A Heavy Fermion Can Create a Soliton: A 1+1 Dimensional Example

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Abstract

We show that quantum effects can stabilize a soliton in a model with no soliton at the classical level. The model has a scalar field chirally coupled to a fermion in 1+1 dimensions. We use a formalism that allows us to calculate the exact one loop fermion contribution to the effective energy for a spatially varying scalar background. This energy includes the contribution from counterterms fixed in the perturbative sector of the theory. The resulting energy is therefore finite and unambiguous. A variational search then yields a fermion number one configuration whose energy is below that of a single free fermion.

I. INTRODUCTION

Scalar field theories can contain spatially varying (but time independent) configurations that are local minima of the classical energy. These solitons are found as solutions to the non–linear classical equations of motion. Sometimes a topological conservation law can be used to show that the soliton is absolutely stable because it is the lowest energy configuration with a given value of a conserved topological charge. When quantum effects are taken into account, the classical description must be re–examined. Now the spatially varying soliton configuration should minimize the “effective energy” which takes into account classical and quantum effects\textsuperscript{1}. Since the effective energy for general configurations is difficult to compute, quantum effects are typically computed as approximate corrections to the classical soliton. In a non–renormalizable theory, these corrections are cutoff dependent and the model must

\textsuperscript{1}By effective energy we mean the effective action per unit time; the term “effective potential” is reserved for spatially constant configurations.

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be redefined to include the cutoff prescription. The hope is that the energy of the soliton is slightly altered by quantum effects but its qualitative features remain.

In this Letter we give an example of a quantum soliton that is not present in the classical theory alone. We examine a renormalizable model in 1 + 1 dimensions where a scalar field is Yukawa coupled to a fermion. Fermion number is conserved. The classical energy is minimized when the scalar field has a constant value $v$. There are no classical solitons. The fermion gets a mass $m = Gv$ through the Yukawa coupling. We calculate exactly the fermion’s properly renormalized one loop contribution to the scalar field effective energy. By “exactly” we mean to all orders in the derivative expansion, which is crucial since we consider configurations varying on the scale $\frac{1}{m}$. We then show that for certain choices of model parameters — in particular with $G$ large — we can exhibit a field configuration that carries fermion number and has energy below $m$. It cannot decay by emitting a free fermion. We search for the lowest energy configuration carrying fermion number using a few parameter variational ansatz. The soliton, which is the actual lowest energy configuration with fermion number one, is presumably not far from our variational minimum and has strictly lower energy. The soliton therefore has energy less than $m$ and is absolutely stable.

The idea that a heavy fermion can create as soliton is not new and has been explored previously [1,2]. What we are adding to the discussion is the ability to exactly calculate the renormalized fermionic one loop effective energy for any spatially varying meson background, which is essential for demonstrating stability at the quantum level. Since we are working in a renormalizable theory, the counterterms in the Lagrangian cancel the cutoff dependent part of the sum over zero–point energies in the explicit evaluation of the effective energy, leaving a finite result. This result is unambiguous because we are able to fix the counterterms in the perturbative sector of the theory. Furthermore, we can choose model parameters to justify neglecting the one loop boson contributions and all higher loop contributions. Thus we conclude that in this 1 + 1 dimensional model, a heavy fermion can create a soliton.

II. THE MODEL

The model we consider has a two–component meson field $\vec{\phi} = (\phi_1, \phi_2)$ coupled equally to $N_F$ fermions. We suppress the fermion flavor label but will keep track of the factor $N_F$ as necessary. The Lagrangian is $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$ with

$$\mathcal{L}_B = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - V(\vec{\phi}).$$

(1)

where

$$V(\vec{\phi}) = \frac{\lambda}{8} \left[ \vec{\phi} \cdot \vec{\phi} - v^2 + \frac{2\alpha v^2}{\lambda} \right]^2 - \frac{\lambda}{2} \left( \frac{\alpha v^2}{\lambda} \right)^2 - \alpha v^3 (\phi_1 - v)$$

