Quadratic supersymmetric transformations of the Dirac Green functions

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Abstract

We consider the quadratic supersymmetric aspect of the Darboux transformation for the Green functions of the one-dimensional Dirac equation with a generalized form of the potential. We obtain the relation between the initial and the transformed Green functions on the whole real line. We also construct the formula for the unabridged trace of the difference of the transformed and the initial Green functions of the boundary problem on the whole real line. We present an example illustrated our developments.

Key words: Dirac equation; Green function; Darboux transformation.

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1 Introduction

The Darboux transformation (DT) [12,3,4] provides a very powerful method for constructing supersymmetric quantum mechanical systems [5,6,7,8] and for finding new solvable potentials both for the Schrödinger equation [9,10,11] and the Dirac equation [12,13,14].

It is well known that supersymmetric quantum mechanics [15] is basically equivalent to DT and the factorization properties of the Schrödinger equation [7,9,10]. The DT of the one-dimensional stationary Dirac equation is equivalent to the quadratic supersymmetry (QS) and the factorization properties of the Dirac equation [13,14,16].

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Despite of the growing number of papers in this area the author are aware of only several papers \cite{17,18,19,20,21,22} devoted to the supersymmetric transformations at the level of the Green functions. In \cite{17} was obtained the integral relation between the Green functions for two supersymmetric partner Hamiltonians of the one-dimensional Schrödinger equation with the discrete spectrum. In \cite{18,19} the integral relation between the Green functions corresponding two supersymmetric partner Hamiltonians of the one-dimensional Schrödinger equation is generated to the case of continuous spectrum. The exact the Schrödinger propagators for the supersymmetric partners are studied \cite{19,20,21}. In \cite{22} the results analogous to \cite{17} was obtained for the Dirac equation. In the present paper, we construct the QS transformations (QST) of the Green function for the whole real line of the one-dimensional stationary Dirac equation with a generalized form of the potential and generalize the results of \cite{22} to the case of continuous spectrum.

The plan of the paper is as follows. In Section 2 we study the QST of the Dirac equation. In Section 3 we consider the Green function of the one-dimensional stationary Dirac equation with a generalized form of the potential for the whole real line. In Section 4 we construct the QST of the Green function. We also obtain the relation between the transformed and the initial Green functions. In Section 5 we get formulas for the unabridged trace of the difference of the modified and the initial Green functions. In Section 6 we illustrate our developments using the free particle examples considered on the whole real line. Consequently, in Section 7 we briefly discuss the results.

2 The QST of the Dirac equation

In this section we will consider the QST of the one-dimensional Dirac equation. We will first look like a particular case of the QST (the matrix $A=I$) \cite{16}:

$$
Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & L^+ \\ 0 & 0 \end{pmatrix}, \quad Q^2 = (Q^+)^2 = 0, \quad (Q, Q^+) \equiv QQ^+ + Q^+Q = (H - \lambda_1 I)(H - \lambda_2 I), \quad H \equiv \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}. \quad (1)
$$

The QST was established on the factorization properties \cite{14,16}:
\[ L^+L = (h_0 - \lambda_1 I)(h_0 - \lambda_2 I), \quad LL^+ = (h_1 - \lambda_1 I)(h_1 - \lambda_2 I), \quad (3) \]

where

\[ L = \partial_x - u_x u^{-1}, h_0 = i\sigma_2 + V_0, h_1 = i\sigma_2 + V_1, V_1 = V_0 + [i\sigma_2, u_x u^{-1}], \quad (4) \]

the matrix \( u \) is the solution of the initial matrix Dirac equation.

Next, let us consider the general case: the matrix \( A \neq I \). Let introduce the matrix with the properties

\[ A^+ = A^{-1}, \quad i\sigma_2 A = A i\sigma_2 \quad (5) \]

and the operator \( \Gamma = AL \).

