European option pricing models described by fractional operators with classical and generalized Mittag-Leffler kernels

Mehmet Yavuz$^{1,2}$

$^1$Department of Mathematics and Computer Sciences, Necmettin Erbakan University, Konya, Turkey
$^2$Department of Mathematics, College of Engineering, Mathematics and Physical Sciences, University of Exeter, Cornwall, TR10 9FE, UK

Correspondence
Mehmet Yavuz, Department of Mathematics and Computer Sciences, Necmettin Erbakan University, 42090 Konya, Turkey.
Email: mehmetyavuz@erbakan.edu.tr, m.yavuz@exeter.ac.uk

Abstract
In this paper, we investigate novel solutions of fractional-order option pricing models and their fundamental mathematical analyses. The main novelties of the paper are the analysis of the existence and uniqueness of European-type option pricing models providing to give fundamental solutions to them and a discussion of the related analyses by considering both the classical and generalized Mittag-Leffler kernels. In recent years, the generalizations of classical fractional operators have been attracting researchers’ interest globally and they also have been needed to describe the dynamics of complex phenomena. In order to carry out the mentioned analyses, we take the Laplace transforms of either classical or generalized fractional operators into account. Moreover, we evaluate the option prices by giving the models’ fractional versions and presenting their series solutions. Additionally, we make the error analysis to determine the efficiency and accuracy of the suggested method. As per the results obtained in the paper, it can be seen that the suggested generalized operators and the method constructed with these operators have a high impact on obtaining the numerical solutions to the option pricing problems of fractional order. This paper also points out a good initiative and tool for
those who want to take these types of options into account either individually or institutionally.

**KEYWORDS**
Atangana–Baleanu fractional operator, Black–Scholes option pricing models, error analysis, existence and uniqueness, generalized Mittag-Leffler kernel

## 1 | INTRODUCTION

Recently, a massive volume of financial commodities have been utilized all around the world. One of the most significant instruments is a share option which permits the holder to buy or sell shares in a specified time. For instance, a European put option gives the right to the owner, but not the obligation, to sell a predetermined amount of an underlying asset at a predetermined price $Q$ within a predetermined time $T$. While the pay-off function $\Theta$ for the European call option is given by

$$\Theta(Y(T)) = \{Y(T) - Q\}_+,$$

the European put option is given by

$$\Theta(Y(T)) = \{Q - Y(T)\}_+,$$

where $Y(T)$ is the asset price at time $T$. Similarly, there are a large number of different kinds of options such as American options which can be exercised at any time before $T$, barrier options which can be exercised only if the asset price reaches a specified level (barrier) and many other types such as bond options, exotic options, foreign exchange options.

The most important problem of financial derivatives is determining the present day value. In order to calculate this, one needs to compute the expected value of the option’s pay-off functions at the exercise time. For that, the process of determining the underlying assets price can be modeled by considering a system of stochastic differential equations. In order to determine the price $Y_t$ of a single security, the geometric Brownian motion with drift $\rho$ and volatility $\sigma$ has been used in the Black–Scholes model [1]. The geometric Brownian motion is

$$dY_t = \rho Y_t dt + \sigma Y_t dW_t,$$

where $W$ is a Wiener process. Not only a stochastic formulation technique, but also a partial differential equation (PDE) formulation has been widely used in order to determine value $R(t, s)$. After necessary calculation we obtain the final value problem for European options with a number of underlying securities with share prices $\tilde{s} = (s_1, s_2, \ldots, s_d)^T$ as

$$\frac{\partial R}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d} [\sigma \sigma^T]_{ij} s_i s_j \frac{\partial^2 R}{\partial s_i \partial s_j} + \rho \sum_{i=1}^{d} s_i \frac{\partial R}{\partial s_i} - \rho R = 0, \quad s \in \mathbb{R}_+^d, \quad t < T,$$

$$R(T, \tilde{s}) = \nu(\tilde{s}), \quad \tilde{s} \in \mathbb{R}_+^d,$$

where $\sigma$ is a matrix formed volatility. In this formula, every stochastic variable represents to one dimension in the PDE.

In 1973, Black and Scholes [1] pointed out a model which can easily evaluate the prices of the options that is now well-known as the Black–Scholes model. This pricing model is one of the most effective mathematical equations in the mathematical finance literature. In Equation (4), we can obtain that $R(0, t) = 0, R(Y, t) \sim Y$ as $Y \to \infty$, and we can find the following payoff functions: $R_c(Y, T) = \max(Y - Q, 0)$ and $R_p(Y, T) = \max(Q - Y, 0)$, where $R_c(Y, T)$ and $R_p(Y, T)$ show the value
of vanilla call and put options, respectively. For transferring to the fractional version of the above equation, we make the following modifications:

\[ Y = Qe^{\xi}, \quad t = T - \frac{2\nu}{\sigma^2}, \quad R = Q\eta(\xi, \nu). \]

This yields the equation:

\[
\frac{\partial^\alpha \eta(\xi, \nu)}{\partial \nu^\alpha} = \frac{\partial^2 \eta(\xi, \nu)}{\partial \xi^2} + (k - 1) \frac{\partial \eta(\xi, \nu)}{\partial \xi} - k\eta(\xi, \nu), \quad \nu > 0, \quad \xi \in \mathbb{R}, \quad 0 < \alpha \leq 1, \tag{5}
\]

with initial condition:

\[ \eta(\xi, 0) = \max(e^{\xi} - 1, 0). \tag{6} \]

Equation (5) is called the Black–Scholes option pricing equation of fractional order. In Equation (5), we define \( k = \frac{2\rho}{\sigma^2} \), where \( k \) shows the balance between the interest rates’ and stock returns’ variability.

Moreover, Cen and Le considered the generalized version of the fractional Black–Scholes equation (GFBSE) [2] by assigning \( \rho = 0.06 \) and \( \sigma = 0.4(2 + \sin \xi) \) in Equation (5):

\[
\frac{\partial^\alpha \eta}{\partial \nu^\alpha} + 0.08\xi^2(2 + \sin \xi)^2 \frac{\partial^2 \eta}{\partial \xi^2} + 0.06\xi \frac{\partial \eta}{\partial \xi} - 0.06\eta = 0, \quad \nu > 0, \quad \xi \in \mathbb{R}, \quad 0 < \alpha \leq 1, \tag{7}
\]

with the initial condition:

\[ \eta(\xi, 0) = \max(\xi - 25e^{-0.06}, 0). \tag{8} \]

Up to here, the emergence of the new fractional operators in the literature can be considered as a result of the reproduction of new problems that model different types of real-life events. Fractional derivative operators that can be stated as nonlocal have been developed to address these kinds of nonlinear differential equations such as the Riemann–Liouville fractional-order [3], Caputo–Fabrizio fractional-order derivative based on the exponential kernel [4], Atangana–Baleanu fractional-order derivative which is based on the generalized Mittag-Leffler (GML) function as nonlocal and nonsingular kernel [5]. One can see solid theoretical results and related essential applications of the mentioned operators in the literature [6–25]. Moreover, in recent years the generalizations of classical fractional operators have been attracting effect all over the world and they are needed to describe the dynamics of complex phenomena. Abdeljawad et al. [26] pointed out very important related results for the generalized fractional operators with and without Mittag-Leffler kernel. Meanwhile, Abdeljawad [27, 28] developed the corresponding fractional integrals of generalized ABC operator with arbitrary order by using the infinite binomial theorem, and studied their semi-group properties and their action on the ABC type fractional derivatives to prove the existence and uniqueness theorem for the ABC-fractional initial value problems.

Acay et al. [29, 30] examined on the fractional falling body problem relied on Newton’s second law and they also studied a certain economic problem by using the ABC operator containing both classical and GML kernels. Ozarslan et al. [31] investigated the fractional form of wind-influenced projectile motion equations with the aid of the ABC operator. There are a number of important related studies which have been done in recent years by considering the ABC operator, its generalized version and other generalized fractional operators such as [32–49].

