Generalized embedding variables for geometrodynamics and spacetime diffeomorphisms: Ultralocal coordinate conditions

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Abstract

We investigate the embedding variable approach to geometrodynamics advocated in work by Isham, Kuchař and Unruh for a general class of coordinate conditions that mirror the Isham-Kuchař Gaussian condition but allow for arbitrary algebraic complexity. We find that the same essential structure present in the ultralocal Gaussian condition is repeated in the general case. The resultant embedding–extended phase space contains a full representation of the Lie algebra of the space-time diffeomorphism group as well as a consistent pure gravity sector.

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1 Introduction

The central beauty of general relativity, general covariance, is both a joy and a curse for those who have chosen to work in the field of geometrodynamics. To study dynamics, we need some form of preferred, God given, time coordinate, but we know that a theory that is invariant under space-time diffeomorphisms cannot have an externally preferred time. In fact, as we shall demonstrate in the next section, fixation of the coordinates in the Einstein-Hilbert action prior to the variation of the dynamical variables (i.e. certain combinations of the components of the metric tensor) does not lead to the full Einstein equations, and thus, at a first glance, it seems that a specification of time will be impossible if we hope to recover general relativity. Furthermore, gravity possesses a far richer concept of time than that which our pre-relativistic ancestors grew up with; the natural structure for time in Lorentzian signature geometrodynamics is the set of all spacelike hypersurfaces in a given spacetime, dubbed hypertime by Kuchař [1]. Hypertime is many-fingered; every point in the spacetime can experience time at its own rate, corresponding to the various ways of pushing forward a hypersurface into the future, and thus time is not some single variable, as in conventional dynamics, but instead can only be represented by the full spacetime coordinates $X^\mu(t, x)$ of a chosen hypersurface in a foliation of the spacetime, where $X^\mu$ denotes a general coordinate on the spacetime manifold, and $x$ denotes a position on the hypersurface, which is selected from the foliation by the label $t$. The nature of hypertime becomes a major problem when we try to quantize gravity via canonical methods, as discussed by Kuchař [1] as well as Unruh and Wald [2]. This is obvious from the structure of the Wheeler-DeWitt equation (cf. Section 3) which has no manifest time parameter and thus no easy interpretation in terms of quantum dynamics, leading to the so-called problem of time. In ground breaking work, Isham, Kuchař and Unruh [3, 4, 5] have taken a bold step into land that most relativists fear to tread, and have indeed advocated the introduction of a preferred time coordinate into geometrodynamics. Their work has indicated the process by which we may do this and still recover normal Einsteinian gravity. The Isham-Kuchař-Unruh (IKU) approach involves adding a set of variables to gravity that represent a preferred coordinate system, bolted firmly in place to the gravitational field by a set of coordinate conditions. These variables can be considered to be specifications of the embedding of hypersurfaces into...
the spacetime, in the sense of the $X^\mu$ described earlier, and thus are a representation of the hypertime concept. The embedding variables are appended to the Einstein-Hilbert action by using a Lagrange multiplier term involving the desired coordinate conditions (cf. next section). The multipliers can then be removed as free variables at the Hamiltonian level by some construction, giving an extended phase space consisting of only the gravitational degrees of freedom plus the embedding variables. The embedding-extended theory can be regarded as one describing a matter field coupled to Einsteinian geometrodynamics. For the Gaussian \cite{4, 6} and harmonic \cite{7, 8} coordinate conditions, it can be demonstrated that one can consistently impose extra constraints on the theory to obtain standard general relativity \textit{in vacuo}. Furthermore, the extended phase space contains a full representation of the Lie algebra of the spacetime diffeomorphism group. However, all work so far has used specific coordinate conditions, and thus a general understanding of the dependence of the resulting formalism on the conditions themselves has been lacking.

With the former in mind, it seems apt to embark on a program in which we try to formulate extremely general coordinate conditions, so that we may understand the general nature of the embedding approach. In this paper, we will study embedding variables corresponding to coordinate conditions that can be written in terms of ultralocal algebraic functions of the metric with respect to the embedding on a spatial hypersurface. This condition includes the Isham-Kuchař Gaussian condition, the Unruh unimodular condition \cite{5} (with added frame choice conditions) as well as the conformal gauge condition \cite{9} (in two dimensions), but does not include the harmonic condition. We shall verify the properties discovered for the Gaussian coordinate condition and shall find that the general algebraic complexity of the description does not damage the geometrical content of the resulting Hamiltonian formulation. We will briefly discuss issues related to the use of this method for canonical quantization of the gravitational field, but will primarily focus on the purely classical geometrodynamical content of the construction.

2 Coordinate conditions in the Lagrangian

We will start by briefly examining the effect of using coordinate conditions in gravity in the manifestly covariant Lagrangian formalism before proceeding, in the next section, with the Hamiltonian analysis, which shall form the major
part of this paper. We will assume for the entire paper that the spacetime $n$-manifold, $\mathcal{M}$, is of the form $M \times \mathcal{T}$, where $M$ is an $(n-1)$-manifold without boundary, and $\mathcal{T}$ is a 1-manifold (possibly with boundary). We shall further assume that $\mathcal{M}$ possesses a metric $g$ of Lorentzian signature, giving it the structure of a pseudo-Riemannian manifold. Such a metric is a map defined in terms of the tangent space at a point $Y \in \mathcal{M}$,

$$g_Y : T_Y(\mathcal{M}) \times T_Y(\mathcal{M}) \rightarrow \mathbb{R}. \quad (1)$$

The foliation embedding of $M$ into $\mathcal{M}$, $i(t) : M \rightarrow \mathcal{M}$, will be restricted to be everywhere spacelike for all values of $t \in \mathcal{T}$. We can then define the (positive definite) metric tensor on each leaf of the foliation by

$$h_{(t,x)} = i^*g_{Y(t,x)}, \quad (2)$$

where $x \in M$, $h : T_x(M) \times T_x(M) \rightarrow \mathbb{R}$, and $Y(t, x) = i(t)x$.

