Optimal measurement problem for semi-linear descriptor system with Showalter–Sidorov condition

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Abstract. A semilinear descriptor system with a linear part and a nonlinear term is considered. This system is not resolved with respect to the derivative of the desired vector function. The article describes the optimal measurement problem for the model under study with the Showalter–Sidorov condition. Based on the methods of the theory of dynamic measurements, sufficient conditions for the existence of a solution to the problem of optimal measurement are found.

Introduction
Dynamic measurements are becoming more and more widespread in technology and scientific research, in such areas as metrology, energy, computational neuroscience, mathematical economics and other areas. Changing requirements for measurement results is a consequence of improved test quality and production efficiency. So, one of the most significant issues in the theory of dynamic measurements is the problem of reconstructing the measured signal. In the last three decades, the problems of dynamic measurement for linear and nonlinear descriptor systems have become especially popular. Nonlinear descriptor systems have found their application in modeling the motion of aircraft [1], chemical processes [2], in technical systems [3], in robotics [4], etc. To achieve the proximity of the values of virtual observations and real observations of the sensor, methods of the theory of optimal control are used [5]. As a result, the determined virtual measurement – the input signal of the model – is a solution to the mathematical problem of optimal control, and the resulting optimal dynamic measurement most accurately reflects the input signal of the sensor. This approach assumes minimization of the penalty functional, with the help of which the smallest discrepancy is achieved as the value of the output signal to the output signal of the sensor model, and their derivatives. This problem is also called the optimal measurement problem [6], and the found minimum is the optimal measurement. In all of the above works, linear descriptor systems or systems of Leontief-type equations were considered, the proposed work is devoted to the study of optimal measurement problems specifically for semilinear descriptor systems. Consideration of semi-linear descriptor systems makes it possible to generalize the Shestakov–Sviridyuk model [5], based on linear systems, to a more general case, which will allow taking into account nonlinear connections in sensors.

As an example of a sensor model operating on the basis of a nonlinear descriptor system, consider the FitzHugh–Nagumo oscillator, a simplified diagram of which is shown in Fig. 1.

It is an oscillatory RLC circuit with a nonlinear element $R^0$ connected to it (in the classical version it is a tunnel diode), which is acted upon by the sum of voltages: a constant voltage from the circuit power supply $V_c$ and an alternating external signal $\xi(t)$. The current-voltage characteristic of the
element (Fig. 2) can be approximated by a cubic polynomial: \( I = F(U) = aU^3 - gU \), where \( g \) is the negative differential conductivity, the presence of which allows the “pumping of energy” into the oscillatory circuit. Using Kirchhoff’s laws, you can get the circuit equations:

\[
\frac{dI}{dt} = V_c + \varepsilon(t) - RI - U;
\]
\[
\frac{dU}{dt} = I - F(U).
\]

Let us introduce the dynamic variables \( x_1 = -U \) and \( x_2 = I / g \). For new variables, the equations will be written in the following form:

\[
\begin{align*}
\varepsilon_1 \dot{x}_1 &= x_1 - ax_1^3 - x_2; \\
\varepsilon_2 \dot{x}_2 &= \gamma x_1 - \delta x_2 + \beta + \eta(t).
\end{align*}
\]

Here \( \varepsilon = C / g \), \( \alpha = ag \), \( \beta = Vc / L \), \( \delta = Rg / L \), and \( \gamma = I / L \) are parameters that control the behavior of the oscillator, and the value of \( \varepsilon \) is assumed to be small: \( \varepsilon \ll 1 \); \( \eta(t) = \xi / L \) is external influence; dot means time differentiation. This system is degenerate. Depending on the values of the parameters, system can be in three different dynamic modes: self-oscillating, excitable, and bistable. In the first of them, it acts as a generator of undamped periodic oscillations, in the second as a pulse signal generator, and in the third as a trigger. Then, the model of the measuring device will be represented by the system

\[
\begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2
\end{pmatrix}
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix}
=
\begin{pmatrix}
1 & -1 \\
\gamma & -\delta
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
+
\begin{pmatrix}
-a \cdot x_1^3 \\
0
\end{pmatrix}
+
\begin{pmatrix}
\beta + \eta(t)
\end{pmatrix}.
\]