Consider the operator multiplications \( \Gamma^+ \Gamma, \Gamma \Gamma^+ \):

\[ \Gamma^+ \Gamma = (h_0 - \lambda_1 I)(h_0 - \lambda_2 I), \quad \Gamma \Gamma^+ = A(h_1 - \lambda_1 I)(h_1 - \lambda_2 I)A^+ \quad (6) \]

Taking into account (5), we see that the expression (6) is as follows:

\[ \Gamma \Gamma^+ = A(h_1 - \lambda_1 I)A^{-1}A(h_1 - \lambda_2 I)A^{-1}, \quad (7) \]

where

\[ A(h_1 - \lambda_1 I)A^{-1} = Ah_1 A^{-1} - \lambda_1 I, \quad (8) \]
\[ Ah_1 A^{-1} = i\sigma_2 + AV_0 A^{-1} + A[i\sigma_2, u_x u^{-1}]A^{-1} - i\sigma_2 A_x A^{-1}. \quad (9) \]

Let us denote

\[ H_0 = i\sigma_2 + W_0, H_1 = i\sigma_2 + W_1, \quad (10) \]
\[ W_1 = AW_0 A^{-1} + A[i\sigma_2, u_x u^{-1}]A^{-1} - i\sigma_2 A_x A^{-1}, \quad (11) \]

\[ \mathcal{H} = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 \\ \Gamma & 0 \end{pmatrix}, \quad \Omega^+ = \begin{pmatrix} 0 & \Gamma^+ \\ 0 & 0 \end{pmatrix}. \quad (12) \]

Now it is easy to obtained that the properties for the operators \( \Gamma, \Gamma^+, \mathcal{H}, \Omega, \Omega^+ \) are analogous to the properties for operators \( L, L^+, H, Q, Q^+ \).
\[ \Gamma^+ \Gamma = (H_0 - \lambda_1 I)(H_0 - \lambda_2 I), \quad \Gamma \Gamma^+ = (H_1 - \lambda_1 I)(H_1 - \lambda_2 I), \quad (13) \]

\[ \Omega^2 = (\Omega^+)^2 = 0, \quad \{\Omega, \Omega^+\} \equiv \Omega \Omega^+ + \Omega^+ \Omega = (H - \lambda_1 I)(H - \lambda_2 I). \quad (14) \]

The QS transformed potential (11) early was introduced in [16] as the Darboux transformed potential of the one-dimensional Dirac equation.

3 Green function of the one-dimensional Dirac equation

In this section we consider some properties of the Green functions of the one-dimensional Dirac equation with the generalized form of the potential. We will reconcentrate mostly the case on the whole real line \((a = -\infty, b = \infty)\).

Let us consider the one-dimensional Dirac equation

\[ H_0 \psi = E \psi, \quad x \in (a,b). \quad (15) \]

The Hamiltonian \(H_0\) include the generalized form of the potential

\[ W_0(x) = \omega(x) I + (m + S(x)) \sigma_3 + q(x) \sigma_1, \quad (16) \]

where \(\omega(x) = \phi_{el}(x), S(x) = \phi_{sc}(x), q(x) = k_j/x + \phi_{am}, k_j\) are the eigenvalues of the spin-orbit operator [23], \(m\) is the mass of a particle, \(\sigma_1, \sigma_3\) are usual Pauli matrices. The generalized potential is the self-adjoint potential.

There are exist two representation of the Green function of the one-dimensional Dirac equation. The first representation is obtained with the help of two real solutions \(\psi, \varphi\). The Green function in this case is

\[ G_0(x, y, E) = \frac{1}{W(E)} \begin{cases} 
\psi(x, E) \varphi^T(y, E), & y \leq x, \\
\varphi(x, E) \psi^T(y, E), & x < y,
\end{cases} \quad (17) \]

where \(W(E) = W\{\psi(x, E), \varphi(x, E)\} = \text{const} \) is the Wronskian of the two functions \(\psi(x, E)\) and \(\varphi(x, E)\). Early similar view of the Green functions was obtained in [23] for the Dirac equation with a particular potential.

If we consider the Green functions on the whole real line, then suppose that \(\psi, \varphi\) satisfy the zero boundary conditions: \(\varphi(a, E) = 0, \psi(b, E) = 0\).

