Gaussian approximation of the (2+1) dimensional Thirring model in the functional Schrödinger picture

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Abstract

The (2+1)-dimensional Thirring model is studied by using the Gaussian approximation method in the functional Schrödinger picture. Although the dynamical symmetry breaking does not occur in the large $N$ limit, it does occur in the Gaussian approximation which includes the higher order contributions in $1/N$.

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I. INTRODUCTION

The Gaussian variational method in the functional Schrödinger-picture has been shown
to be useful for the study of quantum structures of field theories [1]. This method has been
shown to be well-suited not only for the study of bosonic field theories, but also for the study
of fermionic field theories [2]. The Gaussian method is non-perturbative from the viewpoint
of both the ordinary weak coupling expansion and also the 1/N expansion, and hence it can
give the better informations than the large-N expansion in principle, although it has not
been realized yet. It is the purpose of this paper to give an example where the Gaussian
approximation method provides better information than the large N approximation.

In the Floreanini-Jackiw representation the fermion field operators [3], which satisfy the
anticommutation relations

$$\{\psi_i(x, t), \psi^\dagger_j(y, t)\} = i\delta_{ij}\delta(x, y), \quad (1)$$

are realized as

$$\psi(x) = \frac{1}{\sqrt{2}}[u(x) + \frac{\delta}{\delta u^\dagger(x)}], \quad (2)$$

$$i\psi^\dagger(x) = i\frac{1}{\sqrt{2}}[u^\dagger(x) + \frac{\delta}{\delta u(x)}]$$

with anticommuting Grassmann variables $u$ and $u^\dagger$. For the variational approximation, we
take the trial wavefunctional in the Gaussian form

$$|\Psi| \rightarrow |G| = \frac{1}{(\text{det} G)^{1/4}} \exp[\int_{x,y} u^\dagger(x) G(x, y) u(y)], \quad (3)$$

$$<\Psi | \rightarrow <G| = \frac{1}{(\text{det} G)^{1/4}} \exp[\int_{x,y} u^\dagger(x) \overline{G}(x, y) u(y)], \quad (4)$$

where

$$\overline{G} = (G^\dagger)^{-1}. \quad (5)$$

This prescription has been used in [2] to study the Gross-Neveu model. In the present
paper, we apply this Gaussian variational method to study some aspects of the $(2 + 1)$-
dimensional Thirring model. In the models considered so far, the Gaussian variational
method gives qualitatively the same results as the large $N$ approximation. We will show that the Gaussian approximation method applied to the (2+1)-dimensional Thirring model provides new informations that cannot be obtained in the large $N$ limit.

The massless Thirring model has, at the classical level, parity and chiral symmetry in two and four component representations of fermions, respectively. Recently it has been shown [4,5] that in this model, dynamical breaking of parity (chiral) symmetry occurs as a cooperative effect of different orders in $1/N$ expansion. In this paper we consider this problem in the context of Gaussian approximation scheme. It is shown that the symmetry breaking occurs in the Gaussian approximation although it does not in the large N limit. This agrees qualitatively with the results of [4,5].

II. DYNAMICAL SYMMETRY BREAKING IN TWO-COMPONENT REPRESENTATION

A. Schrödinger picture Gaussian approximation

The (2+1)-dimensional massless Thirring model is described by the Lagrangian density

$$\mathcal{L}_0 = i\bar{\psi}_a \gamma^\mu \psi_a - \frac{g}{2N} (\bar{\psi}_a \gamma^\mu \gamma^\nu \psi_a) (\bar{\psi}_b \gamma_\mu \psi_b),$$  \hspace{1cm} (6)

where $a = 1, 2, \cdots, N$. In (2+1) dimensions the $\gamma$ matrices for two-component representations of fermions can be represented by the $2 \times 2$ Pauli matrices,

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2.$$  \hspace{1cm} (7)

As is well-known, the model has parity invariance at the classical level, under which the fermions transform as

$$\psi(x^0, x^1, x^2) \rightarrow \gamma^1 \psi(x^0, -x^1, x^2).$$  \hspace{1cm} (8)

The mass term $\bar{\psi}\psi$ changes sign under the transformation and thus breaks the symmetry.
In order to compute the effective potential and see if dynamical symmetry breaking occurs, we introduce the vacuum condensate of fermion bilinear,
\[ \sigma_a \equiv -\frac{g}{2} \langle \overline{\psi}_a \psi_a \rangle, \tag{9} \]
where \( a \) is a color index, and compute the effective potential as a function of \( \sigma_a \). One can achieve this by writing the Lagrangian as
\[ \mathcal{L} \equiv \mathcal{L}_0 + \alpha_a (\sigma_a + \frac{g}{2} \overline{\psi}_a \psi_a). \tag{10} \]

