Derivation of Number Based Size Distribution from Modified Mass Based Rosin-Rammler Distribution and Estimation of the Various Mean Particle Diameters of Powder*

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The present study deals with the characterization in the particle size distributions of metallic powders. The Rosin-Rammler distribution is used as a particle size distribution function for its good applicability. First, a new number based expression is introduced for a modified Rosin-Rammler distribution which has been proposed in a previous paper, and compared with the previous expression. Next, the calculating equations are presented for the various mean particle diameters of the powders, having the distributions. Furthermore, the equations can be applied to the cases of classification and blending of powders.

The results are summarized as follows:
1. New number based approximate equation is at least ten times more accurate for the expression of the modified Rosin-Rammler distribution than the previous equation.
2. The equations for the calculation of the various mean particle diameters of the distributions have been derived.
3. The various mean particle diameters can be also calculated in the cases of classification and blending of the powders.

The present results can be also applicable for other distributions such as grain and pore sizes, etc.

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I. Introduction

Particle size distribution is one of the most fundamental and important characteristics of industrial powders. For the mathematical expression of the particle size distribution, the Rosin-Rammler distribution(1) has been widely applied as well as the log-normal one. As has been concluded in the previous work(2), the Rosin-Rammler distribution (described hereafter as the R-R distribution) has much better applicability for the mass based size distributions of real metal powders. There has been, however, no satisfactory study in the R-R distribution hitherto, not only for the interrelation between weight-based and population-based expressions but also for the estimation of the various mean particle diameters.

In the present work, the following two analyses are carried out to evaluate adequately industrial metal powders through the characterization of the particle size distribution. The first is deriving a new number based expression from the modified mass based R-R distribution which has been proposed in the previous paper(2). The second is intended for the presentation of the equations through which mean particle diameters can be determined for the powders. The result will be applied to the size evaluation in the classification and blending of powders.

II. A New Number Based Expression for a Modified R-R Distribution

1. Theoretical derivation to the number based distribution equation from the mass based one

A modified mass based R-R distribution has been defined(2) as eq. (1) normalized over the
finite range between two size limits, $D_{\text{min}}$ and $D_{\text{max}}$.

$$R(D_p) = \frac{100}{\alpha - \beta} \left[ \exp \left\{ -\left( \frac{D_p}{D_v} \right)^m \right\} - \beta \right]$$

(1)

where $R(\%)$ is the cumulative oversize mass fraction, $D_p(\mu m)$ the particle size, $m(-)$ the distribution constant, and $D_v(\mu m)$ the absolute size constant corresponding to 36.8% of the cumulative fraction. Both of the constants, $m$ and $D_v$ are the parameters which characterize the distribution. The particle sizes, $D_{\text{min}}$ and $D_{\text{max}}(\mu m)$, are of minimum and maximum, corresponding to 99.9% ($\alpha=0.999$) and 0.1% ($\beta=0.001$) of the cumulative fraction for the conventional R-R distribution, respectively. Figure 1 shows a modified Rosin-Rammler-Sperling diagram (R.R.S. diagram) based on eq. (1) in the case of $m=2.5$ and $D_v=70\mu m$, where the broken line indicates the conventional R-R distribution and the straight solid line the modified one. Although there seems to appear obvious difference between the two lines in the diagram because of the exaggerated enlargement in the end ranges near 100% and 0% of the cumulative fraction, the real difference is negligibly small in the usual measurement of the actual size distribution: i.e., the modification for the R-R distribution gives no essential changes in the actual application.

In order to theoretically obtain a number based expression from the mass based distribution formula, eq. (1) is differentiated with regard to the particle size into

$$\frac{dR}{dD_p} = \frac{100}{\alpha - \beta D_v} \left( \frac{D_p}{D_v} \right)^{m-1} \exp \left\{ -\left( \frac{D_p}{D_v} \right)^m \right\}.$$  

(2)

There is also a differential relation between the mass and number based fractions as follows:

$$dR = \left( \frac{D_p}{D_v} \right)^3 \, dY$$  

(3)

where $D_v(\mu m)$ is the mean volume diameter, $Y(\%)$ the cumulative oversize population fraction. In the case of the modified R-R distribution, the mean volume diameter can be expressed with eq. (4) according to the definition,

$$\left( \frac{1}{D_v} \right)^3 = \frac{1}{\alpha - \beta D_v} \left( \frac{D_p}{D_v} \right)^{m-4} \exp \left\{ -\left( \frac{D_p}{D_v} \right)^m \right\} dD_p.$$  

