\textbf{W-ALGEBRA CONSTRAINTS AND TOPOLOGICAL RECURSION FOR $A_N$-SINGULARITY}

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\textbf{Abstract.} We derive a Bouchard–Eynard type topological recursion for the total descendant potential of $A_N$-singularity. Our argument relies on a certain twisted representation of a Heisenberg Vertex Operator Algebra (VOA) constructed via the periods of $A_N$-singularity. In particular, our approach allows us to prove that the topological recursion for the total descendant potential is equivalent to a certain generating set of $W$-algebra constraints.

\section{Introduction}

Motivated by his work in Gromov–Witten theory, Givental has introduced the notion of a total descendant and a total ancestor potential (see \cite{7}). The definition makes sense for every conformal semi-simple Frobenius manifold. The main input is the so-called $R$-matrix and several copies of the Witten–Kontsevich $\tau$-function normalised in an appropriate way (see \cite{7,8}). On the other hand, it was proved by \cite{5} and \cite{11} that the total ancestor potential can be reconstructed only in terms of the $R$-matrix by using the \textit{local} Eynard–Orantin recursion. The main problem addressed in this paper is to find a topological recursion for the total descendant potential. The first step in solving this problem was suggested by Bouchard and Eynard in \cite{2}. Their construction was successfully applied to obtain a recursion for the total descendant potential of $A_N$-singularity in \cite{6} (see Section 7). In general however, the method of Bouchard and Eynard is not directly applicable, because the spectral curve is an infinite sheet covering, i.e., not a Riemann surface. In this paper we would like to suggest an approach based on the VOA construction of \cite{3}. We will focus on the case of $A_N$-singularity and hence we will recover Theorem 7.3 in \cite{6}. Furthermore, our approach allows us to compare the topological recursion and the $W$-constraints for the total descendant potential of $A_N$-singularity (see \cite{3}). More precisely, we prove that the so called \textit{dilaton shift} identifies the differential operators of the topological recursion with states in the $W$-algebra corresponding to the elementary symmetric polynomials. Constructing explicitly elements of the $W$-algebra is in general very difficult problem. It would be interesting to find out other examples in which the topological recursion can be used to construct generators of a $W$-algebra.

\subsection{Results.} Our main result will be stated entirely in terms of the root system of type $A_N$. The formulation in terms of vertex algebras requires a little bit more notation, so it will be given later on in Section \cite{3}. Let us fix the notation and recall the necessary background. Let $\mathfrak{h} \subset \mathbb{C}^{N+1}$ be the hyper-plane $\chi_1 + \cdots + \chi_{N+1} = 0$, where $\chi_i$ are the standard coordinate.

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functions on \( \mathbb{C}^{N+1} \). Recall that the root system of type \( A_N \) can be realised as
\[
\Delta = \{ \chi_i - \chi_j \mid 1 \leq i \neq j \leq N \} \subset h^*.
\]
The corresponding Weyl group is the symmetric group on \( N + 1 \) elements, while its action on \( h^* \) is induced from the standard action on \( (\mathbb{C}^{N+1})^* \) given by permuting \( \chi_1, \ldots, \chi_{N+1} \).
Furthermore, the unique \( W \)-invariant bilinear form \( (\mid) \) for which \( (\alpha|\alpha) = 2 \) for all \( \alpha \in \Delta \) is induced by the following bilinear form on \( (\mathbb{C}^{N+1})^* \):
\[
(\chi_i|\chi_j) = -\frac{1}{h} + \delta_{ij},
\]
where \( h := N + 1 \).

We define a set of differential operators on the infinitely many variables
\[
t = \{ t_{k,a} \}, \quad 1 \leq a \leq N, \quad k \geq 0.
\]
Sometimes it is convenient to rescale the above variables and to work with
\[
x_{k,a} = \frac{t_{k,a}}{(-a + h)(-a + 2h) \cdots (-a + kh)}, \quad 1 \leq a \leq N, \quad k \geq 0.
\]
First, we define a set of linear differential operators
\[
\Phi_a(\lambda) := \sum_{m=0}^{\infty} \left( \lambda^m x_{m,a} h^{-1/2} + \lambda^{-m-1} (a + mh) h^{1/2} \partial/\partial x_{m,h-a} \right), \quad 1 \leq a \leq h,
\]
where \( h \) is a formal parameter. Next we introduce the so called propagators
\[
P_{ij}(\lambda) := \frac{\eta^{j+i}}{(\eta^i - \eta^j)^2} \lambda^{-2}, \quad 1 \leq i \neq j \leq h,
\]
where \( \eta = e^{2\pi \sqrt{-1}/h} \). Finally, the differential operators that we need are
\[
X_j(\lambda) = \sum_{a=1}^{N} \eta^{-ja} \Phi_a(\lambda) \lambda^{-a/h}, \quad 1 \leq j \leq h
\]
and
\[
(1) \quad X_{j_1, \ldots, j_r}(\lambda) = \sum_{i_1, \ldots, i_{r'}} \left( \prod_{s=1}^{r'} P_{i_s}(\lambda) \right) : \prod_{j \in J \setminus I} X_j(\lambda) ;,
\]
where the sum is over all disjoint pairs \( i_s = (i_s^{(1)}, i_s^{(2)}) \), \( 1 \leq s \leq r' \), s.t.,
\[
1 \leq i_s^{(1)} < i_s^{(2)} \leq h, \quad i_1^{(1)} < \cdots < i_r^{(1)},
\]
we have used the notation
\[
I = \bigcup_{s=1}^{r'} \{ i_s^{(1)}, i_s^{(2)} \}, \quad J = \{ j_1, \ldots, j_r \}, \quad P_{i_s}(\lambda) = P_{i_s^{(1)}, i_s^{(2)}}(\lambda),
\]
and \( : \) is the normal ordering in which all differentiation operations are applied before the multiplication ones.

The total descendant potential is a formal series of the type
\[
\mathcal{D}(h; t) = \exp \left( \sum_{g=0}^{\infty} h^{g-1} \mathcal{F}_g(t) \right),
\]
where $F^{(g)}$ are formal power series in $t$. We refer to [8] for the precise definition. Let us define $\Omega_{j_1,\ldots,j_r}^{(g)}$ by the following identity

$$X_{j_1,\ldots,j_r} D(h; t) = \left( \sum_{g=0}^{\infty} \hbar^{g-r/2} \Omega_{j_1,\ldots,j_r}^{(g)}(\lambda; t) \right) D(h; t),$$

where $1 \leq j_1 < \cdots < j_r \leq h$.

**Theorem 1.1.** The following identity holds:

$$(-a + (m + 1)\hbar) \frac{\partial F^{(g)}}{\partial x_{m,a}} = -\text{Res}_{\lambda=0} \sum_{i=1}^{h} \sum_{j_1,\ldots,j_r} \frac{\eta^{-ia} \lambda^{m+1-\frac{1}{2}(a+r)}}{\prod_{s=1}^{r} (\eta^{s} - \eta^{j_s})} \Omega_{j_1,\ldots,j_r}^{(g)}(\lambda; t)d\lambda,$$

where the 2nd sum is over all non-empty subsets $\{j_1,\ldots,j_r\}$ of $\{1,\ldots,i-1,i+1,\ldots,h\}$.

It is not hard to see that if we give an appropriate weight to each variable $x_{k,i}$, so that the functions $F^{(g)}$ are homogeneous, then the identity in Theorem 1.1 will give us a recursion that uniquely determines $F^{(g)}$ for all $g \geq 0$.

