ON THE SPECTRUM OF THE SUPERPOSITION OF SEPARATED POTENTIALS

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Abstract. Suppose that $V(x)$ is an exponentially localized potential and $L$ is a constant coefficient differential operator. A method for computing the spectrum of $L + V(x) + ... + V(x-x_N)$ given that one knows the spectrum of $L + V(x)$ is described. The method is functional theoretic in nature and does not rely heavily on any special structure of $L$ or $V$ apart from the exponential localization. The result is aimed at applications involving the existence and stability of multi-pulses in partial differential equations.

1. Introduction. In systems of nonlinear partial differential equations which admit coherent structures (for instance traveling waves or stationary solutions) which are localized in the sense that the tails are exponentially decaying, it is natural to inquire about solutions which are roughly the superposition two or more such solutions. Supposing that the linearization about one structure has the form

$$L + V(x)$$

where $L$ is a constant coefficient differential operator and $V(x)$ is a localized function, then when considering the existence and stability of the superposition of several structures one will have to deal with the operator

$$L + V(x-x_0) + V(x-x_1) + ... + V(x-x_N).$$

In this document we consider the following question: can we determine the spectrum of $L + V(x-x_0) + V(x-x_1) + ... + V(x-x_N)$ if we know the spectrum of $L + V(x)$. Our main result says (in essence) that the spectrum of the second is a small perturbation of $N$ copies of the former.

Theorems similar to this result have been proven in a number of contexts using a variety of tools (see, for instance, [10, 7]). Our approach is relatively simple, works in arbitrary dimension and for arbitrary numbers of potentials, applies to systems as well as scalar equations and does not rely strongly on any underlying structure of $L$ or the $V_j$ apart from the exponential localization.

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2. Preliminaries. Suppose that $L$ is a constant-coefficient differential linear operator of order $s \geq 1$ which acts on functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^d$:

$$L : (W^{s,p}(\mathbb{R}^n))^d \rightarrow (L^p(\mathbb{R}^n))^d.$$ 

We further assume that $L$ has the following estimate:

$$C^{-1}||u||_{(W^{s',p}(\mathbb{R}^n))^d} \leq ||u||_{(L^p(\mathbb{R}^n))^d} \leq ||Lu||_{(L^p(\mathbb{R}^n))^d} \leq C||u||_{(W^{s,p}(\mathbb{R}^n))^d}. \quad (1)$$

Here $1 \leq s' \leq s$. This assumption is satisfied, for instance, if $L$ is elliptic. Henceforth, for simplicity, we write $W^{s,p}$ instead of $(W^{s,p}(\mathbb{R}^n))^d$, $L^p$ instead of $(L^p(\mathbb{R}^n))^d$ and so on.

Additionally assume that for $j = 1, \ldots, N$ we have the matrix valued “potential” functions

$$V_j : \mathbb{R}^n \rightarrow GL(\mathbb{R}^d) \quad (2)$$

which we assume are $C^\infty$ and are “exponentially localized at 0”. That is to say, there are $\beta_0, \Omega > 0$ so that for $j = 1, \ldots, N$ we have

$$\sup_{x \in \mathbb{R}^n} |\cosh(\beta_0|x|)V_j(x)| \leq \Omega. \quad (3)$$

Set

$$A := L + \sum_{j=1}^N V_j(x - x_j).$$

Our goal is the determine the spectrum of $A$ given that we know the spectra of

$$A_j := L + V_j(x).$$

In particular we are interested in the case when the $V_j$ are well-separated, i.e. when

$$\Gamma := \max_{j \neq k} |x_j - x_k| \gg 1.$$

We denote by $\sigma(T)$ the spectrum of an operator $T$ when viewed as an unbounded operator on $L^p$. Its point spectrum is denoted as $\sigma_{pt}(T)$ and its essential spectrum as $\sigma_{ess}(T)$. Note that in this document, we define the essential spectrum of an operator $T$ as those points $\lambda \in \mathbb{C}$ at which $T - \lambda$ fails to be a Fredholm operator.

