EDGE UNIVERSALITY FOR A CLASS OF REPULSIVE PARTICLE SYSTEMS

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ABSTRACT. We study a class of interacting particle systems on \( \mathbb{R} \) which was recently investigated by F. Götzte and the second author ([GV14]). These ensembles generalize eigenvalue ensembles of Hermitian random matrices by allowing different interactions between particles. Although these ensembles are not known to be determinantal one can use the stochastic linearization method of [GV14] to represent them as averages of determinantal ones. We prove that the local correlations of the particles at the edge of the support of the limiting measure are universal and equal those for the eigenvalue ensembles. In particular, they are given in terms of the Airy kernel. Moreover, we prove that the largest particle converges, appropriately rescaled, to the Tracy-Widom distribution. In the regime of the Limit Law the averaging only shows through a weaker bound on the rate of convergence. We also obtain the leading order behavior of the upper tail of the distribution of the largest particle in a significant part of the regime of moderate deviations. For the averaging procedure we need detailed asymptotic information on the behavior of Christoffel-Darboux kernels, uniformly for a perturbative family of weights. Such results have been provided by K. Schubert, K. Schüler and the authors in [KSSV14].

1. INTRODUCTION AND MAIN RESULTS

In this article we concentrate on the distributions arising as correlations of eigenvalues of Hermitian random matrices at the edge of the spectrum, in particular the famous Tracy-Widom distribution. In [GV14], a class of particle systems on \( \mathbb{R} \) was studied which generalized certain random eigenvalue ensembles. It was shown there that the local bulk statistics, measured by the correlation functions, were in the limit of infinitely many particles the same as for those from random eigenvalues. The purpose of this paper is to extend the results of [GV14] to the edge of the “spectrum” and to prove that the fluctuations of the largest particle are in the limit given by the famous Tracy-Widom distribution.

We consider particle ensembles on \( \mathbb{R} \) with density proportional to

\[
\prod_{i<j} \varphi(x_i - x_j) e^{-N \sum_{j=1}^{N} Q(x_j)},
\]

where \( Q: \mathbb{R} \to \mathbb{R} \) is a continuous function of sufficient growth at infinity compared to the continuous function \( \varphi: \mathbb{R} \to [0, \infty) \). Apart from some technical conditions we will assume

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that

$$\varphi(0) = 0, \quad \varphi(t) > 0 \text{ for } t \neq 0 \quad \text{and} \quad \lim_{t \to 0} \frac{\varphi(t)}{|t|^2} = c > 0,$$

(2)

or, in other terms, 0 is the only zero of $\varphi$ and it is of order 2. The connection to random matrix theory is as follows: If $\varphi(t) = t^2$, then we have the unitary invariant ensembles $P_{N,Q}$ given by

$$P_{N,Q} := \frac{1}{Z_{N,Q}} \prod_{i<j} |x_i - x_j|^2 e^{-N \sum_{j=1}^{N} Q(x_j)},$$

(3)

$P_{N,Q}$ is the distribution of the eigenvalues for the random $(N \times N)$ Hermitian matrix with density proportional to

$$e^{-N \text{Tr}(Q(X))}$$

w.r.t. the Lebesgue measure on the space of $(N \times N)$ Hermitian matrices. Here $\text{Tr}$ denotes the trace and $Q(X)$ is defined by spectral calculus. This matrix distribution is invariant under unitary conjugations. The unitary invariant ensembles (3) have been well-understood in terms of asymptotic behavior of the eigenvalues. A common way to describe their properties is to use correlation functions. For a probability measure $P_N(x)dx$ on $\mathbb{R}^N$ and a natural number $k \leq N$ define

$$\rho^k_N(x_1, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} P_N(x)dx_{k+1} \ldots dx_N.$$ 

The $k$-th correlation function is a probability density on $\mathbb{R}^k$, the corresponding measure is called $k$-th correlation measure. Note that in the Random Matrix literature the $k$-th correlation function often differs from the definition above by the combinatorial factor $N!/(N-k)!$.

The key for the detailed understanding of the local eigenvalue statistics of unitary invariant ensembles $P_{N,Q}$ is that they are determinantal ensembles with a kernel that can be expressed in terms of orthogonal polynomials. More precisely, denote by $K_N$ the Christoffel-Darboux kernel of degree $N$ associated to the orthogonal polynomials with respect to the measure $e^{-NQ(x)}dx$. Then, for all $1 \leq k \leq N$ one can express the $k$-th correlation function by

$$\frac{N!}{(N-k)!} \rho^k_N(x_1, \ldots, x_k) = \det ((K_N(x_i, x_j))_{1 \leq i,j \leq k}).$$

(4)

To state our results, it is convenient to rewrite (1). Let $h : \mathbb{R} \to \mathbb{R}$ be continuous, even and bounded below. Let $Q : \mathbb{R} \to \mathbb{R}$ be continuous, even and of sufficient growth at infinity. Consider the probability density on $\mathbb{R}^N$ given by

$$P^h_{N,Q}(x) := \frac{1}{Z^h_{N,Q}} \prod_{i<j} |x_i - x_j|^2 \exp\{-N \sum_{j=1}^{N} Q(x_j) - \sum_{i<j} h(x_i - x_j)\},$$

(5)

where $Z^h_{N,Q}$ is the normalizing constant. Choosing $\varphi(t) := t^2 \exp\{-h(t)\}$, we see that $P^h_{N,Q}$ is in the form (1). The conditions (2) are satisfied as $h$ has no singularities.
G. Borot has pointed out to the second author that $P_{N,Q}^h$ is the eigenvalue distribution of the unitary invariant ensemble of Hermitian matrices with density proportional to
\[ e^{-N \text{Tr}(Q(X)) - \text{Tr}(h(X \otimes I - I \otimes X))}, \]
where $I$ denotes the identity matrix (cf. [BEO13]). However, we will rather view $P_{N,Q}^h$ as particle ensemble and therefore speak of particles instead of eigenvalues. Furthermore, the ensemble does not seem to be determinantal, in contrary to $P_{N,Q}^h$.

Our method of proof requires the external field $Q$ to be sufficiently convex. To quantify this, we define for $Q$ being twice differentiable and convex $\alpha_Q := \inf_{t \in \mathbb{R}} Q''(t)$. In the following we will denote by $\rho_{N,Q}^{h,k}$ the $k$-th correlation function or measure of $P_{N,Q}^h$.

The first information we need is the global behavior of the particles, i.e. their asymptotic location and density. To this end, we state the following theorem which was proved in [GV14].

**Theorem 1 ([GV14, Theorem 1]).** Let $h$ be a real analytic and even Schwartz function. Then there exists a constant $\alpha^h \geq 0$ such that for all real analytic, strictly convex and even $Q$ with $\alpha_Q > \alpha^h$, the following holds:

|\[
\lim_{N \to \infty} \int g \rho_{N,Q}^{h,k} d^k t = \int g d \left( \mu_Q^h \right)^{\otimes k},
|\]

for any bounded and continuous function $g : \mathbb{R}^k \to \mathbb{R}$.

**Remark 2.**

a) If $h$ is positive definite (i.e. with nonnegative Fourier transform), $\alpha^h$ can be chosen as $\alpha^h = \sup_{t \in \mathbb{R}} -h''(t)$. This remark extends to all results in this paper.

b) It was shown in [GV14] that the measure $\mu_Q^h$ is the equilibrium measure with respect to the external field $V(x) = Q(x) + \int h(x-t) d\mu_Q^h(t)$, i.e. it is the unique Borel probability measure on $\mathbb{R}$ which minimizes the functional

|\[
\mu \mapsto \int \int \log |t - s|^{-1} d\mu(t)d\mu(s) + \int V(t)d\mu(t).
|\]

Its density can be written as

|\[
\frac{d\mu_Q^h(t)}{dt} = \frac{1}{2\pi b^2} G_V(t/b) \sqrt{b^2 - t^2} \mathbb{1}_{[-b,b]}(t),
|\]

where $G_V$ (see (11)) is a real-analytic function which is strictly positive on a neighborhood of $[-1,1]$.

c) In [BdMPS95], ensembles with many-body interactions are considered, replacing $h$ in [5]. Here global asymptotics but not local correlations are discussed. In the case of pair interactions, the classes of admissible interactions in [BdMPS95] and in this paper are different. In [BdMPS95], a convexity condition is posed, depending solely on the additional interaction potentials where our conditions depend on both $Q$ and $h$. The characterisation of the limiting measure is different, too.
Asymptotic expansions of the partition functions can be found in the recent paper [BEO13]. Further global asymptotics for repulsive particle systems have been studied in [CGZ14, CP14].