We can work in a coordinate patch of $\mathcal{M}$, $U \times V$, where $U \subseteq M$ and $V \subseteq \mathcal{T}$. We will denote the resulting coordinates by $x^a(Y)$ and $t(Y)$, where lower case Latin characters range from one to $n-1$ and will denote the $n$-dimensional coordinate $(t, x^a)$ by $x^\mu$, with lower case greek letters running from zero to $n-1$. We can use the push-forward, $Y_*$, derived from the embedding to isomorphically map $T_{(t,x)}(M \times \mathcal{T})$ to $T_Y(\mathcal{M})$, and shall therefore not worry about distinguishing between objects on the various bundles constructed over the manifold in either language. The condition that the embedding be spacelike and that $g$ be Lorentzian is that for all points $Y \in \mathcal{M}$ we have

$$g_Y \left( v^a \frac{\partial}{\partial x^a}, v^b \frac{\partial}{\partial x^b} \right) > 0, \quad (3)$$

for all $v^a \in \mathbb{R}^{n-1}$ with $v^a \neq 0$, and

$$g_Y \left( t^\mu \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^a} \right) = 0, \forall a \Rightarrow g_Y \left( t^\mu \frac{\partial}{\partial x^\mu}, t^\nu \frac{\partial}{\partial x^\nu} \right) < 0, \quad (4)$$

where $t^\mu \in \mathbb{R}^n$, with $t^\mu \neq 0$. We can express the metric in terms of these coordinates by

$$g_Y = g_{\mu\nu}(Y) \, dx^\mu \otimes dx^\nu, \quad (5)$$

and similarly for $h$ in its coordinate system, as well as other objects. The standard Einstein-Hilbert action for general relativity is then

$$S^G[g] = \int_{\mathcal{M}} dt \, d^3x \sqrt{-g} R[g, Y] - 2 \int_{\partial\mathcal{M}} d^3x \sqrt{h} K[g, x], \quad (6)$$

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where $R[\mathbf{g}, Y]$ is the curvature scalar of $\mathbf{g}$, $K[\mathbf{g}, x]$ is the trace of the extrinsic curvature tensor of $\mathbf{g}$ relative to the embedding of the boundary, $\partial \mathcal{M} = M \times \partial T$, and $g$ and $h$ are the determinants of the metrics in the $x^\mu$ and $x^a$ coordinate systems respectively. The style of brackets is a reminder of the functional dependence on the metric. We then write this in terms of a gravitational Lagrangian, $L^G[\mathbf{g}]$, by defining

$$S^G[\mathbf{g}] = \int_T dt L^G[\mathbf{g}, t],$$

where $L^G$ only depends on $K$, $R$ and $g$ on the leaf of the foliation labeled by $t$. We can now define the usual Einstein tensor by

$$G_{\mu\nu}[\mathbf{g}, Y] = \frac{1}{\sqrt{-g(Y)}} \frac{\delta}{\delta g^{\mu\nu}(Y)} S[\mathbf{g}],$$

where $g^{\mu\nu}$ is the standard inverse to $g$, expressed in terms of the $x^\mu$ coordinate system. The resulting Einstein equations are, as always

$$G_{\mu\nu} = 0,$$

as long as we can freely vary $g^{\mu\nu}$, which corresponds to varying the embedding, $i$, as well as the metric, $h$, on each leaf.

However, what happens if we wish to keep the embedding fixed, and work in a given coordinate system? In this case, $g^{\mu\nu}$ cannot be varied freely, and we will not obtain the full Einstein equations. Consider a general coordinate system $(U_\alpha, X(\alpha))$, where $\mathcal{M} = \bigcup U_\alpha$, and $X(\alpha)$ takes $U_\alpha$ injectively into $\mathbb{R}^n$, with image $X^\mu_{(\alpha)}$ and with the map $X(\alpha)X^{-1}(\beta)$ being a $C^\infty$ diffeomorphism on $X(\beta)(U_\alpha \cap U_\beta)$; For this paper, we will define a preferred coordinate system to be one in which we choose ‘almost one’ $X(\alpha)$ in the following sense: the only maps allowed in the preferred atlas are a subset with $dX(\alpha) = dX(\beta)$ in $U_\alpha \cap U_\beta$. Therefore, the transition function derivatives,

$$X^\mu_{\nu(\alpha)} = \frac{\partial X^\mu_{(\alpha)}}{\partial x^\nu},$$

between an arbitrary coordinate system $x^\nu$ and $X^\mu_{(\alpha)}$, are independent of $\alpha$. We will denote this common value by $X^\mu_\nu$ and denote the images of the coordinate maps by $X^\mu$, and call these variables "embedding variables," from
now onwards. It is important to note that such a coordinate system will usually be singular. To select the \( n \) \( X^\mu \)'s, we need \( n \) independent conditions on the components of \( g \) in the \( X^\mu \) system, \( \bar{g}^{\mu \nu} \),

\[
F^A (\bar{g}^{\mu \nu}, Y) = 0,
\]

where

\[
\bar{g}^{\mu \nu} (Y) = \left[ g_Y \left( \frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial X^\nu} \right) \right]^{-1} = X^\mu_r (Y) X^\nu_s (Y) g^{\rho \sigma} (Y).
\]

Here, \( A \) runs from zero to \( n - 1 \) and \( g^{\mu \nu} \) is, as before, the inverse metric in the \( x^\mu \) coordinate system. We will restrict ourselves to ultralocal coordinate conditions, which are defined as follows:

**Definition 1 (Ultralocal coordinate conditions)** An ultralocal coordinate condition is a specification of \( n \) independent conditions, \( F^A (\bar{g}^{\mu \nu}) = 0 \), with the property that \( F^A (\bar{g}^{\mu \nu}) \), for each value of \( A \), is an ultralocal function of only the induced metric, \( \bar{g}^{\mu \nu} \). We will further require that each \( F^A (\bar{g}^{\mu \nu}) \) is a smooth function of \( \bar{g}^{\mu \nu} \) everywhere, except possibly for a discrete set of metrics.

Notice that we have two important invariant behaviors here; \( \bar{g}^{\mu \nu} \) is invariant to changes of \( x^\mu \) coordinates as well as to the choice of \( \alpha \) in \( X^\mu_\alpha \). Thus, each \( F^A \) transforms as a scalar under changes of the \( x^\mu \) coordinates.

