The integral Cauchy problem for generalized Boussinesq equations with general leading parts

Veli B. Shakhmurov
Department of Mechanical Engineering, Okan University, Akfirat, Tuzla 34959
Istanbul, E-mail: veli.sahmurov@okan.edu.tr;
Rishad Shahmurov
shahmurov@hotmail.com
University of Alabama Tuscaloosa USA, AL 35487

Abstract

In this paper, the integral initial value problems for Boussinesq type equations are studied. The equation include the general differential operators. The existence, uniqueness and regularity properties of solution of these problems are obtained. By choosing differential operators including in the equation, the regularity properties of the Cauchy problem for different type of Boussinesq equations are studied.

Key Word: Boussinesq equations, Hyperbolic equations, differential operators, Fourier multipliers

AMS: 35Lxx, 35Qxx, 47D

1. Introduction

The aim in this paper is to study the existence and uniqueness of solution of the integral initial value problem (IVB) for the generalized Boussinesq equation

\[ u_{tt} + L_0 u_{tt} + L_1 u = L_2 f(u), \quad x \in \mathbb{R}^n, \quad t \in (0, T), \]

\[ u(x, 0) = \varphi(x) + \int_0^T \alpha(\sigma) u(x, \sigma) \, d\sigma, \]

\[ u_t(x, 0) = \psi(x) + \int_0^T \beta(\sigma) u_t(x, \sigma) \, d\sigma, \]

where \( L_i \) are differential operators with constant coefficients, \( f(u) \) is the given nonlinear function, \( \varphi(x) \) and \( \psi(x) \) are the given initial value functions, \( \alpha \) and \( \beta \) are measurable functions on \((0, T)\).

Remark 1.1. Note that particularly, the condition (1.2) can be expressed as the following multipoint initial condition

\[ u(x, 0) = \varphi(x) + \sum_{k=1}^l \alpha_k u(x, \lambda_k), \quad u_t(0, x) = \psi(x) + \sum_{k=1}^l \beta_k u_t(x, \lambda_k). \]

By choosing the operators \( L_i \) we obtain numerous classes of generalized Boussinesq type equations which occur in a wide variety of physical systems, such as
in the propagation of longitudinal deformation waves in an elastic rod, hydro-
dynamical process in plasma, in materials science which describe spinodal de-
composition and in the absence of mechanical stresses (see [1 − 4]). For exam-
ple, if we choose \( L_0 = L_1 = L_2 = -\Delta \), where \( \Delta \) is \( n \)−
dimensioned Laplace, we obtain the Cauchy problem for the Boussinesq equation

\[
\begin{aligned}
&u_{tt} - \Delta u_{tt} - \Delta u = \Delta f(u), \quad x \in \mathbb{R}^n, \ t \in (0, T), \\
&u(x, 0) = \varphi(x), \ u_t(x, 0) = \psi(x).
\end{aligned}
\] (1.4)

The equation (1.4) arises in different situations (see [1, 2]). For example, for
\( n = 1 \) it describes a limit of a one-dimensional nonlinear lattice [3], shallow-
water waves [4, 5] and the propagation of longitudinal deformation waves in an
elastic rod [6]. Rosenau [7] derived the equations governing dynamics of one,
two and three-dimensional lattices. One of those equations is (1.4). Note that,
the existence of solutions and regularity properties for different type Boussinesq
equations are considered e.g. in [8-15]. In [8] and [9] the existence of the global
classical solutions and the blow-up of the solutions of the initial boundary value
problem (1.4) are studied. In this paper, we obtain the existence and uniqueness
of solution and regularity properties of the problem (1.1) – (1.2). The strategy
is to express the Boussinesq equation as an integral equation. To treat the
nonlinearity as a small perturbation of the linear part of the equation, the
contraction mapping theorem is used. Also, a priori estimates on \( L^p \) norms of
solutions of the linearized version are utilized. The key step is the derivation of
the uniform estimate of the solutions of the linearized Boussinesq equation.
The methods of harmonic analysis, operator theory, interpolation of Banach Spaces
and embedding theorems in Sobolev spaces are the main tools implemented to
carry out the analysis.

In order to state our results precisely, we introduce some notations and some
function spaces.

**Definitions and Background**

Let \( E \) be a Banach space. \( L^p(\Omega; E) \) denotes the space of strongly measurable
\( E \)-valued functions that are defined on the measurable subset \( \Omega \subset \mathbb{R}^n \) with the
norm

\[
\|f\|_{L^p} = \|f\|_{L^p(\Omega; E)} = \left( \frac{1}{\Omega} \int_{\Omega} \|f(x)\|^p_E \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
\]

\[
\|f\|_{L^\infty(\Omega)} = \text{ess sup}_{x \in \Omega} \|f(x)\|_E.
\]

Let \( \mathbb{C} \) denote the set of complex numbers. For \( E = \mathbb{C} \) the \( L^p(\Omega; E) \) denotes
by \( L^p(\Omega) \).

Let \( E_1 \) and \( E_2 \) be two Banach spaces. \( (E_1, E_2)\theta,p \) for \( \theta \in (0, 1), \ p \in [1, \infty] \)
denotes the interpolation spaces defined by \( K \)-method [17, §1.3.2].
Let $m$ be a positive integer. $W^{m,p}(\Omega)$ denotes the Sobolev space, i.e. space of all functions $u \in L^p(\Omega)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k}$ in $L^p(\Omega)$, $1 \leq p \leq \infty$ with the norm

$$
\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{k=1}^{n} \left\| \frac{\partial^m u}{\partial x_k} \right\|_{L^p(\Omega)} < \infty.
$$

Let $L^{s,p}(\mathbb{R}^n)$, $-\infty < s < \infty$ denotes Liouville-Sobolev space of order $s$ which is defined as:

$$
L^{s,p} = L^{s,p}(\mathbb{R}^n) = (I - \Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^n)
$$

with the norm

$$
\|u\|_{L^{s,p}} = \left\| (I - \Delta)^{-\frac{s}{2}} u \right\|_{L^p(\mathbb{R}^n)} < \infty.
$$

It clear that $L^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. It is known that $L^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$ for the positive integer $m$ (see e.g. [18, § 15]). Let $S(\mathbb{R}^n)$ denote Schwartz class, i.e., the space of rapidly decreasing smooth functions on $\mathbb{R}^n$, equipped with its usual topology generated by seminorms. Let $S'(\mathbb{R}^n)$ denote the space of all continuous linear operators $L : S(\mathbb{R}^n) \to \mathbb{C}$, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n)$ is norm dense in $L^p(\mathbb{R}^n)$ when $1 \leq p < \infty$.

Let $1 \leq p \leq q < \infty$. A function $\Psi \in L_\infty(\mathbb{R}^n)$ is called a Fourier multiplier from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if the map $B : u \to F^{-1} \Psi(\xi) Fu$ for $u \in S(\mathbb{R}^n)$ is well defined and extends to a bounded linear operator

$$
B : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n).
$$

Let $L^*_q(E)$ denote the space of all $E$--valued function space such that

$$
\|u\|_{L^*_q(E)} = \left( \int_0^\infty \|u(t)\|^q_E \frac{dt}{t} \right)^{\frac{1}{q}} < \infty, \quad 1 \leq q < \infty, \quad \|u\|_{L^*_q(E)} = \sup_{t \in (0,\infty)} \|u(t)\|_E.
$$

Here, $F$ denote the Fourier transform. Fourier-analytic representation of Besov spaces on $\mathbb{R}^n$ is defined as:

$$
B^s_{p,q}(\mathbb{R}^n) = \left\{ u \in S'(\mathbb{R}^n), \right. \left. \|u\|_{B^s_{p,q}(\mathbb{R}^n)} = \left\| F^{-1} t^{\kappa-s} \left( 1 + |\xi|^\kappa \right) e^{-t|\xi|^2} Fu \right\|_{L_q^*(L^p(\mathbb{R}^n))}, \right. \left. |\xi|^2 = \sum_{k=1}^n \xi_k^2, \xi = (\xi_1, \xi_2, ..., \xi_n), p \in (1, \infty), q \in [1, \infty], \kappa > s \right\}.
$$

It should be note that, the norm of Besov space does not depends on $\kappa$ (see e.g. [17, § 2.3]). For $p = q$ the space $B^s_{p,q}(\mathbb{R}^n)$ will be denoted by $B^s_p(\mathbb{R}^n)$.
Note that integral conditions for hyperbolic equations were studied e.g. in [21, 22]. In a similar way as [21] we obtain

**Lemma 1.1.** Let \(C(\xi, t)\) be continuous uniformly bounded in \(\xi \in \mathbb{R}^n\) such that \(|C(t)| \leq 1\) and

\[
\left| 1 + \int_0^T \alpha (\sigma) \beta (\sigma) d\sigma \right| > \int_0^T (|\alpha (\sigma)| + |\beta (\sigma)|) d\sigma.
\]

(1.1)

Then the function \(O = O(\xi)\) defined by

\[
O(\xi) = 1 + \int_0^T \int_0^T C(\xi, \sigma - \tau) \alpha (\sigma) \beta (\tau) d\sigma d\tau - \int_0^T [\alpha (s) + \beta (s)] C(\xi, s) ds
\]

has an uniformly bounded inverse.

