THE FIRST LAW OF BLACK BRANE MECHANICS

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ABSTRACT

We obtain ADM and Komar surface integrals for the energy density, tension and angular momentum density of stationary $p$-brane solutions of Einstein’s equations. We use them to derive a Smarr-type formula for the energy density and thence a first law of black brane mechanics. The intensive variable conjugate to the worldspace $p$-volume is an ‘effective’ tension that equals the ADM tension for uncharged branes, but vanishes for isotropic boost-invariant charged branes.
1 Introduction

The status of energy in General Relativity is rather special because, for a general spacetime, there is no coordinate-independent meaning to the energy in a given local region of a spacelike hypersurface. However, if the spacetime is asymptotically flat then it is possible to define a total energy, which can be expressed as an integral at spatial infinity: the ADM integral \( \int \). Consider an asymptotically flat spacetime, of \( d = 1 + n \) dimensions, for which the asymptotic form of the metric, in cartesian coordinates \( x^M = (t, x^i) \) \( (i = 1, \ldots, n) \) is

\[
g_{MN} \sim \eta_{MN} + h_{MN} \tag{1}\]

where \( \eta \) is the Minkowski metric and \( h \) is a perturbation with the appropriate fall-off at spatial infinity needed to make the energy integral converge. We shall suppose that the action takes the form\( ^1 \)

\[
I = \frac{1}{2} \int d^d x \sqrt{-\det g} \, R + I^{(\text{mat})}. \tag{2}\]

The matter stress-tensor is

\[
T^{(\text{mat})}_{MN} = -\frac{2}{\sqrt{-\det g}} \frac{\delta I^{(\text{mat})}}{\delta g^{MN}}, \tag{3}\]

and the Einstein field equation is

\[
G_{MN} = T^{(\text{mat})}_{MN}. \tag{4}\]

The ADM energy integral in these conventions is

\[
E = \frac{1}{2} \oint dS_i \left( \partial_j h_{ij} - \partial_i h_{jj} \right). \tag{5}\]

In the special case of a stationary asymptotically flat spacetime, for which there exists a normalized timelike Killing vector field \( k \), the energy in any volume \( V \) with boundary \( \partial V \) can be expressed as the Komar integral \( ^2 \)

\[
E(V) = -\left( \frac{n-1}{4(n-2)} \right) \int_{\partial V} dS_{MN} D^M k^N. \tag{6}\]

where \( dS_{MN} \) is the co-dimension 2 surface element on the \( (n-1) \)-dimensional boundary \( \partial V \) of \( V \). By taking \( V \) to be a spacelike Cauchy surface \( \Sigma \) with boundary at spatial infinity, we get an alternative expression for the total energy \( E \) in which \( k \) need be only asymptotically Killing. Taking \( \Sigma \) to be a surface of constant \( t \), in coordinates for which \( k = \partial/\partial t \), we thus find that

\[
E = -\left( \frac{n-1}{2(n-2)} \right) \oint \, dS_i \partial_i h_{00}, \tag{7}\]

which shows that the energy can be read off from the asymptotic behaviour of the \( g_{00} \) component of the metric.

One aim of this paper is to generalize these surface integrals to energy densities of ‘p-brane spacetimes’, which are not asymptotically flat but rather ‘transverse asymptotically flat’, and to demonstrate their equivalence. One difference between the \( p = 0 \) and \( p > 0 \) cases is that for \( p > 0 \) the energy is replaced by a stress-tensor density, which has energy per unit p-volume \( \mathcal{E} \) (the energy density) and \( p \) tensions as its \( (p+1) \) diagonal entries. For

\(^1\)This corresponds to a choice of units for which \( 8\pi G = 1 \), where \( G \) is the \( d \)-dimensional Newton constant.
an isotropic brane all $p$ tensions are equal, and equal by definition to the brane tension $T$. Although isotropy it is not really necessary to our analysis we will mostly consider isotropic branes for the sake of simplicity. It should be noted that the energy density is sometimes referred to in the brane literature as the ‘tension’, but it actually equals the tension only for boost-invariant isotropic branes, examples of which are provided by many extremal brane spacetimes. Here we wish to consider the more general case of non-extremal, and non boost-invariant, brane spacetimes, so we must distinguish between tension $T$ and energy density $E$.

The ADM integral for the energy of transverse asymptotically flat spacetimes was first obtained by Deser and Soldate [3] for $p = 1$, using the method introduced by Abbott and Deser [4], and the result for general $p$ was given in [5]. We will present here the extension of the argument of [3] to general $p$. We also obtain ADM-type integrals for the tension and angular momentum. We then use these results to find the corresponding Komar-type integrals for these quantities, generalizing the standard covariant Komar surface integrals of the $p = 0$ case. For $p = 0$ these integrals are unique up to normalization, but this feature does not extend to $p > 0$, which is why we need to start from the ADM integrals. A corollary of our results is that when $p > 0$ neither the energy density nor the brane tension can be read off from the $g_{00}$ component of the metric, an observation that seems to have first been made (for the energy density) by Lu [6] in the context of a special class of brane spacetimes.

Another aim of this paper is to derive a first law for black branes analogous to the first law of black hole mechanics [7]. For black holes the first law can be derived from a generalization of the Smarr formula for the energy of a black hole spacetime [8], which can itself be deduced from the Komar integrals for black hole energy and angular momentum [9]. The covariant surface integrals for the energy and angular momentum densities of brane spacetimes therefore constitute a natural starting point for a derivation of the first law of black brane mechanics. There is one version of this law in which the intensive variables conjugate to the surface horizon $\kappa$, the angular velocities of the horizon $\Omega_H$ and the (electric) potential of the the horizon $\Phi_H$ are densities. This law relates a change of the energy density to changes in other densities such as the angular momentum densities $J$ and the (electric) charge density $Q$, and it turns out to take the form

$$dE = \kappa dA_{eff} + \Omega_H \cdot dJ + \Phi_H dQ,$$

where $A_{eff}$ is an ‘effective horizon area’. Because of some simplifying assumptions that we make, there are brane spacetimes to which our derivation of this law does not apply (although the law itself may remain valid). These would include, for example, 5-branes of D=11 supergravity carrying both 5-brane and 2-brane charge. More significantly, we consider only translationally-invariant brane spacetimes, although our covariant surface integrals are valid for brane spacetimes that are only asymptotically translationally invariant. Even so, our analysis applies to many cases of interest, including that of the rotating non-extremal membrane solutions of D=11 supergravity obtained in [9]. Recent work of Traschen and Fox [10] in which a version of the first law of black brane mechanics is derived by Hamiltonian methods appears to go some way towards relaxing these restrictions.

There is another version of the first law that applies to toroidally-compactified branes in which a change in the total energy $E$, integrated over the $p$-dimensional worldspace, is related to changes in the total horizon area $A$, total angular momenta $J$ and total charge $Q$, as for black holes. This is hardly surprising because a toroidally-compactified black brane can be viewed as a black hole in the lower dimension. From the higher-dimensional

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2We became aware of this paper after an earlier submission to the archives of the work reported here; the precise connection to the work reported here is not yet clear to us.
point of view, however, the worldspace p-volume $V_p$ is a new extensive variable. If we call its conjugate variable the ‘effective’ tension $T_{\text{eff}}$ then the first law becomes

$$dE = \kappa dA + \Omega_H \cdot J + \Phi_H dQ + T_{\text{eff}} dV_p. \quad (9)$$

One might expect that the effective tension appearing in this law should equal the ADM tension $T$, but this turns out to be the case only for uncharged branes. For charged branes we find that

$$T_{\text{eff}} = T - \Phi_H Q, \quad (10)$$

so that the effective tension is reduced by a pressure due to the gauge fields produced by the charge on the black brane. Of course, the variable $V_p$ also has a ‘lower dimensional’ interpretation as a scalar field expectation value, in which case its conjugate variable $T_{\text{eff}}$ could be called a ‘scalar charge’ [11]. An extension of the first law to accommodate scalar fields and their scalar charges has been presented previously by Gibbons et al. [12]; the novelty of our formulation of this version of the first law is the interpretation of scalar charge in terms of ADM tension.

