Simple Non Linear Klein-Gordon Equations in 2 space dimensions, with long range scattering

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Abstract
We establish that solutions, to the most simple NLKG equations in 2 space dimensions with mass resonance, exhibits long range scattering phenomena. Modified wave operators and solutions are constructed for these equations. We also show that the modified wave operators can be chosen such that they linearize the non-linear representation of the Poincaré group defined by the NLKG.

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1 Introduction
The purpose of this article is to study Non-Linear Klein-Gordon Equations in 2 space dimensions with a finite number of masses $m_i > 0$, having a mass resonance of the following kind, introduced in [ST85]: For some $j$, $j_1$, $j_2$, there exists numbers $\epsilon_{j_1}, \epsilon_{j_2} = \pm 1$ such that

$$m_j = \epsilon_{j_1} m_{j_1} + \epsilon_{j_2} m_{j_2}.$$  

The equations for the real valued functions $\varphi_i$ are:

$$(\Box + m_i^2)\varphi_i = F_i(\varphi, \partial\varphi).$$

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where $\varphi$ is the vector with components $\varphi_i$, $t \in \mathbb{R}$, $x \in \mathbb{R}^2$, $\varphi_i(t, x) \in \mathbb{R}$, $\partial = (\partial_0, \partial_1, \partial_2)$, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$ for $j = 1, 2$, $\Delta = \sum_{i=1}^{2} \partial_i^2$, $\Box = (\partial_0)^2 - \Delta$. The $F_i$ are real $C^\infty$ functions, vanishing together with their first derivative at the origin.

In this paper we shall study the simplest cases of eq. (2), when condition (1) is satisfied. For a given mass $m > 0$, we consider the following two systems of non-linear Klein-Gordon (NLKG) equations, each containing one of the basic critical terms of (2):

$$\Box + m^2 \varphi_1 = 0, \quad (\Box + (2m)^2) \varphi_2 = (\varphi_1)^2 \quad (3)$$

and

$$\Box + m^2 \varphi_1 = \varphi_1 \varphi_2, \quad (\Box + (2m)^2) \varphi_2 = 0. \quad (4)$$

It easily follows that the Cauchy problem for each of the system of equations (3) and (4) has global solutions for large initial data (see Theorem 1 for a precise formulation). The scattering problem is more interesting, since it is only the quadratic terms in (2) which can give rise to long-range phenomenas:

1) We establish (Theorem 1) that the systems (3) and (4) have “long range” modified wave operators and that they fail to have “short range” wave operators. This is due to the second degree “mass resonance”, defined by (1), which is present in these systems together with the $t^{-1}$ time decrease of the $L^\infty(\mathbb{R}^2)$-norm of solutions of the linear K-G equation. This should be compared with the small-data Cauchy and scattering problem for the NLKG

$$\Box + m^2 \varphi = F(\varphi, \partial \varphi), \quad (5)$$

with only one mass $m > 0$. For $n \geq 2$ space dimensions, the scattering theory of (5) is short range [ST92] (see also [H97] and references therein for further developments), which reflects the fact that there is no second degree “mass resonance”. However, there is a third degree “mass resonance” which for $n = 1$ together with the $t^{-1}$ time decrease of the $L^\infty(\mathbb{R})$-norm of $\varphi^2$ gives rise to the “long range” behavior treated in [D01], [LS1-05] and [LS2-05] for the cubic NLKG. We note that the asymptotic completeness of the modified wave operators for (1) is not studied in this paper. The methods in [LS2-05], adapted to spaces of initial conditions like Schwartz spaces, seem to give a promising departure for such future studies. For (3) the asymptotic completeness is a trivial consequence of Theorem 1 and Theorem 4.

2) For $n \geq 1$ space dimensions, all formal nonlinear representations of the Poincaré group only involving massive fields are (at least formally) linearizable (see [T84] where the corresponding cohomology was proved to be trivial). Then a natural question is: can modified wave operators be chosen such that they intertwine the non-linear representation, of the Poincaré

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group (and its Lie algebra) naturally defined on initial conditions for (3) and (4), and the linear representation defined by their linear part i.e.

\[(\Box + m^2)\varphi_1 = 0, \quad (\Box + (2m)^2)\varphi_2 = 0. \tag{6}\]

We prove that the answer is yes (Theorem 4). This is not at all automatic. For example, it is not possible for the Maxwell-Dirac equations in three space dimensions. In fact, as was proved in [FST97], MD is non-linearizable, on natural spaces of initial conditions.

We next write equations (3) and (4) as evolution equations in a Hilbert space \(E\). The variable \(a(t) = (a_{1,+}(t), a_{1,-}(t), a_{2,+}(t), a_{2,-}(t))\) is defined by:

\[a_{\epsilon}(t) = \varphi_\epsilon(t) + \epsilon i \omega_m (-i \nabla) \varphi_\epsilon(t), \quad \epsilon = \pm 1, \tag{7}\]

where \(\omega_m(p) = (M^2 + |p|^2)\) and \(\varphi_\epsilon(t, x) = \frac{\partial}{\partial t} \varphi_\epsilon(t, x)\). The inverse of the transformation (7) is

\[\varphi_\epsilon(t) = (2i \omega_m (-i \nabla))^{-1}(a_{\epsilon,+}(t) - a_{\epsilon,-}(t)), \quad \dot{\varphi}_\epsilon(t) = 2^{-1}(a_{\epsilon,+}(t) + a_{\epsilon,-}(t)). \tag{8}\]

Equations (3) and (4) then reads

\[
\begin{align*}
\frac{d}{dt} a_1(t) &= i \omega_m (-i \nabla)(a_{1,+}(t), -a_{1,-}(t)) + (F_1(a(t)), F_1(a(t))) \\
\frac{d}{dt} a_2(t) &= i \omega_m (-i \nabla)(a_{2,+}(t), -a_{2,-}(t)) + (F_2(a(t)), F_2(a(t))),
\end{align*} \tag{9}\]

where in the case of equation (3)

\[F_1 = 0, \quad F_2(a(t)) = ((2i \omega_m (-i \nabla))^{-1}(a_{1,+}(t) - a_{1,-}(t)))
\]

\[= ((2i \omega_m (-i \nabla))^{-1}(a_{1,+}(t) - a_{1,-}(t))). \tag{10}\]

and in the case of equation (4)

\[F_2 = 0, \quad F_1(a(t)) = ((2i \omega_m (-i \nabla))^{-1}(a_{1,+}(t) - a_{1,-}(t)))
\]

\[= ((2i \omega_m (-i \nabla))^{-1}(a_{1,+}(t) - a_{1,-}(t))). \tag{11}\]

The real Hilbert space \(E\) is defined by \(E = E_{(1)} \oplus E_{(2)}\) with norm \(\|f\|_E = (\sum_{j=1,2} \|f_j\|_{E_{(j)}})^{1/2}\), where \(E_{(j)}\) is the real subspace of \(E_{(j)}^C = E_{(j,+)} \oplus E_{(j,-)}\) such that the image of the transformation (8) only contains real functions. The norms in the complex Hilbert spaces \(E_{(j)}^C\) and \(E_{(j,-)}\) are given by

\[\|f_j\|_{E_{(j)}} = (\sum_{\epsilon = \pm} \|f_{j,\epsilon}\|_{E_{(j,\epsilon)}})^{1/2}\]

\[\text{and} \quad \|f_j\|_{E_{(j,-)}} = \|(\omega_m(-i \nabla))^{-1/2} f_{j,\epsilon}\|_{L^2}. \tag{12}\]

We shall define modified out and in wave operators \(\Omega_+ : \mathcal{O}^+ \to \mathcal{O}^0\) and \(\Omega_- : \mathcal{O}^- \to \mathcal{O}^0\) respectively, by introducing, for given scattering data \(f \in \mathcal{O}^\delta\),
\( \delta = \pm \), an approximate solution \( a^{(\delta)}(f) \) satisfying for some initial condition \( a(0) \) of equation (13) and for \( \alpha = 0 \):

\[
\lim_{t \to \delta \infty} (1 + |t|)^\alpha \| a(t) - (a^{(\delta)}(f))(t) \|_E = 0. \tag{13}
\]