Sometimes we use one and the same symbol \(C\) without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say \(\alpha\), we write \(C_\alpha\).

The paper is organized as follows: In Section 1, some definitions and background are given. In Section 2, we obtain the existence of unique solution and a prioriy estimates for solution of the linearized problem (1.1) – (1.2). In Section 3, we show the existence and uniqueness of local strong solution of the problem (1.1) – (1.2). In the Section 4 we show same applications of the problem (1.1) – (1.2).

Sometimes we use one and the same symbol \(C\) without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say \(h\), we write \(C_h\).

### 2. Estimates for linearized equation

In this section, we make the necessary estimates for solutions of integral IVB for the linearized Boussinesq equation

\[
u_{tt} + L_0 u_{tt} + L_1 u = L_2 g(x, t), \quad x \in \mathbb{R}^n, \quad t \in (0, T),
\]

(2.1)

\[
u(x, 0) = \varphi (x) + \int_0^T \alpha (\sigma) u(x, \sigma) d\sigma,
\]

(2.2)

\[
u_t (x, 0) = \psi (x) + \int_0^T \beta (\sigma) u_t (x, \sigma) d\sigma,
\]

where

\[
L_i u = \sum_{|\alpha| \leq m_i} a_{\alpha} D^\alpha u, \quad a_{\alpha} \in \mathbb{C}, \quad i = 0, 1, 2,
\]
= (α_1, α_2, ..., α_n), α_k are natural numbers, |α| = \sum_{k=1}^{n} \alpha_k and m_i are positive integers. Let
\[
L_i (\xi) = \sum_{|\alpha| \leq m_i} a_{\alpha} \xi_1^{\alpha_1} \xi_2^{\alpha_2} ... \xi_n^{\alpha_n}, \quad i = 0, 1, 2.
\]

Here,
\[
X_p = L^p (\mathbb{R}^n), \quad 1 \leq p \leq \infty, \quad Y^{s,p} = L^{s,p} (\mathbb{R}^n), \quad Y_i^{s,p} = L^{s,p} (\mathbb{R}^n) \cap L^1 (\mathbb{R}^n), \quad q^{s,p} = L^{s,p} (\mathbb{R}^n) \cap L^{\infty} (\mathbb{R}^n),
\]
\[
Q = Q (\xi) = L_1 (\xi) [1 + L_0 (\xi)]^{-1}, \quad L (\xi) = L_2 (\xi) [1 + L_0 (\xi)]^{-1}. \quad (2.3)
\]

**Condition 2.1.** Let (1.1) holds and \( s > \frac{n}{p} \) for \( 1 < p < \infty \). Assume that \( L_1 (\xi) \neq 0, L_0 (\xi) \neq -1 \) and there exist a positive constants \( M_1 \) and \( M_2 \) depend only on \( a_\alpha \) such that
\[
\left| Q^{1/2} (\xi) \right| \leq M_1 (1 + |\xi|)^{s - \frac{n}{p}}, \quad \left| L (\xi) Q^{-1/2} (\xi) \right| \leq M_2 (1 + |\xi|)^{s - \frac{n}{p}}
\]
for all \( \xi \in \mathbb{R}^n \).

**Remark 2.1.** The Condition 2.1 means that there exists positive constants \( M_1 \) and \( M_2 \) depend only on \( a_\alpha \) such that
\[
\left| Q^{-1/2} (\xi) \right| = \left| \left[ |L_1^{-1} (\xi) + L_0 (\xi) L_1^{-1} (\xi)\right| \right|^{1/2} \leq M_1 (1 + |\xi|)^{(s - \frac{n}{p})},
\]
\[
\left| L_2 (\xi) L_1^{-1} (\xi) \right| |1 + L_0 (\xi)|^{-1/2} \leq M_2 (1 + |\xi|)^{(s - \frac{n}{p})}
\]
for all \( \xi \in \mathbb{R}^n \). By Condition 2.1, \( L_1^{-1} (\xi) \) and \( [1 + L_0 (\xi)]^{-1/2} \) are uniformly bounded. Therefore, the inequalities (2.3) are satisfied if
\[
m_0 - m_1 \leq 2 \left( s - \frac{n}{p} \right), \quad m_2 - m_1 \leq 2 \left( s - \frac{n}{p} \right)
\]
respectively, i.e.(2.3) is hold trivially if \( m_0 = m_1 = m_2 \).

First we need the following lemmas

**Lemma 2.1.** Let the Conditions 2.1 be satisfied.Then problem (2.1) – (2.2) has a generalized solution.

**Proof.** By using of Fourier transform we get from (2.1) – (2.2):
\[
\dot{u}_{tt} (\xi, t) + Q (\xi) \dot{u} (\xi, t) = L (\xi) \dot{g} (\xi, t),
\]
\[
\dot{u} (\xi, 0) = \dot{\phi} (\xi) + \int_0^\tau \alpha (\sigma) \dot{u} (\xi, \sigma) d\sigma,
\]
\[
\ddot{u} (\xi, t) + Q (\xi) \ddot{u} (\xi, t) = \ddot{g} (\xi, t),
\]
\[
\dddot{u} (\xi, 0) = \dddot{\phi} (\xi) + \int_0^\tau \alpha (\sigma) \dddot{u} (\xi, \sigma) d\sigma,
\]
\[
\dddot{u} (\xi, t) + Q (\xi) \dddot{u} (\xi, t) = \dddot{g} (\xi, t),
\]
\[
\dddot{u} (\xi, 0) = \dddot{\phi} (\xi) + \int_0^\tau \alpha (\sigma) \dddot{u} (\xi, \sigma) d\sigma,
\]
\[
\dddot{u} (\xi, t) + Q (\xi) \dddot{u} (\xi, t) = \dddot{g} (\xi, t),
\]
\[
\dddot{u} (\xi, 0) = \dddot{\phi} (\xi) + \int_0^\tau \alpha (\sigma) \dddot{u} (\xi, \sigma) d\sigma,
\]
\[
\dddot{u} (\xi, t) + Q (\xi) \dddot{u} (\xi, t) = \dddot{g} (\xi, t),
\]
\[
\dddot{u} (\xi, 0) = \dddot{\phi} (\xi) + \int_0^\tau \alpha (\sigma) \dddot{u} (\xi, \sigma) d\sigma,
\]
\[
\dddot{u} (\xi, t) + Q (\xi) \dddot{u} (\xi, t) = \dddot{g} (\xi, t),
\]
\[
\dddot{u} (\xi, 0) = \dddot{\phi} (\xi) + \int_0^\tau \alpha (\sigma) \dddot{u} (\xi, \sigma) d\sigma,
\]
\[
\dddot{u} (\xi, t) + Q (\xi) \dddot{u} (\xi, t) = \dddot{g} (\xi, t),
\]
\[
\dddot{u} (\xi, 0) = \dddot{\phi} (\xi) + \int_0^\tau \alpha (\sigma) \dddot{u} (\xi, \sigma) d\sigma,
\[ \hat{u}_t (\xi, 0) = \hat{\psi} (\xi) + \int_0^T \beta (\sigma) \hat{u}_t (\xi, \sigma) d\sigma, \]

where \( \hat{u} (\xi, t) \) is a Fourier transform of \( u (x, t) \) with respect to \( x \) and \( \hat{\varphi} (\xi), \hat{\psi} (\xi) \) are Fourier transform of \( \varphi, \psi \), respectively.

