How to Secure Matchings Against Edge Failures

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Abstract

Suppose we are given a bipartite graph that admits a perfect matching and an adversary may delete any edge from the graph with the intention of destroying all perfect matchings. We consider the task of adding a minimum cost edge-set to the graph, such that the adversary never wins. We provide efficient exact and approximation algorithms. In particular, for the unit-cost problem, we provide a $\log_2 n$-factor approximation algorithm and a polynomial-time algorithm for chordal-bipartite graphs. Furthermore, we give a fixed parameter algorithm for the problem parameterized by the treewidth of the input graph. For general non-negative weights we settle the approximability of the problem and show a close relation to the Directed Steiner Forest Problem. Additionally we prove a dichotomy theorem characterizing minor-closed graph classes which allow for a polynomial-time algorithm. Our methods rely on a close relationship to the classical strong connectivity augmentation problem and directed Steiner problems.

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1 Introduction

An augmentation problem asks for a minimum-cost set of edges to be added to a graph in order to establish a certain property. We say that a bipartite graph is robust if it admits a perfect matching after the removal of any edge. Our goal is to make a bipartite graph robust at minimal cost and we study the complexity of the corresponding augmentation problem. We refer to this problem informally as robust matching augmentation. As a motivation, note that in many situations some kind of infrastructure is already available, so we may prefer upgrading it instead of designing robust infrastructure from scratch. Assume we have some assignment-type application, such as staff or task scheduling, so our infrastructure is given in terms of a bipartite graph. The application requires that we choose a perfect matching that assigns, say, tasks to machines. By buying additional edges, we would like to ensure that no matter which edge fails, the resulting graph has a perfect matching, i.e., the infrastructure remains useable. In such an application, buying edges may correspond for example to training staff or upgrading machines.

A complementary approach to creating robust infrastructure is captured by design problems. A design problem asks for a minimum-cost subgraph with a certain property, for instance a minimum-cost k-edge-connected subgraph \[11, 21\]. Robust matching augmentation can be stated also as a design problem, where the given infrastructure is available at zero cost and the host graph is a complete bipartite graph. In fact, our problem is a special case of the bulk-robust assignment problem, a design problem introduced in [2]. Bulk-robustness is a redundancy-based robustness concept proposed by Adjiashvili, Stiller and Zenklusen [3], which allows to specify a list of failure scenarios. The bulk-robust assignment problem is known to be \( \text{NP} \)-hard even if only one of two fixed edges may fail [2]. Here we consider the setting that any single edge may fail [2].

A central theme in our algorithmic results is the occurrence of the classical strong connectivity augmentation problem, which asks for the minimal number of arcs that are needed to make a given digraph strongly connected. It was shown by Eswaran and Tarjan that this problem admits a polynomial-time algorithm, but its edge-weighted variant is \( \text{NP} \)-hard \[15\]. We show that also for robust matching augmentation the weighted problem is much harder than its cardinality version. To this end, we give a log \( n \)-factor approximation algorithm for the cardinality version which is essentially tight and prove that the weighted problem admits no log\(^2\) \( n \)-factor approximation under standard complexity assumptions.

Our Contribution Recall that we call a graph robust if it admits a perfect matching after the removal of any single edge. For a bipartite graph \((V, E)\), we denote by \(\overline{E} \) the edge-set of its bipartite complement. We provide algorithms and hardness results for several restrictions of the following problem.

Robust Matching Augmentation

instance: Undirected bipartite graph \( G = (U + W, E) \) that admits a perfect matching.

task: Find a set \( L \subseteq \overline{E} \) of minimum cardinality, such that the graph \( G + L \) is robust.
By a close relation of robust matching augmentation and connectivity augmentation, we provide a deterministic log₂ n-factor approximation for Robust Matching Augmentation, as well as a fixed parameter tractable (FPT) algorithm for the same problem parameterized by the treewidth of the input graph. We also give a polynomial-time algorithm for instances on chordal-bipartite graphs, which are bipartite graphs without induced cycles of length at least six. Furthermore, we show that Robust Matching Augmentation admits no polynomial-time sublogarithmic-factor approximation algorithm unless P = NP, so our approximation guarantee is essentially tight.

Let us give an overview of the high-level ideas behind our algorithmic results and make some connections to other problems. We first show that we may restrict our attention to an arbitrary fixed perfect matching of the input graph. That is, it suffices to prevent the adversary from destroying a given fixed matching. From the input graph and the perfect matching we construct an auxiliary digraph. In this digraph we select certain sources and sinks which we connect using the Eswaran-Tarjan algorithm to obtain a strongly connected subgraph. It turns out that strong connectivity in the auxiliary digraph implies robustness in the original graph. We obtain an optimal solution to our Robust Matching Augmentation instance if the selection of sources and sinks was optimal.

We model the task of properly selecting sources and sinks as a variant of the Set Cover problem with some additional structure. Given an acyclic digraph, the task is to select a minimum-cardinality subset of the sources, such that each sink is reachable from at least one of the selected sources. We refer to this problem as Source Cover and remark that its complexity may be of independent interest, since it generalizes Set Cover but is a special case of Directed Steiner Tree. We give an FPT algorithm for the Source Cover problem parameterized by the treewidth of the input graph (ignoring orientations). This FPT algorithm is single exponential in the treewidth. As a by-product, we obtain FPT algorithms for the node-weighted and arc-weighted versions of the Directed Steiner Tree problem on acyclic digraphs, which are exponential in the treewidth and linear in the number of nodes of the input graph.

Finally, we relax the requirement of having a perfect matching to having a matching of cardinality at least k. In fact, all of our algorithmic results for Robust Matching Augmentation generalize to the setting where we desire to have a matching of cardinality k after deleting any single edge from a graph.

We refer by Weighted Robust Matching Augmentation to the generalization of Robust Matching Augmentation, where each edge e ∈ E has a non-negative cost c_e. The task is to find a minimum-cost set L ⊆ E, such that G + L is robust. First, we show that the approximability of Weighted Robust Matching Augmentation is closely linked to that of Directed Steiner Forest. In particular we show that an f(n)-factor approximation algorithm for Weighted Robust Matching Augmentation implies an f(n+k)-factor approximation algorithm for Directed Steiner Forest, where k is the number of terminal pairs. By a result of Halperin and Krauthgamer [22] it follows that there is no log²⁻ε(n)-factor approximation for Weighted Robust Matching Augmentation, unless NP ⊆ ZTIME(n polylog(n)). On the positive side, we show that an f(k)-factor approximation for the Directed Steiner Forest problem yields an (f(k)+1)-factor approximation Weighted Robust Match-
Hence, the algorithms from [9, 17] give an approximation guarantee of \(1 + n^{1/2 + \varepsilon}\) for Weighted Robust Matching Augmentation, for every \(\varepsilon > 0\).

Second, we prove a complexity dichotomy based on graph minors. Let \(\mathcal{T}\) be a class of connected graphs closed under connected minors. We show that Weighted Robust Matching Augmentation restricted to input graphs from \(\mathcal{T}\) is \(\mathsf{NP}\)-complete if \(\mathcal{T}\) contains at least one of two simple graph classes, which will be defined in Section 5 and admits a polynomial-time algorithm otherwise. The polynomial-time algorithm for the remaining instance classes uses a reduction to the Directed Steiner Forest problem with a constant number of terminal pairs, which in turn admits a (slice-wise) polynomial-time algorithm due to a result by Feldman and Ruhl [16]. The terminal pairs of the instance are computed by the Eswaran-Tarjan algorithm.

Related work

Adjiashvili, Binnewald and Michaels in [2] proposed an LP-based randomized algorithm for the bulk-robust assignment problem. They claim an \(O(\log n)\)-factor approximation guarantee for their algorithm. Since the robust assignment problem generalizes Weighted Robust Matching Augmentation, an \(O(\log n)\)-factor approximation for our problem is implied. However, due to our inapproximability result for Weighted Robust Matching Augmentation, this can not be true, unless \(\mathsf{NP} \subseteq \mathsf{ZTIME}(n^{\text{polylog}(n)})\). The authors of [2] agree that their analysis is incorrect.

A connectivity augmentation problem related to strong connectivity, but of a different flavor, is the tree augmentation problem (TAP). The TAP asks for a minimum-cost edge-set that increases the edge-connectivity of a given tree from one to two. In contrast to robust matching augmentation, the TAP admits a constant-factor approximation [20]. The constant has recently been lowered to \(3/2 + \varepsilon\) for bounded-weight instances [11, 18]. Similar to robust matching augmentation, the input graph is available at zero cost. Let us briefly remark that there is more conceptual similarity. The matching preclusion number of a graph is the minimal number of edges to be removed, such that the remaining graph has no perfect matching. Robust matching augmentation can be stated as the task of finding a minimum-cost edge-set that increases the matching preclusion number of a bipartite graph from one to two, while the TAP aims to increase connectivity from one to two. The matching preclusion number is considered to be a measure of robustness of interconnect networks [8, 10].

Determining the matching preclusion number of a graph is \(\mathsf{NP}\)-hard [13, 24].

Robust perfect matchings with a given recovery budget were studied by Dourado et al. in [13]. Our notion of robustness corresponds to 1-robust \(\infty\)-recoverable in their terminology. They provide hardness results and structural insights mainly for fixed recovery budgets, which bound the number of edges that can be changed in order to repair a matching, after a certain number of edges has been removed from the graph.

Notation

Undirected and directed graphs considered here are simple. For sets \(U, W\), we denote by \(U + W\) their disjoint union. For an undirected bipartite graph \(G = (U + W, E)\) with bipartition \((U, W)\), we denote by \(\overline{E}\) the edge-set of its bipartite complement. Let \(D = (V, A)\) be a directed graph. We refer by \(\overline{A}\) to the arcs not present in \(D\). That is, we let \(\overline{A} \subseteq (V \times V) \setminus A\). By \(\mathcal{U}(D)\)
we refer to the underlying undirected graph of $D$. For $L \subseteq E$, we write $G + L$ for the graph $G' = (V(G), E(G) \cup L)$. Simple paths in graphs are given by a sequence of vertices. For graphs $G, H$ we write $H \subseteq G$ if $H$ is a subgraph of $G$. Recall that a graph $H$ is an induced minor of a graph $G$ if it arises from $G$ by a sequence of vertex deletions and edge contractions. Similarly, the graph $H$ is a minor of $G$ if we additionally allow edge deletion. Furthermore, the graph $H$ is a connected minor of $G$ if $H$ is connected and a minor of $G$. In general, contractions may result in parallel edges or loops, which we simply discard in order to keep our graphs simple. Let $G$ be a class of graphs. We will refer to the restriction of (Weighted) Robust Matching Augmentation to instances where the graph $G$ is bipartite, admits a perfect matching, and belongs to the class $G$ as (Weighted) Robust Matching Augmentation on $G$. Given a set of items $X$ and sets $S \subseteq 2^X$, the Set Cover problem asks for a minimum-cardinality subset $C \subseteq S$ such that each $x \in X$ is contained in some $s \in C$. The incidence graph $G(I)$ of a Set Cover instance $I = (X, S)$ is an undirected bipartite graph on the vertex set $X + S$ that has an edge $xs$ if and only if the item $x \in X$ is contained in the set $s \in S$.

Organization of the Paper The remainder of the paper is organized as follows. We illustrate the relation between robust matching augmentation and strong connectivity augmentation in Section 2. Algorithms for the Source Cover problem are given in Section 3. Based on the results from Sections 2 and 3, we present our results on robust matching augmentation with unit costs in Section 4. In Section 5 we give the complexity classification for the weighted version of the problem and Section 6 concludes the paper.

2 Robust Matchings and Strong Connectivity Augmentation

In this section we give some preliminary observations on the close relationship between robust matching augmentation with unit costs and strong connectivity augmentation. For this purpose, we fix an arbitrary perfect matching and construct an auxiliary digraph that is somewhat similar to the alternating tree used in Edmond’s blossom algorithm. We show that the original graph is robust if the auxiliary graph is strongly connected (but not vice versa). Furthermore, we show that there is an optimal edge-set making the given graph robust, that corresponds to a set of arcs connecting sources and sinks in the auxiliary digraph. Finally, if no source or sink of the auxiliary digraph corresponds to a non-trivial robust part of the original graph, then we may use the algorithm for strong connectivity augmentation by Eswaran and Tarjan [15] to make the original graph robust. As a consequence, we have that Robust Matching Augmentation on trees can be solved efficiently by using the Eswaran-Tarjan algorithm. In Section 4 we will generalize this result.

