ON A NEW APPROACH TO THE PROBLEM OF THE ZERO DISTRIBUTION OF HERMITE–PADÉ POLYNOMIALS FOR A NIKISHIN SYSTEM

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ABSTRACT. A new approach to the problem of the zero distribution of Hermite–Padé polynomials of type I for a pair of functions $f_1, f_2$ forming a Nikishin system is discussed. Unlike the traditional vector approach, we give an answer in terms of a scalar equilibrium problem with harmonic external field, which is posed on a two-sheeted Riemann surface.

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Contents

1. Introduction and statement of the problem 1
2. Proof of Theorem 1 8
3. Proof of Theorem 2 9
References 19

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

1.1. Let

$$f_1(z) := \frac{1}{(z^2 - 1)^{1/2}}, \quad f_2(z) := \int_{-1}^{1} \frac{h(x)}{(z - x) \sqrt{1 - x^2}} \, dx, \quad z \in D := \mathbb{C} \setminus E;$$

(1)

here $E := [-1, 1]$, $h$ is a holomorphic function on $E$ (written $h \in \mathcal{H}(E)$) of the form $h(z) = \hat{\sigma}(z)$, where

$$\hat{\sigma}(z) := \int_F \frac{d\sigma(t)}{z - t}, \quad z \in \mathbb{C} \setminus F, \quad F := \bigcup_{j=1}^{p} [c_j, d_j] \subset \mathbb{R} \setminus E,$$

(2)

c_j < d_j, $\sigma$ is a positive Borel measure with support in $F$ and such that $\sigma' := d\sigma/dx > 0$ almost everywhere (a.e.) on $F$. Functions $\hat{\sigma}(z)$ in (2) are called Markov functions. Regarding the choice of branches of the function $(\cdot)^{1/2}$ and of the root $\sqrt{\cdot}$ in (1), see § 1.2 below.

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For a tuple $[1, f_1, f_2]$ of three functions, where $f_1$ and $f_2$ are given by (1), and an arbitrary $n \in \mathbb{N}$, Hermite–Padé polynomials of type I $Q_{n,0}, Q_{n,1}, Q_{n,2}$, deg $Q_{n,j} \leq n$, $Q_{n,j} \neq 0$, of order $n$ are defined (not uniquely) from the relation

$$R_n(z) := (Q_{n,0} \cdot 1 + Q_{n,1}f_1 + Q_{n,2}f_2)(z) = O\left(\frac{1}{z^{2n+2}}\right), \quad z \to \infty. \quad (3)$$

The purpose of the present paper is to put forward and discuss, on an example of a pair of functions of the form (1), a new approach to the study of the limit distribution of the zeros for Hermite–Padé polynomials of type I as defined by (3). As it is our intention to apply, in subsequent studies, this approach to fairly general classes of analytic functions (see the result announced in [50] and Remark 1 below), we shall first give the notation to be used below (in this respect, see [46], [32], [49]).

Let $\Sigma \subset \mathbb{C}$ be an arbitrary finite set, card $\Sigma < \infty$. We let $A^o(\Sigma)$ denote the class of all analytic functions which are holomorphic at each point $z_0 \in \mathbb{C} \setminus \Sigma$, admit analytic continuation from $z_0$ along any path $\gamma$ in $\mathbb{C}$ disjoint from $\Sigma$, and such that at least one point of the set $\Sigma$ is a branch point of this function. For $f_1, f_2 \in A^o(\Sigma)$ (under the assumption that the functions $1, f_1, f_2$ are independent over the field $\mathbb{C}(z)$ of rational functions of $z$ with complex coefficients), the problem of the limit distribution of the zeros of Hermite–Padé polynomials has a long history and in general is still unsolved (see [36], [43], [3], [41]). There is also no complete understanding what terms should be employed to solve this problem. At present, the answer to the problem of the limit distribution of the zeros of Hermite–Padé polynomials is available only for some particular classes of analytic functions (see [17], [34], [38], [19], [2], [4], [40], [32]). As a rule, the limit distribution of the zeros of Hermite–Padé polynomials for a pair of functions $f_1, f_2$ can be described following the approach first proposed by Nuttall (see [36], [38]) in terms related to some three-sheeted Riemann surface which in a certain sense is “associated” with the pair of functions $f_1, f_2$ (for the relation between the three-sheeted Riemann surface with the asymptotics of Hermite–Padé polynomials, see also [25], [6], [26].)

For a pair of functions $f_1, f_2$ of form (1) the above problem was solved by Nikishin [34] in 1986 (see also [33], [35], [7]). Note that in [34] the problem was solved for an arbitrary number of functions $f_1, f_2, \ldots, f_m$ forming a Nikishin system; a pair of functions (1) is a particular case of such a system. The solution of the problem of the distribution of the zeros of Hermite–Padé polynomials in [34] is based on the potential theory approach developed by Gonchar and Rakhmanov [17] in 1981 for the purposes of solving the zero distribution problem for Hermite–Padé polynomials of type II forming an Angelesco system (a particular case of an arbitrary number of functions $f_1, f_2, \ldots, f_m$ was also considered in the paper [17], in which, in particular, the effect of pushing of the support of the equilibrium measure inside the original orthogonality interval was discovered; see also [42]). Within the framework of this vector approach, the answer for a pair of functions (1) is

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1Similarly to the way the strong asymptotics of Padé polynomials is described in terms related to the two-sheeted Riemann surface associated (in accordance with the Stahl theory) with an arbitrary function from the class $A^o(\Sigma)$; see [37], [5], [31].
given in terms of a vector-equilibrium measure \( \vec{\lambda} = (\lambda_1, \lambda_2) \) supported on the vector-compact set \((E, F)\) (that is, \(\text{supp} \lambda_1 \subset E, \text{supp} \lambda_2 \subset F\)). The equilibrium conditions are determined by the interaction matrix of measures \( M_{\text{Nik}} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \), which is known as the Nikishin matrix. The solution of the problem is a unique vector-measure \( \vec{\lambda} = (\lambda_1, \lambda_2) \) with support on the vector-compact set \((E, F)\); this measure is extremal for the energy functional defined by the logarithmic kernel and the interaction matrix \( M_{\text{Nik}} \) (see [35], [19], [3], [28]). The extremal vector-measure \( \vec{\lambda} = (\lambda_1, \lambda_2) \) is completely characterized by the equilibrium condition for the corresponding vector potential and the vector-compact set \((E, F)\); for more details, see [3], [28].

