Improved Kato’s lemma on ordinary differential inequality and its application to semilinear wave equations *

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Keywords: ordinary differential inequality, semilinear wave equation, lifespan

MSC2010: primary 35L71, 34A40, 35E15, secondary 35A01, 35B44

Abstract

We are interested in estimating the lifespan of solutions of semilinear wave equations from above. For the sub-critical case in high dimensions, it has been believed that the basic tools of its analysis are Kato’s lemma on ordinary differential inequalities and the rescaling argument in the functional method. But there is a small lack of delicate analysis and no published paper about this. Here we give a simple alternative proof by means of improved Kato’s lemma without any rescaling argument.

1 Introduction

We consider the initial value problem,

\[
\begin{cases}
  u_{tt} - \Delta u = |u|^p, & \text{in } \mathbb{R}^n \times [0, \infty), \\
  u(x, 0) = \varepsilon f(x), & \text{in } \mathbb{R}^n, \\
  u_t(x, 0) = \varepsilon g(x)
\end{cases}
\]

(1.1)

*This work is partially supported by the Grant-in-Aid for Scientific Research (C) (No. 24540183), Japan Society for the Promotion of Science.
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assuming that $\varepsilon > 0$ is “small.” Let us define a lifespan $T(\varepsilon)$ of a solution of (1.1) by

$$T(\varepsilon) = \sup \{ t > 0 : \exists \text{a solution } u(x, t) \text{ of (1.1) for arbitrarily fixed } (f, g).\},$$

where “solution” means classical one when $p \geq 2$. When $1 < p < 2$, it means weak one, but sometimes the one of a solution of associated integral equations to (1.1) by standard Strichartz’s estimate. See Sideris [11] or Georgiev, Takamura and Zhou [2] for example on such an argument.

When $n = 1$, we have $T(\varepsilon) < \infty$ for any power $p > 1$ by Kato [6]. When $n \geq 2$, we have the following Strauss’ conjecture on (1.1) by Strauss [12].

$$T(\varepsilon) = \infty \quad \text{if } p > p_0(n) \text{ and } \varepsilon \text{ is “small” (global-in-time existence)},$$

$$T(\varepsilon) < \infty \quad \text{if } 1 < p \leq p_0(n) \quad \text{(blow-up in finite time)},$$

where $p_0(n)$ is so-called Strauss’ exponent defined by positive root of the quadratic equation,

$$\gamma(p, n) = 2 + (n + 1)p - (n - 1)p^2 = 0. \quad (1.2)$$

That is,

$$p_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}. \quad (1.3)$$

We note that $p_0(n)$ is monotonously decreasing in $n$. This conjecture had been verified by many authors of partial results. All the references on the final result in each part can be summarized in the following table.

| $n$     | $p < p_0(n)$                  | $p = p_0(n)$                  | $p > p_0(n)$                  |
|---------|-------------------------------|-------------------------------|-------------------------------|
| $n = 2$ | Glassey [3]                   | Schaeffer [10]               | Glassey [4]                   |
| $n = 3$ | John [5]                      | Schaeffer [10]               | John [5]                      |
| $n \geq 4$ | Sideris [11]               | Yordanov & Zhang [14]       | Georgiev & Lindblad & Sogge [1] |
|         | Zhou [18], indep.             |                               |                               |

In the blow-up case of $1 < p \leq p_0(n)$, we are interested in the estimate of the lifespan $T(\varepsilon)$. From now on, $c$ and $C$ stand for positive constants but independent of $\varepsilon$. When $n = 1$, we have the following estimate of the lifespan $T(\varepsilon)$ for any $p > 1$.

$$\begin{cases} 
  c e^{-(p-1)/2} \leq T(\varepsilon) \leq C e^{-(p-1)/2} & \text{if } \int_{\mathbb{R}} g(x)dx \neq 0, \\
  c e^{-p(p-1)/(p+1)} \leq T(\varepsilon) \leq C e^{-p(p-1)/(p+1)} & \text{if } \int_{\mathbb{R}} g(x)dx = 0.
\end{cases} \quad (1.4)$$
This result has been obtained by Zhou [15]. Moreover, Lindblad [8] has obtained more precise result for \( p = 2 \),

\[
\begin{align*}
\exists \lim_{\varepsilon \to +0} \varepsilon^{1/2} T(\varepsilon) &> 0 \quad \text{for } \int_{\mathbb{R}} g(x) dx \neq 0, \\
\exists \lim_{\varepsilon \to +0} \varepsilon^{2/3} T(\varepsilon) &> 0 \quad \text{for } \int_{\mathbb{R}} g(x) dx = 0.
\end{align*}
\]

Similarly to this, Lindblad [8] has also obtained the following result for \((n, p) = (2, 2)\),

\[
\begin{align*}
\exists \lim_{\varepsilon \to +0} \varepsilon T(\varepsilon) &> 0 \quad \text{for } \int_{\mathbb{R}^2} g(x) dx \neq 0, \\
\exists \lim_{\varepsilon \to +0} \varepsilon T(\varepsilon) &> 0 \quad \text{for } \int_{\mathbb{R}^2} g(x) dx = 0,
\end{align*}
\]

where \( a = a(\varepsilon) \) is a number satisfying

\[
a^2 \varepsilon^2 \log(1 + a) = 1.
\]

