Comment on Two schemes for Secure Outsourcing of Linear Programming

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\textbf{Abstract.} Recently, Wang et al. [IEEE INFOCOM 2011, 820-828], and Nie et al. [IEEE AINA 2014, 591-596] have proposed two schemes for secure outsourcing of large-scale linear programming (LP). They did not consider the standard form: minimize $c^T x$, subject to $Ax = b, x \geq 0$. Instead, they studied a peculiar form: minimize $c^T x$, subject to $Ax = b, Bx \geq 0$, where $B$ is a non-singular matrix. In this note, we stress that the proposed peculiar form is unsolvable and meaningless. The two schemes have confused the \textit{functional inequality constraints} $Bx \geq 0$ with the \textit{nonnegativity constraints} $x \geq 0$ in the linear programming model. But the condition $x \geq 0$ is indispensable to the simplex method. Therefore, both two schemes failed.

\textbf{Keywords.} Cloud computing, confidentiality-preserving image search, additive homomorphic encryption, symmetric key encryption.

1 Introduction

Recently, Wang et al. \cite{1}, and Nie et al. \cite{2} have proposed two schemes for secure outsourcing of large-scale linear programming (LP). They did not consider the standard form:

\[
\text{minimize } c^T x, \quad \text{subject to } Ax = b, x \geq 0,
\]

Instead, they studied a peculiar form:

\[
\text{minimize } c^T x, \quad \text{subject to } Ax = b, Bx \geq 0
\]

where $A$ is an $m \times n$ matrix, $c$ is an $n \times 1$ vector, $b$ is an $m \times 1$ vector, $x$ is an $n \times 1$ vector of variables, and $B$ is an $n \times n$ non-singular matrix.

In this note, we would like to stress that the proposed peculiar form is unsolvable and meaningless. The two schemes have confused the \textit{functional inequality constraints} $Bx \geq 0$ with \textit{nonnegativity constraints} $x \geq 0$ in the linear programming model. In nature, the condition $x \geq 0$ is indispensable to the simplex method. Thus, both two schemes failed.

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2 Preliminaries

The standard form for a linear programming problem can be described as follows. Select the values for \(x_1, \ldots, x_n\) so as to

\[
\text{maximize } \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n,
\]

subject to the restrictions

\[
\begin{align*}
\text{subject to } & \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
& \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
& \quad \vdots \\
& \quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m
\end{align*}
\]

and

\[
x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0.
\]

\(c_1x_1 + c_2x_2 + \cdots + c_nx_n\) is called the objective function. The first \(m\) constraints are sometimes called functional constraints. The \(x_j \geq 0\) restrictions are called nonnegativity constraints.

The simplex method, a general procedure for solving linear programming problems, is based on solving systems of equations. Therefore, it has to firstly convert the functional inequality constraints to equivalent equality constraints. This conversion is accomplished by introducing slack variables. After the conversion, the original linear programming model can now be replaced by the equivalent model (called the augmented form).

Using matrices, the standard form for the general linear programming model becomes

\[
\begin{align*}
\text{maximize } \quad & c^T x, \quad \text{subject to } \quad Ax \leq b, x \geq 0 \\
\end{align*}
\]

where \(A\) is an \(m \times n\) matrix, \(c\) is an \(n \times 1\) vector, \(b\) is an \(m \times 1\) vector, and \(x\) is an \(n \times 1\) vector of variables. To obtain the augmented form of the problem, introduce the column vector of slack variables \(x_s = (x_{n+1}, \ldots, x_{n+m})^T\) so that the constraints become

\[
[A, I] \begin{bmatrix} x \\ x_s \end{bmatrix} = b \quad \text{and} \quad \begin{bmatrix} x \\ x_s \end{bmatrix} \geq 0,
\]

where \(I\) is the \(m \times m\) identity matrix, and the null vector \(0\) now has \(n + m\) elements.

Notice that the nonnegativity constraints are left as inequalities because they are used to determine the leaving basic variable according to the minimum ratio test [3].

