Aging at the edge of chaos: Glassy dynamics and nonextensive statistics

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Abstract

We go over our finding that the dynamics at the noise-perturbed edge of chaos in logistic maps is comparable to that observed in supercooled liquids close to vitrification. That is, the three major features of glassy dynamics in structural glass formers, two-step relaxation, aging, and a relationship between relaxation time and configurational entropy, are displayed by orbits with vanishing Lyapunov exponent. The known properties in control-parameter space of the noise-induced bifurcation gap play a central role in determining the characteristics of dynamical relaxation at the chaos threshold. Time evolution is obtained from the Feigenbaum RG transformation, it is expressed analytically via $q$-exponentials, and described in terms of nonextensive statistics.

Key words: Glassy dynamics, ergodicity breakdown, edge of chaos, external noise, nonextensive statistics

PACS: 64.70.Pf, 64.60.Ak, 05.10.Cc, 05.45.Ac, 05.40.Ca

1 Introduction

The problem of understanding the dynamics of glass formation is known to be difficult and challenging and therefore continues to be a frontier topic in
statistical physics [1]. It is generally thought that ordinary phase-space mixing is not wholly realized during glass forming dynamics, i.e. upon cooling, caged molecules rearrange so slowly that they cannot sample configurations in the available time allowed by the process [1]. The advance to the glass transition is signaled by very pronounced deceleration of relaxation processes since the characteristic times for their completion increase several orders of magnitude within a small interval of temperatures. Because of this extreme circumstances an important theoretical issue is to determine whether under conditions of ergodicity malfunction, and, as a final point, downright failure, the Boltzmann-Gibbs (BG) statistical mechanics is still capable of describing stationary states on the point of glass formation or those representing the glass itself. This question and also that about the possible applicability of generalizations of the BG statistics, such as the nonextensive statistics [2] [3], to glass formation is a subject of current concern.

As a fresh approach we study the dynamics of a model in which only the essential ergodicity breakdown feature is present and where there is neither molecular structure nor molecular interactions. The general aim is to learn if the occurrence of the key properties of glassy dynamics are due primarily to the approach to a nonergodic state and therefore are observable in completely different types of systems, hence suggesting some degree of universality. Our minimal model for glass dynamics is the prototypical logistic map at the edge of chaos but perturbed by small amplitude noise. It has only one degree of freedom but the consideration of external noise could be thought to represent the effect of many other systems coupled to it, like in the so-called coupled map lattices [4]. Recently [5], we have proved that the dynamics of this map shows robust similarities with that of supercooled liquids close to vitrification. This is so since two-step relaxation, aging, and a relationship between relaxation time and configurational entropy, are the three main (phenomenologically established) characteristics of glassy dynamics in structural glass formers [1]. The basic ingredient of ergodicity failure is obtained for orbits at the onset of chaos in the limit towards vanishing noise amplitude. Here expand the discussion of our results [5].

Our view is that glass formation is one in which the system is driven gradually into a nonergodic state by reducing its ability to pass through phase-space-filling configurational regions until it is only possible to go across a (multi)fractal subset of phase space. This situation is generated in the
logistic map with additive external noise,

\[ x_{t+1} = f_\mu(x_t) = 1 - \mu x_t^2 + \chi_t \sigma, \quad -1 \leq x_t \leq 1, \quad 0 \leq \mu \leq 2, \]

where \( \chi_t \) is Gaussian-distributed with average \( \langle \chi_t \chi_{t'} \rangle = \delta_{t,t'} \), and \( \sigma \) measures the noise intensity \([12] [13]\). Notice that the formula (1) can be written as a discrete form for a Langevin equation.

