EFFECT OF RELIABILITY ON VARYING DEMAND AND HOLDING COST ON INVENTORY SYSTEM INCORPORATING PROBABILISTIC DETERIORATION

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Abstract. This paper presents a mathematical framework to derive an inventory model for time, reliability, and advertisement dependent demand. This paper considers the demand rate is high initially, and then the demand rate reduces later stage, which reflects the situation related to cash in hand. The uncertain deterioration of the product presents through Uniform, Triangular, and Double Triangular probability distributions. The holding cost of the proposed inventory system is dependent on the reliability of the item to make this study a more realistic one. This proposed inventory system allows the situation of shortage and partially backlogged at a fixed rate. Numerical examples, along with managerial implications and sensitivity analysis of the inventory parameters, discuss to examine the effect of changes on the optimal total inventory cost.

1. Introduction. In the study of the inventory system, many researchers have been considered classical inventory models with constant demand rate or linearly increasing or decreasing. Several researchers have considered the demand rate is decreasing function of time or stock dependent and price dependent. But it has been noticing in the market that the demand pattern does not precisely represent certain items such as newly launched products, hardware devices, cosmetics, fashionable garments, mobiles, electronic items, etc. increases up to some time. The most important and real in practice for the inventory system is cash in hand, which is a vital fact for demand patterns. For a salaried person, they will get a salary in starting of the month/fortnight and so the cash in hand will be maximum at that time. Hence they will try to buy more necessary items during the initial time of the cycle; therefore, the demand will automatically increase. From the middle of the period, the cash in hand decreases and so they will buy limited and necessary items. Thus the demand will automatically decrease due to restricted buying

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decisions. Hence cash in hand plays a vital role in the field of inventory system. Several researchers constructed inventory models considering different types of demand, such as inventory level based demand [70], time-varying demand ([7], [14], [15], [25], [57], [60], [61], [64]), price-sensitive demand model ([9], [20], [36]), permissible delay in payments and stock dependent demand ([3], [10], [11], [13], [16], [17], [19], [26], [30], [33], [34], [39], [43], [44], [66]), Poisson demand ([18], [35], [62]), and ramp type demand rate ([12], [37], [69]). Several researchers ([22], [23], [24], [58], [67]) discussed inventory model with a fuzzy demand rate and fuzzy deterioration rate. Barzegar et al. [5] studied vendor-managed inventory systems with partial back ordering under stochastic demand and limited storage capacity. Modak and Kelle [32] presented price and delivery-time dependent stochastic demand. Also, many researchers ([29], [31], [38], [49]) developed an inventory model with variable demand. Shah and Vaghe [59] studied imperfect production inventory model for time and effort dependent demand under-inflation and reliability. Pakhira et al. [41] proposed a deteriorating seasonal item with time, price, and promotional cost dependent demand under a finite time horizon. Pervin et al. [42] developed an inventory model with a declining demand market for deteriorating items under a trade credit policy. Roy et al. [50] discussed a two-warehouse probabilistic model with a price discount on backorders under two levels of trade-credit policy. Pervin et al. [45] studied an integrated vendor-buyer model with quadratic demand under inspection policy and preservation technology. Lotfi et al. [28] proposed interdependent demand in the two-period newsvendor problem. To the authors’ best knowledge, no researchers have considered variable demand related to cash in hand; this paper discusses variable demand dependent on the time that referred to money in hand.

Deterioration plays a significant role in most of the industry or medical sector or education sector, even also in the banking sector, etc. The rate of deterioration for each product will vary because the items in the inventory become obsolete, devalued, damaged, or decayed depending on the type of goods. Sana [51] developed an inventory system with time-varying deterioration. Sanni and Chukwu [52] presented Weibull distribution deterioration in the inventory system. Many researchers ([27], [56]) discussed on a probabilistic inventory model with ramp type demand rate for deteriorating items. Chan et al. [6] studied an inventory model for deteriorating items with consideration of the optimal production rate and deterioration during delivery. Sundararajan and Uthayakumar [63] constructed replenishment policies for instantaneous deterioration with backlogging and permissible delay in payment. A few inventory models ([1], [40]) studied with non-instantaneous deteriorating items. Tiwari et al. [68] constructed an inventory model for deteriorating items with expiration dates and partial backlogging under the supply chain. Sanni et al. [53] discussed an economic order quantity model with a reverse logistics program. Barman et al. [4] presented a back-ordered inventory model with inflation in a cloudy-fuzzy environment.

Many researchers were developed different inventory models under the assumption that for the entire inventory cycle, the holding cost is constant, but it is right to a limited extend. In the present day, customers want to buy such products whose reliability is high; therefore, the demand is automatically increased and holding time decreased. That means a product whose reliability is high, the holding cost automatically decreases, and so the total cost decreases. Many researchers discussed on inventory model with nonlinear holding cost ([8], [21], [54], [55]), and time-varying
holding cost ([2], [46]). Pervin et al. [47] discussed a two-echelon inventory model with stock-dependent demand and variable holding cost for deteriorating items. Taleizadeh et al. [65] studied on supplier-retailer supply chain under noise effect with bundling and inventory strategies. Pervin et al. [48] developed an integrated inventory model with variable holding cost under two levels of trade-credit policy. However, it is very rare reliability depended on holding cost in the modeling of inventory system. This study of inventory system considers holding charge is reliability dependent.

From the above discussion, the following Table 1 gives an overview of the contribution of this paper relevant to the contributions of previous literature. Note that “NA” stands for Not Applicable.

