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\textbf{Thermodynamic stability of charged BTZ black holes: Ensemble dependency problem and its solution}

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\textbf{Abstract} Motivated by the wide applications of thermal stability and phase transition, we investigate thermodynamic properties of charged BTZ black holes. We apply the standard method to calculate the heat capacity and the Hessian matrix and find that thermal stability of charged BTZ solutions depends on the choice of ensemble. To overcome this problem, we take into account cosmological constant as a thermodynamical variable. By this modification, we show that the ensemble dependency is eliminated and thermal stability conditions are the same in both ensembles. Then, we generalize our solutions to the case of nonlinear electrodynamics. We show how nonlinear matter field modifies the geometrical behavior of the metric function. We also study phase transition and thermal stability of these black holes in context of both canonical and grand canonical ensembles. We show that by considering the cosmological constant as a thermodynamical variable and modifying the Hessian matrix, the ensemble dependency of thermal stability will be eliminated.

\textbf{1 Introduction}

Discovery of the three dimensional BTZ (Banados-Teitelboim-Zanelli) black holes is accounted as one of the greatest achievements in gravitational framework \cite{1}. These solutions provide simplified machinery that enables us to investigate different aspects of the black objects, such as their thermodynamics \cite{2,3,4}. The generalization of these black holes to higher dimensions has been employed to investigate the black holes solutions and their thermodynamics \cite{5}. In addition, BTZ black holes have been used to expand our understanding of gravitational interaction in low dimensional spacetimes \cite{6}. In context of the string theory and quantum gravity, there have been several studies regarding the effective action of the string theory and BTZ black holes \cite{7}. Due to the fact that these solutions being asymptotically AdS, there has been various studies regarding AdS/CFT correspondence \cite{8}. In addition, the noncommutative geometry in three dimensions has been of interest recently \cite{9} and it was shown that gravitational Aharonov-Bohm effect could be originated from noncommutative BTZ black holes \cite{10}. Moreover, in order to obtain quantum aspect of three dimensional gravity, entanglement and quantum loop entropy of the BTZ black holes have been investigated in literatures \cite{11}. Also, the holography of the BTZ black holes has been investigated in details \cite{12}.

Nonlinear theories have been of interest due to their special properties. Most of natural systems have nonlinear behavior and usually governed by nonlinear theories. Among these theories, the nonlinear

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electrodynamics, specially Born-Infeld (BI) types, have been of interest [13]. BI electrodynamics first has been introduced in order to solve the self energy of charged point like particles in Maxwell theory [14]. It was also shown that this theory enjoys the absence of the shockwave [15]. On the other hand, it was also proven that there is no birefringence phenomena in this nonlinear theory [16]. Another important property of the BI types of electromagnetic field is in context of the string theory. It was shown that the Lagrangian of these theories may be arisen in limit of low energy effective of heterotic string theory [18]. One of the interesting properties of the BI types theories is the fact that their series expansion for large values of the nonlinearity parameter yields the same structure; The first two terms of their series expansions are Maxwell invariant and a quadratic Maxwell invariant coupled with first order nonlinearity parameter with different coefficients [19]. Therefore, one can construct a generalized theory of nonlinear electromagnetic field consisting the quadratic Maxwell invariant in addition to the Maxwell Lagrangian [20]. Besides, it is arguable that in generalizing linear Maxwell theory to nonlinear one, we should take only small values of nonlinearity into consideration. Also, in context of experimental results, only small power of nonlinearity are applicable. Another interesting property of this theory is the fact that we are dealing with nonlinearity as a perturbation. In other words, we are considering additional term as a perturbation to the Maxwell Lagrangian. Therefore, we are dealing not only with a nonlinear theory but also a perturbative one too.

The concept of black holes as thermodynamical systems enables one to consider various thermodynamical aspects of solutions. One of the most important thermodynamical properties of the black holes is their thermal stability. For black holes being physical objects, they should be stable in context of dynamical and thermodynamical frameworks. The instability of black holes means whether the system is completely non physical or it may have phase transition. In other words, the system may go under phase transition to acquire a stable state. The stability of BTZ black holes and black string have been studied in several articles [21]. Myung showed that there is a possible phase transition between non-rotating BTZ black holes and a thermal AdS spacetime [22]. In Ref. [23] Hawking-Page phase transition of BTZ black holes and thermal soliton were studied. Regarding the BTZ black holes with torsion, it was shown that phase transition depends on theory of gravity under consideration [24].

The rest of the paper is organized as follows. In the next section, we present a brief review of the Lagrangian and related field equations of Einstein-Maxwell gravity as well as Einstein-nonlinear Maxwell theory. In Sec. 3 we investigate thermodynamics properties of charged BTZ black holes and discuss ensemble dependency. Then, we present a suggestion for overcoming the mentioned dependency. We generalize our results in Sec. 4 to the case of nonlinear electromagnetic field. It means that we study thermal stability of these black holes in context of both canonical and grand canonical ensembles, and suggest that in order to avoid ensemble dependency, we should consider the cosmological constant as a thermodynamical variable. We finish our paper with the highlight of conclusions.

