Note on the nonexpansive operators based on arbitrary variable metric\(^*\)

Feng Xue\(\dagger\)

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Abstract

In this note, we study the nonexpansive properties equipped with arbitrary variable metric and explore the connections between firm nonexpansiveness, cocoerciveness and averagedness. A convergence rate analysis for the associated fixed-point iterations is presented by developing the global ergodic and non-ergodic iteration-complexity bounds in terms of metric distances. The obtained results are finally exemplified with the metric resolvent, which provides a unified framework for many existing first-order operator splitting algorithms.

Keywords Nonexpansiveness, \(\alpha\)-averaged, variable metric, convergence rates, metric resolvent

AMS subject classifications 68Q25, 47H05, 90C25, 47H09

1 Introduction

1.1 Nonexpansive operators

The nonexpansive mappings were extensively studied in some early works, e.g. [2, 9, 21], and the generalizations have recently been discussed in [8, 31, 20]. Refer to [4] for the comprehensive treatment of the nonexpansive mappings.

The notion of nonexpansiveness arises primarily in connection with the study of fixed-point theory, and underlies the convergence analysis of various fixed-point iterations. Nowadays, there has been a revived interest in the design and analysis of the first-order operator splitting methods [33], of which many algorithms can be interpreted by the nonexpansive mappings, e.g. proximal forward-backward splitting algorithms [1, 16, 23], Douglas-Rachford splitting [18], primal-dual splitting methods [34, 6]. An overview of the operator splitting algorithms from the perspective of nonexpansive mappings is given by [24], which reinterprets a variety of algorithms by a simple Krasnosel’skii-Mann iteration built from a nonexpansive operator.

Recently, the nonexpansive mappings have been extended to arbitrary variable metric in various contexts, e.g. generalized proximity operator [23], Bregman-based proximal operator [13, 26], generalized resolvent [32], which lay the foundation for analyzing proximal mapping [14], variable metric proximal point method [7, 28], Bregman-based proximal schemes [33], variable metric Fejér sequence [15]. This trend necessitates a systematic study of the metric-based nonexpansive mappings.

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\(\dagger\)National Key Laboratory of Science and Technology on Test Physics and Numerical Mathematics, Beijing, 100076, China (E-mail: fxue@link.cuhk.edu.hk)
1.2 Contributions

In this note, we study the nonexpansive properties (e.g. firm nonexpansiveness, cocoerciveness and averagedness) of the nonexpansive mapping in the context of arbitrary variable metric $Q$. In particular, the $\alpha$-averagedness is generalized to $Q$-based $\xi\text{-}Lipschitz \alpha\text{-}averaged$. Then, for the fixed-point Banach-Picard and Krasnosel’skii-Mann iterations associated with the nonexpansive operator, we establish the global pointwise/nonergodic and ergodic convergence rates in terms of the $Q$-based solution distance and sequential error.

A prominent example of the $Q$-based nonexpansive operator is the metric resolvent. The convergence property of the associated fixed-point iterations, being equivalent to the (relaxed) variable metric proximal point algorithm, can be easily obtained from the above results.

1.3 Notations

We use standard notations and concepts from convex analysis and variational analysis, which, unless otherwise specified, can all be found in the classical and recent monographs [29, 30, 4, 5].

A few more words about our notations are in order. The classes of positive semi-definite (PSD) and positive definite (PD) matrices are denoted by $\mathcal{M}^+$ and $\mathcal{M}^{++}$, respectively. The classes of symmetric, symmetric and PSD, symmetric and PD matrices are denoted by $\mathcal{M}_S$, $\mathcal{M}^+_S$, and $\mathcal{M}^{++}_S$, respectively. For our specific use, the $Q$-based inner product (where $Q$ is an arbitrary square matrix) is defined as:

$$\langle a | b \rangle_Q := \langle Qa | b \rangle = \langle a | Q^\top b \rangle,$$

for all $(a, b) \in \mathcal{H} \times \mathcal{H}$; the $Q$-norm is defined as: $\|a\|_Q^2 := \langle Qa | a \rangle$, $\forall a \in \mathcal{H}$. Note that unlike the conventional treatment in the literature, $Q$ is not assumed to be symmetric and PSD here, and hence, $\| \cdot \|_Q$ is not always well-defined.

2 The nonexpansiveness based on arbitrary variable metric

2.1 Definitions based on arbitrary variable metric

Inspired by the work of [11], the following definitions extend the classical notions of resolvent [4, Definition 23.1, Corollary 23.10], Lipschitz continuity [4, Definition 1.46], nonexpansiveness [4, Definition 4.1], cocoerciveness [4, Definition 4.4] and averagedness [4, Definition 4.23] to arbitrary (not necessarily symmetric and PSD) metric $Q$.

**Definition 2.1.** Let $Q$ be arbitrary variable metric, then, the operator $T : \mathcal{H} \mapsto \mathcal{H}$ is ($\forall (b_1, b_2) \in \mathcal{H} \times \mathcal{H}$):

(i) $Q$-partly nonexpansive, if:

$$\langle b_1 - b_2 | Tb_1 - Tb_2 \rangle_Q \geq \|Tb_1 - Tb_2\|_Q^2,$$

(ii) $Q$-nonexpansive, if:

$$\|Tb_1 - Tb_2\|_Q^2 \leq \|b_1 - b_2\|_Q^2$$

(iii) $Q$-based $\xi$-Lipschitz continuous, if:

$$\|Tb_1 - Tb_2\|_Q^2 \leq \xi^2\|b_1 - b_2\|_Q^2$$
(iv) \( Q \)-firmly nonexpansive, if:

\[
\|T b_1 - T b_2\|_Q^2 + \|(I - T)b_1 - (I - T)b_2\|_Q^2 \leq \|b_1 - b_2\|_Q^2
\]

(v) \( Q \)-based \( \beta \)-cocoercive, if \( \beta T \) is \( Q \)-partly nonexpansive:

\[
\langle b_1 - b_2 | T b_1 - T b_2 \rangle_Q \geq \beta \| T b_1 - T b_2 \|_Q^2
\]

(vi) \( Q \)-based \( \alpha \)-averaged with \( \alpha \in ]0, 1[ \), if there exists a \( Q \)-nonexpansive operator \( K : \mathcal{H} \mapsto \mathcal{H} \), such that \( T = (1 - \alpha)I + \alpha K \).

\[\text{Remark 1. The notions of partly nonexpansive and } \beta \text{-cocoercive are based on arbitrary metric } Q, \text{ not limited to symmetric and PSD case. Definition 2.1–(ii) and (iii) use } \| \cdot \|_Q^2 \text{ rather than } \| \cdot \|_Q, \text{ because } \| \cdot \|_Q^2 \text{ is always well defined for arbitrary } Q, \text{ even if } Q \text{ is not PSD.}\]

The \( Q \)-based \( \alpha \)-averaged is further generalized as follows.

