Eddy memory as an explanation of intraseasonal periodic behaviour in baroclinic eddies

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Funding information
Swedish Research Council United States Office of Naval Research Netherlands Earth System Science Centre,
Grant/Award Numbers: 638-2013-9243, N00014-19-1-2421, 024.002.001

Abstract
The baroclinic annular mode (BAM) is a leading-order mode of the eddy kinetic energy in the Southern Hemisphere exhibiting oscillatory behaviour at intraseasonal time-scales. The oscillation mechanism has been linked to transient eddy–mean flow interactions which remain poorly understood. Here we demonstrate that the finite memory effect in eddy-heat flux dependence on the large-scale flow can explain the origin of the BAM’s oscillatory behaviour. We represent the eddy memory effect by a delayed integral kernel that leads to a generalized Langevin equation for the planetary-scale heat equation. Using a mathematical framework for the interactions between planetary- and synoptic-scale motions, we derive a reduced dynamical model of the BAM – a stochastically forced oscillator with a period proportional to the geometric mean between the eddy memory time-scale and the diffusive eddy equilibration time-scale. Our model provides a formal justification for the previously proposed phenomenological model of the BAM and could be used to explicitly diagnose the memory kernel and improve our understanding of transient eddy–mean flow interactions in the atmosphere.

KEYWORDS
Eddy heat fluxes, eddy memory, low-frequency variability, Southern Hemisphere baroclinic annular mode (BAM), quasi-periodic oscillation, zonal index

1 INTRODUCTION

Large-scale atmospheric dynamics at midlatitudes are significantly influenced by the baroclinic wave life cycle initiated by shear instability (Pierrehumbert and Swanson, 1995). Constrained by geostrophic and hydrostatic balance, a meridional temperature gradient imposed by the heat imbalance between Tropics and high latitudes is proportional to the vertical shear of the midlatitude jet, which acts as a key parameter controlling the growth rate of baroclinic waves (Charney, 1947; Eady, 1949; Phillips, 1951). During the baroclinic growth of synoptic waves, poleward heat flux increases and the waves propagate upward, after which the waves break near critical latitudes while propagating meridionally (Thorncroft et al., 1993). The whole life cycle of unstable baroclinic waves could be represented by energy exchange between zonal mean zonal wind and synoptic eddies (Lorenz, 1955; Simmons and Hoskins, 1978; Edmon et al., 1980; Moon and Feldstein, 2009). This cycle is reflected in daily weather at midlatitudes and understood as an engine to transfer residual energy from the Tropics to high latitudes. It determines...
many aspects of large-scale circulations in extratropical areas (Held and Hoskins, 1985).

The life cycle of baroclinic waves is an essential part of variability in midlatitudes with its time-scale being around 3–4 days. However, the low-frequency intraseasonal variability with a time-scale longer than weather but shorter than seasonal (Hartmann, 1974; Branstator, 1992) is not entirely explained by the traditional baroclinic wave life cycle (Pratt, 1977; Barnston and Livezey, 1987). For example, the North Atlantic Oscillation (NAO) (Hurrell et al., 2003), which represents the fluctuation of the difference of atmospheric pressure between the Icelandic Low and the Azores High, contains the intraseasonal time-scale of around 10 days. Similarly, the Arctic Oscillation (AO; Thompson and Wallace 1998 or the Southern Annular Mode (SAM; Limpasuvan and Hartmann 1999, representing the zonally symmetric seesaw between sea level pressures in polar and temperate latitudes in Northern and Southern Hemispheres respectively, also vary on intraseasonal time-scales.

The variability of the midlatitude westerlies is commonly defined using the zonal index (Namias, 1950) which was originally defined by the pressure difference between 35 and 55°N, but there are many variants (Robinson, 1996; Li and Wang, 2003). The zonal index, especially in the Southern Hemisphere, shows strong intraseasonal and interannual variability based on several observational data (Hartmann and Lo, 1998). On intraseasonal time-scales, it is hard to identify a major external forcing and hence the midlatitude jet variability could be controlled by internal dynamics (Branstator, 1992). Indeed, the low-frequency variability of the zonal index has been identified in simple numerical models with zonally symmetric thermal forcing (Robinson, 1991), implying that variability can be caused by the interaction between synoptic eddies and the zonal mean field.

The eddy–mean flow interaction is a highly nonlinear process involving turbulent mixing of synoptic waves leading to energy transfer among different scales which are difficult to accurately parametrize. However, phenomenological models of eddy–mean flow interactions exhibiting oscillatory behaviour have been put forward. Thompson and Barnes (2014) introduced a stochastic model to explain the quasi-oscillatory behaviour of the poleward heat flux in the Southern Hemisphere, also referred to as the Southern Hemisphere baroclinic annular mode (BAM) introduced by Thompson and Woodworth (2014). Specifically, they suggest a two-dimensional stochastic model representing interactions between the poleward heat flux and the meridional temperature gradient to capture the quasi-oscillatory variability with a dominant time period around 25 days. In this model, the increase of the poleward heat flux is proportional to the growth rate of eddies generated by baroclinic instability, which is proportional to the meridional temperature gradient. At the same time, the time evolution of the anomalous meridional temperature gradient is controlled by the poleward heat flux with a damping. This phenomenology results in a two-dimensional stochastic dynamical system which contains oscillatory solutions depending on the choice of parameters. With this idealized description of the variability, Thompson and Barnes (2014) suggest that the quasi-oscillatory low-frequency variability can be caused by nonlinear eddy–mean flow interaction.

The BAM is a companion index of the Southern Annular mode (SAM) in the Southern Hemisphere representing the characteristics of large-scale low frequencies in the atmosphere. The SAM is defined by zonal mean kinetic energy describing the variability of zonal mean wind mainly influenced by the momentum flux. The BAM is constructed by eddy kinetic energy mainly controlled by the meridional heat flux at lower levels (Thompson and Woodworth, 2014). The two processes, meridional heat flux at lower levels and momentum flux at upper ones, show different characteristics in their variabilities (Pfeffer, 1992; Thompson and Woodworth, 2014; Wang and Nakamura, 2015; 2016; Boljka et al., 2018). The SAM is well approximated by an autoregressive model of order 1 (AR(1) process), having a red-noise spectrum, but the BAM shows a distinct peak around 25 days in its power spectrum. The stochastic oscillatory behaviour shown in the BAM is worth investigating in detail due to the expectation that it leads to significant progress in predictability in midlatitudes on sub-seasonal time-scales. Furthermore, Thompson and Li (2015) have shown that the BAM also exists in the Northern Hemisphere. This implies that the oscillatory behaviour in meridional heat flux or eddy kinetic energy is an intrinsic feature of eddy–mean flow interactions in large-scale atmospheric dynamics in both hemispheres. Therefore, the major question is how the meridional heat flux at lower levels can induce the oscillatory behaviour in the interaction with the mean field.

Generation of low-frequency oscillations in a rotating fluid does not seem to be limited to the large-scale atmosphere as similar internal oscillations were also found in large-scale ocean currents. An eddy-resolving ocean model simulation with a repeated annual cycle forcing reveals an intrinsic mode in the Southern Ocean with a period of 40–50 years (Le Bars et al., 2016; van Westen and Dijkstra, 2017). An idealized model of the surface-stress-driven Beaufort Gyre in the Arctic Ocean with mesoscale eddies as a key equilibration process (Manucharyan and Spall, 2016) generates an oscillation with a period of 50 years (Manucharyan et al., 2017). To explain the oscillation, Manucharyan et al. (2017) introduce eddy memory into
the relation between the eddy buoyancy flux and the mean
buoyancy gradient, which leads to a modification of the
commonly used Gent–McWilliams (GM) parametrization
(Gent and McWilliams, 1990). The inclusion of eddy mem-
ory leads to a stochastic oscillation if the ratio between
the memory time-scale and the diffusible equilibration
time-scale reaches a certain threshold (Manucharyan
et al., 2017).

