ON DIRICHLET’S LAMBDA FUNCTIONS

SU HU AND MIN-SOO KIM

Abstract. Let

\[ \lambda(s) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} \]

be Dirichlet lambda function,

\[ \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \]

be its alternating form, and

\[ \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \]

be Dirichlet eta function. According to a recent historical book by V.S. Varadarajan ([23, p. 70]), the above series of three functions have been investigated by L. Euler long time ago under the notations \( N(s) \), \( L(s) \) and \( M(s) \), respectively.

In this paper, we shall further present some properties for them. That is, a number of infinite families of linear recurrence relations of \( \lambda(s) \) at positive even integer arguments, \( \lambda(2m) \), has been obtained, the convolutional identities for the special values of \( \lambda(s) \) at even arguments and for the special values of \( \beta(s) \) at odd arguments have been proved, and a power series expansion for the alternating Hurwitz zeta function \( J(s, a) \) has also been given, which involves a known one for \( \eta(s) \).

1. Introduction

1.1. Background. According to Varadarajan [23, p. 59], long time ago, Pietro Mengoli (1625–1686) posed the problem of finding the sum of the series

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots. \]  

This problem was first solved by Euler in a letter to Daniel Bernoulli as

\[ 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}. \]  

More generality, let

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

be Dirichlet lambda functions, Dirichlet eta functions, Recurrences, Euler polynomials.
be Riemann’s zeta function, Euler also proved the following formula

\[(1.3) \quad \zeta(2k) = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \cdots = \frac{(-1)^{k-1}B_{2k}2^{2k}}{2(2k)!}\pi^{2k},\]

where the \(B_{2k}\) are the Bernoulli numbers defined by the generating function

\[(1.4) \quad \frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k}}{(2k)!}.\]

One of Euler’s proofs of (1.2) around 1742 is based on the following infinite product expansion of \(\frac{\sin x}{x}\),

\[(1.5) \quad \frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).\]

Euler also gave another proof of (1.2) by starting from the formula

\[(1.6) \quad \frac{1}{2} (\arcsin x)^2 = \int_0^x \frac{\arcsin t}{\sqrt{1 - t^2}} dt,\]

that is, by letting \(x = 1\) in the left hand side, we get \(\frac{\pi^2}{8}\), and in the right hand side, by expanding \(\arcsin t\) as a power series and integrating term by term, we get

\[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2},\]

which arrives the summation formula

\[(1.7) \quad \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} = \frac{\pi^2}{8}.\]

Then by noticing that

\[(1.8) \quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \zeta(2) - \frac{1}{2^2} \zeta(2) = \frac{3}{4} \zeta(2),\]

we recover (1.2) (see [23, p. 62–63]).

Let

\[(1.9) \quad \lambda(s) = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^s} = (1 - 2^{-s})\zeta(s), \quad \text{Re}(s) > 1,\]

be Dirichlet lambda function according to Abramowitz and Stegun’s handbook [11, pp. 807–808], which has also been studied by Euler under the notation \(N(s)\) (see [23, p. 70]). The above equation (1.7) has given the special value of \(\lambda(s)\) at 2, and the special values of \(\lambda(s)\) at any even positive integer \(2m\) is calculated by the following formula ([10 (2.15)])

\[(1.10) \quad \lambda(2m) = (-1)^m \frac{\pi^{2m}}{4(2m - 1)!} E_{2m-1}(0), \quad m \geq 1,\]
where the Euler polynomials $E_n(x)$ are defined by the following generating function

$$\frac{2}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.11}$$

The integers $E_n = 2^n E_n(1/2), n \in \mathbb{N}_0,$ are called Euler numbers, which can also be defined as the coefficients of $t^n/n!$ in the Taylor expansion of $\text{sech}(t), \ |t| < \pi/2.$ For example, $E_0 = 1, E_2 = -1, E_4 = 5,$ and $E_6 = -61.$ Euler numbers and polynomials (so called by Scherk in 1825) were introduced in Euler’s famous book, Institutiones Calculi Differentials (1755, pp. 487–491 and p. 522).

Recently, the Euler polynomials have been applied in several different areas, such as the semi-classical approximations to quantum probability distributions (see [3]), various approximations and expansion formulas in discrete mathematics and number theory (see for instance [1, 18]). Euler polynomials can be defined by various methods depending on their applications. For instance, the explicit formula

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} E_k(0) \tag{1.12}$$

shows that $E_n(x)$ is a polynomial of degree $n.$ Different approaches to the theory of Euler polynomials are well known and can be found in the classical papers by Euler [6], Nörlund [17] and Raabe [19].

It may be interesting to point out that there is also a connection between the generalized Euler numbers and the ideal class group of the $p^{n+1}$-th cyclotomic field when $p$ is a prime number. For details, we refer to a recent paper [9], especially [9, Proposition 3.4].

In addition, Euler also studied the following alternating form of $\lambda(s)$ (in his notation, $L(s),$ see [23, p. 70])

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \quad \text{Re}(s) > 0, \tag{1.13}$$

and $\beta(2) = G$ is the Catalan’s constant (see [25], [11 p. 807] and [7, p. 53]). It is known that the special values of $\beta(s)$ at odd positive integers $2n + 1$ are given by

$$\beta(2n + 1) = (-1)^n E_{2n} \frac{E_{2n}}{2^{2n+1} (2n)!} \left(\frac{\pi}{2}\right)^{2n+1}, \tag{1.14}$$

where $E_n$ are the Euler numbers. $\beta(s)$ has the following interesting integral representation by a trigonometric function

$$\beta(s) = \frac{1}{2 \Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{\cosh(t)}, \quad \text{Re}(s) > 0,$$

where $\Gamma$ denotes the Gamma function (see [7, p. 56]).
1.2. **Main results.** In 1987, Song [21] found the following linear recurrence relation for Riemann’s zeta function at even arguments \( \zeta(2n) \) by using Fourier series expansion of periodic functions,

\[
(-1)^{n+1} \frac{\pi^{2n} \cdot n}{(2n + 1)!} + \sum_{k=1}^{n-1} \frac{\pi^{2k}}{(2k + 1)!} \zeta(2n - 2k) = 0.
\]

Recently, Merca [14] obtained another proof of the above equality by using the generating function of Bernoulli numbers, and he [15] also obtained a homogeneous linear recurrence relation for \( \zeta(2n) \) by using several tools from symmetric functions theory. Extending the above works, in 2017, Merca [16] also introduced a number of infinite families of linear recurrence relations between \( \zeta(2n) \).