1.2. **Genus-0.** Since the propagators do not contribute to genus 0, the genus-0 reduction of the identity in Theorem 1.1 takes a very simple form. Put

$$p_{m,a} = (-a + (m + 1)\hbar) \frac{\partial F^{(0)}}{\partial x_{m,a}}, \quad 1 \leq a \leq N, \quad m \geq 0,$$

and define the following numbers

$$\Phi_{a}^{(0)}(\lambda, t) := \sum_{m=0}^{\infty} \left( x_{m,a} \lambda^{m} + p_{m,h-a} \lambda^{-m-1} \right),$$

and define the following numbers

$$C(a_1,\ldots,a_r) = \sum_{1 \leq j_1 < \cdots < j_r \leq h-1} \frac{\eta^{-j_1a_1}}{1 - \eta^{j_1}} \cdots \frac{\eta^{-j_ra_r}}{1 - \eta^{j_r}}, \quad 1 \leq a_1,\ldots,a_r \leq N.$$

**Corollary 1.2.** The following identity holds

$$p_{m,a} = -\text{Res}_{\lambda=0} \sum_{a_1,\ldots,a_r=1}^{h-1} C(a_1,\ldots,a_r) \Phi_{a_0}^{(0)}(\lambda, t) \Phi_{a_1}^{(0)}(\lambda, t) \cdots \Phi_{a_r}^{(0)}(\lambda, t) \lambda^{m+n+1} d\lambda,$$

where the numbers $n \in \mathbb{Z}$ and $a_0, 0 \leq a_0 \leq h - 1$ are defined by

$$-(a + r + a_1 + \cdots + a_r) = nh + a_0$$

and if $a_0 = 0$ then we set $\Phi_{a_0}^{(0)} = 0$.

If we set $x_{0,a} := t_a$ and $x_{m,a} = 0$ for $m > 0$, then the identity in Corollary 1.2 allows us to compute the primary potential of the Frobenius structure.
1.3. $\mathcal{W}$-constraints. Recall that the vector space $\mathcal{F} := \text{Sym}(\mathfrak{h}[\zeta^{-1}][\zeta^{-1}])$ has the structure of a highest weight $\hat{\mathfrak{h}}$-module, where $\hat{\mathfrak{h}} := \mathfrak{h}[\zeta, \zeta^{-1}] \oplus \mathbb{C}$ is the Heisenberg Lie algebra with Lie bracket defined via the invariant bi-linear form $(|\ )$ (see Section 3). Following the construction in [3] we define a state-field correspondence $v \mapsto X(v)$, which to every $v \in \mathcal{F}$ associates a twisted field $X(v)$. The latter is a differential operator on a set of formal variables $q_{k,i}, 1 \leq i \leq N, k \geq 0$ whose coefficients are Laurent polynomials in $\lambda^{1/h}$. Let us point out that under the dilaton shift

$$t_{k,i} = q_{k,i} + \delta_{k,0} \delta_{i,N}, \quad 1 \leq i \leq N, \quad k \geq 0,$$

the differential operators

$$X_{j_1,\ldots,j_r}(\lambda) = X(\chi_{j_1} \cdots \chi_{j_r}, \lambda),$$

where $X(v, \lambda)$ denotes the value of $X(v)$ at the point $\lambda$ and we identify $\mathfrak{h} \subset \mathcal{F}$ via $a \mapsto a \zeta^{-1}$.

Let $e_r \in \text{Sym}(\mathfrak{h}), 2 \leq r \leq h$, be the degree-$r$ elementary symmetric polynomials in $\chi_1, \ldots, \chi_h$. Note that from the topological recursion in Theorem 1.1 we get a set of differential operators that annihilates the total descendant potential $D(\hbar; t)$.

**Theorem 1.3.** Under the dilaton shift (2) the set of differential constraints corresponding to the topological recursion turns into

$$\text{Res}_{\lambda=0} \lambda^m X(e_{h+1-a}, \lambda) D(\hbar; q) = 0, \quad 1 \leq a \leq N, \quad m \geq 0.$$

The proof of Theorem 1.3 will be reduced to a combinatorial identity, whose proof will be given in the Appendix. It is easy to check that all $e_r, 2 \leq r \leq h$, are in the kernel of the screening operators $e_{(\beta)}^\beta, \beta \in \Delta$. Therefore the main result in [3] and Theorem 1.3 give an alternative proof of Theorem 1.1. Let us point out that in general the invariant polynomials are not in the $\mathcal{W}$-algebra, so at least to the author, it is a little bit surprising that the elementary symmetric polynomials have this property.

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2. CONFORMAL FROBENIUS STRUCTURE

Let us recall the construction of a Frobenius structure on the space of miniversal unfolding of $A_N$-singularity (see [1, 9, 13]). Let

$$F(s, x) = \frac{x^{N+1}}{N+1} + s_1 x^{N-1} + \cdots + s_N$$

be a miniversal unfolding of singularity of type $A_N$. The deformation parameters are allowed to take arbitrary complex values, i.e.,

$$s = (s_1, \ldots, s_N) \in B := \mathbb{C}^N.$$
The space $B$ is equipped with a semi-simple Frobenius structure as follows. Using the so-called Kodaira–Spencer isomorphism

$$T_s B \cong \mathbb{C}[x]/(\partial_x F(s, x)), \quad \partial/\partial s_i \mapsto \partial_{s_i} F \quad (\text{mod } \partial_x F)$$

we can equip each tangent space $T_s B$ with a multiplication $\bullet_s$ and with a residue pairing

$$\left( \partial/\partial s_i, \partial/\partial s_j \right) = \frac{1}{2\pi i} \oint_C \frac{\partial_s F \partial_{s_j} F}{\partial_x F} dx,$$

where the contour of integration $C$ is a big loop enclosing the critical points of $F$. The main property of the above pairing and multiplication is that the family of connections

$$\nabla = \nabla^{LC} - z^{-1} \sum_{i=1}^{N} (\partial_{s_i} \bullet_s) ds_i$$

is flat. Here $z$ is a formal parameter, $\nabla^{LC}$ is the Levi–Civita connection of the residue pairing, and $\partial_{s_i} \bullet_s$ denotes the linear operator in $T_s B$ of multiplication by the tangent vector $\partial/\partial s_i$.

The flatness of $\nabla$ implies the flatness of $\nabla^{LC}$. We construct a trivialisation of the tangent and the cotangent bundle as follows. Let us denote by $H = \mathbb{C}[x]/x^N$ the local algebra of $F(0, x)$. Then we have the following identifications

$$T^* B \cong TB \cong B \times T_0 B \cong B \times H,$$

where the first isomorphism is given by the residue pairing, the second one uses the parallel transport with $\nabla^{LC}$, and the last one is the Kodaira–Spencer isomorphism. Let us choose a flat coordinate system $t = (t_1, \ldots, t_N)$, s.t., the point $t = 0$ corresponds to $s = 0$, and the vector fields $\partial/\partial t_i$ correspond to the basis $\phi_i(x) = x^{N-i}$ ($1 \leq i \leq N$) of $H$.

The connection (5) can be extended also in the $z$-direction

$$\nabla_{\partial/\partial z} = \frac{\partial}{\partial z} - \theta z^{-1} + (E \bullet) z^{-2},$$

where $\theta$ is the so-called Hodge grading operator and $E$ is the Euler vector field. Recall that via the Kodaira-Spencer isomorphism (3) $E$ corresponds to $F$. In flat coordinates we have

$$E = \sum_{i=1}^{N} (1 - d_i) t_i \partial/\partial t_i,$$

where $d_i = \deg(\phi_i) = (N - i)/(N + 1)$ ($1 \leq i \leq N$) is the so-called degree spectrum. The maximal degree $D = d_1 = (N - 1)/(N + 1)$ is called the conformal dimension of the Frobenius manifold. The operator $\theta = \frac{D}{2} - \deg$, i.e.,

$$\theta : H \to H, \quad \theta(\phi_i) = (D/2 - d_i) \phi_i = \left( -\frac{1}{2} + \frac{i}{N+1} \right) \phi_i.$$
2.1. The periods of $A_N$-singularity. Put $X = B \times \mathbb{C}$ and let

$$\varphi : X \to B \times \mathbb{C}, \quad \varphi(t, x) = (t, F(t, x)).$$

The non-singular fibers $X_{t,\lambda} := \varphi^{-1}(t, \lambda)$ form a smooth fibration called the Milnor fibration. Let us choose a solution to Dubrovin’s connection in the form $\Phi(\eta)$ where $\eta$ is a fundamental solution to $\nabla$, the transposition with respect to the residue pairing. The function $\Phi$ is obtained from $\nabla$ by $\varphi = dx$ and $d^{-1}\omega = x$ (this is a 0-form), the integration cycle $a_{t,\lambda}$ is obtained from $a$ after choosing a reference path in $(B \times \mathbb{C})'$ from $(0, 1)$ to $(t, \lambda)$ and using the parallel transport with respect to the corresponding Gauss–Manin connection. Finally, $d_t$ is the De Rham differential on $B$.

