Gauge Theory in $d = 2 + 1$ at High Temperature: $Z_N$ interface

C. Korthals Altes$^a$, A. Michels$^b$, M. Stephanov$^{c,†}$ and M. Teper$^b$

$^a$Centre Physique Theorique au CNRS, Luminy, B.P. 907, 13288 Marseille, France

$^b$Theoretical Physics, U. of Oxford, 1 Keble Rd., Oxford OX1 3NP, UK

$^c$Dept. of Physics, U. of Illinois at Urbana-Champaign, 1110 W. Green St., Urbana 61801–3080, USA

We calculate on the lattice the interface tension in the $SU(2)$ pure gauge theory in $d = 2 + 1$ at high temperature. The result is compared to the perturbative prediction. The agreement confirms applicability of the perturbation theory in this case.

1. Hot QCD and $Z_N$ symmetry

Studies of QCD at high temperature attracted considerable attention recently (see, e.g., [1,2] and refs. therein). One of the interesting questions concerns the existence and physical significance of the so-called $Z_N$ ($N = 3$) phases of quark–gluon plasma at high temperature ($T > T_c$). These phases appear naturally in Euclidean formulation of QCD. In such a formulation the system lives in a 4-dimensional box. The extent of the three spatial dimensions is large compared to the physical scale and the extent of the fourth (Euclidean time) dimension is $1/T$. The boundary conditions in this time direction are periodic (for gluon fields).

Consider a hypothetical QCD without quarks and generalize it to an $SU(N)$ gauge theory in $d$ space–time dimensions. An order parameter distinguishing between the $Z_N$ phases is the value of the Polyakov loop: a path ordered exponent along a path in the time direction at fixed spatial coordinate $x$:

$$P(x) = \frac{1}{N} \text{Tr} P \exp \left\{ i g \int_0^{1/T} dt A_\mu(t, x) \right\} , \quad (1)$$

where $A_\mu = A_\mu^a T^a$ is the $N \times N$ matrix of the gauge potential. It is gauge invariant due to periodic boundary conditions provided that the gauge transformation $U(x, t)$ is also periodic in $t$. However, there are also gauge transformations which are periodic up to an element of the center of the gauge group:

$$U(x, t + 1/T) = z U(x, t) , \quad (2)$$

where $z = I \exp( i 2 \pi k /N )$, $k = 0, 1, 2, \ldots, N - 1$; $I$ is a unit $N \times N$ matrix. The periodic b.c. are not affected by such a gauge transformation (if there are no fermions) as $z$ commutes with all matrices in the gauge group. However, the Polyakov loop acquires a phase $\exp( i 2 \pi k /N )$.

If one integrates out all degrees of freedom except for the Polyakov loops the resulting effective theory of the complex scalar field $P(x)$ in $d - 1$ dimensions will have a global $Z_N$ symmetry: $P \rightarrow \exp( i 2 \pi k /N ) P$. It turns out [3] that at high $T$ this system is ferromagnetically ordered: $\langle P \rangle \neq 0$ (because an effective ferromagnetic coupling grows with $T$). The value of $\langle P \rangle$ can be related to the free energy $F$ of a static quark inserted in the hot gluon plasma: $\langle P \rangle = \exp( - F(T) /T )$. Thus confinement corresponds to $\langle P \rangle = 0$, i.e., $F = \infty$. In this way one can relate the breaking of the $Z_N$ symmetry to the deconfinement transition.

2. Interfaces and perturbation theory

There are $N$ possible complex directions where the vacuum expectation value $\langle P \rangle$ can point. If two parts of space are occupied by domains with different values of $\langle P \rangle$ an interface occurs between the domains. Such interfaces might have interesting cosmological consequences [4].

Perturbation theory can be used to calculate properties of such interfaces at very high tem-
temperature when the effective gauge coupling becomes small. Integrating out quadratic fluctuations around constant homogeneous configuration \(A_0\) one obtains the effective potential as a function of \(A_0\). Consider, for example, an \(SU(2)\) gauge theory. One can rotate the constant field \(A_0\) to some direction in the matrix space, say, \(A_0 = A_0^0 \tau_3 / 2\). Then the effective potential has the form as shown in Fig. 1 [5].

\[
V_{\text{eff}}(0, 2\pi T/g, A_0^3)
\]

Figure 1. \(V_{\text{eff}}\).

The periodicity of \(V_{\text{eff}}\) reflects the \(Z_2\) symmetry of the effective theory of the Polyakov loops: \(P \rightarrow -P\). Indeed, for our constant \(A_0\): \(P = \cos(gA_0^0 / 2T)\). At high \(T\) the \(Z_2\) symmetry will be spontaneously broken: minima of \(V_{\text{eff}}\) correspond to \(P = \pm 1\).

One can calculate the interface tension between two phases by considering a soliton-like solution which starts in one minimum, goes through the barrier and ends in another minimum. The action on such a trajectory would give the interface tension. In \(d = 4\) one obtains [6]: \(\sigma = cT^2 / g\), where \(c\) is a numerical constant. This result is easy to understand: the width of the interface is of order \(1 / gT\) – Debye screening length, and the energy density inside the interface is of order \(T^4\).

