ON THE ANTI-INVARIANT COHOMOLOGY OF ALMOST COMPLEX MANIFOLDS

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Abstract. We study the space of closed anti-invariant forms on an almost complex manifold, possibly non compact. We construct families of (non integrable) almost complex structures on \( \mathbb{R}^4 \), such that the space of harmonic \( J \)-anti-invariant forms is infinite dimensional respectively 1-dimensional. In the compact case, we construct 6-dimensional almost complex manifolds with arbitrary large anti-invariant cohomology and a 2-parameter family of almost complex structures on the Kodaira-Thurston manifold whose anti-invariant cohomology group has maximum dimension.

1. Introduction

Cohomological properties provide a connection between analytical and topological features of complex manifolds. Indeed for a given complex manifold \((M, J)\), natural complex cohomologies are defined, e.g., the Dolbeault, Bott-Chern and Aeppli cohomology groups, given by

\[
H^{p,q}_d(M) = \frac{\text{Ker} \partial}{\text{Im} \partial}, \quad H^{p,q}_{BC}(M) = \frac{\text{Ker} \partial \cap \text{Ker} \overline{\partial}}{\text{Im} \partial \cap \text{Im} \overline{\partial}}, \quad H^{p,q}_A(M) = \frac{\text{Ker} \partial \partial}{\text{Im} \partial + \text{Im} \overline{\partial}}.
\]

Furthermore, if \((M, J)\) is a compact complex manifold admitting a Kähler metric, that is a \( J \)-Hermitian metric whose fundamental form is closed, as a consequence of Hodge theory, the complex de Rham cohomology groups decompose as the direct sum of \((p, q)\)-Dolbeault groups and strong topological restrictions on \( M \) are derived.

For an almost complex manifold \((M, J)\) the exterior differential \( d \) acting on the space of complex valued \((p, q)\)-forms splits as

\[
d = \mu + \partial + \overline{\partial} + \overline{\mu},
\]

where \( \overline{\partial} \), respectively \( \overline{\mu} \) are the \((p, q + 1)\) respectively the \((p - 1, q + 2)\) components of \( d \). It turns out that the almost complex structure \( J \) is integrable if and only if \( \overline{\mu} = 0 \). Consequently, in the non integrable case, \( \overline{\partial} \) is not a cohomological operator.

In \cite{Li-Zhang} Li and Zhang, motivated by the study of comparison of tamed and compatible symplectic cones on a compact almost complex manifold, introduced the \( J \)-anti-invariant and \( J \)-invariant cohomology groups as the (real)

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de Rham 2-classes represented by \(J\)-anti-invariant, respectively \(J\)-invariant forms and the notion of \(C^\infty\)-pure-and-full almost complex structures, namely those ones such that the second de Rham cohomology group decomposes as the direct sum of the \(J\)-anti-invariant and \(J\)-invariant cohomology groups.

In [3], Dr˘a˘gici, Li and Zhang proved that an almost complex structure on a compact 4-dimensional manifold is \(C^\infty\)-pure-and-full.

In [5] and [6], the same authors studied the \(J\)-anti-invariant cohomology of an almost complex manifold. In particular, writing \(h^-_J\) for the dimension of this cohomology, they stated the following

**Conjecture 2.4.** For generic almost complex structures \(J\) on a compact 4-manifold \(M\), \(h^-_J = 0\).

**Conjecture 2.5.** On a compact 4-manifold, if \(h^-_J \geq 3\), then \(J\) is integrable.

For other results on \(C^\infty\)-pure-and-full and \(J\)-anti-invariant closed forms see [2, 3, 7, 9].

In this note, motivated by the previous questions, we study the anti-invariant cohomology of an almost complex manifold, possibly non compact. We construct a family of (non integrable) almost complex structures on \(\mathbb{R}^4\), such that the space of harmonic \(J\)-anti-invariant forms is infinite dimensional (Theorem 3.6). In other words, compactness is essential for Conjecture 2.5.

Then we show the following (see Theorem 3.7)

**Theorem** There exists a family of non integrable almost complex structures \(\{J_f\}\) on \(\mathbb{C}^2\) such that

- \(J_f \rightarrow J_0 = i\) on \(\mathbb{C}^2\);
- \(J_f = i\) outside of \(B(1)\);
- \(h^-_{J_f} = \dim_{\mathbb{R}} \mathcal{H}^-_{J_f}(\mathbb{R}^4) = 1\).

That is, an arbitrarily small, compactly supported, perturbation of a complex structure having an infinite dimensional space of anti-invariant forms may admit only a single such form up to scale. This provides supporting evidence for Conjecture 2.5, showing that typically anti-invariant forms do not persist under nonintegrable perturbations.