**Definition 2.2.** An operator \( T : \mathcal{H} \mapsto \mathcal{H} \) is said to be \( Q \)-based \( \xi \)-Lipschitz \( \alpha \)-averaged with \( \xi \in ]0, \infty[ \) and \( \alpha \in ]0, 1[ \), if there exists a \( Q \)-based \( \xi \)-Lipschitz continuous operator \( K : \mathcal{H} \mapsto \mathcal{H} \), such that \( T = (1 - \alpha)I + \alpha K \). In particular, if \( \xi \in ]1, \infty[ \), \( T \) is \( Q \)-weakly averaged; if \( \xi \in ]0, 1[ \), \( T \) is \( Q \)-strongly averaged.

To lighten the notation, we denote a family of \( Q \)-based \( \xi \)-Lipschitz \( \alpha \)-averaged operators by \( F_{\xi,\alpha}^Q \), and thus, an operator \( T \) belonging to this family is simply expressed by \( T \in F_{\xi,\alpha}^Q \). Obviously, Definition 2.1–(vi) is a special case of Definition 2.2 with \( \xi = 1 \), and thus, can be denoted by \( T \in F_{1,\alpha}^Q \).

### 2.2 Nonexpansiveness of operator

This section presents the nonexpansive properties in the context of arbitrary various metric \( Q \). First, Definition 2.1 is connected via the following results.

**Lemma 2.3.** \( T : \mathcal{H} \mapsto \mathcal{H} \) is \( Q \)-partly nonexpansive,

(i) if and only if \( T \) is \( Q \)-based 1-cocoercive;

(ii) if \( T \) is \( Q \)-based \( \beta \)-cocoercive with \( Q \in \mathcal{M}^+ \) and \( \beta \in ]1, \infty[ \);

(iii) if and only if \( T \) is \( Q \)-firmly nonexpansive with \( Q \in \mathcal{M}_S \).

**Proof.** Definition 2.1 and the conditions of \( Q \). \( \square \)

**Lemma 2.4.** \( T : \mathcal{H} \mapsto \mathcal{H} \) is \( Q \)-nonexpansive,

(i) if and only if \( T \) is \( Q \)-based 1-Lipschitz continuous;

(ii) if \( T \) is \( Q \)-based \( \xi \)-Lipschitz continuous with \( Q \in \mathcal{M}^+ \) and \( \xi \in ]0, 1[ \);

(iii) if \( T \) is \( Q \)-firmly nonexpansive with \( Q \in \mathcal{M}^+ \).

**Proof.** Definition 2.1 and the conditions of \( Q \). \( \square \)

\[\text{[4, Corollary 2.14] is also valid for arbitrary (not necessarily symmetric and PSD) metric } Q, \text{ as stated below.}\]
Lemma 2.5. The following identity holds for any $\kappa \in \mathbb{R}$ and arbitrary $Q$:
\[ \| \kappa b_1 + (1 - \kappa) b_2 \|_Q^2 = \kappa \| b_1 \|_Q^2 + (1 - \kappa) \| b_2 \|_Q^2 - \kappa(1 - \kappa) \| b_1 - b_2 \|_Q^2 \]

Proof. Noting that $\langle b_1 | b_2 \rangle_{Q^\top} = \| b_1 \|_Q^2 + \| b_2 \|_Q^2 - \| b_1 - b_2 \|_Q^2$, we have:
\[
\begin{align*}
\| \kappa b_1 + (1 - \kappa) b_2 \|_Q^2 &= \kappa^2 \| b_1 \|_Q^2 + (1 - \kappa)^2 \| b_2 \|_Q^2 + \kappa(1 - \kappa) \langle b_1 | b_2 \rangle_{Q^\top} \\
&= \kappa^2 \| b_1 \|_Q^2 + (1 - \kappa)^2 \| b_2 \|_Q^2 + \kappa(1 - \kappa) (\| b_1 \|_Q^2 + \| b_2 \|_Q^2 - \| b_1 - b_2 \|_Q^2) \\
&= \kappa \| b_1 \|_Q^2 + (1 - \kappa) \| b_2 \|_Q^2 - \kappa(1 - \kappa) \| b_1 - b_2 \|_Q^2
\end{align*}
\]
which completes the proof. \qed

Lemma 2.6 is an extended version of [4, Proposition 4.2] for the case of arbitrary metric $Q$, which shows the equivalence between partly nonexpansive and firmly nonexpansive, in case of symmetric $Q$.

Lemma 2.6. Let $T : H \mapsto H$, then, the following are equivalent:
(i) $T$ is $Q$–partly nonexpansive;
(ii) $T$ is $Q$–firmly nonexpansive with $Q \in M_S$;
(iii) $I - T$ is $Q$–firmly nonexpansive with $Q \in M_S$;
(iv) $2T - I$ is $Q$–nonexpansive with $Q \in M_S$;
(v) $I - T$ is $Q^\top$–partly nonexpansive.

Proof. (i)$\leftrightarrow$(ii)$\leftrightarrow$(iii)$\leftrightarrow$(iv): Definition 2.1, [4, Proposition 4.2] and Lemma 2.5.
(i)$\leftrightarrow$(v): By Definition 2.1–(i), we have:
\[ \langle (I - T)b_1 - (I - T)b_2 | Tb_1 - Tb_2 \rangle_Q \geq 0; \ \forall (b_1, b_2) \in H \times H \]
Adding $\| (I - T)b_1 - (I - T)b_2 \|_Q^2$ on both sides, we obtain:
\[ \langle (I - T)b_1 - (I - T)b_2 | b_1 - b_2 \rangle_Q \geq \| (I - T)b_1 - (I - T)b_2 \|_Q^2 \]
which leads to (v), by the definitions of $\langle \cdot | \cdot \rangle_Q$ and $\| \cdot \|_Q$, specified in Section 1.3. \qed

The following results extend [4, Proposition 4.25, Remark 4.24, Remark 4.27, Proposition 4.33, Proposition 4.28] to arbitrary metric $Q$, which build the connections of $Q$–based 1–Lipschitz $\alpha$–averagedness (i.e. $F^Q_{1,\alpha}$) to other concepts.

Lemma 2.7. Let $T : H \mapsto H$, then, the following hold.
(i) $T \in F^Q_{1,\alpha}$ with $\alpha \in ]0, 1[$, if and only if $\forall (b_1, b_2) \in H \times H$:
\[ \| Tb_1 - Tb_2 \|_Q^2 + \frac{1 - \alpha}{\alpha} \| (I - T)b_1 - (I - T)b_2 \|_Q^2 \leq \| b_1 - b_2 \|_Q^2 \]
(ii) $T$ is $Q$–firmly nonexpansive, if and only if $T \in F^{\infty}_{1,\frac{1}{2}}$.
(iii) If $T \in F^Q_{1,\alpha}$ with $Q \in M^+$ and $\alpha \in ]0, \frac{1}{2}]$, then $T$ is $Q$–firmly nonexpansive.
(iv) Let \( \alpha \in ]0,1[, \gamma \in ]0, \frac{1}{\alpha}[, \) then, \( T \in F_{\alpha,1}^Q, \) if and only if \( (1 - \gamma)I + \gamma T \in F_{1,\gamma}^Q. \)

(v) \( T \) is \( Q \)-based \( \beta \)-cocoercive with \( Q \in M_S, \) if and only if \( \beta T \in F_{1,\frac{1}{\beta}}^Q. \)

(vi) Let \( T \) be \( Q \)-based \( \beta \)-cocoercive with \( Q \in M_S. \) If \( \gamma \in ]0,2\beta[ \), then, \( I - \gamma T \in F_{1,\frac{1}{2\beta}}^Q. \)

**Proof.** (i) [4, Proposition 4.25]—(iii) and Lemma 2.5;
(ii) [4, Remark 4.24]—(iii);
(iii) [4, Remark 4.27];
(iv) [4, Proposition 4.28].
(v) Lemma 2.7—(ii), Lemma 2.3—(iii) and Definition 2.1—(v).
(vi) Lemma 2.7—(v) and [4, Proposition 4.33].