We will make use of the exponentially weighted spaces:

$$W_b^{s,p}(\mathbb{R}^n) := \{f(x) : \cosh(b|x|)f(x) \in W^{s,p}(\mathbb{R}^n)\}$$

and $L_b^p(\mathbb{R}^n) := W_b^{0,p}(\mathbb{R}^n)$. At times we will choose to view our operators as acting on the weighted spaces, and in that case we denote the spectrum by $\sigma^b(T)$, $\sigma_{ess}^b(T)$ and $\sigma_{pt}^b(T)$. We likewise define the resolvent sets $\rho(T)$ and $\rho^b(T)$.

The exponential localization of the $W_b^{s,p}$ spaces together with the Kondrashov embedding theorem imply, for any $s > 0$, $b > 0$,

$$W_b^{s,p} \subset \subset L^p.$$ 

Therefore (1) implies that the $V_j$ are relatively compact with respect to $L$. And so

$$\sigma_{ess}^b(A) = \sigma_{ess}^b(A_j) = \sigma_{ess}^b(L)$$

for $0 \leq b < \beta_0$. One can compute $\sigma_{ess}^b(L)$ explicitly by means of the Fourier transform (see, e.g., [4] Theorem 6.2, p 430) and the methods described in [11] (specifically Theorem A2, p 140) can be used to find the essential spectrum in the weighted spaces. Moreover, because $L$ is translation invariant we have $\sigma^0(L) = \sigma_{ess}^0(L)$, and arguments in Lemma 4.2 show the same is true for $\sigma^b(L)$. Thus we are primarily interested in determining the point spectrum of $A$. 

Our main result is:

**Theorem 2.1.** Suppose that (1) and (3) are satisfied. Let \( K \) be a compact subset of \( \mathbb{C} \) such that \( K \cap \sigma_{ss}(A) = \emptyset \). Moreover suppose that \( A_j \) (for \( j = 1, \ldots, N \)) when viewed as an unbounded operator on \( L^p \), has finitely many eigenvalues of total multiplicity \( m(j) < \infty \) within \( K \). Then there exists \( 0 < \beta < \beta_0, \Gamma_0 > 0 \) and \( c > 0 \) such that for all \( \Gamma \geq \Gamma_0 \) the spectrum of \( A \) within \( K \) consists only of a set of eigenvalues, of total multiplicity \( M = \sum_{j=1}^{N} m(j) \), which are \( O(e^{-c\Gamma}) \) close to the union of the eigenvalues of the \( A_j \) in \( K \).

The proof of this result is based upon methods developed in [8, 12] (and which are related to methods used in [9, 3]) to study the existence and stability of multipulse solutions to reaction-diffusion equations. There are three main steps. The first is to embed the eigenvalue problem for \( A \) into a larger system in which \( A \) can be viewed, in some sense, as a perturbation of the \( A_j \). The second step is to prove that this perturbation is in fact small when applied to localized functions and then to apply classical linear perturbation theory in the spaces \( L^p \). The final step is to prove that the restriction to the weighted spaces can in fact be removed.

3. **The embedding.** To compute the spectrum of \( A \) we need to understand when we can solve the equation

\[
(A - \lambda)f = g.
\]

We would like to view \( A \) as being some sort of superposition of the \( A_j \), and so we chop up \( \mathbb{R}^n \) around each of the \( x_j \). Let

\[
U_j := \left\{ x : |x - x_j| = \min_k |x - x_k| \right\},
\]

the set of points which are nearest to \( x_j \). Then let

\[
\hat{U}_j := \{ x : \text{dist}(x, U_j) \leq 1/2 \}.
\]

be thickened versions of the \( U_j \). We assume that \( \Gamma_0 \geq 1 \) so that \( x_j \in U_j \) if and only if \( j = k \). One should think of \( V_j(x - x_j) \) as being localized to \( \hat{U}_j \). Let \( \chi_1, \ldots, \chi_N \) be a \( C^\infty \) partition of unity subordinate to the \( \hat{U}_j \).