In the compact case, we construct a 2-parameter family of (non integrable) almost complex structures on the Kodaira-Thurston manifold, depending on two smooth functions, for which the anti-invariant cohomology group has maximum dimension equal to 2 (see Proposition 4.2). In the last section, we give a simple construction to obtain 6-dimensional compact almost complex manifolds with arbitrary large anti-invariant cohomology (see Proposition 5.1). Hence dimension 4 is also an essential part of Conjecture 2.5.

For almost-complex structure on a 4-manifold which are tamed by a symplectic form, Dr˘a˘gici, Li and Zhang show in [11], Theorem 3.3, that \(h^-_J \leq b^+-1\). Thus any counterexamples to Conjecture 2.5 cannot come from tame almost-complex structures on symplectic 4-manifolds with \(b^+ \leq 3\). Moreover T.-J. Li in [10], Theorem 1.1, shows that symplectic 4-manifolds of Kodaira dimension 0 all have \(b^+ \leq 3\). We thank Weiyi Zhang for pointing this out.
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2. Anti-Invariant cohomology

In this Section we will fix some notation and recall the generalities on anti-invariant forms and some notion about the cohomology of almost complex manifolds. Let $M$ be a smooth $2n$-dimensional manifold. We will denote by $J$ a smooth almost complex structure on $M$, that is a smooth $(1,1)$-tensor $J$ field satisfying $J^2 = -\text{id}$. The almost complex structure $J$ is said to be integrable if its Nijenhuis tensor, that is the $(1,2)$-tensor given by

$$N_J(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY],$$

According to Newlander-Nirenberg Theorem, $J$ is integrable if and only if $J$ is induced by a structure of complex manifold on $M$. Let $J$ be a smooth almost-complex structure on $M$ and denote by $\Lambda^r(M)$ the bundle of $r$-forms on $M$; let $\Omega^r(M) := \Gamma(M, \Lambda^r(M))$ be the space of smooth global sections of $\Lambda^r(M)$ and let $\Lambda^r(M; \mathbb{C}) = \Lambda^r(M) \otimes \mathbb{C}$. Then $J$ acts in a natural way on the space $\Omega^r(M; \mathbb{C})$ of smooth sections of $\Lambda^r(M; \mathbb{C})$ giving rise to the following bundle decomposition

$$\Lambda^r(M; \mathbb{C}) = \bigoplus_{p+q=r} \Lambda^{p,q}_J(M).$$

Accordingly, $\Omega^r(M; \mathbb{C})$ and $\Omega^r(M)$ decompose respectively as

$$\Omega^r(M; \mathbb{C}) = \bigoplus_{p+q=r} \Omega^{p,q}_J(M),$$

and

$$\Omega^r(M) = \bigoplus_{p+q=r, p \leq q} \Omega^{(p,q),(q,p)}(M)_{\mathbb{R}},$$

where, for $p < q$

$$\Omega^{(p,q),(q,p)}(M)_{\mathbb{R}} = \{ \alpha \in \Omega^{p,q}_J(M) \oplus \Omega^{q,p}_J(M) \mid \alpha = \overline{\alpha} \}$$

and

$$\Omega^{(p,p)}(M)_{\mathbb{R}} = \{ \beta \in \Omega^{p,p}_J(M) \mid \beta = \overline{\beta} \}$$

In particular for $r = 2$, $J$ acts as involution on $\Omega^2(M)$ by

$$J\alpha(X,Y) = \alpha(JX,Y),$$

for every pair of vector fields $X, Y$ on $M$. Then we denote as usual by $\Lambda^+_J(M)$ (respectively $\Lambda^-_J(M)$) the $+1$ (resp. $-1$)-eigenbundle; then the space of corresponding sections $\Omega^+_J(M)$ (respectively $\Omega^-_J(M)$) are, namely the spaces of $J$-anti-invariant, (respectively $J$-invariant) forms, i.e.,

$$\Omega^+_J(M) = \{ \alpha \in \Omega^2(M) \mid J\alpha = \pm \alpha \}$$

$$\Omega^{(2,0),(0,2)}(M)_{\mathbb{R}} = \Omega^-_J(M), \quad \Omega^{1,1}(M)_{\mathbb{R}} = \Omega^+_J(M)$$

Let

$$\Omega^+_J(M) = \Omega^2(M) \cap \Omega^+_J(M) = \{ \alpha \in \Omega^+_J(M) \mid d\alpha = 0 \}$$.Phi
If \( \{ \varphi^1, \ldots, \varphi^n \} \) is a local coframe of \((1,0)\)-forms on \((M,J)\), then \( \Lambda^{-}_J(M) \) is locally spanned by
\[
\{ \Re(\varphi^r \wedge \varphi^s), \ \Im(\varphi^r \wedge \varphi^s), \ 1 \leq r < s \leq n \}.
\]
Then, according to the previous decomposition on forms, T.-J. Li and W. Zhang [11] defined the following cohomology spaces
\[
H^\pm_J(X) = \{ a \in H^2_{dR}(X; \mathbb{R}) \mid \exists \alpha \in Z^\pm_J | a = [\alpha] \}
\]
and they gave the following (see [11, Definition 4.12])

