Mutual space-frequency distribution of Gaussian signal

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Abstract

Mutual space-frequency distribution is proposed and it is shown that Wigner and Weyl distribution functions are only particular cases of these distribution. Mutual distribution for Gaussian signal is analytically obtained. The simple connection between Wigner and Weyl distributions is established. It is shown that Wigner distribution forms as the rotational displacement of Weyl distribution on informational diagram of conjugate coordinates \((x;p)\) on an angle proportional to the mutual parameter \(t\). The results of direct calculations of mutual distribution for Gaussian signal in the mutual domain are presented.

1 Introduction

The main aim of space-frequency analysis is elaboration of distributions in order to get the information about signal simultaneously in coordinate and frequency domains. As a rule the Fourier transform is used to receive the signal frequency spectrum. This well-known transformation is good tool for the analysis of signal intensity distribution in the frequency plane. Such analysis foresees the calculation of Fourier-spectrum for constant coordinates. Practically, we have to deal with a certain momentary value of coordinates for which the signal and their Fourier-image are simultaneously determinated. With the coordinate variation the appropriate conversion of Fourier-spectrum also takes place and at once the problem of signal analysis occurs; the latter contains frequency components that are variable in accordance with coordinate. In such a case it is important to know the value of coordinate at which the corresponding transformation of frequency spectrum takes place. In order to investigate the variation of signal spectrum with the variation of its coordinate as far back as in 60-80-ies of the previous century was propound a new approach. It unites the information about coordinate and frequency constituent of the signal in so-called space-frequency representations. In such representations under consideration is a certain mutual function of coordinate and frequency. The idea...
of construction of mutual representations originates in works of E. Wigner (1932) [1], D. Gabor (1946) [2] and J. Weyl (1932) [3]. Before the 80-ies of the previous century tens of space-frequency representations of such case were taken under consideration [7, 8, 10]. However, the Wigner and Weyl distributions that are the most used for the present day remained the prerogative of quantum mechanics and they have not have precisely expressed use. Only in 1980 T. Klasen and B. Mecklenbrauker worked out the theory of application of Wigner distribution for space-frequency analysis of signals. Its main results were published in the series of works under the title ”Wigner distribution - the instrument for space-frequency analysis of signals” [4, 5, 6]. The successful use of the Wigner distribution in the theory of signals was stipulated by its ”good” mathematical characteristics, especially by its representative characteristics that is basic in the restoration of signal intensity distribution.

Within the framework of the given investigation among the variety of space-frequency representations we single out two basic distributions by Wigner and Weyl that are widely used by the theory of signals in solving inverse physical problems [8, 11]. The investigation of the signal characteristics takes place on the basis of comparison with its displaced analogues. The shift within a time results in subtraction of a specific value from the signal argument

$$\rightarrow x_{\tau}(t) = x(t + \tau).$$

The suitable displacement in accordance with frequency results in displacement of argument of the Fourier-spectrum signal, what equals to multiplication by phase multiplier in coordinate plane.

$$X_\omega \rightarrow X_\omega(t) = x(t)e^{i\omega t}. \quad (2)$$

Similar correlations are well-known from classical analysis [11]. Within the limits of space-frequency analysis we are interested in the signals displaced simultaneously with time and frequency, namely

$$x_{-\frac{T}{2}, -\frac{\nu}{2}} = x\left(t - \frac{T}{2}\right)e^{-i\omega t/2}, \quad (3)$$

$$x_{\frac{T}{2}, \frac{\nu}{2}} = x\left(t + \frac{T}{2}\right)e^{i\omega t/2}. \quad (4)$$

We can easily calculate the value of displacement between the signals

$$d(x_{-\frac{T}{2}, -\frac{\nu}{2}}, x_{\frac{T}{2}, \frac{\nu}{2}})^2 = 2||x||^2 - 2\Re\{A_{xx}(\omega, \tau)\}. \quad (5)$$

$A_{xx}(\omega, \tau)$ in this correlation plays a part of the distance and is called a time-frequency autocorrelation function or the ambiguity function

$$A_{xx}(\omega, \tau) = \int x^*\left(t - \frac{T}{2}\right)x\left(t + \frac{T}{2}\right)e^{-i\omega t}dt. \quad (6)$$

In accordance to the Parceval theory we can rewrite this correlation by Fourier-images of displaced signals

$$A_{xx}(\omega, \tau) = \frac{1}{2\pi} \int X\left(\nu - \frac{\omega}{2}\right)X^*\left(\nu + \frac{\omega}{2}\right)e^{i\omega \tau}d\omega. \quad (7)$$
The function in such a form was for the first time set by J. Weyl [3] and for the present time is known in the theory of signals under the name of Weyl distribution or the ambiguity function. Having realized the direct and inverse Fourier-transform we get the value

\[ W_{xx}(t, \nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_{xx}(\omega, \tau) e^{i\omega t} e^{-i\nu \tau} d\omega d\tau. \]  

(8)

that in explicit form has the following notation

\[ W_{xx}(t, \nu) = \int x^* \left( t - \frac{\tau}{2} \right) x \left( t + \frac{\tau}{2} \right) \exp (-i\nu \tau) d\tau, \]  

(9)

or by the Fourier-image of the function \( x(t) \)

\[ W_{xx}(t, \nu) = \frac{1}{2\pi} \int X \left( \nu - \frac{\omega}{2} \right) X^* \left( \nu + \frac{\omega}{2} \right) \exp (i\omega t) d\omega. \]  

(10)

The function was firstly introduced by E. Wigner and is named after him - the Wigner function of distribution or just Wigner distribution [1].