Table 1. Contributions of the proposed model with compare to previous studies

| Author’s                      | Cash in hand | Demand depend on | Holding Cost depend on | Deterioration | Backlog |
|-------------------------------|--------------|------------------|------------------------|---------------|---------|
| Giri & Chaudhuri (1998)       | NA           | stock            | Non linear             | NA            | No      |
| Chang (2004)                  | NA           | stock            | Non linear             | constant      | No      |
| Shonri et al. (2009)          | NA           | ramp type        | NA                     | Weibull       | Yes     |
| Sana (2010)                   | NA           | stock            | NA                     | probabilistic | No      |
| Sett et al. (2012)            | NA           | time demand      | NA                     | NA            | No      |
| Sarkar & Sarkar (2013)        | NA           | time             | NA                     | probabilistic | No      |
| Choudhury et al. (2014)       | NA           | time-quadratic   | NA                     | time demand   | Yes     |
| Ghorbani et al. (2014)        | NA           | price & time     | NA                     | non-instantaneous | Yes |
| Ghosh et al. (2015)           | NA           | stock            | constant               | constant      | No      |
| Wu & Zhao (2015)              | NA           | inventory & time | NA                     | constant      | No      |
| Bhunia et al. (2015)          | NA           | time, advertisement | NA                     | constant      | Yes     |
| Alfares & Ghaihan (2016)      | NA           | price            | time                   | NA            | No      |
| Chandra & Kumar (2016)        | NA           | advertising & price | NA                     | NA            | No      |
| Sanni & Chukwu (2016)         | NA           | deterministic    | NA                     | Weibull       | Yes     |
| Shahi & Vagela (2016)         | NA           | time & advertisement | NA                     | constant      | No      |
| Mahapatra et al. (2017)       | NA           | time & reliability | NA                     | constant      | Yes     |
| Mokhtari et al. (2017)        | NA           | stochastic       | NA                     | constant      | Yes     |
| Pervin et al. (2018)          | NA           | time             | time                   | Weibull       | Yes     |
| Laj et al. (2018)             | NA           | independent      | NA                     | NA            | Yes     |
| Dey et al. (2019)             | NA           | selling price    | NA                     | NA            | Yes     |
| Pervin et al. (2019)          | NA           | price and stock  | purchasing cost        | constant      | Yes     |
| Pervin et al. (2020)          | NA           | time & price     | constant               | constant      | Yes     |
| Roy et al. (2020)             | NA           | probabilistic    | Constant               | Weibull       | Yes     |
| This paper                    | Consider     | time, reliability, advertisement | reliability | probabilistic | Yes |

In this study, the inventory model has these advantages: (i) As demand rate is dependent on cash in hand, and initially, the demand rate is high for a specified period. Therefore, companies are prepared to produce many items within the initial time of the cycle. Thus, initially, shortage does not occur. (ii) After that, the rate of demand will reduce compared to the initial part of the time and also related to cash in hand. So, in that period, companies are produced fewer items, and for that reason, the holding cost should be less. (iii) Deterioration function follows probability distribution such as (a) Uniform, (b) Triangular, and (c) Double Triangular, to make the research a more realistic and comparative one. (iv) Holding cost dependent on reliability. So, companies are decided and produce those items whose reliability is high.

From the literature, it is infrequent that an EPQ model with a reliability dependent holding cost. A variable demand relating to cash in hand, a probabilistic deterioration of a product, and reliability dependent holding cost have not considered in the study of inventory system. Numerical examples and sensitivity analysis
for the different parameters, and managerial implications for interpreting the effective angle of the inventory model are presented. The concluding remarks with the limitations of this study and scope for future research is also discussed.

2. Notations, assumptions and problem definition. The objective of the proposed study of the inventory system to find the optimal average total inventory cost per unit time. The proposed inventory model will develop with following notations, assumptions:

Notations:

Parameters
$C_h$ : Holding cost per unit item.
$C_s$ : Shortage cost per unit item.
$C_d$ : The cost of each deteriorated item.
$C_l$ : The penalty cost for lost sale per unit item.
$T$ : Replenishment cycle.
$S$ : The initial inventory level.
$A_0$ : Ordering cost.
$A$ : Frequency of advertisement.
r : reliability.
$a, b, c, d, \nu, \alpha, k, u, \lambda$ : Shape parameters.

Variables:
$T_1$ : Inventory level rapidly decreased up to time.
$T_2$ : When the inventory level vanishes and shortages start occurring.

Assumptions:
1) Deterioration starts as soon as items are received into inventory, and the deterioration rate $\theta$ is probabilistic.
2) Shortages are allowed.
3) The demand rate depends on time, reliability, and advertisement, i.e., $R(t, r, A) = \begin{cases} ae^{bt}A^\nu r^k & \text{for } 0 < t < T_1 \\ (c + \frac{d}{t})A^\nu r^k & \text{for } T_1 < t < T \end{cases}$ where $a, b, c, d, \nu, \alpha, k > 0$ are parameters.
4) The holding cost is a function of reliability and consider as $HC = r^{-u}$ where $u > 0$.
5) The demand during a shortage is partially lost and partially backorder. The backlog function taken is $\beta(x) = e^{-\lambda x}$ with $\beta(0) = 1$ and $\beta(T) \geq 0$ where $\lambda$ is a parameter.
6) The inventory level rapidly decreased in $(0, T_1)$, and in the range $(T_1, T_2)$ the inventory level gradually decreased. At the time $T_2$ when the inventory level vanishes and shortages start occurring and continue up to time $T$.

Problem Definition:
This study presents a deterministic inventory model for time, reliability, and advertisement dependent variable demand. This demand considers in two types based on the financial situation, firstly the demand rate is high for a certain period, and secondly, the demand rate is reduced and less compare to initial part of time both are related to cash in hand. The system is allowed shortages with the second type of demand pattern and allowed partially backlogged at a fixed rate. The deterioration function follows probability distribution such as (a) Uniform, (b) Triangular, and (c) Double Triangular, to make the research a more realistic and comparative one. Based on the assumption, the holding cost is reliability dependent.
The inventory level decrease due to demand and deterioration of items. The inventory level rapidly decreased up to time $t = T_1$, and at time $t = T_2$ it becomes zero, then shortages are accumulated up to time $t = T$, which are partially backlogged. Let us consider inventory level at time $t$ ($0 \leq t \leq T$) be $Q(t)$. The differential equations for the immediate state $Q(t)$ over $(0, T)$ are given by

\[
\frac{dQ}{dt} + \theta Q = -ae^{bt}A^v r^k, \text{ for } 0 \leq t \leq T_1
\]  

\[
\frac{dQ}{dt} + \theta Q = -(c + t^{-\alpha} d)A^v r^k, \text{ for } T_1 \leq t \leq T_2
\]  

\[
\frac{dQ}{dt} = -(c + t^{-\alpha} d)A^v r^k \beta(T-t), \text{ for } T_2 \leq t \leq T
\]

with the conditions are $Q(0) = S$ and $Q(T_2) = 0$.