The following theorem, as a main result of this note, collects the key results of \( F_{\xi,\alpha}^Q. \)

**Theorem 2.8.** Let \( T \in F_{\xi,\alpha}^Q \) with \( \xi \in ]0, +\infty[ \) and \( \alpha \in ]0,1[. \) Then, the following hold.
(i) \( T \) satisfies \( \forall (b_1, b_2) \in H \times H \):

\[
\|Tb_1 - Tb_2\|_Q^2 \leq (1 - \alpha + \alpha\xi^2)\|b_1 - b_2\|_Q^2 - \frac{1 - \alpha}{\alpha}\|(I - T)b_1 - (I - T)b_2\|_Q^2
\]

(ii) If \( Q \in M^+, \) \( 0 < \xi \leq \min\{\frac{1 - \alpha}{\alpha}, 1\}, \) then \( T \) is \( Q \)-firmly nonexpansive.
(iii) If \( Q \in M_S^+, \xi \in ]0, \frac{1 - \alpha}{\alpha}[ \) then \( T \) is \( Q \)-based \( \beta \)-cocoercive, with \( \beta = \frac{1}{2}(1 + \frac{1}{1 - \alpha + \alpha\xi^2}). \)
(iv) \( I - \gamma T \in F_{\xi,\alpha,\gamma(1-\alpha)}^Q, \) if \( \gamma \in ]0, \frac{1 - \alpha}{\alpha}]. \)
(v) If \( Q \in M^+, \gamma \in ]0, \frac{1 - \alpha}{\alpha}[ \), \( \xi \leq \min\{\frac{1 - \alpha}{\alpha}, \frac{1 - \alpha - 1 - \alpha}{\alpha}\}, \) then \( I - \gamma T \) is \( Q \)-firmly nonexpansive.
(vi) If \( Q \in M_S^+, \gamma \in ]0, \frac{1 - \alpha}{\alpha}[ \), \( \xi \in ]0, \frac{1 - \alpha}{\alpha} - \frac{1 - \alpha - 1 - \alpha}{\alpha}\), then \( I - \gamma T \) is \( Q \)-based \( \beta \)-cocoercive with \( \beta = \frac{1}{2}(1 + \frac{1 - \alpha}{1 - \alpha - \gamma(1-\alpha)^2+\alpha\xi^2}). \)
(vii) The reflected operator of \( T \) follows \( 2T - I \in F_{\xi,2\alpha}^Q, \) if \( \alpha \in ]0, \frac{1}{2}[. \)

**Proof.** (i) By Definition 2.2, there exists a \( Q \)-based \( \xi \)-Lipschitz continuous operator \( K : H \mapsto H, \) such that \( T = (1 - \alpha)I + \alpha K, \) and thus, \( K = \frac{1}{\alpha}T + (1 - \frac{1}{\alpha})I. \) By Lemma 2.5, we have:

\[
\|Kb_1 - Kb_2\|_Q^2 = (1 - \frac{1}{\alpha})\|b_1 - b_2\|_Q^2 + \frac{1}{\alpha}\|Tb_1 - Tb_2\|_Q^2 + \frac{1 - \alpha}{\alpha^2}\|(I - T)b_1 - (I - T)b_2\|_Q^2
\]

which yields the desired inequality, after simple rearrangements.

(ii) If \( Q \in M^+, \) to ensure that \( T \) is \( Q \)-firmly nonexpansive, we need to let \( 1 - \alpha + \alpha\xi^2 \leq 1 \) and \( \frac{1 - \alpha}{\alpha} \geq 1, \) by (1) and Definition 2.1—(iv). It yields \( \xi \in ]0,1[ \) and \( \alpha \in ]0, \frac{1}{2}[, \)

On the other hand, rewrite (1) as:

\[
\frac{\alpha}{1 - \alpha}\|Tb_1 - Tb_2\|_Q^2 \leq \frac{\alpha}{1 - \alpha}(1 - \alpha + \alpha\xi^2)\|b_1 - b_2\|_Q^2 - \|(I - T)b_1 - (I - T)b_2\|_Q^2
\]

The firm nonexpansiveness of \( T \) requires \( \frac{\alpha}{1 - \alpha} \geq 1 \) and \( \frac{\alpha}{1 - \alpha}(1 - \alpha + \alpha\xi^2) \leq 1, \) i.e. \( \xi \in ]0, \frac{1}{1 - \alpha}[ \) and \( \alpha \in ]\frac{1}{2}, 1[. \) Finally, combining both conditions yields \( 0 < \xi \leq \min\{\frac{1 - \alpha}{\alpha}, 1\}. \)
(iii) If $Q \in \mathcal{M}_S$, expanding $\| (I - T) b_1 - (I - T) b_2 \|_Q^2$, (1) is equivalent to:

$$
\frac{2(1 - \alpha)}{\alpha} \langle b_1 - b_2 | T b_1 - T b_2 \rangle_Q \geq \frac{1}{1 - \alpha} \| T b_1 - T b_2 \|_Q^2 - (2 - \alpha - \frac{1}{\alpha} + \alpha \xi^2) \| b_1 - b_2 \|_Q^2
$$

which yields:

$$
\langle b_1 - b_2 | T b_1 - T b_2 \rangle_Q \geq \frac{1}{2(1 - \alpha)} \| T b_1 - T b_2 \|_Q^2
$$

(3)

if $Q \in \mathcal{M}^+$ and $2 - \alpha - \frac{1}{\alpha} + \alpha \xi^2 \leq 0$, i.e. $\xi \leq \frac{1 - \alpha}{\alpha}$.

