Median Statistics Estimate of the Galactic Rotational Velocity

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Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT
We compile a complete collection of 19 recent (since 2000) measurements of Θ₀, the rotational velocity of the Milky Way at R₀ (the radial distance of the Sun from the Galactic center). These measurements use tracers that are believed to more accurately reflect the systematic rotation of the Milky Way. Unlike other recent compilations of Θ₀, our collection includes only independent measurements. We find that these 19 measurements are distributed in a mildly non-Gaussian fashion and a median statistics estimate indicates Θ₀ = 220 ± 10 km s⁻¹ (2σ error) as the most reliable summary, at R₀ = 8.0 ± 0.3 kpc (2σ error).

Key words: (cosmology:) cosmological parameters – Galaxy: fundamental parameters – Galaxy: kinematics and dynamics – Galaxy: structure – methods: data analysis – methods: statistical

1 INTRODUCTION
A more accurate model of the Milky Way will improve the accuracy of inter- and extra-galactic measurements. Two constants play a fundamental role in describing the current model of the Milky Way: R₀ (the radial distance of the Sun to the Galactic center, Sgr A*) and Θ₀ (the rotational velocity of the Milky Way at R₀). Camarillo et al. (2018, hereafter C18) have recently measured R₀ from a carefully compiled set of independent R₀ data points. Earlier compilations of data used to estimate R₀ included non-independent measurements. C18 found that the data set they put together of 28 independent R₀ results published during 2011–17 was somewhat non-Gaussian and the results of their median statistics analysis are reasonably well summarized as R₀ = 8.0 ± 0.3 kpc (2σ error). After C18 appeared, J. Vallée encouraged us to perform a similar analysis for Θ₀.

There have been three recent attempts at measuring Θ₀ from compilations of measurements: Vallée (2017, hereafter V17), de Grijs & Bono (2017, hereafter dGB17), and Rajan & Desai (2018, hereafter RD18). These analyses use compilations that include non-independent measurements which can significantly affect the results and render them unreliable. In this paper we first put together a collection of 29 independent estimates of Θ₀ that have been published in 2000 or later. Of these 29 measurements, 19 correspond to tracers (such as CO and H I gas clouds) that are believed to more accurately reflect the systematic rotation of the Milky Way; these are the ones we use to estimate Θ₀. We find that this collection of 19 measurements is somewhat non-Gaussian so a median statistics analysis (Gott et al. 2001) is needed for a more reliable estimate of Θ₀. Using median statistics we find Θ₀ = 219.70 ± 8.67 ± 8.77 km s⁻¹ (1σ and 2σ error bars) which for most purposes can be summarized as Θ₀ = 220 ± 10 km s⁻¹. Given the extent to which our data compilation is only mildly non-Gaussian, it is likely that undiscovered systematic errors will not significantly change these estimates and Θ₀ = 220±10 km s⁻¹ (2σ error) probably provides the most reliable estimate.

In § 2 we discuss our compilation of recent independent Θ₀ measurements and how it differs from those of V17 and dGB17. In § 3 we summarize the central estimates statistics and the tests of Gaussianity. We present and discuss our results § 4. We conclude in § 5.

2 DATA
Table 1 lists the Θ₀ data we use in our analyses here. These are from measurements published in or after 2000 and we believe this is an exhaustive list of all such independent measurements.

In all cases the angular velocity ω₀i = Θ₀i/R₀i was what was measured, so we list Θ₀i ± σΘ₀i and R₀i ± σR₀i in this table. σΘ₀i and/or σR₀i are not listed in Table 1 if these are not given in the cited reference. In C18 we measure R₀±σR₀ = 7.96 ± 0.17 kpc. We use these measurements to compute the
Table 1. Independent $\Theta_0$ measurements since 2000