On the local scale, there are basically two different regimes of interest. The local regime in the bulk of the particles has been investigated in [GV14], yielding asymptotics in terms of the sine kernel \( \sin(\pi(t-s))/\pi(t-s) \).

The asymptotics at the edge are governed in terms of the Airy kernel. The Airy function \( \text{Ai} : \mathbb{R} \rightarrow \mathbb{R} \) is uniquely determined by the requirements that it is a solution to the differential equation \( f''(t) = tf(t) \) and has the asymptotics \( \text{Ai}(t) \sim (4\pi t)^{-1/2} e^{-\frac{2}{3}t^{3/2}} \) as \( t \to \infty \). The Airy kernel is defined by

\[
K_{\text{Ai}}(t, s) := \frac{\text{Ai}(t)\text{Ai}'(s) - \text{Ai}'(t)\text{Ai}(s)}{t-s}
\]

extended continuously onto the diagonal \( s = t \). The first main result of this work is the edge universality of the correlation functions of \( \rho^h_{N,Q} \).

**Theorem 3.** Let \( h \) and \( Q \) satisfy the assumptions of Theorem 1 and assume furthermore that the Fourier transform of \( h \) decays exponentially fast. Let \( [-b, b] \) denote the support of \( \mu^h_Q \). Set \( c^* := \frac{2^{-1/3}}{b}G_V(1)^{2/3} \) with \( G_V \) from (7). Let \( q < 0 < p \) and \( 0 < \sigma < 1/3 \) be given. Then we have

\[
\left( \frac{N^{1/3}}{c^*} \right)^k \rho^h_{N,Q}(b + \frac{t_1}{c^*N^{2/3}}, \ldots, b + \frac{t_k}{c^*N^{2/3}}) = \det [K_{\text{Ai}}(t_i, t_j)]_{1 \leq i, j \leq k} + O(N^{-\sigma}). \tag{8}
\]

with the \( O \) term being uniform in \( N \) and \( t \in [q, pN^{4/15}]^k \).

Let us remark that the Airy kernel determinant is the leading term in (8) only for \( t \in [q, o((\log N)^{2/3})]^k \). The condition of exponential decay of the Fourier transform of \( h \) allows for writing \( h \) as difference of positive definite and real-analytic functions. This technical condition is needed and explained in Section 4.

The next theorem shows that the distribution of the rescaled largest particle converges weakly to the Tracy-Widom distribution.

**Theorem 4.** Let \( x_{\text{max}} \) denote the largest component of a vector \( x \). Under the conditions and with the notation of Theorem 3 we have for any \( s \in \mathbb{R} \) and any \( 0 < \sigma < 1/3 \)

\[
F^h_{N,Q} \left( (x_{\text{max}} - b) c^*N^{2/3} \leq s \right) = F_2(s) + O(N^{-\sigma}),
\]

where \( F_2 \) is the distribution function of the (\( \beta = 2 \)) Tracy-Widom distribution,

\[
F_2(s) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{(s, \infty)^k} \det \left( K_{\text{Ai}}(t_i, t_j) \right)_{i,j \leq k} \, dt.
\]

The distribution function \( F_2 \) can also be expressed as

\[
F_2(s) = \exp \left\{ -\int_s^\infty (t - s)q(t)^2 \, dt \right\},
\]
where \( q \) is the solution of the differential equation
\[
q''(t) = tq(t) + 2q(t)^3
\]
which has the asymptotic \( q(t) \sim Ai(t) \) as \( t \to \infty \).

**Remark.** If the repulsion exponent 2 in (3) is replaced by an arbitrary \( \beta > 0 \), we arrive at the so-called \( \beta \)-ensembles. The local correlations in the bulk are, at least in the one-cut case, known to be universal, i.e. independent of \( Q \) (see [BEY14b] [BEY12] [Shc14] [BFG13]). Bulk universality for particle systems with general repulsion \( \beta \) (the \( \beta \)-analog of (5)) has been shown in [Ven13] under the same assumptions on \( Q \) and \( h \) as in the present article, except for the decay condition on \( h \). Edge universality for \( \beta \)-ensembles with one-cut support has been proved in the recent works [BEY14a] and [BFG13].

Complementary to Theorem 4, we provide results on moderate deviations for the upper tail of the largest particle. In order to state it, we introduce the function \( \eta_V : \mathbb{R} \to \mathbb{R} \),
\[
\eta_V(t) := \int_1^t \sqrt{s^2 - 1} \ G_V(s)ds \quad \text{for} \quad t > 1,
\]
where the function \( G_V \) is defined in (11) below for the external field \( V \) introduced in Remark 2 above.

**Theorem 5.** Let \( x_{\max} \) denote the largest component of a vector \( x \). Under the conditions and with the notation of Theorem 3 we have that for any choice of \( 2/5 < \alpha < \varepsilon < 2/3 \) and \( t \in (b + N^{-\varepsilon}, b + 2^{-\alpha}) \)
\[
P_{N,Q}^h (x_{\max} > t) = \frac{b^3}{4\pi} \frac{e^{-N\eta_V(t/b)}}{N((t^2 - b^2)^{3/2}G_V(t/b))} (1 + o(1))
\]
as \( N \to \infty \) with an \( o(1) \) bound that is uniform in \( t \).

Observe that Theorem 5 does not cover all of the regime of moderate deviations since e.g. the values \( \alpha \in (0, 2/5) \) for the exponent are not included in the statement. However, until the very recent work [Sch15] even in the most studied case of the Gaussian Unitary Ensemble (GUE) the leading order behavior of the upper tail of the largest eigenvalue had not been proven anywhere in the regime of Theorem 5.

The paper is organized as follows. In Section 2 we collect all results from [KSSV14] on determinantal ensembles that are needed in this paper and bring them in a form that is suitable for our purposes. Section 3 provides a sketch of the central ideas introduced in [GV14] that guide the proofs of all our main results. Then, in Section 4 we derive our universality result on the correlation functions. The last section deals with the distribution of the largest particle and we show the Tracy Widom Limit Law and obtain the leading order behavior of the upper tail for a significant part of the moderate deviations’ regime.

## 2. Asymptotics for determinantal ensembles

The analysis of \( P_{N,Q}^h \) uses a representation of the new pair interaction \( \sum_{i<j} h(x_i - x_j) \) in terms of random linear statistics \( \sum_i f(x_i) \). Results for \( P_{N,Q}^h \) can then be deduced from corresponding results for eigenvalue ensembles of the form \( P_{N,V+f/N} \). Here \( V \) is an external field depending on \( Q \) and \( h \) (see Remark 2). For obtaining rates of convergence as in Theorem 3 the corresponding rates for ensembles \( P_{N,V+f/N} \) have to be uniform in \( f \). As a truncation
of $P^h_{N,Q}$ to some compact $[-L,L]^N$ will be made later on, we treat ensembles with support $[-L,L]$ only.

As $f$ will in general not be symmetric around 0, let us recall the notion of the equilibrium measure with respect to a general external field $V$. By this we mean the unique Borel probability measure $\mu_V$ which minimizes the energy functional

$$\mu \mapsto \int \int \log |t-s|^{-1} d\mu(t)d\mu(s) + \int V(t)d\mu(t).$$

A general reference on equilibrium measures with external fields is [SI97]. It is known that under mild assumptions such a unique minimizer exists. In our case $V$ will be convex and the density of $\mu_V$ can be described as follows (see e.g. [KSSV14, Sec. 2]). There exist real numbers $a_V < b_V$ that are uniquely determined by the two equations for $a$ and $b$,

$$\int_a^b \frac{V'(t)}{\sqrt{(b-t)(t-a)}} dt = 0, \quad \int_a^b \frac{tV'(t)}{\sqrt{(b-t)(t-a)}} dt = 2\pi.$$

