APPENDIX SUBLOOPS AND FREIMAN’S THEOREM IN FINITELY GENERATED COMMUTATIVE MOUANG LOOPS

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Abstract. Fix a parameter $K \geq 1$. A $K$-approximate subgroup is a finite set $A$ in a group $G$ which contains the identity, is symmetric and such that $A.A$ can covered by $K$ left translates of $A$. This article deals with the generalisation of the concept of approximate groups in the case of loops which we call approximate loops and the description of $K$-approximate subloops when the ambient loop is a finitely generated commutative moufang loop. Specifically we have a Freiman type theorem where such approximate subloops are controlled by arithmetic progressions defined in the commutative moufang loops.

1. Introduction

This article deals with the study of sets of small tripling in non-associative, commutative Moufang loops. We study the connections between groups, quasigroups, loops and combinatorics. We shall make the relevant ideas more precise in the following sections but first let us recall the known literature in the case of groups.

1.1. History and Background. The formal study of the structure of approximate subgroups began with the celebrated theorem of Freiman

**Theorem 1.1** (Freiman [Fre64]). Let $A \subseteq \mathbb{Z}$ be a finite set of integers with small sumset $|A + A| \leq \alpha |A|

Then $A \subseteq P$ where $P$ is a $d$–dimensional arithmetic progression with $d \leq d(\alpha)$ and $|P| \leq C(\alpha)|A|$ (i.e., the length and dimension of the progression is a constant depending only on $\alpha$).

The above was a qualitative statement about the subsets of integers. The quantitative bounds on the dimension and size of the progression $P$ were established by Chang in [Cha02].

A natural question was the extension of this result for arbitrary abelian groups. However $\mathbb{Z}$ is a torison free abelian group and it was clear that Freiman’s theorem cannot hold exactly as stated for $\mathbb{Z}$, since if $G$ is a finite abelian group of high rank then $G$ itself satisfies the small doubling condition for every $\alpha \geq 1$, but it cannot be contained in any set $P$ of the form $\{x_0 + l_1x_1 + ... + l_rx_r : 0 \leq l_i \leq L_i\}$ and $r \leq C(\alpha)$.

Green and Ruzsa showed that this is essentially the only hindrance to Freiman’s theorem in an arbitrary abelian group in the sense that if $G$ is such a group and $A \subset G$ has doubling constant $\alpha$ then $A \subset H + P$ where $H$ is a subgroup and $P$ is an arithmetic progression with $|H + P| \leq C(\alpha)|A|$. $H + P$ is called a coset progression.

**Key words and phrases.** approximate groups, approximate loops, commutative Moufang loops.
Theorem 1.2 (Green-Ruzsa [GR07]). Suppose that $X$ is a symmetric subset of an abelian group $G$ with doubling constant at most $K$. Then there is a subgroup $H \subset 4X$ and an arithmetic progression $P = \{l_1x_1 + \ldots + l_rx_r : 1 \leq l_i \leq L_i, x_i \in G \forall i = 1, \ldots, r\}$ of rank $r$ at most $K^{O(1)}$ such that

$$x_i \in 4X \quad (i = 1, \ldots, r)$$

and

$$X \subset H + P \subset K^{O(1)}X$$

Extension of Freiman’s theorem for arbitrary groups and also establishing better quantitative bounds for particular classes of groups rapidly followed. Some of the notable works in this direction are by Tao - [Tao10], Sanders - [San12], Breuillard, Green, Tao - [BGT12], Tointon - [Toi14] etc to name a few. As our goal here is different so we shall not go into further details in those directions.

The formal definition of an approximate group was introduced by Tao in [Tao08] and was in part motivated by its use in the work of Bourgain-Gamburd [BG08] on super-strong approximation for Zariski dense subgroups of $SL_2(\mathbb{Z})$. Approximate groups were also used extensively in Helfgott’s seminal paper [Hel08].

Definition 1.3 (Approximate subgroup). Let $G$ be some group and $K \geq 1$ be some parameter. A finite set $A \subseteq G$ is called a $K$-approximate subgroup if

1. Identity of $G$, $e \in A$.
2. It is symmetric, i.e. if $a \in A$ then $a^{-1} \in A$.
3. There is a symmetric subset $X$ lying in $A.A$ with $|X| \leq K$ such that $A.A \subseteq X.A$

The notion of groups can be generalised to loops. Briefly speaking, a loop is a quasi-group with an identity element but not necessarily being associative. However loops are considered to have very little in the way of algebraic structure and it is for that reason we sometimes limit our investigations to loops which satisfy a weak form of associativity. Common examples are the Moufang loops. We leave the general definition of a Moufang loop for the next section and state the definition of a commutative moufang loop directly.

Definition 1.4 (commutative moufang loop). A loop $L$ is called a commutative moufang loop if

$$(1.1) \quad xy = yx,$$

$$x^2(yz) = (xy)(xz)$$

are satisfied for every $x, y, z \in L$.

In this article we generalize the notion of approximate subgroups in the case of loops. We call them approximate subloops (see 2.14) and show a structure theorem for approximate subloops of finitely generated commutative moufang loops. We shall formally define all the terms in the next section.

1.2. Main Result. (Freiman’s Theorem in finitely generated CML) Let $M$ be a $n$ generated commutative moufang loop. Let $A$ be a $K$-approximate subloop $A^2$ of $M$. Then $A^2$ is contained in a coset progression in $M$ of dimension (rank) $(K|M'|)^{O(1)}$ and of size of progression at most $\exp((K|M'|)^{O(1)}) |A^2|$. 
1.3. **Outline of the paper.** The paper is divided into the following sections.

(1) Introduction
(2) Preliminaries
(3) Properties of associator subloops
(4) Progressions in loops - Here we define the effective notion of arithmetic progressions in CMLs.
(5) Freiman’s theorem in CMLs - This section contains the lemmas and propositions for the main result of Freiman’s theorem for finitely generated CMLs
(6) Concluding remarks

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2. **Preliminaries**

To begin with we state the definitions and properties of groupoids, quasigroups, loops and related structures.

2.1. **Groupoids, Quasigroups, Loops, Moufang loops.**

**Definition 2.1** (Groupoids). Fix a positive integer $n$. An $n$-ary groupoid $(G, T)$ is a non-empty set $G$ together with an $n$-ary operation $T$ defined on $G$.

The order of an $n$-ary groupoid $(G, T)$ is the cardinality of $G$. An $n$-ary groupoid is said to be finite whenever its order is finite. We shall be dealing with binary groupoids and then we denote the operation $T$ as $\ast$.

**Definition 2.2** (Translation maps). Let $(G, \ast)$ be a groupoid and let $a$ be a fixed element in $G$. Then the translation maps $L(a) : G \to G$ and $R(a) : G \to G$ for each $a \in G$ are defined by

$$L(a)x = a \ast x, \quad R(a)x = x \ast a \quad \forall x \in G$$

**Definition 2.3** (Cancellation groupoids). A groupoid $(G, \ast)$ is called left cancellation (resp. right cancellation) if the left (resp. right) translation map $L(a)$ (resp. $R(a)$) is injective for any $a \in G : a \ast x = a \ast y \Rightarrow x = y$ for all $a, x, y \in G$ (resp. $x \ast a = y \ast a \Rightarrow x = y$ for all $a, x, y \in G$).

A groupoid $(G, \ast)$ is called cancellation if it is both left and right cancellation.

**Definition 2.4** (Division groupoids). A groupoid $(G, \ast)$ is called left division (resp. right division) if the left (resp. right) translation map $L(a)$ (resp. $R(a)$) is surjective for any $a \in G : a \ast x = b$ has solutions for any ordered pair of elements $a, b \in G$ (resp. $y \ast a = b$ has solutions for any ordered pair of elements $a, b \in G$).

A groupoid $(G, \ast)$ is called division groupoid if it is both left and right division.

**Definition 2.5** (Idem groupoids). An element $x$ such that $x \ast x = x$ is called an idempotent element of the binary groupoid $(G, \ast)$. A groupoid which has idempotent elements is called an idem groupoid.

**Definition 2.6** (Quasigroup). An $n$-ary groupoid $(G, T)$ with $n$-ary operation $T$ is called a quasigroup if in the equality $T(x_1, x_2, ..., x_n) = x_{n+1}$, knowledge of any $n$ elements among $x_1, x_2, ..., x_n, x_{n+1} \in G$ uniquely specifies the remaining one.
In binary case, this is equivalent to the following:

**Definition 2.7** (Binary quasigroup). A *binary groupoid* \((Q, \ast)\) is called a binary *quasigroup* if for all ordered pairs \((a, b) \in Q^2\) there exist unique solutions \(x, y \in Q\) to the equations \(x \ast a = b\) and \(a \ast y = b\).

From here onwards we shall be only concerned with binary operations. It is easy to see the following equivalent criteria for a quasigroup.

**Remark 2.8.** A *groupoid* \((G, \ast)\) is a *quasigroup* iff the maps \(L(a) : G \to G, R(a) : G \to G\) are bijections for all \(a \in G\).

