Lattice Linear Problems vs Algorithms

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ABSTRACT
Modelling problems using predicates that induce a partial order among global states was introduced as a way to permit asynchronous execution in multiprocessor systems. A key property of such problems is that the predicate induces one lattice in the state space which guarantees that the execution is correct even if nodes execute with old information about their neighbours. Unfortunately, many interesting problems do not exhibit lattice linearity. This issue was alleviated with the introduction of eventually lattice linear algorithms. Such algorithms induce a partial order in a subset of the state space even though the problem cannot be defined by a predicate under which the states form a partial order.

This paper focuses on analyzing and differentiating between lattice linear problems and algorithms. It also introduces a new class of algorithms called fully lattice linear algorithms. These algorithms partition the entire reachable state space into one or more lattices and the initial state locks into one of these lattices. Thus, under a few additional constraints, the initial state can uniquely determine the final state. For demonstration, we present lattice linear self-stabilizing algorithms for minimal dominating set and graph colouring problems, and a parallel processing lattice linear 2-approximation algorithm for vertex cover.

The algorithm for minimal dominating set converges in \( n \) moves, and that for graph colouring converges in \( n + 2m \) moves. These algorithms preserve this time complexity while allowing the nodes to execute asynchronously and take actions based on old or inconsistent information about their neighbours. They present an improvement to the existing algorithms present in the literature. The algorithm for vertex cover is the first lattice linear approximation algorithm for an NP-Hard problem; it converges in \( n \) moves.

KEYWORDS
self-stabilization, lattice linear problems, lattice linear algorithms, minimal dominating set, graph colouring, vertex cover.

1 INTRODUCTION
A concurrent algorithm can be viewed as a loop where in each step/iteration, a node reads the shared memory, based on which it decides to take action and update its own memory. As an example, in an algorithm for graph colouring, in each step, a node reads the colour of its neighbours and, if necessary, updates its own colour. Execution of this algorithm in a parallel or distributed system requires synchronization to ensure correct behaviour. For example, if two nodes, say \( i \) and \( j \) change their colour simultaneously, the resulting action may be incorrect. Synchronization has been studied under various titles such as mutual exclusion, transactions, (cache) coherency, dining philosopher, and locking.

Let us consider that we allow such algorithms to execute in asynchrony: consider that \( i \) and \( j \) from above execute asynchronously. Let us suppose that the \( i \) updates first. The action of \( i \) is acceptable till now. However, when we observe \( j \), it is possible that \( j \) has already read the variables of \( i \), and consequently, these values are now old/inconsistent, as \( i \) has updated its variables. The synchronization primitives discussed in the previous paragraph aim to eliminate such behaviour. However, they introduce an overhead.

Generally, reading old values in this manner causes the algorithm to fail. However, if the correctness of the algorithm can be proved even when a node executes based on old information, then such an algorithm can benefit from asynchronous execution; such execution would not suffer from synchronization overheads and each node can execute independently.

In [10], Garg introduced a class of problems which can be modelled such that the execution stays correct even if a node reads old values from other nodes. In this class of problems, denoted as lattice linear problems, the predicate representing the problem induces a partial order, resulting in one lattice among the states. Although highly useful, the requirements of lattice linear problems are very stringent. Various problems (e.g., minimal dominating set, graph colouring) are not lattice linear. In [13], the authors introduced eventually lattice linear algorithms for non-lattice linear problems. Such algorithms induce a partial order in a subset, say \( S_f \), of the state space \( S \). Thus, these algorithms can benefit from asynchronous execution if the program starts in \( S_f \) (or it is guaranteed to reach \( S_f \) if it starts from a state outside of \( S_f \)) and it is guaranteed that it will never leave \( S_f \). In other words, the work presented in [13] increases the applicability of lattice linearity to a larger class of problems. However, the problems studied in [13] require self-stabilization, and therefore, to benefit from those algorithms, additional proof requirements (namely, ensuring that the program reaches \( S_f \) and never leaves \( S_f \)) have to be fulfilled.

In this paper, we introduce fully lattice linear algorithms (as opposed to lattice linear problems or eventually lattice linear algorithms) that are capable of imposing a partial order on the entire state space even if the underlying problem does not possess a lattice linear predicate. These algorithms overcome the limitations of [10], as they have wider applications, as well as [13], as they permit asynchronous execution in the entire state space rather than a limited subset of states. We also show that with fully lattice linear algorithms, it is possible to combine lattice linearity with self-stabilization, which ensures that the system converges to a legitimate state even if it starts from an arbitrary state.

1.1 Contributions of the paper
- We alleviate the limitations of inducing a partial order using a predicate by introducing fully lattice linear algorithms
where a partial order is imposed by the algorithm even when it cannot be induced by a predicate.

- We provide upper bounds to the convergence time for an arbitrary algorithm traversing a lattice of states.
- We bridge the gap between [10] and [13] by introducing fully lattice linear algorithms. The former creates a single lattice among the states whereas the latter creates multiple lattices in a subset of the state space. Fully lattice linear algorithms induce one or more lattices (depending on the underlying problem) among the reachable states. This overcomes the limitations of [10] and [13].
- We present fully lattice linear self-stabilizing algorithms for the minimal dominating set and graph colouring problems, and a lattice linear 2-approximation algorithm for vertex cover.
- The algorithm for minimal dominating set converges in \( n \) moves, and the algorithm for graph colouring converges in \( n + 2m \) moves. These algorithms are fully tolerant to consistency violations and asynchronous parallel processing systems. Thus, they present an improvement over the existing algorithms present in the literature. This is the first lattice linear approximation algorithm for an NP-complete problem.
- The algorithm for vertex cover is the first lattice linear approximation algorithm for an NP-Hard problem; it converges in \( n \) moves.

### 1.2 Organization of the paper

In Section 2, we elaborate the preliminaries. In Section 3, we discuss some background results related to lattice linearity that are present in the literature. In Section 4, we describe the general structure of a (fully) lattice linear algorithm. In Section 5, we present a fully lattice linear algorithm for minimal dominating set, and that for graph colouring in Section 6. We present a lattice linear 2-approximation algorithm for vertex cover in Section 7. In Section 8, we provide an upper bound to the number of moves required for convergence in an algorithm traversing a lattice of states. We discuss related works in Section 9 and conclude in Section 10.