Joint space-frequency representations are widely used not only in the theory of signals. They have a number of practical use in different fields of physics, geology, seismology, etc. Within the limits of the given research we are interested in the use of such distributions in the region of representations treatment and recognition of images. This field of the physics imposes a set of demands that the generalized distribution of signals have to meet. For the efficient use in the theory of representatives of space-frequency distribution they have to be characterized by representative property; for the limiting values of the variable \( t \) they have to develop into known distributions; to have high distributive capacity in the field of Wigner distribution as well as in Weyl distribution; to take positive values.

Representations that the most precisely meet the demands stated in the theory of image are basic distributions by Wigner and Weyl. These distributions also have their own peculiarities. Unfortunately, up to the present there exists no simple deduction about the expediency of use of this or that distribution. There exist a number of signals for which Wigner and Weyl distributions proceed into the negative region. In such cases these distributions are interpreted as quasiprobable. In spite of the external resemblance of the properties of Wigner and Weil distributions they have the peculiarity of principle in the mechanism of renewal of the entrance signal according to the known distribution. Wigner formalism allows renewing the signal according to the so-called marginal distribution

\[ |x(t)|^2 = \int_{-\infty}^{\infty} W_{xx}(t, \nu) d\nu. \]  

(11)

In the frame of Weyl formalism signal reconstruction takes place using signal restoration scheme

\[ |x(t)|^2 = \int_{-\infty}^{\infty} A_{xx}(0, \omega) e^{i\omega t} d\omega. \]  

(12)
Sometimes working with the scheme of renewal according to the Weyl distribution is to the great extent more easily (due to the integrating of its crosscut) than in the case of marginal distribution. For the present Wigner distribution is more commonly applied as it uses marginal distributions that can be measured by experiment. Though, both approaches have the right to existence.

Within the framework of the given experiment we investigate uninterrupted transition between these distributions by means of introduction of a generalized common function of time and frequency that depends from a variable \( t \).

For present day various types of space-frequency distributions are successfully used for the analysis of nonstationary signals [7, 8]. Many of such distributions are characterized by advantages as well as by disadvantages in use in various fields of physics. Wigner and Weyl distributions are widely used in space-frequency analysis and in particular in optical information processing systems. Well known is a fact, that the very distributions possess such charateristics, that are successfully used for description of many optical systems. The spectrum of appliance of these distributions is extremely wide. They are used, in particular, in the theory of optical lens system, theory of communication, hydrolocation and other fields [9, 10, 11]. The researches of last year proved the efficiency of use of space-frequency distributions in biology and medicine; especially Wigner distribution was successfully used for renewal of volumetric structure of objects within the framework of optical tomography [12, 13, 14]. One of the promising investigation directions within the space-frequency processing of signals is studying the properties of novel space-frequency representations of the distributions, with the aim of their further applications in different areas of physics and medicine. Unfortunately, it often happens that some space-frequency distributions do not meet demands raised by one or another specific application. In this relation, many of the existing distributions need generalization or improvements when applied to a given problem. During the second half of the past century and the beginning of this one, a cleartendency has been observed towards generalization of different space-frequency distributions. The first attempt of such a generalization has been due to L.Cohen [15] as long ago as in 1966. The author has introduced a number of quasiprobable distributions that provide proper quantum mechanical marginal distributions. Within the limits of this research the Wigner distribution was examined as a separate case. The next step has been done by N. De Brujin in 1973 [16]. His work has been devoted to elaboration of theory of generalized functions, with application concerned with Wigner and Weil distributions. Summarizing the results of numerous investigations L.Cohen [7, 10] has suggested to has suggested a generalized distribution involving a certain kernel

\[
C(x, \omega, \Phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y + \frac{x_0}{2}) f^*(y - \frac{x_0}{2}) \Phi(x, \omega) e^{-i(\omega x_0 - \omega_0 y + \omega_0 y_0)} dy dx_0 d\omega_0.
\]

Depending on the form of the kernel \( \Phi(x, \omega) \), this distribution degenerates into one of the known distributions (Wigner, Weyl, Woodward, Kirkwood, Page, Mark, etc.). In the general case expression (13) describes the class of space-frequency distribu-
tion later named Cohen’s class. Members of this class are known distributions as well as a set of still unknown distributions, that also satisfy all necessary requirements of existing distributions. The theory of generalization of space-frequency distributions has been developed by also famous specialist in the theory of signals by A. Mertins. In his monograph ”Signal Analysis” he has singled out this topic into a separate section “General space-frequency distributions”. The author has stated that the Wigner distribution serves as an excellent tool for space-frequency analysis as long as a linear dependence is kept between the instantaneous coordinates and-frequencies. Otherwise, a need in generalizing appears, whose general principles are described in detail in the mentioned work.

Among numerous recent studies related to generalizing space-frequency distributions, we should mention only the most typical ones. The PhD Thesis by L. Durak “Novel time-frequency analysis technique for deterministic signals” is one of such studies, where a close attention has been paid to generalizing distributions and introducing their additional parameters.