2 General formalism and field equations

In order to study 3-dimensional black holes in the presence of a matter field, we employ the following Lagrangian

\[ L_{\text{tot}} = R - 2\Lambda + L_m, \]

where \( R \) and \( \Lambda \) are, respectively, the Ricci scalar and the (negative) cosmological constant. The last term in Eq. (1) is the Lagrangian of matter field, which we choose an electromagnetic field. The usual linear electrodynamics is Maxwell field with the following Lagrangian

\[ L(F) = -F, \]

where the Maxwell invariant is \( F = F_{ab}F^{ab} \) in which \( F_{ab} = \partial_a A_b - \partial_b A_a \) is the electromagnetic field tensor and \( A_a \) is the gauge potential. In addition to the linear Maxwell field, one can use the nonlinear models. Nonlinear models of electrodynamics were introduced with different motivations. In this paper, we consider a quadratic correction in addition to the Maxwell Lagrangian to obtain the following nonlinear electrodynamics

\[ L(F) = -F + \beta F^2 + O(\beta^2), \]
where \( \beta \) is nonlinearity parameter. As one can see for vanishing nonlinearity parameter, Maxwell theory of electromagnetic field is recovered. Using variational principle and varying Lagrangian \( L \) with respect to metric tensor and gauge potential, we can find the following field equations

\[
R_{ab} - \frac{1}{2} R g_{ab} + A g_{ab} = \frac{1}{2} g_{ab} L(F) - 2 L g_{ac} F_{bc},
\]

\[
\partial_a \left( \sqrt{-g} L F^{ab} \right) = 0,
\]

where \( L \) is derivation with respect to Maxwell invariant. In order to obtain 3-dimensional solutions, one can employ the following static metric ansatz

\[
ds^2 = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2.
\]

Regarding the structure of the electromagnetic field, we are interested in radial electric field. Therefore, we consider the gauge potential as

\[
A^a = h(r) \delta^a_0,
\]

which results into the following nonzero components of electromagnetic field

\[
F_{tr}(r) = - F_{rt}(r).
\]

### 3 Thermodynamic properties of charged BTZ black hole

As we mentioned in Sec. 1, static BTZ black hole solutions in the presence of Maxwell field have been obtained before \[1\]. In addition, thermodynamic properties \[2\] and heat capacity \[3\] of charged BTZ black holes have been investigated in literature. In other words, thermal stability of the solutions has been investigated in canonical ensemble. In this section, we review the BTZ black hole solutions in the presence of Maxwell theory. Then, we show that there is a case of ensemble dependency in studying thermal stability of the solutions which will be removed by special consideration of cosmological constant as a thermodynamical variable.

In order to obtain electric potential, one should use Eqs. \(5\) and \(7\) with considering the linear Lagrangian of Maxwell theory, \(2\) and 3-dimensional metric \(6\). These considerations yield the following forms for the electromagnetic potential and the electric field

\[
h(r) = q \ln \left( \frac{r}{l} \right),
\]

\[
F_{tr} = E(r) = \frac{q}{r},
\]

in which \( q \) is integration constant related to total charge of black holes. It is notable that we inserted a scale factor, \( l \), into Eq. \(9\) to obtain dimensionless argument for the logarithmic function. Although this scale parameter does not affect the consistency of field equations, we will find that it has a decisive rule for removing ensemble dependency. Considering obtained electromagnetic tensor, one can show that \( tt \) and \( rr \) components of Eq. \(4\) yield the same differential equations. We find

\[
\dot{j}_{rr} = \dot{j}_{tt} = r^2 f' + 2 Ar^3 + 2q^2 r = 0,
\]

\[
\dot{j}_{tt} = r^2 f'' + 2 Ar^2 - 2q^2 = 0,
\]

where prime and double prime denote first and second derivative with respect to \( r \), respectively. It is notable that \( j_{tt} = j_{tr} - \frac{1}{2} j_{tt} \), and therefore, the solutions of Eq. \(11\) satisfy Eq. \(12\), simultaneously. The consistent solution is \[1\]

\[
f(r) = -Ar^2 - 2q^2 \ln \left( \frac{r}{l} \right) - m,
\]

where the integration constant \( m \) is the geometrical mass of black holes.
In order to interpret the solution as a black hole, we should obtain its singularity(ies) and horizon(s). For investigating the existence of the singularity, we obtain the Kretschmann and Ricci scalars which yield the following relations

\[ R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 12\Lambda^2 + \frac{8q^2A}{r^2} + \frac{12q^4}{r^4}, \]  
\[ R = 6\Lambda + \frac{2q^2}{r^2}. \]

Taking into account Eqs. (14) and (15), it is evident that there is an essential singularity at the origin. Besides, considering the behavior of the Kretschmann and Ricci scalars at large values of \( r \), one can confirm that obtained solution is asymptotically AdS. Now we should discuss the existence of horizon. Plotting metric function versus \( r \), we find that the number of the metric function’s roots (horizons) is a function of free parameters (see Fig. 1). Fixing \( l \) and \( q \), we find that for sufficiently small geometrical mass, there is no root and obtained solutions are presenting naked singularity. There is an extremal mass \( m_{ext} \) in which only one extreme root is observed. Increasing this value lead to a black hole with two horizons (one inner (Cauchy) horizon and one outer (event) horizon).

In order to find the values of the horizon radius, one should consider the vanishing metric function at the horizon. Therefore, for the case of \( f(r_+) = 0 \), one can find the following relation

\[ m + Ar_\pm^2 + 2q^2\ln\left(\frac{r_+}{r_-}\right) = 0, \]  

in which \( r_- \) and \( r_+ \) are the horizon radii with the following explicit forms

\[ r_\pm = l\exp\left[-\frac{m}{2q^2} - \frac{L_{W_{\pm}}}{2}\right], \]

where \( L_{W_{\pm}} = LambertW\left[\frac{4\Lambda^2}{q^4}\exp\left(-\frac{m}{q^4}\right)\right] \) and \( L_{W_-} = LambertW\left[-1, \frac{4\Lambda^2}{q^4}\exp\left(-\frac{m}{q^4}\right)\right] \) (for more details about \( LambertW(x) \) function see Ref. [25]).