For isotropic brane solutions of supergravity theories there is a bound on the energy density [13], which is saturated by supersymmetric solutions. We shall show that this energy bound implies the bound $T \leq E$, which is also saturated by supersymmetric solutions. Thus, supersymmetric isotropic branes are boost-invariant. The energy bound also implies the bound $T_{\text{eff}} \geq 0$, which is again saturated by supersymmetric solutions. This can be understood from a worldvolume perspective; the effective tension is the vacuum energy, which has $T$ as a gravitational contribution and the term proportional to $Q$ as an ‘electrostatic’ contribution, the two cancelling in the supersymmetric case.

## 2 ADM integrals

Our first task is to obtain the ADM integrals for brane energy density, tension and angular momentum. We shall do this using a generalization of the method of Abbott and Deser [4]. This method can be used to define the energy as a surface integral at infinity in any spacetime that is asymptotic to some background spacetime admitting a timelike Killing vector field. Here we shall apply it to spacetimes that are transverse asymptotically flat, so that the background spacetime is a Minkowski spacetime of $D = (n+p+1)$ dimensions, to which the full metric $g_{MN}$ is asymptotic at transverse spatial infinity. We may write

$$g_{MN} = \eta_{MN} + h_{MN} \quad (11)$$

where $h_{MN}$ is a (not necessarily small) perturbation about the Minkowski background metric $\eta$. We can now rewrite the Einstein equation as

$$G_{MN}^{(\text{lin})} = T_{MN}^{(\text{mat})} + \tau_{MN} \quad (12)$$

where $G_{MN}^{(\text{lin})}$ is the linearized Einstein tensor and $\tau_{MN}$ is the gravitational stress pseudotensor; i.e., the stress tensor of the gravitational field relative to the $D$-dimensional Minkowski background. Let

$$H_{MN} = h_{MN} - \frac{1}{2} \eta_{MN} h, \quad (13)$$

where $h = h^{M}M$, and define

$$K^{MPNQ} = \frac{1}{2} \left[ \eta^{MQ} H^{NP} + \eta^{NP} H^{MQ} - \eta^{MN} H^{PQ} - \eta^{PQ} H^{MN} \right]. \quad (14)$$
Note that $K$ has the algebraic symmetries of the Riemann tensor. We can now rewrite the linearized Einstein tensor, in cartesian coordinates, as

$$G_{(\text{lin})}^{MN} = \partial_P \partial_Q K^{MPNQ}.$$  (15)

Using this, we can rewrite the Einstein equation (12) in the form

$$\partial_P \partial_Q K^{MPNQ} = T^{MN},$$  (16)

where

$$T^{MN} = T_{(\text{mat})}^{MN} + \tau^{MN}$$  (17)

is the ‘total’ stress pseudo-tensor. Consistency of (16) requires that

$$\partial_M T^{MN} = 0.$$  (18)

Let $X^M = (t, y^m, x^i)$ ($i = 1, \ldots, n; m = 1, \ldots, p$) be cartesian coordinates for the background Minkowski spacetime, so that

$$ds^2(\eta) = -dt^2 + \sum_{m=1}^{p} dy^m dy^m + \sum_{i=1}^{n} dx^i dx^i.$$  (19)

Note that

$$\bar{k} = \frac{\partial}{\partial t}, \quad \bar{\ell}_{(m)} = \frac{\partial}{\partial y^m},$$  (20)

are commuting Killing vector fields of this background metric, which we may write collectively as $\bar{k}(\mu)$, $\mu = 0, 1, \ldots, p$. We will assume that the full metric $g$ is asymptotic to $\eta$ as $|x| \to \infty$; it is thus ‘transverse asymptotically’ flat, stationary and translationally invariant (in brane directions). Equivalently, we assume that $T^{MN}$ vanishes in the limit $|x| \to 0$ (sufficiently rapidly for convergence of the energy integral to be discussed below).

Because of the condition (18), the currents

$$J_{(\mu)}^M = T_{MN} \bar{k}^N_{(\mu)}$$  (21)

are divergence-free (where indices are raised or lowered with the background Minkowski metric). We may also construct an analogous set of p-form currents:

$$A^{P_1 \ldots P_p}_{(0)} = \bar{\ell}^{[P_1}_{(1)} \ldots \bar{\ell}^{P_p]}_{(p)}$$
$$A^{P_1 \ldots P_p}_{(m)} = \bar{\ell}^{[P_1}_{(1)} \ldots \bar{k}^{P_m} \ldots \bar{\ell}^{P_p]}_{(p)},$$  (22)

where $\bar{k}$ replaces $\bar{\ell}_{(m)}$ in the second expression. Because the Killing vector fields commute, the p-form currents $A_{(\mu)}$ are also divergence free, in the sense that (in cartesian coordinates)

$$\partial_S A_{(\mu)}^{S_{P_1} \ldots P_{p-1}} = 0 \quad (\mu = 0, 1, \ldots, p).$$  (23)

We now define the brane stress tensor density as

$$\theta_{\mu\nu} = -\int_{\Sigma} dS_{MP_1 \ldots P_p} J_{(\mu)}^M A^{P_1 \ldots P_p}_{(\nu)},$$  (24)

where $dS_{MP_1 \ldots P_p}$ is the co-dimension $(2+p)$ surface element for $\Sigma$, normal to $\bar{k}$ and the $p$ vector fields $\{\bar{\ell}\}$. In standard cartesian coordinates for the background Minkowski metric, we have

$$\theta_{\mu\nu} = \int d^n x T_{\mu\nu}$$  (25)
where $T_{\mu\nu}$ are the worldvolume components of $T_{MN}$, and the integral is over the transverse Euclidean n-space. Clearly, $\theta_{\mu\nu}$ can be written as a surface integral at transverse spatial infinity only if it is independent of the worldvolume coordinates $y^\mu = (t, y^m)$, since any surface integral will have this property as a consequence of the assumption of asymptotic time and translational invariance. But this condition is met only if $J_{(\mu \wedge A_\nu)}$ is a divergence-free $(p + 1)$-current, and this requires both

$$\mathcal{L}_k T_{MN} = 0 \quad (26)$$

and

$$\mathcal{L}_{i(m)} T_{MN} = 0 \quad (m = 1, \ldots, p). \quad (27)$$

These conditions are quite severe as they effectively restrict us to spacetimes that are everywhere both stationary and translationally-invariant, rather than just asymptotically so. If we consider only the component $\theta_{00}$ then we need only require $\mathcal{L}_{0} T_{MN} = 0$ but this still restricts us to translationally-invariant brane spacetimes. To circumvent this restriction we shall impose periodicity in the brane directions, with total p-volume $V_p$, and define

$$\langle A \rangle = \frac{1}{V_p} \int d^p y \, A \quad (28)$$

for any function $A$ on the background Minkowski spacetime. The average energy density is then

$$\mathcal{E} \equiv \langle \theta_{00} \rangle = \int d^n x \, \langle T^{00} \rangle . \quad (29)$$

For stationary spacetimes we may similarly define the average tension $\mathcal{T}$ as

$$\mathcal{T} \equiv \frac{1}{p} \sum_m \langle \theta_{mm} \rangle = -\frac{1}{p} \sum_m \int d^n x \, \langle T^{mm} \rangle . \quad (30)$$

An analogous procedure yields the expression

$$\mathcal{L}^{ij} = 2 \int d^n x \, x^{[i} \langle T^{j]0} \rangle . \quad (31)$$

for the average p-brane angular momentum 2-form density.