Generally, according to the Mori–Zwanzig formal-
ism (Zwanzig, 1961), the dynamic interactions between
slow-evolving modes and fast ones are reflected as a
delayed integral on the time evolution of the slowly vary-
ing modes. The memory effect, normally represented by
a delayed term or integral, is not new in climate science.
One of the simplest models for the El Niño Southern
Oscillation (ENSO) is a delayed ordinary differential
equation (Suarez and Schopf, 1988). Even with such a
one-dimensional model, it was found that the delayed
model contains chaotic dynamics under seasonal forcing
(Tziperman et al., 1994). Recently, such delay equation
climate models were derived using the Mori–Zwanzig for-
amalism (Falkena et al., 2019). Considering the complex
interactions in fluid dynamical systems, the memory effect
is an intrinsic characteristic leading to various internal
variables.

Our research focuses on the memory effect in the
interaction between synoptic eddies and a zonal mean
zonal wind and its relation to low-frequency modes on
time-scales longer than the weather. More specifically, we
propose a formalized explanation of the quasi-oscillatory
behaviour of the BAM (Thompson and Barnes, 2014)
by incorporating the eddy memory (Manucharyan et al.,
2017) into multi-scale equations of atmospheric dynam-
ics (Moon and Cho, 2020). We outline the multi-scale
atmospheric model in Section 2 and we introduce the
eddy memory effect into the planetary-scale equations in
Section 3, where we formally derive the reduced model for
the zonal mean flow, the stochastic oscillator. We conclude
in Section 4.

2 | MULTISCALE MODEL OF
ATMOSPHERIC MOTIONS

The basic governing equation for Earth’s climate starts
from the simplest heat flux balance between short-wave
radiative flux and outgoing long-wave radiative flux, which
gives us a global average temperature as an equilibrium.
This is possible when we consider the Earth as a point
object in the universe maintaining a thermal equilibrium
state. If we magnify the size of the Earth from a point to
a sphere, we can see that there is an imbalance of heat
flux from the Tropics to higher latitudes, which requires
transferral of heat from the Tropics to higher latitudes. A
simple approximation of the poleward heat flux is a tur-
bulent diffusion with a constant eddy diffusivity. In an
even more magnified view of the Tropics and polar areas,
the dominant physics shift from energy flux balance to
potential vorticity conservation, which acts as a theoreti-
cal foundation for the generation of midlatitude weather
systems. There are unclearly defined time-scales between
the weather dominated by the vorticity dynamics and the
season controlled by external heat flux. These time-scales
lie at around 10 to 30 days, where numerous variabilities
have been detected in climate data. There should be an
intermediate framework lying between the energy flux bal-
ance and the vorticity dynamics to investigate causes of the
variability.

The large-scale atmosphere is governed by the primi-
tive equations which are comprised of three momentum
balances, the continuity equation and the heat budget.
In Cartesian coordinates, the three momentum equations are

\[
\begin{align*}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho} \frac{\partial P}{\partial x}, \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho} \frac{\partial P}{\partial y}, \\
\frac{Dw}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial z} - g,
\end{align*}
\]

where \( P \) and \( \rho \) are atmospheric pressure and density,
respectively, and the velocities in the zonal, meridional,
and vertical directions are \( u, v, \) and \( w \). The Coriolis param-
eter \( f \) is approximated by \( f_0 + \beta y \), where \( f_0 \) is a Coriolis
parameter calculated at a reference latitude \( \phi_0 \) and \( \beta y \) is
the latitudinal variation of the Coriolis parameter where
\( \beta = \partial f / \partial y \). The material derivative \( D/Dt \) is defined as

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}.
\]

Along with the momentum equations, the continuity
equation is

\[
\frac{1}{\rho} \frac{D\rho}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\] (2)

Finally, to complete the equations, we need the ideal
gas law \( P = \rho RT \) and thermodynamic equation

\[
\frac{D\theta}{Dt} = \frac{\Theta}{C_pT} \left( \frac{k}{\rho} \frac{\partial^2 T}{\partial y^2} + Q \right),
\] (3)

where \( \Theta \) is the potential temperature, that is,

\[
\Theta \equiv T \left( \frac{P_0}{P} \right)^{\kappa/C_p},
\] (4)
where \( P \) and \( T \) represent the pressure and the temperature, respectively, \( P_0\) is a reference pressure, typically 1000 mb, \( R \) is the gas constant for air, and \( C_P \) is the specific heat at constant pressure. The thermal diffusion \( (\kappa/P)\nabla^2T \) and the thermal forcing \( Q \) lead to the time-evolution of the potential temperature \( \Theta \). The full equations are highly nonlinear and complicated and, to explain the BAM, we need to simplify the equations using an appropriate scaling.

The scale of atmospheric motions is distinguished by the magnitude of the Rossby number, \( U/fL \), where \( U \) is a horizontal velocity-scale, and \( L \) is a horizontal length-scale. When the Rossby number is asymptotically small, the geostrophic balance becomes a dominant force balance in the horizontal momentum equations. The large-scale atmospheric flows satisfying the geostrophic and hydrostatic balances are called geostrophic motions.

There are two kinds of geostrophic motions depending on the horizontal length-scale (Phillips, 1963). If the length-scale is similar to the internal Rossby deformation radius \( (L \sim 1,000 \text{ km}) \), the motion is called quasi-geostrophic and is described by the conservation of potential vorticity,

\[
\frac{D_H}{Dt} \left[ \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \frac{\rho_s \partial \Psi}{S \partial z} \right) + \beta y \right] = 0, \tag{5}
\]

where

\[
\frac{D_H}{Dt} = \frac{\partial}{\partial t} - \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y}. \tag{6}
\]

\( \Psi \) is a leading-order pressure field acting as a stream function, \( \rho_s \) is a mean vertical density profile, \( S \) represents the average vertical stability, and

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \frac{\rho_s \partial \Psi}{S \partial z} \right) + \beta y
\]

is the potential vorticity. All the variables are non-dimensionalized (Pedlosky, 1987). On the other hand, if the horizontal length-scale is close to the external Rossby deformation radius \( (L_D \sim 3000 \text{ km}) \), the governing equations become

\[
u_L = -\frac{\partial P_L}{\partial y}, \tag{7}
\]

\[
u_L = \frac{\partial P_L}{\partial x} \tag{8}
\]

\[
\rho_L = \frac{1}{\rho_s} \frac{\partial}{\partial z} (\rho_s P_L), \tag{9}
\]

\[
\frac{1}{\rho_s} \frac{\partial}{\partial z} (\rho_s w_L) - \beta_L v_L = 0. \tag{10}
\]

where \( P_L \) is the planetary-scale pressure, \( u_L, v_L \) and \( w_L \) are the zonal, the meridional and the vertical velocity, respectively, \( \Theta_s \) is the hemispheric average potential temperature, \( \Theta_L \) is an anomalous potential temperature, \( \beta_L \) is the planetary-scale beta effect, and \( Q_L \) represents radiative processes. The thermodynamic forcing \( Q_L \) is a residual of local radiative fluxes driving a local temperature change. The subscript \( L \) implies planetary-scale variables. Equations (7)–(11) are usually referred to as planetary geostrophy, where the heat equation with advection is constrained by the geostrophic and hydrostatic balances, together with the Sverdrup relation (Moon and Cho, 2020). Similar results can be found in Dolaptchiev and Klein (2009, 2013), where the same leading-order equations are derived. They used a different small parameter \( \varepsilon = (a\Omega^2/g)^{1/3} \) instead of the Rossby number \( U/fL \), where \( a \) is the Earth’s radius and \( \Omega \) is the earth’s rotation frequency, thus the detailed derivations toward the leading-order equations are different.

