ISOMORPHIC CLASSIFICATION OF PROJECTIVE TENSOR PRODUCTS OF SPACES OF CONTINUOUS FUNCTIONS

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ABSTRACT. We prove that for infinite, countable, compact, Hausdorff spaces $K,L$, $C(K)\hat{\otimes}_\pi C(L)$ is isomorphic to exactly one of the spaces $C(\omega^{\omega^\xi})\hat{\otimes}_\pi C(\omega^{\omega^\xi})$, $0 \leq \zeta < \xi < \omega_1$. We also prove that $C(K)\hat{\otimes}_\pi C(L)$ is not isomorphic to $C(M)$ for any compact, Hausdorff space $M$.

1. Introduction

A classical result of Banach space theory, due to Bessaga and Pełczyński, is that if $K$ is an infinite, countable, compact, Hausdorff space, then $C(K)$, the space of scalar-valued, continuous functions on $K$, is isomorphic to $C(\omega^{\omega^\xi})$ for exactly one ordinal $\xi < \omega_1$, where $\omega_1$ denotes the smallest uncountable ordinal. From this it follows that for infinite, countable, compact, Hausdorff spaces $K, L$, the projective tensor product $C(K)\hat{\otimes}_\pi C(L)$ is isomorphic to at least one of the spaces $C(\omega^{\omega^\xi})\hat{\otimes}_\pi C(\omega^{\omega^\xi})$, $\zeta \leq \xi < \omega_1$. However, due to the somewhat intractable nature of projective tensor products, it has not previously been known whether $C(K)\hat{\otimes}_\pi C(L)$ was isomorphic to exactly one such space. Solving this problem is equivalent to asking whether the spaces $C(\omega^{\omega^\xi})\hat{\otimes}_\pi C(\omega^{\omega^\xi})$, $\zeta \leq \xi < \omega_1$, are mutually non-isomorphic. The goal of this work is to provide the following answer to that question, and provide an isomorphic classification of projective tensor products of separable, Asplund $C(K)$ spaces analogous to the result of Bessaga and Pełczyński’s classification of the spaces themselves.

Theorem 1.1. The spaces $C(\omega^{\omega^\xi})\hat{\otimes}_\pi C(\omega^{\omega^\xi})$, $\zeta \leq \xi < \omega_1$, are mutually non-isomorphic. Consequently, if $K, L$ are infinite, countable, compact, Hausdorff spaces, then $C(K)\hat{\otimes}_\pi C(L)$ is isomorphic to exactly one of the spaces $C(\omega^{\omega^\xi})\hat{\otimes}_\pi C(\omega^{\omega^\xi})$, $\zeta \leq \xi < \omega_1$.

The typical way to show that the spaces $C(\omega^{\omega^\xi})$, $\xi < \omega_1$, are mutually non-isomorphic, is to use the Szlenk index, which is an isomorphic invariant. The third named author showed in [25] that the Szlenk index of $C(\omega^{\omega^\xi})$ is $\omega^{\xi+1}$, which immediately yields that $C(\omega^{\omega^\xi})$ cannot be isomorphic to $C(\omega^{\omega^\xi})$ when $\zeta, \xi < \omega_1$ are distinct. However, it was recently shown in [10] that for any ordinals $\zeta \leq \xi$, the Szlenk index of $C(\omega^{\omega^\xi})\hat{\otimes}_\pi C(\omega^{\omega^\xi})$ is equal to the Szlenk index of $C(\omega^{\omega^\xi})$. Therefore the Szlenk index is an insufficiently granular tool to provide an isomorphic classification of the projective tensor products. Therefore we introduce a family of properties parameterized by two ordinals, $\Phi_{\alpha,\beta}$, $\alpha, \beta < \omega_1$. We define these properties in Section 5. As with the Szlenk index, these properties are isomorphic invariants. Let us define $O_2(X)$ to be the lexicographically minimal $(\alpha, \beta)$ such that $X$ has property $\Phi_{\alpha,\beta}$, assuming such an $(\alpha, \beta) \in [0, \omega_1)^2$ exists. Otherwise we define $O_2(X) = \infty$. We will prove that the spaces $C(\omega^{\omega^\xi})\hat{\otimes}_\pi C(\omega^{\omega^\xi})$, $\zeta \leq \xi < \omega_1$, are mutually non-isomorphic with the

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following result. In Section 2, we provide a detailed presentation of the geometry underlying these properties and how we will verify that the spaces \( C(\omega^\xi) \dot{\otimes}_\pi C(\omega^\zeta) \) enjoy these properties.

**Theorem 1.2.** For \( \zeta \leq \xi < \omega_1 \), \( O_\zeta(C(\omega^\xi) \dot{\otimes}_\pi C(\omega^\zeta)) = (\xi, 1 + \zeta) \).

Throughout, all Banach spaces are over the field \( \mathbb{K} \), which is either \( \mathbb{R} \) or \( \mathbb{C} \). By subspace, we shall mean closed, linear subspace, and by operator, we shall mean bounded, linear operator. If \( X \) is a Banach space, \( x \in X \) and \( x^* \in X^* \) we denote \( x^*(x) \) by \( \langle x^*, x \rangle \).

Let \( X \) and \( Y \) be two Banach spaces. \( B(X, Y) \) denotes the space of bounded, bilinear forms on \( X \times Y \). The space \( B(X, Y) \) is isometrically isomorphic to the space \( L(X, Y^*) \) of operators from \( X \) to \( Y^* \). Fix \( u = \sum_{i=1}^m x_i \otimes y_i \in X \otimes Y \). The norm \( \| u \|_\varepsilon \) is defined on the tensor product \( X \otimes Y \) by

\[
\| u \|_\varepsilon = \sup \left\{ \left\| \sum_{i=1}^m x^*(x_i)y^*(y_i) \right\| : x^* \in B_{X^*}, \text{and } y^* \in B_{Y^*} \right\}.
\]

The norm \( \| \cdot \|_\pi \) is defined on the tensor product \( X \otimes Y \) by

\[
\| u \|_\pi = \sup \left\{ \left\| \sum_{i=1}^m b(x_i, y_i) \right\| : b \in B_{B(X, Y)} \right\} = \sup \left\{ \left\| \sum_{i=1}^m T(x_i)(y_i) \right\| : T \in B_{L(X, Y^*)} \right\}.
\]

We denote by \( X \dot{\otimes}_\varepsilon Y \) the completion of \( X \otimes Y \) with respect to \( \| \cdot \|_\varepsilon \) and by \( X \dot{\otimes}_\pi Y \) the completion of \( X \otimes Y \) with respect to \( \| \cdot \|_\pi \).

2. **1-Weakly Summing Sequences and 2-Weakly Summing Sequences in \( C(\omega^\xi) \dot{\otimes}_\pi C(\omega^\zeta) \)**

The Szlenk index can be characterized as the complexity which is required to obtain arbitrarily small convex (or \( \ell^+_p \)) combinations of the branches of weakly null trees in the unit ball of an Asplund space. Here, complexity is an ordinal-quantified notion. Roughly speaking, a complexity means trivial convex combinations, by which we mean a single vector with coefficient 1. Having Szlenk index \( 1 = \omega^0 \) is equivalent to the property that every weakly null net in the unit ball of the space is norm null, since in this case the tree consists of many incomparable nodes and the convex combinations of a branch are single vectors with coefficient 1. The next level, complexity 1, means that among the branches of a weakly null tree in the ball of the space, we can find branches, say \( (x_1, \ldots, x_n) \), such that \( \| \frac{1}{n} \sum_{i=1}^n x_i \| \leq \theta_n \), where \( \theta_n^n = \infty \) is a null sequence which depends on the space, but not on the particular tree. For example, in \( \ell_p \), \( 1 < p < \infty \), we obtain such convex combinations with \( \theta_n = n^{1/p-1} + \varepsilon \). In other words, complexity 1 consists of flat averages. Complexity 2 consists of flat averages of flat averages. That is, convex combinations of the form \( \frac{1}{n} \sum_{i=1}^n \frac{1}{m} \sum_{j=1}^{n_i} x_{ij} \), where in order to obtain small norms, \( n_i \) must be chosen sufficiently large in a way that depends on the previously chosen vectors \( x_{kj} \), \( k < i \). Such vectors appear naturally, for example, in classical computations in Schreier space or in Schlumprecht space. Complexity 3 is flat averages of flat averages of flat averages, etc. When dealing with the repeated averages hierarchy, which is not the only way to extend this quantification of complexity beyond the finite ordinals, but which is our approach in this paper, we diagonalize at limit ordinals.

It is a classical result of Bessaga and Pelczyński that if \( K \) is infinite, countable, compact, Hausdorff, then \( C(K) \) is isomorphic to one of the spaces \( C(\omega^\xi) \), \( 0 \leq \xi < \omega_1 \). Of course, they showed that \( C(K) \) is isomorphic to exactly one of these spaces, but we only wish to cite one direction of this
result, while we will discuss the use of the Szlenk index to deduce the other direction. The reason for this is because the cited direction immediately yields that if $K, L$ are both infinite, countable, compact, Hausdorff spaces, then $C(K) \hat{\otimes}_\pi C(L)$ is isomorphic to one of the spaces $C(\omega^{\omega^\xi}) \hat{\otimes}_\pi C(\omega^\xi)$, $0 \leq \zeta \leq \xi < \omega_1$. The goal of this work is to prove that $C(K) \hat{\otimes}_\pi C(L)$ is isomorphic to only one of the spaces $C(\omega^{\omega^\xi}) \hat{\otimes}_\pi C(\omega^\xi)$. So we first indicate how the Szlenk index is used for the case of a single space, and then indicate how our approach to characterize the projective tensor products is analogous to the case of a single space.

The Szlenk index is an isomorphic invariant, which tautologically means that spaces with different Szlenk indices cannot be isomorphic. Therefore in order to show that $C(K)$ cannot be isomorphic to $C(\omega^{\omega^\xi})$ for two different $\xi < \omega_1$, it is sufficient to show that no two of the spaces $C(\omega^{\omega^\xi})$, $\xi < \omega_1$, have the same Szlenk index. Since it was shown by Samuel the third named author that $\mathcal{Sz}(C(\omega^{\omega^\xi})) = \omega^{\xi+1}$ for each $\xi < \omega_1$, we obtain the other direction of the isomorphic classification.

In fact, due to properties of the Szlenk index which we will later recall and analogize, it follows from the fact that $\mathcal{Sz}(C(\omega^{\omega^\xi})) = \omega^{\xi+1}$ for $\xi < \omega_1$ that $C(\omega^\xi)$ is not isomorphic to any subspace of any quotient of $C(\omega^\xi)$ for $\zeta < \xi$. The computation $\mathcal{Sz}(C(\omega^{\omega^\xi})) = \omega^{\xi+1}$ can be cleanly and conceptually separated into the upper and lower estimates $\mathcal{Sz}(C(\omega^{\omega^\xi})) \leq \omega^{\xi+1}$ and $\mathcal{Sz}(C(\omega^{\omega^\xi})) \geq \omega^{\xi+1}$, which involve, respectively, showing that every weakly null tree of rank $\omega^{\xi+1}$ in $B_{C(\omega^{\omega^\xi})}$ has a branch with an arbitrarily small convex combination, and that there exists a weakly null tree of rank $\omega^\xi$ in $B_{C(\omega^{\omega^\xi})}$ such that all the convex combinations of the branches are uniformly bounded away from 0. In this work, we proceed in the same way, proving in Section 6 that $C(\omega^{\omega^\xi}) \hat{\otimes}_\pi C(\omega^\xi)$ has a certain property quantified by a pair of ordinals, and in Section 7 that it does not have any quantitatively better property.

In this work, the Szlenk index is too coarse a tool to deduce that $C(K) \hat{\otimes}_\pi C(L)$ is isomorphic to at most one of the spaces $C(\omega^{\omega^\xi}) \hat{\otimes}_\pi C(\omega^\xi)$, $0 \leq \zeta \leq \xi < \omega_1$. Indeed, we will show that

$$\mathcal{Sz}(C(\omega^{\omega^\xi}) \hat{\otimes}_\pi C(\omega^\xi)) = \omega^{\xi+1} = \mathcal{Sz}(C(\omega^\xi))$$

for any $0 \leq \zeta \leq \xi < \omega_1$. So we introduce properties which depend on two ordinals. More precisely, we will define for each Banach space $X$ a (possibly infinite) quantity $\mathfrak{g}_{\xi,\zeta}(X)$, and $X$ will have the $\mathfrak{G}_{\xi,\zeta}$ property provided that $\mathfrak{g}_{\xi,\zeta}(X) < \infty$. Moreover, if $X$ has a separable dual, then $\mathfrak{g}_{\xi,\zeta}(X) = 0$ for some (equivalently, every) $\zeta < \omega_1$ if and only if $\mathcal{Sz}(X) \leq \omega^\xi$, and if $\mathfrak{g}_{\xi,\zeta}(X) < \infty$ for some $\zeta < \omega_1$, then $\mathcal{Sz}(X) \leq \omega^{\xi+1}$. Therefore for a fixed $\xi < \omega_1$ and a Banach space $X$ with separable dual, $X$ can only have one of the properties $\mathfrak{G}_{\xi,\zeta}$, $\zeta < \omega_1$ in a non-trivial way (that is, $\mathfrak{g}_{\xi,\zeta}(X) \in (0, \infty)$) if and only if $\mathcal{Sz}(X) = \omega^{\xi+1}$. In this way, the $\xi$ can be thought of as quantifying some coarser quantification, and the ordinal $\zeta$ provides some finer gradations within the coarser classes.

As mentioned above, the Szlenk index can be thought of as the complexity required to obtain $\ell_1^+$ combinations of the branches of weakly null trees in the ball of a given space. In our properties $\mathfrak{G}_{\xi,\zeta}$, and some associated auxiliary properties $\mathfrak{G}_\xi$, the coarser ordinal $\xi$ will quantify the complexity required of $\ell_1^+$ combinations of the branches of weakly null trees to guarantee that when we obtain an appropriately complex sequence of $\ell_1^+$ combinations of a branch of a weakly null tree in the ball of our space, that sequence will be 2-weakly summing with 2-weakly summing norm depending only on the space. Of course, we can then begin to block the sequence of $\ell_1^+$ combinations. Since the sequence of $\ell_1^+$ combinations is 2-weakly summing, we can block it with not into further $\ell_1^+$
combinations, but into $\ell_2^+$ combinations. The purpose of the coarser property is only to guarantee that such $\ell_2^+$ combinations remain bounded. The complexity of $\ell_2^+$ combinations can be defined as we did with $\ell_1^+$ combinations. A level 0 combination is no blocking, which would be the situation of performing only the initial $\ell_1^+$ combinations and no further $\ell_2^+$. Level 1 would be to first obtain a sequence $(x_i)_{i=1}^n$ of $\ell_1^+$ combinations, and then further combine them into a vector of the form $\sqrt{n} \sum_{i=1}^n x_i$. Level 2 would be vectors of the form $\sqrt{n} \sum_{i=1}^n \sqrt{n} \sum_{j=1}^{n_i} x_{ij}$, where the $x_{ij}$ are obtained by the initial $\ell_1^+$ combinations. In general, the coefficients are simply the 2-concavifications of the corresponding level of the repeated averages hierarchy. The finer ordinal, $\zeta$, is determined by the complexity of the $\ell_2^+$ combinations, performed on the $\ell_1^+$ combinations, which are required to obtain sequences which are 1-weakly summing. Roughly speaking, among the branches of a weakly null tree in the unit ball of our space, we first take $\ell_1^+$ combinations of the required complexity $\xi$ such that the resulting combinations of the branches begin to have $\ell_2^+$ upper estimates, and then take $\ell_2^+$ combinations of the $\ell_1^+$ combinations until the resulting branches begin to have $c_0$ upper estimates.

Let us now discuss how to obtain sequences with $\ell_2$ and $c_0$ upper estimates in the spaces $C_0(\omega^{\omega^\omega}) \hat{\otimes} C_0(\omega^{\omega^\omega})$. Note the use of $C_0(\omega^{\omega^\omega})$ in place of $C(\omega^{\omega^\omega})$. In the subsequent discussion, we assume $\zeta \leq \xi$. The statements we make here are exactly what we will show in Section 6. We think about tensors in $U := C_0(\omega^{\omega^\omega}) \hat{\otimes} C_0(\omega^{\omega^\omega})$ as block matrices. Any sequence $(u_n)_{n=1}^\infty \subset B_U$ supported as in Figure 1 is 2-weakly summing with $\| (u_n)_{n=1}^\infty \|_2 \leq k_G$, where $k_G$ is Grothendieck’s constant. This is shown for elementary tensors in Proposition 3.1 and extends to the general by noting that any sequence $(u_n)_{n=1}^\infty$ supported as in Figure 1 has the property that each $u_n$ lies in the closed, convex hull of elementary tensors supported in the same rectangles and using the triangle inequality. Here we are using the fact that ordinal intervals $(\alpha_{n-1}, \alpha_n]$ are clopen, and so vectors supported in Cartesian products of such intervals lie in 1-complemented subspaces of $U$. Of course, by symmetry, a sequence $(v_n)_{n=1}^\infty$ supported as in $B_U$ is also 2-weakly summing with norm not more than $k_G$.

From this it follows that a sequence $(w_n)_{n=1}^\infty \subset B_U$ supported as in Figure 3 is 2-weakly summing with 2-weakly summing norm not exceeding $2k_G$, by decomposing this sequence as in Figure 4. This is the reason for the factor of 2 in our results. Sequences of the form depicted in Figure 3
are exactly the ones we will find with $\ell_2$ upper estimates after taking $\ell^+_1$ combinations of sufficient complexity.

Our method of obtaining sequences of the form $(w_n)_{n=1}^\infty$ as in Figure 3 will be to use an inductive hypothesis on the Szlenk index of $C_0(\omega^\xi) \widehat{\otimes}_\pi C_0(\omega^\xi)$ which guarantees that for $\alpha < \omega^\xi$ and $\beta < \omega^\xi$, $C(\alpha) \widehat{\otimes}_\pi C(\beta)$ has Szlenk index not exceeding $\omega^\xi$, from which it follows that a $\ell^+_1$ combinations of complexity $\xi$ can be made to be arbitrarily small on regions such as the shaded region in Figure 5. We then apply this recursively to obtain the sequence $(w_n)_{n=1}^\infty$ as depicted in Figure 6. This is the reason we use $C_0(\omega^\xi) \widehat{\otimes}_\pi C_0(\omega^\xi)$ in place of $C(\omega^\xi) \widehat{\otimes}_\pi C(\omega^\xi)$. This will give our coarser results, which are determined by the maximum of $\xi$ and $\zeta$. For a concrete example, in $c_0 \widehat{\otimes}_\pi c_0$, which corresponds to $\xi = \zeta = 0$, the gray region in Figure 5 corresponds to a finite dimensional subspace. Therefore any weakly null sequence will have terms arbitrarily small on the gray region, and therefore we can obtain vectors which are small perturbations of a sequence of the form of $(w_n)_{n=1}^\infty$ by taking $\ell^+_1$ combinations at the 0 level, which means single vectors with coefficients 1.

For $\kappa > 0$ and any $(x_n)_{n=1}^\infty \subset \kappa B_\ell$ supported as in Figure 7, $\|(x_n)_{n=1}^\infty\|_w^w \leq \kappa$. This is trivial for elementary tensors, since in any Banach spaces $Y, Z$ and $(y_n)_{n=1}^\infty \in \ell^w_1(Y)$, $(z_n)_{n=1}^\infty \in \ell^w_1(Z)$,
We next discuss how to further block the \( \ell_1^+ \) combinations using \( \ell_2^+ \) coefficients to obtain sequences as in Figure 7. We must take a sufficiently large combination to guarantee that the resulting vector is essentially zero in the gray region of Figure 8. Of course, by symmetry, we would also like have the vector be essentially zero in the gray region in Figure 9. To return to our concrete example of \( c_0 \otimes_\pi c_0 \), the gray region in Figure 8 can be identified with \( \ell_1^+ \otimes_\pi c_0 \), which is isomorphic to \( c_0 \). This means that sufficiently well-separated vectors \( (x_i)_{i=1}^n \subset B_{c_0 \otimes_\pi c_0} \) will satisfy the estimate \( \| P \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \| \leq C/\sqrt{n} \), where \( P \) is the projection onto the gray region of Figure 8. Here, \( C \) is a constant that depends on \( \gamma \), and in our recursive onto the gray region of Figure 8. Here, \( C \) is a constant that depends on \( \gamma \), and in our recursive selections, \( \gamma \) will depend on the supports of previously chosen vectors. Therefore we must take larger and larger \( n \), depending on the choices already made, which is characteristic of level 1 averaging.

Actually, it is essential that we be able to find a single \( \ell_2^+ \) combination which is simultaneously essentially zero on the gray regions in both Figures 8 and 9. We will define our properties \( \mathcal{G}_{\xi,\zeta} \) and
frame our induction in such a way that we can easily choose our $\ell_2^+$ combinations to be simultaneously small on both gray regions, and therefore on the gray region in Figure 10. From this we can inductively choose our sequence $(x_n)_{n=1}^{\infty}$ as depicted in Figure 11.

In summary, Figures 6 and 11 demonstrate how we will choose our blockings, first $\ell_1^+$ combinations which vanish in the gray regions as in Figure 6 and are therefore 2-weakly summing, and then $\ell_2^+$ combinations of the $\ell_1^+$ combinations which vanish in the gray regions as in Figure 11 and are therefore 1-weakly summing. We will show that, with our quantifications described above, we must take $\ell_1^+$ combinations of complexity $\xi$ and then $\ell_2^+$ combinations of complexity $1 + \zeta$.

The result above is analogous to the upper estimates $S_2(C(\omega^\omega)) \leq \omega^{\xi+1}$. In Section 7, we show the analogue of the lower estimate, which involves constructing sufficiently large trees without the requisite upper estimates. For this, we use a tree of trees approach. In the dual of $C(\omega^\omega)$, we obtain a normalized, weak*-null, separated tree, which can be used to well-norm a weakly null tree of rank $\omega^\omega$ in $B(C(\omega^\omega))$. We also build a large tree in the dual of $C(\omega^\omega)$ whose branches are Rademacher systems, and therefore have $\ell_2$ upper estimates. We then create a tree in the dual of $C(\omega^\omega) \hat{\otimes} C(\omega^\omega)$ of the appropriate rank, with the inner trees coming from our tree in $C(\omega^\omega)^*$ and the outer tree coming from $C(\omega^\omega)^*$. We then use these trees to norm the branches of a carefully chosen weakly null predual tree to prove that insufficiently large $\ell_2^+$ combinations of $\ell_1^+$ combinations have $\ell_2$ lower estimates, and therefore do not have $c_0$ upper estimates.

