Using the Topology of Large Scale Structure to constrain Dark Energy

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ABSTRACT
The use of standard rulers, such as the scale of the Baryonic Acoustic oscillations (BAO), has become one of the more powerful techniques employed in cosmology to probe the entity driving the accelerating expansion of the Universe. In this paper, the topology of large scale structure (LSS) is used as one such standard ruler to study this mysterious ‘dark energy’. By following the redshift evolution of the clustering of luminous red galaxies (LRGs) as measured by their 3D topology (counting structures in the cosmic web), we can chart the expansion rate and extract information about the equation of state of dark energy. Using the technique first introduced in (Park \& Kim (2009)), we evaluate the constraints that can be achieved using 3D topology measurements from next-generation LSS surveys such as the Baryonic Oscillation Spectroscopic Survey (BOSS). In conjunction with the information that will be available from the Planck satellite, we find a single topology measurement on 3 different scales is capable of constraining a single dark energy parameter to within 5% and 10% when dynamics are permitted. This offers an alternative use of the data available from redshift surveys and serves as a cross-check for BAO studies.

1 INTRODUCTION
In the late 90’s, the expansion of the Universe, first detected by Hubble, was confirmed from the precise measurements of SNIa and astonishingly found to be accelerating (Riess et al. (1998); Perlmutter et al. (1999); Lange et al. (2001); Hoekstra et al. (2002); Riess et al. (2004); Cole et al. (2005); Astier et al. (2006); Spergel et al. (2006); Riess et al. (2006)). This provided convincing evidence for the presence of an unidentified entity in the Universe which (in the context of General Relativity) must act against the gravitational attraction of ordinary matter. Indirect yet also compelling evidence came from the missing energy density inferred from the discrepancy between the measurements of matter density (from direct measurements and observation of the Integrated Sachs-Wolfe effect) and the indications of spatial flatness from the CMB anisotropy spectrum, as well as the level of the initial inhomogeneity measured in the CMB compared with large scale structure today. Although the presence of this ‘dark energy’ is now well-established, its nature still evades us and characterizing it has become one of the most important topics in cosmology today. This is evidenced by the large number of experiments which have been proposed and designed with this question in mind.

One such effort is the Baryon Oscillations Spectroscopic Survey (BOSS) which plans to map the spatial distribution of luminous red galaxies (LRG) and quasars over 10,000 sq. deg. of the sky. The survey hopes to detect the excess of galaxy clustering at 100 Mpc/h separations left over from the acoustic oscillations in the baryon distribution at the time of last scattering. The change in the characteristic scale of this phenomenon from the time of the CMB to today is encapsulated by the diameter angular distance $d_A(z) = (1 + z)^{-1}r(z)$, which is related to the expansion rate of space via the comoving distance

$$r(z) = \int_0^z \frac{dz'}{H(z')}.$$  \hspace{0.5cm} (1)

where

$$H(z) = H_0 \sqrt{\Omega_m (1+z)^3 + \Omega_X} \exp \left( \int_0^z \frac{1 + w(z')} {1 + z'} dz' \right)$$ \hspace{0.5cm} (2)

where $H_0$ is the Hubble parameter today (note that flatness is assumed), $c$ is the speed of light, $\Omega_m$ is the current matter density and $\Omega_X$ is the current density of dark energy. We focussing on measuring the equation of state $w(z)$ of the dark energy component, which describes the ratio of its pressure to its energy density as a function of redshift. Because the scale of the oscillations at the time of last scattering is measured precisely from the CMB peak morphology, the BAO scale becomes a standard ruler. BOSS is forecasted to measure $d_A(z)$ to 1% at various redshifts (Schlegel et al. (2009)), placing constraints on the equation of state of dark energy $w$ in Eqn. 2. In this paper, we will use the topology of large scale structure as another such standard ruler with which to get a handle on $w(z)$. Being a tracer of the primordial density perturbations, LSS provides a record of the initial conditions and so its topology has been used extensively to test our current assumptions of the earliest epoch. For example, the measured topology of the Sloan Sky Digital Survey (SDSS) traced by LRGs was shown to be consistent with expectations from a Universe with Gaussian random phase initial conditions (Gott et al. (2009)). In this paper, we recognize that topology is another measure of clustering or the num-
ber of structures per unit volume and thus by mapping how galaxy clustering per unit volume evolves with redshift, the expansion history of the Universe can be studied. The potential of this dataset in dark energy studies was first recognized in (Park & Kim (2009)), in which the genus statistic used to characterize LSS is adapted to map \( r(z) \) at various epochs, thereby probing \( w(z) \). Using the method presented in (Park & Kim (2009)), we build on this analysis with the aim of determining how effective the measurement of topology from future LSS surveys such as BOSS will be at elucidating the nature of dark energy. We find that the topology is indeed effective in placing constraints of the average equation of state (at the 5% level). The constraints do however weaken when a smoothly evolving equation of state is considered, with very little information on a second dark energy parameter delivered. In Section 2 the genus statistics are reviewed, followed by a discussion of their application in Cosmology in Section 3. Section 4 gives the details of the analysis. In Section 5 the method is applied to the BOSS data as well as information about the matter density and Hubble parameter from the Planck satellite.