By using the Gaussian trial wave functional (3) and (4), we calculate the vacuum expectation value of the Hamiltonian of the model,
\[ \langle G|H|G \rangle = \frac{1}{2} \int d^2x d^2y Tr[(-i\gamma^0 \gamma^i \partial_i(x, y) + \frac{g}{2N} \delta(x, y) \gamma^0 \gamma^\mu \gamma^0 \gamma_\mu + \frac{g}{2} \alpha_a(x) \delta(x, y) \gamma^0 \Omega(y, x)] \]
\[ - \frac{g}{8N} Tr[\gamma^0 \gamma^\mu \Omega(x, y) \gamma^0 \gamma^\mu \Omega(y, x) \delta(x, y)] \]
\[ + \int d^2x \{ \frac{g}{8N} Tr[\gamma^0 \gamma^\mu \Omega(x, x)] Tr[\gamma^0 \gamma_\mu \Omega(x, x)] + \alpha_a(x) \sigma_a(x) \}, \tag{11} \]
where
\[ \Omega(x, y) \equiv 2 \langle G|\psi \psi^\dagger|G \rangle = \langle x | (1 + G) S^{-1} (1 + \overline{G}) | y \rangle, \tag{12} \]
\[ S \equiv G + \overline{G}, \tag{13} \]
\[ \partial_i(x, y) \equiv \frac{\partial}{\partial x^i} (\delta(x^i - y^i)), \tag{14} \]
and \( Tr \) denotes the trace over Dirac spinor and color indices. In order to calculate the Gaussian approximation, it is convenient to introduce the current \( A_\mu \equiv \frac{g}{2\sqrt{N}} \langle \overline{\psi} \gamma_\mu \psi \rangle \),
which can be realized by introducing the Lagrange’s multiplier into Hamiltonian,
\[ H_{eff} \equiv \frac{1}{2} \int d^2x d^2y Tr[h_\Omega(x, y) \Omega(y, x)] \]
\[ + \int d^2x \{ \frac{1}{2g} A_\mu(x) A_\mu(x) + \beta_\mu(A_\mu(x) - \frac{g}{2\sqrt{N}} Tr[\gamma^0 \gamma^\mu \Omega(x, x)]) + \alpha_a \sigma_a \}, \tag{15} \]
where
\[ h_\Omega(x, y) \equiv -i \gamma^0 \gamma^i \partial_i(x, y) \]
\[ + \frac{g}{4N} \gamma^0 \gamma^\mu (2I(x, y) - \Omega(x, y)) \delta(x, y) \gamma^0 \gamma_\mu + \frac{g}{2} \alpha_a(x) \delta(x, y) \gamma^0. \tag{16} \]
The effective potential is obtained by minimizing \( H_{eff} \) with respect to the kernel \( G(x, y) \).
B. Effective Potential in the Large N Limit

Since the second term in the $h_\Omega$ of Eq\((\text{16})\) is of $1/N$ order, the effective potential in the large N limit is obtained by considering

$$H_{eff} = \frac{1}{2} Tr h N \Omega + \frac{1}{2g} A^\mu A_\mu + \beta^\mu A_\mu + \alpha a \sigma_a,$$  \hspace{1cm} (17)

where

$$h_N \equiv -i\gamma^0 \gamma^i \partial_i (x, y) + \frac{g}{2} \alpha_a (x) \delta(x, y) \gamma^0 - \frac{g}{2\sqrt{N}} \beta_\mu(x) \gamma^0 \gamma^\mu \delta(x, y).$$  \hspace{1cm} (18)

In order to obtain the effective potential of the system, we take variations on $<G | H | G>$ with respect to $G$ and $\overline{G}$. The invariance of $<G | H | G>$ under these variations yields the equations

$$(I - G)h_N(I + G) = 0$$  \hspace{1cm} (19)

$$(I + \overline{G})h_N(I - \overline{G}) = 0.$$  \hspace{1cm} (20)

These two conditions are shown to be equivalent under the condition of (3), and can be solved by the well-known method [2]. The Eq.\((\text{19})\) can be rewritten in the form

$$h_N^2 - K_N^2 + [h, K_N] = 0$$  \hspace{1cm} (21)

where $h_N$ and $K_N \equiv h_N G_N$ are $2 \times 2$ matrices in the Dirac spinor space. Any $2 \times 2$ matrix can be expressed as a linear combination of $\Gamma^a$:

$$\Gamma^0 = I, \quad \Gamma^1 = -i\gamma^1, \quad \Gamma^2 = -i\gamma^2, \quad \Gamma^3 = \gamma^0.$$  \hspace{1cm} (22)

