THE NUMBER OF INTERLACING EQUALITIES RESULTING FROM REMOVAL OF A VERTEX FROM A TREE*

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Abstract. We consider the set of Hermitian matrices corresponding to a given graph, that is, Hermitian matrices whose nonzero entries correspond to the edges of the graph. When a particular vertex is removed from a graph a number of eigenvalues of the resulting principal submatrix may coincide with eigenvalues of the original Hermitian matrix. Here, we count the maximum number of “interlacing equalities” when the graph is a tree and the original matrix has distinct eigenvalues. We provide an upper bound and lower bound for the count and discuss some conditions under which the count is equal to the upper bound.

Key words. eigenvalues, interlacing equalities, parter vertices, trees

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1. Introduction. Let $G$ be an undirected graph on $n$ vertices, numbered $v_1, \ldots, v_n$, and let $\mathcal{H}(G)$ be the set of $n$-by-$n$ Hermitian matrices with graph $G$. In particular, if $A = (a_{ij}) \in \mathcal{H}(G)$, then for $i \neq j$, $a_{ij} = 0$ if and only if $\{v_i, v_j\}$ is not an edge of $G$. No restriction is placed on the diagonal entries of $A$. For an eigenvalue $\lambda$ of $A$, we denote by $m_\lambda(A)$ the multiplicity of $\lambda$ as an eigenvalue of $A$. We denote by $\sigma(A)$ the spectrum of $A$.

The removal of a vertex $v$ from $G$ results in an induced subgraph $G'$ that we write as $G' = G - v$. If $G$ is a tree, the graph $G - v$ consists of several (at least two) connected components. We call each such connected component a branch of $G - v$, or, equivalently, a branch of $G$ at $v$. The removal of a vertex $v$ from $G$ corresponds to the extraction of an $(n-1)$-by-$(n-1)$ principal submatrix from $A$, which we denote by $A(v)$. It is well known [2] that, ordering the eigenvalues of $A(v)$ as $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ and the eigenvalues of $A$ as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, we have

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$ 

We note that, of course, when $\lambda_i = \lambda_{i+1}$, $\lambda_i = \mu_i = \lambda_{i+1}$, and that similarly when $\mu_j = \mu_{j+1}$, $\mu_j = \lambda_{j+1} = \mu_{j+1}$. Even when the eigenvalues of $A$ and $A(v)$ are distinct, however, it can happen that $\lambda_i = \mu_{i+1}$ or $\lambda_i = \mu_i$. We call any such occurrence, in which a real number $\lambda$ satisfies $\lambda \in \sigma(A) \cap \sigma(A(v))$ (\sigma(A) is the spectrum of $A$), an interlacing equality for $G$ at $v$. Every real number $a$ that appears both in the spectrum of $A$ and in the spectrum of $A(v)$ is counted as one interlacing equality, regardless of the amount of times it appears in $\sigma(v)$. (Throughout the paper, we deal only with matrices $A$ in which all the eigenvalues are different, so in $A$ every eigenvalue appear exactly once.) One would expect that in the general case, no interlacing equality

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occur, beyond those guaranteed by the interlacing theorem. Hence, the following result, which appeared in [7] was very surprising: If $T$ is a tree and $A \in H(T)$ and $m_A(\lambda) \geq 2$, then there is a vertex $i$ such that $m_{A(i)}(\lambda) \geq 3$ and $\lambda$ is an eigenvalue of at least three components (branches) of $A(i)$. In particular, if $m_A(\lambda) = 2$, then $m_{A(i)}(\lambda) = m_A(\lambda) + 1$. This result was further generalized in [8]. More results regarding interlacing equalities in trees can be found in [1, 3, 4]. In view of the mentioned results, it became natural to consider the case in which $G$ is a tree, as the mentioned results suggest that trees exhibit very special properties related to interlacing equalities.

Our primary interest is to determine the maximum number of interlacing equalities for each graph $G$ and each vertex $v$ among all matrices $A \in H(G)$ with distinct eigenvalues. We make the assumption of distinct eigenvalues in analogy to the path case, discussed below. We denote this maximum number of interlacing equalities by $e_G(v)$. One can imagine labeling each vertex $v$ of $G$ with the integer $e_G(v)$; for this reason, we refer to $e_G(v)$ as the interlacing label, or simply the label, of $v$ in $G$.

It is well known that for each path $P$ on $n$ vertices, every matrix in $H(P)$ has distinct eigenvalues and that $P$ is the only such tree on $n$ vertices with this property [3]. Furthermore, if $v$ is a pendant vertex of $P$ (i.e., the degree of $v$ is 1), then $e_P(v) = 0$. Indeed, this is the only appearance of 0 as a label in $P$, and [3] gives the labeling of paths (of any length). The label of each vertex turns out to be its least distance from a pendant vertex. For example, the vertices of the path on seven vertices are labeled as described in Figure 1.

In the next section, we give a brief discussion of necessary background and then provide a generalization to one of the major results in the area of interlacing equalities, which is called the Parter–Wiener theorem. This generalization proves to be very useful for our results on interlacing labels. Next, we provide an upper bound and a lower bound on the labeling of any vertex in any tree. Then, assuming a certain conjecture to be true, we show that the upper bound is the exact labeling. Some sample applications to particular classes of trees are given. In a further section, more results about labeling are described, and an appendix containing all labeled trees on fewer than seven vertices is attached.

2. Parter vertices and types of interlacing equalities. Let $A$ be an $n$-by-$n$ Hermitian matrix, and let $\alpha \subset \{1, 2, \ldots, n\}$. We denote by $A[\alpha] (A(\alpha))$ the submatrix of $A$ whose rows and columns sets are $\alpha (\{1, 2, \ldots, n\} \setminus \alpha)$. For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix}$$

and $\alpha = \{1, 3\}$, then $A[\alpha] = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ and $A(\alpha) = \begin{pmatrix} 2 & 5 \\ 6 & 7 \end{pmatrix}$. We begin by describing a generalization of Parter–Wiener theorem [3].

**Theorem 2.1.** Let $A$ be a Hermitian matrix whose graph is a tree $T$, and suppose that there exists a vertex $v$ of $T$ and a real number $\lambda$ such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. 