Now, we can use the standard trick to calculate the variation of \( S^G \) whilst maintaining the conditions given to us by Equation (11); we append a Lagrange multiplier term to \( L^G \),

\[
L(t) = L^G(t) + L^R(t),
\]

with

\[
L^R(t) = -\frac{1}{2} \int_M d^3 x \sqrt{-g(t,x)} \lambda_A (t,x) F^A (\bar{g}^{\mu \nu}, Y(t,x)) .
\]

We then make the following obvious definitions:

\[
S [g] = S^G [g] + S^R [g], \quad S^R [g] = \int_T dt L^R [g, t] .
\]

Variation of \( \lambda_A \) then gives us Equation (11), and we may fix \( \lambda_A \) by varying \( X^\mu \). Notice that we have used general covariance to write the integration in
the Lagrange multiplier term as one over the \(x^a\) variables. Similarly, we can also write \(L^G\) purely in terms of \(x^\mu\) objects, and thus the \(X^\mu\)'s only appear through the \(\bar{g}^{\mu\nu}\) term in the conditions. It is then a simple matter to see that our new equations of motion corresponding to variation of \(g^{\mu\nu}\) are

\[
G_{\mu\nu} = T_{\mu\nu},
\]

where

\[
T_{\mu\nu}(t, x) = -\frac{1}{\sqrt{-g(t, x)}} \frac{\delta}{\delta g^{\mu\nu}(t, x)} L^R(t).
\]

Now, \(T_{\mu\nu}\) will transform as a rank two covariant tensor under general transformations of the \(x^\mu\) coordinates. Furthermore, it is well-known that diffeomorphism invariance is revealed through the contracted Bianchi identities \(G_{\mu\nu}^{;\nu} = 0\), which force the solutions to obey \(T_{\mu\nu}^{;\nu} = 0\). These relations give us differential equations for \(\lambda_A\) which are first order in time. However, we see that these are exactly the equations we get from the variation of \(X^\mu\) from the following theorem:

**Theorem 1 (\(X^\mu\) and the Bianchi identities)** The variation of the embedding variables, \(X^\mu\), in the extended action generates the contracted Bianchi identities for the energy-momentum tensor, \(T_{\mu\nu}\).

**Proof:** Now, \(T_{\mu\nu}^{;\nu}\) is a covector and thus we can evaluate it in any coordinate system and foliation; Let us choose \(x^\mu = X^\mu\). We then have \(\bar{g}^{\mu\nu} = g^{\mu\nu}\) and

\[
\delta \bar{g}^{\mu\nu} = 2\delta X^{(\mu;\nu)}.
\]

Thus we have

\[
\delta S = \delta S^R = 2 \int_M dt \, d^3x \left( \frac{\delta}{\delta g^{\mu\nu}(Y)} S^R \right) \delta X^{(\mu;\nu)}(Y).
\]

Therefore, integration by parts gives us

\[
\delta S = 2 \int_M dt \, d^3x \sqrt{-g(Y)} T_{\mu\nu}^{;\nu}(Y) \delta X^\nu.
\]

Hence we arrive at our final destination,

\[
\frac{\delta S}{\delta X^\nu(Y)} = 0 \Leftrightarrow T_{\mu\nu}^{;\nu}(Y) = 0,
\]

Thus we have
which is then true in all coordinate systems by general covariance. QED

Thus the action of spacetime diffeomorphism is now contained within the extended Lagrangian, $L$. This procedure allows us to consider the $X^\mu$ variables as *scalar matter fields* on $\mathcal{M}$ with a conserved energy-momentum tensor, with the caveat that they are to have the multivalued property discussed above. These fields may or may not behave like physical matter fields, as has been discussed elsewhere [4, 5].

To finish this section, we will briefly examine some possible ultralocal coordinate conditions:

### 2.1 Gaussian condition

Here, for any dimension $n$, we pick the Gaussian time condition,

$$F^0(\bar{g}^{\mu\nu}) = \bar{g}^{00} + 1,$$

and the Gaussian frame condition,

$$F^i(\bar{g}^{\mu\nu}) = \bar{g}^{0i}.$$  \hfill (23)

This condition forms the basis for the original work by Isham and Kuchař [4]. The solutions are metrics of the block form

$$\bar{g}^{\mu\nu}(Y) = \left( \begin{array}{cc} -1 & 0 \\ 0 & \gamma^{ij}(Y) \end{array} \right),$$

where $\gamma^{ij}$ is the metric on the $X^0 = constant$ surfaces, with respect to the $X^a$ coordinate system.

### 2.2 Conformal gauge condition

This is a condition that we see primarily in string theory, in $n = 2$. We have

$$F^0(\bar{g}^{\mu\nu}) = \bar{g}^{12}, \quad F^1(\bar{g}^{\mu\nu}) = \bar{g}^{11} + \bar{g}^{22}. $$

This puts the metric into conformal form,

$$\bar{g}^{\mu\nu}(Y) = \Omega^2(Y) \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right),$$

where $\Omega$ is the conformal factor. This condition has been studied by Kuchař and Torre [3].
2.3 Conditions of unimodular type

Here we will introduce a generalization of the unimodular condition proposed by Unruh [3], which corresponds to,

\[ F^0(\bar{g}^{\mu\nu}) = \det(\bar{g}^{\mu\nu}) + 1. \]  

(27)

We will call this condition the unimodular time condition. However, Equation (27) does not provide enough information to specify all \( n \) embedding variables, as has been discussed by Kuchař [4]. To fully encode hypertime, we will need extra frame conditions. For each choice of frame condition, we will have a corresponding generalization of the unimodular condition. One simple choice is to use the Gaussian frame conditions given by Equation (23), giving us a metric of the form

\[ \bar{g}^{\mu\nu}(Y) = \begin{pmatrix} -\det^{-1}(\gamma^{ij}(Y)) & 0 \\ 0 & \gamma^{ij}(Y) \end{pmatrix}, \]  

(28)

where \( \gamma^{ij} \) has the same interpretation as before. Again, this general class of conditions will work for any value of \( n \).