By the uniqueness of the solution \( a \) we can now define

\[
a(0) = \Omega_\delta(f). \tag{14}
\]

Since the cases \( \delta = \pm \) are so similar, we limit ourselves to \( \delta = + \). A study of the large time behavior of solutions of \( g \) by stationary phase methods and the use of (184) to construct linearization maps of nonlinear representations of the Poincaré group leads to a choice of approximate solutions \( a^{(+)}(f) \). With the notation \( V(t); \alpha, \epsilon \) we have \( \exp(i \omega_{jm} (-i \nabla) t) \) we define \((a^{(+)}(f))(t) = V(t)(b^{(+)}(f))(t)\), where in the case of (10)

\[
\begin{align}
\left\{ 
\begin{array}{l}
\dot{b}^{(+)}_1(t) = f_1 \\
(\dot{b}^{(+)}_2(t))(k) = \hat{f}_2, \\
(\dot{b}^{(+)}_2(t))(k) = i \epsilon \ln (1 + \frac{(t^2m)^2}{\omega_m(k)}) \frac{1}{m} (\hat{f}_1(k/2))^2 \\
\end{array}
\right.
\tag{15}
\end{align}
\]

\((f \text{ in } (b^{(+)}(f))(t) \text{ has here been omitted}) \text{ and in the case of } (11)

\[
\begin{align}
\left\{ 
\begin{array}{l}
\dot{b}^{(+)}_1(t) = \exp \left( \frac{1}{m} LI f_2 \ln (1 + t m^2(\omega_m(-i \nabla))^{-1}) \right) f_1 \\
\dot{b}^{(+)}_2(t) = f_2,
\end{array}
\right.
\tag{16}
\end{align}
\]

where for \( g \in E_{(1),\infty} \) and \( h \in E_{(2),\infty} \), \( E_{(1),\infty} = S(\mathbb{R}^2, \mathbb{C}) \oplus S(\mathbb{R}^2, \mathbb{C}) \),

\[
((L(h)g)_{\epsilon})(k) = (L_{\epsilon}(h)g_{\epsilon})(k) = i \epsilon \hat{g}_{\epsilon}(2k) \hat{g}_{\epsilon}(-k). \tag{17}
\]

Then, for a given \( f \in E_{\infty} = E_{(1),\infty} \oplus E_{(2),\infty} \), where \( E_{(j),\infty} = E_{(1)} \cap E_{(1),\infty} \), \( a \) is formally a solution of

\[
a(t) = (a^{(+)}(f))(t) - \int_{t}^{\infty} V(t - s) \left( T_{P_0}^{a}(a(s)) - V(s)(\dot{b}^{(+)}(f))(s) \right) ds, \tag{18}
\]

where \( \dot{b}^{(+)}(f))(t) = \frac{d}{dt}(b^{(+)}(f))(t) \) and see (25) for \( T_{P_0}^{a} \). A rigorous study of this equation in next section, will lead to the construction and covariance properties of modified wave operators (Theorem 1).

The construction of modified wave operators and solutions of more general evolution equations also leads to an equation analog to (18), where the recipe (usually based on an iteration starting with a free solution) how to find an

\[\text{The Fourier transformation } f \mapsto \hat{f} \text{ is here defined by } \hat{f}(k) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ikx} f(x)dx.\]
approximate solution \( a^{(+)}(f) \) for given scattering data \( f \) has to be specified in each particular case. In the case of relativistic covariant equations, this was accomplished for the MD eq. in three space dimensions \([FST87]\) (see also \([FST97]\) for asymptotic completeness) and for NLKG in one space dimension \([D01], [LS1-05] \) and \([LS2-05]\). For NLS it was accomplished in \([O-91]\) and for several other non-relativistic equations in \([GB03]\) and references therein to related papers by the same authors.

The Poincaré group \( \mathcal{P} = \mathbb{R}^3 \ltimes SO(2,1) \) acts on elements \( y = (y^0, y^1, y^2) \) in the 3-dimensional Minkowski space by \( g \in SO(2,1) \) and \( a \in \mathbb{R}^2 \). \( \mathcal{P} \) acts on real functions \( f \) on the Minkowski space by a linear representation \( R \):

\[
(R_g f)(y) = f(g^{-1} y), \quad y \in \mathbb{R}^3.
\]  

Covariance of the NLKG under the representation \( R \) leads to nonlinear representations of \( \mathcal{P} \). \( \Pi = \{ P_0, P_1, P_2, R, N_1, N_2 \} \) denotes an ordered standard basis of the Poincaré Lie algebra \( p = \mathbb{R}^3 \ltimes so(2,1) \) in 3 dimensions. Here \( P_0, P_1, P_2, R, N_1 \) and \( N_2 \) are respectively, the time translation, the two space translation, the space rotation and the two boost generators. We define a linear representation \( T^1 \) of \( p \) in the Schwartz space \( E_\infty \) of elements \( f = (f_{1,+}, f_{1,-}, f_{2,+}, f_{2,-}) \) by:

\[
(T^1_{P_0} f)_j = i\omega jm (-i\nabla)(f_{j,+}, -f_{j,-}), \quad j = 1, 2,
\]

\[
T^1_{P_n} f = \partial_n f, \quad n = 1, 2,
\]

\[
T^1_{R} f = m_{12} f, \quad m_{12} = x_1 \partial_2 - x_2 \partial_1,
\]

\[
(T^1_{N_n} f)_j(x) = (i\omega jm (-i\nabla)x_n f_{j,+}, -i\omega jm (-i\nabla)x_n f_{j,-}), \quad j, n = 1, 2.
\]

The non-linear representation \( T \) of \( p \) on \( E_\infty \) (see \([FSP77]\)), is obtained by the fact that equations (3) and (4) are manifestly covariant:

\[
T_X = T^1_X + T^2_X, \quad X \in p,
\]

where for \( f \in E_\infty \) the quadratic term \( T^2 \) is given by

\[
T^2_{P_0}(f) = (F_1(f), F_1(f), F_2(f), F_2(f)),
\]

\[
T^2_{P_1} = T^2_{P_2} = T^2_{R} = 0,
\]

\[
(T^2_{N_n}(f))(x) = x_n (T^2_{P_0}(f))(x), \quad n = 1, 2.
\]

In particular, equation (12) reads

\[
\frac{d}{dt} a(t) = T_{P_0}(a(t)).
\]
The representation $T^1$ is the differential of a unitary representation $U^1$ of the Poincaré group $\mathcal{P}$ in the Hilbert space $E$. Let $\Pi'$ be the standard basis of the universal enveloping algebra $\mathcal{U}(\mathfrak{p})$ of $\mathfrak{p}$ corresponding to $\Pi$. We give $\Pi'$ its lexicographic order with respect to the ordered basis $\Pi$. Let $|Y|$ be the degree of $Y \in \Pi'$. The space $E_n$ of $n$-differentiable vectors for the representation $U^1$ in $E$ coincides with the Hilbert space obtained by the completion of $E_\infty$ with respect to the norm (summing over $Y \in \Pi'$ and $|Y| \leq n$)

$$
\|u\|_{E_n} = \left( \sum_T \|T^1 Y u\|_E^2 \right)^{1/2},
$$

where $T^1_Y$ is defined by the canonical extension of $T^1$ from $\mathfrak{p}$ to $\mathcal{U}(\mathfrak{p})$. We have $E_\infty \subset E_j \subset E_i \subset E_0 = E$ for $i \leq j$. $U^1_j$ and $T^1_j$ denote the representations obtained by restricting $U^1$ and $T^1$ to $E_j$ and $E_{(j),\infty}$ respectively. Here $E_{(j),n}$ and $E_{(j),\infty}$ denotes the image of the canonical projection of $E_n$ and $E_\infty$ on $E_j$. We note the well-known fact that the norms $\|\cdot\|_{E_n}$ and $q_n$ are equivalent, where (summing over multi-indices with $x^\mu = x_{1,2}^\mu$ and $\nabla^\mu = \partial_1^\mu \partial_2^\mu$)

$$
q_n(f) = \bar{q}_n((I - \Delta)^{-1/4} f) \quad \text{and} \quad \bar{q}_n(f) = \left( \sum_{|\mu|,|\nu| \leq n} \|x^\mu \nabla^\nu f\|_{L^2}^2 \right)^{1/2}.
$$

