RIGIDITY OF CAUSAL MAPS
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Abstract. We show that order-invariant injective maps on the noncompactly causal symmetric space \( \text{SO}_0(1,n)/\text{SO}_0(1,n-1) \) belong to \( O(1,n)^+ \).

1. Introduction

Let \( G = \text{SO}_0(1,n) \), let \( H = \text{SO}_0(1,n-1) \), which we identify with a subgroup of \( G \) in the usual way, and let \( \mathcal{M} \) be the quotient space \( G/H \). Then \( \mathcal{M} \) is a noncompactly causal semisimple symmetric space, that is, there exists a \( G \)-invariant global partial order determined infinitesimally by an \( H \)-invariant cone \( C_{eH} \) in the tangent space \( T_{eH}G/H \) at the origin \( eH \). More precisely, \( C_{eH} \) is the Lorentz cone, and the order is defined by \( yH \preceq xH \) if and only if \( yx^{-1} \in \exp(C_{eH}) \), where \( x,y \in G \). If we identify \( \mathcal{M} \) with \( H_n \), the hyperboloid with one sheet in \( \mathbb{R}^{n+1} \):

\[
H_n := \{ x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : -x_0^2 + x_1^2 + \cdots + x_n^2 = 1 \},
\]

then the partial order is given by

\[
y \geq x \iff (y_0 \geq x_0 \quad \text{and} \quad -x_0y_0 + x_1y_1 + \cdots + x_ny_n \geq 1).
\]

The space \( \mathcal{M} \) may also be realised as a Makarevič space, that is, as an open symmetric orbit in the conformal compactification \( V^c \) of a semisimple Euclidean Jordan algebra \( V \). Then \( \exp V \), which will be written \( \Omega \), is a symmetric convex cone that defines the (flat) invariant causal structure on \( \mathcal{M} \).

In this paper we characterise the order-invariant (or, cone-preserving) injective maps \( f \) on \( \mathcal{M} \), without making any continuity or differentiability assumptions, nor assuming that \( f \) is surjective. Our strategy is to study the behaviour of \( f \) at the

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future boundary of $\mathcal{M}$ (as $x_0 \to \infty$), which in the Makarevič picture can be identified with $\mathbb{R}^{n-1} \cup \{\infty\}$. We define an (unique) extension of $f$ at $\infty$, and show that this is a Möbius map, which in turn implies that $f$ is causal, and hence birational and smooth.

Geometrically, noncompactly causal symmetric spaces may be viewed as generalisations of the space-times arising in general relativity. In this physical set-up, Alexandrov [1], Zeeman [13] and others have showed that bijective maps on Minkowski space preserving causality (or preserving timelike or null cones) are given by Lorentz transformations, translations and dilations. There are generalisations to other space-times, and Alexandrov [2] and Lester [12] have also considered conformal spaces.

2. The imaginary Lobachevski space $\mathcal{M} \simeq \text{SO}_0(1, n)/\text{SO}_0(1, n-1)$

In the following two sections, we summarise the notions and results that we need about causal symmetric spaces and Makarevič spaces. For details, we refer to [5, 6, 8, 9, 10]. In particular, we use the examples in [8, §8] and [9, §10].

Let $V$ be the Euclidean space $\mathbb{R}^n$, equipped with the usual Euclidean scalar product, and write $z \in \mathbb{R}^n$ in coordinates as $(z_0, z_1, \ldots, z_{n-1})$. Define the product $z = xy$ by

$$z_0 = x_0y_0 + \cdots + x_{n-1}y_{n-1} \quad \text{and} \quad z_j = x_0y_j + x_jy_0 \quad \text{if} \quad j = 1, \ldots, n-1.$$ 