Let $G = (U + W, E)$ be a bipartite graph that admits a perfect matching and let $M$ be an arbitrary but fixed perfect matching of $G$. We call an edge $e \in M$ critical if $G - e$ admits no perfect matching. Observe that an edge $e \in M$ is critical if and only if it is not contained in an $M$-alternating cycle. Furthermore, no edge in $E \setminus M$ is critical. Since $M$ is perfect, each edge $e \in M$
is incident to a unique vertex $u_e$ of $U$. We consider the following auxiliary digraph $D(G, M) = (U, A)$, whose arc-set $A$ is given by

$$A := \{ uu' \mid u, u' \in U : \text{there is a vertex } w \in W \text{ such that } uw \in M \text{ and } wu' \in E \setminus M \}.$$  

We first note that the choice of the bipartition of $G$ is irrelevant.

**Fact 1.** Let $G' = (U' + W', E)$, where $(U', W')$ is a bipartition of $G$. Then $D(G, M)$ is isomorphic to $D(G', M)$.

Note that we may perform the reverse construction as well. That is, from any digraph $D'$ we may obtain a corresponding undirected graph $G$ and a perfect matching $M$ of $G$ such that $D(G, M) = D'$. In fact, augmenting edges to $G$ is equivalent to augmenting arcs to $D(G, M)$.

**Fact 2.** Let $\overline{A}$ be the set of arcs that are not present in $D(G, M)$. Then there is a 1-to-1 correspondence between $E$ and $\overline{A}$. An example of the correspondence mentioned in Fact 2 is shown in Figure 1.

In order to keep our notation tidy, we will make implicit use of Fact 2 and refer to $\overline{A}$ and $\overline{E}$ interchangeably. Observe that for edges $e, f \in M$ there is an $M$-alternating path containing $e$ and $f$ in $G$ if and only if $u_e$ is connected to $u_f$ in $D(G, M)$. This implies the following characterization of robustness.

**Fact 3.** $G$ is robust if and only if each strongly connected component of $D(G, M)$ is non-trivial, that is, it contains at least two vertices.

Let $D'$ be a digraph. A vertex of $D'$ is called a source (sink) if it has no incoming (outgoing) arc. We refer to the set of sources (sinks) of $D'$ by $V^+(D')$ ($V^-(D')$). Furthermore, we denote by $C(D')$ the condensation of $D'$, that is, the directed acyclic graph of strongly connected components of $D'$. We call a source or sink of $C(D')$ strong if the corresponding strongly connected component of $D'$ is non-trivial. From Fact 3 it follows that a subgraph of $G$ that corresponds to a strong source or a strong sink is robust against the failure of a single edge. Furthermore, observe that the choice of the perfect matching $M$ of $G$ is irrelevant in the following sense.

**Fact 4.** Let $M$ and $M'$ be perfect matchings of $G$. Then $C(D(G, M))$ is isomorphic to $C(D(G, M'))$. 

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**Figure 1:** Illustration of the correspondence between the dotted edges in and .
Fact 4 is of key importance for our algorithmic results, for which we generally assume that some fixed perfect matching is given. Next, we observe that for unit costs we may restrict our attention to connecting sources and sinks of \( C(D) \) in order to make \( G \) robust. It is easy to check that this does not hold for general non-negative costs.

**Fact 5.** Let \( L \subseteq E \) such that \( G + L \) is robust. Then there is some \( L' \subseteq \overline{E} \) of cardinality at most \( |L| \), such that \( G + L' \) is robust and \( L' \) connects only sinks to sources of \( C(D(G, M)) \).

We remark that the construction of \( L' \) given in the proof of Fact 5 can be performed in polynomial time.

We denote by \( \gamma(D') \) the minimal number of arcs to be added to a digraph \( D' \) in order to make it strongly connected. Eswaran and Tarjan have proved the following min-max relation [15].

**Fact 6.** Let \( D' \) be a digraph. Then \( \gamma(D') = \max\{ |V^+(D')|, |V^-(D')| \} \).

From the proof of Fact 6 it is easy to obtain a polynomial-time algorithm that, given a digraph \( D' \), computes an arc-set \( L \) of cardinality \( \gamma(D') \) such that \( D' + L \) is strongly connected [19]. We will refer to this algorithm by Eswaran-Tarjan. The following proposition illustrates the usefulness of the algorithm Eswaran-Tarjan for Robust Matching Augmentation, and at the same time its limitations.

**Fact 7.** Suppose that \( C(D(G, M)) \) contains no strong sources or sinks. Then Eswaran-Tarjan computes a set \( L \subseteq E \) of minimum cardinality such that \( G + L \) is robust.

Fact 7 implies that Eswaran-Tarjan solves Robust Matching Augmentation on trees. If strong sources or sinks are present in \( D(G, M) \), then we may or may not need to consider them in order to make \( G \) robust. This is precisely what makes the problem Robust Matching Augmentation hard. We will formalize the task of selecting strong sources and sinks in terms of the Source Cover problem, which is discussed in the next section.

### 3 The Source Cover Problem

To present our algorithmic results in Section 4 in a concise fashion it will be convenient to introduce the Source Cover problem. Given an acyclic digraph, the Source Cover problem asks for a minimum-cardinality subset of its sources, such that each sink is reachable from at least one selected source. It is easy to see that Source Cover is a special case of the Directed Steiner Tree problem and that it generalizes Set Cover. We give a simple polynomial-time algorithm for Source Cover if the input graph is chordal-bipartite (ignoring orientations). Furthermore, we show that Source Cover parameterized by treewidth (again ignoring orientations) is FPT. As a by-product, we obtain a simple FPT algorithm for the arc-weighted and node-weighted versions of the Directed Steiner Tree problem on acyclic digraphs, whose running time is linear in the size of the input graph and exponential in the treewidth of the underlying undirected graph. To the best of our knowledge, the parameterized
(a) A digraph $D$ such that $U(D)$ is balanced, but $U(F(D))$ is not.

(b) Digraphs $D$ such that $U(D)$ has treewidth one, but the treewidth of $U(F(D))$ is unbounded.

Figure 2: Examples showing that flattening does not preserve balancedness or bounded treewidth.

complexity of the general Directed Steiner Tree problem with respect to treewidth is open. For the corresponding undirected Steiner Tree problem, an FPT algorithm was given by Bodlaender et al. in [7].

The Source Cover problem is formally defined as follows.

**Source Cover**

**instance:** Weakly connected acyclic digraph $D = (V, A)$ with at least one arc.

**task:** Find a minimum-cardinality subset $S$ of the sources $V^+(D)$ of $D$, such that for each sink $t \in V^-(D)$ there is an $S$-$t$-path in $D$.

The assumptions that $D$ is connected and contains at least one arc are present only for technical reasons. By “flattening” the input digraph, we can turn an instance $I = (D)$ of Source Cover into a Set Cover instance as follows. Let $F(D) = (V^+(D) \cup V^-(D), A')$ be an acyclic digraph, where $A'$ is given by

$$A' := \{st \mid s \in V^+(D), t \in V^-(D), t \text{ is reachable from } s \text{ in } D\}.$$  

Then $U(F(D))$ is the incidence graph of a Set Cover instance $A$ on $V^-(F(D))$, such that the feasible solutions of $I$ and $A$ are in 1-to-1 correspondence.

As illustrated in Figure 2, useful properties of the input digraph may not be present in the corresponding flattened digraph. In particular, if $U(D)$ has treewidth at most $r$, then the treewidth of $U(F(D))$ cannot be bounded by a constant in general. Furthermore, the graph $U(F(D))$ is not necessarily balanced (or planar) if $U(D)$ is. Therefore, we cannot take advantage of polynomial-time algorithms for Set Cover on balanced incidence graphs or incidence graphs of bounded treewidth. Motivated by the example in Figure 2b we leave as an open question, whether Source Cover on balanced graphs admit polynomial-time algorithms. By Theorem 11, the existence of such an algorithm implies a polynomial-time algorithm for Robust Matching Augmentation on balanced graphs.

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1 A graph is called balanced if the length of each induced cycle is divisible by four.
3.1 Source Cover on Chordal Bipartite Graphs

We show that in contrast to the treewidth and balancedness, chordal-bipartiteness is indeed preserved by the flattening operation introduced above. From this we obtain the following result.

**Theorem 8.** Source Cover on chordal-bipartite graphs admits a polynomial-time algorithm.

To prove the theorem, we show that if \( \mathcal{U}(D) \) is chordal-bipartite, so is \( \mathcal{U}(F(D)) \). The graph \( \mathcal{U}(F(D)) \) is the incidence graph of a Set Cover instance, whose optimal solutions correspond canonically to the optimal solutions of the Source Cover instance \( (D) \). It is known that Set Cover on chordal-bipartite incidence graphs (and more generally, balanced graphs) admits a polynomial-time algorithm: It is possible to use LP-methods and the fact that covering polyhedra of balanced matrices are integral, see [25, pp. 562-573]. Alternatively we can use a combinatorial algorithm by Hoffman et al. [23].

3.2 Source Cover on Graphs of Bounded Treewidth

We provide a fixed-parameter algorithm for Node Weighted Directed Steiner Tree on acyclic digraphs that is single-exponential in the treewidth of the underlying undirected graph and linear in the instance size. Since Source Cover is a restriction of Node Weighted Directed Steiner Tree on acyclic graphs, we have a polynomial-time algorithm for Source Cover parameterized by the treewidth of the underlying undirected graph. Let us first recall some definitions related to Steiner problems and tree decompositions.

**Node Weighted Directed Steiner Tree**

- **instance:** Acyclic digraph \( D = (V, A) \), costs \( c \in \mathbb{R}^V \geq 0 \), terminals \( T \subseteq V \), root \( r \in V \).
- **task:** Find a minimum-cost subset \( F \subseteq V \), such that \( r \) is connected to each terminal in \( (F, E(F)) \).

**Arc Weighted Directed Steiner Tree** is the corresponding problem, where the costs are on the arcs of the graph. A tree decomposition of a graph \( G = (V, E) \) is a tree \( T \) as follows. Each node \( x \in V(T) \) of \( T \) has a bag \( B_x \subseteq V \) of vertices of \( G \) such that the following properties hold.

- \( \bigcup_{x \in V(T)} B_x = V \).
- If \( B_x \) and \( B_y \) both contain a vertex \( v \in V \), then the bags of all nodes of \( T \) in the path between \( x \) and \( y \) contain \( v \) as well. Equivalently, the tree nodes containing vertex \( v \) form a connected subtree of \( T \).
- For each edge \( vw \) in \( G \) there is some bag that contains both \( v \) and \( w \). That is, for vertices adjacent in \( G \), the corresponding subtrees have a node in common.

The width of a tree decomposition is the size of its largest bag minus one. The treewidth \( \text{tw}(G) \) of \( G \) is the minimum width among all possible tree decompositions of \( G \).
To solve the **Node Weighted Directed Steiner Tree** on acyclic digraphs, we use a simple dynamic-programming algorithm over the tree decomposition of the underlying undirected graph of the input digraph $D$ with $n$ vertices.

**Theorem 9.** **Node Weighted Directed Steiner Tree** on acyclic digraphs can be solved in time $O(5^w \cdot w \cdot n)$ if a tree decomposition of $U(D)$ of width $w$ is provided.

Note that an optimal tree-decomposition of a graph $G$ can be computed in time $O(2^{O(tw(G)^3)} \cdot n)$ by Bodlaender’s famous theorem [6]. Our algorithm intuitively works in the following way and is similar to the dynamic programming algorithm for **Dominating Set** (see, e.g., [12, Section 7.3.2]). We interpret a solution to **Node Weighted Directed Steiner Tree** as follows: each vertex of $D$ may be active or not. Each active vertex needs a predecessor that is also active, unless it is the root. The cost to activate a vertex is given by the cost function of the **Node Weighted Directed Steiner Tree** instance. Starting with all terminals active, it is easy to see that **Node Weighted Directed Steiner Tree** on acyclic graphs is equivalent to the problem of finding a minimum cost active vertex set satisfying the above conditions. We compute an optimal solution in a bottom-up fashion using a so-called nice tree decomposition of the input graph.

By a simple reduction, we also obtain an **FPT**-time algorithm for **Arc Weighted Directed Steiner Tree** on acyclic digraphs. We just subdivide each arc and assign the cost of the arc to the corresponding new vertex. Each old vertex receives cost zero. This transformation does not increase the treewidth.

