On the Auslander-Reiten quiver for the category of representations of partially ordered sets with an involution.

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In this article we describe the Auslander-Reiten quiver for some posets with an involution, that we call types $U_n$ and $U_\infty$. These posets appear in the differentiation III of Zavadskij [see, [12]]. We follow the approach to classical Auslander-Reiten theory due to Auslander, Reiten and Smalø [1]. For this purpose, we give a natural exact structure for the category of representations of a partially ordered set with an involution. We describe the projective and injective representations.

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1 Introduction

Matrix representations of a partially ordered set (poset) with an equivalence relation were introduced by Nazarova and Roiter in [8]. A particular but important case is when each equivalence class consists of at most two elements. In this case we say that the poset has an involution. Following the ideas presented by Gabriel in [6], Zavadskij (see section 9 of [12]) consider filtered $k$-linear representations of a poset with an involution ($k$ a field) with the purpose of giving a strict foundation to the several differentiation algorithms introduced by him. These algorithms were conceived in terms of matrices.

The different differentiation algorithms are a powerful tool for classifying posets with an involution of finite and tame type. These algorithms can be applied when there is an appropriated subposet of the poset with an involution, then the differentiation consists in the change of this subposet obtaining a new poset. After this, is defined a functor from the representations of the original poset into the new one in such a way that with the exception of a finite number of isomorphism classes of indecomposables, one obtains a bijection between the isomorphism classes of the indecomposable representations of the original poset and the isomorphism classes of indecomposables of the new poset.

For some cases it is well known that the functor given by the differentiation, induces an equivalence between the corresponding categories of representations modulo the ideal generated by a finite number of indecomposable objects. Here we are interested in the appropriated subposet appearing in the differentiation III of Zavadskij. We will describe its Auslander-Reiten quiver. To do that we first make some general considerations on a natural exact structure on the category of vector space representations of a partially ordered set with an involution and their Auslander-Reiten sequences.

In section 3 we will see that the above exact structure has enough projectives. Taking the endomorphism ring of the direct sum of a representative set of the indecomposable projectives we obtain a right-peak algebra in the sense of Simson (see, [11]). The category of representations of the poset with involution is equivalent with the category
of the socle projective modules of this last algebra. Posets with an equivalence relation are a special case of completed posets introduced by Nazarova and Roiter. A further generalization is the notion of stratified posets introduced by Simson (see, 17.8 of [10]). A vector space category in the sense of Ringel (see, [9]) is associated to a stratified poset. Given a vector space category \( \mathbb{K}_F \), there are two equivalent categories of representations, the subspace category \( \mathcal{U}(\mathbb{K}_F) \), and \( \mathcal{V}(\mathbb{K}_F) \) the factor space category. Moreover a right-peak algebra \( \mathbb{R}_{\mathbb{K}_F} \) is associated to any vector space category \( \mathbb{K}_F \). The category \( \text{mod}_{sp}(\mathbb{R}_{\mathbb{K}_F}) \) is equivalent to the category \( \mathcal{V}(\mathbb{K}_F) \) modulo those morphisms which are factorized through finite direct sums of a finite family of objects. In our case if \( \mathbb{K}_F \) is the vector space category associated to a poset with involution, the endomorphism algebra of the projective generator of the category of representations of our poset with involution is isomorphic to \( \mathbb{R}_{\mathbb{K}_F} \).

Now, we give the layout of the contents of this paper. In section 2 we introduce the main definitions. In section 3 we present an exact structure for the category of representations of a partially ordered set with involution. In the second part of this section we consider representations by factor spaces instead of subspaces. This allows us to describe the injective objects in the category of representations. In section 4 we take a projective generator for the category of representations of a poset with involution, obtaining a right-peak algebra. We prove that the category of the socle-projective modules of this algebra is equivalent to the category of representations of the poset with involution. Finally in section 5 we describe the Auslander-Reiten quiver of a poset with involution \( \Lambda_n \) and its extension \( \Lambda_\infty \).

2 Partially ordered set with an involution

In this section we define partially ordered sets with an involution and their category of representations.

**Definition 1.** A partially ordered set with an equivalence relation is a triple \((\mathcal{P}, \leq, \theta)\), where \((\mathcal{P}, \leq)\) is a partially ordered set and in \(\mathcal{P}\) there is an equivalence relation whose equivalence classes is \(\theta\). If the cardinality of each equivalence class is less than or equal to two, we will say that triple \((\mathcal{P}, \leq, \theta)\) is a partially ordered set with an involution. If \(x \in \mathcal{P}\) we will denote by \([x]\) its equivalence class.

**Remark 1.** From now on we will omit the order relation in the notation for poset with an involution, that is, we will write \((\mathcal{P}, \theta)\) instead of \((\mathcal{P}, \leq, \theta)\).

**Example 1.** Let \((\mathcal{P}, \theta)\) be a poset with involution where \(\mathcal{P}\) is as in Figure 1 with \(a < b, c < b, c < a^*\) and \(\theta = \{(a, a^*), b, c\}\).

![Figure 1: Diagram of a poset with an involution](image)
2.1 Vector space representations for posets with an involution

Zavadskij introduced filtered $k$-linear representation of posets with an involution $(\mathcal{P}, \theta)$ [12]. Here we introduce an equivalent definition to the one given by him. For this, we consider $(\mathcal{P}, \theta)$ a poset with an involution. We take $V_0$ a $k$-vector space and $z \in \theta$, take $V_z^0$ the $k$-vector space consisting of all functions $h : z \to V_0$. For $x \in z$, we have the inclusion: $i_x : V_0 \to V_z^0$, defined by

$$i_x(v)(y) = \begin{cases} 
0, & \text{if } y \neq x, \\
v, & \text{otherwise.}
\end{cases}$$

and the projection in the summand $x$ of $V_z^0$, $\pi_x : V_z^0 \to V_0$, that is, for $h \in V_z^0$, $\pi_x(h) = h(x)$.

In the following, if $V$ is a $k$-vector subspace of $V_z^0$ and $x \in z$,

$$V_z^- = i_x^{-1}(V) = \{v \in V_0 \mid i_x(v) \in V\},$$

$$V_z^+ = \pi_x(V) = \{h(x) \mid h \in V\}.$$

Definition 2. A vector space representation $V = (V_0, V_z)_{z \in \theta}$ of $(\mathcal{P}, \theta)$ is given by:

1. a finite-dimensional $k$-vector space $V_0$,
2. for each $z \in \theta$, a vector subspace $V_z$ of $V_z^0$ such that if $y < x$ then $V_y^+ \subset V_x^-.$

Example 2. Let $(\mathcal{P}, \theta)$ be a poset with an involution where $\mathcal{P}$ is as in Figure 2 with $a < b^*$, $a^* < b$, $a^* < b^*$ and $\theta = \{(a, a^*), (b, b^*)\}$.

$$\mathcal{P} = \begin{array}{ccc}
\bullet & \times & \circ \\
\downarrow & & \\
\bullet & & \bullet
\end{array}$$

Figure 2: Diagram of a poset with an involution.

We will show that $V = (V_0, V_{(a, a^*)}, V_{(b, b^*)})$ is a vector space representation of $(\mathcal{P}, \theta)$, where $V_0 = \mathbb{R}^3$, $\mathcal{B} = \{e_1, e_2, e_3\}$ is the canonical basis of $V_0$ and $V_{(a, a^*)} = \langle h \rangle$, with $h : (a, a^*) \to \mathbb{R}^3$

$$h : \begin{cases} 
(a, a^*) \to \mathbb{R}^3 \\
a \mapsto e_1 \\
a^* \mapsto e_2
\end{cases}$$

and $V_{(b, b^*)} = \langle h_1, h_2, h_3, h_4 \rangle$, with
have the morphism $\varphi \rightarrow \{(R^V)\}$.

With the definitions above, the vector space representation $s$ of a poset with an involution can also be viewed categorically. We denote by $\text{Rep}(\mathcal{P}, \varnothing)$ the category of pairs of morphisms $\mathcal{P} \rightarrow \mathcal{P}$.

Definition 3. If $V = (V_0, V_2)_{z \in \theta}$ and $W = (W_0, W_2)_{z \in \theta}$ are two representations in $(\mathcal{P}, \varnothing)$, and $\varphi : V_0 \rightarrow W_0$ is a morphism of vector spaces, such that for each $z \in \theta$, we have the morphism $\varphi^2 : V_0 \rightarrow W_0$ and for $h : z \rightarrow V_0$, $\varphi^2(h) = \varphi h$. Then a morphism $V \rightarrow W$ consists of a morphism of vector space $\varphi : V_0 \rightarrow W_0$ such that, for all $z \in \theta$

$$\varphi^2(V_z) \subset W_z.$$  

With the definitions above, the vector space representations of a poset with an involution can also be viewed categorically. We denote by $\text{Rep}(\mathcal{P}, \varnothing)$ this category.

Definition 4. If $V = (V_0, V_2)_{z \in \theta}$ and $W = (W_0, W_2)_{z \in \theta}$ are two representations of $\text{Rep}(\mathcal{P}, \varnothing)$ then their direct sum is $V \bigoplus W = (V_0 \bigoplus W_0, V_2 \bigoplus W_2)_{z \in \theta}$.

