Coordinate transformations make perfect invisibility cloaks with arbitrary shape

Wei Yan, Min Yan, Zhichao Ruan and Min Qiu

Laboratory of Optics, Photonics and Quantum Electronics, Department of Microelectronics and Applied Physics, Royal Institute of Technology, 164 40 Kista, Sweden
E-mail: min@kth.se

New Journal of Physics 10 (2008) 043040 (13pp)
Received 14 December 2007
Published 23 April 2008
Online at http://www.njp.org/
doi:10.1088/1367-2630/10/4/043040

Abstract. By investigating wave properties at cloak boundaries, invisibility cloaks with arbitrary shape constructed by general coordinate transformations are confirmed to be perfectly invisible to the external incident wave. The differences between line transformed cloaks and point transformed cloaks are discussed. The fields in the cloak medium are found analytically to be related to the fields in the original space via coordinate transformation functions. At the exterior boundary of the cloak, it is shown that no reflection is excited even though the permittivity and permeability do not always have a perfectly matched layer form, whereas at the inner boundary, no reflection is excited either, and in particular no field can penetrate into the cloaked region. However, for the inner boundary of any line transformed cloak, the permittivity and permeability in a specific tangential direction are always required to be infinitely large. Furthermore, the field discontinuity at the inner boundary always exists; the surface current is induced to make this discontinuity self-consistent. A point transformed cloak does not experience such problems. The tangential fields at the inner boundary are all zero, implying that no field discontinuity exists.
1. Introduction

The recent exciting development of invisibility cloaks has attracted intense attention and discussions [1]–[16]. Theoretically, the cloaks are constructed easily based on a coordinate transformation method as proposed in [1]. The object inside the cloak is invisible to the outside observer, because the light is excluded from the object and the exterior field is not perturbed. The invisibility of linearly radially transformed cylindrical and spherical cloaks has been confirmed by both numerical calculations [3, 4] and analytical solutions [5, 6]. Experimentally, an invisibility cloak with simplified material parameters has been implemented by Schurig et al [7] at the microwave frequency. Inspired by the idea of the invisibility cloak, some interesting applications, such as field concentration [9], field rotation [10] and electromagnetic (EM) wormholes [11], have been proposed.

Up to now, most of the discussions on invisibility cloaks have focused on the cylindrical and spherical cloaks produced by a coordinate transformation only in the radial direction. For instance, in [1], linearly radially transformed cylindrical and spherical cloaks are discussed in detail, and their invisibility is confirmed by ray tracing. In [5, 6], the invisibility performances of such cylindrical and spherical cloaks are further confirmed by obtaining the exact fields in the cloak medium directly from Maxwell’s equations. In practice, it is sometimes desirable to have invisibility cloaks whose shapes are tailored for the objects to be cloaked. Thus, one needs to understand well the properties of invisibility cloaks with arbitrary shape produced by general coordinate transformations. However, investigations on an invisibility cloak with arbitrary shape are only seen in a few papers [13, 14]. The mechanism by which the invisibility of a general cloak produced by compressing space is ensured is still unclear. In this paper, we investigate the EM properties of invisibility cloaks with arbitrary shape constructed by general coordinate transformations, and we confirm their perfect invisibility. To figure out the main physical properties of invisibility cloaks, we focus only on the ideal case, without considering the practical implementation, in this paper.
The paper is organized as follows. In section 2, Maxwell’s equations in a curved coordinate system are derived. In section 3, we show how to construct an invisibility cloak by compressing space in a general manner. In section 4, the wave behaviors and the medium properties at the exterior boundary of the cloak are investigated. In section 5, we study the wave behaviors and the medium properties at the inner boundary of the cloak. Through sections 4 and 5, the invisibility of cloaks with arbitrary shape is confirmed, and the fields in the cloak medium are derived with simple expressions. In section 6, the cloak parameters and the fields in the cloak are derived when the transformed space is described under an arbitrary coordinate system. In section 7, two examples of invisibility cloaks, i.e. cylindrical and spherical invisibility cloaks, are investigated. In section 7, the paper is summarized.

2. Maxwell’s equations in a curved coordinate system

Maxwell’s equations in a Cartesian \((x, y, z)\) space take the form

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \quad \nabla \cdot \mathbf{D} = \rho, \quad \nabla \cdot \mathbf{B} = 0,
\]

with

\[
\mathbf{D} = \varepsilon_0 \varepsilon \cdot \mathbf{E}, \quad \mathbf{B} = \mu_0 \mu \cdot \mathbf{H}.
\]

Consider the transformation from Cartesian space to an arbitrary curved space described by coordinates \(q_1, q_2, q_3\) with

\[
x = f_1(q_1, q_2, q_3), \quad y = f_2(q_1, q_2, q_3), \quad z = f_3(q_1, q_2, q_3).
\]

The length of a line element in the transformed space is given by

\[
dl^2 = \left[ dq_1, dq_2, dq_3 \right] Q \left[ dq_1, dq_2, dq_3 \right]^T,
\]

where \(Q = gg^T\) with

\[
g = \begin{bmatrix}
\frac{\partial f_1}{\partial q_1} & \frac{\partial f_2}{\partial q_1} & \frac{\partial f_3}{\partial q_1} \\
\frac{\partial f_1}{\partial q_2} & \frac{\partial f_2}{\partial q_2} & \frac{\partial f_3}{\partial q_2} \\
\frac{\partial f_1}{\partial q_3} & \frac{\partial f_2}{\partial q_3} & \frac{\partial f_3}{\partial q_3}
\end{bmatrix}.
\]

The volume of a space element is expressed as

\[
dv = \text{det}(g) \, dq_1 \, dq_2 \, dq_3,
\]

where \(\text{det}(g)\) represents the determinant of \(g\). Here, it should be noted that the way of describing the space transformation in this paper is similar to that in [13], where the time transformation is also taken into account. The space-time metric tensor \(g_{\alpha\beta}\) defined in [13] is \(g_{\alpha\beta} = \text{diag}[1, -Q]\) in the present paper, where only space transformation is considered.

