Research Article

Jordan Form-Based Algebraic Conditions for Controllability of Multiagent Systems under Directed Graphs

Shuai Liu, Zhijian Ji, and Huizi Ma

1Institute of Complexity Science, College of Automation, Qingdao University, Qingdao 266071, China
2College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

Correspondence should be addressed to Zhijian Ji; jizhijian@pku.org.cn

Received 10 August 2019; Revised 16 January 2020; Accepted 31 January 2020; Published 28 February 2020

Academic Editor: Saleh Mobayen

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Based on the Jordan form of system matrix, this paper discusses algebraic conditions for the controllability of the multiagent network system with directed graph from two aspects: leader-follower network attribute and coupling input disturbance. Leader-follower network attribute refers to the topology structure and information communication among agents. Coupling input disturbance includes the number of external coupling inputs and the selection of leader nodes. When the leader-follower network attribute is fixed, the selection method of coupling input disturbance is studied for the controllability, and when the coupling input disturbance is known, we derive necessity and sufficiency conditions to determine the controllability. The reliability of theoretical results is verified by numerical examples and model simulation. Besides, the generally perfect controllability is introduced, that is, the system is always controllable regardless of the number and the locations of leaders. In practical engineering applications, the perfectly controllable topology can improve the system fault tolerance and accelerate the commercialization process, which has a profound significance for promoting the modernization process.

1. Introduction

In real life, the solution of many complex problems can often get good enlightenment from the natural world. A typical scenario is that bionics has always attracted the attention of a large number of experts and scholars [1–29], who have learned from a series of natural phenomena, such as bees building nests, ants foraging for food, and geese migrating, and realized that clustering behavior may be a reasonable way to solve a class of engineering problems. The concept of distributed multiagent system has been widely discussed and developed since 1970s [30–34]. The distributed multiagent network system is a kind of multiagent cluster which is formed by incomplete information communication under the constraint of fixed neighbor protocol. The information acquisition of a single agent mainly comes from the sensor’s perception and the influence of external inputs, so the information content of a single agent may have one-sidedness and certain falsity compared with the centralized system. Through the communication and mutual restrain of multiagents, the information content of each agent overlaps. So, it can effectively screen out false information, perfect the information content, and make a certain advance judgment to the development trend. The research on the inherent mechanism and application direction of the distributed multiagent network system was initially applied to formation control (such as international robot soccer match, satellite cooperative control, underwater robot, and UAV formation control).

In 2002, Fax and Murray proved a Nyquist criterion by using the eigenvalues of the graph Laplacian matrix to determine the effect of the communication topology on formation stability [35]. Then, Lafferrriere et al. proved that a necessary and sufficient condition for the existence of an appropriate decentralized linear stabilizing feedback is that graph G has a rooted directed spanning tree. They showed a relationship between the rate of formation convergence and the eigenvalues of the (directed) Laplacian of G [36]. In recent years, people have new thinking on formation. Lin et al. introduced a new multiagent control problem, called an
affine formation control problem, with the objective of asymptotically reaching a configuration that preserves col-linearity and ratios of distances with respect to a target configuration [37]. In the paper of Ye and Hu, a distributed Nash equilibrium seeking strategy is firstly proposed under undirected graphs with its convergence property analytically studied [38–40]. With the deepening of research, the distributed multiagent network system has brought commercial value in the coordination of complex traffic conditions, industrial power supply scheduling, and use of medical devices [41, 42].

In 2004, Tanner applied the controllability concept to the distributed multiagent network system for the first time and transformed the multiagent formation control problem into a topological structure problem [43]. This laid the methodological foundation for the discussion of multiagent network systems (MASs): leader-follower topology. Since then, the controllability of MASs has been extensively discussed, respectively, in the single integrator model, the second-order integrator model, the high-order integrator model, and the general linear model. At the same time, the focus of the research has gradually shifted from single-leader situation to multileader situation. Tanner pointed out that for the single-leader time-invariant system, a necessary and sufficient condition for achieving controllability is as follows. (1) The eigenvalues of $F$ are all distinct (where $F$ is the matrix obtained from $\Delta^{-1} L$ ($\Delta$ is the degree matrix and $L$ is the Laplacian matrix) after deleting the last row and column and $r$ is the vector of the first $N-1$ elements of the deleted column). (2) The eigenvectors of $F$ are not orthogonal to $r$ [43]. Ji et al. discussed the multileader case in depth and obtained some algebraic criteria based on graph theory. That is, the system $\mathcal{G}_{f, l}$ (where $\mathcal{G}_{f} = L_fL_f^T$, $I_{f} = L_f^T$, $L_f \in \mathbb{R}^{N_f \times M}$, and $L_f \in \mathbb{R}^{N_f \times M}$) is controllable if $G$ is connected and $\mathcal{H}(\mathcal{G}_{f}) \subseteq \mathcal{H}(\mathcal{G}_{f})$ (where $\mathcal{H}(\cdot)$ denotes the null space and $1$ is the vector with all entries being one) [44]. Jiang et al. have expounded the controllability equivalence between the first-order system model and the general linear system model in detail. The conclusion shows that under certain conditions, the controllability of MASs is not affected by its individual attributes which is completely determined by the topology graph (communication information and topology structure) [45, 46].

In addition, the results of controllability under undirected and unweighted graphs produce a large number of outputs. Since the system Laplacian matrix is symmetric in undirected graphs, the information communication between MASs is strongly constrained. As a consequence, the controllability problem under the directed graph has aroused more and more experts’ interest. Jiang et al. established a unified model of controllability in the case of undirected and directed topology and analyzed the influence of the weight value on controllability [45, 46]. In 2007, Ji and Egerstedt gave a necessary condition for distributed multiagent systems. They pointed out that for a given connected graph $G$ and the induced follower graph $G_f$, the system $(\mathcal{G}_f, I_f)$ is not completely controllable if there exist NEPs on $G$ and $G_f$, say $\Pi$: and $\Pi_f$, such that all the nontrivial cells of $\Pi$ are contained in $\Pi_f$ [47]. Then, Ji et al. pointed out that based on the basis of the abovementioned research, a sufficient and necessary condition for the controllability of the multiagent system is that there is no zero element in the leader position of the eigenvector in the graph $G$. It also gives an important guidance to the research of the multiagent problem [48]. In 2008, Liu et al. were focused on the advantages of the switching topology scheme. Based on switched control system theory, they derived a simple controllability condition for the network with switching topology, which indicates that the controllability of the whole network does not need to rely on the specific topology for each network [49]. For the controllability problem of complex networks, Cai et al. conducted an in-depth discussion and pointed out that almost any weighted complex network with noise on the strength of communication links is controllable in the sense of Kalman controllability [50]. In the recent question of research, in contrast to the existing results that mainly focus on unsigned networks, She et al. characterized controllability and developed leader selection methods for signed networks. It is important to note that the developed results are generic, in the sense that they are not only applicable to signed networks but also to unsigned networks [51].