We now aim to rewrite these transverse-space integrals as surface integrals using (16). Having done so we may then omit the averaging because of the translational invariance at transverse spatial infinity. For the energy density we thus find that

$$\mathcal{E} = \oint_{\infty} dS_i \partial_j K^{ij00} , \quad (32)$$

and for the angular momentum 2-form density we find that

$$\mathcal{L}^{ij} = 2 \oint_{\infty} dS_k \left[ K^{ij00} + x^{[i} \partial_k K^{j]00} + x^{[i} \partial_0 K^{j]00} \right] , \quad (33)$$

where the integrals are over an $(n-1)$-sphere at transverse spatial infinity. These expressions are the basis for the ADM-type formulae for brane energy and angular momentum. Specifically, the surface integral (32) is equivalent to

$$\mathcal{E} = \frac{1}{2} \oint_{\infty} dS_i \left( \partial_j h_{ij} - \partial_i h_{jj} - \partial_i h_{mm} \right) . \quad (34)$$

3In the non-isotropic case this is also an average over brane directions.
This is the result given in [3, 5]. For asymptotically stationary spacetimes, for which time derivatives vanish at infinity, the integral (33) is similarly equivalent to

\[ L^{ij} = \oint_{\infty} dS_i \left[ 2\delta^{[i} h^{j]} - x^{[i} \partial_{[i} h^{j]} + x^{[i} \partial_{j]} h^{0]} \right]. \] (35)

This result implies that the angular momentum density can be read off from the coefficient of \( h_{0i} \) according to the formula

\[ h_{0i} = \frac{L^{ij} x^j}{\text{vol}(n-1) \nu} \] (36)

where \( \text{vol}_k \) is the volume of the unit \( k \)-sphere.

For future use we note that for constants \( \Omega_{ij} = -\Omega_{ji} \) we have

\[ \frac{1}{2} \Omega_{ij} L^{ij} = \oint_{\infty} dS_i \left[ m_{ij} h_{0j} - \frac{1}{2} (h_{0i,j} - h_{0j,i}) m_j \right] \] (37)

where

\[ m_i = \Omega_{ij} x^j, \] (38)

which are the components of a rotational Killing vector field \( m \) of \( E^n \).

3 Komar integrals

The ADM-type surface integral (34) can be written as the following bulk integral

\[ \mathcal{E} = \frac{1}{2} \int d^n x \left[ h_{ij,ij} - h_{jj,ii} - h_{mm,ii} \right], \] (39)

where repeated indices are summed. This assumes that the metric perturbation is not so large in the interior that it changes the topology of space; otherwise we cannot view the metric perturbation as a field on Minkowski spacetime. However, as long as we express any final result as a surface integral at infinity, this restriction can make no difference to the result. We may now use the Einstein equations in the form (12) to deduce alternative bulk energy integrals, some of which may then be re-expressed as surface integrals; this is how one gets from the ADM energy integral (5) to the Komar formula (7). We shall now explain this point in some detail for the general \( p \geq 0 \) case.

The equations (12) are equivalent to the Pauli-Fierz equations

\[ \Box h_{MN} + h_{MN} = 2h_{(M,N)} = -2 \left( T_{MN} - \frac{1}{D-2} \eta_{MN} T \right) \] (40)

where \( T = \eta^{MN} T_{MN} \). For a time-independent metric perturbation these equations imply that

\[ (D - 3) \left[ \langle h_{ij},ij \rangle - \langle h_{jj},ii \rangle - \langle h_{mm},ii \rangle \right] = - \left[ (D - 2) \nabla^2 h_{00} + 2 (\langle T_{ii} \rangle + \langle T_{mm} \rangle) \right]. \] (41)

Integrating both sides over the \( n \)-dimensional transverse Euclidean space we deduce (as long as \( D > 3 \) that

\[ \mathcal{E} = - \left( \frac{D - 2}{2(D - 3)} \right) \oint_{\infty} dS_i \partial_i h_{00} + \frac{p}{D-3} \mathcal{T} - \frac{1}{D-3} \int d^n x \langle T_{ii} \rangle, \] (42)

\(^4\text{See [14] for a discussion of angular momentum in General Relativity.}\)
where $E$ is the ADM energy density (39) and $T$ is the tension, as given in (30). Now, the conservation condition (18) implies that
\[ \partial_i \langle T_{ij} \rangle = 0, \] (43)
and hence that
\[ \int d^p x \langle T_{ii} \rangle = \oint dS_i \langle x^j T_{ij} \rangle = 0, \] (44)
where the last equality follows from the assumption that $T_{ij}$ vanishes at transverse spatial infinity. Thus, the $T_{ii}$ term in (42) does not contribute. Consequently, (42) reduces to (7) for $p = 0$. For $p > 0$, however, the tension contributes to the energy density and we need to rewrite it too as a surface term. Using the Pauli-Fierz equation again we deduce, after further use of (44), that
\[ T = - \left( \frac{1}{2p(n-2)} \right) \oint dS_i \partial_i [p h_{00} - (n + p - 2) h_{mm}], \] (45)
where we have again omitted the averaging since this makes no difference at transverse spatial infinity. Combining this formula with (42), we obtain the alternative energy surface integral (summed over the index $m$)
\[ E = - \left( \frac{1}{2(n-2)} \right) \oint dS_i \partial_i [(n - 1) h_{00} - h_{mm}]. \] (46)
This is equivalent to the formula given in [6], but now derived for a much more general class of spacetimes (even within the subclass of translationally invariant ones). In addition (45) provides a similar surface integral for the tension $T$. Note that for isotropic boost invariant p-brane spacetimes we have $h_{mm} = - h_{00}$ for each $m$, so that $T = E$, as claimed earlier.

We now aim to rewrite these surface integrals in covariant form. Let $k$ and $\ell_{(m)}$ be vector fields that are asymptotic to $\vec{k}$ and $\vec{\ell}_{(m)}$ as $|x| \to \infty$. Then, as may be verified,
\[ E = - \left( \frac{1}{2(n-2)} \right) \oint dS_{MN P_1 \ldots P_p} \left[ (n - 1)(D^M k^N) \ell_{P_1}^{(1)} \cdots \ell_{P_p}^{(p)} + k^N D^M \ell_{P_1}^{(1)} \cdots \ell_{P_p}^{(p)} \right], \] (47)
where $D$ is the usual gravitational covariant derivative associated with the full metric $g$.