Next we decompose the functions \( f \) and \( g \) as:

\[
f(x) = \sum_{i=1}^{N} f_i(x - x_i) \quad \text{and} \quad g(x) = \sum_{i=1}^{N} g_i(x - x_i)
\]

where the \( f_i \) and \( g_i \) are unspecified functions. Then \( (A - \lambda)f \) can be rewritten:

\[
(A - \lambda)f(x) = (L - \lambda) \sum_{i} f_i(x - x_i) + \sum_{i} \sum_{j} V_j(x - x_j)f_i(x - x_i)
\]

In the second sum, we split off the diagonal terms where \( i = j \) to get:

\[
\sum_{i} (L + V_i(x - x_i) - \lambda)f_i(x - x_i) + \sum_{j \neq k} \sum_{i} V_j(x - x_j)f_k(x - x_k).
\]

Since \( \sum_i \chi_i(x) = 1 \) for any \( x \), we have:

\[
\sum_{i} (L + V_i(x - x_i) - \lambda)f_i(x - x_i) + \sum_{i} \chi_i(x) \sum_{j \neq k} V_j(x - x_j)f_k(x - x_k)
\]
A rearrangement of terms here and the fact that we are trying to solve \((A - \lambda)f = g\) gives us:

\[
\sum_i \left[ (L + V_i(x - x_i) - \lambda) f_i(x - x_i) + \chi_i(x) \sum_{k \neq j} V_j(x - x_j) f_k(x - x_k) \right] = \sum_i g_i(x - x_i).
\]

This last equation we “decompose by hand”—for each \(i\):

\[
(L + V_i(x - x_i) - \lambda) f_i(x - x_i) + \chi_i(x) \sum_{k \neq j} V_j(x - x_j) f_k(x - x_k) = g_i(x - x_i),
\]

or rather

\[
(L + V_i(x) - \lambda) f_i(x) + S(x_i) \left[ \chi_i(x) \sum_{k \neq j} V_j(x - x_j) f_k(x - x_k) \right] = f_i(x),
\]

where

\[
S(y) f(x) := f(x + y).
\]

If we set \(f(x) := (f_1(x), f_2(x), \ldots, f_N(x))^t\), \(g(x) := (g_1(x), g_2(x), \ldots, g_N(x))^t\),

\[
A_0 := \text{diag} \{A_1, A_2, \ldots, A_N\}
\]

and define the operator \(B\) component-wise as

\[
(Bf)_i := S(x_i) \left[ \chi_i(x) \sum_{k \neq j} V_j(x - x_j) f_k(x - x_k) \right]
\]

then, with \(A := A_0 + B\), we have the augmented version of (4):

\[
(A - \lambda)f := (A_0 + B - \lambda)f = g.
\]

Let

\[
Rf(x) := \sum_i f(x - x_i).
\]

By construction, we have:

\[
(A - \lambda)f = g \implies (A - \lambda)Rf = Rg.
\]

It may seem that some spectral information is changed in going from \(A\) to \(A\), given that choices are made in choosing the sets \(U_j\) and the partition of unity \(\chi_j\). Nevertheless the spectrum of \(A\) is nearly identical to that of \(A\), and we have a characterization of any differences which occur. Specifically:

**Lemma 3.1.** We have \(\sigma(A) \subseteq \sigma(A)\). If in addition we have \(Rf \neq 0\) for each eigenfunction \(f\) of \(A\), then \(\sigma(A) = \sigma(A)\).

**Remark 1.** Note that this lemma remains true when we view \(A\) and \(A\) as operators on the weighted spaces \(L^p_b\).
Proof. First,
\[ \sigma_{ess}(A) = \sigma_{ess}(A) \]
because \( A \) is a relatively compact perturbation of \( L \) and \( A \) is a relatively compact perturbation of \( \text{diag} \{ L, L, ..., L \} \).