**Definition 2.1.** An almost complex structure \( J \) on \( M \) is said to be
- \( C^\infty \)-pure if \( H^+_J(M) \cap H^-_J(M) = \{0\} \).
- \( C^\infty \)-full if \( H^2_{dR}(M; \mathbb{R}) = H^+_J(M) + H^-_J(M) \).
- \( C^\infty \)-pure-and-full if \( H^2_{dR}(M; \mathbb{R}) = H^+_J(M) \oplus H^-_J(M) \).

Given an almost complex manifold \((M,J)\), we set, as usual
\[
h^\pm_J(M) = \dim \mathbb{R} H^\pm_J(M).
\]

**Remark 2.2.** It has to be remarked see e.g., [5, 8, Prop.2.4] that once fixed a \( J \)-Hermitian metric \( g_J \) on the almost complex manifold \((M,J)\), the space \( Z^-_J(M) \) is contained in the kernel of a second order elliptic differential operator \( E \), that is \( Z^-_J(M) \hookrightarrow \text{Ker} E \). Explicitly,
\[
E\alpha = \Delta \alpha + \frac{1}{(n-2)!} d((\alpha \wedge d(\omega^{n-2}))),
\]
where \( \omega \) is the fundamental form of \( g_J \). Therefore, in view of [1] on a connected almost complex manifold \( M \), if \( \alpha \) is any closed anti-invariant form vanishing to infinite order at some point \( p \), then \( \alpha = 0 \). Furthermore, if \( M \) is a compact \( 2n \)-dimensional almost complex manifold, then \( Z^-_J(M) \) has finite dimension. In particular, if \( 2n = 4 \), then \( E = \Delta \) and any \( J \)-anti-invariant closed form on \((M,J)\) is harmonic with respect to the Hodge Laplacian.

Hence, denoting by \( H^+_J(M; \mathbb{R}) \) the space of self-dual harmonic forms, we have that \( Z^-_J(M) \subset H^+_J(M; \mathbb{R}) \), since any \( J \)-anti-invariant form on a 4-dimensional almost complex manifold is auto-dual, so that \( h^+_J(M) \leq b^+(M) \) and \( Z^-_J(M) \hookrightarrow H^2_{dR}(M; \mathbb{R}) \). More generally, if \( J \) is compatible with a symplectic structure \( \omega \), namely \((M,J,\omega)\) is an almost Kähler manifold, then (see e.g., [5] or [8] Proposition 2.2, Corollary 2.3) again we have an injection \( Z^-_J(M) \hookrightarrow H^2_{dR}(M; \mathbb{R}) \).

3. Closed \( J \)-anti-invariant forms and an integrability condition

Let \( J \) be an almost complex structure on a 4-dimensional manifold. Let \( \omega \neq 0 \) be a closed \( J \)-anti-invariant form on \( M \). Then, according to [4, Lemma 2.6] (see also [8] Prop. 2.6) the zero’s set \( \omega^{-1}(0) \) of \( \omega \) has empty interior, so that \( M \setminus \omega^{-1}(0) \) is open and dense. Since \( M \setminus \omega^{-1}(0) \) coincides
with the subset of $M$ where $\omega$ is non degenerate (see [4 Lemma 2.6] or [8, Lemma 1.1]), we have the following

**Lemma 3.1.** Let $(M, J)$ be a 4-dimensional almost complex manifold and $0 \neq \omega \in \mathbb{Z}_J$. Then $\omega$ is a symplectic form on the open dense set $M \setminus \omega^{-1}(0)$.

Let $J_0$ be the standard complex structure on the vector space $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ induced by the multiplication by $i$, that is,

$$J_0(z_1, \ldots, z_n) = (e^{i\pi/2}z_1, \ldots, e^{i\pi/2}z_n).$$

Then, for every given real number $r$, define $J^r_0 \in \text{End}(\mathbb{C}^n)$, by setting

$$J^r_0(z_1, \ldots, z_n) = (e^{i\pi r}z_1, \ldots, e^{i\pi r}z_n).$$

Let now $J$ be any almost complex structure on the manifold $\mathbb{C}^n \simeq \mathbb{R}^{2n}$; then there exists $A : \mathbb{R}^{2n} \to \text{GL}(2n, \mathbb{R})$ such that $J$ is conjugated to the standard complex structure $J_0$, i.e.,

$$J_x = A(x)J_0A^{-1}(x).$$

For $r = r(x) \in \mathbb{R}$, define

$$J^r_x := A(x)J^r_0A^{-1}(x).$$

**Lemma 3.2.** The local 2-form $\theta$ is $J$-anti-invariant.

**Proof.** For any given pair of tangent vectors $v, w$ at $x$,

$$J_x \theta_x(v, w) = J_x[w, J_x v] = \omega_x(J_x v, J^r_x w) = -\omega_x(v, J^r_x w) = -\theta_x(v, w),$$

that is $J \theta = -\theta$. □

The last Lemma allows to produce anti-invariant forms starting from an anti-invariant one. The next result gives an integrability condition in the 4-dimensional case.