The matrix (17) can be written in the form:
\[ G_0(x, y, E) = \frac{\psi(x)\varphi^T(y)\Theta(x - y) + \varphi(x)\psi^T(y)\Theta(y - x)}{W\{\psi(x), \varphi(x)\}}, \] 
\[ (18) \]

where \( \Theta(x - y), \Theta(y - x) \) are the Heaviside step functions.

We suppose that discrete spectrum eigenfunctions \( \{\psi_n\} \), together with continuous spectrum eigenfunctions \( \{\psi_k\} \), \( E = k^2 \), is complete in the Hilbert space

\[ \sum_n \psi^T_n(x)\psi_n(y) + \int dk\psi^T_k(x)\psi_k(y) = \delta(x - y). \]
\[ (19) \]

The spectral representation of the Green function may be found in terms of the complete set as follows:

\[ G_0(x, y, E) = \sum_n \frac{\psi_n(x)\psi^T_n(y)}{E_n - E} + \int \frac{\psi_k(x)\psi^T_k(y)}{k^2 - E} dk. \]
\[ (20) \]

For the spectral problem on the whole real axis the integrals over \( k \) run from minus infinity to infinity.

4 The QST of the Green functions

In this Section using spectral the representation we obtain a relation between the Green functions of the initial and the transformed problems. With the help of the properties of the QST we check that this relation is correct for usual representation of the Green functions of the one-dimensional stationary initial and transformed Dirac equations.

The functions \( \tilde{\psi}_n = \Gamma \psi_n, n = 1, 2, ..., \) describe (unnormalized) bound states and \( \tilde{\psi}_E = \Gamma \psi_E \) correspond to (unnormalized) scattering states of \( H_1 \). The normalization constants are easily obtained with the help of the factorization property (13). The functions

\[ \phi_n = \frac{\Gamma \psi_n}{\sqrt{(E_n - \lambda_1)(E_n - \lambda_2)}}, \quad \phi_E = \frac{\Gamma \psi_E}{\sqrt{(E_n - \lambda_1)(E_n - \lambda_2)}}. \]
\[ (21) \]

form an orthonormal set.

It is well known [16] that there exist kinds of the QST: (i) the transformation deleting the state levels, (ii) the isospectral transformation, (iii) and the transformation creating new state levels.
In the case (i) the spectral representation of the transformed Green function is given by:

\[ G_1(x, y, E) = \sum_{n_1} \frac{\phi_{n_1}(x)\phi_{n_1}^T(y)}{E_n - E} + \int \frac{\phi_k(x)\phi_k^T(y)}{k^2 - E} \, dk. \] (22)

The designation \( n_1 < n_0 \) correspond to numbers of the Hamiltonian’s \( H_1 \) discrete eigenvalues. In the case (ii) \( n_1 \) is equal \( n_0 \). In the case (iii), when the columns of matrix \( (u^+)^{-1} \) are square-integrable, the spectral representation of the transformed Green function is as follows:

\[ G_1(x, y, E) = \sum_{n_1} \frac{\phi_{n_1}(x)\phi_{n_1}^T(y)}{E_n - E} + \sum_m \frac{\phi_{\lambda_m}(x)\phi_{\lambda_m}^T(y)}{\lambda_m - E} + \int \frac{\phi_k(x)\phi_k^T(y)}{k^2 - E} \, dk. \] (23)

If the transformation creates one new state level, then \( m = 1 \). If the transformation creates two new state levels, then \( m = 2 \).

For all free case (i), (ii), (iii) the transformed and initial Green functions are interrelated:

\[ G_1(x, y, E) = \frac{L_x G_0(x, y, E)L_y^T - (\gamma \partial_x + V_1 + E - \lambda_1 - \lambda_2)\delta(x - y)}{(E - \lambda_1)(E - \lambda_2)}. \] (24)

In the fist representation the transformed Green functions looks like as

\[ G_1(x, y, E) = \frac{\tilde{\psi}(x)\tilde{\varphi}^T(y)\Theta(x - y) + \tilde{\varphi}(x)\tilde{\psi}^T(y)\Theta(y - x)}{W\{\tilde{\psi}(x), \tilde{\varphi}(x)\}}. \] (25)

If we consider the Green functions on the whole real line, then suppose that \( \tilde{\psi}, \tilde{\varphi} \) satisfy the zero boundary conditions: \( \tilde{\varphi}(a, E) = 0, \tilde{\psi}(b, E) = 0 \).