The linear map $X \mapsto T_X$, from $\mathfrak{p}$ to the vector space of all $C^\infty$ maps from $E_\infty$ to $E_\infty$, extends to $\mathcal{U}(\mathfrak{p})$ by defining inductively (see [ST92]): $T_1 = I$, where $I$ is the identity element in the enveloping algebra, and

$$
T_{XY} = DT_Y T_X, \quad X, Y \in \mathfrak{p},
$$

where $(DA.B)(f) = (DA)(f; B(f))$ is the the Fréchet derivative of the map $A$ at the point $f$ in the direction $B(f)$. Suppose for the moment that the nonlinear Lie algebra representation $X \mapsto T_X$ is (locally) integrable, i.e. in this case $\forall X \in \mathfrak{p}$ and $\forall f \in E_\infty$ there exists $c > 0$ such that for $|t| < c$

$$
\frac{d}{dt} U_{g(t)}(f) = T_X(U_{g(t)}(f)), \quad g(t) = \exp(tX).
$$

Then, for an element $Y \in \mathcal{U}(\mathfrak{p})$ (see [ST92], [ST95])

$$
\frac{d}{dt} T_{Ad_{g(t)}(Y)}(U_{g(t)}(f)) = T_{X Ad_{g(t)}(Y)}(U_{g(t)}(f)),
$$

where the adjoint representation is given by

$$
\frac{d}{dt} Ad_{g(t)} Y = [X, Ad_{g(t)} Y], \quad Ad_{g(0)} Y = Y.
$$
2 Main Results

Since the equation given by (9) for $a_2$ (resp. $a_1$) in the case of (10) (resp. (11)) is simply a linear K-G, with an inhomogeneous (resp. linear potential) term, we easily prove the following theorem:

**Theorem 1** i) There exists $N_0$ such that for $N \geq N_0$, $T$ is integrable to a unique global nonlinear analytic group representation $U$ of $\mathcal{P}$ on $E_N$ and $U : \mathbb{R} \times E_\infty \rightarrow E_\infty$ is $C^\infty$.

ii) For all initial conditions $f \in E_\infty$, equation (9) has a unique $C^\infty$ solution $a: \mathbb{R} \rightarrow E_\infty$.

iii) For all initial conditions $(\varphi_1(0), \dot{\varphi}_1(0), \varphi_2(0), \dot{\varphi}_2(0)) \in S(\mathbb{R}^2, \mathbb{R}^4)$, there exists a unique solution $(\varphi_1, \varphi_2) \in C^\infty(\mathbb{R}^3, \mathbb{R}^2)$ of eq. (3) (resp. (7)).

**Outline of proof:** Proceeding as in [ST92] and [ST95], for $Y \in U(p)$ and $X \in p$ introduce

$$u_Y(t) = T_{\text{Ad}_{\exp(tX)}}(u(t)).$$

Let $u(0) = f \in E_\infty$. According to equation (33)

$$\frac{d}{dt} u_Y(t) = u_{XY}(t), \quad u_Y(0) = T_Y(f). \quad (35)$$

Let $I < Y_1 < \ldots < Y_{c(k)}$ be the lexicographic ordering of the set of $Y \in \Pi'$ such that $|Y| \leq k$, and let

$$v_N(t) = (u_{Y_0}(t), u_{Y_1}(t), \ldots, u_{Y_{c(n)}}(t)), N \geq 0. \quad (36)$$

According to formula (2.23a) of [ST95], (35) leads to an equation for $v_N$,

$$v_N(t) = U^1_{\text{exp}(tx)}v_N(0) + \int_0^t U^1_{\text{exp}((t-s)x)}G_N(v_N(s))ds, \quad (37)$$

for some, in this case, quadratic forms $G_N$ depending on $X$. We define, for a function $d: \Pi' \rightarrow E$ and for $n \in \mathbb{N}$:

$$\mathcal{P}_n(d) = \left( \sum_{Y \in \Pi' \atop |Y| \leq n} ||d_Y||_E^2 \right)^{1/2}. \quad (38)$$

Choosing $N_0$ sufficiently large, one obtains from (10), (11) and (37) using the unitarity of $U^1$ that

$$\mathcal{P}_N(u(t)) \leq \mathcal{P}_N(T(f)) + \int_0^t C_N \mathcal{P}_N(T(f))(1+s)^{-1}\mathcal{P}_N(u(s)) ds, \quad N \geq N_0.$$
Then by Grönwall’s lemma \( \mathcal{P}_N(u(t)) \leq \mathcal{P}_N(T(f))(1 + t)^{C \cdot \mathcal{P}_N(T(f))} < \infty \) for \( t \geq 0 \). Statement (i) now follows by using Theorem 6 of \[ST95\]. Statements (ii) and (iii) are direct consequences of (i). QED

The following two lemmas give time decrease of \( b^{(+)}(t) \) and its derivatives.

**Lemma 2** Let \( f \in E_\infty \). Then \( t \mapsto b^{(+)}(t) \) is a \( C^\infty \) mapping from \([0, \infty[\) to \( E_\infty \) and there exists constants \( C \) independent of \( f \) and \( C_{N,n} \) such that for all \( f \in E_\infty \), \( t \geq 0 \), \( n \geq 0 \) and \( N \geq 2 \)

i) if \( F \) is given by (10) then, \( C_{N,n} \) is independent of \( f \),

\[
\| b_2^{(+)}(t) \|_{E(2),N} \leq \| f_2 \|_{E(2),N} + C_{N,0} \ln (2 + tm) \| f_1 \|_{E(1),N} \frac{\| f_1 \|_{E(1),2}}{E(1,2)}
\]  

(39) and for \( n \geq 1 \)

\[
\| \frac{d^n}{dt^n} b_2^{(+)}(t) \|_{E(2),N} \leq C_{N,n} (1 + t)^n \| f_1 \|_{E(1),N} \frac{\| f_1 \|_{E(1),2}}{E(1,2)}
\]  

(40)

ii) if \( F \) is given by (11) then, \( C_{N,n} \) only depends on \( \| f_2 \|_{E(2),3} \) and

\[
\| \frac{d^n}{dt^n} b_1^{(+)}(t) \|_{E(2),N} \leq C_{N,n} (1 + t)^n \| f_2 \|_{E(2),1} \ln (2 + tm)^{2N+1} \| f_1 \|_{E(1),N} \| f_2 \|_{E(2),N} + \| f_1 \|_{E(1),N} \| f_2 \|_{E(2),3}.
\]  

(41)

**Proof:** We only consider the more difficult case (ii). Expression (16) gives

\[
(b_1^{(+)}(t))_r(k) = \cosh (S(t,k)) \hat{f}_{1,r}(k) + \frac{\sinh (S(t,k))}{S(t,k)} T(t,k),
\]  