By means of linear transformation, this paper discusses the strict requirements of the main constraints (topology structure and number of external inputs) for the controllability of the system. The influence on controllability of the distributed multiagent network system is analyzed when there are repeated eigenvalues in the Laplacian matrix. On the one hand, when the system is controllable under the condition of fixed topology and the Laplacian has repeated eigenvalues, the specific requirements of the number of external inputs are further studied. In addition, the concept of the minimum number of external inputs is proposed, and the coupling relationship between the minimum number of external inputs and the topology is given when the system is controllable. On the other hand, for a given number of external inputs, a Laplacian matrix with repeated eigenvalues is discussed to determine how to make the MASs achieve controllability. For multiple inputs, the concept of perfectly controllable is generalized to generally perfectly controllable, and necessary and sufficient conditions are derived for achieving generally perfectly controllable. The innovation and main contributions of this paper can be described as follows:

(i) The influence of repeated eigenvalues of the Laplacian matrix on controllability of the multiagent network system is discussed and necessary and sufficient conditions for global controllability are given.

(ii) The coupling relationship between the number of external inputs and the eigenvalues is analyzed. Moreover, the minimum number of external inputs to achieving controllability of multiagent systems is given.

(iii) The problem of determining the controllability of the system is transformed into determining whether
H_{\lambda_p} has a d_{\lambda_p}-order cofactor that is not zero, where $H_{\lambda_p}$ is obtained by newly introduced partitioning matrix $H$ based on eigenvalue $\lambda_p$, and $d_{\lambda_p}$ is the number of primary elementary corresponding to eigenvalue $\lambda_p$.

(iv) The Jordan standard form of the Laplacian matrix is used to discuss the necessary and sufficient condition for achieving perfect controllability.

(v) A generally perfectly controllable system is defined, which is a supplementary definition of perfect controllability under multiple external inputs. Necessary and sufficient conditions are derived for this kind of controllability.

Based on the Jordan form of the system matrix under directed graphs, we consider the influence of the Laplacian matrix with repeated eigenvalues on the controllability when the coupling input disturbances are fixed. We also pay attention to the relationship between the number of externally coupling inputs and the controllability of the system when the leader-follower network attribute is fixed.

In particular, focusing on the Laplacian matrix with repeated eigenvalues is in fact an in-depth discussion of the more general system, involving the relevant issues of eigenvalue distribution, primary elementary distribution, perfectly controllable topology, and so on. Related to this are the number of externally coupling inputs and the selection of leader nodes, which may also affect the controllability of the system. Besides, the generally perfectly controllable system is defined, which is a supplementary definition of perfect controllability under multiple external inputs.

The concrete structure of this article is as follows. Section 2 describes the multiagent system and distributed neighbor protocols discussed in this article. Section 3 gives the concrete content of the basic knowledge that needs to be used. Section 4 elaborates the main conclusions and the proof process under detailed argumentation. Section 5 validates the objectivity of the conclusion of the paper through example analysis and simulation. Section 6 summarizes the work of the paper and gives the prospect of the future application.

2. Problem Statement

Graph theory is a basic tool to study the multiagent network system. The definition of graphs represents the basic information of the multiagent system. The complex changes of communication state and velocity information between multiagents are abstracted into the mathematical evolution of multinodes and weighted edges.

Define $G = [V, E, \mathcal{A}]$ as a directed graph of the distributed multiagent system, with $V = \{v_1, v_2, \ldots, v_n\}$ as the node set, and $v_i (i, j = 1, 2, \ldots, n)$ is a single agent. $E = \{(i, j) \in V \times V\}$ is the set of edges that describes the communication state between multiagents, where $i \neq j$ does not consider the self-loop case. The adjacency matrix of the system is described by $\mathcal{A} = (a_{ij})_{n \times n}$ where $a_{ij}$ is the communication intensity of node $i$ receiving $j$. Then, the value of $a_{ij}$ represents the magnitude of edge weight. Suppose $a_{ij}a_{ji} \geq 0$, where $a_{ij} > 0$ represents a cooperative relationship between agents $i$ and $j$ and $a_{ij} < 0$ represents a competitive relationship. When there is no communication information between multiagents, $a_{ij} = 0$.

Given a graph $G$ of $n$ nodes, its Laplacian matrix $L(G) = (l_{ij})_{n \times n}$ is defined as $L(G) = \Delta(G) - \mathcal{A}(G)$, where $\Delta(G)$ (diagonal matrix) is the degree matrix of $G$ and the diagonal element is $\sum_{j \in N}a_{ij}$. $j \in N^i$ represents a neighbor node $j$ of $i$, and $N^i$ is the neighbor set of node $i$. Consider the leader-follower multiagent system:

$$\begin{align*}
\dot{x}_i &= u_i + y_i, \\
\dot{x}_f &= u_i, & i = 1, \ldots, n,
\end{align*}$$

where $y_i = c_i r = c_i r_1 + c_i r_2 + \cdots + c_i w r_w$, $x_d$ and $x_i$ are the state of leaders and followers, respectively. $u_i$ is the control input. $x_d$ represents the $i$th leader node, which can receive the information communication between the external input $r_d$ and its neighbors. $x_{if}$ represents the $i$th follower node, which only receives the information from neighbor nodes. In particular, $y_i$ is the total input gain of the external input $r$; $c_i$ and $r$ are taken, respectively, as row vectors and column vectors ($c_i = [c_{i1}, c_{i2}, \ldots, c_{iw}]$, $r = [r_1, r_2, \ldots, r_w]^T$, where $r_i$ describes different external input signals). $C^T = (c_{iw})_{n \times w} = [c_1, c_2, \ldots, c_w]^T$ is a real matrix, where $c_{ie}$ represents the connection between the external input and leader node.

Remark 1. It should be noticed that the number of leaders is less than or equal to $w$, $c_{ie}$ is binary and can only be chosen as 0 or 1. In fact, the value of $\sum_{i \in [1,w]} c_{ie}$ determines whether node $i$ is the leader node. $c_{ie} = 1$ indicates that the $i$th node is selected as the leader node and accepts the eth external input signal. $c_{ie} = 0$ indicates that node $i$ is not affected by the eth external input signal. Only in case $\sum_{i \in [1,w]} c_{ie} = 0$, the $i$th node plays as a follower.