### 2 PRELIMINARIES

In this paper, we are mainly interested in graph algorithms where the input is a graph \( G \), \( V(G) \) is the set of its nodes and \( E(G) \) is the set of its edges. For a node \( i \in V(G) \), \( Adj_i \) is the set of nodes connected to \( i \) by an edge, and \( Adj_i^x \) is the set of nodes within \( x \) hops from \( i \), excluding \( i \). While writing the time complexity of the algorithms, we notate \( n \) to be \( |V(G)| \) and \( m \) to be \( |E(G)| \). For a node \( i \), \( deg(i) = |Adj_i| \).

Each node in \( V(G) \) stores a set of variables which represent its local state. A global state, say \( s \), is the union of local states of all nodes. We use \( S \) to denote the state space, the set of all global states, of a given problem.

Each node in \( V(G) \) is also associated with actions. Each action at node \( i \) checks values of the variables of the nodes in \( Adj_i^x \cup \{i\} \) (where the value of \( x \) is problem dependent) and updates the variables of \( i \). An action at a node \( i \) is of the form \( g \rightarrow a_c \) where \( g \), a guard, is a Boolean expression over variables in \( Adj_i^x \cup \{i\} \) and \( a_c \) is a set of instructions that updates the variables of \( i \) if \( g \) is true. A node is enabled if at least one of its guards is true, otherwise it is disabled. A move is an event in which an enabled node updates its variables.

**Execution without synchronization.** Typically, we view a sequence of states as a computation \((s_0, s_1, \cdots)\) of the given algorithm where each \( s_j + 1 (\forall j \geq 0) \) is obtained by executing some action by one or more nodes in \( s_j \). Under proper synchronization, node \( i \) evaluates its guards on the current local states of its neighbours.

Let that \( x(s) \) denotes the value of some variable \( x \) in state \( s \), and let \( \forall j \geq 0, s_{j+1} \) be the state when all nodes perform execution asynchronously on the current local states in \( s_j \). If \( g \) is evaluated without synchronization, then, say node \( i \), may read old values of some variables. In such a case, if we consider the computation prefix \((s_0, s_1, \cdots, s_j)\) and evaluate \( g \) in state \( s_j \), then the state may appear to \( i \) as \( s' \neq s_j \), where some values are old. Specifically, the state \( s_j \) will appear as \( s' \) where \( x(s') \in \{x(s_0), x(s_1), \cdots, x(s_j)\} \).

Now, if \( g \) evaluates to true then node \( i \) will execute \( a_c \). The state \( s'' \) obtained by executing \( a_c \) in \( s' \) is not necessarily equal to \( s_{j+1} \).

In this paper, we introduce algorithms that are tolerant of such actions.

**Self-stabilization:** An algorithm \( A \) is self-stabilizing for the tuple \((S_0, S_f)\), where \( S_0 \) and \( S_f \) are sets of states and \( S_0 \subseteq S_f \), iff

- **Convergence:** Starting from an arbitrary state, any sequence of computations of \( A \) reaches a state in \( S_0 \).
- **Closure:** Any computation of \( A \) starting from \( S_f \) always stays in \( S_f \).

In this work, we are mainly interested in the convergence aspect; after reaching the legitimate states (e.g., the minimal dominating set is found), the algorithm does not have any actions to execute.

**Embedding a \&<\&\text{-lattice in global states.** Next, we discuss the structure of a partial order in the state space which, under proper constraints, allows an algorithm to converge to an optimal state. To describe the embedding, let that \( s \) denotes a global state, and let that \( s[i] \) denotes the state of node \( i \) in \( s \). First, we define a total order \&<\&\text{\&-lattice} among global states as follows:

We say that \( s <_g s' \) iff \((\forall i: s[i] = s'[i] \vee s[i] <_l s'[i]) \wedge (\exists i: s[i] <_l s'[i])\). Also, \( s = s' \) iff \( s[i] = s'[i] \). For brevity, we use \&<\& to denote \&<\&\text{\&} and \&<\&\text{\&-lattice} corresponds to \&<\& while comparing local states, and \&<\&\text{\&-lattice} corresponds to \&<\& while comparing global states. We use the symbol ‘\&\&’ which is the opposite of \&<\&\text{\&-lattice}, i.e., \( s >_g s' \). Similarly, we use symbols ‘\&\&\&’ and ‘\&\&\&\&’; e.g., \( s \leq s' \) iff \( s = s' \vee s < s' \).

We call the lattice, formed from such partial order, a \&<\&\text{- lattice.}

**Definition 1. \&<\&\text{-lattice.** Given a total relation \&<\&\text{\&} that orders the values of \( s[i] \) (the local state of node \( i \) in state \( s \)), the \&<\&\text{-lattice corresponding to \&<\&\text{\&} is defined by the following partial order: \( s < s' \) iff \((\forall i: s[i] \leq s'[i]) \wedge (\exists i: s[i] < s'[i])\).

**Remark:** The \&<\&\text{-lattice alone is insufficient for permitting asynchronous execution. Additional constraints are identified in Section 3.**

In the \&<\&\text{-lattice discussed above, we can define the meet and join of two states in the standard way (respectively, join), of
two states $s_1$ and $s_2$ is a state $s_3$ where $\forall i, s_3[i]$ is equal to $\min(s_1[i], s_2[i])$ (respectively, $\max(s_1[i], s_2[i])$).

By varying $\prec$ that identifies a total order among the states of a node, one can obtain different lattices. A $\prec$-lattice, embedded in the state space, is useful for permitting the algorithm to execute asynchronously. Under proper constraints on how the lattice is formed, convergence is ensured.

Remark: It is not necessary that a $\prec$-lattice is of finite size, in which case, there will be no supremum, and possibly no infimum. We discuss more on this in Section 3.1. In such cases, we can still use Definition 1 as well.