$K_N$ and $h_N$ is then decomposed as

$$K_N(x, y) = \sum_{a=0}^3 \Gamma^a \int \frac{d^2p}{(2\pi)^2} e^{-ip \cdot (x-y)} K_{Na}(p)$$  \hspace{1cm} (23)

$$h_N(x, y) = \int \frac{d^2p}{(2\pi)^2} e^{-ip \cdot (x-y)} [-g\beta_0 \Gamma^0$$

$$- (p_2 + g\beta_2) \Gamma^1 + (p_1 + g\beta_1) \Gamma^2 + g\alpha_a \Gamma^3].$$  \hspace{1cm} (24)
The Eqs. (23), (24) and (21) yield

\[ K_{N_0}^2 + K_{N_i} K_{N_i} = (p_1 + g \beta_1)^2 + (p_2 + g \beta_2)^2 + g^2 \alpha_\alpha^2 + g^2 \beta_0^2, \]

\[ K_{N_0} K_{N_1} = g \beta_0 (p_2 + g \beta_2) + i(p_1 + g \beta_1) K_{N_3} - ig \alpha_a K_{N_2}, \]

\[ K_{N_0} K_{N_2} = -g \beta_0 (p_1 + g \beta_1) + i(p_2 + g \beta_2) K_{N_3} + ig \alpha_a K_{N_1}, \]

\[ K_{N_0} K_{N_3} = -g^2 \beta_0 \alpha_a - i(p_2 + g \beta_2) K_{N_2} - i(p_1 + g \beta_1) K_{N_1}, \] (25)

where the summation convention for the index \( i(=1,2,3) \) is implied. The Eq. (25) has the non-trivial solutions

\[ K_{N_0} = \pm \sqrt{h_i h_i}, \]

\[ K_{N_i} = h_0 h_i / K_{N_0} \] (26)

as well as the trivial solution \( K_{N_0} = \pm g \beta_0 \), and this gives the solution for the kernel

\[ G_N(x, y) = (h_N^{-1} K_N)(x, y) \]

\[ = \pm \int \frac{d^2p}{(2\pi)^2} e^{-ip(x-y)} \frac{1}{\sqrt{h_i h_i}} h_j \Gamma^j. \] (27)

Following the similar procedure as in the case of Gross-Neveu model [2], the effective potential \( V_{eff} \), which is defined by

\[ H_{eff} \equiv \int d^2x V_{eff}, \] (28)

can be determined as

\[ V_{eff} = N \int d^2x \left[ -\int \frac{d^2p}{(2\pi)^2} \left( g \beta_0 + \sqrt{-(p_1 + g \beta_1)^2 - (p_2 + g \beta_2)^2 + g^2 \alpha^2} \right) \right. \]

\[ + \frac{A^2}{2Ng} + \frac{1}{N} \beta^\mu A_\mu + \alpha \sigma \right] \]

\[ = -\frac{gN}{2\pi} I_1 \beta_0 - \frac{N}{2\pi} I_2 - \frac{N}{4\pi} g^2 \alpha^2 I_0(M) + \frac{N}{6\pi} g^3 | \alpha |^3 \]

\[ + \frac{1}{2g} A^\mu A_\mu + \beta^\mu A_\mu + N\alpha \sigma, \] (29)

where
Here we have taken the solution with minus sign in Eq.(26) and (27), which corresponds to lowest energy. Since we want to see whether dynamical symmetry breaking occurs, the auxiliary field variables $A_\mu$ in Eq.(30) should be eliminated via variational method. If we neglect the irrelevant infinity, the effective potential becomes

$$V_{eff} = -\frac{N}{4\pi} g^2 \alpha^2 I_0(M) + \frac{N}{6\pi} g^3 |\alpha|^3 + N\alpha \sigma.$$  \hspace{1cm} (32)

By eliminating $\sigma$, the effective potential can be written as

$$V_{eff} = -\frac{N}{4\pi} g^2 \alpha^2 (I_0(M) - \frac{4}{3} g |\alpha|).$$  \hspace{1cm} (33)

This effective potential can easily be renormalized in the form,

$$V_{eff} = -\frac{N}{4\pi} g_r^2 \alpha^2,$$  \hspace{1cm} (34)

where $g_r^2 \equiv g^2 I_0(M)$ is the renormalized coupling constant. This implies that the symmetry breaking does not occur in the large $N$ limit.