Consider the problem
\[ \hat{u}_{tt} (\xi, t) + Q (\xi) \hat{u} (\xi, t) = L (\xi) \hat{g} (\xi, t), \quad (2.6) \]
\[ \hat{u} (\xi, 0) = u_0 (\xi), \quad \hat{u}_t (\xi, 0) = u_1 (\xi), \quad \xi \in \mathbb{R}^n, \quad t \in [0, T]. \]

By using the variation of constants it is not hard to see that problem (2.6) has a unique solution for \( \xi \in \mathbb{R}^n \) and the solution can be expressed as
\[ \hat{u} (\xi, t) = C (\xi, t) u_0 (\xi) + S (\xi, t) u_1 (\xi) + \int_0^t S (\xi, t - \tau) \hat{\Phi} (\xi, \tau) d\tau, \quad t \in (0, T), \quad (2.7) \]

where,
\[ C (t) = C (\xi, t) = \cos \left( Q^{\frac{1}{2}} t \right), \quad S (t) = S (\xi, t) = Q^{-\frac{1}{2}} \sin \left( Q^{\frac{1}{2}} t \right), \]
\[ \hat{\Phi} (\xi, t) = L (\xi) Q^{-\frac{1}{2}} (\xi) \sin \left( Q^{\frac{1}{2}} t \right) \hat{g} (\xi, t). \quad (2.8) \]

By using (2.7) and the condition
\[ u_0 (\xi) = \hat{\varphi} (\xi) + \int_0^T \alpha (\sigma) \hat{u} (\xi, \sigma) d\sigma, \]

we get
\[ u_0 (\xi) = \hat{\varphi} (\xi) + \int_0^T \alpha (\sigma) [C (\xi, \sigma) u_0 (\xi) + S (\xi, \sigma) u_1 (\xi)] d\sigma + \]
\[ \int_0^T \int_0^\sigma S (\xi, \sigma - \tau) \hat{\Phi} (\xi, \tau) d\tau d\sigma, \quad t \in (0, T). \]

Then,
\[ \left[ I - \int_0^T \alpha (\sigma) C (\xi, \sigma) d\sigma \right] u_0 (\xi) - \left[ \int_0^T \alpha (\sigma) S (\xi, \sigma) d\sigma \right] u_1 (\xi) = \]
\[ \int_0^T \int_0^\sigma \alpha (\sigma) S (\xi, \sigma - \tau) \hat{\Phi} (\xi, \tau) d\tau d\sigma + \hat{\varphi} (\xi). \quad (2.9) \]
Differentiating both sides of formula (2.7) and in view of (2.8) we obtain
\[ \hat{u}_t (\xi, t) = -Q (\xi) S (\xi, t) u_0 (\xi) + C (\xi, t) u_1 (\xi) + \int_0^t C (\xi, t - \tau) \hat{\Phi} (\xi, \tau) d\tau, \quad t \in (0, \infty). \] (2.10)

Using (2.10) and the integral condition
\[ u_1 (\xi) = \hat{\psi} (\xi) + \int_0^T \beta (\sigma) \hat{u}_t (\xi, \sigma) d\sigma \]
we obtain
\[ u_1 (\xi) = \hat{\psi} (\xi) + \int_0^T \beta (\sigma) \left[ -Q (\xi) S (\xi, \sigma) u_0 (\xi) + C (\xi, \sigma) u_1 (\xi) \right] d\sigma + \int \int_0^T C (\xi, \sigma - \tau) \hat{\Phi} (\xi, \tau) d\tau d\sigma. \]

Thus,
\[ \int_0^T \beta (\sigma) Q (\xi) S (\xi, \sigma) d\sigma u_0 (\xi) + \left. \int_0^T \beta (\sigma) C (\xi, \sigma) d\sigma \right. \] u_1 (\xi) =
\[ \int \int_0^T \beta (\sigma) C (\xi, \sigma - \tau) \hat{\Phi} (\xi, \tau) d\tau d\sigma + \hat{\psi} (\xi). \] (2.11)

Now, we consider the system of equations (2.9), (2.11) in \( u_0 (\xi) \) and \( u_1 (\xi) \). The determinant of this system is
\[ D (\xi) = \begin{vmatrix} \alpha_{11} (\xi) & \alpha_{12} (\xi) \\ \alpha_{21} (\xi) & \alpha_{22} (\xi) \end{vmatrix}, \]
where
\[ \alpha_{11} (\xi) = I - \int_0^T \alpha (\sigma) C (\xi, \sigma) d\sigma, \quad \alpha_{12} (\xi) = - \int_0^T \alpha (\sigma) S (\xi, \sigma) d\sigma, \]
\[ \alpha_{21} (\xi) = \int_0^T \beta (\sigma) Q (\xi) S (\xi, \sigma) d\sigma, \quad \alpha_{22} (\xi) = I - \int_0^T \beta (\sigma) S (\xi, \sigma) d\sigma. \]
Then by using the properties
\[ [C(\sigma)C(\tau) + Q(\xi)S(\sigma)S(\tau)] = C(\sigma - \tau) \]
we obtain
\[ D(\xi) = I - \int_0^T [\alpha(\sigma) + \beta(\sigma)]C(\sigma)\,d\sigma + \]
\[ \int_0^T \int_0^T \alpha(\sigma)\beta(\tau)[C(\xi,\sigma)C(\xi,\tau) + Q(\xi)S(\xi,\sigma)S(\xi,\tau)]\,d\sigma\,d\tau = \]
\[ I - \int_0^T [\alpha(\sigma) + \beta(\sigma)]C(\xi,\sigma)\,d\sigma + \int_0^T C(\xi,\sigma - \tau)\alpha(\sigma)\beta(\tau)\,d\sigma\,d\tau = O(\xi). \]

By Lemma 1.1, \( D^{-1}(\xi) = O^{-1} \) is uniformly bounded. Solving the system (2.10) – (2.11), we get
\[ u_0(\xi) = D^{-1}(\xi) \left\{ \left[ I - \int_0^T \beta(\sigma)C(\xi,\sigma)\,d\sigma \right] f_1 + \int_0^T \alpha(\sigma)S(\xi,\sigma)\,d\sigma f_2 \right\}, \]
\[ u_1(\xi) = D^{-1}(\xi) \left\{ \left[ I - \int_0^T \alpha(\sigma)C(\xi,\sigma)\,d\sigma \right] f_2 - \int_0^T \beta(\sigma)Q(\xi)S(\xi,\sigma)\,d\sigma f_1 \right\}, \]
where
\[ f_1 = \int_0^T \int_0^\sigma \alpha(\sigma)S(\xi,\sigma - \tau)\Phi(\xi,\tau)\,d\tau d\sigma + \Phi(\xi), \]
\[ f_2 = \int_0^T \int_0^\sigma \beta(\sigma)C(\xi,\sigma - \tau)\Phi(\tau,\xi)\,d\tau d\sigma + \Psi(\xi). \]

From (2.7), (2.12) and (2.13) we get that the solution of (2.4) – (2.5) can be expressed as
\[ \hat{u}(\xi, t) = D^{-1}(\xi) \left\{ C(\xi, t) \left[ I - \int_0^T \beta(\sigma)C(\xi,\sigma)\,d\sigma \right] f_1 + \right. \]
\[ \int_0^T \alpha(\sigma)S(\xi,\sigma)\,d\sigma f_2 \right\} + S(t, \xi) \left[ \left( I - \int_0^T \alpha(\sigma)C(\xi,\sigma)\,d\sigma \right) f_2 - \right. \]
\[ \left. \int_0^T \alpha(\sigma)S(\xi,\sigma)\,d\sigma f_1 \right\].
\[
\int_0^T \beta(\sigma) Q(\xi) S(\xi, \sigma) \, d\sigma \mathcal{F}_1 \} + \int_0^t S(t - \tau, \xi) \Phi(\tau, \xi) \, d\tau, \; t \in (0, T). \tag{2.14}
\]