From now on we shall drop the reference to the binary operation \(\ast\) and denote algebraic objects \((Q, \ast)\) as \(Q\) and operations \(a \ast x = y\) as \(ax = y\).

**Definition 2.9** (Loop). A quasigroup \(Q\) is a *loop* if \(Q\) possesses a neutral element, i.e., if there exists \(e\) such that

\[ ae = a = ea \]

for every \(a \in Q\).

Clearly the neutral element if it exists is idempotent so loops are idempotent groupoids. Also the neutral element is unique (but there might exist other idempotent elements).

Neither quasigroups nor loops are necessarily associative and so care needs to be taken when writing down complex expressions. We employ the following evolution rules, juxtaposition has the highest priority followed by \(\ast\) and then parentheses.

One usually studies loops satisfying some weak form of associativity. This leads us to the notion of *moufang loops*.

**Definition 2.10** (Moufang loop). A loop \(L\) is called a Moufang loop if the *Moufang identities*

\[
\begin{align*}
xy.zx &= x(yz.x), \\
x(y.xz) &= (xy.x)z, \\
x(y.zy) &= (x.yz)y
\end{align*}
\]

are satisfied for every \(x, y, z \in L\).

It must be noted that any one of the three identities implies the other two. Our main object of interest will be commutative moufang loops. For commutative moufang loops the above set of identities reduce to a single one

\[ x^2(yz) = (xy)(xz) \]

Let us give some examples.

**Example 2.11.** To begin with any group is trivially a Moufang loop. For the non-trivial case, we first give examples of non-commutative moufang loops and then commutative ones.

1. Octonions over the real numbers form a non-commutative, non-associative moufang loop.
(2) (Zorn’s vector matrix algebra). Let \( T \) be the set of matrices \((\alpha \ a, \ b \ \beta)\) with real scalars \(\alpha, \beta\) and real 3-vectors \(a, b\). Define the product of two elements of \( T \) by

\[
(\alpha \ a, \ b \ \beta) \cdot (\gamma \ c, \ d \ \delta) = (\alpha.\gamma + a.d, \ \alpha c + \delta a - b \times d).
\]

using the scalar product \(x, y\) and cross-product \(x \times y\) of 3-vectors. Let us define the determinant of a matrix by

\[
\det (\alpha \ a, \ b \ \beta) = \alpha \beta - a.b
\]

Let \( Q \) be the subset of \( T \) consisting of matrices whose determinant is 1. Let us set

\[
I = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right).
\]

Then \((Q, I)\) is a non-commutative and non-associative Moufang loop.

(3) (Zassenhaus’s Commutative Moufang loop) Let \( \mathbb{F}_3 \) denote the finite field with 3 elements. Let \( Q \) be the set \( \mathbb{F}_3^4 \). Define a new multiplication \( \circ \) on \( \mathbb{F}_3^4 \) for \( x = (x_1, x_2, x_3, x_4) \) and \( y = (y_1, y_2, y_3, y_4) \), by

\[
x \circ y = x + y + (0, 0, 0, (x_3 - y_3)(x_1y_2 - x_2y_1))
\]

Then \((Q, \circ)\) is a commutative moufang loop that is not associative.

(4) This construction is due to J.D.H. Smith. Let \( \mathbb{F}_3^4 \) and \( \mathbb{F}_3^4 \).

Let \( G \) be a group of exponent 3 with involution operator \( \sigma \), \( M(G) = \{x \in G : x^\sigma = x^{-1}\} \). Then \( M(G) \) is a commutative moufang loop of exponent 3 (i.e. \( x^3 = 1 \) for all \( x \) in \( M(G) \)) with respect to multiplication defined by \( x \ast y = x^{-1}y^3\). It is also clear that every 3 exponent CML can be obtained in this way.

Proof. It is clear to see that \( M(G) \) is a loop with identity element 1 \( \in G \). Also the Moufang identity is satisfied,

\[
x^3 \ast (y \ast z) = x^{-1} \ast (y^{-1}z^{-1}) = x(y^{-1}z^{-1})x
\]

\[
= (xy^{-1}x)(x^{-1}z^{-1}x)(xy^{-1}x) = x^{-1}y^{-1}x \ast (x^{-1}z^{-1}) = (x \ast y) \ast (x \ast z)
\]

(5) This is Bruck [Bru46] and has also been mentioned by L. Bénétou in [Ben84]. A group \( G \) is said to be 3-abelian if for all \( x, y \in G \) we have \( x^3y^3 = (xy)^3 \) identically. In such a group, any cube \( x^3 \) lies in the centre \( Z(G) \) of the group.

(Generic CML) If \( G \) is a non-trivial 3-abelian group ,

(a) the binary law \( x \ast y = x^{-1}yx^2 \) makes the set \( G \) a commutative moufang loop (denoted by \( G_\ast \)).

(b) the set product \((\mathbb{Z}/3\mathbb{Z}) \times G = \{(p, x) : p \in \mathbb{Z}/3\mathbb{Z}, x \in G\}\) together with the law of composition :

\[
(p, x) \ast (q, y) = (p + q, z_{p,q}(x, y))
\]

where \( z_{-1}(x, y) = xy, z_0(x, y) = x^{-1}yx^2, z_1(x, y) = xy \) becomes a CML \( \tilde{G}_\ast \) which contains \( \{0\} \times G \simeq G_\ast \) as a maximal subloop of index 3.

If we suppose that \( G \) is non-abelian then the associative centre of \( \tilde{G}_\ast \) consists of elements of the form \( (0, z) \) where \( z \in Z(G) \) while if \( G \) is abelian then \( \tilde{G}_\ast = \mathbb{Z} \times G \)

(c) When \( G \) is of exponent 3', the loops \( G_\ast \) and \( \tilde{G}_\ast \) also have the same form of exponent.
2.2. Approximate groups and approximate loops. To motivate the discussion on approximate loops, let us first recall Tao’s definition of approximate groups already stated in the introduction [1.3].

A finite set $A \subseteq G$ is called a $K$-approximate subgroup if

1. Identity of $G$, i.e. $e \in A$.
2. It is symmetric, i.e. if $a \in A$ then $a^{-1} \in A$.
3. There is a symmetric subset $X$ lying in $A.A$ with $|X| \leq K$ such that $A.A \subseteq X.A$

Remark 2.12. From the above definition we see that the property of associativity is not needed. But in the study of approximate groups we are mostly interested in the growth of iterated powers of sets. For example $A^2 \subseteq XA$ in $G$ automatically guarantees the fact that $A^n \subseteq X^{n-1}A$ in $G$ which implies that $|A^n| \leq K^{n-1}|A|$. In a non-associative loop it is not clear how to define the powers of sets. The fact that $AA \subseteq XA$ doesn’t guarantee $A(AA) \subseteq (X^2)A$ and also the balls $A, A^2, \ldots, A^n$ are not well defined, let alone the growth of them to be bounded in terms of a constant times $|A|$ (an essential condition for the study of approximate sets). To fix this problem we define the powers of a set $A$ in a loop $L$ as follows.

Notation 1. Let $L$ be a loop, $n \in \mathbb{N}$ and $A \subseteq L$ be a finite set. Let $C_n = \frac{1}{n+1} \binom{2n}{n}$ denote the $n$th Catalan number, which represents the number of ways of inserting parentheses in the expression $A.A.A.\ldots$. Let each $B_i$ for $i = 1, \ldots, n$ represent one such set. Then we define

$$A^{(n+1)} := \bigcup_{k=1}^{n+1} B_k$$

and

$$A^{n+1} := \cap_{k=1}^{n+1} B_k.$$

For example if $n = 3$ we have $C_3 = 5$, $B_1 = (A^2).(A^2)$, $B_2 = ((A^2)A)A$, $B_3 = (A(A^2))A$, $B_4 = A((A^2)A)$, $B_5 = A(A(A^2))$ and

$$A^{(4)} = ((A^2).(A^2)) \cup (((A^2)A)A) \cup ((A(A^2))A) \cup (A((A^2)A)) \cup (A(A(A^2)))$$

$$A^4 = ((A^2).(A^2)) \cap (((A^2)A)A) \cap ((A(A^2))A) \cap (A((A^2)A)) \cap (A(A(A^2)))$$

Remark 2.13. (1) If $1 \in A$ then $A^2 \subseteq A^n$ and hence $A^n$ is non-empty.
(2) In the case of groups we have $A^{(n)} = A^n$.
(3) In a commutative Moufang loop we have the identification $B_2 = B_3 = B_4 = B_5$ and hence the number reduces.

We are now in a position to define approximate loops.

Definition 2.14 (Approximate subloops). Let $L$ be a loop with $e$ the neutral element. Let $K \geq 1$ be a parameter and $A \subseteq L$ be a finite set. We say that $A$ is a $K$ approximate subloop of $L$ if

(i) $1 \in A$,
(ii) Symmetricity - $A$ is symmetric: $A = A^{-1}$,
(iii) Bounded growth condition - $|A^{(3)}| < K|A|$.

