On Error Bounds for a Pseudo-Formal Linearization of Chebyshev Interpolation Type

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Abstract This paper is concerned with the error bounds of a pseudo-formal linearization method based on Chebyshev interpolation for nonlinear dynamic systems. First, an error bound formula is derived for a functional approximation of this linearization, in which a min-max theory is well applied to it. Then, by using this formula, another error bound formula for nonlinear dynamic systems is nicely obtained. They are more specific formulas than the general error bounds of polynomials type. Numerical experiments demonstrate the usefulness of these formulas.

Keywords: nonlinear system, pseudo-formal linearization, Chebyshev interpolation, error bound, linearization function

1. Introduction

Linearization method for nonlinear dynamic systems employed to apply linear system theories \cite{1}-\cite{5} have been well studied for several decades. A formal linearization method \cite{6}-\cite{8} is considered as one of them. And then it has been expanded into a pseudo-formal linearization method \cite{9}-\cite{11}. For this pseudo-formal linearization of polynomials type, the general error bounds have been already studied \cite{9}. One of the best error bounds is acquired in the case of Chebyshev polynomial type \cite{10}. And then it has been shown that this approach could reduce the computational burden of the linearization when used Chebyshev interpolation technique \cite{11}.

In this paper, we consider the more specific error bounds than them for the pseudo-formal linearization of Chebyshev interpolation type \cite{11} as follows. At first, an error bound formula is derived for a functional approximation of this pseudo-formal linearization by making good use of a min-max theory \cite{4}. Next, by based on this result, another error bound formula for nonlinear dynamic systems is obtained successfully. Obtained error bounds show that the presented method may improve the accuracy of linearization when parameters are appropriately chosen. Numerical experiments show the effectiveness of these error bound formulas.

2. Pseudo-Formal Linearization

A nonlinear dynamic system is assumed to have the form

$$\Sigma_1: \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in D$$ (1)

where \(t\) denotes time, \(\dot{\cdot} = \frac{d}{dt}\), \(x\) is an \(n\)-dimensional state vector, and \(f \in \mathbb{R}^n\) is a sufficiently smooth nonlinear vector-valued function. \(D\) is a domain denoted by the Cartesian product

$$D = \prod_{i=1}^{n}[m_i - p_i, m_i + p_i] \quad (m_i \in \mathbb{R}, p_i > 0)$$

To apply a pseudo-formal linearization method, a vector-valued separable function is introduced as

$$C : D \rightarrow R^L$$ (2)

which is continuously differentiable, and here we let \(C(x) = [I : 0]x\) (\(I: L \times L\) unit matrix) for simplicity. Considering the nonlinearity of the given nonlinear system, this \(C(x)\) can be determined. We let \(D\) be a domain of \(C^{-1}\) and divide into \((M+1)\) subdomains:

$$D = \bigcup_{k=0}^{M} D_k$$ (3)

where \(D_M = D - \bigcup_{k=0}^{M-1} D_k\) and \(C^{-1}(D_0) \ni 0\) (see Fig. 1). \(D_k (0 \leq k \leq M - 1)\) endowed with a lexicographic
order is the Cartesian product

\[ D_k = \prod_{j=1}^{L} (a_{kj}, b_{kj}) \quad (a_{kj} < b_{kj}) \]

![Figure 1 Pseudo-formal linearization](image)

A pseudo-formal linearization uses an automatic choosing function of the signomial type as follows:

\[ I_k(\zeta) = \prod_{j=1}^{L} \left\{ 1 - \frac{1}{1 + \exp(2\mu(\zeta_j - a_{kj}))} \right\} \quad (0 \leq k \leq M - 1) \tag{4} \]

\[ I_M(\zeta) = 1 - \sum_{k=0}^{M-1} I_k(\zeta) \]

so that

\[ \sum_{k=0}^{M} I_k(\zeta) = 1 \tag{5} \]

where

\[ \zeta = [\zeta_1, \ldots, \zeta_L]^T = C(x) \]

and \( \mu \) is a positive real value.