With this additional structure, $V$ is the Euclidean Jordan algebra $\mathbb{R}^{1,n-1}$, and its neutral element $e$ is $(1, 0, \ldots, 0)$. Let $\Delta$ denote the quadratic Lorentz form:

$$\Delta(x) = x_0^2 - x_1^2 - \cdots - x_{n-1}^2.$$ 

The associated open symmetric cone $\Omega$ is the Lorentz cone:

$$\Omega = \{x^2\}^o = \{x \in V : \Delta(x) > 0, x_0 > 0\}.$$ 

Let $j : x \mapsto x^{-1}$ be the inversion map on $V$, let $N_+ \simeq V$ be the translation group $\{\tau_v : v \in V\}$, where $\tau_v(x) := x + v$, and let $\text{Str}(V)$ denote the structure group:

$$\text{Str}(V) := \{g \in \text{GL}(V) : j \circ g \circ j \in \text{GL}(V)\}.$$
Note that $\text{Str}(V) = G(\Omega) \times \{\pm 1\}$, where

$$
G(\Omega) = \{g \in \text{GL}(V) : g(\Omega) \subseteq \Omega\} = \mathbb{R}_+ \times O(1, n-1)^+.
$$

The conformal group $\text{Co}(V)$ of $V$ is the group of rational maps generated by $N_+$, $\text{Str}(V)$ and the inversion map $j$, and the causal group $\text{Co}(G(\Omega))$ is the subgroup generated by $N_+$, $G(\Omega)$ and $j$.

Define $N_- := j \circ N_+ \circ j$, then $\text{Str}(V) \rtimes N_-$ is a parabolic subgroup of $\text{Co}(V)$. The conformal compactification $V^c$ of $V$ is defined by $V^c := \text{Co}(V)/\text{Str}(V) \rtimes N_-$. We equip $V^c$ with the flat $\text{Co}(G(\Omega))$-invariant causal structure defined by $-\Omega \subseteq T_x V^c = V$.

Define the involution $\alpha$ on $V$ by

$$
\alpha(x_0, x_1, \ldots, x_{n-1}) = (x_0, -x_1, \ldots, -x_{n-1});
$$

the eigenspaces of $\alpha$ are $V_+$ and $V_-$, where $V_+ = \{(x_0, 0, \ldots, 0) : x_0 \in \mathbb{R}\}$ and

$$
V_- = \{(0, x_1, \ldots, x_{n-1}) : (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\} \cong \mathbb{R}^{n-1}.
$$

Further, let

$$
\Omega_+ = \Omega \cap V_+ = \{(x_0, 0, \ldots, 0) : x_0 \in \mathbb{R}_+\},
$$

and define $x_+ := (x + \alpha(x))/2$ and $x_- := (x - \alpha(x))/2$, for all $x \in V$.

Let $G$ be the group

$$
G := \text{Co}(V)^{(-\alpha)}_0 = \{g \in \text{Co}(V) : (-\alpha) \circ g \circ (-\alpha) = g\}_0
$$

(the subscript $0$ indicates the identity component). Then $G$ is generated by translations by elements of $V_-$, dilations $x \mapsto rx$, where $r > 0$, the orthogonal maps in $O(n-1)$ and $j$. The orbit $\mathcal{M} = Ge$ is open in $V^c$, and $\mathcal{M} \cong G/H$, where $H := \{g \in G : ge = e\}$. The involution $\sigma : g \mapsto j \circ g \circ j$ makes $\mathcal{M} = G/H$ into a symmetric space of Makarevič type, with flat causal structure inherited from $V^c$.

Further, $G \cong SO_0(1, n)$ and $H \cong SO_0(1, n-1)$.

Let $\mathcal{H}_n$ be the one-sheeted hyperboloid $\{y \in \mathbb{R}^{n+1} : -y_0^2 + y_1^2 + \cdots + y_n^2 = 1\}$ in $\mathbb{R}^{n+1}$. The connected component of $e$ of the subset $\mathcal{M} \cap V$ is given by

$$
(\mathcal{M} \cap V)_e = \{x \in V : x + \alpha(x) \in \Omega\}_e = V_- + \Omega_+,
$$
and the map from $\mathcal{M}$ to $H_n$ determined by
\begin{equation}
(1) \quad y_0 = \frac{1 - \Delta(x)}{2x_0}, \quad y_1 = \frac{x_1}{x_0}, \quad \ldots, \quad y_{n-1} = \frac{x_{n-1}}{x_0}, \quad y_n = \frac{1 + \Delta(x)}{2x_0}
\end{equation}
for all $x \in (\mathcal{M} \cap V)_e$ is a causal isomorphism, with inverse given by
\begin{equation}
(2) \quad x_0 = \frac{1}{y_0 + y_n}, \quad x_1 = \frac{y_1}{y_0 + y_n}, \quad \ldots, \quad x_{n-1} = \frac{y_{n-1}}{y_0 + y_n}.
\end{equation}