(4)

The frequency distribution function based on the population can be formulated by substituting eqs. (3) and (4) into eq. (2). Then the integration over the interval from $D_p$ to $D_{\text{max}}$ and the rearrangement give the cumulative oversize number fraction $Y$,

$$Y(D_v) = \frac{100}{\alpha - \beta D_v} \left[ \exp \left\{ -\left( \frac{D_p}{D_v} \right)^m \right\} \right]_{D_p}^{D_{\text{max}}}$$  

(5)

which is the theoretical expression for the modified number based R-R distribution. Figure 2 shows an example of converting the mass basis into the number one for a modified R-R distribu-
tion through eqs. (1) and (5). The solid line indicates the mass based distribution and the broken line the number based one. The definite integral in eq. (5) can be calculated automatically by the numerical method developed by Takahashi and Mori[3][4], based on the double exponential formula.

2. Other expressions for the number based equation of the modified mass based R-R distribution

(1) Analytically derivable expressions

Instead of eq. (5), two alternative expressions are mathematically derived for the number based equation of the modified mass based R-R distribution. One is the equation which includes the second type incomplete gamma functions, and from eq. (5),

\[ Y(D_p) = \frac{\Gamma \left( \frac{3}{m}, \left( \frac{D_{\text{max}}}{D_e} \right)^m \right) - \Gamma \left( \frac{3}{m}, \left( \frac{D_p}{D_e} \right)^m \right)}{\Gamma \left( \frac{3}{m}, \left( \frac{D_{\text{max}}}{D_e} \right)^m \right) - \Gamma \left( \frac{3}{m}, \left( \frac{D_{\text{min}}}{D_e} \right)^m \right)} \]

where \( X_{\text{min}} = \frac{(D_{\text{min}}/D_e)^m}{-\ln(\alpha)} \), \( X_{\max} = \frac{(D_{\text{max}}/D_e)^m}{-\ln(\beta)} \). As \( \exp(-X) \) is independent on the constant \( m \) for the integration in eq. (7) and can be mathematically developed into the following summation of infinite progression:

\[ \exp(-X) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(i-1)!} X^{i-1}. \]

Then the expression

\[ Y(D_p) = 100 \int_{X_{\text{min}}}^{X_{\text{max}}} \frac{X^{-3/m} \exp(-X) \, dX}{\int_{X_{\text{min}}}^{X_{\text{max}}} X^{-3/m} \exp(-X) \, dX} \]

\[ \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(i-1)!} \frac{1}{-3/m+i} \left\{ \left( \frac{D_{\text{max}}}{D_e} \right)^{-3+m-i} - \left( \frac{D_p}{D_e} \right)^{-3+m-i} \right\} - \left\{ \left( \frac{D_{\text{min}}}{D_e} \right)^{-3+m-i} - \left( \frac{D_{\text{max}}}{D_e} \right)^{-3+m-i} \right\} \]

can be deduced by substituting eq. (8) into eq. (7), integrating the resultant equation with \( X \), and replacing \( X \), \( X_{\text{min}} \) and \( X_{\text{max}} \) into \((D_p/D_e)^m\), \((D_{\text{min}}/D_e)^m\) and \((D_{\text{max}}/D_e)^m\), respectively. If \(-3/m + i = 0\), only the concerned term must be replaced by the following form:

\[ \frac{(-1)^{i-1}}{(i-1)!} \frac{1}{-3/m+i} \left\{ \left( \frac{D_{\text{max}}}{D_e} \right)^{-3+m-i} - \left( \frac{D_p}{D_e} \right)^{-3+m-i} \right\} \rightarrow \frac{(-1)^{i-1}}{(i-1)!} \left[ \ln \left( \frac{D_{\text{max}}}{D_e} \right) - \ln \left( \frac{D_p}{D_e} \right) \right] \]

where \( D_x \) is equal to \( D_p \) or \( D_{\text{min}} \). Equation (9) is of a expression with a type of primitive functions and can be applied for any values of \( m \). It is, however, not so practical due to much troublesome calculation.