2 THE GENUS AND RELATED STATISTICS

Because the perturbations in the primordial density field on scales larger than the correlation scale underwent linear growth, the pattern of matter overdensities today should reflect the distribution of these seeds from which they formed - - high density regions such as galaxies and clusters of galaxies, are in fact amplifications of the primordial density perturbations. This means that the distribution of the structure today on large scales gives us a window to the conditions of the Universe in a much younger state. As pointed out in (Gott et al. 1988), this initial state of the perturbations is not recorded only in the pattern of the overdense regions; the cosmic underwater also form part of entire structure and it is the cosmic sponge on scales larger than the RMS displacement of the matter that remains preserved (Park & Kim 2009)). The topology of LSS can therefore be used to directly test predictions of our current theories describing the initial conditions. To measure the topology, the number density distribution is smoothed with a Gaussian smoothing ball of radius \( R_{s} \). We then find the iso-density contours of the smoothed distribution which divides the space into two, where the fraction volume on the high density side given by

\[
f(\nu) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x^2/2} dx.
\]

Here \( \nu \) merely a label for the contours. For example, \( \nu = 0 \) denotes the iso-contour of the median density which encloses f = 50% of the volume, while \( \nu = 1.5 \) labels the f = 75% contour. One can then define the genus as a function of \( \nu \), given by the difference between the number of donut-like holes and isolated regions;

\[
G(\nu) = \text{No. of holes} - \text{No. of isolated regions}.
\]

For a random Gaussian phase density field, the genus per unit volume, \( g(\nu) = G(\nu)/V \), is predicted from theory to be (c.f. Hamilton et al. 1986; Gott et al. 1987))

\[
g(\nu) = A(1 - \nu^2) e^{-\nu^2/2}
\]

where the amplitude is given by

\[
A = \frac{\langle k^3 \rangle^{3/2}}{2\pi^2}.
\]

Here \( \langle k^3 \rangle \) is the average value of \( k^3 \) in the smoothed power spectrum. \( A \) is sensitive to the slope of the power spectrum near the selected smoothing scale and is independent of the amplitude of \( P(k) \). Recent studies of the topology of LSS traced by LRGs in the SDSS survey (Gott et al. 2009b) at smoothing lengths of 21 h\(^{-1}\) Mpc and 34 h\(^{-1}\) Mpc show the amplitude of the genus curve, \( A \), is modeled very well by N-body simulations and closely follows that of the initial conditions. Note that the topology of a non-Gaussian density field at any scale within the linear regime is also preserved in comoving space.

The currently favoured model for the beginnings of structure formation, called Inflation, assumes that the primordial fluctuations to be Gaussian random phase, which has been shown to lead to a sponge-like topology (Gott et al. (1987, 1988)). The medium density contours measured for various LSS samples over the past few years have been found to be consistent with expectations of this type of topology (see Gott et al. (2009b); Park et al. 2005a,b; Gott et al. 2009a, 1989; Hamilton et al. 1986). In addition, the theoretical prediction for Gaussian random phase initial conditions in Eqn. 5 was shown to provide a suitable fit to the observed genus curves for these data sets. In this paper, we focus of whether the same data can be used to study the late-time behavior of the Universe.

3 USING TOPOLOGY TO CONSTRAIN COSMOLOGY

The comoving distance \( r(z) \) in Eqn. 1 tells us how scales change as a function of redshift due to the expansion of space. This means that comparison of characteristic scales in cosmology, such as features in the power spectrum or the correlation function, from one epoch to the another, provides a handle on \( r(z) \). Because the shape of the primordial power spectrum \( P(k) \) on large scales (where we are still in the linear regime) should be conserved in redshift space (the transfer function does not depend on \( z \)), we can determine \( P(k) \) for a given set of cosmological parameters from two measurements of \( P(k, z) \) and expect to find it unchanged. If the power spectrum does change, the \( r(z) \) relation in Eqn. 1 and hence the chosen cosmology must be wrong.

