1. INTRODUCTION

We present combinatorial rules (one theorem and two conjectures) concerning three bases of \( \text{Pol} = \mathbb{Z}[x_1, x_2, \ldots] \).

Consider a basic question (studied for example in [L13+]):

How does one lift properties of the ring \( \Lambda \) of symmetric functions (and its Schur basis) to the entirety of \( \text{Pol} \)?

The bases below lift the Schur polynomials. However, one wishes to analogize the relationship in \( \Lambda \) between rules for Schur polynomials and Littlewood-Richardson rules. For these bases, no rule has yet provided a parallel, explaining a desire for alternative forms.

First, we prove a “splitting” rule for the basis of key polynomials \( \{ \kappa_{\alpha} | \alpha \in \mathbb{Z}_{\geq 0}^\infty \} \), thereby establishing a new positivity theorem about them. This family was introduced by [D74] and first studied combinatorially in [LS89, LS90]. Combinatorial rules for their monomial expansion are known, see, e.g., [LS89, LS90, RS95, HHL09]. Our rule refines [RS95, Theorem 5(1)] and is compatible with the splitting rule [BKTY04, Corollary 3] for the basis of Schubert polynomials \( \{ S_w | w \in S_\infty \} \).

Second, we investigate a basis \( \{ \Omega_{\alpha} | \alpha \in \mathbb{Z}_{\geq 0}^\infty \} \) defined by [L01] that deforms the key basis. By extending the Kohnert moves of [K90] we conjecturally give the first combinatorial rule for the \( \Omega \)-polynomials.

Third, in [K90], the Kohnert moves were used to conjecture the first combinatorial rule for Schubert polynomials (a proof was later presented in [W03]). Similarly, we use the extended Kohnert moves to give a conjecture for the basis of Grothendieck polynomials \( \{ G_w | w \in S_\infty \} \) [LS82]. This rule appears significantly different than earlier (proved) rules, such as those in [FK94, L01, BKTY05, LRS06].

1.1. Splitting key polynomials. Let \( S_\infty \) be the group of permutations of \( \mathbb{N} \) with finitely many non-fixed points. This acts on \( \text{Pol} \) by permuting the variables. Let \( s_i \) be the simple transposition interchanging \( x_i \) and \( x_{i+1} \). The divided difference operator acts on \( \text{Pol} \) by

\[
\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}.
\]

Define the Demazure operator by setting

\[
\pi_i(f) = \partial_i(x_i \cdot f), \text{ for } f \in \text{Pol}.
\]

For \( \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_{\geq 0}^\infty \), the key polynomial \( \kappa_\alpha \) is

\[
\kappa_\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots, \text{ if } \alpha \text{ is weakly decreasing.}
\]
Otherwise, \( \kappa_\alpha = \pi_i(\kappa_{\tilde{\alpha}}) \) where \( \tilde{\alpha} = (\ldots, \alpha_{i+1}, \alpha_i, \ldots) \) and \( \alpha_{i+1} > \alpha_i \).

Since the leading term of \( \kappa_\alpha \) is \( x_1^{\alpha_1} x_2^{\alpha_2} \cdots \), the key polynomials form a \( \mathbb{Z} \)-basis of \( \text{Pol} \).

The key polynomials lift the Schur polynomials: when

(1) \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t, 0, 0, 0, \ldots) \), where \( \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_t \), then

(2) \( \kappa_\alpha = s_{(\alpha_t, \ldots, \alpha_2, \alpha_1)}(x_1, \ldots, x_t) \).