We denote by $\lambda_V$ the linear rescaling that maps the interval $[-1,1]$ onto $[a_V, b_V]$, 

$$\lambda_V(t) := \frac{b_V - a_V}{2} t + \frac{a_V + b_V}{2}, \quad (10)$$

and introduce a function $G_V$ on $\mathbb{R}$ by

$$G_V(t) := \frac{1}{\pi} \int_{-1}^1 \int_0^1 \frac{(V \circ \lambda_V)'(t + s(t-t))}{\sqrt{1-s^2}} duds, \quad (11)$$

that inherits real analyticity and postivity from $V''$. The density of the equilibrium measure is then given by

$$d\mu_V(t) = \frac{8}{(b_V - a_V)^2 \pi} \sqrt{(t - a_V)(b_V - t)} G_V(\lambda_V^{-1}(t)) 1_{[a_V,b_V]}(t) dt,$$

As mentioned above, the linearization technique of [GV14] requires to consider unitary invariant ensembles $P_{N,V,f/N,L}$ that arise from restricting $P_{N,V,f/N}$ to $[-L,L]^N$ and renormalizing them as a probability measures. This will always be done in such a way that the support $[a_V, b_V]$ of $\mu_V$ is contained in the interior of $[-L,L]$. The functions $f$ will be defined on some fixed domain $D$ in the complex plane that contains $[-L,L]$. Moreover, they will be chosen from the real Banach space $X_D := \{ f : D \to \mathbb{C} : f \text{ analytic and bounded}, f(D \cap \mathbb{R}) \subset \mathbb{R} \}$, equipped with the norm $\|f\| := \|f\|_{D,\infty} := \sup_{z \in D} |f(z)|$. By [KSSV14] Lemma 2.4, the maps $\hat{V} \mapsto a_V, \hat{V} \mapsto b_V$ that relate the external fields to the endpoints of the support of their equilibrium measures are $C^1$ with bounded derivatives on a neighborhood of $V$ in $X_D$. Thus, for sufficiently large values of $N$,

$$a_{V,f} - a_V, b_{V,f} - b_V = O\left(\frac{\|f\|}{N}\right), \quad (12)$$

where the subscript $V,f$ is short for $V+f/N$. We can now formulate a first result on kernels $K_{N,V,f/L}$ associated to determinantal ensembles of the form $P_{N,V,f/L}$ that has essentially been proved in [KSSV14].
Proof. It is immediate from [KSSV14, Theorem 1.8] that

\[ \inf_{G} \text{the positivity of } K \text{ kernels deviations (Theorem 5) we need the following asymptotics for the diagonal entries of the} \]

\[ \text{determine the leading order of the distribution of the largest particle in the regime of moderate} \]

\[ \text{Proposition 6 is an essential ingredient to the proofs of Theorems 3 and 4. In order to} \]

\[ \text{now follows from the global Lipschitz continuity of the Airy kernel in the domain considered} \]

\[ \text{relations (10)-(12), the thrice differentiability of } V \text{ on } [-L,L], \text{ the analyticity of } f \text{ in } D, \text{ and} \]

\[ \text{the positivity of } G, \text{ one obtains} \]

\[ G_{V,f}(t) = G_{V}(t)(1 + \mathcal{O}(\|f\|/N)) \] (14)

\[ \text{uniformly for } f \in X_D \text{ with } \|f\|_{D,\infty} \leq N^\kappa, \text{ and also uniformly in } t \in \mathbb{R} \text{ with both } \lambda_V(t), \lambda_V(t) \in [-L,L]. \text{ This yields in particular } \gamma_{V,f} = \gamma_{V}(1 + \mathcal{O}(\|f\|/N)). \text{ The proposition now follows from the global Lipschitz continuity of the Airy kernel in the domain considered} \]

\[ b_V + \frac{t}{N^{2/3}} = b_{V,f} + \frac{t'}{N^{2/3}} \]

\[ \text{leads for all relevant values of } f \text{ and } t \text{ to the uniform estimate} \]

\[ t' = t + tO(\|f\|/N) + N^{2/3} (b_V - b_{V,f}) = t + \mathcal{O}(N^{\kappa-1/3}). \] (15)

Proposition 6 is an essential ingredient to the proofs of Theorems 3 and 4. In order to determine the leading order of the distribution of the largest particle in the regime of moderate deviations (Theorem 5) we need the following asymptotics for the diagonal entries of the kernels $K_{N,V,f,L}(t, t)$ which have essentially been proved in [KSSV14]. Due to the assumed evenness of $Q$ and $h$ it suffices to formulate them for the symmetric situation $a_V = -b_V$ only.

Proposition 7 (cf. [KSSV14, Theorem 1.5]). Let $V$ satisfy the assumptions of Proposition 6 and assume in addition $a_V = -b_V$. Recall the notations introduced in the beginning of this
section and the definiintion of the function \( \eta_V \) in \([9]\). There exists a positive constant \( c \) only depending on \( V \) such that for \( b_V + cN^{-2/3} < t < L \):

\[
K_{N,V,f:L}(t,t) = \frac{b_V}{4\pi} e^{-N\eta_V(t/b_V)} \frac{1}{t^2 - b_V^2} \times \left[ 1 + \mathcal{O} \left( e^{\mathcal{O}(\sqrt{t-b_V} ||f||)} - 1 \right) + \mathcal{O} \left( \frac{||f||}{N(t-b)} \right) + \mathcal{O} \left( \frac{1}{N||t-b_V||^{3/2}} \right) \right]
\]

with all \( \mathcal{O} \) terms being uniform in \( N, t \in (b_V+cN^{-2/3}, L) \), and in \( f \in X_D \) with \( ||f||_{D,\infty} \leq N^\kappa \), \( 0 < \kappa < 1/3 \).

Proof. From the arguments given in the beginning of the proof of Proposition \([5]\) it follows that we can apply the result \([KSSV14]\) Theorem 1.5 (ii) to the ensembles \( P_{N,V+f/N:L} \) provided that \( N \) is sufficiently large. Hence there exists a positive number \( \tilde{c} \) such that for all \( x > 1+cN^{-2/3} \) with \( \lambda_{V,f}(x) \leq L \)

\[
K_{N,V,f:L}(\lambda_{V,f}(x), \lambda_{V,f}(x)) = \frac{1}{2\pi(b_{V,f} - a_{V,f})} e^{-N\eta_{V,f}(x)} \frac{1}{x^2 - 1} \left[ 1 + \mathcal{O} \left( \frac{1}{N(x-1)^{3/2}} \right) \right],
\]

holds with the required uniformity. Setting \( x = \lambda_{V,f}^{-1}(t) \) we immediately arrive at

\[
K_{N,V,f,L}(t,t) = \frac{1}{2\pi} \frac{b_{V,f} - a_{V,f}}{4(t-b_{V,f})(t-a_{V,f})} \left[ 1 + \mathcal{O} \left( \frac{1}{N||t-b_{V,f}||^{3/2}} \right) \right].
\]  \hspace{1cm} (16)

Using \( \kappa < 1/3 \) and the lower bound on \( t \), we learn from \([12]\) that for real exponents \( \alpha \) one has

\[
(t - b_{V,f})^\alpha = (t - b_V)^\alpha \left[ 1 + \mathcal{O} \left( \frac{||f||}{N(t-b)} \right) \right]
\]  \hspace{1cm} (17)

and consequently

\[
\frac{b_{V,f} - a_{V,f}}{4(t-b_{V,f})(t-a_{V,f})} \left[ 1 + \mathcal{O} \left( \frac{1}{N||t-b_{V,f}||^{3/2}} \right) \right] = \frac{b_V}{2(t^2 - b_V^2)} \left[ 1 + \mathcal{O} \left( \frac{||f||}{N(t-b)} \right) + \mathcal{O} \left( \frac{1}{N||t-b_V||^{3/2}} \right) \right].
\]  \hspace{1cm} (18)

We now turn to the exponential term. Applying \([14]\) to the definition of \( \eta_V \) in \([9]\), we see \( \eta_{V,f} = \eta_V(1 + \mathcal{O}(||f||/N)) \) uniformly on the domain of interest. Thus

\[
\eta_{V,f}(\lambda_{V,f}^{-1}(t)) - \eta_{V,f}(\lambda_{V,f}^{-1}(t)) = \eta_V \left( \lambda_{V,f}^{-1}(t) \right) \mathcal{O}(||f||/N).
\]

