We show that a one-dimensional differential equation depending on a parameter $\mu$ with a saddle-node bifurcation at $\mu = 0$ can be modelled by an extended normal form

$$\dot{y} = v(\mu) - y^2 + a(\mu)y^3,$$

where the functions $v$ and $a$ are solutions to equations that can be written down explicitly. The equivalence to the original equations is a local differentiable conjugacy on the basins of attraction and repulsion of stationary points in the parameter region for which these exist, and is a differentiable conjugacy on the whole local interval otherwise. (Recall that in standard approaches local equivalence is topological rather than differentiable.) The value $a(0)$ is Takens’ coefficient from normal form theory. The results explain the sense in which normal forms extend away from the bifurcation point and provide a new and more detailed characterization of the saddle-node bifurcation. The one-dimensional system can be derived from higher dimensional equations using centre manifold theory. We illustrate this using two examples from climate science and show how the functions $v$ and $a$ can be determined analytically in some settings and numerically in others.
1. Introduction

The saddle-node bifurcation is one of the two generic codimension-one bifurcations of stationary points of differential equations (the other being a Hopf bifurcation). It occurs at parameters where the Jacobian matrix of a stationary point has a zero eigenvalue and two genericity conditions hold. Near a saddle-node bifurcation the local dynamics can be reduced to a one-dimensional differential equation on the associated extended centre manifold [1,2].

As parameters are varied to pass through the bifurcation, a pair of stationary points is created or destroyed. Restricted to the centre manifold, one stationary point is stable, while the other is unstable. On the centre manifold the system is locally topologically equivalent (in ways that will be made more explicit in §3) to the truncated normal form

\[ \dot{y} = v - y^2. \] (1.1)

This truncated normal form is often introduced via coordinate transformations which push the other terms in the Taylor expansion of the family of systems beyond quadratic. These coordinate transformations can be given explicitly ([2], section 3.3), yet the local equivalence is topological rather than differentiable except in the non-hyperbolic case when the eigenvalue of the Jacobian matrix is zero. In this case, successive coordinate changes can be made to remove nonlinear terms of all orders except quadratic and cubic [3], see also §2. This begs an obvious question: Why is it possible to obtain stronger results (differentiable rather than topological) in the apparently more exceptional non-hyperbolic case?

In this article, we show that the differentiable result extends beyond the bifurcation value provided a parameter-dependent cubic term is added to equation (1.1):

\[ \dot{y} = v(\mu) - y^2 + a(\mu)y^3. \] (1.2)

For this modified equation, the equivalence to the original equation restricted to the basin of attraction of the stable stationary point and, separately, to the basin of repulsion of the unstable stationary point is differentiable provided the parameter-dependent coefficients \( v \) and \( a \) are chosen appropriately.

Here and later, we assume a reduction to the centre manifold has been made, so stability statements refer to stability restricted to the centre manifold. To be more precise, suppose the restriction to the centre manifold is

\[ \dot{x} = f(x, \mu), \] (1.3)

and the saddle-node bifurcation occurs at \( x = 0 \) when \( \mu = 0 \). Then \( f(0,0) = f_x(0,0) = 0 \), and for genericity, we require \( f'_x(0,0) \neq 0 \) and \( f_{xx}(0,0) \neq 0 \). Without loss of generality, by changing of sign of the parameter and variable where necessary, we assume

\[ f(0,0) = 0, \quad f_x(0,0) = 0, \quad f'_x(0,0) > 0, \quad f_{xx}(0,0) < 0. \] (1.4)

As shown in §5, to obtain a differentiable conjugacy between (1.2) and (1.3), it is necessary to have

\[ a(0) = \frac{2f_{xxx}}{3y^2_{xx}}, \] (1.5)

evaluated at \( x = \mu = 0 \). This is exactly the quantity obtained for the non-hyperbolic case \( \mu = 0 \) by Takens [3], see §2, so we refer to equation (1.5) as Takens' coefficient for the saddle-node bifurcation. It provides a quantitative measure of how far the system is from the truncated normal form (1.1) and breaks the time-reversing symmetry \((y, t) \to (-y, -t)\) of (1.1). It also reflects the degree of asymmetry in the two branches of stationary points in bifurcation diagrams, and a large value of \( a(0) \) indicates proximity to a cusp bifurcation.

The remainder of this article is organized as follows. In §2, we revisit the formal power series approach to differentiable equivalence at \( \mu = 0 \) and Takens' theorem for non-hyperbolic stationary points. Then, in §3, we bring together the main theoretical results on local equivalence of [4–6] in the form that will be needed in later sections. Thus, the results of §§2 and 3 are not
new. In §§4 and 5, we use these results to modify the approach of [7] for bifurcations of maps to the continuous-time setting of saddle-node bifurcations. In §4, theorem 4.1 describes the smooth local equivalence in the case that no stationary points exist locally, proving that there is a \( C^k \) conjugacy between the flow of \( f \) and the appropriate normal form. In §5, theorem 5.1 completes our analysis by showing that the conjugacy is \( C^k \) on basins of attraction and repulsion of fixed points and collecting all the previous results for the saddle-node bifurcation into one statement. The new technical part of this result is obtained by using the implicit function theorem to match eigenvalues of the two stationary points to those of the extended normal form (1.2) and then using the classic differentiable conjugacy results of §3. This matching process determines the functions \( \nu \) and \( a \) from \( f \) and its derivatives.

To illustrate the results, we have chosen two applications in climate modelling. In §6, we consider a simple temperature model of Fraedrich [8] and show how Takens’ coefficient can be expressed in terms of physical attributes of the system. Then, in §7, we consider a version of Stommel’s box model for ocean circulation [9]. This model is two-dimensional but has a slow-fast time-scale structure, so perturbation techniques can be used to describe the leading order behaviour by a one-dimensional system for which Takens’ coefficient can be evaluated explicitly. In §8, we explain how the reduction to a centre manifold can be performed, and in §9, we use this to evaluate Takens’ coefficient for Stommel’s model numerically. Finally, §10 provides concluding remarks.

2. Coordinate transformations: power series

In standard normal form analysis, the first step is to simplify when parameters are zero and then use a versal deformation argument to introduce general parameters to obtain local behaviour [1,10]. Coordinate changes are not performed away from the bifurcation point; instead topological equivalence arguments are used, often without detailed justification. On the other hand, to analyze saddle-node bifurcations using the implicit function theorem, the full strength of normal form theory is unnecessary and a shift of origin and scaling are used to remove the linear term in \( x \) from the Taylor expansion of the differential equation and then it is pointed out that the resulting system is locally topologically equivalent to the truncated normal form (1.1), see, e.g. [2].