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**Fig. 1. The labeled path on seven vertices.**
Then

(a) there is a vertex \( v' \) of \( T \) such that \( m_{A(v')}(\lambda) = m_A(\lambda) + 1 \); 
(b) if \( m_A(\lambda) \geq 2 \), then \( v' \) can be chosen so that \( \deg v' \geq 3 \) and so that there are at least three components \( T_1, T_2, \) and \( T_3 \) of \( T - v' \) such that \( m_{A[T_i]}(\lambda) \geq 1, i = 1, 2, 3 \); 
(c) if \( m_A(\lambda) = 1 \), then \( v' \) can be chosen so that \( \deg v' \geq 2 \) and so that there are two components \( T_1 \) and \( T_2 \) of \( T - v' \) such that \( m_{A[T_i]}(\lambda) = 1, i = 1, 2 \).

A vertex \( v' \) satisfying property (a) in Theorem 2.1 will be called a Parter vertex for \( \lambda \) in \( T \). We note that \( v \) itself can be an example of such a \( v' \) and that \( v' \) need not be unique. Furthermore, we see by interlacing that when \( m_A(\lambda) \geq 2, \lambda \in \sigma(A) \cap \sigma(A(v)) \) for all \( v \in V(T) \).

Our first result is a generalization of Theorem 2.1, part (c).

**Theorem 2.2.** Let \( A \in \mathcal{H}(T) \) be a matrix with distinct eigenvalues, where \( T \) is a tree, and suppose that there exists a vertex \( v \) of \( T \) and a real number \( \lambda \) such that \( m_A(\lambda) = m_A(v)(\lambda) = 1 \). Then the Parter vertex \( v' \) can be chosen so that \( \deg v' \geq 3 \) and so that there are two connected components \( T_1 \) and \( T_2 \) of \( T - v' \) that do not contain \( v \) for which \( m_{A[T_i]}(\lambda) = 1, i = 1, 2 \).

We can state the theorem equivalently as follows.

**Theorem 2.3.** Let \( A \in \mathcal{H}(T) \) be a matrix with distinct eigenvalues, where \( T \) is a tree, and suppose that there exists a real number \( \lambda \) such that \( m_A(\lambda) = 1 \) and a vertex \( v \) of \( T \) such that \( \lambda \in \sigma(A(v)) \). Then either \( v \) is Parter, or there exists a Parter vertex \( v' \neq v \) for which,

- \( \deg v' \geq 3 \),
- there are two connected components \( T_1 \) and \( T_2 \) of \( T - v' \) that do not contain \( v \), and they satisfy \( m_{A[T_i]}(\lambda) = 1 \) for \( i = 1, 2 \).

Note that the labeling of a path is easily derived from Theorem 2.2. Before stating the proof of the theorem, we present the following lemma.

**Lemma 2.4.** Let \( A \in \mathcal{H}(T) \) be a matrix with distinct eigenvalues, \( T \) being a tree, and suppose that there exists a vertex \( v \in V(T) \) and a real number \( \lambda \) such that \( m_A(\lambda) = m_{A[T-v]}(\lambda) = 1 \). Let the single branch of \( T - v \) for which \( \lambda \) is an eigenvalue be \( T^* \). Then \( v' \), the Parter vertex whose two relevant branches do not contain \( v \), lies in \( T^* \).

**Proof.** We recall that the relevant components \( T_1 \) and \( T_2 \) of \( T - v' \) do not contain \( v \). Defining \( T^* \) as the branch of \( T - v \) which contains \( v' \), it follows that they remain branches of \( T - v' \). By interlacing, this means that \( T^* \) has \( \lambda \) as an eigenvalue, and so \( T' = T^* \).

We are now ready to present the proof of Theorem 2.2.

**Proof.** We proceed using induction on the number of vertices, \( n \). For \( n < 5 \), \( T \) is either a path or \( K_{1,3} \). For paths, the statement follows from [3], and the proof for \( K_{1,3} \) is trivial. Assume that \( n \geq 5 \) and that the theorem holds for trees on fewer than \( n \) vertices. Let \( T \) be a tree of order \( n \), \( A \in \mathcal{H}(T) \), and \( v \in V(T) \) such that \( m_A(\lambda) = m_{A[T-v]}(\lambda) = 1 \). Let \( T' \) be a branch of \( T - v \) with eigenvalue \( \lambda \). By Theorem 2.1, there exists a Parter vertex \( v^* \), and since \( v \neq v^* \), it must exist in some branch of \( T - v \). Thus, we have four cases:

1. \( v^* \in V(T') \), \( \{v, v^*\} \notin E(T) \),
2. \( v^* \in V(T') \), \( \{v, v^*\} \in E(T) \),
3. \( v^* \notin V(T') \), \( \{v, v^*\} \notin E(T) \),
4. \( v^* \notin V(T') \), \( \{v, v^*\} \in E(T) \).

We discuss each one of them separately.
1. By Theorem 2.1, there exist two branches \(T_1\) and \(T_2\) of \(T - v^*\) which contain \(\lambda\) as eigenvalues. If neither contain \(v\), we are done, so let us assume, without loss of generality, that \(v \in V(T_1)\). It follows that \(\lambda \in \sigma(T') \cap \sigma(T' - v^*)\), and thus by Theorem 2.1 there exists a Parter vertex \(w \in V(T')\). We now have two cases: Either \(w = v^*\), or \(w \neq v^*\).

(a) If \(w = v^*\), then the two branches of \(T' - v^*\) which contain \(\lambda\) as an eigenvalue must be \(T_2\) and \(T_1 \cap T'\) (as every other branch of \(T' - v^*\) with \(\lambda\) as an eigenvalue would also be a branch of \(T - v^*\)). Since \(T_1 - v\) has \(T_1 \cap T'\) as a branch, we have \(\lambda \in \sigma(T_1) \cap \sigma(T_1 - v)\). Thus, by Theorem 2.1, there exists a vertex \(x\) Parter for \(\lambda\) in \(T_1\). If \(x = v\), then we get a contradiction, as at least one branch of \(T - v\) (which is not equal to \(T'\)) must have \(\lambda\) as an eigenvalue. Thus \(x \neq v\). By the induction hypothesis, we conclude that \(x\) must be Parter for two branches of \(T_1\) that do not contain \(v\). Additionally, applying Lemma 2.4, we note that \(x\) must be on the branch \(T_1 \cap T'\), and thus \(v\) and \(v^*\) are on separate branches of \(T - x\). Let us examine the branches \(T_3\) and \(T_4\) of \(T_1 - x\) that have \(\lambda\) as eigenvalues and that do not contain \(v\). If \(T_5\) is the branch of \(T - x\) containing \(w\), then the only branch of \(T_1 - x\) that does not remain a branch of \(T - x\) is \(T_3 \cap T_1\). Thus, if neither of \(T_3, T_4\) is equal to \(T_5 \cap T_1\), then \(x\) is also Parter for \(\lambda\) in \(T\) for the two branches which do not contain \(v\), and hence \(x = v'\) and we are done. We therefore assume that, without loss of generality, \(T_3 = T_5 \cap T_1\). Therefore, \(m_{T_3 - v^*}(\lambda) = 2\), as both \(T_2\) and \(T_5 \cap T_1\) have \(\lambda\) as eigenvalues. So by interlacing, \(T_5\) has \(\lambda\) as an eigenvalue, and therefore \(m_{T - v}(\lambda) = 2\), as both \(T_5\) and \(T_4\) have \(\lambda\) in their spectra. Since \(T_5\) and \(T_4\) do not contain \(v\), \(x = v'\) we are done.

