Fractality and geometry in ultra-relativistic nuclear collisions

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Abstract

Assuming fractality of hadronic constituents, we argue that asymmetry of space-time can be induced in the ultra-relativistic interactions of hadrons and nuclei. The asymmetry is expressed in terms of the anomalous fractal dimensions of the colliding objects. Besides state of motion, the relativistic principle is applied to the state of asymmetry as well. Such realization of relativity concerns scale dependence of physical laws emerging at small distances. We show that induced asymmetries of space-time are a priori not excluded by the Michelson’s experiment even at large scales.

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I. INTRODUCTION

At sufficient high energies, the interactions of hadrons and nuclei can be considered as an ensemble of individual interactions of their constituents. The constituents are partons in the parton model or quarks and gluons which are the building blocks in the theory of QCD. Production of particles with large transverse momenta from such reactions has relevance to fundamental principles of physics at small interaction distances. One of the expressions aiming to reflect this relation is the $z$ scaling observed in differential cross sections for the inclusive reactions at high energies. The hypothesis of the scaling states that cross sections of particles with large transverse momentum $Q_\perp$ produced in relativistic collisions of hadrons and nuclei depend on the single variable

$$z = z_0 \Omega^{-1},$$

where

$$\Omega(x_1, x_2) = (1 - x_1)^{\delta_1} (1 - x_2)^{\delta_2}.$$

The scaling variable $z$ has character of a fractal measure. For a given production process, its finite part $z_0$ is proportional to the transverse energy released in the underlying collision of constituents. The divergent part $\Omega^{-1}$ describes the resolution at which the collision of the constituents can be singled out of this process. The $\Omega(x_1, x_2)$ represents relative number of all initial configurations containing the constituents which carry the fractions $x_1$ and $x_2$ of the incoming momenta. In such a picture, we consider the ultra-relativistic collisions of hadrons and nuclei as collisions of parton fractals with the anomalous fractal dimensions $\delta_1$ and $\delta_2$.

The goal of the paper is to focus on the general premisses of the $z$ scaling in view of fundamental principles of physics at small interaction distances. It concerns scale dependence of physical laws gradually emerging in various experimental and theoretical investigations. Such extension of physics is intrinsically linked to the evolution of the concept of space-time. Its structure is characterized by explicitly scale dependent metric potentials. Asking questions about the metrics leads one to question the relativity. The relativistic principle besides motion, applies also to the laws of scale. The basic assumption tackled in the paper is breaking of the reflection invariance which is the characteristic feature of fractality at small distances. One of the consequences is change of the standard dispersion relation between the energy and momentum having implications to non-standard expressions of these quantities in terms of the velocity. New dispersion relation is Lorentz transformation of the free energy with respect to change of the state of scale. The scale changes are expressed in terms of the "scale velocity" which is property of space-time induced by the interaction. We express the scale velocity connected with the space-time asymmetry in terms of the anomalous fractal dimensions of the interacting fractal objects. The geometrical objects model the internal parton structure of hadrons and nuclei revealed in their interactions at high energies. Dealing with asymmetry of space-time, we show that anisotropic propagation of light signals need not to be necessary in contradiction with Michelson’s experiment concerning the light interference.

II. CONSTITUENT INTERACTIONS

Ultra-relativistic interactions of hadronic constituents are local relative to the resolution which depends on the kinematical characteristics of particles produced in the collisions. In accordance with the property of locality it has been suggested that gross features of the single-inclusive particle distributions for the reaction

$$M_1 + M_2 \rightarrow m_1 + X$$

(3)

can be described in terms of the corresponding kinematical characteristics of the interaction

$$(x_1 M_1) + (x_2 M_2) \rightarrow m_1 + (x_1 M_1 + x_2 M_2 + m_2).$$

(4)

The $M_1$ and $M_2$ are masses of the colliding hadrons or nuclei and $m_1$ is the mass of the inclusive particle. The parameter $m_2$ is used in connection with internal conservation laws (for isospin,
baryon number, and strangeness). The \( x_1 \) and \( x_2 \) are the scale-invariant fractions of the incoming four-momenta \( P_1 \) and \( P_2 \) of the colliding objects. We have determined the momentum fractions \( x_1 \) and \( x_2 \) in the way to minimize the fractal resolution \( \Omega^{-1}(x_1, x_2) \) accounting simultaneously for the recoil mass condition

\[
(x_1 P_1 + x_2 P_2 - Q)^2 = (x_1 M_1 + x_2 M_2 + m_2)^2.
\]

The \( Q \) is the four-momentum of the inclusive particle with the mass \( m_1 \). The momentum fractions resulting from these requirements represent the sum

\[
x_1 = \lambda_1 + \chi_1, \quad x_2 = \lambda_2 + \chi_2.
\]

The parts \( \lambda_i \) are connected with the inclusive particle and the parts \( \chi_i \) with the creation of its recoil. According to the decomposition, the binary subprocess (4) can be rewritten to the symbolic form

\[
(\lambda_1 + \chi_1) + (\lambda_2 + \chi_2) \rightarrow (\lambda_1 + \lambda_2) + (\chi_1 + \chi_2).
\]

The explicit formulae for the momentum fractions \( \lambda_i \) read

\[
\lambda_1 = \frac{(P_2 Q) + M_2 m_2}{(P_1 P_2) - M_1 M_2}, \quad \lambda_2 = \frac{(P_1 Q) + M_1 m_2}{(P_1 P_2) - M_1 M_2}.
\]

Their combinations

\[
\sqrt{\lambda_1 \lambda_2} \sim \frac{E_\perp}{\sqrt{s}}, \quad \sqrt{\frac{\lambda_2}{\lambda_1}} \sim \tan(\theta/2)
\]

are related to the transverse energy \( E_\perp \) and the centre-of-mass angle \( \theta \) of the inclusive particle \( m_1 \). Now let us turn to the recoil part in the constituent subprocess and examine it in more detail. Suppose both colliding objects possess similar fractal structures in the sense that their anomalous fractal dimensions are equal \( \delta_1 = \delta_2 \). In that case, the momentum fractions \( \chi_1 \) and \( \chi_2 \) take the form

\[
\chi_1 = \sqrt{\mu_1 \lambda_2 + \omega_1}, \quad \chi_2 = \sqrt{\mu_2 \lambda_1 + \omega_2},
\]

where

\[
\mu = \sqrt{\lambda_1 \lambda_2 + \lambda_0}, \quad \lambda_0 = \frac{0.5(m_2^2 - m_1^2)}{(P_1 P_2) - M_1 M_2}.
\]

Similar as for the inclusive particle, the combinations

\[
\sqrt{\mu_1 \mu_2} \sim \frac{E'_\perp}{\sqrt{s}}, \quad \sqrt{\frac{\mu_2}{\mu_1}} \sim \tan(\theta'/2)
\]

are related to the transverse energy \( E'_\perp \) and the centre-of-mass angle \( \theta' \) of the recoil in the binary subprocess (4). The center-of-mass angels do not comply the back-to-back correlation \( \theta + \theta' = \pi \) between the inclusive particle and its recoil for \( x_1 \neq x_2 \). This is because the center-of-mass system of the interacting constituents is generally not equal to the center-of-mass system of the reaction. The transverse energy balance is expressed by the relation \( \mu^2 = \mu_1 \mu_2 \).

Now suppose the colliding objects possess mutually different fractal structures in the sense that their anomalous fractal dimensions are not equal, \( \delta_1 \neq \delta_2 \). In that case, the momentum fractions \( \chi_1 \) and \( \chi_2 \) have more complicated form

\[
\chi_1 = \sqrt{\mu_1^2 + \omega_1^2 - \omega_1}, \quad \chi_2 = \sqrt{\mu_2^2 + \omega_2^2 + \omega_2},
\]

where
\[
\mu_1 = \mu_1 \sqrt{\alpha}, \quad \mu_2 = \mu_2 \frac{1}{\sqrt{\alpha}},
\]
\[
\omega_1 = \mu_1 \bar{a}, \quad \omega_2 = \mu_2 \bar{a}.
\]

The parameter \(\alpha = \delta_2/\delta_1\) is ratio of the anomalous (fractal) dimensions of the fractal objects (hadrons and nuclei) colliding at high energy. The symbol \(\bar{a}\) is given by the formula
\[
\bar{a} = a \frac{\alpha - 1}{2 \sqrt{\alpha}} \xi
\]

where
\[
\xi = \frac{\pi}{\sqrt{(1 - \lambda_1)(1 - \lambda_2)}}
\]

is a scale factor from the interval \(0 \leq \xi \leq 1\). When approaching the phase-space limit, the scale factor \(\xi\) tends to unity. Along the phase-space limit \(\xi = 1\) and \(x_1 = x_2 = 1\). The phase-space boundary thus corresponds to the fractal limit characterized by infinite value of the fractal measure \(\xi\), i.e. by infinite value of the scaling variable \(z\). For collisions of the asymmetric objects with \(\delta_1 \neq \delta_2\), we denote the center-of-mass angle of the recoil in the subprocess (4) by \(\theta'\). The transverse energy \(E'_\perp\) and the angle \(\theta'\) correspond to the following combinations of the momentum fraction
\[
\sqrt{\chi_1 \chi_2} \sim \frac{E'_\perp}{\sqrt{s}}, \quad \sqrt{\frac{\chi_2}{\chi_1}} \sim \tan(\theta'/2).
\]

The conservation of transverse degrees of freedom is given by \(\mu^2 = \chi_1 \chi_2\). It can be shown from the above relations that
\[
\theta' \leq \theta \quad \text{for} \quad \delta_1 \leq \delta_2
\]

and vice versa. We thus make the following conclusion. In the collision of fractal objects with mutually different anomalous dimensions, the momentum of the recoil produced in a constituent collision is shifted towards the fractal object with richer fractal structure expressed by larger anomalous dimension. In this sense the interactions of constituents are influenced by the asymmetric fractal background created in collisions of parton fractals (hadrons or nuclei) with \(\delta_1 \neq \delta_2\). The asymmetry of such background represents a sort of "medium" with scale properties characterized by a scale velocity \(v\).

There exists analogy of such situation which is the propagation of light in moving refracting media. If a medium with the refractive index \(n\) moves with the velocity \(v\), the elementary waves are dragged along the medium with the velocity \(\bar{a}\)
\[
a = \frac{v}{1 - v^2/n^2} \left(1 - \frac{1}{n^2}\right).
\]

There is, however, significant difference between this analogy and the situation we consider in ultra-relativistic nuclear collisions. The difference is because of the basic property of fractals - never ending structure at any resolution. Basic assumption here is that fractality of the interaction distorts the very structure of space-time in the interaction region. As a result the space-time becomes polarized with the metric undergoing change. Background metric changes have been considered as "recoil" effects modifying the relativistic momentum-energy dispersion relation \[14\]. The particle feels an 'unusual' metric which is a constant of motion. Fractalization of space-time was considered in Ref. \[5,6\] and its properties have been studied by others \[7\]. One of its basic properties is breakdown of the reflection invariance which depends on scale.