Furthermore, we can reduce **Source Cover** to **Node Weighted Directed Steiner Tree** by adding a new vertex $r$ and connecting $r$ to each source by an arc. The sources have cost one, while all other vertices have cost zero. The root vertex is $r$ and the set of terminals is the set of sinks. By adding only one new vertex, the treewidth is increased by at most one. As a consequence of this reduction and Theorem 9, we obtain the following result.

**Corollary 10.** **Source Cover** can be solved in time $O(5^w \cdot w \cdot n)$ if a tree-decomposition of $U(D)$ of width $w$ is provided.

### 4 Robust Matching Augmentation

In this section we present our main results on the problem **Robust Matching Augmentation**. Let us first redefine the problem in a slightly different way.

**Robust Matching Augmentation**

**instance:** Bipartite graph $G = (U + W, E)$ and perfect matching $M$ of $G$.

**task:** Find a minimum-cardinality set $L \subseteq \overline{E}$ such that $G + L$ is robust.

Fixing the perfect matching $M$ in the instance is just for notational convenience, since we can compute a perfect matching in polynomial time and our results do not depend on the exact choice of $M$, according to the discussion in
Section 2. The next theorem is our main technical result of this section. By combining the theorem with the results in Section 3 we obtain our algorithmic results.

**Theorem 11.** There is a polynomial-time algorithm that, given an instance \( I = (G, M) \) of Robust Matching Augmentation, computes two instances \( A_1 = (S_1) \) and \( A_2 = (S_2) \) of Source Cover such that the following holds.

1. \( U(S_1) \) and \( U(S_2) \) are induced minors of \( U(D(G, M)) \).

2. \( \text{OPT}(I) = \max\{\text{OPT}(A_1), \text{OPT}(A_2)\} \)

3. From a solution \( C_1 \) of \( A_1 \) and a solution \( C_2 \) of \( A_2 \) we can construct in polynomial time a solution \( L \) of \( I \) of cardinality \( \max\{|C_1|, |C_2|\} \).

**Proof.** Let \( I = (G, M) \) be an instance of Robust Matching Augmentation, where \( G = (U + W, E) \). Our goal is to obtain from solutions of the Source Cover instances a suitable selection of sources and sinks of \( C(D(G, M)) \), such that we can make \( M \) robust by connecting the selected sources and sinks, using the algorithm Eswaran-Tarjan. Let us denote by \( u_e \) the vertex in \( U \) that is incident to an edge \( e \in M \). Furthermore, let \( D := D(G, M) \). We construct the Source Cover instance \( A_1 \) as follows. For each critical edge \( e \in M \), we remove from \( D \) each vertex \( v \in U - u_e \), such that \( v \) is reachable from \( u_e \) in \( D \). Let \( D' \) be the resulting graph and let the Source Cover instance \( A_1 \) be given by \( A_1 := (C(D')) \). The construction of \( A_2 \) is as for \( A_1 \), but with the arcs of \( D \) reversed. This turns the sources of \( D \) into sinks. Clearly, the acyclic digraphs of \( A_1 \) and \( A_2 \) are induced minors of \( U(D) \), since they were constructed by deleting vertices of \( U(D) \) and contracting strong components. By Fact 5 the set of critical edges can be obtained efficiently by Tarjan’s classical algorithm for computing strongly connected components. In order to generate \( A_1 \) and \( A_2 \), observe that \( D' \) and \( C(D') \) can both be obtained by applying a breadth-first search starting at each vertex of \( D \) or \( D' \), respectively. So it remains to prove Statement 2 and 3.

Let \( C_1 \) (\( C_2 \)) be a solution to \( A_1 \) (\( A_2 \)). We show how to construct in polynomial time a solution \( L \) of \( I \) of cardinality \( \max\{|C_1|, |C_2|\} \). Let \( X \subseteq V(\hat{D}) \) be the set of vertices incident to critical edges. Moreover, let \( \hat{D} \subseteq C(D) \) be the graph induced by the vertices of \( C(D) \) that are on \( C_1 \) \( X \)-paths or on \( X \)-\( C_2 \)-paths in \( C(D) \). Note that \( \hat{D} \) can be computed by a depth-first search applied on each source and sink. By running Eswaran-Tarjan on \( \hat{D} \) we obtain an arc-set \( L^* \) such that \( \hat{D} + L^* \) is strongly connected. Hence, each \( u \in X \) is on some directed cycle in \( \hat{D} + L^* \). From \( L^* \) we can obtain in a straight-forward way an arc-set \( L \) of the same cardinality, such that each \( u \in X \) is on some directed cycle of \( D + L \). For each \( ss' \in L^* \), we add to \( L \) an arc \( uu' \), where \( u \) (\( u' \)) is some vertex in the strong component \( s (s') \) of \( D \). By the construction of \( L \), each \( u \in X \) is on some directed cycle of \( D \). By Fact 2 and 3 we have constructed a solution \( L \) of \( I \) of cardinality \( |L| = |L^*| = \max\{|C_1|, |C_2|\} \). This completes the proof of Statement 3.

It remains to prove that \( \text{OPT}(I) \geq \max\{\text{OPT}(A_1), \text{OPT}(A_2)\} \). Suppose for a contradiction that \( \text{OPT}(I) < \max\{\text{OPT}(A_1), \text{OPT}(A_2)\} \). Without loss of generality, let \( \text{OPT}(A_1) \) attain the maximum. Due to Fact 5 we may assume that an optimal solution \( L \) of \( I \) connects sources and sinks of \( C(D) \). Let \( R \subseteq V(C(D)) \)
be the corresponding sources of $C(D)$. Then for each critical edge $e \in M$, the vertex $u_e$ must be reachable from some source $s \in R$. But then $R$ is a solution of $A_1$ of cardinality $|R| = \text{OPT}(I) < \text{OPT}(A_1)$, a contradiction.

As a first consequence of Theorem 11 we obtain a simple $\log_2 n$-factor approximation algorithm for ROBUST MATCHING AUGMENTATION. We “flatten” the graph of the SOURCE COVER instances as described in Section 5 to obtain SET COVER instances and then use the classic Greedy-Algorithm to achieve a $\log_2 n$-factor approximation.

**Corollary 12.** ROBUST MATCHING AUGMENTATION admits a polynomial-time $\log_2 n$-factor approximation algorithm, where $n$ is the number of vertices of the input graph.

In a similar fashion we obtain a polynomial-time algorithm on chordal-bipartite graphs by combining Theorems 11 and 8 and the observation that $\mathcal{U}(D(G, M))$ is chordal-bipartite if $G$ is. Furthermore, we give an FPT algorithm parameterized by the treewidth by combining Theorems 11 and Corollary 10 and the observation that treewidth is monotone under taking minors.

**Corollary 13.** ROBUST MATCHING AUGMENTATION admits a polynomial-time algorithm on chordal-bipartite graphs and an FPT algorithm parameterized by the treewidth of the input graph.

We now show that our algorithms are also applicable in the following more general setting. Suppose we would like to have a matching of a given cardinality in the graph, no matter which edge is deleted by the adversary.

**Robust $k$-Matching Augmentation**

**Instance:** Bipartite graph $G = (U + W, E)$ that admits a matching of size $k$.

**Task:** Find a minimum-cardinality set $L \subseteq E$ such that for $e \in E$, the graph $G + L - e$ admits a matching of size $k$.

Note that if $k$ is not the size of a maximum matching, then $L = \emptyset$ is feasible due to the existence of a larger matching. We give a polynomial-time reduction from ROBUST $k$-MATCHING AUGMENTATION to ROBUST MATCHING AUGMENTATION that increases the treewidth by at most two. On the other hand, chordal-bipartiteness of the input graph is not preserved. However, the corresponding digraph contains no induced cycle of length at least six, so Theorem 8 is still applicable. By Proposition 14 and the previous corollaries, we obtain for ROBUST $k$-MATCHING AUGMENTATION a $\log_2 n$-factor approximation algorithm, a polynomial-time algorithm on chordal-bipartite graphs, and an FPT algorithm parameterized by the treewidth.

**Proposition 14.** There is a polynomial-time reduction $f$ from ROBUST $k$-MATCHING AUGMENTATION to ROBUST MATCHING AUGMENTATION, such that the following holds. Let $I := (G)$ be an instance of ROBUST $k$-MATCHING AUGMENTATION and let $f(I) = (G')$. Then

1. $\text{OPT}(f(I)) = \text{OPT}(I)$ and from a solution $L'$ of $f(I)$ we can construct in polynomial time a solution $L$ of $I$ such that $|L| \leq |L'|$.
2. $\text{tw}(G') \leq \text{tw}(G) + 2$
3. If $G$ is chordal-bipartite then $U(D(G', M'))$ has no induced cycle of length at least six.

5 Weighted Robust Matching Augmentation

As shown above, Robust Matching Augmentation is tightly linked to Set Cover in terms of approximation. Our first result in this section shows that Weighted Robust Matching Augmentation is substantially more complicated, as its approximability is closely linked to Directed Steiner Forest. This problem is formally defined as follows:

**Directed Steiner Forest**

instance: Directed graph $G = (V, A)$, $k$ terminal pairs $(s_i, t_i)_{1 \leq i \leq k}$, costs $w \in \mathbb{Z}_{\geq 0}$.

task: Find a minimum-cost subgraph $G' \subseteq G$ such that for each $1 \leq i \leq k$, the vertex $s_i$ is connected to $t_i$ in $G'$.

**Proposition 15.** Let $n'$ be the number of vertices of the Weighted Robust Matching Augmentation instance and $n$ and $k$ be the number of vertices and terminals of the Directed Steiner Forest instance, respectively.

An $f(n')$-factor approximation algorithm for Weighted Robust Matching Augmentation implies an $f(n + k)$-factor approximation algorithm for Directed Steiner Forest. An $f(n)$- or an $f(k)$-factor approximation algorithm for Directed Steiner Forest imply an $(f(n) + 1)$- or $(f(k) + 1)$-factor approximation algorithm for Weighted Robust Matching Augmentation, respectively.

On the one hand this result implies an $(n^{1/2 + \varepsilon} + 1)$-factor approximation algorithm for Weighted Robust Matching Augmentation for every $\varepsilon > 0$, due to [9] [17], who achieve a guarantee of $k^{1/2 + \varepsilon}$, for every $\varepsilon > 0$. On the other hand, an algorithm achieving a guarantee of $n^{1/3}$ or better for Weighted Robust Matching Augmentation implies a better approximation algorithm for Directed Steiner Forest, as the number $k$ of distinct terminal pairs is at most $O(n^2)$ and the current best approximation factor in terms of $n$ is $n^{2/3 + \varepsilon}$ due to Berman et al. [4]. Additionally, by a result of Halperin and Krauthgamer [22], the above proposition implies the following lower bound.

**Corollary 16.** For every $\varepsilon > 0$ Weighted Robust Matching Augmentation does not admit a $\log^{2-\varepsilon}(n)$-factor approximation algorithm unless $\text{NP} \subseteq \text{ZTIME}(n^{\text{polylog}(n)})$.

Given this negative result we proceed to the analysis of structural restrictions that make Weighted Robust Matching Augmentation more accessible. The main result of this section is a classification of the complexity of the problem Weighted Robust Matching Augmentation on minor-closed graph classes. In particular we show that the problem is NP-hard on a minor-closed class $\mathcal{G}$ of graphs if and only if $\mathcal{G}$ contains at least one of the two graph classes $K$ and $P^*$, which we will define next. Let $K_{1,r}$ be the star graph with $r$ leaves and let $P_r$ be the path on $r$ vertices. For any graph $H$ let $H^*$ be the graph obtained by attaching a leaf to each vertex of $H$. Then $K^* := \{K_{1,r} \mid r \in \mathbb{N}\}$ and $P^* := \{P_r^* \mid r \in \mathbb{N}\}$. Note that each graph in $K^*$ and $P^*$ has a unique perfect matching. See Figure 3 for an illustration of the graphs $K_{1,3}^*$ and $P_3^*$. 

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Lemma 17. **Weighted Robust Matching Augmentation** is NP-hard on each of the classes $K^*$ and $P^*$.

We complement Lemma 17 by showing that **Weighted Robust Matching Augmentation** on a class $G$ of graphs admits a polynomial-time algorithm if $G$ contains neither $K^*$ nor $P^*$.

Theorem 18. Let $G$ be a class of connected graphs that is closed under connected minors. Then **Weighted Robust Matching Augmentation** on $G$ admits a polynomial-time algorithm if and only if there is some $r \in \mathbb{N}$ such that $G$ contains neither the graph $K^*_{1,r}$ nor $P^*_{r}$. The only if part holds under the assumption that $P \neq \text{NP}$.