To make use of Chebyshev interpolation, the state vector \( x \) is changed into \( y \), so that \( y \) has the basic domain of the Chebyshev polynomials \( D_0 = \prod_{i=1}^{n} [-1, 1] \) and \( y \) is rewritten as

\[ y = \mathcal{P}^{(-1)}(x - \mathcal{M}(k)) \in D_0 \tag{6} \]

where

\[ y = \begin{bmatrix} y_1 \\ \vdots \\ y_L \\ y_{L+1} \\ \vdots \\ y_n \end{bmatrix}, \quad \mathcal{M}(k) = \begin{bmatrix} m_{1}^{(k)} \\ \vdots \\ m_{L}^{(k)} \\ m_{L+1}^{(k)} \\ \vdots \\ m_{n}^{(k)} \end{bmatrix} \]

\[ \mathcal{P}^{(k)} = \begin{bmatrix} p_1^{(k)} \\ \vdots \\ p_L^{(k)} \\ p_{L+1} \\ \vdots \\ p_n \end{bmatrix} \]

\[ m_i^{(k)} = \frac{1}{2} (a_{ki} + b_{ki}), \quad p_i^{(k)} = \frac{1}{2} (b_{ki} - a_{ki}) \]

The given dynamic system (Eq. (1)) becomes

\[ \dot{y}(t) = \mathcal{P}^{(k)} \cdot \mathcal{F}(\mathcal{P}^{(k)} y(t) + \mathcal{M}(k)) \tag{7} \]

Here we define an \( N \)th order formal linearization function that consists of polynomials defined by

\[ \phi(x) \triangleq [x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, x_2^2, \ldots, x_1^2 x_2^2, \ldots, x_1^n x_2^n, \ldots, \frac{x_1^N x_2^N \cdots x_n^N}{N! \cdots N!}]^T \]

\[ = [\phi(10^{-s})(x), \ldots, \phi(1 \cdots n)(x), \ldots, \phi(N \cdots N)(x)]^T \tag{8} \]

From Eq. (1), the derivative of each element of \( \phi \) becomes

\[ \dot{\phi}(r_{1 \cdots n})(x) = \frac{\partial}{\partial x} \phi(r_{1 \cdots n})(x) \cdot \dot{x} \]

\[ = \frac{\partial}{\partial x} T \phi(r_{1 \cdots n})(x) \cdot \mathcal{F}(x) = \frac{\partial}{\partial y} \mathcal{P}^{-1} \mathcal{P}^{(k)} \cdot \mathcal{F}(\mathcal{P}^{(k)} y + \mathcal{M}(k)) \]

\[ \triangleq G_{(r_{1 \cdots n})}^{(k)}(y) \tag{9} \]

To this \( G_{(r_{1 \cdots n})}^{(k)}(y) \), we apply Chebyshev interpolation \([4],[11]\) as follows. The Chebyshev polynomials \( \{T_q()\} \) are defined as

\[ T_q(y_i) = \cos(q \cdot \cos^{-1} y_i) \quad (q = 0, 1, 2, \cdots) \tag{10} \]

or

\[ T_0(y_i) = 1, \quad T_1(y_i) = y_i, \quad T_2(y_i) = 2y_i^2 - 1, \]

\[ T_3(y_i) = 4y_i^3 - 3y_i, \quad T_4(y_i) = 8y_i^4 - 8y_i^2 + 1, \cdots \]

and their polynomial representation in terms of powers of \( y_i \) is given by

\[ T_q(y_i) = \sum_{s=0}^{[q/2]} d_s^{(q)} y_i^{q-2s} \tag{11} \]

where

\[ d_s^{(q)} = \begin{cases} (-1)^s 2^{q-2s-1} \prod_{j=0}^{s-1} (q - s - j) & (2s < q) \\ (-1)^s & (q = 2s) \end{cases} \]
and \(| \cdot |\) is the floor function. Applying Chebyshev interpolation up to the Nth order on each subdomain \(D_k\), \(G^{(k)}(r_1 \cdots r_n)(y)\) is expanded as

\[
G^{(k)}(r_1 \cdots r_n)(y) = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C^{(k)}(q_1 \cdots q_n) T(q_1 \cdots q_n)(y)
+ R^{(k)}(N+1)(r_1 \cdots r_n)(x(y))
\]

where

\[
T(q_1 \cdots q_n)(y) = T_{q_1}(y_1) T_{q_2}(y_2) \cdots T_{q_n}(y_n),
\]