On the other hand, if $Q \in \mathcal{M}^+_S$, (2) becomes:

$$
\frac{\alpha}{1 - \alpha} \| T b_1 - T b_2 \|_Q^2 \\
\leq \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1 - \alpha} \| b_1 - b_2 \|_Q^2 - \| (I - T) b_1 - (I - T) b_2 \|_Q^2 \\
= \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1 - \alpha} \| (I - T) b_1 - (I - T) b_2 + T b_1 - T b_2 \|_Q^2 - \| (I - T) b_1 - (I - T) b_2 \|_Q^2 \\
= \left( \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1 - \alpha} - 1 \right) \| (I - T) b_1 - (I - T) b_2 \|_Q^2 - \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1 - \alpha} \| T b_1 - T b_2 \|_Q^2 \\
+ 2 \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1 - \alpha} \langle b_1 - b_2, T b_1 - T b_2 \rangle_Q
$$

If $\frac{\alpha(1 - \alpha + \alpha \xi^2)}{1 - \alpha} \leq 1$, i.e. $\xi \leq \frac{1 - \alpha}{\alpha}$, it yields:

$$
\frac{2 \alpha(1 - \alpha + \alpha \xi^2)}{1 - \alpha} \langle b_1 - b_2, T b_1 - T b_2 \rangle_Q \geq \left( \frac{\alpha}{1 - \alpha} + \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1 - \alpha} \right) \| T b_1 - T b_2 \|_Q^2
$$

i.e.

$$
\langle b_1 - b_2, T b_1 - T b_2 \rangle_Q \geq \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha + \alpha \xi^2} \right) \| T b_1 - T b_2 \|_Q^2
$$

(4)

Finally, (iii) follows by comparing (3) with (4), and noting that: $\frac{1}{1 - \alpha} \leq 1 + \frac{1}{1 - \alpha + \alpha \xi^2}$, if $\xi \leq \frac{1 - \alpha}{\alpha}$.

(iv) Expanding $\| (I - T) b_1 - (I - T) b_2 \|_Q^2$, (1) is equivalent to:

$$
\langle b_1 - b_2 | T b_1 - T b_2 \rangle_Q \geq \frac{1}{1 - \alpha} \| T b_1 - T b_2 \|_Q^2 - \frac{\alpha^2 \xi^2 - (1 - \alpha)^2}{1 - \alpha} \| b_1 - b_2 \|_Q^2
$$

Then, we have:

$$
\| (I - \gamma T) b_1 - (I - \gamma T) b_2 \|_Q^2 \\
= \| b_1 - b_2 \|_Q^2 - \gamma \langle b_1 - b_2 | T b_1 - T b_2 \rangle_Q \| T b_1 - T b_2 \|_Q^2 + \| T b_1 - T b_2 \|_Q^2 \\
\leq \| b_1 - b_2 \|_Q^2 - \gamma \frac{1}{1 - \alpha} \| T b_1 - T b_2 \|_Q^2 + \gamma \frac{\alpha^2 \xi^2 - (1 - \alpha)^2}{1 - \alpha} \| b_1 - b_2 \|_Q^2 + \| T b_1 - T b_2 \|_Q^2 \\
= \left( 1 + \gamma \frac{\alpha^2 \xi^2 - (1 - \alpha)^2}{1 - \alpha} \right) \| b_1 - b_2 \|_Q^2 - \left( \frac{1}{\gamma(1 - \alpha)} - 1 \right) \| T b_1 - T b_2 \|_Q^2
$$

Let $I - \gamma T \in \mathcal{Q}_{\xi', \alpha'}$, then, by (1), we have $\frac{1 - \alpha'}{\alpha'} = \frac{1}{\gamma(1 - \alpha)} - 1$ and $1 - \alpha' + \alpha' \xi'^2 = 1 + \gamma \frac{\alpha^2 \xi^2 - (1 - \alpha)^2}{1 - \alpha}$, which yields $\alpha' = \gamma(1 - \alpha)$ and $\xi' = \frac{\alpha \xi}{1 - \alpha}$.  

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(v) Theorem 2.8–(ii) and (iv).
(vi) Theorem 2.8–(iii) and (iv).
(vii) We deduce that:

\[
\begin{align*}
\| (2T-I)b_1 - (2T-I)b_2 \|^2_Q &= 2\| Tb_1 - Tb_2 \|^2_Q - \| b_1 - b_2 \|^2_Q + 2\| (I-T)b_1 - (I-T)b_2 \|^2_Q \\
&\leq 2(1 - \alpha + \alpha \xi^2)\| b_1 - b_2 \|^2_Q - \frac{2(1 - \alpha)}{\alpha}\| (I-T)b_1 - (I-T)b_2 \|^2_Q \\
&= (1 - 2\alpha + 2\alpha \xi^2)\| b_1 - b_2 \|^2_Q - \frac{1 - 2\alpha}{2\alpha}\| (2I - 2T)b_1 - (2I - 2T)b_2 \|^2_Q
\end{align*}
\]

Let $2T - I \in F_{\xi,\alpha}'$. Thus, we have $\frac{1 - \alpha'}{\alpha'} = \frac{1 - 2\alpha}{2\alpha}$ and $1 - \alpha' + \alpha' \xi^2 = 1 - 2\alpha + 2\alpha \xi^2$, i.e. $\alpha' = 2\alpha$, and $\xi' = \xi$. \hfill $\square$

**Corollary 2.9.** [Further results of Theorem 2.8–(iii)] Let $T \in F_{\xi,\alpha}'$ with $\xi \in [0, +\infty]$ and $\alpha \in [0, 1]$, then, the following hold.

(i) If $\xi \leq \min\{\frac{1 - \alpha}{\alpha}, 1\}$, then, $T$ is $Q$-based $\beta$-cocoercive with $\beta \in [1, +\infty]$, strongly $\alpha$-averaged, and $Q$-firmly nonexpansive.

(ii) If $\alpha \in [0, \frac{1}{2}], \xi \in [1, \frac{1 - \alpha}{\alpha}]$, then, $T$ is $Q$-based $\beta$-cocoercive with $\beta \in [0, 1]$, and weakly $\alpha$-averaged.

**Proof.** By Theorem 2.8–(iii), $T$ is $\beta$-cocoercive, with $\beta = \frac{1}{2}\left(1 + \frac{1}{1 - \alpha + \alpha \xi^2}\right)$. The proof is completed by comparing $\beta$ with 1. \hfill $\square$

**Corollary 2.10.** [Further results of Theorem 2.8–(vi)] Let $T \in F_{\xi,\alpha}'$ with $\xi \in [0, +\infty]$ and $\alpha \in [0, 1]$. If $\gamma \in [0, \frac{1}{1 - \alpha}]$, then, the following hold.

(i) If $\xi \leq \min\{\frac{1 - \alpha}{\alpha}, \frac{1 - \alpha}{1 - \alpha}, 1 - \frac{1 - \alpha}{\alpha}\}$, then, $I - \gamma T$ is $Q$-based $\beta$-cocoercive with $\beta \in [1, +\infty]$, strongly $\alpha$-averaged, and $Q$-firmly nonexpansive.

(ii) If $\xi \in [0, 1 - \frac{1 - \alpha}{\alpha}, 1 - \frac{1 - \alpha}{\alpha}, \alpha^{-1} - \frac{1 - \alpha}{\alpha}]$, then, $I - \gamma T$ is $Q$-based $\beta$-cocoercive with $\beta \in [0, 1]$, and weakly $\alpha$-averaged.

