Coassociative grammar, periodic orbits and quantum random walk over $\mathbb{Z}^1$

Philippe LEROUX

Institut de Recherche Mathématique, Université de Rennes I and CNRS UMR 6625
Campus de Beaulieu, 35042 Rennes Cedex, France, philippe.leroux@univ-rennes1.fr

Abstract: This work will be devoted to the quantisation of the classical Bernoulli random walk over $\mathbb{Z}$. As this random walk is isomorphic to the classical chaotic dynamical system $x \mapsto 2x \mod 1$ with $x \in [0,1]$, we will explore the rôle of classical periodic orbits of this chaotic map in relation with a non commutative algebra associated with the quantisation of the Bernoulli walk. In particular we show that the set of periodic orbits, $\text{PO}$, of the map $x \mapsto 2x \mod 1$ can be embeded into a language equipped with a coassociative grammar and for any fixed time, that any vertex of $\mathbb{Z}$ is in one to one with of a subset of $\text{PO}$. The reading and the contraction maps applied to these periodic orbits allow us to recover the combinatorics generated by the quantum random walk over $\mathbb{Z}$.

1 Introduction and Notation

Motivated by the success of classical random walks and chaotic dynamical systems, we study the quantisation of the random walk over $\mathbb{Z}$ and its relationships with a classical chaotic system $x \mapsto 2x \mod 1$ with $x \in [0,1]$ . The first part will be devoted to notation and to the introduction of notions required in the sequel. We recall the notion of quantum graphs, unistochastic processes, directed graphs and extension, $L$-coalgebras and coproducts, the relation between the Bernoulli walk and the map $x \mapsto 2x \mod 1$ and a quantisation of this walk proposed by Biane [4]. Section 2 displays the relationships between some coassociative coalgebras, chaotic maps $x \mapsto nx \mod 1$, with $n > 1$ and $(n,1)$-De Bruijn graphs. Section 3 yields a bridge between quantum channels and quantum graphs. Section 4 defines the quantum random walk over $\mathbb{Z}$ and section 5 explains the link between periodic orbits of $x \mapsto 2x \mod 1$, their coassociative language and the combinatorics generated by the quantum random walk over $\mathbb{Z}$.

1.1 Quantum graphs

Let $B$ be a bistochastic matrix representing a directed graph, i.e. two vertices $x_i$ and $x_j$ are linked iff $B_{ij} \neq 0$. $B$ is said unistochastic if it exists an unitary matrix $U$ such that $B_{ij} = |U_{ij}|^2$. In this case, we

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say that the graph can be quantized. The notion of quantum graphs were introduced as a toy model for studying quantum chaos by Kottos and Smilansky [11], [12]. This notion was also studied by Tanner [19] and by Barra and Gaspard [2] [3]. In this article we will follow another approach leading to quantum graphs put forward in [18] concerning the one dimensional dynamical systems. They consider an one-dimensional mapping \( f \) acting on \( I = [0,1] \) such that \( f : I \to I \), is piecewise linear. Moreover \( f \) verifies the following three conditions:

1. There exists a Markov partition of the interval \( I \) into \( M \) equal cells \( E_i := \left( \frac{i-1}{M}, \frac{i}{M} \right) \), \( i = 1 \ldots M \), with \( M \) a positive integer and \( f \) is linear on each cell \( E_i \).

2. For all \( y \in I \),

\[
\sum_{x \in f^{-1}(y)} \frac{1}{f'(x)} = 1,
\]

where \( f' \) is the right derivative, (defined almost everywhere on \( I \)).

3. The finite transfer matrix \( B \), describing the action of \( f \) on the cells \( E_i \) is unistochastic.

**Remark:** On each cell, \( f \) coincides with the function \( f_i : \left( \frac{i-1}{M}, \frac{i}{M} \right) \to I, x \mapsto c_i x + b_i \) where the \( c_i \) have to be non zero integers and the \( b_i \) are rational. The unistochastic matrix \( B \) is a \( M \) by \( M \) matrix and \( B_{ij} = \frac{1}{|c_i|} \). Thus the probability of visiting the cell \( E_j \) from \( E_i \) is equal to \( \frac{1}{f'(x)} \), with \( x \in E_i \) and \( f(x) \in E_j \).

**Remark:** The Kolmogorov-Sinai-entropy of the Markov chain generated by the bistochastic matrix \( B \) is

\[
H_{KS} = -\sum_{i=1}^{M} \tilde{p}_i \sum_{j=1}^{M} B_{ij} \log B_{ij},
\]

where \( \tilde{p} \) is the normalized left eigenvector of \( B \) such that \( \tilde{p}B = \tilde{p} \), with \( \sum_{i=1}^{M} \tilde{p}_i = 1 \). This equation gives the dynamical entropy of the system since the Markov partition on \( M \) equal cells is a generating partition of the system. As the transition matrix is bistochastic, all the components of \( \tilde{p} \) are equal to \( \frac{1}{M} \). Thus \( H_{KS} = 0 \) iff all the \( B_{ij} \in \{0,1 \} \). This entails that \( |f'(x)| = 1 \), i.e. the system is regular. With the conditions stated above, the converse is true.

**Example 1.1 [Regular system [18]]**

Here is an example of such piecewise linear map. The associated bistochastic matrix is \( B_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \).

In the sequel we will be interested by one of the simplest 1D maps displaying chaotic dynamics.
Example 1.2 [Chaotic system, the Bernoulli shift]
The Bernoulli shift is described by $f : x \mapsto 2x \mod 1$. This map is associated with the unistochastic matrix $B_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

We recall an important theorem in [18].

**Theorem 1.3** With the assumptions described in [18] on the piecewise linear map $f$, to every periodic orbit of period $n$ of the dynamical system described by $f$, corresponds an unique periodic orbit of period $n$ of the directed graph describing by the associated unistochastic matrix.

1.2  $L$-coalgebras

To unify the directed graphs framework, even equipped with a family of probability vectors, with coassociative coalgebra theory, we are led to introduce the notion of $L$-coalgebra over a field $k$, i.e. a $k$-vector space equipped with two coproducts $\Delta$ and $\tilde{\Delta}$ which obey the coassociativity breaking equation $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$. If $\Delta = \tilde{\Delta}$, the $L$-coalgebra is said degenerate. Moreover a $L$-coalgebra can have two counits, the right counit $\epsilon : G \rightarrow k$ which verifies $(id \otimes \epsilon)\Delta = id$ and the left counit $\tilde{\epsilon} : G \rightarrow k$, which verifies $(\tilde{\epsilon} \otimes id)\tilde{\Delta} = id$.

One of the interests of such a formalism is to describe directed graphs equipped with a family of probability vectors or not, thanks to their coproducts instead of the classical source and terminus mappings. For building such a directed graph for each $L$-coalgebra, we associate with each tensor product $x \otimes y$ appearing in the definition of the coproducts a directed arrow $x \rightarrow y$. For instance, here is the directed graph associated with $Sl_q(2)$.

**Example 1.4** [$Sl_q(2)$] The well-known coassociative coalgebra structure is:

$$
\Delta a = a \otimes a + b \otimes c, \quad \Delta b = a \otimes b + b \otimes d, \quad \Delta c = d \otimes c + c \otimes a, \quad \Delta d = d \otimes d + c \otimes b.
$$

and its directed graph is $G(Sl_q(2))$:  

\footnote{Here supposed to be with no sources and no sinks, see [3] otherwise.}
Remark: In the sequel, we denote by $\mathcal{E}$ the coassociative coalgebra generated by $a, b, c$ and $d$, having the same coproduct and counit definitions as $Sl_q(2)$.

Definition 1.5 [Markov $L$-coalgebra] A Markov $L$-coalgebra $C$ is a $L$-coalgebra such that for all $v \in C$, $\Delta v = \sum_{i \in I} \lambda_i v \otimes v_i$ and $\tilde{\Delta} v = \sum_{j \in J} \mu_j v_j \otimes v$, with $v_i, v_j \in C, \lambda_i, \mu_j \in k$ and $I, J$ are finite sets.