Similarly, (for $p > 0$) we have
\[ T = - \left( \frac{1}{2p(n-2)} \right) \oint dS_{MN P_1 \ldots P_p} \left[ p (D^M k^N) \ell_{P_1}^{(1)} \cdots \ell_{P_p}^{(p)} + (n + p - 2) k^N D^M \ell_{P_1}^{(1)} \cdots \ell_{P_p}^{(p)} \right]. \] (48)

The angular momentum per unit p-volume $J$ associated with a rotational Killing vector field $m$ may be similarly expressed as a covariant surface integral. Let $m$ have the asymptotic form (38) in cartesian coordinates at transverse spatial infinity, and define
\[ J(m) = \frac{1}{2} \Omega_{ij} L^{ij}, \quad \left( m \sim \Omega_{ij} x^j \frac{\partial}{\partial x^i} \right). \] (49)
It may be verified that
\[ J(m) = \frac{1}{2} \oint dS_{MN P_1 \ldots P_p} (D^M m^N) \ell_{P_1}^{(1)} \cdots \ell_{P_p}^{(p)}. \] (50)
4 Smarr-type formula

We now aim to use the covariant surface integrals for brane tension and angular momentum to derive an analog for branes of the Smarr formula for the mass of a stationary black hole [8]. We will suppose that a stationary p-brane spacetime is regular on and outside a Killing horizon of the Killing vector field

\[ \xi = k + \Omega_H \cdot m \]  

(51)

where \( k \) is the unique normalized Killing vector field that is timelike at transverse spatial infinity (normalized such that \( k^2 \to -1 \)) and \( m = (m_1, \ldots) \) is a set of commuting rotational Killing vector fields, normalized such that their orbits have length \( 2\pi R \) as \( R \to \infty \), where \( R \) is distance from the horizon; the constants \( \Omega_H \) are the corresponding components of the angular velocity of the horizon. Now use (51) to rewrite the expression (17) as

\[ \mathcal{E} = -\frac{1}{2(n-2)} \int_\infty dS_{MNP_1\ldots P_p} \left\{ (n-1)(D^M \xi^N) \ell_{(1)}^{P_1} \ldots \ell_{(p)}^{P_p} \right. 
\]

\[ + \xi^N D^M \left( \ell_{(1)}^{P_1} \ldots \ell_{(p)}^{P_p} \right) \left\} + \frac{(n-1)}{n-2} \Omega_H \cdot J \right. 
\]

\[ + \frac{1}{2(n-2)} \Omega_H \cdot \int_\infty dS_{MNP_1\ldots P_p} m^N D^M \left( \ell_{(1)}^{P_1} \ldots \ell_{(p)}^{P_p} \right), \]  

(52)

where

\[ J = \frac{1}{2} \int_\infty dS_{MNP_1\ldots P_p} (D^M m^N) \ell_{(1)}^{P_1} \ldots \ell_{(p)}^{P_p}. \]  

(53)

By means of Gauss’s law, we can rewrite the surface integral at infinity for \( \mathcal{E} \) as the sum of a surface integral over the horizon and a bulk integral over the region bounded by the horizon and transverse spatial infinity on an \( n \)-surface \( \Sigma \) having \( k \) as one of its \( (p+1) \) normals. In doing so we shall assume that the vector fields \( \ell_m \) are everywhere Killing, and not just asymptotically Killing, which means that we now make the restriction to brane spacetimes that are translationally invariant as well as stationary. This means that there are \( p \) translational Killing vector fields \( \{ \ell \} \) in addition to the timelike Killing vector field \( k \), and we assume that all are mutually commuting.

We begin by spelling out the conditions that the rotational Killing vector fields \( m \) must satisfy, corresponding to the assumption that they generate rotations in the transverse space. We assume that each of the vector fields \( \{ \ell \} \) commutes with \( m \). We shall also assume that the rotational Killing vector fields \( m \) satisfy

\[ m^M dS_{MN_1\ldots N_p} = 0, \]  

(54)

where \( dS_{MN_1\ldots N_p} \) is the volume element of \( \Sigma \). This condition is automatically satisfied for \( p = 0 \) as long as \( \Sigma \) is spacelike; if, for \( p > 0 \), the Killing vector fields \( \{ \ell \} \) are normal to \( \Sigma \) then it is equivalent to \( m \cdot \ell_{(m)} = 0 \). Given these restrictions on \( m \), an application of Gauss’s law yields

\[ \int_\infty dS_{MNP_1\ldots P_p} m^N D^M \left( \ell_{(1)}^{P_1} \ldots \ell_{(p)}^{P_p} \right) = \int_H dS_{MNP_1\ldots P_p} m^N D^M \left( \ell_{(1)}^{P_1} \ldots \ell_{(p)}^{P_p} \right). \]  

(55)

An application of Gauss’s law to the other surface integral on the right hand side of (52), yields both an integral over the horizon and a bulk integral over \( \Sigma \). Rewriting the latter in terms of the matter stress tensor by means of the Einstein equation

\[ R_{MN} = T_{MN}^{(mat)} - \frac{1}{D-2} g_{MN} T^{(mat)} \quad (T^{(mat)} = g^{MN} T_{MN}^{(mat)}), \]  

(56)
we deduce that
\[
E = -\frac{1}{2(n-2)} \int_H dS_{\mathcal{M}N_1...N_p} \left\{ (n-1)(D^M \xi^N) \ell^{P_1}_{(1)} \cdots \ell^{P_p}_{(p)} + \xi^N D^M \left( \ell^{P_1}_{(1)} \cdots \ell^{P_p}_{(p)} \right) \right\} \\
+ \frac{n-1}{n-2} \Omega_H \cdot \mathcal{J} + \frac{1}{2(n-2)} \Omega_H \cdot \int_H dS_{\mathcal{M}N_1...N_p} \mathbf{m}^N D^M \left( \ell^{P_1}_{(1)} \cdots \ell^{P_p}_{(p)} \right) \\
+ E^{(\text{mat})},
\]
where
\[
E^{(\text{mat})} = \frac{1}{n-2} \int_{\Sigma} dS_{\mathcal{M}N_1...N_{p-1}} \left\{ \ell^{P_1}_{(1)} \cdots \ell^{P_{p-1}}_{(p-1)} \ell^N_{(p)} \left[ (n-1) \left( T^{(\text{mat})} \right)^M Q \xi^Q \right] \right. \\
- \left. \left( T^{(\text{mat})} \right)^M \xi^M \right\} - p \ell^{P_1}_{(1)} \cdots \ell^{P_{p-1}}_{(p-1)} \ell^Q_{(p)} \left( T^{(\text{mat})} \right)^M Q \xi^N.
\]
For the moment we will assume that the matter stress tensor vanishes. This means, in particular, that we are now considering uncharged black branes since a charge would produce ‘electric’ or ‘magnetic’ fields with a non-vanishing stress tensor; we will consider the case of electrically-charged branes in the following section.

Given that \(E^{(\text{mat})}\) vanishes, we have only to simplify and interpret the horizon integrals in (57) to obtain a brane analogue of the Smarr formula for neutral black branes. On the horizon, the orientation of the surface element \(dS_{\mathcal{M}N_1...N_p}\) is determined by \(\xi\) itself, \(p\) other normals \(\zeta_{(m)}\) and one other null vector \(\ell\) which we may choose such that
\[
\xi \cdot n = -1.
\]
Since \(\xi\) is tangent to the horizon, as well as normal to it, it must be orthogonal to the other \(p\) normals, so
\[
\xi \cdot \zeta_{(m)} = 0.
\]
Given that the horizon surface area element (per unit \(p\)-volume) has magnitude \(dA\), we then have
\[
dS_{\mathcal{M}N_1...N_p} = [(p+2)!] v^{-1}(\zeta) dA \xi_{[M} n_{N} \zeta_{(1)}^{P_1} \cdots \zeta_{(p)}^{P_p]},
\]
where
\[
v(\zeta) = |\zeta_{(1)} \wedge \cdots \wedge \zeta_{(p)}|.
\]
For a generic \(p\)-brane spacetime we may choose
\[
\zeta_{(m)} = \ell_{(m)},
\]
although this will not be possible in those cases in which one or more of the vector fields \(\ell_{(m)}\) has a fixed point on the horizon, as happens for certain extremal \(p\)-brane solutions. However, such non-generic \(p\)-brane spacetimes can be considered as limits of generic spacetimes, so we shall assume here that the choice (63) is possible. In this case \(v(\zeta)\) equals \(v(\ell)\), which is the ratio of the worldspace volume on the horizon to the worldspace volume \(V_p\) at infinity.