Now, we are in position to investigate the thermodynamical properties of the solution and study its thermal stability in context of both canonical ensemble (heat capacity) and grand canonical ensemble (Hessian matrix).
3.1 Thermodynamics and conserved quantities

Using area law for calculating the entropy of the system in Einstein gravity \[19\], leads to the following result

\[ S = \frac{\pi r_+^2}{2}. \quad (18) \]

On the other hand, considering the Gauss’s law for obtaining total electric charge leads to

\[ Q = \frac{q}{2}. \quad (19) \]

while for the electric potential, one can use the following standard relation \[19\]

\[ \phi = A_{\mu} \chi^\mu|_{r \to \text{reference}} - A_{\mu} \chi^\mu|_{r \to r_+} = -q \ln \left( \frac{r}{r_+} \right). \quad (20) \]

In order to calculate the temperature of the black hole, we use the definition of the surface gravity, which in the case of our static black hole will be \[2\]

\[ T_\gamma = \frac{f'(r_+)}{4\pi} = \frac{Ar_+^2 + q^2}{2\pi r_+}. \quad (21) \]

Due to our interest in AdS solutions and inspired by AdS/CFT correspondence, we use the counterterm method in order to calculate finite total mass of black hole. As one can see, evaluating action of this configuration at infinity, yields infinite value. In order to overcome this problem, we add additional terms to action. One can show that for the obtained solution with flat boundary, \( R_{abcd}(\gamma) = 0 \), the finite action is

\[ I_{\text{finite}} = I_G + I_b + I_{ct}, \quad (22) \]

where the bulk, boundary and counterterm actions are, respectively,

\[ I_G = -\frac{1}{16\pi} \int_M d^3x \sqrt{-g} L_{\text{tot}}, \quad (23) \]
\[ I_b = -\frac{1}{8\pi} \int_{\partial M} d^2x \sqrt{-\gamma} K, \quad (24) \]
\[ I_{ct} = -\frac{1}{8\pi} \int_{\partial M} d^2x \sqrt{-\gamma} \left( \sqrt{-\Lambda} \right), \quad (25) \]

which \( \gamma \) and \( K \) are, respectively, the traces of the induced metric, \( \gamma_{ab} \), and the extrinsic curvature, \( K^{ab} \), on the boundary \( \partial M \). Using the Brown–York method of quasilocal definition with Eq. \[22\] - \[25\], one can introduce divergence-free stress-energy tensor as follow

\[ T^{ab} = \frac{1}{8\pi} \left[ \sqrt{-\gamma} \left( K^{ab} - \gamma^{ab} \right) + \sqrt{-\Lambda} \gamma^{ab} \right]. \quad (26) \]

Then, the quasilocal conserved quantities associated with the stress tensor of Eq. \[26\] can be written as

\[ Q(\xi) = \int_B d\theta \sqrt{-\gamma} T_{ab} n^a \xi^b, \quad (27) \]

where \( n^a \) is the timelike unit normal vector to the boundary \( B \). Taking into account Eq. \[27\] with \( \xi = \partial / \partial t \) as a Killing vector, we find following result for obtained solution

\[ M = \frac{m}{8}. \quad (28) \]
3.2 Thermal stability of charged BTZ black hole

In order to investigate thermal stability of the black holes, one can adopt two different approaches to the matter at hand. In one, the electric charge is considered as a fixed parameter and heat capacity of the black hole will be calculated. The positivity of the heat capacity is sufficient to ensure the local thermal stability of the solutions. This approach is known as canonical ensemble. In this case, the system which is unstable may go under phase transition to stabilize. The phase transition points are where the heat capacity has root(s) or diverges. Another approach for studying thermal stability of the black holes is grand canonical ensemble. In this approach, the thermal stability is investigated by calculating the determinant of Hessian matrix of \( M(S, Q) \) with respect to its extensive variables. The positivity of this determinant also represents the local stability of the solutions. In what follows, we study stability of the solution in context of both ensembles. Before we conduct our study, it is worthwhile to mention an important point. In some cases, these two approaches may admit the stable phase for black holes. But studying the positivity of these two quantities is not sufficient for ensuring the physical thermodynamical behavior of the system. In order to have more realistic results and also enriching them, studying the behavior of the temperature is necessary, simultaneously. In other words, positivity of the temperature may denote the systems being physical whereas its negativity is representing non-physical systems.

For the canonical ensemble, the heat capacity is \( C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q \), where by using chain rule, one can rewrite it in the following form

\[
C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = T \left( \frac{\partial S}{\partial r} \right)_Q + \left( \frac{\partial T}{\partial r} \right)_Q \left( \frac{\partial T}{\partial r} \right)_Q.
\]

(29)

It is a matter of calculation to show that the heat capacity for charged BTZ black hole will be

\[
C_Q = \pi r_+ \left( \frac{Ar_+^2 + q^2}{2 (Ar_+^2 - q^2)} \right).
\]

(30)

It is evident from the obtained relation for heat capacity that, in case of the asymptotically AdS spacetime \( (\Lambda < 0) \), the denominator of \( C_Q \) is negative. Therefore, in order to have positive heat capacity (stable charged BTZ black hole), the following inequality must be hold

\[
\Lambda < -\frac{q^2}{r_+^2}.
\]

(31)

In order to have type one phase transition \((C_Q = 0)\), we obtain the following phase transition point

\[
r_{1+} = \frac{q}{\sqrt{-\Lambda}}.
\]

(32)

In addition, we should note that only in case of asymptotically AdS spacetime the phase transition type one exists. As for the phase transition type two \((C_Q \to \infty)\), one can find divergence points of the heat capacity by the following relation

\[
r_{2+} = \frac{q}{\sqrt{\Lambda}}.
\]

(33)

To summarize, we interestingly find that the phase transitions for asymptotically AdS or dS solutions are, respectively, the first type (vanishing point of heat capacity) or second type (singular point of heat capacity). In other words, there is no divergency for the heat capacity in case of asymptotically AdS spacetime.