Then Maxwell’s equations in the curved space take the form [1, 13]

\[
\nabla_q \times \hat{\mathbf{E}} = -\frac{\partial \hat{\mathbf{B}}}{\partial t}, \quad \nabla_q \times \hat{\mathbf{H}} = \frac{\partial \hat{\mathbf{D}}}{\partial t} + \hat{\mathbf{j}}, \quad \nabla_q \cdot \hat{\mathbf{D}} = \hat{\rho}, \quad \nabla_q \cdot \hat{\mathbf{B}} = 0,
\]

with

\[
\hat{\mathbf{D}} = \varepsilon_0 \varepsilon \cdot \hat{\mathbf{E}}, \quad \hat{\mathbf{B}} = \mu_0 \mu \cdot \hat{\mathbf{H}},
\]

_new journal of physics_ 10 (2008) 043040 (http://www.njp.org/)
Figure 1. The cross section of the cloak (shaded region).

\[ \hat{\varepsilon} = \det(g)(g^T)^{-1}\varepsilon^{-1}, \quad \hat{\mu} = \det(g)(g^T)^{-1}\mu^{-1}, \quad (7) \]

\[ \hat{j} = \det(g)(g^T)^{-1}j, \quad \hat{\rho} = \det(g)\rho, \quad (8) \]

\[ \hat{E} = gE, \quad \hat{H} = gH, \quad (9) \]

where the superscript ‘$^{-1}$’ denotes the inverse of the matrix.

The permittivity and permeability $\hat{\varepsilon}$ and $\hat{\mu}$ in the Cartesian space are considered for a general case, i.e. they can be tensors. It is seen above that Maxwell’s equations in curved space have the same form as in Cartesian space. However, the definitions of the permittivity, permeability, current density and electric charge density are different, as shown in equations (7) and (8).

3. Construction of invisibility cloaks

To construct a cloak, one usually starts by compressing an enclosed space with the exterior boundary unchanged [1]. As seen in figure 1, the region enclosed by boundary $S_1$ is compressed to the region bounded by the exterior boundary $S_1$ and the interior boundary $S_2$. Such a space compression can be viewed as a certain coordinate transformation described by equation (3), which makes a connection between the points in the compressed space with coordinates $(q_1, q_2, q_3)$ and the points with Cartesian coordinates $(x, y, z)$ in the original space. The exterior boundary $S_1$ satisfies $q_1 = x, q_2 = y$ and $q_3 = z$. Notice the interior boundary $S_2$ is obtained by blowing up a line or a point [15]. Thus, the cloaks can be divided into two classes: line transformed cloaks and point transformed cloaks. The compressed shaded region in figure 1 is the desired cloak.

The permittivity and permeability tensors of the cloak in the Cartesian coordinate system are given in equation (7). It seems that the cloak medium is very complex, whose permittivity and permeability are tensors and their values vary with the spatial location. However, the
eigenfunctions of the wave equations in the cloak medium are quite simple, and are related to the eigenfunction of the uncompressed space by equation (9).

In order to achieve invisibility, the cloak should be able to exclude light from a protected object without perturbing the exterior field. Thus, for the above cloak, at the exterior boundary $S_1$, external incident light should excite no reflection, whereas at the interior boundary $S_2$, no reflection is excited either and light cannot penetrate into the cloaked region. In the following sections, we will prove the invisibility of the cloak by investigating the wave behaviors at the cloak’s exterior and inner boundaries. For the simplicity of our discussions and considering the practical application, the invisibility cloak is considered to be placed in air. Then the permittivity and permeability of the cloak in equation (7) will be simplified to

$$\hat{\varepsilon} = \hat{\mu} = \text{det}(g)(g^T)^{-1}g^{-1},$$

which is the same as proposed in [1, 13]. The cloak is considered to be lossless at the working frequency.

4. The cloak’s exterior boundary

In this section, we will prove that no reflection is excited at the exterior boundary. The transmitted electric field $\hat{E}^i$ and magnetic field $\hat{H}^i$ without interacting with the inner boundary $S_2$ are expressed as

$$\hat{E}^i = gE^i, \quad \hat{H}^i = gH^i,$$

where $E^i$ and $H^i$ represent the electric and magnetic fields of the external incident waves. According to equation (9), it is easily seen that the fields expressed in equation (11) satisfy Maxwell’s equations in the cloak medium. Thus, in order to prove that no reflection is excited at $S_1$, one needs only to confirm that tangential components of $\hat{E}^i$ ($\hat{H}^i$) and $\hat{E}^i$ ($\hat{H}^i$) are continuous across $S_1$.