Such a structure reproduces locally what we have in mind when we speak about random walks on a directed graph if the scalars are positive and the right counit $v \mapsto 1$ exists. We recall that the right counit is a map $\epsilon : C \to k$ verifying $(id \otimes \epsilon)\Delta = id$. A Markov $L$-bialgebra is a Markov $L$-coalgebra and an unital algebra such that its coproducts and counits are homomorphisms.

Example 1.6 [The graph of the map $x \mapsto 2x \mod 1$]

\[ E_1 \quad E_2 \]

With the $(2, 1)$-De Bruijn graph, we associate its natural Markov $L$-coalgebra by defining: $\Delta E_1 = E_1 \otimes E_1 + E_1 \otimes E_2$ and $\Delta E_2 = E_2 \otimes E_1 + E_2 \otimes E_2$, $\tilde{\Delta} E_1 = E_2 \otimes E_1 + E_1 \otimes E_1$ and $\tilde{\Delta} E_2 = E_1 \otimes E_2 + E_2 \otimes E_2$.

Example 1.7 [Unital algebra] Let $A$ be an unital algebra. $A$ carries a non-trivial Markov $L$-bialgebra called the flower graph with coproducts $\delta(a) = a \otimes 1$ and $\tilde{\delta}(a) = 1 \otimes a$, for all $a \in A$. We call such a Markov $L$-coalgebra a flower graph because it is the concatenation of petals:

An other example is the directed triangle graph:
Indeed consider the iid process defined by \( Y \). It is well known that this process and the symbolic dynamics generated by \( \Delta \) is commutative. Moreover we have \( \Phi(\theta Y) = 2\Phi(\theta \omega') \mod 3 \).

1.3 Classical random walk over \( \mathbb{Z} \) and the Bernoulli shift

We consider the random walk over \( \mathbb{Z} \), i.e. we consider \( \Omega = \{-1,1\}^\mathbb{N} \) equipped with the product measure \( \mu \otimes \mu' \), where \( \mu = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1 \). We consider the sequence of iid random variables \( (X_n)_{n \in \mathbb{N}} \), with:

\[
X_n : \Omega \to \{-1,+1\}, \text{ such that } X_n(\omega) = \omega_n.
\]

It is well known that this process and the symbolic dynamics generated by \( x \mapsto 2x \mod 1 \) are isomorphic. Indeed consider the iid process defined by \( Y_n : \Omega \to \{0,+1\} \), such that \( Y_n(\omega) = \omega'_n := \frac{\omega_n+1}{2} \). \( \Omega = \{-1,1\}^\mathbb{N} \) becomes \( \Omega' = \{0,1\}^\mathbb{N} \) and \( \mu \) becomes \( \mu' = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1 \). Define the measurable function:

\[
\Phi : \Omega' \to [0,1[, \omega' \mapsto \sum_{n=0}^{\infty} \frac{\omega'_n}{2^{n+1}}.
\]

The cylinder \( C = \{Y_0 = \omega'_0; \ldots; Y_l = \omega'_l\} \), being the set of the sequences starting by \( (\omega'_0; \ldots; \omega'_l) \), will cover the interval \( \left[ \sum_{n=0}^{l} \frac{\omega'_n}{2^{n+1}}; \sum_{n=0}^{l} \frac{\omega'_n}{2^{n+1}} + \sum_{n=l+1}^{\infty} \frac{1}{4^{n+1}} \right] \). We notice that \( \text{Leb}(C) = \frac{1}{4^{l+1}} \), where \( \text{Leb} \) is the Lebesgue measure. Let us consider the shift \( \theta \) such that \( \theta(\omega') = \omega'_{l+1} \). This shift lets the Lebesgue measure of the cylinder \( C \) invariant if we denote \( \theta([Y_0 = \omega'_0; \ldots; Y_l = \omega'_l]) = ([Y_1 = \omega'_0; \ldots; Y_{l+1} = \omega'_l]) \).

Moreover we have \( \Phi(\theta(\omega')) = 2\Phi(\omega') \mod 1 \).

The random walk over \( \mathbb{Z} \), described by \( \Omega = \{-1,1\}^\mathbb{N}, \mu \otimes \theta \) is isomorphic to \( \Omega' = \{0,1\}^\mathbb{N}, \mu' \otimes \theta' \) which is isomorphic to the chaotic system \( ([0,1[, \beta [0,1[, \text{Leb}, f : x \mapsto 2x \mod 1) \). As \( \Phi \circ \theta' = f \circ \Phi \), the following diagram,

\[
\begin{array}{ccc}
(\{0,1\}^\mathbb{N}, \mu \otimes \theta) & \xrightarrow{\theta} & (\{0,1\}^\mathbb{N}, \mu \otimes \theta') \\
\Phi \downarrow \quad & & \quad \downarrow \Phi \\
([0,1[, \text{Leb}) & \xrightarrow{f} & ([0,1[, \text{Leb})
\end{array}
\]

is commutative.
1.4 quantisation of the classical Bernoulli walk

In [4], Biane purposes a non commutative version of the Bernoulli process. Set \( \Omega := \{+1, -1\} \), the probability space and define the probability \( \mathbb{P}(\{+1\}) = p \) and \( \mathbb{P}(\{-1\}) = q \). The process \( X : \Omega \to \mathbb{R} \) is defined as \( X(+1) = +1 \) and \( X(-1) = -1 \). By identifying \((1, 0) \) with \((4p)^{-\frac{1}{2}} (1 + X)\) and \((0, 1) \) with \((4q)^{-\frac{1}{2}} (1 - X)\), the space \( L^2(\Omega, \mathbb{P}) \) is isomorphic to \( \mathbb{C}^2 \). We notice that the algebra \( L^\infty(\Omega, \mathbb{P}) \), acting on \( L^2 \) can be identified with diagonal matrix of \( M_2(\mathbb{C}) \). A natural non commutative generalisation consists in lifting this commutative algebra into a bigger non commutative one, i.e. \( M_2(\mathbb{C}) \). In the sequel, we will not follow the Biane’s work. Nevertheless, we keep in mind that \( M_2(\mathbb{C}) \) is a suitable algebra and we will see later how to rediscover it.

2 Relationships between coassociative coalgebras and chaotic maps \( x \mapsto nx \mod 1 \)

Definition 2.1 [De Bruijn graph] A \((p,n)\) De Bruijn sequence on the alphabet \( \Sigma = \{a_1, \ldots, a_p\} \) is a sequence \((s_1, \ldots, s_m)\) of \( m = p^n \) elements \( s_i \in \Sigma \) such that subsequences of length \( n \) of the form \((s_{i}, \ldots, s_{i+n-1})\) are distinct, the addition of subscripts being done modulo \( m \). A \((p,n)\)-De Bruijn graph is a directed graph whose vertices correspond to all possible strings \( s_1s_2\ldots s_n \) of \( n \) symbols from \( \Sigma \). There are \( p \) arcs leaving the vertex \( s_1s_2\ldots s_n \) and leading to the adjacent node \( s_2s_3\ldots s_n\alpha, \alpha \in \Sigma \). Therefore the \((p,1)\)-De Bruijn graph is the directed graph with \( p \) vertices, complete, with a loop at each vertex.

Definition 2.2 [Extension] The extension of a directed graph \( G \), with vertex set \( J_0 = \{j_1, \ldots, j_n\} \) and edges set \( A_0 \subseteq J_0 \times J_0 \) is the directed graph with vertex set \( J_1 = A_0 \) and the edge set \( A_1 \subseteq J_1 \times J_1 \) defined by \( ((j_k, j_l), (j_e, j_f)) \) iff \( j_l = j_e \). This directed graph, called the line directed graph, is denoted \( E(G) \).

Recall that the definition of an associative dialgebra is a notion due to Loday [15]. Here, we are interested in the notion of a coassociative co-dialgebra.

Definition 2.3 [Coassociative co-dialgebra of degree \( n \)] Let \( D \) be \( K \)-vector space, where \( K \) denotes the real or complex field. For every \( n > 0 \), Let \( \Delta \) and \( \tilde{\Delta} \) be two linear mapping \( D^\otimes n \to D^\otimes n+1 \). \( D \) is said a coassociative co-dialgebra of degree \( n \) if the following axioms are verified:

1. \( \Delta \) and \( \tilde{\Delta} \) are coassociative,
2. \( (id \otimes \Delta)\Delta = (id \otimes \tilde{\Delta})\Delta \),
3. \( (\tilde{\Delta} \otimes id)\tilde{\Delta} = (\Delta \otimes id)\Delta \),
4. \( (\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta} \).