The proposed inventory model presents the concept of two types of demand in a single cycle and allowing shortages but incorporating partially backlogged at a fixed rate. The graphical sketch for the proposed system (Figure 1) for inventory level vs. time as follows:

\section{Mathematical analysis of proposed inventory system.}

The solution of the proposed inventory model described by the differential equation (1), after applying the initial condition $Q(0) = S$ is

\[
Q(t) = \left(Se^{-\theta t} - aA^v r^k t\right), \text{ for } 0 \leq t \leq T_1
\]

At $t = T_1$, we have from equation (4) as follows:

\[
Q(T_1) = \left(Se^{-\theta T_1} - aA^v r^k T_1\right)
\]

The above equation (5) is the initial condition for $Q(t)$ at $t = T_1$ of the differential equation (2).

\textbf{Lemma 3.1.} The demand rate $R(t, r, A)$ is an increasing function and the rate of change of demand rate increases to the time where $0 < t < T_1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Proposed inventory model with inventory vs time}
\end{figure}
Proof. The demand rate decreases with respect to time if \( \frac{dR(t,r,A)}{dt} > 0 \) for all \( t \). Now, \( \frac{dR(t,r,A)}{dt} = abe^{bt}A^{\nu r k} > 0 \) for all \( t \) where \( 0 < t < T_1 \), \( a > 0, b > 0 \). Hence the demand rate \( R(t,r,A) \) increases with the increase of the time \( t \). The demand rate increases with respect to time if \( \frac{dR(t,r,A)}{dt} > 0 \) for all \( t > 0 \). The rate of change of demand increases for time if \( \frac{d^2R(t,r,A)}{dt^2} > 0 \) for all \( t > 0 \).

Now, \( \frac{dR(t,r,A)}{dt} = abe^{bt}A^{\nu r k} > 0 \) for all \( t \) and \( a > 0, b > 0 \) and \( \frac{d^2R(t,r,A)}{dt^2} = ab^2e^{bt}A^{\nu r k} > 0 \) for all \( t \) and \( a > 0 \). Hence the rate of change of demand rate increases with the increase of the time.

Now, the proof is complete.

Lemma 3.2. The demand rate \( R(t,r,A) \) is a decreasing function, and the rate of change of demand rate increases with respect to time for \( T_1 < t < T \).

Proof. The demand rate decreases with respect to time if \( \frac{dR(t,r,A)}{dt} < 0 \) for all \( t \).

Now, \( \frac{dR(t,r,A)}{dt} = -\alpha dA^{\nu r k} \frac{1}{b^{\nu r k}} < 0 \) for all \( t \) and \( d > 0, \alpha > 0 \) where \( T_1 < t < T \). Hence the demand rate \( R(t,r,A) \) decreases with the increase of the time \( t \).

The demand rate decreases with respect to time if \( \frac{dR(t,r,A)}{dt} < 0 \) for all \( t > 0 \). The rate of change of demand with respect to time increases if \( \frac{d^2R(t,r,A)}{dt^2} > 0 \).

Now, \( \frac{dR(t,r,A)}{dt} = -\alpha dA^{\nu r k} \frac{1}{b^{\nu r k}} < 0 \) for all \( t \) and \( \alpha > 0, d > 0 \) and \( \frac{d^2R(t,r,A)}{dt^2} = \alpha \frac{(\alpha + 1)}{d}A^{\nu r k} \frac{1}{b^{\nu r k}} > 0 \) for all \( t \) and \( \alpha > 0, d > 0 \). Hence the rate of change of demand rate increases with the decrease of the time.

Now, the proof is complete.

Now the solutions of the differential equations (2) and (3), after applying the initial conditions (5) and \( Q(T_2) = 0 \) are obtained as follows:

\[
Q(t) = \left[ S + A^{\nu r k} \left\{ c(T_1 - t) - a T_1 e^{\theta T_1} \right\} \right] e^{-\theta t}, \quad \text{for } T_1 \leq t \leq T_2 \tag{6}
\]

\[
Q(t) = c e^{-\lambda t} A^{\nu r k} (T_2 - t), \quad \text{for } T_2 \leq t \leq T \tag{7}
\]

Therefore the equation (4), (6), and (7) represents three different inventory situation of the proposed inventory system.

Lemma 3.3. The maximum inventory level \( S \) must satisfy the relation \( S = A^{\nu r k} [a T_1 e^{\theta T_1} + c(T_2 - T_1)] \) in terms of the shape parameter \( a, \nu, c, k \), and other inventory parameters.

Proof. Using the boundary condition \( Q(T_2) = 0 \) on (6) we get

\[
\left[ S + A^{\nu r k} \left\{ c(T_1 - T_2) - a T_1 e^{\theta T_1} \right\} \right] e^{-\theta T_2} = 0
\]

\[
S = A^{\nu r k} [a T_1 e^{\theta T_1} + c(T_2 - T_1)]
\]

Thus, the maximum inventory level \( S \) must satisfy the relation \( S = A^{\nu r k} [a T_1 e^{\theta T_1} + c(T_2 - T_1)] \) in terms of the shape parameter \( a, \nu, c, k, \) and other inventory parameters.