Note that the space $C(\omega^\omega) \hat{\otimes} \mathbb{K}$ fits naturally into this hierarchy as well, since we can obtain 1-weakly summing combinations by taking $\ell_1^+$ combinations of complexity $\xi$ and then $\ell_2^+$ combinations of complexity 0, which simply means without any further blocking. Therefore, with the establishment of the convention $C(\omega^{-1}) = \mathbb{K}$, for $0 \leq \xi < \omega_1$ and $-1 \leq \zeta \leq \xi$, the space $C(\omega^\omega) \hat{\otimes} C(\omega^\omega)$ will have property $\mathfrak{S}_{\xi,1+\zeta}$ and no better. Since we will show that these properties are isomorphic invariants, these spaces are mutually non-isomorphic.
3. Tensor Products and p-Weakly Summable Sequences

We recall that for a Banach space $E$, $1 \leq p < \infty$, a sequence $(e_i)_{i=1}^\infty$ is said to be \textit{weakly $p$-summing} provided that

$$\|(e_i)_{i=1}^\infty\|_p^w := \sup \left\{ \left( \sum_{i=1}^\infty |\langle e^*, e_i \rangle|^p \right)^{1/p} : e^* \in B_{E^*} \right\}$$

is finite. In this case, $\|(e_i)_{i=1}^\infty\|_p^w$ denotes the \textit{weakly $p$-summing} norm of $(e_i)_{i=1}^\infty$. We note that if $1/p + 1/q = 1$,

$$\|(e_i)_{i=1}^\infty\|_p^w = \sup \left\{ \left\| \sum_{i=1}^\infty a_i e_i \right\| : \|(a_i)_{i=1}^\infty\|_{\ell_1} = 1 \right\}. \quad \square \tag{3.1}$$

The following is a consequence of Grothendieck’s theorem.

**Proposition 3.1.** \cite[Lemma 1.1]{29} Let $K, L$ be compact, Hausdorff spaces. If $(f_n)_{n=1}^\infty$ is a sequence in $B_{C(K)}$ such that $\|(f_n)_{n=1}^\infty\|_2 \leq 1$, then for any sequence $(g_n)_{n=1}^\infty$ in $B_{C(L)}$, $\|(f_n \otimes g_n)_{n=1}^\infty\|_1^w \leq k_G$.

The following facts regarding the tensorization of weakly $p$-summing norms behave under the formation of projective and injective tensor products.

**Proposition 3.2.** Let $X, Y$ be Banach spaces and fix a sequence $(x_n)_{n=1}^\infty$ in $X$ and a sequence $(y_n)_{n=1}^\infty$ in $Y$.

(i) $\|(x_n \otimes y_n)_{n=1}^\infty\|_{1, 1, \pi}^w \leq \|(x_n)_{n=1}^\infty\|_1^w \|(y_n)_{n=1}^\infty\|_1^w$.

(ii) $\|(x_n \otimes y_n)_{n=1}^\infty\|_{2, 2, \varepsilon}^w \leq \|(x_n)_{n=1}^\infty\|_{2}^w \|(y_n)_{n=1}^\infty\|_{2}^w$.

Here, $\| \cdot \|_{1, 1, \pi}^w$ denotes the weakly 1-summing norm of the sequence of tensors considered as members of $X \hat{\otimes}_\pi Y$ and $\| \cdot \|_{2, 2, \varepsilon}^w$ denotes the weakly 2-summing norm of the sequence of tensors considered as members of $X \hat{\otimes}_\varepsilon Y$.

\textbf{Proof.} (i) Let $(\varepsilon_i)_{i=1}^\infty$ be a Rademacher sequence. Fix $m \in \mathbb{N}$ and scalars $(\delta_i)_{i=1}^m$ such that $|\delta_i| = 1$ for all $1 \leq i \leq m$,

$$\sum_{i=1}^m \delta_i x_i \otimes y_i = \mathbb{E} \left( \left( \sum_{i=1}^m \delta_i \varepsilon_i x_i \right) \otimes \left( \sum_{i=1}^m \varepsilon_i y_i \right) \right),$$

where $\mathbb{E}$ is the expectation with respect to $(\varepsilon_i)_{i=1}^\infty$, so

$$\left\| \sum_{i=1}^m \delta_i x_i \otimes y_i \right\|_{1, \pi} \leq \mathbb{E} \left( \left\| \sum_{i=1}^m \delta_i \varepsilon_i x_i \right\| \left\| \sum_{i=1}^m \varepsilon_i y_i \right\| \right) \leq \|(x_i)_{i=1}^\infty\|_1^w \|(y_i)_{i=1}^\infty\|_1^w.$$

(ii) Fix $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$. Fix $m \in \mathbb{N}$ and scalars $(a_i)_{i=1}^m$ such that $1 = \sum_{i=1}^m |a_i|^2$. Then

$$\left| \langle x^* \otimes y^*, \sum_{i=1}^m a_i x_i \otimes y_i \rangle \right| \leq \left\| \sum_{i=1}^m a_i (y^*, y_i) x_i \right\| \leq \|(x_i)_{i=1}^\infty\|_2^w \|(y_i)_{i=1}^\infty\|_{\ell_1} \leq \|(x_i)_{i=1}^\infty\|_2^w \|(y_i)_{i=1}^\infty\|_{\ell_2} \leq \|(x_i)_{i=1}^\infty\|_2^w \|(y_i)_{i=1}^\infty\|_{\ell_2}. \quad \square \tag{3.2}$$
4. Trees and indices

If $\Omega$ is a topological space and $\rho$ is a metric on $\Omega$ (not necessarily compatible with the topology on $\Omega$), $\varepsilon > 0$, and if $L$ is a subset of $\Omega$, we can define the derived set $\iota(L, \rho, \varepsilon)$ of $L$ to be the set of all $\varpi \in L$ such that for any open neighborhood $U$ of $\varpi$, $\text{diam}_\rho(U \cap L) > \varepsilon$. We define the higher order derived sets by

$$\iota^0(L, \rho, \varepsilon) = L,$$
$$\iota^{\xi+1}(L, \rho, \varepsilon) = \iota(\iota^\xi(L, \rho, \varepsilon), \rho, \varepsilon),$$

and if $\xi$ is a limit ordinal,

$$\iota^\xi(L, \rho, \varepsilon) = \bigcap_{\zeta < \xi} \iota^\zeta(L, \rho, \varepsilon).$$

We say a subset $L$ of $\Omega$ is $\rho$-fragmentable provided that for any $\varepsilon > 0$, there exists an ordinal $\xi$ such that $\iota^\xi(L, \rho, \varepsilon) = \emptyset$. If $L$ is $\rho$-fragmentable, we define $\text{Frag}(L, \rho, \varepsilon)$ to be the minimum ordinal $\xi$ such that $\iota^\xi(L, \rho, \varepsilon) = \emptyset$, and we define $\text{Frag}(L, \rho) = \sup_{\varepsilon > 0} \text{Frag}(L, \rho, \varepsilon)$. We refer to this as the $\rho$-fragmentation index of $L$. Formally speaking, this terminology should also reference the underlying topology, but the topology under consideration will be clear from context in all cases.

Given a set $\Lambda$, we let $\Lambda^{< \omega} = \bigcup_{n=0}^{\infty} \Lambda^n$ denote the set of all finite sequences of elements in $\Lambda$. We note that $\Lambda^{< \omega}$ also includes the empty sequence, which we denote by $\emptyset$. For $t \in \Lambda^{< \omega}$, we let $|t|$ denote the length of $t$. For $t \in \Lambda^{< \omega}$ and $0 \leq i \leq |t|$, we let $t|_i$ denote the initial segment of $t$ having length $i$. We endow $\Lambda^{< \omega}$ with the initial segment order, denoted $\preceq$. We note that $s \preceq t$ if $|s| \leq |t|$ and $s = t|_{|s|}$.

A non-empty subset $T$ of $\Lambda^{< \omega}$ is called a tree on $\Lambda$ (or just a tree) provided that if $s \preceq t$ and $t \in T$, then $s \in T$. For a tree $T$, we let $\text{MAX}(T)$ denote the set of all maximal members of $T$ with respect to the initial segment ordering. For a tree $T$, we define the derived tree of $T$, denoted by $T'$, by

$$T' = T \setminus \text{MAX}(T).$$

We define the transfinite derived trees by

$$T^0 = T,$$
$$T^{\xi+1} = (T^\xi)',$$

and if $\xi$ is a limit ordinal,

$$T^\xi = \bigcap_{\zeta < \xi} T^\zeta.$$

We say $T$ is well-founded provided there exists an ordinal $\xi$ such that $T^\xi = \emptyset$. In this case, the rank of $T$ is defined to be the minimum $\xi$ such that $T^\xi = \emptyset$. We say $T$ is ill-founded if it is not well-founded. We note that if $\Omega = \Lambda^{< \omega}$ is endowed with the (non-Hausdorff) topology whose subbase consists of all wedges $V_t = \{ s \in \Lambda^{< \omega} : t \preceq s \}$, and if $\rho$ is the discrete metric on $\Lambda^{< \omega}$, then $T \subseteq \Lambda^{< \omega}$ is well-founded if and only if it is $\rho$-fragmentable, and the rank of the tree is its $\rho$-fragmentation index.

We denote by $[\mathbb{N}]^{< \omega}$ the set of finite subsets of $\mathbb{N}$. We denote by $[\mathbb{N}]$ the set of infinite subsets of $\mathbb{N}$. Similarly, for $M \in [\mathbb{N}]$, we let $[M]^{< \omega}$ and $[M]$ denote the sets of finite and infinite subsets of $M$, respectively. By an abuse of notation, we identify subsets of $\mathbb{N}$ with finite or infinite strictly
increasing sequences. We use set and sequential notation interchangeably. More precisely, we identify a subset $E$ of $\mathbb{N}$ with the sequence obtained by listing the members of $E$ in increasing order. With this identification, $[\mathbb{N}]^{<\omega}$ can be treated as a subset of $\mathbb{N}^{<\omega}$. For a finite subset $E$ of $\mathbb{N}$, the notation $|E|$ for the cardinality of $E$ is equal to the length of $E$ when treated as a sequence, so the notation is unambiguous. For $E \subseteq \mathbb{N}$ and $1 \leq n \leq |E|$, we let $E(n)$ denote the $n^{th}$ smallest member of $E$. Similarly, for an infinite subset $M$ of $\mathbb{N}$ and $n \in \mathbb{N}$, we let $M(n)$ denote the $n^{th}$ smallest member of $M$. We let $E < F$ denote the relation that either $E = \emptyset$, $F = \emptyset$, or $\max E < \min F$. For $E < F$, we let $E \cap F$ denote the concatenation of $E$ with $F$.

We say a set $F \subseteq [\mathbb{N}]^{<\omega}$ is hereditary provided that if $E \subseteq F$ and $F \in F$, then $E \in F$. Using our identification of subsets of $\mathbb{N}$ with sequences in $\mathbb{N}$, each hereditary subset of $[\mathbb{N}]^{<\omega}$ can be treated as a tree on $\mathbb{N}$.

We endow the power set $2^\mathbb{N}$ of $\mathbb{N}$ with the Cantor topology, which is the topology making the identification $2^\mathbb{N} \ni E \leftrightarrow 1_E \in \{0, 1\}^\mathbb{N}$ a homeomorphism, where $\{0, 1\}^\mathbb{N}$ is endowed with the product of the discrete topology. We say a subset $F$ of $2^\mathbb{N}$ is compact if it is compact in the Cantor topology.

For $(m_i)_{i=1}^p, (n_i)_{i=1}^p \in [\mathbb{N}]^{<\omega}$, we say $(n_i)_{i=1}^p$ is a spread of $(m_i)_{i=1}^p$ provided that $m_i \leq n_i$ for all $1 \leq i \leq p$. We say $F \subseteq [\mathbb{N}]^{<\omega}$ is spreading provided $F$ contains all spreads of its members. We say a subset $F$ of $[\mathbb{N}]^{<\omega}$ is regular provided it is compact, spreading, and hereditary.

For a compact, Hausdorff topological space $K$ and a subset $L$ of $K$, we let $L'$ denote the set of members of $L$ which are not relatively isolated in $L$. The set $L'$ is the Cantor-Bendixson derivative of $L$. We define

$$K^0 = K,$$

$$K^{\xi+1} = (K^\xi)',$$

and if $\xi$ is a limit ordinal, we let

$$K^\xi = \bigcap_{\zeta < \xi} K^\zeta.$$

We say that $K$ is scattered (or dispersed) provided that there exists some ordinal $\xi$ such that $K^\xi = \emptyset$ (equivalently, every non-empty subset of $K$ has an isolated point). In this case, we let $CB(K)$ denote the minimum ordinal $\xi$ such that $K^\xi = \emptyset$. The ordinal $CB(K)$ is the Cantor-Bendixson index of $K$. We note that if $\rho$ is the discrete metric on $K$, then $K$ is scattered if and only if it is $\rho$-fragmentable, and the Cantor-Bendixson index is its $\rho$ fragmentation index.

We recall the coarse wedge topology on a tree $T$, which is the topology for which wedges $V_t$ and their complements form a subbase. Let us suppose that a tree $T$ has the following property: For each ordinal $\xi$ and each $t \in T^\xi$, either $t \in \text{MAX}(T^\xi)$ or $\{s \in T^\xi : t \prec s\}$ is infinite. If the tree $T$ has this property, then the derived trees $T^\xi$ coincide with the Cantor-Bendixson derivatives, the tree is well-founded if and only if it is compact and scattered with the coarse wedge topology, and the rank of the tree is equal to its Cantor-Bendixson index.

We last define the Szlenk index, which is the fragmentation of the dual ball $B_{X^*}$ of a Banach space with the weak* topology and the metric induced by the norm. More precisely, for $\varepsilon > 0$ and $K \subseteq X^*$, we let $s_\varepsilon(K)$ denote the set of $x^* \in K$ such that for any weak*-neighborhood $V$ of $x^*$, $\text{diam}(V \cap K) > \varepsilon$. We define

$$s_\varepsilon^0(K) = K,$$
\[ s_{\xi}^{\xi+1}(K) = s_{\xi}(s_{\xi}(K)), \]

and if \( \xi \) is a limit ordinal,

\[ s_{\xi}(K) = \bigcap_{\zeta < \xi} s_{\zeta}(K). \]

We let \( S_{\xi}(K, \varepsilon) \) denote the minimum \( \xi \) such that \( s_{\xi}(K) = \emptyset \), if such a \( \xi \) exists, and otherwise we write \( S_{\xi}(K, \varepsilon) = \infty \). We let \( S_{\xi}(K) = \sup_{\varepsilon > 0} S_{\xi}(K, \varepsilon) \) if \( S_{\xi}(K, \varepsilon) \) is an ordinal for all \( \varepsilon > 0 \), and we write \( S_{\xi}(K) = \infty \) if \( S_{\xi}(K, \varepsilon) = \infty \) for some \( \varepsilon > 0 \). For a Banach space \( X \), we let \( S_{\xi}(X, \varepsilon) = S_{\xi}(B_{X^\ast}, \varepsilon) \) and \( S_{\xi}(X) = S_{\xi}(B_{X^\ast}) \).

We next recall the definition of the Schreier families, which are regular families of subsets of \([\mathbb{N}]^{<\omega}\). We let

\[ \mathcal{S}_0 = \{\emptyset\} \cup \{(n) : n \in \mathbb{N}\}. \]

Assuming that \( \mathcal{S}_\xi \) has been defined for some \( \xi < \omega_1 \), we let

\[ \mathcal{S}_{\xi+1} = \{\emptyset\} \cup \left\{ \bigcup_{n=1}^m E_n : E_1 < \ldots < E_m, \emptyset \neq E_n \in \mathcal{S}_\xi, m \leq E_1 \right\}, \]

and if \( \xi \) is a limit ordinal, fix a sequence \( \xi_n \uparrow \xi \) and let

\[ \mathcal{S}_\xi = \{\emptyset\} \cup \left\{ E : \exists n \leq E \text{ and } E \in \mathcal{S}_{\xi_n+1} \right\}. \]

We refer to this sequence as the characteristic sequence in \( \xi \). We note that the sequence \( (\xi_n)_{n=1}^{\infty} \) can be chosen to satisfy the property that \( \mathcal{S}_{\xi_{n+1}} \subset \mathcal{S}_{\xi_n+1} \) for all \( n \in \mathbb{N} \), which was shown in [5]. We assume these sequences are chosen in this way, from which it follows that

\[ \mathcal{S}_\xi = \{\emptyset\} \cup \left\{ E : \emptyset \neq E \in \mathcal{S}_{\xi_{\text{min}}+1} \right\}. \]

We also recall a common method for constructing new families from old. If \( \mathcal{F}, \mathcal{G} \subset [\mathbb{N}]^{<\omega} \),

\[ \mathcal{F}[\mathcal{G}] = \{\emptyset\} \cup \left\{ \bigcup_{n=1}^m E_n : E_1 < \ldots < E_m, \emptyset \neq E_n \in \mathcal{G}, (\min E_n)_{n=1}^m \in \mathcal{F} \right\}. \]

Our main interest with this construction will be \( \mathcal{S}_\zeta[\mathcal{S}_\xi] \) for \( \zeta, \xi < \omega_1 \), in which case

\[ \mathcal{S}_\zeta[\mathcal{S}_\xi] = \{\emptyset\} \cup \left\{ \bigcup_{n=1}^m E_n : E_1 < \ldots < E_m, \emptyset \neq E_n \in \mathcal{S}_\zeta, (\min E_n)_{n=1}^m \in \mathcal{S}_\xi \right\}. \]

We note that \( \mathcal{S}_{\xi+1} = \mathcal{S}_1[\mathcal{S}_\xi] \) for all \( \xi < \omega_1 \).

We let \( \text{MAX}(\mathcal{S}_\xi) \) denote the set of all maximal elements of \( \mathcal{S}_\xi \) with respect to the initial segment order. We note that since \( \mathcal{S}_\xi \) is spreading and hereditary, \( \text{MAX}(\mathcal{S}_\xi) \) is also the set of maximal elements of \( \mathcal{S}_\xi \) with respect to inclusion. Similarly, we denote by \( \text{MAX}(\mathcal{S}_\zeta[\mathcal{S}_\xi]) \) the set of all maximal elements of \( \mathcal{S}_\zeta[\mathcal{S}_\xi] \) with respect to the initial segment ordering (equivalently, with respect to inclusion).

We isolate the following important properties of the Schreier families. Each of these properties can be found in [5].

**Lemma 4.1.** Fix \( \xi, \zeta < \omega_1 \).

(i) \( \mathcal{S}_\xi \) is a regular family with rank \( \omega^\xi + 1 \). More generally, \( \mathcal{S}_\zeta[\mathcal{S}_\xi] \) is a regular family with rank \( \omega^{\xi + \zeta} + 1 \).
(ii) If $\xi \leq \zeta$, there exists $k = k(\xi, \zeta) \in \mathbb{N}$ such that if $k \leq E \in S_\xi$, then $E \in S_\xi$.

(iii) For $E \in S_\xi$, either $E \in \text{MAX}(S_\xi)$ or $E \sim (n) \in S_\xi$ for all $E < n \in \mathbb{N}$.

(iv) For $E \in \text{MAX}(S_\xi)$, if $E \preceq F \in S_\xi$, then $E = F$.

Remark 4.2. Fix $\xi, \zeta < \omega_1$. For each infinite subset $M$ of $\mathbb{N}$, there exist a unique sequence $(F_k)_{k=1}^\infty$ of successive members in MAX$(S_\xi)$ and a unique sequence $(E_k)_{k=1}^\infty$ of successive members in MAX$(S_\xi[S_\xi])$ such that $M = \bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty F_k$. The sequence $(F_k)_{k=1}^\infty$ is called the $S_\xi$-decomposition of $M$ and the sequence $(E_k)_{k=1}^\infty$ is called the $S_\xi[S_\xi]$-decomposition of $M$. Since $S_0[S_\xi] = S_\xi$, it is sufficient to prove the $S_\xi[S_\xi]$-decomposition. We let $M|_{\xi,\zeta,0} = M|_{\xi,0} = \emptyset$, $M|_{\xi,\zeta,n} = \bigcup_{k=1}^n E_k$ and $M|_{\xi,n} = \bigcup_{k=1}^n F_k$.

In order to see that each $M$ admits such a decomposition, it is sufficient to note that there exists a (necessarily unique) initial segment of $M$ which is a maximal member of $S_\xi[S_\xi]$, which we denote by $M|_{\xi,\zeta,1}$. We then note if $E_1 = M|_{\xi,\zeta,1}$, $E_2 = (M \setminus E_1)|_{\xi,\zeta,1}$, ..., $E_{n+1} = (M \setminus \bigcup_{k=1}^n E_k)|_{\xi,\zeta,1}$, then $M = \bigcup_{k=1}^\infty E_k$ is the unique decomposition of $M$ into successive, maximal member of $S_\xi[S_\xi]$.

Let us explain how to deduce the existence of $M|_{\xi,\zeta,1}$. We let

$$A = \{ m \in \mathbb{N} : (M(1), \ldots, M(m)) \in S_\xi[S_\xi] \}.$$ 

Since $S_\xi[S_\xi]$ contains all singletons, $1 \in A \neq \emptyset$. Since $S_\xi[S_\xi]$ is hereditary, $A$ is an interval. Since $S_\xi[S_\xi]$ is compact, the set $A$ is finite. Indeed, if $A$ were infinite, then it would be the case that $A = \mathbb{N}$, which means $F_m := (M(1), M(2), \ldots, M(m)) \in S_\xi[S_\xi]$ for all $m \in \mathbb{N}$. But the sequence $(F_m)_{m=1}^\infty$ converges to $M$ and so $M \in S_\xi[S_\xi]$, contradicting the fact that $S_\xi[S_\xi]$ consists only of finite sets. Therefore the set $A$ is finite and non-empty so it has a maximum, call it $m$.

By definition of $A$ and maximality of $m$, $E_1 := (M(1), \ldots, M(m)) \in S_\xi[S_\xi]$ and $E_1 \sim (M(m + 1)) = (M(1), \ldots, M(m + 1)) \notin S_\xi[S_\xi]$. It remains to show that $E_1$ is a maximal member of $S_\xi[S_\xi]$. Here we recall that maximality with respect to inclusion and initial segment ordering are equivalent. If it were not maximal, then it would be a proper initial segment of some other member of $S_\xi[S_\xi]$, and since $S_\xi[S_\xi]$ is hereditary, $E_1 \sim (p) \in S_\xi[S_\xi]$ for some $E_1 < p$. By by Lemma 11(iii), this would imply that $E_1 \sim (M(m + 1)) = (M(1), \ldots, M(m + 1)) \in S_\xi[S_\xi]$, contradicting the maximality of $m$. This shows that $E_1$ is maximal in $S_\xi[S_\xi]$.

Note also that non-empty members of $S_\xi[S_\xi]$ admit a canonical representation: For $\emptyset \neq E \in S_\xi[S_\xi]$, we can write $E = \bigcup_{n=1}^m E_n$ with $E_n \in S_\xi$ for each $1 \leq n \leq m$, and $E_n \in \text{MAX}(S_\xi)$ for $1 \leq n < m$. If $E \in S_\xi[S_\xi]$ and if $E = \bigcup_{n=1}^m E_n$ is the canonical representation of $E$, then $(\min E_n)_{n=1}^m \in S_\xi$. The existence of this representation is established similarly to the previous paragraphs. If $F \in [\mathbb{N}]^{<\omega}$, we let $E_1$ be the maximal initial segment of $F$ which is in $S_\xi[S_\xi]$. Then either $E_1 = F$, in which case $F \in S_\xi[S_\xi]$ may be non-maximal in $S_\xi[S_\xi]$, or $E_1 < F$ and $E_1$ is maximal in $S_\xi[S_\xi]$, with the same proof as in the last paragraphs. In the latter case, we let $E_2$ denote the maximal initial segment of $F \setminus E_1$ which is a member of $S_\xi[S_\xi]$, etc. We arrive at the advertised decomposition where each set $E_k$, except possibly the last, is maximal in $S_\xi[S_\xi]$. 


We also note that the convolution sets $S_\zeta[S_\xi]$ satisfy the following identity for all countable $\zeta, \xi$:

\[
S_{\zeta+1}[S_\xi] = (S_1[S_\zeta])[S_\xi] = S_1[S_\zeta[S_\xi]]
\]

\[
= \{\emptyset\} \cup \left\{ \bigcup_{n=1}^{m} E_n : E_1 < \ldots < E_m, \emptyset \neq E_n \in S_\zeta[S_\xi], m \leq \min E_n \right\}.
\]

Note that members of $S_1[S_\zeta[S_\xi]]$ admit a canonical decomposition.

