All static and electrically charged solutions with Einstein base manifold of Einstein-Maxwell massless scalar field system in arbitrary dimensions

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Abstract

We present a simple and complete classification of static solutions in the Einstein-Maxwell system with a massless scalar field in arbitrary $n(\geq 3)$ dimensions. We consider spacetimes which correspond to a warped product $M^2 \times K^{n-2}$, where $K^{n-2}$ is a $(n-2)$-dimensional Einstein space. The scalar field is assumed to depend only on the radial coordinate and the electromagnetic field is purely electric. The general solution with a non-constant real scalar field consists of seven solutions for $n \geq 4$ and three solutions for $n = 3$. None of them is endowed of a Killing horizon in accordance with the no-hair theorem.
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1 Introduction

The role of exact solutions in physics is to understand the properties of physical phenomena in a variety of situations. Even a particular solution may play a large role for our understanding of nature. In gravitational physics, good examples of such exact solutions are Schwarzschild and Kerr solutions. By the black hole uniqueness theorem, asymptotically flat static black holes in vacuum are represented by Schwarzschild solution \(^1\) and this result has been extended to the stationary case in which Kerr solution represents the unique black hole \(^2\). This implies that, in the asymptotically flat and stationary spacetime in vacuum, all the properties of a black hole are encoded in the Kerr black hole and it lead us to the discovery of the black-hole mechanics \(^3\) and then its celebrated thermodynamics description \(^4\).

However, this strong uniqueness result does not mean that the final state of gravitational collapse is always a Kerr black hole because there might be other solutions which do not represent a black hole but a star or naked singularity. In this context, the cosmic censorship hypothesis has been proposed, which asserts that the final state in physically reasonable and generic situations cannot be a naked singularity \(^5\). Although the generic proof of this hypothesis is far from complete, it has been studied in the systems with symmetry which makes the problem tractable. Among others, in the spherically symmetric spacetime, the cosmic censorship hypothesis was shown to be false, where a naked singularity is formed in the dynamical region with matter \(^7\). On the other hand, it was shown to be true in the case with a massless scalar field \(^8\).

These results suggest that the final static configuration after the collapse may be not a Schwarzschild black hole but a naked singularity and motivate us to classify all the spherically symmetric and static solutions to find the candidates of the final configuration. It was shown \(^9\) that the general spherically symmetric and static solution for a massless scalar field is the Janis-Newman-Winicour solution \(^10\) which contains one additional parameter to the mass parameter in the Schwarzschild solution.

This classification problem can be naturally extended in the presence of the Maxwell field in addition. The unique static black hole in this system is the Reissner-Nordström black hole \(^11\) and the exterior of this charged black hole is stable against linear perturbations \(^12\) similar to the Schwarzschild black hole \(^13\). However, it suffers from different kinds of instability. One is the so-called mass inflation instability of the inner horizon \(^14\) which transform a part of the inner horizon into a curvature singularity in gravitational collapse of a massless scalar field with the Maxwell field \(^15\). It was also found that extreme Reissner-Nordström black hole suffers from a different type of instability at the extremal horizon where the second derivative of a massless scalar field generically grows with time \(^16\) (see also \(^17\)).
These instability motivates us to classify all the spherically symmetric and static solutions with a massless scalar field and purely electric Maxwell field, however, it is not an easy task. For example, while the Janis-Newman-Winicour solution can be obtained by a certain solution-generating method from the Schwarzschild solution \[18\], such a method has not been found yet in the presence of the Maxwell field. Actually in \[19\], all the asymptotically flat and static solutions have been classified in four dimensions, however, the metric functions and the scalar field are given in terms of the electric potential. In the present paper, we present all static solutions in a closed form in a more general setup and in arbitrary dimensions.

In the next section we introduce the action and the corresponding field equations. From the general expressions for a static metric and for a radial scalar and electric field, it is shown the absence of a Killing horizon in presence a non-constant scalar field, in agreement with the no-hair theorem. In Sec. 3, the field equations are solved for four and higher dimensions and the complete set of the solutions is expressed in a very simple closed form. The three-dimensional case is also solved in Sec. 4. After some concluding remarks, an appendix is included. The appendix contains a simple derivation of the full set of solutions, which is obtained by using an alternative radial coordinate.

Our basic notations follow \[20\]. The conventions of curvature tensors are \([\nabla_\rho, \nabla_\sigma]V^\mu = R^\mu_{\nu \rho \sigma}V^\nu\) and \(R_{\mu \nu} = R^p_{\mu p \nu}\). The Minkowski metric is taken to be the mostly plus sign, and Roman indices run over all spacetime indices. The \(n\)-dimensional gravitational constant is denoted by \(\kappa\), and the electromagnetic field strength is given by \(F_{\mu \nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu\).

