Phantom crossing and quintessence limit in extended nonlinear massive gravity

Emmanuel N Saridakis

Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece
CASPER, Physics Department, Baylor University, Waco, TX 76798-7310, USA
Institut d’Astrophysique de Paris, UMR 7095-CNRS, Université Pierre & Marie Curie, 98bis boulevard Arago, F-75014 Paris, France
E-mail: Emmanuel_Saridakis@baylor.edu

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Abstract
We investigate the cosmological evolution in a universe governed by the extended, varying-mass, nonlinear massive gravity, in which the graviton mass is promoted to a scalar field. We find that the dynamics, both in flat and open universe, can lead the varying graviton mass to zero at late times, offering a natural explanation for its hugely constrained observed value. Despite the limit of the scenario toward standard quintessence, at early and intermediate times it gives rise to an effective dark-energy sector of a dynamical nature, which can also lie in the phantom regime, from which it always exits naturally, escaping a Big Rip. Interestingly enough, although the motivation of massive gravity is to obtain an IR modification, its varying-mass extension in cosmological frameworks leads to early and intermediate times modification instead.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The idea of adding mass to the graviton is quite old [1], but the necessary nonlinear terms [2] that can give rise to the continuity of the observables [3, 4] lead also to Boulware–Deser (BD) ghosts [5], making the theory unstable. However, recently, a nonlinear extension of massive gravity has been constructed [6, 7] such that the BD ghost is systematically removed (see [8] for a review). The theoretical and phenomenological advantages, amongst which is the universe self-acceleration arising exactly from this IR gravity modification, brought this theory to a significant attention [9–51].

Despite the successes of massive gravity, in the case where the physical and the fiducial metrics have simple homogeneous and isotropic forms the theory proves to be unstable at the
perturbation level [40], which led some authors to start constructing less symmetric models [13, 41]. However, in [52] a different approach was followed, that is expected to be free of the above instabilities, namely to extend the theory in a way that the graviton mass is varying, and this was achieved by introducing an extra scalar field which coupling to the graviton potentials produces an effective, varying, graviton mass.

In this work, we desire to explore the cosmological implications of this ‘extended’, varying-mass, massive gravity, in both flat and open universes. As we show, at least in simple cosmological ansatz, the dynamics leads the varying graviton mass to zero, or to a suitably chosen very small value in agreement with observations, at late times, and thus the theory has as a limit the standard quintessence paradigm. However, at intermediate times the varying graviton mass leads to very interesting behavior, with a dynamical effective dark-energy sector which can easily lie in the phantom regime. Strictly speaking, although the motivation of massive gravity is to obtain an IR modification, its extension in cosmological frameworks leads rather to early and intermediate times modification, and thus to a radical UV modification instead.

2. Extended nonlinear massive gravity

Let us briefly review the ‘mass-varying massive gravity’ that was recently presented in [52]. Their construction is based on the promotion of the graviton mass to a scalar-field function (potential), with the additional insertion in the action of this scalar field’s kinetic term and standard potential. Since such a modification is deeper than allowing for a varying mass, we prefer to call it ‘extended’ nonlinear massive gravity.

In such a construction, the action is written as

\[ S = \int \mathcal{D}x \sqrt{-g} \left[ \frac{M_p^2}{2} R + V(\psi)(U_2 + \alpha_3 U_3 + \alpha_4 U_4) - \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - W(\psi) \right], \]  

(2.1)

where \( M_p \) is the Planck mass, \( R \) is the Ricci scalar and \( \psi \) is the new scalar field with \( W(\psi) \) being its standard potential and \( V(\psi) \) its coupling potential which spontaneously breaks general covariance. Furthermore, as usual \( \alpha_3 \) and \( \alpha_4 \) are dimensionless parameters, and the graviton potentials are given by

\[ U_2 = \mathcal{K}_{\mu}^\nu \mathcal{K}_{\nu}^\mu, \quad U_3 = \mathcal{K}_{\mu}^\nu \mathcal{K}_{\nu}^\rho \mathcal{K}_{\rho}^\mu, \quad U_4 = \mathcal{K}_{\mu}^\nu \mathcal{K}_{\nu}^\rho \mathcal{K}_{\rho}^\sigma \mathcal{K}_{\sigma}^\mu, \]  

