A\(_\infty\)-STRUCTURES AND DIFFERENTIALS OF THE ADAMS SPECTRAL SEQUENCE

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The Adams spectral sequence was invented by J.F.Adams [1] almost fifty years ago for calculations of stable homotopy groups of topological spaces and in particular of spheres. The calculation of differentials of this spectral sequence is one of the most difficult problem of Algebraic Topology. Here we consider an approach to solve this problem in the case of \(\mathbb{Z}/2\) coefficients and find inductive formulas for the differentials. It is based on the \(A_\infty\)-structures [2], operad methods [3], [4], [5], [6] and functional homology operations [7], [8], [9].

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1. The Bousfield-Kan spectral sequences

Consider the Bousfield-Kan spectral sequence [10], which is one of the most general spectral sequence of the homotopy groups.

Let \(R\) be a field. Given a simplicial set \(Z\) denote by \(RZ\) the free simplicial \(R\)-module generated by \(Z\). There is the cosimplicial resolution

\[
R^\ast Z : \quad Z \xrightarrow{\delta^0} RZ \xrightarrow{\delta^0, \delta^1} R^2 Z \to \cdots \to R^n Z \xrightarrow{\delta^0, \ldots, \delta^n} R^{n+1} Z \to \ldots.
\]

This resolution was used by Bousfield and Kan [10] to construct the spectral sequence of the homotopy groups of \(Z\) with coefficients in \(R\).

The \(E^1\)-term of this spectral sequence is expressed by the complex

\[
H_\ast(Z; R) \to H_\ast(RZ; R) \to \cdots \to H_\ast(R^{n-1} Z; R) \to H_\ast(R^n Z; R) \to \ldots.
\]

Higher differentials are expressed by the homology operations

\[
d_m : H_\ast(R^{n-1} Z; R) \to H_\ast(R^{n+m-1} Z; R).
\]

In [7], [8] the homology operations were defined as partial and multi-valued mappings. However there is a general method to choose the homology operations to be usual homomorphisms. The corresponding theory was developed in [9]. Recall main definitions.
For a chain complex $X$ denote by $X_*$ its homology, $X_* = H_*(X)$. Fix chain mappings $\xi: X_* \to X$, $\eta: X \to X_*$ and chain homotopy $h: X \to X$ satisfying the relations

$$\eta \circ \xi = \text{Id}, \quad d(h) = \xi \circ \eta - \text{Id}, \quad h \circ \xi = 0, \quad \eta \circ h = 0, \quad h \circ h = 0.$$ 

Consider a sequence of mappings

$$f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1}$$

and define functional homology operations

$$H_*(f^n, \ldots, f^1): X^1_* \to X^{n+1}_*$$

putting

$$H_*(f^n, \ldots, f^1) = \eta \circ f^n \circ h \cdots \circ f^1 \circ \xi.$$ 

Direct calculations show that the following relations are satisfied

$$\sum_{i=1}^{n-1} (-1)^{n-i+1} H_*(f^n, \ldots, f^{i+1} \circ f^i, \ldots, f^1) = \sum_{i=1}^{n-1} (-1)^{n-i} H_*(f^n, \ldots, f^{i+1}) \circ H_*(f^i, \ldots, f^1).$$

Functional homology operations may be defined not only for the category of chain complexes but in some other situations, for example, for the category of simplicial modules.

Directly from the definition it follows that higher differentials of the Bousfield-Kan spectral sequence are expressed by the functional homology operations

$$H_*(\delta, \ldots, \delta): H_*(R^{n-1}Z; R) \to H_*(R^{n+m-1}Z; R).$$

So we have

**Theorem 1.** The differentials of the Bousfield-Kan spectral sequence are expressed by the functional homology operations

$$H_*(\delta, \ldots, \delta): H_*(R^{n-1}Z; R) \to H_*(R^{n+m-1}Z; R).$$

These operations determine on the $E^1$-term new differential. The homology of the corresponding complex is isomorphic to the $E^\infty$-term of the spectral sequence.

As it was proved in [5] instead of the Bousfield-Kan cosimplicial object we may consider the following cosimplicial object

$$F^*(C, RZ) : \quad RZ \xrightarrow{\delta^0, \delta^1} CRZ \to \cdots \to C^{n-1}RZ \xrightarrow{\delta^0, \ldots, \delta^n} C^nRZ \to \cdots,$$

where $CRZ$ is the free commutative simplicial coalgebra generated by $RZ$.

The $E^1$-term of the corresponding spectral sequence is expressed by the complex

$$H_*(Z; R) = \sigma_\ast (RZ) \to \sigma_\ast (CRZ) \to \cdots \to \sigma_\ast (C^{n-1}RZ) \to \sigma_\ast (C^nRZ) \to \cdots,$$
Directly from the definition it follows that higher differentials of this spectral sequence are expressed by the functional homology operations
\[ H_\ast[\delta, \ldots, \delta] : \pi_\ast(C^m RZ) \to \pi_\ast(C^{m+m} RZ). \]

Moreover there is a cosimplicial mapping
\[ RZ \longrightarrow R^2 Z \longrightarrow \cdots \longrightarrow R^{n+1} Z \longrightarrow \cdots = \]
\[ RZ \longrightarrow CRZ \longrightarrow \cdots \longrightarrow C^n RZ \longrightarrow \cdots, \]
inducing the isomorphism of the corresponding spectral sequences. So we have

**Theorem 2.** The differentials of the Bousfield-Kan spectral sequence of the cosimplicial object

\[ F^\ast(C, RZ) : \quad RZ \xrightarrow{\delta^0, \delta^1} CRZ \longrightarrow \cdots \longrightarrow C^{n-1} RZ \xrightarrow{\delta^0, \ldots, \delta^n} C^n RZ \longrightarrow \cdots \]

are expressed by the functional homology operations
\[ H_\ast(\delta, \ldots, \delta) : \pi_\ast(C^m RZ) \to \pi_\ast(C^{m+m} RZ). \]

These operations determine on the \( E^1 \)-term a new differential. The homology of the corresponding chain complex is isomorphic to the \( E^\infty \)-term of this spectral sequence.

### 2. \( E_\infty \)-algebras and \( E_\infty \)-coalgebras

Recall that an operad in the category of chain complexes is a family \( E = \{E(j)\}_{j \geq 1} \) of chain complexes \( E(j) \) together with given actions of symmetric groups \( \Sigma_j \) and operations
\[ \gamma : E(k) \otimes E(j_1) \otimes \cdots \otimes E(j_k) \to E(j_1 + \cdots + j_k) \]
compatible with these actions and satisfying associativity relations [3], [4].

An operad \( E = \{E(j)\} \) for which complexes \( E(j) \) are acyclic and symmetric groups act on them freely is called an \( E_\infty \)-operad.

A chain complex \( X \) is called an algebra (a coalgebra) over an operad \( E \) or simply an \( E \)-algebra (an \( E \)-coalgebra) if there are given mappings
\[ \mu : E(k) \otimes_{\Sigma_k} X^\otimes k \to X \quad (\tau : X \to Hom_{\Sigma_j}(E(k); X^\otimes j)), \]
satisfying some associativity relations.

Algebras (coalgebras) over an \( E_\infty \)-operad are called \( E_\infty \)-algebras (\( E_\infty \)-coalgebras).

Any operad in the category of chain complexes determines a monad \( E \) and a comonad \( \overline{E} \) by the formulas
\[ E(X) = \sum_j E(j, X), \quad E(j, X) = E(j) \otimes_{\Sigma_j} X^\otimes j; \]
\[ \overline{E}(X) = \prod_j \overline{E}(j, X), \quad \overline{E}(j, X) = Hom_{\Sigma_j}(E(j), X^\otimes j). \]
An operad structure \( \gamma \) induces natural transformations

\[
\gamma : E \circ E \to E, \quad \tau : E \to E \circ E.
\]

An \( E \)-algebra (an \( E \)-coalgebra) structure on a chain complex \( X \) induces a mapping

\[
\mu : E(X) \to X \quad (\tau : X \to E(X)).
\]

So to give on a chain complex \( X \) an \( E \)-algebra (\( E \)-coalgebra) structure is the same as to give on \( X \) an algebra (coalgebra) structure over the monad \( E \) (the comonad \( E \)).

One of the most important example of an \( E_\infty \)-algebra is the singular cochain complex \( C^*(Y; R) \) of a topological space \( Y \).

Dually, the singular chain complex \( C_*(Y; R) \) of a topological space \( Y \) and the chain complex \( N(RZ) \) of a simplicial set \( Z \) are examples of \( E_\infty \)-coalgebras.

The homotopy theory of \( E_\infty \)-coalgebras was constructed in [5]. There were defined the homotopy groups of \( E_\infty \)-coalgebras. For the chain complex \( N(RZ) \) of a simplicial set \( Z \) these homotopy groups are isomorphic to the homotopy groups of \( Z \) with coefficients in \( R \).

For an \( E \)-coalgebra \( X \), using cosimplicial resolution

\[
F^*(E, E, X) := X \xrightarrow{\tau} E(X) \to \cdots \to E^{n-1}(X) \to E^n(X) \to \ldots,
\]

there was constructed the spectral sequence of the homotopy groups of the \( E \)-coalgebra \( X \), [5].

Denote by \( X_* \) the homology of the complex \( X \) and by \( \overline{E}_* \) the homology of a comonad \( E \). \( X_* \) will be \( \overline{E}_* \)-coalgebra. There is the cosimplicial resolution

\[
F^*(\overline{E}_*, \overline{E}_*, X_*): \overline{E}_*(X_*) \to \overline{E}_*^2(X_*) \to \cdots \to \overline{E}_*^n(X_*) \to \overline{E}_*^{n+1}(X_*) \to \ldots.
\]

The \( E^1 \) term of the spectral sequence is expressed by the cobar construction

\[
F(\overline{E}_*, X_*): X_* \to \overline{E}_*(X_*) \to \cdots \to \overline{E}_*^n(X_*) \to \overline{E}_*^{n+1}(X_*) \to \ldots,
\]

obtained from the resolution by taking primitive elements. So there is the inclusion \( F(\overline{E}_*, X_*) \to F(\overline{E}_*, \overline{E}_*, X_*) \).

The functional homology operations

\[
H_*(\delta, \ldots, \delta): \overline{E}_*^n(X_*) \to \overline{E}_*^{n+m}(X_*)
\]

determine new differential in the resolution and in the cobar construction. The corresponding complexes denote by \( \widetilde{F}(\overline{E}_*, \overline{E}_*, X_*) \), \( \widetilde{F}(\overline{E}_*, X_*) \).

Note that the complex \( \widetilde{F}(\overline{E}_*, \overline{E}_*, X_*) \) is a resolution of the complex \( X_* \) and there is the inclusion \( \widetilde{F}(\overline{E}_*, X_*) \to \widetilde{F}(\overline{E}_*, \overline{E}_*, X_*) \).

**Theorem 3.** Differentials of the spectral sequence of the homotopy groups of an \( E \)-coalgebra \( X \) are determined by the functional homology operations

\[
H_*(\delta, \ldots, \delta): \overline{E}_*^n(X_*) \to \overline{E}_*^{n+m}(X_*)
\]
The homology of $\tilde{F}(E_*, X_*)$ is isomorphic to the $E\infty$-term of the spectral sequence.