The period integrals are solutions to a connection $\nabla^{(n)}$, which is a Laplace transform of the Dubrovin’s connection

$$\nabla^{(n)}_{\partial_i} = \partial_i + \frac{\phi_i}{\lambda - E} \left( \theta - \frac{1}{2} - n \right), \quad 1 \leq i \leq N,$$

$$\nabla^{(n)}_{\partial_\lambda} = \partial_\lambda - \frac{1}{\lambda - E} \left( \theta - \frac{1}{2} - n \right).$$

The above system of equations can be solved in a neighbourhood of $\lambda = \infty$ in the following way. Let us choose a solution to Dubrovin’s connection in the form $\Phi(t, z) = S(t, z)z^\theta$, where $S(t, z) = 1 + S_1(t)z^{-1} + \cdots$ is an operator series whose coefficients $S_k(t) \in \text{End}(H)$. Such a solution is unique and it satisfies the symplectic condition $S(t, z)S(t, -z)^T = 1$, where $^T$ is transposition with respect to the residue pairing. The function

$$Y^{(n)}(t, \lambda) = S(t, -\partial_\lambda^{-1}) \frac{\lambda^{\theta-n-1/2}}{\Gamma(\theta - n + 1/2)}$$

is a fundamental solution to $\nabla^{(n)}$. Moreover, the reference point $(0, 1)$ is within the range of convergence (because $S(0, z) = 1$), therefore we can define an isomorphism $\mathfrak{h} \cong H$, s.t.,

$$I^{(n)}_a(t, \lambda) = Y^{(n)}(t, \lambda)a$$

for all $(t, \lambda)$ sufficiently close to $(0, 1)$.

2.2. Monodromy representation. Let us denote by $\Delta \subset H$ the set of vanishing cycles.

Lemma 2.1. The set of vanishing cycles $\Delta = \{ \chi_i - \chi_j \mid 1 \leq i \neq j \leq N + 1 \}$, where

$$\chi_i = \sum_{a=1}^{N} \eta^{-ia}(N + 1)^{-a/(N+1)}\Gamma\left(1 - \frac{a}{N+1}\right)\phi_{N+1-a},$$

where $\eta = e^{2\pi\sqrt{-1}/(N+1)}$. 

Proof. The fiber $X_{t,\lambda}$ consists of the zeroes $x_i(t, \lambda)$ $(1 \leq i \leq N+1)$ of the equation $F(s, x) = \lambda$. The vanishing cycles have the form $\alpha = [x_i(0, 1)] - [x_j(0, 1)]$, where $x_i(0, 1) = (N + 1)^{1/(N+1)}\eta^i$. By definition
\[
I^{(0)}_\alpha(t, \lambda) = -dt \int_{a(t, \lambda)} x = -dt(x_i(t, \lambda) - x_j(t, \lambda)).
\]
Furthermore,
\[
-dt \; x_i(t, \lambda) = \frac{\sum a=1^N x_i(t, \lambda)^{N-a}}{\partial F(t, x_i)} ds_a.
\]
On the other hand, note that the residue pairing has the form
\[
(\partial/\partial t_a, \partial/\partial t_b) = (x^{N-a}, x^{N-b}) = \delta_{a+b, N+1}.
\]
Therefore, at $t = 0$ we have
\[
I^{(0)}_\alpha(0, \lambda) = \sum a=1^N(x_i(0, \lambda)^{-a} - x_j(0, \lambda)^{-a}) ds_a
\]
and since at $t = 0$: $ds_a = dt_a = x^{a-1} = \phi_{N+1-a}$, we get

\[
\sum a=1^N(N+1)^{-a/(N+1)}(\eta^{-ia} - \eta^{-ja}) \lambda^{-a/(N+1)} \phi_{N+1-a} = Y^{(0)}(0, \lambda)(\chi_i - \chi_j). \quad \Box
\]

Using the above Lemma we can verify Saito’s formula for the intersection pairing (see [12]), i.e., the bi-linear form
\[
(a|b) := (I_a^{(0)}(t, \lambda), (\lambda - E\bullet)I_b^{(0)}(t, \lambda))
\]
coincides with the intersection pairing in $\tilde{H}_0(X_{0,1})$. The Picard–Lefschetz formula for the monodromy of the Gauss–Manin connection (see [1]) takes the form
\[
w_\alpha(y) = y - (\alpha|y)\alpha, \quad y \in H,
\]
where $w_\alpha$ is the image of the monodromy representation of $\nabla^{(n)}$
\[
\pi_1(B \times \mathbb{C})' \rightarrow GL(H)
\]
of a simple loop around the discriminant corresponding to a path along which the cycle vanishes. In particular, we get that the monodromy group is the symmetric group $S_{N+1}$ acting by permutation on the set $(\chi_1, \ldots, \chi_{N+1})$, while $w_\alpha$ for $\alpha = \chi_i - \chi_j$ is just the transposition swapping $\chi_i$ and $\chi_j$.

3. HEISENBERG VERTEX OPERATOR ALGEBRA

Let us denote by $\widehat{\mathfrak{h}}$ the Heisenberg Lie algebra $H[\zeta, \zeta^{-1}] \oplus \mathbb{C}$ with bracket
\[
[f(\zeta), g(\zeta)] = \text{Res}_{\zeta=0}(f'(\zeta)|g(\zeta))d\zeta.
\]
It is convenient to denote $a_{(n)} = a\zeta^n$ for $a \in H$ and $n \in \mathbb{Z}$. Then the above formula is equivalent to
\[
[a_{(m)}, b_{(n)}] = m(a|b)\delta_{m+n,0}, \quad m, n \in \mathbb{Z}.
\]
The vector space $\mathcal{F} = \text{Sym}(H[\zeta^{-1}]\zeta^{-1})$ has a natural structure of a highest-weight $\widehat{\mathfrak{h}}$-module, s.t., $a_{(n)}1 = 0$ for all $a \in H$ and $n \geq 0$. 
3.1. **The tame Fock space.** Given a commutative ring \(R\), let us denote by \(\hat{\mathcal{V}}_R\) the space of formal series of the form

\[
\sum_{g \in \mathbb{Z}} \sum_{K=((k_1,i_1),\ldots,(k_s,i_s))} c^{(g)}_{K,I} h^{g-1} t_{k_1,i_1} \cdots t_{k_s,i_s}, \quad c^{(g)}_{K,I} \in R,
\]

where the 2nd sum is over all lexicographically increasing sequences \(K\) of pairs \((k,i), k \geq 0, 1 \leq i \leq N\), i.e., either \(k_p < k_{p+1}\) or \(k_p = k_{p+1}\) and \(i_p \leq i_{p+1}\) if \(R = \mathbb{C}\), then we simply put \(\hat{\mathcal{V}} := \hat{\mathcal{V}}_\mathbb{C}\). Let us denote by \(\mathcal{V}_{\text{tame}} \subset \mathcal{V}\) the subspace of formal series satisfying the tameness condition: if \(c^{(g)}_{K,I} \neq 0\), then

\[
k_1 + \cdots + k_s \leq 3g - 3 + s.
\]

Let us denote by \(\mathcal{O}\) the algebra of holomorphic functions on the monodromy covering space of \((B \times \mathbb{C})'\). A twisted field on \((B \times \mathbb{C})\) is a \(\mathbb{C}\)-linear map \(\mathcal{V}_{\text{tame}} \to \hat{\mathcal{V}}_\mathcal{O}\). The space of all twisted fields will be denoted by \(\text{Hom}_\mathbb{C}(\mathcal{V}_{\text{tame}}, \hat{\mathcal{V}}_\mathcal{O})\).