2 Dynamics of glass formation

We recall the main dynamical properties displayed by supercooled liquids on approach to glass formation. One is the growth of a plateau and for that reason a two-step process of relaxation, as presented by the time evolution of correlations e.g. the intermediate scattering function \( F_k(\Delta t) [1]\). This consists of a primary power-law decay in time difference \( \Delta t \) (so-called \( \beta \) relaxation) that leads into the plateau, the duration \( t_x \) of which diverges also as a power law of the difference \( T - T_g \) as the temperature \( T \) decreases to a glass temperature \( T_g \). An ‘ideal’ glass transition has been conjectured to occur at a \( T_c \) which is not experimentally accessible but that approaches \( T_g \) at infinitely slow cooling in fragile glass formers \([14]\). After \( t_x \) there is a secondary power law decay (so-called \( \alpha \) relaxation) away from the plateau \([1]\). A second important (nonequilibrium) dynamic property of glasses is the loss of time translation invariance observed for \( T \) below \( T_g \), a characteristic known as aging \([6]\), that is due to the fact that properties of glasses depend on the procedure by which they are obtained. Remarkably, the time fall off of relaxation functions and correlations display a scaling dependence on the ratio \( t/t_w \) where \( t_w \) is a waiting time. A third notable property is that the experimentally observed relaxation behavior of supercooled liquids is effectively described, via reasonable heat capacity assumptions \([1]\), by the so-called Adam-Gibbs equation, 

\[ t_x = A \exp(B/TS_c), \]

where the relaxation time \( t_x \) can be identified with the viscosity or the inverse of the diffusivity, and the configurational entropy \( S_c \) is related to the number of minima of the fluid’s potential energy surface (and \( A \) and \( B \) are constants) \([1]\). A first-principles derivation of this equation has not been developed at present. As the counterpart to the Adam-Gibbs formula, we show below that our one-dimensional map model for glassy dynamics exhibits a relationship between the plateau duration \( t_x \), and the entropy \( S_c \) associated to the iterate positions
(configurations) within the largest number of phase-space bands allowed by the bifurcation gap - the noise-induced cutoff in the period-doubling cascade [12].

Our results suggest that the properties known to be basic of glassy dynamics [1] are likely to manifest too in completely different physical problems, such as in nonlinear dynamics, e.g. the mentioned coupled map lattices [7], in critical dynamics [8], and other fields. This hints that new predictions might be encountered in the studies, experimental or otherwise, of slow dynamics displayed by systems other than liquids close to the glass transition. In relation to this, aspects of glassy dynamics have been observed in metastable quasistationary states in microcanonical Hamiltonian systems of \(N\) classical rotors with homogeneous long-ranged interactions. For special types of initial conditions it has been found that both two-step relaxation [9], where the length of the metastable plateau diverges with infinite size \(N \to \infty\), and aging [10], [11] are present in these systems. In our simpler one-dimensional dissipative map the amplitude \(\sigma\) plays a role parallel to \(T - T_g\) or \(T - T_c\) in the supercooled liquid or \(1/N\) in the system of rotors. Notice that the equivalence between \(\sigma\) and \(T - T_g\) is not literal as \(\sigma\) cannot take negative values.

3 Dynamics at the edge of chaos

In the absence of noise \(\sigma = 0\) the Feigenbaum attractor at \(\mu = \mu_c(0) = 1.40115...\) is the accumulation point of both the period doubling and the chaotic band splitting sequences of transitions [12]. Except for a set of zero measure, all the trajectories with \(\mu_c(0)\) and initial condition \(-1 \leq x_{in} \leq 1\) fall into the attractor with fractal dimension \(d_f = 0.5338...\). These trajectories represent nonergodic states, since as \(t \to \infty\) only a Cantor set of positions is accessible out of the total phase space \(-1 \leq x \leq 1\). For \(\sigma > 0\) the noise fluctuations wipe the sharp features of the periodic attractors as these widen into bands similar to those in the chaotic attractors, nevertheless there remains a well-defined transition to chaos at \(\mu_c(\sigma)\) where the Lyapunov exponent \(\lambda_1\) changes sign. The period doubling of bands ends at a finite value \(2^{N(\sigma)}\) as the edge of chaos transition is approached and then decreases at the other side of the transition. This effect displays scaling features and is referred to as the bifurcation gap [12] [13]. When \(\sigma > 0\) the trajectories visit sequentially a set
of $2^n$ disjoint bands or segments leading to a cycle, but the behavior inside each band is completely chaotic. These trajectories represent ergodic states as the accessible positions have a fractal dimension equal to the dimension of phase space. Thus the removal of the noise $\sigma \to 0$ leads to an ergodic to nonergodic transition in the map and we compare its properties with those known for the process of vitrification of a liquid as $T \to T_g$.