In this section, we will recall the changes of coordinate at the bifurcation point for the saddle-node bifurcation leading to Takens’ normal form theorem (theorem 2.1).

(a) Scaling the quadratic coefficient

With \( \mu = 0 \), the general system (1.3) with (1.4) has the form

\[
\dot{x} = \frac{1}{2} f_{xx} x^2 + \frac{1}{6} f_{xxx} x^3 + O(x^4),
\]

where here, and unless otherwise stated, derivatives are evaluated at \( x = \mu = 0 \). To reach the modified normal form (1.2) with \( \nu = 0 \), we first perform a linear change of coordinates, \( y = \alpha x \). We have

\[
\dot{y} = \alpha \left( \frac{1}{2} f_{xx} y^2 + \frac{1}{6} f_{xxx} y^3 \right) + O(y^4)
\]

\[
= \frac{1}{2\alpha} f_{xx} y^2 + \frac{1}{6\alpha^2} f_{xxx} y^3 + O(y^4).
\]

Thus, the choice \( \alpha = -\frac{1}{2} f_{xx} \) yields

\[
\dot{y} = -y^2 + a y^3 + O(y^4), \quad a = \frac{2f_{xxx}}{3f_{xx}^2}.
\]

Already this simple calculation shows the origin of Takens’ coefficient. Notice our coordinate change \( y = \alpha x \) is orientation-preserving because \( f_{xx} < 0 \) by assumption.
(b) Persistence of the cubic term

To the system (2.1), it is instructive to try and remove the cubic term via a subsequent coordinate change of the form:

\[ z = y + \beta y^2. \]

This inverts to \( y = z - \beta z^2 + O(z^3) \), so

\[
\dot{z} = (1 + 2\beta y) \dot{y} \\
= -y^2 + (a - 2\beta)y^3 + O(y^4) \\
= -z^2 + az^3 + O(z^4),
\]

and notice \( \beta \) is absent from the cubic term. Thus, the cubic term cannot be removed by this change of coordinates.

(c) Removal of quartic and higher order terms

However, it is possible to remove higher order terms. Write (2.1) as

\[ \dot{y} = -y^2 + ay^3 + by^k + O(y^{k+1}), \]

where \( k \geq 4 \). Then with

\[ z = y + \beta y^{k-1}, \quad \text{so} \quad y = z - \beta z^{k-1} + O(z^k), \]

we have

\[
\dot{z} = (1 + \beta (k - 1)y^{k-2}) \dot{y} \\
= -y^2 + ay^3 + (b - \beta (k - 1))y^k + O(y^{k+1}) \\
= -z^2 + az^3 + (b - \beta (k - 3))z^k + O(z^{k+1}).
\]

Since \( k > 3 \), we can choose

\[ \beta = \frac{b}{k - 3}, \]

which removes the \( z^k \) term. This can be repeated for successively larger values of \( k \) removing all nonlinear terms of order four and above.

(d) Takens’ theorem and general remarks

It is one thing to show that there is a formal power series which removes all terms higher than cubic, and it is quite another to show that this formal power series converges on a neighbourhood of the stationary point. Takens resolved this issue in the theorem quoted later. The aim of this article is to determine the correct formulation which allows us to accommodate \( \mu \neq 0 \).

**Theorem 2.1.** (Takens [3]) If \( f \) is \( C^\infty \) and satisfies (1.4), then on a neighbourhood of zero, the sequence of coordinate transformations defined earlier converges to a \( C^\infty \) change of variables in which the equation takes the normal form of

\[ \dot{y} = -y^2 + ay^3, \quad a = \frac{2f_{xxx}}{3f_{xx}^2}. \]  

(2.2)

We are not aware of explicit \( C^k \) versions of this theorem, but there are corresponding discrete-time formulations [11], see also [7].
(e) Bifurcation theorems

In higher codimension problems, e.g. the Takens–Bogdanov bifurcation, and in some approaches to the Hopf bifurcation, a normal form is used at the bifurcation point and then a versal deformation argument is used to identify those small low-order terms (unfoldings) in parameter and phase space that imply all the topological behaviours close to the bifurcation point are realised [1,2,10]. For the simple codimension-one bifurcations, this approach is not necessary because topological equivalence is such a weak requirement. Presumably it is for this reason that theorem 2.1 is not often stated in the literature.

3. Equivalences and conjugacy

Formal proofs of even the local topological conjugacy results of bifurcation theorems are rarely given in textbooks, so in this section, we gather together the definitions and conjugacy results needed in the remainder of this article and state them in a form which simplifies their application in later sections. We emphasize that the results in this section are not new, and the novelty of this article lies in how these results determine properties of the saddle-node bifurcation described in §§4 and 5.

(a) Smooth conjugacies

Let $U, V \subset \mathbb{R}$ be open intervals and $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ be $C^k$ with $k \geq 2$. The differential equations

$$\dot{x} = f(x) \quad \text{and} \quad \dot{y} = g(y),$$

are said to be $C^r$-conjugate ($r \geq 1$) if there exists a $C^r$ diffeomorphism $h : U \to V$ such that

$$g(h(x)) = h'(x)f(x), \quad \text{for all } x \in U. \quad (3.1)$$

This expresses the fact that $y = h(x)$ is a change of coordinates with

$$\dot{y} = h'(x)\dot{x} = h'(x)f(x) = g(h(x)) = g(y).$$

Now let $\phi_t(x)$ and $\psi_t(y)$ denote the flows induced by $\dot{x} = f(x)$ and $\dot{y} = g(y)$, respectively. An equivalent formulation of (3.1) is

$$h(\phi_t(x)) = \psi_t(h(x)). \quad (3.2)$$

(b) Linearization

If $x^*$ is a stationary point of $f$, then $y^* = h(x^*)$ is a stationary point of $g$. By differentiating (3.1) and setting $x = x^*$, we see that the two stationary points have the same stability coefficient, i.e.

$$f'(x^*) = g'(y^*).$$

**Theorem 3.1.** ($C^k$-linearization theorem) Suppose $f : U \to \mathbb{R}$ is $C^k$ ($k \geq 2$) and $x^* \in U$ is a stationary point of $\dot{x} = f(x)$ with $f'(x^*) = \lambda \neq 0$. Then there exist neighbourhoods $U_0 \subseteq U$ of $x^*$ and $V_0 \subset \mathbb{R}$ of 0 such that $\dot{x} = f(x)$ on $U_0$ is $C^k$-conjugate to $\dot{y} = \lambda y$ on $V_0$.