From (2.14) we get that there is a generalized solution of (2.1) – (2.2) given by

\[
u(x, t) = S_1(t) \varphi(x) + S_2(t) \psi(x) + \Phi(x, t), \tag{2.15}
\]

where \(S_1(t)\) and \(S_2(t)\) are defined by

\[
S_1(t) \varphi = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^n} \left\{ e^{i\xi \cdot \xi} D^{-1}(\xi) C(t, \xi) \left( I - \int_0^T \beta(\sigma) C(\xi, \sigma) \right) \right\} \, d\xi,
\]

\[
S_2(t) \psi = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^n} \left\{ e^{i\xi \cdot \xi} D^{-1}(\xi) C(t, \xi) \right\} \, d\xi,
\]

\[
\Phi(x, t) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^n} \left\{ e^{i\xi \cdot \xi} \left\{ \int_0^T S(\xi, t - \tau) \Phi(\xi, \tau) \, d\tau + \int_0^T \alpha(\sigma) S(\xi, \sigma) \, d\sigma \right\} \right\} \, d\xi.
\]

here

\[
g_1(\xi) = \int_0^T \alpha(\sigma) S(\xi, \sigma - \tau) \hat{g}(\xi, \tau) \, d\tau d\sigma, \tag{2.14}
\]

\[
g_2(\xi) = \int_0^T \beta(\sigma) C(\xi, \sigma - \tau) \hat{g}(\xi, \tau) \, d\tau d\sigma.
\]
Theorem 2.1. Let the Condition 2.1 be hold. Then for $\varphi, \psi, g(x,t) \in Y^s_{1,p}$ the solution (2.1) – (2.2) satisfies the following estimate
\[
\|u\|_{X_\infty} + \|u_t\|_{X_\infty} \leq C \left( \|\varphi\|_{Y^s_{r,p}} + \|\varphi\|_{X_1} + \right. \\
\left. \|\psi\|_{Y^s_{r,p}} + \|\psi\|_{X_1} + \int_0^t (\|g(.,\tau)\|_{Y^s_{r,p}} + \|g(.,\tau)\|_{X_1}) \, d\tau \right)
\] (2.15)
uniformly with respect to $t \in [0,T]$.

Proof. Let $N \in \mathbb{N}$ and
\[
\Pi_N = \{ \xi : \xi \in \mathbb{R}^n, \ |\xi| \leq N \}, \ \Pi'_N = \{ \xi : \xi \in \mathbb{R}^n, \ |\xi| \geq N \}.
\]
It is clear to see that
\[
\left\| F^{-1} C(\xi, t) \hat{\varphi}(\xi) \right\|_{X_\infty} + \left\| F^{-1} S(\xi) \hat{\psi}(\xi, t) \right\|_{X_\infty} \leq \\
\left\| \int_{\mathbb{R}^n} e^{ix\xi} C(\xi, t) \varphi(x) \, dx \right\|_{L^\infty(\Pi_N)} + \left\| \int_{\mathbb{R}^n} e^{ix\xi} S(\xi, t) \psi(x) \, dx \right\|_{L^\infty(\Pi_N)} + (2.16)
\]
\[
\left\| F^{-1} C(\xi, t) \hat{\varphi}(\xi) \right\|_{L^\infty(\Pi'_N)} + \left\| F^{-1} S(\xi, t) \hat{\psi}(\xi) \right\|_{L^\infty(\Pi'_N)}.
\]
By using the Minkowski’s inequality for integrals and in view of the uniformly boundedness of $C(\xi, t), S(\xi, t)$ on $\Pi_N$ we have
\[
\left\| \int_{\mathbb{R}^n} e^{ix\xi} C(\xi, t) \varphi(x) \, dx \right\|_{L^\infty(\Pi_N)} + \left\| \int_{\mathbb{R}^n} e^{ix\xi} S(\xi, t) \psi(x) \, dx \right\|_{L^\infty(\Pi_N)} \leq (2.17)
\]
\[
C \left[ \|\varphi\|_{X_1} + \|\psi\|_{X_1} \right].
\]
Hence,
\[
\| F^{-1} C(\xi, t) \hat{\varphi}(\xi) \|_{L^\infty(\Pi'_N)} + \| F^{-1} S(\xi, t) \hat{\psi}(\xi) \|_{L^\infty(\Pi'_N)} = \\
\| F^{-1} \left( 1 + |\xi|^2 \right)^{-\frac{\alpha}{2}} C(\xi, t) \left( 1 + |\xi|^2 \right)^{\frac{\alpha}{2}} \hat{\varphi}(\xi) \|_{L^\infty(\Pi'_N)} + (2.18)
\]
\[
\| F^{-1} \left( 1 + |\xi|^2 \right)^{-\frac{\alpha}{2}} S(\xi, t) \left( 1 + |\xi|^2 \right)^{\frac{\alpha}{2}} \hat{\psi}(\xi) \|_{L^\infty(\Pi'_N)}.
\]
By using (2.3) and the first estimate in Condition 2.1 we get
\[
\sup_{\xi \in \mathbb{R}^n, t \in [0,T]} |\xi|^{|\alpha| + \frac{\alpha}{2}} D^\alpha \left[ \left( 1 + |\xi|^2 \right)^{-\frac{\alpha}{2}} C(\xi, t) \right] \leq C_2,
\]
10
for \( s \geq \frac{n}{p} \), \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \), \( \alpha_k \in \{0, 1\} \), \( \xi \in \mathbb{R}^n \) and \( \xi \neq 0 \) uniformly in \( t \in [0, T] \). By multiplier theorems (see e.g. [16]) from (2.19) we get that the functions \( \left(1 + |\xi|^2\right)^{-\frac{n}{2}} C(\xi, t), \left(1 + |\xi|^2\right)^{-\frac{n}{2}} S(\xi, t) \) are \( L^p(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \) Fourier multipliers. Then by Minkowski's inequality for integrals, from (2.17) – (2.18) we obtain

\[
\|F^{-1} C(\xi, t) \hat{\varphi}(\xi)\|_{L^\infty(W^s_{p, \alpha})} + \|F^{-1} S(\xi, t) \hat{\psi}(\xi)\|_{L^\infty(W^s_{p, \alpha})} \leq \tag{2.20}
\]

\[
C(\|\varphi\|_{\dot{Y}^{s, p}} + \|\psi\|_{\dot{Y}^{s, p}}).
\]

By using the representation of \( \hat{\Phi}(\xi, t) \) in (2.8) and the second inequality in Condition 2.1 we get the uniform estimate

\[
\sup_{\xi \in \mathbb{R}^n, t \in [0, T]} |\xi| \left| |\alpha|^\frac{p}{2} D^\alpha \left[ \left(1 + |\xi|^2\right)^{-\frac{n}{2}} \right] C(\xi, t) \right| \leq C_3. \tag{2.21}
\]

By reasoning as the above we have

\[
\left\| F^{-1} \int_0^t \hat{\Phi}(t - \tau, \xi) \hat{\gamma}(\xi, \tau) \, d\tau \right\|_{X^\infty} \leq C \int_0^t (\|g(\cdot, \tau)\|_{\dot{Y}^s} + \|g(\cdot, \tau)\|_{X^1}) \, d\tau.
\]

Hence, we obtain the estimate

\[
\|u\|_{X^\infty} \leq C(\|\varphi\|_{\dot{Y}^{s, p}} + \|\varphi\|_{X^1} + \|\psi\|_{\dot{Y}^{s, p}} + \|\psi\|_{X^1} + \int_0^t (\|g(\cdot, \tau)\|_{\dot{Y}^{s, p}} + \|g(\cdot, \tau)\|_{X^1}) \, d\tau).
\]

By using (2.3) and Condition 2.1 in view of (2.20) in similar way, we deduced the estimate of type (2.22) for \( u_t \), i.e. we obtain the assertion.