In order to prove Lemma 17 we first show that **Weighted Robust Matching Augmentation** is NP-hard for graphs consisting only of a perfect matching by a reduction from **Robust Matching Augmentation**. The hardness of **Weighted Robust Matching Augmentation** on $K^*$ and $P^*$ follows from this result.

Before we give the proof of Theorem 18 we need the following key lemma.

The polynomial-time algorithm described in the proof of the lemma uses the fact that **Directed Steiner Forest** can be solved in polynomial time if the number of terminal pairs is constant [16].

Lemma 19. Let $r \in \mathbb{N}$ be constant and let $T$ be a class of perfectly matchable trees, each with at most $r$ leaves. Then **Weighted Robust Matching Augmentation** on $T$ admits a polynomial-time algorithm.

We remark that the running time of the algorithm given in Lemma 19 slice-wise polynomial in the number of leaves of the input graph. We can now state the proof of our main result.

**Proof of Theorem 18.** According to Lemma 17, **Weighted Robust Matching Augmentation** is NP-hard if $G$ completely contains the class $K = \{K^*_{1,r} \mid r \in \mathbb{N}\}$ or the class $P = \{P^*_{r} \mid r \in \mathbb{N}\}$. Assuming $P \neq \text{NP}$, this proves the only if statement of the theorem.

To see the if statement, let us consider $r \in \mathbb{N}$ such that $G$ does not contain $K^*_{1,r}$ or $P^*_{r}$. First we will reduce the problem to the case when $G$ contains only trees. For this, let $T$ be the class of all trees in $G$ that admit a perfect matching. **Claim 1.** There is a polynomial time reduction of **Weighted Robust Matching Augmentation** on $G$ to **Weighted Robust Matching Augmentation** on $T$. 
The key idea for the proof is to define an equivalent instance on an arbitrary tree of $G$ on an adapted cost function. We may hence restrict our attention to Weighted Robust Matching Augmentation on the class $T$. As the next claim shows, the relevant trees contained in $T$ have a bounded number of leaves.

Claim 2. There is some number $f(r)$ depending only on $r$ such that every tree in $T$ has at most $f(r)$ many leaves.

According to the above claims, there is a polynomial reduction of Weighted Robust Matching Augmentation on $G$ to Weighted Robust Matching Augmentation on a class of trees with a bounded number of leaves. Hence, Lemma 19 implies that Weighted Robust Matching Augmentation on $G$ can be solved in polynomial time. ■

6 Conclusion

We presented algorithms for the task of securing matchings of a graph against the failure of a single edge. For this, we established a connection to the classical strong connectivity augmentation problem. Not surprisingly, the unit weight case is more accessible, and we were able to give a $\log_2 n$-factor approximation algorithm, as well as polynomial-time algorithms for graphs of bounded treewidth and chordal-bipartite graphs. For general non-negative weights, we showed a close relation to Directed Steiner Forest in terms of approximability and gave a dichotomy theorem characterizing minor-closed graph classes which allow a polynomial-time algorithm.

In our opinion, the case of a single edge failure is well understood now and so one might go for the case of a constant number of edge failures next. Let us remark that if the number of edge failures is a part of the input, even checking feasibility is $\text{NP}$-hard [14, 24].

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A Omitted Proofs

A.1 Proofs Omitted from Section 2

Proof of Fact 4 Let $M$ and $M'$ be two distinct perfect matchings of $G$. Then their symmetric difference $M \Delta M'$ is a sum of $(M, M')$-alternating cycles. But each cycle is in some strong component of $D(G, M)$ and $D(G, M')$, so both condensations must be isomorphic.

Proof of Fact 5 Let $vw$ be an arc in $L$. Let $L'$ be a copy of $L$, where the arc $vw$ is replaced by an arc $v'w'$ from a sink $v'$ of $C(D(G, M))$ reachable from $v$ to a source $w'$ of $C(D(G, M))$ from which $w$ is reachable. We show that $G + L'$ is robust. Suppose for a contradiction that this is not the case. Then there is some edge $xy \in M$, such that $x \in U, y \in W$, and $xy$ is not on an $M$-alternating cycle in $G + L'$. Equivalently, $x$ is not contained in a directed cycle of $D + L'$. However, since $G + L$ is robust, we have that $x$ and the arc $vw$ are contained in some directed cycle $C = v_1, v_2, \ldots, v_k$ of $D + L$. That is, there are $1 \leq i, j < k$, such that $x = v_i$, $v = v_j$, and $w = v_{j+1}$. Let $P(Q)$ be a path connecting $v$ and $v'$ ($w'$ and $w$). Then $C' := v_1, v_2, \ldots, v_j, P, Q, v_{j+2}, \ldots, v_k$ is a closed walk that contains a simple directed cycle visiting $x$. This contradicts our assumption that $x$ is not on a directed cycle in $G + L'$. By iterating this argument we obtain an arc-set $L'$ such that $|L'| \leq |L|$ and $G + L'$ is robust. By construction, $L'$ contains only arcs that connect sources and sinks of $C(D(G, M))$.

Proof of Fact 6 By assumption, we have that $C(D(G, M))$ contains no strong sources or sinks. Therefore, each source and each sink of $C(D(G, M))$ corresponds to a critical edge of the matching $M$. Let $L' \subseteq F$ of minimum cardinality, such that $G + L'$ is robust. By Fact 5 we may assume that $L'$ connects only sinks to sources of $C(D(G, M))$. If $|L'| < \gamma(D(G, M)) = \max\{|V^+(C(D(G, M)))|, |V^-(C(D(G, M)))|\}$, then at least one sink or at least one source is not incident to an arc of $L'$. Therefore, the graph $G + L'$ is not robust.

A.2 Proofs Omitted from Section 3

Proof of Theorem 8 Let $(D)$ be a Source Cover instance such that $U(D)$ is connected, has at least one arc, and $U(D)$ contains no induced cycle of length at least six. If $U(F(D))$ is chordal-bipartite, then we can apply the polynomial-time algorithm for SET COVER on chordal-bipartite incidence graphs, see [25, pp. 562-573] and [24]. It remains to show that $U(F(D))$ is chordal-bipartite. Suppose for a contradiction, that $U(F(D))$ contains an induced cycle $C_{FD} = \{s_1, t_1, \ldots, s_k, t_k, s_{k+1} = s_1\}$, where $s_1, s_2, \ldots, s_k \in V^+(F(D))$ and $t_1, t_2, \ldots, t_k \in V^-(F(D))$, and $k \geq 3$. In order to keep the notation concise, let $t_0 := t_k$.

Since $C_{FD}$ is a cycle in $U(F(D))$ connecting sources and sinks, we have that for $1 \leq i \leq k$, there are directed paths $P_{i}^{-1}$ and $P_{i}$ in $D$ such that $P_{i}^{-1}$ connects $s_i$ to $t_{i-1}$ and $P_{i}$ connects $s_i$ to $t_i$. We now construct a cycle $C$ in $U(D)$ and then show that $C$ is chordless and has length at least $k$. Let $Q^i_{j}$ be any shortest path from $s_1$ to $t_1$ in $D$. Let us assume we already constructed the paths $Q^i_{j}$ and $Q^{i-1}_{j}$ for $1 \leq j \leq i \leq k - 1$. We now define the paths $Q^{i+1}_{j}$ and $Q^{i}_{j+1}$ in the following way: $Q^{i+1}_{j}$ is a shortest path from $s_{i+1}$ to $Q^i_{j}$ in $D$. If there exist more than one shortest path, then we pick the one whose endpoint
We now present the \( t \) whose starting point is closest to \( y_i \). We refer to this endpoint by \( t'_i \). Similarly, \( Q^i_{t_{i+1}} \) is a shortest path from \( Q^k_{t_{i+1}} \) to \( t_{i+1} \) in \( D \). If there is more than one shortest path, then we pick the one whose starting point is closest to \( t_k \) on \( Q^k \). We refer to this starting point by \( s'_i \). Finally \( Q^i_1 \) is a shortest path from \( Q^1_1 \) to \( Q^k_1 \). Again, if there is more than one such shortest path, then we first pick the one whose starting point is closest to \( t_1 \) on \( Q^1_1 \) and then whose endpoint is closest to \( t_k \) on \( Q^k_1 \). We refer to these two vertices by \( s'_i \) and \( t'_k \), respectively. Now let \( C = \{ Q_1^1, Q_2^1, \ldots, Q^{k-1}_k, Q_k^k, Q^k_1 \} \).

We have that \( C \) is by construction a cycle in \( U(D) \). Note that \( s'_i \neq t'_{i-1} \) and \( s'_i \neq t'_{i} \), since otherwise \( s_{i-1} \) were adjacent to \( t_i \) or \( s_{i+1} \) were adjacent to \( t_{i-1} \) in \( U(F(D)) \). Therefore, \( C \) is simple and has length at least \( k \). Now assume for a contradiction that \( C \) has some chord \( a \). Observe that \( a \) connects two distinct paths \( Q^i_1 \) and \( Q^j_1 \) (without loss of generality, \( i \leq k \) and \( j \leq \ell \) only if \( i = k \) and \( j = \ell - 1 \) or \( i = k - 1 \) and \( j = \ell \), respectively, since otherwise \( C_{FD} \) is not chordless. On the other hand \( i = k \) and \( j = \ell - 1 \) contradicts the choice of the starting vertex of \( Q^i_1 \) on \( Q^{i-1}_1 \). Similarly, \( i = k - 1 \) and \( j = \ell \) contradicts the choice of the endvertex of \( Q^i_{t_{i+1}} \) on \( Q^1_1 \). Therefore, \( C \) is an induced cycle in \( U(D) \) of length at least \( k \), which contradicts our assumption that \( U(D) \) has no induced cycles of length \( \geq 6 \).

\[ \square \]

### A.3 Source Cover on graphs with bounded treewidth

We now present the FPT-time algorithm for **Node Weighted Directed Steiner Tree** on acyclic digraphs that is single-exponential in the treewidth of the underlying undirected graph and linear in the instance size. Let us first again recall some definitions. The problem node-weighted Directed Steiner Tree problem is defined as follows.

**Node Weighted Directed Steiner Tree**

**instance:** Acyclic digraph \( D = (V, A) \), costs \( c \in \mathbb{R}_{\geq 0} \), terminals \( T \subseteq V \), root \( r \in V \).

**task:** Find a minimum-cost subset \( F \subseteq V \) such that \( r \) is connected to each terminal in \( (F, E(F)) \).

Our algorithm is presented best using a so-called nice tree decomposition. This kind of decomposition limits the structure of the difference of two adjacent nodes in the decomposition. Formally, consider a tree decomposition \( T \) of a graph \( G \), rooted in a leaf of \( T \). We say that \( T \) is a nice tree decomposition if every node \( x \in V(T) \) is of one of the following types.

- **Leaf:** \( x \) has no children and \( B_x = \emptyset \).
- **Introduce:** \( x \) has exactly one child \( y \) and there is a vertex \( v \notin B_y \) of \( G \) with \( B_x = B_y \cup \{ v \} \).
- **Forget:** \( x \) has exactly one child \( y \) and there is a vertex \( v \notin B_x \) of \( G \) with \( B_y = B_x \cup \{ v \} \).
- **Join:** \( x \) has two children \( y \) and \( z \) such that \( B_x = B_y = B_z \).

Such a nice decomposition is easily computed given any tree decomposition of \( G \). We define \( x^+ \) to be the subtree of \( T \) rooted in \( x \): the tree of all vertices not
connected to the root in the forest $T - x$, together with $x$. By $B^+_x$ we denote the set of vertices contained in all bags of nodes in $x^+$.

A coloring of a bag $B_x$ is a mapping $f : B_x \to \{1, 1\gamma, 0\}^{|B_x|}$, where the individual colors have the following meaning.

- Active and already covered, represented by a 1, means that the vertex is active and that there is at least one predecessor of it that is either labeled 1 or $1\gamma$.
- Active and not yet covered, represented by a $1\gamma$, means that the vertex is active but every predecessor is labeled 0.
- Not active, represented by a 0, means that the vertex is not contained in the solution.

Note that there are $3^{|B_x|}$ colorings of the bag $B_x$. For a coloring $f$ of $x$ we denote by $OPT(f, x)$ the minimum cost $\sum$ of a coloring $B^+_x \to \{1, 1\gamma, 0\}$ satisfying the following conditions.

(a) each vertex in $B_x$ is colored 1, $1\gamma$ or 0 according to $f$.
(b) every vertex of $B^+_x \setminus B_x$ is colored 0 or 1.
(c) each sink $v \in V^- \cap B^+_x$ is colored either 1 or $1\gamma$.
(d) each $v \in B_x$ with $f(v) = 1$ is either a source or at least one predecessor of $v$ in $D(B^+_x)$ is colored either 1 or $1\gamma$.

