Loan transactions associated to continuous distributions of capital

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Abstract

The objective of this paper is to design a loan repayment plan adapted to the financed company with the aim to avoid a possible situation of illiquidity. More specifically, in this paper the sequence of payments which amortizes a loan is determined according to the distribution of the future income expected by the borrower. To do this, we will start from the concept of a continuous distribution of capital or, equivalently, a continuous annuity. The continuous distribution of capital will be derived from a continuous distribution of probability, to then obtain a discrete annuity which fits the future income of the debtor. On the other hand, it is noteworthy the analysis of the particular case of financial transactions with interest periods of different amplitude, which can facilitate the adjustment of the sequence of instalments to the timing of the real income of the borrower in loan transactions.

Keywords: Mathematics, Economics

1. Introduction

Usually, the instalments used to amortize a loan follow a given regularity in their money amounts and maturities. In most cases, these amounts are constant or variable
in arithmetic or geometric progression whilst the interest periods are of the same length. Nevertheless, in order to avoid a possible situation of illiquidity in the financed company, it is sometimes required an adjustment of payments to the future distribution of expected income by the borrower. For instance, this would be the case of a hotel located in a coast area where most of its revenues are obtained in summer.

Thus, as indicated by Kryklii and Krukhmal (2014), it is determinant to define the problem of the loan management system in the borrowers from the structural and functional integration, trying to minimize the losses which can derive from the relations with the borrowers.

Roughly speaking, Eroglu and Ozturk (2016) stated that the problem of periodic payments in a loan transaction is on the present value of the debt being equal the net present value of periodic payments. Thus, the contrast between loan repayment models is due to the alternation of distribution of periodic payments. That is to say, in most used payment loan methods in which periodic payments are constant, a geometric alternating and an arithmetic alternating series appear. However, borrowers do not wishing to make payments in some periods due to the variability of income over time. In most loan transactions, discrete annuities are used for repaying the principal. These annuities, depending on the finite amplitude of interest periods, can be (Cruz Rambaud and Valls Martinez, 2014) periodic when the amplitude is the same for all periods, or non-periodic when the amplitude of at least one of the interest periods is different from the rest. Contrary to discrete annuities, continuous annuities have infinitesimal interest periods.

In the context of annuities (Gerber, 1997) calculation, most papers have been focused on the interest rate which is considered as a parameter of great relevance because small changes in it may cause big changes in the total annuity value. Consequently, many scholars have concentrated their attention in the randomness of the interest rate when determining the annuities involved in the valuation process. In particular, Liu et al. (2011) analyze the present value of constant, increasing, decreasing, immediate annuities with a random interest rate. However, at this point, we have to highlight the work by Zaks (2001) who modifies some of his previous formulas on annuities under random rates of interest. In the same vein, Date et al. (2007) and Dufresne (2007) state that the use of interest rates, as a part of the stochastic discount factors, allows a more accurate pricing of any operation.

Alternatively and in order to approximate the present value of an annuity, some previous research obtains the moment generating function (De Schepper et al., 1992) and some moment values (Ouyang and Yan, 2003). This will be the research line followed in the present manuscript.
Therefore, the main objective of this paper is to derive the sequence of installments to amortize a loan starting from the concept of a continuous distribution of capital. In this context, a distribution of capital is a set of amounts whose maturities constitute a set $S$ of Borel, contained in the space of all possible maturities (Gil Luezas and Gil Peláez, 1988). On the other hand, a distribution of capital is defined by a function $R(t)$ which indicates the mass of money amount maturing before $t$:

$$R(t) = M(-\infty,t] = M(x \leq t).$$

This mathematical function allows to determine the amount associated to a half-open interval of the type $[t, t + h]$: $$M(t < x \leq t + h] = R(t + h) - R(t).$$

The distributions of capital can be:

- **Discrete**, consisting in a finite set of amounts such as $(a_0, 0), (a_1, 1), (a_2, 2), ..., (a_n, n), ...$

- **Continuous**, defined by a distribution of capital, denoted by $R(t)$, continuous at every instant and whose derivative is continuous, except for a finite number of moments. In continuous distributions, the amount associated to each instant is zero whilst the difference $R(t + h) - R(t)$ defines amount corresponding to the interval $[t, t + h]$.