The definition is consistent as the usual operation of taking inverses is valid in loops in general. So for all subsets $S \subseteq L$ we know that $S^{-1}$ exists and $|S^{-1}| = |S|$. In this chapter we shall be mainly concerned with the growth of approximate subloops of commutative Moufang loops (CMLs in short).
**Remark 2.15.** In the case of approximate groups bounded growth condition and subset criterion are equivalent. We have seen that the subset criterion implies bounded growth of sets $A^n \forall n \in \mathbb{N}$. The other direction can be seen from - If $|A^3| \leq K|A|$, then $B := (A \cup A^{-1} \cup \{1\})^2$ is a $O(K^{O(1)})$-approximate group. Later we shall see how this result can be extended in the setting of finitely generated commutative moufang loops.

2.3. Properties of moufang loops.

2.4. Moufang loops. As a starting point we give some definitions and recall some properties of Moufang loops.

**Definition 2.16 (Associator).** Let $L$ be a loop and $x, y, z \in L$. Then the associator of $x, y, z$ denoted by $(x, y, z)$ is defined by

$$xy.z = (x.yz).(x, y, z)$$

or equivalently

$$(x, y, z) := (x.yz)^{-1}(xy.z)$$

The associator measures the deviation of a loop from associativity.

**Definition 2.17 (Commutator).** In a loop $L$, the commutator of $x, y \in L$, denoted by $[x, y]$ is defined as

$$[x, y] := x^{-1}y^{-1}xy$$

The commutator measures the deviation of a loop from commutativity.

**Definition 2.18 (Nucleus or centre).** The nucleus $Z(M)$ of a moufang loop $M$ (called centre in case of CMLs) is the set of elements in $M$ which commute and associate with all other elements in $M$,

$$Z(M) := \{ z \in M : [z, x] = 1, (z, x, y) = 1 \forall x, y \in M \}$$

**Remark 2.19.** The centre $Z(M(G))$ of the moufang loop $G$ is the set

$$\{ p \in M(G) : (p, x, y) = 1 \forall x, y \in G \} = Z(G)$$

**Definition 2.20 (power associativity).** A groupoid is said to be power associative if every element generates an associative subgroupoid.

Due to power associativity, the expression $x^n$ has a unique interpretation for every non-negative integer $n$ and every $x \in G$. In a power associative loop the identity is the unique idempotent element.

**Definition 2.21 (diassociativity).** A groupoid is said to be diassociative if every two elements generate an associative subgroupoid.

Due to diassociativity, we may omit parentheses in expressions involving only powers of two elements.

**Proposition 2.22.** In a diassociative quasigroup if there exists an idempotent element then it is unique.

**Proof.** Let us assume that there exists two idempotent elements $x$ and $y$. Then $x^2 = x$ and $y^2 = y$. Consider the well defined product $xy^2$. We get that $xy^2 = xy$ which implies $xy = x$. Similarly considering $x^2y$ we get that $xy = y$ (we use the property of cancellation in quasigroups). Hence $x = y$. \qed
Definition 2.23 (Homomorphisms and kernels). Let $M_1$ and $M_2$ be moufang loops. A single valued mapping $\theta$ of $M_1$ into $M_2$ is said to be a homomorphism of moufang loops if

$$\theta(ab) = \theta(a)\theta(b) \quad \forall a, b \in M_1$$

The kernel $K$ of $\theta$ is the set $\{k \in M_1 : \theta(k) = 1 \in M_2\}$

Definition 2.24 (semi-endomorphism). A single valued mapping $\theta$ of the Moufang loop $G$ into itself is called a semi-endomorphism of $G$ provided that

$$\theta(xyx) = (\theta x)(\theta y)(\theta x), \quad \theta 1 = 1.$$ for all $x, y \in G$.

Definition 2.25 (subloops and normal subloops). Let $M$ be a loop. A subset $H \subseteq M$ is a subloop of $M$ iff $H \subseteq M$ and $a.b \in H$ for all $a, b \in H$. A subloop $N$ of $M$ which satisfies $xN = Nx, \ (Nx)y = N(xy), \ y(xN) = (yx)N$ is called a normal subloop of $M$. We shall denote $N$ normal in $M$ by $N \trianglelefteq M$.

Remark 2.26. We note that

1. The intersection of subloops of a loop $M$ is also a subloop of $M$.
2. In the loop $M$ for $A \subseteq M$, the subloop generated by $A$ is denoted by $\langle A \rangle$ and defined to be the intersection of all subloops of $M$ containing $A$. It is the smallest subloop containing $A$.
3. Any subgroup of the nucleus $Z(M)$ of a loop $M$ is a normal subloop of $M$.
4. Let $N$ be a normal subloop of $M$. The right cosets $Nx$ of $M$ modulo $N$ partition $M$. That is

$$y \in Nx \iff Ny = Nx$$

Definition 2.27 (quotient loops). Let $M$ be a CML and $N \trianglelefteq M$. Then $M/N$ has the structure of a loop and is called the quotient loop of $M$ modulo $N$. The right cosets $Nx$ of $M \mod N$ partitions $M$ and $x \equiv y \mod N$ iff $xy^{-1} \in N$.

The fundamental theorem for Moufang loops is the following which ensures that we don’t need to bother to give parentheses when evaluating expressions in moufang loops involving only two elements and their powers. This was proved by Moufang in [Mou35] simultaneously for Moufang loops and alternative division rings.

Theorem 2.28 (Moufang’s theorem). Moufang loops are diassociative.

Proof. See [Mou35]. □

2.5. Properties of commutative moufang loops. The following lemma forms the basis for calculation in commutative moufang loops. It can be found in any standard text on commutative moufang loops (for example [Bru58]) but since we shall be using the identities repetitively so we state it here.

Lemma 2.29 (Identities involving associators in case of commutative moufang loops). In a commutative moufang loop $M$ with $x, y, z, w \in M$ the associator has the following properties

1. $(x, y, z) = (y, z, x) = (y, x, z)^{-1}$
2. $(x^{-1}, y, z) = (x, y, z)^{-1}$ and $\forall m, n, p \in \mathbb{Z}, (x^m, y^n, z^p) = (x, y, z)^{mnp}$
3. $(xy, z, w) = \{(x, z, w)(y, z, w)\}\{(x, z, w), x, y\}\{(y, z, w), y, x\}$.
4. $(x, y, z)^3 = 1$
Proof. The proof depends on several associator results. Let $L(x, y)$ and $R(x, y)$ denote the mappings

$$L(x, y) = L(yx)^{-1}L(y)L(x), \quad R(x, y) = R(yx)^{-1}R(y)R(x)$$

We have $L(z, y)(x) = x(x, y, z)^{-1}$.

Proof :- By definition, $(xy)z = [x(yz)](x, y, z)$. Hence $(x, y, z)^{-1} = [z^{-1}(y^{-1}x^{-1})][x(yz)]$ and $x(x, y, z)^{-1} = \{x[z^{-1}(y^{-1}x^{-1})]x\}(yz) = [(x^{-1}y^{-1})(yz)] = R(z^{-1}, y^{-1})(x) = L(z, y)(x)$.

Since Moufang loops are power associative for every semi-endomorphism $(\theta, \phi)$ and for all $x \in M$ we have $\theta(x^n) = (\theta(x))^n$. Now $L(z, y)$ is a semi-endomorphism of $M$ and hence we have $x^n(x^n, y, z)^{-1} = [x(x, y, z)^{-1}]^n$ for all integers $n$. Taking $n = -1$ we have the first result of (2) and $(x^n, y, z)^{-1} = (x, y, z)^{-n}$ from which we get that $(x^n, y, z) = (x, y, z)^n \forall n \in \mathbb{Z}$. The rest of (2) follows.

For (1) we first prove the fact that $(x, y, z) = (x, y, z)$. This is a direct consequence of $L(x, y) = L(xy, y) = L(x, yx)$ and $L(z, y)x = x(x, y, z)^{-1}$. Using this we get that $(x, y, z) = (xy, z, y)^{-1}$. By (2) we have $(x, y, z) = (x^{-1}, y, z)^{-1} = (x^{-1}y, z, y) = (y^{-1}x, z, y)^{-1} = (y^{-1}x, z, x)^{-1} = (y^{-1}, x, z) = (y, x, z)^{-1}$; this, applied to $(x, y, z) = (x, z, y)^{-1}$, gives $(x, y, z) = (x, z, y)^{-1} = (z, x, y)^{-1} = (z, y, x)^{-1} = (y, z, x)$. This shows (1).