Generally, fractal approach to the ultra-relativistic interactions of hadrons and nuclei needs profound understanding. It concerns the deformation of space-time at small scales in the interaction region and attributes additional meaning to the physical quantities such as the momentum, mass, energy or velocity. They may be defined from parameters of the fractal objects in terms of the
fractal geometry. This includes extension of the relativity principles to the relativity of scales as well to more comprehensive scale-motion relativistic concepts.

III. BREAK DOWN OF THE REFLECTION INVARIANCE, THE WAY TOWARDS SCALE-MOTION RELATIVITY

General solution to the theory of the special relativity is the Lorenz transformation. As demonstrated by Nottale, it can be obtained under minimal number of three successive constraints. They are (i) homogeneity of space-time translated as the linearity of the transformation, (ii) the group structure defined by the internal composition law and (iii) isotropy of space-time expressed as the reflection invariance. Let us consider the relativistic boost along the x-axis. Without any loss of generality, the linearity of the transformation can be expressed in the form

\[ x' = \gamma(u)[x - ut], \]
\[ t' = \gamma(u)[A(u)t - B(u)x], \]

where \( \gamma, A, \) and \( B \) are functions of a parameter \( u \). The parameter represents usual velocity in the motion relativity or the "scale velocity" used, e.g., in the concept of the scale relativity concerning fractal dimensions and fractal measures. The principle of relativity tells us that these equations keep the same form whatever the state of motion. The third constraint, the isotropy of space-time, results in the requirement that change of orientation of the variable axis does not change the form of the transformations, provided \( u' = -u \). As considered in the previous section, the fractal approach to the ultra-relativistic interaction of hadrons and nuclei leads to the space-time isotropy breakdown in the interaction region. This is translated as breaking of the reflection invariance at the infinitesimal level.

A. Space-time asymmetry in 3+1 dimensions

We now turn to the question how to express breaking of the reflection invariance in the framework of special relativity. Let us describe a point \( P \) in two Cartesian reference systems \( S \) and \( S' \). We assume that the systems are oriented parallel to each other and that \( S' \) is moving relative to \( S \) with the velocity \( u \) in the direction of the positive x-axis. We suppose that the asymmetry expressed by a parameter \( a \) is parallel to the velocity \( u \). As demonstrated in Appendix, the relativistic transformations of the coordinates and time are given by

\[ x'_1 = \gamma(u)[x_1 - ut], \quad t' = \gamma(u)[(1 - 2au)t - u x_1], \]

where

\[ \gamma(u) = \frac{1}{\sqrt{1 - 2au - u^2}} \]

The violation of the space-time reflection invariance is expressed by a non-zero value of \( a \). For the vanishing value of \( a \), the transformations turn into the usual relativistic transformations of the coordinates and time. The inverse relations

\[ x_1 = \gamma(u)[(1 - 2au)x'_1 + ut'], \quad t = \gamma(u)[t' + ux'_1] \]

are obtained as the solution of Eqs. (23) and (24) with respect to the unprimed variables. They can be also derived by the interchange \( x_1 \leftrightarrow x'_1, t \leftrightarrow t', u \leftrightarrow u' \), and by the relation

\[ u' = -\frac{u}{1 - 2au}. \]

This formula connects the velocity \( u' \) of the system \( S \) in the \( S' \) frame with the velocity \( u \) of the system \( S' \) in the \( S \) reference system. Because of the asymmetry parameter \( a \), the magnitudes of the two velocities are not equal. The invariant of the transformations (23) is
\[ t^2 - x^2 - 2tx_1. \]  

(27)

In more general case, when the space-time anisotropy \( a \) acquires an arbitrary direction, we write
the invariant in the form
\[ \eta_{\mu\nu}(a)x^\mu x^\nu = t^2 - x^2 - 2ta \cdot x - (a \times x)^2 \equiv \tau^2. \]  

(28)

Besides the diagonal part, it has extra terms given by a non-zero values of the vector \( a \). Similar
extra terms of the relativistic invariant were considered in Ref. [7] and associated with breaking of
the reflection invariance assumed at the infinitesimal level. In the four dimensional notation, the
invariant (28) corresponds to the metrics
\[ \eta_{\mu\nu}(a) = \left( \begin{array}{cc} \eta_{ij} - a_i - a_j \\ 1 \end{array} \right), \quad \eta_{ij} = -(1 + a^2)\delta_{ij} + a_i a_j. \]  

(29)

Here the indices \( i \) and \( j \) numerate the first three rows and columns of the matrix \( \eta \), respectively. The
\( \delta_{ij} \) is the Kronecker’s symbol. Next we present the explicit form for the relativistic transformations of
the coordinates and time in the considered case. They must be linear and homogeneous, preserving
the invariant (28). The transformations have to possess an internal group structure required by
the principle of relativity. We denote the parameter of the group by the symbol \( u \). The parameter
is the velocity of the system \( S' \) with respect to the \( S \) reference frame. In connection with the
transformation formulae, it is convenient to introduce the notations
\[ \gamma = \frac{1}{\sqrt{1 - u^2 - 2a \cdot u - (a \times u)^2}}, \]  

(30)

and
\[ g = (1 - a \cdot u)\gamma - 1. \]  

(31)

Here \( a^2 = a \cdot a \) and \( u^2 = u \cdot u \). We define the following combinations of \( g \) and \( \gamma \),
\[ \gamma_\pm = g \pm \gamma a \cdot u, \quad g_\pm = \gamma(1 + a^2) \pm ga \cdot u/u^2. \]  

(32)

Let us consider the relativistic transformations
\[ x' = x - u \left[ \gamma(t - a \cdot x) - gu \cdot x/u^2 \right], \]  

(33)

\[ t' = t + \left[ \gamma_+(t - a \cdot x) - g_+u \cdot x \right]. \]  

(34)

They generalize the special transformations (23) which are recovered by \( u = (u, 0, 0) \) and \( a = (a, 0, 0) \). The inverse relations are obtained by the interchange \( x \leftrightarrow x', t \leftrightarrow t', u \leftrightarrow u' \), where
\[ u' = -\frac{u}{1 - 2a \cdot u}. \]  

(35)

According to the substitution, there exist the symmetry properties
\[ \gamma(u') = (1 - 2a \cdot u)\gamma(u), \quad \gamma_\pm(u') = \gamma_\mp(u), \]  

(36)

\[ g(u') = g(u), \quad g_\pm(u') = (1 - 2a \cdot u)g_\mp(u). \]  

(37)

Exploiting the properties, the inverse transformations
\[ x = x' + u \left[ \gamma(t' - a \cdot x') + gu \cdot x'/u^2 \right], \]  

(38)

\[ t = t' + \left[ \gamma_+(t' - a \cdot x') + g_+u \cdot x' \right]. \]  

(39)
follow immediately. The relativistic transformations can be expressed in a more compact form

\[ x' = D(u, a)x, \]  

(40)

where

\[ D(u, a) = \begin{pmatrix} \delta_{ij} + gu_iu_j/u^2 + \gamma u_j & -\gamma u_i \\ -g_ju_i - \gamma a_j & 1 + \gamma \end{pmatrix}. \]  

(41)

The inverse matrix reads

\[ D^{-1}(u, a) = \begin{pmatrix} \delta_{ij} + gu_iu_j/u^2 - \gamma u_j & +\gamma u_i \\ +g_ju_i - \gamma a_j & 1 + \gamma \end{pmatrix}. \]  

(42)

The transformation matrices can be decomposed into the product

\[ D(u, a) = A^{-1}_x(a)\Lambda(\beta)A_x(a), \]  

(43)

Here

\[ A_x(a) = \begin{pmatrix} \sqrt{1 + a^2}\delta_{ij} & 0 \\ -a_j & 1 \end{pmatrix}, \]  

(44)

and

\[ \Lambda(\beta) = \begin{pmatrix} \delta_{ij} + g_0\beta_i\beta_j/\beta^2 & -\gamma_0\beta_i \\ -\gamma_0\beta_j & \gamma_0 \end{pmatrix}, \]  

(45)

with

\[ \gamma_0 = \frac{1}{\sqrt{1 - \beta^2}}, \quad g_0 = \gamma_0 - 1. \]  

(46)

The matrix \( \Lambda \) depends on the vector

\[ \beta = \sqrt{1 + a^2} \frac{u}{1 - a \cdot u}. \]  

(47)

Let us notice that the interchange \( u \leftrightarrow u' \) is equivalent to the symmetry \( \beta \leftrightarrow -\beta \). The relativistic transformations preserve the invariant (28). This follows from the relation

\[ D^\dagger(u, a)\eta(a)D(u, a) = \eta(a) = A^\dagger_x(a)\eta_0 A_x(a), \]  

(48)

where \( \eta_0 \) stands for the diagonal matrix \( \eta_0 = \text{diag}(-1, -1, -1, +1) \).

The transformations comply the principle of relativity. Mathematically it is expressed by their group properties. Let \( D(u, a) \) and \( D(v', a) \) be two successive relativistic transformations represented by the matrices \( \Lambda \). The composition of the transformations has the form

\[ \Omega_x(\phi, a)D(v, a) = D(v', a)D(u, a), \]  

(49)

provided

\[ v = \frac{v' + u \left[ \gamma(1 - a \cdot v') + gu \cdot v'/u^2 \right]}{1 + \gamma(u \cdot v') + g_+ u \cdot v'}. \]  

(50)

One can obtain the above relations by exploiting the decomposition (43) and using the structure of the Lorentz group expressed by the formula

\[ R(\phi)\Lambda(\beta_v) = \Lambda(\beta_v')\Lambda(\beta_a). \]  

(51)

The matrix
\[ R(\phi) = \begin{pmatrix} r_{ij} & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi = v \times u \]  

(52)

describes the Thomas precession \[1\] around the vector \(\phi\) known in the theory of relativity. The angle of the precession \(\varphi\) depends on the vectors \(\beta_v\) and \(\beta_u\). It remains to identify

\[ \Omega_z(\phi, a) = A_z^{-1}(a)R(\phi)A_z(a) \]  

(53)

and we get Eq. \([19\]). The relativistic transformation of the coordinates and time with rotation of the coordinate axes has the structure

\[ D(u, a)\Omega_z(\phi, a) = \left(\begin{array}{ccc} r_{ij} + gu_{ik}r_{kj}/u^2 + \gamma u_i a_j & -\gamma u_j \\
-g_{-u_k}r_{kj} + a_k r_{kj} - (1 + \gamma) a_j & 1 + \gamma \end{array}\right), \]  

(54)

provided the asymmetry of space-time is expressed by the vector \(a\). As concerns Eq. \([30\]), it can be obtained from the usual relativistic composition of the factors \(\beta\) given by Eq. \([47\]). The inverse relation

\[ v' = \frac{v - u [\gamma(1 - a \cdot v) - gu \cdot v/u^2]}{1 + \gamma - (1 - a \cdot v) - gu \cdot v} \]  

(55)

corresponds to the composition of the transformations in the following form

\[ \Omega_z(-\phi, a)D(v', a) = D(v, a)D^{-1}(u, a). \]  