(2)

and

$$\mathcal{L}_F = \frac{1}{2} \left( i \left[ \bar{\Psi}, \partial \Psi \right] - G \left( \left[ \bar{\Psi}, \Psi \right] \phi_1 + i \left[ \bar{\Psi}, \gamma_5 \Psi \right] \phi_2 \right) \right).$$

(3)

(The reason for the commutators in eq. (3) will be explained later.) Note that with $\alpha$ set to zero, the theory has a global $U(1)$ invariance.
\[ \phi_1 + i \phi_2 \rightarrow e^{i\varphi} (\phi_1 + i \phi_2) \quad \text{and} \quad \Psi \rightarrow e^{-i\varphi_\gamma/2} \Psi. \]  

\( \psi \rightarrow \psi \) and \( \psi \rightarrow e^{i\phi_1} (\phi_1 + i \phi_2) \) \( \rightarrow \psi \rightarrow e^{-i\varphi_\gamma/2} \Psi \).

Naïvely, one would imagine that spontaneous symmetry breaking occurs with \( \alpha = 0 \). Then we could pick a classical vacuum, say \( \vec{\phi}_{\text{cl}} = (v, 0) \), and expand the theory about this point. But in \( 1 + 1 \) dimensions, the massless mode that corresponds to motion along the chiral circle, \( \vec{\phi} \cdot \vec{\phi} = v^2 \), gives rise to infra–red singularities and there is no spontaneous symmetry breaking \[3\]. By introducing \( \alpha \neq 0 \) we have tilted the potential to eliminate the massless mode. For \( \alpha \) large enough it is legitimate to expand about \( \vec{\phi}_{\text{cl}} \). There are two massive bosons, which we call \( \sigma \) and \( \pi \), with \( m^2_\sigma = (\lambda + \alpha) v^2 \) and \( m^2_\pi = \alpha v^2 \). The classical bosonic theory governed by \( L_B \) has no classical soliton.

The fermions get mass through their Yukawa coupling to \( \vec{\phi} \). In the perturbative vacuum, expanding about \( \vec{\phi}_{\text{cl}} \), the fermion has mass \( m = Gv \). One could imagine that various distortions of \( \vec{\phi} \) would affect the fermion spectrum. For example, one could keep \( \phi_2 = 0 \) and let \( \phi_1 \rightarrow \phi_1(x) \) with \( \lim_{x \rightarrow \pm \infty} \phi_1(x) = v \), but \( \phi_1(x) < v \) over some region in \( x \) of order \( w \). Alternatively, one could keep \( \vec{\phi} \cdot \vec{\phi} = v^2 \), but let \( \vec{\phi} = v(\cos \theta(x), \sin \theta(x)) \), where \( \theta(x) \rightarrow 0 \) as \( x \rightarrow -\infty \) and \( \theta(x) \rightarrow 2\pi \) as \( x \rightarrow +\infty \). Again the deviation of \( \vec{\phi} \) from \( \vec{\phi}_{\text{cl}} \) occurs in a region of order \( w \). In both cases, if \( w \) is of order \( 1/m \), then there are bound state solutions of the single–particle Dirac equation associated with eq. \( \psi \) that have binding energies of order \( m \), so that a fermion bound to the \( \vec{\phi} \) field has an energy below \( m \). (Because of its topological properties, the latter configuration is especially efficient at binding a fermion \[4\].) However, there is an energy cost from the gradient and potential terms. Still, considering just the single bound fermion and the classical scalar energy, we might expect a total energy below \( m \) for \( G \) large enough.