\[
C^{(k)}(q_1 \cdots q_n) = \frac{2^{n-\gamma}}{n} \prod_{i=1}^{n} (N+1) \sum_{j_1=0}^{N} \sum_{j_2=0}^{N} \cdots \sum_{j_n=0}^{N} \sum_{j_{k-1}=0}^{N} C^{(k)}(r_1 \cdots r_n) \times T_{q_1}(y_{j_1}) T_{q_2}(y_{j_2}) \cdots T_{q_n}(y_{j_n})
\]

\[
\gamma = \{\text{the number of } q_i = 0 : 1 \leq i \leq n\}
\]

\[
R^{(k)}_{N+1}(r_1 \cdots r_n)(x(y)) \text{ is an expansion error. The interpolating points } \{y_{ji}\} \text{ are set to be}
\]

\[
y_{ji} = \cos \frac{2j_i + 1}{2N + 2} \pi \quad (i = 1, \cdots, n, \ j_i = 0, \cdots, N)
\]

This \(G^{(k)}_{r_1 \cdots r_n}(y)\) in Eq. (12) is expressed by the use of the polynomial representation of the Chebyshev polynomials (Eq. (11)) and the binomial theorem in terms of \(x_k (i = 1, \cdots, n)\) as

\[
G^{(k)}_{r_1 \cdots r_n}(y) = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C^{(k)}(q_1 \cdots q_n)
\]

\[
\left\{ \sum_{s_1=0}^{\lfloor q_1/2 \rfloor} d^{(q_1)}_{s_1} y_{s_1}^{q_1-2s_1} \cdots \sum_{s_n=0}^{\lfloor q_n/2 \rfloor} d^{(q_n)}_{s_n} y_{s_n}^{q_n-2s_n} \right\}
\]

\[
= \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C^{(k)}(q_1 \cdots q_n) \left\{ \sum_{s_1=0}^{\lfloor q_1/2 \rfloor} \cdots \sum_{s_n=0}^{\lfloor q_n/2 \rfloor} d^{(q_s)}_{s_1} \frac{x_1 - M^{(k)}_{s_1}}{P^{(k)}_1} y_{s_1}^{q_1-2s_1} \cdots d^{(q_n)}_{s_n} \frac{x_1 - M^{(k)}_{s_n}}{P^{(k)}_n} y_{s_n}^{q_n-2s_n} \right\}
\]

\[
= \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} \sum_{s_1=0}^{\lfloor q_1/2 \rfloor} \cdots \sum_{s_n=0}^{\lfloor q_n/2 \rfloor} C^{(k)}(r_1 \cdots r_n) \frac{d^{(q_1)}_{s_1} \cdots d^{(q_n)}_{s_n}}{P^{(k)}_1 y_{s_1}^{q_1-2s_1} \cdots P^{(k)}_n y_{s_n}^{q_n-2s_n}}
\]

\[
\frac{q_1-2s_1}{j_1} \sum_{j_1=0}^{N} \frac{q_1-2s_1}{j_1} (-M^{(k)}_{s_1})^{q_1-2s_1-j_1} \cdots \frac{q_n-2s_n}{j_n} \sum_{j_n=0}^{N} \frac{q_n-2s_n}{j_n} (-M^{(k)}_{s_n})^{q_n-2s_n-j_n}
\]

\[
\times x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}
\]

(15)

where \(\binom{i}{j} = \frac{i!}{j!(i-j)!}\) is the binomial coefficient. Substituting this \(G^{(k)}_{r_1 \cdots r_n}(y)\) into Eq. (9) yields

\[
\phi(r_1 \cdots r_n)(x) = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} A^{(k)}(r_1 \cdots r_n) \phi(q_1 \cdots q_n)(x)
+ R^{(k)}_{N+1}(r_1 \cdots r_n)(x)
\]

where

\[
A^{(k)}(r_1 \cdots r_n) = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} \sum_{s_1=0}^{\lfloor q_1/2 \rfloor} \cdots \sum_{s_n=0}^{\lfloor q_n/2 \rfloor} C^{(k)}(r_1 \cdots r_n)
\]

\[
\frac{d^{(q_1)}_{s_1} \cdots d^{(q_n)}_{s_n}}{P^{(k)}_1 y_{s_1}^{q_1-2s_1} \cdots P^{(k)}_n y_{s_n}^{q_n-2s_n}} \left( \frac{q_1-2s_1}{j_1} \left( -M^{(k)}_{s_1} \right)^{q_1-2s_1-j_1} \cdots \frac{q_n-2s_n}{j_n} \left( -M^{(k)}_{s_n} \right)^{q_n-2s_n-j_n} \right)
\]