Fix \( \lambda \in \rho(A) \). We will show that \( \lambda \in \rho(A) \). First we claim \( A - \lambda \) is onto. Fix \( g \in L^p \) arbitrary and set \( g(x) = (g(x + x_1), 0, ..., 0)^t \). Since \( \lambda \in \rho(A) \) there exists \( f \) so that
\[ (A - \lambda)f = g \]
and thus (6) implies \( (A - \lambda)Rf = g \), and the claim is shown.

It is not obvious that \( A - \lambda \) is one to one. Suppose the contrary. Then there exists a function \( f \neq 0 \) in the kernel of \( A - \lambda \). Let \( g_0 = (f(x + x_1), 0, ..., 0)^t \). Since \( \lambda \in \rho(A) \) there exists \( f_0 \) so that
\[ (A - \lambda)f_0 = g_0. \]
Thus \( (A - \lambda)Rf_0 = f \). And so \( \lambda \) is of algebraic multiplicity at least 2. Now set \( g_1 = (Rf_0(x + x_1), 0, ..., 0)^t \) and \( f_1 \) be the function for which
\[ (A - \lambda)f_1 = g_1. \]
Which implies \( (A - \lambda)^2Rf_1 = (A - \lambda)Rf_0 = f \). Thus so \( \lambda \) is of algebraic multiplicity at least 3. We can repeat this process an arbitrary number of times and conclude that \( \lambda \) is of infinite multiplicity. But this implies that \( A - \lambda \) is not a Fredholm operator. This implies that \( \lambda \in \sigma_{ess}(A) \), which in turn implies \( \lambda \in \sigma_{ess}(A) \). Thus we have a contradiction. Therefore we have \( \rho(A) \subseteq \sigma(A) \) which implies \( \sigma(A) \subseteq \sigma(A) \).

Finally, suppose that \( \lambda \) is an eigenvalue of \( A \) with eigenfunction \( f \). By (6) we have \( A\lambda = \lambda Rf \). If \( Rf \neq 0 \) then \( \lambda \) is an eigenvalue of \( A \). Thus if for all eigenvalues of \( A \) we have \( Rf \neq 0 \), we have \( \sigma_{pt}(A) \subseteq \sigma_{pt}(A) \). This concludes the proof. \( \square \)

4. The proof of Theorem 2.1. Lemma 3.1 tells us that we can compute \( \sigma(A) \) by computing the spectrum of \( \sigma(A) = \sigma(A_0 + B) \). We know the spectrum of \( A_0 \) exactly:
\[ \sigma(A_0) = \bigcup_j \sigma(A_j). \]
By hypothesis, in the compact set \( K \), for each \( j \), the operator \( A_j \) has finitely many eigenvalues of total multiplicity \( m(j) \). Thus the spectrum of \( \sigma(A_0) \) within \( K \) is just the union of these eigenvalues; there are a finite number of them and their total multiplicity is \( M = \sum_j m(j) \). Consequently our plan is to treat \( A \) as a perturbation of \( A_0 \). However, \( B \) is not small, at least not when it acts upon functions in \( W^{s,p} \). However, it is small when acting upon localized functions. We have our key lemma:

**Lemma 4.1.** For all \( 0 \leq \beta \leq \beta_0 \)
\[ \|Bf\|_{L^p_\beta} \leq C\|f\|_{L^p} \]
and
\[ \|Bf\|_{L^p_\beta} \leq C\, e^{-\frac{d}{2}t} \|f\|_{L^p_\beta}. \]
The constant \( C \) depends only on \( N \) and the \( L^\infty_\beta \) norms of the \( V_j \), not on the particular configuration of the \( x_j \).

We prove this lemma in the final section. First we use it to finish off the proof of Theorem 2.1.
4.1. **The spectrum of** $A_0 + B$ **in weighted spaces.** The second estimate in Lemma 4.1 tells us that $B$ is a bounded operator which can be made arbitrarily small by taking $\Gamma$ large. Thus classical perturbation theory will apply.