5. ProPERTIES OF INTEREST

We next define the weights which we will use throughout this work. The weights in question are the coefficients coming from the repeated averages hierarchy, and their square roots, but we find it convenient to depart somewhat from the usual notation. We begin by defining the weights $p^E_\xi$ for each $\xi < \omega_1$ and $\emptyset \neq E \in [N]^{<\omega}$. We let $p^0_E = 1$ for all $\emptyset \neq E \in [N]^{<\omega}$. Next, assume that $p^E_\zeta$ has been defined for some $\xi < \omega_1$ and every $\emptyset \neq E \in [N]^{<\omega}$. For $\emptyset \neq E \in [N]^{<\omega}$, we can uniquely write $E = \bigcup_{n=1}^{m} E_n$, $E_1 \ldots E_m$, where $E_n \in \text{MAX}(S_{\xi+1})$ for each $1 \leq n < m$ and $\emptyset \neq E_m \in S_{\xi+1}$. Since $E_m \in S_{\xi+1}$, we can uniquely write $E_m = \bigcup_{i=1}^{l} F_i$, $F_1 \ldots F_l$, with $F_n \in \text{MAX}(S_\xi)$ for each $1 \leq n < l$ and $\emptyset \neq F_l \in S_\xi$. We define $p^{E_{\xi+1}}_E = p_{F_1} \min E_m$. Assume $\xi < \omega_1$ is a limit ordinal and $p^E_\xi$ has been defined for each $\nu < \xi$ and $\emptyset \neq E \in [N]^{<\omega}$. Let $\xi \uparrow \xi$ be the characteristic sequence of $\xi$ and fix $\emptyset \neq E \in [N]^{<\omega}$. We can uniquely write $E = \bigcup_{n=1}^{m} E_n$, $E_1 \ldots E_m$, $E_n \in \text{MAX}(S_\xi)$ for each $1 \leq n < m$ and $E_m \in S_\xi$. Then $E_m \in S_{\min E_m+1}$. We define $p^E_\xi = p^{E_{\min E_m+1}}_E$.

Next, for $\xi, \zeta < \omega_1$, we define coefficients $q^E_\xi$ for each $\emptyset \neq E \in [N]^{<\omega}$. The definition is by induction on $\zeta$ for $\xi$ held fixed. We let $q^{E,0}_\xi = 1$ for all $\emptyset \neq E \in [N]^{<\omega}$. Assume that $q^{E,\nu}_\xi$ has been defined for some $\zeta < \omega_1$ and all $\emptyset \neq E \in [N]^{<\omega}$. We can uniquely write $E = \bigcup_{n=1}^{m} E_n$, $E_1 \ldots E_m$, $E_n \in \text{MAX}(S_{\zeta+1}[S_\xi])$ for each $1 \leq n < m$, and $E_m \in S_{\zeta+1}[S_\xi] = S_1[S_\zeta[S_\xi]]$. We then express $E_m = \bigcup_{i=1}^{l} F_i$, $F_1 \ldots F_l$, $F_n \in \text{MAX}(S_{\zeta}[S_\xi])$ for each $1 \leq n < l$, and $F_l \in S_{\zeta}[S_\xi]$. We then define

\[
q^{E,\zeta+1}_\xi = q^{E,\zeta}_\xi / \sqrt{\min E_m}.
\]

Assume $\zeta < \omega_1$ is a limit ordinal and $q^{E,\nu}_\xi$ has been defined for each $\nu < \zeta$ and $\emptyset \neq E \in [N]^{<\omega}$. Let $\zeta_n \uparrow \zeta$ be the characteristic sequence of $\zeta$ and fix $\emptyset \neq E \in [N]^{<\omega}$. We can uniquely write $E = \bigcup_{n=1}^{m} E_n$, $E_1 \ldots E_m$, $E_n \in \text{MAX}(S_{\zeta}[S_\xi])$ for each $1 \leq n < m$, and $E_m \in S_{\zeta}[S_\xi]$. Then $E_m \in S_{\zeta_{\min E_m+1}}$. We define

\[
q^{E,\zeta}_\xi = q^{E,\zeta_{\min E_m+1}}_E.
\]

We next isolate the permanence and convexity properties of the coefficients defined above.

**Proposition 5.1.**  
(i) Suppose $E = \bigcup_{n=1}^{m} E_n$, where $E_1 \ldots E_n$ are successive members of $S_\xi$ such that $E_m \in \text{MAX}(S_\xi)$ for all $1 \leq m < n$ and $E_n \in S_\xi$. Then $p^E_\xi = p^E_\xi$.

(ii) Suppose $E = \bigcup_{n=1}^{m} E_n$, where $E_1 \ldots E_n$ are successive members of $S_\xi[S_\xi]$ such that $E_m \in \text{MAX}(S_\xi[S_\xi])$ for each $1 \leq m < n$ and $E_n \in S_\xi[S_\xi]$. Then $q^{E,\zeta}_\xi = q^{E,\zeta}_E$.

(iii) Assume $E = \bigcup_{n=1}^{m} E_n$, $E_1 \ldots E_n$, $E_m \in \text{MAX}(S_\xi)$ for all $1 \leq m \leq n$. Then for each $1 \leq l \leq n$,

\[
1 = \sum_{E_m \preceq E \preceq \bigcup_{m=1}^{l-1} E_m} p^E_\xi.
\]
(iv) Assume \( E = \bigcup_{m=1}^{n} E_m, E_1 < \ldots < E_n, E_m \in \text{MAX}(S_{\xi|S_{\xi}}) \) for all \( 1 \leq m \leq n \). Then for each \( 1 \leq l \leq n \), if \( E_l = \bigcup_{r=1}^{m} F_r, \) where \( F_1 < \ldots < F_s, F_r \in \text{MAX}(S_{\xi}), 1 \leq r \leq s \), then

\[
1 = \sum_{r=1}^{s} \sum_{G \cup \bigcup_{r=1}^{m} F_r} q_{F_r}^{\xi, \xi} p_{F_r}^{\xi, \xi},
\]

where \( G = \bigcup_{m=1}^{l-1} E_m \).

Proof. (i) We work by induction on \( \xi \). For \( \xi = 0 \), \( p_{E_0}^0 = 1 = p_{E_0}^E \). Next, assume the result holds for \( \xi \). By the definition of the weights, \( p_{E_1}^{\xi} = p_{E_1}^{\xi}/\min E_n = p_{E_n}^{\xi+1} \), where \( E_n = \bigcup_{k=1}^{l} F_k, F_1 < \ldots < F_l, F_k \in \text{MAX}(S_{\xi}) \) for \( 1 \leq k < l \), and \( F_1 \in S_{\xi} \). If \( \xi \) is a limit ordinal, then \( p_{E_1}^{\xi} = p_{E_n}^{\xi+1} = p_{E_n}^{\xi} \), where \( \xi_m \uparrow \xi \) is the characteristic sequence of \( \xi \).

(ii) We work by induction on \( \zeta \) with \( \xi \) held fixed. The \( \zeta = 0 \) case is trivial, since \( q_{E_0}^{\xi,0} = 1 \) for all \( E \). For the successor case, \( q_{E_1}^{\xi,\xi+1} = q_{E_1}^{\xi,\xi}/\sqrt{\min E_n} = q_{E_n}^{\xi+1} \), where \( E_n = \bigcup_{k=1}^{l} F_k, F_1 < \ldots < F_l, F_k \in \text{MAX}(S_{\xi|S_{\xi}}) \) for all \( 1 \leq k < l \), and \( F_1 \in S_{\xi}(S_{\xi}) \). If \( \zeta \) is a limit ordinal, \( q_{E_1}^{\xi,\xi} = q_{E_n}^{\xi+1} = q_{E_n}^{\xi} \), where \( \zeta_m \uparrow \xi \) is the characteristic sequence of \( \zeta \).

(iii) By (i), it is sufficient to prove the result in the case \( n = 1 \). We prove this result by induction on \( \xi \). For the case \( \xi = 0 \), \( E \) is a singleton and \( p_{E_1}^{E} = 1 \), so the sum is of a single term equal to 1. For the successor case, let \( E \in \text{MAX}(S_{\xi+1}) \) and \( (F_k)_{k=1}^{l} \) be the \( S_{\xi} \)-decomposition of \( E \). Then \( l = \min E = \min F_1 \). Then by the inductive hypothesis,

\[
\sum_{F \subseteq E} p_{E_1}^{\xi} = \sum_{k=1}^{l} \sum_{E|\xi,k} = \sum_{k=1}^{l} \sum_{E \subseteq G} p_{E_1}^{\xi} = \sum_{k=1}^{l} \min E \sum_{E \subseteq G} p_{E}^{\xi} = 1.
\]

If \( \xi \) is a limit ordinal, then since \( E \in \text{MAX}(S_{\xi}), E \in \text{MAX}(S_{\xi}^{\min,E+1}) \), where \( \xi_m \uparrow \xi \) is the characteristic sequence of \( \xi \). Then

\[
\sum_{F \subseteq E} p_{E}^{\xi} = \sum_{F \subseteq E} q_{E}^{\xi+1} = 1.
\]

(iv) By (ii), it is sufficient to assume \( n = 1 \). In the \( n = 1 \) case, \( G = \emptyset \). We work by induction on \( \zeta \). The \( \zeta = 0 \) case is (iii). We next prove the successor case. Let \( E \in \text{MAX}(S_{\xi+1}|S_{\xi}) \), and write \( E = \bigcup_{r=1}^{m} F_r \) where \( F_1, \ldots, F_s \) are successive, maximal members of \( S_{\xi} \). We note that necessarily \( \min F_r \in \text{MAX}(S_{\xi+1}) = \text{MAX}(S_{\xi}|S_{\xi}) \). Therefore there exist \( 0 = t_0 < t_1 < \ldots < t_j \) such that for each \( 1 \leq m \leq j \), \( \min F_r \in \text{MAX}(S_{\xi}|S_{\xi}) \), and such that \( j = \min E = \min F_1 \). Therefore with \( H_m = \bigcup_{r=m+1}^{t_m} F_r \), it holds that \( H_m \in \text{MAX}(S_{\xi}|S_{\xi}) \) for each \( 1 \leq m \leq j \). Note that \( \min H_1 = \min E = j \). Define \( G_m = \bigcup_{r=m}^{t_m-1} F_r \) for \( 1 \leq m \leq j \). By the definition of \( q_{E_1}^{\xi,\xi} \) and (ii), it holds that \( q_{G_m \cup F}^{\xi,\xi} = q_{G_m \cup F}^{\xi,\xi}/\sqrt{f} \) for each \( 1 \leq m \leq j \) and \( E \subseteq H_m \). By the inductive hypothesis,

\[
\sum_{r=1}^{s} \sum_{\bigcup_{r=1}^{m} F_r \subseteq \bigcup_{r=1}^{m+1} F_r} q_{F_r}^{\xi+1} \left[ \frac{p_{F}^{\xi,\xi}}{p_{F_r}^{\xi,\xi}} \right]^2 = \sum_{m=1}^{j} \sum_{r=m+1}^{t_m} \left[ \sum_{\bigcup_{r=1}^{m} F_r \subseteq \bigcup_{r=1}^{m+1} F_r} q_{F_r}^{\xi,\xi} \right]^2 = \sum_{m=1}^{j} \frac{1}{j} = 1.
\]

For the limit ordinal case, \( E \in \text{MAX}(S_{\xi}|S_{\xi}) \) implies that \( E \in \text{MAX}(S_{\xi}^{\min,E+1}|S_{\xi}) \). If \( E = \bigcup_{r=1}^{m} F_r \), where \( F_1, \ldots, F_r \) are successive, maximal members of \( S_{\xi} \), then \( q_{E_1}^{\xi,\xi} = q_{E}^{\xi,\xi} \) for each
Given ordinals $\xi, \zeta < \omega_1$, a positive integer $n \in \mathbb{N}$, and an infinite set $N \in [\mathbb{N}]$, we define

$$\mathbb{E}_{\xi, N} u(n) = \sum_{N|\xi, n-1 < E \leq N|\xi, n} p_E^\xi u_E$$

and $\mathbb{E}_{\xi, Nu} = (\mathbb{E}_{\xi, N} u(n))_{n=1}^{\infty}$. We define

$$\mathbb{E}_{2}^{(2)} \xi, \zeta, N u(n) = \sum_{N|\xi, \zeta, n-1 < E \leq N|\xi, \zeta, n} q_{E}^\xi p_E^\xi u_E$$

and $\mathbb{E}_{2}^{(2)} \xi, \zeta, Nu = (\mathbb{E}_{2}^{(2)} \xi, \zeta, N u(n))_{n=1}^{\infty}$.

If $V$ is a Banach space and $A : U \rightarrow V$ is an operator, then for $\mathbb{E} = (u_E)_{\varnothing \neq E \in [\mathbb{N}] < \omega}$ of $U$, $\xi, \zeta < \omega_1$, and $N \in [\mathbb{N}]$, we let $A\mathbb{E}_{\xi, N} u = (A\mathbb{E}_{\xi, N} u(n))_{n=1}^{\infty}$ and $A\mathbb{E}_{2}^{(2)} \xi, \zeta, N u = (A\mathbb{E}_{2}^{(2)} \xi, \zeta, N u(n))_{n=1}^{\infty}$.

For a Banach space $U$ and a collection $\mathbb{E} = (u_E)_{E \in [\mathbb{N}] < \omega}$ in $U$, we say $\mathbb{E}$ is weakly null provided that for each $F \in [\mathbb{N}]^{<\omega}$, $(u_{F \cap (k)})_{F \prec k}$ is weakly null.

Given an ordinal $\xi < \omega_1$, a constant $\gamma \geq 0$, Banach spaces $U, V$, and an operator $A : U \rightarrow V$, we say $A$ has the $\gamma$-S$_\xi$ Property provided that for any weakly null collection $\mathbb{E} = (u_E)_{\varnothing \neq E \in [\mathbb{N}] < \omega}$ in $B_U$ and any $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that $\|A\mathbb{E}_{\xi, M} u\|_2^w \leq \gamma$. We say the Banach space $U$ has the $\gamma$-S$_\xi$ Property if $I_X$ does. We say $A$ (resp. $X$) has the $S_\xi$ Property if it has the $\gamma$-S$_\xi$ Property for some $\gamma \geq 0$.

Given ordinals $\xi, \zeta < \omega_1$, a constant $\gamma \geq 0$, Banach spaces $U, V$, and an operator $A : U \rightarrow V$, we say $A$ has the $\gamma$-G$_{\xi, \zeta}$ Property provided that for any weakly null collection $\mathbb{E} = (u_E)_{\varnothing \neq E \in [\mathbb{N}] < \omega}$ in $B_U$ and any $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that $\|A\mathbb{E}_{2}^{(2)} \xi, \zeta, M u\|_1^w \leq \gamma$. We say the Banach space $U$ has the $\gamma$-G$_{\xi, \zeta}$ Property if $I_X$ does. We say $A$ (resp. $X$) has the $G_{\xi, \zeta}$ Property if it has the $\gamma$-G$_{\xi, \zeta}$ Property for some $\gamma \geq 0$.

For Banach spaces $U, V$, we let $\mathcal{G}_\xi(U, V)$ denote the class of all operators $A : U \rightarrow V$ which have property $S_\xi$. We let $\mathcal{G}_\xi$ denote the class $\bigcup_{U, V \in \text{Ban}} \mathcal{G}_\xi(U, V)$, where $\text{Ban}$ is the class of all Banach spaces over $\mathbb{K}$. For $A \in \mathcal{G}_\xi$, we let $s_\xi(A)$ denote the infimum of $\gamma \geq 0$ such that $A$ has the $S_\xi$ Property. We similarly define $\mathcal{G}_{\xi, \zeta}$ and $g_{\xi, \zeta}$. If $A$ is an operator which fails the $S_\xi$ Property (resp. the $G_{\xi, \zeta}$ Property), we define $s_\xi(A) = \infty$ (resp. $g_{\xi, \zeta}(A) = \infty$). If $U$ is a Banach space, we write $s_\xi(U)$ (resp. $g_{\xi, \zeta}(U)$) instead of $s_\xi(I_U)$ (resp. $g_{\xi, \zeta}(I_U)$).

We recall that an operator $A : U \rightarrow V$ is completely continuous provided that it maps weakly null sequences to norm null sequences.

**Example 5.1.** Let $A : U \rightarrow V$ be a completely continuous operator, then $A$ has the $\gamma$-S$_\xi$ Property and $s_\xi(A) = 0$. In particular, $\mathcal{G}_\xi$ contains all compact operators, and in particular all finite rank operators.
Fix $\gamma > 0$. Let $u = (u_E)_{\sigma \neq E \in [\mathbb{N}]}$ be a weakly null collection in $B_U$ and $L \in [\mathbb{N}]$. Since $A$ sends weakly null sequences to norm null sequences, we can select $M = (l_1, l_2, \ldots) \in [L]$ such that $\sum_{n=1}^{\infty} \|A u(n)\| < \gamma$.

Since

$$AE_{\xi,M}u(n) = \sum_{M | \xi, n \prec E \preceq M | \xi, n} p_E^\xi A u_E \in \left\{ \sum_{E \preceq M} a_E A u_E : (a_E)_{E \preceq M} \in c_00 \cap B_{\ell_\infty} \right\},$$

for all $n \in \mathbb{N}$, it holds that

$$\|AE_{\xi,M}u\|_2 \leq \sum_{m=1}^{\infty} \|A u_M(m)\| < \gamma.$$ 

Since $\gamma$ was arbitrary, we have shown that $s_\xi(A) = 0$.

**Remark 5.2.** As shown in Proposition 5.1, for any any Banach spaces $U, V$, any operator $A : U \rightarrow V$, any collection $u = (u_E)_{\sigma \neq E \in [\mathbb{N}]}$ in $U$, any ordinals $\xi, \zeta < \omega_1$, any $M \in [\mathbb{N}]$, and any $m \in \mathbb{N}$,

$$AE^{(2)}_{\xi,\zeta,M}u(m) \in \left\{ A \sum_{n=1}^{\infty} a_n E_{\xi,M}u(n) : (a_n)_{n=1}^{\infty} \in c_00, \sum_{n=1}^{\infty} |a_n|^2 = 1 \right\}.$$ 

Therefore if $\|AE_{\xi,M}u\|_2 \leq \gamma$, $\|AE^{(2)}_{\xi,\zeta,M}u(m)\| \leq \gamma$ for all $\zeta < \omega_1$ and $m \in \mathbb{N}$. One reason we isolate the property $S_\xi$ is to guarantee that the $\ell_2$ averaging in the definition of $E^{(2)}_{\xi,\zeta,M}u$ results in sequences that remain bounded.

Let us recall a special case of the infinite Ramsey theorem, the proof of which was achieved in steps by Nash-Williams [19], Galvin and Prikry [14], Silver [28] and Ellentuck [13].

**Theorem 5.3.** If $V \subset [\mathbb{N}]$ is closed, then for any $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that either $[M] \subset V$ or $[M] \cap V = \emptyset$.

We deduce the following easy consequence.

**Proposition 5.4.** Let $\zeta, \xi$ be countable ordinals, $\gamma \geq 0$ a constant, $U, V$ Banach spaces, and $A : U \rightarrow V$ an operator.

(i) If $A$ has the $\gamma$-$S_\xi$ Property, then for any weakly null collection $u = (u_E)_{\sigma \neq E \in [\mathbb{N}]}$ in $B_U$ and any $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that for all $N \in [M]$, $\|AE_{\xi,N}u\|_2 \leq \gamma$.

(ii) If $A$ has the $\gamma$-$G_{\xi,\zeta}$ Property, then for any weakly null collection $u = (u_E)_{\sigma \neq E \in [\mathbb{N}]}$ in $B_U$, any $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that for all $N \in [M]$, $\|AE^{(2)}_{\xi,\zeta,N}u\|_1 \leq \gamma$.

**Proof.** (i) Assume $A : U \rightarrow V$ has the $\gamma$-$S_\xi$ Property. Fix a weakly null collection $u = (u_E)_{\sigma \neq E \in [\mathbb{N}]}$ in $B_U$ and $L \in [\mathbb{N}]$. For $m \in \mathbb{N}$, let $V_m$ denote the set of all $M \in [\mathbb{N}]$ such that $\|(AE_{\xi,M}u(n))_{n=1}^{m}\|_2 \leq \gamma$. Let $V = \cap_{m=1}^{\infty} V_m$. Note that the conclusion of (i) is that there exists $M \in [L]$ such that $[M] \subset V$. The hypothesis that $A$ has the $\gamma$-$S_\xi$ Property implies that for any $M \in [L]$, $[M] \cap V \neq \emptyset$. Therefore in order to deduce the existence of the desired $M$, we need only to show that each set $V_m$ is closed and by Theorem 5.3 there must exist some $M \in [L]$ such that either $[M] \subset V$ or $[M] \cap V = \emptyset$. But the $\gamma$-$S_\xi$ Property precludes the latter alternative. Let us show that $V_m$ is closed. By the permanence property, if $M \in [\mathbb{N}]$ is fixed and $E_1 \prec \ldots \prec E_m$ are successive, maximal members of
\( \mathcal{S}_\xi \) such that \( \bigcup_{n=1}^m E_n \) is an initial segment of \( M \), then \( E_{\xi,M} u(n) = E_{\xi,N} u(n) \) for each \( 1 \leq n \leq m \) and any \( N \in [N] \) such that \( \bigcup_{n=1}^m E_n \) is an initial segment of \( N \). From this it follows that for a fixed \( m \), the map \( M \mapsto \| (A E_{\xi,M} u(n))_{n=1}^m \|_2^\alpha \) is locally constant, and therefore continuous. From this it follows that the preimage of \( [0, \gamma] \) under this map, which is \( \mathcal{V}_m \), is closed.

\( (ii) \) The proof of \( (ii) \) is an inessential modification of the proof of \( (i) \).

\[ \square \]

**Remark 5.5.** It is quite clear that if \( Y \) is a subspace of \( X \), then \( \mathcal{S}_\xi(Y) \leq \mathcal{S}_\xi(X) \) and \( \mathcal{G}_{\xi,\zeta}(Y) \leq \mathcal{G}_{\xi,\zeta}(X) \) for any countable \( \xi, \zeta \). Moreover, it is easy to see that for any countable \( \xi, \zeta \), any Banach spaces \( W, X, Y, Z \), and any operators \( C : W \to X, B : X \to Y, \) and \( A : Y \to Z, \mathcal{S}_\xi(ABC) \leq \| A \| \| \mathcal{G}_{\xi}(B) \| C \| \) and \( \mathcal{G}_{\xi}(ABC) \leq \| A \| \| \mathcal{G}_{\xi}(B) \| C \| \), and in fact we will prove this in Lemma 5.8. From this it follows that if \( A : X \to Y \) is an operator and \( | \cdot |_X, | \cdot |_Y \) are equivalent norms on \( X \) and \( Y \), respectively, then \( A : X \to Y \) lies in \( \mathcal{S}_\xi \) (resp. \( \mathcal{G}_{\xi,\zeta} \)) if and only if \( A : (X, | \cdot |_X) \to (Y, | \cdot |_Y) \) does. Therefore when considering membership of an operator in the classes \( \mathcal{S}_\xi, \mathcal{G}_{\xi,\zeta} \), we can renorm the domain and codomain for convenience.

Let us note the following easy fact.

