On the Class of Similar Square \([-1, 0, 1]\)-Matrices Arising from Vertex maps on Trees

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Abstract

Let \( n \geq 2 \) be an integer. In this note, we show that the oriented transition matrices over the field \( \mathbb{R} \) of all real numbers (over the finite field \( \mathbb{Z}_2 \) of two elements respectively) of all continuous vertex maps on all oriented trees with \( n + 1 \) vertices are similar to one another over \( \mathbb{R} \) (over \( \mathbb{Z}_2 \) respectively) and have characteristic polynomial \( \sum_{k=0}^{n} x^k \). Consequently, the unoriented transition matrices over the field \( \mathbb{Z}_2 \) of all continuous vertex maps on all oriented trees with \( n + 1 \) vertices are similar to one another over \( \mathbb{Z}_2 \) and have characteristic polynomial \( \sum_{k=0}^{n} x^k \). Therefore, the coefficients of the characteristic polynomials of these unoriented transition matrices, when considered over the field \( \mathbb{R} \), are all odd integers (and hence nonzero).

Keywords: Similar matrices, oriented trees, (un)oriented transition matrices, vertex maps

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Let \( n \geq 2 \) be an integer and let \( T \) be a tree with \( n + 1 \) vertices \( V_1, V_2, \ldots, V_{n+1} \). The tree \( T \) has \( n \) edges, say, \( E_1, E_2, \ldots, E_n \). If vertices \( V_{j_1} \) and \( V_{j_2} \) are endpoints of an edge \( E \), then, we let \( [V_{j_1} : V_{j_2}] \) denote the edge \( E \), i.e., the set of all points in \( E \) and, following Bernhardt [1], we denote the positively oriented edge from \( V_{j_1} \) to \( V_{j_2} \) as \( \overrightarrow{[V_{j_1}, V_{j_2}]} \) and call \( V_{j_1} \) the first vertex of \( \overrightarrow{[V_{j_1}, V_{j_2}]} \) and \( V_{j_2} \) the second. We also define \( -[V_{j_1}, V_{j_2}] \) by putting \( -[V_{j_1}, V_{j_2}] = [V_{j_2}, V_{j_1}] \) and call it the negatively oriented edge from \( V_{j_1} \) to \( V_{j_2} \). So, the first vertex of \( -[V_{j_1}, V_{j_2}] \) is \( V_{j_2} \) and the second is \( V_{j_1} \). Thus, both \([V_{j_1}, V_{j_2}]\) and \( -[V_{j_1}, V_{j_2}] \) represent the same edge \([V_{j_1}, V_{j_2}]\), but with the opposite orientations. In the sequel, we denote these \( n \) positively oriented edges of \( T \) as \( \overrightarrow{E_1}, \overrightarrow{E_2}, \ldots, \overrightarrow{E_n} \) and call the resulting tree oriented tree and denote it as \( \overrightarrow{T} \). It is clear that there are exactly \( 2^n \) distinct such oriented trees \( \overrightarrow{T} \). Later, we shall see that different choices of orientations on the edges of \( T \) will not affect our main results. When no confusion arises, we shall always use \( V_1, V_2, \ldots, V_{n+1} \) and \( \overrightarrow{E_1}, \overrightarrow{E_2}, \ldots, \overrightarrow{E_n} \) to denote respectively the vertices and the oriented edges of any tree with \( n + 1 \) vertices.
Following [1], for any two vertices $V_i$ and $V_j$ in the oriented tree $\overrightarrow{T}$, a path from $V_i$ to $V_j$ is a sequence of oriented edges $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_m}$, where $\overrightarrow{e_k} \in \{\overrightarrow{E_s}, -\overrightarrow{E_s} : 1 \leq s \leq n\}$ for all $1 \leq k \leq m$, and the first vertex of $\overrightarrow{e_1}$ is $V_i$, the second vertex of $\overrightarrow{e_m}$ is $V_j$, and the second vertex of $\overrightarrow{e_\ell}$ is equal to the first vertex of $\overrightarrow{e_{\ell+1}}$ for all $1 \leq \ell \leq m - 1$. It is clear that, for any two vertices, $V_i$ and $V_j$, there is a unique shortest path from $V_i$ to $V_j$ in $\overrightarrow{T}$ which will be denoted as $[V_i, V_j]$. We also denote $-[V_i, V_j] = [V_j, V_i]$ as the shortest path from vertex $V_j$ to vertex $V_i$. From now on, when we write the shortest path $[V_i, V_j]$, we always mean the shortest path from vertex $V_i$ to vertex $V_j$ in $\overrightarrow{T}$. We also let $[V_i : V_j]$ denote the collection of all points in all (oriented) edges in the shortest path $[V_i, V_j]$.

Following [1, 2], let $f : T \rightarrow T$ be a continuous vertex map, i.e., $f$ is a continuous map such that the $n + 1$ vertices of $T$ form a periodic orbit and, for each $1 \leq i \leq n$, $f$ is monotonic on the unoriented edge $E_i = [V_{i+1} : V_i]$, meaning that, as the point $x$ moves from vertex $V_{i+1}$ to vertex $V_i$ monotonically along the edge $E_i$, the point $f(x)$ moves monotonically from vertex $f(V_{i+1})$ to vertex $f(V_i)$ along the shortest path $[f(V_{i+1}), f(V_i)]$ from $f(V_{i+1})$ to $f(V_i)$. We now let $\mathcal{F}$ denote the field of all real numbers and let $\mathbb{Z}_2$ denote the finite field $\{0, 1\}$ of two elements and let $\mathcal{F}$ be any field with unity $1$. We define the associated oriented transition $n \times n \{−1, 0, 1\}$-matrix $A_n(f) = (\alpha_{i,j})$ over $\mathcal{F}$ by putting the positively oriented edge $\overrightarrow{E_i} = [V_i, V_{i+1}]$ and putting

$$\alpha_{i,j} = \begin{cases} 1, & \text{if $\overrightarrow{E_j}$ appears in the shortest path $[f(V_{i+1}), f(V_i)]$ from vertex $f(V_{i+1})$ to vertex $f(V_i)$} \\ -1, & \text{if $-\overrightarrow{E_j}$ appears in the shortest path $[f(V_{i+1}), f(V_i)]$ from vertex $f(V_{i+1})$ to vertex $f(V_i)$} \\ 0, & \text{otherwise.} \end{cases}$$

and define the associated unoriented transition $n \times n \{0, 1\}$-matrix $B_n(f) = (\beta_{i,j})$ over $\mathcal{F}$ by putting, for all $1 \leq i \leq n$ and all $1 \leq j \leq n$, $\beta_{i,j} = 1$ if $\alpha_{i,j} \neq 0$ and $\beta_{i,j} = 0$ otherwise, or equivalently,

$$\beta_{i,j} = \begin{cases} 1, & \text{if the set inclusion $f(E_i) \supset E_j$ holds,} \\ 0, & \text{otherwise.} \end{cases}$$

There are exactly $2^n$ such oriented transition matrices $A_n(f)$ for each $f$ and yet they all have the same unoriented transition matrix $B_n(f)$. Later, we shall see that the determinant of $A_n(f)$ is $(-1)^n$ and that of $B_n(f)$ is an odd integer. In [4], we study the special case when $T$ is a compact interval in the real line and $f$ is a continuous vertex map on $T$. In this note, we generalize the main results in [4] for interval maps to vertex maps on trees. Surprisingly, the arguments used there almost work for vertex maps on trees. For completeness, we include the proofs.

