THE GENERALIZED CASSELS-TATE DUAL EXACT SEQUENCE FOR 1-MOTIVES

CRISTIAN D. GONZÁLEZ-AVILÉS AND KI-SENG TAN

Abstract. We establish a generalized Cassels-Tate dual exact sequence for 1-motives over global fields. We thereby extend the main theorem of [4] from abelian varieties to arbitrary 1-motives.

1. Introduction

Let \( K \) be a global field and let \( M = (Y \to G) \) be a (Deligne) 1-motive over \( K \), where \( Y \) is étale-locally isomorphic to \( \mathbb{Z}^r \) for some \( r \geq 0 \) and \( G \) is a semiabelian variety over \( K \). Let \( M^* \) be the 1-motive dual to \( M \). If \( B \) is a topological abelian group, \( B^\wedge \) will denote the completion of \( B \) with respect to the family of open subgroups of finite index. Let \( \mathfrak{H}^1(M) \) (resp. \( \mathfrak{H}^1_\omega(M) \)) denote the subgroup of \( \mathfrak{H}^1(K,M) \) of all classes which are locally trivial at all (resp. all but finitely many) primes of \( K \). There exists a canonical exact sequence of discrete torsion groups

\[
0 \to \mathfrak{H}^1(M) \to \mathfrak{H}^1_\omega(M) \to \bigoplus_{\text{all } v} \mathfrak{H}^1(K_v,M) \to \mathfrak{U}^1(M) \to 0,
\]

where we have written \( \mathfrak{U}^1(M) \) for the cokernel of the middle map. By the local duality theorem for 1-motives [7, Theorem 2.3 and Proposition 2.9], the Pontryagin dual of the above exact sequence is an exact sequence

\[
0 \to \mathfrak{U}^1(M)^D \to \prod_{\text{all } v} \mathfrak{H}^0(K_v,M^*^\wedge) \to \mathfrak{H}^1_\omega(M)^D \to \mathfrak{H}^1(M)^D \to 0,
\]

where each group \( \mathfrak{H}^0(K_v,M^*) \) is endowed with the topology defined in [7, p.99]. A fundamental problem is to describe \( \mathfrak{U}^1(M)^D \). This problem was first addressed in the case of elliptic curves \( E \) over number
fields \( K \) (i.e., \( Y = 0 \) and \( G = E \) above), by J.W.S.Cassels (see [2, Theorem 7.1] and [3, Appendix 2]). Cassels showed that \( \Psi^1(E^*)^D \) is canonically isomorphic to the pro-Selmer group \( T\text{Sel}(E) \) of \( E \). This result was extended to abelian varieties \( A \) over number fields \( K \) by J.Tate, under the assumption that \( \text{X}_1(A) \) is finite (unpublished). In this case \( T\text{Sel}(A) \) is isomorphic to \( H^0(K, A) \) and \( \text{X}_1(A^*) = H^1(K, A^*) \) for any \( v \) since \( H^0(K, A) \) is profinite. Further, \( \text{X}_1(\text{A}^*) = H^1(K, A^*) \) and \( \text{III}^1(\text{A}^*) = \text{III}^1(\text{A}) \). The exact sequence obtained by Tate, now known as the Cassels-Tate dual exact sequence, is

\[
0 \to H^0(K, A) \to \prod_{v} H^0(K_v, A) \to H^1(K, A^*)^D \to \text{III}^1(\text{A}) \to 0. \tag{1}
\]

Further, the image of \( H^0(K, A) \) is isomorphic to the closure \( \overline{H^0(K, A)} \) of the diagonal image of \( H^0(K, A) \) in \( \prod_{v} H^0(K_v, A) \). See [11, Remark I.6.14(b), p.102]. The preceding exact sequence was recently extended to arbitrary 1-motives over number fields by D.Harari and T.Szamuely [7, Theorem 1.2], again under the assumption that \( \text{III}^1(\text{M}) \) is finite. These authors established the exactness of the sequence

\[
0 \to \overline{H^0(K, M)} \to \prod_{v} \overline{H^0(K_v, M)} \to \text{III}^1(\text{M}^*)^D \to \text{III}^1(\text{M}) \to 0,
\]

where the middle map is induced by the local pairings of [7, §2]. This natural analogue of (1) was used in [op.cit., §6] to study weak approximation on semiabelian varieties over number fields. However, it does not provide a description of \( \Psi^1(M)^D \) when \( \text{III}^1(M) \) is infinite. Our objective in this paper is to describe \( \Psi^1(M)^D \) for any \( K \) independently of the finiteness assumption on \( \text{III}^1(M) \). In order to state our main result, let

\[
\text{Sel}(M) = \text{Ker} \left[ H^1(K, T_{Z/n}(M)) \to \prod_{v} \text{H}^1(K_v, M)_n \right]
\]

be the \( n \)-th Selmer group of \( M \), where \( n \) is any positive integer and \( T_{Z/n}(M) \) is the \( n \)-adic realization of \( M \). Let \( T\text{Sel}(M) = \lim_{\longrightarrow n} \text{Sel}(M)_n \) be the pro-Selmer group of \( M \). Our main theorem is the following result.