(42)

where, for given \( \epsilon, S(t,k) = (1/4m) |\hat{f}_{2,\epsilon}(2k)| \ln (1 + \frac{tm^2}{\omega_m(k)}) \) and \( T(t,k) = \frac{\imath}{4m} \hat{f}_{2,\epsilon}(2k) \ln (1 + \frac{tm^2}{\omega_m(k)}) f_{1,-\epsilon}(-k) \). Let \( F_r(z) = \sum_{n \geq 0} z^n/(2^n + r)! \), \( r = 0,1 \). Then \( F_0(z^2) = \cosh (z) \) and \( F_1(z^2) = \sinh (z)/z \). The \( n \)-th derivative satisfies \( |F_r^{(n)}(z)| \leq 1 \) \( F_r^{(n)}(|z|) \) and \( F_r^{(n+1)}(x) \leq F_r^{(n)}(x), x \geq 0 \). We define the norms \( Q_N \) and \( Q_N' \), \( N \geq 1 \) by

\[
Q_N(a) = \| a \|_{L^\infty} + Q_N'(a), \quad Q_N'(a) = ( \sum_{0 \leq |\mu| \leq N} \| \frac{x^\mu}{(1 + |x|^2)^{1/4}} \nabla^\nu a \|_{L^2}^2 )^{1/2}.
\]  

(43)

Let \( h_r(t,k) = F_r(g(t,k)) \), where \( g(t,k) = (S(t,k))^2 \). Applying \( k^\alpha (\partial/\partial k)^\beta \) on \( h_r(t,k) \), for multi-indices \( \alpha \) and \( \beta \), and using the above properties of \( F_r \), the expression (30), Plancherel’s theorem and interpolation, give for \( N \geq 3 \)

\[
Q_N(h_r(t,\cdot)) \leq C_N F_r(||g(t,\cdot)||_{L^\infty})(1 + Q_3'(g(t,\cdot)))^{N-1}(1 + Q_N'(g(t,\cdot))).
\]
We note that \( \|g(t, \cdot)\|_{L^\infty} \leq C^2 \|f_2\|_{E_{(2),1}}^2 \) \((2 + tm))^2 \), and that by interpolation \( Q_N^c(g(t, \cdot)) \leq C^N Q_N^c((f_2))\) \((2 + tm))^2 \leq C^N \|f_2\|_{E_{(2),3}}\|f_2\|_{E_{(2),N}} \((2 + tm))^2 \). Since \( F_r(x^2) \leq e^x, x \geq 0 \), we obtain for \( N \geq 3 \):

\[
Q_N(h_r(t, \cdot)) \leq C_N(1 + tm)^C\|f_2\|_{E_{(2),3}}\|f_2\|_{E_{(2),N}}^N (1 + \|f_2\|_{E_{(2),3}})^{-1} (1 + \|f_2\|_{E_{(2),N}}).
\]

Interpolation then gives, with \( (H_0(t))(k) = h_0(t, k)f_{1,h}(k) \) and \( (H_1(t))(k) = h_1(t, k)T(t, k) \):

\[
\|H_0(t)\|_{E_{(1),N}} + \|H_1(t)\|_{E_{(1),N}} \leq C_N(1 + tm)^C\|f_2\|_{E_{(2),3}}\|f_2\|_{E_{(2),N}}^N (1 + \|f_2\|_{E_{(2),3}})^{-1} (1 + \|f_2\|_{E_{(2),N}}).
\]

which proves (11) in the case of \( n = 0 \). Repeated use of

\[
\frac{d^n}{dt^n} b_1^{(+)}(t) = \frac{1}{4} L(f_2)(\omega m(-i\nabla)/m + tm)^{-1} b_1^{(+)}(t)
\]

and interpolation leads to, for \( N \geq 3 \) and \( n \geq 1 \):

\[
\|\frac{d^n}{dt^n} b_1^{(+)}(t)\|_{E_{(1),N}} \leq C_N, n(1 + tm)^{-n} (1 + \|f_2\|_{E_{(2),3}})^{-n} (\|b_1^{(+)}(t)\|_{E_{(1),N}} \|f_2\|_{E_{(2),3}} + \|b_1^{(+)}(t)\|_{E_{(1),N}} \|f_2\|_{E_{(2),N}}).
\]

The case \( n = 0 \) of (11) and inequality (17) prove statement (ii) of the lemma. QED

Lemma 3

For all \( f \in E_{\infty}, \ t \geq 0 \) and \( n, N \geq 0 \) there exists a constant \( C \) independent of \( f \) and constants \( C_{N,n} \) and \( C' \) such that

i) if \( F \) is given by (10) then, \( C_{N,n} \) is independent of \( f \) and

\[
q_N \left( \frac{d^n}{dt^n} \left( e^{-i\omega m(-i\nabla)} t \left( (2i\omega m(-i\nabla))^{-1} a_{1,\epsilon}^{(+)}(t) \right)^2 - b_{2,\epsilon}^{(+)}(t) \right) \right) \leq C_{N,n}(1 + t)^{-n-2} \|f_1\|_{E_{(1),N}}^2,
\]

ii) if \( F \) is given by (11) then, \( C_{N,n} \) only depends on \( \|f\|_{E_3} \) and

\[
q_N \left( \frac{d^n}{dt^n} \left( e^{-i\omega m(-i\nabla)} t \left( (2i\omega m(-i\nabla))^{-1} a_{1,-\epsilon}^{(+)}(t) \right)(2i\omega m(-i\nabla))^{-1} a_{2,\epsilon}^{(+)}(t) \right)^2 - b_{1,\epsilon}^{(+)}(t) \right) \leq C_{N,n}(1 + tm)^{C\|f_2\|_{E_{(2),3}}^{-n-2} (\|f_2\|_{E_{(2),N}})^{2N+1} \|f_1\|_{E_{(1),N}} \|f_2\|_{E_{(2),N}}^N}.
\]
Proof: We only consider the case (ii). Define $h$, $I$ and $J$ by

$$(h(s))(k) = \frac{i\epsilon}{4m}(b_{1,-\epsilon}^{(+)}(s))(-k)f_{2,\epsilon}(2k), \quad I(t) = \hat{b}_{1,\epsilon}^{(+)}(t) - h(t)/t, \quad J(t, s) = -h(s)/t$$

$$- e^{-i\omega_m(-i\nabla)t}((2i\omega_m(-i\nabla))^{-1}e^{-i\omega_m(-i\nabla)t}\hat{b}_{1,-\epsilon}^{(+)}(s))((2i\omega_m(-i\nabla))^{-1}e^{-i\omega_m(-i\nabla)t}f_{2,\epsilon}).$$

We have to prove that $q_N\left(\frac{dm}{dt}(I(t,t) - I(t))\right)$ is majorized by the right hand side of inequality (49). Let $J_{n_1,n_2}(t, s) = (d/(dt))^{n_1}(d/(ds))^{n_2}J(t, s)$. Theorem 7 (with $n = 0$ and $(d/(ds))^{n_2}h(s)$ instead of $f_0$), gives

$$q_N(J_{n_1,n_2}(t, s)) \leq C_{N,n}(1 + tm)^{-n_2}q_{N'}((\frac{d}{ds})^{n_2}b_{1,\epsilon}^{(+)}(s))\|f_2\|_{E(2),N'}.$$ 

Inequality (41) then gives, with $n = n_1 + n_2$ and new $C_{N,n}$ and $N'$:

$$q_N(J_{n_1,n_2}(t, s)) \leq C_{N,n}(1 + mt)^{C\|f_2\|_{E(2),1}^{-n_2}}(\ln (2 + tm))^{2N'+1}\|f_1\|_{E(2),N'}\|f_2\|_{E(2),N'}. $$

Summing over $n_1 + n_2 = n$, it follows that $q_N\left(\frac{dm}{dt}J(t, t)\right)$ is majorized by the right hand side of inequality (49). This is also the case for $q_N\left(\frac{dm}{dt}I(t, t)\right)$. In fact, according to (10), $(I(t))(k) = (i\epsilon/(mt))(1 + \omega_m(k)/(tm^{\alpha}))^{-1} - 1)(b_{1,-\epsilon}^{(+)}(s))(-k)f_{2,\epsilon}(2k)$. Derivation in $t$ and application of inequality (11) now give the result. QED

To state the main result on the existence of covariant modified wave operators for equation (9), with the nonlinearities (10) and (11), we define $\mathcal{O}^+ = E_\infty$ in the case of (10) and $\mathcal{O}^+ = \{f \in E_\infty \mid C\|f_2\|_{E(2),1} < 1\}$ in the case of (11), where $C > 0$ as in Lemma 3.

