Geometry of quantum state space and entanglement

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Abstract

Recently, an explicit relation between a measure of entanglement and a geometric entity has been reported in Deb (Quantum Inf Process 15:1629–1638, 2016). It has been shown that if a qubit gets entangled with another ancillary qubit, then negativity, up to a constant factor, is equal to the square root of a specific Riemannian metric defined on the metric space corresponding to the state space of the qubit. In this article, we consider different class of bipartite entangled states and show explicit relation between two measures of entanglement and Riemannian metric.

Keywords Entanglement · Riemannian metric · State space · Negativity · Concurrence

1 Introduction

Nowadays, geometric tools are often used in quantum information theory because of the fact that these tools provide advantage to find out less trivial and robust physical constraints on physical systems. Among such various tools, differential geometry is an important one. Before quantum information, it was applied in classical information theory. As a result of which a new discipline, called Information Geometry emerged and it got maturity through the works of Amari, Nagaoka and other mathematicians in the 1980s [1]. Initially, the goal of information geometry was to understand the interplay between the information-theoretic quantities and the geometry of probability space by constructing a Riemannian space corresponding to probability space. Later, Morozova and Čencov [2] extended the geometric formulation of probability space to quantum setting by proposing Riemannian metrics on the space of density matrices. Their study

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gradually progressed through the works of Petz and other authors [3–9]. The monotone Riemannian metric corresponding to Wigner–Yanase–Dyson skew information [10, 11] was found out in [7] which expresses the relation between geometry of space and an information-theoretic quantity of great importance. Geometric distances (metrics) are also shown to be useful in quantum state discrimination problem [12,13]. In [14], the authors have demonstrated that a lower bound for quantum coherence measure can be found out using Riemannian monotone metric. Apart from applications in information theory, metric finds its usefulness in quantum optics also. One such example is the Pancharatnam or Berry phase, which arises when a quantum system is subjected to cyclic adiabatic processes [15]. This phase is a geometric one, resulting from the curvature in Hilbert space which can be expressed via metric.

On the other hand, quantum correlation is a resource in quantum information processing. Though there are different aspects of quantum correlations, entanglement and discord are the two aspects which have been extensively studied. However, till date, quantum correlation is not fully understood. So, the study of quantum correlations demands importance in quantum information theory. Here, we consider entanglement because of the fact that all the measures of entanglement are monotonic in nature. Quantum entanglement [16–18] is one of the bizarre phenomena exhibited by composite quantum systems. It is a resource for quantum information processing tasks, such as teleportation [19], dense coding [20], quantum cryptography [21,22], state merging [23], quantum computation and many more. A composite quantum system $\rho_{AB}$ is said to be entangled if it cannot be written as $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$, where $p_i$ are probabilities, $\rho_A$ and $\rho_B$ are, respectively, the density matrices of subsystem $A$ and $B$. If the subsystems are two-level quantum states, then these are termed as qubits [24] in analogy with classical bits. Qubits are the fundamental units in quantum information theory.

Entanglement is the most studied form of quantum correlations. However, geometry of quantum state space has not been applied in the study of quantum entanglement till date. Therefore, study of quantum correlations using geometry of quantum state space may be an interesting area of research. To begin with, one may address the problem of finding unique Riemannian metrics corresponding to different measures of entanglement. Interestingly, a problem close to this has recently been addressed in [25]. An explicit relation between negativity, a measure of entanglement, and monotone Riemannian metric was established in that article. The author considered entanglement generation between two qubits and calculated the negativity of the generated entangled state to establish the relation between a geometric entity (Riemannian metric) and an entanglement measure (negativity). A specific unitary operation was considered which can create entanglement between two qubits, initially in a product form. However, one question remained unanswered in that paper: given an entangled state, is there any explicit relation between measures of entanglement and the Riemannian metrics? In this article, we focus exactly on this question, because we think that answering this question will enable us to gain some insights which will further help to find out unique Riemannian metrics corresponding to different measures of entanglement. To find out the answer of the question, we consider two measures of entanglement, namely concurrence ($C$) [26] and negativity ($N$) [27,28]. We take different class of bipartite entangled states whose subsystems are non-maximally mixed qubits. Then,
we determine a particular Riemannian metric corresponding to those states using a theorem proposed by Morozova and Ėncov. Finally, we show that negativity and concurrence of such states are explicitly related to their corresponding Riemannian metrics.