What is the point spectrum $\sigma_{pt}^b(A_0)$? It is not immediately evident that it is the same as $\sigma_{pt}(A_0)$, since $L^p_b \subset L^p$. Clearly, if $\lambda \in \sigma_{pt}^b(A)$ and $\lambda \notin \sigma_{ess}(A)$, then $\lambda \in \sigma_{pt}(A)$. On the other hand we have the following Lemma:

**Lemma 4.2.** There exists $\beta_1 > 0$ so that for all $0 \leq b \leq \beta_1$ we have $\sigma_{pt}(A_0) \cap K \subseteq \sigma_{pt}^b(A_0) \cap K$.

**Proof.** We have

$$\sigma^b(A_0) = \sigma^0(\cosh(b|x|) \circ A_0 \circ \operatorname{sech}(b|x|))$$

That is to say, the spectrum of $A_0$ on the weighted space is equal to the spectrum of the conjugation of $A_0$ by the weight $\cosh(b|x|)$ viewed as an operator on $L^p$.

(See, for instance, [6].) One can show that $A_{0,b} := \cosh(b|x|) \circ A_0 \circ \operatorname{sech}(b|x|)$ is a small relatively bounded perturbation of $A_0$ for $b \sim 0$. Thus the eigenvalues of $A_{0,b}$ are small perturbations of the those of $A_0$. Fix attention, therefore, on a particular eigenvalue, $\lambda_b$, of $A_0$ which perturbs to the eigenvalue $\lambda_b$ of $A_{0,b}$. Let us assume that $\lambda$ is simple and so therefore is $\lambda_b$. Let $f_b \in L^p$ be its corresponding eigenfunction.

Unwinding the definition of $A_{0,b}$ shows that $A_0(\operatorname{sech}(b|x|)f) = \lambda_b(\operatorname{sech}(b|x|)f)$ and so $\lambda_b$ is an eigenvalue of $A_0$ as an operator in $L^p_b$. Clearly, then $\operatorname{sech}(b|x|)f \in L^p_b$ and thus $\lambda_b$ is an eigenvalue of $A_0$ viewed as an operator on that space. That is to say, $\lambda_b$ is an eigenvalue of $A$ viewed as an operator on $L^p$ which is nearby $\lambda_0$. But $\lambda_b$ is a simple eigenvalue and therefore isolated. Thus $\lambda_b = \lambda_0$. This same argument can be modified to apply to eigenvalues of higher multiplicity.

**Remark 2.** We sketch an alternate proof of Lemma 4.2 here. Suppose that $\lambda_0 \in \sigma_{pt}(A_0)$, with eigenfunction $f_0 \in L^p$. Specifically, we know that $\lambda_0 \notin \sigma_{ess}(A_0) = \sigma_{ess}(L)$. For $b$ sufficiently small, a straightforward perturbation argument shows that $\lambda_0 \notin \sigma_{ess}^b(L)$ as well, and as discussed above the translation invariance of $L$ gives $\sigma_{pt}(L) = \emptyset$. So $(L - \lambda_0)^{-1}$ is a bounded map from $L^p_b$ into $W^{s,p}_b$. Since $A_0 = \operatorname{diag}\{L + V_1, \ldots, L + V_N\} =: L + V$ the eigenvalue relation $A_0 f_0 = (L + V) f_0 = \lambda_0 f_0$ can be rewritten as

$$(L - \lambda_0) f_0 = -V f_0.$$ 

We now $f_0$ is in $L^p$, and since the functions $V_j$ are assumed to decay to zero exponentially we have $V f_0 \in L^p_b$. Since $(L - \lambda_0)^{-1}$ is a bounded map from $L^p_b$ into $W^{s,p}_b$, we can invert the operator on the left hand side of the last displayed equation to get

$$f_0 = -(L - \lambda_0)^{-1} V f_0.$$ 

This implies $f_0 \in W^{s,p}_b$ and thus $\lambda_0 \in \sigma_{pt}^b(A_0)$.

