New hierarchy of multiple soliton solutions for the (2+1)-dimensional Sawada-Kotera equation

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Abstract A new transformation $u = 4(\ln(f))_x$ that can formulate a quintic linear equation and a pair of Hirota bilinear equations for the (2+1)-dimensional Sawada-Kotera (2DSK) equation is reported firstly, which enables one to obtain a new hierarchy of multiple soliton solutions of it. It tells a crucial fact that a nonlinear partial differential equation could possess two hierarchies of soliton solutions and the 2DSK equation is the first and only one found in this paper. The quintic linear equation can be written as a pair of Hirota bilinear equations, of which one is the SK bilinear equation coming from $u = 2(\ln(f))_x$, and the other is the KdV bilinear equation. The (1+1)-dimensional SK equation does not possess this property. As another example, a (3+1)-dimensional nonlinear partial differential equation possessing a pair of Hirota bilinear equations is studied.

Keywords (2+1)-dimensional Sawada-Kotera equation; Multiple soliton solutions; Quintic linear equation; Bilinear Hirota equations; Dependent variable transformation.

1 Introduction

Nonlinear problems has received a great amount of attention of physicists, mathematicians, engineers and other scientists because most of them are inherently nonlinear in the nature, such as chaos, singularities, solitons, dynamic of population and the organization of nature [15]. Many engineers would like to discuss systematically the complex nonlinear phenomena in engineering nonlinear systems, including the periodically forced Duffing oscillator, nonlinear self-excited systems, nonlinear parametric systems and nonlinear rotor systems. To do so, they usually discuss various analytical solutions of periodic motions to chaos or quasi-periodic motions in nonlinear dynamical systems in engineering and consider engineering applications, design, and control. Solitary wave solutions of nonlinear evolution equations have been playing important roles in nonlinear science and engineering, and applications in related areas of science, especially in nonlinear physical science. The solitary wave solution as well as the soliton solution that reflect a common nonlinear phenomenon in nature provide physical information and more insight into the physical aspects of the problem thus leading to further applications [11]. In the last two decades, an increasing number of researchers have taken more attention to the study of various kinds of exact solutions,

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such as the travelling wave solutions, multiple soliton solutions, rational solutions and the rogue waves etc., which are exponentially or algebraically localized solutions in certain directions.

We know that a partial differential equation (PDE) usually possesses a hierarchy of multiple soliton solutions, however we find that the (2+1)-dimensional Sawada-Kotera (2DSK) hierarchy possesses two families of multiple soliton solutions. This very special property is novel and first proposed by us. We guess that there are other types of high dimensional and high-order nonlinear PDEs bearing such properties. It is worth for us to explore more and take more attention on this question.

In this paper, we will concentrate on the following 2DSK equation

\[ v_t + v_5x + 15v_xv_{xx} + 15v_y + 5v_{xy} + 15v_y + 15v_x \int v_y dx - 5 \int v_{yy} dx = 0. \]

where \( v \) is a function of the variables \( x, y, \) and \( t, \) was first proposed by Konopelchenko and Dubrovsky, and has been proposed as a \((2+1)\)-dimensional integrable generalization of the \((1+1)\)-dimensional Sawada-Kotera equation [13, 21]. Now they are known also as a member of the so-called CKP hierarchy. Eq. (1.1) was widely used in many branches of physics, such as two-dimensional quantum gravity gauge field, conformal field theory and nonlinear science Liouville flow conservation equations. In Ref. [2], the equation was decomposed into three \((0 + 1)\)-dimensional Bargmann flows and its explicit algebraic-geometric solution was reported. In Ref. [13], four sets of bilinear Bäcklund transformations were constructed to derive multiple soliton solutions. In Ref. [23], the multi-wave method was used to seek for \( w \)-type wave solutions, periodic soliton wave solutions, and three soliton wave solutions. In Ref. [20], the equation has been studied in the view point of Bell polynomials and its Lax pair can be found there.