The deterioration is defined as damage, change, spoilage obsolescence, decay, and loss of utility that consequences in the decreasing usefulness from the original of the
product. The deterioration cost ($DC$) during the time $[0, T_2]$ is

$$
DC = C_d \left[ Q(0) - \int_0^{T_2} R(t, r, A) \, dt \right] \quad \text{Neglecting cubic and higher terms}
$$

$$
= C_d A^v r^k \left[ aT_1^2 \left( 1 + \frac{\theta T_1}{2} \right) - \frac{bT_1^2}{2} (a + c) + \frac{bcT_2^2}{2} \right]
$$

Shortage cost ($SC$) is the costs incurred by an organization when it has no inventory in stock, the shortage cost of the proposed inventory system during $[T_2, T]$ is

$$
SC = -C_s \int_{T_2}^{T} Q(t) \, dt = \frac{1}{2} C_s c e^{-\lambda T} A^v r^k (T - T_2)^2
$$

Holding cost ($HC$) is the cost associated with storing inventory that remains unsold. The proposed inventory system’s holding cost includes the price of goods damaged or spoiled, storage space, labor, and insurance during the time $[0, T_2]$ is

$$
HC = H_c \int_0^{T_2} r^{-u} \, Q(t) \, dt \quad \text{(Neglecting cubic and higher terms)}
$$

$$
= \frac{1}{4} C_h r^{-u} A^v r^k \left[ 2c (T_2^2 - T_1^2) + aT_1^2 \left( 2 + 2\theta T_1 - \theta^2 T_1^2 \right) \right],
$$

The lost sale cost ($LS$) is the costs that have been lost because an item was out of stock or without carrying a particular brand or for any reason that caused it to miss the opportunity to sell. The backlog or lost sale cost of the proposed inventory system during $[T_2, T]$ is

$$
LS = C_l \int_{T_2}^{T} \left\{ 1 - \beta(T - t) \right\} R(t, r, A) \, dt, \quad \text{(Neglecting cubic and higher terms)} \quad (8)
$$

$$
= cC_l A^v r^k (T - T_2) \left\{ \lambda T - \frac{\lambda^2 T_2^2}{2} - \frac{b}{2} (T + T_2) \right\}
$$

The expenses incurred to create and process an order to a supplier in the economic order quantity system is called the ordering cost ($OC$), the proposed inventory system consider the fixed $OC = A_o$

The total inventory cost is the accumulation of the ordering cost ($OC$), shortage cost ($SC$), holding cost ($HC$), deterioration cost ($DC$), and lost sale cost ($LS$). i.e., the total inventory cost per item per unit time is given by

$$
TC = \frac{A_o}{T} + \frac{A^v r^k}{T} \left\{ C_d \left[ aT_1^2 \left( 1 + \frac{\theta T_1}{2} \right) - \frac{b(a+c)}{2} T_1^2 + \frac{bc}{2} T_2^2 \right] + \frac{1}{2} C_h c e^{-\lambda T} (T - T_2)^2 \right\} + \frac{C_s}{4r^k} \left\{ 2c (T_2^2 - T_1^2) + aT_1^2 \left( 2 + 2\theta T_1 - \theta^2 T_1^2 \right) \right\} + cC_l \left\{ \lambda T - \frac{\lambda^2 T_2^2}{2} - \frac{b}{2} (T + T_2) \right\}
$$

(9)

4. **Optimization of total cost of proposed inventory system.** From a management point of view, the above total cost should be optimized, and hence the objective is to minimize the total inventory cost $TC$ per unit time. The total inventory cost $TC$ from equation (9) of the inventory model will give the optimal solution based on the following lemma. The condition for the optimal total average cost is

The average cost $TC$ is a function of two independent variables $T_1$ and $T_2$. Now it is supposed that for some parametric values involved in the system of equations
\( \frac{\partial TC}{\partial T_1} = 0 \) and \( \frac{\partial TC}{\partial T_2} = 0 \), there exists at least one positive point \((T_1^*, T_2^*)\) at which (i) \( \frac{\partial^2 TC}{\partial T_1^2} \) and \( \frac{\partial^2 TC}{\partial T_2^2} \) both are positive and (ii) \( \frac{\partial^2 TC}{\partial T_1^2} \frac{\partial^2 TC}{\partial T_2^2} - \left( \frac{\partial^2 TC}{\partial T_1 \partial T_2} \right)^2 > 0 \), then we obtain the minimum average cost at the point \((T_1^*, T_2^*)\).

**Lemma 4.1.** The total average cost is minimum if (i) \( 2a \theta C_d + aC_h r^{-u} > b(a + c) C_d + C_h r^{-u} c \), (ii) \( r^uC_d \theta + C_h > 5 \theta^2 C_h T_1^2 \), (iii) \( b(C_d + C_l) + C_e e^{-\lambda T} + C_h r^{-u} > 0 \)

**Proof.** The total inventory cost from equation (9) differentiating with respect to two variables \( T_1 \) and \( T_2 \) as follows:

\[
\frac{\partial TC}{\partial T_1} = \frac{A^{r_{\theta}^*}}{T} T_1 \left[ C_d \left( 2a \theta + \frac{3}{2} a \theta^2 T_1 - b(a + c) \right) + C_h r^{-u} \left( a - c + \frac{3}{2} \theta T_1 - \frac{5}{4} a \theta^3 T_1^3 \right) \right]
\]

(10)

\[
\frac{\partial TC}{\partial T_2} = \frac{A^{r_{\theta}^*}}{T} T_c \left[ C_d b T_2 - C_s e^{-\lambda T} (T-T_2) + C_h r^{-u} T_2 + C_l \left( b T_2 - \lambda T + \frac{\lambda^2 T^2}{2} \right) \right]
\]

(11)

Now differentiating (10) and (11) with respect to \( T_1 \) and \( T_2 \) respectively, we get

\[
\frac{\partial^2 TC}{\partial T_1^2} = \frac{A^{r_{\theta}^*}}{T} \left[ C_d \left( 2a \theta + 3a \theta^2 T_1 - b(a + c) \right) + C_h r^{-u} \left( 3a \theta T_1 + a - c - 5a \theta^3 T_1^3 \right) \right]
\]

(12)

\[
\frac{\partial^2 TC}{\partial T_2^2} = \frac{A^{r_{\theta}^*}}{T} \left[ b(C_d + C_l) + C_s e^{-\lambda T} + C_h r^{-u} \right]
\]

(13)

Again, we get \( \frac{\partial^2 TC}{\partial T_1 \partial T_2} = 0 \).