(2) Approximate expressions

Practical application demands the approximation of eq. (5) through which the easier calculation is possible for the wide region of the distribution constant \( m \). The following ex-
pression

\[ Y(D_p) \approx \frac{100}{\alpha' - \beta'} \left[ \exp \left\{ - \frac{(D_p)}{D_e} \right\} - \beta \right] \]

\[ \equiv Y(D_p) \]  

(10)

has been already proposed in the previous work\(^{(2)}\), where \( n \) and \( D_e \) are the distribution constant and the absolute size constant of number basis, and \( \alpha' \) and \( \beta' \) are determined by the equations

\[ \alpha' = \exp \left\{ - \left( \frac{D_{\min}}{D_e} \right)^n \right\}, \]

\[ \beta' = \exp \left\{ - \left( \frac{D_{\max}}{D_e} \right)^n \right\}, \]

respectively. The parameters \( n \) and \( D_e \) have been also given with two equations which are the functions of \( m \) and \( D_e \). These equations are practical rather than theoretical, and adaptable within a region between 1 and 5 of \( m \). More precisely, the equation

\[ Y(D_p) \approx \frac{100}{\alpha' - \beta'} \left[ \exp \left\{ - \frac{(D_p)}{D_e} \right\} - \beta \right] \]

\[ \equiv Y(D_p) \]  

(11)

has been also prepared for a correction term to eq. (10), where \( y' = Y'/100 \), and \( a \) and \( b \) are constants which are dependent only upon the distribution constant \( m \). The dependencies have been illustrated in the previous work\(^{(2)}\). Even though eq. (11) is a better approximate expression than eq. (10), the accuracy is not enough for the calculation of mean particle diameters, resulting in significant error as shown later.

New approaches are then tried for the approximation of eq. (5) according to the similar way of the derivation to eq. (9) by searching for a polynomial expression which is appropriate for \( \exp (-X) \). First, instead of eq. (8),

\[ \exp (-X) \approx \sum_{i=1}^{k} \frac{(-1)^{i-1}}{(i-1)!} X^{i-1} \]  

(12)

can be applied. This approximate expression is almost similar to eq. (8), but has a restriction to the first \( k \) terms on the summation of the infinitely developed progression series. An examination is then carried out about the effect of the number of the term on the approximation precision. The result is shown in Fig. 3, where the abscissa indicates the number of the term \( k \) and the ordinate in the logarithmic scale the standard deviation (the error of mean square) \( \sigma_{12} \) as follows:

\[ \sigma_{12}^2 = \frac{1}{X_{\max} - X_{\min}} \left( \sum_{i=1}^{k} \frac{(-1)^{i-1}}{(i-1)!} X^{i-1} \right) \exp (-X) \]  

\[ \int dX. \]  

(13)

The integration in eq. (13) can be also evaluated by the numerical method similar to that in eq. (5)\(^{(3)(4)}\). According to Fig. 3, eq. (12), for example, requires the summation of the first twenty terms for 0.003 of the standard deviation. It is troublesome to estimate the cumulative oversize number fraction \( Y \) by using eq. (12) because of the so many progression terms.

Some calculations are tried to fairly approx-
imate exp \((-X)\) in the region between \(X_{\text{min}}\) and \(X_{\text{max}}\) with a polynomial expression which consists of the terms as few as possible. Several trials result that six terms are enough to satisfy both simplification and accuracy in expression. Then the expression with six terms is defined as follows:

\[ \exp (-X) \approx \sum_{i=1}^{6} a_{i-1} X^{i-1} \]  

where \(a_{i-1}\) are constants. Since \(\exp (-X) = 1\) at \(X = 0\), it holds always that \(a_0 = 1\). The other constants \(a_{i-1}\) \((i = 2 \sim 6)\) can be evaluated by the successive least square approximation. The results are shown in Table 1.

As for the precision of eq. (14) using those constant values, the standard deviation \(\sigma_{14}\) can be defined as well as \(\sigma_{12}\) as follows:

\[
\sigma_{14}^2 = \frac{1}{X_{\text{max}} - X_{\text{min}}} \int_{X_{\text{min}}}^{X_{\text{max}}} \left( \sum_{i=1}^{6} a_{i-1} X^{i-1} - \exp (-X) \right)^2 dX.
\]  

The resultant standard deviation \(\sigma_{14} = 0.005\) is as the same magnitude as obtained from eq. (10) by the summation up to the 20 terms. The adaptability of eq. (14) for \(\exp (-X)\) is also shown in Fig. 4, where both values based on \(\exp (-X)\) and eq. (14) are plotted against \(X\).

