1. Introduction

Convex stochastic processes are introduced by Nikodem [1] in 1980. Skowroński [2] generalized certain well-known properties of convex functions to convex stochastic processes. In [3], Kotrys investigated the Hermite–Hadamard inequality involving convex stochastic processes. In recent works on convex stochastic processes, many authors studied different integral inequalities, see [3–9].

Motivated from the above works, we study the Hermite–Hadamard type inequalities for \( r \)-convex stochastic processes. We also consider the case of the product of an \( r \)-convex and \( s \)-convex stochastic process.

2. Preliminaries

Let \((\Omega, \mathcal{F}, P)\) be an arbitrary probability space. A \(\mathcal{F}\)-measurable function \(X : \Omega \rightarrow \mathbb{R}\) is said to be a random variable. Let \(I \subset \mathbb{R}\) be an interval. Then a function \(X : I \times \Omega \rightarrow \mathbb{R}\) is said to be the stochastic process if the function \(X(t,.)\) is a random variable for all \(t\) in \(I\).

Let \(P\) and \(E[X]\) denote the limit in probability and the expected value of \(X\), respectively. A stochastic process \(X : I \times \Omega \rightarrow \mathbb{R}\) is said to be continuous in probability if \(P\)–lim \(X(t,.) = X(t_0,.)\) for all \(t_0 \in I\) while it is said to mean-square continuous in \(I\) if

\[
\lim_{t \to t_0} E[(X(t) - X(t_0))^2] = 0, \quad \forall t_0 \in I.
\]

It is worthy to note that if \(X : I \times \Omega \rightarrow \mathbb{R}\) is mean-square continuous, then it is continuous in \(I\) but the converse does not hold.

The mean-square integral is defined as: A random variable \(Y : \Omega \rightarrow \mathbb{R}\) is said to be the mean-square integral of the stochastic process \(X : I \times \Omega \rightarrow \mathbb{R}\) on \([a, b] \subset I\) with \(E[(X(t))^2] < \infty\) \(\forall t \in I\), if for every normal sequence of partitions of \([a, b]\), the following relation holds

\[
\lim_{n \to \infty} E\left[\left(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2\right] = 0,
\]

where \(\Theta_k \in [t_{k-1}, t_k], \ k = 1, 2, 3, \ldots, n\) and \(a = t_0 < t_1 < t_2 < \ldots < t_n = b\) is the partition of \([a, b]\). Thus, we can write

\[
Y(.) = \int_a^b X(t,.) \, dt \quad (a.e).
\]

The assumption of the mean-square continuity of the stochastic process is enough for the mean-square integral to exist.

From the definition of mean-square integral, we immediately have the following relation.

\[
X(t,.) \leq Y(t,.) \quad (a.e), \text{for} \ t \in [a, b] \implies \int_a^b X(t,.) \, dt \leq \int_a^b Y(t,.) \, dt \quad (a.e).
\]
That is a mean-square integral is monotonic. Throughout the entire paper, the monotonicity of mean-square integral and positivity of the stochastic process will be frequently used. Now, we define the following.

**Definition 2.1:** A stochastic process $X : l \times \Omega \to [0, \infty)$ is said to be $r$-convex, if for each $u, v \in l$ and $\lambda \in [0, 1]$

$$X(\lambda u + (1 - \lambda)v, \cdot) \leq \begin{cases} (\lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot))^{1/r}, & r \neq 0 \\ (X(u, \cdot)^{1/r^2}(X(v, \cdot))^{1 - 1/r}, & r = 0 \end{cases} \text{ (a.e.)},$$

(1)

Note that 0-convex stochastic processes are logarithmic convex (see [9]) and 1-convex stochastic processes are the classical convex functions. Note that if $X$ is $r$-convex, then $X^r$ is convex stochastic process ($r > 0$).

The above definition is analogue of the $r$-convex functions in the classical convex functions, see [10, 11].

Kotrys [3] studied the following well-known Hermite–Hadamard type inequality

$$X\left(\frac{u + v}{2}\right) \leq \frac{1}{v - u} \int_u^v X(t, \cdot)dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \text{ (a.e.)},$$

where $X : l \times \Omega \to \mathbb{R}$ is Jensen convex and mean-square continuous stochastic process. Hermite–Hadamard type inequalities for log-convex functions was investigated by Dragomir and Mond [12].

Pachpatte [13, 14] also gave some other refinements of these inequalities related with differentiable log-convex functions.