Recall that $\mathcal{F}$ denotes a field with unity $1$. Let $W_{\mathcal{F}}^n(\mathcal{E}) = \{\sum_{i=1}^n r_i \overrightarrow{E_i} : r_i \in \mathcal{F}, 1 \leq i \leq n\}$ denote the $n$-dimensional vector space over $\mathcal{F}$ with $\mathcal{E} = \{\overrightarrow{E_j} : 1 \leq j \leq n\}$ as a basis. In the sequel, when there is no confusion, we shall write $W_{\mathcal{F}}^n$ instead of $W_{\mathcal{F}}^n(\mathcal{E})$. So, now we regard each positively oriented edge $\overrightarrow{E_j}$ as a basis element of the vector space $W_{\mathcal{F}}^n$ and regard the negatively oriented edge $-\overrightarrow{E_j}$ of $\overrightarrow{E_j}$ as an element in $W_{\mathcal{F}}^n$ such that $\overrightarrow{E_j} + (-\overrightarrow{E_j}) = \mathbf{0}$. Let $\sum_{i=1}^n r_i \overrightarrow{E_i}$ be an element of $W_{\mathcal{F}}^n$ such that $r_i \in \{-1, 0, 1\}$ for all $1 \leq i \leq n$. If there exist two vertices $V_i$ and $V_j$ such that $r_k = 1$ if and only if $\overrightarrow{E_k}$ appears in the shortest path $[V_i, V_j]$ from vertex $V_i$ to vertex
Let \( V_j \) and \( r_k = -1 \) if and only if \( -E_i^j \) appears in the shortest path \([V_i, V_j]\) from vertex \( V_i \) to vertex \( V_j \), then we define \([V_i, V_j] = \sum^n_{i=1} r_i E_i^j\) and \([V_j, V_i] = -[V_i, V_j] = -(\sum^n_{i=1} r_i E_i^j)\). In particular, if \([V_i, V_j] = e_1^i e_2^i \cdots e_m^i\) is the shortest path in the oriented tree \(\overrightarrow{T}\) defined as above, then, as elements of \(\overrightarrow{W}_x^n\), we have \([V_i, V_j] = \sum_{k=1}^m e_k\). Therefore, for any two vertices \( V_i \) and \( V_j \), the notation \([V_i, V_j]\) will have two meanings: It represents the unique shortest path from vertex \( V_i \) to vertex \( V_j \) in the oriented tree \(\overrightarrow{T}\) on the one hand, and represents the element of \(\overrightarrow{W}_x^n\) which is a sum of those oriented (positively or negatively) edges which appear in the unique shortest path \([V_i, V_j]\) from vertex \( V_i \) to vertex \( V_j \) on the other. There should be no confusion from the texts. With respect to the oriented transition \( n \times n \{-1, 0, 1\}\)-matrices \(A_n(f) = (\alpha_{i,j})\) of the continuous vertex tree map \( f \), we define a linear transformation \( \Phi_f \) from \(\overrightarrow{W}_x^n\) into itself such that, for each \( 1 \leq i \leq n \),

\[
\Phi_f([E_i]) = \sum_{j=1}^n \alpha_{i,j} [E_j].
\]

Therefore, if \(\overrightarrow{E_i} = [V_{i_1}, V_{i_2}]\) is a positively oriented edge of \(\overrightarrow{T}\) from vertex \( V_{i_1} \) to vertex \( V_{i_2} \), then, when considered as an element of \(\overrightarrow{W}_x^n\), we have, by definition of \(A_n(f)\) and \(\Phi_f\), \(\Phi_f(\overrightarrow{E_i}) = \sum_{j=1}^n \alpha_{i,j} [E_j] = [f(V_{i_1}), f(V_{i_2})]\) which also represents the unique shortest path from vertex \( f(V_{i_1}) \) to vertex \( f(V_{i_2}) \) in \(\overrightarrow{T}\).

We shall need the following fundamental result.

**Lemma 1.** For any distinct vertices \( V_i \) and \( V_j \) of the oriented tree \(\overrightarrow{T}\), we have \(\Phi_f([V_i, V_j]) = [f(V_i), f(V_j)]\). That is, if \([V_i, V_j]\) is the unique shortest path from vertex \( V_i \) to vertex \( V_j \) in \(\overrightarrow{T}\), then \(\Phi_f([V_i, V_j]) = [f(V_i), f(V_j)]\) is the unique shortest path from vertex \( f(V_i) \) to vertex \( f(V_j) \) in \(\overrightarrow{T}\). Similarly, if \( V_{i_1}, V_{i_2}, \ldots, V_{i_m} \) are vertices of \(\overrightarrow{T}\), then \(\sum_{k=1}^{m-1} [V_{i_k}, V_{i_{k+1}}] = [V_{i_1}, V_{i_m}]\).

**Proof.** Let \( V_i, V_k, V_j \) be three distinct vertices of the tree \(\overrightarrow{T}\). Assume that both \([V_i, V_k]\) and \([V_k, V_j]\) are positively oriented edges of \(\overrightarrow{T}\). If the set intersection \( f([V_i : V_k]) \cap f([V_k : V_j]) = \{f(V_k)\} \), then the concatenation of the shortest path \([f(V_i), f(V_k)]\) and the shortest path \([f(V_k), f(V_j)]\) becomes the shortest path \([f(V_i), f(V_j)]\) from vertex \( f(V_i) \) to vertex \( f(V_j) \). Therefore, we have \(\Phi_f([V_i, V_j]) = [f(V_i), f(V_j)]\). On the other hand, if the set intersection \( f([V_i : V_k]) \cap f([V_k : V_j]) = \{V_\ell : f(V_\ell) \neq f(V_k)\} \) for some vertex \( V_\ell \neq f(V_k) \), then the shortest path \([f(V_\ell), f(V_k)]\) in the shortest path \([f(V_i), f(V_k)]\) and the shortest path \([f(V_k), f(V_j)]\) in the shortest path \([f(V_k), f(V_\ell)]\) cancel out. So, \(\Phi_f([V_i, V_j]) = \Phi_f([V_i, V_k] + [V_k, V_j]) = \Phi_f([V_i, V_k]) + \Phi_f([V_k, V_j]) = [f(V_i), f(V_k)] + [f(V_k), f(V_j)] = ([f(V_i), V_\ell] + [V_\ell, f(V_k)]) + ([f(V_k), V_\ell] + [V_\ell, f(V_j)]) = [f(V_i), V_\ell] + [V_\ell, f(V_j)] = [f(V_\ell), f(V_j)] = [f(V_i), f(V_j)]\).