Different types of generalizations of space-frequency distributions have been thoroughly considered in the book by B. Boashash. Among a number of studies included in it, we would like to emphasize the works by R. Baraniuk (p. 123), X. Xia (p. 223) and A. Papandreou-Suppappola (p. 643). Within the mentioned collection, the work by G. Matz by F. Hlawatsch (p. 400) is of particular interest in relation to the problem of distributions generalization. It considers methodology for constructing generalized distributions on the basis of both the Wigner distribution and the ambiguity function (i.e. the Weyl distribution).

In the present work we try to use interlinks between the Wigner and Weyl distributions with the aim of joining them into a single, more general distribution. Up to date, it has been revealed that the two distributions are related by a double Fourier transform. The results obtained by us allow tracing transformation of one of the distributions into the other, while changing the distribution parameter. This generalized distribution generates a whole set of new distributions formed in the process of switching between the basic distributions. The latter fact may be important from the viewpoint of possible practical applications. For the present day a choice between the Weyl and Wigner distributions remains ambiguous. Each of them has its own scheme for reconstruction of signal intensity distribution. The scheme adopted for the Wigner distribution includes calculating the marginal distributions. The Weyl distribution provides much simpler reconstruction scheme, owing to simpler mathematical transformations. Traditionally, the Wigner distribution has been used in a large majority of studies performed within the field. Introduction of the mutual distribution would mean a possibility for calculating ‘mixed’ states and determining the necessary contributions of each of the limiting distributions. As stated above, there appears a possibility for generalization of distributions concerning various applied problems. However, only S. Chountasis has suggested the approach that enables transitions between the Wigner and Weyl distributions. Such distributions play an important role in the analysis of phase space and, moreover, can be immediately applied in the Wigner tomography. In 1999 S. Chountasis and co-authors have developed a general distribution based on
the Wigner formalism, which involves an additional parameter \( \theta \). This study has been performed in frame of quantum-mechanical formalism. It allows passing the Wigner and Weyl distributions into each other by means of changing the common parameter.

The problem of calculation of a classical analogue of this generalized distribution remains urgent. It may be constructed based on the results [19] or using the formalism of Weyl distribution, as has been done by the present author when studying the properties of fractional Fourier transform [23]. Similarly to the works [18] [22], the author has employed peculiarities of reconstruction of signal intensity based upon the Weyl distribution. Meanwhile, it is just this reconstruction scheme is realized experimentally in the real optical schemes [23].

2 Mutual distribution: basic relations

2.1 Theoretical statements

In the present work we propose to use a type of generalized distribution with parameter \( t \) based upon the Weyl distribution function. The use of Weyl distribution has a peculiarity of principle in comparison with the function of Wigner distribution, that consists of the possibility to renew the intensity of distribution. The latter is registered experimentally at the output of the optical system. Common distribution of two signals \( f_1(x) \) and \( f_2(x) \) may be written as follows [25]

\[
K_{f_1 f_2}^{(t)}(x; p) = \frac{C_t}{1 + t} \int dx_0 d\omega_0 \exp \left\{ i [x_0 p - \omega_0 x] \right\} \\
\times \exp \left\{ -i \frac{(x - x_0)^2 + (p - \omega_0)^2}{\tan(\theta/2)} \right\} \\
\times \int f_1 \left( z + \frac{x_0}{2} \right) f_2^* \left( z - \frac{x_0}{2} \right) \exp (-i\omega_0 z) dz. \tag{14}
\]

Constant \( C_t \) and variable of the generalized distribution \( t \) are determined by the expressions

\[
C_t = \frac{2}{\pi} \frac{1}{1 - \exp i\theta}, \quad t = \frac{\theta}{\pi}. \tag{15}
\]

Distribution (14) is called the mutual space-frequency distribution, or concisely the mutual distribution. Limiting cases of distribution (14) is Weyl distribution (ambiguity function) (6) with the value of the parameter \( t = 0 \) and Wigner distribution (9) with the value of parameter \( t = 1 \). Accordingly, two known distributions (6) and (9) have plenty of alternative distributions and to each of them corresponds a specific value of parameter \( t \).

The expression of the mutual distribution (14) may be also represented in the simplified form using Weyl distribution (6)

\[
K_{f_1 f_2}^{(t)}(x; p) = \frac{C_t}{1 + t} \int dx_0 d\omega_0 A_{f_1 f_2}^*(x_0; \omega_0) \\
\times \exp \left\{ i [x_0 p - \omega_0 x] \right\} \exp \left\{ -i \frac{(x - x_0)^2 + (p - \omega_0)^2}{\tan(\theta/2)} \right\} . \tag{16}
\]
Performing the converted transformation we can render the Weyl distribution by means of the above introduced function of the mutual distribution

\[
\mathcal{A}_{f_1f_2}^t(x_0';\omega_0') = \frac{1 + t}{C_t} \int \int dxdp \mathcal{K}^{(t)}_{f_1f_2}(x;p) \times \exp \left\{ -i [x_0'p - \omega_0'x] \right\} \exp \left\{ i \frac{(x - x_0')^2 + (p - \omega_0')^2}{\tan(\theta/2)} \right\}. \tag{17}
\]

Formula (17) constitutes the inverse connection between the simple \( \mathcal{A}_{f_1f_2}^t(x_0;\omega_0) \) and generalized \( \mathcal{K}^{(t)}_{f_1f_2}(x;p) \) Weyl distributions. This makes the possibility of restoring the distribution of signal intensity according to the mutual distribution what has not been established when using the generalized Wigner distribution [19].