To show (3), we first prove the following lemma:

**Lemma 2.30.** If $w, x, a, b$ are elements of a commutative Moufang loop $M$, then $(wa)(xb) = (wxc)$ where $c = pq^{-1}$ and

$$p = [R(w, x)a][R(x, w)b] = [a(a, w, x)][b(b, x, w)], \quad q = (w^{-1}x, R(w, x)a, R(x, w)b)$$

Proof. We multiply the equation $(wx)c = (wa)(xb)$ by $w^{-2}$ to get $x(w^{-1}c) = a[w^{-1}(xb)]$. If $\theta = R(w, x)$ then $R(w^{-1}, x) = \theta^{-1}, R(x, w^{-1}) = \theta$. Hence $x(w^{-1}c) = (xw^{-1})(\theta^{-1}(c))$ and $w^{-1}(xb) = (w^{-1}x)(\theta(b))$, so the equation becomes $(w^{-1}x)(\theta^{-1}(c)) = a[(w^{-1}x)(\theta(b))]$. Also $\theta^3 = I$ gives us $\theta^2 = \theta$. Moreover, $\theta$ leaves $w$ and $x$ fixed. Hence

$$(w^{-1}x)c = \theta((w^{-1}x)(\theta^{-1}(c))) = \theta(a[(w^{-1}x)(\theta(b))]) = (\theta(a))[(w^{-1}x)(\theta^{-1}(b))] = [(w^{-1}x)(\theta^{-1}(b))](\theta(a)).$$

From here we get

$$(w^{-1}x)c = (w^{-1}x)[((\theta^{-1}(b))(\theta(a)))(w^{-1}x, \theta^{-1}(b), \theta(a))],$$

whence

$$c = [(\theta(a))(\theta^{-1}(b))] \times (w^{-1}x, \theta(a), \theta^{-1}(b))^{-1}.$$

Since $\theta = R(w, x)$, $\theta^{-1} = R(x, w)$, the proof is complete.

Now we deduce the expansion formula for $(wx, y, z)$. Since $\phi = R(y, z)$ is an automorphism,

$$(wx)(wx, y, z) = \phi(wx) = \phi(\phi(x)) = [w(w, y, z)][x(x, y, z)].$$

We apply the above lemma with $a = (w, y, z), b = (x, y, z), c = (wx, y, z)$. In this case, replacement of $y$ by $y^{-1}$ replaces $a, b, c, p$ by their inverses but leaves $q$ fixed. Hence we have both $c = pq^{-1}$ and $c^{-1} = p^{-1}q^{-1}$. By multiplication, $1 = q^{-2}$. However, $q^3 = 1$ and so $q = 1$. Therefore $c = p, 1 = q$ that is,$$
(w, y, z) = [R(w, x)(w, y, z)][R(x, w)(x, y, z)] = [(w, y, z)(w, y, z), w, x][[(x, y, z)(x, y, z), x, w]$$

To show (4), we use the fact that $(xy)^3 = x^3y^3$ by Moufang’s theorem and commutativity for all $x, y \in M$. For each $x$ in $M$, the mapping $T(x) = R(x)L(x)^{-1} = I$ and also $T(x)$ is a
The lower central series
{and the upper central series
{element
c
M
x, y
for all
mapping group. Each element
x
be the subloop generated by all associators
(associator subloops, central series and derived series)
Definition 3.1
denoted by
⟨
M
x
endomorphism
1
pseudo-automorphism
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while the derived series is defined as
Theorem 2.31 (Bruck-Slaby). If M is a commutative moufang loop generated by n elements
then M is centrally nilpotent and if n > 2 then the nilpotency class (size of the lower central series) is bounded by n − 1.
Proof. See [Bru58]. □

3. Associator subloops
In this section we collect some of the important definitions and results concerning associator subloops following Bruck [Bru58] which we shall need later. Let us first recall the definition of a normal subloop of a CML M. N is said to be normal in M (denoted by
N ⋖ M) if it’s a subloop and it satisfies

\[ xN = Nx, \ (Nx)y = N(xy), \ y(xN) = (yx)N \]

for all \( x, y \in M \).

Definition 3.1 (associator subloops, central series and derived series). If A, B, C are normal subloops of a commutative moufang loop M one defines the associator subloop of A, B, C denoted by \( \langle (A, B, C) \rangle \) [or sometimes just as \( (A, B, C) \) when it’s clear from the context] to be the subloop generated by all associators \( (a, b, c) \) with \( a \in A, b \in B, c \in C \).
The lower central series \( \{ M_i \} \) of a commutative moufang loop M is defined by

\[ M_0 = M, M_{i+1} = \langle (M_i, M, M) \rangle, \ i = 0, 1, \ldots; \]

and the upper central series \( \{ Z_\alpha \} \),

\[ Z_0 = 1, \ Z_{i+1}/Z_i = Z(M/Z_i) \forall i = 0, 1, \ldots; \]

while the derived series is defined as

\[ M^{(0)} = M, M^{(i+1)} = (M^{(i)})' = \langle (M^{(i)}, M^{(i)}, M^{(i)}) \rangle, \ i = 0, 1, \ldots; \]

The first lemma is the normality of the associator subloop \( \langle (A, B, C) \rangle \) when A, B, C are normal in M.

Lemma 3.2. Let M be a CML and A ⋖ M, B ⋖ M, C ⋖ M. If A, B, C are generated by the self-conjugate subsets \( U, V, W \) respectively, then their associator subloop \( \langle (A, B, C) \rangle \) is generated by the set of all \( (u, v, w) \) with \( u \in U, v \in V, w \in W \). In particular \( \langle (A, B, C) \rangle \leq M \).
Proof. We shall briefly state the steps in the proof (for details see [Bru58]): Let H be the subloop generated by the set P consisting of all associators \( (u, v, w) \), \( u \in U, v \in V, w \in W \). Then the fact that \( U, V, W \) are self conjugate and every inner mapping of M is an automorphism of M implies P is also self conjugate and H is normal in M. So we can consider the quotient loop G/H which is a commutative moufang loop. The set X of all \( x \in M \) such that \( (x, V, W) = 1 \ mod \ H \) is a subloop of M. Since X contains U so X contains

1A permutation S of a loop M is called a pseudo-automorphism of M provided there exists at least one element c of M, called a companion of S, such that \( (xS)(yS.c) = (xy)S.c \) holds for all \( x, y \) of M.

1A non-empty subset S of a loop M is called self-conjugate in M if \( S^G \subset S \) where \( G = G(M) \) is the inner mapping group. Each element \( x \in M \) determines a self-conjugate subset \( x^G \) called the conjugate class of \( x \) in M.
Lemma 3.4. If $A, B, C, X, Y$ are normal subloops of the commutative moufang loop $M$, then
\[
\langle\langle (A, B, C), X, Y \rangle \rangle \subseteq \langle\langle (A, X, Y), B, C \rangle \rangle \langle\langle (B, Y, X), C, A \rangle \rangle \langle\langle (C, X, Y), A, B \rangle \rangle
\]

Proof. By the previous lemma, $\langle\langle (A, B, C), X, Y \rangle \rangle$ is generated by the set of all elements $(a, b, c, x, y)$ with $a, b, c, x, y$ in $A, B, C, X, Y$ respectively. If $a$ is in $A$ and $p, q$ are in $M$, then $(a, p, q)$ is in $A$. Using the product rule for associators
\[
(a, x, y) = (a, (x, y)) = (a, x, y)
\]
and the fact that every CML satisfies $(p, (p, w, x), (p, y, z)) = 1$ we have, $(a(a, x, y), b, c) = [(a, b, c)((a, x, y), b, c)]p$ where $p = (((a, x, y), b, c), (a, y, a))$. Therefore $(a(a, x, y), b, c) \equiv (a, b, c)$ mod $\langle\langle (A, X, Y), B, C \rangle \rangle$. If $P$ denotes
\[
\langle\langle (A, X, Y), B, C \rangle \rangle \langle\langle (B, Y, X), C, A \rangle \rangle \langle\langle (C, X, Y), A, B \rangle \rangle
\]
continuing as above we have
\[
(3.1) \quad (a(a, x, y), b, c) \equiv (a, b, c) \mod P
\]
and similarly for any permutation of $a, b, c$. If $\theta = R(x, y)$, then
\[
(a, b, c)((a, b, c), x, y) = \theta(a, b, c) = \theta(a) \theta(b) \theta(c) = (a(a, x, y), b(b, x, y), c(c, y)).
\]
Therefore, using (3.1) thrice, $(a, b, c)((a, b, c), x, y) \equiv (a, b(b, x, y), c(c, x, y)) \equiv (a, b, c(c, x, y)) \equiv (a, b, c) \mod P$ which implies $(a(a, c), x, y) \equiv 1 \mod P$. We are done.

We come to one of the main lemmas in this section which relates elements of the lower central series, the upper central series and the derived series.

Lemma 3.3. If $A, B, C, X, Y$ are normal subloops of the commutative moufang loop $M$, then
\[
\langle\langle (A, B, C), X, Y \rangle \rangle \subseteq \langle\langle (A, X, Y), B, C \rangle \rangle \langle\langle (B, Y, X), C, A \rangle \rangle \langle\langle (C, X, Y), A, B \rangle \rangle
\]

The next lemma gives us an inclusion between associator subloops which is required to deduce relations between lower and upper central series in CMLs.