Assume that both \([V_i, V_k]\) and \([V_j, V_k]\) (\(= -[V_k, V_j]\)) are positively oriented edges of the oriented tree \(\overrightarrow{T}\). Then the shortest path \([V_i, V_j]\) is the concatenation of the positively oriented edge \([V_i, V_k]\) and the negatively oriented edge \((-[V_j, V_k])\). Thus, as elements of \(\overrightarrow{W}_x^n\), we have \([V_i, V_j] = [V_i, V_k] - [V_j, V_k]\). So, \(\Phi_f([V_i, V_j]) = \Phi_f([V_i, V_k] - [V_j, V_k]) = \Phi_f([V_i, V_k]) - \Phi_f([V_j, V_k]) = [f(V_i), f(V_k)] - [f(V_j), f(V_k)]\). If the set intersection \( f([V_i : V_k]) \cap f([V_j : V_k]) = \{f(V_k)\} \), then the concate-
nation of the shortest path $[f(V_i), f(V_k)]$ and the shortest path $-f(V_s), f(V_j)]$ becomes the shortest path $[f(V_i), f(V_j)]$ from vertex $f(V_i)$ to vertex $f(V_j)$. Therefore, we obtain that $\Phi_f([V_i, V_j]) = [f(V_i), f(V_j)]$. On the other hand, if the set intersection $f([V_i : V_k]) \cap f([V_j : V_k]) = [V_k : f(V_k)] \neq \{f(V_k)\}$ for some vertex $V_k \neq f(V_k)$, then the shortest path $[V_k, f(V_k)]$ in the shortest path $[f(V_i), f(V_k)]$ and the shortest path $-f(V_s), f(V_j)]$ in the shortest path $-f(V_j), f(V_k)]$ cancel out. Therefore, $\Phi_f([V_i, V_j]) = \Phi_f([V_i, V_k] - [V_j, V_k]) = \Phi_f([V_i, V_k]) - \Phi_f([V_j, V_k]) = [f(V_i), f(V_k)] - [f(V_j), f(V_k)] = ([f(V_i), V_i] + [V_k, f(V_k)]) - ([f(V_j), V_i] + [V_k, f(V_k)]) = [f(V_i), V_i] - [f(V_j), V_i] = [f(V_i), V_i] + [V_k, f(V_i)] = [f(V_i), f(V_j)]$.

If both $[V_k, V_i]$ or $[V_k, V_j]$ or, both $[V_k, V_i]$ and $[V_j, V_k]$ are positively oriented edges of the tree $\hat{T}$, then, by discussing cases depending on the set intersections $f([V_i : V_k]) \cap f([V_j : V_k])$ as above, we obtain that $\Phi_f([V_i, V_j]) = [f(V_i), f(V_j)]$. We omit the details.

So far, we have shown that $\Phi_f([V_i, V_j]) = [f(V_i), f(V_j)]$ as long as the shortest path $[V_i, V_j]$ consists of exactly two oriented edges. Now, if the shortest path $[V_i, V_j] = e_1^1 e_2^1 e_3^1$ consists of exactly three oriented edges $e_1^1, e_2^1, e_3^1$. Let the second vertex of $e_2^1$ be $V_k$. Then $[V_i, V_k] = e_1^1 e_2^1$. It follows from what we just proved above that $\Phi_f([V_i, V_k]) = [f(V_i), f(V_k)]$. Therefore, $\Phi_f([V_i, V_j]) = \Phi_f(e_1^1 e_2^1 e_3^1) = \Phi_f(e_1^1 e_2^1) + \Phi_f(e_3^1) = \Phi_f([V_i, V_k]) + \Phi_f([V_j, V_k]) + \Phi_f([V_i, V_k]) = \Phi_f([V_i, V_k]) + \Phi_f([V_j, V_k])$ whenever $[V_i, V_j] = e_1^1 e_2^1 e_3^1$ consists of exactly three oriented edges $e_1^1, e_2^1, e_3^1$. The general case when $[V_i, V_j]$ consists of more than 3 oriented edges can be proved similarly by induction. Therefore, $\Phi_f([V_i, V_j]) = [f(V_i), f(V_j)]$ as long as $V_i$ and $V_j$ are any two distinct vertices of $\hat{T}$.

Finally, if $V_{i_1}, V_{i_2}, \cdots, V_{i_m}$ are $m \geq 2$ distinct vertices of $\hat{T}$, then similar arguments show that $\sum_{k=1}^{m-1} [V_{i_k}, V_{i_{k+1}}] = [V_{i_1}, V_{i_m}]$. This completes the proof.

Lemma 2. $\Phi_f$ is an isomorphism from $\mathcal{W}^n_\mathcal{F}$ onto itself.

Proof. Let $\hat{f}$ be any continuous vertex map on the tree $T$ such that the composition $\hat{f} \circ f$ is the identity map on the vertices of $T$. Then, by Lemma 1, for each positively oriented edge $\hat{E}_i = [V_{i_1}, V_{i_2}]$, we have $(\Phi_f \circ \Phi_f)(\hat{E}_i) = \Phi_f(\Phi_f([V_{i_1}, V_{i_2}]]) = \Phi_f([f(V_{i_1}), f(V_{i_2}]]) = ([f \circ f](V_{i_1}), (f \circ f)(V_{i_2})] = [V_{i_1}, V_{i_2}] = \hat{E}_i$. Therefore, $\Phi_f$ is the inverse of $\Phi_f$.

We shall need the following result which is proved in [4]. For completeness, we include its proof.