Decompose $\hat{E}^i$ and $\hat{H}^i$ into $\hat{E}^i = [\hat{E}^i_n, \hat{E}^i_{t_1}, \hat{E}^i_{t_2}]$ and $\hat{H}^i = [\hat{H}^i_n, \hat{H}^i_{t_1}, \hat{H}^i_{t_2}]$, where the subscripts ‘$n$’ represent $S_1$’s normal direction pointing outward from the cloak; ‘$t_1$’ and ‘$t_2$’ represent $S_1$’s two tangential directions, which are vertical to each other. Thus, equation (11) can also be expressed as

$$\begin{bmatrix} \hat{E}^i_n \\ \hat{E}^i_{t_1} \\ \hat{E}^i_{t_2} \end{bmatrix} = [\hat{n}, \hat{t}_1, \hat{t}_2]^{-1}gE^i, \quad \begin{bmatrix} \hat{H}^i_n \\ \hat{H}^i_{t_1} \\ \hat{H}^i_{t_2} \end{bmatrix} = [\hat{n}, \hat{t}_1, \hat{t}_2]^{-1}gH^i,$$

where $\hat{n}, \hat{t}_1$ and $\hat{t}_2$ represent the unit vectors in the $n, t_1$ and $t_2$ directions, respectively.

At the exterior boundary $S_1$, $q_1 = x, q_2 = y$ and $q_3 = z$. So $f_i(q_1, q_2, q_3) - q_i = 0$ ($i = 1, 2$ and 3) characterizes the exterior boundary $S_1$. Therefore, it is obvious that the vectors $\nabla_qf_i - \hat{C}_i$ ($i = 1, 2, 3$) lie in the same line as the normal direction $n$ of $S_1$, where $\hat{C}_1 = \hat{x}, \hat{C}_2 = \hat{y}$ and $\hat{C}_3 = \hat{z}$. For the special case when $\nabla_qf_i - \hat{C}_i = 0, \nabla_qf_i - \hat{C}_i$ can be expressed as $0\hat{n}$, i.e. the vector with zero magnitude in the $n$ direction. Therefore, $g$ on $S_1$ can be expressed as

$$g = [F_1\hat{n} + \hat{x}, \ F_2\hat{n} + \hat{y}, \ F_3\hat{n} + \hat{z}],$$

with

$$|F_i| = \sqrt{\left(\frac{\partial f_i}{\partial q_1} - 1\right)^2 + \left(\frac{\partial f_i}{\partial q_2}\right)^2 + \left(\frac{\partial f_i}{\partial q_3}\right)^2},$$
where \( i, j, k = 1, 2, 3 \) and \( i \neq j \neq k \); \( F_i = |F_i| \) when the direction of \( \nabla_q f_i - \hat{C}_i \) is the same as the \( n \) direction and \( F_i = -|F_i| \) if the direction of \( \nabla_q f_i - \hat{C}_i \) is opposite to the \( n \) direction. Substituting equation (13) into (12) and noticing that \( \hat{n}, \hat{t}_1 \) and \( \hat{t}_2 \) are orthogonal to each other, it is easily obtained that at \( S_1 \)

\[
\hat{E}_{1n}^1 = \hat{E}_n^i \cdot \hat{t}_1, \quad \hat{H}_{1n}^1 = \hat{H}_n^i \cdot \hat{t}_1, \quad (15)
\]

\[
\hat{E}_{2n}^1 = \hat{E}_n^i \cdot \hat{t}_2, \quad \hat{H}_{2n}^1 = \hat{H}_n^i \cdot \hat{t}_2, \quad (16)
\]

which indicates that the tangential components of \( \hat{E}_n^i (\hat{H}_n^i) \) and \( \hat{E}^i (\hat{H}^i) \) are continuous across \( S_1 \). Thus, it is proved that no reflection is excited at the exterior boundary.

Consider the permittivity and permeability at \( S_1 \) for the transformed cloak. It should be noticed that no reflection being excited at the exterior boundary does not imply that the exterior boundary is a perfectly matched layer (PML), where the permittivity and permeability at \( S_1 \) have the PML form \( \hat{\varepsilon} = \hat{\mu} = \text{diag} [u, 1/u, 1/u] \) with the principal axes in \( n, t_1 \) and \( t_2 \) directions, respectively. We find that parameters at \( S_1 \) have the PML form only when \( g \) is a symmetric matrix. Observing equation (13), we have \( g^T \hat{t}_1 = \hat{t}_1 \) and \( g^T \hat{t}_2 = \hat{t}_2 \), indicating that \( \hat{t}_1 \) and \( \hat{t}_2 \) are the eigenvectors of \( g^T \) with the same eigenvalue 1. Considering that \( g \) is a symmetric matrix, we can know that the other eigenvector of \( g^T \) is \( \hat{n} \) with eigenvalue \( \det(g) \). Thus, \( \hat{n} \), \( \hat{t}_1 \) and \( \hat{t}_2 \) are eigenvectors of \( Q^{-1} = (g^T)^{-1}g^{-1} \), with eigenvalues \( 1/\det(g)^2 \), 1 and 1, respectively. Thus, based on equation (10), \( \hat{\varepsilon} \) and \( \hat{\mu} \) for a symmetric \( g \) can be expressed as

\[
\hat{\varepsilon} = \hat{\mu} = \text{diag} \left[ \frac{1}{\det(g)}, \det(g), \det(g) \right], \quad (17)
\]

where the diagonal elements correspond to the principal axes \( \hat{n}, \hat{t}_1 \) and \( \hat{t}_2 \), respectively. The radially transformed cylindrical and spherical cloaks fall into this category [1].