Thus, \(\phi(r_1 \cdots r_n)(x)\) on a subdomain \(D_k\) is presented by the formal linearization function as

\[
\phi(r_1 \cdots r_n)(x) = \left[ A^{(k)}(10 \cdots 0), A^{(k)}(01 \cdots 0), \cdots, A^{(k)}(j_1 \cdots j_n), \cdots, A^{(k)}(N-N) \right] \phi(x) + A^{(k)}(00 \cdots 0) + R^{(k)}_{N+1}(r_1 \cdots r_n)(x)
\]

(16)

Therefore, a linear dynamic equation with respect to \(\phi\) is obtained as

\[
\dot{\phi}(x) = A^{(k)} \phi(x) + b^{(k)} + R^{(k)}_{N+1}(x)
\]

(17)

where

\[
A^{(k)} = \left[ A^{(k)}(r_1 \cdots r_n) \right], \quad b^{(k)} = \left[ A^{(k)}(00 \cdots 0) \right]
\]

\[
R^{(k)}_{N+1}(x) = \left[ R^{(k)}_{N+1}(r_1 \cdots r_n)(x) \right]
\]

We unite \((M+1)\) linearized systems (Eq. (17)) on subdomains into a single linear system on the whole domain by using Eq. (5) as

\[
\dot{\phi}(x) = \sum_{k=0}^{M} \phi(x) I_k(\zeta)
\]

\[
= \sum_{k=0}^{M} (A^{(k)} \phi(x) + b^{(k)} + R^{(k)}_{N+1}(x)) I_k(\zeta)
\]
where
\[
\bar{A}(\zeta) = \sum_{k=0}^{M} A^{(k)} I_k(\zeta), \quad \bar{b}(\zeta) = \sum_{k=0}^{M} b^{(k)} I_k(\zeta)
\]
\[
\bar{R}_{N+1}(x, \zeta) = \sum_{k=0}^{M} R^{(k)}(x) I_k(\zeta)
\]

Finally, a pseudo-formal linearization system is defined as
\[
\Sigma_2: \dot{z}(t) = \bar{A}(\zeta) z(t) + \bar{b}(\zeta), \quad z(0) = \phi(x(0))
\]
(19)

From Eq. (8), its inversion is carried out using
\[
\dot{x}(t) = [I, 0, \ldots, 0] z(t)
\]
(20)
as the approximated value of \( x(t) \), where \( I \) is the \( n \times n \) unit matrix.

3. Error Bounds

Let \( \| \cdot \| \) denote the norm \( \| Y \| = \sqrt{Y^T Y} \) to a vector \( Y \) and \( \| A \| = \sqrt{\lambda_{\text{max}}(A^T A)} \) to a matrix \( A \).

Define
\[
\pi(y_i) \triangleq (y_i - y_{i_0})(y_i - y_{i_1}) \cdots (y_i - y_{i_N})
\]
(21)
using the zeros \( \{y_{i_j} : j = 0, 1, \ldots, N\} \) of \( T_{N+1}(y_i) \) which is given by Eqs. (14) and (21). Then it holds [4,8] that
\[
\min_{\{y_{i_j}\}} \max_{y_i \in D_0} \| \pi(y_i) \| = 2^{-N}
\]
(22)