The two geostrophic motions coexist in the large-scale atmosphere. Hence, the large-scale atmospheric dynamics should be represented by interactions between the two scales. Because these two scales are asymptotically separate, we can use a multi-scale analysis in spatial and temporal domains. Based on the Rossby number in the planetary scale \( \varepsilon = U/(f_0 L_D) \), we can introduce the two scales, \((X,Y,\tilde{t})\) for the planetary scale and \((x,y,t)\) for the quasi-geostrophic (QG) one, where \((x,y,t) = \varepsilon^{1/2}(X,Y,\tilde{t})\). Thus, the time and spatial derivatives are scaled as

\[
\frac{\partial}{\partial \tilde{t}} \rightarrow \frac{\partial}{\partial t} + \varepsilon^{-1/2} \frac{\partial}{\partial \tilde{t}},
\]

\[
\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial X} + \varepsilon^{-1/2} \frac{\partial}{\partial x},
\]

\[
\frac{\partial}{\partial y} \rightarrow \frac{\partial}{\partial Y} + \varepsilon^{-1/2} \frac{\partial}{\partial y}.
\]

The two scales are separated by \( \varepsilon^{1/2} \) which comes from the estimation that \( L/L_D \sim \varepsilon^{1/2} \) in the large-scale atmosphere on the Earth.

A regular perturbation analysis of the primitive Equations (1)–(4) (Moon and Cho, 2020) leads to

\[
u_L = -\frac{\partial P_L}{\partial Y}, \quad v_L = \frac{\partial P_L}{\partial X}, \quad \rho_L = \frac{1}{\rho_s} \frac{\partial}{\partial z} (\rho_s P_L), \quad \tag{12}
\]

\[
u_0 = -\frac{\partial P_0}{\partial Y}, \quad v_0 = \frac{\partial P_0}{\partial X}, \quad \rho_0 = \frac{1}{\rho_s} \frac{\partial}{\partial z} (\rho_s P_0), \quad \tag{13}
\]

\[
\frac{\partial \Theta_L}{\partial \tilde{t}} + u_L \frac{\partial \Theta_L}{\partial x} + v_L \frac{\partial \Theta_L}{\partial y} + w_L \frac{\partial \Theta_L}{\partial z} + \frac{w_L}{\epsilon \Theta_s} \frac{\partial \Theta}{\partial z} = Q_L, \tag{14}
\]

\[
\frac{\partial w_L}{\partial \tilde{z}} - \frac{1}{H} w_L = \beta_L v_L,
\]
\[
\frac{\partial \Theta_L}{\partial t} + u_L \frac{\partial \Theta_L}{\partial x} + v_L \frac{\partial \Theta_L}{\partial y} + w_L \left( \frac{\partial \Theta_L}{\partial z} + S \right)
\]
\[
= - \left( \frac{\partial}{\partial t} + u_L \frac{\partial}{\partial x} + v_L \frac{\partial}{\partial y} \right) \Theta_0 - \left( u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y} \right) \Theta_L
\]
\[
- \frac{\partial}{\partial x} \left( u_0 \Theta_0 \right) - \frac{\partial}{\partial y} \left( v_0 \Theta_0 \right) - w_1 \left( \frac{\partial \Theta_L}{\partial z} + S \right) + Q_L,
\]
(15)
\[
\frac{\partial}{\partial t} \nabla^2 P_0 + (u_L + u_0) \frac{\partial}{\partial x} \nabla^2 P_0 + (v_L + v_0) \frac{\partial}{\partial y} \nabla^2 P_0 + \beta v_0
\]
\[
= \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) w_1,
\]
(16)

where
\[
\frac{1}{H} = - \frac{1}{\rho_s} \frac{d \rho_s(z)}{dz} \quad \text{and} \quad S = \frac{1}{F \Theta_k} \frac{d \Theta_k(z)}{dz}.
\]

\( L \) and the numbers 0 and 1 are used as the subscripts to represent planetary-scale and synoptic-scale variables, respectively. The number 0 (1) represents the leading-order (the next order) in the synoptic scale. It is assumed that planetary-scale (synoptic-scale) variables are only dependent upon planetary-scale (synoptic-scale) coordinates \( X, Y, \) and \( t \). The vertical coordinate \( z \) is used for both scales. Equations (12)–(16) show the dynamics of the two scales and their mutual interactions. The detailed derivation and discussions are found in Moon and Cho (2020). Equations (12)–(14) describe the two scales satisfying the basic (geostrophic and hydrostatic) balances. The continuity Equation (14) on the planetary scale includes the \( O(1) \) beta effect, which is known as the Sverdrup relation. The two scales contribute to the energy flux balance in the heat Equation (15) with their own temporal and spatial scales. The last Equation (16) is the vorticity equation governing the quasi-geostrophic dynamics. Here the planetary geostrophic motion provides a mean field for the development of vorticity on the Rossby deformation scale. Dolaptchiev and Klein (2013) also considered a similar multi-scale analysis to represent the interactions between planetary and synoptic scales, which leads to equivalent results. However, their main focus lies on vorticity dynamics at the two scales, and hence a planetary vorticity equation was derived from the combination of the heat equation and the Sverdrup relation along with the quasi-geostrophic potential vorticity equation. Our focus is in the planetary-scale heat equation with thermal forcing \( Q_L \). Lying between the simple energy flux balance model (North et al., 1981) and the quasi-geostrophic dynamics, the planetary heat equation together with basic dynamic balances connects the planetary thermal heat flux to fluid dynamics on planetary scales.

The above equations become simpler when the planetary-scale thermodynamic forcing \( Q_L \) is zonally homogeneous. The planetary scale preserves that symmetry, in which case all \( X \)-derivative terms vanish. Hence, the zonal means of the planetary meridional and vertical velocities \( u_L \) and \( w_1 \) become zero. This yields

\[
u_L = - \frac{\partial P_L}{\partial Y},
\]
(17)
\[
\frac{\partial \Theta_L}{\partial t} = - \left( \frac{\partial}{\partial t} + u_L \frac{\partial}{\partial x} \right) \Theta_0 - \frac{\partial}{\partial y} \left( u_0 \Theta_0 \right)
\]
\[
- \frac{\partial}{\partial x} \left( u_0 \Theta_0 \right) - \frac{\partial}{\partial y} \left( v_0 \Theta_0 \right) - w_1 \left( \frac{\partial \Theta_L}{\partial z} + S \right) + Q_L.
\]
(18)
\[
\frac{\partial}{\partial t} \nabla^2 P_0 + (u_L + u_0) \frac{\partial}{\partial x} \nabla^2 P_0 + (v_L + v_0) \frac{\partial}{\partial y} \nabla^2 P_0 + \beta v_0
\]
\[
= \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) w_1.
\]
(19)

