Existence and asymptotic properties for the solutions to nonlinear SFDEs driven by G-Brownian motion with infinite delay

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Abstract

The aim of this paper is to present the analysis for the solutions of nonlinear stochastic functional differential equation driven by G-Brownian motion with infinite delay (G-SFDEwID). Under some useful assumptions, we have proved that the G-SFDEwID admits a unique local solution. The mentioned theory has been further generalized to show that G-SFDEwID admits a unique strong global solution. The asymptotic properties, mean square boundedness and convergence of solutions with different initial data have been derived. We have assessed that the solution map $X_t$ is mean square bounded and two solution maps from different initial data are convergent. In addition, the exponential estimate for the solution has been studied.

Key words: Existence, uniqueness, local and global solutions, boundedness, convergence, exponential estimate, solution maps, stochastic functional differential equations, G-Brownian motion.

1 Introduction

The stochastic dynamical systems, in which the future state of the systems not only relies on the current state but also on its past history, lead to stochastic functional differential equations with delays. These equations have tremendous applications in diverse areas of sciences and engineering such as population dynamics [1 24], epidemiology [3], gene expression [25], financial assets [1 33 37] and neural networks [18]. There is now a rather comprehensive mathematical literature on existence, uniqueness, stability, moment estimates and other related results of solutions for stochastic functional differential equations [21 22 23 26 27]. In the framework of G-Brownian motion, the existence and uniqueness theorem for solutions to stochastic functional differential equations with infinite delay has been given by Ren, Bi and Sakthivel. Under the linear growth and Lipschitz conditions, they have used the Picard approximation technique [31] while Faizullah

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has used the Cauchy-Maruyama approximation scheme \cite{12} to develop the mentioned theory. The idea has been extended to non-Lipschitz conditions by Faizullah to prove the existence-uniqueness theorem \cite{9} and \( p \)-th moment estimates for the solutions to these equations \cite{10, 11}. Recently, Faizullah et. al., \cite{7, 8} has generalized the theory to determine existence, stability and the \( p \)-th moment estimates for solutions to neutral stochastic functional differential equations in the G-framework (G-SFDEs). However, to the best of our knowledge, no literature can be found on stochastic functional differential equations driven by G-Brownian motion with infinite delay (G-SFDEwID) in the phase space \( C_q((-\infty, 0]; \mathbb{R}^d) \) defined below. This article will contribute to fill the mentioned gap. By using the truncation method, the global strong solutions for G-SFDEwID will be explored. Furthermore, this article will present a systematic study of the asymptotic properties for the solutions as well as solution maps for G-SFDEwID. The \( L^2_G \) and exponential estimates will also be investigated. Let \( \mathbb{R}^d \) and \( A^r \) denote \( d \)-dimensional Euclidean space and transpose of a matrix or vector \( A \) respectively. Let \( C((-\infty, 0]; \mathbb{R}^d) \) be the collection of continuous functions from \( (-\infty, 0] \) to \( \mathbb{R}^d \), then for a given number \( q > 0 \) we define the phase space with the fading memory \( C_q((-\infty, 0]; \mathbb{R}^d) \) by

\[
C_q((-\infty, 0]; \mathbb{R}^d) = \{ \psi \in C((-\infty, 0]; \mathbb{R}^d) : \lim_{\alpha \to -\infty} e^{\alpha \alpha} \psi(\alpha) \text{exists in } \mathbb{R}^d \}.
\]

The space \( C_q((-\infty, 0]; \mathbb{R}^d) \) is complete with norm \( \| \psi \|_q = \sup_{-\infty < \alpha < 0} e^{\alpha \alpha} |\psi(\alpha)| < \infty \). This is a Banach space of continuous and bounded functions and for any \( 0 < q_1 \leq q_2 < \infty \), \( C_{q_1} \subseteq C_{q_2} \) \cite{14, 36}. Let \( \mathcal{B}(C_q) \) be the \( \sigma \)-algebra generated by \( C_q \) and \( C_{q_0} = \{ \psi \in C_q : \lim_{\alpha \to -\infty} e^{\alpha \alpha} \psi(\alpha) = 0 \} \). Denote by \( L^2(C_q) \) (resp. \( L^2(C_{q_0}) \)) the space of all \( \mathcal{F} \)-measurable \( C_q \)-valued (resp. \( C_{q_0} \)-valued) stochastic processes \( \psi \) such that \( E\| \psi \|^2 < \infty \). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space, \( B(t) \) be a \( d \)-dimensional G-Brownian motion and \( \mathcal{F}_t = \sigma \{ B(s) : 0 \leq s \leq t \} \) be the natural filtration. Let the filtration \( \{ \mathcal{F}_t \} \) satisfies the usual conditions. Let \( \mathcal{P} \) be the collection of all probability measures on \( (C_q, \mathcal{B}(C_q)) \) and \( L_b(C_q) \) be the set of all bounded continuous functionals. Let \( N_0 \) be the set of probability measures on \( (-\infty, 0] \) such that for any \( \mu \in N_0 \), \( \int_{-\infty}^{0} \mu(da) = 1 \). For any \( m > 0 \) we define \( N_m \) by

\[
N_m = \{ \mu \in N_0 : \mu^{(m)} = \int_{-\infty}^{0} e^{-ma} \mu(da) < \infty \},
\]

where for any \( m \in (0, m_0) \), \( N_{m_0} \subset N_m \subset N_0 \) \cite{36}. Let \( g : C_q((-\infty, 0]; \mathbb{R}^d) \to \mathbb{R}^d \), \( h : C_q((-\infty, 0]; \mathbb{R}^d) \to \mathbb{R}^{d \times m} \) and \( \gamma : C_q((-\infty, 0]; \mathbb{R}^d) \to \mathbb{R}^{d \times m} \) be Borel measurable. Consider the following stochastic functional differential equation driven by G-Brownian motion with infinite delay

\[
dX(t) = g(X_t)dt + h(X_t)d(B,B)(t) + \gamma(X_t)dB(t), \\
\text{on } t \geq 0 \text{ with the given initial data } X_0 = \zeta \in C_q((-\infty, 0]; \mathbb{R}^d) \text{ and } X_t = \{ X(t + \alpha) : -\infty < \alpha \leq 0 \}.
\]

**Definition 1.1.** A continuous \( \mathbb{R}^d \)-valued and \( \mathcal{F}_t \) adapted process \( X(t), -\infty < t < \theta_e \) is called a local strong solution of problem (1.1) with initial data \( \zeta \in C_q((-\infty, 0]; \mathbb{R}^d) \) if \( X(t) = \zeta(t) \) on \( -\infty < t \leq 0 \) and for all \( t \geq 0 \),

\[
X(t) = \zeta(0) + \int_{0}^{t} g(X_s)ds + \int_{0}^{t} h(X_s)d(B,B)(s) + \int_{0}^{t} \gamma(X_s)dB(s),
\]

holds q.s. for each \( m \geq 1 \), where \( \{ \theta_m ; m \geq 1 \} \) is a nondecreasing sequence of stopping times such that \( \theta_m \to \theta_e \) quasi-surely as \( m \to \infty \).
In addition, if \( \limsup_{t \to \theta_{e}} |X(t)| = \infty \) holds q.s. when \( \theta_{e} < \infty \) q.s., then \( X(t) \), \( -\infty < t < \theta_{e} \) is called a maximal local strong solution and \( \theta_{e} \) is called the explosion time. If \( \theta_{e} = \infty \), then it is called a global solution. A maximal local strong solution \( X(t) \), \( -\infty < t < \theta_{e} \) is said to be unique if for any other maximal local strong solution \( Y(t) \), \( -\infty < t < \theta_{e} \), we have \( \theta_{e} = \theta_{e} \) and \( X(t) = Y(t) \) for \( -\infty < t < \theta_{e} \) quasi-surely. The rest of the paper is organized as follows. Section 2 is devoted to some basic concepts required for the subsequent sections of this paper. Section 3 introduces the existence and uniqueness theory of local and global solutions for stochastic functional differential equations driven by G-Brownian motion with infinite delay. Section 4 describes that G-SFDEwID has a bounded solution. Moreover, it shows that two solutions of G-SFDEwID with distinct initial data converge. Section 5 studies the asymptotic properties such as boundedness and convergence of the solutions map \( X_{t} \) of G-SFDEwID. The \( L_{G}^{2} \) and exponential estimates are included in section 6.

2 Preliminaries

Building on the concepts of G-Brownian motion theory, this section includes the basic notions, results and definitions needed for the further study of the subject. For more details on the concepts briefly discussed, readers are refer to the papers [2, 14, 17, 20, 28, 29, 32, 34]. Let \( \Omega \) be a given basic non-empty set. Assume \( \mathcal{H} \) be a space of real functions defined on \( \Omega \). Then \((\Omega, \mathcal{H}, \hat{E})\) is a sublinear expectation space, where \( \hat{E} \) is a sub-expectation defined as the following.

**Definition 2.1.** A functional \( \hat{E} : \mathcal{H} \to \mathbb{R} \) satisfying the following four characteristics is known as a sub-expectation. Let \( X, Y \in \mathcal{H} \), then

1. **Monotonicity:** \( \hat{E}[X] \geq \hat{E}[Y] \) if \( X \geq Y \).

2. **Constant preservation:** \( \hat{E}[K] = K \), for all \( K \in \mathbb{R} \).

3. **Positive homogeneity:** \( \hat{E}[\alpha X] = \alpha \hat{E}[X] \), for all \( \alpha \in \mathbb{R}^{+} \).

4. **Sub-additivity:** \( \hat{E}[X] + \hat{E}[Y] \geq \hat{E}[X + Y] \).

Let \( \hat{E}[Y] = \hat{E}[-Y] = 0 \), \( K \in \mathbb{R} \) and \( \alpha \in \mathbb{R}^{+} \) then \( \hat{E}[K + \alpha Y + X] = K + \hat{E}[X] \). Furthermore, assume that \( \Omega \) be the space of all \( \mathbb{R}^{d} \)-valued continuous paths \( (w(t))_{t \geq 0} \) starting from zero equipped with the norm

\[
\rho(w^{1}, w^{2}) = \sum_{i=1}^{\infty} \frac{1}{2^{i}} \left( \max_{t \in [0,i]} |w^{1}(t) - w^{2}(t)| \wedge 1 \right),
\]

then for any fixed \( T \in [0, \infty) \),

\[
L_{ip}^{0}(\Omega_{T}) = \left\{ \phi(B(t_{1}), B(t_{2}), \ldots, B(t_{d})) : d \geq 1, t_{1}, t_{2}, \ldots, t_{d} \in [0, T], \phi \in C_{b,Lip}(\mathbb{R}^{d \times n}) \right\},
\]

where \( C_{b,Lip}(\mathbb{R}^{d}) \) is a space of bounded Lipschitz functions, for \( w \in \Omega, t \geq 0, B(t) = B(t, w) = w(t) \) is the canonical process, \( L_{ip}^{0}(\Omega_{t}) \subseteq L_{ip}^{0}(\Omega_{T}) \) for \( t \leq T \) and \( L_{ip}^{0}(\Omega) = \cup_{n=1}^{\infty} L_{ip}^{0}(\Omega_{n}) \). The completion of \( L_{ip}^{0}(\Omega) \) under the Banach norm \( \hat{E}[|.|^{p}]^{\frac{1}{p}} \), \( p \geq 1 \) is denoted by \( L_{G}^{p}(\Omega) \), where \( L_{G}^{p}(\Omega_{t}) \subseteq L_{G}^{p}(\Omega_{T}) \subseteq L_{G}^{p}(\Omega) \) for \( 0 \leq t \leq T < \infty \). Generated by the canonical process \( \{B(t)\}_{t \geq 0} \), the filtration is given by
\[ F_t = \sigma \{ B(s), 0 \leq s \leq t \}, \quad F = \{ F_t \}_{t \geq 0}. \]

Let \( \pi_T = \{ t_0, t_1, ..., t_N \}, 0 \leq t_0 \leq t_1 \leq ... \leq t_N \leq \infty \) be a partition of \([0, T]\). Choose \( p \geq 1 \), let \( M^{p, 0}_G(0, T) \) denotes a collection of the following type processes

\[
\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t),
\]

where \( \xi_i \in L^p_G(\Omega_t), i = 0, 1, ..., N - 1 \). Moreover, the completion of \( M^{p, 0}_G(0, T) \) with the norm given below is denoted by \( M^p_G(0, T) \), \( p \geq 1 \)

\[
\| \eta \| = \left\{ \int_0^T \mathbb{E}[|\eta_s|^p] ds \right\}^{1/p}.
\]

**Definition 2.2.** A d-dimensional stochastic process \( \{ B(t) \}_{t \geq 0} \) satisfying the following features is called a G-Brownian motion

1. \( B(0) = 0 \).
2. The increment \( B(t+s) - B(t) \) is \( N(0, [s\sigma^2, s\bar{\sigma}^2]) \)-distributed.
3. The increment \( B(t+s) - B(t) \) is independent of \( B(t_1), B(t_2), ..., B(t_d) \), for every \( d \in \mathbb{Z}^+ \) and \( 0 \leq t_1 \leq t_2 \leq ... \leq t_d \leq t \).