Now, we plot various diagrams in order to investigate thermal stability of charged BTZ solution in context of heat capacity. Fig. \( \text{[2]} \) shows that there is a critical value \((r_{1+})\) in which for \( r_+ < r_{1+} \) the temperature (and the heat capacity) is negative, and therefore, the black hole is non-physical. Otherwise, the system has positive heat capacity and temperature. In other words, the black hole is physical and in thermally stable phase for \( r_+ > r_{1+} \).
Fig. 2  \(C_Q\) and \(T\) (bold lines) versus \(r_+\) for \(A = -1\), and \(q = 1\) (solid line), \(q = 1.2\) (dotted line) and \(q = 1.4\) (dashed line).

Now, we study the stability of solutions in the case of grand canonical ensemble. In this case, the standard extensive parameters are charge and entropy. Therefore, one can see the mass of the black hole will be in the following form

\[
M(S, Q) = -\frac{1}{2} \left[ \frac{\Lambda S^2}{\pi^2} + 2Q^2 \ln \left( \frac{2S}{\pi l} \right) \right],
\]

(34)

and one can obtain related Hessian matrix with the following form

\[
\mathbf{H}^M_{S,Q} = \begin{bmatrix}
-\frac{\Lambda S^2 + Q^2 \pi^2}{2Q^2} & -\frac{2Q}{S^2} \\
-\frac{2Q}{S^2} & \ln \left( \frac{\pi^2 l^2}{2Q^2} \right)
\end{bmatrix},
\]

(35)

which by using Eqs. 18 and 19 its determinant will be

\[
|\mathbf{H}^M_{S,Q}| = \frac{(Ar_+^2 - q^2)}{\pi^2 r_+^4} \ln \left( \frac{r_+^2}{l^2} \right) - \frac{4q^2}{\pi^2 r_+^4}.
\]

(36)

Now, we plot Fig. 3 for studying thermal stability in context of grand canonical ensemble (Eq. 36).

It is evident that there is a case of ensemble dependency. In other words, considering the same parameters, these two approaches for studying thermal stability do not yield the same result.

The ensemble dependency may be originated from one of the following reasons: first of all, because of the dependency of the Hessian matrix to thermodynamical variables, our choices of thermodynamical variables may be wrong. In other words, total mass of the black hole may depend on more extensive parameters that we had taken into account. Second, the system may be constructed in a way that being ensemble dependent is a part of its fundamental properties. In this paper, we take the first assumption into account and try to solve the ensemble dependency problem.

Motivated by recent analogy in which (negative) cosmological constant is considered as a thermodynamical variable, we take cosmological constant as an extensive parameter. Therefore, the Hessian matrix from \(2 \times 2\) changes into a \(3 \times 3\) one. As we pointed out, we inserted the scale factor, \(l\), into the solutions for obtaining dimensionless argument for the logarithmic function and in general it is independent of cosmological constant. But since dimension of cosmological constant is \((\text{length})^{-2}\) and in order to overcome ensemble dependency, we use the following logical relation for the scale factor

\[
l^{-1} = \sqrt{-\Lambda}.
\]

(37)
By this consideration, the structure of Hessian matrix for our study will be

$$H_{S,Q,\Lambda}^M = \begin{bmatrix}
\frac{-A S^2 + Q^2 \pi^2}{\pi^2 S^2} & \frac{-2Q}{S} & \frac{-S}{\pi^2} \\
\frac{-2Q}{S} & \ln \left( \frac{-A r^2 + q^2}{4\pi}\right) & \frac{-Q}{A} \\
\frac{-S}{\pi^2} & \frac{-Q}{A} & \frac{Q^2}{4\pi^2}
\end{bmatrix},$$

(38)

with the following determinant

$$|H_{S,Q,\Lambda}^M| = \frac{\left(2A r^2 - q^2\right) \ln \left(-A r^2 + q^2\right) - 6q^2}{8\pi^2 A^2 r^2},$$

(39)

Now, in order to study the behavior of Eq. (39), we plot Fig. 4. It is clearly seen that the ensemble dependency is solved by this consideration. It is true that one may argue that for small values of the horizon radius there is a region in which the determinant of the hessian matrix is positive, hence black hole is stable which is in contradiction to earlier result that was derived for heat capacity. It is notable that this region is located at the place where the system has negative temperature. Therefore, we take away this region for the reason of non-physical black hole solution.

### 4 Generalization to nonlinear electrodynamics

In this section, we consider the generalization of linear electrodynamics to nonlinear one by adding the quadratic Maxwell invariant to the Maxwell Lagrangian. Using Eqs. (5) and (7) with Lagrangian of nonlinear electromagnetic field (3) and metric (6), one finds

$$h(\rho) = q \ln \left(\frac{r}{\rho} \right) + \frac{2q^3}{r^2} \beta + O \left( \beta^2 \right),$$

(40)

in which $q$ is integration constant related to total electric charge of the solution. Using obtained function for the electric potential results into electric field $F_{tr}$ to be

$$F_{tr} = E(r) = \frac{q}{r} - \frac{4q^3}{r^3} \beta + O \left( \beta^2 \right),$$

(41)
which for small values of $\beta$, obtained relations reduce to linear Maxwell case.