Given that
\[
(\xi \cdot D\xi)^M = \kappa \xi^M
\]
on the horizon, for surface gravity \(\kappa\), we then find that
\[
dS_{\mathcal{M}N_1...N_p} (D^M \xi^N) \ell^{P_1}_{(1)} \cdots \ell^{P_p}_{(p)} = -2\kappa v(\ell) dA.
\]
Given that \(\xi\) commutes with the Killing vector fields \(\{\ell\}\), we find similarly that
\[
dS_{\mathcal{M}N_1...N_p} \xi^N D^M \left( \ell^{P_1}_{(1)} \cdots \ell^{P_p}_{(p)} \right) = 0.
\]
Finally, the Killing vector fields $\mathbf{m}$ are necessarily tangent to the horizon (as for black holes) and hence orthogonal to its normals:

$$\xi \cdot \mathbf{m} \big|_H = 0, \quad \ell_{(m)} \cdot \mathbf{m} \big|_H = 0. \quad (67)$$

Since $\mathcal{L}_\xi \ell_{(m)} = 0$, it then follows that the horizon integral involving $\mathbf{m}$ vanishes. Putting together these results for the horizon integrals, and assuming that the zeroth law of black hole mechanics (constancy of $\kappa$ on the horizon) continues to apply to black branes, we can simplify (57) to

$$\mathcal{E} = \left(\frac{n-1}{n-2}\right) \left[ \kappa \mathcal{A}_{\text{eff}} + \mathbf{\Omega}_H \cdot \mathbf{J} \right], \quad (68)$$

where

$$\mathcal{A}_{\text{eff}} \equiv \oint_H v(\ell) \, dA \quad (69)$$

is the ‘effective $(n-1)$-area’ of the horizon. This generalizes the formula of [13] for D-dimensional stationary black holes to stationary and translationally-invariant black branes. Note that the total horizon $(n+p-1)$-area is

$$A = \mathcal{A} v(\ell) V_p = \mathcal{A}_{\text{eff}} V_p. \quad (70)$$

An entirely analogous computation, applied to the Komar-type formula (48) for the brane tension yields the result

$$\mathcal{T} = \left(\frac{1}{n-1}\right) \mathcal{E}. \quad (71)$$

An immediate consequence of this is that an uncharged isotropic stationary black brane cannot be boost invariant, since boost-invariance implies $\mathcal{T} = \mathcal{E}$. The qualification ‘black’ is essential here since there is no general principle that would forbid uncharged boost-invariant branes; Dirac’s relativistic membrane action and its p-brane generalization provides an effective (worldvolume) description of such an object.

### 5 Behaviour under dimensional reduction

Given the black hole result of [13], the Smarr-type formula (68) for black branes is essentially what one would expect from the fact that black holes are obtainable from black branes by dimensional reduction, except possibly for the appearance of the factor $v(\ell)$ in the ‘effective’ horizon area. We will now confirm this intuition, and find an interpretation for $\mathcal{A}_{\text{eff}}$ and a check on the relation (71).

Consider the black hole solution of the $d = (n+1)$ dimensional Einstein equations obtained by dimensional reduction of a black $p$-brane solution of the $D = (d+p)$ dimensional Einstein equation along orbits of the vector fields $\{\ell\}$. In coordinates $(x^I, y^m)$, where $I = (0, i)$ and $\partial/\partial y^m$ are the $p$ commuting translational Killing vector fields, the general translationally-invariant p-brane D-metric takes the form

$$ds_D^2 = g_{IJ} dx^I dx^J + g_{mn} \left(dy^m + (p)g^{mp}g_{pK} dX^K\right) \left(dy^n + (p)g^{nq}g_{qL} dX^L\right) \quad (72)$$

where $(p)g^{mn}$ is the $p \times p$ matrix inverse of $g_{mn}$, and all components are $y$-independent. Note that

$$v(\ell) = \sqrt{\det g_{mn}}, \quad (73)$$

and that

$$\sqrt{-\det g_{MN}} = v(\ell) \sqrt{-\det g_{IJ}}. \quad (74)$$
We will assume that the p-brane has finite total p-volume $V_p$. Integrating over the p-surface spanned by orbits of the Killing vector fields $\{\ell_\ell\}$, and then dividing by $V_p$, reduces the D-dimensional Einstein action to the $d$-dimensional one
\[
I = \frac{1}{2} \int d^d x \sqrt{-\det \tilde{g}} \tilde{R} + \tilde{I}^{(\text{mat})},
\]
where
\[
\tilde{g}_{IJ} = [v(\ell)]^{-1} g_{IJ},
\]
and $\tilde{I}^{(\text{mat})}$ now includes the vector fields $g_{Lm}$ and the scalar fields $g_{mn}$. It follows immediately from (76) that the black hole horizon $(n-1)$-area element equals $v(\ell)$ times the $(n-1)$-area element of the p-brane horizon, which is precisely the ‘effective’ area density $A_{\text{eff}}$.

It also follows from (76) that
\[
\tilde{h}_{IJ} = h_{IJ} + \frac{1}{n-1} \eta_{IJ} h_{mm} + O(h^2),
\]
(77)

The ADM energy on constant time surfaces of the $d$-dimensional spacetime is
\[
E = \frac{1}{2} \oint \infty dS_i \left[ \partial_j \tilde{h}_{ij} - \partial_i \tilde{h}_{jj} \right]
= \frac{1}{2} \oint \infty dS_i \left[ \partial_j \tilde{h}_{ij} - \partial_i \left( \tilde{h}_{jj} + h_{mm} \right) \right],
\]
in agreement with the ADM integral (34) for $p$-brane energy density. Similarly, the Komar energy integral for the $d$-dimensional spacetime is equivalent to
\[
E = -\left( \frac{n-1}{2(n-2)} \right) \oint \infty dS_i \partial_i \tilde{h}_{00}
= -\left( \frac{1}{2(n-2)} \right) \oint \infty dS_i \partial_i [(n-1)h_{00} - h_{mm}],
\]
in agreement with the formula (46) for $p$-brane energy density.

The above analysis is as valid for charged branes, which we will consider in the following sections, as uncharged ones, but it is instructive to consider the special case of uncharged black p-branes of the vacuum D-dimensional Einstein equations. These are found by a trivial lift of stationary black hole solutions of the vacuum $d$-dimensional Einstein equations, and their D-metric takes the simple form
\[
ds^2_D = g_{IJ} dx^I dx^J + dy^m dy^n.
\]
(80)

It follows that $h_{mn} = 0$ and hence, from (43) and (46), that $E = (n-1)T$, in agreement with (44). It is worth noting here that charged black p-branes cannot be obtained from charged black holes in the quite the same way because these are not solutions of the vacuum Einstein equations, and even if the components of the stress tensor in the extra dimensions are zero we still have $R_{mn}$ proportional to the trace of the matter stress tensor. Thus, charged black p-branes will not generally have vanishing $h_{mn}$, and hence the relation (71) need not hold for them.