We can introduce a temporal- and spatial-scale average to the \( QG \) variables under the assumption that the time and spatial average of a \( QG \) variable is close to zero. This implies that the overall effect of synoptic scales on the planetary-scale motions is represented by the temporal and spatial average of \( QG \)-scale fluxes. In particular, it is important to consider a planetary-scale spatial average over the terms containing the \( QG \)-scale spatial derivative such as \( \partial/\partial x \) and \( \partial/\partial y \). Let us consider a large horizontal length \( l \) for the spatial average on the synoptic scale; then the meridional average of \( (\partial F/\partial y) \big|_{Y_0} \), where \( Y_0 \) represents a position in the planetary-scale coordinate, is then interpreted as

\[
\frac{\partial F}{\partial y} \big|_Y = \frac{1}{2l} \int_{Y_0-l}^{Y_0+l} \frac{\partial F}{\partial y} \, dy
\]
\[
\approx \lim_{\Delta Y \rightarrow 2\Delta Y} \left( F(Y_0 + \Delta Y) - F(Y_0 - \Delta Y) \right)
\]
\[
= e^\frac{1}{l} \frac{\partial F}{\partial Y}.
\]
(20)

Here \( l \) and \( \Delta Y \) are non-dimensional lengths on the synoptic and planetary scales, respectively. The same length is expressed in two length-scales, that is, \( l^* = \Delta Y^* \) where \( * \) is used to represent dimensional quantities, which is the same as \( LI = LD \Delta Y \). Thus, \( l = \Delta Y / e^{1/2} \), where \( L/L_D = e^{1/2} \) is used. Due to the asymptotic difference between the two scales, the length \( l \) on the synoptic scale is approximated by the limit of the planetary-scale length \( \Delta Y \) toward zero.

The heat Equation (18) and the \( QG \) vorticity Equation (19), after taking the fast time (synoptic-scale) average (\( \bar{\cdot} \)), become

\[
\frac{\partial \Theta_L}{\partial t} = Q_L - \frac{\partial}{\partial x} \left( u_0 \Theta_0 \right) - \frac{\partial}{\partial y} \left( v_0 \Theta_0 \right) - w_1 \left( \frac{\partial \Theta_L}{\partial z} + S \right),
\]
(21)
where the time average of linear terms on synoptic scales such as $\partial\Theta_0/\partial t$ and $\partial\Theta_0/\partial x$ is assumed to be zero and the nonlinear terms representing heat and momentum fluxes are considered as non-zero average terms. Using $\nabla^2 P_0 = \partial v_0/\partial x - \partial u_0/\partial y$, we find that the Equation (22) is equivalent to

$$\frac{1}{\rho_s} \frac{\partial}{\partial z} (\rho_s \tilde{w}_1) = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) u_0 v_0 - \frac{\partial^2}{\partial x \partial y} u_0^2 - v_0^2. \quad (23)$$

Now, we can take the planetary-scale spatial average over Equations (21) and (22), and combine the two by replacing $\tilde{w}_1$ by the horizontal vorticity convergence, which leads to

$$\frac{\partial \Theta_L}{\partial t} = Q_L - e^{1/2} \left( \frac{\partial}{\partial x} u_0 \Theta_0 + \frac{\partial}{\partial Y} v_0 \Theta_0 \right)$$

$$+ \epsilon \left( \frac{\partial \Theta_L}{\partial z} + S \right) \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial Y^2} \right) \left( \frac{1}{\rho_s} \int_0^z \rho_s (u_0^2 - v_0^2) \, dz' \right)$$

$$- \epsilon \left( \frac{\partial \Theta_L}{\partial z} + S \right) \frac{\partial^2}{\partial x \partial Y} \left( \frac{1}{\rho_s} \int_0^z \rho_s (u_0^2 - v_0^2) \, dz' \right). \quad (24)$$

Equation (24), after taking the zonal average $\langle \cdot \rangle$, becomes

$$\frac{\partial \Theta_L}{\partial t} = Q_L - e^{1/2} \frac{\partial}{\partial Y} \left( v_0 \Theta_0 \right)$$

$$- \epsilon \left( \frac{\partial \Theta_L}{\partial z} + S \right) \frac{\partial^2}{\partial Y^2} \left( \frac{1}{\rho_s} \int_0^z \rho_s (u_0^2 - v_0^2) \, dz' \right). \quad (25)$$

where the dominant balance, after ignoring the momentum flux contribution in the $O(e)$ order, is

$$\frac{\partial \Theta_L}{\partial t} \approx Q_L - e^{1/2} \frac{\partial}{\partial Y} \left( v_0 \Theta_0 \right). \quad (26)$$

In Equation (26), we can consider an asymptotic solution $\Theta_L \approx [\Theta_L] + e^{1/2} \eta$, in which case the radiative process $Q_L$ is represented approximately as

$$Q_L(\tilde{t}, \Theta_L, Y) \approx Q_L(\tilde{t}, [\Theta_L], Y) + e^{1/2} a(Y, \tilde{t}) \eta,$$

where $a(Y, \tilde{t}) = \partial Q_L(\tilde{t}, \Theta_L, Y) / [\Theta_L]_{\Theta_L = [\Theta_L]}$.

The leading-order balance is

$$\frac{\partial [\Theta_L]}{\partial \tilde{t}} \approx [Q_L]. \quad (27)$$

This is understood as a local radiative energy flux balance, and $[\Theta_L]$ represents the seasonal mean of the potential temperature mainly determined by the seasonally varying radiative flux balance $[Q_L]$. The simplest representation of $[Q_L]$ is $S_0 (1 - a) - \sigma [\Theta_L]^4$, where $S_0$ is the short-wave radiative flux, $a$ is the local albedo, and $\sigma [\Theta_L]^4$ is the outgoing long-wave radiative flux, but we have to consider other heat fluxes such as incoming long-wave radiative flux and turbulent sensible and latent heat flux. In the planetary-scale heat equation, the leading-order governing physics is the local energy flux balance contributing mainly to a seasonal cycle of the potential temperature.

The $O(e^{1/2})$ order balance is

$$\frac{\partial \eta}{\partial \tilde{t}} = a(Y, \tilde{t}) \eta - \frac{\partial}{\partial Y} \left( v_0 \Theta_0 \right). \quad (28)$$

The fluctuation $\eta$ around the seasonal cycle $[\Theta_L]$ is controlled by the seasonal sensitivity of the radiative processes $a(Y, \tilde{t})$ and the meridional heat flux convergence induced by synoptic eddies. Moon and Wettlaufer (2017) focus on the monthly average variability of surface air temperature based on a periodic non-autonomous stochastic differential equation $\eta = a(\tilde{t}) \eta + N(\tilde{t}) \xi(\tilde{t})$, where $a(\tilde{t})$ is equivalent to $a(Y, \tilde{t})$ in Equation (28), $N(\tilde{t}) \xi(\tilde{t})$ is a white noise mimicking the overall effect of weather-related processes and $N(\tilde{t})$ is a monthly varying amplitude of noise. The noise forcing $N(\tilde{t}) \xi(\tilde{t})$ can be understood as an approximation of the residual forcing $R(Y, \tilde{t})$ which can be considered as a contribution of short-time processes in Equation (28). The monthly statistics including variance and lagged correlations in a given monthly averaged data, such as surface sea temperature or tropical climate indexes, is regenerated by the periodic non-autonomous stochastic model with an appropriate choice of the two periodic functions $a(\tilde{t})$ and $N(\tilde{t})$. In particular, the positive sign of the $a(\tilde{t})$ implies existence of positive feedbacks which magnifies the magnitude of a given perturbation. For example, when the stochastic model is applied to the Niño 3.4 index, the $a(\tilde{t})$ is positive from July to November, which shows the seasonality of the Bjerknes feedback. The construction of the monthly sensitivity $a(\tilde{t})$ enables us to detect how a positive feedback shapes the monthly statistics of a climate variable. It is essential to understand how phase locking and seasonal predictability barrier of climate phenomena are associated with the magnitude and timing of positive feedback.

