ON WEIGHTED $L^p$-HARDY INEQUALITY ON DOMAINS IN $\mathbb{R}^n$

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Dedicated to Professor Shmuel Agmon

ABSTRACT. We consider weighted $L^p$-Hardy inequalities involving the distance to the boundary of a domain in the $n$-dimensional Euclidean space with nonempty boundary. Using criticality theory, we give an alternative proof of the following result of F. G. Avkhadiev (2006)

**Theorem.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an arbitrary domain, $1 < p < \infty$ and $\alpha + p > n$. Let $d_\Omega(x) = \text{dist}(x, \partial \Omega)$ denote the distance of a point $x \in \Omega$ to $\partial \Omega$. Then the following Hardy-type inequality holds

$$
\int_{\Omega} |\nabla \varphi|^p \, dx \geq \left( \frac{\alpha + p - n}{p} \right)^p \int_{\Omega} \frac{|\varphi|^p}{d_\Omega^{p\alpha}} \, dx \quad \forall \varphi \in C^\infty_c(\Omega),
$$

and the lower bound constant $\left( \frac{\alpha + p - n}{p} \right)^p$ is sharp.

1. Introduction

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$ with nonempty boundary, and let $d_\Omega(x) = \text{dist}(x, \partial \Omega)$ denote the distance of a point $x \in \Omega$ to the boundary of $\Omega$. Fix $p \in (1, \infty)$. We say that the $L^p$-Hardy inequality is satisfied in $\Omega$ if there exists $c > 0$ such that

$$
\int_{\Omega} |\nabla \varphi|^p \, dx \geq c \int_{\Omega} \frac{|\varphi|^p}{d_\Omega^{p\alpha}} \, dx \quad \forall \varphi \in C^\infty_c(\Omega).
$$

The $L^p$-Hardy constant of $\Omega$ is the best constant $c$ for inequality (1.1) which is denoted here by $H_p(\Omega)$. It is a classical result that goes back to Hardy himself (see for example [4, 13]) that if $n = 1$ and $\Omega \subset \mathbb{R}$ is a bounded or unbounded interval, then the $L^p$-Hardy inequality holds and $H_p(\Omega)$ coincides with the widely known constant

$$
c_p = \left( \frac{p - 1}{p} \right)^p.
$$

Recall that if $\Omega$ is bounded and has a sufficiently regular boundary in $\mathbb{R}^n$, then the $L^p$-Hardy inequality holds and $0 < H_p(\Omega) \leq c_p$ (for instance, see [10, 16]). Moreover, if $\Omega$ is convex, or more generally, if it is weakly mean convex, i.e., if $\Delta d_\Omega \leq 0$ in the distributional sense in $\Omega$ (see [10, 11, 15, 18]),

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then \( H_p(\Omega) = c_p \). On the other hand, it is also well-known (see for example [4, 13]) that if \( \Omega = \mathbb{R}^n \setminus \{0\} \) and \( p \neq n \), then the \( L^p \)-Hardy inequality holds and \( H_p(\Omega) \) coincides with the other widely known constant

\[
c_{p,n} = \left| \frac{p - n}{p} \right|^p,
\]

which indicates that the \( L^p \)-Hardy inequality does not hold for \( \mathbb{R}^n \setminus \{0\} \) if \( p = n \).

In the present paper we study a weighted \( L^p \)-Hardy inequality involving the distance function to the boundary. We give a new proof for the following result.

**Theorem 1.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be an arbitrary domain, where \( n \geq 2 \). Fix \( 1 < p < \infty \) and \( \alpha + p > n \). Then

\[
\int_{\Omega} |\nabla \varphi|^p \, d\alpha \geq \left( \frac{\alpha + p - n}{p} \right)^p \int_{\Omega} |\varphi|^p \, d\alpha \quad \forall \varphi \in C_c^{\infty}(\Omega),
\]

and the lower bound constant

\[
c_{\alpha,p,n} := \left( \frac{\alpha + p - n}{p} \right)^p
\]

is sharp. In particular, for \( p > n \) we have

\[
H_p(\Omega) \geq c_{p,n} = \left( \frac{p - n}{p} \right)^p
\]

for any domain \( \Omega \subseteq \mathbb{R}^n \).