On the other hand, an annuity is a distribution of capital where each amount is associated with the interest period where it has been generated (Gil Luezas and Gil Peláez, 1988). Consequently, discrete annuities exhibit finite interest periods whilst in continuous annuities the duration is divided into infinitesimal intervals. In the context of this paper, a continuous distribution of capital can be assimilated to a continuous annuity with infinitesimal intervals of the type $[t, t + dt]$ whose associated amount is $R(t + dt) - R(t) = R'(t) dt$

In summary, a continuous annuity is given by a continuous distribution of capital defined in a closed interval $[0, n]$, where $0$ is the commencement and $n$ is the end of the annuity. Assume that the distribution of capital, denoted by $R(t)$, is continuous in the interval $[0, n]$ and its derivative, $R'(t) := C(t)$, labelled as the density function of the annuity, is continuous except for a finite number of instants. Before continuing with the description of our methodology, we are going to introduce some general concepts which are necessary for the development of this manuscript. Let $I$ be an interval (bounded or not) of $\mathbb{R}$ and let us consider the subset $X$ of $I$ given by:
\[ X = \{(t,p) \in \mathbb{R}^2 / t \leq p\}. \]

A capitalizing function is a continuous real-valued function:

\[ F : X \rightarrow \mathbb{R}^+ \]

defined by

\[ (t,p) \mapsto F(t,p), \]

which satisfies the following conditions:

1. \( F(t,t) = 1 \),
2. \( F \) is increasing with respect to \( p \) (called the focal date).

Analogously, let \( I \) be an interval of \( \mathbb{R} \) and let us consider the subset \( Y \) of \( I^2 \) given by:

\[ Y = \{(t,p) \in I^2 / t \geq p\}. \]

A discount function is a continuous real-valued function:

\[ F : Y \rightarrow \mathbb{R}^+ \]

defined by

\[ (t,p) \mapsto F(t,p), \]

which satisfies the following conditions:

1. \( F(t,t) = 1 \),
2. \( F \) is decreasing with respect to \( p \) (called the focal date).

Observe that both the capitalization and the discount function depend on two instants but, usually, this kind of functions is defined by using the time as an interval. However, this is not a problem because we can denote \( p - t \) (or \( t - p \), in the case of a discount function) as \( a \), by obtaining a capitalization (discount) function \( F(t,a) \) depending on \( a \).

Finally, observe that the presence of a focal date is completely necessary when capitalizing or discounting a given amount according a certain criterion. Nevertheless, in the rest of this manuscript this element will not be taken into account because the possible consideration of a focal date is irrelevant when working with the exponential discounting.

Thus, the value of this continuous annuity, defined by its density function \( C(t) \) in the interval \([0,n]\), at a generic instant \( s \) by using a valuation (capitalizing/discount) function \( F(t,p) \) with focal date at \( p \), is:
\[ V_s = \frac{1}{F(s,p)} \int_0^n F(t,p)C(t)\,dt. \]

In particular, the present value \((s = 0)\) is:

\[ V_0 = \frac{1}{F(0,p)} \int_0^n F(t,p)C(t)\,dt, \]

whilst its final value \((s = n)\) is:

\[ V_n = \frac{1}{F(n,p)} \int_0^n F(t,p)C(t)\,dt. \]

After this brief theoretical introduction, let us start with a discrete, periodic, variable and ordinary annuity whose amounts \(a_s (s = 1, 2, \ldots, n)\) are different (see Fig. 1).

The aim of this paper is to increase the cases in which there is certain regularity in the money amounts of an annuity, apart from the traditionally well-known annuities whose amounts are constant or variable in arithmetic or geometric progression. To provide a general solution to this question, we are going to construct a continuous annuity starting from the probability density function \(f(t)\) of a continuous random variable (Mood et al., 1974). As known, the area enclosed by the curve and the abscissa axis is one. Thus, by multiplying the function \(f(t)\) by an amount \(C\), we obtain an annuity whose density function is \(Cf(t)\). Thus, a continuous distribution of capital can be obtained with the shape of a well-known probability distribution (normal, triangular, exponential, etc.) (Papoulis, 1980). The so-constructed distribution of capital has the advantage that the final value of the distribution is given by the product of the amount \(C\) by the moment-generating function, \(MT(x)\), of the random variable \(T\) (Ríos, 1975). Later, the practical application we are going to implement is based on the relationship between the most commonly used continuous probability distributions (rectangular, normal, etc.) and the structure of the instalments of a loan or savings transaction.

![Fig. 1. Graphic representation of a discrete, periodic, variable and ordinary annuity. Source: Own elaboration.](https://doi.org/10.1016/j.heliyon.2018.e00859)
Consequently, it is useful to accurately estimate borrowing constraints and to determine the availability of credit, the “loan frontier”, depending on the characteristics of the borrower. This loan will be in line with the patterns of institutional constraints and other aggregate measures of availability, having an outstanding role in economic activity (Anenberg et al., 2017).

On the other hand, Chakroun and Abid (2016) have observed the importance in bank risk management, selecting an optimal initial loan portfolio based on the possibility of default by the borrower and, thus, the need to invest in assets with an acceptable level of risk.

Thus, Amat et al. (2017) evaluate the probability of default on loans to help financial institutions to classify their clients and improve the efficiency of lending. In addition, improving the probability of default can bring benefits to the lender. Therefore, statistical techniques are used basing on information about the borrower and the characteristics of specific loans in previous transactions, such as payments made or default, in order to predict the risk of the financial transaction.