### 3 Exact structure on the category $\text{Rep}(\mathcal{P}, \varnothing)$.

Let $\mathcal{A}$ be an additive category in which all idempotents split, and let $\varepsilon$ be a collection of pairs of morphisms $M \rightarrow E \rightarrow N$. A morphism $u : M \rightarrow E$ is called an $\varepsilon$-inflation if there exists a morphism $v : E \rightarrow N$ such that $(u, v) \in \varepsilon$. A morphism $v : E \rightarrow N$ is called an $\varepsilon$-deflation if there exists a morphism $u : M \rightarrow E$ such that $(u, v) \in \varepsilon$.

The pair $(\mathcal{A}; \varepsilon)$ will be called exact structure (see, [7], [5], and [3]) if the following conditions are satisfied:

1. The family $\varepsilon$ is closed under isomorphisms; that is, if there exists a commutative diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{u} & E \\
\downarrow & & \downarrow v \\
M_1 & \xrightarrow{u_1} & E_1
\end{array}
$$

where $s, t, r$ are isomorphisms and the top row is in $\varepsilon$, then the bottom row belongs to $\varepsilon$.

2. If $(u, v) \in \varepsilon$, then $u$ is a kernel of $v$ and $v$ is a cokernel of $u$.

3. $\text{id}_M : M \rightarrow M$ is both $\varepsilon$-inflation and $\varepsilon$-deflation.

4. a. For each $\varepsilon$-sequence $M \xrightarrow{f} E \xrightarrow{\lambda} N$ and each morphism $w : X \rightarrow N$ there are morphisms $\beta : F \rightarrow X$ and $\lambda : F \rightarrow E$ such that the pair $(\lambda, \beta)$ is a pullback of the pair $(g, w)$ and $\beta$ is an $\varepsilon$-deflation.


b. For each ε-sequence \( M \xrightarrow{f} E \xrightarrow{g} N \) and each morphism \( u : M \to X \) there are morphisms \( \alpha : X \to F \) and \( \lambda : E \to F \) such that the pair \((\alpha, \lambda)\) is a pushout of the pair \((u, f)\) and \(\alpha\) is an ε-inflation.

5. The composition of ε-inflations (ε-deflations, respectively) is again an ε-inflation (ε-deflation, respectively).

6. If \( u_2u_1 \) is an ε-inflation then \( u_1 \) is an ε-inflation. If \( v_2v_1 \) is an ε-deflation then \( v_2 \) is an ε-deflation.

In our case, let \((\mathcal{P}, \theta)\) be a poset with an involution and let us consider ε the family of sequences of morphisms:

\[
(V_0, V) \xrightarrow{u} (E_0, E) \xrightarrow{v} (W_0, W),
\]

in the category \( \text{Rep}(\mathcal{P}, \theta) \) such that:

1. The sequence \( 0 \to V_0 \xrightarrow{u} E_0 \xrightarrow{v} W_0 \to 0 \) is exact.
2. For all \( z \in \theta \), the sequence \( 0 \to V_z \xrightarrow{u_z} E_z \xrightarrow{v_z} W_z \to 0 \) is exact.

From now on we will denote a representation \((U_0, U_z)_{z \in \theta}\) in \( \text{Rep}(\mathcal{P}, \theta) \) by \( U \), unless it is necessary to specify the subspaces \( U_z \).

**Definition 5.** A morphism \( f : V \to W \) in the category \( \text{Rep}(\mathcal{P}, \theta) \) will be called a proper epimorphism if \( f : V_0 \to W_0 \) is an epimorphism and for each \( z \in \theta \), \( f^z : V_0^z \to W_0^z \) induces an epimorphism \( f^z : V_z \to W_z \).

**Proposition 1.** Let \( f : V \to W \) be a proper epimorphism. If \( U_0 = \ker(f) \) and \( U_z = \ker(f^z) \cap V_z \), then \( U = (U_0, U_z)_{z \in \theta} \) is a representation of \((\mathcal{P}, \theta)\).

**Proof.** In the first place, we observe that for each \( z \in \theta \), \( \ker(f^z) = U_0^z \), then \( U_z = U_0^z \cap V_z \). We suppose now that \( a < a_1 \) and \((x, y) \in U_{(a, b)} \) so, \((x, y) \in V_{(a, b)} \) then if \( z_1 = (a_1, b_1), (x, 0) \in V_{(a_1, b_1)} \) as, \((x, y) \in U_{(a, b)}, (f(x), f(y)) = (0, 0) \) then \( f(x) = 0 \); therefore \((x, 0) \in U_{(a_1, b_1)} \). This proves that in effect \((U_0, U_z)_{z \in \theta} \) is a representation of \((\mathcal{P}, \theta)\).

**Corollary 1.** If \( f : V \to W \) is a proper epimorphism and \( U \) is as in the previous proposition, then an ε-sequence

\[
U \xrightarrow{f} V \xrightarrow{g} W,
\]

is obtained. Therefore, \( f \) is an ε-deflation if and only if \( f \) is a proper epimorphism.

**Definition 6.** A morphism \( f : U \to V \) in the category \( \text{Rep}(\mathcal{P}, \theta) \) will be called a proper monomorphism if \( f : U_0 \to V_0 \) is a monomorphism and for each \( z \in \theta \), \( f^z(U_z) = U_0 \cap f(U) \).

**Proposition 2.** If \( f : U \to V \) is a proper monomorphism then there exists a sequence in ε:

\[
U \xrightarrow{f} V \xrightarrow{g} W.
\]

Therefore \( f \) is an ε-inflation if and only if \( f \) is a proper monomorphism.
Proof. Let \( g : V_0 \rightarrow W_0 \) be the cokernel of \( f \). For \( z \in \Theta \) we define \( W_z = f^z(V_z) \). We will check that \((W_0, W_z)_{z \in \Theta}\) is a representation of \((\mathcal{P}, \Theta)\). Indeed, let \((x, y) \in W_{(a,b)}\) and \((a_1, b_1) \in \Theta\) with \( a < a_1 \). Then \( x = g(x_1) \), \( y = g(y_1) \) with \((x_1, y_1) \in V_{(a_1,b_1)}\) so, \((x, 0) \in V_{(a_1,b_1)}\), therefore \((x, 0) = g(x_1, 0) \in W_{(a_1,b_1)}\). This proves that \((W_0, W_z)_{z \in \Theta}\) is a representation. We prove now that for each \( z \in \Theta \), the sequence:

\[
0 \rightarrow U_z \xrightarrow{f_z} V_z \xrightarrow{g_z} W_z \rightarrow 0,
\]

is exact. Since \( f_z \) is a monomorphism, \( g_z \) is an epimorphism and \( g_z f_z = 0 \). It only remains to prove that if \((x, y) \in V_z\) is such that \((g(x), g(y)) = (0, 0)\) then \((x, y) \in f^z(U_z)\). Since the sequence

\[
0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0,
\]

is exact, then \((x, y) \in f(U)\). Also \((x, y) \in V_z\) and \( f \) is a proper monomorphism, it follows that \((x, y) \in f^z(U_z)\). This proves our claim.

Proposition 3. The pair \((\text{Rep}(\mathcal{P}, \Theta), \varepsilon)\) is an exact category.

Proof. Conditions 1, 2 and 3 are verified directly; our characterization of \( \varepsilon \)-deflations and \( \varepsilon \)-inflations implies items 5 and 6. Thus, it remains to prove item 4. Let us show condition 4.a. Let

\[
U \xrightarrow{h} E \xrightarrow{v} W,
\]

be an \( \varepsilon \)-sequence and let \( f : W \rightarrow V \) be a morphism. Consider the morphism:

\[
\phi = (v, -f) : E \bigoplus W \rightarrow V,
\]

here \( v : W_0 \rightarrow V_0 \) and \( v^z : W_z \rightarrow V_z \) are epimorphisms for all \( z \in \Theta \). Then \((v, -f) : E_0 \bigoplus W_0 \rightarrow V_0 \) and \( \phi_z = (v^z, f^z) : E_z \bigoplus W_z \rightarrow V_z \) are epimorphisms for all \( z \in \Theta \), so \( \phi = (v, -f) \) is a deflation, therefore there is an \( \varepsilon \)-sequence.

\[
L \xrightarrow{(h_1, -h_2)^f} E \bigoplus W \xrightarrow{\phi} V.
\]

This implies we have pull-back diagrams:

\[
L \xrightarrow{h_2} W \xrightarrow{f} V \quad L_0 \xrightarrow{h_2} W_0 \xrightarrow{f_0} V_0 \quad L_z \xrightarrow{h_2^z} W_z \xrightarrow{f^z} V_z
\]

here \( v : E_0 \rightarrow V_0 \) and \( v^z : E_z \rightarrow V_z \) are epimorphisms, then \( h_2 : L_0 \rightarrow W_0 \) and \( h_2^z : L_z \rightarrow W_z \) are epimorphisms for all \( z \in \Theta \), this implies that \( h_2 \) is a deflation. This proves 4.a; item 4.b is proved in a similar way.
3.1 $\varepsilon$-projective representations

**Definition 7.** A representation $P$ of $(\mathcal{P}, \theta)$ is called $\varepsilon$-projective if given an $\varepsilon$-deflation $g : E \rightarrow V$ and a morphism $f : P \rightarrow V$, there exists a morphism $h : P \rightarrow E$ such that $gh = f$.