## 2 Preliminaries

### 2.1 System

We consider the Einstein-Maxwell system in \(n(\geq 3)\) spacetime dimensions with a massless scalar field, which is defined by the action

\[
S[g_{\mu \nu}, A_\mu, \phi] = \int d^n x \sqrt{-g} \left( \frac{1}{2\kappa} R - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} (\nabla \phi)^2 \right). \tag{2.1}
\]

This action gives the following field equations:

\[
E_{\mu \nu} := R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - \kappa \left( T_{\mu \nu}^{(em)} + T_{\mu \nu}^{(\phi)} \right) = 0, \quad \nabla_\nu F^{\mu \nu} = 0, \quad \Box \phi = 0, \tag{2.2}
\]
where the energy-momentum tensors for the Maxwell field and the massless Klein-Gordon field are

\[ T^{(\text{em})}_{\mu\nu} := F_{\mu\rho} F^\rho_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}, \quad (2.3) \]
\[ T^{(\phi)}_{\mu\nu} := (\nabla_{\mu} \phi)(\nabla_{\nu} \phi) - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2, \quad (2.4) \]

respectively.

In the present paper, we consider static spacetimes which correspond to a warped product \( M^2 \times K^{n-2} \), where \( K^{n-2} \) is a \((n-2)\)-dimensional Einstein space. A general metric in such a spacetime can be written as

\[ ds^2 = g_{tt}(x) dt^2 + g_{xx}(x) dx^2 + R(x)^2 \gamma_{ij}(z) dz^i dz^j. \quad (2.5) \]

Here \( \gamma_{ij}(z) \) is the metric on the \((n-2)\)-dimensional Einstein space \( K^{n-2} \), whose Ricci tensor is given by \((n-2)\)R\(_{ij} = k(n-3)\gamma_{ij} \), where \( k = 1, 0, -1 \). In addition, we assume \( \phi = \phi(x) \) and \( A_\mu dx^\mu = A_t(x) dt \). Then, the Maxwell equation and the Klein-Gordon equation (2.2) are integrated to give

\[ F_{xt} = q \left( -\frac{g_{tt} g_{xx}}{R^2(n-2)} \right)^{1/2}, \quad \frac{d\phi}{dx} = \phi_1 \left( -\frac{g_{xx}}{g_{tt} R^2(n-2)} \right)^{1/2}, \quad (2.6) \]

where \( q \) and \( \phi_1 \) are integration constants. We assume that \( \phi_1 \) is real and non-zero, namely we are considering a non-constant real scalar field throughout this paper.

### 2.2 Absence of a Killing horizon

In the spacetime (2.5), \( g_{tt}(x) = 0 \) corresponds to a Killing horizon if it is regular and not infinity. However, it is shown that there is no Killing horizon in the present system unless \( \phi_1 = 0 \).

Here we adopt the following coordinates:

\[ ds^2 = -f(x) e^{-2\delta(x)} dt^2 + f(x)^{-1} dx^2 + R(x)^2 \gamma_{ij}(z) dz^i dz^j. \quad (2.7) \]

If \( R \) is constant, the Einstein equations give

\[ 0 = E^t_t - E^x_x = -\frac{\phi_1^2 e^{2\delta}}{f R^{2(n-2)}} \quad (2.8) \]

from which \( \phi_1 = 0 \) is concluded. Since we consider the case with \( \phi_1 \neq 0 \), \( R \) is not constant and then we can choose coordinates such that \( R(x) = x \) without loss of generality. Then, the trace of the Einstein equations gives

\[ (n-2)\mathcal{R} = \kappa \left\{ -\frac{(n-4)q^2}{x^{2(n-2)}} + \frac{(n-2)\phi_1^2 e^{2\delta}}{f x^{2(n-2)}} \right\}. \quad (2.9) \]
f(x_h) = 0 with |δ(x_h)| < ∞ defines a Killing horizon if x = x_h(> 0) is not infinity. Equation \( \text{(2.9)} \) shows that \( \lim_{x \to x_h} R \to \infty \) unless \( \phi_1 = 0 \). Therefore, it is concluded that there is no Killing horizon (and then no event horizon exists), in the presence of a non-constant scalar field.

3 General solution in four and higher dimensions

In this section, we present all the static solutions in the present system in four and higher dimensions. We perform the complete classification in the following coordinates:

\[
ds^2 = -F(x)^{-2}dt^2 + F(x)^{2/(n-3)}G(x)^{-(n-4)/(n-3)} \left( dx^2 + G(x)\gamma_{ij}(z)dz^idz^j \right),\]

(3.1)
in which we have

\[
F_{xt} = \frac{q}{F^2 G}, \quad \frac{d\phi}{dx} = \frac{\phi_1}{G}.
\]

(3.2)

First we derive the basic equations. The combination \( E^t_t + E^i_i = 0 \) gives the master equation for \( G(x) \):

\[
\frac{d^2G}{dx^2} - 2k(n - 3)^2 = 0.
\]

(3.3)