Now, we are in a position to obtain metric function of this nonlinear theory up to first order of nonlinearity parameter. To do so, we use Eq. (4) by applying electromagnetic field tensor (41) which will lead to the following differential equations

$$\begin{align*}
e_{rr} &= e_{tt} = r^2 f'' + 2 A r^3 + 2 q^2 r - \frac{4 q^4}{r} \beta + O (\beta^2), \\
e_{\theta\theta} &= r^2 f'' + 2 A r^2 - 2 q^2 + \frac{12 q^4}{r^2} \beta + O (\beta^2).
\end{align*}$$

(42)

(43)

It is easy to show that $e_{\theta\theta} = e_{tt} - \frac{2}{r} e_{rr}$, and therefore, it is sufficient to solve Eq. (42). One can show that metric function for this configuration will be

$$f(r) = - m - A r^2 - 2 q^2 \ln \left( \frac{r}{l} \right) - \frac{2 q^4}{r^2} \beta + O (\beta^2),$$

(44)

where $m$ is an integration constant which is related to total mass.

The next step will be devoted to study the possibility of existence of singularity and horizon for the solution. In other words, obtained solution may be interpreted as a black hole if there exists a curvature singularity covered with horizon. In order to investigate the existence of singularity, one can study curvature scalars such as Kretschmann and Ricci scalars. Studying these scalars enable us to investigate the asymptotical behavior of the solution too. Evaluating Kretschmann and Ricci scalars result into

$$R_{\alpha\beta\gamma\delta} R^\alpha^\beta^\gamma^\delta = 12 A^2 + \frac{8 q^2}{r^2} A + \frac{12 q^4}{r^4} + \frac{16 q^6}{r^6} \left( A - \frac{5 q^2}{r^2} \right) \beta + O (\beta^2),$$

(45)

$$R = 6 A + \frac{2 q^2}{r^2} + \frac{4 q^4}{r^4} \beta + O (\beta^2),$$

(46)

which confirm that there is an essential singularity located at $r = 0$. In addition, for large values of $r$, the Kretschmann and Ricci scalars yield $12 A^2$ and $6 A$, respectively, which confirm that obtained solution is asymptotically AdS. In order to investigate the existence of horizon, one should consider vanishing metric function at the horizon. Considering $f (r_+) = 0$, one can write

$$m + A r_+^2 + 2 q^2 \ln \left( \frac{r_+}{l} \right) + \frac{2 q^4}{r_+^2} \beta = 0.$$  

(47)
Obtaining analytical solutions of Eq. (47) with respect to $r+$ is not easy. So in order to investigate the horizons, we plot some graphs for the metric function (see Figs. 5 and 6). In addition, for studying the effects of the additional correction ($\beta$-term), we plot Fig. 6 for metric function versus radial coordinate.

As one can see, for small values of geometrical mass (Fig. 5), the metric function has only one root which is located at small values of $r$. In general the effect of mass on number of horizon(s) could be divided into three regions that are denoted by two critical values for the mass parameter; $m_1$ and $m_2$ in which $m_1 < m_2$. For $m = m_1$ and $m = m_2$, two roots exist. In case of $m_1 < m < m_2$, there are three roots for metric function. Otherwise, metric function has only one root. The location of outer (event) horizon is an increasing functions of geometrical mass.

Next, due to our interest in nonlinear theory that was used in this paper, we plot the metric function diagrams for variation of the $\beta$ (Fig. 6). It is evident from studying Fig. 6 that the effect of the nonlinearity parameter is the same as geometrical mass and similar behavior is observed.
4.1 Thermodynamics and conserved quantities

Here, we are interested in studying thermodynamic properties of obtained black hole and calculating conserved quantities. First of all as one can see, the metric, that was employed for obtaining solution, contains a Killing vector field which is temporal ($\chi^\mu = \delta^0_\mu$). Considering the concept of surface gravity, one finds the temperature of obtained black hole solution

$$T = \frac{1}{2\pi} \sqrt{-\frac{1}{2} \left( \nabla_\mu \chi^\nu \right) \left( \nabla^\mu \chi^\nu \right)} = -\frac{r_+ A + r_+^2 q^2 + 2q^4 \beta}{2\pi r_+} + O(\beta^2). \quad (48)$$

The next step is devoted to calculate the entropy of black holes. Since we are working in the Einstein gravity, it is allowed to employ the area law for calculating the entropy. By doing so, one can find the same relation such as that in Eq. (18). It is notable that although entropy has the same form in both Maxwell theory and NLED, $r_+$ in NLED is different from that in Maxwell one. Considering the flux of the electric field at infinity leads to similar relation for the electric charge as we obtained in Eq. (19). It means that the nonlinearity does not affect the total electric charge of black hole. This behavior was also observed in other BI type theories of electrodynamics [19].

Next, we calculate the electric potential, $\Phi$, at the horizon with respect to a reference with vanishing electric potential

$$\Phi = A_\mu \chi^\mu|_{r \rightarrow r_{\text{reference}}} - A_\mu \chi^\mu|_{r \rightarrow r_+} = -q \left( \ln \left( \frac{r_+}{r} \right) - \frac{2q^2}{r_+} \beta \right) + O(\beta^2). \quad (49)$$

Following the same approach and applying the same counterterm action, we find that the nonlinearity does not change the form of finite mass. Therefore, we obtain the finite mass as

$$M = \frac{m}{8}, \quad (50)$$

where one can obtain $m$ through Eq. (17).

Now, we are in a position to study the validation of the obtained thermodynamic quantities, through the first law of thermodynamics. Using Eqs. (18), (19), (50) and considering $M$ as a function of extensive parameters $S$ and $Q$, we obtain

$$M(S, Q) = -\frac{1}{2} \left[ \frac{A S^2}{\pi^2} + 2Q^2 \ln \left( \frac{2S}{\pi l} \right) + \frac{2\pi^2 Q^4}{S^2} \beta \right] + O(\beta^2), \quad (51)$$

where we considered $\beta$, $l$ and $\Lambda$ as a fixed parameter. According to the first law of thermodynamics, we should check the following relation

$$dM = T dS + \Phi dQ. \quad (52)$$

It is straightforward to calculate $(\partial M / \partial Q)_S$ and $(\partial M / \partial S)_Q$ and check their results to be in agreement with what were previously obtained in Eqs. (18) and (19), respectively. Therefore, obtained conserved and thermodynamic quantities satisfy the first law of thermodynamics.