Because the value of \( \varepsilon \) is assumed to be small we consider the limiting situation when \( \varepsilon_1 = 0 \) or \( \varepsilon_2 = 0 \). In this case, the system of equations is degenerate and belongs to the class of equations of the Sobolev type [7–9]. In the case when the phase space of the system is a simple Banach \( C^\infty \)-manifold [10], the problem has a unique solution. In particular, Leontief type systems are considered as a finite-dimensional analogue of the linear Sobolev type equations.

1. Main Results.

Consider a semilinear descriptor system of the form

\[
Lx(t) + Mx(t) + N(x(t)) = u(t), \quad \det L = 0,
\]

(1)
and

\[ y(t) = Dx(t) \]  

with the Showalter–Sidorov initial conditions

\[ L(x(0) - x_0) = 0. \]  

Here \( L, M \) are matrices of order \( n \) \((\langle Lx, x \rangle \geq 0 \) and \( \langle Mx, x \rangle > 0 \)), representing the mutual influence of the state and state velocities of the measuring device, respectively, and \( \text{dim ker } L = \text{dim coker } L \); \( D \) is a square matrix of order \( n \), \( N \) is a nonlinear operator defined by the formula \( <N(x(t)), x(t)> = a_1x_1^t + a_2x_2^t + \ldots + a_nx_n^t, a_i \geq 0 \), where \( \langle , \rangle \) is the Euclidean scalar product in \( R^n \), \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) and are vector functions of the state and rate of change of the state of the measuring device, respectively; \( u(t) = (u_1(t), u_2(t), \ldots, u_n(t)) \) and \( y(t) = (y_1(t), y_2(t), \ldots, y_n(t)) \) are measurement vector functions and observations of the measuring device, respectively.

The main goal of this work is to minimize the values of the penalty functional

\[ J(y) = J(x, u) = \int_0^T \| y(t) - y_0(t) \|^2 dt, \]  

where \( \| \cdot \| \) is the norm in \( R^n \), \( U = \{ u \in L^4/3((0, T), R^n) \} \) is the measurement space, \( y_0(t) = (y_{01}(t), y_{02}(t), \ldots, y_{0n}(t)) \) is an observation obtained during a full-scale experiment. Note that, by virtue of (1), (2), the vector function \( y \) depends on the functions \( x, u \); therefore, we can assume that \( J(y) = J(x, u) \).

Consider \( \text{coim } L = \{ x \in \mathfrak{X} : \langle x, \varphi \rangle = 0 \forall \varphi \in \ker L \setminus \{0\} \} \), where \( \langle , \rangle \) is a scalar product in \( L_2((0, T), R^n) \). Let \( \mathfrak{X} = \{ x \in L_4((0, T), R^n) : \dot{x} \in \text{coim } L \} \) be the state space, \( \mathcal{R} = \mathfrak{X} \) be the observation space for some fixed \( T \in R^+ \). Let us single out in \( U \) a closed, convex subset \( \mathcal{U}_\beta \), the set of admissible measurements.

Consider the set

\[ M = \{ x \in \mathfrak{X} : (I - Q)(Mx + N(x)) = (I - Q)u \}. \]  

By virtue of the properties of the matrix \( L \), there exists a projector \( Q \) along coker \( L \) onto im \( L \). Note that if \( x = x(t) \) is a solution to equation (1), then it necessarily lies in the set \( M \). The set \( M \) will be called the phase manifold of equation (1).