**Theorem 3.3.** Let $(M, J)$ be a 4-dimensional almost complex manifold. Let $\omega \in \Omega^2_J(M)$. Assume that, around at any $x \in M \setminus \omega^{-1}(0)$, the local form $\theta_x(\cdot, \cdot) = \omega_x(\cdot, J_x \cdot)$ is closed. Then $J$ is integrable.

**Proof.** By Lemma 3.1 the 2-form $\omega$ is a symplectic structure on $M \setminus \omega^{-1}(0)$. Let $x \in M \setminus \omega^{-1}(0)$ and $U$ be a coordinate neighbourhood of $x$ contained in $M \setminus \omega^{-1}(0)$. Define a local complex 2-form on $(M, J)$ by setting, for every $x \in U$,

$$\Psi_x = \omega_x - i\theta_x.$$
We show that $\Omega$ is of type $(2,0)$. Indeed, for every given $v, w$,
$$\Psi_x(v-iJv, w+iJw) = (\omega_x - i\theta_x)(v-iJv, w+iJw)$$
$$= \omega_x(v, w) + \omega_x(Jv, Jw) - i(\theta_x(v, w) - \theta_x(Jv, Jw))$$
$$+ i(\omega_x(v, Jw) - \omega_x(Jv, w) - i(\theta_x(v, Jw) - \theta_x(Jv, w)))$$
$$= 0,$$

since $\omega$ and $\theta$ are $J$-anti-invariant. Therefore, $\Psi$ vanishes on any pair of complex vectors of type $(1,0)$, respectively, that is
$$\Psi \in \Omega^{2,0}_J(U) \oplus \Omega^{0,2}_J(U).$$

Similarly,
$$\Psi_x(v+iJv, w+iJw) = (\omega_x - i\theta_x)(v+iJv, w+iJw)$$
$$= \omega_x(v, w) - \omega_x(Jv, Jw) - i(\theta_x(v, w) - \theta_x(Jv, Jw))$$
$$+ i(\omega_x(v, Jw) + \omega_x(Jv, w) - i(\theta_x(v, Jw) + \theta_x(Jv, w)))$$
$$= 2(\omega_x(u, v) - i\theta_x(u, v)) + 2i(\omega_x(v, Jw) - i\theta_x(v, Jw))$$
$$= 2(\omega_x(u, v) - i\theta_x(u, v)) + 2i(\theta_x(v, w) + i\omega_x(v, w))$$
$$= 0.$$

Therefore, $\Psi \in \Omega^{2,0}_J(U)$ is nowhere vanishing and closed. Let $\alpha$ be any local complex $(1,0)$-form. Then, by type reason, $\alpha \wedge \Psi = 0$. Hence, at $x$,
$$0 = d(\alpha \wedge \Psi) = d\alpha \wedge \Psi = (d\alpha)^{0,2} \wedge \Psi,$$

which implies that the $(0,2)$-part $(d\alpha)^{0,2}$ of $d\alpha$ vanishes.

Therefore, $N_J = 0$, at any point of the dense subset $M \setminus \omega^{-1}(0)$. Hence $N_J = 0$ on the whole $M$ and $J$ is integrable. \hfill $\square$

Let $(x_1, x_2, y_1, y_2)$ be natural coordinates on $\mathbb{R}^4$ and $f = f(x_1, x_2, y_1, y_2)$ be a smooth $\mathbb{R}$-valued function on $\mathbb{R}^4$. Define $J \in \text{End}(T\mathbb{R}^4)$ by setting

$$J = \frac{\partial}{\partial x_1} f \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1}, \quad J = \frac{\partial}{\partial x_2} f \frac{\partial}{\partial y_1}, \quad J = -f \frac{\partial}{\partial x_1} - f \frac{\partial}{\partial y_2}, \quad J = -f \frac{\partial}{\partial x_2} - f \frac{\partial}{\partial y_1},$$

and extend it $C^\infty(\mathbb{R}^4)$-linearly. Then $J$ gives rise to an almost complex structure on $\mathbb{R}^4$.