Due to its significance and wide applications in quantum gravity field theory, conformal field theory and conserved current of Liouville equation [22], the 2DSK equation has been studied extensively and intensively in a number of literatures. For example, Darboux transformation was discussed in [4]. Painlevé property and pseudo potentials for the 2DSK equation were investigated in [4]. Lie symmetry structures were analyzed in [17]. Multi-soliton solutions were derived via truncated Painlevé expansion [19] and reduced bilinear direct method [24]. Analytical solutions of the 2DSK equation were obtained via the Lie symmetry method [17]. Several types of exact solutions were given via the \( \partial \)-dressing method [3] and Wronskian technique [11]. Important contributions were also made by Ma, Li and Chen et al. on lump solutions to the 2DSK equation in [26, 10, 14].

\section{2 New transformation and a pair bilinear equations of the \((2 + 1)\)-dimensional Sawada-Kotera equation}

Introducing a potential variable transformation \( v = u_x, \) \( u = u(x, y, t), \) Eq. (1.1) reduces to

\[ u_{xt} + u_{6x} + 5u_{3xy} - 5u_{yy} + 15u_xu_{3x} + 15u_xu_{4x} + 15u_xu_{xy} + 15u_{xx}u_y + 45u_x^2u_{xx} = 0, \]

where \( u \) is a scalar potential function, \( x \) and \( y \) are respectively the longitudinal and transverse spatial coordinates, subscripts \( x, y, t \) denote partial derivatives, i.e. \( u_{xt} = \frac{\partial^2 u}{\partial x \partial t} \). Eq. (2.1) is the two (space)-dimensional extension of the famous KP equation, which originates from a 1970 paper by Boris Kadomtsev and Vladimir Petviashvili [12].

In this paper, we mainly consider the reduced Eq. (2.1) of Eq. (1.1). A 2DSK equation usually can be rewritten in bilinear form by expressing solutions in terms of a \( \tau \)-function. If \( u = 2(\ln(\tau))_x \) then \( \tau(x, y, t) \) satisfies the Hirota bilinear equation

\[ (D_x^6 + D_xD_t - 5D_x^3D_y + 5D_y^2)\tau \cdot \tau = 0, \]
where \( D_x^0, D_x D_t, D_x^2 D_y \) and \( D_y^2 \) are all the Hirota bilinear derivative operators \(^9\) defined by

\[
D_x^\alpha D_y^\beta D_u^\gamma f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^\alpha \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^\beta \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^\gamma \times f(x, y, t) g(x', y', t') \big|_{x'=x, y'=y, t'=t}.
\]  

(2.3)

This formula provides the basis for using Hirota method to obtain solutions of the BKP equation \(^9\) and 2DSK equation. In Refs. \(^5\) \(^6\) \(^7\) \(^8\), the author Hietarinta J. provided a complete classification for KdV-type, MKdV-type, sine-Gordon-type and Complex bilinear equations passing Hirota’s three-soliton condition. For the results in this paper, we have not seen any report therein or other references. We take account of a different alternative representation being not the Hirota bilinear form but a pair of two bilinear equations for Eq. (2.1), based on which we obtain new results in this paper.

Taking a dependent variable transformation

\[
u = p(\ln(f))_x
\]  

(2.4)

with \( f = f(x, y, t) \) and combining with the assumption

\[
f \equiv f_1 = 1 + \exp(\xi_1), \quad \xi_1 = k_1 x + l_1 y + w_1 t,
\]  

(2.5)

instead of using the classical Hirota bilinear method, we obtain the following results first (see Ref. \(^25\) for more details).

\[
p = 2, \quad w_1 = -(k_1^6 + 5k_1^3l_1 - 5l_1^2)/k_1, \quad (2.6)
\]

\[
p = 4, \quad l_1 = -k_1^3, \quad w_1 = 9k_1^2. \quad (2.7)
\]

and an alternative quintic linear equation for Eq. (2.1) later. The advantage is that this direct method could avoid constructing bilinear form which usually playing an important role for deriving multiple soliton solutions. Actually we prove that even a nonlinear PDE does not possesses the so-called Hirota bilinear form apparently, we could obtain a hierarchy of related soliton solutions.