### 1.1 A summary of numerical-approximate solutions in option pricing models

There exist a number of solutions which give us closed results to determine the price of a few options. However, since European-type options modeled Black–Scholes equation based on some underlying assets, one need to use numerical methods for valuation.
One of the most commonly used methods for valuation option prices is Monte Carlo simulation (MCS) method which simulates trajectories [50]. The major benefits are that the MCS method is easy to apply comparing with the other methods even with several underlying securities and that the implementation cost increases linearly with many securities. However, MCS methods are considerably slow to converge. In general, in order to decrease the variance in the estimates, MCS methods require simulating millions of trajectories. Another downside with MCS methods is that MCS is not applicable for Greeks. They might be obtained through additional MCS simulations presuming that the pay-off continuously differentiable.

In order to avoid high computational cost when pricing options, it can be addressed by raising hardware performance. In financial word, a number of people try to find a better computer clusters that provide more accurate and faster result via parallel programming. On the other hand, a large number of approximate-series solutions to option pricing models have been examined. Among them, Yavuz et al. [51–54] presented the solutions of fractional order BSE by using different types of fractional kernels and methods. Fall et al. [55] obtained an approximate solution to the mentioned problem by using the Caputo generalized fractional derivative. In [56] the authors have obtained the semi-analytical solution to the Ivancevic option pricing model of fractional order, which is an alternative of the standard Black–Scholes pricing equation and signifies a controlled Brownian motion related to the nonlinear Schrodinger equation. Additionally, one can see the related studies for the BSE of fractional order in [57–62].

The remaining parts of the study have been outlined as the following: in Section 2, we introduce the fractional-order derivatives, GML kernel, and their Laplace transformation which have been used in the paper. In Section 3, we present the mathematical investigation of the fractional model. In Section 4, we provide the existence and uniqueness of the solutions to fractional BSE. In Section 5, we describe the method by using the GML kernel for obtaining the approximate solutions of the models. Moreover, in this section, we give their corresponding solutions. In Section 6, we provide the error analysis of the approximate solutions we have obtained. In Section 7, we illustrate the main results by graphical representations and discuss the impact of the fractional order \( \alpha \) when we fix the other parameters \( \mu, \gamma \). We give the conclusions and perspectives in Section 8.

\section{Some Preliminaries}

In this section, we summarize the following fundamental definitions of fractional calculus which are used further in the present paper. In fractional calculus, most frequently used definition of integration of non-integer order comes from the extension of the well-known formula for the \( n \)-fold integration and is named Riemann–Louville operators. Moreover, in this section, we recall the definition of the fractional operators with singular kernel and their generalized versions which have been recently proposed in the literature by Abdeljawad et al. [26, 28]. Before giving the fractional operator with their generalized form we begin with Mittag-Leffler function.

**Definition 1** The Mittag-Leffler function with the parameters \( \alpha \) and \( \mu \) is defined as following series

\[
E_{\alpha,\mu}(q) = \sum_{m=0}^{\infty} \frac{q^m}{\Gamma(am + \mu)},
\]

where \( \alpha > 0, \mu \in \mathbb{R}, \) and \( q \in \mathbb{C} \). It is well known that the convergence of this series results from the assumptions \( \alpha > 0, \) and \( \mu > 0 \).
Definition 2 [26, 28]. The GML function is defined as following series

$$\mathbb{E}_{a,\mu}^\gamma(q) = \sum_{m=0}^{\infty} \frac{q^m(\gamma)_m}{\Gamma(am + \mu + 1)},$$

where $Re(\alpha) > 0$, $\mu, \alpha, \gamma \in \mathbb{C}$, and $q \in \mathbb{C}$. In Equation (10), $(\gamma)_m = \gamma(\gamma + 1) \cdots (\gamma + m - 1)$ is the rising factorial. It is clear that $(1)_m = m!$ and $\mathbb{E}_{a,\mu}^1(q) = \mathbb{E}_{a,\mu}(q)$.

Definition 3 [26, 28]. The ML function for a certain function is defined by

$$E_\alpha(\lambda, q) = \sum_{m=0}^{\infty} \frac{\lambda^m q^m}{\Gamma(am + 1)},$$

and

$$E_{a,\mu}(\lambda, q) = \sum_{m=0}^{\infty} \frac{\lambda^m q^{am+\mu-1}}{\Gamma(am + \mu)},$$

where $Re(\alpha) > 0$, $0 \neq \lambda \in \mathbb{R}$, and $q, \mu \in \mathbb{C}$. In addition, the GML function is given as

$$E_{a,\mu}^\gamma(\lambda, q) = \sum_{m=0}^{\infty} \frac{\lambda^m q^{am+\mu-1}(\gamma)_m}{\Gamma(am + \mu + 1)},$$

where $Re(\alpha) > 0$, $0 \neq \lambda \in \mathbb{R}$, and $q, \mu \in \mathbb{C}$.

Definition 4 [26, 28]. The left and right Atangana–Baleanu (ABC) fractional derivatives which have been constructed with the GML function defined in Definition 2 are given as, respectively:

$$\frac{^ABC}{a}D_{a}^{\alpha}(g(\nu)) = \frac{A(\alpha)}{1-\alpha} \int_{\alpha(a)}^{\nu} g'(k)E_{\alpha,\mu}(\lambda, \nu - k)dk,$$

and

$$\frac{^ABC}{b}D_{\nu}^{\alpha}(g(\nu)) = -\frac{A(\alpha)}{1-\alpha} \int_{\nu}^{b} g'(k)E_{a,\mu}(\lambda, k - \nu)dk,$$

where $0 < \alpha < 1$, $A(\alpha)$ is an arrangement function so that $A(0) = A(1) = 1$, and $\lambda = -\frac{\alpha}{1-\alpha}$.

Definition 5 [26, 28]. The left and right ABC fractional derivatives which have been constructed with the GML function defined in Definition 2 are given as, respectively:

$$\frac{^ABC}{a}D_{\nu}^{\alpha,\mu,\gamma}(g(\nu)) = \frac{A(\alpha)}{1-\alpha} \int_{\alpha(a)}^{\nu} E_{\alpha,\mu}^\gamma(\lambda, \nu - k)g'(k)dk,$$

and

$$\frac{^ABC}{b}D_{\nu}^{\alpha,\mu,\gamma}(g(\nu)) = -\frac{A(\alpha)}{1-\alpha} \int_{\nu}^{b} E_{a,\mu}^\gamma(\lambda, k - \nu)g'(k)dk.$$

Definition 6 [26, 28]. Assume that $g(\nu)$ is defined on $[a, b]$. Then the generalized left and right Riemann–Liouville fractional integrals of the AB operator of order $0 < \alpha \leq 1$, $\gamma > 0$, $Re(1 - \mu) > 0$ are given as, respectively:

$$\frac{^AB}{a}I_{a}^{\alpha,\mu,\gamma}(g(\nu)) = \sum_{p=0}^{\infty} \left(\frac{\gamma}{p}\right) \frac{\alpha^p}{A(\alpha)(1-\alpha)^{p-1}} (a^{\mu p+1-\alpha} g(\nu)),$$

and

$$\frac{^AB}{b}I_{\nu}^{\alpha,\mu,\gamma}(g(\nu)) = \sum_{p=0}^{\infty} \left(\frac{\gamma}{p}\right) \frac{\alpha^p}{A(\alpha)(1-\alpha)^{p-1}} (b^{\mu p+1-\alpha} g(\nu)),$$
where \( aI^\alpha \{ g(v) \} \) and \( I^\alpha_b \{ g(v) \} \) are the left and right Riemann–Liouville fractional integrals, respectively.

