UNIQUENESS OF QUANTIZATION OF COMPLEX CONTACT MANIFOLDS

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Abstract. Using the language of algebroid stacks, we will show that Kashiwara’s quantization of a complex contact manifold is unique.

Introduction

Let \(M\) be a complex manifold and \(P^*M\) its projective cotangent bundle, endowed with the canonical contact structure. Let \(E_M\) be the sheaf of algebras on \(P^*M\) of microdifferential operators. Recall that the order of the operators defines a filtration on \(E_M\) such that its associated graded algebra is isomorphic to \(\bigoplus_{m \in \mathbb{Z}} O_{P^*M}(m)\). (Here \(O_{P^*M}(m)\) denotes the \(m\)-th tensor power of the dual of the tautological bundle \(O_{P^*M}(-1)\)). The product on \(E_M\) is given by the Leibniz rule and it is compatible with the Jacobi structure on \(O_{P^*M}(1)\) induced by the Poisson bracket on \(T^*X\). Hence this algebra provides a quantization of \(P^*M\). Any filtered sheaf of algebras which has \(\bigoplus_{m \in \mathbb{Z}} O_{P^*M}(m)\) as graded algebra and which is locally isomorphic to \(E_M\) gives another quantization of \(P^*M\). We call such an object an \(E\)-algebra.

On a complex contact manifold \(Y\) there may not exist an \(E\)-algebra, that is, a filtered sheaf of algebras which has \(\bigoplus_{m \in \mathbb{Z}} L^{\otimes m}\) as graded algebra (here \(L\) is the line bundle associated to the contact structure) and which is locally isomorphic to \(i^{-1}E_M\), for any contact local chart \(i: Y \supset U \to P^*M\). However, Kashiwara [10] proved that the stack (sheaf of categories) \(\mathcal{M}_E(Y)\) of modules over these locally defined sheaves of algebras is always defined.

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This quantization of $Y$ is better understood using the language of algebroid stacks. In the spirit of Mitchell’s notion of “algebra with several objects”, we may say that an algebroid stack is a sheaf of algebras with locally several (locally) isomorphic objects. With this notion at hand, we may reformulate Kashiwara’s result by saying that there exists an algebroid stack $\mathcal{E}_Y$ over $Y$ such that $\text{Mod}(\mathcal{E}; Y)$ is equivalent to the stack of $\mathcal{E}_Y$-modules. The advantage is that $\mathcal{E}_Y$ has now similar properties to that of an $\mathcal{E}$-algebra: it is filtered with $\bigoplus_{m \in \mathbb{Z}} \mathcal{L}^{\otimes m}$ as associated graded trivial algebroid and locally equivalent to the trivial algebroid $i^{-1}\mathcal{E}_M$, for any contact local chart $i: Y \supset U \to P^*M$.

The purpose of this paper is to show that the algebroid stack $\mathcal{E}_Y$ is the unique quantization of $Y$ endowed with an anti-involution. (We note here that the complex symplectic case behaves differently. Reference is made to [17].)

The paper is organized as follows: in Section 1 we recall the definition of microdifferential operator and that of $\mathcal{E}$-algebra on $P^*M$. In Section 2 we give the main definitions and properties of filtered and graded stacks. In Section 3 we prove the uniqueness of the algebroid stack $\mathcal{E}_Y$ (Theorem 3.3). In Appendix A we recall some basic facts about stacks of 2-groups, which are needed in the proof of Theorem 3.3.

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Notations and conventions All the filtrations are intended to be over $\mathbb{Z}$, increasing and exhaustive. If $A$ (resp. $\mathcal{A}$) is a filtered algebra (resp. sheaf of filtered algebras), we will denote by $\text{Gr}(A)$ (resp. $\mathcal{G}r(\mathcal{A})$) its associated graded algebra (resp. sheaf of graded algebras), and by $\text{Gr}_0(A)$ (resp. $\mathcal{G}r_0(\mathcal{A})$) the algebra (resp. sheaf of algebras) of homogeneous elements of degree 0. We will use similar notations for morphisms.

If $\mathcal{A}$ is a sheaf of algebras, we will denote by $\mathcal{A}^\times$ the sheaf of groups of its invertible elements and, for each section $a \in \mathcal{A}^\times$, by $\text{ad}(a): \mathcal{A} \to \mathcal{A}$ the algebra isomorphism $b \mapsto aba^{-1}$.

We will use the upper index $\text{op}$ to denote opposite structure, when referring either to (sheaves of) algebras, to categories or to stacks.
1. $\mathcal{E}$-algebras on $P^*M$

We recall here the basic properties of the algebra of microdifferential operators. References are made to [19, 9, 11] (see also [20] for an exposition).

Let $M$ be a complex manifold, and $\pi: P^*M \to M$ its projective cotangent bundle. Denote by $\mathcal{E}_M$ the sheaf of microdifferential operators, considered as a sheaf on $P^*M$. In a local coordinate system $(x)$ on $M$, with associated local coordinates $(x; [\xi])$ on $P^*M$, a microdifferential operator $P$ of order $m$ defined on an open subset $U$ of $P^*M$ has a total symbol

$$\sigma_{\text{tot}}(P) = \sum_{j=-\infty}^{m} p_j(x; \xi),$$

where the $p_j$'s are sections on $U$ of $\mathcal{O}_{P^*M}(j)$, subject to the estimates

$$\left\{ \begin{array}{l}
\text{for any compact subset } K \text{ of } U \text{ there exists a constant } 
C_K > 0 \text{ such that for all } j < 0, \sup_K |p_j| \leq C_K^{-j}(-j)!.
\end{array} \right.$$  

(Here $\mathcal{O}_{P^*M}(j)$ denotes the $j$-th tensor power of the dual of the tautological bundle $\mathcal{O}_{P^*M}(-1)$). The product structure on $\mathcal{E}_M$ is given by the Leibniz formula: if $Q$ is another microdifferential operator defined on $U$ of total symbol $\sigma_{\text{tot}}(Q)$, then

$$\sigma_{\text{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{\text{tot}}(P) \partial_x^\alpha \sigma_{\text{tot}}(Q).$$

Recall that the center of $\mathcal{E}_M$ is the constant sheaf $\mathbb{C}_{P^*M}$ and that $\mathcal{E}_M$ is filtered. One denotes by $\mathcal{E}_M(m)$ the sheaf of operators of order less than or equal to $m$ and by

$$\sigma_m(\cdot): \mathcal{E}_M(m) \to \mathcal{E}_M(m)/\mathcal{E}_M(m-1) \simeq \mathcal{O}_{P^*M}(m)$$

the symbol map of order $m$, which does not depend on the local coordinate system on $P^*M$. If $\sigma_m(P)$ is not identically zero, then one says that $P$ has order $m$ and $\sigma_m(P)$ is called the principal symbol of $P$. In particular, an element $P$ in $\mathcal{E}_M$ is invertible if and only if its principal symbol is nowhere vanishing.