**Proposition 5.6.** If \( X \) is a Banach space not containing \( \ell_1 \), \( Y \) a subspace of \( X \), and \( (z_n)_{n=1}^\infty \) a weakly null sequence in \( B_X \), then for any \( L \in [N] \), there exist \( M \in [L] \) and a weakly null sequence \( (x_n)_{n=1}^\infty \) in \( 3B_X \) such that \( Q x_n = z_n \) for all \( n \in M \). Here, \( Q : X \to X/Y \) denotes the quotient map.

**Proof.** Fix a sequence \( (u_n)_{n=1}^\infty \) in \( (4/3)B_X \) such that \( Qu_n = z_n \) for all \( n \in N \). Fix \( L \in [N] \). Since \( X \) does not contain a copy of \( \ell_1 \), there exists \( M \in [L] \) such that the sequence \( (u_n)_{n \in M} \) is weakly Cauchy. Since \( (z_n)_{n=1}^\infty \) is weakly null, we may choose finite sets \( I_1 < I_2 < \ldots, I_n \subset M \), and positive numbers \( (w_i)_{i \in \bigcup_{n=1}^\infty I_n} \) such that \( 1 = \sum_{i \in I_n} w_i \) and \( \| \sum_{i \in I_n} w_i z_i \| < 2^{-n}/3 \). We may choose for each \( n \in N \) some \( v_n \in (2^{-n}/3)B_X \) such that \( Q v_n = \sum_{i \in I_n} w_i z_i. \) Let \( x_n = u_n - \sum_{i \in I_n} w_i u_i + v_n \) for \( n \in M \) and let \( x_n = 0 \) for \( n \in N \setminus M \). Since \( (u_n)_{n \in M} \) is weakly Cauchy, \( (u_n - \sum_{i \in I_n} w_i u_i)_{n \in M} \) is weakly null and since \( (v_n)_{n \in M} \) is norm null, the sequence \( (x_n)_{n \in M} \) is weakly null. Since \( x_n = 0 \) for all \( n \in N \setminus M \), \( (x_n)_{n=1}^\infty \) is weakly null. Note that

\[ Q x_n = Q u_n - \sum_{i \in I_n} w_i Q x_i + Q v_n = z_n - \sum_{i \in I_n} w_i z_i + \sum_{i \in I_n} w_i z_i = z_n. \]

For each \( n \in N \), \( \| x_n \| = 0 \) if \( n \in N \setminus M \), and for \( n \in M \),

\[ \| x_n \| \leq \| u_n \| + \sum_{i \in I_n} w_i \| u_i \| + \| v_n \| < 4/3 + 4/3 + 1/3 = 3. \]

\[ \square \]

**Proposition 5.7.** Let \( X \) be a Banach space not containing \( \ell_1 \), \( Y \) a subspace of \( X \), and let \( (x_E)_{E \in [N] < \omega \setminus \emptyset} \) be a weakly null collection in \( B_X \). Then for any \( L_0 \in [N] \), there exist a weakly null collection \( (x_E)_{E \in [N] < \omega \setminus \emptyset} \) in \( 3B_X \) and \( L \in [L_0] \) such that for any \( M \in [L] \), there exists \( N \in [M] \) such that \( Q x_N/n = u_N/n \) for all \( n \in N \).

**Proof.** In the proof, recall the convention that \( k > \emptyset \) for all \( k \in N \).
Let $E_1, E_2, \ldots$ be an enumeration of $[N]^{<\omega}$. By Proposition 5.6, for each $n \in \mathbb{N}$ and $L \in [N]$, there exist $M \in [L]$ and a weakly null collection $(x_{E_n^<(k)})_{k \geq E_n}$ in $3B_X$ such that $Q x_{E_n^<(k)} = u_{E_n^<(k)}$ for all $E_n < k \in M$. By Proposition 5.6, we can recursively select $L_0 \supset L_1 \supset L_2 \supset \ldots$ and weakly null sequences $(x_{E_n^<(k)})_{k \geq E_n}$ in $3B_X$ such that for all $n \in \mathbb{N}$, for all $k$, $E_n < k \in L_n$, $Q x_{E_n^<(k)} = u_{E_n^<(k)}$. This defines the entire collection $(x_E)_{\sigma \neq E \in [N]^{<\omega}}$, which is a weakly null collection, since $(x_{E_n^<(k)})_{k \geq E_n}$ is weakly null for each $n \in \mathbb{N}$. Choose $l_1 < l_2 < \ldots$ such that $E_n < l_n \in L_n$ and let $L = (l_1, l_2, \ldots)$. Note that $(l_n, l_{n+1}, \ldots) \subset L_n$ for all $n \in \mathbb{N}$, from which it follows that for any $M \in [L]$, $k, m \in \mathbb{N}$, there exist $n \in (m, \infty) \cap M \cap L_k$. Fix $k_1 \in \mathbb{N}$ such that $\emptyset = E_{k_1}$ and fix $n_1 \in M \cap L_{k_1}$. Note that $Q x_{(n_1)} = Q x_{E_{k_1}^<(n_1)} = u_{E_{k_1}^<(n_1)} = u_{(n_1)}$, since $E_{k_1} < n_1 \in L_{k_1}$. Now assuming $n_1 < \ldots < n_t$ have been chosen, fix $k_{t+1} \in \mathbb{N}$ such that $(n_1, \ldots, n_t) = E_{k_{t+1}}$. Fix $n_{t+1} \in (n_t, \infty) \cap M \cap L_{k_{t+1}}$. Then $Q x_{(n_1, \ldots, n_{t+1})} = Q x_{E_{k_{t+1}}^<(n_{t+1})} = u_{E_{k_{t+1}}^<(n_{t+1})} = u_{(n_1, \ldots, n_{t+1})}$, since $E_{k_{t+1}} < n_{t+1} \in L_{k_{t+1}}$. This completes the recursive choice. Let $N = (n_1, n_2, \ldots)$. □

Lemma 5.8. Fix $\xi, \zeta < \omega_1$. Let $U, V$ be Banach spaces and $A : U \to V$ be an operator.

(i) The functionals $s_\xi$ and $g_{\xi, \zeta}$ satisfy the triangle inequality on $G_\xi(U, V)$ and $G_{\xi, \zeta}(U, V)$, respectively.

(ii) The classes $G_\xi$, $G_{\xi, \zeta}$ are operator ideals.

(iii) If $X$ is a Banach space which does not contain an isomorphic of $\ell_2$ and $Q : X \to U$ is a surjection such that $AQ \in G_\xi$ (resp. $AQ \in G_{\xi, \zeta}$), then $A \in G_\xi$ (resp. $G_{\xi, \zeta}$).

(iv) If $X$ is a Banach space and $j : V \to X$ is an isomorphic embedding such that $jA \in G_\xi$ (resp. $jA \in G_{\xi, \zeta}$), then $A \in G_\xi$ (resp. $A \in G_{\xi, \zeta}$).

Proof. (i) Fix $A, B \in G_\xi(U, V)$. Fix $\alpha > s_\xi(A)$ and $\beta > s_\xi(B)$. By the definition of $s_\xi$, $A$ has the $\alpha$-$S_\xi$ Property and $B$ has the $\beta$-$S_\xi$ Property. Fix $L \in [N]$. Let $u = (u_E)_{\sigma \neq E \in [N]^{<\omega}}$ be a weakly null collection in $B_U$. By Proposition 5.3, there exists $M_1 \in [L]$ such that for all $N \in [M_1]$, $\| A E_{\xi, \zeta} u \|_2^w \leq \alpha$. Since $B$ has the $\beta$-$S_\xi$ Property, there exists $M \in [M_1]$ such that $\| B E_{\xi, \zeta} u \|_2^w \leq \beta$. Therefore

$$\|(A + B) E_{\xi, \zeta} u \|_2^w \leq \| A E_{\xi, \zeta} u \|_2^w + \| B E_{\xi, \zeta} u \|_2^w \leq \alpha + \beta.$$ 

Since this holds for any such $u$ and $L$, $s_\xi(A + B) \leq \alpha + \beta$. Since $\alpha > s_\xi(A)$ and $\beta > s_\xi(B)$ were arbitrary, $s_\xi(A + B) \leq s_\xi(A) + s_\xi(B)$. The proof of the analogous statement for $g_{\xi, \zeta}$ is similar.

(ii) By (i), if $A, B \in G_\xi(U, V)$, then $s_\xi(A + B) \leq s_\xi(A) + s_\xi(B)$, so $A + B \in G_\xi(U, V)$. It is quite clear that $s_\xi(cA) = |c| s_\xi(A)$ for any $A : U \to V$ and any scalar $c$.

Next, we show the ideal property. We recall that $G_\xi$ contains all compact operators and, in particular, all finite rank operators (5.1). Fix Banach spaces $T, U, V, W$ and operators $C : T \to U$, $B : U \to V$, $A : V \to W$ such that $B \in G_\xi$, $\| A \|, \| C \| \leq 1$. Then for any weakly null collection $t = (t_E)_{\sigma \neq E \in [N]^{<\omega}}$ in $B_T$ and $L \in [N]$, since $u := (C t_E)_{\sigma \neq E \in [N]^{<\omega}}$ is a weakly null collection in $B_U$, for any $\varepsilon > 0$, there exists $M \in [L]$ such that $\| B C E_{\xi, \zeta} t \|_2^w = \| B E_{\xi, \zeta} u \|_2^w \leq s_\xi(B) + \varepsilon$. Then $\| A B C E_{\xi, \zeta} t \|_2^w \leq \| A \| \| B C E_{\xi, \zeta} t \|_2^w \leq s_\xi(B) + \varepsilon$. Since such $t$, $L$, and $\varepsilon > 0$ were arbitrary,
\[ \mathbf{g}_\xi(ABC) \leq \mathbf{g}_\xi(B), \text{ and } ABC \in \mathfrak{S}_\xi. \] If we remove the assumption that \( \|A\|, \|C\| \leq 1 \), then by homogeneity, we obtain the estimate \( \mathbf{g}_\xi(ABC) \leq \|A\| \mathbf{g}_\xi(B) \|C\| \). This concludes (ii) for \( \mathfrak{S}_\xi \).

For \( \mathfrak{S}_{\xi,\zeta} \), the proof is an inessential modification. The only difference is that in the proof that \( \mathfrak{S}_{\xi,\zeta} \) contains all completely continuous operators, we use the membership

\[ A \mathbf{E}^{(2)}_{\xi,\zeta,M} u(n) \in \left\{ \sum_{m=1}^{\infty} a_m Au_M|m : (a_m)_{m=1}^{\infty} \in C_{00} \cap B_{\ell_\infty} \right\} \]

for all \( n \in \mathbb{N} \).

(iii) Assume \( AQ \in \mathfrak{S}_\xi \). If \( Q : X \to U \) is a surjection, then \( U \) is isomorphic to \( X/\ker(Q) \). By renorming \( U \), we can assume \( Q : X \to U \) is a quotient map. We note that this does not affect the conclusion of (iii), as noted in Remark 5.5. Assume \( \varepsilon > 0 \) is such that \( \mathbf{g}_\xi(A) > \varepsilon \). Then there exist a weakly null collection \( u = (u_E)_{E \neq E \in [N]^{<\omega}} \) in \( B_U \) and \( L_0 \in [N] \) such that for all \( M \in [L_0] \), \( \|A \mathbf{E}_{\xi,M} u\| > \varepsilon \). By Proposition 5.7, there exist a weakly null collection \( r = (x_E)_{E \neq E \in [N]^{<\omega}} \subset 3B_X \) and \( L \in [L_0] \) such that for any \( M \in [L] \), there exists \( N \in [M] \) such that \( Q x_{N^{|n|}} = u_{N^{|n|}} \) for all \( n \in \mathbb{N} \). The latter condition implies that for any \( M \in [L] \), there exists \( N \in [M] \) such that \( A Q \mathbf{E}_{\xi,N} r = A \mathbf{E}_{\xi,N} u \). Since \( AQ \in \mathfrak{S}_\xi \) and since \( (x_E/3)_{E \neq E \in [N]^{<\omega}} \subset 3B_X \) is a weak null collection, for any \( \gamma > \mathbf{g}_\xi(AQ) \), there exists \( M \in [L] \) such that for all \( N \in [M] \), \( \|A Q \mathbf{E}_{\xi,N} r\|_2 \leq 3\gamma \). In particular, if \( N \in [M] \) is chosen such that \( A Q \mathbf{E}_{\xi,N} r = A \mathbf{E}_{\xi,N} u \), then

\[ \varepsilon \leq \|A \mathbf{E}_{\xi,N} u\|_2^w = \|A Q \mathbf{E}_{\xi,N} r\|_2 \leq 3\gamma. \]

From this it follows that \( \mathbf{g}_\xi(A) \leq 3\mathbf{g}_\xi(AQ) \). The proof for \( \mathfrak{g}_{\xi,\zeta} \) is similar.

(iv) As noted in Remark 5.5, by renorming \( V \), we can assume \( j \) is an isometric embedding. Then since \( \|j A \mathbf{E}_{\xi,M} u\|_2^w = \|A \mathbf{E}_{\xi,M} u\|_2^w \) for any countable \( \xi \), any \( M \in [N] \), and any weakly null collection \( u \subset B_U \), it follows immediately that \( \mathbf{g}_\xi(A) = \mathbf{g}_\xi(jA) \). The proof for \( \mathfrak{g}_{\xi,\zeta} \) is similar.

\[ \Box \]

**Corollary 5.9.** Fix ordinals \( \xi, \zeta < \omega_1 \). Let \( X, X_1, Y, Y_1 \) be Banach spaces and let \( A : X \to X_1 \), \( B : Y \to Y_1 \) be operators such that \( B \) is finite rank. Let \( A \otimes B : X \hat{\otimes}_\pi Y \to X_1 \hat{\otimes}_\pi Y_1 \) be the product of \( A \) and \( B \). Then if \( \mathfrak{g}_{\xi,\zeta}(A) < \infty \) (resp. \( \mathfrak{g}_{\xi,\zeta}(A) = 0 \)) then \( \mathfrak{g}_{\xi,\zeta}(A \otimes B) < \infty \) (resp. \( \mathfrak{g}_{\xi,\zeta}(A \otimes B) = 0 \)).

**Proof.** If \( B \) is the zero operator, then so is \( A \otimes B \), and \( \mathfrak{g}_{\xi,\zeta}(A \otimes B) = 0 < \infty \).

If \( B \) is rank 1, then \( B = y^* \otimes y_1 \) for some \( y^* \in Y^* \) and \( y_1 \in Y_1 \). Define \( L : X \hat{\otimes}_\pi Y \to X \) to be the linear extension of the map defined on elementary tensors by \( L(x \otimes y) = y^*(y)x \). Let \( R : X_1 \to X_1 \hat{\otimes}_\pi Y_1 \) be the linear extension of the map defined on elementary tensors by \( R(x_1) = x_1 \otimes y_1 \). Then \( A \otimes B \) factors through \( LR \), where \( \|L\| \|R\| = \|y^*\| \|y_1\| \). By Lemma 5.8(iii), \( \mathfrak{g}_{\xi,\zeta}(A \otimes B) \leq \mathfrak{g}_{\xi,\zeta}(A) \|y^*\| \|y_1\| \). This inequality implies the result in the case that \( B \) is rank 1.

In the rank of \( B \) is \( n > 1 \), then we can write \( B = \sum_{i=1}^n B_i \), where \( B_i \) is rank 1 for each \( 1 \leq i \leq n \). Then

\[ \mathfrak{g}_{\xi,\zeta}(A \otimes B) = \mathfrak{g}_{\xi,\zeta}\left( A \otimes \sum_{i=1}^n B_i \right) = \mathfrak{g}_{\xi,\zeta}\left( \sum_{i=1}^n A \otimes B_i \right) \leq \sum_{i=1}^n \mathfrak{g}_{\xi,\zeta}(A \otimes B_i). \]

We conclude by the previous paragraph.

\[ \Box \]
Lemma 5.10. Suppose that $U, V$ are Banach spaces, $\xi, \zeta < \omega_1$ are ordinals, and $A : U \to V$ is an operator such that for each weakly null collection $u = (u_E)_{E \not\in \mathbb{N}^\omega} \in B_U$, each $\varepsilon > 0$, and each $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that $\|A E_{\xi, \zeta, M,u}(1)\| < \varepsilon$.

(i) For any sequence $(\varepsilon_n)_{n=1}^\infty$ in $(0, \infty)$, any weakly null collection $u = (u_E)_{E \not\in \mathbb{N}^\omega} \in B_U$, and any $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that for all $n \in \mathbb{N}$, $\|A E_{\xi, \zeta, M,u}(n)\| < \varepsilon_n$.

(ii) $g_{\xi, \zeta}(A) = 0$.

Proof. (i) Fix $u = (u_E)_{E \not\in \mathbb{N}^\omega} \in B_U$, $(\varepsilon_n)_{n=1}^\infty$ in $(0, 1)$, and $L \in [\mathbb{N}]$. By hypothesis, we can select $M_1 \in [L]$ such that $\|A E_{\xi, \zeta, M_1,u}(1)\| < \varepsilon_1$. Now assuming that $E_1 < \ldots < E_n$, $E_i \in \text{MAX}(S_\zeta[S_\xi])$, $M_1 \supset \ldots \supset M_n \in [\mathbb{N}]$ have been chosen with $E_i < M_i$, we define $v = (v_E)_{E \not\in \mathbb{N}^\omega} \in B_U$ by letting $v_E = u_{E_1 \cup \ldots \cup E_n \cup E}$ if $E_n < E$, and $v_E = 0$ otherwise. Then $(v_E)_{E \not\in \mathbb{N}^\omega}$ is weakly null.

By hypothesis, we can select $M_{n+1} \in [(\max E_n, \infty) \cap M_n]$ such that $\|A E_{\xi, \zeta, M_{n+1},v}(1)\| < \varepsilon_{n+1}$. Let $E_{n+1} \prec M_{n+1}$ be such that $E_{n+1} \in \text{MAX}(S_\zeta[S_\xi])$ and note that $E_n < E_{n+1}$ and, by the permanence property,

$$E_{\xi, \zeta, N,v}(1) = E_{\xi, \zeta, M_{n+1},v}(1) = E_{\xi, \zeta, E_1 \cup \ldots \cup E_n \cup M_{n+1}}u(n+1) = E_{\xi, \zeta, E_1 \cup \ldots \cup E_n \cup M_{n+1}}u(n+1)$$

for any $N \in [M_{n+1}]$ with $E_{n+1} < N$. From this it follows that for any $N \in [\mathbb{N}]$ with $\bigcup_{m=1}^{n+1} E_m < N$ and $N \setminus \bigcup_{m=1}^{n+1} E_m \in [M_{n+1}]$, with $P = N \setminus \bigcup_{m=1}^{n+1} E_m$,

$$\|A E_{\xi, \zeta, N,u}(n+1)\| = \|A E_{\xi, \zeta, P,u}(1)\| \leq \varepsilon_{n+1}.$$ 

This completes the recursive construction. The conclusion of (i) is reached with $M = E_1 \cup E_2 \cup \ldots$.

(ii) By (i), for any $\varepsilon > 0$, any weakly null $u = (u_E)_{E \not\in \mathbb{N}^\omega} \in B_U$, and $L \in [\mathbb{N}]$, by (i), we can fix $M \in [L]$ such that $\|A E_{\xi, \zeta, M,u}(n)\| \leq \varepsilon/2^n$ for all $n \in \mathbb{N}$. Therefore

$$\|A E_{\xi, \zeta, M,u}(n)\|^\omega \leq \sum_{n=1}^{\infty} \|A E_{\xi, \zeta, M,u}(n)\| < \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon.$$

Since $\varepsilon > 0$, $u = (u_E)_{E \not\in \mathbb{N}^\omega} \in B_U$, and $L \in [\mathbb{N}]$ were arbitrary, $g_{\xi, \zeta}(A) = 0$.

We next show that once the prescribed linear combinations of branches of weakly null trees become weakly 1-summing, higher linear combinations can be made arbitrarily close to 0 in norm.

Lemma 5.11. Let $U, V$ be Banach spaces and fix $\xi, \zeta < \omega_1$. If $A : U \to V$ is an operator such that $g_\xi(A) < \infty$ and $g_{\xi, \zeta}(A) < \infty$, then for any $0 < \eta < \omega_1$, any $\varepsilon > 0$, any $u = (u_E)_{E \not\in \mathbb{N}^\omega} \in B_U$ and any $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that for all $N \in [M]$, $\|A E_{\xi, \zeta, \eta, M,u}(1)\| \leq \varepsilon$. In particular, $g_{\xi, \zeta, \eta}(A) = 0$ for any $0 < \eta < \omega_1$.

Proof. We note that the last sentence of the lemma follows from the preceding sentence together with Lemma 5.10. Therefore we prove only the first assertion.

Also, by the infinite Ramsey theorem, it is sufficient to prove that if $A : U \to V$ is an operator such that $g_\xi(A) < \infty$ and $g_{\xi, \zeta}(A) < \infty$, then for any $0 < \eta < \omega_1$, any $\varepsilon > 0$, any $u = (u_E)_{E \not\in \mathbb{N}^\omega} \in B_U$ and any $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that $\|A E_{\xi, \zeta, \eta, M,u}(1)\| \leq \varepsilon$. 

□
We prove by induction on $\eta \in (0, \omega_1)$ that for any Banach spaces $U, V$, any $\xi, \zeta < \omega_1$, and any operator $A : U \to V$ such that $g_{\xi, \zeta}(A) < \infty$, any $\varepsilon > 0$, any $u = (u_E)_{E \not\in \mathbb{N}^\omega}$ in $B_U$, and any $L \in \mathbb{N}$, there exists $M \in [L]$ such that $\|A\epsilon_{\xi, \zeta + \mu, M}^u(1)\| < \varepsilon$.

We first prove the $\eta = 1$ case. Fix Banach spaces $U, V$, ordinals $\xi, \zeta < \omega_1$, and an operator $A : U \to V$ with $g_{\xi, \zeta}(A) < \infty$. Fix $\gamma > g_{\xi, \zeta}(A)$. Fix $u = (u_E)_{E \not\in \mathbb{N}^\omega}$ in $B_U$ weakly null and fix $L \in \mathbb{N}$. For $\varepsilon > 0$ and $L \in \mathbb{N}$, we can choose $M \in [L]$ such that $\gamma/\sqrt{\min L} < \varepsilon$ and $\|A\epsilon_{\xi, \zeta, M}^u\| < \gamma$. Then

$$
\|A\epsilon_{\xi, \zeta + 1, M}^u(1)\| = \frac{1}{\sqrt{\min M}} \left\| \sum_{n=1}^M A\epsilon_{\xi, \zeta, M}^u(n) \right\| \leq \gamma/\sqrt{\min M} < \varepsilon.
$$

This completes the $\eta = 1$ case.

Assume the result holds for $\nu$. Then for any Banach spaces $U, V$, ordinals $\xi, \zeta < \omega_1$, any operator $A : U \to V$ with $g_{\xi, \zeta}(A) < \infty$, it follows that $g_{\xi, \zeta + \nu}(A) = 0 < \infty$. Applying the $\eta = 1$ case to the ordinals $\xi, \zeta + \nu$ yields the result for $\eta = \nu + 1$.