As described in Section 2 the genus statistic is related to the entire shape of the power spectrum and essentially measures its slope near the smoothing scale. The statistic \( g(\nu) \) is a measure of the number of structures after smoothing on a certain scale, per unit volume, out to a given distance. The measure therefore relies on a choice of \( r(z) \) relation (and hence cosmology) to compute the enclosed volume. Assuming the incorrect cosmology and hence expansion rate will lead to an incorrect estimate of the volume in which one is counting structures. In so doing, a different amplitude of the genus curve from the true value will be measured. Furthermore, the chosen smoothing scale which determines the size of the structures will be incorrect. For example, in the case where one overestimates the expansion rate, what is assumed to be a unit volume will enclose less structure than at \( z = 0 \). As a result, the measured amplitude of structure on the smoothing scale will be lower. At the same time, because the smoothing scale per unit volume is kept fixed, smaller structures are effectively smoothed over so these two effects partially cancel. Fortunately, there is a net effect because the density perturbation field is not scale free and the power spectrum has a shape with a varying slope over the region of interest. Fig. 1 shows the genus curves for two different cosmological models. In this paper, we will exploit the dependence of the genus curve on \( r(z) \) in order to constrain the equation of state of dark energy, as explained in the next section.

We consider a flat Universe filled with cold dark matter,
baryons and dark energy. The cosmological parameters of interest are the matter density, $\Omega_m$ (baryonic and dark), the Hubble parameter $H_0$ and the equation of state of the dark energy component, which has been shown to provide a suitable phenomenological description of dark energy (Frieman et al. 2008). To start, we assume the simplest parameterization, namely a constant equation of state $\omega$. The comoving distance along the line of sight, $r_{||}$ (in redshift space, and in the traverse direction, $r_\perp$, of a feature sitting at a redshift $z$, are related to the redshift range $\Delta z$ covered and the angle subtended $\Delta \Theta$ respectively, by

$$r_{||} = c\frac{\Delta z}{H(z)}, \quad r_\perp = c(1+z)d_A(z)\Delta \Theta.$$  

Suppose a length along the line of sight spans a redshift range of $\Delta z$. The comoving distance of the length is given by

$$\lambda_x = r_{||}(z) = \frac{c\Delta z}{H(z, \Omega_m, w, H_0)}.$$  

If we then perturb the cosmology, the same redshift range $\Delta z$ will correspond to a different comoving distance given by

$$\lambda'_x = \frac{c\Delta z}{H(z, \Omega'_m, w, H'_0)}.$$  

Thus the different comoving distances are related by

$$\lambda_x = r_{||}(z) = (1+z)d_A(z, \Omega_m, w, H_0) \Delta \Theta.$$  

If we again perturb the cosmology, the same angle $\Delta \Theta$ will sub tend a new comoving distance of

$$\lambda'_x = (1+z)d_A(z, \Omega'_m, w', H'_0) \Delta \Theta.$$  

So we can write

$$\lambda'_x = \lambda_x \frac{(1+z)d_A(z, \Omega'_m, w', H'_0)}{(1+z)d_A(z, \Omega_m, w, H_0)}.$$  

Similarly, the comoving distances subtended by the angle $\Delta \Theta$ in the $y$ direction for two different cosmologies are related by

$$\lambda'_y = \lambda_y \frac{d_A(z, \Omega'_m, w', H'_0)}{d_A(z, \Omega_m, w, H_0)}.$$  

We can then write the volume of the corresponding ellipsoid

$$V = \frac{4\pi}{3} \lambda_x \lambda_y \lambda_z.$$  

so we can relate the comoving volumes in the two different cosmologies via

$$V' = \frac{4\pi}{3} \lambda_x' \lambda_y' \lambda_z' \left[ \lambda_x d_A(z, \Omega'_m, w', H'_0) \right]^2 \left( \frac{d_A(z, \Omega'_m, w', H'_0)}{d_A(z, \Omega_m, w, H_0)} \right)^2 = \frac{4\pi}{3} \lambda_x \lambda_y \lambda_z H(z, \Omega_m, w, H_0) d_A(z, \Omega_m, w, H_0)^2 \frac{d_A(z, \Omega'_m, w', H'_0)}{d_A(z, \Omega_m, w, H_0)}.$$  

We denote the factor relating the volumes in the different cosmologies as

$$V' = \frac{V \frac{d_A(z, \Omega'_m, w', H'_0)^2}{H(z, \Omega'_m, w', H'_0)^2}}{\frac{d_A(z, \Omega_m, w, H_0)^2}{H(z, \Omega_m, w, H_0)^2}} = \frac{d_A(z, \Omega_m, w, H_0)^2}{d_A(z, \Omega'_m, w', H'_0)^2}.$$  