**Definition 7** We assume \( f, g : [0, \infty) \rightarrow \mathbb{R} \), then the convolution of these functions is given as

\[
(f \ast g)(v) = \int_0^v f(v - k)g(k)dk,
\]

and the following property holds

\[
\mathcal{L}\{f \ast g(v)\} = \mathcal{L}\{f(v)\} \mathcal{L}\{g(v)\},
\]

where \( \mathcal{L} \) represents the usual Laplace transform.

**Definition 8** [5] The Laplace transform of the ABC fractional operator is given by

\[
\mathcal{L}\{_{0}^{ABC}D^\alpha \xi \{ g(\nu) \} \}(s) = \frac{A(\alpha)}{s^\alpha \Gamma(\alpha + 1)} - s^\alpha g(0).
\]

**Definition 9** [26]. The generalized Laplace transform of the ABC fractional operator is given by

\[
\mathcal{L}\{_{0}^{ABC}D^{\alpha,\mu,\gamma} \xi \{ g(\nu) \} \} = \frac{A(\alpha)}{1 - \alpha} s^{-\mu}(1 - \lambda s^{-\alpha})^{-\gamma} (s\mathcal{L}\{g(\nu)\} - g(0)).
\]

**Lemma 1** [27]. We assume \( \text{Re}(\mu) > 0, \mu, \alpha, \gamma, \lambda, s \in \mathbb{C}, \text{Re}(s) > 0, \) and \( |\lambda s^{-\alpha}| < 1 \). Then the Laplace transform of \( _{a}^{E_{a,\mu}}(\lambda t^\alpha) \) is given as

\[
\mathcal{L}\{_{a}^{E_{a,\mu}}(\lambda, t - a)\}(s) = s^{-\mu}(1 - \lambda s^{-\alpha})^{-\gamma}.
\]

### 3 Mathematical Investigation of the Fractional Option Pricing Model

In this section, we give the mathematical perspective of the fractional Black–Scholes model by considering the generalized ABC fractional operator. First, we present the fundamental instruments of the investigation:

Consider the fractional BSE with the generalized ABC derivative given in Equation (5):

\[
_{0}^{ABC}D^{\alpha,\mu,\gamma} \xi \{ \eta(\xi, v) \} = \frac{\partial^2 \eta(\xi, v)}{\partial \xi^2} + (k - 1)\frac{\partial \eta(\xi, v)}{\partial \xi} - k\eta(\xi, v), \quad 0 < \alpha \leq 1,
\]

subject to the initial condition

\[
\eta(\xi, 0) = \max(e^{\xi} - 1, 0).
\]

Operating Equation (18) of order \( \alpha, \mu, \gamma \) on Equation (25), we get

\[
\eta(\xi, v) = \sum_{p=0}^{\gamma} \left( \begin{array}{c} \gamma \\ p \end{array} \right) \frac{\alpha^p}{A(\alpha)(1 - \alpha)^{p-1}} \frac{1}{\Gamma(\alpha p + 1 - \mu)} \times \int_0^v (v - h)^{\alpha p - \mu}[\eta_{\xi \xi} + (k - 1)\eta_{\xi} - k\eta]dh.
\]
Assigning \( \varpi(\xi, \nu, \eta) = \eta \xi + (k - 1) \eta \xi - k \eta \), Equation (27) turns to

\[
\eta(\xi, \nu) - \eta(\xi, 0) = \sum_{\rho=0}^{y} \left( \frac{\gamma}{p} \right) \frac{\alpha^{p}}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(\alpha p + 1 - \mu)} \times \int_{0}^{v} (v - h)^{\alpha p - \mu} \varpi(\xi, h, \eta) d h.
\] (28)

The function \( \eta(\xi, \nu) \) has an upper bound if the kernel \( \varpi(\xi, \nu, \eta) \) satisfies the Lipschitz condition. Exclusively, \( \eta(\xi, \nu) \) has an upper bound, since we have

\[
\| \varpi(\xi, \nu, \eta) - \varpi(\xi, \nu, \kappa) \| = \left\| \eta_1 \xi + (k - 1) \eta_1 \xi - k \eta_1 \right\| \\
\leq \left\| \eta_1 \right\| + \left\| k - 1 \right\| \left\| \eta_1 \right\| + \left\| k \right\| \left\| \eta_1 \right\| \\
\leq (n_1 \xi_1 + n_2 \xi_2 + \xi_3) \left\| \eta_1 \right\|.
\]

Taking \( N = n_1 \xi_1 + n_2 \xi_2 + \xi_3 \), this intends that

\[
\| \varpi(\xi, \nu, \eta) - \varpi(\xi, \nu, \kappa) \| \leq N \left\| \eta_1 \right\|. 
\] (29)

Accordingly, \( \varpi(\xi, \nu, \eta) \) satisfies the Lipschitz condition. Therewith, the function \( \eta(\xi, \nu) \) is bounded. We then give the following theorems:

**Theorem 1**  Assume that \( \eta(\xi, \nu) \) is a bounded function, then the operator \( \Psi(\eta(\xi, \nu)) \) given by

\[
\Psi(\eta(\xi, \nu)) = \eta(\xi, 0) + \sum_{\rho=0}^{y} \left( \frac{\gamma}{p} \right) \frac{\alpha^{p}}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(\alpha p + 1 - \mu)} \times \int_{0}^{v} (v - h)^{\alpha p - \mu} \varpi(\xi, h, \eta) d h,
\] (30)

satisfies the Lipschitz condition.

**Proof.** Postulate that \( \eta(\xi, \nu) \) and \( \kappa(\xi, \nu) \) are bounded functions, then

\[
\| \Psi(\eta(\xi, \nu)) - \Psi(\kappa(\xi, \nu)) \| = \left\| \sum_{\rho=0}^{y} \left( \frac{\gamma}{p} \right) \frac{\alpha^{p}}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(\alpha p + 1 - \mu)} \times \int_{0}^{v} (v - h)^{\alpha p - \mu} \left( \varpi(\xi, h, \eta) - \varpi(\xi, h, \kappa) \right) d h \right\|
\]

\[
\leq \sum_{\rho=0}^{y} \left( \frac{\gamma}{p} \right) \frac{\alpha^{p}}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(\alpha p + 1 - \mu)} \times \int_{0}^{v} (v - h)^{\alpha p - \mu} \left\| \varpi(\xi, h, \eta) - \varpi(\xi, h, \kappa) \right\| d h
\]

\[
\leq \sum_{\rho=0}^{y} \left( \frac{\gamma}{p} \right) \frac{\alpha^{p}}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(\alpha p + 1 - \mu)} \times \int_{0}^{v} (v - h)^{\alpha p - \mu} \left\| \eta - \kappa \right\| d h
\]

\[
\leq \Lambda \left\| \eta - \kappa \right\|,
\] (31)
where \( \Lambda = N_1 N \) and \( N_1 = \sum_{p=0}^{\gamma} \left( \frac{\gamma}{p} \right) \frac{\alpha^p}{A(\alpha(1-\alpha)^{p-1} \Gamma(ap + 1 - \mu)} \right). \) Then we have

\[
\left\| \Psi(\eta(\xi, \nu)) - \Psi(\kappa(\xi, \nu)) \right\| \leq \Lambda \| \eta - \kappa \|.
\]  

(32)

Thus, the operator \( \Psi(\eta(\xi, \nu)) \) satisfies the Lipschitz condition. This proves the theorem.

\[\Box\]

**Theorem 2**

Suppose that \( \eta(\xi, \nu) \) is a bounded function, then the operator given by

\[\varphi(\eta) = \eta(\xi) + (k - 1)\eta(\xi) - k\eta,\]

satisfies the condition

\[| \langle \varphi(\eta) - \varphi(\kappa), \eta - \kappa \rangle | \leq N \| \eta - \kappa \|^2.\]

(34)