A descent of \( \alpha \) is an index \( i \) such that \( \alpha_i \geq \alpha_{i+1} \); a strict descent is an index \( i \) such that \( \alpha_i > \alpha_{i+1} \). Fix descents \( d_1 < d_2 < \ldots < d_k \) of \( \alpha \) containing all strict descents of \( \alpha \). Since \( \pi_i \) symmetrizes \( \{x_i, x_{i+1}\} \), \( \kappa_\alpha \) is separately symmetric in each collection:

\[
X_1 = \{x_1, x_2, \ldots, x_{d_1}\}, \quad X_2 = \{x_{d_1+1}, x_{d_1+2}, \ldots, x_{d_2}\}, \ldots, \quad X_k = \{x_{d_{k-1}+1}, x_{d_{k-1}+2}, \ldots, x_{d_k}\}.
\]

(The variables \( x_{d_k+1}, x_{d_k+2}, \ldots \) do not appear in \( \kappa_\alpha \).) Therefore, uniquely:

(3) \( \kappa_\alpha(X) = \sum_{\lambda^1, \ldots, \lambda^k} \mathcal{E}^\alpha_{\lambda^1, \ldots, \lambda^k} s_{\lambda^1}(X_1) \cdots s_{\lambda^k}(X_k) \),

where each \( \lambda^i \) is a partition. A priori one only knows \( \mathcal{E}^\alpha_{\lambda^1, \ldots, \lambda^k} \in \mathbb{Z} \).

Given \( \alpha \in \mathbb{Z}_{\geq 0}^\infty \), there is a unique \( w[\alpha] \in S_\infty \) such that \( \text{code}(w[\alpha]) = \alpha \) (see, e.g., [M01, Proposition 2.1.2]). Here \( \text{code}(w[\alpha]) \in \mathbb{Z}_{\geq 0}^\infty \) counts the number of boxes in columns of \( \text{Rothe}(w[\alpha]) \). We will need a special tableau coming from [S84, Section 4]:

The tableau \( T[\alpha] \): Given \( w[\alpha] \), \( i_1 < i_2 < \ldots < i_a \) in the first column of \( T[\alpha] \) are given by having \( i_j \) be the largest descent position smaller than \( i_{j+1} \) in the permutation \( w_{s_i_{a} \cdots s_i_{i-1} s_{i+1}} \).

The next column of \( T[\alpha] \) is similarly determined, starting from \( w_{s_i_{a} \cdots s_i_{i-1} s_{i+1}} \), etc.

An increasing tableau \( T \) of shape \( \lambda \) is a filling with strictly increasing rows and columns. (In fact, \( T[\alpha] \) is an increasing tableau.) Let \( \text{row}(T) \) be the reading word of \( T \), obtained by reading the entries of \( T \) along rows, from right to left, and from top to bottom. Let \( \min(T) \) be the smallest label in \( T \). Finally, given a reduced word \( a = a_1 a_2 \ldots a_m \), let \( \text{EGLS}(a) \) be the output of the Edelman-Greene correspondence (see Section 2.1).

The following result shows \( \mathcal{E}^\alpha_{\lambda^1, \ldots, \lambda^k} \in \mathbb{Z}_{\geq 0} \). It is analogous to one on Schubert polynomials [BKY04, Corollary 3] (which our proof uses).

**Theorem 1.1.** The number \( \mathcal{E}^\alpha_{\lambda^1, \ldots, \lambda^k} \) counts sequences of increasing tableaux \( (T_1, T_2, \ldots, T_k) \) where

- \( T_i \) is of shape \( \lambda^i \);
- \( \min(T_1) > 0, \min(T_2) > d_1, \min(T_3) > d_2, \ldots, \min(T_k) > d_{k-1} \); and
- \( \text{row}(T_1) \cdot \text{row}(T_2) \cdots \text{row}(T_k) \) is a reduced word of \( w[\alpha] \) such that \( \text{EGLS}(\text{row}(T_1) \cdot \text{row}(T_2) \cdots \text{row}(T_k)) = T[\alpha] \).

When \( d_j = j \) for all \( j \geq 1 \), Theorem 1.1 specializes to an instance of the monomial expansion formula [RS95, Theorem 5(1)] for \( \kappa_\alpha \) (restated as Theorem 2.5 below). Also, when (1) holds, \( k = 1, d_1 = t \) and thus Theorem 1.1 gives (2).