**Remark 3.** At this point we fix $\beta$ so that

$$0 < \beta < \min \{\beta_0, \beta_1\}.$$ 

This lemma assures that if we work in the weighted spaces that the point spectrum of $A_0$ in $K$ is the same as in the unweighted space. We know that (see [5] pp 212-215) the eigenvalues in $\sigma_{pt}^b(A_0) \cap K$ will perturb just as the eigenvalues of a matrix would. That is, there exists $\delta_0$ such that if $\|B\|_{W^{s,p} \to L^p_b} = \delta \leq \delta_0$ then the eigenvalues of $A_0 + B$ in $K$ have total multiplicity $M$ and lie within $O(\beta^{1/M')}$.
of the eigenvalues of $A_0$. Here $M'$ is maximal algebraic multiplicity of any single eigenvalue of $A_0$ in $K$.

Therefore Lemma 4.1 confirms for us that the eigenvalues of $A$ in $K$ are within $O(e^{-\frac{a}{2\rho}})$ of those of the union of the eigenvalues of the $A_i$ in $K$. These eigenvalues are in $L^p_{\beta}$, since that is the space within which we have done the perturbation. As this space is contained within $L^p$, these eigenvalues are true eigenvalues for $A$ with respect to $L^p$. Note that a (generalized) eigenfunction $f$ of $A_0 + B$, after normalization, must be a small perturbation of a (generalized) eigenfunction of $A_0$, denoted $\tilde{f}$. Since no eigenfunction $\tilde{f}$ of $A_0$ has $R\tilde{f} = 0$, therefore neither can $Rf = 0$.

In this way, we see that no eigenvalues of $A$ are “lost” when we pass back to $A$. In conclusion we see that $\sigma^b(A_0 + B) \cap K$ (and therefore $\sigma^b(A) \cap K$) consists of a set of eigenvalues, of total multiplicity $M$, which are $O(e^{-ct})$ close to $\bigcup_j \sigma_{pt}(A_j)$.

### 4.2. The spectrum of $A_0 + B$ in unweighted spaces.

That the above section works in the weighted spaces is unsatisfying as we would like to know $\sigma(A_0 + B)$. The following argument shows that we can in fact pass from $\sigma^b(A_0 + B)$ to $\sigma(A_0 + B)$.

First of all, it is clear that $\sigma^b_{pt}(A_0 + B) \subset \sigma_{pt}(A_0 + B)$. Are there any additional eigenvalues? Suppose that $\lambda \in \rho^b(A) \cap \rho(A_0)$. We will show that $\lambda \in \rho(A_0 + B)$. The following operators are well-defined and bounded:

\[(A_0 - \lambda)^{-1} : L^p \rightarrow L^p,\]
\[(A_0 + B - \lambda)^{-1} : L^p_{\beta} \rightarrow L^p_{\beta}.\]

Now consider

\[\Psi := (A_0 - \lambda)^{-1} - (A_0 + B - \lambda)^{-1} \circ B \circ (A_0 - \lambda)^{-1}.\]

This is a bounded operator from $L^p$ to itself. The first estimate in Lemma 4.1 implies that

\[B \circ (A_0 - \lambda)^{-1} : L^p \rightarrow L^p_{\beta}\]

and so

\[(A_0 + B - \lambda)^{-1} \circ B \circ (A_0 - \lambda)^{-1} : L^p \rightarrow L^p_{\beta} \subset L^p\]

is perfectly well-defined and bounded. Moreover

\[(A_0 + B - \lambda)\Psi f = (A_0 + B - \lambda) \circ (A_0 - \lambda)^{-1}f - B \circ (A_0 - \lambda)^{-1}f = f\]

Thus $\Psi$ is a bounded inverse of $A + B - \lambda$ on $L^p$ and consequently this operator is onto. Thus the cokernel of this operator is trivial. The Fredholm indices of $A_0 - \lambda$ and $A_0 + B - \lambda$ must agree since $B$ is a compact operator. Since $\lambda$ is in the resolvent set of $A_0$, we must have that the index of $A_0 - \lambda$ is zero. Thus the kernel of $A_0 + B - \lambda$ is trivial. And therefore so $\lambda \in \rho(A_0 + B)$.