(2.2)

with \( \mathcal{K}_{\mu}^\nu \) being a symmetric tensor. For example, \( \mathcal{K}_{\mu}^\nu \) is given by

\[ \mathcal{K}_{\mu}^\nu = \frac{\delta^\mu_\nu - g^\mu_\nu f_{AB} \partial_\nu \phi^A \partial_\mu \phi^B}{\sqrt{g}}. \]  

(2.3)

As in standard massive gravity \( f_{AB} \) is a fiducial metric, and the four \( \phi^A(x) \) are the St"uckelberg scalars introduced to restore general covariance [53], and in the particular case where the \( f_{AB} \) is the Minkowski metric they form Lorentz 4-vectors in the internal space and the theory presents a global Poincaré symmetry, too. Finally, one can show that the above-extended massive gravity is still free from the BD ghost [52].

3. Cosmological equations

Let us now examine cosmological scenarios in a universe governed by the extended nonlinear massive gravity. Firstly, in order to obtain a realistic cosmology one includes the usual matter
action $S_m$, coupled minimally to the dynamical metric, corresponding to energy density $\rho_m$ and pressure $p_m$. Now, for simplicity, we consider the fiducial metric to be Minkowski\(^1\)

$$f_{AB} = \eta_{AB},$$

and without loss of generality we assume that the dynamical and fiducial metrics are diagonalized simultaneously. For the dynamical metric, one can either consider for simplicity a flat Friedmann–Robertson–Walker (FRW) form or he/she can apply an open geometry. In the following two subsections, we examine these two cases separately.

### 3.1. Flat universe

We consider a flat FRW physical metric of the form

$$d^2s = -N(\tau)^2 \, d\tau^2 + a(\tau)^2 \delta_{ij} \, dx^i \, dx^j,$$

with $N(\tau)$ being the lapse function and $a(\tau)$ the scale factor, and for simplicity for the St"uckelberg fields we choose the ansatz

$$\phi^0 = b(\tau), \quad \phi^i = a_{\text{ref}} x^i,$$

with $a_{\text{ref}}$ being a constant coefficient. Although the above specific application is only a simple subclass of the rich set of possible scenarios, it proves to exhibit very interesting cosmological behavior.

The variation of the total action $S + S_m$ with respect to $N$ and $a$ provides the two Friedmann equations\(^{[52]}\):

$$3M_P^2 H^2 = \rho_{DE} + \rho_m,$$

$$-2M_P^2 \dot{H} = p_{DE} + \rho_{DE} + \rho_m + p_m,$$

where we have defined the Hubble parameter $H = \dot{a} / a$, with $\dot{a} = da / (N \, d\tau)$, and in the end we set $N = 1$. In the above expressions, we have defined, respectively, the energy density and pressure of the effective dark-energy sector as

$$\rho_{DE} = \frac{1}{2} \dot{\psi}^2 + W(\psi) + V(\psi) \left( \frac{a_{\text{ref}}}{a} - 1 \right) \left[ f_3(a) + f_4(a) \right]$$

$$p_{DE} = \frac{1}{2} \dot{\psi}^2 - W(\psi) - V(\psi) f_3(a) - V(\psi) \dot{b} f_1(a),$$

having also introduced the convenient functions

$$f_1(a) = 3 - \frac{2 a_{\text{ref}}}{a} + a_3 \left( 3 - \frac{a_{\text{ref}}}{a} \right) \left( 1 - \frac{a_{\text{ref}}}{a} \right)^2 + a_4 \left( 1 - \frac{a_{\text{ref}}}{a} \right)^3$$