To present the individual steps of the algorithm, assume that we are given a nice tree decomposition of our input graph. Let us say we are currently considering the node $x$ in $T$ and distinguish between the type of node $x$.

- **Leaf:** put $OPT(f, x) = 0$ if it is not the root.
- **Introduce:** let $y$ be the unique child of $x$ and let $v \notin B_y$ such that $B_x = B_y \cup \{v\}$. The value $OPT(f, x)$ depends on the type of vertex $v$ is and on the coloring $g$ of $y$. By definition, sinks have to be active and therefore the optimal value is $\infty$ if $f(v) = 0$. The same is true for sources labeled $1\gamma$ in $f$ (those do not have predecessors and need to be labeled either 1 or 0). Finally, we set the cost to be $\infty$ if $v$ is labeled 1 in $f$ and not a source, but non of its predecessors is active in $f$. Thus we set

\[
OPT(f, x) = \begin{cases} 
\infty, & \text{if } v \in V^- \text{ and } f(v) = 0 \\
\infty, & \text{if } v \in V^+ \text{ and } f(v) = 1\gamma \\
\infty, & \text{if } v \notin V^+ \text{ and } f(v) = 1 \text{ and } (\delta^-(v) \cap B_y) \subseteq f^{-1}(0) \\
\min\{OPT(g, x) : (g, y) \text{ is compatible to } (f, x)\}, & \text{if } f(v) = 0 \\
\min\{OPT(g, x) : (g, y) \text{ is compatible to } (f, x)\} + c(v), & \text{else,}
\end{cases}
\]

where the pair $(g, y)$ is compatible to $(f, x)$ if the following conditions hold.

- If $f(v) = 0$, then $g = f|_{B_y}$. As the introduced vertex is not considered to be part of the solution, we can simply keep the coloring of the child node.

\[^2\text{Here, a vertex } v \text{ has a cost } c(v) \text{ if it is colored } 1 \text{ or } 1\gamma \text{ and } 0 \text{ otherwise.}\]
If \( f(v) = 1 \), then \( f^{-1}(0) = g^{-1}(0) \), \( f^{-1}(1) = g^{-1}(1) \cup (g^{-1}(1) \cap \delta^+(v)) \), and \( \delta^-(v) \subseteq g^{-1}(0) \). This condition makes sure that the introduced vertex can only be labeled 1 if none of its predecessors is labeled 1 or \( 1^? \).

If \( f(v) = 1 \), then \( f^{-1}(0) = g^{-1}(0) \), \( f^{-1}(1) = g^{-1}(1) \cup (g^{-1}(1) \cap \delta^+(v)) \), and, moreover, \( \delta^-(v) \setminus g^{-1}(0) \neq \emptyset \) or \( v \in V^+ \). This condition says that the introduced vertex can only be labeled 1 if at least one of its predecessors is labeled 1 or \( 1^? \), unless it is a source.

• **Forget**: let \( y \) be the unique child of \( x \) and let \( v \notin B_x \) such that \( B_y = B_x \cup \{v\} \). Then we put

\[
\text{OPT}(f, x) = \min \{ \text{OPT}(g, y) : f = g|B_x \} \text{ if } g(v) \neq 1^?.
\]

We do not allow a vertex labeled \( 1^? \) to be forgotten, as we cannot assure to cover it in later bags. For the remaining cases we simply keep the optimal value.

• **Join**: let \( y \) and \( z \) be the two children of the join node \( x \) with \( B_x = B_y = B_z \). We put

\[
\text{OPT}(f, x) = \min \{ \text{OPT}(g, y) + \text{OPT}(h, z) - \sum_{v \in B_x \setminus (g^{-1}(0) \cap h^{-1}(0))} c(v) \},
\]

where the minimum runs over all colorings \( g \) of \( y \) and \( h \) of \( z \) with \( f^{-1}(0) = g^{-1}(0) = h^{-1}(0) \) and \( f^{-1}(1) = g^{-1}(1) \cup h^{-1}(1) \).

• **Root**: as the graph is connected and the root node is a leaf, the root node is a forget node, where its child node contains exactly one vertex in its bag. The algorithm terminates with the output

\[
\text{OPT} = \text{OPT}(f, x),
\]

where \( f \) is the unique coloring of the empty bag \( x \).

Having presented the algorithm, we need to prove Theorem 10 by showing the correctness and bounding the running time of the algorithm.

*Proof of Theorem 10.* We need to show that the algorithm works correctly and is fixed parameter tractable when parameterized by the treewidth of the underlying graph. Let \( T \) be a nice tree decomposition of \( \mathcal{U}(D) \) of width \( w \) with \( t \) nodes.

**Claim 1.** The algorithm correctly computes an optimal solution to **Node Weighted Directed Steiner Tree in Acyclic Graphs**.

*Proof.* We show the statement by a straightforward inductive proof on the decomposition tree. The induction hypothesis states that \( \text{OPT}(f, x) \) is the minimum cost of a solution induced by the vertices of \( B_x^+ \), satisfying the conditions (a)-(d) (see p. 20). The base case are the leaf nodes where the hypothesis clearly holds. Now let the induction hypothesis be true for all descendants of \( x \). We distinguish between the remaining three node types and argue that the induction hypothesis holds in \( x \).
• **Introduce:** let $y$ be the unique child of the introduce node $x$ and let $v \notin B_y$ such that $B_x = B_y \cup \{v\}$. Clearly (a) holds and (b) holds by the induction hypothesis. By putting $\text{OPT}(f, x)$ to $\infty$ if $f(v) = 0$ for a sink $v \in V^-$, (c) also holds.

For (d) observe that the notion of compatibility is defined correctly. If $f(v) \in \{1,0\}$ this is trivial. For $f(v) = 1$ observe that $v$ has to satisfy the condition that $\delta^-(v) \setminus g^{-1}(0) \neq \emptyset$. Thus the condition (d) holds for $x$. Now for a given coloring $f$ we have to check if $\text{OPT}(f, x)$ is calculated correctly. This is true for the cases in which $\text{OPT}(f, x)$ is set to $\infty$. So it remains to show that we identify all compatible colorings $g$ for $y$ to calculate the minimum. The case $f(v) = 0$ is trivial. For the cases $f(v) \in \{1,1\}$ observe that $g$ has to satisfy $f^{-1}(0) = g^{-1}(0)$ and $f^{-1}(1) = g^{-1}(1) \cup (g^{-1}(1) \cap \delta^+(v))$. Calculating the minimum over all pairs $(g, y)$ compatible to $(f, x)$ is hence correct. Finally it is clear that $\text{OPT}(f, y)$ that we have to add $c(v)$ to the minimum of all compatible colorings $(g, y)$ for $(f, x)$ if $f(v) \neq 0$.

• **Forget:** let $y$ be the unique child of $x$ and let $v \notin B_x$ such that $B_y = B_x \cup \{v\}$. For a forget node we put $\text{OPT}(f, x) = \min \{\text{OPT}(g, y) : f = g|_{B_x}\}$ if $g(v) \neq 1_\gamma$. Clearly (a), (c) and (d) hold by the induction hypothesis. (b) also holds as we only allow colorings that satisfy $f(v) \neq 1_\gamma$. Finally it is easily verified that the calculation of $\text{OPT}(f, x)$ is correct.

• **Join:** let $y$ and $z$ be the two children of the join node $x$ with $B_x = B_y = B_z$. By (2), a vertex $v \in B_x$ may only be colored 1 if it is colored 1 either in $B_y$ or $B_z$. As the induction hypothesis holds for $y$ and $z$, (a)-(d) also hold for $x$. It remains to show that $\text{OPT}(f, x)$ is calculated correctly. The considered colorings $g$ and $h$ of $y$ and $z$ have to satisfy $f^{-1}(0) = g^{-1}(0) = h^{-1}(0)$ and $f^{-1}(1) = g^{-1}(1) \cup h^{-1}(1)$. By adding $\text{OPT}(g, y) + \text{OPT}(h, z)$ we count the vertices in the set $B_X \setminus (g^{-1}(0) \cap h^{-1}(0))$ twice. Thus we obtain $\text{OPT}(f, x) = \text{OPT}(g, y) + \text{OPT}(h, z) - \sum_{v \in B_X \setminus (g^{-1}(0) \cup h^{-1}(0))} c(v)$.

**Claim 2.** Given $T$, the running time of the dynamic programming algorithm is bounded by $O(5^w t)$.

**Proof.** In each node $x$ of the nice tree decomposition $T$ we consider $O(3^{|B_x|})$ many different colorings $f$. We bound the running time for a bag by considering the different kinds of bags. For this, note that the interesting steps are the computation of the pairs $(g, y)$ compatible to $(f, x)$ for the minimum in (1), and the computation of the minimum in (2).

Consider an introduce node $x$ with its unique child $y$ and let $v \notin B_y$ such that $B_x = B_y \cup \{v\}$. Let $f$ and $g$ be colorings for $x$ and $y$, respectively. For a vertex $u \in B_y$ we consider all possible combinations $(f(u), g(u))$ for the three possible values of $f(v)$ which are given by (1).

- In the case $f(v) = 0$ we have that $g = f|_{B_x}$, that is, $(f(u), g(u)) = (g(u), g(u))$. 

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• In the case \( f(v) = 1 \) we have that

\[
(f(u), g(u)) \in \begin{cases} 
\{(0, 0), (1, 1), (1, 1)\}, & \text{if } u \in \delta^+(v), \\
\{(0, 0), (1, 1), (1, 1)\}, & \text{if } u \notin \delta^+(v),
\end{cases}
\]  

(3)

and \((f, x)\) and \((g, y)\) are not compatible unless \( \delta^- (v) \subseteq g^{-1} (0) \).

• In the case \( f(v) = 1 \) we allow the same pairs \((f(u), g(u))\) like in the case \( f(v) = 1 \), but \((f, x)\) and \((g, y)\) are not compatible if \( \delta^- (v) \subseteq g^{-1} (0) \).

We basically have three different options for the pairs \((f, g)\). Processing through \( f \) and \( g \) at the same time leads to the total running time for an introduce node of at most \( O(3^w) \).

For a join node, let \( y \) and \( z \) be the two children of \( x \) with \( B_x = B_y = B_z \). Let \( f, g, h \) be colorings of \( x, y \) and \( z \), respectively. For a vertex \( u \in B_x \) we consider all possible combinations \((f(u), g(u), h(u))\) with

\[
(f(u), g(u), h(u)) \in \{(0, 0, 0), (1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)\}.
\]  

(4)

Here we are given five different options for the triples \((f, g, h)\), and so the total computation time is at most \( O(5^w) \).

The overall bottleneck case is when \( x \) is a join-node since we need to compute \( 2 \). As we just said, this can be done in \( O(5^w) \) time. Since we have \( t \) nodes, the total running time is \( O(5^wt) \).

By storing the best current solution alongside the OPT\((f, x)\)-values we can compute an optimal solution together with OPT. We do not give details here since this is standard. Finally observe that the algorithm is indeed fixed parameter tractable when parameterized by the treewidth of the underlying graph. This completes the proof.

Given a graph on \( n \) vertices of treewidth \( w \), one can compute a tree decomposition of width \( w \) in time \( O(2^{O(w^3)}n) \) by Bodlaender’s famous theorem [6]. Given a tree decomposition of width \( w \) with \( t \) nodes, one can compute a nice tree decomposition of width \( w \) on \( O(wt) \) nodes in \( O(w^2t) \) time in a straightforward way. We thus arrive at an algorithm that, given a tree decomposition of width \( w \), runs in \( O(5^w w^2 |V|) \) time.