5. The cloak’s inner boundary

In this section, we will prove that at the inner boundary \( S_2 \), no reflection is excited and no field can penetrate into the cloaked region. As discussed in section 3, the inner boundary is constructed by blowing up a line or a point, as seen in figures 2(a) and (b). So in the following, two cases, (1) line transformed cloaks and (2) point transformed cloaks, will be discussed separately.

5.1. Case 1: line transformed cloaks

Assume that \( x = b_1(s), y = b_3(s) \) and \( z = b_3(s) \) characterize the line that is mapped to the inner boundary \( S_2 \). We have \( f_1(q_1, q_2, q_3) = b_1(s), f_2(q_1, q_2, q_3) = b_2(s) \) and \( f_3(q_1, q_2, q_3) = b_3(s) \) at \( S_2 \). Each point \( (b_1, b_2, b_3) \) on the line maps to a closed curve on \( S_2 \). The parameter \( s \) can be expressed as a function of \( q_1, q_2 \) and \( q_3 \) with \( s = u(q_1, q_2, q_3) \). \( \nabla_q s = \partial u/\partial q_1 \hat{x} + \partial u/\partial q_2 \hat{y} + \partial u/\partial q_3 \hat{z} \) is the gradient of \( s \), which points in the direction of the greatest increase rate of \( s \). For \( \nabla_q b_i \), we have \( \nabla_q b_i = \partial b_i/\partial s \nabla_q s \), where \( i = 1, 2 \) and 3. Thus, \( \nabla_q b_1 \) and \( \nabla_q s \) have the same direction. For ease of discussion, we again decompose the incident fields at the inner boundary as \( \hat{\mathbf{E}} = [\hat{E}_n^i, \hat{E}_t_1^i, \hat{E}_t_2^i], \hat{\mathbf{H}} = [\hat{H}_n^i, \hat{H}_t_1^i, \hat{H}_t_2^i] \), where the subscripts ‘\( n \)’ denote \( S_2 \)’s normal direction, which points outward from the cloaked region; ‘\( t_1 \)’ and ‘\( t_2 \)’ denote the

New Journal of Physics 10 (2008) 043040 (http://www.njp.org/)
However, the other components of fields are not zero. In particular, $F_i$, where $|F_i| = \sqrt{\left(\partial f_i/\partial q_1 - \partial b_i/\partial q_1\right)^2 + \left(\partial f_i/\partial q_2 - \partial b_i/\partial q_2\right)^2 + \left(\partial f_i/\partial q_3 - \partial b_i/\partial q_3\right)^2}$, where $F_i = |F_i|$ when the direction of $\nabla_q f_i - \nabla_q b_i$ is the same as the $n$ direction, and $F_i = -|F_i|$ when the direction of $\nabla_q f_i - \nabla_q b_i$ is opposite to the $n$ direction.

Notice that $t_1$ is orthogonal to both $n$ and $s_q$, where $\hat{s}$ denotes the unit of the vector in the $\nabla_q s$ direction. Substituting equation (18) into equation (12), it is easily derived that

$$\hat{E}_{i1} = \hat{H}_{i1} = 0.$$  \hfill (20)

However, the other components of fields are not zero. In particular,

$$\hat{E}_{i2} = (\hat{s} \cdot \hat{t}_2)[B_1, B_2, B_3]E^i,$$  \hfill (21)

$$\hat{H}_{i2} = (\hat{s} \cdot \hat{t}_2)[B_1, B_2, B_3]H^i,$$  \hfill (22)

$$\hat{E}_{i3} = [F_1 + B_1(\hat{s} \cdot \hat{n}), F_2 + B_2(\hat{s} \cdot \hat{n}), F_3 + B_3(\hat{s} \cdot \hat{n})]E^i,$$  \hfill (23)

$$\hat{H}_{i3} = [F_1 + B_1(\hat{s} \cdot \hat{n}), F_2 + B_2(\hat{s} \cdot \hat{n}), F_3 + B_3(\hat{s} \cdot \hat{n})]H^i,$$  \hfill (24)

with

$$B_i = \sqrt{\partial b_i/\partial q_1^2 + \partial b_i/\partial q_2^2 + \partial b_i/\partial q_3^2}.$$  \hfill (25)
To further investigate how the waves interact with the inner boundary, the values of the permittivity and permeability at $S_2$ are needed. Observing $g$ expressed in equation (18), it is easily obtained that $g^T \hat{t}_1 = 0$, indicating that $Q \hat{t}_1 = gg^T \hat{t}_1 = 0$. Thus, one of $Q$’s eigenvectors is $\hat{t}_1$ with the eigenvalue $\lambda_{t_1} = 0$, implying $\det(g) = 0$. Because $Q$ is a symmetric matrix, the other two eigenvectors denoted by $\hat{a}$ and $\hat{b}$ should be orthogonal to each other and in the $n-t_2$ plane. The corresponding eigenvalues are denoted by $\lambda_a$ and $\lambda_b$, respectively, with

$$\lambda_a, \lambda_b = |\hat{n} \times \hat{s}|^2 |\hat{F} \times \hat{B}|^2,$$

where $\hat{F} = F_1 \hat{x} + F_2 \hat{y} + F_3 \hat{z}$, $\hat{B} = B_1 \hat{x} + B_2 \hat{y} + B_3 \hat{z}$.