**Lemma 3.4.** The almost complex structure $J$ is integrable if and only if
$$f_{x_2} = 0, \quad f_{y_2} = 0.$$

**Proof.** It is enough to show that $N_J(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) = 0$ if and only if
$$f_{x_2} = 0, \quad f_{y_2} = 0.$$

We easily compute
$$N_J(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) = [J \frac{\partial}{\partial x_1}, J \frac{\partial}{\partial x_2}] - [J \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}] - J[J \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}] - J[J \frac{\partial}{\partial x_1}, J \frac{\partial}{\partial x_2}]$$
$$= [f \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}] - J[f \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}] - J[f \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}]$$
$$= -f y_2 \frac{\partial}{\partial x_2} + f x_2 \frac{\partial}{\partial y_2}.$$

Lemma is proved. \hfill $\square$
ON THE ANTI-IN Variant COHOMOLOGY OF ALMOST COMPLEX MANIFOLDS

According to the definition of $J$, the induced almost complex structure $J$ on $T^\ast \mathbb{R}^4$ is given by

\begin{equation}
Jdx_1 = -dy_1, \quad Jdx_2 = fdx_1 - dy_2, \quad Jdy_1 = dx_1, \quad Jdy_2 = -fdy_1 + dx_2.
\end{equation}

Consequently, setting

$\varphi^1 = dx_1 + idy_1, \quad \varphi^2 = dx_2 + i(-fdx_1 + dy_2),$

then \{\varphi^1, \varphi^2\} is a complex $(1,0)$-coframe on the almost complex manifold $(\mathbb{R}^4, J)$, so that

$\beta = \text{Re}(\varphi^1 \wedge \varphi^2), \quad \gamma = \text{Im}(\varphi^1 \wedge \varphi^2),$

is a global frame of $\Lambda^\ast_\mathbb{J}(\mathbb{R}^4)$. Explicitly,

\begin{equation}
\beta = dx_1 \wedge dx_2 - fdx_1 \wedge dy_1 - dy_1 \wedge dy_2, \quad \gamma = dx_1 \wedge dy_2 - dx_2 \wedge dy_1.
\end{equation}

**Lemma 3.5.** Let $\alpha$ be an arbitrary smooth section of $\Lambda^\ast_\mathbb{J}(\mathbb{R}^4)$. Set $\alpha = a\beta + b\gamma$, for $a, b$ smooth $\mathbb{R}$-valued functions on $\mathbb{R}^4$. Then $d\alpha = 0$ if and only if the following condition holds

\begin{equation}
\begin{cases}
a_{y_1} - b_{x_1} + (fa)_{x_2} & = 0 \\
a_{x_1} + b_{y_1} + (fa)_{y_2} & = 0 \\
a_{y_2} - b_{x_2} & = 0 \\
a_{x_2} + b_{y_2} & = 0
\end{cases}
\end{equation}

**Proof.** Expanding $d\alpha$ we get:

\[d\alpha = da \wedge \beta - adf \wedge dx_1 \wedge dy_1 + db \wedge \gamma \]

\begin{align*}
&= (a_{x_1}dx_1 + a_{x_2}dx_2 + a_{y_1}dy_1 + a_{y_2}dy_2) \wedge (dx_1 \wedge dx_2 - fdx_1 \wedge dy_1 - dy_1 \wedge dy_2) \\
&\quad - a(f_{x_1}dx_1 + f_{x_2}dx_2 + f_{y_1}dy_1 + f_{y_2}dy_2) \wedge dx_1 \wedge dy_1 + \\
&\quad + (b_{x_1}dx_1 + b_{x_2}dx_2 + b_{y_1}dy_1 + b_{y_2}dy_2) \wedge (dx_1 \wedge dy_2 - dx_2 \wedge dy_1) \\
&\quad = -a_{x_1}dx_1 \wedge dy_1 \wedge dy_2 + a_{x_2}fdx_1 \wedge dx_2 \wedge dy_1 - a_{x_2}dx_2 \wedge dy_1 \wedge dy_2 + \\
&\quad + a_{y_1}dx_1 \wedge dx_2 \wedge dy_1 + a_{y_2}dx_1 \wedge dx_2 \wedge dy_2 - a_{y_2}fdx_1 \wedge dy_1 \wedge dy_2 \\
&\quad + af_{x_2}dx_1 \wedge dx_2 \wedge dy_1 - af_{y_2}dx_1 \wedge dy_1 \wedge dy_2 - b_{x_1}dx_1 \wedge dx_2 \wedge dy_1 + \\
&\quad - b_{x_2}dx_1 \wedge dx_2 \wedge dy_2 - b_{y_1}dx_1 \wedge dy_1 \wedge dy_2 - b_{y_2}dx_2 \wedge dy_1 \wedge dy_2 \\
&\quad = (a_{y_1} - b_{x_1} + (af)_{x_2})dx_1 \wedge dx_2 \wedge dy_1 + (a_{y_2} - b_{x_2})dx_1 \wedge dx_2 \wedge dy_2 + \\
&\quad - (a_{x_1} + b_{y_1} + (af)_{y_2})dx_1 \wedge dy_1 \wedge dy_2 - (a_{x_2} + b_{y_2})dx_2 \wedge dy_1 \wedge dy_2.
\end{align*}