**Remark 1.2.** Theorem 1.1 was proved by F. G. Avkhadiev in [3] using a cubic approximation of \( \Omega \). One should note that J. L. Lewis [14] proved that (1.1) holds true (for \( \alpha = 0 \)) with a fixed positive constant independent on \( \Omega \), and in [19], A. Wannebo generalized Lewis’ result to the case \( \alpha + p > n \).

We need the following version of the Harnack convergence principle which will be used several times throughout the paper.

**Proposition 1.3** (Harnack convergence principle). Consider an exhaustion \( \{\Omega_i\}_{i=1}^{\infty} \) of smooth bounded domains such that

\[
\left\{ x \in \overline{\Omega} : d_{\Omega}(x) > \frac{1}{i} \right\} \subseteq \Omega_i \subseteq \Omega_{i+1}, \text{ and } \cup_{i \in \mathbb{N}} \Omega_i = \Omega.
\]

For each \( i \in \mathbb{N} \), let \( u_i \) be a positive (weak) solutions of the equation

\[
-\text{div} \left( d_{\Omega_i}^{-\alpha} |\nabla u_i|^{p-2} \nabla u_i \right) - \mu_i \frac{|u_i|^{p-2} u_i}{d_{\Omega_i}^{\alpha+p}} = 0 \quad \text{in } \Omega_i
\]

such that \( u_i(x_0) = 1 \), where \( x_0 \in \Omega_1 \), and \( \mu_i \in \mathbb{R} \).

If \( \mu_i \to \mu \), then there exists \( 0 < \beta < 1 \) such that, up to a subsequence, \( \{u_i\} \) converges in \( C_c^{0,\beta}(\Omega) \) to a positive (weak) solution \( u \in W_{\text{loc}}^{1,p}(\Omega) \) of the equation

\[
-\text{div} \left( d_{\Omega}^{-\alpha} |\nabla u|^{p-2} \nabla u \right) - \mu \frac{|u|^{p-2} u}{d_{\Omega}^{\alpha+p}} = 0 \quad \text{in } \Omega.
\]
Proof. Since \( d_{\Omega_i} \to d_{\Omega} \), the theorem follows directly from [12, Proposition 2.7].

The paper is organized as follows: In Section 2 we give our proof of Theorem 1.1 while in Appendix we outline two alternative proofs.

2. Proof of Theorem 1.1

Our proof of Theorem 1.1 is based on a simple construction of a (weak) positive supersolutions to the associated Euler-Lagrange Lagrange equations

\[
- \text{div} \left( d_{\Omega}^{-\alpha} |\nabla u|^{p-2} \nabla u \right) - \mu \frac{|u|^{p-2} u}{d_{\Omega}^{\alpha+p}} = 0 \quad \text{in } \Omega,
\]

for any \( \mu < c_{a,p,n} \). Theorem 1.1 then follows from the Harnack convergence principle (Proposition 1.3) together with the Agmon-Allegretto-Piepenbrink-type (AAP) theorem [17, Theorem 4.3] which asserts that the Hardy inequality (1.2) holds true if and only if (2.1) admits a positive (super)solution for \( \mu = c_{a,p,n} \). It seems that the method of the proof can be used to prove lower bounds for the best Hardy constant in different situations.

Proof of Theorem 1.1. A direct computation shows that for any \( y \in \mathbb{R}^n \), the function

\[
u_y(x) := |x - y|^{(\alpha + p - n)/(p - 1)}
\]

is a positive solution of the equation

\[- \text{div} \left( d_{\Omega_y}^{-\alpha} |\nabla u|^{p-2} \nabla u \right) = 0 \quad \text{in } \Omega_y := \mathbb{R}^n \setminus \{y\},
\]

where \( d_{\Omega_y}(x) = |x - y| \).