(b) If \(w \neq v^*\), then \(m_{T' - v^*}(\lambda) = 1\). Applying the induction hypothesis, we conclude that the two branches of \(T' - w\) which contain \(\lambda\) as an eigenvalue do not contain \(v^*\). Therefore, by Lemma 2.4, \(w \in V(T_2)\). The two relevant branches of \(T' - w\) cannot contain \(v^*\) (since \(v\) and \(v^*\) are on the same branch of \(T - w\), and so they must remain branches in \(T - v\)). Thus, \(w\) is Parter for two branches that do not contain \(v\) and we are done.

2. If \(v \notin V(T_1)\) and \(v \notin V(T_2)\) then we are done. Thus, without loss of generality, \(v \in V(T_1)\). Therefore \(T' - v\) has \(T_2\) as a branch, and either \(v^*\) is Parter, or by the induction hypothesis some other vertex \(w\) is Parter for two branches that do not contain \(v\). If \(v^*\) were Parter in \(T'\), then the two relevant branches would also be Parter in \(T\) and would not contain \(v\). Similarly, if \(w\) were Parter for two branches that do not contain \(v^*\), those same two branches would remain branches of \(T - w\) and would not contain \(v\) (as \(v^*\) and \(v\) would be on the same branch). So the theorem holds when \(\{v, v^*\} \in E(T)\).

3. Let \(v^* \notin T''\) for some branch \(T''\), \(T'' \neq T'\). Let \(T_1\) and \(T_2\) be the two branches of \(T - v^*\) which contain \(\lambda\) as eigenvalues. If neither \(T_1\) nor \(T_2\) contains \(v\) as a vertex, then they remain branches of \(T'' - v^*\), and therefore \(T''\) would have \(\lambda\) in its spectrum by interlacing, which is a contradiction. Hence, without loss of generality, let \(v \in V(T_1)\). Then \(\lambda \in \sigma(T_1) \cap \sigma(T_1 - v)\), and so either \(v\) is Parter or some other vertex \(w\) is Parter. If \(v\) is Parter, then since \(m_{T - v}(\lambda) = 1\), the two branches of \(T_1 - v\) containing \(\lambda\) as an eigenvalue must be \(T'\) and \(T_1 \cap T''\). Therefore, \(T'' - v^*\) has both \(T_1 \cap T''\) and \(T_2\) as branches with \(\lambda\) in their spectra, and by interlacing, \(\lambda\) is an eigenvalue of \(T''\), contradicting our original assumption. So some other vertex \(w\) must be Parter in \(T_1\). By
the induction hypothesis, its two relevant branches must not contain \( v \), and by Lemma 2.4, \( v \in T' \). Thus, those branches must also remain branches in \( T - w \), and so \( w \) is Parter for two branches which do not contain \( v \). Thus, we can take \( w = v' \) and we are done.

4. We refer to the branch on which \( v^* \) lies as \( T'' \), and the two relevant branches of \( T - v^* \) as \( T_1 \) and \( T_2 \). As before, without loss of generality let \( v \in V(T_1) \).

Then \( \lambda \in \sigma(T'') \cap \sigma(T'' - v) \), and so either \( v \) is Parter for \( \lambda \) or, by the induction hypothesis, some other vertex \( w \) is Parter in \( T'' \) for two branches that do not contain \( v \). If \( v \) is Parter in \( T'' \), then the two relevant branches would also remain branches of \( T - v \), and so we get a contradiction. So the latter case must be true, and the two branches of \( T_1 - w \) must remain branches of \( T - w \), as the only one that does not remain a branch contains \( v \). We conclude that the theorem still holds in this final case.

In all possible cases, we note that the relevant Parter vertex \( v' \) must be of degree at least three, as there must exist at least three branches of \( T - v' \): two branches which have \( \lambda \) as an eigenvalue and do not contain \( v \), and a third branch containing \( v \).

This result will prove to be important to our method of labeling trees. In particular, we see from Theorem 2.2 that there is a fundamental difference between interlacing equalities in which \( v \) itself is Parter, and interlacing equalities in which some other vertex \( v' \) is Parter. We note that when \( v \) is Parter, two separate branches of \( T - v \) have \( \lambda \) as an eigenvalue, forcing two of the \( n - 1 \) eigenvalues of \( A[T - v] \) to be \( \lambda \). On the other hand, when, instead, some other vertex \( v' \) is Parter for two branches that do not contain \( v \), only one of the eigenvalues of \( A[T - v] \) is fixed. We call an interlacing equality in which \( v \) itself is Parter \( \text{Type 1} \) and an interlacing equality in which some other vertex \( v' \) is Parter, and \( v \) is not, \( \text{Type 2} \). Note, furthermore, that if we were to utilize all \( n - 1 \) eigenvalues of \( A[T - v] \), then we would get more interlacing equalities by having more Type 2 interlacing equalities.

Before continuing with our discussion of interlacing labels, we give a few generalizations of Theorem 2.2.

Remark 2.5. Let \( A \in H(T) \), \( T \) being a tree, and suppose that there exists a vertex \( v \in V(T) \) and a real number \( \lambda \) such that \( m_A(\lambda) = m_{A[T - v]}(\lambda) = m \geq 1 \). Then the Parter vertex \( v' \) can be chosen so that \( \deg v' \geq 3 \) and so that there are two components \( T_1 \) and \( T_2 \) of \( T - v' \) that do not contain \( v \) such that \( m_{A[T_i]}(\lambda) \geq 1, i = 1, 2 \).

When \( m = 1 \), this statement is simply Theorem 2.2. When \( m > 1 \), it trivially follows from Theorem 2.1.