When \( 1 < p < p_0(n) \) \((n \geq 3)\) or \( 2 < p < p_0(2) \) \((n = 2)\), we have the following conjecture.

\[
c \varepsilon^{-2p(p-1)/\gamma(p,n)} \leq T(\varepsilon) \leq C \varepsilon^{-2p(p-1)/\gamma(p,n)},
\]

where \( \gamma(p,n) \) is defined by (1.2). We note that (1.8) coincides with the second line in (1.4) if we define \( \gamma(p,n) \) by (1.2) even for \( n = 1 \). All the results verifying this conjecture are summarized in the following table.

| \( n \)   | lower bound of \( T(\varepsilon) \) | upper bound of \( T(\varepsilon) \) |
|-----------|-----------------------------------|-------------------------------|
| \( n = 2 \) | Zhou [17]                         | Zhou [17]                     |
| \( n = 3 \) | Lindblad [8]                      | Lindblad [8]                  |
| \( n \geq 4 \) | Lai & Zhou [7]                  | (rescaling argument of Sideris [11]) |

We note that, for \( n = 2, 3, \)

\[
\exists \lim_{\varepsilon \to +0} \varepsilon^{2p(p-1)/\gamma(p,n)} T(\varepsilon) > 0
\]

is established in this table. Moreover, it has been believed that the upper bound in the case where \( n \geq 4 \) easily follows from the rescaling method applied to the proof in Sideris [11] which proves \( T(\varepsilon) < \infty \). Such an argument is actually employed in Georgiev, Takamura and Zhou [2] for the analysis on the system. But it requires more delicate analysis in the following sense. The
conclusion of the original Kato’s lemma in [6] is that the integral in full space of unknown function blows-up in finite time. The rescaling argument uses this blow-up time as a coefficient in front of the order of $\varepsilon$ with a negative power. To do this, we have to clarify that such a blow-up time does not depend on the size of the initial data. Of course, it is true as in Lemma 2.1 in Section 2 below. But we don’t need the rescaling argument as we see later if we employ such an improved Kato’s lemma. The purpose of this paper is to show this story.

When $p = p_0(n)$, we have the following conjecture.

$$\exp\left(c\varepsilon^{-p(p-1)}\right) \leq T(\varepsilon) \leq \exp\left(C\varepsilon^{-p(p-1)}\right).$$

(1.9)

All the results verifying this conjecture are also summarized in the following table.

| $n = 2$ | lower bound of $T(\varepsilon)$ | upper bound of $T(\varepsilon)$ |
|--------|-------------------------------|-------------------------------|
| Zhou [17] | Zhou [17] | |
| $n = 3$ | Zhou [16] | Zhou [16] |
| $n \geq 4$ | Lindblad & Sogge [9] | Takamura & Wakasa [13] |
| $: n \leq 8$ or radially symm. sol. |

Our motivation of this work comes from [13], in which the improved Kato’s lemma in the critical case is the one of keys for the success to finalize this part.

This paper is organized as follows. In the next section, we improve Kato’s lemma showing estimates of the existence time of unknown functions in terms of the holding time of the key inequality. In Section 3, improved Kato’s lemma is applied to semilinear wave equations. In Section 4 or 5, the improvement of the estimate is given in two or one dimensional case respectively.

## 2 Improved Kato’s lemma

Kato’s lemma on ordinary differential inequality in [6] is improved here.

**Lemma 2.1** Let $p > 1, a > 0, q > 0$ satisfy

$$M := \frac{p-1}{2}a - \frac{q}{2} + 1 > 0.$$  

(2.1)

Assume that $F \in C^2([0, T))$ satisfies

$$F(t) \geq At^a \quad \text{for } t \geq T_0,$$  

(2.2)

$$F''(t) \geq B(t + R)^{-q}|F(t)|^p \quad \text{for } t \geq 0,$$  

(2.3)

$$F(0) \geq 0, \quad F'(0) > 0,$$  

(2.4)
where $A, B, R, T_0$ are positive constants. Then, there exists a positive constant $C_0 = C_0(p, a, q, B)$ such that

$$T < 2^{2/M}T_1$$

holds provided

$$T_1 := \max \left\{ T_0, \frac{F(0)}{F'(0)} R \right\} \geq C_0 A^{-(p-1)/(2M)}.$$  

(2.6)

**Remark 2.1** The statement of the original version of this lemma in Kato [6] is simply that $T_0$ is large and $T < \infty$ without (2.2), but there is no restriction on $F(0)$. See also Lemma 4 in Sideris [11]. The first paper to put an additional assumption (2.2) and to get a sharp estimate of the lifespan of solutions of semilinear wave equations is Glassey [3] in two space dimensions.

**Proof of Lemma 2.1.** Since (2.3) and (2.4) imply $F'(t) \geq F'(0) > 0$, we have

$$F(t) \geq F'(0)t + F(0) \geq F(0) \geq 0 \quad \text{for } t \geq 0. \quad (2.7)$$

Multiplying (2.3) by $F'(t) > 0$, we get

$$\left( \frac{1}{2} F'(t)^2 \right)' \geq B(t + R)^{-q} F(t)^p F'(t) \quad \text{for } t \geq 0.$$  

It follows from this inequality and (2.7) that

$$\frac{1}{2} F'(t)^2 \geq \frac{1}{2} F'(0)^2 + B(t + R)^{-q} \int_0^t F(s)^p F'(s) ds$$

$$\geq \frac{B}{p + 1} (t + R)^{-q} \left\{ F(t)^{p+1} - F(0)^{p+1} \right\}$$

$$\geq \frac{B}{p + 1} (t + R)^{-q} F(t)^{p} \left\{ F(t) - F(0) \right\}$$

for $t \geq 0$.