Using Eq. (3.3), we write the Einstein equations as

\[
0 = E^t_t = -\frac{n - 2}{8(n - 3)} F^{-2/(n-3)}G^{(n-4)/(n-3)} \left\{ -8F^{-1}\frac{d^2F}{dx^2} + 4F^{-2}\left( \frac{dF}{dx} \right)^2 \right. \\
- 8F^{-1}\frac{dF}{dx} \frac{dG}{dx} + G^{-2}\left( \frac{dG}{dx} \right)^2 - 4k(n - 3)^2G^{-1} \right\} \\
- \kappa F^{-2/(n-3)}G^{(n-4)/(n-3)} \left( -\frac{q^2}{2} F^{-2}G^{-2} - \frac{\phi_1^2}{2} G^{-2} \right),
\]

(3.4)

\[
0 = E^i_x = -\frac{n - 2}{8(n - 3)} F^{-2/(n-3)}G^{(n-4)/(n-3)} \left\{ 4F^{-2}\left( \frac{dF}{dx} \right)^2 - G^{-2}\left( \frac{dG}{dx} \right)^2 + 4k(n - 3)^2G^{-1} \right\} \\
- \kappa F^{-2/(n-3)}G^{(n-4)/(n-3)} \left( -\frac{q^2}{2} F^{-2}G^{-2} + \frac{\phi_1^2}{2} G^{-2} \right),
\]

(3.5)

from which we obtain the master equation for \( F(x) \):

\[
F^{-1}\frac{d^2F}{dx^2} + F^{-1}G^{-1}\frac{dF}{dx} \frac{dG}{dx} - \frac{1}{4} G^{-2}\left( \frac{dG}{dx} \right)^2 + k(n - 3)^2G^{-1} + \frac{(n - 3)\kappa \phi_1^2}{n - 2} G^{-2} = 0.
\]

(3.6)
We are now ready to perform the classification. \( G(x) \) and \( F(x) \) are obtained from Eqs. (3.3) and (3.6), respectively, and Eq. (3.4) is a constraint on them.

In order to find the location of the (naked) curvature singularities in this coordinate system, we use the trace of the Einstein equations:

\[
(n-2)R = \kappa \left\{ -\frac{(n-4)q^2}{(F^2G)^{(n-2)/(n-3)}} + \frac{(n-2)\phi_1^2}{(F^2G^{n-2})^{1/(n-3)}} \right\}
= \kappa \left\{ -(n-4)q^2 + (n-2)\phi_1^2 F^2 \right\} / (F^2G)^{(n-2)/(n-3)}.
\] (3.7)

The above expression shows that the real zeros of \( F^2G = 0 \) correspond to curvature singularities. Note that where the numerator of Eq. (3.7) vanishes, \( F \) has a finite value. Then, the denominator of (3.7) can be zero at that point only if \( G \) vanishes there. However, for all our solutions in which \( G \) can be zero at some point, the function \( F \) diverges there. Therefore, the numerator and denominator in (3.7) do not vanish simultaneously. Consequently, in all the solutions presented in this paper, there appear two classes of curvature singularities: One is given by \( G = 0 \) with infinite \( F \) satisfying \( F^2G = 0 \) and the other is given by \( F = 0 \) with finite \( G \). At the singularity in the first class, the scalar field diverges, while it remains finite at the singularity in the second class.

### 3.1 General solution for \( k = 1, -1 \) when \( G(x) \) has real roots

First we consider the case of \( k = 1, -1 \) where \( G(x) \) has real roots. In this case, Eq. (3.3) is integrated to give

\[
G(x) = k(n-3)^2(x-a)(x-b),
\] (3.8)

where \( a, b \) are constants. We can assume \( a \geq b > 0 \) without loss of generality. Then the scalar field is given by

\[
\phi(x) = \phi_0 + \frac{\phi_1}{k(n-3)^2(a-b)} \ln \left( \frac{\varepsilon x - a}{x - b} \right)
\] (3.9)

for \( a \neq b \) and

\[
\phi(x) = \phi_0 - \frac{\phi_1}{k(n-3)^2(x-a)}
\] (3.10)

for \( a = b \). The scalar field diverges only at \( x = a \) and \( x = b \). We have put \( \varepsilon = \pm 1 \) in order to make inside the bracket being positive depending on the domain of \( x \).