Equation (28) is an extension of the one-dimensional stochastic model considering meridional variation. Especially, it represents a strong influence from synoptic-scale eddies on the planetary-scale temperature. It is also an extension of Boljka and Shepherd (2018) who provide similar budget equations on the planetary scale based on the multi-scale analysis suggested by Dolaptchiev and Klein (2013). In their budget equations, the planetary
scale is independent from the synoptic scale (and similar to Equation (27) here), thus the two scales in their model interact indirectly through source terms or through higher-order equations. Here we show that only a slightly higher-order equation (i.e., Equation (28)) is necessary to obtain an interaction between planetary and synoptic waves. Equations (27) and (28) show a hierarchical structure of physics controlling the local temperature. The largest contribution comes from the local energy flux balance (Equation (27)) and the poleward heat flux induced by synoptic eddies contributes to the temperature in the next higher order (Equation (28)). In particular, Equation (28) shows an interaction between thermodynamics and large-scale atmospheric dynamics on the time evolution of the local temperature. Here, the impact of synoptic eddies on the temperature is controlled by the local stability of energy flux balance \(a(Y, \tilde{t})\). One remaining step to close the planetary-scale equations is to parametrize the meridional heat flux \(\langle v_0 \Theta_0 \rangle\) using the planetary-scale temperature \(\Theta_L\).

3 | EMERGENCE OF A GENERALIZED LANGEVIN EQUATION

3.1 | Fickian diffusion model

The meridional heat flux at midlatitudes under the growth of synoptic waves is the major consequence of baroclinic instability. The baroclinic instability is initiated from an unstable mean state measured by the vertical shear of zonal mean wind \(\partial u_L / \partial z\). The growth of baroclinic waves by a baroclinic instability starts from near the surface, which induces a meridional heat flux. Hence, the meridional gradient of the planetary temperature \(\partial \theta_L / \partial Y\) is strongly related to the meridional heat flux \(\langle v_0 \Theta_0 \rangle\).

The simplest representation of the mutual relationship between \(\partial \theta_L / \partial Y\) and \(\langle v_0 \Theta_0 \rangle\) is a turbulent flux gradient parametrization, \(\langle v_0 \Theta_0 \rangle \approx -K \partial \theta_L / \partial Y\), where \(K\) is a constant (North et al., 1981). If we use this parametrization in Equation (26), this leads to

\[
\frac{\partial \theta_L}{\partial t} \approx c^{1/2} K \frac{\partial^2 \theta_L}{\partial Y^2} + Q_L. \tag{29}
\]

This simple energy flux balance model was first introduced by Sellers (1969) and North (1975) to include the effect of large-scale atmospheric dynamics upon the local energy flux balance. Equation (29) is the combination of Equations (27) and (28) without distinguishing terms by order. The consequence of complicated large-scale atmospheric dynamics especially in midlatitudes is to transfer the surplus of energy in low latitudes to high latitudes, which is simply approximated by a turbulent meridional diffusion.

The temporal and spatial evolution of the perturbation \(\eta\) governed by Equation (28) is represented by

\[
\frac{\partial \eta}{\partial t} = K \frac{\partial^2 \eta}{\partial Y^2} + a(Y, \tilde{t}) \eta + R(Y, \tilde{t}), \tag{30}
\]

where we include the \(R(Y, \tilde{t})\) for the contribution of short-time processes. We can consider two boundary conditions in meridional coordinate,

\[
\frac{\partial \eta}{\partial Y}(\cdot, Y = 0) = \frac{\partial \eta}{\partial Y}(\cdot, Y = 1) = 0,
\]

implying there is no meridional heat flux near the pole and the tropical areas.

For the \(Y\)-direction diffusion operator, consider the eigenvalue problem

\[
K \frac{\partial^2}{\partial Y^2} H_n = -\lambda_n H_n \tag{31}
\]

with

\[
\frac{\partial H_n}{\partial Y}(Y = 0) = \frac{\partial H_n}{\partial Y}(Y = 1) = 0.
\]

Here, \(H_n = A_n \cos(n \pi Y)\) with \(\lambda_n = K n^2 \pi^2\) and \(n = 0, 1, 2, \ldots\). Hence, the temporal and spatial perturbation of the potential temperature anomaly \(\eta\) can be represented by an infinite series of eigenfunctions with time-varying coefficients

\[
\eta = \sum_{n=0}^{\infty} x_n(\tilde{t}) H_n(Y). \tag{32}
\]

Consider first the case that \(a(Y, \tilde{t})\) is a constant \(-\gamma\) and that the sum of short-time processes \(R(Y, \tilde{t})\) is represented as a sum of the same eigenfunctions, \(R = \sum_{n=0}^{\infty} R_n(\tilde{t}) H_n(Y)\).

The time-varying coefficient \(x_n(\tilde{t})\) then satisfies

\[
\frac{dx_n(\tilde{t})}{dt} = (-\lambda_n + \gamma) x_n(\tilde{t}) + R_n(\tilde{t}). \tag{33}
\]

If we consider the last term as a stochastic noise, in particular a Gaussian white noise, this becomes a Langevin equation with a decay time-scale \(1/(\lambda_n + \gamma)\). The time-scale becomes shorter as \(n\) increases, which indicates that the first several modes representing large-scale motions dominate in the overall fluctuations. When \(n\) is equal to 0, the Langevin equation becomes \(d \tilde{x}_0 / dt = -\gamma \tilde{x}_0 + R_0(\tilde{t})\), where \(\gamma\) represents the seasonal sensitivity of local energy flux balance mainly dominated by
radiative-convective equilibrium in land and ocean heat flux in the ocean boundary layer. The formalism used in this argument was first introduced by Manucharyan et al. (2016) to explain the internal variability caused by interactions between mesoscale eddies and mean fields under the framework of the Gent–McWilliam parametrization.

The simple Langevin equation was introduced to climate science by Hasselmann (1976) to interpret climate variability in terms of stochastic dynamics. The deterministic part is understood as a stabilizing process around a mean state and the stochastic noise as the effect of weather. Climate is understood as a combination of the usage of an AR1 process explaining the zonal index variability (Lorenz and Hartmann, 2001).

### 3.2 Quasi-oscillatory behaviour of the BAM

Thompson and Barnes (2014) (TB) studied quasi-periodic variability of the Southern Hemisphere BAM. Instead of the red-noise spectrum originated from a simple linear Langevin equation, the BAM shows a clear sign of quasi-oscillation in its power spectrum. This indicates that the simple Langevin equation leading to a red-noise spectrum is not adequate to describe the quasi-periodic fluctuation.

