Decomposition of compact exceptional Lie groups into their maximal tori

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Abstract. In this paper we treat the intersection of fixed point subgroups by the involutive automorphisms of exceptional Lie group $G = F_4, E_6, E_7$. We shall find involutive automorphisms of $G$ such that the connected component of the intersection of those fixed point subgroups coincides with the maximal torus of $G$.

1. Introduction

It is known that the involutive automorphisms of the compact Lie groups play an important role in the theory of symmetric space (c.f. Berger [1]). In [8],[9] Yokota showed that the exceptional symmetric spaces $G/H$ are realized definitely by calculating the fixed point subgroup of the involutive automorphisms $\tilde{\gamma}, \tilde{\gamma}', \tilde{\sigma}, \tilde{\sigma}'$ of $G$, where $\tilde{\gamma}, \tilde{\gamma}', \tilde{\sigma}, \tilde{\sigma}'$ are induced by $\mathbb{R}$-linear transformations $\gamma, \gamma', \sigma, \sigma'$ of $J$ and $\tilde{\iota}$ is induced by $C$-linear transformation $\iota$ of $\mathfrak{P}^C$. Here $\gamma, \gamma' \in G_2 \subset F_4 \subset E_6 \subset E_7$, $\sigma, \sigma' \in F_4 \subset E_6 \subset E_7$ and $\iota \in E_7$. For the cases of the graded Lie algebras $\mathfrak{g}$ of the second kind and third kind, the corresponding subalgebras $\mathfrak{g}_0, \mathfrak{g}_{ev}, \mathfrak{g}_{od}$ of $\mathfrak{g}$ are realized as the intersection of those fixed point subgroups of the commutative involutive automorphisms ([3],[6],[7],[10],[11],[12]).

In [2],[4],[5] we determined the intersection of those fixed point subgroups of the involutive automorphisms of $G$ when $G$ is a compact exceptional Lie group. We remark that those intersection subgroups are maximal rank of $G$.

In general, let $G$ be a connected compact Lie group and $\sigma_1, \sigma_2, \cdots, \sigma_m$ commutative automorphism elements of $G$. Set $G^{\sigma_1, \sigma_2, \cdots, \sigma_k} = \{ \alpha \in G \mid \sigma_i \alpha = \alpha \sigma_i, i = 1, \cdots, k \}$. We expect that the group $G^{\sigma_1, \sigma_2, \cdots, \sigma_k}$ is a maximal rank subgroup of $G$. Consider the following degreasing sequence of subgroups of $G$:

$$ G^{\sigma_1} \supset G^{\sigma_1, \sigma_2} \supset \cdots \supset G^{\sigma_1, \cdots, \sigma_m}. $$

Let $T^l$ be the maximal tours of $G$. In this paper we would like to find $\sigma_1, \sigma_2, \cdots, \sigma_m$ such that the connected component subgroup $(G^{\sigma_1, \sigma_2, \cdots, \sigma_k})_0$ of the group $G^{\sigma_1, \sigma_2, \cdots, \sigma_k}$
is isomorphic to $T^4$ when $G$ is simply connected compact exceptional Lie groups $G_2, F_4, E_6$ or $E_7$. For the case $G = G_2$, we prove that the group $((G_2)^{\gamma, \gamma'})_0 \cong T^2$ by [5], Theorem 1.1.3. Then we shall prove the following:

1. $((F_4)^{\gamma, \gamma', \sigma, \sigma'})_0 \cong T^4$.
2. $((E_6)^{\gamma, \gamma', \sigma, \sigma'})_0 \cong T^6$.
3. $((E_7)^{\gamma, \gamma', \sigma, \sigma'})_0 \cong T^7$.

For the case $G = E_8$, we conjecture that the group $((E_8)^{\gamma, \gamma', \sigma, \sigma', \nu_3})_0 \cong T^8$, where $\lambda' \in E_8$ (As for $\nu_3$, see [3]).

2. Group $F_4$

The simply connected compact Lie group $F_4$ is given by the automorphism group of the exceptional Freudenthal algebra $\mathfrak{f}$:

$$F_4 = \{ \alpha \in \text{Iso}_\mu(\mathfrak{f}) | \alpha(X \times Y) = \alpha X \times \alpha Y \}.$$ 

We shall review the definitions of $R$-linear transformations $\gamma, \gamma', \sigma, \sigma'$ of $\mathfrak{f}$([8], [10], [12]).