Moreover, it is straightforward to see that \( \lambda_{V,f}(t) = t/b_V \), \( \lambda_{V,f}(t) - \lambda_{V,f}(t) = \mathcal{O}(||f||/N) \), \( \eta_V(x) = \mathcal{O}((x-1)^{3/2}) \), and \( \eta_V(x) = \mathcal{O}((x-1)^{1/2}) \). Using in addition \([17]\) we obtain

\[
\eta_{V,f}(\lambda_{V,f}^{-1}(t)) - \eta_{V,f}(\lambda_{V,f}^{-1}(t)) = \mathcal{O} \left( (t - b_V)^{3/2} ||f||/N \right),
\]

\[
\eta_{V,f}(\lambda_{V,f}^{-1}(t)) - \eta_{V,f}(t/b_V) = \mathcal{O} \left( (t - b_V)^{1/2} ||f||/N \right),
\]

that leads to

\[
e^{-N\eta_{V,f}(\lambda_{V,f}^{-1}(t))} = e^{-N\eta_V(t/b_V)} \left[ 1 + \left( e^{\mathcal{O}(\sqrt{t-b_V} ||f||)} - 1 \right) \right]
\]  \hspace{1cm} (19)

Combining \([16]\) with \([18]\) and \([19]\) completes the proof. \( \square \)
3. Outline of the proof of Theorem 3

We first mention the basic steps from [GV14] in the analysis of $P_{N,Q}^h$. The additional interaction term $\sum_{i<j} h(x_i - x_j)$ is in general of order $N^2$, so it may influence the limiting measure. The idea is to split it into a term contributing to the formation of the limiting measure and a perturbation term of lower order. To this end, let us introduce for a probability measure $\mu$ on $\mathbb{R}$ the notation $h_\mu(t) := \int (t - s) d\mu(s)$ and $h_{\mu\mu} := \int \int (t - s) d\mu(t) d\mu(s)$. We can then write

$$\sum_{i<j} h(x_i - x_j) = \frac{1}{2} \sum_{i,j} h(x_i - x_j) - \frac{N}{2} h(0)$$

$$= N \sum_{j=1}^N h_\mu(x_j) + \frac{1}{2} \sum_{i,j} \left[ h(x_i - x_j) - N h_\mu(x_i) - N h_\mu(x_j) + h_{\mu\mu} \right] + C_N,$$

(20)

where $C_N := -(N/2)h(0) - (N^2/2)h_{\mu\mu}$. The term in brackets in (20) is the Hoeffding decomposition of the statistic $\sum_{i<j} h(x_i - x_j)$ w.r.t. the measure $\mu$. The term $N \sum_{j=1}^N h_\mu(x_j)$ will be added to the external field $Q$ forming a new potential $V_\mu := Q + h_\mu$. Our aim is to find a measure $\mu$ such that $P_{N,Q}^h$ and the unitary invariant ensemble $P_{N,V_\mu}$ have the same limiting distribution of particles. For this to work out, we need that the double sum in (20) is small in a probabilistic sense, or in other words, concentrated under the reference measure $P_{N,V_\mu}$. Therefore we seek for a probability measure $\mu$ which is the equilibrium measure to the external field $V_\mu$. This recursive problem was solved in [GV14] by a fixed point argument, yielding existence but not uniqueness of the fixed point. The uniqueness followed later by proving that any fixed point is the limiting measure for $P_{N,Q}^h$. From now on let $\mu$ denote the fixed point, write $V := V_\mu$ and define

$$U(x) := -\frac{1}{2} \sum_{i,j} \left[ h(x_i - x_j) - N h_\mu(x_i) - N h_\mu(x_j) + h_{\mu\mu} \right].$$

(21)

The identity

$$P_{N,Q}^h(x) = \frac{Z_{N,V}}{Z_{N,V,U}} P_{N,V}(x) e^{U(x)}$$

with $Z_{N,V,U} := Z_{N,Q} e^{C_N}$ allows to carry many properties from $P_{N,V}$ over to $P_{N,Q}^h$. At this stage, it can also been proved that the ratio $Z_{N,V}/Z_{N,V,U}$ is bounded in $N$ and bounded away from 0 provided that $\alpha_Q$ is large enough. More precisely, given $\lambda > 0$, there is an $\alpha(\lambda) < \infty$ such that there are constants $0 < C_1 < C_2 < \infty$ such that for $\alpha_Q \geq \alpha(\lambda)$

$$C_1 \leq E_{N,V} e^{\lambda U} \leq C_2$$

(22)

for all $N$. The main tool to this bound is the following concentration of measure inequality for linear statistics.

**Proposition 8.** Let $Q$ be a real analytic external field with $Q'' \geq c > 0$. Then for any Lipschitz function $f$ whose third derivative is bounded on an open interval $I \subset \mathbb{R}$ containing...
the support of $\mu_Q$, we have for any $\varepsilon > 0$

$$\mathbb{E}_{N,Q} \exp \left\{ \varepsilon \left( \sum_{j=1}^{N} f(x_j) - N \int f(t) d\mu_Q(t) \right) \right\} \leq \exp \left\{ \frac{\varepsilon^2 |f|_{L}^2}{2c} + \varepsilon C(\|f\|_{\infty} + \|f^{(3)}\|_{\infty}) \right\},$$

where $C$ is uniform in $f$, $|f|_{L}$ denotes the Lipschitz constant of $f$ and $\| \cdot \|_{\infty}$ denotes the sup norm on $I$.

The proposition can be found in [GV14, Corollary 4.4] and follows from the fact that for strongly convex $Q$, $P_{N,Q}$ fulfills a log-Sobolev inequality which implies concentration of the Lipschitz function $\sum_{j=1}^{N} f(x_j)$ around its expectation (see e.g. [AGZ10, Proposition 4.4.26]). To obtain Proposition 8 we combine this with an estimate about the distance of the expectation to its large $N$ limit from [Shc11, Theorem 1] (see also [KS10, Theorem 1]).

The main idea to tackle the local universality is linearizing the bivariate statistic $U$ to transform it into linear statistics. To this end, assume that $-h$ is positive definite (the general case is reduced to this case) or, in other terms, the covariance function of a centered stationary Gaussian process $(f(t))_{t \in \mathbb{R}}$. Then we can express

$$\exp \left\{ -\frac{1}{2} \sum_{i,j} h(x_i - x_j) \right\} = \mathbb{E} \exp \left\{ \sum_{j=1}^{N} f(x_j) \right\},$$

where the expectation is w.r.t. the probability space underlying the Gaussian process. It is now easily seen that

$$\exp \{ \mathcal{U}(x) \} = \mathbb{E} \exp \left\{ \sum_{j=1}^{N} f(x_j) - N \int f d\mu \right\}$$

holds. The term $\sum_{j=1}^{N} f(x_j) - N \int f d\mu$ can now be added to the confining part $N \sum_{j=1}^{N} V(x_j)$. It is known that such a perturbation has no influence on the limiting measure. The limiting edge correlations are not altered by the function $f$, either, as can be seen from Proposition 6.

It should be noted that in Proposition 8 the scaling of the correlation functions was chosen independently of $f$ at the cost of the quality of the $O$ term. The advantage however is that the ensemble $P^h_{N,Q}$ can now be represented as an average over determinantal ensembles $P_{N,V+f/N}$ where $f/N$ is a small random perturbation. Hence local universality of $P^h_{N,Q}$ can be deduced from local universality of the invariant ensembles $P_{N,V+f/N}$, provided a sufficient uniformity in $f$ can be achieved.

However, in order to use the rich theory on Gaussian processes, in particular the sub-Gaussianity of maxima of Gaussian processes on a compact, we first need to truncate the correlation functions to some compact interval.

4. Detailed proof of Theorem 3

Representation of correlation functions and truncation.