**Definition 1.** A weak solution to equation (1) is a vector function \( x \in \mathfrak{X} \) satisfying the condition

\[ \int_0^T \varphi(t) \left[ \frac{d}{dt} Lx + Mx + N(x), w \right] dt = \int_0^T \varphi(t) \left[ u, w \right] dt, \]  

for any \( w \in \mathfrak{X} \) and any \( \varphi \in L_2(0, T) \). A solution to equation (1) is called a solution to the Showalter–Sidorov problem if it satisfies (3). It is required to find the optimal measurement \( \tilde{u} \in \mathcal{U}_\beta \) that satisfies system (1), (2), the Showalter–Sidorov initial condition (3), and

\[ J(u) = \min_{u \in \mathcal{U}_\beta} J(u). \]  

**Definition 2.** A pair \( (\tilde{x}, \tilde{u}) \in \mathfrak{X} \times \mathcal{U}_\beta \) is called a solution to the optimal measurement problem (1)–(3), (7) if

\[ J(\tilde{x}, \tilde{u}) = \min_{(x, u)} J(x, u), \]  

where the pairs \( (x, u) \in \mathfrak{X} \times \mathcal{U}_\beta \) satisfy (1)–(3) in the sense of Definition 1.
Definition 3. Admissible pairs $\mathcal{A}$ of problem (1)–(3), (7) are a couple $(x, u) \in \mathfrak{X} \times U_\delta$ satisfying problem (1)–(3), for which

$$J(x, u) < +\infty.$$ 

Theorem 1. If $M$ is a simple Banach $C^\infty$-manifold, then for any $x_0 \in \mathfrak{X}$ and $y_0 \in \mathfrak{R}$ there is a unique measurement vector function $\bar{u} \in U_\delta$ for which (7) holds.

Proof. 1. Since the set of admissible pairs $\mathcal{A}$ is not empty, there is a sequence $(x_m, u_m) \in \mathfrak{X} \times U_\delta$ such that

$$\lim_{m \to \infty} J(x_m, u_m) = \min_{(x, u) \in \mathfrak{X} \times U_\delta} J(x, u).$$

Since the functional has the coercivity property, then

$$\|u_m\|_{L^4(0, \tau); \mathbb{R}^n} \leq \text{const},$$

$$\|x_m\|_{L^4(0, \tau); \mathbb{R}^n} \leq \text{const},$$

for all $m \in \mathbb{N}$.

From the estimates above (possibly passing to a sequence), we extract weakly converging sequences in the corresponding spaces $x_m \rightharpoonup \bar{x}$, $u_m \rightharpoonup \bar{u}$, $L \frac{dx_m}{dt} \rightharpoonup L \frac{d\bar{x}}{dt}$, $N(x_m) \rightharpoonup \mu$. By Mazur’s theorem $u \in U_\delta$. Detailed proof can be viewed in [11].

2. We pass to the limit in the equation of state $\left[ L \frac{d\bar{x}}{dt} + M\bar{x} + \mu, \xi \right] = [\bar{u}, \xi]$. Detailed proof can be viewed in [11].

3. Since the operator $N$ is monotonic, we obtain $\mu = N(\bar{x})$, then

$$L \frac{d\bar{x}}{dt} + M\bar{x} + N(\bar{x}) = \bar{u}.$$ 

Detailed proof can be viewed in [11].

4. Hence, $\bar{x} = \bar{x}(u)$, then

$$\lim_{m \to \infty} \inf (x_m, u_m) \geq J(\bar{x}, \bar{u}).$$

It follows that $\bar{u}$ is the optimal control of the problem.

Remark. Using the monotonicity method, similarly to [11], we can prove the existence of a solution to the Showalter–Sidorov problem (1)–(3), and the solution is unique. If $U_\delta \neq \emptyset$, then for any $u \in U_\delta \subset U$ due to the existence of a unique solution it follows that the set of admissible pairs $\mathcal{A}$ is not empty.

The vector function $\tilde{u} = \tilde{u}(t)$ from Theorem 1 is further called the exact optimal measurement.

Note that the vector functions $\tilde{u} = \tilde{u}(t)$ and $y = y(t)$ obtained as a result of applying Theorem 1 are virtual exact optimal measurement and virtual exact optimal observation.

