Exact and Broken Symmetries in Particle Physics

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Abstract

In these lectures, I discuss the role of symmetries in particle physics. I begin by discussing global symmetries and show that they can be realized differently in nature, depending on whether or not the vacuum state is left invariant by the symmetry. I introduce next the notion of local symmetries and show how these symmetries can be implemented through the introduction of gauge fields. Using the simple example of a spontaneously broken $U(1)$ symmetry, I discuss the Higgs mechanism showing that it provides a natural way for the gauge fields to acquire mass. Finally, I show how these concepts are used as the basis for the Standard Model of particle physics, ending with a brief description of some of the salient aspects of Quantum Chromodynamics and of the electroweak theory.

1 Introduction

All experimental evidence points to the strong, weak and electromagnetic interactions of hadrons (strongly interacting particles) and of leptons as being described by a gauge theory, based on the group

$$G_{SM} = SU(3) \times SU(2) \times U(1).$$

The strong interaction theory–QCD–has as fundamental fermionic entities a triplet of quarks, which feel the $SU(3)$ gauge interactions. Both the quarks and the leptons appear in nature in a repetitive fashion, in three distinct families of doublets under the $SU(2) \times U(1)$ electroweak group. Although $G_{SM}$ correctly describes the symmetry of the fundamental interactions among quarks and leptons, only $SU(3)$ is an exact symmetry of the theory. The electroweak group, in fact, suffers a spontaneous breakdown to $U(1)_{em}$:

$$SU(2) \times U(1) \rightarrow U(1)_{em}.$$

In these lectures we will describe the fundamental concepts upon which the theory for these interactions is built upon. These are related to the way in which symmetries are realized in nature and to the role of gauge fields in rendering theories invariant under local transformations. A crucial notion is that of a spontaneously broken symmetry and the effect that this spontaneous breakdown has for the spectrum of excitations in the theory.

2 Global Symmetries in Field Theory

The natural language for elementary particle physics is that of a quantum field theory, where to each fundamental excitation one assigns a corresponding quantum field. Symmetries of nature are incorporated by constructing Lagrangian densities, made up of these quantum fields, which have an action

$$W = \int d^nx\mathcal{L}. $$
explicitly invariant under the symmetry in question:

\[ W \rightarrow W' = W. \]  

In what follows, I will consider only continuous symmetry transformations based on some Lie group \( G \). Let me denote a generic quantum field by \( \chi_\alpha(x) \), which \( x \) being the space-time location of the quantum field and \( \alpha \) being an (internal) index which runs over the possible components of \( \chi \). [For instance, for a quark field which is a triplet of \( SU(3) \) one would have \( q_\alpha(x) \), with \( \alpha = 1, 2, 3 \).] If \( \alpha \) is one of the operations of the symmetry group \( G \) of transformations, and if the quantum fields \( \chi_\alpha \) are members of an (irreducible) multiplet, then under this operation one has

\[ \chi_\alpha(x) \xrightarrow{\alpha} \chi'_\alpha(x) = R_{\alpha\beta}(a)\chi_\beta(x). \]

That is, under the transformation the field \( \chi \) goes into a new field \( \chi' \) whose components are linear combinations of the old components.

Because, by assumption, the quantum fields \( \chi_\alpha \) are members of a multiplet under \( G \), the matrices \( R(a) \) constitute a representation matrix for the group \( G \) and obey a characteristic composition property. This follows from comparing the sequence of transformations

\[ \chi_\alpha(x) \xrightarrow{\alpha} \chi'_\alpha(x) \xrightarrow{\alpha'} \chi''_\alpha(x) \]

to the direct transformation

\[ \chi_\alpha(x) \xrightarrow{\alpha''} \chi''_\alpha(x). \]

Hence, one finds

\[ R_{\alpha\beta}(a')R_{\beta\gamma}(a) = R_{\alpha\gamma}(a'') \]

In the Hilbert space of the quantum field \( \chi_\alpha(x) \) the transformation (5) is induced by a unitary operator \( U(a) \), so that

\[ U^{-1}(a)\chi_\alpha(x)U(a) = \chi'_\alpha(x) = R_{\alpha\beta}(a)\chi_\beta(x). \]

It is easy to see that the composition property (8) has its counterpart in terms of the unitary operators \( U \):

\[ U(a)U(a') = U(a'') \]

Since we are considering continuous symmetry transformations, it suffices to focus only on infinitesimal transformations \( \delta a \), since finite transformations can always be built up via (10) by (infinite) compounding. A given Lie group is characterized by the number of parameters associated with these infinitesimal transformations and, more specifically, by the algebra obeyed by the operators connected to the distinct infinitesimal parameters.

Let us write for an infinitesimal transformation

\[ U(\delta a) = 1 + i\delta a_i G_i \]

where the index \( i \) runs over all the independent infinitesimal parameters of the Lie group \( G \) [e.g. for the rotation group in 3 dimensions \( O(3) \), \( \delta a_i \) would describe the three independent rotations about the \( x, y \) and \( z \) axis]. The operators \( G_i \) are called the group generators and the composition property (10) implies a group algebra for the generators. Without loss of
generality the parameters $\delta a_i$ can be taken as real, so that the $G_i$ are Hermitian. They obey the Lie algebra:

$$[G_i, G_j] = i c_{ijk} G_k .$$  

(12)

The structure constants $c_{ijk}$ characterize the group $G$ and can be chosen so as to be totally antisymmetric in $i, j$ and $k$.

Just as $U(\delta a)$ can be expanded in terms of the generators $G_i$, so can the representation matrices $R_{\alpha\beta}(\delta a)$. One has, for an infinitesimal transformation

$$R_{\alpha\beta}(\delta a) = \delta_{\alpha\beta} + i \delta a_i (g_i)_{\alpha\beta} .$$  

(13)

It is easy to show that the matrices $g_i$ furnish a representation for the generators $G_i$ and so obey themselves Eq. (12). To see this let us use (13) and (11) in the defining equation (9). One has

$$(1 - i \delta a_i G_i) \chi_\alpha(x)(1 + i \delta a_i G_i) = \chi_\alpha(x) + i \delta a_i (g_i)_{\alpha\beta} \chi_\beta$$  

(14)

which implies

$$[G_i, \chi_\alpha(x)] = -(g_i)_{\alpha\beta} \chi_\beta(x) .$$  

(15)

This equation embodies succinctly how the quantum fields $\chi_\alpha$ transform under the group $G$, and will be repeatedly used in what follows. By using (15) in the Jacobi identity

$$[G_i, [G_j, \chi_\alpha]] + [\chi_\alpha, [G_i, G_j]] + [G_j, [\chi_\alpha, G_i]] = 0$$  

(16)

one readily sees that the matrices $g_i$ obey Eq. (12).

Let us explore the consequences of having a theory built out of the quantum fields $\chi_\alpha$ which is invariant under the transformations of the group $G$. As we shall see, the invariance of the action under $G$ implies the existence of conserved currents and a set of constants of the motion, which are nothing else but the generators $G_i$ of the group! Since the Lagrangian density $L$ depends in general on $\chi_\alpha$ and its space-time derivatives $\partial_\mu \chi_\alpha$, the invariance statement (4) implies

$$\int d^4x L(\chi_\alpha, \partial_\mu \chi_\alpha) = \int d^4x L(\chi'_\alpha, \partial_\mu \chi'_\alpha) .$$  

(17)

For $\chi'_\alpha$ infinitesimally different from $\chi_\alpha$, the stationarity of the action implies

$$0 = \delta W = \int d^4x \left[ \frac{\partial L}{\partial \chi_\alpha} \delta \chi_\alpha + \frac{\partial L}{\partial \partial_\mu \chi_\alpha} \delta \partial_\mu \chi_\alpha \right]$$  

(18)

$$= \int d^4x \left\{ \left[ \frac{\partial L}{\partial \chi_\alpha} - \partial_\mu \left( \frac{\partial L}{\partial \partial_\mu \chi_\alpha} \right) \right] \delta \chi_\alpha + \partial_\mu \left[ \frac{\partial L}{\partial \partial_\mu \chi_\alpha} \delta \chi_\alpha \right] \right\} .$$

The first term above in the curly brackets vanishes because of the Euler-Lagrange equations of motion. The second can be rewritten in terms of the generator matrices $g_i$, since

$$\delta \chi_\alpha = \chi'_\alpha - \chi_\alpha = i \delta a_i (g_i)_{\alpha\beta} \chi_\beta .$$  

(19)

Hence
\[ 0 = \delta W = - \int d^4 x \delta a_i \partial_\mu \left( \frac{\partial L}{\partial \partial_\mu \chi_\alpha} \frac{1}{i} (g_i)_{\alpha \beta} \chi_\beta \right) . \quad (20) \]

Since the parameters \( \delta a_i \) are independent, it follows that the currents
\[ J_\mu^i(x) = \frac{\partial L}{\partial \partial_\mu \chi_\alpha(x)} \frac{1}{i} (g_i)_{\alpha \beta} \chi_\beta(x) , \quad (21) \]
as a result of the symmetry, are conserved
\[ \partial_\mu J_\mu^i(x) = 0 . \quad (22) \]

Because of (22)—if one assumes that the fields \( \chi_\alpha \) drop off sufficiently fast at spatial infinity—there exists a set of constants of the motion, given by the space integral of the \( J_\mu^i \). One has
\[ Q_i = \int d^3 x J_\sigma^i(x) \quad (23) \]
with
\[ \frac{d}{dt} Q_i = 0 . \quad (24) \]

It is easy to check—and we shall do so below—that the operators \( Q_i \) are precisely the generators \( G_i \). That is, they obey both Eqs. (12) and (15). If \( H \) is the Hamiltonian of the theory, then Heisenberg’s equation of motion imply
\[ [H, G_i] = 0 \quad (25) \]
which may be a more familiar way to express the invariance of the theory under the transformations of the group \( G \) (e.g. rotational invariance is expressed via the vanishing of the commutator \( [H, L_i] = 0 \)).

Let us verify that indeed
\[ G_i \equiv Q_i = \int d^3 x J_\sigma^i = \int d^3 x \left[ \frac{\partial L}{\partial \partial_\sigma \chi_\alpha} \frac{1}{i} (g_i)_{\alpha \beta} \chi_\beta \right] \quad (26) \]
acts as a generator is supposed to do. For that, remark that the canonical momentum conjugate to \( \chi_\alpha \) is precisely\[ \pi_\alpha(x) = \frac{\partial L}{\partial \partial_\sigma \chi_\alpha(x)} \quad (27) \]
and that (for bosonic fields) one has the equal time commutation relations
\[ [\pi_\alpha(x), \chi_\beta(y)]_{x^a = y^a} = \frac{1}{i} \delta^3(\vec{x} - \vec{y}) \delta_{\alpha \beta} \quad (28) \]
\[ [\pi_\alpha(x), \pi_\beta(y)]_{x^a = y^a} = [\chi_\alpha(x), \chi_\beta(y)]_{x^a = y^a} = 0 . \]

Then
\[ G_i = \int d^3 x \pi_\alpha(x) \frac{1}{i} (g_i)_{\alpha \beta} \chi_\beta(x) . \quad (29) \]
Since $G_i$ is time-independent, in computing the commutator of $G_i$ with $\chi_\gamma(y)$ one can set the time $x^o$ in (29) equal to $y^o$. Using (28) it is then trivial to check that

$$[G_i, \chi_\gamma(y)] = \int d^3x \left[ \pi_\alpha(x) \frac{1}{i} (g_i)_{\alpha\beta} \chi_\beta(x), \chi_\gamma(y) \right]^{x^o = y^o}$$

and

$$[G_i, G_j] = \int d^3xd^3y \left[ \pi_\alpha(x) \frac{1}{i} (g_i)_{\alpha\beta} \chi_\beta(x), \pi_\gamma(y) \frac{1}{i} (g_j)_{\gamma\delta} \chi_\delta(y) \right]^{x^o = y^o}$$

$$= \int d^3x \pi_\alpha(x) \left( \frac{1}{i} [g_i, g_j] \right)_{\alpha\beta} \chi_\beta(x)$$

$$= i c_{ijk} \int d^3x \pi_\alpha(x) \frac{1}{i} (g_k)_{\alpha\beta} \chi_\beta(x) = i c_{ijk} G_k . \quad (31)$$

Up to now in the discussion of symmetries I focussed on the transformation properties of the quantum fields $\chi_\alpha(x)$. What equation (9) says is that under a group transformation the component fields $\chi_\alpha$ transform in a well-defined way. The correspondence between quantum fields and particles makes it natural to suppose that the quantum states associated with the fields $\chi_\alpha(x)$ will transform in an analogous way. Let me denote the one-particle state associated with the field $\chi_\alpha$ by $|p; \alpha\rangle$, where $p^\mu$ is the 4-momentum of the state and, since these states are supposed to describe particles of a given mass, $p^2 = -m_\alpha^2$. Then, corresponding to Eq. (9), one has

$$U^{-1}(a)|p; \alpha\rangle = \mathcal{R}_{\alpha\beta}(a)|p; \beta\rangle . \quad (32)$$

This equation can be used to deduce that all states of the multiplet $|p; \alpha\rangle$ have the same mass.