**Example 1.2.** The (strict) descents of \( \alpha = (1, 3, 0, 2, 2, 1) \) are \( d_1 = 2, d_2 = 5 \), and

\[
\kappa_{1,3,0,2,2,1} = s_{3,2}(x_1, x_2)s_{2,1,1}(x_3, x_4, x_5) + s_{3,2}(x_1, x_2)s_{2,1}(x_3, x_4, x_5)s_1(x_6)
+ s_{3,1}(x_1, x_2)s_{2,2}(x_3, x_4, x_5)s_1(x_6) + s_{3,1}(x_1, x_2)s_{2,2,1}(x_3, x_4, x_5).
\]

exhibits the claimed non-negativity of Theorem 1.1.
Also, \( w[\alpha] = 2516743 \) (one line notation) and \( T[\alpha] = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 \\ 4 & 6 \\ 5 & 6 \end{pmatrix} \). Thus, \( \mathcal{E}^{(1,3,0,2,2,1)} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 \\ 4 & 6 \\ 5 & 6 \end{pmatrix} \).

\[
\mathcal{E}^{(1,3,0,2,2,1)}(3,2,11,1,1) = \mathcal{E}^{(1,3,0,2,2,1)}(3,1,22,1,1) = \mathcal{E}^{(1,3,0,2,2,1)}(3,1,22,1,1,1) = 1 \text{ are respectively witnessed by } \\
\begin{pmatrix} 1 & 3 & 4 & 6 & \emptyset \\ 2 & 5 & 5 & 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 4 & 6 & 6 \\ 2 & 5 & 5 & 6 \\ 6 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 3 & 4 & 6 & \emptyset \\ 2 & 5 & 5 & 6 \\ 6 \end{pmatrix}.
\]

For example, for the leftmost sequence, \( \text{EGLS}(43152 \cdot 6456 \cdot \emptyset) = T[\alpha] \) holds. \( \square \)

1.2. The \( \Omega \) polynomials. \( \)A. Lascoux \[ L01 \] defines \( \Omega_\alpha \) for \( \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_{\geq 0}^\infty \) by replacing \( \pi_i \) in the definition of the key polynomials with the operator defined by

\[ \tilde{\pi}_i(f) = \partial_i(x_i(1 - x_{i+1})f). \]

The initial condition is \( \Omega_\alpha = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} \cdots (= \kappa_\alpha) \), if \( \alpha \) is weakly decreasing.

The skyline diagram is \( \text{Skyline}(\alpha) = \{(i, y) : 1 \leq y \leq \alpha_i \} \subset \mathbb{N}^2 \). Graphically, it is a collection of columns \( \alpha_i \) high. For instance,

\[
\text{Skyline}(1, 3, 0, 2, 2, 1) = \begin{pmatrix} . & + & . & \cdots & . \\ . & + & . & + & . \\ . & + & . & + & + \end{pmatrix}
\]

Beginning with \( \text{Skyline}(\alpha) \), Kohnert’s rule \[ K90 \] generates diagrams \( D \) by sequentially moving any + at the top of its column to the rightmost open position in its row and to its left. (The result of such a move need not be the skyline of any \( \gamma \in \mathbb{Z}_{\geq 0}^\infty \).) Let \( x^D = \prod_i x_i^d_i \) be the column weight where \( d_i \) is the number of +’s in column \( i \) of \( D \). If the same \( D \) results from a different sequence of moves, it only counts once. Kohnert’s theorem states \( \kappa_\alpha = \sum x^D \), where the sum is over all such \( D \). Extending this, we introduce:

The \( K \)-Kohnert rule: Each + either moves as in Kohnert’s rule, or stays in place and moves. In the latter case, mark the original position with a “g”. The g’s are unmovable, but a given + treats g the same as other +’s when deciding if it can move, and to where. Diagrams with the same occupied positions but different arrangements of +’s and g’s are counted separately.