3 Embeddings in the Hamiltonian formalism

We will now proceed to our main purpose; for geometrodynamics, we are mainly interested in the Hamiltonian evolution of the theory. Let us start by looking at pure Einstein gravity, without extra variables. We will use the standard ADM technique [10], in which we write the inverse of the action of \( i^* \) on \( g \) as

\[ g_Y = -(N^2 - N_iN^i)dt \otimes dt + N_idx^i \otimes dt + h_{ij}dx^i \otimes dx^j, \]  

(29)

in terms of the \( x^\mu \) coordinates. \( N \) is the standard lapse function and \( N_i \in T(M) \) is the standard shift vector field. If we now use an overdot to denote differentiation with respect to the \( t \) coordinate, then we can define the momenta conjugate to the \( h_{ij} \) by

\[ \pi^{ij}(t, x) = \frac{\delta L_G(t)}{\delta \dot{h}_{ij}(t, x)}. \]  

(30)
The lapse and shift have no conjugate momenta, and thus are not true dynamical variables. Thus we can define coordinates \((h_{ij}, \pi^{ij})\) on the actual gravitational phase space \(T^*(C_h)\), the cotangent bundle over the space of positive definite metrics on \(M, C_h\). It is in this space that the usual dynamics of gravity take place. However, the action in these coordinates is now
\[
S^G[h_{ij}, \pi^{ij}, N, N^i] = \int_M dt d^3x \left( \pi^{ij} \dot{h}_{ij} - NH^G - N^i H^G_i \right),
\] (31)
and where the superhamiltonian is given by
\[
H^G = \frac{1}{\sqrt{h}} \left( \pi_{ij} \pi^{ij} - \frac{1}{2} \pi^i_i \pi^j_j \right) - \sqrt{h} R[h],
\] (32)
where the indices are raised and lowered by using \(h_{ij}\), and the supermomentum is given by
\[
H^G_i = -2\pi^{ij}_{ij}.
\] (33)
Now, variation of the redundant lapse and shift functions produces constraints on the dynamical system, given by
\[
H^G = 0 = H^G_i.
\] (34)
The constraints reduce the arena of geometrodynamics down to the constraint manifold \(\Gamma \subset T^*(C_h)\) on which Equations (34) hold. Ideally one would like to then solve these constraints and thus produce a standard, unconstrained, phase space on which normal dynamical evolution would take place. However, the constraints are by no means easy to solve and there may even be topological obstructions to a general solution [11]. Another problem with these constraints is their canonical commutation relations, which are
\[
\{H^G(x), H^G(x')\} = \left( h^{kl} H^G_k(x) - h^{kl} H^G_k(x') \right) \delta_J(x, x'),
\] (35)
\[
\{H^G_k(x), H^G(x')\} = H^G(x) \delta_J(x, x'),
\] (36)
and
\[
\{H^G_k(x), H^G_i(x')\} = H^G_i(x) \delta_{k,l} - H^G_i(x') \delta_{k,l}.
\] (37)
Our problem with these relations is the occurrence of \(h^{kl}\) in Equation (35); This makes it impossible to construct a representation in \(T^*(C_h)\) of the Lie algebra, \(\text{LDiff}(M)\), of the the Lie group of spacetime diffeomorphisms,
Diff(\mathcal{M})$, which is the gauge group of general relativity (cf. Ref. [4] for a discussion). Our third problem arises if we do not wish to solve the constraints and proceed to quantization; In this case, the constraints becomes operators, $\hat{H}$ and $\hat{H}_i$, on the wavefunction, $\psi(h)$. The equation $\hat{H}_i \psi = 0$ then tells us that $\psi$ is invariant under the action of Diff(\mathcal{M}), the group of spatial diffeomorphisms, and is thus only a function of 3-geometries, $^{(3)}G$, with the superhamiltonian constraint becoming the Wheeler-DeWitt equation,

$$\hat{H}_i^{(3)}G = 0.$$  \hspace{1cm} (38)

This equation has no manifest time parameter, thus producing extreme difficulties in the interpretation of quantum geometrodynamics (QGD). Reduction (solving the constraints) would allow us to identify some functional of the phase space coordinate $\tau[h_{ij}, \pi_{ij}]$ as a time variable, and would thus allow us to interpret Equation (38) in terms of a quantum evolution with respect to that variable [12]. However, we will not allow ourselves the luxury of imagining that we may be so lucky as to find such an ‘internal time’ but will, instead, use our embedding variables as a preferred, external, hypertime for the theory. As we have seen in Section 2, these variables change the dynamics of the theory; now we will investigate them at the Hamiltonian level.

The action $S^R$ can be regarded as simply a standard matter action, coupling the fields $\lambda_A$ and $X^\mu$ to gravity. $F^A$ contains no derivatives of $\bar{g}_{\mu\nu}$, and thus we do not change the gravitational velocity-momentum relationships derived from Equation (30). Let us now define an extended phase space $\mathcal{E} = T^*(\mathcal{C}_h \times \mathcal{C}_X)$, where $\mathcal{C}_X$ is the configuration space of the embedding variables on $\mathcal{M}$. We then have a representation of the tangent bundle $T(\mathcal{M})$, restricted to the embedding, through its isomorphic relationship to the velocity bundle $T(\mathcal{C}_X)$. Again, we have no momenta conjugate to the $\lambda_A$ and thus we can simply regard these variables as generating constraints on the extended phase space, as one might expect from their rôle as enforcers of the coordinate conditions.

The Hamiltonian formalism can be expressed in a simpler form if we define

$$n^\mu = \frac{1}{N} \left( \dot{X}^\mu - L_N X^\mu \right),$$  \hspace{1cm} (39)

where $L_N$ is the standard Lie derivative in the direction $N^i$ on $\mathcal{M}$. This object has a simple geometric interpretation as being the normal to the foliation $i$. 