| Radius (kpc) | $\Theta_0$ (km s$^{-1}$) | Rescaled $\Theta_0$ (km s$^{-1}$) | Reference | Tracer Type | Notes |
|-------------|--------------------------|----------------------------------|-----------|-------------|------|
| 6.72 ± 0.39 | 203.35 ± 12.00           | 240.87 ± 20.59                  | Brumlik (2014) | Young | Table 3, Hipparcos catalog, 6208 OB stars |
| 7.62 ± 0.32 | 205.00                   | 214.15 ± 10.09                  | Battinelli et al. (2013) | Old | Figure 3, 4400 carbon stars |
| 7.84 ± 0.32 | 217.00 ± 11.00           | 226.05 ± 15.63                  | Bobylev & Bajkova (2013) | Young | Table 3, Cepheids near SLS UCA4 |
| 7.97 ± 0.15 | 226.80 ± 4.20            | 228.52 ± 7.69                   | McMillan (2017) | Both | In Abstract, from alternative mass model |
| 7.98 ± 0.79 | 238.54 ± 11.66           | 239.84 ± 24.76                  | Shen & Zhang (2010) | Young | From Hipparcos Cepheids |
| 8.00        | 220.80 ± 13.60           | 219.70 ± 14.32                  | Bedin et al. (2003) | Old | From WFCPC2/HST photometry on M4 globular cluster |
| 8.00 ± 0.50 | 202.70 ± 24.70           | 201.69 ± 27.95                  | Kalirai et al. (2004) | Old | From HST on M4 globular cluster, independent of Bedin et al. (2003) |
| 8.00 ± 0.50 | 235.00 ± 15.00           | 234.82 ± 21.52                  | Reid & Brunthaler (2004) | Old | From VLBA proper motion around Sgr A* |
| 8.00 ± 0.40 | 208.50 ± 20.00           | 207.96 ± 20.39                  | Xu et al. (2006) | Old | Averaged from Table 3 for range 7.5–8.5 kpc |
| 8.00 ± 0.40 | 243.50 ± 13.00           | 242.29 ± 15.93                  | Yuan et al. (2008) | Old | From Hipparcos–M giants |
| 8.00        | 226.84                   | 225.71 ± 4.82                   | Sharms et al. (2011) | Old | From comparing galaxy model to Hipparcos, General-Copenhagen survey, and SDSS and 6DSS catalogs, see Chapter 3, SBEI's 2001–2006 and SDSS G-type dwarfs, SBEI's 2007–2008, 3D-Galaxy data from spectroscopic binaries, in Results section, 3D-Galaxy data from SDSS data, 2013 |
| 8.00        | 218.00 ± 10.00           | 216.91 ± 10.98                  | Boyer & Rix (2013) | Old | Data from spectroscopic binaries, in Results section |
| 8.00 ± 0.40 | 204.00 ± 14.00           | 203.83 ± 18.82                  | Bobylev & Bajkova (2015) | Young | Old From Hipparcos Cepheids, SEGUE and 2 CO cloud catalogs, independent of any of the individual measurements, but like v17 they also do not study a compilation of independent measurements, this lack of independence can bias results. Here we have invested significant effort in compiling a collection of independent $\Theta_0$ measurements published during 2000–2017. |
| 8.00 ± 0.40 | 200.00 ± 18.55           | 200.95 ± 19.43                  | Bobylev & Bajkova (2015) | Young | Old From 20 figure 8, from thin disk stars in Gaia-ESO survey |
| 8.00 ± 0.40 | 236.00 ± 18.55           | 237.42 ± 19.52                  | Aumer & Schönrich (2015) | Young | Old From 2004 on SDSS photometry and spectrometry, USN0-B stars and range 7.5–8.5 kpc |
| 8.00 ± 0.40 | 226.00 ± 17.00           | 227.42 ± 18.92                  | Aumer & Schönrich (2015) | Young | Old From SDSS photometry and spectrometry, USN0-B stars and range 7.5–8.5 kpc |
| 8.00 ± 0.40 | 215.00 ± 24.00           | 208.71 ± 29.67                  | Nikiforov (2000) | Old | Old From 2005 on SDSS photometry and spectrometry, USN0-B stars and range 7.5–8.5 kpc |
| 8.20        | 238.00                   | 231.01 ± 4.93                   | Portail et al. (2017) | Old | Old From 2016 on SDSS photometry and spectrometry, USN0-B stars and range 7.5–8.5 kpc |
| 8.24 ± 0.12 | 256.50 ± 7.20            | 228.46 ± 9.12                   | Baertogues et al. (2017) | Old | Old From Palomar 5 globular cluster, page 20, error is average of upper and lower error bars, from red giant branches, independent of any of the individual measurements, but like v17 they also do not study a compilation of independent measurements, this lack of independence can bias results. Here we have invested significant effort in compiling a collection of independent $\Theta_0$ measurements published during 2000–2017. |
| 8.30 ± 0.25 | 233.00 ± 11.35           | 224.36 ± 13.66                  | Küpper et al. (2015) | Old | Old From 2017 on SDSS photometry and spectrometry, USN0-B stars and range 7.5–8.5 kpc |
| 8.30 ± 0.20 | 236.00 ± 6.00            | 226.33 ± 9.29                   | Bobylev & Bajkova (2016) | Young | Old From Palomar 5 globular cluster, page 20, error is average of upper and lower error bars, from red giant branches, independent of any of the individual measurements, but like v17 they also do not study a compilation of independent measurements, this lack of independence can bias results. Here we have invested significant effort in compiling a collection of independent $\Theta_0$ measurements published during 2000–2017. |
| 8.34 ± 0.16 | 240.00 ± 6.82            | 229.06 ± 8.72                   | Huang et al. (2016) | Young | Old From 2015 on SDSS photometry and spectrometry, USN0-B stars and range 7.5–8.5 kpc |
| 8.40 ± 0.40 | 224.00 ± 12.50           | 212.27 ± 16.22                  | Koposov et al. (2010) | Old | Old From SDSS photometry and spectrometry, USN0-B stars and range 7.5–8.5 kpc |
| 8.50        | 226.00                   | 211.64 ± 4.52                   | Martínez-Barcena et al. (2017) | Old | Old From SDSS photometry and spectrometry, USN0-B stars and range 7.5–8.5 kpc |