**Definition 2.3.** Let \( \eta_t \in M^{2,0}_G(0, T) \) be given by (2.1). Then the G-Itô’s integral \( I(\eta) \) is defined by

\[
I(\eta) = \int_0^T \eta(s) dB^a(s) = \sum_{i=0}^{N-1} \xi_i \left( B^a(t_{i+1}) - B^a(t_i) \right).
\]

A mapping \( I : M^{2,0}_G(0, T) \rightarrow L^2_G(F_T) \) can be continuously extended to \( I : M^2_G(0, T) \rightarrow L^2_G(F_T) \) and for \( \eta \in M^2_G(0, T) \) the G-Itô integral is still defined by

\[
\int_0^T \eta(s) dB^a(s) = I(\eta).
\]

**Definition 2.4.** The G-quadratic variation process \( \{ \langle B^a \rangle(t) \}_{t \geq 0} \) of G-Brownian motion is defined by

\[
\langle B^a \rangle(t) = \lim_{N \to \infty} \sum_{i=0}^{N-1} \left( B^a(t^N_{i+1}) - B^a(t^N_i) \right)^2 = B^a(t)^2 - 2 \int_0^t B^a(s) dB^a(s),
\]

which is an increasing process with \( \langle B^a \rangle(0) = 0 \) and for any \( 0 \leq s \leq t \),

\[
\langle B^a \rangle(t) - \langle B^a \rangle(s) \leq \sigma_{aa} \tau(t-s).
\]
Assume that $a, \hat{a} \in \mathbb{R}^d$ be two given vectors. Then the mutual variation process of $B^a$ and $B^{\hat{a}}$ is defined by $\langle B^a, B^{\hat{a}} \rangle = \frac{1}{4}(\langle B^a + B^{\hat{a}} \rangle(t) - \langle B^a - B^{\hat{a}} \rangle(t))$. A mapping $H_{0,T} : M_G^{0,1}(0, T) \mapsto L_G^2(\mathcal{F}_T)$ is defined by

$$H_{0,T}(\eta) = \int_0^T \eta(s)d\langle B^a \rangle(s) = \sum_{i=0}^{N-1} \xi_i \left( \langle B^a \rangle(t_{i+1}) - \langle B^a \rangle(t_i) \right),$$

which can be continuously extended to $M_G^1(0, T)$ and for $\eta \in M_G^1(0, T)$ this is still denoted by

$$\int_0^T \eta(s)d\langle B^a \rangle(s) = H_{0,T}(\eta).$$

The G-Itô integral and its quadratic variation process satisfies the following properties [30, 35].

**Proposition 2.5.**

1. $\hat{\mathbb{E}}[\int_0^T \eta(s)dB(s)] = 0$, for all $\eta \in M_G^p(0, T)$.

2. $\hat{\mathbb{E}}[(\int_0^T \eta(s)dB(s))^2] = \mathbb{E}[\int_0^T \eta^2(s)(B, B)(t)] \leq \sigma^2 \mathbb{E}[\int_0^T \eta^2(s)dt]$, for all $\eta \in M_G^2(0, T)$.

3. $\hat{\mathbb{E}}[\int_0^T |\eta(s)|^pdt] \leq \int_0^T \hat{\mathbb{E}}|\eta(s)|^pdt$, for all $\eta \in M_G^p(0, T)$.

The concept of G-capacity and lemma 2.8 can be found in [5].

**Definition 2.6.** Let $\mathcal{B}(\Omega)$ be a Borel $\sigma$-algebra of $\Omega$ and $\mathcal{P}$ be a collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$. Then the G-capacity denoted by $\hat{C}$ is defined as the following

$$\hat{C}(A) = \sup_{\mathcal{P} \in \mathcal{P}} \mathbb{P}(A),$$

where set $A \in \mathcal{B}(\Omega)$.

**Definition 2.7.** A set $A \in \mathcal{B}(\Omega)$ is said to be polar if its capacity is zero i.e. $\hat{C}(A) = 0$ and a property holds quasi-surely (q.s) if it holds outside a polar set.

**Lemma 2.8.** Let $X \in L^p$ and $\hat{\mathbb{E}}|X|^p < \infty$. Then for each $\delta > 0$, the G-Markov inequality is defined by

$$\hat{C}(|X| > \delta) \leq \frac{\hat{\mathbb{E}}|X|^p}{\delta}.$$ 

For the proof of the following lemmas see [13].

**Lemma 2.9.** Let $p \geq 2$, $\eta \in M_G^2(0, T)$, $a \in \mathbb{R}^d$ and $X(t) = \int_0^t \eta(s)dB^a(s)$. Then there exists a continuous modification $\hat{X}(t)$ of $X(t)$, that is, on some $\Omega \subset \Omega$ with $\hat{C}(\Omega^c) = 0$ and for all $t \in [0, T]$, $C(|X(t) - \hat{X}| \neq 0) = 0$ such that

$$\hat{\mathbb{E}} \left[ \sup_{s \leq t \leq t} |\hat{X}(v) - \hat{X}(s)|^p \right] \leq \hat{K} \sigma_{aa}^p \hat{\mathbb{E}} \left( \int_s^t |\eta(v)|^2dv \right)^{\frac{p}{2}},$$

where $0 < \hat{K} < \infty$ is a positive constant.
Lemma 2.10. Let $p \geq 1$, $\eta \in \mathcal{M}^p(0, T)$ and $a, \hat{a} \in \mathbb{R}^d$, then there exists a continuous modification $X^{a,\hat{a}}(t)$ of $X^{a,\hat{a}}(t) = \int_0^t \eta(s) d(B^a, B^{\hat{a}})(s)$ such that for $0 \leq s \leq t \leq T$,

$$\mathbb{E} \left[ \sup_{0 \leq s \leq v \leq t} |X^{a,\hat{a}}(v) - X^{a,\hat{a}}(s)|^p \right] \leq \left( \frac{1}{4} \sigma_{a+\hat{a}}(a-\hat{a})^r \right)^p (t-s)^{p-1} \mathbb{E} \int_s^t \eta(v)^p dv,$$

The next two lemmas will also be used in the subsequent sections of this article [21].

Lemma 2.11. Let $a, b \geq 0$ and $\epsilon \in (0, 1)$. Then

$$(a + b)^2 \leq \frac{a^2}{\epsilon} + \frac{b^2}{1-\epsilon}.$$

Lemma 2.12. Assume $p \geq 2$ and $\epsilon, a, b > 0$. Then the following two inequalities hold.

(i) $a^{p-1}b \leq \frac{(p-1)\epsilon a^p}{p} + \frac{b^p}{p^{p-1}}$.

(ii) $a^{p-2}b^2 \leq \frac{(p-2)\epsilon a^p}{p} + \frac{2b^p}{p^{p-2}}$.

3 Existence and uniqueness of solutions

In the phase space $BC((\mathbb{R}^d))$, equation (1.1) under the global Lipschitz and growth conditions admits a unique solution [12, 31]. In this section, we study the existence and uniqueness theory for the solutions to (1.1) in the phase space with fading memory $C_q((\mathbb{R}^d))$.

Theorem 3.1. Let there exists two positive constants $L$ and $K$ such that for all $\varphi, \psi \in C_q((\mathbb{R}^d))$ and $t \in [0, T]$, the following conditions hold.

$$|g(\psi) - g(\varphi)|^2 \vee |h(\psi) - h(\varphi)|^2 \vee |\gamma(\psi) - \gamma(\varphi)|^2 \leq L \|\psi - \phi\|_q^2, \quad (3.1)$$

$$|g(\varphi)|^2 \vee |h(\varphi)|^2 \vee |\gamma(\varphi)|^2 \leq K(1 + \|\varphi\|_q^2). \quad (3.2)$$

Then problem (1.1) with initial data $\zeta \in C_q((\mathbb{R}^d))$ admits a unique bounded solution $X(t)$, which is continuous and $\mathcal{F}_t$-adapted on $t \in [0, T]$.

We omit the proof as it can be derived in a similar way like [31]. To extend the above existence-uniqueness result to G-SFDEwID with locally Lipschitz continuous coefficients, we first give the following lemma.

Lemma 3.2. Let $p \geq 1$ and $\lambda < pq$. Then for any $\zeta \in C_q((\mathbb{R}^d))$,

$$\|X(t)\|_q^p \leq e^{-\lambda t} \|\zeta\|_q^p + \sup_{0 \leq s \leq t} |X(s)|^p.$$
Proof. By virtue of the definition of norm $\|\cdot\|$ and observing that $pq > \lambda$ we have
\[
\|X_t\|^p_q = \left[ \sup_{-\infty < \alpha \leq 0} e^{p\alpha} \|X(t + \alpha)\|^p \right] \\
\leq \sup_{-\infty < \alpha \leq 0} e^{\lambda \alpha} \|X(t + \alpha)\|^p \\
\leq \sup_{-\infty < s \leq 0} e^{-\lambda (t-s)} \|X(s)\|^p + \sup_{0 < s \leq t} e^{-\lambda (t-s)} \|X(s)\|^p \\
= e^{-\lambda t} ||\varphi||^p + e^{-\lambda t} \sup_{0 < s \leq t} |X(s)|^p \\
\leq e^{-\lambda t} ||\varphi||^p + \sup_{0 < s \leq t} |X(s)|^p.
\]
The proof is complete. \(\square\)

All through this article we assume that for any $p \geq 1$, $\lambda < pq$.

**Theorem 3.3.** Let for any $m > 0$ there exists a positive constant $K_m$ such that for all $\varphi, \psi \in C_q((-\infty, 0]; \mathbb{R}^d)$ and $t \in [0, T]$, the following local Lipschitz condition hold,
\[
|g(\varphi) - g(\psi)|^2 \vee |h(\psi) - h(\varphi)|^2 \vee |\gamma(\psi) - \gamma(\varphi)|^2 \leq K_m ||\varphi - \psi||^2_q, \quad (3.3)
\]
with $||\varphi|| \vee ||\varphi|| \leq m$. Then the G-SFDEwID (1.1) having the initial data $\zeta \in C_q((-\infty, 0]; \mathbb{R}^d)$ admits a continuous and $\mathcal{F}_t$-adapted unique local solution $X(t)$ quasi-surely on $t \in (-\infty, \theta_e)$, where $\theta_e$ is the potential explosion time.