Hence, using the supersolution construction [9], it follows that

\[
u_y(x) := u_y^{(p-1)/p}(x) = |x - y|^{(\alpha + p - n)/p}
\]

is a positive solution of the equation

\[- \text{div} \left( d_{\Omega_y}^{-\alpha} |\nabla u|^{p-2} \nabla u \right) - c_{a,p,n} \frac{|u|^{p-2} u}{d_{\Omega_y}^{\alpha+p}} = 0 \quad \text{in } \Omega_y.
\]

Moreover, it is known [3] (see also [2]) that \( c_{a,p,n} \) is the best constant for the inequality

\[
\int_{\Omega_y} \frac{|\nabla \varphi|^p}{|x - y|^\alpha} \, dx \geq \mu \int_{\Omega_y} \frac{|\varphi|^p}{|x - y|^{p+\alpha}} \, dx \quad \forall \varphi \in C_c^\infty(\Omega_y).
\]

Hence, the lower bound for the Hardy constant for the functional inequality

\[
\int_{\Omega} \frac{|\nabla \varphi|^p}{d_{\Omega}^{\alpha+p}} \, dx \geq \mu \int_{\Omega} \frac{|\varphi|^p}{d_{\Omega}^{p+\alpha}} \, dx \quad \forall \varphi \in C_c^\infty(\Omega),
\]

in a domain \( \Omega \subseteq \mathbb{R}^n \) is less or equal to \( c_{a,p,n} \).

It remains to prove that (2.1) admits positive supersolutions in \( \Omega \) for

\[
\mu = \mu_\delta := c_{a,p,n} - \delta > 0, \quad \forall \, 0 < \delta < c_{a,p,n},
\]

where \( \Omega \subseteq \mathbb{R}^n \) is an arbitrary domain. We divide the proof into two steps.
Step 1: Assume first that \( \Omega \) is a smooth bounded domain. Fix \( \delta \) as above, and choose \( \varepsilon > 0 \) small enough such that

\[
\varepsilon < \min \left\{ \left( \frac{c_{\alpha,p,n}}{\mu_\delta} \right)^{1/p} - 1, \frac{\mu_\delta (\alpha + p - n)}{p|\alpha|c_{\alpha,p,n}} \right\}.
\]

(2.2)

For \( x \in \Omega \), let \( P(x) \in \partial \Omega \) be the projection \( x \) of into \( \partial \Omega \) which is well defined for a.e. \( x \in \Omega \), that is, \( |x - P(x)| = d_\Omega(x) \). For any \( y \in \partial \Omega \), consider the set

\[
D_{y,\varepsilon} := \left\{ x \in \Omega \mid |x - y| < (1 + \varepsilon) d_\Omega(x), \cos(x - y, x - P(x)) > 1 - \varepsilon, \right. \\
\text{and } d_\Omega(x) > \varepsilon/2 \}.
\]

If

\[
\Omega_\varepsilon = \left\{ x \in \Omega \mid d_\Omega(x) > \varepsilon \right\},
\]

(2.3)

then \( \bigcup_{y \in \partial \Omega} D_{y,\varepsilon} \) is an open covering of the compact set \( \overline{\Omega}_\varepsilon \). Therefore, there exist \( y_i, i = 1, 2, \ldots, m \) such that \( \Omega_\varepsilon \subseteq \bigcup_{i=1}^m D_{y_i,\varepsilon} \) We note that \( u_{y_i} \) is a positive supersolution to the equation

\[
- \text{div} \left( d_\Omega^{-\alpha} |\nabla u_{y_i}|^{p-2} \nabla u_{y_i} \right) + \varepsilon |\alpha| k_{\alpha,p,n} \frac{|u_{y_i}|^{p-2} u_{y_i}}{d_\Omega^{\alpha+p}} = 0 \quad \text{in } D_{y_i,\varepsilon}.
\]

where \( k_{\alpha,p,n} := \left( \frac{\alpha + p - n}{p-1} \right)^{p-1} \). Indeed, for \( \alpha \geq 0 \),

\[
- \text{div} \left( d_\Omega^{-\alpha} |\nabla u_{y_i}|^{p-2} \nabla u_{y_i} \right) = \alpha d_\Omega^{-\alpha} k_{\alpha,p,n} \left( \frac{\nabla d_\Omega \cdot (x - y)|x - y|^{\alpha-n}}{d_\Omega} - |x - y|^{\alpha-n} \right) \\
\geq \alpha d_\Omega^{-\alpha} k_{\alpha,p,n} |x - y|^{\alpha-n} \left( \frac{\nabla d_\Omega||x - y|(1 - \varepsilon)}{d_\Omega} - 1 \right) \\
\geq \alpha d_\Omega^{-\alpha} k_{\alpha,p,n} |x - y|^{\alpha-n} (1 - \varepsilon - 1) \\
= -\varepsilon \alpha d_\Omega^{-\alpha} k_{\alpha,p,n} |x - y|^{\alpha-n} \quad \text{in } D_{y_i,\varepsilon}.
\]