If \( T_1 = T_1^* \) and \( T_2 = T_2^* \) are the solutions of \( \frac{\partial TC}{\partial T_1} = 0 \) and \( \frac{\partial TC}{\partial T_2} = 0 \), then from equations (10) and (11), we get

\[
C_d \left( 2a \theta + \frac{3}{2} a \theta^2 T_1 - b(a + c) \right) + C_h r^{-u} \left( a - c + \frac{3}{2} a \theta T_1 - \frac{5}{4} a \theta^3 (T_1^*)^3 \right) = 0
\]

\[
C_d b T_2^* - C_s e^{-\lambda T} (T-T_2^*) + C_h r^{-u} T_2^* + C_l \left( b T_2^* - \lambda T + \frac{\lambda^2 T^2}{2} \right) = 0
\]

(14)

Solving the above two equations, we will get the optimum value of \( T_1^* \) and \( T_2^* \). From (13) it is clear that the right-hand side is positive. To prove the convexity, the right-hand term of (12) must be positive as well as \( \frac{\partial^2 TC}{\partial T_1^2} \frac{\partial^2 TC}{\partial T_2^2} - \left( \frac{\partial^2 TC}{\partial T_1 \partial T_2} \right)^2 \) will also be a positive.

Now from (12), an auxiliary function say \( f(T_1) \) can be constructed as follows.

\[
f(T_1) = \left[ C_d \left( 2a \theta + 3a \theta^2 T_1 - b(a + c) \right) + C_h r^{-u} \left( 3a \theta T_1 + a - c - 5a \theta^3 T_1^3 \right) \right]
\]

(15)

Now, \( f(0) = C_d (2a \theta - b(a + c)) + C_h r^{-u} (a - c) \). So, \( f(0) \) must be positive if

\[
2a \theta C_d + aC_h r^{-u} > b(a + c) C_d + C_h r^{-u} c
\]

Also, from (15)

\[
\frac{df}{dT_1} = \left[ 3a C_d \theta^2 + C_h r^{-u} \left( 3a \theta - 15a \theta^3 T_1^2 \right) \right]
\]

(16)
Now the function \( f(T_1) \) will be increasing in \([0, 1]\) if \( f(T_1) > 0 \). So, from (16) \( f(T_1) \) will be increasing in \([0, 1]\) if

\[
r^nC_d\theta + C_h > 5\theta^2C_hT_1^2
\]

Thus, the right-hand side of (12) must be positive if both the conditions \( 2a\theta C_d + aC_hr^{-u} > b(a+c)C_d + C_hr^{-u}c \) and \( r^nC_d\theta + C_h > 5\theta^2C_hT_1^2 \) must be hold.

Again, \( \frac{\partial^2TC}{\partial T_1^2} \frac{\partial^2TC}{\partial T_2^2} - \left( \frac{\partial^2TC}{\partial T_1 T_2} \right)^2 = \frac{\partial^2TC}{\partial T_1^2} \frac{\partial^2TC}{\partial T_2^2} > 0 \) means either both are positive or both are negative. But to prove the convexity, we only consider the positivity. Hence, the total average cost is minimum if the following conditions are satisfied:

\[
2a\theta C_d + aC_hr^{-u} > b(a+c)C_d + C_hr^{-u}c, \quad r^nC_d\theta + C_h > 5\theta^2C_hT_1^2 \quad \text{and} \quad b(C_d + C_l) + C_se^{-\lambda T} + C_hr^{-u} > 0.
\]

Therefore, the optimal solution for the proposed inventory system exists.

The computational procedure to find the optimum total average cost from (9) of the proposed inventory model comprises the following steps:

| Computational Steps for optimum average total inventory cost. |
|---------------------------------------------------------------|
| Steps 1: Calculate partial differentiation \( \frac{\partial TC}{\partial T_1} \), \( \frac{\partial TC}{\partial T_2} \), \( \frac{\partial^2TC}{\partial T_1^2} \), and \( \frac{\partial^2TC}{\partial T_2^2} \). |
| Steps 2: Find out the time \( T_1 = T_1^* \) and \( T_2 = T_2^* \) by solving the equations \( \frac{\partial TC}{\partial T_1} = 0 \) and \( \frac{\partial TC}{\partial T_2} = 0 \). |
| Steps 3: Check the positivity of second-order differentiation \( \frac{\partial^2TC}{\partial T_1^2} \) and \( \frac{\partial^2TC}{\partial T_2^2} \). |
| Steps 4: Construct an auxiliary function \( f(T_1) \), which is increasing in \([0, 1]\), and value of \( f(T_1) \) at \( T_1 = 0 \) is positive, then function \( f(T_1) \) must be positive. |
| Steps 5: Calculate the functional value of \( f(T_1) \) at \( T_1 = 0 \). |
| Steps 6: One condition will occur if \( f(T_1) \) at \( T_1 = 0 \) occur positive value. |
| Steps 7: Calculate first-order differentiation \( \frac{\partial f(T_1)}{\partial T_1} \) of the above auxiliary function. |
| Steps 8: Condition for optimal will occur if the auxiliary function is positive. |
| Steps 9: Check the positivity of second-order differentiation \( \frac{\partial^2f(T_1)}{\partial T_1^2} \) and \( \frac{\partial^2f(T_1)}{\partial T_2 T_1} \). |
| Steps 10: Check the positivity of \( \frac{\partial^2TC}{\partial T_1^2} \frac{\partial^2TC}{\partial T_2^2} - \left( \frac{\partial^2TC}{\partial T_1 T_2} \right)^2 \), and if the convexity holds. |
| Steps 11: Evaluate \( TC(T_1^*) \) and \( TC(T_2^*) \) to obtain optimal solution. |

5. Numerical experiments. This section presents numerical examples, comparison of the different cases by graphical presentations for the understanding of the utility of the proposed inventory model. The values of the parameters of the model are not selected from any case study. However, these values have been seemed to be realistic. The following examples have been solved to find the optimal values of time \( T_1^* \) and \( T_2^* \), along with the optimal average cost \( TC(T_1^*, T_2^*) \) of the inventory system.