Note that, for arbitrary functions \( f_1, f_2 \in \mathcal{A}^\circ(\Sigma) \), the problem of the limit distribution of the zeros of the corresponding Hermite–Padé polynomials turns out to be equivalent to the problem of the limit distribution of the zeros of polynomials satisfying some non-Hermitian orthogonality conditions (see [18], [39], [41], [42]). The characteristic feature of non-Hermitian orthogonality conditions is that the contour of integration is not fixed a priori, but rather lies in some class of “admissible” contours. The following heuristic conclusion can be made based on a series of particular cases investigated so far: in this class, there exists a unique “optimal” contour\(^2\) attracting in the limit the zeros of Hermite–Padé polynomials. The “optimality” property of a contour is formulated in terms of the corresponding vector equilibrium problems of potential theory. This optimal contour possesses a certain vector \( S \)-property, which completely characterizes it in the class of admissible vector-contours. In modern terms, such a contour is called an \( S \)-curve or an \( S \)-compact set (see [39]).

The concept of an \( S \)-compact set was first introduced by H. Stahl in the 1985–1986s (see [44] and [45] and there references given therein) when considering the problem of the limit distribution of the zeros and poles of Padé approximants in the class of multivalued analytic functions \( \mathcal{A}^\circ(\Sigma) \). In 1987 Gonchar and Rakhmanov [18], in their solution of the “1/9 conjecture”, developed a different approach to the problem of the limit distribution of the zeros of non-Hermitian orthogonal polynomials. This approach is based on the scalar equilibrium problem, but with the so-called “external field” defined by a harmonic function (more general external fields and the corresponding \( S \)-curves were considered in [39]). This new approach was used in 2012–2015 by Buslaev [9]–[11] to solve the problem of the limit distribution of the zeros and poles of multivalued Padé approximants. Here, the potential of a negative unit charge concentrated at a finite number of interpolation nodes appears naturally as an external field (see also [13], [15], [14]).

The class of methods developed by H. Stahl, A. A. Gonchar, and E. A. Rakhmanov in the 1980s for the purpose of studying the limit distribution of the zeros of non-Hermitian orthogonal polynomials is called at present the

\(^{2}\)Here and below, by a contour we shall mean a composite contour consisting of a finite number of closed curves and splitting the Riemann sphere into a finite number of domains; see [10], [11].
Gonchar–Rakhmanov–Stahl method (or briefly the GRS-method); see [46],[32],[41],[42].

The purpose of the present paper is, by using an example of two functions \(f_1\) and \(f_2\) of the form (1), put forward and discuss a new approach to the problem of the limit distribution of the zeros of Hermite–Padé polynomials, which in a certain sense further develops the approach of A. A. Gonchar and E. A. Rakhmanov employed in their solution of the “1/9 conjecture”. Namely, the limit distribution of the zeros of the polynomial \(Q_{n,2}\) as \(n \to \infty\) will be characterized in terms related to some scalar potential theory equilibrium problem (but with external field), which in addition is posed not on the Riemann sphere \(\mathbb{C}\), but rather on the two-sheeted Riemann surface of the function \(w^2 = z^2 - 1\). This is the principal distinguishing feature of the approach of the present paper from the standard method based on the vector equilibrium problem posed on the Riemann sphere.

Let us clarify the choice of the pair of functions (1) to illustrate the new approach and the fact that here we speak only about the distribution of the zeros of the polynomial \(Q_{n,2}\).

The thing is, on the one hand, as we have already mentioned, in the class \(\mathcal{A}(\Sigma)\) the problem of the distribution of the zeros of Hermite–Padé polynomials for an arbitrary pair of independent functions \(f_1, f_2 \in \mathcal{A}(\Sigma)\) is not yet solved and it is even unclear what terms should be employed to find its solution (for conjectures in this direction, see [36],[3],[43],[41]). In particular, there is no solution in this problem even for a pair of functions with two branch points, of which each is in “the general position”. On the other hand, for the Padé polynomials \(P_{n,0}, P_{n,1}\), \(\deg P_{n,j} \leq n, P_{n,j} \not\equiv 0\), as defined from the relations
\[
(P_{n,0} + P_{n,1}f)(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty,
\]
where \(f \in \mathcal{H}(\infty)\), Stahl’s theory is valid \(^3\) for an arbitrary function \(f\) from the class \(\mathcal{A}(\Sigma)\). This leads to the following fairly natural argument: one of the functions, \(f_1\), say, in the relation (3) defining the Hermite–Padé polynomials, should be taken as simple as possible (retaining the independence of two functions \(f_1, f_2\)) with the aim at maximally extending the calculus to a larger class of functions that contains the second function \(f_2\). In this way, using the approach proposed here, we managed to substantially enlarge the class of functions containing the function \(h\) in representation (1). Namely, for an arbitrary function \(h \in \mathcal{A}(\Sigma)\), where \(\Sigma \subset \mathbb{C} \setminus E\), it is possible to characterize completely the problem of the limit distribution of the zeros of the polynomials \(Q_{n,2}\) in terms of the same scalar potential theory equilibrium problem with an external field. This problem is posed on the same two-sheeted Riemann surface of the function \(w^2 = z^2 - 1\) in a similar way as in the present paper. The principal difference is that in the general case it is first required to establish the existence of an appropriate \(S\)-compact set \(F\) corresponding to the problem under consideration and which replaces the union of a finite number of closed intervals (see (2)). The corresponding

\(^3\)Stahl’s theory is much more general and can be applied to any multivalued analytic function, whose singular set is of zero logarithmic capacity.
result was announced in [50]; the author intends to give the proof of this result in a separate paper.

We note the papers [40], [46] and [24], in which the equilibrium problem for a mixed Green-logarithmic potential was employed for the study of the limit distribution of the zeros of Hermite–Padé polynomials for a tuple $[1, f_1, f_2]$, where a pair of functions $f_1, f_2$ forms a generalized (complex) Nikishin system (see also [12], [47], [32], [41]). The method of investigation proposed in the present paper is different from that of [40], [46] and [24]. Some precursor considerations and results that eventually culminated in the statement of the potential theory equilibrium problem on the Riemann surface $w^2 = z^2 - 1$ were obtained by the author in [48].

It also should be mentioned about the papers by H. Stahl with coauthors [8] and [30], in which potentials pretty close to those used in the present paper were used. However, as far as the author is aware, such potentials have not been applied before in the study of the distribution of the zeros of Hermite–Padé polynomials.

It is worth pointing out that in the present paper we discuss and examine only the case of “diagonal” (that is, of the same degree) Hermite–Padé polynomials of type I. The nondiagonal case, as well as in the case of Hermite–Padé polynomials of type II merits special consideration within the framework of the new approach proposed here (of course, if such a research will prove feasible).

The fact that the problems on the distribution of the zeros of Hermite–Padé polynomials of type I and type II are substantially different and in general call for different approaches and methods of investigation is well illustrated in Figs. 1–2, which were derived for the pair of functions

$$f_1(z) := \frac{1}{(z^2 - 1)^{1/2}}, \quad f_2(z) := \frac{1}{((z - .8 - .5i)(z + .8 - .5i))^{1/2}},$$

forming an Angelesco system.