3 **BACKGROUND: TYPES OF LATTICE LINEAR TRANSITION SYSTEMS**

Lattice linearity has been shown to be induced in problems in two ways, one where the lattice linearity arises due to the problem itself, and another where the problem does not manifest lattice linearity but the algorithm imposes it. We discuss these in Section 3.1 and Section 3.2 respectively.

3.1 **Natural Lattice Linearity: Lattice Linear Problems**

In this subsection, we discuss lattice linear problems, i.e., the problems where the description of the problem statement creates the lattice structure automatically. Such problems can be represented by a predicate under which the states in $S$ form a lattice. These problems include stable (man-optimal) marriage problem, market clearing price and others. These problems have been discussed in [9–11].

In lattice linear problems, a problem $P$ can be represented by a predicate $P$ such that for any node $i$, if it is violating $P$ in some state $s$, then it must change its state, otherwise the system will not satisfy $P$. Let $P(s)$ be true iff state $s$ satisfies $P$. A node violating $P$ in $s$ is called a forbidden node. Formally,

**Definition 2.** [10] Forbidden (local state) Forbidden $(i, s, P) \equiv \neg P(s) \land (\forall s' > s : s'[i] = s[i] \implies \neg P(s'))$.

Definition 2 implies that if a node $i$ is forbidden in some state $s$, then in any state $s'$ such that $(s' > s)$, if the state of $i$ remains the same, then the algorithm will not converge. If no node is forbidden in $s$, then $P(s)$ is true and all nodes are disabled. Based on the above definition and given that $S$ forms a $\prec$-lattice, we have the definition of lattice linear problems as follows.

**Definition 3.** Lattice linear problems. A problem $P$ is lattice linear iff there exists a predicate $P$ and an $\prec$-lattice such that

- $P$ requires that we reach a state where $P$ is true, and
- $P$ is lattice linear with respect to the $\prec$-lattice induced in $S$ [10], i.e., $\forall s : \neg P(s) \implies \exists i : \text{FORBIDDEN}(i, s, P)$.

Remark: We use the term non-lattice linear problem to denote problems that do not meet the constraints of Definition 3.

For a lattice linear problem $P$, we can design an algorithm $A$ by following the following guidelines [10].

- Each node $i$ checks if it is violating $P$ in $s$ and determines if it is forbidden.
- If $i$ is forbidden, then $i$ increments its value with respect to $\prec$, otherwise, no action is taken.

If the above rules are followed and the algorithm initializes in the infimum of lattice then it will either reach a state where $P$ is true or it will try to transition to a state which is not in the lattice because some nodes tries to exceed their upper bound as allowed by the supremum. In this case, it will declare an absence of a solution. Definition 2 ensures that this property is satisfied even if a node reads old values; if $P$ is false in the state observed by the given node, it will also be false in the current state. Thus such an algorithm finds the lowest state in the lattice where $P$ is true.

**Example 1. SMP.** We describe a lattice linear problem, the stable (man-optimal) marriage problem (SMP) from [10]. In SMP, all men (respectively, women) rank women (respectively men) in terms of their preference (lower rank is preferred more). A global state is represented as a vector $s$ where the vector $s[i]$ contains a single value which represents the ID of the woman that man $i$ is proposing to.

The requirements of SMP can be defined as $P_{\text{SMP}} \equiv \forall m_i : m_i \neq m_s \rightarrow s[m] \neq s[m_i]$. $P_{\text{SMP}}$ is true iff no two men are proposing to the same woman. A man $m_i$ is forbidden iff there exists $m_s$ such that $m_i$ and $m_s$ are proposing to the same woman $w$ and $w$ prefers $m_s$ over $m_i$. Thus, $\text{FORBIDDEN-SMP}(m_i, s, P_{\text{SMP}}) \equiv \exists m_s : s[m] = s[m_s] \land \text{rank}(s[m], m_s) < \text{rank}(s[m], m_i)$. If $m_i$ is forbidden, he increments $s[m_i]$ by $1$ until all his choices are exhausted.

A key observation from the stable marriage problem (SMP) and other problems from [10] is that the partial order formed in $S$ contains a global infimum $\ell$ and possibly a global supremum $u$ i.e., $\ell$ and $u$ are the states such that $\forall s \in S, \ell \leq s \land \forall s \in S, u \geq s$. All the states in $S$ form a single lattice.

**Example 2. SMP continuation.** As an illustration of SMP, consider the case where we have $3$ men $m_1, m_2, m_3$ and $3$ women $w_1, w_2, w_3$. The lattice induced in this case is shown in Figure 1. In this figure, every vector represents the global state $s$ such that $s[i]$ represents the index of woman that $m_i$ is proposing to. The algorithm begins in the state $(1, 1, 1)$ (i.e., each man starts with his first choice) and continues its execution in this lattice. The algorithm terminates in the lowest state in the lattice where no node is forbidden.

In SMP and other problems in [10], the algorithm needs to be initialized to $\ell$ to reach an optimal solution. If we start from a state $s : s \neq \ell$, then the algorithm can only traverse the lattice from $s$. Hence, upon termination, it is possible that the optimal solution is not reached. In other words, such algorithms cannot be self-stabilizing [5] unless $u$ is the optimal state.

**Example 3. SMP continuation.** Consider that men and women are $M = (A, J, T)$ and $W = (K, Z, M)$ indexed in that sequence respectively. Let that proposal preferences of men are $A = (Z, K, M)$, $J = (Z, K, M)$ and $T = (K, M, Z)$, and women have ranked men as $Z = (A, J, T)$, $K = (J, T, A)$ and $M = (T, J, A)$. The optimal state (starting from $(1,1,1)$) is $(1,2,2)$. Starting from $(1,2,3)$, the algorithm terminates at $(1,2,3)$ which is not optimal. Starting from $(3,1,2)$, the algorithm terminates declaring that no solution is available.
3.2 Imposed Lattice Linearity: Eventually Lattice Linear Algorithms

Unlike the lattice linear problems where the problem description creates a lattice among the states in $S$, there are problems where the states do not form a lattice naturally. However, a lattice can be imposed by an algorithm by restricting the transitions of the system. Authors of [13] have illustrated this where they impose a lattice on a subset of the state space.