The organization of this paper is as follows: In Section 2, the concepts of the mathematical and statistical magnitudes to be used are presented, in addition to the general setting which will be applied in different cases. In Section 3, the methodology is presented for a discrete income from a constant continuous distribution of capital. In Section 4, a discrete income to repay a loan from a normal probability distribution is obtained. Finally, Section 5 summarizes and concludes.

2. Background

A continuous distribution of capital is a continuous function \( R(t) \) whose derivative \( R'(t) = C(t) \), called the density function of capital, is also continuous except for a finite number of instants. Figs. 2 and 3 display two continuous distributions of capital (in green) and their respective density functions (in red). Observe the similarity of these concepts with the continuous distribution \( F(t) \) and density function of probability \( f(t) \) with the only difference that \( F(-\infty) = 0 \) and \( F(+\infty) = 1 \) (see Fig. 4).

![Fig. 2](https://example.com/f2.png)

**Fig. 2.** A continuous, non-differentiable, distribution and density function of capital.

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2.1. Methodology

The methodology we are going to follow in sections 2 and 3 can be systematized in the following steps:

1. We start from a continuous distribution of capital (or, what is the same, from a continuous annuity) whose density function (Martín-Pliego and Ruiz-Mayá, 2006) is given by a known probability density function multiplied by an amount \( C \). So we will be able to work with continuous annuities which are well adapted to the cash availability of the borrower over time. Recall that the total area enclosed by the density function \( f(t) \) of a continuous probability distribution and the abscissa axis is 1 (see Fig. 5).

As indicated, by multiplying the probability density function by an amount \( C \), we obtain the density function \( C(t) := Cf(t) \), where, obviously, now the area enclosed by the curve and the abscissa axis is \( C \) (see Fig. 6).

2. We will transform the continuous distribution of capital into an equivalent discrete ordinary annuity, using the property of condensation of the total amount

![Fig. 3. A continuous and differentiable distribution and density function of capital.](image1)

![Fig. 4. A continuous distribution and density function of probability.](image2)
enclosed by the distribution in \( n \) intervals (Gil Luezas and Gil Peláez, 1988). In effect, starting from a density function \( C(t) \) (which will be assumed to be asymmetric on the left), we will apply the continuous capitalization at every instant of each subinterval \( [t_{s-1}, t_s] \) of the total interval \( [t_0, t_n] \) to obtain the instalment \( a_{s-1,s} \) corresponding to the upper endpoint of the interest period (see Fig. 7).

Thus, for a generic subinterval \( [t_{s-1}, t_s] \), one has:

\[
a_{s-1,s} = \int_{t_{s-1}}^{t_s} C(t) (1 + i)^{t_s - t} dt, \quad s = 1, 2, \ldots, n,
\]

or

\[
a_{s-1,s} = C \int_{t_{s-1}}^{t_s} f(t) (1 + i)^{t_s - t} dt, \quad s = 1, 2, \ldots, n,
\]  

(1)
where it has been supposed that instants $t_1$, $t_2$, ..., $t_{n-1}$ lead to a partition of the total interval $[t_0, t_n]$ into $n$ subintervals. Moreover, the used valuation function has been the exponential discount of interest rate $i$. Observe that the transformation of a continuous annuity into an equivalent discrete annuity condenses the amount of each subinterval of the distribution of capital at its upper endpoint. However, this methodology can be similarly used by condensing the amount at another point of each subinterval (lower endpoint, middle point, etc.). As indicated, from a practical point of view, this construction is interesting for the borrower or debtor because he/she can adapt the payments to the dynamics of his/her expected income. The same reasoning can be applied to savings transactions.

3. We determine the final value of the income or, what is the same, the final value of the continuous distribution of capital as the moment-generating function of the probability distribution multiplied by the amount $C$. In effect, the moment-generating function of a random variable $T$ with continuous probability distribution in the interval $[t_0, t_n]$ is given by (Billingsley, 1995):

$$M_T(x) = E(e^{xT}) = \int_{t_0}^{t_n} f(t) e^{xt} dt.$$ 

Therefore, the final value of continuous income is:

$$V_n = C \int_{t_0}^{t_n} f(t)(1 + i)^{t_n-t} dt =$$
\[ V_0 = C \int_{t_0}^{t_n} f(t)(1 + i)^{-t} dt = C(1 + i)^b \int_{t_0}^{t_n} f(t)(1 + i)^{-t} dt = \]

\[ = C(1 + i)^b \int_{t_0}^{t_n} f(t) e^{-\ln(1 + i)t} dt = C(1 + i)^b M_T(-\rho), \]

where \( \rho = \ln(1 + i) \) is the instantaneous discount rate of the exponential discount function with interest rate \( i \). On the other hand, the present value of the continuous income could also be calculated as the moment-generating function of the probability distribution multiplied by the amount \( C \), as follows:

\[ V_s = C \int_{t_0}^{t_n} f(t)(1 + i)^{-(t-s)} dt \]

Our paper has been focused on the calculation of the present and the final values but the same reasoning could be applied to calculate the value of the annuity at any instant \( s \), as shown by the following expression:

The advantage of using probability distributions is that the moment-generating functions of the well-known probability distributions are tabulated in textbooks of Mathematical Statistics (Fisz, 1963). Moreover, in this novel context, certain statistical parameters make new sense. For example, if the statistical distribution corresponding to the distribution of capital exhibits a positive (resp. negative) coefficient of asymmetry, the loan repayments are concentrated on the left (resp. on the right) of the time interval. On the other hand, if the statistical distribution shows a positive kurtosis coefficient, the greater repayments are very high in comparison with the others, whilst the repayments in the tails of the time interval are very low. Contrarily, a negative kurtosis coefficient means a certain uniformity of repayments in the entire interval.
4. Making the final value obtained in the step #3 equal to the final value of the loan principal or to the savings amount, the value of $C$ can be determined and then the value of all payments or savings quotas, respectively (because they depend on $C$).

### 2.2. Theory/calculation

Hereinafter, we will focus on loan transactions. In this case, the generic payment

$$a_{r,s} = C \int_{t_r}^{t_s} f(t)(1+i)^{t-s} \, dt$$

is a generalization of Eq. (1) and will be named the *payment from $t_r$ to $t_s$*, or the *payment corresponding to the subinterval $[t_r,t_s]$*, $t_0 \leq t_r \leq t_s \leq t_n$. Observe that Eq. (2) has the advantage that $t_1, t_2, \ldots, t_{n-1}$ can be any points located between $t_0$ and $t_n$ chosen without regularity. The payment from $t_r$ to $t_s$ satisfies the following properties:

- $a_{s,s} = 0$, for every $t_s \in [t_0, t_n]$.
- $a_{0,n} = C_0(1+i)^{t_n-t_0}$.
- $a_{r,s} = \sum_{k=r}^{s-1} a_{k,k+1}(1+i)^{t_k-t_{k+1}}$. In particular, if $t_0 = 0$, $t_n = n$ and the partition is given by the instants 1, 2, ..., $n-1$, one has:

$$a_{r,s} = \sum_{k=r}^{s-1} a_{k+1}(1+i)^{t_{k+1}-t_k} \, ,$$

where $a_{k+1} := a_{k,k+1}$, which coincides with the payment calculated following the traditional process of repayment. The calculation of the outstanding principal (on the right) at instant $t_s$ can be calculated by using the following expressions:

- $C_s = C_0(1+i)^{t_s-t_0} - a_{0,s}$ (retrospective method),
- $C_s = a_{s,s}(1+i)^{t_{s+1}-t_s}$ (prospective method), or
- $C_s = C_r(1+i)^{t_s-t_r} - a_{r,s} (r < s)$ (recursive method).

Finally, according to Eq. (3), one has:

$$C_r - C_s = a_{r,s} - C_r[(1+i)^{t_s-t_r} - 1] \, .$$

Denoting

$$A_{r,s} := C_r - C_s$$

and

$$I_{r,s} := C_r[(1+i)^{t_s-t_r} - 1] \, ,$$

...
one has \( a_{r,s} = A_{r,s} + I_{r,s} \). The parameter \( A_{r,s} \) will be called the repaid principal from \( t_r \) to \( t_s \), and \( I_{r,s} \) will be named the interest due from \( t_r \) to \( t_s \).

3. Methodology

In this section, we are going to calculate the payments derived from a continuous distribution of capital which is constant in its defining interval. To do this, we will strictly follow each of steps described in Section 2.

**Step #1.** We start from the probability density function of the uniform or rectangular distribution defined in the interval \([t_0, t_n] \) (see Fig. 8).

Multiplying \( f(t) \) by \( C \), we obtain a continuous distribution of capital defined in the interval \([t_0, t_n] \) by the constant amount \( kC \).

**Step #2.** Then we are going to calculate the amount associated with each subinterval of the total interval \([t_0, t_n] \) by using the distribution of capital obtained in Step #1, thus obtaining an equivalent constant discrete annuity. To do this and based on the condensation property at an instant of the total amount of a distribution of capital (Gil Peláez, 1987), the process consists in capitalizing the amounts maturing in the subinterval \([t_{s-1}, t_s] \) up to the final instant \( t_s \) (see Fig. 9).

The amount \( a_{s-1,s} \), associated to the subinterval \([t_{s-1}, t_s] \), will be:

\[
a_{s-1,s} = \int_{t_{s-1}}^{t_s} kC(1 + i)^{t_s-t} \, dt = kC \int_{t_{s-1}}^{t_s} (1 + i)^{t_s-t} \, dt =
\]

![Fig. 8. Density function of the uniform distribution. Source: Own elaboration.](https://doi.org/10.1016/j.heliyon.2018.e00859)
\[ kC \left( \frac{1+i}{\ln(1+i)} \right)_{t_{s-1}}^{t_s} = kC \frac{1-i^{h_{s-1}}}{\ln(1+i)} + kC \frac{1-i^{h_{s-1}}}{\ln(1+i)} \]

from which

\[ a_{s-1,s} = kC \frac{(1+i)^{h_{s-1}} - 1}{\rho}, \]

where \( \rho \) is the interest rate in continuous capitalization, which coincides with the result obtained by Gil Luezas and Gil Peláez (1988). It is noteworthy to point out that \( a_{s-1,s} \) takes the same value in all subintervals of the same amplitude, and that it does not have to be greater than \( k \) because its value depends on both \( t_s \), \( t_s \), and \( i \).