The proposed inventory system considers different continuous probabilistic deterioration functions, namely Uniform distribution, Triangular distribution, and Double Triangular distribution. Based on the probabilistic distribution and the others input parameters with proper units \( T = 1 \text{ Year} \), \( A_o = 50\text{$/order} \), \( A = 5 \), \( C_d = 2.28 \text{$/per unit} \), \( C_h = 1.18 \text{$/per unit} \), \( C_s = 68 \text{$/per unit} \), \( C_l = 1.28 \text{$/per unit} \), \( c = 0.7 \), \( \nu = 3.5 \), \( k = 0.8 \), \( b = 1.8 \), \( \lambda = 0.01 \), \( a = 1.7 \), \( \theta = 0.8 \), \( r = 0.75 \), \( u = 2.4 \), for the numerical assessment with the help of MATLAB R2018b to determine the optimal value.

Example 1: The deterioration of items follows Uniform distribution, let \( \theta = E(f(x)) = \frac{m+n}{2} \), where \( m < n \), let have \( \theta = \frac{m+n}{2} = \frac{0.7+0.9}{2} = 0.8 \), along with other input parameters of the system, to find the optimal total cost \( TC(T_1^*, T_2^*) \) for the inventory system.
Figure 2. Total cost vs. $T_1$ vs. $T_2$ for Uniformly distributed deterioration

Figure 2 shows the pictorial representation of the total cost, which is a function of $T_1$ and $T_2$. This figure confirms that the total cost is convex, and hence there exist unique solutions for $T_1$ and $T_2$ that minimize the total cost. The optimal solution for the given parameter set is $T_1^* = 0.166$ year and $T_2^* = 0.418$ year, and the optimum price is $TC(T_1^*, T_2^*) = 151.144$.

Example 2: Deterioration follows Triangular distribution, then $	heta = E(f(x)) = \frac{m+n+l}{3}$, where $m \leq l \leq n$, for numerical illustration, assume that $\theta = \frac{m+n+l}{3} = \frac{0.7+0.85+0.9}{3} = 0.82$, along with values of other parameters of the inventory system, to evaluate the optimal total cost $TC(T_1^*, T_2^*)$ for the proposed inventory system.

Figure 3. Total cost vs. $T_1$ vs. $T_2$ for Triangular distribution deterioration

Figure 3 confirms that the total cost is convex, and hence there exist unique solutions to $T_1$ and $T_2$ that minimize the total cost. In this case, the optimal value of $T_1^* = 0.142$ year, and $T_2^* = 0.418$ year and the optimum cost is $TC(T_1^*, T_2^*) = 151.615$.

Example 3: Let the deterioration of the item follows a Double Triangular distribution, and then $\theta = E(f(x)) = \frac{l+4m+n}{6}$, where $l \leq m \leq n$. Let the distribution is $\theta = \frac{l+4m+n}{6} = \frac{0.7+3.4+0.9}{6} = 0.83$, along with other parameters of the inventory system, to obtain the optimal value of $TC(T_1^*, T_2^*)$. 
Based on the convex nature of Figure 4, there exist unique solutions of $T_1$ and $T_2$ that minimize the total cost, and the optimal value of $T_1^* = 0.108$ year and $T_2^* = 0.418$ year, and optimum total price of inventory system is $TC(T_1^*, T_2^*) = 152.0708$.

From Table 2, it is clear that we consider different intervals for all the three distribution and comparing all the results. From the table, it is seen that the optimal total cost has occurred for Uniform distribution, and the corresponding value of $\theta$ is 0.8 and the optimum total cost is 151.144$, and the optimum time is $T_1 = 0.166$ year and $T_2 = 0.418$ year. Also, we can see that if we increase the value of $\theta$ for all three cases, then the total cost will be increased.

6. Sensitivity analysis. This section presents a sensitivity analysis of the inventory system for the various cases of the different parameters in different labels of percentage. Sensitivity analysis is performed on numerical examples by changing parameters $T, A, C_d, C_b, C_s, C_I, c, r, k, \lambda, \alpha, u$ by $-20\%, -10\%, 10\%, 20\%$, and $r, \theta$, and $b$ by $-10\%, -5\%, 5\%, 10\%$ in numerical values of one parameter at a time for analysis and the value of other parameters remains unchanged.
Figure 5. Percentage change of total profit vs change of parameter

