Degeneration of natural Lagrangians and Prymian integrable systems

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Abstract

Starting from an anti-symplectic involution on a K3 surface, one can consider a natural Lagrangian subvariety inside the moduli space of sheaves over the K3. One can also construct a Prymian integrable system following a construction of Markushevich–Tikhomirov, extended by Arbarello–Saccà–Ferretti, Matteini and Sawon–Shen. In this article we address a question of Sawon, showing that these integrable systems and their associated natural Lagrangians degenerate, respectively, into fix loci of involutions considered by Heller–Schaposnik, García-Prada–Wilkin and Basu–García-Prada. Along the way we find interesting results such as the proof that the Donagi–Ein–Lazarsfeld degeneration is a degeneration of symplectic varieties, a generalization of this degeneration, originally described for K3 surfaces, to the case of an arbitrary smooth projective surface, and a description of the behaviour of certain involutions under this degeneration.

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1 Introduction

1.1 Context and motivation

By means of non-abelian Hodge theory [14, 19, 30, 53, 54], the moduli space of Higgs bundles carries a hyperKähler structure which naturally defines a triple of symplectic structures on it, each holomorphic with respect to one of the three complex structures. For one of these complex structures, the Higgs moduli space is a quasiprojective variety, further equipped with a proper fibration onto a vector space whose generic fibres are abelian Lagrangians with respect to the corresponding holomorphic symplectic form. These data define the Hitchin integrable system [29].

The moduli space of pure dimension 1 sheaves on a symplectic surface (i.e. K3 or abelian) can be equipped with the Mukai [44] holomorphic symplectic form and with the Le Potier (support) fibration [35], whose generic fibres are Jacobians. After Beauville [5], these fibres are also Lagrangians. In the case of K3 surfaces, the base of the Le Potier morphism is a linear system, hence the moduli space of pure dimension 1 sheaves on a K3 becomes a projective integrable system, named the Beauville–Mukai integrable system. Putting aside that it is projective, the Beauville–Mukai integrable system has a similar description to that of the Hitchin system obtained via the spectral correspondence. These similarities become even more explicit with the construction of the Donagi–Ein–Lazarsfeld [18] degeneration of the first integrable system into the latter.

The cohomological structure of the Hitchin system is very rich and many surprising identities occur within this framework, giving rise to a great number of conjectures. One of them is topological Mirror symmetry conjecture [26], predicting the equality between the stringy E-polynomials of Higgs moduli spaces for a pair of Langlands dual groups, proven for $\text{SL}(n, \mathbb{C})$ and $\text{PGL}(n, \mathbb{C})$ [23, 26]. Another conjectural cohomological identity is the P=W conjecture [17], predicting that the morphism in cohomology induced by the non-abelian Hodge correspondence exchanges the preverse filtration associated to the Hitchin fibration with the weight filtration on the associated character variety. This was proven by de Cataldo–Hausel–Migliorini [17] in the rank 2 case, and by de Cataldo–Maulik–Shen [15, 16] in the case of base curves of genus 2. The Donagi–Ein–Lazarsfeld degeneration in the case of abelian surfaces was a key element in the work of de Cataldo–Maulik–Shen. Using this degeneration, they constructed a specialization morphism in cohomology, and, so, they
could apply the powerful machinery developed by Markman [37–39] for the study of the cohomology of the moduli of sheaves on a K3, to the Higgs moduli space.

The cohomological χ -independence is another astonishing property of the moduli spaces of (twisted) Higgs bundles and pure dimension 1 sheaves on del Pezzo surfaces. It states that the intersection cohomology is independent from the Euler characteristic χ of the classified objects and was recently proven by Maulik–Shen [42].

Making use of the hyperKähler structure, [33] Kapustin and Witten introduced branes in the Higgs moduli space, setting that a (BBB)-brane is a hyperholomorphic subvariety supporting a hyperholomorphic sheaf, while a (BAA)-brane is a flat bundle over a complex Lagrangian subvariety for the holomorphic symplectic form associated to the first complex structure. Substituting the first by the second and third complex structures in this definition, we obtain (ABA) and (AAB)-branes, respectively. As indicated in [33], the Mirror symmetry conjecture predicts a duality between (BBB) and (BAA)-branes. This conjecture has motivated many authors to construct and study branes on Higgs moduli spaces. Most of the previous constructions are obtained by considering fixed loci of involutions on the moduli space, we highlight the (BBB) and (BAA)-branes obtained from natural involutions considered by Heller–Schaposnik [28] and García-Prada–Wilkin [22] out of holomorphic involutions on the base curve, the work of García-Prada and Ramanan [21] who classified the involutions obtained out of a combination of outer automorphisms of the group and tensorization, and the recent work [3], where Basu and García-Prada extended this study to include the action of holomorphic involution of the base curve, a set-up which already appeared in [10] for the case of elliptic curves. As indicated by Gukov in [24], Mirror duality between (BBB) and (BAA)-branes would imply certain cohomological relations between their support, a direction that was taken by Hausel–Mellit–Pei [27] to provide strong evidence for the pair of branes considered by Hitchin in [31] which are constructed out of the pair of Nadler–Langlands groups [45] given by Sp(2m, ℂ) and U(m, m).

Kapustin–Witten’s definition of branes extends naturally to hyperKähler varieties other than the Higgs moduli space. The case of the moduli space of sheaves over a symplectic surface was considered by the author, Jardim and Menet in [20], where they constructed branes of any type arising as fixed loci of natural involutions on the moduli induced by involutions on the surface, and studied the behaviour of these natural branes under some correspondences.

Within this setting, the natural involution associated to an anti-symplectic involution on a K3 surface is again anti-symplectic, and its fixed locus defines a complex Lagrangian subvariety of the moduli, which we call natural Lagrangian subvariety. One obtains a symplectic involution by composing this natural anti-symplectic involution with the dualizing involution on the fibres (perhaps tensoring with a line bundle), which is also anti-symplectic. Hence, considering the fixed locus of the symplectic involution constructed out of anti-symplectic involution on a K3 surface, one obtains a class of integrable systems whose Lagrangian fibres are Prym varieties. These Prymian integrable systems on K3 surfaces were first considered by Markushevich–Tikhomirov [40], and extended by Arbarello–Saccà–Ferretti [1], Matteini [41] and Sawon–Shen [50, 52]. In [49], Sawon conjectured that these Prymian integrable systems degenerate into integrable systems related to the Hitchin system, leaving open the description of these conjectural systems.

Sawon’s conjecture was the motivation for our work as such degeneration of Prymian integrable system and that of the associated natural Lagrangian subvarieties, could open the door for a cohomological study of pairs of (BBB) and (BAA)-branes in the Hitchin system, leading perhaps to strong evidence of their duality, by means of the specialization morphism in cohomology given in [15, 16].
In their recent paper [50], Sawon and Shen provided a degeneration of a particular choice of Prymian integrable system into the $\text{Sp}(2m, \mathbb{C})$-Higgs moduli space.

### 1.2 Main results

In this article we extend the Donagi–Ein–Lazarsfeld construction [18] to the case of a curve fitting in an arbitrary smooth projective surface, obtaining a degeneration of the moduli of pure dimension 1 sheaves on the surface into the moduli space of Higgs bundles on the curve, twisted by the normal bundle of the curve inside the surface (Theorem 4.1). We highlight that, in particular, our degeneration connects (Remark 4.2) the two moduli spaces for which the cohomological $\chi$-independence is known to hold [42]. We also study how certain involutions fit into this degeneration, finding that natural involutions and their composition with the dualizing involution on the moduli space of pure dimension 1 sheaves on the surface, degenerate into involutions on the moduli space of twisted Higgs bundles that we describe explicitly (Lemmas 4.6 and 4.8). When the twist is the canonical bundle of the curve, i.e. for Higgs bundles in the usual sense, these involutions coincide with those studied by Heller–Schaposnik [28], García-Prada–Wilkin [22] and Basu–García-Prada [3]. This allow us to construct a degeneration of the subvarieties given by loci fixed by these involutions (Theorems 4.12 and 4.14).

In the case of a K3 surface equipped with an antisymplectic involution the previous results provide degenerations of Prymian integrable system and their associated natural Lagrangian subvarieties of the moduli of pure dimension 1 sheaves on a K3. In particular, this gives a degeneration of the Prymian integrable systems constructed in [1] by Arbarello–Saccà–Ferretti (Sect. 4.4.2), and also (Sect. 4.4.3) by Markushevich–Tikhomirov [40], Matteini [41] and Sawon–Shen [50, 52]. Sawon conjectured in [49] that these Prymian integrable systems degenerate into integrable systems related to the Hitchin system. Every Lagrangian fibration degenerates naturally into the cotangent bundle of one of its smooth fibres, and the later can easily be identified with a rank 1 Hitchin system. Hence, one can always find examples of such degenerations. It is still open whether or not, the previous Prymian integrable systems degenerate into more interesting Hitchin systems. Our work provides an answer to Sawon’s question in the case of sheaves supported on non-primitive curves, obtaining a degeneration of the corresponding Prymian integrable systems into Hitchin systems of higher rank.

We also review the degeneration given by Sawon–Shen [50] into the $\text{Sp}(2m, \mathbb{C})$-Higgs moduli space, studying as well the degeneration of the associated natural Lagrangian (Sect. 4.4.4). It is worth noticing that we find that this natural Lagrangian degenerates into the fixed locus of an involution associated to $U(m, m)$-Higgs bundles, whose Nadler–Langlands group is $\text{Sp}(2m, \mathbb{C})$.

We also show that the Donagi–Ein–Lazarsfeld degenerations in the case of symplectic surfaces is equipped with a deformation of the symplectic structure of the moduli spaces (Theorem 4.4). This allow us to understand the degenerations of natural Lagrangian subvarieties studied in [20] and the Prymian integrable systems as degenerations of (BAA) and (BBB)-branes (Corollary 5.2). Finally, we prove that these branes are dual under a Fourier–Mukai transform restricted to the locus of pure 1 dimension sheaves with smooth support curves (Proposition 5.3 and Corollary 5.4).
1.3 Outline of the paper

The paper is structured as follows. In Sect. 2 we review the necessary background for our work. Section 2.1 contains generalities on the moduli space of pure dimension 1 sheaves on surfaces. In Sect. 2.2 we study the involutions on the moduli space naturally induced by pull-back of involutions on the surface. Section 2.3 is dedicated to the study of the involution on the moduli space of pure dimension 1 sheaves induced by dualizing the restriction of a sheaf to its fitting support. We also study the composition of this involution with a natural involution, and study their fixed loci, which is generically a fibration by Prym varieties. In Sect. 2.4 we consider these involutions starting from an anti-symplectic involution on a K3 surface, revisiting the Markushevich–Tikhomirov construction of Prymian integrable systems. Since we are interested in moduli spaces of sheaves with a non-primitive first Chern class, we treat this case in detail. Also, we study the Lagrangian subvarieties that arise from the fixed point locus of the natural involutions on the moduli constructed out of our anti-symplectic involution on the K3 surface. In Sect. 2.5 we collect the necessary facts about ruled surfaces which will be used in Sect. 2.6 to study twisted Higgs bundles, whose spectral data provide a particularly relevant example of pure dimension 1 sheaves.

We describe in Sect. 3 certain involutions on moduli spaces of twisted Higgs bundles. In Sect. 3.1 we describe some involutions on a ruled surface and study their relation with the Poisson structure. We construct the corresponding involutions on the moduli spaces of spectral data of twisted Higgs bundles in Sect. 3.2 and study their behaviour under the spectral correspondence. We obtain involutions on the moduli space of $L$–Higgs bundles, which in the particular case of the twisting by the canonical bundle, are involutions that have been widely studied by Heller–Schaposnik, García-Prada–Wilkin and Basu–García-Prada.

The main results of the paper are contained in Sect. 4. In Sect. 4.1 we provide a generalization to the case of an arbitrary smooth projective surface of the Donagi–Ein–Lazarsfeld degeneration, originally described for K3 surfaces. Hence, we obtain a degeneration of the moduli space of pure dimension 1 sheaves on a surface into the moduli spaces of Higgs bundles twisted by the normal bundle of a curve inside our surface. In Sect. 4.2 we provide a deformation of the symplectic structure of the moduli spaces involved in the Donagi–Ein–Lazarsfeld degeneration in the case of symplectic surfaces, showing that it provides a degeneration of symplectic varieties. In Sect. 4.3, we study the behaviour under the Donagi–Ein–Lazarsfeld degeneration of the involutions considered in Sects. 2.2 and 2.3. We then obtain a degeneration of the subvarieties described by their fixed loci. In Sect. 4.4 we provide an explicit description of such degenerations in the context of a K3 surface equipped with an anti-symplectic involution. We describe the non-linear degenerations of the Prymian integrable systems constructed by Arbarello–Saccà–Ferretti, Markushevich–Tikhomirov, Matteini and Sawon–Shen, showing that they degenerate into integrable systems related to the Hitchin system. We also describe the degeneration of the associated natural Lagrangian subvarieties.

Finally, in Sect. 5, we consider the natural Lagrangian subvarieties and the Prymian integrable systems in the context of branes, and provide some evidence for their duality.
2 Pure dimension 1 sheaves on surfaces and involutions on their moduli

2.1 Generalities on the moduli space of sheaves on surfaces

Let $S$ be a projective surface and take $H$ to be a polarization on it. We say that a coherent sheaf $F$ on $S$ is pure dimension $d$ if its schematic support has dimension $d$ and every subsheaf is also supported on dimension $d$ subschemes. In that case, its Hilbert polynomial $P(F, H)$ has degree $d$ and we define its H-polarized rank $\text{rk}(F, H)$ to be the leading term of $P(F, H)$ multiplied by $d!$. A coherent sheaf $F$ is H-stable (resp. H-semistable) if it is of pure dimension, and for every proper subsheaf $F' \subset F$ we have that its Hilbert polynomial satisfy $P(F', H)/\text{rk}(F', H) < P(F, H)/\text{rk}(F, H)$ (resp. $P(F', H)/\text{rk}(F', H) \leq P(F, H)/\text{rk}(F, H)$) when $n \gg 0$. As we require $n$ to be big enough, $P(F, H)(n) = \chi(F(nH))$. A semistable sheaf is polystable if it decomposes as a direct sum of stable sheaves. Simpson provided in [53] the existence of the moduli space $M^H_S(P)$ of pure dimension H-semistable sheaves with Hilbert polynomial $P$, whose closed points represent polystable sheaves.

The topological invariants of a sheaf over a smooth surface $S$ determined by the Hilbert polynomial are the H-polarized rank $\text{rk}(F, H)$, the first Chern class $c_1(F)$, and the Euler characteristic $\chi(F)$. To simplify the computations, we introduce the Mukai vector, $\nu(F) := \text{ch}(F) \cdot \sqrt{\text{Todd}(S)}$. Note that $\nu(F) \in H^{2*}(S, \mathbb{Q})$ and the cup product in cohomology provides the Mukai pairing which endows $H^{2*}(S, \mathbb{Q})$ with a lattice structure. We abbreviate by $M^H_S(\nu)$ the moduli space of H-semistable sheaves $S$ with Hilbert polynomial determined by the Mukai vector $\nu \in H^{2*}(S, \mathbb{Q})$.

It follows from deformation theory that the smooth locus of $M_{S,H}(v_a)$ is the locus defined by those sheaves that are simple and that the tangent space to $M_{S,H}(v_a)$ at the point determined by the simple coherent sheaf $F$ corresponds with

$$T_F M_{S,H}(v_a) = \text{Ext}^1_S(F, F),$$

and its cotangent space is

$$T^*_F M_{S,H}(v_a) = \text{Ext}^1_S(F, F \otimes K_S).$$

Hence, $M_{S,H}(v_a)$ is smooth at those points.

If, further, $S$ is a K3 or abelian surface, its canonical bundle $K_S$ is trivial and a choice of a (non-vanishing) section $\Omega_S$ provides a symplectic form on $S$. In both cases we say that $S$ is a symplectic surface. Thanks to Serre duality, $H^2(S, \mathcal{O}_S)$ is dual to $H^0(S, K_S) \cong \mathbb{C}$.

**Theorem 2.1** [44] Let $S$ be a symplectic surface. Then, $M_{S,H}(v_a)$ is equipped with a holomorphic 2-form $\Omega_M$ defined by taking the trace of the Yoneda product composed with $\Omega_S$,

$$\text{Ext}^1_S(F, F) \wedge \text{Ext}^1_S(F, F) \xrightarrow{\circ} \text{Ext}^2_S(F, F) \xrightarrow{\text{tr}} H^2(S, \mathcal{O}_S) \xrightarrow{\Omega_S} \mathbb{C}.$$

Furthermore, $\Omega_M$ is non-degenerate on the smooth locus of $M_{S,H}(v_a)$, defining a symplectic form there.

Bottacin [13] and Markman [36] generalized this construction to the case of Poisson surfaces, i.e. those equipped with Poisson bi-vector, a non-zero section $\Theta_S \in H^0(S, K_S^{-1})$. 

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Theorem 2.2 [13, 36] Let $S$ be equipped with a Poisson bi-vector $\Theta_S$. Then, one can define a (closed although possibly degenerate) Poisson structure $\Theta_M$ on the smooth locus of $M_{S, H}(v_a)$ by taking the trace of the Yoneda product composed with $\Theta_S$,

$$\text{Ext}^1_S(F, F \otimes K_S) \wedge \text{Ext}^1_S(F, F \otimes K_S) \xrightarrow{\circ} \text{Ext}^2_S(F, F \otimes K^2_S)$$

$$\xrightarrow{\text{tr}} H^2(S, K^2_S) \xrightarrow{\langle \cdot, \Theta_S \rangle} H^2(S, K_S) \cong \mathbb{C}.$$ 

Note that a symplectic surface is also Poisson, and, in that case, the Bottacin–Markman form coincides with the Mukai form after identifying the tangent and cotangent spaces on the smooth locus of our moduli (via a trivialization of $K_S$).

Proposition 2.3 [13] A smooth Poisson surface is either a symplectic surface (i.e. K3 or abelian) or a ruled surface.

On a smooth surface $S$, every pure dimension 1 sheaf $F$ has a locally free resolution of length 2 and we define the fitting support, $\text{supp}(F)$, of $F$ as the determinant associated to this resolution. Note that the first Chern class is the cohomology class of its fitting support $c_1(F) = [\text{supp}(F)]$.

The Mukai vector of a pure dimension 1 sheaf $F$ on a smooth surface with canonical sheaf $K_S$ takes the form

$$v(F) = \left(0, [\text{supp}(F)] \cdot \chi(F) + \frac{1}{2} [K_S] \cdot [\text{supp}(F)] \right).$$

and the pairing is simply given by the intersection of the supports,

$$\langle v(F), v(F') \rangle = v_2(F) \cdot v_2(F') = [\text{supp}(F)] \cdot [\text{supp}(F')].$$

Pick a smooth curve $C$ of genus $g_C$ in $S$ and consider the associated curve $nC$ which is non-reduced if $n > 1$. For any integer $a$ we define the Mukai vector

$$v_a = \left(0, [nC], a - \frac{n^2(C \cdot C)}{2} \right). \quad (2.1)$$

After Le Poitier [35], the corresponding moduli space can be equipped with a fibration to the Hilbert scheme classifying dimension 1 subschemes of $S$ with first Chern class equal to the second component of the Mukai vector,

$$h_S : M_{S, H}(v_a) \longrightarrow \text{Hilb}_S([nC])$$

$$F \mapsto \text{supp}(F). \quad (2.2)$$

The fibre of $h_S$ over the curve $A \in \text{Hilb}_S([nC])$ is the Simpson compactified Jacobian classifying $H|_A$-semistable pure dimension 1 sheaves on $A$ of rank 1 and degree $a$

$$h_S^{-1}(A) = \text{Jac}_A^H(a).$$

Remark 2.4 The Picard group of a smooth projective K3 surface is discrete and embeds into its second integral cohomology space. Hence, in this case, the second component of the Mukai vector fixes the determinant and $\text{Hilb}_S([nC])$ is the linear system $|nC|$. After the work of Beauville [5] the fibres are Lagrangian with respect to the Mukai form $\Omega_M$. Hence, this is an algebraic completely integrable system called the Beauville–Mukai system.
In general, the natural morphism \( \text{Pic}(S) \to H^2(S, \mathbb{Z}) \) is not an embedding. Furthermore \( \text{Hilb}_S([nC]) \) can be reducible and even non-connected. Choose a curve \( A \) parametrized by \( \text{Hilb}_S([nC]) \), and let us denote by \( \{ A \} \) the connected component of the Hilbert scheme containing the curve \( A \). Accordingly, we denote by

\[ M_{S,H}(v_a, A) = h_S^{-1}(\{ A \}) , \]

the associated connected component of the moduli.