This formulation is due to Sternberg ([6], theorem 4). In the general case of differential equations in $\mathbb{R}^n$, there are extra resonance conditions that need to hold between eigenvalues of the Jacobian matrices of $f$ and $g$ at the corresponding stationary points [4–6]. In the one-dimensional case, there is no resonance, and more generally some differentiability is possible even with resonance if $f$ is sufficiently smooth [12].

Two differential equations satisfying the $C^k$-linearization theorem for the same value of $\lambda \neq 0$ are both conjugate to $\dot{y} = \lambda y$; thus, they are conjugate to each other. That is, we have the following result.
Corollary 3.2. Suppose \( f : U \to \mathbb{R} \) and \( g : V \to \mathbb{R} \) are \( C^k \) \((k \geq 2)\) and \( \dot{x} = f(x) \) and \( \dot{y} = g(y) \) have stationary points \( x^* \in U \) and \( y^* \in V \) with \( f'(x^*) = g'(y^*) \neq 0 \). Then there exist neighbourhoods \( U_0 \subseteq U \) of \( x^* \) and \( V_0 \subseteq V \) of \( y^* \) such that \( \dot{x} = f(x) \) on \( U_0 \) and \( \dot{y} = g(y) \) on \( V_0 \) are \( C^k \)-conjugate.

(c) Extension to basins of attraction and repulsion

We can now prove an adapted version of a theorem of Belitskii [13] for discrete-time dynamical systems that is key to the construction of the differentiable conjugacies for the bifurcations. A similar result is given in [7] for the discrete-time setting.

Theorem 3.3. Suppose \( f : U \to \mathbb{R} \) and \( g : V \to \mathbb{R} \) are \( C^k \) \((k \geq 2)\) and \( \dot{x} = f(x) \) and \( \dot{y} = g(y) \) have precisely \( n \geq 1 \) stationary points, \( x_1 < x_2 < \cdots < x_n \) and \( y_1 < y_2 < \cdots < y_n \) respectively, with \( f'(x_j) = g'(y_j) \neq 0 \) for all \( j \in \{1, 2, \ldots, n\} \). Write \( U = (u_0, u_1) \) and \( V = (v_0, v_1) \). Then there exist \( x_0 \in (u_0, x_1) \), \( y_0 \in (v_0, y_1) \), \( x_{n+1} \in (x_n, u_1) \), and \( y_{n+1} \in (y_n, v_1) \) such that \( \dot{x} = f(x) \) on \((x_{j-1}, x_{j+1})\) and \( \dot{y} = g(y) \) on \((y_{j-1}, y_{j+1})\) are \( C^k \)-conjugate for all \( j \in \{1, 2, \ldots, n\} \).

Proof. As mentioned earlier, let \( \phi_t(x) \) and \( \psi_t(y) \) denote the flows generated by \( f \) and \( g \), respectively. Choose any \( j \in \{1, 2, \ldots, n\} \) and suppose \( f'(x_j) < 0 \) (the case \( f'(x_j) > 0 \) can be treated similarly). By corollary 3.2, there exist open neighbourhoods \((u_-, u_+)\) of \( x_j \) and \((v_-, v_+)\) of \( y_j \) and a \( C^k \) conjugacy \( h : (u_-, u_+) \to (v_-, v_+) \) between \( f \) and \( g \), see figure 1.

Now suppose \( j \notin \{1, n\} \) (the extremal cases will be treated at the end). Our task is to extend \( h \) if necessary to the whole of \((x_{j-1}, x_{j+1})\) and \((y_{j-1}, y_{j+1})\). Since \( f'(x_j) > 0 \), \( f(x) > 0 \) for \( x \in (x_j, x_{j+1}) \) and \( \phi_t(x) \) is an increasing function of \( t \) for all \( x \in (x_j, x_{j+1}) \) and \( y \in (y_j, y_{j+1}) \). If \( u_+ = x_{j+1} \) we are done, since clearly \( \lim_{x \to x_{j+1}} h(x) = y_{j+1} \). So suppose \( u_+ < x_{j+1} \) and let \( v_+ = h(u_+) \) and notice \( v_+ < y_{j+1} \).

Let \( p \in (x_j, u_+) \) and \( q = h(p) \in (y_j, v_+) \). For each \( x \in (p, x_{j+1}) \), there exists \( t_x > 0 \) such that \( \phi_{t_x}(p) = x \) and clearly \( t_x \) is an increasing function of \( x \) with \( t_x \to \infty \) as \( x \to x_{j+1} \). Define \( h(x) \) by using the two flows: from \( x \) evolve back to \( p \), transfer to \( q = h(p) \), then evolve forward to \( h(x) \), i.e.

\[
h(x) = \psi_t(q) = \psi_t(h(\phi_{-t}(x))).
\]

This is illustrated in figure 1. Since the flows are \( C^k \) ([14], theorem 0.8), by construction \( h(x) \) is \( C^k \) and satisfies the conjugacy condition (3.2) for all \( x \in (p, x_{j+1}) \). Thus, we have extended \( h \) to \((u_-, x_{j+1})\). The same argument in \((x_{j-1}, x_j)\) allows us to extend \( h \) to \((x_{j-1}, x_{j+1})\). (If \( f'(x_j) < 0 \), then the argument is similar but with negative time.)

Finally, we treat the case \( j = n \) \((j = 1 \) is similar). As mentioned earlier, let \( p \in (x_n, u_+) \) and \( q = h(p) \in (y_n, v_+) \). As also mentioned earlier, we only treat the case \( f'(x_n) > 0 \). The main difference here is that orbits of \( \dot{x} = f(x) \) reach the right endpoint of \( U \) in finite time, and similarly for \( \dot{y} = g(y) \).


To accommodate this, let \( t_1, s_1 > 0 \) be such that \( \phi_{t_1}(p) = u_1 \) and \( \psi_{s_1}(q) = v_1 \). If \( t_1 \leq s_1 \), let \( x_{n+1} = u_1 \) and \( y_{n+1} = \psi_{t_1}(q) \leq v_1 \), and if \( t_1 > s_1 \), let \( x_{n+1} = \phi_{s_1}(p) \leq u_1 \) and \( y_{n+1} = v_1 \). Then the construction (3.3) produces a \( C^k \) conjugacy for all \( x \in (p, x_{n+1}) \), and as explained earlier, this is readily extended to \( (x_{n-1}, x_{n+1}) \).

Note that the intervals \((x_{j-1}, x_{j+1})\) are precisely the basins of attraction of \( x_j \) if \( f'(x_j) < 0 \) or the basin of repulsion of \( x_j \) if \( f'(x_j) > 0 \). Hence, another way to phrase theorem 3.3 is that there are local \( C^k \) conjugacies on the basins of attraction and repulsion of corresponding stationary points.