Re-arranging the above gives an equation which gives the volume in the new cosmology as a function of the original volume;

$$V' = V \left( \frac{V'}{V_{eff}} \right)^{1/3}.$$  

Suppose we have a galaxy redshift catalogue with which we wish to constrain the underlying cosmology. After smoothing the distribution with a Gaussian smoothing sphere of dimensions $\lambda = \lambda_x = \lambda_y = \lambda_z$ and volume $V = \lambda^3$, we measure an amplitude of $G = g(\lambda, z)V = g(\lambda, z)\lambda^3$. Now suppose that the assumed cosmology for the smoothing process is in fact incorrect. This means that the $r(z)$ relation is wrong. In the true cosmology, the smoothing sphere is in reality an ellipsoid with volume $V' = \lambda'_x \lambda'_y \lambda'_z$. The same amplitude $G = g'(\lambda', z)V'$ is measured by smoothing a different cosmology with an ellipsoid or a sphere of radius $\lambda'$, which is calculated using

$$\frac{\lambda'}{\lambda} = \left( \frac{V'}{V} \right)^{1/3} = \left( \frac{V'_{eff}}{V_{eff}} \right)^{1/3}. $$  

So we can relate the smoothing scales within the two different cosmologies using

$$\lambda' = \lambda \left( \frac{V'_{eff}}{V_{eff}} \right)^{1/3}.$$  

Equating $gV = g'V'$ gives us

$$g' (\lambda', z) = g(\lambda, z) \frac{V}{V'_{eff}} = g(\lambda, z) \frac{V_{eff}}{V'.}$$  

Eqn. 22 is very useful because it allows us to calculate the expected genus curve for any cosmological model provided we know the theoretical $g(\lambda, z)$ at a given scale $\lambda$ for one set of parameter values. This is readily computed using the smoothed matter power spectrum generated using a Boltzmann code such as CAMB (Lewis et al. 2000). The validity of Eqn. 22 depends on how well a galaxy distribution smoothed with a sphere of geometric mean $R_0$ represents that smoothed using an ellipsoid of median radius $R_0$. This is tested and discussed in the next section.

4 Analysis

The aim of the analysis is to perform dark energy forecasts for a set of measurements of the genus at various smoothing scales, based on the 3D distribution of LRGs as it will be measured by BOSS. Using a 2048$^3$ particle cold dark matter N-body simulation, Kim & Park (2009)’s technique which identifies LRG galaxies by selecting the most massive bound halos, correctly reproduces the 3D topology of the LRG galaxies in the SDSS (Gott et al. 2009a).
We apply the same technique to the Horizon Run Simulation, which is the largest N-body simulation to date with a volume of $6.543\, (\text{Gpc}/h)^3$ [Kim et al. (2009)]. Provided we remain in the linear regime, the genus curve for the initial conditions remains essentially unchanged and should provide a suitable representation of the genus that would be measured using LRGs as a probe of LSS today. Hence we use the initial density field from the Horizon Run Simulation to construct a hypothetical data set and subdivide the simulation into 8 cubes. Placing an observer at the centre of each cube, a sphere of comoving radius $1570\, \text{Mpc}/h$ (corresponding to a redshift of $z = 0.6$) is carved around the observer and quartered to produce 4 BOSS mock catalogues of $\pi$ steradians and in total, 32 mock galaxy catalogues with which to test the statistics. The genus per unit smoothing volume is then measured for each catalogue at 3 different smoothing scales; 15, 21, 34 $\text{Mpc}/h^{-1}$. These scales were selected because they are sufficiently disparate that different types of structures are being smoothed and can, for statistical purposes, be regarded as independent. This gives three independent data points $g_{\text{data}}(\bar{\chi}, \chi)$ where $i = 1 - N_z$ per redshift bin. The noise is taken to be the standard deviation of the measured $g$ values of the 32 simulated maps.

First, we compare the genus per unit smoothing volume measured in our mock surveys to the theoretical values from Fig. 1. As it should, the shape of the genus curves follows that of Gaussian random fluctuations (Eqn. 5), but we systematically find slightly lower amplitudes than predicted from theory. In particular, the suppression is systematically larger with larger smoothing length, and in redshift bins with more surface pixels. We believe this is due to two related effects: pixels on the edges are not counted by the program measuring the genus, and structures near the edges are suppressed by the smoothing and "cut off", both lowering the value of the genus measured. To account for this, we introduce correction factors for each redshift bin, so that for the original density field, the mean values for the given bin correspond to the theoretical value.