The author is grateful to the referee for the many helpful comments and suggestions which led to a great improvement in the presentation of the paper and for calling his attention to the papers [1] and [29].

1.2. We shall require the following notation and definitions. We set $D := \mathbb{C} \setminus E$,

$$\varphi(z) := z + (z^2 - 1)^{1/2}, \quad z \in D,$$

where we choose the branch of the root function such that $(z^2 - 1)^{1/2}/z \to 1$ as $z \to \infty$. For $x \in (-1, 1)$, by $\sqrt{1 - x^2}$ we shall understand the positive square root: $\sqrt{b^2} = b$ for $b \geq 0$.

Given an arbitrary polynomial $Q \in \mathbb{C}[z]$, $Q \not\equiv 0$, by

$$\chi(Q) := \sum_{\zeta: Q(\zeta) = 0} \delta_{\zeta},$$

we shall mean the counting measure of the zeros of the polynomial $Q$ (counting multiplicities). In what follows, given an arbitrary $n \in \mathbb{N}$, we denote by $\mathbb{P}_n := \mathbb{C}_n[z]$ the class of all algebraic polynomials of degree $\leq n$ with complex coefficients.
We let \( \mathcal{R}_2 \) denote the two-sheeted Riemann surface of the function \( w^2 = z^2 - 1 \) regarded as a two-sheeted covering of the extended complex plane \( \mathbb{C} \) with branch points at \( z = \pm 1 \). Each (open) sheet of the Riemann surface \( \mathcal{R}_2 \) is the Riemann sphere cut along the interval \( E \), the opposite sides of cuts from different sheets being identified. The first (open) sheet \( \mathcal{R}^{(1)} \) of the Riemann surface \( \mathcal{R}_2 \) is that on which \( w = (z^2 - 1)^{1/2} \sim z \) as \( z \to \infty \); on the second sheet \( \mathcal{R}^{(2)} \) \( w = -(z^2 - 1)^{1/2} \sim -z \) as \( z \to \infty \). A point \( z \) on the Riemann surface \( \mathcal{R}_2 \) is the pair \( (z, w) = z \in \mathcal{R}_2 \). The canonical projection \( \pi, \pi: \mathcal{R}_2 \to \mathbb{C} \), is defined in the standard way: \( \pi(z) = z \). Note that \( \mathcal{R}_2 \) is a Riemann surface of zero genus, and so any divisor \( d \) of degree 0 on \( \mathcal{R}_2 \) is a principal one; that is, there exists a meromorphic function on \( \mathcal{R}_2 \) whose divisor of the zeros and poles coincides with \( d \). From a given divisor of degree 0 such a meromorphic function is defined uniquely up to a nontrivial multiplicative constant (for a more detailed account of these and other aspects of Riemann surfaces, see [16]).

Thus, the function \( z + w \), which is meromorphic on the Riemann surface \( \mathcal{R}_2 \), will be denoted by \( \Phi(z) := z + w \). Points of the Riemann surface lying on the first (open) sheet \( \mathcal{R}^{(1)} \) will be denoted by \( z^{(1)} \); by \( z^{(2)} \) we denote points from the second sheet \( \mathcal{R}^{(2)} \). So, \( z^{(1)} = (z, (z^2 - 1)^{1/2}) \), \( z^{(2)} = (z, -(z^2 - 1)^{1/2}) \), \( \pi(\mathcal{R}^{(1)}) = \pi(\mathcal{R}^{(2)}) = D \).

The following identity\(^4\) is easily verified for \( z, a \in \mathcal{R}_2 \setminus \Gamma \)

\[
\frac{z - a}{2\Phi(z)} = \frac{\Phi(z) - \Phi(a)[1 - \Phi(z)\Phi(a)]}{2\Phi(z)\Phi(a)},
\]

Indeed, each of the functions on the right and left of (6) is meromorphic on the Riemann surface \( \mathcal{R}_2 \). The divisor \( z - a \) of the left-hand side can be easily evaluated to be equal to \( d_1 = -\infty^{(1)} - \infty^{(2)} + a^{(1)} + a^{(2)} \). For the divisor of the right-hand side, we also have \( d_2 = -\infty^{(1)} - \infty^{(2)} + a^{(1)} + a^{(2)} \). Hence, these two functions are identically equal except for a multiplicative constant, which can be easily calculated.

From (6) we have, in particular, the identity

\[
\frac{z - a}{2\varphi(z)} = \frac{\varphi(z) - \varphi(a)[1 - \varphi(z)\varphi(a)]}{2\varphi(z)\varphi(a)},
\]

which holds for \( z, a \in D \). The following identity

\[
\Phi(z^{(1)})\Phi(z^{(2)}) \equiv 1, \quad z \in D,
\]

can also be easily verified.

Let \( M_1(F) \) be the space of all unit positive Borel measures supported on a compact set \( F \). Given an arbitrary measure \( \mu \in M_1(F) \), we define by

\[
V^\mu(z) := \int_F \log \frac{1}{|z - t|} d\mu(t)
\]

the logarithmic potential of \( \mu \),

\[
I(\mu) := \int_{F \times F} \log \frac{1}{|z - t|} d\mu(z) d\mu(t) = \int_F V^\mu(z) d\mu(z)
\]

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\(^4\)This identity, for all its undoubted simplicity, was first used very effectively in [20], formulas (15), (67).
is the corresponding energy functional. By $M^n_0(F) \subset M^1_1(F)$ we shall denote the space of measures with finite energy, $I(\mu) < \infty$. We recall the positivity property of logarithmic energy\(^5\) with respect to neutral charges:

$$I(\mu - \nu) \geq 0 \quad \forall \mu, \nu \in M^n_0(F) \quad \text{and} \quad I(\mu - \nu) = 0 \iff \mu = \nu. \quad (11)$$

For an account of these and other properties of logarithmic potentials employed in the present paper, see [27].