Specifically, the algorithms presented in [13] partition the state space into two parts: feasible and infeasible states, and induce multiple lattices among the feasible states. These algorithms work in two phases. The first phase takes the system from an infeasible state to a feasible state, where the graph starts to exhibit the desired property. In the second phase, only a forbidden node can change its state. This phase takes the system from a feasible state to an optimal state. (The processes themselves do not know which phase they are in.) These algorithms converge starting from an arbitrary state; they are called eventually lattice linear self-stabilizing algorithms.

Example 2. MDS. In the MDS problem, the task is to choose a minimal set of nodes $D$ in a given graph $G$ such that for every node in $V(G)$, either it is present in $D$, or at least one of its neighbours is in $D$. Each node $i$ stores a variable $st_i$ with domain $\{IN, OUT\}$; $i \in D$ iff $st_i = IN$.

Remark: The minimal dominating set (MDS) problem is not a lattice linear problem. This can be illustrated through a simple instance of a 2 node connected network with nodes $A$ and $B$, initially both in the dominating set. Here, MDS can be obtained without removing $A$. Thus, $A$ is not forbidden. The same applies to $B$.

Example MDS continuation 1. Even though the MDS problem is not lattice linear, lattice linearity can be imposed on it by the algorithm. Algorithm 1 is based on the algorithm in [13] for a more generalized version of the problem, the service demand based minimal dominating set problem. Algorithm 1 consists of two phases.

For the subsequent discussion, we only focus on phase 2 of Algorithm 1 where the property of lattice linearity is satisfied.

Example MDS continuation 2. To illustrate the partial order imposed by phase 2 of Algorithm 1, consider an example of graph $G_4$ containing four nodes connected in such a way that they form two disjoint edges, i.e., $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{\{v_1, v_2\}, \{v_3, v_4\}\}$. Assume that $G$ is initialized in a feasible state.

The lattices formed in this case are shown in Figure 2. We write a state $s$ of this graph as $(st_{v_1}, st_{v_2}, st_{v_3}, st_{v_4})$. As shown in this figure, a subset of the global states, i.e., only the feasible states, participate in the partial order.

Figure 2: The lattices induced in the problem instance in Example MDS continuation 2.

Remark: Partial order imposed by Algorithm 1. In Algorithm 1, a subset of the state space forms multiple lattices. After the execution of phase 1, the algorithm locks into one of these lattices. Thereafter in phase 2, the algorithm executes lattice linearly to reach the supremum of that lattice. Since the supremum of every lattice represents a minimal dominating set, this algorithm will always
converge to an optimal state. A key disadvantage of Algorithm 1 and other eventually lattice linear algorithms is that you need separate analysis of Phase 1 as well as interference between the two phases. Only if these phases do not interfere, and the first phase guarantees reaching the lattice from an arbitrary state outside of the lattice, asynchronous executions can be permitted. Without that, they cannot benefit from asynchronous execution available to the lattice linear phase. The goal of this paper is to overcome these limitations.

4 INTRODUCING FULLY LATTICE LINEAR ALGORITHMS: OVERCOMING LIMITATIONS OF [10] AND [13]

In this section, we introduce the notion of fully lattice linear algorithms that induce multiple lattices in the entire reachable state space. While defining these algorithms, we also distinguish them from the closely related work in [10] and [13]. Specifically, we discuss why developing algorithms for non-lattice linear problems (such that the algorithms are lattice linear, i.e., they induce a lattice in the reachable state space) requires the innovation presented in this paper. We also discuss how we extend the notion of [13], where only a subset of state space forms a lattice, and develop algorithms such that a lattice is induced among all the reachable states.

In [10], authors consider lattice linear problems. Here, the state space is induced under a predicate and forms one lattice. Such problems possess only one optimal state, and hence a violating node must change its state. The acting algorithm simply follows that lattice to reach the optimal state.

Certain problems, e.g., dominating set, are not lattice linear because if one of the nodes is restricted to changing its state, then also an optimal solution can be achieved (cf. the remark below Example 2).

Such problems are studied in [13]. An interesting observation on the algorithms studied in [13] is that they induce multiple lattices in a subset of the state space. For example, as presented in Figure 2, multiple lattices are induced in a subset of the state space, where the nodes form a (possibly suboptimal) dominating set.

Limitations of [10]: From the above discussion, we note that the general approach presented in [13] is applicable to a wider class of problems. Additionally, many lattice linear problems do not allow self-stabilization. In such cases, e.g., in SMP, if the algorithm starts in, e.g., the supremum of the lattice, then it may terminate declaring that no solution is available. Unless the supremum is the optimal state, the acting algorithm cannot be self-stabilizing.

Limitations of [13]: The approach in [13] suffers from two issues. First, the lattice structure is imposed only on a subset of states. Thus, by design, the algorithm has actions, say $A_1$, that operate in the part of the state space where the lattice structure does not exist, and actions, say $A_2$, that operate in the part of the state space where the lattice is formed. For example, in MDS, actions in $A_1$ (phase 1 in Algorithm 1) are those that add nodes to ensure that the set of selected nodes form a valid dominating set whereas actions in $A_2$ (phase 2 in Algorithm 1) are those that remove nodes from the dominating set to make the current set optimal. Since actions in $A_1$ are operating outside the lattice structure, the developer must guarantee that if the system is initialized outside the lattice structure, then $A_1$ converges the system to one of the states participating in the lattice (from where $A_2$ will be responsible for the traversal of the system through the lattice) and thus the developer faces an extra proof obligation. Furthermore, a node does not know if the current global state is in one of the lattices or whether it is outside all the lattice structures. This means that it is possible that node $i$ is executing an action in $A_2$ while its neighbor is executing an action in $A_1$. Thus, to ensure correctness, another proof obligation is created to ensure that actions in $A_2$ do not interfere with actions of $A_1$. E.g., one has to make sure that actions in $A_2$ do not perturb the node to a state where the selected nodes do not form a dominating set.