3. The future

A continuous, piecewise differentiable curve $\gamma : [a, b] \to \mathcal{M}$ is causal, if the (right) derivative $\gamma'(t)$ belongs to $\gamma(t) - \overline{\Omega}$ for all $t \in (a, b)$. We define the partial order $\leq$ on $\mathcal{M}$ by $x \leq y$ if and only if there exists a causal curve from $x$ to $y$. For each $z \in \mathcal{M}$, we say that $x$ belongs to the future of $z$ if $x \geq z$, and we write $\mathcal{M}_z^+ := \{ x \in \mathcal{M} : x \geq z \}$.

We now show that with our choice of causal structure, the future of $z \in (\mathcal{M} \cap V)_e$ does not contain ‘points at $\infty$’.

**Lemma 1.** Let $z \in (\mathcal{M} \cap V)_e$. The ‘future of $z$’ is given by
\[ \mathcal{M}_z^+ = \{ x \in \mathcal{M} : x \geq z \} = (z - \overline{\Omega}) \cap (x_+ \in \Omega_+). \]

This is a bounded set.

**Proof.** Let $x \in (z - \overline{\Omega}) \cap (x_+ \in \Omega_+)$, and define $\gamma : [0, 1] \to V$ by $\gamma(t) = tx + (1 - t)z$. Then $(\gamma(t))_+ = (\gamma(t) + \alpha(\gamma(t)))/2 \in \Omega$ by convexity, and $\gamma'(t) = x - z \in -\overline{\Omega}$. So $\gamma$ is a nontrivial causal curve in $\mathcal{M}$ from $z$ to $x$, and $x$ belongs to the future of $z$.

Conversely, let $\gamma : [a, b] \to \mathcal{M}$ be a causal curve, with $\gamma(a) = z$, that contains ‘points at $\infty$’, and define
\[ \kappa := \inf \{ t \in [a, b] : \gamma(t) \notin \mathcal{M} \cap V \}. \]

The curve $\gamma : [a, \kappa] \to V$ is causal (with causal structure given by $-\overline{\Omega}$) and $\gamma \subseteq z - \overline{\Omega}$. Now $\gamma$ is contained in $\{ x \in V : x + \alpha(x) \in \Omega \}$, and so $\gamma \subseteq (z - \overline{\Omega}) \cap (x_+ \in \Omega_+)$, which is a bounded set, and we have a contradiction. \qed

The case in which $z = e$ was proved in [4, Proposition 5.2]. We define the ‘future boundary’ of $\mathcal{M}_z^+$ in $V^c$ by $\partial_\infty \mathcal{M}_z^+ := \overline{\mathcal{M}_z^+} \setminus \mathcal{M}_z^+$. 
Corollary 2. Let \( z \in (\mathcal{M} \cap V)_e \). The future boundary of \( \mathcal{M}_z^+ \) is given by
\[
\partial_{\infty} \mathcal{M}_z^+ = \overline{\mathcal{M}_z^+} \setminus \mathcal{M}_z^+ = (z - \overline{1}) \cap (x_+ \in \overline{\Omega} \setminus \Omega_+).
\]

Proof. Suppose that \( x \in (z - \overline{1}) \cap (x \in \overline{\Omega} \setminus \Omega_+) \); define the curve \( \gamma: [0, 1] \to V^c \) by \( \gamma(t) = tx + (1-t)z \). Then \( \gamma(t) \in \mathcal{M}_z^+ \) for all \( t \in [0, 1] \), whence \( x \in \overline{\mathcal{M}_z^+} \), but \( x \notin \mathcal{M}_z^+ \).