**Theorem 2.2.** Let the Conditions 2.1 be hold. Then for \( \varphi, \psi, g(x, t) \in \dot{Y}^{s, p} \) the solution of the problem (2.1) – (2.2) satisfies the following uniform estimate

\[
(\|u\|_{\dot{Y}^{s, p}} + \|u_t\|_{\dot{Y}^{s, p}}) \leq C \left( \|\varphi\|_{\dot{Y}^{s, p}} + \|\psi\|_{\dot{Y}^{s, p}} + \int_0^t \|g(\cdot, \tau)\|_{\dot{Y}^{s, p}} \, d\tau \right). \tag{2.23}
\]

**Proof.** From (2.7) we have the following uniform estimate

\[
\left( \left\| F^{-1} \left(1 + |\xi|^2\right)^{\frac{n}{2}} \hat{u} \right\|_{X^p} + \left\| F^{-1} \left(1 + |\xi|^2\right)^{\frac{n}{2}} \hat{u}_t \right\|_{X^p} \right) \leq \tag{2.24}
\]
\[
\begin{align*}
C \left\{ \left\| F^{-1} (1 + |\xi|)^{\frac{2}{m}} C (\xi, t) \hat{\varphi} \right\|_{X_p} + \left\| F^{-1} (1 + |\xi|)^{\frac{2}{m}} S (\xi, t) \hat{\psi} \right\|_{X_p} + \\
\int_0^t \left\| (1 + |\xi|)^{\frac{2}{m}} \Phi (\xi, t - \tau) \hat{g} (\cdot, \tau) \right\|_{X_p} d\tau \right\}.
\end{align*}
\]

By Condition 2.1 and by virtue of Fourier multiplier theorems (see [17, § 2.2]) we get that \( C (\xi, t) \), \( S (\xi, t) \) and \( \Phi (\xi, t) \) are Fourier multipliers in \( L^p (\mathbb{R}^n) \) uniformly with respect to \( t \in [0, T] \). So, the estimate (2.24) by using the Minkowski’s inequality for integrals implies (2.23).

3. Initial value problem for nonlinear equation

In this section, we will show the local existence and uniqueness of solution for the Cauchy problem (1.1) – (1.2). For the study of the nonlinear problem (1.1) – (1.2) we need the following lemmas

Lemma 3.1 (Nirenberg’s inequality) [19]. Assume that \( u \in L^p (\Omega) \), \( D^m u \in L^q (\Omega) \), \( p, q \in (1, \infty) \). Then for \( i \) with \( 0 \leq i \leq m \), \( m > \frac{n}{q} \) we have

\[
\left\| D^i u \right\|_p \leq C \left\| u \right\|_p^{1-\mu} \sum_{k=1}^n \left\| D^m_{k} u \right\|_q^\mu,
\]

where

\[
\frac{1}{r} = \frac{i}{m} + \mu \left( \frac{1}{q} - \frac{m}{n} \right) + (1 - \mu) \frac{1}{p} \quad \frac{i}{m} \leq \mu \leq 1.
\]

Lemma 3.2 [20]. Assume that \( u \in W^{m,p} (\Omega) \cap L^\infty (\Omega) \) and \( f (u) \) possesses continuous derivatives up to order \( m \geq 1 \). Then \( f (u) - f (0) \in W^{m,p} (\Omega) \) and

\[
\left\| f (u) - f (0) \right\|_p \leq \left\| f^{(1)} (u) \right\|_\infty \left\| u \right\|_p,
\]

\[
\left\| D^k f (u) \right\|_p \leq C_0 \sum_{j=1}^k \left\| f^{(j)} (u) \right\|_\infty \left\| u \right\|_\infty^{j-1} \left\| D^k u \right\|_p, \quad 1 \leq k \leq m,
\]

where \( C_0 \geq 1 \) is a constant.

Let

\[
X_p = L^p (\mathbb{R}^n), \quad Y = W^{2,p} (\mathbb{R}^n), \quad E_0 = (X_p, Y)_{\frac{1}{p}}^{\frac{1}{p}} = B_{p}^{2-\frac{1}{p}} (\mathbb{R}^n).
\]

Remark 3.1. By using J.Lions-I. Petree result (see e.g [17, § 1.8.]) we obtain that the map \( u \to u (t_0), t_0 \in [0, T] \) is continuous and surjective from \( W^{2,p} (0, T) \) onto \( E_0 \) and there is a constant \( C_1 \) such that

\[
\left\| u (t_0) \right\|_{E_0} \leq C_1 \left\| u \right\|_{W^{2,p} (0,T)}, \quad 1 \leq p \leq \infty.
\]
First of all, we define the space $Y ( T ) = C \left( [0, T ]; Y _{\infty}^{2,p} \right)$ equipped with the norm defined by

$$
\|u\|_{Y ( T )} = \max_{t \in [0, T]} \|u\|_{Y_{2,p}} + \max_{t \in [0, T]} \|u\|_{X_{\infty}}, \quad u \in Y ( T ).
$$

It is easy to see that $Y ( T )$ is a Banach space. For $\varphi, \psi \in Y_{2,p}$, let

$$
M = \|\varphi\|_{Y_{2,p}} + \|\varphi\|_{X_{\infty}} + \|\psi\|_{Y_{2,p}} + \|\psi\|_{X_{\infty}}.
$$

**Definition 3.1.** For any $T > 0$ if $\varphi, \psi \in Y_{\infty}^{2,p}$ and $u \in C \left( [0, T ]; Y_{\infty}^{2,p} \right)$ satisfies the equation (1.1) – (1.2) then $u(x, t)$ is called the continuous solution or the strong solution of the problem (1.1) – (1.2). If $T < \infty$, then $u(x, t)$ is called the local strong solution of the problem (1.1) – (1.2). If $T = \infty$, then $u(x, t)$ is called the global strong solution of the problem (1.1) – (1.2).

**Condition 3.1.** Assume:

1. The Condition 2.1 holds, $\varphi, \psi \in Y_{\infty}^{2,p}$ for $1 < p < \infty$ and $\frac{2}{p} < 2$;
2. the function $u \rightarrow f(x, t, u)$: $R^n \times [0, T] \times E_0 \rightarrow E$ is a measurable in $(x, t) \in R^n \times [0, T]$ for $u \in E_0$; $f(x, t, u)$. Moreover, $f(x, t, u)$ is continuous in $u \in E_0$ and $f(x, t, u) \in C^3(E_0; E)$ uniformly with respect to $x \in R^n$, $t \in [0, T]$. Main aim of this section is to prove the following result:

**Theorem 3.1.** Let the Condition 3.1 hold. Then problem (1.1) – (1.2) has a unique local strange solution $u \in C^2 \left( [0, T_0]; Y_{\infty}^{2,p} \right)$, where $T_0$ is a maximal time interval that is appropriately small relative to $M$. Moreover, if

$$
\sup_{t \in \left[0, T_0 \right]} \left( \|u\|_{Y_{2,p}} + \|u\|_{X_{\infty}} + \|u_t\|_{Y_{2,p}} + \|u_{tt}\|_{X_{\infty}} \right) < \infty \quad \text{(3.3)}
$$

then $T_0 = \infty$.