If $X$ is the normalized chain complex of a simplicial set $Z$, i.e. $X = N(RZ)$, then there is a mapping of cosimplicial objects

$$
N(RZ) \longrightarrow N(CRZ) \longrightarrow \ldots \longrightarrow N(C^nRZ) \longrightarrow \ldots
= \downarrow \quad \downarrow \quad \downarrow
X \longrightarrow \tilde{E}(X) \longrightarrow \ldots \longrightarrow \tilde{E}^n(X) \longrightarrow \ldots,
$$

inducing the isomorphism of the corresponding spectral sequences. So we have

**Theorem 4.** The differentials of the Bousfield-Kan spectral sequence of the homotopy groups of a simplicial set $Z$ are determined by the functional homology operations

$$
H_*(\delta, \ldots, \delta): \tilde{E}_*^n(Z_*) \to \tilde{E}_*^{n+m}(Z_*).
$$

The homology of $\tilde{F}(\tilde{E}_*, Z_*)$ is isomorphic to the $E\infty$-term of the spectral sequence.

Note that the suspension $SX$ over an $E$-coalgebra $X$ is an $SE$-coalgebra and the following diagrams commute

$$
\tilde{E} \xrightarrow{\tau} \tilde{E} \circ \tilde{E} \quad \quad SX \xrightarrow{\tau} \tilde{S}(E(SX))
$$

Moreover from the expression of the homology $\tilde{E}_*$ of the comonad $\tilde{E}$ (see below) it follows that the mappings $\xi: \tilde{E}_* \to \tilde{E}$, $\eta: \tilde{E} \to \tilde{E}_*$, $h: \tilde{E} \to \tilde{E}$ may be chosen permutable with the suspension homomorphism $\tilde{S}E \to \tilde{E}$. Therefore constructed functional homology operations permute with the suspension homomorphism. Hence the following theorem is taken place.

**Theorem 5.** Functional homology operations giving higher differentials of the Bousfield-Kan spectral sequence permute with the suspension and hence are stable. They induce the differentials of the Adams spectral sequence of stable homotopy groups of a topological space.

3. $A\infty$-cosimplicial objects

Make more precise the form of higher differentials of the Bousfield-Kan spectral sequence using the notion of an $A\infty$-cosimplicial object.

A family $X^* = \{X^n\}_{n \geq 0}$ of objects $X^n$ of a category $\mathcal{X}$ will be called a pre-cosimplicial object if there are given coface and codegeneracy operators

$$
\delta^i: X^n \to X^{n+1}, \quad 1 \leq i \leq n + 1;
$$

$$
\sigma^i: X^n \to X^{n-1}, \quad 0 \leq i \leq n - 1,
$$

satisfying the following relations

$$
\delta^j \delta^i = \delta^i \delta^{j-1}, \quad i < j,
$$

$$
\sigma^j \sigma^i = \sigma^i \sigma^{j+1}, \quad i \leq j,
$$

$$
\sigma^j \delta^i = \begin{cases} 
\delta^i \sigma^{j-1}, & i < j, \\
Id, & i = j, i = j + 1, \\
\delta^i \sigma^{j-1}, & i > j + 1.
\end{cases}
$$
Thus a precosimplicial object differs a cosimplicial objects only by a coface operators $\delta^0$. A cosimplicial object has such operator but a precosimlicial object hasn’t.

A mapping $f^*: X \to Y$ of a precosimplicial objects is a family $f^* = \{f^n\}_{n \geq 0}$ of mappings $f^n: X^n \to Y^n$ commuting with coface and codegeneracy operators, i.e. satisfying the following relations

$$\delta^i f_n = f^{n+1}\delta^i;$$
$$\sigma^i f_n = f_{n-1}\sigma^i.$$

Now we define the notion of an $A_\infty$-cosimplicial object of the category of topological spaces.

Let $I^n$ be the unite cube, $I^n = \{(t_1, \ldots, t_n)| 0 \leq t_i \leq 1\}$. Denote by

$$u_i^\epsilon: I^n \to I^{n+1}, \epsilon = 0, 1; 1 \leq i \leq n + 1;$$
$$v_i: I^n \to I^{n-1}, 0 \leq i \leq n,$$

the mappings defined by the formulas

$$u_i^\epsilon(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \epsilon, t_i, \ldots, t_n);$$
$$v_i(t_1, \ldots, t_n) = \begin{cases} (t_2, \ldots, t_n), & i = 0, \\
(t_1, \ldots, t_i \ast t_{i+1}, \ldots, t_n), & 1 \leq i \leq n - 1, \\
(t_1, \ldots, t_{n-1}), & i = n,
\end{cases}$$

where $t_i \ast t_{i+1} = t_i + t_{i+1} - t_i \cdot t_{i+1}$.

A precosimplicial object $X^* = \{X^n\}$ of the category of topological spaces will be called an $A_\infty$-cosimplicial object or simply an $A_\infty$-cosimplicial space if there are given coface operators

$$\delta^0_m: X^n \times I^m \to X^{n+m+1},$$

satisfying the relations

$$\sigma^0 \delta^0 = Id;$$
$$\delta^0_m(1 \times u^0_i) = \delta^i \delta^0_{m-1}, 1 \leq i \leq m;$$
$$\delta^0_m(1 \times u^1_i) = \delta^0_{i-1}(\delta^0_{m-1} \times 1), 1 \leq i \leq m;$$
$$\delta^0_{m-1}(1 \times v_i) = \sigma^i \delta^0_m, 0 \leq i \leq m, m \geq 1;$$
$$\delta^0_m(\delta^i \times 1) = \delta^{i+m+1} \delta^0_m, i \geq 1;$$
$$\delta^0_m(\sigma^i \times 1) = \sigma^{i+m+1} \delta^0_m, i \geq 0.$$

It is clear that a cosimplicial object $X^* = \{X^n\}$ of the category of topological spaces may be concidered as an $A_\infty$-cosimplicial object with trivial operators

$$\delta^0_m: X^n \times I^m \to X^{n+m+1}, m \geq 1.$$

Note that the family $I^* = \{I^n\}$ itself may be considered as an $A_\infty$-cosimplicial space for which

$$\delta^i = u^0_i: I^n \to I^{n+1}, 1 \leq i \leq n + 1;$$
$$\delta^0_m = u^1_{n+1}: I^n \times I^m = I^{n+m} \to I^{n+m+1};$$
$$\sigma^i = v_i: I^n \to I^{n-1}, 0 \leq i \leq n.$
Define also the notion of an $A_\infty$-mapping.

Let $X^* = \{X^n\}$ be an $A_\infty$-cosimplicial space and $Y^* = \{Y^n\}$ be a cosimplicial space. Then $A_\infty$-mapping from $X^*$ to $Y^*$ is a precosimplicial mapping $f^* = \{f^n\}$, $f^n: X^n \to Y^n$ together with the family of mappings

$$f^n_m: X^n \times I^m \to Y^{n+m}, \quad 0 \leq m \leq n,$$

satisfying the following relations

$$
\begin{align*}
  f^n_0 &= f^n; \\
  f^n_m(1 \times u^0_i) &= \delta^{i-1} f^n_{m-1}, \quad 1 \leq i \leq m; \\
  f^n_m(1 \times u^1_i) &= f^{n+m-i-1}_i (\delta^0_{m-i} \times 1), \quad 1 \leq i \leq m; \\
  f^n_{m-1}(d_i \times 1) &= d_{i-m} f^n_m, \quad i > m; \\
  f^n_{m-1}(1 \times v_i) &= \sigma^i f^n_m, \quad 0 \leq i < m; \\
  f^n_m(\delta^i \times 1) &= \delta^{i+m} f^{m-1}_m, \quad i \geq 1; \\
  f^n_{m-1}(\sigma^i \times 1) &= \sigma^{i+m} f^n_m, \quad i \geq 0.
\end{align*}
$$

An $A_\infty$-mapping will be called an $A_\infty$-homotopy equivalence if the corresponding mappings $f^n_m$ are homotopy equivalences.

Of course the notion of an $A_\infty$-cosimplicial object may be defined not only for the category of topological spaces but also in some other situations. For example it may be defined for the category of simplicial sets, for the category of chain complexes and so on. To do it we need to use the analog of the $n$-dimensional cube $I^n$ for these categories.

Consider more precisely $A_\infty$-cosimplicial objects for the category of chain complexes. In the capacity of $I^n$ we take the normalized chain complex of $n$-dimensional cube.

The definitions of an $A_\infty$-cosimplicial object and an $A_\infty$-mappings may be reformulated in the following form.

A precosimplicial object $X^* = \{X^n\}$ of the category of chain complexes is an $A_\infty$-cosimplicial object or simply an $A_\infty$-cosimplicial complex if there are given mappings

$$\delta^0_m: X^n \to X^{n+m+1},$$

increasing dimensions by $m$ and satisfying the following relations

$$
\begin{align*}
  d(\delta^0_m) &= \sum_{i=1}^{n} (-1)^{i-1}(\delta^i \delta^0_{m-1} - \delta^0_{i-1} \delta^0_{m-i}); \\
  \delta^0_m \delta^i &= \delta^{i+m} \delta^0_m, \quad i \geq 1; \\
  \sigma^i \delta^0_m &= \begin{cases} 
  0, & 0 \leq i \leq m, \quad m \geq 1, \\
  \text{Id}, & i = m = 0, \\
  \delta^0_m \sigma^{i-m-1}, & i > m.
\end{cases}
\end{align*}
$$

An $A_\infty$-mapping from an $A_\infty$-cosimplicial complex $X^*$ to a cosimplicial complex $Y^*$ is a precosimplicial mapping $f^* = \{f^n\}$, $f^n: X^n \to Y^n$ together with a family of mappings

$$f^n_m: X^n \times I^m \to Y^{n+m}, \quad 0 \leq m \leq n,$$

satisfying the following relations

$$
\begin{align*}
  f^n_0 &= f^n; \\
  f^n_m(1 \times u^0_i) &= \delta^{i-1} f^n_{m-1}, \quad 1 \leq i \leq m; \\
  f^n_m(1 \times u^1_i) &= f^{n+m-i-1}_i (\delta^0_{m-i} \times 1), \quad 1 \leq i \leq m; \\
  f^n_{m-1}(d_i \times 1) &= d_{i-m} f^n_m, \quad i > m; \\
  f^n_{m-1}(1 \times v_i) &= \sigma^i f^n_m, \quad 0 \leq i < m; \\
  f^n_m(\delta^i \times 1) &= \delta^{i+m} f^{m-1}_m, \quad i \geq 1; \\
  f^n_{m-1}(\sigma^i \times 1) &= \sigma^{i+m} f^n_m, \quad i \geq 0.
\end{align*}
$$

An $A_\infty$-mapping will be called an $A_\infty$-homotopy equivalence if the corresponding mappings $f^n_m$ are homotopy equivalences.
mappings $f^n_m: X^n \to Y^{n+m}$ increasing dimensions by $m$ and satisfying the following relations

$$
f^n_0 = f^n; \\
\delta (f^n_m) = \sum_{i=1}^{n} (-1)^{i-1} (\delta^{i-1} f^n_{m-1} - f^{n+m-i-1}_m); \\
f^{n+1}_m \delta^i = \delta^{i+m} f^n_m; \quad i \geq 1; \\
\sigma^i f^n_m = \begin{cases} 0, & i < m; \\ f^{n+1}_m \sigma^{i-m}, & i \geq m. \end{cases}
$$

It is clear that a cosimplicial complex $X^* = \{X^n\}$ may be considered as an $A_\infty$-cosimplicial complex with trivial operators $\delta^0_m: X^n \otimes I^m \to X^{n+m+1}$ if $m \geq 1$.