**Proof.** For any $m \geq m_0$, we define the following stopping time
\[
\theta_m = \inf\{t \geq 0, |X(t)| > m\},
\]
with $\inf \emptyset = \infty$. Let equation (1.1) has two solutions $X(t)$ and $Y(t)$ where
\[
\theta_m = \inf\{t \geq 0, |X(t)| > m\} \wedge \inf\{t \geq 0, |Y(t)| > m\}.
\]
By using the inequality $(\sum_{i=1}^3 a_i)^2 \leq 3 \sum_{i=1}^3 a_i^2$, from (1.1) we have
\[
|Y(t \wedge \theta_m) - X(t \wedge \theta_m)|^2 \leq 3 \int_0^{s \wedge \theta_m} |g(Y_s) - g(X_s)|^2 ds + 3 \int_0^{s \wedge \theta_m} |h(Y_s) - h(X_s)|^2 ds + 3 \int_0^{s \wedge \theta_m} |\gamma(Y_s) - \gamma(X_s)|^2 ds \\
+ 3 \int_0^{s \wedge \theta_m} |\gamma(Y_s) - \gamma(X_s)|^2 ds.
\]
Taking the G-expectation on both sides, using the Holder inequality, lemma 2.9, lemma 2.10 and condition (3.3), there exist positive constants $c, c_1$ and $c_2$ such that
\[
\tilde{\mathbb{E}} \left[ \sup_{0 \leq s \leq t} |Y(s \wedge \theta_m) - X(s \wedge \theta_m)|^2 \right] \leq 3c_1 \tilde{\mathbb{E}} \int_0^{t \wedge \theta_m} |g(Y_s) - g(X_s)|^2 ds + 3c_1 \tilde{\mathbb{E}} \int_0^{t \wedge \theta_m} |h(Y_s) - h(X_s)|^2 ds \\
+ 3c_2 \tilde{\mathbb{E}} \int_0^{t \wedge \theta_m} |\gamma(Y_s) - \gamma(X_s)|^2 ds \\
\leq 3K_m(c + c_1 + c_2) \tilde{\mathbb{E}} \int_0^{t \wedge \theta_m} |Y_s - X_s|^2 ds \\
\leq 3K_m(c + c_1 + c_2) \int_0^t \tilde{\mathbb{E}} \left[ \sup_{0 \leq s \leq t} |Y(s \wedge \theta_m) - X(s \wedge \theta_m)|^2 \right] ds.
\]
The above last inequality is obtained by using lemma (3.2). Finally, the Gronwall inequity yields
\[ \hat{E} \left[ \sup_{0 \leq s \leq t} |Y(s) - X(s)|^2 \right] = 0, \]
which gives that \( Y(t) = X(t) \) quasi-surely. According to the definition of \( \theta_m \), \( \{ \theta_m : m \geq m_0 \} \) is a non-decreasing sequence and as \( m \to \infty \) quasi-surely \( \theta_m \to \theta_\infty \leq \theta_e \). We therefore have \( Y(t) = X(t) \) for all \( t \in (-\infty, \theta_e) \). The uniqueness has been proved. To prove the existence of solutions, for any sufficiently large \( m_0 \) and \( m \geq m_0 \), we set the truncation functions \( g_m, h_m \) and \( \gamma_m \) as follows
\[
\begin{align*}
g_m(\varphi) &= \begin{cases} g(\varphi), & \text{if } \|\varphi\|_q \leq m ; \\
g\left(\frac{m \varphi}{\|\varphi\|_q}\right), & \text{if } \|\varphi\|_q > m, \end{cases} \\
h_m(\varphi) &= \begin{cases} h(\varphi), & \text{if } \|\varphi\|_q \leq m ; \\
h\left(\frac{m \varphi}{\|\varphi\|_q}\right), & \text{if } \|\varphi\|_q > m, \end{cases} \\
\gamma_m(\varphi) &= \begin{cases} \gamma(\varphi), & \text{if } \|\varphi\|_q \leq m ; \\
\gamma\left(\frac{m \varphi}{\|\varphi\|_q}\right), & \text{if } \|\varphi\|_q > m. \end{cases}
\end{align*}
\]

Then \( g_m, h_m \) and \( \gamma_m \) satisfy the assumptions (3.1) and (3.2). By virtue of theorem 3.1, problem
\[ X^{(m)}(t) = \zeta(0) + \int_0^t g_m(X^{(m)}_s)ds + \int_0^t h_m(X^{(m)}_s)d(B,B)(s) + \int_0^t \gamma_m(X^{(m)}_s)d\hat{B}(s), \quad (3.4) \]
adopts a unique bounded solution \( X^{(m)}(t) \), which is continuous and \( \mathcal{F}_t \)-adapted. Notice that for \( 0 \leq t \leq \theta_m \), we have \( g_n(\varphi^n) = g(\varphi^n) \), \( h_n(\varphi^n) = h(\varphi^n) \) and \( \gamma_n(\varphi^n) = \gamma(\varphi^n) \). By using the inequality \( \sum_{i=1}^n a_i^2 \leq \sum_{i=1}^n a_i^2 \), from (1.1) and (3.4) for each \( t \in [0, \theta_m] \), we get
\[
\begin{align*}
|X^{(m)}(t) - X(t)|^2 &\leq 3 \left( \int_0^t |g(X^{(m)}_s) - g(X_s)|ds \right)^2 + 3 \left( \int_0^t |h(X^{(m)}_s) - h(X_s)|d(B,B)(s) \right)^2 \\
&\quad + 3 \left( \int_0^t |\gamma(X^{(m)}_s) - \gamma(X_s)|dB(s) \right)^2.
\end{align*}
\]
Taking the G-expectation on both sides, using the Holder inequality, lemma 2.9, lemma 2.10, condition (3.3) and lemma (3.2), by straightforward calculations in a similar way as above, we derive
\[
\begin{align*}
\hat{E} \left[ \sup_{0 \leq s \leq t} |X^{(m)}(s) - X(s)|^2 \right] &\leq 3c_1 \hat{E} \int_0^t |g(X^{(m)}_s) - g(X_s)|^2 ds + 3c_1 \hat{E} \int_0^t |h(X^{(m)}_s) - h(X_s)|^2 ds \\
&\quad + 3c_2 \hat{E} \int_0^t |\gamma(X^{(m)}_s) - \gamma(X_s)|^2 ds \\
&\leq 3K_n(c + c_1 + c_2) \hat{E} \int_0^t |X^{(n)}_s - X_s|^2 ds \\
&\leq 3K_m(c + c_1 + c_2) \hat{E} \int_0^t \hat{E} \left[ \sup_{0 \leq s \leq t} |X^{(m)}(s) - X(s)|^2 \right] ds
\end{align*}
\]
By virtue of the Grownwall inequity, it follows

$$\mathbb{E}\left[ \sup_{0 \leq s \leq t} |X^{(m)}(s) - X(s)|^2 \right] = 0, \quad 0 \leq t \leq \theta_m, \quad (3.5)$$

The above expression means that for all $t \in [0, \theta_m]$, $X^{(m)}(t) = X(t)$ quasi-surely. Therefore we have $X^{(m)}(t) = X(t)$, for all $t \in (-\infty, \theta_m)$ quasi-surely. Also, by using lemma 3.2 and expression (3.3) we have

$$\mathbb{E}\|X^{(m)}_t - X_t\|_q^2 \leq \mathbb{E}\left[ \sup_{0 \leq s \leq t} |X^{(m)}(s) - X(s)|^2 \right] = 0, \quad 0 \leq t \leq \theta_m,$$

which implies that for all $t \in [0, \theta_m]$, $X^{(m)}_t = X_t$ quasi-surely. Next to show that $X(t)$ is the solution of (1.1), by $X^{(m)}(t \wedge \theta_m) = X(t \wedge \theta_m)$, $X^{(m)}_{t \wedge \theta_m} = X_{t \wedge \theta_m}$ and (3.4), it follows

$$X(t \wedge \theta_m) = \zeta(0) + \int_0^{t \wedge \theta_m} g_m(X_s)ds + \int_0^{t \wedge \theta_m} h_m(X_s)d[B,B](s) + \int_0^{t \wedge \theta_m} \gamma_m(X_s)dB(s)$$

$$= \zeta(0) + \int_0^t g(X_s)ds + \int_0^t h(X_s)d[B,B](s) + \int_0^t \gamma(X_s)dB(s),$$

letting $m \to \infty$, it follows that for any $t \in [0, \theta_e)$,

$$X(t) = \zeta(0) + \int_0^t g(X_s)ds + \int_0^t h(X_s)d[B,B](s) + \int_0^t \gamma(X_s)dB(s),$$

that is, $X(t), t \in (-\infty, \theta_e)$ is a local solution of equation (1.1). The proof stands completed. □

To examine the global existence and uniqueness of solutions for problem (1.1), we assume the following conditions.

(A₁) For any probability measure $\mu_1, \mu_2, \mu_3 \in N_{2q}$ there exists positive constants $\lambda_i$, $i = 1, 2, .., 5$ such that for any $\psi, \varphi \in C_q((\infty, 0]; \mathbb{R}^d)$, we have

$$[\psi(0) - \varphi(0)]^T [g(\psi) - g(\varphi)] \leq -\lambda_1 |\psi(0) - \varphi(0)|^2 + \lambda_2 \int_{-\infty}^0 |\psi(\alpha) - \varphi(\alpha)|^2 \mu_1(\alpha), \quad (3.6)$$

$$[\psi(0) - \varphi(0)]^T [h(\psi) - h(\varphi)] \leq -\lambda_3 |\psi(0) - \varphi(0)|^2 + \lambda_4 \int_{-\infty}^0 |\psi(\alpha) - \varphi(\alpha)|^2 \mu_2(\alpha), \quad (3.7)$$

and

$$|\gamma(\psi) - \gamma(\varphi)|^2 \leq \lambda_5 \int_{-\infty}^0 |\psi(\alpha) - \varphi(\alpha)|^2 \mu_3(\alpha). \quad (3.8)$$

The upcoming lemma will be used in several places all through this article.