The last equation is called the coassociativity breaking equation in [13].

Proposition 2.4 The \((n,1)\)-De Bruijn graph describes the chaotic dynamics \( x \mapsto nx \mod 1 \).
Proof: The map \( x \mapsto nx \mod 1 \) is coded by the unistochastic \( n \times n \) matrix \( B_n \), with \( B_{ij} = \frac{1}{n} \), for all \( i, j = 1, \ldots, n \). The associated directed graph is then the \((n,1)\)-De Bruijn graph. \( \square \)

**Proposition 2.5** The coproducts of the Markov \( L \)-coalgebra \( G \) of the \((n,1)\)-De Bruijn graph define a coassociative co-dialgebra of degree 1.

**Proof:** The coproducts are coassociative and send the Markov \( L \)-coalgebra \( G \) into \( G \otimes^2 \). The other axioms are straightforward. There is a right counit \( \epsilon \) which maps each vertex into \( \frac{1}{n} \) for the coproduct \( \Delta \) and a left counit \( \tilde{\epsilon} \) which maps each vertex into \( \frac{1}{n} \) for the coproduct \( \tilde{\Delta} \). \( \square \)

**Proposition 2.6** The extension of the \((n,1)\)-De Bruijn graph, can be equipped with a coassociative coproduct.

**Proof:** Let us denote the edge emerging from a given vertex \( i \), with \( i = 1, \ldots, n \) of the \((n,1)\)-De Bruijn graph \( G_n \) by \( a_{ij} \). The new vertex of the extension of \( G_n \) is denoted by \( a_{ij} \) and the edges are denoted by \( ((i,j),(j,k)) \). By denoting \( \Delta a_{ij} = \sum_l a_{il} \otimes a_{lj} \), this coproduct is coassociative and the graph associating with the coassociative coalgebra \( \{a_{ij}\}_{i,j=1,\ldots,n}, \Delta \) is easily seen to be \( E(G_n) \). It has an obvious counit, \( a_{ij} \mapsto 0 \) if \( i \neq j \) and \( a_{ij} \mapsto 1 \) otherwise. \( \square \)

**Corollary 2.7** The extension of the \((2,1)\)-De Bruijn graph can be equipped with the coassociative coproduct of the coassociative coalgebra \( E \).

**Proof:** Straightforward. \( \square \)

We have shown that each Markov \( L \)-coalgebra described by the \((n,1)\)-De Bruijn graph is an unistochastic map associated with the classical chaotic map \( x \mapsto nx \mod 1 \). It has a structure of coassociative co-dialgebra. Its extension yields a coassociative coalgebra. The relationships between the \((n,1)\)-De Bruijn graphs and their extensions are treated in more details in [14].

### 3 Quantum channels from quantum graphs

Let \( B \) be a \( n \times n \) unistochastic matrix associating with a classical dynamical system satisfying the condition [14]. We introduce now the quantisation and deformation of the Markov partition associated with the Markov family \( E_i, i = 1, \ldots, M \) and the unistochastic matrix. The unistochastic matrix \( B \) can be decomposed into \( n \) independent matrices \( X_h, h = 1, \ldots, n \) defined by \( (X_h)_{ij} = B_{ij} \) if \( h = i \) and 0 otherwise. We get \( B = \sum_{h=1}^k X_h \) and the \( X_h \) verify the algebraic relations \( X_h X_l = B_{hl} X_l \). When we quantify the directed graphs associated with the unistochastic matrix \( B \), i.e. when we choose an unitary matrix \( U \) satisfying \( B_{ij} = |U_{ij}|^2 \), we deform the algebraic relations between the \( X_h \) to have \( U = \sum_{h=1}^k Q_h \), with \( (X_h)_{ij} = |(Q_h)_{ij}|^2 \) and \( Q_h Q_l = U_{hl} Q_l \). We can say that we have quantised the classical dynamics described by the unistochastic matrix \( B \).

**Example 3.1** The operators \( X_h, h = 1, \ldots, k \), associated with the chaotic map \( x \mapsto nx \mod 1 \) obey the algebraic laws \( X_h^2 = \frac{1}{n^2} X_h \) and \( X_h X_l X_h = \frac{1}{n^2} X_h \).
Proposition 3.2 Let $X_h, h = 1, \ldots, k$, be the operators associated with the unistochastic matrix $B$. Its quantisation yields the operators $Q_h$. We get $\sum_{h=1}^k Q_h Q_h^\dagger = \text{Id}$ and $\sum_{h=1}^k Q_h^\dagger Q_h = \text{Id}$, i.e. the linear map $\rho \mapsto \sum_{h=1}^k Q_h^\dagger Q_h \rho Q_h$ are quantum channels. Moreover $Q_l Q_h^\dagger = 0$ if $h \neq l$, i.e. the quantum channel $\rho \mapsto \sum_{h=1}^k Q_h^\dagger \rho Q_h$ is an homomorphism.

Proof: This is due to the fact that the columns of $U$ are orthonormal. \hfill \square

In the following we will focus on the $(2,1)$-De Bruijn graph, associated with the decomposition of the unistochastic map $B_2$ and hence with the chaotic map $x \mapsto 2x \mod 1$.

Remark: Let $X_1, X_2$ be the operators associated with the chaotic map $x \mapsto 2x \mod 1$. One of the possible quantisation of this chaotic map is the Hadamard matrix, $U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. It is decomposed into two operators, $P := Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $Q := Q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$. These different choices lead to a set of suitable unitary matrices. However, for the study of quantum random walk, we can enlarge this set to include all possible unitary matrices. In general, such an unitary matrix reads $U = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, with obvious conditions on $\alpha, \beta, \gamma, \delta$. Its decomposition is denoted by: $P = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$ and $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ \gamma & \delta \end{pmatrix}$ and verify the following algebraic relations $P^2 = \alpha P$, $Q^2 = \delta Q$, $PQP = \beta \gamma P$, $QQP = \beta \gamma Q$.

Proposition 3.3 Suppose $\alpha \delta \neq 0$. Consider $e_1 := \frac{1}{\alpha} P$ and $e_2 := \frac{1}{\delta} Q$. We get $e_1 e_2 e_1 = \lambda e_1$ and $e_2 e_1 e_2 = \lambda e_2$, where $\lambda := \frac{2 \beta}{\beta \gamma}$, i.e. the algebra generated by $e_1, e_2$ is a Jones algebra [7].

Proof: Straightforward. \hfill \square

4 Quantum random walk over $\mathbb{Z}$

In the physics literature, quantum random walk has been studied for instance, by Ambainis and al [1], Konno and al, in [9] [10]. Here we propose a more mathematical framework for the quantum random walk over $\mathbb{Z}$ and show that the combinatorics of this walk can be recovered by using the coproduct of $\mathcal{E}$. We will show that this walk is closely related to the quantisation of the $(2,1)$-De Bruijn graph and that the polynomials involved in each vertex of $\mathbb{Z}$ are related, for a given time, in a bijection way, to periodic orbits of the chaotic map $x \mapsto 2x \mod 1$ and that these periodic orbits can be manipulated by the coproduct of $\mathcal{E}$.