According to the mentioned discussion for considering cosmological constant as a thermodynamical variable, one should generalize $M(S, Q)$ to $M(S, Q, \Lambda)$. Therefore Eqs. (51) and (52) will be modified as

$$M(S, Q, \Lambda) = -\frac{1}{2} \left[ \frac{A S^2}{\pi^2} + 2Q^2 \ln \left( \frac{2S}{\pi \sqrt{-\Lambda}} \right) + \frac{2\pi^2 Q^4}{S^2} \beta \right] + O(\beta^2). \quad (53)$$

and

$$dM = T dS + \Phi dQ + \Theta d\Lambda, \quad (54)$$

where

$$T = \left( \frac{\partial M}{\partial S} \right)_{Q, \Lambda}, \quad \Phi = \left( \frac{\partial M}{\partial Q} \right)_{S, \Lambda}, \quad \Theta = \left( \frac{\partial M}{\partial \Lambda} \right)_{S, Q}.$$
4.2 Thermal stability

Using Eqs. (18), (19) and (51), one can find heat capacity of nonlinearly charged black hole solution

$$C_Q = \frac{\pi r_+ \left( A r_+^2 + q^2 - \frac{2q^4 \beta}{r_+^2} \right)}{2 \left( A r_+^2 - q^2 + \frac{6q^4 \beta}{r_+^2} \right)}.$$  \hspace{1cm} (55)

In order to investigate the type one phase transition points, one should find the roots of the numerator of the heat capacity

$$A r_+^4 + q^2 r_+^2 - 2q^4 \beta = 0.$$  \hspace{1cm} (56)

Solving Eq. (56), one finds two real positive roots in which type one phase transition takes place

$$r_+|_{C_Q=0} = q \left( \sqrt[4]{\frac{\pm \chi - 1}{2 \Lambda}} \right).$$  \hspace{1cm} (57)

where $\chi = \sqrt{1 + 8A \beta}$. Due to square root function and our interest in AdS spacetime, we have the restriction in which $\beta \leq -1/8A$. Next, as for the type two phase transitions, one can find roots of denominator of the heat capacity

$$A r_+^4 - q^2 r_+^2 + 6q^4 \beta = 0,$$  \hspace{1cm} (58)

which results into one real positive root

$$r_+|_{C_Q\to\infty} = q \left( \sqrt[4]{\frac{1 - \lambda}{2A}} \right),$$  \hspace{1cm} (59)

where $\lambda = \sqrt{1 - 24A \beta}$. Overall by what was mentioned, one can conclude that there are three phase transition points for these black holes: two of these phase transitions are type one and the other one is type two. We plot two diagrams for studying thermal stability in context of canonical ensemble (Figs. 7 and 8).

It is evident from studying the diagrams (Figs. 7 and 8) that in this case (NLED source), we are dealing with two types of phase transitions. One is related to the root(s) of the heat capacity and the other one is related to divergence points of the heat capacity. As one can see, there is a region...
Fig. 8 Different scales of $C_Q$ and $T$ (bold lines) versus $r_+$ for $q = 1$, $\Lambda = -1$, and $\beta = 0.001$ (solid line), $\beta = 0.003$ (dotted line) and $\beta = 0.006$ (dashed line).

$(r_+ < r_{1+}$ where $r_{1+}$ is the smaller root of the heat capacity) in which heat capacity is negative and system is unstable. In this region the temperature is positive. In $r_{1+} < r_+ < r_{d+}$ (where $r_{d+}$ is divergence point), the heat capacity is positive but the temperature is negative. Therefore, the system is not in a real physical stable condition. Both the temperature and the heat capacity are negative in the interval $r_{d+} < r_+ < r_{2+}$ in which $r_{2+}$ is the larger root of heat capacity. Finally, the system acquires a physically thermal stable state in $r_{2+} < r_+$. In other words, in this region both heat capacity and temperature are positive and the system is in thermal stable state. It is notable that the divergence point of the heat capacity is located in the region in which temperature is negative. Therefore, it is not a physical phase transition. As one can see, in case of variation of electric charge (Fig. 7), the roots and the divergence points of the heat capacity are increasing functions of electric charge. For the nonlinearity parameter, the smaller root and divergence point are increasing functions of $\beta$ (Fig. 8: left) whereas the larger root is a decreasing function of $\beta$ (Fig. 8: right). This shows that for increasing $\beta$, the interval between larger root and divergence point ($r_d < r_+ < r_2$) decreases and a compactification takes place.

Next, we investigate determinant of Hessian matrix in the context of the grand canonical ensemble, in which finite mass of the black hole is a thermodynamical potential with two extensive parameters, $S$ and $Q$. This assumption leads to following form of Hessian matrix

$$H_{S,Q}^M = \begin{bmatrix}
-\frac{A S^2 + Q^2 S^2 \pi^2 \sigma^2 - 6 \sigma^4 Q^4 \beta}{\pi^2 S^4} & \frac{2Q(4\pi^2 Q^2 \beta - S^2)}{S^2} \\
2Q(4\pi^2 Q^2 \beta - S^2) & \frac{S^2 \ln \left( \frac{S^2}{\pi^2} \right) - 12 \pi^2 Q^2 \beta}{S^2}
\end{bmatrix},$$

(60)

which by using Eqs. (18) and (19), we find

$$|H_{S,Q}^M| = \frac{(Ar_+^2 - q^2 r_+^2 + 6q^4 \beta)}{\pi^2 r_+^4} \ln \left( \frac{r_+^2}{\sqrt{2}} \right) + \frac{4q^2 \left[ (3Ar_+^2 + 5q^2) \beta - r_+^2 \right]}{\pi^2 r_+^4}.$$

(61)

Using obtained determinant, one can plot its diagrams (Figs. 9 and 10) in order to study thermal stability of the system.