**Proof.** The proof is completed by comparing $\beta$ in Theorem 2.8–(vi) with 1. \hfill $\square$

Part of the results in Theorem 2.8, Corollary 2.9 and Corollary 2.10 is summarized in Fig.1, where FNE stands for ‘$Q$-firmly nonexpansive’. From Fig.1, we can see that $T \in F_{\xi,\alpha}'$ could be $Q$-firmly nonexpansive for $\alpha > 1/2$, at the expense of stricter condition on the Lipschitz constant $\xi \leq \frac{1 - \alpha}{\alpha} < 1$. 

\[7\]
Lemma 2.11. Let the operator $\mathcal{T} : \mathcal{H} \mapsto \mathcal{H}$ be $Q$-based $\beta$-cocoercive with $Q \in \mathcal{M}_S$ and $\beta \in \left[\frac{1}{2}, +\infty\right]$. Then, the following hold.

(i) If $\gamma \in [0, 2\beta]$, then, $\mathcal{T} \in \mathcal{F}_Q^{\beta} \cap \left[\frac{1}{2}, 1 - \frac{1}{2\beta}\right]$, $I - \gamma \mathcal{T} \in \mathcal{F}_Q^{\beta}$. 

(ii) If $Q \in \mathcal{M}_S$, $\gamma \in [0, \beta]$, $I - \gamma \mathcal{T}$ is $Q$-based 1-cocoercive (i.e. $Q$-partly nonexpansive).

Proof. (i) If $Q \in \mathcal{M}_S$, we have:

\[
\begin{align*}
\| (I - \mathcal{T})b_1 - (I - \mathcal{T})b_2 \|^2_Q &= \| b_1 - b_2 \|^2_Q - 2\langle b_1 - b_2, T(b_1 - T(b_2) \rangle_Q + \| T(b_1 - T(b_2) \|^2_Q \\
&\leq \| b_1 - b_2 \|^2_Q - 2\beta \| T(b_1 - T(b_2) \|^2_Q + \| T(b_1 - T(b_2) \|^2_Q \\
&= \| b_1 - b_2 \|^2_Q - (2\beta - 1) \| T(b_1 - T(b_2) \|^2_Q
\end{align*}
\]

which yields:

\[\| T(b_1 - T(b_2) \|^2_Q \leq \frac{1}{2\beta - 1} \| b_1 - b_2 \|^2_Q - \frac{1}{2\beta - 1} \| (I - \mathcal{T})b_1 - (I - \mathcal{T})b_2 \|^2_Q \quad (5)\]

Thus, if $\mathcal{T} \in \mathcal{F}_Q^{\beta_{\alpha', \xi'}}$, by (1), we have: $\frac{1 - \alpha'}{\alpha} = \frac{1}{2\beta - 1}$ and $1 - \alpha' + \alpha'\xi'^2 = \frac{1}{2\beta - 1}$, i.e. $\alpha' = 1 - \frac{1}{2\beta}$ and $\xi' = \frac{1}{2\beta - 1}$.

$I - \gamma \mathcal{T}$ follows from Theorem 2.8–(iv).

(ii) Theorem 2.8–(iii) and Lemma 2.11–(i). \qed

3 The associated fixed-point iterations

Given the $Q$-based nonexpansiveness of $\mathcal{T}$, we are going to show the convergence properties of the fixed-point iteration in terms of solution distance and sequential error. Here, we define the solution as $b^* \in \text{Fix} \mathcal{T}$, and the ($Q$-based) solution distance of the $k$-th iterate is defined by $\| b^k - b^* \|_Q$. The ($Q$-based) sequential error is defined by $\| b^{k+1} - b^k \|_Q$.

We need the following definition, which is an extended version of asymptotically regular [22, 3].
Definition 3.1. A mapping $T: \mathcal{H} \mapsto \mathcal{H}$ is $Q$–asymptotically regular, if $\|T^k b - T^{k+1} b\|_Q \to 0$, as $k \to \infty$, $\forall b \in \mathcal{H}$. Here, $T^k$ is defined as: $T^k := T \circ \cdots \circ T$, $k$ times.

Clearly, if $T$ is $Q$–asymptotically regular, the sequential error of the fixed point iteration tends to 0, as $k \to \infty$.

We also need the well-known Opial’s lemma [27] as a basic tool for convergence analysis. Also see [4, Lemma 2.39] for the proof.

Lemma 3.2. [Opial’s lemma [27]] Let $\{b^k\}_{k \in \mathbb{N}}$ be a sequence in $\mathcal{H}$, and let $C$ be a nonempty set $C \subset \mathcal{H}$. Suppose that:

(a) for every $b^* \in C$, $\{\|b^k - b^*\|\}_{k \in \mathbb{N}}$ converges, i.e. $\lim_{k \to \infty} \|b^k - b^*\|$ exists;
(b) every sequential cluster point of $\{b^k\}_{k \in \mathbb{N}}$ belongs to $C$.

Then, $\{b^k\}_{k \in \mathbb{N}}$ converges to a point in $C$.

3.1 Banach–Picard iteration

The scheme is given as:

$$b^{k+1} := Tb^k$$

(6)

where $T \in \mathcal{F}^Q_{\xi, \nu}$ with $Q \in \mathcal{M}^+_5$, $\alpha \in [0,1]$ and $\xi \in [0, +\infty]$. The convergence properties of (6) are presented as follows.

Theorem 3.3 (Convergence in terms of $Q$–based distance). Let $b^0 \in \mathcal{H}$, $\{b^k\}_{k \in \mathbb{N}}$ be a sequence generated by (6). Denote $\nu := 1 - \alpha + \alpha \xi^2$. If $\xi \in [0, 1]$, then, the following hold.

(i) $T$ is $Q$–asymptotically regular.
(ii) [Basic convergence] There exists $b^* \in \text{Fix} T$, such that $b^k \to b^*$, as $k \to \infty$.
(iii) [Sequential error] $\|b^{k+1} - b^k\|_Q$ has the pointwise sublinear convergence rate of $O(1/\sqrt{k})$:

$$\|b^{k+1} - b^k\|_Q \leq \frac{1}{\sqrt{k + 1}} \sqrt{\frac{\alpha}{1 - \alpha}} \|b^0 - b^*\|_Q, \forall k \in \mathbb{N}$$

(iv) [$q$–linear convergence] If $\xi \in [0, 1]$, both $\|b^k - b^*\|_Q$ and $\|b^k - b^{k+1}\|_Q$ are $q$–linearly convergent with the rate of $\sqrt{\nu}$.

(v) [$r$–linear convergence] If $\alpha \in \left[\frac{3 - \sqrt{5}}{2}, 1\right]$, $\xi \in \left[0, \sqrt{1 - \frac{3 - \sqrt{5}}{2\alpha}}\right]$, $\|b^k - b^{k+1}\|_Q$ is globally $r$–linearly convergent w.r.t. $\|b^0 - b^*\|_Q$:

$$\|b^k - b^{k+1}\|_Q \leq \sqrt{\frac{\alpha(1 - \nu)}{(1 - \alpha)\nu}} \cdot \nu^{k+1} \|b^0 - b^*\|_Q$$

The above inequality is also locally satisfied, for $k \geq \frac{\ln((1 + \sqrt{5})/2) - 1}{\ln(1/\sqrt{\nu})}$, if one of the following conditions hold: (1) $\alpha \in \left[\frac{3 - \sqrt{5}}{2}, 1\right]$, $\xi \in \left[\sqrt{1 - \frac{3 - \sqrt{5}}{2\alpha}}, 1\right]$; (2) $\alpha \in \left[0, \frac{3 - \sqrt{5}}{2}\right]$, $\xi \in [0, 1]$. 

