The wrought iron beauty of Poncelet loci

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Abstract. We’ve built a web-based tool for the real-time interaction with loci of Poncelet triangle families. Our initial goals were to facilitate exploratory detection of geometric properties of such families. During frequent walks in my neighborhood, it appeared to me Poncelet loci shared a palette of motifs with those found in wrought iron gates at the entrance of many a residential building. As a result, I started to look at Poncelet loci aesthetically, a kind of generative art. Features were gradually added to the tool with the sole purpose of beautifying the output. Hundreds of interesting loci were subsequently collected into an online “gallery”, with some further enhanced by a graphic designer. We will tour some of these byproducts here. An interesting question is if Poncelet loci could serve as the basis for future metalwork and/or architectural designs.

Keywords: triangle · inversive · locus · porism · Poncelet

1 Introduction

The front doors of apartment buildings lining many a street in Rio de Janeiro, Brazil, are bedecked with wrought iron\textsuperscript{1} masterpieces. Their curvaceous designs seem to strike a perfect balance between beauty and sturdiness.

As one experiments with the curves swept by points attached to Poncelet triangle families (more details in Section 2), one becomes aware of certain common motifs linking these two worlds, e.g., four-fold symmetry, harmonious spirals, tiling, recursion, etc., see Figure 1.

We had built a web-based tool to interact with Poncelet triangles and their loci, see [7,27]. As new experiments were set up, we would often stumble upon a new ornate locus. We quickly recognized in Poncelet loci a kind of aesthetic talent. Gradually, we added features geared at beautifying, coloring, and sharing such curves, see Figure 2. At the same time, we started to collect hundreds of aesthetically-pleasing finds into slideshow and video “galleries” [24,25,26]. Leafing through these gives one an idea of the wide design palette possible with Poncelet loci.

\textsuperscript{1} Iron was popular for ornamental applications in the 19th and early 20th century, given its high ductility and structural capabilities. It has now been by and large replaced by steel.
Fig. 1: Wrought iron gates and loci of Poncelet triangles, hinting at common design motifs.

Fig. 2: Beautifying loci: from wireframe (left), to a thick curve against a dark background (middle), to a region-colored design (right). Experimental parameters used to produce them can be shared as a URL and/or exported as vector graphics for further processing by a graphic artist.
Our goal here is to explore the ingredients involved in generating such loci. In the next sections we (i) review Poncelet’s porism, (ii) review the basic geometry of a triangle’s notable points (whose loci we will sweep), (iii) explore some features of our locus-rendering app, and (iv) tour a few the artful byproducts of Poncelet loci.

Related work The fields of kinetic art [23] and computer-generated (generative) art [1] have been evolving for several decades. Works that explore the connection between classical art and geometry include [15,18,20]. The usage of dynamic geometry tools for mathematical discovery is beautifully expounded in [33]. In [21,22,34,37,38] loci of triangle centers are studied over various 1d triangle families, Poncelet or otherwise. Works [29,30] follow in their footsteps and identify new curious properties and invariants of Poncelet families. Proofs that loci of certain triangle centers in over the confocal family are ellipses appear in [9,10,13,32]. A theory of locus ellipticity is slowly emerging, see [16,17]. In [11,12,28], similar geometric properties and invariants are used to cluster Poncelet triangles families. Proofs of experimentally-detected invariants of Poncelet families appear in [2,3,6,35].

2 Loci of Poncelet Triangles

Poncelet’s closure theorem is illustrated in Figure 3. It states that given two real conics\(^2\) \(C, C'\), if one can draw a polygon with all vertices on \(C\) and with all sides tangent to \(C'\), then a one-dimensional family of such polygons exists. [5,8,4].

We will herein focus on families of Poncelet triangles families. In Figure 4, six examples are shown of such families interscribed\(^3\) between two concentric, axis-aligned ellipses\(^4\).

Referring to Figure 5, a first natural question is: over some particular triangle family, what are curves swept by a notable point? Recall the four classical notable points of a triangle, shown in Figure 6, namely, (i) the incenter \(X_1\), (ii) the barycenter \(X_2\), (iii) the circumcenter \(X_3\), and (iv) the orthocenter \(X_4\). The \(X_k\) notation conforms with [19], where thousands of such points, known as triangle centers, are specified.

Referring to Figure 6(left), an early observation was that over the confocal family, the loci of the four notable points \(X_1, X_2, X_3, X_4\) are ellipses. Formal proofs appeared in [32,9,10]. Interestingly, other centers can sweep quartics, self-intersecting curves, segments, and be stationary points, see Figure 6(right), and [13].

3 The Locus Rendering App

Referring to Figure 8, the default view of our tool is the elliptic locus of the incenter over the confocal family. The app can render loci of the first 1000

\(^2\) Recall these comprise ellipses, hyperbolas, parabolas, as well as some degenerate specimens [14].