Next, to test the validity of Eqn. 22, we remap the density field in each survey, according to two different values of constant $w$: $-0.9$ and $-0.8$. For each cosmology, we construct a new density field from the original, based on the distances the observers would infer according to those cosmologies. We then again compute the genus per unit volume for the same three smoothing lengths and redshift bins, in our 32 mock surveys. Each bin is corrected for edge effects by the factors calculated above, and then compared to the expected number based on the geometric mean volume (Eqn. 22). We find that for both cosmologies, in all of the redshift bins, for all smoothing lengths, the differences between the observed values in the simulated cosmologies (with $w = -0.8$ and $w = -0.9$) and theoretical values using Eqn. 22 are well within one standard deviation of the mean. We conclude that as long as edge effects are taken into account, the theoretical value based on effective smoothing length (Eqn. 22) agrees excellently with the observations. Therefore, we go ahead and use Eqn. 22 to compute genus curve amplitudes in different cosmologies for the likelihood analysis.

We now wish to evaluate the likelihood of the parameters $\vec{p} = (H_0, \Omega_m, w)$ in light of the data. This involves computing the expected values of the genus per unit volume at the point $\vec{p}_i$ in parameter space using Eqn. 22 for a given smoothing length and comparing them with the data using

$$\chi^2 = \sum_{i,j} \frac{(g_{\text{data}}(z_j, \chi_i) - g_{\text{ref}}(z_j, \chi_i))^2}{\sigma_{ij}}$$

where

$$g_{\text{trial}}(\lambda, z) = g_{\text{ref}}(\lambda', z) \left(\frac{V_{\text{ref}}}{V_{\text{trial}}}ight)^{1/3}$$

and $\sigma_{ij}$ is the uncertainty associated with the data smoothed on the $i$th scale in the $j$th redshift bin. This is computed from variance measured from the 32 mock surveys. The likelihood of the trial point is then $L(\vec{p}_i | g_{\text{data}})$.

In the above, $g_{\text{ref}}(\lambda, z)$ are the expected genus values for our reference cosmology, calculated from the linear matter power spectrum generated using CAMB and smoothed on the scale

$$\lambda' = \lambda \left(\frac{V_{\text{ref}}}{V_{\text{trial}}}ight)^{1/3}.$$  

The value of $g_{\text{ref}}(\lambda', z)$ at this new smoothing scale $\lambda'$ is found using interpolation. We use the WMAP 3-year best-fit cosmological parameters to generate $g_{\text{ref}}(\lambda, z)$ shown in Fig. 1. This procedure is repeated at each point in parameter space, which is efficiently sampled using a Monte Carlo Markov chain (MCMC) algorithm until convergence. The 68% and 95% confidence intervals for the parameters $(H_0, \Omega_m, w)$ are then computed from the chains.

5 RESULTS

Assuming a completed BOSS survey, we constructed a hypothetical data set consisting of three genus measurements, each at one of the three selected smoothing scales. We pick a dark energy model with $w \neq -1$ to test the ability of the data to distinguish such a model from cosmological constant. We simulate a flat Universe with a dark energy component described by $w = -0.9$.

Fig. 2 shows the 68% and 95% confidence regions in parameter space when (a) $H_0$ is fixed while $\Omega_m$ and $w$ are allowed to vary and (b) $\Omega_m$ is fixed while $H_0$ and $w$ are allowed to vary. The elongated contours exhibit strong degeneracy between the matter density and Hubble parameter and the equation of state. For a more negative choice of $w$ than the true value, the dark energy component is less important in the past while the matter density today is

![Figure 1. Plot of the amplitude of the genus curve $g R^3$ as a function of the smoothing length $R_h$ for the best-fit parameters values for WMAP 3-year ($\Omega_m = 0.24, H_0 = 72$) and WMAP 5-year data ($\Omega_m = 0.26, H_0 = 74$) assuming a flat $\Lambda$CDM cosmology.](image-url)
where \( H \) is the expansion rate, dictating the scales on which the acoustic features appear in the CMB spectrum today. The size of the largest full acoustic compression is related to that in the topology data. As \( w \) is perpendicular to the surface of last scatter, subtending smaller angles, manifesting as a shift in the acoustic peaks to lower angular scales. The information regarding the expansion rate and hence dark energy available in the CMB data comes primarily from the distance to the last scattering surface \( d_{\text{sls}} \).