**Remark 2.** The representation $S = (S_0, S_z)_{z \in \theta}$ with $S_0 = k$ and $S_z = 0$ for all $z \in \theta$, is a projective representation.

Let $w = (a, b) \in \theta$, we will define the representation $P(w) = (P(w)_0, P(w)_z)_{z \in \theta}$, where $P(w)_0 = k(e_1, e_2)$ the vector space of dimension two with bases $e_1, e_2$. If $a$ and $b$ are incomparable $P(w)_w = \langle (e_1, e_2) \rangle$, while if $a < b$ then $P(w)_w = \langle (0, e_1), (e_1, e_2) \rangle$.

Henceforth, we will use the following notation, if $d_1, d_2 \in \mathcal{P}$ then

$$\lambda(d_1, d_2) = \begin{cases} 1 & \text{if } d_1 < d_2, \\ 0 & \text{otherwise.} \end{cases}$$

If $z = (a_1, b_1)$, $P(w)_z = \langle (\lambda(a, a_1)e_1, 0), (0, \lambda(a, b_1)e_1), (\lambda(b, a_1)e_2, 0), (0, \lambda(b, b_1)e_2) \rangle$.

In case that, $w = \{a\}$ then $P(w)_0 = k(e)$ and for $z = (a_1, b_1)$

$$P(w)_z = \langle (\lambda(a, a_1)e, 0), (0, \lambda(a, b_1)e) \rangle.$$  

It can be verified that $P(w) = (P(w)_0, P(w)_z)_{z \in \theta}$ is in fact a representation.

**Definition 8.** The element $\langle e_1, e_2 \rangle \in P(w)$ will be called the generator of the representation $(P(w)_0, P(w)_z)_{z \in \theta}$, when $w = (a, b)$ while the element $e \in P(w)$ is the generator when $w$ consists of a single point.

**Proposition 4.** Let $(V_0, V_z)_{z \in \theta}$ be a representation of $(\mathcal{P}, \theta)$, then if $w = (a, b) \in \theta$ and $v \in V_w$ there exists an unique morphism $f : P(w) \rightarrow (V_0, V_z)_{z \in \theta}$ such that $f(\langle e_1, e_2 \rangle) = v$. If $w = \{a\}$ and $v \in V_w$ there exists an unique morphism as before such that $f(e) = v$.

**Proof**

1. If $w = (a, b)$ and $a < b$. Let $v = (v_1, v_2) \in V_{(a,b)}$ and $f : P(0) \rightarrow V_0$ with $f(e_1) = v_1; f(e_2) = v_2$. Since $(0, v_1) \in V_{(a,b)}$, then $f_w(0, e_1) = (0, v_1)$ and $f_w((e_1, e_2)) = (v_1, v_2) \in V_{(a,b)}$; therefore $f_w(P(w)_w) \in V_w$. Let $(a_1, b_1) \in \theta$, then

$$P(w)_{(a_1, b_1)} = \langle (\lambda(a, a_1)e_1, 0), (0, \lambda(a, b_1)e_1), (\lambda(b, a_1)e_2, 0), (0, \lambda(b, b_1)e_2) \rangle.$$  

If $\lambda(a, a_1) \neq 0$, then $a < a_1$ and therefore $(v_1, 0) \in V_{(a_1, b_1)}$ and $f_z(\lambda(a, a_1)e_1, 0) = (v_1, 0) \in V_{(a,b)}$. In the same way, it is seen that $f_z$, sends each generator from $P(w)_{(a_1, b_1)}$ into $V_{(a_1, b_1)}$. The uniqueness of $f$ is clear.

2. If $w = (a, b)$ and $a, b$ are incomparable. In this case, $P(w)_{(a,b)} = \langle (e_1, e_2) \rangle$ and $f_w((e_1, e_2)) = (v_1, v_2) \in V_{(a,b)}$. Therefore, $f_w(P(w)_{(a,b)}) \in V_{(a,b)}$. For the rest it is checked as in the previous case.

3. If $w = \{a\}$, the proof is similar to the previous cases.
Proposition 5. The representations $P(w) = (P(w)_0, P(w)_z)_{z \in \theta}$ have the following properties:

1. $P(w)$ is an $\varepsilon$-projective representation.
2. $\text{End}(P(w)) \cong k$ if $w = (a, b)$ with $a$ and $b$ incomparable or when $w$ consists of a single element. If $w = (a, b)$ with $a < b$ then $\text{End}(P(w)) \cong k[x]/x^2$. Therefore $P(w)$ is indecomposable for all $w \in \theta$.
3. For any representation $(V_0, V_z)_{z \in \theta}$, there exists an $\varepsilon$-deflation $g : (Q_0, Q_z)_{z \in \theta} \rightarrow (V_0, V_z)_{z \in \theta}$, where $(Q_0, Q_z)_{z \in \theta}$ is $\varepsilon$-projective.
4. If $(Q_0, Q_z)_{z \in \theta}$ is an indecomposable projective representation of $\text{Rep}(P, \theta)$, with $Q_z \neq 0$ for some $z \in \theta$, then $(Q_0, Q_z)_{z \in \theta} \cong P(w)$ for some $w \in \theta$.

Proof

1. Let $f : (E_0, E_z)_{z \in \theta} \rightarrow (V_0, V_z)_{z \in \theta}$ be a morphism given by $w \in V_w$. We define $f^w(v) = v$ such that $f^w$ is a generator of $(P(w)_0, P(w)_z)_{z \in \theta}$. Since $f^w$ is surjective there exists $v_1 \in V_{w}$ such that $f^w(v_1) = v$. By Proposition 3 there exists a morphism $h : (P(w)_0, P(w)_z)_{z \in \theta} \rightarrow (E_0, E_z)_{z \in \theta}$, such that $h(v) = v_1$, so $fh(v) = g(v)$. By the uniqueness in the Proposition 3, the uniqueness in the Proposition 3 is obtained for all $v \in V_v$. Therefore, $(P(w)_0, P(w)_z)_{z \in \theta}$ is a projective representation.

2. If $w = (a, b)$ or $w = \{a\}$ then $P(w)_w = \langle \varepsilon \rangle$ with $\varepsilon$ the generator of $(P(w)_0, P(w)_z)_{z \in \theta}$; therefore, if $f : (P(w)_0, P(w)_z)_{z \in \theta} \rightarrow (P(w)_0, P(w)_z)_{z \in \theta}$ then $f(\varepsilon) = c\varepsilon$ with $c \in k$. Hence, $f = c(id_{P(w)_0})$. This proves that $\text{End}(P(w)_0, P(w)_z)_{z \in \theta} = k(id_{P(w)_0}) \cong k$.

We suppose now that $w = (a, b)$ with $a < b$, then if $g = (e_1, e_2)$ is the generator of $P(w)_w$ we have that $P(w)_w = \langle (e_1, e_2), (0, e_1) \rangle$. Let $f$ be an endomorphism of $(P(w)_0, P(w)_z)_{z \in \theta}$, then $f_{a,b}((e_1, e_2)) = c(e_1, e_2) + d(0, e_1)$ with $c, d \in k$. Therefore $f(e_1) = ce_1, f(e_2) = ce_2 + de_1$. In view of Proposition 3 the morphism $f$ is completely determined by the matrix $M(f) = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}$. If $f_1$ is another automorphism of $(P(w)_0, P(w)_z)_{z \in \theta}$, then $M(f_1f) = M(f_1)M(f)$. Hence, $\text{End}(P(w)_0, P(w)_z)_{z \in \theta} \cong \left\{ \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \mid c, d \in k \right\} \cong k[x]/(x^2)$.

3. For $V$ we choose a basis $B(0)$ and for each $z \in \theta$ such that $V_z \neq 0$, we choose $B(z)$ a $k$-basis of $V_z$. For each $v \in B(0)$ we take the morphism $f_v : S \rightarrow (V_0, V_z)_{z \in \theta}$ which sends $1 \in k$ in $v \in V$ and for $v \in B(z)$ we have a morphism $f_v : (P(w)_0, P(w)_z)_{z \in \theta} \rightarrow (V_0, V_z)_{z \in \theta}$, such that $f_v(\varepsilon) = v$ where $\varepsilon$ is the generator of $(P(w)_0, P(w)_z)_{z \in \theta}$. Let $B = \bigcup \limits_{z} B(z)$, then we have a morphism $f = (f_v)_{v \in B} : \bigoplus \limits_{v \in B(0)} S \bigoplus \limits_{z \in B(z)} P(w) \rightarrow (V_0, V_z)_{z \in \theta}$; clearly this morphism is an $\varepsilon$-deflation and the representation $\bigoplus \limits_{v \in B(0)} S \bigoplus \limits_{z \in B(z)} P(w)$, is $\varepsilon$-projective.
4. Let \((Q_0, Q_z)_{z \in \theta}\) be a projective representation, such that for some \(z \in Q_z \neq 0\). From the above, we have a deflation:

\[
P \xrightarrow{f} (Q_0, Q_z)_{z \in \theta},
\]

then there exists a morphism \(h : (Q_0, Q_z)_{z \in \theta} \to P\) such that \(fh = id_Q\). This implies that \((Q_0, Q_z)_{z \in \theta}\) is a direct sum of \(P\). The last representation is a direct sum of representations \(S\) and \(P(z)\); therefore, our representation is isomorphic to one of these, and as for some \(z \in \theta, Q_z \neq 0\), then \((Q_0, Q_z)_{z \in \theta} \cong P(w)\) for some \(w \in \theta\).

