W. Hawking and G. F. R. Ellis, "The Larger Scale Structure of Spacetime", (Cambridge Uni. Press, Cambridge).

[16] R. D’Inverno, "Introducing Einstein’s Relativity", Oxford University Press, New York, (1992).

[17] M. Abdel-Megied and R. M. Gad, Chaos, Solitons and Fractals, 23, 313 (2005).

[18] R. Kerner, J. Martin, S. Mignemi and J. E. Van Holten, Phys. Rev. D 63, 027502 (2001).

[19] G. F. R. Ellis and H. van Elst, "Deviation of geodesics in FLRW spacetime geometries", arXiv: gr-qc/9709060.

[20] R. Koley, S. Pal and S. Kar, Am. j. Phys., 71, 1037 (2003)

[21] S. Ghosh and Sayan Kar, "Geodesics and geodesic deviation in warped spacetime with a time dependent extra dimension", arXiv:0904.2321v1[gr-qc].

[22] A. Shapere, S. Trivedi and F. Wilczek, Mod. Phys. Lett. A6, 2677 (1991).

[23] C. J. S. Clark, (1979), "Elementary General relativity", (Edward Arnold, London).

[24] R. M. Wald, (1984), "General Relativity", (Chicago and London).

[25] R. Penrose, (1972), "Techniques of Differential Topology in Relativity", (SIAM, Philadelphia, p. 60).

[26] F. J. Tipler, Phys. Lett. 64A, 8 (1977).