It is evident from studying these diagrams (Figs. 9 and 10), that there is a critical horizon radius, $r_{+c}$, in which for $r_+ < r_{+c}$ the determinant of the Hessian matrix is positive which means the system is in thermal stable state. For the case of $r_+ > r_{+c}$, the determinant of the Hessian matrix is negative.
\( r_{+c} \) is an increasing function of the electric charge and nonlinearity parameter. It is worthwhile to mention that for a region of the \( r_+ \), the temperature and determinant of the Hessian matrix are both positive which in that region the heat capacity is negative. As one can see in this case, thermal stability of the system in canonical and grand canonical ensembles does not have the same result. Therefore, we encounter a case of ensemble dependency.

In order to remove ensemble dependency, we consider the assumption that was used in case of linearly charged BTZ black hole. One can find the mass of black hole as a function of three extensive parameters: electric charge, entropy and cosmological constant. Taking into account this assumption,

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**Fig. 9** Different scales of \( \frac{H_{10}}{10^5} \) and \( T \) (bold lines) versus \( r_+ \) for \( \beta = 0.001, l = 1, A = -1, \) and \( q = 1 \) (solid line), \( q = 1.2 \) (dotted line) and \( q = 1.4 \) (dashed line).

**Fig. 10** Different scales of \( \frac{H_{10}}{10^5} \) and \( 10T \) (bold lines) versus \( r_+ \) for \( q = 1, l = 1, A = -1, \) and \( \beta = 0.001 \) (solid line), \( \beta = 0.003 \) (dotted line) and \( \beta = 0.006 \) (dashed line).
the Hessian matrix for nonlinear charged black hole will be as follow
\[
H^{M}_{S,Q,Λ} = \begin{pmatrix}
-\frac{A S^4 + Q^2 S^2 + 6 \pi^4 Q^4 \beta}{\pi^2 \Lambda^4} & \frac{2Q(4\pi^2 Q^2 - S^2)}{S^4} & -\frac{S}{\pi^2} \\
\frac{2Q(4\pi^2 Q^2 - S^2)}{S^4} & \frac{S^2 \ln \left(\frac{S}{\pi^2} + 12\pi^2 Q^2\beta\right)}{S^4} & -\frac{Q}{\Lambda} \\
-\frac{S}{\pi^2} & -\frac{Q}{\Lambda} & \frac{Q^2}{2\Lambda^2}
\end{pmatrix},
\]

where its determinant is
\[
|H^{M}_{S,Q,Λ}| = \frac{A \ln (-Ar^2_+)^2}{8\pi^2 \Lambda^2 r^4_+},
\]

in which
\[
A = 2\Lambda^2 r^6_+ - q^2 (q^2 r^2_+ - \Lambda r^4_+ - 6q^4 \beta),
\]
\[
B = 16q^6 \beta + (22\Lambda \beta - 3) q^1 r^4_+ + 3 (4\Lambda \beta - 1) Aq^2 r^4_+.
\]

Now, we plot the related diagrams of Eq. (63) and compare them with the results of the heat capacity (see Figs. 7 to 9). As one can see, considering this modification changes the number of the roots of the determinant of the Hessian matrix. In other words, contrary to previous case (Figs. 9 and 10) in which the determinant of the Hessian matrix had only one root, in this case it has two roots (Figs. 11 to 13). We name these two roots \(r_{1+}\) and \(r_{2+}\) in which \(r_{1+} < r_{2+}\). For \(r_{1+} < r_+ < r_{2+}\), determinant of the Hessian matrix is negative, otherwise (\(r_+ < r_{1+}\) and \(r_+ > r_{2+}\)) it is positive. Interestingly, by considering cosmological constant as a thermodynamical variable, the larger root of the heat capacity, temperature and determinant of the Hessian matrix coincide. In other words, for \(r_+ > r_{2+}\) in both canonical and grand canonical ensembles, the system is thermally stable and temperature is positive. Therefore, the ensemble dependency is removed by this consideration.

Considering the cosmological constant also enable us to consider the variation of it in plotting diagrams. As one can see in this case, the smaller root of the determinant of the Hessian matrix and temperature are decreasing functions of the cosmological constant (Fig. 13 left). On the other hand,
the larger root of the determinant of the Hessian matrix and temperature of the system are increasing functions of \( \Lambda \).

Here, we discuss the effects of considering the nonlinearity parameter, \( \beta \), as a thermodynamical variable. In other words, taking into account the Hessian matrix as a function of \( \beta \), one can regard \( H = H(S, Q, \beta) \) or \( H = H(S, Q, A, \beta) \). Considering \((S, Q, \beta)\) and \((S, Q, A, \beta)\) as extensive parameters,
Fig. 14 Different scales of $10^k \times |H_{S,Q,\beta}^M|$ and $T$ (bold lines) versus $r_+$ for $\beta = 0.001$, $l = 1$, $\Lambda = -1$, and $q = 1$ (solid line), $q = 1.2$ (dotted line) and $q = 1.4$ (dashed line). "In order to plot clear figures, we set $k = -1$ and $k = -5$ for left and right figures, respectively"