Firstly we define $R$-linear transformations $\gamma, \gamma'$ of $\mathfrak{f}$ by

$$\gamma(X + M) = X + \gamma(m_1, m_2, m_3) = X + (\gamma m_1, \gamma m_2, \gamma m_3),$$

$$\gamma'(X + M) = X + \gamma'(m_1, m_2, m_3) = X + (\gamma' m_1, \gamma' m_2, \gamma' m_3),$$

$$\gamma_1(X + M) = X + \gamma(m_1, m_2) = X + M \in \mathfrak{f} \oplus M(3, C) = \mathfrak{f},$$

respectively, where $\mathfrak{f} = \{ X \in M(3, C) | X^* = X \}$, the right-hand side transformations $\gamma, \gamma' : C^3 \to C^3$ are defined by

$$\gamma \left( \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} ight) = \begin{pmatrix} n_1 \\ -n_2 \\ -n_3 \end{pmatrix}, \quad \gamma' \left( \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} ight) = \begin{pmatrix} -n_1 \\ n_2 \\ -n_3 \end{pmatrix}, \quad n_i \in C.$$ 

Then $\gamma, \gamma', \gamma_1 \in G_2 \subset F_4$, and $\gamma^2 = \gamma'^2 = \gamma_1^2 = 1$.

Further we define $R$-linear transformations $\sigma$ and $\sigma'$ of $\mathfrak{f} \oplus M(3, C) = \mathfrak{f}$ by

$$\sigma(X + M) = \sigma X + (m_1, -m_2, -m_3),$$

$$\sigma'(X + M) = \sigma' X + (-m_1, -m_2, m_3), \quad X + M \in \mathfrak{f} \oplus M(3, C) = \mathfrak{f},$$

respectively, where the right-hand side transformations $\sigma, \sigma' : \mathfrak{f} \to \mathfrak{f}$ are defined by

$$\sigma X = \sigma \left( \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ x_2 \\ x_1 \\ \xi_3 \end{pmatrix} \right) = \begin{pmatrix} \xi_1 \\ -\xi_2 \\ -\xi_3 \\ x_2 \\ \xi_1 \\ \xi_3 \end{pmatrix},$$

$$\sigma' X = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ -x_2 \\ -\xi_1 \\ \xi_3 \end{pmatrix}.$$ 

Then $\sigma, \sigma' \in F_4$ and $\sigma^2 = \sigma'^2 = 1$. 
The group $Z_2 = \{1, \gamma_1\}$ acts on the group $U(1) \times U(1) \times SU(3)$ by
\[\gamma_1(p, q, A) = (p, \overline{q}, A).\]
Hence the group $Z_2 = \{1, \gamma_1\}$ acts naturally on the group $(U(1) \times U(1) \times SU(3))/Z_3$.

Let $(U(1) \times U(1) \times SU(3)) \cdot Z_2$ be the semi-direct product of those groups under this action.

Hereafter, $\omega_1$ denotes $\frac{1}{2} + \frac{\sqrt{3}}{2} e_1 \in C$.

**Proposition 2.1.** $(F_4)^{\gamma, \gamma'} \cong ((U(1) \times U(1) \times SU(3))/Z_3) \cdot Z_2$, $Z_3 = \{(1, 1, E), (\omega_1, \omega_1, \omega_1 E), (\omega_1^2, \omega_1^2, \omega_1^2 E)\}$.

**Proof.** We define a mapping $\varphi_4 : (U(1) \times U(1) \times SU(3)) \cdot Z_2 \to (F_4)^{\gamma, \gamma'}$ by
\[\varphi_4((p, q, A), 1)(X + M) = AXA^* + D(p, q)MA^*,\]
\[\varphi_4((p, q, A), \gamma_1)(X + M) = \overline{AXA^*} + D(p, q)\overline{MA^*},\]
where $D(p, q) = \text{diag}(p, q, \overline{q}) \in SU(3)$. Then $\varphi_4$ induces the required isomorphism (see [5] for details). \(\Box\)

**Lemma 2.2.** The mapping $\varphi_4 : (U(1) \times U(1) \times SU(3)) \cdot Z_2 \to (F_4)^{\gamma, \gamma'}$ satisfies
\[\sigma = \varphi_4((1, 1, E_{1,-1}), 1), \quad \sigma' = \varphi_4((1, 1, E_{-1,1}), 1),\]
where $E_{1,-1} = \text{diag}(1, -1, -1)$, $E_{-1,1} = \text{diag}(-1, -1, 1) \in SU(3)$.