To explain a periodic behaviour of meridional temperature gradient and poleward heat flux, TB introduce mutual interaction and feedback between the meridional temperature gradient \( b \equiv \partial \eta / \partial Y \) and the poleward heat flux \( \langle \bar{v}_0 \Theta_0 \rangle \). Baroclinic instability theory tells us that the growth rate of the baroclinic waves is proportional to the meridional temperature gradient (Lindzen and Farrell, 1980), which could be represented by

\[
\frac{d}{dt} \langle \bar{v}_0 \Theta_0 \rangle = -\alpha(b) + c(t), \tag{34}
\]

where \( \alpha \) is a constant representing the amplitude of the feedback between the baroclinicity and the meridional heat flux (Hoskins and Valdes, 1990). Here, the noise forcing \( c(t) \) is added to include the effect of chaotic weather. At the same time, TB consider a feedback of the poleward heat flux on the meridional temperature gradient, which simply assumes that the meridional temperature gradient \( \langle b \rangle \) increases linearly with respect to the poleward heat flux \( \langle \bar{v}_0 \Theta_0 \rangle \). Thus, the second equation in TB is

\[
\frac{db}{dt} = \mu \langle \bar{v}_0 \Theta_0 \rangle - \frac{b}{r}, \tag{35}
\]

where \( \mu \) is the degree of the feedback and \( r \) is a recovering time-scale of the meridional temperature gradient due to diabatic processes and vertical motions. The combination of Equations (34) and (35) could generate an oscillation whose period is determined by the relevant coefficients. While ignoring the noise forcing in Equation (34), combining the two equations (Thompson and Barnes, 2014) leads to

\[
\frac{d^2 b}{d t^2} + \frac{b}{r} \frac{db}{dt} + a \mu b = G, \tag{36}
\]

where \( G \) is the time-derivative of the short-time-scale forcing \( c(t) \). Oscillatory behaviour comes out when \((1/4r^2) - \alpha \mu < 0\). TB used \( r \approx 4 \) days and \( \alpha \mu \approx 0.05 \) day^{-2} to generate a dominant peak of the power spectrum of poleward heat flux around 20 days. They suggest a mutual interaction between \( \langle \bar{v}_0 \Theta_0 \rangle \) and \( \partial \eta / \partial Y \) with the damping represented by a Newtonian cooling. In the model, it is hard to guess what determines the damping time-scale \( r \) and the feedback parameter \( \alpha \mu \).

By a simple comparison, Equation (35) is equivalent to the anomalous heat Equation (28) and then Equation (34) provides a relationship between the poleward heat flux and the meridional temperature gradient. Even though Equation (35) is likely to be derived from the anomalous heat equation, the damping term \( -b/r \) is not clearly understood physically. Understanding the relationship between \( b \) and \( \langle \bar{v}_0 \Theta_0 \rangle \) on intraseasonal time-scales demands recognition of what determines the damping time-scale \( r \).

We can incorporate Equation (34) into Equation (28) after taking time-derivative and \( Y \)-derivative on Equation (28). This results in

\[
\frac{\partial^2 b}{\partial t^2} = \frac{\partial^2 b}{\partial Y^2} - \gamma \frac{\partial b}{\partial t} + \frac{\partial R}{\partial t}, \tag{37}
\]

where \( -\gamma \) is used instead of \( a(Y, \bar{t}) \) and we ignore \( \partial a / \partial Y \) for simplicity. The constant \( \gamma \) implies the damping time-scale of seasonal energy flux balance. On land, the outgoing long-wave radiative flux dominantly controls the time-scale. Similarly, we can consider the eigenvalue problem for the diffusion operator,

\[
\alpha \frac{\partial^2}{\partial Y^2} H_n = -\lambda_n H_n, \tag{38}
\]
where \( b = \sum_{n=0}^{\infty} R_n(t) H_n(Y) \). We obtain the time-evolution equation for \( x_n(t) \),

\[
\frac{d^2}{dt^2} x_n + \gamma \frac{dx_n}{dt} + \lambda_n x_n = \frac{dR_n}{dt},
\]

(39)

where \( R = \sum_{n=0}^{\infty} R_n(t) H_n(Y) \) is used. The equation for \( x_n \) is similar to Equation (36), which suggests that the damping time-scale is equivalent to \( 1/\gamma \). Physically, \( \gamma \) is introduced as a sensitivity of the radiative energy flux balance. In terms of time-scale, it could be interpreted as a time-scale to return to a climatological seasonal cycle. Moon and Wetlaufer (2017) developed a time-series method to construct the \( \gamma \) from monthly averaged surface air temperature. The time-scale of the surface energy flux balance in the Southern Hemisphere is around 1.5 month, which is much larger than 4 days. Therefore, the damping time-scale \( r \) does not come from diabatic processes related to radiative fluxes.

### 3.3 Eddy memory and generalized Langevin equation

The damping time-scale introduced to explain the oscillatory behaviour of the baroclinic mode in the Southern Hemisphere is much shorter than that of the mean seasonal cycle of radiative processes. It is plausible that the time-scale is related to synoptic eddies, rather than any external influences which have longer time-scales. Synoptic eddies generated from baroclinic instability due to an unstable background undergoes a specific energy cycle with a zonal mean steady state. The baroclinic instability enables the synoptic eddies to extract energy from the zonal mean state, from which the poleward heat flux increases near the surface. The growing waves propagate upward and equatorward and meet critical latitudes where phase speeds of the waves are the same as the mean wind. The waves break and give their energy back to the mean state by momentum flux. This baroclinic wave life cycle is complete in a few days.

The overall effect of the baroclinic wave life cycle on the meridional heat flux is represented by the planetary potential temperature anomaly \( \eta \) as \( \langle v_0 \Theta_0 \rangle = -K(\partial \eta / \partial Y) \). This Fickian diffusion approximation is well applied when the time evolution of the planetary variable is much slower than the turnover time-scale of synoptic eddies. On intraseasonal time-scales, however, the time-scale for the evolution of planetary-scale variables is not clearly separated from that of the baroclinic wave life cycle. The advective time-scale for the synoptic waves is around 1.2 day (where we use \( L_D = 3,000 \text{ km} \) and \( U = 10 \text{ m/s}^{-1} \)) and the same time-scale for the planetary waves is around 3.5 day (where we use \( L_D = 3,000 \text{ km} \) and \( U = 10 \text{ m/s}^{-1} \)). A complete baroclinic wave life cycle takes around 3–4 days, hence the time-scale for the planetary-scale motion is comparable with that for the baroclinic wave life cycles. It is questionable to apply the Fickian diffusion as a parametrization of the poleward heat flux at these time-scales.

Non-Fickian approximation for turbulent transport or diffusion has been a central topic in turbulent closure problems (Orszag, 1970). For a transient and chaotic turbulence, the Fickian approximation is not enough to capture the non-local and non-Gaussian nature of turbulence. One of the simplest approach is called the minimal \( \tau \) approximation, where the third-order momentum represented as a forcing for the time evolution of second-order moments is approximated as a damping term with the time-scale \( \tau \) (Brandenburg et al., 2004; Brandenburg and Subramanian, 2005). This is equivalent to applying an integral kernel instead of a constant diffusivity with a finite memory (Hubbard and Brandenburg, 2009).