The coupling protocol between nodes in system (1) is

$$u_i = \sum_{j \in N^i} a_{ij} (x_j - x_i).$$

where $x_j - x_i$ represents the signal input from neighbor nodes, which indicates the information error between nodes $i$ and $j$ in the current time. Actually, protocol (2) shows that the feedback input control of the node $i$ is realized by accumulating the communication errors of all the nodes in the neighbor set. Considering both system (1) and protocol (2), one has

$$\begin{bmatrix}
\dot{x}_{if} \\
\dot{x}_{il}
\end{bmatrix} = \begin{bmatrix}
L(G)_{ij} & L(G)_{jl} \\
L(G)_{il} & L(G)_{ll}
\end{bmatrix} \begin{bmatrix}
x_{if} \\
x_{il}
\end{bmatrix} + C^T r,$$

where $L(G)_{ij}, L(G)_{jl}$ represent the communication between the follower nodes and the input information that the follower nodes receive from the leader nodes, respectively. Similarly, $L(G)_{il}$ represents the input information that the leader nodes receive from the follower nodes and the information communication between the leader nodes, respectively. Simplify (3) so that it takes the form of

$$\dot{X} = -L(G)X + y,$$

where $y = [y_1, y_2, \ldots, y_n]^T = C^T r$. Complexity

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3
3. Methodology

Considering the Laplacian matrix $L(G)$ of the graph. Since the matrix is symmetric and semidefinite, its real eigenvalues can be sorted as $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$, where $\lambda_1(G) = 0$. A necessary and sufficient condition for $G$ to be connected is $\lambda_2(G) > 0$. In this paper, we consider the controllability of the multiagent system under connected conditions. The eigenvalues of the Laplacian matrix $L(G)$ are sorted from small to large, where the minimum eigenvalue is 0 and the minor eigenvalue must be greater than 0. Therefore, 0 is a single eigenvalue of the Laplacian matrix.

Let $A(\lambda)$ be an $n$-dimensional $\lambda$-matrix and $k$ be a positive integer less than or equal to $n$. If the greatest common divisor of all $k$-order subformulas (which is the first polynomial) of $A(\lambda)$ is not equal to zero, then this polynomial is called the $k$-order determinant divisor of $A(\lambda)$ and is denoted as $D_k(\lambda)$. If all the $k$-order subformulas of $A(\lambda)$ are equal to zero, the $k$-order determinant divisor of $A(\lambda)$ is 0. Let $D_1(\lambda), D_2(\lambda), \ldots, D_r(\lambda)$ be the nonzero determinant divisor of $\lambda$-matrix $A(\lambda)$ [52]. Then, each of $g_1(\lambda) = D_1(\lambda), g_2(\lambda) = D_2(\lambda)/D_1(\lambda), \ldots, g_r(\lambda) = D_r(\lambda)/D_{r-1}(\lambda)$ is called an invariant factor of $A(\lambda)$. Decompose the invariant factor of the Laplacian matrix. We see that the order of each invariant factor is greater than zero. The invariant factor is written as the product of the first-order factorial powers whose first term is 1, and each factorial power is required to be different. All of these first-order factorial powers (the same must be calculated as the number of occurrences) are called the elementary factors of the matrix $L(G)$.

The controllability problem is a theoretical prerequisite for the study of the practical application of the multiagent system, which guarantees the stability of the multiagent system under certain external input conditions to achieve the control objective. The specific definition of controllability is as follows:

**Definition 1.** For system,
\[
\dot{x} = Ax + Bu,
\]
where $x$ is state, $u$ is control input, and $A, B$ denote coupling gain matrices. If an initial state of the system is recorded as $x_0$ and there exists a control signal $u$ defined on $[0, T]$, such that the state of the system is transferred from $x_0$ to $x(T) = 0$ under the effect of the control signal, we say that $x_0$ can be controlled. If all states of the system are controllable, the system is said to be controllable [53].

In fact, in the case of the single external input, we may find a topology which can be controlled when any vertex on the topology is chosen as the leader node. For this particular topology, we defined it specifically.

**Definition 2.** A multiagent system is said to be perfectly controllable if it is controllable under any selection of leaders. Here, both the number and the locations of leaders are arbitrary [54].

For the discussion of the controllability of the multiagent system, the conditions of its controllability are very important. This paper is mainly based on the following three lemmas to verify the reliability of the results.

**Lemma 1.** For the system defined in Definition 1, a necessary and sufficient condition for controllability is
\[
\text{rank}[sI - A, B] = n, \tag{6}
\]
where $\forall s$ is in the complex field. Or
\[
\text{rank}[(\lambda_i I - A, B) = n \quad i = 1, 2, \ldots, n, \tag{7}
\]
where $\lambda_i$ is the characteristic value of the system [53].

**Lemma 2.** The linear transformation does not change the controllability of the multiagent network system [48].

**Lemma 3.** The multiagent system (1) is controllable if and only if there is no eigenvector of $G$ taking 0 on the elements corresponding to the leaders [48].

In addition, other symbol definitions in this paper are as follows. $R^{m\times n}$ and $R^{n\times m}$ denote the $n$-dimensional square matrix and $n \times m$-dimensional matrix, respectively. $P^{-1}$ and $P^T$ are the inverse and transpose matrices of the matrix $P$, respectively, and $\otimes$ is the Kronecker product. $\lambda_1, \lambda_2, \ldots, \lambda_n$ is marked as $\lambda_1 \sim \lambda_n$. For the sake of a more in-depth discussion, let the Jordan standard form of the Laplacian matrix $L(G)$ be

\[
J(G) = \begin{bmatrix}
J_1 & & \\
& J_2 & \\
& & \ddots \\
& & & J_p \\
\end{bmatrix},
\]
where $J(G)$ contains $p$ Jordan blocks, which correspond to primary elementary of the matrix. For example, $J_m$ corresponds to the $m$th element, $n_m$ is the dimension of the Jordan block, and $m = 1, 2, \ldots, p$ and $n_1 + n_2 + \cdots + n_p = n$.

Suppose $J(G)$ has $l$ different eigenvalues, denoted as $\lambda_1, \lambda_2, \ldots, \lambda_l$, and the number of primary elementary corresponding to eigenvalue $\lambda_i (g = 1, 2, \ldots, l)$ is $d_g$. The Jordan block corresponding to primary elementary in $J_{\lambda_i}$ is denoted as $J_i(1, 2, \ldots, d_g)$. Let the total number of primary elementary corresponding to $g$ eigenvalues of the rang $\lambda_i \sim \lambda_n$ be expressed as $s_g = d_1 + d_2 + \cdots + d_g (g = 1, 2, \ldots, l)$. Then, $s_1 = d_1 + d_2 + \cdots + d_l = p$. The number of rows in the first $m$ primary elementary is given as $h_m = n_1 + n_2 + \cdots + n_m$. We also have $h_p = h_t = n$. It is worth noting that the Jordan block corresponding to the same eigenvalue is written as $J_{\lambda_i}$. To make it easier to discuss our problems, $J(G)$ is rewritten as
Define the similar transformation matrix $P$ of the Laplacian matrix $L(G)$ as

$$
P = \begin{bmatrix}
\zeta_{11}^T & \zeta_{12} & \cdots & \zeta_{1n}^T \\
\zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_{h_{n1}} & \zeta_{h_{n2}} & \cdots & \zeta_{h_{nn}}^T
\end{bmatrix},
$$

such that $L(G) = P^{-1}J(G)P$, where $\zeta_i = [\zeta_{1i}, \zeta_{2i}, \ldots, \zeta_{hi}]^T$.