**Step #3.** As the moment-generating function of the uniform distribution is \( M_T(x) = \frac{e^{xt_0} - e^{xt_1}}{x(t_1 - t_0)} \), the final value of the annuity is:

\[ V_n = C(1+i)^{h_n} M_T(-\rho) = C \frac{(1+i)^{h_n} - 1}{\rho(t_n - t_0)}. \]

We can also obtain the present value of the annuity, as follows:

\[ V_0 = C(1+i)^{h_0} M_T(\rho) = C \frac{1 - (1+i)^{-(t_n-t_0)}}{\rho(t_n - t_0)}. \]

**Step #4:** Let us calculate the loan instalments in the particular case in which \( t_0 = 0, \quad t_1 = 1, \quad \ldots, \quad t_n = n \). In this case, making the final value obtained in Step #3 equal to the final value of the loan principal, one has:
\[
C \frac{(1+i)^n - 1}{\rho n} = C_0 (1+i)^n,
\]
from which
\[
C = \frac{C_0 (1+i)^n \rho n}{(1+i)^n - 1}.
\]
Consequently, as \( k = 1/n \), the value of the constant instalment is:
\[
a = \frac{kCi}{\rho} = \frac{C_0 i}{1 - (1+i)^{-n}},
\]
expression which coincides with the constant instalment obtained by the French method of loan amortization. In the same way, the instalments could have calculated starting from the present value of the annuity:
\[
C \frac{1-(1+i)^{-n}}{\rho n} = C_0.
\]
As indicated in the Introduction, the methodology presented here has the advantage that the loan interest periods do not necessarily have the same amplitude.

### 3.1. Example

Let us suppose that the length of each interest period is the double of that the former one (if this is possible, that is to say, when the upper endpoint of the last interest period coincides with the end of the loan transaction) (see Fig. 10).

In this case, by writing the expression of two consecutive instalments, one has:
\[
a_{s-1,s} = kC \frac{(1+i)^{t_{s-1} - t_s} - 1}{\rho}
\]
and
\[
a_{s,s+1} = kC \frac{(1+i)^{t_{s+1} - t_s} - 1}{\rho}.
\]

By dividing the respective hand-left and hand-right sides of the former equalities, one has:
\[
\frac{a_{s,s+1}}{a_{s-1,s}} = \frac{(1+i)^{t_{s+1} - t_s} - 1}{(1+i)^{t_{s-1} - t_s} - 1}.
\]

\[
\begin{align*}
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad \ldots \\
a_{0,1} & \quad a_{1,2} & \quad a_{2,3} & \quad \ldots
\end{align*}
\]

Fig. 10. Loan schedule with interest periods of variable amplitude. Source: Own elaboration.
from which, by simplifying and according to hypothesis about the lengths of the interest periods, we obtain the following recurrence relationships:

$$a_{s+1} = a_{s-1}[(1+i)^r - t_{r-1} + 1].$$

Therefore, once determined the first instalment:

$$a_{0,1} := a = \frac{kCi}{\rho} = \frac{C_0i}{1 - (1+i)^{-\pi}},$$

it follows:

$$a_{1,2} = a_{0,1}[(1+i) + 1],$$
$$a_{2,3} = a_{1,2}[(1+i)^2 + 1],$$
$$a_{3,4} = a_{2,3}[(1+i)^4 + 1],$$

Writing the obtained instalments depending on $a$, we can observe that this loan category is a variant of the French method where several instalments have been condensed to obtain a new one:

$$a_{1,2} = a(1+i) + a,$$
$$a_{2,3} = a(1+i)^3 + a(1+i)^2 + a(1+i) + a,$$

In effect, $a_{1,2}$ is the condensation at instant 3 of instalments $(a, 2)$ and $(a, 3)$. Likewise, $a_{2,3}$ is the condensation at instant 7 of $(a, 4)$, $(a, 5)$, $(a, 6)$ and $(a, 7)$, and so on. Finally, for the case when $C_0 = $100,000 and $i = 3\%$, the loan amortization schedule (see Table 1) can be obtained by using the mathematical formulas derived in this section and in Section 2.

Analogously, by applying the steps from #1 to #4, we can obtain the instalments for a linear and an exponential distribution. Example 1 could be of particular interest for those companies which obtain most profits at the beginning of their economic activity, moderating later their outcomes. This could be the case of companies belonging to the artificial intelligence sector. This way these companies may fit the receipt of profits to the loan payments by increasing the amplitude of intervals.