Let $|p; \alpha\rangle_{\text{rest}}$ denote the state corresponding to 4-momentum $p^\mu = (0, m_\alpha)$. Then, by definition, the action of the Hamiltonian on this state is just

$$H|p; \alpha\rangle_{\text{rest}} = m_\alpha |p; \alpha\rangle_{\text{rest}} . \quad (33)$$

However, if the theory is invariant under the group $G$, so that $H$ commutes with all the generators (c.f. Eq. (25)) it follows also that

$$[H, U^{-1}(a)] = 0 . \quad (34)$$

Applying this equation on the rest state proves our contention, since

$$0 \equiv [H, U^{-1}(a)]|p; \alpha\rangle_{\text{rest}} = (HU^{-1}(a) - U^{-1}(a)H)|p; \alpha\rangle_{\text{rest}}$$

$$= \mathcal{R}_{\alpha\beta}(a)(m_\beta - m_\alpha)|p; \beta\rangle_{\text{rest}} . \quad (35)$$

Because $\mathcal{R}_{\alpha\beta}(a)$ is arbitrary, it follows that $m_\alpha = m_\beta$.

One says that a symmetry is realized in a Wigner-Weyl way if the invariance of the action under $G$ leads to the appearance in nature of particle multiplets with the same mass. A well known example of an (approximate) Wigner-Weyl symmetry is strong isospin. This approximate global $SU(2)$ symmetry of the strong interaction leads to a nearly degenerate nucleon doublet ($m_\pi \simeq m_\alpha$) and a pion triplet ($m_{\pi^-} = m_{\pi^0} \simeq m_{\pi^+}$). Remarkably, however, the Wigner-Weyl way is not the only way in which a symmetry can be realized in nature!
The Nambu-Goldstone Realization

It is possible that the action is invariant under a symmetry group $G$ but that the physical states of the theory show no trace of this symmetry. This happens in the case in which, although

$$[H, U^{-1}(a)] = 0 ,$$

the vacuum state is not invariant under $G$. Such symmetries are called spontaneously broken, or realized in a Nambu-Goldstone way.

Eq. (32), which lead to the deduction that all states in a multiplet $|p, \alpha\rangle$ have the same mass, can be derived from the transformation properties of the quantum fields $\chi_\alpha$, provided one assumes that the vacuum state is $G$ invariant:

$$U(a)|0\rangle = |0\rangle .$$

The one particle states $|p, \alpha\rangle$ are constructed by the action of the (asymptotic) creation operators for the field $\chi_\alpha$. For a scalar field $\chi_\alpha(x)$ one writes in the usual way

$$\chi_\alpha(x) = \int \frac{d^3p}{(2\pi)^32p^3} [e^{ipx}a_\alpha(p,t) + e^{-ipx}a_\alpha^\dagger(p,t)] .$$

Then, one has

$$|p; \alpha\rangle = \lim_{t \to \pm \infty} a_\alpha^\dagger(p,t)|0\rangle = \lim_{x^0 \to \pm \infty} \int d^3xe^{ipx} \frac{1}{i} \partial_\alpha \chi_\alpha(x) |0\rangle ,$$

where

$$A \partial_\alpha B = A\partial_\alpha B - (\partial_\alpha A)B .$$

Consider then, as in Eq. (32), the action of $U^{-1}(a)$ on the state $|p; \alpha\rangle$

$$U^{-1}(a)|p, \alpha\rangle = \lim_{x^0 \to \pm \infty} \int d^3xe^{ipx} \frac{1}{i} \partial_\alpha U^{-1}(a)\chi_\alpha(x)|0\rangle .$$

If (37) holds, one can write

$$U^{-1}(a)\chi_\alpha(x)|0\rangle = U^{-1}(a)\chi_\alpha(x)U(a)|0\rangle = R_{\alpha\beta}(a)\chi_\beta(x)|0\rangle$$

which immediately establishes (32). However, if the vacuum is not left invariant by a $G$-transformation— i.e. if the vacuum state is degenerate or not unique— then even though the fields $\chi_\alpha$ transform according to some irreducible representation, there are no longer degenerate multiplets in the spectrum.

When a symmetry is realized in a Nambu-Goldstone way, instead of having multiplets of particles with the same mass, there appear in the theory massless excitations— the so-called Goldstone bosons. To see how these ensue consider again the fields $\chi_\alpha$ and take the vacuum expectation value of Eq. (15)

$$(0|G_i, \chi_\alpha(x)|0) = -(g_i)_{\alpha\beta}\langle 0|\chi_\beta(x)|0\rangle .$$
If the vacuum is invariant under $G$ transformations it follows from Eq. (37) that

$$G_i|0\rangle = 0. \quad (44)$$

It is immediate from (43) then that the vacuum expectation values of the fields $\chi_{\alpha}$ must vanish. However, if (44) does not hold, and $\chi_{\alpha}$ are scalar fields, there is no argument why one cannot have

$$\langle 0|\chi_{\alpha}(x)|0 \rangle \neq 0. \quad (45)$$

[If $\chi_{\alpha}$ correspond to fields with spin then the equivalent of Eq. (43) for Lorentz transformations, along with the invariance of the vacuum under these transformations, informs one that the vacuum expectation value of these fields must vanish.]

A symmetry is realized in a Nambu-Goldstone way if there exist some scalar field (which may not necessarily be elementary) with non-zero vacuum expectation value. Imagine that this is so in Eq. (43). Then using the definition of the generators $G_i$ (Eq. (26)) one has

$$0 \neq -(g_{i})_{\alpha\beta}\langle 0|\chi_{\beta}(x)|0 \rangle = \int d^3y\langle 0|J_{\alpha}^\sigma(y)\chi_{\alpha}(x) - \chi_{\alpha}(x)J_{\alpha}^\sigma(y)|0 \rangle . \quad (46)$$

This equation can be written in a more interesting way by inserting a complete set of states $|n\rangle$ and making use of translational invariance on the currents $J_{\alpha}(y)$

$$J_{\alpha}^\sigma(y) = e^{-iP\cdot y}J_{\alpha}^\sigma(0)e^{iP\cdot y} . \quad (47)$$

Then the RHS of Eq. (46) reads

$$\text{RHS} = \sum_n \int d^3y \left\{ \langle 0|e^{-iP\cdot y}J_{\alpha}^\sigma(0)e^{iP\cdot y}|n\rangle\langle n|\chi_{\alpha}(x)|0 \rangle \\
- \langle 0|\chi_{\alpha}(x)|n\rangle\langle n|e^{-iP\cdot y}J_{\alpha}^\sigma(0)e^{iP\cdot y}|0 \rangle \right\} \\
= \sum_n \int d^3y e^{iP\cdot y}\langle 0|J_{\alpha}^\sigma(0)|n\rangle\langle n|\chi_{\alpha}(x)|0 \rangle \\
- \sum_n \int d^3y e^{-iP\cdot y}\langle 0|\chi_{\alpha}(x)|n\rangle\langle n|J_{\alpha}^\sigma(0)|0 \rangle \\
= \sum_n (2\pi)^3\delta^3(\vec{p}_n) \left\{ e^{-iP\cdot y\cdot \sigma}\langle 0|J_{\alpha}^\sigma(0)|n\rangle\langle n|\chi_{\alpha}(x)|0 \rangle \\
- e^{+iP\cdot y\cdot \sigma}\langle 0|\chi_{\alpha}(x)|n\rangle\langle n|J_{\alpha}^\sigma(0)|0 \rangle \right\} . \quad (48)$$

By assumption this expression does not vanish and, furthermore, since the LHS is independent of $y^\sigma$ it must also be independent of $y^\sigma$. Clearly this can only happen if in the theory there exist some massless one-particle states $|n\rangle$ and only these states contribute to the sum in (48). These zero mass states are the Goldstone bosons.

It is not difficult to convince oneself that for each generator $G_i$ that does not annihilate the vacuum there is a corresponding Goldstone boson (after all the action of $G_i$ on the vacuum must give some state—and these states are associated with the Goldstone bosons!). Let us write the Goldstone boson states as $|p^2, j\rangle$, where $p^2 = 0$. Then it follows that the
matrix element of the currents associated with the broken generators between the vacuum and these states are non-vanishing:

\[ \langle 0 | J_\mu^i(0) | p; j \rangle = i f_j \delta_{ij} p^\mu \]  \hspace{1cm} (49)

where \( f_j \) are some non-vanishing constants, which are related to the vacuum expectation values of the fields \( \chi_\alpha \). Indeed, remembering that for a one-particle state

\[ \sum_n \equiv \int \frac{d^3 p_n}{(2\pi)^3 2p_n^0} \]  \hspace{1cm} (50)

it follows from Eqs. (46) and (48) that

\[ i(g_{i\alpha\beta}) \langle 0 | \chi_\beta(0) | 0 \rangle = \lim_{p^\mu \to 0} \frac{1}{2} \left[ f_i \langle p; i | \chi_\alpha(0) | 0 \rangle + f_i^* \langle 0 | \chi_\alpha(0) | p; i \rangle \right] . \]  \hspace{1cm} (51)

Because the Nambu-Goldstone realization of a symmetry is so much less familiar, it is instructive to illustrate it with a very simple example. For these purposes consider the following Lagrangian density describing the interaction of a complex scalar field

\[ L = -\partial_\mu \phi^\dagger \partial^\mu \phi - \lambda \left( \phi^\dagger \phi - \frac{1}{2} f \right)^2 . \]  \hspace{1cm} (52)

Obviously this theory is invariant under a \( U(1) \) transformation (phase transformation)

\[ \phi(x) \to \phi'(x) = e^{i\alpha} \phi(x) \]
\[ \phi^\dagger(x) \to \phi^\dagger'(x) = e^{-i\alpha} \phi^\dagger(x) . \]  \hspace{1cm} (53)

The conserved current associated with this symmetry is easily constructed from our general formula (21)

\[ J_\mu = \frac{\partial L}{\partial \partial_\mu \phi} \frac{1}{i} (1) \phi + \frac{\partial L}{\partial \partial_\mu \phi^\dagger} \frac{1}{i} (-1) \phi^\dagger = i \left[ (\partial_\mu \phi^\dagger) \phi - (\partial_\mu \phi) \phi^\dagger \right] . \]  \hspace{1cm} (54)

The corresponding generator

\[ G = \int d^3 x J^\alpha = i \int d^3 x \left[ (\partial^\alpha \phi^\dagger) \phi - (\partial^\alpha \phi) \phi^\dagger \right] \]  \hspace{1cm} (55)

obeys the commutation relations (15)

\[ [G, \phi(x)] = -\phi(x) \]  \hspace{1cm} (56)
\[ [G, \phi^\dagger(x)] = +\phi^\dagger(x) . \]

In a classical sense, the second term in the Lagrangian correspond to a potential for the fields \( \phi, \phi^\dagger \):

\[ V(\phi, \phi^\dagger) = \lambda \left( \phi^\dagger \phi - \frac{1}{2} f \right)^2 . \]  \hspace{1cm} (57)

Obviously, to guarantee the positivity of the theory, one needs that \( \lambda > 0 \). However, the physics is very different depending on the sign of \( f \). If \( f < 0 \) the potential has a unique
minimum at $\phi = \phi^\dagger = 0$ and the theory is realized in a Wigner-Weyl way, leading to a
degenerate multiplet of massive states. If $f > 0$, on the other hand, the potential has an
infinity of minima characterized by the condition $\phi^\dagger \phi = \frac{1}{2} f$. The theory is realized in a
Nambu-Goldstone way and there is both a massless and a massive state in the theory.

Quantum mechanically, if $f < 0$, it is sensible to expand the potential about $\phi = 0$,
since this is the minimum of the potential. One has

$$V = \lambda \left( \phi^\dagger \phi - \frac{1}{2} f \right)^2 = \frac{1}{4} \lambda f^2 - \lambda f \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 . \quad (58)$$

The quadratic term $-\lambda f \phi^\dagger \phi$, since $f < 0$, is a perfectly good mass term for the fields $\phi$ and
$\phi^\dagger$ and one identifies

$$m^2_\phi = m^2_{\phi^\dagger} = -\lambda f > 0 . \quad (59)$$

In this case, one has a degenerate multiplet of two charge-conjugate particles interacting via
the $\lambda (\phi^\dagger \phi)^2$ term.

If $f > 0$, on the other hand, an expansion about $\phi = 0$ makes no sense as the potential
has a local maximum. The only sensible point to expand the potential is about its minimum
value which occurs at $\phi_{\min} = \sqrt{\frac{f}{2}} e^{i\theta}$. In fact since $f > 0$ there is no way that the quadratic
term in $\phi^\dagger \phi$ can represent a mass term.

Quantum mechanically the non-zero value of $\phi_{\min}$ implies that $\phi$ has a non-vanishing vacuum expectation value

$$\langle \theta | \phi(x) | \theta \rangle = \sqrt{\frac{f}{2}} e^{i\theta} . \quad (60)$$

The phase $\theta$, characterizing the vacuum state $|\theta\rangle$, is in fact irrelevant and can be rotated away. It is a reflection of the non-uniqueness of the vacuum state of the theory. Since under a $U(1)$ transformation

$$U^{-1}(\alpha) \phi(x) U(\alpha) = e^{i\alpha} \phi(x) \quad (61)$$

it is clear that the expectation of $\phi(x)$ between the states $U(-\theta) | \theta \rangle$ is purely real

$$\langle \theta | U^{-1}(-\theta) \phi(x) U(-\theta) | \theta \rangle = e^{-i\theta} e^{i\theta} \sqrt{\frac{f}{2}} = \sqrt{\frac{f}{2}} . \quad (62)$$

Obviously $U(-\theta) | \theta \rangle \equiv |0\rangle$ is just as good a vacuum as $|\theta\rangle$.