For Dirichlet lambda function \( \lambda(s) \), in 2013, M.C. Lettington [13, (1.24)] proved the following linear recurrence relation between its special values \( \lambda(2m) \).

**Theorem 1.1** (Lettington). Let \( m \) be a positive integer. Then

\[
\lambda(2m) = (-1)^{m-1} \left( \frac{\pi^{2m}}{4(2m)!} + \sum_{k=1}^{m-1} \frac{(-1)^{m-k} \pi^{2k}}{(2k + 1)!} \lambda(2m - 2k) \right).
\]

In this paper, based on the integral representation of

(1.15)

\[
J(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + a)^s}
\]

(see (2.3)) and an analogue of the Hermite formula for \( J(s, a) \) (see (2.4)), we give another proof of Theorem 1.1.

In analogue with Merca’s work [16], we further prove a number of infinite families of linear recurrence relations of \( \lambda(2m) \) by using the generating function of Euler polynomials and the Euler-type formula (1.10). (See Theorems 1.2, 1.3, 1.7, 1.9, 1.10, 1.11 and Corollaries 1.4 and 1.5).

Moreover, we also prove the convolutional identities for the special values of \( \lambda(s) \) at even arguments and for the special values of \( \beta(s) \) at odd arguments, respectively. (See Theorems 5.1 and 5.2).

Finally, we also show a power series expansion for the function \( J(s, a) \) (1.15), which involves a known one by Coffey for the Dirichlet eta function,

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \text{Re}(s) > 0.
\]

(See Proposition 6.1).

In what follows, we shall use the usual convention that an empty sum is taken to be zero. For example, if \( m = 1 \), we understand \( \sum_{k=1}^{m-1} = 0 \).

1.3. **Linear recurrence relations.**
Theorem 1.2. Let $m$ be a positive integer, and $\alpha$ complex numbers such that $\alpha \neq \frac{1}{2}$. Then
\[
(-1)^{m+1} \frac{\alpha^{2m} - (\alpha - 1)^{2m}}{4} \cdot \frac{\pi^{2m+2}}{(2m)!} + \sum_{k=1}^{m} (-1)^k (\alpha^{2k-1} + (\alpha - 1)^{2k-1}) \frac{\pi^{2k}}{(2k-1)!} \lambda(2m - 2k + 2) = 0.
\]

Theorem 1.3. Let $m$ be a nonnegative integer, and $\alpha$ complex numbers. Then
\[
(-1)^{m+1} \frac{\alpha^{2m+1} - (\alpha - 1)^{2m+1}}{4} \cdot \frac{\pi^{2m+2}}{(2m + 1)!} + \sum_{k=0}^{m} (-1)^k (\alpha^{2k} + (\alpha - 1)^{2k}) \frac{\pi^{2k}}{(2k)!} \lambda(2m - 2k + 2) = 0,
\]
where define $0^0 = 1$.

Corollary 1.4. Let $m$ be a nonnegative integer. Then
\[
(-1)^{m+1} \frac{\pi^{2m+2}}{4(2m + 1)!} + \sum_{k=0}^{m} (-1)^k (1 + \delta_k) \frac{\pi^{2k}}{(2k)!} \lambda(2m - 2k + 2) = 0,
\]
where $\delta_k = 0$ if $k \neq 0$ and $1$ if $k = 0$.

Corollary 1.5. Let $m$ be a nonnegative integer. Then
\[
(-1)^{m+1} \frac{\pi^{2m+2}}{(2m + 1)!} + \sum_{k=0}^{m} (-1)^k \left( \frac{1}{2} \right)^{-2m+2k-3} \frac{\pi^{2k}}{(2k)!} \lambda(2m - 2k + 2) = 0.
\]

Remark 1.6. If $m$ is a positive integer, then the above identity may be written as
\[
\lambda(2m) = (-1)^{m-1} \left( \frac{\pi^{2m}}{2^{2m+1}(2m - 1)!} + \sum_{k=1}^{m-1} (-1)^{m-k} \frac{\pi^{2k}}{2^{2k}(2k)!} \lambda(2m - 2k) \right),
\]
which is similar to the work of Lettington (see Theorem 1.1).

Theorem 1.7. Let $m$ and $\alpha$ be a positive integer. Then
\[
(-1)^{m+1} \frac{\alpha^{2m} - (\alpha - 1)^{2m}}{4} \cdot \frac{\pi^{2m+2}}{(2m)!} + \sum_{k=1}^{m} (-1)^k \alpha^{2k-1} \frac{\pi^{2k}}{(2k-1)!} \lambda(2m - 2k + 2) = 0.
\]

Remark 1.8. We see that this linear recurrence relation does not require a priori knowledge of the Euler numbers. If $m$ is small, then it is easy to
evaluate $\lambda(2m)$ as follows

$$
\lambda(2) = \frac{\pi^2}{8},
$$

$$
\lambda(4) = \frac{1}{2^5} \left( -\frac{\pi^4}{3!} + 2^3 \frac{\pi^2}{2!} \lambda(2) \right) = \frac{\pi^4}{96},
$$

$$
\lambda(6) = \frac{1}{2^7} \left( \frac{\pi^6}{5!} - 2^3 \frac{\pi^4}{4!} \lambda(2) + 2^5 \frac{\pi^2}{2!} \lambda(4) \right) = \frac{\pi^6}{960},
$$

$$
\lambda(8) = \frac{1}{2^9} \left( -\frac{\pi^8}{7!} + 2^3 \frac{\pi^6}{6!} \lambda(2) - 2^5 \frac{\pi^4}{4!} \lambda(4) + 2^7 \frac{\pi^2}{2!} \lambda(6) \right) = \frac{17\pi^8}{161280}.
$$