Let $X'^*, X''^*$ are $A_\infty$-cosimplicial complexes. Then there is defined the tensor product $X'^* \otimes X''^*$. It is an $A_\infty$-cosimplicial complex $X^*$ for which

$$
X^n = X'^m \otimes X''^m; \\
\delta^i = \delta'^i \otimes \delta''^i, \quad i \geq 1; \\
\sigma^i = \sigma'^i \otimes \sigma''^i, \quad i \geq 0.
$$

Operations $\delta^0_m: X^n \to X^{n+m+1}$ are the compositions

$$
X'^m \otimes X''^m \otimes I^m \xrightarrow{1 \otimes 1 \otimes \nabla} X'^m \otimes X''^m \otimes I^m \otimes I^m \xrightarrow{T} \\
\to X'^m \otimes I^m \otimes X''^m \otimes I^m \xrightarrow{\delta'^0 \otimes \delta''^0} X'^{m+1} \otimes X''^{n+m+1},
$$

where $\nabla: I^m \to I^m \otimes I^m$ is a comultiplication in the coalgebra $I^m$.

Given $A_\infty$-cosimplicial complex $X^*$ define its realization $|X^*|$ putting

$$
|X^*| = Hom(I^*; X^*),
$$

where $Hom$ is considered in the category of $A_\infty$-cosimplicial complexes.

Similary as for the usual realization for given $A_\infty$-cosimplicial complexes $X'^*$, $X''^*$ there is a chain equivalence

$$
|X'^* \otimes X''^*| \simeq |X'^*| \otimes |X''^*|.
$$

Transfer the Perturbation Theory [11] to $A_\infty$-cosimplicial objects.

We say that a chain complex $\tilde{X}$ is a deformation retract of a chain complex $X$ if there are chain mappings $\xi: \tilde{X} \to X$, $\eta: X \to \tilde{X}$ and a chain homotopy $h: X \to X$ such that

$$
\eta \circ \xi = Id, \quad d(h) = Id - \xi \circ \eta.
$$

In this case we may additionally assume that

$$
h \circ \xi = 0, \quad \eta \circ h = 0, \quad h \circ h = 0, \quad [10].
$$

A precosimplicial complex $\tilde{X}^*$ will be called a deformation retract of a precosimplicial complex $X^*$ if for any $n$ a chain complex $\tilde{X}^n$ is a deformation retract of a
complex $X^n$ and the corresponding chain mappings and chain homotopies commute with a precosimplicial structure.

The next theorem is analog of the main lemma of the Perturbation Theory [11].

**Theorem 6.** Let $X^* = \{X^n\}$ be a cosimplicial complex and a precosimplicial complex $\tilde{X}^* = \{\tilde{X}^n\}$ is a deformation retract of $X^*$ considered as a precosimplicial complex. Then on $X^*$ there is an $A_\infty$-cosimplicial structure and an $A_\infty$-cosimplicial homotopy equivalence between $X^*$ and $\tilde{X}^*$.

**Proof.** Let $\xi^*: \tilde{X}^* \rightarrow X^*$, $\eta^*: X^* \rightarrow \tilde{X}^*$ and $h^*: X^* \rightarrow X^*$ be the corresponding mappings. Define operators $\delta^0_m: \tilde{X}^n \rightarrow \tilde{X}^{n+m+1}$ putting

$$\delta^0_m = \eta \delta^0 h^{n+m} \ldots \delta^0 h^{n+1} \delta^0 \xi.$$

Direct calculations show that the required relations are satisfied.

Define also mappings $\xi^m_n: \tilde{X}^n \rightarrow X^{n+m}$ putting

$$\xi^m_n = h^{n+m} \delta^0 \ldots h^{n+1} \delta^0 \xi.$$

Direct calculations show that these mappings give us the $A_\infty$-cosimplicial homotopy equivalence between $\tilde{X}^*$ and $X^*$.

**Theorem 7.** Let $f^*: X^* \rightarrow Y^*$ be a mapping of cosimplicial complexes and precosimplicial complexes $\tilde{X}^*, \tilde{Y}^*$ are deformation retracts of $X^*, Y^*$, considered as precosimplicial complexes. Then for above defined $A_\infty$-cosimplicial structures on $\tilde{X}^*, \tilde{Y}^*$ there is an $A_\infty$-mapping $\tilde{f}^*: \tilde{X}^* \rightarrow \tilde{Y}^*$ such that the following diagram commutes up to homotopy

$$
\begin{array}{ccc}
X^* & \xrightarrow{f^*} & Y^* \\
\uparrow \xi^* & & \uparrow \xi^* \\
\tilde{X}^* & \xrightarrow{\tilde{f}^*} & \tilde{Y}^*
\end{array}
$$

The proof is analog to the previous theorem.

Taking into account that the $E^1$-term of the Bousfield-Kan spectral sequence, considered as a precosimplicial object, is a deformation retract of the initial cosimplicial object we obtain

**Theorem 8.** The functional homology operations giving higher differentials of the Bousfield-Kan spectral sequence may be chosen in such a way to form on the $E^1$-term an $A_\infty$-cosimplicial structure.

4. $E_\infty$-structure on the Bousfield-Kan spectral sequence

Our aim here is to define $E_\infty$-structure on the Bousfield-Kan spectral sequence. To do it consider the following additional propety of the operad $E$.

**Theorem 9.** Given $E_\infty$-operad $E$ there is a permutation mapping

$$T: E \circ E \rightarrow E \circ E,$$

commuting the following diagrams

$$
\begin{array}{cccc}
E^2 \circ E & \xrightarrow{T_{E \circ E} \circ E} & E \circ E^2 & \xrightarrow{T} & E \circ E \\
T_{E \circ E} \downarrow & & T \downarrow & & \downarrow \tau E \\
E \circ E & \xrightarrow{T} & E \circ E & \xrightarrow{T_{E \circ E} \circ E} & E \circ E^2
\end{array}
$$
Proof. Given an $E_\infty$-operad $E$ we may construct an operad mapping $\nabla: E \to E \otimes E$ consisting of mappings $\nabla(j): E(j) \to E(j) \otimes E(j)$ This mapping give on $E$ a Hopf operad structure. Denote $\nabla(j,i): E(j) \to E(j)^{\otimes i}$ the iterations of these mappings $\nabla(j,2) = \nabla(j)$. They are $\Sigma_j$-mappings, i.e.

$$\nabla(j)(x \sigma) = \nabla(j)(x)\sigma^{\otimes j}, \quad \sigma \in \Sigma$$

but are not commuting with permutations of factors of $E(j)^{\otimes i}$. However they may be extended till the mappings $\nabla(j,i): E(i) \otimes E(j) \to E(j)^{\otimes i}$, compatible with the actions of symmetric groups $\Sigma_i$ and $\Sigma_j$.

Rewrite these mappings in the form

$$\nabla(j,i): E(i) \otimes E(j)^{\otimes i} \to E.$$ 

If the operad $E$ is chosen freely then they may be done compatible with an operad structure.

Passing to the dual mappings we obtain mappings

$$\nabla(j,i): E(j)^{\otimes i} \otimes E(i) \to E.$$ 

Define mappings

$$T(j,i): E(j)^{\otimes i} \otimes E(i) \to E(e)^{\otimes i}$$

as the compositions

$$E(j) \otimes E(i)^{\otimes j} \xrightarrow{\nabla(j) \otimes 1^{\otimes j}} E(j) \otimes E(j) \otimes E(i)^{\otimes j} \xrightarrow{1 \otimes \nabla(j,i)} E(i) \otimes E(i) \otimes E(j)^{\otimes i} \xrightarrow{\nabla(i) \otimes 1^{\otimes i}} E(i) \otimes E(j)^{\otimes i}.$$ 

The family $T(j,i)$ determines the required permutation mapping

$$T: E \circ E \to E \circ E.$$ 

A chain complex $X$ will be called an $E_\infty$-Hopf algebra if there are given an $E_\infty$-algebra structure $\mu: E(X) \to X$ and an $E_\infty$-coalgebra structure $\tau: X \to E(X)$ such that the following diagram commutes

$$\begin{array}{ccc}
E(X) & \xrightarrow{\mu} & X \\
\uparrow \tilde{E} & & \quad \downarrow \tau \\
E(\tau) & \xrightarrow{T} & E(X) \\
\end{array}$$
Let \( X \) be an \( E_\infty \)-Hopf algebra. Then there is a mapping of augmented cosimplicial objects

\[
\begin{align*}
E(X) &\longrightarrow E \bar{E}(X) \longrightarrow \ldots \longrightarrow E \bar{E}^n(X) \longrightarrow \ldots \\
\downarrow &\quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \downarrow \\
X &\longrightarrow \bar{E}(X) \longrightarrow \ldots \longrightarrow \bar{E}^n(X) \longrightarrow \ldots
\end{align*}
\]

Then the complex \( F(\bar{E}, \bar{E}, X) \) will be \( E_\infty \)-algebra. Passing to the homology we obtain \( E_\infty \)-algebra structure on the complex \( \tilde{F}(E_*, \bar{E}_*, X_*) \). Thus we have

**Theorem 10.** If \( X \) is an \( E_\infty \)-Hopf algebra, then the complex \( \tilde{F}(E_*, \bar{E}_*, X_*) \) possesses \( E_\infty \)-algebra structure.

This structure will be used in the further calculations of higher differentials of the Bousfield-Kan spectral sequence.

Note that on the cobar construction \( \tilde{F}(E_*, X_*) \) there is no \( E_\infty \)-algebra structure.

5. The homology of an \( E_\infty \)-operad and the Milnor coalgebra

Let \( E \) be an \( E_\infty \)-operad, \( M \) be a graded module (over \( \mathbb{Z}/2 \)). As it is known (see for example [6]), the homology \( E_*(M) \) of the complex \( E(M) \) is the polynomial algebra generated by the elements \( e_{i_1} \cdots e_{i_k} x_m, 1 \leq i_1 \leq \cdots \leq i_k, x_m \in M \) of dimensions \( i_1 + 2i_2 + \cdots + 2^{k-1}i_k + 2^k m \).

The elements \( e_{i_1} \cdots e_{i_k} x_m \) of \( E_*(M) \) may be rewritten in the form

\[
Q^{j_1} \cdots Q^{j_k} \otimes x_m; \quad j_1 \leq 2j_2, \ldots, j_{k-1} \leq 2j_k, m \leq j_k,
\]

where

\[
\begin{align*}
\dot{j}_k &= i_k + m, \\
\dot{j}_{k-1} &= i_{k-1} + i_k + 2m, \\
\cdots \\
\dot{j}_1 &= i_1 + i_2 + 2i_3 + \cdots + 2^{k-2}i_k + 2^{k-1}m.
\end{align*}
\]

The sequences \( Q^{j_1} \cdots Q^{j_k} \) give the elements of the Dyer-Lashof algebra \( \mathcal{R} \), [12], [13].

Given a graded module \( M \) denote by \( \mathcal{R} \times M \) the quotient module of the tensor product \( \mathcal{R} \otimes M \) under the submodule generated by the elements \( Q^{j_1} \cdots Q^{j_k} \otimes x_m, j_k < m \). The correspondence \( M \mapsto \mathcal{R} \times M \) determines the monad in the category of graded modules.