Let $\mathcal{H}$ be a separable Hilbert space of infinite dimension with $(|e_n\rangle)_{n \in \mathbb{Z}}$ as an orthonormal basis. In directed graphs have been put on algebraic structures such as $k$-vector spaces, algebras, coalgebras and so on. We decide to put on $\mathcal{H}$, the following directed graph $G_\mathbb{Z}$:

\[
\begin{array}{ccccccc}
\cdots & \xrightarrow{e_{-1}} & |e_{-1}\rangle & \xrightarrow{e_{0}} & |e_{0}\rangle & \xrightarrow{e_{1}} & |e_{1}\rangle & \cdots
\end{array}
\]

We keep the notation of [11].
With this directed graph, we can talk about quantum random walk over the Hilbert space \( \mathcal{H} \), the classical analogue being the classical random walk over \( \mathbb{Z} \), and embed \( \mathcal{H} \) into a Markov \( L \)-coalgebra, \( \mathcal{H}_Z \). Now we introduce over the basis \( \mathcal{H}_Z \), the fiber \( M_2(\mathbb{C}) \) to get the trivial tensor bundle \( \mathcal{H}_Z \otimes M_2(\mathbb{C}) \). We have said that the classical random walk over \( \mathbb{Z} \) is isomorphic to the symbolic dynamic described by the map \( x \mapsto 2x \mod 1 \) over the interval \([0, 1]\). By fixing the Markov partition \((E_1, E_2)\) leading to the \((2, 1)\)-De Bruijn graph, we code the periodic orbits of the classical dynamical system in an one-to-one correspondence with the periodic orbits of this graph. The strategy now consists to fix an unitary matrix \( U \) and to consider the operators \( P \) and \( Q \), such that \( U = P + Q \), with:

\[
P = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ \gamma & \delta \end{pmatrix}.
\]

We consider non commutative polynomials in \( P \) and \( Q \), i.e. the algebra \( \mathbb{C}(P, Q) \) and denote by \( \mathcal{D}_-, \mathcal{D}_+ \) the dispersion operators, i.e. the linear maps

\[
\mathcal{D}_-, \mathcal{D}_+: \mathcal{H}_Z \otimes \mathbb{C}(P, Q) \to \mathcal{H}_Z \otimes \mathbb{C}(P, Q)
\]

which are defined for all \( k \in \mathbb{Z} \) and for all discrete time \( n \in \mathbb{Z} \),

\[
\mathcal{D}_- (|e_{k+1}\rangle \otimes \Xi_{[k+1:n]}) = |e_k\rangle \otimes \Xi_{[k+1:n]} P, \quad \mathcal{D}_+ (|e_{k-1}\rangle \otimes \Xi_{[k-1:n]}) = |e_k\rangle \otimes \Xi_{[k-1:n]} Q,
\]

where \( \Xi_{[0:n]} = id, \Xi_{[-1:1]} = P, \Xi_{[+1:1]} = Q \) and so on. The dynamics is defined by:

\[
|e_k\rangle \otimes \Xi_{[n,n+1]} := \mathcal{D}_- (|e_{k+1}\rangle \otimes \Xi_{[k+1:n]}) + \mathcal{D}_+ (|e_{k-1}\rangle \otimes \Xi_{[k-1:n]}).
\]

That is:

![Diagram](https://via.placeholder.com/150)

Quantum random walk over \( \mathbb{Z} \), up to \( t = 3 \).

**Example 4.1** We yield here the non null polynomials \( \Xi_{[k,n+1]} \) for time \( t = 0, \ldots, 4 \). At time \( t = 0 \), we get \( \Xi_{[0:0]} = id \). At time \( t = 1 \), \( \Xi_{[-1:1]} = P, \Xi_{[+1:1]} = Q \). At time \( t = 2 \), \( \Xi_{[-2:2]} = P^2, \Xi_{[0:2]} = PQ+QP \) and \( \Xi_{[1:2]} = Q^2 \). At time \( t = 3 \), \( \Xi_{[-3:3]} = P^3, \Xi_{[-1:3]} = QP^2+QPQ+P^2Q, \Xi_{[+1:3]} = PQ^2+QPQ+Q^2P \) and \( \Xi_{[3:3]} = Q^3 \). At time \( t = 4 \), \( \Xi_{[-4:4]} = P^4, \Xi_{[-2:4]} = QP^3+QPQ+Q^2P+Q^3Q, \Xi_{[0:4]} = P^2Q^2+PQ^2P+Q^2P^2+QPQ+PQ^2Q, \Xi_{[+2:4]} = PQ^2+Q^2PQ+Q^2P+Q^3P \) and \( \Xi_{[4:4]} = Q^4 \).

Let us denote by \( S(\mathbb{C}^2) \), the set of vectors \( \psi \) of \( S(\mathbb{C}^2) \) such that \( \psi^\dagger \psi = 1 \). The quantum random walk over a state \( \psi \in S(\mathbb{C}^2) \) is defined by the initial condition \( \Psi_{space=0, time=0} := |e_0\rangle \otimes \psi \). At time \( n \), this state
will spread and the probability amplitude at position \( k \) described by \( |e_k \rangle \) will be \( \Psi_{k,n} := |e_k \rangle \otimes \Xi_{[k;n+1]} \psi \), (since \( P^i P + Q^i Q = 1 \), the norm of the initial state is preserved.). We have an action from the bundle \( \mathcal{H}_Z \otimes \mathbb{C} \langle P, Q \rangle \) on \( \mathbb{C}^2 \) described by

\[
\mathcal{R}W : \mathcal{H}_Z \otimes \mathbb{C} \langle P, Q \rangle \times \mathbb{C}^2 \to \mathcal{H}_Z \otimes \mathbb{C}^2; \quad (|e_k \rangle \otimes \Xi_{[k;n+1]} \psi) \mapsto |e_k \rangle \otimes \Xi_{[k;n+1]} \psi.
\]

The total state is \( \Psi_{total}^n := \sum_k \Psi_{k,n} \).

**Proposition 4.2** For all \( x \in \mathbb{C} \langle P, Q \rangle \), we define the right polynomial multiplication \( R_x : \mathbb{C} \langle P, Q \rangle \to \mathbb{C} \langle P, Q \rangle, y \mapsto yx \), we have, \([D_+, D_-] = \text{id} \otimes R_{[Q,P]}\].

**Proof:** Straightforward. \( \square \)

**Remark:** [On the importance of discrete quantum random walk]

Let us denote by \( S(M_2(\mathbb{C})) \), the set of density matrices \( \rho \) of \( S(M_2(\mathbb{C})) \), i.e. the set of trace one positive matrices. We know that such a density matrix can be written by \( \rho = \sum_i p_i |\psi_i \rangle \langle \psi_i| \), where \( \psi_i \in S(\mathbb{C}^2) \) and \( \sum_i p_i = 1 \). We can apply the quantum random walk over each \( \psi_i \), with different initial spacial conditions. By a suitable normalisation, we recover a quantum channel which scatters each initial component of the initial density matrix \( \rho \).

5 Non commutative polynomials and the reading of periodic orbits of \( G(\mathcal{E}) \)

**Remark:** From now on, we forget the algebraic relation between \( P, Q \) and their powers. These monomials will be treaded simply as non commutative words.

The aim of this section is to recover the polynomials \( \Xi_{[k;n]} \), involved in the quantum random walk, from the periodic orbits of the chaotic map \( x \mapsto 2x \mod 1 \). We have seen that the classical random walk over \( \mathbb{Z} \) is isomorphic to the chaotic map \( x \mapsto 2x \mod 1 \), that all periodic orbits of this map were coded bijectively into the \( (2,1) \)-De Bruijn graph, associated with the unistochastic process \( B_2 \) and that the extension of this graph leads to the directed graph associated with the coalgebra \( \mathcal{E} \).

We will show that the periodic orbits of this directed graph allow us to recover the combinatorics generated by the quantum random walk over \( \mathbb{Z} \) and that this combinatorics is governed by the coassociative
coproduct of $E$. We recall that this coproduct is defined by:

$$\Delta a = a \otimes a + b \otimes c, \quad \Delta b = a \otimes b + b \otimes d, \quad \Delta c = d \otimes c + c \otimes a, \quad \Delta d = d \otimes d + c \otimes b.$$ 

However this directed graph can also be embedded into its natural Markov $L$-coalgebra. We define only the right coproduct $\Delta_M$. We recall that by definition,

$$\Delta_M a = a \otimes (a + b), \quad \Delta_M b = b \otimes (c + d), \quad \Delta_M c = c \otimes (a + b), \quad \Delta_M d = d \otimes (c + d).$$

Let us start with the definition of a language.

**Definition 5.1 [Language]** We denote by $F$ the free vector space generated by all the words constructed from the alphabet $a, b, c, d$ and representing a path of the graph $G(E)$, for instance $aabdcabcb$. (It is also called the path space). We will call such a space a language. Its grammar, i.e. the laws allowing us to construct words longer and longer is constructed from the Markovian coproduct of $G(E)$, i.e. the substitution rules are:

$$a \mapsto ab, \quad a \mapsto aa, \quad b \mapsto bc, \quad b \mapsto bd, \quad c \mapsto ca, \quad c \mapsto cb, \quad d \mapsto dc, \quad d \mapsto dd.$$ 

**Remark:** In (classical) theoretical computer science the definition of a grammar is more restrictive. Here, as in [17], we prefer view a grammar as a way to produce sequences of strings, words, via substitution rules. The language so obtained, is then the space generated by such a grammar.