**Table 1. Amortization schedule of Example 1. Source: Own elaboration.**

| $t_s$ | $a_{s-1}s$ | $A_{s-1}s$ | $L_{s-1}s$ | $C_s$ | $M_s$ |
|-------|-----------|-----------|-----------|-------|-------|
| 0     | -         | -         | -         | 100,000.00 | 0.00 |
| 1     | 11,682.95 | 3,682.95 | 8,000.00 | 96,317.05 | 3,682.95 |
| 3     | 24,300.55 | 8,273.39 | 16,027.16 | 88,043.66 | 11,956.34 |
| 7     | 52,644.70 | 20,905.94 | 31,738.77 | 67,137.72 | 32,862.28 |
| 15    | 124,267.24 | 67,137.72 | 57,129.52 | 0.00 | 100,000.00 |
4. Model

In this section, we are going to apply the general methodology described in Section 2 to a normal probability distribution defined by its density function.

**Step #1:** We start from the probability density function of the normal distribution, defined by
\[ f(t) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(t-m)^2}{2\sigma^2}}, \]
where \( t \) is the instant at which the payment will be made, and whose parameters are the mean time \((m)\), which coincides with the modal value and the median of the variable, such that \(-\infty < m < +\infty\), and the standard deviation \( \sigma \) which represents the volatility of the variable, being \( \sigma > 0 \) (see Fig. 11).

In this case, the random variable “time” follows a normal distribution denoted by \( N(m, \sigma) \), where \( m \) is the mean and \( \sigma \) is the standard deviation. As observed in Fig. 8, the normal curve is symmetric and asymptotic, indicating that the curve extends infinitely in both directions of the abscissa axis. Multiplying \( f(t) \) by an amount \( C \), we obtain the density function \( C(t) \), whose graphic representation has been displayed in Fig. 12. We would be assuming, for instance, the case of the distribution of payments for a six-year loan transaction.

**Step #2:** We can calculate the instalments associated to all subintervals. In this case, the first payment, \( a_{-\infty,1} \), has been calculated by condensing, at time \( t_1 \), the amount corresponding to the subinterval \([-\infty, t_1]\). Thus, one has:
\[ a_{-\infty,1} = \frac{1}{\sigma \sqrt{2\pi}} C \int_{-\infty}^{t_1} e^{-\frac{(t-m)^2}{2\sigma^2}} e^{(t_1-t)} dt, \]

or equivalently:
\[ f(t) \]

![Fig. 11. Density function of the normal distribution with mean \( m \) and standard deviation \( \sigma \). Source: Own elaboration.](https://doi.org/10.1016/j.heliyon.2018.e00859)
Observe that, in order to easily calculate with the density function of a normal distribution, we have used the discount factor in the format $e^{-kt}$ instead of $(1+i)^{-t}$. It is well known that the relationship between $k$ and $i$ is $k = \ln(1+i)$.

Therefore, leaving provisionally aside the factor $\frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(t_0-\mu)^2}{2\sigma^2}}$, we are going to proceed to solve the integral by the method of obtaining a perfect square in a generic subinterval $[t_r, t_s]$:

$$\int_{t_r}^{t_s} e^{-\frac{(t_0-\mu)^2}{2\sigma^2}} e^{-kt} \, dt = \int_{t_r}^{t_s} e^{-\frac{t^2-2tm(t_0-\mu)^2+2tm\mu}{2\sigma^2}} \, dt =$$

$$= \int_{t_r}^{t_s} e^{-\frac{(-t_0+\mu)^2}{2\sigma^2}} \, dt = \int_{t_r}^{t_s} e^{-\frac{(-t_0+\mu)^2}{2\sigma^2}} \, dt =$$

$$= \int_{t_r}^{t_s} e^{-\frac{(t_0-\mu)^2}{2\sigma^2}-2\mu \frac{t_0-\mu}{\sigma^2}} \, dt = \int_{t_r}^{t_s} e^{-\frac{(t_0-\mu)^2}{2\sigma^2}} \, dt =$$

$$= \int_{t_r}^{t_s} e^{-\frac{(t_0-\mu)^2}{2\sigma^2} + \frac{(2tm)(t_0-\mu)^2}{2\sigma^2}} \, dt = \int_{t_r}^{t_s} e^{-\frac{(t_0-\mu)^2}{2\sigma^2}} \, dt =$$

$$= \int_{t_r}^{t_s} e^{-\frac{(t_0-\mu)^2}{2\sigma^2} + \frac{2tm(t_0-\mu)^2}{\sigma^2}} \, dt = \int_{t_r}^{t_s} e^{-\frac{(t_0-\mu)^2}{2\sigma^2}} \, dt =$$

$$= e^{-\frac{(t_0-\mu)^2}{2\sigma^2}} \int_{t_r}^{t_s} e^{-\frac{(t_0-\mu)^2}{2\sigma^2}} \, dt,
$$

from which, by considering the first summand and the factor, one has:
where the integrand multiplied by the factor $\frac{1}{\sqrt{2\pi}}$ is the density function of a normal distribution $N(t_m - k\sigma^2, \sigma)$. Definitively, the instalment is:

$$a_{rs} = Ce^{-\frac{(2t_m - 2t_r - k\sigma^2)^2}{2\sigma^2}} \int_{t_r}^{t_m} e^{-\frac{(t - t_m)^2}{2\sigma^2}} \, dt,$$

where $N$ denotes the distribution function of the normal $N(t_m - k\sigma^2, \sigma)$. Therefore, the instalments, from $a_{-\infty, 1}$ to $a_{4.5}$, are as follow:

$$a_{-\infty, 1} = Ce^{-\frac{(2t_m - 2t_1 - k\sigma^2)^2}{2\sigma^2}} [N(t_1) - N(-\infty)],$$

$$a_{1.2} = Ce^{-\frac{(2t_m - 2t_2 - k\sigma^2)^2}{2\sigma^2}} [N(t_2) - N(t_1)],$$

$$\vdots$$

$$a_{4.5} = Ce^{-\frac{(2t_m - 2t_5 - k\sigma^2)^2}{2\sigma^2}} [N(t_5) - N(t_4)],$$

The last instalment is the sum of $a_{n-1,n}$, which is the result of capitalizing the elemental amounts from $t_{n-1}$ to $t_n$, and $a_{n,+\infty}$, which is the result of discounting the elemental amounts from $t_n$ to $+\infty$. Thus, the general expression is:

$$a_{n-1,n} + a_{n,+\infty} = Ce^{-\frac{(2t_m - 2t_n - k\sigma^2)^2}{2\sigma^2}} [N(+\infty) - N(t_{n-1})].$$

In our example, one has:

$$a_{5.6} + a_{6,+\infty} = Ce^{-\frac{(2t_m - 2t_6 - k\sigma^2)^2}{2\sigma^2}} [N(+\infty) - N(t_5)].$$

**Step #3:** The moment-generating function of a normal distribution takes the following expression:

$$M_T(k) = e^{t_m k + \frac{k^2\sigma^2}{2}}.$$ 

Therefore, by applying the formula obtained in Section 2, the final value of the annuity would be:

$$V_n = C(1 + i)^{t_n} e^{t_m k + \frac{k^2\sigma^2}{2}} = C(1 + i)^{t_n} e^{k \left( t_m + \frac{\sigma^2}{2} \right)},$$

or equivalently:

$$V_n = C(1 + i)^{t_n - t_m} e^{\frac{2\ln(1+i)}{2}}.$$
Moreover, the capitalized value of the loan principal is:

\[ V_n = C_0(1 + i)^{n-t_0}. \]

**Step 4:** Making equal both expressions of the final value, one has:

\[ C_0(1 + i)^{n-t_0} = C(1 + i)^{n-t_m + \frac{\sigma^2 \ln(1+i)}{2}}, \]

from which \( C \) can be obtained according to \( C_0 \):

\[ C = \frac{C_0(1 + i)^{n-t_0}}{(1 + i)^{n-t_m + \frac{\sigma^2 \ln(1+i)}{2}}} \]

which allows us to substitute its value in the previously calculated instalments, and to translate the results to the amortization schedule just like in previous sections.

### 4.1. Example

Let us suppose that the six-year amortization schedule follows a normal probability distribution. The probability \( N(1) \) of the normal distribution

\[ N(t_m - k\sigma^2, \sigma) \approx N(3 - 1.71^2 \ln(1.03); \ 1.71) \]

is

\[ P(T \leq 1) = P\left( \frac{T - t_m}{\sigma} \leq \frac{1 - (3 - 1.71^2 \ln(1.03)}{1.71} \right) = \]

\[ = P\left( Z \leq \frac{-1.91}{1.71} \right) = P(Z \leq -1.12) = 1 - P(Z \leq 1.12) = 0.1314. \]

As \( N(2) = 0.2946 \), \( N(3) = 0.5199 \), \( N(4) = 0.7389 \), \( N(5) = 0.8888 \), \( N(6) = 0.9649 \), \( N(+ \infty) = 1 \), and \( N(- \infty) = 0 \), then we can calculate the ordinary instalments associated to all subintervals. Thus, the instalment \( a_{\infty,1} \), associated to the upper endpoint of the subinterval \([ - \infty, 1] \), (interest rate: 3%) is:

\[ a_{\infty,1} = 0.1314 C(1.03)^\frac{-6-2-1.71^2 \ln(1.03)}{2}, \]

from which \( a_{\infty,1} = \$0.12401502 \ C \). The amount \( a_{1,2} \) associated to the upper endpoint of the subinterval \([1, 2]\) is:

\[ a_{1,2} = C(1.03)^\frac{-6-4-1.11^2 \ln(1.03)}{2}(0.2946 - 0.1314) = \$0.15864862 \ C. \]