Assume $\eta$ is a limit ordinal and the result holds for all $\nu < \eta$. Fix Banach spaces $U, V$, ordinals $\xi, \zeta < \omega_1$, and an operator $A : U \to V$ such that $g_{\xi, \zeta}(A) < \infty$ and $s_{\xi}(A) < \infty$. Fix $u = (u_E)_{E \not\in \mathbb{N}^\omega}$ in $B_U$ weakly null and fix $L \in \mathbb{N}$. Fix $s > s_{\xi}(A)$ and note that, by replacing $L$ with a subset thereof and relabeling, we can assume $\|A\epsilon_{\xi, \nu, M}^u(1)\| < s$ for all $M \in [L]$ and all countable ordinals $\nu$. Note that $\zeta + \eta$ is a limit ordinal, from which it follows that there exists a sequence $\eta_n \uparrow \zeta + \eta$ such that

$$
S_{\zeta + \eta} = \{\emptyset\} \cup \{E : \emptyset \neq E \in S_{\eta_n + E + 1}\}.
$$

By replacing $L$ with a subset, we can assume $s/\sqrt{\min L} < \varepsilon/2$ and $\zeta + 1 < \eta_{\min L}$. Let $F_1$ be the maximal initial segment of $L$ which lies in $S_{\eta_{\min L} + \zeta}$. Define $v = (v_E)_{E \not\in \mathbb{N}^\omega}$ by letting $v_E = u_{F_1 \cap E}$ if $F_1 < E$ and $v_E = 0$ otherwise. Note that this collection is weakly null. By the inductive hypothesis combined with Lemma 5.10 there exists $M \in [L]$ such that for all $N \in [M]$ and $n \in \mathbb{N}$, $\|A\epsilon_{\xi, \eta_{\min L} + \zeta}^v(n)\| < \varepsilon/2 \min L$. Without loss of generality, we assume $F_1 < M$. Let $N = F_1 \cap M$ and let $(F_n)_{n=2}^\infty$ be the $S_{\eta_{\min L}[S_{\xi}]}$-decomposition of $M$. By the properties of $S_{\zeta + \eta}$, if $G \subset \mathbb{N}$ is any finite set with $\min G = \min L$, then $G \in S_{\zeta + \eta}$ if and only if $G \in S_{\eta_{\min L} + 1}$. Applying this to initial segments of the sequence $(\min F_0, \min F_1, \ldots)$, we deduce that the maximal initial segment of $N$ which lies in $S_{\zeta + \eta}[\mathcal{S}_{\xi}]$ is $\bigcup_{n=1}^{\min L} F_n = \bigcup_{n=1}^{\min L} F_n$. Therefore

$$
\|A\epsilon_{\xi, \eta_{\min L} + \zeta}^u(1)\| = \|A\epsilon_{\xi, \eta_{\min L} + \zeta, N}^u(1)\| = \frac{1}{\sqrt{\min L}} \left\| A \sum_{m=1}^{\min L} A\epsilon_{\xi, \eta_{\min L}, N}^u(m) \right\|
$$

$$
\leq \frac{\|A\epsilon_{\xi, \eta_{\min L}, N}^u(1)\|}{\sqrt{\min L}} + \sum_{m=2}^{\min L} \|A\epsilon_{\xi, \eta_{\min L}, N}^u(1)\|
$$

$$
= \frac{\|A\epsilon_{\xi, \eta_{\min L}, N}^u(1)\|}{\sqrt{\min L}} + \sum_{m=2}^{\min L - 1} \|A\epsilon_{\xi, \eta_{\min L}, N}^u(1)\|
$$

$$
< \frac{s}{\sqrt{\min L}} + \sum_{m=1}^{\min L - 1} \varepsilon/2 \min L < \varepsilon.
$$

□
While weak nullity is a condition only on sequences of immediate successors, we can show that in Banach spaces with separable dual, these trees have subtrees which have a stronger property.

**Proposition 5.12.** Let $U$ be a Banach space such that $U^*$ is separable. If $u = (u_E)_{E \notin [N]^{<\omega}}$ is a weakly null collection in $B_U$, then for any $L \in [N]$, there exists $M \in [L]$ such that if $F, F_1, F_2, \ldots \in [M]^{<\omega}$ are such that $F < F_n \neq \emptyset$ for all $n \in \mathbb{N}$ and $\lim_n \max F_n = \infty$, then $(u_{F_n})_{n=1}^\infty$ is weakly null.

**Proof.** Let $(u_n^\infty)_{n=1}^\infty$ be a dense sequence in $U^*$ and define $d : U \to \mathbb{R}$ by $d(u) = \sum_{n=1}^\infty \frac{|(u^*_n, u)|}{1 + 2^n \|u\|}$. Note that a bounded sequence $(u_n)_{n=1}^\infty$ in $U$ is weakly null if and only if $\lim_n d(u_n) = 0$. Fix $L \in [N]$ and a weakly null collection $u = (u_E)_{E \notin [N]^{<\omega}}$ in $B_U$. Since $(u_m)_{m=1}^\infty$ is weakly null, there exists $m_1 \in L$ so large that $d(u_{(m_1)}) < 1/2$. Then we construct by induction on the integer $n$ a strictly increasing sequence $(m_n)_{n=1}^\infty$ in $L$ such that for every integer $n$ and every subset $E \subset \{m_1, \ldots, m_n\}$, $d(u_{E \setminus (m_{n+1})}) > 1/2^{n+1}$. Let $M = \{m_n : n = 1, 2, \ldots\}$. Let $F_1, F_2, \ldots$ be finite subsets of $M$ such that $F < F_n, F_n \neq \emptyset$ for all $n \in \mathbb{N}$ and $\lim_n \max M_n = \infty$. Next, we show that $\lim_n d(u_{F \setminus F_n}) = 0$. Let $\varepsilon > 0$. Fix an integer $N$ such that $1/2^N < \varepsilon$ and an integer $n_0$ such that for any $n \geq n_0$, $\max F_n > n_0$. Fix $n \geq n_0$, then $F \setminus F_n$ is the strictly increasing sequence $(m_i)_{i=1}^k$. We have $m_i = \max F_n > n_0$ so $d(u_{F \setminus F_n}) < 1/2^k < 1/2^N$ and we are done.

We conclude this section by showing that for $\xi, \zeta < \omega_1$, a separable Banach space $U$ containing no isomorph of $\ell_1$ can only enjoy the $\mathcal{G}_{\xi,\zeta}$ Property in a non-trivial way if $Sz(U) = \omega^{\xi+1}$.

**Lemma 5.13.** Fix $\xi < \omega_1$. Let $U$ be a separable Banach space not containing an isomorphic copy of $\ell_1$.

(i) If $Sz(U) > \omega^{\xi+1}$, then for every $\zeta < \omega_1$, $s_\zeta(U) = g_{\xi,\zeta}(U) = \infty$.

(ii) If $Sz(U) \leq \omega^{\xi}$ if and only if for any $\varepsilon > 0$, any weakly null collection $u = (u_E)_{E \notin [N]^{<\omega}}$ in $B_U$, and any $L \in [N]$, there exists $M \in [L]$ such that $\|E_{\xi, M} u(1)\| < \varepsilon$.

(iii) If $Sz(U) \leq \omega^{\xi}$ if and only if $s_\zeta(U) = 0$ if and only if $g_{\xi,0}(U) = 0$ if and only if for every $\zeta < \omega_1$, $g_{\xi,\zeta}(U) = 0$.

**Proof.** (i) It is known (see [20, Proposition 5 and Theorem 12]) that if $U$ is a separable Banach space containing no isomorph of $\ell_1$ such that $Sz(U) > \omega^{\xi+1}$, then there exist $\varepsilon > 0$ and a weakly null collection $u = (u_E)_{E \in S_{\xi+1} \setminus \{\emptyset\}}$ in $B_U$ such that

(a) for any $E \in S_{\xi+1} \setminus \text{MAX}(S_{\xi+1})$, $(u_{E \setminus (n)})_{E \in \mathbb{N}}$ is a weakly null sequences,

(b) for each $E \in S_{\xi+1}$ and each $x \in \text{co}(u_F : \emptyset < F \leq E)$, $\|x\| \geq \varepsilon$.

Let $u_E = 0$ for all $E \in [N]^{<\omega} \setminus S_{\xi+1}$, so $u = (u_E)_{E \in [N]^{<\omega} \setminus \{\emptyset\}}$ is a weakly null collection in $B_U$. Fix $n \in \mathbb{N}$ and let $L \in [N]$ be such that $n = \min L$. Fix $M \in [L]$, so that $m := \min M \geq n$. Moreover, since $E = \bigcup_{i=1}^m M_{\xi,i} \in S_{\xi+1}$, it follows that

$$\frac{1}{m} \sum_{i=1}^m E_{\xi, M} u(i) \in \text{co}(u_F : \emptyset < F \leq E).$$
Therefore \( \varepsilon \leq \left\| \frac{1}{m} \sum_{i=1}^{m} E_{\xi,M}u(i) \right\| \), so \( \left\| E_{\xi,M}u \right\| \leq \varepsilon m^{1/2} \geq \varepsilon n^{1/2} \). Since \( n \) was arbitrary, \( s_{\xi}(U) = \infty \). It also follows from this that \( \left\| E_{\xi,0,M}u \right\|_{1} \geq \left\| E_{\xi,M}u \right\|_{2} \geq \varepsilon m^{1/2} \geq \varepsilon n^{1/2} \). This shows that \( g_{\xi,0}(U) = \infty \). By Lemma 5.11, \( g_{\xi,\zeta}(U) = \infty \) for all \( 0 < \zeta < \omega_1 \).

(ii) As noted in (i), if \( Sz(U) > \omega^\xi \), then there exist \( \varepsilon > 0 \) and a collection \( u = (u_E)_{E \in S_\xi \setminus \{\varnothing\}} \) in \( B_U \) such that

(a) for any \( E \in S_\xi \setminus MAX(S_\xi) \), \( u_{E\setminus M} <_n \) is a weakly null sequences,
(b) for each \( E \in S_\xi \) and each \( x \in co(u_F : \varnothing < F \leq E) \), \( \|x\| \geq \varepsilon \).

Let \( u_E = 0 \) for all \( E \in [N]^{<\omega} \setminus S_\xi \), so \( u = (u_E)_{E \neq F} [N]^{<\omega} \subset B_U \) is weakly null. Then for any \( M \in [N] \), since \( M|_{\xi,1} \in S_\xi \) and \( E_{\xi,M}u(1) \in co(u_F : \varnothing < F \leq M_{\xi,1}) \), \( \varepsilon \leq \|E_{\xi,M}u(1)\| \geq \varepsilon \). By contraposition, if for every \( \varepsilon > 0 \), every weakly null collection \( u = (u_E)_{E \neq F} [N]^{<\omega} \subset B_U \), and every \( L \in [N] \), there exists \( M \in [L] \) such that \( \|E_{\xi,M}u(1)\| \leq \varepsilon \), it holds that \( Sz(U) \leq \omega^\xi \).

For the converse, assume \( Sz(U) \leq \omega^\xi \) and that there exist \( \varepsilon > 0 \), a weakly null collection \( u = (u_E)_{E \in [N]^{<\omega} \setminus \{\varnothing\}} \) in \( B_U \), and \( L \in [N] \) such that for all \( M \in [L] \), \( \|E_{\xi,M}u(1)\| \geq \varepsilon \). Since \( U \) is separable and \( Sz(U) \) is countable, \( U^* \) is separable [15 Theorem 1]. Therefore by Proposition 5.12 by replacing \( L \) with a subset thereof and relabeling, we can assume that if \( F, F_1, F_2, \ldots \in [L]^{<\omega} \) are such that \( F < F_n \neq \varnothing \) for all \( n \in N \) and \( \lim_n \max F_n = \infty \), then \( u_{F\setminus F_n)^\infty \in [N]^{<\omega} \) is weakly null. For each \( M \in [N] \), fix \( u_M \in B_{U^*} \) such that \( \Re \langle u_M, E_{\xi,M}u(1) \rangle = \|E_{\xi,M}u(1)\| \). Fix \( \Phi : MAX(S_\xi) \to [N] \) such that for each \( E \in MAX(S_\xi) \), \( E < \Phi(E) \in [N] \) and such that if \( E \in [L]^{<\omega} \), then \( \Phi(E) \in [L] \). Define \( f : N \times MAX(S_\xi) \to [-1, 1] \) by

\[
f(n, E) = \begin{cases} \Re \langle u_{\Phi(E)}, u_{[1,n]\cap E} \rangle & : n \in E \\ 0 & : n \in N \setminus E. \end{cases}
\]

Then for any \( M \in [L] \),

\[
\sum_{\varnothing < F \leq M_{\xi,1}} p^\xi_F \max F, M_{\xi,1} = \sum_{\varnothing < F \leq M_{\xi,1}} \Re \langle u_{\Phi(M_{\xi,1})}, p^\xi_F u_F \rangle = \Re \langle u_{\Phi(M_{\xi,1})}, E_{\xi,M}u(1) \rangle \geq \varepsilon.
\]

In the terminology of [11], the \( \xi^\text{th} \) level of the repeated averages hierarchy considered as a family of probability measures on \( S_\xi \) are \( \xi \)-sufficient, and therefore \( \xi \)-regulatory probability block (see [11 Corollary 4.8]). For \( 0 < \varepsilon_1 < \varepsilon \), there exists \( N \in [N] \) such that for each \( F \in S_\xi \), there exists \( E \in MAX(S_\xi) \cap [L]^{<\omega} \) such that for all \( i \in F \),

\[
\Re \langle u_{\Phi(E)}, u_{[1,N(i)]\cap E} \rangle = f(N(i), E) \geq \varepsilon_1
\]

(see the definition of \( \xi \)-full and the definitions of the sets \( E(f, \Psi, P, \varepsilon) \) and \( G(f, P, M, \varepsilon) \) preceding [11 Lemma 4.4]). Since \( f(N(i), E) \geq \varepsilon_1 > 0 \), then \( N(i) \in E \).

Fix \( 0 < \varepsilon_2 < \varepsilon_1 \). We claim that for any \( 0 \leq \zeta \leq \omega^\xi \) and for any \( F \in MAX(S_\xi) \), there exists \( u^* \in s_{\zeta}^\xi (B_{U^*}) \) and, if \( F \neq \varnothing \), there also exists \( E \in S_\xi \) such that for each \( i \in F \), \( N(i) \in E \) and \( \varepsilon_1 \leq \Re \langle u^*, u_{[1,N(i)]\cap E} \rangle \). We prove this by induction on \( \zeta \). The \( \zeta = 0 \) case is precisely the last two sentences of the preceding paragraph with \( u^* = u_{\Phi(E)} \), using the fact that \( s_{\zeta}^\xi (B_{U^*}) = B_{U^*} \).

Assume \( \zeta \) is a limit ordinal and the conclusion holds for each \( \mu < \zeta \). Fix \( F \in MAX(S_\xi) \). If \( \zeta = \omega^\xi \), then \( F = \varnothing \), and we need to show that \( s_{\omega^\xi}^\xi (B_{U^*}) \neq \varnothing \). In this case, by the inductive hypothesis, \( s_{\omega^\xi}^\xi (B_{U^*}) \neq \varnothing \) for each \( \mu < \omega^\xi \). From this and weak*-compactness, \( s_{\omega^\xi}^\xi (B_{U^*}) \neq \varnothing \). If \( \zeta < \omega^\xi \), then \( F \neq \varnothing \). In this case, for each \( \mu < \zeta \), there exists \( F_\mu \in MAX(S_\xi) \) such that
$F \succeq F_\mu$. By the inductive hypothesis, there exist $u^*_\mu \in s^{\mu}_{\varepsilon_2}(B_{U^*})$ and $E_\mu \in S_\xi$ such that for each $i \in F$, $N(i) \in E_\mu$ and $\text{Re} \left\langle u^*_\mu, u_{[1,N(i)] \cap E_\mu} \right\rangle \geq \varepsilon_1$. Fix $\mu_n \uparrow \xi$ and note that since $B_{U^*} \times S_\xi$ is compact metrizable, then by passing to a subsequence and relabeling, we can assume $(u^*_{\mu_n}, E_{\mu_n})_{n=1}^\infty$ is convergent to some $(u^*, E) \in B_{U^*} \times S_\xi$. By passing to a subsequence again, we can assume that $[1, N(\max F)] \cap E_{\mu_n} = [1, N(\max F)] \cap E$ for all $n \in \mathbb{N}$. For each $i \in F$, since $N(i) \in E_{\mu_n}$ for all $n \in \mathbb{N}$, $N(i) \in E$. Moreover, since $[1, N(\max F)] \cap E_{\mu_n} = [1, N(\max F)] \cap E$ for all $n \in \mathbb{N}$, it follows that $u_{[1,N(i)] \cap E} = u_{[1,N(i)] \cap E_{\mu_n}}$ for all $n \in \mathbb{N}$. Since $(u^*_n)_{n=1}^\infty$ is weak*-convergent to $u^*$, it holds that, for each $i \in F$,

$$\text{Re} \left\langle u^*, u_{[1,N(i)] \cap E} \right\rangle = \lim_n \text{Re} \left\langle u^*_n, u_{[1,N(i)] \cap E} \right\rangle = \lim_n \text{Re} \left\langle u^*_n, u_{[1,N(i)] \cap E_{\mu_n}} \right\rangle \geq \varepsilon_1.$$  

Since $u^*_n \in s^{\mu_n}_{\varepsilon_2}(B_{U^*})$, it follows that, $u^* \in \cap_{n=1}^\infty s_{\varepsilon_2}^{\mu_n}(B_{U^*}) = s_{\varepsilon_2}^\xi(B_{U^*})$. This completes the limit ordinal case of the proof.

Next, assume that the result holds for some $\xi$ and assume $F \in \text{MAX}(S_\xi^{\xi+1})$. In this case, $F \cap (n) \in \text{MAX}(S_\xi^{\xi})$ for all $F \subset n \in \mathbb{N}$. By the inductive hypothesis, for each $F < n$, there exists $u^*_n \in s_{\varepsilon_2}^\xi(B_{U^*})$ and $E_n \in S_\xi$ such that for each $i \in F \cup (n)$, $N(i) \in E_n$ and $\text{Re} \left\langle u^*_n, u_{[1,N(i)] \cap E_n} \right\rangle \geq \varepsilon_1$. We can select an infinite subset $R$ of $(\max F, \infty)$ such that $(u^*_n)_{n \in R}$ is weak*-convergent to some $u^* \in s_{\varepsilon_2}^\xi(B_{U^*})$ and $E_\xi \in S_\xi$ such that $([1, N(n)] \cap E_n)_{n \in R}$ converges to $E$. By passing to a further subsequence and relabeling, we can assume that for each $i \in F$ and $n \in R$, $[1, N(i)] \cap E_n = [1, N(i)] \cap E$. Since $N(i) \in E_n$ for all $i \in F$ and $n \in R$, $N(i) \in E$. From this it follows that $N(i) \in E$ for all $i \in F$. Moreover, since $u_{[1,N(i)] \cap E_n} = u_{[1,N(i)] \cap E}$ for all $i \in F$ and $n \in R$ and since $(u^*_n)_{n \in R}$ is weak*-convergent to $u^*$, it holds that

$$\text{Re} \left\langle u^*, u_{[1,N(i)] \cap E} \right\rangle = \lim_n \text{Re} \left\langle u^*_n, u_{[1,N(i)] \cap E} \right\rangle = \lim_n \text{Re} \left\langle u^*_n, u_{[1,N(i)] \cap E_{\mu_n}} \right\rangle \geq \varepsilon_1.$$  

It remains to show that $u^* \in s_{\varepsilon_2}^\xi(B_{U^*})$. We note that since $([1, N(n)] \cap E_n)_{n \in R}$ is convergent to $E$ and since $n \leq N(n) \in [1, N(n)] \cap E_n$ for all $n \in R$, by replacing $R$ with a subset thereof and relabeling if necessary, we can assume that for each $n \in R$, $[1, N(n)] \cap E_n = E \cap F_n$ for some $E \subset F_n \neq \emptyset$ such that $\lim_n \max F_n = \infty$. By our choice of $L$, $(u_{[1,N(n)] \cap E_n})_{n \in R} = (u_{E \cap F_n})_{n \in R}$ is weakly null. From this it follows that

$$\liminf_{n \in R} \|u^* - u^*_n\| \geq \liminf_{n \in R} \text{Re} \left\langle u^*_n - u^*, u_{[1,N(n)] \cap E_n} \right\rangle \geq \varepsilon_1 - 0 > \varepsilon_2.$$  

Combining this with the fact that $(u^*_n)_{n \in R} \subset s_{\varepsilon_2}^\xi(B_{U^*})$, we deduce that $u^* \in s_{\varepsilon_2}^{\xi+1}(B_{U^*})$. This concludes the proof by induction. Applying the $\xi = \omega^\xi$ case yields that $s_{\varepsilon_2}^{\omega^\xi}(B_{U^*}) \neq \emptyset$, and $S_{\omega^\xi}(U) > \omega^\xi$. This contradiction finishes $(iii)$.

(iii) If $S_{\omega^\xi}(U) \leq \omega^\xi$, then by $(ii)$, for any $\varepsilon > 0$, any weakly null collection $u = (u_E)_{E \neq F \subset [N]\subset \omega}$ in $B_{U^*}$, and any $L \subset [N]$, there exists $M \subset [L]$ such that $\|E_{\xi,M}u(1)\| < \varepsilon$. Since $E_{\xi,M}u = E_{\xi,0,M}^{(2)}u$, and in particular $E_{\xi,0,M}u(1) = E_{\xi,0,M}^{(2)}u(1)$, it follows from Lemma 5.10 that $g_{\xi,0}(U) = 0$.

Next, suppose that $g_{\xi,0}(U) = 0$. By Lemma 5.11 $g_{\xi,\xi}(U) = 0$ for all $0 < \xi < \omega_1$, and therefore for all $0 \leq \xi < \omega_1$.

Next, suppose that $g_{\xi,\xi}(U) = 0$ for all $\xi < \omega_1$. Then since $g_{\xi,0}(U) = 0$, it holds that for any $\gamma > 0$, any weakly null collection $u = (u_E)_{E \neq F \subset [N]\subset \omega}$ in $B_{U^*}$, and any $L \subset [N]$, there exists $M \subset [L]$ such that $\|E_{\xi,0,M}^{(2)}u\|_1^w \leq \gamma$. Since $E_{\xi,0,M}^{(2)}u = E_{\xi,M}u$, it holds that $\|E_{\xi,M}u\|_2^w \leq \|E_{\xi,0,M}^{(2)}u\|_1^w \leq \gamma$. This yields that $g_{\xi}(U) = 0$. 


Suppose that $\mathfrak{g}_\xi(U) = 0$. Then for any $\varepsilon > 0$, any weakly null collection $u = (u_E)_{E \not\in \mathbb{N}^\omega} \subset B_U$, and any $L \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that $\|\mathbb{E}_{\xi,M}u\|_w^w < \varepsilon$. Since $\|\mathbb{E}_{\xi,M}u(1)\| \leq \|\mathbb{E}_{\xi,M}u\|_2^w < \varepsilon$, (ii) yields that $Sz(U) \leq \omega^\xi$.

\[ \square \]

6. Positive results

In this section, for a compact, Hausdorff space $K$ and $f \in C(K)$, we let

$$\text{supp}(f) = \{ \bar{\omega} \in K : f(\bar{\omega}) \neq 0 \}.$$ 

For convenience, we define $(-1, \beta) = [0, \beta]$ for any ordinal $\beta$.

**Proposition 6.1.** For $\xi < \omega_1$ and $\beta < \omega$, $\mathfrak{g}_{\xi,0}(C_0(\omega^\xi) \hat{\otimes}_\pi C(\beta)) < \infty$.