Remark 1.1. The algebra $\mathcal{E}_M$ is a quantization of $P^*M$ in the following sense. Denote by $\mathcal{S}_{P^*M}$ the sheaf on $P^*M$ whose local sections are symbols, that is, series $\sum_{j=-\infty}^{m} f_j$ with $f_j \in \mathcal{O}_{P^*M}(j)$ satisfying the
estimates (1.1). It is endowed with a natural filtration and the associated graded sheaf is $\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_{P^*M}(m)$. Then the sheaf $\mathcal{E}_M$ is locally isomorphic to $\mathcal{S}_{P^*M}$ as filtered $\mathbb{C}_{P^*M}$-modules (via the total symbol), and the product on $\mathcal{S}_{P^*M}$ induced by the Leibniz rule is unitary, associative and compatible with the Jacobi structure on $\mathcal{O}_{P^*M}(1)$, that is, the following diagram commutes

\[(1.2)\]

\[
\begin{array}{ccc}
\mathcal{E}_M(1) \times \mathcal{E}_M(1) & \xrightarrow{[\cdot, \cdot]} & \mathcal{E}_M(1) \\
\sigma_1 \times \sigma_1 & \downarrow & \sigma_1
\end{array}
\]

\[
\mathcal{O}_{P^*M}(1) \times \mathcal{O}_{P^*M}(1) \xrightarrow{\{\cdot, \cdot\}} \mathcal{O}_{P^*M}(1).
\]

(Here $[\cdot, \cdot]$ denotes the commutator and $\{\cdot, \cdot\}$ is induced by the Poisson bracket on $T^*M$.)

Let $\Omega_M$ be the canonical line bundle on $M$, that is, the sheaf of forms of top degree. Recall that each locally defined volume form $\theta \in \Omega_M$ gives rise to a local isomorphism $*_{\theta}: \mathcal{E}^{op}_M \simeq \mathcal{E}_M$, which sends an operator $P$ to its formal adjoint $P^*_{\theta}$ with respect to $\theta$. In a local coordinate system $(x)$ satisfying $\theta = dx$, with associated local coordinates $(x; [\xi])$, one has

$$\sigma_{\text{tot}}(P^*_{\theta}) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \sigma_{\text{tot}}(P)(x, -\xi).$$

Twisting $\mathcal{E}_M$ by $\Omega_M$, one then gets a globally defined isomorphism of algebras

$$\mathcal{E}^{op}_M \simeq \pi^{-1}\Omega_M \otimes \mathcal{E}_M \otimes \pi^{-1}\Omega_M^{-1} \quad P \mapsto \theta \otimes P^*_{\theta} \otimes \theta^{-1},$$

which does not depend on the choice of the volume form. (Here the tensor product is over $\pi^{-1}\mathcal{O}_M$ and $\theta^{-1}$ denotes the unique section of $\Omega_M^{-1}$, the dual of $\Omega_M$, satisfying $\theta^{-1} \otimes \theta = 1$.) This leads to replace the algebra $\mathcal{E}_M$ by its twisted version by half-forms

$$\mathcal{E}^{\sqrt{\nu}}_M = \pi^{-1}\Omega_M^{\otimes_{1/2}} \otimes \mathcal{E}_M \otimes \pi^{-1}\Omega_M^{-\otimes_{1/2}}.$$

The sheaf $\mathcal{E}^{\sqrt{\nu}}_M$ is a $\mathbb{C}$-algebra on $P^*M$ locally isomorphic to $\mathcal{E}_M$, and it has the following properties:

\[^1\text{Recall that the sections of } \mathcal{E}^{\sqrt{\nu}}_M \text{ are locally defined by } \theta^{\otimes_{1/2}} \otimes P \otimes \theta^{-\otimes_{1/2}} \text{ for a volume form } \theta \text{ and a microdifferential operator } P, \text{ with the equivalence relation } \theta_1^{\otimes_{1/2}} \otimes P_1 \otimes \theta_1^{-\otimes_{1/2}} = \theta_2^{\otimes_{1/2}} \otimes P_2 \otimes \theta_2^{-\otimes_{1/2}} \text{ if and only if } P_2 = (\theta_1/\theta_2)^{1/2} P_1 (\theta_1/\theta_2)^{-1/2}.\]
(i) it is filtered;
(ii) there is an isomorphism of graded algebras
\[
\sigma: \mathcal{G}r(\mathcal{E}_M^{\sqrt{\nu}}) \xrightarrow{\sim} \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_{P^*M}(m);
\]

(iii) it is endowed with an anti-involution *, i.e. an isomorphism of algebras
\[
*: (\mathcal{E}_M^{\sqrt{\nu}})^{op} \xrightarrow{\sim} \mathcal{E}_M^{\sqrt{\nu}} \quad \text{such that } *^2 = \text{id}.
\]

Moreover, these data are compatible: the anti-involution respects the filtration and \(\mathcal{G}r_0(*)\) is the identity. Note that one has \(\sigma_m(P^*) = (-1)^m \sigma_m(P)\) for all \(P \in \mathcal{E}_M^{\sqrt{\nu}}(m)\).

This suggests the following

**Definition 1.2.** An \(\mathcal{E}\)-algebra with anti-involution on \(P^*M\), an \((\mathcal{E},\ast)\)-algebra for short, is a sheaf of \(\mathbb{C}\)-algebras \(\mathcal{A}\) together with

(i) a filtration \(\{F_mA\}_{m \in \mathbb{Z}}\);

(ii) an isomorphism of graded algebras \(\nu: \mathcal{G}r(\mathcal{A}) \xrightarrow{\sim} \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_{P^*M}(m)\);

(iii) an anti-involution \(\iota\);

such that the triplet \((\mathcal{A},\nu,\iota)\) is locally isomorphic to \((\mathcal{E}_M^{\sqrt{\nu}},\sigma,\ast)\).