\textbf{C. Beyond large N limit}

To compute the full Gaussian effective potential, it is convenient to rewrite the original effective hamiltonian density $\mathcal{H}$ in the form

$$H_{eff} = \frac{1}{2} Tr[h\Omega] + \frac{g}{8N} Tr[\gamma^0 \gamma^\mu \Omega \gamma^0 \gamma^\mu \Omega] + \frac{1}{2g} A^\mu A_\mu + \beta_\mu A^\mu + \alpha_\alpha \sigma.$$  \hspace{1cm} (35)

where

$$h \equiv h_N + \frac{g}{2N} \gamma^0 \gamma^\mu (I - \Omega(x,x)) \gamma^0 \gamma_\mu.$$  \hspace{1cm} (36)

Taking variations on $H_{eff}$ with respect to $G$ and $\bar{G}$ yield
\[(I - G)h(I + G) = 0. \quad (37)\]

This equation has the same form as Eq. (21) with \(h_N\) replaced by \(h\). Therefore we can solve

\[h^2 - K^2 + [h, K] = 0 \quad (38)\]

where \(K \equiv hG\). We can follow the same procedure as in the case of large \(N\)-limit and obtain the nontrivial solution for \(K\)

\[
\tilde{K}(p) = -\frac{1}{\sqrt{h_i h_i}} (h_j h_j \Gamma^0 + h_0 h_j \Gamma^j),
\]

and

\[
\tilde{G}(p) = -\frac{1}{\sqrt{h_i h_i}} h_j \Gamma^j,
\]

which corresponds to the minimum energy solution. We also obtain the solution for \(\Omega\), \(\Omega = I + G\), from the relations among \(G, G^\dagger, K, K^\dagger, h,\) and \(h^\dagger\). Then using the Eq. (36) one finds

\[
h(x, y) = \int \frac{d^2 p}{(2\pi)^2} e^{-ip(x-y)} \left[ -\frac{g\beta_0}{2\sqrt{N}} \Gamma^0 + (-p_2 + \frac{g\beta_2}{2\sqrt{N}} - \frac{g}{2N} G_1(0)) \Gamma^1 
+ (p_1 - \frac{g\beta_1}{2\sqrt{N}} - \frac{g}{2N} G_2(0)) \Gamma^2 + (\frac{g}{2} - \frac{3g}{2N} G_3(0)) \Gamma^3 \right],
\]

where

\[
G_i(0) = \int \frac{d^2 p}{(2\pi)^2} \tilde{G}_i(p).
\]

From Eq. (41) and (41), we obtain the consistency condition

\[
G_i(0) = -\int \frac{d^2 p}{(2\pi)^2} \frac{1}{\sqrt{h_i h_j}} h_i,
\]

and therefore,

\[
G_1(0) = G_2(0) = 0,
\]

\[
G_3(0) = -m \int \frac{d^2 p}{(2\pi)^2} [(p_1 - \frac{g\beta_1}{2\sqrt{N}})^2 + (p_2 - \frac{g\beta_2}{2\sqrt{N}})^2 + m^2]^{-\frac{1}{2}}
= -\frac{m}{2\pi} (I_0 - m),
\]

(44)
where

\[ m \equiv \frac{g}{2} \alpha - \frac{3g}{2N} G_3(0). \] (45)

Extremizing the effective potential with respect to \( G_3(0) \), we obtain

\[ V_{\text{eff}} = -\frac{N}{4\pi} I_0(M)m^2 + \frac{N}{6\pi} |m|^3 + \frac{g}{12} N^2 \alpha^2 + \frac{N^2}{3g} m^2 \]

\[ -\frac{N^2}{3} \alpha m + N\alpha \sigma \] (46)

Eq. (45) can then be written as

\[ \alpha - \frac{2m}{g} + 3m2\pi N(I_0 - m) = 0, \] (47)

which can be understood as \( \frac{\partial}{\partial m} V_{\text{eff}} = 0 \).