**Theorem 1.1.** (The generalized Cassels-Tate dual exact sequence for 1-motives). Let \( M \) be a 1-motive over a global field \( K \). Then there
exists a canonical exact sequence of profinite groups
\[
0 \rightarrow \mathbb{H}^2(M^*)^D \rightarrow T\text{Sel}(M)^\wedge \rightarrow \prod_{v} \mathbb{H}^0(K_v, M)^\wedge \\
\rightarrow \mathbb{H}^1(M^*)^D \rightarrow \mathbb{H}^1(M)^D \rightarrow 0.
\]

If \( M = (0 \rightarrow A) \) is an abelian variety, then \( \mathbb{H}^2(M^*)^D = 0 \) and we recover the main theorem of [4]. Applications of Theorem 1.1 will be given in [6].

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2. Preliminaries

Let \( K \) be a global field, i.e. \( K \) is a finite extension of \( \mathbb{Q} \) (the “number field case”) or is finitely generated and of transcendence degree 1 over a finite field of constants \( k \) (the “function field case”). For any prime \( v \) of \( K \), \( K_v \) will denote the completion of \( K \) at \( v \) and \( \mathcal{O}_v \) will denote the corresponding ring of integers. Thus \( \mathcal{O}_v \) is a complete discrete valuation ring. Further, \( X \) will denote either the spectrum of the ring of integers of \( K \) (in the number field case) or the unique smooth complete curve over \( k \) with function field \( K \) (in the function field case).

All cohomology groups below are flat (fppf) cohomology groups.

For any topological abelian group \( B \), we set \( B^D = \text{Hom}_{\text{cont.}}(B, \mathbb{Q}/\mathbb{Z}) \) and endow it with the compact-open topology, where \( \mathbb{Q}/\mathbb{Z} \) carries the discrete topology. If \( n \) is any positive integer, \( B/n \) will denote \( B/nB \) with the quotient topology. Let \( B_\wedge = \lim_{\longrightarrow} B/n \) with the inverse limit topology. Further, define \( B^\wedge = \lim_{U \in \mathcal{U}} B/U \), where \( \mathcal{U} \) denotes the family of open subgroups of finite index in \( B \). If \( B_\sim = \lim_{\longleftarrow} B/n \), where \( nB \) denotes the closure of \( nB \) in \( B \), then there exists a canonical isomorphism \( (B_\sim)^\wedge = B^\wedge \). Consequently, there exists a canonical map \( B_\wedge \to B^\wedge \). If \( B \) is discrete (or compact), then \( B_\sim = B_\wedge \) and therefore \( (B_\sim)^\wedge = B^\wedge \). We also note that \( B^\wedge = B \) if \( B \) is profinite (see, e.g., [14, Theorem 2.1.3, p.22]). For any positive integer \( n \), \( B_n \) will denote the \( n \)-torsion subgroup of \( B \) and \( TB = \lim_{n \in \mathbb{N}} B/nB \) is the total Tate module of \( B \). Note that \( TB = 0 \) if \( B \) is finite.

Let \( M = (Y \to G) \) be a Deligne 1-motive over \( K \), where \( Y \) is étale-locally isomorphic to \( \mathbb{Z}^r \) for some \( r \) and \( G \) is a semiabelian variety (for basic information on 1-motives over global fields, see [7, §1] or [5, §3].
Let $n$ be a positive integer. The $n$-adic realization of $M$ is a finite and flat $K$-group scheme $T_{\mathbb{Z}/n}(M)$ which fits into an exact sequence

$$0 \to G_n \to T_{\mathbb{Z}/n}(M) \to Y/n \to 0.$$ 

There exists a perfect pairing

$$T_{\mathbb{Z}/n}(M) \times T_{\mathbb{Z}/n}(M^*) \to \mu_n,$$

where $\mu_n$ is the sheaf of $n$-th roots of unity. Further, given positive integers $n$ and $m$ with $n | m$, there exist canonical maps $T_{\mathbb{Z}/n}(M) \to T_{\mathbb{Z}/m}(M)$ and $T_{\mathbb{Z}/m}(M) \to T_{\mathbb{Z}/n}(M)$. Let $T(M)_{\text{tors}} = \lim \limits_{\longrightarrow} T_{\mathbb{Z}/n}(M)$.

Further, for any $i \geq 0$, define

$$H^i(K, T(M)) = \lim \limits_{\longleftarrow} H^i(K, T_{\mathbb{Z}/n}(M)).$$

If $v$ is archimedean and $i \geq -1$, $\mathbb{H}^i(K_v, M)$ will denote the (finite, 2-torsion) reduced (Tate) hypercohomology groups of $M_{K_v}$ defined in [7, p.103]. All groups $\mathbb{H}^i(K_v, M)$ will be given the discrete topology, except for $\mathbb{H}^0(K_v, M)$ for non-archimedean $v$. The latter group will be given the topology defined in [7, p.99]. Thus there exists an exact sequence $0 \to I \to \mathbb{H}^0(K_v, M) \to F \to 0$, where $F$ is finite and $I$ is an open subgroup of $\mathbb{H}^0(K_v, M)$ which is isomorphic to $G(K_v)/L$ for some finitely generated subgroup $L$ of $G(K_v)$. If $n$ is a positive integer, $G(K_v)/n$ is profinite (see [5, beginning of §5]). Thus the exactness of

$$L/n \to G(K_v)/n \to I/n \to 0$$

shows that $I/n$ is profinite as well. Now the exactness of

$$F_n \to I/n \to \mathbb{H}^0(K_v, M)/n \to F/n \to 0$$

shows that $\mathbb{H}^0(K_v, M)/n$ is profinite (see [14, Proposition 2.2.1(e), p.28]). The latter also holds if $v$ is archimedean. We conclude that $\mathbb{H}^0(K_v, M)_\wedge$ is profinite for every $v$ (see [14, Proposition 2.2.1(d), p.28]).