A graded module \( M \) is called an unstable module over the Dyer-Lashof algebra if it is an algebra over the corresponding monad.

Dually, the homology \( \bar{E}_*(M) \) of the complex \( \bar{E}(M) \) is the free commutative coalgebra generated by the elements

\[
e_{i_1} \cdots e_{i_k} x^m, \quad 1 \leq i_1 \leq \cdots \leq i_k, \quad x^m \in M
\]

of dimensions \( 2^k m - (i_1 + 2i_2 + \cdots + 2^{k-1}i_k) \).
Regrading of the elements of $\overline{E}_n(M)$ leads to the Milnor coalgebra $K$. By definition $K$ is the polynomial algebra generated by the elements $\xi_i$, $i \geq 0$ of dimensions $2^i - 1$. A comultiplication

$$\nabla: K \to K \otimes K$$
on the generators $\xi_i$ is given by the formula

$$\nabla(\xi_i) = \sum_k \xi_{i-k}^{2^k} \otimes \xi_k.$$ 

On the other elements the comultiplication is determined by the Hopf relation.

Define the grading $\deg(x)$ of elements $x \in K$ putting $\deg(\xi_i) = 1$ and the grading of the product equal to the sum of the gradings of factors.

Given graded module $M$ denote by $K \times M$ the submodule of the tensor product $K \otimes M$ generated by the elements $x \otimes y$, $\deg(x) = \dim(y)$. The correspondence $M \mapsto K \times M$ determines the comonad $K$ in the category of graded modules.

A graded module $M$ is called an unstable comodule over the Milnor coalgebra if it is a coalgebra over the corresponding comonad.

Let $M$ be an unstable module over the Milnor coalgebra. There is a cosimplicial resolution

$$F^*(K, K, M) : M \to K \times M \to \cdots \to K^{\times n} \times M \to K^{\times n+1} \times M \to \cdots$$

If $Y$ is a "nice" (in the sense of Massey-Peterson) space then the Bousfield-Kan spectral sequence turns to the Massey-Peterson spectral sequence. The $E^1$-term of this spectral sequence may be written in the form

$$F^*(K, Y) : Y_* \to K \times Y_* \to \cdots \to K^{\times n} \times Y_* \to \cdots,$$

where $Y_* = H_*(Y; \mathbb{Z}/2)$.

From the previous theorems it follows the next theorem.

**Theorem 11.** The functional homology operations determining higher differentials of the Massey-Peterson spectral sequence of a "nice" space $Y$ may be chosen in such a way to form on $F^*(K, K, Y_*)$ an $A_\infty$-cosimplicial structure. The homology of the corresponding cobar construction $\tilde{F}(K, Y_*)$ is isomorphic to the $E_\infty$ term of the Massey-Peterson spectral sequence. If $Y$ is an $E_\infty$-space then the complex $\tilde{F}(K, K, Y_*)$ is $E_\infty$-algebra.

Note that there is an inclusion $\tilde{F}(K, Y_*) \to \tilde{F}(K, K, Y_*)$. However on the $\tilde{F}(K, Y_*)$ there is no $E_\infty$-algebra structure.

Besides the Milnor coalgebra $K$ we will consider the stable Milnor coalgebra $K_s$ for which $\xi_0 = 1$.

Given comodule $M$ over the stable Milnor coalgebra there is a cosimplicial resolution

$$F^*(K_s, K_s, M) : K_s \otimes M \to K_s^{\otimes 2} \otimes M \to \cdots \to K_s^{\otimes n} \otimes M \to \cdots$$

Stabilization of the Bousfield-Kan spectral sequence leads to the Adams spectral sequence of stable homotopy groups of a topological space $Y$. $E^1$-term of this spectral sequence may be written in the form

$$F^*(K_s, K_s, Y_*) : Y_* \to K_s \otimes Y_* \to \cdots \to K_s^{\otimes n} \otimes Y_* \to \cdots.$$
Thus we have

**Theorem 12.** Functional homology operations determining higher differentials of the Adams spectral sequence of stable homotopy groups of a topological space $Y$ may be chosen in such a way to form on $F^*(K_*, K_*, Y_*)$ the structure of an $A_\infty$-cosimplicial object. The homology of the corresponding cobar construction $\tilde{F}(K_*, Y_*)$ is isomorphic to the $E^\infty$ term of the Adams spectral sequence.

Let us calculate the $E_\infty$-algebra structure on the Milnor coalgebra. As it was pointed out above for an $E_\infty$-operad $E$ there is the permuting mapping $T_*: E_\infty \circ E_\infty \to E_\infty \circ E_\infty$. It induces the permuting mapping $T^*: E_\infty \circ E_\infty \to E_\infty \circ E_\infty$ commuting diagrams

$$
\begin{align*}
E_\infty \circ E_\infty & \xrightarrow{\gamma} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
E_\infty \circ E_\infty & \xrightarrow{T^*} E_\infty \circ E_\infty \\
\end{align*}
$$

The permuting mapping $T^*$ induces the action $\mu*: E_\infty \circ E_\infty \to E_\infty$ and the dual coaction $\tau*: E_\infty \to E_\infty \circ E_\infty$.

Denote by $e_i: \mathcal{K} \to \mathcal{K}$ the operation on the Milnor coalgebra inducing by the restriction of $\mu*$ on the elements $e_i$. From the commutative diagrams for the permuting mapping $T^*$ it follows

**Theorem 13.** The operations $e_i: \mathcal{K} \to \mathcal{K}$ satisfy the relations:

1. $e_0(x) = x^2$.
2. $e_i(xy) = \sum e_k(x) e_{i-k}(y)$.
3. $\nabla e_i(x) = \sum \xi_0^{i-1} e_{i-1}(\xi_0 x) \otimes e_k(x''', y)$, where $\sum x' \otimes x''' = \nabla(x)$.

Using these relations to calculate the operations $e_i$ it is sufficient to calculate only $e_i(\xi_0)$. Direct calculations show that $e_1(\xi_0) = \xi_1 \xi_0$. From the third relation it follows

**Theorem 14.** There are the following formulas

$$
e_i(\xi_k) = \begin{cases} 
\xi_{m+k} \xi_k, & i = 2^{m+k} - 2^k; \\
\xi_{m+k} \xi_{k-1}, & i = 2^{m+k} - 2^k - 2^{k-1}; \\
\ldots & \ldots \\
\xi_{m+k} \xi_0, & i = 2^{m+k} - 2^k - \ldots - 1; \\
0, & \text{in other cases.}
\end{cases}
$$

Using the second relation we may obtain formulas for the operations $e_i$ on the products of the elements $\xi_k$.

Passing from the elements $e_i$ to the elements of the Dyer-Lashof algebra we obtain the action of the Dyer-Lashof algebra on the Milnor coalgebra. On the generators $\xi_i$ it is given by the formulas

$$
Q_i + 2^k - 1(\xi_k) = \begin{cases} 
\xi_{m+k} \xi_k, & i = 2^{m+k} - 2^k; \\
\xi_{m+k} \xi_{k-1}, & i = 2^{m+k} - 2^k - 2^{k-1}; \\
\ldots & \ldots \\
\xi_{m+k} \xi_0, & i = 2^{m+k} - 2^k - \ldots - 1; \\
0, & \text{in other cases.}
\end{cases}
$$
On the other elements this action is determined by the Hopf relations
\[ Q^i(xy) = \sum Q^k(x)Q^{i-k}(y). \]

Besides the action of the Dyer-Lashof algebra on the Milnor coalgebra \( K \) there are \( \cup_i \)-products and an \( E_\infty \)-algebra structure. On the generators \( x \in K \) it is defined by the formulas
\[ x \cup_i x = e_i(x). \]
On the other elements \( \cup_i \)-products are defined by the relations
\[ x \cup_i y = y \cup_i x, \]
\[ (x_1x_2) \cup_i y = x_1(x_2 \cup_i y) + (x_1 \cup_i y)x_2. \]

Note that the stable Milnor coalgebra \( K_s \) has no action of the Dyer-Lashof algebra and has no \( E_\infty \)-algebra structure.

6. Degenerating \( A_\infty \)-structures

As it was pointed out above the Milnor coalgebra \( K \) is \( E_\infty \)-algebra and in particular \( A_\infty \)-algebra. Here we show that this \( A_\infty \)-structure on the Milnor coalgebra is degenerated.

Recall that a chain complex \( A \) is called an \( A_\infty \)-algebra \([2]\) (over \( \mathbb{Z}/2 \)) if there are given operations
\[ \pi_n: A^\otimes n+2 \to A, \quad n \geq 0, \]
increasing dimensions by \( n \) and satisfying the following relations
\[ d(\pi_{n+1}) = \sum_{i=0}^{n} \pi_i(1 \otimes \cdots \otimes \pi_{n-i} \otimes \cdots \otimes 1), \]
where the sum is taken also over all places of \( \pi_{n-i} \).

In particular a graded module \( A_\ast \) is an \( A_\infty \)- if there are given operations
\[ \pi_n: A_\ast^\otimes n+2 \to A_\ast, \quad n \geq 0, \]
increasing dimensions by \( n \) and satisfying the following relations
\[ \sum_{i=0}^{n} \pi_i(1 \otimes \cdots \otimes \pi_{n-i} \otimes \cdots \otimes 1) = 0. \]

Let \( A', A'' \) be \( A_\infty \)-algebras. A family of mappings
\[ f_n: A'^\otimes n+1 \to A'', \quad n \geq 0, \]
increasing dimensions by \( n \) is called an \( A_\infty \)-mapping from \( A' \) to \( A'' \) if the following relations are satisfied
\[ d(f_{n+1}) = \sum_{i=0}^{n} f_i(1 \otimes \cdots \otimes \pi_{n-i} \otimes \cdots \otimes 1) - \]
\[ - \sum \pi_i''(f_{n_1} \otimes \cdots \otimes f_{n_{i+2}}). \]
Two $A_\infty$-structures $\{\pi'_n\}$ and $\{\pi''_n\}$ are called equivalent if there exists an $A_\infty$-mapping $f: (A, \pi') \rightarrow (A, \pi'')$, for which $f_0 = Id: A \rightarrow A$.

An $A_\infty$-structure is called degenerated if it is equivalent to usual algebra structure.

$A_\infty$-algebra structure appears by a natural way on the homology $A_*$ of a differential algebra $A$, [14]. Namely, we fix the homomorphism $\xi: A_* \rightarrow A$ of choosing representatives in homology classes, fix the inverse homomorphism $\eta: A \rightarrow A_*$ and a chain homotopy $h: A \rightarrow A$ such that

$$\eta \circ \xi = Id, \quad d(h) = \xi \circ \eta - Id, \quad h \circ \xi = 0, \quad \eta \circ h = 0, \quad h \circ h = 0.$$ 

Then $A_\infty$-algebra structure on $A_*$ is given by the formula

$$\pi_n = \eta \pi(h \pi \otimes 1 + 1 \otimes h \pi) \ldots (h \pi \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes h \pi)(\xi \otimes \cdots \otimes \xi).$$

It is easy to see that if the homology $A_*$ of a differential algebra $A$ is isomorphic to the tensor algebra $TX$ generated by a graded module $X$, then $A_\infty$-algebra structure on $A_*$ is degenerated. In this case there is an algebra mapping $\xi: A_* \rightarrow A$ giving by the formula

$$\xi(x_1 \otimes \cdots \otimes x_n) = \xi(x_1) \cdot \ldots \cdot \xi(x_n),$$

where $x_i \in X$, $\xi(x_i)$ - representatives of elements $x_i$.