The main idea is applied to the time evolution of the poleward heat flux of synoptic-scale waves, that is,

\[
\frac{\partial}{\partial t} \langle v_0 \Theta_0 \rangle = \langle \frac{\partial v_0}{\partial \Theta_0} \rangle + \langle v_0 \frac{\partial \Theta_0}{\partial t} \rangle.
\]

(40)

The time evolution equations for the synoptic-scale meridional velocity \( v_0 \) and potential temperature \( \Theta_0 \) (Moon and Cho, 2020) are

\[
\begin{align*}
\frac{\partial v_0}{\partial t} &= -(u_L + u_0) \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_0}{\partial y} - u_1 - \frac{\beta y}{e^{1/2}} u_0 - \frac{\partial P_1}{\partial x}, \\
\frac{\partial \Theta_0}{\partial t} &= -\frac{\partial \eta}{\partial t} -(u_L + u_0) \frac{\partial \Theta_0}{\partial x} - v_0 \frac{\partial \eta}{\partial y} - v_0 \frac{\partial \Theta_0}{\partial y} - w_1 \left( \frac{\partial \Theta_L}{\partial z} + S \right).
\end{align*}
\]

(41)

Hence,

\[
\frac{\partial}{\partial t} \langle v_0 \Theta_0 \rangle = -\langle v_0^2 \rangle \frac{\partial \eta}{\partial Y} + D.
\]

(42)

where \( D \) contains the terms representing the contribution from synoptic eddies. After taking synoptic-time averages on both sides, we obtain

\[
\frac{\partial}{\partial t} \langle v_0 \Theta_0 \rangle = -\langle v_0^2 \rangle \frac{\partial \eta}{\partial Y} + \overline{D}.
\]

(43)

The main assumption of the minimal \( \tau \) approximation is that the contribution of the synoptic eddies \( \overline{D} \) is approximated by a damping of the \( \langle v_0 \Theta_0 \rangle \) with the designated time-scale \( r \). Therefore,

\[
\frac{\partial}{\partial t} \langle v_0 \Theta_0 \rangle = -K \frac{\partial \eta}{\partial Y} - \frac{\langle v_0 \Theta_0 \rangle}{r}.
\]

(44)
where $K \equiv \langle \nu_0 \rangle$, suggesting how the eddy diffusivity $K$ is related to the second-order statistics of synoptic eddies. The minimal $\tau$ approximation for $\langle \nu_0 \Theta_0 \rangle$ seems to be equivalent to Equation (34) in TB. The feedback parameter $a$ could be understood as $K$. TB suggest the relationship based on the result from the baroclinic instability. On the other hand, we derived a similar one by the simplest non-Fickian approximation as a closure of turbulent eddies. TB included a random forcing in the relationship to include unresolved processes, but the randomness coming from short-time processes is considered in the heat equation in our derivation. The above parametrization can be represented by an integral form, that is,

$$
\langle \nu_0 \Theta_0 \rangle = -K \frac{\partial}{\partial Y} \int_{-\infty}^{\tau} \eta \exp \left( -\frac{t-t'}{\tau} \right) dt'.
$$

(45)

The integral form originates from integrating Equation (44) for $\langle \nu_0 \Theta_0 \rangle$ with respect to time $\tau$. From this integral form, we can see that the poleward heat flux is the result of accumulating the baroclinicity from a certain time until the present. It represents the memory effect caused by synoptic eddies.

The minimal $\tau$ approximation was tested in direct simulations of isotropic 3D turbulence (Brandenburg et al., 2004). The statistics of a passive scalar are compared between direct simulations and the parametrized equations, where decent matches are obtained. In particular, the parametrization transforms the main parabolic tracer equation to a wave equation leading to an oscillatory behaviour. The simulations show decayed oscillation with a certain choice of the damping time-scale $\tau$. In an idealized Beaufort Gyre numerical simulation, Manucharyan et al. (2017) introduced a finite-memory kernel instead of a constant diffusivity in the Gent–McWilliam parametrization, which generates the quasi-periodic variability in the eddy–mean interaction. The minimal $\tau$ approximation can be understood as a finite memory effect represented by an integral kernel.

Taking account of the memory effect of eddies, we can introduce an integral kernel $\kappa(t-t')$ on the parametrization of the meridional heat flux,

$$
\langle \nu_0 \Theta_0 \rangle = -K \frac{\partial}{\partial Y} \int_{-\infty}^{\tau} \eta(t') \kappa(t-t') dt',
$$

$$
= -K \frac{\partial \eta^*}{\partial Y},
$$

(46)

which is a generalization of the minimal $\tau$ approximation in Equation (45) where $\eta^*$ is an effective temperature anomaly defined by

$$
\eta^* \equiv \int_{-\infty}^{\tau} \eta(t') \kappa(t-t') dt'.
$$

(47)

Thus, Equation (30) becomes

$$
\frac{\partial \eta}{\partial t} = K \frac{\partial^2 \eta^*}{\partial Y^2} - \gamma \eta + R(Y, \eta),
$$

(48)

where we assume that $a(Y, \tau) = -\gamma$ for simplicity. Furthermore, if we expand $\eta^*$ using the eigenfunctions of the diffusion operator, $\eta^* = \sum_{n=0}^{\infty} \gamma_n \kappa^*(\tau) H_n(Y)$, and the residual forcing, $R = \sum_{n=0}^{\infty} \gamma_n \kappa^*(\tau) H_n(Y)$, each time-dependent coefficient $\gamma_n$ satisfies

$$
\frac{d\gamma_n}{d\tau} = -\gamma_n \gamma_n - \gamma x_n + R_n
$$

$$
= -\lambda_n \int_{-\infty}^{\tau} \gamma_n \kappa^*(\tau) \kappa^*(\tau) d\tau - \gamma x_n + R_n,
$$

(49)

which is a generalized Langevin equation (Zwanzig, 1973). The generalized Langevin equation contains memory terms representing the effect of past states, which is generally originating from interactions with short-time-scale components.

The complicated chaotic processes to generate memory are captured by the memory kernel $\kappa^*(\tau-t')$. The dynamics represented by a generalized Langevin equation are dependent upon the choice of an integral kernel. We can follow the theme of the $\tau$ approximation, using an integral kernel with a finite-time memory. It has been suggested that baroclinic eddies generated by baroclinic instability are eventually organized to maintain a midlatitude jet characterized by a meridional temperature gradient (Robinson, 2000; Lorenz and Hartmann, 2001). This implies that the effect of baroclinic eddies upon the meridional heat flux is limited within a finite-time-scale, which leads to an approximation

$$
\kappa^*(\tau-t') = \exp \left( -\frac{\tau-t'}{\tau} \right).
$$

(50)

Here $r$ represent the finite memory of the baroclinic eddies, which is equivalent to $dx_n^2/\tau = -x_n^2/r + x_n/r$. Based on the specific memory kernel, Equation (49) becomes

$$
\frac{d^2 x_n}{d\tau^2} + \left( \frac{1}{r} + \gamma \right) \frac{dx_n}{d\tau} + \frac{\gamma + \lambda_n}{r} x_n = \frac{R_n}{r} + \frac{dR_n}{d\tau}.
$$

(51)

There are two damping time-scales in the above equation defined by $r$ and $1/\gamma$. The $1/\gamma$ comes from the seasonal heat flux balance such that it is approximately 1.5 month in the Southern Hemisphere (Moon and Wettlauffer, 2017). This implies that $\gamma$ is around 0.02 day$^{-1}$. Thompson and Barnes (2014) use around 4 days as a damping time-scale, which is equivalent to the eddy memory. Hence, $r$ is at most several days. It follows
\(1/r \gg \gamma\). The eigenvalues \(\lambda_n\) come again from (31) with \(Qn'(0) = Qn'(1) = 0\) implying no heat flux at both ends, which gives us \(\lambda_n = Kn^2\pi^2\) and \(Q_n(Y) = \cos(n\pi Y)\). The BAM is defined by a dominant mode of the meridional heat flux maximized around the centre of the midlatitudes. This is similar to the mode with \(n = 1\), \(Q_1(Y)\), whose derivative \(Q_1'(Y)\), has its maximum at the centre. The time-dependent coefficient of the mode, \(x_1\), satisfies

\[
\frac{d^2x_1}{dt^2} + \frac{1}{r} \frac{dx_1}{dt} + \frac{\gamma + K\pi^2}{r} x_1 = \frac{R_1}{r} + \frac{dR_1}{dt},
\]

(52)

which should be understood as an equivalent form to Equation (36). The damping time-scale \(r = 4\) days comes from the memory effect of synoptic eddies on the meridional heat flux, which seems to be strongly related to the baroclinic wave life cycle. Moreover, \(a\mu\) in Equation (36) is the same as \((\gamma + K\pi^2)/r\). Following the feedback parameter \(a\beta\) used in Thompson and Barnes (2014), we can deduce that \(K \approx 1/5\pi^2\), where we ignored the contribution of the small parameter \(\gamma\).