### 2.2 Representation in the terms of Wigner distribution

In order to make a comparison of the results with the existing analogues it is indispensable to have possibility to precisely calculate the limiting cases of the mutual distribution (17). Calculation of the limiting case \( t = 1 \) by means of the formula (16) may be conducted precisely, and in the case \( t = 0 \) such transition is not a trivial matter. To make the calculations simpler we introduce the representation of the mutual space-frequency distribution by means of Wigner distribution. We make use of the known identity

\[
\mathcal{A}_{f_1f_2}^t(x_0;\omega_0) = \frac{1}{2\pi} \int \int d\xi d\eta \mathcal{W}_{f_1f_2}(\eta;\xi) \exp(-i\omega_0\eta) \exp(i\xi x_0), \tag{18}
\]

that connects Weyl and Wigner distributions.

Placing (18) into the expression (16) and making a set of conversions we receive a formula describing the mutual space-frequency distribution in the terms of Wigner distribution

\[
\mathcal{K}^{(t)}_{f_1f_2}(x;p) = \tilde{C}_t \int \int dx_0 d\omega_0 \mathcal{W}_{f_1f_2}(x_0;\omega_0) \exp \left\{ -i [x_0p + \omega_0x] \right\} \exp \left\{ \frac{i}{4} \tan \frac{\theta}{2} \left[ (x - x_0)^2 + (p - \omega_0)^2 \right] \right\}, \tag{19}
\]

where the constants \( \tilde{C}_t \) are determined by means of the correlation

\[
\tilde{C}_t = \frac{-i}{\pi} \frac{1}{1 - \exp i\theta} \tan \frac{\theta}{2} \frac{1}{1 + t}. \tag{20}
\]

As it can be easily seen in the case of representation of the mutual distribution by means of the Wigner distribution peculiarities in the point \( t = 0 \) and around it disappear, however, there appear peculiarities around the point \( t = 1 \). Thereby the pair of representations: (16) and (19) complement one another and completely describe mutual space-frequency distribution in the region \( t = [0, 1] \).
2.3 Limiting cases

The aim of this work is to determine the mechanism of re-distribution between Wigner and Weyl distributions and to investigate the peculiarities of mutual distributions describing the region of values $t = [0, 1]$. Consequently the investigation of limiting cases (16) and (19) of mutual distribution is of peculiar importance. Let us study the limiting cases.

**Case $t = 1$.**

To describe this case we make use of coordinate representation of the mutual distribution (16). Placing $t = 1$ (or $\theta = \pi$) under (16) we arrive to the following result

$$K_{f_1 f_2}^{t=1}(x; p) = \frac{1}{2\pi} \int \int dx_0 d\omega_0 A_{f_1 f_2}(x_0; \omega_0) \exp(ix_0 p - i\omega_0 x).$$  \hspace{1cm} (21)

Accordingly to (8) we have

$$K_{f_1 f_2}^{t=1}(x; p) = \mathcal{W}_{f_1 f_2}(x; p).$$  \hspace{1cm} (22)

The limiting case $t = 1$ of mutual space-frequency distribution corresponds to the function of Wigner distribution.

**Case $t = 0$.**

To describe this case we make use of coordinate representation of the mutual distribution (19). Placing here $t = 0$ (or $\theta = 0$) we arrive to the following result

$$K_{f_1 f_2}^{t=0}(x; p) = \frac{1}{2\pi} \int \int dx_0 d\omega_0 \mathcal{W}_{f_1 f_2}(x_0; \omega_0) \exp(-i\omega_0 x + ix_0 p).$$  \hspace{1cm} (23)

Accordingly to the formula connecting Wigner and Weyl distributions we obtain

$$K_{f_1 f_2}^{t=0}(x; p) = \mathcal{A}_{f_1 f_2}(x; p).$$  \hspace{1cm} (24)

Expression (24) is identical with the function of Weyl distribution. Hereby, introduced by us distribution (14) in the values of the limiting cases of the parameter $t$ describes known distributions (6) and (9). This distribution has two equivalent representations: by means of the function of Wigner distribution (19) or function of Weyl distribution (16). Having at least one from the basic functions of distribution we can obtain the image of mutual distribution. The object of the further investigation is to study the peculiarities of the set of intermediate distributions when $(0 < t < 1)$. In order to display the peculiarities of the mutual distribution and to visually demonstrate the results we illustrate it on the example of Gaussian signal. This is one of the simplest types of signals that allows us to make calculations in the explicit form. We shall notice that similar calculations may be also made with other signals (in peculiar with orthogonal impulse, etc.). Choice of the Gaussian signal is related to explicit form of the common distribution. Herewith we can trace the mechanism of transition of Weyl distribution into Wigner distribution and vice versa.
3 Mutual distribution of Gaussian signal

3.1 Basic relations

In the work [24] have been investigated the main peculiarities of generalized coordinate-frequency distribution on the example of Gaussian signal. Obtained results were based on the frequent calculations of the proper distributions that allow to properly evaluate processes of distribution. In the present work we conduct an analytical calculation of the mutual space-frequency distribution on the example of Gaussian signal that is represented in the following way

\[ g(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right). \]  

(25)

Fourier image of these function is

\[ \hat{F}[g(x)] = \int_{-\infty}^{\infty} g(x)e^{i\omega x}dx = \exp\left(-\frac{p^2\sigma^4}{2}\right). \]  

(26)

Such selection form of Gaussian functions are stipulated by the fact that its integrating according to all values \( x \) results in value

\[ \int_{-\infty}^{\infty} g(x)dx = 1. \]  

(27)

As investigations of the mutual space-frequency distribution provides the study of re-distribution between Wigner and Weyl distributions it is reasonable to present the explicit form of these distributions for the case of Gaussian signal [24].