Conversely, \( \partial_{\infty} \mathcal{M}_z^+ \subseteq (z - \overline{1}) \cap (x \in \overline{\Omega}+) \). \( \square \)

In coordinates, we may write
\[
\mathcal{M}_z^+ = \{ x \in V_+ + \Omega_+ : \Delta(z - x) \geq 0, z_0 - x_0 \geq 0, x > 0 \}
\]
\[
= \{ (z_0 - x_0, z_1 - x_1, \ldots, z_{n-1} - x_{n-1}) : x_1^2 + \cdots + x_{n-1}^2 \leq x_0^2, 0 < z_0 - x_0 \leq z_0 \},
\]
for all \( z \in (\mathcal{M} \cap V)_e \), and the future boundary \( \partial_{\infty} \mathcal{M}_z^+ \) of \( \mathcal{M}_z^+ \) is
\[
\partial_{\infty} \mathcal{M}_z^+ = \{ (0, z_1 - x_1, \ldots, z_{n-1} - x_{n-1}) : x_1^2 + \cdots + x_{n-1}^2 \leq z_0^2 \},
\]
the ball around \( (z_1, \ldots, z_{n-1}) \) with radius \( z_0 \). Here and later in this paper, a ball means a closed ball of finite but strictly positive radius. We note that any ball in \( \mathbb{R}^{n-1} \) can be represented as the future \( \partial_{\infty} \mathcal{M}_z^+ \) of an element \( z \in (\mathcal{M} \cap V)_e \).

4. Möbius maps in \( \mathbb{R}^{n-1} \cup \{ \infty \} \)

We denote a point \( x \) in \( \mathbb{R}^{n-1} \) by \( (x_1, \ldots, x_{n-1}) \), the Euclidean distance between \( x \) and \( y \) in \( \mathbb{R}^{n-1} \) by \( |x - y| \), and the one-point compactification \( \mathbb{R}^{n-1} \cup \{ \infty \} \) by \( \mathbb{R}^{n-1}_\infty \).

The hyperplane in \( \mathbb{R}^{n-1}_\infty \) determined by distinct points \( a, b \in \mathbb{R}^{n-1} \) is the set of points \( x \in \mathbb{R}^{n-1}_\infty \) (including \( \infty \)) such that \( |x - a| = |x - b| \); the reflection in the hyperplane is the isometry of \( \mathbb{R}^{n-1} \) that fixes the points in the hyperplane and exchanges \( a \) and \( b \); it is extended to \( \mathbb{R}^{n-1}_\infty \) in the obvious way.

The hypersphere (or Euclidean sphere) determined by a point \( a \) in \( \mathbb{R}^{n-1} \) and a positive number \( r \) is the set of points \( x \in \mathbb{R}^{n-1} \) satisfying \( |x - a| = r \); the reflection (or inversion) in this sphere is the map \( x \mapsto a + (r/|x - a|)^2(x - a) \), defined on \( \mathbb{R}^{n-1}_\infty \) in the obvious way. In particular, let \( j_-(x) := |x|^{-2}x \) be the reflection in the unit hypersphere.
We define a sphere to be either a hyperplane or a hypersphere. A Möbius map \( \phi \) on \( \mathbb{R}_\infty^{n-1} \) is a composition of a finite number of reflections in spheres, and the group of Möbius maps on \( \mathbb{R}_\infty^{n-1} \) is denoted by \( \text{Möb}(\mathbb{R}_\infty^{n-1}) \). We refer to [3] for a treatment of the general theory of Möbius maps.

A map \( \phi \) of \( \mathbb{R}^{n-1} \) is a similarity if there exists a positive number \( r \) such that 
\[
|\phi(y) - \phi(x)| = r|y - x|
\]
for all \( x, y \in \mathbb{R}^{n-1} \). Given such a map \( \phi \), there exists an orthogonal matrix \( A \) such that \( \phi(x) = rAx + x_0 \), where \( x_0 \in \mathbb{R}^{n-1} \). Similarities are Möbius maps, and the Möbius maps are generated by similarities and the reflection \( j_- \).