**Theorem 1.9.** Let $m$ be a positive integer, and $\alpha$ complex numbers. Then

$$
(-1)^m \frac{\pi^{2m+2}}{4(2m)!} ((\alpha - 1)^{2m} + (\alpha + 1)^{2m} - 2\alpha^{2m})
$$

$$
+ \sum_{k=1}^{m} (-1)^k ((\alpha - 1)^{2k-1} - (\alpha + 1)^{2k-1}) \frac{\pi^{2k}}{(2k-1)!} \lambda(2m - 2k + 2) = 0.
$$

**Theorem 1.10.** Let $m$ be a nonnegative integer, and $\alpha$ complex numbers. Then

$$
(-1)^m \frac{\pi^{2m+2}}{4(2m+1)!} ((\alpha - 1)^{2m+1} - (\alpha + 1)^{2m+1})
$$

$$
+ \sum_{k=0}^{m} (-1)^k ((\alpha - 1)^{2k} + (\alpha - 1)^{2k} + 2\alpha^{2k}) \frac{\pi^{2k}}{(2k)!} \lambda(2m - 2k + 2) = 0,
$$

where define $0^0 = 1$.

**Theorem 1.11.** Let $m$ be a nonnegative integer. Then

$$
(-1)^{m+1} 2(1 - 3^{-2m-1}) \lambda(2m + 2)
$$

$$
= \frac{\pi^{2m+2}}{3^{2m+1}(2m+1)!} + 4 \sum_{k=0}^{m} (-1)^{k+1} \frac{\pi^{2m-2k}}{3^{2m-2k}(2m-2k)!} \lambda(2k + 2).
$$

2. Another proof of Theorem 1.11

To our purpose, we need the following lemmas.

**Lemma 2.1** ([2, p. 4, (9)]). Let $i = \sqrt{-1}$. Then

$$
(a + ix)^{-s} - (a - ix)^{-s} = \frac{2}{i(a^2 + x^2)^{s/2}} \sin\left(s \tan^{-1}\left(\frac{x}{a}\right)\right).
$$

**Proof.** The proof is based upon the relation $\sin y = \frac{1}{2i}(e^{iy} - e^{-iy})$ with $y = -s \tan^{-1}\left(\frac{x}{a}\right)$ and $\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{2i} \left(\frac{1 - \frac{x^2}{a^2}}{1 + \frac{x^2}{a^2}}\right)$. □

**Lemma 2.2.** Let $m$ be a positive integer and $i = \sqrt{-1}$. Then

$$
\sum_{k=1}^{m} (-1)^k \left(\frac{2m}{2k-1}\right) \left(\frac{t}{\pi}\right)^{2k-1} = \frac{i}{2\pi^{2m}} \left[(\pi + it)^{2m} - (\pi - it)^{2m}\right].
$$
Proof. From the expansion \((\pi + it)^{2m} = \pi^{2m} \sum_{k=0}^{2m} \binom{2m}{k} (\frac{it}{\pi})^k\), we have

\[
i \left[(\pi + it)^{2m} - (\pi - it)^{2m}\right] = i \pi^{2m} \sum_{k=0}^{2m} (1 - (-1)^k) \binom{2m}{k} (\frac{it}{\pi})^k
= i \pi^{2m} \sum_{k=1}^{m} 2 \binom{2m}{2k - 1} (\frac{it}{\pi})^{2k - 1}
= 2 \pi^{2m} \sum_{k=1}^{m} (-1)^k \binom{2m}{2k - 1} (\frac{t}{\pi})^{2k - 1}.
\]

This completes the proof. □

Lemma 2.3. Let \(m\) be a positive integer. Then

\[
\sum_{k=1}^{m} (-1)^k \frac{\pi^{2m-2k}}{(2m-2k+1)! (2k-1)!} \frac{t^{2k-1}}{(2m)!} = -\frac{(\pi^2 + t^2)^m}{\pi (2m)!} \sin \left(2m \tan^{-1} \left(\frac{t}{\pi}\right)\right).
\]

Proof. By manipulating binomial expansions, we may calculate the following summation

\[
\sum_{k=1}^{m} (-1)^k \frac{\pi^{2m-2k}}{(2m-2k+1)! (2k-1)!} \frac{t^{2k-1}}{(2m)!}
= \sum_{k=1}^{m} (-1)^k \frac{(2m)! \pi^{-2k+1} t^{2k-1}}{(2m-2k+1)! (2k-1)! (2m)!}
= \frac{\pi^{2m-1}}{(2m)!} \sum_{k=1}^{m} (-1)^k \binom{2m}{2k - 1} \left(\frac{t}{\pi}\right)^{2k - 1}
= \frac{i}{2\pi (2m)!} \left[(\pi + it)^{2m} - (\pi - it)^{2m}\right]
= -\frac{1}{\pi (2m)!} (\pi^2 + t^2)^m \sin \left(2m \tan^{-1} \left(\frac{t}{\pi}\right)\right)
\]

(by Lemma 2.2)

(by Lemma 2.1 with \(a = \pi, x = t\) and \(s = -2m\)).

This completes the proof. □

Lemma 2.4. Let \(n\) be a positive integer. Then

\[
\int_0^{\infty} (\pi^2 + t^2)^n \sin \left(2m \tan^{-1} \left(\frac{t}{\pi}\right)\right) \frac{e^t dt}{e^{2t} - 1} = \frac{\pi^{2m+1}}{4}.
\]

Proof. Recall that the function \(J(s, a)\) is defined as follows:

(2.1) \(J(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + a)^s}\),

where \(0 < a \leq 1\) (see [26, (1.1)]). As the classical Hurwitz zeta functions

(2.2) \(\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}\),
we also have an integral representation of $J(s, a)$,

$$
\Gamma(s)J(s, a) = \int_0^\infty \frac{e^{(1-a)t}x^{s-1}}{e^t + 1} dt, \quad \text{Re}(s) > 0,
$$