The general solution for Eq. (3.6) with the constraint (3.4) in the case of \( a \neq b \) and \( \phi_1^2 \neq (n-2)(n-3)^3(a-b)^2/4\kappa \) (corresponding to \( a \neq 0 \)) is

\[
F(x) = A \left( \frac{\varepsilon}{\varepsilon} \right)^{\alpha/2} + B \left( \frac{x - a}{x - b} \right)^{-\alpha/2},
\] (3.11)
where constants $\alpha$, $A$, and $B$ satisfy
\begin{align}
AB &= -\frac{\kappa q^2}{(n-2)(n-3)^3(a-b)^2\alpha^2}, \\
\phi_1^2 &= \frac{(n-2)(n-3)^3(1-\alpha^2)(a-b)^2}{4\kappa}.
\end{align}
(3.12)

This is the generalization in arbitrary dimensions of the Penney solution ($n = 4$ and $k = 1$) found in [21]. For this solution, we compute
\begin{equation}
F^2G = k(n-3)^2 \left\{ A\frac{(x-a)^{(a+1)/2}}{(x-b)^{(a-1)/2}} + B\frac{(x-a)^{-a/2}}{(x-b)^{-a/2}} \right\}^2,
\end{equation}
where we have set $\varepsilon = 1$ for simplicity. Since reality of the scalar field requires $-1 < \alpha < 1$ by Eq. (3.13), $F^2G = 0$ holds at both $x = a$ and $x = b$ and hence they are curvature singularities. A solution of $F(x) = 0$ for $AB < 0$ necessarily satisfies $x \neq a, b$ and it corresponds to a curvature singularity with finite $\phi$.

In the case of $a \neq b$ and $\phi_1^2 = (n-2)(n-3)^3(a-b)^2/4\kappa$, the general solution is
\begin{equation}
F(x) = A\ln\left(\frac{x-a}{x-b}\right) + B,
\end{equation}
where
\begin{equation}
A^2 = \frac{\kappa q^2}{(n-2)(n-3)^3(a-b)^2}.
\end{equation}
(3.16)

In this solution, both $x = a$ and $x = b$ correspond to curvature singularities because the following expression
\begin{equation}
F^2G = k(n-3)^2(x-a)(x-b)\left\{ A\ln\left(\frac{x-a}{x-b}\right) + B \right\}^2
\end{equation}
shows that $F^2G = 0$ holds there. A solution of $F(x) = 0$ for $A \neq 0$ satisfies $x \neq a, b$ and it corresponds to a curvature singularity with finite $\phi$.

Lastly, in the case of $a = b$, the general solution is
\begin{equation}
F(x) = A\sin\left(\sqrt{\frac{\kappa}{(n-2)(n-3)^3k(n-3)(x-a)}} \phi_1\right) + B\cos\left(\sqrt{\frac{\kappa}{(n-2)(n-3)^3k(n-3)(x-a)}} \phi_1\right),
\end{equation}
(3.18)
where
\begin{equation}
q^2 = (A^2 + B^2)\phi_1^2.
\end{equation}
(3.19)

Since the metric function $F$ is finite in this solution, the zero of $G$, $x = a$, corresponds to a curvature singularity. Moreover, in this special case, there is an infinite number of solutions for $F(x) = 0$, which are all different from $x = a$ and represent curvature singularities with finite $\phi$. 

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3.2 General solution for $k = 1, -1$ when $G(x)$ has no real root

Next we consider the case of $k = 1, -1$ where $G(x)$ has no real root. In this case, Eq. (3.3) is integrated to give

$$G(x) = k(n - 3)^2 x^2 + G_0,$$  \hspace{1cm} (3.20)

where $G_0$ is a constant satisfying $kG_0 > 0$ and we have used the degree of freedom to change the origin of $x$. Then the scalar field is given by

$$\phi(x) = \phi_0 + \frac{\phi_1}{(n - 3)\sqrt{kG_0}} \arctan \left( \frac{(n - 3)kx}{\sqrt{kG_0}} \right).$$  \hspace{1cm} (3.21)

Remarkably, the scalar field is finite everywhere in this class of solutions.

The general solution for Eq. (3.6) with the constraint (3.4) is given by

$$F(x) = A \sin \left\{ \sqrt{\frac{\kappa \phi_1^2 + (n - 2)(n - 3)kG_0}{(n - 2)(n - 3)kG_0}} \arctan \left( \frac{(n - 3)kx}{\sqrt{kG_0}} \right) \right\}$$
$$+ B \cos \left\{ \sqrt{\frac{\kappa \phi_1^2 + (n - 2)(n - 3)kG_0}{(n - 2)(n - 3)kG_0}} \arctan \left( \frac{(n - 3)kx}{\sqrt{kG_0}} \right) \right\},$$  \hspace{1cm} (3.22)

where constants $A$ and $B$ satisfy

$$A^2 + B^2 = \frac{\kappa q^2}{(n - 2)(n - 3)kG_0 + \kappa \phi_1^2}.\hspace{8cm} (3.23)$$

In this case, there is a finite number of solutions for $F(x) = 0$, which all correspond to curvature singularities. For example, the solution for $F(x) = 0$ with $A \neq 0$ is given in the following form:

$$\arctan \left( \frac{(n - 3)kx}{\sqrt{kG_0}} \right) = \sqrt{\frac{(n - 2)(n - 3)kG_0}{\kappa \phi_1^2 + (n - 2)(n - 3)kG_0}} \left( \pi N - \arctan \frac{B}{A} \right),$$  \hspace{1cm} (3.24)

where $N$ is an integer. Since the absolute value of the left-hand side is less than $\pi/2$, this algebraic equation has solutions only for some values of $N$ depending on the integration constants.