**Weyl distribution for Gaussian signal**

\[ \mathcal{A}_{ff^*}(x_0;\omega_0) = \frac{1}{2\sqrt{\pi\sigma}} \exp\left(-\frac{x_0^2}{4\sigma^2} - \frac{\omega_0^2\sigma^2}{4}\right). \]  

(28)

**Wigner distribution for Gaussian signal**

\[ \mathcal{W}_{ff^*}(x;\omega) = \frac{1}{\sqrt{\pi\sigma}} \exp\left(-\frac{x^2}{\sigma^2} - \omega^2\sigma^2\right). \]  

(29)

It is known [7], that one of the basic characteristics of the signal is its representativity that is the possibility to restoring the signal according to its distribution. Schemes of renewing for Weyl and Wigner distributions are well known. For the case of Gaussian signal they have the following form

**restoration scheme for Gaussian signal by Weyl distribution**

\[ |g(x)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{A}_{ff^*}(0;\omega_0)e^{i\omega_0 x}d\omega_0 = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2}{\sigma^2}\right). \]  

(30)

**restoration scheme for Gaussian signal by Wigner distribution**
\[ |g(x)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{W}_{ff^*}(x; \omega) d\omega = \frac{1}{2\pi \sigma^2} \exp \left(-\frac{x^2}{\sigma^2}\right). \] (31)

Independently from the choice of distribution form Gaussian signal is precisely renewed using both schemes of renewing.

We find explicit analytical form of the mutual distribution for the case of Gaussian signal. For this reason we will place into the formula of mutual distribution (16) the Weyl distribution of Gaussian signal (16). We obtain expression

\[ K_{ff^*}(x; p) = \frac{C_t}{1 + t \sigma \sqrt{\pi}} \exp \left\{ -\frac{i}{T} (x^2 + p^2) \right\} I_1(x, p) I_2(x, p), \] (32)

where

\[ I_1(x, p) = \int dx_0 \exp \left\{ ix_0 p + \frac{i}{T} 2x x_0 - \frac{i}{T} x_0^2 - \frac{x_0^2}{4\sigma^2} \right\}, \] (33)

\[ I_2(x, p) = \int d\omega_0 \exp \left\{ -i \omega_0 x + \frac{i}{T} 2p \omega_0 - \frac{i}{T} \omega_0^2 - \frac{\omega_0^2 \sigma^2}{4} \right\}. \] (34)

Expressions (33) and (34) may be written in the following form

\[ I_1(x, p) = \frac{\sqrt{\pi}}{\sqrt{a_1}} \exp \left\{ -\frac{1}{a_1} \left( \frac{x}{T} + \frac{p}{2} \right)^2 \right\}, \] (35)

\[ I_2(x, p) = \frac{\sqrt{\pi}}{\sqrt{a_2}} \exp \left\{ -\frac{1}{a_2} \left( \frac{p}{T} - \frac{x}{2} \right)^2 \right\}, \] (36)

where the following symbols are introduced

\[ a_1 = \frac{T + 4\sigma^2}{4T \sigma^2}, \quad a_2 = \frac{T \sigma^2 + 4i}{4T}, \quad T = \tan \left( \frac{\theta}{2} \right). \] (37)

The mutual distribution of Gaussian signal (32) has the form

\[ K_{ff^*}(x; p) = C_t^\sigma \exp \left\{ -\frac{i}{T} (x^2 + p^2) \right\} \times \exp \left\{ -\frac{1}{a_1} \left( \frac{x}{T} + \frac{p}{2} \right)^2 \right\} \exp \left\{ -\frac{1}{a_2} \left( \frac{p}{T} - \frac{x}{2} \right)^2 \right\}. \] (38)

where constant

\[ C_t^\sigma = \frac{C_t}{1 + t \sigma \sqrt{\pi}} \frac{1}{\sqrt{a_1 a_2}} \] (39)

depends from dispersion of Gaussian distribution of signal \( \sigma \) and from the values \( a_1, a_2 \). From the values \( a_1^{-1} \) and \( a_2^{-1} \) depend also expressions in the index of the expression exponential curve (41). We depict them in the form

\[ a_1^{-1} = a_{11} + i a_{12} : \quad a_{11} = \frac{4T^2 \sigma^2}{16\sigma^4 + T^2}, \quad a_{12} = -\frac{16T \sigma^4}{16\sigma^4 + T^2}, \] (40)
\[
\begin{align*}
\frac{1}{a_2^{-1}} &= a_{22} + ia_{21}, \quad a_{22} = \frac{4T^2\sigma^2}{16 + T^2\sigma^4}, \quad a_{21} = -\frac{16T}{16 + T^2\sigma^4}. 
\end{align*}
\]