1 More properly one would use the rescaled angular velocities in the analysis and then convert the resulting angular velocity central value to a linear velocity central value. However, the uncertainty on $R_0$ is small and so results from the two different approaches will only differ slightly.

2 Perhaps the most famous example is the Hubble constant (Chen et al. 2003; Chen & Ratra 2011a). For other examples in astronomy, cosmology, and physics see Farooq et al. (2013); Cranell et al. (2015); Farooq et al. (2017); Bailey (2017); Zhang (2017), and references therein. Significant effort is devoted to testing for intrinsic non-Gaussianity in physical systems (e.g. Park et al. 2001; Planck Collaboration 2016), as opposed to measurement induced non-Gaussianity, since Gaussianity is usually assumed in parameter estimation (e.g. Samushia et al. 2007; Chen & Ratra 2011b; Ooba et al. 2017).
the probability of the median being between $\Theta_i$ and $\Theta_{i+1}$ is given by the binomial distribution

$$P = \frac{2^{-N} N!}{i!(N-i)!}.$$  

(2)

The 1σ error about the median is then defined by the range about it such that 68.27% of the probability is included. This can be extended to finding the 2σ error about the median, where instead 95.45% of the probability would be enclosed. We refer to the median of the Galactic rotational velocity at $R_0$ as $\Theta_{0,median}$. The weighted mean comes with the benefit of additional information in the errors, at the potential expense of including inaccurate uncertainties (Podariu et al. 2001). The weighted mean of the Galactic rotational velocity is

$$\Theta_{0}^{wm} = \frac{\sum_{i=1}^{N} \Theta_{0i} / \sigma_{0i}^2}{\sum_{i=1}^{N} 1 / \sigma_{0i}^2}$$  

(3)

where $\Theta_{0i}$ and $\sigma_{0i}$ are the rotational velocities and errors. The weighted mean standard deviation is

$$\sigma_{0}^{wm} = \frac{1}{\sqrt{\sum_{i=1}^{N} 1 / \sigma_{0i}^2}}$$  

(4)

The next step in analyzing the data is to construct error distributions of the data based on the chosen central estimate. For a central estimate $\Theta_{0,CE}$ independent of the data $\Theta_{0i}$, the number of standard deviations that each value deviates from the central estimate is

$$N_{\Theta_{0i}} = \frac{\Theta_{0i} - \Theta_{0,CE}}{\text{Var} (\Theta_{0i} - \Theta_{0,CE})}$$  

(5)

where $\text{Var} (\Theta_{0i} - \Theta_{0,CE})$ is the variance between the independent measurement, $\Theta_{0i}$ and the central estimate, $\Theta_{0,CE}$.