**Remark:** As we associate with each tensor product $x \otimes y$ a directed arrow $x \to y$, the relationship between the $(2, 1)$-De Bruijn graph, whose vertex set is $\{P, Q\}$, and its extension is given by identifying $a := P \otimes P$, $b := P \otimes Q$, $c := Q \otimes P$ and $d := Q \otimes Q$.

Often, for simplifying notation and only in the language $F$, we will write $xy$ instead of $x \otimes y$. No confusion is possible since, in the sequel we forgot the algebraic relations between letters.

We define now the contraction map. It will help us to go from the language $F$ to the $(2, 1)$-De Bruijn graph.

**Definition 5.2 [Contraction map]** For all $n > 2$, the contraction map is the linear map

$$C : F \to \mathbb{C}\langle P, Q \rangle, (y_1 \otimes y_2) \otimes (y_3 \otimes y_4) \otimes \ldots \otimes (y_{n-1} \otimes y_n) \mapsto y_1 y_2 \ldots y_n,$$

where the $y_i$ stands for $P$ and $Q$. For $n = 2$, the contraction is by convention equal to the usual product of $M_2(\mathbb{C})$.

**Example 5.3** For instance, the contraction of $a \otimes b \otimes c := (P \otimes P) \otimes (P \otimes Q) \otimes (Q \otimes P)$ is equal to $PPQP$.

**Remark:** The contraction is associative and generalize the usual product on $M_2(\mathbb{C})$ denoted by $m$. In our example, it symbolizes graphically that the gluing between the loop indexed by $P$ and the edge $P \to Q$ at the vertex $P$ in the $(2, 1)$-De Bruijn graph is represented by the arrow $a \otimes b$ in $F$. 

11
Proposition 5.4 For a time \( t > 1 \), to any monomial \( \Xi \) constructed from \( P, Q \) in the algebra \( \mathbb{C}(P, Q) \), excepted of course any linear superposition of \( P \) and \( Q \), corresponds an unique word \( \omega \) in the language \( F \) such that \( C(\omega) = \Xi \).

Proof: Any sequence \( \Xi \) constructed from \( P, Q \) in the algebra \( \mathbb{C}(P, Q) \) corresponds to an unique path of the (2,1)-De Bruijn graph, i.e. a unique path of its extension. \( \square \)

Denote by \( (\Delta_M)_0 = id, (\Delta_M)_1 := \Delta_M, (\Delta_M)_2 := (id \otimes \Delta_M)\Delta_M \), more generally, for all \( n > 1 \), \( (\Delta_M)_n := (id \otimes \ldots \otimes id \otimes \Delta_M)(\Delta_M)_{n-1} \), similarly for \( \Delta \).

Let us show that the combinatorics generated by the quantum random walk can be obtained by contraction of all the words from the language \( F = \{(\Delta_M)_n(a + b + c + d), n \in \mathbb{N}\} \) and that \( F \) can be also viewed as equipped by the grammar generated by \( \Delta \), i.e.

\[
a \mapsto aa, a \mapsto bc, b \mapsto ab, b \mapsto bd, c \mapsto dc, c \mapsto ca, d \mapsto dd, d \mapsto cb,
\]

i.e. \( F = \{\Delta_n(a + b + c + d), n \in \mathbb{N}\} \). That is we will show that all the words from the language \( F \), present at a fixed time \( t = n > 1 \) and thus obtained by \( (\Delta_M)_{n-2} (a + b + c + d) \), can be also obtained by computing \( \Delta_{n-2} (a + b + c + d) \).

Lemma 5.5 \( \Delta_M(a + b) = \Delta(a + b) \) and \( \Delta_M(c + d) = \Delta(c + d) \).

Proof: Straightforward. \( \square \)

Lemma 5.6 If \( x \) stands for \( a, b, c \) or \( d \), we have the following equalities:

\[
\begin{align*}
C(x \otimes a)P &= C(x \otimes a \otimes a); & C(x \otimes a)Q &= C(x \otimes a \otimes b) \\
C(x \otimes b)P &= C(x \otimes b \otimes c); & C(x \otimes b)Q &= C(x \otimes b \otimes d) \\
C(x \otimes c)P &= C(x \otimes c \otimes a); & C(x \otimes c)Q &= C(x \otimes c \otimes b) \\
C(x \otimes d)P &= C(x \otimes d \otimes c); & C(x \otimes d)Q &= C(x \otimes d \otimes d)
\end{align*}
\]

Proof: Straightforward. \( \square \)

Corollary 5.7 Let \( x \) stands for \( a, b, c, d \). We have \( C(x)(P + Q) = C(\Delta_M x) \).

Proof: Straightforward. \( \square \)

These lemmas claim that for a fixed time \( t = n > 1 \) and a given vertex \( k \) of \( \mathbb{Z} \), all the polynomials present at this vertex can be either recovered by those which were at the vertex \( k - 1 \) and \( k + 1 \) at time \( t = n - 1 \) by multiplying by \( P \) and \( Q \) or by computing the walk starting from \( a, b, c, d \) and generated by the Markov coproduct \( \Delta_M \), the expected polynomials in \( P, Q \) being obtained by contraction.

\[\text{With these substitution laws, coming from a coassociative coproduct, the language } F \text{ will be also said a coassociative language.}\]
Theorem 5.8 \((id \otimes \Delta)\Delta_M = (id \otimes \Delta_M)\Delta_M\).

Proof: Straightforward, thanks to the lemma \[5.3\]. For instance \(a \xrightarrow{\Delta_M} a \otimes (a+b) \xrightarrow{id \otimes \Delta_M(a+b)} a \otimes \Delta(a+b)\). \(\square\)

Corollary 5.9 \((id \otimes \Delta)\Delta(a+b) = (id \otimes \Delta_M)\Delta_M(a+b)\), and \((id \otimes \Delta)\Delta(c+d) = (id \otimes \Delta_M)\Delta_M(c+d)\). This implies that \(\forall n (\Delta_M)_n(a+b+c+d) = (\Delta)_n(a+b+c+d)\).

Proof: Straightforward, thanks to the lemma \[5.5\]. \(\square\)

Remark: This corollary means that all the polynomials, in \(P,Q\) created by the quantum random walk can be obtained by contraction of the markovian walk over the directed graph \(G(\mathcal{E})\) or can be also obtained by contraction of the coassociative walk over \(G(\mathcal{E})\). Thus we have proved that the combinatorics generated by the quantum random walk can be viewed by a coassociative coproduct point of view. We represent here the beginning of the walk coded in terms of the language \(\mathcal{F}\).

The quantum random walk, coded in terms of the language \(\mathcal{F}\).

For the moment, we get all the words created by the coassociative language. But if a word is picked up from this language how can we say that it has to belong to such or such vertex? We have to enlarge the definition of the language \(\mathcal{F}\) by defining an index map and an index language. From now on, we denote, by convention, \(x_{-1,-1} := a, x_{-1,+1} := b, x_{+1,-1} := c, x_{+1,+1} := d\). Notice then that a word from the language \(\mathcal{F}\) can be written like \(\omega := x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_n-1}i_n\). The index language \(\hat{\mathcal{F}}\) is by definition \(\mathbb{Z} \otimes \mathcal{F}\).

Definition 5.10 [Index map] Let \(\omega\) be a word from the language \(\mathcal{F}\), i.e., \(\omega := x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_n-1}i_n\). We define the (linear) index map as: \(\hat{\text{ind}} : \mathcal{F} \to \hat{\mathcal{F}}, \omega \mapsto (\text{ind}(\omega) \otimes \omega)\), with \(\text{ind}(\omega) = \text{ind}(x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_n-1}i_n) := \sum_{k=1}^{n} i_k\).