(56)

When using Eqs. \([61\] and \([63\]), we get

\[ 1 - a \cdot v = \gamma \frac{(1 - u \cdot a)(1 - a \cdot v') + (1 + a^2)u \cdot v'}{1 + \gamma(1 - a \cdot v') + gu \cdot v'}, \]  

(57)

\[ 1 - a \cdot v' = \gamma \frac{(1 - a \cdot u)(1 - a \cdot v) - (1 + a^2)u \cdot v}{1 + \gamma - (1 - a \cdot v) - gu \cdot v}. \]  

(58)

It follows from the relations that

\[ \gamma(v) = \gamma(v') [1 + \gamma(1 - a \cdot v') + gu \cdot v'], \]  

(59)

\[ \gamma(v') = \gamma(v) [1 + \gamma - (1 - a \cdot v) - gu \cdot v]. \]  

(60)

Region of the accessible values of the velocities is given by the factor \(\gamma\). The boundary of the region is fixed by the condition \(\gamma(v) = \infty\). For a given value of \(a\), it is an ellipsoid

\[ (v_\parallel + a)^2 + (1 + a^2)v_\perp^2 = 1 + a^2 \]  

(61)

in the velocity space. The focus of the ellipsoid is situated into the point \(v = 0\). The \(v_\parallel\) and \(v_\perp\) denote the velocity components which are parallel and perpendicular to the vector \(a\), respectively. The ellipsoid is invariant under the relativistic transformations \([50\] and \([52\). In the case of \(u = (u, 0, 0)\) and \(a = (a, 0, 0)\), the composition of the velocities has the simple form

\[ v_1' = \frac{v_1 - u}{1 - 2au - uv_1}, \quad v_i' = v_i \sqrt{1 - 2au - uv_i}/(1 - 2au - uv_i), \quad i = 2, 3. \]  

(62)

The inverse relations can be obtained by the interchange \(v \leftrightarrow v'\) and \(u \leftrightarrow u'\). Using Eq. \([20\), they can be written as follows

\[ v_1 = \frac{v_1' + u - 2auv_1'}{1 + uv_1'}, \quad v_i = v_i' \sqrt{1 - 2au - uv_i}/(1 + uv_i'), \quad i = 2, 3. \]  

(63)
B. Energy and momentum

Consider a material particle in space-time. In relativistic mechanics, the position and momentum of the particle are given by the four-vectors \( x^\mu = \{x, t\} \) and \( p^\mu = \{p, E\} \), respectively. Let us define an "elementary" particle as an object which reveals no internal structure at any resolution considered. We comprehend the notion of elementarity as a relative concept which relies on the scales we are dealing with. For the infinite resolution it should be a perfect point whose trajectory is a fractal curve. For an arbitrary small but still finite resolution, the perfect point is approximated by a particle which we call "elementary" with respect to this resolution. It is therefore natural to assume that the concepts of the momentum, energy, mass and the velocity of the "elementary" particle have good physical meaning also at the scales where space-time is expected to break down its isotropy. We impose the following requirements on the energy and momentum.

1. The energy of a free particle cannot be pumped from the structure of space-time. This condition reads

\[
E = E_{\text{min}} \quad \text{for} \quad v = 0
\]  

where \( v \) is velocity of the particle.

2. Rate of clocks is slowest in the centre of gravity system. The only source of gravity is the free particle itself and not the structure of space-time. This condition reads

\[
dt = dt_{\text{min}} \quad \text{for} \quad p = 0
\]  

where \( p \) is the momentum of a free particle.

3. The transverse mass expressed relative to the space-time asymmetry \( a \) is conserved. The requirement states that energy of a particle moving perpendicular to the asymmetry \( a \) is equal to the transverse mass of the particle given by the transverse component of the momentum \( p_\perp \) with respect to \( a \). This is

\[
E_\perp^2 = p_\perp^2 + m_0^2 \quad \text{for} \quad v = v_\perp.
\]  

The physical requirements are obvious in the Minkovski space-time. In the case the asymmetry occurs, they lead to specific constraints on the energy and momentum. We denote the values of the momentum and the energy of a free particle by \( (p, E) \) or \( (p', E') \) in the reference systems \( S \) or \( S' \), respectively. In consistence with the principle of relativity and the ideas presented above, we search for relations connecting these quantities. In order to do that, let us first determine the 4-momentum \( \pi^\mu = \{\pi, \pi_0\} \) with space component in the direction of the asymmetry \( a \). The components of the variables determined relative to the systems \( S \) and \( S' \) transform in the way

\[
\pi' = \Pi(u, a)\pi,
\]  

where \( \Pi(u, a) = D^\dagger(u, -a) \).

The explicit form of the transformation matrix is

\[
\Pi(u, a) = \begin{pmatrix}
\delta_{ij} + gu_iu_j / u^2 - \gamma a_i u_j & -g_+ u_i + \gamma a_i \\
-\gamma u_j & 1 + \gamma_+
\end{pmatrix}.
\]  

The inverse transformation is given by

\[
\Pi^{-1}(u, a) = \begin{pmatrix}
\delta_{ij} + gu_iu_j / u^2 + \gamma a_i u_j & +g_+ u_i - \gamma a_i \\
+\gamma u_j & 1 + \gamma_-
\end{pmatrix}.
\]  

According to the relation (68), the matrix \( \Pi \) can be expressed in the form

\[
\Pi(u, a) = A^{-1}_\pi(a)\Lambda(\beta)A_\pi(a)
\]  

where

\[
A_\pi(a) = \begin{pmatrix}
\delta_{ij} + gu_iu_j / u^2 & -g_+ u_i \\
-\gamma u_j & 1 - \gamma_+
\end{pmatrix},
\]  

\[
\Lambda(\beta) = \begin{pmatrix}
\gamma & 0 \\
0 & 1 / \gamma
\end{pmatrix},
\]  

\[
A^{-1}_\pi(a) = \begin{pmatrix}
\delta_{ij} - gu_iu_j / u^2 & g_+ u_i \\
\gamma u_j & 1 + \gamma_-
\end{pmatrix}.
\]
where

$$A_s^{-1}(a) = A_s^1(-a).$$  \hfill (72)

The group properties of the transformations (67) are given by the composition

$$\Omega_\pi(\phi, a)\Pi(v, a) = \Pi(v', a)\Pi(u, a),$$  \hfill (73)

provided the velocities $u, v'$, and $v$ satisfy the relation (50). Here

$$\Omega_\pi(\phi, a) = A_s^{-1}(a)R(\phi)A_s(a).$$  \hfill (74)

We show that Eqs. (49) and (73) are consistent with relation (68). Let us transpose Eq. (49). Exploiting the correspondence (68) and using Eqs. (53), (72), and (74), we can write

$$\Pi(v, a)\Omega_\pi(-\phi, a) = \Pi(v, a)\Omega_\pi^1(\phi, -a) = \Pi(u, a)\Pi(v', a).$$  \hfill (75)

We apply the transposition operation on Eq. (51) too. As the matrices $\Lambda$ are invariant under the operation, we obtain the composition of the parameters $\beta_u$ and $\beta_{v'}$ in the mutual reverse order.

From the symmetry reasons, the composition must be of the same form as Eq. (51). We have therefore

$$R(-\phi)\Lambda(\beta_w) = \Lambda(\beta_v)R(-\phi) = \Lambda(\beta_u)\Lambda(\beta_{v'}).$$  \hfill (76)

The vector $\beta_w$ corresponds to the velocity $w$ according to Eq. (47). The velocity is given by the formula (50) in which the velocities $u$ and $v'$ are mutually interchanged. Multiplying Eq. (76) by the $A_s^{-1}$ from the left and by the $A_s$ from the right, we get

$$\Omega_\pi(-\phi, a)\Pi(w, a) = \Pi(v, a)\Omega_\pi(-\phi, a).$$  \hfill (77)

Together with Eq. (73) one has

$$\Omega_\pi(-\phi, a)\Pi(w, a) = \Pi(u, a)\Pi(v', a).$$  \hfill (78)

After performing the interchange $u \leftrightarrow v'$, we obtain Eq. (73). It was thus shown that the composition of two successive transformations of the variables $\pi$ follows from the composition of the corresponding transformations of the coordinates and time, provided their transformation matrices are connected by the relation (68).

Now we determine the momentum $p$ of a free particle which in general is not parallel to the asymmetry $a$. Definition of the momentum has to be in consistence with the condition 3 that requires preservation of the transverse mass defined by the momentum components perpendicular to the asymmetry $a$. There exists two sets of the variables $p^s = (p_s, E), s = L, R$ which comply the requirement. They are determined by the relation

$$\pi = A_s(a)p_s,$$  \hfill (79)

where

$$A_s(a) = \begin{pmatrix} \delta_{ij} \pm \varepsilon_{ijk}a_k & 0 \\ 0 & 1 \end{pmatrix}.$$  \hfill (80)

Here $\varepsilon_{ij}$ is the Levi-Civita symbol. The plus (in the next every upper) sign and the minus (in the next every lower) sign corresponds to $s = L$ and $s = R$, respectively. We attribute the first set of the variables ($s = L$) to the particle which we call left-handed. The second set ($s = R$) corresponds to the particle with right-handed type of motion. The explicit relations between the momenta $p_s$ and $\pi$ read

$$\pi = p_s \pm p_s \times a, \quad p_s = \frac{\pi \mp \pi \times a + (a \cdot \pi)a}{1 + a^2}.$$  \hfill (81)
In the context of these definitions, it is suitable to determine the associative variables \( x_s = \{x_s, x_0\} \), \( s = L, R \), to the coordinates and time by the formula

\[
x_s = A_s(-a)x.
\] (82)

The quantity \( u_s = du_s/dx_0 \) is the velocity of a particle for the special case when the particle moves in the direction of \( a \). It is related to the velocity \( u \) as follows

\[
u_s = u \pm a \times u, \quad u = \frac{u_s \mp a \times u_s + (a \cdot u_s)a}{1 + a^2}.
\] (83)

Exploiting the additional notations

\[
h \equiv \frac{g}{1 + (a \cdot u)^2/u_s^2}, \quad h_s \equiv \frac{g_s}{1 + a^2},
\] (84)

the relativistic transformations of the energy and momentum take the form

\[
p'_s = \Delta(u_s, a)p_s,
\] (85)

where

\[
\Delta(u, a) = \begin{pmatrix} \delta_{ij} + hu_iu_j/u^2 - h - a_iu_j & -h + u_i + ha_i \\ -\gamma u_j & 1 + \gamma \end{pmatrix}.
\] (86)

The inverse matrix reads

\[
\Delta^{-1}(u, a) = \begin{pmatrix} \delta_{ij} + hu_iu_j/u^2 + h + a_iu_j & h - u_i + ha_i \\ +\gamma u_j & 1 + \gamma \end{pmatrix}.
\] (87)

The transformation matrices can be written in the way

\[
\Delta(u_s, a) = A^{-1}_{ps}(a)A_{ps}(a),
\] (88)

where

\[
A_{ps}(a) = A_x(a)A_s(a) = \frac{1}{\sqrt{1 + a^2}} \begin{pmatrix} \delta_{ij} \pm \varepsilon_{ijk} a_k & -a_i \\ 0 & \sqrt{1 + a^2} \end{pmatrix}
\] (89)

and \( A \) is given by Eq. (83).