For a measure $\mu \in M^1_1(F)$, we set\(^6\)

$$P^\mu(z) := \int_F \log \frac{|1 - \varphi(z)\varphi(t)|}{|z - t|^2} \, d\mu(t), \quad \psi(z) := \log |\varphi(z)|, \quad (12)$$

and define

$$J(\mu) := \iint_{F \times F} \log \frac{|1 - \varphi(z)\varphi(t)|}{|z - t|^2} \, d\mu(z) \, d\mu(t)$$

$$= \int_F P^\mu(z) \, d\mu(z),$$

$$J_\psi(\mu) := \iint_{F \times F} \left\{ \log \frac{|1 - \varphi(z)\varphi(t)|}{|z - t|^2} + \psi(z) + \psi(t) \right\} \, d\mu(z) \, d\mu(t)$$

$$= \int_F P^\mu(z) \, d\mu(z) + 2 \int_F \psi(z) \, d\mu(z). \quad (13)$$

From identity (7) we have the following equality, which holds for $z, \zeta \in D$,

$$\log \frac{|1 - \varphi(z)\varphi(\zeta)|}{|z - \zeta|^2} = \log \frac{1}{|z - \zeta|} + \log \frac{1}{|\varphi(z) - \varphi(\zeta)|} + \log 2 + \psi(z) + \psi(\zeta). \quad (14)$$

Potentials with kernels of the form

$$\log \frac{1}{|z - \zeta|} + \log \frac{1}{|\varphi(z) - \varphi(\zeta)|},$$

where $z, \zeta \in [A, B] \subset \mathbb{R}$, $\nu(z)$ is an arbitrary nondecreasing function on $[A, B]$, were considered in the paper [30], however, the author of the present paper is unaware of any applications of such potentials in the theory of Hermite–Padé polynomials.

1.3. The main results of the present paper are Theorems 1 and 2.

**Theorem 1.** In the class $M^n_1(F)$, there exists a unique measure $\lambda = \lambda_F \in M^n_1(F)$ such that

$$J_\psi(\lambda) = \min_{\mu \in M^1_1(F)} J_\psi(\mu). \quad (15)$$

The measure $\lambda$ is completely characterized by the following equilibrium condition:

$$P^\lambda(z) + \psi(z) \equiv w_F, \quad z \in S(\lambda), \quad (16)$$

**Theorem 2.** Let $f_1$ and $f_2$ be functions given by the representations (1) and let $Q_{n,2}$ be the Hermite–Padé polynomial defined by (3). Then

$$\frac{1}{n} \chi(Q_{n,2}) \to \lambda, \quad n \to \infty. \quad (17)$$

\(^5\)More precisely, the positivity of the logarithmic kernel.

\(^6\)It is clear that $\psi(z) = \log \varphi(z)$ for $z \in \mathbb{R} \setminus E$. 
The convergence in (17) shall be understood in the sense of weak convergence in the space of measures. It may be pointed out once more that the assertion of Theorem 2 on the existence of the limit distribution of the zeros of the polynomials $Q_n$ is not new (see, first of all, [34], and also [19], [4]). The new point here is the characterization of this limit distribution in terms the scalar equilibrium problem (15)–(16). This was achieved by posing the corresponding potential theory problem not on the Riemann sphere, but on the two-sheeted Riemann surface of the function $w^2 = z^2 - 1$.

2. PROOF OF THEOREM 1

2.1. Let $U \supset F$ be some neighborhood of the compact set $F$ such that $U \cap E = \emptyset$. For all $\mu \in M_1(F)$, the function

$$ \int_F \log |1 - \varphi(z)\varphi(t)| \, d\mu(t) $$

is harmonic in $U$ and the potential $V^\mu(z)$ is a superharmonic function in $U$, and hence, since $\operatorname{cap} F > 0$, $M_1(F)$ is compact in the weak topology, and using the principle of descent for logarithmic potentials (see [27], Ch. I, §3, Theorem 1.3), we see that there exists a measure $\lambda \in M_1^\circ(F)$ satisfying equality (15). Using identity (14), one can easily prove the convexity of the energy functional $J_\varphi(\cdot)$,

$$ J_\varphi \left( \frac{\mu + \nu}{2} \right) \leq \frac{1}{2} \left[ J_\varphi(\mu) + J_\varphi(\nu) \right] \quad \forall \mu, \nu \in M_1(F); \quad (18) $$

moreover,

$$ J(\mu - \nu) = 2J_\varphi(\mu) + 2J_\varphi(\nu) - 4J_\varphi \left( \frac{\mu + \nu}{2} \right), \quad (19) $$

$$ J(\mu - \nu) = \iint_{F \times F} \left\{ \log \frac{1}{|z - \zeta|} + \log \frac{1}{|\varphi(z) - \varphi(\zeta)|} \right\} \left( d(\mu - \nu)(z) d(\mu - \nu)(\zeta) \right), \quad (20) $$

for all $\mu, \nu \in M_1^\circ(F)$. As a direct corollary of (18)–(20) we see that the functional $J(\cdot)$ is positive on neutral charges (cf. (11)),

$$ J(\mu - \nu) \geq 0 \quad \forall \mu, \nu \in M_1^\circ(F) \quad \text{and} \quad J(\mu - \nu) = 0 \iff \mu = \nu. $$

Furthermore, the following equalities are easily verified:

$$ J(\mu) = \iint_{F \times F} \left\{ \log \frac{1}{|z - \zeta|} + \log \frac{1}{|\varphi(z) - \varphi(\zeta)|} \right\} + \log 2 + 2 \int \varphi(z) \, d\mu(z), $$

$$ J_\varphi(\mu) = \iint_{F \times F} \left\{ \log \frac{1}{|z - \zeta|} + \log \frac{1}{|\varphi(z) - \varphi(\zeta)|} \right\} + \log 2 + 4 \int \varphi(z) \, d\mu(z). \quad (21) $$

2.2. Arguing as in Lemma 6 of [18], we can now prove the equilibrium property (16) of the extremal measure $\lambda$ by using the above equalities and the positivity property of the functional $J(\cdot)$.
Indeed\(^7\) one verifies directly that
\[ J_\psi(\varepsilon \nu + (1 - \varepsilon)\lambda) - J_\psi(\lambda) = 2\varepsilon \int_F (P^\lambda + \psi)(z) \, d(\nu - \lambda) + \varepsilon^2 J(\nu - \lambda) \quad (22) \]
for any \(\varepsilon > 0\) and measure \(\nu \in M_1^n(F)\). It follows that the minimizing measure \(\lambda\) is the only measure from \(M_1^n(F)\) satisfying the condition
\[ \int_F (P^\lambda + \psi) \, d(\nu - \lambda) \geq 0 \quad \forall \nu \in M_1^n(F). \quad (23) \]
In the actual fact, (23) is an immediate consequence of (18), (11) and (22) as \(\varepsilon \to 0\). On the other hand, since the energy functional \(J(\cdot)\) is positive on neutral charges, we have \(J(\nu - \lambda) \geq 0\) for any measure \(\nu \in M_1^n(F)\). An appeal to (22) with \(\varepsilon = 1\) shows that any measure \(\nu \in M_1^n(F)\) satisfying (23) minimizes the energy integral \(J_\psi(\cdot)\). If a measure \(\lambda\) satisfies condition (23), then it obeys the equilibrium relations (16) with
\[ w_F := \int_F (P^\lambda + \psi) \, d\lambda. \]
Indeed, if \(P^\lambda(x) + \psi(x) < w_F\) on a closed set \(c \subset F\), \(\text{cap}(c) > 0\), then there exists \(\nu \in M_1^n(e)\), for which \(\int_F (P^\lambda + \psi)(x) \, d\nu(x) < w_F\), which shows that (23) is violated. Hence \((P^\lambda + \psi)(x) \geq w_F\) everywhere on the (regular) compact set \(F\). If \((P^\lambda + \psi)(x) > w_F\) on a nonempty set \(e \subset S(\mu)\), then the inequality \(\int_F (P^\lambda + \psi)(x) \, d\lambda(x) > w_F\) is secured by the lower semi-continuity of the function \((P^\lambda + \psi)(z)\), contradicting the definition of \(w_F\).