Analogously, we can deduce that the third instalment is \( a_{2,3} = \$0.22558726 \ C \), the instalment associated to the subinterval \([3, 4]\) is \( a_{3,4} = \$0.2258576 \ C \), and the fifth
instalment is \( a_{4,5} = 0.15923167 \) C. Finally, the last instalment, \( a_{5,6} \), associated to the upper endpoint of the subinterval \([5, 6]\) is:

\[
a_{5,6} + a_{6, +\infty} = C(1.03)^{-\frac{6-2.172\ln(1.03)}{2}}(1 - 0.8888),
\]

from which \( a_{5,6} + a_{6, +\infty} = 0.12166617 \) C. Now, \( C \) can be calculated according to \( C_0 \):

\[
C = \frac{1.03^6 C_0}{1.03^{6-3} \ln(1.03)^{1}}
\]

that is to say

\[
C = 1.09133556 \ C_0.
\]

By substituting the value of \( C \) in the previously calculated instalments, one has:

\[
a_{-\infty,1} = 0.1353420000 \ C_0.
\]

\[
a_{1,2} = 0.1731388800 \ C_0.
\]

\[
a_{2,3} = 0.2461913931 \ C_0.
\]

\[
a_{3,4} = 0.2464864294 \ C_0.
\]

\[
a_{4,5} = 0.1737751837 \ C_0.
\]

\[
a_{5,6} + a_{6, +\infty} = 0.1327786154 \ C_0.
\]

Table 2 shows the loan amortization schedule for \( C_0 = 100,000 \) by using the mathematical formulas obtained in this section.

Definitively, in the former examples it has been proved that, starting from a normal probability distribution, we have determined an equivalent discrete annuity with an

**Table 2. Amortization schedule of Example 2. Source: Own elaboration.**

| S  | \( a_{-\infty,1} \) | \( A_{1,1} \) | \( I_{1,1} \) | \( C_1 \) | \( M_1 \) |
|----|---------------------|--------------|-------------|--------|--------|
| 0  | -                   | -            | -           | 100,000.00 | 0.00   |
| 1  | 13,534.20           | 10,534.20    | 3,000.00    | 89,465.80 | 10,534.20 |
| 2  | 17,313.89           | 14,629.91    | 2,683.97    | 74,835.89 | 25,164.11 |
| 3  | 24,619.14           | 22,374.06    | 2,245.08    | 52,461.82 | 47,538.18 |
| 4  | 24,648.64           | 23,074.79    | 1,573.85    | 29,387.04 | 70,612.96 |
| 5  | 17,377.52           | 16,495.91    | 881.61      | 12,891.13 | 87,108.87 |
| 6  | 13,277.86           | 12,891.13    | 386.73      | 0.00     | 100,000.00 |
even number of instalments with which to repay the loan principal. On the other hand, when the annuity has an odd number of instalments, the procedure would be similar to the case of an even number of instalments.

By continuing with the case of an even number of instalments, observe that instalments in Table 2 are not symmetric. However, we can use the following instalments instead:

\[
da'_1 = a'_6 = \frac{(1 + i)^5 a_1 + a_6}{(1 + i)^5 + 1},
\]

\[
da'_2 = a'_5 = \frac{(1 + i)^4 a_2 + (1 + i)a_5}{(1 + i)^4 + (1 + i)},
\]

and

\[
da'_3 = a'_4 = \frac{(1 + i)^3 a_3 + (1 + i)^2a_4}{(1 + i)^3 + (1 + i)^2},
\]

where \(a'_1 = a_{-1,1}, a'_2 = a_{1,2}, a'_3 = a_{2,3}, a_4 = a'_{3,4}, a'_5 = a_{4,5}, \text{ and } a'_6 = a_{5,6} + a_{6,\infty} \).

Table 3 shows the loan amortization schedule for the case \(C_0 = $100,000\) by using the expressions to obtain the symmetric instalments.

### 5. Conclusion

Loan transactions associated to continuous distributions of capital offer an alternative to companies and individuals with financial difficulties to pay for the regular instalments to amortize a loan due to the seasonality of certain businesses or economic activities.
In order to solve this situation, in this paper we have obtained a discrete income adapted to the cash availability of the borrower over time. In this way, the loan will be amortized starting from a continuous distribution of capital (or, what is the same, from a continuous annuity) $C(t)$ obtained by multiplying a well-known probability distribution $f(t)$ by a given amount $C$, resulting in $C(t) = Cf(t)$.

Based on this general approach, we have obtained the amounts of a discrete annuity fitted to a uniform and a normal distribution. This methodology allows us to obtain other repayment models, in addition to those models in which annuities are constant or variable in arithmetic and geometric progression.

This solves the problem of randomness in the amounts of the annuity used to repay the loan due to the irregular revenues expected by the borrower. In this paper, we obtain the mathematical expressions of the corresponding instalments and the expression of payments when the interest periods have different amplitude.

**Declarations**

**Author contribution statement**

Salvador Cruz Rambaud, María del Carmen Valls Martínez, Emilio Abad Segura: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

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The authors declare no conflict of interest.

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