**Proof.** First, we are going to prove the existence and the uniqueness of the local continuous solution of the problem (1.1) – (1.2) by contraction mapping principle. Consider a map $G$ on $Y ( T )$ such that $G(u)$ is the solution of the Cauchy problem

$$
G_{tt} (u) + L_0 G_{tt} (u) + L_1 G (u) = L_2 f (G(u)), \quad x \in R^n, \quad t \in (0, T), \quad \text{(3.4)}
$$

$$
G(u)(x, 0) = \varphi (x) + \int_0^T \alpha (\sigma) G(u)(x, \sigma) \, d\sigma,
$$

$$
G(u)_t(x, 0) = \psi (x) + \int_0^T \beta (\sigma) G(u)_t(x, \sigma) \, d\sigma.
$$

From Lemma 3.2 we know that $f(u) \in L^p \left( 0, T; Y_{\infty}^{2,p} \right)$ for any $T > 0$. Thus, by Theorem 2.1, problem (3.4) has a unique solution which can be written as

$$
G(u)(x, t) = S_1 (t) \varphi (x) + S_2 (t) \psi (x) +
$$
where $S_1 (t), S_2 (t)$ are linear operators in $L^p (R^n)$ defined by (2.15). From Lemma 3.2 it is easy to see that the map $G$ is well defined for $f \in C^2 (X_0 ; \mathbb{C})$. We put

$$Q (M; T) = \{ u \mid u \in Y (T), \|u\|_{Y(T)} \leq M + 1 \} .$$

First, by reasoning as in [9] let us prove that the map $G$ has a unique fixed point in $Q (M; T)$. For this aim, it is sufficient to show that the operator $G$ maps $Q (M; T)$ into $Q (M; T)$ and $G : Q (M; T) \to Q (M; T)$ is strictly contractive if $T$ is appropriately small relative to $M$. Consider the function $\tilde{f} (\xi): [0, \infty) \to [0, \infty)$ defined by

$$\tilde{f} (\xi) = \max_{|x| \leq \xi} \left\{ \left\| f^{(1)} (x) \right\|_C, \left\| f^{(2)} (x) \right\|_C \right\}, \quad \xi \geq 0.$$

It is clear to see that the function $\tilde{f} (\xi)$ is continuous and nondecreasing on $[0, \infty)$. From Lemma 3.2 we have

$$\| f (u) \|_{Y, p} \leq \left\| f^{(1)} (u) \right\|_{X, \infty} \|u\|_{X, p} + \left\| f^{(1)} (u) \right\|_{X, \infty} \|Du\|_{X, p} + C_0 \left[ \left\| f^{(1)} (u) \right\|_{X, \infty} \|u\|_{X, p} + \left\| f^{(2)} (u) \right\|_{X, \infty} \|u\|_{X, \infty} \|D^2 u\|_{X, p} \right] \leq 2C_0 \tilde{f} (M + 1) (M + 1) \|u\|_{Y, p} .$$

By using the Theorem 2.1 we obtain from (3.5):

$$\|G(u)\|_{X, \infty} \leq \|\varphi\|_{X, \infty} + \|\psi\|_{X, \infty} + \int_0^t \|f(x, \tau, u(\tau))\|_{X, \infty} , \quad (3.7)$$

$$\|G(u)\|_{Y, p} \leq \|\varphi\|_{Y, p} + \|\psi\|_{Y, p} + \int_0^t \|f(x, \tau, u(\tau))\|_{Y, p} d\tau . \quad (3.8)$$

Thus, from (3.6) – (3.8) and Lemma 3.2 we get

$$\|G(u)\|_{Y(T)} \leq M + T \left( M + 1 \right) \left[ 1 + 2C_0 \left( M + 1 \right) \tilde{f} (M + 1) \right] .$$

If $T$ satisfies

$$T \leq \left\{ \left( M + 1 \right) \left[ 1 + 2C_0 \left( M + 1 \right) \tilde{f} (M + 1) \right] \right\}^{-1} , \quad (3.9)$$

then

$$\|Gu\|_{Y(T)} \leq M + 1 .$$
Therefore, if (3.9) holds, then $G$ maps $Q(M; T)$ into $Q(M; T)$. Now, we are going to prove that the map $G$ is strictly contractive. Assume $T > 0$ and $u_1$, $u_2 \in Q(M; T)$ given. We get

$$G(u_1) - G(u_2) = \int_0^t F^{-1}S(t - \tau, \xi)L(\xi) \left[ \hat{f}(u_1)(\xi, \tau) - \hat{f}(u_2)(\xi, \tau) \right] d\tau, \; t \in (0, T).$$

By using the assumption (2) and the mean value theorem, we obtain

$$\hat{f}(u_1) - \hat{f}(u_2) = \hat{f}^{(1)}(u_1 + \eta_1 (u_1 - u_2))(u_1 - u_2),$$

$$D_\xi \left[ \hat{f}(u_1) - \hat{f}(u_2) \right] = \hat{f}^{(2)}(u_2 + \eta_2 (u_1 - u_2))(u_1 - u_2) D_\xi u_1 + \hat{f}^{(1)}(u_1 - u_2) D_\xi u_2,$$

$$D_\xi^2 \left[ \hat{f}(u_1) - \hat{f}(u_2) \right] = \hat{f}^{(3)}(u_2 + \eta_3 (u_1 - u_2))(u_1 - u_2) (D_\xi u_1)^2 + \hat{f}^{(2)}(u_1 - u_2) (D_\xi u_1 - D_\xi u_2) (D_\xi u_1 + D_\xi u_2) + \hat{f}^{(1)}(u_2) (D_\xi^2 u_1 - D_\xi^2 u_2),$$

where $0 < \eta_i < 1$, $i = 1, 2, 3, 4$. Thus, using Hölder’s and Nirenberg’s inequality, we have

$$\left\| \hat{f}(u_1) - \hat{f}(u_2) \right\|_{X_p} \leq \hat{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p},$$

$$\left\| \hat{f}(u_1) - \hat{f}(u_2) \right\|_{X_p} \leq \hat{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p},$$

$$\left\| D_\xi \left[ \hat{f}(u_1) - \hat{f}(u_2) \right] \right\|_{X_p} \leq (M + 1) \hat{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} + \hat{f}(M + 1) \left\| D_\xi u_1 \right\|_{X_p}^2 + \hat{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} \left\| D_\xi^2 u_1 \right\|_{X_p} + \hat{f}(M + 1) \left\| D_\xi u_1 - u_2 \right\|_{X_p} \leq$$

$$C^2 \hat{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} \left\| u_1 \right\|_{X_p} \left\| D_\xi^2 u_1 \right\|_{X_p} +$$

$$C^2 \hat{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} \left\| D_\xi^2 (u_1 - u_2) \right\|_{X_p} \left\| u_1 + u_2 \right\|_{X_p} \left\| D_\xi^2 (u_1 + u_2) \right\|_{X_p} \left\| D_\xi^2 u_1 \right\|_{X_p} \leq$$

$$3C^2 (M + 1)^2 \hat{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} + 2C^2 (M + 1) \hat{f}(M + 1) \left\| D_\xi^2 (u_1 - u_2) \right\|_{X_p},$$

15
where $C$ is the constant in Lemma 3.1. From (3.10) – (3.11), using Minkowski’s inequality for integrals, Fourier multiplier theorems for operator-valued functions in $X_p$ spaces and Young’s inequality, we obtain

$$
\|G(u_1) - G(u_2)\|_{Y(T)} \leq \int_0^t \|u_1 - u_2\|_{X_\infty} \, d\tau + \int_0^t \|u_1 - u_2\|_{Y_{2,p}} \, d\tau + 
$$

$$
\int_0^t \|f(u_1) - f(u_2)\|_{X_\infty} \, d\tau + \int_0^t \|f(u_1) - f(u_2)\|_{Y_{2,p}} \, d\tau \leq 
$$

$$
T \left[ 1 + C_1 (M + 1)^2 \bar{f}(M + 1) \right] \|u_1 - u_2\|_{Y(T)},
$$

where $C_1$ is a constant. If $T$ satisfies (3.9) and the following inequality holds

$$
T \leq \frac{1}{2} \left[ 1 + C_1 (M + 1)^2 \bar{f}(M + 1) \right]^{-1}, \quad (3.14)
$$

then

$$
\|G u_1 - G u_2\|_{Y(T)} \leq \frac{1}{2} \|u_1 - u_2\|_{Y(T)}. 
$$

That is, $G$ is a constructive map. By contraction mapping principle we know that $G(u)$ has a fixed point $u(x, t) \in Q(M; T)$ that is a solution of (1.1) – (1.2). From (2.5) we get that $u$ is a solution of the following integral equation

$$
u(t, x) = S_1(t) \varphi(x) + S_2(t) \psi(x) + 
$$

$$
+ \int_0^t F^{-1} \left[ S(t - \tau, \xi) L(\xi) \hat{f}(u)(\xi, \tau) \right] \, d\tau, \quad t \in (0, T). 
$$