Of course, \( \Psi \) describes a quantum field and any distortion of the background \( \vec{\phi}(x) \) away from \( \vec{\phi}_{\text{cl}} \) will cause shifts in the zero–point energies of the fermion fluctuations. To form a self–consistent approximation, we must compute the effect of these shifts as well, since they are of the same order in \( \hbar \) as the bound state contribution. In general the sum over zero–point energies diverges. In order to proceed we must regularize and renormalize the calculation. We are working in a renormalizable field theory so we know that the counterterms that are implicit in \( L_B \) will cancel these divergences. We want to compare the energy of non–trivial configurations with the perturbative spectrum of the model, therefore we fix the counterterms by standard renormalization conditions on the Green’s functions. The Green’s functions are evaluated perturbatively so the counterterms have an expansion in Feynman diagrams.

Regularization and renormalization of the sum over zero–point energies has been problematic in the past. We work in the continuum, where the sum is replaced by an integral over scattering phase shifts. In Ref. \[4\] we show that it is possible to analytically continue this integral to \( d \) spatial dimensions, where it converges. Then we are able to identify potential divergences with low orders in the Born expansion for the phase shifts, and, in turn, with specific Feynman diagrams. We subtract the low order Born terms from the integral, which then remains finite when analytically continued back to \( d = 1 \). We then add back in the corresponding Feynman diagrams, which combine with the counterterms in the usual way to yield a finite and unambiguous result in \( d = 1 \). In the next section we show how we evaluate this contribution to the energy of a static configuration.
III. THE ONE FERMION LOOP EFFECTIVE ENERGY

We have written eq. (3) as a commutator to ensure that the Lagrangian is invariant under the charge conjugation operation $\Psi \rightarrow C\Psi^*$ and $(\phi_1, \phi_2) \rightarrow (\phi_1, -\phi_2)$. As a result, the vacuum energy gets contributions from both the positive and negative energy eigenvalues of the single particle Dirac Hamiltonian

$$H[\vec{\phi}] = i\sigma_1 \frac{d}{dx} + G (\sigma_2 \phi_1 + \sigma_3 \phi_2). \quad (5)$$

Here we are using a Majorana representation of the Dirac matrices, $\gamma_0 = \sigma_2$, $\gamma_1 = i\sigma_3$, and $\gamma_5 = \sigma_1$, which implies that $C = 1$. For one flavor of fermions the energy of the lowest energy state is

$$E_{\text{vac}} = \frac{1}{2} \left\{ - \sum_{\omega_n > 0} \omega_n + \sum_{\omega_n < 0} \omega_n \right\} = -\frac{1}{2} \sum_n |\omega_n|, \quad (6)$$

where the $\omega$’s are the eigenvalues of $H[\vec{\phi}]$. (For a charge conjugation invariant background, for each eigenvalue $\omega$ there is an eigenvalue $-\omega$, so the two sums in eq. (6) are the same, and $E_{\text{vac}}$ reduces to the sum over the Dirac sea, $E_{\text{vac}} = \sum_{\omega_n < 0} \omega_n$.) We will restrict our attention to background fields that obey $\phi_1(x) = \phi_1(-x)$ and $\phi_2(x) = -\phi_2(-x)$. In this case, $H[\vec{\phi}]$ commutes with the parity operator $P = \sigma_2 \Pi$, where $\Pi$ is the coordinate reflection operator that transforms $x$ to $-x$. We can thus decompose the solutions of eq. (6) into separate parity channels.

For a given background, we wish to evaluate eq. (3) and subtract from it what we get in the free case, $\vec{\phi} = (\nu, 0)$. Following Ref. [5], we use the relationship between the change in the density of states and the derivative of the phase shift

$$\rho(k) - \rho_0(k) = \frac{1}{\pi} \frac{d\delta(k)}{dk} \quad (7)$$

to write the change in the vacuum energy as

$$\Delta E^F = -\frac{1}{2} \sum \omega_i |\omega_i| - \int_0^\infty \frac{dk}{2\pi} \omega(k) \frac{d}{dk} \delta_F(k) + \frac{m}{2} + E_{\text{ct}}, \quad (8)$$

where

$$\delta_F(k) = \delta_+(\omega(k)) + \delta_+(-\omega(k)) + \delta_-(\omega(k)) + \delta_-(\omega(k)). \quad (9)$$