**Example 1.3.** Below, we give all \( K \)-Kohnert moves one step from \( D \):

\[
D = \begin{pmatrix} + & . & g & + & . \\ . & + & + & + & + \end{pmatrix} \mapsto \begin{pmatrix} + & . & g & + & . \\ . & + & + & + & + \end{pmatrix}, \begin{pmatrix} + & . & g & + & . \\ + & g & + & + & + \end{pmatrix}, \begin{pmatrix} + & + & g & . \\ . & + & + & + & + \end{pmatrix}, \begin{pmatrix} + & . & g & + & . \\ + & + & + & + & . \end{pmatrix}, \begin{pmatrix} + & + & g & . \\ + & + & + & + & . \end{pmatrix}, \begin{pmatrix} + & . & g & + & . \\ + & + & + & + & g \end{pmatrix}.
\]

Let

\[
J^{(\beta)}_\alpha = \sum \beta(\# g \text{’s appearing in } D) x^D.
\]

**Conjecture 1.4.** \( J^{(-1)}_\alpha = \Omega_\alpha \).
Thus, we show in Section 4.2 that the infinite isobaric divided difference operator Grothendieck polynomials.

1.3. Grothendieck polynomials. The Grothendieck polynomial [LS82] is defined using the isobaric divided difference operator whose action on $f \in \text{Pol}$ is given by:

$$\pi_i(f) = \partial_i((1 - x_{i+1}) f).$$

 Declare $\mathfrak{G}_{w_0}(X) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ where $w_0$ is the long element in $S_n$. Set $\mathfrak{G}_w(X) = \pi_i(\mathfrak{G}_{w_0})$ if $i$ is an ascent of $w$. The Grothendieck polynomials are known to lift $\{s_\lambda\}$ to Pol.

 One has $\mathfrak{G}_w = \mathfrak{G}_w + (\text{higher degree terms})$. We now state the A. Kohnert’s conjecture [K90] for $\mathfrak{G}_w$. The Rothe diagram is $\text{Rothe}(w) = \{(x, y) | y < w(x) \text{ and } x < w^{-1}(y)\} \subset [n] \times [n]$ (indexed so that the southwest corner is labeled $(1, 1)$). Starting with $\text{Rothe}(w)$, the Kohnert’s rule generates diagrams $D$ by applying the same rules as described for his rule for $\kappa_\alpha$. Then $\mathfrak{G}_w = \sum x^D$, the sum is over all such $D$.

 Analogously, we define

$$K_w^{(\beta)} = \sum_D \beta(\# g’s \text{ appearing in } D) x^D$$

where the sum is over all diagrams $D$ generated by the $K$-Kohnert rule. For example, if $w = 3142$ the diagrams contributing to $K_w^{(3)}$ are

$$\text{Rothe}(3142) = \left( \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ + & + & + & + \\ + & + & + & + \end{array} \right) \left( \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ + & + & + & + \\ + & + & + & + \end{array} \right) \left( \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ + & + & + & + \\ + & + & + & + \end{array} \right) .$$

and hence correspondingly, $K_w^{(1)} = (x_1^2 x_3 + x_1^2 x_2) - (x_1^2 x_2 x_3)$.

Conjecture 1.6. $K_w^{(1)} = \mathfrak{G}_w$.

Note, $K_w^{(0)} = \mathfrak{G}_w$ is precisely Kohnert’s conjecture. Conjecture [L.6] has been checked by computer for $n \leq 7$, and extensively for larger $n$. While Kohnert’s rule for $\mathfrak{G}_w$ is handy, it remains mysterious, even after [W03]. Conjectures [L.4] and [L.6] return to Kohnert’s conjecture (albeit with a parameter $\beta$).
2. Proof of Theorem 1.1

2.1. Reduced word combinatorics. Given \( w \in S_n \), let \( a = (a_1, a_2, \ldots, a_{\ell(w)}) \) and \( i = (i_1, i_2, \ldots, i_{\ell(w)}) \).