Since $\lambda_i, \lambda_a, \lambda_b = \det(Q) = \det(g)^2$, $\lambda_{t_1} = \det(g)^2 / (\lambda_a, \lambda_b)$. Observing equation (10), it is obtained that $Q \hat{e} = Q \hat{v} = \det(g)$. Therefore, it is known that $\hat{t}_1$, $\hat{a}$ and $\hat{b}$ are the principal axes of the cloaked medium at $S_2$ with $\hat{e}$ and $\hat{v}$ expressed as

$$\hat{e} = \hat{v} = \text{diag} [\lambda_a, \lambda_b / \det(g), \det(g) / \lambda_a, \det(g) / \lambda_b],$$

where the diagonal elements correspond to the principal axes $\hat{t}_1$, $\hat{a}$ and $\hat{b}$, respectively. Since $\det(g) = 0$, we have $\epsilon_a = \mu_a = \epsilon_b = \mu_b = 0$, indicating that the cloaked medium at $S_2$ is isotropic in the $n-t_2$ plane. Therefore, $\hat{n}$ and $\hat{t}_2$ can be considered as the principal axes with $\epsilon_n = \mu_n = \epsilon_{t_2} = \mu_{t_2} = 0$. For $\epsilon_{t_1}$ and $\mu_{t_1}$, it is seen that they have infinitely large values. Thus, the inner boundary operates similarly as a combination of the perfect electric conductor (PEC) and perfect magnetic conductor (PMC), which can support both electric and magnetic surface displacement currents in the $t_1$ direction [8, 12, 15]. In order to have zero reflection at $S_2$, the boundary conditions at this PEC and PMC combined layer require that the incident electric (magnetic) fields in the $t_1$ direction and normal electric (magnetic) displacement fields are all zero. From equation (20), we have $\hat{E}^{i}_{t_1} = \hat{H}^{i}_{t_1} = 0$. Since $\epsilon_n = \mu_n = 0$, it is obtained that $\hat{D}^i_n = \hat{B}^i_n = 0$. Therefore, it is achieved that no reflection is excited at $S_2$, and $\hat{E}^i$ and $\hat{H}^i$ expressed in equation (11) are just the total fields in the cloak medium. The PEC and PMC combined layer guarantees that no field can penetrate into the cloaked region. It is worth noting that the induced displacement surface currents in the $t_1$ direction make $\hat{E}^{i}_{t_2}$ and $\hat{H}^{i}_{t_2}$ at $S_2$ decrease to zero. However, $\hat{E}^{i}_{t_2}$ and $\hat{H}^{i}_{t_2}$ are not zero at a location approaching $S_2$ in the cloak medium. Thus, $\hat{E}^{i}_{t_2}$ and $\hat{H}^{i}_{t_2}$ are discontinuous across the inner boundary $S_2$ [8, 12, 15].

5.2. Case 2: point transformed cloaks

In this case, a point with the coordinate $(c_1, c_2, c_3)$ maps to the inner boundary. At $S_2$, we have $f_1(q_1, q_2, q_3) = c_1$, $f_2(q_1, q_2, q_3) = c_2$ and $f_3(q_1, q_2, q_3) = c_3$. The incident electric and magnetic fields at the inner boundary can be decomposed into $\hat{E}^i = [\hat{E}^i_n, \hat{E}^i_{t_1}, \hat{E}^i_{t_2}]$ and $\hat{H}^i = [\hat{H}^i_n, \hat{H}^i_{t_1}, \hat{H}^i_{t_2}]$, where the definition of the subscript ‘$n$’ denotes $S_2$’s normal direction, which directly outward from the cloaked region; ‘$t_1$’ and ‘$t_2$’ represent $S_2$’s two tangential directions, which are vertical to each other, as shown in figure 2(b). Consider $g$ at the inner boundary, which can be expressed as

$$g = \text{diag} [F_1 \hat{n}, F_2 \hat{n}, F_3 \hat{n}],$$

with

$$|F_i| = \sqrt{\left(\partial f_i / \partial q_1\right)^2 + \left(\partial f_i / \partial q_2\right)^2 + \left(\partial f_i / \partial q_3\right)^2},$$
where \( i = 1, 2 \) and 3. Then, substituting equation (27) into (11), we derive that at \( S_2 \)
\[
\hat{E}^i_{t_1} = \hat{H}^i_{t_1} = 0,
\]
\[
\hat{E}^i_{t_2} = \hat{H}^i_{t_2} = 0,
\]
\[
\hat{E}^i_n = [F_1, F_2, F_3] E^i, \quad \hat{H}^i_n = [F_1, F_2, F_3] H^i.
\]

Unlike case 1, in this case tangential fields are all zero, implying that no field discontinuity exists at \( S_2 \).

Analyzing \( Q \) similarly as in case 1, we obtain that \( \hat{n} \) is an eigenvector of \( Q \) with the eigenvalue \( \lambda_n = F_1^2 + F_2^2 + F_3^2 \), while the other two eigenvectors are \( \hat{i}_1 \) and \( \hat{i}_2 \) with the corresponding eigenvalues \( \lambda_{i1} = \lambda_{i2} = 0 \), indicating \( \det(g) = 0 \). Considering \( \lambda_n \lambda_{i1} \lambda_{i2} = \det(g)^2 \), we have \( \lambda_{i1} = \lambda_{i2} = \det(g) / \sqrt{(F_1^2 + F_2^2 + F_3^2)} \). Therefore, \( \hat{n}, \hat{i}_1 \) and \( \hat{i}_2 \) are the principal axes of the cloak medium at \( S_1 \) with \( \hat{\varepsilon} \) and \( \hat{\mu} \) given by
\[
\hat{\varepsilon} = \hat{\mu} = \text{diag} \left[ \frac{\det(g)^2}{(F_1^2 + F_2^2 + F_3^2)}, \sqrt{F_1^2 + F_2^2 + F_3^2}, \sqrt{F_1^2 + F_2^2 + F_3^2} \right],
\]
where the diagonal elements are in the principal axes \( \hat{n}, \hat{i}_1 \) and \( \hat{i}_2 \), respectively. Since \( \det(g) = 0 \), \( \varepsilon_n = \mu_n = 0 \). Considering that \( \varepsilon_n = \mu_n = 0 \) and tangential components of incident fields at \( S_2 \) are zero, it can be concluded that no reflection is excited at \( S_2 \), and no field penetrates into the cloaked region. The fields expressed in equation (11) are the total fields in the cloak medium.