Lemma 3. Let $1 \leq j \leq n$ be any fixed integer and let $b$ denote the greatest common divisor of $j$ and $n + 1$. Let $s = (n + 1)/b$. For every integer $1 \leq k \leq s - 1$, let $1 \leq m_k \leq n$ be the unique integer such that $k b \equiv m_k \pmod{n + 1}$. Then the $m_k$’s are all distinct and $\{m_k : 1 \leq k \leq s - 1\} = \{kb : 1 \leq k \leq s - 1\}$.
Proof. Let \( B = \{m_k : 1 \leq k \leq s-1\} \) and \( C = \{kb : 1 \leq k \leq s-1\} \). For every integer \( 1 \leq k \leq s-1 \), since \( j/b \) and \( (n + 1)/b \) are relatively prime, the congruence equation \( (j/b)x \equiv k \pmod{(n + 1)/b} \) has an integer solution \( x \) such that \( 1 \leq x \leq s-1 = [(n + 1)/b] - 1 \). Consequently, for every integer \( 1 \leq k \leq s-1 \), the congruence equation \( (mx \equiv jx \equiv kb \pmod{n + 1}) \) has an integer solution \( x \) such that \( 1 \leq x \leq s-1 \). Since \( 1 \leq kb \leq (s-1)b \leq n \) and \( 1 \leq m_k \leq n \) for every integer \( 1 \leq k \leq s-1 \), we obtain that \( C \subseteq B \). Since both \( B \) and \( C \) consist of exactly \( s-1 \) elements, we obtain that \( B = C \). That is, \( \{m_k : 1 \leq k \leq s-1\} = \{kb : 1 \leq k \leq s-1\} \). This completes the proof. \( \Box \)

Let \( M_1 \) and \( M_2 \) be two \( n \times n \) matrices over the field \( \mathcal{F} \). We say that \( M_1 \) is similar to \( M_2 \) through the invertible matrix \( G \) if \( M_1 \cdot G = G \cdot M_2 \). We can now prove our main result.

**Theorem 1.** Let \( n \geq 2 \) be an integer. Let \( T \) be any tree with \( n+1 \) vertices. Let \( f \) be a continuous vertex map on \( T \). Let \( R, \ Z_2, \ F, \ W^\rightarrow_n, \ W^\rightarrow_{Z_2}^n, \ \Phi_f, \ \mathcal{A}_n(f) \) and \( \mathcal{B}_n(f) \) be defined as above. Then the following hold:

1. For each integer \( 1 \leq i \leq n \), \( \sum_{k=1}^{n} \Phi^k_f(W^\rightarrow_i) = 0 \) and so, \( \sum_{k=1}^{n} \Phi^k_f(w) = 0 \) for all \( w \) in \( W^\rightarrow_n \).

2. Let \( i \) and \( j \) be two integers in the interval \([1, n]\) and let \( \vec{J} \) denote the shortest path \([V_i, f^j(V_i)]\) in \( \vec{T} \). If \( j \) and \( n+1 \) are relatively prime, then the set \( \mathcal{W}_f = \{\Phi^k_f(\vec{J}) : 0 \leq k \leq n-1\} \) is a basis for \( W^\rightarrow_{Z_2}^n \) and for \( W^\rightarrow_n \) when \( \mathcal{F} \) is a field with characteristic zero or the determinant of the matrix \( \mathcal{M}_f \) of the set \( \mathcal{W}_f \) with respect to the basis \( \vec{E} = \{\vec{E}_1, \vec{E}_2, \ldots, \vec{E}_n\} \) is not "divisible" by the finite characteristic of \( \mathcal{F} \). Furthermore, when \( T \) is a tree in the real line with \( n+1 \) vertices and \( f \) is a continuous vertex map on \( T \), then the constant term of the characteristic polynomial of the matrix \( \mathcal{M}_f \) is \( \pm 1 \) and hence the set \( \mathcal{W}_f \) is a basis of \( W^\rightarrow_n \) for any field \( \mathcal{F} \) (however, not all coefficients of the characteristic polynomial of the matrix \( \mathcal{M}_f \) are odd integers (see, for example, Figure 1(a) with \( \vec{J} = [1, 2] \) where the corresponding characteristic polynomial is \( x^5 - x^4 - x + 1 \)).

3. Over any field \( \mathcal{F} \) with characteristic zero (\( \mathbb{Z}_2 \) respectively), the oriented transition matrix \( \mathcal{A}_n(f) \) and its inverse \([\mathcal{A}_n(f)]^{-1}\), as \([-1, 0, 1]\)-matrices, are similar to the following companion matrix:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & -1 & \cdots & -1
\end{bmatrix}
\]

of the polynomial \( \sum_{k=0}^{n} x^k \) through invertible \([-1, 0, 1]\)-matrices over \( \mathcal{F} \) (\( \mathbb{Z}_2 \) respectively) and have the same characteristic polynomial \( \sum_{k=0}^{n} x^k \) while the unoriented transition matrices of all continuous vertex maps on \( T \) may not be similar to each other over the same field \( \mathcal{F} \) (\( \mathbb{Z}_2 \) respectively) (see Figures 1 - 3). Furthermore, if \( T \) is a tree in the real line, then the oriented transition matrices, when considered over any field, of all continuous vertex maps on \( T \) with \( n+1 \) vertices and their inverses are similar to one another through invertible \([-1, 0, 1]\)-matrices and have the same characteristic polynomial \( \sum_{k=0}^{n} x^k \).
(4) The coefficients of the characteristic polynomial of the unoriented transition matrix $B_n(f)$, when considered as a matrix over $\mathcal{R}$, are all odd integers (see Figures 1 - 4). Furthermore, the unoriented transition matrices over any field of all continuous vertex maps on all trees with $n + 1$ vertices, when considered as matrices over $\mathbb{Z}_2$, are similar to one another and have characteristic polynomial $\sum_{k=0}^{n} x^k$, but may not be similar to each other when considered over the finite field $\mathbb{Z}_p = \{0, 1, 2, \cdots, p - 1\}$, where $p \geq 3$ is a prime number (see Figures 1 - 3).