Placing (40) and (41) under (38) we find explicit analytical form of the mutual distribution of the Gaussian signal (14)

\[
\mathcal{K}^t_{ff}(x; p) = \frac{C_t}{1 + t} \sqrt{\frac{\pi}{2\sigma} \frac{1}{\sqrt{|a_1a_2|}}} \exp \left\{ -\frac{i}{T} \left( x^2 + p^2 \right) \right\} \times \exp \left\{ -a_{11} \left( \frac{x}{T} + \frac{p}{2} \right)^2 - ia_{12} \left( \frac{x}{T} + \frac{p}{2} \right)^2 \right\} \times \exp \left\{ -a_{22} \left( \frac{p}{T} - \frac{x}{2} \right)^2 - ia_{21} \left( \frac{p}{T} - \frac{x}{2} \right)^2 \right\}. 
\]

We may check that expression (42) in limiting cases \( t = 0 \) and \( t = 1 \) changes into Weyl and Wigner distributions respectively. In order to ascertain the circumstance we investigate the conduct of the expression (42) in the limit \( t \to 1 \). Parameter \( t \) is determined according to (15) by value \( \theta \).

\[
\theta = \pi - \alpha. 
\]

Region of small values \( \alpha \) corresponds to quasi-Wigner region of mutual function of distribution. Then the value \( C_t \) that is a part of \( C_t^\sigma \) in case of small values \( \alpha \) can be depicted as follows

\[
C_t \approx \frac{1}{\pi} \left( 1 + \frac{i\alpha}{2} \right). 
\]

Asymptotics of the values \( (a_1a_2)^{-1/2} \) may be easily find when use its module \( r_{12} \) and argument \( \varphi_{12} = \varphi_1 + \varphi_2 \)

\[
(a_1a_2)^{-1/2} = r_{12} e^{i(\varphi_1 + \varphi_2)/2}, \quad (45)
\]

where

\[
r_{12} = \pm \frac{4T\sigma}{(16 + T^2\sigma^4)^{1/4}(16\sigma^4 + T^2)^{1/4}}, \\
\varphi_1 = \arctan \left( -\frac{4\sigma^2}{T} \right), \quad \varphi_2 = \arctan \left( -\frac{4}{T\sigma^2} \right). 
\]

In the case of small values \( \alpha \) \( (T \gg 1) \) we have such approximate form for the value \( r_{12} \)

\[
r_{12} = 4 \left( 1 - \frac{4}{T^2} \left( \sigma^4 + \sigma^{-4} \right) \right), \quad (47)
\]

and for value \( \varphi_1 \) and \( \varphi_2 \) we find

\[
\varphi_{12} = \varphi_1 + \varphi_2 = -\frac{4}{T} \left( \sigma^2 + \sigma^{-2} \right). 
\]

Thereby, the constant \( C_t^\sigma \) from (42) in quasi-Wigner region \( \alpha \ll 1 \) has the form

\[
C_t^\sigma(1) = \frac{1}{\sigma\sqrt{\pi}} \left( 1 + \frac{i\alpha}{2} \right) \left( 1 - \frac{4}{T^2} \left( \sigma^4 + \sigma^{-4} \right) \right) e^{-\frac{2\pi i}{T}(\sigma^2 + \sigma^{-2})}. 
\]
Index 1 in the value $C_t^\sigma$ denotes a condition $\alpha \ll 1$. Naturally, that in the $\alpha \to 0$ we has the value

$$C_t^\sigma = \frac{1}{\sigma \sqrt{\pi}},$$

that precisely corresponds to the amplitude of the expression for Wigner distribution (29).

Let us observe the coefficient $C_t^\sigma$ from (42) in the case of small values $\theta (t \to 0)$. We shall call the region of parameter $t$ values quasi-Weyl region as far as in the value $t = 0$ we acquire Weyl distribution. Similarly as in the case $t = 1$, we find for small values the following expressions

$$C_t = \frac{i}{\pi T}(1 - iT), \quad r_{12} = T, \quad \varphi_1 = \varphi_2 = -\pi/2, \quad C_t^{\sigma(2)} = \frac{1}{2\sqrt{\pi} \sigma}.$$ (50)

Such value of constant $C_t^{\sigma(2)}$ exactly corresponds to the amplitude of the value from Weyl distribution of Gaussian signal (28). Thus, the mutual distribution of Gaussian signal in limiting cases according to amplitude precisely coincides with the known distributions of Wigner and Weyl.

Calculation of real and imaginary part of constant $C_t^\sigma$ depicted on the Fig.1. for $C_t^\sigma$ provides an exact correspondence according to amplitude with Wigner and Weyl distributions. In the common region appears imaginary part $C_t^\sigma$ that is inherent only in mutual distribution. In the limiting cases $t = 0(\theta = 0^0)$ and $t = 1(\theta = 180^0)$ imaginary part dissappears what corresponds to the cases of basic distributions. Worth mentioning is also peculiar conduct of the imaginary part of constant $C_t^\sigma$ having maximum in the quasi-Wigner region.