Theorem 2.6. Let \( A \in H(T) \), \( T \) being a tree, and suppose that there exists a vertex \( v \in V(T) \) and a real number \( \lambda \) such that \( m_A(\lambda) = m_{A[T - v]}(\lambda) = m \geq 1 \). Suppose further that the Parter vertex \( v' \) guaranteed by Theorem 2.1 has all of its relevant branches not including \( v \). Then \( T - v \) has a single branch \( T' \) such that \( m_{A[T']}(\lambda) = m \), with no other branches having nonzero multiplicity for \( \lambda \), and \( v' \in V(T') \).

Proof. Let \( T'' \) be the branch of \( T - v \) which contains \( v' \). Since none of the branches of \( T - v' \) contain \( v \), each of them will also be branches of \( T' - v' \). So, by interlacing, \( \lambda \) is an eigenvalue of \( T' \) with multiplicity at least \( m \). Indeed, it cannot have multiplicity greater than \( m \), as \( m_{A[T - v]}(\lambda) = m \), so it has multiplicity exactly \( m \). Furthermore, if any other branch of \( T - v \) has nonzero multiplicity for \( \lambda \), then \( m_{A[T - v]}(\lambda) > m \), which is a contradiction.

Theorem 2.7. Let \( A \in H(T) \), \( T \) being a tree, and suppose that there exists a vertex \( v \in V(T) \) and a real number \( \lambda \) such that \( m_A(T)(\lambda) = m \geq 1 \) and such that there
exist \( m \) branches \( T_1, \ldots, T_m \) of \( T - v \) such that \( m_{A[T_i]}(\lambda) = 1 \) for \( i = 1, \ldots, m \). Then there exists a Parter vertex \( v_i \in T_i \) for \( i = 1, \ldots, m \).

**Proof.** We induct on \( m \). The statement holds for \( m = 0 \), and \( m = 1 \) is simply Lemma 2.4. Now assume that the theorem holds for all multiplicities smaller than \( m \geq 2 \), and consider a matrix \( A \in \mathcal{H}(T) \), and \( v \in V(T) \) that satisfies the hypotheses of the theorem. Then by Theorem 2.1, there exists a vertex \( v_1 \) such that there are three branches \( B_1, B_2, B_3 \) of \( T - v_1 \) with \( m_{A[T_1]}(\lambda) + m_{A[T_2]}(\lambda) + m_{A[T_3]}(\lambda) = m + 1 \).

There are two cases:

1. \( \{v_1, v\} \notin E(T) \),
2. \( \{v_1, v\} \in E(T) \).

We discuss each one of them separately.

1. Let \( T' \) be the branch of \( T - v \) containing \( v_1 \). Since at least two of \( B_1, B_2, B_3 \) must be branches of \( T' - v_1 \), without loss of generality \( B_1 \) and \( B_2 \) are those branches. Note that by Theorem 2.6, \( B_3 \) is not a branch of \( T' - v_1 \). Hence, \( T' \) must have \( \lambda \) as an eigenvalue, and thus without loss of generality we assume that \( T' = T_1 \). Furthermore, if \( m_{A[B_1]}(\lambda) + m_{A[B_2]}(\lambda) > 2 \), then by interlacing \( m_{A[T_1]}(\lambda) > 1 \), which is a contradiction. Therefore, \( m_{A[B_1]}(\lambda) = m_{A[B_2]}(\lambda) = 1 \) and \( m_{A[B_3]}(\lambda) = m - 1 \).

\( \lambda \) is not an eigenvalue of \( T_1 \cap B_3 \) (since otherwise \( T_1 - v_1 \) would produce \( \lambda \) on three branches, implying that \( T_1 \) would have \( \lambda \) of multiplicity at least two, which contradicts the assumptions). Therefore, \( B_3 - v \) has branches \( T_2, T_3, \ldots, T_m \) and no other branches with \( \lambda \) as an eigenvalue. So, \( m_{A[B_3]}(\lambda) = m - 1 \) and \( m_{A[B_3 - v]}(\lambda) = m - 1 \) with \( m - 1 \) branches each of multiplicity one.

By the induction hypothesis, there exists a Parter vertex \( v_2, \ldots, v_m \) for \( \lambda \) in \( B_3 \) in each of the branches \( T_2, \ldots, T_m \).

We claim that for each of these vertices \( v_i \), the branch of \( T - v_i \) containing \( v \), denoted by \( B'_i \), has \( \lambda \) of multiplicity one more than the corresponding branch of \( B_i \cap B_3 \). First, note that \( B'_i \) must have \( \lambda \) of multiplicity at least \( m - 2 \), as the removal of \( v \) results in \( \lambda \) having multiplicity \( m - 1 \) (since \( T_i \cap B'_i \) cannot be Parter for \( \lambda \) without making \( m_{A[T_i]}(\lambda) > 1 \)). Furthermore, \( m_{A[B'_i]} \neq m \), as interlacing and the removal of \( v_i \) would imply that \( T \) would have \( \lambda \) with multiplicity at least \( m + 1 \). Thus, \( B'_i \) has \( \lambda \) with multiplicity either \( m - 2 \) or \( m - 1 \). Since removing \( v_i \) from \( B'_i \) gives us at least \( m \) eigenvalues (one for \( B_1 \), one for \( B_2 \), and \( m - 2 \) for \( B_3 \cap B'_i \)), it follows that \( B'_i \) must have multiplicity \( m - 1 \). Thus, each of the \( v_i \) are also Parter for \( \lambda \) in \( T \).

2. The essence of the proof still holds. Note that we need not worry about whether \( T_1 \cap B_3 \) has \( \lambda \) as an eigenvalue, as it is the empty set. \( \Box \)

3. **The \( r \)-function.** We define here a function that is useful in our study of interlacing equalities and labels. Let \( \{L_i\}_{i=1}^k \) be a collection of \( k \) sets, and for each \( 1 \leq i \leq n \) let \( l_i = |L_i| \). We are interested in the largest possible number of pairings between elements of different sets, which we denote by \( r(l_1, l_2, \ldots, l_k) \). An alternative way to define \( r \) is through maximum matchings, that is, \( r(l_1, l_2, \ldots, l_k) \) is the cardinality of the maximum matching in the complete \( k \)-partite graph \( K_{l_1, l_2, \ldots, l_k} \). Note that the definition of \( r \) implies that for any permutation \( \omega \) on the elements \( \{1, 2, \ldots, k\} \), we have \( r(l_{\omega(1)}, l_{\omega(2)}, \ldots, l_{\omega(k)}) \), and thus we can assume without loss of generality that \( l_1 \geq l_2 \geq \cdots \geq l_k \).