### 6 Extension to charged branes

We now aim to generalize the Smarr-type formula (88) to allow for (electrically) charged branes. A charged p-brane is a source for a $(p+1)$ form potential $A$ with $(p+2)$-form
field strength $F = dA$, so we must now allow for a matter stress tensor of the form

$$ T^{(mat)}_{MN} = \frac{1}{(p+1)!} \left[ F_{MNP_1...P_p} F_{NQ_1...Q_{p+1}} - \frac{1}{2(p+2)} g_{MN} F^2 \right] $$

(81)

where $F^2 = F_{MNP_1...P_p} F^{MNP_1...P_p}$. In this case

$$ \mathcal{E}^{(mat)} = \left( \frac{1}{(n-2)(p+1)!} \right) \int_{\Sigma} dS_{MNP_1...P_p} \left\{ \ell_{(1)}^{P_1} \cdots \ell_{(p-1)}^{P_{p-1}} \ell_{(p)}^{N} \right. $$

$$ \times \left. \left[ (n-1) F^{MR_1...R_{p+1}} Q F_{QR_1...R_{p+1}} - \frac{(p+1)}{p+2} F^2 \xi^M \right] - p \ell_{(1)}^{P_1} \cdots \ell_{(p-1)}^{P_{p-1}} \ell_{(p)}^{Q} F_{QR_1...R_{p+1}} F^{MR_1...R_{p+1}} \xi^N \right\} . $$

(82)

We will assume that the field strength $F$ is invariant under the symmetries generated by the Killing vector fields $\xi$ and $\ell_{(m)}$, and we choose a gauge for $A$ such that

$$ \mathcal{L}_\xi A = 0, \quad \mathcal{L}_{(m)} A = 0. $$

(83)

It then follows that

$$ F^{MR_1...R_{p+1}} Q F_{QR_1...R_{p+1}} = - (p+1) D_P \left[ F^{MPR_1...R_{p}} Q A_{QR_1...R_{p}} \right] $$

$$ + (p+1) \left( D_P F^{MPR_1...R_{p}} \right) \left( \xi^P A_{QR_1...R_{p}} \right), $$

(84)

and similar expressions with $\xi$ replaced by a brane translation Killing vector field $\ell$. It also follows that

$$ \xi^M F^2 = 2(p+2) D_P \left[ \xi^M F^{P[Q_1...Q_{p+1}} A_{Q_1...Q_{p+1}] Q} \right] $$

$$ - (p+2) \xi^M (D_P F^{P[Q_1...Q_{p+1}} A_{Q_1...Q_{p+1}]}). $$

(85)

Using these results, and the field equation

$$ D_M F^{MN_1...N_{p+1}} = 0, $$

(86)

we deduce that

$$ \mathcal{E}^{(mat)} = - \frac{(p+2)}{2(n-2)!} \int_{\Sigma} dS_{MNP_1...P_p} \left\{ \ell_{(1)}^{P_1} \cdots \ell_{(p-1)}^{P_{p-1}} \ell_{(p)}^{N} \right. $$

$$ \times \left. \left[ (n-1) F^{MSR_1...R_{p}} Q \xi^Q A_{QR_1...R_{p}} - p \ell_{(1)}^{P_1} \cdots \ell_{(p-1)}^{P_{p-1}} \ell_{(p)}^{Q} \xi^Q F^{MSR_1...R_{p}} A_{QR_1...R_{p}} \right] \right\}. $$

(87)

This bulk integral equals the difference of surface integrals at infinity and the horizon. If we choose a gauge for which $A$ vanishes at infinity then the surface term at infinity also vanishes and we are left with the following surface integral over the horizon:

$$ \mathcal{E}^{(mat)} = \frac{1}{2(n-2)!} \int_{H} dS_{MNP_1...P_p} \left\{ \ell_{(1)}^{P_1} \cdots \ell_{(p-1)}^{P_{p-1}} \ell_{(p)}^{N} \times \right. $$

$$ \left. \left[ (n-1) F^{MSR_1...R_{p}} \xi^Q + 2 F^{SQR_1...R_{p}} \xi^M \right] A_{QR_1...R_{p}} \right. $$

$$ \left. - p \ell_{(1)}^{P_1} \cdots \ell_{(p-1)}^{P_{p-1}} \ell_{(p)}^{Q} \xi^Q F^{MSR_1...R_{p}} A_{QR_1...R_{p}} \right\}. $$

(88)

We now have to evaluate this integral and interpret the result.
We begin by observing that
\[ \xi M \left( \xi^M F^{NQR_1 \ldots R_p} \right) \big|_H = 0. \] (89)

This can be proved as follows. Both \( \xi^2 \) and \( R_{MN} \xi^M \xi^N \) vanish on a Killing horizon of \( \xi \), so the Einstein equation (56) implies that the \((p+1)\)-form \( i_\xi F \) has vanishing norm on the horizon. However, we also have \( i_\xi (i_\xi F) \equiv 0 \). These properties of \( i_\xi F \) and \( \xi \) imply that \( \xi \wedge i_\xi F = 0 \) on the horizon (where \( \xi \) is here the 1-form constructed from the timelike Killing vector field and the metric \( g \)) and hence (since \( \xi \) is null on the horizon) that \( i_\xi (\xi \wedge F) \) vanishes on the horizon, but this is just (89). Given the form (61) of the horizon surface element, and defining the set of rank \((p+3)\) antisymmetric tensors
\[ K_{(m)}^{M_1 \ldots N_{p+2}} \equiv (p+3) \ell_{(m)}^M F_{N_1 \ldots N_{p+2}}, \] (90)

we can now rewrite (88) as
\[ E^{(\text{mat})} = -\frac{1}{2(n-2)p!} \oint_H dS_{MN1 \ldots P_{p-1}} \ell_{(1)}^{P_1} \ldots \ell_{(p-1)}^{P_{p-1}} F^{MSR_1 \ldots R_p} \Phi_{R_1 \ldots R_p}, \] (91)

where
\[ \Phi_{R_1 \ldots R_p} = -\xi^Q A_{QR_1 \ldots R_p}. \] (92)

We shall now make the simplifying assumption that
\[ K_{(m)}^{M_1 \ldots M_{p+3}} = 0 \quad (m = 1, \ldots, p). \] (93)

We postpone discussion of the significance of this constraint. For the moment we need only observe that it allows us to simplify the horizon integral (91) to
\[ E^{(\text{mat})} = \frac{1}{(p+2)!} \oint_H dS_{MN1 \ldots P_p} F^{MNP_1 \ldots P_p} \Phi_H, \] (94)

where \( \Phi_H \) is the electric potential of the horizon, given by
\[ \Phi_H = \ell_{(1)}^{P_1} \ldots \ell_{(p)}^{P_p} \Phi_{R_1 \ldots R_p} \big|_H. \] (95)

The electric potential is constant over the horizon. To see this we first recall that \( i_\xi F \wedge \xi = 0 \) on the horizon. This implies, since \( \xi \cdot \ell_{(m)} = 0 \) on the horizon, that
\[ (i_{(1)} \ldots i_{(p)} F) \wedge \xi = 0 \] (96)
on the horizon, where \( i_{(m)} \) indicates contraction with the vector field \( \ell_{(m)} \). It follows that the 1-form \( i_{(1)} \ldots i_{(p)} F \) is proportional to \( \xi \) on the horizon, and hence that
\[ i_t (i_{(1)} \ldots i_{(p)} F) \big|_H = 0 \] (97)
for any tangent \( t \) to the horizon. Since \( A \) is assumed to be invariant under the symmetries generated by \( \xi \) and \( \ell_{(m)} \) this is equivalent to
\[ i_t d\Phi_H = 0, \] (98)

which states that \( \Phi_H \) is constant on the horizon. We can therefore take \( \Phi_H \) outside the integral to get
\[ E^{(\text{mat})} = \frac{1}{(p+2)!} \Phi_H \oint_H dS_{MN1 \ldots P_p} F^{MNP_1 \ldots P_p}. \] (99)
Given the field equation (86) we can move the integration surface from the horizon to infinity to get
\[ E^{(mat)} = \Phi_H Q \]
where
\[ Q = \frac{1}{(p+2)!} \oint_\infty dS_{MN\ldots P} F^{MNP_{\ldots P}} \]
is the total 'electric' charge per unit p-volume. Using this result in (88) we arrive at the following Smarr-type formula for the energy per unit p-volume of a charged black p-brane:
\[ E = \left( \frac{n-1}{n-2} \right) \kappa A_{eff} + \left( \frac{n-1}{n-2} \right) \Omega_H \cdot J + \Phi_H Q . \]
This generalizes the result of [16] for charged black holes. An entirely analogous calculation, applied to the Komar-type formula (45) for the brane tension yields the relation
\[ T - \Phi_H Q = \left( \frac{1}{n-1} \right) [E - \Phi_H Q] , \]
which generalizes (71).