(see [26 (3.1)]) and an analogue of Hermite’s formula for $J(s, a)$,

$$
J(s, a) = \frac{a^{-s}}{2} + 2\int_0^\infty (a^2 + y^2)^{-s/2} \sin \left( s \tan^{-1} \frac{y}{a} \right) \frac{e^{\pi y} dy}{e^{2\pi y} - 1}.
$$

The above formula enables $J(s, a)$ to be continued analytically to the whole complex plane. The special values of $J(s, a)$ at non-positive integers can be represented by Euler polynomials as follows,

$$
J(-m, a) = (-1)^m \frac{2}{2} E_m(1-a) = \frac{1}{2} E_m(a) \quad (m \geq 0)
$$

(see [26 (3.8)]). Letting $t = \pi y, a = 1$ and $s = -2m$ in (2.4), we have

$$
J(-2m, 1) = -\frac{1}{2} \frac{2}{\pi} \int_0^\infty \left( 1 + \left( \frac{t}{\pi} \right)^2 \right)^m \sin \left( 2m \tan^{-1} \left( \frac{t}{\pi} \right) \right) \frac{e^{t} dy}{e^{2t} - 1}
$$

$$
= -\frac{2}{\pi^{2m+1}} \int_0^\infty (\pi^2 + t^2)^m \sin \left( 2m \tan^{-1} \left( \frac{t}{\pi} \right) \right) \frac{e^{t} dy}{e^{2t} - 1}.
$$

Replacing $-m$ by $-2m$ and setting $a = 1$ in (2.5), we have

$$
J(-2m, 1) = (-1)^{-2m} \frac{2}{2} E_{2m}(0) = \frac{1}{2} E_{2m}(1) = 0 \quad (m \geq 1),
$$

since $E_{2m}(0) = 0$ for $m > 1$. Combining (2.6) and (2.7), we obtain the desired result. $\Box$

**Proof of Theorem 1.1.** Start with the formula

$$
\frac{\Gamma(s)}{(2n + 1)^s} = \int_0^\infty e^{-(2n+1)t}t^{s-1}dt, \quad \text{Re}(s) > 0.
$$

Summing both sides over $n$, we have

$$
\Gamma(s) \sum_{n=0}^\infty \frac{1}{(2n + 1)^s} = \sum_{n=0}^\infty \int_0^\infty e^{-(2n+1)t}t^{s-1}dt
$$

$$
= \int_0^\infty \sum_{n=0}^\infty e^{-(2n+1)t}t^{s-1}dt
$$

$$
= \int_0^\infty \frac{e^{-t}t^{s-1}}{1 - e^{-2t}} dt
$$

for $\text{Re}(s) > 0$, that is,

$$
\Gamma(s)\lambda(s) = \int_0^\infty \frac{e^{t}t^{s-1}}{e^{2t} - 1} dt, \quad \text{Re}(s) > 0.
$$
Then applying the integral representation (2.10), we have

\[
\sum_{k=1}^{m-1} \frac{(-1)^k \pi^{2m-2k}}{(2m-2k+1)!} \lambda(2k) = \sum_{k=1}^{m-1} \frac{(-1)^k \pi^{2m-2k}}{(2m-2k+1)!} \frac{1}{\Gamma(2k)} \int_0^\infty \frac{e^{t2k-1}}{e^{2t} - 1} dt
\]

\[
= \int_0^\infty \left[ \sum_{k=1}^{m-1} \frac{(-1)^k \pi^{2m-2k}}{(2m-2k+1)!} \frac{t^{2k-1}}{(2k-1)!} \right] \frac{e^t}{e^{2t} - 1} dt
\]

To obtain the desired result, we need to eliminate the last line of (2.11). This is done by applying Lemma 2.3 and (2.10) with \(s = 2m\) to the last line of (2.11), then we have

\[
\sum_{k=1}^{m-1} \frac{(-1)^k \pi^{2m-2k}}{(2m-2k+1)!} \lambda(2k) = -\frac{1}{\pi(2m)!} \int_0^\infty \left[ (\pi^2 + t^2)^m \sin \left( 2m \tan^{-1} \left( \frac{t}{\pi} \right) \right) \right] \frac{e^t}{e^{2t} - 1} dt + (-1)^{m+1} \pi^{2m} \lambda(2m),
\]

and the desired result follows from a direct manipulation. \(\square\)

3. Proofs of Theorem 1.2, 1.3, 1.7 and Corollary 1.4, 1.5

First we prove the following lemma.

**Lemma 3.1.** Let \(n\) be a nonnegative integer, and \(x\) and \(\alpha\) complex numbers. Then

\[
\sum_{k=0}^{n} \left( \alpha^{n-k} - (-1)^k (2x - 1 + \alpha)^{n-k} \right) E_k(x) = 0.
\]

**Proof.** We consider (1.11) and the series \(\sum_{n=0}^\infty \alpha^n t^n / n! = e^{\alpha t}\). For \(|t| < \pi\), it holds that

\[
2e^{tx} \frac{t}{e^t + 1} e^{\alpha t} = \left( \sum_{n=0}^\infty E_n(x) \frac{t^n}{n!} \right) \left( \sum_{n=0}^\infty \alpha^n \frac{t^n}{n!} \right),
\]
which can be written as
\[
\frac{2e^{tx}}{e^t + 1}e^{at} = \frac{2e^{x(t-a)}}{1 + e^{-t}e^{(2x+a-1)t}}
\]
(3.2)
\[
= \left( \sum_{n=0}^{\infty} (-1)^n E_n(x) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} (2x + \alpha - 1)^n \frac{t^n}{n!} \right).
\]
Applying Cauchy’s rule for the multiplication of two power series to the right hand sides of (3.1) and (3.2), then comparing the coefficients of \( t^n \), we obtain the relation
\[
\sum_{k=0}^{n} \left( \binom{n}{k} E_k(x) \alpha^{n-k} \right) = \sum_{k=0}^{n} (-1)^k \left( \binom{n}{k} \right) E_k(x)(2x + \alpha - 1)^{n-k}
\]
for \( n \geq 0 \), and the desired result comes from a direct manipulation. \( \square \)