The current constraints from the WMAP 7-year data are \( \theta_s = 0.597 \pm 0.0016 \) with the matter and baryon densities estimated to be \( \Omega_m h^2 = 0.1098 \pm 0.0058 \) and \( \Omega_b h^2 = 0.0250 \pm 6.3 \times 10^{-4} \) respectively (Komatsu et al. 2010). The higher resolution and sensitivity of the Planck Satellite lead to tighter forecasted constraints of \( \theta_s = 0.597 \pm 3.1 \times 10^{-4} \), \( \Omega_m h^2 = 0.1098 \pm 1.4 \times 10^{-3} \) and \( \Omega_b h^2 = 0.0250 \pm 1.6 \times 10^{-4} \) (Colombo et al. 2009). As a result of the degeneracy between \( w \) and \( H_0 \), permitting non-\( \Lambda \) models and allowing \( w \) to vary significantly degrades the constraints on \( H_0 \). We thus include the current measurement of the Hubble constant from \( H_0 = 74 \pm 3.6 \) km s\(^{-1}\) Mpc\(^{-1}\) (Riess et al. 2009).

Including information from the CMB is highly advantageous as the direction of the degeneracy between \( \Omega_m \) and \( w \) is perpendicular to that in the topology data. As \( w \) becomes less negative, its contribution to the overall density at earlier epoch increases. As a result, the expansion rate is reduced, with the features on the last scattering surface subtending smaller angles, manifesting as a shift in the acoustic peaks to lower \( \ell \). This effect on the CMB anisotropy spectrum can however be countered by an increase in the matter density which delays recombination and leads to a larger sound horizon \( r_s \).

\[ r_s(z_{\text{cmb}}, \Omega_b, \Omega_r) = \int_0^{z_{\text{cmb}}} c_s dz \]  

(26)

where \( c_s \) is the sound speed defined by

\[ c_s = c \sqrt{3 \left( 1 + 3\Omega_b \right)^{-1/2}} \]  

(27)

Since the baryon to photon ratio can be precisely measured from the acoustic peak morphology in the CMB, we can predict \( c_s \), and thus the scale of the first peak. The angular scale on which this feature appears today, \( \theta_s \), depends on the expansion history and the distance to the surface of last scattering, \( d_{\text{sls}} \).

\[ \theta_s = \frac{180^\circ}{\pi} \frac{r_s}{d_{\text{sls}}} \]  

(28)

where

\[ d_{\text{sls}} = c \int_0^{z_{\text{cmb}}} \frac{dz'}{H(z')} \]  

(29)

Assuming parameter values near the \( \Lambda \) concordance model, we use the following formula

\[ r_s = 144.4 \text{Mpc} \left( \frac{\Omega_b h^2}{0.024} \right)^{-0.252} \left( \frac{\Omega_m h^2}{0.14} \right)^{-0.083} \]  

(30)

This assumes that energy density contribution from the dark energy component during this epoch is sufficiently small such that it does not drastically affect the size of the acoustic features at the surface of last scattering. The information regarding the expansion rate and hence dark energy available in the CMB data comes primarily from the distance to the last scattering surface \( d_{\text{sls}} \).

In addition to the simulated data from BOSS, we choose to include information from the Cosmic Microwave Background (CMB). The presence of dark energy impacts the CMB primarily through the distance to the surface of last scatter, with the expansion rate dictating the scales on which the acoustic features appear in the CMB spectrum today. The size of the largest full acoustic compression which manifests as the first peak in the CMB angular power spectrum, is determined solely by the distance that these waves could have traveled in the time before recombination, namely the sound horizon \( r_s \) at last scattering:

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Including information from the CMB is highly advantageous as the direction of the degeneracy between \( \Omega_m \) and \( w \) is perpendicular to that in the topology data. As \( w \) becomes less negative, its contribution to the overall density at earlier epoch increases. As a result, the expansion rate is reduced, with the features on the last scattering surface subtending smaller angles, manifesting as a shift in the acoustic peaks to lower \( \ell \). This effect on the CMB anisotropy spectrum can however be countered by an increase in the matter density which delays recombination and leads to a larger sound horizon \( r_s \). Fig. 4 shows the allowed regions in parameter space for the simulated BOSS topology data in conjunction with the WMAP 7-year (left) and the Planck data (right). The change in the direction of \( w, \Omega_m \) ellipse points to degeneracy breaking. There is negligible improvement on the dark energy EOS constraint from \( w = -0.92^{+0.5}_{-0.6} \) to \( w = -0.92^{+0.05}_{-0.05} \) when the CMB data set is updated from WMAP-7 year to the Planck data, despite the clear reduction in the confidence interval evident in Fig. 4. This indicates
the topology data is primarily responsible for the constraints on $w$.