We denote $U(1) \times \cdots \times U(1)$, $(1, \cdots , 1)$ and $(\omega_k, \cdots , \omega_k)$ ($l$-times) by $U(1)^{\times l}$, $(1)^{\times l}$ and $(\omega_k)^{\times l}$, respectively.

Now, we determine the structures of the group $(F_4)^{\gamma, \gamma', \sigma, \sigma'} = ((F_4)^{\gamma, \gamma'})^{\sigma, \sigma'}$.

**Theorem 2.3.**
\[((F_4)^{\gamma, \gamma', \sigma, \sigma'})_0 \cong U(1)^{\times 4}.\]

**Proof.** For $\alpha \in (F_4)^{\gamma, \gamma', \sigma, \sigma'} \subset (F_4)^{\gamma, \gamma'}$, there exist $p, q \in U(1)$ and $A \in SU(3)$ such that $\alpha = \varphi_4((p, q, A), 1)$ or $\alpha = \varphi_4((p, q, A), \gamma_1)$ (Proposition 2.1). For the case of $\alpha = \varphi_4((p, q, A), 1)$, by combining the conditions of $\sigma \alpha \sigma = \alpha$ and $\sigma' \alpha \sigma' = \alpha$ with Lemma 2.2, we have
\[\varphi_4((p, q, E_{1,-1}AE_{1,-1}), 1) = \varphi_4((p, q, A), 1)\]
and
\[\varphi_4((p, q, E_{-1,1}AE_{-1,1}), 1) = \varphi_4((p, q, A), 1).\]
Hence
\[(i) \quad E_{1,-1}AE_{1,-1} = A, \quad (ii) \quad \begin{cases} p = \omega_1 p \\ q = \omega_1 q \end{cases} \quad (iii) \quad \begin{cases} p = \omega_1^2 p \\ q = \omega_1^2 q \end{cases} \quad E_{1,-1}AE_{1,-1} = \omega_1 A, \quad E_{1,-1}AE_{1,-1} = \omega_1^2 A\]
and

(iv) \( E_{-1,1}AE_{-1,1} = A \)

\[
\begin{cases}
  p = \omega_1 p \\
  q = \omega_1 q \\
  E_{-1,1}AE_{-1,1} = \omega_1 A,
\end{cases}
\]

\[
\begin{cases}
  p = \omega_1^2 p \\
  q = \omega_1^2 q \\
  E_{-1,1}AE_{-1,1} = \omega_1^2 A.
\end{cases}
\]

We can eliminate the case (ii), (iii), (v) or (vi) because \( p \neq 0 \) or \( q \neq 0 \). Hence we have \( p, q \in U(1) \) and \( A \in SU(1) \times U(1) \times U(1) \). Since the mapping \( U(1) \times U(1) \rightarrow SU(1) \times U(1) \times U(1) \),

\[ h(a_1, a_2) = (a_1, a_2, \bar{a}_1a_2) \]

is an isomorphism, the group satisfying with the conditions of case (i) and (iv) is \((U(1)^4)/Z_3\). For the case of \( \alpha = \varphi_4((p, q, A), \gamma_1) \), from \( \varphi_4((p, q, A), \gamma_1) = \varphi_4((p, q, A), 1)\gamma_1, \varphi_4((1, 1, E_1, -1), 1)\gamma_1 = \gamma_1\varphi_4((1, 1, E_1, -1), 1) \) and \( \varphi_4((1, 1, E_1, -1), 1)\gamma_1 = \gamma_1\varphi_4((1, 1, E_1, -1), 1) \), this case is in the same situation as above. Thus we have \( (F_4)\gamma, \gamma', \sigma, \sigma' \cong \{(U(1)^4)/Z_3\} \cdot Z_2 \), \( Z_3 = \{(w_1)^4, (w_1^2)^4\} \). The group \((U(1)^4)/Z_3\) is naturally isomorphic to the torus \((U(1)^4)\), hence we obtain \((F_4)\gamma, \gamma', \sigma, \sigma' \cong \{(U(1)^4)\} \cdot Z_2 \).

Therefore we have the required isomorphism of the theorem.