Without loss of generality we can set $\theta = 0$ and expand $\phi$ as

$$\phi = \sqrt{\frac{f}{2}} + \chi \quad (63)$$

where the quantum field $\chi$, by assumption, has a vanishing vacuum expectation value. The potential in terms of $\chi$ reads

$$V = \lambda \left( \phi^\dagger \phi - \frac{f}{2} \right)^2 = \lambda \left( \chi^\dagger \chi + \sqrt{\frac{f}{2}} (\chi + \chi^\dagger) \right)^2$$
$$= \frac{\lambda f}{2} (\chi + \chi^\dagger)^2 + \sqrt{2f} \lambda (\chi + \chi^\dagger) \chi^\dagger \chi + \lambda^2 (\chi^\dagger \chi)^2 . \quad (64)$$
Obviously, it appears that a linear combination of $\chi$ and $\chi^\dagger$ has a mass, while its orthogonal combination is massless. Let us write

$$\chi^+ = \frac{1}{\sqrt{2}}(\chi + \chi^\dagger) ; \quad \chi^- = \frac{i}{\sqrt{2}}(\chi^\dagger - \chi).$$  \hspace{1cm} (65)$$

Then

$$m_+^2 = 2\lambda f > 0 \; ; \; \; m_-^2 = 0.$$  \hspace{1cm} (66)$$

Even though the Langragian (52) is $U(1)$ symmetric, this symmetry is not reflected in the spectrum, when the theory is realized in the Nambu-Goldstone manner!

The above identification of $\chi$ as the Goldstone boson field also follows directly from the commutators (56). Since $f$ is real by assumption, one has

$$\chi^- = \frac{i}{\sqrt{2}}(\chi^\dagger - \chi) = \frac{i}{\sqrt{2}}(\phi^\dagger - \phi)$$ \hspace{1cm} (67)

and hence

$$[G, \chi^-] = \frac{i}{\sqrt{2}}(\phi^\dagger + \phi) = i \left[ \sqrt{f} + \chi^+ \right].$$ \hspace{1cm} (68)$$

Whence, taking expectation values, one obtains

$$\langle 0 | [G, \chi^-] | 0 \rangle = i \sqrt{f}.$$ \hspace{1cm} (69)$$

This equation clearly singles out $\chi^-$ as the Goldstone boson field.

If $|p\rangle$ is the state corresponding to this Goldstone boson then, neglecting non-linearities, one expects

$$\langle 0 | \chi^- (0) | p \rangle = 1$$ \hspace{1cm} (70)$$

Eq. (69) then gives, in the same approximation,

$$\langle 0 | J^\mu (0) | p \rangle = i \sqrt{f} p^\mu.$$ \hspace{1cm} (71)$$

The decay constant $f_i$ of Eq. (49) here is just $\sqrt{f}$ and is related to the vacuum expectation value of $\phi$, as expected from Eq. (51). There is an alternative way to accomplish this identification by using directly the current $J^\mu$ and rewriting it in terms of the fields $\chi^+$ and $\chi^-$. One has

$$J^\mu = i[(\partial^\mu \phi^\dagger)\phi - (\partial^\mu \phi)\phi^\dagger]$$ \hspace{1cm} (72)$$

$$= i \left[ (\partial^\mu \chi^\dagger) \left( \sqrt{\frac{f}{2}} + \chi \right) - (\partial^\mu \chi) \left( \sqrt{\frac{f}{2}} + \chi^\dagger \right) \right]$$

$$= i \sqrt{f} \frac{1}{\sqrt{2}} \partial^\mu (\chi^\dagger - \chi) + i[(\partial^\mu \chi^\dagger)\chi - (\partial^\mu \chi)\chi^\dagger]$$

$$= \sqrt{f} \partial^\mu \chi^- + \text{non-linear terms}$$

which directly implies (71).

To summarize, there are two ways in which symmetries ($[H, U] = 0$) can be realized in nature. If the vacuum state is unique ($U | 0 \rangle = | 0 \rangle$), then we have a Wigner-Weyl realization
with degenerate particle multiplets. If, on the other hand, the vacuum state is not unique 
\( U |0 \rangle \neq |0 \rangle \), then we have a Nambu-Goldstone realization with a number of massless 
excitations, one for each of the generators of the group which does not annihilate the vacuum. 
In this latter case one often refers to the phenomena as spontaneous symmetry breaking 
because, although the symmetry exists, it is not reflected in the spectrum of the states of 
the theory.

3 Local Symmetries in Field Theory

In all the preceding discussion I have talked implicitly only about \textbf{global} symmetry transfor-
mations. That is the parameters \( \delta a_i \) were assumed to be independent of space-time. Clearly 
in this case fields at different space-time points are transformed all in the same way. One 
may well ask what happens if the group parameters are space-time dependent. In this case 
the fields \( \chi_\alpha(x) \) and \( \chi_\alpha(x') \) would be rotated in a different way by the group transformation. 
Transformations where this happens are called \textbf{local} symmetries, to distinguish them from the 
global case when \( \delta a_i \) is \( x \)-independent.

Under a \textbf{local} transformation one has

\[
\chi_\alpha(x) \rightarrow \chi'_\alpha(x) = R_{\alpha\beta}(a(x))\chi_\beta(x) .
\]  

(73)

Because \( R \) is now space-time dependent, even though the action

\[
W = \int d^4x L(\partial_\mu \chi_\alpha, \chi_\alpha)
\]  

(74)

was invariant under \textbf{global} \( G \) transformations, this action will fail to be invariant under \textbf{local} \( G \) transformations. Because of the kinetic energy terms, which depends on \( \partial_\mu \chi_\alpha \), 
there will be pieces in \( W \) which are no longer invariant. Indeed, it is easy to identify what 
destroys the possibility of local invariance of the action. Consider the transformation of the 
derivative term \( \partial_\mu \chi_\alpha \) under local transformations. One has

\[
\partial_\mu \chi_\alpha(x) \rightarrow \partial_\mu \chi'_\alpha(x) = \partial_\mu [R_{\alpha\beta}(a(x))\chi_\beta(x)]
\]

\[
= R_{\alpha\beta}(a(x))\partial_\mu \chi_\beta(x) + \partial_\mu R_{\alpha\beta}(a(x))\chi_\beta(x) .
\]  

(75)

The presence of the second term above destroys the local invariance of the action. However, 
one can compensate for the appearance of this term by adding to the, globally invariant, 
Lagrangian additional fields (gauge fields) which cancel this contribution. It is clear that to 
make a Lagrangian locally invariant necessarily involves the introduction of more degrees of 
freedom in the theory.

Before giving a general prescription of how to make a globally invariant Lagrangian 
locally invariant, it is useful to illustrate this procedure with a simple example. Consider a 
free Dirac field with Lagrangian density

\[
L = -\bar{\psi}(x) \left( \gamma^\mu \frac{1}{i} \partial_\mu + m \right) \psi(x) .
\]  

(76)

Clearly \( L \) is invariant under the \( U(1) \) transformation

\[
\psi(x) \rightarrow \psi'(x) = e^{i\alpha} \psi(x)
\]

\[
\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = e^{-i\alpha} \bar{\psi}(x) ,
\]  

(77)

11
which leads to the associated current:

\[ J^\mu(x) = \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}(x)} \frac{1}{i} \psi(x) = \bar{\psi}(x) \gamma^\mu \psi(x) . \] (78)

It is clear, however, that if \( \alpha = \alpha(x) \) the Lagrangian (76) ceases to be invariant, since

\[ \partial_\mu \psi(x) \rightarrow \partial_\mu \psi'(x) = e^{i\alpha(x)} \partial_\mu \psi(x) + i(\partial_\mu \alpha(x)) \psi(x)e^{i\alpha(x)} . \] (79)

Thus

\[ \mathcal{L}(x) \xrightarrow{a(x)} \mathcal{L}'(x) = \mathcal{L}(x) - (\partial_\mu \alpha(x)) \bar{\psi}(x) \gamma^\mu \psi(x) \]

\[ = \mathcal{L}(x) - J^\mu(x) \partial_\mu \alpha(x) . \] (80)

One may get rid of the additional contribution in (80) by augmenting the Lagrangian (76) by an additional term

\[ \mathcal{L}_{\text{extra}} = e A^\mu(x) J_\mu(x) \] (81)

involving a vector field \( A^\mu(x) \), which under a local \( U(1) \) transformation translates by an amount \( \partial_\mu \alpha(x) \):

\[ A^\mu(x) \xrightarrow{a(x)} A'^\mu(x) = A^\mu(x) + \frac{1}{e}(\partial^\mu \alpha(x)) . \] (82)

Of course, if this field \( A^\mu(x) \) is to have a dynamical role, and one wants to preserve the local invariance, the kinetic energy term for \( A^\mu(x) \) should also be invariant under (82). This is easily accomplished by introducing the field strengths:

\[ F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \] (83)

which are clearly invariant under (82). Hence, the total Lagrangian

\[ \mathcal{L} = -\bar{\psi}(x) \left( \gamma^\mu \frac{1}{i} \partial_\mu + m \right) \psi(x) + e A^\mu(x) \bar{\psi}(x) \gamma_\mu \psi(x) \]

\[ - \frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) \] (84)

involving the additional gauge field \( A^\mu \) is locally \( U(1) \) invariant:

\[ \mathcal{L}(x) \xrightarrow{a(x)} \mathcal{L}'(x) = \mathcal{L}(x) \] (85)

when

\[ \psi(x) \xrightarrow{a(x)} \psi'(x) = e^{i\alpha(x)} \bar{\psi}(x) \]

\[ A^\mu(x) \xrightarrow{a(x)} A'^\mu(x) = A^\mu(x) + \frac{1}{e} \partial^\mu \alpha(x) . \] (86)

Note that to make the Lagrangian (76) locally \( U(1) \) invariant it was necessary to introduce an interaction term between the gauge fields \( A^\mu \) and the globally conserved \( U(1) \) current \( J_\mu \). There is a more geometrical way to see how the interaction (81) is necessary to guarantee local invariance. As (79) demonstrates, the reason that the original Lagrangian
(76) is not locally invariant is because the derivative of the \( \psi \) field transforms inhomogeneously under a local \( U(1) \) rotation. If one could construct a modified derivative, \( D_\mu \psi \), which under local transformations transformed in the same way that \( \partial_\mu \psi \) transforms under global transformations, then the original Lagrangian could be trivially made locally invariant by the replacement

\[
\mathcal{L}(\partial_\mu \psi, \psi) \rightarrow \mathcal{L}(D_\mu \psi, \psi).
\]

(87)

Using Eq. (82), it is clear that for the case in question this modified derivative— so called, covariant derivative— is

\[
D_\mu \psi = \partial_\mu \psi - ieA_\mu \psi,
\]

(88)
since

\[
D_\mu \psi \rightarrow D_\mu' \psi' = e^{i\alpha} \partial_\mu \psi + i(\partial_\mu \psi)e^{i\alpha} - i(\partial_\mu \psi)e^{i\alpha}
\]

\[
= e^{i\alpha(x)}[\partial_\mu \psi - ieA_\mu \psi] = e^{i\alpha(x)} D_\mu \psi.
\]

(89)

Obviously

\[
\mathcal{L} = -\bar{\psi}(x) \left( \gamma^\mu \frac{1}{4} D_\mu + m \right) \psi(x) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}
\]

(90)
is locally \( U(1) \) invariant and coincides with the expression (84).

Viewed from this perspective, the demand of local invariance of a Lagrangian is a marvelous prescription to fix the interactions of the globally invariant fields with the gauge fields. Furthermore, the gauge transformation (82) does not allow the introduction of a mass term for the \( A_\mu \) field, since

\[
\mathcal{L}_{\text{mass}} = -\frac{1}{2} m^2 A_\mu(x) A_\mu(x)
\]

(91)
brakes the local \( U(1) \) transformation. So local invariance of a theory severely restricts the dynamics. In the example in question, it will be recognized that the demand that a Dirac field be described by a Lagrangian that is locally \( U(1) \) invariant has produced the QED Lagrangian! To guarantee local \( U(1) \) transformations it is necessary to introduce a massless gauge field \( A^\mu(x) \)— the photon field— interacting with strength \( e \)— the electric charge— with the conserved current \( J^\mu \).