$$f_2(a) = 1 - \frac{a_{\text{ref}}}{a} + a_3 \left( 1 - \frac{a_{\text{ref}}}{a} \right)^2 + \frac{a_4}{3} \left( 1 - \frac{a_{\text{ref}}}{a} \right)^3$$

$$f_3(a) = 3 - \frac{a_{\text{ref}}}{a} + a_3 \left( 1 - \frac{a_{\text{ref}}}{a} \right)$$

$$f_4(a) = -\left[ \frac{6 - 6 a_{\text{ref}}}{a} + \left( \frac{a_{\text{ref}}}{a} \right)^2 + a_3 \left( 1 - \frac{a_{\text{ref}}}{a} \right) \left( 4 - \frac{2 a_{\text{ref}}}{a} \right) + a_4 \left( 1 - \frac{a_{\text{ref}}}{a} \right)^2 \right].$$

Note that from the above expressions we observe that $a_{\text{ref}}$ plays the role of a reference scale factor that can be arbitrary.

\(^1\) Note that this case includes the subclasses where $f_{AB}$ can be brought to the Minkowski metric by general coordinate transformation, as we can always choose a gauge for the St"uckelberg fields $\phi^A$\(^{[52]}\).
One can easily verify that the dark-energy density and pressure satisfy the usual evolution equation
\[ \dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0, \] (3.9)
and we can also define the dark-energy equation-of-state parameter as usual as
\[ w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}}. \] (3.10)

Note that in [52] the authors had named the aforementioned ‘dark-energy’ sector as ‘massive gravity’ one, and the quantities \( \rho_{DE} \) and \( p_{DE} \) as \( \rho_{MG} \) and \( p_{MG} \). However, since in this work we focus to late-time cosmological behavior, we prefer the above name.

The variation of the total action \( S + S_m \) with respect to \( \psi \) provides the scalar-field evolution equation
\[ \ddot{\psi} + 3H \dot{\psi} + \frac{dW}{d\psi} + \frac{dV}{d\psi} \left( \frac{\dot{a}_{\text{ref}}}{a} - 1 \right) [f_3(a) + f_1(a)] + 3bf_2(a) = 0. \] (3.11)

Furthermore, the variation of \( S + S_m \) with respect to \( b \) provides the constraint equation
\[ V(\psi)Hf_1(a) + \dot{V}(\psi)f_2(a) = 0. \] (3.12)

Finally, one can also extract the matter evolution equation
\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0. \] (3.13)

### 3.2. Open universe

Let us now consider an open\(^2\) FRW physical metric of the form
\[ d^2s = -N(\tau)^2 d\tau^2 + a(\tau)^2 \delta_{ij} dx^i dx^j - a(\tau)^2 \frac{k^2(\delta_{ij}x^i x^j)^2}{1 + k^2(\delta_{ij}x^i x^j)}, \] (3.14)
with \( N(\tau) \) being the lapse function and \( a(\tau) \) the scale factor, and \( K < 0 \) with \( k = \sqrt{|K|} \). For simplicity, for the St"uckelberg fields we choose [52]
\[ \phi^0 = b(\tau)\sqrt{1 + k^2(\delta_{ij}x^i x^j)}, \quad \phi^i = kb(\tau)x^i. \] (3.15)

Variations of the action with respect to \( N \) and \( a \) give rise to the following Friedmann equations:
\[ 3M_p^2 \left( H^2 - \frac{k^2}{a^2} \right) = \rho_{DE} + \rho_m, \] (3.16)
\[ -2M_p^2 \left( \dot{H} + \frac{k^2}{a^2} \right) = \rho_{DE} + p_{DE} + \rho_m + p_m, \] (3.17)
where the effective dark-energy density and pressure are given by
\[ \rho_{DE} = \frac{1}{2} \dot{\psi}^2 + W(\psi) + V(\psi) \left( \frac{kb}{a} - 1 \right) [f_3(a) + f_1(a)] \] (3.18)
\[ p_{DE} = \frac{1}{2} \dot{\psi}^2 - W(\psi) - V(\psi)f_3(a) - V(\psi)bf_1(a), \] (3.19)