If $F(0) > 0$, then we shall assume that $t \geq F(0)/F'(0)$ which implies $F(t) \geq 2F(0)$. Hence we have

$$F'(t) > \sqrt[2(p+1)]{\frac{B}{p + 1} (t + R)^{-q/2} F(t)^{(p+1)/2}} \quad \text{for } t \geq \frac{F(0)}{F'(0)}.$$  

We note that it is trivial that this inequality also holds for $F(0) = 0$.  

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From now on, we assume that $t \geq T_1$. Then, (2.2) is available, so that taking $\delta$ as $0 < \delta < (p - 1)/2$, we obtain

$$\frac{F'(t)}{F(t)^{1+\delta}} > 2^{-q/2} \sqrt{\frac{B}{p+1}} t^{-q/2} F(t)^{(p-1)/2-\delta}$$

$$> 2^{-q/2} \sqrt{\frac{B}{p+1}} A^{(p-1)/2-\delta} t^{M-1-\delta a}$$

for $t \geq T_1$. Integrating this inequality on $[T_1, t]$, we get

$$\frac{1}{\delta} \left( \frac{1}{F(T_1)^{\delta}} - \frac{1}{F(t)^{\delta}} \right) > 2^{-q/2} \sqrt{\frac{B}{p+1}} A^{(p-1)/2-\delta} \int_{T_1}^{t} s^{M-1-\delta a} ds.$$  

Neglecting the second term in the left-hand side and restricting $\delta$ further as

$$0 < \delta < \min \left( \frac{p-1}{2}, \frac{M}{2a} \right),$$

we have

$$\frac{1}{\delta F(T_1)^{\delta}} > 2^{-q/2} \sqrt{\frac{B}{p+1}} A^{(p-1)/2-\delta} \frac{t^{M-\delta a} - T_1^{M-\delta a}}{M - \delta a}$$

for $t \geq T_1$. Then, it follows from (2.2) with $t = T_1$ that

$$\frac{1}{T_1^{\delta a}} \left( \frac{2^{-q/2} \sqrt{B}}{M - \delta a} A^{(p-1)/2} T_1^M + \frac{1}{\delta} \right) > 2^{-q/2} \sqrt{\frac{B}{p+1}} A^{(p-1)/2} t^{M-\delta a}$$

for $t \geq T_1$. If we assume further that

$$\frac{2^{-q/2}}{M - \delta a} \sqrt{\frac{B}{p+1}} A^{(p-1)/2} T_1^M \geq \frac{1}{\delta},$$

namely (2.6) with

$$C_0 = \left( \frac{2^{-q/2} \delta}{M - \delta a} \sqrt{\frac{B}{p+1}} \right)^{-1/M} > 0,$$

we obtain $2T_1^{M-\delta a} > t^{M-\delta a}$, that is

$$2^{2/M} T_1 > 2^{1/(M-\delta a)} T_1 > t.$$  

Therefore the proof of the lemma is now completed.

We have one more lemma with a different assumption on the initial data.
Lemma 2.2 Assume that (2.4) is replaced by
\[ F(0) > 0, \quad F'(0) = 0 \] (2.8)
and additionally that there is a time \( t_0 > 0 \) such that
\[ F(t_0) \geq 2F(0). \] (2.9)
Then, the conclusion of Lemma 2.1 is changed to that there exists a positive constant \( C_0 = C_0(p, a, q, B) \) such that
\[ T < 2^{2/M} T_2 \] (2.10)
holds provided
\[ T_2 := \max \{ T_0, t_0, R \} \geq C_0 A^{-(p-1)/(2M)}. \] (2.11)

Remark 2.2 The statement of the original version of this lemma in Kato [6] is simply that \( T_0 \) is large and \( T < \infty \), but the assumption on the data is only \( F(0) \neq 0 \) and there is no restriction such as (2.9).

Proof of Lemma 2.2. It follows from (2.3) with \( t = 0 \) and (2.8) that \( F''(0) > 0 \) which implies \( F'(t) > F'(0) = 0 \) for small \( t > 0 \). Hence \( F''(t) \geq 0 \) for \( t \geq 0 \) from (2.3) yields that
\[ F'(t) > 0 \quad \text{and} \quad F(t) > F(0) > 0 \quad \text{for} \quad t > 0. \]
Hence, similarly to the proof of Lemma 2.1, we obtain that
\[ \frac{1}{2} F'(t)^2 \geq \frac{B}{p+1} (t + R)^{-q} F(t)^p \left\{ F(t) - F(0) \right\} \quad \text{for} \quad t > 0. \]
Then taking into account of (2.9) and \( F(t) \geq F(t_0) \) for \( t \geq t_0 \), we have that
\[ F'(t) > \sqrt{\frac{B}{p+1}} (t + R)^{-q/2} F(t)^{(p+1)/2} \quad \text{for} \quad t \geq t_0. \]
Therefore, after assuming \( t \geq T_2 \), one can readily establish the same argument as in the proof of Lemma 2.1 in which \( T_1 \) is replaced by \( T_2 \). □
3 Upper bound of the lifespan

In this section, we prove the expected theorem on the upper bound of the lifespan in high dimensional case.