If \(\lambda\) is an equilibrium measure, then \(P^\lambda + \psi \leq w_F\) everywhere on \(S(\lambda)\), which shows that \(\lambda \in M_1^n(F)\). Since the sets of zero inner capacity play no role in integration with respect to measures in \(M_1^n(F)\), we obtain (23). Finally, \((P^\lambda + \psi)(z) \equiv w_F\) on \(S(\lambda)\), because \(F\) is a regular compact set.

Thus, the extremal measure \(\lambda\), and only this measure, satisfies the equilibrium conditions (16). This proves Theorem 1.

Note that \(F\) is a regular compact set, and hence the equilibrium measure is characterized by the equality
\[ \min_{z \in F} (P^\lambda + \psi)(z) = \max_{\mu \in M_1(F)} \min_{z \in F} (P^\mu + \psi)(z). \]

3. Proof of Theorem 2

3.1. From (3) we have the relation
\[ 0 = \int_\gamma (Q_{n,0} + Q_{n,1}f_1 + Q_{n,2}f_2)(z)q(z) \, dz = \int_\gamma (Q_{n,1}f_1 + Q_{n,2}f_2)(z)q(z) \, dz, \quad (24) \]
which holds for any polynomial \(q \in \mathbb{P}_{2n}\); in (24) \(\gamma\) is an arbitrary contour separating the interval \(E\) from the infinity point \(z = \infty\).

\(^7\) For completeness of presentation, we give the proof of (16), cf. Lemma 6 of [18].
Let \( P_n \) and \( P_{n,1} \) be the Padé polynomials for the function \( f_1 \); that is, \( \deg P_{n,j} \leq n \), \( P_{n,j} \neq 0 \), and

\[
H_n(z) := (P_{n,0} + P_{n,1}f_1)(z) = O \left( \frac{1}{z^{n+1}} \right), \quad z \to \infty. \tag{25}
\]

It is known that \( P_{n,1} = T_n \) are Chebyshev polynomials of the first kind that are orthogonal on the interval \( E \) with the weight \( 1/\sqrt{1-x^2} \); \( H_n \) is the corresponding function of the second kind. We shall assume that the Chebyshev polynomials are normalized as follows: \( T_n(z) = 2^n z^n + \cdots \). Hence, for the functions of the second kind \( H_n \), we have

\[
H_n(z) = \frac{\varphi_{n+1}'(z)}{\varphi^{n+1}(z)}, \quad \varphi_n \neq 0, \quad H_n(z) = \frac{1}{2\pi i} \int_E \frac{T_n(x)\Delta f_1(x)}{x - z} \, dx, \quad z \in D, \tag{26}
\]

\[
\Delta H_n(x) : = H_n(x + i0) - H_n(x - i0) = T_n(x)\Delta f_1(x) = T_n(x) \frac{2}{i\sqrt{1-x^2}}, \quad x \in (-1, 1). \tag{27}
\]

Besides, the polynomials \( T_n \) and the functions of the second kind \( H_n \) satisfy the same second-order recurrence relation, but with different initial data

\[
y_k = 2z y_{k-1} - y_{k-2}, \quad k = 1, 2, \ldots, \tag{28}
\]

where one should put \( y_{-1} \equiv 1 \), \( y_0 \equiv 1 \) for the polynomials \( T_k \) and \( y_{-1} \equiv 1 \), \( y_0 = f_1(z) = 1/(z^2 - 1)^{1/2} \) for the functions of the second kind \( H_k \). We have

\[
\int_\gamma p(z)f_1(z)T_{n+j}(z) \, dz = 0, \quad j = 1, 2, \ldots, n
\]

for any polynomial \( p \in \mathbb{P}_n \), and so from (24) with \( q = T_{n+1}, \ldots, T_{2n} \) it follows that

\[
\int_\gamma Q_{n,2}(z)f_2(z)T_{n+j}(z) \, dz = 0, \quad j = 1, 2, \ldots, n. \tag{29}
\]

Next, using (29) and the definition (1) of the function \( f_2 \), we have

\[
\int_E Q_{n,2}(x)T_{n+j}(x) \frac{1}{\sqrt{1-x^2}} h(x) \, dx = 0, \quad j = 1, \ldots, n. \tag{30}
\]

In view of (27), the above relation is equivalent to the relation

\[
\int_\gamma Q_{n,2}(z)H_{n+j}(z)h(z) \, dz = 0, \quad j = 1, \ldots, n, \tag{31}
\]

where \( \gamma \) is an arbitrary contour separating the interval \( E \) from the compact set \( F \). Since \( h(z) = \tilde{h}(z) \), relation (31) can be easily written in the form

\[
\int_F Q_{n,2}(x)H_{n+j}(x) \, d\sigma(x) = 0, \quad j = 1, \ldots, n. \tag{32}
\]

These orthogonality relations\(^8\) will play a key role in the subsequent analysis of the limit distribution of the zeros of the polynomials \( Q_{n,2} \).

Let \( N, 0 \leq N \leq n \), be an arbitrary natural number. We shall assume without loss of generality that \( N = 2m \) is an even number (the case of an

\(^8\)In view of the representation \( H_n(z) = \varphi'(z)/\varphi^{n+1}(z) \), the orthogonality relations (32) are similar to those considered in [30].
odd $N$ is treated similarly). Given arbitrary complex numbers $c_1, \ldots, c_N \in \mathbb{C}$, consider the sum

$$\sum_{j=1}^{N} c_j H_{n+j}(z).$$

By using the recurrence relations (28), this sum can be easily written as

$$\sum_{j=1}^{N} c_j H_{n+j}(z) = q_{m,1}(z)H_{n+m+1}(z) + q_{m,2}(z)H_{n+m}(z), \quad (33)$$

where $q_{m,1}, q_{m,2} \in \mathbb{P}_{m-1}$ are polynomials of degree $\leq m - 1$. Since the constants $c_1, \ldots, c_N$ in (33) are arbitrary, it is easily verified that the polynomials $q_{m,1}$ and $q_{m,2}$ can also be chosen arbitrarily. So, using (33), relations (32) can be written in the following equivalent form