**Theorem 3.1.** For the function \( r(l_1, l_2, \ldots, l_k) \) defined above, we have the following:
Let \( \lambda \) and such that there is a vertex proving the lower bound. Those cases in the next section. Bound, and hence for such cases the theorem provides an exact labeling. We discuss the main result in this section, we provide a lower bound and upper bound for the labeling of a vertex in a tree. There are some cases in which the upper bound equals the lower bound.  

We now introduce the conjecture. Let \( v \in H \) such that \( \sigma(A) = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \). Theorem 4.2 (see \[6\]). Let \( T \) be a tree on \( n \) vertices and \( v \) be a vertex of \( T \). Let \( \lambda_1 < \cdots < \lambda_n \) and \( \mu_1 < \cdots < \mu_{n-1} \) be real numbers. If \( \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{n-1} < \lambda_n \), then there exists a matrix \( A \in \mathcal{H}(T) \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \) and such that \( A(v) \) has eigenvalues \( \mu_1, \ldots, \mu_{n-1} \).

We now introduce the conjecture. Let \( T \) be a tree and \( A \in \mathcal{H}(T) \). An eigenvalue \( \lambda \in \sigma(A) \) is upward at a vertex \( v \) of \( T \) if \( m_A(v)(\lambda) = m_A(\lambda) - 1 \). We further call \( \lambda \) upward for \( A \) if \( \lambda \) is upward for some vertex \( v \) of \( T \).

**Conjecture 4.3.** Suppose that a tree \( T \) on \( n \) vertices permits \( n - k \) upward eigenvalues for some \( A \in \mathcal{H}(T) \) with distinct eigenvalues, and let \( C \) be a set of \( k \) different real numbers. Then, there is a \( B \in \mathcal{H}(T) \) with distinct eigenvalues of which \( k \) are the numbers in \( C \) and the remaining \( n - k \) are upward.

We note that this statement is valid for paths \[5\]. It is also not difficult to prove for stars. Furthermore, cases \( k = n, 0, 1, 2 \) can be proved for any tree: The case \( k = n \) follows from Theorem 4.2. The case \( k = 0 \) is clear. If \( k = 1 \), there exists a real number \( \alpha \) such that \( B = A + \alpha I \), and if \( k = 2 \), there exist real numbers \( \alpha, \beta \) such that \( B = \beta A + \alpha I \).

We are now ready to present our main result.

**Theorem 4.4 (recursive labeling algorithm).** Let \( T \) be a tree on \( n \) vertices and \( v \) be a vertex of \( T \). Furthermore, let the \( k \) branches of \( T - v \) be \( T_1, \ldots, T_k \), and let the vertices adjacent to \( v \) on each of those branches be \( u_1, \ldots, u_k \). Then

\[
r(|T_1|, \ldots, |T_k|) \leq e_T(v) \leq \sum e_{T_i}(u_i) + r(|T_1| - e_{T_1}(u_1), \ldots, |T_k| - e_{T_k}(u_k)).
\]
If Conjecture 4.3 holds for $T$ and all its subtrees, the second inequality becomes equality, that is, $e_T(v) = \sum e_{T_i}(u_i) + r([T_1] - e_{T_i}(u_1), \ldots, [T_k] - e_{T_i}(u_k))$.

**Proof.** We start from proving the lower bound. By Theorem 4.2, we can choose a set $S$ of $r([T_1], \ldots, [T_k])$ different real numbers, such that each one of those numbers is an eigenvalue of exactly two different branches $T_i$ and $T_j$. Moreover, by the same theorem, for each $1 \leq i \leq k$, we can construct $A[T_i]$ in such a way that the eigenvalues of $A[T_i]$ and $A[T_i - v_i]$ strictly interlace, and $A[T_i], A[T_j]$ have no common eigenvalues other than those in $S$. The interlacing theorem implies that each one of the numbers in $S$ is an eigenvalue of at least two different branches $T_i$ and $T_j$. Hence, in order to conclude that $r([T_1], \ldots, [T_k])$ interlacing equalities. Thus, in order to conclude that $r([T_1], \ldots, [T_k]) \leq e_T(v)$, we just need to show that all the eigenvalues of $A$ are different. Assume in contradiction that this is not the case, and there exists $\lambda \in \sigma(A)$ of multiplicity at least two, and hence $\lambda \in \sigma(A) \cup \sigma(A(v))$. If $\lambda \in S$, then take a branch $T_j$ that has $\lambda$ as an eigenvalue, and by our construction we get that $m_A[T_j - u_j](\lambda) = m_A[T_j](\lambda) - 1$. Hence, by Lemma 4.1, $m_A(\lambda) = m_A(v)(\lambda) - 1 = 1$, a contradiction. Therefore, $\lambda \notin S$, and by our construction there exists exactly one branch $T_j$ that has $\lambda$ as an eigenvalue, and by the same lemma we get a contradiction again. Hence, all the eigenvalues of $A$ are different, and therefore $r([T_1], \ldots, [T_k]) \leq e_T(v)$.

We now show that if Conjecture 4.3 holds for $T$ and all its subtrees, then the number $\sum e_{T_i}(u_i) + r([T_1] - e_{T_i}(u_1), \ldots, [T_k] - e_{T_i}(u_k))$ is attainable, and next we show that regardless of the conjecture, $e_{T_i}(u_i) + r([T_1] - e_{T_i}(u_1), \ldots, [T_k] - e_{T_i}(u_k))$ is an upper bound. In particular, we will prove that the maximal number of Type 2 interlacing equalities possible is given by $\sum e_{T_i}(u_i)$ and that attaining a maximal number of Type 2 interlacing equalities does indeed give us a maximal number of interlacing equalities.

Consider each label $e_{T_i}(u_i)$. Let $\lambda$ be an eigenvalue of the $e_{T_i}(u_i)$ interlacing equalities which are obtained in the optimal situation for $u_i$ in $T_i$. Either $u_i$ is Parter for $\lambda$ in $T_i$, or some other vertex $u_i'$ will be Parter for $\lambda$ in $T_i$ for two branches which do not contain $u_i$.