Therefore, $d\alpha = 0$ if and only if (4) holds. \[\square\]

It is immediate to note that, condition (4) of Lemma 3.5 can be rewritten as

\[db = (a_{y_1} + (af)_{x_2})dx_1 + a_{y_2}dx_2 - (a_{x_1} + (af)_{y_2})dy_1 - a_{x_2}dy_2.\]

Therefore, given $a$, there exists a $b$ such that $\alpha = a\beta + b\gamma$ is a closed $J$-anti-invariant form on $(\mathbb{R}^4, J)$ if and only if the differential form

\[(a_{y_1} + (af)_{x_2})dx_1 + a_{y_2}dx_2 - (a_{x_1} + (af)_{y_2})dy_1 - a_{x_2}dy_2\]
is closed. The latter condition is equivalent to the following PDEs system:

$$\begin{align*}
\alpha_{x_1y_2} - \alpha_{x_2y_1} - (af)_{x_2x_2} &= 0 \\
\alpha_{x_1y_2} - \alpha_{x_2y_1} + (af)_{y_2y_2} &= 0 \\
\alpha_{x_1x_1} + \alpha_{y_1y_1} + (af)_{x_2y_1} + (af)_{x_1y_2} &= 0 \\
\alpha_{x_1x_2} + \alpha_{y_1y_2} + (af)_{x_2y_2} &= 0
\end{align*}$$

(5)

Let $g_J$ be a $J$-Hermitian metric on $\mathbb{R}^4$. Denote by $H^J_1(\mathbb{R}^4)$ the space of harmonic $J$-anti-invariant forms on $\mathbb{R}^4$. We are ready to state and prove the following

**Theorem 3.6.** Let $f(x_1,x_2,y_1,y_2) = x_2$ and $J$ be defined as in (1). Let

$$\beta = dx_1 \wedge dx_2 - f dx_1 \wedge dy_1 - dy_2, \quad \gamma = dx_1 \wedge dy_2 - dx_2 \wedge dy_1.$$

Then

1. $J$ is a non-integrable almost complex structure on $\mathbb{R}^4$.

2. For every given $k \in \mathbb{R}$,

$$\alpha_k = (e^{-y_1} - k)\beta + kx_1\gamma$$

is a $J$-anti-invariant closed form. Therefore, $H^J_1(\mathbb{R}^4)$ has infinite dimension.

**Proof.**

(1) In view of Lemma 3.4, $J$ is integrable if and only if $f_{x_2} = f_{y_2} = 0$. By assumption, we get $f_{x_2} = 1$. Therefore $J$ is not integrable.

(2) We rewrite the PDEs system (1) in complex notation.

Set $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,

$$\partial z_1 = \frac{i}{2}(\partial x_1 - i\partial y_1), \quad \partial z_2 = \frac{i}{2}(\partial x_2 - i\partial y_2)$$

$$\overline{\partial z_1} = \frac{i}{2}(\partial x_1 + i\partial y_1), \quad \overline{\partial z_2} = \frac{i}{2}(\partial x_2 + i\partial y_2)$$

A pair of real valued functions $(a,b)$ on $\mathbb{R}^4$ is a solution of (1) if and only if the complex valued function $w = a - ib$ solves the following

$$\begin{align*}
\partial z_1 w - \frac{i}{4}(z_2 + \overline{z_2})(\partial z_2 - \partial \overline{z_2}) w + \frac{i}{4}(w + \overline{w}) &= 0 \\
\partial z_2 w &= 0
\end{align*}$$

that is,

$$\begin{align*}
\partial z_1 w + \frac{i}{4}(z_2 + \overline{z_2})\partial z_2 w + \frac{i}{4}(w + \overline{w}) &= 0 \\
\partial z_2 w &= 0
\end{align*}$$

(6)

The system above is a perturbed Cauchy-Riemann PDEs system. Indeed, for $f = 0$, we have that $J$ is the standard complex structure in $\mathbb{C}^2$ and $\bar{z}$ are the usual Cauchy-Riemann conditions. A straightforward computation shows that, given any $k \in \mathbb{R}$ the complex function

$$w = (e^{-y_1} - k) + ikx_1$$

solves (6). This ends the proof.

**Theorem 3.7.** There exists a family of non integrable almost complex structures $\{J_f\}$ on $\mathbb{C}^2$ such that

1. $J_f \to J_0 = i$ on $\mathbb{C}^2$;
Lemma 4.1. The almost complex structure \( J_f \) is nonintegrable.