$$\int_{F} Q_{n,2}(x) \left\{ q_{m,1}(x)H_{n+m+1}(x) + q_{m,2}(x)H_{n+m}(x) \right\} d\sigma(x) = 0 \quad (34)$$

with arbitrary polynomials $q_{m,1} \in \mathbb{P}_{m-1}$ and $q_{m,2} \in \mathbb{P}_{m-1}$. Now, from (34) and the available properties of the functions of the second kind (see (26)), we have

$$0 = \int_{F} Q_{n,2}(x) \left\{ q_{m,1}(x)\frac{H_{n+m+1}(x)}{H_{n+m}} + q_{m,2}(x) \right\} H_{n+m}(x) d\sigma(x)$$

$$\quad = \int_{F} Q_{n,2}(x) \left\{ q_{m,1}(x)\frac{\kappa_{n+m+1}}{\kappa_{n+m}} + q_{m,2}(x) \right\} \frac{\kappa_{n+m} \varphi'(x)}{\varphi^{n+m+1}(x)} d\sigma(x). \quad (35)$$

Now, using the definition of the function $\Phi(z)$ (see sec. 1.2), which is meromorphic on the Riemann surface $\mathcal{R}_2$, we get the following orthogonality relation

$$\int_{F} Q_{n,2}(x) \left\{ q_{m,1}(x)\Phi(x^{(2)}) + q_{m,2}(x) \right\} \varphi'(x)\Phi(x^{(2)})^{n+m+1} d\sigma(x) = 0, \quad (36)$$

which holds for any polynomials $q_{m,1}, q_{m,2} \in \mathbb{P}_{m-1}$.

3.2. We now set

$$g_n(z) := q_{m,1}(z)\Phi(z) + q_{m,2}(z), \quad (37)$$

where it is assumed that $\deg q_{m,1} = \deg q_{m,2} = m - 1$. Then, for the divisor of the function $g_n$ we have

$$\text{div}(g_n) = -m\infty^{(1)} - (m - 1)\infty^{(2)} + \sum_{j=1}^{N-1} a_{N,j}, \quad (38)$$

where, as is clear, the zeros $a_{N,j}$ of the function $g_n$ can be chosen arbitrarily, because the polynomials $q_{m,1}, q_{m,2}$ are arbitrary. Next, the function $g_n$ is meromorphic on $\mathcal{R}_2$ and the genus of the Riemann surface $\mathcal{R}_2$ is zero, and hence the function $g_n$ is completely defined by its divisor (38) (of the zeros and poles). As a result, from (38) we have the following explicit representation for the function $g_n$:

$$g_n(z) = C_N \cdot \prod_{j=1}^{N-1} \left[ \Phi(z) - \Phi(a_{N,j}) \right] \cdot \Phi(z)^{-m+1}, \quad C_N \neq 0. \quad (39)$$
Indeed, it is easily checked that the divisor of the zeros and poles of the right-hand side of (39) coincides with that of (38). Below, in accordance with (36), we shall need to consider only the case when all points \(a_{N,j}\) lie on the second sheet of the Riemann surface \(\mathcal{R}_2\), \(a_{N,j} = a_{N,j}^{(2)} \in \mathcal{R}^{(2)}\). More precisely, the zeros \(a_{N,j}\) should be as follows: they should lie on the second list and be such that \(\pi(a_{N,j}) \in \hat{F} \setminus E\), where \(\hat{F}\) is the convex hull of \(F\). In this case, it follows from (39) that

\[
g_n(z^{(2)})\Phi(z^{(2)})^{n+m+1} = C_N \cdot \prod_{j=1}^{N-1} \left[ \Phi(z^{(2)}) - \Phi(a_{N,j}^{(2)}) \right] \cdot \Phi(z^{(2)})^{n+2}. \tag{40}\]

We now consider the product \(g_n(z)\Phi(z)^{n+m+1}\). Using identities (6) and (8), we write it as

\[
g_N(z)\Phi(z)^{n+m+1} = C_N \cdot \prod_{j=1}^{N-1} \left[ \Phi(z) - \Phi(a_{N,j}) \right] \cdot \Phi(z)^{-m+1}\Phi(z)^{n+m+1} = \tilde{C}_N \cdot \prod_{j=1}^{N-1} \frac{z - a_{N,j}}{1 - \Phi(z)\Phi(a_{N,j})} \cdot \Phi(z)^{N+n+1}, \tag{41}\]

where \(\tilde{C}_N \neq 0\) and it is assumed that all \(a_{N,j} \neq \infty^{(1)}, \infty^{(2)}\). In accordance with (36), we shall require representation (41) only in the case when \(z = z^{(2)}\) and all \(a_{N,j} = a_{N,j}^{(2)}\). In this setting, we have by (41)

\[
g_N(z^{(2)})\Phi(z^{(2)})^{n+m+1} = \tilde{C}_N \prod_{j=1}^{N-1} \frac{z - a_{N,j}}{1 - \Phi(z^{(2)})\Phi(a_{N,j}^{(2)})} \cdot \Phi(z^{(2)})^{N+m+1}. \tag{42}\]

Since \(\Phi(z^{(2)}) = 1/\varphi(z)\) for all \(z \in D\), the last relation can be written as

\[
g_N(z^{(2)})\Phi(z^{(2)})^{n+m+1} = C_3(N) \prod_{j=1}^{N-1} \frac{z - a_{N,j}}{1 - \varphi(z)\varphi(a_{N,j})} \cdot \frac{1}{\varphi^{n+2}(z)}. \tag{43}\]

Using (43), the orthogonality relation (36) can be put in the form

\[
\int_{\hat{F}} Q_{n,2}(x) \prod_{j=1}^{N-1} \frac{x - a_{N,j}}{1 - \varphi(x)\varphi(a_{N,j})} \cdot \frac{\varphi'(x)}{\varphi^{n+2}(x)} \, d\sigma(x) = 0, \tag{44}\]

where the number \(N \leq n\) is arbitrary and all points \(a_{N,j}\) lie in \(D\). From (44), it follows that \(\deg Q_{n,2} = n\), all zeros of the polynomial \(Q_{n,2}\) lie on \(\hat{F}\) (which is the convex hull of the compact set \(F\)); besides, the gap with number \((p-1)\) between the intervals \([c_j, d_j], j = 1, 2, \ldots, p\), may contain at most \(p-1\) zeros of this polynomial. The orthogonality relations (44), which are defined for an arbitrary \(N \leq n\) and arbitrary points \(a_{N,j} \in \hat{F} \setminus E\), will underlie our further analysis.
3.3. As usual, when applying the GRS-method, we assume that
\[
\frac{1}{n} \chi(Q_{n,2}) \not\rightarrow \lambda = \lambda_F
\]  
(45)
as \(n \to \infty\). We shall arrive at a contradiction by using the orthogonality relations (44) and condition (45).