If $u_i$ is Parter, then the two relevant branches are clearly still branches of $T - u_i$ and do not contain $v$. Thus, any Type 1 interlacing equality that arises in the optimal situation for $u_i$ in $T_i$ becomes a Type 2 interlacing equality for $v$. If some other vertex $u_i'$ is Parter for $\lambda$ in $T_i$ for two branches which do not contain $u_i$, then the two branches will remain branches of $T - u_i'$ that do not contain $v$. Hence, the Type 2 interlacing equalities that arise in the optimal situation for $u_i$ in $T_i$ remain Type 2 interlacing equality for $v$. It follows that by allowing a submatrix for the optimal number of interlacing equalities for each $u_i$ in each $T_i$, we can get $\sum e_{T_i}(u_i)$ Type 2 interlacing equalities for $v$.

Therefore, each branch $T_i$ will have $|T_i| - e_{T_i}(u_i)$ remaining free eigenvalues. By Conjecture 4.3, we can make these be whatever we want, so we can get $r([T_1] - e_{T_1}(u_1), \ldots, [T_k] - e_{T_k}(u_k))$ Type 1 interlacing equalities for $v$. This proves that $\sum e_{T_i}(u_i) + r([T_1] - e_{T_i}(u_1), \ldots, [T_k] - e_{T_i}(u_k))$ is an attainable number of interlacing equalities for $v$.

We next show that one can get no more Type 2 interlacing equalities. Consider an eigenvalue $\lambda$ which is of a Type 2 interlacing equality for $v$ in $T$. It has some vertex, $v'$, which is Parter for $\lambda$, and is on some branch $T'$ of $T - v$ with vertex $u'$ adjacent to $v$. By Theorem 2.2, the two branches of $T - v'$ which have $\lambda$ as an eigenvalue do not contain $v$, and thus they will also be branches of $T' - v'$. There are two possibilities: Either $u' = v'$, or it does not. If it does, then we have a Type 1 interlacing equality for $u'$. If it does not, then since $u'$ and $v$ are adjacent, the two relevant branches of $T' - v'$
also do not contain $u'$. So in this case, we have a Type 2 interlacing equality for $u'$.

In either case, we recognize that a Type 2 interlacing equality for $v$ must also produce an interlacing equality for some adjacent vertex of $v$ within its branch. It follows that $\sum e_T(u_i)$ is an upper bound for the number of Type 2 interlacing equalities.

Last, we will show that the maximum number of interlacing equalities is obtained by maximizing Type 2 equalities first. It is clear that by using at most $\sum e_T(u_i) + 2 \cdot r([T_1] - e_T(u_1), \ldots, [T_k] - e_T(u_k))$ eigenvalues of $A[T - v]$ for interlacing equalities, $\sum e_T(u_i) + r([T_1] - e_T(u_1), \ldots, [T_k] - e_T(u_k))$ is indeed our maximum, since this maximizes the number of Type 2 interlacing equalities, and Type 2 equalities fix only a single eigenvalue (as opposed to two for Type 1). If we use more than $\sum e_T(u_i) + 2 \cdot r([T_1] - e_T(u_1), \ldots, [T_k] - e_T(u_k))$ of the eigenvalues of $A[T - v]$ for interlacing equalities, then we must have more Type 1 interlacing equalities. In such a case, however, we would have to have fewer than $\sum e_T(u_i)$ Type 2 interlacing equalities in order to free up extra eigenvalue slots to create more Type 1 interlacing equalities. This does not create more interlacing equalities overall, as each Type 2 interlacing equality which could be attained, but is not, only allows for at most a single additional Type 1 equality. Therefore, $\sum e_T(u_i) + r([T_1] - e_T(u_1), \ldots, [T_k] - e_T(u_k))$ is an upper bound for the labeling of $v$. \hfill \Box

Let us give an example of the labeling procedure given above. Let us denote by $T$ the graph in Figure 2 and denote by $v$ the central vertex in $T$. Denote by $u_1$ and $u_2$ the neighbors of $v$ in $T$. The graph $T - v$ consisted of two paths $T_1$ and $T_2$ of length 3 (that contains $u_1$ and $u_2$, respectively), and hence $e_T(u_1) = e_T(u_2) = 1$. Now, $r([T_1] - e_T(u_1), [T_2] - e_T(u_2)) = r(2, 2) = 2$, and hence we get that $e_T(u_1) + e_T(u_2) + r([T_1] - e_T(u_1), [T_2] - e_T(u_2)) = 4$ is an upper bound for $e_T(v)$. Since Conjecture 4.3 holds for paths, this upper bound is tight, and we get that $e_T(v) = 4$.

5. Some formulas. Our recursive formula, as well as our understanding of the types of interlacing equalities, allows us to now give explicit formulas for the labels in several simple types of trees. We define a star-like tree to be a tree with exactly one vertex whose degree is larger than 2. (We call such a vertex a high-degree vertex.) See Figure 3 for an example.

Let $T$ be a star-like tree with a high-degree vertex $v$. Furthermore, let $w$ be an arbitrary degree-two vertex of $T$, and let $y$ be an arbitrary pendant vertex (assuming they exist). We write the $k$ branches of $T - v$ as $T_1 \ldots T_k$, and the branches on which $w$ and $y$ are on as $T_w$ and $T_y$, respectively. Finally, we write the two branches of
$T - w$ as $T'$ and $T''$, and we let $T'$ be the branch which contains $v$. Then

\[ e_T(v) = r(|T_1|, \ldots, |T_k|), \]
\[ e_T(y) = r(|T_1|, \ldots, |T_{y-1}|, |T_{y+1}|, \ldots, |T_k|), \]
\[ r(|T'|, |T''|) \leq e_T(w) \leq r(|T_1|, \ldots, |T_{w-1}|, |T_{w+1}|, \ldots, |T_k|) \]
\[ + r(|T'| - r(|T_1|, \ldots, |T_{w-1}|, |T_{w+1}|, \ldots, |T_k|), |T''|). \]

These formulas arise by applying Theorem 4.4 together with some simple observations. First, by Lemma 2.4, the only Parter vertex available for Type 2 interlacing equalities is $v$. So $e_T(v)$ cannot be the result of any Type 2 interlacing equalities, and there is only one kind of Type 2 interlacing equalities for $w$ and $y$. Second, being a pendent vertex, $y$ cannot produce any Type 1 interlacing equalities. The equalities in the first two formulas follow from the fact that the lower bound and the upper bound given in Theorem 4.4 are equal.

We may also label double stars. A double star is a tree that has exactly two vertices of degree at least two (we call them the stars center); these two vertices are connected, and all other vertices are of degree one. See Figure 4 for an example.