Lemma 2 indicates that the linear transformation does not affect the controllability of the system. So, let us take the transformation $\bar{X} = PX$. Then, (3) can be written as

$$
P^{-1}\dot{\bar{X}} = -L(G)P^{-1}\bar{X} + \gamma.
$$

(11)

It follows from (4) and (11) that

$$\dot{\bar{X}} = J(G)\bar{X} + \bar{X}r,
$$

(12)

where $\bar{X} = PC^T = \begin{bmatrix} \bar{X}_1^T & \bar{X}_2^T & \cdots & \bar{X}_n^T \end{bmatrix}^T$. Define

$$
H = \begin{bmatrix}
\zeta_{11}^T & \zeta_{12} & \cdots & \zeta_{1n}^T \\
\zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_{h_{n1}} & \zeta_{h_{n2}} & \cdots & \zeta_{h_{nn}}^T
\end{bmatrix},
$$

(13)

where $k = 1, 2, \ldots, d_{max}$ and $d_{max} (d_{max} \geq 1)$ is the number of primary elementary corresponding to the eigenvalue $\lambda_{max}$. We specify that the number of primary elementary corresponding to the eigenvalue $\lambda_i (i = 1, 2, \ldots, l)$ is less than or equal to $d_{max}$. Then, $\text{rank}(H) = d_{max}$. In particular, it should be noted that each element in $H$ satisfies

$$
\zeta_{h_{i,k}}^T \in \{a\zeta_{h_{1,i}} + \beta\zeta_{h_{n,i}} + \cdots + \delta\zeta_{h_{n,i}} | a, \beta, \delta \in \{0, 1\}\}.
$$

(14)

Partition matrix $H$ based on eigenvalues:

$$
H = \begin{bmatrix}
H_{\lambda_1}^T & H_{\lambda_2}^T & \cdots & H_{\lambda_l}^T
\end{bmatrix}^T.
$$

(15)

The matrix block corresponding to the eigenvalue $\lambda_g$ satisfies

$$
\begin{bmatrix}
J_{\lambda_g}^1 \\
J_{\lambda_g}^2 \\
\vdots \\
J_{\lambda_g}^d
\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix}
\zeta_{h_{g,1}} & \zeta_{h_{g,2}} & \cdots & \zeta_{h_{g,d_{max}}}
\end{bmatrix} \\
\begin{bmatrix}
\zeta_{h_{g+1,1}} & \zeta_{h_{g+1,2}} & \cdots & \zeta_{h_{g+1,d_{max}}}
\end{bmatrix} \\
\vdots \\
\begin{bmatrix}
\zeta_{h_{g+2,1}} & \zeta_{h_{g+2,2}} & \cdots & \zeta_{h_{g+2,d_{max}}}
\end{bmatrix}
\end{bmatrix}.
$$

(16)

In the discussion that follows, system (12) is rewritten as

$$
\dot{X} = \begin{bmatrix}
J_{\lambda_1} \\
J_{\lambda_2} \\
\vdots \\
J_{\lambda_l}
\end{bmatrix}X + \begin{bmatrix}
\bar{X}_{\lambda_1} \\
\bar{X}_{\lambda_2} \\
\vdots \\
\bar{X}_{\lambda_l}
\end{bmatrix}r.
$$

(17)

4. Results

For the controllability problem of the distributed multi-agent network system, we consider the matrix pair $(J_{\lambda_g}, \bar{X}_{\lambda_g})$, which is composed of Jordan block $J_{\lambda_g}$ and the corresponding coupling matrix $\bar{X}_{\lambda_g}$. Next, we give the following definition.

Definition 3. If the matrix pair $(J_{\lambda_g}, \bar{X}_{\lambda_g})$ corresponding to the eigenvalue $\lambda_g$ is controllable, the Jordan block $J_{\lambda_g}$ corresponding to the same eigenvalue $\lambda_g$ in $J(G)$ is said to be controllable.

Then, we have the following conclusions to illustrate the relationship between Jordan block controllability and system controllability.

Lemma 4. The necessary and sufficient condition for the controllability of system (17) should be that the Jordan block $J_{\lambda_g}$ corresponding to the same eigenvalue $\lambda_g (g = 1, 2, \ldots, l)$ is controllable.

Proof (sufficiency). It is assumed that the Jordan block $J_{\lambda_g}$ corresponding to $\lambda_g (g = 1, 2, \ldots, l)$ of system (17) is controllable. Then, $\text{rank} (\lambda_gI - J_{\lambda_g}, \bar{X}_{\lambda_g}) = h_{\lambda_g} - h_{\lambda_{g-1}}$. We have

$$
\left(\lambda_gI - J_{\lambda_g}, \bar{X}_{\lambda_g}\right) = \left(K + (\lambda_gI - J_{\lambda_g}, \bar{X}_{\lambda_g})\right),
$$

(18)

where
In $J(G)$, the eigenvalues corresponding to different Jordan blocks $J_{\lambda_g}$ ($g = 1, 2, \ldots, l$) are different from each other (e.g., $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_l$). It is ready to deduce that each Jordan block satisfies the full rank. Then, rank $(K) = h_{s_{g-1}} + h_{s_g} - h_{\lambda_g} = n - h_{s_g} + h_{s_{g+1}}$. We assume from the abovementioned problem that the Jordan block $J_{\lambda_g}$ is controllable (rank $(\lambda_g I - J_{\lambda_g}, \bar{\chi}_{\lambda_g}) = h_{s_g} - h_{s_{g+1}}$). To sum up, (17) satisfies rank $(\lambda_g I - J_{\lambda_g}, \bar{\chi}_{\lambda_g}) = n$. According to the PBH criterion, system (17) is controllable.

Necessity: suppose that system (17) is controllable and $J_{\lambda_g}$ is uncontrollable. Then,

$$\text{rank}(\lambda_g I - J_{\lambda_g}, \bar{\chi}_{\lambda_g}) = n,$$

namely,

$$\text{rank}(K) + \text{rank}(\lambda_g I - J_{\lambda_g}, \bar{\chi}_{\lambda_g}) = n. \tag{21}$$

From equation (21),

$$\text{rank}(\lambda_g I - J_{\lambda_g}, \bar{\chi}_{\lambda_g}) = n - \text{rank}(K)$$

$$= n - (n - h_{s_g} + h_{s_{g+1}})$$

$$= h_{s_g} - h_{s_{g+1}}. \tag{22}$$

It is further known that the Jordan block $J_{\lambda_g}$ corresponding to the eigenvalue $\lambda_g$ is controllable, and the assumption is not valid. Therefore, for any $\lambda_g$ ($g = 1, 2, \ldots, l$), there is rank $(\lambda_g I - J_{\lambda_g}, \bar{\chi}_{\lambda_g}) = h_{s_g} - h_{s_{g+1}}$, that is, the Jordan block $J_{\lambda_g}$ is controllable. To summarize the above, for (17), the Jordan block $J_{\lambda_g}$ is controllable and accordingly the system is controllable. $\Box$

**Corollary 1.** In the case of the single external input, a necessary and sufficient condition for the controllability of system (17) is that only one primary elementary corresponds to one eigenvalue $\lambda_g$ ($g = 1, 2, \ldots, l$), and the element in the $h_g$th row of the input matrix $\bar{\chi}$ is not 0.