Here $\omega(k) = \sqrt{k^2 + m^2}$, $\delta_{\pm}$ is the scattering phase shift for the positive (negative) parity channel, the $\omega_i$ are the discrete bound state energy levels, and $E_{\text{ct}}$ is the counterterm contribution, which is fixed by renormalization conditions discussed below. The extra $m/2$ reflects an important subtlety in one dimension: in the non–interacting case ($\delta_F(k) = 0$) there are bound states exactly at the continuum thresholds, $\omega = \pm m$, which count as $1/2$ in the sum in eq. (3) [3]. Levinson’s theorem,

$$\delta_+(m) + \delta_+(-m) = \pi (n^+ - \frac{1}{2}), \quad (10)$$
relates the phase shift at threshold to the number of bound states \( n_{\pm} \), with threshold bound states again counting as \( 1/2 \). It allows us to rewrite eq. (8) as

\[
\Delta E^F = -\frac{1}{2} \sum_{l} (|\omega_l| - m) - \int_{0}^{\infty} \frac{dk}{2\pi} (\omega(k) - m) \frac{d}{dk} \delta_F(k) + E_{ct},
\]

(11)

which is convenient because it makes it clear that as we increase the background and a bound state appears, there are no discontinuities in \( \Delta E^F \).

Of course, \( \Delta E^F \) given by eq. (11) is formally infinite. For large \( k \), the phase shifts go to zero like \( 1/k \) so the integral is divergent. To regulate this divergence and allow us to identify it unambiguously with specific Feynman diagrams, we have extended the method of dimensional regularization to the density of states written in terms of phase shifts. The details are presented in Ref. [4]. Once continued to \( d \)-dimensions, where all quantities are finite, we can identify the leading large \( k \) behavior of \( \delta_F(k) \) with the contribution of the first Born approximation plus the piece of the second Born approximation related to it by chiral symmetry, which we call \( \hat{\delta}(k) \). We also identify it unambiguously with the coefficient of the Lagrangian counterterm, \( v^2 - \vec{\phi} \cdot \vec{\phi} \), evaluated by standard Feynman perturbation theory. The renormalization conditions that fix the counterterm in perturbation theory here translate into the statement that in evaluating eq. (11) we should subtract \( \hat{\delta}(k) \) from \( \delta_F(k) \). After this subtraction the integral can be analytically continued back to \( d = 1 \) to give a result that is finite and unambiguous,

\[
\Delta E^F = -\frac{1}{2} \sum_{l} (|\omega_l| - m) - \int_{0}^{\infty} \frac{dk}{2\pi} (\omega(k) - m) \frac{d}{dk} \left( \delta_F(k) - \hat{\delta}(k) \right),
\]

(12)

where

\[
\hat{\delta}(k) = \frac{2G^2}{k} \int_{0}^{\infty} dx \left( v^2 - \vec{\phi}(x)^2 \right).
\]

(13)

We can solve numerically for the phase shift \( \delta_F(k) \) for any background \( \vec{\phi}(x) \). We make use of the fact that \( P \) commutes with \( H[\vec{\phi}] \) so that positive and negative parity channels are decoupled. The positive (negative) parity states obey

\[
\psi_{\pm}(-x) = \pm \sigma_2 \psi_{\pm}(x)
\]

(14)

and therefore

\[
\psi_{\pm}(0) \propto \begin{pmatrix} 1 \\ \pm i \end{pmatrix}.
\]

(15)

Any state defined for \( x \geq 0 \) and obeying one of the boundary conditions in eq. (15) can be extended via eq. (14) to the whole line, so we need only consider the half line \( x \geq 0 \) with the boundary conditions eq. (15). Consider the free case, \( \vec{\phi} = (v, 0) \). For each energy \( \omega \), both positive and negative, the right moving eigenstate of eq. (5) is

\[
\varphi_{k}^0(x) = \frac{1}{\omega} \begin{pmatrix} \omega \\ -k + im \end{pmatrix} e^{ikx}
\]