We define a product of the operator by
\[
\omega^{[i]} \triangleq \omega_1 \circ \cdots \circ \omega_2 \circ \omega_1 \quad (0 \leq i \leq n)
\]
(23)
in which \( \omega^{[k]}(y_1) \) in Eq. (12) becomes
\[
\omega^{[n]}(G^{[k]}(r_1 \ldots r_n))(y) = \sum_{q_i=0}^{N} (G^{[k]}(r_1 \ldots r_n))_{q_i} T_{q_i}(y_i) = G^{[k]}(r_1 \ldots r_n)(y)
\]
(24)

where
\[
\omega^{[0]}(G^{[k]}(r_1 \ldots r_n))(y) = G^{[k]}(r_1 \ldots r_n)(y)
\]
\[
\omega^{[1]}(G^{[k]}(r_2 \ldots r_n))(y) = \omega_1 G^{[k]}(r_1 \ldots r_n)(y)
\]
\[
\omega^{[n]}(G^{[k]}(r_1 \ldots r_n))(y) = \sum_{q_i=0}^{N} (G^{[k]}(r_1 \ldots r_n))(y_i) T_{q_i}(y_i)
\]

The norm of Eq. (27) by Eq. (22) is
\[
\| \varepsilon^{(y_i)(k)}(N+1(r_1 \ldots r_n))(y) \| = \| \frac{1}{(N+1)!} \]
}\| T_{q_i}(y_i) \| \leq \| \pi(y_i) \| \cdot \| \omega^{[i]} \| \cdot \| \frac{1}{(N+1)!} \| + \frac{\Lambda^{(k)}(r_1 \ldots r_n)}{2^{N}(N+1)!}
\]
(29)
where
\[ K_{(r_1\cdots r_n)}^{(k)} = \max_{1 \leq i \leq n} \sup \left\{ \left\| \frac{\partial^{N+1}}{\partial t^{N+1}} \left( (\omega^{[i]} \cdot 1) C_{(r_1\cdots r_n)}^{(k)} (y) \right) \right\| : y \in D_0 \right\} \]

Thus it yields
\[ \| R_{N+1}^{(k)} (x) \| = \left\{ \sum_{(r_1\cdots r_n)=1}^{N} \sum_{r_1\cdots r_n}^{N} \left( \sum_{i=1}^{n} \varepsilon_{N+1}^{(r_1\cdots r_n)} (y) \right)^2 \right\}^{\frac{1}{2}} \]

\[ \leq \left\{ \sum_{(r_1\cdots r_n)=1}^{N} \sum_{r_1\cdots r_n}^{N} \left( \frac{K_{(r_1\cdots r_n)}^{(k)}}{2} \right)^2 \right\}^{\frac{1}{2}} \]

\[ = \frac{\sum_{(r_1\cdots r_n)=1}^{N} \sum_{r_1\cdots r_n}^{N} \left( K_{(r_1\cdots r_n)}^{(k)} \right)^{\frac{1}{2}}}{2^{N+1} (N + 1)!} \]

\[ \leq \varepsilon_{N+1} \|

where
\[ \varepsilon_{N+1} = \max_k \left\{ \varepsilon_{N+1}^{(k)} : 0 \leq k \leq M \right\} \]

Therefore, we have the following error bounds.

**Theorem 1**

An error bound when approximated by the pseudo-formal linearization is
\[ \varepsilon_{N+1} = \max_{0 \leq k \leq M} \frac{n}{2^{N+1} (N + 1)!} \left\{ \sum_{(r_1\cdots r_n)=1}^{N} \sum_{r_1\cdots r_n}^{N} \left( K_{(r_1\cdots r_n)}^{(k)} \right)^{\frac{1}{2}} \right\} \]

where
\[ K_{(r_1\cdots r_n)}^{(k)} = \max_{1 \leq i \leq n} \left\{ \left\| \frac{\partial^{N+1}}{\partial t^{N+1}} \left( (\omega^{[i]} \cdot 1) C_{(r_1\cdots r_n)}^{(k)} (y) \right) \right\| : y \in D_0 \right\} \]

**Proof**

Note that \( 0 \leq I_k (\zeta) \leq 1 \). At any fixed \( \zeta \), the norm of the function error of Eq. (18) by Eq. (32) is
\[ \| R_{N+1} (x, \zeta) \| = \left\| \sum_{k=0}^{M} R_{N+1}^{(k)} (x) I_k (\zeta) \right\| \]

\[ \leq \sum_{k=0}^{M} \| R_{N+1}^{(k)} (x) I_k (\zeta) \| = \sum_{k=0}^{M} \| R_{N+1}^{(k)} (x) \| I_k (\zeta) \]

\[ \leq \sum_{k=0}^{M} \varepsilon_{N+1} I_k (\zeta) \leq \varepsilon_{N+1} \sum_{k=0}^{M} I_k (\zeta) \]

\[ = \varepsilon_{N+1} \]