Hence,

\[
- \text{div} \left( d_\Omega^{-\alpha} |\nabla u_{y_i}|^{p-2} \nabla u_{y_i} \right) + \varepsilon |\alpha| k_{\alpha,p,n} \frac{|u_{y_i}|^{p-2} u_{y_i}}{d_\Omega^{\alpha+p}} \\
\geq d_\Omega^{-\alpha} k_{\alpha,p,n} |x - y|^{\alpha-n} \left( -\varepsilon |\alpha| \frac{|x - y|^{p}}{d_\Omega^{p}} \right) \\
\geq d_\Omega^{-\alpha} k_{\alpha,p,n} |x - y|^{\alpha-n} (\varepsilon |\alpha|) = 0 \quad \text{in } D_{y_i,\varepsilon}.
\]
Similarly, for \( \alpha < 0 \)

\[
-\text{div} \left( d_\Omega^{-\alpha} |\nabla u_y|^{p-2} \nabla u_y \right) + \varepsilon |\alpha| k_{\alpha,p,n} \frac{|u_y|^{p-2} u_y}{d_\Omega^{\alpha+p}} \\
\geq d_\Omega^{-\alpha} k_{\alpha,p,n} |x-y|^{\alpha-n} \left( \alpha \nabla d_\Omega \cdot (x-y) - \alpha + \varepsilon |\alpha| \frac{|x-y|^p}{d_\Omega} \right) \\
\geq d_\Omega^{-\alpha} k_{\alpha,p,n} |x-y|^{\alpha-n} \left( \alpha |\nabla d_\Omega| |x-y| - \alpha + \varepsilon |\alpha| \right) \\
\geq d_\Omega^{-\alpha} k_{\alpha,p,n} |x-y|^{\alpha-n} (\alpha(1+\varepsilon) - \alpha + \varepsilon |\alpha|) \\
\geq d_\Omega^{-\alpha} k_{\alpha,p,n} |x-y|^{\alpha-n} (\varepsilon|\alpha| + \varepsilon |\alpha|) = 0 \text{ in } D_{y,\varepsilon}.
\]

Now, the weak comparison principle \cite{17} Lemma 5.1 implies that

\( u_\delta := \min \{ u_i \mid 1 \leq i \leq m \} \)

is a supersolution to the equation

\[
(2.4) \quad -\text{div} \left( d_\Omega^{-\alpha} |\nabla u|^{p-2} \nabla u \right) + \varepsilon |\alpha| k_{\alpha,p,n} \frac{|u|^{p-2} u}{d_\Omega^{\alpha+p}} = 0 \text{ in } \Omega_\varepsilon.
\]

**Claim 1:** There exists a positive solution to the following equation

\[
(2.5) \quad -\text{div} \left( d_\Omega^{-\alpha} |\nabla u|^{p-2} \nabla u \right) - \left( \mu_\delta - \frac{\varepsilon p|\alpha| k_{\alpha,p,n}}{\alpha + p - n} \right) \frac{|u|^{p-2} u}{d_\Omega^{\alpha+p}} = 0 \text{ in } \Omega_\varepsilon.
\]

Employing the AAP-type theorem \cite{17} Theorem 4.3, it is enough to prove that there exists a positive supersolution to (2.5) in \( \Omega_\varepsilon \). We use the supersolution construction \cite{9} and prove that \( v_\delta := u_\delta^{(p-1)/p} \) is a supersolution to (2.5). Using the fact that \( u_\delta \) is a supersolution to (2.4), we deduce that