Theorem 3.1 Let $1 < p < p_0(n)$ for $n \geq 2$. Assume that both $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ are non-negative and have compact support, and that $g$ does not vanish identically. Suppose that the problem (1.1) has a solution $(u, u_t) \in C([0, T(\varepsilon)), H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$ with

$$\text{supp}(u, u_t) \subset \{(x, t) \in \mathbb{R}^n \times [0, \infty) : |x| \leq t + R\}. \quad (3.1)$$

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(f, g, n, p, R)$ such that $T(\varepsilon)$ has to satisfy

$$T(\varepsilon) \leq C\varepsilon^{-2p/(p-1)} \gamma(p, n) \quad (3.2)$$

for $0 < \varepsilon \leq \varepsilon_0$, where $C$ is a positive constant independent of $\varepsilon$.

Remark 3.1 In view of (1.6), (3.2) is not optimal for $n = 2$ and $1 < p \leq 2$. This part can be covered by better estimate than (3.5) in Proposition 3.1 below. See Section 4. For the excluded case $n = 1$, see Section 5.

Proof of Theorem 3.1. In order to employ Lemma 2.1, let us set

$$F(t) = \int_{\mathbb{R}^n} u(x, t)dx.$$ 

Then, (2.4) immediately follows from the assumption of the theorem as

$$F(0) = \varepsilon \int_{\mathbb{R}^n} f(x)dx \geq 0, \quad F'(0) = \varepsilon \int_{\mathbb{R}^n} g(x)dx > 0. \quad (3.3)$$

For (2.3), we shall employ the same argument as (13)-(15) in Sideris [11]. Then it follows that

$$F''(t) = \int_{\mathbb{R}^n} |u(x, t)|^p dx \geq \frac{|F(t)|^p}{\left(\int_{|x|\leq t+R} 1dx\right)^{p-1}}.$$ 

This inequality means that (2.3) is available with

$$B = \left(\int_{|x|\leq 1} dx\right)^{1-p} > 0, \quad q = n(p-1) > 0. \quad (3.4)$$

For the key inequality (2.2), we employ the following proposition.
Proposition 3.1 Suppose that the assumption in Theorem 3.1 is fulfilled. Then, there exists a positive constant \( C_1 = C_1(f, g, n, p, R) \) such that \( F(t) = \int_{\mathbb{R}^n} u(x, t) \, dx \) satisfies

\[
F''(t) \geq C_1 \varepsilon^p t^{(n-1)(1-p/2)} \quad \text{for } t \geq R.
\]  

This estimate is valid also for the case where \( f(x) \geq 0(\neq 0) \) and \( g(x) \equiv 0 \).

Remark 3.2 This is a slightly modified estimate of (2.5') in Yordanov and Zhang [14] in which \( C_1 = 0 \) when \( f(x) \equiv 0 \).

Proof of Proposition 3.1. It follows from Lemma 2.2 in [14] that

\[
F_1(t) \geq \frac{1}{2}(1 - e^{-2R}) \int_{\mathbb{R}^n} \{ \varepsilon f(x) + \varepsilon g(x) \} \phi_1(x) \, dx \quad \text{for } t \geq R,
\]

where

\[
\phi_1(x) = \int_{S^{n-1}} e^{x \cdot \omega} d\omega \quad \text{and} \quad F_1(t) = \int_{\mathbb{R}^n} u(x, t) \phi_1(x) e^{-t} \, dx.
\]

Combining this estimate and (2.4) in [14], we immediately obtain (3.5). We note that there is no restriction on \( n \) in the argument.

Let us continue to prove the theorem. Integrating (3.5) in \( [R, t] \), we have

\[
F'(t) \geq C_1 \frac{\varepsilon^p t^{(n-1)(1-p/2)}}{n - (n-1)p/2} e^p \left(t^{n-(n-1)p/2} - R^{n-(n-1)p/2} \right) + F'(R)
\]

for \( t \geq R \) because of

\[
1 < p < p_0(n) < \frac{2n}{n-1} \quad \text{for } n \geq 2.
\]

Note that the same argument as in (2.7) and (2.8) yields \( F'(R) > 0 \). Hence we obtain that

\[
F'(t) > C_1 \frac{(1 - 2^{-n+(n-1)p/2})}{n - (n-1)p/2} \varepsilon^p t^{n-(n-1)p/2} \quad \text{for } t \geq 2R.
\]  

Integrating this inequality in \( [2R, t] \) together with \( F(0) \geq 0 \), we get

\[
F(t) > C_2 \varepsilon^p t^{n+1-(n-1)p/2} \quad \text{for } t \geq 4R,
\]

(3.7)
where
\[ C_2 = \frac{C_1}{\{n/(n-1)p/2\} \{n+1-(n-1)/2\}} > 0. \]

We are now in a position to apply our result here to Lemma 2.1 with special choices on all positive constants except for \( T_0 \) as
\[ A = C_2 \varepsilon^p, \quad B = \left( \int_{|x|\leq 1} dx \right)^{1-p}, \quad a = n + 1 - \frac{n-1}{2}p, \quad q = n(p-1) \]
which imply that (2.1) yields
\[ M = \frac{p-1}{2}a - \frac{q}{2} + 1 = \frac{\gamma(p,n)}{4} > 0. \]