Proof. We choose an \( f \) which vanishes outside of the ball \( \mathbb{B}(1) \) but is not identically 0. Then neither \( f_{x_2} \) nor \( f_{y_2} \) can vanish identically and so by Lemma 3.3 we see that \( J_f \) is nonintegrable. We determine the anti-holomorphic forms by finding solutions to the system (4).

First note that the first two equations in (5) imply that \( af \) is a harmonic function of \( x_2, y_2 \), which is identically 0 outside of a compact set (since \( f \) is). Hence \( af \) is identically 0 everywhere.

Fix \( x_1, y_1 \), say \( x_1 = s, y_1 = t \), so that \( f \) does not vanish identically on the corresponding \( x_2, y_2 \) plane. Working in this plane, as \( af \) is identically 0 it follows that \( a \) is identically 0 on the open set where \( f \) is nonzero. But the final equation in (5) says that \( a \) is also harmonic in \( x_2, y_2 \), hence \( a \) vanishes identically on the whole plane, and similarly on all nearby \( x_2, y_2 \) planes.

Next we look at \( x_1, y_1 \) planes. As \( af = 0 \) the third equation in (5) says that \( a \) is harmonic. But as we know that \( a \) is 0 close to \( (s, t) \) we can conclude that \( a = 0 \) everywhere.

Therefore the only closed anti-invariant forms \( a\beta + b\gamma \) are of the form \( a = 0 \) and \( b \) constant, showing that \( h_f = 1 \) as required. \( \square \)

4. Families of non-integrable almost complex structures with \( h_f = 2 \) on the Kodaira-Thurston manifold

We will recall the construction of the Kodaira-Thurston manifold. Let \( \mathbb{R}^4 \) be the Euclidean space with coordinate \( (x_1, \ldots, x_4) \) endowed with the following product \( \ast \): given any \( a = (x_1, \ldots, x_4), y = (y_1, \ldots, y_4) \in \mathbb{R}^4 \), define
\[
x \ast y = (x_1 + y_1, x_2 + y_2, x_3 + x_1 y_2 + y_3, x_4 + y_4).
\]
Then \( (\mathbb{R}^4, \ast) \) is a nilpotent Lie group and
\[
\Gamma = \left\{ (\gamma_1, \ldots, \gamma_4) \in \mathbb{R}^4 \mid \gamma_j \in \mathbb{Z}, j = 1, \ldots, 4 \right\}
\]
is a uniform discrete subgroup of \( (\mathbb{R}^4, \ast) \), so that \( M = \Gamma \backslash \mathbb{R}^4 \) is a 4-dimensional compact manifold. Setting,
\[
E^1 = dx_1, \quad E^2 = dx_2, \quad E^3 = dx_3 - x_1 dx_2, \quad E^4 = dx_4,
\]
then it is immediate to check that \( E^1, E^2, E^3, E^4 \) are \( \Gamma \)-invariant 1-forms on \( \mathbb{R}^4 \), and, consequently, they give rise to a global coframe on \( M \). Then the following structure equations hold
\[
dE^1 = 0, \quad dE^2 = 0, \quad dE^3 = -E^1 \wedge E^2, \quad dE^4 = 0.
\]
Denoting by \( \{E_1, \ldots, E_4\} \) the dual global frame on \( M \), then
\[
[E_1, E_2] = E_3,
\]
the other brackets vanishing. Let \( \lambda = \lambda(x_4), \mu = \mu(x_4) \) be non constant \( \mathbb{R} \)-valued smooth \( \mathbb{Z} \)-periodic functions. Define an almost complex structure \( J = J_{\lambda, \mu} \) on \( M \) by setting
\[
JE_1 = e^{\lambda(x_4)} E_2, \quad JE_2 = -e^{-\lambda(x_4)} E_1, \quad JE_3 = e^{\mu(x_4)} E_4, \quad JE_4 = -e^{-\mu(x_4)} E_3.
\]

Lemma 4.1. The almost complex structure \( J \) is non integrable.
Proof. We compute
\[ N_J(E_1, E_3) = [JE_1, J E_3] - [E_1, E_3] - J[J E_1, E_3] - J[E_1, J E_3] \]
\[ = [e^{\lambda(x_4)}E_2, e^{\mu(x_4)}E_4] - J[e^{\lambda(x_4)}E_2, E_3] - J[E_1, e^{\mu(x_4)}E_4] \]
\[ = -E_4(e^{\lambda(x_4)})E_2 = -e^{\lambda(x_4)}\lambda'(x_4)E_2 \neq 0 \]
\[ \square \]

Proposition 4.2. Let \( J = J_{\lambda, \mu} \) be the family of the (non invariant) almost complex structures on the Kodaira-Thurston manifold defined as in (7). Then \( h^{-J}(M) = 2 \).