In connection to [BJS93], we say the pair \((a, i)\) is a stable compatible pair for \( w \) if \( s_{a_1} \cdots s_{a_{\ell(w)}} \) is a reduced word for \( w \) and the following two conditions on \( i \) hold:

\[ \begin{align*}
(\text{cs.1}) & \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{\ell(w)} < n; \\
(\text{cs.2}) & \quad a_j < a_{j+1} \implies i_j < i_{j+1}.
\end{align*} \]

We will identify \( w \) with \( a \) and the associated reduced word.

The Edelman-Greene correspondence [EG87] (the same basic construction is used in [LS82]) is a bijection

\[ \text{EGLS} : (a, i) \mapsto (T, U) \]

where

- \( T \) is an increasing tableau such that \( \text{row}(T) \) is a reduced word for \( a \); and
- \( U \) is a semistandard tableau whose multiset of labels is precisely those in \( i \), and which has the same shape as \( T \).

EGLS (column) insertion: Initially insert \( a_j \) into the leftmost column (of what will be \( T \)). If there are no labels strictly larger than \( a_j \), we place \( a_j \) at the bottom of that column. If \( a_j + t \) for \( t > 2 \) appears, we bump this \( a_j + t \) to the next column to the right, replacing it with \( a_j \). The same holds if \( a_j + 1 \) appears but not \( a_j \). Finally, if both \( a_j + 1 \) and \( a_j \) already appear, we insert \( a_j + 1 \) into the next column to the right. Since \( a \) is assumed to be reduced, the above enumerates all possibilities. Finally at step \( j \) a new box is created at a corner; in what will be \( U \) we place \( i_j \).

Mildly abusing terminology, let \( \text{EGLS}(a) = T \).

2.2. Formulas for Schubert polynomials. A stable compatible pair \((a, i)\) is a compatible pair for \( w \) if in addition to (cs.1) and (cs.2) the following holds:

\[ \begin{align*}
(\text{cs.3}) & \quad i_j \leq a_j.
\end{align*} \]

Let \( \text{Compatible}(w) \) be the set of compatible sequences for \( w \). A rule of [BJS93] states:

\[ \mathcal{S}_w(X) = \sum_{(a, i) \in \text{Compatible}(w)} x^i. \]

A descent of \( w \) is an index \( j \) such that \( w(j) > w(j+1) \). Let \( \text{Descents}(w) \) be the set of descents of \( w \). The following is [BKTY04, Corollary 3]:

**Theorem 2.1.** Let \( w \in S_n \) and suppose \( \text{Descents}(w) \subseteq \{d_1 < d_2 < \ldots < d_k\} \). Then

\[ \mathcal{S}_w(X) = \sum_{\lambda^1, \ldots, \lambda^k} c^w_{\lambda^1, \ldots, \lambda^k} s_{\lambda^1}(X_1) \cdots s_{\lambda^k}(X_k) \]

where \( c^w_{\lambda^1, \ldots, \lambda^k} \) counts the number of tuples of increasing tableaux \((T_1, \ldots, T_k)\) where

\[ \begin{align*}
(i) & \quad T_i \text{ has shape } \lambda^i; \\
(ii) & \quad \min T_1 > 0, \min T_2 > d_1, \ldots, \min T_k > d_{k-1}; \text{ and} \\
(iii) & \quad \text{row}(T_1) \cdots \text{row}(T_k) \text{ is a reduced word of } w.
\end{align*} \]
Assume for the remainder of the proof that
\[ \text{Descents}(w) \subseteq \{d_1 < d_2 < \ldots < d_k\}. \]

Let
\[ \text{Tuples}(w) = \{(T_1, U_1), (T_2, U_2), \ldots, (T_k, U_k)\} \]
where the \( T_i \)'s satisfy (i), (ii) and (iii) from Theorem 2.1 and each \( U_i \) is a semistandard tableau of shape \( \lambda^i \) using the labels \( d_{i-1} + 1, d_{i-1} + 2, \ldots, d_i \) \((d_0 = 0)\).