Lemma 3.4. If $M$ is a commutative moufang loop,
\[
\langle\langle M_i, M_j, M_k \rangle \rangle \subseteq M_{i+j+k+1}
\]
\[
\langle\langle M_i, M_j, Z_k \rangle \rangle \subseteq Z_{k-i-j-1}
\]
\[
M^{(i)} \subseteq M_{(3^j-1)/2}
\]
for all $i, j, k \geq 0$.

Proof. We proceed by induction on $j$. Start with $j = 0$ so that $M_0 = M$. The identity
\[
\langle\langle M_i, M_0, M \rangle \rangle \subseteq M_{i+1}
\]
holds for all non-negative integers $i$. Let us assume that it holds for all $i \geq 0$ and some $j$. Then for $j + 1$ we have
\[
\langle\langle M_i, M_{j+1}, M \rangle \rangle = \langle\langle M_i, (M_j, M, M), M \rangle \rangle = \langle\langle (M_j, M, M), M_i, M \rangle \rangle
\]
\[
\subseteq \langle\langle (M_j, M, M), M_i, M \rangle \rangle \langle\langle (M_i, M_j, M), M \rangle \rangle \langle\langle (M_i, M_j, M), M_i \rangle \rangle \subseteq M_{i+j+2},
\]
This implies that the statement holds for all $i, j > 0, k = 0$. We now proceed by induction on $k$ to get the first result.
Next we show that $M^{(i)} \subseteq M_{(3^i-1)/2}$. The result holds for $i = 0, 1$. Let us assume that it holds for some fixed $i$. Then using the first result we have that

$$M^{(i+1)} = \langle (M^i, M^i, M^i) \rangle \subseteq \langle (M_{(3^i-1)/2}, M_{(3^i-1)/2}, M_{(3^i-1)/2}) \rangle \subseteq M_{3^i+3^{i+1} - 1} = M_{(3^{i+1}-1)/2}$$

The result follows by induction.

Finally, from the definition of the upper central series, for $i = j = 0$ and for all $k$ we have $\langle (M, M, Z_k) \rangle \subseteq Z_{k-1}$. We use the previous lemma and induction to show this for all non-negative $i, j$.

**Proposition 3.5.** Let $M$ be a CML generated by $n$ elements. Then the derived subloop $M' = \langle (M, M, M) \rangle$ of $M$ is finite.

**Proof.** Let $M$ be generated by a set $S$ with $|S| = n$. By Theorem 2.31, we know that $M$ is centrally nilpotent of class $t$ where $t \leq n - 1$. For $t = 0, 1$, we have $M'$ is either trivial or a finitely generated group of exponent 3 and hence is finite (by the restricted Burnside theorem). The rest of the proof is based on induction. Let $t = k + 1$ with $k \geq 1$ and let us assume that the statement is true for commutative moufang loops of class at most $k$. The quotient loop $H = M/M_k$ has nilpotency class $k$ and is finitely generated. This implies by the induction hypothesis that $H' = M'/M_k$ is finite. All that now remains to be shown is that $M_k$ is finite.

If $k > 1$ then $M_{k-1} \subseteq M'$ and hence $M_{k-1}/M_k$ is a subloop of $H'$, implying that $M_{k-1}/M_k$ is finite. If $k = 1$ then $M_{k-1}/M_k = M/M_k$ is finitely generated. We can thus find a finite non-empty subset $T$ of $M$ such that $T$ and $M_k$ generate $M_{k-1}$. We also have the fact that $S$ generates $M$.

**Claim 1.** $\langle (T, S, S) \rangle$ is finite and $M_k = \langle (T, S, S) \rangle$.

**Proof of claim.** Clearly $J = \langle (T, S, S) \rangle$ is a finitely generated subloop of $M_k$. $M_k \subset Z(M)$ where $Z(M)$ denotes the centre of $M$ and hence $J$ is a normal subloop of $M$. $J$ is in fact a finitely generated abelian group of exponent 3 and so $J$ is finite. Using the fact that $S$ generates $M$, we have $\langle (T, M, M) \rangle = \langle (T, S, S) \rangle = J$. Now $\langle (M_k, M, M) \rangle = M_{k+1} = 1$. But $T$ and $M_k$ generate $M_{k-1}$ which means that $\langle (M_{k-1}, M, M) \rangle \subseteq \langle (T, M, M) \rangle \cup \langle (M_k, M, M) \rangle \subseteq J$.

Thus we have

$$M_k = \langle (M_{k-1}, M, M) \rangle \subset J \subset M_k.$$

Thus $M_k$ is finite which implies $M'$ is finite.

4. Progressions in commutative moufang loops

We follow the notation of group theorists with respect to mappings. Let us define a notion of arithmetic progressions (APs) in commutative moufang loops. We recall the notion of arithmetic progressions in case of arbitrary abelian groups.

**Definition 4.1** (arithmetic progression in groups). Let $G$ be an abelian group. An arithmetic progression of dimension $d$ and size $L$ is a set of the form

$$P = \{v_0 + l_1v_1 + \ldots + l_dv_d : 0 \leq l_j < L_j \}$$

where $l_1l_2\ldots l_d = L$. $P$ is said to be proper if all of the sums in the above set are distinct, in which case $|P| = L$. 
The dimension of an arithmetic progression is the measure of its linear independence. An equivalent notion of arithmetic progressions of dimension \(d\) can be defined in case of commutative moufang loops but care must be taken because the lack of associativity implies that even if the dimension is fixed the same "formal" expression for APs can have multiple values depending on the position of the parentheses. A naive way to define an AP in case of a commutative moufang loop is to look at the inverse images of the projection onto the quotient loop. Formally,

**Definition 4.2** (Generalised arithmetic progression in a CML). Let \(M\) be a commutative moufang loop. Let \(M' = (M, M, M)\) denote its associator subloop which is a normal subloop of \(M\). Let \(\pi : M \rightarrow M/M'\) denote the projection map onto the abelian group \(M/M'\). Let \(P\) be an arithmetic progression of dimension \(d\) in \(M/M'\). Then \(\pi^{-1}(P)\) is defined to be the generalised arithmetic progression of dimension \(d\) in \(M\).

**Remark 4.3.** Let \(P = \{l_1 v_1 + l_2 v_2 + l_3 v_3 : |l_j| < 5\}\). A typical element of a generalised AP in a CML looks like \(((x_1^j x_2^j) x_3) m\) where \(m \in M'\) and \(x_i = \pi^{-1}(v_i), i = 1, 2, 3\).

**Definition 4.4** (Equality of generalised AP's). Two elements \(R_1\) and \(R_2\) of a generalised AP are said to be equal if \(\pi(R_1) = \pi(R_2)\). Alternatively if \(\exists m \in M'\) such that \(R_1 = R_2 m\). This an equivalence relation which partitions the set of all generalised AP's into equivalence classes.

We shall state a special form of APs called canonical APs

**Definition 4.5** (Canonical form of generalised arithmetic progressions). Let \(x_1, x_2, ..., x_r\) be fixed elements in a commutative moufang loop \(M\) and \(L = (L_1, ..., L_r)\) be a vector of positive integers. Then the set of all products in the \(x_i\) and their inverses which are of the form

\[
\{(...(x_1^{l_1} x_2^{l_2} x_3^{l_3})...) x_r^{l_r} : |l_i| < L_i \forall i = 1, 2, ..., r\}
\]

is a canonical form of an AP of rank \(r\) and side lengths \(L\).

Intuitively the canonical form of a generalised AP is easier to handle given that the position of the parentheses are fixed and so two APs in canonical form can be easily compared. But we need a method to convert each generalised AP into its canonical form. This process is called associator collection.

**Lemma 4.6** (associator collection). Let \(x_1, ..., x_r\) be elements in a commutative moufang loop \(M\) and let \(S\) be a finite string defined on \(M\) containing the elements \(x_1, ..., x_r\) which occur \(l_1, ..., l_r\) times respectively. Then \(S\) can be written as

\[
S = (((...(x_1^{l_1} x_2^{l_2} x_3^{l_3})...) x_r^{l_r}) m
\]

with \(m \in M' = ((M, M, M))\)

**Proof.** Use lemma 3.4 to see that \(((M_i, M_j, M_k)) \subseteq M'\) for all \(i, j, k \geq 0\). Consider two strings \(S_1, S_2\) having the same elements, the same number of times and differing only in the placement of the parentheses. Then it is direct to see that \(S_1 = S_2 m\) where \(m \in M'\). For example take \(S_1 = (x_1^2 x_2^2) x_3, S_2 = (x_1^2 (x_2 x_3)) x_2\). Then

\[
S_2 = (x_1^2 (x_2 (x_2 x_3)))(x_1^2, x_2, x_2 x_3)
\]

\[
= (((x_1^2 x_2^2) x_3)(x_1^2, x_2, x_3))(x_1^2, x_2, x_2 x_3)
\]

\[
= S_1((x_1^2 x_2^2) x_3, (x_1^2, x_2, x_3), (x_1^2, x_2, x_2 x_3))
\]
where \( m_1 \in M' \). Taking \( m = m_1^{-1} \) we get the result. \( \square \)

**Lemma 4.7.** Each generalised AP of dimension \( d \) can be written as \( Rm \) where \( R \) is an AP in canonical form of dimension (rank) \( d \) and \( m \in M' \).