The transformations (85) possess group properties. Let us consider two successive transformations expressed by the matrices \( \Delta(u_s, a) \) and \( \Delta(u'_s, a) \). The resultant transformation is given by

\[
\Omega_{ps}(\phi, a)\Delta(v_s, a) = \Delta(v'_s, a)\Delta(u_s, a),
\] (90)

provided

\[
v_s = \frac{v'_s - u_s \left[ \gamma - h - a \cdot v'_s + h u_s \cdot v'_s/u'_s \right]}{1 + \gamma - h a \cdot v'_s + h u_s \cdot v'_s}. \] (91)

Formula (91) is consequence of Eqs. (91) and (88). The matrix \( \Omega_{ps} \) has the structure

\[
\Omega_{ps}(\phi, a) = A^{-1}_{ps}(a)R(\phi)A_{ps}(a) = A^{-1}(a)\Omega(s)(-\phi, a)A_s(a).
\] (92)

The inverse relation to Eq. (91) reads

\[
v'_s = \frac{v_s - u_s \left[ \gamma - h + a \cdot v'_s - h u_s \cdot v'_s/u'_s \right]}{1 + \gamma - h a \cdot v_s - h u_s \cdot v_s}.
\] (93)

The composition rules (91) and (93) are obtained by substituting Eq. (83) into the formulae (91) and (93), respectively.
The four-momenta and position four-vectors shall not be treated on the same footing at small scales in the region where the asymmetry of space-time is induced by the interaction. For a non-zero value of the asymmetry \( a \) there exist two sets of the mechanical variables \( p^\mu, s = L, R \) attributed to the kinematical variables \( x^\mu \). Single sets of the mechanical variables correspond to the right-handed and left-handed types of motion, respectively. Both of them have either positive or negative energy.

The sign of the energy is conserved in whatever reference frame. The variables \( x^\mu \) and \( p^\mu \) are transformed according to Eq. (85). In the special case, when the velocity \( u \) is parallel to the vector \( a, u = u_a \) and the transformation matrices take the simple form

\[
D(u, a) = \left( \begin{array}{cc} \delta_{ij} + \gamma_+ u_i u_j/u^2 & -\gamma u_i \\ -\gamma u_j & 1 + \gamma_+ \end{array} \right), \quad \Delta(u, a) = \left( \begin{array}{cc} \delta_{ij} + \gamma_- u_i u_j/u^2 & -\gamma u_i \\ -\gamma u_j & 1 + \gamma_- \end{array} \right).
\]

(94)

The inverse matrices read

\[
D^{-1}(u, a) = \left( \begin{array}{cc} \delta_{ij} + \gamma_+ u_i u_j/u^2 & \gamma u_i \\ \gamma u_j & 1 + \gamma_+ \end{array} \right), \quad \Delta^{-1}(u, a) = \left( \begin{array}{cc} \delta_{ij} + \gamma_- u_i u_j/u^2 & \gamma u_i \\ \gamma u_j & 1 + \gamma_- \end{array} \right).
\]

(95)

As follows from the relation

\[
\Delta^\dagger(u_s)\eta(-a)\Delta(u_s) = \eta(-a) = (1+a^2)A_{ps}(a)\eta_0A_{ps}(a),
\]

the relativistic transformations (87) preserve the expression

\[
p^2 = \frac{1}{1+a^2}\eta_{\mu\nu}(-a)p^\mu p^\nu = \frac{1}{1+a^2} \left[ E^2 - \mathbf{p}^2 + 2E\mathbf{a}\cdot\mathbf{p} - (\mathbf{a}\times\mathbf{p})^2 \right] \equiv m_0^2.
\]

(97)

The invariant is equal to \( m_0^2 \), which is square of the rest mass of a particle in the non-fractal and non-relativistic mechanics. Equation (87) implies the dependence of the total energy of a particle on its momentum in the following way

\[
E = \sqrt{1+a^2}\mathcal{E} - \mathbf{a}\cdot\mathbf{p},
\]

(98)

where the symbol

\[
\mathcal{E} = \sqrt{p^2 + m_0^2}
\]

(99)

stands for the particle’s free energy. Here we do not consider the minus sign before the square root relevant for anti-particles. The energy (98) is positive for arbitrary values of the asymmetry \( a \) and the momentum \( p \). It has a single minimum corresponding to the momentum and energy

\[
p_0 = m_0a, \quad E(p_0) = m_0.
\]

(100)

respectively. Beyond the minimum, as the momentum increases, the energy tends to infinity. The energy \( E \) consists of two terms. The first term is the free energy multiplied by the factor \((1+a^2)^{1/2}\). The second term, \( V = -\mathbf{a}\cdot\mathbf{p} \), plays the role of a potential induced by the asymmetry of space-time. The asymmetry does not violate the energy momentum conservation. We demonstrate it on a closed system with the mass \( m_0 \) which splits into two parts. Denoting the four-momenta of the decay products by \( p_1 \) and \( p_2 \), one can write

\[
m_0^2 = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1p_2
\]

\[
= \frac{1}{1+a^2} \left[ E_1^2 - p_1^2 + 2E_1\mathbf{a}\cdot\mathbf{p}_1 - (\mathbf{a}\times\mathbf{p}_1)^2 \right] + \frac{1}{1+a^2} \left[ E_2^2 - p_2^2 + 2E_2\mathbf{a}\cdot\mathbf{p}_2 - (\mathbf{a}\times\mathbf{p}_2)^2 \right]
\]

\[+
\frac{2}{1+a^2} \left[ E_1E_2 - p_1\cdot\mathbf{p}_2 + E_1\mathbf{a}\cdot\mathbf{p}_2 + E_2\mathbf{a}\cdot\mathbf{p}_1 - (\mathbf{a}\times\mathbf{p}_1)(\mathbf{a}\times\mathbf{p}_2) \right]
\]

\[=
\frac{1}{1+a^2} \left[ (E_1 + E_2)^2 - (p_1 + p_2)^2 + 2(E_1 + E_2)\mathbf{a}\cdot(p_1 + p_2) - (\mathbf{a}\times(p_1 + p_2))^2 \right].
\]

(101)
We see that if the four-momenta \( p_1 \) and \( p_2 \) are characterized by the invariant (97), their sum \( p_1 + p_2 \) possesses this property too. This implies the conservation of the total energy and momentum, \( E = E_1 + E_2 \) and \( p = p_1 + p_2 \), which results in the conservation of the free energy

\[
\sqrt{p^2 + m_0^2} = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} \tag{102}
\]
as well.

C. Relations of the kinematical and mechanical variables

Fundamental concepts of the special theory of relativity lead us to the relation between the energy/momentum of a particle and its velocity. The velocity is limited within the sphere of the radius \( c = 1 \) in every system of reference and is oriented in the direction of the particle momentum. This concerns the homogeneous and isotropic space-time. We show how the relations change when we abandon the space-time isotropy which we expect to break down at small scales.

Let us first determine how depends the four-momentum \( \pi^\mu \) on the velocity \( v \). According to the definition (67), the space component of \( \pi^\mu \) is parallel to the asymmetry \( a \). We search for the functions \( f_1 \) and \( f_2 \),

\[
\pi^0 = f_1(v, a), \quad \pi^0 = f_2(v, a), \tag{103}
\]

which are form invariant with respect to the relativistic transformations of the four-momentum \( \pi^\mu \) and the velocity \( v \). One can convince itself that the expressions

\[
\pi = [(1 + a^2)v + (1 - a \cdot v)a] \gamma(v)m_0, \tag{104}
\]

\[
\pi_0 = (1 - a \cdot v)\gamma(v)m_0 \tag{105}
\]

fulfill the requirements. Really, substituting the expressions into the transformation formula (67), one arrives at the system

\[
\left( \begin{array}{c}
(1 + a^2)v' + (1 - a \cdot v')a_i \\
1 - a \cdot v'
\end{array} \right) \gamma(v') = \\
\left( \begin{array}{c}
\delta_{ij} + gu_{ij}/u^2 - \gamma a_iu_j - g + u_i + \gamma a_i \\
- \gamma u_j
\end{array} \right) \left( \begin{array}{c}
(1 + a^2)v_i + (1 - a \cdot v)a_i \\
1 - a \cdot v
\end{array} \right) \gamma(v) \tag{106}
\]

consisting of four equations. The last one is identity. This can be shown by using Eqs. (58) and (61). In the same manner one can convince itself, that the first three equations of the system are consistent with Eq. (55). Now we substitute Eq. (79) into the formulae (104) and (105) and get

\[
p_s = m(v + a) \pm m(a \times v), \tag{107}
\]

\[
E = m(1 - a \cdot v). \tag{108}
\]

These are the expressions for the momentum and the energy of a left handed \( (s = L \text{ with upper sign}) \) and right handed \( (s = R \text{ with lower sign}) \) "elementary" particle moving with the velocity \( v \) in space-time characterized by the vector anisotropy \( a \). As can be seen by direct calculation, the formulae are consistent with the invariant (28) and (97). The coefficient of the proportionality between the momentum \( p_s \) and the velocity \( v \) is denoted by the symbol \( m \) and represents the inertial mass of the particle. The inertial mass depends on the velocity in the way

\[
m = m_0 \gamma(v), \tag{109}
\]

where \( m_0 \) is the rest mass of the particle corresponding to the minimal energy (100). Another relation which is invariant under the transformations (40) and (85) is action of the free particle
\[ S_{\text{as}} = -\tau m_0 = -tE + x \cdot p_s \pm a \cdot (x \times p_s). \] (110)

In classical mechanics, it determines the particle trajectory expressed in terms of its momentum, energy, and the mass \( m_0 \). The trajectory is given by \( x = vt \), where

\[ v = \frac{p_s \pm p_s \times a - aE}{E + a \cdot p_s}. \] (111)

If we substitute this expression into Eq. (110) and exploit the formulae (98), (107), and (108), we get

\[ t = \tau \gamma. \] (112)

The proportionality relates the time \( t \) recorded in the observer’s system \( S \) to the particle’s proper time \( \tau \). We see from Eq. (111) that for the zero value of the momentum there exists the non-zero value of the velocity

\[ v_0 = -a \] (113)

which minimizes the relation (112). This gives the slowest rate of clocks in the center of gravity system \( p = 0 \) in consistence with the requirement 2. As concerns the requirement 1, the minimal energy (100) corresponds to \( v = 0 \).