Proposition 5.11 Let \(\omega := x_{i_1} \ldots x_{i_n-1}i_n\) be a word from the language \(\mathcal{F}\). The index \(\text{ind}(\omega)\) is equal to the number of \(Q\) minus the number of \(P\) obtained in the contraction of the word \(\omega\). Therefore, the index map fixes the vertex attributed by the quantum random walk over \(\mathbb{Z}\).

Proof: We will proceed by recurrence. It is true for \(n = 2\), i.e., for \(a,b,c,d\). Suppose \(\omega\), a word present at vertex \(k\) and at time \(t = n > 2\). We have \(\omega = x_{i_1} \ldots x_{i_n-1}i_n\), and the index \(\text{ind}(\omega) = k\) does indicate the number of \(Q\) minus the number of \(P\) obtained in the contraction of this word. At time \(t = n + 1\),
ω ↦ ω ⊗ x_{i_n, i_{n+1}}. By definition of the quantum random walk this word will be at vertex $k + 1$ if $x_{i_n, i_{n+1}}$ is equal to $Q$, or $k - 1$ if $x_{i_n, i_{n+1}}$ is equal to $P$. Now $\text{ind}(\omega \otimes x_{i_n, i_{n+1}}) = \text{ind}(\omega) + i_{n+1}$. By definition, $i_{n+1} = +1$ for $b$ and $d$, which are words finishing by $Q$ and $i_{n+1} = -1$ for $a$ and $c$ which are words finishing by $P$. □

**Example 5.12** $\text{ind}(a) = -2$ and $C(a) = P^2$. Hence the word $a$ has to be present at time $t = 2$. Moreover its contraction yielding the monomial $P^2$, $a$ is at vertex $-2$, as expected.

The next question is how can we produce all these words, coming from the language $F$, from the notion of periodic orbits of the classical chaotic map $x \mapsto 2x \mod 1$.

**Definition 5.13 [Periodic orbits, pattern]** We define an equivalence relation $\sim$ inside $F$ by saying that $\omega_1 \sim \omega_2$ iff $\omega_1 = x_{i_1} \cdots x_{i_{n-1}}$, for some $n$ and $\exists m$, $\tau^m(\omega_1) = \omega_2$. The set $\PO = F/\sim$ is the set of the periodic orbits of the directed graph $G(\mathcal{E})$. We denote by $< \omega >$ the pattern of an equivalence classe associated with $\omega$ and its permutations, i.e. $< \omega > := < x_{i_1} x_{i_2} \cdots x_{i_{n-1}} >$. A periodic orbit is just the graphical representation of the pattern. Often, we will confound the two words.

**Remark:** Fix a time $t$. We will see later that the length of the pattern of a periodic orbit $< \omega >$ present at $t$ denoted by $l(< \omega >)$, will be equal to $t$.

**Example 5.14** We have $a \otimes b \otimes c \sim c \otimes a \otimes b \sim b \otimes c \otimes a$. The equivalent classe is designed by the pattern $< a \otimes b \otimes c >$ and the associated periodic orbit is $\ldots abcabcabcabc\ldots$.

This very periodic orbit can be also represented by the pattern $< a \otimes b \otimes c \otimes a \otimes b \otimes c >$, i.e we cover two times the triangle and that is all.

Similarly with the language $F$, we have to enlarge the vector space of the periodic orbits $\PO$ to keep the notion of vertex attributed by the quantum walk to each periodic orbit. We denote by $\widehat{\PO} := \mathbb{Z} \otimes \PO$ such a set.

**Definition 5.15 [Index map in $\widehat{\PO}$]** We define the (linear) index map $\widehat{\text{Ind}} : \PO \to \widehat{\PO}$, $< \omega >\mapsto \text{Ind}(< \omega >) \otimes < \omega >$ with,

$$\text{Ind}(< \omega >) := \text{Ind}(< x_{i_1} x_{i_2} \cdots x_{i_{n-1}} >) := \frac{1}{2}(i_1 + i_2) + (i_2 + i_3) + \ldots (i_{n-1} + i_1) = \sum_{k=1}^{n-1} i_k.$$
Example 5.16 \( \text{Ind}(x_{-1,-1}) = -1, \text{Ind}(x_{-1,+1}) = 0, \text{Ind}(x_{+1,-1}) = 0, \text{Ind}(x_{+1,+1}) = +1. \)

This definition does not depend on the choice of the representative of the equivalent classe. Once we have the definition of periodic orbits, we have to read them to obtain information.

Definition 5.17 [Reading map] Let \(<\omega>:=<x_{i_1,i_2}x_{i_2,i_3} \ldots x_{i_n,i_1}>\) be a periodic orbit. The reading map is denoted \(R: \text{PO} \rightarrow F\) with \(x_{i_1,i_2}x_{i_2,i_3} \ldots x_{i_n,i_1} \mapsto \sum_{k=1}^{n} x_{i_k,i_{k+1}}x_{i_{k+1},i_{k+2}} \ldots x_{i_{n+k-2},i_{k+n-1}},\) the labels being understood modulo \(n.\) The reading map does not depend on the choice of the representative of the equivalent classe.

Remark: As we are interested in the notion of information generated by the reading of periodic orbits, we wish to avoid the repetition of words, because knowing that some words are present several times in the reading of a periodic orbit does not bring more information. That is why we define the map \(J: F \rightarrow F, \sum_{k} \lambda_k \omega_k \mapsto \sum_{k} \omega_k,\) where the \(\lambda_k\) are scalars (integers). From now on, the (linear) reading map \(R\) will be always composed with the (non-linear) map \(J.\) This composition will be still denoted by \(R.\)

Example 5.18 Consider the periodic orbit \(<abc>\). Its index is \(-1\) and its reading yields \(ab + bc + ca.\) By contraction we obtain \(PPQ + PQP + QPP,\) which is exactly the polynomial expected at time \(t = 3\) and at vertex \(-1.\)

Example 5.19 The reading of the periodic orbit \(\langle b \otimes c \otimes b \otimes c \rangle,\) with a pattern of length 4, yields \(b \otimes c \otimes b + c \otimes b \otimes c + b \otimes c \otimes b + c \otimes b \otimes c.\) Composed by the map \(J\) we get \(R(b \otimes c \otimes b \otimes c) := b \otimes c \otimes b + c \otimes b \otimes c.\) By contraction, we obtain \(PQPQ + QPQP.\) Moreover its index is 0.

\[
\begin{array}{c}
\text{Periodic orbits of period 4, } <bcbc>, \text{ with pattern of length 4.}
\end{array}
\]

The reading of the periodic orbit \(\langle a \otimes b \otimes d \otimes c \rangle,\) with pattern of length 4, yields \(a \otimes b \otimes d + b \otimes d \otimes c + d \otimes c \otimes a + c \otimes a \otimes b.\) By contraction, we obtain \(PPQQ + PQQP + QQPP + QPPQ.\) Its index is 0.

\[
\begin{array}{c}
\text{Periodic orbit, } <abdc>, \text{ with pattern of length period 4.}
\end{array}
\]
That is the sum of these two periodic orbits yield the polynomials expected at time \( t = 4 \) and at vertex 0.

So as to give a minorant of the number of periodic orbits at a given time \( n \) at vertex \( k \), we have to link the number \( k \) to the number \( x_P \) of \( P \) and the number \( x_Q \) of \( Q \). Suppose \( k \) positive \(^5\). This means that \( x_Q = x_P + k \). As \( x_P + x_Q = n \), we get \( x_P = \frac{n-k}{2} \). As this solution has to be an integer, this will fix the possibilities of \( k \), i.e. the possible vertices reached by the quantum random walk at time \( n \). We denote \( \kappa := \frac{n-k}{2} \).

**Proposition 5.20** Set \( \varpi := \frac{1}{n \kappa! (n-\kappa)!} \). The number of periodic orbits at the vertex \( k \) and at time \( n \) is greater or equal to \( \varpi \) if this number is an integer and greater or equal to the integer part of \( \varpi + 1 \) if \( \varpi \) is not an integer.

**Proof:** For a given vertex \( k \) and a given time \( n \), we have to have \( \frac{n!}{\kappa! (n-\kappa)!} \) polynomials in \( P \) and \( Q \) and by definition we know that the reading of these periodic orbits yields at most \( n \) different words. \( \square \)

In the following picture, we indicate the classical periodic orbits involved into the quantum walk over \( \mathbb{Z} \).