3.2. **Twisted representation.** Following Givental \[8\], we introduce the symplectic vector space \(\mathcal{H} = H((z^{-1}))\) with the symplectic form

\[
\Omega(f(z), g(z)) = \text{Res}_{z=0}(f(-z), g(z))dz.
\]

Recall, also the following quantisation rules

\[
(\phi_i z^k)^\sim = -\hbar^{1/2} \partial_{k,i}, \quad (\phi_i^* (-z)^{-k-1})^\sim = \hbar^{1/2} t_{k,i},
\]

where \(\phi_i^* := \phi_{N+1-i}\) is the dual to \(\phi_i\) with respect to the residue pairing. These rules extend by linearity to define a representation of the Poisson Lie algebra of linear and constant functions on \(\mathcal{H}\). We define a *State-Field* correspondence

\[
X : \mathcal{F} \to \text{Hom}_\mathbb{C}(\mathcal{V}_{\text{tame}}, \hat{\mathcal{V}}_\mathcal{O})
\]

as follows

\[
X(a\varsigma^{-1}) := \phi_a(t, \lambda) := (\phi_a(t, \lambda; z))^\sim, \quad a \in H, \quad n \in \mathbb{Z}_{\geq 0},
\]

where

\[
\phi_a(t, \lambda; z) = \sum_{n \in \mathbb{Z}} f^{(n+1)}_a(t, \lambda) (-z)^n.
\]

For the remaining states the definition is such that

\[
X_t(a_{(-n-1)}v, \lambda) = \text{Res}_{\lambda'}=\lambda \left( X_t(a, \lambda')X_t(v, \lambda) \frac{d\lambda'}{\lambda'-\lambda} \right)_{n+1},
\]

where we denoted by \(X_t(v, \lambda)\) the value of the field \(X(v)\) at a point \((t, \lambda) \in (B \times \mathbb{C})'\).

More explicitly, if \(v = \alpha_{-k_1-1}^1 \cdots \alpha_{-k_r-1}^r \in \mathcal{F}\), then the field \(X(v)\) can be computed explicitly in terms of the generating fields \(\phi_\alpha(t, \lambda)\) and the so called *propagators*

\[
P_{\alpha,\beta}^{(k)}(t, \lambda) \in \mathcal{O}, \quad \alpha, \beta \in H, \quad k \in \mathbb{Z}_{\geq 0}
\]

defined by the Laurent series expansion

\[
\Omega(\phi_\alpha^+(t, \lambda_1; z), \phi_\beta(t, \lambda_2; z)) = \frac{(\alpha|\beta)}{(\lambda_1 - \lambda_2)^2} + \sum_{k=0}^{\infty} P_{\alpha,\beta}^{(k)}(t, \lambda_2)(\lambda_1 - \lambda_2)^k.
\]
The formula for the field $X_t(v, \lambda)$ is reminiscent of the Whick formula

$$X_t(v, \lambda) = \sum_J \left( \prod_{(i,j) \in J} \partial^{(k)}_{\lambda} \right) P_{\alpha', \alpha}^{(k)}(t, \lambda) \left( \prod_{l \in J'} \partial^{(k)}_{\lambda} X_{s, \lambda}(\alpha^l) \right),$$

where $\partial^{(k)}_{\lambda} := \frac{\partial^{(k)}}{\partial v}$ and the sum is over all collections $J$ of disjoint ordered pairs $(i_1, j_1), \ldots, (i_s, j_s) \subset \{1, \ldots, r\}$ such that $i_1 < \cdots < i_s$ and $i_t < j_t$ for all $t$, and $J' = \{1, \ldots, r\} \setminus \{i_1, \ldots, i_s, j_1, \ldots, j_s\}$.

It is proved in [10] that the analytic continuation of the propagators is compatible with the monodromy action on $\alpha$ and $\beta$. Moreover, we have the following explicit formulas

$$\Omega(\phi_\alpha^+(t, \lambda_1; z), f_\beta(t, \lambda_2; z)) = \frac{1}{\lambda_1 - \lambda_2} (I^{(0)}_{\alpha}(t, \lambda_1), (\lambda_2 - E \cdot I^{(0)}_{\beta}(t, \lambda_2))$$

and

$$P^{(0)}_{\alpha, \beta}(t, \lambda) = \frac{1}{2} ((\lambda - E \cdot I^{(1)}_{\alpha}(t, \lambda), I^{(1)}_{\beta}(t, \lambda)),$$

where

$$f_\beta(t, \lambda; z) = \sum_{n \in \mathbb{Z}} I^{(n)}_{\beta}(t, \lambda) (-z)^n.$$

The monodromy representation extends naturally to $\mathcal{F}$. It follows from formula (7) that the analytic continuation of $X_t(v, \lambda)$ in $(t, \lambda)$ is compatible (or equivalent) to the monodromy action on $v$.

3.3. Global recursion. If $(t, \lambda) \in (B \times \mathbb{C})'$ and $c^1, \ldots, c^r \in H$, then we define

$$\Omega^{(g)}_{c^1, \ldots, c^r}(t, \lambda; t) \in \mathbb{C}[t_0, t_1, t_2, \ldots]$$

by the following equation

$$X_t(c^1 \times \cdots \times c^r 1, \lambda) A_t(h; t) = \sum_{g=0}^{\infty} h^{g-\frac{1}{2}} \Omega^{(g)}_{c^1, \ldots, c^r}(t, \lambda; t) A_t(h; t).$$

Recall also, that the total ancestor potential has the form

$$A_t(h; t) = \exp \left( \sum_{n, g=0}^{\infty} \frac{h^{g-1}}{n!} \langle t(\psi), \ldots, t(\psi) \rangle_{g,n}(t) \right)$$

where $t(\psi) = \sum_{k=0}^{\infty} \sum_{a=1}^{N} t_{k,a} \phi_a \psi^k$ and the correlator has the form

$$\langle \phi_{a_1} \psi^{k_1}, \ldots, \phi_{a_n} \psi^{k_n} \rangle_{g,n}(t) = \int_{\mathcal{M}_{g,n}} \Lambda^t_{g,n}(\phi_{a_1}, \ldots, \phi_{a_n}) \psi_1^{k_1} \cdots \psi_n^{k_n},$$

where $\Lambda^t_{g,n} : H^{\otimes n} \to H^*(\mathcal{M}_{g,n}; \mathbb{C})$ is a certain Cohomological Field Theory defined through the Frobenius structure. According to the main result in [11], the total ancestor potential is uniquely determined by the following recursion:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle \phi_a \psi^m, \ldots, \psi \rangle_{g,n+1}(t) = \frac{1}{4} \sum_{i=1}^{N} \text{Res}_{\lambda=u_i} \frac{(f^{(m-1)}_{\beta_i}(t, \lambda), \phi_a)}{(f^{(1)}_{\beta_i}(t, 1), 1)} \Omega^{(g)}_{\beta_i, \beta_i}(t, \lambda; t) d\lambda,$$

where $u_i (1 \leq i \leq N)$ are the critical values of $F(t, x)$ and $\beta_i$ is a cycle vanishing over $\lambda = u_i$. 
Let $C$ be a loop that encloses all critical values. Motivated by the work of Bouchard and Eynard [2], we would like to compare the RHS of (8) with the following integral

$$\frac{1}{2\pi\sqrt{-1}} \oint_C \sum_{i=1}^{N+1} \sum_J (I_{x_i}^{(-m-1)}(t, \lambda), \phi_a) \prod_{j \in J} (I_{x_j}^{(-1)}(t, \lambda), 1) \Omega^{(g)}_{x_i, x_1, \ldots, x_r}(t, \lambda; t)d\lambda,$$

where the 2nd sum is over all non-empty subsets $J \subset \{1, \ldots, N+1\} \setminus \{i\}$ and $j_1, \ldots, j_r$ are the elements of $J$.