**Remark 3.** An exact category is said to have enough projectives if it satisfies property 3 of Proposition 5.

**Definition 9.** A representation \((I_0, I_z)_{z \in \theta}\) is called \(\varepsilon\)-injective if given an \(\varepsilon\)-inflation \(f : (V_0, V_z)_{z \in \theta} \to (E_0, E_z)_{z \in \theta}\) and a morphism \(g : (V_0, V_z)_{z \in \theta} \to (I_0, I_z)_{z \in \theta}\), there exists a morphism \(h : (E_0, E_z)_{z \in \theta} \to (I_0, I_z)_{z \in \theta}\) such that \(hf = g\).

Henceforth it is convenient to use the following notation to represent poset with an involution: the pair \((P, \theta)\) if and only if for each \(z \leq y\) in \(P\), there exists a linear transformation \(\tau : V_{[z]} \to V_{[y]}\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
V_{[z]} & \xrightarrow{i_z} & V_0^{[z]} \\
\downarrow \tau & & \downarrow \pi_x \\
V_{[y]} & \xrightarrow{i_y} & V_0^{[y]}
\end{array}
\]

where \(i_z : V_{[z]} \to V_0^{[z]}\) and \(i_y : V_0 \to V_0^{[y]}\) are the inclusions and \(\pi_x : V_0^{[z]} \to V_0\) is the projection.

### 3.2 Representations by quotients

Let \((P, \theta)\) be a poset with an involution and \(k\) be a field. A *representation by quotient* \((V_0, j_z)_{z \in \theta}\), consists of a \(k\)-vector space \(V_0\) and for each \(z \in \theta\) an epimorphism \(j_z : V_0^z \to V_z\) such that if \(a_1 < a\) and \(z = (a, b), z_1 = (a_1, b_1)\) then there exists a morphism \(\tau : V_z \to V_{z_1}\) such that

\[
\tau j_z = j_{a_1} i_{a_1} \pi_{a_1}.
\]

A morphism \(f : (V_0, j_z)_{z \in \theta} \to (V_0', j_z')_{z \in \theta}\) consists of a linear transformation \(f_0 : V_0 \to V_0'\) and for each \(z \in \theta\) a linear transformation \(f^z : V_z \to V_z'\) such that the following diagram commutes:

\[
\begin{array}{ccc}
V_0^z & \xrightarrow{j_z} & V_z \\
\downarrow f^z & & \downarrow f_z \\
(V_0')^z & \xrightarrow{j_z'} & V_z'
\end{array}
\]

We denote by \(\text{Repq}(P, \theta)\) the category of quotient representations.
Proof. We identify $D$ and $D$ therefore, Proposition 7. There are contravariant functors thus, $\mathcal{V}$ defined by $C(V_0, V_1)_x \in \Theta$, where $i_z : V_z \to V'_z$ is the inclusion and $K(V_0, j_z)_x \in \Theta$. Further, $CK \cong \text{id}_{\text{Rep}(\mathcal{P}, \theta)}$ and $KC \cong \text{id}_{\text{Rep}(\mathcal{P}, \theta)}$ therefore Rep$(\mathcal{P}, \theta)$ is equivalent to Rep$(\mathcal{P}_o, \theta)$. Let $(V_0, V_1)_x \in \Theta$ be an object of Rep$(\mathcal{P}, \theta)$ and we take $j_z : V'_z \to V'_z$ the cokernel of $i_z$. We suppose that $x \in z$ and $y \in z_1$ with $x < y$ then we obtain the morphism $i_y \pi_x : V'_0 \to V'_0$ and a morphism $\tau : V_z \to V_{z_1}$ such that $i_y \tau = i_y \pi_x i_z$. Therefore there exists a morphism $\tau' : V'_z \to V'_{z_1}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
V_z & \xrightarrow{i_z} & V'_0 \\
\downarrow{\tau} & & \downarrow{\tau'} \\
V_{z_1} & \xrightarrow{i_{z_1}} & V'_{z_1}
\end{array}
$$

This proves that $(V_0, j_z)_x \in \Theta \in \text{Rep}(\mathcal{P}_o, \theta)$. Now, let $f : (V_0, V_1)_x \in \Theta \to (W_0, W_1)_x \in \Theta$ be a morphism in Rep$(\mathcal{P}, \theta)$; we denote $r_z : W_z \to W'_z$ the inclusion and by $r'_z : W'_z \to W''_z$ its cokernel. The morphism $g_z : V_z \to W_z$ is obtained, and it is such that $f'_z i_z = r_z g_z$. Therefore there exists a morphism $f'_z : V'_z \to W'_z$ such that the following diagram is commutative

$$
\begin{array}{ccc}
V_z & \xrightarrow{i_z} & V'_0 \\
\downarrow{g_z} & & \downarrow{f'_z} \\
W_z & \xrightarrow{r_z} & W'_z
\end{array}
$$

thus, $f'_z$ is a morphism of $C((V_0, V_z)_x \in \Theta)$ in $C((W_0, W_z)_x \in \Theta)$. We define $C(f) = f'_z$.

Now, if $(V_0, j_z)_x \in \Theta$ is an object of Rep$(\mathcal{P}_o, \theta)$, by using diagram A, is obtained that $K((V_0, j_z)_x \in \Theta) \in \text{Rep}(\mathcal{P}, \theta)$. If $f : (V_0, j_z)_x \in \Theta \to (W_0, r'_z)_x \in \Theta$ is a morphism in Rep$(\mathcal{P}_o, \theta)$ such that $f_0 : V_0 \to W_0$ then by using B is obtained that $f_0$ produces a morphism of $K((V_0, j_z)_x \in \Theta)$ in $K((W_0, r'_z)_x \in \Theta)$. The rest of the proof is clear. □

Henceforth, if $W$ is a $k$-vector space $D(W) = \text{Hom}_k(W, k)$.

Proposition 7. There are functors
defined by $D_i : \text{Rep}(\mathcal{P}, \theta) \to \text{Rep}(\mathcal{P}_o, \theta)$, and $D_2 : \text{Rep}(\mathcal{P}, \theta) \to \text{Rep}(\mathcal{P}, \theta)$,

Further $D_2 D_1 \cong \text{id}_{\text{Rep}(\mathcal{P}, \theta)}$ and $D_1 D_2 \cong \text{id}_{\text{Rep}(\mathcal{P}_o, \theta)}$.

Proof. We identify $D(V'_0) = D(V_0)^x$. Let $(V_0, V_z)_x \in \text{Rep}(\mathcal{P}, \theta)$, then $D_z((V_0, V_z)_x \in \Theta) = (D(V_0), D(i_z))_x \in \Theta$ where $i_z : V_z \to V'_z$ is the inclusion. Then if $a \in z$, $a_1 \in z_1$ with $a_1 < a$. Hence there exists a morphism $\tau : V_{z_1} \to V_z$ such that $i_z \tau = i_a \pi_{a_1} i_{z_1}$; therefore,

$$
D(\tau)D(i_z) = D(i_{z_1})D(\pi_{a_1})D(i_a).
$$

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We observe that $D(i_{a_1}) : D(V_0) \rightarrow D(V_0)$ is equal to $\pi_{a_1}$ and $D(\pi_a) : D(V_0) \rightarrow D(V_0)$ is equal to $i_{a_1}$; therefore

$$D(\tau)D(i_{a}) = D(i_{a_1})i_{a_1}\pi_{a}.$$ 

The above implies that $D_1((V_0, V_z)_{\theta} \in \text{Rep}(\mathcal{P}, \theta)$. It is clear that

$$f : (V_0, V_z)_{z \in \theta} \rightarrow (W_0, W_z)_{z \in \theta}$$

is a morphism in $\text{Rep}(\mathcal{P}, \theta)$, then $D(f_0) : D(W_0) \rightarrow D(V_0)$ determines a morphism $D_1(f) : D_1((W_0, W_z)_{z \in \theta}) \rightarrow D_1((V_0, V_z)_{z \in \theta})$. The rest of the proposition proceeds in a similar way.  

**Definition 10.** We consider $\varepsilon_q$ the class of sequences in $\text{Rep}(\mathcal{P}, \theta)$ which have the form

$$(V^1_0, j^1_x)_{z \in \theta} \overset{j^1_z}{\rightarrow} (V^2_0, j^2_x)_{z \in \theta} \overset{j^2_z}{\rightarrow} (V^3_0, j^3_x)_{z \in \theta},$$

such that

$$0 \rightarrow V^1_z \overset{j^1_0}{\rightarrow} V^2_z \overset{j^2_0}{\rightarrow} V^3_z \rightarrow 0,$$

and

$$0 \rightarrow V^1_z \overset{j^1_z}{\rightarrow} V^2_z \overset{j^2_z}{\rightarrow} V^3_z \rightarrow 0,$$

are exact, where $j^1_z : (V^1_0) \rightarrow V^2_z$.

