A new stress-dilatancy framework for the modelling of rocks and rock masses

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Abstract. It is known that the stress-dilatancy behaviour of soils and rock masses is an integral aspect of their deformation behaviour; and that stress-dilatancy relations for soils and rocks and rock masses have strong similarities. However, when stress-dilatancy relations tuned for soils are directly adopted in constitutive models for rocks and rock masses without due consideration for differences in their yielding characteristics, some unintended assumptions are carried over. This has consequences on the qualitative and quantitative aspects of the predicted stress-strain behaviour. In this paper, a new stress-dilatancy relation tuned towards the modelling of deformation behaviour of rocks and rock masses is presented. The formulation is established in both plane strain and full 3D and accommodates both loading and unloading in shear and effect possible non-coaxiality between eigen directions of plastic strain increments and stresses.

1. Introduction

The stress-dilatancy behaviour of rocks, rock masses and soils is an integral part of their deformation behaviour under shearing. It is therefore a continuous interest to find a better and capable stress-dilatancy framework. The stress-dilatancy behaviour of rocks is seen in the same light as that of soils. Among the stress-dilatancy theories frequently used for soils and rock masses, Rowe’s [1] stress-dilatancy theory and Taylor’s [2] work hypothesis have been frequently applied as they are or with some modification, e.g., Vermeer and De Borst [3]. There have also been stress-dilatancy formulations established on a purely empirical basis, e.g., Alejano and Alonso [4].

The stress-dilatancy behaviour of rocks, as in soils, is the integral aspect of their plastic dissipation mechanism. From this point of view, the former approaches have been appealing. However, there have been four limitations that have in the past been overlooked and were lingering when adopting stress-dilatancy relationships developed for soils to rocks and rock masses. These were:
- The Coulomb friction rule which is less suitable for rocks and rock masses is assumed
- Only loading is considered
- Conditions of coaxiality between Eigen directions of stresses and plastic strains.
- Shear mode dependency of the stress-dilatancy relationship

The first limitation has been lifted in Tsegaye and Benz [5] who formulated a stress-dilatancy relationship for the Hoek-Brown material assuming the Hoek-Brown criterion holds good for rocks and rock masses. The objective of this paper is to further extend the framework such that the stress-dilatancy relation is relived of the mentioned limitations.
2. The coaxial stress-dilatancy relation for the Hoek-Brown material

The Hoek-Brown criterion [6-9] is one of the criteria frequently used for rocks and rock masses. The principal stress ratio, \( N_{\sigma}^{HB} = \frac{\sigma_1}{\sigma_3} \), and the modified residual strength derived from the generalized Hoek-Brown criterion may be written as

\[
N_{\sigma-peak}^{HB} = 1 + \Gamma_{\text{peak}}, \quad \Gamma_{\text{peak}} = \bar{b} \left( \frac{m_b}{b} + s \right) a, \quad \bar{b} = \frac{\sigma_{ci}}{\sigma_3}
\]

and

\[
K_{\sigma}^{HB} = 1 + \bar{\Gamma}, \quad \bar{\Gamma} = f_{sd} \bar{b} \left( \frac{\bar{m}_b}{b} + \bar{s} \right) \bar{a}
\]

respectively, where \( \sigma_1 \) and \( \sigma_3 \) are respectively the major and the minor principal effective stresses. \( \sigma_{ci} \) is the uniaxial compressive strength of intact rock. The parameters \( m_b, s \) and \( s \) are constants which depend on the rock mass characteristics-the Geological Strength Index (GSI) and disturbance factor (D), Appendix 1. The circumflex letters, \( \bar{m}_b, \bar{s} \) and \( \bar{a} \), in Equation 2 are Hoek-Brown parameters at the residual state. Eq. (2) is assuming that the Hoek-Brown criterion governs the stress ratio since the onset of plastic deformation to the residual state. It is further assumed that plastic volumetric changes cease at the residual state, i.e., constant volume condition is reached.

The following stress-dilatancy relation was then established in Tsegaye and Benz [5]

\[
K_{\sigma}^{HB} m_s N_{\psi=\text{peak}} = N_{\sigma-peak}^{HB},
\]

where \( N_{\psi} = -\frac{\varepsilon_P^p}{\varepsilon_1^p} \) is the negative of the ratio of the minor principal plastic strain rate \( (\varepsilon_P^p) \) and the major plastic strain rate \( (\varepsilon_1^p) \), \( m_s \) is a number related to the mode of shearing \( (m_s = 2 \) for triaxial compression, \( m_s = 1 \) for plane strain and \( m_s = 1/2 \) for triaxial extension).