Consider the case when $A_*$ is the polynomial algebra $PX$ generated by a graded module $X$.

If $A$ is a commutative algebra then $A_\infty$-algebra structure on $A_* = PX$ degenerated. The corresponding algebra mapping $\xi: A_* \rightarrow A$ is given by the formula

$$\xi(x_1 \cdot \ldots \cdot x_n) = \xi(x_1) \cdot \ldots \cdot \xi(x_n).$$

In the case of a none commutative algebra $A$ the mapping $\xi: A_* = PX \rightarrow A$ may be choosen in such a way that $\xi(x \cdot y) = \xi(x) \cdot \xi(y)$ if $x \leq y$.

From the above formula for $\pi_n$ in this case it follows that $\pi_n(x_1 \otimes \cdots \otimes x_{n+2}) = 0$, if $x_1 \leq x_2$ or $\ldots$ or $x_{n+1} \leq x_{n+2}$. If $x_1 > \cdots > x_{n+2}$ then $\pi_n(x_1 \otimes \cdots \otimes x_{n+2}) \neq 0$ in general.

So $A_\infty$-algebra structure on $A_* = PX$ may be not degenerated. As it was shown in [15], on the polynomial algebra $P$ with $n$ generators, $n \geq 3$, there really exists none trivial $A_\infty$-algebra structures. In the case $n = 3$ they are in one to one correspondence with the Hochschild homology $H^3(P; P) \cong P$.

So none trivial elements $y \in P$ degenerate none degenerated $A_\infty$-algebra structures. The corresponding $A_\infty$-algebra denote by $\tilde{P}$.

In the capacity of $A$ we may take the cobar construction over the bar construction over $\tilde{P}$, i.e. $A = FB\tilde{P}$. Then $A$ will be a differential algebra which homology is isomorphic to $\tilde{P}$, so $A$ is the desired algebra.

Consider the question about what additional conditions must be satisfied for a differential algebra $A$ to be the $A_\infty$-algebra structure on $A_*$ degenerated. To give answer to this question we introduce the notion of a homotopy trivial Lie algebra.

A Lie algebra $L$ with a multiplication $\mu: L \otimes L \rightarrow L$ will be called homotopy trivial if there are given mappings

$$\mu: L \otimes^{n+1} L \rightarrow L.$$
increasing dimensions by $n$ and satisfying the following relations

$$d(\mu_{n+1})(x_0 \otimes \cdots \otimes x_{n+1}) = \sum_{p+q=n} \mu(\mu_p(x_{i_0} \otimes \cdots \otimes x_{i_p}) \otimes \mu_q(x_{j_0} \otimes \cdots \otimes x_{j_q})),$$

where $\mu_0(x) = x$ and the sum is taken over all shuffles $I = \{i_0, \ldots, i_p\}$, $J = \{j_0, \ldots, j_q\}$ of the set $0, \ldots, n+1$, such that $I < J$.

Consider examples of homotopy trivial Lie algebras. Let $A$ be a differential algebra. It may be turned to a Lie algebra by introducing a new multiplication $\mu: A \otimes A \to A$, by the formula

$$\mu(x \otimes y) = x \cdot y - y \cdot x.$$

Suppose the algebra $A$ possesses a $\cup_1$-product $\cup_1: A \otimes A \to A$, satisfying the relations

$$d(x \cup_1 y) = d(x) \cup_1 y + x \cup_1 d(y) + x \cdot y - y \cdot x;$$

$$x \cup_1 (y_1 \cdot y_2) = (x \cup_1 y_1) \cdot y_2 + y_1 \cdot (x \cup_1 y_2).$$

Show that in this case the corresponding Lie algebra will be homotopy trivial.

Define mappings $\mu_n: A^{\otimes n+1} \to A$ putting

$$\mu_n(x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1}) = x_1 \cup_1 (x_2 \cup_1 (\cdots \cup_1 x_{n+1}) \cdots).$$

Direct calculations show that the required relations are satisfied.

Note that instead of the distributivity relation for the $\cup_1$-product we may demand the homotopy distributivity relation. Namely, let besides the operation $\cup_1$ there are given operations

$$\phi_n: A^{\otimes n+1} \to A, \quad n \geq 1, \quad \phi_1 = \cup_1,$$

satisfying the relations

$$d(\phi_{n+1})(x \otimes y_1 \otimes \cdots \otimes y_n) = y_1 \cdot \phi_n(x \otimes y_2 \otimes \cdots \otimes y_n) +$$

$$+ \sum_{i=1}^{n-1} \phi(x \otimes y_1 \otimes \cdots \otimes y_i \cdot y_{i+1} \otimes \cdots \otimes y_n) +$$

$$+ \phi_n(x \otimes y_1 \otimes \cdots \otimes y_{n-1}) \cdot y_n.$$

Then on $A$ we also may define the structure of a homotopy trivial Lie algebra putting

$$\mu_n(x_0 \otimes \cdots \otimes x_n) = x_0 \cup_1 (x_1 \cup_1 (\cdots (x_{n-1} \cup_1 x_n) \cdots) +$$

$$+ x_0 \cup_1 (\cdots \cup_1 \phi_2(x_{n-2} \otimes (x_{n-1} \otimes x_n - x_n \otimes x_{n-1})) \cdots) +$$

$$+ \cdots + \sum \phi_n(x_0 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}),$$

where the last sum is taken over all permutations of the collection $1, \ldots, n$. 

Direct calculations show that the required relations are satisfied.

From here it follows that the cochain complex $C^*(X)$ of a topological space $X$ gives us the example of a homotopy trivial Lie algebra. Indeed $C^*(X)$ is an $E_\infty$-algebra [4] and hence the Lie algebra structure on $C^*(X)$ will be homotopy trivial.

Another example of a homotopy trivial Lie algebra gives us the cobar construction $\mathcal{F} K$ over a Hopf algebra $K$. Indeed in this case on the cobar construction $\mathcal{F} K$ there is defined distributive $\cup_1$-product. On generators it is given by the formula

$$[x] \cup_1 [y] = [x \cdot y].$$

Therefore the Lie algebra structure on $\mathcal{F} K$ will be homotopy trivial.

**Theorem 15.** Let $A$ be a differential algebra for which the corresponding Lie algebra structure is homotopy trivial and the homology $A_* = H_*(A)$ is the polynomial algebra. Then $A_\infty$-structure on $A_*$ is degenerated.

**Proof.** Let $A_* = PX$. Put in order generators $x \in X$ and denote by $\xi(x) \in A$ their representatives. Define a mapping $\xi: A_* \rightarrow A$ putting

$$\xi(x_{i_1} \cdots x_{i_n}) = \xi(x_{i_1}) \cdots \xi(x_{i_n}), \quad i_1 \leq \cdots \leq i_n.$$

Of course in the case when the algebra $A$ is not commutative this mapping is not an algebra mapping. Our task is to add on the mapping $\xi$ till an $A_\infty$-mapping. Defining mappings $\xi_n: A_* \otimes^{n+1} \rightarrow A$, putting on generators

$$\xi_n(x_{i_0} \otimes \cdots \otimes x_{i_n}) = \begin{cases} \mu_n(\xi(x_{i_0}) \otimes \cdots \otimes \xi(x_{i_n})), & i_0 > \cdots > i_n, \\ 0, & \text{in other cases}. \end{cases}$$

If the elements $u_0, \ldots, u_n \in A_*$ satisfy one of the inequalities $u_0 \leq u_1, \ldots u_{n-1} \leq u_n$ then we put

$$\xi_n(u_0 \otimes u_1 \otimes \cdots \otimes u_n) = 0.$$

Show that on the other elements the mappings $\xi_n$ is determined by the relations

$$d(\xi_{n+1})(u_0 \otimes \cdots \otimes u_{n+1}) = \sum_{i=0}^n \xi_n(u_0 \otimes \cdots \otimes u_i \cdot u_{i+1} \cdots \otimes u_{n+1}) -$$

$$- \sum_{i=0}^n \xi_i(u_0 \otimes \cdots \otimes u_i) \cdot \xi_{n-i}(u_{i+1} \otimes \cdots \otimes u_{n+1}),$$

meaning that the family $\{\xi_n\}$ is an $A_\infty$-mapping from $A_*$ to $A$.

The definition of $\xi_n(u_0 \otimes \cdots \otimes u_n)$ is inductive over the total numbers of factors of $u_0, \ldots, u_n$.

Denote by $x$ the minimal generator of the elements $u_0, \ldots, u_n$. Suppose that $u_0$ contains $x$. If $x = u_0$ then from the inequality $x \leq u_1$ it follows that

$$\xi_n(x \otimes \cdots \otimes u_n) = 0.$$

If $u_0 = x \cdot u_0'$ then from the above relations it follows that

$$\xi_n(x \otimes u_1 \otimes \cdots \otimes u_n) = \xi(x) \xi_n(x' \otimes \cdots \otimes u_n).$$
So the value of $\xi_n$ is determined by the value on the element of lesser filtration.

Similarly if $u_k$ contains $x$, i.e. $u_k = x \cdot u_k'$ then we have

$$\xi_n(u_0 \otimes \cdots \otimes x \cdot u_k' \otimes \cdots \otimes u_n) = \xi_n(u_0 \otimes \cdots \otimes x \cdot u_{k-1} \otimes u_k' \otimes \cdots \otimes u_n) +$$

$$+ \xi_k(u_0 \otimes \cdots \otimes u_{k-1} \otimes x) \cdot \xi_{n-k}(u_k' \otimes \cdots \otimes u_n).$$

So the value of $\xi_n$ is determined by the value on the elements of lesser filtrations and by the value on the element in which $x$ is contained by $x \cdot u_{k-1}$.

Repeating this procedure we come to the values of $\xi_n$ on the elements of lesser filtration and the value on the element in which $x$ is contained by the first factors. It remains the case when $x = u_n$ and elements $u_0, \ldots, u_{n-1}$ do not contain $x$. In this case we need to consider the minimal generator contained by $u_0, \ldots, u_{n-1}$ and repeat described above procedure. Thus we obtain that the values of $\xi_n$ are determined by the given values and values on the elements of lesser filtrations.

For example direct inductive calculations show that there is the following formula

$$\xi_1(x_{i_1} \cdots x_{i_n} \otimes x_{j_1} \cdots x_{j_m}) =$$

$$\sum_{k,l} \xi(x_{i_1} \cdots x_{i_{k-1}} x_{j_1} \cdots x_{j_{l-1}}) \xi(x_{i_k} \otimes x_{j_l}) \xi(x_{i_{k+1}}) \cdots \xi(x_{i_n}) \xi(x_{j_{l+1}}) \cdots \xi(x_{j_m}).$$

Since the Milnor coalgebra may be obtained as the cohomology of an $E_\infty$-operad, we will have

**Corollary.** The Milnor coalgebra $K$ has degenerated $A_\infty$-algebra structure.

7. Functorial homology operations

Let $\Delta^* = \{\Delta^n\}$ denotes the cosimplicial object of the category of chain complexes, consisting of the chain complexes of the standard $n$-dimensional simplices. Let further $F$ be a functor in the category of chain complexes for which there are given transformations

$$\Delta^n \otimes F(X) \to F(\Delta^n \otimes X),$$

permuting with coface and codegeneracy operators. Such functor $F$ will be called a chain functor.