Depending on the choice of the parameters \(r\) and \(K\), Equation (52) can show decayed oscillatory behaviour or exponential decay. Under the assumption that \(R_1/r + dR_1/dt\) is approximately a Gaussian white noise, we can simulate the equation to generate a stochastic realization. Figure 1 shows several power spectra of stochastic realizations from Equation (52) with different parameters \(r\) and \(K\) along with the power spectrum of the BAM index (Thompson and Woodworth, 2014). With a fixed \(K\), four different memory time-scales from 1 to 4 days are used for stochastic realization. In Figure 1a, we see that longer memory (4 days) leads to oscillatory behaviour, which has a similar power spectrum to that of the BAM index, and a shorter memory (1 day) than a red-noise spectrum. Similarly, we fix \(r = 4\) days and vary \(K\), which also shows the characteristics of stochastic oscillation (Figure 1b). Moreover, when \(K\pi^2/r = 1\) day\(^{-2}\), the power spectrum is also close to that of the BAM index. If we lower the value of \(K\pi^2/r\) further, the oscillatory behaviour will become a red-noise process.

The baroclinic wave life cycle results in recovering the vertical shear of the mean jet. This could be an origin of the finite memory effect in meridional heat transfer on the planetary scale, which leads to oscillatory behaviour of the zonal mean potential temperature. Following the above formalism, the fluctuation of potential temperature anomaly \(\eta\) is represented as

\[
\eta = \sum_{n=0}^{\infty} x_n(t)Q_n(Y), \quad \text{hence} \quad \frac{\partial \eta}{\partial Y} = \sum_{n=0}^{\infty} x_n(t) \frac{dQ_n}{dY}.
\]

By the thermal wind balance,

\[
\frac{\partial U}{\partial z} \propto \sum_{n=0}^{\infty} x_n(t) \frac{dQ_n}{dY},
\]
which implies that the vertical shear of the jet at lower levels could show quasi-oscillatory behaviour depending on the time-scale of the eddy memory and relevant eigenvalues defining the decay time-scale of a specific normal mode.

The main focus lies on how to parametrize the meridional heat flux induced by synoptic eddies \( \langle v_0 \theta_0 \rangle \) using

\[
\text{the planetary-scale potential temperature } \Theta_L. \quad \text{Considering that the time-scale for synoptic eddies is not much smaller than that of planetary-scale motions, the meridional heat flux is not entirely determined by an instantaneous potential temperature gradient; instead it is dependent upon the temporal history of the potential temperature gradient. Hence, the Fickian diffusion approximation is not appropriate to parametrize the meridional heat flux induced by synoptic eddies. The simplest way of considering the temporal history of the planetary-scale potential temperature is to introduce a time integration of the potential temperature with an appropriate memory kernel, which enables us to include the past influence of planetary-scale mean fields on the turbulent heat flux. As a result, fluctuations of potential temperature can be described approximately by a generalized Langevin equation containing a memory term.}

The time-delayed effect represented as an integral kernel in the generalized Langevin equation can generate various types of variability ranging from a simple red-noise process to quasi-oscillations and chaos. Hence, an appropriate choice of integral kernel is crucial to define the statistical characteristics of planetary-scale variability.

### 4 | CONCLUSION

The main goal of the study was to rationalize the quasi-oscillatory variability of the BAM (Thompson and Barnes, 2014) on intraseasonal time-scales using the multi-scale representation of primitive equations of the large-scale atmospheric flow (Moon and Cho, 2020). Conceptually, the multi-scale approximation is a combination of the energy flux balance in the heat equation and the potential vorticity conservation. We found that the critical issue under the framework is the specification of an appropriate parametrization of the horizontal heat flux convergence from the synoptic- and planetary-scale eddy dynamics. If the heat fluxes are represented using the simplest diffusive (Fickian) parametrization with a constant eddy diffusivity, the planetary-scale equations simplify to an energy flux balance with a meridional turbulent diffusion, similar to North (1975). Other types of eddy parametrizations are also possible, of which the most notable is the residual mean theory of Andrews and McIntyre (1976) with a diffusive parametrization of isopycnal eddy fluxes of potential vorticity. While the residual mean formulation gives a different perspective on the mean flow dynamics compared to the direct diffusive closure of the heat fluxes (Schneider and Dickinson, 1974), our study questions their equilibrium-type Fickian assumption for the relation between the eddy fluxes and gradients of the considered tracer fields. Under the equilibrium-type eddy parametrizations, the main equation in the planetary-scale atmosphere is the heat equation with the basic balances resulting in a parabolic partial differential equation which does not support natural oscillations. The resulting temporal variability of observables obeys the Langevin equation leading to a red-noise spectrum which does not explain the quasi-oscillatory behaviour of the BAM. Thus the equilibrium-type eddy parametrizations are likely not appropriate at the intraseasonal time-scales and can only be used for representing the relatively long-term average of the planetary-scale dynamics.

We find that using a non-equilibrium eddy parametrization with the eddy memory effect represented by the delayed integral in the flux-gradient relation, as was proposed for mesoscale ocean eddies (Manucharyan et al., 2017), can explain the oscillatory behaviour of the BAM (Thompson and Barnes, 2014) on intraseasonal time-scales using the multi-scale representation of primitive equations of the large-scale atmospheric flow (Moon and Cho, 2020). Conceptually, the multi-scale approximation is a combination of the energy flux balance in the heat equation and the potential vorticity conservation. We found that the critical issue under the framework is the specification of an appropriate parametrization of the horizontal heat flux convergence from the synoptic- and planetary-scale eddy dynamics. If the heat fluxes are represented using the simplest diffusive (Fickian) parametrization with a constant eddy diffusivity, the planetary-scale equations simplify to an energy flux balance with a meridional turbulent diffusion, similar to North (1975). Other types of eddy parametrizations are also possible, of which the most notable is the residual mean theory of Andrews and McIntyre (1976) with a diffusive parametrization of isopycnal eddy fluxes of potential vorticity. While the residual mean
and eddy heat fluxes using atmospheric models would be the next crucial step towards validating our theoretical arguments about the BAM. Furthermore, understanding the nature of the eddy memory and how its time-scale depends on the mean flow or eddy characteristics would be necessary to understand the climate conditions under which the BAM could exhibit oscillatory behaviour. Finally, our results emphasize the importance of the external noise in driving the variance of the BAM and the need to understand whether it is dependent on the mean flow or acts as an external/independent source of energy.

ACKNOWLEDGEMENTS
W.M. acknowledges the support of Swedish Research Council grant no. 638-2013-9243. G.E.M. acknowledges support from the United States Office of Naval Research award N00014-19-1-2421. H.A.D. acknowledges support from the United States Office of Naval Research Council grant no. 638-2013-9243. G.E.M. acknowledges support from the Netherlands Earth System Science Centre (NESSC), financially supported by the Ministry of Education, Culture and Science (OCW), grant no. 024.002.001.

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### How to cite this article:
Moon W, Manucharyan GE, Dijkstra HA. Eddy memory as an explanation of intraseasonal periodic behaviour in baroclinic eddies. *QJR Meteorol. Soc.* 2021;147:2395–2408. https://doi.org/10.1002/qj.4030