\(^2\) Similar to usual massive gravity, closed FRW solutions are not possible since the fiducial Minkowski metric cannot be foliated by closed slices [16, 52].
but now the functions become

\[
\begin{align*}
    f_1(a) &= 3 - \frac{2kb}{a} + \alpha_3 \left( 3 - \frac{kb}{a} \right) \left( 1 - \frac{kb}{a} \right) + \alpha_4 \left( 1 - \frac{kb}{a} \right)^2 \\
    f_2(a) &= 1 - \frac{kb}{a} + \alpha_3 \left( 1 - \frac{kb}{a} \right)^2 + \frac{\alpha_4}{3} \left( 1 - \frac{kb}{a} \right)^3 \\
    f_3(a) &= 3 - \frac{kb}{a} + \alpha_3 \left( 1 - \frac{kb}{a} \right) \\
    f_4(a) &= -\left[ 6 - 6 \frac{kb}{a} + \left( \frac{kb}{a} \right)^2 + \alpha_3 \left( 1 - \frac{kb}{a} \right) \left( 4 - \frac{2kb}{a} \right) + \alpha_4 \left( 1 - \frac{kb}{a} \right)^2 \right].
\end{align*}
\]

These verify the usual evolution equation

\[
\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0.
\]

The variation of the action with respect to the scalar field \( \psi \) provides its evolution equation

\[
\dot{\psi} + 3H\dot{\psi} + \frac{dW}{d\psi} + \frac{dV}{d\psi} \left\{ \left( \frac{kb}{a} - 1 \right) [f_3(a) + f_1(a)] + 3bf_2(a) \right\} = 0.
\]

Finally, variation with respect to \( b \) provides the constraint equation

\[
V(\psi) \left( H - \frac{k}{a} \right) f_1(a) + V(\psi) f_2(a) = 0.
\]

4. Cosmological behavior

The cosmological implications of extended nonlinear massive gravity prove to be very interesting; however, at least in its present simple but general example, it can be radically different than the usual massive gravity. In the following two subsections, we examine the flat and open geometries separately.

4.1. Flat universe

In the case of a flat FRW universe, the cosmological equations are (3.4), (3.5) or (3.11) and (3.12), and the reason that these equations lead to a different behavior comparing to the usual massive gravity is the constraint equation (3.12). In order to elaborate the equations, we have to consider at will \( W(\psi) \) and \( V(\psi) \) and solve the equations to obtain \( a(\tau), \psi(\tau) \) and \( b(\tau) \), that is, the Stückelberg scalars are suitably reconstructed in order to correspond to a consistent solution.

A crucial observation is that for \( f_2(a) \neq 0 \) (which is the case in general) the constraint equation (3.12) can be explicitly solved giving\(^3\)

\[
V(a) = C_0 e^{\int \frac{db}{a^2}(\alpha_3 a^2 - (3\alpha_3 + 2\alpha_4) a_0 + (3 + 3\alpha_3 + \alpha_4) a^2)},
\]

where we have used definitions (3.8), with \( C_0 \) being a positive integration constant. Thus, since from the known \( V(\psi) \) we can straightforwardly obtain \( \psi(V) \) as a function of \( V \), relation (4.1) eventually provides \( \psi(a) \). Then one can insert the known \( \psi(a) \) into the Friedmann equation (3.4) which becomes a simple differential equation for \( a(\tau); \langle b(\tau) \rangle \) does not appear

\(^3\) The importance of the constraint equation (3.12) was not revealed in [52], where all the specific examples that the authors considered were exactly those fine-tuned parameter choices that lead to \( f_1(a) = f_2(a) = 0 \) and thus to a trivial satisfaction of the constraint (3.12).
Finally, with $a(\tau)$ known and therefore $\psi(a(\tau))$ known, one can use (3.11) to find $\dot{b}$ as

$$
\dot{b}(\tau) = \frac{1}{3 f_2(a(\tau))} \left\{ -\frac{\dot{\psi}(\tau) + 3H(\tau)\psi(\tau) + \frac{dV}{d\psi}(\tau)}{\frac{d\psi}{d\tau}(\tau)} - \left( \frac{a_{\text{ref}}}{a(\tau)} - 1 \right) \left[ f_3(a(\tau)) + f_1(a(\tau)) \right] \right\},
$$