**Proof:** Let the Jordanian block $J_{\lambda_g}$ and the associated external input coupling matrix $\bar{\chi}_{\lambda_g}$ be $J_{\lambda_g} = I_g$ and $\bar{\chi}_{\lambda_g} = \begin{bmatrix} 0 & 0 & \cdots & \bar{\chi}_{h_{s_g}} \end{bmatrix}^T$, respectively. Verify the following equation:

$$\text{rank}(\lambda_g I - J_{\lambda_g}, \bar{\chi}_{\lambda_g}) = \text{rank} \begin{bmatrix} 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = h_{s_g} - h_{s_{g+1}}. \tag{23}$$

The element in the $h_g$th row of the coupling matrix is not 0. Therefore, the matrix pair $(J_{\lambda_g}, \bar{\chi}_{\lambda_g})$ can be controlled. According to Lemma 4, we get the necessary and sufficient condition for the system to be controllable under a single input. That is, only one primary elementary is associated
with the one eigenvalue \( \lambda_g \) (\( g = 1, 2, \ldots, l \)), and the element in the \( h_g \) th row of the input matrix \( \bar{X} \) is not 0.

For a distributed multiagent network system, its controllability is not only related to Jordan blocks corresponding to repeated eigenvalues but also dependent on the number of external inputs. Numerous simulation examples show that the more external inputs, the easier to control the multiagent system. In other words, the smaller the number of external inputs, the more difficult it is for the system to achieve controllability. But in engineering applications, the more the number of external inputs, the larger the production costs, and the production costs can be well controlled when the number of external control inputs is small. However, considering most of the system, single external input cannot complete the control task effectively. Therefore, in the following, we consider the correlation between the number of external inputs and the controllability of the system. This paper attempts to discuss the number of external inputs and find out the minimum input requirements for the controllable system, so as to solve the optimal control problem in practical production with reasonable production cost.

**Theorem 1.** For the Laplacian matrix of (17), there is an eigenvalue \( \lambda_g \), with the number of primary elementary corresponding to eigenvalue \( \lambda_g \) (\( g = 1, 2, \ldots, l \)) being less than or equal to the number of primary elementary corresponding to the eigenvalue \( \lambda_g \). Note that the number of primary elementary corresponding to the eigenvalue \( \lambda_g \) is \( d_{\text{max}} \) (\( d_{\text{max}} \geq 1 \)), where \( d_{\text{max}} \) is the minimum number of external inputs that the system is controllable.

\[
\begin{bmatrix}
\lambda_g I - J_{\beta_g} & \bar{X} \\
\bar{X} & 0
\end{bmatrix}
\]

**Proof.** Suppose that system (17) is controllable, and the number of external inputs is \( \omega \) (\( d_{\text{max}} > \omega \)). There is

\[
\bar{X} = \begin{bmatrix}
\bar{X}_{1,1} & \bar{X}_{1,2} & \cdots & \bar{X}_{1,\omega} \\
\bar{X}_{2,1} & \bar{X}_{2,2} & \cdots & \bar{X}_{2,\omega} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{X}_{n,1} & \bar{X}_{n,2} & \cdots & \bar{X}_{n,\omega}
\end{bmatrix}. \quad (24)
\]

Let the number of primary elementary corresponding to eigenvalue \( \lambda_g \) be \( d_{\text{max}} \), and consider the matrix pair \( (\lambda_g I - J_{\beta_g}, \bar{X}_g) \), where

\[
\bar{X}_g = \begin{bmatrix}
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega} \\
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega}
\end{bmatrix}. \quad (25)
\]

The form of \( J_{\beta_g} \) is

\[
J_{\beta_g} = \begin{bmatrix}
J_{g+1} & & & \\
& J_{g+2} & & \\
& & \ddots & \\
& & & J_{\text{max}}
\end{bmatrix}, \quad (26)
\]

and one has

\[
\begin{bmatrix}
\lambda_g I - J_{\beta_g} & \bar{X}_g \\
\bar{X}_g & 0
\end{bmatrix} = \begin{bmatrix}
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega} \\
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega}
\end{bmatrix}.
\]

Take the \( h_{g+1}, h_{g+2}, \ldots, h_{\text{max}} \) -row elements of the matrix pair \( (\lambda_g I - J_{\beta_g}, \bar{X}_g) \) to form the new matrix pair \( (\lambda_g I - J_{\beta_g}, \bar{X}_g) \), then

\[
\begin{bmatrix}
\lambda_g I - J_{\beta_g} & \bar{X}_g \\
\bar{X}_g & 0
\end{bmatrix} = 0,
\]

\[
\begin{bmatrix}
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega} \\
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega}
\end{bmatrix} = \bar{X}_g.
\]

\[
\begin{bmatrix}
\lambda_g I - J_{\beta_g} & \bar{X}_g \\
\bar{X}_g & 0
\end{bmatrix} = \begin{bmatrix}
0, \bar{X}_g \\
\bar{X}_g & 0
\end{bmatrix} = \bar{X}_g.
\]

\[
\begin{bmatrix}
\lambda_g I - J_{\beta_g} & \bar{X}_g \\
\bar{X}_g & 0
\end{bmatrix} = \begin{bmatrix}
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega} \\
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{X}_{h_{g+1},g} & \bar{X}_{h_{g+2},g} & \cdots & \bar{X}_{h_{g+1},\omega}
\end{bmatrix}.
\]

\[
\begin{bmatrix}
\lambda_g I - J_{\beta_g} & \bar{X}_g \\
\bar{X}_g & 0
\end{bmatrix} = \begin{bmatrix}
0, \bar{X}_g \\
\bar{X}_g & 0
\end{bmatrix} = \bar{X}_g.
\]

\[
\begin{bmatrix}
\lambda_g I - J_{\beta_g} & \bar{X}_g \\
\bar{X}_g & 0
\end{bmatrix} = \begin{bmatrix}
0, \bar{X}_g \\
\bar{X}_g & 0
\end{bmatrix} = \bar{X}_g.
\]
We have
\[ \text{rank}(\lambda \mathbf{I} - J_{\lambda} \bar{\mathbf{x}}_{\lambda}) = \text{rank}(\lambda \mathbf{I} - J_{\lambda} \bar{\mathbf{x}}_{\lambda}) + h_{s_j} - h_{s_{j+1}} - d_{\text{max}} \]
\[ = h_{s_j} - h_{s_{j+1}} - d_{\text{max}} + \omega \]
\[ < h_{s_j} - h_{s_{j+1}}. \]
(29)

It is known from equation (29) that when the number of external inputs is \( \omega (d_{\text{max}} > \omega) \), the matrix pair \((\lambda \mathbf{I} - J_{\lambda}, \bar{\mathbf{x}}_{\lambda})\) is uncontrollable. According to Lemma 4, the system cannot be controlled under this condition, so the assumption is not true. Therefore, \( d_{\text{max}} \) is the minimum number of external inputs for the controllability of the system.