A transformation $\alpha: F' \to F''$ of chain functor is a transformations of functors, commuting the diagrams

$$\begin{array}{ccc}
\Delta^n \otimes F'(X) & \longrightarrow & F'(\Delta^n \otimes X) \\
\downarrow 1 \otimes \alpha & & \downarrow \alpha \\
\Delta^n \otimes F''(X) & \longrightarrow & F''(\Delta^n \otimes X)
\end{array}$$

Given chain functor $F$ we may consider mappings

$$F(f): F(X) \to F(Y),$$

induced not only by chain mappings $f: X \to Y$ of dimension zero but dimension $n$ also. Namely given mapping $f: X \to Y$ of dimension $n$ we represent as the
restriction of the mapping \( \tilde{f}: \Delta^n \otimes X \to Y \) on the \( n \)-dimensional generator \( u_n \in \Delta^n \). Then the required mapping

\[ F(f): F(X) \to F(Y) \]

of dimension \( n \) will be the restriction of the composition

\[ \Delta^n \otimes F(X) \to F(\Delta^n \otimes X) \xrightarrow{F(\tilde{f})} F(Y) \]

on the \( n \)-dimensional generator \( u_n \in \Delta^n \).

Given chain functor \( F \) denote by \( F_* \) the functor, corresponding to a chain complex \( X \) the graded module of its homology \( F_*(X) = H_*(F(X)) \). The functor \( F_* \) is not only a functor but an \( A_\infty \)-functor, i.e. there are functional homology operations which assigns to sequences of chain mappings \( f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1} \) the mapping

\[ F_*(f^n, \ldots, f^1) = H_*(F(f^n), \ldots, F(f^1)): F_*(X^1) \to F_*(X^{n+1}), \]

of dimension \( n - 1 \).

A transformation \( \alpha: F' \to F'' \) of chain functors induces an \( A_\infty \)-transformation of the \( A_\infty \)-functor \( F'_* \) to the \( A_\infty \)-functor \( F''_* \).

**Theorem 16.** Let \( F \) be a chain functor. Then for any sequence of chain mappings \( f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1} \) the following formula is taken place

\[ H_*(F(f^n), \ldots, F(f^1)) = \sum (-1)^c F_*(H_*(f^n, \ldots, f^{n+m+1}), \ldots, H_*(f^n, \ldots, f^1)), \]

where the sum is taken over \( m \) and \( n_1, \ldots, n_m \) such that \( 1 \leq n_1 < \cdots < n_m < n \).

**Proof.** Let \( X \) be chain complex. We take the mapping \( F_*(X_*) \to F(X) \) of choosing representatives as the composition

\[ \xi(F): F_*(X_*) \to F(X_*) \]

\[ F(\xi): F(X_*) \to F(X). \]

Similary we take the projection \( F(X) \to F_*(X_*) \) as the composition

\[ F(\eta): F(X) \to F(X_*) \]

\[ \eta(F): F(X_*) \to F_*(X_*). \]

We take the homotopy \( H: F(X) \to F(X) \) as the sum

\[ F(\xi) \circ h(F) \circ F(\eta) + F(h). \]

Substituting these mappings to the formula of functional homology operation we obtain the required formula.

Similary there is the following theorem.

**Theorem 17.** Let \( \alpha: F' \to F'' \) be a transformation of chain functors. Then for any sequence of chain mappings \( f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1} \) the following formula is taken place

\[ H_*(F''(f^n), \ldots, \alpha(X^i), \ldots, F'(f^1)) = H_*(F''(f^n), \ldots, \alpha(X^i), \ldots, F'(f^1)). \]
A functor $F$ will be called formal if homotopies $h(F)$ may be chosen in such a way that for any mapping of graded modules $f: M' \to M''$ the following relation is satisfied

$$h''(F) \circ F(f) = F(f) \circ h'(F).$$

Note that from the last relation it follows

$$\eta''(F) \circ F(f) = F_*(f) \circ \eta'(F), \ F(f) \circ \xi'(F) = \xi''(F) \circ F_*(f).$$

From the definition of functorial homology operations directly follows that if $F$ is a formal functor restricted on the category of graded modules then the $A_\infty$-structure on $F_*$ is degenerated. In this case there is the formula

$$F_*(f^n, \ldots, f^1) = F_*(H_*(f^n, \ldots, f^1)).$$

A transformation $\alpha: F' \to F''$ of chain functors will be called formal if homotopies $h(F')$ and $h(F'')$ may be chosen in such a way that the following relation is satisfied

$$h''(F) \circ \alpha = \alpha \circ h'(F).$$

If $\alpha: F' \to F''$ is a formal transformation then the $A_\infty$-transformation structure from $F'_*$ to $F''_*$ is degenerated.

In this case there is the formula

$$\alpha_*(X_{n+1}) \circ F'_*(f^n, \ldots, f^1) = F''_*(f^n, \ldots, f^1) \circ \alpha_*(X_1).$$

8. Homology operations for the operad $E$

Show that the functors $\underline{E}, \underline{E}$ corresponding to an $E_\infty$-operad $E$ are chain. To do it we define the family of mappings

$$\Delta^n \otimes \underline{E}(j; X) \to \underline{E}(j; \Delta^n \otimes X)$$

to be the compositions

$$\Delta^n \otimes \underline{E}(j; X) = \Delta^n \otimes E(j) \otimes_{\Sigma_j} X^{\otimes j} \overset{\Delta^n \otimes E(j) \otimes \Sigma_j X^{\otimes j}}{\longrightarrow} \Delta^n \otimes E(j) \otimes E(j) \otimes_{\Sigma_j} X^{\otimes j} \overset{\tau \otimes_{\Sigma_j} 1}{\longrightarrow} \Delta^n \otimes E(j) \otimes \Sigma_j X^{\otimes j} \overset{\Delta^n \otimes \Sigma_j X^{\otimes j}}{\longrightarrow} \Delta^n \otimes X^{\otimes j} = \underline{E}(j; \Delta^n \otimes X),$$

where $\tau: \Delta^n \otimes E(j) \to \Delta^n \otimes j$ is an $E$-coalgebra structure on the complex $\Delta^n$.

Direct verification show that the required relations are satisfied.

Similary define mappings

$$\Delta^n \otimes \underline{E}(j; X) \to \underline{E}(j; \Delta^n \otimes X)$$

or, that is the same, mappings

$$E(j) \otimes \Delta^n \otimes \text{Hom}_j(F(j); X^{\otimes i}) \to (\Delta^n \otimes X)^{\otimes i}.$$
to be the compositions

\[
E(j) \otimes \Delta^n \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j}) \xrightarrow{\nabla \otimes 1 \otimes 1} E(j) \otimes E(j) \otimes \Delta^n \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j}) \\
\rightarrow E(j) \otimes \Delta^n \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j}) \rightarrow (\Delta^n \otimes X)^{\otimes j}.
\]

Direct verification shows that the required relations are satisfied. So we have

**Theorem 18.** The functors $E_*, \overline{E}_*$ are $A_\infty$-functors.

Our aim is to calculate the functional homology operations for the functors $E_*, \overline{E}_*$. It means that for any sequence

\[
f^1 : X^1 \rightarrow X^2, \ldots, f^n : X^n \rightarrow X^{n+1}
\]

of chain mappings we need to calculate the mappings

\[
E_*(f^n, \ldots, f^1); E_*(X^1) \rightarrow E_*(X^{n+1}), \quad \overline{E}_*(f^n, \ldots, f^1); \overline{E}_*(X^1) \rightarrow \overline{E}_*(X^{n+1}).
\]

Consider firstly the functor $E(2; -)$ corresponding to a complex $X$ the complex

\[
E(2; X) = E(2) \otimes_{\Sigma_2} X \otimes X,
\]

where $E(2)$ is a $\Sigma_2$-free and acyclic complex with generators $e_i$ of dimensions $i$. A differential is defined by the formula

\[
d(e_i) = e_{i-1} + e_{i-1}T, \quad T \in \Sigma_2.
\]

The homology $E_*(2; -)$ of this functor, as it was pointed out above, is not only a functor but an $A_\infty$-functor. It means that for any sequence of chain mappings

\[
f^1 : X^1 \rightarrow X^2, \ldots, f^n : X^n \rightarrow X^{n+1}
\]

there is the operation

\[
E_*(2; f^n, \ldots, f^1); E_*(2; X^1) \rightarrow E_*(2; X^{n+1}).
\]

Let us calculate these operations.

Note that for a chain complex $X$ there is an isomorphism

\[
E_*(2; X) \cong E_*(2; X_*).
\]

If $X_*$ is a graded module then $E_*(2; X_*)$ is the sum of two factors. The first factor is the quotient module $X_* \cdot X_*$ of the tensor product $X_* \otimes X_*$ up to permutation of factors. The second factor is the module generated by the elements of the form $e_i \times y_n$, $i \geq 1$ of dimensions $i + 2n$. The elements $y_n \cdot y_n \in X_* \cdot X_*$ will be also denoted as $e_0 \times y_n$.

Let $\xi: X_* \rightarrow X$, $\eta: X \rightarrow X_*$, $h: X \rightarrow X$ are mappings giving a chain equivalence between $X$ and $X_*$. Denote by

\[
E(\xi); E(2; X_*) \rightarrow E(2; X_*), \quad E(\eta); E(2; X) \rightarrow E(2; X_*) \quad E(h); E(2; X) \rightarrow E(2; X_*)
\]
the mappings defined by the formulas

\[ E(\xi)(e_i \otimes y_1 \otimes y_2) = e_i \otimes \xi(y_1) \otimes \xi(y_2), \quad E(\eta)(e_i \otimes x_1 \otimes x_2) = e_i \otimes \eta(x_1) \otimes \eta(x_2), \]
\[ E(h)(e_i \otimes x_1 \otimes x_2) = e_i \otimes (x_1 \otimes h(x_2) + h(x_1) \otimes \xi\eta(x_2)) + e_{i-1} \otimes h(x_1) \otimes h(x_2). \]

It is clear they give a chain equivalence \( E(2, X) \simeq E(2, X_*) \).

Define mappings

\[ \xi(E): E_*(2; X_*) \rightarrow E(2; X_*) \]
\[ \eta(E): E(2; X_*) \rightarrow E_*(2; X_*) \]
\[ h(E): E(2; X_*) \rightarrow E(2; X_*) \]

To do it firstly we we choose an ordering basis \( \{y\} \) in \( X_\ast \). Then define the mapping \( \xi(E) \) putting

\[ \xi(E)(e_i \otimes y) = e_i \otimes y \otimes y; \quad \xi(E)(y_1 \cdot y_2) = e_0 \otimes (y_1 \otimes y_2), \; y_1 \leq y_2. \]

Define the mapping \( \eta(E) \) putting

\[ \eta(E)(e_i \otimes y_1 \otimes y_2) = \begin{cases} 
  e_i \otimes y_1, & y_1 = y_2 \\
  y_1 \cdot y_2, & y_1 < y_2, i = 0 \\
  0, & \text{in other cases}
\end{cases} \]

Define the mapping \( h(E) \) putting

\[ h(E)(e_i \otimes y_1 \otimes y_2) = \begin{cases} 
  e_{i+1} \otimes y_2 \otimes y_1, & y_1 > y_2, \\
  0, & \text{in other cases}
\end{cases} \]

Direct calculations show that the required relations are satisfied.