**Proof.** Assume that \( \eta(\xi, \nu) \) is a bounded function, then

\[
| \langle \varphi(\eta) - \varphi(\kappa), \eta - \kappa \rangle | = | \langle \eta(\xi) - \kappa(\xi), (k - 1)(\eta(\xi) - \kappa(\xi)) - k(\eta - \kappa), \eta - \kappa \rangle |
\]

\[
\leq | \langle \eta - \kappa, \eta - \kappa \rangle | + | \langle (k - 1)(\eta - \kappa), \eta - \kappa \rangle | + | \langle k(\eta - \kappa), \eta - \kappa \rangle |
\]

\[
\leq N \| \eta - \kappa \| + \| (k - 1) \| (\eta - \kappa) \| \| \eta - \kappa \|
\]

\[
\leq (n_1 \zeta_1 + n_2 \zeta_2 + \zeta_3) \| \eta - \kappa \|^2.
\]

It implies that

\[| \langle \varphi(\eta) - \varphi(\kappa), \eta - \kappa \rangle | \leq N \| \eta - \kappa \|^2.\]

(35)

This gives the proof.

\[\Box\]

**4 | EXISTENCE AND UNIQUENESS OF THE SOLUTION**

In this section, we analyze the existence and uniqueness of the solution of the problem which is given by Equation (25). Taking the unknown function \( \eta(\xi, \nu) \) into consideration, we can generate the following iterative formula:

\[
\eta_{z+1}(\xi, \nu) = \sum_{p=0}^{\gamma} \left( \frac{\gamma}{p} \right) \frac{\alpha^p}{A(\alpha(1-\alpha)^{p-1} \Gamma(ap + 1 - \mu)} 1 \right) \times \int_0^\nu (\nu - h)^{ap-\mu} \sigma(\xi, h, \eta_z) dh,
\]

(36)

where \( \eta_0 = \eta(\xi, 0) \).

The difference between iterative terms can be given as

\[
\chi_z(\xi, \nu) = \eta_z(\xi, \nu) - \eta_{z-1}(\xi, \nu)
\]

\[
= \sum_{p=0}^{\gamma} \left( \frac{\gamma}{p} \right) \frac{\alpha^p}{A(\alpha(1-\alpha)^{p-1} \Gamma(ap + 1 - \mu)} 1 \right) \times \int_0^\nu (\nu - h)^{ap-\mu} (\sigma(\xi, h, \eta_{z-1}) - \sigma(\xi, h, \eta_{z-2})) dh.
\]

(37)
We need to consider the following in order to proceed
\[ \eta_z(\xi, \nu) = \sum_{j=0}^{z} \chi_j(\xi, \nu). \] (38)

Taking the norm of both sides of Equation (37), we obtain
\[
\left\| \chi_z(\xi, \nu) \right\| = \left\| \eta_z(\xi, \nu) - \eta_{z-1}(\xi, \nu) \right\|
= \left\| \sum_{p=0}^{\gamma} \binom{\gamma}{p} \alpha^p \frac{1}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(ap + 1 - \mu)} \right.
\times \int_0^\nu (\nu - \tilde{h})^{ap-\mu} (\sigma(\xi, h, \eta_{z-1}) - \sigma(\xi, h, \eta_{z-2})) \, d\tilde{h} \right\|.
\] (39)

By benefiting from the triangular inequality, we can conclude that
\[
\left\| \chi_z(\xi, \nu) \right\| \leq \sum_{p=0}^{\gamma} \binom{\gamma}{p} \alpha^p \frac{1}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(ap + 1 - \mu)} \right.
\times \int_0^\nu (\nu - \tilde{h})^{ap-\mu} \left\| (\sigma(\xi, h, \eta_{z-1}) - \sigma(\xi, h, \eta_{z-2})) \right\| \, d\tilde{h}.
\] (40)

Since the kernel satisfies the Lipschitz condition, then we get
\[
\left\| \chi_z(\xi, \nu) \right\| \leq \sum_{p=0}^{\gamma} \binom{\gamma}{p} \alpha^p \frac{1}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(ap + 2 - \mu)} \times N \int_0^\nu (\nu - \tilde{h})^{ap-\mu} \left\| \eta_{n-1} - \eta_{n-2} \right\| \, d\tilde{h}.
\] (41)

Taking Equations (36)–(41) into consideration, the following theorem is formed:

**Theorem 3** Suppose that \( \eta(\xi, \nu) \) is a bounded functions, then Equation (25) is said to have a solution if there exists \( v_0 \) and the following inequality holds:
\[
\sum_{p=0}^{\gamma} \binom{\gamma}{p} \alpha^p \frac{1}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(ap + 2 - \mu)} \times N \int_0^\nu (\nu - \tilde{h})^{ap-\mu} \left\| \eta_{n-1} - \eta_{n-2} \right\| \, d\tilde{h} < 1.
\] (42)

**Proof.** Assume that \( \eta(\xi, \nu) \) is a bounded function. Using the fact that in Equation (41), and considering the recursive relation, we get
\[
\left\| \chi_z(\xi, \nu) \right\| \leq \sum_{p=0}^{\gamma} \binom{\gamma}{p} \alpha^p \frac{1}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(ap + 2 - \mu)} \times N \int_0^\nu (\nu - \tilde{h})^{ap-\mu} \left\| \eta_{n-1} - \eta_{n-2} \right\| \, d\tilde{h}.
\] (43)

Thus, Equation (38) exists and it is smooth. We then show that Equation (38) is a solution to Equation (25).

Assume that \( \eta(\xi, \nu) - \eta(\xi, 0) = \eta_z(\xi, \nu) - \Delta_z(\xi, \nu) \), then we have
\[
\left\| \Delta_z(\xi, \nu) \right\| = \left\| \sum_{p=0}^{\gamma} \binom{\gamma}{p} \alpha^p \frac{1}{A(\alpha)(1 - \alpha)^{p-1} \Gamma(ap + 1 - \mu)} \right.
\times \int_0^\nu (\nu - \tilde{h})^{ap-\mu} \left\| \sigma(\xi, h, \eta) - \sigma(\xi, h, \eta_{z-1}) \right\| \, d\tilde{h}
\]
\[ \sum_{p=0}^{\gamma} \left( \begin{array}{c} \gamma \\ p \end{array} \right) \frac{\alpha^p}{A(\alpha)(1-\alpha)^{p-1} \Gamma(ap + 1 - \mu)} \times \frac{\nu^{ap+1-\mu}}{ap + 1 - \mu} N \left| \phi - \phi_{n-1} \right|. \] \quad (44)

Using the recursive scheme, we achieve

\[ \left\| \Delta_z(\xi, \nu) \right\| \leq \left[ \sum_{p=0}^{\gamma} \left( \begin{array}{c} \gamma \\ p \end{array} \right) \frac{\alpha^p}{A(\alpha)(1-\alpha)^{p-1} \Gamma(ap + 2 - \mu)} \right]^{\zeta+1} N^{\zeta+1} \xi, \] \quad (45)

at \( \nu = \nu_0 \), Equation (45) turns to

\[ \left\| \Delta_z(\xi, \nu) \right\| \leq \left[ \sum_{p=0}^{\gamma} \left( \begin{array}{c} \gamma \\ p \end{array} \right) \frac{\alpha^p}{A(\alpha)(1-\alpha)^{p-1} \Gamma(ap + 2 - \mu)} \nu^{ap+1-\mu} \right]^{\zeta+1} N \] \quad (46)

as \( z \to 0 \) reaches that \( \left\| \Delta_z(\xi, \nu) \right\| \to 0. \) This proves the theorem.

We now keep going to show the uniqueness of the solution to Equation (25).