The manner in which the attractor at the onset of chaos $\mu_c(0)$ is visited in time was analyzed recently with the use of the initial condition $x_{in} = 0$ [15]. It was found that the absolute values for the positions $x_\tau$ of this trajectory at time-shifted $\tau = t + 1$ has an structure consisting of subsequences with a common power-law decay of the form $\tau^{-1/1-q}$ with $q = 1 - \ln 2 / \ln \alpha \simeq 0.24449$, where $\alpha = 2.50290...$ is the Feigenbaum universal constant that measures the period-doubling amplification of iterate positions [15]. That is, the Feigenbaum attractor can be decomposed into position subsequences generated by the time subsequences $\tau = (2k + 1)2^n$, each obtained by proceeding through $n = 0, 1, 2, ...$ for a fixed value of $k = 0, 1, 2, ...$. The $k = 0$ subsequence can be written as $x_t = \exp_{2-q}(-\lambda_q t)$ with $\lambda_q = \ln \alpha / \ln 2$, and where $\exp_q(x) \equiv [1 - (q-1)x]^{1/1-q}$ is the $q$-exponential function. These properties follow from the use of $x_{in} = 0$ in the scaling relation [15]

$$x_\tau = \left| g^{(r)}(x_{in}) \right| = \tau^{-1/1-q} \left| g(\tau^{1/1-q} x_{in}) \right|. \quad \text{(2)}$$

In the presence of noise ($\sigma$ small) one obtains instead [5]

$$x_\tau = \tau^{-1/1-q} \left| g(\tau^{1/1-q} x_{in}) + \chi \sigma \tau^{1/1-r} G_\Lambda(\tau^{1/1-q} x_{in}) \right|, \quad \text{(3)}$$

where $G_\Lambda(x)$ is the first order perturbation eigenfunction, and where $r = 1 - \ln 2 / \ln \kappa \simeq 0.6332$. Use of $x_{in} = 0$ yields $x_\tau = \tau^{-1/1-q} \left| 1 + \chi \sigma \tau^{1/1-r} \right|$ or $x_t = \exp_{2-q}(-\lambda_q t) \left[ 1 + \chi \sigma \exp_{r} (\lambda_r t) \right]$ where $t = \tau - 1$ and $\lambda_r = \ln \kappa / \ln 2$.

At each noise level $\sigma$ there is a 'crossover' or 'relaxation' time $t_x = \tau_x - 1$ when the fluctuations start suppressing the fine structure of the orbits with $x_{in} = 0$. This time is given by $\tau_x = \sigma^{-1}$, the time when the fluctuation term in the perturbation expression for $x_\tau$ becomes unbounded by $\sigma$, i.e. $x_{\tau_x} = \tau_x^{-1/1-q} \left| 1 + \chi \right|$. There are two regimes for time evolution at $\mu_c(\sigma)$. When $\tau < \tau_x$ the fluctuations are smaller than the distances between neighboring subsequence positions of the $\sigma = 0$ orbit at $\mu_c(0)$, and the iterate positions with $\sigma > 0$ fall within small non overlapping bands each around
the $\sigma = 0$ position for that $\tau$. Time evolution follows a subsequence pattern analogous to that in the noiseless case. When $\tau \sim \tau_x$ the width of the noise-generated band reached at time $\tau_x = 2^N$ matches the distance between adjacent positions where $N \sim -\ln \sigma / \ln \kappa$, and this implies a cutoff in the progress along the position subsequences. At longer times $\tau > \tau_x$ the orbits no longer follow the detailed period-doubling structure of the attractor. The iterates now trail through increasingly chaotic trajectories as bands merge with time. This is the dynamical image - observed along the time evolution for the orbits of a single state $\mu_c(\sigma)$ - of the static bifurcation gap originally described in terms of the variation of the control parameter $\mu$ [13], [16], [17]. The plateau structure of relaxation and the crossover time $t_x$ can be clearly observed in Fig. 1b in Ref. [18] where $< x_t^2 > - < x_t >^2$ is shown for several values of $\sigma$.