3. The group \( E_6 \)

The simply connected compact Lie group \( E_6 \) is given by

\[ E_6 = \{ \alpha \in \text{Iso}_C(3^C) \mid \alpha X \times \alpha Y = \tau_\alpha X \times Y, (\alpha X, \alpha Y) = (X, Y) \}. \]

\( R \)-linear transformations \( \gamma, \gamma', \gamma_1, \sigma \) and \( \sigma' \) of \( \mathfrak{j} = \mathfrak{j}_C \oplus \mathfrak{M}(3, C) \) are naturally extended to the \( C \)-linear transformations of \( \gamma, \gamma', \gamma_1, \sigma \) and \( \sigma' \) of \( \mathfrak{j}_C \oplus \mathfrak{M}(3, C)^C \).

Then we have \( \gamma, \gamma', \gamma_1, \sigma, \sigma' \in E_6 \).

The group \( Z_2 = \{1, \gamma_1\} \) acts on the group \( U(1) \times U(1) \times SU(3) \times SU(3) \) by

\[ \gamma_1(p, q, A, B) = (p, q, B, \bar{A}). \]

Hence the group \( Z_2 = \{1, \gamma_1\} \) acts naturally on the group \( (U(1) \times U(1) \times SU(3) \times SU(3))/Z_3 \).

Let \( (U(1) \times U(1) \times SU(3) \times SU(3))/Z_3 \cdot Z_2 \) be the semi-direct product of those groups under this action.

**Proposition 3.1.** \( (E_6)\gamma, \gamma' \cong \{(U(1) \times U(1) \times SU(3) \times SU(3))/Z_3 \} \cdot Z_2 \), \( Z_3 = \{(1, 1, E, E), (\omega_1, \omega_1, \omega_3 \epsilon, \omega_1 E), (\omega_1^2, \omega_1^2, \omega_1^2 E, \omega_1^2 E)\} \).

**Proof.** We define a mapping \( \varphi_6 : (U(1) \times U(1) \times SU(3) \times SU(3))/Z_2 \rightarrow (E_6)\gamma, \gamma' \) by

\[ \varphi_6((p, q, A, B), 1)(X + M) = h(A, B)Xh(A, B)^* + D(p, q)M\tau h(A, B)^*, \]

\[ \varphi_6((p, q, A, B), \gamma_1)(X + M) = h(A, B)Xh(A, B)^* + D(p, q)M\tau h(A, B)^*, \]

\[ X + M \in (\mathfrak{j}_C)^C \oplus \mathfrak{M}(3, C)^C = \mathfrak{j}_C. \]
Here $D(p, q) = \text{diag}(p, q, \overline{pq}) \in SU(3)$ and $h : M(3, C) \times M(3, C) \rightarrow M(6, C)^C$ is defined by
\[
h(A, B) = \frac{A + B}{2} + i\frac{A - B}{2}e_1.
\]
Then $\varphi_6$ induces the required isomorphism (see [5] for details). \hfill \Box

**Lemma 3.2.** The mapping $\varphi_6 : (U(1) \times U(1) \times SU(3) \times SU(3)) \cdot Z_2 \rightarrow (E_6)^{\gamma, \gamma'}$ satisfies
\[
\sigma = \varphi_6((1, 1, E_{1,-1}, E_{1,-1}), 1), \quad \sigma' = \varphi_6((1, 1, E_{-1,1}, E_{-1,1}), 1).
\]

The group $Z_2 = \{1, \gamma_1\}$ acts on the group $U(1)^6$ by
\[
\gamma_1(p, q, a_1, a_2, a_3, a_4) = (\bar{p}, \bar{q}, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4).
\]
Let $(U(1)^6) \cdot Z_2$ be the semi-direct product of those groups under this action.

Now, we determine the structures of the group $(E_6)^{\gamma, \gamma', \sigma, \sigma'} = ((E_6)^{\gamma, \gamma'})^{\sigma, \sigma'}$.