**Theorem 3.1.** The RHS of the local recursion (8) coincides with the integral (9).

**Proof.** The integral (9) can be evaluated with the residue theorem. It is a sum of the residues at the critical values. Let us verify that the residue at $\lambda = u_1$ coincides with the corresponding residue in the local recursion (8). Similar argument applies to the remaining critical values. Let us verify that the residue at $\lambda = u_1$. The terms with $r = 1$ contribute to the residue only if the set $J \cup \{i\}$ contains 1 or 2, otherwise $(\chi_i | \beta) = (\chi_j | \beta) = 0$ and the entire expression is analytic at $\lambda = u_1$. The 2 terms for which $i = 1$, $J = \{2\}$ and $i = 2$, $J = \{1\}$ add up to

$$- \text{Res}_{\lambda = u_1} \left( \frac{I_{\chi_1 - \chi_2}^{(-m-1)}(t, \lambda), \phi_a}{I_{\chi_1 - \chi_2}^{(-1)}(t, \lambda), 1} \right) \Omega^{(g)}_{\chi_1 \chi_2}(t, \lambda; t)d\lambda.$$

However, $-\Omega^{(g)}_{\chi_1 \chi_2} = \frac{1}{3}(\Omega^{(g)}_{\beta, \beta} - \Omega^{(g)}_{\chi_1 + \chi_2, \chi_1 + \chi_2})$ and since $(\chi_1 + \chi_2 | \beta) = 0$, the form $\Omega^{(g)}_{\chi_1 + \chi_2, \chi_1 + \chi_2}$ is analytic at $\lambda = u_1$, so it does not contribute to the residue. The above residue coincides with the residue contribution at $\lambda = u_1$ of (8).

We claim that the terms for which the set $J \cup \{i\}$ contains precisely one of the elements 1 or 2 cancel with the terms for which $J \cup \{i\}$ contains both 1 and 2. To avoid cumbersome notation put $\chi_i := \chi_i \zeta^{-1} \in F$ and

$$X_I(t, \lambda) := X_I(\chi_i \cdots \chi_i, 1, \lambda)$$

where $I = \{i_1, \ldots, i_s\} \subset \{1, 2, \ldots, N + 1\}$. Let us compute

$$- \text{Res}_{\lambda = u_1} \sum_{s=1}^{2} \sum_{i_1=1}^{N+1} \frac{(I_{\chi_i}^{(-m-1)}(t, \lambda), \phi_a)}{(I_{\chi_i}^{(-1)}(t, \lambda), 1)} X_{J \cup \{i\}}(t, \lambda) A_I(h; t).$$

We may replace $X_{J \cup \{i\}}(t, \lambda) A_I$ by

$$X_{(J \cup \{i\}) \setminus \{s\}}(t, \lambda) (\phi_{\chi_i}^+(t, \lambda; z))^\wedge A_I$$

because the remaining terms do not contribute to the residue. Note that

$$(\phi_{\chi_i}^+(t, \lambda; z))^\wedge A_I = -\hbar^{1/2} \sum_{g,n=0}^{\infty} \frac{h^{g-1}}{n!} \langle \phi_{\chi_i}^+(t, \lambda; \psi), t, \ldots, t \rangle_{g,n+1}.$$

Recalling the local recursion (8), the expression (11) is transformed into

$$\frac{1}{4} \hbar^{1/2} \sum_{k=1}^{N} \text{Res}_{\lambda = u_k} \Omega(\phi_{\chi_k}^+(t, \lambda; z), f_{\beta_k}(t, \lambda'; z)) \frac{(I_{\beta_k}^{(-1)}(t, \lambda'), 1)}{(I_{\beta_k}^{(0)}(t, \lambda), \lambda') X_{(J \cup \{i\}) \setminus \{s\}}(t, \lambda) X_I(\beta_k^2, \lambda') A_I(h; t)}.$$

On the other hand

$$\Omega(\phi_{\chi_k}^+(t, \lambda; z), f_{\beta_k}(t, \lambda'; z)) = \frac{1}{\lambda - \lambda'} (I_{\chi_k}^{(0)}(t, \lambda), (\lambda' - E \bullet) I_{\beta_k}^{(0)}(t, \lambda')) = \frac{(\chi_k | \beta_k)}{\lambda - \lambda'} + \cdots ,$$

which completes the proof.
where the dots stand for a term analytic at \( \lambda' = \lambda \). The sum \((10)\) turns into

\[
-\frac{1}{4} \hbar^{1/2} \sum_{k=1}^{N} \text{Res}_{\lambda = u_1} \text{Res}_{\lambda' = u_k} \sum_{s=1}^{2} \sum_{i=1}^{N+1} \frac{1}{\lambda - \lambda'} (\mathcal{I}_{\lambda^s}(t, \lambda), (\lambda' - E \bullet) \mathcal{I}^0_{\beta_k}(t, \lambda'))
\]

\[
\frac{(I_{x_1}^{(-m-1)}(t, \lambda), \phi_a)}{(I_{\beta_1}^{(-1)}(t, \lambda'), 1) \prod_{j \in J} (I_{x_j - \chi_j}^{(-1)}(t, \lambda), 1)} X_{(J \cup \{i\}) \setminus \{s\}}(t, \lambda) X_t(\beta_1^0, \lambda') A_t(h; t) d\lambda' d\lambda.
\]

Note that if we compute first the residue with respect to \( \lambda = u_1 \) we would get 0. Furthermore, the two residue operations commute unless \( k = 1 \). If \( k = 1 \), then

\[
\text{Res}_{\lambda = u_1} \text{Res}_{\lambda' = u_1} = \text{Res}_{\lambda' = u_1} \text{Res}_{\lambda = \lambda'}.
\]

Recalling the definition of the State-Field correspondence we get

\[
-\frac{1}{4} \hbar^{1/2} \text{Res}_{\lambda = u_1} \sum_{s=1}^{2} \sum_{i=1}^{N+1} (\chi_s | \beta_1) \times
\]

\[
\frac{(I_{x_1}^{(-m-1)}(t, \lambda), \phi_a)}{(I_{\beta_1}^{(-1)}(t, \lambda), 1) \prod_{j \in J} (I_{x_j - \chi_j}^{(-1)}(t, \lambda), 1)} X_{(J \cup \{i\}) \setminus \{s\}}(t, \lambda) X_t(\beta_1^0, \lambda') A_t(h; t) d\lambda,
\]

where \( j_1, \ldots, j_r \) are the elements of the set \( J' := (J \cup \{i\}) \setminus \{s\} \). Just like before we can replace \(-\frac{1}{4} \beta_1^2\) with \( \chi_1 \chi_2 \). Rearranging the sum so that the summation over \( J' \) is first we get

\[
\hbar^{1/2} \text{Res}_{\lambda = u_1} \sum_{J' = (j_1, \ldots, j_r)} \left( \sum_{s=1}^{2} \sum_{i=1}^{N+1} (\chi_s | \beta) \times
\right)
\]

\[
\frac{(I_{x_1}^{(-m-1)}(t, \lambda), \phi_a)}{(I_{x_1 - \chi_2}^{(-1)}(t, \lambda), 1) \prod_{j \in J'} (I_{x_j - \chi_j}^{(-1)}(t, \lambda), 1)} X_{J' \cup \{1, 2\}}(t, \lambda) A_t(h; t),
\]

where the outer sum is over all subsets \( J' \) that do not contain 1 and 2. Note that the sum over \( s \) and \( J' \) in the brackets yields

\[
\frac{(I_{x_1}^{(-m-1)}(t, \lambda), \phi_a)}{(I_{x_1 - \chi_2}^{(-1)}(t, \lambda), 1) \prod_{j \in J'} (I_{x_j - \chi_j}^{(-1)}(t, \lambda), 1)} - \frac{(I_{x_2}^{(-m-1)}(t, \lambda), \phi_a)}{(I_{x_2 - \chi_1}^{(-1)}(t, \lambda), 1) \prod_{j \in J'} (I_{x_j - \chi_j}^{(-1)}(t, \lambda), 1)} + \]

\[
\sum_{J' \cup \{i\}} \frac{1}{(I_{x_i - \chi_1}^{(-1)}(t, \lambda), 1)} - \frac{1}{(I_{x_i - \chi_2}^{(-1)}(t, \lambda), 1)} \frac{(I_{x_1}^{(-m-1)}(t, \lambda), \phi_a)}{(I_{x_1 - \chi_2}^{(-1)}(t, \lambda), 1) \prod_{j \in J' \cup \{i\}} (I_{x_j - \chi_j}^{(-1)}(t, \lambda), 1)}.
\]