### 4. Differentiable conjugacy and saddle-node bifurcations

The saddle-node bifurcation is associated with differential equations with a one-dimensional centre manifold reduction on which the equation \( \dot{x} = f(x, \mu) \) satisfies the conditions (1.4). In this section, we sketch the conditions that need to hold in order for \( C^k \) conjugacies to exist, with proofs for the cases with no periodic orbits locally. Then, in §5, we apply these ideas to the normal form (1.2).

By the implicit function theorem, there exists a unique local branch of stationary points of the form

\[
\mu = -\frac{f_{xx}}{2f_\mu} x^2 + O(|x|^3). \tag{4.1}
\]

This has two solutions in \( \mu > 0 \) and none in \( \mu < 0 \) for the choice of \( f_\mu > 0 \) and \( f_{xx} < 0 \) made in (1.4).

Now consider two \( C^k \) families of differential equations, \( f \) parametrized by \( \mu \), and \( g \) parametrized by \( v \), both of which satisfy (1.4). Let \( \phi_t(x, \mu) \) and \( \psi_t(y, v) \) denote their flows. If \( \mu \) and \( v \) are positive, then by theorem 3.3, there are local conjugacies if it is possible to choose \( v = N(\mu) \) such that the multipliers of the corresponding stationary points are equal. This is not a trivial condition. In §5, we use the implicit function theorem to prove that the coefficient \( a \) of the normal form (1.2) and the parameter \( v \) can be chosen as functions of \( \mu \) so that these conditions do hold, and so theorem 3.3 can be applied.

If instead \( \mu = v = 0 \), then both systems satisfy Takens’ theorem (theorem 2.1) provided \( f \) and \( g \) are \( C^\infty \). The following theorem accommodates the case in which \( \mu \) and \( v \) are negative.

**Theorem 4.1.** Suppose that \( f \) and \( g \) are \( C^k \) families of differential equations, \( f \) parametrized by \( \mu \), and \( g \) parametrized by \( v \), both of which satisfy (1.4). There exist neighbourhoods \( U \) and \( V \) of zero and a \( C^k \) function \( N(\mu) \) with \( N(0) = 0 \) such that for all \( \mu < 0 \) with \( |\mu| \) sufficiently small \( \dot{x} = f(x, \mu) \) on \( U \) is \( C^k \)-conjugate to \( \dot{y} = g(y, N(\mu)) \) on \( V \).

**Proof.** If \( \mu < 0 \) and \( v < 0 \) with \( |\mu| \) and \( |v| \) sufficiently small, then the flows are decreasing in a neighbourhood of the origin. Given two such neighbourhoods \( U = (u_0, u_1) \) and \( V = (v_0, v_1) \), there exists \( T > 0 \) such that for all \( t > T \) there are functions \( \mu(t) \) and \( v(t) \) for which

\[
\phi_t(u_1, \mu(t)) = u_0 \quad \text{and} \quad \psi_t(v_1, v(t)) = v_0,
\]

for the first time. Moreover, since the flows are \( C^k \) functions of the time and parameter (e.g. [15], sections 7.5 and 7.6 or [16], theorems 1.1.2 and 1.1.4), \( \mu(t) \) and \( v(t) \) are \( C^k \) functions which tend to zero from below as \( t \) tends to infinity. Moreover, after further restricting the size of the neighbourhoods \( U \) and \( V \) and increasing \( T \) if necessary, standard comparison theorems (e.g. [15], section 2.7) imply that \( \mu'(t) \) and \( v'(t) \) are strictly positive. Hence, by the inverse function theorem [17], \( \mu(t) \) can be inverted to obtain a \( C^k \) correspondence \( v = v(t(\mu)) = N(\mu) \) between \( \mu \) and \( v \).

At corresponding parameters, the systems are \( C^k \)-conjugate on \( U \) and \( V \) by the \( C^k \) flow box theorem ([14], theorem 1.1). Equivalently, as this is trivial in one dimension, for all \( x \in U \), there exists \( \tau \) such that \( x = \phi_\tau(u_1, \mu) \), and the \( C^k \) conjugating function \( y = h(x) \) can be defined by

\[
h(x) = \psi_\tau(v_1, N(\mu)).
\]
5. Normal forms and Takens’ coefficient

In this section, we will prove the claims in §1. The analysis closely follows those in [7] for the discrete-time setting. There are four parts to the calculation: an asymptotic computation of the position of the stationary points of the general system (1.4), the equivalent computation for the normal form (1.2), the use of the implicit function theorem to identify functions $\nu(\mu)$ and $a(\mu)$ for the parameter and coefficient of the normal form at which the corresponding stationary points have equal multipliers, and finally the use of theorem 3.3 to establish the existence of local differentiable conjugacies. Throughout this section, equation (1.4) can be used to write the differential equation for $f$ (with all partial derivatives evaluated at $x = \mu = 0$ and assuming that $f$ is at least $C^3$) as follows:

$$
\dot{x} = f_x + 1 \left( f_{xxx}^2 + 2f_x x + f_{x\mu} \mu^2 \right) + \frac{1}{6} f_{xxxx}^3 + \cdots, \quad (5.1)
$$

and by assumption,

$$
f_{\mu} > 0 \quad \text{and} \quad f_{xx} < 0. \quad (5.2)
$$

(a) Stationary points of $f$

From (4.1), we know that there are two stationary points if $\mu > 0$ and that their positions are functions of $m = \sqrt{\mu}$. Expanding in power series in $m$, substituting into (5.1) and setting the right-hand side equal to zero gives the two solutions $x_{1,2}$ as follows (for $r = 1, 2$ below):

$$
x_r = (-1)^r \sqrt{-\frac{2f_{\mu}}{f_{xx}}} m + \left( \frac{f_{f_{xxx}^2} - 3f_{x\mu} f_{xx}}{3f_{xx}^2} \right) m^2 + O(m^3), \quad (5.3)
$$

assuming $f$ is at least $C^4$. Differentiating (5.1) with respect to $x$ and evaluating at $x_r$ gives the multipliers

$$
f'(x_r) = (-1)^{r+1} \sqrt{-2f_{\mu} f_{xx} m - \frac{2f_{f_{xxx}^2}}{3f_{xx}} m^2 + O(m^3)} . \quad (5.4)
$$