3.3 General solution for $k = 0$

In the case of $k = 0$, Eq. (3.3) is integrated to give

$$G(x) = G_1 x + G_0,$$  \hspace{1cm} (3.25)
where \( G_0 \) and \( G_1 \) are constants. Then the scalar field is given by

\[
\phi(x) = \phi_0 + \frac{\phi_1}{G_1} \ln \left\{ \varepsilon(G_1 x + G_0) \right\}
\]

(3.26)

for \( G_1 \neq 0 \) and

\[
\phi(x) = \phi_0 + \frac{\phi_1}{G_0} x
\]

(3.27)

for \( G_1 = 0 \).

The general solution for Eq. (3.6) with the constraint (3.4) in the case of \( G_1 \neq 0 \) and \( \phi_1^2 \neq (n-2)G_1^2/4(n-3)\kappa \) is given by

\[
F(x) = A \left\{ \varepsilon(G_1 x + G_0) \right\}^{\alpha/2} + B \left\{ \varepsilon(G_1 x + G_0) \right\}^{-\alpha/2},
\]

(3.28)

where constants \( \alpha, A, \) and \( B \) satisfy

\[
\phi_1^2 = \frac{(n-2)(1-\alpha^2)G_1^2}{4(n-3)\kappa},
\]

(3.29)

\[
AB = -\frac{(n-3)\kappa q^2}{\{(n-2)G_1^2 - 4(n-3)\kappa \phi_1^2\}}.
\]

(3.30)

In this case, we compute

\[
F^2 G = \left\{ A(G_1 x + G_0)^{(1+\alpha)/2} + B(G_1 x + G_0)^{(1-\alpha)/2} \right\}^2,
\]

(3.31)

where we have set \( \varepsilon = 1 \) for simplicity. Since reality of the scalar field requires \(-1 < \alpha < 1\) by Eq. (3.29), \( F^2 G = 0 \) holds at \( G = 0 \), namely \( x = -G_0/G_1 \). Hence it corresponds to a curvature singularity. Also, a solution of \( F(x) = 0 \) for \( AB < 0 \) corresponds to a curvature singularity but with finite \( \phi \).

The general solution for Eq. (3.6) in the case of \( G_1 \neq 0 \) and \( \phi_1^2 = (n-2)G_1^2/4(n-3)\kappa \) is given by

\[
F(x) = A \ln \left\{ \varepsilon(G_1 x + G_0) \right\} + B,
\]

(3.32)

where

\[
A^2 = \frac{(n-3)\kappa q^2}{(n-2)G_1^2}.
\]

(3.33)

Also in this case, \( G(x) = 0 \) corresponds to a curvature singularity since \( F^2 G = 0 \) holds there. In addition, a solution of \( F(x) = 0 \) for \( A \neq 0 \) corresponds to a curvature singularity with finite \( \phi \).

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Lastly, the general solution in the case of $G_1 = 0$ is given by

$$F(x) = A \sin \left( \sqrt{\frac{(n-3)\kappa}{n-2}} \frac{\phi_1}{G_0} x \right) + B \cos \left( \sqrt{\frac{(n-3)\kappa}{n-2}} \frac{\phi_1}{G_0} x \right), \quad (3.34)$$

where

$$A^2 + B^2 = \frac{q^2}{\phi_1^2}. \quad (3.35)$$

Since $F$ is finite everywhere, $G(x) = 0$ corresponds to a curvature singularity. Additionally, in this special case, there is an infinite number of zeros of $F(x) = 0$ which are all curvature singularities with finite $\phi$.

### 4 General solution in three dimensions

In this section, we present the classification in three dimensions. We will use the Einstein equations in the form of $\mathcal{E}^\mu_\nu = 0$, where

$$\mathcal{E}_{\mu\nu} := R_{\mu\nu} - \kappa \left\{ F_{\mu\rho}F_{\nu}\rho - \frac{1}{2(n-2)} g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \right\} - \kappa (\nabla_\mu \phi)(\nabla_\nu \phi). \quad (4.1)$$

Clearly the gauge $\text{[3.1]}$ does not work for $n = 3$. For the three-dimensional case, we adopt the following coordinates:

$$ds^2 = -e^{-2\Phi(r)} dt^2 + e^{2\Psi(r)} (dr^2 + e^{2\Phi(r)} d\theta^2). \quad (4.2)$$

In this coordinate system, the scalar field is integrated to give

$$\phi(r) = \phi_0 + \phi_1 r, \quad (4.3)$$

where $\phi_0$ and $\phi_1$ are constants. Moreover, the field strength is given by

$$F_{rt} = q e^{-2\Phi(r)}, \quad (4.4)$$

where $q$ is an integration constant.