**Proof.** Let $I$ denote the identity on $C_0(\omega^\xi)$. Let $u = (u_E)_{E \not\in \mathbb{N}^\omega} \subset B_{C_0(\omega^\xi)}$ and fix $L \in \mathbb{N}$. Fix $\varepsilon > 0$. Let $\beta_0 = -1$. Let $M_1 = L$ and let $E_1 \in \text{MAX}(S_\xi)$ be such that $E_1 < M_1$. We can select $g_1 \in B_{C_0(\omega^\xi)}$ and $\beta_1 < \omega^\xi$ such that $\text{supp}(g_1) \subset (\beta_0, \beta_1)$ and

$$\|g_1 - \sum_{\varnothing < F \subseteq E_1} p^\xi_F u_F\| < \varepsilon/2.$$ 

By the permanence property, $\sum_{\varnothing < F \subseteq E_1} p^\xi_F u_F = \mathbb{E}_{\xi,M}u(1)$ for all $M \in \mathbb{N}$ such that $E_1 < M$.

Next assume that $\beta_1 < \ldots < \beta_n$, $E_1 < \ldots < E_n$, $M_1 < \ldots < M_n$ have been chosen. Define $v = (v_E)_{E \not\in \mathbb{N}^\omega} \subset B_{C(\beta_n)}$ by letting $v_E = u_E \cup \ldots \cup u_{E \cup \mathbb{N}}$ if $E_n < E$ and $v_E = 0$ otherwise. Then $v \subset B_{C(\beta_n)}$ is weakly null. Since $Sz(C(\beta_n)) \leq \omega^\xi$, we can select $M_{n+1} \in [(\max E_n, \infty) \cap M_n]$ such that for any $M \in [M_{n+1}]$,

$$\|\mathbb{E}_{\xi,M}v(1)\| < \varepsilon/2^{n+1}.$$ 

Let $E_{n+1} < M_{n+1}$ be such that $E_{n+1} \in \text{MAX}(S_\xi)$ and select $g_{n+1} \in B_{C_0(\omega^\xi)}$ and $\beta_{n+1} \in (\beta_n, \omega^\xi)$ such that $\text{supp}(g_{n+1}) \subset (\beta_n, \beta_{n+1})$ and

$$\|g_{n+1} - \sum_{E_1 \cup \ldots \cup E_{n+1} < F \subseteq E_1 \cup \ldots \cup E_{n+1}} p^\xi_F u_F\| = \|g_{n+1} - \sum_{\varnothing < F \subseteq E_1} p^\xi_F v_F\| < \varepsilon/2^{n+1}.$$ 

The equality here is due to the permanence property. Again by the permanence property, it follows that for any $M \in \mathbb{N}$ such that $\cup_{m=1}^{n+1} E_m < M$,

$$\|g_{n+1} - \mathbb{E}_{\xi,M}u(n+1)\| = \|g_{n+1} - \sum_{E_1 \cup \ldots \cup E_{n+1} < F \subseteq E_1 \cup \ldots \cup E_{n+1}} p^\xi_F u_F\| < \varepsilon/2^{n+1}. $$

This completes the recursive construction. We let $M = \cup_{n=1}^\infty E_n$. We note that $(g_n)_{n=1}^\infty \subset B_{C_0(\omega^\xi)}$ have pairwise disjoint supports, so $\|g_n\|_w^w < 1$. Since $\|g_n - \mathbb{E}_{\xi,M}u(n)\| < \varepsilon_n$ for all $n \in \mathbb{N}$, which follows from the recursive construction, it follows that

$$\|\mathbb{E}_{\xi,M}u\|_1^w \leq \|g_n\|_1^w + \sum_{n=1}^\infty \|g_n - \mathbb{E}_{\xi,M}u(n)\| < 1 + \varepsilon.$$ 

From this it follows that $\mathfrak{g}_{\xi,0}(I) \leq 1$.

For $\beta < \omega$, since $C_0(\omega^\xi) \hat{\otimes}_\pi C(\beta)$ is isomorphic to $C_0(\omega^\xi)$, we deduce that $\mathfrak{g}_{\xi,0}(C_0(\omega^\xi) \hat{\otimes}_\pi C(\beta)) < \infty$. 

$\square$
Let $H$ denote the set of all ordered pairs $(\xi, \zeta)$ of countable ordinals such that $\zeta < \xi$. Let $k_G$ denote Grothendieck’s constant. For each $(\xi, \zeta) \in H$, let $h(\xi, \zeta)$ be the proposition
\[
 h(\xi, \zeta) = \begin{cases} 
 \max \{s_\xi(C_0(\omega^\xi) \hat{\otimes}_\pi C_0(\omega^\xi)), s_{\xi,1+\zeta}(C_0(\omega^\xi) \hat{\otimes}_\pi C_0(\omega^\xi))\} \leq k_G : \zeta < \xi \\
 \max \{s_\xi(C_0(\omega^\xi) \hat{\otimes}_\pi C_0(\omega^\xi)), s_{\xi,1+\zeta}(C_0(\omega^\xi) \hat{\otimes}_\pi C_0(\omega^\xi))\} \leq 2k_G : \zeta = \xi.
 \end{cases}
\]

For $\xi < \omega_1$, let $h(\xi)$ be the proposition that $h(\xi, \zeta)$ holds for all $\zeta < \xi$. Let $h$ be the proposition that $h(\xi)$ holds for all $\xi < \omega_1$.

**Theorem 6.2.** The proposition $h$ holds.

**Proof.** We first isolate the following consequence of the hypotheses $h(\xi)$, $\xi < \omega_1$.

**Claim 1.** Suppose that $\xi < \omega_1$ is such that $h(\eta)$ holds for all $\eta < \xi$. Then for any $\alpha, \beta < \omega^\xi$, $Sz(C(\alpha) \hat{\otimes}_\pi C(\beta)) \leq \omega^\xi$.

First, note that since $C(\alpha) \hat{\otimes}_\pi C(\beta)$ is isometrically embeddable into $C(\max\{\alpha, \beta\}) \hat{\otimes}_\pi C(\max\{\alpha, \beta\})$, it is sufficient to deduce that $Sz(C(\alpha) \hat{\otimes}_\pi C(\alpha)) \leq \omega^\xi$ for each $\alpha < \omega^\xi$.

If $\xi = 0$, which implies that $\alpha < \omega$, or if $0 < \xi < \omega_1$ and $\alpha < \omega$, then $C(\alpha) \hat{\otimes}_\pi C(\alpha)$ is finite dimensional and has Szlenk index $1 = \omega^0 \leq \omega^\xi$.

Now we suppose $0 < \xi < \omega_1$ and $\omega \leq \alpha < \omega^\xi$. Then for some $\eta < \xi$, $C(\alpha)$ is isomorphic to $C_0(\omega^\eta)$ [23 Theorem 2.14].

Therefore $C(\alpha) \hat{\otimes}_\pi C(\alpha)$ is isomorphic to $C_0(\omega^\eta) \hat{\otimes}_\pi C_0(\omega^\eta)$.

Since we have assumed $h(\eta, \eta)$ holds, it follows that $s_\eta(C_0(\omega^\eta) \hat{\otimes}_\pi C_0(\omega^\eta)) \leq 2k_G < \infty$. It follows from the proof of Lemma 5.11 that $s_{\eta+1}(C_0(\omega^\eta) \hat{\otimes}_\pi C_0(\omega^\eta)) = 0$. Since the Szlenk index is an isomorphic invariant, and by Lemma 6.1 $Sz(C(\alpha) \hat{\otimes}_\pi C(\alpha)) = Sz(C_0(\omega^\eta) \hat{\otimes}_\pi C_0(\omega^\eta)) \leq \omega^{\eta+1} \leq \omega^\xi$.

This yields the claim if $0 < \xi < \omega_1$.

Therefore Claim 1 holds.

We next prove that $h(\xi)$ holds assuming that $h(\eta)$ holds for all $\eta < \xi$. For this, we will prove that for $\zeta < \xi$, $h(\xi, \zeta)$ holds assuming $h(\xi, \nu)$ holds for all $\nu < \zeta$. To that end, fix $\zeta < \xi < \omega_1$ and assume $h(\eta)$ holds for all $\eta < \xi$ and $h(\xi, \zeta)$ holds for all $\nu < \zeta$. Let $U = C_0(\omega^\xi) \hat{\otimes}_\pi C_0(\omega^\xi)$. Fix $\kappa \in \mathbb{R}$ such that $\kappa > k_G$ if $\zeta < \xi$ and $\kappa > 2k_G$ if $\zeta = \xi$. We will first prove that $s_\xi(U) \leq \kappa$, and we will then show that $g_{\xi,1+\zeta}(U) \leq \kappa$.

In the proof, for an ordinal $\alpha$, we let $R_\alpha f = f|_{[0,\alpha]}$ so $R_\alpha$ is an operator into $C(\alpha)$. By an abuse of notation, we use this notation both for the restriction operators on both $C_0(\omega^\xi)$ and $C_0(\omega^\xi)$. We let $P_\alpha$ be the operator (either from $C_0(\omega^\xi)$ to itself or from $C_0(\omega^\xi)$ to itself) given by letting

$P_\alpha f(\varpi) = f(\varpi)$ for $\varpi < \alpha$ and $P_\alpha f(\varpi) = 0$ for $\varpi > \alpha$.

We first consider the case $\zeta < \xi$. Fix a weakly null collection $u = (u_E)_{E \in \mathbb{N} \setminus \{\emptyset\}}$ in $B_U$ and $L \in [\mathbb{N}]$. Fix $\varepsilon > 0$ such that $k_G + \varepsilon < \kappa$. Let $\alpha_0 = -1$. Let $\alpha = L$ and let $E_1 < L_1$ be such that $E_1 \in Max(S_\xi)$. There exist finite sets $F_1 \subset B_{C_0(\omega^\xi)}$, $G_1 \subset B_{C_0(\omega^\xi)}$, and $u \in \mathcal{C}\{f \otimes g : (f, g) \in F_1 \times G_1\}$ such that

$$\|v_1 - \mathbb{E}_{\xi,L_1}u(1)\| < \varepsilon/2.$$
By replacing the members of $F_1$ with projections thereof, we can assume there exists $\alpha_1 < \omega^{\omega^\xi}$ such that $\text{supp}(f) \subset (\alpha_0, \alpha_1]$ for all $f \in F_1$. Next, assume $E_1 \leq \ldots < E_n$, $M_1 \supset \ldots \supset M_n \in [L]$, $\alpha_0 < \ldots < \alpha_n < \omega^{\omega^\xi}$, $F_1, \ldots, F_n$, and $G_1, \ldots, G_n$ have been chosen. Since $\alpha_n, \omega^{\omega^\xi} < \omega^{\omega^\xi}$, we deduce from Claim 1 that $C(\alpha_n) \assign \omega^{\omega^\xi}$ has Szlenk index not exceeding $\omega^\xi$. Define the collection $v = (v_E)_{E \neq \emptyset \in [N]^\omega} \in B_{C(\alpha_n) \assign \omega^{\omega^\xi}}$ by letting $v_E = (R_{\alpha_n} \otimes I) u_{E_1 \cup \ldots \cup E_n \cup E}$ if $E_n < E$ and $v_E = 0$ otherwise. Then $v$ is weakly null. Since $C(\alpha_n) \assign \omega^{\omega^\xi}$ has Szlenk index not exceeding $\omega^\xi$, there exists $M_{n+1} \in [(\max E_n, \infty) \cap M_n]$ such that $\|(R_{\alpha_n} \otimes I) \xi, M_{n+1} v(1)\| < \varepsilon / 3 \cdot 2^{n+1}$. Let $E_{n+1} = M_{n+1} \setminus \xi, 1$ and note that for any $E_{n+1} < \emptyset \in [N]$ 

$$E_{\xi, E_{1 \cup \ldots \cup E_{n+1 \cup M}} u(n + 1) = E_{\xi, M_{n+1}} v(1)$$

and

$$\|(R_{\alpha_n} \otimes I) E_{\xi, E_{1 \cup \ldots \cup E_{n+1 \cup M}} u(n + 1)\| = \|(R_{\alpha_n} \otimes I) E_{\xi, M_{n+1}} v(1)\| < \varepsilon / 3 \cdot 2^{n+1}.$$ 

We can choose finite sets $F' \subset B_{C(\omega^{\omega^\xi})}$, $G_{n+1} \subset B_{C(\omega^{\omega^\xi})}$ and

$$v' \in \co \{ f \otimes g : (f, g) \in F' \times G_{n+1} \}$$

such that $\|v' - E_{\xi, M_{n+1}} v(1)\| < \varepsilon / 3 \cdot 2^{n+1}$. By replacing the members of $F'$ with small perturbations thereof, we can assume there exists $\alpha_{n+1} \in (\alpha_n, \omega^{\omega^\xi})$ such that $\text{supp}(f) \subset [0, \alpha_{n+1}]$ for all $f \in F'$. Define

$$F_{n+1} = \{(I - P_{\alpha_n}) f : f \in F'\} \subset B_{C(\omega^{\omega^\xi})}$$

and

$$v_{n+1} = ((I - P_{\alpha_n}) \otimes I)v'.$$

Note that $\text{supp}(f) \subset (\alpha_n, \alpha_{n+1}]$ for all $f \in F_{n+1}$. Note also that

$$\|v_{n+1} - E_{\xi, M_{n+1}} v(1)\| = \|v' - E_{\xi, M_{n+1}} v(1) - (P_{\alpha_n} \otimes I)v'\|$$

$$\leq \|v' - E_{\xi, M_{n+1}} v(1) - (P_{\alpha_n} \otimes I)(v' - E_{\xi, M_{n+1}} v(1))\| + \|(P_{\alpha_n} \otimes I) E_{\xi, M_{n+1}} v(1)\|$$

$$= \|v' - E_{\xi, M_{n+1}} v(1) - (P_{\alpha_n} \otimes I)(v' - E_{\xi, M_{n+1}} v(1))\| + \|(R_{\alpha_n} \otimes I) E_{\xi, M_{n+1}} v(1)\|$$

$$\leq 2\|v' - E_{\xi, M_{n+1}} v(1)\| + \varepsilon / 3 \cdot 2^{n+1}$$

$$< \varepsilon / 2^{n+1}.$$ 

This completes our recursive construction. Let $M = \cup_{n=1}^\infty E_n$. By the recursive construction, $\|v_n - E_{\xi, M} u(n)\| < \varepsilon_n$ for all $n \in \mathbb{N}$. Therefore we can deduce that $\|E_{\xi, M} u\|_2 < k_G + \varepsilon$ by showing that $\|\|v_n\|_2\|_2^\infty < k_G$. Since $v_n \in \co \{ f \otimes g : (f, g) \in F_n \times G_n \}$ for all $n \in \mathbb{N}$, by the triangle inequality, we only need to observe that $\|(f_n \otimes g_n)_{n=1}^\infty\|_2 < k_G$ for all $(f_n, g_n)_{n=1}^\infty \in \prod_{n=1}^\infty F_n \times G_n$. For this, we apply Proposition 3.1. Here we note that for any $(f_n)_{n=1}^\infty \in \prod_{n=1}^\infty F_n$, $\|f_n\|_2^\infty \leq \|(f_n)_{n=1}^\infty\|_1 \leq 1$, since $(f_n)_{n=1}^\infty$ have pairwise disjoint supports. This shows that $s_{\xi}(U) \leq k_G$ if $\zeta < \xi$.

We next show that $s_{\xi}(U) \leq 2k_G$ if $\zeta = \xi$. The proof proceeds as in the previous paragraph, except that we recursively choose $E_1, E_2, \ldots, M_1, M_2, \ldots, -1 = \alpha_0 < \alpha_1 < \ldots, F_1, \Phi_n, G_n, \Gamma_n \subset B_{C_n(\omega^{\omega^\xi})}$, and

$$v_n \in \co \{ f \otimes g : (f, g) \in F_n \times G_n \},$$

$$w_n \in \co \{ f \otimes g : (f, g) \in \Phi_n \times \Gamma_n \}$$

and
such that, with \( M = \bigcup_{n=1}^{\infty} E_n \), \( \| v_n + w_n - E_{\xi,M} u(n) \| < \varepsilon_n \) for all \( n \in \mathbb{N} \), and such that \( \text{supp}(f) \subseteq \{\alpha_{n-1}, \alpha_n\} \) for all \( n \in \mathbb{N} \) and \( f \in F_n \cup \Gamma_n \). From here, we argue as in the previous paragraph to deduce that \( \|(f_n \otimes g_n)_{n=1}^{\infty}\|_2 \leq k_G \) for all \((f_n, g_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \times G_n \), since in this case the functions \((f_n)_{n=1}^{\infty} \) have pairwise disjoint supports. Similarly, \( \|(f_n \otimes g_n)_{n=1}^{\infty}\|_2 \leq k_G \) for all \((f_n, g_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \Phi_n \times \Gamma_n \), since \((g_n)_{n=1}^{\infty} \) have pairwise disjoint supports. Again, by the triangle inequality, \( \|(v_n)_{n=1}^{\infty}\|_2, \|(w_n)_{n=1}^{\infty}\|_2 \leq k_G \), and

\[
\|E_{\xi,M} u\| \leq \|(v_n)_{n=1}^{\infty}\|_2 + \|(w_n)_{n=1}^{\infty}\|_2 + \sum_{n=1}^{\infty} \|v_n + w_n - E_{\xi,M} u(n)\| \leq 2k_G + \varepsilon.
\]

The base step of making this sequence of choices is arbitrary, as in the previous paragraph. We indicate how to complete the recursive step. Assume \( E_1, \ldots, E_n, M_1, \ldots, M_n, \alpha_0, \ldots, \alpha_n \) have been chosen. We now consider the collection \( v = (v_E)_{E \neq \emptyset \in [\mathcal{N}]^{<\omega}} \) of \( B_{C(\alpha_n) \oplus \pi C(\alpha_n)} \) given by \( v_E = (R_{\alpha_n} \otimes R_{\alpha_n}) u_{E_1 \cup \cdots \cup E_n \cup E} \) if \( E_n < E \) and \( v_E = 0 \) otherwise. By Claim 1, \( C(\alpha_n) \oplus \pi C(\alpha_n) \) has Szlenk index not exceeding \( \omega^\delta \), so as in the previous paragraph, we can choose \( M_{n+1} \) such that, if \( E_{n+1} \prec M_{n+1} \) is such that \( E_{n+1} \in \text{MAX}(\mathcal{S}_\xi) \), then for any \( M \in [M_{n+1}] \) with \( E_{n+1} \prec M \),

\[
E_{\xi,E_1 \cup \cdots \cup E_{n+1} \cup M} u(n + 1) = E_{\xi,M_{n+1}} v(1)
\]

and

\[
\|(R_{\alpha_n} \otimes R_{\alpha_n}) E_{\xi,M_{n+1}} v(1)\| < \varepsilon/3 \cdot 2^{n+1}.
\]

We then choose a finite set \( F \subset B_{C_0(\omega^\delta)} \) and \( \alpha_{n+1} \in (\alpha_n, \omega^\delta) \) such that \( \text{supp}(f) \subseteq [0, \alpha_{n+1}] \) for all \( f \in F \) and

\[
v' \in \text{co}\{f \otimes g : (f, g) \in F \times F\}
\]

such that \( \|v' - E_{\xi,M_{n+1}} v(1)\| < \varepsilon/3 \cdot 2^{n+1} \). We then let

\[
F_{n+1} = \Gamma_{n+1} = \{(I - P_{\alpha_n}) f : f \in F\},
\]

\[
G_{n+1} = F,
\]

\[
\Phi_{n+1} = \{P_{\alpha_n} f : f \in F\},
\]

\[
v_{n+1} = ((I - P_{\alpha_n}) \otimes I)v',
\]

and

\[
w_{n+1} = (P_{\alpha_n} \otimes (I - P_{\alpha_n})) v'.
\]

We compute

\[
\|v_{n+1} - w_{n+1} - E_{\xi,M_{n+1}} v(1)\| = \|v' - E_{\xi,M_{n+1}} v(1) + (P_{\alpha_n} \otimes P_{\alpha_n}) v'\|
\leq 2\|v' - E_{\xi,M_{n+1}} v(1)\| + \|(P_{\alpha_n} \otimes P_{\alpha_n}) E_{\xi,M_{n+1}} v(1)\|
< \varepsilon/2^{n+1}.
\]

This completes the proof that \( s_\xi(U) \leq 2k_G \) in the case that \( \zeta = \xi \).

We next isolate the analogue of Claim 1 which will be used to prove our results about \( g_{\xi,\zeta} \).
Claim 2. Still assuming that $\zeta < \xi$ are such that $h(\xi, \nu)$ holds for all $\nu < \zeta$ and $h(\eta)$ holds for any $\eta < \xi$, then for any $\alpha < \omega^\zeta$, any $\beta < \omega^\zeta$, any $\varepsilon > 0$, any weakly null collection $v = (v_E)_{E \not= E \in [N]^{<\omega}}$ of $B_U$, and any $L \in [N]$, there exists $M \in [L]$ such that

$$\|E^{(2)}_{\xi,1+\xi,M}v(1) - ((I - P_\alpha) \otimes (I - P_\beta))E^{(2)}_{\xi,1+\xi,M}v(1)\| < \varepsilon.$$  

Let us show the claim. We first consider the operator $I \otimes R_\beta : C_0(\omega^\xi) \otimes_\pi C(\omega^\zeta) \to C_0(\omega^\xi) \otimes_\pi C(\beta)$ for $\beta < \omega^\zeta$. We claim that there exists $\mu < 1 + \zeta$ such that $g_{\xi,\mu}(C_0(\omega^\xi) \otimes_\pi C(\beta)) < \infty$. If $\beta$ is finite, then $g_{\xi,0}(C_0(\omega^\xi) \otimes_\pi C(\beta)) < \infty$ by Lemma 5.11. In this case, let $\mu = 0 < 1 + \zeta$. If $\beta$ is infinite, then $C(\beta)$ is isomorphic to $C_0(\omega^\nu)$ for some $\nu < \zeta$. In this case, $C_0(\omega^\xi) \otimes_\pi C(\beta)$ is isomorphic to $C_0(\omega^\xi) \otimes_\pi C_0(\omega^\nu)$. Since $h(\xi, \nu)$ is assumed to hold, it follows that $g_{\xi,1+\nu}(C_0(\omega^\xi) \otimes_\pi C(\beta)) < \infty$, and therefore $g_{\xi,1+\mu}(C_0(\omega^\xi) \otimes_\pi C(\beta)) < \infty$. Note also that since $\nu < \zeta$, $1 + \nu < 1 + \zeta$. In this case, let $\mu = 1 + \nu < 1 + \zeta$.

It follows from Lemma 5.11 combined with the infinite Ramsey theorem that for any $\varepsilon > 0$, any weakly null collection $w = (w_E)_{E \not= E \in [N]^{<\omega}}$ in $B_{C_0(\omega^\xi) \otimes_\pi C(\beta)}$, and $L \in [N]$, there exists $M \in [L]$ such that for all $N \in [M]$, $\|E^{(2)}_{\xi,1+\xi,N}w(1)\| < \varepsilon$. Therefore for any $\varepsilon > 0$, any weakly null collection $v = (v_E)_{E \not= E \in [N]^{<\omega}}$ in $B_U$, and $L \in [N]$, by letting $w = (w_E)_{E \not= E \in [N]^{<\omega}} = ((I \otimes R_\beta)v_E)_{E \not= E \in [N]^{<\omega}}$, we deduce that there exists $M \in [L]$ such that for all $N \in [M]$, $\|(I \otimes R_\beta)E^{(2)}_{\xi,1+\xi,N}v(1)\| < \varepsilon$.