In the supergravity context, the energy density of any 'physical' isotropic brane solution of the vacuum supergravity equations can be shown to satisfy the bound
\[ E \geq \Phi_H Q , \]
with equality for solutions that are 'supersymmetric'. From (103) it can be seen that this condition is equivalent to vanishing \(\kappa\) and \(\Omega_H\): 'supersymmetric' is therefore a stronger condition than 'extremal' (\(\kappa = 0\)). When combined with (103), the bound on the energy density implies the bounds
\[ T \leq E , \]
and
\[ T \geq \Phi_H Q , \]
both of which are saturated by supersymmetric (i.e., isotropic boost-invariant) branes.

7 The first law of black branes

So far we have derived a Smarr-type formula for the energy per unit p-volume of a stationary and translationally invariant black p-brane that solves the Euler-Lagrange equations of the action
\[ I = \frac{1}{2} \int d^D x \sqrt{-\det g} \left[ R - \frac{1}{(p+2)!} F^2 \right] , \]
for \((p + 2)\) form field strength \(F = dA\). If the metric and the \((p + 1)\)-form potential \(A\) are assigned dimension zero then the dimension of each term in the action is determined by the number of derivatives. Since both terms in (107) have two derivatives they also have the same dimension, and it follows from this that the field equations are invariant under a constant rescaling of the coordinates, which therefore takes one black p-brane solution into another one via a rescaling of any dimensionful parameters on which the solution depends. Thus any dimensionful quantity, such as the horizon area, must be

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5The proof of this bound in [13] does not assume supersymmetry but it turns out that, for \(p > 0\), the conditions required to establish it are satisfied only by theories that are 'supersymmetrizable'. As the proof does not assume the existence of a horizon, the coefficient of proportionality was not identified in [13] as the horizon potential; but this was pointed out for D=5 black holes in [16].
a weighted homogeneous function of any set of independent parameters of the solution, the weights being determined by the dimensions of these parameters. In the black hole \((p = 0)\) case, the uniqueness theorems imply that the only independent parameters of the solution are the mass \(M\), charge \(Q\) and angular momenta \(J\), so the horizon area \(A\) is a weighted homogeneous function of them. Equivalently, \(M\) is a weighted homogeneous function of \(Q, J\) and \(A\). Given this, the first law may be derived by an application of Euler’s theorem on weighted homogeneous functions. This method was used in [16] to derive the first law for charged black holes in an arbitrary spacetime dimension from a Smarr-type relation that is formally identical to (102). The only difference is that for \(p > 0\) the extensive variables \(M, Q, J, A\) must be replaced by the densities \(E, Q, J, A_{eff}\). The first three of these are parameters that determine the coefficients of the asymptotic metric and \((p + 1)\)-form potential, and their dimensions are therefore determined by the requirement that the fields be dimensionless. This yields:

\[
\begin{align*}
[Q] &= [E] = L^{n-2}, & [J] &= L^{n-1} \\
\end{align*}
\]

In addition, we have

\[
[A_{eff}] = L^{n-1}
\]

because it is a measure of the ‘area’ of an \((n - 1)\)-surface. We thus have, for all \(p\),

\[
\begin{align*}
[Q] &= [E], & [J] &= [A_{eff}] = [E]^{(n-1)/(n-2)}.
\end{align*}
\]

If we know that \(E\) is some function of \(Q, J\) and \(A_{eff}\) (which presumably follows from some extension to \(D > 4\) and \(p > 0\) of the uniqueness theorems for \(D=4\) black holes) then Euler’s theorem implies that

\[
E = Q \frac{\partial E}{\partial Q} + \left(\frac{n-1}{n-2}\right) J \cdot \frac{\partial E}{\partial J} + \left(\frac{n-1}{n-2}\right) A_{eff} \frac{\partial E}{\partial A_{eff}}.
\]

Comparing this with (102) we see that

\[
\begin{align*}
\frac{\partial E}{\partial Q} &= \Phi_H, & \frac{\partial E}{\partial J} &= \Omega_H, & \frac{\partial E}{\partial A_{eff}} &= \kappa
\end{align*}
\]

and hence the first law,

\[
dE = \kappa dA_{eff} + \Phi_H dQ + \Omega_H \cdot dJ.
\]

For even \(p\) and spacetime dimension \(D = 3p + 5\) one can add to the action an additional ‘FFA’ Chern-Simons (CS) term. This leads to a modification of the field equation (86) to one of the form

\[
D_ME^{M_1...N_{p+1}} \propto Z^{N_1...N_{p+1}},
\]

where

\[
Z^{N_1...N_{p+1}} = \varepsilon^{N_1...N_{p+1}M_1...M_{p+2}L_1...L_{p+2}}F_{M_1...M_{p+2}}F_{L_1...L_{p+2}}
\]

is a topological \((p + 1)\)-form current. As the CS term has the same dimension as the other terms in the action, the modified field equations again involve no dimensionless parameters and the Euler’s theorem argument still applies. For \(p = 0\) and \(p = 2\) this possibility is relevant to supergravity theories since both the pure \(D=5\) supergravity and \(D=11\) supergravity are of this form, with a definite non-zero coefficient for the CS term.

\footnote{We believe that this argument was originally due to G.W. Gibbons; dimensional considerations also play a part in the proof of the first law developed by Wald.}

\footnote{It might seem odd that the dimensions of \(E\) and \(J\), for example, are not related in the way one would expect for energy and angular momentum, but this is due to implicit factors of Newton’s constant \(G\).}
For $p = 0$ the previous derivation of the Smarr-type formula (102) requires modification to take into account the fact that $D_M F^{MN}$ no longer vanishes. This was done in [16], where it was shown that all the additional terms cancel. For $p > 0$ the analogous computation does not lead to a similar cancellation, but for $p > 0$ our assumption that $F$ is restricted by (93) implies that the topological current $Z$ vanishes, so in either case the formula (102) is unchanged.

At this point, we should address the physical meaning of the condition (93). It implies

$$F_{MNP_1...P_p} = f_{[MN} e^{(1)}_{P_1} \cdots e^{(p)}_{P_p]}$$  (116)

for some 2-form $f$, so (93) is essentially a truncation to those fields that would yield a pure Einstein-Maxwell system upon dimensional reduction to a $d = n + 1$ dimensional spacetime. In view of this, it is not surprising that the first law of black brane mechanics (113) is just such that it reduces to the first law for charged black holes of Einstein-Maxwell theory, and it could have been deduced in this way. However, this fact alone does not provide much insight into the status of (93). Some insight can be had from specific examples. Consider the rotating non-extremal membrane solutions of D=11 supergravity of [9]; it may be verified that they satisfy (93). In the non-rotating case $F$ is purely electric. A magnetic component is induced by rotation but $F$ remains of the form (116). Obviously, there can be no net magnetic charge because the magnetic object is a fivebrane. More generally, whenever $D = 3p + 5$ the electric $p$-brane spacetime can carry no net magnetic charge, and one might expect, by some extension of the black hole uniqueness theorems, that all stationary and translationally-invariant $p$-brane spacetimes will be determined by their energy density angular momentum density and electric charge density. In this case, the above example suggests that (93) will be a consequence of the assumptions of stationarity and translational invariance for actions of the form assumed here, at least in those spacetime dimensions $D$ for which a CS term is possible.