Remark 2. The Jordan form is an important tool to discuss the distributed multiagent network system. By using the Jordan canonical form, we can clearly study the effect of repeated eigenvalues on the controllability of the system. At the same time, we introduce the concept of the minimum number of externally coupled inputs, which is a constraint optimization for multiagent system problems. It can be used as a critical constraint to discuss the controllability, consensus, and energy of the system. Therefore, by defining this concept, we can more simply and effectively discuss the impact of the leader-follower network attribute (including topology structure and information communication) on controllability.

Theorem 1 shows that when the system is controllable, the minimum number of external inputs is determined by the maximum number of primary elementary corresponding to the same eigenvalues in the Laplacian matrix. This conclusion is helpful for us to judge whether the system can achieve controllability when the number of external inputs is fixed and to find the minimum number of external inputs which can achieve controllability according to the attributes of the system Laplacian matrix. The discovery of the coupling relationship between topology and external inputs will provide a new direction for the future research on controllability of the distributed multiagent network system. (1) Discuss the condition of determining the controllability of the system under the minimum number of external inputs; (2) the method of realizing the controllability of the system under the fixed topological structure by changing the number and position of external inputs. Next, we discuss the conditions for determining the controllability of the system with minimal external inputs.

Theorem 2. When the minimum number of external inputs is \( d_{\text{max}} (d_{\text{max}} \geq 1) \), system (17) is controllable if and only if there is a matrix \( H \) such that the \( d_{\text{max}} \)-order cofactor of submatrix \( H_{\lambda_i} (i = 1, 2, \ldots, l) \) is not 0.

Proof. Necessity: let eigenvalue \( \lambda_i \) correspond to the maximum number \( d_{\text{max}} \) of primary elementary. The external input matrix is
There must be a $H_{\lambda_i}$ that satisfies $H_{\lambda_i} = \tilde{\chi}_P$, and we conclude that system (17) is controllable. Then, there exists a matrix $H$ such that the $d_i$-order cofactor corresponding to the submatrix $H_{\lambda_i}(i = 1, 2, \ldots, l)$ is not 0.

**Sufficiency 1.** Assume that system (17) is controllable and matrix $H$ has a submatrix $H_{\lambda_i}(i = 1, 2, \ldots, l)$, with all of its $d_i$-order cofactor being 0. In other words, there is a submatrix $H_{\lambda_i} = \tilde{\chi}_P(i = 1, 2, \ldots, l)$ in matrix $H$ whose row rank is always less than $d_i$, then system (17) is not controllable. That is, when the matrix pair $(\lambda, I - J_{\lambda, \tilde{\chi}})$ is not controllable, system (17) is controllable. This is contrary to Lemma 4, and accordingly the assumption does not hold.

To sum up, when the minimum number of external inputs is $d_{\text{max}}$ ($d_{\text{max}} \geq 1$), a necessary and sufficient condition for system (17) to be controllable is that there is a matrix $H$ such that the $d_i$-order cofactor of submatrix $H_{\lambda_i}(i = 1, 2, \ldots, l)$ is not 0.

**Remark 3.** Corollary 1 states that when system (17) with a single external input is controllable, the same eigenvalue $\lambda_i(i = 1, 2, \ldots, l)$ corresponds to only one primary elementary. We consider the problem of controllability in Theorem 2 with single external input, where $H$ is a matrix of dimensional $l \times 1$. Then, the one-order cofactor of the submatrix $H_{\lambda_i}$ should not be zero, that is, each element in the matrix $H$ is not 0. Considering comprehensively, in the case of the single input, a necessary and sufficient condition for (17) to be controllable is that the eigenvalue $\lambda_i$ is associated with one primary elementary and there is no element 0 in the column vector $H = [\tilde{\chi}_{h_1}, \tilde{\chi}_{h_1}, \cdots, \tilde{\chi}_{h_1}]^T$.

We discuss perfect controllability of the system, which is to design the communication method between neighbor nodes under certain external input conditions, and then to find a topology structure, which can ensure that the controllability can be achieved when the position and number of leader nodes, are chosen arbitrarily. In this paper, we discuss the problem of perfect controllability for system (1) with protocol (2) from the Jordan standard form of the Laplacian matrix and give the corresponding condition to realize perfect controllability.

**Corollary 2.** A necessary and sufficient condition for the perfect controllability of system (17) is that the matrix $H$ satisfies that the element $\tilde{\chi}_{h_1}^{m \times 1} \neq 0$ ($\tilde{\chi}_{h_1}^{m \times 1} \in \{\alpha \tilde{\chi}_{h_1}^{m \times 1} + \beta \tilde{\chi}_{h_2}^{m \times 1} + \cdots + \delta \tilde{\chi}_{h_n}^{m \times 1} \big| \alpha, \beta, \delta \in \{0, 1\}, m = 1, 2, \ldots, p \}$).

**Proof.** In the case of the single external input, if system (17) is controllable, then

$$\text{rank}(\lambda I - J_{\lambda, \tilde{\chi}}) = n,$$

where $\tilde{\chi}^T = [\tilde{\chi}_{1,1} \ \tilde{\chi}_{2,1} \ \cdots \ \tilde{\chi}_{h,n}]$. Theorem 1 points out that the minimum number of inputs in a controllable system is the maximum number of primary elementary corresponding to the eigenvalue $\lambda_i(i = 1, 2, \ldots, l)$. Therefore, in the case of single input, if system (17) is perfectly controllable, the same eigenvalue corresponds to only one primary elementary. In Corollary 1, we show that the necessary and sufficient condition for the controllability of the multiagent network system with single input is that the eigenvalue $\lambda_i$ is associated with only one primary elementary and the element of $h_i$-row of input matrix $\tilde{\chi}$ is not 0. Because $\tilde{\chi} = PC^T$, we see that $\tilde{\chi}_{h_i} = a\tilde{\chi}_{h_1} + \beta\tilde{\chi}_{h_2} + \cdots + \delta\tilde{\chi}_{h_n} \neq 0$, $\alpha, \beta, \delta \in \{0, 1\}$. Let $\tilde{\chi}^{m \times 1}_{h_i} \in \{\alpha \tilde{\chi}^{m \times 1}_{h_1} + \beta \tilde{\chi}^{m \times 1}_{h_2} + \cdots + \delta \tilde{\chi}^{m \times 1}_{h_n} \big| \alpha, \beta, \delta \in \{0, 1\}, m = 1, 2, \ldots, p \}$.

**Definition 4.** In the case of multi-input, system (17) is said to be generally perfectly controllable if the system is always controllable regardless of the number and locations of leaders.

Combining Theorem 2 with Corollary 2, we can arrive at a further conclusion. A necessary and sufficient condition for system (17) to achieving the generally perfectly controllable system is that all $d_i$-order cofactor corresponding to the submatrix $H_{\lambda_i}(i = 1, 2, \ldots, l)$ of the matrix $H$ are not 0.

