Quantum Anatomy of the Classical Interference of $n$-Photon States in a Mach-Zehnder Interferometer

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Abstract. In this work we present the theory for the quantum interference of states with an arbitrary number of photons in a Mach-Zehnder interferometer. We express the mathematical description of the interference in an algebraic way. We show the interference pattern of an average of a $n$ photons input state corresponds to the classical interference pattern, which tells us the last comes from a quantum interference statistical average. Then, we propose to use this scheme to study the statistical transition from quantum to classical interference.

1. Introduction

The idea of classical interference is deep-rooted in our intuition. For example, the double-slit experiment made by Young with light [1] can be elegantly explained by thinking the light as a wave. However, when we consider a quantum description of the same experiment it is necessary to include photon interference. A study of the transition between quantum interference of photons and a wave description of this interference could help to better understand the transition between quantum and classical optics. The fact that a train of single photons can produce an interference pattern represents the main difference between both descriptions of the interference. Therefore, there is not much difference among them except for the auto-interference [2–4], a paradigm for quantum interference.

In this work we discuss the problem of recovering the classical interference of light from the quantum interference of $|n\rangle$ photon states in a Mach-Zehnder interferometer (MZI). First, we show there is no difference between the quantum and classical geometric patterns of interference due to the randomness of the photon output states $|n - k\rangle$ and $|k\rangle$, the combination produced by the beam splitters in the MZI. We will see that what we know as classical interference is no more than the statistical average of the different quantum interferences.

Finally, the output photon statistics can be selected ‘on demand’ among poissonian, sub-poissonian and super-poissonian quantum statistics as a function of the dephasing parameter.
2. \( n \)-photon interference in a Mach-Zehnder interferometer

The quantum interference’s hearth lies in the state of a photon going through a beam-splitter. This state is represented by

\[
|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|1\rangle^T + i|1\rangle^R),
\]

where \( T \) and \( R \) mean transmitted and reflected, respectively. We have selected the reflection phase as \( \pi/2 \). For \( n \) photons the quantum state is limited to the quantum components resultant of expanding a \( n \)-dimensional binomial,

\[
|n\rangle \rightarrow \frac{1}{\sqrt{2^n}}(|1\rangle^T + i|1\rangle^R)^n.
\]

If the transmitted and reflected states are redirected to a second beam-splitter, the expression for the normalized probability amplitudes of \( n-k \) photons arriving to detector \( D_1 \) and \( k \) photons arriving to detector \( D_2 \) (Fig. 1) is

\[
|n\rangle \rightarrow \frac{1}{(\sqrt{2})^n} \sum_{k=0}^{n} i^k \left( \frac{n!}{k!(n-k)!} \right)^\frac{1}{2} (e^{i\theta} - 1)^{n-k} (e^{i\theta} + 1)^k |n-k\rangle_{D1}|k\rangle_{D2}.
\]

For \( n = 1 \) we recover the well-known case where \( P_{D1} = \frac{1}{2} [1 + \cos(\theta)] \) and \( P_{D2} = \frac{1}{2} [1 - \cos(\theta)] \) [5, 6]. See Figure 2. The photon itself interferes. The output photons arrive to the detectors with a sinusoidal probabilistic distribution.

Figure 2. Interference of single photons: (Black) Probability of one photon on \( D_1 \) an zero photons on \( D_2 \) and (Red) probability of one photon on \( D_1 \) an zero photons on \( D_2 \).

This case represents a paradigm linking quantum and classic interference. A time series of single photon states \( |1\rangle \), has been performed experimentally many times in a MZI [7,8], validating the shape of the interference. If we increase the photon number to the state \( |2\rangle \), we note an important change in the probabilities. See Figure 3. Now, in the expansion of (3), the probability amplitude implies a cosine sum, whose arguments are \( \theta \) and \( 2\theta \). For this reason, the probability amplitude has an asymmetry between the maximum and minimum among the components (2,0)
and (0, 2). The minimum is more widespread than the maximum, which becomes narrower. Note that we draw the interference patterns for both ports.