### A.4 Proofs Omitted from Section 4

**Proof of Corollary 2** Let \( I = (G, M) \) be an instance of ROBUST MATCHING AUGMENTATION. We use Theorem 11 to obtain from \( I \) in polynomial time the SOURCE COVER instances \( A_1 \) and \( A_2 \) such that OPT\((I) = OPT(I') = \max \{A_1, A_2\} \). Let \( i \in \{1, 2\} \). Let \( S_i \) be the acyclic input graph of \( A_i \). We “flatten” the graph \( S_i \) as described in Section 3 to obtain a SET COVER instance \( B_i \) on the incidence graph \( U(F(S_i)) \). The classical greedy algorithm for SET COVER yields ((\( \ln |M| \) + 1))-approximate cover \( C_i \) for \( B_i \). By Theorem 11 we can construct from \( C_1 \) and \( C_2 \) in polynomial time a solution \( L \) of \( I \). By recalling that \( n = |V(G)| \geq |M|/2 \) and some simple calculations, we conclude that \( L \) is \( \log_2 n \)-approximate.
Proof of Corollary 13. Let $k \in \mathbb{N}$ and $I = (G, M)$ be an instance of ROBUST MATCHING AUGMENTATION such that the graph $G = (U + W, E)$ has treewidth at most $k$. We then use Theorem 11 to construct in polynomial time the SOURCE COVER instances $A_1 = (S_1)$ and $A_2 = (S_2)$ from $I'$. By Theorem 11, $U(S_1)$ and $U(S_2)$ are minors of $G$. Since treewidth is monotone under taking minors, we have that $U(S_1)$ and $U(S_2)$ have treewidth at most $k$. Hence, by Theorem 10, optimal solutions of $A_1$ and $A_2$ can be computed in polynomial time. By Theorem 11, we can obtain in polynomial time from these two solutions a solution $L$ of $I$, such that $|L| = \text{OPT}(I) = \max\{\text{OPT}(A_1), \text{OPT}(A_2)\}$.

Now let $I = (G, M)$ be an instance of ROBUST MATCHING AUGMENTATION such that $G$ is chordal-bipartite. Then $U(D(G, M))$ contains no induced cycle of length at least six. To see this, note that this is a special case in the proof of Claim 3 in the proof of Proposition 14. We use Theorem 11 to construct in polynomial time the SOURCE COVER instances $A_1 = (S_1)$ and $A_2 = (S_2)$ from $I$. In order to obtain the source cover instances, we simply contract all edges of a strong component of $D(G, M)$ to a single vertex. As the contraction of edges only reduces the size of cycles, the underlying undirected graphs occurring in the source cover instances cannot have induced cycles of length at least six. Hence, by Theorem 10, optimal solutions of $A_1$ and $A_2$ can be computed in polynomial time. By Theorem 11, we can obtain in polynomial time from these two solutions a solution $L$ of $I$, such that $|L| = \text{OPT}(I) = \max\{\text{OPT}(A_1), \text{OPT}(A_2)\}$. ■

Proof of Proposition 14. Let $M$ be a maximum matching of $G = (U + W, E)$. Without loss of generality, we assume that $M$ is $U$-perfect, so $|U| \leq |W|$. Otherwise, adding an edge joining two unmatched vertices solves the problem. We construct the graph $G' = f(G)$ as follows. Let $G'$ be a copy of $G$ to which we add a leaf to each unmatched vertex of $W$. We then add a vertex $z$ to $U$ joined to each vertex of the other part of the bipartition. Finally, we add a vertex $z'$ joined to $z$ and each leaf added in the previous step. Furthermore, we extend the matching $M$ of $G$ to a perfect matching $M'$ of $G'$ by adding the edges between the leaves and the previously unmatched vertices to $M'$. Note that by construction, if $e$ is a critical edge of $G'$ then $G - e$ does not admit a matching of cardinality $|M|$. We prove the statements one by one.

Claim 1. $\text{OPT}(I') = \text{OPT}(I)$ and from a solution $L'$ of $I'$ we can construct in polynomial time a solution $L$ of $I$ such that $|L| \leq |L'|$.

Proof. Let $(U, W)$ be the bipartition of $G$ as chosen in the construction, i.e., such that $z \in U$. Note that since $z$ is joined to each vertex $w \in W$, there is an arc from each vertex of $D(G', M')$ to $z$. Therefore, $C(D(G', M'))$ has a single strong sink, say $\hat{S}$, originated from the vertex set $\hat{Y} \subseteq V(D(G', M'))$. Observe that $z, u'_1, u'_2, \ldots, u'_k \in \hat{Y}$. For a strong component $s$ of $D(G', M')$, let $Y_s$ be the set of vertices of $V(D(G', M'))$ in the component $s$.

We first show that $\text{OPT}(I) \leq \text{OPT}(I')$. Let $\hat{L}$ be a solution of $I'$. According to Fact 5 and the algorithm contained in its proof we may construct from $\hat{L}$ a solution $L'$ to $I'$ of cardinality at most $|\hat{L}|$, such that $L'$ connects only sources and sinks of $C(D(G', M'))$. Since there is only the sink $\hat{S}$, we may further assume that $L'$ connects $\hat{S}$ to a selection $S \subseteq V^+(D(G', M'))$. Let $x \in W$ be $M$-exposed. We construct a solution $L$ of $I$ as follows. For each source $s \in S$, we pick a vertex $u \in U$ in the corresponding component in $D(G', M')$. Now let $1 = (G, M)$ be an instance of ROBUST MATCHING AUGMENTATION such that the graph $G = (U + W, E)$ has treewidth at most $k$. We then use Theorem 11 to construct in polynomial time the SOURCE COVER instances $A_1 = (S_1)$ and $A_2 = (S_2)$ from $I'$. By Theorem 11, $U(S_1)$ and $U(S_2)$ are minors of $G$. Since treewidth is monotone under taking minors, we have that $U(S_1)$ and $U(S_2)$ have treewidth at most $k$. Hence, by Theorem 10, optimal solutions of $A_1$ and $A_2$ can be computed in polynomial time. By Theorem 11, we can obtain in polynomial time from these two solutions a solution $L$ of $I$, such that $|L| = \text{OPT}(I) = \max\{\text{OPT}(A_1), \text{OPT}(A_2)\}$.

Now let $I = (G, M)$ be an instance of ROBUST MATCHING AUGMENTATION such that $G$ is chordal-bipartite. Then $U(D(G, M))$ contains no induced cycle of length at least six. To see this, note that this is a special case in the proof of Claim 3 in the proof of Proposition 14. We use Theorem 11 to construct in polynomial time the SOURCE COVER instances $A_1 = (S_1)$ and $A_2 = (S_2)$ from $I$. In order to obtain the source cover instances, we simply contract all edges of a strong component of $D(G, M)$ to a single vertex. As the contraction of edges only reduces the size of cycles, the underlying undirected graphs occurring in the source cover instances cannot have induced cycles of length at least six. Hence, by Theorem 10, optimal solutions of $A_1$ and $A_2$ can be computed in polynomial time. By Theorem 11, we can obtain in polynomial time from these two solutions a solution $L$ of $I$, such that $|L| = \text{OPT}(I) = \max\{\text{OPT}(A_1), \text{OPT}(A_2)\}$. ■

Proof of Proposition 14. Let $M$ be a maximum matching of $G = (U + W, E)$.

Without loss of generality, we assume that $M$ is $U$-perfect, so $|U| \leq |W|$. Otherwise, adding an edge joining two unmatched vertices solves the problem. We construct the graph $G' = f(G)$ as follows. Let $G'$ be a copy of $G$ to which we add a leaf to each unmatched vertex of $W$. We then add a vertex $z$ to $U$ joined to each vertex of the other part of the bipartition. Finally, we add a vertex $z'$ joined to $z$ and each leaf added in the previous step. Furthermore, we extend the matching $M$ of $G$ to a perfect matching $M'$ of $G'$ by adding the edges between the leaves and the previously unmatched vertices to $M'$. Note that by construction, if $e$ is a critical edge of $G'$ then $G - e$ does not admit a matching of cardinality $|M|$. We prove the statements one by one.

Claim 1. $\text{OPT}(I') = \text{OPT}(I)$ and from a solution $L'$ of $I'$ we can construct in polynomial time a solution $L$ of $I$ such that $|L| \leq |L'|$.

Proof. Let $(U, W)$ be the bipartition of $G$ as chosen in the construction, i.e., such that $z \in U$. Note that since $z$ is joined to each vertex $w \in W$, there is an arc from each vertex of $D(G', M')$ to $z$. Therefore, $C(D(G', M'))$ has a single strong sink, say $\hat{S}$, originated from the vertex set $\hat{Y} \subseteq V(D(G', M'))$. Observe that $z, u'_1, u'_2, \ldots, u'_k \in \hat{Y}$. For a strong component $s$ of $D(G', M')$, let $Y_s$ be the set of vertices of $V(D(G', M'))$ in the component $s$.

We first show that $\text{OPT}(I) \leq \text{OPT}(I')$. Let $\hat{L}$ be a solution of $I'$. According to Fact 5 and the algorithm contained in its proof we may construct from $\hat{L}$ a solution $L'$ to $I'$ of cardinality at most $|\hat{L}|$, such that $L'$ connects only sources and sinks of $C(D(G', M'))$. Since there is only the sink $\hat{S}$, we may further assume that $L'$ connects $\hat{S}$ to a selection $S \subseteq V^+(D(G', M'))$. Let $x \in W$ be $M$-exposed. We construct a solution $L$ of $I$ as follows. For each source $s \in S$, we pick a vertex $u \in U$ in the corresponding component in $D(G', M')$.
and add the edge $ux$ to $L$. We now show that $G + L$ is robust. Recap that by construction, the critical edges of $(G', M')$ are precisely the critical edges of $(G, M)$. Let $e \in M$ be a critical edge of $(G, M)$. Since $L'$ is feasible for $I'$, any vertex $u \in U$ that is incident to a critical edge of $(G', M')$ is reachable from some $s \in S$ by a directed path in $C(D(G', M'))$. This directed path corresponds to an $M$-alternating path in $G$ starting from any vertex $u \in Y_s$ with an $M$-edge. Therefore, the edge $e$ is not critical in $(G + ux, M)$ for any $u \in Y_s$. Hence, $(G + L, M)$ has no critical edges and from $|L| = |L'| \leq |L|$ we conclude that $OPT(I) \leq OPT(I')$. Moreover, we can construct $L$ from $L'$ in polynomial time.

It remains to show that $OPT(I') \leq OPT(I)$. Let $L$ be an optimal solution of $I$. Note that each critical edge of $(G, M)$ is on an $M$-alternating cycle or a maximal even-length $M$-alternating path in $G + L$. We construct from $L$ a solution $L'$ to $I'$. Let $x \in W$ be $M$-exposed. For each $u \in U$ and $w \in W$ such that $uw \in L$, we add the edge $ux$ to $L'$. We show that $L'$ is feasible for $I'$. Let $uw \in L$ and let $e \in M$ be a critical edge of $(G, M)$ on a maximal $M$-alternating path $P$ of even length. By replacing $uw$ by $ux$, we split $P$ into at most two maximal $M$-alternating paths of even length. Oh the other hand, suppose that $e$ is on some $M$-alternating cycle involving $uw$. Replacing $uw$ by $ux$ yields a maximal $M$-alternating path containing $e$. Therefore, each critical edge of $(G, M)$ is on some maximal $M$-alternating path of even length. Oh the construction above, each critical edge of $(G', M')$ is hence on some maximal $M'$-alternating cycle of $(G' + L', M')$, so $M'$ is robust in $G' + L'$. Since $|L| = |L'|$, we have that $OPT(I) \leq OPT(I')$.

Claim 2. $tw(G') \leq tw(G) + 2$.

Proof. To prove Claim 2 observe that adding a single vertex to a graph increases its treewidth by at most one. Furthermore, adding a leaf vertex to a graph does not increase its treewidth. We obtain $G'$ from $G$ by adding leaf vertices to each exposed vertex and finally add two more vertices. Therefore, $tw(G') \leq tw(G) + 2$.

Claim 3. If $G$ is chordal-bipartite then $U(D(G', M'))$ has no induced cycle of length at least six.