We begin with a different representation of the correlation functions. Let us introduce a generalized invariant ensemble: Define for a continuous $Q$ of sufficient growth, continuous $f$
of moderate growth and $M \in \mathbb{N}$ the density on $\mathbb{R}^N$ by
\[
P_{N,Q,f}^M(x) := \frac{1}{Z_{N,Q,f}^M} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 e^{-M \sum_{j=1}^N Q(x_j) + \sum_{j=1}^N f(x_j)}.
\] (25)

We will usually have $M = N + k$ for some $k$. If $M = N$, we will abbreviate $P_{N,Q,f} := P_{N,Q,0}^M$ and $P_{N,Q} := P_{N,Q,0}^M$ if $f = 0$. If $f = 0$ and $M = N$, we have $P_{N,Q}$. The $k$-th correlation function of $P_{N,V}$ at points $t_1, \ldots, t_k$ can be written as
\[
\rho^k_{N,V}(t_1, \ldots, t_k)
= \int_{\mathbb{R}^{N-k}} \frac{1}{Z_{N,V}} \exp \left\{ -N \sum_{j=k+1}^N V(x_j) + 2 \sum_{i<j; i,j>k} \log |x_j - x_i| \right\}
\times \exp \left\{ -N \sum_{j=1}^k V(t_j) + 2 \sum_{i<j; i,j<k} \log |t_i - t_j| \right\}
\times \exp \left\{ 2 \sum_{i \leq k, j > k} \log |t_i - x_j| \right\} dx_{k+1} \ldots dx_N
\times \exp \left\{ 2 \sum_{i \leq k, j \geq k} \log |t_i - x_j| \right\},
\]
where
\[
F(t) := \exp \left\{ -N \sum_{j=1}^k V(t_j) + 2 \sum_{i<j; i,j \leq k} \log |t_i - t_j| \right\}
\] (26)
is the factor [26], which depends only on the fixed particles. We label the random eigenvalues of the ensemble $P_{N-k,V}$ by $x_{k+1}, \ldots, x_N$. Setting
\[
R := R_{N-k,V}^N(t,x) := 2 \sum_{i \leq k, j > k} \log |t_i - x_j| + \log [F(t) \frac{Z_{N-k,V}^N}{Z_{N,V}}],
\] (27)
we arrive at the shorthand
\[
\rho^k_{N,V}(t_1, \ldots, t_k) = \mathbb{E}_{N-k,V}^N \exp \{ R \}.
\] (28)

Similarly, we see that the $k$-th correlation function $\rho^h_{N,Q}$ of $P_{N,Q}^h$ at $t_1, \ldots, t_k$ can be written as
\[
\rho^h_{N,Q}(t_1, \ldots, t_k) = \frac{1}{\mathbb{E}_{N,V} \exp \{ U(x) \}} \mathbb{E}_{N-k,V}^N \exp \{ U(t, x) + R \},
\] (29)
where we abbreviated $U(t_1, \ldots, t_k, x_{k+1}, \ldots, x_N)$ by $U(t, x)$. Now, [GV14 Lemma 28] tells us that we can assume that $x_{k+1}, \ldots, x_N \in [-L, L]$ for $L$ large enough. More precisely, the lemma states that for each $k$ there are $L, C > 0$ such that for all $N$ and for all $t_1, \ldots, t_k$
\[
\left| \rho^h_{N,Q}(t_1, \ldots, t_k) - \frac{1}{\mathbb{E}_{N,V,L} \exp \{ U(x) \}} \mathbb{E}_{N-k,V,L}^N \exp \{ U(t, x) + R_L \} \right| \leq e^{-CN},
\] (30)
where \( \mathbb{E}^M_{N,V:L} \) denotes expectation w.r.t. the ensemble \( P^M_{N,V:L} \) obtained by normalizing the ensemble \( P^M_{N,V} \) restricted to \([-L,L]^N \) and \( R_L \) is the analog of \( R \) in which all integrations over \( \mathbb{R} \) have been replaced by integrations over \([-L,L] \). The same holds for \( \rho^k_{N,Q} \).

**Linearization, uniform convergence and Vitali’s Theorem.**

We now give a more detailed description of the linearization procedure. If \( -h \) is a positive definite function, we can indeed view it as covariance function of a centered stationary Gaussian process on \( \mathbb{R} \) which is equivalent to the property that finite linear combinations of the family of random variables \( \{B_t : t \geq 0\} \) are Gaussian. By a limit argument we see that (31) is a Gaussian random variable and by elementary computations we find that the mean is 0 and the variance is \( \int_0^\infty g(s)^2 ds \).

Now, for the representation of \( f \), let \( (B^1_t)_t, (B^2_t)_t \) denote two independent Brownian motions. Then we have a.s.

\[
    f(t) = (2\pi)^{-1/4} \int_0^\infty \cos(ts) \sqrt{-\hat{h}(s)} dB^1_s + (2\pi)^{-1/4} \int_0^\infty \sin(ts) \sqrt{-\hat{h}(s)} dB^2_s. \tag{32}
\]

To verify this representation, it is enough to note that the right hand side of (32) forms a Gaussian process on \( \mathbb{R} \) (which can be easily checked using the characterization mentioned above) with mean 0 and covariance function \(-h\). By the assumption on the exponential decay of \( h \), representation (32) continues to hold for \( z \) from a strip \( D := \{ x + iy : x \in \mathbb{R}, |y| < c \} \) for some \( c > 0 \). We thus see that \( f \) is analytic on \( D \) a.s.. It follows also from (32) that the extended process \( (f(w))_{w \in D} \) is a complex-valued centered Gaussian process and it is straightforward to show that the covariance function is \( \mathbb{E}(f(w_1)\overline{f(w_2)}) = h(w_1 - \overline{w_2}) \).

If \(-h\) is not positive definite, we may write it as a difference of positive definite functions. Denoting by \( g_{\pm} \) nonnegative and negative part of a function, we first write \( \hat{h} = (\hat{h})_+ - (\hat{h})_- \).

Setting \( h^\pm := (\hat{h})^\pm \), we get a decomposition \( h = h^+ - h^- \) of \( h \) into positive definite, real-analytic functions. It is this step where the assumption of exponential decay of \( \hat{h} \) is needed. Exponential decay of \( \hat{h} \) is equivalent to \( h^\pm \) being real-analytic. Sufficiency is easily seen as the exponential decay allows for an extension of the Fourier representation of \( h \) to a strip \( D \) from which analyticity of \( h^\pm \) can be deduced. For the necessity we remark that any real-analytic positive-definite function has an exponentially decaying Fourier transform \([LS52\, Theorem 2] \) and thus with \( h^\pm \) also \( h \) must have this property.
Define for a complex parameter $z \in \mathbb{C}$

$$U_z(x) = \frac{z}{2} \left( \sum_{i,j=1}^{N} h^+(x_i - x_j) - \left[ h^+_\mu(x_i) + h^+_\mu(x_j) - h^-_{\mu\mu} \right] \right)$$

(33)

$$+ \frac{1}{2} \left( \sum_{i,j=1}^{N} h^-(x_i - x_j) - \left[ h^-_{\mu}(x_i) + h^-_{\mu}(x_j) - h^-_{\mu\mu} \right] \right).$$

(34)

We have $U_{-1} = U$. Let us introduce the abbreviation

$$A\mathbb{I}_k(t) := \det [K_{A_i}(t_i, t_j)]_{1 \leq i,j \leq k}.$$  

(35)

To prove Theorem 3 in view of (30) we have to show

$$(c^*)^{-k}N^{k/3}E_{N-k,V;L}^N \exp \left\{ U_z(b + \frac{t}{c^*N^{2/3}}, x) + RL \right\} - E_{N,V;L} \exp \left\{ U_z(x) \right\} A\mathbb{I}_k(t)$$

$$= O \left( N^{-\sigma} \right)$$

(36)

for $z = -1$ in the prescribed uniformity. Here we used that by (22), $\exp \{ U(x) \}$ is bounded and bounded away from 0, which carries over to the truncated setting. Equation (36) makes sense for all $z \in \mathbb{C}$. As our linearization procedure only allows for nonnegative real $z$, we will prove (36) for positive real $z$ and use complex analysis to deduce (36) also for $z = -1$. Recall Vitali’s Theorem (a consequence of Montel’s Theorem and the Identity Theorem) for instance from [Tit39, 5.21]: Let $(f_n)_n$ be a sequence of analytic functions on a region $U \subset \mathbb{C}$ with $|f_n(z)| \leq M$ for all $n$ and all $z \in U$. Assume that $\lim_{n \to \infty} f_n(z)$ exists for a set of $z$ having a limit point in $U$. Then $\lim_{n \to \infty} f_n(z)$ exists for all $z$ in the interior of $U$ and the limit is an analytic function in $z$.

To capture the uniformity of the convergence in (36) in a way which preserves analyticity in $z$, we use the following obvious characterisation. A sequence of continuous real-valued functions $(f_n)_n$, defined on $\mathbb{R}^l$, converges uniformly on the sequence of compact sets $(A_n)_n$, $A_n \subset \mathbb{R}^l$ towards a continuous function $f$ if and only if for all sequences $(n_m)_m \subset \mathbb{N}$ with $\lim_{m \to \infty} n_m = \infty$ and all sequences $(t_m)_m$ with $t_m \in A_{n_m}$ we have $\lim_{m \to \infty} f_{n_m}(t_m) - f(t_m) = 0$.