The mappings

\[ E(\xi) \circ \xi(E): E_*(2, X_*) \rightarrow E(2, X), \quad \eta(E) \circ E(\eta): E(2, X) \rightarrow E_*(2, X_*) \]
\[ E(\xi) \circ h(E) \circ E(\eta) + E(h): E(2, X) \rightarrow E(2, X) \]

give us a chain equivalence between \( E(2, X) \) and \( E_*(2, X_*) \).

From the general formula of functional homology operations for a chain functor it follows that for the functor \( E(2, -) \) the following formula is taken place

\[ E_*(2, f^n, \ldots, f^1) = \sum E_*(2, H_*(f^n, \ldots, f^{n+1}), \ldots, H_*(f^n, \ldots, f^1)), \]

where the sum is taken over all \( m \) and \( n_1, \ldots, n_m \) such that \( 1 \leq n_1 < \cdots < n_m < n \).

For a graded module \( X \) with fixed ordering basis \( \{x_i\} \) define mappings \( p: X \otimes X \rightarrow X, \; q: X \rightarrow X \otimes X, \; r: X \otimes X \rightarrow X \otimes X \) putting \( q(x_i) = x_i \otimes x_i \) and

\[ p(x_i \otimes x_j) = \begin{cases} 
  x_i, & i = j, \\
  0, & i \neq j
\end{cases} \]
\[ r(x_i \otimes x_j) = \begin{cases} 
  x_j \otimes x_i, & i > j, \\
  0, & i \geq j
\end{cases} \]

For a sequence of mappings \( f^1: X^1 \rightarrow X^2, \ldots, f^n: X^n \rightarrow X^{n+1} \) of graded modules with ordering basises define the mapping \((f^n, \ldots, f^1): X^1 \rightarrow X^{n+1} \) putting

\[ (f^n, \ldots, f^1) = p \circ (f^n) \circ q \circ (f^{n-1}) \circ q \circ \cdots \circ p \circ (f^1) \]
Directly from the definition of the homology operations it follows

**Theorem 19.** For a sequence of mappings \( f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1} \) of graded modules the following formula is taken place

\[
E_*(2; f^n, \ldots, f^1)(e_i \times x) = e_{i+n-1} \times (f^n, \ldots, f^1)(x).
\]

To obtain the corresponding formula for the functor \( E_* \) it needs to use a monad structure \( \gamma_*: E_* \circ E_* \to E_* \) and the formula

\[
E_*(f^n, \ldots, f^1) \circ \gamma_* = \sum E_*(E_*(f^n, \ldots, f^{n_m+1}), \ldots, E_*(f^n, \ldots, f^1)),
\]

where the sum is taken over all \( m \) and \( n_1, \ldots, n_m \) such that \( 1 \leq n_1 < \cdots < n_m < n \).

Passing to the Dyer-Lashof algebra \( R \) we obtain the operations

\[
R(f^n, \ldots, f^1): R \times X^1 \to R \times X^{n+1},
\]

which on the generators \( Q^i \) are expressed by the formulas

\[
R(f^n, \ldots, f^1)(Q^i \otimes x) = Q^{i+n-1} \otimes (f^n, \ldots, f^1)(x).
\]

Dually for the functor \( E_* \) there is

**Theorem 20.** For a sequence of mappings \( f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1} \) of graded modules the following formula is taken place

\[
E_*(2; f^n, \ldots, f^1)(e_i \times x) = \begin{cases} 
  e_{i-n+1} \times (f^n, \ldots, f^1)(x), & i \geq n-1 \\
  0, & \text{in other cases}
\end{cases}
\]

To obtain the corresponding formula for the functor \( E_* \) it needs to use a comonad structure \( \gamma_*: E_* \to E_* \circ E_* \) and the formula

\[
\gamma_* \circ E_*(f^n, \ldots, f^1) = \sum E_*(E_*(f^n, \ldots, f^{n_m+1}), \ldots, E_*(f^n, \ldots, f^1)) \circ \gamma_*,
\]

where the sum is taken over all \( m \) and \( n_1, \ldots, n_m \) such that \( 1 \leq n_1 < \cdots < n_m < n \).

Passing to the Milnor coalgebra \( K \) we obtain the operations

\[
K(f^n, \ldots, f^1): K \times X^1 \to K \times X^{n+1},
\]

which are expressed by the formulas

\[
K(f^n, \ldots, f^1)(y \otimes x) = y \cdot \xi_{i}^{n-1} \otimes (f^n, \ldots, f^1)(x).
\]

Consider operations associated with a comultiplication \( \nabla \) of the Milnor coalgebra \( K \). Denote

\[
\nabla(n) = \nabla \otimes 1 \cdots \otimes 1 - \cdots - (-1)^{n-1} 1 \otimes \cdots \otimes 1 \otimes \nabla: K^\times n \to K^\times n+1.
\]

Direct calculations show that the operations

\[
\langle \nabla(n), \nabla \rangle: K \to K^\times n+1, n \geq 2,
\]
are trivial on the elements $\xi_i^{2k}$. However on the other elements these operations in general are not trivial. For example there is the formula

$$(\nabla(2), \nabla)(\xi_i \xi_j) = \xi_{j-i}^2 \xi_0^2 \otimes \xi_i \xi_0^2 \otimes \xi_i, \ i < j.$$ 

Denote by $\tilde{\nabla}$ a comultiplication in the tensor product $\mathcal{K} \otimes \mathcal{K}$,

$$\tilde{\nabla} = (1 \otimes T \otimes 1)(\nabla \otimes \nabla).$$

Put

$$\tilde{\nabla}(n) = \tilde{\nabla} \otimes \cdots \otimes 1 - \cdots + (-1)^{n-1} 1 \otimes \cdots \otimes 1 \otimes \tilde{\nabla} : (\mathcal{K} \otimes \mathcal{K})^{\times n} \to (\mathcal{K} \otimes \mathcal{K})^{\times n+1}.$$

Consider the operations

$$(\pi^{\times n+1}, \tilde{\nabla}(n), \ldots, \tilde{\nabla)) : \mathcal{K}^{\otimes 2} \to \mathcal{K}^{\times n+1}.$$ 

Its restriction on the elements $x \otimes x \in \mathcal{K} \otimes \mathcal{K}$ we denote by

$$\Psi^n : \mathcal{K} \to \mathcal{K}^{\times n+1}.$$ 

From the formula of a comultiplication in the Milnor coalgebra directly follows the formula

$$\Psi^1(\xi_n) = \sum_{i < j} \xi_{n-i}^2 \xi_{n-j}^2 \otimes \xi_i \xi_j,$$

or in more general case

$$\Psi^1(\xi_n^m) = \sum_{i < j} \xi_{n-i}^{2i+m} \xi_{n-j}^{2j+m} \otimes \xi_i^m \xi_j^m.$$ 

In particular for the primitive elements $\xi_1^{2m} \in \mathcal{K}$ we have the formula

$$\Psi^1(\xi_1^{2m}) = \xi_1^{2m} \xi_0^m \otimes \xi_1^{2m}.$$ 

Similary for the operation $\Psi^2$ we have the formula

$$\Psi^2(\xi_n^{2m}) = \sum_{i < j \atop k > l} \xi_{n-i}^{2i+m} \xi_{n-j}^{2j+m} \otimes \xi_i^{2k+m} \xi_j^{2i+m} \otimes \xi_k^m \xi_l^m.$$ 

In particular for the primitive elements $\xi_1^{2m} \in \mathcal{K}$ we have the formula

$$\Psi^2(\xi_1^{2m}) = 0.$$ 

And so on.

9. $\cup_\infty - A_\infty$-Hopf algebras

To calculate higher differentials of the Adams spectral sequence we need to use not only the action of the Dyer-Lashof algebra, but the $E_\infty$-structure. However
this structure is too complicated. Some of the calculations were made in [6]. Here we’ll use only a part of the \( E_\infty \)-structure consisting of \( \cup_i \)-products.

A chain complex \( A \) will be called a \( \cup_\infty \)-algebra if there are given operations \( \cup_i: A \otimes A \to A, i \geq 0, \) called \( \cup_i \)-products, increasing dimensions by \( i \) and satisfying the relation

\[
d(x \cup_i y) = d(x) \cup_i y + x \cup_i d(y) + x \cup_{i-1} y + y \cup_{i-1} x.
\]

A differential coalgebra \( K \) will be called a \( \cup_\infty \)-Hopf algebra if there are given \( \cup_i \)-products \( \cup_i: K \otimes K \to K \) satisfying the distributivity relation

\[
\nabla(x \cup_i y) = \sum_k (x' \cup_{i-k} T^k y') \otimes (x'' \cup_k y''),
\]

where \( \nabla(x) = \sum x' \otimes x'' \), \( \nabla(y) = \sum y' \otimes y'' \), \( T: K \otimes K \to K \otimes K \) is the permutation mapping, \( T^k \) its \( k \)-th iteration.

**Theorem 21.** The cobar construction \( FK \) over a \( \cup_\infty \)-Hopf algebra \( K \) is a \( \cup_\infty \)-algebra. Moreover \( \cup_i \)-products \( \cup_i: FK \otimes FK \to FK \) uniquely determined by the formula

\[
[x] \cup_i [y] = \begin{cases} [x \cup_{i-1} y], & i \geq 1, \\ [x, y], & i = 0. \end{cases}
\]

and the relations

\[
(x_1 x_2) \cup_i [y] = (x_1 \cup_i [y]) x_2 + x_1 (x_2 \cup_i [y]),
\]

\[
(x_1 x_2) \cup_i (y_1 y_2) = \sum_k (x_1 \cup_{i-k} T^k y_1)(x_2 \cup_k y_2) +
\]

\[
+ (x_1 \cup_i (y_1 y_2)) x_2 + x_1 (x_2 \cup_i (y_1 y_2)) +
\]

\[
+ ((x_1 x_2) \cup_i y_1) y_2 + y_1 ((x_1 x_2) \cup_i y_2) +
\]

\[
+ x_1 (x_2 \cup_i y_1) y_2 + y_1 (x_1 \cup_i y_2) x_2,
\]

where \( x_1, x_2, y_1, y_2 \in FK, y \in K, i \geq 1. \)

Indeed, the products \( [x_1, \ldots, x_n] \cup_i [y] \) are determined by the first relation

\[
[x_1, \ldots, x_n] \cup_i [y] = \sum_{k=1}^{n} [x_1, \ldots, x_k \cup_i y, \ldots, x_n]
\]

From the second relation it follows that to define \( \cup_i \)-products in general case, i.e.

\[
[x_1, \ldots, x_n] \cup_i [y_1, \ldots, y_m]
\]

it is sufficient to define only \( \cup_i \)-products \( [x] \cup_i [y_1, \ldots, y_m] \).

We have

\[
d([y_1, \ldots, y_m] \cup_{i+1} [x]) = \sum_{k=1}^{m} [y_1, \ldots, d(y_k), \ldots, y_m] \cup_{i+1} [x] +
\]

\[
+ \sum_{k=1}^{m} [y_1, \ldots, y'_k, y''_k, \ldots, y_m] \cup_{i+1} [x] +
\]

\[
+ [y_1, \ldots, y_m] \cup_{i+1} ([d(x) + [x', x''] +
\]

\[
+ [x, y_1], \ldots, [x, y_m] + [x] + [y_1, \ldots, y_m]].
\]
Thus the product \([x] \cup [y_1, \ldots, y_m]\) is expressed through already defined products and products of the elements lesser dimensions. Hence \(\cup_i\)-products are determined by induction.