For median statistics when the central estimate is assumed to not be directly correlated with the data itself we have

$$N_{\Theta_{0i}}^{med} = \frac{\Theta_{0i} - \Theta_{0,med}}{\sqrt{(\sigma_{0i})^2 + (\sigma_{0,med})^2}}$$  

(6)

For Gaussianly distributed measurements and the weighted mean central estimate estimated from the data (and so correlated with the data) we instead have (see the Appendix of C18)³

$$N_{\Theta_{0i}}^{wm} = \frac{\Theta_{0i} - \Theta_{0,wm}}{\sqrt{(\sigma_{0i})^2 + (\sigma_{0,wm})^2}}$$  

(7)

The two error distributions, $N_{\Theta_{0i}}^{med}$ and $N_{\Theta_{0i}}^{wm}$, can be analyzed with a non-parametric Kolmogorov-Smirnov (KS) test.

We compare these error distributions to a few standard functional forms (e.g. Crandall & Ratra 2015; C18). The four probability distribution functions (PDFs) we consider here are the standard Gaussian PDF, the Cauchy (Lorentzian) PDF, the Student’s $t$ PDF, and the Laplacian (double exponential) PDF. The KS test allows us to quantify the level of deviation from Gaussianity of the error distributions by examining the outputs of this test, the D-statistic and the p-value. For a 95% confidence level in the probability that we cannot reject a specific PDF as describing the data, we look for two requirements for a data set of 29 measurements: $D \rightarrow 0$ but $\leq 0.246$, and $p \rightarrow 1$. For the Old tracers (19 measurements) we require almost the same: $D \rightarrow 0$ but $\leq 0.301$, and $p \rightarrow 1$, and for Young tracers (11 measurements) we require $D \rightarrow 0$ but $\leq 0.391$, and $p \rightarrow 1$.⁴

The standard Gaussian PDF is

$$P(|X|) = \frac{1}{\sqrt{2\pi}} \exp (-|X|^2/2),$$  

(8)

and is characterized by 1σ (2σ), or 68.27% (95.45%), of the data falling within $|X| \leq 1 \ (<|X| \leq 2$).

The Cauchy PDF has higher probability in the tails of the curve, and 1σ (2σ), or 68.27% (95.45%), of the data falls within $|X| \leq 1.8 \ (<|X| \leq 14)$. The Cauchy PDF is

$$P(|X|) = \frac{1}{\pi(1 + |X|^2)}$$  

(9)

The Student’s $t$ distribution also has widened tails, and involves an additional parameter $n$. A Student’s $t$ of $n = 1$ is the Cauchy PDF, and as $n \rightarrow \infty$, it becomes the Gaussian. The PDF is

$$P(|X|) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi} (n/2)^{(n/2)+1/2}} \left(1 + |X|^2/n\right)^{-(n+1)/2}$$  

(10)

and the limits for 1σ (2σ), or 68.27% (95.45%), of the data varies with $n$. Here $\Gamma$ is the gamma function and the addition of $n$ (positive integer parameter) decreases the total degrees of freedom by one.

The last PDF we consider is the Laplacian

$$P(|X|) = \frac{1}{2} \exp (-|X|).$$  

(11)

It is characterized by 1σ (2σ), or 68.27% (95.45%), of the data falling within $|X| \leq 1.2 \ (<|X| \leq 3.1).$ This results in a more sharply peaked distribution than either a Cauchy or a Student’s $t$.

In eqs. (8), (9), (10), and (11) the value $|X| = |\Theta_0|/S$. $S$ is a scale factor width for each distribution ($S = 1$ for a Gaussian PDF represents the standard normal distribution). We allow $S$ to vary in small increments of 0.001 from $S = 0$ to 5, and for the Student’s $t$ we do this for every value of $n$ from $n = 2$ to 100. The KS test, being non-parametric, makes no assumptions about the data.