Proof. Now suppose that $G$ is chordal-bipartite. Assume for a contradiction that $H = U(D(G', M'))$ has an induced cycle $C'$ of length at least six. It is easy to see that $z$ is not contained in $C'$ since $z$ is adjacent to all $v \in H$. In order to obtain a cycle $C$ in $G$, for every edge $e$ in $H[C']$, replace $e$ by the unique corresponding path $P_e$ in $G'$ consisting of a matching edge and a non-matching edge. If two consecutive paths $P_e$ and $P_{e'}$ use the same matching edge, simply delete those matching edges in $C$ such that $C$ is a cycle. Note that all edges in $H[C']$ incident to $U'$ are directed from $U'$ to $U$ in $D(G', M')$. Hence consecutive edges to vertices in $U'$ use the same matching edges, which are then deleted. Therefore $V(C) \subseteq V(G)$. Now if $G[C]$ contains a chord then $H[C']$ also contains a chord due to fact 2. Therefore $C$ is an induced cycle in $G$ (since $z \notin C'$) and $|C| \geq |C'| \geq 6$, a contradiction. ■
A.5 Proofs Omitted from Section 5

Proof of Proposition 5.5: We first prove the following statement: An \( f(n') \)-factor approximation algorithm for Weighted Robust Matching Augmentation implies an \( f(n + k) \)-factor approximation algorithm for Directed Steiner Forest. Let \( I \) be a feasible instance of Directed Steiner Forest with input graph \( D = (V, A) \), \( |V| = n \), costs \( c \in \mathbb{Z}_{\geq 0}^{A} \) and \( k \) terminal pairs \( (s_1, t_1), \ldots, (s_k, t_k) \in V \). Without loss of generality, let \( S = \bigcup_{i \in [k]} \{ s_i \} \) be the set of sources and let \( T = \bigcup_{i \in [k]} \{ t_i \} \) be the set of sinks of \( D \). We may also assume that \( (s_i, t_i) \notin A \) for all \( i \in [k] \). In the reduction it is important that each terminal is a unique vertex, i.e. \( t_i \neq t_j \) for all \( i \neq j, i, j \in [k] \). We ensure this by introducing a copy of each terminal \( t_i \) and then connect it to all neighbors of the original vertex, resulting in a graph of at most \( n + k \) vertices.

To obtain an instance \( I' \) of Weighted Robust Matching Augmentation, we create a bipartite graph \( G = (U + W, E) \), a cost function \( c' \in \mathbb{Z}_{\geq 0}^{E} \), and a perfect matching \( M \) of \( G \) in the following way.

For each \( v \in V \) we add the vertices \( u_v \) and \( w_v \), and the edge \( u_vw_v \) to \( G \) and \( M \). For each \( i \in [k] \) we additionally add the edge \( u_{s_i}w_{t_i} \) to \( E \). For each matching edge \( u_vw_v \in M \) with \( v \notin \{ t_1, \ldots, t_k \} \) we add the vertices \( u'_v \) and \( w'_v \) and the path \( u_vw'_v \) \( u'_v \) to \( G \), and we add the edge \( w'_v \) \( u'_v \) to \( M \). Observe that \( n' = |U| + |W| = O(n + k) \).

For each \( a = vv' \in A \), note that \( e_a = u_{v}w_{v'} \) is an element of \( E \) and put \( c'(e_a) = c(a) \). Let \( E_A := \{ e_a \mid a \in A \} \) be this set of edges. Every remaining edge \( e \in E \) has cost \( c'(e) = 1 + f(n + k) \cdot \sum_{a \in A} c(a) \) such that this edge is not contained in any \( f(n') \)-approximative solution. This completes the construction of \( G \), \( c' \) and \( M \). Observe that this transformation can be performed in polynomial time and that \( M \) is indeed a perfect matching of \( G \). Additionally, there is a one-to-one correspondence between arcs in \( A \) to edges in \( E_A \) as stated in Fact 2. We now show that \( M \) is feasible. Now let \( X' \) be a solution to \( I' \) of cost \( c'(X') \). Let \( X \subseteq A \) be the edges corresponding to the edge set \( X' \cap E_A \). Observe that \( c(X) = c(X') \). We now show that \( X \) is feasible. Let \( X' \) be a solution to \( I' \) of cost \( c'(X') \). Let \( X \subseteq A \) be the edges corresponding to the edge set \( X' \cap E_A \). Observe that \( c(X) = c(X') \). We now show that \( X \) is a feasible solution to \( I \) of cost \( c(X) \). We have by Fact 3 that every vertex is contained in a directed cycle in \( D(G + X, M) \). As a directed cycle through \( u_{t_i} \), we use the edge \( u_{t_i}u_{s_i} \) (since no terminal vertex appears more than once in \( I \)), the directed cycle in \( D(G + X, M) \) has to go through \( u_{s_i} \). This implies that there is a directed path from \( s_i \) to \( t_i \) in \( D[X] \) for each \( i \in [k] \) and therefore the feasibility of \( X \). Finally, as \( n' = |U| + |W| = O(|V| + k) \), we have proved the first part of the proposition.

We now prove the second part: An \( f(n') \)- or \( f(k) \)-factor approximation algorithm for Directed Steiner Forest implies an \( f(n + 1) \)- or \( f(k + 1) \)-factor approximation algorithm for Weighted Robust Matching Augmentation, respectively. Let \( I \) be an instance of Weighted Robust Matching Augmentation with \( G = (U + W, E) \) and \( c \in \mathbb{Z}_{\geq 0}^{E} \). We set \( c^* \in \mathbb{Z}_{\geq 0}^{E} \) to \( c^*(e) = c(e) \) if
e ∈ \overline{E} and c(e) = 0, otherwise. Let \( M = \{u_1w_1, \ldots, u_nw_n\} \) be any cost minimal perfect matching with respect to \( e^* \), where \( n = |U| = |W| \). We construct the Directed Steiner Forest instance \( I' \) with \( D = (V', A) \), the terminal pairs \((s_1, t_1), \ldots, (s_k, t_k)\) and the cost function \( c' \in \mathbb{Z}^A_{\geq 0} \) in the following way. We set \( V' = V \) and add an arc \( a_e = uw \) to \( A \) if \( e = uw \in M \) and add an arc \( a_e = uu \) if \( e = uu \in (E \cup \overline{E}) \setminus M \). In other words, we direct the matching edges from \( U \) to \( W \) and the non-matching edges from \( W \) to \( U \). The terminal pairs are defined according to the matching, i.e., we let \( s_i := w_i \) and \( t_i := u_i \). Finally, for every \( a \in A \), we let \( c'(a) = 0 \) if \( e \in M \) and \( c'(a_i) := c(e) \) otherwise. This completes the construction of \( I' \).

Let \( X' \) be a feasible solution to \( I' \) of cost \( c'(X') \). Observe that by the chosen orientations of the arcs in \( A \), any path from \( w_i \) to \( u_i \) in \( X \) implies that there is an alternating path in the corresponding undirected graph with edge set \( X \). Hence \( X \cup M \) is feasible for \( I \). Finally, as \( M \) is a cost minimal matching with respect to \( e^* \) and \( k = \mathcal{O}(n) \), we have that \( M \cup X \) is an \((f(n) + 1)\)- or \((f(k) + 1)\)-factor approximation for Weighted Robust Matching Augmentation if \( X' \) is an \((f(n))\)- or \((f(k))\)-factor approximation for Directed Steiner Forest. \(\square\)

Proof of Corollary 16. Observe that by the construction in the proof of Proposition 15, we have that the number of vertices in the Weighted Robust Matching Augmentation instance is quadratic in the number of vertices from the Directed Steiner Forest instance. Hence, by [22] and Proposition 15, we have that for every \( \varepsilon > 0 \) Weighted Robust Matching Augmentation does not admit a \( \log^{2-\varepsilon}(n) \)-factor approximation algorithm unless \( \text{NP} \subseteq \text{ZTIME}(n^{\text{polylog}(n)}) \). \(\square\)

Proof of Lemma 17. The result follows in large parts from Lemma 20. The main idea is that any instance of Weighted Robust Matching Augmentation on independent edges can be embedded in a sufficiently large member of \( \mathcal{K}^* \) or \( \mathcal{P}^* \). More formally, consider an instance \( I = (G, M, c) \) of Weighted Robust Matching Augmentation, where \( G \) consists of independent edges. Let \((U, W)\) be any bipartition of \( V(G) \).

We first prove the statement for the class \( \mathcal{K}^* \). We construct an instance \( I' = (G', M', c') \) of Weighted Robust Matching Augmentation from \( I \), where \( G' = \mathcal{K}^*_n \). Let \( G' \) contain the independent edges \( M \) and a path \( P = v_1, v_2, v_3, v_4 \), where \( v_1, v_2, v_3, v_4 \) are new vertices. For each \( u \in U \), connect \( v_2 \) to \( u \) by an edge. Observe that \( M' := M \cup \{v_1v_2, v_3v_4\} \) is a perfect matching of \( G' \). The costs \( c' \in \mathbb{Z}^A_{\geq 0} \) are given by

\[
c'_e := \begin{cases} 
c_e, & \text{if } e \in E(G), \\
0, & \text{if } e = v_1v_4, \\
K, & \text{if otherwise},
\end{cases}
\]

where \( K \) is chosen such that no optimal solution contains an edge of weight \( K \), for example, \( K := |V(G')| \cdot \max_{e \in \overline{E}} c_e \). Since we may add \( v_1v_4 \) to any solution at no cost, we assume that it is present in any solution. Now, from the definition of \( c' \) it follows that an optimal solution to \( I \) is also an optimal for \( I' \) and vice versa.

It remains to prove the statement for the class \( \{\mathcal{P}^*_r \mid r \in \mathbb{N}\} \). In the following, let \( n := |M| \). We construct an instance \( I'' = (G'', M'', c'') \) of Weighted
Robust Matching Augmentation from I, where $G'' = P_{2n}$. Let $G''$ contain the independent edges $M$ and join the vertices $U$ in any order by a path $P = v_1, u_1, v_2, u_2, \ldots, v_n, u_n$, where $u_1, u_2, \ldots, u_n \in U$ and $v_1, v_2, \ldots, v_n$ are new vertices. Finally, for each $1 \leq i \leq n$, add a new vertex $v'_i$ to $G'$ and join it to $v_i$ by an edge. Let $M' := M \cup \{v_iv'_i \mid 1 \leq i \leq n\}$ and let $c'' \in \mathbb{Z}_{\geq 0}^{E(G)}$ be given by

$$c''_e := \begin{cases} c_e & \text{if } e \in E(G) \\ 0 & \text{if } e = v_iv'_i \text{ or } e = v_iv'_{i+1}, 1 \leq i < n \\ K & \text{otherwise}, \end{cases}$$

where $K$ is again chosen such that no optimal solution contains an edge of weight $K$, for example, $K := |V(G'')| \cdot \max_{e \notin T} c_e$. By the choice of $c''$, we may assume that each edge in $M'' \setminus M$ is contained in an alternating cycle. Furthermore, since no optimal solution to $I''$ connects $V(G'') \setminus V(G)$ to $V(G)$, we have that any optimal solution to $I''$ is optimal for $I$ and vice versa. —

Full Proof of Theorem 18. According to Lemma 14, Weighted Robust Matching Augmentation is NP-hard if $G$ completely contains the class $K = \{K^*_{1,r} \mid r \in \mathbb{N}\}$ or the class $P = \{P^* \mid r \in \mathbb{N}\}$. Assuming $P \neq \text{NP}$, this proves the only if statement of the theorem.

To see the if statement, let us consider $r \in \mathbb{N}$ such that $G$ does not contain $K^*_{1,r}$ or $P^*$. First we will reduce the problem to the case when $G$ contains only trees. For this, let $\mathcal{T}$ be the class of all trees in $G$ that admit a perfect matching.

Claim 1. There is a polynomial time reduction of Weighted Robust Matching Augmentation on $G$ to Weighted Robust Matching Augmentation on $\mathcal{T}$.

Proof. To see this, consider an input $(G, M, c)$ of Weighted Robust Matching Augmentation on $G$, consisting of a bipartite graph $G \in \mathcal{G}$, a perfect matching $M$ of $G$, and costs $c$ of edges in the bipartite complement of $G$. We first compute a spanning tree $T$ of $G$ that contains all edges of $M$ using, e.g., Kruskal’s algorithm. We extend the costs $c$ to all edges in the set $E(G) \setminus E(T)$ by setting $c_e = 0$.

Note that $(T, M, c)$ is an instance of Weighted Robust Matching Augmentation on $\mathcal{T}$. Moreover, for every optimal solution $S$ of the instance $(T, M, c)$, $S \setminus E(G)$ is an optimal solution of the instance $(G, M, c)$. —

We may hence restrict our attention to Weighted Robust Matching Augmentation on the class $\mathcal{T}$. As the next claim shows, the relevant trees contained in $\mathcal{T}$ have a bounded number of leaves.

Claim 2. There is some number $f(r)$ depending only on $r$ such that every tree in $\mathcal{T}$ has at most $f(r)$ many leaves.

Proof. Let $T \in \mathcal{T}$ be arbitrary, and let $\ell$ be the number of leaves of $T$. Let us first show that the maximum degree of $T$ is bounded by $r$. Fix any perfect matching $M$ of $T$. Consider a vertex $v$ of $T$, and let $X$ be the set of all neighbors of $v$ together with their matching partners. Note that $T[X]$ is isomorphic to $K^*_{1,d(v)}$, where $d(v)$ denotes the degree of $v$. Since $G$ is closed under taking connected minors, $K^*_{1,d(v)} \in G$, and hence $d(v) < r$.
Next, we show that the number of vertices of degree at least 3 is bounded. Since the maximum degree of $T$ is bounded by $r$, the following holds for the number of leaves in $T$:

$$\ell = 2 + \sum_{j=3}^{r} (j - 2)|V_j|,$$

where $V_j = \{ v \in V(T) : d(v) = 3 \}$.