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Proof. (i) Taking $b_1 = b^k$, $b_2 = b^* \in \text{Fix} Tr$ in (1), we obtain:

\[
\|b^{k-1} - b^*\|_Q^2 \leq \left(1 - \alpha + \alpha \xi^2\right)\|b^k - b^*\|_Q^2 - \frac{1-\alpha}{\alpha}\|b^k - b^{k+1}\|_Q^2
\]

\[
\leq \|b^k - b^*\|_Q^2 - \frac{1-\alpha}{\alpha}\|b^k - b^{k+1}\|_Q^2 \quad \text{by } \xi \in ]0,1]
\]

(7)

Summing up (7) from $k = 0$ to $K$ yields:

\[
\sum_{k=0}^K \|b^k - b^{k+1}\|_Q^2 \leq \frac{\alpha}{1-\alpha}\|b^0 - b^*\|_Q^2
\]

(8)

Taking $K \to \infty$, we have: $\sum_{k=0}^\infty \|b^k - b^{k+1}\|_Q^2 \leq \frac{\alpha}{1-\alpha}\|b^0 - b^*\|_Q^2 < \infty$. This implies that $b^k - b^{k+1} \to 0$, as $k \to \infty$. Then, (i) follows, by noting that $b^k = T^k b^0$ and $b^{k+1} = T^{k+1} b^0$.

(ii) In view of (7), the condition $Q \in \mathcal{M}_S^+$ guarantees that $\{\|b^k - b^*\|_Q\}_{k \in \mathbb{N}}$ is non-increasing, and bounded from below (always being non-negative), and thus, convergent, i.e. $\lim_{k \to \infty} \|b^k - b^*\|_Q$ exists. Thus, the condition (a) of Opial’s lemma (see Lemma 3.2) is satisfied.

By Theorem 3.3-(i), we have: $b^k \to b^{k+1} = T b^k$, as $k \to \infty$, i.e. the cluster point of $\{b^k\}_{k \in \mathbb{N}}$ belongs to Fix$T$. The condition (b) of Opial’s lemma is satisfied. Finally, the convergence of $\{b^k\}_{k \in \mathbb{N}}$ is established by Lemma 3.2.

(iii) Taking $b_1 = b^k$ and $b_2 = b^{k+1}$ in (1), we have:

\[
\|b^{k+1} - b^{k+2}\|_Q^2 \leq (1 - \alpha + \alpha \xi^2)\|b^k - b^{k+1}\|_Q^2 - \frac{1-\alpha}{\alpha}\|(I - T)b^k - (I - T)b^{k+1}\|_Q^2
\]

\[
\leq \|b^k - b^{k+1}\|_Q^2 \quad \text{by } \xi \in ]0,1] \text{ and } \alpha \in ]0,1[\]

(9)

which implies that $\|b^k - b^{k+1}\|_Q$ is non-increasing. Then, (iii) follows from (8).

(iv) If $\xi \in ]0,1[$, (7) yields that

\[
\|b^{k+1} - b^*\|_Q^2 \leq \nu \|b^k - b^*\|_Q^2
\]

where $\nu \in ]1 - \alpha, 1[$.

Regarding the sequential error, (9) yields that

\[
\|b^{k+1} - b^{k+2}\|_Q^2 \leq \nu \|b^k - b^{k+1}\|_Q^2
\]

(10)

(v) Combining (10) with (8) yields:

\[
\left(\nu^{-k} + \nu^{-(k-1)} + \ldots + 1\right)\|b^k - b^{k+1}\|_Q^2 \leq \frac{\alpha}{1-\alpha}\|b^0 - b^*\|_Q^2
\]

which leads to:

\[
\|b^k - b^{k+1}\|_Q^2 \leq \frac{\alpha(1-\nu)}{(1-\alpha)^2} \frac{1}{\nu^{-(k+1)} - 1}\|b^0 - b^*\|_Q^2
\]

Clearly, if $\frac{1}{\nu^{-(k+1)} - 1} \leq \nu^{\frac{k+1}{\nu}}$, (i.e. $k \geq \frac{\ln(1+\sqrt{\nu})/2}{\ln(1/\nu)} - 1$), $\|b^k - b^{k+1}\|_Q^2$ is $r$-linearly convergent w.r.t. $\|b^0 - b^*\|_Q^2$:

\[
\|b^k - b^{k+1}\|_Q^2 \leq \frac{\alpha(1-\nu)}{(1-\alpha)^2} \nu^{\frac{k+1}{\nu}} \|b^0 - b^*\|_Q^2
\]
Furthermore, if \( \frac{\ln(1+\sqrt{5})}{\ln(1/\nu)} - 1 \leq 1 \), the \( r \)-linear convergence is globally valid for \( \forall k \in \mathbb{N} \). This condition can be simplified as \( \xi^2 \leq 1 - \frac{3 - \sqrt{5}}{2\alpha} \).

The following results build the connection of the convergence properties with the cocoerciveness of an operator.

**Proposition 3.4** (Convergence in terms of \( Q \)-based distance). Let \( b^0 \in H \), \( \{b^k\}_{k \in \mathbb{N}} \) be a sequence generated by (6), with \( T \) being \( Q \)-based \( \beta \)-cocoercive with \( \beta \in [1, +\infty] \). Then, the following hold.

(i) \( T \) is \( Q \)-asymptotically regular.

(ii) [Basic convergence] There exists \( b^* \in \text{Fix} T \), such that \( b^k \to b^* \), as \( k \to \infty \).

(iii) [Sequential error] \( \|b^{k+1} - b^k\|_Q \) has the pointwise sublinear convergence rate of \( O(1/\sqrt{k}) \):

\[
\|b^{k+1} - b^k\|_Q \leq \frac{1}{\sqrt{k+1}} \cdot \sqrt{2\beta - 1} \|b^0 - b^*\|_Q, \forall k \in \mathbb{N}
\]

(iv) [\( q \)-linear convergence] If \( \beta \in [1, +\infty] \), both \( \|b^k - b^*\|_Q \) and \( \|b^k - b^{k-1}\|_Q \) are \( q \)-linearly convergent with the rate of \( \frac{1}{\sqrt{2q^2 - 1}} \).

(v) [\( r \)-linear convergence] If \( \beta \in [\frac{3+\sqrt{5}}{4}, +\infty] \), \( \|b^k - b^{k-1}\|_Q \) is globally \( r \)-linearly convergent w.r.t. \( \|b^0 - b^*\|_Q \):

\[
\|b^k - b^{k-1}\|_Q \leq \sqrt{2(\beta - 1)} \cdot (2\beta - 1)^{-\frac{k-1}{2}} \|b^0 - b^*\|_Q
\]

The above inequality is also locally satisfied, for \( k \geq \frac{\ln(1+\sqrt{5})}{\ln(2\beta - 1)} - 1 \), if \( \beta \in [1, \frac{3+\sqrt{5}}{4}] \).