(b) Stationary points of the normal form

The normal form (1.2) is $\dot{y} = g(y)$ with

$$
g(y) = v - y^2 + ay^3. \quad (5.5)
$$

Hence,

$$
g_v = 1, \quad g_{yy} = -2, \quad g_{gyy} = 6a,
$$

and all other derivatives are zero. From (5.3) and (5.4), we can read off the values of the stationary points and their multipliers with $v = n^2 > 0$:

$$
y_r = (-1)^r n + \frac{1}{2} am^2 + O(n^3) \quad (5.6)
$$

and

$$
g'(y_r) = 2(-1)^{r+1} n + 2an^2 + O(n^3). \quad (5.7)
$$

(c) Equality of the multipliers

We require $f'(x_r) = g'(y_r)$ for $r = 1$ and $r = 2$. These two equations are enough to determine $v$ and $a$ as functions of $\mu$. However, their leading order terms are equal (up to sign), and so to use the implicit function theorem, it is necessary to solve two related functions.
Anticipating that \( n \) is proportional to \( m \), define \( p \) by \( n = mp \). Now (again a standard trick, see, e.g. [7,18]), let \( G_r(a,p,m) = f'(x_r) - g'(y_r) \) and define

\[
F_1(a,p,m) = \begin{cases} 
G_1(a,p,m) & \text{if } m \neq 0, \\
\frac{\partial G_1}{\partial m}(a,p,m) & \text{if } m = 0.
\end{cases}
\]  

(5.8)

From (5.4) and (5.7) with \( n = pm \),

\[
F_1(a,p,m) = \sqrt{-2f_{xx} - 2p + O(m)},
\]

(5.9)

and observe \( F_1(a,p,0) = 0 \) if \( p = p_0 \) with

\[
p_0 = \sqrt{-\frac{1}{2} f_{xx}}.
\]

(5.10)

Now let

\[
F_2(a,p,m) = \begin{cases} 
\frac{G_1(a,p,m) + G_2(a,p,m)}{m^2} & \text{if } m \neq 0, \\
\frac{1}{2} \frac{\partial^2}{\partial m^2} (G_1(a,p,m) + G_2(a,p,m)) & \text{if } m = 0.
\end{cases}
\]

(5.11)

Once again, from (5.4) and (5.7) with \( n = pm \),

\[
F_2(a,p,m) = -\frac{4 f_{xx}}{3} f_{xxx} - 4ap^2 + O(m),
\]

(5.12)

and observe \( F_1(a,p,0) = 0 \) if \( p = p_0 \) and \( a = a_0 \) with

\[
a_0 = \frac{1}{p_0} \frac{f_{xxx}}{3f_{xx}} = \frac{2f_{xxx}}{3f_{xx}^2}.
\]

(5.13)

With these choices of \( p_0 \) and \( a_0 \)

\[
F_1(a_0,p_0,0) = 0 \quad \text{and} \quad F_2(a_0,p_0,0) = 0.
\]

To apply the implicit function theorem, the functions \( F_1 \) and \( F_2 \) must be at least \( C^1 \); hence, \( f \) must be at least \( C^4 \). To obtain a unique local solution (valid in \( m > 0 \)), the determinant of the matrix \( M \) of partial derivatives of \( F_1 \) and \( F_2 \) must not vanish. From (5.9) and (5.12),

\[
M = \begin{bmatrix} \frac{\partial F_1}{\partial a} & \frac{\partial F_1}{\partial p} \\ \frac{\partial F_2}{\partial a} & \frac{\partial F_2}{\partial p} \end{bmatrix}_{(a,p,m)=(a_0,p_0,0)} = \begin{bmatrix} 0 & -2 \\ -4p_0^2 & -8a_0p_0 \end{bmatrix},
\]

so \( \det(M) = -8p_0^2 \neq 0 \). Thus, indeed there is a unique local curve of solutions with \( n = p_0m + O(m^2) \), i.e. \( v = p_0^2m + O(m^{3/2}) \), and \( a = a_0 + O(m) \).

(d) Local differentiable conjugacies

We can now put these calculations together with the strategy of §4 to determine the relation between the original map \( f \) and the normal form \( g \) (1.2), which depends on the parameters \( \nu \) and \( a \).

**Theorem 5.1.** Suppose \( f \) is \( C^k \), \( k \geq 4 \), and satisfies (1.4) and let \( p_0 \) and \( a_0 \) be given by (5.10) and (5.13). There exist neighbourhoods \( U \) and \( V \) of zero, continuous functions \( a(\mu) \) and \( \nu(\mu) \) such that \( a(0) = a_0 \) and \( \nu(0) = 0 \), and \( \mu_0 > 0 \) with the following properties.

(i) If \( \mu \in (-\mu_0,0) \), then \( a(\mu) \) and \( \nu(\mu) \) are \( C^k \) and \( \dot{x} = f(x) \) on \( U \) at \( \mu \) is \( C^k \) conjugate to \( \dot{y} = g(y) \) on \( V \).

(ii) If \( \mu = 0 \) and \( f \) is \( C^\infty \), then \( \dot{x} = f(x) \) on \( U \) is \( C^\infty \) conjugate to \( \dot{y} = g(y) \) on \( V \).
If $\mu \in (0, \mu_0)$, then $a$ and $\nu$ are $C^{k-3}$ functions with
\[ \nu = p_0^2 \mu + O(\sqrt{\mu}^3) \quad \text{and} \quad a = a_0 + O(\sqrt{\mu}), \]
and $\dot{x} = f(x)$ is $C^k$ conjugate to $\dot{y} = g(y)$ on the basins of attraction and repulsion of the corresponding stationary points in $U$ and $V$. 

This simply collects together the results established earlier. Case (i) is theorem 4.1, note that if $\mu < 0$, then any $C^k$ function could be used and $a(\mu) = a_0$ would be a particularly simple choice. The condition $a(0) = a_0$ ensures continuity of the function with the solution to the implicit function theorem from case (iii). Case (ii) is Takens’ theorem (theorem 2.1), and case (iii) follows from the results in this section together with theorem 3.3. The regularity of the parametrization follows from the fact that the implicit function theorem determining the functions $\nu$ and $a$ is applied to a system of equations that is $C^{k-3}$. Note that this does not imply that these functions are $C^{k-3}$ at the origin as the explicit function theorem is applied using the variable $\sqrt{\mu}$.

We conjecture that theorem 5.1(ii) remains true in the $C^k$ case, $k \geq 4$. This conjecture is certainly true in the discrete-time case [7,11].

### (e) A remark on signs

Although $f_\mu(0,0) > 0$ and $f_{xx}(0,0) < 0$ may be assumed without loss of generality for a generic saddle-node bifurcation, it is useful to have the constants $p_0$ and $a_0$ in a form which can be applied without making the explicit transformation to this case. Clearly
\[ p_0^2 = \frac{1}{2} |f_\mu f_{xx}|, \tag{5.14} \]
regardless of the signs of $f_\mu$ and $f_{xx}$, but the sign of $a_0$ needs a little more thought. By considering the transformation $x \to -x$, we see that in the differential equation, $f_{xx} \to -f_{xx}$ and $f_{xxx} \to f_{xxx}$. Hence, $a_0$ is invariant under the transformation and no adjustment is necessary.