**Theorem 3.3.** \((E_6)^{\gamma, \gamma', \sigma, \sigma'})_0 \cong U(1)^6\).

**Proof.** For $\alpha \in (E_6)^{\gamma, \gamma', \sigma, \sigma'} \subset (E_6)^{\gamma, \gamma'}$, there exist $p, q \in U(1)$ and $A, B \in SU(6)$ such that $\alpha = \varphi_6((p, q, A, B), 1)$ or $\alpha = \varphi_6((p, q, A, B), \gamma_1)$ (Proposition 3.1). For the case of $\alpha = \varphi_6((p, q, A, B), 1)$, by combining the conditions $\sigma \alpha \sigma = \alpha$ and $\sigma' \alpha \sigma' = \alpha$ with Lemma 3.2, we have
\[
\varphi_6((p, q, E_{1,-1}AE_{1,-1}, E_{1,-1}BE_{1,-1}), 1) = \varphi_6((p, q, A, B), 1)
\]
and
\[
\varphi_6((p, q, E_{-1,1}AE_{-1,1}, E_{-1,1}BE_{-1,1}), 1) = \varphi_6((p, q, A, B), 1).
\]
Hence
\[
(i) \quad \begin{cases} E_{1,-1}AE_{1,-1} = A \\ E_{1,-1}BE_{1,-1} = B \end{cases}, \quad (ii) \quad \begin{cases} p = \omega_1 p \\ q = \omega_1 q \end{cases}, \quad (iii) \quad \begin{cases} p = \omega_1^2 p \\ q = \omega_1^2 q \end{cases}
\]
\begin{align*}
E_{1,-1}AE_{1,-1} &= \omega_1 A \\ E_{1,-1}BE_{1,-1} &= \omega_1 B
\end{align*}
and
\[
(iv) \quad \begin{cases} E_{-1,1}AE_{-1,1} = A \\ E_{-1,1}BE_{-1,1} = B \end{cases}, \quad (v) \quad \begin{cases} p = \omega_1 p \\ q = \omega_1 q \end{cases}, \quad (vi) \quad \begin{cases} p = \omega_1^2 p \\ q = \omega_1^2 q \end{cases}
\]
\begin{align*}
E_{-1,1}AE_{-1,1} &= \omega_1 A \\ E_{-1,1}BE_{-1,1} &= \omega_1 B
\end{align*}

We can eliminate the case (ii), (iii), (v) or (vi) because $p \neq 0$ or $q \neq 0$. Thus we have $p, q \in U(1)$ and $A, B \in SU(1)^3$. Since the mapping $U(1)^4 \rightarrow SU(1)^5$,
\[
h(a_1, a_2, a_3, a_4) = (a_1, a_2, a_3, a_4, \overline{a_1a_2a_3a_4})
\]
is an isomorphism, the group satisfying with the conditions of case (i) and (iv) is \((U(1)^{\times 6})/\mathbb{Z}_3\). For the case of \(\alpha = \varphi_6((p, q, A, B), \gamma_1)\), from \(\varphi_6((p, q, A, B), 1)\gamma_1 = \varphi_6((1, 1, E_{1,-1}, E_{1,-1}), 1)\gamma_1 = \gamma_1\varphi_6((1, 1, E_{1,-1}, E_{1,-1}), 1)\) and \(\varphi_6((1, 1, E_{-1,-1}, E_{-1,-1}), 1)\gamma_1 = \gamma_1\varphi_6((1, 1, E_{-1,-1}, E_{-1,-1}), 1)\), this case is in the same situation as above. Thus we have \((E_0)^{\gamma, \gamma', \sigma, \sigma'} \cong \left((U(1)^{\times 6})/\mathbb{Z}_3\right) \cdot \mathbb{Z}_2, \mathbb{Z}_2 = \{(1)^{\times 6}, (w_1)^{\times 6}, (w_1^2)^{\times 6}\}\). The group \((U(1)^{\times 6})/\mathbb{Z}_3\) is naturally isomorphic to the torus \(U(1)^{\times 6}\), hence we obtain \((E_0)^{\gamma, \gamma', \sigma, \sigma'} \cong (U(1)^{\times 6}) \cdot \mathbb{Z}_2\). Therefore we have the required isomorphism of the theorem.

4. Group \(E_7\)

Let \(\mathfrak{g}^C = \mathfrak{g}^C \oplus \mathfrak{g}^C \oplus C \oplus C\). The simply connected compact Lie group \(E_7\) is given by

\[
E_7 = \{ \alpha \in \text{Iso}_C(\mathfrak{g}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle \}.
\]