**Proof.** It is sufficient to collect the terms in the formal expression of a generalised AP of dimension \( d \). \( \square \)

We also have an implicit notion of an AP inside a CML (without using the abelian quotient). We call this a “usual” arithmetic progression.

**Definition 4.8** (Usual arithmetic progression in a CML). Let \( x_1, x_2, \ldots, x_r \) be fixed elements in a commutative moufang loop \( M \) and \( L = (L_1, \ldots, L_r) \) be a vector of positive integers. Then the set of all products in the \( x_i \) and their inverses, in which each \( x_i \) and \( x_i^{-1} \) appear at most \( L_i \) times between them and also the position of the parentheses vary is called a “usual” arithmetic progression of rank \( r \) and side lengths \( L_1, \ldots, L_r \). It is denoted by \( P^{\ast}_{\text{cml}}(x_1, \ldots, x_r; L) \).

**Lemma 4.9.** Let \( M \) be a CML. Each usual AP in \( M \) can be reduced to an element of the canonical form of a generalised AP and two usual APs are said to be equivalent (denoted by \( \equiv \)) if they have the same canonical portion after reduction.

**Proof.** Start with any usual AP of rank (dimension) \( r \) and apply the process of collection. \( \square \)

**Remark 4.10.** Take \( P^{\ast}_{\text{cml}}(x_1, x_2, x_3; 4, 4, 4) \) of rank 3. Two typical elements of this usual arithmetic progression look like

\[
((x_1x_2^2)(x_1^{-1}(x_3^{-1}x_2^{-1}))(x_3^2(x_1^2x_2))) \quad \text{and} \quad ((x_1x_2^2)(x_1^{-1}(x_3^{-1}x_2^{-1}))x_3^2(x_1^2x_2))
\]

Note that they are the same in case of groups but not in case of commutative moufang loops. The above two elements have the same canonical portion after reduction (which is equal to \( (x_1^2x_2^2)x_3 \)).

**Lemma 4.11.** Let \( M \) be an \( n \)-generated CML. If we have two usual APs, \( P^{\ast}_{\text{cml}} \) and \( Q^{\ast}_{\text{cml}} \) with \( P^{\ast}_{\text{cml}} \equiv Q^{\ast}_{\text{cml}} \) then \( |P^{\ast}_{\text{cml}}| \leq |M'||Q^{\ast}_{\text{cml}}| \) and \( |Q^{\ast}_{\text{cml}}| \leq |M'||P^{\ast}_{\text{cml}}| \).

**Proof.** \( M \) is an \( n \)-generated commutative moufang loop. Hence \( |M'| \) is finite. Take two equivalent usual arithmetic progressions and reduce it to the canonical generalised form. Let them be \( R_1m_1 \) and \( R_2m_2 \) where \( R_1 = R_2 \) (since they are equivalent) and \( m_1, m_2 \in M' \). As \( M' \) is a subloop, \( m_1m_2^{-1} \in M' \) and the result follows. \( \square \)

5. Structure of approximate subloops of finitely generated commutative moufang loops

We start with generalisation of some well known results in groups adapted in the case of commutative moufang loops.

**Lemma 5.1** (Ruzsa’s covering lemma for loops). Let \( S, T \) be subsets of a loop \( L \) such that \( |S.T| \leq K|S| \). Then there is a set \( X \subset T \) with \( |X| \leq K \), such that \( T \subset S^{-1}(SX) \).
Lemma 5.2 (Ruzsa inequality for CML). Let $A, Y, Z$ be finite sets in a commutative moufang loop $M$. Let $[A, Y, Z] = \{(a, y, z) : a \in A, y \in Y, z \in Z\}$. Then we have
\[ |A||YZ^{-1}| \leq |A^{-1}Y| \times |AZ^{-1}| \times ||A, Y, Z|| \]

Proof. If $yz^{-1}$ is an element of $YZ^{-1}$ with $y \in Y$ and $z \in Z$ then use \ref{2.30} to see that
\[ (ya^{-1})(az^{-1}) = (yz^{-1})(y, a, z) \]
Hence we have that $yz^{-1}$ can be written as a product of an element of $A^{-1}Y$ and an element of $AZ^{-1}$ in at least $|A|/||A, Y, Z||$ ways.
\[
\square
\]

Remark 5.3. A special case is when one of $A, Y, Z$ is in the centre $Z(M)$ of the Moufang loop $M$. For example in a CML one has $\forall x \in M x^3 \in Z(M)$. So if $A(3) = \{a^3 : a \in A\}$ has size $|A(3)| \geq |A|/C$ where $C$ is an absolute constant and if $A$ satisfies a small tripling condition ($A$ is an approximate loop) then a large portion of $A$ has bounded growth in terms of $|A|$.

An interesting result in the growth of finite sets in group theory is Petridis’s lemma \cite{Pet12}. It is a direct consequence to see that in the setting of finitely generated commutative moufang loops the same result holds with an extra term of the size of a finite set which depends on the loop.

Lemma 5.4 (Petridis type lemma for Loops). Let $X$ and $B$ be finite sets in a Moufang loop. Suppose that
\[ K := \frac{|XB|}{|X|} \leq \frac{|ZB|}{|Z|}, \forall Z \subseteq X. \]
Then for all finite sets $C$
\[ |(CX)B| \leq K||C, X, B|||CX| \]
where $[A, B, C] := \{(a, b, c) : a \in A, b \in B, c \in C\}$ and $(a, b, c) := (ab.c)(a.bc)^{-1}$

Proof. In the proof of \cite{Pet12} we need to add the associator subset $[C, X, B]$. Let $C = \{c_1, ..., c_r\}$ be ordered arbitrarily.
\[ CX = \bigcup_{i=1}^{r} (c_i X_i) \]
where $X_1 = X$ and for $i > 1$
\[ X_i = \{x \in X : c_i x \notin \{c_1, ..., c_{i-1}\} X\} \]
Then for all $j$
\[ \{c_1, ..., c_j\} X = \bigcup_{i=1}^{j} (c_i X_i) = \bigcup_{i=1}^{j} (c_i X_i) \]
The sets $c_i X_i$ are all disjoint and for all $j$, we have
\[ \{|c_1, ..., c_j\} X| = \sum_{i=1}^{j} |c_i X_i| = \sum_{i=1}^{j} |X_i| \]
We now apply induction on $r$.
Step 1: For $r = 1$ we have
\[ |(c_1 X)B| \leq |c_1(XB)||[c_1, X, B]| = |X||[c_1, X, B]| = K||[c_1, X, B]| |c_1 X| \]
Step 2: For $r > 1$ let us write $X^c_r = X \setminus X_r$, for the complement of $X_r$ in $X$. By definition of $X_r$, $c_r X^c_r \subseteq \{c_1, c_2, ..., c_{r-1}\} X$ and $(c_r X^c_r) B \subseteq (\{c_1, c_2, ..., c_{r-1}\} X) B$

$$(CX) B = ((c_1, c_2, ..., c_r) X) B = [(c_1, c_2, ..., c_{r-1}) X] B \cup [(c_r) X B \setminus (c_r) X^c_r) B]$$

We note that $|(c_r X B \setminus (c_r X^c_r) B) | \leq |c_r (XB) \setminus c_r (X^c_r B)| ||c_r, X, B|| = (|XB| - |X^c_r B|)(||c_r, X, B||)$

This implies that

$$|(CX) B| \leq |\{(c_1, c_2, ..., c_{r-1}) X\} B| + (|XB| - |X^c_r B|)(||c_r, X, B||)$$

The first summand is bounded by the inductive hypothesis on $r$ namely

$$|\{(c_1, c_2, ..., c_{r-1}) X\} B| \leq K|\{c_1, ..., c_{r-1}\} X||C, X, B|| = |C(XB)| \leq K \max_{c \in C} ||c, X, B|| |CX|$$

and the second summand is at most $K |X_r||[c_r, X, B]|$

Combining the above we get that

$$|(CX) B| \leq K ||C, X, B|| |CX|$$

\[ \square \]

**Remark 5.5.** The proof of the above inequality shows that the following statements are also true -

$$|(CX) B| \leq K \max_{c \in C} ||c, X, B|| |CX|$$

and

$$|C(XB)| \leq K \max_{c \in C} ||c, X, B|| |CX|$$

The above two lemmas suggest that if we have a $K$ approximate subloop $A$ inside a commutative moufang loop such that the size of the associator set $|[A, A, A]|$ is independent of the size of $|A|$ then we have the following $|A^{(n)}| \leq C(n, K)|A|$ where $C$ is a constant independent of the size of $A$.