2.3. “Splitting” the EGLS correspondence. Assuming (6) we define:
\[ \Phi : \text{Compatible}(w) \rightarrow \text{Tuples}(w). \]

Description of \( \Phi \) (using EGLS): Uniquely split \((a, i) \in \text{Compatible}\) as follows
\[ ((a^{(1)}, i^{(1)}), (a^{(2)}, i^{(2)}), \ldots, (a^{(k)}, i^{(k)})) \]
where
- \( a = a^{(1)} \cdots a^{(k)} \) and \( i = i^{(1)} \cdots i^{(k)} \) (“...” means concatenation); and
- the entries of \( i^{(j)} \) are contained in the set \( \{d_{j-1} + 1, d_{j-1} + 2, \ldots, d_j\} \).

Now define
\[ \Phi((a, i)) := (\text{EGLS}(a^{(1)}, i^{(1)}), \ldots, \text{EGLS}(a^{(k)}, i^{(k)})). \]

**Proposition 2.2.** The map \( \Phi : \text{Compatible}(w) \rightarrow \text{Tuples}(w) \) is well-defined and a bijection.

**Proof.** \( \Phi \) is well-defined: The condition (i) is just says \( T_j \) and \( U_j \) have the same shape, which is true by EGLS’s description. For (ii), the splitting says each label in \( i^{(j)} \) is strictly bigger than \( d_{j-1} \). Now by (cs.3), each label in \( a^{(j)} \) is strictly bigger than \( d_{j-1} \) as well. By EGLS’s definition, the set of labels appearing in \( T_j \) is the same as that of \( a^{(j)} \); hence (ii) holds. Lastly, \( \text{row}(T_j) \) is a reduced word for \( a^{(j)} \). Then (iii) is clear.

\( \Phi \) is a bijection: Since EGLS is a bijective correspondence, clearly \( \Phi \) is an injection. Consider the weight function on \( \text{Compatible}(w) \) that assigns \((a, i)\) weight \( x^i \) and assigns \([(T_1, U_1), \ldots, (T_k, U_k)]\) the weight \( x^{U_1} \cdots x^{U_k} \), where \( x^{U_i} \) is the usual monomial associated to the tableau \( U_i \). Then clearly \( \Phi \) is a weight-preserving map (since EGLS is similarly weight-preserving). Hence the surjectivity of \( \Phi \) holds by (4) and Theorem 2.1. \( \square \)

See [L04, Section 5] for a proof of Theorem 2.1 which is close to the study of the split EGLS correspondence (the argument constructs certain crystal operators).

2.4. The tableau \( T[\alpha] \). Recall \( w[\alpha] \in S_\infty \) satisfies \( \text{code}(w[\alpha]) = \alpha \). Let \( < \) be the pure reverse lexicographic total ordering on monomials. The Schubert polynomial \( S_{w[\alpha]} \) has leading term \( x^\alpha \) (with respect to \( < \)). The same is true of \( \kappa_\alpha \) (see [RS95, Corollary 7]) so
\[ S_{w[\alpha]} = \kappa_\alpha + \text{linear combination of other key polynomials}. \]

Given an increasing tableau \( U \), the nil left key \( K^0(U) \) is defined by [LS89] (cf. [RS95, p.111–114]). Let \( \text{sort}(\alpha) \) be the partition obtained by rearranging \( \alpha \) into weakly decreasing order. Also let \( \text{content}(T) \) the usual content vector of a semistandard tableau \( T \). This is a result of A. Lascoux-M.-P. Schützenberger (cf. [RS95, Theorem 4]):
Theorem 2.3.

\[ \mathcal{S}_w(X) = \sum \kappa_{\text{content}}(K^\alpha_0(U)) \]
where the sum is over all increasing tableaux \( U \) of shape \( \text{sort}(\alpha) \) with \( \text{row}(U) = w \).