**Proposition 8.** The functor $D_1$ sends $\varepsilon$-sequences to $\varepsilon_q$-sequences and the functor $D_2$ sends $\varepsilon_q$-sequences in $\varepsilon$-sequences. In particular, a morphism $f : (V_0, V_z)_{z \in \theta} \rightarrow (V'_0, V'_z)_{z \in \theta}$ in $\text{Rep}(\mathcal{P}, \theta)$ is an $\varepsilon$-inflation ($\varepsilon$-deflation, respectively) if and only if $D_1(f)$ is an $\varepsilon$-deflation ($\varepsilon$-inflation, respectively).

**Corollary 2.** The class of morphisms $\varepsilon_q$ is an exact structure. Further the category $\text{Rep}(\mathcal{P}, \theta)$ has enough injectives.

**Corollary 3.** The exact category $(\text{Rep}(\mathcal{P}, \theta), \varepsilon)$ has enough injectives. The indecomposable injectives of this category have the form $KD_1(P_z)$ for $z \in \theta$ and $KD_1((k, 0_z)_{z \in \theta})$, where $P_z$ and $(k, 0_z)_{z \in \theta}$ are projectives in $\text{Rep}(\mathcal{P}^{op}, \theta)$.

**Proof.** The indecomposable injectives of $\text{Rep}(\mathcal{P}^{op}, \theta)$ have the form $D_1(P_z)$ and $D_1((k, 0_z)_{z \in \theta})$. Since the functor $K$ is an equivalence of categories such that sends $\varepsilon_q$-sequences in $\varepsilon$-sequences then the injectives indecomposables of $\text{Rep}(\mathcal{P}, \theta)$ are the form $KD_1(P_z)$ for $z \in \theta$ and $KD_1((k, 0_z)_{z \in \theta})$.

**Remark 4.** $KD_1((k, 0_z)_{z \in \theta}) = J$ is the representation $J = (J_0, J_z)_{z \in \theta}$ such that $J_0 = k$ and $J_z = k^z$. 

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4 The endomorphism algebra

Let $(\mathcal{P}, \theta)$ be a poset with an involution. We know that in the exact category $(\text{Rep}(\mathcal{P}, \theta), \varepsilon)$ a system of representatives of isomorphism classes of indecomposable projectives is given by $P(z)$ for $z \in \theta$ and $P(0) = S$, as in Remark 2. We take $\mathcal{P} = \bigoplus_{z \in \theta} P(z) \bigoplus P(0)$ and $A = \text{End}_{\text{Rep}(\mathcal{P}, \theta)}(\mathcal{P})$. Take $P(j)$ with $j = z \in \theta$ or $j = 0$. If we consider the projection $\pi_j : \mathcal{P} \to P(j)$ and the inclusion $\sigma_j : P(j) \to \mathcal{P}$ we obtain the idempotent $e_j = \sigma_j \pi_j \in A$. Then $1_A = \sum_{z \in \theta} e_z + e_0$, and $A = \bigoplus_{z \in \theta} e_z A \bigoplus e_0 A$.

Now we recall that an algebra $B$ is called right-peak algebra if $\text{soc}(B) = 0$ is a sum of copies of a simple projective module.

**Proposition 9.** The algebra $A = \text{End}_{\text{Rep}(\mathcal{P}, \theta)}(\mathcal{P})$ is a right peak algebra and the only simple projective right $A$-module up to isomorphism is $e_0 A$.

**Proof** We have:

$$A = \begin{pmatrix} A_1 & M \\ 0 & k \end{pmatrix}$$

where $A_1 = \text{End}_{\text{Rep}(\mathcal{P}, \theta)}(\bigoplus_{z \in \theta} P(z))$ and $M = \text{Hom}_{\text{Rep}(\mathcal{P}, \theta)}(P(0), \bigoplus_{z \in \theta} P(z))$. Here $M$ is a faithful left $A$-module so by [10], $A$ is a right peak algebra and the only simple projective $A$-module up to isomorphism is $e_0 A$.

**Remark 5.** The injective envelope of $e_0 A$ is $E = D(Ae_0)$ and

$$\dim_k E e_z = \text{card}(z), \quad \dim_k E e_0 = 1$$

**Proof** Here

$$\dim_k E e_z = \dim_k D(Ae_0) e_z = \dim_k D(e_z Ae_0) = \dim_k e_z Ae_0 = \dim_k \text{Hom}_{\text{Rep}(\mathcal{P}, \theta)}(P(0), P(z)) = \text{card}(z).$$

Now, we consider the full and faithful functor $H = \text{Hom}_{\text{Rep}(\mathcal{P}, \theta)}(\mathcal{P}, -) : \text{Rep}(\mathcal{P}, \theta) \to \text{mod} A$. Since $\mathcal{P}$ is projective, this functor sends $\varepsilon$-sequences in exact sequences in $\text{mod} A$. Moreover, $H(\mathcal{P}) = A$ and $H(P(z)) = e_z A$.

**Remark 6.** If $f : V \to W$ is a morphism in $\text{Rep}(\mathcal{P}, \theta)$, with $f : V_0 \to W_0$ a monomorphism then $H(f) : H(V) \to H(W)$ is also a monomorphism.

**Lemma 1.** If $V \in \text{Rep}(\mathcal{P}, \theta)$, then $\text{soc} H(V)$ is projective. Moreover, $\dim_k (\text{soc} H(V)) = \dim_k V_0$.

**Proof** Take $V \in \text{Rep}(\mathcal{P}, \theta)$ and consider $\text{soc} H(V)$, if this is not a direct sum of copies of $e_0 A$, there is a non zero morphism $u : e_z A \to \text{soc} H(V)$ with $z \in \theta$, such that $u(\text{rad} e_z A) = 0$. We have $u = H(u')$ with $u' : P(z) \to V$, so $u' : P(z)_0 \to V_0$ is a non-zero $k$-linear map, then there is a non zero $k$-linear map $\lambda : S_0 \to P(z)_0$ such that $u' \lambda \neq 0$, but $\lambda$ is a morphism $S \to P(z)$ then $H(u') H(\lambda) \neq 0$. Here the image of $H(u')$ lies in the radical of $e_z A$, so $H(u') H(\lambda) = u(\text{rad} H(\lambda)) = 0$ a contradiction. This implies that $\text{soc} H(V)$ is projective. Then $H(V) \in \text{mod}_{\text{proj}}(A)$. Moreover, $\dim_k (\text{soc} H(V)) = \dim_k \text{Hom}_A(e_0 A, H(V)) = \dim_k \text{Hom}_{\text{Rep}(\mathcal{P}, \theta)}(S, V) = \dim_k V_0$.

**Lemma 2.** Consider $J \in \text{Rep}(\mathcal{P}, \theta)$, the representation of Remark 5, then $H(J) \cong E$ the injective envelope of $e_0 A$. 
Proof Since $J_0 = k$, then $\text{soc}H(J)$ is a simple right $A$-module, the injective envelope of $H(J)$ is of the form $u : H(J) \to E$. Here $\dim_k H(J) e_z = \dim_k \text{Hom}_{\text{Rep}(\mathcal{P}, \theta)}(P(z), J) = \dim_k (J_z) = \dim_k (k^z) = \text{card}(z)$. But because the above $\dim_k E e_z = \dim_k H(J) e_z$, therefore $H(J) \cong E$. ■

Theorem 1. The functor $H$ induces an equivalence of categories $H : \text{Rep}(\mathcal{P}, \theta) \to \text{mod}_{\text{sp}}(A)$ where $\text{mod}_{\text{sp}}(A)$ is the full subcategory of $\text{mod}A$ whose objects are the right $A$-modules $M$ with $\text{soc}M$ projective.

Proof We know that $H$ induces a full and faithful functor $H : \text{Rep}(\mathcal{P}, \theta) \to \text{mod}_{\text{sp}}(A)$. Now we are going to prove that the functor $H$ is dense. For this take $X \in \text{mod}_{\text{sp}}A$, then the injective envelope of $X$ is of the form $X \to E^n \cong H(J^n)$ so we have a monomorphism $f : X \to H(J^n)$. Take $g : Q \to X$ a projective cover of $X$ then we may assume $Q = H(L)$ with $L$ a projective in $\text{Rep}(\mathcal{P}, \theta)$. In this way we obtain a morphism $gf : H(L) \to H(J^n)$, since $H$ is a full functor there is a morphism $h_1 : L \to J_n$ with $H(h) = gf$. The morphism $h = h_2h_1$ with $h_1 : L \to h(L)$ a deflation and $h_2 : h(L) \to J^n$ such that $h_2 : h(L)_0 \to (J^n)$ a monomorphism. Therefore $gf = H(h) = H(h_2)H(h_1)$ where $H(h_1)$ an epimorphism and $H(h_2)$ a monomorphism. From this we infer that $\text{Im}(gf) = \text{Im}(f) \cong \text{Im}(H(h_1))$. Here $\text{Im}(f) = X$ and $\text{Im}(h_1) = H(\text{Im}h_1)$. This proves our proposition. ■

Remark 7. By a result of D. Simson [see, [10]], the category $\text{mod}_{\text{sp}}(A)$ has almost split sequences. Then Theorem 1 implies that the exact category $(\text{Rep}(\mathcal{P}, \theta), \varepsilon)$ has also almost split sequences.