Now the Einstein equations are written as

$$\frac{d^2 \Phi}{dr^2} = 0, \quad (4.5)$$

$$2 \frac{d \Phi}{d r} \frac{d \Psi}{d r} + 2 \left( \frac{d \Phi}{d r} \right)^2 + \frac{d^2 \Psi}{d r^2} = -\kappa \phi_1^2, \quad (4.6)$$

$$\frac{d^2 \Psi}{d r^2} = -\kappa q^2 e^{-2\Phi}. \quad (4.7)$$
The general solution of Eq. (4.5) is given by
\[ e^{-2\Phi} = c_0^2 e^{-2\Phi_1 r}, \]  
where \( c_0 \) and \( \Phi_1 \) are constants. The classification is rather simple: We solve Eq. (4.7) for \( \Psi(r) \) and use Eq. (4.6) as a constraint.

### 4.1 General solution for \( \Phi_1 = 0 \)

If \( \Phi_1 = 0 \), the general solution for \( \Psi(r) \) is
\[ \Psi(r) = -\frac{1}{2} \kappa \phi_1^2 (r - a)(r - b) \]  
(4.9)
if \( \Psi(r) \) has real roots and
\[ \Psi(r) = -\Psi_0 - \frac{1}{2} \kappa \phi_1^2 r^2 \]  
(4.10)
if \( \Psi(r) \) has no root, where \( a, b, \) and \( \Psi_0 (> 0) \) are constants. In both cases, \( \phi_1 \) is given by
\[ \phi_1^2 = c_0^2 q^2. \]  
(4.11)
These solutions acquire a simple form after a rescaling of \( t \) and \( \theta \);
\[ ds^2 = -dt^2 + e^{-\kappa \phi_1 (r-a)(r-b)} (dr^2 + d\theta^2) \]  
(4.12)
and
\[ ds^2 = -dt^2 + e^{-2\Psi_0 - \kappa \phi_1^2 r^2} (dr^2 + d\theta^2). \]  
(4.13)
In both cases, \( \phi(r) \) and \( F_{rt} \) are given by
\[ \phi(r) = \phi_0 \pm \sqrt{qr}, \quad F_{rt} = q. \]  
(4.14)
In these solutions, \( \phi \) becomes constant in the limit of \( q \to 0 \).

Since the Kretschmann invariant \( K := R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) is given by
\[ K = 4\kappa^2 \phi_1^4 e^{2\kappa \phi_1^2 (r-a)(r-b)} \]  
(4.15)
for the metric (4.12) and
\[ K = 4\kappa^2 \phi_1^4 e^{4\Psi_0 + 2\kappa \phi_1^2 r^2} \]  
(4.16)
for the metric (4.13), curvature singularities are located at \( r \to \pm \infty \) in both cases.
4.2 General solution for $\Phi_1 \neq 0$

If $\Phi_1 \neq 0$, the general solution for $\Psi(r)$ is

$$
\Psi(r) = \Psi_0 - \left( \Phi_1 + \frac{\kappa \phi_1^2}{2 \Phi_1} \right) r - \frac{\kappa q c_0^2}{4 \Phi_1^2} e^{-2\Phi_1 r}. \tag{4.17}
$$

After the coordinate transformations $x = c_0 \Phi_1^{-1} e^{-\Phi_1 r}$ and $\Phi_1 t \rightarrow t$, we obtain the solution in the simplest form:

$$
d s^2 = -x^2 d t^2 + x^{\kappa \phi_1^2} \exp \left( 2 \Psi_0 - \frac{1}{2} \kappa q^2 x^2 \right) (d x^2 + d \theta^2), \tag{4.18}
$$

$$
\phi(r) = \phi_0 + \phi_1 \ln |x|, \quad F_{xt} = qx, \tag{4.19}
$$

where $\phi_0$, $\phi_1$, and $\Psi_0$ have been redefined. In this case, the scalar field is not constant in the limit of $q \rightarrow 0$.

The Kretschmann invariant $K := R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}$ is given by

$$
K = \kappa^2 (3q^4 x^4 - 2q^2 \phi_1^2 x^2 + 3\phi_1^4) x^{-2\kappa \phi_1^2 - 4} e^{\kappa q^2 x^2 - 4\Psi_0}. \tag{4.20}
$$

Hence, curvature singularities are located at $x = 0, \pm \infty$.

5 Concluding remarks

In the present paper, we have presented a complete classification of static solutions in the Einstein-Maxwell system with a non-constant massless scalar field in arbitrary $n(\geq 3)$ dimensions. We have considered a warped product spacetimes $M^2 \times K^{n-2}$, where $K^{n-2}$ is a $(n-2)$-dimensional Einstein space and assumed that the scalar field depends only on the radial coordinate and the electromagnetic field is purely electric.

The general solution consists of seven solutions for $n \geq 4$ and three solutions for $n = 3$, which are all written by elementary functions and summarized in Table 1. None of them is endowed of a Killing horizon in accordance with the no-hair theorem. The solutions in four and higher dimensions are also obtained in a different but useful coordinate system, which are presented in Appendix A.