Eq.(3) was then employed in back calculating the peak dilatancy angle of triaxial tests of predominantly weak samples of sandstone and mudstone by Farmer [10]. The tests were performed at different confining pressures. Published test data included peak strength, residual strength, and volumetric strains. Peak strength and residual strength for the rock types considered are shown in Figure 6. Continuous lines in the same figure depict Hoek-Brown criteria calculated from material parameters given in Table 1. For intact samples, GSI = 100 and D = 0 are assumed. Residual strength is fitted by reducing GSI to GSIres as given in Table 1.

It should be noted that GSI was originally not intended to be used as a parameter to describe residual rock strength. Nevertheless, the fit to existing data is reasonable; although not particularly good at low confining pressures (Figure 1). Eq. (3) is used in the following and as only peak dilatation angle will be derived, the function \( f_{sd} \) is considered constant and its value is derived assuming \( \psi_{\text{peak}} = \varphi_{\text{peak}} \) at confining stress \( \sigma_3 = 0 \), which requires that \( f_{sd} = \frac{\varphi_{\text{peak}}}{\psi_{\text{peak}}} \). This assumption is due to Alejano and Alonso [4] who observed associated plasticity in many rock samples at low confining pressures.

Using Eqs. (1) and (2) into Eq. (3), the dilatancy angle in triaxial compression condition, i.e., \( m_s = 2 \), can be obtained as

\[
\psi_{\text{peak}} = \sin^{-1} \left( \frac{\Gamma_{\text{peak}} - \bar{\Gamma}}{2 + \Gamma_{\text{peak}} + \bar{\Gamma}} \right).
\]

A comparison between calculated and experimentally obtained dilatancy angles is shown in Figure 2. The measured dilatancy angles depicted in Figure 2 were interpreted from Farmer’s tests by Alejano.
and Alonso [4], who also proposed an empirical peak dilatancy angle-peak friction angle relationship for rock given by

\[
\psi_{\text{peak}} = \frac{\varphi_{\text{peak}}}{1 + \log_{10} \frac{\sigma_{ci}}{\sigma_3 + 0.1}},
\]

(5)

where \(\varphi_{\text{peak}}\) is the peak friction angle derived from the slope of the non-linear Hoek-Brown criterion at \(\sigma_3 = 0\).

For comparison, results from Eq.(5) are also shown in Figure 2. The Hoek-Brown criteria tuned for the peak strength shown in Figure 1 were used to evaluate \(\psi_{\text{peak}}\). The comparison shown in Figure 2 revealed that the flow rule obtained from the stress-dilatancy formulation in Eq.(4) is in reasonable agreement with experimental data. However, the prediction capabilities of the formulation depend on the appropriateness of the underlying function that governs the stress ratio as can be seen in the mudstone example; at low confining pressures, the dilatancy angle is underestimated because of overestimation of the residual stress ratio by the Hoek-Brown criterion.

The reduction in the dilatancy angle with confining pressure comes naturally in the stress-dilatancy relation presented in Eq.(3). It can also be seen in Figure 2 that the higher the minor principal stress, the more ductile is the sample. Consequently, the maximum dilatancy angle goes to zero. For \(f_{sd} = 1\), this can be proven mathematically as the limit, \(b \to 0\), \(m_{\psi_{\text{peak}}} = 1\) into Eq.(4), one obtains a peak dilatancy angle of zero. This corresponds to a ductile (contractive) behaviour thus in line with the observation that ductility increases with confining pressure [11, 12].

**Table 1.** Hoek-Brown parameters for Farmer’s (1993) tests[10], Tsegaye and Benz [5]

| Material | \(\sigma_{ci}\) | GSI | \(m_i\) | D | GSI\(_{res}\) |
|----------|----------------|-----|--------|---|------------|
| Sandstone | 93            | 100 | 19     | 0 | 45         |
| Mudstone  | 50            | 100 | 4      | 0 | 80         |

![Figure 1](image-url). Peak and residual stress ratios for Sandstone and Mudstone from tests by Farmer [10]. Continuous lines give Hoek-Brown curves for the material parameters in Table 1, Tsegaye and Benz [5]
3. Non-coaxial stress-dilatancy relation under loading and unloading of a Hoek-Brown material

3.1 Plane strain

The preliminary validation of the stress-dilatancy relationship deduced in Tsegaye and Benz [5] and presented in Section 2 implies that the framework captures the required characteristics of the stress-dilatancy relationship for rocks and rock masses. We are going to follow it up next, mainly from theoretical point of view and relive the stress-dilatancy relation from the assumption of coaxiality while at the same time both loading and unloading are explicitly considered. We employ the non-coaxial stress-dilatancy relation in [13, 14] written here as

\[ N_\psi = \frac{N_\sigma + K_\sigma \tan^2 \Delta}{K_\sigma + N_\sigma \tan^2 \Delta'} \]  

where \( \Delta \) is the degree of non-coaxiality between eigen directions of stresses and strains. Eq. (6) has been explored for a Mohr-Coulomb material, [13, 14] where also the \( K_\sigma \) for loading and unloading are related according to \( K_\sigma^L = 1/K_\sigma^U \). This assumption is considered here as well.