Lemma 3.4. Let $p \geq 2$, $\lambda < pq$ and for any $i \in \mathbb{Z}^+$, $\mu_i \in N_t$. Then for any $\zeta \in C_q((\infty, 0]; \mathbb{R}^d)$,

$$\int_0^t \int_{-\infty}^0 |X(s + \alpha)|^p \mu_i(\alpha)ds \leq \frac{\mu_i^{(2q)}}{2q} \|\zeta\|_q^p + \int_0^t |X(s)|^p ds, \quad (3.9)$$

$$\int_0^t \int_{-\infty}^0 e^{\lambda s} |X(s + \alpha)|^p \mu_i(\alpha)ds \leq \frac{\mu_i^{(pq)}}{2q - \lambda} \|\zeta\|_q^p + \mu_i^{(pq)} \int_0^t e^{\lambda s} |X(s)|^p ds. \quad (3.10)$$
Proof. Noticing that \( \zeta \in C_q((\infty, 0]; \mathbb{R}^d) \) and for any \( i \in \mathbb{Z}^+ \), \( \mu_i \in N_{pq} \), using the definition of norm and the Fubini theorem, we derive

\[
\int_0^t \int_{-\infty}^0 |X(s + \alpha)|^p \mu_i(\alpha) d\alpha ds \\
= \int_0^t \left[ \int_{-\infty}^{-s} e^{pq(s+\alpha)} |X(s + \alpha)|^p e^{-pq(s+\alpha)} \mu_i(\alpha) + \int_{-s}^0 |X(s + \alpha)|^p \mu_i(\alpha) \right] ds \\
\leq \| \zeta \|^p q \int_0^t e^{-pq \alpha} ds \int_{-\infty}^0 e^{-pq \alpha} \mu_i(\alpha) + \int_{-\infty}^0 \mu_i(\alpha) \int_0^t |X(s)|^p ds,
\]

by noticing that \( \int_{-\infty}^0 \mu_i(\alpha) = 1 \) and \( \int_{-\infty}^0 e^{-pq \alpha} \mu_i(\alpha) = \mu_i^{(pq)} \), \( i \in \mathbb{Z}^+ \), we derive

\[
\int_0^t \int_{-\infty}^0 |X(s + \alpha)|^p \mu_i(\alpha) d\alpha ds \leq \frac{\mu_i^{(pq)}}{pq} \| \zeta \|^p + \int_0^t |X(s)|^p ds.
\]

The proof of (3.9) is complete. To prove (3.10), we use similar arguments as used above and proceed as follows

\[
\int_0^t \int_{-\infty}^0 e^{\lambda s} |X(s + \alpha)|^p \mu_i(\alpha) d\alpha ds \\
= \int_0^t e^{\lambda s} ds \left[ \int_{-\infty}^{-s} |z(s + \alpha)|^p \mu_i(\alpha) + \int_{-s}^0 |X(s + \alpha)|^p \mu_i(\alpha) \right] \\
= \int_0^t e^{\lambda s} ds \int_{-\infty}^{-s} |X(s + \alpha)|^p \mu_i(\alpha) + \int_{-\infty}^0 \mu_i(\alpha) \int_0^t e^{\lambda s} |X(s + \alpha)|^p ds \\
\leq \int_0^t e^{\lambda s} ds \int_{-\infty}^{-s} e^{2q(s+\alpha)} |X(s + \alpha)|^p e^{-2q(s+\alpha)} \mu_i(\alpha) + \int_{-\infty}^0 \mu_i(\alpha) \int_0^t e^{\lambda(s-\alpha)} |X(s)|^p ds \\
\leq \| \zeta \|^p q \int_0^t e^{-(pq+\lambda) s} ds \int_{-\infty}^0 e^{-pq \alpha} \mu_i(\alpha) + \int_{-\infty}^0 e^{-\lambda \alpha} \mu_i(\alpha) \int_0^t e^{\lambda s} |X(s)|^p ds,
\]

by using the definition \( \mu_i^{(m)} = \int_{-\infty}^0 e^{-ma} \mu_i(\alpha) \) and noticing that \( pq > \lambda \), we have

\[
\int_0^t \int_{-\infty}^0 e^{\lambda s} |X(s + \alpha)|^p \mu_i(\alpha) d\alpha ds \leq \frac{\mu_i^{(pq)}}{pq} \| \zeta \|^p q + \mu_i^{(pq)} \int_0^t e^{\lambda s} |X(s)|^p ds.
\]

The proof of (3.10) stands completed. \( \square \)

**Theorem 3.5.** Let assumptions (3.3) and \( A_1 \) hold. Then the G-SFDEwID (1.1) has a continuous and \( F_t \)-adapted global solution.

**Proof.** For any initial data \( \zeta \in C_q \), in view of local Lipschitz condition \( A_1 \), theorem 3.3 gives that (1.1) admits a unique maximal local strong solution \( X(t) \) on \( t \in (-\infty, \theta_\epsilon) \) and this solution is continuous for any \( t \in (-\infty, \theta_\epsilon) \) and \( F_t \)-adapted. To prove that this solution is global, we only need to show that \( \theta_\epsilon = \infty \) q.s. Note that \( \theta_m \) is increasing as \( m \to \infty \) and \( \theta_m \to \theta_\epsilon \leq \theta_\epsilon \) q.s. If we can prove that \( \theta_\epsilon = \infty \) q.s., then \( \theta_\epsilon = \infty \) q.s., which implies that \( X(t) \) is global. This is equivalent to proving that as \( m \to \infty \), for any \( T > 0, C(\theta_m \leq T) \to 0 \). Applying the G-Itô formula to \( |X(t)|^2 \),
From result (3.9) of lemma 3.4, we have

\[ \mathbb{E}|X(t \wedge \theta_m)|^2 \leq \mathbb{E}|X(0)|^2 + \mathbb{E} \int_0^{t \wedge \theta_m} 2X^\tau(s)g(X_s)ds \]
\[ + k_1 \mathbb{E} \int_0^{t \wedge \theta_m} \left[ 2X^\tau(s)h(X_s) + |\gamma(X_s)|^2 \right] ds. \]  

(3.11)

By using condition (3.6) and the fundamental inequality \(2a_1a_2 \leq \sum_{i=1}^2 a_i^2\) we derive

\[ X^\tau(t)g(X_t) \leq -\left(\lambda_1 - \frac{1}{2}\right)|X(t)|^2 + \frac{1}{2}|g(0)|^2 + \lambda_2 \int_{-\infty}^0 |X(t + \alpha)|^2 \mu_1(d\alpha). \]  

(3.12)

Similar arguments follows

\[ X^\tau(t)h(X_t) \leq -\left(\lambda_3 - \frac{1}{2}\right)|X(t)|^2 + \frac{1}{2}|h(0)|^2 + \lambda_4 \int_{-\infty}^0 |X(t + \alpha)|^2 \mu_2(d\alpha). \]  

(3.13)

By using condition (3.8) and the basic inequality \((\sum_{i=1}^2 a_i^2) \leq 2 \sum_{i=1}^2 a_i^2\) we get

\[ |\gamma(X_t)|^2 \leq 2|\gamma(0)|^2 + 2\lambda_5 \int_{-\infty}^0 |X(t + \alpha)|^2 \mu_3(d\alpha). \]  

(3.14)

On substituting (3.12), (3.13) and (3.14) in (3.11), we derive

\[ \mathbb{E}|X(t \wedge \theta_m)|^2 \leq \mathbb{E}|X(0)|^2 + [|g(0)|^2 + k_1|h(0)|^2 + 2k_1|\gamma(0)|^2]T \]
\[ + (k_1 - 2\lambda_1 - 2k_1\lambda_3 + 1)\mathbb{E} \int_0^{t \wedge \theta_m} |X(s)|^2 ds + 2\lambda_2 \mathbb{E} \int_0^{t \wedge \theta_m} \int_{-\infty}^0 |X(s + \alpha)|^2 \mu_1(d\alpha)ds \]
\[ + 2k_1\lambda_4 \mathbb{E} \int_0^{t \wedge \theta_m} \int_{-\infty}^0 |X(s + \alpha)|^2 \mu_2(d\alpha)ds \]
\[ + 2k_1\lambda_5 \mathbb{E} \int_0^{t \wedge \theta_m} \int_{-\infty}^0 |X(s + \alpha)|^2 \mu_3(d\alpha)ds, \]

letting \(K_1 = \mathbb{E}|X(0)|^2 + [|g(0)|^2 + k_1|h(0)|^2 + 2k_1|\gamma(0)|^2]T\), we obtain

\[ \mathbb{E}|X(t \wedge \theta_m)|^2 \leq K_1 + (-2\lambda_1 - 2k_1\lambda_3 + k_1 + 1)\mathbb{E} \int_0^{t \wedge \theta_m} |X(s)|^2 ds \]
\[ + 2\lambda_2 \mathbb{E} \int_0^{t \wedge \theta_m} \int_{-\infty}^0 |X(s + \alpha)|^2 \mu_1(d\alpha)ds + 2\lambda_4 k_1 \mathbb{E} \int_0^{t \wedge \theta_m} \int_{-\infty}^0 |X(s + \alpha)|^2 \mu_2(d\alpha)ds \]
\[ + 2k_1\lambda_5 \mathbb{E} \int_0^{t \wedge \theta_m} \int_{-\infty}^0 |X(s + \alpha)|^2 \mu_3(d\alpha)ds. \]  

(3.15)

From result (3.9) of lemma 3.4, we have

\[ \int_0^{t \wedge \theta_m} \int_{-\infty}^0 |X(s + \alpha)|^2 \mu_i(d\alpha)ds \leq \frac{1}{2q} ||\zeta||_q^2 \mu_i^{(2q)} + \int_0^t |X(s \wedge \theta_m)|^2 ds. \]  

(3.16)
Substitutions for $i = 1, 2, 3$ in (3.15) give
\[
\hat{E}|X(t \land \theta_m)|^2 \leq K_1 + \frac{1}{q} [\lambda_2 \mu_1^{(2q)} + k_1 \lambda_4 \mu_2^{(2q)} + k_1 \lambda_5 \mu_3^{(2q)}] \hat{E}||\zeta||^2_q \\
+ (k_1 + 1 - 2\lambda_1 + 2\lambda_2 - 2k_1 \lambda_3 + 2k_1 \lambda_4 + 2k_1 \lambda_5) \hat{E} \int_0^t |X(s \land \theta_m)|^2 ds \\
= K_2 + K_3 \int_0^t \hat{E}|X(s \land \theta_m)|^2 ds
\]
where $K_2 = K_1 + \frac{1}{q} [\lambda_2 \mu_1^{(2q)} + k_1 \lambda_4 \mu_2^{(2q)} + k_1 \lambda_5 \mu_3^{(2q)}] ||\zeta||^2_q$ and $K_3 = k_1 + 1 - 2\lambda_1 + 2\lambda_2 - 2k_1 \lambda_3 + 2k_1 \lambda_4 + 2k_1 \lambda_5$. By virtue of the Grownwall inequality,
\[
\hat{E}|X(t \land \theta_m)|^2 \leq K_2 e^{K_3 t},
\]
taking $t = T$ yields
\[
\hat{E}|X(T \land \theta_m)|^2 \leq K_2 e^{K_3 T}.
\]
By using lemma 2.8, the definition of $\theta_m$ and the above inequality, it follows
\[
\hat{C}(\theta_m \leq T) = \hat{C}(\theta_m \leq T, |X(T \land \theta_m)| > m) \\
\leq \frac{1}{m^2} \hat{E}|X(\theta_m)|^2 1_{\theta_m \leq T} \\
= \frac{1}{m^2} \hat{E}|X(T \land \theta_m)|^2 1_{\theta_m \leq T} \\
= \frac{1}{m^2} \hat{E}|X(T \land \theta_m)|^2 \\
\leq \frac{1}{m^2} K_2 e^{K_3 T}.
\]
Taking limits $m \to \infty$ gives
\[
\lim_{m \to \infty} \hat{C}(\theta_m \leq T) = 0,
\]
which implies that G-SFDEwID (1.1) admits a unique global solution $X(t)$ on $(-\infty, \infty)$ quasi-surely. The proof stands completed. \qed

4 Asymptotic properties of solution

In this section, the mean square boundedness for solution of the G-SFDEwID is proved. Under different initial data, the convergence of solutions is derived.