The above simple example can be generalized to theories where the global symmetry group is bigger than the \( U(1) \) phase symmetry, where the structure constants vanish (Abelian group). For these purposes, consider again a Lagrangian density \( \mathcal{L}(\partial_\mu \chi_\alpha, \chi_\alpha) \) composed of fields which transform irreducibly under a non-Abelian group \( G \) (a group where the structure constants \( c_{ijk} \neq 0 \)). Under global \( G \) transformations, one has

\[
\chi_\alpha(x) \rightarrow \chi'_\alpha(x) = R_{\alpha\beta}(a)\chi_\beta(x)
\]

(92)

\[
\partial_\mu \chi_\alpha(x) \rightarrow \partial_\mu \chi'_\alpha(x) = R_{\alpha\beta}(a)\partial_\mu \chi_\beta(x),
\]

If this Lagrangian density is invariant under these transformations then

\[
\mathcal{L}(\partial_\mu \chi_\alpha, \chi_\alpha) \rightarrow \mathcal{L}'(\partial_\mu \chi'_\alpha, \chi'_\alpha) = \mathcal{L}(\partial_\mu \chi_\alpha, \chi_\alpha).
\]

(93)
Suppose one were able to introduce appropriate gauge fields to construct a **covariant derivative**, $D_{\mu}\chi_\alpha(x)$, which under **local** $G$ transformations transformed as $\partial_{\mu}\chi_\alpha(x)$ does under **global** transformations. That is,

$$D_{\mu}\chi_\alpha(x) \xrightarrow{a(x)} D'_{\mu}\chi'_\alpha(x) = \mathcal{R}_{\alpha\beta}(a(x))D_{\mu}\chi_\beta(x) \ .$$

(94)

Then, clearly, the Lagrangian $\mathcal{L}(D_{\mu}\chi_\alpha, \chi_\alpha)$ would be **locally** $G$ invariant

$$\mathcal{L}(D_{\mu}\chi_\alpha, \chi_\alpha) \xrightarrow{a(x)} \mathcal{L}'(D'_{\mu}\chi'_\alpha, \chi'_\alpha) = \mathcal{L}(D_{\mu}\chi_\alpha, \chi_\alpha) \ .$$

(95)

For the theory to be physical, in addition, of course, one must also provide appropriate locally invariant field strengths for the gauge fields entering in the covariant derivatives $D_{\mu}\chi_\alpha$.

By assumption, the covariant derivatives required must transform under local transformations as $\partial_{\mu}\chi_\alpha$ does in Eq. (92). In analogy to what was done for the simple $U(1)$ example, it is suggestive to introduce one gauge field $A^\mu_\alpha$ for each of the parameters $\delta a_i$ of the group $G$. After all, the gauge fields are supposed to compensate for the local variations of the fields $\chi_\alpha$, and so there should be a gauge field for each of the parameters $\delta a_i(x)$ of the Lie group $G$. Taking the field $\chi_\alpha(x)$ transformations under $G$ to be those of Eq. (15)

$$[G_i, \chi_\alpha(x)] = -(g_i)_{\alpha\beta}\chi_\beta$$

(96)

the $U(1)$ example suggest writing for the covariant derivative $D_{\mu}\chi_\alpha$ the expression

$$D_{\mu}\chi_\alpha(x) = [\delta_{\alpha\beta}\partial_{\mu} - ig(g_i)_{\alpha\beta}A^\mu_\beta(x)]\chi_\beta(x) \ ,$$

(97)

where $g$ is some coupling constant.

For Eq. (92) to be satisfied for $D_{\mu}\chi_\alpha$, the gauge fields must respond appropriately under local transformations. To determine what this behavior should be, let us compute $D'_{\mu}\chi'_\alpha$ and compare it to what we expect from (92). One has

$$D'_{\mu}\chi'_\alpha(x) = \partial_{\mu}\chi'_\alpha(x) - ig(g_i)_{\alpha\beta}A^\mu_\beta(x)\chi'_\beta(x)$$

$$= \partial_{\mu}[\mathcal{R}_{\alpha\beta}(a(x))\chi_\beta(x)] - ig(g_i)_{\alpha\beta}A^\mu_\beta(x)\mathcal{R}_{\beta\gamma}(a(x))\chi_\gamma(x)$$

$$= \mathcal{R}_{\alpha\beta}(a(x))\partial_{\mu}\chi_\beta(x) + (\partial_{\mu}\mathcal{R}_{\alpha\gamma}(a(x)))\chi_\gamma(x)$$

$$- ig(g_i)_{\alpha\beta}A^\mu_\beta(x)\mathcal{R}_{\beta\gamma}(a(x))\chi_\gamma(x) \ .$$

(98)

By definition we want

$$D'_{\mu}\chi'_\alpha(x) = \mathcal{R}_{\alpha\beta}(a(x))D_{\mu}\chi_\beta(x)$$

$$= \mathcal{R}_{\alpha\beta}(a(x))\partial_{\mu}\chi_\beta - ig\mathcal{R}_{\alpha\beta}(a(x))(g_i)_{\beta\gamma}A^\mu_\gamma(x)\chi_\gamma(x) \ .$$

(99)

It follows, therefore, that one must require that

$$- ig(g_i)_{\alpha\beta}A^\mu_\beta(x)\mathcal{R}_{\beta\gamma}(a(x)) + \partial_{\mu}\mathcal{R}_{\alpha\gamma}(a(x)) = -ig\mathcal{R}_{\alpha\beta}(a(x))(g_i)_{\beta\gamma}A^\mu_\gamma(x) \ .$$

(100)

Multiplying the above by $\mathcal{R}^{-1}$ finally gives the transformation required for the gauge field:

$$(g_i)_{\alpha\beta}A^\mu_\beta(x) = \frac{1}{ig}[\partial_{\mu}\mathcal{R}_{\alpha\gamma}(a(x))] [\mathcal{R}^{-1}(a(x))]_{\gamma\beta}$$

$$+ \mathcal{R}_{\alpha\gamma}(a(x))(g_i)_{\gamma\delta}[\mathcal{R}^{-1}(a(x))]_{\delta\beta}A^\mu_\beta(x) \ .$$

(101)
It is easy to check that this formula agrees with Eq. (82) in the Abelian $U(1)$ case when $R = e^{i\alpha}$, $g_i = 1$ and $g = e$. In principle, however, Eq. (101) has a very troublesome aspect, since it appears that the transformation properties of the gauge fields $A'_\mu$ depend on how the field $\chi_\alpha$ transforms under $G$. If this were to be really the case it would be disastrous, because to obtain a locally invariant theory one would need to introduce a separate compensating gauge field for each matter field in the theory. Fortunately, although (101) as written appears to depend on $R$ explicitly, this dependence is in fact illusory. The transformation properties of gauge fields depend only on the group $G$ and not on how the matter fields transform.

To prove this very important point, it is useful to consider Eq. (101) for infinitesimal transformations, where

$$\mathcal{R}_{\alpha\beta}(\delta a(x)) = \delta_{\alpha\beta} + i \delta a_i(g_i)_{\alpha\beta} .$$

(102)

Using the above in (101), and employing an obvious matrix notation, one has

$$g_k A'_\mu k(x) = \frac{1}{ig} [\delta_{\mu}(1 + i\delta a_k(x)g_k)] [1 - i\delta a_i(x)g_i]$$

$$+ [1 + i\delta a_j(x)g_j] g_i [1 - i\delta a_k(x)g_k] A_\mu i(x)$$

$$\simeq g_k A_{\mu k}(x) + i\delta a_j(x) [g_j, g_i] A_\mu i(x) + \frac{1}{g} [\partial_{\mu} \delta a_k(x)] g_k .$$

(103)

Using the commutation relations for the matrices $g_i$

$$[g_j, g_i] = ic_{ijk} g_k = -ic_{ijk} g_k$$

(104)

it is easy to see that the RHS of (103) is simply proportional to $g_k$

$$\text{RHS} = g_k \left[ A_{\mu k}(x) + c_{ijk} \delta a_j(x) A_\mu i(x) + \frac{1}{g} [\partial_{\mu} \delta a_k(x)] \right] .$$

(105)

Thus, as anticipated, the transformation properties of the gauge fields are independent of the representation matrices $g_k$ associated with the fields $\chi_\alpha(x)$ and depend only on the structure constants of the group $c_{ijk}$:

$$A'_{\mu k}(x) = A_{\mu k}(x) + \delta a_j(x) c_{ijk} A_\mu i(x) + \frac{1}{g} \partial_{\mu} (\delta a_k(x)) ,$$

(106)

For global transformations, where the parameters $\delta a_k$ are $x$-independent, the last term in (106) does not contribute and the transformation of the gauge fields can be written in the standard form one expects for a quantum field:

$$A'_{\mu k}(x) = A_{\mu k}(x) + i\delta a_j(\tilde{g}_j)_{ki} A_\mu i(x) .$$

(107)

Here the “generator” matrices appropriate for the gauge fields, $\tilde{g}$, are expressible in terms of the structure constants of the group

$$(\tilde{g}_j)_{ki} = -i c_{ijk} = -i c_{kji} .$$

(108)

It is not hard to show (by using the Jacobi identity for $\tilde{g}_i$, $\tilde{g}_j$, and $\tilde{g}_k$) that the matrices $\tilde{g}$ in Eq. (108) indeed obey the group algebra of $G$

$$[\tilde{g}_i, \tilde{g}_j] = i c_{ijk} \tilde{g}_k .$$

(109)
The above discussion makes it clear that the gauge fields $A_i^\mu$ introduced in the covariant derivative (97) transform according to a special representation of the group $G$, the adjoint representation. If $G$ has $n$ parameters, then the matrices $\hat{g}_i$ are $n \times n$ matrices, whose elements are related to the structure constants $c_{ijk}$. Their transformation has no connection with how the matter fields $\chi_i$ transform, but is intimately connected with the key parameters of $G$, its structure constants.

Having made $\mathcal{L}$ locally invariant through the replacement of derivatives by covariant derivatives, it remains to construct the field strengths for the fields $A_i^\mu$, so as to be able to incorporate into the theory the kinetic energy terms for the gauge fields. It is easy to check that the naive generalization of the Abelian example

$$F_k^{\mu\nu} = \partial^\mu A_k^\nu - \partial^\nu A_k^\mu$$

(110)
will not work, since its transformation will still contain derivatives of the parameters $\delta a_i$. Indeed, using Eq. (106) one sees that

$$F_k^{\mu\nu} = \partial^\mu A_k^\nu - \partial^\nu A_k^\mu + \hat{F}_k^{\mu\nu} + \delta a_i c_{ijk} \tilde{F}_i^{\mu\nu}$$

(111)

What one wants to do to obtain the correct field strengths is to augment (110) so as to eliminate altogether the last term in (111). Since this term contains both $\partial^\mu \delta a_j$ and $A_i^\mu$ in an antisymmetric fashion, one is led, after a bit of reflection, to try the following ansatz for the non-Abelian field strengths:

$$F_k^{\mu\nu} (x) = \partial^\mu A_k^\nu (x) - \partial^\nu A_k^\mu (x) + g c_{kij} A_i^\mu (x) A_j^\nu (x).$$

(112)

Let us check that Eq. (112) has the right properties. Using (106), the third term in (112) transforms as

$$g c_{kij} A_i^\mu (x) A_j^\nu (x) \rightarrow g c_{kij} A_i^\mu (x) A_j^\nu (x)$$

$$= g c_{kij} \left[ A_i^\mu (x) + \delta a_i (x) c_{mij} A_m^\nu (x) + \frac{1}{g} \partial^\mu \delta a_i (x) \right] \cdot \left[ A_j^\nu + \delta a_j (x) c_{mnj} A_m^\nu (x) + \frac{1}{g} \partial^\nu \delta a_j (x) \right]$$

$$\simeq g c_{kij} A_i^\mu (x) A_j^\nu (x) + c_{kij} \left[ (\partial^\mu \delta a_i) A_j^\nu + (\partial^\nu \delta a_j) A_i^\mu \right]$$

$$+ \delta a_i (x) \left[ g c_{kij} c_{mij} A_m^\mu A_j^\nu + g c_{kij} c_{mij} A_i^\mu A_m^\nu \right].$$

(113)

However, making use of the antisymmetry of the structure constants, one has:

$$c_{kij} (\partial^\mu \delta a_i) A_j^\nu = c_{kji} (\partial^\mu \delta a_j) A_i^\nu = -c_{ijk} (\partial^\mu \delta a_j) A_i^\nu$$

$$c_{kij} (\partial^\nu \delta a_j) A_i^\mu = c_{ijk} (\partial^\nu \delta a_j) A_i^\mu,$$

(114)

and one sees that the last term in (111) precisely cancels the second term in (113).

It is also not hard to check that the last term in (113) can be written in a much more interesting form by making use of (109). Relabeling dummy indices and using (109) one
obtains

\[ \text{3rd term} = g \delta \alpha \ell \left[ c_{kij} c_{m \ell i} A^\mu_m A^\nu_j + c_{km \ell i} c_{j \ell i} A^\mu_m A^\nu_j \right] \]
\[ = g \delta \alpha \ell \left[ -c_{jki} c_{m \ell i} + c_{mki} c_{j \ell i} \right] A^\mu_m A^\nu_j \]
\[ = g \delta \alpha \ell \left[ \bar{g}_j [g_m]_{k\ell} A^\mu_m A^\nu_j \right] \]
\[ = ig \delta \alpha c_{j \rho m} [g_p]_{k\ell} A^\mu_m A^\nu_j \]
\[ = g \delta \alpha c_{j \rho \ell} [\bar{g}^\mu_m A^\nu_j] \]
\[ = \delta \alpha c_{i\ell k} \left[ g c_{imj} A^\mu_m A^\nu_j \right]. \] (115)

Using the above one sees that what remains of (113) transforms in precisely the same way as the second term of \( \tilde{F}^{\mu\nu k} \) [cf. Eq. (111)].