Table 3. Sensitivity analysis of the proposed inventory system for parameters

| Parameter | Change(%) | $T_1$  | $T_2$  | $TC$  | Change $TC$(%) |
|-----------|-----------|--------|--------|-------|----------------|
| $T$       | -20       | 0.166  | 0.334  | 143.064 | -5.346         |
|           | -10       | 0.166  | 0.376  | 146.444 | -3.110         |
|           | 10        | 0.166  | 0.459  | 156.796 | 3.739          |
|           | 20        | 0.166  | 0.501  | 163.153 | 7.946          |
| $A$       | -20       | 0.166  | 0.418  | 96.319  | -36.274        |
|           | -10       | 0.166  | 0.418  | 119.950 | -20.639        |
|           | 10        | 0.166  | 0.418  | 191.193 | 26.498         |
|           | 20        | 0.166  | 0.418  | 241.458 | 59.754         |
| $C_d$     | -20       | 0.085  | 0.442  | 140.954 | -6.742         |
|           | -10       | 0.127  | 0.430  | 146.384 | -3.149         |
|           | 10        | 0.201  | 0.406  | 155.207 | 2.688          |
|           | 20        | 0.235  | 0.396  | 158.566 | 4.911          |
| $C_h$     | -20       | 0.250  | 0.431  | 142.313 | -5.843         |
|           | -10       | 0.205  | 0.424  | 147.077 | -2.691         |
|           | 10        | 0.131  | 0.411  | 154.715 | 2.363          |
|           | 20        | 0.099  | 0.405  | 157.929 | 4.489          |
| $C_s$     | -20       | 0.166  | 0.365  | 116.975 | -22.607        |
|           | -10       | 0.166  | 0.392  | 134.803 | -10.812        |
|           | 10        | 0.166  | 0.441  | 166.178 | 9.947          |
|           | 20        | 0.166  | 0.462  | 180.055 | 19.128         |
| $C_l$     | -20       | 0.166  | 0.430  | 178.469 | 18.079         |
|           | -10       | 0.166  | 0.424  | 164.852 | 9.069          |
|           | 10        | 0.166  | 0.411  | 137.350 | -9.126         |
|           | 20        | 0.166  | 0.405  | 123.473 | -18.308        |
| $r$       | -10       | 0.074  | 0.400  | 151.598 | 0.390          |
|           | -5        | 0.121  | 0.409  | 151.493 | 0.231          |
|           | 5         | 0.210  | 0.425  | 150.459 | -0.453         |
|           | 10        | 0.252  | 0.431  | 149.375 | -1.171         |
| $\theta$ | -10       | 0.283  | 0.418  | 146.868 | -2.829         |
|           | -5        | 0.220  | 0.418  | 149.608 | -1.016         |
|           | 5         | 0.119  | 0.418  | 151.946 | 0.530          |
|           | 10        | 0.079  | 0.418  | 152.317 | 0.776          |
Based on sensitivity analysis represents by Table 3 and Figure 5, the proposed model is highly sensitive to cycle length \((T)\). The total cost \(TC\) increases with time \((T)\); however, time \(T_1\) unchanged and \(T_2\) increases. With an increase, the value of \((T)\), the time \(T_1\) does not change, since \(T_1\) is the time where the demand is increasing due to cash in hand. At the time \(T_2\), where the shortage will occur is increasing as cycle length increases mean a lot of time remaining in that cycle period, so the demand will automatically decrease in that period and hence the shortage time will increases. Also, shortage of time increases means holding time increases as well as lost sale cost decreases and so the total cost increases.

Figure 5 shows that the inventory model is highly sensitive to the frequency of advertisement \((A)\), since the total cost automatically increases if the frequency of advertisement increases. From Table 3 and Figure 8, the inventory system also moderately sensitive to the reliability \((r)\), this is a real fact of an inventory management system that the demands are automatic increases for more reliable products, and therefore the total cost decreases.

From Table 3 and Figure 6, the model is highly sensitive to deterioration cost \((C_d)\), with the practical fact as the deterioration cost \((C_d)\) increases, then the total cost also increases with time \(T_1\), and hence \(T_2\) decreases. Therefore, this study will help the decision-maker regarding the holding time of the products such as vegetables, fruits, foodstuffs, etc. This study highlights the situation while reserving in store for a longer time the products such as vegetables, fruits, foodstuffs are subject to direct spoilage. Electronic goods, radioactive substances, photographic film, grain, etc. deteriorate through a gradual loss of potential or utility with time. Also, the model is very highly sensitive to the shortage cost \((C_s)\), the total inventory cost increases for increasing value of \((C_s)\) along with \(T_2\), but the time \(T_1\) remains unchanged. Therefore if the shortage cost \((C_s)\) increases, then the total inventory cost will also increase, and the value of time \(T_2\) increased by some percentage, i.e., the shortage period will be decreased. The proposed model is very highly sensitive to the lost sale cost \((C_l)\), as the lost sale cost \((C_l)\) increases by some percentage the total cost decreases.
Table 4. Sensitivity analysis of the proposed inventory model for parameters

| Parameter | Change(%) | $T_1$ | $T_2$ | $TC$ | Change $TC$(%) |
|-----------|-----------|-------|-------|------|----------------|
| $a$       | −20       | 0.306 | 0.418 | 145.956 | −3.432         |
|           | −10       | 0.227 | 0.418 | 149.419 | −1.141         |
|           | 10        | 0.116 | 0.418 | 151.970 | 0.547          |
|           | 20        | 0.076 | 0.418 | 152.329 | 0.784          |
| $b$       | −10       | 0.047 | 0.436 | 160.574 | 6.239          |
|           | −5        | 0.106 | 0.427 | 156.287 | 3.403          |
|           | 5         | 0.227 | 0.409 | 144.764 | −4.221         |
|           | 10        | 0.289 | 0.400 | 136.737 | −9.532         |
| $c$       | −20       | 0.058 | 0.418 | 131.920 | −12.713        |
|           | −10       | 0.111 | 0.418 | 141.825 | −6.166         |
|           | 10        | 0.221 | 0.418 | 159.594 | 5.591          |
|           | 20        | 0.278 | 0.418 | 166.861 | 10.399         |
| $\nu$     | −20       | 0.166 | 0.418 | 82.784  | −45.784        |
|           | −10       | 0.166 | 0.418 | 107.584 | −28.820        |
|           | 10        | 0.166 | 0.418 | 227.656 | 50.622         |
|           | 20        | 0.166 | 0.418 | 362.046 | 139.537        |
| $k$       | −20       | 0.166 | 0.418 | 155.908 | 3.152          |
|           | −10       | 0.166 | 0.418 | 153.499 | 1.558          |
|           | 10        | 0.166 | 0.418 | 148.843 | −1.523         |
|           | 20        | 0.166 | 0.418 | 146.594 | −3.010         |
| $\lambda$ | −20       | 0.166 | 0.418 | 151.242 | 0.065          |
|           | −10       | 0.166 | 0.418 | 151.193 | 0.032          |
|           | 10        | 0.166 | 0.417 | 151.095 | −0.032         |
|           | 20        | 0.166 | 0.417 | 151.046 | −0.065         |
| $u$       | −20       | 0.218 | 0.426 | 145.779 | −3.550         |
|           | −10       | 0.192 | 0.422 | 148.496 | −1.752         |
|           | 10        | 0.140 | 0.413 | 153.738 | 1.716          |
|           | 20        | 0.115 | 0.408 | 156.298 | 3.410          |

Figure 7. Percentage change of total profit vs change of parameter
From the above Table 4, Figures 5, 6, 7, and 8, it is presented that the parameter $(\nu)$ is a very high sensitivity, $(b)$ and $(c)$ are high sensitivity, $(a)$, $(u)$, and $(k)$ are moderately sensitivity, and $(\lambda)$ is less sensitivity. The sensitivity analysis will help to take proper managerial decision making on changes in inventory costs and the parameters.