If we set
\[ T_0 = C_0 A^{-(p-1)/(2M)} = C_0 C_2^{-2(p-1)/\gamma(p,n)} \varepsilon^{-2p(p-1)/\gamma(p,n)}, \tag{3.8} \]
we then find that there is an \( \varepsilon_0 = \varepsilon_0(f,g,n,p,R) > 0 \) such that
\[ T_0 \geq \max \left\{ \frac{F(0)}{F'(0)}, 4R \right\} \quad \text{for } 0 < \varepsilon \leq \varepsilon_0 \]
because \( F(0)/F'(0) \) does not depend on \( \varepsilon \). This means that \( T_1 = T_0 \) in (2.6). Therefore the conclusion of Lemma 2.1 implies that the maximal existence time \( T \) of \( F(t) \) has to satisfy
\[ T(\varepsilon) \leq T \leq C_3 \varepsilon^{-2p(p-1)/\gamma(p,n)} \quad \text{for } 0 < \varepsilon \leq \varepsilon_0, \]
where
\[ C_3 = 2^{4/\gamma(p,n)} C_0 C_2^{-2(p-1)/\gamma(p,n)} > 0. \]

The proof of the theorem is now completed. \( \square \)

**Theorem 3.2** Let \( 1 < p < p_0(n) \) for \( n \geq 2 \). Assume that \( f \in H^1(\mathbb{R}^n) \) is non-negative, does not vanish identically and \( g \equiv 0 \). Suppose that the problem (1.1) has a solution \((u, u_t) \in C([0,T(\varepsilon)), H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \) with (3.1). Then, there exists a positive constant \( \varepsilon_1 = \varepsilon_1(f, n, p, R) \) such that \( T(\varepsilon) \) has to satisfy
\[ T(\varepsilon) \leq C \varepsilon^{-2p(p-1)/\gamma(p,n)} \tag{3.9} \]
for \( 0 < \varepsilon \leq \varepsilon_1 \), where \( C \) is a positive constant independent of \( \varepsilon \).

**Remark 3.3** We have no new estimate for high dimensional case in this theorem, but it especially covers the optimality on \( n = 2 \) and \( p = 2 \). See (1.6).
Proof of Theorem 3.2. Note that Proposition 3.1 is also available in this case. Hence we have (3.7) again. Assume that
\[ C_2 \varepsilon t_0^{n+1-(n-1)p/2} = 2F(0) = 2\|f\|_{L^1(\mathbb{R}^n)} \varepsilon \quad \text{and} \quad t_0 \geq 4R. \]
Then (2.9) in Lemma 2.2 is fulfilled with
\[ t_0 = C_4 \varepsilon^{-(p-1)/(n-(n-1)p/2)} \]
if \( t_0 \geq 4R \), where
\[ C_4 = \left\{ \frac{2\|f\|_{L^1(\mathbb{R}^n)}}{C_2} \right\}^{(n+1-(n-1)p/2)^{-1}} > 0. \]

Since
\[ \frac{1}{n+1-(n-1)p/2} \leq \frac{2p}{\gamma(p,n)} \]
is equivalent to the trivial condition:
\[ p \geq \frac{2}{n+1}, \]
we obtain that there is an \( \varepsilon_1 = \varepsilon_1(f, n, p, R) > 0 \) such that
\[ T_0 \geq t_0 \quad \text{and} \quad t_0 \geq 4R \quad \text{for} \quad 0 < \varepsilon \leq \varepsilon_1 \]
with the same choice of \( T_0 \) as in (3.8). This means \( T_2 = T_0 \) in (2.11), so that the same conclusion as in Theorem 3.2 holds. \( \square \)

4 Note on 2 dimensional case

The optimal estimate in \( n = 2 \) and \( p = 2 \) in the case where \( g(x) \geq 0(\neq 0) \) is obtained by better estimate than (3.5) as announced in Remark 3.1.

Theorem 4.1 Let \( n = 2, 1 < p \leq 2 \) and \( f \equiv 0 \). Assume that \( g \in C^1(\mathbb{R}^2) \) is non-negative, does not vanish identically, and have compact support as \( \text{supp} \ g \subset \{x \in \mathbb{R}^2 : |x| \leq R\} \). Suppose that the integral equation associated with the problem (1.1) has a solution \( u \in C^1([0,T(\varepsilon)) \times \mathbb{R}^2) \) with \( \text{supp} \ u(x,t) \subset \{x \in \mathbb{R}^2 : |x| \leq t + R\} \). Then, there exists a positive constant \( \varepsilon_0 = \varepsilon_0(g, n, p, R) \) such that \( T(\varepsilon) \) has to satisfy
\[ T(\varepsilon) \leq \begin{cases} Ca(\varepsilon) & \text{when} \ p = 2, \\ C\varepsilon^{-(p-1)/(3-p)} & \text{when} \ 1 < p < 2 \end{cases} \]
for \( 0 < \varepsilon \leq \varepsilon_2 \), where \( a(\varepsilon) \) is defined in (1.7) and \( C \) is a positive constant independent of \( \varepsilon \).
Remark 4.1 (4.1) is a weaker result than (1.6) in the sense that the constant $C$ is not of the optimal choice and that the assumption on the data is stronger than Lindblad [8]. Moreover, we note that the optimality of (4.1) for $1 < p < 2$ is still open, but may true. Because

$$\frac{1}{3-p} < \frac{2p}{\gamma(p, 2)}$$

is equivalent to $1 < p < 2$. But it is out of the main purpose of this paper. So we shall take another opportunity to study this part.