5 The Auslander-Reiten quiver of posets with an involution of type $\mathcal{U}_n$.

By using the results from the previous section, we construct the Auslander-Reiten quiver for a poset type that we will denote by $\mathcal{U}_n$. In this section we will assume that $k$ is an algebraically closed field.

5.1 Poset with an involution of Type $\mathcal{U}_n$

We denote by $\mathcal{U}_n$ to the poset with an involution $(\mathcal{P}, \leq, \theta)$ where $(\mathcal{P}, \leq) = \{a_n < a_{n-1} < \cdots < a_1 < b_1 < b_2 < \cdots < b_{n-1} < b_n\}$ and $\theta = \{(a_i, b_j)\}_{i=1, \ldots, n}$. We denote by $\text{Rep}(\mathcal{U}_n)$ the category of representations of the poset $\mathcal{U}_n$.

The Hasse diagram of the poset $\mathcal{U}_n$ is as follows

We consider the following representations of $\mathcal{U}_n$:

- $\mathcal{L}_{1, i} = (\mathcal{L}_0, \mathcal{L}_{(a_j, b_j)})_{j \geq 1}$ where $\mathcal{L}_0 = k\{e\}$ and
  \[
  \mathcal{L}_{(a_j, b_j)} = \begin{cases} 
  (0, 0), & \text{if } j < i, \\
  (0, e), & \text{if } j \geq i.
  \end{cases}
  \]

- $\mathcal{L}_{2, i} = (\mathcal{L}_0, \mathcal{L}_{(a_j, b_j)})_{j \geq 1}$ where $\mathcal{L}_0 = k\{e\}$ and
  \[
  \mathcal{L}_{(a_j, b_j)} = \begin{cases} 
  ((0, e), (e, 0)), & \text{if } j \leq i, \\
  ((0, e)), & \text{if } j > i.
  \end{cases}
  \]

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A direct sum of some representations which are isomorphic to the trivial representation of the trivial indecomposable representation of \( U \) in this case. Take composable representations of \( U \) that are non trivial. Proposition 10. The representations above is the complete list of non trivial indecomposable representations of \( \mathfrak{U}_n \).

**Remark 8.** The representation \( S = (k, S_z)_{z \in \theta} \) with \( S_z = 0 \) for all \( z \in \theta \), is called a trivial indecomposable representation of \( \mathfrak{U}_n \).

**Proof.** We will prove by induction on \( n \) that any representation of \( \mathfrak{U}_n \) can be written as a direct sum of some representations which are isomorphic to the trivial representation and to some representations in the previous list. We first consider the case \( n = 1 \), in this case \( \mathfrak{U}_1 = (P, \theta) \), with \( P = \{a, b\} \) with \( a < b \) and \( \theta = z \), where \( z = (a, b) \). Take \( V = (V_0, V_2) \) a representation of \( \mathfrak{U}_1 \). Here we can take \( V_0 = V_0 \oplus V_0 \) where for \( w = (w_1, w_2) \in V_0 \), \( \pi_1(w) = w_1 \), and \( \pi_0(w) = w_2 \). Then by definition if \( (w_1, w_2) \in V_0, (0, w_1) \in V_2 \). Take \( u(1), \ldots, u(n) \) a basis for \( V_0 \) and \( e_1, \ldots, e_m \) a basis for \( V_0 \), then we have \( u(s) = \left( \sum_{j=1}^{m} a_j^1 e_j, \sum_{j=1}^{m} a_j^2 e_j \right) \) for \( s = 1, \ldots, n \). We have the matrices \( T_a = (\alpha_{a,1}^1) \) and \( T_b = (\alpha_{a,1}^2) \) Choosing some other bases, the matrices \( T_a \) and \( T_b \) change to \( ST_a R \) and \( ST_b R \) where \( S \) and \( R \) are non singular matrices. Here we are assuming that \( k \) is an algebraically closed field, then using the Kronecker decomposition theorem we may find bases for \( V_0 \) and \( V_0 \) such that the corresponding matrices \( T_a, T_b \) with respect to the chosen bases is a direct sum of matrices of the form:

\[
\begin{align*}
\text{a. } & \left( \begin{array}{c} E_n \\ 0_{1,n} \end{array} \right), \left( \begin{array}{c} 0_{1,n} \\ E_n \end{array} \right) & \text{c. } & (0_{n,1}, E_n), (E_n, 0_{n,1}) & \text{e. } & J_{\lambda,n}, E_n \\
\text{b. } & \left( \begin{array}{c} 0_{1,n} \\ E_n \end{array} \right), \left( \begin{array}{c} E_n \\ 0_{1,n} \end{array} \right) & \text{d. } & (E_n, 0_{n,1}), (0_{n,1}, E_n) & \text{f. } & E_n, J_{\lambda,n}
\end{align*}
\]

Therefore we have a decomposition \( V_0 = V_0^{(1)} \oplus \cdots \oplus V_0^{(j)} \) and \( V_0 = V_0^{(1)} \oplus \cdots \oplus V_0^{(l)} \) with \( V_0^{(j)} \subset (V_0^{(j)})^2 \) for \( j = 1, \ldots, l \). Moreover, there are bases for each \( V_0^{(j)} \) and \( V_0^{(j)} \).
such that the corresponding matrices $T'_j, T'_j$ have one of the forms $a, b, c, d, e, f$. Since $(V'_0, V_z)$ is a representation, each $(V'_0, V'_z)$ is a representation of $(\mathcal{F}, \theta)$.

Therefore we may assume that there is a basis for $V'_0$ and $V_z$ such that the pair of matrices $T_a, T_b$ have one of the forms $a, b, c, d, e, f$. Here $\pi_a(V'_z) \subset \pi_b(V_z)$, then the only pair of matrices satisfying this condition are $c, d, e, f$.

Consider the case $c$:
Here for $n = 1, V_z$ is generated by the vectors $u(1) = (0, e_1), u(2) = (e_1, 0)$, therefore $(V'_0, V_z) = L_{2,1}$. Now if $n = 2$, then $V_z$ is generated by the vectors $u(1) = (0, e_1), u(2) = (e_1, e_2), u(3) = (e_2, e_1), ..., u(n) = (e_{n-1}, e_n), u(n + 1) = (e_n, 0)$. Observe that for $n = 2, (e_2, 0) \in V_z$ but the vector $(0, e_2)$ is not in $V_z$. For $n = 3$, the vector $(e_2, e_3) \in V_z$ but $(0, e_2)$ is not in $V_z$. Therefore the only possibility is $n = 1$ and in this case $(V'_0, V_z) = L_{2,1}$. Similarly in case $d$, $n = 1$ and $(V'_0, V_z) = L_{2,1}$.

In cases $e$ and $f$ if $\lambda \neq 0$, the vectors in $V_z$ have the form $(\lambda x, x)$ in case $e$ and $(x, \lambda x)$ in case $f$. In the first case for $x \neq 0, \lambda x \in \pi_a(V_z)$ but $(0, x)$ is not in $V_z$; similarly in the second case there is $x \in \pi_a(V_z) \setminus (0, x)$ is not in $V_z$. Suppose now $\lambda = 0$. In case $e$, for $n = 1, V_z = \langle (0, e_1) \rangle$, therefore $V = (V'_0, V_z) = L_{1,1}$. For $n = 2, V_z = \langle (0, e_1), (e_1, e_2) \rangle$, in this case $V = (V'_0, V_z) = L_{3,1}$. For $n > 2$, we have $(0, e_1), (e_1, e_2), (e_2, e_3) \in V_z$, but $(0, e_1)$ is not in $V_z$. Therefore this case can not happen. For $f$ with $\lambda = 0$, we have that $(e_1, 0)$ is in $V_z$ but $(0, e_1)$ is not in $V_z$, therefore this case can not happen. We have proved that the indecomposable representations of $(\mathcal{F}, \theta)$ are $L_{1,1}, L_{2,1}$ and $L_{3,1}$. This shows our result for case $n = 1$.

Now, assume our result true for $u_{n-1}$. Let $V = (V'_0, V_{(a_j, b_j)})_{1 \leq n}$ be a representation of $U_n$. We take $U_1 = (a_n < b_n; \{a_n, b_n\})$. Then $(V'_0, V_{(a_n, b_n)})$ is a representation of $U_1$. Therefore $(V'_0, V_{(a_n, b_n)}) = S(W'_0) \oplus V$ with $V'_0 = V_1 \oplus V_2 \oplus V_3 \oplus V_z'$ and

$$\tilde{V}_{(a_n, b_n)} = V_{(a_n, b_n)} = (0, V_1) \oplus (V_2, 0) \oplus (0, V_2) \oplus (0, V_3) \oplus H_\phi(V_3, V_z')$$

where, $V_0 = W_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_z, \phi : V \rightarrow V_z'$ is an isomorphism and $H_\phi(V_3, V_z') = \{(\phi(u)w) \mid u \in V_3\}$.