Eq. (6) implies that non-coaxiality decreases plastic dissipation. This seems to be the case in soil tests that involve principal stress rotation, Figure 3. Here, it is assumed that the principle holds for rocks and soils alike. In the following, the stress-dilatancy relation is derived for loading and unloading separately.
i. Loading in shear

Loading in shear is here defined as mobilizing away from isotropy. Substitution of Eqs. (1) and (2) into Eq. (6) leads to [13, 16]

$$N^L_H = \frac{N^L_H \tan^2 \Delta}{K^L_H + N^L_H \tan^2 \Delta}$$  \hspace{1cm} (7)

Then, the dilatancy angle, \(\psi_m^{HB} \), is obtained as

$$\sin \psi_m^{HB} = -\frac{N^L_H \tan \Delta}{N^L_H + 1}.$$  \hspace{1cm} (8)

Some further rearrangement of Eq.(7) using the definition in Eq.(8) leads to:

$$\sin \psi_m^{HB} \tan \Delta = -c \cos 2\Delta$$  \hspace{1cm} (9)

in which \(c = \cos 2\Delta \) is the degree of coaxiality. The relationship is illustrated in Figure 4 for different values of degrees of non-coaxiality, \(\Delta\).
Figure 4. Plots of dilatancy ratio and dilatancy angle against normalized stress ratio for different values of degree of non-coaxiality for loading in shear.

ii. Unloading in shear

Unloading in shear is here defined as mobilizing towards isotropy. For obtaining the stress-dilatancy relation during unloading in shear $K_{U}^\sigma = 1/K_{N}^{HB}$ is considered into in Eq. (6). Accordingly,

$$N_{\psi}^{HB,U} = \frac{K_{HB}^{HB}N_{\psi}^{HB} + \tan^{2}\Delta}{1 + K_{HB}^{HB}N_{\psi}^{HB}\tan^{2}\Delta}$$

(10)

is obtained. Further substituting Eqs. (1) and (2) into Eq. (10) and considering the definition in Eq. (8), one obtains

$$\sin \psi_{m}^{HB,U} = -c_{\Delta} \frac{\Gamma + \hat{\Gamma} + \hat{\Gamma}^*}{2 + \Gamma + \hat{\Gamma} + \hat{\Gamma}^*}$$

(11)

The coaxial version is recovered for $c_{\Delta} = 1$. The relationship is illustrated in Figure 5 for different values of degrees of non-coaxiality. Note that unloading in shear, according to this scheme is necessarily contractive.
Figure 5. Plots of dilatancy angle and dilatancy ratio against the stress ratio scaled up by $K_{NB}$ for different values of degrees of non-coaxiality for unloading in shear.

The resulting non-coaxial stress-dilatancy relation for the Hoek-Brown material implies that the higher the degree of non-coaxiality, the lower is the dilatancy angle. In soils, it is known that the degree of coaxiality increases with increasing stress ratio. A similar trend may be expected for rocks and rock masses as well. However, experimental evidences are lacking for rocks and rock masses.

3.2 Stress-dilatancy formulation in the generalized stress-plastic strain increment

So far specific modes of shear were considered, and the influence of the intermediate stress was not considered. In this section, the stress-dilatancy relationship is generalised for arbitrary modes of shear, defined by a Lode angle $\theta$, considering the non-coaxial plastic dissipation in Tsegaye [13],

$$D_M^\theta = p \dot{\varepsilon}_q^P c \alpha M_M^\theta \geq 0,$$

(12)

where $p$ is the effective confining stress, $\dot{\varepsilon}_q^P$ is the magnitude of the deviatoric strain rate, $c$ is the degree of coaxiality ($0 \leq c \leq 1$), $\alpha M_M^\theta$ is a material constant, which is likely to depend on the mode of shear, $M_M^\theta$ is the stress ratio defined as the ratio of deviatoric stress to the effective confining stress, and $s = 1$ for loading in shear and $s = -1$ for unloading in shear.

The plastic dissipation equation in Eq. (12) implies a stress-dilatancy relation

$$M^\theta = -\dot{\varepsilon}(s M_M^\theta - C_M^\theta).$$

(13)

where $M_M^\theta$ is the ratio of the negative of the plastic volumetric rate ($\dot{\varepsilon}_V^P$) to the magnitude of plastic deviatoric strain rate ($\dot{\varepsilon}_q^P$).