Tomar et al. [9] proved the following inequalities:

**Theorem 3.1:** Let $X : l \times \Omega \to [0, \infty)$ be a $r$-convex stochastic process with mean-square continuity in $l$. Then for $u, v \in l$ with $u < v$, the below inequality holds

$$\frac{1}{v - u} \int_u^v X(t, \cdot)dt \leq \left[\frac{X(u, \cdot) + X(v, \cdot)}{2}\right]^{1/r} \text{ (a.e.)},$$

(2)

**Proof:** From Jensen inequality, we obtain

$$\left(\frac{1}{v - u} \int_u^v X(t, \cdot)dt\right)^r \leq \frac{1}{v - u} \int_u^v X^r(t, \cdot)dt \text{ (a.e.)}.$$ 

Since $X^r$ is convex, then Hermite–Hadamard type inequality for convex stochastic processes yields us (see [3])

$$\frac{1}{v - u} \int_u^v X^r(t, \cdot)dt \leq \left[\frac{X(u, \cdot) + X(v, \cdot)}{2}\right] \text{ (a.e.)}.$$ 

Hence,

$$\frac{1}{v - u} \int_u^v X(t, \cdot)dt \leq \left[\frac{X(u, \cdot) + X(v, \cdot)}{2}\right]^{1/r} \text{ (a.e.)}.$$ 

This completes the proof.

**Corollary 3.1:** Let $X : l \times \Omega \to [0, \infty)$ be 1-convex stochastic process with mean-square continuity in $l$. Then for $u, v \in l$ with $u < v$, the following inequality holds.

$$\frac{1}{v - u} \int_u^v X(t, \cdot)dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}.$$ 

**Theorem 3.2:** Let $X : l \times \Omega \to (0, \infty)$ be $r$-convex stochastic process ($r \geq 0$) with mean-square continuity in $l$. Then for $u, v \in l$ with $u < v$, the following inequalities hold

$$\frac{1}{v - u} \int_u^v X(t, \cdot)dt \leq \begin{cases} \frac{r}{r + 1} \left[\frac{X^{r+1}(v, \cdot) - X^{r+1}(u, \cdot)}{X(v, \cdot) - X(u, \cdot)}\right], & r \neq 0 \\ \frac{X(v, \cdot) - X(u, \cdot)}{\log X(v, \cdot) - \log X(u, \cdot)}, & r = 0 \end{cases} \text{ (a.e.)}.$$ 

(3)

**Proof:** For $r = 0$, Tomar et al. [9] proved this result. We proceed for the case $r > 0$. Since $X$ is $r$-convex stochastic...
process, for all \( \lambda \in [0, 1] \), we have
\[
X(\lambda u + (1 - \lambda)v,)
\leq (\lambda X'(u,.) + (1 - \lambda)X'(v,))^{1/r}
\quad ('a.e.').
\]

Therefore by using the same method as of [3], we have
\[
\frac{1}{v - u} \int_u^v X(t,.)dt
\leq \int_0^1 \left( X'(v,.) + \lambda(X'(u,.) - X'(v,)) \right)^{1/r} d\lambda
\quad ('a.e.').
\]

Putting \( \tau = X'(v,.) + \lambda(X'(u,.) - X'(v,)) \), we have
\[
\frac{1}{v - u} \int_u^v X(t,.)dt \leq \frac{1}{X'(v,.) - X'(u,.)} \int_{X'(u,.)}^{X'(v,.)} (\tau)^{1/r} d\tau
\]
\[
= \frac{r}{r + 1} \left[ \frac{X'(v,.) - X'(u,.)}{X'(v,.) - X'(u,.)} \right],
\quad ('a.e.').
\]

which completes the proof. \( \square \)

Note that for \( r = 1 \), in the above theorem, we have the same inequality again as in Corollary 3.1.

\begin{theorem}
Let \( X : I \times \Omega \to (0, \infty) \) be \( r \)-convex \((0 \leq r \leq s)\) stochastic process with mean-square continuity in \( I \). Then \( X \) is \( s \)-convex stochastic process.
\end{theorem}

\begin{proof}
To prove this, we need the following inequality for non-negative real numbers \( x, y \)
\[
x^{1 - \lambda}y^{\lambda} \leq ((1 - \lambda)x + \lambda y)^{1/r}
\]
\[
\leq ((1 - \lambda)x + \lambda y)^{1/s},
\quad ('a.e.').
\]

(4)
\end{proof}

where \( 0 \leq \lambda \leq 1, \ 0 \leq r \leq s \). Since \( X \) is \( r \)-convex stochastic process, by inequality (4) for all \( u, v \in I, \lambda \in [0, 1] \), we obtain
\[
X(\lambda u + (1 - \lambda)v,)
\leq \begin{cases} 
(\lambda X'(u,.) + (1 - \lambda)X'(v,))^{1/r} \leq \lambda X'(u,.) \\
(1 - \lambda)X'(v,.)^{1/r}, \quad 0 < r \leq s \\
\lambda X'(u,.)^{1/s} \leq (\lambda X'(u,.) \\
(1 - \lambda)X'(v,.)^{1/s}, \quad 0 = r \leq s.
\end{cases}
\]

Hence, \( X \) is a \( s \)-convex stochastic process. \( \square \)

As a special case of the above theorem, we deduce the following results.