One has to take perturbative results at high temperature with caution, however. Corrections from higher orders of the perturbative expansion contain infrared divergencies. For example, it was suggested in [7] that infrared divergencies might result in vanishing interface tension.

The aim of this paper is to study a pure gauge theory nonperturbatively (on the lattice), determine the interface tension at very high temperatures \((T \gg T_c)\) and compare these results with perturbation theory.

3. Lattice study

We study \(SU(2)\) pure gauge theory in \(d = 2 + 1\). The gauge variables — \(SU(2)\) matrices — are defined on the links of a simple cubic \(L_x \times L_y \times L_t\) lattice. The action is a sum over the plaquette products \(\Box_P\) of the gauge matrices:

\[
S = \sum_P (1 - \frac{1}{2} \text{Tr} \Box_P)
\]

The partition function is \(Z = \exp(-\beta S)\).

To enforce an interface we use a twist [8]. The action of a system with the twist differs from the untwisted action in that the contribution of a set of plaquettes pierced by a line in \(y\) direction changes sign. One can check that this is equivalent to antiperiodic b.c. in \(x\) direction on the Polyakov loops.

We simulate the system with and without the twist and measure the average action in the two cases. The difference \(\Delta S\) is related to the excess free energy \(\Delta F = -\ln(Z_{\text{tw}} / Z)\) due to the interface and hence to the interface tension \(\sigma\):

\[
\Delta S = \frac{\partial \Delta F}{\partial \beta} = \frac{\partial}{\partial \beta} \left( \frac{\sigma A}{T} \right),
\]

where \(A = L_y a\) is the “area” of the interface, \(T = 1 / (L_t a)\) is the temperature in physical units and \(a\) is the lattice spacing.

We calculated the interface tension for this model in continuum to the leading order in \(g^2\):

\[
\sigma = \alpha_0 \frac{T^{5/2}}{g}, \quad \alpha_0 = 5.104\ldots
\]

The interface width is of order \(g \sqrt{T}\). The \(g^2\) is the (dimensionful) bare coupling constant for the \(SU(2)\) gauge theory in continuum. It is related to the lattice parameter \(\beta\) as: \(\beta = 4 / (g^2 a)\). The dimensionless small expansion parameter of the perturbation theory is \(g^2 / T\), which in terms of the lattice parameters is: \(4L_t / \beta\).

We can now substitute (5) into (4) and express \(A, T\) and \(g\) through the lattice parameters \(L_y, L_t\) and \(\beta\) to get the leading large \(\beta\) behavior of \(\Delta S\) in perturbation theory:

\[
\Delta S = \alpha_0 \frac{L_y}{4L_t^{3/2}} \frac{1}{\sqrt{\beta}}.
\]
Figure 2. The value of $\alpha \equiv 4L_t^{3/2} \Delta S \beta/L_y$ as a function of $1/\beta$ at a given $L_t$ and a linear extrapolation to $\beta = \infty$. (For comparison: $1/\beta_c \approx 0.29$ [9]).

We perform measurements at different values of $\beta$ at given $L_t = 2, 3, 4$. The spatial sizes $L_x, L_y$ are chosen after test runs to study finite size dependence. For $L_t = 2$ and $\beta = 15$ we chose $L_x = 36$ and $L_y = 12$ and then scaled the sizes with the interface width (roughly as $\sqrt{L_t \beta}$). We plot $4L_t^{3/2} \Delta S \beta/L_y$ as a function of $1/\beta$ and extrapolate to $\beta = \infty$ (see, e.g., Fig. 2). The resulting $\alpha_0$ depends on $L_t$ which is the cutoff $1/a$ in units of the temperature $T$. The continuum limit is $L_t \to \infty$. It is possible, however, to calculate $\alpha$ to the leading order in $g^2/T \sim 1/\beta$ in perturbation theory for finite $L_t$. We compare our results to perturbation theory at each $L_t$ in Fig. 3.

The extrapolation to $1/\beta = 0$ (i.e., $T = \infty$) in Fig. 2 has an ambiguity as we do not know the form of the perturbative correction to $\sigma$ in (5). The linear $1/\beta$ extrapolation we used assumes this correction to be $O(g^4/T^2 \beta)$. While we do not know its precise form, a simple analysis shows that this correction could contain additional powers of $\ln T/g^2 \sim \ln \beta$ which might also build up to a power of $\beta$. Such an ambiguity is of the same order as the deviation of the $1/\beta$ extrapolation from the leading perturbative result in Fig. 3. We checked this extrapolating with $1/\beta^{3-\epsilon}$, $0 < \epsilon < 0.5$.\footnote{Such logarithms are due to the IR divergencies specific to $d=2+1$ and appear, e.g., in the calculation of the Debye mass in this theory [10].}

\footnote{This results in smaller extrapolated $\alpha_0$ values.}

In conclusion, while our work is at a preliminary stage both analytically and numerically, we can already see that perturbation theory does indeed describe the properties of the interface very accurately.

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