Proof. To prove Part (1), recall that $f$ is a continuous vertex map on the tree $T_1$. For any fixed integer $1 \leq i \leq n$, let $\overrightarrow{E}_i = [\overrightarrow{V}_i, \overrightarrow{V}_{i+1}]$ and let $1 \leq j \leq n$ be the unique integer such that $f^j(\overrightarrow{V}_i) = \overrightarrow{V}_{i+1}$. So, $\overrightarrow{E}_i = [\overrightarrow{V}_i, \overrightarrow{V}_{i+1}] = [\overrightarrow{V}_i, f^j(\overrightarrow{V}_i)]$. Let $b$ be the greatest common divisor of $j$ and $n + 1$ and let $s = (n + 1)/b$. So, $sj = (j/b)(sb) = (j/b)(n + 1)$. For every integer $1 \leq k \leq s - 1$, let $1 \leq m_k \leq n$ be the unique integer such that $kj \equiv m_k \pmod{n + 1}$. Then, by Lemma 3, we obtain that $\{m_k : 1 \leq k \leq s - 1\} = \{kj : 1 \leq k \leq s - 1\}$. Let $m_0 = 0$. Then $\{m_k : 0 \leq k \leq s - 1\} = \{kj : 0 \leq k \leq s - 1\}$. Hence, the set $\{1, 2, \cdots, n - 1, n\}$ is the disjoint union of the sets $\{m_k + m : 0 \leq k \leq s - 1\}, 0 \leq m \leq b - 1$. Therefore, by Lemma 1, $\sum_{k=0}^{s-1} \Phi_{m_k}(\overrightarrow{E}_i) = \sum_{k=0}^{s-1} \Phi_{kj}(\overrightarrow{E}_i)$ (since $kj \equiv m_k \pmod{n + 1}$) $= [\overrightarrow{V}_i, f^j(\overrightarrow{V}_i)] + [f^j(\overrightarrow{V}_i), f^{2j}(\overrightarrow{V}_i)] + [f^{2j}(\overrightarrow{V}_i), f^{3j}(\overrightarrow{V}_i)] + \cdots$ $+ [f^{(s-2)j}(\overrightarrow{V}_i), f^{(s-1)j}(\overrightarrow{V}_i)] + [f^{(s-1)j}(\overrightarrow{V}_i), \overrightarrow{V}_{i+1}] = 0$. Thus, $\sum_{k=0}^{b-1} \Phi_{m_k}(\overrightarrow{E}_i) = \sum_{m=0}^{b-1} \Phi_m(\sum_{k=0}^{s-1} \Phi_{m_k}(\overrightarrow{E}_i)) = 0$. Therefore, $\sum_{k=1}^{n} \Phi_f(w) = 0$ for all vectors $w$ in $W_{\mathbb{Z}_2}^n$. This establishes Part (1).

For the proof of Part (2), we first consider those $\Phi_f$ over the field $\mathbb{Z}_2$. Now we want to show that if $\mathcal{N}$ is a nonempty subset of $\{1, 2, \cdots, n - 1, n\}$ such that $\overrightarrow{J} + \sum_{k \in \mathcal{N}} \Phi_f(J) = \mathbf{0}$, then $\mathcal{N} = \{1, 2, \cdots, n - 1, n\}$. Indeed, for every integer $1 \leq k \leq n$, let $1 \leq m_k \leq n$ be the unique integer such that $kj \equiv m_k \pmod{n + 1}$. Assume that $(j) = m_1 \notin \mathcal{N}$. Then, for any $m \in \mathcal{N}$, $m \neq j$. So, $f^m(V_i) \neq f^j(V_i)$. If $(f^m(f^j(V_i))) = f^{m+j}(V_i) = f^j(V_i)$, then the least period of $f^j(V_i)$ under $f$ divides $m$ ($< n + 1$) which contradicts the fact that its least period under $f$ is $n + 1$. Therefore, the shortest path $\Phi^m_f(J) = \Phi^m_f([V_i, f^j(V)] = [f^j(V_i), f^{m+j}(V_i)]$ either contains the vertex $f^j(V_i)$ in its "interior" or does not contain it. So, in the expression of the element $\Phi^m_f(J) = \Phi^m_f([V_i, f^j(V_i)]) = [f^m(V_i), f^{m+j}(V_i)]$ as a sum of the basis elements $\overrightarrow{E}_k$'s, the number of the basis elements $\overrightarrow{E}_k$ which contain the vertex $f^j(V_i)$ as an endpoint is either 0 or 2. Since $\overrightarrow{J} = [V_i, f^j(V_i)]$ contains exactly one basis element $\overrightarrow{E}_k$ which has the vertex $f^j(V_i)$ as an endpoint, there are an odd number of basis elements $\overrightarrow{E}_k$'s which has the vertex $f^j(V_i)$ as an endpoint in the expression of the element $\overrightarrow{J} + \sum_{k \in \mathcal{N}} \Phi_f(J)$ as a sum of the basis elements $\overrightarrow{E}_k$'s. Consequently, $\overrightarrow{J} + \sum_{k \in \mathcal{N}} \Phi_f(J) \neq \mathbf{0}$. This is a contradiction. So, $(j) = m_1 \in \mathcal{N}$ and

$$0 = \overrightarrow{J} + \sum_{k \in \mathcal{N}} \Phi_f(J) = \overrightarrow{J} + \sum_{k \in \mathcal{N} \setminus \{m_1\}} \Phi_f(J)$$

$$= [V_i, f^j(V_i)] + [f^j(V_i), f^{2j}(V_i)] + \sum_{k \in \mathcal{N} \setminus \{m_1\}} \Phi_f(J).$$
Now assume that $m_2 \notin \mathcal{N} \setminus \{m_1\}$. Then, for any $m \in \mathcal{N} \setminus \{m_1\}$, $m \notin \{m_1, m_2\} \subset \{1, 2, \ldots, n\}$. If $f^m(V_i) = f^{2j}(V_i) = f^{m+j}(V_i)$, then $m \equiv m_2 \pmod{n+1}$. Since both $m$ and $m_2$ are integers in the set $\{1, 2, \ldots, n\}$ such that $m \equiv m_2 \pmod{n+1}$, we have $m = m_2$. This is a contradiction. If $f^{m+j}(V_i) = f^{2j}(V_i)$, then $m + j \equiv 2j \pmod{n+1}$ and so, $m \equiv j \pmod{n+1}$. Since both $m$ and $m_1$ are integers in the set $\{1, 2, \ldots, n\}$ such that $m \equiv m_1 \pmod{n+1}$, we have $m = m_1$. This is again a contradiction. Therefore, in the expression of the element $\Phi_f^m(\mathcal{J}) = \Phi_f([V_i, f^j(V_i)]) = [f^m(V_i), f^{m+j}(V_i)]$ as a sum of the basis elements $E_k$'s, the number of the basis elements $E_k$ which contain the vertex $f^{2j}(V_i)$ as an endpoint is either 0 or 2. Since $[V_i, f^{2j}(V_i)]$ contains exactly one basis element $E_k$ which has the vertex $f^{2j}(V_i)$ as an endpoint, there are an odd number of basis elements $E_k$'s which has the vertex $f^{2j}(V_i)$ as an endpoint in the expression of the element $[V_i, f^{2j}(V_i)] + \sum_{k \in \mathcal{N}\setminus\{m_1\}} \Phi^k_f(\mathcal{J})$ as a sum of the basis elements $E_k$'s. Consequently, $\mathcal{J} + \sum_{k \in \mathcal{N}} \Phi^k_f(\mathcal{J}) = [V_i, f^{2j}(V_i)] + \sum_{k \in \mathcal{N}\setminus\{m_1\}} \Phi^k_f(\mathcal{J}) \neq 0$. This is a contradiction. So, $m_2 \in \mathcal{N} \setminus \{m_1\}$ and

$$0 = \mathcal{J} + \sum_{k \in \mathcal{N}} \Phi^k_f(\mathcal{J}) = [V_i, f^{2j}(V_i)] + \sum_{k \in \mathcal{N}\setminus\{m_1\}} \Phi^k_f(\mathcal{J}).$$

$$= [V_i, f^{2j}(V_i)] + \Phi^{m_2}_f(\mathcal{J}) + \sum_{k \in \mathcal{N}\setminus\{m_1, m_2\}} \Phi^k_f(\mathcal{J})$$

$$= [V_i, f^{2j}(V_i)] + [f^{2j}(V_i), f^{3j}(V_i)] + \sum_{k \in \mathcal{N}\setminus\{m_1, m_2\}} \Phi^k_f(\mathcal{J})$$

$$= [V_i, f^{3j}(V_i)] + \sum_{k \in \mathcal{N}\setminus\{m_1, m_2\}} \Phi^k_f(\mathcal{J}).$$