\(^3\) This is shorthand for “inscribed while simultaneously circumscribing”.

\(^4\) In general, the pair of Poncelet conics need not be concentric nor axis aligned.
Fig. 3: A heptagon $P_1, ..., P_7$ (blue) is shown inscribed in an outer ellipse and circumscribing an inner one. In such a case, Poncelet's theorem guarantees that any point of the outer ellipse can be used to construct a 7-sided polygon $P'_i$ similarly inscribed/circumscribed about the two conics. A smooth traversal of the entire family can be seen here.

Fig. 4: Six Poncelet triangle families inscribed in an ellipse and circumscribing a concentric, axis-parallel inellipse or caustic. We call these “confocal”, “incircle”, “circumcircle”, “homothetic”, “dual”, “excentral”. In each case, a certain triangle center remains stationary at the common center, namely, $X_k$, $k = 9, 1, 3, 2, 4, 6$, respectively. Video
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Fig. 5: **Left:** The incenter $X_1$ of a triangle inscribed in an ellipse and circumscribing a confocal one (shown on the right). **Right:** As one sweeps the triangle family, the incenter changes position: $X_1, X'_1, X''_1$, sweeping a locus. Video 1, Video 2

Fig. 6: The four notable points of a triangle: (i) top left: the incenter $X_1$, where angular bisectors meet, also the center of the inscribed circle; (ii) top right: the barycenter $X_2$, where medians meet; bottom right: (iii) the circumcenter $X_3$, where perpendicular bisectors meets, also the center of the circumscribed circle; (iv) bottom right: the orthocenter $X_4$, where altitudes meet.
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Fig. 7: **Left:** elliptic loci of the incenter $X_1$ (red), barycenter $X_2$ (green), circumcenter $X_3$, and orthocenter $X_4$ over Poncelet triangles between two confocal ellipses. **Right:**
Over the same family, the mittenpunkt $X_9$ (purple) is stationary [30], the symmedian point $X_6$ (red) sweeps a quartic [13], the Feuerbach point $X_{11}$ (green) is a curve identical to the caustic (inner ellipse), and $X_{59}$ sweeps a self-intersected curve (blue), studied in [29].

triangle centers in [19], over one dozen Poncelet families, including the ones of Figure 3. The user can interactively change parameters of the simulation (e.g., aspect ratio of Poncelet ellipses, triangle center tracked, derived triangle being used, etc.), and observe topological changes in the loci being studied. As an example see Figure 9.

Loci of triangle centers of derived triangles can be studied as well, e.g., the orthic, medial excentral, triangles etc., see [36,19]. As shown in Figure 10, the locus of vertices of the main or derived triangles can be studied as well, further expanding the palette of obtainable curves.

4 Aestheticizing Poncelet

As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired from ideas coming from reality, it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely l’art pour l’art. – John von Neumann, “the Mathematician”

Not exactly following von Neumann’s advice, we added features to our locus tool to beautify loci: dark backgrounds, thick outlines, and automatic color-filling of connected regions with a random palette of pastel colors, see Figures 11 to 13. As mentioned above, 100s of such designs are now collected in [24,25].

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5 These correspond to triangles with vertices which are the feet of altitudes, medians, and where external bisectors meet, respectively.
Fig. 8: Locus Visualization app to explore 3-periodic families. Shown are the loci of $X_k$, $k = 1, 2, 3, 4$, over billiard 3-periodics. The “(E)” suffix indicated they are numerically ellipses. Live; Also see our tutorial playlist.

Fig. 9: Observing transitions in the topology of the locus of the Feuerbach point $X_1$ of the orthic triangle (smaller dashed red) as the aspect ratio of the outer ellipse in the confocal pair is smoothly altered; live.
Fig. 10: Loci of vertices of well-known derived triangles over the confocal family. Their construction is defined in [36].

Fig. 11: Two “leafy” Poncelet loci.

Fig. 12: Dragonfly-like loci of Poncelet triangles.
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Fig. 13: A collage of color-filled loci which can be produced with our web-based tool. For the complete list, see [25,24,26].

Interaction with a digital designer Some outlines of interesting loci are shown in Figure 14. These were sent in vector format to a graphic designer (my sister [31]). Using digital image-editing tools and much creativity, a sample of her work appears in Figure 15.

5 Conclusion

Art, architecture, and design have been inspired by geometry and mathematics and vice-versa. Common design motifs between wrought iron gates and Poncelet loci have stimulated us to look at the latter both from a geometric and an aesthetic perspective. As seen in Figure 16, wrought iron kraftwerk is on a league of its own. An interesting question is if Poncelet loci could ever be used as a basis for new metalwork and/or architectural design.

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Fig. 14: Raw loci sent to digital designer for further coloring and manipulation.

Fig. 15: Samples of Regina Reznik’s artwork [31].
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Fig. 16: More wrought iron masterpieces from the streets of Flamengo (left) and Copacabana (right).

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