\begin{corollary}
Let \( X : I \times \Omega \to (0, \infty) \) be \( r \)-convex \((0 \leq r \leq 1)\) stochastic process with mean-square continuity in the interval \( I \). Then \( X \) is a convex stochastic process.
\end{corollary}

\begin{corollary}
Let \( X : I \times \Omega \to (0, \infty) \) be \( r \)-convex \((0 \leq r \leq s)\) stochastic process with mean-square continuity in the interval \( I \). Then the following inequalities holds:
\[
\frac{1}{v - u} \int_u^v X(t,.)dt
\leq \begin{cases} 
\frac{r}{r + 1} \left[ \frac{X'(v,.) - X'(u,.)}{X'(v,.) - X'(u,.)} \right], \quad 0 < r \leq s \\
\frac{s}{s + 1} \left[ \frac{X'(v,.) - X'(u,.)}{X'(v,.) - X'(u,.)} \right], \quad 0 = r \leq s.
\end{cases}
\quad ('a.e.').
\]

\begin{proof}
The proof follows at once by using Theorem 3.3 and proceeding on similar lines as Theorem 3.1. \( \square \)
\end{proof}

\begin{theorem}
Let \( X, Y : I \times \Omega \to (0, \infty) \) be \( r \)-convex and \( s \)-convex stochastic process with mean-square continuity respectively. Then for \( u, v \in I \) with \( u < v \), the following inequalities hold:
\[
\frac{1}{v - u} \int_u^v X(t,.)Y(t,.)dt
\leq \begin{cases} 
\frac{r}{r + 2} \left[ \frac{X'(v,.) - X'(u,.)}{X'(v,.) - X'(u,.)} \right] \\
\frac{s}{s + 2} \left[ \frac{Y'(v,.) - Y'(u,.)}{Y'(v,.) - Y'(u,.)} \right]
\end{cases}
\quad (X(u,.) \neq X(v,.), \ Y(u,.) \neq Y(v,.),)
\quad ('a.e.').
\]

\begin{proof}
Since \( X \) is \( r \)-convex stochastic process and \( Y \) is \( s \)-convex stochastic convex, for all \( \lambda \in [0, 1] \), we have
\[
X(\lambda u + (1 - \lambda)v,.) \leq (\lambda X'(u,.) \\
+ (1 - \lambda)X'(v,.)^{1/r}
\quad ('a.e.),
\]
\[
Y(\lambda u + (1 - \lambda)v,.) \leq (\lambda Y'(u,.) \\
+ (1 - \lambda)Y'(v,.)^{1/s}
\quad ('a.e.).
\]

Therefore,
\[
\frac{1}{v - u} \int_u^v X(t,.)Y(t,.)dt
\leq \begin{cases} 
\int_0^1 X(\lambda u + (1 - \lambda)v,.)Y(\lambda u + (1 - \lambda)v,.)d\lambda \\
\int_0^1 (\lambda X'(u,.) + (1 - \lambda)X'(v,.)^{1/r}(\lambda Y'(u,.) \\
+ (1 - \lambda)Y'(v,.)^{1/s}d\lambda.
\end{cases}
\quad ('a.e.).
\]
\end{proof}

\]
Now applying Cauchy’s inequality, we obtain
\[
\int_0^1 \left( X'(v, \lambda) + \lambda (X'(u, \lambda) - X'(v, \lambda)) \right)^{1/\tau} \times (Y^2(v, \lambda) + \lambda (Y^2(u, \lambda) - Y^2(v, \lambda)))^{1/\tau} \, d\lambda \\
\leq \frac{1}{2} \int_0^1 \left[ X'(v, \lambda) + \lambda (X'(u, \lambda) - X'(v, \lambda)) \right]^{2/\tau} \, d\lambda \\
+ \frac{1}{2} \int_0^1 \left[ Y^2(v, \lambda) + \lambda (Y^2(u, \lambda) - Y^2(v, \lambda)) \right]^{2/\tau} \, d\lambda
\] (a.e.)

If we choose \( \tau = X'(v, \lambda) + \lambda (X'(u, \lambda) - X'(v, \lambda)), \) \( \eta = Y^2(v, \lambda) + \lambda (Y^2(u, \lambda) - Y^2(v, \lambda)), \) then we obtain the following inequality
\[
\int_0^1 \left[ X'(v, \lambda) + \lambda (X'(u, \lambda) - X'(v, \lambda)) \right]^{1/\tau} \times (Y^2(v, \lambda) + \lambda (Y^2(u, \lambda) - Y^2(v, \lambda)))^{1/\tau} \, d\lambda \\
\leq \frac{1}{2} \int_0^1 \left[ X'(v, \lambda) - X'(u, \lambda) \right]^{2/\tau} \, d\tau \\
+ \frac{1}{2} \int_0^1 \left[ Y^2(v, \lambda) - Y^2(u, \lambda) \right]^{2/\tau} \, d\eta
\] (a.e.),

which leads us to the required result.

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No potential conflict of interest was reported by the authors.

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