**Remark 2.5** Our notation differs from that of Matteini in [41], where, given a smooth curve \( A \), he defined \( \{ A \} \) to be the irreducible component of \( \text{Hilb}_S([nC]) \) containing \( A \). Since we will allow \( A \) to be singular, and even non-reduced, \( A \) may be contained in several irreducible components, although it determines uniquely its connected component.

**Remark 2.6** When the irregularity of the surface \( S \) vanishes, i.e. \( h^1(S, O_S) = 0 \), we have that \( \text{Hilb}_S([nC]) \) is a discrete union of connected components. In this case, \( \{ A_i \} = |A_i| \), each being a linear system.

Let us denote by \( \mathcal{A} \) the restriction to \( \{ A \} \times S \) of the universal subscheme associated to \( \text{Hilb}_S([nC]) \). Note that \( \mathcal{A} \) is a family of locally planar curves parametrized by \( \{ A \} \). The polarization \( H \) on \( S \) provides a relative polarization on \( \mathcal{A} \) that we still denote by \( H \). By means of the Le Poitier fibration (2.2), one can identify the component \( M_{S,H}(v_a, A) \) of our moduli space of pure dimension 1 sheaves on \( S \) with the relative compactified Jacobian over \( \mathcal{A} \) of relative torsion free sheaves of rank 1 and degree \( a \),

\[ M_{S,H}(v_a, A) \cong \overline{\text{Jac}}_{\mathcal{A}/\{ A \}}(a) . \]

If \( \{ A \}^{\text{sm}} \) denotes the open set of smooth curves, and \( \mathcal{A}^{\text{sm}} \) the restriction of \( \mathcal{A} \) there, one has the following identification of open subsets

\[ M_{S,H}(v_a, A) |_{\{ A \}^{\text{sm}}} \cong \overline{\text{Jac}}_{\mathcal{A}^{\text{sm}}/\{ A \}^{\text{sm}}}(a) , \]

contained in the smooth locus.

**Remark 2.7** We have dropped the polarization from the notation on the Jacobian as pure dimension 1 sheaves whose restriction to its support have rank 1, are always stable, regardless of the choice of polarization.

### 2.2 Natural involutions from involutions on the surface

In this section we study the involutions on the moduli space naturally induced by pull-back of involutions on the surface.

Suppose that our smooth projective surface \( S \) is equipped with an involution

\[ \zeta_S : S \to S. \]

Let \( C \) be a smooth projective curve in \( S \) which is preserved under \( \zeta_S \) and consider a Mukai vector \( v_a \) of the form specified in (2.1) built out the \( \zeta_S \)-invariant curve \( C \). Then, one trivially has that \( v_a \) is \( \zeta_S \)-invariant as well, so one can construct a birational involution on the moduli space,

\[ \widehat{\zeta}_S : M_{S,H}(v_a) \dashrightarrow M_{S,H}(v_a) \]

\[ \mathcal{F} \quad \dashrightarrow \quad \zeta_S^*\mathcal{F}, \]
that we call natural involution associated to $\zeta_S$. We denote the closure of its fixed point locus by

$$N_{S,H}(v_a) := \text{Fix}(\zeta_S).$$

If, further, the polarization $H$ is $\zeta_S$-invariant, $\hat{\zeta}_S$ is a biregular morphism and its fixed locus is already closed. The support morphism (2.2) restricts to $N_{S,H}(v_a)$ with image $\text{Hilb}_S([nC])^{\zeta_C}$ of curves preserved by the involution, which may have several components.

Note that every curve $A'$ in $\text{Hilb}_S([nC])^{\zeta_C}$ inherits an involution

$$\zeta_{A'} : A' \to A',$$

inducing another birational involution on the compactified Jacobian,

$$\hat{\zeta}_{A'} : \text{Jac}^H_A(a) \dashrightarrow \text{Jac}^H_{A'}(a),$$

As the fibres of (2.2) are identified with the corresponding compactified Jacobian, one has

$$h_{S}^{-1}(A') \cap N_{S,H}(v_a) = \text{Fix}(\hat{\zeta}_{A'}).$$

It has been shown (see [2, Lemma 9] or [20] for instance) that the involution induced on the moduli space has the same behaviour under the symplectic form as the starting one. The generalization to the case of Poisson surfaces is straight-forward.

**Proposition 2.8** Consider a Poisson surface $S$ and suppose it is equipped with a Poisson involution $\zeta_S^+$ (resp. an anti-Poisson involution $\zeta_S^-$). Then, the natural (birational) involution $\hat{\zeta}_S^\pm$ is also Poisson (resp. $\hat{\zeta}_S^\pm$ is anti-Poisson).

**Proof** We have to prove that whenever our involution $\zeta_S^\pm$ preserves the (powers of the) canonical bundle $K_S$ and commutes or anti-commutes with $\Theta_S$, so does $\hat{\zeta}_S^\pm$ with respect to $\Theta_M$ over the open subset $U \subset M_S^H(v_a)$ of smooth points where $\hat{\zeta}_S^\pm$ is biregular.

The proof is very similar to that of [20, Theorem 3.4]. Suppose that $F$ represents a point in $U$. Then, $(\zeta_S^\pm)^*F$ lies in $U$ too. For each $i \geq 0$, consider the induced morphism in cohomology $R^i(\zeta_S^\pm)^*$, and note that it commutes with the Yoneda product. Also, one has that $R^i(\zeta_S^\pm)^*$ commutes with the trace morphism and Serre duality, appearing in the definition of the Bottacin–Markman form. The Poisson involution $\zeta_S^+$ commutes with the remaining morphism in that definition, $\langle \cdot, \Theta_S \rangle$, while the anti-Poisson involution $\zeta_S^-$ anticommutes with it. Then, we can see that $\zeta_S^+$ commutes with the composition of all these morphisms, which define the Poisson form $\Theta_M$ while $\zeta_S^-$ anticommutes with it. Hence, the proof is complete. $\square$

Suppose now that the quotient surface $T := S/\zeta_S$ is smooth and denote the quotient map by

$$q_S : S \to T.$$

Suppose as well that

$$C := D \times_T S$$

for some smooth curve $D \subset T$. We have already seen that the Mukai vector $v_a$ is $\zeta_S$-invariant as so is $C$. 
Remark 2.9 A Mukai vector of the form $v_{2b}$ is the pull-back of the Mukai vector in $T$,

$$w_b := \left(0, \lbrack nD\rbrack, b - \frac{n^2(D \cdot D)}{2}\right).$$

Pick a curve $B \subset T$ in the class $\lbrack nD\rbrack$ and consider its lift $A = B \times_T S$ to $S$, lying in the class $\lbrack nC\rbrack$. Recall that we denote by $\{B\}$ and $\{A\}$ the connected components of $\text{Hilb}_T(\lbrack nD\rbrack)$ and $\text{Hilb}_S(\lbrack nC\rbrack)$ containing $B$ and $A$, respectively. Note that $\hat{\zeta}_S$ restricts to $\{A\}$ although the fixed locus $\{A\}^{\hat{\zeta}_S}$ might be disconnected. We shall denote by $q^*_S\{B\}$ the connected subset of $\{A\}^{\hat{\zeta}_S}$ given by those curves obtained by lifting curves in $\{B\}$. In principle, $q^*_S\{B\}$ is not a connected component of $\{A\}^{\hat{\zeta}_S}$, only a union of some irreducible components.

Remark 2.10 We have seen Remark 2.6 whenever the surface $S$ has vanishing irregularity, the $\{A_i\}$ are the linear systems $|A_i|$. In this case, the locus $\{A_i\}^{\hat{\zeta}_S}$ of curves preserved by $\zeta_S$ is disconnected and amounts to the union of $\{A_i\}^{\hat{\zeta}_S}$ and $\{A_i\}^{\hat{\zeta}_S}$, each being the projectivization of the $+1$ and $-1$ eigenspaces for the action of $\zeta_S$ on $H^0(S, O_S(A_i))$. In this context,

$$\{A\}^{\hat{\zeta}_S} = q^*_S\{B\}.$$ 

We consider the subvariety

$$N_{S,H}(v_a, B) := N_{S,H}(v_a) \cap h^{-1}_S \left(q^*_S\{B\} \right). \quad (2.4)$$

By restriction of (2.2), $N_{S,H}(v, B)$ is equipped with a fibration,

$$N_{S,H}(v, B) \xrightarrow{\nu} M_{S,H}(v, A) \cong \text{Jac}_{A/\{A\}}(a) \quad (2.5)$$

whose fibres are

$$\nu^{-1}(A) = \text{Fix}(\hat{\zeta}_A).$$

Consider over $T$ the moduli space $M_{T,1}(w_b)$ of sheaves with a Mukai vector $w_b$ and a polarization $I$. As before, we denote by $\{B\}$ the connected component of the Hilbert scheme in $T$ containing $B$, and by $\mathcal{B}$ the family of curves parametrized by $\{B\}$ that we obtain from restricting there the universal subscheme associated to $\text{Hilb}(\lbrack nD\rbrack)$. Recall that one has the identification

$$M_{T,1}(w_b, B) \cong \text{Jac}_{\mathcal{B}/\{B\}}(b).$$

Denote the pull-back of the polarization by

$$\hat{\mathcal{I}} = q^*_S\mathcal{I}.$$ 

For these choices, the involution $\hat{\zeta}_S$ is biregular and the image of the pull-back morphism under the corresponding quotient map lies in its fixed locus,

$$\tilde{q}_S : M_{T,1}(w_b, B) \cong \text{Jac}_{\mathcal{B}/\{B\}}(b) \to N_{S,\tilde{\mathcal{I}}}(v_{2b}, B) \subset M_{S,\tilde{\mathcal{I}}}(v_{2b}, A) \cong \text{Jac}_{A/\{A\}}(2b)$$

$$\cong q^*_S\mathcal{F}. \quad (2.6)$$
Given the curves \( B \subset T \) and \( A \subset S \) as before, denote by
\[ q_A : A \to B, \]
the projection induced by \( q_S \). Observe that the restriction of (2.6) to the Hitchin fibres gives,
\[ \hat{q}_A : h_T^{-1}(B) \cong \text{Jac}_B(b) \to v^{-1}(A) \subset h_S^{-1}(A) \cong \text{Jac}_A(2b), \]
where \( \hat{q}_A := q_A^* \).

Let us denote by \( \{B\}^{ssl} \) the locus of \( \{B\} \) parametrizing smooth curves \( B' \) whose lift \( A' = B' \times_T S \subset \{A\} \) is also smooth. The notation stands for smooth and smooth lift. Recall the family of curves \( B \to \{B\} \) and denote by \( B^{ssl} \) its restriction to \( \{B\}^{ssl} \). Set also \( A^{ssl} \) to be the restriction to \( q_S^*(\{B\})^{ssl} \) of the family of curves \( A \to \{A\} \), and observe that it comes equipped with a projection induced by \( q_S \).

\[
\begin{array}{ccc}
A^{ssl} & \xrightarrow{q_A^{ssl}} & B^{ssl} \\
\downarrow & & \downarrow \\
\{B\}^{ssl} & & \\
\end{array}
\]

One can see that \( q_{A^{ssl}} \) is ramified whenever the intersection of the branching locus \( \Delta \) of \( q_S \) with generic elements of \( \{B\} \) is non-empty, and, when \( \zeta_S \) is without fixed points, \( q_{A^{ssl}} \) is unramified. Let us denote by \( \hat{q}_{A^{ssl}} \) the restriction of \( \hat{q}_S \) to \( q_S^*(\{B\})^{ssl} \). Recall from Remark 2.7, that the sheaves supported on smooth (hence irreducible) curves are always stable,
\[
\hat{q}_{A^{ssl}} : \text{Jac}_{B^{ssl}/\{B\}^{ssl}}(b) \to N_{S,H}(v_{2b}, B) \subset \text{Jac}_{A^{ssl}/q_S^*(\{B\})^{ssl}}(2b).
\]

This provides a description of an open subset of \( N_{S,H}(v_{2b}, B) \).

**Proposition 2.11** Suppose that \( q_{A^{ssl}} \) is ramified. Then
\[
N_{S,H}(v_{2b}, B)|_{\{B\}^{ssl}} \cong \text{Jac}_{B^{ssl}/\{B\}^{ssl}}(b)
\]
and (2.6) is, generically, a 1:1 morphism.

If, on the contrary, \( q_{A^{ssl}} \) is an unramified 2 : 1 cover associated to the relative 2-torsion line bundle \( L \to B^{ssl} \), one has that \( \hat{q}_{A^{ssl}} \) factors through the quotient by \( \mathbb{Z}_2 \) acting by tensor product with \( L \), giving
\[
N_{S,H}(v_{2b}, B)|_{\{B\}^{ssl}} \cong \text{Jac}_{B^{ssl}/\{B\}^{ssl}}(b)/\mathbb{Z}_2
\]
and (2.6) is, generically, a 2:1 morphism.

**Proof** In the first case, by Kempf descent lemma, \( \text{Im}(\hat{q}_{A^{ssl}}) \) coincides with the left-hand side of (2.9). Since, by hypothesis, \( q_{A^{ssl}} \) is ramified, \( \hat{q}_{A^{ssl}} \) is an embedding. As we find ourselves within the smooth locus of \( M_H^S(v_{2b}, A) \), \( \hat{q}_{A^{ssl}} \) provides an isomorphism of its source with its image. As \( \{B\}^{ssl} \) is a dense open subset of \( \{B\} \), this is, generically, 1 : 1 morphism.

The second case follows since the pull-back of \( L \) is trivial over \( A^{ssl} \).

**2.3 The dualizing involution and Prymian fibrations**

In this section we study the involution on the moduli space of pure dimension 1 sheaves induced by dualizing the restriction of a sheaf to its fitting support. We also study the composition of this involution with a natural involution, and study their fixed loci, which is generically a fibration by Prym varieties.
Given a Mukai vector \( v_a \) on \( S \) as defined in \((2.1)\), whose second component is the class \([nC]\), let us choose a line bundle \( J \in \text{Pic}(S) \) whose intersection with the second component of the Mukai vector satisfies
\[
\frac{2a}{n} = J \cdot C + n(C \cdot C).
\] (2.11)
Note that not every value of \( a \) and \( n \) allow for such a \( J \). It is discussed in [1, Sections 3.3 and 3.5], that the moduli space \( M_{S,H}(v_a) \) is equipped with an involution
\[
\xi_{J,S} : M_{S,H}(v_a) \to M_{S,H}(v_a)
\]
\( \mathcal{F} \to \text{Ext}^1_S(\mathcal{F}, J). \) (2.12)
Pick \( A = \text{supp}(\mathcal{F}) \), which is Gorenstein with canonical line bundle \( K_S|_A \), Grothendieck–Verdier duality implies that,
\[
\text{Ext}^1_S(\mathcal{F}, J) \cong \text{Hom}_A(\mathcal{F}|_A, J(A)|_A).
\] (2.13)
Since \( A \) belongs to the class \([nC]\), equation \((2.11)\) implies \( 2a = \text{deg}(J(A)|_A) \) and this ensures that \( \xi_{J,S} \) preserves the Mukai vector. Observe that \( \xi_{J,S} \) restricts to the fibres of the Mukai fibration, \( h_S^{-1}(A) \cong \text{Jac}_A^{d}(A) \), giving the dualizing involution composed with an appropriate tensor product,
\[
\xi_{J,A} : \text{Jac}_A^H(a) \to \text{Jac}_A^H(a)
\]
\( \mathcal{E} \to \mathcal{E}^\vee \otimes J(A)|_A \), (2.14)
where we recall that the dual of a rank 1 torsion-free sheaf on a Gorenstein curve is always well defined and \( H \)-stability is preserved.

**Remark 2.12** Condition \((2.11)\) is always satisfied when we choose \( v_0 \) and \( J_0 = \mathcal{O}_S(-nC) \), for instance. In that case \( \xi_{J_0,A} \) is just the dualizing involution \( \mathcal{E} \mapsto \mathcal{E}^\vee \).

Recall that we have chosen the Mukai vector \( v_a \) to be \( \xi_S \)-invariant. If further the line bundle \( J \) on \( S \) satisfying \((2.11)\) is \( \xi_S \)-invariant too, \( \xi_S \) and \( \xi_{J,S} \) commute and, following [1], we consider their composition
\[
\lambda_{J,S} := \hat{\xi}_S \circ \xi_{J,S} : M_{S,H}(v_a) \to M_{S,H}(v_a),
\]
which defines a birational involution on \( M_{S,H}(v_a) \). We denote the closure of its fixed locus by
\[
P_{S,H}(v_a, J) := \text{Fix}(\lambda_{J,S}).
\]
We now study the relation of these involutions with the Poisson (and symplectic) structure. It was proved in [1, Proposition 3.11] that \( \xi_{J,S} \) is anti-symplectic under the assumptions of \( S \) being a K3 surface, \( v_0 \) a Mukai vector with \( n = 1 \) and \( J = \mathcal{O}_S(-C) \).

**Proposition 2.13** Given a Poisson surface \( S \), for any choice of Mukai vector \( v_a \) and \( J \in \text{Pic}(S) \) satisfying \((2.11)\), the involution \( \xi_{J,S} \) is anti-Poisson with respect to the Bottacin–Markman Poisson form \( \Theta_M \) on \( M_{S,H}(v_a) \).
\[
\xi_{J,S}^* \Theta_M = -\Theta_M.
\]
**Proof** The proof of [1, Proposition 3.11] extends to the Poisson case with minor changes. \( \square \)
As an immediate consequence of Propositions 2.8 and 2.13 one derives the following corollary which will be used in Sect. 4.4.

**Corollary 2.14** Let $S$ be a Poisson surface and $\xi^+_S$ a Poisson involution on it (resp. $\xi^-_S$ an anti-Poisson involution). For any choice of Mukai vector $v_a$ and a $\xi^+_S$-invariant (resp. $\xi^-_S$-invariant) $J \in \text{Pic}(S)$ satisfying (2.11), the involution $\lambda^+_J, S = \xi^+_S \circ \xi^+_J, S$ is a birational anti-Poisson involution (resp. $\lambda^-_J, S = \xi^-_S \circ \xi^-_J, S$ is a birational Poisson involution) on $M_{S, H}(v_a)$.

Let us now assume that the quotient surface $T$ is smooth, and recall the notation introduced in Sect. 2.2. Consider the closed subvariety

$$P_{S, H}(v_a, J, B) := P_{S, H}(v_a, J) \cap h^{-1}_S(\xi^+_S(B)).$$

Whenever $H$ is $\xi^+_S$-invariant, $\lambda^+_J, S$ is well defined as a regular involution and its fixed locus is already closed.