Thus, by (8) combined with Theorem 2.3 there exists a unique increasing tableau \( U[\alpha] \) of shape \( \text{sort}(\alpha) \) with \( \text{row}(U[\alpha]) = w[\alpha] \) and such that \( \alpha = \text{content}(K^\alpha_0(U[\alpha])) \).

Let \( F_w = \lim_{k \to \infty} \mathcal{S}_{1 \times k, w} \) be the stable Schubert polynomial associated to \( w \). This is a symmetric polynomial in infinitely many variables. So therefore one has an expansion

\[ (9) \quad F_w = \sum_{\lambda} a_{w, \lambda} \frak{s}_\lambda, \]
where the \( a_{w, \lambda} \in \mathbb{Z}_{\geq 0} \) are counted by increasing tableaux \( A \) of shape \( \lambda \) with \( \text{row}(A) = w \).

In [S84, Theorem 4.1], it is shown \( a_{w, \mu(w)'} = 1 \) for a certain explicitly described “maximal” \( \mu'(w) \). Moreover a simple description of the witnessing tableau \( A[\alpha] \) is given. Straightforwardly, \( \mu'(w[\alpha]) = \text{sort}(\alpha) \). Then \( T[\alpha] \) is precisely the witnessing tableau \( A[\alpha] \) for \( a_{w[\alpha], \lambda(w[\alpha])] \) (after accounting for the fact that [S84]'s conventions use \( F_{w[\alpha]} \) for what we call \( F_{w[\alpha]-1} \)). We leave the details to the reader.

Finally, the expansion of Theorem 2.3 refines (9); see, e.g., [RS95]. Hence, \( T[\alpha] = A[\alpha] = U[\alpha] \). So, \( T[\alpha] \) is an increasing tableau of shape \( \text{sort}[\alpha] \) with \( \text{row}(T[\alpha]) = w[\alpha] \) and \( \text{content}(K_-(T[\alpha])) = \alpha \).

2.5. Conclusion of the proof of Theorem 1.1: From the definition of \( \text{Rothe}(w[\alpha]) \):

Lemma 2.4. The descents of \( w[\alpha] \) are contained in the set of descents \( d_1 < d_2 < \ldots < d_k \) of \( \alpha \).

Thus,

\[ (10) \quad \mathcal{S}_{w[\alpha]}(X) = \sum_{(a, i)} x^i = \sum_{\lambda^1, \ldots, \lambda^k} c_{\lambda^1, \ldots, \lambda^k}^{w[\alpha]} s_{\lambda^1}(X_1) \cdots s_{\lambda^k}(X_k). \]

We recall a formula [RS95, Theorem 5]:

Theorem 2.5. Fix an increasing tableau \( T \) with \( \text{content}(K^0_-(T)) = \alpha \). Then

\[ \kappa_\alpha = \sum_{(a, i)} x^i \]
where the sum is over compatible sequences \( (a, i) \) satisfying (cs.1), (cs.2), (cs.3) and \( \text{EGLS}(a) = T \).

Two reduced words \( a \) and \( a' \) for the same permutation are in the same Coxeter-Knuth class if \( \text{EGLS}(a) = \text{EGLS}(a') = T \). This \( T \) represents the class. This equivalence relation \( \sim \) on reduced words is defined by the symmetric and transitive closure of the relations:

\[ (11) \quad A i(i + 1)iB \sim A (i + 1)i(i + 1)B \]
\[ AacbB \sim AcabB \]
\[ AbacB \sim AbcaB \]

where \( a < b < c \). In particular, it is true that \( a \sim \text{row}(\text{EGLS}(a)) \).