Let us show that this solution is a unique in $Y(T)$. Let $u_1, u_2 \in Y(T)$ are two solution of the problem (1.1) – (1.2). Then

$$
u_1 - u_2 = \int_0^t F^{-1} \left[ S(t - \tau, \xi) L(\xi) \hat{f}(u_1)(\xi, \tau) - \hat{f}(u_2)(\xi, \tau) \right] \, d\tau. \quad (3.15)
$$

By the definition of the space $Y(T)$, we can assume that

$$
\|u_1\|_{X_\infty} \leq C_1(T), \quad \|u_2\|_{X_\infty} \leq C_1(T).
$$

Hence, by Minkowski’s inequality for integrals and Theorem 2.2 we obtain from (3.15)

$$
\|u_1 - u_2\|_{Y_{2,p}} \leq C_2(T) \int_0^t \|u_1 - u_2\|_{Y_{2,p}} \, d\tau. \quad (3.16)
$$
From (3.16) and Gronwall’s inequality, we have \( \|u_1 - u_2\|_{Y^2, \rho} = 0 \), i.e. problem (1.1)–(1.2) has a unique solution which belongs to \( Y(T) \). That is, we obtain the first part of the assertion. Now, let \([0, T_0)\) be the maximal time interval of existence for \( u \in Y(T_0) \). It remains only to show that if (3.3) is satisfied, then \( T_0 = \infty \). Assume contrary that, (3.3) holds and \( T_0 < \infty \). For \( T \in [0, T_0) \), we consider the following integral equation

\[
\varphi (x, t) = S_1 (t) u (x, T) + S_2 (t) u_t (x, T) + \int_0^t F^{-1} \left[ S (t - \tau, \xi) L (\xi) \hat{f} (\xi, \tau) \right] d\tau, \quad t \in (0, T).
\]

By virtue of (3.3), for \( T' > T \) we have

\[
\sup_{t \in [0, T]} \left( \|u\|_{Y^2, \rho} + \|u\|_{X^\infty} + \|u_t\|_{Y^2, \rho} + \|u_t\|_{X^\infty} \right) < \infty.
\]

By reasoning as a first part of theorem and by contraction mapping principle, there is a \( T^* \in (0, T_0) \) such that for each \( T \in [0, T_0) \), the equation (3.17) has a unique solution \( u \in Y(T^*) \). The estimates (3.9) and (3.14) imply that \( T^* \) can be selected independently of \( T \in [0, T_0) \). Set \( T = T_0 - \frac{T^*}{2} \) and define

\[
\tilde{u} (x, t) = \begin{cases} 
  u (x, t), & t \in [0, T] \\
  v (x, t - T), & t \in \left[ T, T_0 + \frac{T^*}{2} \right]
\end{cases}
\]

By construction \( \tilde{u} (x, t) \) is a solution of the problem (1.1)–(1.2) on \( \left[ T, T_0 + \frac{T^*}{2} \right] \) and in view of local uniqueness, \( \tilde{u} (x, t) \) extends \( u \). This is against to the maximality of \([0, T_0)\), i.e we obtain \( T_0 = \infty \).

4. Applications

In this section we give some application of Theorem 3.1.

1. Let

\[
L_0 = L_1 = L_2 = A_1 = \sum_{|\alpha| \leq 2} a_\alpha D^\alpha,
\]

where \( a_\alpha \) are complex numbers.

Then the problem (1.1) – (1.2) is reduced to the Cauchy problem for the following Boussinesq equation

\[
u_{tt} + A_1 u_{tt} + A_1 u = A_1 f (x, t, u), \quad x \in R^2, \quad t \in (0, T),
\]

\[
u (x, 0) = \varphi (x) + \int_0^T \alpha (\sigma) u (x, \sigma) d\sigma, \quad u_t (x, 0) = \psi (x) + \int_0^T \beta (\sigma) u_t (x, \sigma) d\sigma,
\]

where \( \alpha, \beta \) are given functions.
here
\[ \varphi, \psi \in W^{s,p}(\mathbb{R}^2), \ s > \frac{2}{p}, \ p \in (1, \infty). \]

Assume
\[ A_1(\xi) = \sum_{|\alpha| \leq 4} a_\alpha \xi_1^{\alpha_1} \xi_2^{\alpha_2} > 0 \text{ for } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \]

Then it is not hard to see that
\[ A_1(\xi) \neq 0, -1 \]\nand
\[ |Q^{-\frac{1}{p}}(\xi)| \leq 1 \text{ for } \xi \in \mathbb{R}^2, \]
where
\[ Q(\xi) = A_1(\xi) \left[ 1 + A_1(\xi) \right]^{-1}. \]

Hence, the Condition 2.1 is satisfied. Let
\[ X_p = L^p(\mathbb{R}^2), \ 1 \leq p \leq \infty, \ Y^{s,p} = L^{s,p}(\mathbb{R}^2). \]

Hence, from Theorem 3.1 we obtain:

**Theorem 4.1.** Assume that the function \( u \to f(x,t,u): \mathbb{R}^2 \times [0,T] \times B_{p^{-\frac{1}{p}}}^2(\mathbb{R}^2) \to L^p(\mathbb{R}^2) \) is measurable in \((x,t) \in \mathbb{R}^2 \times [0,T]\) for \( u \in B_{p^{-\frac{1}{p}}}^2(\mathbb{R}^2) \).

Moreover, \( f(x,t,u) \) is continuous in \( u \in B_{p^{-\frac{1}{p}}}^2(\mathbb{R}^2) \) and \( f(x,t,u) \in C^{(3)} \left( B_{p^{-\frac{1}{p}}}^2(\mathbb{R}^2) ; \mathbb{C} \right) \)

uniformly with respect to \((x,t) \in \mathbb{R}^2 \times [0,T]\). Then for \( \varphi, \psi \in Y^{2,p}_\infty \) and \( p \in (1, \infty) \) problem (4.1) has a unique local strange solution \( u \in C^{(2)} \left( [0, T_0); Y^{2,p}_\infty \right) \), where \( T_0 \) is a maximal time interval that is appropriately small relative to \( M \).

Moreover, if
\[ \sup_{t \in [0, T_0)} \left( \|u\|_{Y^{2,p}_\infty} + \|u_t\|_{X_\infty} + \|u_{tt}\|_{Y^{2,p}_\infty} + \|u_{ttt}\|_{X_\infty} \right) < \infty \]

then \( T_0 = \infty \).

2. Let
\[ L_0 = L_1 = L_2 = A_2 = \sum_{|\alpha| \leq 4} a_\alpha D^\alpha, \]
where \( a_\alpha \) are complex numbers, \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \), \( \alpha_k \) are natural numbers and \( |\alpha| = \sum_{k=1}^{3} \alpha_k \).

Then the problem (1.1) – (1.2) is reduced to the Cauchy problem for the following Boussinesq equation
\[ u_{tt} + A_2 u_{ttt} + A_2 u = A_2 f(x,t,u), \ x \in \mathbb{R}^3, \ t \in (0,T), \quad (4.2) \]
\[ u(x, 0) = \varphi(x) + \int_0^T \alpha(\sigma) u(x, \sigma) \, d\sigma, \quad u_t(x, 0) = \psi(x) + \int_0^T \beta(\sigma) u_t(x, \sigma) \, d\sigma, \]

where

\[ \varphi, \psi \in W^{s,p}(\mathbb{R}^3), \quad s > \frac{3}{p}, \quad p \in (1, \infty). \]

Assume

\[ A_2(\xi) = \sum_{|\alpha| \leq 4} a_0 \xi_1^\alpha \xi_2^\alpha \xi_3^\alpha \neq 0, \quad -1 \text{ for all } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \]

Then it is not hard to see that, there exists a positive constant \( M \) such that

\[ \left| Q_0^\xi(\xi) \right| \leq 1 \text{ for } \xi \in \mathbb{R}^3 \text{ and } p \in (1, \infty), \]

where

\[ Q(\xi) = A_2(\xi) [1 + A_2(\xi)]^{-1}. \]

Therefore, the Condition 2.1 is satisfied.