The above sum is precisely

\[
\sum_{J' \cup \{i\} \cup \{1, 2\}} \frac{(I_{x_1}^{(-m-1)}(t, \lambda), \phi_a)}{\prod_{j \in J' \cup \{i\} \cup \{1, 2\}} (I_{x_j - \chi_j}^{(-1)}(t, \lambda), 1)}.
\]

Note that the sum over \( J \) and \( i \) of the terms in \((9)\) for which \( J \cup \{i\} \) contains precisely one of the elements 1 or 2 coincides with \((10)\). While the above argument shows that the sum \((10)\) cancels with the sum over \( J \) and \( i \) of the terms in \((9)\) for which \( J \cup \{i\} \) contains both 1 and 2. Therefore our claim follows and the proof of the Theorem is completed. \(\square\)
Therefore a straightforward computation gives

\[ A = \text{Example.} \]

3.4. Example. Let us use Corollary 1.2 to compute the primary genus-0 potential of the \( A_3 \)-singularity. Put \( p_a := p_{0,a}, \) \( x_{m,a} = 0 \) for \( m > 0, \) and \( t_a := x_{0,a} = t_{0,a}. \) Note that

\[ \Phi_a^{(0)} = t_a + p_{4-a} \lambda^{-1}. \]

The identities in Corollary 1.2 yield

\[
\begin{align*}
    p_3 &= t_1 t_3 + \frac{1}{2} t_2^2, \\
    p_2 &= 2 t_2 t_3 - (C(1, 1) + C(1, 2) + C(2, 1)) t_1^2 t_2, \\
    p_1 &= -t_1 p_3 + \frac{3}{2} t_2^2 - C(1, 1) t_1^2 t_3 - (C(1, 2) + C(2, 1) + C(2, 2)) t_1 t_2^2 \\
    & \quad - (C(1, 3) + C(3, 1)) t_1^2 t_3 - C(1, 1, 1) t_1^4.
\end{align*}
\]

A straightforward computation gives

\[
\begin{align*}
    C(1, 1) &= C(2, 2) = 0, \\
    C(1, 2) &= 1/2, \quad C(2, 1) = 1/2, \quad C(1, 3) = (\eta - 1)/2, \quad C(3, 1) = (-\eta - 1)/2, \\
    C(1, 1, 1) &= -1/4.
\end{align*}
\]

Therefore

\[
\begin{align*}
    p_3 &= t_1 t_3 + \frac{1}{2} t_2^2, \quad p_2 = 2 t_2 t_3 - t_1 t_2, \quad p_1 = -\frac{3}{2} t_1 t_2 + \frac{3}{2} t_3^2 + \frac{1}{4} t_1^4,
\end{align*}
\]

i.e.,

\[
\begin{align*}
    \frac{\partial F}{\partial t_3} &= t_1 t_3 + \frac{1}{2} t_2^2, \quad \frac{\partial F}{\partial t_2} = t_2 t_3 - \frac{1}{2} t_1^2 t_3, \quad \frac{\partial F}{\partial t_1} = -\frac{1}{2} t_1 t_2 + \frac{1}{2} t_3^2 + \frac{1}{12} t_1^4,
\end{align*}
\]

where \( F \) is the restriction of \( F^{(0)} \) to \( t_{0,a} = t_a, \) \( t_{m,a} = 0 \) for \( m > 0. \) Now it is easy to find that

\[ F(t_1, t_2, t_3) = \frac{1}{2} (t_1 t_2^2 + t_2 t_3^2) - \frac{1}{4} t_1^2 t_2 + \frac{1}{60} t_1^5. \]

4. The Topological Recursion and \( \mathcal{W} \)-Constraints

The goal in this section is to prove Theorem 1.3. Note that the differential operators corresponding to the topological recursion have the form

\[
\sum_{r=0}^{h-1} \text{Res}_{\lambda=0} \sum_{i=1}^{h} \sum_{\substack{1 \leq j_1 < \cdots < j_r \leq h \atop j_s \neq i}} \frac{(I_{\chi_i}^{(m-1)}(0, \lambda), \phi_a)}{\prod_{s=1}^{r} (I_{\chi_{j_s} - \chi_{j_s} - \chi_{j_s}}^{(1, 1)}(0, \lambda), 1)} h^{r-1/2} X_0(\chi_i \chi_{j_1} \cdots \chi_{j_r}, \lambda).
\]

Note that by definition if \( r = 0, \) then the product over \( s \) is 1 and the corresponding contribution to the sum is \( \partial_{t_{m,a}}. \) To avoid cumbersome notation we set \( X(v, \lambda) := X_0(v, \lambda). \) It is convenient to rewrite the above differential operator in terms of the cycles

\[ \gamma_a := h^{-a/h} \Gamma(1 - a/h) \phi_{h-a}, \quad 1 \leq a \leq N. \]

Note that \( \chi_i = \sum_{a=1}^{N} \eta^{-ia} \gamma_a \) and that

\[ I_{\gamma_a}^{(0)}(0, \lambda) = (h\lambda)^{-a/h} \phi_{h-a}. \]
We get
\begin{equation}
(I_{\chi_i - \chi_j}^{(-1)}(0, \lambda), 1) = (\eta^i - \eta^j) I_{\gamma N}^{(-1)}(0, \lambda), 1) = (\eta^i - \eta^j) (\hbar \lambda)^{1/\hbar}
\end{equation}
and
\begin{equation}
(I_{\gamma a}^{(-m-1)}(0, \lambda), \phi_a) = \frac{(\hbar \lambda)^{m+1-a/\hbar}}{(-a + \hbar) \cdots (-a + (m+1)\hbar)}.
\end{equation}

The differential operator (12) takes the form
\[\sum_{0 \leq r \leq h-1} \text{Res}_{\lambda=0} \ h d\lambda \frac{(I_{\gamma a}^{(-m-1)}(0, \lambda), \phi_a)}{(I_{\gamma N}^{(-1)}(0, \lambda), 1)^r} \ h^{(r-1)/2} \times \sum_{a_0, \ldots, a_r=1}^{\sum_{a_0}^{h-1}} \sum_{\sum_{i=1}^{h} \sum_{j_s \neq i}^{1 \leq j_1 < \cdots < j_r \leq h} \eta^{-r+a+a_0+a_1+\cdots+a_r} \left( \prod_{s=1}^{r} \frac{\eta^{-(j_s-i)a_s}}{1-\eta^{j_s-i}} \right) X(\gamma a_0 a_1 \cdots a_r, \lambda).
\]