In the above sections, it has been proved that no reflection is excited at both the exterior boundary and the inner boundary of the cloak, and no field can penetrate into the cloaked region. Therefore, the invisibility of the invisibility cloaks with arbitrary shape constructed by general coordinate transformations is confirmed.

6. Transformation under an arbitrary coordinate system

The cloak parameters and the fields inside the cloak are expressed in equations (10) and (11), respectively. These results are expressed under the Cartesian coordinate system, i.e. \((q_1, q_2, q_3)\) representing Cartesian coordinates in the transformed space. However, sometimes, it is much easier to discuss cloaks under other coordinate systems, such as the cylindrical cloak under the cylindrical coordinate system \((r, \phi, z)\). Thus, it is necessary to obtain the corresponding expressions for cloak parameters and the fields in a cloak under an arbitrary coordinate system, which has also been discussed in [13].

Consider coordinate transformation, where \((q_1, q_2, q_3)\) denotes the coordinates of an arbitrary coordinate system \((u, v, w)\) in the transformed space. The spatial metric tensor of such a coordinate system is \(Q_u = g_u g_u^T\), where \(g_u\) can be obtained easily by considering the relationship between the Cartesian coordinate system and this arbitrary coordinate system. For the cylindrical coordinate system and the spherical coordinate system, the metric tensors are \(\text{diag}[1, r^2, 1]\) and \(\text{diag}[1, r^2, r^2 \sin^2 \theta]\), respectively. The spatial metric tensor of \((q_1, q_2, q_3)\) is expressed as \(Q_q = g_q g_q^T\), where \(g_q\) is shown in equation (4). Assuming that \((x_1, y_1, z_1)\) are the corresponding Cartesian coordinates of \((q_1, q_2, q_3)\) in the transformed space, the spatial metric tensor of \((x_1, y_1, z_1)\) is then obtained as \(Q_e = g_e g_e^T\), where \(g_e = g_u^{-1} g_q\), and \(g_u^{-1}\) represents \(g_u\) expressed under the coordinates \((q_1, q_2, q_3)\). Thus, from equation (10), the permittivity and
permeability of the cloak in Cartesian coordinates are obtained as
\[ \hat{\varepsilon} = \frac{\det(g_u)}{\det(g_{\hat{u}})} g_{u1}^T Q_q^{-1} g_{u1}. \]
Then, expressing \( \hat{\varepsilon} \) and \( \hat{\mu} \) in the \((u, v, w)\) coordinate system, we easily obtain
\[ \hat{\varepsilon} = \frac{\det(g_q)}{\det(g_{\hat{u}})} P_1 Q_u^{-1} Q_u P_1^{-1}, \]
where \( P_1 = \text{diag}[p_1', p_2', p_3'] \) with \( p_i' = \sqrt{g_{u11} + g_{u22} + g_{u33}} \), and \( Q_u = g_{u1} g_{u1}^T \). As an example, \( P_1 = \text{diag}[1, r, 1] \) for the cylindrical coordinate system.

Consider the fields in the cloak. It is easy to know that the fields expressed in the Cartesian coordinate system are \( \hat{\mathbf{E}} = g_u \mathbf{E} \) and \( \hat{\mathbf{H}} = g_u \mathbf{H} \). Thus, the fields expressed in the \((u, v, w)\) coordinate system are expressed as follows:
\[ \hat{\mathbf{E}} = P_1 Q_u^{-1} g_u(g_{u0})^T P_0^{-1} \mathbf{E}', \quad \hat{\mathbf{H}} = P_1 Q_u^{-1} g_u(g_{u0})^T P_0^{-1} \mathbf{H}', \]
where \( g_{u0} \) represents \( g_u \) expressed under the coordinates \((q_1', q_2', q_3')\) and \( Q_{u0} = g_{u0} g_{u0}^T \). \( P_0 = \text{diag}[p_0', p_0', p_0'] \) with \( p_i' = \sqrt{g_{u01} + g_{u02} + g_{u03}} \), \( \mathbf{E}' \) and \( \mathbf{H}' \) represent incident electrical and magnetic field vectors expressed under the \((u, v, w)\) coordinate system.