The above formula is a standard graph theory exercise. As $r$ is constant, this implies $\sum_{j=3}^{r} |V_j| = \Omega(\ell)$. Again since $r$ is constant, there is a path in $T$ containing $\Omega(\log \ell)$ many vertices of degree at least 3 in $T$. Let $T'$ be this path together with all vertices adjacent to it.

Note that $P_r$ is a minor of $T'$ where $t + 2$ is the number of vertices of degree at least 3 on $T$. Since $G$ is closed under connected minors and $P_r \notin G$, we have $t < r$. Consequently, $t \in \Omega(\log \ell)$ implies that $\ell \leq f(r)$ for some number $f(r)$ depending only on $r$. ■

According to the above claims, there is a polynomial reduction of Weighted Robust Matching Augmentation on $G$ to Weighted Robust Matching Augmentation on a class of trees with a bounded number of leaves. Hence, Lemma 19 implies that Weighted Robust Matching Augmentation on $G$ can be solved in polynomial time. ■

Proof of Lemma 19. Let $I = (G, M, c)$ be an instance of Weighted Robust Matching Augmentation, where $G = (V, E) \in T$ is a tree with at most $r$ leaves and a given bipartition $(U, W)$. Moreover, let $M$ be the unique perfect matching of $G$. We say that an arc $xy$ is a shortcut if there is an additional directed path from $x$ to $y$ in $D(G, M)$.

Claim 1. Let $L$ be an optimal solution to $I$. Then we may assume that $D(G + L, M)$ contains no shortcut.

Proof. By Fact 3, each strongly connected component of $D(G + L, M)$ is non-trivial. Suppose for a contradiction that $D(G + L, M)$ contains a shortcut arc $a$ and let $e \in E$ be the edge corresponding to $a$. Then each strongly connected component of $D(G + (L - e), M)$ is non-trivial. Since the costs $c$ are non-negative, we conclude that $L - e$ is solution of weight at most $\text{OPT}(I)$. ■

By Claim 1 we only need to augment edges that do not correspond to shortcuts in $D(G, M)$. So let $\hat{E} \subseteq \overline{E}$ be the subset of edges that are useful for augmentation, that is,

$$\hat{E} := \{ uw \in \overline{E} \mid D(G + uw, M) \text{ has no shortcut} \}.$$

For $F \subseteq E$, we denote by $F_{WU}$ the set of arcs obtained from $F$ by directing all edges from $W$ to $U$. We construct a new directed graph $D'$ on the vertices $V$ by directing all $M$-edges from $U$ to $W$ and making each edge in $E \setminus M$ bidirected.

Claim 2. Let $L' \subseteq \hat{E}$. Then $G + L'$ is robust if and only if $D' + L'_{WU}$ is strongly connected.

Proof. First assume that $G + L'$ is robust and let $e = uw \in M$. Then $e$ is contained in some $M$-alternating cycle $C$ in $G + L'$. It is readily verified that there is a corresponding directed cycle in $D' + L'_{WU}$ containing the arc $uw$. 29
Therefore, there is a path from \( w \) to \( u \) in \( D' \). Since the edges in \( E \setminus M \) are undirected in \( D' \), it follows that \( D' + L_{WU} \) is strongly connected. Now suppose that \( D' + L_{WU} \) is strongly connected. Thus, each \( M \)-edge is contained in some cycle. Since \( L' \subseteq \hat{E} \), each \( M \)-edge is contained in an \( M \)-alternating cycle of \( G + L' \), so \( G + L' \) is robust.

Using the two claims above we finish the proof of the lemma. By Claim 2, our task is to find a minimum-weight set \( L' \subseteq \hat{E} \), such that \( D' + L' \) is strongly connected. For this purpose, we construct in polynomial time an instance \( I' \) of \textsc{Directed Steiner Forest} with at most \( r \) terminal pairs, such that from an optimal solution of \( I' \) we obtain an optimal solution of \( I \) in a straight-forward manner. Since the number of terminals \( r \) is constant, we can solve the \textsc{Directed Steiner Forest} instance \( I' \) in polynomial using the algorithm from [16] and obtain a solution of \( I \) in polynomial time.

The digraph of the instance \( I' \) is \( D' + \hat{E}_{WU} \) and the arc-costs \( c' \) of \( I' \) are given as follows. For each arc \( uw \) of \( D' + \hat{E}_{WU} \), let \( c'_{uw} \) be

\[
c'_{uw} := \begin{cases} c_{uw}, & \text{if } uw \in \hat{E}_{WU}, \\ 0, & \text{otherwise.} \end{cases}
\]

The terminal pairs of \( I' \) are given as follows. We run the algorithm Eswaran-Tarjan on \( D(G, M) \) and obtain an arc-set \( L \) such that \( D(G, M) + L \) is strongly connected. By Fact 8 we have \( |L| = \max\{|V^+(D)|, |V^-(D)|\} \leq r \). Each arc \( e \in L \) corresponds to a pair of terminals we wish to connect. This completes the construction of \( I' \).

We now show that optimal solutions of \( I' \) correspond to optimal solutions of \( I \). Let \( L' \) be an optimal solution to \( I' \). That is, \( D' + L' \) is strongly connected. We may assume that \( L' \) contains all arcs of \( D' \), since each of them has weight zero. Since \( L' \subseteq \hat{E}_{WU} \), we invoke Claim 2 and have that \( G + \mathcal{U}(L') \) is strongly connected, where \( \mathcal{U}(L') \) are the undirected edges corresponding to \( L' \). Let \( L \subseteq \hat{E} \) such that \( G + L \) is robust and assume that \( c(L) < c(\mathcal{U}(L')) \). Then \( L_{WU} \) is a solution of \( I' \) and \( c'(L_{WU}) > c'(L') \). This contradicts the optimality of \( L' \), so \( \mathcal{U}(L) \) is optimal for \( I \).

**Lemma 20.** \textbf{Weighted Robust Perfect Matching Augmentation on independent edges is \textsc{NP}-hard.}

**Proof.** We reduce from \textsc{Robust Matching Augmentation}, which was proved to be \textsc{NP}-hard in Proposition 21. Let \( I = (G, M) \) be an instance of \textsc{Robust Matching Augmentation}, where \( G = (V, E) \). We construct an instance \( I' = (G', M, c) \) of \textsc{Weighted Robust Matching Augmentation} as follows: Let \( G' := (V, M) \) consist only of the edges from the perfect matching \( M \). Furthermore, let the costs \( c \in \mathbb{Z}_{\geq 0}^{E(G)} \) be given by

\[
c_e := \begin{cases} 0, & \text{if } e \in E(G) \setminus M, \\ 1, & \text{otherwise.} \end{cases}
\]

Clearly, the construction can be performed in polynomial time. The solutions of \( I \) and \( I' \) are in 1-to-1 correspondence and the costs are preserved by the transformation.
B  Hardness Results

We show that the problem Robust Matching Augmentation is NP-hard, even on (bipartite) graphs of maximum degree three. Furthermore, it is NP-hard to find a $o(\log n)$-approximate solution in polynomial time. The result mainly follows from the results of [5] and an additional lemma. Nevertheless, we give the full proof here.

Proposition 21. Robust Matching Augmentation parameterized by the solution size is W[2]-hard, even on graphs of maximum degree three.

Proof. We give a parameterized reduction from Set Cover, which is W[2]-hard. Let $(X, S)$ be an instance of Set Cover. We construct an instance $(G, M)$ of Robust Matching Augmentation as follows. Let $d$ be the maximal cardinality of the sets in $S$. For each set $S \in S$, we add a cycle $C_S$ of length $2d$ on the vertices $c_S^1, c_S^2, \ldots, c_S^{2d}$ and for each item $u \in X$, we add an edge $u_1u_2$ to $G$. For each $u \in X$ and $S \in S$, if $u \in S$, we join $u_1$ to $c_S^i$ by an edge, such that $i$ is odd and the vertex $c_S^i$ has maximum degree three. This is possible since $C_S$ has length $2d$. Finally, we add two vertices $t_1$ and $t_2$ to $G$, join them by an edge, and connect for each $u \in X$, $u_2$ to $t_1$. The matching $M$ contains for each $S \in S$ the edges $c_S^1c_S^2, c_S^3c_S^4, \ldots, c_S^{2d-1}c_S^{2d}$ and for each $u \in X$ the edge $u_1u_2$, and also $t_1t_2$. It is readily verified that $M$ is a perfect matching of $G$. Let us choose the bipartition $(U, W)$ of $G$ such that $u_1 \in U$ for some $u \in X$.

Claim 1. $C(D(G, M))$ contains a single sink $t_1$ and for each $S \in S$ its node-set $V(C_S)$ defines a strong source.

Proof. Clearly, the vertices of each cycle $C_S$ are in a strong component of $D(G, M)$. Observe that by the construction of $G$, any maximal $M$-alternating path that leaves a cycle $C_S$ terminates in $t_2$. It follows that $t_1$ is the only sink of $C(D(G, M))$. Moreover, no two distinct cycles $C_S$ and $C_S'$ are in the same strong component of $C(D(G, M))$.

Let $L \subseteq \overline{C}$ be an optimal solution to $(G, M)$. By Fact 5 we can assume that $L$ connects sources to the unique sink of $C(D(G, M))$. Let

$$C_L := \{ S \in S \mid L \text{ connects } C_S \text{ to } t_1 \}.$$

Next we prove that $L$ is a solution of size $\ell$ if and only if $C_L$ is a solution of size $\ell$. For the only if part, assume this is not true and let $u \in X$ be not covered by $C_L$. Thus none of the sets containing $u$ is contained in $C_L$, meaning that $L$ does not connect $t_1$ to a strong source that is a predecessor of $u_1$ in $D(G, M)$ (as $L$ only connects $t_1$ to strong sources). Hence $u_1u_2$ is not contained in an alternating cycle, a contradiction. For the if part, let $C_L$ be a cover of size $\ell$ and let $L$ be the corresponding arcs in $D$. Assume $u_1$ is not contained in a strong component in $D(G + L, M)$. As $L$ only connects strong sources to sinks, no predecessor of $u_1$ has an edge to $t_1$. This is a contradiction to $C_L$ being a cover.

We now describe how to reduce the degree of the constructed graph. Note that the only vertices with degree possibly greater than 3 are $t_1$ and $u_1$, $u \in X$. Both of them are in $U$. Consider a vertex $u \in U$ of degree at least $q > 3$ with its neighbors $w_1, \ldots, w_q$. We do not connect the vertices $w_i, 1 \leq i \leq q$ directly to $u$. Instead we add a path $P = \{ u'i'u_2w'2, \ldots, u'q = u \}$, where for $1 \leq i < q$ the edges $u'i'u_2$ are matching edges. Instead of $w_iu$ we add the edges $w_iu_i'$ for
\(1 \leq i \leq q\). Observe that we still have the same properties as before but each vertex in \(G\) has degree at most 3.

**Proposition 22.** Robust Matching Augmentation admits no polynomial time \(o(\log n)\)-factor approximation algorithm unless \(P = NP\), where \(n\) is the number of critical edges of the input graph.

Proof. Assume for a contradiction that there is a polynomial-time algorithm \(A\) that computes a \(f(n)\)-approximate solution of Robust Matching Augmentation, where \(f(n) = o(\log n)\). Let \(I' = (X, \mathcal{S})\) be an instance of Set Cover and construct from \(I'\) in polynomial time an instance \(I\) of Robust Matching Augmentation as in the proof of Proposition 21. We now also have that \(\text{OPT}(I) = \text{OPT}(I')\) and \(n = |X|\). Applying algorithm \(A\) on \(I\) yields a solution \(L\) of cardinality at most \(f(n) \cdot \text{OPT}(I)\). Without loss of generality, we may assume that \(L\) only connects sources and sinks due to Fact 5. We now set

\[C_L := \{ S \in \mathcal{S} \mid L \text{ connects } C_S \text{ to } t_1 \}.\]

By the same arguments as in the proof of Proposition 21 we observe that \(C_L\) is a feasible solution to \(I'\) of cardinality at most \(f(n) \cdot \text{OPT}(I')\). This contradicts an inapproximability result of Dinur and Steurer for Set Cover [13].

\[\Box\]