The rest of the article is arranged as follows. In Sect. 2, we first provide an overview on Riemannian metric and Riemannian metrics on matrix space (quantum state space). Then, we discuss an important theorem on Riemannian metric. Section 3 is dedicated to show our results. In Sect. 4, we conclude our work with discussions.

2 Riemannian geometry of quantum state space

Riemannian geometry is a branch of differential geometry which includes Riemannian manifolds and Riemannian metrics. Riemannian manifold is a real and smooth differentiable manifold embedded with an inner product at each point of the tangent space, and the inner product varies smoothly from point to point. More precisely, if \( M \) is a differentiable manifold, \( X \) and \( Y \) are two vectors on the tangent space \( T_x M \) passing through \( x \), and \( g_x \) is the inner product on the tangent space at each point \( x \), then \( x \mapsto g_x\{X(x), Y(x)\} \) is a smooth function. Riemannian metric on a manifold \( M \) is the family of \( g_x \). Morozova and Ėncov initiated the study of monotone Riemannian metrics on the space of matrices. The motivation behind their work was to extend the geometric approach to quantum setting. They proposed the problem of finding Riemannian monotone metrics on the quantum state space which is endowed with a metric structure.

Quantum state space is identified with the set \( \mathcal{M}_n \) of positive \( n \times n \) matrices of trace one; they are termed as density matrices. This space of density matrices forms differential manifold on which a differentiable metric determines a Riemannian metric. On the other hand, the operators that act on the quantum states are expressed by \( n \times n \) complex Hermitian matrices. The space of quantum operators is an inner product space, and the simplest inner product is the Hilbert–Schmidt one, defined as

\[
\langle X, Y \rangle = \text{Tr}(X^\dagger Y),
\]

where \( \text{Tr} \) is as usual matrix trace and \( X, Y \in \mathcal{B}_n(\mathbb{C}) \), \( \mathcal{B}_n(\mathbb{C}) \) being the set of complex self-adjoint matrices. This inner product is unitarily invariant, that is, \( \langle X, Y \rangle = \langle UXU^\dagger, UYYU^\dagger \rangle \) for every unitary \( U \). This invariance property is so strong that it determines the Hilbert–Schmidt inner product up to a constant multiple.

Now, by making the inner products depending on quantum states \( (\rho) \), Riemannian monotone metric can be determined on the quantum state space in the following way. Assume that for every \( A, B \in \mathcal{B}_n(\mathbb{C}) \), for every \( \rho \in \mathcal{M}_n \), and for every \( n \in \mathbb{N} \), a complex quantity \( K_\rho(A, B) \) is given. The complex quantity \( K_\rho(A, B) \) will be a metric if the following conditions hold [3]:

(a) \( (A, B) \mapsto K_\rho(A, B) \) is sesquilinear.
(b) \( K_\rho(A, A) \geq 0 \), and the equality holds iff \( A = 0 \)
(c) \( \rho \mapsto K_\rho(A, A) \) is continuous on \( \mathcal{M}_n \) for every \( A \).
The family of $K_\rho(A, B)$ with the above-mentioned properties constitute Riemannian metric on the differentiable manifold formed by the density matrices. The Riemannian metric will be monotone if

(d) under completely positive trace preserving (CPTP) map, $K_\rho(A, A)$ is contractive, i.e., $K_{A\rho}(A, A) \leq K_\rho(A, A)$ for every $A$, $\rho$ and $A$; $A(\cdot)$ being the CPTP map.

For clear illustration of the metric $K_\rho(A, B)$, it is important to focus on the geometry of the quantum state space. Here, $\mathcal{M}_n$ is the differential manifold and the self-adjoint operators $A$ and $B$ are the tangent vectors on the tangent space $T_\rho$. Therefore, $K_\rho(A, B)$ is the inner product on the tangent space $T_\rho$ at point $\rho$. Considering the quantum state space to be finite dimensional, let us denote the set of all Hermitian operators by

$$\mathcal{A} = \left\{ A | A = A^\dagger \right\}$$

and from the definition of $\mathcal{M}_n$,

$$\mathcal{M}_n = \left\{ \rho | \rho = \rho^\dagger \geq 0 \text{ and } \text{Tr}\rho = 1 \right\}.$$  

The tangent space $T_\rho(\mathcal{M}_n)$ of each point $\rho$ may then be identified with [1]

$$\mathcal{A}_0 = \{ A | A \in \mathcal{A} \text{ and } \text{Tr}A = 0 \}.$$  

It can be shown that if $K$ is an operator and $K \in \mathcal{A}_0$, then $i[\rho, K]$ will be an ordinary element of the tangent space $T_\rho(\mathcal{M}_n)$, that is, $i[\rho, K] \in \mathcal{A}_0$ [1]. Therefore, by identifying tangent vectors Riemannian metric can be defined on the differential manifold formed by the density matrices, and if the metric satisfies condition (d), then the metric will be called monotone Riemannian metric.

