LAGRANGIAN SUBMANIFOLDS AND DYNAMICS ON LIE AFFGEBROIDS

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Abstract. We introduce the notion of a symplectic Lie affgebroid and their Lagrangian submanifolds in order to describe the Lagrangian (Hamiltonian) dynamics on a Lie affgebroid in terms of this type of structures. Several examples are discussed.

1. Introduction

Recently, there has been a lot of interest in the study of Lie algebroids, which can be thought of as “generalized tangent bundles”, since they generalize Lie algebras as well as (regular) integrable distributions. From the Physics point of view, Lie algebroids can be used to give geometric descriptions of Hamiltonian and Lagrangian Mechanics. In [23], A. Weinstein introduces “Lagrangian systems” on a Lie algebroid $E$ by means of the linear Poisson structure on the dual $E^*$ and a Legendre-type map from $E$ to $E^*$, associated to a given Lagrangian function $L$ on $E$, provided that $L$ is regular. In that paper, he also asks about the possibility to develop a geometric formalism on Lie algebroids similar to Klein’s formalism in ordinary Lagrangian Mechanics. An answer for this question was given by Martinez in [12] (see also [13, 19]), using the notion of prolongation of a Lie algebroid over a mapping [8].

In [21, 22] an interpretation of the Classical Lagrangian and Hamiltonian dynamics as Lagrangian submanifolds of convenient special symplectic manifolds is described. In doing so, one introduces certain canonical isomorphisms which allow to build the so-called Tulczyjew’s triple of Classical Mechanics (see [21, 22]). In [10], this line of research has been followed. More precisely, the authors introduce the notion of a symplectic Lie algebroid and their Lagrangian submanifolds in order to give an interpretation of Lagrangian and Hamiltonian Mechanics on Lie algebroids in terms of Lagrangian submanifolds of symplectic Lie algebroids. In addition, as an application, the authors recover Lagrange-Poincaré [2] (respectively, Hamilton-Poincaré [11]) equations associated with a $G$-invariant Lagrangian (respectively, Hamiltonian) on a principal $G$-bundle $p : Q \rightarrow M$ as the Euler-Lagrange (respectively, Hamilton) equations on the Atiyah algebroid $TQ/G$.

On the other hand, in [4, 10] (see also [5, 15, 17]) a possible generalization of the concept of a Lie algebroid to affine bundles is introduced in order to create a geometric model which...
provides a natural framework for a time-dependent version of Lagrange (Hamilton) equations on Lie algebroids. These new structures are called Lie affgebroid structures (in the terminology of [3]).

The main aim of this paper is to introduce the notion of a Lagrangian submanifold of a symplectic Lie affgebroid and, then, to use this type of geometric objects to describe the Lagrangian (Hamiltonian) dynamics on Lie affgebroids.

The paper is organized as follows. In Section 2.1 we recall the notion of a Lie algebroid and several constructions related with them, in particular the prolongation \( L^f \) of a Lie algebroid \( \tau : E \to M \) over a smooth map \( f : M' \to M \) and the definition of the Lie algebroid structure of an action Lie algebroid. In Section 2.2 we describe the concept of a Lie affgebroid structure on an affine bundle \( \tau_A : \tilde{A} \to M \) modelled on a vector bundle \( \tau_V : V \to M \). We remark that if \( \tau_{A^+} : A^+ = \text{Aff}(A, \mathbb{R}) \to M \) is the dual bundle to \( A \) and \( \hat{A} = (A^+)^* \) is the bidual, then a Lie affgebroid structure on \( A \) is equivalent to a Lie algebroid structure on \( \hat{A} \) such that the distinguished section \( 1_A \) of \( \tau_{A^+} : A^+ \to M \) (corresponding to the constant function 1 on \( \hat{A} \)) is a 1-cocycle for the Lie algebroid cohomology.