In this paper, we adopt the determined dependent variable transformation

\[
\nu = 4(\ln(f))_x
\]  

(2.8)

instead of \( \nu = 2(\ln(f))_x \) to seek for new soliton solutions of Eq. (2.1) first and then using \( \nu \to \nu_x \) to obtain the multiple soliton solutions for Eq. (1.1). Substituting (2.8) into Eq. (2.1) yields the following quintic linear equation

\[
P \equiv f^4(f_{xx} + 5f_{4xy} + f_{2xt} - 5f_{x2y}) + f^3(25f_{4x}f_{3x} + 39f_{5x}f_{2xx} - 5f_{4x}f_{y} - f_{2x}f_{x} - 7f_{6x}f_{x} - 20f_{5xy}f_{x} + 40f_{3x}f_{xy} + 5f_{x}f_{2y} + 30f_{2x}f_{xy} + 10f_{xy}f_{y} - 2f_{xt}f_{x}) + 2f^2(-10f_{3x}f_{x}f_{y} + f_{x}^2f_{x} - 50f_{2x}^2f_{x} - 135f_{4x}f_{x}f_{2x} - 9f_{5x}f_{x}^2 - 90f_{2x}f_{x}f_{xy} - 5f_{x}f_{y}^2 - 15f_{2x}f_{y} + 5f_{3x}f_{2x}^2) + 2f(90f_{2x}f_{x}f_{y} + 270f_{3x}f_{x}^2f_{x} + 60f_{3x}f_{xy} - 225f_{4x}f_{x} + 105f_{4x}f_{3x}^2) - 40f(12f_{3x}f_{x}^4 + 9f_{2x}^2f_{x}^3 - 3f_{x}^4f_{y}) = 0,
\]  

(2.9)
Hence, if \( f = f(x, y, t) \) solves Eqs. (2.9), then \( u = u(x, y, t) \) is a solution of Eq. (2.1) by Eq. (2.8) and a solution of Eq. (1.1) by computing once derivative of the obtained solution \( u \) with respect to \( x \). Then we put forward a theorem, to prove that, some bilinear identities are listed firstly. From the definition (2.3), we have

\[
\begin{align*}
D_x^6 f \cdot f &= 2f_{6x}f - 12f_{5x}f_x + 30f_{4x}f_{xx} - 20f_{3x}^2, \\
D_x^4 f \cdot f &= 2f_{4x}f - 8f_{3x}f_x + 6f_{2x}^2, \\
D_x^2 f \cdot f &= 2f_{yy}f - 2f_y^2, \\
D_x D_t f \cdot f &= 2f_{xt}f - 2f_{xt}, \\
D_x D_y f \cdot f &= 2f_{xy}f - 2f_y f_x, \\
D_x^3 D_y f \cdot f &= 2f_{3xy}f - 6f_{2xy}f_x + 6f_{xy}f_{2x} - 2f_y f_{3x}, \quad (2.10)
\end{align*}
\]

**Theorem 1** If \( f = f(x, y, t) \) solves a couple of two bilinear form equations

\[
(D_x^6 + D_x D_t + 5D_x^3 D_y - 5D_y^2)f \cdot f = 0,
\]

\[
(D_x^4 + D_x D_y)f \cdot f = 0,
\]

where Eq. (2.11) is of the standard (2+1)-dimensional bilinear SK equation, and Eq. (2.12) is of the standard bilinear KdV equation, then \( f \) is also a solution of the quintic linear equations (2.9), and \( u = 4(\ln(f))_x \) a solution of Eq. (2.7).

**Proof** The quintic linear equation (2.9) according to the identities (2.10) can be written as the following form

\[
P = 2(\Delta_{sk})_x/f^2 - 4[\Delta_{sk}f_x - 15\Delta_{kdv}(f_{xx} + f_{3x})]/f^3 - 60\Delta_{kdv}\left[(f_{x}^{2} + 5f_{xx}f_x)/f^4 - 4f_{3x}/f^5\right],
\]

where

\[
\Delta_{kdv} = (D_x^4 + D_x D_y)f \cdot f, \\
\Delta_{sk} = (D_x^6 + D_x D_t + 5D_x^3 D_y - 5D_y^2)f \cdot f.
\]

It shows that if \( \Delta_{sk} = 0, \Delta_{kdv} = 0 \), i.e.,

\[
(D_x^6 + D_x D_t + 5D_x^3 D_y - 5D_y^2)f \cdot f = 0, \quad (D_x^4 + D_x D_y)f \cdot f = 0,
\]

then \( P = 0 \). Accordingly, when \( f = f(x, y, t) \) solves Eqs. (2.11) and (2.12), it also solves Eq. (2.9). Obviously, Eq. (2.11) is the Hirota bilinear form of the 2DSK equation starting with \( v = 2(\ln(f))_x \), and Eq. (2.12) is the known one of the KdV equation.