Let $U_{n-1} = \{a_n < 1 < \cdots < a_i < b_i < \cdots < b_{n-1} ; \{a_n, b_n, \ldots, (a_{n-1}, b_{n-1})\}\}$ and $\mathcal{L} = (V'_0, V_{(a_i, b_i)})_{1 \leq n-1}$ be the restriction of the representation $V$ to $U_{n-1}$.

**Remark 9.** If $u \in V_2 \oplus V_3$ then $(u, 0)$ and $(0, u)$ are in $V_{(a_i, b_i)}$ for all $i < n$. Indeed, if $u \in V_2, (u, 0) \in V_{(a_i, b_i)}$. As $a_n < a_i, (u, 0) \in V_{(a_i, b_i)}$ and as $a_i < b_i, (0, u) \in V_{(a_i, b_i)}$. If $u \in V_3$ then $(u, \phi(u)) \in V_{(a_n, b_n)}$. As $a_n < a_i, (u, 0) \in V_{(a_i, b_i)}$, therefore $(0, u) \in V_{(a_i, b_i)}$.

**Remark 10.** If $(u, v) \in V_{(a_i, b_i)}$ with $i < n$ then $u$ and $v$ are in $V_1 \oplus V_2 \oplus V_3$. Indeed, as $a_i < b_i, (0, u) \in (a_i, b_i)$, therefore $u \in V_1 \oplus V_2 \oplus V_3$. Analogously, as $b_i < b_n$ then $(0, v) \in V_{(a_i, b_n)}$, thus $v \in V_1 \oplus V_2 \oplus V_3$.

We consider $V'' = (V_1, V''_{(a_i, b_i)})_{1 \leq i \leq n-1}$, where $V''_{(a_i, b_i)} = \{(u, v) \in V_{(a_i, b_i)} \mid u, v \in V_1\}$, and

$$V''' = (L, V'''_{(a_i, b_i)})_{1 \leq i \leq n-1}.$$
with $L = V_2 \oplus V_3 \oplus V_3' \oplus W'$ and $V''_{(a_i, b_i)} = \{(u, v) \in V_{(a_i, b_i)} \mid u, v \in L\}$. By Remarks 9 and 10 for all $i$, $1 \leq i \leq n - 1$:

$$V''_{(a_i, b_i)} = (V_2, 0) + (0, V_2) + (V_3, 0) + (0, V_3).$$

Both $V'$ and $V''$ are representations of $\mathcal{U}_{n-1}$.

**Claim** $V = V' \oplus V''$.

We just have to prove that for all $i < n$,

$$V_{(a_i, b_i)} = V'_{(a_i, b_i)} \bigoplus V''_{(a_i, b_i)}.$$ 

Let $(u, v) \in V_{(a_i, b_i)}$, due to Remark 10 $u = u_1 + u_2$, $v = v_1 + v_2$ where $v_1, v_2$ are in $V_1$, and $u_2, v_2$ are in $V_2 \oplus V_3$; thus $(u_2, v_2) = (u_2, 0) + (0, v_2) \in V_{(a_i, b_i)}$, so $(u_1, v_1) \in V'_{(a_i, b_i)}$ and since $u_2, v_2$ are in $V_1$, it is obtained that $(u_1, v_1) \in V''_{(a_i, b_i)}$ and clearly $(u_2, v_2) \in V''_{(a_i, b_i)}$. This proves our claim for $n = 1$. We now suppose proved our result for $\mathcal{U}_{n-1}$, for proving it for $\mathcal{U}_n$ we denote by $L_{1,s}^{n-1}, L_{2,s}^{n-1}$ and $L_{3,s}^{n-1}$ the representations given in (1), (2) and (3) for $1 \leq s \leq n - 1$. In a similar way, we have the corresponding representations for $u_n$. $L_{1,t}, L_{2,t}, L_{3,t}$ for $1 \leq t \leq n$. Then observe that for $1 \leq s, t \leq n - 1$ one has:

$$L_{1,s}^{n-1}(e)(a_i, b_i) = L_{1,s}^{a_i}(e)(a_i, b_i); \quad L_{1,s}^{n-1}(e)(a_i, b_i) = \langle 0, e \rangle;$$

$$L_{2,s}^{n-1}(e)(a_i, b_i) = L_{2,s}^{a_i}(e)(a_i, b_i); \quad L_{2,s}^{n-1}(e)(a_i, b_i) = \langle 0, e \rangle;$$

$$L_{3,s}^{n-1}(e, f)(a_i, b_i) = L_{3,s}^{a_i}(e, f)(a_i, b_i); \quad L_{3,s}^{n-1}(e, f)(a_i, b_i) = \langle 0, e, (0, f) \rangle.$$

By application of the induction hypothesis we obtain:

$$V' = \left( \bigoplus_{e \in B_{1,s}, 1 \leq s \leq n-1} L_{1,s}^{a_i}(e) \right) \bigoplus \left( \bigoplus_{e \in B_{2,t}, 1 \leq t \leq n-1} L_{2,t}^{a_i}(e) \right) \bigoplus \left( \bigoplus_{e \in B_{3,r}, 1 \leq r \leq n-1} L_{3,r}^{a_i}(e, \psi(e)) \right) \bigoplus S(W_0')$$

where $B_{1,s}, 1 \leq s \leq n - 1, B_{2,t}, 1 \leq t \leq n - 1, B_{3,r}, 1 \leq r \leq n - 1$ are subsets of $V_1$, $\psi : B_{3,r} \rightarrow B_{3,r}'$ is a bijection, and $B = \bigcup B_{1,s} \bigcup B_{2,t} \bigcup B_{3,r} \bigcup B_{3,r}'$ is a set of linearly independent elements and $V_1 = \langle B \rangle \oplus W''_0$.

Take now $h_1, \ldots, h_l$ a $k$-basis for $V_2$ and $g_1, \ldots, g_m$ a $k$-bases for $V_3$. Then for any $i$ with $1 \leq i \leq n - 1$, we have

$$V_{(a_i, b_i)} = V''_{(a_i, b_i)} \bigoplus \bigoplus_{j=1}^{m} (g, 0, 0, g_j).$$
Take $B_{1,n}$ a bases for $W_0$. We have:

$$V(a_i, b_i) = \bigoplus_{e \in B_{1,s}, 1 \leq s \leq n-1} \mathcal{L}_{1,s}^n(e)_{(a_n, b_n)} \bigoplus \bigoplus_{e \in B_{2,t}, 1 \leq t \leq n-1} \mathcal{L}_{2,t}^n(e)_{(a_n, b_n)} \bigoplus \bigoplus_{e \in B_{3,r}, 1 \leq r \leq n-1} \mathcal{L}_{3,r}^n(e, \psi(e))_{(a_n, b_n)} \bigoplus \bigoplus_{e \in B_{4,s}, 1 \leq s \leq n-1} \mathcal{L}_{3,s}^n(e)_{(a_n, b_n)} \bigoplus \bigoplus_{j=1}^m \mathcal{L}_{2,1}^n(h_j)_{(a_n, b_n)} \bigoplus \bigoplus_{j=1}^m \mathcal{L}_{3,n}^n(g_j, \phi(g_j))_{(a_n, b_n)} \bigoplus S(W_0).$$

Here $\mathcal{L}_{3,n}^n(g_j, \phi(g_j))_{(a_n, b_n)} = ((g_j, 0), (0, g_j))$. We obtain:

$$V = \bigoplus_{e \in B_{1,s}, 1 \leq s \leq n} \mathcal{L}_{1,s}^n(e) \bigoplus \bigoplus_{e \in B_{2,t}, 1 \leq t \leq n-1} \mathcal{L}_{2,t}^n(e) \bigoplus \bigoplus_{e \in B_{3,r}, 1 \leq r \leq n-1} \mathcal{L}_{3,r}^n(e, \psi(e)) \bigoplus \bigoplus_{j=1}^m \mathcal{L}_{2,1}^n(h_j) \bigoplus \bigoplus_{j=1}^m \mathcal{L}_{3,n}^n(g_j, \phi(g_j)) \bigoplus S(W_0).$$

This proves our result.

Using the notation in $a$, $b$ and $c$ of 5.1 by means of a direct calculation, the following Lemmas can be proved.

**Lemma 3.** (a.) For all $i$, \( \text{Hom}(\mathcal{L}_{3,i}, \mathcal{L}_{3,i}) = \left\langle \phi = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \ \psi = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle. \)

(b.) For $i > j$, \( \text{Hom}(\mathcal{L}_{3,i}, \mathcal{L}_{3,j}) = \left\langle \phi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \ \psi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle. \)

(c.) For $i < j$, \( \text{Hom}(\mathcal{L}_{3,i}, \mathcal{L}_{3,j}) = \left\langle \phi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \ \psi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle. \)

**Lemma 4.** (a.) If $i > j$, \( \text{Hom}(\mathcal{L}_{2,i}; \mathcal{L}_{3,j}) = \left\langle \phi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \ \psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle. \)

(b.) If $i \leq j$, \( \text{Hom}(\mathcal{L}_{1,i}, \mathcal{L}_{3,j}) = \left\langle \phi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle. \)

**Lemma 5.** If $\text{Hom}(\mathcal{L}_{2,i}, \mathcal{L}_{3,j}) \neq 0$, then $i < j$ and this $k$-vector space is generated by $\phi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

**Theorem 2.** 1. A representative set of the isomorphism classes of indecomposable projectives in $\text{Rep}(\mathcal{U}_n)$ is given by $\mathcal{L}_{3,r}$ with $1 \leq r \leq n$ and the trivial representation $S$.