From the above discussions, it can be said that eq. (14) is a simple and accurate expression of \(\exp (-X)\) for the purpose. Instead of eq. (9), eq. (14) leads to

\[
Y(D_p) = \frac{100}{\sum_{i=1}^{6} a_{i-1} \left( \left( \frac{D_{\text{max}}}{D_e} \right)^{-3/m + i} - \left( \frac{D_{\text{p}}}{D_e} \right)^{-3/m + i} \right)} \left( \ln \left( \frac{D_{\text{max}}}{D_e} \right) - \ln \left( \frac{D_{\text{p}}}{D_e} \right) \right)
\]

If \(-3/m + i = 0\), only the term to satisfy the condition should be replaced with the following form:

\[
\frac{a_{i-1}}{-3/m + i} \left( \left( \frac{D_{\text{max}}}{D_e} \right)^{-3/m + i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-3/m + i} \right) \rightarrow a_{i-1} \{ \ln \left( \frac{D_{\text{max}}}{D_e} \right) - \ln \left( \frac{D_{\text{min}}}{D_e} \right) \}
\]

where \(D_e\) is equal to \(D_p\) or \(D_{\text{min}}\). Equation (16), which approximates eq. (5) of a definite integral formula, is of the form of primitive function and can be adaptable for any values of the distribution constant \(m\). The application of the equation, furthermore, is possible for the relatively easy and accurate estimation of the cumulative oversize population fraction \(Y\). Figure 5 shows accuracy of eqs. (11) and (16) in comparison with eq. (5), where the accuracy are estimated with the standard deviations expressed as follows:

\[
\sigma_{11}^2 = \frac{1}{X_{\text{max}} - X_{\text{min}}} \int_{X_{\text{min}}}^{X_{\text{max}}} \left( Y_{11}(D_p) - Y(D_p) \right)^2 dD_p
\]

and

\[
\sigma_{16}^2 = \frac{1}{X_{\text{max}} - X_{\text{min}}} \int_{X_{\text{min}}}^{X_{\text{max}}} \left( Y_{16}(D_p) - Y(D_p) \right)^2 dD_p
\]

Figure 5 reveals that eq. (16) is ten times or more accurate than eq. (11) for the approximation, even though the accuracy are dependent on the distribution constant \(m\).

| \(a_0\) | \(a_1\) | \(a_2\) | \(a_3\) | \(a_4\) | \(a_5\) |
|--------|--------|--------|--------|--------|--------|
| \(-9.2135 \times 10^{-1}\) | \(3.6011 \times 10^{-1}\) | \(-7.2132 \times 10^{-2}\) | \(7.2526 \times 10^{-3}\) | \(-2.8966 \times 10^{-4}\) |
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Mean Particle Diameters of a Modified R-R Distribution

In Powder Metallurgy, a median diameter (i.e., 50% particle diameter) has been often used as the representative mean particle diameter of a size-distributed powder. The median diameter, however, has a difference in values according to the bases, such as number and mass bases, when the particles size and the size distribution are measured. Whenever a certain characteristic of a powder is discussed in relation to the size and the size distribution, a suitable representative mean particle diameter must be selected according to the physical meaning. Thus it is important to know the various mean particle diameters.

There have been no suitable expressions with which the mean diameters can be analytically calculated for any based R-R distribution law or the modified one. In the present work, examinations are carried out to drive the expressions appropriate for the evaluation of the various mean particle diameters in the R-R distribution measured on the mass basis.

1. Theoretical expressions of the various mean particle diameters

Table 2 shows the names, definitions and physical meanings of the major mean particle diameters\(^5\)-\(^7\). The definitions are based on the assumption that a powder is composed of the particles similar in shape but varying in size. In the definition formulae, \( n (-) \) and \( w (kg) \) are the population and the weight of a group of the particles having the diameter \( d \), respectively. In the table, \( \rho_p \) (kg/m\(^3\)) indicates the density of the particle material, \( S_w \) (m\(^2\)/kg) the specific surface area, and \( \phi_s, \phi_v \) and \( \phi (-) \) are the surface, volume and specific surface shape factors, respectively.