**Theorem 4**  
Equation (25) is said to have a unique solution if

\[ \left( 1 - \sum_{p=0}^{\gamma} \left( \begin{array}{c} \gamma \\ p \end{array} \right) \frac{\alpha^p}{A(\alpha)(1-\alpha)^{p-1} \Gamma(ap + 1 - \mu)} \right) > 0. \] \quad (47)

**Proof.**  
Assume that Equation (25) has two solutions, namely, \( \eta_1(\xi, \nu) \) and \( \eta_2(\xi, \nu) \), then we can write

\[
\eta_1(\xi, \nu) - \eta_2(\xi, \nu) = \sum_{p=0}^{\gamma} \left( \begin{array}{c} \gamma \\ p \end{array} \right) \frac{\alpha^p}{A(\alpha)(1-\alpha)^{p-1} \Gamma(ap + 1 - \mu)} \times \int_0^\nu (\nu - h)^{ap-\mu} \left( \sigma(\xi, h, \eta_1) - \sigma(\xi, h, \eta_2) \right) dh. \]

(48)

Taking norm of both sides of Equation (48), we have

\[
\left\| \eta_1(\xi, \nu) - \eta_2(\xi, \nu) \right\| = \sum_{p=0}^{\gamma} \left( \begin{array}{c} \gamma \\ p \end{array} \right) \frac{\alpha^p}{A(\alpha)(1-\alpha)^{p-1} \Gamma(ap + 1 - \mu)} \times \int_0^\nu (\nu - h)^{ap-\mu} \left\| \sigma(\xi, h, \eta_1) - \sigma(\xi, h, \eta_2) \right\| dh. \]

(49)

It follows that

\[
\left\| \eta_1(\xi, \nu) - \eta_2(\xi, \nu) \right\| \leq \sum_{p=0}^{\gamma} \left( \begin{array}{c} \gamma \\ p \end{array} \right) \frac{\alpha^p}{A(\alpha)(1-\alpha)^{p-1} \Gamma(ap + 1 - \mu)} \times \frac{\nu^{ap+1-\mu}}{ap + 1 - \mu} N \left\| \eta_1(\xi, \nu) - \eta_2(\xi, \nu) \right\|. \]

(50)
Therefore,
\[
\| \eta_1(\xi, \nu) - \eta_2(\xi, \nu) \| \left( 1 - \sum_{p=0}^{\gamma} \left( \frac{\gamma}{p} \right) \frac{\alpha^p}{\Gamma(\alpha a + 2 - p)} \right) \leq 0. \tag{51}
\]
If the condition given in Equation (51) holds, then we have
\[
\| \eta_1(\xi, \nu) - \eta_2(\xi, \nu) \| = 0, \tag{52}
\]
which gives \( \eta_1(\xi, \nu) = \eta_2(\xi, \nu) \). Hence, Equation (25) has a unique solution. \qed

5 | MAIN RESULTS

In this section, we give the solutions of the fractional order classical and generalized Black–Scholes option pricing models. In order to achieve that, we consider the fractional operators with both of the classical and GML kernels.

5.1 | Fundamental solutions via the classical Mittag-Leffler kernel

In this subsection, we first define the suggested method by using the Laplace transformation to solve the Black–Scholes option pricing problems mentioned in Section 1. This method is combined with the classical homotopy method and Laplace transform. Now, we consider the FBSE which is given by Equation (5) and its initial condition in Equation (6):
\[
ABC_0^\alpha D_v^{\alpha, \mu, \nu} \{ \eta(\nu) \} = \frac{\partial^2 \eta(\xi, \nu)}{\partial \xi^2} + (k - 1) \frac{\partial \eta(\xi, \nu)}{\partial \xi} - k \eta(\xi, \nu), \tag{53}
\]
with initial condition given by
\[
\eta(\xi, 0) = \max (e^\xi - 1, 0), \tag{54}
\]
where \( ABC_0^\alpha D_v^{\alpha, \mu, \nu} \{ \eta(\nu) \} \) shows the generalized ABC operator. Other variables stated in Equation (53) are the same with those which were defined in Section 3. Using the LT, we define the \( \mathcal{L} \{ \eta(\xi, \nu) \} = H(\xi, s) \).

We here construct the solution steps only according to the generalized ABC operator. For classical ABC operator one can obtain the method by using the similar steps as in this section. Then applying the homotopy to Equation (53) we derive the homotopies for the generalized ABC operator. Taking the LT of both sides of Equation (53), yields
\[
H(\xi, s) = \frac{1 - \alpha}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-a})^{-\gamma}} \mathcal{L} \left\{ \frac{\partial^2 \eta}{\partial \xi^2} + (k - 1) \frac{\partial \eta}{\partial \xi} - k \eta \right\} + \frac{1}{s} \eta(\xi, 0). \tag{55}
\]
We assume that the solution is given by the following series
\[
H(\xi, s) = \sum_{m=0}^\infty \omega^m H_m(\xi, s), \tag{56}
\]
then substituting Equation (56) into Equation (55) and applying the homotopy steps, we have
\[
\sum_{m=0}^\infty \omega^m H_m(\xi, s) = \omega \left[ \frac{1 - \alpha}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-a})^{-\gamma}} \mathcal{L} \left\{ \sum_{m=0}^\infty \omega^m \left\{ \frac{\partial^2 \eta}{\partial \xi^2} + (k - 1) \frac{\partial \eta}{\partial \xi} - k \eta \right\} \right\} \right] + \frac{1}{s} \eta(\xi, 0). \tag{57}
\]
Then the approximate solution is given as

$$\eta(\xi, \nu) = L^{-1}\left\{\sum_{m=0}^{\infty} H_m(\xi, s)\right\},$$

(58)

where

$$H(\xi, s) = \frac{1}{s} \eta(\xi, 0) + \left[1 - \frac{1}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-\nu})^{-\gamma}} L\left\{\eta(\xi, \nu)_{0\xi} + (k - 1)\eta(\xi, \nu)_{1\xi} - k\eta(\xi, \nu)_{0}\right\}\right]$$

\begin{align*}
&+ \left[1 - \frac{1}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-\nu})^{-\gamma}} L\left\{\eta(\xi, \nu)_{1\xi} + (k - 1)\eta(\xi, \nu)_{2\xi} - k\eta(\xi, \nu)_{1}\right\}\right] \\
&+ \left[1 - \frac{1}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-\nu})^{-\gamma}} L\left\{\eta(\xi, \nu)_{2\xi} + (k - 1)\eta(\xi, \nu)_{3\xi} - k\eta(\xi, \nu)_{2}\right\}\right] \\
&\vdots \\
&+ \left[1 - \frac{1}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-\nu})^{-\gamma}} L\left\{\eta(\xi, \nu)_{n\xi} + (k - 1)\eta(\xi, \nu)_{(n+1)\xi} - k\eta(\xi, \nu)_{n}\right\}\right] + \cdots. \\
(59)
\end{align*}

By following the similar solution steps we can obtain the solution of the generalized fractional Black–Scholes equation which is given in Equations (7) and (8).

$$^{\alpha, \mu, \gamma}D^\nu_{t^*} \{g(\nu)\} = -0.08\xi^2(2 + \sin\xi)^2 \frac{\partial^2 \eta}{\partial \xi^2} - 0.06\xi \frac{\partial \eta}{\partial \xi} + 0.06\eta,$$

(60)

with initial condition given by

$$\eta(\xi, 0) = \max(\xi - 25e^{-0.06}, 0),$$

(61)

where \(^{\alpha, \mu, \gamma}D^\nu_{t^*} \{g(\nu)\}\) shows the GABC operator. Then applying the homotopy to Equation (60) we construct the homotopies for the ABC operator. Taking the LT of both sides of Equation (60), yields

$$H(\xi, s) = \frac{1 - \alpha}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-\nu})^{-\gamma}}$$

$$\times L\left\{-0.08\xi^2(2 + \sin\xi)^2 \frac{\partial^2 \eta}{\partial \xi^2} - 0.06\xi \frac{\partial \eta}{\partial \xi} + 0.06\eta\right\}$$

$$+ \frac{1}{s} \eta(\xi, 0).$$

(62)