4 Parallels with glassy dynamics

At noise level $\sigma$ the orbits visit points within the set of $2^N$ bands and, as explained in Ref. [5], this takes place in time in the same way that period doubling and band merging proceeds in the presence of a bifurcation gap when the control parameter is run through the interval $0 \leq \mu \leq 2$. Namely, the trajectories starting at $x_{in} = 0$ duplicate the number of visited bands at times $\tau = 2^n$, $n = 1, \ldots, N$, the bifurcation gap is reached at $\tau_x = 2^N$, after which the orbits fall within bands that merge by pairs at times $\tau = 2^{N+n}$, $n = 1, \ldots, N$. The sensitivity to initial conditions grows as $\xi_t = \exp_q(\lambda_t t)$ ($q = 1 - \ln 2/\ln \alpha < 1$) for $t < t_x$, but for $t > t_x$ the fluctuations dominate and $\xi_t$ grows exponentially as the trajectory has become chaotic ($q = 1$) [5]. This behavior was interpreted [5] to be the dynamical system analog of the $\alpha$ relaxation in supercooled fluids. The plateau duration $t_x \to \infty$ as $\sigma \to 0$. Additionally, trajectories with initial conditions $x_{in}$ not belonging to the attractor exhibit an initial relaxation process towards the plateau as the orbit approaches the attractor. This is the map analog of the $\beta$ relaxation in supercooled liquids.

The entropy $S_c(\mu_c(\sigma))$ associated to the distribution of iterate positions (configurations) within the set of $2^N$ bands was determined in Ref. [5]. This entropy has the form $S_c(\mu_c(\sigma)) = 2^N \sigma s$, since each of the $2^N$ bands contributes with an entropy $\sigma s$, where $s = -\int_1^{1/p(\chi)} p(\chi) \ln p(\chi)d\chi$ and where
\( p(\chi) \) is the distribution for the noise random variable. Given that \( 2^N = 1 + t_x \) and \( \sigma = (1 + t_x)^{-1/1-r} \), one has \( S_c(\mu_c, t_x)/s = (1 + t_x)^{-r/1-r} \) or, conversely, \( t_x = (s/S_c)^{(1-r)/r} \). (4)

Since \( t_x \simeq \sigma r^{-1} \), \( r - 1 \simeq -0.3668 \) and \( (1 - r)/r \simeq 0.5792 \) then \( t_x \to \infty \) and \( S_c \to 0 \) as \( \sigma \to 0 \), i.e. the relaxation time diverges as the 'landscape' entropy vanishes. We interpret this relationship between \( t_x \) and the entropy \( S_c \) to be the dynamical system analog of the Adam-Gibbs formula for a supercooled liquid. Notice that Eq.(4) is a power law in \( S_c^{-1} \) while for structural glasses it is an exponential in \( S_c^{-1} \) [1]. This difference is significant as it indicates how the superposition of molecular structure and dynamics upon the bare ergodicity breakdown phenomenon built in the map modifies the vitrification properties.

The aging scaling property of the trajectories \( x_t \) at \( \mu_c(\sigma) \) was examined in Ref. [5]. The case \( \sigma = 0 \) is readily understood because this property is actually built into the position subsequences \( x_\tau = |g^{(r)}(0)|, \tau = (2k + 1)2^n, k, n = 0, 1, ... \) referred to above. These subsequences can be employed for the description of trajectories that are first held at a given attractor position for a waiting period of time \( t_w \) and then released to the normal iterative procedure. For illustrative purposes we select the holding positions to be any of those for a waiting time \( t_w = 2k + 1, k = 0, 1, ... \) and notice that for the \( x_{in} = 0 \) orbit these positions are visited at odd iteration times. The lower-bound positions for these trajectories are given by those of the subsequences at times \( (2k + 1)2^n \). See Fig. 1 in Ref. [5]. Writing \( \tau \) as \( \tau = t_w + t \) we have that \( t/t_w = 2^n - 1 \) and \( x_{t+t_w} = g^{(t_w)}(0)g^{(t/t_w)}(0) \) or

\[ x_{t+t_w} = g^{(t_w)}(0) \exp\left(-\lambda_1 t/t_w\right). \] (5)

This fully developed aging property is gradually modified when noise is turned on. The presence of a bifurcation gap limits its range of validity to total times \( t_w + t < t_x(\sigma) \) and so progressively disappears as \( \sigma \) is increased. Recently [19] aging has been observed in the subdiffusion associated to a nonlinear map that, as it is the case here, has \( \lambda_1 = 0 \) [20].