Let $T$ be a double star, and denote by $v_1$ and $v_2$ its star centers. Denote by $k_1$ and $k_2$ the number of pendent vertices that attached to $v_1$ and $v_2$, respectively. Then, by simple application of the recursive formula and the fact that Conjecture 4.3 holds for paths, we have

\[ e_T(v_1) = \left\lfloor \frac{k_2}{2} \right\rfloor + r \left( k_2 + 1 - \left\lfloor \frac{k_2}{2} \right\rfloor, 1, 1, \ldots, 1 \right). \]

Similarly, the label of $v_2$ is

\[ e_T(v_2) = \left\lfloor \frac{k_1}{2} \right\rfloor + r \left( k_1 + 1 - \left\lfloor \frac{k_1}{2} \right\rfloor, 1, 1, \ldots, 1 \right). \]

Several other kinds of simple trees can also be explicitly labeled using our recursive labeling formula. One could give a formula for double star-like, for instance. Nevertheless, the notation quickly becomes cumbersome, and the recursive algorithm is always applicable to any particular case.

6. Further results about labeling.

6.1. Continuity. Given any two adjacent vertices $v$ and $w$ of a tree on fewer than ten vertices $T$, we note that $e_T(v)$ and $e_T(w)$ differ by no more than one. We call this property continuity. It can be shown that continuity does not hold in general. For instance, given a double star with one hundred pendent vertices coming off one vertex and two hundred coming off the other, the labels of the two central vertices differ by 25. Indeed, by varying $n$ pendent vertices to come off one vertex and $2n$ off the other, one can obtain a difference between labels of the two central vertices of any magnitude.

There are, however, several interesting cases in which continuity does hold. In this section, we discuss only graphs for which the upper bound in Theorem 4.4 is equal.
to \(e_T(v)\), or alternatively, graphs for which Conjecture 4.3 holds. (Here, we require Conjecture 4.3 to hold also for all the subgraphs.) For such graphs, pendent vertices and adjacent degree-two vertices always obey continuity. Both of these results follow naturally from our recursive labeling algorithm.

**Theorem 6.1.** Let \(T\) be a tree with a pendent vertex \(v\), and let \(\{v, w\} \in E(T)\). Then \(|e_T(v) - e_T(w)| \leq 1\).

**Proof.** Let \(\text{deg}(w) = k\) in \(T\), and let \(T_1, \ldots, T_k\) be the \(k\) branches of \(T - w\), such that \(T_k\) is the branch that includes only the vertex \(v\). Let \(u_i\) be the vertex adjacent to \(w\) on branch \(T_i\), \(1 \leq i \leq k\), and hence \(u_k = v\). Then

\[
e_T(w) = \sum_{i=1}^{k-1} e_{T_i}(u_i) + e_{T_k}(u_k) + r(|T_1| - e_{T_1}(u_1), \ldots, |T_k| - e_{T_k}(u_k)).
\]

Since \(u_k\) is a pendent vertex, \(e_{T_k}(u_k) = 0\) and \(|T_k| = 1\). So we have

\[
e_T(w) = \sum_{i=1}^{k-1} e_{T_i}(u_i) + r(|T_1| - e_{T_1}(u_1), \ldots, |T_{k-1}| - e_{T_{k-1}}(u_{k-1}), 1).
\]

Let \(T' = T - v\). By our recursive formula, the label \(e_T(v) = e_{T'}(w)\), and

\[
e_T(v) = \sum_{i=1}^{k-1} e_{T_i}(u_i) + r(|T_1| - e_{T_1}(u_1), \ldots, |T_{k-1}| - e_{T_{k-1}}(u_{k-1})).
\]

Using these explicit formulas as well as the properties of the \(r\) function, we conclude that \(|e_T(v) - e_T(w)| \leq 1\). \(\square\)

**Theorem 6.2.** Let \(T\) be a tree with two adjacent degree-two vertices \(v\) and \(w\). Then \(|e_T(v) - e_T(w)| \leq 1\).

**Proof.** Let \(\{u_2, w\}, \{u_1, v\} \in E(T)\), \(u_2 \neq v, u_1 \neq w\), let \(T_1\) be the branch of \(T - v\) containing \(u_1\), and let \(T_2\) be the branch of \(T - w\) containing \(u_2\).

By the recursive formula,

\[
e_T(v) = e_{T_1}(u_1) + e_{T_2 \cup w}(w) + r(|T_1| - e_{T_1}(u_1), |T_2| + 1 - e_{T_2 \cap w}(w)).
\]

Since \(w\) is a pendent vertex in \(T_2 \cup w\), \(e_{T_2 \cup w}(w) = e_{T_2}(u_2)\). Therefore,

\[
e_T(v) = e_{T_1}(u_1) + e_{T_2}(w) + r(|T_1| - e_{T_1}(u_1), |T_2| + 1 - e_{T_2}(u_2)).
\]

Similarly,

\[
e_T(w) = e_{T_1}(u_1) + e_{T_2}(w) + r(|T_1| + 1 - e_{T_1}(u_1), |T_2| - e_{T_2}(u_2)).
\]

Hence by definition of \(r\), \(|e_T(v) - e_T(w)| \leq 1\). \(\square\)

**6.2. \(M(T)\).** In this section, we discuss relative sizes of labelings. Clearly, the label of a vertex in a tree of order \(n\) cannot be more than \(n\). Now, in order to compare labelings of vertices in trees of different orders, it is natural to consider the parameter \(\frac{e_T(v)}{n}\), which can be viewed as the normalized labeling of a vertex in the graph. This parameter if bounded between 0 and 1. It is easy to find trees of any order for which \(\frac{e_T(v)}{n} = 0\) for some vertex \(v\). For example, one can take a path of order \(n\), in which \(v\) is a pendent vertex. How about the upper bound? Is it possible that the normalized label will be equal to 1, or at least limit to 1 in some infinite family of trees? In order
to discuss this problem, we first define the following two parameters:

\[ M(T) = \max_{v \in V(T)} \frac{e_T(v)}{n}, \]

\[ \hat{M} = \sup_T M(T), \]

where \( T \) runs over all the trees.