The weak compactness of the space of measures \(M_1(\hat{F})\) shows that
\[
\frac{1}{n} \chi(Q_{n,2}) \rightarrow \mu \neq \lambda, \quad n \in \Lambda, \quad n \to \infty
\]  
(46)
for some infinite subsequence \(\Lambda \subset \mathbb{N}\); besides, \(S(\mu) \subset F\), \(\mu \in M_1(F)\), \(\mu(1) = 1\) by the above properties of the polynomial \(Q_{n,2}\). We claim that relation (46) and the orthogonality relation (44) contradict each other.

Setting
\[
\tilde{V}^\mu(z) := \int_F \log \left| \frac{1}{1 - \varphi(z)\varphi(t)} \right| d\mu(t),
\]
we have
\[
P^\mu(z) = 2V^\mu(z) - \tilde{V}^\mu(z).
\]
Since \(\mu \neq \lambda\), it follows that, for \(z \in S(\mu) \subset F\),
\[
P^\mu(z) + \psi(z) \neq m_0 := \min_{z \in F} (P^\mu(z) + \psi(z)) = P^\mu(x_0) + \psi(x_0),
\]  
(47)
where \(x_0 \in F\). Hence there exists a point \(x_1 \in S(\mu)\), \(x_1 \neq x_0\), and a number \(\varepsilon > 0\) such that
\[
P^\mu(x_1) + \psi(x_1) = m_1 > m_0 + \varepsilon.
\]  
(48)
Further, since the function \(\psi(z)\) is harmonic and the potential \(P^\mu\) is lower semi-continuous, the same inequality (48) holds in some \(\delta\)-neighbourhood \(U_\delta(x_1) := (x_1 - \delta, x_1 + \delta) \ni x_0, \delta > 0\), of the point \(x_1\). We have \(x_1 \in S(\mu)\), and so \(\mu(U_\delta(x_1)) > 0\). Hence, for all sufficiently large \(n \geq n_0, n \in \Lambda\), there exists a polynomial \(p_n(z) = (z - \zeta_{n,1})(z - \zeta_{n,2})\) such that \(\zeta_{n,1}, \zeta_{n,2} \in U_\delta(x_1)\) and \(p_n\) divides the polynomial \(Q_{n,2}\); that is, \(Q_{n,2}/p_n \in \mathbb{P}_{n-2}\). We set
\[
\tilde{Q}_n(z) := \frac{Q_{n,2}(z)}{p_n(z)} = \prod_{j=1}^{n-2} (z - x_{n,j}).
\]  
(49)
We may assume in what follows that, for \(n \in \Lambda\), all zeros of the polynomial \(Q_{n,2}\) lie in the set \(\hat{F} \setminus E\). Indeed, there is at most one gap between the intervals \([c_j, d_j]\) that may contain the interval \(E\), in each gap lying at most one zero of the polynomial \(Q_{n,2}\). If some zero of the polynomial \(Q_{n,2}\) lies on the interval \(E\), then in definition (49) of the polynomial \(\tilde{Q}_n\) one should replace the corresponding factor \((z - x_{n,j_0}, \text{say})\) by the factor \((z - \tilde{x}_{n,j_0})\), where the point \(\tilde{x}_{n,j_0}\) still lies in the (open) gap, but it is not lying in \(E\) anymore.

\(\text{Note that, under the hypotheses of Theorem 2, the GRS-method is much easier to deal with, because an } S\text{-compact set } F\text{ is a finite union of intervals of the real line and } \sigma\text{ is a positive measure on } F;\text{ cf. } [44], [18], [42].\)}}
Now in the orthogonality relation (44) we put $N = n - 1$ and take the zeros $x_{n,j}$ of the polynomial $\tilde{Q}_n$ as points $a_{n,j}$ (with the possible correction mentioned above), relation (44) assuming the form

\[
0 = \int_{F \setminus U_\delta(x_1)} \frac{Q_{n,2}^2(x)}{p_n(x)} \prod_{j=1}^{n-2} \frac{1}{1 - \varphi(x)\varphi(x_{n,j})} \cdot \varphi'(x) \varphi^{n+2}(x) d\sigma(x) + \int_{U_\delta(x_1)} \frac{Q_{n,2}^2(x)}{p_n(x)} \prod_{j=1}^{n-2} \frac{1}{1 - \varphi(x)\varphi(x_{n,j})} \cdot \varphi'(x) \varphi^{n+2}(x) d\sigma(x). \tag{50}
\]

We denote by $I_{n,1}$ and $I_{n,2}$, respectively, the first and second integrals in (50). Since the integrand in $I_{n,1}$ has constant sign for $x \in F \setminus U_\delta(x_1)$, we have

\[
|I_{n,1}| = \int_{F \setminus U_\delta(x_1)} \left| \frac{Q_{n,2}^2(x)}{p_n(x)} \prod_{j=1}^{n-2} \frac{1}{1 - \varphi(x)\varphi(x_{n,j})} \cdot \varphi'(x) \varphi^{n+2}(x) \right| d\sigma(x) = \int_{F \setminus U_\delta(x_1)} |Q_{n,2}(x)| \prod_{j=1}^{n-2} \frac{x - x_{n,j}}{1 - \varphi(x)\varphi(x_{n,j})} \cdot \varphi'(x) \varphi^{n+2}(x) d\sigma(x). \tag{51}
\]

A similar analysis (see Lemma 7 of [18]) with the use of standard machinery of the logarithmic potential theory shows that

\[
\lim_{n \to \infty} |I_{n,1}|^{1/n} = \exp \left\{ - \min_{x \in F \setminus U_\delta(x_1)} \left( P^\mu(x) + \psi(x) \right) \right\} = e^{-m_0}. \tag{52}
\]

We give a proof of (52) for completeness (cf. Lemma 7 of [18])