Theorem 4.1. Let $X(t)$ be the unique solution of problem (1.1) with initial data $\zeta \in C_q((-\infty, 0]; \mathbb{R}^d)$. Assume assumption $A_1$ holds. Let $\lambda_i$, $i = 1, 2, \ldots, 5$ satisfy $2\lambda_1 > 2\lambda_2 \mu_1^{(2q)} + 2k_1 \lambda_4 \mu_2^{(2q)} + k_1 \lambda_5 \mu_3^{(2q)} - 2k_1 \lambda_3$. Then there exists $\lambda \in (0, (2\lambda_1 + 2k_1 \lambda_3 - 2\lambda_2 \mu_1^{(2q)} - 2k_1 \lambda_4 \mu_2^{(2q)} - k_1 \lambda_5 \mu_3^{(2q)}) \land 2q)$ such that
\[
\hat{E}[|X(t)|^2] \leq K_4 + K_5 e^{-\lambda t},
\]
where
\[
K_4 = \frac{1}{\lambda} \left( \frac{1}{\epsilon} |g(0)|^2 + \frac{k_1}{\epsilon_1} |h_1(0)|^2 + \frac{k_1}{\epsilon_2} |\gamma(0)|^2 \right)
\]
and
\[ K_5 = \mathbb{E}[|X(0)|^2] + \frac{2\lambda_2 \mu_1}{2q-\lambda} \mathbb{E}\|\zeta\|^2_q + \frac{2k_1 \lambda_3 \mu_2}{2q-\lambda} \mathbb{E}\|\zeta\|^2_q + \frac{k_1 \lambda_5 \mu_3}{(2q-\lambda)(1-\epsilon_2)} \mathbb{E}\|\zeta\|^2_q. \]
and \( \epsilon, \epsilon_1 \) and \( \epsilon_2 \) are sufficiently small such that
\[ 2\lambda_1 - \epsilon - \lambda - k_1 \epsilon_1 + 2k_1 \lambda_3 - 2\lambda_2 \mu_1 - 2k_1 \lambda_4 \mu_2 - \frac{k_1 \lambda_5}{1-\epsilon_2} \mu_3 > 0. \]

**Proof.** Applying the G-Itô formula to \( e^{\lambda t}|X(t)|^2 \), taking the G-expectation on both sides, using properties of G-Itô integral and lemma 2.10, there exists a positive constant \( k_1 \) such that
\[ \mathbb{E}[e^{\lambda t}|X(t)|^2] \leq \mathbb{E}|X(0)|^2 + \mathbb{E}\int_0^t e^{\lambda s} \left[ \lambda |X(s)|^2 + 2X^\tau(s)g(X_s) \right] ds + k_1 \mathbb{E}\int_0^t e^{\lambda s} \left[ 2X^\tau(s)h(X_s) + |\gamma(X_s)|^2 \right] ds. \]

(4.2)

By using condition (3.6) and Lemma 2.12 we derive
\[ X^\tau(t)g(X_t) \leq \left( \frac{\epsilon}{2} - \lambda_1 \right)|X(t)|^2 + \frac{1}{2\epsilon_1} |g(0)|^2 + \lambda_2 \int_{-\infty}^0 |X(t+\alpha)|^2 \mu_1(d\alpha). \]

Similar arguments yield
\[ X^\tau(t)h(X_t) \leq \left( \frac{\epsilon_1}{2} - \lambda_3 \right)|X(t)|^2 + \frac{1}{2\epsilon_2} |h(0)|^2 + \lambda_4 \int_{-\infty}^0 |X(t+\alpha)|^2 \mu_2(d\alpha). \]

In view of condition (3.8) and lemma 2.11 we obtain
\[ |\gamma(X_t)|^2 \leq \frac{1}{\epsilon_2} |\gamma(0)|^2 + \frac{\lambda_5}{1-\epsilon_2} \int_{-\infty}^0 |X(t+\alpha)|^2 \mu_3(d\alpha). \]

By substituting the above obtained inequalities, (4.2) takes the following form
\[ \mathbb{E}[e^{\lambda t}|X(t)|^2] \leq \mathbb{E}|X(0)|^2 + \frac{1}{\lambda \epsilon_1} |g(0)|^2 + \frac{k_1}{\epsilon_1} |h(0)|^2 + \frac{k_1}{\epsilon_2} |\gamma(0)|^2 \left( e^{\lambda t} - 1 \right) + (\epsilon + \lambda - 2\lambda_1 + k_1 \epsilon_1 - 2k_1 \lambda_3) \mathbb{E}\int_0^t e^{\lambda s} |X(s)|^2 ds \]
\[ + 2\lambda_2 \mathbb{E}\int_0^t e^{\lambda s} \int_{-\infty}^0 |X(s+\alpha)|^2 \mu_1(d\alpha) ds \]
\[ + 2k_1 \lambda_4 \mathbb{E}\int_0^t e^{\lambda s} \int_{-\infty}^0 |X(s+\alpha)|^2 \mu_2(d\alpha) ds \]
\[ + k_1 \lambda_5 \frac{1}{1-\epsilon_2} \mathbb{E}\int_0^t e^{\lambda s} \int_{-\infty}^0 |X(s+\alpha)|^2 \mu_3(d\alpha) ds. \]

(4.3)
By result (3.10) in lemma 3.4, we have
\[
\int_0^t \int_{-\infty}^0 e^{\lambda s} |X(s + \alpha)|^2 \mu_i(\alpha) d\alpha ds \leq \frac{1}{2q - \lambda} \|\zeta\|_q^2 \mu_i^{(2q)} + \mu_i^{(2q)} \int_0^t e^{\lambda s} |X(s)|^2 ds,
\] (4.4)
which on substituting in (4.3) for \(i = 1, 2, 3\) follows
\[
\hat{E}[e^{\lambda t} |X(t)|^2] \leq \hat{E}[|X(0)|^2] + \frac{2\lambda_2 \mu_1^{(2q)}}{2q - \lambda} \hat{E}[\|\zeta\|_q^2] + \frac{2k_1 \lambda_4 \mu_2^{(2q)}}{2q - \lambda} \hat{E}[\|\zeta\|_q^2] + \frac{k_1 \lambda_5 \mu_3^{(2q)}}{(2q - \lambda)(1 - \epsilon_2)} \hat{E}[\|\zeta\|_q^2]
\]
\[+ \frac{1}{\lambda} \left( \frac{1}{\epsilon} |g(0)|^2 + \frac{k_1}{\epsilon_1} |h(0)|^2 + \frac{k_1}{\epsilon_2} |\gamma(0)|^2 \right) (e^{\lambda t} - 1)
\]
\[- (2\lambda_1 - \epsilon - \lambda - k_1 \epsilon_1 + 2k_1 \lambda_3 - 2\lambda_2 \mu_1^{(2q)} - 2k_1 \lambda_4 \mu_2^{(2q)} - \frac{k_1 \lambda_5 \mu_3^{(2q)}}{1 - \epsilon_2} > 0,
\]
we have
\[
E[|X(t)|^2] \leq K_4 + K_5 e^{-\lambda t},
\]
where
\[
K_4 = \frac{1}{\lambda} \left( \frac{1}{\epsilon} |g(0)|^2 + \frac{k_1}{\epsilon_1} |h(0)|^2 + \frac{k_1}{\epsilon_2} |\gamma(0)|^2 \right)
\]
and
\[
K_5 = \hat{E}[|X(0)|^2] + \frac{2\lambda_2 \mu_1^{(2q)}}{2q - \lambda} \hat{E}[\|\zeta\|_q^2] + \frac{2k_1 \lambda_4 \mu_2^{(2q)}}{2q - \lambda} \hat{E}[\|\zeta\|_q^2] + \frac{k_1 \lambda_5 \mu_3^{(2q)}}{(2q - \lambda)(1 - \epsilon_2)} \hat{E}[\|\zeta\|_q^2].
\]
The proof stands completed. \(\square\)

**Remark 4.2.** The above theorem 4.1 shows that the solution of initial value problem (1.1) with given initial data \(\zeta \in C_q((-\infty, 0]; \mathbb{R}^d)\) is mean square bounded.

**Theorem 4.3.** Let all the assumptions of theorem 4.1 hold. Let equation (1.1) has two different solutions \(X(t)\) and \(Y(t)\) corresponding to distinct initial data \(\zeta\) and \(\xi\) respectively. Then
\[
\hat{E}[(X(t) - Y(t))^2] \leq K_6 \hat{E}[\|\zeta - \xi\|_q^2 e^{-\lambda t}],
\] (4.5)
where \(K_6 = 1 + \frac{1}{2q - \lambda} (2\lambda_2 \mu_1^{(2q)} + 2k_1 \lambda_4 \mu_2^{(2q)} + k_1 \lambda_5 \mu_3^{(2q)}).\)

**Proof.** First we define \(\Lambda(t) = X(t) - Y(t), \hat{g}(t) = g(X_t) - g(Y_t), \hat{h}(t) = h(X_t) - h(Y_t)\) and \(\hat{\gamma}(t) = \gamma(X_t) - \gamma(Y_t)\). Then applying the G-Itô formula to \(e^{\lambda t} |\Lambda(t)|^2\), taking the G-expectation on both sides, using properties of G-Itô integral and lemma 2.10 there exists a positive constant \(k_1\) such that
\[
e^{\lambda t} E[|\Lambda(t)|^2] \leq \hat{E}[|\zeta(0) - \xi(0)|^2] + \hat{E} \int_0^t e^{\lambda s} |\Lambda(s)|^2 + 2\Lambda^r(s) \hat{g}(s) ds \]
\[+ k_1 \hat{E} \int_0^t e^{\lambda s} [2\Lambda^r(s) \hat{h}(s) + |\hat{\gamma}(s)|^2] ds.
\] (4.6)
From assumption $A_1$, we have
\[
\Lambda^\tau(t) \dot{g}(t) \leq -\lambda_1 |\Lambda(t)|^2 + \lambda_2 \int_{-\infty}^{0} \Lambda(t + \alpha) \mu_1(d\alpha),
\]
\[
\Lambda^\tau(t) \dot{h}(t) \leq -\lambda_3 |\Lambda(t)|^2 + \lambda_4 \int_{-\infty}^{0} \Lambda(t + \alpha) \mu_2(d\alpha)
\]
and
\[
|\dot{\gamma}(t)|^2 \leq \lambda_5 \int_{-\infty}^{0} \Lambda(t + \alpha) \mu_3(d\alpha).
\]
In view of the above inequalities, (4.6) takes the following form
\[
e^{\lambda t} \hat{E} |\Lambda(t)|^2 \leq \hat{E} |\zeta(0) - \xi(0)|^2 + (\lambda - 2\lambda_1 - 2k_1 \lambda_3) \hat{E} \int_{0}^{t} e^{\lambda s} |\Lambda(s)|^2 ds
\]
\[
+ 2\lambda_2 \hat{E} \int_{0}^{t} \int_{-\infty}^{0} e^{\lambda s} \Lambda(s + \alpha) \mu_1(d\alpha) ds + 2k_1 \lambda_4 \hat{E} \int_{0}^{t} \int_{-\infty}^{0} e^{\lambda s} \Lambda(s + \alpha) \mu_2(d\alpha) ds \quad (4.7)
\]
\[
+ k_1 \lambda_5 \hat{E} \int_{0}^{t} \int_{-\infty}^{0} e^{\lambda s} \Lambda(s + \alpha) \mu_3(d\alpha) ds.
\]
From the result [3.10] of lemma [3.2] for $i = 1, 2, 3$ we have
\[
\int_{0}^{t} \int_{-\infty}^{0} e^{\lambda s} |\Lambda(s + \alpha)|^2 \mu_i(d\alpha) ds \leq \frac{1}{2q - \lambda} \|\zeta - \xi\|_q^2 \mu_i^{(2q)}(0) + \mu_i^{(2q)} \int_{0}^{t} e^{\lambda s} |\Lambda(s)|^2 ds \quad (4.8)
\]
By substituting (4.8) in (4.7) we obtain
\[
e^{\lambda t} \hat{E} |\Lambda(t)|^2 \leq \hat{E} |\zeta(0) - \xi(0)|^2 + \frac{1}{2q - \lambda} \left[ 2\lambda_2 \mu_1^{(2q)} + 2k_1 \lambda_4 \mu_2^{(2q)} + k_1 \lambda_5 \mu_3^{(2q)} \right] \hat{E} \|\zeta - \xi\|_q^2
\]
\[- (2\lambda_1 + 2k_1 \lambda_3 - \lambda - 2\lambda_2 \mu_1^{(2q)} - 2k_1 \lambda_4 \mu_2^{(2q)} - k_1 \lambda_5 \mu_3^{(2q)}) \hat{E} \int_{0}^{t} e^{\lambda s} |\Lambda(s)|^2 ds.
\]
In view of the conditions $2\lambda_1 > 2\lambda_2 \mu_1^{(2q)} + 2k_1 \lambda_4 \mu_2^{(2q)} + k_1 \lambda_5 \mu_3^{(2q)} - 2k_1 \lambda_3$ and $\lambda \in (0, (2\lambda_1 + 2k_1 \lambda_3 - 2\lambda_2 \mu_1^{(2q)} - 2k_1 \lambda_4 \mu_2^{(2q)} - k_1 \lambda_5 \mu_3^{(2q)}) \land 2q)$ it follows
\[
\hat{E} |\Lambda(t)|^2 \leq \left[ 1 + \frac{1}{2q - \lambda} (2\lambda_2 \mu_1^{(2q)} + 2k_1 \lambda_4 \mu_2^{(2q)} + k_1 \lambda_5 \mu_3^{(2q)}) \right] \hat{E} \|\zeta - \xi\|_q^2 e^{-\lambda t},
\]
consequently,
\[
\hat{E} |X(t) - Y(t)|^2 \leq K_6 \|\zeta - \xi\|_q^2 e^{-\lambda t},
\]
where $K_6 = 1 + \frac{1}{2q - \lambda} (2\lambda_2 \mu_1^{(2q)} + 2k_1 \lambda_4 \mu_2^{(2q)} + k_1 \lambda_5 \mu_3^{(2q)})$. The proof is complete. \qed

**Remark 4.4.** The above theorem [4.3] describes that two distinct solutions of the initial value problem (1.1) with two distinct initial data are convergent.