By construction, one has the following commuting diagram

$$P_{S, H}(v_a, J, B) \xrightarrow{\mu} M_{S, H}(v_a, A) \cong \text{Jac}^H_{A|A}(a)$$

Observe that, over $A' \in \xi^+_S(B)$,

$$\mu^{-1}(A') = \text{Fix}(\lambda^+_J, A'),$$

where

$$\lambda^+_J, A' := \xi^+_A \circ \xi^+_J, A' : \text{Jac}^H_{A}(a) \rightarrow \text{Jac}^H_{A}(a)$$

is the induced birational involution on $h^{-1}_S(A')$. If $A'$ is irreducible, in particular when it is smooth and connected, every pure dimension sheaf is stable so $\xi^+_J, A'$ is a biregular involution and so is $\lambda^+_J, A'$. Hence, the associated fixed locus is already closed.

Recall that we denoted by $\{ B \}^{ssl}$, the open subset of $\{ B \}$ parametrizing smooth curves, whose lift to $S$ is smooth too.

**Remark 2.15** For $B' \in \{ B \}^{ssl}$ lifting to $A'$ smooth one has that $h^{-1}_S(A') = \text{Jac}_{A'}(a)$ is a torsor for the abelian variety $\text{Jac}_{A'}(0)$. After (2.14), one has that the fibre of $\mu$ is

$$\mu^{-1}(A') = \ker (J(A')|A' + \xi^+_A),$$

whose connected components are torsors for the Prym variety

$$\text{Prym}(q_{A'}) = \ker (J_{\text{Jac}} + \xi^+_A)_{0}.$$
Since the kernel $1_{\text{Jac}} + \zeta A'$ has two connected components if $q_{A'}$ is an unramified $2 : 1$ cover. Then, in that case, $P_{S,H}(v_0,J_0,B)|_{\{B\}^\text{ssl}}$ has 2 connected components and (2.17) holds after restricting ourselves to the connected component of the identity.

Whenever $\{B\}^\text{ssl}$ is a dense open subset of $\{B\}$, by Remark 2.16 the generic fibres of $\mu : P_{S,H}(v_a,J,B) \to q_\ast \{B\}$ are (non-canonically) isomorphic to Prym varieties. In that case, we refer to it as the Prymian fibration associated to the involution $\zeta_S$.

### 2.4 Natural Lagrangian and Prymian integrable systems

Following Sawon–Shen [50] (see also [52]) and Matteini [41], we provide a straight-forward adaptation of the Markushevich–Tikhomirov construction of Prymian integrable systems to the case of non-primitive first Chern classes.

Prymian integrable systems are integrable systems fibered by Prym varieties. The class that we consider here arises from an anti-symplectic involution on a K3 surface. These systems were first constructed by Markushevich and Tikhomirov [40] and later generalized by Arbarello, Saccà and Ferretti [1], Matteini [41], Sawon and Shen [50, 52] and others. The case considered in [40] corresponds to an anti-symplectic involution on a smooth K3 surface, whose associated quotient is a del Pezzo surface of degree 2. In [1], Arbarello, Saccà and Ferretti constructed a Prymian integrable system from an anti-symplectic involution on a K3 giving an Enriques surface. Matteini [41] generalized the construction of [40] to the case of anti-symplectic involutions on K3 associated to a del Pezzo surface of degree 3. Sawon and Shen [50, 52] studied the case of an antisymplectic involution on a K3 whose quotient is a del Pezzo surface of degree 1.

Let $X$ denote a smooth K3 surface, $\zeta_X$ an antisymplectic involution on it, and $Y$ the quotient $Y = X/\zeta_X$ which is a either a smooth rational surface or an Enriques surface. Denote by $\Delta_Y$ the branch locus of the cover $X \to Y$, which is a subcurve of $Y$ giving a point in the linear system $|-2KY|$. Over $Y$, pick a smooth connected curve $D \subset Y$ of genus $g_D$ not tangent to the branch locus $\Delta_Y$, and consider

$$C = D \times_Y X = q_S^{-1}(D), \quad (2.18)$$

which is smooth, and it is equipped with a double cover $q_C : C \to D$ ramified at $\Delta_Y \cdot D$ and with a Galois involution $\zeta_C : C \to C$ induced by $\zeta_X$. Similarly, consider curves $nD$ and $nC = nD \times_Y X$ which are non-reduced for $n > 1$, and denote by $g_{nD}$ and $g_{nC}$ the genus of the smooth curves in the linear systems $|nD|$ and $|nC|$. We have for all $n > 0$, that

$$g_{nC} = 2g_{nD} - 1 - nD \cdot KY, \quad (2.19)$$

where,

$$g_{nD} = 1 + \frac{nD \cdot (nD + KY)}{2},$$

by the genus formula. Since $\chi(Y) = 1$ in all possible cases,

$$\chi(nD) = 1 + \frac{nD \cdot (nD - KY)}{2},$$

so

$$\chi(nD) = g_{nD} - nD \cdot KY.$$
Combining this with \((2.19)\) one has
\[
g_{nC} - g_{nD} = \chi(nD) - 1. \tag{2.20}
\]

Since the irregularity of \(Y\) vanishes, \(h^1(Y, O_Y) = 0\), Hilb\(_Y([nD])\) consists of a disjoint union of linear systems, for instance \([nD] = |nD|\).

Consider the natural (birational) anti-symplectic involution \(\hat{\xi}_X\) and, having picked \(J\) for which \((2.11)\) holds with respect to \(v_a\), the (birational) involution \(\lambda_{J,X} = \hat{\xi}_X \circ \xi_{J,X}\), which is symplectic. Pick the subvarieties \(N_{X,H}^{-}(v_a, nD)\) and \(P_{X,H}^{-}(v_a, J, nD)\) defined as the closure of the fixed locus of \(\hat{\xi}_X\) and \(\lambda_{J,X}\) respectively, given by those sheaves supported on curves parametrized by \(\eta_X[nD]\).

Recall that we denoted by \(|nD|^\ssl\) the locus of smooth curves in \(|nD|\) with smooth lift in \(|nC|\).

**Lemma 2.17** Let \(D\) be a smooth curve in a smooth surface \(Y\) and let \(q_X : X \to Y\) be a 2 : 1 covering by the smooth surface \(X\). Then, \(|nD|^\ssl\) is an open subset in \(|nD|\).

**Proof** By Bertini’s theorem, the locus of smooth curves, \(|nD|^\sm\), is an open subset in \(|nD|\). If the curve \(B'\) is smooth and does not intersect tangentially the branch locus \(\Delta\) of \(q_X\), the lifted curve \(A' = B' \times_Y X\) is smooth by construction. The latter is an open condition, so \(|nD|^\ssl\) is an open subset of \(|nD|\).

Hence \(\eta_X[nD]^\ssl\) is open in \(\eta_X[nD]\), and so are \(N_{X,H}^{-}(v_a, nD)|_{\eta_X[nD]^\ssl}\) in \(N_{X,H}^{-}(v_a, nD)\) and \(P_{X,H}^{-}(v_a, J, nD)|_{\eta_X[nD]^\ssl}\) in \(P_{X,H}^{-}(v_a, J, nD)\), respectively. In that case, \(h_X^{-1}(A)\) classifies line bundles over \(A\), being stable and smooth as points in \(M_{X,H}(v_a)\). Then, both \(\hat{\xi}_X\) and \(\lambda_{J,X}\) restrict to birational involutions over the locus of smooth curves in \(\eta_X[nD]\). This allow us to prove the following.

**Proposition 2.18** The projective subvariety \(N_{X,H}^{-}(v_a, nD) \subset M_{X,H}(v_a)\) is Lagrangian with respect to \(\Omega_M\), in particular,
\[
\dim \left( N_{X,H}^{-}(v_a, nD) \right) = n^2(g_C - 1) + 1.
\]

**Proof** We have seen that \(M_{X,H}(v_a)\) is smooth over \(\eta_X[nD]^\ssl\). Also, \(\hat{\xi}_X\) is a birational anti-symplectic involution there. It then follows that its fixed point locus \(N_{X,H}^{-}(v_a, nD)|_{\eta_X[nD]^\ssl}\) is smooth and Lagrangian. As it is an open subset of \(N_{X,H}^{-}(v_a, nD)\), it follows that the latter is a Lagrangian subvariety as well.

In view of Proposition 2.18, we refer to \(N_{X,H}^{-}(v_a, nD)\) as the *natural Lagrangian* subvariety of \(M_{X,H}(v_a)\) associated to \(\xi_X^{-}\).

We study now the fixed locus of \(\lambda_{J,X}\).

**Proposition 2.19** The projective subvariety \(P_{X,H}^{-}(v_a, J, nD) \subset M_{X,H}(v_a)\) has dimension
\[
\dim \left( P_{X,H}^{-}(v_a, J, nD) \right) = \chi(nD) + h^0(Y, nD) - 2 = 2h^0(Y, nD) - 2 + (h^2(Y, nD) - h^1(Y, nD)).
\]

**Proof** Following Remark 2.16 one has a description of the open subset \(P_{X,H}^{-}(v_a, J, nD)|_{\eta_X[nD]^\ssl}\) of \(P_{X,H}^{-}(v_a, J, nD)\) in terms of a Prymian fibration constructed with a projection of curves with genus \(g_{nC}\), to curves of genus \(g_{nD}\), having fibres of dimension \(g_{nC} - g_{nD}\). After (2.20), the statement follows.
Remark 2.20 Following Proposition 2.19, whenever
\[ h^1(Y, nD) = h^2(Y, nD), \]  
(2.21)
the dimension of \( P_{X, H}^{-}(v_a, J, nD) \) is \( 2\chi(nD) - 2 \) and the generic fibres of \( \mu \) are half-dimensional.

We next see that the Prymian fibration
\[ \mu : P_{X, H}^{-}(v_a, J, nD) \to q_X^*[nD] \]
ends \( P_{X, H}^{-}(v_a, J, nD) \) with the structure of an integrable system, that we call the **Prymian integrable system** associated to the anti-symplectic involution \( \zeta_X^{-} \). This construction has been considered by several authors \([1, 40]\) in the case of a primitive first Chern class, and later in \([41, 50]\) including the non-primitive case. For completion, we include a proof of it instead of just citing the previous articles.

**Theorem 2.21** Let \( X \) be a smooth K3 surface \( X \) equipped with an anti-symplectic involution \( \zeta_X^{-} \) giving the quotient \( Y = X/\zeta_X^{-} \). Pick a smooth curve \( D \subset Y \) of genus \( g_D \) and \( n \geq 1 \) such that (2.21) holds. Take \( C \) as in (2.18), a Mukai vector \( v_a \) as in (2.1) and \( J \in \text{Pic}(X) \) satisfying (2.11) with respect to \( v_a \). Then, one has that

(1) \( P_{X, H}^{-}(v_a, J, nD) \) is a projective variety of dimension \( 2\chi(nD) - 2 = n^2 D^2 - nD \cdot KY \);
(2) the smooth locus of \( P_{X, H}^{-}(v_a, J, nD) \) carries a holomorphic 2-form \( \Omega_{P} \) which is symplectic on a dense open subset where the fibres of \( \mu \) are Lagrangian; and
(3) the generic fibre of \( \mu \) is a a torsor over a smooth abelian variety of dimension \( \chi(nD) - 1 \)
\[ \frac{1}{2}(n^2 D^2 - nD \cdot KY), \]
being the Prym of a double cover of smooth curves.

**Proof** (1) is an immediate consequence of Proposition 2.19 and Remark 2.20.

Over the smooth locus of \( P_{X, H}^{-}(v_a, J, nD) \), its tangent space embeds into the tangent space of \( M_{X, H}(v_a) \) and one can define the holomorphic 2-form by restricting the Mukai form \( \Omega_{M} \) there. Also, \( \lambda_{J,X}^{-} \) restricts to a biregular symplectic involution over the locus of smooth curves in \( q_X^*[nD] \), where the symplectic form \( \Omega_{M} \) is well defined. There, the open subset of \( P_{X, H}^{-}(v_a, J, nD) \) given by the fixed locus of \( \lambda_{J,X}^{-} \) inherits a symplectic form which obviously coincides with the restriction of \( \Omega_{P} \) whenever we find ourselves in the smooth locus of \( P_{X, H}^{-}(v_a, J, nD) \). By [5], \( \Omega_{M} \) vanishes on the fibres of \( h_X \) as we have seen in Remark 2.4. As \( \Omega_{P} \) is obtained from restricting \( \Omega_{M} \), it follows that \( \Omega_{P} \) vanishes on the fibres of \( \mu \) over smooth curves. Equivalently, for every \( A \in q_X^*[nD] \) smooth, \( \mu^{-1}(A) \) is isotropic with respect to \( \Omega_{P} \). Recalling from Remark 2.20 that these fibres are half-dimensional, we conclude the proof of (2).

Finally (3) follows from Remark 2.15. \( \square \)

### 2.5 Ruled surfaces

Let us recall in this section some facts about ruled surfaces that will be used in Sect. 2.6.

Given a smooth projective curve \( C \) of genus \( g_C \) and a line bundle \( L \) on \( C \), consider the total space of our line bundle \( \text{Tot}(L) \) and the obvious projection \( \rho : \text{Tot}(L) \to C \). Set \( \ell := \text{deg}(L) \), assuming \( \ell \geq 0 \), and consider the projective compactification of \( \text{Tot}(L) \), namely the ruled surface with topological invariant \( \ell \),
\[ \mathbb{L} := \mathbb{P}(\mathcal{O}_C \oplus L), \]  
(2.22)
naturally equipped with the projection that we still denote by \( p : \mathbb{L} \rightarrow C \).

One has, of course,

\[
\text{Tot}(L) = \mathbb{L} - \{ \sigma_\infty \},
\]

where \( \sigma_\infty \equiv C \) is the curve in \( \mathbb{L} \) associated to the 0 section of \( O_C \). Let us briefly recall some properties of \( \mathbb{L} \) following [25, Sect. 2, Chap. V]. The curve \( \sigma_\infty \) has negative self-intersection, hence

\[
H^0(\mathbb{L}, O_{\mathbb{L}}(m \sigma_\infty)) = 1
\]

for \( m \geq 0 \). Along with \( \text{Pic}(C) \), \( \sigma_\infty \) generates the Picard group of \( \mathbb{L} \),

\[
\text{Pic}(\mathbb{L}) = (\mathbb{Z} \cdot \sigma_\infty) \oplus p^* \text{Pic}(C),
\]

so any divisor \( D \) on \( \mathbb{L} \) is numerically equivalent to \( b_1 \sigma_\infty + b_2 F \), where \( F \) denotes the fibre and \( \sigma_\infty \cdot \sigma_\infty = -\ell, \sigma_\infty \cdot F = 1 \) and \( F \cdot F = 0 \). A divisor is ample if and only if \( b_1 > 0 \) and \( b_2 > \ell b_1 \), and only exist irreducible curves in those classes with \( b_1 > 0 \) and \( b_2 \geq \ell b_1 \), with the exception of the infinity section \( \sigma_\infty \). The zero section of \( L \) defines \( \sigma_0 : C \rightarrow \mathbb{L} \) whose image is linearly equivalent to \( \sigma_\infty + \ell F \) and its normal bundle returns

\[
L \equiv O_{\mathbb{L}}(\sigma_0) |_{\sigma_0}.
\]

Observe that any divisor \( D \) with null intersection with \( \sigma_\infty \) is numerically equivalent to a multiple of \( \sigma_0 \). Also, for \( n \geq 0 \), it is linearly equivalent to \( n \sigma_0 \) if and only if \( O_{\mathbb{L}}(D) \) has a non-zero section, as by scaling the fibres of \( \text{Tot}(L) \), \( D \) deforms linearly to a multiple curve supported on \( \sigma_0 \). Hence, the linear system \( |n \sigma_0| \) has an open subset classified by linear deformations of the curve, which coincides with those curves not intersecting \( \sigma_\infty \). Linear deformations of curves are classified by the sections of the normal bundle restricted to the curve. Hence, one has the identification

\[
|n \sigma_0|_{\text{supp} \sigma_\infty = \emptyset} = H^0(n \sigma_0, O_{\mathbb{L}}(n \sigma_0)|_{n \sigma_0}).
\]

Let \( r \) denote the obvious projection of the multiple curve onto its reduced support and observe that \( r_* O_{\mathbb{L}}(n \sigma_0)|_{n \sigma_0} \) amounts to \( O_{\mathbb{L}}(n \sigma_0)|_{\sigma_0} \otimes r_* O_{n \sigma_0} \) by the projection formula. The structural sheaf \( O_{n \sigma_0} \cong O_C / I_C^2 \) decomposes, as an \( O_C \)-module, into \( O_C \oplus \bigoplus (I_C / I_C^2) \oplus \cdots \oplus (I_C / I_C^{n-1}) \). The intersection of \( \sigma_0 \) with the canonical divisor of the surface (see (2.27) below) is \( L^{-1} \) times the canonical bundle of the curve (i.e. the cotangent bundle of the curve). It then follows that the normal bundle is \( I_C / I_C^2 \cong \bigoplus (n \sigma_0)|_{\sigma_0} \cong L^{-1} \). Therefore,

\[
|n \sigma_0|_{\text{supp} \sigma_\infty = \emptyset} \cong \bigoplus_{i=1}^n H^0(C, L^\otimes i).
\]

The canonical line bundle of \( \mathbb{L} \) is

\[
K_{\mathbb{L}} \cong O_{\mathbb{L}}(-2 \sigma_\infty) \otimes p^* K_C \otimes p^* L^{-1},
\]

with canonical class

\[
[K_{\mathbb{L}}] = -2 \sigma_\infty + (2g - 2 - \ell) F.
\]

If \( \ell > 2g_C - 2 \) or \( L = K_C \), the inverse of the canonical \( K_{\mathbb{L}}^{-1} \cong \bigwedge^2 T \mathbb{L} \) always has non-zero sections and we can equip \( \mathbb{L} \) with a Poisson structure by picking a Poisson bi-vector given by a non-zero section \( \Theta_{\mathbb{L}} \in H^0(\mathbb{L}, K_{\mathbb{L}}^{-1}) \). In general, the Poisson structure of \( \mathbb{L} \) is not uniquely defined.
2.6 Twisted Higgs bundles and their spectral data

In our last preliminary section we introduce \((L\text{-twisted})\) Higgs bundles, whose spectral data are pure dimension 1 sheaves on ruled surfaces.

As before, \(C\) denotes a smooth projective curve and \(L\) a line bundle on it. An \(L\text{-twisted Higgs bundle}\) over \(C\) is a pair \((E, \varphi)\), where \(E\) is a holomorphic vector bundle on \(C\), and \(\varphi \in H^0(C, \text{End}(E) \otimes L)\) is a holomorphic section of the endomorphisms bundle, twisted by \(L\). When \(L = \mathcal{K}_C\) is the canonical bundle \(\mathcal{K}_C\) of the curve, we refer to \(\mathcal{K}_C\)-Higgs bundles, simply, as Higgs bundles. An \(L\text{-twisted Higgs bundle} (E, \varphi)\) is \((semi)stable\) if every \(\varphi\)-invariant subbundle \(F \subset E\) satisfies

\[
\frac{\deg F}{\text{rk} F} < ( \leq ) \frac{\deg E}{\text{rk} E}.
\]

A semistable \(L\text{-twisted Higgs bundle} (E, \varphi)\) is \(polystable\) if it is a direct sum of stable bundles \((E, \varphi) = \bigoplus (E_i, \varphi_i)\) (all with the same slope \(\deg E_i / \text{rk} E_i\)). It is possible to construct [46] the moduli space of rank \(n\) and degree \(d\) semistable (resp. stable) \(L\text{-twisted}\) Higgs bundles on \(C\) which we denote by \(\mathcal{M}_C^L(n, d)\) (resp. \(\mathcal{M}_C^L(n, d)\)\(^{\text{st}}\)). When \(L = \mathcal{K}_C\), we will write \(\mathcal{M}_C(n, d)\) (resp. \(\mathcal{M}_C(n, d)\)\(^{\text{st}}\)) for the moduli space of semistable (resp. stable) Higgs bundles whose construction follows from [30, 54], being a connected, normal and irreducible variety of dimension \(2n^2(g_C - 1) + 2\) [54]. When \(\deg(L) > \deg(K)\) or \(\deg(L) = \deg(K_C)\) but \(L^n \not\cong K_C^n\), the moduli space \(\mathcal{M}_C^L(n, d)\) is a quasi-projective variety of dimension \(2n^2 \deg(L) + 1\) [46].