To show that \( f = f(x, y, t) \) solving Eqs. (2.11) and (2.12) is a solution of Eq. (2.9), one should prove the consistency of Eqs. (2.11) and (2.12), say, of which the solution set is not empty. For simplicity and without loss of generality, setting \( f = \exp(v), v = v(x, y, t) \) in Eqs. (2.11), (2.12) and solving \( v_{xt}, v_{yx} \) respectively yields

\[
v_{xt} = 120v_{2x}^3 - 5v_{3x} + 5v_{2y} - v_{6x}, \quad v_{xy} = -(v_{4x} + 6v_{2x}^2). \quad (2.14)
\]

Next, one should prove \( v_{xt,xy} = v_{xy,xt} \).

\[
v_{xy,xt} = -(v_{5xt} + 12v_{3x}v_{2xt} + 12v_{2x}v_{3xt}),
\]

\[
v_{xt,xy} = 720v_{2x}v_{2xy}v_{3x} + 360v_{2x}v_{3xy} - 5v_{4x} + 5v_{3x} - v_{7xy}. \quad (2.15)
\]
Computing $v_{xy,xt} - v_{xt,xy}$ and substituting $v_{xt}$ and $v_{xy}$ with the forms (2.4), one can obtain

$$v_{xy,xt} - v_{xt,xy} = 0.$$ 

Theorem 1 shows that the obtained quintic linear equation starting with $u = 4(\ln(f))_x$ of Eq. (2.1) can be written as the pair of bilinear equations which serve as a role to construct new hierarchy of high-order soliton solutions for the 2DSK equation. It is well known that Eq. (2.11) ensures the $N$-soliton solutions of the 2DSK equation starting from $u = 2(\ln(f))_x$, and Eq. (2.12) ensures the $N$-soliton solutions for the KdV equation. An explicit connection between the KdV equation and the 2DSK equation is built, from which we can infer that Eq. (2.1) possesses a new hierarchy of $N$-soliton solutions starting from the new transformation (2.8).

3 New multiple soliton solutions of the (2 + 1)-dimensional Sawada-Kotera equation

To construct multiple soliton solutions of the 2DSK Eq. (2.1), we start with exponential function with the following forms

$$f_1 = 1 + \exp(\xi_1),$$
$$f_2 = 1 + \exp(\xi_1) + \exp(\xi_2) + h_{1,2}\exp(\xi_1 + \xi_2),$$
$$f_3 = 1 + \exp(\xi_1) + \exp(\xi_2) + \exp(\xi_3) + h_{1,2}\exp(\xi_1 + \xi_2) + h_{1,3}\exp(\xi_1 + \xi_3) + h_{2,3}\exp(\xi_2 + \xi_3) + h_{1,2}h_{1,3}h_{2,3}\exp(\xi_1 + \xi_2 + \xi_3),$$

$$\ldots,$$
$$f_n = \sum_{\rho \subseteq \{1, \ldots, n\}} \exp \left( \sum_{i=1}^{n} \mu_i \xi_i + \sum_{i,j} \mu_i \mu_j H_{ij} \right),$$
$$= \sum_{\rho \subseteq \{1, \ldots, n\}} \left( \prod_{(i,j) \subseteq \rho} h_{i,j} \right) \exp \left( \sum_{k \in \rho} \xi_k \right),$$

where $\xi_i = \xi_i(x, y, t) = k_i x + l_i y + w_i t$, $(i = 1, \ldots, n)$, and $k_i, l_i, w_i$ are arbitrary constants for all $i$. It is actually a simplified direct Hirota method.