For our application, we take $A_N := [q, pN^{4/15}]^k$. Let $(N_m)_m \subset \mathbb{N}$ be a sequence going to infinity and $(t_m)_m$ be a sequence with $t_m \in A_{n_m}$. Define $W_m : \mathbb{C} \to \mathbb{C}$ by

$$W_m(z) := (c^*)^{-k}N_m^{k/3}E_{N_m-k,V;L}^N \exp \left\{ U_z(b + \frac{t_m}{c^*N_m^{2/3}}, x) + RL \right\}$$

$$- E_{N_m,V;L} \exp \left\{ U_z(x) \right\} A\mathbb{I}_k(t_m).$$

(37)

Note that we suppressed the dependence of $U_z$ and $R$ on $m$. It is clear from (33) and (34) that $(W_m)_m$ is a sequence of analytic functions.

If we can show for some $\sigma' > \sigma$ that $W_m(z) = O \left( N_m^{-\sigma'} \right)$ for all $z \in (0, \delta)$ for $\delta$ sufficiently small, then the desired result follows by Vitali’s Theorem provided $(W_m)_m$ is uniformly bounded on a region including $-1$ and $(0, \delta)$.
Uniform boundedness and proof of Theorem 3

Proof of Theorem 3. We will mostly omit the index \(m\) in the following. It is easily seen from (33) that it suffices to show uniform boundedness of \(W_m(z)\) for (small) positive \(z\). Indeed, the imaginary part of \(z\) only gives a phase. For negative \(z\), the term (33) is nonpositive as \[\sum_{i,j=1}^{N\cdot m} h^+(x_i - x_j) - [h^+_{+}(x_i) + h^+_{-}(x_j) - h^+_+]\] is the variance of a Gaussian random variable (cf. (24)).

So let \(z > 0\) and fix some \(c' \in (1/3,1)\). For the linearization, let \(f^\pm\) denote two independent, centered Gaussian processes on \([-L,L]\) with covariance functions \(zh^+\) and \(h^-\), respectively. Let \(f := f^+ - \int f^+ d\mu + f^- - \int f^- d\mu\) and let \(\mathbb{E}\) denote expectation w.r.t. the underlying probability measure. Then we can write

\[
\mathbb{E}^N_{N-k,V;L} \exp \left\{ \mathcal{U}(b + \frac{t}{c^*N^{2/3}}, x) + R_l \right\} = \mathbb{E} \left[ \mathbb{E}^N_{N-k,V;L} \exp \left\{ \sum_{j=1}^{N} f((b + \frac{t}{c^*N^{2/3}}, x_j) + R_l \right\} \right]
\]

\[
\mathbb{E} \left[ \mathbb{E}^N_{N,V;L} \exp \left\{ \sum_{j=1}^{N} f(x_j) \right\} \rho^k_{N,V,f;L}(b + \frac{t_1}{c^*N^{2/3}}, \ldots, b + \frac{t_k}{c^*N^{2/3}}) \right]
\]

where the last equality is similar to (29) and \(\rho^k_{N,V,f;L}\) is the \(k\)-th correlation function of the unitary invariant ensemble \(P_{N,V+f/N;L} =: P_{N,V,f;L}\) defined on \([-L,L]\). We thus get that \(W_m(z)\) is equal to

\[
\mathbb{E} \left[ \mathbb{E}^N_{N,V;L} \exp \left\{ \sum_{j=1}^{N} f(x_j) \right\} \left( (c^*)^{-k}N^{k/3}\rho^k_{N,V,f;L}(b + \frac{t_1}{c^*N^{2/3}}, \ldots, b + \frac{t_k}{c^*N^{2/3}}) - A\mathbb{I}_k(t) \right) \right].
\]

(39)

By Proposition 3, by the determinantal relations (4), and by the almost sure analyticity of \(f\) the term in the inner parenthesis is \(O(N^{-\sigma'})\) for almost all \(f\) with the \(O\)-term uniform for \(\|f\| \leq N^\kappa\) with \(\kappa := 1/3 - \sigma'\). Our next aim is to show that we can neglect those \(f\) for which \(\|f\| > N^\kappa\). We will show that

\[
\mathbb{E} \mathbb{1}_{\{\|f\| > N^\kappa\}} \left[ \mathbb{E}^N_{N,V;L} \exp \left\{ \sum_{j=1}^{N} f(x_j) \right\} \right. 
\times \left( (c^*)^{-k}N^{k/3}\rho^k_{N,V,f;L}(b + \frac{t_1}{c^*N^{2/3}}, \ldots, b + \frac{t_k}{c^*N^{2/3}}) - A\mathbb{I}_k(t) \right) = O(c^{-cN^{2\kappa}})
\]

(40)

for some \(c > 0\). This bound will follow from an application of Hölder’s inequality to separate the expectations of \(\mathbb{1}_{\{\|f\| > N^\kappa\}}\), of \(\mathbb{E}^N_{N,V;L} \exp \left\{ \sum_{j=1}^{N} f(x_j) \right\}\) and of

\[
(c^*)^{-k}N^{k/3}\rho^k_{N,V,f;L}(b + \frac{t_1}{c^*N^{2/3}}, \ldots, b + \frac{t_k}{c^*N^{2/3}}) - A\mathbb{I}_k(t).
\]

Let us exemplarily consider the process \(f^+\). It follows readily from (32) that real and imaginary parts of \(f^+\) on \(D\) are (real-valued) centered Gaussian processes. Their covariance
functions are easily shown to be
\[
\mathbb{E}(\text{Re}f(w_1) + \text{Re}f(w_2)^+) = \frac{z}{2} (\text{Re}h^+(w_1 - w_2) + \text{Re}h^+(w_1 - \overline{w_2})),
\]
\[
\mathbb{E}(\text{Im}f(w_1) + \text{Im}f(w_2)^+) = \frac{z}{2} (\text{Re}h^+(w_1 - w_2) - \text{Re}h^+(w_1 - \overline{w_2})), \quad w_1, w_2 \in D,
\]
in particular the variances are
\[
\mathbb{E}(\text{Re}f(w)^+)^2 = \frac{z}{2} (\text{Re}h^+(0) + \text{Re}h^+(2\text{Im}w)),
\]
\[
\mathbb{E}(\text{Im}f(w)^+)^2 = \frac{z}{2} (\text{Re}h^+(0) - \text{Re}h^+(2\text{Im}w)), \quad w \in D.
\]
By Borell’s inequality (see e.g. [AT07 Theorem 2.1.1]) the supremum \(\|X\|_{\infty}\) of a continuous centered Gaussian process \(X_t\) over a compact \(K\) is sub-Gaussian, i.e. dominated by a Gaussian random variable with a certain expectation and variance \(\sigma := \sup_{t \in K} \mathbb{E}X_t^2\). As the sum of sub-Gaussian random variables is also sub-Gaussian, we see using \(\sup_{w \in D} |f^+(w)| \leq \sup_{w \in D} |\text{Re}f^+(w)| + \sup_{w \in D} |\text{Im}f^+(w)|\) that \(\sup_{w \in D} |f^+(w)|\) has sub-Gaussian tails. By the same reasoning we conclude that \(\sup_{w \in D} |f(w)|\) is sub-Gaussian, which implies
\[
P\{\|f\|_D > N^\kappa\} = \mathcal{O}(e^{-cN^{2\kappa}})
\]
for some \(c > 0\).

We deal next with \(\mathbb{E}_{N,V;L} \exp \{ \sum_{j=1}^N f(x_j) \}\). Proposition 8 extends easily to the case of a truncated ensemble, yielding an error of exponential order which we omit in the following. It is important to note that in the truncated case the Lipschitz constant is evaluated over \([-L,L]\) instead of \(\mathbb{R}\). With the proposition, we get by definition of \(f\)
\[
\mathbb{E}_{N,V;L} \exp \{ \sum_{j=1}^N f(x_j) \} \leq \exp \left\{ \frac{|f|^2}{2\alpha_V^2} + C(||f||_{\infty} + ||f^{(3)}||_{\infty}) \right\}.
\]

The derivative processes \((f^+)’\) are centered stationary Gaussian processes with covariance functions \(-z(h^+)^{''}\) and \(-(h^-)^{''}\), respectively. Borell’s inequality yields again sub-Gaussianity of \(|f|\) and analogous arguments also sub-Gaussianity of \(||f^{(3)}||_{\infty}\). We conclude that for any \(\lambda \geq 1\) there is a constant \(C = C(\lambda)\) such that for all \(N\)
\[
\mathbb{E} \left[ \mathbb{E}_{N,V;L} \exp \{ \sum_{j=1}^N f(x_j) \} \right]^\lambda < C,
\]
provided that \(\alpha_Q\) (and hence \(\alpha_V\)) is large enough.