**Proposition 5.6** (Lemmas on approximate loops). Let $M$ be a finitely generated commutative moufang loop having derived subloop $M'$. Fix $K > 0$.

1. Let there exist $X, A \subseteq M$ with $|X| \leq K$, $A$ finite and $A^2 \subseteq X.A$. Then $|A^{(3)}| = |A^3| \leq K^2 |M'||A|$

2. Suppose $A \subseteq M$ be a symmetric set, $|A^3| \leq K |A|$. Then $|A^{(n)}| \leq \left[ \frac{C_{n-1}}{2^n} \right] K^{n-2} |M'||2^{n-6}|A|$

   (where $C_n$ denotes the $n$th Catalan number)

3. Let $\pi: M \to M'$ be a homomorphism of loops having a finite kernel (normal subloop) $N$. Let $A \subseteq M'$ be such that $A^2 \subseteq X.A$ with $|X| \leq K$. Then $(\pi^{-1}(A))^2 \subseteq Y.\pi^{-1}(A)$ where $Y = Y^{-1} \subseteq M$ and $|Y| \leq 2K$.

4. Let the notations be as in (2), then $\forall n \geq 3$ the following holds: if $T \subseteq A^{(n)}$ and $|TA| \leq K|A|$ then there exists $X \subseteq T$ with $|X| \leq K$ and $T \subseteq A^{-1}(AX)$

**Proof.**

1. It is clear to see that $A^2 \subseteq X.A$ implies $A^3 \subseteq (XA)A \subseteq (X(A^2))X, A, A) \subseteq (X(XA))M' \subseteq ((X^2)A)M'$ and we have the cardinality inequality.

2. We recall the Ruzsa inequality for CML,

$$|A||YZ^{-1}| \leq |A^{-1}Y| \times |AZ^{-1}| \times ||A, Y, Z|| \leq |M'||A^{-1}Y| \times |AZ^{-1}|.$$
Proposition 5.7. Let $M/M$ canonical homomorphism on both sides we have $\pi$ is an abelian group we see that $A$ derived subloop of $\pi$. For $n = 4$ we have $A^{(4)} = (A^3)A \cup (A^2).A$. Let $T = (A^2)^2$, choose $Y = Z = A^2$ and we have

$$|A||T| \leq |M'||A^3|^2 \leq K^2|M'|A^2 \Rightarrow |T| \leq K^2|M'||A|$$

Noting that $((A^3)A) \subseteq ((A^2)^2)M'$ we get that $|A^{(4)}| \leq 2(K|M'|)^2|A|$.

For $n = 5$ we have $A^{(5)} = (A^3)A^2 \cup ((A^2)^2)A \cup (A^3)A$ Applying Ruzsa’s lemma for CML to $(A^3)A^2 = A^3.A^2$ we get that

$$|A||A^{(3)}| \leq |M'||A^3|A^3| \leq K^3|M'|^3|A|^2,$$

hence $|A^{(3)}| \leq (K|M'|)^3|A|$. Considering the fact that the other elements in the union are subsets of $(A^3.A^2)M'$ and there are at most $\lceil \frac{2n}{K} \rceil$ sets in the union we get that $|A^{(5)}| \leq \lceil \frac{C_{12}}{2^k} \rceil K^3|M'|^4|A|$ (where $C_n = \frac{1}{n+1} \binom{2n}{n}$ denotes the $n$th Catalan number.)

Fix $k \in \mathbb{N}$. Let us assume that the result is true for all $n \leq k - 1$. For $n \leq k - 1$ by the induction hypothesis, $|A^{(k-1)}| \leq \lceil \frac{C_{12}}{2^{k-4}} \rceil K^{k-3}|M'|^{2(k-4)}|A|$. This implies that

$$|A||A^{(k-2)}|A^2| \leq |M'||A^{(k-2)}|A^3| \leq \lceil \frac{C_{12}}{2^{k-3}} \rceil K^{k-2}|M'|^{2(k-4)+1}|A|^2$$

Now writing $A^{(k)}$ in the union form and noting that $C_{k-1} = \Sigma_{i=0}^{k-2}C_iC_{k-i}$ we have $|A^{(k)}| \leq \lceil \frac{C_{12}}{2^{k-3}} \rceil K^{k-2}|M'|^{2(k-3)}|A|^2$.

(3) The fact that $K$ is finite implies that $\pi^{-1}(A)$ is finite. It remains to check the existence of a set $Y \subseteq M$ of cardinality at most $2K$ such that $(\pi^{-1}(A))^2 \subseteq Y.\pi^{-1}(A)$. By hypothesis there is a set $X \subseteq M'$ of cardinality at most $K$ such that $A^2 \subseteq XA$. For each $x \in X$ select an element $\omega(x) \in \pi^{-1}(x)$. Set $Y = \{\omega(x) : x \in X\} \cup \{\omega^{-1}(x) : x \in X\}$, so that $Y$ is symmetric and of cardinality at most $2K$. Given $a_1, a_2 \in \pi^{-1}(A)$, note that by definition of $X$ there exist $x \in X$ and $a \in \pi^{-1}(A)$ such that $\pi(a_1)\pi(a_2) = x\pi(a)$. This implies that there exists $b \in ker\pi$ such that $a_1a_2 = \omega(x)ab$. However, $ab\in \pi^{-1}(A)$, and so $a_1a_2 \in Y\pi^{-1}(A)$, as desired.

(4) The last one is a direct consequence of Ruzsa covering lemma.

\[\square\]

**Proposition 5.7.** Let $M$ be an $n$ generated CML. There exists a correspondence between approximate subloops of $M$ and approximate subgroups of $M/M'$ where $M'$ denotes the derived subgroup of $M$. If $A$ is an approximate subgroup in $M$ then the canonical homomorphism $\pi : M \rightarrow M/M'$ turns $\pi(A^2)$ into an approximate subgroup of $M/M'$ and if $B$ is an approximate subgroup inside the abelian group $M/M'$ then $\pi^{-1}(B)$ is an approximate subgroup inside $M$.

**Proof.** Let $A \subseteq M$ be a $K$-approximate subgroup in the CML $M$. We have $|A^3| \leq K|A|$. We use Ruzsa’s inequality for loops to conclude that $|A(A^3)| \leq K|M'||A|$ and Ruzsa’s covering lemma for loops to conclude that $A^3 \subseteq (XA)A$ where $X$ is of size at most $K|M'|$. Taking the canonical homomorphism on both sides we have $\pi(A^3) \subseteq \pi(X)\pi(A)^2$ in $M/M'$. As $M/M'$ is an abelian group we see that $\pi(A^3)^2 = \pi(A^3)\pi(A) \subseteq \pi(X)\pi(A^2)\pi(A) \subseteq \pi(X)\pi(A^2) \subseteq \pi(X)\pi(A)^2$. Hence $\pi(A^2)$ is a $\pi(X)$ approximate subgroup in $M/M'$ where $|X| \leq K|M'|$. 
In the other direction, consider the inverse image, \( \pi^{-1} : M/M' \to M \). We know that \( M \) is a \( n \) generated CML, hence \( M' \) is a finite loop (also it is the kernel of the homomorphism \( \pi \)). If we have a \( K \)-approximatif subloop (subgroup) \( B \) in \( M/M' \) (abelian group) then \( B^2 \subseteq Y.B \) (with \( |Y| \leq K \)). We can look at the inverse images \( \pi^{-1}(B) \) and conclude that \( \pi^{-1}(B) \) is a subloop such that \( (\pi^{-1}(B))^2 \subseteq Z.\pi^{-1}(B) \) (with \( |Z| \leq 2K \) and \( Z \in M \)) and \( \pi^{-1}(B) \subseteq M, \) \( |\pi^{-1}(B)| < \infty \) (we use the finiteness of the kernel \( M' \) here). Now from (1) of the previous proposition we can conclude that \( \pi^{-1}(B) \) is an approximate subloop of \( M \).

\[ \square \]

**Remark 5.8.** The essential point in the above proposition is the finiteness of \( M' \) which is guaranteed in a finitely generated commutative moufang loop. For moufang loops having approximate subloops \( A \) such that \( |[A, A, A]| \leq C \) where \( C \) is a constant independent of \( A \), we also have a same sort of structure theorem in the sense that these approximate subloops are essentially controlled by approximate subgroups in \( M/M' \). Note that commutativity is not required in this case.

**Definition 5.9** (coset progression in a CML). By a coset progression in a CML \( M \), we mean sets of the form \( \pi^{-1}(H)\pi^{-1}(P) \subseteq M \) where \( \pi \) is the projection map from \( M \) to \( M/M' \), \( H \) is a subgroup of \( M/M' \) and \( P \) is an arithmetic progression in \( M/M' \). The dimension or rank of the coset progression is the dimension of \( P \).

We finally state a sort of structure theorem for finitely generated commutative moufang loops.