A morphism of \((\mathcal{E},\ast)\)-algebras is a \(\mathbb{C}\)-algebra morphism compatible with the structures (i), (ii) and (iii).

**Remark 1.3.** Definition 1.2 is adapted from [1]. See also [2] for similar definitions in the framework of real manifolds, where \(\mathcal{E}\)-algebras are replaced by Toeplitz algebras.

By definition, a morphism \(\varphi: \mathcal{A}_1 \to \mathcal{A}_2\) of \((\mathcal{E},\ast)\)-algebras is a \(\mathbb{C}\)-algebra morphism commuting with the anti-involutions, mapping \(F_m\mathcal{A}_1\) to \(F_m\mathcal{A}_2\) in such a way that \(\nu^i_m(\varphi(P)) = \nu^i_m(P)\) for all \(P \in F_m\mathcal{A}_1\). (Here \(\nu^i_m\) denotes the symbol map \(F_m(\mathcal{A}_i) \to F_m(\mathcal{A}_i)/F_{m-1}(\mathcal{A}_i) \simeq \mathcal{O}_{P^*M}(m)\) of order \(m\), for \(i = 1,2\).) It follows that any \((\mathcal{E},\ast)\)-algebra provides a quantization of \(P^*M\).

**Example 1.4.** Let \(f: P^*M \to P^*M\) be a contact transformation. Then \(f^{-1}\mathcal{E}_M^{\sqrt{\nu}}\) inherits an anti-involution and a filtration from \(\mathcal{E}_M^{\sqrt{\nu}}\), in such a way that the associated graded algebra is isomorphic (via \(f\)) to \(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_{P^*M}(m)\). By a result of [11], for each \(p \in P^*M\) there exist an
open neighborhood $V$ of $p$ and an isomorphism\(^2\) \(f^{-1}\mathcal{E}_M^{\sqrt{\Sigma}}|_V \xrightarrow{\sim} \mathcal{E}_M^{\sqrt{\Sigma}}|_V\) respecting (i), (ii) and (iii). It follows that \(f^{-1}\mathcal{E}_M^{\sqrt{\Sigma}}\) is an \((\mathcal{E}, \ast)\)-algebra.

Denote by \(\text{Aut}_{\mathcal{E}, \ast}(\mathcal{E}_M^{\sqrt{\Sigma}})\) the group of automorphisms of \(\mathcal{E}_M^{\sqrt{\Sigma}}\) as an \((\mathcal{E}, \ast)\)-algebra and set
\[
\mathcal{E}_M^{\sqrt{\Sigma}, \ast} = \{P \in \mathcal{E}^{\sqrt{\Sigma}}; \ P \text{ has order 0, } \sigma_0(P) = 1 \text{ and } PP^* = 1\} \subset (\mathcal{E}_M^{\sqrt{\Sigma}})^{\times}.
\]

**Lemma 1.5 (cf \cite{10}).** The assignment \(P \mapsto \text{ad}(P)\) defines an isomorphism of sheaves of groups on \(P^*M\)
\[
\text{ad} : \mathcal{E}_M^{\sqrt{\Sigma}, \ast} \to \text{Aut}_{\mathcal{E}, \ast}(\mathcal{E}_M^{\sqrt{\Sigma}}).
\]

The set of isomorphism classes of \((\mathcal{E}, \ast)\)-algebras on \(P^*M\) is in bijection with \(H^1(P^*M; \text{Aut}_{\mathcal{E}, \ast}(\mathcal{E}_M^{\sqrt{\Sigma}}))\).

**Corollary 1.6.** The \((\mathcal{E}, \ast)\)-algebras on \(P^*M\) are classified by the pointed set \(H^1(P^*M; \mathcal{E}_M^{\sqrt{\Sigma}, \ast})\).

Adapting a result in \cite{1}, we get the following

**Proposition 1.7.** If \(\dim M \geq 3\), then \(\mathcal{E}_M^{\sqrt{\Sigma}}\) is the unique, up to isomorphism, \((\mathcal{E}, \ast)\)-algebra on \(P^*M\).

## 2. Filtered and Graded Stacks

According to Kashiwara’s result \cite{10}, one has to replace sheaves by stacks in order to quantize a contact complex manifold. It becomes thus necessary to define the notions of filtration and graduation of a stack. We start here by recalling what a filtered (resp. graded) category is and how to associate a graded category to a filtered one. Then we stackify these definitions. We assume that the reader is familiar with the basic notions from the theory of stacks which are, roughly speaking, sheaves of categories. (The classical reference is \cite{5}, and a short presentation is given \textit{e.g.} in \cite{10 5}.)

Let \(R\) be a commutative ring.

**Definition 2.1.** An \(R\)-category\(^3\) \(\mathcal{C}\) is filtered (resp. graded) if the following properties are satisfied:

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\(^2\)This isomorphism is called Quantized Contact Transformation over \(f\).

\(^3\)An \(R\)-category is a category whose sets of morphisms are endowed with an \(R\)-module structure, so that the composition is bilinear. An \(R\)-functor is a functor between \(R\)-categories which is linear at the level of morphisms.
• for any objects $P, Q \in C$, the $R$-module $\text{Hom}_C(P, Q)$ is filtered (resp. graded);

• for any $P, Q, R \in C$ and any morphisms $f$ in $F_m\text{Hom}_C(Q, R)$ (resp. in $G_n\text{Hom}_C(Q, R)$) and $g$ in $F_n\text{Hom}_C(P, Q)$ (resp. in $G_n\text{Hom}_C(P, Q)$), the composition $f \circ g$ is in $F_{m+n}\text{Hom}_C(P, R)$ (resp. in $G_{m+n}\text{Hom}_C(P, R)$);

• for each $P \in C$, the identity morphism $\text{id}_P$ is in $F_0\text{Hom}_C(P, P)$ (resp. in $G_0\text{Hom}_C(P, P)$).