The only place left in $K$ where an additional eigenvalue of $A_0 + B$ could be lurking is in $\sigma_{pt}(A_0)$. However, we already know that in the neighborhood of any member of $\sigma_{pt}(A_0)$ there are perturbed eigenvalues $O(e^{-ct})$ close by, and with the same total multiplicity. These eigenvalues are frozen with respect to varying the decay rate $b$. A straightforward argument about the continuous dependence of eigenvalues on a parameter means that there is no way for an additional eigenvalue to appear nearby as $b$ tends to zero.
5. The proof of Lemma 4.1. From the definition of $Bf$, we see that we need only to estimate terms of the form

$$I := \| S(x_i) \chi_i(\cdot) V_j(\cdot - x_j) f_k(\cdot - x_k) \|_L^p$$

where $j \neq k$. Using the definition of the $L^p$ norm and the translation invariance of the $L^p$ norm:

$$\| S(x_i) \chi_i(\cdot) V_j(\cdot - x_j) f_k(\cdot - x_k) \|_L^p = \| \cosh(\beta) \cdot x_j \|_L^p \cdot \| \chi_i(\cdot) V_j(\cdot - x_j) f_k(\cdot - x_k) \|_L^p$$

Now we utilize (3), the definition of $L^p$ norm and the $L^\infty - L^p$ Holder estimate to get:

$$I \leq \Omega \| \cosh(\beta) \cdot x_j \|_L^p \| \chi_i(\cdot) \cosh(\beta) \cdot x_j \|_L^p \leq \Omega \| \cosh(\beta) \cdot x_j \|_L^p \| \chi_i(\cdot) \cosh(\beta) \cdot x_j \|_L^p$$

There is a constant $C > 1$ so that $C^{-1} e^{|y|} \leq \cosh(|y|) \leq C e^{|y|}$. Also, the support of $\chi_i$ is inside the set $\tilde{U}_i$ and so:

$$I \leq C \| \exp \{ \beta \cdot x_j - x_i \} \|_L^\infty(\tilde{U}_i) \| f_k \|_L^p.$$

Given the definition of the sets $\tilde{U}_i$, it is straightforward to show that for $x \in \tilde{U}_i$

$$|x - x_i| - |x - x_j| \leq 1 \quad \text{and} \quad |x - x_i| - |x - x_k| \leq 1.$$

Therefore

$$\mu := \| \exp \{ \beta \cdot x_i - x_j - b \cdot x_k \} \|_L^\infty(\tilde{U}_i) \leq C \| \exp \{ -b \cdot x_k \} \|_L^\infty(\tilde{U}_i).$$

In the case when $b = 0$ its obvious that $\mu \leq C$ and the first estimate in the Lemma follows.

When $b = 0$ then the fact that $j \neq k$ allows us to assume without loss of generality that $k \neq i$. The definition of $\Gamma$ then implies that

$$\min_{x \in \tilde{U}_i} |x - x_k| \geq \Gamma/2 - 1/2$$

and so

$$\| \exp \{ -b \cdot x_k \} \|_L^\infty(\tilde{U}_i) \leq C e^{-\beta \Gamma/2}$$

This then implies the second estimate in the Lemma and we are done. \[\Box\]

6. Conclusions. Exponential localization of coherent structures is common, particularly in problems where the coherent structure is given by the solution of an ordinary differential equation. Such localization is a very strong property, and allows for results of the type we demonstrate here. It is natural to ask if weaker (say algebraic) localization would result in a similar result. While we suspect that answer is yes, doing so would require a reformulation of Lemma 4.1 for algebraically weighted spaces.

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