(QED)

**Theorem 2**

An error bound of the pseudo-formal linearization for nonlinear dynamic system is
\[ \| x(t) - \dot{x}(t) \| \leq \| \phi (x(t)) - z(t) \| \]

\[ \leq e^{A_{max} t} \| \phi (x(0)) - z(0) \| + \frac{\varepsilon_{N+1} \| A_{max} t - 1 \|}{\| A_{max} \|} \]

Equation (34) indicates that the first term comes from the initial error, and the second term is made from the functional approximation error by this approach.

**Proof**

A differential equation of difference between \( \phi \) in Eq. (18) and \( z \) in Eq. (19) is
\[ \frac{d}{dt} (\phi (x(t)) - z(t)) = A(\zeta) (\phi (x) - z) + R_{N+1} (x, \zeta) \]

\[ = \sum_{k=0}^{M} \left\{ A^{(k)} (\phi (x) - z) + R_{N+1}^{(k)} (x) \right\} I_k (\zeta) \]

so that
\[ \sum_{k=0}^{M} \left\{ \frac{d}{dt} (\phi (x) - z) - A^{(k)} (\phi (x) - z) - R_{N+1}^{(k)} (x) \right\} I_k (\zeta) = 0 \]

or
\[ \frac{d}{dt} (\phi (x) - z) = A^{(k)} (\phi (x) - z) + R_{N+1}^{(k)} (x) \]

for all \( k \). Its solution is
\[ \phi (x(t)) - z(t) = \sum_{k=0}^{M} \left[ \phi (x(t)) - z(t) \right] I_k (\zeta) \]

\[ = \sum_{k=0}^{M} \left( e^{A^{(k)} t} I_k (\zeta) \right) (\phi (x(0)) - z(0)) \]

\[ + \sum_{k=0}^{M} \left( \int_{0}^{t} e^{A^{(k)} (t - \tau)} R_{N+1}^{(k)} (x(\tau)) d\tau \right) I_k (\zeta) \]

The norm of Eq. (35) at any fixed \( \zeta \) is
\[ \| \phi (x(t)) - z(t) \| \leq \sum_{k=0}^{M} e^{A_{max} t} I_k (\zeta) \| \phi (x(0)) - z(0) \| \]

\[ + \sum_{k=0}^{M} \left( \int_{0}^{t} e^{A_{max}(t - \tau)} \| R_{N+1}^{(k)} (x(\tau)) \| d\tau \right) I_k (\zeta) \]

\[ \leq e^{A_{max} t} \| \phi (x(0)) - z(0) \| + \varepsilon_{N+1} \int_{0}^{t} e^{A_{max}(t - \tau)} d\tau \]
\[
e^{|\mathbf{A}|\max t}\|\phi(x(0)) - z(0)\| + \varepsilon_{N+1} \frac{e^{\|\mathbf{A}\|\max t} - 1}{\|\mathbf{A}\|\max}
\]  
(37)

where

\[
\|\mathbf{A}\|\max = \max\{\|\mathbf{A}^{(k)}\| : 0 \leq k \leq M\}
\]

From Eq. (20),

\[
x(t) - \hat{x}(t) = [I, 0, \cdots, 0](\phi(x(t)) - z(t))
\]

so that

\[
\|x(t) - \hat{x}(t)\| \leq \|\phi(x(t)) - z(t)\|
\]

(38)

Thus we have Eq. (34) from Eqs. (37) and (38).

(QED)

Theorems 1 and 2 show that an accuracy of the pseudo-formal linearization system can be improved by increasing the order \(N\) of a linearization function and \(M\). On the other hand, the computational complexity of the linearization increases with the order \(N\) and \(M\) as shown in Eqs. (16) and (17). When the method is applied to practical systems, one might consider the approximation errors and its computational burden.