For a modified R-R distribution, the mean diameter can be analytically calculated with the formula which is derived from the definition through the following procedures. In the case of the number mean diameter \( D_1 \), for instance, both the denominator and the numerator in the equation of the number based definition can be expressed with

\[
\sum n = \frac{N}{100} \int_{D_{\text{min}}}^{D_{\text{max}}} \left( \frac{dY}{dD_p} \right) dD_p \quad (19)
\]

and

\[
\sum (nd) = \frac{N}{100} \int_{D_{\text{min}}}^{D_{\text{max}}} \left( \frac{dY}{dD_p} \right) dD_p \quad (20)
\]

respectively, where \( N \) is the total number of the particles in the powder. By substituting eq. (5) into these two equations, the diameter is

\[
D_1 = D_c \cdot \exp \left\{ - \left( \frac{D_p}{D_c} \right)^m \right\} \quad (21)
\]

The same result, of course, can be obtained by...
the substitution of eq. (1) into the mass based definition formula. The similar procedures result in the other 8 kinds of the mean particle diameters. The resultant formulae are summarized as the theoretical expressions in Table 3. The attention is paid to the five mean particle diameters having the obvious physical meanings, i.e., the number mean diameter $D_1$, the surface mean diameter $D_3$, the mean surface diameter $D_s$, the mean volume diameter $D_v$ and the harmonic mean diameter $D_h$. Figure 6(a) and (b) show the differences among these mean particle diameters estimated with the theoretical expressions in Table 3. The figure (a) illustrates the relation between the distribution constant $m$ and the ratio of the each mean particle diameter $D_i$ to the absolute size constant $D_0$, while the figure (b) reveals the connection between $m$ and the cumulative oversize mass fraction corresponding to $D_i$ $(i=1, 3, s, v, h)$. In both figures, the scales on the ordinates are logarithmic on the left hand side for the range between 0.2 and 1.0 of the distribution constant $m$, and ordinary on the right hand side for the range between 1.0 and 5.0. For a comparison, figure (a) contains the relation for the median diameter $D_{50}$, which can be expressed in the case of the modified R-R distribution as follows:

$$D_{50} = D_e \left\{ -\ln \left( \frac{\alpha + \beta}{2} \right) \right\}^{1/m}. \quad (22)$$

In the actual estimation of the each mean particle diameter with the theoretical expression shown in Table 3, the definite integral has been numerically calculated in a similar manner to that in eq. (5).

According to Fig. 6 for the modified R-R distribution,

1. The values of $D_i/D_0$ and $R(D)$ are dependent upon the distribution constant $m$, and large different from the median diameter $D_{50}$.

2. There is a certain order in size among the various mean particle diameters.
The thorough examination in size among all the 9 mean diameters results in the following order:

\[ D_4 > (D_3, D_w, D_vd) > (D_1, D_2) > D_6 > D_1 > D_h, \]

where two groups \((D_3, D_w, D_vd)\) and \((D_1, D_2)\) have the different orders in size which depend upon the range of \(m\). When examined in details,

\[ 0.2 < m < 0.3520 \quad D_6 > D_1 > D_vd, \quad D_1 > D_2 \]
\[ 0.3520 < m < 1.1827 \quad D_3 > D_v > D_vd, \quad D_3 < D_2 \]
\[ 1.1827 < m < 1.6976 \quad D_1 > D_vd > D_w, \quad D_1 > D_2 \]
\[ 1.6976 < m < 5.0 \quad D_3 > D_vd > D_w, \quad D_2 > D_v \]

### 2. Approximate expressions of the various mean particle diameters

To facilitate the estimation of the mean particle diameters, an attempt is made at deriving the approximate expressions from the theoretical ones by using eq. (14). As for the number mean diameter \(D_1\), for instance, the substitution of the equation \(X=(D_p/D_e)^m\) reforms the theoretical expression eq. (21) into

| Symbol | Theoretical expression | Approximate expression |
|--------|------------------------|-----------------------|
| \(D_1\) | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] |
| \(D_2\) | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] |
| \(D_3\) | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] |
| \(D_4\) | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] |
| \(D_5\) | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] |
| \(D_6\) | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] | \[ \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} \left( \frac{D_{\text{max}}}{D_e} \right)^{-1+m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-1+m+i} \] |

Table 3 Theoretical and approximate expressions for nine kinds of mean particle diameters.
As for the other mean diameters, the approximate expressions are also able to be derived in the similar way, and summarized in Table 3.