For the solutions to the Dirac equation the following properties hold:

\[ \psi\varphi^T - \varphi\psi^T = -W\{\varphi, \psi\}\gamma, \] (26)

\[ W\{\tilde{\varphi}, \tilde{\psi}\} = (E - \lambda_1)(E - \lambda_2)W\{\varphi, \psi\}, \] (27)

\[ (u_xu^{-1} - (u_xu^{-1})^T)\gamma = \lambda_1 + \lambda_2. \] (28)

Since the differential of the Heaviside step function is the delta-function, the properties (26), (27), (28) are fulfilled. Substituting (18) into (24) we obtain Eq. (25).
5 Unabridged trace for the difference of the modified and the initial Green functions

In this Section we consider the unabridged trace for the Green function defined as

$$tr \int_{a}^{b} G(x, y, E)|_{x=y} dy.$$ 

It can be represented in the form:

$$tr \int_{a}^{b} G_1(x, y, E)|_{x=y} dy = \frac{tr \int_{a}^{b} \tilde{\psi} \tilde{\phi} T|_{x=y} dy}{W\{\tilde{\psi}\}}. \quad (29)$$

It is known that the action of the transformation operator and the conjugate transformation one on the spinors can be written in the following way [16]:

$$L \psi = u \frac{d}{dx}(u^{-1} \psi), \quad (30)$$

$$L^+ \tilde{\psi} = -(u^+)^{-1} \frac{d}{dx}(u^{-1} \tilde{\psi}). \quad (31)$$

Accounting these properties, we can write (29) in the form:

$$tr \int_{a}^{b} G_1(x, y, E)|_{x=y} dy = tr \int_{a}^{b} \frac{d}{dx}(u^{-1} \psi) \tilde{\phi} T u|_{x=y} dy / W\{\tilde{\psi}\}. \quad (32)$$

We integrate (32) by parts and apply the trace property

$$tr AB = tr BA \quad (33)$$

to obtain the relation

$$tr \int_{a}^{b} G_1(x, y, E)|_{x=y} dy = \frac{tr \psi \tilde{\phi} T|_{a}^{b} + \int_{a}^{b} tr \psi (L^+ L \phi) T|_{x=y} dy}{W\{\tilde{\psi}\}}. \quad (34)$$

Due to (27) and the factorization property from [16] we obtain:

$$\int_{a}^{b} tr G_1(x, y, E)|_{x=y} dy = \frac{tr \psi \tilde{\phi} T|_{a}^{b}}{W\{\tilde{\psi}\}} + \int_{a}^{b} tr G_0(x, y, E)|_{x=y} dy. \quad (35)$$

Similarly, we would like to write

$$\int_{a}^{b} tr G_1(x, y, E)|_{x=y} dy = \frac{tr \tilde{\psi} \phi T|_{a}^{b}}{W\{\tilde{\psi}\}} + \int_{a}^{b} tr G_0(x, y, E)|_{x=y} dy. \quad (36)$$
Finally, from (35), (36) the formulae for the unabridged trace of the difference of the modified and the initial Green functions are the following:

\[
\int_{a}^{b} (G_1(x, y, E) - G_0(x, y, E))|_{x=y}dy = \frac{1}{W(\psi, \varphi)} \text{tr}(\psi\varphi^T)|_{a}^{b},
\]

\[
\int_{a}^{b} (G_1(x, y, E) - G_0(x, y, E))|_{x=y}dy = \frac{1}{W(\tilde{\psi}, \tilde{\varphi})} \text{tr}(\tilde{\psi}\tilde{\varphi}^T)|_{a}^{b}.
\]

Here \(a = -\infty, b = \infty\) are the limits of integration.