Putting everything together, one sees that under a local transformation the field strength \( F_{k}^{\mu\nu} \) transforms as

\[ F_{k}^{\mu\nu}(x) \rightarrow F_{k}^{\mu\nu}(x) = F_{k}^{\mu\nu}(x) + \delta \alpha (x) c_{ij k} F_{i}^{\mu\nu}(x). \] (116)

The above is the desired result. Namely, that under local transformations the field strengths should transform as a quantum field which belongs to the adjoint representation of the group.

In view of (116), it is easy to show that \( F_{k}^{\mu\nu} F_{k\mu\nu} \) is \( G \)-invariant. One has

\[ F_{k}^{\mu\nu} F_{k\mu\nu} \rightarrow F_{k}^{\mu\nu} F_{k\mu\nu}' = (F_{k}^{\mu\nu} + \delta \alpha c_{ij k} F_{i}^{\mu\nu}) (F_{k\mu\nu} + \delta \alpha c_{ij k} F_{i\mu\nu}) \]
\[ = F_{k}^{\mu\nu} F_{k\mu\nu} + \delta \alpha (c_{ij k} F_{i}^{\mu\nu} F_{k\mu\nu} + c_{ij k} F_{k}^{\mu\nu} F_{i\mu\nu}) \]
\[ = F_{k}^{\mu\nu} F_{k\mu\nu}', \] (117)

since the 2nd term vanishes because of the antisymmetry of \( c_{ij k} : c_{ij k} = -c_{jik} \).

Let us recapitulate our results. The Lagrangian density \( \mathcal{L}(\partial_{\nu} \chi_{\alpha}, \chi_{\alpha}) \) — assumed to be invariant under \textbf{global} \( G \) transformations — can be made locally invariant by introducing gauge fields \( A_{\mu}^{i} \), which enter in the covariant derivatives \( D_{\mu} \chi_{\alpha} \) and the field strengths \( F_{i}^{\mu\nu} \). The locally invariant Lagrangian density is simply:

\[ \mathcal{L}_{\text{local}} = \mathcal{L}(\partial_{\nu} \chi_{\alpha}, \chi_{\alpha}) - \frac{1}{4} F_{\mu\nu}^{i} F_{i\mu\nu} \] (118)

and is \textbf{completely} determined from a knowledge of the global invariant Lagrangian \( \mathcal{L} \)

Three remarks are in order:

i) Again, as in the Abelian case, no mass term for the gauge fields \( A_{\mu}^{i} \) are allowed if one wants to preserve the local invariance (106).

ii) The pure gauge Lagrangian

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{i} F_{i\mu\nu} \] (119)

which contains the kinetic energy terms for the gauge fields \( A_{\mu}^{i} \) is already a \textbf{nonlinear} field theory, since \( F_{\mu\nu}^{i} \) contains terms quadratic in the gauge fields \( A_{\mu}^{i} \). For the Abelian case, where the structure constants vanish, these nonlinear terms are absent.
iii) Because the gauge fields transform nontrivially under the group $G$, as far as global transformations go, the symmetry currents of the full theory given by Eq. (118) now also get a contribution from the gauge fields. That is, one has

$$J^\mu_i = \frac{\partial L}{\partial \partial^\mu \chi^\alpha_i} (g_i)^{\alpha\beta} \chi^\beta + \frac{\partial L}{\partial \partial^\mu A^\nu_j} (\tilde{g}_i)^{jk} A^\nu_k .$$

(120)

4 The Higgs Mechanism

We saw earlier that in the case of global symmetries, these symmetries could be realized either in a Wigner-Weyl or Nambu-Goldstone way, depending on whether the vacuum state was left, or not left, invariant by the group transformations. It is clearly of interest to know what happens in each of these cases when the global symmetry is made local, via the introduction of gauge fields. For the Wigner-Weyl case, nothing very much happens. Besides the various degenerate multiplets of particles of the global symmetry there is now also a degenerate zero mass multiplet of gauge field excitations. In the Nambu-Goldstone case, however, some remarkable things happen. When the global symmetry is gauged, the Goldstone bosons associated with the broken generators disappear and the corresponding gauge fields acquire a mass! This is the celebrated Higgs mechanism.

To understand this phenomena, it is useful to return to the simple $U(1)$ model discussed earlier and see what obtains when one tries to make the $U(1)$ global symmetry also a local symmetry of the Lagrangian. Recall that the Lagrangian density of the model was

$$L = -\partial^\mu \phi^\dagger \partial^\mu \phi - \lambda \left( \phi^\dagger \phi - \frac{1}{2} f \right)^2 ,$$

(121)

and that the sign of $f$ determined whether one had a Wigner-Weyl realization ($f < 0$) or a Nambu-Goldstone realization ($f > 0$). To make the above Lagrangian locally $U(1)$ invariant it suffices to replace $\partial^\mu \phi$ by a covariant derivative $D^\mu \phi$ involving a gauge field $A^\mu$, and include in the theory a kinetic energy term for this gauge field.

If under local $U(1)$ transformations one assumes that

$$\phi(x) \rightarrow \phi'(x) = e^{i\alpha(x)} \phi(x) \quad \text{and} \quad A^\mu(x) \rightarrow A^\mu'(x) = A^\mu(x) + \frac{1}{g} \partial^\mu \alpha(x) ,$$

(122)

then the covariant derivative

$$D^\mu \phi(x) = (\partial^\mu - ig A^\mu) \phi$$

(123)

clearly transforms just like $\phi$ does

$$D^\mu \phi(x) \rightarrow D^\mu' \phi'(x) = e^{i\alpha(x)} (D^\mu \phi(x)) .$$

(124)

Whence the augmented Lagrangian

$$\mathcal{L} = -(D^\mu \phi)^\dagger (D^\mu \phi) - \lambda \left( \phi^\dagger \phi - \frac{1}{2} f \right)^2 - \frac{1}{4} F^\mu\nu F_{\mu\nu}$$

(125)

with

$$F_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

(126)
is clearly locally $U(1)$ invariant.

If $f < 0$, so that the global symmetry is Wigner-Weyl realized, the above Lagrangian is suitable for computation as is. It describes the interaction of a degenerate multiplet of scalar fields ($\phi$ and $\phi^\dagger$) both with themselves and with a massless gauge field $A_\mu$. These latter interactions–since the $\phi$’s are scalar fields and hence have quadratic kinetic energies–contain both a linear term in the gauge fields:

$$L^{(1)}_{\text{int}} = gA_\mu \left[ i(\partial^\mu \phi^\dagger)\phi - i\phi^\dagger \partial^\mu \phi \right] = gA_\mu J^\mu$$  \hspace{1cm} \text{(127)}$$

as well as a quadratic–so called “sea-gull” term–contribution:

$$L^{(1)}_{\text{sea-gull}} = -g^2 A^\mu A^\mu \phi^\dagger \phi .$$  \hspace{1cm} \text{(128)}$$

These interactions follow directly from the gauge invariant replacement $\partial_\mu \phi \rightarrow D_\mu \phi = (\partial_\mu - igA_\mu)\phi$.

If $f > 0$, on the other hand, so that the global $U(1)$ symmetry is realized in a Nambu-Goldstone way, one must reparametrize the theory in terms of fields with vanishing expectation value (c.f. Eq. (63)). This reparametrization is such that one is computing oscillations around the minimum of the potential $V(\phi)$. That is, one replaces

$$\phi^\dagger \phi = \frac{f}{2} + \text{quantum fields} .$$  \hspace{1cm} \text{(129)}$$

This necessary shift implies that the seagull term of Eq. (128) gives rise to a mass term for the $A_\mu$ field!

$$L_{\text{mass}} = -g^2 f A^\mu A_\mu \equiv -\frac{1}{2} m_A^2 A^\mu A_\mu .$$  \hspace{1cm} \text{(130)}$$

If the gauge field acquires mass, it follows that it cannot be purely transverse (like the photon) but must also have a longitudinal polarization component. This extra degree of freedom must come from somewhere. It is not difficult to show that it arises from the disappearance of the Nambu-Goldstone excitation, which would ordinarily arise from the spontaneous breakdown of the global $U(1)$ symmetry.

To check this assertion, it is convenient to reparametrize the field $\phi$, in the case $f > 0$, in a somewhat different way than that chosen before. [The physics of the theory is, in fact, independent of the parametrization one chooses, but certain parametrizations are more directly physical. Different choices for $\phi$ are akin to choosing different gauges for $A_\mu$.] Let us write $\phi$ in the following exponential parametrization:

$$\phi(x) = \frac{1}{\sqrt{2}} \left[ \sqrt{f} + \rho(x) \right] \exp \left[ \frac{i\xi(x)}{\sqrt{f}} \right] .$$  \hspace{1cm} \text{(131)}$$

Here $\rho(x)$ and $\xi(x)$ are real fields, with $\xi(x)$–the phase field–being connected to the Goldstone boson. This last assertion is easy to understand since $\xi(x)$ vanishes altogether from the potential $V$, and so obviously cannot have any mass term. One has simply

$$V = \lambda \left( \phi^\dagger \phi - \frac{f}{2} \right)^2 = \lambda \left( \frac{\rho^2}{2} + \sqrt{f} \rho \right)^2 ,$$  \hspace{1cm} \text{(132)}$$
so that the $\rho$ field has a mass
\[ m_\rho^2 = 2\lambda f \]  
(133)
in agreement with the value obtained earlier (cf. (66)).

It is easy to check that the phase field $\xi$ enters in the covariant derivative in a trivial way, so that it can also be eliminated from the kinetic energy term by an appropriate gauge choice. Thus, as advertised, the Nambu-Goldstone boson plays no role in the local theory. It is “eaten” to give mass to the gauge fields. To prove this assertion, let us consider $D_\mu \phi$

\[ D_\mu \phi = (\partial_\mu - igA_\mu)\phi = (\partial_\mu - igA_\mu) \frac{1}{\sqrt{f}}(\sqrt{f} + \rho) \exp \left[ -\frac{i\xi}{\sqrt{f}} \right] \]  
(134)

Obviously the factor in front of the $[\ ]$ bracket in (134) involving $\exp \left[ -\frac{i\xi}{\sqrt{f}} \right]$ will not appear in the Lagrangian (125), since the Lagrangian involves $(D_\mu \phi)^\dagger (D^\mu \phi)$. Furthermore the $\xi$ dependence in the curly bracket is also spurious, since it can be eliminated via a gauge transformation of the field $A^\mu$

\[ A^\mu \rightarrow B^\mu = A^\mu - \frac{1}{g} \partial^\mu \frac{\xi}{\sqrt{f}}. \]  
(135)

If the $U(1)$ global symmetry is spontaneously broken ($f > 0$) the Lagrangian (125) can be rewritten entirely in terms of a massive vector field $B^\mu$ and a massive real scalar field $\rho$. The resulting Lagrangian

\[ \mathcal{L} = -\frac{1}{2} \partial_\mu \rho \partial^\mu \rho - \frac{1}{2} m_\rho^2 \rho^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m_A^2 B^\mu B_\mu - g^2 \left( \sqrt{f} \rho + \frac{1}{2} \rho^2 \right) B^\mu B_\mu - \lambda \left( \sqrt{f} \rho^3 + \frac{1}{4} \rho^4 \right) \]  
(136)

where

\[ m_\rho^2 = 2\lambda f ; \quad m_A^2 = g^2 f \]  
(137)

shows no explicit traces of the original $U(1)$ symmetry, except that certain parameters in the interactions have particular interrelations. I remark that, although we demonstrated the absorption of the Goldstone boson to produce a massive gauge field only in the Abelian case, this same phenomenon also occurs in the non-Abelian case.

Let me close this section by discussing the two versions of the model [Wigner-Weyl $f < 0$; Nambu-Goldstone $f > 0$] in terms of the degrees of freedom present in the theory. In the Wigner-Weyl case the theory has a complex scalar field $\phi$ (2 degrees of freedom) plus a massless gauge field $A^\mu$ (2 degrees of freedom, corresponding to the two transverse polarizations). In the Nambu-Goldstone case in the theory there is a real scalar field $\rho$ (1 degree of freedom) plus a massive spin 1 field $B_\mu$ (3 degrees of freedom). Clearly both versions of the theory have the same number of degrees of freedom. However the spectrum of the excitations is completely different!
5 The Structure of Quantum Chromodynamics

As a first illustration, I want to describe very briefly the structure of Quantum Chromodynamics (QCD), the theory that describes the strong interactions. As we shall see, although QCD is a local gauge theory realized in a Wigner-Weyl way, it possesses also a set of approximate global symmetries. It turns out that some of these global symmetries are realized in a Wigner-Weyl way, while some others are realized in a Nambu-Goldstone manner. Hence, QCD provides a nice practical example of the more formal considerations we have discussed up to now.

We know in nature of the existence of six different types—flavors—of quarks: u, d, s, c, b, and t. Each flavor of quark is actually a triplet of fields, since the quarks transform irreducibly under the $SU(3)$ symmetry group that characterizes QCD. This $SU(3)$ symmetry is a local symmetry, so besides quarks in QCD one must introduce the $3^2 - 1 = 8$ gauge fields which are associated with the local $SU(3)$ symmetry. These 8 gauge fields are known as gluons, since they help bind quarks into hadrons—like protons and π-mesons.