**Theorem 4** If $f \in \mathcal{O}^+$ then, there exists a unique solution $a \in C(\mathbb{R}, (I - \Delta)^{-1}E)$ of equation (18), such that the asymptotic condition (13) is satisfied with $\alpha = 0$. This solution satisfies (14) for an $\alpha > 0$ and $a \in C^\infty(\mathbb{R}, E_\infty)$ and defines by (14) a $C^\infty$ modified wave operator $\Omega_+ : \mathcal{O}^+ \to E_\infty$. $\Omega_+$ intertwines the linear and nonlinear representations of $\mathcal{P}$, i.e. for all $f \in \mathcal{O}^+$ there exists a neighborhood of the identity in $\mathcal{P}$ of elements $g$ such that $U_g(\Omega_+(f)) = \Omega_+(U^*_g f)$.

**Proof:** We only consider the case of the nonlinearity (11), since the case (10) is easier. Let $f \in \mathcal{O}^+$. For $j = 1, 2$, $T_{(j)}^2$ and $T_{(j,\epsilon)}^2$ be the orthogonal projections of $T^2$ on $E_{(j)}$ and $E_{(j,\epsilon)}$ respectively. We shall use the following
notations, where \( g, h(t) \in E_{(1), N} \), for some \( N \):

\[
(H(h))(t) = - \int_t^\infty V_1(-s)T_{(1)}^2 p_0(V(s)(h_1(s), f_2)) \, ds,
\]

\[
I_\epsilon(t) = - \int_t^\infty \sum_{\epsilon_1 + 2\epsilon_2 \neq \epsilon} V_{1, \epsilon}(-s)T_{(1, \epsilon)}^2 p_0(V_{1, \epsilon, 1}(s)(b_{1, \epsilon, 1}(f))(s), V_{2, \epsilon, 2}(s)f_{2, \epsilon, 2}) \, ds,
\]

\[
J_\epsilon(t) = - \int_t^\infty (V_{1, \epsilon}(-s)T_{(1, \epsilon)}^2 p_0(V_{1, -\epsilon}(s)(b_{1, -\epsilon}(f))(s), V_{2, \epsilon, 2}(s)f_{2, \epsilon, 2})) - (b_{1, \epsilon}(f))(s) \, ds,
\]

\[
(K_\epsilon(g, f_2))(k) = \frac{i}{2\pi} \int_{\mathbb{R}^2} \sum_{\epsilon_1 + 2\epsilon_2 \neq \epsilon} d_{\epsilon_1, \epsilon_2}(p, k - p) \hat{g}_{\epsilon_1}(p) \frac{\hat{f}_{2, \epsilon_2}(k - p)}{2i\omega_m(p)}.
\]

\[
d_{\epsilon_1, \epsilon_2}(p_1, p_2) = (\epsilon_1\omega_m(p_1 + p_2) - \epsilon_2\omega_m(p_1) - \epsilon_2\omega_m(p_2))^{-1}.
\]

\[
(50)
\]

Given \( c > 0 \) let \( M_\tau \), where \( \tau > 0 \), be the Banach space of functions \( h \in C([\tau, \infty], (I - \Delta)^{-1}E_{(1)}) \) with norm \( ||h|| = \sup_{t \geq \tau} (1 + t)^{-1}||I - \Delta\|h(t)\|_{E_{(1)}} < \infty \). Using that \( ||V_2(t)(\omega_{2m}(\cdot)\nabla))^{-1/2}f_2||_{L^\infty} \leq C'(1 + t)^{-1}||f_2||_{E_{(2), N_0}} \) for some \( N_0 \) it follows that \( |||H(h)||| \leq C_\tau |||h||| ||f||_{E_{(2), N'}} \) for some \( N' \) and \( C_\tau \). To estimate \( J \), for the given \( f \in \mathcal{O}^+ \) we choose \( c \) such that \( 0 < c < 1 - C||f_2||_{E_{(2), 1}} \). Inequality \( (49) \) of Lemma \( \textbf{3} \) with \( N = 2 \) and \( n = 0 \), then gives that \( |||J||| \leq C''||f_1||_{E_{(1), N'}}||f_2||_{E_{(2), N'}} \) for some new \( N' \) and \( C'' \). To estimate the non-resonant terms \( I(t) \) we proceed, with minor changes, as in §3 of \( \textbf{ST}, \). We obtain (see Corollary 3.8 of \( \textbf{ST}, \)) \( ||I - \Delta)K(V_1(t)g, V_2(t)f_2)||_{E_{(1)}} \leq C'(1 + t)^{-1}||g||_{E_{(1), N_0}}||f_2||_{E_{(2), N_0}} \) for some \( C', N_0 \). Partial integration gives

\[
I(t) = K(V_1(t)(b_{1}^{+}(f))(t), V_2(t)f_2) + \int_t^\infty K(V_1(s)(b_{1}^{+}(f))(s), V_2(s)f_2) \, ds
\]

\[
(51)
\]

By Lemma \( \textbf{2} \) we now obtain (with new constants) that \( ||I|| \leq C''||f_1||_{E_{(1), N'}}||f_2||_{E_{(2), N'}} \). These estimates give, with \( G(h, f) = H(h) + I - J \) that \( ||G(h, f)|| \leq C''||h|| ||f_2||_{E_{(2), N'}} + ||f_1||_{E_{(1), N'}}||f_2||_{E_{(2), N'}} \) and \( ||G(h, f) - G(h', f')|| \leq C'||h - h'|| ||f_2||_{E_{(2), N'}} \). Let \( f_2 \) be such that \( C''||f_2||_{E_{(2), N'}} < 1 \), so \( G \) is a contraction. The equation \( (18) \) for \( t \geq \tau \) is, since \( (b_2(f))(t) = f_2 \), equivalent to

\[
b_1 - b_1^{(0)}(f) = G(b_1 - b_1^{(0)}(f), f).
\]

\[
(52)
\]

This equation has a unique solution \( b - b_1^{(0)}(f) \in M_\tau \). It follows using Grönwall's lemma that, there is a unique continuation of \( V(\cdot)b(\cdot) \) to a solution \( a \in C(\mathbb{R}, (I - \Delta)^{-1}E) \) of the integrated version of \( (25) \) and that \( b_1 - b_1^{(0)}(f) \in M_0 \). Similarly, one establish that the mappings \( \mathcal{O}^+ \ni f \mapsto b(0) = a(0) = \Omega_+(f) \in (I - \Delta)^{-1}E \) and \( f \mapsto b, a \in M_\tau, \tau \in \mathbb{R} \) are \( C^\infty \).
We next turn to the covariance properties of \( \Omega_+ \). For given \( f \in \mathcal{O}^+ \), we consider an open neighborhood of the identity of elements \( g \in \mathcal{P} \) such that \( U^1 g f \in \mathcal{O}^+ \). \( a^g \) denotes the solution in \( C(\mathbb{R}, (I - \Delta)^{-1} E) \), given by the above construction, of (18) with scattering data \( U^1 g f \) and \( b^g(t) = V(-t) a^g(t) \).