Proof of Theorem 4.1. Under the assumption of the theorem, it is well-known that our integral equation is of the form,

$$u(x, t) = \frac{\varepsilon}{2\pi} \int_{|x-y| \leq t} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy + \frac{1}{2\pi} \int_0^t d\tau \int_{|x-y| \leq t-\tau} \frac{|u(y, \tau)|^p}{\sqrt{(t-\tau)^2 - |x-y|^2}} dy. \quad (4.2)$$

Note that $|y| \leq R$ and $|x| \leq t + R$ due to the support property. Then, neglecting the second term in the right-hand side and making use of inequalities;

$$t - |x - y| \leq t - ||x| - |y|| \leq t - |x| + R \quad \text{for } |x| \geq R,$$

$$t + |x - y| \leq t + |x| + R \leq 2(t + R),$$

we obtain that

$$u(x, t) \geq \frac{\varepsilon}{2\sqrt{2\pi t + R} \sqrt{t - |x| + R}} \int_{|x-y| \leq t} g(y) dy \quad \text{for } |x| \geq R.$$ If we assume $|x| + R \leq t$ which implies $|x - y| \leq t$ for $|y| \leq R$, we get

$$\int_{|x-y| \leq t} g(y) dy = \|g\|_{L^1(\mathbb{R}^2)}.$$ Therefore we have that

$$u(x, t) \geq \frac{\|g\|_{L^1(\mathbb{R}^2)}}{2\sqrt{2\pi t + R} \sqrt{t - |x| + R}} \varepsilon \quad \text{for } R \leq |x| \leq t - R. \quad (4.3)$$

It follows from the same argument as in high dimensional case that $F(t) = \int_{\mathbb{R}^2} u(x, t) dx$ satisfies

$$F''(t) = \int_{\mathbb{R}^2} |u(x, t)|^p dx \geq \int_{R \leq |x| \leq t-R} |u(x, t)|^p dx \quad \text{for } t \geq 2R.$$
Plugging (4.3) into the right-hand side of this inequality, we have that

\[ F''(t) \geq \left( \frac{\|g\|_{L^1(R^2)}}{2\sqrt{2}\pi t + R} \right)^p \int_{R \leq |x| \leq t - R} \frac{1}{(t - |x| + R)^{p/2}} dx, \]

that is

\[ F''(t) \geq \frac{2\pi \|g\|^p_{L^1(R^2)}}{(2\sqrt{2}\pi)^p (t + R)^{p/2}} \int_{R}^{t-R} \frac{r}{(t - r + R)^{p/2}} dr \quad \text{for } t \geq 2R. \quad (4.4) \]

**Case of** \( p = 2. \) Making use of integration by parts, we have that

\[ \int_{R}^{t-R} \frac{r}{t - r + R} dr = R \log t - (t - R) \log 2R + \int_{R}^{t-R} \log(t - r + R) dr. \]

Without loss of the generality, we may assume that

\[ R \geq 1. \quad (4.5) \]

Note that \( (t - R)/2 \geq R \) is equivalent to \( t \geq 3R. \) Hence, diminishing the domain of the integral to \( [R, (t - R)/2] \), we get that

\[ \int_{R}^{t-R} \log(t - r + R) dr \geq \left( \frac{t - R}{2} - R \right) \log \frac{t + 3R}{2} \quad \text{for } t \geq 3R. \]

Therefore we obtain that

\[ \int_{R}^{t-R} \frac{r}{t - r + R} dr \geq R \left( \log t - \log \frac{t + 3R}{2} \right) + (t - R) \left( \frac{1}{2} \log \frac{t + 3R}{2} - \log 2R \right) \quad \text{for } t \geq 3R. \]

Then it follows from this inequality,

\[ \log \frac{t + 3R}{2} \leq \log t \quad \text{for } t \geq 3R \]

and

\[ \frac{1}{4} \log \frac{t + 3R}{2} \geq \log 2R \quad \text{for } t \geq 16R^4 - 3R \]

that

\[ \int_{R}^{t-R} \log(t - r + R) dr \geq \frac{1}{6} t \log \frac{t + 3R}{2} \quad \text{for } t \geq 16R^4 - 3R \]
because (4.5) implies $16R^4 - 3R \geq 3R$. Combining this estimate with (4.4), one can obtain that

$$F''(t) \geq \frac{\|g\|_{L^1(R^2)}^2}{48\pi} \varepsilon^2 \log \frac{t}{2} \quad \text{for } t \geq 16R^4.$$ 

This is a better estimate than (3.5) as the extra factor $\log(t/2)$ appears, which leads to the optimal order of the lifespan in this case as follows.