### 6. Fraedrich’s temperature model

Saddle-node bifurcations are a mechanism for tipping points in climate dynamics. One such saddle-node bifurcation is exhibited by Fraedrich’s model for temperature, $T(t)$; see refs. [8,19] for details and justification. The temperature variation has a black body radiation term of order $T^4$, a constant solar warming term, and a $T^2$-term describing the variation of ice albedo with temperature. The one-dimensional model is\(^1\)
\[ \dot{T} = a(-T^4 + b\mu T^2 - d\mu), \tag{6.1} \]
and the parameters are
\[ a = \frac{e_{SA}\sigma}{c}, \quad b = \frac{b_2 I_0}{4 e_{SA}\sigma}, \quad d = \frac{(a_2 - 1)I_0}{4 e_{SA}\sigma}. \]

Here, the more fundamental constants are the Stefan–Boltzmann constant $\sigma$, the insolation $I_0$, the oceanic thermal capacity $c$ and the effective emissivity $e_{SA}$. The constants $a_2 > 1$ and $b_2$ model the ice albedo effect: the albedo is $a_2 - b_2 T^2$, and $\mu$ is a parameter that describes variations due to changes in the planetary orbit or insolation values. Reasonable values (taken from [19]) are
\[ I_0 = 1366 \text{ Wm}^{-2}, \quad \sigma = 5.6704 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}, \quad c = 108 \text{ kgKs}^{-2}, \quad e_{SA} = 0.62, \quad a_2 = 1.6927, \quad b_2 = 1.690 \times 10^{-5} \text{ K}^{-2}. \tag{6.2} \]

The parameter $\mu$ is order one and acts as the bifurcation parameter.

\(^1\)Ashwin et al. [19] include an extra factor $1/c$ in the ordinary differential equation which we believe should be absent, likely a simple typographical error that arose when simplifying equation (4.1) of [8].
Stationary points are solutions of the quadratic equation for $T^2$,

$$T^4 - b\mu T^2 + d\mu = 0,$$

so the solutions are

$$T^2 = \frac{1}{2} \left( b\mu \pm \sqrt{b^2\mu^2 - 4d}\mu \right).$$

Two solutions are created in a saddle-node bifurcation as $\mu$ increases through $\mu_c = 4d/b^2$ with temperature $T_c = \sqrt{\frac{1}{2}b\mu_c} = \sqrt{2d/b}$. Now let $f(T, \mu)$ denote the right-hand side of (6.1). A routine calculation shows that at the bifurcation point

$$f_\mu = ad, \quad f_{TT} = -\frac{16ad}{b}, \quad f_{TTT} = -24aT_c.$$ 

This means that the two quantities, $p_0$ and $a_0$, that drive the normal form are related to the basic parameters by

$$p_0^2 = -\frac{1}{2}f_{\mu f_{TT}} = \frac{8d^2a^2}{b} = \frac{2e_{SA}\sigma I_0(a_2 - 1)^2}{b_2c^2}$$

and

$$a_0 = \frac{2f_{TTT}}{3f_{TT}} = -\frac{1}{8\sqrt{2a}} \left( \frac{b}{d} \right)^{3/2} = -\frac{c}{8\sqrt{2e_{SA}}} \left( \frac{b_2}{a_2 - 1} \right)^{3/2}.$$ 

An interesting feature of this analysis is that Takens’ coefficient, $a_0$ has two factors. The factor $1/a$ represents a scaling in time since in the new time $\tau = at$ the multiplicative factor in (6.1) is scaled to unity. The second (and indeed the last ratio in the expression for $p_0^2$) shows a nonlinear dependence on the level and sensitivity of the albedo effect to changes in global temperature.

### 7. Thermohaline circulation: perturbation theory

Stommel’s two-box model of the oceanic circulation [9] remains a useful motivating example for the possibility of dramatic non-reversible changes in the climate due to global temperature rise. In the version used by Cessi [20] this can be written using a cubic nonlinearity instead of the modulus in the original model, so the non-dimensionalized temperature gradient $\dot{x}$ and salinity gradient $\dot{y}$ between the two boxes evolves according to the following equations:

$$\dot{x} = -\alpha(x - 1) - x (1 + m(x - y)^2)$$

and

$$\dot{y} = p - y (1 + m(x - y)^2).$$

Here, $\alpha$ is a non-dimensionalized time constant, the ratio between the diffusive and temperature relaxation time scales, and is approximately 3600, $m$ is the ratio of the diffusive time scale to the advective times scale and is approximately 7.5 and $p$ is the non-dimensional freshwater flux with an average of order one. These estimates of magnitude are taken from [20], where $p$ is allowed to oscillate stochastically about its mean and $m$ is denoted by $\mu_2$, which we have changed to avoid confusion with general bifurcation parameters. Here, we take $p$ to be constant and treat it as the bifurcation parameter (cf. [9]). Kuehn [21] described a slow-fast manifold approach to the same problem.

Since $\alpha$ is much larger than $m$ and $p$, a perturbation theory approach can be taken with solutions expanded as power series in $\alpha^{-1}$ giving $x = 1 + O(\alpha^{-1})$ and

$$\dot{y} = p - y (1 + m(1 - y)^2) + O(\alpha^{-1}).$$

Writing $y = y_0 + O(\alpha^{-1})$, the perturbation equation for $y_0$ is obtained by ignoring the order $\alpha^{-1}$ terms, so, dropping the subscript 0 from here on, saddle-node bifurcations of (7.2) at leading order occur if
Figure 2. A bifurcation diagram of the box model (7.1) in the limit $\alpha \to \infty$ and with $m = 7.5$. Branches of stable (unstable) equilibria are indicated by solid (dashed) curves. Two saddle-node bifurcations are indicated. (Online version in colour.)

$$p = y(1 + m(1 - y)^2) \quad \text{and} \quad 1 + m - 4my + 3my^2 = 0. \quad (7.3)$$

The second equation of (7.3), which comes from setting the derivative of the right hand side of (7.2) equal to zero, is a quadratic for $y = y(m)$. By solving it for $y$, we obtain

$$y_{\pm}(m) = \frac{1}{3} \left(2 \pm \sqrt{1 - \frac{3}{m}}\right), \quad m > 3,$$

and by substituting this into the first equation of (7.3), we obtain (after simplification)

$$p_{\pm} = \frac{2}{3} + \frac{2m}{27} \left[1 \mp \left(1 - \frac{3}{m}\right)^{3/2}\right]. \quad (7.4)$$

Note that $p_+ < p_-$. If $p \in (p_+, p_-)$ and then there are three solutions, see figure 2. The two stable solutions correspond to those with highest and lowest salinity gradient. The ocean circulation currently lies on the solution with higher salinity gradient, and if $p > p_-$, this is the only stable stationary point, lending weight to arguments that higher freshwater flux stabilizes the ocean circulation. However, if $p$ decreases through $p_+$, then this solution disappears and the solution would move rapidly to the stationary point with lower gradient and weaker circulation. This would have rapid and severe consequences for weather on the eastern European Atlantic coast.