Another issue in [13] is the absence of a deterministic output. This is because $A_1$ does not execute in a lattice of global states and thus can face cycles while it transitions the system through the state space. By contrast, this is not observed in algorithms developed for lattice linear problems. E.g., if we run MDS on two different computer systems, their output may differ, even if they start from the same initial state. On the other hand, if we independently run SMP on two different computer systems then their output would be the same. A deterministic output is important in various applications for mundane reasons such as automating the grading of a programming assignment, or in applications such as blockchains where the deterministic output is used to validate the work done by others.

Alleviating the limitations of [10] and [13]: In this paper, we investigate if we can benefit from advantages of both [10] and [13]. We study if there exist fully lattice linear algorithms where a partial order can be imposed on all reachable states. In the case that there are multiple optimal states and the problem requires self-stabilization, it would be necessary that multiple disjoint lattices (no common states) are formed where the supremum of each lattice is an optimal state. Self-stabilization also requires that these lattices are exhaustive, i.e., their union is equal to the entire state space.

This ensures that asynchronous execution can be permitted in the entire state space, i.e., it allows the system to initialize in any state. Additionally, the initial state locks into one of the lattices thereby ensuring deterministic output. It would also permit multiple optimal states. There will be no need to deal with interference between actions.

**Definition 4.** A lattice linear algorithm (LLA) is an FLLA for a problem $P$, represented by a predicate $P$, if it induces a $<\cdot$-lattice among the states of $S_1, ..., S_w (w \geq 1)$ that:

- State space $S$ of $P$ contains mutually disjoint lattices, i.e., $S_1, S_2, ..., S_w \subseteq S$ are pairwise disjoint.
- $S_1 \cup S_2 \cup \cdots \cup S_w$ contains all the reachable states.
- Lattice linearity is satisfied in each subset, i.e., $P$ requires that the system reaches a state where $P$ is true.
- For any $k$, $1 \leq k \leq w$, $P$ is lattice linear with respect to the partial order induced in $S_k$ by $A$, i.e., $\forall s \in S_k : P(s) \Rightarrow \not\exists i, s, P(i)$.
- Any move will make the system move up in the same lattice, i.e.,
In this paper, we demonstrate that fully lattice linear algorithms for minimal dominating set (Section 5) and graph colouring (Section 6) problems. In addition, we present a lattice linear approximation algorithm developed for an NP-Hard problem under the paradigm of lattice linearity.

5 FULLY LATTICE LINEAR ALGORITHM FOR MINIMAL DOMINATING SET (MDS)

In this section, we present a lattice linear self-stabilizing algorithm for MDS. MDS has been defined in Example 2. We describe the algorithm as Algorithm 2.

In Algorithm 2, the first two macros are the same as Example 2. The definition of a node being forbidden (FORBIDDEN-II-DS(i)) is changed to make the algorithm fully lattice linear. Specifically, even allowing a node to enter into the DS is restricted such that only the nodes with the highest ID in their distance-2 neighbourhood can enter the DS. Any node i which is addable or removable will toggle its state if it is forbidden, i.e., any other node j ∈ Adj^2_i : id_j > id_i is neither addable or removable. In the case that i is forbidden, if i is addable, then we call it addable-forbidden, if it is removable, then we call it removable-forbidden.

Algorithm 2. Algorithm for MDS.

To demonstrate that Algorithm 2 is lattice linear, we define state value and rank as follows:

State-Value-DS(i, s) =

\[
\begin{cases}
1 & \text{if Unsatisfied-DS(i) in state } s \\
0 & \text{otherwise}
\end{cases}
\]

Rank-DS(s) = \sum_{i \in V(G)} State-Value-DS(i, s).

To understand the lattice imposed by Algorithm 2, we need to consider State-Value-DS(i, s) as an auxiliary variable associated with node i. The lattice is formed with respect to this variable: State-Value-DS(i, s) can change from 1 to 0 but not vice versa.

Hence, the lattices imposed by Algorithm 3 have the property that RANK-DS always decreases until it becomes zero. At the supremum of a lattice, this value is 0.

Lemma 1. Any node in an input graph does not revisit its older state while executing under Algorithm 2.

Proof. Let s be the state at time step t while Algorithm 2 is executing. We have from Algorithm 2 that if a node i is addable-forbidden or removable-forbidden, then no other node in Ad^2_j changes its state.

If i is addable-forbidden at t, then any other node in Ad^2_j is out of the DS. After when i moves in, then any other node in Ad^2_j is no longer addable, so they do not move in after t. As a result, i does not have to move out after moving in.

If otherwise i is removable-forbidden at t, then all the nodes in Ad^2_j are being dominated by some node other than i. So after when i moves out, then none of the nodes in Ad^2_j, including i, becomes unsatisfied. Also, the nodes in Ad^2_j do not change their state unless i changes its state.

Let that is dominated and out, and some j ∈ Ad^2_j is removable forbidden. j will change its state to OUT only if i is being covered by another node. Also, while j turns out of the DS, no other node in Ad^2_j, and consequently in Ad^2_j, changes its state. As a result, j does not have to turn itself in because of the action of j.

From the above cases, we have that i does not change its state to st. i after changing its state from st. i to st’ i throughout the execution of Algorithm 2. □

Theorem 1. Algorithm 2 is self-stabilizing and lattice linear.

Proof. We have from the proof of Lemma 1 that if G is in state s and RANK-DS(s) is non-zero, then at least one node will be forbidden. For any node i, we have that State-Value-DS(i) decreases whenever i is forbidden and never increases. So RANK-DS monotonically decreases throughout the execution of the algorithm until it becomes zero. This shows that Algorithm 2 is self-stabilizing.

Next, we show that Algorithm 2 is fully lattice linear. We claim that there is one lattice corresponding to each optimal state. So if there are w optimal states for a given instance, then there are w disjoint lattices S_1, S_2, · · · , S_w formed in the state space S. We show this as follows.

We observe that an optimal state (manifesting a minimal dominating set) is at the supremum of its respective lattice, as there are no outgoing transitions from an optimal state.