Let $q^f_\alpha(x)$ stand for a quark field, with the index $f$ denoting the various flavors $f = \{u, d, s, c, b, t\}$ and $\alpha = \{1, 2, 3\}$ being an $SU(3)$ index. Under local infinitesimal $SU(3)$ transformation then one has:

$$q^f_\alpha(x) \rightarrow q'^f_\alpha(x) = \left[ \delta_{\alpha\beta} + i \delta a_i(x) \left( \frac{\lambda_i}{2} \right)_{\alpha\beta} \right] q^f_\beta(x).$$

(In the above, the $\lambda_i$ matrices $i = 1, \ldots, 8$ are the $3 \times 3$ Gell-Mann matrices transforming as the 3 representation of $SU(3)$. The $SU(3)$ structure constants—denoted here by $f_{ijk}$—are easily found by using the explicit form of the $\lambda$-matrices given below:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}; \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}; \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$}

One has

$$\left[ \frac{\lambda_i}{2}, \frac{\lambda_j}{2} \right] = i f_{ijk} \frac{\lambda_k}{2}. \quad (140)$$

The gauge fields $A^\mu_k(x)$ under a local infinitesimal $SU(3)$ transformation rotate into each other with coefficients proportional to the structure functions $f_{ijk}$ and shift by the gradient of the $SU(3)$ parameters $\delta a_k(x)$:

$$A^\mu_k(x) \rightarrow A'^\mu_k(x) = A^\mu_k(x) + \delta a_j f_{ijk} A^\mu_i(x) + \frac{1}{g} \partial^\mu \delta a_k(x). \quad (141)$$
The field strengths

\[ F_{\mu \nu}^i = \partial_\mu A^\nu_i - \partial_\nu A^\mu_i + g f_{ijk} A^\mu_j A^\nu_k \]  

(142)

transform in the same way as the \( A^\mu_i \) fields do but have no inhomogeneous contribution proportional to derivatives of \( \delta a_k(x) \). Finally, the covariant derivatives of the quark fields

\[ D^\mu_{\alpha \beta} q^f_\beta = \left[ \partial^\mu \delta_{\alpha \beta} - i g \left( \frac{\lambda_i}{2} \right)_{\alpha \beta} A^\mu_i \right] q^f_\beta \]  

(143)

transform under local SU(3) transformations precisely as the quark fields themselves do.

Using the above equations, it is easy to see that the QCD Lagrangian

\[ \mathcal{L}_{\text{QCD}} = \sum_f -\bar{q}^f_\alpha \left[ \gamma^\mu \frac{1}{i} [D_\mu]_{\alpha \beta} + m_f \delta_{\alpha \beta} \right] q^f_\beta - \frac{1}{4} F_{\mu \nu}^i F^i_{\mu \nu} \]  

(144)

is locally SU(3) invariant. In the above, the parameters \( m_f \) are mass terms for each flavor \( f \) of quarks. If these terms were absent, that is if one could set \( m_f \to 0 \), it is clear that the QCD Lagrangian has a large global symmetry in which quarks of one flavor are changed into quarks of another flavor. For six flavors of quarks, it is not difficult to show that, in the limit \( m_f \to 0 \), the QCD Lagrangian is invariant under a \( U(6) \times U(6) \) group of global transformations.

Physically, it turns out that whether one can, or one cannot, approximately neglect the quark mass terms \( m_f \) depends on whether the mass \( m_f \) is much smaller, or much greater, than the dynamical scale, \( \Lambda_{\text{QCD}} \), associated with QCD. This latter scale is of order 300 MeV which is, in fact, much greater than both the \( u \)- and \( d \)-quark masses. Although these, so-called light quarks have masses much smaller than \( \Lambda_{\text{QCD}} \): 

\[ m_{u,d} \ll \Lambda_{\text{QCD}} , \]  

(145)

it turns out that \( m_c \sim \Lambda_{\text{QCD}} \), while \( \Lambda_{\text{QCD}} \) is much smaller than the masses of the \( c \)-, \( b \)- and \( t \)-quarks. For this reason, in what follows, I will consider only the QCD piece of the Lagrangian involving the \( u \)- and \( d \)-light quarks. This, of course, is particularly interesting since these quarks are the ones that make up ordinary hadrons, like the proton, neutron and the pions.

The QCD Lagrangian for this 2-flavor case, if we neglect for the moment altogether \( m_u \) and \( m_d \), reads

\[ \mathcal{L}_{\text{QCD}}^{2\text{-flavor}} \bigg|_{m_u=m_d=0} = -\bar{u}_\alpha \gamma^\mu \frac{1}{i} [D_\mu]_{\alpha \beta} u_\beta - \bar{d}_\alpha \gamma^\mu \frac{1}{i} [D_\mu]_{\alpha \beta} d_\beta - \frac{1}{4} F_{\mu \nu}^i F^i_{\mu \nu} . \]  

(146)

Let us organize the \( u \)- and \( d \)-quarks into a doublet

\[ Q_\alpha = \left( \begin{array}{c} u \\ d \end{array} \right)_\alpha , \]  

(147)

then the above Lagrangian can be written simply as

\[ \mathcal{L}_{\text{QCD}}^{2\text{-flavor}} \bigg|_{m_u=m_d=0} = -\bar{Q}_\alpha \gamma^\mu \frac{1}{i} [D_\mu]_{\alpha \beta} Q_\beta - \frac{1}{4} F_{\mu \nu}^i F^i_{\mu \nu} . \]  

(148)
In this case, to a good approximation one can neglect both breakdown of $U$ better both the origin of the approximate symmetry and the mechanism which causes the should. For instance, at low energy their couplings vanish linearly with energy. show that, dynamically, these states really behave as approximate Nambu-Goldstone states as the likely candidate for these approximately Nambu-Goldstone states. Indeed, one can of the same strong forces that confine quarks into hadrons, condensates of pions ($\pi$) expect some (nearly) massless Nambu-Goldstone states to appear in the theory. The triplet $\text{U}_\text{A}$ Goldstone way. Indeed, if $\text{U}_\text{A}$ this approximate doublet. Because this additional degenerate doublet was not seen in the spectrum of baryons, doublet of states, of opposite parity, approximately degenerate with the neutron-proton would one expect a degenerate neutron-proton doublet but also one should have another even if $\text{U}_\text{A}$ even if they are not equal!) and the strong interactions are then invariant under the $\text{SU}(2)$ isospin group.

The approximate $\text{U}(2)_V \times \text{U}(2)_A$ invariance of the strong interactions was discovered in the 1960’s even before QCD was put forth as the theory of the strong interactions. It was realized then, however, that while the $\text{U}(2)_V$ global symmetry appeared to be realized in nature as a Wigner-Weyl symmetry, the $\text{U}(2)_A$ symmetry was realized in a Nambu-Goldstone way. Indeed, if $\text{U}(2)_A$ were an approximate Wigner-Weyl symmetry, not only would one expect a degenerate neutron-proton doublet but also one should have another doublet of states, of opposite parity, approximately degenerate with the neutron-proton doublet. Because this additional degenerate doublet was not seen in the spectrum of baryons, this approximate $\text{U}(2)_A$ symmetry must be spontaneously broken. In this case, one would expect some (nearly) massless Nambu-Goldstone states to appear in the theory. The triplet of pions ($\pi^+, \pi^-, \pi^0$), which are much lighter than any other meson states, were suggested as the likely candidate for these approximately Nambu-Goldstone states. Indeed, one can show that, dynamically, these states really behave as approximate Nambu-Goldstone states should. For instance, at low energy their couplings vanish linearly with energy.

Matters were clarified further with the advent of QCD, since one was able to understand better both the origin of the approximate symmetry and the mechanism which causes the breakdown of $\text{U}(2)_V \times \text{U}(2)_A$. Let me briefly comment on this last point. In QCD, because of the same strong forces that confine quarks into hadrons, condensates of $u$- and $d$-quarks can form. These condensates are nothing but non-zero expectation values of quark bilinears in the QCD vacuum. Clearly if

$$ \langle 0 | \bar{u}(0) u(0) | 0 \rangle = \langle 0 | \bar{d}(0) d(0) | 0 \rangle \neq 0 , $$

although $\text{U}(2)_V \times \text{U}(2)_A$ is an (approximate) symmetry of the QCD Lagrangian, only $\text{U}(2)_V$ remains as a true symmetry of the spectrum. That is, the above condensates breaks

$$ \text{U}(2)_V \times \text{U}(2)_A \to \text{U}(2)_V $$
Naively one would expect as a result of the above spontaneous breakdown that four Nambu-Goldstone bosons should appear in the theory. In fact, the $U(1)_A$ subgroup of the $U(2)_A$ group, although it is a symmetry at the Lagrangian level, can be shown not to be a real quantum symmetry of QCD. Radiative effects cause the divergence of the $U(1)_A$ current not to vanish. Unfortunately, the argument why the $U(1)_A$ symmetry acquires an anomalous divergence—a so-called chiral anomaly—is too complex to enter upon here. Nevertheless, taking this result at face value, one expects that the formation of the condensates above should produce three Nambu-Goldstone bosons, associated with the breakdown of the $SU(2)_A$ symmetry. These states are the pions. Indeed, one can show that the pion mass attains a finite value once one turns on the $u$- and $d$-quark masses, but vanishes in the limit as $m_u, m_d \to 0$. I will not pursue this point further here, but note only how simply one can understand the approximate symmetry properties of the strong interactions, deduced in the 1960s after much hard work, directly from the QCD Lagrangian and a few dynamical assumptions, Eqs. (145) and (151).

6 The Structure of the $SU(2) \times U(1)$ Theory

The ideas we have just discussed of a spontaneously broken gauge theory have found a spectacular application in the $SU(2) \times U(1)$ model of the electroweak interactions of Glashow, Salam and Weinberg. At first sight, it appears that weak and electromagnetic interactions have little in common, so that a combined gauge model of these forces does not appear very natural. However, there were at least two phenomenological similarities which hinted at a common link, and which helped in the formulation of the $SU(2) \times U(1)$ model.

The first of these similarities is that in both weak and electromagnetic interactions currents are involved. In the electromagnetic case the interaction Lagrangian

\[ \mathcal{L}_{em} = e A^\mu J_{\mu}^{em} \]  

(153)

gives rise to long-range forces between charged particles due to the exchange of a massless photon field. The $1/r$ potential between charged particles follows from the $1/q^2$ propagator for the photon field. The effective action among charged particles due to (153) is simply

\[ W_{em}^{\text{eff}} = \frac{i}{2} \int d^4x \ e^{i J_{\mu}^{em}(x)} \langle T(A_\mu(x) A_\nu(y)) \rangle \ d^4y \ e^{i J_{\nu}^{em}(y)} \]  

(154)

\[ = \frac{1}{2} \int d^4x \ e^{i J_{\mu}^{em}(x)} D_{\mu\nu}(x - y) d^4y \ e^{i J_{\nu}^{em}} , \]

where $D_{\mu\nu}$ is the photon propagator. Since the currents $J_{\mu}^{em}$ are conserved, one can take effectively

\[ D_{\mu\nu}(x - y) = \eta_{\mu\nu} \int \frac{d^4q}{(2\pi)^4} e^{iq(x-y)} \frac{1}{q^2 - i\epsilon} , \]  

(155)

where

\[ \eta_{\mu\nu} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \]  

(156)
is the metric tensor. Hence

\[ W_{em}^{\text{eff}} = \frac{1}{2} \int d^4x d^4y \frac{d^4q}{(2\pi)^4} e^{iq(x-y)} \frac{1}{q^2 - i\epsilon} e_j e_{j\mu}(y) \]  

(157)

Thus, in momentum space, one has simply

\[ L_{em}^{\text{eff}}(q) = \frac{1}{2} \left[ e_j e_{j\mu}(q) \frac{1}{q^2 - i\epsilon} e_j e_{j\mu}(-q) \right]. \]

(158)

For the charged current weak interactions, which are responsible for the rather long lived nuclear disintegrations, like neutron $\beta$ decay, one has known for a long time that they could be described by an effective current-current theory, the Fermi theory:

\[ L_{\text{Fermi}} = \frac{G_F}{\sqrt{2}} \left[ J_\mu(x)J_{-\mu}(x) \right]. \]

(159)

Here $G_F$—the Fermi constant—has dimensions of (mass)$^{-2}$ and $G_F \sim 10^{-5}$ (GeV)$^{-2}$. In momentum space (159) looks like the e.m. case, except that the photon propagator $1/q^2$ is replaced by the constant $G_F/\sqrt{2}$. In momentum space, one has

\[ L_{\text{eff}}^{cc}(q) = \frac{G_F}{\sqrt{2}} \left[ J_\mu(q)J_{-\mu}(q) \right]. \]

(160)

This phenomenological resemblance can be sharpened by imagining that the contact nature of the charged current weak interactions is due to the exchange of a very heavy “weak boson”. For low momentum transfer processes, the propagator of the weak boson would be effectively constant

\[ \frac{1}{q^2 + M_W^2} \geq \frac{1}{M_W^2}. \]

(161)

So Eq. (159) could arise from an interaction Lagrangian very similar to that of electromagnetism:

\[ L_{\text{weak}} = \tilde{g}[J_\mu^W(x)W_{-\mu}(x) + J_\mu^W(x)W_{+\mu}(x)] \]

(162)

involving some spin one bosons $W_{\pm}^\mu$. Then one could obtain, for $q^2 \ll M_W^2$, $L_{\text{eff}}^{cc}$ from the exchange of these massive fields.

\[ L_{\text{eff}}^{cc}(q) \geq \frac{\tilde{g}^2}{M_W^2} \left[ J_\mu^W(q)J_{-\mu}(-q) \right]. \]

(163)

which identifies the Fermi constant as

\[ \frac{G_F}{\sqrt{2}} = \frac{\tilde{g}^2}{M_W^2}. \]

(164)

Note that if $\tilde{g}^2 \sim e^2$ then from the value of $G_F$ one infers that the masses of the weak bosons are really heavy: $M_W \sim 100$ GeV!
The second similarity between weak and electromagnetic processes is that the charged currents that enter in weak decays appear to be related to the electromagnetic current—at least as far as the strongly interacting particles go. This interrelation was discussed long ago by Feynman and Gell-Mann, and by Marshak and Sudarshan.