Morozova and Čencov tried to describe monotone metrics on the space of self-adjoint matrices, but they were unable to show any metric. However, they proposed several candidates and provided an useful theorem. Later, Petz and other authors were able to find monotone metrics by introducing operator monotone functions. Their works showed that there is an abundance of monotone metrics on the space of self-adjoint matrices [4–6]. For our purpose, we will make use of the theorem provided by Morozova and Čencov

**Theorem [2,3]** Assume that for every $D \in \mathcal{M}_n$ a real bilinear form $K'_D$ is given on the $n$-by-$n$ self-adjoint matrices such that the conditions (b), (c) and (d) are satisfied for self-adjoint $A$. Then, there exists a positive continuous function $c(\lambda, \mu)$ and a constant $C$ with the following property: If $D$ is diagonal with respect to the matrix units $E_{ij}$, i.e., $D = \sum_i \lambda_i E_{ii}$, then

$$K'(A, A) = C \sum_{i=1}^n \lambda_i^{-1} A_{ii}^2 + 2 \sum_{i<j} |A_{ij}|^2 c(\lambda_i, \lambda_j)$$

for every self-adjoint $A = (A_{ij})$. Moreover if $c$ is symmetric in its two variables, $c(\lambda, \lambda) = C\lambda^{-1}$ and $c(t\lambda, t\mu) = t^{-1} c(\lambda, \mu)$.

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The function \( c(\lambda, \mu) \) is termed as Morozova–Čencov function. The theorem tells that when \( \mathcal{M}_n \) is considered as a differentiable manifold, the Riemannian metric must be a real bilinear form and the tangent vectors may be identified with self-adjoint matrices. Moreover, for all \( D \) and for all self-adjoint operator \( A \) the metric \( K'(A, A) \) can be determined using the above theorem.

### 3 Results

It is already mentioned that in [25] an explicit relation between negativity and Riemannian metric has been established. In brief, if a qubit \( \rho_S \) interacts with an ancillary qubit \( \rho_A \) and an entangled state \( \rho_{SA} \) is produced, then negativity \( N' \) of the entangled state is, up to a constant factor, equal to the square root of the Riemannian metric defined for \( \rho_S \). A specific unitary \( U_{SA} \) produces the entangled state by acting on the initial product state \( \rho_S \otimes |0\rangle_A \). Here, we will show that for different class of entangled states explicit relations between some measures of entanglement and Riemannian metric exist. To begin with, let us consider an entangled state \( \rho_{A_1A_2} \) consisting of subsystems \( A_1 \) and \( A_2 \). Negativity [28] of this state is given by

\[
N(\rho_{A_1A_2}) = \frac{\| T_{A_1} \rho_{A_1A_2} \|_1 - 1}{2},
\]

where \( T_{A_1} \rho_{A_1A_2} \) denotes the partial transpose with respect to the subsystem \( A_1 \) and \( \| T_{A_1} \rho_{A_1A_2} \|_1 \) denotes the trace norm of the matrix. Concurrence [26] of the state \( \rho_{A_1A_2} \) is given by

\[
C = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},
\]

where \( \lambda 's \) are the square root of the eigenvalues of \( \rho_{A_1A_2} \bar{\rho}_{A_1A_2} \) in decreasing order. The spin-flipped density matrix \( \bar{\rho}_{A_1A_2} \) is defined as

\[
\bar{\rho}_{A_1A_2} = \sigma_y^{A_1} \otimes \sigma_y^{A_2} \rho^* \sigma_y^{A_1} \otimes \sigma_y^{A_2},
\]

where \( \ast \) denotes the complex conjugate in the computational basis. In order to establish explicit relation between these two measures and Riemannian metric, we need to determine Riemannian metric corresponding to any of the subsystem state. For this purpose, we will use the theorem mentioned earlier. Now, we consider different class of entangled states and show the proposed results.