4 RESULTS

We provide in Table 2 the central estimate statistics for the data listed in column 3 of Table 1. In Table 2, column 2 shows the median (with 1σ and 2σ error ranges) and weighted mean results for all 29 values. Column 3 shows the results of only analyzing the 18 Old tracer references, plus

³ An analogous equation for median statistics, for the case when the median is estimated from the data and so is correlated with the data, is not yet known.

⁴ See C18 and Appendix 3 of O’Connor & Kleyner (2012) for more detailed discussion of the outputs of the KS tests and the critical values for below which $D$ must fall.
the mixed tracer type of McMillan (2017). Column 4 shows the results of the 10 Young tracer types, plus the mixed tracer type as well.

While Table 3 shows the highest probabilities for Young tracer types, with all probabilities $p \geq 0.99$, the scale factors for all these PDFs are very non-Gaussian with all of them having $1\sigma$ ranges requiring $|X| \leq 0.5$. The All tracers compilation is also fairly non-Gaussian.

For the Old tracers collection with the median as the central estimate, $p = 0.83$ while $S = 1.08$ for the Gaussian PDF, indicating not unreasonable consistency with Gaussianity. This is also supported by the weighted mean result for the Gaussian PDF. Together these results indicate that the weighted mean summary for $\Theta_0$ is less appropriate than our median statistics one of $\Theta_0 = 219.70 \pm 6.67 + 8.77 \text{ km s}^{-1}$ (1$\sigma$ and 2$\sigma$ errors), which for most purposes can be taken to be $\Theta_d = 220 \pm 7 \pm 10 \text{ km s}^{-1}$. In summary, for practical purposes, we find at 1$\sigma$: $\Theta_d = 220 \pm 7 \text{ km s}^{-1}$ $R_0 = 7.96 \pm 0.17 \text{ kpc}$ $\omega = \Theta_d/R_0 = 27.6 \pm 1.1 \text{ km s}^{-1} \text{ kpc}^{-1}$

where the angular speed $\omega$ error is determined by adding the fractional uncertainties of $\Theta_d$ and $R_0$ in quadrature.

Table 1 of V17 lists 28 measurements of $\Theta_0$ from mid-2012 to 2017. V17 arrives at a $\Theta_0$ close to $230 \text{ km s}^{-1}$, median value $\Theta_{0,\text{med}} = 232 \text{ km s}^{-1}$, weighted mean value $\Theta_{0,\text{wm}} = 228 \pm 2 \text{ km s}^{-1}$, and an arithmetic mean value $\Theta_{0,\text{mean}} = 229 \pm 3 \text{ km s}^{-1}$. He recommends the set of Galactic constants:

$\Theta_0 = 230 \pm 3 \text{ km s}^{-1}$ $R_0 = 8.0 \pm 0.2 \text{ kpc}$ $\omega = \Theta_0/R_0 = 29 \pm 1 \text{ km s}^{-1} \text{ kpc}^{-1}$

We emphasize that several of the V17 Table 1 data are repeats of prior publications, big offenders being masers, OB stars, and Cepheids. Less than half of V17 Table 1 measurements are included in our list of independent measurements. V17 also does not distinguish between Old and Young tracer measurements of $\Theta_0$. These are probably the most reliable.

dGB17 on the other hand do note that Old tracers provide a better estimate of $\Theta_0$ and their recommended set of Galactic constants are (when their statistical and systematic errors are added in quadrature):

$\Theta_0 = 225 \pm 10 \text{ km s}^{-1}$ $R_0 = 8.3 \pm 0.4 \text{ kpc}$ $\omega = \Theta_0/R_0 = 27.1 \pm 1.8 \text{ km s}^{-1} \text{ kpc}^{-1}$

While their Old tracers compilation includes non-independent data points, dGB17 add on rather large undiscovered systematic errors and so their results are not inconsistent with our results. We note, in particular, as described in C18, that their estimate of $R_0$ is based on a very small set of data points (that are also not all independent). We emphasize that from our analysis of the Gaussianity of our $R_0$ and $\Theta_0$ compilations, here and in C18, we do not see strong evidence for large undiscovered systematic errors that dGB17 advocate for.