## 5 Discussion

Thus, the dynamics of noise-perturbed logistic maps at the chaos threshold exhibits the most prominent features of glassy dynamics in supercooled li-
uids. The existence of this analogy cannot be considered accidental since
the limit of vanishing noise amplitude $\sigma \to 0$ (the counterpart of the limit
$T - T_g \to 0$ in the supercooled liquid) entails loss of ergodicity. The incidence
of these properties in such simple dynamical systems, with only a few degrees
of freedom and no reference to molecular interactions, suggests a universal
mechanism underlying the dynamics of glass formation. As definitely proved
[15], the dynamics of deterministic unimodal maps at the edge of chaos is
a genuine example of the pertinence of nonextensive statistics in describing
states with vanishing ordinary Lyapunov exponent $\lambda_1$. Here we have shown
that this nonergodic state corresponds to the limiting state, $\sigma \to 0$, $t_x \to \infty$,
for a family of small $\sigma$ noisy states with glassy properties, that are noticeably
described for $t < t_x$ via the $q$-exponentials of the nonextensive formalism [15].

To ascertain the degree of parallelism between the map at $\mu_c(\sigma)$ and a
thermal system we keep in mind the following criteria: The exponential $\xi_t$
(or $\lambda_1 > 0$) of a chaotic state is the counterpart of the BG equilibrium state.
On the other hand, a power-law $\xi_t$ (or $\lambda_1 = 0$) of an incipient chaotic state
represents an ”anomalous stationary state”, $q \neq 1$, and in the case of $\mu_c(0)$
a nonergodic state. Only when $\sigma = 0$ we have a true anomalous stationary
state (of infinite duration) that exhibits a precise aging property. In the map
at $\mu_c(\sigma)$ we find two regimes separated by $t_x$ with presence of aging when
$t < t_x$. In glass formers the two-step relaxation is seen in equilibrium two-
time correlations while aging is a manifestation of the system having fallen
out of equilibrium. Because of these apparent differences a more detailed
comparison between the map and the thermal system properties is required,
and it may be necessary to consider neighbouring states around $\mu_c(\sigma)$. These
states would be expected to display also slow dynamics. When $\mu > \mu_c(\sigma)$
one has $\lambda_1 > 0$ and the analog of BG equilibrium, whereas for $\mu < \mu_c(\sigma)$ one
has $\lambda_1 < 0$ and a situation ”out of equilibrium”.

The anomalous stationary state has properties akin to both equilibrium
and out of equilibrium. Equilibrium-like since it is stationary, an orbit that
starts in the attractor remains there forever. Nonequilibrium-like because
the finite-time Lyapunov exponents fluctuate and take negative values (im-
plying confinement in phase space). When $\sigma > 0$ it would be expected that
the frequency of these negative values decreases sharply after $t_x$. Further
studies are needed here. Also, there remains the question about whether
some features of glassy dynamics, such as loss of time translation invariance,
are due to nonergodicity or to merely being out of equilibrium (i.e. in a
non-stationary state).

It has been suggested on several occasions \cite{3} that the setting in which nonextensive statistics appears to emerge is linked to the incidence of nonuniform convergence, such as that involving the thermodynamic $N \to \infty$ and very large time $t \to \infty$ limits. For example, in the rotor problem mentioned above - for specific choices of initial conditions - if $N \to \infty$ is taken before $t \to \infty$ the anomalous metastable states with noncanonical properties appear to be the only observable stationary states, whereas if $t \to \infty$ is taken before $N \to \infty$ the usual BG equilibrium states are obtained. Here it is clear that a similar situation takes place, that is, if $\sigma \to 0$ is taken before $t \to \infty$ a nonergodic orbit confined to the Feigenbaum attractor is obtained, whereas if $t \to \infty$ is taken before $\sigma \to 0$ a typical $q = 1$ chaotic (ergodic) orbit is observed.

Finally, it is worth mentioning that while the properties displayed by the map capture in a qualitative, heuristic, way the phenomenological issues of vitrification, they are obtained in a quantitative and rigorous manner as the map is concerned. Our map setup is a rarely available 'laboratory' where every aspect of the dynamics can be studied analytically.

Acknowledgments. I am grateful to Piero Tartaglia for informative discussions and also for his kind hospitality at Dipartamento di Fisica, Università degli Studi di Roma "La Sapienza". I thank an anonymous referee for important criticism. Work partially supported by CONACyT grant P-40530-F.

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