Integrating the inequality above in $[16R^4, t]$ and making use of $F'(0) = \|g\|_{L^1(R^2)} \varepsilon > 0$, we get

$$F'(t) > \frac{\|g\|_{L^1(R^2)}^2}{96\pi} \varepsilon^2 \int_{t/2}^{t} \log \frac{s}{2} ds \geq \frac{\|g\|_{L^1(R^2)}^2}{256\pi} \varepsilon^2 t \log \frac{t}{4} \quad \text{for } t \geq 32R^4.$$ 

The same procedure with $F(0) = 0$ in $[32R^2, t]$ gives us that

$$F(t) > \frac{\|g\|_{L^1(R^2)}^2}{96\pi} \varepsilon^2 \int_{t/2}^{t} s \log \frac{s}{4} ds \geq \frac{\|g\|_{L^1(R^2)}^2}{256\pi} \varepsilon^2 t^2 \log \frac{t}{8} \quad \text{for } t \geq 64R^4.$$ 

We are now in a position to apply our situation to Lemma 2.1 with

$$a = p = q = 2, B = \pi^{-1}$$

which imply $M = 1$. If we can make constants $A$ and $T_0$ to satisfy

$$\frac{\|g\|_{L^1(R^2)}^2}{256\pi} \varepsilon^2 \log \frac{T_0}{8} \geq A \geq T_0^{-2}C_0^2 \quad \text{for } t \geq T_0 \geq \max \left\{ \frac{F(0)}{F'(0)}, 64R^2 \right\},$$

then we find that (2.2) and (2.6) are automatically satisfied with $T_1 = T_0$.

For this possibility, one can set

$$\frac{T_0}{16} = a(\varepsilon) \quad \text{and} \quad A = T_0^{-2}C_0^2,$$

where $a(\varepsilon)$ is defined in (1.7) provided

$$\frac{\|g\|_{L^1(R^2)}^2}{\pi C_0^2} \geq 1.$$ 

Because there exists a positive constant $\varepsilon_2 = \varepsilon_2(g, R)$ such that

$$a(\varepsilon) \geq \max \left\{ \frac{F(0)}{16F'(0)}, 4R^2 \right\} \quad \text{for } 0 < \varepsilon \leq \varepsilon_2,$$
due to the fact that $a(\varepsilon)$ is monotonously decreasing function of $\varepsilon$ and
\[ \lim_{\varepsilon \to +0} a(\varepsilon) = \infty. \]
This gives us
\[ \frac{T_0}{8} = 2a(\varepsilon) \geq 4R^4 + a(\varepsilon) \geq 1 + a(\varepsilon) \quad \text{for } 0 < \varepsilon \leq \varepsilon_2. \]
On the counter part,
\[ \frac{\|g\|_{L^1(R^2)}^2}{\pi C_0^2} \leq 1, \]
one can set
\[ \frac{T_0}{16} = a(\varepsilon) \quad \text{and} \quad A = T_0^{-2}C_0^2 \]
because it also gives us
\[ \frac{T_0}{8} = 2\sqrt{\pi C_0} - a(\varepsilon) \geq 2a(\varepsilon) \geq 1 + a(\varepsilon) \quad \text{for } 0 < \varepsilon \leq \varepsilon_2. \]
Therefore it follows from Lemma 2.1 that
\[ T(\varepsilon) \leq 32 \max\left\{ 1, \frac{\sqrt{\pi C_0}}{\|g\|_{L^1(R^2)}} \right\} a(\varepsilon) \quad \text{for } 0 < \varepsilon \leq \varepsilon_2. \]

**Case of $1 < p < 2$.** Turning back to (4.4), we find that
\[ \int_{t-R}^{t-R} \frac{r}{(t - r + R)^{p/2}} dr \geq \frac{1}{2t^{p/2}} \left\{ (t - R)^2 - R^2 \right\}. \]
Hence it follows that
\[ \int_{t-R}^{t-R} \frac{r}{(t - r + R)^{p/2}} dr \geq \frac{1}{6} t^{2-p/2} \quad \text{for } t \geq 3R \]
which yields
\[ F''(t) \geq \frac{\|g\|_{L^1(R^2)}^p}{3 \cdot 2^{3/p-1}} \varepsilon^{p/2-p} \quad \text{for } t \geq 3R. \]
We note that this is a better estimate than (3.5) because
\[ 1 - \frac{p}{2} < 2 - p \]
is equivalent to $p < 2$. Integrating this inequality in $[3R, t]$ and making use of $F'(0) = \|g\|_{L^1(R^2)}\varepsilon > 0$, we get
\[ F'(t) > \frac{\|g\|_{L^1(R^2)}^p(1 - (3/4)^3-p)}{3(3 - p)2^{3/p-1}} \varepsilon^{p/2-p} \quad \text{for } t \geq 4R. \]
Hence we obtain that

\[ F(t) > C_5 \varepsilon^p t^{4-p} \quad \text{for} \quad t \geq 5R, \]

where

\[ C_5 = \frac{\|g\|_{L^1(\mathbb{R})}^p (1 - (3/4)^{3-p})(1 - (4/5)^{4-p})}{3(3-p)(4-p)2^p \pi^{p-1}} > 0. \]

We are in a position to apply our situation to Lemma 2.1 with

\[ A = C_5 \varepsilon^p, \quad B = \pi^{1-p}, \quad a = 4-p, \quad q = 2(p-1) \]

which imply

\[ M = \frac{p-1}{2}(4-p) - \frac{2(p-1)}{2} + 1 = \frac{p(3-p)}{2} > 0. \]

Therefore the theorem follows from setting

\[ T_0 = C_0 A^{-(p-1)/(2M)} = C_0 C_5^{-(p-1)/(p(3-p))} \varepsilon^{-(p-1)/(3-p)} \]

as before. \( \square \)

5 Note on one dimensional case

The optimal estimate in \( n = 1 \) in the case where \( g(x) \geq 0(\neq 0) \) is also obtained by better estimate than (3.5) as announced in Remark 3.1.