All groups $\mathbb{H}^i(K, M)$ will be endowed with the discrete topology.

**Lemma 2.1.** $\mathbb{H}^0(K, M)_\wedge$ is Hausdorff, locally compact and $\sigma$-compact.

**Proof.** This follows from the fact that $\mathbb{H}^0(K, M)_\wedge$ is topologically isomorphic to a countable direct limit of compact groups. Indeed, there exists a canonical isomorphism

$$\mathbb{H}^0(K, M)_\wedge = \lim \limits_{(U, \mathcal{M}) \in \mathcal{F}} \mathbb{H}^0(U, \mathcal{M})_\wedge,$$

where $\mathcal{F}$ is the set of all pairs $(U, \mathcal{M})$ such that $U$ is a nonempty open affine subscheme of $X$ and $\mathcal{M}$ is a 1-motive over $U$ which extends $M$ (cf. [5, proof of Lemma 2.3]). By [7, Lemma 3.2(3), p.107], each
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\[ H_0(U, M) \wedge \text{is profinite. Further, since the complement of } U \text{ in } X \text{ is a finite set of primes of } K \text{ and } K \text{ has only countably many primes, } \mathcal{F} \text{ is countable.} \]

For each \( i \geq 0 \), let \( \mathbb{P}^i(M) \) be the restricted direct product over all primes of \( K \) of the groups \( H^i(K_v, M) \) with respect to the subgroups

\[ H^i_{nr}(K_v, M) = \text{Im} \left[ H^i(O_v, \mathcal{M}) \to H^i(K_v, M) \right] \]

for \( v \in U \), where \( U \) is any nonempty open subscheme of \( X \) such that \( M \) extends to a 1-motive \( M \) over \( U \). The groups \( \mathbb{P}^i(M) \) are defined similarly for any abelian fppf sheaf \( F \) on \( \text{Spec } K \). By [7, Lemma 5.3], \( \mathbb{P}^0(M) \wedge \) is the restricted direct product of the groups \( H^0(K_v, M) \wedge \) with respect to the subgroups \( H^0_{nr}(K_v, M) \wedge \). It is therefore Hausdorff, locally compact and \( \sigma \)-compact (see [10, 6.16(c), p.57]). Further, by [7, Theorems 2.3 and 2.10], the dual of \( \mathbb{P}^0(M) \wedge \) is \( \mathbb{P}^1(M^*_{\text{tors}}) \).

Recall that a morphism \( f : A \to B \) of topological groups is said to be \textit{strict} if the induced map \( A/\text{Ker } f \to \text{Im } f \) is an isomorphism of topological groups. Equivalently, \( f \) is strict if it is open onto its image [1, §III.2.8, Proposition 24(b), p.236]. We will need the following

**Lemma 2.2.** Let \( A \xrightarrow{f} B \xrightarrow{g} C \) be an exact sequence of abelian topological groups and strict morphisms. If \( C \to C^\wedge \) is injective, then \( A^\wedge \xrightarrow{\tilde{f}} B^\wedge \xrightarrow{\tilde{g}} C^\wedge \) is also exact.

**Proof.** The map \( A \to \text{Im } f \) induced by \( f \) is an open surjection, so \( A^\wedge \to (\text{Im } f)^\wedge \) is surjective as well. Further, since \( B \to \text{Im } g \) is an open surjection, the sequence \( (\text{Im } f)^\wedge \to B^\wedge \to (\text{Im } g)^\wedge \to 0 \) is exact [7, Appendix]. Finally, since \( C \) injects into \( C^\wedge \), \( (\text{Im } g)^\wedge \) is the closure of \( \text{Im } g \) in \( C^\wedge \), whence \( (\text{Im } g)^\wedge \to C^\wedge \) is injective. \( \square \)

3. THE POITOU-TATE EXACT SEQUENCE FOR 1-MOTIVES OVER FUNCTION FIELDS

For any positive integer \( n \), there exists a canonical exact commutative diagram

\[
\begin{array}{c}
0 \longrightarrow \mathbb{H}^0(K, M)/n \longrightarrow H^1(K, T_{\mathbb{Z}/n}(M)) \longrightarrow \mathbb{H}^1(K, M)_n \longrightarrow 0 \\
0 \longrightarrow \mathbb{P}^0(M)/n \longrightarrow \mathbb{P}^1(T_{\mathbb{Z}/n}(M)) \longrightarrow \mathbb{P}^1(M)_n \longrightarrow 0,
\end{array}
\]

\[ ^1 \text{This result and its proof remain valid in the function field case, using the fact that } H^1_v(O_v, T_{\mathbb{Z}/p^m}(M)) = 0 \text{ for any } m \text{ by [13, beginning of §7, p.349].} \]
whose vertical maps are induced by the canonical morphisms $\text{Spec } K_\nu \to \text{Spec } K$. For the exactness of the rows, see [7, p.109]. Now, for any $i \geq -1$, set

$$\mathbb{II}^i(M) = \text{Ker} \left[ H^i(K, M) \to \mathbb{P}^i(M) \right].$$

Further, define

$$\text{Sel}(M)_n = \text{Ker} \left[ H^1(K, T \mathbb{Z}/n(M)) \to \mathbb{P}^1(M)_n \right],$$