Every Möbius map \( \phi \) has the property that either \( \phi^{-1}(\mathbb{R}^{n-1}) = \mathbb{R}^{n-1} \), in which case \( \phi \) is a similarity, or there is a point \( x_0 \) in \( \mathbb{R}^{n-1} \) such that \( \phi(x_0) = \infty \). In the first case, \( \phi \) maps spheres to spheres and the interiors of spheres (the bounded connected component of the complement of the sphere) to the interiors of spheres. In the second case, \( \phi \) maps spheres and their interiors to spheres and their interiors, as long as \( x_0 \) is not an element of the sphere or its interior.

Möbius maps have a unique continuation property, namely, if \( f \) is a map of a domain \( D \) in \( \mathbb{R}_\infty^{n-1} \) into \( \mathbb{R}_\infty^{n-1} \) and for each point \( a \) in \( D \) there is a ball \( B \) containing \( a \) such that the restriction of \( f \) to \( B \) coincides with the restriction to \( B \) of an element of \( \text{Möb}(\mathbb{R}_\infty^{n-1}) \), then \( f \) is the restriction to \( D \) of an element of \( \text{Möb}(\mathbb{R}_\infty^{n-1}) \). This follows immediately from the fact that Möbius maps are completely determined by their restrictions to arbitrarily small nonempty open sets.

The next result [11, Theorem 2.1] generalises a theorem of Carathéodory [7], which treats the case in which \( n = 2 \).

**Theorem 3.** Let \( D \) be a domain in \( \mathbb{R}^{n-1} \), where \( n \geq 3 \). Let \( f : D \to \mathbb{R}^{n-1} \) be an injective map such that \( f(H) \) is a hypersphere whenever \( H \) is a hypersphere contained in \( D \) whose interior is contained in \( D \). Then \( f \) is the restriction of a Möbius map.

Here, the ‘interior’ means the bounded component of the complement of the hypersphere. Our next result is a corollary.
Corollary 4. Let $D$ be a domain in $\mathbb{R}^{n-1}$, where $n \geq 3$. Let $f : D \to \mathbb{R}^{n-1}$ be an injective map such that $f(B)$ is a ball whenever $B$ is a ball contained in $D$. Then $f$ is the restriction to $D$ of a Möbius map.

Proof. Let $B$ be a ball in the domain $D$. It is easy to verify that a point $x$ in $B$ is an interior point if and only if there are two balls $B_1$ and $B_2$ contained in $B$ such that $B_1 \cap B_2 = \{x\}$, and is a boundary point if and only if there is a ball $B_3$ contained in $D$ such that $B \cap B_3 = \{x\}$.

Now let $B$ be a ball contained in $D$. Then $f(B)$ is a ball in $f(D)$; it must be infinite and cannot be all of $\mathbb{R}^{n-1}$ since $f$ is injective. From the characterisation above, $f$ maps interior points of $B$ to interior points of $f(B)$ and boundary points to boundary points. Since $f(B)$ is a ball, the restriction of $f$ to the interior of $B$ maps onto the interior of $f(B)$, and the restriction of $f$ to the boundary of $B$ maps onto the boundary of $f(B)$. So $f$ satisfies the hypothesis of Theorem 3 and hence is a Möbius map. □

5. THE BOUNDARY MAP

Definition 5. An map $f : \mathcal{M} \to \mathcal{M}$ is said to be conal if it is injective and maps $\mathcal{M}_z^+$ onto $\mathcal{M}_{f(z)}^+$ for all $z \in \mathcal{M}$.

In other words, a conal map is an injective order-preserving map. We note that the composition of two conal maps again is a conal map.

Definition 6. A sequence $\{z_i\}_{i \in \mathbb{N}} \subseteq \mathcal{M}$ is said to be increasing if $z_{i+1} \geq z_i$ for all $i \in \mathbb{N}$.