Having determined the action \( S_{\text{as}} \), we re-examine the Klein-Gordon equation for free particle in space-time with the asymmetry \( a \). It follows from Eq. (40) that the derivation operator \( \partial' = (\partial, \partial_0) \) transforms in the way

\[ \partial' = (D\dagger)^{-1} \partial. \] (114)

The Dalambertian operator is modified as follows

\[ \Box_a = \partial' \eta^{-1} \partial. \] (115)

Its invariance with respect to the relativistic transformations is seen from the relation

\[ \Box_a = \partial' \eta^{-1} \partial = \partial' \eta^{-1} \partial' = \Box'_a. \] (116)

Here we have exploited the decomposition

\[ \eta^{-1} = D\eta^{-1} D\dagger. \] (117)

The explicit form of the operator (115) reads

\[ \Box_a = \partial_0^2 - \frac{1}{1+a^2} [\partial + a \partial_0]^2. \] (118)

The corresponding modified Klein-Gordon equation

\[ -\Box_a \psi_{\text{as}} = m_0^2 \psi_{\text{as}} \] (119)

has the solution

\[ \psi_{\text{as}} = \exp(iS_{\text{as}}), \] (120)

where \( S_{\text{as}} \) is the action given by Eq. (110). After inserting this solution into Eq. (119), one arrives at the equation (17) relating the energy and momentum of a free particle in space-time with the asymmetry \( a \).

According to our opinion, the parameter could have relevance to more deeper context of the metric potentials which have relation to the intimate structure of space-time. It may be connected with a "field of the space-time asymmetries" reflecting its structure at small scales. Existence of such a "field" would result into a disparity between the energy-momentum and the coordinates and time. Here the disparity is demonstrated by the commutation relation.
\[
A^\dagger_{ps} \eta_0 A_x - A^\dagger_x \eta_0 A_{ps} = \left( \begin{array}{cc} \pm 2 \varepsilon_{ijk} a_k & 0 \\ 0 & 0 \end{array} \right)
\]

(121)

between the matrices representing fluctuations in momenta and coordinates, respectively. The commutator is non-zero provided the non-zero value of the "field". As a first step, one can approximate the "field of space-time asymmetries" in terms of the asymmetry vector \( \mathbf{a} \) and consider it as a random and chaotic quantity. The investigations in this direction require, however, more detailed and fundamental study.

**D. Relativistic properties of the space-time asymmetry**

In this section we focus on specific properties of the space-time asymmetries such as their group structure and the composition rules. The properties are explicitly seen in terms of the scale velocity \( \nu \) defined in the way

\[
\nu = \frac{a}{\sqrt{1 + a^2}}
\]

(122)

The scale velocity does not represent state of real motion but characterizes the state of scale. In terms of the scale velocity \( \nu \), the total energy \( \mathcal{E} \) of a free particle can be expressed as follows

\[
E = \frac{1}{\sqrt{1 - \nu^2}} (\mathcal{E} - \nu \cdot \mathbf{p}).
\]

(123)

The dispersion relation (98) can be thus considered as the Lorenz transformation of energy expressed in terms of the scale velocity \( \nu \). Exploiting the expressions (107) and (109), the free energy \( \mathcal{E} \) can be rewritten in the way

\[
\mathcal{E}^2 \equiv p^2 + m_0^2 = p^2_\nu + m^2 \equiv \mathcal{E}_\nu^2
\]

(124)

where

\[
p_\nu = \frac{m}{\sqrt{1 - \nu^2}} \nu.
\]

(125)

If the motion velocity \( \mathbf{v} \) is null, the momentum \( p_\nu \) corresponds to the minimal energy given by Eq. (100). Inserting Eqs. (107) and (108) into Eq. (123) and performing some elementary algebra we get

\[
m = \frac{1}{\sqrt{1 - \nu^2}} (\mathcal{E}_\nu - \nu \cdot \mathbf{p}_\nu),
\]

(126)

\[
0 = p_\theta = p_\nu + \frac{\nu (\nu \cdot \mathbf{p}_\nu)}{\nu^2} \left( \frac{1}{\sqrt{1 - \nu^2}} - 1 \right) - \frac{\nu \mathcal{E}_\nu}{\sqrt{1 - \nu^2}}.
\]

(127)

These equations represent the Lorenz transformation \( \Lambda(\nu) \) of free energy \( \mathcal{E}_\nu \) and the momentum \( p_\nu \) in dependence on the scale velocity \( \nu \). Because the same is valid for any other velocity \( \nu' \),

\[
p_\theta = \Lambda(\nu') p_{\nu'},
\]

(128)

we can write

\[
p_\nu = \Lambda(-\nu) \Lambda(\nu') p_{\nu'} = R(\phi) \Lambda(\nu'') p_{\nu''}.
\]

(129)

The matrix \( R \) formally describes the Thomas precession around the axis \( \phi = \nu' \times \nu \). The relativistic transformation (129) connects the mechanical variables expressed relative to space-time with different asymmetries. We conclude from the character of the transformation that the scale velocities comply the standard relativistic composition rule.
\[
\nu'' = \frac{\nu' \sqrt{1 - \nu'^2} + \nu \left( (1 - \sqrt{1 - \nu'^2}) \frac{\nu \cdot \nu' / \nu^2 - 1}{1 - \nu \cdot \nu'} \right)}{1 - \nu \cdot \nu'}
\]  
(130)

According to the relation (122), this determines the composition of the corresponding space-time asymmetries \(a\) as well.

Consequently, we consider two types of the velocities. First one is the motion velocity \(v\) which determines the state of particle motion. The velocity represents change of the particle position with a change of time in space-time in which the particle is embedded. The motion velocities are composed according to Eq. (50) and (55). The second type of velocity is the scale velocity \(\nu\) which characterizes state of scale of the particle with respect to the state of scale of a reference system. Structures of both the particle and the reference system are considered to be fractals of various fractal dimensions. The reference system can be another particle being a fractal object or the fractal structure of the (QCD) vacuum itself. The intimate structure of the later consists of the infinite number of gluons and quark-antiquark pairs at small scales.

The scale velocities \(\nu\) depend on fractal characteristics of both particle and the reference system (see the next section). They are composed according to Eq. (130) which represents the special scale relativity law. In this sense, the scale relativity concerns the transformations of particle characteristics with respect to self-similar structures which are fractals of various fractal dimensions. Single fractal structures have different anomalous fractal dimensions and play analogous role as the inertial systems in the motion relativity. The above scale relativistic transformations reflect the fact that there does not exist an absolute scale reference system connected either with a fractal object or with any particular vacuum fractal structure.

The invariants of the scale relativistic transformations are the scalar products

\[
(p_1 p_2)_a = \frac{1}{1 + a^2} [E_1 E_2 - p_1 \cdot p_2 + E_1 a \cdot p_2 + E_2 a \cdot p_1 - (a \times p_1) \cdot (a \times p_2)].
\]  
(131)

It follows from the previous sections that the scalar products do not change under the relativistic transformations (53) concerning the relativity of motion. Moreover, the scalar products are invariant under the scale transformations (129) as well. Really, if we insert Eq. (98) for the energies \(E_1\) and \(E_2\) into the relation (131), we get

\[
(p_1 p_2)_a = \mathcal{E}_1 \mathcal{E}_2 - p_1 \cdot p_2.
\]  
(132)

The scalar product is expressed in the standard way in terms of free energies and momenta and is manifestly invariant under the scale transformations (129). In other words we can write

\[
(p_1 p_2)_a = (p'_1 p'_2)_b.
\]  
(133)

We have to answer one more question which is how does the motion velocity \(v\) change with a change of the scale velocity \(\nu\). The answer follows from the Lorenz transformation

\[
t = \frac{1}{\sqrt{1 - \nu^2}} (T + \nu \cdot x),
\]  
(134)

\[
x_0 = x + \frac{\nu (\nu \cdot x)}{\nu^2} \left( \frac{1}{\sqrt{1 - \nu^2}} - 1 \right) + \frac{\nu T}{\sqrt{1 - \nu^2}}
\]  
(135)

of the coordinates and time with respect to the scale velocity \(-\nu\). The transformation is identical with the scale transformation of the energy (126) and momentum (127) except the opposite sign of the scale velocity \(\nu\). According to Eq. (124), this corresponds to the opposite signs of the asymmetry \(a\) in the invariants (28) and (57), respectively.

The treatment of the symbols here should be taken carefully. Canonical variables associated with the particle momentum \(p\) and the energy \(E\) are the particle position \(x\) and the time \(t\). The symbol \(T\) stands for the time parameter associated with the particle free energy \(\mathcal{E}\). We have to realize that rate of time which determines purely motion is given by the variable \(t\). Exactly this time determinates rate of motion contrary to the time \(T\) which is influenced by the rate of scale
change. Therefore, any motion velocity is given as a derivative with respect to \( t \). In this sense, the variables

\[
v = \frac{dx}{dt}, \quad \text{and} \quad v_o = \frac{dx_o}{dt}
\]  

(136)

are the motion velocities in space-time with and without the asymmetry \( a \), respectively. Differentiating Eq. (133) with respect to \( t \) and using Eq. (134) one gets

\[
v_o = v + \nu + \frac{\nu(\nu - v)}{\nu^2} \left( \sqrt{1 - \nu^2} - 1 \right).
\]  

(137)

As the relation holds for any asymmetry, the change of the asymmetry \( \nu \) results in the change of the velocity \( v \) in the way which corresponds to the relation

\[
x_\nu = \Lambda(\nu)\Lambda(-\nu')x_{\nu'} = R(\phi)\Lambda(-\nu'')x_{\nu''}.
\]  

(138)

Exploiting Eq. (137), one can show that

\[
m = \frac{\tilde{m}_0}{\sqrt{1 - \nu_0^2}},
\]  

(139)

where

\[
m_0 = \frac{\tilde{m}_0}{\sqrt{1 - \nu^2}}.
\]  

(140)

From the relations we see that the total mass of a particle \( m \) is, beside the state of motion, determined also by a state of scale. If the motion diminishes, \( v_0 \rightarrow \nu \), and the mass of the particle \( m \) becomes its rest mass \( m_0 \). The rest mass is given by a mass \( \tilde{m}_0 \) in terms of the scale velocity \( \nu \). If one admits the asymmetry of space-time fluctuates in dependence on scale, the rest mass \( m_0 \) becomes function of the scale velocity fluctuations. If the fluctuations of \( \nu \) have fractal character, the particle may be identified with fluctuations of a point-like object with the mass \( \tilde{m}_0 \). The energy \( \tilde{E} \) and the momentum \( \tilde{p} \) of the point-like object are related to the energy \( E \) and the momentum \( p \) of the particle as follows

\[
E = \frac{\tilde{E}}{\sqrt{1 - \nu^2}}, \quad p = \frac{\tilde{p}}{\sqrt{1 - \nu^2}}.
\]  

(141)

Inserting this into equation (137), we get the invariant in the compact metric form

\[
p^2 = \eta_{\mu\nu}( - \mathbf{a})\tilde{p}^\mu \tilde{p}^\nu = \tilde{E}^2 - \tilde{p}^2 + 2\tilde{E}\mathbf{a} \cdot \tilde{p} - (\mathbf{a} \times \tilde{p})^2 = m_0^2.
\]  

(142)

**IV. INTERACTIONS OF FRACTAL OBJECTS**

The ability of fractals to structure space-time was discussed in Ref. [6]. Such approach gives us possibility to attribute geometrical notions to the structural parameters characterizing fractal properties of free particles. The need to satisfy the principles of the scale-motion relativity implies replacement of the scale independent physical laws by the scale dependent equations. This concerns the energy and momentum which in the presence of the space-time anisotropy \( a \) are converted to the variables satisfying the dispersion relation (18). Intuitively, the anisotropy of space-time could be induced in the interaction region at small scales as a result of the fractality of the interacting constituents.