Equation (18) gives

\[
b^g(t) - (b^{(+)}(U^1 g f))(t) = \int_{-\infty}^{t} (V(-s) T^2_{P_0} (V(s)b^g(s)) - (b^{(+)}(U^1 g f))(s)) \, ds.
\]

Let \( R' \) be the representation of \( \mathcal{P} \), on tempered distributions \( F \in S'(\mathbb{R}^3, \mathbb{C}^4) \), defined by the representation \( R \) in (19) on \( \phi_1, \phi_2 \in S'(\mathbb{R}^3, \mathbb{C}) \) and the transformations \( \{ 1 \} \) and \( F(t) = V(-t)a(t) \). For translations, i.e. \( g = (I, (s_0, s_1, s_2)) \), formula (19) shows that \( b^{(+)}(U^1 g f) = U^1 g b^{(+)}(f) \). Let \( \delta_g(f) = R'_g b^{(+)}(f) - b^{(+)}(U^1 g f) \). If \( g \) is a space translation, i.e. \( s_0 = 0 \), then \( R'_g(t, x) = F(t, x + s_1, x_2 + s_2) \), so \( \delta_g(f) = 0 \). If \( g \) is a time translation, i.e. \( s_1 = s_2 = 0 \), then \( R'_g F(t, x) = V(s_0) F(t + s_0, x) \), so \( R'_g b^{(+)}(f))(t) = (b^{(+)}(U^1 g f))(t + s_0) \) and \( (\delta_g(f))(t) = V(s_0)((b^{(+)}(f))(t + s_0) - (b^{(+)}(f))(t))) \). With \( c' < c \), above, let \( 0 < c' < c \). Then Lemma 3 gives that \( \| d_{\mathbb{R}^3}(\delta_g(f))(t) \|_{E_N} \leq |s_0| C_N(1 + t)^{-c' - 1} \), for some \( C_N \). We can now integrate \( d_{\mathbb{R}^3}(\delta_g(f))(t) \) in formula (53), which shows that for sufficiently small translations

\[
b^g(t) - (R'_g b^{(+)}(f))(t) = \int_{-\infty}^{t} (V(-s) T^2_{P_0} (V(s)b^g(s)) - \frac{d}{ds} (R'_g b^{(+)}(f))(s)) \, ds.
\]

It follows that (54) holds true with \( b^g = R'_g b \) and this solution is unique. If \( g \) is a space rotation, then similarly one finds that \( b^g = R'_g b \). This shows the intertwining property, \( U_g(\Omega_+(f)) = \Omega_+(U^1 g f) \), for \( g \) in a neighborhood of the identity in the \( \mathbb{R}^3 \mathfrak{so}(2) \) subgroup of \( \mathcal{P} \).

For the case of a Lorentz transformation, let \( g(s) = \exp(s N_j) \), \( j = 1, 2 \) and \( X(t) = N_j + t P_j \). The already proved intertwining property shows that \( a^{g(s)}(t) = \Omega_+(V(t) U^1 g(s) f) \). This function is \( C^\infty \) in \( (t, s, f) \), since \( \Omega_+ \) is \( C^\infty \). Suppose for the moment that, for \( t = s = 0 \),

\[
\frac{d}{ds} a^{g(s)}(t) = T_{X(t)}(a^{g(s)}(t)). \tag{55}
\]

Then one obtains \( D\Omega_+(f; T_{N_j} f) = T_{N_j}(a^{g(0)}(0)) = T_{N_j}(\Omega_+(f)), \) which shows that the intertwining property holds true for a neighborhood of the identity in \( \mathcal{P} \). Successive differentiation of \( g \mapsto \Omega_+(U^1 g(s) f) \in (I - \Delta)^{-1} E \) gives that \( T_Y(\Omega_+(f)) \in E \) for all \( Y \in \Pi' \). Now according to Theorem 2 of [ST95], \( \Omega_+(f) \in E_\infty \) and this mapping from \( \Omega_+ \) to \( E_\infty \) is \( C^\infty \).

To complete the proof we shall prove formula (55) for \( t \geq 0 \). The differentiability of \( b \) in \( f \), justifies to differentiate in \( s \) both sides of formula

\[
\frac{d}{ds} a^{g(s)}(t) = T_{X(t)}(a^{g(s)}(t)). \tag{55}
\]
Using Lemmas 2 and 3, one establishes, with $b = b^0 = b$ and $b' = \frac{db^0}{ds}|_{s=0}$ and $b^{(+)} = \frac{db^{(+)}}{ds}|_{s=0}:

$$b'(t) - b^{(+)}(t) = \int_{-\infty}^{t} (V(-s)2(DT_{P_0}^2)(V(s)b(s); V(s)b'(s)) - \frac{d}{ds}b^{(+)}(s)) \, ds.$$ (56)

The generator $\Xi_{N_j}$ of $s \mapsto R'_{g(s)}$, is given by its component in $E_{(1,e)}$:

$$((\Xi_{(1,e)N_j}b)(t))(k) = (-\epsilon \omega_m(k) \frac{\partial}{\partial k_j} + \frac{k_j t}{\omega_m(k)} \frac{\partial}{\partial t}) + i(\frac{\partial}{\partial k_j}-\frac{k_j}{(\omega_m(k))^2}) \frac{\partial}{\partial t} (h(t))(k).$$ (57)

Using Lemmas [2] and [3], one establishes, with $c'$ as above, that $\|\frac{d}{dt}((\Xi_{N_j}b^{(+)})(t) - b^{(+)}(t))\|_{L^2} \leq |s_0|C_N'(1 + t)^{-c' - 1}$, for some $C_N'$. This shows that we can replace $b^{(+)}$ by $\Xi_{N_j}b^{(+)}$ on both sides of (56). Observing that $V(t) \Xi_{N_j}b(t) = T_{X(t)}(a(t))$ and that $T_{P_0} V(t) \Xi_{N_j}b(t) + 2(DT_{P_0}^2)(V(t)b(t); V(t)\Xi_{N_j}b(t)) = T_{P_0} T_{X(t)}(a(t))$ we can now identify $b'(t)$ with $V(-t)T_{X(t)}(a(t))$ which satisfies the equality

$$V(-t)T_{X(t)}(a(t)) - (\Xi_{N_j}b^{(+)})(t) = \int_{-\infty}^{t} (V(-s)2(DT_{P_0}^2)(a(s); T_{X(t)}(a(s))) - \frac{d}{ds}(\Xi_{N_j}b^{(+)})(s)) \, ds.$$

QED

3 The linear K-G equation

We shall here give certain results on phase and decrease properties of solutions of linear Klein-Gordon equations, which we have used to study resonant terms. They are adapted from Appendix [HST97] to our situation and are based on the symbolic calculus developed in [H87]. Let $M > 0$. For given $\epsilon = \pm$ and $f \in S(\mathbb{R}^2, \mathbb{C})$,

$$\varphi(t) = e^{i\epsilon M(-i\nabla)}f,$$ (58)

defines a solution $\varphi$ of

$$(\Box + M^2)\varphi = 0.$$ (59)

The forward light-cone is denoted $\Gamma_+ = \{(t,x) \in \mathbb{R}^3 \mid t^2 - |x|^2 \geq 0, t \geq 0\}$ and let $\rho = (t^2 - |x|^2)^{1/2}$ for $(t,x) \in \Gamma_+$. The sequence of functions $g_l \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^2 - \{0\})$, with support in the forward light-cone, is defined by

$$g_0(t,x) = i\epsilon(Mt/\rho^2)\hat{f}(-\epsilon Mx/\rho), \quad g_l = \frac{\rho}{2i\epsilon M} \Box g_{l-1}, \quad l \geq 1,$$ (60)
for \((t, x) \in \Gamma_+\), \(g_t\) is homogeneous of degree \(-1 - l\). The solution \(\varphi = \varphi_0\) has an asymptotic expansion with rest-term \(\varphi_n\):

\[
\varphi_n = \varphi_0 - e^{i\kappa M\rho} \sum_{0 \leq l \leq n-1} g_l, \quad n \geq 1.
\]

(61)

Define \(\lambda(t)\) and \(\delta(t)\) for \(t \geq 0\) by

\[
(\lambda(t))(x) = t/(1 + t - |x|) \quad \text{for } 0 \leq |x| \leq t,
\]

(62)

\[
(\lambda(t))(x) = |x| \quad \text{for } 0 \leq t \leq |x|,
\]

(63)

\[
(\delta(t))(x) = 1 + t + |x|.
\]

(64)

We introduce the representation \(X \mapsto \xi_X\) of the Poincaré Lie algebra \(\mathfrak{p}\) by:

\[
\xi_{P_0} = \frac{\partial}{\partial t}, \quad \xi_{P_i} = \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq 2,
\]

(65)

\[
\xi_{N_i} = x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq 2,
\]

(66)

\[
\xi_R = -x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_1}.
\]

(67)

We define, for a function \(d : \Pi' \to S(\mathbb{R}^2, \mathbb{C})\) and for \(n \in \mathbb{N}\):

\[
p^{(s)}_n(d) = \sum_{Y \in \Pi'} \left( M\|d_Y\|_{L^s} + \sum_{0 \leq \mu \leq 2} \|d_{P_{\mu}Y}\|_{L^s} \right).
\]

(68)

The following theorem gives decrease properties of the solution \(\varphi\) and the rest terms \(\varphi_n\). We omit its proof, since its so similar to that of Theorem A.1 in [FST97], considering the case of the Dirac equation in 3-space dimensions. Given an ordering on the basis \(Q = \{N_1, N_2, R\}\) of \(\mathfrak{so}(2, 1)\), let \(Q'\) be the corresponding standard basis of the enveloping algebra \(U(\mathfrak{so}(2, 1))\) of \(\mathfrak{so}(2, 1)\).