For using the Hoek-Brown criterion in this framework, the stress ratios for the triaxial compression mode and the triaxial extension mode of shear are defined as

$$M_{\sigma}^{H.B,C} = \frac{N_{\sigma}^{H.B}}{N_{\sigma}^{H.B} + 2} = \frac{3\Gamma}{\Gamma + 3},$$

(14)
\[ M_{\sigma}^{HB,E} = 3 \frac{N_{\sigma}^{HB-1}}{2N_{\sigma}^{HB+1}} = \frac{3\Gamma}{2\Gamma+3} \]  

where \( N_{\sigma}^{HB} \) is as defined in Eq.(1). The stress ratio at the constant volumetric strain condition is to be defined accordingly by replacing \( N_{\sigma}^{HB} \) by \( K_{\sigma}^{HB} \). This is assuming that, during continuous shearing, the stress ratio in Eq. (2) is reached at constant volumetric strain condition where the tendency for further dilation ceases. The superscript \( C/E \) indicate triaxial compression/extension. The shear mode dependency for an arbitrary mode of shear is considered through a convenient Lode angle dependency function say

\[ M_{\phi}^{HB} = -\tilde{c}(s l_{\theta} M_{\sigma}^{HB,C} - \bar{l}_{\theta} M_{K}^{HB,E}) = -3\tilde{c} \left( s l_{\theta} \frac{p}{\Gamma+3} - \bar{l}_{\theta} \frac{p}{\Gamma+3} \right), \]  

where \( l_{\theta} \) and \( \bar{l}_{\theta} \) are appropriate Lode angle dependency functions for the current stress ratio and for residual stress ratio respectively. Both have, by definition, a value of 1 for the triaxial compression, and respectively

\[ l_{\theta=\pi/6} = M_{\sigma}^{HB,E} M_{\sigma}^{HB,C} = \frac{N_{\sigma}^{HB}+2}{2N_{\sigma}^{HB+1}} = \frac{\Gamma+3}{2\Gamma+3} \]  

and

\[ \bar{l}_{\theta=\pi/6} = M_{K}^{HB,E} M_{K}^{HB,C} = \frac{R_{K}^{HB}+2}{2R_{K}^{HB+1}} = \frac{\bar{\Gamma}+3}{2\bar{\Gamma}+3} \]  

for the triaxial extension condition. Eq.(16) is now the desired non-coaxial stress-dilatancy relationship for a general stress-strain condition. The relationship is illustrated in Figure 6 for the triaxial compression case considering both loading and unloading conditions. There are several Lode angle dependency functions developed for soils, e.g., [13, 17]. It has also seen that the yielding stresses for rocks and rock masses are dependent on the mode of shear, e.g., [18, 19]. Lode angle dependency functions that are appropriate for rocks and rock masses need to be investigated further.

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**Figure 6.** Plots of normalized dilatancy ratio against normalized stress ratio for triaxial compression mode of shear and different values of the degree of coaxiality according to the stress dilatancy relation in Eq.(16) a) loading and b) unloading.
4. Summary and Conclusions

In this paper, a stress-dilatancy framework for the Hoek-Brown material is presented. The framework is developed for both plane strain and general stress-strain conditions and is intended for use in constitutive modelling of rocks and rocks masses. Loading and unloading are explicitly considered and effect of non-coaxiality between eigen directions of plastic strain increments and stresses is considered. It is seen that

- preliminary evaluation of the coaxial version using triaxial tests show that the framework is promising.
- the function that governs the stress-ratio reflects in the stress dilatancy relation.
- the framework presented is flexible and can be used for any other function that governs the stress-ratio.
- unloading in shear leads to plastic contraction.
- the degree of non-coaxiality has a strong influence on the stress-dilatancy relation and the plastic dissipation.

The framework is not limited to the Hoek-Brown criterion. Following a similar procedure, it can be applied for any other sensible function that describes the stress ratio. Supplementary experiments are lacking on the non-coaxial deformation behaviour of rocks and rock masses. This needs to be considered in future studies.

Appendix

The Hoek-Brown constant $m_l$ for rock mass given by

$$m_b = m_l \exp \left( \frac{GSI - 100}{28 - 14D} \right),$$

(1)

The constants $s$ and $a$ are model parameters that depend upon the rock mass characteristics and are given by empirical equations as

$$s = m_l \exp \left( \frac{GSI - 100}{9 - 3D} \right),$$

(2)

$$a = \frac{1}{2} + \frac{1}{6} \exp \left( - \frac{GSI}{15} \right) - \exp \left( - \frac{20}{3} \right).$$

(3)

GSI is Geological Strength Index and $D$ is a factor that quantifies the disturbance of the rock mass (See [20] for details and ranges of the parameters).
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