**Remark 4.** The perfectly controllable problem is defined in a multiagent network system with a single external input. On this basis, we consider the multiinput system and extend the definition to generally perfectly controllable problem. The number of external inputs must be greater than or equal to the minimum number of controllable inputs. That is, the generally perfectly controllable problem is realized if the maximum number of primary elementary corresponding to the same eigenvalue in the Laplacian matrix of the system is less than or equal to the number of external inputs.

5. Example and Simulation

This section is divided into two parts to verify the conclusions above under the fixed topological structure. In the first part, we write out the Laplacian matrix $L(G)$, its Jordan standard form $J(G)$, and the similarity transformation matrix $P$ by multiagent communication. According to the conclusion of Theorem 1, the minimum number of external inputs is obtained when the system is controllable. Then, a decision matrix $H$ is found by Theorem 2, and the reasonable leader nodes are selected to make the whole system controllable. Finally, the PBH criterion is used to verify whether the controllability of the system is achieved by the existing leaders, and the feasibility of Theorems 1 and 2 is verified. In the second part, according to the definition of perfect controllability, we choose any position and number of leader nodes to judge whether the system is controllable under the structure of Figure 1. According to the topology of Figure 1, the decision matrix $H$ is found and verified to be consistent with the conclusion in Theorem 2. In this section, the communication weight between multiagent nodes is 1 by default, and all nodes receive input from neighbor information.
5.1. Algebraic Determination of Controllability of Multiagent System. Consider a multiagent network system with eight nodes, whose communication topology is shown in Figure 2. Under system (1) and protocol (2), the Laplacian matrix is

$$L(G) = \begin{bmatrix}
-3 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & -3 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -3 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & -3 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -3 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & -3 \\
\end{bmatrix}. \tag{37}$$

The Jordan standard form for $L(G)$ is

$$J(G) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
\end{bmatrix} \tag{38}$$

The similarity transformation matrix is

$$P = \begin{bmatrix}
0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\
0.125 & -0.125 & 0.125 & -0.125 & -0.125 & 0.125 & -0.125 & 0.125 \\
-0.125 & 0.125 & -0.125 & -0.125 & 0.375 & 0.125 & -0.125 & -0.125 \\
-0.375 & -0.125 & 0.125 & -0.125 & -0.125 & 0.375 & 0.125 & -0.125 \\
-0.125 & -0.375 & -0.125 & 0.125 & 0.125 & -0.125 & 0.375 & -0.125 \\
0.375 & -0.125 & -0.125 & -0.125 & 0.375 & -0.125 & 0.375 & -0.125 \\
0.125 & -0.375 & -0.125 & -0.125 & -0.125 & 0.375 & -0.125 & -0.125 \\
0.125 & 0.375 & -0.125 & -0.125 & -0.125 & 0.375 & -0.125 & -0.125 \\
\end{bmatrix}. \tag{39}$$

In the Jordan standard form $J(G)$ of $L(G)$, the maximum number of primary elementary corresponding to the same eigenvalue is 3. Theorem 1 points out that there is a maximum number of eigenvalues $\lambda_i (i = 1, 2, \ldots, l)$ corresponding to primary elementary, which is denoted as $d_{\max} (d_{\max} \geq 1)$ and $d_{\max}$ is the minimum number of external inputs that the system can be controllable. Therefore, when system (17) is controllable, the minimum number of external coupling inputs is 3. If the 1, 2, 3th columns of $P$ are selected to form the matrix $H$, then
A complex network system is of the cofactor of $H$ external inputs is if and only if there is a matrix $H$ such that the $d_i$-order cofactor of submatrix $H_i$ ($i = 1, 2, \ldots, l$) is not 0. The order of the cofactor of $H_0, H_2, H_4,$ and $H_6$ is 1, 3, 3, and 1, respectively, which satisfies the condition of nonzero. Therefore, when the leader nodes are 1, 2, and 3, the system is controllable. The dynamic equation of the equivalent system with controllability is

\[
\hat{X} = H_0 \hat{X} + r.
\]

The dynamic equation of the distributed multiagent network system is

\[
\dot{X} = H_0 X + r.
\]

In formula (43),
and it was verified that
\[
\text{rank}\,(4I_8 - A, B) = \text{rank}\,(2I_8 - A, B) = \text{rank}\,(6I_8 - A, B) = \text{rank}\,(-A, B) = 8.
\]  
Therefore, Theorem 2 holds.

Again, for Figure 3, we have
\[
L(G) = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\
\end{bmatrix}.
\]  
The Jordan standard form of \( L(G) \) is
\[
J(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 - i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 + i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\end{bmatrix}.
\]  
The similarity transformation matrix \( P \) is
\[
P = \begin{bmatrix} 0 & -1 & -1 & -2 & -3 & -3 & 0 & 0 \\ 0 & -1 & 1 - i & 1 + i & i & -1 - i & 0 & 0 \\ 0 & -1 & 1 + i & 1 - i & -i & -1 + i & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & -2 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & -1 & 0 & -1 & 1 & 2 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]  
The maximum number of primary elementary corresponding to the same eigenvalues in \( J(G) \) is 2. It is known that the minimum number of external inputs which make the system controllable is \( d_{\max} = 2 \). Therefore, when system (17) is controllable, the minimum number of coupling inputs is 2. If the first and second columns of \( P \) are selected to form the matrix \( H_i \), then
\[
H = \begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ -1 & 1 \\
\end{bmatrix}.
\]  
Dividing \( H \) according to the corresponding eigenvalues, we have \( H = [H_0^T, H_{2-i}^T, H_{2+i}^T, H_{1}^T, H_{2}^T]^T \), where
\[
H_0 = [0 \ 1] ,
H_{2-i} = [0 \ -1] ,
H_{2+i} = [0 \ -1] ,
H_1 = [0 \ -1] ,
H_2 = [0 \ -1].
\]  
The order of the cofactor of \( H_0, H_{2-i}, H_{2+i}, H_1, \) and \( H_2 \) is 1, 1, 1, 1, and 2, respectively, and satisfies Theorem 2. Therefore, when the leader nodes are 1 and 2, the system is controllable. The dynamic equation of the equivalent system with the same controllability is
\[
\dot{\tilde{X}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 - i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 + i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\end{bmatrix} \tilde{X} + \begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ -2 & 1 \\ 0 & -1 \\ -1 & 1 \\
\end{bmatrix} r.
\]  
The dynamic equation of the distributed multiagent network system is
In equation (52),

\[
\begin{bmatrix}
2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
X \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8
\end{bmatrix}
\]

\[
X + \begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8
\end{bmatrix}
\]

and it was verified that

\[
\text{rank}(-A, B) = \text{rank}(2 - i) \cdot I_8 - A, B) = \text{rank}(2 + i) \cdot I_8 - A, B)
\]

\[
= \text{rank}(I_8 - A, B) = 8.
\]