\[
-\text{div} \left( d_\Omega^{-\alpha} |\nabla v_\delta|^{p-2} \nabla v_\delta \right) - \left( \mu_\delta - \frac{\varepsilon p|\alpha| k_{\alpha,p,n}}{\alpha + p - n} \right) \frac{|v_\delta|^{p-2} v_\delta}{d_\Omega^{\alpha+p}} \\
= - \left( \frac{p-1}{p} \right)^{p-1} \text{div} \left( d_\Omega^{-\alpha} |\nabla u_\delta|^{p-2} \nabla u_\delta u_\delta^{-(p-1)/p} \right) - \left( \mu_\delta - \frac{\varepsilon p|\alpha| k_{\alpha,p,n}}{\alpha + p - n} \right) \frac{|u_\delta|^{(p-1)^2/p}}{d_\Omega^{\alpha+p}} \\
\geq \left( \frac{p-1}{p} \right)^p d_\Omega^{-\alpha} |\nabla u_\delta|^{p-2} u_\delta^{-(2p-1)/p} - \mu_\delta \frac{|u_\delta|^{(p-1)^2/p}}{d_\Omega^{\alpha+p}} \\
= \left| u_\delta \right|^{(p-1)^2/p} - \mu_\delta \frac{|\nabla u_\delta|^{p/p}}{u_\delta^{p}} \text{ in } \Omega_\varepsilon.
\]

Therefore, we need to prove that \( \left( \frac{p-1}{p} \right)^p \frac{|\nabla u_\delta|^{p/p}}{u_\delta^{p}} - \mu_\delta \geq 0 \). Indeed, for a.e. \( x \in \Omega_\varepsilon \), \( u_\delta = u_{y_0} \) for some \( y_0 \) in a neighborhood of \( x \). Using the
definition of $\varepsilon$ and $D_{\varepsilon, \delta}$, we get
\[
\left( \frac{p-1}{p} \right)^p \frac{\lvert \nabla u \rvert^p d^p}{u^p_{\delta}} - \mu_{\delta} = \left( \frac{p-1}{p} \right)^p \left( \frac{\alpha + p - n}{p - 1} \right)^p \frac{d^p}{|x - y_0|^p} - \mu_{\delta} \geq \frac{c_{\alpha, p, n}}{(1 + \varepsilon)^p} - \mu_{\delta} > 0.
\]
Hence, Claim 1 is proved.

Claim 2: There exists a positive solution to the following equation
\[(2.6) - \text{div} \left( d^{-\alpha}_\Omega |\nabla u|^{p-2} \nabla u \right) - \mu_{\delta} \frac{|u|^{p-2} u}{d^{\alpha+p}_\Omega} = 0 \text{ in } \Omega.\]

Let $\varepsilon_0 > 0$ be small enough such that (2.2) holds, and set $\varepsilon_i := \min\{\varepsilon_0, \frac{1}{i}\}$. Clearly, $\Omega_{\varepsilon_i} \Subset \Omega_{\varepsilon_{i+1}}$ for $i$ large enough, and $\Omega = \bigcup_{i=1}^{\infty} \Omega_{\varepsilon_i}$, where $\Omega_{\varepsilon_i}$ is defined in (2.3). Employing Claim 1, it follows that for $i \geq 1$ there exists a positive solution $u_i$ to (2.5) in $\Omega_{\varepsilon_i}$ satisfying $u_i(x_0) = 1$. In light of the Harnack convergence principle (Proposition 1.3), it follows that Claim 2 holds.

Step 2: Assume now that $\Omega$ is an arbitrary domain. Choose a smooth compact exhaustion $\{\Omega_i\}$ of $\Omega$. That is, $\{\Omega_i\}$ is a sequence of smooth bounded domains such that $\Omega_i \Subset \Omega_{i+1} \Subset \Omega$, $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, and
\[
\max_{x \in \partial \Omega_i \cap B_i} \{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega_i)\} < \frac{1}{i},
\]
where $B_i = \{|x| < i\}$. Observe that $d_{\Omega_i} \to d_{\Omega}$ a.e. in $\Omega$. Indeed, for $x \in \overline{\Omega_i} \cap B_i$ one has
\[
|d_{\Omega_i}(x) - d_{\Omega}(x)| = |\text{dist}(x, \partial \Omega) - \text{dist}(x, \partial \Omega_i)| < \frac{1}{i}.
\]
Invoking Claim 2, it follows that for each $i \geq 1$, there exists $u_i > 0$ satisfying $u_i(x_0) = 1$ and the equation
\[- \text{div} \left( d^{-\alpha}_{\Omega_i} |\nabla u_i|^{p-2} \nabla u_i \right) - \mu_{\delta} \frac{|u_i|^{p-2} u_i}{d^{\alpha+p}_{\Omega_i}} = 0 \text{ in } \Omega_i.
\]