A useful task would be to determining the constraining power of this dataset as a function of redshift. This can be studied by performing principal component analysis, which essentially identifies the directions in the data in which the variation is maximal. These directions are captured by $z$-dependent eigenvectors, with the eigenvalue of mode representing how well it is constrained (i.e. the strength of the information it represents). This will reveal where in redshift space this particular dataset is most sensitive to.

We start by dividing the redshift region in 10 equally sized bins centered at redshifts $z_i$. We parameterize the EOS as a piecewise constant function, with a constant $w_i$ in each bin where $z_i - \Delta z/2 < z < z_i + \Delta z/2$. We then sample the posterior probability distribution an MCMC algorithm, marginalizing over the other cosmological parameters. Using the chain of samples, we construct the covariance matrix as follows:

$$C_{ij} = \frac{1}{N-1} \sum_{k=1}^{N} (X_i - \bar{X}_i)(X_j - \bar{X}_j) \tag{31}$$

where $X_i$ is the value of the parameter at the $i^{th}$ point in the MCMC chain and $\bar{X}_i$ is the sample mean. We find the eigenvectors by diagonalizing the covariance matrix and decomposing it as follows:

$$C = W^TAW \tag{32}$$

where the diagonal matrix $\Lambda$ contains the eigenvalues corresponding to the eigenvectors $\phi_i$ in the rows of matrix $W$. We follow (Tang et al. (2008)) and track the redshift sensitivity of each mode by plotting the quantity $\phi_i(z)$:

$$\phi_i(z) = N | \sqrt{\lambda_i} \phi_i(z) | \tag{33}$$

where the amplitude is proportional to the accuracy with which the eigenvector $\phi_i(z)$ can be measured. $N$ is included to lessen the dependence on the number of bins. Fig. 3 plots $\phi_i(z)$ as a function of $z$. The behaviour of the first and best-constrained mode (red) indicates that the constraining power of the survey is a decreasing function of $z$, with most of the information about $w$ delivered at low redshift. All other eigenvectors are relatively noisy with no $z$-dependent shape distinct shape. We infer that the redshift sensitivity of this data set is likely to behave as $\phi_1(z)$ (shown in red).

### 5.1 Redshift bins

Up to this point we have smoothed the entire observed volume on three different scales, constructing a dataset comprising of three data points. In this section we consider the effect of dividing the observed space into redshift bins of equal volume, thereby increasing the number of independent measurements three fold. The characteristics of the redshift shells in the case of one, two and three bins are summarized in table 5.1.

The results of the analysis are summarized in the first section of table 5.2. An increase in the number of redshift shells is shown to make little difference to the constraints when a constant equation of state is considered. The increase in the number of data points is clearly countered by the reduction in signal-to-noise in each redshift shell over a single bin. In light of the finding that the sensitivity of the topology data to the dark energy model is a decreasing function of redshift (shown in Fig. 1), it is likely that most of the information available about $w$ is concentrated in the first redshift bin, with the high-$z$ bins adding little to the analysis.

![Figure 3](image-url) Figure 3. Figure showing a plot of $\phi_1(z)$ as a function of reshift. The first eigenvector $\phi_1(z)$ is shown in red, the second $\phi_2(z)$ in blue and with the remaining 8 shown in black. This illustrates the redshift sensitivity of this dataset.

![Figure 4](image-url) Figure 4. 95% confidence regions for $(w, H_0)$ for the topology data including (a) WMAP 7 year data and (b) Planck data. Plots of $(w, \Omega_m)$ for the same combinations of data sets are shown in (c) and (d). The true parameter values are shown by the yellow crosses.

| Bins | Redshift range | Distance (Mpc/h) | Volume (Gpc/h)$^3$ |
|------|----------------|-----------------|------------------|
| 1    | $z < 0.1$      | $r < 1570$      | 4.036            |
| 2    | $0.1 < z < 0.46$ | $r < 1246$      | 2.016            |
|      | $0.46 < z < 0.6$ | $1246 < r < 1570$ | 1.997            |
| 3    | $0.396 < z < 0.513$ | $r < 1089$      | 1.342            |
|      | $0.513 < z < 0.6$ | $1089 < r < 1371$ | 1.326            |
|      | $0.6 < z$       | $1371 < r < 1570$ | 1.321             |

Table 1. Table summarizing the partitioning of redshift space.
5.2 Constraining dynamical dark energy