\(^5\)As the walk is symmetric, we have as many polynomials at the vertex \( k \) as at the vertex \(-k\).
Periodic orbits and their associated pictures.

**Proposition 5.21** Let $< \omega > := (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ be the pattern of a periodic orbit. Its reading yields $R < \omega > := \sum X_k$, where $X_k$ is the word $x_{ik+1}, x_{ik+2}, \ldots, x_{ik+n-1}$ mod $n$. We have $\text{Ind}(< \omega >) := \text{ind}(X_k)$, for all $k = 1, \ldots, n$ which proves that the reading map decomposes the periodic orbit into words with same index.

**Proof:** Let $< \omega > := (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ be a periodic orbit and the $X_k$ its decomposition under the reading map. We have $s_1 := \text{ind}(X_1) := \text{ind}(x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}+1}) = \sum_{k=1}^{n} i_k$. $s_2 := \text{ind}(X_2) := \text{ind}(x_{i_2}, x_{i_3}, \ldots, x_{i_{n-1}+1}) = \sum_{k=2}^{n} i_k + i_1$. Thus $s_1 = s_2$, $s_k := \text{ind}(X_k) := i_k + i_{k+1} + i_{k+2} + \ldots + i_n + i_1 + \ldots + i_{k-1} = s_1$. Besides, by definition $\text{Ind}(< \omega >) := \frac{1}{2}(2\text{ind}(\omega))$, whatever the choice of the representative is. □
Remark: A periodic orbit with index $k$, will be affected to the vertex $k$. Its reading will yield words of index $k$.

Example 5.22 The reading of the periodic orbit $<abc>$ is: $R(<abc>) := ab + bc + ca$. We have $\text{Ind}(<abc>) = -1$ and $\text{ind}(ab) = \text{ind}(ca) = \text{ind}(bc) = -1$.

Definition 5.23 [Completion] For a time $t = n$ and a given vertex $k$, all the polynomials can be recovered by the contraction of words coming from the coassociative language $F$ and having the same index $k$. Suppose we pick up one of the word present at this vertex $k$, say $x_{i_1j_1}x_{i_2j_2} \ldots x_{i_{n-1}j_{n-1}}x_{i_nj_n}$. The completion $\text{Comp}$ maps $x_{i_1j_1}x_{i_2j_2} \ldots x_{i_{n-1}j_{n-1}}x_{i_nj_n}$ into $<x_{i_1j_1}x_{i_2j_2} \ldots x_{i_{n-1}j_{n-1}}x_{i_nj_n}>$, thus it is a map from $F$ to $PO$.

Proposition 5.24 $\text{Ind}(\text{Comp}(x_{i_1j_1}x_{i_2j_2} \ldots x_{i_{n-1}j_{n-1}})) = \text{ind}(x_{i_1j_1}x_{i_2j_2} \ldots x_{i_{n-1}j_{n-1}})$.

Proof: Straightforward. □

Remark: The reading of the completion of a word present at vertex $k$ will yield many words present at this vertex. By contraction we will recover polynomials in $P, Q$ present at vertex $k$. As all these polynomials are coded bijectively by a word of $F$ present at the vertex $k$, the completion of all these words will form all the necessary periodic orbits whose the reading will yield all the words present at $k$. For the time being, we started from the polynomial algebra $C(P, Q)$ and arrived at the periodic orbits set of the chaotic map $x \mapsto 2x \mod 1$. We showed that the reading of all the patterns of periodic orbits present at a vertex $k$ yielded all the words of the language $F$ present at $k$ and whose the contraction gave the polynomials in $P, Q$.

The following idea is to equip the periodic orbits with the coproduct of $E$. Thanks to this coproduct, we will be able to speak of the growth of periodic orbits and to recover the quantum random walk over $\mathbb{Z}$ from the reading of them.

Remark: To avoid redundancy, we define the map $J_* : PO \to PO$, $\sum_k \lambda_k <\omega_k> \mapsto \sum_k <\omega_k>$, where $\lambda_k$ are scalars (integers).

Definition 5.25 [Growth of periodic orbit] We denote by $\mathcal{G} : PO \to PO$ the growth operator:

$$\mathcal{G}(<\omega>) = \sum_{k=1}^{l(<\omega>)} <^k\delta(\omega)>,$$

where $^k\delta = \text{id} \otimes \ldots \otimes \Delta_k \otimes \ldots \otimes \text{id}$ and $l(<\omega>)$ is the length of the pattern of the periodic orbit $<\omega>$.

Remark: As $\Delta x_{ij} := \sum_{k=+1,-1} x_{ik} \otimes x_{kj}$, it is easy to see that $\mathcal{G} : PO \to PO$. We take into account all the possible substitutions coming from the coassociative grammar.

Remark: Notice also that the coproduct leaves a letter of index $k$ into two letters of index $k + 1$ and $k - 1$, i.e the coassociative coproduct let the index invariant. For instance $\Delta b = ab + bd$ and $\text{Ind}(b) = 0 \mapsto (\text{Ind}(ab) = -1) + (\text{Ind}(bd) = +1)$.
Theorem 5.26  The growth operator applied on all the periodic orbits at time \( t = n \) will yield all the periodic orbits at time \( t = n + 1 \). By applying the operator \( J_* \), we will recover exactly the number of periodic orbits present at time \( t = n + 1 \).

Proof: Let \( \omega = x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_{n-1}} \) a word present at time \( t = n \), its completion yields the periodic orbit \(< x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_{n-1}}x_{i_1} >\) whose reading will give us the words \( x_{i_k}x_{ik+1}x_{ik+2} \ldots x_{ik+n-2} \mod n \), \( k = 1, \ldots, n \). By definition of the quantum random walk, we have to multiply them by \( P \) and \( Q \) to have the new polynomials present at time \( t = n + 1 \). Let see how it works on \( \omega \) itself. We get \( C(x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_{n-1}})P \) and \( C(x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_{n-1}})Q \). These two polynomials come from the contraction of two words present at time \( t = n + 1 \), \( x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_{n-1}}x_{i_n} \) and \( x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_{n-1}}x_{i_n} \). The completion of these two words comes from the periodic orbit \(< x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_{n-1}}\Delta x_{i_{n+1}} >\). Now, for some \( k \) and by reading the labels \( \mod n \), the contraction of the word \( x_{i_k}x_{ik+1}x_{ik+2} \ldots x_{ik+n-2}x_{i_{n+1}} \) by multiplication by \( P \) and \( Q \) will come from the contraction of the word \( x_{i_k}x_{ik+1}x_{ik+2} \ldots x_{ik+n-2}x_{i_{n+1}} \) and the word \( x_{i_k}x_{ik+1}x_{ik+2} \ldots x_{i_{k+n-2}}x_{i_{k+n-1}} \). The completion of these two words comes from the periodic orbit \(< x_{i_k}x_{ik+1}x_{ik+2} \ldots x_{i_{k+n-2}}x_{i_{k+n-1}} \Delta x_{i_{k+n-1}} >\). The last term \( \Delta x_{i_{k+n-1}} \) is equal to \( \Delta x_{i_k} \). That is why the growth operator works on periodic orbits to recover the words at time \( t = n + 1 \) from the periodic orbits at time \( t = n \). We have just obtained all the periodic orbits with repetition, by applying \( J_* \), we get the theorem. □

Remark: It is worth to notice that the markovian coproduct is closely related to the coassociative coproduct. Indeed for a word \( x_{i_1}x_{i_2}x_{i_3} \ldots x_{i_{n-1}} \), the product by \( P \) and \( Q \) will yield the words \( x_{i_1}x_{i_2}x_{i_3} \ldots \Delta_M(x_{i_{n-1}}) \), see lemma 5.6 whereas its completion will use the coassociative coproduct.

Example 5.27  Consider the word \( ab \). The contraction of \( ab \) multiplied by \( P \) and \( Q \) yields \( abc \) and \( abd \). Its completion will yield \( abca \) and \( abdc \), which is \(< ab\Delta c >\). Thus \( ab \xrightarrow{P,Q} abc + abd = a\Delta_M(b) \xrightarrow{\text{Comp}} < ab\Delta c >\).