**Proof of Theorem 1.2.** Setting \( x = 0 \) and \( n = 2m \) in Lemma 3.1, we have
\[
\sum_{k=0}^{2m} \left( \alpha^{2m-k} - (-1)^k (\alpha - 1)^{2m-k} \right) \left( \binom{2m}{k} \right) E_k(0) = 0
\]
for \( m \geq 0 \), which can be written as
\[
\sum_{k=1}^{m} \frac{\alpha^{2k-1} + (\alpha - 1)^{2k-1}}{(2k-1)!(2m-2k+1)!} E_{2m-2k+1}(0) = -\frac{\alpha^{2m} - (\alpha - 1)^{2m}}{(2m)!}
\]
for \( m > 0 \), since \( E_0(0) = 1 \). Then using Euler’s formula (1.10) for the Dirichlet lambda functions, we derive the relation
\[
\sum_{k=1}^{m} \frac{\alpha^{2k-1} + (\alpha - 1)^{2k-1}}{(2k-1)!(2m-2k+1)!} (-1)^{m-k+1} \frac{4}{\pi^{2m-2k+2}} \lambda(2m-2k+2)
\]
\[
= -\frac{\alpha^{2m} - (\alpha - 1)^{2m}}{(2m)!},
\]
where \( \alpha \neq \frac{1}{2} \), and a direct manipulation leads to the desired result. \( \square \)

**Proof of Theorem 1.3.** Setting \( x = 0 \) with \( n = 2m + 1 \) in Lemma 3.1, we have
\[
\sum_{k=0}^{2m+1} \left( \alpha^{2m+1-k} - (-1)^k (\alpha - 1)^{2m+1-k} \right) \left( \binom{2m+1}{k} \right) E_k(x) = 0
\]
for \( m \geq 0 \), which can be written as
\[
\sum_{k=0}^{m} \frac{\alpha^{2k} + (\alpha - 1)^{2k}}{(2k)!(2m-2k+1)!} E_{2m-2k+1}(0) = -\frac{\alpha^{2m+1} - (\alpha - 1)^{2m+1}}{(2m+1)!}
\]
for $m \geq 0$, since $E_0(0) = 1$. Then using Euler’s formula (1.10) for the Dirichlet lambda functions, we derive the relation
\[
\sum_{k=0}^{m} \frac{\alpha^{2k} + (\alpha - 1)^{2k}}{(2k)!} (-1)^{m-k+1} \frac{4}{\pi^{2m-2k+2}} (2m - 2k + 2) = -\frac{\alpha^{2m+1} - (\alpha - 1)^{2m+1}}{(2m+1)!},
\]
where $m \geq 0$ (define $0^0 = 1$), and the desired result comes from a direct manipulation. □

**Proofs of Corollaries 1.4 and 1.5** The first corollary is the case $\alpha = 1$ and the second is the case $\alpha = \frac{1}{2}$ of Theorem 1.3, respectively. □

**Proof of Theorem 1.7** First, it is easy to observe that
\[
\sum_{j=1}^{\alpha} (-1)^j (j^{2m} - (j - 1)^{2m}) = 2 \sum_{j=1}^{\alpha-1} (-1)^j j^{2m} + (-1)^{\alpha} \alpha^{2m}
\]
and
\[
\sum_{j=1}^{\alpha} (-1)^j (j^{2k-1} + (j - 1)^{2k-1}) = (-1)^{\alpha} \alpha^{2k-1}.
\]
Therefore, applying Theorem 1.2 we have
\[
\sum_{j=1}^{\alpha} (-1)^j \left( (-1)^{m+1} \frac{j^{2m} - (j - 1)^{2m}}{4} \cdot \frac{\pi^{2m+2}}{(2m)!} \right) + \sum_{k=1}^{m} (-1)^k (j^{2k-1} + (j - 1)^{2k-1}) \frac{\pi^{2k}}{(2k-1)!} \lambda(2m - 2k + 2) = 0,
\]
thus
\[
\frac{(-1)^{m+1}}{4} \left( 2 \sum_{j=1}^{\alpha-1} (-1)^j j^{2m} + (-1)^{\alpha} \alpha^{2m} \right) \frac{\pi^{2m+2}}{(2m)!} \right) + (-1)^{\alpha} \sum_{k=1}^{m} (-1)^k \alpha^{2k-1} \frac{\pi^{2k}}{(2k-1)!} \lambda(2m - 2k + 2) = 0.
\]
This completes the proof. □

**4. Proofs of Theorem 1.9, 1.10, and 1.11**

In this section, we use the generating functions of Euler polynomials (1.11) to derive another three families of infinite recurrence relations for the Dirichlet lambda functions at positive even integer arguments.

**Lemma 4.1.** Let $n$ be a nonnegative integer, and $x$ and $\alpha$ complex numbers. Then
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{(2x + \alpha - 1)^k + (2x - \alpha - 1)^k}{2} E_{n-k}(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} E_{n-2k}(x) \alpha^{2k},
\]
where \( \lfloor \cdot \rfloor \) is the greatest integer function.

**Proof.** Using the generating function of Euler polynomials \( E_n(x) \) \((1.11)\) and the power series expansion of \( e^{xt} \), we obtain

\[
(4.1) \quad \left( \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} \alpha^{2n} \frac{t^{2n}}{(2n)!} \right) = \frac{1}{2} \left( \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} \alpha^n \frac{t^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \alpha^n \frac{t^n}{n!} \right)
\]

\[
= \frac{1}{2} \left( \frac{2e^{xt}}{e^x+1} \right) \left( e^{(2x+\alpha-1)t} + e^{(2x-\alpha-1)t} \right)
\]

\[
= \frac{1}{2} \left( \sum_{n=0}^{\infty} (-1)^n E_n(x) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} ((2x+\alpha-1)^n + (2x-\alpha-1)^n) \frac{t^n}{n!} \right)
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} \frac{(2x+\alpha-1)^k + (2x-\alpha-1)^k}{(n-k)!k!} E_{n-k}(x) t^n,
\]

where the last equality is obtained by using Cauchy’s rule for multiplying power series. Also applying Cauchy’s rule, we have the following relation

\[
(4.2) \quad \left( \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} \alpha^{2n} \frac{t^{2n}}{(2n)!} \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\alpha^{2k}}{(n-2k)!(2k)!} E_{n-2k}(x) \right) t^n,
\]

where \( \lfloor \cdot \rfloor \) is the greatest integer function. Finally, comparing the coefficients of \( t^n \) in \((4.1)\) and \((4.2)\), we have the desired result. \( \square \)