Using again the Harnack convergence principle (Proposition 1.3), we obtain a positive solution $u_\delta$ to (2.6) satisfying $u_\delta(x_0) = 1$. Letting $\delta \to 0$, we get by Harnack convergence principle a positive solution $u_0$ to the equation
\[(2.7) - \text{div} \left( d^{-\alpha}_\Omega |\nabla u|^{p-2} \nabla u \right) - c_{\alpha, p, n} \frac{|u|^{p-2} u}{d^{\alpha+p}_\Omega} = 0 \text{ in } \Omega
\]
that satisfies $u_0(x_0) = 1$. In light of the AAP-type theorem we obtain the Hardy inequality
\[
\int_{\Omega} \frac{\lvert \nabla \varphi \rvert^p}{d^{\alpha}_\Omega} \, dx \geq \left( \frac{\alpha + p - n}{p} \right)^p \int_{\Omega} \frac{\lvert \varphi \rvert^p}{d^{\alpha+p}_\Omega} \, dx \quad \forall \varphi \in C^\infty_c(\Omega). \quad \square
\]
Appendix A. Different Proofs

Here we give two alternative proofs of Theorem 1.1, both of them do not use an exhaustion argument. On the other hand, both rely on the following folklore lemma which is of independent interest, see for example, propositions 1.1.3. and 2.2.2. in [6] (cf. [15, Theorem 1.6], where the case of $C^2$-domains is discussed).

Lemma A.1. Let $\Omega \subseteq \mathbb{R}^n$ be a domain.

(i) The inequality

\[-\Delta d_\Omega \geq -\frac{n-1}{d_\Omega},\]

holds true in the sense of distributions in $\Omega$.

(ii) Moreover,

\[(A.1) \int_{\Omega} \nabla \psi \cdot \nabla d_\Omega \, dx \geq - (n-1) \int_{\Omega} \frac{\psi}{d_\Omega} \, dx \quad \forall \, \psi \in C^\infty_c(\Omega), \psi \geq 0.\]

Proof. (i) Since the function $\left| x \right|^2 - d_\Omega^2(x)$ is convex, it follows that its distributional Laplacian is a nonnegative Radon measure (see [8, Theorem 2-§6.3] and [18, Lemma 2.1] for the details). Hence,

\[\langle (n-1) - d_\Omega \Delta d_\Omega, \varphi \rangle = \int_{\Omega} \varphi \, d\nu \quad \forall \, \varphi \in C^\infty_c(\Omega),\]

where $\nu$ is a nonnegative Radon measure, and $\langle \cdot, \cdot \rangle : D'(\Omega) \times D(\Omega)$ is the canonical duality pairing between distributions and test functions. Consequently, the distributional Laplacian of $-d_\Omega$ is itself a signed Radon measure $\mu$. Thus,

\[-\langle \Delta d_\Omega, \psi \rangle = - \int_{\Omega} \Delta \psi \, d\Omega \, dx = \int_{\Omega} \psi \, d\mu \geq -(n-1) \int_{\Omega} \frac{\psi}{d_\Omega} \, dx \quad \forall \, \psi \in C^\infty_c(\Omega), \psi \geq 0.\]

(ii) Since $\nabla d_\Omega \in L^\infty(\Omega, \mathbb{R}^n)$, it follows that

\[-\langle (d_\Omega)_{x_i,x_i}, \psi \rangle = \int_{\Omega} (d_\Omega)_{x_i,x_i} \psi_{x_i} \, dx.\]

Therefore, $\Delta d_\Omega$, the distributional divergence of $\nabla d_\Omega$, satisfies

\[-\langle \Delta d_\Omega, \psi \rangle = \int_{\Omega} \nabla d_\Omega \cdot \nabla \psi \, dx \quad \forall \, \psi \in C^\infty_c(\Omega).\]