Furthermore, given a state s, we can uniquely determine the optimal state, say s_{opt}, that would be reached from s. This is because in any given non-optimal state, there will be some forbidden nodes.

The non-forbidden nodes will not move even if they are unsatisfied. Hence the forbidden nodes must move in order for the global state to reduce the rank of that state (and thus move towards an optimal state). Since the forbidden nodes in any state can be uniquely identified, the optimal state reached from a given state s can also be uniquely identified.

This implies that starting from a state s in S_k(1 ≤ k ≤ w), the algorithm cannot converge to any state other than the supremum of S_k. Thus, the state space of the problem is partitioned into
6 FULLY LATTICE LINEAR ALGORITHM FOR GRAPH COLOURING (GC)

In this section, we present a lattice linear self-stabilizing algorithm for graph colouring.

**Definition 6.** Graph colouring: In the GC problem, the input is a graph \( G \) with possibly some initial colour assignment \( \forall i \in V(G) : \text{colour}_i \in \mathbb{N} \). The task is to (re)assign the colour values to each node such that there should be no conflict between adjacent nodes, i.e., \( \forall i \in V(G), \forall j \in \text{Adj}_i : \text{colour}_i \neq \text{colour}_j \), and there should not be an assignment such that it can be reduced without conflict, i.e., \( \forall c < \text{colour}_i, \exists j \in \text{Adj}_i : c = \text{colour}_j \).

We describe the algorithm as Algorithm 3. Any node \( i \), which has a conflicting colour with any of its neighbours, or if its colour value is reducible, is an unsatisfied node. A node having a conflicting or reducible colour changes its colour to the lowest non-conflicting value if it is forbidden, i.e., \( \forall j \in \text{Adj}_i : \text{id}_j > \text{id}_i \) is not unsatisfied. If the case that \( i \) is forbidden, if \( i \) has a conflict with any of its neighbours, then we call it conflict-forbidden, if its colour is reducible, then we call it reducible-forbidden.

**Algorithm 3.** Algorithm for GC.
7 LATTICE LINEAR 2-APPROXIMATION ALGORITHM FOR VERTEX COVER

It is highly alluring to develop parallel processing approximation algorithms for NP-Hard problems under the paradigm of lattice linearity. In fact, it has been an open question if this is possible [10]. We observe that this is possible. In this section, we present a lattice linear 2-approximation algorithm for VC.

The following algorithm is the classic 2-approximation algorithm for VC. Choose an uncovered edge \( \{A, B\} \), select both \( A \) and \( B \), repeat until all edges are covered. Since the minimum VC must contain either \( A \) or \( B \), the selected VC is at most twice the size of the minimum VC.

While the above algorithm is sequential in nature, we demonstrate that we can transform it into a distributed algorithm under the paradigm of lattice linearity as shown in Algorithm 4 (we note that this algorithm is not self-stabilizing). The symbol "⊤" stands for null.

In Algorithm 4, all nodes are initially out of the VC. A node is called done if it has already evaluated if all the edges incident on it are covered. A node is forbidden if it is not done yet and it is the highest ID node in its distance-3 neighbourhood which is not done. If a forbidden node has an uncovered edge, and assume that \( \{i, j\} \) is an uncovered edge with \( j \) being the highest ID node which is not (note that \( i = \text{out} \)), then \( i \) turns both \( i \) and \( j \) into the VC. If otherwise \( i \) evaluates that all its edges are covered, then it declares that it is done (\( i \) sets done.\( i \) to true), while staying out of the VC.

This is straightforward from the 2-approximation algorithm for VC. We have chosen 3-neighbourhood to evaluate forbidden to ensure that no conflicts arise while execution from the perspective of the 2-approximation algorithm for VC.

**ALGORITHM 4. A 2-approximation lattice algorithm for VC.**

```
Init: \( V_1 \in V(G), st.i = \text{OUT}, \text{done}.i = \text{false} \).

FORBIDDEN-VC(\( i \)) \( \ni \) done.\( i \) = \text{false} \( \land \) (\( \forall j \in \text{Adj}_i \) : id.\( j \) < id.\( i \) \( \lor \) done.\( j \) = true).

Rules for node \( i \).

FORBIDDEN-VC(\( i \)) \( \rightarrow \)

if (\( \forall k \in \text{Adj}_i \) \( \land \) st.\( k \) = IN), then done.\( i \) = true. else, then

\( j = \text{arg max}\{\text{id.}\( x \) : x \in \text{Adj}_i \land \text{done}.\( x \) = \text{false}\} \).

st.\( i \) = IN.

if \( j \neq \top \), then st.\( j \) = IN, done.\( j \) = true.

\( \text{done}.\( i \) = \text{true} \).
```

Observe that the action of node \( i \) is looking at the edge \( \{i, j\} \) and adding \( i \) and \( j \) to the VC. This follows straightforwardly from the classic 2-approximation algorithm.

7.1 Lattice Linearity of Algorithm 4

To demonstrate that Algorithm 4 is lattice linear, we define the state value and rank as follows.

\[
\text{State-Value-VC}(i, s) =
\begin{cases}
\{j \in \text{Adj}_i : \text{st}.\( j \) = \text{OUT}\} & \text{if } \text{st}.\( i \) = \text{OUT} \\
0 & \text{otherwise}
\end{cases}
\]

\[
\text{Rank-VC}(s) = \sum_{i \in V(G)} \text{State-Value-VC}(i, s)
\]

Similar to the proofs elaborated in Section 5 and Section 6, in order to understand the lattice imposed by Algorithm 4, we need to consider \( \text{State-Value-VC}(i, s) \) as an auxiliary variable associated with node \( i \). \( \text{State-Value-VC}(i, s) \) is initialized at a value equal to the degree of \( i \). Then, as Algorithm 4 executes, \( \text{State-Value-VC}(i, s) \) monotonously decreases until it becomes zero. Hence, \( \text{Rank-VC} \) always decreases until it becomes zero.

In the following theorem, we explore the lattice linearity and 2-approximability of Algorithm 4.