An $R$-functor $\Phi : C \to C'$ between filtered (resp. graded) categories is filtered (resp. graded) if for any objects $P, Q \in C$, the $R$-module morphism $\text{Hom}_C(P, Q) \to \text{Hom}_{C'}(\Phi(P), \Phi(Q))$ is filtered (resp. graded).

A natural transformation $\alpha = (\alpha_P)_{P \in C} : \Phi \Rightarrow \Phi'$ between filtered (resp. graded) functors is filtered (resp. graded) if for each $P \in C$ the morphism $\alpha_P$ is in $F_0\text{Hom}_C(\Phi(P), \Phi'(P))$ (resp. in $G_0\text{Hom}_C(\Phi(P), \Phi'(P))$).

If $\Phi, \Phi' : C \to C'$ are filtered $R$-functors, one defines $F_n\text{Hom}(\Phi, \Phi')$ as the set of those natural transformations of functors $\alpha = (\alpha_P)_{P \in C} : \Phi \Rightarrow \Phi'$ with $\alpha_P \in F_n\text{Hom}_{C'}(\Phi(P), \Phi'(P))$ for each $P \in C$. The category $\text{Hom}_{C'}(C, C')$ thereby obtained is then filtered. A similar remark holds also for the category of graded functors between graded categories.

To any filtered $R$-category $C$ there is an associated graded $R$-category $\text{Gr}(C)$, whose objects are the same of those of $C$ and for any objects $P, Q$ the set of morphisms is defined by $\text{Hom}_{\text{Gr}(C)}(P, Q) = \text{Gr}(\text{Hom}_C(P, Q))$. Similarly, to any filtered functor $\Phi$, one associates a graded functor $\text{Gr}(\Phi)$. In this way, we get a functor $\text{Gr}(\cdot)$ from filtered $R$-categories to graded ones. Note that, if $\Phi, \Phi'$ are filtered functors, then there is a natural injective morphism $\text{Gr}_n(\text{Hom}(\Phi, \Phi')) \to G_n\text{Hom}(\text{Gr}(\Phi), \text{Gr}(\Phi'))$ for each $n \in \mathbb{Z}$. We will denote by $\text{Gr}(\alpha)$ the graded natural transformation $\text{Gr}(\Phi) \Rightarrow \text{Gr}(\Phi')$ associated to the filtered natural transformation $\alpha : \Phi \Rightarrow \Phi'$ via the previous morphism for $n = 0$.

When we restrict to morphisms homogeneous of degree 0, we will use the notation $\text{Gr}_0$ (for categories, functors and natural transformations).

Following the presentation in [5], recall that there is a fully faithful functor $(\cdot)^+ : \text{Gr}(\cdot)$ from filtered (resp. graded) $R$-algebras to filtered (resp. graded) $R$-categories, which sends an $R$-algebra $A$ to the category $A^+$ with a single object $\bullet$ and $\text{End}(\bullet) = A$ as set of morphisms. If

4It follows that $\text{Gr}(\cdot)$ defines a 2-functor from the 2-category of filtered categories, filtered functors and filtered natural transformation to that of graded categories, graded functors and graded natural transformations.
Let $A$ be a topological space, and $\mathcal{R}$ a sheaf of commutative rings. As for categories, there are natural notions of filtered (resp. graded) $\mathcal{R}$-stack, of filtered (resp. graded) $\mathcal{R}$-functor between filtered (resp. graded) stacks and of filtered (resp. graded) natural transformations between filtered (resp. graded) functors.

As above, we denote by $(\cdot)^+$ the faithful and locally full functor from filtered (resp. graded) $\mathcal{R}$-algebras to filtered (resp. graded) $\mathcal{R}$-stacks, which sends an $\mathcal{R}$-algebra $A$ to the stack $A^+$ defined as follows: it is the stack associated with the pre-stack $X \supset U \mapsto A(U)^+$. If $f,g: A \to B$ are filtered (resp. graded) $\mathcal{R}$-algebra morphisms, filtered (resp. graded) natural transformations $f^+ \Rightarrow g^+$ are locally described as above.

Let $\mathcal{A}$ be a filtered $\mathcal{R}$-algebra. The stack $\text{Mod}_F(\mathcal{A})$ of filtered left $\mathcal{A}$-modules is filtered and equivalent to the stack $\text{Hom}_F(\mathcal{A}^+, \text{Mod}_F(\mathcal{R}))$ of filtered functors. Moreover the Yoneda embedding gives a fully faithful functor

$$\mathcal{A}^+ \to \text{Hom}_F((\mathcal{A}^+)^{\text{op}}, \text{Mod}_F(\mathcal{R})) \approx \text{Mod}_F(\mathcal{A}^{\text{op}})$$

into the stack of filtered right $\mathcal{A}$-modules. This identifies $\mathcal{A}^+$ with the full substack of filtered right $\mathcal{A}$-modules which are locally isomorphic
to $\mathcal{A}$. As above, everything remains true replacing filtered algebras and stacks by graded ones.

Let $\mathcal{G}$ be a filtered $\mathcal{R}$-stack. We denote by $\text{Gr}(\mathcal{G})$ the graded stack associated to the pre-stack $X \supset U \mapsto \text{Gr}(\mathcal{G}(U))$. As above, this defines a functor from filtered $\mathcal{R}$-stacks to graded ones (which is, in fact, a 2-functor). As before, we will make use of the notation $\text{Gr}_0$.

**Proposition 2.2.** Let $\mathcal{A}$ be a filtered $\mathcal{R}$-algebra and $\text{Gr}(\mathcal{A})$ its associated graded algebra. Then there is an equivalence of graded stacks $\text{Gr}(\mathcal{A}^+) \approx \text{Gr}(\mathcal{A})^+$.