5 CONCLUSION

The data listed in Table 1 is the first compilation of independent $\Theta_0$ measurements published during 2000-2017. Given the mild non-Gaussianity of the Old tracer measurements, we favor a median statistics value of $\Theta_0 = 219.70 \pm 6.67 + 8.77 \text{ km s}^{-1}$ (1$\sigma$ and 2$\sigma$ errors). For most purposes this can be summarized as $\Theta_0 = 220 \pm 7 \pm 10 \text{ km s}^{-1}$. Given that the measured non-Gaussianity is mild, we believe most current $\Theta_0$ error bars are reasonable and that at present there is no strong evidence for large undiscovered systematic errors. In summary our recommended set of Galactic constants, with $1\sigma$ error bars,

$\Theta_0 = 220 \pm 7 \text{ km s}^{-1}$ $R_0 = 7.96 \pm 0.17 \text{ kpc}$ $\omega = \Theta_0/R_0 = 27.6 \pm 1.1 \text{ km s}^{-1} \text{ kpc}^{-1}$

are probably the most reliable.

ACKNOWLEDGEMENTS

We thank A. Quinnell and J. Vallée. This research was supported in part by DOE grant DE-SC0011840.

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Table 2. Central estimates of rescaled $\Theta_0$ data (in km s$^{-1}$)

| Statistic | All Tracers | Old Tracers | Young Tracers |
|-----------|-------------|-------------|---------------|
| Median    | 226 ± 1.12  | 219 ± 1.17  | 228 ± 1.09    |
| 1st range | 223 ± 1.24  | 212 ± 1.27  | 226 ± 1.23    |
| 2nd range | 217.9 ± 1.22| 204.9 ± 1.26| 226.9 ± 1.19  |

Weighted Mean

| Value     | All Tracers | Old Tracers | Young Tracers |
|-----------|-------------|-------------|---------------|
| Median    | 224.36 ± 1.67| 222.01 ± 1.99| 230.05 ± 3.09|
| 1st range | 222.89 ± 225.06| 220.82 ± 224.86| 226.95 ± 223.14|

Table 3. $N_{\sigma}$ KS test results for rescaled $\Theta_0$

| Type   | PDF       | $p^a$ | $\delta^b$ | PDF       | $p^a$ | $\delta^b$ |
|--------|-----------|-------|------------|-----------|-------|------------|
| All    | Gaussian  | 0.49  | 0.75       | Gaussian  | 0.20  | 0.73       |
|        | Cauchy    | 0.70  | 0.46       | Cauchy    | 0.34  | 0.45       |
| $n = 2$ Student’s t | 0.59 | 0.59 | $n = 2$ Student’s t | 0.26 | 0.58 |
| Laplace| 0.62      | 0.69  |            | Laplace   | 0.29  | 0.68       |
| Old    | Gaussian  | 0.83d | 1.08       | Gaussian  | 0.78  | 1.26       |
|        | Cauchy    | 0.71  | 0.70       | Cauchy    | 0.81  | 0.89       |
| $n = 36$ Student’s t | 0.83d | 1.07 | $n = 2$ Student’s t | 0.80 | 1.06 |
| Laplace| 0.76      | 1.04  |            | Laplace   | 0.81  | 1.27       |
| Young  | Gaussian  | 0.99  | 0.35       | Gaussian  | 0.99  | 0.38       |
|        | Cauchy    | 0.99  | 0.18       | Cauchy    | 0.99  | 0.20       |
| $n = 2$ Student’s t | 0.99 | 0.23 | $n = 2$ Student’s t | 0.99 | 0.26 |
| Laplace| 0.99      | 0.27  |            | Laplace   | 0.99  | 0.30       |

$^a$ The probability ($p$-value) that the input data doesn’t not come from the PDF.
$b$ The scale factor $S$ that maximizes $p$.
$c$ More precisely, $p = 0.82817$.
$d$ More precisely, $p = 0.82811$.

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