**Theorem 5.1** Let \( p > 1 \) for \( n = 1 \). Assume that both \( f \in C^2(\mathbb{R}) \) and \( g \in C^1(\mathbb{R}) \) have compact support as \( \text{supp} (f, g) \subset \{ x \in \mathbb{R} : |x| \leq R \} \). Suppose that the problem (1.1) has a solution \( u \in C^2([0, T(\varepsilon)) \times \mathbb{R}) \). Then, there exists a positive constant \( \varepsilon_0 = \varepsilon_0(f, g, n, p, R) \) such that \( T(\varepsilon) \) has to satisfy

\[ T(\varepsilon) \leq C \varepsilon^{-(p-1)/2} \quad \text{if} \quad \int_{\mathbb{R}} g(x)dx > 0, \]

\[ T(\varepsilon) \leq C \varepsilon^{-p(p-1)/(p+1)} \quad \text{if} \quad f \geq 0(\neq 0), \quad g \equiv 0 \]

for \( 0 < \varepsilon \leq \varepsilon_0 \), where \( C \) is a positive constant independent of \( \varepsilon \).

**Remark 5.1** The assumption of this theorem is stronger than Zhou [15]. But for the sake of completeness of applications of our lemma, we shall prove it here.
Proof of Theorem 5.1. First, we note that the assumption on the support of the initial data implies that the solution $u \in C^2(\mathbb{R} \times [0, T(\varepsilon))]$ of (1.1) has to satisfy

$$\text{supp } u(x, t) \subset \{ x \in \mathbb{R} : |x| \leq t + R \}.$$  

Hence, integrating the equation in $\mathbb{R}$, we have that $F(t) = \int_{\mathbb{R}} u(x, t) dx$ satisfies

$$F''(t) = \int_{\mathbb{R}} |u(x, t)|^p dx = \int_{-(t+R)}^{-t} |u(x, t)|^p dx$$  \hspace{1cm} (5.2)

which yields

$$F''(t) \geq 2^{1-p}(t + R)^{1-p} |F(t)|^p \quad \text{for } t \geq 0.$$  \hspace{1cm} (5.3)

It is well-known that $u$ has a representation formula of the form,

$$u(x, t) = \frac{f(x + t) + f(x - t)}{2} + \varepsilon \int_{x-t}^{x+t} g(\xi) d\xi + \frac{1}{2} \int_{t}^{t+\tau} d\tau \int_{x-t}^{x-t+\tau} |u(\xi, \tau)|^p d\xi.$$  \hspace{1cm} (5.4)

Case of $\int_{\mathbb{R}} g(x) dx > 0$. In view of (5.4), we find that the support condition implies

$$u(x, t) \geq G\varepsilon \quad \text{for } x + t \geq R \text{ and } x - t \leq -R,$$

where

$$G = \frac{1}{2} \int_{\mathbb{R}} g(x) dx > 0.$$  

Plugging this estimate into (5.2), we have that

$$F''(t) \geq (G\varepsilon)^p \int_{0}^{t-R} dx = G^p \varepsilon^p (t - R) \quad \text{for } t \geq R.$$  

This is a better estimate than (3.5). Integrating this inequality twice in $[R, t]$ and making use of

$$F'(R) \geq F'(0) > 0, \quad F(R) > F(0) \geq 0,$$

we obtain that

$$F(t) > \frac{G^p}{6} \varepsilon^p (t - R)^3 \quad \text{for } t \geq R$$

which implies that

$$F(t) > \frac{G^p}{48} \varepsilon^p t^3 \quad \text{for } t \geq 2R.$$
In view of this estimate and (5.3), we are in a position to apply our situation to Lemma 2.1 with
\[ A = \frac{G^p}{48} \varepsilon^p, \quad B = 2^{1-p}, \quad a = 3, \quad q = p - 1. \]
Noticing that
\[ M = \frac{p - 1}{2} - 3 - \frac{p - 1}{2} + 1 = p, \]
once can set
\[ T_0 = C_0 A^{-(p-1)/(2M)} = C_0 \frac{G^p}{48} \varepsilon^{-(p-1)/2}. \]
Therefore the first conclusion of the theorem follows.

Case of \( f \geq 0 (\not\equiv 0), g \equiv 0 \). In view of (5.4), we find that the support condition implies
\[ u(x, t) \geq \frac{f(x - t)}{2} \varepsilon \quad \text{for} \quad x + t \geq R \quad \text{and} \quad -R \leq x - t \leq R. \]
Plugging this estimate into (5.2), we have that
\[ F''(t) \geq \frac{\varepsilon^p}{2p} \int_{t-R}^{t+R} f(x - t)^p dx = \frac{\|f\|_{L^p(R)}^p \varepsilon^p}{2^p} \quad \text{for} \quad t \geq R. \]
This is the same estimate as (3.5). Therefore the second conclusion of the theorem immediately follows by same manner in the proof of Theorem 3.2 because of \( \gamma(p, 1) = 2 + 2p \).

Acknowledgment. The author is grateful to members of a private seminar held at Future University Hakodate on 24-25 Jan. 2015, Professors A. Hoshiga, K. Yokoyama, Y. Kurokawa and Dr. Wakasa, for discussions which lead to improvement of the assumption of Lemma 2.2.

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