Proceeding in this manner finitely many times, we obtain that $\{m_1, m_2, \ldots, m_{n-1}, m_n\} \subset \mathcal{N}$. Since $j$ and $n+1$ are relatively prime, we see that, by Lemma 3, $\{m_1, m_2, \ldots, m_n\} = \{1, 2, \ldots, n-1, n\}$. Since $\{m_1, m_2, \ldots, m_n\} \subset \mathcal{N} \subset \{1, 2, \ldots, n-1, n\}$, we obtain that $\mathcal{N} = \{1, 2, \ldots, n-1, n\}$. This proves our assertion.

Now assume that $\sum_{k=0}^{n-1} r_k \Phi^k_f(\mathcal{J}) = 0$, where $r_k = 0$ or 1 in $\mathcal{Z}_2$, for all $0 \leq k \leq n-1$. If $r_0 = 0$ and $r_\ell \neq 0$ for some integer $1 \leq \ell \leq n-1$, we may assume that $\ell$ is the smallest such integer. Since $\Phi_f$ is invertible on $\mathcal{W}_{\mathcal{Z}_2} n$, we obtain that $\mathcal{J} + \sum_{k=1}^{n-1-\ell} r_k \Phi^k_f(\mathcal{J}) = 0$. So, without loss of generality, we may assume that $r_0 \neq 0$. That is, we may assume that $\mathcal{J} + \sum_{k=1}^{n-1} r_k \Phi^k_f(\mathcal{J}) = 0$. Let $\mathcal{N} = \{k : 1 \leq k \leq n-1 \text{ and } r_k \neq 0\}$. Then we have $\mathcal{J} + \sum_{k \in \mathcal{N}} \Phi^k_f(\mathcal{J}) = 0$. However, it follows from what we just proved above that $\mathcal{N} = \{1, 2, \ldots, n-1, n\}$. This contradicts the assumption that $\mathcal{N} \subset \{1, 2, \ldots, n-1\}$. Therefore, the set $\{\Phi^k_f(\mathcal{J}) : 0 \leq k \leq n-1\}$ is linearly independent in the $n$-dimensional vector space $\mathcal{W}_{\mathcal{Z}_2} n$ and hence is a basis for $\mathcal{W}_{\mathcal{Z}_2} n$. Consequently, the matrix of the basis $\{\Phi^k_f(\mathcal{J}) : 0 \leq k \leq n-1\}$ over $\mathcal{Z}_2$ with respect to the basis $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\}$ of $\mathcal{W}_{\mathcal{Z}_2} n$, denoted as $[\mathcal{J}, \Phi_f(\mathcal{J}), \Phi^2_f(\mathcal{J}), \ldots, \Phi^{n-1}_f(\mathcal{J}) : \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n]$ (over $\mathcal{Z}_2$),

7
has nonzero determinant and hence equals 1. This implies that, over the general field \( \mathcal{F} \) with unity 1, the \( \{-1,0,1\} \)-matrix

\[
\mathcal{M}_f = [\vec{j}, \Phi_f(\vec{j}), \Phi_f^2(\vec{j}), \cdots, \Phi_f^{n-1}(\vec{j}) : \vec{E}_1, \vec{E}_2, \cdots, \vec{E}_n] \text{ (over } \mathcal{F})
\]

of the set \( \{\Phi_f^k(\vec{j}) : 0 \leq k \leq n - 1\} \) with respect to the basis \( \{\vec{E}_1, \vec{E}_2, \cdots, \vec{E}_n\} \) of \( \overrightarrow{W}_x^n \) also has nonzero determinant if the characteristic of \( \mathcal{F} \) is zero or the determinant of \( \mathcal{M}_f \) is not divisible by the finite characteristic of \( \mathcal{F} \). In particular, if \( \mathcal{F} = \mathbb{R} \) or \( \mathcal{F} = \mathbb{Z}_2 \), then the set \( \{\Phi_f^k(\vec{j}) : 0 \leq k \leq n - 1\} \) is a basis for \( \overrightarrow{W}_x^n \) [7]. Note that when \( \mathcal{F} = \mathbb{R} \), the determinant of \( \mathcal{M}_f \) is an odd integer. We do not know if it is always equal to \( \pm 1 \). However, when \( T \) is a tree in the real line, by choosing all orientations on the edges same direction, the matrix \( \mathcal{M}_f \) over \( \mathcal{R} \) is a Petrie matrix, i.e., in any row, nonzero entries are consecutive and are all equal to 1 or to \( -1 \). It follows from easy induction [6] that the determinant of any Petrie matrix is 0 or \( \pm 1 \). Since the determinant of \( \mathcal{M}_f \) over \( \mathcal{R} \) is nonzero, we obtain that the determinant of \( \mathcal{M}_f \) is \( \pm 1 \). The rest is easy and omitted. This confirms Part (2).