**Proof.** (i)–(ii): By Lemma 2.11–(i), we have: \( T \in \mathcal{F}^Q_{\beta\text{-based}} \). Taking \( b_1 = b^k \) and \( b_2 = b^* \) in (5), we have:

\[
\|b^{k+1} - b^*\|_Q \leq \frac{1}{2\beta - 1} \|b^k - b^*\|_Q - \frac{1}{2\beta - 1} \|b^k - b^{k+1}\|_Q \\
\leq \|b^k - b^*\|_Q - \frac{1}{2\beta - 1} \|b^k - b^{k+1}\|_Q \quad \text{by } \beta \geq 1
\]

The rest of proof is similar to Theorem 3.3–(i) and (ii).

(iii) Similar Theorem 3.3–(ii).

(iv) If \( \beta > 1 \), (11) yields:

\[
\|b^{k+1} - b^*\|_Q \leq \frac{1}{2\beta - 1} \|b^k - b^*\|_Q
\]

Regarding the sequential error, we have:

\[
\|b^{k+1} - b^k\|_Q \leq \frac{1}{2\beta - 1} \|b^k - b^{k+1}\|_Q
\]

(v) Combining (iv) with \( \sum_{k=0}^{K} \|b^k - b^{k+1}\|_Q \leq (2\beta - 1) \|b^0 - b^*\|_Q \) yields:

\[
(2\beta - 1)^k + (2\beta - 1)^{k-1} + \cdots + 1) \|b^k - b^{k+1}\|_Q \leq (2\beta - 1) \|b^0 - b^*\|_Q
\]
which leads to:
\[
\|b_k - b_{k+1}\|^2 \leq \frac{2(2\beta - 1)(\beta - 1)}{(2\beta - 1)^{k+1} - 1} \cdot \|b^0 - b^*\|^2.
\]

If \( \frac{1}{(2\beta - 1)^{k+1} - 1} \leq (2\beta - 1)^{-k+1} \), (i.e. \( k \geq \frac{\ln(1+\sqrt{5}/2)}{\ln\sqrt{2\beta-1}} - 1 \)), \( \|b_k - b_{k+1}\|^2 \) is \( r \)-linearly convergent w.r.t. \( \|b^0 - b^*\|^2 \):
\[
\|b_k - b_{k+1}\|^2 \leq 2(2\beta - 1) \cdot (2\beta - 1)^{-k+1} \|b^0 - b^*\|^2.
\]

Furthermore, if \( \frac{\ln(1+\sqrt{5}/2)}{\ln\sqrt{2\beta-1}} - 1 \leq 1 \), the \( r \)-linear convergence is globally valid. \( \square \)

Theorem 3.3 is closely linked to Proposition 3.4, if \( T \in F^Q_{\xi,\alpha} \) is also \( \beta \)-cocoercive. This connection can be immediately obtained by Theorem 2.8–(iii). Indeed, if \( \xi \leq \min\{\frac{1-\alpha}{\alpha}, 1\} \), \( T \in F^Q_{\xi,\alpha} \) is \( Q \)-firmly nonexpansive (by Theorem 2.8–(ii)), and also \( \beta \)-cocoercive with \( \beta = \frac{1}{2}(1 + \frac{1}{1-\alpha+\alpha^2}) \geq 1 \) (by Theorem 2.8–(iii)). According to Theorem 3.3, \( \xi \in [0,1] \) is sufficient to guarantee the convergence, while \( T \) is not necessarily \( Q \)-firmly nonexpansive. This implies that the \( Q \)-firm nonexpansiveness of \( T \) is an over–sufficient condition for the convergence of (6).

If \( T \in F^Q_{\xi,\alpha} \) is \( \beta \)-cocoercive, \( \xi \leq \min\{\frac{1-\alpha}{\alpha}, 1\} \) guarantees the convergence (by Proposition 3.4), while \( T \) is also \( Q \)-firmly nonexpansive (by Theorem 2.8–(ii)). In this sense, Proposition 3.4 is somewhat a special case of Theorem 3.3. Note that in Theorem 3.3, the convergence condition \( \xi \in [0,1] \) cannot guarantee the cocoerciveness of \( T \). For instance, when \( \alpha \in [\frac{1}{2},+\infty[ \) and \( \xi \in ]\frac{1-\alpha}{\alpha}, 1[ \), (6) is convergent, but \( T \) is not cocoercive.

### 3.2 Krasnosel’skiĭ–Mann algorithm

Consider the iteration:
\[
b_{k+1} := b_k + \gamma(Tb_k - b^*) := T_{\gamma}b_k
\]
where \( T_{\gamma} = I - \gamma(I - T) \), \( T \in F^Q_{\xi,\alpha} \) with \( Q \in M_{\xi,\alpha}, \alpha \in ]0,1[, \xi \in ]0, +\infty[ \).

**Corollary 3.5** (Convergence in terms of \( Q \)-based distance). Let \( b^0 \in H \), \( \{b^k\}_{k \in \mathbb{N}} \) be a sequence generated by (12). Denote \( \nu := 1 - \gamma\alpha + \gamma\alpha^2 \). If \( \xi \in ]0,1[ \), \( \gamma \in ]0, \alpha[ \), then, the following hold.

(i) \( T_{\gamma} \) is \( Q \)-asymptotically regular.

(ii) [Basic convergence] There exists \( b^* \in \text{Fix}T \), such that \( b_k \to b^* \), as \( k \to \infty \).

(iii) [Sequential error] \( \|b^{k+1} - b^k\|_Q \) has the pointwise sublinear convergence rate of \( O(1/\sqrt{k}) \):
\[
\|b^{k+1} - b^k\|_Q \leq \frac{1}{\sqrt{k+1}} \sqrt{\frac{\gamma\alpha}{1 - \gamma\alpha}} \|b^0 - b^*\|_Q, \forall k \in \mathbb{N}
\]

(iv) [\( q \)-linear convergence] If \( \xi \in ]0,1[ \), both \( \|b^k - b^*\|_Q \) and \( \|b^k - b^{k+1}\|_Q \) are \( q \)-linearly convergent with the rate of \( \sqrt{q} \).