**Proof of Theorem 1.9.** Setting \( x = 0 \) and \( n = 2m \) in Lemma 4.1, we have

\[
\alpha^{2m} = \frac{1}{2} \sum_{k=0}^{2m} (-1)^{2m-k} \binom{2m}{k} \left( (\alpha-1)^k + (-\alpha-1)^k \right) E_{2m-k}(0)
\]

for \( m \geq 0 \). Since \( E_{2m}(0) = 0 \) \((m > 1)\) and \( E_{0}(0) = 1 \), after moving the \( k = 2m \) term in the right hand side of the above equality to the left hand side, we have

\[
\alpha^{2m} - \frac{1}{2} \left( (\alpha-1)^{2m} + (\alpha+1)^{2m} \right)
\]

\[
= \frac{1}{2} \sum_{k=1}^{m} \binom{2m}{2k-1} \left( (\alpha-1)^{2k-1} + (-\alpha-1)^{2k-1} \right) E_{2m-(2k-1)}(0)
\]

for \( m \geq 1 \). Finally, by applying Euler’s formula \((1.10)\) for the Dirichlet lambda functions to the above equality, we obtain the desired result. \( \square \)
Proof of Theorem 1.10. Setting $x = 0$ and $n = 2m + 1$ in Lemma 4.1, we have
\[
\left\lfloor \frac{2m+1}{2} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{2m+1}{2} \right\rfloor} \binom{2m+1}{2k} E_{2m+1-2k}(0) \alpha^{2k}
\]
\[
= \frac{1}{2} \sum_{k=0}^{2m+1} (-1)^{2m+1-k} \binom{2m+1}{k} ((\alpha - 1)^k + (-\alpha - 1)^k) E_{2m+1-k}(0).
\]
for $m \geq 0$. Since $E_{2m}(0) = 0 \ (m > 1)$ and $E_0(0) = 1$, after moving the $k = 2m + 1$ term in the right hand side of the above equality to the left hand side, we have
\[
\sum_{k=0}^{m} \binom{2m+1}{2k} E_{2m+1-2k}(0) \alpha^{2k} - \frac{1}{2} ((\alpha - 1)^{2m+1} + (-\alpha - 1)^{2m+1})
\]
\[
= \frac{1}{2} \sum_{k=0}^{m} (-1)^{2m+1-2k} \binom{2m+1}{2k} ((\alpha - 1)^{2k} + (-\alpha - 1)^{2k}) E_{2m+1-2k}(0).
\]
Finally, by applying Euler’s formula (1.10) for the Dirichlet lambda functions to the above equation, we obtain the relation
\[
\sum_{k=0}^{m} (-1)^{k+1} \frac{2m+1}{(2k)!} \left( \frac{1}{2} ((\alpha - 1)^{2k} + (\alpha + 1)^{2k} + \alpha^{2k}) \right) \lambda(2m - 2k + 2)
\]
\[
= \frac{1}{8} (-1)^m \pi^{2m+2} ((\alpha - 1)^{2m+1} - (\alpha + 1)^{2m+1}),
\]
which is the desired result. \(\square\)

Proof of Theorem 1.11. From the expression for Euler polynomials
\[
E_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} E_k(0),
\]
we have
\[
E_{2m+1} \left( \frac{1}{3} \right) = \sum_{k=0}^{2m+1} \binom{2m+1}{k} E_k(0) \left( \frac{1}{3} \right)^{2m+1-k}
\]
\[
= \left( \frac{1}{3} \right)^{2m+1} + \sum_{k=0}^{m} \binom{2m+1}{2k+1} E_{2k+1}(0) \left( \frac{1}{3} \right)^{2m-2k}
\]
since $E_{2m}(0) = 0 \ (m > 1)$ and $E_0(0) = 1$. It is known from [22] Theorem 3.3] that
\[
E_n(x) = m^n \sum_{k=0}^{m-1} (-1)^k E_n \left( \frac{x+k}{m} \right) \quad \text{if } 2 \nmid m.
\]
Putting $x = 0$ and $m = 3$ in (4.4), and noticing that $E_{2m+1} \left( \frac{2}{3} \right) = -E_{2m+1} \left( \frac{1}{3} \right)$, we have
\[
E_{2m+1} \left( \frac{1}{3} \right) = \frac{1}{2} (1 - 3^{-2m-1}) E_{2m+1}(0).
\]
Comparing (4.3) and (4.5), we find that

\[
\frac{1}{2}(1 - 3^{-2m-1})E_{2m+1}(0) = \left(\frac{1}{3}\right)^{2m+1} + \sum_{k=0}^{m} \left(\frac{2m + 1}{2k + 1}\right)E_{2k+1}(0) \left(\frac{1}{3}\right)^{2m-2k}.
\]

Then applying Euler’s formula (1.10) for the Dirichlet lambda functions to the above equality, we obtain the desired result. \(\square\)

5. Convolutional identities

Euler found the following beautiful convolutional identity for Bernoulli numbers

\[
\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n - 1)B_n \quad (n \geq 1),
\]

which has been generalized by many authors from different directions (see for instance [5] and [11]). In this section, we prove the convolutional identities for the special values of \(\lambda(s)\) at even arguments and for the special values of \(\beta(s)\) at odd arguments, respectively.

**Theorem 5.1.** Let \(m\) be a positive integer. Then we have

\[
\sum_{k=1}^{m} \lambda(2k)\lambda(2m - 2k + 2) = \left(\frac{m + 1}{2}\right)\lambda(2m + 2).
\]

**Proof.** Setting \(x = y = 0\) in \([5, (4.2)]\), we have

\[
\sum_{k=1}^{m} \left(\frac{2m}{2k - 1}\right)E_{2k-1}(0)E_{2m-2k+1}(0) = 2E_{2m+1}(0).
\]

Then applying Euler’s formula (1.10), we obtain the desired result. \(\square\)

The following theorem has been proved by G.T. Williams many years ago (see [25, p. 22, Theorem II]). Here we show another proof.