From theorem [4.3] we get the following stability result.

**Corollary 4.5.** Let all conditions of theorem [4.3] hold. If $g(0) = h(0) = \gamma(0) = 0$, then the trivial solution of problem (1.1) is mean square exponentially stable.
5 Convergence and boundedness of the solution map

This section examines the asymptotic properties of the solution map \( X_t \). First we find that the solution map \( X_t \) is mean square bounded. Then we show the convergence of distinct solution maps having distinct initial data. Let problem (1.1) with the given initial data \( \zeta \in C_q((\infty, 0]; \mathbb{R}^d) \) has a unique solution \( X(t) \).

**Theorem 5.1.** Let assumption \( A_1 \) holds. Assume that \( \lambda_i, i = 1, 2, \ldots, 5 \) satisfy
\[
2\lambda_1 > 2\lambda_2 \mu_1^{2q} + 1 + 2k_1\lambda_4 \mu_2^{2q} - 2k_1\lambda_5 + k_1 + 2(k_1 \mu_2^{2q} + 2k_3 \mu_3^{2q})\lambda_5 \quad \text{and} \quad \lambda \in \left(0, (2\lambda_1 - 2\lambda_2 \mu_1^{2q}) - 1 - 2k_1\lambda_4 \mu_2^{2q} + 2k_1\lambda_3 - k_1 - 2k_1 \mu_2^{2q} + 2k_3 \mu_3^{2q})\lambda_5 \right) \wedge 2q.
\]
For any initial data \( \zeta \in C_q((\infty, 0]; \mathbb{R}^d) \), we then have
\[
\mathbf{E}\|X_t\|^2 \leq K_7 + K_8e^{-\lambda t},
\]
where \( K_7 = \frac{2}{\lambda} \left((g(0))^2 + k_1|h(0)|^2 + 2(k_1 + 2k_3)|\gamma(0)|^2\right) \) and \( K_8 = 3 + \frac{4}{2q - \lambda}[\lambda_2 \mu_1^{2q} + k_1(\lambda_4 + \lambda_5)\mu_2^{2q} + 2k_3 \lambda_5 \mu_3^{2q}] \).

**Proof.** Applying the G-Itô formula to \( e^{\lambda t}|X(t)|^2 \) and taking the G-expectation on both sides, we have
\[
\mathbf{E}\left[ \sup_{0 < s \leq t} e^{\lambda s}|X(s)|^2 \right] \leq \mathbf{E}|\zeta(0)|^2 + \mathbf{E}\left[ \sup_{0 < s \leq t} \int_0^t e^{\lambda s}\left(\lambda |X(s)|^2 + 2X^T(s)g(X_s)\right) ds \right]
+ \mathbf{E}\left[ \sup_{0 < s \leq t} \int_0^t e^{\lambda s}\left(2X^T(s)h(X_s) + |\gamma(X_s)|^2\right) d\langle B, B \rangle(s) \right]
+ 2\mathbf{E}\left[ \sup_{0 < s \leq t} \int_0^t e^{\lambda s}X^T(s)\gamma(X_s) dB(s) \right].
\]
(5.1)

By straightforward calculations, using (3.12) and then (4.1) we obtain
\[
\mathbf{E}\left[ \sup_{0 < s \leq t} \int_0^t e^{\lambda s}\left(\lambda |X(s)|^2 + 2X^T(s)g(X_s)\right) ds \right]
\leq \frac{1}{\lambda}|g(0)|^2(e^{\lambda t} - 1) + \frac{2\lambda_2}{2q - \lambda} \mathbf{E}\|\zeta\|^{2q} + \frac{2\lambda_1 + 2\lambda_2 \mu_1^{2q} + 1}{\lambda} \mathbf{E} \int_0^t e^{\lambda s}|X(s)|^2 ds.
\]
(5.2)
By using (3.13), (3.14), (4.4) and lemma 2.10 there exists a positive constant $k_1$ such that

$$
\mathbb{E} \left[ \sup_{0 < s \leq t} \int_0^t e^{\lambda s} (2X^\tau(s)h(X_s) + |\gamma(X_s)|^2) d\langle B, B \rangle(s) \right]
$$

$$
\leq \frac{1}{\lambda} k_1 (|h(0)|^2 + 2|\gamma(0)|^2)(e^{\lambda t} - 1) - k_1 (2\lambda_3 - 1) \mathbb{E} \int_0^t e^{\lambda s} |X(s)|^2 ds
$$

$$
+ 2k_1 \lambda_4 \mathbb{E} \int_0^t \int_{-\infty}^0 e^{\lambda s} |X(s + \alpha)|^2 \mu_2(d\alpha) ds
$$

$$
+ 2k_1 \lambda_5 \mathbb{E} \int_0^t \int_{-\infty}^0 e^{\lambda s} |X(s + \alpha)|^2 \mu_3(d\alpha) ds
$$

simplification yields

$$
\mathbb{E} \left[ \sup_{0 < s \leq t} \int_0^t e^{\lambda s} (2X^\tau(s)h(X_s) + |\gamma(X_s)|^2) d\langle B, B \rangle(s) \right]
$$

$$
\leq \frac{1}{\lambda} k_1 (|h(0)|^2 + 2|\gamma(0)|^2)(e^{\lambda t} - 1) - k_1 (2\lambda_3 - 1) \mathbb{E} \int_0^t e^{\lambda s} |X(s)|^2 ds
$$

$$
+ \frac{2k_1 \lambda_4}{2q - \lambda} \mathbb{E}\|\zeta\|^2 \mu_2(\zeta) + 2k_1 \lambda_4 \mu_2(2q) \mathbb{E} \int_0^t e^{\lambda s} |X(s)|^2 ds
$$

$$
+ \frac{2k_1 \lambda_5}{2q - \lambda} \mathbb{E}\|\zeta\|^2 \mu_3(\zeta) + 2k_1 \lambda_5 \mu_3(2q) \mathbb{E} \int_0^t e^{\lambda s} |X(s)|^2 ds,
$$

By utilizing (3.14), the inequality $a_1 a_2 \leq \frac{1}{2} \sum_{i=1}^2 a_i$ and lemma 2.10 there exists a positive constant $k_2$ such that

$$
2 \mathbb{E} \left[ \sup_{0 < s \leq t} \int_0^t e^{\lambda s} X^\tau(s) \gamma(X_s) dB(t) \right] \leq 2k_2 \mathbb{E} \left[ \int_0^t e^{\lambda s} |X(s)|^2 e^{\lambda s} |\gamma(X_s)|^2 ds \right]^\frac{1}{2}
$$

$$
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{0 < s \leq t} e^{\lambda s} |X(s)|^2 \right] + 2k_2^2 \mathbb{E} \int_0^t e^{\lambda s} |\gamma(X_s)|^2 ds
$$

$$
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{0 < s \leq t} e^{\lambda s} |X(s)|^2 \right] + 4k_2^2 \frac{1}{\lambda} |\gamma(0)|^2 (e^{\lambda t} - 1)
$$

$$
+ 4k_2^2 \lambda_5 \mathbb{E} \int_0^t \int_{-\infty}^0 e^{\lambda s} |X(s + \alpha)|^2 \mu_3(d\alpha),
$$

by using lemma 3.4 we get

$$
2 \mathbb{E} \left[ \sup_{0 < s \leq t} \int_0^t e^{\lambda s} X^\tau(s) \gamma(X_s) dB(t) \right] \leq \frac{1}{2} \mathbb{E} \left[ \sup_{0 < s \leq t} e^{\lambda s} |X(s)|^2 \right] + 4k_3 \frac{1}{\lambda} |\gamma(0)|^2 (e^{\lambda t} - 1)
$$

$$
+ \frac{4k_3 \lambda_5}{2q - \lambda} \mathbb{E}\|\zeta\|^2 \mu_3(\zeta) + 4k_3 \lambda_5 \mu_3(2q) \mathbb{E} \int_0^t e^{\lambda s} |X(s)|^2 ds,
$$

(5.4)
where $k_3 = k_2^2$. Noticing that $\zeta(0) \leq \sup_{-\infty < \alpha \leq 0} e^{q\alpha} |\zeta(\alpha)| = \|\zeta\|_q$ and substituting (5.2), (5.3) and (5.4) in (5.1), it follows

$$
\mathbb{E} \left[ \sup_{0 < s \leq t} e^{\lambda s} |X(s)|^2 \right] \leq \frac{2}{\lambda} \left( (|g(0)|^2 + k_1|h(0)|^2 + 2(k_1 + 2k_3)|\gamma(0)|^2) (e^{\lambda t} - 1) + \frac{2}{2q - \lambda} \left( 2q - \lambda + 2\lambda_2 \mu_1^{(2q)} + 2k_1(\lambda_4 + \lambda_5)\mu_2^{(2q)} + 4k_3\lambda_5\mu_3^{(2q)} \right) \mathbb{E} \|\zeta\|_q^2 
- 2 \left( 2\lambda_1 - 2\lambda_2 \mu_1^{(2q)} \right) - 1 - \lambda \right) - 2k_1\lambda_4\mu_2^{(2q)} + 2k_1\lambda_3 - k_1 - 2(k_1\mu_1^{(2q)} + 2k_3\mu_3^{(2q)}) \lambda_5 \right) \mathbb{E} \int_0^t e^{\lambda s} |X(s)|^2 ds.
$$

By using the assumptions $2\lambda_1 > 2\lambda_2 \mu_1^{(2q)} + 1 + 2k_1\lambda_4\mu_2^{(2q)} - 2k_1\lambda_3 + k_1 + 2(k_1\mu_1^{(2q)} + 2k_3\mu_3^{(2q)}) \lambda_5$ and $\lambda \in \left( 0, (2\lambda_1 - 2\lambda_2 \mu_1^{(2q)} - 1 - 2k_1\lambda_4\mu_2^{(2q)} + 2k_1\lambda_3 - k_1 - 2(k_1\mu_1^{(2q)} + 2k_3\mu_3^{(2q)}) \lambda_5) \wedge 2q \right)$, we get

$$
\mathbb{E} \left[ \sup_{0 < s \leq t} e^{\lambda s} |X(s)|^2 \right] \leq \frac{2}{\lambda} \left( (|g(0)|^2 + k_1|h(0)|^2 + 2(k_1 + 2k_3)|\gamma(0)|^2) (e^{\lambda t} - 1) + \frac{2}{2q - \lambda} \left( 2q - \lambda + 2\lambda_2 \mu_1^{(2q)} + 2k_1(\lambda_4 + \lambda_5)\mu_2^{(2q)} + 4k_3\lambda_5\mu_3^{(2q)} \right) \mathbb{E} \|\zeta\|_q^2 \right).
$$