Along the text we have considered a real scalar field. However, one can consider also a phantom scalar field. This case follows from our solutions by including the condition $\phi_1^2 < 0$. In such a case with a phantom scalar field, a complete classification requires an additional
| Name                   | Metric functions | \( \phi \) | Phantom allowed? | Comment                                   |
|------------------------|------------------|-------------|------------------|-------------------------------------------|
| Type-I \((k = \pm 1)\) | (3.8), (3.11)   | (3.9)       | Yes              | \( n = 4, k = 1 \) given in [21]          |
| Type-II \((k = \pm 1)\) | (3.8), (3.15)   | (3.9)       | No               | \( \phi \rightarrow \text{constant not allowed} \) |
| Type-III \((k = \pm 1)\) | (3.8), (3.18)  | (3.10)      | Yes              |                                           |
| Type-IV \((k = \pm 1)\) | (3.20), (3.22) | (3.21)      | Yes              | \( q \rightarrow 0 \) not allowed         |
| Type-V \((k = 0)\)    | (3.25), (3.28)  | (3.26)      | Yes              |                                           |
| Type-VI \((k = 0)\)   | (3.25), (3.32)  | (3.28)      | No               | \( \phi \rightarrow \text{constant not allowed} \) |
| Type-VII \((k = 0)\)  | (3.25), (3.34)  | (3.27)      | Yes              |                                           |
| Type-VIII \((n = 3)\) | (4.12), (4.14)  | (4.14)      | No               |                                           |
| Type-IX \((n = 3)\)   | (4.18)          | (4.19)      | Yes              |                                           |
| Type-X \((n = 3)\)    | (4.18)          |             |                  |                                           |

Table 1: Classification of the static solutions. The limit \( q \rightarrow 0 \) is allowed in the solutions I–III, and V–X, where \( \phi(x) \) then necessarily becomes constant in the solutions III and VII–IX. The limit to constant \( \phi \) is allowed in the solutions I, III–V, and VII–X, where \( q \) necessarily reduces to zero in the solutions III and VII–IX.

The general solution in the case of \( \kappa \phi_1^2 = -(n-2)(n-3)kG_0(< 0) \) is given by

\[
F(x) = A \arctan\left( \frac{(n-3)kx}{\sqrt{kG_0}} \right) + B, \tag{5.1}
\]

where

\[
A^2 = \frac{\kappa q^2}{(n-2)(n-3)kG_0}. \tag{5.2}
\]

This corresponds to an arbitrary dimensional generalization of the Ellis wormhole \[23\] with a Maxwell field. This solution with \( n = 4 \) and \( k = 1 \) was given in \[22\].

One of the possible generalization of the present work is to add a cosmological constant. Even without the Maxwell field, a complete classification of the static solutions has not been performed yet. Only the general solution for the case of Ricci flat base manifolds \((k = 0)\), and in presence of a negative cosmological constant, is known in any spacetime dimension \[24\]. Another possible generalization is to consider the dilatonic coupling of the scalar field to the Maxwell field. We will address these problems elsewhere.

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A Other useful gauge in four and higher dimensions

A.1 Basic equations

In four and higher dimensions we consider a new radial coordinate $r$, such as $dx = Gdr$. In this gauge the metric (3.1) reads now

$$ds^2 = -F(r)^{-2}dt^2 + F(r)^2/(n-3)G(r)^{1/(n-3)}\left(G(r)dr^2 + \gamma_{ij}(z)dz^idz^j\right),$$  \hspace{1cm} (A.1)

so that the field equations (2.2) yield

$$F_{rt} = \frac{q}{F^2}, \quad \frac{d\phi}{dr} = c,$$  \hspace{1cm} (A.2)

where $c$ and $q$ are constants.

Now, we write the Einstein equations as

$$\mathcal{E}^\mu_\nu \equiv R^\mu_\nu - \kappa \left(T^\mu_\nu - \frac{1}{n-2}T \delta^\mu_\nu\right) = 0,$$  \hspace{1cm} (A.3)

where the energy-momentum tensor is $T_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(em)}_{\mu\nu}$ and $T$ is its trace.