where the map involved is the composite

$$H^1(K, T \mathbb{Z}/n(M)) \to P^1(T \mathbb{Z}/n(M)) \to \mathbb{P}^1(M)_n.$$

Now set

$$T\text{Sel}(M) = \lim_{\leftarrow} \text{Sel}(M)_n$$

$$P^1(T(M)) = \lim_{\leftarrow} \mathbb{P}^1(T \mathbb{Z}/n(M)).$$

Since $(\mathbb{H}^0(K, M)/n)$ and $(\mathbb{P}^0(M)/n)$ are inverse systems with surjective transition maps, the inverse limit of (2) is an exact commutative diagram

\[
0 \longrightarrow \mathbb{H}^0(K, M) \longrightarrow H^1(K, T(M)) \longrightarrow \mathbb{T} \mathbb{H}^1(K, M) \longrightarrow 0
\]

\[
0 \longrightarrow \mathbb{P}^0(M) \longrightarrow P^1(T(M)) \longrightarrow \mathbb{T} \mathbb{P}^1(M) \longrightarrow 0
\]

(see, e.g., [9, Example 9.1.1, p.192]). The above diagram yields an exact sequence

\[
0 \longrightarrow \mathbb{H}^0(K, M) \longrightarrow T\text{Sel}(M) \longrightarrow \mathbb{T} \mathbb{II}^1(M) \longrightarrow 0.
\]

Thus, if $\mathbb{II}^1(M)$ is finite, then $T\text{Sel}(M)$ is canonically isomorphic to $\mathbb{H}^0(K, M)$. In particular, $T\text{Sel}(M)^\wedge = (\mathbb{H}^0(K, M))^\wedge = \mathbb{H}^0(K, M)^\wedge$.

**Lemma 3.1.** $T\text{Sel}(M)$ is locally compact and $\sigma$-compact.

**Proof.** By [7, Lemma 3.2(2)] and [5, Lemma 6.5], $\mathbb{II}^1(M)_n$ is finite for any $n$. Thus $T \mathbb{II}^1(M)$ is profinite and the lemma follows from (4), Lemma 2.1 and [15, Theorem 6.10(c), p.57].

**Lemma 3.2.** The canonical map $H^1(K, T(M)) \to T \mathbb{H}^1(K, M)$ appearing in diagram (3) induces an isomorphism

$$H^1(K, T(M))/T\text{Sel}(M) \simeq T \mathbb{H}^1(K, M)/T \mathbb{II}^1(M).$$

**Proof.** This is immediate from diagram (3) and the definitions of $\mathbb{II}^1(M)$ and $T\text{Sel}(M)$. 

\[\square\]
By definition of $\text{TSel}(M)$, diagram (3) induces a canonical map

$$\theta_0 : \text{TSel}(M) \to \mathbb{P}^0(M)_\wedge.$$ 

**Proposition 3.3.** There exists a perfect pairing

$$\text{Ker} \theta_0 \times \text{III}^2(M^*) \to \mathbb{Q}/\mathbb{Z},$$

where the first group is profinite and the second is discrete and torsion.

**Proof.** (Cf. [7, proof of Proposition 5.1, p.119]) There exists a canonical exact commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \text{TSel}(M) & \to & H^1(K, T(M)) & \to & T \mathbb{H}^1(K, M)/T \text{III}^1(M) \\
\downarrow \theta_0 & & \downarrow \theta & & \downarrow & \\
0 & \to & \mathbb{P}^0(M)_\wedge & \to & P^1(T(M)) & \to & T \mathbb{P}^1(M).
\end{array}
\]

The top row is exact by Lemma 3.2. Clearly, $\text{Ker} (\theta_0) = \text{Ker} (\theta)$. Now, by Poitou-Tate duality for finite modules ([13, Theorem I.4.10, p.70] and [5, Theorem 4.9]), for each $n$ there exists a perfect pairing of finite groups

$$\text{III}^1(T_{\mathbb{Z}/n}(M)) \times \text{III}^2(T_{\mathbb{Z}/n}(M^*)) \to \mathbb{Q}/\mathbb{Z}.$$ 

We conclude that $\text{Ker} (\theta) = \varprojlim_n \text{III}^1(T_{\mathbb{Z}/n}(M))$ is canonically dual to $\text{III}^2(T(M^*)_{\text{tors}}) := \varprojlim_n \text{III}^2(T_{\mathbb{Z}/n}(M^*))$. But [5, proof of Lemma 5.8(a)] shows that $\text{III}^2(T(M^*)_{\text{tors}}) = \text{III}^2(M^*)$, which completes the proof. \hfill \square

**Remark 3.4.** In the number field case, $\text{III}^2(M^*)$ is known to be finite [12]. Further [op.cit., proof of Theorem 9.4], the finite group $\text{Ker} (\theta_0) = \varprojlim_n \text{III}^1(T_{\mathbb{Z}/n}(M))$ is canonically isomorphic to

$$\text{Ker} \left[ \mathbb{H}^0(K, M) \to \prod \mathbb{H}^0(K_v, M)_\wedge \right],$$

which conjecturally is the same as $\text{III}^0(M)$.