#### 3.1 Pure entangled state

Let us consider a Bell-like state

\[
|\psi\rangle_{A_1A_2} = \alpha|00\rangle + \beta|11\rangle,
\]

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where $|\alpha|^2 + |\beta|^2 = 1$. The density matrix corresponding to this state is $\rho_{A_1 A_2}$. Tracing out subsystem $A_2$, we get the state $\rho_{A_1}$ of the subsystem $A_1$ as:

$$
\rho_{A_1} = \text{Tr}_{A_2}(\rho_{A_1 A_2}) = |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|.
$$

(10)

Our aim is to find the metric $K_{\rho_{A_1}}(A, B)$. From the definition of the metric, $A$ and $B$ are traceless self-adjoint operators and they act as tangent vectors corresponding to the Riemannian manifold formed by single-qubit density matrices. In order to construct $A$ and $B$ with their respective properties, we use the fact that the Pauli matrices are self-adjoint traceless operators and define them as $A = B = i[\rho_{A_1}, \sigma_l | l = x, y, z]$, where ‘i’ stands for imaginary. Matrix representation of these operators is given by:

$$
i [\rho_{A_1}, \sigma_x] = i \begin{pmatrix} 0 & (|\alpha|^2 - |\beta|^2) \\ (|\beta|^2 - |\alpha|^2) & 0 \end{pmatrix}
$$

(11)

and

$$
i [\rho_{A_1}, \sigma_y] = \begin{pmatrix} 0 & (|\alpha|^2 - |\beta|^2) \\ (|\beta|^2 - |\alpha|^2) & 0 \end{pmatrix}.
$$

(12)

It is easy to verify from Eqs. (11) and (12) that trace of the two self-adjoint operators is zero. Moreover, the diagonal elements of the matrix corresponding to the operators (tangent vectors) $i[\rho_{A_1}, \sigma_x]$ and $i[\rho_{A_1}, \sigma_y]$ are zero, i.e., $A_{11} = A_{22} = 0$. On the other hand, as the state $\rho_{A_1}$ is a subsystem of a two-qubit entangled state, it is always a mixed state having two nonzero eigenvalues. From the above arguments, it can be easily verified that the first summation term of Eq. (5) is zero, i.e.,

$$
C \sum_{i=1}^{2} \lambda_i^{-1} A_{ii}^2 = 0.
$$

(13)

Therefore, for operator $i[\rho_{A_1}, \sigma_x]$ we get Riemannian metric as:

$$
K_{\rho_{A_1}}(i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x]) = 2 \sum_{i<j} |A_{ij}|^2 c(\lambda_i, \lambda_j),
$$

(14)

where $A_{ij}$ are the off-diagonal elements of the matrix given in Eq. (11) and $c(\lambda_i, \lambda_j)$ is the Morozova–Čencov function [2,3]. For our calculation, we take one of the functions proposed originally by Morozova and Čencov [2]:

$$
c(\lambda_i, \lambda_j) = \frac{2}{\lambda_i + \lambda_j}.
$$

(15)
Using Eqs. (11) and (13–15), we get the Riemannian metric $K_{\rho_{A_1}} (i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x])$ corresponding to the state $\rho_{A_1}$ as

$$K_{\rho_{A_1}} (i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x]) = 2|A_{12}|^2 \frac{2}{\lambda_1 + \lambda_2} = 2(|a|^2 - |\beta|^2)^2 \frac{2}{|a|^2 + |\beta|^2} = 4(1 - 4|a|^2|\beta|^2),$$  

(16)

where $\lambda_1$ and $\lambda_2$ are eigenvalues of the density matrix $\rho_{A_1}$ and $A_{12}$ is the off-diagonal element of the operator $i[\rho_{A_1}, \sigma_x]$. Now, using Eqs. (6, 7) we get the negativity and concurrence of the state $\rho_{A_1A_2}$ as

$$N(\rho_{A_1A_2}) = |a||\beta|$$  

and

$$C = 2|a||\beta|.$$  

(17)  

(18)

Finally, from Eqs. (16) and (17, 18) we get

$$K_{\rho_{A_1}} (i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x]) = 4(1 - 4N^2) = 4(1 - C^2).$$  

(19)

The above equation is valid for $\sigma_y$ as well, i.e.,

$$K_{\rho_{A_1}} (i[\rho_{A_1}, \sigma_y], i[\rho_{A_1}, \sigma_y]) = 4(1 - 4N^2) = 4(1 - C^2).$$  

(20)

Instead of $i[\rho_{A_1}, \sigma_x]$ or $i[\rho_{A_1}, \sigma_y]$ if we consider a general element of tangent space, i.e., $i[\rho_{A_1}, \hat{n} \cdot \vec{\sigma}]$, then we get the same relation as that of Eqs. (19) and (20), which means

$$K_{\rho_{A_1}} (i[\rho_{A_1}, \hat{n} \cdot \vec{\sigma}], i[\rho_{A_1}, \hat{n} \cdot \vec{\sigma}]) = 4(1 - 4N^2) = 4(1 - C^2).$$  