Now, for the general field \( \mathcal{F} \) with unity 1, let \( \overrightarrow{T} \) be the oriented tree on the interval \([1, n + 1]\) in the real line with \( n + 1 \) vertices \( \hat{V}_i = i, 1 \leq i \leq n + 1 \) and \( n \) positively oriented edges \( \overrightarrow{D}_j = [j, j + 1], 1 \leq j \leq n \). Let \( h \) be the continuous vertex map on \( \overrightarrow{T} \) such that \( h(x) = x + 1 \) for all \( 1 \leq x \leq n \) and \( h(x) = -nx + n^2 + n + 1 \) for all \( n \leq x \leq n + 1 \). Then \( \Phi_h(\overrightarrow{D}_k) = \overrightarrow{D}_{k+1} = \Phi_f^k(\overrightarrow{D}_1) \) for all \( 1 \leq k \leq n - 1 \) and \( \Phi_h(\overrightarrow{D}_n) = \Phi_h([\hat{V}_n, \hat{V}_{n+1}]) = [\hat{V}_{n+1}, \hat{V}_1] = -[\hat{V}_1, \hat{V}_{n+1}] = - \sum_{k=1}^n \overrightarrow{D}_k \). By definition, the set \( \mathcal{D} = \{\Phi_h^k(\overrightarrow{D}_1) : 0 \leq k \leq n - 1\} = \{\overrightarrow{D}_1, \overrightarrow{D}_2, \cdots, \overrightarrow{D}_n\} \) is a basis for \( \overrightarrow{W}_x^n(\mathcal{D}) \). Let \( 1 \leq i \leq n \) be a fixed integer and choose a fixed integer \( 1 \leq j \leq n \) such that \( j \) and \( n + 1 \) are relatively prime and let \( \overrightarrow{J} = [V_i, \overrightarrow{J}^j(V_i)] \).

Suppose the set \( \{\Phi_f^k(\overrightarrow{J}) : 0 \leq k \leq n - 1\} \) is a basis for \( \overrightarrow{W}_x^n(\mathcal{E}) \). (*)

Let \( \phi : \overrightarrow{W}_x^n(\mathcal{D}) \longrightarrow \overrightarrow{W}_x^n(\mathcal{E}) \) be the linear transformation defined by

\[
\phi(\overrightarrow{D}_k) = \Phi_f^{k-1}(\overrightarrow{J}) \text{ for all } 1 \leq k \leq n.
\]

Then \( \phi \) is an isomorphism and the matrix of the basis \( \{\Phi_f^k(\overrightarrow{J}) : 0 \leq k \leq n - 1\} \) with respect to the basis \( \mathcal{E} = \{\overrightarrow{E}_1, \overrightarrow{E}_2, \cdots, \overrightarrow{E}_n\} \) of \( \overrightarrow{W}_x^n(\mathcal{E}) \) is an \( n \times n \) \( \{-1,0,1\} \)-matrix. Furthermore,

\[
(\phi \circ \Phi_h)(\overrightarrow{D}_n) = \phi(\Phi_h(\overrightarrow{D}_n)) = \phi(- \sum_{k=1}^n \overrightarrow{D}_k) = - \sum_{k=1}^n \phi(\overrightarrow{D}_k) = - \sum_{k=1}^n \Phi_f^{k-1}(\overrightarrow{J}) = \Phi_f^n(\overrightarrow{J}) \text{ (by Part (1))}
\]

and, for every integer \( 1 \leq k \leq n - 1 \),

\[
(\phi \circ \Phi_h)(\overrightarrow{D}_k) = \phi(\Phi_h(\overrightarrow{D}_k)) = \phi(\overrightarrow{D}_{k+1}) = \Phi_f^k(\overrightarrow{J}) = \Phi_f(\Phi_f^{k-1}(\overrightarrow{J})) = \Phi_f(\phi(\overrightarrow{D}_k)) = (\Phi_f \circ \phi)(\overrightarrow{D}_k).
\]

Therefore, \( \Phi_f \) is similar to \( \Phi_h \) through \( \phi \). Similarly, \( \Phi_f \) is similar to \( \Phi_h \), where \( \hat{f} \) is any continuous vertex map on the oriented tree \( \overrightarrow{T} \) such that the composition \( \hat{f} \circ f \) on the vertices of \( \overrightarrow{T} \) is the identity.
map. So, the matrices $A_n(f)$ and $A_n(f) = [A_n(f)]^{-1}$ are similar to $A_n(h)$ over $\mathcal{F}$. Similarly, the matrices $A_n(g)$ and $[A_n(g)]^{-1}$ are similar to $A_n(h)$ over $\mathcal{F}$. Consequently, we obtain that the matrices $A_n(f)$, $[A_n(f)]^{-1}$, $A_n(g)$, and $[A_n(g)]^{-1}$ are similar to one another over $\mathcal{F}$. By Part (2), the above (*) holds for $\mathcal{F} = \mathbb{Z}_2$ and for any field $\mathcal{F}$ with $\text{char}(\mathcal{F}) = 0$. Therefore, the matrices $A_n(f)$, $[A_n(f)]^{-1}$, $A_n(g)$, and $[A_n(g)]^{-1}$ are similar to one another over $\mathbb{Z}_2$ and over any field $\mathcal{F}$ with $\text{char}(\mathcal{F}) = 0$. On the other hand, let $P_n(x) = x^n + \cdots$ denote the characteristic polynomial of $A_n(f)$ over $\mathbb{Z}_2$ or over a field $\mathcal{F}$ with $\text{char}(\mathcal{F}) = 0$. By Part (2), the degree of the minimal polynomial of the element $[V_i, f(V_i)]$ is at least $n$. It follows from Part (1) that the polynomial $\sum_{k=0}^n x^k$ is the minimal polynomial of $[V_i, f(V_i)]$. By the well-known Cayley-Hamilton theorem on matrices, we see that the element $[V_i, f(V_i)]$ also satisfies the polynomial $P_n(x) - \sum_{k=0}^n x^k$ whose degree is at most $n - 1 (< n)$. Therefore, $P_n(x) - \sum_{k=0}^n x^k = 0$, i.e., the characteristic polynomial of $A_n(f)$ is $\sum_{k=0}^n x^k$. This proves Part (3).

Let $A_n(f)$ and $B_n(f)$ be the oriented and unoriented transition matrices of $\Phi_f$ over $\mathcal{R}$ respectively. Then it follows from Part (3) that the characteristic polynomial of $A_n(f)$ is $\sum_{k=0}^n x^k$. When we consider $A_n(f)$ as a matrix over $\mathbb{Z}_2$, we obtain that $B_n(f) = A_n(f)$ and the characteristic polynomial of $B_n(f)$ is $\sum_{k=0}^n x^k$ over $\mathbb{Z}_2$. Consequently, the coefficients of the characteristic polynomial of $B_n(f)$ over $\mathcal{R}$ are all odd integers (see Figures 1 - 4). Furthermore, we see that $B_n(f) = A_n(f)$ over $\mathbb{Z}_2$. So, it follows from Part (3) that $B_n(f)$ and $B_n(g)$, when considered as matrices over $\mathbb{Z}_2$, are similar to each other. This proves Part (4) and completes the proof of the theorem. \hfill \square

Remark. Let $f$ be a continuous vertex map on the tree $T (= T_1)$ with $n+1 \geq 3$ vertices. For $\mathcal{F} = \mathcal{R}$ and any choices of orientations on the edges of $T$, it follows from Theorem 1(1) that the determinant of the corresponding oriented transition matrix $A_n(f)$ is $(-1)^n$ while, by Theorem 1(4), that of the corresponding unoriented transition matrix $B_n(f)$ is an odd integer which is not necessarily equal to $\pm 1$. See Figure 2 for some examples. In the following, we present two sufficient conditions which guarantee that the determinant of the corresponding unoriented transition matrix $B_n(f)$ is $\pm 1$. For other related problems regarding the unoriented transition matrices $B_n(f)$, we refer to [5] where (new) notions of one-sided and two-sided similarities or weak similarities of square $\{0, 1\}$-matrices are introduced and examples are presented. It is clear that notions of various similarities of the unoriented transition matrices similar to those considered in [5] can be generalized from trees in the real line (i.e., compact intervals) to arbitrary trees.