Up to this point, we have selected the simplest description of the equation of state, a constant $w$. Although it has been shown to be a good approximation to a quintessence component obeying tracker solutions (Efstathiou (1999)) while accommodating the vacuum energy ($w_0 = -1$) as one of its solutions, it cannot however, be used to characterize the effect of scalar field models, in general, or modified gravity models on the observable Universe (Frieman et al. (2008)). In this Section, we wish to determine the effectiveness of the topology data in constraining a second dark energy parameter. A popular two-parameter ansatz for $w(z)$ was introduced in (Chevallier & Polarski (2001); Linder (2003)) with the following form:

$$w(z) = w_0 + w_a (1 - a).$$

where $a = 1/(1 + z)$. This particular function is a favourite in the cosmology community because it solves the divergence problem at high redshift accompanying other parameterizations, but at the cost of a more rigid assumption of the behavior a priori (Riess et al. (2004)). It is also limited in how well it can cope with a rapidly evolving equation of state (Liddle et al (2003)). We simulate a set of genus values for a dark energy model with $w_0 = -1$ and $w_a = 0.5$ in three different binning schemes and add Gaussian noise. Table 5 summarizes the results of the likelihood analysis when the simulated topology data in conjunction with the Planck priors is used to constrain $\rho = \Omega_m H_0, w_0, w_a$). The constraints on $w_0$ weaken as expected towards 10%. We find that the topology data is relatively uninformative with regards to a second dark energy parameter, regardless of the number of redshift bins used. In fact, given that data is too noisy to detect any variation in $w$ with redshift, partitioning the data into redshift bins serves only to weaken the constraints on the average $w$. Figure 5 shows the 68% and 95% confidence regions in $(w_0, w_a)$ when the data is binned into three redshift shells of equal volume and serves to illustrate the increase in the confidence region accompanying the increased freedom in the dark energy model. Table 5 also indicates that the use of a single redshift bin is preferred overall.

| Number of $z$ bins | $w_0$ | $w_a$ |
|--------------------|-------|-------|
| **Constant EOS:**  | $w_0 = -0.9$ |       |
| 1                  | $-0.92_{-0.05}^{+0.05}$ | -     |
| 2                  | $-0.92_{-0.04}^{+0.05}$ | -     |
| 3                  | $-0.92_{-0.04}^{+0.05}$ | -     |
| **Dynamical:**     | $w_0 = -1$ | $w_a = 0.5$ |
| 1                  | $-1.04_{-0.1}^{+0.1}$ | $0.45_{-0.5}^{+0.3}$ |
| 2                  | $-0.93_{-0.1}^{+0.1}$ | $0.23_{-0.6}^{+0.5}$ |
| 3                  | $-0.98_{-0.2}^{+0.1}$ | $0.41_{-0.6}^{+0.7}$ |

Table 2. Table summarizing the constraints on the dark energy EOS parameters for different binning schemes using the topology data and the Planck data. We assume the EOS to be constant from the last redshift bin at $z = 0.6$ to the last scattering surface at $z = 1089$.

6 DISCUSSION

The question of the nature of dark energy is one of profound importance in cosmology today. The currently favoured model proposes that it is energy density associated with the vacuum, with a constant EOS of $w = -1$, identified as the mathematical equivalent of cosmological constant $\Lambda$ in the Einstein field equation. In the context of our current theories of structure formation, $\Lambda$ appears to be most successful in reproducing a wide range of present-day observations. However, it faces serious theoretical opposition and several recent works have highlighted the biases towards $w = -1$ that could potentially arise from the inclusion of a priori information when fitting the current data (Bassett et al. (2004); Linder (2004)). Future experiments offer the exciting prospect of determining whether the cosmological constant is indeed the correct model. BOSS is forecasted to place constraints on $w$ at the level of a few percent using clustering of LRGs on the scale of 100 Mpc. Park & Kim (2009) proposed a way to use the nature of LRG clustering on a wider range of scales as measured by the 3D topology to probe the expansion rate. Because genus statistics relate to the overall shape of the power spectrum as opposed to a single scale, they are robust against fortuitous noise measurements. Furthermore, the topology measurements offer an alternative use of the data from redshift surveys and may be used as a cross check of the conclusions drawn from other techniques, such as the BAO method which uses the oscillations in the matter power spectrum to extract information.

In this paper, we have evaluated the constraints on $w(z)$ that are achievable from the BOSS survey using the method presented in (Park & Kim (2009)). We found that the BOSS survey using the topology method alone is capable of placing constraints of 5% on a constant equation of state. Allowing for the possibility of time variation in $w(z)$ degrades the constraints on $w_0$ to 12%, while providing weak evidence for a second dark energy parameter. Although the topology measurement of BOSS may not be capable of testing for dynamical behaviour in $w(z)$, it has been shown to provide a robust measurement of the average equation of state, which may be sufficient to rule out non-$\Lambda$ models.

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