We regard that the solution of the GFBSE is given by the following series

$$H(\xi, s) = \sum_{m=0}^{\infty} \omega^m H_m(\xi, s),$$

(63)

then substituting Equation (63) into Equation (62) and applying the homotopy steps, we have

$$\sum_{m=0}^{\infty} \omega^m H_m(\xi, s) = \omega \left[\frac{1 - \alpha}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-\nu})^{-\gamma}} L\left\{\sum_{m=0}^{\infty} \omega^m \right\}

\times \left\{-0.08\xi^2(2 + \sin\xi)^2 \frac{\partial^2 \eta}{\partial \xi^2} - 0.06\xi \frac{\partial \eta}{\partial \xi} + 0.06\eta\right\}\right]

+ \frac{1}{s} \eta(\xi, 0).$$

(64)
Then the approximate solution of the mentioned problem is given as

\[ \eta(\xi, \nu) = \mathcal{L}^{-1} \left\{ \sum_{m=0}^{\infty} H_m(\xi, s) \right\}, \]

where

\[ H(\xi, s) = \frac{1}{s} \eta(\xi, 0) + \frac{1 - \alpha}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-\alpha})^{-\gamma}} \]

\[ \times \mathcal{L} \{ -0.08 \xi^2 (2 + \sin \xi)^2 \eta(\xi, \nu)_{\nu \xi} - 0.06 \xi \eta(\xi, \nu)_{\nu \xi} + 0.06 \eta(\xi, \nu) \} + \frac{1 - \alpha}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-\alpha})^{-\gamma}} \]

\[ \times \mathcal{L} \{ -0.08 \xi^2 (2 + \sin \xi)^2 \eta(\xi, \nu)_{\nu \xi} - 0.06 \xi \eta(\xi, \nu)_{\nu \xi} + 0.06 \eta(\xi, \nu) \} + \frac{1 - \alpha}{A(\alpha)} \frac{s^{\mu-1}}{(1 - \lambda s^{-\alpha})^{-\gamma}} \]

\[ \times \mathcal{L} \{ -0.08 \xi^2 (2 + \sin \xi)^2 \eta(\xi, \nu)_{\nu \xi} - 0.06 \xi \eta(\xi, \nu)_{\nu \xi} + 0.06 \eta(\xi, \nu) \} + \cdots. \]

Now we achieve the solutions of the FBSE and GFBSE by considering the solution methods that have been constructed by using the fractional operators with both of the classical and GML kernels. First, we use the method for the FBSE with the classical Mittag-Leffler kernel operator: if we apply the Laplace transform of the ABC in Definition 9 to Equation (53) and considering its initial condition in Equation (54), we have the following relations:

\[ \sum_{m=0}^{\infty} \omega^m H_m(\xi, s) = \omega \left[ \frac{\alpha + (1 - \alpha)s^\alpha}{s^\alpha A(\alpha)} \right] \mathcal{L} \left\{ \sum_{m=0}^{\infty} \omega^m \left\{ \frac{\partial^2 \eta}{\partial \xi^2} + (k - 1) \frac{\partial \eta}{\partial \xi} - k \eta \right\} \right\} + \frac{1}{s} \eta(\xi, 0). \]

By equating the powers of \( \omega \), we get the approximate solution as the following

\[ \eta(\xi, \nu) = \mathcal{L}^{-1} \left\{ \sum_{m=0}^{\infty} H_m(\xi, s) \right\} \]

\[ \approx \max(0, e^\xi - 1) + \frac{k(e^\xi - \max(0, e^\xi - 1))}{A(\alpha)} \left( -\alpha + \frac{\alpha^\nu}{\Gamma(\alpha + 1)} + 1 \right) \]

\[ - \frac{k^2(e^\xi - \max(0, e^\xi - 1))}{A(\alpha)} \left( \frac{\alpha - 1}{2} + \frac{\alpha^\nu}{\Gamma(2\alpha + 1)} - \frac{2(\alpha - 1)\alpha^\nu}{\Gamma(3\alpha + 1)} \right) \]

\[ + \frac{k^3(e^\xi - \max(0, e^\xi - 1))}{A(\alpha)} \frac{3(\alpha - 1)^2 \alpha \Gamma(2\alpha + 1) \Gamma(3\alpha + 1)\nu^\alpha}{A(\alpha)^3 \Gamma(\alpha + 1) \Gamma(2\alpha + 1) \Gamma(3\alpha + 1)} \]

\[ + \frac{k^3(e^\xi - \max(0, e^\xi - 1))}{A(\alpha)} \frac{\Gamma(\alpha + 1) (-3(\alpha - 1) \alpha^2 \Gamma(3\alpha + 1)\nu^2\alpha)}{A(\alpha)^3 \Gamma(\alpha + 1) \Gamma(2\alpha + 1) \Gamma(3\alpha + 1)} \]

\[ + \frac{k^3(e^\xi - \max(0, e^\xi - 1))}{A(\alpha)} \frac{\Gamma(\alpha + 1) (-3(\alpha - 1)^3 \Gamma(3\alpha + 1))}{A(\alpha)^3 \Gamma(\alpha + 1) \Gamma(2\alpha + 1) \Gamma(3\alpha + 1)} \]

\[ \cdots. \]
By following similar steps above one can obtain the other parts of the series and then for the special case of the fractional parameter $\alpha = 1$, the exact solution of the problem is obtained as 

$$\eta(\xi, \nu) = e^\xi (1 - e^{-k\nu}) + e^{-k\nu} \max(e^\xi - 1, 0).$$

Second, we will also give the solution of the generalized FBSE by considering the classical ABC operator. Applying the LT of the ABC to Equation (60) and considering its initial condition in Equation (61), we have the following relations:

$$\sum_{m=0}^{\infty} \omega^m H_m(\xi, s) = \omega \left[ \frac{\alpha + (1 - \alpha)x^\alpha}{s^\alpha A(\alpha)} \sum_{m=0}^{\infty} \omega^m \right. \times \left\{ -0.08\xi^2(2 + \sin(\xi))^2 \frac{\partial^2 \eta}{\partial \xi^2} - 0.06\xi \frac{\partial \eta}{\partial \xi} + 0.06\eta \right\} \right] + \frac{1}{s^\alpha} \eta(\xi, 0). \quad (69)$$

Then by taking into account the powers of $\omega$, we have the approximate solution as the following

$$\eta(\xi, \nu) = \mathcal{L}^{-1} \left\{ \sum_{m=0}^{\infty} H_m(\xi, s) \right\}$$

$$\approx \max \left( \xi - \frac{25}{e^{0.06}}, 0 \right) - \frac{0.06 \left( \xi - \max \left( \xi - \frac{25}{e^{0.06}}, 0 \right) \right)}{A(\alpha)} \left( -\alpha + \frac{a^2 v^a}{\Gamma(2\alpha + 1)} + 1 \right)$$

$$- \frac{(0.06)^2 \left( \xi - \max \left( \xi - \frac{25}{e^{0.06}}, 0 \right) \right)}{A(\alpha)^2} \left( \frac{3(\alpha - 1)^2 a\Gamma(2\alpha + 1)\Gamma(3\alpha + 1)v^a}{A(\alpha)^2 \Gamma(\alpha + 1)\Gamma(2\alpha + 1)\Gamma(3\alpha + 1)} \right)$$

$$- \frac{(0.06)^3 \left( \xi - \max \left( \xi - \frac{25}{e^{0.06}}, 0 \right) \right)}{A(\alpha)^3} \left( \frac{\Gamma(\alpha + 1)(-3(\alpha - 1)a^2\Gamma(3\alpha + 1)v^{2a})}{A(\alpha)^3 \Gamma(\alpha + 1)\Gamma(2\alpha + 1)\Gamma(3\alpha + 1)} \right)$$