**Theorem 5** There exists \(C_i \in \mathbb{R}, i \geq 0, \text{ and } \kappa_i \in \mathbb{N}, 1 \leq i \leq 4\), such that for all \(j, k, n \in \mathbb{N}, t > 0, f \in S(\mathbb{R}^2, \mathbb{C}), X \in \Pi' \cap U(\mathbb{R}^3)\) and \(Y \in Q'\):

\[
p^{(2)}_j((1 + \lambda(t))^{k/2}(\xi \varphi_0)(t)) \leq C_{j+k} q_j + \sum_{1 \leq i \leq 2} q_{j+i}(\partial_i f),
\]

(69)

\[
p^{(\infty)}_j((\delta(t)(1 + \lambda(t))^{k/2}(\xi \varphi_0)(t)) \leq C_{j+k} q_j + \kappa_2(f),
\]

(70)

\[
p^{(2)}_j((1 + \lambda(t))^{k/2}(\xi \varphi_{n+1})(t)) \leq C_{j+k+n} t^{-n-1} q_{3(j+k+n)+\kappa_1}(f),
\]

(71)

\[
p^{(\infty)}_j((\delta(t)(1 + \lambda(t))^{k/2}(\xi \varphi_{n+1})(t)) \leq C_{j+k+n} t^{-n-1} q_{3(j+k+n)+\kappa_3}(f),
\]

(72)

\[
\|\rho^{-j}\xi_X Y g_n(t)\|_{L^2} \leq C_{j+|X|+n} t^{-j-n-|X|} q_{3|X|+3n+|Y|+j}(f),
\]

(73)

\[
\|\rho^{-j}\xi_X Y g_n(t)\|_{L^\infty} \leq C_{j+|X|+n} t^{-1-j-n-|X|} q_{3|X|+3n+|Y|+j}(f).
\]

(74)
The development defined by (60) and (61) can be inverted. Given a homogeneous function \( g \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^2) - \{0\}) \) of degree \(-1\) with support in \( \Gamma_+ \), we construct by iteration \( f_0, \ldots, f_n \in \mathcal{D}_\infty \):

\[
\hat{f}_l(k) = -i\epsilon(M/(\omega(k))^2)g_{l,0}(1,-\epsilon k/\omega_M(k)), \quad 0 \leq l \leq n, \\
g_{0,0} = g, \quad g_{l,0}(t, x) = -\sum_{1 \leq j \leq l} t^j g_{l-j,0}(t, x), \quad 1 \leq l \leq n, \\
g_{l,j} = \frac{\rho}{2t \epsilon M} \square g_{l,j-1}, \quad 1 \leq j \leq n - l.
\]

By this construction \( g_{l,j} \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^2) - \{0\}) \) is homogeneous of degree \(-1-j\) with support in \( \Gamma_+ \). Reformulation in two space dimensions of Theorem A.2 [FST97], (there proved in the case of three space dimensions), gives:

**Theorem 6** Let \( g \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^2) - \{0\}) \) be a homogeneous function of degree \(-1\) with supp \( g \subset \Gamma_+ \). If \( f_0, \ldots, f_n \) are given by the construction \( 75, 77 \) and

\[
u_n(t) = e^{i\epsilon M t} g(t) - \sum_{0 \leq l \leq n} t^{-l} e^{i\epsilon \omega_M (-i\nabla)t} f_l,
\]

then there exists \( C_i \in \mathbb{R}, \ i \geq 0, \) independent of \( g \), such that for all \( j, k, n \in \mathbb{N} \) and \( t > 0 \) and with \( \kappa_1 \) and \( \kappa_3 \) as in Theorem 5:

\[
\tilde{q}_j(f_n) \leq C_{n+j} \sum_{q+|Y| \leq j+2n} \|(m/\rho(1, \cdot))^{q+n}(\xi Y g)(1, \cdot)\|_{L^2},
\]

\[
p_j^{(2)}((1 + \lambda(t))^{k/2}(\xi u_n)(t)) \leq C_{j+k+n} \sum_{q+|Y| \leq 3(j+k+n)+\kappa_1} \|(m/\rho(1, \cdot))^{q}(\xi Y g)(1, \cdot)\|_{L^2} t^{-n-1},
\]

\[
p_j^{(\infty)}(\delta(t)(1 + \lambda(t))^{k/2}(\xi u_n)(t)) \leq C_{j+k+n} \sum_{q+|Y| \leq 3(j+k+n)+\kappa_3} \|(m/\rho(1, \cdot))^{q}(\xi Y g)(1, \cdot)\|_{L^2} t^{-n-1}.
\]

Theorem 5 and Theorem 6 permit to find the asymptotic behavior and estimates of resonant terms.

**Theorem 7** Let \( M, M_1, M_2 > 0 \) and \( \epsilon, \epsilon_1, \epsilon_2 \in \{-1, 1\} \) be such that \( \epsilon M = \epsilon_1 M_1 + \epsilon_2 M_2 \) and let \( f^{(1)}, f^{(2)} \in S(\mathbb{R}^2, \mathbb{C}) \). There exists a unique sequence of functions \( f_l \in S(\mathbb{R}^2, \mathbb{C}) \), such that if

\[
\delta_l(t) = e^{-i\epsilon \omega_M (-i\nabla)t} \left( (e^{i\epsilon_1 \omega M_1 (-i\nabla)t} f^{(1)}) (e^{i\epsilon_2 \omega M_2 (-i\nabla)t} f^{(2)}) \right) - \sum_{0 \leq l \leq n} t^{-1-l} f_l,
\]

15
then for all \( N, j, n \in \mathbb{N} \) there are \( C \) and \( N' \) such that

\[
\tilde{q}_N((\frac{d}{dt})^j \delta_n(t)) \leq C(1 + t)^{-2-j-n} \tilde{q}_N'(f^{(1)}) \tilde{q}_N'(f^{(2)}).
\] (83)

Moreover

\[
\hat{f}_0(k) = i \frac{\epsilon_1 M_1 \epsilon_2 M_2}{\epsilon M} \left( \frac{\omega_M(k)}{M} \right)^2 (f^{(1)}) (\frac{\epsilon_1 M_1}{\epsilon M} k) (f^{(2)}) (\frac{\epsilon_2 M_2}{\epsilon M} k)
\] (84)

and

\[
\tilde{q}_N(f_j) \leq C \tilde{q}_N'(f^{(1)}) \tilde{q}_N'(f^{(2)}).
\] (85)