(16)

5
and the left moving free eigenstate is

$$\varphi_{-k}(x) = \frac{1}{\omega} \left( k + i m \right) e^{-ikx}$$

(17)

where \( k = \sqrt{\omega^2 - m^2} \) is positive. For backgrounds \( \phi(x) \) that are not everywhere equal to \( \phi_{cl} \), we still impose the requirement that \( \phi(x) \) approaches \( \phi_{cl} \) as \( x \) gets large. For these non–trivial backgrounds we call \( \varphi_{k}(x) \) the eigenstate of eq. (3) that approaches \( \varphi_{k}(x) \) as \( x \to \infty \) and \( \varphi_{-k}(x) \) the eigenstate that approaches \( \varphi_{-k}(x) \) as \( x \to \infty \). Note that \( \varphi_{k}, \varphi_{-k}, \varphi_{k} \), and \( \varphi_{-k} \) are not eigenstates of \( P \). For \( x \geq 0 \) let

$$\psi_{\pm}(x) = \varphi_{-k}(x) \pm \frac{-ik + m}{\omega} e^{2id_{\pm}(\omega)} \varphi_{k}(x)$$

(18)

be the parity eigenstates of \( H[\phi] \) with energy \( \omega \). This defines the phase shifts \( d_{\pm}(\omega) \). The \( 2\pi \) ambiguity in this definition is resolved by requiring that the phase shifts be smooth and go to zero as \( \omega \to \pm\infty \). Note that in the free case \( d_{\pm}(\omega) = 0 \). For any value of \( \omega \), we can solve numerically for the eigenstates of eq. (3) in both parity channels. Using eq. (18) we can then extract the phase shifts. Our method for computing the phase shift actually allows us to resolve the \( 2\pi \) ambiguity for each \( \omega \) individually and has other numerical advantages, which are elaborated in Ref. [5].

For any value of \( k \) we can obtain \( d_{F}(k) \), so we can compute the integral in eq. (12). To find the bound state energies we solve the eigenvalue problem numerically. Levinson’s theorem tells us how many bound states to search for. For a fixed background \( \phi \), the numerical evaluation of eq. (12) can be done quickly and with high accuracy. This allows us to search over a class of \( \phi \)'s for the configuration with the lowest total energy.

IV. THE TOTAL ENERGY

We are interested in calculating the total one loop effective energy of a static configuration \( \bar{\phi}(x) \). We take \( \bar{\phi}(x) \) to be specified by a short list of parameters \( \{\zeta_{i}\} \). We measure energy in units of the fermion mass \( m = Gv \) and use a dimensionless distance \( \xi = mx \). In \( 1 + 1 \) dimensions \( \bar{\phi}(x) \) and \( v \) are dimensionless. We rescale \( \bar{\phi}(x) \) by \( v \) so that \( \bar{\phi}(x) \to (1, 0) \) as \( |\xi| \to \infty \), and define dimensionless couplings

$$\tilde{\alpha} = \frac{\alpha}{G^2} \quad \text{and} \quad \tilde{\lambda} = \frac{\lambda}{G^2}.$$  

(19)

By this rescaling, using eq. (4) and eq. (2), we have

$$\frac{E_{cl}[\bar{\phi}]}{m} = v^2 \int_{-\infty}^{\infty} d\xi \left( \frac{1}{2} \bar{\phi}' \cdot \bar{\phi}' + \frac{\tilde{\lambda}}{8} [\bar{\phi} \cdot \bar{\phi} - 1 + \frac{2\tilde{\alpha}}{\lambda} \phi_1 - 1] - \frac{\tilde{\lambda}}{2} \left( \frac{\tilde{\alpha}}{\lambda} \right)^2 - \tilde{\alpha} \phi_1 \right)$$

$$= v^2 E_{cl}(\tilde{\alpha}, \tilde{\lambda}, \{\zeta_{i}\}),$$

(20)

where prime denotes differentiation with respect to \( \xi \).