Usually, researchers proceed with the studies such as constructing multiple soliton solutions by taking $p = 2$ in Eq. (2.4) because that it is often obtained by using the Hirota method and the transformation $u = 2(\ln(f))_x$ enables one to obtain a corresponding Hirota bilinear form, and then soliton solutions. It is deemed that one could not obtain high-order multiple soliton solutions if $u = 4(\ln(f))_x$ does not make the considered PDE to transform into bilinear form. Truthfully, this simplified direct Hirota method could produce different results, alternatively, when we move forward by choosing $u = 4(\ln(f))_x$ alternatively. It is crucial that we indeed obtain a new hierarchy of multiple soliton solutions by this new transformation. Class of nonlinear PDEs which can be linearized is small, and it is rare that they can be rewritten into a quintic linear form equation. So far, we have not seen any similar discussions on the quintic linear equation and the corresponding soliton solutions for the 2DSK equation (2.1).

One-soliton solutions of the 2DSK (2.1) are

$$u = 4(\ln(f_{\text{new}}))_x \text{ with } f_{\text{new}} = 1 + \exp(k_1 x - k_1^3 y + 9 k_1^3 t),$$
$$u = 2(\ln(f_{\text{old}}))_x \text{ with } f_{\text{old}} = 1 + \exp \left( k_1 x + l_1 y + \frac{5 k_1^2 - k_1^3 + 5 k_1^3 t}{k_1} \right),$$

(3.5)
where \( k_1 \neq 0, l_1 \neq 0 \) are free parameters. The first one of Eqs. (3.5) is a new one-soliton solution, whereas the second is the known one. They can be rewritten in terms of hyperbolic cosine function [16].

\[
\begin{align*}
    u_{\text{new}} &= 4 \ln \left[ \cosh\left( \frac{k_1}{2} x - \frac{k_1^3}{2} y + \frac{9k_1^3}{2} t \right) \right], \\
    u_{\text{old}} &= 2 \ln \left[ \cosh\left( \frac{p_1+q_1}{2} x - \frac{p_1^3+q_1^3}{2} y + \frac{9(p_1^3+q_1^3)}{2} t \right) \right],
\end{align*}
\]

(3.6)

where \( p_1, q_1 \) are arbitrary constants.

3.1 Two hierarchies of two-soliton solutions of the 2DSK equation

As for two-soliton solutions, starting from \( u = 4(\ln(f_{\text{2new}}))_x \) for Eq. (2.1) we obtain two-soliton solution with \( f_{\text{2new}} \) being the form

\[
f_{\text{2new}} = a_{1,2}\cosh\left( \frac{\xi_1}{2} + \frac{\xi_2}{2} \right) + b_{1,2}\cosh\left( \frac{\xi_1}{2} - \frac{\xi_2}{2} \right), \quad \xi_i = k_i x + l_i y + w_i t,
\]

(3.7)

and \( l_i = -k_i^3, w_i = 9k_i^3, \ (i = 1, 2), \ a_{1,2} = k_1 - k_2, \ b_{1,2} = k_1 + k_2. \)

As a comparison, we give the two-soliton solution in terms of \( \cosh \) function obtained from \( u = 2(\ln(f_{\text{2old}}))_x \), where \( f_{\text{2old}} \) for the known two-soliton solution is the same with Eq. (3.7), whereas

\[
\begin{align*}
    k_i &= p_i + q_i, \quad l_i = -(p_i^3 + q_i^3), \quad w_i = 9(p_i^5 + q_i^5), \quad (i = 1, 2), \\
    a_{1,2} &= \sqrt{(q_1 - q_2)(q_1 - p_2)(p_1 - q_2)(p_1 - p_2)}, \quad b_{1,2} = \sqrt{(q_1 + q_2)(q_1 + p_2)(p_1 + q_2)(p_1 + p_2)}.
\end{align*}
\]

(3.8)

with \( p_i \neq q_i, \ (i = 1, 2) \) are arbitrary constants and have same properties from now on.