If \((q_1', q_2', q_3')\) denotes the corresponding coordinates of the coordinate system \((u, v, w)\) in the original space, then \( g_u \) can be written as \( g_u = g_s g_{u0} \), where
\[ g_s = \begin{bmatrix} \frac{\partial q_1'}{\partial q_1} & \frac{\partial q_2'}{\partial q_1} & \frac{\partial q_3'}{\partial q_1} \\ \frac{\partial q_1'}{\partial q_2} & \frac{\partial q_2'}{\partial q_2} & \frac{\partial q_3'}{\partial q_2} \\ \frac{\partial q_1'}{\partial q_3} & \frac{\partial q_2'}{\partial q_3} & \frac{\partial q_3'}{\partial q_3} \end{bmatrix}. \]

Then equations (34) and (35) can be expressed as
\[ \hat{\varepsilon} = \frac{\det(g_s)}{\det(g_{\hat{u}})} P_1 (g_s^T)^{-1} Q_{u0}^{-1} g_s^{-1} Q_u P_1^{-1}, \]
\[ \hat{\mathbf{E}} = P_1 Q_u^{-1} g_s(Q_{u0})^T P_0^{-1} \mathbf{E}', \quad \hat{\mathbf{H}} = P_1 Q_u^{-1} g_s(Q_{u0})^T P_0^{-1} \mathbf{H}'. \]

7. Examples: Cylindrical and Spherical Cloaks

In this section, based on the results obtained above, the well-known radially transformed cylindrical and spherical cloaks will be discussed as examples.

7.1. Cylindrical Cloaks

A two-dimensional cylindrical cloak is constructed by compressing EM fields in a cylindrical region \( r' < b \) into a concentric cylindrical shell \( a < r < b \). Its inner boundary is blown up by a straight line. Thus, a cylindrical cloak is actually a line transformed cloak. Here consider a generalized coordinate transformation that \( r' = f(r) \) with \( f(a) = 0 \) and \( f(b) = b \), while \( \theta \) and \( z \) are kept unchanged. Thus, \( Q_{u0} \) and \( Q_u \) defined in the above section are \( \text{diag}[1, f(r)^2, 1] \) and \( \text{diag}[1, r^2, 1] \), respectively, which indicates that \( \det(g_{u0}) = f(r) \) and \( \det(g_{u1}) = r \). \( P_0 \) and \( P_1 \)
are \( \text{diag}[1, f(r), 1] \) and \( \text{diag}[1, r, 1] \), respectively. \( g_r \) is equal to \( \text{diag}[f'(r), 1, 1] \). Substituting these expressions into equation (37), the permittivity and permeability of the cloak expressed in the cylindrical coordinate system are obtained easily:

\[
\epsilon_r = \mu_r = \frac{f(r)}{r f'(r)}, \quad \epsilon_\theta = \mu_\theta = \frac{r f'(r)}{f(r)}, \quad \epsilon_z = \mu_z = \frac{f(r) f'(r)}{r}.
\] (39)

It is seen that at the exterior boundary \( r = b \), the cloak medium has the PML form with \( \epsilon_r = \mu_r = 1/f'(b) \) and \( \epsilon_\theta = \mu_\theta = \epsilon_z = \mu_z = f'(b) \), which results from the symmetry of \( g_c \), which can be calculated easily.

Consider the fields \( \hat{E}^i \) and \( \hat{H}^i \) incident upon the cloak. It is derived that \( P_1 Q_{31}^1 Q_{30}^1 P_0^{-1} = \text{diag}[f'(r), f(r)/r, 1] \). Then, the fields in the cloaked medium can be obtained directly from equation (38):

\[
\hat{E}_r(r, \theta, z) = f'(r) E_r^i(f(r), \theta, z), \quad \hat{H}_r(r, \theta, z) = f'(r) H_r^i(f(r), \theta, z),
\] (40)

\[
\hat{E}_\theta(r, \theta, z) = \frac{f(r)}{r} E_\theta^i(f(r), \theta, z), \quad \hat{H}_\theta(r, \theta, z) = \frac{f(r)}{r} H_\theta^i(f(r), \theta, z),
\] (41)

\[
\hat{E}_z(r, \theta, z) = E_z^i(f(r), \theta, z), \quad \hat{H}_z(r, \theta, z) = H_z^i(f(r), \theta, z),
\] (42)

where \( [E_r^i, E_\theta, E_z] \) and \( [H_r^i, H_\theta, H_z] \) are components of the incident fields expressed in the cylindrical coordinate system. When \( f(r) = b(r-a)/(b-a) \), substituting the expression of \( f(r) \) into equations (40)–(42), we obtain the fields in the cloak medium, which is just the result in [6].

At the inner boundary, one can easily see that \( s_q \) and \( t_2 \) are both in the \( z \) direction, \( t_1 \) is in the \( \theta \) direction and \( n \) is in the \( r \) direction. Therefore, no matter what \( f(r) \) is, \( \epsilon_\theta \) and \( \mu_\theta \) are infinitely large, and the other components are zero, as seen in equation (39). \( E_\theta \) and \( H_\theta \), \( D_r \) and \( B_r \) are all zero at \( S_2 \), which guarantees that no reflection is excited at \( S_2 \) as analyzed above. The surface displacement currents are induced to make \( E_z \) and \( H_z \) down to zero at \( S_2 \), whereas \( E_z \) and \( H_z \) are not zero at the locations approaching \( S_1 \) in the cloak medium. Thus, \( E_z \) and \( H_z \) are discontinuous across the inner boundary [8, 12, 15].