We therefore concentrate on the tipping point $p_+$. If $f$ represents the leading order term in the differential equation (7.2), then

$$f_p = 1, \quad f_{yy} = 4m - 6my, \quad f_{yyy} = -6m.$$

Observe $f_p > 0$ and $f_{yy} < 0$ at $y_+$, and thus, the bifurcation occurs as $p$ increases. The determining bifurcation numbers (evaluating at $y_+$) are

$$p_0^2 = -\frac{1}{2} f_p f_{yy} = \sqrt{m(m - 3)}$$

and

$$a_0 = \frac{2f_{yyyy}}{3f_{yy}^2} = \frac{-1}{m - 3}. \quad (7.5)$$

Thus, Takens’ coefficient is large if $m$ is only slightly greater than 3, and if $m = 7.5$, it is approximately 0.22 which is fairly small and so shows that at the currently assumed parameter values, the bifurcation is relatively symmetric.

The bifurcation numbers are plotted in figure 3 for both $y_-$ and $y_+$. The solid curves in figure 3c are the two branches $p_{\pm}^2 (7.4)$ of saddle-node bifurcations in the $(m, p)$ parameter plane. These meet in a cusp bifurcation at $(m, p) = (3, \frac{8}{3})$. The other plots show $p_0^2$ and $1/a_0$ which are identical for the two branches. Notice both $p_0^2$ and $1/a_0$ converge to 0 at the cusp bifurcation.
Figure 3. (a, b) Graphs of $1/a_0$ and $p_0^2 = \frac{1}{2} |f_{p_{yy}}|$ against $m$, where $a_0 = 2f_{yy} / 3f_{y}^2$. (c) A two-parameter bifurcation diagram showing branches of saddle-node bifurcations for the box model (7.1). Solid curves correspond to the limit $\alpha \to \infty$ (given explicitly by (7.4)); dashed curves are for $\alpha = 36$ (computed by numerical continuation).

8. Centre manifolds

In higher dimensional problems, the one-dimensional system describing the saddle-node bifurcation is the projection of the flow onto a centre manifold. In this section, we use a centre manifold reduction to show how the bifurcation numbers $p_0$ and $a_0$ can be computed for a two-dimensional system.

Consider a $C^k$ ($k \geq 4$) system

$$\dot{x} = F(x, y, \mu) \quad \text{and} \quad \dot{y} = G(x, y, \mu),$$

where $F(0, 0, 0) = G(0, 0, 0) = 0$ and the eigenvalues of the Jacobian matrix of (8.1) evaluated at $(x, y, \mu) = (0, 0, 0)$ are 0 and $\lambda \neq 0$. If $\lambda < 0$, the local centre manifold is stable. Assume a coordinate transformation has been done so that the Jacobian matrix is in Jordan form, specifically

$$\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}.$$

Then the system can be written as follows:

$$\begin{align*}
\dot{x} &= b_0 \mu + b_1 x^2 + b_2 xy + b_3 y^2 + b_4 \mu x + b_5 \mu y + b_6 \mu^2 + b_7 x^3 + \cdots \\
\dot{y} &= \lambda y + c_1 x^2 + c_2 xy + c_3 y^2 + c_4 \mu x + c_5 \mu y + c_6 \mu^2 + c_7 x^3 + \cdots,
\end{align*}$$

(8.2)
where the neglected terms are cubic or higher order in $x$, $y$ and $\mu$, except we have explicitly written the $x^3$ terms as one of them will be needed later. There is an extended centre manifold $y = h(x, \mu)$ obtained formally by adding the trivial equation $\dot{\mu} = 0$. This manifold has the following form

$$y = d_1x^2 + d_2\mu x + d_3\mu^2 + O((|x| + |\mu|)^3),$$

and invariance implies

$$\dot{y} = (2d_1x + d_2\mu)\dot{x} + O((|x| + |\mu|)^2).$$

By substituting (8.2) and equating the coefficients of $x^2$, we find $d_1 = -(c_1/\lambda)$ (formulas for $d_2$ and $d_3$ will not be needed). Then by substituting our expression for the extended centre manifold into (8.2), we obtain the leading order equation on the centre manifold:

$$\dot{x} = b_0\mu + b_1x^2 + b_4\mu x + b_6\mu^2 + \left(b_7 - \frac{b_2c_1}{\lambda}\right)x^3 + \cdots .$$

Finally, by using the formulas given earlier, where now $f$ denotes the right-hand side of (8.3),

$$p_0^2 = \frac{1}{2}\left|f_{x2}\right| = |b_0b_1| = \frac{1}{2}\left|F_2x\right|$$

and

$$a_0 = \frac{2f_{xxx}}{3f_{xx}^2} = \frac{1}{b_1^2}\left(b_7 - \frac{b_2c_1}{\lambda}\right) = \frac{1}{3f_{xx}^2}\left(2F_{xxx} + \frac{6F_{xy}G_{xx}}{\lambda}\right).$$

Notice (8.3) has an additional term in the cubic coefficient and hence Takens’ coefficient $a_0$ is modified by the centre manifold reduction even though the coefficients $b_0$ and $b_1$ are unaltered. In other words, Takens’ coefficient contains information about the curvature of the centre manifold and the strength of the linear contraction or expansion in coordinates orthogonal to the centre subspace.

9. Thermohaline circulation: centre manifold analysis

Returning to the model of thermohaline convection (7.1) of §7, the centre manifold analysis of §8 makes it possible to extend the more refined description of the saddle-node bifurcations away from the large $\alpha$ limit. This involves three steps. First, the bifurcation values of the parameters and variables need to be determined. Then a centre manifold reduction is calculated, and finally, the coefficients of the equation on the centre manifold need to be evaluated. Section 8 shows that for two-dimensional systems, the last step is simple once the transformation to Jordan normal form is made at the bifurcation point: $p_0$ and $a_0$ can be evaluated directly using (8.4). However, as in many problems, the first step is either algebraically messy or impossible. Therefore, in this section, we will also use numerical methods to evaluate $p_0$ and $a_0$.