We can translate our question above to the following question: Is \( \hat{M} = 1 \)? In this section, we will show that \( \frac{1}{2} \leq \hat{M} \leq 1 \) and that if Conjecture 4.3 holds, then \( \hat{M} = 1 \). First, it is clear that \( \hat{M} \leq 1 \). Now, let us examine the sequence of trees \( \{T_i\} \) constructed recursively as follows. Let \( T_1 \) be the graph that has one vertex, and let us denote this vertex by \( v_1 \). Let \( T_2 \) be the graph obtained by taking two copies of \( T_1 \), adding a new vertex \( v_2 \), and adding an edge between \( v_2 \) and each one of the two vertices \( v_1 \) (that correspond to the two copies of \( T_1 \)). Similarly, in order to construct \( T_k \), we take two copies of \( T_{k-1} \), add a new vertex \( v_k \), and connect it to the two vertices \( v_{k-1} \). It is therefore apparent that the total number of vertices in \( T_k \) is \( 2^k - 1 \).

Figure 5 describes the graphs \( T_i \) for \( 1 \leq i \leq 4 \) including their labelings. Using induction and the recursive labeling formula, we can find a formula for the label of the vertex \( v_k \) for each \( T_k \), and we call this vertex the central vertex in \( T_k \).

**Theorem 6.3.** The label of the central vertex in \( T_k \) is bounded from below by \( 2^k - k - 1 \) and from above by \( 2^k - k - 1 \). If Conjecture 4.3 holds for \( T_k \) and all its subgraphs, then the label is exactly \( 2^k - k - 1 \).

**Proof.** We note that the statement is true for the first graph in the sequence: \( 2^1 - 1 - 1 = 0 \). By Theorem 4.4, \( 2^{k-1} - 1 = r(T_{k-1}, |T_{k-1}|) \leq e_{T_k}(v_k) \). We now prove the upper bound by induction. Assume that the theorem holds for the \( T_k \). We note that the two branches \( T_{k+1} - v_{k+1} \) are both equal to \( T_k \) and that the adjacent vertices to \( v_{k+1} \) in these two branches are \( v_k \) within their branches. So,

\[ e_{T_{k+1}}(v_{k+1}) = e_{T_k}(v_{k}) + r(2^k - 1 - e_{T_k}(v_k), 2^k - 1 - e_{T_k}(v_k)). \]

Substituting in, we get

\[ e_{T_{k+1}}(v_{k+1}) \leq 2(2^k - k - 1) + r(2^k - 1 - (2^k - k - 1), 2^k - 1 - (2^k - k - 1)), \]

which simplifies further to

\[ e_{T_{k+1}}(v_{k+1}) \leq 2^{k+1} - (k + 1) - 1. \]

The result follows by induction. \( \square \)
In conclusion, \( \frac{2^k-1}{2^k-1} \leq M(T_k) \leq \frac{2^k-k-1}{2^k-1} \), and hence \( \frac{1}{2} \leq \hat{M} \leq 1 \), and if Conjecture 4.3 holds for trees of the form \( T_k \) and their subtrees, then \( \hat{M} = 1 \).

6.3. Other kinds of counting. A natural question arises: What if we do not assume that the eigenvalues of our matrix \( A \) are distinct? How do the labels change in that case? In order to answer this question, we must first generalize our notion of an interlacing equality. There are two natural generalizations, which we now define.

Let \( v \in V(T) \), \( T \) being a tree. Define \( c_T(v) \) to be equal to the maximum total number of distinct eigenvalues, taken over all elements of \( \mathcal{H}(T) \), that appear both in \( \sigma(A) \) and \( \sigma(A[T-v]) \). We might consider this type of counting a conservative count. We note that when we restrict all eigenvalues to be distinct for \( A \), we have \( c_T(v) = e_T(v) \).

Let \( v \) be a vertex of a tree \( T \). Define \( l_T(v) \) to be equal to the maximum total number of eigenvalues, taken over all elements of \( \mathcal{H}(T) \), that appear both in \( \sigma(A) \) and \( \sigma(A[T-v]) \). We might refer to this type of counting as a liberal count. For instance, if \( \lambda \) is of multiplicity 2 and \( v \) is a neutral vertex for \( \lambda \), we count two interlacing equalities. Similarly, restricting all eigenvalues of \( A \) to be distinct implies \( l_T(v) = e_T(v) \).

We begin by noting, first, that our liberal count may not be very interesting. Consider, for example, the star on \( n \) vertices. By letting each of its pendent vertices have \( \lambda \) as an eigenvalue, one can guarantee \( n - 2 \) interlacing equalities. That a tree as simple as the star attains the maximum possible count suggests that this system of counting may not be very meaningful, though it may be that its behavior becomes more interesting with more complicated trees.

On the other hand, we can also examine our conservative count. Intuition tells us that the label \( e_T(v) \) may in fact be equal to \( c_T(v) \), for the introduction of a multiple eigenvalue seems to add only additional restrictions; when \( \lambda \) is a multiple eigenvalue, it will appear as an interlacing equality for \( v \) (by interlacing), so there must exist a Parter vertex somewhere which is Parter for two branches that do not contain \( v \), as well as a third branch which may or may not contain \( v \). This additional restriction appears to only constrain our ability to maximize interlacing equalities.

Appendix.

Fig. 6. Labeled trees with at most 5 vertices.
REFERENCES

[1] C. M. da Fonseca, Interlacing properties for Hermitian matrices whose graph is a given tree, SIAM J. Matrix Anal. Appl., 27 (2005), pp. 130–141.
[2] R. Horn and C. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[3] C. Johnson, A. Duarte, and C. Saiago, The Parter–Wiener theorem: Refinement and generalization, SIAM J. Matrix Anal. Appl., 2 (2003), pp. 352–361.
[4] C. Johnson, A. Duarte, and C. Saiago, Inverse eigenvalue problems and lists of multiplicities of eigenvalues for matrices whose graph is a tree: The case of generalized stars and double generalized stars, Linear Algebra Appl., 373 (2003) pp. 311–330.
[5] C. Johnson and A. Leal-Duarte, On the possible multiplicities of the eigenvalues of a Hermitian matrix whose graph is a tree, Linear Algebra Appl., 348 (2002) pp. 7–21.
[6] A. Leal-Duarte, Construction of acyclic matrices from spectral data, Linear Algebra Appl., 113 (1989) pp. 173–182.
[7] S. Parter, On the eigenvalues and eigenvectors of a class of matrices, J. Soc. Indust. Appl. Math., 8 (1960), pp. 376–388.
[8] G. Wiener, Spectral multiplicity and splitting results for a class of qualitative matrices, Linear Algebra Appl., 61 (1984), pp. 15–29.