**Lemma 3.5.** $\theta_0$ is a strict morphism.

**Proof.** By Lemma 3.1, [10, Theorem 5.29, p.42] and [15, Theorem 4.8, p.45], it suffices to check that $\text{Im} \theta_0$ is a closed subgroup of the locally compact Hausdorff group $\mathbb{P}^0(M)_\wedge$. The image of the map $\theta$ in diagram (5) can be identified with the kernel of the map

$$P^1(T(M)) \to H^1(K, T^*(M^*)_{\text{tors}})^D.$$
coming from the Poitou-Tate exact sequence for finite modules ([13, Theorem I.4.10, p.70] and [5, Theorem 4.12]). Now diagram (5) shows that $\text{Im} \theta_0$ can be identified with the kernel of the continuous composite map

$$\mathbb{P}^0(M) \to P^1(T(M)) \to H^1(K, T(M^*)_{\text{tors}})^D.$$ 

Thus $\text{Im} \theta_0$ is indeed closed in $\mathbb{P}^0(M)$. \hfill \Box

There exists a natural commutative diagram

$$\begin{array}{ccc}
TSel(M) & \xrightarrow{\theta_0} & \mathbb{P}^0(M) \\
\downarrow & & \downarrow \\
TSel(M)^\wedge & \xrightarrow{\beta_0} & \mathbb{P}^0(M)^\wedge,
\end{array}$$

where $\beta_0 = \hat{\theta}_0$.

**Lemma 3.6.** The vertical maps in the preceding diagram are injective.

**Proof.** (Cf. [7, proof of Proposition 5.4, p.119]) Let $\xi = (\xi_n) \in T\text{Sel}(M)$ be nonzero. Then, for some $n$, $\xi_n \in \text{Sel}(M)_n$ is nonzero. Since the canonical map $\text{Sel}(M)_n \to \text{Sel}(M)^\wedge_n$ is injective by [7, Lemma 5.5], we conclude that the image of $\xi_n$ in $\text{Sel}(M)^\wedge_n$ is nonzero. Consequently, there exists a subgroup $N$ of $\text{Sel}(M)_n$, of finite index, such that $\xi_n \notin N$. It follows that $\xi$ is not contained in the inverse image of $N$ under the canonical map $T\text{Sel}(M) \to \text{Sel}(M)_n$, which is an open subgroup of finite index in $T\text{Sel}(M)$. We conclude that the image of $\xi$ in $T\text{Sel}(M)^\wedge$ is nonzero. This proves the injectivity of the left-hand vertical map in diagram (6). To prove the injectivity of the right-hand vertical map, let $x = (x_v) \in \mathbb{P}^0(M)^\wedge$ be nonzero. Then $x \notin n\mathbb{P}^0(M)$ for some $n$, whence $x_v \notin n\mathbb{H}^0(K_v, M)$ for some $v$ (see [7, Lemma 5.3, p.118]). Thus the image of $x$ under the canonical map

$$\mathbb{P}^0(M)^\wedge \to \mathbb{H}^0(K_v, M)/n = (\mathbb{H}^0(K_v, M)/n)^\wedge$$

is nonzero, where the equality comes from the fact that $\mathbb{H}^0(K_v, M)/n$ is profinite. But the preceding map factors through $\mathbb{P}^0(M)^\wedge$, so the image of $x$ in $\mathbb{P}^0(M)^\wedge$ is nonzero. \hfill \Box

**Proposition 3.7.** The map $\text{Ker} \theta_0 \to \text{Ker} \beta_0$ induced by diagram (6) is an isomorphism.

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\[2\] This uses the fact that $\lim_{\rightarrow} (1) \mathbb{H}^1(Z_n/M) = 0$, which holds since each $\mathbb{H}^1(Z_n/M)$ is finite. See [11, Proposition 2.3, p.14].
Proof. The injectivity of the above map is immediate from Lemma 3.6. Now, by Lemmas 2.2, 3.5 and 3.6, the exact sequence

$$\text{Ker } \theta_0 \to T \text{Sel}(M) \xrightarrow{\theta_0} P^0(M)$$

induces an exact sequence

$$(\text{Ker } \theta_0)^{\wedge} \to T \text{Sel}(M)^{\wedge} \xrightarrow{\beta_0} P^0(M)^{\wedge}.$$ 

But $$(\text{Ker } \theta_0)^{\wedge} = \text{Ker } \theta_0$$ since $\text{Ker } \theta_0$ is profinite by Proposition 3.3, so $\text{Ker } \theta_0 \to \text{Ker } \beta_0$ is indeed surjective. \qed

For each $v$ and any $n \geq 1$, there exists a canonical pairing

$$(-, -)_v : H^0(K_v, M) / n \times H^1(K_v, M^*)_n \to \mathbb{Q} / \mathbb{Z}$$

which vanishes on $H^0_{nr}(K_v, M) / n \times H^1_{nr}(K_v, M^*)_n$. See [7, p.99 and proof of Theorem 2.10, p.104]. Let $\gamma_{0,n} : P^0(M) / n \to (H^1(K, M^*)_n)^D$ be defined as follows. For $x = (x_v) \in P^0(M) / n$ and $\xi \in H^1(K, M^*)_n$, set

$$\gamma_{0,n}(x)(\xi) = \sum_{v} (x_v, \xi|_{K_v}),$$

where $\xi|_{K_v}$ is the image of $\xi$ under the canonical map $H^1(K_v, M^*)_n \to H^1(K_v, M^*)_n$ (the sum is actually finite since $x_v \in H^0_{nr}(K_v, M) / n$ and $\xi|_{K_v} \in H^1_{nr}(K_v, M)_n$ for all but finitely many primes $v$). Consider the map

$$\gamma'_{0,n} := \lim_{\leftarrow n} \gamma_{0,n} : P^0(M)^{\wedge} \to H^1(K, M^*)^{\wedge}.$$ 

By [7, p.122], the sequence

$$(7) \quad T \text{Sel}(M) \xrightarrow{\theta_0} P^0(M)^{\wedge} \xrightarrow{\gamma'_{0,n}} H^1(K, M^*)^{\wedge}$$

is a complex.