Hence,

\[\int_{\Omega} \nabla d_\Omega \cdot \nabla \psi \, dx = -\langle \Delta d_\Omega, \psi \rangle \geq -(n-1) \int_{\Omega} \frac{\psi}{d_\Omega} \, dx \quad \forall \, \psi \in C^\infty_c(\Omega), \psi \geq 0. \quad \square\]

Lemma A.2. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let

\[1 < p < \infty, \quad \alpha \in \mathbb{R}^n, \quad \text{and} \quad 0 < \gamma < \frac{\alpha + p - n}{p - 1}.\]
Then \( d_\Omega^\gamma \) is a (weak) positive supersolution of the equation
\[
-\text{div} \left( d_\Omega^{-\alpha} |\nabla u|^{p-2} \nabla u \right) - C_{\alpha,p,n,\gamma} \frac{|u|^{p-2} u}{d_\Omega^{p+\alpha}} = 0 \quad \text{in } \Omega,
\]
where \( C_{\alpha,p,n,\gamma} := |\gamma|^{p-1}(\alpha - n + 1 - (\gamma - 1)(p - 1)) > 0. \)

**Proof.** Using (A.1) we obtain
\[
\int_{\Omega} d_\Omega^{-\alpha} |\nabla (d_\Omega^\gamma)|^{p-2} \nabla (d_\Omega^\gamma) \cdot \nabla \varphi dx = |\gamma|^{p-2} \int_{\Omega} (d_\Omega^{(\gamma-1)(p-1)-\alpha}) \nabla d_\Omega \cdot \nabla \varphi dx
\]
\[
= |\gamma|^{p-1} \int_{\Omega} ((\gamma-1)(p-1) - \alpha) d_\Omega^{(\gamma-1)(p-1)-\alpha-1} \varphi dx \geq C_{\alpha,p,n,\gamma} \int_{\Omega} d_\Omega^{(\gamma-1)(p-1)-\alpha-1} \varphi dx.
\]

**Remark A.3.** Observe that
\[
C_{\alpha,p,n} = \max \left\{ C_{\alpha,p,n,\gamma} \mid \gamma \in \left( 0, \frac{\alpha + p - n}{p - 1} \right) \right\},
\]
and the maximum is obtained with \( \gamma = (\alpha + p - n)/p. \)

**Alternative proof of Theorem 1.1 I.** Using Lemma A.2 for \( \gamma = (\alpha + p - n)/p, \) we deduce that \( d_\Omega^{(\alpha+p-n)/p} \) is positive (weak) supersolution to (2.7). Consequently, the AAP-type theorem [17, Theorem 4.3] implies the Hardy-type inequality (1.2).

**Alternative proof of Theorem 1.1 II.** Let \( \Omega \subset \mathbb{R}^n \) be a domain, and fix \( s > n. \) Using Lemma A.1 the following \( L^1 - \text{Hardy inequality is proved in [18, Theorem 2.3]}: \)
\[
(A.2) \quad \int_{\Omega} \frac{|\nabla \varphi|}{d_\Omega^{s/p}} dx \geq (s - n) \int_{\Omega} \frac{|\varphi|}{d_\Omega} dx \quad \forall \varphi \in C_c^\infty(\Omega).
\]
Substituting \( \varphi = |\psi|^p \) in (A.2) and using Hölder inequality, we obtain
\[
\frac{s - n}{p} \int_{\Omega} \frac{|\psi|^p}{d_\Omega^s} dx \leq \int_{\Omega} \frac{|\psi|^{p-1} |\nabla \psi|}{d_\Omega^{s/p-1}} dx = \int_{\Omega} \frac{d_\Omega^{s/p-1} |\nabla \psi|}{d_\Omega^{s/p-1}} \frac{|\psi|}{d_\Omega} dx
\]
\[
\leq \left( \int_{\Omega} \frac{|\psi|^p}{d_\Omega^s} \right)^{1/p} \left( \int_{\Omega} \frac{|\nabla \psi|^p}{d_\Omega^{s/p}} \right)^{1/p} \quad \forall \psi \in C_c^\infty(\Omega).
\]
Hence, for \( s = \alpha + p, \) we get (1.2).

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