Let us proceed the investigation of limiting cases of expressions placed in the index of the exponent on a curve formula (42). As was shown above the amplitude of mutual distribution in limiting cases coincides with the amplitude of known distributions of Weyl and Wigner. Providing that the form of these distributions will also coincide functionally (as functions $x$ and $p$), mutual distribution may be considered as generalization of well-known distributions of Wigner and Weyl. Having made a number of mathematical transformations the expression (42) aquires the following form

$$K^{(i)}_{\text{rr}}(x; p) = C_t^\sigma \exp \left\{ -gx^2 - fp^2 - dxp \right\},$$ (51)

where coefficients $g$, $p$ and $d$ are certain complex values

$$g = g_0 + ig_1, \quad f = f_0 + if_1, \quad d = d_0 + id_1$$ (52)

moreover

$$g_0 = \frac{\sigma^2}{\mu}(64 + 20T^2\sigma^4 + T^4), \quad g_1 = \frac{T}{\mu}(16[1 - 4\sigma^4] + T^2[\sigma^4 - 4]);
$$

$$f_0 = \frac{\sigma^2}{\mu}(64\sigma^4 + 20T^2 + T^4\sigma^4), \quad f_1 = \frac{T\sigma^4}{\mu}(16[\sigma^4 - 4] + T[1 - 4\sigma^4]);$$

$$d_0 = \frac{4T\sigma^2}{\mu}(1 - \sigma^4)(16 - T^2), \quad d_1 = 16T^2\frac{1 - \sigma^8}{\mu}.$$ (53)
where the value $\mu$ has the form

$$
\mu = 16^2\sigma^4 + 16T^2(1 + \sigma^8) + \sigma^4T^4
$$

Constant $C'_t$ from (39) may be represented as

$$
C'_t = \frac{i}{\pi} \frac{1 + e^{-ig}}{\sin \theta} \frac{1}{1 + t} \frac{\sqrt{\pi}}{2\sigma} r_{12} e^{i(\varphi_1 + \varphi_2)/2},
$$

where

$$
r_{12} = \frac{4\sigma T}{(16 + T^2\sigma^4)^{1/4}(16\sigma^4 + T^2)^{1/4}}
$$

$$
\varphi_1 = \arctan\left(-\frac{4\sigma^2}{T}\right), \quad \varphi_2 = \arctan\left(-\frac{4}{T\sigma^2}\right)
$$

Representation (51) is an explicit form of the mutual distribution of Gaussian signal. Let us consider asymptotic values of $g$, $f$ and $d$ in the limiting cases $t = 0$ and $t = 1$.

In the case $t = 0$ ($T \to 0$) from (53), (54) we find

$$
g_0 = (4\sigma^2)^{-1}, \quad f_0 = \sigma^2/4.
$$

The rest of coefficients turns into zero

$$
g_1 = f_1 = d_0 = d_1 = 0.
$$

In the limiting case $t = 1$ ($T \to \infty$) we receive

$$
g_0 = \sigma^{-2}, \quad f_0 = \sigma^2.
$$

Other coefficients dissipate as in (58). We shall notice that imaginary part of cross-expressions in (51) real part dissipates in $t \to 0$ as well as in $t \to 1$.

What concerns the values $g$ and $f$, their imaginary parts also tend to zero in the case $t = 0$ and $t = 1$. Thus, the very quadratical members are responsible for forming of the distribution in limiting cases as they are included into known Wigner and Weyl distributions, and cross-representations that are inherent only in mutual distribution dissipate.

### 4 Signal restoration scheme in mutual distribution domain

As it is known, restoration of the signal according to Weyl distribution takes place correspondingly to (30), and according to Wigner distribution - according to (31). Above mentioned schemes of signal restoration may be united into one formula using the mutual space-frequency distribution suggested above. We shall introduce the value

$$
f_\theta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\rho w_\theta (x \sin(\theta/2)p) e^{ipx \cos(\theta/2)}.
$$
It can be easily seen, that when $\theta = 0$ we arrive at Weyl renewal scheme, and when $\theta = \pi$ - we acquire Wigner renewal signal. Taking into consideration (51), for the function $w_\theta$ we have

$$w_\theta(x \sin(\theta/2), p) = C_t^\sigma \exp\left\{-g \sin^2(\theta/2)x^2 - fp^2 - d \sin(\theta/2)xp\right\}. \quad (61)$$

In the result of integration of (60) we acquire

$$f_\theta(x) = \frac{C_t^\sigma \sqrt{\pi}}{2\pi (f_0 + if_1)^{1/2}} e^{-x^2 \sin \theta/2(g_0 + ig_1)} e^{-\frac{1}{\mu}x^2} \exp\left\{\frac{1}{4} \left(\frac{x^2}{f_0 + if_1}\right) \left(\sin \frac{\theta}{2}(d_0 + id_1) - i \cos \frac{\theta}{2}\right)^2\right\}. \quad (62)$$

It can be easily seen that in limiting cases function $f_\theta(x)$ transforms into expressions (30) and (31). Taking into account expressions for real and imaginary part of coefficients $g$, $f$ and $d$ we arrive at

$$f_\theta(x) = \frac{1}{2\sqrt{\pi T_f}} e^{-\frac{i}{4} \varphi_f} e^{-G x^2}, \quad (63)$$

where

$$r_f = \frac{\sigma_f^2}{\mu} \left\{64\sigma^4 + 20T^2 + \sigma^4T^4\right\} + T^2\sigma^4 \left\{16(\sigma^4 - 4) + T^2(1 - 4\sigma^4)\right\}^{1/2},$$

$$\varphi_f = \arctan\left(\sigma^2T \frac{16(\sigma^4 - 4) + T^2(1 - 4\sigma^4)}{64\sigma^4 + 20T^2 + T^4\sigma^4}\right),$$

$$G = r_g \sin \frac{\theta}{2} e^{i\varphi_g} - \frac{1}{4} \frac{1}{4T_f} \left\{r_a \sin^2 \frac{\theta}{2} e^{i(\varphi_f + 2\varphi_d)} - 2r_d \sin \frac{\theta}{2} e^{i(\varphi_f + \varphi_d + \frac{\pi}{2})} - \cos^2 \frac{\theta}{2} e^{i\varphi_f}\right\}.$$