From lemma (5.2), we have

$$
\mathbb{E} \|X_t\|^2 \leq e^{-\lambda t} \mathbb{E} \|\zeta\|_q^2 + e^{-\lambda t} \mathbb{E} \left[ \sup_{0 < s \leq t} e^{\lambda s} |X(s)|^2 \right].
$$

Substituting (5.5) in the above inequality yields

$$
\mathbb{E} \|X_t\|^2 \leq e^{-\lambda t} \mathbb{E} \|\zeta\|_q^2 + \frac{2}{\lambda} \left( (|g(0)|^2 + k_1|h(0)|^2 + 2(k_1 + 2k_3)|\gamma(0)|^2) (1 - e^{-\lambda t}) + \frac{2}{2q - \lambda} \left( 2q - \lambda + 2\lambda_2 \mu_1^{(2q)} + 2k_1(\lambda_4 + \lambda_5)\mu_2^{(2q)} + 4k_3\lambda_5\mu_3^{(2q)} \right) \mathbb{E} \|\zeta\|_q^2 e^{-\lambda t} \right)
\leq K_7 + K_8 e^{-\lambda t}
$$

where $K_7 = \frac{2}{\lambda} \left( (|g(0)|^2 + k_1|h(0)|^2 + 2(k_1 + 2k_3)|\gamma(0)|^2 \right)$ and $K_8 = 3 + \frac{4}{2q - \lambda} \left( \lambda_2 \mu_1^{(2q)} + k_1(\lambda_4 + \lambda_5)\mu_2^{(2q)} + 2k_3\lambda_5\mu_3^{(2q)} \right)$. The proof is complete.

**Remark 5.2.** Theorem 5.1 states that the solution map $X_t$ of the initial value problem (1) with given initial data $\zeta \in C_q((-\infty, 0]; \mathbb{R}^d)$ is mean square bounded.

**Theorem 5.3.** Let assumptions $A_1$ hold. Assume that $\lambda_i$, $i = 1, 2, \ldots, 5$ satisfy $2\lambda_1 > 2\lambda_2 \mu_1^{(2q)} - 2k_1\lambda_3 + 2k_1\lambda_4\mu_2^{(2q)} + (k_1 + 2k_3)\mu_3^{(2q)} \lambda_5$ and $\lambda \in \left( 0, (2\lambda_1 - 2\lambda_2 \mu_1^{(2q)} + 2k_1\lambda_3 - k_1 - 2k_1\lambda_4\mu_2^{(2q)} - (k_1 + 2k_3)\mu_3^{(2q)} \lambda_5) \wedge 2q \right)$. Then for distinct initial data $\zeta, \xi \in C_q$ the respective solution maps $X_t^\zeta$ and $Y_t^\xi$ satisfy

$$
\mathbb{E} \|X_t^\zeta - Y_t^\xi\|_q^2 \leq K_9 \mathbb{E} \|\zeta - \xi\|_q^2 e^{-\lambda t},
$$

where $K_9 = 1 + \frac{4}{2q - \lambda} [\lambda_2 \mu_1^{(2q)} + k_1(2\lambda_4\mu_2^{(2q)} + \lambda_5\mu_3^{(2q)}) + k_3\lambda_5\mu_3^{(2q)}]$. 18
Proof. Keeping in mind the definitions of $\Lambda(t)$, $\hat{g}(t)$, $\hat{h}(t)$ and $\hat{g}(t)$, we apply the G-Itô formula to $e^{\lambda t}|\Lambda(t)|^2$ and then taking G-expectation on both sides to obtain

\[
\begin{align*}
\hat{E}[ \sup_{0<s\leq t} e^{\lambda s}|\Lambda(s)|^2 ] &\leq \hat{E}[ \xi(0) - \xi(0)]^2 + \hat{E}[ \sup_{0<s\leq t} \int_0^t e^{\lambda s}|\Lambda(s)|^2 + 2\Lambda^2(s) \hat{g}(s) ] ds \\
&+ \hat{E}[ \sup_{0<s\leq t} \int_0^t e^{\lambda s}[2\Lambda^2(s) \hat{h}(s) + |\hat{g}(s)|^2]d\langle B, B \rangle(s)] \\
&+ 2\hat{E}[ \sup_{0<s\leq t} \int_0^t e^{\lambda s}\Lambda^2(s) \gamma(X_s) d\langle B(s) \rangle].
\end{align*}
\]

By using (3.6) and then lemma 3.4 we have

\[
\begin{align*}
\hat{E}[ \sup_{0<s\leq t} \int_0^t e^{\lambda s}[2\Lambda^2(s) \hat{h}(s) + |\hat{g}(s)|^2]d\langle B, B \rangle(s)] \\
&\leq 2\lambda_2 \mu_1(2q) \hat{E}\|\hat{\zeta} - \xi\|_q^2 - (2\lambda_1 - \lambda - 2\lambda_2 \mu_1(2q)) \hat{E}\int_0^t e^{\lambda s}|\Lambda(s)|^2 ds.
\end{align*}
\]

By using (3.7), (3.8), lemma 3.4 and lemma 2.10 there exists a positive constant $k_1$ such that

\[
\begin{align*}
\hat{E}[ \sup_{0<s\leq t} \int_0^t e^{\lambda s}[2\Lambda^2(s) \hat{h}(s) + |\hat{g}(s)|^2]d\langle B, B \rangle(s)] \\
&\leq -2k_1 \lambda_3 \hat{E}\int_0^t e^{\lambda s}|\Lambda(s)|^2 ds + 2k_1 \lambda_4 \hat{E}\int_0^t \int_{-\infty}^0 e^{\lambda s}|\Lambda(s + \alpha)|^2 \mu_2(d\alpha) ds \\
&+ k_1 \lambda_5 \hat{E}\int_0^t \int_{-\infty}^0 e^{\lambda s}|\Lambda(s + \alpha)|^2 \mu_3(d\alpha) ds \\
&\leq \frac{1}{2q - \lambda} k_1 (2\lambda_4 \mu_2(2q) + \lambda_5 \mu_3(2q)) \hat{E}\|\hat{\zeta} - \xi\|_q^2 - (2k_1 \lambda_3 - 2k_1 \lambda_4 \mu_2(2q) - \lambda_5 k_1 \mu_3(2q)) \hat{E}\int_0^t e^{\lambda s}|\Lambda(s)|^2 ds.
\end{align*}
\]

By utilizing (3.11), the inequality $a_1 a_2 \leq \frac{1}{2} \sum_{i=1}^2 a_i$ and lemma 2.9 there exists a positive constant $k_2$ such that

\[
\begin{align*}
2\hat{E}[ \sup_{0<s\leq t} \int_0^t e^{\lambda s}\Lambda^2(s) \gamma(X_s) d\langle B(s) \rangle] &\leq 2k_2 \hat{E}\int_0^t e^{\lambda s}|\Lambda(s)|^2 e^{\lambda s}|\gamma(X_s)|^2 ds \\
&\leq \hat{E}(\sup_{0<s\leq t} e^{\lambda s}|F(s)|^2)^\frac{1}{2} \hat{E}(4k_2^2 \int_0^t e^{\lambda s}|\gamma(X_s)|^2 ds)^\frac{1}{2} \\
&\leq \frac{1}{2} \hat{E}(\sup_{0<s\leq t} e^{\lambda s}|\Lambda(s)|^2)^\frac{1}{2} + 2k_2^2 \lambda_5 \hat{E}\int_0^t \int_{-\infty}^0 e^{\lambda s}|\Lambda(s + \alpha)|^2 \mu_3(d\alpha) ds,
\end{align*}
\]

by using lemma 3.4 we get

\[
\begin{align*}
2\hat{E}[ \sup_{0<s\leq t} \int_0^t e^{\lambda s}\Lambda^2(s) \gamma(X_s) d\langle B(s) \rangle] \\
&\leq \frac{1}{2} \hat{E}(\sup_{0<s\leq t} e^{\lambda s}|\Lambda(s)|^2)^\frac{1}{2} + \frac{2k_3 \lambda_5}{2q - \lambda} \hat{E}\|\hat{\zeta} - \xi\|_q^2 + 2k_3 \lambda_5 \mu_3(2q) \hat{E}\int_0^t e^{\lambda s}|X(s)|^2 ds,
\end{align*}
\]
where \( k_3 = k_2^2 \). Substituting all the above derived inequalities in (5.6) we derive

\[
\hat{E}[\sup_{0 < s \leq t} e^{\lambda t} |\Lambda(t)|^2] \leq \frac{1}{2q - \lambda} [2\lambda_2 \mu_1^{(2q)} + k_1(2\lambda_4 \mu_2^{(2q)} + \lambda_5 \mu_3^{(2q)}) + 2\lambda_3 \lambda_5 \mu_3^{(2q)}] \hat{E}\| \zeta - \xi \|_q^2
\]

\[
- [2\lambda_1 - \lambda - 2\lambda_2 \mu_1^{(2q)} + 2k_1 \lambda_3 - 2k_1 \lambda_4 \mu_2^{(2q)}] \hat{E}\int_0^t e^{\lambda s} |\Lambda(s)|^2 ds
\]

\[
+ \frac{1}{2} \hat{E} \left( \sup_{0 < s \leq t} e^{\lambda s} |\Lambda(s)|^2 \right),
\]

simplification yields

\[
\hat{E}[\sup_{0 < s \leq t} e^{\lambda t} |\Lambda(t)|^2] \leq \frac{2}{2q - \lambda} [2\lambda_2 \mu_1^{(2q)} + 2k_1(2\lambda_4 \mu_2^{(2q)} + \lambda_5 \mu_3^{(2q)}) + 2\lambda_3 \lambda_5 \mu_3^{(2q)}] \hat{E}\| \zeta - \xi \|_q^2
\]

\[
- 2[2\lambda_1 - \lambda - 2\lambda_2 \mu_1^{(2q)} + 2k_1 \lambda_3 - 2k_1 \lambda_4 \mu_2^{(2q)}] - (k_1 + 2k_3) \mu_3^{(2q)} \hat{E}\int_0^t e^{\lambda s} |\Lambda(s)|^2 ds.
\]

By using the assumptions \( 2\lambda_1 > 2\lambda_2 \mu_1^{(2q)} - 2k_1 \lambda_3 + 2k_1 \lambda_4 \mu_2^{(2q)} + (k_1 + 2k_3) \mu_3^{(2q)} \lambda_5 \) and \( \lambda \in (0, (2\lambda_1 - 2\lambda_2 \mu_1^{(2q)} + 2k_1 \lambda_3 - 2k_1 \lambda_4 \mu_2^{(2q)} - (k_1 + 2k_3) \mu_3^{(2q)} \lambda_5) \wedge 2q) \), we get

\[
\hat{E}[\sup_{0 < s \leq t} e^{\lambda s} |\Lambda(s)|^2] \leq \frac{2}{2q - \lambda} [2\lambda_2 \mu_1^{(2q)} + 2k_1(2\lambda_4 \mu_2^{(2q)} + \lambda_5 \mu_3^{(2q)}) + 2\lambda_3 \lambda_5 \mu_3^{(2q)}] \hat{E}\| \zeta - \xi \|_q^2
\]

(5.8)

In view of lemma 3.2 we have

\[
\hat{E}\| X_t^\xi - Y_t^\xi \|_q^2 \leq e^{-\lambda t} \hat{E}\| \zeta - \xi \|_q^2 + e^{-\lambda t} \hat{E} \left( \sup_{0 < s \leq t} e^{\lambda s} |\Lambda(s)|^2 \right),
\]

on substituting (5.8) in the above inequality, we derive

\[
\hat{E}\| X_t^\xi - Y_t^\xi \|_q^2
\]

\[
\leq e^{-\lambda t} \hat{E}\| \zeta - \xi \|_q^2 + \frac{4}{2q - \lambda} [\lambda_2 \mu_1^{(2q)} + k_1(2\lambda_4 \mu_2^{(2q)} + \lambda_5 \mu_3^{(2q)}) + k_3 \lambda_5 \mu_3^{(2q)}] \hat{E}\| \zeta - \xi \|_q^2 e^{-\lambda t}
\]

\[
= K_9 \hat{E}\| \zeta - \xi \|_q^2 e^{-\lambda t},
\]

where \( K_9 = 1 + \frac{4}{2q - \lambda} [\lambda_2 \mu_1^{(2q)} + k_1(2\lambda_4 \mu_2^{(2q)} + \lambda_5 \mu_3^{(2q)}) + k_3 \lambda_5 \mu_3^{(2q)}] \). The proof stands completed. \( \square \)

**Remark 5.4.** Theorem 5.3 indicates that two distinct solution maps \( X_t \) and \( Y_t \) from the respective distinct initial data \( \zeta \in C_q((0, 0]; \mathbb{R}^d) \) and \( \xi \in C_q((0, 0]; \mathbb{R}^d) \) are convergent.