(21)

where $\hat{n}$ is an arbitrary unit vector on the Bloch sphere. The above three Eqs. (19–21) represent explicit relation between measures of quantum correlations (entanglement) and Riemannian metric for two-qubit pure entangled states or Bell-like states. Due to unitary invariance property of concurrence, negativity and Riemannian metric, the above equations are also unitary invariant. In the next subsection, we will consider mixed entangled states.
3.2 Mixed entangled state

In this section, we are going to show the relation between the Riemannian metric and measures of entanglement for two-qubit mixed entangled state. We consider two-qubit maximally entangled mixed state (MEMS) [29] and non-maximally entangled mixed state.

3.2.1 Maximally entangled mixed state

Maximally entangled mixed states (MEMS) are those states that have the maximum possible entanglement for a given mixedness. MEMS were first introduced by Ishizaka and Hiroshima [29] in a way that their entanglement is maximized by fixing the eigenvalues of the density matrices. The amount of entanglement of these states cannot be increased by any global unitary transformation, and this property will hold for states having rank less than 4. Later, Munro et al. [30] had derived an analytical form of MEMS and showed that these states are optimal for the entanglement (concurrence) and purity measure. Wei et al. [31] further showed that MEMS depend on the measures one uses to quantify entanglement. For different entanglement measures and mixedness, there are different forms of maximally entangled mixed state (MEMS). Here, we will show relation between Riemannian metric and entanglement measures for different forms of maximally entangled mixed state (MEMS).

**MEMS of Ishizaka and Hiroshima** Ishizaka and Hiroshima have proposed maximally entangled mixed states whose entanglement is maximized at fixed eigenvalues for different rank.

- **Rank-4 state:**
  The states they have proposed are those which can be obtained by applying any local unitary transformation on the state

\[
\rho_{A_1 A_2} = p_1 |\psi^-\rangle\langle\psi^-| + p_2 |00\rangle\langle00| + p_3 |\psi^+\rangle\langle\psi^+| + p_4 |11\rangle\langle11|,
\]

(22)

where \( |\psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2} \) are the Bell states and \( |00\rangle, |11\rangle \) are product states orthogonal to \( |\psi^\pm\rangle \). \( p_i \) are the eigenvalues of \( \rho_{A_1 A_2} \) in decreasing order \( (p_1 \geq p_2 \geq p_3 \geq p_4) \) and \( p_1 + p_2 + p_3 + p_4 = 1 \). Ishizaka and Hiroshima have shown that Rank-4 states will be MEMS if the following relation is satisfied:

\[
p_3 = p_2 + p_4 - \sqrt{p_2 p_4}.
\]

(23)

Now, concurrence of the state given in Eq. (22) is found to be

\[
C = p_1 - p_3 - 2\sqrt{p_2 p_4}.
\]

(24)
The state of subsystem $A_1$ is
\[
\rho_{A_1} = \left( \frac{p_1 + p_3}{2} + p_2 \right) |0\rangle \langle 0| + \left( \frac{p_1 + p_3}{2} + p_4 \right) |1\rangle \langle 1|. \tag{25}
\]

Riemannian metric corresponding to this state will be
\[
K_{\rho_{A_1}} \left( i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x] \right) = 2 \sum_{i<j} |A_{ij}|^2 c(\lambda_i, \lambda_j) = 4(p_2 - p_4)^2, \tag{26}
\]
where $c(\lambda_i, \lambda_j) = 2/(\lambda_i + \lambda_j)$.

Using Eqs. (23, 24, 26), we finally get the relation between concurrence and Riemannian metric as
\[
\sqrt{K_{\rho_{A_1}} \left( i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x] \right)} = \frac{2}{3} (1 - C) - 4p_4. \tag{27}
\]

- **Rank-3 state:**
  MEMS of rank-3 can be derived from Eq. (22) by putting $p_4 = 0$:
  \[
  \rho_{A_1A_2} = p_1 |\psi^-\rangle \langle \psi^-| + p_2 |00\rangle \langle 00| + p_3 |\psi^+\rangle \langle \psi^+|, \tag{28}
  \]
  where $p_1 + p_2 + p_3 = 1$. After doing similar calculations as in the previous case, we obtain the relation between Riemannian metric and concurrence as
  \[
  \sqrt{K_{\rho_{A_1}} \left( i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x] \right)} = 2(1 - C) - 4p_3. \tag{29}
  \]