(4.2)

integration of which provides the St"uckelberg scalar function $b(\tau)$. (Note however that in the observables it is $\dot{b}$ and not $b$ that appears.)

The first observation that one can immediately make from (4.1) is that in general at late times the graviton mass always goes to zero, independent of the specific $V(\psi)$ and the model parameters, that is, the evolution of $\psi$ will be such, in order for $V(\psi)$ to go to zero. (If $V(\psi)$ cannot be zero for any $\psi$, then the scenario will break down at some scale factor, since $\psi$ would need to be complex, that is, a solution can no longer be found.) This means that the present scenario of extended nonlinear massive gravity, in a cosmological framework of a flat universe, cannot provide the usual massive gravity, and in contrast it always gives the standard gravity along with the standard quintessence scenario [54, 55]. Similarly, once introduced, the scalar field cannot be set to zero by hand, since this is not a solution of (3.11) and (3.12) (unless we also set $V(\psi) = 0$ but in this case the model coincides completely with standard quintessence), that is, $\psi$ will always have a non-trivial dynamics.

However, although at late times the present scenario coincides with standard quintessence, it can have a very interesting behavior at intermediate times. In particular, the dark-energy sector is not only dynamical, but it can easily lie at the phantom regime [56–61]. This can be seen by observing $\rho_{\text{DE}}$ and $p_{\text{DE}}$ from (3.6) and (3.7), respectively, which using the constraint equation (3.12) give

$$
\rho_{\text{DE}} + p_{\text{DE}} = \dot{\psi}^2 - V(\psi) \left( \dot{b} - \frac{a_{\text{ref}}}{a} \right) f_1(a).
$$

(4.3)

So we can always find regions in the $\alpha_3, \alpha_4$ parameter space that can lead to $p_{\text{DE}} + \rho_{\text{DE}} < 0$ at some stage of evolution (with a potential $W(\psi)$ that will not lead to large $\psi$), even if we require to always have $\rho_{\text{DE}} > 0$ (which does not need to be the case in general). This null energy condition violation is always canceled at late times, where the vanishing of the graviton mass leads to $w_{\text{DE}} \geq -1$.

From the above discussion however one can see that despite the interesting cosmological behavior, in the flat case there is a potential disadvantage, namely that the graviton square mass, as it is given by (4.1), diverges and changes sign at least for one finite scale factor independent of the model parameters. (Even if we choose $\alpha_3, \alpha_4$ in order for the second term in the denominator not to have roots, there is always the point $a(\tau) = a_{\text{ref}}$.4 A negative graviton square mass would make the scenario unstable at the perturbation level and thus its application meaningless; therefore, we desire the observable universe evolution to take place in the regime $V(\psi) \geq 0$. In order to avoid a collapse of the scenario in the future (choosing $a_{\text{ref}}$ larger than the present scale factor) in the following we prefer to choose it suitably small in order not to interfere with the observed thermal history of the universe ($a_{\text{ref}} \lesssim 10^{-9}$ in order to be smaller than the Big Bang nucleosynthesis scale factor). Note also that one could additionally ‘shield’ $a_{\text{ref}}$ with a cosmological bounce, the case in which the universe is always away from it [62], or even choose $a_{\text{ref}}$ to be negative. However, these considerations can

4 Note that in the case where $V(\psi)$ is imposed to be non-negative, the negativiry of $V(a)$ from (4.1) would demand the scalar field to be complex and thus the model can no longer have consistent solutions, too.
only cure the problem phenomenologically, while at the theoretical level it remains unsolved. Clearly, the scenario of a flat universe has serious disadvantages and thus one should look for a more general solution through its generalizations. This will be performed in the next subsection where the addition of curvature makes the graviton mass square always positive. However, for completeness we provide in the present subsection the phenomenologically (but not theoretically) consistent flat analysis, too.