(v) [\( r \)-linear convergence] If \( \gamma \alpha \in ]\frac{3-\sqrt{5}}{2}, 1[ \), \( \xi \in ]0, \sqrt{1 - \frac{3-\sqrt{5}}{2\gamma\alpha}}] \), \( \|b^k - b^{k+1}\|_Q \) is globally
$\nu$-linearly convergent w.r.t. $\|b^0 - b^*\|_Q$:

$$\|b^k - b^{k+1}\|_Q \leq \sqrt{\frac{\gamma \alpha (1 - \nu)}{(1 - \gamma \alpha) \nu}} \cdot \nu^{\frac{k}{1+\nu}} \|b^0 - b^*\|_Q.$$ 

The above inequality is also locally satisfied, for $k \geq \frac{\ln((1 + \sqrt{5})/2)}{\ln(1/\sqrt{\nu})} - 1$, if one of the following conditions hold:

1. $\gamma \alpha \in [\frac{3 - \sqrt{5}}{2}, 1[; \xi \in \sqrt{1 - \frac{3 - \sqrt{5}}{2} \gamma \alpha}, +\infty]$;
2. $\gamma \alpha \in [0, \frac{3 - \sqrt{5}}{2}], \xi \in ]0, 1].$

**Proof.** If $\gamma < \frac{1}{\alpha}$, we obtain by Theorem 2.8–(iv) that:

$$T \in \mathcal{F}_{\xi, \alpha}^Q \Rightarrow R = I - T \in \mathcal{F}_{\frac{1}{2}, 1 - \alpha}^Q \Rightarrow T_\gamma = I - \gamma R \in \mathcal{F}_{\xi, \gamma \alpha}^Q$$

The rest of the proof is very similar to that of Theorem 3.3, just replacing $\alpha$ by $\gamma \alpha$, provided that $\gamma < \frac{1}{\alpha}$.

Note that (ii) follows from the fact that $\text{Fix} T_\gamma = \text{Fix} T$. Indeed, $b^* \in \text{Fix} T_\gamma \iff b^* = b^* - \gamma (b^* - T b^*) \iff b^* = T b^* \iff b^* \in \text{Fix} T$.

### 4 Application to metric resolvent

#### 4.1 Basic properties

Consider the operator:

$$T := (A + Q)^{-1} Q \quad (13)$$

where $A : H \mapsto 2^H$ is a set-valued maximally monotone operator. Rewriting (13) as $T = (I + Q^{-1} \circ A)^{-1}$ and recalling the standard resolvent $\mathcal{J}_A$ [4, Definition 23.1], $T = \mathcal{J}_{Q^{-1} \circ A}$ is a $Q^{-1}$-resolvent of $A$ or called a metric resolvent.

It is easy to show that $T$ is $Q$-partly nonexpansive, $I - T$ is $Q^T$-partly nonexpansive. If $Q \in \mathcal{M}_S$, $T \in \mathcal{F}_{\frac{1}{2}, 1}$, $I - T \in \mathcal{F}_{\frac{1}{2}}$. Furthermore, if $A$ is $\mu$-strongly monotone, if $Q \in \mathcal{M}_S^+$, $T \in \mathcal{F}_{\frac{1}{2}, \|Q\|, 2 \mu + \|Q\|, 2 \mu + \|Q\|}$, $I - T \in \mathcal{F}_{\frac{1}{2}, \|Q\|, \frac{\|Q\|}{\mu + \|Q\|}, \frac{\|Q\|}{\mu + \|Q\|}}$. Then, the convergence properties of the Banach-Picard iteration:

$$b^{k+1} := (A + Q)^{-1} Q b^k \quad (14)$$

immediately follow from Theorem 3.3 and Proposition 3.4, by substituting $\xi$ and $\alpha$ with proper quantities. Note that (14) is equivalent to the monotone inclusion $0 \in A b^{k+1} + Q (b^{k+1} - b^k)$, which is a typical variable metric proximal point algorithm [11, 28, 12, 7].

Considering the Krasnosel’skii-Mann iteration:

$$b^{k+1} := b^k + \gamma (A + Q)^{-1} Q b^k \quad (15)$$

it is easy to show that $T_\gamma \in \mathcal{F}_{\frac{1}{2}, \frac{\|Q\|}{\mu + \|Q\|}, \frac{\gamma (2 \mu + \|Q\|)}{2 \mu + \|Q\|}}$, if $Q \in \mathcal{M}_S^+$. Furthermore, if $A$ is $\mu$-strongly monotone, $T_\gamma \in \mathcal{F}_{\frac{1}{2}, \frac{\|Q\|}{\mu + \|Q\|}, \frac{\gamma (2 \mu + \|Q\|)}{2 \mu + \|Q\|}}$. Then, the convergence properties of (15) follow from Corollary 3.5. Note that (15) is a relaxed version of (14).
4.2 Reinterpretation of primal-dual hybrid gradient algorithm

We take the primal-dual hybrid gradient (PDHG) algorithm as a typical example. Consider the problem:

\[
\min_x f(x) + g(Ax)
\]

where \(x \in \mathbb{R}^N\), \(A : \mathbb{R}^N \mapsto \mathbb{R}^L\), \(f : \mathbb{R}^N \mapsto ] - \infty, +\infty]\) and \(g : \mathbb{R}^L \mapsto ] - \infty, +\infty]\) are proper, lower semi-continuous and convex functions. The PDHG generally first reformulates (16) as (obtained by Legendre-Fenchel transform [30, Chapter 11]):

\[
\min_x \max_s \mathcal{L}(x, s) := f(x) + \langle s, Ax \rangle - g^*(s)
\]

and then performs alternating update (by gradient descent) between primal \(x\) and dual variable \(s\) [17]:

\[
\begin{align*}
\begin{cases}
    s^{k+1} &:= \text{prox}_{\sigma g^*}(s^k + \sigma Ax^k) & \text{dual step} \\
x^{k+1} &:= \text{prox}_{\tau f}(x^k - \tau A^T (2s^{k+1} - s^k)) & \text{primal step}
\end{cases}
\end{align*}
\]

It can be written as a simple fixed-point iteration of metric resolvent (13):

\[
\begin{bmatrix}
s^{k+1} \\
x^{k+1}
\end{bmatrix} = \begin{bmatrix}
    \frac{\partial g^*}{A^T} & -A \\
    \frac{1}{\sigma} I_L & \frac{1}{\tau} I_N
\end{bmatrix}^{-1} \begin{bmatrix}
    \frac{1}{\sigma} I_L & A \\
    \frac{1}{\tau} I_N & \frac{1}{\tau} I_N
\end{bmatrix} \begin{bmatrix}
s^k \\
x^k
\end{bmatrix}
\]

Then, the convergence can be analyzed by the nonexpansiveness of the metric resolvent, according to the regularity conditions of \(f\) and \(g\).

5 Conclusions and further discussions

In this note, we investigated in details the nonexpansive mappings in the context of arbitrary variable metric, from which immediately followed the convergence properties of the associated fixed-point iterations.

Below, we briefly further discuss more prospective applications of the \(Q\)-based nonexpansive mapping. Not limited to the metric resolvent presented in this note, our results can also be applied to analyze other related concepts, e.g. generalized proximity operator, Bregman proximal map, variable metric Fejér sequence.

Since the fixed-point iteration of metric resolvent is essentially the variable metric proximal point algorithm, our results can also be applied to other operator splitting algorithms, e.g. alternating direction of multipliers method [10], Douglas-Rachford splitting algorithm [18], a family of primal-dual splitting algorithms [34, 6], which have recently been shown as the specific applications of the proximal point scheme in the literature, e.g. [19, 25].

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