It is noteworthy that, as the processes are stationary on \(\mathbb{R}\), the variances of \(|f|\), \(||f||_{\infty}\) and \(||f^{(3)}||_{\infty}\) and therefore the required \(\alpha_Q\) are independent of the truncation threshold \(L\). We also remark that this \(\alpha_Q\) is independent of \(h^+\), as \(z > 0\) can be chosen arbitrarily small.

Next, we will estimate \((e^*)^{-k}N^{k/3}\rho_{N,V,f;L}^k(b + \frac{t_1}{c N^{2/3}}, \ldots, b + \frac{t_k}{c N^{2/3}})\).

From [1] we see that \((K_N(t_i, t_j))_{1 \leq i, j \leq k} = A\) (where \(K_N\) is now a shorthand for the kernel \(K_{N,V,f;L}\) associated to \(P_{N,V,f;L}\) evaluated at rescaled variables \(b + \frac{t_j}{c N^{2/3}}, 1 \leq j \leq k\) is positive semi-definite and can hence be written as \(A = B^2\) for some matrix \(B\). Now using
Hadamard’s inequality we obtain
\[
\det A = (\det B)^2 \leq \prod_{j=1}^{k} \sum_{i=1}^{k} |B_{ij}|^2 = \prod_{j=1}^{k} A_{jj}.
\]

In our case this reads
\[
\rho_{N,V,f;L}^k(b + \frac{t_1}{c^* N^{2/3}}, \ldots, b + \frac{t_k}{c^* N^{2/3}}) \leq (N - k)!/(N!) \prod_{j=1}^{k} K_N(t_j, t_j)
\]
\[
\leq e^k \prod_{j=1}^{k} \rho_{N,V,f;L}^1(b + \frac{t_j}{c^* N^{2/3}}),
\]

(44)

Next, we use the well-known (see e.g. [Pre69]) representation
\[
\rho_{N,V,f;L}^1(t) = \frac{e^{-NV+f}}{N\lambda_N(e^{-NV+f}, t)}, \lambda_N(e^{-NV+f}, t) := \inf_{P_{N-1}(t)=1} \int_{-L}^{L} |P_{N-1}(s)|^2 e^{-NV(s)+f(s)} ds,
\]

where the infimum is taken over all polynomials \(P_{N-1}\) of at most degree \(N - 1\) with the property that \(P_{N-1}(t) = 1\). From this it is obvious that
\[
\rho_{N,V,f;L}^1(b + \frac{t_j}{c^* N^{2/3}}) \leq \rho_{N,V;L}^1(b + \frac{t_j}{c^* N^{2/3}}) e^{2\|f\|_\infty},
\]

where \(\|f\|_\infty\) denotes the sup-norm of \(f\) on the set \(D\). By Proposition 6, (4), and by the uniform boundedness of the Airy kernel in the region of interest, we find that
\[
(c^*)^{-N^{1/3}} \rho_{N,V,f;L}^1(b + \frac{t_j}{c^* N^{2/3}}) \leq C e^{2\|f\|_\infty}
\]

and thus
\[
(c^*)^{-k N^{k/3}} \rho_{N,V,f;L}^k(b + \frac{t_1}{c^* N^{2/3}}, \ldots, b + \frac{t_k}{c^* N^{2/3}}) \leq (Ce^{2\|f\|_\infty+1})^k,
\]

where \(C\) does not depend on \(f\). As \(\|f\|_\infty\) is sub-Gaussian, there exists for any \(\lambda' \geq 1\) some \(C' = C'(\lambda')\) such that for all \(N\)
\[
\mathbb{E} \left[ (c^*)^{-k N^{k/3}} \rho_{N,V,f;L}^k(b + \frac{t_1}{c^* N^{2/3}}, \ldots, b + \frac{t_k}{c^* N^{2/3}}) - A I_k(t) \right]^{\lambda'} \leq C'.
\]

(45)

Combining (41), (43) and (45), we can prove (40) via Hölder’s inequality. Using in addition the discussion below (39) we arrive at
\[
\mathbb{E} \left[ \mathbb{E}_{N,V;L} \exp \left\{ \sum_{j=1}^{N} f(x_j) \right\} \times \left( (c^*)^{-k N^{k/3}} \rho_{N,V,f;L}^k(b + \frac{t_1}{c^* N^{2/3}}, \ldots, b + \frac{t_k}{c^* N^{2/3}}) - A I_k(t) \right) \right] = O(N^{-\alpha Q}),
\]
given that \(\alpha_Q\) is large enough. This holds for all \(z \in [0, \varepsilon)\) for some small \(\varepsilon > 0\) (recall that both \(\mathbb{E}\) and \(f\) depend on \(z\)).

As all bounds are uniform in \(z\), we conclude with the estimates (41), (43) and (45) with \(\lambda = \lambda' = 1\) that \(W_m(z)\) (cf. (57)) is uniformly bounded in an interval \([0, \varepsilon)\) for \(\varepsilon\) small.
enough. To summarize, $W_m(z)$ is uniformly bounded for $z$ from some complex domain including $(-\infty, \varepsilon)$ and $W_m(z) = \mathcal{O}(N^{-\sigma'})$ for $z \in [0, \varepsilon)$. Thus Vitali’s Theorem gives

$$W_m(-1) = \mathcal{O}(N^{-\sigma}),$$

concluding the proof. □

5. Proof of Theorems 4 und 5

Proof of Theorem 4. We have to compute the limit of the gap probability

$$P_{N,Q}^h(x_j \notin (b + \frac{s}{c^* N^{2/3}}, \infty), \ j = 1, \ldots, N).$$

This proof uses the same techniques and route as the proof of Theorem 3. However, in that proof the truncation threshold $L$ depends on the order of the correlation function $k$. This leads to an $\mathcal{O}$-term depending on $k$ in a non-obvious way, e.g. in the expectation of quantities like $\|f\|_D$ or $|f|_L$. It is therefore convenient to truncate the event before expanding in terms of correlation functions. In the proof of the truncation lemma [GV14, Lemma 28], the last inequality states

$$\rho_{N,Q}^{h,k}(t_1, \ldots, t_k) \leq \exp\{CN - c_1 N \sum_{i=1}^k [V(t_i) - c_2 \log(1 + t_i^2)]\} \quad (46)$$

for some positive $C, c_1, c_2$. Hence, if $L$ is chosen large enough, we have for some $c > 0$

$$P_{N,Q}^h(x_j \in (L, \infty) \text{ for some } j) \leq N \int_L^\infty \rho_{N,Q}^{h,1}(t) dt = \mathcal{O}(e^{-cN}).$$

From this we also conclude that the replacement of the normalizing constant $Z_{N,Q}^h$ by $Z_{N,Q,L}^h$, the normalizing constant of the to $[-L, L]^N$ truncated ensemble $P_{N,Q,L}^h$, is negligible. We thus have

$$P_{N,Q}^h(x_j \notin (b + \frac{s}{c^* N^{2/3}}, \infty), j = 1, \ldots, N) = P_{N,Q,L}^h(x_j \notin (b + \frac{s}{c^* N^{2/3}}, L), j = 1, \ldots, N) + \mathcal{O}(e^{-cN}).$$