one finds their related hessian matrices are, respectively,

$$
H_{S,Q,\beta}^M = \begin{bmatrix}
-A S^4 + Q^2 S^2 \pi^2 - 6 e^4 Q^4 \\
2 Q (4 \pi^2 Q^2 \beta - S^2) \\
2 Q^4 S^2
\end{bmatrix}
$$

and

$$
H_{S,Q,\beta,\Lambda}^M = \begin{bmatrix}
-A S^4 + Q^2 S^2 \pi^2 - 6 e^4 Q^4 \\
2 Q (4 \pi^2 Q^2 \beta - S^2) \\
2 Q^4 S^2
\end{bmatrix}
- \begin{bmatrix}
-S \\
-\frac{1}{\pi^2} \ln \left( \frac{x^2}{\pi^2} \right) - \frac{12 \pi \beta Q^2}{S^2} - \frac{4 Q^2 S^2}{S^2} \\
-\frac{Q}{4} \\
0
\end{bmatrix}
$$

Straightforward calculations show that

$$
|H_{S,Q,\beta}^M| = \frac{4 A q^6}{\pi^2 r_+^3} + \frac{q^6}{\pi^2 r_+^3} + \frac{q^6}{\pi^2 r_+^3}
$$

and

$$
|H_{S,Q,\beta,\Lambda}^M| = \frac{4 A q^6}{\pi^2 r_+^3} + \frac{3 q^8}{2 \pi^2 r_+^3} + \frac{q^8}{8 \pi^2 r_+^3} + \frac{q^{10} \beta}{2 \pi^2 r_+^3}
$$

Now, we plot Figs. 14 and 15 to discuss ensemble dependency. Regarding Figs. 14 and 15 we find that considering $\beta$ as a thermodynamical variable (with or without regarding $A$ as a thermodynamical parameter) leads to ensemble dependency. In other words, we conclude that $\beta$ is not a thermodynamical variable.
Fig. 15 Different scales of $10^k \times |H_{S,Q,A,\beta}^r|$ and $T$ (bold lines) versus $r_+$ for $\beta = 0.001$, $\Lambda = -1$, and $q = 1$ (solid line), $q = 1.2$ (dotted line) and $q = 1.4$ (dashed line). "In order to plot clear figures, we set $k = -1$ and $k = -5$ for left and right figures, respectively."

5 Summary and Conclusion

In this paper, we have investigated thermodynamic properties of the charged BTZ black hole and studied thermal stability of it in the context of both canonical and grand canonical ensembles. It was observed that this black hole has two types of the phase transition; one was related to the changing in the signature of the heat capacity (vanishing heat capacity). There was also another type of the phase transition which was related to the divergency of the heat capacity. It was shown analytically that there exists a real positive root and also, a divergence point for the heat capacity.

One of the most interesting results of this paper was ensemble dependency of thermal stability. It was seen that considering the total mass of black hole as a function of two extensive parameters ($S$ & $Q$) will lead to a $2 \times 2$ Hessian matrix. The determinant of this matrix was positive only for a region of small values of horizon radius. In other words, the conditions for thermal stability of the black hole for these two ensembles were different. Therefore, there was a case of ensemble dependency. In order to solve this ensemble dependency, we considered cosmological constant as an extensive thermodynamical parameter. This lead to the Hessian matrix being $3 \times 3$. This consideration resulted into removing ensemble dependency and both ensembles yield the same result. In other words, in both pictures, thermally stable physical black hole only observed for large $r_+$. It is notable that the constant $l$ in the gauge potential and logarithmic part of metric function inserted by hand for the reason of obtaining dimensionless argument for the logarithmic function. Calculations showed that this length parameter should be related to cosmological constant ($l = 1/\sqrt{-\Lambda}$) to remove ensemble dependency. Another important property is that our study support the idea that the cosmological constant is not a fixed parameter, but an extensive thermodynamical variable. In other words, the variation of this parameter, should be taken into account in studying thermodynamical behavior of the system.

In the next part of the paper, we generalized Maxwell theory to the case of NLED. The Lagrangian of the mentioned nonlinear model was a quadratic Maxwell invariant term in addition to the Maxwell Lagrangian. It was seen that the structure of the metric function, number of the horizons and the behavior of the metric function were modified. In context of thermodynamical quantities, no effect on the forms of total mass, electric charge and entropy was observed whereas the temperature and electric potential were modified.

In case of the phase transition in AdS spacetime it was seen that: BTZ chargeless black holes have no phase transitions. Linearly charged ones only have a type one phase transition. Whereas, nonlinearly charged black holes enjoy two types of phase transitions. On the other hand, being dS and
AdS spacetime highly modified the number and the type of phase transitions. This emphasizes the contributions and effects of background spacetime on thermodynamical structure and behavior of the system.

In studying the stability, similar to linearly charged BTZ black holes, ensemble dependency was observed for nonlinearly charged BTZ solutions. In order to remove this ensemble dependency, we employed the method that was used in case of linearly charged BTZ black holes. It was seen that this method successfully removed ensemble dependency. This shows that our consideration of cosmological constant as thermodynamical variable is an acceptable one. In other words, cosmological constant indeed is a thermodynamical variable, not a fixed parameter and its variation should be taken into account in the first law of thermodynamics

\[ dM = TdS + \Phi dQ + \Theta d\Lambda, \]

where

\[ \Theta = \left( \frac{\partial M}{\partial \Lambda} \right)_{S,Q}. \]

We also checked the effects of considering the nonlinearity parameter, \( \beta \), as a thermodynamical variable. We found that regarding \( \beta \) as a thermodynamic (extensive) parameter leads to existence of ensemble dependency. Therefore in order to remove ensemble dependency, we should consider \( (S, Q, \Lambda) \) as a set of extensive parameters.

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