3 Analysis of the two schemes for secure outsourcing of LP

3.1 Review

We now take the scheme in Ref.[1] as the example to show the correctness of the proposed peculiar form (see the page 822 of Ref.[1], and the page 592 of Ref.[2]). In the scheme, there are
two entities, the client and the server. The client has the original problem

\[
\text{minimize } \mathbf{c}^T \mathbf{x}, \quad \text{subject to } \mathbf{A} \mathbf{x} = \mathbf{b}, \ B \mathbf{x} \geq 0
\]

(1)

where \( \mathbf{A} \) is an \( m \times n \) matrix, \( \mathbf{c} \) is an \( n \times 1 \) vector, \( \mathbf{b} \) is an \( m \times 1 \) vector, \( \mathbf{x} \) is an \( n \times 1 \) vector of variables, \( \mathbf{B} \) is an \( n \times n \) non-singular matrix.

To ensure the privacy of input and output, the client transforms the original problem into the following problem

\[
\text{minimize } \mathbf{c}'^T \mathbf{y}, \quad \text{subject to } \mathbf{A}' \mathbf{y} = \mathbf{b}', \ B' \mathbf{y} \geq 0
\]

(2)

where

\[
\begin{align*}
\mathbf{A}' &= \mathbf{QAM} \\
\mathbf{B}' &= (\mathbf{B} - \mathbf{PQA}) \mathbf{M} \\
\mathbf{b}' &= \mathbf{Q}(\mathbf{b} + \mathbf{Ar}) \\
\mathbf{c}' &= \gamma \mathbf{M}^T \mathbf{c} \\
\mathbf{y} &= \mathbf{M}^{-1}(\mathbf{x} + \mathbf{r})
\end{align*}
\]

satisfying

\[|\mathbf{B}'| \neq 0, \mathbf{Pb}' = \mathbf{Br}, \mathbf{b} + \mathbf{Ar} \neq 0, \gamma > 0,\]

where \( \mathbf{P} \) is an \( n \times m \) matrix, \( \mathbf{Q} \) is a random \( m \times m \) non-singular matrix, \( \mathbf{M} \) is a random \( n \times n \) non-singular matrix, and \( \mathbf{r} \) is an \( n \times 1 \) vector.

The client then sends the problem (2) to the server, instead of the original problem (1).

3.2 Analysis

When the server receives the problem (2), he has to introduce the nonnegativity conditions \( \mathbf{y} \geq 0 \) into it and solve the following problem

\[
\text{minimize } \mathbf{c}'^T \mathbf{y}, \quad \text{subject to } \mathbf{A}' \mathbf{y} = \mathbf{b}', \ B' \mathbf{y} \geq 0, \ \mathbf{y} \geq 0
\]

(3)

This is because the constraints \( B' \mathbf{y} \geq 0 \) should be viewed as a part of the functional constraints, not the necessary nonnegativity constraints. Unless \( (\mathbf{B} - \mathbf{PQA}) \mathbf{M} \) can be rewritten as a diagonal matrix where the entries on the main diagonal are strictly positive (in such case, \( B' \mathbf{y} \geq 0 \) implies \( \mathbf{y} \geq 0 \)).

Unfortunately, the solution of the following problem

\[
\text{minimize } \mathbf{c}^T \mathbf{x}, \quad \text{subject to } \mathbf{A} \mathbf{x} = \mathbf{b}, \ B \mathbf{x} \geq 0, \ \mathbf{x} \geq 0
\]

(4)

cannot be derived from the solution of the problem (3), because the transformation

\[\mathbf{y} = \mathbf{M}^{-1}(\mathbf{x} + \mathbf{r}), \quad \text{where } \mathbf{x} \geq 0\]

cannot ensure that \( \mathbf{y} \geq 0 \).

Remark 1. The authors of [1][2] have confused the functional inequality constraints \( B \mathbf{x} \geq 0 \) with the nonnegativity constraints \( \mathbf{x} \geq 0 \). In fact, the proposed form is meaningless and unsolvable, unless \( B \mathbf{x} \geq 0 \) can be rewritten as \( \mathbf{x} \geq 0 \).
4 Conclusion

We would like to restate that the procedure for determining the leaving basic variable in the simplex method requires that all the variables have nonnegativity constraints. One must draw a clear distinction between the functional inequality constraints and the nonnegativity constraints. Notice that deriving the augmented form of a standard form for a linear programming problem is very easy. It can be solely done by the client himself even though who is assumed to be of weak computational capability.

References

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