Therefore, Theorem 2 holds. Let the initial state be \([1, -1, 2, 1.8, 0.9, 1.4, 1.6, 1.1]^T\) and the external input be \([0.1, 0.5, -0.3]^T\). The controllability of the system is simulated, as shown in Figure 1:

5.2. Verification of Perfectly Controllable Inference.

Figure 4 shows the communication topology of the multi-agent network system. Under system (1) and protocol (2), the Laplacian matrix of the system is

\[
L(G) = \begin{bmatrix}
3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\
-1 & 0 & -1 & 3 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & -1 & -1 & 0 & -1 & 4 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 2
\end{bmatrix}
\]

The Jordan standard form of \(L(G)\) is

\[
J(G) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.18339 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.64632 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.45815 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2.78532 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3.94675 & 0 & 0 \\
0 & 0 & 0 & 0 & 4.57706 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5.17803 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.22498
\end{bmatrix}
\]

The similarity transformation matrix is \(P\), where
The Laplacian matrix of the system has nine different eigenvalues $\lambda_1 = 0, \lambda_2 = 1.18339, \lambda_3 = 1.64632, \lambda_4 = 2.45815, \lambda_5 = 2.78532, \lambda_6 = 3.94675, \lambda_7 = 4.57706, \lambda_8 = 5.17803,$ and $\lambda_9 = 6.22498$. The minimum number of external inputs is one. Next, we use the PBH criterion to verify the conclusion of Corollary 2. Under the perfectly controllable graph, we can choose any number and position of leaders. So, we take any three groups of leader node, say $\{1, 3, 4\}$, $\{2, 4, 6, 7\}$, and $\{1, 2, 5, 7, 9\}$, then

$$C_a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

At this point, $\text{rank}(\lambda_m I - A, B) = 9, m = 1, 2, \ldots, 9$. When the leader selection is $\{2, 4, 6, 7\}$, the system dynamic equation is

$$\dot{x} = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} R.$$

(58)

At this point, $\text{rank}(\lambda_m I - A, B) = 9, m = 1, 2, \ldots, 9$. When the leader selection is $\{1, 3, 4\}$, the system dynamic equation is

$$\dot{x} = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} R.$$

(59)

When the leader selection is $\{1, 3, 4\}$, the system dynamic equation is

$$P = \begin{bmatrix} 0.1111 & 0.1111 & 0.1111 & 0.1111 & 0.1111 & 0.1111 & 0.1111 & 0.1111 & 0.1111 \\ -0.13033 & -0.10457 & -0.25334 & -0.26249 & 0.13029 & 0.19371 & 0.12313 & -0.09316 & 0.39676 \\ 0.10880 & 0.07312 & 0.00276 & 0.01792 & 0.05623 & -0.01258 & -0.28240 & -0.08730 & 0.12344 \\ -0.08784 & -0.21696 & 0.05358 & 0.13731 & 0.03204 & -0.08330 & -0.05539 & 0.10867 & 0.11187 \\ -0.12245 & 0.06415 & 0.11509 & -0.03853 & -0.05190 & 0.02113 & -0.02575 & -0.00091 & 0.03918 \\ 0.02937 & 0.01870 & -0.03892 & 0.16144 & -0.20795 & -0.00816 & 0.07780 & -0.14330 & 0.11101 \\ 0.11050 & -0.01568 & 0.08991 & -0.12875 & -0.02982 & -0.17567 & 0.06714 & 0.00264 & 0.07974 \\ -0.02833 & 0.07694 & -0.08048 & 0.00888 & -0.02414 & -0.05879 & -0.00965 & 0.08945 & 0.02609 \\ 0.00917 & -0.00682 & 0.00029 & -0.00690 & -0.01586 & 0.01253 & -0.00600 & 0.01280 & 0.00079 \end{bmatrix}.$$
and again rank $(\lambda_m I - A, B) = 9, m = 1, 2, \ldots, 9$. With the choice of the three groups of leaders, the system can be controlled. Next, we verify whether the matrix $H$ satisfies the conclusion of Corollary 2 when the leaders are chosen in the form of $C_a$, $C_b$, and $C_c$, respectively:

$$H_a = \begin{bmatrix} 0.3333 & -0.6462 & 0.1295 & 0.1031 & -0.0459 & 0.1519 & 0.0717 & -0.0999 & 0.0026 \end{bmatrix}^T,$$

$$H_b = \begin{bmatrix} 0.4444 & -0.0502 & -0.2039 & -0.2183 & 0.0210 & 0.2498 & -0.2530 & 0.0174 & -0.0072 \end{bmatrix}^T,$$

$$H_c = \begin{bmatrix} 0.5556 & 0.4153 & 0.0792 & -0.2163 & -0.0968 & 0.0289 & 0.2119 & 0.0409 & -0.0187 \end{bmatrix}^T.$$
We can clearly see that there is no element 0 in $H_a, H_b,$ and $H_c,$ which satisfies the conclusion of Corollary 2. So, the verification is valid. The abovementioned verification fully shows the rationality of Corollary 2 and can be applied to the judgment of the perfect controllability of the multiagent network system.

6. Conclusion

Based on the Jordan standard form of the system Laplacian matrix, the controllability problem of the multiagent network system was discussed in this paper. Necessary and sufficient conditions were derived. On the one hand, in the Jordan standard form with repeated eigenvalues, if the Jordan blocks corresponding to the same eigenvalues are controllable, the system is controllable. The conclusion shows that for the multiagent system with fixed topology, the minimum number of external inputs is required when the system can be controlled under certain communication intensity. On the other hand, when the minimum number of external inputs is $d_{\text{max}} (d_{\text{max}} \geq 1)$, the necessary and sufficient condition for a multiagent network system to be controllable is that there exists a matrix $H$ such that the $d_{\text{max}}$-order cofactor is not zero corresponding to the submatrix $H_{\lambda}$. This conclusion points out the requirement of topology and communication state in the case of the controllable multiagent network system with fixed number of external inputs. The two conclusions explained in detail the interaction of topology structure, communication state, and the number of external inputs in the process of realizing the controllability of the multiagent network system. It provides a theoretical basis for the further study of the multiagent network. In addition, in the end of this paper, a generally perfectly controllable topology is introduced, that is, in the case of multi-input, the topology structure can make the system controllable by choosing the position and number of the leaders arbitrarily. The simulation results verify the existence of the structure.

The necessary communication state and the energy of communication are two important topics in the future for the generally perfect controllable topology. The generally perfectly controllable topology provides a more reasonable technical scheme for the future engineering application and industrial manufacture, which improves the fault tolerance of the system, reduces the difficulty of operation and manufacture, and can successfully avoid the safety risk. The advantages of the multiagent network system with more stability are highlighted. It is believed that the popularization and application of the generally perfectly topological structure will play a significant role in promoting social progress.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (nos. 61873136, 61374062, and 61603288), Science Foundation of Shandong Province for Distinguished Young Scholars (no. JQ201419), and Shandong Provincial Natural Science Foundation, China (no. ZR201709260010).

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