Given a Higgs bundle \((E, \varphi)\), we see that \(\varphi : E \to E \otimes L\) determines uniquely an action of \(\text{Sym}^*(L^*)\) on \(E\), the Higgs bundle \((E, \varphi)\) can be seen as a \(\text{Sym}^*(L^*)\)-module. Since \(\tau\) is an affine morphism and \(\text{Sym}^*(L^*) \cong p_*\mathcal{O}_{\text{Tot}(L)}\), the push-forward under \(\tau\) provides an equivalence of categories between \(\text{Sym}^*(L^*)\)-modules (including Higgs bundles) and \(\mathcal{O}_{\text{Tot}(L)}\)-modules, called the spectral correspondence [8, 29] where the Higgs bundle \((E, \varphi)\) is sent to the pure dimension 1 sheaf \(\mathcal{E}\) on \(\text{Tot}(L)\) having rank 1 on each irreducible component of its support, called the spectral datum of \((E, \varphi)\), and defined by \(\ker(p^*\varphi - \tau)\), where \(\tau\) denotes the tautological section of the pullback bundle \(p^*L \to \text{Tot}(L)\). One can observe that the spectral datum \(\mathcal{E}\) associated to any Higgs bundle \((E, \varphi)\) is supported on a curve in the linear system \(|n\sigma_0|\), that we call the spectral curve of \((E, \varphi)\), and the restriction of \(\mathcal{E}\) to any irreducible component of the spectral curve has rank 1, and over the whole spectral curve is a sheaf with degree

\[
a = d + \delta,
\]

where we denote

\[
\delta := \frac{1}{2}(n^2 - n)\ell,
\]

denoting by \(\ell\) the topological invariant \(\ell\) of the ruled surface \(\mathbb{L}\). It then follows from (2.28) that the spectral datum of a \(L\text{-Higgs bundle}\) of rank \(n\) and degree \(d\) is a sheaf over \(\mathbb{L}\) with Mukai vector \(v_{d+\delta}\), where \(C = \sigma_0\) on its definition. Starting from such \(\mathcal{E}\), one obtains a Higgs bundle \((E, \varphi)\), by setting \(E = p_*\mathcal{E}\) and \(\varphi = p_*(1_\mathcal{E} \otimes \tau)\), where we have considered the morphism given by tensor product with the tautological bundle,

\[
1_\mathcal{E} \otimes \tau : \mathcal{E} \to \mathcal{E} \otimes p^*L.
\]

After [51] one has that the stability notions of a Higgs bundle and the corresponding spectral data coincide. Therefore, the Higgs moduli space is the open and dense subset of the moduli
space of sheaves on $\mathbb{L}$ with Mukai vector $v_{d+\delta}$, where we recall (2.28), given by those sheaves whose restriction to its associated spectral curve has rank 1 on each irreducible component of its support,

$$\mathcal{M}_C^L(n, d) \subset M_{\mathbb{L}, H_0(v_{d+\delta})}|_{\text{supp} \cap \sigma_\infty = \emptyset}.$$  \hspace{1cm} (2.30)

We have seen that $\mathbb{L}$ is equipped with a Poisson structure $\Theta_{\mathbb{L}}$ when $\ell > 2g_C - 2$ or $L = K_C$. In that case, one can define the Bottacin–Markman Poisson structure on $M_{\mathbb{L}, H_0(v_{d+\delta})}$, as we saw in Theorem 2.2, which restricts to a Poisson structure $\Theta_0$ on $\mathcal{M}_C^L(n, d)$. When $L = K_C$, $\Theta_0$ defines a symplectic form $\Omega_0$ on $\mathcal{M}_C(n, d)$. In the recent work [11], Biswas–Bottacin–Gomez showed that this restriction coincides (up to scaling) with that obtained by extending the canonical symplectic form defined on the cotangent of the moduli space of stable vector bundles. Over the locus of smooth spectral curves, such identification is already implicit in [29].

Since any divisor with null intersection with $\sigma_\infty$ is a multiple of $\sigma_0$, the restriction of (2.2) provides a morphism,

$$\mathcal{M}_C^L(n, d) \longrightarrow |n\sigma_0|_{\text{supp} \cap \sigma_\infty = \emptyset},$$  \hspace{1cm} (2.31)

known as the $L$-Hitchin fibration, which takes the form

$$\mathcal{M}_C^L(n, d) \longrightarrow B^L := \bigoplus_{i=1}^n H^0(C, L^{\otimes i}),$$  \hspace{1cm} (2.32)

after the identification 2.26. When $L = K_C$, this provides an algebraic completely integrable system [29] called the Hitchin system.

### 3 Involutions on Higgs moduli spaces from involutions on ruled surfaces

#### 3.1 Some involutions on ruled surfaces

We say that an involution on a variety equipped with a Poisson structure is Poisson if it preserves the Poisson structure under pull-back, and we call it anti-Poisson if the pull-back returns the inverse of the Poisson structure. In this section we study some involutions on $\mathbb{L}$ and their relation with the Poisson structure.

The first involution we consider is

$$\iota : \mathbb{L} \rightarrow \mathbb{L},$$  \hspace{1cm} (3.1)

defined by multiplying by $-1$ the fibres of $L$. Note that its fixed locus is the disjoint union of the zero and infinity sections,

$$\text{Fix} (\iota) = \sigma_\infty \sqcup \sigma_0.$$  \hspace{1cm} (3.2)

**Lemma 3.1** Every line bundle $J \in \text{Pic}(\mathbb{L})$ is preserved by the involution $\iota$, i.e.

$$\iota^* J \cong J.$$  

Suppose further that $\ell \geq 2g_C - 2$, then $\iota$ is an anti-Poisson involution with respect to any Poisson structure $\Theta_{\mathbb{L}}$ defined on the ruled surface.
Proof One naturally has that $i^*O_L(\sigma_{\infty}) \cong O_L(i(\sigma_{\infty}))$, hence $O_L(\sigma_{\infty})$ is preserved by $i$ after (3.2). As $i$ preserves the fibres of $p : L \to C$, every line bundle in $p^*\text{Pic}(C)$ is $i$-invariant and the first statement follows from (2.24). In particular, $K_L^{-1}$ is invariant.

As $i$ is an automorphism of $L$, we have that $dt \wedge dt$ provides an isomorphism between $i^*K_L^{-1}$ and $K_L^{-1}$. Since the eigenvalues of $dt$ are constantly 1 and $-1$, one has

$$dt \wedge dt = -1,$$

and all the sections of $K_L^{-1}$ are $i$-anti-invariant. This proves the second statement. □

Consider an involution on the base curve

$$\zeta_C : C \to C.$$

We say that a line bundle $L$ on $C$ is $\zeta_C$-invariant if there exists an isomorphism

$$f : \zeta_C^*L \cong L.$$

In this case, the action of $\zeta_C$ lifts, via $f$, to an action on $L$ that can be easily extended to its completion $\mathbb{L}_n$,

$$\zeta_f : \mathbb{L} \to \mathbb{L}.$$  (3.3)

Observe that the involution associated to $-f$ coincides with the composition $i \circ \zeta_f$. Denote the associated quotient map by

$$q_f : \mathbb{L} \to \mathbb{L}/\zeta_f.$$  (3.4)

Remark 3.2 Suppose that $q_C : C \to D := C/\zeta_C$ gives a smooth curve. By Kempf’s descent lemma [34], if $\zeta_f$ acts trivially on the fibres of the points fixed under $\zeta_C$, the line bundle $L$ descends to a line bundle $W$ on $D$. Let us denote the associated ruled surface by $\mathbb{W} := \mathbb{P}(O_D \oplus W)$. Under these conditions, the quotient is

$$\mathbb{L}/\zeta_f \cong \mathbb{W}.$$

Lemma 3.3 The involutions $i$ and $\zeta_f$ preserve cohomology class of every line bundle on $\mathbb{L}$. Therefore, every Mukai vector on $\mathbb{L}$ is $i$-invariant and $\zeta_f$-invariant.

Proof After Lemma 3.1 it is enough to prove the lemma for $\zeta_f$. Pull-back under $\zeta_f$ preserves the line bundle $O_L(\sigma_{\infty})$ and those line bundles in $p^*\text{Pic}(\mathbb{L})$ coming from $\zeta_C$-invariant line bundles. Even if not every line bundle in $p^*\text{Pic}(\mathbb{L})$ is preserved by $\zeta_f$, the involution preserves the class of a fibre, $F$. Then, the first statement follows from (2.24). The second follows immediately from the first. □

As $\zeta_C$ is an automorphism of $C$, $\partial \zeta_C$ provides a natural isomorphism between $K_C$ and $\zeta_C^*K_C$. Its compactification, $K_C = \mathbb{P}(O_C \oplus K_C)$ is a ruled surface of topological invariant $2g_C - 2$. Let us denote by

$$\zeta_+^{\uparrow} := \zeta_++\zeta_C,$$  (3.5)

and

$$\zeta_-^{\downarrow} := \zeta_+-\zeta_C.$$  (3.6)
After (2.27) the canonical bundle of $\mathbb{K}$ is

$$K_{\mathbb{K}} \cong \mathcal{O}_{\mathbb{K}}(-2\sigma_{\infty}),$$

which trivializes on $\text{Tot}(K_C)$, the complement of $\sigma_{\infty}$. Recalling (2.23), one has a unique (up to scaling) Poisson structure $\Theta_{\mathbb{K}}$ on $\mathbb{K}$ defining a canonical symplectic structure on $\text{Tot}(K_C)$.

**Lemma 3.4** The involution $\zeta_{\mathbb{K}}^+$ is Poisson while $\zeta_{\mathbb{K}}^-$ is anti-Poisson. Their restriction to $\text{Tot}(K_C)$ provide, respectively, a symplectic and an anti-symplectic involution.

Suppose that $q_C : C \to D := C/\zeta_C$ gives a smooth curve. Then, the quotient $K/\zeta_C$ is smooth.

**Proof** For the first statement, thanks to Lemma 3.1, it is enough to prove that $\zeta_{\mathbb{K}}^+$ is a Poisson involution as $\zeta_{\mathbb{K}}^-$ is the composition of $\zeta_{\mathbb{K}}^+$ with the anti-Poisson involution $\hat{\imath}$. Suppose first that $q_C$ is unramified. On an open subset associated to a trivializing subset of $K_C$, one can easily show that $d\zeta_{\mathbb{K}}^+$ acts like $\partial_{\zeta_C}$ along the curve and like $d\zeta_C$ along the fibres. Hence, $d\zeta_{\mathbb{K}}^+ \wedge d\zeta_{\mathbb{K}}^+ = d\zeta_C \wedge \partial_{\zeta_C} = 1$ and $\zeta_{\mathbb{K}}^+ = \zeta_{\partial_{\zeta_C}}$ is Poisson.

After Remark 3.2, the second statement follows from the observation that $\zeta_{\mathbb{K}}^-$ acts, over an open subset associated to a trivializing subset of $K_C$, as $d\zeta_C$ along the fibres and $-\partial_{\zeta_C}$ along the curve. On the points of $\text{Fix}(\zeta_C)$, one has that $d\zeta_C = -1$ so $-\partial_{\zeta_C} = 1$ and $\zeta_{\mathbb{K}}^-$ acts trivially on the fibres of the line bundle $K_C$ over $\text{Fix}(\zeta_C)$. The statement then follows from Remark 3.2.

### 3.2 Natural involutions under the spectral correspondence

Starting from the involutions $\imath$ and $\zeta_f$ on $L$ defined in Sect. 3.1, one can consider the natural involutions $\hat{\imath}$ and $\hat{\zeta}_f$ on the moduli space of sheaves $M_{L,H_0(\nu_{d+\delta})}$, as defined in Sect. 2.2. Also, we recall the involutions $\xi_{J,L}$ and $\lambda_{J,f} = \xi_{J,L} \circ \hat{\zeta}_f$ defined in full generality in Sect. 3.1. In this section we describe the counterpart of these involutions on the moduli space of $L$-Higgs bundles. We finish this section studying the particular case of $L = K_C$, associated to Higgs bundles (with no twisting). We find in this case that the previous involutions give rise to well known involutions that have been widely studied [3, 21, 22, 28].

**Lemma 3.5** Under the spectral correspondence, the involution $\hat{\imath}$ corresponds to the biregular involution

$$\hat{\imath} : M_L^L(n, d) \longrightarrow M_L^L(n, d)$$

$$(E, \varphi) \longmapsto (E, -\varphi).$$

If, given the involution $\zeta_C : C \to C$, $f$ provides an isomorphism between $L$ and $\zeta_C^*L$, then $\hat{\zeta}_f$ corresponds under the spectral correspondence to the involution

$$\hat{\zeta}_f : M_L^L(n, d) \longrightarrow M_L^L(n, d)$$

$$(E, \varphi) \longmapsto (\zeta_C^*E, (1_{\zeta_C} \otimes f) \circ \zeta_C^* \varphi),$$

which is biregular.

**Proof** We begin by observing that $\hat{\imath}$ and $\hat{\zeta}_f$ preserve (semi)stability of Higgs bundles, so both are regular morphisms within the moduli of Higgs bundles. Recall that the spectral correspondence is realized by taking the push-forward under $p : L \to C$, obtaining the
Higgs field by pushing-forward the morphism given by tensoring by the tautological section \( \tau, \varphi = p_*(1_\mathcal{E} \otimes \tau) \). In the first case, observe that \( p \circ i = p \), so \( \tilde{i}(E) = E \) and

\[
\tilde{i}(\varphi) = p_*(1_{(iL)}\mathcal{E} \otimes \tau) = p_*i_{L*}(1_{(iL)}\mathcal{E} \otimes \tau) = p_*(1_\mathcal{E} \otimes (-\tau)) = -\varphi,
\]

what concludes the proof of the first statement. For the second statement note that \( p \circ \xi_f = \xi_C \circ p \), which implies that \( \tilde{\xi}_f(E) = \xi^*_C E \). Then,

\[
\xi_C, \tilde{\xi}_f(\varphi) = \xi_{C,*}p_*(1_{\tilde{\xi}_C^*\mathcal{E}} \otimes \tau) = p_*\xi_{f,*}(1_{\tilde{\xi}_C^*\mathcal{E}} \otimes \tau) = p_*(1_\mathcal{E} \otimes \xi^*_f \tau)
\]

Note also that the tautological section satisfies that

\[
\tau = p^*f^{-1} \circ \xi^*_f \tau,
\]

so,

\[
\xi_{C,*} \tilde{\xi}_f(\varphi) = p_*(1_\mathcal{E} \otimes p^*f) \circ (1_\mathcal{E} \otimes \tau) = (1_\mathcal{E} \otimes f) \circ \varphi,
\]

from which the proof follows. \( \square \)

Let us denote the fixed point locus of \( \tilde{\xi}_f \) by

\[
\mathcal{N}_{L}^C(n, d) := \text{Fix}(\tilde{\xi}_f),
\]

which is, naturally, a closed subvariety of \( \mathcal{M}_{L}^C(n, d) \).

As we saw in Remark 3.2, Kempf descent lemma says that the line bundle \( L \) descends to a line bundle \( W \) on \( D \) if the action of \( f \) is trivial on the ramification locus of \( q_{\mathcal{C}} : C \to D \). In that case, the quotient \( \mathbb{L}/\xi_f \) is the ruled surface \( \mathbb{W} \). Observe, as well, that in that case \( L \cong q_{\mathcal{C}}^*W \).

**Lemma 3.6** Under the spectral correspondence, \( \mathcal{N}_{L}^C(n, d) \) embeds into \( \mathbb{N}_{L, H_0}^0(v_{d+\delta}, nD) \). Furthermore, the Hitchin fibration restricts there to

\[
\mathcal{N}_{L}^C(n, d) \to \bigoplus_{i=1}^{n} q_{\mathcal{C}}^*H^0(D, W^\otimes i). \tag{3.9}
\]

**Proof** We recall from (2.31) that the image of the support (Hitchin) morphism is contained in (an open subset of) a linear system \( |nD_0| \) in \( \mathbb{L} \). The fixed locus under the action of \( \tilde{\xi}_f \) on a linear system is disconnected and given by the union of the projectivization of the +1 and −1 eigenspaces of the associated space of sections. Observe that the component associated to +1 coincides with \( q_{\mathcal{C}}^+|nD| \), containing the open subset \( \bigoplus_{i=1}^{n} q_{\mathcal{C}}^*H^0(D, W^\otimes i) \). The rest of the proof follows easily from this observation. \( \square \)

One can consider the morphism

\[
\tilde{q}_{\mathcal{C}} : \mathcal{M}_{L}^W(n, d') \to \mathcal{N}_{L}^C(n, 2d') \subset \mathcal{M}_{C}^L(n, 2d') \quad (E, \varphi) \mapsto (q_{\mathcal{C}}^*, E, \xi_{f,*}^*\varphi), \tag{3.10}
\]

which corresponds to (2.6) under the spectral correspondence. If the spectral curves in \( \text{Tot}(L) \) are ramified over the spectral curves in \( \text{Tot}(W) \), then, Proposition 2.11 implies that \( \tilde{q}_{\mathcal{C}} \) is generically 1:1, being an isomorphism when we restrict to the locus of smooth curves.
over smooth curves (corresponding to the locus indexed by ssl). If, on the contrary, \( \zeta_2 \) is unramified, Proposition 2.11 says that \( \hat{q}_C \) factors through

\[
\mathcal{M}_D^W(n, d') \xrightarrow{\hat{q}_s} \mathcal{N}_C^L(n, 2d') \subset \mathcal{M}_C^L(n, 2d')
\]

where \( \mathbb{Z}_2 \) acts under tensorization by the 2-torsion line bundle on \( D \) associated to the unramified 2:1 cover \( q_C : C \to D \).

Suppose one can pick an \( \zeta_f \)-invariant line bundle \( J \) such that condition (2.11) holds with respect to \( v_{d+\delta} \). Observe that, after (2.24) and since \( \sigma_{\infty} \cdot \sigma_0 = 0 \), one has that the restriction of \( J \) to any curve \( A \) on \( |n\sigma_0| \) is

\[
J|_A \cong p^*J_C, \quad (3.11)
\]

for some \( J_C \in \text{Pic}(C) \).

We now consider the involution \( \xi_{J,L} \) and the composition \( \lambda_{J_0,f} = \xi_{J,L} \circ \hat{\zeta}_f \) which is another involution as \( \xi_{J,L} \) and \( \hat{\zeta}_f \) commute due to the \( \zeta_f \)-invariance of \( J \). Let us also describe the counterpart of these involutions in \( \mathcal{M}_C^L(n, d) \).

**Lemma 3.7** Under the spectral correspondence, the involution \( \xi_{J,L} \) corresponds to

\[
\tilde{\xi}_{J,C,L} : \mathcal{M}_C^L(n, d) \to \mathcal{M}_C^L(n, d)
\]

\[
(E, \varphi) \mapsto (E^* \otimes J_C L, \varphi'). \quad (3.12)
\]

Also, \( \lambda_{J,f} \) corresponds to

\[
\tilde{\lambda}_{J,C,f} : \mathcal{M}_C^L(n, d) \to \mathcal{M}_C^L(n, d)
\]

\[
(E, \varphi) \mapsto (\zeta_C^* E^* \otimes J_C L, (1_{\zeta_C} E \otimes f) \circ \zeta_C \varphi'), \quad (3.13)
\]

both being biregular involutions.