- **Rank-2 state:**
  Putting $p_3 = p_4 = 0$ in Eq. (22), we get rank-2 MEMS
  \[
  \rho_{A_1A_2} = p_1 |\psi^-\rangle \langle \psi^-| + p_2 |00\rangle \langle 00|, \tag{30}
  \]
  where $p_1 + p_2 = 1$. The relations between concurrence and Riemannian metric for such states is found to be
  \[
  \sqrt{K_{\rho_{A_1}} \left( i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x] \right)} = 2(1 - C). \tag{31}
  \]

In deriving the relations between Riemannian metric and concurrence for states of different rank, we have considered subsystem $A_1$. However, similar relations can also be derived if we take subsystem $A_2$ as the density matrices of two subsystems are same.
Wei et al. [31] had shown that MEMS have different forms for different entanglement measures. They had derived analytical form of MEMS for different entanglement measures for a given amount of mixedness. Here, we have taken a MEMS for negativity measure

\[
\rho_{A_1 A_2} = \left( \frac{1 + \sqrt{3}r^2 + 1}{6} \right) |00\rangle\langle 00| + \frac{r}{2} |00\rangle\langle 11| + \frac{r}{2} |11\rangle\langle 00| + \left( \frac{1 + \sqrt{3}r^2 + 1}{6} \right) |11\rangle\langle 11| + \left( 4 - 2\sqrt{3}r^2 + 1 \right) |01\rangle\langle 01|,
\]

where \(0 \leq r \leq 1\). Now, the state of subsystem \(A_1\) is

\[
\rho_{A_1} = \left( \frac{5 - \sqrt{3}r^2 + 1}{6} \right) |0\rangle\langle 0| + \left( \frac{1 + \sqrt{3}r^2 + 1}{6} \right) |1\rangle\langle 1|.
\]

After calculating negativity and Riemannian metric, we finally get the relation between them as

\[
\sqrt{K_{\rho_{A_1}}} \left( i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x] \right) = \frac{4}{3} (1 - N).
\]

### 3.2.2 Non-maximally entangled mixed state

We have considered maximally entangled mixed state so far. Now, let us take non-maximally entangled mixed state. One such example is the state

\[
\rho_{A_1 A_2} = p |\psi\rangle\langle \psi| + (1 - p) |01\rangle\langle 01|,
\]

where \( |\psi\rangle = \alpha |00\rangle + \beta |11\rangle \) is a non-maximally entangled pure state and \( |\alpha|^2 + |\beta|^2 = 1 \). For this state, we get the relation between Riemannian metric and concurrence as

\[
\sqrt{K_{\rho_{A_1}}} \left( i[\rho_{A_1}, \sigma_x], i[\rho_{A_1}, \sigma_x] \right) = 2 \left( 1 - 2\frac{\alpha}{\beta} C \right).
\]

### 4 Conclusion

Quantum correlation is a resource for quantum information processing tasks, and entanglement is the most studied form of it. However, till date, no attempt has been taken to study quantum entanglement using geometry of quantum state space. In this article, we have addressed a problem, which we think can shed some light on the study of quantum entanglement using Riemannian metrics on quantum state space.
We have considered different class of two-qubit entangled states, and for each class we have shown an explicit relation between measures of entanglement and Riemannian metric. The measures that we have taken into consideration are negativity and concurrence. Riemannian metric on the differential manifold has been constructed by using a theorem provided by Morozova and Čencov. While calculating the value of the defined metric, we have considered a specific Morozova–Čencov function. However, this function is not unique to show explicit relation between metric and entanglement measures. Application of other functions will also result in some relation between Riemannian metric and entanglement measures but having different forms. These newly derived results show that the metric is changing as amount of entanglement changes, which implies that the curvature of the Riemannian space is changing. In this way, experimentalists can visualize entanglement, which in turn may help in designing their experiment without requiring to resort to lengthy algebra. The entangled states that we have considered have a common feature; the subsystems are non-maximally mixed. For maximally mixed subsystems, the Riemannian metric will be zero. Therefore, in such cases we cannot get explicit relation between measures of entanglement and Riemannian metric. This is in fact a limitation of the process. Though the results of this paper do not mention any unique mapping between Riemannian metric and measures of entanglement, these certainly emphasize that there exists explicit relation between measures of quantum correlation and geometry of quantum state space. We hope that our work will be useful in defining unique Riemannian metrics corresponding to different entanglement measures.

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