**THEOREM 3.** Algorithm 4 is a lattice linear 2-approximation algorithm for VC.

**Proof.** Lattice linearity: We have that the initial state is where every node \( i \) has done.\( i \) = false and st.\( i \) = OUT. Let \( s \) be an arbitrary state at the beginning of some time step while the algorithm is under execution such that \( s \) does not manifest a vertex cover. Let \( i \) be some node such that \( i \) is of the highest ID in its distance 3 neighbourhood such that some of its edges are not covered. Also, let \( j \) be the node of the highest ID in \( \text{Adj}_i \) for which done.\( j \) = false, if one such node exists. We have that \( i \) is the only forbidden node in its distance-3 neighbourhood, and \( j \) is the specific additional node, which \( i \) turns in. This forms a partial order among the states where state \( s \) transitions to another state \( s' \) where \( s < s' \) and for any such \( i \), \( \text{st}.s'[i] = \text{IN} \land \text{done}.s'[i] = \text{true} \) and \( \text{st}.s'[j] = \text{IN} \land \text{done}.s'[j] = \text{true} \).

If \( s \) manifests a vertex cover, then no additional nodes will be turned in, and atmost one additional node (node \( i \), as described in the paragraph above) will have done.\( s'[i] = \text{true} \).

From these observations, we have that Algorithm 4 exhibits properties similar to Algorithm 2 which are elaborated in the proof for its lattice linearity in Theorem 1, however, it induces only one lattice in the state space since the initial state is predetermined, and so \( w = 1 \). Thus, we have that Algorithm 4 also is lattice linear.

2-approximability: If a node \( i \) is forbidden and if one of its edges is uncovered, then it selects an edge (it points to the other node in that edge) and both the nodes in that edge turn in; thus it straightforwardly follows the standard sequential 2-approximation algorithm. If some node \( k \) (at a distance farther than 3 from \( i \)) executes and selects \( k' \in \text{Adj}_k \) to turn in, then and neither \( i \) nor \( j \) cannot be a neighbour of \( k \) or \( k' \). Thus for any pair of edges that are selected together, they will not have a common node, and none of their nodes will be adjacent. This shows that Algorithm 4 preserves the 2-approximability of the classic 2-approximation algorithm for VC.

**Example 3.** In Figure 4, we show a graph (containing eight nodes \( v_1, \ldots, v_8 \)) and the lattice induced by Algorithm 4 in the state space of that graph. We are omitting how the value of done.\( i \) gets modified; we only show how the vertex cover is formed. Only the reachable states are shown. Each node in the lattice represents a tuple of states of all nodes (st.\( v_1, \text{st}.v_2, \ldots, \text{st}.v_8 \)).
In Algorithm 4, local state of any node \( i \) is represented by two variables \( \text{done}.i \) and \( \text{st}.i \). Observe that in this algorithm, the definition of a node being forbidden depends on \( \text{done}.i \) and not \( \text{st}.i \). Therefore the transitions and consequently the lattice depends on \( \text{done}.i \) only, whose domain is of size 2.

In this algorithm, a node \( i \) makes changes to the variables of another node \( j \), which is, in general, not allowed in a distributed system. We observe that this algorithm can be transformed into a lattice linear distributed system algorithm where any node only makes changes only to its own variables. We describe the transformed algorithm in the following subsection.

### 7.2 Distributed Version of Algorithm 4

In Algorithm 4, we presented a lattice linear 2-approximation algorithm for VC. In that algorithm, the states of two nodes \( i \) and \( j \) were changed in the same action.

Here, we present a mapping of that algorithm where \( i \) and \( j \) change their states separately. The key idea of this algorithm is when \( i \) intends to add \( j \) to the VC, \( \text{point}.i \) is set to \( j \). When \( i \) is pointing to \( j \), \( j \) has to execute and add itself to the VC. Thus, the transformed algorithm is as shown in Algorithm 5.

**Algorithm 5.** Algorithm 4 transformed where every node modifies only its own variables.

---

**Remark:** Observe that in this algorithm, any \( j \) chosen throughout the algorithm by some forbidden \( i \) does not move in if it is already covered. Thus, we have that this algorithm computes a minimal as well as a 2-approximate vertex cover. On the other hand, Algorithm 4 is a faithful replication of the classic sequential 2-approximation algorithm for vertex cover.

### 8 CONVERGENCE TIME OF LATTICE LINEAR ALGORITHMS

**Theorem 4.** Given a system of \( n \) processes, with the domain of size not more than \( m \) for each process, the acting algorithm will converge in \( n \times (m - 1) \) moves.

**Proof.** Assume for contradiction that the underlying algorithm converges in \( x \geq n \times (m-1) + 1 \) moves. This implies, by pigeonhole principle, that at least one of the nodes \( i \) is revisiting their states \( \text{st}.i \) after changing to \( \text{st}'.i \). If \( \text{st}.i \) to \( \text{st}'.i \) is a step ahead transition for \( i \), then \( \text{st}'.i \) to \( \text{st}.i \) is a step back transition for \( i \) and vice versa. For a system where the global states form a \( < \)-lattice, we obtain a contradiction since step back actions are absent in such systems.

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**Example MDS continuation 4.** Consider phase 2 of Algorithm 1. As discussed earlier, the execution of phase 2 is lattice linear. The domain of each process \( \langle \text{IN}, \text{OUT} \rangle \) is of size 2. Hence, phase 2 of Algorithm 1 requires at most \( n \times (2 - 1) = n \) moves. (Phase 1 also requires at most \( n \) moves. But this fact is not relevant to the above theorem.)

**Example SMP continuation 3.** Observe from Figure 1 that any system of 3 men and 3 women with arbitrary preference lists will converge in \( 3 \times (3 - 1) = 6 \) moves. This comes from 3 men (resulting in 3 processes) and 3 women (domain size of each man (process) is 3).

**Corollary 1.** Let that the nodes are multivariable. Let that in each node, atmost \( r \) of these variables, \( \text{var}._1,i \),..., \( \text{var}_r,i \) (with domain
sizes $m'_1, \ldots, m'_r$, respectively) contribute independently to the formation of the lattice. Then the LLTS will converge in $n \times \left( \prod_{j=1}^{r} (m'_j - 1) \right)$ moves.