The first law (113) is a statement that relates a change in the energy density to changes in other densities. By setting

$$E = \mathcal{E} V_p, \quad A = A_{\text{eff}} V_p, \quad J = J V_p, \quad Q = Q V_p$$  (117)

we deduce that

$$dE = \kappa dA + \Omega_H \cdot dJ + \Phi_H dQ + T_{\text{eff}} dV_p,$$  (118)

where, after use of both (102) and (103), the ‘effective tension’ is found to be

$$T_{\text{eff}} = T - \Phi_H Q.$$  (119)

For constant $V_p$ this is just the first law of black hole mechanics, since $A$ is equal to the total horizon area, but a change in $V_p$ induce changes in $E$ proportional to what we have called the ‘effective tension’. For neutral branes this is just the ADM tension, but for charged branes there is an additional, negative, contribution to the effective tension proportional to the charge $Q$.

We pointed out in the previous section that $T$ is subject to a lower bound, which we can now write as

$$T_{\text{eff}} \geq 0.$$  (120)

We also pointed out that saturation of this bound occurs only for isotropic boost-invariant branes. In all other cases, the effective tension is strictly positive and the system can lower its energy by reducing $V_p$. This is exactly what one would expect from wrapping a $p$-brane of positive ADM tension on a $p$-torus; the surprise is that this effect of the ADM tension

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*In other words, a positive contribution to the worldspace pressure; this is presumably due to the fact that a charged black brane has a non-zero stress-tensor in the region outside the horizon.
is cancelled for boost-invariant branes. This can be understood from the perspective of the effective worldvolume action. What we have called the effective tension can be identified as the vacuum energy of the brane. There can be a contribution to this vacuum energy from the ‘Wess-Zumino’ term $\int_w \bar{A}$ in the effective action, where the integral is over the worldvolume and $\bar{A}$ is here the pullback to $w$ of the background $(p+1)$-form potential. Since the background is the Minkowski vacuum, $\bar{A}$ is pure gauge. The choice $\bar{A} = -\Phi H vol(w)$ yields a contribution of $-\Phi H Q$ to the vacuum energy; this gauge choice must be the correct one because it ensures that the vacuum energy vanishes when the worldvolume theory is supersymmetric.

8 Summary and Discussion

We hope to have clarified various aspects of the ADM-type integrals for the energy density, tension and angular momentum of ‘brane’ spacetimes that are not, strictly, asymptotically flat but only ‘transverse asymptotically flat’; only the ADM formula for energy density had been previously discussed. We then used these results to find new covariant Komar-type surface integrals for the energy density, tension and angular momentum of transverse asymptotically flat brane spacetimes that are also asymptotically stationary and asymptotically translationally invariant in brane directions. The formula for the energy density, in particular, is not an obvious generalization of the $p=0$ case because the brane tension (which is absent for $p=0$) contributes to the energy density.

As an application of our Komar-type surface integrals, we deduced a Smarr-type formula for the energy density of translationally invariant charged black brane spacetimes, and thence the first law of black brane mechanics. The simplest case of physical interest to which our analysis applies is that of 2-brane solutions of D=11 supergravity, which couple to the 3-form potential of D=11 supergravity. It would be of interest to extend our results to 5-branes of D=11 supergravity but this will require a better understanding of how to include both electric and magnetic charges in theories with CS terms.

From the Smarr-type formula we deduced a version of the first law of black brane mechanics that relates changes in densities; this is formally the same as the first law of black hole mechanics obtained in [1] but the horizon area is now replaced by an ‘effective’ area per unit p-volume. These results can be understood via a dimensional reduction of the p-brane to a black hole, but the first law of black hole mechanics that one obtains this way now involves the worldspace volume and an ‘effective’ tension as a new conjugate pair. For uncharged black holes the effective tension equals the ADM tension, as one might have anticipated, but for charged branes there is an additional negative contribution that causes the effective tension to vanish for boost-invariant branes, which are supersymmetric in the context of supergravity. This cancellation can be understood from the perspective of an effective worldvolume field theory because the effective tension is the vacuum energy of this theory.

Although the surface integrals that we have discussed are valid for branes that are only asymptotically translationally invariant, our derivation of the first law assumed translation invariance. However, the first law of black brane mechanics must be a special case of the first law of thermodynamics, once quantum mechanics is taken into account, so it should be possible to derive it for branes that are not translationally invariant. Some progress on this front has already been made in [14]; this is an important problem because non-extremal stationary translationally invariant brane spacetimes are generically unstable [15, 20], decaying to some stable stationary brane that is not translationally invariant [21]. From the worldvolume perspective this could be seen as a spontaneous breakdown of translational invariance, but the worldvolume theory cannot be Poincaré invariant because non-extremal black branes are not boost invariant.
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References

[1] R. Arnowit, S. Deser and C. Misner, The dynamics of General relativity, in Gravitation: an introduction to current research, ed. L. Witten (Wiley, 1962).

[2] A. Komar, Covariant conservation laws in General Relativity, Phys. Rev. 113 (1959) 934.

[3] S. Deser and M. Soldate, Gravitational energy in spaces with compactified dimensions, Nucl. Phys. B311 (1987) 739.

[4] L.F. Abbott and S. Deser, Stability of gravity with a cosmological constant, Nucl. Phys. 195 (1982) 76.

[5] K.S. Stelle, BPS branes in supergravity, hep-th/9803116.

[6] J.X. Lu, ADM masses for black strings and p-branes, Phys. Lett. 313B (1993) 29.

[7] J.M. Bardeen, B. Carter and S.W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. 31 (1973) 161.

[8] L. Smarr, Mass formula for Kerr black holes, Phys. Rev. Lett. 30 (1973) 71.

[9] M. Cvetič and D. Youm, Rotating intersecting M-branes, Nucl. Phys. B499 (1997) 253.

[10] J. Traschen and D. Fox, Tension perturbations of black brane spacetimes, gr-qc/0103106.

[11] G.W. Gibbons, Antigravitating black hole solutions with scalar hair in N=4 supergravity, Nucl. Phys. 207 (1982) 337.

[12] G.W. Gibbons, R. Kallosh and B. Kol, Moduli, scalar charges and the first law of thermodynamics, Phys. Rev. Lett. 77 (1996) 4992.

[13] G.W. Gibbons, G.T Horowitz and P.K. Townsend, Higher-dimensional resolution of dilatonic black hole singularities, Class. Quantum Grav. 12 (1995) 297.

[14] A. Ashtekar and R.O. Hansen, A unified treatment of null and spatial infinity in general relativity I, J. Math. Phys. 19 (1978) 1542.

[15] R.C. Myers and M.J. Perry, Black holes in higher-dimensional spacetimes, Ann. Phys. (NY) 172 (1986) 304.

[16] J.P. Gauntlett, R.C. Myers and P.K. Townsend, Black Holes of D=5 Supergravity, Class. Quantum Grav. 16 (1999) 1.

[17] R.M. Wald, The first law of black hole mechanics, gr-qc/9305022; Black hole entropy is Noether charge, Phys. Rev. D48 (1993) 3427.

[18] R. Gregory and R. Lafaille, Black strings and p-branes are unstable, Phys. Rev. Lett. 70 (1993) 2837.

[19] S.S. Gubser and I. Mitra, The evolution of unstable black holes in anti-de Sitter space, hep-th/0011127.
[20] H. Reall, *Classical and thermodynamic stability of black branes*, hep-th/0104071.

[21] G. Horowitz and K. Maeda, *Fate of the black string instability*, hep-th/0105111.