Consider an ultra-relativistic collision of two fractal objects (hadrons or nuclei) in their total center-of-energy system \( E_1 = E_2 \). Suppose the fractals have different anomalous fractal dimensions \( \delta_1 < \delta_2 \). According to our working hypothesis, the interaction of the fractals induces a space-time asymmetry \( a = (0, 0, a) \) at small scales. Due to the asymmetry, momenta of the produced particles are shifted against motion of the fractal with larger anomalous dimension (18). Here all the momenta
are considered to be shifted accordingly. As the fractal objects collide at ultra-relativistic energies, one can neglect their masses and write

\[ \sqrt{1+a^2}p_1 - a \cdot p_1 = \sqrt{1+a^2}p_2 - a \cdot p_2 = \sqrt{s}/2. \]  

(143)

Because the incoming momenta \( p_1 \) and \( p_2 \) are aligned with the asymmetry \( a \), the relation implies the following formulae

\[ p_1 + p_2 = a\sqrt{s}, \quad p_1 + p_2 = \sqrt{s}\sqrt{1+a^2}, \quad 4p_1p_2 = s. \]  

(144)

Let us denote the momentum of the recoil particle produced in the constituent interaction as \( q' \). The corresponding momentum fractions are given in the way

\[ \chi_1 = \frac{(p_2 \cdot q')_a}{(p_1 \cdot p_2)_a}, \quad \chi_2 = \frac{(p_1 \cdot q')_a}{(p_1 \cdot p_2)_a}, \]  

(145)

where, according to Eq. (132), the scalar products have the form

\[ (p_i \cdot q')_a = p_i \sqrt{q'^2 + m_{z}'^2} - p_i \cdot q'. \]  

(146)

When using Eq. (144), (146), and (18), one can express the sum of the fractions as follows

\[ \chi_1 + \chi_2 = \frac{2\sqrt{1+a^2}\sqrt{q'^2 + m_{z}'^2} - 2a \cdot q'}{\sqrt{s}} = \frac{2E'}{\sqrt{s}}. \]  

(147)

Now we substitute for the fractions \( \chi_i \) their expressions from Eq. (13) and get

\[ \sqrt{(1+a^2)(\mu_{z}^2 + \mu_{\perp}^2)} - a \mu_{z} = \sqrt{\omega_1^2 + \mu_{1}^2} + \sqrt{\omega_2^2 + \mu_{2}^2} - (\omega_1 - \omega_2). \]  

(148)

Here we have used the notations

\[ \mu_{z} = \frac{2q'_{z}}{\sqrt{s}}, \quad \mu_{\perp} = \frac{2m'_{\perp}}{\sqrt{s}}, \quad m'_{\perp} = \sqrt{q'^{2}_{\perp} + m_{2}^2}. \]  

(149)

The longitudinal and transversal components of the momentum \( q' \) with respect to the collision axis are denoted by \( q'_{z} \) and \( q'_{\perp} \), respectively. As follows from the conservation of the free energy (102), Eq. (148) splits into two parts

\[ \sqrt{(1+a^2)(\mu_{z}^2 + \mu_{\perp}^2)} = \sqrt{\omega_1^2 + \mu_{1}^2} + \sqrt{\omega_2^2 + \mu_{2}^2}, \]  

(150)

\[ a \mu_{z} = \omega_1 - \omega_2. \]  

(151)

The obtained system for the unknown variables \( \mu_{z} \) and \( \mu_{\perp} \) depends on the parameter \( a \). The variation range of the variables is given by the condition \( \chi_1 + \chi_2 \leq 1 \). According to Eqs. (147) and (148), it can be rewritten as follows

\[ (\mu_{z} - a)^2 + (1+a^2)\mu_{\perp}^2 \leq 1+a^2. \]  

(152)

The \( \mu_{z} \) and \( \mu_{\perp} \) are bounded inside the ellipsoid given by the asymmetry \( a \). If we approach the phase-space limit of the reaction (3), the variables tend to their boundary values and satisfy the equation of the ellipsoid. Similar applies for any other particle produced in the elementary interaction. The particle’s momentum \( q' \) and energy \( E' \) are connected by the dispersion relation (98) which can be expressed in the way

\[ \left( \frac{q'}{E'} - a \right)^2 + (1+a^2)\left( \frac{m'_{\perp}}{E'} \right)^2 = 1+a^2. \]  

(153)
Exploiting the relations
\[
\frac{q'_z}{E'} = \frac{\sqrt{s}}{2E'} \mu_z = \frac{\mu_z}{\chi_1 + \chi_2}, \quad \frac{m'_1}{E'} = \frac{\sqrt{s}}{2E'} \mu_\perp = \frac{\mu_\perp}{\chi_1 + \chi_2},
\]
(154)
it can be rewritten into the form
\[
\left( \frac{\mu_z}{\chi_1 + \chi_2} - a \right)^2 + (1+a^2) \left( \frac{\mu_\perp}{\chi_1 + \chi_2} \right)^2 = 1 + a^2.
\]
(155)
The values of \( \mu_z/(\chi_1 + \chi_2) \) are limited within the interval
\[
-a_- \leq \frac{\mu_z}{\chi_1 + \chi_2} \leq a_+ \quad (156)
\]
where
\[
a_\pm = \sqrt{1 + a^2} \pm a.
\]
(157)
According to the kinematics of the process, the maximal value of \( \mu_{z\text{max}}/(\chi_1 + \chi_2) = a_+ \) should correspond to \( \chi_2 = 0 \). The minimal value of \( \mu_{z\text{min}}/(\chi_1 + \chi_2) = -a_- \) is given by \( \chi_1 = 0 \). The maximum of \( \mu_\perp/(\chi_1 + \chi_2) = 1 \) should be achieved for \( \chi_1 = \chi_2 \) and thus for \( \mu_z/(\chi_1 + \chi_2) = a \). The conditions are satisfied by the linear combination
\[
\mu_z = (\chi_1 + \chi_2) a + (\chi_1 - \chi_2) \sqrt{1 + a^2}.
\]
(158)
Substituting the expression (158) into the left side of the relation (151) and using the formulae (13)-(15), one arrives at the quadratic equation for the unknown parameter \( a \). Its solution, which complies the physical requirements on the kinematics of the process, is \( a = \bar{a} \). Reminding Eq. (16), this explicitly reads
\[
a = \frac{\alpha - 1}{2\sqrt{\alpha}} \xi
\]
(159)
where the scale factor \( \xi \) is given by Eq. (17). Using Eqs. (15), (150), and (151), one can express the variables \( \mu_z \) and \( \mu_\perp \) in a simple form
\[
\mu_z = \mu_1 - \mu_2, \quad \mu_\perp = 2\sqrt{\mu_1 \mu_2}.
\]
(160)
In this view, Eq. (158) is the first of the following two equations
\[
\mu_1 - \mu_2 = \frac{1}{\sqrt{1-\nu^2}} \left[ (\chi_1 - \chi_2) + \nu(\chi_1 + \chi_2) \right],
\]
(161)
\[
\mu_1 + \mu_2 = \frac{1}{\sqrt{1-\nu^2}} \left[ (\chi_1 + \chi_2) + \nu(\chi_1 - \chi_2) \right],
\]
(162)
representing the scale transformation of energy and momentum along the \( z \)-axis. This is seen if we realize that the above combinations of fractions are expressed in terms of the corresponding quantities in the way
\[
\mu_z = \frac{2}{\sqrt{s}} q'_z, \quad \mu_1 + \mu_2 = \frac{2}{\sqrt{s}} \mathcal{E}'_z,
\]
(163)
\[
\chi_z = \chi_1 - \chi_2 = \frac{2}{\sqrt{s}} Q'_z, \quad \chi_1 + \chi_2 = \frac{2}{\sqrt{s}} \mathcal{E}'Q'_z.
\]
(164)
The scale transformations connect the momentum and the free energy of the recoil expressed relative to space-time with and without the asymmetry \( a = (0,0,a) \), respectively. The conservation of the transverse mass

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\[ \chi_\perp = 2\sqrt{\chi_1 \chi_2} = 2\sqrt{\mu_1 \mu_2} = \mu_\perp \] (165)

preserves the invariant forms

\[ \mu_\perp^2 + \mu_\perp^2 = (\mu_1 + \mu_2)^2, \quad \chi_\perp^2 + \chi_\perp^2 = (\chi_1 + \chi_2)^2 \] (166)

which are equivalent to the invariant relations

\[ q_z'^2 + (q_\perp'^2 + m_1^2) = \mathcal{E}_q'^2, \quad Q_z'^2 + (Q_\perp'^2 + m_2^2) = \mathcal{E}_{Q'}^2. \] (167)

Similar should apply to the inclusive particle \( m_1 \). The only difference is that the inclusive particle is fixed by its momentum \( Q \) while the momentum of the recoil is determined from the requirement on minimal resolution concerning the fractal measure \( z \). The scale transformations of the energy and momentum of the inclusive particle with the scale velocity \( \nu \) oriented along the \( z \)-axis

\[ \tilde{\mu}_1 - \tilde{\mu}_2 = \frac{1}{\sqrt{1 - \nu^2}} [(\lambda_1 - \lambda_2) + \nu(\lambda_1 + \lambda_2)] \] (168)

\[ \tilde{\mu}_1 + \tilde{\mu}_2 = \frac{1}{\sqrt{1 - \nu^2}} [(\lambda_1 + \lambda_2) + \nu(\lambda_1 - \lambda_2)] \] (169)

have the same form as the transformations (161) and (162). The momentum fractions

\[ \lambda_1 = \sqrt{\tilde{\mu}_1^2 + \tilde{\omega}_1^2} - \tilde{\omega}_1, \quad \lambda_2 = \sqrt{\tilde{\mu}_2^2 + \tilde{\omega}_2^2} + \tilde{\omega}_2, \] (170)

are given in terms of

\[ \tilde{\omega}_1 = \tilde{\mu}_1 a, \quad \tilde{\omega}_2 = \tilde{\mu}_2 a \] (171)

similarly as their counterparts \( \chi_i \). The value of \( a \) is the same as in Eq. (159). The combinations of fractions

\[ \tilde{\mu}_z = \tilde{\mu}_1 - \tilde{\mu}_2 = \frac{2}{\sqrt{s}} q_z, \quad \tilde{\mu}_1 + \tilde{\mu}_2 = \frac{2}{\sqrt{s}} \mathcal{E}_q, \] (172)

and

\[ \lambda_z = \lambda_1 - \lambda_2 = \frac{2}{\sqrt{s}} Q_z, \quad \lambda_1 + \lambda_2 = \frac{2}{\sqrt{s}} \mathcal{E}_Q \] (173)

are given in terms of the momentum and the free energy of the inclusive particle expressed relative to space-time with and without the asymmetry \( a \), respectively. Conservation of the transverse mass

\[ \lambda_\perp = 2\sqrt{\lambda_1 \lambda_2} = 2\sqrt{\mu_1 \mu_2} = \tilde{\mu}_\perp \] (174)

and the relations

\[ \tilde{\mu}_\perp^2 + \tilde{\mu}_\perp^2 = (\tilde{\mu}_1 + \tilde{\mu}_2)^2, \quad \lambda_\perp^2 + \lambda_\perp^2 = (\lambda_1 + \lambda_2)^2 \] (175)

have the same form as Eqs. (165) and (166). The longitudinal momentum balance reads

\[ x_1 a_+ - x_2 a_- = \tilde{\mu}_z + \mu_z \] (176)

or

\[ x_1 - x_2 = \lambda_z + \chi_z \] (177)

relative to the scale reference systems with or without the asymmetry \( a = (0, 0, a) \), respectively.