**Corollary 2.** (From Theorem 1 and Theorem 4) Algorithm 2 converges in $n$ moves.

**Corollary 3.** (From Theorem 2 and Theorem 4) Algorithm 3 converges in $\sum_{i \in V(G)} \deg(i) + 1 = n + 2m$ moves.

**Corollary 4.** (From Theorem 3 and Corollary 1) Algorithm 4 converges in $n$ moves.

9 RELATED WORK

**Lattice linearity:** In [10], the authors have studied lattice linear problems which possess a predicate under which the states naturally form a lattice among all states. Problems like the stable marriage problem, job scheduling, market clearing price and others are studied in [10]. In [11] and [9], the authors have studied lattice linearity in, respectively, housing market problem and several dynamic programming problems.

In [13], the authors have extended the theory presented in [10] to develop eventually lattice self-stabilizing algorithms for some non-lattice linear problems. Such algorithms impose a lattice among the subset of the state space.

In this paper, we developed (fully) lattice linear algorithms that induce a lattice among all reachable states in non-lattice linear problems.

**Dominating set:** Self-stabilizing algorithms for the minimal dominating set problem have been proposed in several works in the literature, for example, in [4, 12, 14, 17, 18]. The best convergence time among these works is $4n$ moves. The eventually lattice linear algorithm presented in [13] for a more generalized version, i.e., the service demand based MDS problem, takes $2m$ moves to converge.

In this paper, the fully lattice linear algorithm that we present converges in $n$ moves and is fully tolerant to consistency violations. This is an improvement as compared to the results presented in the literature.

**Colouring:** Self-stabilizing algorithms for decentralized (where nodes only read from their immediate neighbours) graph colouring have been presented in [1–3, 6–8, 13, 15, 16]. The best convergence time among these algorithms is $\Delta \times \Lambda$ moves, where $\Lambda$ is the maximum degree of the input graph.

The fully lattice linear algorithm for graph colouring that we present is decentralized and converges in $n + 2m$ moves and is fully tolerant to consistency violations. This is an improvement to the results presented in the literature.

10 CONCLUSION

In this paper, we focused on lattice linear problems and algorithms. Lattice linear problems [10] exhibit a lattice among the states. The induction of lattice allows execution of algorithms in concurrent systems [10] such that asynchronous and reading old values can be permitted (indefinitely, so long as every updated value reaches the nodes that are trying to read it). If the problem does not provide a lattice structure, the lattice structure can be induced by constraints imposed by the underlying algorithm. In [13] eventually lattice linear algorithms were introduced where the lattice structure is induced in a subset of the state space.

We bridge the gap between lattice linear problems of [10] and eventually lattice linear algorithms of [13] by introducing fully lattice linear algorithms. Unlike [10], here the lattice structure is not imposed by the problem but by the algorithm. Furthermore, unlike [13], lattice is formed among all reachable states in the state space rather than just a subset of states. We demonstrated the existence of fully lattice linear self-stabilizing algorithms by presenting algorithms for minimal dominating set (MDS) and graph colouring (GC).

Fully lattice linear algorithms presented in this paper overcome the limitations of [10] and [13] while preserving their key benefits. Fully lattice linear algorithms can be developed even for problems that are not lattice linear. This overcomes a key limitation of [10] where a solution could not be obtained for problems such as dominating set and colouring as they are not lattice linear. Specifically, since the lattice structures exist in the entire (reachable) state space, all the actions execute in the lattice structure. This overcomes a limitation of [13]. Fully lattice linear algorithms have a deterministic output. This benefit of [10] (that was lost in [13]) is preserved in fully lattice linear algorithms. Finally, fully lattice linear algorithms preserve a key benefit of [10, 13] that the computation is guaranteed to be correct even if a process is reading old information in its execution.

We analyzed the time complexity bounds of an algorithm traversing a lattice of states (whether present naturally in the problem or imposed by the algorithm). This bound remains valid for all kinds of problems where lattice linearity can be induced. It remains true in all kinds of synchronous and asynchronous settings, e.g., this bound stays true in any kind of distributed scheduler (as long as all nodes get a chance to execute infinitely often), and also if nodes are running asynchronously and reading old values.

A reader may wonder if the approach used for designing lattice linear algorithms can be generalized to all approximation algorithms. Unfortunately, the answer is no. Specifically, it remains an open question as to whether fully lattice linear algorithms can be developed for minimal vertex cover and maximal independent set problems even though eventually lattice linear algorithms exist for them [13]. Another open question is if a lattice linear self-stabilizing 2-approximation algorithm for vertex cover can be developed.

REFERENCES

[1] Apurv Bhartia, Deeparnab Chakrabarty, Krishna Chintalapudi, Lili Qiu, Bozidar Rudanovic, and Ramamohan R. Ramanjee. 2016. IQ-Hopping: Distributed Oblivious Channel Selection for Wireless Networks. In Proceedings of the 17th ACM International Symposium on Mobile Ad Hoc Networking and Computing (Paderborn, Germany) (MobiHoc ’16). Association for Computing Machinery, New York, NY, USA, 81–90. https://doi.org/10.1145/2942358.29423576

[2] Deeparnab Chakrabarty and Paul de Supinski. 2020. On a Decentralized $(\Delta+1)$-Graph Coloring Algorithm. Symposium on Simplicity in Algorithms (SOSA), SIAM, 91–98. https://doi.org/10.1137/1.9781611976014.13 arXiv:https://epubs.siam.org/doi/pdf/10.1137/1.9781611976014.13

[3] Alessandro Checo and Doug J. Leith. 2017. Fast, Responsive Decentralized Graph Coloring. IEEE/ACM Transactions on Networking 25, 6 (2017), 3628–3640. https://doi.org/10.1109/TNET.2017.275544

[4] Well Y. Chiu, Chiyuan Chen, and Shih-Yu Tsai. 2014. A 4n-move self-stabilizing algorithm for the minimal dominating set problem using an unfair distributed daemon. Inform. Process. Lett. 114, 10 (2014), 515–518.
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