**Lemma 3.8.** The sequence (7) is exact.

Proof. As noted in the proof of Lemma 3.5, $\text{Im } \theta_0$ is the kernel of the composite map

$$P^0(M)^{\wedge} \to P^1(T(M)) \to H^1(K, T(M^*)_{\text{tors}})^D.$$ 

Further, there exists a canonical commutative diagram

$$\begin{array}{ccc}
P^0(M)^{\wedge} & \xrightarrow{\theta_0} & P^1(T(M)) \\
\downarrow{\gamma_0} & & \downarrow \\
H^1(K, M^*)^{\wedge} & \xrightarrow{\beta_0} & H^1(K, T(M^*)_{\text{tors}})^D,
\end{array}$$

where the bottom map is the dual of the surjection of discrete groups $H^1(K, T(M^*)_{\text{tors}}) \to H^1(K, M^*)$ (the latter map is the direct limit over
of the surjections appearing on the top row of diagram (2) for $M^\ast$).
We conclude that $\text{Im} \theta_0 = \text{Ker} \gamma'_0$, as claimed. □

**Lemma 3.9.** $\gamma'_0$ is a strict morphism.

**Proof.** Since $P^0(M)_\wedge$ is locally compact and $\sigma$-compact and $H^1(K, M^\ast)^D$ is profinite, it suffices to check, by [10, Theorem 5.29, p.42] and [15, Theorem 4.8, p.45], that $\text{Im} \gamma'_0$ is closed in $H^1(K, M^\ast)^D$ (cf. proof of Lemma 3.5). By Lemma 3.8 and diagram (5), $\text{Im} \gamma'_0 = \text{Coker} \theta_0$ (with the quotient topology) injects as a closed subgroup of $\text{Coker} \theta$. On the other hand, the Poitou-Tate exact sequence for finite modules ([13, Theorem I.4.10, p.70] and [5, Theorem 4.12]) shows that $\text{Coker} \theta$ is a closed subgroup of the compact group $H^1(K, T(M^\ast)_{\text{tors}})^D$. It follows that $\text{Im} \gamma'_0$ is a compact (and hence closed) subgroup of the Hausdorff group $H^1(K, M^\ast)^D$. □

Now consider

$$\gamma_0 = (\gamma'_0)^\wedge : P^0(M)^\wedge \to (H^1(K, M^\ast)^D)^\wedge = H^1(K, M^\ast)^D.$$ 

**Proposition 3.10.** The sequence

$$T\text{Sel}(M)^\wedge \xrightarrow{\beta_0} P^0(M)^\wedge \xrightarrow{\gamma_0} H^1(K, M^\ast)^D,$$

is exact.

**Proof.** This follows by applying Lemma 2.2 to the exact sequence (7) using Lemmas 3.5 and 3.9. □

The following is the main result of this Section. It extends [7, Theorem 5.6, p.120] to the function field case.

**Theorem 3.11.** Let $K$ be a global function field and let $M$ be a 1-motive over $K$. Assume that $\text{III}^1(M)$ is finite. Then there exists a canonical 12-term exact sequence

$$\begin{array}{ccccccccc}
H^{-1}(K, M)^\wedge & \xrightarrow{\gamma'_2} & \prod_{\text{all } v} H^2(K_v, M^\ast)^D & \xrightarrow{\beta_2^D} & H^2(K, M^\ast)^D \\
\downarrow & & \downarrow & & \\
H^1(K, M^\ast)^D & \xrightarrow{\gamma_0} & P^0(M)^\wedge & \xrightarrow{\beta_0} & H^0(K, M)^\wedge \\
\downarrow & & \downarrow & & \\
H^1(K, M) & \xrightarrow{\beta_1} & P^1(M)_{\text{tors}} & \xrightarrow{\gamma_1} & (H^0(K, M^\ast)^D)_{\text{tors}} \\
\downarrow & & \downarrow & & \\
H^{-1}(K, M^\ast)^D & \xrightarrow{\gamma_2} & \bigoplus_{\text{all } v} H^2(K_v, M) & \xrightarrow{\beta_2} & H^2(K, M),
\end{array}$$
where the maps \( \beta_i \) are canonical localization maps, the maps \( \gamma_i \) are induced by local duality and the unlabeled maps are defined in the proof.