In order to present the above behavior in a more transparent way, we consider without loss of generality the graviton mass potential to be
\[ V(\psi) = V_0 e^{-\lambda_V \psi}, \] (4.4)
and the usual scalar-field potential
\[ W(\psi) = W_0 e^{-\lambda_W \psi}. \] (4.5)
In this case \( \psi(a) = -\ln(V(a)/V_0)/\lambda_V \), with \( V(a) \) being given by (4.1), and thus substitution into (3.4) gives a differential equation that can be easily solved numerically to give \( a(\tau) \), while insertion into (4.2) provides \( b \) and therefore all the observables are known. In figure 1, we present the effective dark-energy equation-of-state parameter \( w_{DE} \) as a function of the redshift \( z = a_0/a - 1 \) (with \( a_0 \) being the present scale factor set to 1), with the reference scale factor \( a_{ref} \) set to \( 10^{-9} \), and assuming the matter to be dust (\( w_m \equiv \rho_m/\rho_m = 0 \), that is, \( \rho_m(a) = \rho_{m0}/a^3 \), with \( \rho_{m0} \) being the energy density at present). The parameters \( \alpha_3, \alpha_4, V_0, W_0, \lambda_V, \lambda_W \) are chosen at will\(^5\) (concerning \( \alpha_3 \) and \( \alpha_4 \), we have to ensure that they lead to a positive graviton square mass, that is, especially to a positive last term in the denominator of (4.1)), while we fix \( \rho_{m0} \) and the integration constant \( C_0 \) in order for the present dark-energy density \( \Omega_{DE} \equiv \rho_{DE}/(3M^2_P H^2) \) to be \( \approx 0.72 \) and its initial value to be \( \approx 0 \). (Concerning the observables no more condition is needed since it is \( b \) and not \( b \) that appears in the corresponding relations;\(^5\)

\(^5\) Note that the graviton mass and the usual potential are significantly downgraded by the \( \psi \)-dynamics and thus they are far below \( M^4_P \) even if \( V_0 \) and \( W_0 \) are chosen larger than \( M^4_P \).
however, if one desires to obtain $b(\tau)$ too then he needs to impose an extra condition, for instance the present $b$-value.)

As described above, at early and intermediate times the coupling potential $V(\psi)$ is non-zero leading $w_{\text{DE}}$ to exhibit a dynamical nature, which can lie in the quintessence regime (black-solid curve) or in the phantom regime (red-dashed curve). Additionally, as we said, at late times, where the coupling $V(\psi)$ becomes zero, both sub-cases tend to their usual quintessence limit, where the final $w_{\text{DE}}$ is determined solely from the $W$-potential exponent $\lambda_W$ [63], with the second model experiencing the phantom-divide crossing from below to above.

In summary, as we can see the scenario at hand exhibits very interesting cosmological behavior at early and intermediate times, with a dynamical dark-energy sector which can additionally lie in the phantom regime, before it is limited toward the standard quintessence scenario. Note that despite the phantom realization, at late times we always obtain $w_{\text{DE}} \geq -1$ since the vanishing of the graviton mass restores the null energy condition for the effective dark-energy sector, that is, the universe will always escape from the phantom regime and the Big-Rip future [64, 65] that is common to the majority of phantom models.

However, as we mentioned, the above flat scenario has two significant disadvantages. The first is that not all ansatz for $V(\psi)$ can lead to consistent solutions at all times, since the field $\psi$ would need to become complex at some scale factor, that is, the theory breaks down. Secondly, the appearance of $a_{\text{ref}}$ in the equations leads to scale-factor regions where the graviton mass square becomes negative, and thus the theory becomes unstable at the perturbation level. Although one can still cure the above problems at the phenomenological level, and move them away from the observed universe history, clearly a generalization of the scenario is necessary in order to completely remove these disadvantages. This is performed if one goes beyond the flat case, as we analyze in the next subsection.