2. A representative set of the isomorphism classes of indecomposable injectives in $\text{Rep}(\mathcal{U}_n)$ are given by $\mathcal{L}_{3,r}$ with $1 \leq r \leq n$ and $\mathcal{L}_{2,n}$.
3. The following are all the almost split sequences in Rep(\mathcal{U}_n):

For 2 \leq i \leq n,

\[ \mathcal{L}_{1,1} \xrightarrow{(0,1)^t} \mathcal{L}_{3,1} \xrightarrow{(1,0)} \mathcal{L}_{1,1}. \]

For 1 \leq i \leq n - 1,

\[ \mathcal{L}_{1,1} \xrightarrow{(0,1)^t} \mathcal{L}_{3,i-1} \xrightarrow{(1,0)} \mathcal{L}_{2,i} \]

and

\[ S \xrightarrow{(0,1)^t} \mathcal{L}_{3,n} \xrightarrow{(1,0)} \mathcal{L}_{2,n}. \]

Proof

1. By Proposition 5, \( \mathcal{L}_{3,i} = P(\{a_i, b_i\}) \), and \( S \) are all the projectives.

2. Observe that \( \mathcal{U}_n^{op} \) is also of the form \( \mathcal{U}_n \), then indecomposables \( W \in \text{Rep}(\mathcal{U}_n^{op}) \) with \( \dim W = 2 \) are projectives, this implies by Corollary 3, that indecomposables \( V \in \text{Rep}(\mathcal{U}_n) \) with \( \dim V = 2 \) are injectives, therefore all the \( \mathcal{L}_{3,i} \) are injectives, and \( \mathcal{L}_{2,n} \) coincides with the representation \( J \) of Remark 4, therefore the injectives are the projectives \( \mathcal{L}_{3,i} \) and \( \mathcal{L}_{2,n} \).

3. All sequences in this item are \( \varepsilon \)-sequences. We will prove that the morphism \( \mathcal{L}_{1,i} \xrightarrow{(0,1)^t} \mathcal{L}_{3,i-1} \) is a left almost split morphism. For this it is enough to prove that for any morphism \( v : \mathcal{L}_{1,i} \rightarrow V \), with \( V \) indecomposable non isomorphic to \( \mathcal{L}_{1,i} \) there is a morphism \( u : \mathcal{L}_{3,i-1} \rightarrow V \) with \( u(0,1)^t = v \). We may assume \( v \neq 0 \), and \( v(e) = e \). Then if \( V = \mathcal{L}_{1,j} \) we must have \( j < i \), and the linear map \( u = (0,1) : k(e_1, e_2) \rightarrow k(e) \) is a morphism from \( \mathcal{L}_{3,i-1} \rightarrow \mathcal{L}_{1,j} \) with \( u(0,1)^t = v \). Now if \( V = \mathcal{L}_{2,j} \), again we may suppose that \( v(e) = e \). Then the linear map \( u = (0,1) : k(e_1, e_2) \rightarrow ke \) gives a morphism from \( \mathcal{L}_{3,i-1} \rightarrow \mathcal{L}_{1,i} \) with \( u(0,1)^t = v \). Now suppose \( V = \mathcal{L}_{3,j} \), if \( i - 1 = j \), then \( v = (d, c)^t \). The linear map \( u = \begin{pmatrix} c & d \\ 0 & e \end{pmatrix} : k(e_1, e_2) \rightarrow k(e_1, e_2) \) is a morphism from \( \mathcal{L}_{3,i-1} \rightarrow \mathcal{L}_{3,i-1} \) such that \( u(0,1)^t = v \). Suppose now \( i - 1 > j \), the linear map \( u = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} \) is a morphism \( \mathcal{L}_{3,i-1} \rightarrow \mathcal{L}_{3,j} \) with \( u(0,1)^t = v \). In case \( i < j \), we have \( v = (c, 0)^t \), and there is a morphism \( u : \mathcal{L}_{3,i-1} \rightarrow \mathcal{L}_{3,j} \) given by the linear map \( u = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \), such that \( v = u(0,1)^t \). In a similar way, using the above lemmas one can prove that the first morphism in each of the sequences is a left almost split morphism. Since the other end of the sequence is indecomposable, therefore all these sequences are almost split sequences.

In this way we obtain that the Auslander-Reiten quiver for \( D_n \) is given by
5.2 Poset with an involution of type $\mathfrak{U}_\infty$

The previous results can be extended to posets of type $\mathfrak{U}_\infty$ in the following way. We define the functor

$$\Xi_n : \text{Rep}(\mathfrak{U}_n) \to \text{Rep}(\mathfrak{U}_\infty)$$

such that if $V = (V_0, V_{(a_i, b_i)})_{1 \leq i \leq n}$ then $\Xi(V) = (V_0, V_{(a_i, b_i)})_{1 \leq i \leq n}$ with $V_{(a_j, b_j)} = (0, V_0)$ for $j > n$.

Clearly if $f : V \to W$ is a morphism in $\text{Rep}(\mathfrak{U}_n)$ determined by the morphism $f : V_0 \to$
$W_0$ then this morphism is the same in the category $\text{Rep}(U_\infty)$. So, taking $\Xi_n(f) = f$ we have defined the functor $\Xi_n$. Analogously we define a functor: $\Theta_n : \text{Rep}(U_n) \to \text{Rep}(U_{n+1})$ and it is obtained that $\Xi_{n+1}\Theta_n = \Xi_n$.

**Proposition 11.** The indecomposable representations of $U_\infty$ are the representations $L_{s,i} = \Xi_i(L_{s,i})$ together with the simple trivial representation $S$.

**Proof.** Let $V = (V_0, V_{(a_i, b_i)})_{i \leq i}$ be an indecomposable representation of $U_\infty$. We suppose that $V_{(a_1, b_1)} = 0$, then for each $n$ the restriction of $V$ to $U_n$ is the form $(V)_{U_n} = \bigoplus_{i=1}^s L_i$ with $L_i = \mathcal{L}_{1,i}$ with $i > 1$ or the trivial representation. Since $V_0$ is finite dimensional there exists $n$ and finite sum $W = \bigoplus_{i=1}^a L_i$ in $U_n$ such that for all $m > n$, $V$ restricted to $U_m$ coincides with the restriction of $\Xi_n(W)$ to $U_m$, therefore $V = \Xi_n(W)$ and since $V$ is an indecomposable, then $V = \Xi_n(L_{1,n})$.

Now we suppose that $V_{(a_1, b_1)} \neq 0$ then $V$ restricted to $U_n$ is the form $\bigoplus_{i=1}^a L_i$ where each $L_i$ is the form $\mathcal{L}_{j,i}$ with $j = 1, 2, 3$ and at least one $L_i$ has the form $\mathcal{L}_{2,i}$ or $\mathcal{L}_{3,i}$. As before, there exists $n$ such that for all $m \geq n$ the restriction from $V$ to $U_m$ coincides with the restriction of $\Xi_n(W)$ to $U_m$ therefore $V = \Xi_n(L_{j,n})$ with $j = 2$ or $j = 3$. ■

**Proposition 12.** Let $a : X \xrightarrow{u} Y \xrightarrow{v} Z$ be an almost split sequence in $\text{Rep}(U_n)$ with $X$ different from the trivial representation. Then

$$b : \Xi_n(X) \xrightarrow{\Xi_n(a)} \Xi_n(Y) \xrightarrow{\Xi_n(v)} \Xi_n(Z),$$

is an almost split sequence in $\text{Rep}(U_\infty)$.

**Proof.** The sequence $b$ is a nontrivial $\varepsilon$-sequence in $\text{Rep}(U_\infty)$ whose extremes are indecomposable. Let $h : Y \to \Xi_n(Y)$ be a morphism that is not a retraction in $\text{Rep}(U_\infty)$ with $Y$ indecomposable, then $Y = \Xi_m(W)$ with $W$ indecomposable in $\text{Rep}(U_m)$ for some $m > n$. We have that $\Xi_m(a)$ is an almost split sequence in $\text{Rep}(U_m)$, so we can suppose that $m = n$ and then $h = \Xi_m(w)$ where $w : W \to Z$ is a morphism that is not a retraction. Therefore, there exists $g : W \to Y$ with $vg = w$, thus $h = \Xi_m(v)\Xi_m(g)$. This proves our assertion.

In this way, we obtain that the Auslander-Reiten quiver for $D_\infty$ is given by
Figure 5: Auslander-Reiten quiver of a poset with an involution of type $D_\infty$. 

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