**Proof.** The exactness of the first line follows as in [7, p.122], using [5, Theorem 4.12] and noting that [7, Lemma 5.8] remains valid (with the same proof) in the function field case. The top right-hand vertical map 

\[
H^2(K, M^*)^D \to H^0(K, M)^\wedge
\]

is the composite

\[
H^2(K, M^*)^D \to III^2(M^*)^D \xrightarrow{\sim} \text{Ker } \theta_0 \xrightarrow{\sim} \text{Ker } \beta_0 \to T\text{Sel}(M)^\wedge = H^0(K, M)^\wedge,
\]

where the isomorphisms come from Propositions 3.3 and 3.7 and the equality is a consequence of the finiteness hypothesis on \( III^1(M) \). The exactness of the second line of the sequence of the theorem is the content of Proposition 3.10 (again using the equality \( T\text{Sel}(M)^\wedge = H^0(K, M)^\wedge \)). Since \( \gamma_0 \) is the dual of the natural map \( H^1(K, M^*) \to \mathbb{P}^1(M^*)_{\text{tors}} \) and \( III^1(M^*)^D \simeq III^1(M) \) by [7, Corollary 4.9 and Remark 5.10] and [5, corollary 6.7], we conclude that there exists an exact sequence

\[
0 \to H^{-1}(K, M)^\wedge \xrightarrow{\gamma_0} \prod_{v} H^2(K_v, M^*)_v^D \xrightarrow{\beta_0} H^2(K, M^*)^D
\]

\[
H^1(K, M^*)^D \xrightarrow{\gamma_0} \mathbb{P}^0(M)^\wedge \xrightarrow{\beta_0} H^0(K, M)^\wedge
\]

\[
III^1(M).
\]

The above is an exact sequence of profinite groups and continuous homomorphisms, so each morphism is strict [1, §III.2.8, p.237]. Consequently, the dual of the preceding sequence is also exact [15, Theorem 23.7, p.196]. Exchanging the roles of \( M \) and \( M^* \) in this dual exact sequence and noting that \( (H^0(K, M^*)^\wedge)^D = (H^0(K, M^*)^D)_{\text{tors}} \) and \( (H^{-1}(K, M^*)^\wedge)^D = H^{-1}(K, M^*)^D \) (since \( H^{-1}(K, M^*) \) is finitely
generated by [7, Lemma 2.1, p.98]), we obtain an exact sequence

\[
\begin{array}{cccccc}
\mathbb{H}^1(K, M) & \to & \mathbb{H}^1(M_{\text{tors}}) & \to & (\mathbb{H}^0(K, M^*)^D)_{\text{tors}} \\
\mathbb{H}^{-1}(K, M^*)^D & \to & \bigoplus_{\text{all } v} \mathbb{H}^2(K_v, M) & \to & \mathbb{H}^2(K, M).
\end{array}
\]

The sequence of the theorem may now be obtained by splicing together the preceding two exact sequences. \hfill \Box

4. The generalized Cassels-Tate dual exact sequence

For \( i = 1 \) or \( 2 \), define

\[
\mathbb{III}^i(T(M)) = \text{Ker} \left[ H^i(K, T(M)) \to \prod_{\text{all } v} H^i(K_v, T(M)) \right]
\]

and

\[
\mathbb{III}^i(M) = \text{Ker} \left[ H^i(K, M) \to \prod_{\text{all } v} H^i(K_v, M) \right],
\]

where the \( v \)-component of each of the maps involved is induced by the natural morphism \( \text{Spec } K_v \to \text{Spec } K \).

**Proposition 4.1.** There exists a perfect pairing

\[
\mathbb{III}^1(T(M^*)) \times \mathbb{III}^2(M) \to \mathbb{Q}/\mathbb{Z},
\]

where the first group is profinite and the second is discrete and torsion.

**Proof.** The proof is similar to the proof of Proposition 3.3. \hfill \Box

Let \( S \) be any finite set of primes of \( K \) and define, for \( i = 1 \) or \( 2 \),

\[
\mathbb{III}^i_S(T(M)) = \text{Ker} \left[ H^i(K, T(M)) \to \prod_{v \notin S} H^i(K_v, T(M)) \right]
\]

and

\[
\mathbb{III}^i_S(M) = \text{Ker} \left[ H^i(K, M) \to \prod_{v \notin S} H^i(K_v, M) \right].
\]
Thus $\Pi^i_\emptyset(M) = \Pi^i(T(M))$ and $\Pi^i\emptyset(M) = \Pi^i(M)$. Now partially order the family of finite sets $S$ by defining $S \leq S'$ if $S \subset S'$. Then $\Pi^1_S(M) \subset \Pi^1_{S'}(M)$ for $S \leq S'$. Set

$$\Pi^1\omega(M) = \lim_{\leftarrow S} \Pi^1_S(M) = \bigcup_S \Pi^1_S(M) \subset \mathbb{H}^1(K, M),$$

where the transition maps in the direct limit are the inclusion maps. Thus $\Pi^1\omega(M)$ is the subgroup of $\mathbb{H}^1(K, M)$ of all classes which are locally trivial at all but finitely many places of $K$. Clearly, for each $S$ as above, there exists an exact sequence of discrete torsion groups

$$0 \rightarrow \Pi^1(M^*) \rightarrow \Pi^1_S(M^*) \rightarrow \prod_{v \in S} \mathbb{H}^1(K_v, M^*)$$

whose dual is an exact sequence of profinite groups

$$(8) \quad \prod_{v \in S} \mathbb{H}^0(K_v, M)^\wedge \xrightarrow{\hat{\theta}_S} \Pi^1_S(M^*)^D \rightarrow \Pi^1(M^*)^D \rightarrow 0.$$
is also exact (cf. proof of Theorem 3.11). The above sequence induces
an exact sequence of discrete groups
\[ \varprojlim \mathbb{H}^1(K_v, M^*) \to \prod_{v \in S} \mathbb{H}^1(K_v, M^*) \to (T\text{Sel}(M)^\wedge)^D \]
whose dual is an exact sequence
\[ (T\text{Sel}(M)^\wedge \to \prod_{v \in S} \mathbb{H}^0(K_v, M)^\wedge \overset{\tilde{\beta}_S}{\longrightarrow} \varprojlim \mathbb{H}^1(K_v, M)^\wedge) \to \varprojlim \mathbb{H}^1(M^*)^D. \]
Taking the inverse limit over \( S \) above and noting that the inverse limit
functor is exact on the category of profinite groups [14, Proposition
2.2.4, p.32], we obtain the exact sequence of the proposition. That \( \hat{\phi} \)
has the stated factorization follows from the proof. \( \square \)