3.2 Two hierarchies of three-soliton solutions of the 2DSK equation

Starting from \( u = 4(\ln(f))_x \) yields new three-soliton solution with

\[
\begin{align*}
    f_{\text{3new}} &= K_0\cosh\left( \frac{\xi_1}{2} + \frac{\xi_2}{2} + \frac{\xi_3}{2} \right) + K_1\cosh\left( \frac{\xi_1}{2} - \frac{\xi_2}{2} - \frac{\xi_3}{2} \right) + \\
    &+ K_2\cosh\left( \frac{\xi_1}{2} - \frac{\xi_2}{2} + \frac{\xi_3}{2} \right) + K_3\cosh\left( \frac{\xi_1}{2} + \frac{\xi_2}{2} - \frac{\xi_3}{2} \right),
\end{align*}
\]

(3.9)

where \( \xi_i = k_i x - k_i^3 y + 9k_i^5 t, \ (i = 1, \ldots, 3) \), and

\[
\begin{align*}
    K_0 &= a_{1,2}a_{1,3}a_{2,3}, \quad K_1 = b_{1,2}b_{1,3}a_{2,3}, \quad K_2 = b_{1,2}a_{1,3}b_{2,3}, \quad K_3 = a_{1,2}b_{1,3}b_{2,3}, \\
    a_{i,j} &= k_i - k_j, \quad b_{i,j} = k_i + k_j, \quad k_i \neq \pm k_j, \quad (1 \leq i < j \leq 3).
\end{align*}
\]

(3.10)

For the known three-soliton solution in terms of \( \cosh \) function, \( f_{\text{3old}} \) has the same form with that of Eq. (3.9), \( \xi_i = k_i x + l_i y + w_i t, \ k_i = p_i + q_i, \ l_i = -(p_i^3 + q_i^3), \ w_i = 9(p_i^5 + q_i^5), \ i = 1, \ldots, 3 \) and \( K_i \ (i = 0, \ldots, 3) \) is the same with (3.10), however \( a_{i,j}, b_{i,j} \) have the following different forms

\[
\begin{align*}
    a_{1,2} &= \sqrt{(q_1 - q_2)(q_1 - p_2)(p_1 - q_2)(p_1 - p_2)}, \quad b_{1,2} = \sqrt{(q_1 + q_2)(q_1 + p_2)(p_1 + q_2)(p_1 + p_2)}, \\
    a_{1,3} &= \sqrt{(q_1 - q_3)(q_1 - p_3)(p_1 - q_3)(p_1 - p_3)}, \quad b_{1,3} = \sqrt{(q_1 + q_3)(q_1 + p_3)(p_1 + q_3)(p_1 + p_3)}, \\
    a_{2,3} &= \sqrt{(q_3 - q_2)(q_3 - p_2)(p_3 - q_2)(p_3 - p_2)}, \quad b_{2,3} = \sqrt{(q_2 + q_3)(p_2 + q_3)(q_2 + p_3)(p_2 + p_3)}.
\end{align*}
\]

3.3 Two hierarchies of \( N \)-soliton solutions of the 2DSK equation

The \( N \)-soliton solution \( u = 4(\ln(f_{\text{Nnew}}))_x \) in terms of \( \cosh \) function are

\[
f_{\text{Nnew}} = \sum_{\nu} K_{\nu} \cosh \left( \sum_{i=1}^{N} \frac{\nu_i \xi_i}{2} \right), \quad K_{\nu} = \prod_{i<j} a_{i,j},
\]

(3.11)
where $\xi_i = k_i x - k_i^3 y + 9k_i^5 t$, $(i = 1, \ldots, N)$, $a_{i,j} = k_i - \nu_i \nu_j k_j$, $(1 \leq i < j \leq N)$, and the summation of $\nu = \{\nu_1, \nu_2, \ldots, \nu_N\}$ should be done for all non-dual permutations of $\nu_i = 1, -1, (i = 1, 2, \ldots, N)$, ($\nu \nu'$ are dual if $\nu = -\nu'$).