Theorem 1 establishes the existence of a solution, but does not describe a method for finding it. For a numerical study of problem (1)–(3), (7), we linearize equation (1) using the decomposition method, then problem (1)–(3), (7) is equivalent to the problem

$$L\dot{x}(t) + Mx(t) + N(v(t)) = u(t),$$

$$y(t) = Dx(t),$$

$$x(u, v(t)) = \gamma(t),$$

$$L(x(0) - x_0) = 0.$$ 

(8)

Definition 3. The triple $(\bar{x}, \bar{v}, \bar{u}) \in \mathfrak{X} \times \mathfrak{X} \times U_\delta$ is called the solution of the optimal measurement problem (7), (8) if
where the triple \((x, v, u) \in \mathfrak{X} \times \mathfrak{X} \times \mathfrak{U}_0\) satisfies (8) in the sense of Definition 1. By virtue of problem (8) \(y = y(t, x, u)\).

**Theorem 2.** If \(M\) is a simple Banach \(C^\infty\)-manifold, then for any \(x_0 \in \mathfrak{X}\) and \(y_0 \in \mathfrak{Y}\) there is a unique solution of problem (7), (8).

**Proof.** The main ideas of the proof are based on the reduction of the problem we are investigating to the general problem (2), (3). The conceptual proof of this Theorem is similar to the proof of Theorem 2 in [11].

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**References**

[1] Stevens B. L., Lewis F. L. Aircraft Modeling. Dynamics and Control. New York: Wiley, 1991.

[2] Kumar A. and Daoutidis P. Feedback Control of Nonlinear Differentialalgebraic Equation Systems. AIChE Journal. 1995. Vol. 41. P. 619–636.

[3] Hemami H., Wyman B.F. Modeling and Control of Constrained Dynamic Systems with Application to Biped Locomotion in the Frontal Plane. IEEE Trans. Automat. Contr. 1979. Vol. 24. P. 526–555.

[4] Mills J.K., Goldenberg A.A. Force and Position Control of Manipulators During Constrained Motion Tasks. IEEE Trans. Robot. Automat. 1989. V. 5. P. 30–46.

[5] Shestakov A.L., Sviridyuk G.A. A New Approach to Measuring Dynamically Perturbed Signals. Bulletin of the South Ural State University, Series: Mathematical Modelling, Programming and Computer Software. 2010. № 16(192), Vol. 5. C. 116–120.

[6] Shestakov A.L., Sviridyuk G.A., Keller A.V., Zamyshlyaeva A.A., Khudyakov Y.V. Numerical Investigation of Optimal Dynamic Measurements. Acta IMEKO. 2018. Vol. 7(2). P. 65–72.

[7] Sviridyuk G.A., Zagrebina S.A. Verigin's Problem for Linear Equations of the Sobolev Type with Relatively p-Sectorial Operators. 2002. Vol. 38, no. 12. P. 1745–1752.

[8] Keller A.V., Ebel A.A. The Existence of a Unique Solution to a Mixed Control Problem for Sobolev-Type Equations. Bulletin of the South Ural State University, Series: Mathematical Modelling, Programming and Computer Software. 2014. Vol. 7, no. 3. P. 121–127.

[9] Sviridyuk G.A., Zamyshlyaeva A.A., Zagrebina S.A. Multipoint Initial-Final Problem for One Class of Sobolev Type Models of Higher Order with Additive White Noise. Bulletin of the South Ural State University, Series: Mathematical Modelling, Programming and Computer Software. 2018. Vol.11, no.3, P.103–117.

[10] Sviridyuk, G.A., Kazak, V.O. The Phase Space of an Initial-Boundary Value Problem for the Hoff equation. Mathematical Notes. 2002. Vol. 71, no. 1–2. P. 262–266.

[11] Manakova N.A. Method of Decomposition in the Optimal Control Problem for Semilinear Sobolev Type Models. Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software, 2015. Vol. 8, no. 2, P. 133–137.