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expressed in the $X^\mu$ coordinates, in the sense that we have

$$n = n^\mu \frac{\partial}{\partial X^\mu} = \frac{1}{N} \left( \frac{\partial}{\partial t} - N^i \frac{\partial}{\partial x^i} \right), \quad X^i \frac{\partial}{\partial X^\mu} = \frac{\partial}{\partial x^i};$$  \hspace{1cm} (40)

and therefore that Equation (29) tells us that

$$g_Y(n, n) = -1, \quad g_Y(n, X^\nu_i \frac{\partial}{\partial X^\nu}) = 0$$  \hspace{1cm} (41)

for all $i$. This greatly simplifies the form of the Equation (12) for the ADM form of $g$ in the $X^\mu$ coordinate system, giving

$$\bar{g}^\mu_\nu = -n^\mu n^\nu + h_{ij} X^\mu_i X^\nu_j.$$  \hspace{1cm} (42)

We will now restrict ourselves to $F^A$ functions that correspond to a physical, unique, embedding, and are thus good in the following sense:

**Definition 2 (Good ultralocal coordinate conditions)** A good ultralocal coordinate condition is one with the property that, for a given specification of $X^\mu$ and $h$ on a hypersurface, there are at most two real solutions, $\pm n_S$, with $F^A(n_S) = 0$.

Notice that we do not require that $F^A(n_S) = 0$ has a solution for all values of $X^\mu$ on the hypersurface; some of the embeddings will correspond to non-spacelike hypersurfaces in $\mathcal{M}$ and thus Equation (41) will be violated.

Now we are in a position to calculate the momenta conjugate to the $X^\mu$:

$$P_{\mu}(t, x) = \frac{\delta L(t)}{\delta X^\mu_0(t, x)} = \sqrt{h} \lambda_A F^A_{\mu\nu}(n)n^\nu;$$  \hspace{1cm} (43)

where

$$F^A_{\mu\nu}(n) = \frac{\partial F^A(n)}{\partial \bar{g}^{\mu\nu}},$$  \hspace{1cm} (44)

and where we explicitly indicate the dependence of $F^A_{\mu\nu}$ on $n$ whilst not indicating the other variables for convenience later on. However, these relations immediately present us with a problem; the function

$$\lambda_A F^A_{\mu\nu}(n)n^\nu : T(C_X) \rightarrow T^*(C_X)$$
will not always be injective and therefore not surjective, with the result being that we cannot always invert it to retrieve $n$, and hence $\dot{X}^\mu$, from the momenta. This means that there must be extra constraints in the theory. To solve this problem, and to also make it simple to remove the Lagrange multipliers from our system, we will solve these extra constraints, and work with the remaining freedom in $\lambda_A$. We proceed as follows: Let us define $C_\lambda$ to be the configuration bundle of $\lambda_A$ over $M$, with fiber $C_{\lambda x}$. We find that although Equation (43) is not generally invertible for $n$, as a function of the momenta, it is generally invertible for $\lambda_A \in C_{\lambda x}$, as a function of the momenta and velocities; 

**Theorem 2 (Inversion for Lagrange multipliers)** If the coordinate condition functions $F^A$ are functionally independent, as functions of $n$, then there exists an open set $C'$, of the total configuration bundle, $C = C_h \times C_X \times C_\lambda$, such that the map

$$W^A_\mu(n) = F^A_{\mu \nu}(n)n^\nu: C_{\lambda x} \to T^*_x(C_X)$$

is invertible for all configurations $c \in C'$.

**Proof:** Now, we have

$$W^A_\mu = -\frac{1}{2} \frac{\partial F^A}{\partial n^\nu},$$

and is therefore proportional to the Jacobian matrix for $F^A$ as a function of $n$. Therefore functional independence tells us that there cannot exist a field $k_A \in C_\lambda$ such that $W^A_\mu k_A = 0$ for all of $C$. However, due to the smoothness properties of $F^A$, this implies that there must be an open set $C'$ that is dense in $C$ such that for all $c \in C'$ there does not exist a $k_{Ax} \in T_x(C_\lambda)$ with $W^A_\mu k_{Ax} = 0$. Thus $W^A_\mu$ has no kernel at $c$ and is thus invertible. QED

This theorem therefore allows us to invert Equation (43) to get

$$\lambda_A(n) = \frac{1}{\sqrt{h}} T^\mu_A(n) P_\mu, \quad T^\mu_A(n) = \left( W^A_\mu(n) \right)^{-1},$$

for all $c \in C'$. However, for a fixed $P_\mu$, variation of $n$ does not generally produce all values in $C_\lambda$, but only those in a subspace, denoted by $C_\lambda^\parallel$. Thus, the values of $\lambda_A$ are constrained to lie in this space, and thus we can label these physical values by a new Lagrange multiplier, $\tilde{n}$, with the substitution

$$\lambda_A(\tilde{n}) = \lambda_A(n \to \tilde{n}) = \frac{1}{\sqrt{h}} T^\mu_A(\tilde{n}) P_\mu.$$
The $n$ components of $\bar{n}$ are, of course, an overcounting of the values in $C_\lambda$; let $C_{\bar{n}}$ be the configuration space for $\bar{n}$, and let us define the following equivalence relation:

**Definition 3 (Equivalence on $C_{\bar{n}}$)** We say that $\bar{n}, \bar{n}' \in C_{\bar{n}}$ are equivalent, denoted by $\bar{n} \sim \bar{n}'$, if and only if $\lambda_A(\bar{n}) = \lambda_A(\bar{n}')$. We denote the equivalence classes of $C_{\bar{n}}/\sim$ by $[\bar{n}] \subseteq C_{\bar{n}}$.

We can then obviously see that $C_{\parallel \lambda}$ is isomorphic to the quotient space $C_{\bar{n}}/\sim$. However, for technical reasons, we will have to define a finer equivalence relationship on $C_{\bar{n}}$, to allow full inversion of the velocity-momentum relations; in general, $[\bar{n}]$ will not be a single path component of $C_{\bar{n}}$ and thus we will split up these equivalence classes into a discrete union of path components and define a new relation as follows:

**Definition 4 (Local equivalence on $C_{\bar{n}}$)** We say that $\bar{n}, \bar{n}' \in C_{\bar{n}}$ are locally equivalent, denoted by $\bar{n} \leftrightarrow \bar{n}'$, if and only if there exists a continuous path $\bar{m}(s) \in C_{\bar{n}}$ with $s \in [0, 1]$, $\bar{m}(0) = \bar{n}$, $\bar{m}(1) = \bar{n}'$, and $\bar{m}(s) \sim \bar{n}$ for all $s$. We denote the equivalence classes of $C_{\bar{n}}/\leftrightarrow$ by $[\bar{n}]_L \subseteq C_{\bar{n}}$.