An investigation is subsequently carried out on the accuracy of these approximate expressions. The mean particle diameters are calculated in the range of $0.2 < m < 5.0$ through both the theoretical and approximate expressions shown in Table 3. Then the accuracy of the approximation is verified with the relative error $\delta_1(\%)$ of the approximate value to the theoretical one for each of the mean diameters. Besides, the alternative approximate expressions of the mean diameters can be deduced by substituting eq. (11) instead of eq. (5) into the number based forms defined in Table 2. The mean diameters are also estimated in the range of $1.0 < m < 5.0$ with the latter expressions in the same way as in the calculation of the theoretical ones. Another relative error $\delta_2(\%)$ for each of the mean diameters is then defined and estimated for comparison. These investigations result in the following informations;

1. For the same distribution constant $m$, the relative error $\delta_1$ always shows the maximum in the volume mean diameter $D_4$. As for
2. Substituting eq. (14) into eq. (23), finally eq. (21) turns out

\[
D_1 = \frac{\int_{X_{\text{min}}}^{X_{\text{max}}} X^{-2/m} \exp (-X) \, dX}{X_{\text{max}}}.
\]

\[
\sum_{i=1}^{6} a_{i-1} \left\{ \left( \frac{D_{\text{max}}}{D_e} \right)^{-2/m+i} - \left( \frac{D_{\text{min}}}{D_e} \right)^{-2/m+i} \right\}.
\]

Again by substituting eq. (14) into eq. (23), finally eq. (21) turns out
Fig. 6 The relationship between various mean particle diameters $D_i$ and the distribution constant $m$; (a) figured by the ratio $D_i/D_e$ and (b) the cumulative over $R(D_i)$ ($i = 1, 3, s, v, h, 50$).
the relative error $\delta_2$, which has a maximum value in these mean diameters, is dependent upon $m$.

(2) All over the range of $m$, the relative error $\delta_1$ is not beyond $\pm 1.8\%$ even in the volume mean diameter $D_4$. While $\delta_2$ reaches $\pm 50.2\%$ at maximum in the surface mean diameter $D_3$. Figure 7 shows the relations between the distribution constant $m$ and the values of $D_3$ which have been estimated by four methods, i.e., with the theoretical and approximate expressions and by the other two methods. The solid and broken lines indicate the results through the theoretical and approximate expressions following Table 3, respectively. The one-dot chain shows the approximation following eq. (11), and the two-dot chain is the result obtained through the following approximated expression of the specific surface area $S_w$ (m$^2$/kg):

$$S_w = \frac{\phi}{\rho_0} \frac{1.065}{D_2} \exp\left(\frac{1.795}{m^2}\right)$$

which has been presented by Kiesskalt et al.\(^{(8)}\) for a R-R distribution to be applicable in the same particle size range as the modified one, i.e., $0.1% < R < 99.9\%$, but in the restricted range of $0.6 < m < 2.5$. The discussions and Fig. 7 come to a conclusive proof that the approximate expressions shown in Table 3 have the sufficient accuracy.

### IV. Mean Particle Diameters Relevant to Modified R-R Size Distribution

For the practical application, two attempts are made to obtain the various mean particle diameters of the size-distributed powders relevant to the modified R-R distribution.

#### 1. A powder with a classified particle size distribution

For example, when a powder with the modified R-R size distribution is classified into a range between $D_a$ and $D_b$ of the particle sizes ($D_{\min} < D_a < D_b < D_{\max}$), the number mean diameter $D_1$ of the classified powder is expressed with the following equation similar to eq. (24):

$$D_1 \approx D_c \cdot \left( \frac{D_b}{D_c} \right)^{-2 + \frac{m}{i}} - \left( \frac{D_a}{D_c} \right)^{-2 + \frac{m}{i}} - \left( \frac{D_b}{D_c} \right)^{-3 + \frac{m}{i}} + \left( \frac{D_a}{D_c} \right)^{-3 + \frac{m}{i}}$$

This equation can be deduced by simply exchanging $D_{\min}$ and $D_{\max}$ in eq. (24) for $D_a$ and $D_b$, respectively. As for the other mean particle diameters, the approximate expression can be derived in a similar way. This is the great merit in the approximate expressions above proposed for the easy estimation of the mean diameters, even though a classification restricts the distribution to a limited particle size range.

There seems no such a merit in any expressions for the mean diameters of the conventional distribution laws, log-normal one and so on.