All the expressions are given in terms of the asymmetry \( a \) (159) which depends on the ratio \( \alpha = \delta_2 / \delta_1 \) of the anomalous fractal dimensions of the colliding objects. The collisions of the
asymmetric fractal objects are characterized by the different fractal dimensions and thus with \( a \neq 0 \) \((\alpha \neq 1)\). In the considered scenario, it results in creation of a domain in which the isotropy of space-time is violated. If \( \delta_2 = \delta_1 \), there is no polarization of space-time induced by the interaction \((a = 0)\). This corresponds to the collisions of the fractals possessing equal fractal dimensions. Similar applies to the situation at lower energies where the interacting objects reveal no fractal-like constituent substructure. The asymmetry \( a \) changes its sign if \( \lambda_1 \leftrightarrow \lambda_2 \) and \( \alpha \leftrightarrow \alpha^{-1} \), i.e. if the interacting fractals are mutually interchanged.

The fractal limit is achieved in the phase-space limit at any energy. In this case, the fractions \( \lambda_i \) approach the limiting values which depend on the detection angle \( \theta \) of the inclusive particle in the simple way

\[
\lambda_1 \rightarrow \lambda_1^L = \cos^2(\theta/2), \quad \lambda_2 \rightarrow \lambda_2^L = \sin^2(\theta/2). \tag{178}
\]

The limiting values of the longitudinal and the transversal components of the momentum fractions \( \tilde{\mu} \) and \( \mu \) read

\[
\tilde{\mu}_z^L = \sqrt{\alpha} \cos^2(\theta/2) - \frac{1}{\sqrt{\alpha}} \sin^2(\theta/2), \quad \tilde{\mu}_\perp^L = \sin(\theta), \tag{179}
\]

\[
\mu_z^L = \sqrt{\alpha} \sin^2(\theta/2) - \frac{1}{\sqrt{\alpha}} \cos^2(\theta/2), \quad \mu_\perp^L = \sin(\theta). \tag{180}
\]

Last equations correspond to angular parameterization of the ellipsoid of momentum fractions concerning the inclusive particle and its recoil, respectively. The momenta of both particles are shifted against the motion of the fractal object with larger anomalous fractal dimension \( \delta \). The prolonged form of the ellipsoid means that the momenta of the particles increase when directed against relative more resistant fractal structure.

The scale factor \( \xi \) \((17)\) is unity in the fractal limit and the space-time asymmetry acquires its maximal value

\[
a = \frac{\alpha - 1}{2\sqrt{\alpha}}. \tag{181}
\]

In this case, the expression for the scale velocity \((122)\) takes the simple form

\[
\nu = \frac{\alpha - 1}{\alpha + 1} \tag{182}
\]

which depends only on the ratio \( \alpha = \delta_2/\delta_1 \) of the anomalous fractal dimensions of the colliding objects. The velocity has its origin in the asymmetry of the interaction and vanishes in the collisions of objects which possess equal fractal dimensions. The scale velocity represents a space-time "drift" induced by the interaction of the parton fractals. The quantity represents no real motion but characterizes local polarization of the (QCD) vacuum. The velocity depends on the relative state of scale of the reference systems and satisfies the scale-relativity composition rule

\[
\nu' = \frac{\nu + \nu''}{1 + \nu \nu''}, \tag{183}
\]

which results from one dimensional reduction of Eq. \((130)\). If we insert the expression \((182)\) into this equation, we get

\[
\alpha' = \alpha \alpha''. \tag{184}
\]

As concerns the fractal limit, the correspondence leads us to make the following conclusion. While the composition of scale velocities follows Einstein-Lorenz law, the composition of the corresponding ratio of the anomalous fractal dimensions follows the multiplicative group law. Last equation takes even more pronounced form if reduced to a same anomalous fractal dimension, say \( \delta_2 \). In that case, Eq. \((184)\) gives
\[
\frac{\delta_3}{\delta_1} = \frac{\delta_2}{\delta_1} \frac{\delta_1}{\delta_2}.
\] (185)

The reduction means that the state of scale of the reference system possesses natural scaling property consisting in the following. If studying constituent interactions in the collisions of an fractal object 1 with a fractal probe 2, and then in the interactions of the fractal probe 2 with another fractal object 3, one arrives at the same properties as if examining the fractal 1 by mens of the fractal 3. This is again an explicit expression of the scale relativity in which single fractal structures play analogus roles as the inertial systems in the motion relativity.

V. DOES ASYMMETRY OF SPACE-TIME CONTRADICT WITH MICHELSON’S EXPERIMENT?

The concepts considered in the previous section seem to have general validity at least from the mathematical point of view. However, in consistence with our physical intuition, we expect the space-time asymmetry can be induced mainly at small scales accessible in ultra-relativistic hadron and nuclear collisions. The question of its amount is tightly connected with scales we are dealing with. On the other side, there is no apparent and explicit reason why the asymmetry should be exactly zero at any scales, even in large universe. In general thinking, we show that existence of any tiny portion of such asymmetries is in principle not ruled out neither in the famous Michelson’s experiment [12] concerning the interference of light.

The experiment was accomplished by an interferometer having two arms. Light beam from a light source was divided into two rays, I and II, traveling perpendicular to each other along the arms. The mirrors placed on the ends of the spectrometer arms reflected the light back to the telescope where the rays interfered with each other. Assuming the apparatus is placed in a region where the the propagation of light is not isotropic one could expect existence of a phase difference \( \Delta t \) between the rays I and II which is due to the anisotropy. When the apparatus is rotated through an angle of 90\(^\circ\), the orientation of the spectrometer arms is interchanged and the phase difference becomes \( \Delta t \). Such a rotation of the apparatus should therefore cause a shift of the interference fringes between the two rays. The experiment, however, could not find any such effect at all.

We argue below that this experimental fact alone does not imply absolute absence of any anisotropy in light propagation. Let us assume there exists a space-time asymmetry \( a \) induced by some reasons. One of the reasons we have suggested in previous section can be connected with fractality at whatever scale. The asymmetry results in metric changes \( \eta_{\mu\nu}(a) \) associated with deformation of the circular form of the light front. The light front becomes an ellipsoid (61) with the focus in the point the light was emitted (Fig.1). Consider the Michelson’s spectrometer is placed into such anomaly. The time \( t_{II} \) and \( t_{II} \) which the rays take to travel in spectrometer arms I and II can be expressed as follows

\[
t_{I} = x_{I} \left( \frac{1}{v_{1}(\phi)} + \frac{1}{v_{2}(\phi)} \right), \quad t_{II} = x_{II} \left( \frac{1}{v_{3}(\phi)} + \frac{1}{v_{4}(\phi)} \right).
\] (186)

The angle \( \phi \) describes orientation of the spectrometer with respect to the asymmetry \( a \). Because of the asymmetry, the velocities of light propagation \( v_{i}(\phi) \) in different directions are not equal and depend on the orientation of the spectrometer. On the other hand spatial distances (lengths of spectrometer arms) do not depend on the orientation of the spectrometer. This follows from the known fact [9] that the spatial geometry is not simply given by the spatial part \( \eta_{ij} \) of the four dimensional metric \( \eta_{\mu\nu}(a) \). The metric tensor \( \eta^{*}_{ij} \) which determines the spatial geometry is given by

\[
\eta^{*}_{ij} = -\eta_{ij} + \eta^{*}_i \eta^{*}_j,
\] (187)

where

\[
\eta^{*}_i = \frac{\eta_{i0}}{\sqrt{\eta_{00}}}.
\] (188)

In the case of the four-dimensional metric (29), the spatial metric reads
\[ \eta_{ij} = (1 + a^2) \delta_{ij}. \] (189)

Therefore, the distance

\[ d = \sqrt{\eta_{ij} x^i x^j} = \sqrt{1 + a^2 x} \] (190)

is invariant under space rotations. Let us now exploit the following geometrical property of the velocity ellipsoid (192). While the sections \( v_i(\phi) \) connecting any point of the ellipsoid with the focus depend on their orientation \( \phi \), the combination

\[ \frac{1}{v_1(\phi)} + \frac{1}{v_2(\phi)} = \frac{2a_\parallel}{a_\perp} = 2\sqrt{1 + a^2} \] (191)

is rotationally invariant i.e. does not depend on the angle \( \phi \). The symbols \( a_\parallel \) and \( a_\perp \) denote the big and small semi-axis of the ellipsoid, respectively. After inserting expressions (190) and (191) into Eq. (186), we get

\[ t_I = 2d_I, \quad t_{II} = 2d_{II}. \] (192)

The relations connect time the light rays take to travel in the spectrometer arms with the lengths of the arms. Both physical quantities are expressed in space-time with the asymmetry \( a \) giving the relations which are rotational invariant. Therefore, rotation of the spectrometer apparatus should not cause a shift of the interference fringes even for \( a \neq 0 \).

We could also perform a "gedanken" experiment with tree mirrors which reflect the rays of light along a triangle \( ABC \). Consider the triangle depicted by the full lines in Fig.2a. Suppose a light signal is emitted in the point \( A \) and travels along the path \( d_1, d_2, \) and \( d_3 \). The corresponding time interval

\[ t_{ABC} = \frac{x_1}{v_1(\phi)} + \frac{x_2}{v_2(\phi)} + \frac{x_3}{v_3(\phi)} \] (193)

is function of the velocities \( v_1(\phi), v_2(\phi), \) and \( v_3(\phi) \). The velocities are shown on the velocity diagram in Fig.2b. They depend on the orientation \( \phi \) of the three mirrors setup relative to the asymmetry \( a \). If the experimental setup begins to rotate, the values of \( v_i(\phi) \) will change but the value of \( t_{ABC} \) remains invariant with respect to such rotations. The invariance follows from specific geometrical properties of any rotational ellipsoid which we outline below.