In order to make the formula shorten certain symbols are introduced

$$r_g = \frac{\sigma_g^2}{t_4} \left\{64 + 20T^2\sigma^4 + T^4\right\} + \frac{T^2}{\sigma^4} \left\{16(1 - 4\sigma^4) + T^2(\sigma^4 - 4)\right\}^{1/2},$$

$$\varphi_g = \arctan \frac{g_1}{g_0} = \arctan \left(\frac{T\left[16(1 - 4\sigma^4) + T^2(\sigma^4 - 4)\right]}{\sigma^2(64 + 20T^2\sigma^4 + T^4)}\right),$$

$$r_d = \frac{4T}{t_4} \left[\sigma^4(16 - T^2)(1 - \sigma^4)^2 + 16T^2(1 - \sigma^8)\right]^{1/2},$$

$$\varphi_d = \arctan \left(\frac{16T^2(1 - \sigma^8)}{4T\sigma^2(1 - \sigma^4)(16 - T^2)}\right) = \arctan \left(\frac{1 - \sigma^8}{\sigma^2(1 - \sigma^4)(16 - T^2)}\right).$$

Thus, suggested restoration scheme (63) in limiting cases precisely renew Gaussian signal (25)

$$f_{(t=1)}(x) = f_{(t=0)}(x) = \frac{1}{\sqrt{2\pi \sigma}} \exp \left(-\frac{x^2}{2\sigma^2}\right). \quad (64)$$

Thereby, side by side with known limiting cases arises possibility of signal intensity distribution restoration in the region $t = [0, 1]$. In our opinion investigation of intensity distribution in the mentioned region is an urgent problem, however it passes the limits of the present investigation.
5 Conclusions

In this work we propose a mutual space-frequency distribution as generalization of Weyl distribution (16). The mutual distributions is characterized by a certain parameter $t$. It is a generalization of space-frequency representations suggested by Wigner and Weyl and comprises them as limiting cases. In the course of the investigation it has been determined that transition from Weyl distribution into Wigner one occurs by means of mutual region as a curve at the information diagram of mutual coordinates $(x, p)$ Fig.2. When modifying the mutual parameter $t$ the distribution changes and simultaneously suffers deformation for the angle proportional to parameter $t$. Fig.3 depicts the region of mutual distribution close to Weyl distribution (quasi-Weyl region of distribution). Fig. 3(a) illustrates Weyl distribution of Gaussian signal that is formed from the mutual distribution (51) when the value of mutual parameter $t = 0$. Increase of value of mutual parameter till $t = 0, 1$ leads to the curve of the mutual distribution at informational diagram (Fig.3(b)). When $t = 0, 25$ beside curve also starts the process of deformation that leads to the transformation of Weyl distribution into Wigner distribution (Fig.3(c)). The peculiar is the value of parameter $t = 0, 5$. mutual distribution in this case is placed precisely in the middle between limiting cases of Weyl and Wigner distributions (Fig.5). In the process of increase of the mutual parameter $t$ a transformation of Weyl distribution into Wigner distribution takes place by means of change of the curve counterclockwise to the mutual space-frequency distribution. In the limiting case $t = 1$ from the mutual distribution Wigner distribution is formed (Fig.4(a)). When the mutual parameter $t$ decreases in the region of Wigner distribution the curve at the informational diagram of joined coordinates $(x, p)$ is changed (Fig.4(b,c)). In the region of Wigner distribution when the parameter $t$ decreases the curve of mutual distribution is changed clockwise. When the value $t = 0, 5$ the mutual distribution is formed what can be observed at Fig.5. Thereby, we come to conclusion that in the process of changing the parameter $t$ of the mutual distribution the Weyl and Wigner distributions move towards one another at informational diagram and are put in equilibrium in the point $t = 0, 5$ (Fig.5). It can be stated that Wigner distribution is formed as a change of curve of Weyl distribution at informational diagram for the angle $\theta < 90^0$. Similarly, Weyl distribution is formed as a change of curve of Wigner distribution in the contrary way. It should be noticed that in the mutual region the distribution becomes complex one (Fig.6(b-d)), (Fig.7(b-d)). However, in the known limiting cases only the real part has a contribution: $t = 0$ (Fig.6(a)),(Fig.7(a)) and $t = 1$ (Fig.6(e)),(Fig.7(e)).

In this paper we propose new space-frequency distribution (16), which could play an important role in the optical information processing schemes describing.

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Figure 1: Real and Image parts of the Gaussian mutual distribution constant $C_{\theta}$. 
Figure 2: Redistribution of the Gaussian mutual distribution from Weyl to Wigner domain.
Figure 3: Rotational displayment of the Gaussian mutual distribution in the Weyl domain at different values of mutual parameter $t$ (Real part).
Figure 4: Rotational displacement of the Gaussian mutual distribution in the Wigner domain at different values of mutual parameter $t$ (Real part).
Figure 5: Mutual distribution of the Gaussian signal at the value of mutual parameter $t = 0.5$ (Real part).
Figure 6: Mutual space-frequency distribution in direction $x = 0$ of Gaussian signal at different values of mutual parameter $t$, solid line real part and dot line image part.
Figure 7: Mutual space-frequency distribution in direction $p = 0$ of Gaussian signal at different values of mutual parameter $t$, solid line real part and dot line image part.