The fermion one loop contribution to the energy arises from eq. (5), which with \( \bar{\phi} \) measured in units of \( v \) is
\[ H[\phi] = m \left( i \sigma_1 \frac{d}{d\xi} + \sigma_2 \phi_1(\xi) + \sigma_3 \phi_2(\xi) \right). \] (21)

We see that a single fermion makes a contribution proportional to \( m \) and dependent on the variational parameters \( \{\zeta_i\} \). This means that eq. (12) can be expressed as \( m \mathcal{E}^F(\{\zeta_i\}) \). For \( N_F \) flavors the one loop contribution is therefore \( N_F m \mathcal{E}^F(\{\zeta_i\}) \).

The boson one loop contribution comes from summing the square roots of the eigenvalues of the operator \( -\frac{\partial^2}{\partial x^2} + \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \). Rescaling as before we find that the boson one loop energy can be written as \( m \mathcal{E}^B(\{\zeta_i\}) \). Putting together the classical energy and the one loop energies we get

\[ \frac{E_{\text{tot}}[\phi]}{N_F m} = \frac{v^2}{N_F} \mathcal{E}_{\text{cl}}(\tilde{\alpha}, \tilde{\lambda}, \{\zeta_i\}) + \mathcal{E}^F(\{\zeta_i\}) + \frac{1}{N_F} \mathcal{E}^B(\tilde{\alpha}, \tilde{\lambda}, \{\zeta_i\}) + \text{higher loops}. \] (22)

For \( N_F \) large we can neglect the boson one loop contribution relative to the fermion one loop contribution. Furthermore it can be shown that \( 1/v^2 \) counts boson loops. Taking both \( N_F \) and \( v^2 \) large with the ratio fixed, we can neglect the higher loops entirely and all but the single fermion loop in eq. (22). Therefore we need only consider the contributions from \( \mathcal{E}_{\text{cl}} \) and \( \mathcal{E}^F \).

V. THE FERMION NUMBER

A non–trivial background \( \phi(x) \) distorts the energy levels of the Dirac Hamiltonian eq. (5), possibly introducing single particle bound states (with positive and negative energy). We identify the lowest energy state of the system, the one with the all the negative energy levels filled, as the vacuum. If a level crosses zero as we locally interpolate between \( \phi_{\text{cl}}(x) \) and \( \phi(x) \), this state will have non–zero fermion number. In particular, if \( \phi(x) \) circles \( \phi = (0,0) \) as \( \phi \) goes from \( (1,0) \) at \( x = -\infty \) to \( (1,0) \) at \( x = \infty \), then the vacuum state will carry non–zero fermion number provided that the scale over which \( \phi \) varies, \( w \), is much larger than the fermion Compton wavelength \( 1/m \) \[ \text{(2)} \]. In Ref. 8 we derive a formula for the fermion number of the vacuum, \( Q_{\text{vac}} \), in terms of the positive energy phase shifts at \( k = 0 \) and the number of positive energy bound states,

\[ Q_{\text{vac}} = N_F \left( \frac{1}{\pi} [\delta_+(m) + \delta_-(m)] + \frac{1}{2} - n_{\omega>0} \right) \] (23)

where \( n_{\omega>0} \) is the number of bound states with positive energy. The configurations we look at loop at most once around \( \phi = 0 \), so \( Q_{\text{vac}} \) is either 0 or \( N_F \). We are interested in states with fermion number \( N_F \). If \( Q_{\text{vac}} = N_F \), then the state we want is the vacuum. If \( Q_{\text{vac}} = 0 \), then we build the lowest energy state with fermion number \( N_F \) by filling the lowest positive energy level of eq. (5) with \( N_F \) fermions. Therefore, if \( Q_{\text{vac}} = 0 \), \( \mathcal{E}^F \) appearing in eq. (22) must be augmented by \( \omega_1 \) where \( m\omega_1 \) is the smallest positive eigenvalue of eq. (21).
We want to look for background configurations $\vec{\phi}$ that can produce states with fermion number $N_F$, and whose total energy is below $N_F m$. From eq. (22) with $E^B$ neglected, we define