**Proof.** Let $\mathcal{L}$ be a filtered right $\mathcal{A}$-module locally isomorphic to $\mathcal{A}$, that is, an object of $\mathcal{A}^+$. Its associated graded module $\text{Gr}(\mathcal{L})$ is a graded right $\text{Gr}(\mathcal{A})$-module locally isomorphic to $\text{Gr}(\mathcal{A})$, that is, an object of $\text{Gr}(\mathcal{A})^+$. Hence the assignement $\mathcal{L} \mapsto \text{Gr}(\mathcal{L})$ induces a functor $\text{Gr}(\mathcal{A}^+) \to \text{Gr}(\mathcal{A})^+$ of graded stacks. Since at each $x \in X$ this reduces to the equality $\text{Gr}(\mathcal{A}^+_x) = \text{Gr}(\mathcal{A}_x)$, it follows that it is a global equivalence. □

Recall from [13, 6] that an $\mathcal{R}$-algebroid stack is an $\mathcal{R}$-stack $\mathfrak{A}$ which is locally non-empty and locally connected by isomorphisms. Equivalently, for any $x \in X$ there exist an open neighborhood $U$ of $x$ and an $\mathcal{R}$-algebra $\mathcal{A}$ on $U$ such that $\mathfrak{A}|_U \approx \mathcal{A}^+$. Note that, if there exists a global object $L \in \mathfrak{A}(X)$, then $\mathfrak{A} \approx \text{End}_{\mathfrak{A}}(L)^+$.

**Corollary 2.3.** Let $\mathfrak{A}$ be a filtered $\mathcal{R}$-stack. If $\mathfrak{A}$ is an $\mathcal{R}$-algebroid stack, then it associated graded stack $\text{Gr}(\mathfrak{A})$ is again an $\mathcal{R}$-algebroid stack.

Recall that, given an $\mathcal{R}$-algebroid stack $\mathfrak{A}$, the stack of $\mathfrak{A}$-modules is by definition the $\mathcal{R}$-stack of $\mathcal{R}$-functors $\text{Hom}_{\mathcal{R}}(\mathfrak{A}, \text{Mod}(\mathcal{R}))$. It is an example of stack of twisted sheaves (see for example [5, 12]).

### 3. Quantization of complex contact manifolds

Let $(Y, \mathcal{L}, \alpha)$ be a complex contact manifold of dimension $2n + 1$. This means that $\mathcal{L}$ is a line bundle on $Y$ and $\alpha$ is a global section of $\Omega^n_Y \otimes_{\mathcal{O}_Y} \mathcal{L}$ (an $\mathcal{L}$-valued 1-form) such that $\alpha \wedge (da)^n$ is a nowhere vanishing global section of $\Omega_Y \otimes_{\mathcal{O}_Y} \mathcal{L}^\otimes n+1$. Recall that a local model for $Y$ is an open subset $V$ of $P^* M$, for a complex manifold $M$, equipped with the canonical contact structure $(\mathcal{O}_{P^* M}(1)|_V, \lambda)$, where $\lambda$ is induced by the Liouville form on $T^* M$. Hence we may define an $(\mathcal{E}, *)$-algebra...
on $Y$ as a filtered sheaf of $\mathbb{C}$-algebras $\mathcal{A}$ endowed with an isomorphism of graded algebras $\nu: Gr(\mathcal{A}) \rightarrow \bigoplus_{m \in \mathbb{Z}} \mathcal{L}^\otimes m$ and with an anti-involution $\iota$, such that the triplet $(\mathcal{A}, \nu, \iota)$ is locally isomorphic to $(i^{-1}\mathcal{E}^\sqrt{\nu}_M, \sigma, *)$ for any contact local chart $i: Y \supset V \rightarrow P^*M$.

Although we cannot expect a globally defined $(\mathcal{E}, *)$-algebra on $Y$, Kashiwara [10] proved the existence of a canonical stack $\text{Mod}(\mathcal{E}; Y)$ of microdifferential modules on $Y$. Following [6], this result may be restated as:

**Theorem 3.1.** On any complex contact manifold $Y$ there exists a canonical $\mathbb{C}$-stack $\mathcal{E}_Y$ which is locally equivalent to $(i^{-1}\mathcal{E}^\sqrt{\nu}_M, \sigma, *)$ for any contact local chart $i: Y \supset U \rightarrow P^*M$.

Note that $\mathcal{E}_Y$ is a $\mathbb{C}$-algebroid stack and $\text{Mod}(\mathcal{E}; Y)$ is equivalent to the stack of $\mathcal{E}_Y$-modules.

The following proposition allows us to say that the algebroid stack $\mathcal{E}_Y$ provides a quantization of $Y$.

**Proposition 3.2.** The $\mathbb{C}$-algebroid stack $\mathcal{E}_Y$ has the following properties:

(i) it is filtered;
(ii) there is an equivalence of graded stacks

$$\sigma: Gr(\mathcal{E}_Y) \xrightarrow{\approx} (\bigoplus_{m \in \mathbb{Z}} \mathcal{L}^\otimes m)^+;$$

(iii) it is endowed with an anti-involution $*$, that is, with an equivalence of stacks

$$*: \mathcal{E}_Y^{\text{op}} \xrightarrow{\approx} \mathcal{E}_Y$$

and an invertible transformation $\epsilon: *^2 \Rightarrow \text{id}_{\mathcal{E}_Y}$ such that the transformations $\epsilon \circ \text{id}_*: *^3 \Rightarrow *$ and $\text{id}_* \circ \epsilon: * \Rightarrow *^3$ are inverse one to each other.

Moreover, (i) $(\mathcal{E}_Y)$, (ii) $(\mathcal{E}_Y)$ and (iii) $(\mathcal{E}_Y)$ are compatible: $*$ is a filtered $\mathbb{C}$-functor, $\epsilon$ is a filtered natural transformation and there is an invertible transformation $\delta_0: \sigma_0 \circ \text{Gr}_0(*) \Rightarrow D \circ \sigma_0$, where $D$ denotes the functor $(\mathcal{O}^+_Y)^{\text{op}} \xrightarrow{\approx} \mathcal{O}^+_Y$, which sends a locally free $\mathcal{O}_Y$-module of rank one to its
dual, making the following diagram commutative\(^5\)

\[
\begin{array}{c}
\sigma_0 \circ \mathfrak{Gr}_0(\ast^2) \\
\downarrow_{\text{id}_{\sigma_0} \circ \mathfrak{Gr}_0(\epsilon)} \simeq \downarrow_{\text{id}_D \circ \delta_0} \\
D \circ \sigma_0 \circ \mathfrak{Gr}_0(\ast)
\end{array}
\]

(3.1)

\[
\sigma_0 \circ \mathfrak{Gr}_0(\epsilon^2) \xrightarrow{\delta_0 \circ \text{id}_{\sigma_0}(\ast)} D \circ \sigma_0 \circ \mathfrak{Gr}_0(\ast)
\]

**Theorem 3.3.** The \(\mathbb{C}\)-algebroid stack \(\mathcal{E}_Y\) is the unique, up to equivalence – the equivalence being unique up to a unique isomorphism – \(\mathbb{C}\)-stack on \(Y\) satisfying the properties \((i)^+, (ii)^+\) and \((iii)^+\), and such that for any contact local chart \(i: Y \supset V \rightarrow P^\ast M\), the triplet \((\mathcal{E}_Y, \sigma, \ast)\) is locally equivalent to \(((i^{-1}\mathcal{E}_Y^\ast)^+, \sigma^+, \ast^+)\).