So this theorem gives us the formulas for \(\cup_i\)-product in the cobar construction. However they are inductive and not so simple even in the case when higher \(\cup_i\)-products \((i \geq 1)\) on \(K\) are trivial, i.e. when \(K\) is a commutative Hopf algebra.

An \(\cup_\infty\)-Hopf algebra \(K\) will be called commutative if the coproduct \(\nabla: K \to K \otimes K\) is comutative.

**Theorem 22.** The cobar construction \(F K\) over a commutative \(\cup_\infty\)-Hopf algebra is a commutative \(\cup_\infty\)-Hopf algebra. So the cobar construction over a commutative \(\cup_\infty\)-Hopf algebra may be iterated.

**Proof.** Define the coproduct \(\nabla: FK \to FK \otimes FK\) putting

\[
\nabla[x_1, \ldots, x_n] = \sum [x_{i_1}, \ldots, x_{i_p}] \otimes [x_{j_1}, \ldots, x_{j_q}],
\]

where the sum is taken over all \((p, q)\)-shuffles of \(1, 2, \ldots, n\). Direct calculations show that the required relations are satisfied.

Consider the question about the structure on the homology of a \(\cup_\infty\)-Hopf algebra.

Consider the question about the structure on the homology of a \(\cup_\infty\)-Hopf algebra. It is clear that on the homology \(K_*\) of a \(\cup_\infty\)-Hopf algebra \(K\) there are \(\cup_\infty\)-algebra structure, consisting of the operations

\[
\cup_i: K_* \otimes K_* \to K_*
\]

and \(A_\infty\)-coalgebra structure, consisting of the operations

\[
\nabla_n: K_* \to K_*^{\otimes n+2}.
\]

But besides that there are another operations of the form

\[
\Psi_{i,n}: K_* \otimes K_* \to K_*^{\otimes n+2}.
\]

To describe these operations and relations between them we introduce the notion of a \(\cup_\infty - A_\infty\)-Hopf algebra.

An \(A_\infty\)-coalgebra \(K\) will be called an \(\cup_\infty - A_\infty\)-Hopf algebra if on the cobar construction \(\tilde{F}K\) there is given \(\cup_\infty\)-algebra structure satisfying the relations

\[
\begin{align*}
(x_1 x_2) \cup_i [y] &= (x_1 \cup_i [y]) x_2 + x_1 (x_2 \cup_i [y]), \\
(x_1 x_2) \cup_i (y_1 y_2) &= \sum_k (x_1 \cup_{i-k} T^k y_1) (x_2 \cup_k y_2) + \\
&\quad + (x_1 \cup_i (y_1 y_2)) x_2 + x_1 (x_2 \cup_i (y_1 y_2)) + \\
&\quad + ((x_1 x_2) \cup_i y_1) y_2 + y_1 ((x_1 x_2) \cup_i y_2) + \\
&\quad + x_1 (x_2 \cup_i y_1) y_2 + y_1 (x_1 \cup_i y_2) x_2,
\end{align*}
\]

where \(x_1, x_2, y_1, y_2 \in FK, y \in K, i \geq 1\).

**Theorem 23.** If \(K\) is a \(\cup_\infty\)-Hopf algebra then its homology \(K_* = H_*(K)\) is \(\cup_\infty - A_\infty\)-Hopf algebra and there is an equivalence of \(\cup_\infty\)-algebras \(\tilde{F}K \cong FK\).
Proof. It is known \cite{14} that the homology $K_*$ of a differential coalgebra $K$ is $A_\infty$-coalgebra and there are algebra mappings $\xi: \tilde{FK}_* \to FK$, $\eta: FK \to \tilde{FK}_*$ and an algebra chain homotopy $h: FK \to FK$ such that $\eta \circ \xi = Id$, $d(h) = \xi \circ \eta - Id$. So we need to define on $\tilde{FK}_*$ the $\cup_i$-products. Put on generators $[x] \cup_i [y] = \eta(\xi[x] \cup_i \xi[y])$.

On the other elements the $\cup_i$-products determines by the relations.

Applying this theorem to the Milnor coalgebra we obtain

Theorem 24. The Milnor coalgebra $K$ possesses $\cup_\infty - A_\infty$-Hopf algebra structure. The homology of the corresponding cobar construction $\tilde{FK}$ is isomorphic to the $E^\infty$ term of the Adams spectral sequence.

10. The calculation of the differentials

Applying developed methods to calculate higher differentials of the Adams spectral sequence or, that is the same to calculate the differential in $\tilde{FK}$.

Recall that there are the following formulas for the $\cup_i$-products on the Milnor coalgebra

$$
\xi_n \cup_i \xi_n = \begin{cases} 
\xi_{n+k} \xi_n, & i = 2^{n+k} - 2^n; \\
\xi_{n+k} \xi_{n-1}, & i = 2^{n+k} - 2^n - 2^{n-1}; \\
\ldots & \ldots \\
\xi_{n+k} \xi_0, & i = 2^{n+k} - 2^n - \ldots - 1; \\
0, & \text{in other cases.}
\end{cases}
$$

Thus any element of $K$ may be obtained from $\xi_0$ by applying $\cup_i$-products. Hence we have

Theorem 25. The formulas for $\cup_i$-products in the Milnor coalgebra and the relations for $\cup_\infty$-algebra structure in the cobar construction $\tilde{FK}$ completely determine the differential in $\tilde{FK}$.

However the formulas for the differential are inductive and very complicated. So the next step in the calculation of the differential is to replace the Milnor coalgebra $K$ and the cobar construction $\tilde{FK}$ by more simply objects.

To do it consider the filtration of $\tilde{FK}$ putting the filtration of the elements $\xi_{i_1} \cdots \xi_{i_n} \in K$ to be equal $n$. Then the first term of the corresponding spectral sequence will be isomorphic to the polynomial algebra $PS^{-1}X$ over the module $X$ generated by the elements $\xi_i^{2^k}$. We’ll denote elements of $PS^{-1}X$ as elements of the cobar construction, i.e. $[x_1, \ldots, x_n], x_i \in X$.

Note that there is an algebra mapping $\eta: FK \to PS^{-1}X$, given by the formula

$$
\eta[x] = \begin{cases} 
[\xi_i^{2^k}], & x = \xi_i^{2^k}; \\
0, & \text{in other cases.}
\end{cases}
$$

The inverse mapping $\xi: PS^{-1}X \to FK$ may be given by the formula

$$
\xi([x_1, \ldots, x_n]) = [x_1, \ldots, x_n], \quad x_1 \leq \cdots \leq x_n.
$$

It is not an algebra mapping, but if $x \leq y$ then $\xi(x \cdot y) = \xi(x) \cdot \xi(y)$.

From here and from the Perturbation Theory it follows...
Theorem 26. The polynomial algebra $PS^{-1}X$ possesses a $\cup_\infty$-algebra structure determined on the generators by the formulas

\[
[x] \cup_i [y] = \begin{cases} 
    0, & i \geq 1, \ x < y, \\
    [x \cup_{i-1} x], & i \geq 1, \ x = y, \\
    [x, y], & i = 0, \ x \leq y 
\end{cases}
\]

and satisfying the relations

\[
(x_1 x_2) \cup_i [y] = (x_1 \cup_i [y]) x_2 + x_1 (x_2 \cup_i [y]),
\]

\[
(x_1 x_2) \cup_i (y_1 y_2) = \sum_k (x_1 \cup_{i-k} T^k y_1)(x_2 \cup_k y_2) + \\
+ (x_1 \cup_i (y_1 y_2)) x_2 + x_1 (x_2 \cup_i (y_1 y_2)) + \\
+ ((x_1 x_2) \cup_i y_1) y_2 + y_1 ((x_1 x_2) \cup_i y_2) + \\
+ x_1 (x_2 \cup_i y_1) y_2 + y_1 (x_1 \cup_i y_2) x_2,
\]

where $x_1, x_2, y_1, y_2 \in PS^{-1}X, \ y \in X, \ i \geq 1$.

The homology of the corresponding complex $\tilde{P}S^{-1}X$ is isomorphic to the homology of $\tilde{FK}$ and hence to the $E^\infty$ term of the Adams spectral sequence.

This theorem gives us the inductive formulas for the differential and $\cup_i$-products in $\tilde{P}S^{-1}X$.

Since any element of $\tilde{P}S^{-1}X$ may be obtained from $[\xi_0]$ by applying $\cup_i$-products, we have

Theorem 27. The formulas for $\cup_i$-products in the module $X$ and the relations for $\cup_i$-products in $\tilde{P}S^{-1}X$ completely determine the differential in $\tilde{P}S^{-1}X$.

Denote $[\xi_n^m]$ by $h_{n,m}$. Note that in Adams notation $h_{1,m} = h_m$.

Using the formula $h_{0,0} \cup_n h_{0,0} = h_{0,0}$ we obtain

Theorem 28. For the differential in $\tilde{P}S^{-1}X$ on the elements $h_{n,0}$ there is the next formula

\[
d(h_{n,0}) = \sum_{i=1}^{n-1} h_{n-i,i} h_{i,0},
\]

i.e. there are no higher differentials on the elements $h_{n,0}$.

By induction, using the formula $h_{n,m} \cup_1 h_{n,m} = h_{n,m+1}$, we obtain the following theorem.

Theorem 29. For the differential in $\tilde{P}S^{-1}X$ on the elements $h_{n,m}, \ m \geq 1$, there is the next formula

\[
d(h_{n,m}) = \sum_{i=1}^{n-1} h_{n-i,m+1} h_{i,m} + \sum_{i=1}^{m-1} h_{i,0} h_{n,m-i} + \sum_{i=1}^{m-2} h_{1,i} h_{n,m-i}^2 + \\
+ \sum_{i=2}^{m-2} h_{1,0} h_{n,m-1}^2 + \sum_{i=1}^{n-1} h_{n-i,i} h_{i+m,0} + \\
+ \sum_{i=1}^{n-1} h_{n-i,m+i-1} h_{i+1,0}^{2m-1} \ (m > 1) + \\
+ \sum_{i=1}^{n-1} h_{n-2,i+1} h_{i+1,m-2}^2 \ (m > 2).
\]
In particular, for the Adams elements \( h_m = h_{1,m} \) we have the following formula

\[
d(h_m) = \sum_{i=1}^{m-1} h_{i,0} h_{m-i}^{2i} + \sum_{i=1}^{m-2} h_{i-1,1} h_{m-i}^{2i} + \sum_{i=2}^{m-2} h_{0,1} h_{m-i}^{2i} =
\]

\[
= h_0 h_{m-1}^2 + \text{elements of greater filtration}
\]

The first member \( h_0 h_{m-1}^2 \) corresponds to the second differential of the Adams spectral sequence. So we have the Adams formula

\[
d_2(h_m) = h_0 h_{m-1}^2.
\]

Of course it may be obtained directly without using the general formula. Namely,

\[
d(h_m) = d(h_{m-1} \cup_1 h_{m-1}) =
\]

\[
= (h_{-1} h_{m-1} + \ldots) \cup_2 (h_{-1} h_{m-1} + \ldots) =
\]

\[
= (h_{-1} \cup_2 h_{-1}) h_{m-1}^2 + \ldots =
\]

\[
= h_0 h_{m-1}^2 + \ldots
\]

where dots denote the elements of greater filtration. Hence \( d_2(h_m) = h_0 h_{m-1}^2 \).

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