If $\xi = \zeta$, by symmetry, we can apply the same argument to deduce that for any $\varepsilon > 0$, any weakly null collection $v = (v_E)_{E \not= E \in [N]^{<\omega}}$ in $B_U$, and $L \in [N]$, there exists $M \in [L]$ such that for all $N \in [M]$, $\|(R_\alpha \otimes I)E^{(2)}_{\xi,1+\xi,N}v(1)\| < \varepsilon$.

If $\zeta < \xi$, then $Sz(C(\alpha) \otimes_\pi C_0(\omega^\nu)) \leq \omega^\xi$ by Claim 1, which implies that $g_{\xi,0}(C(\alpha) \otimes_\pi C_0(\omega^\nu)) = 0$. Again by Lemma 5.11, we deduce that for any weakly null collection $w = (w_E)_{E \not= E \in [N]^{<\omega}}$ in $B_{C(\alpha) \otimes_\pi C_0(\omega^\xi)}$, and $L \in [N]$, there exists $M \in [L]$ such that for all $N \in [M]$, $\|E^{(2)}_{\xi,1+\xi,N}w(1)\| < \varepsilon$. Therefore for any $\varepsilon > 0$, any weakly null collection $v = (v_E)_{E \not= E \in [N]^{<\omega}}$ in $B_U$, and $L \in [N]$, by letting $w = (w_E)_{E \not= E \in [N]^{<\omega}} = ((R_\alpha \otimes I)v_E)_{E \not= E \in [N]^{<\omega}}$, we deduce that there exists $M \in [L]$ such that for all $N \in [M]$, $\|(R_\alpha \otimes I)E^{(2)}_{\xi,1+\xi,N}v(1)\| < \varepsilon$.

Fix a weakly null collection $v = (v_E)_{E \not= E \in [N]^{<\omega}}$ in $B_U$, $\varepsilon > 0$, and $L \in [N]$. By the first paragraph of the proof of the claim, there exists $M \in [L]$ such that for all $N \in [M]$, $\|(I \otimes R_\beta)E^{(2)}_{\xi,1+\xi,N}v(1)\| < \varepsilon/2$. By the second paragraph of the proof of the claim, there exists $N \in [M]$ such that $\|(R_\alpha \otimes I)E^{(2)}_{\xi,1+\xi,N}v(1)\| < \varepsilon/2$. Note that

$$\|(I \otimes P_\beta)E^{(2)}_{\xi,1+\xi,N}v(1)\| = \|(I \otimes R_\beta)E^{(2)}_{\xi,1+\xi,N}v(1)\| < \varepsilon/2$$

and

$$\|(P_\alpha \otimes I)E^{(2)}_{\xi,1+\xi,N}v(1)\| = \|(R_\alpha \otimes I)E^{(2)}_{\xi,1+\xi,N}v(1)\| < \varepsilon/2.$$
Therefore

\[ \|E_{\xi,1+\xi,M}^{(2)}v(1) - ((I - P_\alpha) \otimes (I - P_\beta))E_{\xi,1+\xi,M}^{(2)}v(1)\| \leq \|(P_\alpha \otimes (I - P_\beta))E_{\xi,1+\xi,M}^{(2)}v(1)\| \]
\[ + \|(P_\alpha \otimes P_\beta)E_{\xi,1+\xi,M}^{(2)}v(1)\| \]
\[ \leq \|(I \otimes (I - P_\beta))(P_\alpha \otimes I)E_{\xi,1+\xi,M}^{(2)}v(1)\| \]
\[ + \|(I \otimes P_\beta)E_{\xi,1+\xi,M}^{(2)}v(1)\| \]
\[ = \|(R_\alpha \otimes I)E_{\xi,1+\xi,M}^{(2)}v(1)\| \]
\[ + \|(I \otimes R_\beta)E_{\xi,1+\xi,M}^{(2)}v(1)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \]

This finishes the proof of Claim 2.

If \( \zeta < \xi \), fix \( \kappa > k_G \). If \( \zeta = \xi \), fix \( \kappa > 2k_G \). Fix a sequence \((\varepsilon_n)_{n=1}^{\infty}\) in \((0, \infty)\) such that
\[ k_G + \sum_{n=1}^{\infty} \varepsilon_n < \kappa \text{ if } \zeta < \xi \text{ or } 2k_G + \sum_{n=1}^{\infty} \varepsilon_n < \kappa \text{ if } \zeta = \xi. \]

Fix a weakly null collection \( u = (u_E)_{\varnothing \neq E \subseteq [\mathbb{N}]^{\omega}} \) in \( B_U \) and \( L \in [\mathbb{N}] \). Since \( s_\xi(U) < \kappa \), we can, by replacing \( L \) with a subset, assume that for all \( M \in [L] \) and all \( n \in \mathbb{N} \), \( \|E_{\xi,M}^{(2)}u(n)\| \leq \kappa \).

Let \( \alpha_0 = \beta_0 = -1 \). Let \( M_1 = M \) and let \( E_1 < M_1 \) be such that \( E_1 \in \text{MAX}(S_{1+\xi}[S_\xi]) \). Since \( \|E_{\xi,M_1}^{(2)}u(1)\| \leq \kappa \), we can select finite sets \( F_1 \subset \kappa B_{C_0(\omega^\xi)}, G_1 \subset B_{C_0(\omega^\xi)} \), \( \alpha_1 < \omega^\xi, \beta_1 < \omega^\xi \), and
\[ v_1 \in \text{co}\{f \otimes g : (f, g) \in F_1 \times G_1\} \]
such that \( \text{supp}(f) \subset (\alpha_0, \alpha_1] \) for all \( f \in F_1 \), \( \text{supp}(g) \subset (\beta_0, \beta_1] \) for all \( g \in G_1 \), and
\[ \|v_1 - E_{\xi,M_1}^{(2)}u(1)\| < \varepsilon_1. \]

Now assume that \( E_1 < \ldots < E_n, M_1 \supset \ldots \supset M_n, \alpha_0 < \ldots < \alpha_n < \omega^\xi, \beta_0 < \ldots < \beta_n < \omega^\xi, F_1, \ldots, F_n, G_1, \ldots, G_n \), and \( v_1, \ldots, v_n \) have been chosen. Define \( v = (v_E)_{\varnothing \neq E \subseteq [\mathbb{N}]^{\omega}} \subset B_U \) by letting \( v_E = u_{\bigcup_{E \subset E_{n+1}} \bigcup_{E \subset E_n} E} \) if \( E_n < E \) and \( v_E = 0 \) otherwise. Using Claim 2 and our choice of \( M \), we can select \( M_{n+1} \subset [(\max E_n, \infty) \cap M_n] \) such that
\[ \|E_{\xi,1+\xi,M_{n+1}}^{(2)}v(1)\| \leq \kappa \]
and
\[ \|E_{\xi,1+\xi,M_{n+1}}^{(2)}v(1) - ((I - P_{\alpha_n}) \otimes (I - P_{\beta_n}))E_{\xi,1+\xi,M_{n+1}}^{(2)}v(1)\| < \varepsilon_{n+1}/2. \]
We can choose finite sets \( F \subset \kappa B_{C_0(\omega^\xi)} \) and \( G \subset B_{C_0(\omega^\xi)} \), ordinals \( \alpha_{n+1} \subset (\alpha_n, \omega^\xi), \beta_{n+1} \subset (\beta_n, \omega^\xi) \), and
\[ v' \in \text{co}\{f \otimes g : (f, g) \in F \times G\} \]
such that \( \text{supp}(f) \subset [0, \alpha_{n+1}] \) for all \( f \in F \), \( \text{supp}(g) \subset [0, \beta_{n+1}] \) for all \( g \in G \), and \( \|v' - E_{\xi,1+\xi,M_{n+1}}^{(2)}v(1)\| < \varepsilon_{n+1}/2. \) Define
\[ F_{n+1} = \{(I - P_{\alpha_n})f : f \in F\}, \]
\[ G_{n+1} = \{(I - P_{\beta_n})g : g \in G\}, \]
and
\[ v_{n+1} = ((I - P_{\alpha_n}) \otimes (I - P_{\beta_n}))v'. \]
Note that \( \text{supp}(f) \subset (\alpha_n, \alpha_{n+1}] \) for each \( f \in F_{n+1}, \) \( \text{supp}(g) \subset (\beta_n, \beta_{n+1}] \) for each \( g \in G_{n+1}, \) and
\[
\|v_{n+1} - \mathbb{E}_{\xi,1+\zeta,M_{n+1}}^{(2)} v(1)\| \leq \|(I - P_{\alpha_n}) \otimes (I - P_{\beta_n}))(v' - \mathbb{E}_{\xi,1+\zeta,M_{n+1}}^{(2)} v(1))\|
+ \|\mathbb{E}_{\xi,1+\zeta,M_{n+1}}^{(2)} v(1) - ((I - P_{\alpha_n}) \otimes (I - P_{\beta_n}))\mathbb{E}_{\xi,1+\zeta,M_{n+1}}^{(2)} v(1)\|
< \varepsilon_{n+1}/2 + \varepsilon_{n+1}/2 = \varepsilon_{n+1}.
\]
This completes the recursive construction. Let \( N = \bigcup_{n=1}^{\infty} E_n. \)

We note that for each \( (f_n, g_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \times G_n, \) \( \| (f_n)_{n=1}^{\infty} \|_w^w \leq \kappa, \) since the functions \( f_n \in \kappa B_{C_0(\omega^\zeta)} \) have pairwise disjoint supports, and \( \| (g_n)_{n=1}^{\infty} \|_1^1 \leq 1, \) since the functions \( g_n \in B_{C_0(\omega^\zeta)} \) have pairwise disjoint supports. Therefore by Proposition 3.2, \( \| (f_n \otimes g_n)_{n=1}^{\infty} \|_1^w \leq \kappa \) for each \( (f_n, g_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \times G_n. \) By the triangle inequality, \( \| (v_n)_{n=1}^{\infty} \|_1^w \leq \kappa. \) Since \( \mathbb{E}_{\xi,1+\zeta,M_{n+1}}^{(2)} v(1) = \mathbb{E}_{\xi,1+\zeta,N}^{(2)} u(n + 1), \) it follows that
\[
\| \mathbb{E}_{\xi,1+\zeta,N}^{(2)} u \|_w^1 \leq \| (v_n)_{n=1}^{\infty} \|_1^w + \sum_{n=1}^{\infty} \| v_n - \mathbb{E}_{\xi,1+\zeta,N}^{(2)} u(n) \| \leq \kappa + \sum_{n=1}^{\infty} \varepsilon_n.
\]
Since this holds for any \( \{ \varepsilon_n \}_{n=1}^{\infty} \subset (0, \infty), \) any weakly null collection \( u = (u_{\omega})_{\omega \in \mathbb{N} \setminus \{ \emptyset \}} \subset B_{U}, \) and \( L \in [N], \) it follows that \( g_{\xi,1+\zeta}(U) \leq \kappa. \) Since \( \kappa > k_G \) was arbitrary in the \( \zeta < \xi \) case, and \( \kappa > 2k_G \) was arbitrary in the \( \zeta = \xi \) case, we are done.

\[
\Box
\]

7. Sharpness

In this section, let \( K \) be a compact, Hausdorff space. We let \( \mathcal{M}(K) = C(K)^*, \) the space of Radon measures on \( K. \) For each \( \varpi \in K, \) we denote by \( \delta_{\varpi} \) the Dirac evaluation functional given by \( \delta_{\varpi}(f) = f(\varpi). \)

Let \( \Delta_m = \{(\varepsilon_n)_{n=1}^{m} : \varepsilon_n \in \{\pm 1\}\}. \) We let \( \Delta_0 = \{\emptyset\}, \) \( \Delta_{<m} = \bigcup_{n=0}^{m} \Delta_n, \) and \( \Delta_{m} \cup \bigcup_{n=0}^{m-1} \Delta_n. \)

A Cantor scheme on \( K \) will refer to a family of non-empty subsets \( D = (A_d)_{d \in \Delta_{<m}} \) of \( K \) such that for each \( d \in \Delta_{<m}, A_d \supset A_{d-(-1)} \cup A_{d-(-1)} \) and \( A_{d-(-1)} \cap A_{d-(-1)} = \emptyset. \) Note that we do not have any requirement that \( A_{\emptyset} \) be equal to \( K. \) Given a Cantor scheme \( D \) on \( K, \) a \( D \)-selector is a function \( \psi : \Delta_m \to K \) such that for each \( d \in \Delta_m, \) \( \psi(d) \in A_d. \)

Given a Cantor scheme \( D = (A_d)_{d \in \Delta_{<m}} \) on \( K, \) we say that a sequence \( (f_i)_{i=1}^{m} \) in \( C(K) \) is compatible with the Cantor scheme \( D \) provided that for any \( 1 \leq i \leq m \) and any \( d \in \Delta_{i-1}, f_i|_{A_{d-\varepsilon_i}} \equiv \varepsilon \) for \( \varepsilon \in \{\pm 1\}. \) Note that if \( (f_i)_{i=1}^{m} \) is compatible with \( D, \) then for any \( D \)-selector \( \psi \) and \( 1 \leq i \leq m, \) \( f_i(\psi(\varepsilon_1, \ldots, \varepsilon_m)) = \varepsilon_i, \) since \( \psi(\varepsilon_1, \ldots, \varepsilon_m) \in A_{(\varepsilon_1, \ldots, \varepsilon_i)} \) and \( f_i|_{A_{(\varepsilon_1, \ldots, \varepsilon_i)}} \equiv \varepsilon_i. \)

Given a Cantor scheme \( D = (A_d)_{d \in \Delta_{<m}} \) on \( K \) and a \( D \)-selector \( \psi, \) we define the associated Rademacher sequence \( (\mu_i)_{i=1}^{m} \) by letting
\[
\mu_i = \frac{1}{2^{m}} \sum_{(\varepsilon_1, \ldots, \varepsilon_m) \in \Delta_{m}} \varepsilon_i \delta_{\psi(\varepsilon_1, \ldots, \varepsilon_m)}.\]

If \( \mu \) denotes the uniform probability measure on \( \{\psi(d) : d \in \Delta_m\}, \) then \( \mu_i \ll \mu \) for each \( 1 \leq i \leq m. \) Then if \( r_i = \frac{d\mu_i}{d\mu} \) is the Radon-Nikodym derivative of \( \mu_i \) with respect to \( \mu, \) \( (\mu_i)_{i=1}^{m} \) is isometrically equivalent to \( (r_i)_{i=1}^{m} \subset L_1(\mu). \) Moreover, \( (r_i)_{i=1}^{m} \) is an orthonormal system in \( L_2(\mu), \) which means it is weakly 2-summing with 2-weakly summing norm equal to 1 when considered as
a sequence in $L_2(\mu)$. Since $\mu$ is a probability measure, the formal inclusion from $L_2(\mu)$ to $L_1(\mu)$ is norm 1, which means that $(r_i)_{i=1}^m$ is weakly 2-summing in $L_1(\mu)$ with 2-weakly summing norm in $L_1(\mu)$ not exceeding 1. It is clear that the 2-weakly summing norm of $(r_i)_{i=1}^m$, when considered as a sequence in $L_1(\mu)$ is exactly 1, since $\|r_i\|_{L_1(\mu)} = 1$ for each $1 \leq i \leq m$. Therefore $(\mu_i)_{i=1}^m$ satisfies $\|(\mu_i)_{i=1}^m\|_2^2 \leq 1$.

**Remark 7.1.** Note that if $D = (A_d)_{d \in \Delta_{<m}}$ is a Cantor scheme on $K$, $(f_i)_{i=1}^m$ is compatible with $D$, and $\psi$ is a $D$-selector, then $(\mu_i)_{i=1}^m$ and $(\mu_i)_{i=1}^m$ are biorthogonal. Indeed,

$$\langle \mu_j^\psi, f_i \rangle = \frac{1}{2^m} \sum_{(\varepsilon_1, \ldots, \varepsilon_m) \in \Delta_m} \varepsilon_j f_i(\varepsilon_1, \ldots, \varepsilon_m) = \frac{1}{2^m} \sum_{(\varepsilon_1, \ldots, \varepsilon_m) \in \Delta_m} \varepsilon_j \varepsilon_j = \delta_{i,j}.$$

We now arrive at the following consequence for witnessing our eventual lower estimates. The following proposition and its proof deal with both projective and injective tensor norms. Therefore we distinguish between projective and injective tensor norms within the following proposition, and thereafter return to considering only projective tensor norms.

**Proposition 7.2.** Let $K$ be a compact, Hausdorff space and let $X$ be a Banach space. Let $D = (A_d)_{d \in \Delta_{<l}}$ be a Cantor scheme on $K$ and let $(f_i)_{i=1}^l$ be compatible with $D$. Fix $1 \leq s_1 < \ldots < s_m \leq l$ and suppose that $(x_j)_{j=1}^n \subset X$, $(x_j^*)_{j=1}^m \subset B_{X^*}$, and $0 = r_0 < \ldots < r_m = n$ are such that $\langle x_i^*, x_j \rangle = 1$ if $j \in (r_{i-1}, r_i]$, and $\langle x_i^*, x_j \rangle = 0$ otherwise. Then for any sequences $(a_i)_{i=1}^m$, $(b_i)_{i=1}^n$ of positive numbers such that $\sum_{i=1}^m a_i^2 = 1$ and $\sum_{j=r_i-1+1}^{r_i} b_j = 1$ for each $1 \leq i \leq m$,

$$\left\| \sum_{i=1}^m \sum_{j=1}^n a_i b_j f_{s_i} \otimes x_j \right\|_\pi \geq 1.$$

**Proof.** Let $\psi$ be any $D$-selector and let $(\mu_i)_{i=1}^l = (\mu_i^\psi)_{i=1}^l$ be the associated Rademacher sequence. Since $\|(\mu_i)_{i=1}^m\|_2^2 \leq \|(\mu_i^\psi)_{i=1}^m\|_2^2 \leq 1$. By Proposition 3.2

$$\|(\mu_i \otimes x_i^*)_{k=1}^m\|_w \leq \|(\mu_i)_{k=1}^m\|_2 \|(x_i^*)_{k=1}^m\|_\infty \leq 1.$$

Here, $\mu_i \otimes x_i^*$ is treated as a member of $\mathcal{M}(K) \otimes_\varepsilon X^* \subset (C(K) \otimes_\pi X)^*$. Therefore

$$\left\| \sum_{i=1}^m a_i \mu_{s_i} \otimes x_i^* \right\|_{\varepsilon} \leq 1.$$

Note that

$$\left\| \sum_{i=1}^m \sum_{j=1}^n a_i b_j f_{s_i} \otimes x_j \right\|_\pi \geq \left\langle \sum_{i=1}^m a_i \mu_{s_i} \otimes x_i^*, \sum_{i=1}^m \sum_{j=1}^n a_i b_j f_{s_i} \otimes x_j \right\rangle$$

$$= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^m a_i b_j a_k \langle \mu_{s_k}, f_{s_i} \rangle \langle x_k^*, x_j \rangle$$

$$= \sum_{i=1}^m \sum_{j=r_i-1+1}^{r_i} b_j = 1.$$
The following proof is similar to the computation of the Bourgain \(\ell_1\) index of the spaces \(C(\omega^{\omega^i})\) found in the work of Alspach, Judd, and Odell [1].

**Lemma 7.3.** For any ordinal \(\gamma \geq 1\), there exists a tree \(T_\gamma\) on \([0, \omega^\gamma)\) with rank\((T_\gamma) = \omega \cdot 1\) and a collection \((f_t)_{t \in T_\gamma}\) in \(B_{C_0(\omega^\gamma)}\) such that for each \(t \in \text{MAX}(T_\gamma)\), there exists a Cantor scheme on \([1, \omega^\gamma]\) with respect to which \((f_s)_{s \leq t}\) is compatible.

**Proof.** We work by induction. We will also define the functions \((f_t)_{t \in T_\gamma}\) to satisfy \(f_t(0) = 0\).

For the \(\gamma = 1\) case, we first define

\[
T_{0,n} = \{(n-1, n-2, \ldots, n-k) : 1 \leq k \leq n\}
\]

and define \(T_1 = \bigcup_{n=1}^\infty T_{0,n}\). Note that rank\((T_1) = \omega = \omega \cdot 1\), since \(T_1 = \bigcup_{n=1}^\infty T_{0,n}\) is a totally incomparable union and the rank of \(T_{0,n}\) is \(n\). It suffices to define \((f_s)_{s \in T_{0,n}}\) for each \(n \in \mathbb{N}\). To that end, fix \(n \in \mathbb{N}\). We define

\[
f_{(n-1, \ldots, n-k)} = \sum_{i=1}^k (-1)^{i-1}1_{(2^{n-k}(i-1), 2^{n-k}]}[t]
\]

where \(1_{(2^{n-k}(i-1), 2^{n-k}]}\) is the indicator function of \([2^{n-k}(i-1), 2^{n-k}])\ in \([0, \omega]\).

The sequence \((f_{(n-1, \ldots, n-k)})_{k=1}^n\) is compatible with the Cantor scheme defined by \(A_{\varnothing} = (0, 2^n)\) and for \(d = (\varepsilon_1, \ldots, \varepsilon_k) \in \Delta_k\) by

\[
A_{(\varepsilon_1, \ldots, \varepsilon_k)} = \bigcap_{t=1}^k f_{(n-1, \ldots, n-k)}(\varepsilon_t).
\]

Assume the result holds for some ordinal \(\gamma\). Fix a tree \(T_\gamma\) on \([0, \omega^\gamma)\) with rank\((T_\gamma) = \omega \cdot 1\) and a collection \((g_t)_{t \in T_\gamma}\) as in the conclusion, and also satisfying \(g_t(0) = 0\) for all \(t \in T_\gamma\). As in the previous paragraph, it suffices to produce pairwise incomparable trees \(T_\gamma,n\) on \([0, \omega^\gamma+n) \subset [0, \omega(\gamma+1))\) with rank\((T_\gamma,n) = \omega \gamma + n\) and a collection of functions \((f_t)_{t \in T_\gamma,n}\) in \(B_{C_0(\omega^{\gamma+1})}\) satisfying the conclusions, and then let \(T_{\gamma+1} = \bigcup_{n=1}^\infty T_{\gamma,n}\). To that end, fix \(n \in \mathbb{N}\). Let

\[
R_n = \{(\omega^\gamma + n - 1, \ldots, \omega^\gamma + n - k) : 1 \leq k \leq n\},
\]

\[
S_n = \{(\omega^\gamma + n - 1, \ldots, \omega^\gamma) \Join t : t \in T_\gamma\},
\]

and let

\[
T_{\gamma,n} = R_n \cup S_n.
\]

It is clear that rank\((T_{\gamma,n}) = \omega \gamma + n\).