6 Example

First of all, we would like to note the multiplication property \(\psi(x)\varphi(y)^T\):

\[
(\psi(x)\varphi(y)^T)^T = \varphi(y)\psi(x)^T.
\]

The multiplication \(\varphi(x)\psi(y)^T\) is obtained due to the transformation of the multiplication \(\psi(x)\varphi(y)^T\) and the substitution \(x \leftrightarrow y\). Analogously,

\[
\psi(x)\varphi(y)^T = (\varphi(x)\psi(y)^T)^T (x \leftrightarrow y).
\]

Now let us consider the examples illustrating the results obtained above.

1. We assume the free motion case \(V_0 = m\sigma_3\) on whole real line \(x \in [-\infty, \infty]\).

We chose the solutions of initial solutions as follows:

\[
\psi = \begin{pmatrix}
\exp(-ikx) \\
\frac{ik}{E-m}
\end{pmatrix}, \quad \varphi = \begin{pmatrix}
\frac{ik}{E+m} \\
\exp(ikx)
\end{pmatrix}.
\]

Use these solutions we construct the Green function:

\[
G_0(x, y, E) = \frac{1}{2} \left\{ A^{(0)}\Theta(x - y) + B^{(0)}\Theta(y - x) \right\},
\]

where

\[
B^{(0)} = \begin{pmatrix}
-ik/(E - m) & -1 \\
1 & ik/(E + m)
\end{pmatrix} \exp(i(kx - ky),
\]
The matrix solution of the initial Dirac equation can be chosen \[16\] as follows:

\[
  u = \begin{pmatrix}
  1 & \rho \sinh (\rho x)/(\varepsilon - m) \\
  0 & \cosh \rho x
  \end{pmatrix}, \quad (45)
\]

where \(\lambda_1 = m\), \(\lambda_2 = \varepsilon < m\), \(\rho = \sqrt{m^2 - \varepsilon^2}\). The corresponding potential are follows

\[
  V_1(x) = -\varepsilon \sigma_3 + \rho \tanh (\rho x)\sigma_3, \quad (46)
\]

the function \(-\varepsilon\) is playing a part of mass.

Use the QST transformation we construct the solutions of the Dirac equation with potential \([46]\):

\[
  \tilde{\psi} = \frac{ik}{E + m} \begin{pmatrix}
  \varepsilon - E \\
  -ik - \rho \tanh (\rho x)
  \end{pmatrix} \exp(-ikx), \quad (47)
\]

\[
  \tilde{\phi} = \begin{pmatrix}
  \varepsilon - E \\
  ik - \rho \tanh (\rho x)
  \end{pmatrix} \exp(ikx). \quad (48)
\]

Now we construct the QST transformed Green function corresponding the potential \([46]\)

\[
  G_1(x, y, E) = \frac{A^{(1)} \Theta(x - y) + B^{(1)} \Theta(y - x)}{2k(E - m)(E - \varepsilon)}, \quad (49)
\]

where

\[
  B^{(1)} = \begin{pmatrix}
  B_{11}^{(1)} & B_{12}^{(1)} \\
  B_{21}^{(1)} & B_{22}^{(1)}
  \end{pmatrix} \frac{ik}{E + m} \exp ik(x - ky), \quad (50)
\]

\[
  A^{(1)} = (B^{(1)})^T(x \leftrightarrow y), \quad (51)
\]
Apply the formula of the unabridged trace from difference transformed and initial Green functions (37), we compute the following trace:

\[
tr \int_a^b (G_1(x, y, E) - G_0(x, y, E))|_{x=y}dy = -\frac{i\rho}{k(E-\varepsilon)}.
\] (56)

The expression reflect the addition one energy level to the spectrum of the modified Hamiltonian.

7 Conclusion

In this paper we studied the QST of the Green function and the relations between the transformed Green functions and the initial Green function of the one-dimensional Dirac equation for the case the whole real line. We also construct the formulas for the unabridged trace of the difference of the transformed and the initial Green functions of the one-dimensional Dirac equation for the case are the whole real line. The main results of the paper are the relation between the initial and the transformed Green functions (24) and the formulae for an unabridged trace (37), (38) in the whole real line. We case of a half-line is more complicated. We believe that this case will be investigate in a separate publication.

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