Indeed,

\[
- \frac{1}{n} \sum_{j=1}^{n-2} \log |1 - \varphi(x)\varphi(x_{n,j})| \to \int_F \log \frac{1}{|1 - \varphi(x)\varphi(t)|} d\mu(t) = \overline{\nu}(x) \tag{53}
\]

as $n \to \infty$ uniformly in $x \in F$. Hence,

\[
\min_{x \in F} \left\{ - \frac{1}{n} \log \left( |Q_{n,2}(x)| \prod_{j=1}^{n-2} \frac{x - x_{n,j}}{1 - \varphi(x)\varphi(x_{n,j})} \cdot \varphi'(x) \varphi^{n+2}(x) \right) \right\} \to \min_{x \in F} \left\{ P^\mu(x) + \psi(x) \right\} \tag{54}
\]

as $n \to \infty$. As a result, we have

\[
\max_{x \in F} \left\{ |Q_{n,2}(x)| \prod_{j=1}^{n-2} \frac{x - x_{n,j}}{1 - \varphi(x)\varphi(x_{n,j})} \cdot \varphi'(x) \varphi^{n+2}(x) \right\}^{1/n} \to \exp \left\{ - \min_{x \in F} \left[ P^\mu(x) + \psi(x) \right] \right\} \tag{55}
\]

as $n \to \infty$, proving thereby the upper estimate

\[
\lim_{n \to \infty} |I_{n,1}|^{1/n} \leq e^{-m_0}.
\]
Let us now prove the corresponding lower estimate. The potential $P^\mu$ is weakly continuous, and hence the function $P^\mu + \psi$ is approximately continuous with respect to the Lebesgue measure on the compact set $F$. Consequently, for any $\varepsilon > 0$, the set
\[ e = \{ x \in F : (P^\mu + \psi)(x) < m_0 + \varepsilon \} \]
has positive Lebesgue measure. From our assumptions we have
\[ \lim_{n \to \infty} \frac{1}{n} \log \left| Q_{n,2}(x) \right| \prod_{j=1}^{n-2} \left| \frac{x - x_{n,j}}{\varphi(x)\varphi(x_{n,j})} \right| \cdot \frac{\varphi'(x)}{\varphi^{n+2}(x)} \to (P^\mu + \psi)(x) \]
as $n \to \infty$ with respect to the measure on $F$. So, the measure of the set
\[ e_n := \left\{ x \in e : - \frac{1}{n} \log \left| Q_{n,2}(x) \right| \prod_{j=1}^{n-2} \left| \frac{x - x_{n,j}}{\varphi(x)\varphi(x_{n,j})} \right| \cdot \frac{\varphi'(x)}{\varphi^{n+2}(x)} < m_0 + \varepsilon \right\} \]
tends to the measure of $e$ as $n \to \infty$. Hence
\[ \lim_{n \to \infty} |I_{n,1}|^{1/n} \geq e^{-(m_0 + \varepsilon)} \lim_{n \to \infty} \left( \int_{e_n} \varphi'(x) \, d\sigma(x) \right)^{1/n} = e^{-(m_0 + \varepsilon)}, \quad (56) \]
the last equality in (56) holding because $\sigma'(x) > 0$ a.e. on $F$. The lower estimate
\[ \lim_{n \to \infty} |I_{n,1}|^{1/n} \geq e^{-m_0} \]
follows from (56), because $\varepsilon > 0$ is arbitrary. This proves (52).

On the other hand, for the second integral $I_{n,2}$ we have the estimate
\[ |I_{n,2}| \leq \int_{U_\delta(z_1)} |Q_{n,2}(x)| \prod_{j=1}^{n-2} \left| \frac{x - x_{n,j}}{\varphi(x)\varphi(x_{n,j})} \right| \cdot \frac{\varphi'(x)}{\varphi^{n+2}(x)} \, d\sigma(x). \quad (57) \]
Now an analysis similar to that above shows that
\[ \lim_{n \to \infty} |I_{n,2}|^{1/n} \leq \exp \left\{ - \min_{x \in U_\delta(z_1)} (P^\mu(x) + \psi(x)) \right\} \leq e^{-m_1} < e^{-(m_0 + \varepsilon)}. \quad (58) \]
But relations (52) and (58) contradict the equality $I_{n,1} = -I_{n,2}$, which is consequent on the orthogonality relations (44).

This proves Theorem 2.

**Remark 1.** In a certain sense, the above transformations mean the change of the variable $z$ by the variable $\zeta = \varphi(z)$. For the case of the Riemann surface $w^2 = z^2 - 1$ under consideration, the key orthogonality relation (44) can be derived directly from relations (26), properties of functions of the second kind, and identity (7), which is much faster. However, in the present paper we chose a different method of exposition, because in a more general setting, when, for example,
\[ f_1(z) := \int_{-1}^1 \frac{r(x)}{(z - x)\sqrt{1 - x^2}} \, dx, \]
where $r \in \mathbb{C}(z)$ is an arbitrary complex rational function without poles and zeros on $E$, such a simplification does not apply anymore, but the conclusions
of Theorem 2 remain valid. It is also worth pointing out the role of the Riemann surface of the function \( w^2 = z^2 - 1 \) in our analysis, because we also intend to extend both the results from the present paper and those announced in [50] to the hyperelliptic setting, when, instead of the Riemann surface of the function \( w^2 = z^2 - 1 \) of genus \( g = 0 \), use is made of the Riemann surface of the function \( w^2 = (z - e_1) \ldots (z - e_4) \) of genus \( g = 1 \). A generalization of the results obtained here to the elliptic case (of course, if such an extension will come to being) will be of the utmost importance in assessing the potency of the method proposed here when investigating the general case of a pair of functions \( f_1, f_2 \in \mathcal{A}^\circ(\Sigma) \). Of course, in this general case the problem of the formula for strong asymptotics for Padé polynomials valid for an arbitrary function \( f \) from the class \( \mathcal{A}^\circ(\Sigma) \) will have a great value; see [37], [31], [5] in this respect.

**Remark 2.** It is well known (see [1], and also [4] and [29]) that, for a pair of functions \( f_1, f_2 \) forming a Nikishin system, the support of the equilibrium measure \( \lambda \) in the diagonal case (which is considered here) coincides with the entire compact set \( F \); this means that only the case of identical equality in relations (16) is possible. So far, this fact has not yet been proved within the framework of the approach proposed here.

**Remark 3.** It is worth pointing out that the approach proposed in the present paper stems, to some extent, from the analysis of numerical experiments of [21]–[23].

\[^{10}\text{As was already mentioned above, the author intends to investigate this general case in a separate paper; see [50].}\]
Figure 1. Zeros of diagonal Hermite–Padé polynomials of type I $Q_{200,0}$ (blue points), $Q_{200,1}$ (red points), $Q_{200,2}$ (black points) for the tuple of functions $[1, f_1, f_2]$, where $f_1(z) := (z^2 - 1)^{-1/2}$, $f_2(z) := ((z - .8 - .5i)(z + .8 - .5i))^{-1/2}$, forming an Angelesco system. No theoretical justification of such behavior of the zeros of Hermite–Padé polynomials of type I has not yet been found to date.
Figure 2. Zeros of the denominator of diagonal Hermite–Padé approximants of type II $P_{400}$ (light blue points) for the tuple of functions $[1, f_1, f_2]$, where $f_1(z) := (z^2 - 1)^{-1/2}$, $f_2(z) := ((z - .8 - .5i)(z + .8 - .5i))^{-1/2}$, forming an Angelesco system. Theoretical justification of such behavior of zeros of Hermite–Padé polynomials of type II was obtained in [2].
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20 SERGEY P. SUETIN

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