**Proof.** Let \(\text{Aut}_{\mathcal{E}_Y}(\mathcal{E}_Y)^\times\) be the stack of 2-groups\(^6\) of auto-equivalences of \(\mathcal{E}_Y\) preserving the structures \((i)^+, (ii)^+\) and \((iii)^+\). (Here the upper index \(\times\) means that all the non-invertible morphisms have been removed.) More precisely, its objects are triplets \((\Phi, \beta, \gamma)\), where \(\Phi: \mathcal{E}_Y \rightarrow \mathcal{E}_Y\) is an equivalence of filtered \(\mathbb{C}\)-stacks and \(\beta: \sigma \circ \mathfrak{Gr}(\Phi) \Rightarrow \sigma\) and \(\gamma: \ast \circ \Phi \Rightarrow \Phi \circ \ast\) are invertible transformations of functors (\(\beta\) being graded and \(\gamma\) filtered) such that the following diagrams commute:

\[
\begin{array}{c}
\ast \circ \Phi \circ \ast \xrightarrow{\text{id} \circ \gamma} \ast^2 \circ \Phi \\
\downarrow_{\gamma \circ \text{id}_{\ast}} \downarrow_{\text{id} \circ \gamma} \\
\Phi \circ \ast^2 \xrightarrow{\text{id}_{\Phi} \circ \epsilon} \Phi
\end{array}
\]

(3.2)

\[
\sigma_0 \circ \mathfrak{Gr}_0(\ast) \circ \mathfrak{Gr}_0(\Phi) \xrightarrow{\text{id}_{\sigma_0} \circ \mathfrak{Gr}_0(\gamma)} \sigma_0 \circ \mathfrak{Gr}_0(\Phi) \circ \mathfrak{Gr}_0(\ast) \\
\text{id}_{D \circ \delta_0} \downarrow \downarrow_{\text{id}_{D \circ \delta_0}} \text{id}_{D \circ \delta_0} \\
D \circ \sigma_0 \circ \mathfrak{Gr}_0(\Phi) \xleftarrow{\text{id}_{D \circ \delta_0}} D \circ \sigma_0 \xrightarrow{\delta_0} \sigma_0 \circ \mathfrak{Gr}_0(\ast).
\]

\(\text{The compatibility between } \epsilon \text{ and } \delta_0 \text{ is quite redundant for the algebroid stack } \mathcal{E}_Y, \text{ since the data of } (\epsilon, \delta_0) \text{ are induced by those of } (\mathcal{E}_Y^\ast)^+. \text{ Anyway, it becomes necessary for different choices of } (\epsilon, \delta_0).\)

\(\text{See the Appendix } A \text{ for the definition, notations and basic properties of a stack of 2-groups.}\)
A morphism \( \alpha : (\Phi_1, \beta_1, \gamma_1) \Rightarrow (\Phi_2, \beta_2, \gamma_2) \) is a filtered invertible transformation of functors \( \alpha : \Phi_1 \Rightarrow \Phi_2 \) making the following diagrams commutative:

\[
\begin{array}{c}
\Phi_1 \xrightarrow{\Phi_1 \circ \alpha} \Phi_1 \circ \Phi_1 \\
\Phi_2 \xrightarrow{\Phi_2 \circ \Phi_2} \Phi_2 \circ \Phi_2
\end{array}
\]

To prove the theorem, it is then enough to show that this stack is trivial, that is, consists only in the identity functor \( \text{id}_{E_Y} \) and in the identity transformation \( \text{id}_{E_Y} \Rightarrow \text{id}_{E_Y} \).

First step. At any point \( y \in Y \), we may find a contact local chart \( i : Y \supset V \to P^*M \) containing \( y \) such that \( \text{E}_Y|_V \) is equivalent to \( (\text{E}_V^\sqrt{v})^+ = (i^{-1}\text{E}_M^\sqrt{v})^+ \). We thus reduced to the study of the autoequivalences of \( (\text{E}_V^\sqrt{v})^+ \) preserving the structures \( (i)^+, (ii)^+ \) and \( (iii)^+ \). Let \( (\Phi, \beta, \gamma) \) be such an equivalence. Since the functor \( (\cdot)^+ \) is locally full, up to shrinking the open subset \( V \), one may suppose that \( \Phi \) is isomorphic to \( \varphi^+ \) for a filtered \( \mathbb{C} \)-algebra isomorphism \( \varphi : \text{E}_V^\sqrt{v} \to \text{E}_V^{\sqrt{v}} \) and that \( \sigma \) (resp. \( * \)) is isomorphic to \( \sigma^+ \) (resp. \( *^+ \)), so that \( \delta_0 \) (resp. \( \epsilon \)) is the identity morphism. The graded transformation \( \beta \) is thus given by a nowhere vanishing function \( f \in \mathcal{O}_{P^*M}|_V \simeq \text{Gr}_0(\text{E}_V^\sqrt{v}) \) and it implies the equalities \( \sigma_m(\varphi(S)) = \sigma_m(S) \) for all \( S \in \text{E}_V^\sqrt{v}(m) \), so that \( \text{Gr}(\varphi) \) must be the identity. Similarly, the transformation \( \gamma \) is given by an invertible operator \( P \in \text{E}_V^\sqrt{v} \) of order 0 satisfying

\[
\varphi \circ * = \text{ad}(P) \circ * \circ \varphi.
\]

A direct computation shows that the diagram (3.2) corresponds to the equality \( P^* = P \) and the diagram (3.3) to \( f^2\sigma_0(P) = 1 \).