**Theorem 5.10** (Freiman’s theorem for finitely generated commutative moufang loops). Let \( M \) be a \( n \) generated commutative moufang loop. Let \( A \) be a \( K \)-approximate subloop of \( M \). Then \( A^2 \) is contained in a coset progression in \( M \) of dimension (rank) depending on \( K|M'| \) and of size of progression at most \( f(K, |M'|)|A^2| \).

**Proof.** Let \( M \) be a \( n \) generated CML. Let \( A \subseteq M \) be a \( K \)-approximate subloop. Let \( M' = \langle (M, M, M) \rangle \) and

\[ \pi : M \to M/M' \]

be the canonical homomorphism. \( M' \) is finite and by the proposition 5.7 we have \( \pi(A^2) \) is a \( \pi(X) \) approximate group in the abelian group \( M/M' \) with \( |\pi(X)| \leq |X| \leq K|M'| \).

Applying Freiman’s theorem in case of general abelian groups, Theorem 1.2 we have \( \pi(A^2) \) is contained inside a coset progression \( H + P \) where \( P \) is of rank at most \( d(K|M'|) = (K|M'|)^{(1)} \) and size at most \( |P| \leq \exp((K|M'|)^{(1)}))|A^2| \). Thus \( A^2 \) is contained inside \( \pi^{-1}(H + P) \subseteq \pi^{-1}(H)\pi^{-1}(P) \) which has rank at most \( (K|M'|)^{(1)} \) and size of \( \pi^{-1}(P) \) at most \( \exp((K|M'|)^{(1)}))|A^2| \).

\[ \square \]

6. Concluding remarks

The question can be posed in the general setting of infinitely generated CMLs. A loop \( L \) is said to have finite rank if there exists an integer \( r \) such that every finitely generated subloop of \( L \) can be generated by at most \( r \) elements.

**Proposition 6.1.** Let \( M \) be a commutative Moufang loop having finite rank \( r \) associator subloop \( \langle (M, M, M) \rangle \). Let \( A \) be a \( K \)-approximate subloop of \( M \). Then \( A^2 \) is contained in a coset progression in \( M \) of dimension depending on \( C_M K \) and of size of progression at most \( f(K, C_M)|A^2| \) where \( C_M \) is an absolute constant depending on \( M \).
Proof. The proof of this proposition follows from the following lemmas.

**Lemma 6.2.** Every commutative moufang loop without elements of infinite order is locally finite.

*Proof.* Let $M$ be a commutative moufang loop without elements of infinite order and $N$ be a finitely generated subloop of $M$. Then $N/N'$ is a finitely generated abelian group without elements of infinite order. So $N/N'$ is finite. Using the same arguments as in Prop. 3.5 we can show that $N'$ is also finite. This implies that $N$ is finite and we are done.

**Lemma 6.3.** Let $M$ be a commutative moufang loop with associator subloop $M'$ and centre $Z(M)$, then $M'$ and $M/Z(M)$ are locally finite loops of exponent 3.

*Proof.* The mapping $x \rightarrow x^3$ is an endomorphism of $M$ into $Z(M)$. The loops $M'$ and $M/Z(M)$ therefore have exponent 3. Using Lemma 6.2 we have $M'$ and $M/Z(M)$ are locally finite.

**Lemma 6.4.** Let $M$ be commutative moufang loop having finite rank associator subloop $\langle (M, M, M) \rangle$. Let $A \in M$ be a finite set. Then $|\langle (A, A, A) \rangle|$ is finite and independent of $|A|$.

*Proof.* We recall that a loop $M$ is said to have finite rank if there exists an integer $r$ such that every finitely generated subloop of $M$ can be generated by at most $r$ elements. The subloop $\langle (A, A, A) \rangle$ is finitely generated and hence there exists an $r$ (uniform) such that it can be generated by at most $r$ elements. Also $\langle (A, A, A) \rangle$ is a CML of exponent 3 and hence is locally finite. Thus $\langle (A, A, A) \rangle$ is finite and its size is independent of $|A|$.

### 6.1. Manin’s problem and effective bounds on the progressions.

An interesting note to add is the dependency of the size and rank of the progression to the size of the associator subloop. Manin’s problem of specifying the 3 rank of a commutative moufang loop deals with the size of the associator subloops with respect to the number of generators [Man74]. Progress in this area has been pretty slow even for small values of $n$. An important approach to a solution to this question was done by J.D.H. Smith in 1982. He gave a hypothetical formula for calculating the 3 rank of a finitely generated CML provided the triple argument hypothesis holds. Under this assumption he was able to compute the orders of the associator subloops. The details can be found in [Smi82].

But it was shown by N. Sandu in [San87] that triple argument hypothesis doesn’t work for $n \geq 9$. Recent works by Grishkov and Sheshtakov [GST11] showed that the triple argument hypothesis actually fails for $n \geq 7$. Smith’s formula still gives correct results for $n \leq 6$.

### References

[Bén84] Lucien Bénéteteau, 3-abelian groups and commutative Moufang loops, European J. Combin. 5 (1984), no. 3, 193–196. MR 765625

[BG08] Jean Bourgain and Alex Gamburd, Uniform expansion bounds for Cayley graphs of $\text{SL}_2(\mathbb{F}_p)$, Ann. of Math. (2) 167 (2008), no. 2, 625–642. MR 2415383

[BGT12] Emmanuel Breuillard, Ben Green, and Terence Tao, The structure of approximate groups, Publ. Math. Inst. Hautes Études Sci. 116 (2012), 115–221. MR 3090256

[Bru46] R. H. Bruck, Contributions to the theory of loops, Trans. Amer. Math. Soc. 60 (1946), 245–354. MR 0017288

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2 The triple argument hypothesis says that the normal subloop of the loop generated by all associators in with an argument appearing thrice vanishes.
[Bru58] Richard Hubert Bruck, *A survey of binary systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 20. Reihe: Gruppentheorie, Springer Verlag, Berlin-Göttingen-Heidelberg, 1958. MR 0093552

[Cha02] Mei-Chu Chang, *A polynomial bound in Freiman’s theorem*, Duke Math. J. 113 (2002), no. 3, 399–419. MR 1909605 (2003d:11151)

[Fre64] G. A. Freiman, *On the addition of finite sets*, Dokl. Akad. Nauk SSSR 158 (1964), 1038–1041. MR 0168529 (29 #5791)

[GR07] Ben Green and Imre Z. Ruzsa, *Freiman’s theorem in an arbitrary abelian group*, J. Lond. Math. Soc. (2) 75 (2007), no. 1, 163–175. MR 2302736 (2007m:20087)

[GS11] Alexander N. Grishkov and Ivan P. Shestakov, *Commutative Moufang loops and alternative algebras*, J. Algebra 333 (2011), 1–13. MR 2785933

[Hel08] H. A. Helfgott, *Growth and generation in SL_2(\mathbb{Z}/p\mathbb{Z})*, Ann. of Math. (2) 167 (2008), no. 2, 601–623. MR 2415382

[Man74] Yu. I. Manin, *Cubic forms: algebra, geometry, arithmetic*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1974, Translated from the Russian by M. Hazewinkel, North-Holland Mathematical Library, Vol. 4. MR 0460349

[Mou35] Ruth Moufang, *Zur Struktur von Alternativkörpern*, Math. Ann. 110 (1935), no. 1, 416–430. MR 1512948

[Pet12] Giorgis Petridis, *New proofs of Plünnecke-type estimates for product sets in groups*, Combinatorica 32 (2012), no. 6, 721–733. MR 3063158

[Sandu87] N. I. Sandu, *Infinite irreducible systems of identities of commutative Moufang loops and of distributive Steiner quasigroups*, Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987), no. 1, 171–188, 208. MR 887606

[Sandu12] Tom Sanders, *On the Bogolyubov-Ruzsa lemma*, Anal. PDE 5 (2012), no. 3, 627–655. MR 2994508

[Smi78a] J. D. H. Smith, *On the nilpotence class of commutative Moufang loops*, Math. Proc. Cambridge Philos. Soc. 84 (1978), no. 3, 405–415. MR 0498933

[Smi78b] _____, *A second grammar of associators*, Math. Proc. Cambridge Philos. Soc. 84 (1978), no. 3, 405–415. MR 0498933

[Smi82] Jonathan D. H. Smith, *Commutative Moufang loops and Bessel functions*, Invent. Math. 67 (1982), no. 1, 173–187. MR 664331

[Tao08] Terence Tao, *Product set estimates for non-commutative groups*, Combinatorica 28 (2008), no. 5, 547–594. MR 2501249

[Tao10] _____, *Freiman’s theorem for solvable groups*, Contrib. Discrete Math. 5 (2010), no. 2, 137–184. MR 2791295

[Toi14] Matthew C. H. Tointon, *Freiman’s theorem in an arbitrary nilpotent group*, Proc. Lond. Math. Soc. (3) 109 (2014), no. 2, 318–352. MR 3254927

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