**Proof** After (2.13),

\[
\tilde{\xi}_{J,C,L}(E) = p_* \text{Hom}_A(F|_A, \mathcal{J}(A)|_A) \cong p_* \text{Hom}_A(F|_A, p^*(J_C L^n)),
\]

where, in this identification, we have made use of (2.25) and (3.11). Thanks to (2.27), we observe that the relative canonical sheaf \( \omega_{A/C} \) is \( p^nL^{n-1} \). Hence,

\[
\tilde{\xi}_{J,L}(E) \cong p_* \text{Hom}_A(F|_A, p^*(J_C L) \otimes \omega_{A/C}) = p_* \text{Hom}_A(F|_A, p^!(J_C L)) \cong E^* \otimes J_C L,
\]

and the first statement follows. The second statement is a direct consequence of the first and Lemma 3.5. \( \square \)

**Remark 3.8** Recall from Remark 2.12 that, choosing \( v_0 \) as Mukai vector and \( J_0 = O_L(-n\sigma_0) \), one gets \( \tilde{\xi}_{J_0,L} \) is the dualizing involution and

\[
\tilde{\xi}_{J_0,L} : \mathcal{M}_C^L(n, -\delta) \to \mathcal{M}_C^L(n, -\delta)
\]

\[
(E, \varphi) \mapsto (E^* \otimes L^{1-n}, \varphi').
\]
Remark 3.9 For $\delta$ as in (2.29), the pair of Mukai vector $v_\delta$ and line bundle $J_\delta = O_L(-\sigma_0)$ satisfies condition (2.11). In that case, one gets

$$\tilde{\xi}_{J_\delta,C,L} : \mathcal{M}_L^n(n, 0) \to \mathcal{M}_C^n(n, 0)$$

$$E, \varphi \mapsto (E^*, \varphi').$$

Let us denote the fixed point locus of $\tilde{\lambda}_{J_C,f}$ by

$$\mathcal{P}_C^n(n, d, J_C) := \text{Fix}(\tilde{\lambda}_{J_C,f}).$$

Observe that $\mathcal{N}_C^n(n, d)$ and $\mathcal{P}_C^n(n, d, J)$ are given by the restriction of $\mathcal{N}_{L,H_0}(v_{d+\delta})$ and $\mathcal{P}_{L,H_0}(v_{d+\delta}, J)$ in (2.30). Recalling the subvarieties $\mathcal{N}_{L,H_0}(v_{d+\delta}, nD)$ and $\mathcal{P}_{L,H_0}(v_{d+\delta}, J, nD)$ from (2.4) and (2.15), respectively, we study their relation with $\mathcal{N}_C^n(n, d)$ and $\mathcal{P}_C^n(n, d, J)$.

Lemma 3.10 Under the spectral correspondence, $\mathcal{P}_C^n(n, d, J_C)$ embeds into $\mathcal{P}_{L,H_0}(v_{d+\delta}, J, nD)$. Furthermore, the Hitchin fibration restricts to

$$\mathcal{P}^L_C(n, d, J_C) \to \bigoplus_{i=1}^n q_C^* H^0(D, W^\otimes i),$$

(3.14)

where $W$ is described in Remark 3.2.

Proof Mutatis mutandis, the proof follows as the proof of Lemma 3.6. □

Sometimes in the literature, in the particular case of $L = K_C$ and $f = \partial \zeta_C$, one abbreviates $(1_{\zeta_C^*E} \otimes \partial \zeta_C) \circ \zeta_C^* \varphi$ simply by $\zeta_C^* \varphi$. Therefore, the natural involutions associated to $\zeta_{K,C}^\pm = \zeta_{\pm \delta \zeta_C}$, as described in (3.5) and (3.6), would be expressed as

$$\tilde{\zeta}_{K,C}^\pm : \mathcal{M}_C(n, d) \to \mathcal{M}_C(n, d)$$

$$E, \varphi \mapsto (\zeta_C^* E, \pm \zeta_C^* \varphi),$$

(3.15)

under the spectral correspondence. This immediately implies that the counterpart of $\tilde{\lambda}_{J,C}^\pm = \tilde{\xi}_{J,C} \circ \tilde{\zeta}_{K,C}^\pm$ on the moduli space of Higgs bundles is

$$\tilde{\lambda}_{J,C,K}^\pm : \mathcal{M}_C(n, d) \to \mathcal{M}_C(n, d)$$

$$E, \varphi \mapsto (\zeta_C^* E^* \otimes J_C K_C, \pm \zeta_C^* \varphi').$$

(3.16)

The involutions $\tilde{\zeta}_{K,C}^\pm$, along with their fixed point locus, have been studied by Heller–Schaposnik [28] and García-Prada–Wilkin [22]. The involutions $\tilde{\lambda}_{J,C,K}^\pm$ are covered by the general study of Basu and García-Prada [3] of automorphisms of the moduli space of Higgs bundles. We combine their results in the following corollary, that could also be derived from Propositions 2.8 and 2.13.

Corollary 3.11 The involution $\tilde{\zeta}_{K,C}^+$ is symplectic and its fixed locus $\mathcal{N}_{C,K}^+(n, d)$ is a holomorphic symplectic subvariety of the Higgs moduli space $\mathcal{M}_C(n, d)$, while $\tilde{\zeta}_{K,C}^-$ is anti-symplectic, and defines a Lagrangian subvariety $\mathcal{N}_{C,K}^-(n, d)$ of $\mathcal{M}_C(n, d)$.

Similarly, $\tilde{\lambda}_{J,C,K}^+$ is an anti-symplectic involution and its fixed locus $\mathcal{P}_{C,K}^+(n, d, J_C)$ a Lagrangian subvariety of $\mathcal{M}_C(n, d)$, while $\tilde{\lambda}_{J,C,K}^-$ is symplectic and fixes a holomorphic symplectic subvariety $\mathcal{P}_{C,K}^-(n, d, J_C)$.
4 Non-linear degeneration of integrable systems

4.1 Extending the non-linear degeneration of Donagi, Ein and Lazarsfeld

In this section we generalize the Donagi, Ein and Lazarsfeld [18] non-linear degeneration of the moduli space of pure dimension 1 sheaves on a K3 surface into the moduli space of Higgs bundles over a curve inside the K3. Such degeneration was recently generalized to the case of curves in abelian surfaces by de Cataldo, Maulik and Shen [15], being a central step on their proof of the P=W conjecture over genus 2 curves. In this section we provide a degeneration of the moduli space of pure dimension 1 sheaves over any smooth surface, into the moduli space of $L$-Higgs bundles over a smooth subcurve, being $L$ the normal bundle of our curve inside the surface.

Take a smooth surface $S$ and consider a smooth curve $C \subset S$ inside it of genus $g_C$. From now on, let us denote its normal bundle by

$$L := O_S(C)|_C,$$

that coincides with $K_C \otimes (K_S|_C)^{-1}$ by adjunction. The projective completion of the normal cone is isomorphic to the ruled surface $\mathbb{L}$ defined in (2.22). This implies that the canonical divisor of $\mathbb{L}$ is

$$K_{\mathbb{L}} = -2\sigma_\infty + p^*K_C - p^*O_S(C)|_C.$$

By the genus formula, and recalling that $\sigma_0$ does not intersect with $\sigma_\infty$, we get that

$$\sigma_0^2 = \ell = C^2$$

and

$$\sigma_0 \cdot K_{\mathbb{L}} = C \cdot K_S.$$

Hence, the linear systems $|nC|$ and $|n\sigma_0|$ contain generic curves of equal genus,

$$g_{nC} = g_{n\sigma_0},$$

and the topological invariants of the ideal sheaves describing these curves coincide.

Take the trivial family $S \times \mathbb{P}^1$ and consider the blow-up at $C \times \{0\}$,

$$\mathcal{S} := \text{Blow}_{C \times \{0\}} \left( S \times \mathbb{P}^1 \right),$$

which is non-singular as we are blowing-up a non-singular subvariety. Composing the structural morphism of the blow-up with the projection $S \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$, one gets the structural morphism

$$\pi : \mathcal{S} \rightarrow \mathbb{P}^1.$$

This map is flat and its fibres outside $0 \in \mathbb{P}^1$ are not affected by our construction, so for the generic fibre, at $t \neq 0$, we have

$$\mathcal{S}_t \cong S.$$

Meanwhile, the central fibre, at $t = 0$, is the union

$$\mathcal{S}_0 = \mathbb{L} \cup S,$$
where both irreducible components meet transversely at the image of infinity section $\sigma_\infty \subset \mathbb{L}$ identified with the curve $C \subset S$. Note that $\overline{S}_0$ is a complete intersection subvariety. One can also consider the open subset

$$S := \overline{S} - (S \times \{0\}), \quad (4.6)$$

and observe that the restriction there of the structural morphism gives, again, a flat family of surfaces over $\mathbb{P}^1$, but now the central fibre $S_{t=0} = \text{Tot}(L)$ is not projective.

Denote by $p_S$ the composition of the structural morphism of the blow-up with the projection of $S \times \mathbb{P}^1$ onto $S$. Being $\overline{S}$ projective, one may choose a relative polarization

$$H = p_S^* H \otimes \mathcal{O}_S(-\mathbb{L}). \quad (4.7)$$

For all $t \neq 0$ in $\mathbb{P}^1$, one trivially has that $H|_{\overline{S}_t} = H$ while $H_0 := H|_{\mathbb{L}}$ corresponds to $p^*(H|_C) \otimes \mathcal{O}_L(\sigma_\infty)$, since $\mathcal{O}_S(\mathbb{L} \cup S)|_L \cong \mathcal{O}_L$ as $\mathbb{L} \cup S$ is a fibre, so $\mathcal{O}_S(-\mathbb{L})|_L \cong \mathcal{O}_S(S)|_L \cong \mathcal{O}_L(\sigma_\infty)$.

Since $C \times \{0\}$ is a Cartier divisor of $C \times \mathbb{P}^1$, one has that

$$C := \text{Blow}_{C \times \{0\}} \left(C \times \mathbb{P}^1\right) \cong C \times \mathbb{P}^1 \quad (4.8)$$

embeds naturally into $\overline{S}$. One can consider the fibration $C \rightarrow \mathbb{P}^1$, whose generic fibres, at $t \neq 0$, are naturally identified with $C$,

$$C_t \cong C \subset S = \overline{S}_t.$$  

Furthermore, the restriction of $C$ to $t = 0$ coincides with the image of the zero section,

$$C_0 \cong \sigma_0 \cong C,$$

inside $\mathbb{L} \subset \overline{S}_0$. Note that the restriction to every point of $\mathbb{P}^1$ gives $C_t \in H^2(\overline{S}_t, \mathbb{Z})$. Using $C$, consider the Mukai vector

$$v_a = (0, [nC], a - n^2 \ell). \quad (4.9)$$

Let us consider the relative Hilbert scheme parametrizing ideal sheaves of curves in the fibres of $\overline{S} \rightarrow \mathbb{P}^1$ with first Chern class $[nC_t]$ and let us denote by $[nC]$ the connected component containing the family of curves $nC$. Let us consider, inside it, the open subset given by those curves $A \subset \overline{S}_t$ contained in the $S \subset \overline{S}$,

$$[nC]|_{A \subset S} := [nC]|_{A \subset S}. \quad (4.10)$$

Take a point of $[nC]|_S$ representing the ideal sheaf $\mathcal{I}_A \hookrightarrow \mathcal{O}_S$ of a curve $A \subset S_t$. We see that $\mathcal{I}_A$ is of rank 1 and pure dimension 2 over a smooth surface $S_t$ which either is projective (for $t \neq 0$) or an affine subset of a smooth projective surface $\mathbb{L}$ (for $t = 0$) where $\mathcal{I}_A$ extends naturally. It follows that the determinant in cohomology is well define and one can consider the determinant morphism

$$\det_S : [nC]|_S \rightarrow \text{Jac}_{\mathbb{P}^1}(\overline{S}), \quad (4.11)$$

to the relative Jacobian of $\overline{S}$, classifying line bundles on the fibres $\overline{S}_t$ of the family of surfaces.

For $t \neq 0$, the fibre of $(4.11)$ over $L^\otimes n \in \text{Pic}(S)$ corresponds to the linear system $|L^\otimes n|$ in $S$. We recall once again that every irreducible projective curve contained in $\text{Tot}(L)$ can be deformed linearly to a multiple curve supported on $\sigma_0$. Then, the image of $[nC]|_S$ under
(4.11) is $O_{S_0}(n\sigma_0)$ and $\{nC\}_{|t=0}$ is the locus of $|n\sigma_0|$ inside $\mathbb{L}$ which does not intersect $\sigma_\infty$. After (2.26), this is
\[
\{nC\}_{|t=0} \cong \bigoplus_{i=1}^n H^0(C, L^{\otimes i}),
\]
coinciding with the Hitchin base in (2.32).

Following Simpson [53, Theorem 1.21] (see [32, Theorem 4.3.7] for a specific treatment of the relative case), let us consider the moduli space $M_{S/\mathbb{P}^1, H}(v_a)$ of relatively pure dimension 1 $H$-semistable relative sheaves over the $\mathbb{P}^1$ scheme $\mathbb{S}$ having Mukai vector $v_a$. By construction, $M_{S/\mathbb{P}^1, H}(v_a)$ comes equipped with the structural flat morphism to $\mathbb{P}^1$, which trivializes over $\mathbb{P}^1 - \{0\}$ having generic fibres equal to $M_{S,H}(v_a)$, and whose central fibre over $t = 0$ is $M_{L\cup S,H_0}(v_a)$. By specifying the relative (Fitting) support, one has the following surjective morphism
\[
h : M_{S/\mathbb{P}^1, H}(v_a) \longrightarrow \text{Hilb}_{\mathbb{P}^1}(\{nC\})
\]
\[
\mathcal{F} \longmapsto \text{supp}(\mathcal{F}),
\]
commuting with both structural morphisms. Consider the restriction $M_{S/\mathbb{P}^1, H}(v_a)|_{\{nC\}_S}$ having generic fibres $M_{S,H}(v_a, nC)$ over $t \neq 0$ and central fibre $M_{L,H_0}(v_a)|_{\text{supp}\sigma_\infty = \emptyset}$ at $t = 0$. For the description of the central fibre, note that a sheaf in $S_0 = L \cup S$ whose support fits in $\text{Tot}(L) \subset L$ is $H_0$-semistable (resp. $H_0$-stable) as a sheaf in $L$ if and only if it is $H_0$-semistable (resp. $H_0$-stable) as a sheaf in the whole $\mathbb{S}_0$. Recalling (2.30), we define the open subset
\[
M_{S/\mathbb{P}^1, H}(v_a, nC) \subset M_{S/\mathbb{P}^1, H}(v_a)|_{\{nC\}_S},
\]
given by the complement of those sheaves at $t = 0$ having rank different from 1 on some irreducible component of their support.

We summarize all of the above in the following theorem. This is a generalization to arbitrary smooth surfaces of the non–linear deformation that appeared first in [18] for K3 surfaces and in [15] for abelian surfaces.

**Theorem 4.1** [18] Consider a smooth surface $S$ and a genus $g \geq 2$ smooth projective curve $C$ inside $S$. Denote the normal bundle of $C$ in $S$ by $L$. Choose a positive integer $n$ and a Mukai vector $v_a$ as in (2.1) constructed out of $n$ and $C$. One can construct $M_{S/\mathbb{P}^1, H}(v_a, nC)$ and $\{nC\}_S$ flat over $\mathbb{P}^1$, and a morphism between them commuting with the structural morphisms,
\[
M_{S/\mathbb{P}^1, H}(v_a, nC) \longrightarrow \{nC\}_S
\]
\[
\mathbb{P}^1.
\]

The horizontal arrow is a surjective fibration, trivial over $\mathbb{P}^1 - \{0\}$, and the generic fibre over $t \neq 0$ is the Le Poitier morphism (2.2),
\[
M_{S,H}(v_a, nC) \rightarrow \{nC\},
\]
while the central fibre at $t = 0$ gives the Hitchin morphism (2.32),
\[ M_C^i(n, d) \rightarrow \bigoplus_{i=1}^n H^0(C, L^{\otimes i}), \]

where \( d = a - \delta \) and \( \delta = \frac{1}{2}(n^2 - n)e. \)

**Remark 4.2** Since the normal bundle \( L \) of a curve \( C \) in a del Pezzo surface has \( \text{deg}(L) > \text{deg}(K_C) \), we observe that the degeneration given in Theorem 4.1 connects the two moduli spaces considered in the work of Maulik and Shen [42], and for which cohomological \( \chi \)-independence is proven.

### 4.2 Deforming the symplectic structure

In this section we restrict ourselves to the case where \( S \) is a symplectic surface. In this context we study the symplectic structure on the moduli spaces involved in the construction obtained in Sect. 4.1, showing that it provides a deformation of symplectic varieties. Although the results in this section seem to be known to experts, we have not be able to find a good reference for them in the literature.

Since the canonical bundle of \( S \) is trivial, one gets, by adjunction, that the normal bundle of \( C \subset S \) is its canonical bundle \( K \). Recall then that the family of surfaces \( \bar{S} \rightarrow \mathbb{P}^1 \) has central fibre \( S_0 = \mathbb{K} \cup S \) meeting at the identification of \( \sigma_\infty \subset \mathbb{K} \) with \( C \subset S \). We start by studying \( K_{S/\mathbb{P}^1} \) and \( K_{S_0} \).

**Proposition 4.3** Let \( S \) be a symplectic surface and pick \( C \subset S \) smooth curve of genus \( g \geq 2. \) One has the following

1. \( K_{S_0} \) is the line bundle given by \( \mathcal{O}_\mathbb{K}(-\sigma_\infty) \) and \( \mathcal{O}_S(C) \) identified along \( \sigma_\infty \cong C \) by a natural isomorphism between \( K \cong \mathcal{O}_\mathbb{K}(-\sigma_\infty)|_{\sigma_\infty} \) and \( K \cong \mathcal{O}_S(C)|_{C}; \)
2. \( H^0(S_0, K_{S_0}) = \mathbb{C} \) and every non-trivial section of \( K_{S_0} \) vanishes on \( \mathbb{K} \) and only on \( \mathbb{K}; \)
3. \( H^0(S_0, \omega_{S_0}^{-1}) = \mathbb{C} \) and every non-trivial section of \( \omega_{S_0}^{-1} \) vanishes on \( S \) and only on \( S; \)
4. \( \pi_* K_{S/\mathbb{P}^1} \equiv \mathcal{O}_{\mathbb{P}^1}(1); \)
5. \( \pi_* \omega_{S/\mathbb{P}^1}^{-1} \equiv \mathcal{O}_{\mathbb{P}^1}(-1). \)

**Proof** We first observe that \( \bar{S} \) is constructed by blowing-up a smooth subvariety of \( S \times \mathbb{P}^1, \) which is also smooth. Hence \( \bar{S} \) is smooth and has canonical bundle \( K_{\bar{S}} = r^* K_{S_x \times \mathbb{P}^1}(\mathbb{K}), \) where we denote by \( r \) the structural morphism of the blow-up and we recall that \( \mathbb{K} \) is the exceptional divisor. One can deduce that the singular central fibre \( S_0 = \mathbb{K} \cup S, \) where both components meet transversally, is a complete intersection variety, so it is Gorenstein and its dualizing sheaf \( K_{S_0} \) is a line bundle. This implies that the relative dualizing sheaf \( K_{S/\mathbb{P}^1} \) is a line bundle too, indeed it is \( \mathcal{O}_S(\mathbb{K}), \) as \( K_S \) is trivial.