Remark: We have proved that all the polynomials of the quantum random walk over \( Z \) can be obtained by reading periodic orbits of the classical chaotic map \( x \mapsto 2x \mod 1 \), and that PO can be viewed as a coassociative language, since its substitution rules come from a coassociative coproduct.

Example 5.28  [Reconstruction of the quantum walk from classical periodic orbits]  We can start by the two loops \(< aa >\) and \(< dd >\), at \( t = 2 \), because the patterns created by \(< bc >\), with the growth operator can be recovered by these two loops. At time \( t = 3 \), the growth operators yield \(< aa >\xrightarrow{< aab >} + < bca > + < aaa > + < abc >\). To avoid the redundancy of information, i.e. by applying \( J_* \), we consider the periodic orbits \(< aab > + < abc >\). By applying the linear map \( \tilde{\text{Ind}} \), we find that, \( \text{Ind}(< aab >) = -3 \) and \( \text{Ind}(< abc >) = -1 \). Their reading yields \( a + a + b + c + a \) which are separated by their indexes. By applying the linear map \( \tilde{\text{ind}} \), we find that \( aa \) has an index equal to \(-3\) and \( ab, bc \) and \( ca \) are all of index \(-1\). Similarly \( < dd >\xrightarrow{< ddd >} + < cbb > + < ddd > + < cbb >\), therefore we consider the orbit \(< ddd > + < cbb >\). Notice that only these 4 periodic orbits are present at time \( t = 3 \). By applying the growth operator at time \( t = 3 \), we will still obtain all the orbits present at time \( t = 4 \), and so forth.
5.1 Arithmetics from the periodic orbits of the map $x \mapsto 2x \mod 1$

The aim of this section is to show that all these periodic orbits are generated by 6 fundamental orbits, which are:

![Fundamental periodic orbits and their associated pictures.](image)

They will play the same rôle that the prime numbers in the integer arithmetics. For that, we need a product.

**Definition 5.29** Define the *gluing map* $\# : \mathcal{PO} \to \mathcal{PO}$ which glues two periodic orbits with their graphic intersection.

**Example 5.30** We show here how to glue periodic orbits.

![Example of gluing of periodic orbits.](image)

**Proposition 5.31** Any periodic orbits of $\mathcal{PO}$ can be decomposed into fundamental periodic orbits, i.e. the graphical representation of a periodic orbit can be viewed as the gluing of some of these fundamental orbits.

*Proof:* All the periodic orbits are created by the growth operator from the two loops $<aa>$ and $<dd>$. Therefore, it suffices to prove that the growth of any fundamental periodic orbits still yield fundamental orbits. But $<aa>$ yields $<aaa>$ and $<abc>$, $<dd>$ yields $<ddd>$ and $<cbd>$, $<abc>$ yields...
All these periodic orbits are graphically the gluing of fundamentals orbits. For instance \(< abddc >\) is the gluing of the square \(< abdc >\) and the loop \(< dd >\), i.e. \(< abddc > = < abdc > \# < dd >\). \(< abebe >\) is the gluing of the triangle \(< abc >\) with \(< bc >\), i.e. \(< abebe > = < abc > \# < bc >\) and so on.

\[\square\]

6 Conclusion

To quantify the Bernoulli walk over \(\mathbb{Z}\) we choose the following strategy. Firstly we use the isomorphism between this discrete process and the chaotic map \(x \mapsto 2x \mod 1\) with \(x \in [0,1]\). With this chaotic process, an unistochastic matrix \(B_2\) has been associated \([18]\). This matrix is identified with the \((2,1)\)-De Bruijn directed graph which can be viewed as a Markov \(L\)-coalgebra. This allows to code periodic orbits of this chaotic map in one to one way with the periodic orbits of the \((2,1)\)-De Bruijn directed graph. To quantify this system we choose an unitary matrix \(U\) such that the point by point product or Hadamard product between \(U\) and \(U^\dagger\) yields \(B_2\). This leads to the notion of quantum graph. From a choice of matrix \(U = P + Q\) among possible quantisations, we recover a non commutative algebra \(\mathbb{C}\langle P, Q \rangle\), subalgebra of \(M_3(\mathbb{C})\), the decomposition of the unitary matrix yielding the matrices \(P\) and \(Q\) which generate the quantum random walk over \(\mathbb{Z}\) \([21]\). But to study quantum chaos, we learn from \([6]\) that the rôle of the peridic orbits of the associated classical system are capital. Would it be possible to yield a physical interpretation of the notion of coassociative coproduct in the quantisation of a classical dynamical system? We know that a classical dynamical system can be modelized by a commutative \(C^*\)-algebra. To quantify such a system we look for a non commutative algebra embeding it. However, in some quantum chaos experiments \([4]\), the periodic orbits of the classical system play a capital rôle. In fact, periodic orbits seem to be invariant when we quantify it. As we show in the model of the quantum random walk over \(\mathbb{Z}\), these classical periodic orbits (of the chaotic map \(x \mapsto 2x \mod 1\)) are always present, not in the form of a classical trajectory, but in the form of a coassociative language. A physical interpretation of the notion of coassociative coproduct would be to generate the grammar of the language associated with the periodic orbits. In addition to the required non commutative algebra in the quantisation of a chaotic system, a coassociative coalgebra, closely related to a Markov \(L\)-coalgebra would be required to manipulate classical periodic orbits, via a coassociative grammar. To recover the combinatorics involved by the quantum dynamics from the language of periodic orbits, we could use a kind of reading map and contraction map. The link between the combinatorics generated by the quantum system and and the coassociative language of its periodic orbits could be interpreted in terms of a smash biproduct \([16]\). [5].

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References

[1] A. Ambainis and al. One-dimensional quantum walks. Proceedings of the 33rd Annual ACM Symposium on theory of computing, pages 37–49.
[2] F. Barra and P. Gaspard. On the level sparing distribution in quantum graphs. *J. Stat. Phys.*, 101:283–319, 2000.

[3] F. Barra and P. Gaspard. Transport and dynamics on open quantum graphs. *Phys. Rev. E*, 65:066215–066236, 2002.

[4] Ph. Biane. Marche de Bernoulli quantiques. *Seminaire de Probabilites XXIV, Lect. Notes in Maths. Springer*, 1426:329–344, 1990.

[5] S. Caenepeel and al. Factorisation structures of algebras and coalgebras. *eprint arXiv:math.QA/9809063.*

[6] M.C. Gutzwiller. *Chaos in classical and quantum mechanics.* Springer-Verlag, 1990.

[7] V.F.R. Jones. Index for subfactors. *Invent. math.*, pages 1–25, 1983.

[8] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems.* Cambridge University Press, 1995.

[9] N. Konno. A new type of limit theorems for the one-dimensional quantum random walk. *eprint arXiv:quant-ph/0206103.*

[10] N. Konno, T. Namiki, and T. Soshi. Symmetricity of distribution for one-dimensional Hadamard walk. *eprint arXiv:quant-ph/0205065.*

[11] T. Kottos and U. Smilansky. *Phys. Rev. Lett.*, 79:4794, 1997.

[12] T. Kottos and U. Smilansky. *Ann. Phys. NY*, 274:76, 1999.

[13] Ph. Leroux. Coassociativity breaking and oriented graphs. *eprint arXiv:math.QA/0204342.*

[14] Ph. Leroux. Tiling the \((n^2,1)\)-de-bruijn graph with \(n\) coassociative coalgebras. *eprint arXiv:math.QA/0209108.*

[15] J.L Loday. Dialgebras. *eprint arXiv:math.QA/0102053, in Dialgebras and related operads, Lecture Notes in Mathematics*, 1763:7–66, 2001.

[16] S. Majid. *Quantum groups.* Cambridge University Press, 1995.

[17] V.A. Malyshev. Random grammars. *Russian Math. Surveys*, 53:2, 1997.

[18] P. Pakonski, K. Zyczkowski, and M. Kus. Classical 1D maps, quantum graphs and ensembles of unitary matrices. *eprint arXiv:quant-ph/0011050.*

[19] G. Tanner. Spectral statistics for unitary transfer matrices of binary graphs. *J. Phys. A: Math. Gen.*, 33:3567–3585, 2000.