Under the identification \((\mathfrak{g}^C)^C \oplus (M(3, C)^C \oplus M(3, C)^C)\) with \(\mathfrak{g}^C : \left((X, Y, \xi, \eta), (M, N) = (X + M, Y + N, \xi, \eta)\right), C\)-linear transformations of \(\gamma, \gamma', \gamma_1, \sigma\) and \(\sigma'\) of \(\mathfrak{g}^C\) are extended to \(C\)-linear transformations of \(\mathfrak{g}^C\) as

\[
\begin{align*}
\gamma(X + M, Y + N, \xi, \eta) &= (X + \gamma M, Y + \gamma N, \xi, \eta), \\
\gamma'(X + M, Y + N, \xi, \eta) &= (X + \gamma' M, Y + \gamma' N, \xi, \eta), \\
\gamma_1(X + M, Y + N, \xi, \eta) &= (X + M, Y + N, \xi, \eta), \\
\sigma(X + M, Y + N, \xi, \eta) &= (\sigma X + \sigma M, \sigma Y + \sigma N, \xi, \eta), \\
\gamma(X + M, Y + N, \xi, \eta) &= (\gamma' X + \gamma' M, \gamma' Y + \gamma' N, \xi, \eta),
\end{align*}
\]

where \(\gamma M = \text{diag}(1, -1, -1) M, \gamma' M = \text{diag}(-1, -1, 1) M, \sigma M = M \text{diag}(1, -1, -1)\) and \(\sigma' M = M \text{diag}(-1, 1, -1)\).

Moreover we define a \(C\)-linear transformation \(\iota\) of \(\mathfrak{g}^C\) by

\[
\iota((X + M, Y + N, \xi, \eta) = (-iX - iM, iY + iN, -i\xi, i\eta).
\]

The group \(\mathbb{Z}_2 = \{1, \gamma_1\}\) acts the group \((U(1) \times U(1) \times SU(6))\) by

\[
\gamma_1(p, q, A) = (\overline{p}, \overline{q}, \overline{(\text{Ad}J_3)A}), \quad J_3 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.
\]

Hence the group \(\mathbb{Z}_2 = \{1, \gamma_1\}\) acts naturally on the group \((U(1) \times U(1) \times SU(6))/\mathbb{Z}_3\).

Let \((U(1) \times U(1) \times SU(6)) \cdot \mathbb{Z}_2\) be the semi-direct product of those groups under this action.

Proposition 4.1. \((E_7)^{\gamma, \gamma'} \cong ((U(1) \times U(1) \times SU(6))/\mathbb{Z}_3) \cdot \mathbb{Z}_2, \mathbb{Z}_2 = \{(1, 1, E), (\omega_1, \omega_1, \omega_1 E), (\omega_1^2, \omega_1^2, \omega_1^2 E)\}\).
PROOF. We define a mapping \( \varphi_7 : (U(1) \times U(1) \times SU(6)) \cdot \mathbf{Z}_2 \to (E_7)^{\gamma, \gamma'} \) by
\[
\varphi_7((p, q, A), 1)P = f^{-1}((D(p, q), A)(fP)), \\
\varphi_7((p, q, A), \gamma)P = f^{-1}((D(p, q), A)(f\gamma P)), \quad P \in \mathfrak{g}^F.
\]
Here \( D(p, q) = \text{diag}(p, q, \overline{pq}) \in SU(3) \) and the mapping \( f \) is defined in [9], Section 2.4. Then \( \varphi_7 \) induces the required isomorphism (see [5] for details).

\[\square\]

**Lemma 4.2.** The mapping \( \varphi_7 : (U(1) \times U(1) \times SU(6)) \cdot \mathbf{Z}_2 \to (E_7)^{\gamma, \gamma'} \) satisfies
\[
\sigma = \varphi_7((1, 1, F_{1,-1}), 1), \sigma' = \varphi_7((1, 1, F_{-1,1}), 1), \quad \tau = \varphi_7((1, 1, F_{e_1}), 1)
\]
where \( F_{1,-1} = \text{diag}(1, -1, -1, 1, -1, -1) \), \( F_{-1,1} = \text{diag}(-1, -1, 1, -1, -1, 1) \), \( F_{e_1} = \text{diag}(e_1, e_1, -e_1, -e_1, -e_1, -e_1) \in SU(6) \).

The group \( \mathbf{Z}_2 = \{1, \gamma_1\} \) acts on the group \( U(1)^7 \) by
\[
\gamma_1(p, q, a_1, a_2, a_3, a_4, a_5) = (p, q, a_1, a_2, a_3, a_4, a_5) = (p, q, a_1, a_2, a_3, a_4, a_5).
\]

Let \( (U(1)^7) \cdot \mathbf{Z}_2 \) be the semi-direct product of those groups under this action.