Lemma 7. Suppose that $f$ is conal, $z \in (\mathcal{M} \cap V)_e$, and $f(z) \in (\mathcal{M} \cap V)_e$. Let $\{z_i\}_{i \in \mathbb{N}}$ be an increasing sequence in $\mathcal{M}_z^+$ that converges (in the Euclidean sense) to $z_-$ in $V_-$ as $i \to \infty$. Then the (increasing) sequence $\{f(z_i)\}_{i \in \mathbb{N}}$ in $\mathcal{M}_{f(z)}^+$ also converges to an element of $V_-$.

Proof. Take $\xi_- \in \bigcap_{i=1}^{\infty} \partial_\infty \mathcal{M}_{f(z_i)}^+$, which is a nonempty intersection of balls in $V_-$. We claim that $f(z_i) \to \xi_-$ as $i \to \infty$. 

The radius of the ball \( \partial_\infty M^+_{f(z)} \) and the height of the cone \( M^+_{f(z)} \) is equal to \( (f(z_i))_0 \) (the 0th coordinate of \( f(z_i) \)). This gives an upper bound on the Euclidean distance between \( f(z_i) \) and \( \xi_- \), namely, \( |f(z_i) - \xi_-| \leq \sqrt{2}(f(z_i))_0 \). Assume that \( (f(z_i))_0 \) does not converge to zero. Then the intersection of cones \( \bigcap_{i=1}^\infty M^+_{f(z_i)} \subseteq (M \cap V)_e \) is nonempty. Let \( \xi = \bigcap_{i=1}^\infty M^+_{f(z_i)} \). Then \( z_i \leq f^{-1}(\xi) \) for all \( i \in \mathbb{N} \), in particular \( (z_i)_0 \geq (f^{-1}(\xi))_0 > 0 \) for all \( i \in \mathbb{N} \), and \( z_i \) does not converge to an element in \( V_- \), which is a contradiction.

**Definition 8.** Let \( f \) be conal and assume that \( z, f(z) \in (M \cap V)_e \). We define a local boundary map \( f^-_z \) on \( \partial_\infty M^+_z \) by

\[
 f^-_z(z_-) := \lim_{z_i \to z_-} f(z_i)
\]

for all \( z_- \in \partial_\infty M^+_z \), where \( \{z_i\}_{i \in \mathbb{N}} \subseteq M^+_z \) is an arbitrary increasing sequence such that \( z_i \to z_- \) as \( i \to \infty \).

It is clear that \( f^-_z \) is well-defined and injective on \( \partial_\infty M^+_z \). We also note that \( f^-_z \) maps the ball \( \partial_\infty M^+_z \) onto the ball \( \partial_\infty M^+_{f(z')} \) for any \( z' \in M^+_z \).

**Lemma 9.** Let \( f \) be conal and assume that \( z, f(z) \in (M \cap V)_e \). Then \( f^-_z \) is the restriction of a Möbius map on the interior \( (\partial_\infty M^+_z)^\circ \) of \( \partial_\infty M^+_z \).

**Proof.** The result follows from Corollary 4, since \( f^-_z \) maps the ball \( B = \partial_\infty M^+_z \), for \( z' \in (\partial_\infty M^+_z)^\circ \), onto the ball \( f(B) = \partial_\infty M^+_{f(z')} \). \( \square \)

Let \( \varphi \in \text{Co}(G(\Omega))^{(-\alpha)} \). Then \( \varphi \) is a birational function defined on \( V^c \) and \( \varphi^- = \varphi_\nu \) by continuity. Let \( f \) be conal and \( z \in (M \cap V)_e \). By transitivity there exists a causal map \( \varphi \in \text{Co}(G(\Omega))^{(-\alpha)} \) such that \( \varphi \circ f(z) \in (M \cap V)_e \). The map \( \varphi \circ f \) is conal and maps \( \partial_\infty M^+_z \) onto \( \partial_\infty M^+_{\varphi \circ f(z)} \).