Obviously, $C_{\bar{n}}/\leftrightarrow$ is a covering space of $C_{\bar{n}}/\sim$ and therefore of $C_{\parallel \lambda}$. We can now invert the velocity-momentum relations on this covering space to arrive at the Hamiltonian form of $S^R$ restricted to $C_{\bar{n}}/\leftrightarrow$:

$$S^R \left[ h_{ij}, P_\mu, X^\nu, N, N^i, [\bar{n}]_L \right] = \int_M dt \, d^3x \left( \dot{X}^\mu P_\mu - NH^R - N^i H^R_i \right),$$  \hspace{1cm} (49)$$

where

$$H^R = \bar{n}^\mu P_\mu + \frac{1}{2} T^\mu_A(\bar{n}) F^A(\bar{n}) P_\mu,$$  \hspace{1cm} (50)$$

$$H^R_i = X^\mu_i P_\mu.$$  \hspace{1cm} (51)$$

to verify the consistency of this equation, we must check that it is invariant to the choice of representative of $\bar{n}' \in [\bar{n}]_L$;

**Theorem 3 (Consistency of Hamiltonian action on covering space)** If $\bar{n}, \bar{n}' \in C_{\bar{n}}$ and $\bar{n} \leftrightarrow \bar{n}'$ then we have $H^R(\bar{n}) = H^R(\bar{n}')$. 

\hspace{13cm} 13
Proof: We obviously only need to verify that the action is invariant under all small perturbations within $[\bar{n}]_L$. It is a simple matter to show that

$$\frac{\partial H^R}{\partial \bar{n}^\mu} = \frac{1}{2} \sqrt{h} \frac{\partial \lambda_A}{\partial \bar{n}^\mu} F^A.$$  

(52)

Now let $\bar{n} \rightarrow \bar{n} + \delta k$, with $\delta k \in \mathcal{C}_n$ small and $\bar{n} \sim \bar{n} + \delta k$. Thus we have

$$\frac{\partial \lambda_A}{\partial \bar{n}^\mu} \delta k^\mu = 0,$$  

(53)

and therefore $H^R(\bar{n} + \delta k) = H^R(\bar{n})$. Hence the result follows, and $S^R$ is a well-defined functional on $\mathcal{C}_n/\leftrightarrow$. QED

We are now in a position to remove $[\bar{n}]_L$ from the action by solving the dynamics of the covering space extended theory for it;

Theorem 4 (Removal of Lagrange multipliers) Variation of $[\bar{n}]_L \in \mathcal{C}_n/\leftrightarrow$ in the action $S^R[h_{ij}, P_\mu, X^\mu, N, N^i, \bar{n}]_L$ on the covering space extended theory gives us $H^R = \tilde{n}^\mu S P_\mu$, where $\tilde{n}_S \in \mathcal{C}_n$ is one of the $P_\mu$ independent solutions of $F^A(\tilde{n}_S) = 0$. This solution is valid for all of $\mathcal{C}$ if we require continuity of $H^R$.

Proof: From Equation (52) in Theorem 3 we have

$$\frac{\delta S^R}{\delta \bar{n}^\mu} = 0 \iff \frac{\partial H^R}{\partial \bar{n}^\mu} = 0 \iff \frac{\partial \lambda_A}{\partial \bar{n}^\mu} F^A = 0.$$  

(54)

This defines a discrete set of surfaces in $\mathcal{C}_n$ each corresponding to a single point $[\tilde{n}_S]_L \in \mathcal{C}_n/\leftrightarrow$, from Theorem 3. However, these solutions include those of the form $[\tilde{n}_S]_L$, with $F^A(\tilde{n}_S) = 0$. This solution is valid for all of $\mathcal{C}'$, but the closure of $\mathcal{C}'$ is $\mathcal{C}$, and thus, by continuity, the result follows. QED

Now, $\tilde{n}_S^\mu$ is specified precisely, up to sign, by $F^A = 0$ as long as the coordinate conditions are ‘good’ in the sense defined earlier; we will therefore pick out a single component, $[\tilde{n}_S]_L \subseteq [n]_L$, and thus our variation gives a well-defined couplet of solutions on $\mathcal{C}_n^\mu$. Due to the relationship between $\tilde{n}^\mu$ and $n^\mu$, and hence $X^\mu_0, \tilde{n}^\mu_S$ will be independent, as a vector, from the $n-1$ vectors $X^\mu_i$. Thus, we are motivated to define

$$Y^\mu_\nu = \begin{cases} 
\tilde{n}^\mu_S & \text{if } \nu = 0 \\
X^\mu_\nu & \text{if } \nu > 0
\end{cases}$$  

(55)
which is an object defined purely on the bundle $T^*(\mathcal{C}_h \times \mathcal{C}_X)$ and is independent of $P_\mu$. Let us now define $N^\mu = (N, N^i)$, $H^G_\mu = (H^G, H^G_i)$ and $H^R_\mu = (H^R, H^R_i)$. In this notation we have the following simple form for the embedding action without Lagrange multipliers:

$$S^R[h_{ij}, X^\mu, P_\nu, N^\sigma] = \int_{\mathcal{M}} dt \, d^3 x \left( \dot{X}^\mu P_\mu - N^\mu H^R_\mu \right), \quad (56)$$

where

$$H^R_\mu = Y^\nu_\mu P_\nu. \quad (57)$$

The total action for the theory is thus

$$S[h_{ij}, \pi^{lm}, X^\mu, P_\nu, N^\sigma] = \int_{\mathcal{M}} dt \, d^3 x \left( \dot{X}^\mu P_\mu + \dot{h}_{ij} \pi^{ij} - N^\mu H_\mu \right), \quad (58)$$