4.2. Open universe

In the previous section, we investigated extended massive gravity in the case of a flat FRW universe and saw that the resulting cosmological behavior can be very interesting. Although we chose the reference scale factor $a_{\text{ref}}$ to be suitably small in order for the graviton mass square to be always positive during the observed universe history, it is desirable to consider a generalization of the scenario, where the potential problem of the graviton mass square negativity will be completely absent. This is obtained by applying extended massive gravity in a non-flat geometry.

In the case of an open FRW universe, the cosmological equations are (3.15), (3.16) or (3.21) and (3.22). (Note that in this case there is no need for a reference scale factor, since it has been absorbed inside $b(\tau)$.) One difference comparing to the flat case is that the constraint equation (3.22) cannot be solved analytically and thus it has to be considered along the other cosmological equations. Although this brings an additional mathematical complexity, it offers a great physical advantage, since the constraint satisfaction can be obtained by significantly larger solution subclasses, and therefore one can always, and in general, find solutions where the graviton mass square is always positive and finite. Similar to the flat case, in the following we consider at will the usual scalar-field potential $W(\psi)$ and the coupling potential $V(\psi)$ and solve the equations to obtain $a(\tau), \psi(\tau)$ and $b(\tau)$.

Let us consider known forms for $W(\psi)$ and $V(\psi)$. Due to the constraint dependence on $b(\tau)$ it cannot be solved alone, and thus one needs to solve the whole system of equations simultaneously. Since this is not analytically possible, we proceed to a numerical elaboration of a specific example. In particular, we first solve algebraically (and analytically if it is possible)
the constraint (3.22) in order to extract \( b(\tau) \) as a function of \( a(\tau), \dot{a}(\tau), \psi(\tau), \dot{\psi}(\tau) \), and then substituting the resulting (quite complicated) expression into (3.15) and (3.16), we obtain two differential equations for \( a(\tau) \) and \( \dot{\psi}(\tau) \) that do not depend on \( b(\tau) \), which can be numerically solved. Note that contrary to the flat case, \( V(\psi) \) does not need to be able to become zero at some \( \psi \) in order for the equations to be solvable; however, for phenomenological reasons we do consider it to be able to reach zero or very small values chosen at will and in agreement with experimental bounds (thus in this case one can re-obtain the usual non-flat massive gravity, where the graviton mass is very small but non-zero).

We choose both \( V(\psi) \) and \( W(\dot{\psi}) \) to have the exponential forms (4.4) and (4.5), namely
\[
V(\psi) = V_0 \mathrm{e}^{-\lambda \psi} \quad \text{and} \quad W(\dot{\psi}) = W_0 \mathrm{e}^{-\lambda W \dot{\psi}},
\]
respectively, although we could still add a constant in \( V(\psi) \), suitably small in order to be consistent with experimental bounds. We evolve the system numerically, using the redshift \( z = a_0/a - 1 \) as the independent variable (with \( a_0 = 1 \) the present scale factor) and assuming dust matter (\( \rho_m = \rho_{m0}/a^3 \), with \( \rho_{m0} \) the present energy density). The parameters \( a_3, a_4, V_0, \lambda_W, W_0 \) and \( \lambda_W \) are chosen at will, while we fix \( k \) in order for the present curvature density parameter (\( \Omega_k = k^2/(a^2 H^2) \)) to be 0.01, and we fix the present values \( \rho_{m0}, \dot{\psi}_0, \dot{\psi}_0 \) and \( a_0 \) in order for the present dark-energy density \( \Omega_{DE} = \rho_{DE}/(3M_p^2 H^2) \) to be \( \approx 0.72 \), its initial value to be \( \approx 0 \) and the present dark-energy equation-of-state parameter to be between \(-0.9 \) and \(-1 \) in agreement with observations.