**Proposition 1.** Let $f$ be a continuous vertex map on the tree $T (= T_1)$ with $n+1 \geq 3$ vertices. For each integer $1 \leq i \leq n$, let $E_i = [V_{i1}, V_{i2}]$ be a positively oriented edge of the oriented tree $\overrightarrow{T}$. Let the field $\mathcal{F} = \mathcal{R}_+$, Then, by Lemma 1 and the definition of the map $\Phi_f$ on the vector space $\overrightarrow{W_{\mathcal{R}}^n}$, we obtain that $\Phi_f(E_i) = [f(V_{i1}), f(V_{i2})]$. Since $f$ is a continuous vertex map on the connected edge $E_i$ of the tree $T$, we can write $\Phi_f(E_i) = [f(V_{i1}), f(V_{i2})] = \sum_{j=1}^{m_i} r_{i,j} [V_{\ell_{i,j}}, \overrightarrow{V_{\ell_{i,j+1}}}]$, where, for each $1 \leq j \leq m_i$, $r_{i,j} = \pm 1$, $V_{\ell_{i,j}} = f(V_{i1})$, $V_{\ell_{i,m_i+1}} = f(V_{i2})$ and $[V_{\ell_{i,j}}, \overrightarrow{V_{\ell_{i,j+1}}}]$ is a positively oriented edge of the oriented tree $\overrightarrow{T}$. Then the following hold:

1. If, for each $1 \leq i \leq n$, $\Phi_f(E_i)$ has only one sign, i.e., $r_{i,1} = r_{i,2} = \cdots = r_{i,m_i}$ (this includes the cases when $T$ is a compact interval in the real line), then the corresponding oriented tran-
osition matrix $\mathcal{A}_n(f)$ of $f$ can be obtained from that of the corresponding unoriented transition matrix $\mathcal{B}_n(f)$ of $f$ by performing the following row operation: Multiplying one row by $-1$. Consequently, the determinant of the matrix $\mathcal{B}_n(f)$ is equal to $\pm 1$ times that of the matrix $\mathcal{A}_n(f)$ which is $\pm 1$ (see [4]).

(2) If, for each $1 \leq i \leq n$ such that $\Phi_f(\vec{E}_i)$ does not have one sign, there exists an integer $1 \leq k_i < m_i$ such that $r_{i,1} = r_{i,2} = \cdots = r_{i,k_i} \neq r_{i,k_i+1} = r_{i,k_i+2} = \cdots = r_{i,m_i}$ and $|r_{i,1}| = |r_{i,m_i}| = 1$, let $\hat{V}_{t_i,k_i}$ be the unique vertex of $T$ such that $f(\hat{V}_{t_i,k_i}) = V_{t_i,k_i}$ and let $\vec{e}_1 e_2 \cdots e_s$ be the shortest path from either $V_{i_1}$ or $V_{i_2}$ to $\hat{V}_{t_i,k_i}$ which passes through the edge $E_i = [V_{i_1}, V_{i_2}]$ (and so the second vertex of $\vec{e}_s$ is $\hat{V}_{t_i,k_i}$). If each one of $\Phi_f(\vec{e}_2), \Phi_f(\vec{e}_3), \cdots, \Phi_f(\vec{e}_{s-1})$ and $\Phi_f(\vec{e}_s)$ has only one sign, then the corresponding oriented transition matrix $\mathcal{A}_n(f)$ of $f$ can be obtained from that of the corresponding unoriented transition matrix $\mathcal{B}_n(f)$ of $f$ by performing the following two row operations: (i) Multiplying one row by $-1$ and (ii) Multiplying one row by $\pm 2$ and adding to another row. Consequently, the determinant of the matrix $\mathcal{B}_n(f)$ is equal to $\pm 1$ times that of the matrix $\mathcal{A}_n(f)$ which is $\pm 1$ (see Figures 1 & 3).

Remark. Figure 4 demonstrates a case which is not covered by Proposition 1, yet has the same conclusion. We note that although, for a continuous vertex map $f$ on a tree $T$ with $n + 1$ vertices, there are $2^n$ distinct oriented transition matrices, they all have one and the same unoriented transition matrix. Therefore, if we can find an orientation for the tree $T$ so that Proposition 1 applies, then we obtain that the determinant of the unoriented transition matrix is $\pm 1$. Figure 4 is such an example.
Figure 1: The above 6 matrices are part of oriented transition matrices of continuous vertex maps on the oriented trees with 6 vertices right above them. They are all similar to one another over the field \( \mathcal{F} \). The characteristic polynomials of their corresponding unoriented transition matrices are

(a) \( x^5 - 3x^4 + x^3 + x^2 - 3x + 1 \); (b) \( x^5 - x^4 - 3x^3 - 3x^2 + x - 1 \); (c) \( x^5 - x^4 - 3x^3 + x^2 - x + 1 \);

(d) \( x^5 - x^4 - 3x^3 + x^2 + x + 1 \); (e) \( x^5 - 3x^4 + x^3 - 3x^2 - x - 1 \); and (f) \( x^5 - x^4 - x^3 - x^2 - x - 1 \) respectively.
Figure 2: The characteristic polynomials of the above corresponding unoriented transition matrices are $x^{11} - x^{10} - 7x^9 + 7x^8 + 7x^7 - 7x^6 + 3x^5 - 7x^4 - x^3 - x^2 - 3x + 3$ and $x^{11} - x^{10} - 9x^9 + 5x^8 + 25x^7 - x^6 - 25x^5 - 11x^4 + 9x^3 + 9x^2 - x - 3$ respectively.

Figure 3: The characteristic polynomials of the above corresponding unoriented transition matrices are $x^{11} - x^{10} - 7x^9 + 7x^8 + 13x^7 - 13x^6 - 7x^5 + 5x^4 + x^3 - x^2 - x + 1$ and $x^{11} - x^{10} - 7x^9 + 3x^8 + 11x^7 + 5x^6 + x^5 - 5x^4 - 5x^3 - 3x^2 + x + 1$ respectively.
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