By a result of Kashiwara (see [18, Lemma 5.3] for a proof), there exists an invertible operator \( Q \in \text{E}_V^\sqrt{v} \) of order 0 satisfying \( Q = Q^* \) and \( P = Q^2 \). We may choose \( Q \) in such a way that \( f\sigma_0(Q) = 1 \). Set \( \bar{\varphi} = \text{ad}(Q^{-1}) \circ \varphi \). Using (3.5), one easily gets the relation

\[
* \circ \bar{\varphi} = \bar{\varphi} \circ *,
\]

so that \( \bar{\varphi} \) is an \( \mathcal{E} \)-algebra automorphism of \( \text{E}_V^\sqrt{v} \). Moreover, the section \( Q \) defines a morphism \( (\bar{\varphi}^+, \text{id}, \text{id}) \Rightarrow (\Phi, \beta, \gamma) \).
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It follows that the functor of stacks of 2-groups
\[
\left[ \mathcal{E}_V^{\sqrt{\pi},*} \xrightarrow{\text{ad}} \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_V^{\sqrt{\pi}}) \right] \rightarrow \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_V^{\sqrt{\pi}})^{\times} \quad \psi \mapsto (\psi^+, \text{id}, \text{id})
\]
is locally essentially surjective, where the left-hand side denotes the stack of 2-groups associated to the crossed module \( \mathcal{E}_V^{\sqrt{\pi},*} \to \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_V^{\sqrt{\pi}}) \).

Second step. Let \( \psi_1, \psi_2 \) be two sections of \( \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_V^{\sqrt{\pi}}) \) and consider a morphism \( \alpha: (\psi_1^+, \text{id}, \text{id}) \Rightarrow (\psi_2^+, \text{id}, \text{id}) \) in \( \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_V^{\sqrt{\pi}})^{\times} \). Then \( \alpha \) is locally given by an invertible operator \( R \in \mathcal{E}_V^{\sqrt{\pi}} \) of order 0 satisfying
\[
\psi_2 = \text{ad}(R) \circ \psi_1,
\]
where the diagrams (3.4) correspond to the equalities \( RR^* = 1 \) and \( \sigma_0(R) = 1 \). Therefore \( R \) defines a morphism in \( \left[ \mathcal{E}_V^{\sqrt{\pi},*} \xrightarrow{\text{ad}} \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_V^{\sqrt{\pi}}) \right] \).

The above functor is thus locally full. Since it also faithful, we get an equivalence
\[
\left[ \mathcal{E}_V^{\sqrt{\pi},*} \xrightarrow{\text{ad}} \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_V^{\sqrt{\pi}}) \right] \approx \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_V^{\sqrt{\pi}})^{\times}.
\]

Third step. By Lemma 1.5, the morphism \( \text{ad}: \mathcal{E}_V^{\sqrt{\pi},*} \to \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_V^{\sqrt{\pi}}) \) is an isomorphism. Thanks to Proposition A.3, the associated stack of 2-groups is then equivalent to the trivial one. It follows that \( \text{Aut}_{\mathcal{E}_V,*}(\mathcal{E}_Y)^{\times} \) is locally, and hence globally, trivial. \( \square \)

**Corollary 3.4.** There exists an \((\mathcal{E}, \ast)-\text{algebra} on } Y \) if and only if \( \mathcal{E}_Y \) has a global object.

**Proof.** Let \( L \) be a global objet of \( \mathcal{E}_Y \), that is, \( L \in \mathcal{E}_Y(Y) \), and set \( \mathcal{A} = \text{End}_{\mathcal{E}_Y}(L) \), the sheaf of endomorphisms of \( L \). Then \( \mathcal{A} \) is a filtered \( \mathbb{C} \)-algebra locally isomorphic to \( i^{-1} \mathcal{E}_M \), for any contact local chart \( i: Y \supset U \to P^*M \). By \((ii)^+\), its graded algebra is isomorphic to \( \bigoplus_{m \in \mathbb{Z}} \mathcal{L}^{\otimes m} \). By \((iii)^+\), there is an isomorphism \( \ast: \mathcal{A}^{\text{op}} \xrightarrow{\sim} \mathcal{A}^* \), where \( \mathcal{A}^* = \text{End}_{\mathcal{E}_Y}(L^*) \) is an inner form of \( \mathcal{A} \), that is, locally isomorphic to \( \mathcal{A} \) with the glueing automorphisms of the form \( \text{ad}(P) \) for some \( P \in \mathcal{A}^\times \). Then one may replace \( \mathcal{A} \) by a twisted version \( \tilde{\mathcal{A}} \), in such a way that \( \ast \) defines an anti-involution on it. It follows that \( \tilde{\mathcal{A}} \) is an \((\mathcal{E}, \ast)-\text{algebra} on } Y \).

Conversely, let \( \mathcal{A} \) be an \((\mathcal{E}, \ast)-\text{algebra on } Y \). Therefore \( \mathcal{A}^+ \) is a \( \mathbb{C} \)-algebroid stack satisfying \((i)^+, (ii)^+ \) and \((iii)^+ \), and locally isomorphic
to \((i^{-1}\mathcal{E}_M)^+\), for any contact local chart \(i: Y \supset U \to P^*M\). By Theorem 3.3, it is equivalent to \(\mathcal{E}_Y\). Since \(\mathcal{A}^+\) has a global object, so does \(\mathcal{E}_Y\). \(\square\)

Remark 3.5. (1) Let \(Y = P^*M\). If \(\dim M < 3\), there are non-isomorphic \((\mathcal{E}, \ast)\)-algebras (cf [1]), but they all give rise to equivalent algebroid stacks.

(2) Using similar techniques, one may show that Theorems 3.1 and 3.3 hold also in the real case, hence replacing \(\mathcal{E}\)-algebras with Toeplitz algebras. However, in that case the algebroid so obtained has always a global object (cf [7]). It follows that in the real case there are no significative differences between the quantizations given by algebras and those provided by algebroids.

Appendix A. Stacks of 2-groups

In this section we briefly recall the notion of stack of 2-groups, which is, roughly speaking, a stack with group-like properties. Reference is made to [3]7. We assume that the reader is familiar with the notions of monoidal category, monoidal functor and monoidal transformation. (The classical reference is [15] and a more recent one is [14].)