In figure 2, we present \( w_{DE} \) as a function of \( z \), for two choices of the parameters. As we observe, at early and intermediate times the coupling potential \( V(\psi) \) is non-zero leading \( w_{DE} \) to exhibit a dynamical nature, which can lie in the quintessence regime (black-solid curve) or in the phantom regime (red-dashed curve), and it can cross the phantom divide from below to above, before asymptotically it is limited toward the usual quintessence scenario. This behavior is similar to the flat universe; however, as we mentioned, in the present case, the graviton mass square is always finite and positive, independent of the specific solution.
5. Discussion

In this work, we investigated the cosmological evolution in a universe governed by the extended, varying-mass, nonlinear massive gravity. Even for simple ansätze the scenario proves to have a very interesting behavior, compared with standard massive gravity.

The first result is that the dynamics in cosmological frameworks can lead the varying graviton mass to zero at late times, both in flat and open geometries (in the open case one can also obtain at will a non-zero but suitably small value if he/she correspondingly chooses the coupling potential), and thus the theory possesses as a limit the standard quintessence paradigm. This is a great advantage of the present construction, since it offers a natural explanation of the tiny and hugely constrained graviton mass that arises from current observations. The graviton mass does not have to be tuned to an amazingly small number, as it is the case in standard massive gravity, but it is the dynamics that can lead it asymptotically to zero. Additionally, although in the simple flat case one may face the problem of a divergent or negative graviton mass square, which should be then shielded by a cosmological bounce, in the non-flat scenario the graviton mass square is always finite and positive, independent of the specific solution.

Despite the vanishing of the graviton mass at late times, and the limit of the scenario toward standard quintessence, at early and intermediate times it can lead to very interesting behavior. In particular, it can give rise to an effective dark-energy sector of a dynamical nature, which can also lie in the phantom regime. The violation of the null energy condition for the effective dark-energy sector at intermediate times arises naturally for suitable (not fine-tuned) regions in the Lagrangian parameters, and it is always canceled at late times due to the vanishing of the graviton mass. These features are in agreement with observations and they offer an explanation for the dynamical evolution of the dark-energy equation-of-state parameter, for its relaxation close or at the cosmological constant value, and also for the indicated possibility of having crossed the phantom divide. Moreover, even if it enters the phantom regime, the scenario at hand always returns naturally to the quintessence one, offering a solution to the Big-Rip fate of the standard phantom scenarios. The complete investigation of the possible late-time behaviors is performed in [66] through a detailed dynamical analysis.

We mention here that although we performed the above analysis with the fiducial metric to be Minkowski, and with specific ansätze for the potentials and the Stückelberg scalars, qualitatively the obtained behavior is not a result of them, but it arises from the deeper structure of the theory, namely from the scalar-field coupling to the graviton potential. Thus, we do not expect the results to change in more general cases, unless one fine-tunes the theory.

In the above analysis, we remained at the background level, as the first approach to the examination of the properties of the theory. Obviously, a crucial issue is the complete investigation of the perturbations in order to see whether the scenario at hand suffers from instabilities. Although one could be based on similar studies of usual, constant mass, massive gravity [22, 28, 40, 44, 46], and see that the generalized Higuchi bound is satisfied, we mention that since a cosmic scalar is introduced to drive the graviton mass varying along background evolution, the stability issue arisen from this scalar field ought to be taken into account in a global analysis. Such a complete perturbation analysis of the extended nonlinear massive gravity lies beyond the scope of this work and it is left for future investigation.

In conclusion, the extended, varying-mass, nonlinear massive gravity leads to very interesting cosmological behavior at early and intermediate times, while it is limited toward the standard quintessence scenario, where the graviton is massless and the extra scalar is only minimally coupled to gravity. Strictly speaking, although the motivation of massive gravity is to obtain an IR modification, its varying-mass extension in cosmological frameworks leads rather to early and intermediate time modification, and thus to a UV modification instead.
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