Shifting the summation indexes $j_s \mapsto j_s + i$ and summing over $i$ we get
\begin{equation}
\sum_{r=0}^{h-1} \text{Res}_{\lambda=0} \ h d\lambda \frac{(I_{\gamma a}^{(-m-1)}(0, \lambda), \phi_a)}{(I_{\gamma N}^{(-1)}(0, \lambda), 1)^r} \ h^{(r-1)/2} \times \sum_{a_0, \ldots, a_r=1}^{\sum_{a_0}^{h-1}} C(a_1, \ldots, a_r) X(\gamma a_0 a_1 \cdots a_r, \lambda),
\end{equation}
where $a_0$ is such that $0 \leq a_0 \leq h - 1$, $r + a + a_0 + \cdots + a_r \equiv 0 \pmod{h}$, we assume that $\gamma a_0 = 0$ if $a_0 = 0$, and
\[C(a_1, \ldots, a_r) := \sum_{1 \leq j_1 < \cdots < j_r \leq h-1} \prod_{s=1}^{r} \frac{\eta^{-j_s a_s}}{1-\eta^{j_s}}.
\]

where for $r = 0$ the RHS is by definition 1. Since the differential operator $X(\gamma a_0 \cdots a_r, \lambda)$ is invariant under the permutations of $(a_0, \ldots, a_r)$ we can arrange the 2nd sum in (15) to be over all increasing sequences $a_0 \leq a_1 \leq \cdots \leq a_r$, i.e.,
\begin{equation}
\sum_{\sum_{1 \leq a_0 \leq \cdots \leq a_r \leq N}^{a_0}} C[a_0, \ldots, a_r] X(\gamma a_0 a_1 \cdots a_r, \lambda),
\end{equation}
where $'$ means that we allow only sequences $(a_0, \ldots, a_r)$ that satisfy the condition
\[r + a + a_0 + \cdots + a_r \equiv 0 \pmod{h}
\]
and the numbers $C[a_0, \ldots, a_r]$ are defined as follows. If $r = 0$, then we put $C[a_0] := 1$. Otherwise,
\begin{equation}
C[a_0, \ldots, a_r] := \sum_{i=0}^{r} \frac{1}{m_i} \text{SymC}(a_0, \ldots, \hat{a_i}, \ldots, a_r),
\end{equation}
where $m_i$ denotes the multiplicity of $a_i$ in the sequence $(a_0, \ldots, a_r)$ and SymC is the symmetrisation of $C$
\[\text{SymC}(b_1, \ldots, b_r) = \frac{1}{|\text{Aut}(b_1, \ldots, b_r)|} \sum_{\sigma \in S_r} C(a_{\sigma(1)}, \ldots, a_{\sigma(r)}).
\]

Let us fix a summand in the sum (16). The corresponding sequence has the form
\[(a_0, a_1, \ldots, a_r) = (b_1, \ldots, b_{r'}, N, \ldots, N), \quad b_i < N, \quad 1 \leq i \leq r'.
\]
Put $m = r + 1 - r'$. Since the dilaton shift is equivalent to shifting
\[ \gamma_a \mapsto \gamma_a + (\mathcal{I}_N^{(-1)}(0, \lambda), 1) h^{-1/2} \delta_{a,N}, \]
our summand is transformed into
\[
C[b_1, \ldots, b_{r'}, N, \ldots, N] \sum_{m'} \left( \frac{m}{m'} \right) X(b_1 \cdots b_{r'} N \cdots N, \lambda) (\mathcal{I}_N^{(-1)}(0, \lambda), 1)^{m-m'} h^{-(m-m')/2}.
\]
The key step now is the following identity.

**Lemma 4.1.** The following identity holds
\[
C[b_1, \ldots, b_r, N, \ldots, N] = (-1)^m \left( \sum_{i=1}^{r} b_i \bmod m \right) C[b_1, \ldots, b_r],
\]
where $[b]_h$ denotes the remainder of $b$ modulo $h$.

The proof will be given in the appendix. Using this Lemma we get
\[
C[b_1, \ldots, b_{r'}, N, \ldots, N] \left( \frac{m}{m'} \right) = C[b_1, \ldots, b_{r'}, N, \ldots, N] (-1)^{m-m'} \left( \sum_{i=1}^{r'} b_i \bmod m \right). 
\]
Note that in particular, the multiplicity $m$ of $N$ in the sequence $(a_0, \ldots, a_r)$ does not exceed $[\sum_{i=1}^{r'} b_i]_h$. The sum (16) can be written as follows:
\[
\sum_{r'=0}^{r+1} \sum_{1 \leq b_1 \leq \cdots \leq b_{r'} < N} \sum_{m'=0}^{r+1-r'} C[b_1, \ldots, b_{r'}, N, \ldots, N] X(b_1 \cdots b_{r'} N \cdots N, \lambda) \times
\]
\[
(-1)^{r+1-r'-m'} \left( \sum_{i=1}^{r'} b_i \bmod m \right) (\mathcal{I}_N^{(-1)}(0, \lambda), 1)^{r+1-r'-m'} h^{-(r+1-r'-m')/2},
\]
where the $'$ in the summation over $(b_1, \ldots, b_{r'})$ means that
\[ r' - 1 + b_1 + \cdots + b_{r'} + a \equiv 0 \pmod{h}. \]
Substituting the above expression in (15), changing the summation index $r$ via $s = r + 1 - r' - m'$, and changing the order of the summation we get
\[
\sum_{r'=0}^{h} \sum_{1 \leq b_1 \leq \cdots \leq b_{r'} < N} \sum_{m'=0}^{r+1-r'} \text{Res}_{\lambda=0} \, h d\lambda \left( \frac{(\mathcal{I}_N^{(-m-1)}(0, \lambda), \phi_a)}{(\mathcal{I}_N^{(-1)}(0, \lambda), 1)^{r+1-m'} h^{-1+(r'+m')/2} \times
\]
\[
C[b_1, \ldots, b_{r'}, N, \ldots, N] X(b_1 \cdots b_{r'} N \cdots N, \lambda) \times
\]
\[
\left( \sum_{i=1}^{r'} b_i \bmod m' \right) (-1)^s \left( \sum_{i=1}^{r'} b_i \bmod m' \right),
\]
Note that the sum over $s$ on the 3rd line of the above formula is 0 unless $m' = \sum b_i$. Note also that $r' + m' \leq h$, otherwise the 2nd line of the formula vanishes. Recalling (18) we get that $r' + m' = h + 1 - a$. Using formulas (13) and (14) we get
\[
hd\lambda \left( \frac{(\mathcal{I}_a^{(-m-1)}(0, \lambda), \phi_a)}{(\mathcal{I}_N^{(-1)}(0, \lambda), 1)^{r'+m'-1} h^{-1+(r'+m')/2} \times
\]
\[
\text{const} \, h^{(h-a-1)/2} \, d\lambda \lambda^m,
\]
where the value of the constant is not important. We get that up to a constant the dilaton shift transforms the differential operator (12) into

$$\text{Res}_{\lambda=0} d\lambda \lambda^m \sum_{1 \leq b_1 \leq \ldots \leq b_{h+1-a} \leq N}^t C[b_1, \ldots, b_{h+1-a}]X(\gamma_{b_1} \ldots \gamma_{b_{h+1-a}}, \lambda),$$

where the $^t$ indicates that the sum is over $(b_1, \ldots, b_{h+1-a})$, s.t., $\sum b_i \equiv 0 \text{ (mod } h)$. Put $r = h + 1 - a$. We claim that

$$\sum_{1 \leq b_1 \leq \ldots \leq b_r \leq N}^t C[b_1, \ldots, b_r] \gamma_{b_1} \ldots \gamma_r$$

coincides with the elementary symmetric polynomial in $\chi_1, \ldots, \chi_h$ of degree $r$. Similarly to what we did in the beginning of this Section we can rewrite the above sum as

$$\sum_{1 \leq i_1 < \ldots < i_r \leq h} \left( \sum_{s=1}^{r} \eta^{i_s(r-1)} \prod_{t \neq s} (\eta^{i_s} - \eta^{i_t}) \right) \chi_{i_1} \ldots \chi_{i_r}.$$

The coefficient in front of $\chi_{i_1} \cdots \chi_{i_r}$ is 1, because if we introduce the Vandermonde matrix $A_{s,t} := \eta^{(s-1)i_t}, 1 \leq s, t \leq r$, then the sum in the brackets can be interpreted as the quotient of the expansion of $\det(A)$ with respect to the last row and

$$\det(A) = \prod_{1 \leq s < t \leq r} (\eta^{i_s} - \eta^{i_t}).$$

**Appendix A. Proof of Lemma 4.1**

by D. Lewanski

Recall the definition of the numbers

$$C[a_1, \ldots, a_r], \quad 1 \leq r \leq h, \quad 1 \leq a_i \leq N$$
given by formula (17). It is convenient to extend the above definition by setting $C[a_1, \ldots, a_r] = 0$ for $r > h$.