Using the gauge (A.1), and defining (for $n \geq 4$)

$$F = e^{-b} \quad \text{and} \quad G = h^{-2},$$  \hspace{1cm} (A.4)

we obtain

$$\mathcal{E}^r_i = 0 \Rightarrow b'' - \frac{n-3}{n-2}\kappa q^2e^{2b} = 0,$$  \hspace{1cm} (A.5)

$$\mathcal{E}^r_r = 0 \Rightarrow h'' - \frac{h}{n-2} \left\{ (n-3)\left(\kappa c^2 - \frac{n-3}{n-2}\kappa q^2e^{2b}\right) + (n-2)b'^2 - b'' \right\} = 0,$$  \hspace{1cm} (A.6)

$$\mathcal{E}^j_j = 0 \Rightarrow \left\{ (n-3)^2k - h'^2 + \left( b'' - \frac{n-3}{n-2}\kappa q^2e^{2b}\right)h^2 + hh'' \right\} \delta^j_i = 0,$$  \hspace{1cm} (A.7)

where a prime denotes the derivative with respect to $r$. Replacing (A.5) in (A.6) and (A.7), and defining the constants

$$q_n^2 := 2\frac{n-3}{n-2}q^2, \quad c_n^2 := 2\frac{n-3}{n-2}c^2, \quad \gamma_n := (n-3)^2k,$$  \hspace{1cm} (A.8)

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a simple system of differential equations is obtained:

\[ b'' - \frac{1}{2} \kappa q_n^2 e^{2b} = 0, \]  
\[ (A.9) \]

\[ h'' - \left( \frac{1}{2} \kappa c_n^2 + b'^2 - b'' \right) h = 0, \]  
\[ (A.10) \]

\[ hh'' - h'^2 + \gamma_n = 0. \]  
\[ (A.11) \]

Remarkably, this system in arbitrary dimensions \( n \geq 4 \) exactly takes the same form as in four dimensions.

A first integral of Eq. (A.9) can be obtained by setting \( b'' = \frac{b'}{db/db} \). Thus, we have

\[ b'^2 = \frac{1}{2} \kappa q_n^2 e^{2b} + b_1, \]  
\[ (A.12) \]

where \( b_1 \) is an integration constant. From (A.9) and (A.12) we note that \( b'^2 - b'' = b_1 \), which reduces (A.10) to \( h'' - a_1 h = 0 \) with \( a_1 = \kappa c_n^2/2 + b_1 \).

In summary, the system to be solved is takes a very simple form:

\[ b'^2 - \frac{1}{2} \kappa q_n^2 e^{2b} - b_1 = 0, \]  
\[ (A.13) \]

\[ h'' - a_1 h = 0, \]  
\[ (A.14) \]

\[ a_1 h^2 - h'^2 + \gamma_n = 0. \]  
\[ (A.15) \]

### A.2 Solutions

Equation (A.13) is easily solved by direct integration yielding

\[
F^{-2} = e^{2b} = \begin{cases} 
\frac{2}{\kappa q_n^2} (r - r_0)^{-2} & \text{if } b_1 = 0, \\
\frac{2b_1}{\kappa q_n^2} \left( \sinh \sqrt{b_1} (r - r_0) \right)^{-2} & \text{if } b_1 > 0, \\
-\frac{2b_1}{\kappa q_n^2} \left( \sin \sqrt{-b_1} (r - r_0) \right)^{-2} & \text{if } b_1 < 0, \\
\exp \left( 2 \sqrt{b_1} (r - r_0) \right) & \text{if } q_n = 0.
\end{cases}
\]  
\[ (A.16) \]

The solution of (A.14) depends on the sign of \( a_1 \). Thus, the following cases appear:
A.2.1 $a_1 > 0$

This occurs if $b_1 > -\kappa c_n^2/2$. In this case (A.14) gives

$$h = c_1 e^{\sqrt{a_1} r} + c_2 e^{-\sqrt{a_1} r}, \quad (A.17)$$

where the integration constants $c_1, c_2$ are constrained by (A.15) to hold

$$4a_1 c_1 c_2 + \gamma_n = 0. \quad (A.18)$$

Note that if $k = 0$, and hence $\gamma_n = 0$, one of the constants $c_1, c_2$ must be 0.

A.2.2 $a_1 = 0$

This is the case when $b_1 = -\kappa c_n^2/2$. Here, the general solution of (A.14) is

$$h = c_1 r + c_2. \quad (A.19)$$

Now, from (A.15) we note that the integration constant $c_1$ must satisfy $c_1^2 = \gamma_n$. Then, this case is not possible for $k = -1$. There are no restriction on $c_2$ except if $k = 0$. In this situation $c_2$ must be positive.

A.2.3 $a_1 < 0$

The constant $a_1$ is negative if $b_1 < -\kappa c_n^2/2$. The solution of (A.14) for $a_1 < 0$ is

$$h = c_1 \sin(\sqrt{-a_1} r) + c_2 \cos(\sqrt{-a_1} r), \quad (A.20)$$

where the integration constants $c_1, c_2$ are required, from (A.15), to verify

$$a_1 (c_1^2 + c_2^2) + \gamma_n = 0. \quad (A.21)$$

Since $a_1 < 0$, the last equation implies that this case is only compatible with a transverse section chosen as a $(n-2)$-dimensional Einstein space having a positive $k$.

In summary, we have determined all the possible solutions associated to the line element (A.1), where $F$ is given in (A.16), and $G = h^{-2}$. The electric field is $F_{rt} = q/F^2$ and the scalar field $\phi = c r + \phi_0$, with $\phi_0$ an integration constant.
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