Since \( S_0 \) is a complete intersection divisor in \( \bar{S}, \) one has that \( K_{S_0} = K_{S/\mathbb{P}^1}(\mathbb{K} + S)|_{S_0}, \) where we abuse of notation by denoting the total transform of \( S \times \{0\} \) simply by \( S. \) Then,

\[ K_{S_0}|_{\mathbb{K}} \cong \mathcal{O}_\mathbb{K}(2\mathbb{K})|_{\mathbb{K}} \otimes \mathcal{O}_S(S)|_{\mathbb{K}} \cong \mathcal{O}_\mathbb{K}(-2\sigma_\infty) \otimes \mathcal{O}_\mathbb{K}(\sigma_\infty) \cong \mathcal{O}_\mathbb{K}(-\sigma_\infty), \]

where we recall that \( \mathcal{O}_S(-\mathbb{K})|_{\mathbb{K}} \) is the linearization \( \mathcal{O}_\mathbb{K}(1) \cong \mathcal{O}_\mathbb{K}(\sigma_\infty). \) Similarly, the restriction of \( K_{S_0} \) to \( S \) gives

\[ K_{S_0}|_S \cong \mathcal{O}_S(2\mathbb{K})|_S \otimes \mathcal{O}_S(S)|_S \cong \mathcal{O}_S(2C) \otimes \mathcal{O}_S(-C) \cong \mathcal{O}_S(C), \]

where we recall that \( K_S \cong \mathcal{O}_S(\mathbb{K} + S)|_S \cong \mathcal{O}_\mathbb{K}(\mathbb{K})|_S \otimes \mathcal{O}_S(S)|_S \) is trivial, hence \( \mathcal{O}_S(S)|_S \cong (\mathcal{O}_S(\mathbb{K})|_S)^{-1} \). Recalling that \( K_{\mathbb{K}} = \mathcal{O}_\mathbb{K}(-2\sigma_\infty) \) and \( K_S \cong \mathcal{O}_S, \) applying adjunction, one
gets that $\mathcal{O}_K(-\sigma_\infty)|_{\sigma_\infty}$ and $\mathcal{O}_S(C)|_C$ both being isomorphic to $K$, the canonical bundle of $\sigma_\infty \cong C$, so the restriction of the line bundle $K_{S_0}$ provides a natural identification. This finishes the proof of (1).

Since $\mathcal{O}_K(-\sigma_\infty)$ has no non-zero sections, every non-zero section of $K_{S_0}$ vanishes completely on $\mathbb{K}$, hence it is given by a section of $\mathcal{O}_S(C)$ vanishing identically at $C$. The set of those sections determines a 1-dimensional subspace of $H^0(S, \mathcal{O}_S(C))$ and (2) follows.

We prove (3) by observing that $h^0(\mathbb{K}, \mathcal{O}_K(\sigma_\infty)) = 1$ and that $h^0(S, \mathcal{O}_S(-C)) = 0$. It follows that every non-zero section of $K_{S_0}^{-1}$ vanishes on $S$, so it is given by a section of $\mathcal{O}_K(\sigma_\infty)$ which vanishes identically at $\sigma_\infty$.

Recall that $\mathbf{S}|_{\mathbb{P}^1 - \{0\}}$ is the trivial fibration $S \times (\mathbb{P}^1 - \{0\})$, so the restriction there of $K_{S/\mathbb{P}^1}$ is the trivial line bundle. After (2) and (3) and upper semicontinuity of cohomology, one has that $\pi_*K_{S/\mathbb{P}^1}$ and $\pi_*\mathcal{O}_S^{-1}|_{\mathbb{P}^1}$ are line bundles over $\mathbb{P}^1$, inverse to each other. Note that $K_{S/\mathbb{P}^1} \cong \mathcal{O}_S(\mathbb{K})$ comes naturally equipped with a section. Furthermore, $h^0(\mathcal{O}_S(\mathbb{K})) = 1$ by the properties of blow-up. It follows that $\pi_*K_{S/\mathbb{P}^1}$ has a single non-zero section (up to scaling), so (4) follows.

Finally, (5) follows from (4).

In view of (2) and (3) of Proposition 4.3 we shall first construct a Poisson structure on $M_{S/\mathbb{A}^1, H}(v_a)$ and, then, derive the symplectic structure on $M_{S/\mathbb{A}^1, H}(v_a, nC)$. Recall that $\ell = 2g_C - 2$ in this case, so we fix $d = a - (n^2 - n)(g_C - 1)$.

**Theorem 4.4** Consider a smooth symplectic surface $S$ and a genus $g \geq 2$ smooth projective curve $C$ inside $S$. Then, there exists a relative Poisson structure $\Theta$ on $M_{S/\mathbb{A}^1, H}(v_a) \to \mathbb{A}^1$ which defines a relative symplectic form $\Omega$ on $M_{S/\mathbb{A}^1, H}(v_a, nC) \to \mathbb{A}^1$ coinciding (up to scaling) with the Mukai form $\Omega$ over the generic fibre $M_{S, H}(v_a)$ over $t \neq 0$, and, on the central fibre $M_C(n, d)$ at $t = 0$, with $\Omega_0$ obtained by extending the canonical symplectic form on the cotangent of the moduli space of stable vector bundles.

**Proof** Starting from (5) of Proposition 4.3, we pick $\mathbb{A}^1 \subseteq \mathbb{P}^1$ a trivialization of $\pi_*K_{S/\mathbb{P}^1}^{-1}$ containing $0 \in \mathbb{P}^1$ and choose a section $\vartheta \in H^0(\mathbb{A}^1, \pi_*K_{S/\mathbb{P}^1}^{-1})$. Following Theorem 2.2 one can construct a relative Bottacin–Markman Poisson structure $\Theta$ on $M_{S/\mathbb{A}^1, H}(v_a)$,

$$
\text{Ext}^1_{\mathcal{O}_{S_t}}(\mathcal{E}, \mathcal{E} \otimes K_{S_t}) \wedge \text{Ext}^1_{\mathcal{O}_{S_t}}(\mathcal{E}, \mathcal{E} \otimes K_{S_t}) \overset{o}{\to} \text{Ext}^2_{\mathcal{O}_{S_t}}(\mathcal{E}, \mathcal{E} \otimes K^2_{S_t}) \overset{tr}{\to} H^2(\mathbf{S}_t, K^2_{S_t}) \overset{(-, d)}{\to} H^2(\mathbf{S}_t, K_{S_t}) \cong \mathbb{C}.
$$

(4.16)

Thanks to (3) of Proposition 4.3, we can check that the restriction of $\Theta$ to the subset $M_C(n, d)$ of the central fibre is non-degenerate, hence defines a symplectic form there, as the tangent and the cotangent spaces to the moduli can be identified thanks to a trivialization of $K_{S_C} = \mathcal{O}_{K_C}(-2\sigma_C)$ over $\text{Tot}(K_C) = \mathbb{K}_C - \{\sigma_C\}$. Furthermore, the canonical bundle is trivial over the generic fibres, $K_S \cong \mathcal{O}_S$, so in this case the cotangent space of $M_{S/\mathbb{A}^1, H}(v_a)|_t = M^H_S(v_a)$ is identified with the tangent space. Then, the Poisson form $\Theta_t$ over $t \neq 0$ is non-degenerate and defines a symplectic form which coincides with the Mukai form up to scaling. Hence, the restriction of $\Theta$ to $M_{S/\mathbb{A}^1, H}(v_a, nC)$ defines a relative symplectic form $\Omega$. \[\square\]
4.3 Degenerating the fixed locus of involutions

In this section, we study the behaviour under the Donagi–Ein–Lazarsfeld degeneration of the involutions considered in Sections 2.2 and 2.3. By doing so, we generalize to the case of an arbitrary smooth surface equipped with an involution, the construction of Sawon and Shen [50] who treated the case described in Sect. 4.4.4.

Consider a smooth surface $S$ equipped with an arbitrary involution $\zeta_S : S \to S$. Pick a smooth curve $C$ preserved by $\zeta_S$ and denote by $\zeta_C : C \to C$ the restriction of the involution to it. As in (4.1), denote the normal bundle by $L = O_S(C)\vert_C$, and note that $d\zeta_S : \zeta_S^*TS \to TS$ sends $\zeta_C^*TC$ to $TC$ inducing the isomorphism

$$f_0 := [d\zeta_S]_TC : \zeta_C^*L = \zeta_S^*(TS/TC) \to L = TS/TC.$$  \hspace{1cm} (4.17)

Let us also recall from (3.3) the associated involution $\zeta_{f_0} : L \to L$.

One can provide an explicit description of $\zeta_{f_0}$ in the particular case of symplectic surfaces.

**Proposition 4.5** Suppose that $S$ is a symplectic surface and $\zeta_S^+$ a symplectic (resp. $\zeta_S^-$ an antisymplectic) involution on it. Pick a smooth curve $C$ preserved by $\zeta_C$ and denote by $L$ its normal bundle inside $S$. Then $L = KC$ and $\zeta_{f_0} = \zeta_K^+$, as defined in (3.5) (resp. $\zeta_{f_0} = \zeta_K^-$, as defined in (3.6)).

**Proof** By adjunction formula and the triviality of the canonical bundle $K_S \cong O_S$ of a symplectic surface, one gets $L = K_C$ in this case.

Note that the isomorphism $\partial\zeta_C : \zeta_C^*K_C \to K_C$ that lifts the action of $\zeta_C$ to the canonical bundle $K_C$, is the inverse of $d\zeta_C$.

Since $\zeta_S^+$ is symplectic (resp. $\zeta_S^-$ is antisymplectic), $d\zeta_S^+ : \zeta_S^*TS \to TS$ has eigenvalues whose product is 1 (resp. $-1$). Then, over $C$, $d\zeta_S$ has eigenvalues $d\zeta_C$ and $f_0 = \partial\zeta_C$ (resp. $f_0 = -\partial\zeta_C$). This completes the proof after recalling (3.5) and (3.6). \hspace{1cm} □

We defined $\bar{S}$ in (4.4) and $S$ in (4.6), having generic fibres $\bar{S}_t = S_t = S$ and central fibres $\bar{S}_0 = S \cup \mathbb{P}^1$ and $S_0 = \text{Tot}(L)$. As our involution $\zeta_S$ preserves $C$, $\zeta_S \times 1_{\mathbb{P}^1}$ lifts to the blow-up $\bar{S}$ giving another involution $\zeta_{\bar{S}} : \bar{S} \to \bar{S}$ which preserves $C$ as defined in (4.8). Furthermore, at $t = 0$, $\zeta_{\bar{S}}$ preserves the irreducible component $S \subset \bar{S}_0$ and we denote by $\zeta_S : S \to S$ the restriction of this involution to $S$. One has commutativity with the structural morphisms,
Given \( C \subset S \), we construct \( C \) as in (4.8) and we define \( \{ nC \}_S \) as in (4.10). Consider a Mukai vector \( v_a \) as in (4.9). Pick a polarization \( H \) on \( S \), inducing \( H \) as described in (4.7), and consider \( \mathcal{M}_{S/P^1}(v_a, nC) \) as defined in (4.14), classifying \( H \)-semistable relative sheaves on \( \bar{S} \) with Mukai vector \( v_a \), and whose support is parametrized by \( \{ nC \}_S \). In particular, their support restricts to \( S \), so, taking pull-back under \( \zeta_S \), one obtains a birational involution

\[
\hat{\zeta}_S : \mathcal{M}_{S/P^1}(v_a, nC) \to \mathcal{M}_{S/P^1}(v_a, nC)
\]

compatible with the structural morphism to \( \mathbb{P}^1 \).

Consider an \( \zeta_S \)-invariant line bundle \( J \to S \) which satisfies condition (2.11) with respect to \( v_a \). Starting from such \( J \), we construct a line bundle on \( S \)

\[
J = p_\mathcal{S}^* J,
\]

where \( p_\mathcal{S} \) is obtained from inclusion \( S \hookrightarrow \bar{S} \), the structural morphism of the blow-up \( \bar{S} \to S \times \mathbb{P}^1 \) and the projection of \( S \times \mathbb{P}^1 \) onto \( S \). Note as well that, over \( t \neq 0 \), it restricts to the original line bundle,

\[
J|_t \cong J,
\]

while, at the central fibre \( S_0 = \text{Tot}(L) \), one gets

\[
J|_0 \cong p^* J_C,
\]

where \( p \) is the structural projection \( \text{Tot}(L) \to C \), and

\[
J_C := J|_C.
\]

We then observe that (2.11) holds fibrewise for \( J \) and \( v_a \).

The following is inspired by [50, Lemma 4].

**Proposition-definition 4.7** There exists a biregular involution

\[
\xi_{J,S} : \mathcal{M}_{S/P^1}(v_a, nC) \to \mathcal{M}_{S/P^1}(v_a, nC).
\]

given by sending a sheaf \( \mathcal{F} \) supported on a curve inside \( S|_t \) to \( \mathcal{E}xt^2_{S/P^1}(\mathcal{F}, J(-L))|_S \).

The restriction of \( \xi_{J,S} \) to a generic fibre over \( t \neq 0 \) is \( \xi_{J,S} \), while at the central fibre on \( t = 0 \) it restricts to \( \xi_{J_C,L} \).

**Proof** Recall that \( \bar{S} \) is smooth with canonical bundle \( K_{\bar{S}} = r^* K_{S \times \mathbb{P}^1}(L) \) Consider the embedding \( J_t : \bar{S}_t \hookrightarrow \bar{S} \). The canonical bundle for this embedding is

\[
\omega_{J_t} = \mathcal{O}_{\bar{S}(\mathbb{L})}|_{\bar{S}_t},
\]

being trivial over a generic fibre over \( t \neq 0 \). Then, given a sheaf \( \mathcal{F} \) supported on \( S_t \) with \( t \neq 0 \), Grothendieck-Verdier duality implies that

\[
\mathcal{E}xt^2_{S/P^1}(J_t, \ast \mathcal{F}, J) \cong \mathcal{E}xt^1_{\bar{S}_t}(\mathcal{F}, J|_t),
\]

so \( \xi_{J,S} \) restricts to \( \xi_{J,S} \) there.

Also, observe that for a sheaf supported on the open subset of the central fibre \( \text{Tot}(L) = S_0 \subset \bar{S}_0 \), one also has that

\[
\mathcal{E}xt^2_{S/P^1}(J_0, \ast \mathcal{F}, J|_0) \cong \mathcal{E}xt^1_{S_0}(\mathcal{F}, J|_0) \cong \mathcal{E}xt^1_{S_0}(\mathcal{F}, p^* J_C).
\]
so \( \xi_{J,S} \) restricts to \( \xi_{J_0,L} \) which further corresponds to \( \hat{\xi}_{J_C,L} \) after Lemma 3.5.

Finally, since \( \xi_{J,S} \) and \( \hat{\xi}_{J_C,L} \) preserves stability, so does \( \xi_{J,S} \) fibrewise, defining a biregular involution of the moduli space. \( \square \)

Observe that \( J \) is \( \zeta_S \)-invariant. Then, \( \hat{\zeta}_S \) and \( \xi_{J,S} \) commute and one can consider the composition

\[
\lambda_{J,S} := \hat{\zeta}_S \circ \xi_{J,S} : \mathbb{M}_{S/\mathbb{P}^1, H}(v_a, nC) \rightarrow \mathbb{M}_{S/\mathbb{P}^1, H}(v_a, nC).
\]

**Lemma 4.8** With the notation as above, \( \lambda_{J,S} \) coincides with \( \hat{\lambda}_{J_C,f_0} \) over \( t = 0 \), and with \( \lambda_{J,S} \) over a generic fibre at \( t \neq 0 \).

**Proof** The lemma is a consequence of Proposition–definition 4.7 and the fibrewise description of \( J \) and \( v_a \). \( \square \)

We now study the closure of their fixed points and their structure morphism.

**Proposition 4.9** The restriction of the structural morphism to \( \mathbb{A}^1 \) of the closed subvarieties

\[
\overline{\text{Fix}(\hat{\zeta}_S)}, \overline{\text{Fix}(\xi_{J,S})} \text{ and } \overline{\text{Fix}(\lambda_{J,S})}
\]

is flat.

**Proof** Since \( \mathbb{M}_{S/\mathbb{P}^1, H}(v_a, nC) \) is flat over \( \mathbb{P}^1 \), which is an irreducible and one dimensional base scheme, and the structural morphism trivializes (hence, it is flat) over \( \mathbb{P}^1 \setminus \{0\} \), one only needs to check that the fibre at \( t = 0 \) arises as the closure of \( \text{Fix}(\hat{\zeta}_S)|_{\mathbb{P}^1 \setminus \{0\}}, \text{Fix}(\xi_{J,S})|_{\mathbb{P}^1 \setminus \{0\}} \) and \( \text{Fix}(\lambda_{J,S})|_{\mathbb{P}^1 \setminus \{0\}} \) inside \( \mathbb{M}_{S/\mathbb{P}^1, H}(v_a, nC) \). Note that this follows trivially for \( \xi_{J,S} \) as it is biregular, but also for \( \hat{\zeta}_S \) and \( \lambda_{J,S} \), which are biregular at the central fibre. \( \square \)

Suppose now that \( S \) denotes a smooth surface equipped with involution \( \zeta_S : S \rightarrow S \) whose quotient map \( q_S : S \rightarrow T := S/\zeta_S \) gives a smooth surface. Pick a pair of smooth curves \( D \subset T \) and \( C \subset S \) such that \( C \) is as in (2.18). Denote by \( \zeta_C : C \rightarrow C \) the restriction of \( \zeta_S \) to \( C \), and by \( q_C : C \rightarrow D = C/\zeta_C \) the restriction there of \( q_S \). As in (4.1), denote the associated normal bundles by \( L = \mathcal{O}_S(C)|_C \), as well as \( W = \mathcal{O}_T(D)|_D \). Consider as well the ruled surfaces \( L \) and \( W \) associated to \( L \) and \( W \), as defined in (2.22). Note that the composition with \( d\zeta_S : \zeta_S^*TS \rightarrow TS \) provides an isomorphism \( f_0 : \zeta_S^*L \overset{\cong}{\rightarrow} L \) and recall from (3.3) the associated involution \( f_0 \), as well as the projection \( q_{f_0} \) from (3.4).

**Lemma 4.10** Lift, via \( f_0 \) as above, the action of \( \zeta_C \) to the normal line bundle \( L \) of \( C \) inside \( S \). Then, \( L \) is the pull-back under \( q_C \) of the normal bundle \( W \) of \( D \) inside \( T \),

\[
L \cong q_C^*W.
\]

Hence,

\[
W \cong L/\xi_{f_0}.
\]

**Proof** The first statement follows easily as the tangent bundles \( T T \) and \( TD \) coincide, respectively, with the \( \zeta_S \)-invariant and \( \zeta_C \)-invariant subbundles of \( TS \) and \( TC \). The second statement follows from the first and Remark 3.2. \( \square \)

Following (4.4) and (4.6), define \( \overline{T} \) as well as \( T \), having generic fibres \( \overline{T}_t = T_t = T \), over \( t \neq 0 \), and central fibre \( \overline{T} = T \cup \overline{W} \) and \( T_0 = \text{Tot}(W) \). Since the preimage of \( D \times \{0\} \) under the composition of \( \overline{S} \rightarrow S \times \mathbb{P}^1 \) and the projection \( q_S \times 1_{\mathbb{P}^1} : S \times \mathbb{P}^1 \rightarrow T \times \mathbb{P}^1 \) is the divisor \( \mathbb{L} \subset \overline{S} \), one obtains the projection \( q_S : \overline{S} \rightarrow \overline{T} \) commuting with the structural morphisms to
$\mathbb{P}^1$. At $t = 0$, $q_S$ sends the irreducible component $S$ of $S_0$ to the irreducible component $T$ of $T_0$. Then, by restricting $q_S$ to $S$, we can define $q_S : S \rightarrow T$ commuting with the structural morphisms,

$$
\begin{array}{ccc}
S & \xrightarrow{q_S} & T \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & & 
\end{array}
$$

(4.20)

**Lemma 4.11** With the notation as above, $q_S$ coincide with $q_f$ on the generic fibres at $t \neq 0$, and, with the restriction of $q_{f_0}$ to $\text{Tot}(L)$ over $t = 0$. Hence, one has that $q_S$ is the quotient map of the action of $\xi_S$.