**Theorem 5.2** (G.T. Williams). Let \(m\) be a nonnegative integer. Then

\[
\sum_{k=0}^{m} \beta(2k + 1)\beta(2m - 2k + 1) = \left(\frac{m + 1}{2}\right)\lambda(2m + 2).
\]

**Proof.** From the generating function of Euler polynomials, we have

\[
\frac{2^2 e^t}{(e^t + 1)^2} = \left(\sum_{n=0}^{\infty} E_n \left(\frac{1}{2}\right) \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} E_m \left(\frac{1}{2}\right) \frac{t^m}{m!}\right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} E_k \left(\frac{1}{2}\right) E_{n-k} \left(\frac{1}{2}\right) \frac{t^n}{k!(n-k)!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} E_k \left(\frac{1}{2}\right) E_{n-k} \left(\frac{1}{2}\right) \frac{t^n}{n!}.
\]
We also have
\[
\frac{1}{2} \left( \frac{2e^{\frac{1}{2}t}}{e^t + 1} \right)^2 = \frac{d}{dt} \left( \frac{2e^t}{e^t + 1} \right) = \frac{d}{dt} \left( \sum_{n=0}^{\infty} E_n(1) \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} E_n+1(1) \frac{t^n}{n!}. 
\]
Comparing the coefficients of \( t^n \) in the above two equalities, we have
\[
(5.2) \quad \sum_{k=0}^{n} \frac{E_k \left( \frac{1}{2} \right) E_{n-k} \left( \frac{1}{2} \right)}{k!(n-k)!} = 2 \frac{E_{n+1}(1)}{n!}. 
\]
Then putting \( n = 2m \) in (5.2), and noticing that \( E_{2m+1}(1) = (-1)^{2m+1} E_{2m+1}(0) \) and \( E_k = 2^k E_k(1/2) \), we obtain
\[
(5.3) \quad \sum_{k=0}^{2m} \frac{E_k E_{2m-k}}{2^{2m+1} k!(2m-k)!} = - \frac{1}{(2m)!} E_{2m+1}(0). 
\]
That is,
\[
(5.4) \quad \sum_{k=0}^{m} \frac{E_{2k} E_{2m-2k}}{2^{2m+1}(2k)!(2m-2k)!} = - \frac{1}{(2m)!} E_{2m+1}(0) 
\]
since \( E_{2k+1} = 0 \) for \( k \geq 0 \). Applying (1.14) to the above equation, we have the desired result. \( \square \)

6. Power series expansion for \( J(s, a) \)

Start with
\[
\frac{\Gamma(s)}{(n + a)^s} = \int_0^\infty e^{-(n+a)t} t^{s-1} dt, \quad \text{Re}(s) > 0. 
\]
Applying it to (2.1), we obtain
\[
\Gamma(s) J(s, a) = \sum_{n=0}^{\infty} \int_0^\infty (-1)^n e^{-(n+a)t} t^{s-1} dt 
\]
\[
= \int_0^\infty \sum_{n=0}^{\infty} (-1)^n e^{-(n+a)t} t^{s-1} dt 
\]
\[
= \int_0^\infty e^{-(n+a)t} t^{s-1} \frac{dt}{e^{-t} + 1} 
\]
for \( \text{Re}(s) > 0 \), that is,
\[
(6.2) \quad \Gamma(s) J(s, a) = \int_0^\infty \frac{e^{-(1-a)t} t^{s-1}}{e^t + 1} dt, \quad \text{Re}(s) > 0. 
\]
Splitting the above integral at \( x \), and using the generating function of Euler polynomials
\[
(6.3) \quad \frac{2e^{zt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}, \quad |t| < \pi, 
\]
for the integration on \([0, x]\), we have

\[
\Gamma(s)J(s, a) = \int_x^\infty \frac{e^{(1-a)t+s-1}}{e^t + 1} dt + \frac{1}{2} \sum_{n=0}^\infty \frac{E_n(1-a)}{n!} \int_0^x t^{n+s-1} dt.
\]

(6.4)

To handle the first integration in the above equality, we use a standard integral representation for the incomplete Gamma function \(\Gamma(s, x)\) (e.g., [1, p. 260] and [4, (1.1)]), together with a geometric series expansion to get

\[
\int_x^\infty \frac{e^{(1-a)t+s-1}}{e^t + 1} dt = \sum_{n=0}^\infty (-1)^n \int_x^\infty e^{-(n-a+2)t} t^{n+s-1} dt
\]

(6.5)

\[
= \sum_{n=0}^\infty (-1)^n \frac{1}{(n-a+2)^s} \Gamma(s, (n-a+2)x).
\]

From (6.4) and (6.5), we obtain

\[
\Gamma(s)J(s, a) = \sum_{n=0}^\infty (-1)^n \frac{\Gamma(s, (n-a+2)x)}{(n-a+2)^s} + \frac{1}{2} \sum_{n=0}^\infty \frac{E_n(1-a)}{n!} \frac{x^{n+s}}{n+s},
\]

(6.6)

which provides us an analytic continuation of the function \(J(s, a)\) to the whole complex plane.

**Proposition 6.1.** Let \(|x| < \pi\). Then we have

\[
\Gamma(s)J(s, a) = \sum_{n=1}^\infty (-1)^{n-1} \frac{\Gamma(s, (n-a+1)x)}{(n-a+1)^s} + \frac{1}{2} \sum_{n=0}^\infty \frac{E_n(1-a)}{n!} \frac{x^{n+s}}{n+s}.
\]

The above representation holds in all the complex plane \(\mathbb{C}\). And it also implies the special values of \(J(s, a)\) at non-positive integers [26, (3.8)]

\[
J(-n, a) = \frac{(-1)^n}{2} E_n(1-a) = \frac{1}{2} E_n(a)
\]

for \(n \geq 0\).