and $H_\mu$ is given by

$$H_\mu = H^G_\mu + H^R_\mu = H^G_\mu + Y^\nu_\mu P_\nu, \quad (59)$$

with the $N^\mu$ variation giving us constraints $H_\mu = 0$ on the embedding-extended phase space. To bring out the final form of the action, let us now define $Q^\mu_\nu$ to be the inverse of $Y^\mu_\nu$, $Q^\mu_\nu Y^\nu_\sigma = \delta^\mu_\sigma$. We can then define a new, equivalent, set of constraint functions by

$$\Pi^G_\mu = Q^\nu_\mu H_\mu = P_\mu + Q^\nu_\mu H^G_\mu, \quad (60)$$

and redefine the lapse and shift to get

$$S[h_{ij}, \pi^{lm}, X^\mu, P_\nu, N^\sigma] = \int_{\mathcal{M}} dt \, d^3 x \left( \dot{X}^\mu P_\mu + \dot{h}_{ij} \pi^{ij} - \bar{N}^\mu \Pi^G_\mu \right), \quad (61)$$

which corresponds to $N^\mu = \bar{N}^\nu Q^\mu_\nu$. The variation of these new lapse and shift functions then gives us $\Pi^G_\mu = 0$. It is this Hamiltonian formulation that allows us to construct a representation of spacetime diffeomorphisms, and which contains a copy of the standard Einstein geometrodynamics.

We have now made contact with the formalism developed by Isham, Kuchař and Torre [4, 6] and will now proceed to list the generic behavior present, which they derived for the specific case of the Gaussian coordinate condition. The proofs are identical, and a reader familiar with the embedding formalism may now move on to the discussion section. We present them
here purely for completeness. This formalism, like standard geometrodynamics, is constrained so that the physical dynamics lie on $\Gamma_e \subset T^*(C_h \times C_X)$, where $\Pi_\mu(\Gamma_e) = 0$. If we now define $h_\mu = Q_\mu^\nu H_\nu^G$, the ‘unprojected’ pure gravity constraint functions, then $\Gamma_e$ also contains a copy of $\Gamma$, the standard constraint surface, determined by $h_\mu(\Gamma) = 0 = H_\mu^G(\Gamma)$. The latter surface corresponds to the intersection of the surface $\Gamma_H \subset T^*(C_h \times C_X)$, defined by $P_\mu(\Gamma_H) = 0$, and $\Gamma_e$. The total Hamiltonian on $T^*(C_h \times C_X) \times C_{\bar{N}}$, where $C_{\bar{N}}$ is simply the configuration space for $\bar{N}_\mu$, is

$$ H_T = \int_M d^3x \bar{N}^\mu \Pi_\mu. \tag{62} $$

For the dynamics to be consistent, all that we would require would be that $\dot{H}_T = \{H_T, H_T\}$ vanishes weakly (vanishes on $\Gamma_e$), and thus that the constraints are propagated by the dynamics induced by $H_T$. However, we have a far stronger result, namely

**Theorem 5 (Abelian constraint algebra)** $\{\Pi_\mu, \Pi_\nu\}$ vanishes strongly.

**Proof:** Isham and Kuchař originally proved this result with a long and complicated calculation [4], but it can be derived from a simple argument [6]; The original constraint functions, $H_\mu$, obey the Dirac algebra, given by Equations (35) to (37), and thus $\{H_\mu, H_\nu\}$ vanishes weakly. Furthermore, the new constraints are equivalent to the old ones and thus $\{\Pi_\mu, \Pi_\nu\}$ weakly vanishes. However $\{\Pi_\mu, \Pi_\nu\}$ does not include any terms containing $P_\mu$, due in particular to the independence of $Q_\mu^\nu$ on the momenta, and hence the value of the commutator cannot depend of the value of the constraint functions, and thus we must have $\{\Pi_\mu, \Pi_\nu\} = 0$ strongly. QED

We can thus proceed to calculate the dynamics of this general IKU embedding-extended theory by evaluating the rest of the brackets. We get the following evolution equations:

$$ X^\mu(x) = \{X^\mu(x), H_T\} = \bar{N}^\mu(x), \tag{63} $$

$$ \dot{h}_{ij}(x) = \{h_{ij}(x), H_T\} = \int_M d^3x' \bar{N}^\mu(x')\{h_{ij}(x), H_\mu^G(x')\}, \tag{64} $$

$$ \dot{P}_\mu(x) = \{P_\mu(x), H_T\} = \int_M d^3x' \bar{N}^\nu(x')\{P_\mu(x), Q_\nu^\sigma(x')\} H_\sigma^G(x'). \tag{65} $$
\[ \dot{\pi}^{ij}(x) = \{\pi^{ij}, H_T\} \]
\[ = \int_M d^3x' N^\mu(x') \{\pi^{ij}(x), H^G_\mu(x')\} + \]
\[ \int_M d^3x' \bar{N}^\nu(x') \{\pi^{ij}(x), Q_{\mu}^{\nu}(x')\} H^G_{\mu}(x'). \] (66)

Where we see the relation \( n = \bar{n}_S \) showing up in Equation (63), thus verifying
the consistency of the dynamics. The coupling of the embedding field to
gravity shows up in the second term in \( \dot{\pi}^{ij} \), which vanishes when \( H^G_{\mu} = 0 \).

We have the following generalization of the work by Isham and Kuchar [4]:

**Theorem 6 (Einstein-Hilbert sector)** The space \( \Gamma_e \cap \Gamma_H \) is preserved un-
der the dynamical evolution generated by \( H_T \). The resulting dynamics are
those of pure Einstein gravity.

**Proof:** Let the system start with \( \Pi_\mu = H^G_\mu = P_\mu = 0 \). Equation (63)
gives us \( \dot{P}_\mu = 0 \), and Theorem 5 gives us \( \dot{\Pi}_\mu = 0 \). Hence \( h_\mu = 0 \), and thus
\( h_{ij} = 0 = H^G_\mu \) for all of the resulting trajectory in phase space. Equations (64)
and (66) then tell us that \( h_{ij} \) and \( \pi^{ij} \) propagate according to the standard
pure geometrodynamics evolution equations. QED

Thus, once we have \( P_\mu = 0 \), no internal gravitational dynamics can couple
to the embedding variables, and pure gravity results. Therefore, all that we
have left in our advertised itinerary is the following:

**Theorem 7 (Representation of spacetime diffeomorphisms)** The space
\( T^*(C_h \times C_X) \) contains a full homomorphic representation of the action of the
Lie algebra of the spacetime diffeomorphism group.