**Proof** After Lemmas 4.6 and 4.10, one has that the restriction of $q_S$ to $L$ coincides with $q_f$, which is the quotient map associated to $\xi_f$. This concludes the proof, as we knew already that $q_S$ is the quotient map of $\xi_S$ over the complement of the central fibre. \(\square\)

We construct $D$ as in (4.8) starting from the smooth curve $D \subset T$. As defined in (4.10), denote the connected components of the corresponding relative Hilbert schemes containing the family of curves $nD$, once we restrict to $T$. Denote also $q_S(nD)_T \subset (nC)_S$ the locus of curves lifted from $T$ to $S$.

Let us consider the restriction to $q^{-1}_S(nD)_T$ of the closure of the locus fixed by $\hat{\xi}_S$,

$$
N_{S/\mathbb{P}^1,H}(v_a, nD) := \overline{\text{Fix}(\hat{\xi}_S) \cap h^{-1}(q^{-1}_S(nD)_T)}.
$$

This provides a degeneration of the fixed locus of a natural involution on the moduli space of sheaves over $S$.

**Theorem 4.12** Let $S$ be a smooth projective surface equipped with an involution $\xi_S$ whose quotient $T$ is smooth, pick a smooth curve $D \subset T$ and consider its lift $C$ to $S$. Pick as well a positive integer $n$ and, associated to $n$ and $C$, a Mukai vector $v_a$ as in (2.1). Then, there exists a closed subvariety $N_{S/\mathbb{P}^1,H}(v_a, nD) \subset M_{S/\mathbb{P}^1,H}(v_a, nC)$, flat over $\mathbb{A}^1$, and equipped with a surjective support morphism

$$
\begin{array}{ccc}
N_{S/\mathbb{P}^1,H}(v_a, nD) & \xrightarrow{q_S} & q_S(nD)_T \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & & 
\end{array}
$$

(4.21)

such that the structural morphism is trivial over $\mathbb{P}^1 - \{0\}$, the generic fibre over $t \neq 0$ is the closure of the fixed locus of the natural involution on $M_{S,H}(v_a)$ associated to $\xi_S$ equipped with the fibration (2.5),

$$
N_{S,H}(v_a, nD) \rightarrow q^e_S(nD),
$$

while the central fibre at $t = 0$ is the fixed locus of the involution $\tilde{\xi}_{f_0}$ given in (3.8), being $f_0$ induced from $d\xi_S$ as in (4.17), endowed with the Hitchin morphism (3.9),

$$
N^L_C(n, d) \rightarrow \bigoplus_{i=1}^n q^+_C H^0(D, W^\otimes i)
$$

where $d = a - \delta$. 

\(\diamondsuit\) Springer
Proof After Lemma 4.6, the fibrewise description is straight-forward since the restriction of \( \hat{\zeta}_S \) to a generic fibre over \( t \neq 0 \) coincides with \( \zeta_S \), while, at the central fibre over \( t = 0 \), \( \zeta_S \) restricts to \( \hat{\zeta}_{f_0} \). Then, the description of the generic fibres is provided by the definitions (2.4) and (2.5), while the description of the central fibre follows from Lemmas 3.6 and 4.10. Finally, recall Proposition 4.9 for the proof of flatness.

For \( T \), consider the moduli space \( \mathcal{M}_{T/\mathbb{P}^1}(w_b, nD) \) of sheaves with a relative Mukai vector \( w_b \) and a polarization \( I \) inducing \( I \) and \( \tilde{I} = q_S^*I \). Since \( \tilde{I} \) is \( \zeta_S \)-invariant, \( \hat{\zeta}_S \) is biregular and the image of the pull-back morphism under the corresponding quotient map lies in its fixed locus,

\[
\hat{\zeta}_S : \mathcal{M}_{T/\mathbb{P}^1}(w_b, nD) \longrightarrow \mathcal{N}_{T/\mathbb{P}^1}(v_{2b}, nD) \subset \mathcal{M}_{T/\mathbb{P}^1}(v_{2b}, nD)
\]

Note that over the generic fibre, \( t \neq 0 \), (4.22) restricts to (2.6), while on the central fibre, \( t = 0 \), the restriction of (4.22) is (3.10). Both cases are covered by Proposition 2.11 which provides the following corollary.

Corollary 4.13 Hence, (4.22) is generically 1:1 when the ramification locus of \( q_S \) intersects \( C \) non-trivially, and, otherwise (in particular when \( \zeta_S \) is unramified), generically 2 : 1.

As we did in the case of a natural involution, consider

\[
P_{S/\mathbb{P}^1,H}(v_a, J, nD) := \text{Fix}(\lambda_{J,S}) \cap h^{-1}(q_S^{-1}[nD]_T),
\]

which gives a degeneration of the fixed locus of \( \lambda_{J,S} \).

Theorem 4.14 Let \( S \) be a smooth projective surface equipped with an involution \( \zeta_S \) whose quotient \( T \) is smooth, pick a smooth curve \( D \subset T \) and consider its lift \( C \) to \( S \). Choose a positive integer \( n \) and take a Mukai vector \( v_a \) as in (2.1) and a line bundle \( J \) on \( S \) satisfying (2.11).

Then, there exists a closed subvariety \( P_{S/\mathbb{P}^1,H}(v_a, J, nD) \subset M_{S/\mathbb{P}^1,H}(v_a, nC) \), flat over \( \mathbb{P}^1 \), and equipped with a surjective support morphism

\[
P_{S/\mathbb{P}^1,H}(v_a, J, nD) \longrightarrow q_S^*[nD]_T
\]

such that the structural morphism is trivial over \( \mathbb{P}^1 - \{0\} \), the generic fibre over \( t \neq 0 \) is the closure of the fixed point locus of the involution \( \lambda_{J,S} \) on \( M_{S,H}(v_a) \) equipped with the fibration (2.16),

\[
P_{S,H}(v_a, J, nD) \rightarrow q_S^*[nD],
\]

while the central fibre at \( t = 0 \) is the fixed locus of the involution \( \tilde{\lambda}_{C,f_0} \) given in (3.13), being \( f_0 \) induced from \( d\zeta_S \) as in (4.17) and \( J_C \) the restriction of \( J \) to \( C \), endowed with the Hitchin morphism (3.14),

\[
\mathcal{D}_C^n(n, d, J_C) \rightarrow \bigoplus_{i=1}^n q_C^*H^0(D, W^{\otimes i}).
\]

where \( d = a - \delta \).
4.4 Degenerations of natural Lagrangians and Prymian integrable systems

Sawon conjectured in [49] that the Prymian integrable systems constructed by Markushevich–Tikhomirov [40], Arbarello–Saccà–Ferretti [1] and Matteini [41], degenerate into integrable systems related to the Hitchin system, leaving open the description of these conjectural systems. It is well known that every Lagrangian fibration degenerates naturally into the cotangent bundle of one of its smooth fibres, and the later can easily be identified with a rank 1 Hitchin system. Hence, one can always find examples of such degenerations, although, perhaps not very interesting ones.

When the curves appearing in the integrable system are not primitive, it is possible to provide a degeneration of the corresponding Prymian integrable systems into Hitchin systems of higher rank. We provide in this section such a construction, providing an answer of Sawon’s question in the case of sheaves supported on non-primitive curves. The question of whether or not these Prymian integrable systems for primitive curves degenerate into more interesting Hitchin systems remains open.

In this section we provide an explicit description of the degenerations of Sect. 4.3 in the cases studied in Sect. 2.4, focusing on the cases specified in [1, 40, 41, 50, 52], which are constructed using primitive first Chern classes. We also review the degeneration given by Sawon–Shen [50] into the \( \text{Sp}(2m, \mathbb{C}) \)-Higgs moduli space, studying as well the degeneration of the associated natural Lagrangian. It is worth noticing that we find that this natural Lagrangian degenerates into the fixed locus of an involution associated to \( \text{U}(m, m) \)-Higgs bundles, whose Nadler–Langlands group is \( \text{Sp}(2m, \mathbb{C}) \).

Our work provides an answer to the question posed by Sawon in [49] (which, strictly speaking, refers only to the case of primitive first Chern classes) and to the reformulation of Sawon’s question on the non-primitive case.

4.4.1 The general case

Consider a K3 surface \( X \) equipped with an antisymplectic involution \( \zeta_X : X \to X \), whose quotient \( q_X : X \to \bar{Y} := X/\zeta_X \) has ramification divisor \( \Delta_X \). Let \( D \subset \bar{Y} \) be a smooth curve and \( C \) as in (2.18), again smooth. We denoted by \( \zeta_C : C \to C \) and \( q_C : C \to D = C/\zeta_C \) the restriction of \( \zeta_X \) and \( q_X \) to \( C \). Note that the ramification divisor of \( \zeta_C \) is \( \Delta_C = C \cap \Delta_X \).

Since \( K_X \) is trivial, the normal bundle of \( C \) is

\[
L = K_C,
\]
giving the ruled surface \( K_C \), and the normal bundle of \( D \) is

\[
W = K_D K_Y^{-1}\big|_D.
\]

(4.24)

associated to the ruled surface \( \bar{W} \). Observe that \( K_Y^{-1}\big|_D \) is a square-root of the branching divisor of \( q_C \), and its pull-back is the ramification divisor.

Recall from Proposition 4.5, that \( d\zeta_X : (\zeta_X)^*TX \to TX \) gives the isomorphism \( -\partial_C : \zeta_C^* K_C \cong K_C \) giving the associated involution \( \zeta_K \) described in (3.6). The associated natural antisymplectic involution \( \zeta_K \) is sent to \( \tilde{\zeta}_K \) under the spectral correspondence as described in (3.15). As a consequence of Proposition 4.5, Theorems 4.4 and 4.12 and Corollary 4.13 one has the following.

\[\zeta \] Springer
Corollary 4.15 There exists a non-linear degeneration $N_{X,H}(v_a, nD)$ of the natural Lagrangian subvariety $N_{X,H}(v_a, nD) \subset M_{X,H}(v_a, nC)$ obtained from the fixed locus of $\xi_X$, into the Lagrangian subvariety of the Higgs moduli space $N_{C}(n, d) \subset M_{C}(n, d)$, with $d = a - n(n - 1)(g_C - 1)$, given by the fixed locus of the involution

$$\tilde{\xi}_X : M_C(n, d) \longrightarrow M_C(n, d)$$

$$(E, \varphi) \longmapsto (\xi^*_C E, -\xi^*_C \varphi).$$

(4.25)

For $a = 2b$ and a $\xi_X$-invariant polarization $\tilde{\eta} = q^*_X \mathbb{I}$, the morphism

$$\tilde{\eta}_X : M_{Y,1}(w_b, nD) \longrightarrow N_{X/H^1,1}(v_{2b}, nD) \subset M_{X/H^1,1}(v_{2b}, nC),$$

given in (4.22), is generically 1:1 when the ramification locus of $q_X$ intersects $C$ non-trivially, and generically 2:1 otherwise, for instance whenever $\tilde{\xi}_X$ is unramified. The generic fibres of $M_{Y/H^1,1}(w_b, nD)$ are the moduli spaces $M_{Y,1}(w_b, nD)$ and the central fibre is $M^W_D(n, d')$, for $W$ described in (4.24) and $d' = b - n(n - 1)(g_D - 1)$.

As a consequence of Proposition 4.5 and Theorems 4.4 and 4.14 one has the following.

Corollary 4.16 Suppose that (2.11) holds for the choice of a line bundle $J$ and a Mukai vector $v_a$ as in (2.1) for some integer $a$ and let $d = a - \delta = a - n(n - 1)(g_C - 1)$.

There exists a non-linear degeneration $P_{X,H}(v_a, J, nD)$ of the Prymian integrable system $P_{X,H}(v_a, J, nD) \subset M_{X,H}(v_a, nC)$ constructed out of $\tilde{\xi}_X$, into the Prymian integrable system $\mathcal{P}_C(n, d, J_C) \subset M_C(n, d)$ given by the fixed locus of the involution

$$\tilde{\lambda}_{C,K} : M_C(n, d) \longrightarrow M_C(n, d)$$

$$(E, \varphi) \longmapsto (\xi^*_C E, -\xi^*_C \varphi).$$

(4.26)

Remark 4.17 For $n = 1$, $\mathcal{P}_C(n, d, J_C)$ is the cotangent bundle of a Prym abelian variety associated to $\xi_C$ (when $\tilde{\xi}_X$ is ramified) or a disjoint union of them (when $\tilde{\xi}_X$ unramified).

4.4.2 The Arbarello–Saccà–Ferretti systems and Saccà Calabi-Yau’s

In this subsection, we consider a K3 surface endowed with an antisymplectic involution $\xi_X^{-1}$ with empty fixed locus and whose quotient $Y = X/\xi_X$ is a smooth Enriques surface. The canonical divisor of an Enriques surface $Y$ is not trivial but satisfies $2K_Y \sim 0$. Any smooth curve $D \subset Y$ with positive self-intersection is big and nef, and so is $D + K_Y$. Then, by Kodaira vanishing theorem,

$$h^1(Y, D) = h^2(Y, D) = 0,$$

hence

$$h^0(Y, D) = 1 + \frac{1}{2}D^2 = g_D,$$

where $g_D$ is the genus of $D$. The curve $C := D \times_Y X$ is equipped with an unramified $2:1$ cover $q_C : C \to D$ and an involution $\xi_C : C \to C$ without fixed points.

Observe that condition (2.21) is satisfied for every smooth curve $D \subset Y$. Then, whenever (2.11) holds for a pair $J$ and $v_a$, $P_{X,H}(v_a, J, nD)$ has 2 connected components each of them an integrable system of dimension $2n^2(g_D - 1)$ known as the Arbarello–Saccà–Ferretti integrable system, as described by these authors in [1] when $n = 1$. The Lagrangian subvariety $N_{X,H}(v_a, nD)$ has dimension $n^2(g_C - 1) + 1$. Furthermore, Saccà [48] studied the moduli
space of pure dimension 1 sheaves over an Enriques surface $M_{Y, I}(w_b, nD)$ showing that it is Calabi–Yau. By Proposition 2.11, $M_{Y, I}(w_b, nD)/\mathbb{Z}_2$ is birational to $N_{X, T}(v_{2b}, nD)$.

Observe that $L \cong K_C$ in this case, and $W \cong K_DL_y$, where $L_y \in H^1(D, \mathbb{Z}_2)$ is the 2-torsion line bundle obtained by restricting $K_Y$ to $D$. After Corollary 4.15 $M_{Y, I}(w_b, nD)$ degenerates into $M^L_{D, K_D}(n, b - \delta/2)$ and $N_{X, H}(v_a, nD)$ into $N_C(n, a - \delta)$, which is the fixed point locus of the involution (4.25) constructed out of $\zeta_C$ unramified. It follows from Corollary 4.16 that $P_{X, H}(v_a, J, nD)$ degenerates into $\mathcal{P}_C(n, a - \delta, J_C)$, which is the fixed point locus of the involution (4.26), constructed out of $\zeta_C$ unramified.

4.4.3 Markushevich–Tikhomirov, Matteini and Sawon–Shen systems

Consider now an anti-symplectic involution $\tilde{\zeta}_X$ on the K3 surface $X$ whose quotient $Y = X/\tilde{\zeta}_X$ is a del Pezzo surface of degree $d$. In that case, by their defining property, the anti-canonical bundle $-K_Y$ is ample. In particular, $-(m + 1)K_Y$ is big and nef for $m \geq 0$ and by Kodaira vanishing theorem,

$$h^1(Y, -mK_Y) = h^2(Y, -mK_Y) = 0,$$

so

$$h^0(Y, -mK_Y) = 1 + \frac{m^2 + m}{2} \cdot d.$$

Note that the genus of a smooth curve in the linear system $-mK_Y$ is $1 + (m^2 - m)d/2$.

We recall that $\Delta$, the branching locus of $q_X$, lies in $| - 2K_Y|$. Picking a smooth curve $D$ in the linear system $| - K_Y|$, hence $g_D = 1$, pulling-back to another smooth curve $C$ on $X$, we obtain a $2 : 1$ cover, $q_C : C \to D$, ramified at the intersection with the ramification locus of $q_X$, that we denote by $R$. Observe that the length of $R$ coincides with the intersection of $D$ and $\Delta$, so $(-K_Y) \cdot (-2K_Y) = 2d$, from which one can derive that $g_C = 1 + d$.

Markushevich and Tikhomirov [40] gave the first construction of Prymian integrable system $P_{X, H}(v_a, J, nC)$ starting from a del Pezzo of degree $d = 2$ and for $n = 1$. In this case we have that $\zeta_C$ sends a genus 3 curve $C$, to $D$, an elliptic curve and $\zeta_C$ is ramified at 4 points and $P_{X, H}(v_a, J, C)$ is a 4-dimensional symplectic $V$-manifold.

Matteini extended the construction of [40], focusing on the cases of del Pezzo surfaces of degree $d = 1$ and $d = 3$. In the case of $d = 1$, $\zeta_C : C \to D$ is the cover of a genus 2 curve over an elliptic curve, and Matteini found that $P_{X, H}(v_a, J, C)$ is an elliptic K3 surface. In the $d = 3$ case, $\zeta_C : C \to D$ is a $2 : 1$ cover of a genus 4 curve onto an elliptic curve and $P_{X, H}(v_a, J, C)$ is a 6-dimensional symplectic $V$-manifold. Even in both cases only $n = 1$ was considered by the authors, their construction works for general $n$ as indicated by Matteini in [41, Section 3.6].

Another 6-dimensional Prym integrable system is considered by Sawon and Shen in [50, 52]. In this case $Y$ is a del Pezzo surface of degree $d = 1$ and $D \subset Y$ lies in $| - 2K_Y|$, so $g_D = 2$. The ramification divisor $R$ of the $2 : 1$ cover $q_C : C \to D$ has length $(-2K_Y) \cdot (-K_Y) = 2$, hence $g_C = 5$.

After Corollary 4.16 that $P_{X, H}(v_a, J, nD)$ degenerates into $\mathcal{P}_C(n, a - \delta, J_C)$, which is the fixed point locus of the involution (4.26), constructed out of $\zeta_C$ ramified at $\Delta_C = \Delta_X \cap X$.

One can also consider the natural Lagrangian subvarieties $N_{X, H}(v_a, nD)$ associated to the mentioned Prymian integrable systems. Note that in this case $N_{X, T}(v_{2b}, nD)$ are birational to $M_{Y, I}(v_b, nD)$. From Corollary 4.15 we know that $N_{X, H}(v_a, nD)$ degenerates into $N_C(n, a -$
\( \delta \), which is the fixed point locus of the involution \((4.25)\), and \( M_{Y,1}(v_b, nD) \) degenerates into \( \mathcal{M}_D^{K}\mathcal{Q}(n, b - (n^2 - n)(g_D - 1)) \), where \( \mathcal{Q} \) is a line bundle on \( D \) whose square is \( \mathcal{O}_D(\Delta_C) \).