The vector piece of the $J_{\pm}^\mu$ currents are identical to the $1 \pm i2$ components of the strong isospin current. In turn the isovector piece of the electromagnetic current is the 3rd component of this same strong isospin current.

Although the above two points hint at a possible common origin of weak and electromagnetic interactions, they are not per se compelling. The dominant reason for attempting to treat both interactions on the same footing is theoretical. The Fermi theory is actually a very sick theory as it stands, since in higher order in perturbation theory one encounters divergences which one cannot eliminate from the theory. These divergences occur because of the very singular nature of the contact interaction which, in contrast to what happens in QED, is not being damped at all for large $q^2$.

It turns out that matters are not ameliorated even if the Fermi theory is replaced by an interaction like (162), involving mediating heavy vector bosons $W_{\pm}^\mu$. This is because the propagator for such a massive boson contains in the numerator a propagator factor, characteristic of a spin one object, which is badly behaved at large $q^2$:

$$\Delta_{\mu\nu}(q) = \frac{1}{q^2 + M_W^2} \left( \eta_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{M_W^2} \right),$$

Thus it is not possible to add “by hand” an interaction like (162) and hope to obtain a sensible weak interaction theory. If, however, the interaction (162) resulted from making a global symmetry local—so that the $W_{\pm}^\mu$ are gauge fields which are massive because of the Higgs mechanism—then the situation is vastly improved. It turns out that the gauge invariance of the theory allows one to calculate higher order corrections with propagators for the $W$-fields which have only the $\eta_{\mu\nu}$ term. These theories, as first shown by ’t Hooft, have the same good asymptotic behavior as QED. They are renormalizable.

The above argues for a theory of the weak interactions based on some symmetry group $G$ which spontaneously breaks down. Two of the currents associated with $G$ must include $J_{\pm}^\mu$ and $J_{em}^\mu$. However, the generator algebra must close and so one expects naturally also some neutral current. This current, in general, will be related to the electromagnetic current. Thus we see that renormalizability has lead us directly to contemplate models in which at the Lagrangian level, weak and electromagnetic currents enter on the same footing!

The simplest unified model of the electroweak interactions, which contains $J_{\pm}^\mu$, $J_{em}^\mu$ and $J_{em}^\mu$, is based on the group $O(3)$. However, the discovery of weak neutral current processes experimentally argued for at least a 4-parameter group. The suggestion of Glashow, Salam and Weinberg, made well before the discovery of these neutral currents processes, was that the electroweak interactions are based on an $SU(2)\times U(1)$ gauge theory, which suffers spontaneous breakdown to $U(1)_{em}$. This theory has three massive gauge bosons, associated with the broken generators, and a massless gauge field, associated with the photon. The model was built to reproduce the known structure of the charged current weak interactions. It then predicted particular neutral current interactions, whose experimental verification provided a direct test of the model. Furthermore, the model also predicts the masses of
the gauge fields associated with the spontaneous breakdown. The observation at CERN of the $W^\pm$ and $Z^0$ bosons, with the masses predicted by the model, provided the final experimental confirmation of the validity of the $SU(2) \times U(1)$ theory.

To detail the structure of the GSW model, one has to specify how the matter degrees of freedom transform under the $SU(2) \times U(1)$ group. This could be deduced from the form of the charged currents $J_\mu^\pm$, which a long series of experiments in the 1950’s and 1960’s showed to have a (V-A) form. That is, only the left-handed projection of the fermionic fields appear to participate in these interactions. For instance, from a study of $\beta$-decay for the muon one established that the current $J_\mu^+$ had both $\mu - \nu_\mu$ and $e - \nu_e$ terms, in which only the left-handed neutrino fields entered:

$$J_\mu^+ = \bar{e} \gamma^\mu (1 - \gamma_5) \nu_e + \bar{\mu} \gamma^\mu (1 - \gamma_5) \nu_\mu + \ldots .$$  \hspace{1cm} (166)

Writing the projections

$$\psi = \frac{1}{2} (1 - \gamma_5) \psi + \frac{1}{2} (1 + \gamma_5) \psi = \psi_L + \psi_R$$  \hspace{1cm} (167)

and using the properties

$$\{ \gamma_5, \gamma^\mu \} = 0 ; \quad \gamma_5^2 = 1 ; \quad \bar{\psi} = \psi^\dagger \gamma^5 ; \quad \gamma_5^\dagger = \gamma_5 ,$$  \hspace{1cm} (168)

one sees that

$$J_\mu^+ = 2 \bar{e}_L \gamma^\mu \nu_{eL} + 2 \bar{\mu}_L \gamma^\mu \nu_{\mu L} + \ldots .$$  \hspace{1cm} (169)

That is, the charged currents only contain left-handed fields.

The structure of $J_\mu^\pm$, and its complex conjugate $J_\mu^-$, suggests that under $SU(2)$ the $\nu_{eL}$ fields (and the $\nu_{\mu L}$ and $\mu_L$ fields) transform as a doublet. The appropriate generator matrix for an $SU(2)$ doublet is $\frac{1}{2} \tau^i$, where $\tau^i$ are the Pauli matrices. Indeed these matrices obey the $SU(2)$ Lie algebra

$$[\frac{\tau^i}{2}, \frac{\tau^j}{2}] = i \epsilon_{ijk} \frac{\tau^k}{2} .$$  \hspace{1cm} (170)

Hence, if $\left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L$ transforms as a doublet, the relevant piece of the $SU(2)$ current involving these fields is

$$J_i^\mu = (\bar{\nu}_e \bar{e})_L \gamma^\mu \frac{\tau_i}{2} \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L$$  \hspace{1cm} (171)

and one sees that indeed

$$2(J_1^\mu - i J_2^\mu) = 2(\bar{\nu}_e \bar{e})_L \gamma^\mu \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L = 2 \bar{e}_L \gamma^\mu \nu_{eL} = (J_\mu^\mu)_{\nu_e - e}$$  \hspace{1cm} (172)

The fundamental matter entities presently known are quarks and leptons, which appear in a repetitive pattern as far as the $SU(2) \times U(1)$ interactions are concerned [cf. the $\nu_{e} - e$ and $\nu_{\mu} - \mu$ terms in $J_\mu^\mu$ of Eq. (166)]. To date we know of the existence of three generations of quarks and leptons: the electron family: $\left( \nu_e, e; u, d \right)$; the muon family $\left( \nu_{\mu}, \mu; c, s \right)$ and
Table 1: Transformation properties of quarks and leptons

| States          | SU(2) | U(1)  | Q   |
|-----------------|-------|-------|-----|
| $(\nu_e)_L$     | 2     | -1/2  | 0   |
| $(\nu_e)_R$     | 1     | 0     | 0   |
| $(e)_R$         | 1     | -1    | -1  |
| $(u)_R$         | 1     | 2/3   | 2/3 |
| $(d)_R$         | 1     | -1/3  | -1/3|

the $\tau$-lepton family ($\nu_\tau, \tau; t, b$), where to each lepton doublet there are associated a pair of quarks. The quarks in the pair actually are comprised each of three states, since each quark carries a color index $\alpha = 1, 2, 3$. As we just discussed, these color degrees of freedom are associated with the strong interactions of quarks, which are based on an SU(3) gauge theory realized in a Wigner-Weyl way—QCD.

Because all the three families transform in the same way under $SU(2) \times U(1)$, I will only describe the $SU(2) \times U(1)$ properties of the electron family. In view of the preceding discussion, it is clear that $(\nu_e)_L$ transforms as an $SU(2)$ doublet. So does the quark pair $(u \ d)_L$, as an analysis of beta decay of nuclei indicates. Furthermore, since only left-handed fields enter in the weak charged currents, it must be that the right-handed components of the electron family are $SU(2)$ singlets. Since the $SU(2) \times U(1)$ group must eventually break down to $U(1)_{em}$, it follows that the electromagnetic charge must be a linear combination of the $U(1)$ generator and of the neutral $T_3$ generator of $SU(2)$, which is diagonal. Thus one can write

$$Q = T_3 + Y,$$

with $Y$ being the $U(1)$ generator. Hence the $U(1)$ quantum numbers of the fields in the electron family follow from their known charges. These considerations allow us to build the following table for the transformation properties of $\nu_e, e, u$ and $d$ under $SU(2) \times U(1)$. The right-handed neutrino field $\nu_R$ in Table 1 is usually not included as a real excitation, since it is a total $SU(2) \times U(1)$ singlet and so does not participate in these interactions.

Given the transformation properties of the quarks and leptons under $SU(2) \times U(1)$, we may now immediately write down the locally $SU(2) \times U(1)$ invariant Lagrangian which describes their interactions. For that purpose we need only to replace in the free Dirac Lagrangian for the fermion fields the ordinary derivatives $\partial_\mu \psi$ by the appropriate $SU(2) \times$
covariant derivatives $D_\mu \psi$ and add the gauge field interactions. Using Table 1, it is trivial to write down these covariant derivatives. One has

$$D_\mu \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L = \left( \partial_\mu - ig \frac{\tau_i}{2} W_{\mu i} + ig' \frac{1}{2} Y_\mu \right) \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L$$

(174)

$$D_\mu \left( \begin{array}{c} u \\ d \end{array} \right)_L = \left( \partial_\mu - ig \frac{\tau_i}{2} W_{\mu i} - ig' \frac{1}{6} Y_\mu \right) \left( \begin{array}{c} u \\ d \end{array} \right)_L$$

(175)

$$D_\mu \nu_R = (\partial_\mu) \nu_R$$

(176)

$$D_\mu e_R = (\partial_\mu + ig' Y_\mu) e_R$$

(177)

$$D_\mu u_R = (\partial_\mu - ig' \frac{2}{3} Y_\mu) u_R$$

(178)

$$D_\mu d_R = (\partial_\mu + ig' \frac{1}{3} Y_\mu) d_R$$

(179)

Here $g, g'$ are the $SU(2)$ and $U(1)$ coupling constants, respectively, while $W_{\mu i}$ and $Y_\mu$ are the $SU(2)$ and $U(1)$ gauge fields, respectively.

The Lagrangian for the $SU(2) \times U(1)$ model of Glashow, Salam and Weinberg—as far as the interactions among the fermions of the electron family and the gauge fields go—is then simply

$$L_{FG} = - (\bar{\nu}_e \bar{e})_{L, \gamma^\mu} \frac{1}{i} D_\mu \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L - (\bar{u} \bar{d})_{L, \gamma^\mu} \frac{1}{i} D_\mu \left( \begin{array}{c} u \\ d \end{array} \right)_L$$

(180)

$$- \bar{e}_R \gamma^\mu \frac{1}{i} D_\mu e_R - \bar{u}_R \gamma^\mu \frac{1}{i} D_\mu e_R - \bar{u}_R \gamma^\mu \frac{1}{i} D_\mu d_R$$

$$- \frac{1}{4} W^{\mu \nu} W_{\mu \nu} - \frac{1}{4} Y^{\mu \nu} Y_{\mu \nu},$$

where the field strengths $W^{\mu \nu}$ and $Y^{\mu \nu}$ are given by

$$W^{\mu \nu}_i = \partial^\mu W_i^\nu - \partial^\nu W_i^\mu + g \epsilon_{ijk} W^\mu_j W^\nu_k$$

(181)

$$Y^{\mu \nu} = \partial^\mu Y^\nu - \partial^\nu Y^\mu.$$ (182)

Note that the Lagrangian (180) contains no mass terms for the fermion fields. Mass terms involve a left-right transition

$$L_{mass} = -m \bar{\psi} \psi = -m (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L).$$

(183)

Since under $SU(2)$ $\psi_L \sim 2$ and $\psi_R \sim 1$, clearly the $SU(2) \times U(1)$ symmetry permits no fermion mass terms. As I will show later, however, masses can be generated when $SU(2) \times U(1)$ is spontaneously broken down.