Table 4 is an example of the mean particle diameters where the calculations have been performed in the two particle size ranges, namely between $D_a (= 37 \mu m)$ and $D_b (= 88 \mu m)$, and between $D_{\min}$ and $D_{\max}$ for comparison.
2. Blend of two or more powders with the modified R-R distributions

In practice, a more complex condition is the case of blending more than two kinds of powders, each distribution of which can be ruled by a modified R-R distribution law. For \( z \) kinds of the powders with the distribution constants \( m_j \) \((j=1, 2, \ldots, z)\), the absolute size constants \( D_{e,j} \) and the particle size ranges \( D_{a,j} \sim D_{b,j} \) \((D_{a,j} < D_{e,j} < D_{b,j} < D_{max,j})\), each one of the powders is blended with the mixing ratio \( \omega_j \) into a powder, the particle size distribution of which is expressed with the following equation:

\[
R(D_p) = 100 \sum_{j=1}^{z} \omega_j \left[ \int_{D_{a,j}}^{D_{b,j}} \left( \frac{D_p}{D_{e,j}} \right)^{m_j-1} \exp \left\{ - \left( \frac{D_p}{D_{e,j}} \right)^{m_j} \right\} dD_p \right]
\]

where

\[
\sum_{j=1}^{z} \omega_j = 1.
\]

The various mean particles diameters can be also deduced analytically for the size distributed powder corresponding to eq. (27). The number mean diameter \( D_1 \), for example, is

\[
D_1 = \frac{\sum_{j=1}^{z} A_j \left( \frac{1}{D_{e,j}} \right)^2 \int_{D_{a,j}}^{D_{b,j}} \left( \frac{D_p}{D_{e,j}} \right)^{m_j-3} \exp \left\{ - \left( \frac{D_p}{D_{e,j}} \right)^{m_j} \right\} dD_p}{\sum_{j=1}^{z} A_j \left( \frac{1}{D_{e,j}} \right)^3 \int_{D_{a,j}}^{D_{b,j}} \left( \frac{D_p}{D_{e,j}} \right)^{m_j-4} \exp \left\{ - \left( \frac{D_p}{D_{e,j}} \right)^{m_j} \right\} dD_p}
\]

where

\[
A_j = \omega_j \left[ \int_{D_{a,j}}^{D_{b,j}} \left( \frac{D_p}{D_{e,j}} \right)^{m_j-1} \exp \left\{ - \left( \frac{D_p}{D_{e,j}} \right)^{m_j} \right\} dD_p \right].
\]

Equation (28) can be approximated through eq. (14) into

\[
D_1 \approx \frac{\sum_{j=1}^{z} A_j \sum_{i=1}^{6} \frac{a_{i-1}}{2i} \left\{ \left( \frac{D_{b,j}}{D_{e,j}} \right)^{-2+m_j,i} - \left( \frac{D_{a,j}}{D_{e,j}} \right)^{-2+m_j,i} \right\}}{\sum_{j=1}^{z} A_j \sum_{i=1}^{6} \frac{a_{i-1}}{3i} \left\{ \left( \frac{D_{b,j}}{D_{e,j}} \right)^{-3+m_j,i} - \left( \frac{D_{a,j}}{D_{e,j}} \right)^{-3+m_j,i} \right\}}
\]

where

\[
A_j = \omega_j \left[ \frac{\sum_{i=1}^{6} \frac{a_{i-1}}{i} \left\{ \left( \frac{D_{b,j}}{D_{e,j}} \right)^{m_j,i} - \left( \frac{D_{a,j}}{D_{e,j}} \right)^{m_j,i} \right\}}{\sum_{j=1}^{z} A_j \left( \frac{1}{D_{e,j}} \right)^3 \int_{D_{a,j}}^{D_{b,j}} \left( \frac{D_p}{D_{e,j}} \right)^{m_j-4} \exp \left\{ - \left( \frac{D_p}{D_{e,j}} \right)^{m_j} \right\} dD_p} \right].
\]
Table 5  Mean particle diameters in the case of blending of z kinds of powders.