Let \(X\) be a topological space.

Definition A.1. (i) A 2-group is a monoidal category \((\mathcal{G}, \otimes, 1)\) whose morphisms are all invertible and such that for each object \(P \in \mathcal{G}\) there exist an object \(Q\) and natural morphisms \(P \otimes Q \simeq 1\) and \(Q \otimes P \simeq 1\). A functor (resp. a natural transformation of functors) of 2-groups is a monoidal functor (resp. a monoidal transformation of functors) between the underlying monoidal categories.

(ii) A pre-stack (resp. stack) of 2-groups on \(X\) is a pre-stack (resp. stack) \(\mathcal{G}\) such that for each open subset \(U \subset X\), the category \(\mathcal{G}(U)\) is a 2-group, and the restriction functors and the natural transformations between them respect the 2-group structure.

Let \(\mathcal{G}\) be a sheaf of groups on \(X\). We denote by \(\mathcal{G}[0]\) the discrete stack defined by trivially enriching \(\mathcal{G}\) with identity arrows, and by \(\mathcal{G}[1]\)

\[\text{We prefer to follow here the terminology of Baez-Lauda [Higher-dimensional algebra. V. 2-groups, Theory Appl. Categ. 12 (2004), 423–491], which seems to us more friendly than the classical one of } gr\text{-category as in loc. cit.}\]
the stack associated to the pre-stack whose category on an open subset $U \subset X$ has a single object $\bullet$ and $\text{End}(\bullet) = \mathcal{G}(U)$ as set of morphisms. Clearly $\mathcal{G}[0]$ is a stack of 2-groups, while $\mathcal{G}[1]$ defines a stack of 2-groups if and only if $\mathcal{G}$ is commutative.

Recall that a crossed module on $X$ is a complex of sheaves of groups $\mathcal{G}^{-1} \overset{d}{\to} \mathcal{G}^0$ endowed with a left action of $\mathcal{G}^0$ on $\mathcal{G}^{-1}$ such that, for any local sections $g \in \mathcal{G}^0(U)$ and $h, h' \in \mathcal{G}^{-1}$, one has

$$d(gh) = \text{ad}(g)(d(h)) \quad \text{and} \quad d(h')h = \text{ad}(h')(h).$$

A morphism of crossed modules is a morphism of complexes compatible with the actions in the natural way.

To each crossed module $\mathcal{G}^{-1} \overset{d}{\to} \mathcal{G}^0$ there is an associated stack of 2-groups on $X$, which we denote by $[\mathcal{G}^{-1} \overset{d}{\to} \mathcal{G}^0]$, defined as follows: it is the stack associated to the pre-stack of 2-groups whose objects on an open subset $U \subset X$ are the sections $g \in \mathcal{G}^0(U)$, with 2-group law $g \otimes g' = gg'$, and whose morphisms $g \to g'$ are given by sections $h \in \mathcal{G}^{-1}(U)$ satisfying $g' = d(h)g$. The 2-group structure for morphisms is given by the rule

$$(g_1 \xrightarrow{h_1} g'_1) \otimes (g_2 \xrightarrow{h_2} g'_2) = g_1g_2 \xrightarrow{h_1h_2} g'_1g'_2.$$ 

In a similar way, a morphism of crossed modules induces a functor of the associated stacks of 2-groups.

**Remark A.2.** In fact, it is true that any stack of 2-groups on $X$ comes from a crossed module. However, this result is not of practical use. We refer to [SGA4] for the proof of this fact in the commutative case and to [4] for the non commutative case on $X = \text{pt}$.

By definition, an object $P$ of $[\mathcal{G}^{-1} \overset{d}{\to} \mathcal{G}^0]$ on an open subset $U \subset X$ is described by an open covering $U = \bigcup_i U_i$ and sections $\{g_i\} \in \mathcal{G}^0(U_i)$, subject to the relation $g_i = d(h_{ij})g_j$ on double intersections $U_{ij}$, for given sections $\{h_{ij}\} \in \mathcal{G}^{-1}(U_{ij})$ satisfying $h_{ij}h_{jk} = h_{ik}$ on triple intersections $U_{ijk}$.

If $\mathcal{G}$ is a sheaf of groups on $X$, then the stack of 2-groups defined by the crossed module $1 \to \mathcal{G}$ is naturally identified with $\mathcal{G}[0]$. If moreover $\mathcal{G}$ is commutative, the complex $\mathcal{G} \to 1$ is a crossed module and the associated stack of 2-groups is identified with $\mathcal{G}[1]$.

---

8Here we use the convention as in [3] for which $\mathcal{G}^i$ is in $i$-th degree.
Proposition A.3. Let $G^{-1} \xrightarrow{d} G^0$ be a crossed module.

1. If $d$ is surjective, then $[G^{-1} \xrightarrow{d} G^0] \approx \ker d[1]$ as stacks of 2-groups.

2. If $d$ is injective, then $[G^{-1} \xrightarrow{d} G^0] \approx \coker d[0]$ as stacks of 2-groups.

In particular, if $d$ is an isomorphism, then $[G^{-1} \xrightarrow{d} G^0]$ is trivial, i.e. it is equivalent to the stack with one object and one morphism.

Proof. (1) Consider the fully faithful functor of stacks of 2-groups $\ker d[1] \rightarrow [G^{-1} \xrightarrow{d} G^0]$, which sends $\bullet$ to the identity object $1 = 1 \in G^0$. Let us check that is is locally essentially surjective. Let $g \in G^0$ be a local section. We may suppose that there exists a local section $h \in G^{-1}$ satisfying $d(h) = g$. Hence $h$ defines a morphism $1 \rightarrow g$ in $[G^{-1} \xrightarrow{d} G^0]$. It follows that any other object of $[G^{-1} \xrightarrow{d} G^0]$ is locally isomorphic to $1$.

(2) Consider the locally essentially surjective functor of stacks of 2-groups $[G^{-1} \xrightarrow{d} G^0] \rightarrow \coker d[0]$, which sends an object to its isomorphism class. It remains to check that $[G^{-1} \xrightarrow{d} G^0]$ has only trivial arrows. But this is clear, since for any object there are no automorphisms other than the identity $1 \in G^{-1}$. □

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