Let

\[ X_p = L^p(\mathbb{R}^3), \quad 1 \leq p \leq \infty, \quad Y^{s,p} = L^{s,p}(\mathbb{R}^3). \]

Hence, from Theorem 3.1 we obtain:

**Theorem 4.2.** Suppose that the function \( u \rightarrow f(x, t, u): \mathbb{R}^3 \times [0, T] \times B_p^{2-\frac{1}{p}}(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3) \) is measurable in \((x, t) \in \mathbb{R}^3 \times [0, T]\) for \( u \in B_p^{2-\frac{1}{p}}(\mathbb{R}^3) \). Moreover, \( f(x, t, u) \) is continuous in \( u \in B_p^{2-\frac{1}{p}}(\mathbb{R}^3) \) and

\[ f(x, t, u) \in C(3) \left( B_p^{2-\frac{1}{p}}(\mathbb{R}^3); \mathbb{C} \right) \]

uniformly with respect to \((x, t) \in \mathbb{R}^3 \times [0, T]\). Then for \( \varphi, \psi \in Y_{\infty}^{2,p} \) and \( p \in (1, \infty) \) problem (4.1) has a unique local strange solution

\[ u \in C(2) \left( [0, T_0]; Y_{\infty}^{2,p} \right), \]

where \( T_0 \) is a maximal time interval that is appropriately small relative to \( M \). Moreover, if

\[ \sup_{t \in [0, T_0]} \left( \| u \|_{Y_{2,p}} + \| u \|_{X_{\infty}} + \| u_t \|_{Y_{2,p}} + \| u_t \|_{X_{\infty}} \right) < \infty \]

then \( T_0 = \infty \).

3. Let

\[ L_0 = \sum_{|\alpha| \leq 4} a_{0\alpha} D^\alpha, \quad L_1 = \sum_{|\alpha| \leq 2} a_{1\alpha} D^\alpha, \quad L_2 = \sum_{|\alpha| \leq 4} a_{2\alpha} D^\alpha, \]
where \(a_{\alpha} \) are complex numbers, \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \), \( \alpha_k \) are natural numbers and \( |\alpha| = \sum_{k=1}^{3} \alpha_k \).

Then the problem (1.1)–(1.2) is reduced to Cauchy problem for the following Boussinesq equation

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} + L_0 \frac{\partial^2 u}{\partial t^2} + L_1 u &= L_2 f(x,t,u), \quad x \in \mathbb{R}^3, \quad t \in (0,T), \\
u(x,0) &= \varphi(x) + \int_0^T \alpha(\sigma) u(x,\sigma) \, d\sigma, \quad u_t(x,0) = \psi(x) + \int_0^T \beta(\sigma) u_t(x,\sigma) \, d\sigma,
\end{align*}
\]

where

\[
\varphi, \psi \in W^{s,p}(\mathbb{R}^3), \quad s > \frac{3}{p}, \quad p \in (1, \infty).
\]

Assume

\[
L_0(\xi) > 0, \quad L_1(\xi) > 0 \quad \text{for} \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
\]

Since \(m_0 - m_1 = m_2 - m_3 = 2\), for \(s \geq 1 + \frac{3}{p}\) the Condition 2.1 is satisfied. Let

\[
X_p = L^p_0(\mathbb{R}^3), \quad 1 \leq p \leq \infty, \quad Y^{s,p} = L^{s,p}(\mathbb{R}^3).
\]

Hence, from Theorem 3.1 we obtain:

**Theorem 4.3.** Suppose that the function \( u \rightarrow f(x,t,u): \mathbb{R}^3 \times [0,T] \times B^{2-\frac{1}{p}}_{p}(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3) \) is measurable in \((x,t) \in \mathbb{R}^3 \times [0,T]\) for \(u \in B^{2-\frac{1}{p}}_{p}(\mathbb{R}^3)\). Moreover, \( f(x,t,u) \) is continuous in \(u \in B^{2-\frac{1}{p}}_{p}(\mathbb{R}^3)\) and

\[
f(x,t,u) \in C^{(3)}\left(B^{2-\frac{1}{p}}_{p}(\mathbb{R}^3) ; \mathcal{C}\right)
\]

uniformly with respect to \((x,t) \in \mathbb{R}^3 \times [0,T]\). Then for \( \varphi, \psi \in Y^{2,p}_\infty \) and \( s \geq 1 + \frac{3}{p}, \quad p \in (1, \infty) \) problem (4.3) has a unique local strange solution

\[
u \in C^{(2)}([0, T_0); Y^{2,p}_\infty),
\]

where \(T_0\) is a maximal time interval that is appropriately small relative to \(M\).

Moreover, if

\[
\sup_{t \in [0, T_0]} (\|u\|_{Y^2,p} + \|u\|_{X_\infty} + \|u_t\|_{Y^2,p} + \|u_t\|_{X_\infty}) < \infty
\]

then \(T_0 = \infty\).

**References**
1. V.G. Makhankov, Dynamics of classical solutions (in non-integrable systems), Phys. Lett. C 35, (1978) 1–128.

2. G.B. Whitham, Linear and Nonlinear Waves, Wiley–Interscience, New York, 1975.

3. N.J. Zabusky, Nonlinear Partial Differential Equations, Academic Press, New York, 1967.

4. C. G. Gal and A. Miranville, Uniform global attractors for non-isothermal viscous and non-viscous Cahn–Hilliard equations with dynamic boundary conditions, Nonlinear Analysis: Real World Applications 10 (2009) 1738–1766.

5. T. Kato, T. Nishida, A mathematical justification for Korteweg–de Vries equation and Boussinesq equation of water surface waves, Osaka J. Math. 23 (1986) 389–413.

6. A. Clarkson, R.J. LeVeque, R. Saxton, Solitary-wave interactions in elastic rods, Stud. Appl. Math. 75 (1986) 95–122.

7. P. Rosenau, Dynamics of nonlinear mass-spring chains near continuum limit, Phys. Lett. 118A (1986) 222–227.

8. S. Wang, G. Chen, Small amplitude solutions of the generalized IMBq equation, J. Math. Anal. Appl. 274 (2002) 846–866.

9. S. Wang, G. Chen, The Cauchy Problem for the Generalized IMBq equation in $W^{s,p}(\mathbb{R}^n)$, J. Math. Anal. Appl. 266, 38–54 (2002).

10. J.L. Bona, R.L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, Comm. Math. Phys. 118 (1988) 15–29.

11. F. Linares, Global existence of small solutions for a generalized Boussinesq equation, J. Differential Equations 106 (1993) 257–293.

12. Y. Liu, Instability and blow-up of solutions to a generalized Boussinesq equation, SIAM J. Math. Anal. 26 (1995) 1527–1546.

13. S. Piskarev and S.-Y. Shaw, Multiplicative perturbations of semigroups and applications to step responses and cumulative outputs, J. Funct. Anal. 128 (1995), 315-340.

14. N. Kutev, N. Kolkovska, and M. Dimova, “Global existence of Cauchy problem for Boussinesq paradigm equation,” Computers and Mathematics with Applications, 65(3) (2013), 500–511.

15. S. Lai, Y.H. Wu, The asymptotic solution of the Cauchy problem for a generalized Boussinesq equation, Discrete Contin. Dyn. Syst. Ser. B 3 (2003).
16. Girardi, M., Lutz, W., Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces, J. Funct. Anal., 204(2), 320–354, 2003.

17. H. Triebel, Interpolation theory, Function spaces, Differential operators, North-Holland, Amsterdam, 1978.

18. H. Triebel, Fractals and spectra, Birkhauser Verlag, Related to Fourier analysis and function spaces, Basel, 1997.

19. L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa 13 (1959), 115–162.

20. S. Klainerman, Global existence for nonlinear wave equations, Comm. Pure Appl. Math. 33 (1980), 43–101.

21. A. Ashyralyev, N. Aggez, Nonlocal boundary value hyperbolic problems involving integral conditions, Bound.Value Probl., 2014 V. 2014:214.

22. L. S. Pulkina, A nonlocal problem with integral conditions for hyperbolic equations, Electron.J.Differ.Equ.1999,45 (1999)