Taking \(a = \frac{1}{2}\) in (2.1), by (1.13), we have

\[
J\left(s, \frac{1}{2}\right) = 2^s \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^s} = 2^s \beta(s),
\]

(6.7)

thus the above proposition also implies the following result.

**Corollary 6.2.** We have

\[
\Gamma(s)\beta(s) = \frac{1}{2^s} \sum_{n=1}^\infty (-1)^{n-1} \frac{\Gamma(s, (n+1/2)x)}{(n+1/2)^s} + \frac{1}{2^{s+1}} \sum_{n=0}^\infty \frac{E_n}{2^n n!} \frac{x^{n+s}}{n+s}
\]

with free parameter \(x \in (0, \pi]\). In particular, we obtain the special values of \(\beta(s)\) at non-positive integers: \(\beta(-n) = E_n/2\) for \(n \geq 0\).

Let \(\eta(s) = (1 - 2^{1-s})\zeta(s)\)
be alternating zeta function (in Euler’s notation, \( M(s) \), see \([23, \text{p. 70}]\)), that is,

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \text{Re}(s) > 0
\]

(see \([1, \text{pp. 807–808}]\)). Then taking \( a = 1 \) in \((2.1)\), we obtain \( J(s, 1) = \eta(s) \).

The first one is sometimes called Lerch’s eta function and second one is sometimes named Dirichlet’s eta function \([24]\). Proposition 6.1 may have many interesting applications. For example, letting \( a = 1 \) in it, we recover the following result by Coffey.

**Corollary 6.3** (Coffey, \([4, \text{p. 1384, Proposition 1}]\)).

\[
\Gamma(s)\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\Gamma(s, nx)}{n^s} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{E_n(0)}{n!} \frac{x^{n+s}}{n+s},
\]

with free parameter \( x \in (0, \pi] \). In particular, we obtain the exact evaluations \( \eta(-n) = (-1)^n E_n(0)/2 \) for \( n \geq 0 \).

**References**

[1] M. Abramowitz and I. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs and mathematical tables*, Dover, New York, 1972.

[2] V.S. Adamchik, *On the Hurwitz function for rational arguments*, Appl. Math. Comput. 187 (2007), no. 1, 3–12.

[3] L.E. Ballentine and S.M. McRae, *Moment equations for probability distributions in classical and quantum mechanics*, Phys. Rev. A 58 (1998) 1799–1809.

[4] M.W. Coffey, *Series representation of the Riemann zeta function and other results: complements to a paper of Crandall*, Math. Comp. 83 (2014), no. 287, 1383–1395.

[5] K. Dilcher, *Sums of products of Bernoulli numbers*, J. Number Theory 60 (1996), no. 1, 23–41.

[6] L. Euler, *Remarques sur un beau rapport entre les sérées des puissances tant directes que réciproques*, 1768, E352 (Eneström Index), The Euler Archive.

[7] S.R. Finch, *Mathematical Constants*, Cambridge Univ. Press, Cambridge, 2003.

[8] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products*, Academic Press, New York, 1980.

[9] S. Hu and M.-S. Kim, *The (S, \{2\})-Iwasawa theory*, J. Number Theory 158 (2016), 73–89.

[10] S. Hu, D. Kim and M.-S. Kim, *Special values and integral representations for the Hurwitz-type Euler zeta functions*, J. Korean Math. Soc. 55 (2018), no. 1, 185–210.

[11] M.-S. Kim and S. Hu, *Sums of products of Apostol-Bernoulli numbers*, Ramanujan J. 28 (2012), no. 1, 113–123.

[12] M.C. Lettington, *Fleck’s congruence, associated magic squares and a zeta identity*, Funct. Approx. Comment. Math. 45 (2011) 165–205.

[13] M.C. Lettington, *A trio of Bernoulli relations, their implications for the Ramanujan polynomials and the special values of the Riemann zeta function*, Acta Arith. 158 (2013) 1–31.

[14] M. Merca, *On the Song recurrence relation for the Riemann zeta function*, Miskolc Math. Notes 17 (2016), no. 2, 941–945.

[15] M. Merca, *Asymptotics of the Chebyshev-Stirling numbers of the first kind*, Integral Transforms Spec. Funct. 27 (2016), no. 4, 259–267.

[16] M. Merca, *On families of linear recurrence relations for the special values of the Riemann zeta function*, J. Number Theory 170 (2017) 55–65.
[17] N.E. Nörlund, Vorlesungen ber Differenzenrechnung, Springer, Berlin, 1924.
[18] Á. Pintér and C. Rakaczki On the decomposability of linear combinations of Euler polynomials, Miskolc Math. Notes 18 (2017), no. 1, 407–415.
[19] J.L. Raabe, Zurückführung einiger Summen und bestimmtem Integrale auf die JacobBernoullische Function, J. Reine Angew. Math. 42 (1851) 348–367.
[20] T. Rivoal and W. Zudilin, Diophantine properties of numbers related to Catalan’s constant, Math. Ann. 326 (2003) 705–721.
[21] I. Song, A recursive formula for even order harmonic series, J. Comput. Appl. Math. 21 (1988) 251–256.
[22] Z.-W. Sun, Introduction to Bernoulli and Euler polynomials, a lecture given in Taiwan on June 6, 2002, http://maths.nju.edu.cn/~zwsun/BerE.pdf
[23] V.S. Varadarajan, Euler through time: a new look at old themes, American Mathematical Society, Providence, RI, 2006.
[24] https://en.m.wikipedia.org/wiki/Dirichlet_eta_function
[25] G.T. Williams, A new method of evaluating $\zeta(2n)$, Amer. Math. Monthly 60 (1953) 19–25.
[26] K.S. Williams and N.Y. Zhang, Special values of the Lerch zeta function and the evaluation of certain integrals, Proc. Amer. Math. Soc. 119 (1993), no. 1, 35–49.

Department of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China
E-mail address: mahusu@scut.edu.cn

Division of Mathematics, Science, and Computers, Kyungnam University, 7(Woryeong-dong) kyungnamdaehak-ro, Masanhappo-gu, Changwon-si, Gyeongsangnam-do 51767, Republic of Korea
E-mail address: mskim@kyungnam.ac.kr