### 6 The exponential estimate

To establish the exponential estimate, assume that equation (1.1) with initial data \( \zeta \in C_q((0, 0]; \mathbb{R}^d) \) has a unique solution \( X(t) \) on \( t \in [0, \infty) \). First, we derive the \( L^2_G \) and then the exponential estimates as follows.

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Theorem 6.1. Let $E\|\zeta\|_q^2 < \infty$ and assumption $A_1$ hold. Then for all $t \geq 0,$

$$
\hat{E}\left[ \sup_{-\infty < s \leq t} |X(t)|^2 \right] \leq \left[ \hat{E}\|\zeta\|_q^2 + L_1 \right]e^{L_2 t},
$$

where $L_1 = \hat{K} + \frac{2}{q}(q + \lambda_2 \mu_1^{(2q)} + k_1(\lambda_5 \mu_2^{(2q)} + \mu_2^{(2q)}) + 2k_3 \lambda_5 \mu_3^{(2q)})) \hat{E}\|\zeta\|_q^2$, $\hat{K} = 2\|g(0)\|^2 + k_1(|h(0)|^2 + 2|\gamma(0)|^2) + 4k_3|\gamma(0)|^2)$ and $L_2 = 2[2\lambda_2 - 2\lambda_1 + 1 + k_1(2\lambda_5 - 2\lambda_3 + 3) + 4k_3\lambda_5].$

Proof. Applying the G-Itô formula to $|X(t)|^2$ and taking the G-expectation on both sides

$$
\hat{E}\left[ \sup_{0 \leq s \leq t} |X(t)|^2 \right] \leq \hat{E}|X(0)|^2 + 2\hat{E}\left[ \sup_{0 \leq s \leq t} \int_0^t X^\tau(s)g(X_s)ds \right] + \hat{E}\left[ \sup_{0 \leq s \leq t} \int_0^t (2X^\tau(s)h(X_s) + |\gamma(X_s)|^2)d\langle B, B \rangle(s) \right] + 2\hat{E}\left[ \sup_{0 \leq s \leq t} \int_0^t X^\tau(s)\gamma(X_s)dB(s) \right] \tag{6.1}
$$

By using (3.12) and then (4.4) we obtain

$$
2\hat{E}\left[ \sup_{0 < s \leq t} \int_0^t X^\tau(s)g(X_s)ds \right] \leq g(0)^2 T + \frac{\lambda_2}{q}\hat{E}\|\zeta\|_q^{2q} + (2\lambda_2 - 2\lambda_1 + 1)\hat{E}\int_0^t |X(s)|^2ds.
$$

By using (3.13), (3.14), (4.4) and lemma 2.10 there exists a positive constant $k_1$ such that

$$
\hat{E}\left[ \sup_{0 \leq s \leq t} \int_0^t (2X^\tau(s)h(X_s) + |\gamma(X_s)|^2)d\langle B, B \rangle(s) \right] \leq k_1\hat{E}\left[ \int_0^t (2X^\tau(s)h(X_s) + |\gamma(X_s)|^2) \right]ds
\leq k_1|h(0)|^2 + 2|\gamma(0)|^2 T + k_1\frac{1}{q}(\lambda_5 \mu_3^{(2q)} + \mu_2^{(2q)})\hat{E}\|\zeta\|_q^2
+ k_1(2\lambda_5 - 2\lambda_3 + 3)\hat{E}\int_0^t |X(s)|^2ds.
$$

By utilizing (3.13), the inequality $a_1a_2 \leq \frac{1}{2}\sum_{i=1}^2 a_i$ and lemma 2.9 straightforward calculations give

$$
2\hat{E}\left[ \sup_{0 < s \leq t} \int_0^t X^\tau(s)\gamma(X_s)dB(t) \right] \leq 2k_2\hat{E}\left[ \int_0^t |X^\tau(s)\gamma(X_s)|^2ds \right]^{\frac{1}{2}}
\leq \frac{1}{2}\hat{E}\left[ \sup_{0 < s \leq t} |X(s)|^2 \right] + 4k_2^2|\gamma(0)|^2 T + 4k_2^2\lambda_5\hat{E}\int_0^t \int_{-\infty}^0 |X(s + \alpha)|^2\mu_3(d\alpha)ds,
$$

by using lemma 3.4 we get

$$
2E\left[ \sup_{0 < s \leq t} \int_0^t X^\tau(s)\gamma(X_s)dB(s) \right]
\leq \frac{1}{2}\hat{E}\left[ \sup_{0 < s \leq t} |X(s)|^2 \right] + 4k_3|\gamma(0)|^2 T + \frac{4k_3\lambda_5}{2q}\hat{E}\|\zeta\|_q^2 + 4k_3\lambda_5\hat{E}\int_0^t |X(s)|^2ds, \tag{6.2}
$$

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where \( k_3 = k_2^2 \). Substituting the above inequalities in (6.1) and then by straightforward calculations, we derive

\[
\hat{E}\left[ \sup_{0 \leq s \leq t} |X(t)|^2 \right] \leq L_1 + L_2 \int_0^t \hat{E}\left[ \sup_{0 \leq s \leq t} |X(s)|^2 \right] ds,
\]

(6.3)

where \( L_1 = \hat{K} + \frac{2}{q}(q + 2\mu_1^{(2q)} + k_1(\lambda_5\mu_3^{(2q)} + \mu_2^{(2q)})) + 2k_3\lambda_5\mu_3^{(2q)}\hat{E}\|\zeta\|_q^2, \hat{K} = 2\left[ |g(0)|^2 + k_1(|h(0)|^2 + 2|\gamma(0)|^2) + 4k_3|\gamma(0)|^2 \right] T \) and \( L_2 = 2[2\lambda_2 - 2\lambda_1 + 1 + k_1(2\lambda_5 - 2\lambda_3 + 3) + 4k_3\lambda_5] \). By noticing that

\[
\hat{E}\left[ \sup_{-\infty < s \leq t} |X(s)|^2 \right] \leq \hat{E}\|\zeta\|_q^2 + \hat{E}\left[ \sup_{0 \leq s \leq t} |X(s)|^2 \right],
\]

it follows

\[
\hat{E}\left[ \sup_{-\infty < s \leq t} |X(s)|^2 \right] \leq \hat{E}\|\zeta\|_q^2 + L_1 + L_2 \int_0^t \hat{E}\left[ \sup_{0 \leq s \leq t} |X(s)|^2 \right] ds
\]

\[
\leq \hat{E}\|\zeta\|_q^2 + L_1 + L_2 \int_{-\infty}^t \hat{E}\left[ \sup_{-\infty < s \leq t} |X(s)|^2 \right] ds.
\]

By applying the Grownwall inequality, we get the desired expression. \( \square \)

**Theorem 6.2.** Let \( \hat{E}\|\zeta\|_q^2 < \infty \) and assumption \( A_1 \) hold. Then for all \( t \geq 0 \),

\[
\lim_{t \to \infty} \sup \frac{1}{t} \log |X(t)| \leq M,
\]

where \( M = 2\lambda_2 - 2\lambda_1 + 1 + k_1(2\lambda_5 - 2\lambda_3 + 3) + 4k_3\lambda_5 \).

**Proof.** Applying the Grownwall inequality from (6.3), it follows

\[
\hat{E}\left[ \sup_{0 \leq s \leq t} |X(s)|^2 \right] \leq L_1 e^{L_2 t},
\]

(6.4)

where \( L_1 = \hat{K} + \frac{2}{q}(q + 2\mu_1^{(2q)} + k_1(\lambda_5\mu_3^{(2q)} + \mu_2^{(2q)})) + 2k_3\lambda_5\mu_3^{(2q)}\hat{E}\|\zeta\|_q^2, \hat{K} = 2\left[ |g(0)|^2 + k_1(|h(0)|^2 + 2|\gamma(0)|^2) + 4k_3|\gamma(0)|^2 \right] T \) and \( L_2 = 2[2\lambda_2 - 2\lambda_1 + 1 + k_1(2\lambda_5 - 2\lambda_3 + 3) + 4k_3\lambda_5] \). By virtue of the above result (6.4), for each \( m = 1, 2, 3, \ldots \), we have

\[
\hat{E}\left[ \sup_{m-1 \leq t \leq m} |X(t)|^2 \right] \leq L_1 e^{L_2 m}.
\]

For any \( \epsilon > 0 \), by using lemma 2.8 we get

\[
\hat{C}\{ w : \sup_{m-1 \leq t \leq m} |X(t)|^2 > e^{(L_2 + \epsilon)m} \} \leq \frac{\hat{E}\left[ \sup_{m-1 \leq t \leq m} |X(t)|^2 \right]}{e^{(L_2 + \epsilon)m}} \leq \frac{L_1 e^{L_2 m}}{e^{(L_2 + \epsilon)m}} = L_1 e^{-\epsilon m}.
\]

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But for almost all \( w \in \Omega \), the Borel-Cantelli lemma yields that there exists a random integer \( m_0 = m_0(w) \) so that
\[
\sup_{m-1 \leq t \leq m} |X(t)|^2 \leq e^{(L_2 + \epsilon)m}, \quad \text{whenever } m \geq m_0,
\]
which implies
\[
\lim_{t \to \infty} \sup_{t} \frac{1}{t} \log |X(t)| \leq \frac{L_2 + \epsilon}{2} = 2\lambda_2 - 2\lambda_1 + 1 + k_1(2\lambda_5 - 2\lambda_3 + 3) + 4k_3\lambda_5 + \frac{\epsilon}{2},
\]
but \( \epsilon \) is arbitrary and the above result reduces to
\[
\lim_{t \to \infty} \sup_{t} \frac{1}{t} \log |X(t)| \leq M,
\]
where \( M = 2\lambda_2 - 2\lambda_1 + 1 + k_1(2\lambda_5 - 2\lambda_3 + 3) + 4k_3\lambda_5 \). The proof is complete. \( \square \)

**Remark 6.3.** The above lemma states that the second moment of Lyapunov exponent \([6, 15]\) \( \lim_{t \to \infty} \sup_{t} \frac{1}{t} \log |X(t)| \) is bounded with upper bound \( M \).

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