By eliminating \( \alpha \) by \( \frac{\partial}{\partial \alpha} V_{\text{eff}} = 0 \) and \( \sigma \) by \( \frac{\partial}{\partial \sigma} V_{\text{eff}} = 0 \), the effective potential becomes

\[ V_{\text{eff}} = \frac{N}{3g}(N - \frac{3I_0}{4\pi} g)m^2 + \frac{N}{6\pi} |m|^3, \] (48)

which needs to be renormalized. \( V_{\text{eff}} \) can be made finite by defining the renormalized coupling constant \( g_r \) by

\[ -\frac{1}{g_r} \equiv \frac{1}{g}(N - \frac{3I_0}{4\pi} g). \] (49)

Then, the renormalized effective potential can be written as

\[ V_{\text{eff}} = -\frac{N}{3g_r} m^2 + \frac{N}{6\pi} |m|^3. \] (50)

This implies that the symmetry breaking occurs if \( g_r \geq 0 \). Since \( \sigma = Nm/3 \) in the extremum, the second term of Eq. (50) is smaller than the first term by \( 1/N \). Eq. (50) shows, in a straightforward manner, why symmetry breaking phenomenon cannot be seen in the large \( N \) limit.
III. DYNAMICAL SYMMETRY BREAKING IN FOUR-COMPONENT REPRESENTATION

To restore parity symmetry even with the explicit mass term, we consider four-component representation of Dirac spinor. This can be achieved by considering a doublet of two component spinors,

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

for which we define the $4 \times 4$ $\gamma$-matrices

$$\gamma^1 = i \Gamma_{13} = i \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad \gamma^2 = i \Gamma_{23}, \quad \gamma^0 = i \Gamma_{33}, \quad \gamma^5 \gamma_5 = \Gamma_{02}. \quad (52)$$

where $\Gamma_{ij} \equiv \sigma_i \otimes \sigma_j$. In this representation, mass term does not spoil parity symmetry, but break the continuous $U(1) \times U(1)$ chiral symmetry \cite{6},

$$\psi_L \rightarrow e^{i\alpha_L} \psi_L, \quad \psi_R \rightarrow e^{i\alpha_R} \psi_R, \quad (53)$$

down to the $U(1)$ subgroup.

Thus the existence of mass term signals the breaking of such chiral symmetry. We adopt the same procedure, as in two-component case, to obtain the effective potential, and introduce two vacuum condensates of fermion bilinear

$$\sigma \equiv -\frac{g}{2} < \bar{\psi} \psi >, \quad \pi \equiv \frac{ig}{2} < \bar{\psi} \gamma_5 \psi >. \quad (54)$$

Then the effective Hamiltonian can be written as

$$H_{\text{eff}} = \frac{1}{2} Tr h \Omega + \frac{1}{2Ng} A^\mu A_\mu + \beta^\mu A_\mu + \alpha_a \sigma_a + \tau_a \pi_a \quad (55)$$

where

$$h = -i \gamma^0 \gamma^\mu \partial_\mu(x,y) + \frac{g}{2} \alpha(x) \delta(x,y) \gamma^0 - \frac{g}{2\sqrt{N}} \beta_\mu \gamma^0 \gamma^\mu$$

$$+ \frac{g}{2N} \gamma^0 \gamma^\mu (1 - \Omega(x,x)) \gamma^0 \gamma_\mu - \frac{ig}{2} \gamma^0 \gamma_5. \quad (56)$$
By following the same procedure as in the two component case, we obtain the effective potential

$$V_{eff} = -\frac{3}{2Ng_r}\phi^2 + \frac{9}{8\pi N^2} |\phi|^3$$

(57)

where $\phi^2 = \sigma^2 + \pi^2$. The model exhibits the dynamical breaking of the chiral symmetry in the exactly same manner as in the two-component case.

IV. CONCLUSION

The Gaussian approximation method has been shown to be useful in studying the non-perturbative aspects of quantum field theories. In the models considered so far [2], the Gaussian approximation method gives qualitatively the same results as those of large N approximation. This is partly due to the fact that the main non-perturbative feature of those models appears at the leading order in $1/N$ expansion. On the other hand (2+1)-dimensional Thirring model has the dynamical symmetry breaking not in the leading order of N, but in the combination of different orders in $1/N$ [4,5]. In two component formalism, this non-perturbative phenomena, from the point of view of large N expansion, occur in the range of $g_R$, $0 \leq g_R \leq g_c \equiv \frac{1}{16}exp\left(-\frac{N\pi^2}{16}\right)$ [3]. In this paper we use the Gaussian approximation scheme to reveal this. We have shown that in this scheme the dynamical symmetry breaking does occur as a cooperative effect of the leading and next-to-leading orders in $1/N$. Since the Gaussian approximation gives better results than the large N approximation, we believe that the Gaussian approximation is well-suited for the difficult non-perturbative problems such as the existence of bound state.
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