Again from Table 3 and Figure 7, the model is highly sensitive to holding cost $(C_h)$, and $(C_h)$ as well as the total cost both are increases for the time $T_1$ and $T_2$ decrease. If the inventory holds for a longer time with a high-cost of holding charge, then evidently, the impact on the optimal total cost and it will increase. Therefore, with the increase in holding cost, the managerial decision is to keep the items for less time, to obtain the minimum total inventory cost.

From Table 4, Figure 2, and 5, it is seen that the model is highly sensitive to $(c)$ and moderately sensitive to $(a)$. Also, if we increase the value of $(a)$ and $(c)$, the total cost increases respectively, and time $T_1$ decreases for both the cases. Because, if we increase the value $(a)$ and $(c)$, the demand rate automatically increases, hence the time $T_1$ gradually decreases. Thus, the shortages will occur very first and so, the total cost increases for both the cases. Here $T_2$ remains the same for both the cases.

Also, from Table 4 and Figure 6, the model is very high sensitivity to $(\nu)$ and if we increase the value of $(\nu)$ the total cost rapidly increased. Because the increase of $(\nu)$ means advertisement increases and so the demand rate increases, hence the total cost will increases. Again, from Figure 6 and Table 4, it is clear that the model is less sensitive to backlog factor $(\lambda)$ and if we increase the value, the total cost decreases. If we increase the value $(\lambda)$, the backlog function decreases that means lost sale increases, and thus the total cost decreases. As backlog starting after shortage time, so time $T_1$ remains unchanged.

From Table 4 and Figure 6, it is seen that the model is moderately sensitive to shape parameter $(k)$, and if we increase the value $(k)$, the total cost gradually decreases. As $r^k$ decreases due to increase of $(k)$, so the demand rate will decrease, and hence the total cost is decreasing. Also, from Figure 5 and Table 4, it is clear that the model is moderately sensitive to $(u)$ and the total cost gradually increases due to an increase in the value of $(u)$. Holding cost increases as we increase the value of $(u)$ and so the total cost increases.
From Table 3 and Figure 8, the model is moderately sensitive to rate of deterioration ($\theta$), because deterioration rate increases mean the products durable, as well as semi-durable, are deteriorate gradually, and so, in that case the total cost increases, and vice versa.

6.1. Managerial implication. Managerial implication is an integral and important part of any inventory or supply chain problem. Based on our inventory model some implications are there

- This study has considered two types of demand based on the financial situation
- It has presented a deterioration function that follows a probability distribution to make the research a more realistic and comparative one.
- To make the proper decision by the decision-maker, three types of distribution functions (Uniform distribution, Triangular distribution, and Double Triangular distribution) are used to improve the optimal total cost and optimal time.
- Industries can consider the best distribution out of the three distribution for their product to optimize the total profit.
- The inventory level rapidly decreased in the interval $(0, T_1)$ and in the range $(T_1, T_2)$ the inventory level gradually decreased. Industries can identify those times where the demand rate rapidly increased or steadily increased so that the managerial decision can be taken at an appropriate time.
- The proposed inventory system considers holding cost is reliability dependent; the managerial decision will be easy to minimize the holding time for the more reliable product.

7. Concluding remarks. The proposed inventory system has discussed a deterministic inventory model for time, reliability, and advertisement dependent variable demand with shortages due to deterioration and demand. This study has considered two types of demand based on the financial situation, firstly the demand rate is high for a certain period, and secondly, the demand rate is reduced and less compare to initial part of time both are related to cash in hand. This study has presented deterioration function follows probability distribution such as (i) Uniform, (ii) Triangular, and (iii) Double Triangular, to make the research a more realistic and comparative one. The more realistic concept has introduced that the holding cost is reliability dependent, and details discussion and analysis have presented that the reliability dependent holding cost has very sensitive characteristics and impact on the optimal inventory system. The system is allowed shortages with the second type of demand pattern and allowed partially backlogged at a fixed rate.

This paper has considered a deterministic inventory model based on a real management situation for finding the optimal total cost where the time is a decision variable. The main contribution of this inventory model is to find out the minimum cost with two types of variable demand, based on the concept of cash in hand is available. As the demand rate is high for a specified period and less for after period, thus companies are prepared to produce many items for certain period and fewer items for the next period, respectively. So, the companies avoided shortages or less the holding cost. There are two useful lemmas to present the theoretical ideas of the real system that gives the demand rate, and the rate of change of demand rate increases. The third lemma represents the relationship between the initial inventory and the time $T_1$, $T_2$, and the last one gives the condition when the total average cost is minimum. Here three suitable numerical examples have presented,
which contain three different continuous probabilistic deterioration functions, and based on the three numerical examples. To increase profit, companies are decided to reliability dependent holding cost and produce those items whose reliability is high. A numerical comparison between the three models is shown graphically. Another extension of the proposed inventory model is the probabilistic deterioration in the deterministic inventory model, where the variable demand rate is dependent on reliability and advertisement. Therefore, this model will help for the real fact of the manufacturing industry to control their production and gain more profit at the optimum level. Sensitivity analysis reflects that the optimal total cost is very high sensitive on the change in the parameter $C_s$, $C_l$, $A$ and $\nu$, high sensitivity on $T$, $C_d$, $C_h$, $b$, $c$, moderately sensitive $a$, $k$, $\theta$, $u$, $r$ and less sensitivity on $\lambda$.

Furthermore, one can extend the multi-item EPQ model with constraints, probabilistic demand, fuzzy demand, shortages, availability of current stock, supply chain and delay-in-payments, etc. It can further be improved by organizing the defective items that follow some continuous or discrete distribution and the study under fuzzy, interval, or other impreciseness of the parameters of the proposed inventory model.

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