To evaluate the latter probability, we proceed as in the proof of Theorem 3. In order to show

$$P_{N,Q,L}^h(x_j \notin (b + \frac{s}{c^* N^{2/3}}, L), j = 1, \ldots, N) - F_2(s) = \mathcal{O}(N^{-\sigma})$$

it suffices to prove that for some $\sigma' \in (\sigma, 1/3)$ the bound

$$\mathbb{E} \left[ \mathbb{E}_{N,V,L} \exp\left\{ \sum_{j=1}^N f(x_j) \right\} \{P_{N,V,f,L}^h(x_j \notin (b + \frac{s}{c^* N^{2/3}}, L), j = 1, \ldots, N) - F_2(s)\} \right] = \mathcal{O}(N^{-\sigma'})$$

holds uniformly for all sufficiently small positive $z$ (cf. (39)). The main difference to the proof of Theorem 3 is that we now need to provide an estimate

$$P_{N,V,f,L}^h(x_j \notin (b + \frac{s}{c^* N^{2/3}}, L), j = 1, \ldots, N) - F_2(s) = \mathcal{O}(N^{-\sigma'}) \quad (47)$$
for all \( f \in X_D \) with \( \|f\| \leq N^\kappa \) where we choose again \( \kappa := 1/3 - \sigma' \). To this end, we will represent both probabilities as Fredholm determinants. It is well-known (see e.g. [TW98] for a nice derivation) that

\[
P_{N,V,f;L}(x_j \notin (b + \frac{s}{c^* N^{2/3}}, L)) = \det(I - \mathcal{K}_{N,V,f;L}) =: \Delta(\mathcal{K}_{N,V,f;L}),
\]

where \( \mathcal{K}_{N,V,f;L} \) denotes the integral operator on \( L^2((s, (L - b)c^* N^{2/3})) \) with kernel

\[
(t_1, t_2) \mapsto \frac{1}{N^{2/3} c^*} K_{N,V,f;L}(b + \frac{t_1}{c^* N^{2/3}}, b + \frac{t_2}{c^* N^{2/3}}).
\]

Similarly, we have

\[
P_{\text{TW}}((s, (L - b)c^* N^{2/3})^c) = \Delta(\mathcal{K}_{\text{Ai}}),
\]

where \( \mathcal{K}_{\text{Ai}} \) is the integral operator on \( L^2((s, (L - b)c^* N^{2/3})) \) with respect to the Airy kernel \( K_{\text{Ai}} \) and \( P_{\text{TW}} \) denotes the Tracy-Widom distribution. From the known asymptotics of \( F_2 \),

\[
1 - F_2(s) = \frac{1}{16\pi} s^{-3/2} e^{-\frac{4}{3} s^{3/2}} \left( 1 + \mathcal{O}(s^{-3/2}) \right), \quad s \to \infty,
\]

we conclude

\[
F_2(s) = P_{\text{TW}}((s, \infty)^c) = P_{\text{TW}}((s, (L - b)c^* N^{2/3})^c) + \mathcal{O}(e^{-cN}),
\]

for some \( c > 0 \). Thus it suffices to estimate the difference \( \Delta(\mathcal{K}_{N,V,f;L}) - \Delta(\mathcal{K}_{\text{Ai}}) \). Here we use the classic series representation of Fredholm determinants [AGZ10, Def. 3.4.3] and the inequality [AGZ10] Lemma 3.4.5] on the difference between the Fredholm determinants of two integral operators \( S, Q \) with bounded kernels \( S, Q \) on an interval \((c, d)\). Let \( \nu \) be a finite measure on \((c, d)\) with total mass \( \|\nu\|_1 \). Let \( \Delta(S, \nu) := \det(I - S) \) denote the Fredholm determinant of \( S \) on \( L^2((c, d), \nu) \). Then

\[
|\Delta(S) - \Delta(Q)| \leq \left( \sum_{k=1}^{\infty} \frac{k^{1+k/2} \|\nu\|_1^k}{k!} \max(\|S\|_\infty, \|Q\|_\infty)^{k-1} \right) \|S - Q\|_\infty,
\]

(48)

where \( \|S\|_\infty := \sup_{x,y \in (c,d)} |S(x,y)| \). A natural choice would be to use the Lebesgue measure for \( \nu \) on the intervals \((s, (L - b)c^* N^{2/3})\). However, \( \|\nu\|_1 \) would then be of order \( N^{2/3} \) causing the series in (48) to diverge. For our application one can circumvent this problem by transferring the fast decay of the kernels onto the measure \( \nu \). In order to do so we work with \( L^2((s, (L - b)c^* N^{2/3}), d\nu) \), \( \nu(dt) = e^{-2t} dt \) and set

\[
S_N(t_1, t_2) := \frac{1}{N^{2/3} c^*} K_{N,V,f;L}(b + \frac{t_1}{c^* N^{2/3}}, b + \frac{t_2}{c^* N^{2/3}}) e^{t_1+t_2}, \quad Q(t_1, t_2) := K_{\text{Ai}}(t_1, t_2) e^{t_1+t_2}.
\]

Using the above mentioned definition of the Fredholm determinant it is straightforward to see that \( \Delta(K_{N,V,f;L}) = \Delta(K_{N,V,f;L}, dt) = \Delta(S_N, \nu) \) and \( \Delta(K_{\text{Ai}}) = \Delta(Q, \nu) \). Observe that \( \|\nu\|_1 \) is now bounded uniformly in \( N \), as well as \( \|Q\|_\infty \) which follows from the asymptotics of the Airy kernel as presented e.g. in [KSSV14 (4.23)]). The uniform boundedness of \( \|S_N\|_\infty \) can be derived from Proposition [7] and from

\[
|S_N(t_1, t_2)|^2 \leq S_N(t_1, t_1) S_N(t_2, t_2),
\]

(49)

which is a consequence of the positivity of the 2-point correlation function combined with its determinantal representation. We have now shown that the series on the right hand side
of (48) is uniformly bounded in $N$ if we choose $S_N$ for $S$. Establishing (47) thus reduces to proving $\|S_N - Q\|_\infty = O(N^{-\sigma'})$. We need to distinguish three different regions to do so. On bounded sets the required estimate follows from Proposition 6. In the case that one of the variables is larger than, say, $\log N$, the crude estimate $|S_N - Q| \leq |S_N| + |Q|$ together with (49) and Proposition 7 are sufficient. In the remaining region one can use KSSV14 Theorem 1.8 and (15). Asymptotic information on the Airy kernel and its derivatives that are useful along the way can be found e.g. in KSSV14 Sec. 4.

We have finally arrived at (47), that plays the same role as the observation below (39) in the proof of Theorem 3. To complete the proof of 3 we can use (41), (43) and the corresponding version of (45) which can be derived with the same techniques that helped to prove (47).

We conclude this paper with the proof of our result on moderate deviations for the upper tail of the distribution of the largest particle, which combines the analysis devised in Sch15 with the procedure of the previous proof.

Proof of Theorem 5. We begin by performing the truncation as in the proof of Theorem 4. It follows from Proposition 7 and from Laplace’s method (see e.g. Sch15 for a similar analysis) that one has

$$\int_I K_{N,V,f;L}(x,x)dx = \frac{b^2}{4\pi} \frac{e^{-N\eta_V(t/b)}}{N(t^2 - b^2)\eta_V'(t/b)}\left(1 + O(N^{-\alpha/2+k}) + O(N^{-1+3\epsilon/2})\right)$$

uniformly for $t \in (b + N^{-\epsilon}, b + N^{-\alpha})$ and for all $f \in X_D$ with $\|f\| \leq N^\kappa$ for any fixed $0 < \kappa < \alpha/2$. The definition of the Fredholm determinant as in AGZ10 Def. 3.4.3 together with an application of Hadamard’s inequality (see (44)) and (9) imply for the corresponding Fredholm determinant $\Delta(K_{N,V,f;L})$ on $L^2((t,L),dx)$ that

$$1 - \Delta(K_{N,V,f;L}) = \frac{b^3}{4\pi} \frac{e^{-N\eta_V(t/b)}}{N(t^2 - b^2)^{3/2}G_V(t/b)}\left(1 + O(N^{-\alpha/2+k}) + O(N^{-1+3\epsilon/2})\right)$$

with the same uniformity of the $O$ terms as above. This is the central estimate analogue to (47) in the proof of Theorem 4. Taking into account the various truncation procedures, the integral over those $f \in X_D$ with $\|f\| > N^\kappa$ and the application of Vitali’s theorem, we see that the claim is true if the leading order term dominates $O(e^{-cN^{2\kappa}}) + O(e^{-cN})$. Using a lower bound on $\eta_V(x)$ of the form $c'(x - 1)^{3/2}$ we see that this condition is satisfied if $2\kappa > 1 - 3\alpha/2$. Recall that $0 < \kappa < \alpha/2$ needs to hold as well. An elementary consideration yields that the choice $\kappa := 1/5$ is optimal for allowing small values for $\alpha$. Indeed $\alpha > 2/5$ then suffices.

We have finally arrived at (47), that plays the same role as the observation below (39) in the proof of Theorem 3. To complete the proof of 3 we can use (41), (43) and the corresponding version of (45) which can be derived with the same techniques that helped to prove (47).

□

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