Restrict \( \Phi \) to those \( (a, i) \in \text{Compatible}(w[\alpha]) \) such that \( \text{EGLS}(a) = T[\alpha] \). Consider \( \Phi(a, i) = [(T_1, U_1), \ldots, (T_k, U_k)] \). Since \( \text{EGLS}(a^{(i)}) \sim \text{row}(T_i) \), by (11) we see

\[ (12) \quad \text{row}(T_1) \cdots \text{row}(T_k) \sim a^{(1)} \cdots a^{(k)} = a. \]
However, since we have assumed $\text{EGLS}(a) = T[\alpha]$, therefore:

\begin{equation}
\text{EGLS}(\text{row}(T_1) \cdots \text{row}(T_k)) = T[\alpha],
\end{equation}

The other two requirements on $(T_1, \ldots, T_k)$ hold since $\Phi$ is well-defined.

Conversely, suppose $\Phi^{-1}((T_1, U_1), \ldots, (T_k, U_k)) = (a, i) \in \text{Compatible}(w[\alpha])$. Also, by (12), $a \sim \text{row}(T_1) \cdots \text{row}(T_k)$. Now, we assumed (13) holds. Hence, $\text{EGLS}(a) = T[\alpha]$ as desired. This completes the proof of the Theorem 1.1. □

3. Additional remarks

3.1. Comments on Theorem 1.1. Since $\kappa_\alpha$ specialize non-symmetric Macdonald polynomials (see, e.g., [HHL09, Section 5.3]), can one extend Theorem 1.1 in that direction?

Theorem 1.1 implies that the key module of [RS95, Section 5] should have an action of $GL(d_1) \times GL(d_2 - d_1) \times \cdots \times GL(d_k - d_{k-1})$ such that the character is $\kappa_\alpha$.

V. Reiner suggests a variation of Theorem 1.1 using the plactic theory. The derivation should be similar, using formulas from [RS94]. However, we are missing the analogue of [BKTY04, Corollary 4]; cf. [KMS06, Sections 7, 8]. Theorem 1.1 naturally generalizes to Grothendieck polynomials, using [BKTY05, BKSTY08]; details may appear elsewhere.

3.2. $J_\alpha$’s form a (finite) basis of $\text{Pol}$. Clearly, $J_\alpha(X) = x^\alpha + \sum_{\beta \prec \alpha} c_\beta x^\beta$. One decomposes $f \in \text{Pol}$ into a possibly infinite sum of $J_\alpha$’s:

\begin{equation}
 f = \sum_{\alpha} q_\alpha J_\alpha
\end{equation}

That is, find the $\prec$-largest monomial $x^{\theta_0}$ appearing in $f^{(0)} := f$ (say with coefficient $c_{\theta_0}$) and let $f^{(1)} := f - c_{\theta_0} \cdot J_{\theta_0}$. Thus $f^{(1)}$ only contains monomials strictly smaller in the $\prec$ ordering. Now repeat, defining $f^{(t+1)} := f^{(t)} - c_\alpha \cdot J_\alpha$, where $x^{\theta_t}$ is the $\prec$-largest monomial appearing in $f^{(t)}$ etc. Since $J_\alpha$ is not homogeneous, each step $t$ potentially introduces $\prec$-smaller monomials but of higher degree. However, we claim:

**Proposition 3.1.** The expansion (14) is finite.

**Proof.** By the $K$-Kohnert rule, each $\beta$ that appears in $J_\alpha$ is contained in the smallest rectangle $R$ that contains $\alpha$. So the above procedure only involves the finitely many diagrams contained in $R$ for one of the finitely many initial $\alpha \in \mathbb{Z}_\geq 0$ such that $x^\alpha$ is in $f$. □

3.3. More on the interplay of Grothendieck and the $\Omega$ polynomials. M. Shimozono has suggested that the expansion of $\Omega_\alpha$ into $\Omega_\alpha$ should alternate in sign, by degree. An explicit rule exhibiting this has been conjectured by V. Reiner and the second author.

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