7.2. Spherical cloaks

A three-dimensional spherical cloak can be constructed by compressing EM fields in a spherical region \( r' < b \) into a spherical shell \( a < r < b \). Its inner boundary is blown up by a point. Thus, a spherical cloak is actually a point transformed cloak. Here, a generalized radial coordinate transformation that \( r' = f(r) \), with \( f(a) = 0 \) and \( f(b) = b \), is considered. Similarly to the above example of the cylindrical cloak, the permittivity and permeability of the spherical cloak expressed in the spherical coordinate system are derived:

\[
\epsilon_r = \mu_r = \frac{f(r)^2}{r^2 f'(r)}, \quad \epsilon_\theta = \mu_\theta = \epsilon_\phi = \mu_\phi = f'(r).
\] (43)

At the exterior boundary \( r = b \), the cloak medium has the PML form with \( \epsilon_r = \mu_r = 1/f'(b) \) and \( \epsilon_\theta = \mu_\theta = \epsilon_\phi = \mu_\phi = f'(b) \), due to the symmetry of \( g_c \).

Consider the incident fields \( \mathbf{E}^i = [E_r^i, E_\theta, E_z^i] \) and \( \mathbf{H}^i = [H_r^i, H_\theta, H_z^i] \) incident upon the cloak; from equation (39), the fields in the cloaked medium are obtained directly:

\[
\hat{E}_r(r, \theta, \phi) = f'(r) E_r^i(f(r), \theta, \phi), \quad \hat{H}_r(r, \theta, \phi) = f'(r) H_r^i(f(r), \theta, \phi),
\] (44)
\[
\hat{E}_\phi(r, \theta, \phi) = \frac{f(r)}{r} E^i_\phi(f(r), \theta, \phi), \quad \hat{H}_\phi(r, \theta, \phi) = \frac{f(r)}{r} H^i_\phi(f(r), \theta, \phi),
\]

(45)

\[
\hat{E}_\phi(r, \theta, \phi) = \frac{f(r)}{r} E^i_\phi(f(r), \theta, \phi), \quad \hat{H}_\phi(r, \theta, \phi) = \frac{f(r)}{r} H^i_\phi(f(r), \theta, \phi).
\]

(46)

When \( f(r) = b(r - a)/(b - a) \), the fields obtained from the above equations agree with the results in [5]. However, the process of the calculation here is simpler.

At the inner boundary, observing equations (45) and (46), the tangential components of fields are zero [5, 15, 16]. Combining with \( \epsilon_n = \mu_n = 0 \), it is known that no field can penetrate into the cloaked region.

8. Conclusions

In this paper, we have studied the properties of invisibility cloaks constructed by general coordinate transformations. The invisibility of cloaks is confirmed by proving that no reflection is excited at both the exterior and interior boundaries of the cloak, and no field can penetrate into the cloaked region. The fields in the cloak medium are related to the fields in the original EM space through \( \hat{E} = gE^i \) and \( \hat{H} = gH^i \). Therefore, to calculate fields in the cloak medium, there is no need to process tedious calculations from the complex material parameters. At the exterior boundary, when \( g \) is a symmetric matrix, the permittivity and permeability of the cloak medium have the PML form, which is just the case for our well-known radially transformed cylindrical and spherical cloaks. At the interior boundary, the properties of the cloak for line and point transformed invisibility cloaks are quite different. For a line transformed cloak, the components of the permittivity and permeability in the \( t_1 \) direction (defined in section 5) are infinitely large, while the other components are all zero. The fields in the \( t_2 \) direction are discontinuous across the inner boundary. The surface displacement currents in the \( t_1 \) direction are induced to make this discontinuity self-consistent. For any point transformed cloak, at the inner boundary, the components of the permittivity and permeability do not have infinitely large components, and the permittivity and permeability in the normal direction are zero. The tangential fields in the inner boundary are zero, implying no discontinuity exists. Therefore, compared to line transformed cloaks, point transformed cloaks are more practical due to the absence of the singularity of the cloak medium.

Acknowledgments

This work is supported by the Swedish Foundation for Strategic Research (SSF) through the Future Research Leaders programme, the SSF Strategic Research Center in Photonics and the Swedish Research Council (VR), and ESA/ESTEC.

References

[1] Pendry J B, Schurig D and Smith D R 2006 Science 312 1780
[2] Leonhardt U 2006 Science 312 1777
[3] Cummer S A, Popa B I, Schurig D, Smith D R and Pendry J B 2006 Phys. Rev. E 74 036621
[4] Zolla F, Guenneau S, Nicolet A and Pendry J B 2007 Opt. Lett. 32 1069
[5] Chen H S, Wu B I, Zhang B L and Kong J A 2007 Phys. Rev. Lett. 99 063903

New Journal of Physics 10 (2008) 043040 (http://www.njp.org/)
[6] Ruan Z C, Yan M, Neff C W and Qiu M 2007 Phys. Rev. Lett. 99 113903
[7] Schurig D, Mock J J, Justice B J, Cummer S A, Pendry J B, Starr A F and Smith D R 2006 Science 314 977
[8] Zhang B L, Chen H S, Wu B I, Luo Y, Ran L X and Kong J A 2007 Phys. Rev. B 76 121101
[9] Rahm M, Schurig D, Roberts D A, Cummer S A, Smith D R and Pendry J B 2007 Preprint 0706.2452v1 [physics.optics]
[10] Chen H Y and Chan C T 2007 Appl. Phys. Lett. 90 241105
[11] Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2007 Phys. Rev. Lett. 99 183901
[12] Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2007 Opt. Express 15 12717
[13] Leonhardt U and Philbin T G 2006 New J. Phys. 8 247
[14] Leonhardt U 2006 New J. Phys. 8 118
[15] Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2007 Commun. Math. Phys. 275 749
[16] Webber R 2007 Preprint 0711.0507 [physics.optics]