Even though it is very complicated to obtain experimentally a quantum state with \( n \geq 2 \) [9], however, it is possible to validate the theory in the lower limit by using a two-photon source. In reference [8] a two-photon source coming from spontaneous parametric down-conversion (SPDC) it is used. Both photons are pumped collinearly into a MZI. In this way, the asymmetry of both states (2, 0) and (0, 2) can be tested. We can predict the interference pattern for quantum states with many photons, even if nowadays there are no sources which can produce those states. Figure 4 shows the probability distribution for a photon state \(|10\rangle\) distributed in the different possible outcomes. As we can observe from the figure, the asymmetries between maximum and minimum increases as the photon number increases. The mixed probabilities for \( P_{n-k} \) and \( P_k \) outline an envelope which amplitude decreases with \( n \) i.e. the individual interference patterns from different combinations begin to form an envelope.

On the other hand, there are well-defined phase shifts where the state is conserved as fully quantum, \( \theta = 0, \pi, 2\pi \ldots \) in the detector \( D_1 \) direction and \( \theta = \pi/2, 3\pi/2, 5\pi/2 \ldots \) in the detector \( D_2 \) direction. Outside these angles, the states lose the phase coherence rapidly building statistical mixes from the states with combinations that sum up ten photons. In order to appreciate better the fore mentioned effect, we calculate the interference of quantum states with one hundred photons at a time.

Figure 5 shows the probability distribution of finding \( n = 100 \) photons in both detectors, as a function of the dephasing angle \( \theta \). We can say the set of individual interference patterns should built up classical interference. We will show it in the next section.
3. Mixture of states and the passage to classical interference

The 50 : 50 beam splitter is a random photons generator. Since the sum of photons in both outputs should be always the same number, a dephasing element in one arm can control the statistics between quantum and poisson behavior. In order to prove this, we calculate the expectation value for the photon number $N(\theta)$ in the $D_1$ outcome of the interferometer,

$$N_1(\theta) = \sum_{n'_1=1}^{n} n'_1 P_{n'_1}(\theta), \quad (4)$$

where $N_1(\theta)$ represents the photon number average in the port $D_1$ of the interferometer and $P_{n'_1}$ represents the probability of $n'_1$ photons coming out from this port.

**Figure 6.** Average number of photons $N_1$ in the $D_1$ outcome of the IMZ. The pattern is fitted by the classic form $\frac{100}{\pi} [1 + \cos(\theta)]$.

In Figure 6 we show the average photon numbers in the $D_1$ port as a function of $\theta$. It is straightforward to recognize in this figure the classical interference pattern $P_{D_1} = \frac{N_1}{\pi} [1 + \cos(\theta)]$, and $P_{D_2} = \frac{N_1}{\pi} [1 - \cos(\theta)]$ [10]. Now, we discuss the meaning of the transition between a quantum state with $n$ photons to a classical state with an average $N$ of $n$ photons in one of the interferometer’s output. Evidently, the sum of the photons on both outcomes must be always $n$. In this part, we realized that the interference of a $n$-photon quantum state in the MZI depends on the interferometer’s characteristics. First, the $n$ photon train is forced to divide itself and reunite again randomly through de two option optical elements with the same probability. This creates two complementary interference patterns, each one composed of the statistical sum of the quantum interference patterns of the different probabilities. We can say this represents some kind of quantum decoherence originated in the geometry of the imposed trajectories of the quantum photon beam.

**Figure 7.** Widths as a function of $n$ and $\theta$ for $n = 1, 2, 10, 100$ and $1000$ photons, from small to bigger curves.

Now, we analyze the photon statistics. There is an average photon number and a standard deviation around this average, because for every dephasing angle, several probability curves of different amounts of photons converge. We show in Figure 7 the behavior of the standard deviation (distribution’s width) $\sigma(n, \theta)$ as a function of the dephasing angle and the photon number [10,11].