| Symbol | Approximate expression |
|--------|------------------------|
| $D_1$  | \[
\sum_{i=1}^{z} A_i \left( \frac{1}{D_{ij}} \right)^{2} \sum_{j=1}^{6} \frac{a_{i-1}}{-2/m_{j}+i} \left\{ \frac{(D_{bj})^{2+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{2+m_{j}}}{D_{ij}} \right\} \]
| $D_2$  | \[
\sum_{i=1}^{z} A_i \left( \frac{1}{D_{ij}} \right)^{3} \sum_{j=1}^{6} \frac{a_{i-1}}{-1/m_{j}+i} \left\{ \frac{(D_{bj})^{3+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{3+m_{j}}}{D_{ij}} \right\} \]
| $D_3$  | \[
\sum_{i=1}^{z} A_i \left( \frac{1}{D_{ij}} \right)^{4} \sum_{j=1}^{6} \frac{a_{i-1}}{-2/m_{j}+i} \left\{ \frac{(D_{bj})^{4+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{4+m_{j}}}{D_{ij}} \right\} \]
| $D_4$  | \[
\sum_{i=1}^{z} A_i \sum_{j=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{1+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{1+m_{j}}}{D_{ij}} \right\} \]
| $D_5$  | \[
\sum_{i=1}^{z} A_i \sum_{j=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{2+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{2+m_{j}}}{D_{ij}} \right\} \]
| $D_6$  | \[
\sum_{i=1}^{z} A_i \sum_{j=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{3+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{3+m_{j}}}{D_{ij}} \right\} \]
| $D_7$  | \[
\sum_{i=1}^{z} A_i \sum_{j=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{4+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{4+m_{j}}}{D_{ij}} \right\} \]
| $D_8$  | \[
\sum_{i=1}^{z} A_i \sum_{j=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{5+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{5+m_{j}}}{D_{ij}} \right\} \]
| $D_9$  | \[
\sum_{i=1}^{z} A_i \sum_{j=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{6+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{6+m_{j}}}{D_{ij}} \right\} \]
| $D_{10}$ | \[
\sum_{i=1}^{z} A_i \sum_{j=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{7+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{7+m_{j}}}{D_{ij}} \right\} \]
| $D_{11}$ | \[
\sum_{i=1}^{z} A_i \sum_{j=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{8+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{8+m_{j}}}{D_{ij}} \right\} \]
| $D_{12}$ | \[
\sum_{i=1}^{z} A_i \sum_{j=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{9+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{9+m_{j}}}{D_{ij}} \right\} \]

$A_j \approx a_j \left[ \sum_{i=1}^{6} \frac{a_{i-1}}{D_{ij}} \left\{ \frac{(D_{bj})^{m_{j}}}{D_{ij}} - \frac{(D_{bj})^{m_{j}}}{D_{ij}} \right\} \right]$  

but if $h/m_{j}+i=0 (h=-4, -3, -2, -1, 0, 1)$  

then $\frac{a_{i-1}}{-h/m_{j}+i} \left\{ \frac{(D_{bj})^{h+m_{j}}}{D_{ij}} - \frac{(D_{bj})^{h+m_{j}}}{D_{ij}} \right\} \rightarrow a_{i-1} \left\{ \ln \frac{(D_{bj})^{b+m_{j}}}{D_{ij}} - \ln \frac{(D_{bj})^{b+m_{j}}}{D_{ij}} \right\}$
The other mean particle diameters can be also deduced into the similar approximate expressions applicable for the general calculations of the any blended powder as summarized in Table 5.

V. Conclusion

In the present work, careful attention is paid to the particle size distribution as one of the most important characteristics for industrial powders, especially the modified R-R one for the metal powders. As for the various attributes of a size distribution function, firstly a more accurate number based expression has been introduced for the one which has been presented according to the modified R-R distribution law in the previous report\(^{(2)}\). Secondly attempts have been made for the easy determination of the various mean particle diameters with the approximate expressions derived in the different ways. Furthermore, these expressions have been also extended to some practical applications for the industrial powders. The results are summarized as follows:

(1) The number based expression eq. (16) is much more accurate for a modified R-R distribution than eq. (11) which has been introduced in the previous paper.

(2) The various mean particle diameters can be determined for a modified R-R size distribution through the calculating equations totally listed in Table 3.

(3) These mean diameters can be also estimated for the powder which has a size distribution altered by the treatment of classification or blending.

Besides the particle size distribution, the above results are commonly applicable for the analyses of the other distributions of grain and pore size.

All the numerical calculations have been computed in the present work with a machine FACOM M-382 in Nagoya University Computation Center.

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