**Theorem 4.3.**
\[
((E_7)^{\gamma, \gamma', \sigma, \sigma', \iota})_0 \cong U(1)^7.
\]

**Proof.** For \( \alpha \in (E_7)^{\gamma, \gamma', \sigma, \sigma', \iota} \), there exist \( p, q \in U(1) \) and \( A \in SU(6) \) such that \( \alpha = \varphi_7((p, q, A), 1) \) or \( \alpha = \varphi_7((p, q, A), \gamma) \) (Proposition 4.1). For the case of \( \alpha = \varphi_7((p, q, A), 1) \), by combining the conditions \( \sigma a \sigma = \alpha, \sigma' a \sigma' = \alpha \) and \( \iota a \iota^{-1} = \alpha \) with Lemma 4.2, we have
\[
\varphi_7((p, q, F_{1,-1}AF_{1,-1}), 1) = \varphi_7((p, q, A), 1), \varphi_7((p, q, F_{-1,1}AF_{-1,1}), 1) = \varphi_7((p, q, A), 1)
\]
an and
\[
\varphi_7((p, q, F_{e_1}\text{AF}_{e_1}^{-1}), 1) = \varphi_7((p, q, A), 1).
\]

Hence
\[
\text{(i) } F_{1,-1}AF_{1,-1} = A, \quad \text{(ii)} \quad \begin{cases} p = \omega_1 p \\ q = \omega_1 q \end{cases}, \quad \text{(iii)} \quad \begin{cases} p = \omega_1^2 p \\ q = \omega_1^2 q \end{cases}, \quad \begin{cases} F_{1,-1}AF_{1,-1} = \omega_1 A, \\ F_{1,-1}AF_{1,-1} = \omega_1^2 A, \end{cases}
\]
\[
\text{(iv) } F_{-1,1}AF_{-1,1} = A, \quad \text{(v)} \quad \begin{cases} p = \omega_1 p \\ q = \omega_1 q \end{cases}, \quad \text{(vi)} \quad \begin{cases} p = \omega_1^2 p \\ q = \omega_1^2 q \end{cases}, \quad \begin{cases} F_{-1,1}AF_{-1,1} = \omega_1 A, \\ F_{-1,1}AF_{-1,1} = \omega_1^2 A, \end{cases}
\]
and
\[
\text{(vii) } F_{e_1}\text{AF}_{e_1}^{-1} = A, \quad \text{(viii)} \quad \begin{cases} p = \omega_1 p \\ q = \omega_1 q \end{cases}, \quad \text{(ix)} \quad \begin{cases} p = \omega_1^2 p \\ q = \omega_1^2 q \end{cases}, \quad \begin{cases} F_{e_1}\text{AF}_{e_1}^{-1} = \omega_1 A, \\ F_{e_1}\text{AF}_{e_1}^{-1} = \omega_1^2 A. \end{cases}
\]
We can eliminate the case (ii), (iii), (v), (vi), (viii) or (ix) because \( p \neq 0 \) or \( q \neq 0 \). Thus we have \( p, q \in U(1) \) and \( A \in S(U(1)^{\times 6}) \). Since the mapping \( U(1)^{\times 5} \to S(U(1)^{\times 6}) \),

\[
h(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_3, a_4, a_5, a_1a_2a_3a_4a_5)
\]
is an isomorphism, the group satisfying with the conditions of case (i),(iv) and (vii) is \( (U(1)^{\times 7})/\mathbb{Z}_3 \). For the case of \( \alpha = \varphi_7((p, q, A), \gamma_1) \), from \( \varphi_7((p, q, A), \gamma_1) = \varphi_7((p, q, A), 1))_1, \varphi_7((1, 1, F_{1-1}), 1))_1 = \gamma_1 \varphi_7((1, 1, F_{1-1}), 1))_1 = \gamma_1 \varphi_7((1, 1, F_{1-1}), 1) \) and \( \varphi_7((1, 1, F_{1-1}), 1))_1 = \gamma_1 \varphi_7((1, 1, F_{1-1}), 1) \), this case is in the same situation as above. Thus we have \( (E_7)^{\gamma_1, \sigma, \sigma', \iota} \cong (U(1)^{\times 7})/\mathbb{Z}_3 \cdot \mathbb{Z}_2, \mathbb{Z}_3 = \{(1)^{\times 7}, (w_1)^{\times 7}, (w_1^2)^{\times 7}\} \). The group \( (U(1)^{\times 7})/\mathbb{Z}_3 \) is naturally isomorphic to the torus \( U(1)^{\times 7} \), hence we obtain \( (E_7)^{\gamma_1, \sigma, \sigma', \iota} \cong (U(1)^{\times 7})/\mathbb{Z}_2 \). Therefore we have the required isomorphism of the theorem. \( \square \)