In order to define the collection \((f_t)_{t \in T_{\gamma,n}}\), we first define \((f_t)_{t \in R_{\gamma,n}}\) by

\[
f_{(\omega^\gamma+n-1, \ldots, \omega^\gamma+n-k)} = \sum_{i=1}^k (-1)^{i-1}1_{(\omega^\gamma 2^{n-k}(i-1), \omega^\gamma 2^{n-k}]}[t]
\]

where, for \(k = 1, \ldots, n\) and \(i = 1, \ldots, 2^k\), \(1_{(\omega^\gamma 2^{n-k}(i-1), \omega^\gamma 2^{n-k}]}\) denotes the indicator function of the interval \([\omega^\gamma 2^{n-k}(i-1), \omega^\gamma 2^{n-k}])\ in \([0, \omega^{\gamma+1}])\). For \(t = (\omega^\gamma + n - 1, \ldots, \omega^\gamma) \Join s\ with \(s \in T_\gamma\), we define \(f_t\) on each interval \((\omega^\gamma(i-1), \omega^\gamma]t)\ for \(1 \leq i \leq 2^m\) by letting \(f_t(\omega^\gamma(i-1) + \eta) = g_\eta(\eta), \eta \in [0, \omega^\gamma]\) and we define \(f_t\) on \((\omega^\gamma 2^n, \omega^\gamma + 1]\) by letting \(f_t(\eta) = 0\) for all \(\eta \in (\omega^\gamma 2^n, \omega^\gamma + 1]\). It is easy to see that each \(f_t \in C_0(\omega^{\gamma+1})\). Let \(t = (\omega^\gamma + n - 1, \ldots, \omega^\gamma) \Join s \in \text{MAX}(T_{\gamma,n})\). By construction \((f_r)_{r \leq t}\) is compatible with a Cantor scheme on \([0, \omega^{\gamma+1}]\).
Now assume that $\gamma$ is a limit ordinal and the result holds for all $1 \leq \zeta < \gamma$. For each $1 \leq \zeta < \gamma$, let $T_\zeta$ be a tree on $[1, \omega^{\zeta}]$ with rank($T_\zeta$) = $\omega^\zeta$ and let $(g_t)_{t \in T_\zeta}$ be a family in $B_{C_0(\omega^{\zeta})}$ be as in the statement of the lemma. Let

$$U_{\zeta+1} = \{(\omega^{\zeta} + \nu_t)_{i=1}^n : (\nu_t)_{i=1}^n \in T_\zeta\}.$$  

Note that the function $\eta \in [0, \omega^{\zeta+1}] \rightarrow \omega^\zeta + \eta \in [\omega^\zeta, \omega^{\zeta+1}]$ is bijective. Therefore the function function $\nu = (\nu_t)_{i=1}^n \in T_{\zeta+1} \mapsto \phi_\zeta(\nu) = (\omega^\zeta + \nu_t)_{i=1}^n \in U_{\zeta+1}$ is a tree isomorphism, and $U_{\zeta+1}$ is a tree on $[\omega^\zeta + 1, \omega^{\zeta+1}]$ with rank $(\zeta + 1)$. This construction guarantees that the trees $(U_{\zeta+1})_{\zeta < \gamma}$ are pairwise incomparable. From this it follows that $T_\gamma = \cup_{\zeta < \gamma} U_{\zeta+1}$ has rank $\omega^\gamma$.

For $t \in U_{\zeta+1}$, define $f_t \in C_0(\omega^{\gamma+1})$ to be the function defined by

$$f_t(\eta) = \begin{cases} g_{\zeta+1}(t)(\eta) : \eta \leq \omega^{\zeta+1} \\ 0 : \omega^{\zeta+1} < \eta \leq \omega^\gamma. \end{cases}$$

It is easy to see that $(f_t)_{t \in T_\gamma}$ satisfies the conclusion. Indeed, for any $t \in MAX(T_\gamma)$, $\phi_\zeta^{-1}(t) \in MAX(T_{\zeta+1})$ for some $\zeta < \gamma$. By construction, there exists a Cantor scheme on $[1, \omega^{\zeta+1}]$ with respect to which $(g_s)_{s \leq \phi_\zeta^{-1}(t)}$ is compatible. Since $f_s|_{[1, \omega^{\zeta+1}]} = g_{\phi_\zeta^{-1}(s)}$ for each $s \leq t$, it follows that if $D$ is a Cantor scheme on $[1, \omega^{\zeta+1}]$ with respect to which $(g_s)_{s \leq \phi_\zeta^{-1}(t)}$ is compatible, then $D$ is also a Cantor scheme on $[1, \omega^\gamma]$ with respect to which $(f_s)_{s \leq t}$ is also compatible.

\[ \square \]

Lemma 7.4. For any $1 \leq \xi, \zeta < \omega_1$ and any tree $T$ with rank$(T) = \omega^{1+\zeta}$, there exists a monotone map $\Phi : S_{1+\zeta}[S_\xi] \setminus \emptyset \rightarrow T$ such that for each $E \in MAX(S_{1+\zeta}[S_\xi])$ if $E = \bigcup_{n=1}^m E_n$ with $E_1 < \cdots < E_m$ and $E_n \in MAX(S_\xi)$ for each $1 \leq n \leq m$, then, for each $1 \leq i \leq m$, $\Phi(\bigcup_{n=1}^i E_n) = t_i$ and $\Phi(F) = t_i$ for each $\bigcup_{n=1}^{i-1} E_n < F \preceq \bigcup_{n=1}^m E_n$.

**Proof.** In the proof, we use the following fact about rank functions on trees. For a well-founded tree $S$ and $s \in S$, let $g(s) = \sup\{\mu : s \in S^\mu\}$. First, note that the supremum exists, since $S^{\text{rank}(S)} = \emptyset$. Second, the preceding fact shows that $g(s) < \text{rank}(S)$. Next, it is easy to see that this supremum is actually a maximum. In fact, there is only something to be shown if $g(s)$ is a limit ordinal, in which case, since $s \in S^\mu$ for all $\mu < g(s)$, it holds that $s \in \cap_{\mu < g(s)} S^\mu = S^{g(s)}$. We also note that if $s < s'$, then $g(s) > g(s')$. More precisely, $g(s) = \sup\{g(s') + 1 : s < s' \in S\}$, with the convention that the supremum of the empty set is 0.

We note that rank($S_{1+\zeta}$) = $\omega^{1+\zeta}+1$ and rank($S_{1+\zeta}\setminus\emptyset$) = $\omega^{1+\zeta}$. Define $h : S_{1+\zeta}\setminus\emptyset \rightarrow [0, \omega^{1+\zeta})$ by $h(E) = \max\{\mu : E \in S_{1+\zeta}^\mu\}$ and define $g : T \rightarrow [0, \omega^{1+\zeta})$ by $g(t) = \max\{\mu : t \in T^\mu\}$.

We will define $\Phi(E)$ by induction on the cardinality of $E$ to have the property that for $E = \bigcup_{n=1}^m E_n \in S_{1+\zeta}[S_\xi]$ such that $E_1 < \cdots < E_m$ and $E_n \in MAX(S_\xi)$ for $1 \leq n < m$, $h((\min E_n)_{n=1}^m) \leq g(\Phi(E))$.

The case $|E| = 1$. For $(n) \in S_{1+\zeta}[S_\xi]$, $h((n)) < \omega^{1+\zeta}$, so there exists $t \in T$ such that $h((n)) \leq g(t)$. Define $\Phi((n)) = t$.

Fix $G \in S_{1+\zeta}[S_\xi]$ with $|G| > 1$ and assume that $\Phi(E)$ has been defined for each $E$ with $|E| < |G|$. Let $p = \max G$ and let $E = G \setminus \{p\}$. Then $G = E \wedge (p)$ and $E \in S_{1+\zeta}[S_\xi] \setminus MAX(S_{1+\zeta}[S_\xi])$. Write
If \( E = \bigcup_{n=1}^{m} E_n \) with \( E_1 < \cdots < E_m, E_n \in \text{MAX}(S_\xi) \) for \( 1 \leq n < m \) and \( E_m \in S_\xi \), if \( E_m \notin \text{MAX}(S_\xi) \) we define \( \Phi(G) = \Phi(E) \). If \( E_m \in \text{MAX}(S_\xi) \), let \( t_m = \Phi(E) \). Let \( H = (\min E_n)_{n=1}^{m} \) and note that, by construction, \( h(H) \leq g(t_m) \). Note that \( h(H \cap (p)) < h(H) \leq g(t_m) \). Therefore there exists \( s \in T \) such that \( t < s \) and \( g(s) \geq h(H \cap (p)) \). Define \( \Phi(G) = s \). This complete the recursive construction. It is clear that the conclusions are satisfied in this case. \( \square \)

For a compact, Hausdorff space \( K \), we let \( d(K) \) denote the Cantor-Bendixson derivative of \( K \), which consists of the points of \( K \) which are not isolated in \( K \). We let \( d(\emptyset) = \emptyset \). Note that either \( d(K) \) is empty, or \( d(K) \) is also compact, Hausdorff. We define

\[
d^0(K) = K, \\
d^{n+1}(K) = d(d^n(K)),
\]

and if \( \eta \) is a limit ordinal,

\[
d^\eta(K) = \bigcap_{\nu < \eta} d^\nu(K).
\]

By the Baire category theorem, every non-empty subset of a countable, compact, Hausdorff space \( K \) has an isolated point. From this and a cardinality argument, for each countable, compact, Hausdorff space, there exists a countable ordinal \( \eta \) such that \( d^\eta(K) \neq \emptyset \) and \( d^{\eta+1}(K) = \emptyset \). Moreover, \( d^\eta(K) \) is finite. The ordinal \( \eta + 1 \) is called the Cantor-Bendixson index of \( K \). We denote the Cantor-Bendixson index of \( K \) by \( CB(K) \), and since \( CB(K) \) is a successor, we let \( CB(K) - 1 \) denote the immediate predecessor of \( CB(K) \). We let

\[
C_0(K) = \{ f \in C(K) : f|_{d^{CB(K)-1}(K)} \equiv 0 \}.
\]

It is easy to see that \([0, \omega^\xi]^\xi = \{ \omega^\xi \} \), so this notation is consistent with the usual notation of \( C_0(\omega^\xi) \).

By classical facts about countable, compact, Hausdorff spaces, two countable, compact, Hausdorff spaces \( K, L \) are homeomorphic if and only if \( CB(K) = CB(L) \) and \( K^{CB(K)-1} \) and \( L^{CB(L)-1} \) have the same cardinality. Moreover, any homeomorphism between \( K \) and \( L \) must map \( K^{CB(K)-1} \) to \( L^{CB(L)-1} \). In this case, \( C_0(K) \) is isometrically isomorphic to \( C_0(L) \). Indeed, if \( \phi : L \to K \) is a homeomorphism, then the restriction of the map \( \Phi : C(K) \to C(L) \), \( \Phi f = f \circ \phi \), to \( C_0(K) \) is an isometric isomorphism between \( C_0(K) \) and \( C_0(L) \).

We also note that \( S_\xi \) with the Cantor topology is compact, and \( d^{\omega^\xi}(S_\xi) = \{ \emptyset \} \). From this and the preceding discussion, it follows that \( C_0(\omega^\xi) \) is isometrically isomorphic to

\[
C_0(S_\xi) = \{ f \in C(S_\xi) : f(\emptyset) = 0 \}.
\]

We will use this in the following result.

**Theorem 7.5.** For each \( \zeta \leq \xi < \omega_1 \), \( g_{\xi,1+\xi}(C_0(\omega^\xi) \otimes_\pi C_0(\omega^\xi)) \geq 1 \).

**Proof.** As noted prior to the statement of the theorem, \( C_0(\omega^\xi) \) is isometrically isomorphic to \( C_0(S_\xi) \). We will use \( C_0(S_\xi) \) in place of \( C_0(\omega^\xi) \).

Define the normalized sequence \( (e_n)_{n=1}^{\infty} \subset C_0(S_\xi) \) by letting \( e_n(E) = 1_E(n) \), where \( 1_E \) is the indicator function of \( E \). Since \( e_n(\emptyset) = 0 \), \( e_n \in C_0(S_\xi) \). Since each \( E \in S_\xi \) is finite, \( \lim_n e_n(E) = 0 \) for all \( n \in \mathbb{N} \). Since \( (e_n)_{n=1}^{\infty} \) is bounded and pointwise null, it is weakly null.
By Lemma 7.3 there exist a tree $T$ with rank $(T) = \omega^{1+\zeta}$ and a collection $(f_t)_{t \in T}$ in $B_{C_0(\omega^{\omega^\zeta})}$ such that for every $t \in \text{MAX}(T)$, $(f_s)_{s \leq t}$ is compatible with a Cantor scheme on $[0, \omega^{\omega^\zeta}]$.

By Lemma 7.4 there exists a map $\Phi : S_{1+\zeta}[S_\xi] \setminus \{\emptyset\} \rightarrow T$ such that for each $E \in \text{MAX}(S_{1+\zeta}[S_\xi])$, if $E = \bigsqcup_{n=1}^{m} E_n$ with $E_1 < \ldots < E_m$ and $E_n \in \text{MAX}(S_\xi)$ for each $1 \leq n \leq m$, then there exist $s_1 < \ldots < s_m$, $s_n \in T$, such that for each $1 \leq i \leq m$, $\Phi(F) = s_i$ whenever $\bigsqcup_{n=1}^{i-1} E_n < F \leq \bigsqcup_{n=1}^{i} E_n$.

For each $E \in [N]^{<\omega}$, define

$$u_E = \begin{cases} 
\epsilon_{\text{max}}E \otimes f_{\Phi(E)} & \text{if } E \notin S_{1+\zeta}[S_\xi] \setminus \{\emptyset\} \\
0 & \text{otherwise}
\end{cases}$$

**Claim 3.** The collection $\mathbf{u} = (u_E)_{\emptyset \neq E \in [N]^{<\omega}}$ in $B_{C_0(S_\xi) \otimes_\pi C_0(\omega^{\omega^\zeta})}$ is weakly null and $\|E^{(2)}_{\xi,1+\zeta,M}\mathbf{u}(1)\| \geq 1$ for all $M \in [N]$.

It is obvious that the claim gives the theorem.

In order to prove the claim we first show that $\mathbf{u}$ is weakly null. Note that $(C(S_\xi) \otimes_\pi C(\omega^{\omega^\zeta}))^* = B(C(S_\xi) \times C(\omega^{\omega^\zeta})) = L(C(S_\xi), \ell_1(\omega^{\omega^\zeta}))$. Fix $E \in [N]^{<\omega}$, $b \in B(C(S_\xi) \times C(\omega^{\omega^\zeta}))$, $T \in L(C(S_\xi), \ell_1(\omega^{\omega^\zeta}))$ the operator given by $T(f)(g) = b(f, g)$.

By standard properties of $S_{1+\zeta}[S_\xi]$, if $E \setminus (n) \in S_{1+\zeta}[S_\xi]$ for some $m > E$, then $E \setminus (n) \in S_{1+\zeta}[S_\xi]$ for all $n > E$. In this case, for each $E < n$, $u_{E-(n)} = e_n \otimes f_{\Phi(E-(n))}$ and $b(u_{E-(n)}) = T(e_n)(f_{\Phi(E-(n))}).$ The sequence $(T(e_n))_{n=1}^\infty$ is weakly null in the Schur space $\ell_1(\omega^{\omega^\zeta})$, so $\lim_n \|T(e_n)\| = 0$. Since the sequence $(f_{\Phi(E-(n))})_{n=1}^\infty$ is bounded, $\lim_n b(u_{E-(n)}) = 0$. Therefore $(u_{E-(n)})_{E<n}$ is weakly null in the case that $E \setminus (n) \in S_{1+\zeta}[S_\xi]$ for some (equivalently, every) $E < n$.

If $E \setminus (n) \in [N]^{<\omega} \setminus S_{1+\zeta}[S_\xi]$ for all $n > E$, then the sequence $(u_{E-(n)})_{n>E}$ is identically zero. Therefore $\mathbf{u}$ is weakly null.

Next, fix $M \in [N]$ and let $E \in \text{MAX}(S_{1+\zeta}[S_\xi])$ be such that $E \prec M$. We can write $E = \bigsqcup_{n=1}^{m} E_n$, where $E_1 < \ldots < E_m$ and $E_n \in \text{MAX}(S_\xi)$ for each $1 \leq n \leq m$.

For $1 \leq i \leq m$, let $a_i$ be the common value of $q F^{\xi,1+\zeta}_F$ for $\bigsqcup_{n=1}^{i} E_n \prec F \leq \bigsqcup_{n=1}^{i} E_n$. Note that $a_i$ are positive and $\sum_{i=1}^{m} a_i^2 = 1$.

For $1 \leq i \leq m$, let $x_i^* = \delta_{E_i} \in \mathcal{M}(S_\xi)$. Note that for $\emptyset \prec F \preceq E_i$, $\langle x_i^*, e_{\text{max},F} \rangle = 1$ if and only if $\bigsqcup_{n=1}^{i} E_n \prec F \leq \bigsqcup_{n=1}^{i} E_n$, and $\langle x_i^*, e_{\text{max},F} \rangle = 0$ otherwise.

Therefore

$$E^{(2)}_{\xi,1+\zeta,M}\mathbf{u}(1) = \sum_{i=1}^{m} \sum_{\bigsqcup_{n=1}^{i-1} E_n \prec F \leq \bigsqcup_{n=1}^{i} E_n} q F^{\xi,1+\zeta}_F p_F e_{\text{max},F} \otimes f_{\Phi(f)} = \sum_{i=1}^{m} a_i \sum_{\bigsqcup_{n=1}^{i} E_n \prec F \leq \bigsqcup_{n=1}^{i} E_n} p_F e_{\text{max},F} \otimes f_{s_i}$$

For $\bigsqcup_{n=1}^{i} E_n \prec F \leq \bigsqcup_{n=1}^{i} E_n$, $p_F \geq 0$ and $\sum_{\bigsqcup_{n=1}^{i} E_n \prec F \leq \bigsqcup_{n=1}^{i} E_n} p_F = 1$ by (3.1). By Proposition 7.2, $\|E^{(2)}_{\xi,1+\zeta,M}\mathbf{u}(1)\| \geq 1$. This finishes the claim and the theorem.

\[\square\]

8. Conclusion

We conclude by describing the isomorphism classes of $C(K) \otimes_\pi C(L)$ for countable, compact, Hausdorff spaces $K, L$. We note that the spaces $C(K)$, $K$ countable, compact, Hausdorff fit into the classification as $C(K) \otimes_\pi C(\{0\})$. Of course, the situation in which $K$ and $L$ are both finite and isomorphism is determined by the dimension of $C(K) \otimes_\pi C(L)$, which is $|K||L|$. Therefore we omit
the situation in which both $K$ and $L$ are finite, but include the case in which at most one of the spaces $K, L$ is finite in our classification.

**Theorem 8.1.** Let $K, L, M, N$ be countable, compact, Hausdorff spaces such that $K, M, N$ are infinite, $CB(L) \leq CB(K)$, and $CB(N) \leq CB(M)$. The following are equivalent.

(i) $C(K) \hat{\otimes}_p C(L)$ and $C(M) \hat{\otimes}_p C(N)$ are isomorphic to subspaces of quotients of each other.

(ii) $C(K) \hat{\otimes}_p C(L)$ and $C(M) \hat{\otimes}_p C(N)$ are isomorphic.

(iii) $C(K)$ is isomorphic to $C(M)$ and $C(L)$ is isomorphic to $C(N)$.

**Proof.** Of course, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Assume (i) holds. This implies that for any countable $\alpha, \beta$, $g_{\alpha, \beta}(C(K) \hat{\otimes}_p C(L)) = 0$ if and only if $g_{\alpha, \beta}(C(M) \hat{\otimes}_p C(N)) = 0$, and $g_{\alpha, \beta}(C(K) \hat{\otimes}_p C(L)) \leq \infty$ if and only if $g_{\alpha, \beta}(C(M) \hat{\otimes}_p C(N)) < \infty$.

Let $\xi < \omega_1$ be such that $C(K)$ is isomorphic to $C(\omega^\xi)$. Let $\mu < \omega_1$ be such that $C(M)$ is isomorphic to $C(\omega^\mu)$. Note that $Sz(C(K) \hat{\otimes}_p C(L)) = \omega^{\xi+1}$ and $Sz(C(M) \hat{\otimes}_p C(N)) = \omega^{\mu+1}$, since $CB(L) \leq CB(K)$ and $CB(N) \leq CB(M)$. By properties of the Szlenk index, since $C(K) \hat{\otimes}_p C(L)$ and $C(M) \hat{\otimes}_p C(N)$ are isomorphic to subspaces of each other, $\omega^{\xi+1} \leq \omega^{\mu+1} \leq \omega^{\xi+1}$, and $\xi = \mu$.

Therefore $C(K)$ and $C(M)$ are isomorphic to $C(\omega^{\omega^\xi}) = C(\omega^{\omega^\mu})$, and are therefore also isomorphic to each other.

We first show that $L$ must be infinite. We work by contradiction. Assume that $L$ is finite, in which case $C(K) \hat{\otimes}_p C(L)$ is isomorphic to $C(K)$ and $g_{\xi, 0}(C(K) \hat{\otimes}_p C(L)) < \infty$. Therefore $g_{\xi, 0}(C(M) \hat{\otimes}_p C(N)) < \infty$, which implies that $g_{\xi, \beta}(C(M) \hat{\otimes}_p C(N)) = 0$ for all $\beta > 0$. Since $N$ is infinite, $C(N)$ is isomorphic to $C(\omega^{\omega^\nu})$ for some $\nu < \omega_1$, and $g_{\xi, 1+\nu}(C(M) \hat{\otimes}_p C(N)) > 0$. But this is a contradiction, since $1 + \nu > 0$. Therefore $L$ must be infinite.

Let $\zeta < \omega_1$ be such that $C(L)$ is isomorphic to $C(\omega^{\omega^{\xi}})$ and let $\nu < \omega_1$ be such that $C(N)$ is isomorphic to $C(\omega^{\omega^{\nu}})$. Then $g_{\xi, 1+\zeta}(C(K) \hat{\otimes}_p C(L)) < \infty$, and therefore $g_{\xi, 1+\zeta}(C(M) \hat{\otimes}_p C(N)) < \infty$. This implies that $g_{\xi, \beta}(C(M) \hat{\otimes}_p C(N)) = 0$ for all $\beta > 1 + \zeta$. But we also know that $g_{\xi, 1+\nu}(C(M) \hat{\otimes}_p C(N)) > 0$, so $1 + \nu \leq 1 + \zeta$, and $\nu \leq \zeta$. By symmetry, $\zeta \leq \nu$, and $C(L)$ is isomorphic to $C(N)$.

**Theorem 8.2.** Let $K, L$ be countable, compact, Hausdorff spaces such that $K$ is infinite and $CB(L) \leq CB(K)$. The following are equivalent.

(i) There exists a compact, Hausdorff space $M$ such that $C(K) \hat{\otimes}_p C(L)$ and $C(M)$ are isomorphic to subspaces of quotients of each other.

(ii) $L$ is finite.

**Proof.** As in the proof of Theorem 8.1 we deduce that $C(K)$ is isomorphic to $C(M)$, and each is isomorphic to $C(\omega^{\omega^\xi})$ for some countable ordinal $\xi$. If $L$ were infinite, then $g_{\xi, 1}(C(K) \hat{\otimes}_p C(L)) > 0$. But since $g_{\xi, 0}(C(M)) < \infty$, $g_{\xi, 1}(C(M)) = 0$. Since $C(K) \hat{\otimes}_p C(L)$ and $C(M)$ are isomorphic to subspaces of quotients of each other, $g_{\xi, 1}(C(K) \hat{\otimes}_p C(L)) = 0$, a contradiction.

**Corollary 8.3.** If $K, L, M$ are infinite, countable, compact, Hausdorff spaces, then $C(K) \hat{\otimes}_p C(L)$ and $C(M)$ are not isomorphic.
The following corollary includes the promised isomorphic classification, and contains within it the classical result of Bessaga and Pełczyński.

**Corollary 8.4.** Let \( C(\omega^{\omega^{-1}}) = C(\{0\}) \). Then for each countable, compact, Hausdorff spaces \( K, L \) such that \( K \cup L \) is infinite, \( C(K) \widehat{\otimes}_\pi C(L) \) is isomorphic to exactly one of the spaces \( C(\omega^\xi) \widehat{\otimes}_\pi C(\omega^\zeta) \), \( \xi < \omega_1, -1 \leq \zeta \leq \xi \).

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