We start by numerically continuing the branches of saddle-node bifurcations. With $\alpha = 3600$, the value of $\alpha$ used by Cessi [20], we observed these branches to be indistinguishable, on the scale of figure 3, from the curves (7.3) obtained earlier. So for illustration, we have instead used $\alpha = 36$, which results in the dashed curves shown in figure 3c.

At points on these branches, the derivatives of the right-hand sides of the differential equation can be evaluated. Then the eigenvectors of the linear part can be used to bring the system into Jordan form and hence to compute derivatives in the coordinates of (8.2). Finally, (8.4) is evaluated numerically.

The values of $\frac{1}{a_0}$ and $p_0^2$ obtained using this process are shown in figure 3a, b. Overall the results with the smaller value $\alpha = 36$ are still reasonably close to the $\alpha \to \infty$ limit; notice the values of $a_0$ and $p_0$ now differ for the two branches.

10. Conclusion

This article describes a deeper connection between truncated normal forms and bifurcations. As in the discrete-time case [7], the local conjugacy can be made differentiable on the basins...
of attraction and repulsion of the stationary solutions provided the cubic coefficient of the normal form is chosen judiciously. This means that (for example) rates of convergence are preserved. At the bifurcation point, the cubic term is described by Takens’ coefficient, and this contains extra information about asymmetry between the two branches and proximity to cusp bifurcations. We believe that Takens’ coefficient should be calculated as a matter of course in many analyses of saddle-node bifurcations for this extra information. The other coefficient identified, \( p_0 \), describes the speed of the bifurcation. The extra information in \( p_0 \) and \( a_0 \) allows different saddle-node bifurcations to be compared in a mathematically meaningful way. Since the parameter dependence of these coefficients can be calculated explicitly using the functional relationships of §5, this variation, and particularly the speed of variation, can also provide information about the evolution of solutions away from the bifurcation point. The coefficients of the normal form vary smoothly with parameters except possibly at the bifurcation point, but we have not considered the regularity of the conjugating function itself (which is \( C^r \) in \( x \) for appropriate \( r \geq 1 \) by theorem 5.1) with changes to the parameter. This would be an interesting question for future work.

The other bifurcations on one-dimensional centre manifolds, the transcritical and pitchfork bifurcations, can be approached in the same way. The details are similar to the discrete-time case [7]; we have concentrated on the generic case here. Note that for the pitchfork bifurcation, two extra terms are needed in the truncated normal form (in addition to the standard cubic term) so that stability coefficients can be matched at three stationary points.

The extended truncated normal forms provide significantly more information than the standard simple truncations and have the added advantage that the techniques to identify the relationship between the parameterizations of the original system and the model are purely algebraic. They rely on solving two or three additional equations, solutions of which can be guaranteed by the implicit function theorem. Since the implicit function theorem is already a standard tool in bifurcation theory, little extra information beyond Sternberg’s linearization theorem has been needed to introduce the extended analysis.

**Note added in proof**: Since completing this work we have found that there is some overlap with results in Y. S. Il’Yashenko and S. Y. Yakovenko (1993) *Nonlinear stokes phenomena in smooth classification problems*, Advances in Soviet Mathematics, 14, pp. 235-287.

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All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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**References**

1. Guckenheimer J, Holmes P. 1983 *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Appl. Math. Sci., vol. 42. New York: Springer.
2. Kuznetsov YA. 1995 *Elements of applied bifurcation theory*. Appl. Math. Sci., vol. 112. New York: Springer.
3. Takens F. 1973 Normal forms for certain singularities of vectorfields. *Ann. Inst. Fourier* 2, 163–195. (doi:10.5802/aif.467)
4. Belitskii GR. 1978 Equivalence and normal forms of germs of smooth mappings. *Russ. Math. Surv.* 3, 107–177. (doi:10.1070/RM1978v033n01ABEH002237)
5. Sell GR. 1985 Linearization near a fixed point. *Am. J. Math.* 10, 1035–1091. (doi:10.2307/2374346)
6. Sternberg S. 1957 Contractions and a theorem of poincaré. *Am. J. Math.* 7, 809–824. (doi:10.2307/2372437)
7. Glendinning PA, Simpson DJW. 2022 Normal forms, differentiable conjugacy, and elementary bifurcations of maps. (http://arxiv.org/abs/2206.04840).
8. Fraedrich K. 1979 Catastrophes and resilience of a zero-dimensional climate system with ice-albedo and greenhouse feedback. *Q. J. R. Meteorol. Soc.* 10, 147–167. (doi:10.1002/qj.49710544310)

9. Stommel H. 1961 Thermohaline convection with two stable regimes of flow. *Tellus* 1, 224–230.

10. Wiggins S. 2003 *Introduction to applied nonlinear dynamical systems and chaos.* Texts in Appl. Math. 2, 2nd edn. New York: Springer.

11. Kuczma M, Choczewski B, Ger R. 1990 *Iterative functional equations.* Cambridge: CUP.

12. Guysinsky M, Hasselblatt B, Rayskin V. 2003 Differentiability of the Hartman-Grobman Linearization. *Disc. Cont. Dyn. Syst.* 9, 979–984. (doi:10.3934/dcds.2003.9.979)

13. Belitskii GR. 1986 Smooth classification of one-dimensional diffeomorphisms with hyperbolic fixed points. *Sibirskii Matematicheskii Zhurnal* 2, 25–27. (Translated in Siberian Mathematical Journal, 27, pp. 801–804).

14. Palis Jr. J, de Melo W. 1982 *Geometric theory of dynamical systems.* New York: Springer.

15. Arnold VI. 1973 *Ordinary differential equations.* Cambridge: MIT Press.

16. Wiggins S. 1988 *Global bifurcations and chaos.* Applied Mathematical Sciences, 73. New York: Springer.

17. Blackadar B. 2015 A general implicit/inverse function theorem. (http://arxiv.org/abs/1509.06025).

18. Devaney RL. 1989 *An introduction to chaotic dynamical systems,* 2nd edn. Redwood: Addison-Wesley.

19. Ashwin P, Wieczorek S, Vitolo R, Cox P. 2012 Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system. *Proc. R. Soc. A* 37, 1166–1184. (doi:10.1098/rsta.2011.0306)

20. Cessi P. 1994 A simple box model of stochastically forced thermohaline flow. *J. Phys. Oceanol.* 2, 1911–1920. (doi:10.1175/1520-0485(1994)024<1911:ASBMOS>2.0.CO;2)

21. Kuehn C. 2013 A mathematical framework for critical transitions: normal forms, variance and applications. *J. Nonl. Sci.* 2, 457–510. (doi:10.1007/s00332-012-9158-x)