3. The group \( E_8 \)

In the \( C \)-vector space \( \mathfrak{e}_8^C \):

\[
\mathfrak{e}_8^C = \mathfrak{e}_{7}^C \oplus \mathfrak{p}_{1}^C \oplus \mathfrak{p}_{2}^C \oplus C \oplus C \oplus C,
\]
if we define the Lie bracket \([R_1, R_2]\) by

\[
[(\Phi_1, P_1, Q_1, r_1, u_1, v_1), (\Phi_2, P_2, Q_2, r_2, u_2, v_2)] = (\Phi, P, Q, r, u, v),
\]

\[
\left\{ \begin{array}{l}
\Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1 \\
P = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + u_1 Q_2 - u_2 Q_1 \\
Q = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + v_1 P_2 - v_2 P_1 \\
r = -\frac{1}{8}\{P_1, Q_2\} + \frac{1}{8}\{P_2, Q_1\} + u_1 v_2 - u_2 v_1 \\
u = \frac{1}{4}\{P_1, P_2\} + 2r_1 u_2 - 2r_2 u_1 \\
v = -\frac{1}{4}\{Q_1, Q_2\} - 2r_1 v_2 + 2r_2 v_1,
\end{array} \right.
\]

then, \( \mathfrak{e}_8^C \) becomes a simple \( C \)-Lie algebra of type \( E_8 \).

The group \( E_8^C \) is defined to be the automorphism group of the Lie algebra \( \mathfrak{e}_8^C \):

\[
E_8^C = \{ \alpha \in \text{Iso}_C(\mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2] \}.
\]

We define \( C \)-linear transformations \( \sigma, \sigma', \bar{\lambda} \) of \( \mathfrak{e}_8^C \) respectively by

\[
\sigma(\Phi, P, Q, r, u, v) = (\sigma P, \sigma Q, r, u, v),
\]

\[
\sigma'(\Phi, P, Q, r, u, v) = (\sigma P, \sigma Q, r, u, v),
\]

\[
\bar{\lambda}(\Phi, P, Q, r, u, v) = (\lambda P, \lambda Q, \lambda r, -r, -v, -u).}
\]
where
\[
\sigma \Phi(\phi, A, B, \nu) \sigma = \Phi(\sigma \phi \sigma, \sigma A, \sigma B, \nu),
\]
\[
\sigma' \Phi(\phi, A, B, \nu) \sigma' = \Phi(\sigma' \phi \sigma', \sigma' A, \sigma' B, \nu),
\]
\[
\lambda \Phi(\phi, A, B, \nu) \lambda^{-1} = \Phi(-\lambda \phi, -B, -A, -\nu).
\]
\((\sigma, \sigma', \lambda)\) of the left sides are the same ones used in [3]. Moreover, the complex conjugation in \(\mathfrak{c}_8\) is denoted by \(\tau\):
\[
\tau \Phi(\phi, P, Q, r, u, v) = (\tau \Phi \tau, \tau P, \tau Q, \tau r, \tau u, \tau v),
\]
where \(\tau \Phi(\phi, A, B, \nu) \tau = \Phi(\tau \phi \tau, \tau A, \tau B, \tau \nu)\).

Now, we define the Lie group \(E_8\) as a compact form of the complex Lie group \(E_8^C\) by
\[
E_8 = \{ \alpha \in E_8^C \mid \tau \lambda \alpha = \alpha \tilde{\lambda} \tau \}.
\]
Then, \(E_8\) is a simply connected compact simple Lie group of type \(E_8\). Note that \(\sigma, \sigma', \tilde{\lambda} \in E_8\). The Lie algebra \(\mathfrak{c}_8\) of the Lie group \(E_8\) is given by
\[
\mathfrak{c}_8 = \{ R \in \mathfrak{c}_8^C \mid \tau \lambda R = R \} = \{ (\Phi, P, -\tau \lambda P, r, u, -\tau u) \in \mathfrak{c}_8^C \mid \Phi \in \mathfrak{c}_7, P \in \mathfrak{c}_7^C, r \in iR, u \in C \}.
\]
Now, we will investigate the Lie algebra \((E_8)^{\sigma, \sigma'}\) of the group
\[
(E_8)^{\sigma, \sigma'} = ((E_8)^{\sigma})^{\sigma'} = (E_8)^{\sigma} \cap (E_8)^{\sigma'}.
\]

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