Before we discuss the breakdown of $SU(2) \times U(1)$ it is useful to organize a bit the interaction terms which emerge from the Lagrangian (180). These take the simple form

$$L_{int} = g W_i^\mu J_{\mu i} + g' Y^\mu J_{\mu Y}$$

(184)

where the $SU(2)$ and $U(1)$ currents, $J_i^\mu$ and $J_Y^\mu$ are readily seen to be

$$J_i^\mu = (\bar{\nu}_e \bar{e})_{L, \gamma^\mu} \frac{1}{i} \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L + (\bar{u} \bar{d})_{L, \gamma^\mu} \frac{1}{i} \left( \begin{array}{c} u \\ d \end{array} \right)_L$$

(185)
\[ J^\mu_Y = \frac{1}{2}(\bar{\nu}_e e)_L \gamma^\mu \left( \nu_e \right)_L + \frac{1}{6}(\bar{u} d)_L \gamma^\mu \left( u \right)_L \]
\[ - \bar{e}_R \gamma^\mu e_R + \frac{2}{3} \bar{u}_R \gamma^\mu u_R - \frac{1}{3} \bar{d}_R \gamma^\mu d_R . \] (186)

I note that since in the model the electromagnetic current is given by [cf. Eq. (173)]
\[ J^\mu_{em} = J^\mu_3 + J^\mu_Y \] (187)

the phenomenological observation mentioned earlier, that the vector piece of the weak charged currents and the isovector piece of \( J^\mu_{em} \) are related, is built in already in (187).

It is convenient to rewrite (184) in terms of physical fields. If the model is to reproduce the weak interactions, the \( SU(2) \times U(1) \) symmetry must suffer a spontaneous breakdown to \( U(1)_{em} \). This means that of the four gauge fields \( W_\mu, Y^\mu \), three must acquire a mass and one will remain massless. Now, in general, \( U(1)_{em} \) is a linear combination of an \( U(1) \subset SU(2) \) and \( U(1)_Y \), so that one expects the photon fields to be a linear combination of \( W_\mu \) and \( Y^\mu \).

The orthogonal combination then corresponds to a massive neutral field—the \( Z^0 \) boson. It has become conventional to parametrize these linear combinations in terms of an angle \( \theta_W \)— the Weinberg angle.

\[ W^\mu_3 = \cos \theta_W Z^\mu + \sin \theta_W A^\mu \]
\[ Y^\mu = - \sin \theta_W Z^\mu + \cos \theta_W A^\mu \] (188)

It proves useful also to rewrite \( W^\mu_1 \) and \( W^\mu_2 \) in terms of fields of definite charge
\[ W^\mu_\pm = \frac{1}{\sqrt{2}}(W^\mu_1 \mp i W^\mu_2) \] (189)

and use the charged currents \( J^\mu_\pm \), which enter in the Fermi theory [cf. Eq. (159)]
\[ J^\mu_\pm = 2(J^\mu_1 \mp i J^\mu_2) . \] (190)

With all these definitions the interaction Lagrangian of Eq. (184) becomes
\[ \mathcal{L}_{int} = \frac{g}{2\sqrt{2}}[W^\mu_+ J^-_\mu + W^\mu_- J^+\mu] \]
\[ \{(g \cos \theta_W + g' \sin \theta_W)J^\mu_3 - g' \sin \theta_W J^\mu_{em}\} Z_\mu \]
\[ + \{g' \cos \theta_W J^\mu_{em} + (g' \cos \theta_W - g \sin \theta_W)J^\mu_3\} A_\mu . \] (191)

In the above, I have made use of (187) to eliminate altogether \( J^\mu_Y \) in favor of \( J^\mu_{em} \).

The above interaction is supposed to reproduce both the electromagnetic interaction (153) and the charged current weak interaction (162). It predicts as well a new neutral current weak interaction involving the \( Z^0 \) boson. Since the photon field is supposed to only interact with \( J^\mu_{em} \) with strength \( e \), one sees that one must require the Weinberg angle to obey the \textit{unification} condition
\[ g' \cos \theta_W = g \sin \theta_W = e . \] (192)
Using this information to eliminate $g$ and $g'$ in terms of $\theta_W$ and $e$ allows one to write for the interaction Lagrangian the expression:

$$L_{\text{int}} = \frac{e}{2\sqrt{2}\sin \theta_W} (W_+^{\mu} J_{\mu-} + W_-^{\mu} J_{\mu+}) + \frac{e}{2 \cos \theta_W \sin \theta_W} J_{\mu}^{\text{NC}} Z_{\mu}.$$  \hspace{1cm} (193)

Here the neutral current $J_{\mu}^{\text{NC}}$ which interacts with the $Z_{\mu}$ field is

$$J_{\mu}^{\text{NC}} = 2[J_{3}^{\mu} - \sin^2 \theta_W J_{\mu}^{\text{em}}].$$  \hspace{1cm} (194)

Comparing this result with our earlier discussion, the coupling $\tilde{g}$ of Eq. (162) is seen to be

$$\tilde{g} = \frac{e}{2\sqrt{2}\sin \theta_W}.$$ \hspace{1cm} (195)

Hence, the comparison with the Fermi theory [cf. Eq. (164)] gives for the Fermi constant the expression

$$\frac{G_F}{\sqrt{2}} = \frac{\tilde{g}^2}{M_W^2} = \frac{e^2}{8 \sin^2 \theta_W M_W^2}.$$ \hspace{1cm} (196)

One sees that a knowledge of the Weinberg angle–which enters in the neutral current–gives direct information on the mass of the heavy weak boson which mediates the charged current weak interactions. One finds experimentally that $\sin^2 \theta_W \simeq 1/4$, which predicts for $M_W$ a value of around 80 GeV. This prediction has been spectacularly confirmed by the discovery at the CERN Collider of a particle of this mass with all the characteristic of the $W$ boson.

Just as charged current interactions, for processes where the momentum transfer $q^2 \ll M_W^2$, can be described by the Fermi theory, one can arrive at a similar structure for neutral current interactions. In the same approximation, $q^2 \ll M_Z^2$, one has

$$L_{\text{NC Fermi}} \simeq \frac{1}{2} \left[ \frac{e}{2 \sin \theta_W \cos \theta_W} \right]^2 \frac{1}{M_Z^2} J_{\mu}^{\text{NC}} J_{\mu}^{\text{NC}}.$$ \hspace{1cm} (197)

Using the identification (195) of the Fermi constant, one has

$$L_{\text{NC Fermi}} = \frac{G_F}{\sqrt{2}} \left[ \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} \right] J_{\mu}^{\text{NC}} J_{\mu}^{\text{NC}} = \frac{G_F}{\sqrt{2}} \rho J_{\mu}^{\text{NC}} J_{\mu}^{\text{NC}},$$ \hspace{1cm} (198)

where the ratio

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W}$$ \hspace{1cm} (199)

gives the relative strength of neutral to charged current weak processes.

To summarize, the weak interactions in the Glashow Salam Weinberg model, in the limit in which $q^2 \ll M_W^2, M_Z^2$ can be written in a current-current form

$$L_{\text{Weak eff}} = \frac{G_F}{\sqrt{2}} [J^{\mu}_{\mu} J_{\mu-} + \rho J_{\mu}^{\text{NC}} J_{\mu}^{\text{NC}}].$$ \hspace{1cm} (200)
The charged current weak interactions by construction agree with experiment. Neutral current weak interactions test the model, since all experiments must be describable by the only two free parameters $\rho$ and $\sin^2 \theta_W$, which enters in the definition of $J^\mu_{NC}$, present in (194). All neutral current experiments, indeed, can be fitted with a common value of $\sin^2 \theta_W \simeq 1/4$ and of $\rho \simeq 1$, thereby providing strong confirmation of the validity of the GSW model. Furthermore, given $\rho$ and $\sin^2 \theta_W$, one can determine the mass of the $Z^0$ and $W^\pm$ bosons from Eqs. (196) and (199). The discovery at the CERN collider of the $W^\pm$ bosons and, soon thereafter, of a neutral heavy particle of mass around 90 GeV, in agreement with the value predicted by the GSW model, provided a splendid confirmation of the model.

To complete the GSW model, it is necessary to describe briefly the mechanism by which the $W^\pm$ and $Z^0$ bosons get mass. The idea here is very much like that described in the last section, when I discussed the Higgs mechanism for the Abelian $U(1)$ model. Namely, one introduces some scalar field whose self interactions cause the $SU(2) \times U(1)$ symmetry to break down. Since we want $SU(2)$ to break down, the scalar field introduced into the theory must carry $SU(2)$ quantum numbers. The simplest possibility is afforded by an $SU(2)$ doublet. Furthermore, since we want also to break the $U(1)$ symmetry, this doublet must be complex. Thus the simplest agent to carry through the desired breakdown is the complex doublet.

$$\Phi = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix},$$

(201)

where $\phi^0$ and $\phi^-$ are complex fields, and the charge assignments identify $Y_\phi = -1/2$.

To accomplish the breakdown we consider a potential analogous to that in (58). In addition we must introduce an appropriately $SU(2) \times U(1)$ covariant kinetic energy term for the field $\Phi$, using the covariant derivative

$$D_\mu \Phi = \left( \partial_\mu - ig\frac{\tau_i}{2} W^i_\mu + ig'\frac{Y}{2} Y^0_\mu \right) \Phi.$$

(202)

The interaction Lagrangian involving the scalar field $\Phi$—the Higgs field—is just then:

$$\mathcal{L}_{HG} = - (D_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda \left( |\Phi|^2 - \frac{v^2}{2} \right)^2.$$

(203)

It is clear that the potential term in (203) will cause $SU(2) \times U(1)$ to break down. The choice of vacuum expectation value

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

(204)

guarantees that $SU(2) \times U(1) \rightarrow U(1)_{em}$. [Actually, with just one doublet $\Phi$ one can always define $U(1)_{em}$ as the $U(1)$ left unbroken in $V$. The choice (204) is dictated by our definition of charge. Any other choice would do, but it would change what we called $Q$.]

Given the vacuum expectation value (204), the mass terms for the gauge fields are read off immediately from the seagull terms:

$$\mathcal{L}_{seagull} = - \left( g\frac{\tau_i}{2} W^i_\mu - g'\frac{1}{2} Y^\mu \right) \Phi \right]^\dagger \left[ \left( g\frac{\tau_i}{2} W^i_\mu - g'\frac{1}{2} Y^\mu \right) \Phi \right].$$

(205)
Using Eq. (188), the gauge field matrix in (205) is easily seen to be
\[
g\tau^i W_\mu^2 - g' \frac{1}{2} Y^\mu = \begin{bmatrix}
g \sqrt{2} W_3^\mu - g' Y^\mu \\
g \sqrt{2} W_-^\mu - g' Y^\mu \
- g \sqrt{2} W_3^\mu - g' Y^\mu
\end{bmatrix} (206)
\]

(206)

Since the vacuum expectation value (204) only has an upper component, one sees that only \( Z^\mu \) and not \( A^\mu \) acquires a mass, confirming our previous identification of this latter field as the photon field. Replacing in (205) \( \Phi \rightarrow \langle \Phi \rangle \) gives the following mass terms for the gauge fields:
\[
\mathcal{L}_{\text{mass}} = - \left( \frac{gv}{2} \right)^2 W_+^\mu W_-^\mu - \frac{1}{2} \left( \frac{gv}{2 \cos \theta_W} \right)^2 Z^\mu Z_\mu . (207)
\]

Hence
\[
M_W^2 = \frac{1}{4} (gv)^2 ; \quad M_Z^2 = \frac{1}{4 \cos^2 \theta_W} (gv)^2 . (208)
\]

We see that the simplest choice of Higgs field to give the \( SU(2) \times U(1) \rightarrow U(1)_{\text{em}} \) breaking predicts that the parameter \( \rho \) in the neutral current interactions is unity!
\[
\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} = 1 . (209)
\]

The experimental indications that \( \rho \approx 1 \) suggest therefore that nature has chosen (again) the simplest course. Using Eq. (208) and the relation of \( M_W \) to the Fermi constant identifies the scale parameter \( v \) in the Higgs potential as
\[
v = (\sqrt{2} G_F)^{-1/2} \approx 250 \, \text{GeV} . (210)
\]

The introduction of a doublet Higgs field \( \Phi \) into the theory has another salutary effect—it allows for the possibility of generating masses for the quarks and leptons! I will illustrate the idea with the up quark. Since \( \Phi \) carries hypercharge \( Y_\Phi = -1/2 \) and is an \( SU(2) \) doublet, the interaction (Yukawa interaction) of \( \Phi \) with \( u^R \) and the \( (\bar{u} \bar{d})_L \) doublet is allowed by the \( SU(2) \times U(1) \) symmetry:
\[
\mathcal{L}_{\text{Yukawa}} = -h (\bar{u} \bar{d})_L \Phi u^R - h^* \bar{u} R \Phi^1 \begin{pmatrix} u \\ d \end{pmatrix}_L . (211)
\]

Obviously, when \( \Phi \) has a vacuum expectation value this interaction will generate a mass term for the \( u \) quark. Taking \( h \) real, one has
\[
\mathcal{L}_{\text{mass}} = - \frac{h v}{\sqrt{2}} \bar{u} u = -m_u \bar{u} u , (212)
\]
so that \( m_u \) is also related to the breakdown parameter \( v \). Unfortunately since \( h \) is not known, no predictions follow. This same mass generation procedure holds for all quarks and leptons.
Acknowledgements

I am extremely grateful to J. Tran Than Van for having invited me to lecture in the fifth Vietnam School of Physics. The extremely friendly atmosphere of the school and of all the participants made my stay in Hanoi a real pleasure. This work was supported in part by the department of energy under contract No. DE-FG03-91ER40662, Task C.

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