As it can be observed, the width is minimum for the angles where we have certainty of the photon counting and maximum for angles where the counting becomes random. If we normalize this quantity with the average photon square root, we find the graphics for every photon number.
have the same behaviour. Figure 8 shows the behavior of $F = \sigma/\sqrt{N}$. From the photon statistics theory we know if $F = 1$ the photon statistics is poissonian, in the other hand, if $F < 1$ the photon statistics is sub-poissonian.

The limit $F = 0$ corresponds to a pure quantum states $|n > |0 >$ or $|0 > |n >$, because for some values of $\theta$ the interferometer only has deviated the incoming photons to only one of the two outputs. The maximal unpredictability corresponds to the other limit $F = 1$, i.e. combination $|n - k >_{D_1} |k >_{D_2}$.

4. Discussion and Conclusions

We have obtained the quantum interference of the $n$-photon quantum states different combinations in a Mach-Zehnder interferometer. Given the randomly splitting of the quantum state in two states $|n - k > |k >$, the interference pattern is a statistical mixture with different levels of anticorrelation in terms of the dephasing parameter between both interferometer’s arms. In order to contrast this result with a classical source’s one, we know that if we introduce a coherent state $|\alpha >$ in the same interferometer, we obtain as a result

$$P_{D_1} = \frac{n}{2}[1 + \cos(\theta)], \quad P_{D_2} = \frac{n}{2}[1 - \cos(\theta)],$$

(5)

where $\bar{n}$ represents $|\alpha|^2$, the average photon number of the coherent state [10]. In this case $F_\alpha(\theta) = 1$. The coherent state’s statistics for any dephasing angle is constant and equal to the Poisson statistics. Then, an important result is that, even if the interference patterns of quantum light and classical light look very similar, their punctual statistics are different. Evidently, both statistics agree on the point of maximum mixture of the quantum states. We can consider this analysis as an exercise of controlled decoherence, because the quantum states come and go as the interferometer’s phase changes.

The fact the quantum interference pattern is similar to the classical interference pattern tells us that we cannot control each of the multiple components of the interference at the same time, because that is what happens with the poissonian light. As we cannot control the statistics, we cannot control the interference. Here, we have another important result: to introduce quantum sates in a Mach-Zehnder interferometer produces a classical interference pattern (in both outcomes), unless we are capable of isolate some of the interference components of the combination $|n - k > |k >$. In Figure 9 it is shown the probability curve of finding at the MZI
output the photon combination $|100⟩|0⟩$. As this one is not mixed with another combination, when we multiply it by 100 (the photon number at this output) it gives us directly the interference pattern for that individual combination. In this sense, this pattern represents one of the 101 quantum interference patterns. Experimentally it could be obtained through energy detectors with enough resolution (still under investigation [12]). Another option, as an state of the art, it could be an arrangement of beam splitters.

Finally, if we compare the interference patterns from Figs. 6 and 9, we can say that if there is a difference between the classical interference and the quantum interference, such difference grows with the photon number. The quantum interference has more spacial resolution, as it was expected.

In conclusion we study the interference of N photons in a IMZ. In order to determinate the output states of the interferometer we analyze the factor $F = \sigma / \sqrt{N}$, and we have quantum behaviour for some angles and a behaviour close to classic for others angles. That is, we can study whether it is possible to output states with some statistics on demand. On the other hand, we are interested in the transition from quantum to classical world. Our conclusion at this point is that classical interference is formed by the average of the quantum interference, this as seen in the average number of photons (Figure 6). This brings us to better understand this transition. The mixing of states is inevitable if we use classical detectors, but is possible to isolate some quantum quantum states using some arrays of beam splitters. This kind of isolated states can be so useful like NOON states [13–15], in quantum metrology.

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