For the known $N$-soliton solution in terms of $cosh$ function, $f_{N,old}$ is the same with that of (3.11), and $\xi_i = k_i x + l_i y + w_i t$, $(i = 1, \ldots, N)$, $k_i = p_i + q_i$, $l_i = -(p_i^3 + q_i^3)$, $w_i = 9(p_i^5 + q_i^5)$ with

$$a_{i,j}^2 = (q_i - \nu_i \nu_j q_j)(q_i - \nu_i \nu_j p_j)(p_i - \nu_i \nu_j q_j)(p_i - \nu_i \nu_j p_j),$$

and $p_i$, $p_j$, $q_i$, $q_j$, $(i, j = 1, \ldots, N)$ are free parameters.

### 4 Concluding remarks

In short, this paper is devoted to exploring the new dependent variable transformation to construct the quintic linear equations that could be rewritten into a couple of two bilinear forms relating to obtaining new hierarchy of multiple soliton solutions for the $(2 + 1)$-dimensional Sawada-Kotera equation. Up to now, the 2DSK is the only nonlinear PDE found by us that possesses two hierarchies of multiple soliton solutions. Abundant results are presented and as one can see, they have gone beyond the known results. The new transformation is missed or has not been given more attention and ignored by researchers before may because it can not provide bilinear form directly by using the classical Hirota bilinear method. We implement this newly found simple but novel transformation to the 2DSK equation to find its another missed hierarchy of multiple soliton solutions. Starting from the transformation, we study the famous 2DSK equation successfully, and we hope to give a category of nonlinear PDEs that also possess a set of soliton solutions which can be solved by two bilinear equations. Actually, there indeed exist such nonlinear partial differential equations. As an example, we further consider a (3+1)-dimensional nonlinear PDE

$$v_{2x} v_{yz} + 2 v_{2x} v_{3xz} + 12 v_{xx}^2 v_{xx} + v_{xy} v_{xt} - 120 v_{xy} v_{4x} v_{2x} - 4 v_{xy} v_{6x} - 240 v_{xy} v_{2x}^3 - 10 v_{xy} v_{3xy} - 60 v_{xy} v_{2x} + 5 v_{xy} v_{2y} = 0. \tag{4.1}$$

Likewise, substituting $v = \ln(f)$ into the Eq. (3.3), we obtain the following pair of bilinear equations

$$(D_y D_z + 2 D_{y}^2 D_z) f \cdot f = 0, \quad (D_x D_t - 4 D_{x}^2 D_t + 10 D_{x}^3 D_t + 5 D_{y}^2) f \cdot f = 0. \tag{4.2}$$

Before constructing soliton solutions using Eqs. (4.2), one should discuss the consistency of them first. From the bilinear equations in (4.2), we get

$$-8 f_{6x} f + 48 f_{5x} f_x - 120 f_{4x} f_{2x} + 80 f_{3x} - 20 f_{3xy} f + 60 f_{2xy} f_x - 60 f_{xy} f_{2x} - 20 f_{y} f_{3x} + 2 f_{xt} f - 2 f_{t} f = 0,
-2 f_x f_t - 10 f_{2y} f + 10 f_y^2 = 0,
4 f_{3xz} f - 12 f_{2xz} f_x + 12 f_{xz} f_{2x} - 4 f_{3x} f_z + 2 f_{yz} f - 2 f_{y} f_z = 0. \tag{4.3}$$

Same as the Theorem 1, we get

$$v_{xt} = 240 v_{2x}^3 + 10 v_{3xy} + 120 v_{4x} v_{2x} - 5 v_{2y} + 4 v_{6x}, \quad v_{yz} = -2 v_{3xz} - 12 v_{2x} v_{xz}. \tag{4.4}$$

It is easy to prove that $(v_{xt})_{yz} - (v_{yz})_{xt} = 0$ after simple computations, which completes the proof of the consistency. Thus, three-soliton solution of Eq. (4.1) by $v = \ln(f)$ with $f$ being the same form given in Eq. (3.3), whereas, $\xi_i = k_i x + p_i y + q_i z + w_i t$, $(i = 1, \ldots, 3)$, and

$$p_i = -2k_i^5, \quad w_i = -36k_i^5, \quad h_{i,j} = (q_i - q_j)(k_i - k_j), (1 \leq i < j \leq 3)$$

with $k_i \neq k_j, q_i \neq q_j$ are nonzero constants for all $i$. 


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