Now let \( z_1, z_2 \in (M \cap V)_e \), and \( \varphi_1, \varphi_2 \in \text{Co}(G(\Omega))^{(-\alpha)} \). Assume that \( (\varphi_1 \circ f)^-_{z_1} \) and \( (\varphi_2 \circ f)^-_{z_1} \) are both defined in a small neighbourhood around \( z_- \in \partial_\infty M^+_{z_1} \cap \partial_\infty M^+_{z_2} \).
The continuity of the causal maps implies that
\[
(\varphi_1 \circ f)^-(z_-) = \lim_{z_i \to z_-} \varphi_1 \circ f(z_i)
\]
\[
= \lim_{z_i \to z_-} \varphi_1 \circ \varphi_2^{-1} \circ \varphi_2 \circ f(z_i)
\]
\[
= \varphi_1 \circ \varphi_2^{-1} \circ \varphi_2 \circ f(z_i)
\]
\[
= \varphi_1 \circ \varphi_2^{-1} \circ (\varphi_2 \circ f)^-(z_-).
\]

Now we define the boundary map globally.

**Definition 10.** Let \( f \) be conal. We define a (global) boundary map \( f^- \) on \( \mathbb{V}_- \) by

\[
f^-(z_-) := \varphi^{-1} \circ (\varphi \circ f)^-(z_-), \quad (z_- \in \partial_\infty \mathcal{M}_z^+ \subset \mathbb{V}_-),
\]

where \( \varphi \) is any map in \( \text{Co}(G(\Omega))^{(-\alpha)} \) such that \( \varphi \circ f(z) \in (\mathcal{M} \cap \mathbb{V})_e \).

**Lemma 11.** Let \( f \) be conal. Then the boundary map \( f^- \) is a Möbius map.

**Proof.** Let \( B \) be a ball in \( \mathbb{V}_- = \mathbb{R}^{n-1} \). Then \( B = \partial_\infty \mathcal{M}_z^+ \), for some \( z \in (\mathcal{M} \cap \mathbb{V})_e \).

The restriction of \( f^- \) to \( (\partial_\infty \mathcal{M}_z^+)^o \) coincides with the restriction of a Möbius map. The unique continuation property of Möbius maps gives the result. \( \square \)

It is worth noting that we have only used the behaviour of the conal functions on the subset \( (\mathcal{M} \cap \mathbb{V})_e = \mathbb{V}_- + \Omega_+ \), so our result holds for injective order-preserving maps from \( (\mathcal{M} \cap \mathbb{V})_e \) into \( \mathcal{M} \). Actually, the boundary map \( f^- \) only depends on the behaviour of \( f \) (arbitrarily) close to the boundary \( \mathbb{V}_- \). Let \( \{U_i\}_{i \in I} = \{\partial_\infty \mathcal{M}_z^+\}_{i \in I} \) be a covering of \( \mathbb{V}_- \) consisting of closed balls of radius less than some \( \varepsilon > 0 \). Then \( f \) is determined by its values on \( \bigcup_{i \in I} \mathcal{M}_z^+ \), which lie inside an \( \varepsilon \)-band around \( \mathbb{V}_- \).

6. **Order invariant injective maps are causal**

Since different conal maps define different boundary maps, that is, the transformation \( f \mapsto f^- \) is injective on the set of conal maps, we have the following rigidity theorem.
Theorem 12. Let $f$ be conal. Then $f$ extends to an element in $\text{Co}(G(\Omega))^{(-\alpha)}$. In particular, $f$ extends to a birational function on $V^c$.

Finally we transfer our results to the hyperboloid with one sheet in $\mathbb{R}^{n+1}$, using the isomorphism given by (1) and (2). With notation as in [9, §10], it is easily seen that $\mathbb{R}^+$ (dilations) corresponds to the hyperbolic multiplication by the Abelian group $A$, the nilpotent groups $N^\circ_\alpha$ and $N^\circ_\alpha$ correspond to the nilpotent groups $N$ and $\overline{N}$ respectively and the (connected) component of $\text{O}(n-1)$ corresponds to the compact group $K \cap H$, where $K = \text{SO}(n)$ is the maximal compact subgroup of $G$.

Corollary 13. Let $f$ be an order-preserving injective map on the noncompactly causal symmetric space $\text{SO}_0(1,n)/\text{SO}_0(1,n-1)$. Then $f$ extends to an element in $\text{O}(1,n)^+$.

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