4. Numerical Experiments

Consider a simple nonlinear dynamic system

\[
\dot{x} = x - x^2, \quad D = [-\frac{1}{4}, \frac{5}{4}] \subset \mathbb{R}
\]

(39)

The parameters are set as \(M = 2\), \(\mu = 50\), and \(\zeta = x\) in Eqs. (4) and (5). \(D\) is divided into \(D = \bigcup_{k=0}^{2} D_k\) where

\[
D_0 = [-\frac{1}{4}, \frac{1}{4}), D_1 = [\frac{1}{4}, \frac{3}{4}), D_2 = [\frac{3}{4}, \frac{5}{4}]
\]

Taylor expansion points are set at

\[
\hat{x}_0 = 0, \hat{x}_1 = 0.5, \hat{x}_2 = 1
\]

for the method of Taylor type [9], and other parameters are

\[
\mathcal{M}^{(0)} = 0, \mathcal{M}^{(1)} = 0.5, \mathcal{M}^{(2)} = 1
\]

\[
\mathcal{P}^{(k)} = 0.25 \quad (k = 0, \cdots, 2)
\]

for the one of Chebyshev type in Eq. (6).

We have followings from Theorem 1. Table 1 shows the error bounds of the proposed approach of Eq. (33) [Chebyshev\((M = 2)\)], and the formal linearization method of Chebyshev type [8] [Chebyshev \((M=0)\)] which is equivalent to \(M = 0\), when the order \(N\) is varied from 1 to 3. Moreover, it includes the \(\varepsilon_{N+1}\) by the pseudo-formal linearization method of Taylor type [9] [Taylor \((M=2)\)] for comparison with them.

Beside we have followings from Theorem 2. Figure 2 shows the pseudo-formal linearization error

\[
e(t) \triangleq \|x(t) - \hat{x}(t)\|
\]

and the error bound

\[
EB(t)(M, N) \triangleq e^{\|\mathbf{A}\|\max t}\|\phi(x(0)) - z(0)\| + \varepsilon_{N+1} \frac{e^{\|\mathbf{A}\|\max t} - 1}{\|\mathbf{A}\|\max}
\]

of Eq. (34) for various \(M\) and \(N\), when \(x(0) = 0.32\), and \(\hat{x}(0) = 0.32\). \(EB(t)(\text{Taylor})\) is by Taylor type [9] for comparison.

Table 1 and Fig. 2 indicate that the pseudo-formal linearization method of Chebyshev type has higher accuracy than that of the Taylor method, and the formal linearization method. These linearization methods accurately improve as the order of the linearization function increases.

![Figure 2 The error bounds](image)

5. Conclusions

We have studied the error bounds of a pseudo-formal linearization of Chebyshev interpolation type. They are specific formulas which enable easy implementation. They could be effective when making use of this pseudo-formal linearization approach for practical systems such as electric power systems.
sider the approximation errors and its computational complexity of the linearization increases with the order $M$ and $N$ of a linearization function. Thus we have Eq. (34) from Eqs. (37) and (38).

Consider a simple nonlinear dynamic system

$$f(x(t)) = 0,$$

where $x(t)$ is the state vector, $f(\cdot)$ is the nonlinear function, and $x(0)$ is the initial condition. The parameters are set as $M = 2$, and $N = 3$.

From Eq. (20),

$$\| \dot{x}(t) \| \leq \| \ddot{x}(t) \| \leq \| \dddot{x}(t) \|,$$

for the one of Chebyshev type in Eq. (6).

The parameters are set as $M = 2$, and $N = 3$. Theorems 1 and 2 show that an accuracy of the formal linearization of Chebyshev interpolation type [8] for comparison.

We have followings from Theorem 1. Table 1 shows the pseudo-formal linearization error $\| \epsilon \|$ which is equivalent to $\| \epsilon \| = \| \epsilon \|$.

Thus we have Eq. (34) for various $A$ and $\hat{y}$.

The formal linearization of Chebyshev interpolation type. They could be effective when making use of Eq. (34) for various $A$ and $\hat{y}$.

Table 1 Error bounds of $A$
\begin{tabular}{|c|c|c|c|}
\hline
$A$ & $\| \epsilon \|$ & $\| \epsilon \|$ & $\| \epsilon \|$ \\
\hline
$2$ & $0.062500$ & $0.015625$ & $0.001953$ \\
$3$ & $0.043944$ & $0.012628$ & $0.001697$ \\
$4$ & $0.032768$ & $0.010102$ & $0.000839$ \\
\hline
\end{tabular}

(QED)

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