$$B = \frac{v^2}{N_F} E_{\text{cl}} + E^F - 1 ,$$  \hspace{1cm} (24)

which is the energy of the fermionic configuration minus the energy of $N_F$ free fermions in units of $m N_F$. For our numerical computations, we take the ansatz

$$\phi_1 + i \phi_2 = 1 - R + R e^{i \Theta} \quad \text{where} \quad \Theta = \pi (1 + \tanh(\xi/w)) .$$ \hspace{1cm} (25)

The two variational parameters are $R$ and $w$. As $\xi$ goes from $-\infty$ to $\infty$, $\vec{\phi}$ moves in a circle of radius $R$ in the $(\phi_1, \phi_2)$ plane, starting and ending at $(1,0)$. The scale over which $\vec{\phi}$ varies is $w$.

For fixed $\tilde{\alpha}$ and $\tilde{\lambda}$ we vary $R$ and $w$ until we produce the configuration with the smallest $B$. The results are shown in Fig. 1. We see that it is possible to find a configuration whose total energy is below $N_F m$. Since we are minimizing $B$ subject to the constraint that $\phi$ is of the form eq. (25), we know that the true minimum of $B$ in the fermion number $N_F$ sector also has an energy below $N_F m$. This is the stable soliton.

In Fig. 2 we show the width, $w_{\text{sol}}$, and the radius, $R_{\text{sol}}$, for the minimum energy configuration as a function of $\tilde{\alpha}$ for various values of $\tilde{\lambda}$. Note that the size of the soliton grows like $1/\sqrt{\tilde{\alpha}}$ as $\tilde{\alpha}$ goes to zero. In that region, $R_{\text{sol}}$ approaches 1, so the $\vec{\phi}$ configuration approaches the chiral circle. In fact, the energy of the fermion number $N_F$ soliton goes to zero as $\tilde{\alpha}$ goes to zero. However for $\tilde{\alpha}$ very small the bosonic quantum fluctuations restore the classically

FIG. 1. $B$ as a function of $v/\sqrt{N_F}$ for various values $\tilde{\lambda}$ with $\tilde{\alpha} = 0.25$ (left panel) and for various values $\tilde{\alpha}$ with $\tilde{\lambda} = 1.0$ (right panel). $B$ negative corresponds to binding.

VI. RESULTS

We want to look for background configurations $\vec{\phi}$ that can produce states with fermion number $N_F$, and whose total energy is below $N_F m$. From eq. (22) with $E^B$ neglected, we define

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broken symmetry. Thus we can not trust our results for $\tilde{\alpha}$ very small and we do not believe that this large and light soliton is a reliable consequence of this model. For moderate values of $\tilde{\alpha}$, where the width of the soliton is not controlled by $1/\sqrt{\tilde{\alpha}}$, we do trust our results. For the value of $v/\sqrt{N_F}$ shown in Fig. 2, the model becomes trustworthy for $\tilde{\alpha} \approx 0.3$. For further discussion of this point, see Ref. [4].

We have developed and applied a variational technique for renormalizable quantum field theories through one loop order. Because we have applied unambiguous, standard perturbative renormalization procedures, we have been able to hold the theory (i.e. the renormalized masses and coupling constants) fixed, while searching over a variational ansatz. Here we have used these methods to demonstrate the existence of a stable fermionic soliton, stabilized by quantum effects, in a model with no soliton at the classical level. The result suggests that similar phenomena might persist in 3+1 dimensions, and no obstacles stand in the way of generalizing the method to that case.

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