$$- \frac{(0.06)^3 \left( \xi - \max \left( \xi - \frac{25}{e^{0.06}}, 0 \right) \right)}{A(\alpha)^3} \left( \frac{\Gamma(\alpha + 1)(\Gamma(2\alpha + 1)(a^3 v^{3a} - (\alpha - 1)^3\Gamma(3\alpha + 1)))}{A(\alpha)^3 \Gamma(\alpha + 1)\Gamma(2\alpha + 1)\Gamma(3\alpha + 1)} \right)$$

$$\vdots \quad (70)$$

By following these steps one can obtain the other parts of the series and then for the special case of the fractional parameter $\alpha = 1$, the exact solution of the problem is obtained as 

$$\eta(\xi, \nu) = \max(\xi - 25e^{-0.06}, 0)e^{0.06\nu} + x(1 - e^{0.06\nu}).$$
In this subsection, we aim to obtain a fundamental solution to both of the FBSE and GFBSE by considering the generalized fractional operator with Mittag-Leffler kernel. For this purpose, first, we use the method for the FBSE via the fractional operator with the GML kernel. We start to the solution by applying the LT of the generalized ABC in Definition 9 to Equation (53) and considering its initial condition in Equation (54), we have the following relations:

\[
\sum_{m=0}^{\infty} \omega^m H_m(\xi, s) = \omega \left[ \frac{1 - \alpha}{A(\alpha)} \left( \frac{s^{\mu-1}}{1 - \lambda s^{-\eta}} \right)^\gamma \left\{ \sum_{m=0}^{\infty} \omega^m \right\} \left\{ \frac{\partial^2 \eta}{\partial \xi^2} + (k - 1) \frac{\partial \eta}{\partial \xi} - k \eta \right\} \right] + \frac{1}{s} \eta(\xi, 0). \tag{71}
\]

By expanding the series according to the powers of \( \omega \), we point out the following decomposition

\[
\eta(\xi, \nu) = \mathcal{L}^{-1} \left\{ \sum_{m=0}^{\infty} H_m(\xi, s) \right\}
\]
\[
\approx \max(0, e^\xi - 1) + \frac{k(e^\xi - \max(0, e^\xi - 1))(1 - \alpha)}{A(\alpha)} E_{a,2-\mu}^{-\gamma} (\lambda v^a) \\
- \frac{k^2(e^\xi - \max(0, e^\xi - 1))(1 - \alpha)^2}{A(\alpha)^2} E_{a,3-2\mu}^{-2\gamma} (\lambda v^a) \\
+ \frac{k^3(e^\xi - \max(0, e^\xi - 1))(1 - \alpha)^3}{A(\alpha)^3} E_{a,4-3\mu}^{-3\gamma} (\lambda v^a) \\
\vdots \\
+ (-1)^{(m+1)} k^m(e^\xi - \max(0, e^\xi - 1))(1 - \alpha^m) E_{a,m+1-\mu}^{-\gamma} (\lambda v^a) \\
= \max(0, e^\xi - 1) + (e^\xi - \max(0, e^\xi - 1))
\]
where $\mathbb{E}_{a,\mu}^\gamma(q)$ is the Mittag-Leffler function with three parameters which has been given in Definition 2.

Second, we will also have the solution of the generalized FBSE by considering the generalized ABC operator. The later step is to apply the LT of the generalized ABC to Equations (60) and (61), we obtain the following results:

\[
\sum_{m=0}^{\infty} \omega^m H_m(\xi, s) = \omega \left[ \frac{1 - \alpha}{A(\alpha)} \frac{s^\mu - 1}{(1 - \lambda s^{-\alpha}) - \gamma} \right] \left( \sum_{m=0}^{\infty} \omega^m \right)
\]
Then, by taking cognizance of the powers of \( \omega \), we reveal the approximate solution as

\[
\eta(\xi, \nu) = \mathcal{L}^{-1}\left\{ \sum_{m=0}^{\infty} H_m(\xi, s) \right\}
\approx \max \left( \xi - \frac{25}{e^{0.06}}, 0 \right) - \frac{0.06 \left( \xi - \max \left( \xi - \frac{25}{e^{0.06}}, 0 \right) \right)(1 - \alpha)}{A(\alpha)} \begin{pmatrix} 1 e^{-\gamma} \end{pmatrix}_{\alpha, 2-\mu} (\lambda^\nu)
\]

FIGURE 7  Numerical solution of the FBSE equation with the generalized MLF kernel

FIGURE 8  Option prices varying according to fractional parameter values for Equation (5)
where $\mathbb{E}^\gamma_{a,\mu}(q)$ is the GML function with three parameters.

6 | ERROR ANALYSIS OF THE METHOD

In this section, we point out the error norms of the suggested method by considering $L_2$ and $L_\infty$ error norms. By the aid of a computer package program, we have calculated the error norms which are given. In numerical computations, it is well known that these $L_2$ and $L_\infty$ error norms have been used to test the accuracy of the numerical results to the exact solution. The formulas of these error norms are given as:
The $L_2$ error norm for the BSE of fractional order which has been constructed with the ABC operator can be defined as \[63\]

$$L_2 = \|\eta^{\text{exact}} - \eta^{\text{num}}\|_2 = \sqrt{\sum_{j=0}^{N} |\eta_j^{\text{exact}} - \eta_j^{\text{num}}|^2} = 1.6428884236525338 \times 10^{-9},$$

and $L_\infty$ error norm can be defined as

$$L_\infty = \|\eta^{\text{exact}} - \eta^{\text{num}}\|_\infty = \max_j |\eta_j^{\text{exact}} - \eta_j^{\text{num}}| = 5.221442167524515 \times 10^{-10}.$$

The error rates we have obtained are quite meaningful and they point out the method we have used is very accurate and effective even we have used only first few components of the series. Therefore using this mentioned method is extremely suggested for obtaining the approximate solution of the fractional PDE.

In Table 1 and Figure 1, we have depicted the series solution results and comparison of the solution with the exact solution which we have obtained in Section 5.1. In addition, we have shown the absolute error values that are very lower even they have been provided by only first four components of the series solution.

### 7 | GRAPHICAL REPRESENTATIONS AND DISCUSSION

In this section, we figure out the solutions we have obtained in Section 3 and we discuss the results that these figures reveal out. In Figures 2 and 3, we have depicted the numerical and exact solutions of the model with respect to the ABC operator which is defined by the classical Mittag-Leffler kernel. In Figure 4, one can see the exact solution which has been obtained by considering the classical Mittag-Leffler kernel. In Figure 5, numerical solutions have been shown according to different values of the fractional parameter. It can be concluded from this figure that as the fractional order increases, option values approach to the exact solution. Figures 6 and 7 represent the exact solution and the numerical solution of the fractional order option pricing problem which includes the GML kernel, respectively. One can understand from Figure 7 that the numerical solutions are totally agreement to the exact solution. Figure 8 depicts the option prices varying according to fractional parameter values for Equation (5). It points out that the fractional parameter has an important effect on the option prices. In addition to these results, we can represent the effects of the volatility values on the option prices with Figure 9.

### 8 | CONCLUSION

In this paper, the existence and uniqueness have been investigated for the fractional option pricing models which have been described by a specific-type fractional derivative operator with the both classical and GML kernel. Moreover, novel solutions of the mentioned problems have been achieved numerically, the successive approximation method has been also discussed and the computational simulations have been depicted graphically. We also have pointed out the error analysis of the method and according to the error values we can conclude that the method we have used is totally agree with the exact
solution. With the aid of the results of this paper, some effective studies can be achieved by considering different types of fractional operators. These results can be regarded as the main novelties of the paper. These results have also shown that both of the classical and GML kernels are quite compatible to reveal the option prices. It is clear that when $\alpha, \mu, \gamma \to 1$, the ABC operator which is defined by the GML kernel turns to the integer order ordinary differential equation. Moreover, although the generalized type fractional derivatives have singular kernels for $0 < \mu < 1$, the one parameter ML function has a nonsingular kernel.