Proposition 4.3. There exists a canonical isomorphism
\[ \text{Ker} \left[ T\text{Sel}(M)^\wedge \overset{\hat{\phi}}{\longrightarrow} \prod_{v \in S} \mathbb{H}^0(K_v, M)^\wedge \right] = \varprojlim \mathbb{H}^2(M^*)^D, \]
where \( \hat{\phi} \) is the map of Proposition 4.2.

Proof. By Proposition 4.2 and the fact that \( \text{Ker} \beta_0 = \text{Ker} \theta_0 = \varprojlim \mathbb{H}^2(M^*)^D \)
by Propositions 3.3 and 3.7, it suffices to check that the canonical map
\( \mathbb{P}^0(M)^\wedge \to \prod_{v \in S} \mathbb{H}^0(K_v, M)^\wedge \) is injective. The argument is similar to
that used in the proof of Lemma 3.6. Let \( x \in \mathbb{P}^0(M)^\wedge \) be nonzero.
There exists an open subgroup \( U \subset \mathbb{P}^0(M) \) of finite index \( n \) (say) such
that the \( U \)-component of \( x \), \( x_U + U \in \mathbb{P}^0(M) / U \) is nonzero, i.e., \( x_U \notin U \).
Then \( x_U \notin n\mathbb{P}^0(M) \), whence \( (x_U)_v \notin n\mathbb{H}^0(K_v, M) \) for some \( v \). Thus
the image of \( x \) in \( \mathbb{H}^0(K_v, M) / n = (\mathbb{H}^0(K_v, M) / n)^\wedge \) is nonzero. Since
the map \( \mathbb{P}^0(M)^\wedge \to (\mathbb{H}^0(K_v, M) / n)^\wedge \) factors through \( \mathbb{H}^0(K_v, M)^\wedge \),
the image of \( x \) in \( \mathbb{H}^0(K_v, M)^\wedge \) is nonzero. \( \square \)

As noted earlier, the inverse limit functor is exact on the category of
profinite groups, so the inverse limit over \( S \) of (8) is an exact sequence
\[ \prod_{v \in S} \mathbb{H}^0(K_v, M)^\wedge \overset{\tilde{\beta}}{\longrightarrow} \varprojlim \mathbb{H}^1(M^*)^D \to \varprojlim \mathbb{H}^1(M^*)^D \to 0. \]
We now use Propositions 4.2 and 4.3 to extend the above exact sequence
to the left and obtain

Theorem 4.4. (The generalized Cassels-Tate dual exact sequence)
There exists a canonical exact sequence of profinite groups
\[ 0 \to \varprojlim \mathbb{H}^2(M^*)^D \to \varprojlim \mathbb{H}^1(M)^\wedge \to \prod_{v \in S} \mathbb{H}^0(K_v, M)^\wedge \]
\[ \to \varprojlim \mathbb{H}^1(M^*)^D \to \varprojlim \mathbb{H}^1(M^*)^D \to 0. \]
Corollary 4.5. There exists a canonical exact sequence of discrete torsion groups
\[ 0 \to \mathcal{H}^1(M) \to \mathcal{H}^1_\omega(M) \to \bigoplus_{v} \mathbb{H}^1(K_v, M) \]
\[ \to (T\text{Sel}(M^*)^D \to \mathcal{H}^2(M) \to 0. \]
\[ \square \]

We conclude this paper with the following result, which extends [8, Theorem 1.2] to the function field case.

Theorem 4.6. Let \( K \) be a global function field and let \( M \) be a 1-motive over \( K \). Assume that \( \mathcal{H}^1(M) \) is finite. Then there exists an exact sequence
\[ 0 \to \overline{\mathbb{H}^0(K, M)} \to \prod_{v} \mathbb{H}^0(K_v, M) \to \mathcal{H}^1_\omega(M^*)^D \to \mathcal{H}^1(M) \to 0, \]
where \( \overline{\mathbb{H}^0(K, M)} \) denotes the closure of the diagonal image of \( \mathbb{H}^0(K, M) \) in \( \prod_{v} \mathbb{H}^0(K_v, M) \).

Proof. The proof is essentially the same as that of [8, Theorem 1.2], noting that \( T\text{Sel}(M^*)^D = \overline{\mathbb{H}^0(K, M)} \) if \( \mathcal{H}^1(M) \) is finite and using Proposition 4.2 in place of [8, Proposition 5.3(1)]. \[ \square \]

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Departamento de Matemáticas, Universidad de La Serena, Chile
E-mail address: cgonzalez@userena.cl

Department of Mathematics, National Taiwan University, Taipei 10764, Taiwan
E-mail address: tan@math.ntu.edu.tw