**Lemma A.1.** Let $r \geq 1$ and $1 \leq a_1 \leq \cdots \leq a_r \leq N - 1$ be an arbitrary sequence. The following identity holds:

$$\sum_{m=0}^{\infty} \text{Sym} C[a_1, \ldots, a_r, N, \ldots, N](1 - Y)^m =$$

$$= \frac{1 - Y^h}{h(1 - Y)} \sum_{k_1, \ldots, k_r = 0}^{\infty} \sum_{|I| = r} \left( \prod_{j=1}^{r} \eta^{-i_j(a_j - k_j)} \right) Y^{\sum_{i=1}^{r} k_i},$$

where the 2nd sum on the RHS is over all sequences $I = (i_1, \ldots, i_r)$ of pairwise different numbers.
Proof. Let us use the notation $I \subset \{1,2,\ldots,N\}$ to denote that $I$ is a sequence $(i_1,\ldots,i_r)$ of pairwise distinct numbers, while $I \subset (1,2,\ldots,N)$ is a subsequence, i.e., a sequence of increasing numbers $i_1 < \cdots < i_r$. Recalling the definition of $\text{SymC}$ we get

$$
\sum_{m=0}^{\infty} \text{SymC}[a_1,\ldots,a_r,N,\ldots,N](1-Y)^m = 
\sum_{m=0}^{\infty} \sum_{I \subset \{1,\ldots,N\}} \prod_{s=1}^{r} \frac{\eta^{i_s}}{1-\eta^{i_s}} \sum_{J \subset (1,\ldots,N) \setminus I} \prod_{t=1}^{m} \frac{\eta^{j_t}}{1-\eta^{j_t}} (1-Y)^m
$$

where $\zeta = \eta^{-1}$. Observe that for the function

$$f_I(x) := \prod_{i \in \{1,\ldots,N\} \setminus I} (x - \zeta^i) = \frac{x^h - 1}{x - 1} \prod_{i \in I} \frac{1}{x - \zeta^i}
$$

we have

$$
\frac{1}{m!} \left. \frac{\partial^m f_I(Y)}{f_I(Y)} \right|_{Y=1} = \sum_{J \subset (1,\ldots,N) \setminus I} \prod_{t=1}^{m} \frac{1}{1-\zeta^j}
$$

contracting the Taylor expansion the initial term is:

$$
\sum_{I \subset \{1,\ldots,N\}} \prod_{s=1}^{r} \frac{-\zeta^{i_s}(a_s+1)}{1-\zeta^{i_s}} \frac{f_I(Y)}{f_I(1)} = \frac{Y^h - 1}{h(Y - 1)} \sum_{I \subset \{1,\ldots,N\}} \prod_{s=1}^{r} \frac{-\zeta^{i_s}(a_s+1)}{Y - \zeta^{i_s}}
$$

Substituting back $\eta = \zeta^{-1}$ and expanding in geometric power series in the variables $Y \eta^{i_s}$ proves the lemma.

□

The statement in Lemma 4.1 is equivalent to the following identity.

Lemma A.2. We have

$$
\sum_{m=0}^{\infty} \text{C}[a_1,\ldots,a_r,N,\ldots,N](1-Y)^m = Y^{[\sum_{i=1}^{r} a_i]_h} C[a_1,\ldots,a_r],
$$

where $[a]_h$ denotes the remainder of $a$ modulo $h$. 

Proof. By definition
\[
\sum_{m=0}^{\infty} C[a_1, \ldots, a_r, N, \ldots, N] (1 - Y)^m = (1 - Y) \sum_{m=0}^{\infty} \text{Sym}C[a_1, \ldots, a_r, N, \ldots, N] (1 - Y)^m + \sum_{i=1}^{r} \text{Sym}C[a_1, \ldots, \hat{a}_i, \ldots, a_r, N, \ldots, N] (1 - Y)^m.
\]
Let us substitute Equation (19) in the right hand side: the factor \((1 - Y)^{-1}\) cancels out in the first summand, while in the \(i\)-th summand can be expanded as \(\sum_{k_i=0}^{\infty} \eta^{-0(a_i-k_i)} Y^{k_i}\). Thus the first summand collects all the subsets of \(\{0, \ldots, N\}\) of cardinality \(r\) not containing zero while the second summand collects all the subsets containing zero with the same cardinality \(r\). Hence we get
\[
\frac{(1 - Y^h)}{h} \sum_{k_1, \ldots, k_r=0}^{\infty} \sum_{|I|=r}^{\infty} \prod_{j=1}^{r} \eta^{-i_j(a_j-k_j)} Y^{\sum k_i}
\]
Now the set \(\{0, 1, \ldots N\}\) is symmetric with respect to the shift \(i_j \mapsto i_j + 1\) simultaneously for all \(j\). This implies \(\eta^{-\sum(a_j-k_j)} = 1\), hence \(\sum k_i = [\sum a_i]_h + hl\), for \(l \in \mathbb{Z}_{\geq 0}\). The initial term can now be expanded in powers of \(Y\) as
\[
Y^{[\sum a_i]_h} (1 - Y^h)^{\frac{1}{h}} \sum_{l=0}^{\infty} c_l (Y^h)^l
\]
Since the expression is polynomial in \(Y\), we should have \(c_l = c_{l+1} = c\) for all indexes \(l \geq 0\). We showed:
\[
\sum_{m=0}^{\infty} C[a_1, \ldots, a_r, N, \ldots, N] (1 - Y)^m = Y^{[\sum a_i]_h} \frac{c}{h}
\]
Now evaluating at \(Y = 1\) gives \(c/h = C[a_1, \ldots, a_r]\) as desired.

\[\square\]

References

[1] V. Arnold, S. Gusein-Zade, A. Varchenko. *Singularities of Differentiable maps*. Vol. II. Monodromy and Asymptotics of Integrals. Boston, MA: Birkhäuser Boston, 1988. viii+492 pp

[2] V. Bouchard and B. Eynard. *Think globally, compute locally*. J. of High Energy Phys. (2013), no. 2, Article 143.

[3] B. Bakalov, T. Milanov. *W-constraints for the total descendant potential of a simple singularity*. Compositio Math. 149 (2013), no. 5, 840–888.

[4] B. Dubrovin. *Geometry of 2d Topological Field Theories*. Integrable Systems and Quantum Groups. Lecture Notes in Math. 1620: Springer, Berlin(1996): 120-148.

[5] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz. *Identification of the Givental formula with the spectral curve topological recursion procedure*. Comm. in Math. Phys. 328 (2014), no. 2, 669–700.

[6] P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, and S. Shadrin. *Dubrovin’s superpotential as a global spectral curve*. arXiv: 1509.06954.
[7] A. Givental. *Semisimple Frobenius structures at higher genus*. Internat. Math. Res. Notices 2001, no. 23, 1265-1286.

[8] A. Givental. *Gromov-Witten invariants and quantization of quadratic Hamiltonians*. Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. Mosc. Math. J. 1 (2001), no. 4, 551-568, 645.

[9] C. Hertling. *Frobenius Manifolds and Moduli Spaces for Singularities*. Cambridge Tracts in Mathematics, 151. Cambridge University Press, Cambridge, 2002. x+270 pp.

[10] T. Milanov. *The phase factors in Singularity theory*. arXiv: 1502.07444.

[11] T. Milanov. *The Eynard–Orantin recursion for the total ancestor potential*. Duke Math. J. 163 (2014), no. 9, 1795–1824.

[12] K. Saito. *Primitive forms for a universal unfolding of a function with an isolated critical point*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), no. 3, 775-792 (1982).

[13] K. Saito and A. Takahashi. *From primitive forms to Frobenius manifolds*. From Hodge theory to integrability and TQFT tt*-geometry, 31-48, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008.

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