**Proof:** The Lie algebra of \( \text{Diff}(\mathcal{M}) \) is just the space of complete vector fields
on \( \mathcal{M} \). Therefore, let

\[ u(X^\alpha(x)) = u_\mu \frac{\partial}{\partial X^\mu} \in \text{LDiff}(\mathcal{M}), \]

then we define

\[ \Pi(u) = \int_M d^3x u^\mu \Pi_\mu. \] (67)
Therefore, we have

\[
\{\Pi(u), \Pi(v)\} = \int_M d^3 x \Pi_\mu \left( u^\nu \frac{\partial u^\mu}{\partial X^\nu} - v^\nu \frac{\partial v^\mu}{\partial X^\nu} \right),
\]

(68)

Where we have used

\[
\{u^\mu(X^\alpha(x)), \Pi_\nu(x')\} = \delta(x - x') \frac{\partial u^\mu(x)}{\partial X^\nu(x')},
\]

(69)

However, the Lie bracket \([uv]\) on \(\text{LDiff}(\mathcal{M})\) is simply

\[
[uv] = -[u, v] = \left( v^\nu \frac{\partial u^\mu}{\partial X^\nu} - u^\nu \frac{\partial v^\mu}{\partial X^\nu} \right) \frac{\partial}{\partial X^\mu},
\]

(70)

and therefore

\[
\{\Pi(u), \Pi(v)\} = \Pi([uv]),
\]

(71)

and thus we have our desired result. \textbf{QED}

We therefore have our desired Hamiltonian theory containing a representation of the Lie algebra of spacetime diffeomorphisms as well as a sector that obeys standard pure geometrodynamics.

4 Discussion

To end this paper, we shall mention a few important issues that crop up in the IKU formalism. The primary motivation behind this construction is the hope of formulating a consistent theory of quantum geometrodynamics. The quantum mechanical formalism is achieved by replacing the phase space coordinates with operators with the standard canonical rules, in which configuration variables are replaced by

\[
X^\mu \rightarrow \hat{X}^\mu = X^\mu \times, \quad h_{ij} \rightarrow \hat{h}_{ij} = h_{ij} \times,
\]

(72)

and the momenta are replaced by

\[
P_\mu \rightarrow \hat{P}_\mu = -i \frac{\delta}{\delta X^\mu}, \quad \pi^{ij} \rightarrow \hat{\pi}^{ij} = -i \frac{\delta}{\delta h_{ij}},
\]

(73)
acting on an embedded wavefunction \( \psi(h, X^\mu) \). Our first major improvement is that the IKU theory describes a wavefunction propagating from one spacelike surface to another, labeled by the preferred hypertime,

\[
\frac{\delta}{\delta X^\mu} \psi(h, X^\mu) = \hat{h}_\mu \psi(h, X^\mu),
\]

in comparison to the standard QGD Wheeler-DeWitt equation, given by Equation (38), which contains no mention of time. Thus the IKU construction gives us a natural multifingered Schrödinger equation. The second point is that the quantum theory then carries a representation of LDiff(\( M \)), as long as Theorems 5 and 7 have quantum equivalents. The latter condition is equivalent to asking for an ordering of the operators in \( \hat{\Pi}_\mu \) so that

\[
[\hat{\Pi}_\mu, \hat{\Pi}_\nu] = 0
\]

where \([,]\) denotes the standard group commutator. Unfortunately, achieving this can be a significant technical problem, as discussed by Kuchař and Torre [6], and may limit the applicability of this work in QGD. The most obvious remaining problem is that the IKU states corresponding to Equation (74) do not correspond classically to pure geometrodynamics, but to the theory given by Equation (16). If we wish to recover Einstein gravity, we must further require \( \hat{P}_\mu \psi = 0 \), which corresponds to \( \psi \) being independent of hypertime, and seemingly returns us to the problematic world of the Wheeler-DeWitt equation. Thus the ‘real’ time variable becomes inaccessible, and we, trapped inside the universe, cannot use it in the so-called Heraclitian sense (cf. Ref. [2]) to provide an ordering of events for us. There are two possible exits to this problem; The first is that the embedding fields may correspond to actual physical fields, as discussed by Kuchař and Torre [6, 7], and thus the constraint \( \hat{P}_\mu \psi = 0 \) is not needed. The second exit is that we can, in principle, construct \( \hat{P}_\mu \psi = 0 \) states out of a superposition of general solutions to the IKU equation, and may thus be able to use the external hypertime as an aid to interpretation. This latter point has been partially discussed by Halliwell and Hartle [13, 14].

The other issue that the Author wishes to raise is the generic nature of this construction; Although we have confined ourselves to Einstein gravity, it is obvious that the construction is valid for any theory of quantum gravity that has constraints similar to the Dirac algebra. We can quite easily couple
matter fields, or even make canonical transformations on the geometrodynam-ic phase space, without changing our results. In fact, the formalism is only dependent on our ability to construct the ‘unprojected’ constraints given by Equation (60) in such a way that $P_\mu$ appears through the linear term only, and does not appear in $h_\mu$. It is this essential property that leads to the representation of spacetime diffeomorphisms, as well as to the existence of a consistent, classical, copy of the original theory inside the extended phase space.

5 Conclusion

We have constructed a general version of the IKU embedding-extended formalism for quantum geometrodynamics that solves the problem of time whilst exchanging it for a collection of extra physical fields. The classical theory includes a copy of pure geometrodynamics, but one cannot be sure whether it is possible for this sector to be realized consistently in the quantum theory. The classical theory contains a representation of the Lie algebra of the spacetime diffeomorphism group, and this result may also extend to the quantum theory if certain operator ordering problems can be solved.

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