Let us denote the internal angles of the triangle \( ABC \) as \( \alpha_1, \alpha_2, \) and \( \alpha_3 \). They comply the elementary geometrical property written in the way

\[ \frac{d_1}{\sin \alpha_1} = \frac{d_2}{\sin \alpha_2} = \frac{d_3}{\sin \alpha_3} = \sqrt{1 + a^2 x_{ABC}}, \] (194)

where \( d_i \) are the opposite sides of the triangle. The angles among the corresponding velocities are denoted by \( \beta_1, \beta_2, \) and \( \beta_3 \) and shown in Fig.2b. In a specific mirror setup, they are fixed by the relation

\[ \beta_i = \pi - \alpha_i, \quad i = 1, 2, 3. \] (195)

The angles \( \beta_i \) do not depend on the orientation \( \phi \) and thus do not depend on the rotation of the apparatus as the whole. Using the above formulae, Eq. (193) takes the form

\[ t_{ABC} = x_{ABC} \left( \frac{\sin \beta_1}{v_1(\phi)} + \frac{\sin \beta_2}{v_2(\phi)} + \frac{\sin \beta_3}{v_3(\phi)} \right). \] (196)

One can convince itself that there exists following geometrical property of the rotational ellipsoids. Consider an ellipse which represents a section of the ellipsoid with a plane passing though the focus of the ellipsoid. The focus is common for this ellipse and the ellipsoid. Let us denote by \( v_1(\phi), v_2(\phi), \) and \( v_3(\phi) \) the sections connecting three different points of the ellipse with the common focus. While magnitudes of the sections \( v_i(\phi) \) depend on the orientation \( \phi \) of the ellipsoid, the combination
\[
\frac{\sin \beta_1}{v_1(\phi)} + \frac{\sin \beta_2}{v_2(\phi)} + \frac{\sin \beta_3}{v_3(\phi)} = \frac{2a_\parallel}{a_\perp^2} (\sin \beta_1 + \sin \beta_2 + \sin \beta_3) .
\]

remains invariant under any rotation of the ellipsoid. The symbols \(a_\parallel\) and \(a_\perp\) denote the big and small semi-axis of the ellipsoid, respectively. Exploiting Eqs. (194) - (197), one arrives at the relation

\[
t_{ABC} = d_1 + d_2 + d_3
\]

which does not depend on the orientation \(\phi\). Therefore, arbitrary rotation of a three mirror setup will not cause a shift of the interference fringes of light for \(a \neq 0\).

Let us now imagine a light signal traveling along the triangles \(ABC\) and \(ACD\) depicted in Fig.2a in the following order. Suppose the signal is emitted in the point \(A\), then travels along the lines \(d_1\), \(d_2\), and \(d_3\). The signal is reflected in the point \(A\) back and travels the distances \(d_4\), \(d_5\), and \(d_6\). The light takes to travel the whole path during the time

\[
t_{ABC} + t_{ACD} = d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = t_{\text{out}} + t_{\text{int}}.
\]

As follows the above considerations, this relation does not depend on the particular choice of the angle \(\phi\). Here we have denoted by \(t_{\text{int}}\) the time the light ray travels along the internal line \(CA\) to and fro. According to Eq. (191), the relation between the internal time \(t_{\text{int}}\) and the distance \(d_3 + d_4\) is invariant under space rotations. Consequently, we get the formula

\[
t_{\text{out}} = d_1 + d_2 + d_5 + d_6
\]

which does not depend on the space-time asymmetry \(a\). It is possible to think of various trajectories from the point \(A\) to the point \(B\). One of the trajectories is line connecting the points \(ADCB\). A light signal traveling along this trajectory takes the time

\[
t_{tr} = d_{tr} + \Delta_{AB}
\]

where

\[
d_{tr} = d_6 + d_5 + d_2, \quad \Delta_{AB} = d_1 - \frac{x_1}{v_1}
\]

are the length of the trajectory and a specific time difference, respectively. The time difference \(\Delta_{AB}\) is due to the space-time asymmetry \(a\) but does not depend on the shape and length of the trajectory \(ADCB\). Therefore, the light signal takes the same time \(t_{tr}' = t_{tr}\) when traveling along another trajectory \(AD'C'B\) with the same length \(d_{tr}' = d_{tr}\). If the trajectory becomes more complicated its length increases so that \(d_{tr} \gg \Delta_{AB}\). In the fractal limit one gets the asymptotic relation

\[
t_{tr} = d_{tr}
\]

which does not depend on any particular value of the space-time asymmetry \(a\) corresponding to a specific resolution.

VI. SUMMARY

The questions addressed in the paper concern ultra-relativistic nuclear interactions at constituent level. They are connected with the notions such as locality, self-similarity, fractality and the scale relativity.

We have discussed some aspects of the relation between the fractality of the interacting objects and the properties of space-time induced by their interactions. The relation is relevant for small scales where the parton composition of the hadron objects is supposed to reveal a fractal-like substructure. The assumption has fundamental consequence which is breaking of the reflection invariance in dependence on scale. We have elaborated the formalism concerning the special relativity in space-time with broken reflection invariance. Our treatment accounts for change in the dispersion
relation including change of the metrics in space-time. If we ignore quantum uncertainties in the energies and momenta we obtain explicit relations between the energy/momentum and the velocity in space-time characterized by the asymmetry $a$. The asymmetry was shown to be a relative quantity governed by scale relativistic principles. This concerns Lorenz invariance with respect to the scale velocity $\nu$ connected with the asymmetry $a$. If considering the asymmetry as fluctuating intrinsic property of space-time itself, local structure of the quantity should depend on scale and the relativistic invariants only. This underlines application of the functional self-similarity to the expression

$$a[\xi, \tau^2] = a[\xi, t^2 - x^2 - 2t \cdot a \cdot x - (a \times x)^2].$$

(204)

In this vision, the asymmetry should depend on a scale parameter $\xi$ and through invariants of motion on the space-time positions and its own values at another scales. Adequate description one has to search for is, in our opinion, within the renorm group approach to the self-similar systems. This should include elementary quantum fields as source of the space-time fluctuations $a$. In view of our results, increase of stochasticity of the space-time asymmetry with decreasing scales would result in fractal-like motion of "elementary" point-like objects with respect to their momenta. This implies change of their rest mass $m_0$ in dependence on the value of $\nu$.

In the paper we have considered space-time asymmetry $a$ and have supposed it can be induced in the ultra-relativistic nuclear collisions. The asymmetry was expressed by the anomalous fractal dimensions of the colliding objects. The fractal dimensions characterize hadronic constituent sub-structure revealed at high energies. The relation is illuminated with respect to maximal resolution in the definition of the scaling variable $z$. Existence of even tiny portion of space-time asymmetry at any scales have been discussed within the framework of the Michelson’s experiment. We have shown that such asymmetry is in principle not ruled out by similar experiments concerning interference of light signals.

Presented approach to the $z$ scaling shows that the observed regularity can have relevance to fundamental principles of physics at small scales. The general assumptions and ideas discussed here underline need of searching for new approaches to physics at ultra-relativistic energies. This concerns better understanding of scale dependence of physical laws in the domain tested by large accelerators of hadrons and nuclei.

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VII. APPENDIX

Here we present derivation of the relativistic transformations in one dimensional case in detail. Without any loss of generality, the linearity of the transformations can be expressed as follows

\[ x' = \gamma(u)[x - ut], \quad (205) \]
\[ t' = \gamma(u)[A(u)t - B(u)x], \quad (206) \]

where \( \gamma, A, \) and \( B \) are unknown functions of a velocity \( u \). Let us compose the transformation with the successive one

\[ x'' = \gamma(v')[x' - v't'], \quad (207) \]
\[ t'' = \gamma(v')[A(v')t' - B(v')x']. \quad (208) \]

The result can be written in the form

\[ x'' = \gamma(u)\gamma(v')[1 + B(u)v'] \left[ x - \frac{u + A(u)v'}{1 + B(u)v'}t' \right], \quad (209) \]
\[ t'' = \gamma(u)\gamma(v')[A(u)A(v') + B(v')u] \left[ t - \frac{A(v')B(u) + B(v')}{A(u)A(v') + B(v')u}t' \right]. \quad (210) \]

The principle of relativity is expressed by the group structure of the transformations. The condition tells us that Eqs. (209) and (210) keep the same form as the initial ones in terms of the composed velocity

\[ v = \frac{u + A(u)v'}{1 + B(u)v'}. \quad (211) \]

The requirement can be satisfied under the following conditions

\[ \gamma(v) = \gamma(u)\gamma(v')[1 + B(u)v'], \quad (212) \]
\[ \gamma(v)A(v) = \gamma(u)\gamma(v')[A(u)A(v') + B(v')u], \quad (213) \]
\[ \frac{B(v)}{A(v)} = \frac{A(v')B(u) + B(v')}{A(u)A(v') + B(v')u}. \quad (214) \]

The isotropy of space-time results in the requirement that the change of orientations of the variable axis are indistinguishable, provided \( u' = -u \). This leads to the parity relations \( \gamma(-u) = \gamma(u), \ A(-u) = A(u), \) and \( B(-u) = -B(u) \). The relations are sufficient for the derivation of the Lorenz transformation. The theory of relativity tells us that the velocity of a physical object can not exceed the value of \( c = 1 \), the velocity of light in the vacuum. The expression of this statement is the Lorenz transformation which yields the limitation of any velocity.

If we leave out the constraint on the reflection invariance, the unknown functions \( \gamma, A, \) and \( B \) do not obey the parity relations resulting from the isotropy requirement. Let us combine Eqs. (211), (212), and (213) into the expression

\[ A \left( \frac{u + A(u)v'}{1 + B(u)v'} \right) = \frac{A(u)A(v') + B(v')u}{1 + B(u)v'}. \quad (215) \]

Its solution has the form
\[ A(u) = 1 - 2au, \quad (216) \]

provided \( B(u)v' = B(v')u \). The condition gives the function \( B(u) = u \) with the normalization constant \( c \) included already in the definition of the variable \( u \). The solution satisfies Eq. (214) as well. The violation of the space-time reflection invariance is expressed by a non-zero value of the parameter \( a \). In terms of \( a \), the composed velocity (211) can be written as follows

\[
v = \frac{v' + u - 2auv'}{1 + uv'}.
\]

(217)

Using this relation, Eq. (212) becomes

\[
\gamma\left(\frac{v' + u - 2auv'}{1 + uv'}\right) = \gamma(u)\gamma(v')(1 + uv').
\]

(218)

Its solution, which for \( a = 0 \) is given by the standard \( \gamma \) factor, has the form

\[
\gamma(u) = \frac{1}{\sqrt{1 - 2au - u^2}}.
\]

(219)

The detailed classification of the linear transformations (205) and (206) in 1+1 dimensions was performed in Ref. [13].

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FIG. 1. The velocity diagram in space-time with the asymmetry $a$. The lines I. and II. correspond to the orientation of the spectrometer arms in the Michelson’s experiment.
FIG. 2. (a) The space diagram of a multi-mirror setup. The mirrors are considered in the points $A$, $B$, $C$, and $D$ reflecting the light signal along the sketched lines. (b) The velocity diagram corresponding to the mirrors setup shown in Fig.2a.