Proof. By the definition of \( J \), the following
\[ \psi^1 = E^1 + ie^{\lambda(x_4)}E_2, \quad \psi^2 = E^3 + ie^{\mu(x_4)}E_4 \]
is a global \((1,0)\)-coframe on \((M, J)\). Then
\[ \theta^1 = E^1 \wedge E^3 - e^{-(\lambda(x_4) + \mu(x_4))}E_2 \wedge E_4, \quad \theta^2 = e^{-\mu(x_4)}E^1 \wedge E_4 + e^{-\lambda(x_4)}E_2 \wedge E_3 \]
globally span \( \Lambda^{-J}(M) \). We immediately obtain
\[ d\theta^1 = 0, \quad d(e^{\lambda(x_4)}\theta^2) = 0, \]
that is \( \theta^1, e^{\lambda(x_4)}\theta^2 \) are closed \( J \)-anti-invariant forms, hence harmonic, which span \( \Lambda^{-J}(M) \). Since \( b^+(M) = 2 \) and \( h^{-J}(M) \leq b^+(M) \) for every compact almost complex manifold, we conclude that \( h^{-J}(M) = 2 \) and
\[ H^{-J}_\top(M) \simeq \text{Span}_\mathbb{R}\langle \theta^1, e^{\lambda(x_4)}\theta^2 \rangle. \]
\[ \square \]

5. 6-DIMENSIONAL COMPACT ALMOST COMPLEX MANIFOLDS WITH ARBITRARILY LARGE ANTI-INARIANT COHOMOLOGY

In this Section we provide simple examples of compact 6-dimensional manifolds endowed with a non integrable almost complex structure with arbitrary large anti-invariant cohomology.

Let \( \Sigma_g \) be a compact Riemann surface of genus \( g \geq 2 \). On the differentiable product \( X = \Sigma_g \times \Sigma_g \), denote by \( J \) the complex product structure. Let \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) be the real 2-torus, where we indicate with \((t_1, t_2)\) global coordinates on \( \mathbb{R}^2 \) and let \( f : X \to \mathbb{R} \) be a smooth positive non constant function. Let \( M = X \times T^2 \). Define \( J \in \text{End}TM \) by setting
\[ J(V, a \frac{\partial}{\partial t_1} + b \frac{\partial}{\partial t_2}) = (JV, -\frac{b}{f} \frac{\partial}{\partial t_1} + f a \frac{\partial}{\partial t_2}) \]
Then, we have the following

Proposition 5.1. \( J \) is a non integrable almost complex structure on \( M = X \times T^2 \) such that
\[ h^{-J}(M) \geq 2g^2. \]
ON THE ANTI-INVARIANT COHOMOLOGY OF ALMOST COMPLEX MANIFOLDS

Proof. It is immediate to check that $J^2 = -\text{id}$. Let $p \in X$ such that $df(p) \neq 0$ and let $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$ be local holomorphic coordinates on $X$ around $p$. We may assume that $\frac{\partial}{\partial z_1} f(p) \neq 0$. We have:

$$\begin{align*}
N_J(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial t_1}) &= [J \frac{\partial}{\partial x_1}, J \frac{\partial}{\partial t_1}] - [\frac{\partial}{\partial x_1}, \frac{\partial}{\partial t_1}] - J[J \frac{\partial}{\partial x_1}, J \frac{\partial}{\partial t_1}]
&= J\left[\frac{\partial}{\partial x_1}, J \frac{\partial}{\partial t_1}\right] - J\left[\frac{\partial}{\partial x_1}, f \frac{\partial}{\partial t_1}\right]
&= f_x(p) \frac{\partial}{\partial t_1} + f_y(p) \frac{\partial}{\partial t_2} \neq 0
\end{align*}$$

Denote by $\{\gamma_1, \ldots, \gamma_g\}, \{\gamma'_1, \ldots, \gamma'_g\}$, respectively be a basis of $H_{\bar{\partial}}^{1,0}(X)$ on the first and on the second copy of $\Sigma_g$, respectively. Then

$$H_{\bar{J}}^{2,0}(X) \simeq \text{Span}_\mathbb{C} \langle \gamma_r \wedge \gamma'_s, \quad 1 \leq r, s \leq g \rangle$$

and clearly $d(\gamma_r \wedge \gamma'_s) = 0$, for every $1 \leq r, s \leq g$. Then $h_{\bar{J}}(X) = 2g^2$. Therefore,

$$h_{\bar{J}}(M) \geq 2g^2.$$

Remark 5.2. The previous Proposition gives a positive answer to the question raised in [2, Question 5.2] where it was asked for examples of non integrable almost complex structures $J$ on a compact $2n$-dimensional manifold with $h_{\bar{J}}(M) > n(n - 1)$.

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