In recent years, the generalizations of classical fractional operators have been attracting researchers’ interest globally and they are needed to describe the dynamics of complex phenomena. In order to carry out the mentioned items, we have taken the Laplace transforms of the either classical or generalized fractional operators into account. Moreover, we have evaluated the option prices by giving the models’ fractional versions and presenting their series solutions. In addition to all results, this paper has pointed out a good initiative and tool for those who want to take these types of options into account either individually or institutionally. For future studies, different types of options such as American options, barrier options, interest options, bond options, exotic options, foreign exchange options, and so on, or various applications of the mentioned method constructed with the GML kernel can be considered.

CONFLICT OF INTEREST
The authors declare no conflicts of interest.

ORCID
Mehmet Yavuz https://orcid.org/0000-0002-3966-6518

REFERENCES
[1] F. Black and M. Scholes, The pricing of options and corporate liabilities, J. Polit. Econ. 81 (1973), 637–654.
[2] Z. Cen and A. Le, A robust and accurate finite difference method for a generalized Black–Scholes equation, J. Comput. Appl. Math. 235 (2011), 3728–3733.
[3] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional integrals and derivatives, Vol 1, Gordon and Breach Science Publishers, Yverdon-les-Bains, Switzerland, 1993.
[4] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl. 1 (2015), 1–13.
[5] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, Therm. Sci. 20 (2016), 763–769.
[6] A. Atangana, Modelling the spread of COVID-19 with new fractal-fractional operators: Can the lockdown save mankind before vaccination? Chaos Solitons Fractals 136 (2020), 109860.
[7] D. Kaya et al., Solutions of the fractional combined Kdv–mKdv equation with collocation method using radial basis function and their geometrical obstructions, Adv. Differ. Eq. 2018 (2018), 77.
[8] W. Gao et al., New numerical results for the time-fractional Phi-four equation using a novel analytical approach, Symmetry 12 (2020), 478.
[9] F. Jarad et al., More properties of the proportional fractional integrals and derivatives of a function with respect to another function, Adv. Differ. Eq. 2020 (2020), 1–16.
[10] P. A. Naik et al., Chaotic dynamics of a fractional order HIV-1 model involving AIDS-related cancer cells, Chaos Solitons Fractals 140 (2020), 110272.
[11] P. A. Naik, M. Yavuz, and J. Zu, The role of prostitution on HIV transmission with memory: A modeling approach, Alexandria Eng. J. 59 (2020), 2513–2531.
[12] P. A. Naik, et al., Modeling and analysis of COVID-19 epidemics with treatment in fractional derivatives using real data from Pakistan, The European Physical Journal Plus 135 (2020), (no. 10). 795, https://doi.org/10.1140/epjp/s13360-020-00819-5.
[13] I. Ahmad et al., Solution of multi-term time-fractional PDE models arising in mathematical biology and physics by local meshless method, Symmetry 12 (2020), 1195.
A. Fernandez and D. Baleanu, *New approach to a generalized fractional integral Analysis and numerical computations of the fractional regularized long-wave equation with the harvesting rate*, Fractal Fract. 4 (2020), 35.

P. A. Naik, *Global dynamics of a fractional order SIR epidemic model with memory*, Int. J. Biomath. 13 2020 (no. 8), 2050071. https://doi.org/10.1142/S1793524520500710.

P. A. Naik, J. Zu, and M. Ghereishi, *Stability analysis and approximate solution of SIR epidemic model with Crowley–Martin type functional response and Holling type-II treatment rate by using homotopy analysis*, J. Appl. Anal. Comput. 10 (2020), 1482–1515.

M. Yavuz and T. Abdeljawad, *Nonlinear regularized long-wave models with a new integral transformation applied to the fractional derivative with power and Mittag-Leffler kernel*, Adv. Differ. Eq. 2020 (2020), 1–18.

M. Yavuz and N. Özdemir, *Analysis of an epidemic spreading model with exponential decay law*, Math. Sci. Appl. E-Notes 8 (2020), 142–154.

F. Evirgen and M. Yavuz, *An alternative approach for nonlinear optimization problem with Caputo–Fabrizio derivative*, in *ITM Web of Conferences*, Vol 22, EDP Sciences, Les Ulis Cedex A, France, 2018, 01009.

A. Keten, M. Yavuz, and D. Baleanu, *Nonlocal Cauchy problem via a fractional operator involving power kernel in Banach spaces*, Fractal Fract. 3 (2019), 27.

P. A. Naik, J. Zu, and K. M. Owolabi, *Modeling the mechanics of viral kinetics under immune control during primary infection of HIV-1 with fractional order*, Physica A 545 (2020), 123816.

Z. Zhang et al., *Dynamics of a fractional order mathematical model for COVID-19 epidemic*, Adv. Differ. Eq. 2020 (2020), 1–16.

A. K. Alomari et al., *An approximate solution method for the fractional version of a singular BVP occurring in the electrohydrodynamic flow in a circular cylindrical conduit*, Eur. Phys. J. Plus 134 (2019), 158.

T. Abdeljawad and D. Baleanu, *On fractional derivatives with generalized Mittag-Leffler kernels*, Adv. Differ. Eq. 2018 (2018), 468.

T. Abdeljawad, *Fractional difference operators with discrete generalized Mittag–Leffler kernels*, Chaos, Solitons Fractals 126 (2019), 315–324.

T. Abdeljawad, *Fractional operators with generalized Mittag-Leffler kernels and their iterated differintegrals*, Chaos 29 (2019), 023102.

B. Acay, R. Ozarslan, and E. Bas, *Fractional physical models based on falling body problem*,AIMS Math. 5 (2020), 2608–2628.

B. Acay, E. Bas, and T. Abdeljawad, *Fractional economic models based on market equilibrium in the frame of different type kernels*, Chaos Solitons Fractals 130 (2020), 109438.

R. Ozarslan et al., *Fractional physical problems including wind-influenced projectile motion with Mittag-Leffler kernel*,AIMS Math. 5 (2020), 467–481.

U. N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl. 6 (2011), 1–15.

F. Jarad and T. Abdeljawad, *A modified Laplace transform for certain generalized fractional operators*, Results Nonlinear Anal. 1 (2018), 88–98.

A. Atangana et al., *Fractional differential and integral operators with non-singular and non-local kernel with application to nonlinear dynamical systems*, Chaos Solitons Fractals 132 (2020), 109493.

U. N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput. 218 (2011), 860–865.

A. Fernandez and D. Baleanu, *On a new definition of fractional differintegrals with Mittag-Leffler kernel*, Filomat 33 (2019), 245–254.

A. Atangana, *Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties*, Physica A 505 (2018), 688–706.

T. Abdeljawad and D. Baleanu, *Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels*, Adv. Differ. Eq. 2016 (2016), 232.

M. Yavuz et al., *Analysis and numerical computations of the fractional regularized long-wave equation with damping term*, Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6343.

M. Yavuz, *Characterizations of two different fractional operators without singular kernel*, Math. Model. Nat. Phenom. 14 (2019), 302.

M. Yavuz and N. Özdemir, *Comparing the new fractional derivative operators involving exponential and Mittag-Leffler kernel*, Discrete Contin. Dyn. Syst. S 13 (2020), 995–1006.

M. Yavuz and E. Bonyah, *New approaches to the fractional dynamics of schistosomiasis disease model*, Physica A 525 (2019), 373–393.

S. Uçar et al., *Mathematical analysis and numerical simulation for a smoking model with Atangana–Baleanu derivative*, Chaos Solitons Fractals 118 (2019), 300–306.

M. Yavuz, N. Özdemir, and H. M. Baskonus, *Solutions of partial differential equations using the fractional operator involving Mittag-Leffler kernel*, Eur. Phys. J. Plus 133 (2018), 215.
Yavuz M. European option pricing models described by fractional operators with classical and generalized Mittag-Leffler kernels. Numer Methods Partial Differential Eq. 2022;38:434–456. https://doi.org/10.1002/num.22645