### 4.4.4 Sawon–Shen degeneration into the \( \text{Sp}(2m, \mathbb{C}) \)-Hitchin system

When \( \tilde{\zeta}_X \) is an anti-symplectic involution on the K3 surface \( X \) giving a del Pezzo surface \( Y = X/\tilde{\zeta}_X \) of degree \( d \), the ramification divisor of \( q_X : X \to Y \) is a connected smooth projective curve \( \Delta \) of genus \( g_\Delta = d + 1 \). Trivially, the restriction of \( \zeta_X \) to \( \Delta \) is the identity \( \zeta_\Delta = 1_\Delta \). Setting \( v_0 = (0, 2m \Delta, -4m^2(g_\Delta - 1)) \), \( H = \mathcal{O}_X(\ell \Delta) \) and \( J = \mathcal{O}_X(-2m \Delta) \), Sawon and Shen \([50]\) have constructed a degeneration of the Prymian integrable system \( P_{X,H}(v_0, J, 2m \Delta) \) into a subvariety of the Higgs moduli space \( \mathcal{M}_\Delta(n, -\delta) \) isomorphic to the moduli space of \( \text{Sp}(2m, \mathbb{C}) \)-Higgs bundles, where \( \delta = 2m(2m - 1)(g_\Delta - 1) \). Shifting in degree, taking \( v_\delta = (0, 2m \Delta, -2m(g_\Delta - 1)) \), Sawon and Shen’s construction becomes a degeneration of the Prymian integrable system \( P_{X,H}(v_\delta, J, 2m \Delta) \) into a compactification of the \( \text{Sp}(2m, \mathbb{C}) \)-Hitchin system over \( \Delta \), given by the fixed locus of the involution

\[
\lambda_{K,2m,\delta} : \mathcal{M}_\Delta(n, 0) \to \mathcal{M}_\Delta(n, 0) \\
(E, \varphi) \mapsto (E^*, \varphi^*).
\]

Note that Corollary 4.15 shows that the natural Lagrangian \( N_{X,H}(v_\delta) \) degenerates into the fixed locus of the involution

\[
\tilde{\zeta}_K : \mathcal{M}_\Delta(n, 0) \to \mathcal{M}_\Delta(n, 0) \\
(E, \varphi) \mapsto (E, -\varphi).
\]

The morphism obtained by extension of structure group sends the moduli space of \( \text{U}(m,m) \)-Higgs bundles maps to the fixed locus of \( \tilde{\zeta}_K \). It is worth noticing that \( \text{U}(m,m) \) and \( \text{Sp}(2m, \mathbb{C}) \) are related under the Nadler-Langlands correspondence \([45]\). In \([31]\) Hitchin proposed that a (BBB)-brane constructed over the moduli space of \( \text{Sp}(2m, \mathbb{C}) \)-Higgs bundles is Mirror dual to a (BAA)-brane supported on the moduli space of \( \text{U}(m,m) \)-Higgs bundles (see Section 5). Evidence for this duality was given by Hitchin himself in \([31]\) and by Hausel, Mellit and Pei in \([27]\).

### 5 Branes and duality

Non-abelian Hodge theory \([14, 19, 30, 53, 54]\) implies that \( \mathcal{M}_C(n, d) \) is equipped with a hyperkähler structure \((g, \Gamma_1, \Gamma_2, \Gamma_3)\), where \( \Gamma_1 \) the complex structure coming from the moduli space of Higgs bundles and \( \Gamma_2 \) from the moduli of flat connections. The associated Kahler forms \( \omega_i(\cdot, \cdot) := g(\cdot, \Gamma_i(\cdot)) \) combine into a \( \Gamma_1 \)-holomorphic symplectic form \( \omega_0 + i\omega_3 \), which coincides with \( \Omega_0 \) up to scaling. Similar constructions can be given for \( i = 2 \) and \( i = 3 \).

Following \([33]\), a (BAA)-brane in \( \mathcal{M}_C(n, d) \) is a pair \((N, \mathcal{G}, \nabla_\mathcal{G})\) where \( N \) is a subvariety which is complex Lagrangian with respect to \( \Omega_0 \), and \((\mathcal{G}, \nabla_\mathcal{G})\) is a flat bundle over \( N \). A (BBB)-brane in \( \mathcal{M}_C(n, d) \) is given by a pair \((P, \mathcal{F}, \nabla_\mathcal{F})\), where \( P \) is a hyperholomorphic subvariety \((i.e. \) holomorphic with respect to \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \)) and \((\mathcal{F}, \nabla_\mathcal{F})\) a hyperholomorphic sheaf on \( P \). This means that the connection \( \nabla_\mathcal{G} \) on the sheaf \( \mathcal{F} \) is of type \((1, 1)\) with respect to all three Kähler structures \( \Gamma_1, \Gamma_2, \Gamma_3 \). It is conjectured in \([33]\) that mirror symmetry interchanges (BBB)-branes with (BAA)-branes within the Higgs moduli space and in a
certain limit, mirror symmetry is enhanced via a Fourier-Mukai transform relative to the Hitchin fibration.

**Remark 5.1** When \((\mathcal{G}, \nabla_{\mathcal{G}})\) appearing in the definition of a \((\text{BAA})\)-brane (resp. \((\text{BBB})\)-brane) in the definition of a \((\text{BBB})\)-brane is the trivial bundle, we simply denote the brane indicating the Lagrangian subvariety \(N\) (resp. the hyperholomorphic subvariety \(P\)). Equivalently, when no bundle is specified in the description of a brane, we are implicitly assuming that it is the trivial one.

Note that the definition of \((\text{BAA})\) and \((\text{BBB})\)-branes extends naturally to any other hyperkähler varieties. The case of the moduli space \(M_{S, H}(v_a)\) of sheaves over a smooth projective symplectic surface \(S\) was considered in [20] by the author, Jardim and Menet. It is also described in [20] the construction of the \((\text{BBB})\)-brane \(N_{S, H}^+(v_a)\) arising from the fixed locus of a natural involution associated to a symplectic involution on the surface. Also, the authors studied the behaviour of these natural branes under some correspondences. As a consequence of Propositions 4.5 and 4.9, we can describe these degenerations in the context of branes, extending the description to \(P_{S, H}(v_a, J)\).

**Corollary 5.2** The \((\text{BBB})\)-brane \(N_{S, H}^+(v_a)\) (resp. the \((\text{BAA})\)-brane \(N_{S, H}^-(v_a)\)) obtained by the fixed locus of a natural involution associated to a symplectic (resp. anti-symplectic) involution on a symplectic surface degenerate into the \((\text{BBB})\)-brane (resp. \((\text{BAA})\)-brane) inside the Hitchin system \(\mathcal{N}_C^+(n, d)\) (resp. \(\mathcal{N}_C^-(n, d)\)), obtained as the respective fixed locus of the involution \(\tilde{\xi}_K^+\) (resp. \(\tilde{\xi}_K^-\)), described in (3.15).

Similarly, the \((\text{BBB})\)-brane obtained from \(P_{S, H}(v_a, J)\) degenerates into the \((\text{BBB})\)-brane given by \(P_C(n, d, J_C)\).

We now discuss the duality under Mirror symmetry of the branes associated to a Prymian integrable system and the natural Lagrangian.

Consider the Mukai vector \(v_0\) on \(S\), where we assume \(a = 0\), and recall from Remark 2.9 that it is the pull-back of the Mukai vector \(w_0\) on \(T\). Following Remark 2.12, choose \(J_0 = \mathcal{O}_S(-nC)\). In this section we show that certain open subsets of the subvarieties \(N_{S, H}(v_0, B)\) and \(P_{S, H}(v_0, J_0, B)\) are dual under Fourier–Mukai transform.

Observe that the diagram,
restricts, after Proposition 2.11 and Remark 2.16, to

\[ \text{Jac}_{A^\text{sm}/[A]^\text{sm}} \xrightarrow{\text{Prym}} \text{Jac}_{B^\text{ssl}/[B]^\text{ssl}} \]

\[ \text{Jac}_{B^\text{ssl}/[B]^\text{ssl}} \xrightarrow{\hat{\Psi}_{A^\text{ssl}}} \text{Jac}_{A^\text{sm}/[A]^\text{sm}} \]

where we have dropped the systematic reference to degree 0 in our notation for Jacobians.

Let us recall that one can equip \( \text{Jac}_{A^\text{sm}/[A]^\text{sm}} \times [A]^\text{sm} \) \( \text{Jac}_{A^\text{sm}/[A]^\text{sm}} \) with a relative Poincaré bundle \( \mathcal{U}_{A^\text{sm}} \). Denoting by \( \pi_i \) the projection from \( \text{Jac}_{A^\text{sm}/[A]^\text{sm}} \times [A]^\text{sm} \) \( \text{Jac}_{A^\text{sm}/[A]^\text{sm}} \) to the \( i \)-th factor, one can consider the relative Fourier–Mukai transform \([43]\)

\[ \Psi_{A^\text{sm}} : \mathcal{D}^b(\text{Jac}_{A^\text{sm}/[A]^\text{sm}}) \longrightarrow \mathcal{D}^b(\text{Jac}_{A^\text{sm}/[A]^\text{sm}}) \]

\[ \mathcal{F} \longmapsto R\pi_{2,*}(\pi_1^*\mathcal{F} \otimes \mathcal{U}_{A^\text{sm}}) , \]

and, after the choice of \( J_0 \), its inverse

\[ \Psi_{A^\text{sm}}^{-1} : \mathcal{D}^b(\text{Jac}_{A^\text{sm}/[A]^\text{sm}}) \longrightarrow \mathcal{D}^b(\text{Jac}_{A^\text{sm}/[A]^\text{sm}}) \]

\[ \mathcal{G} \longmapsto \xi_{J_0}^* R\pi_{1,*}(\pi_1^*\mathcal{G} \otimes \mathcal{U}_{A^\text{sm}}^{-1}) [-g_A]. \]

One can provide a similar construction for \( B^\text{ssl} \) in \( T \), giving rise to the Fourier–Mukai transform \( \hat{\Psi}_{B^\text{ssl}} \). Recalling the morphism \( \hat{\Psi}_{A^\text{ssl}} \) from (2.8), both transforms satisfy (see [47, (11.3.3)] for instance),

\[ \Psi_{A^\text{sm}} \circ R\hat{\Psi}_{A^\text{ssl}}^* \cong L\text{Nm}(q_{A^\text{ssl}})^* \circ \Psi_{B^\text{ssl}} , \quad (5.1) \]

where the norm map \( \text{Nm}(q_{A^\text{ssl}}) \) is the dual of \( \hat{\Psi}_{A^\text{ssl}} \).

**Proposition 5.3** Whenever \( q_{A^\text{ssl}} \) has ramification, the structural sheaves of the subvarieties \( N_{S,H}(v_0, B)[B]^\text{ssl} \) and \( P_{S,H}(v_0, J_0, B)[B]^\text{ssl} \) are dual under a relative Fourier–Mukai transform.

If \( q_{A^\text{ssl}} \) is unramified, the structural sheaf on a connected component of \( P_{S,H}(v_0, J_0, B)[B]^\text{ssl} \) is Fourier–Mukai dual to \( N_{S,H}(v_0, B)[B]^\text{ssl} \) equipped with the pair of line bundles obtained from pushing forward the structural sheaf under \( \hat{\Psi}_{A^\text{ssl}} \) direct sum of the trivial sheaf and the pull-back of a torsion 2 line bundle \( L \rightarrow B^\text{ssl} \).

**Proof** We start by the ramified case. Observe that \( R\hat{\Psi}_{A^\text{ssl}}^* \mathcal{O}_{Jac_{B^\text{ssl}}} \) coincides with the trivial sheaf over \( N_{S,H}(v_0, B)[B]^\text{ssl} \) after Proposition 2.11. Also, \( \Psi_{B^\text{ssl}}(\mathcal{O}_{Jac_{B^\text{ssl}}}) \) is the relative skyscraper sheaf at the identity, so its pull-back under \( \text{Nm}(q_{A^\text{ssl}}) \) is precisely the trivial sheaf over \( P_{S,H}(v_0, J_0, B)[B]^\text{ssl} \). Then, the first statement follows easily from (5.1).

The second statement follows from the diagram (2.10) and the fact that the preimage of the Norm map has two connected components. \( \Box \)

Recalling Proposition 5.3, one immediately deduces the following.

**Corollary 5.4** Whenever \( q_C \) has ramification, the structural sheaves of \( N_{C}(n, d)[B]^\text{ssl} \) and \( P_{C}(n, d, J)[B]^\text{ssl} \) are dual under a relative Fourier–Mukai transform.
If $q_C$ is unramified, the latter is Fourier–Mukai dual to the sheaf supported on $\mathcal{N}_C(n, d)|_{[p]}^\psi$ given by the direct sum of the trivial sheaf and the line bundle associated to the unramified 2 : 1 cover of the family of spectral curves.

In this context, Corollaries 4.15 and 4.16 can be seen, respectively, as a degeneration of (BAA) and (BBB)-branes, from the moduli space of sheaves on symplectic surfaces into the Hitchin system.

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**References**

1. Arbarello, E., Saccà, G., Ferretti, A.: Relative Prym varieties associated to the double cover of an Enriques surface. J. Differ. Geom. 100, 191–250 (2015)
2. Baraglia, D., Schaposnik, L.P.: Higgs bundles and (A, B, A)-branes. Comm. Math. Phys. 331, 1271–1300 (2014)
3. Basu, S., García-Prada O.: Finite group actions on Higgs bundle moduli spaces and twisted equivariant structures. arXiv:2011.04017
4. Beauville, A.: Variétés kählériennes dont la première classe de Chern est nulle. J. Differ. Geom. 18, 755–782 (1983)
5. Beauville, Systèmes hamiltoniens complètement intégrables associés aux surfaces K3. Symp. Math. (1991)
6. Beauville, A.: Complex Algebraic Surfaces, 2nd edn. Cambridge University Press, Cambridge (1996)
7. Beauville, A.: Symplectic singularities. Invent. Math. 139, 541–549 (2000)
8. Beauville, A., Narasimhan, M.S., Ramanan, S.: Spectral curves and the generalised theta divisor. J. Reigne Angew. Math. 398, 169–179 (1989)
9. Biswas, I., Ramanan, S.: An infinitesimal study of the moduli of Hitchin pairs. J. Lond. Math. Soc. 2(49), 219–231 (1994)
10. Biswas, I., Calvo, L., Franco, E., García-Prada, O.: Involutions of Higgs moduli spaces over elliptic curves and pseudo-real Higgs bundles. J. Geom. Phys. 142, 47–65 (2019)
11. Biswas, I., Bottacin, F., Gomez, T.L.: Comparison of Poisson structures on moduli spaces. Rev. Mat. Complut. (2021). https://doi.org/10.1007/s13163-021-00418-7
12. Bottacin, F.: Symplectic geometry on moduli spaces of stable pairs 121, 421–436 (1995)
13. Bottacin, F.: Poisson structures on moduli spaces of sheaves over Poisson surfaces. Invent. math. 121, 421–436 (1995)
14. Corlette, K.: Flat G-bundles with canonical metrics. J. Differ. Geom. 28(3), 361–382 (1988)
15. de Cataldo, M.A., Maulik, D., Shen, J.: Hitchin fibrations, abelian surfaces and the P=W conjecture. arXiv:1909.11885
16. de Cataldo, M.A., Maulik, D., Shen, J.: On the P=W conjecture for SL_n, to appear at Selecta Math
17. de Cataldo, M.A., Hausel, T., Migliorini, L.: Exchange between perverse and weight filtration for the Hilbert schemes of points of two surfaces. J. Singul. 7, 23–38 (2013)
18. Donagi, R.: Ein and Lazarsfeld, Nilpotent cones and sheaves on K3 surfaces. Contemp. Math (1997)
19. Donaldson, S.: Twisted harmonic maps and the self-duality equations. Proc. Lond. Math. Soc. (3) 55(1), 127–131 (1987)
20. Franco, E., Menet, G., Jardim, M.: Brane involutions on irreducible holomorphic symplectic manifolds. Kyoto J. Math (2019)
21. García-Prada, O., Ramanan, S.: Involutions and higher order automorphisms of Higgs moduli spaces. Proc. London Math. Soc. (2019). https://doi.org/10.1112/plms.12242
22. García-Prada, O., Wilkin, G.: Action of Mapping class group on character varieties and Higgs bundles. Doc. Math. 5, 841–868 (2020)
23. Groechenig, M., Wyss, D., Ziegler, P.: Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration. Invent. Math. 221, 505–596 (2020)
24. Gukov, S.: Quantization via Mirror symmetry. Jpn. J. Math. 6, 65–119 (2011)
25. Hartshorne, R.: Algebraic Geometry, Graduate Texts in Mathematics, vol. 52. Springer, Berlin (1977)
26. Hausel, T., Thaddeus, M.: Mirror symmetry, Langlands duality, and the Hitchin system. Invent. Math. 153, 197–229 (2003)
27. Hausel, T., Mellit, A., Du, P.: Mirror symmetry with branes by equivariant Verlinde formulae, in Geometry and Physics: vol. 1. A Festschrift in honour of Nigel Hitchin, Oxford Scholarship Online (2018)
28. Heller, S., Schaposnik, L.P.: Branes through finite group actions. J. Geom. Phys. 129, 279–293 (2018)
29. Hitchin, Stable bundles and integrable systems, Duke Math. J. (1987)
30. Hitchin, N.J.: The self-duality equations on a Riemann surface. Proc. Lond. Math. Soc. 55, 59–126 (1987)
31. Hitchin, N.J.: Higgs bundles and characteristic classes, Arbeitstagung Bonn, Progr. Math., 319. Birkhäuser/Springer, Cham, 2016–2017 (2013)
32. Huybrechts, D., Lehn, M.: The geometry of moduli spaces of sheaves. Cambridge University Press, Cambridge (2010)
33. Kapustin, A., Witten, E.: Electric-magnetic duality and the geometric Langlands program. Comm. Numb. Th, Phys (2007)
34. Kempf, G.: Hochster-Roberts theorem in invariant theory. Mich. Math. J. 26, 19 (1979)
35. Le Potier, J.: Faisceaux semi-stables de dimension 1 sur le plan projectif. Rev. Roumaine Math. Pures Appl. 38(7–8), 635–678 (1993)
36. Markman, E.: Spectral curves and integrable systems. Comp. Math. 93(3), 255–290 (1994)
37. Markman, E.: Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces. J. Reine Angew. Math. 544, 61–82 (2002)
38. Markman, E.: On the monodromy of moduli spaces of sheaves on K3 surfaces. J. Algebraic Geom. 17, 29–99 (2008)
39. Markman, E.: A survey of Torelli and monodromy results for holomorphic-symplectic varieties, Complex and differential geometry, 257–322, Springer roc. Math., 8, Springer, Heidelberg (2011)
40. Markushevich, D., Tikhomirov, A.: New symplectic V-manifolds of dim. 4 via relative compactified Prymian. Int. J. Math. 18(10), 1187–1224 (2007)
41. Matteini, T.: Holomorphically symplectic varieties with Prym Lagrangian fibrations, PhD thesis, University of Lille
42. Maulik, D., Shen, J.: Cohomological -independence for moduli of one-dimensional sheaves and moduli of Higgs bundles, to appear inGeom. Topol.
43. Mukai, S.: Duality between and with its application to Picard sheaves. Nagoya Math. J. 81, 153–175 (1981)
44. Mukai, S.: Symplectic structure on the module space of sheaves on an abelian or K3 surface. Invent. Math. 77, 101–116 (1984)
45. Nadler, D.: Perverse sheaves on real loop Grassmannians. Invent. math. 159, 1–73 (2005)
46. Nitsure, N.: Moduli of semistable pairs on a curve. Proc. Lond. Math. Soc. 62, 275–300 (1991)
47. Polishchuk, A.: Abelian varieties, theta functions and the Fourier transform. Cambridge University Press, Cambridge (2003)
48. Saccà, G.: Relative compactified Jacobians of linear systems on Enriques surfaces. Trans. Am. Math. Soc. 371(11), 7791–7843 (2019)
49. Savon, J.: Lagrangian fibrations by Prym varieties, Matemática Contemporânea, 47, 182–227
50. Savon, J., Shen, C.: Deformations of compact Prym fibrations to Hitchin systems. Bull. London Math. Soc. (2022). https://doi.org/10.1112/blms.12643
51. Schaub, D.: Courbes spectrales et compactifications de Jacobiannes. Mathematische Zeitschrift 227(2), 295–312 (1998)
52. Shen, C.: Lagrangian fibrations by Prym varieties, PhD thesis, U. North Carolina at Chapel Hill
53. Simpson, C.T.: Moduli of representations of the fundamental group of a smooth projective variety I. Publ. Math. Inst. Hautes Etud. Sci. 79, 47–129 (1994)
54. Simpson, C.T.: Moduli of representations of the fundamental group of a smooth projective variety II. Publ. Math. Inst. Hautes Etud. Sci. 80, 5–79 (1995)

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