**Proof:** With \( f^{(i)} \) instead of \( f \), we define \( \varphi^{(i)}_n, g^{(i)}_n \) and \( \varphi^{(i)}_n, g^{(i)}_n \) by formulas (58)–(61). Given \( J \in \mathbb{N} \), formula (82) can be written

\[
\delta_n(t) = e^{i \omega M (-i \nabla) t} \left( \sum_{0 \leq i \leq J} (\varphi^{(1)}_{j+1}(t) + e^{i \epsilon M_1 \rho} \sum_{0 \leq i \leq J} g^{(1)}_i(t)) (\varphi^{(2)}_{j+1}(t) + e^{i \epsilon M_2 \rho} \sum_{0 \leq i \leq J} g^{(2)}_i(t)) \right)
\] 

\[
- \sum_{0 \leq i \leq n} t^{-1-i} f_i,
\] (86)

where the functions \( f_i \) will be defined later in this proof. We define

\[
g_i(t, \cdot) = t^{1+i} \sum_{l_1+l_2 = i \atop 0 \leq l_1 \leq J} g^{(1)}_{l_1}(t, \cdot) g^{(2)}_{l_2}(t, \cdot),
\] (87)

\[
I_J(t) = \varphi^{(1)}_{j+1}(t) \varphi^{(2)}_{j+1}(t) + \sum_{0 \leq i \leq J} (e^{i \epsilon M_1 \rho} g^{(1)}_i(t) \varphi^{(2)}_{j+1}(t) + e^{i \epsilon M_2 \rho} g^{(2)}_i(t) \varphi^{(1)}_{j+1}(t))
\] (88)

and

\[
v_n^J(t) = e^{-i \omega M (-i \nabla) t} \sum_{0 \leq i \leq 2J} t^{-1-i} e^{i \epsilon M \rho} g_i(t, \cdot) - \sum_{0 \leq i \leq n} t^{-1-i} f_i.
\] (89)

Then

\[
\delta_n(t) = e^{-i \omega M (-i \nabla) t} I_J(t) + v_n^J(t).
\] (90)

The function \( g_i \) is homogeneous of degree \(-1\). We note that, according to (73) and (74) of Theorem 5, if \( Z \in \Pi' \) then

\[
\| (\rho^{-1} Z g_i)(1, \cdot) \|_{L^2} \leq C \tilde{q}_N'(f^{(1)}) \tilde{q}_N'(f^{(2)}),
\] (91)

where \( C \) and \( N' \) depend on \( |Z| \) and \( j \). Also, a straight forward application of (71)–(74) gives, with new \( C \) and \( N' \) depending on \( N, k \) and \( J \)

\[
p_N^{(2)}((1 + \lambda(t))^{k/2}(\xi I_J)(t)) \leq C t^{-2-j} \tilde{q}_N'(f^{(1)}) \tilde{q}_N'(f^{(2)}).
\] (92)
With $g_i$ instead of $g$ and $J$ instead of $n$, we define $f_{l,k}$ and $u_{l,J}$ by formulas (75) and (78). Then, according to Theorem 5 and (91),

$$u_{l,J}(t) = e^{iM \rho t}g_i(t) - \sum_{0 \leq k \leq J} t^{-k} e^{i\omega_M(-i\nabla)t} f_{l,k}$$

(93)
satisfies, with $C$ and $N'$ depending on $J$, $N$ and $k$,

$$t^{J+1} \left( p_N^{(2)}((1 + \lambda(t))^{k/2}(\xi u_{l,J})(t)) + p_N^{(\infty)}(\delta(t)(1 + \lambda(t))^{k/2}(\xi u_{l,J})(t)) \right)$$

$$+ \bar{q}_N(f_{l,k}) \leq C \bar{q}_N(f^{(1)})\bar{q}_N(f^{(2)}).$$

(94)

Formulas (89) and (93) give

$$v_n^J(t) = \sum_{0 \leq l \leq 2J} t^{-1-l} \left( e^{-i\omega_M(-i\nabla)t}u_{l,J}(t) + \sum_{0 \leq k \leq J} t^{-k} f_{l,k} \right) - \sum_{0 \leq l \leq n} t^{-1-l} f_l.$$  

(95)

In the sequel of this proof we suppose that $J \geq n$ and define

$$f_l = \sum_{k_1+k_2=l} f_{k_1,k_2}, \quad A_J(t) = \sum_{0 \leq l \leq 2J} t^{-1-l} u_{l,J}(t), \quad B_n(t) = \sum_{(l,k_1,k_2) \in D(n,J)} t^{-1-l} f_{k_1,k_2},$$

(96)

where $D(n,J) = \{(l,k_1,k_2) | l > n, 0 \leq l \leq 2J, 0 \leq k_2 \leq J, k_1 + k_2 = l\}$.

Then $v_n^J(t) = e^{-i\omega_M(-i\nabla)t}A_J(t) + B_n(t)$, so by (90)

$$\delta_n(t) = e^{-i\omega_M(-i\nabla)t}h_J(t) + B_{n,J}(t), \quad \text{where} \quad h_J(t) = I_J(t) + A_J(t).$$

(97)

Inequalities (91) and (94) and the fact that $\xi_N t^{-1-l} = -(1 + l)t^{-2-l}x_i$ give

$$t^{J+2} p_N^{(2)}((1+\lambda(t))^{k/2}(\xi A_J)(t)) + t^{n+j+2} \bar{q}_N((\frac{d}{dt})^j B_n(t)) \leq C \bar{q}_N(f^{(1)})\bar{q}_N(f^{(2)}).$$

(98)

Since $(x_ji\epsilon\omega_M(-i\nabla) - t\partial_x) e^{-i\omega_M(-i\nabla)t} = e^{-i\omega_M(-i\partial)t}x_ji\epsilon\omega_M(-i\nabla)$, it follows from Leibniz’s rule and (30) that

$$\bar{q}_N((\frac{d}{dt})^j(e^{-i\omega_M(-i\nabla)t}h_J(t))) \leq C_{J,N} \sum_{j_1+j_2=j} \|x^{\alpha} \nabla^\beta \bar{q}_N((\frac{d}{dt})^{j_2}h_J(t))\|_{L^2}$$

$$\leq C'_{J,N} \sum_{\{\alpha,\beta\} \subseteq N, j_1+j_2=j} t^k \|x^{\alpha} \nabla^\beta (\frac{d}{dt})^{j_2}h_J(t)\|_{L^2}.$$

(99)
Let $U(\mathbb{R}^3)$ be the enveloping algebra of the translation subalgebra of $p$. Inequalities (92), (94) and (99) give that for $t \geq 1$:

$$\bar{q}_N((\frac{d}{dt})^j(e^{-i\omega M(\xi)\cdot t}h_j(t))) \leq C_{j,N}^n \sum_{X \in U(\mathbb{R}^3) \cap \Pi} t^N \| (1 + \lambda(t))^N (\xi h_j(t)) \|_{L^2}$$

$$\leq C_{j,N}^n t^N p_{N+j}((1 + \lambda(t))^N (\xi h_j(t))(t)) \leq C_{j+N}^n t^{N-2-J} \bar{q}_{N'}(f^{(1)}(t)) \bar{q}_{N'}(f^{(2)}).$$

(100)

Inequality (83), for $t \geq 1$, now follows by choosing $J \geq N + n + j$. The definition of $f_i$ in formula (96) and inequality (94) give inequality (85).

To prove formula (84), we observe that $g_0^j(t, x) = i\epsilon_j(M_j t/\rho^2)\hat{f}(-\epsilon_j M_j x/\rho)$, $j = 1, 2$, according to (60). By (87), $g_0^j(t, x) = t g_0^{(1)}(t, x) g_0^{(2)}(t, x)$. By (96) $f_0 = f_0^0$, so by formulas (93) and (105) $\hat{f}_0(k) = -i\epsilon(M/\omega(k))^2 g_0^0(1, -\omega(k))$. The result now follows using that $\rho(1, -\omega(k)) = M/\omega(k)$. QED

Note added in the proofs: I learned later from H. Sunagawa, after the acceptance of the paper, that he has related results in Hokkaido Math. Journ. 33, 457–472 (2004), which cover some of those for the easier case (3) but not for the case (4), and the method is different.

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