The description of the Hamiltonian dynamics on a Lie affgebroid is presented in Section 3.1 following [14]. This geometric framework allows to write, for a Lie affgebroid \( \tau_A : \tilde{A} \to M \), the Hamilton equations associated with a Hamiltonian section \( h : V^* \to A^+ \) of the canonical projection \( \mu : A^+ \to V^* \). In doing so, we introduce a cosymplectic structure \((\Omega_h, \eta)\) on the prolongation \( L^{\tau_A} \tilde{A} \) of \( \hat{A} \) over \( \tau_V^* : V^* \to M \). Then, the integral curves of the Reeb section \( R_h \) of \((\Omega_h, \eta)\) are just the solutions of the Hamilton equations. Alternatively, one may prove that the solutions of the Hamilton equations are just the integral curves of the Hamiltonian vector field \((\Omega^*)^\tau : V^* \to A^+ \) of \( h \) with respect to the canonical aff-Poisson structure on the line affine bundle \( \mu : A^+ \to V^* \) (see Theorem 3.2). Aff-Poisson structures were introduced in [1] (see also [3]) as the affine version of standard Poisson structures. On the other hand, in Section 3.2 it is developed the corresponding Lagrangian formalism for a Lagrangian function \( L : A \to \mathbb{R} \) (see [14]). In this case, we work on the prolongation \( L^{\tau_A} \tilde{A} \) of \( \hat{A} \) over \( \tau_A \).

We define the Poincaré-Cartan 2-section \( \Omega_L \) (as a section of the vector bundle \( \wedge^2(L^{\tau_A} \tilde{A})^* \to A \)) and the vertical endomorphism \( S \) (as a section of \( L^{\tau_A} \tilde{A} \otimes (L^{\tau_A} \tilde{A})^* \to A \)). These objects allow us to write the Euler-Lagrange equations for \( L \) in an intrinsic way. In the particular case, when \( L \) is regular the pair \((\Omega_L, \phi_0)\) is a cosymplectic structure on \( L^{\tau_A} \tilde{A} \) and the integral curves of the Reeb section \( R_L \) of \((\Omega_L, \phi_0)\) are just the solutions of the Euler-Lagrange equations for \( L \). In Section 3.3 it is stated the equivalence between both formalisms using the Legendre transformation \( leg_L : A \to V^* \), provided that \( L \) is hyperregular (that is, \( leg_L : A \to V^* \) is a global diffeomorphism). In fact, we may construct a Hamiltonian section \( h_L : V^* \to A^+ \) and we have a Lie algebroid morphism \( leg_L : L^{\tau_A} \tilde{A} \to L^{\tau_A} \hat{A} \) over \( leg_L : A \to V^* \) which in the hyperregular case is an isomorphism and, in addition, connects the Lagrangian and Hamiltonian formalism. Conversely, if \( h : V^* \to A^+ \) is a Hamiltonian section, one may introduce a map \( \mathbb{F}h : V^* \to A \) such that \( \tau_A \circ \mathbb{F}h = \tau_V^* \). Furthermore, if \( h \) is hyperregular (that is, \( \mathbb{F}h : V^* \to A \) is a global diffeomorphism) then there exists a hyperregular Lagrangian \( L : A \to \mathbb{R} \) such that \( h_L = h \) and \( \mathbb{F}h = leg_{L_{\mathbb{F}h}}^{-1} \).
In Sections 4 and 5, we extend the construction of Tulczyjew’s triple to the Lie affgebroid setting. More precisely, for a Lie affgebroid $\tau_A : A \to M$ we consider the space

$$\mathcal{J}^A A = \{(a, v) \in A \times TA / \rho_A(a) = (T\tau_A)(v)\},$$

where $\rho_A : A \to TM$ is the anchor map of $A$, and we prove that $\mathcal{J}^A A$ admits two Lie affgebroids structures. These structures are isomorphic under the so-called canonical involution $\sigma_A : \mathcal{J}^A A \to \mathcal{J}^A A$ associated with $A$. In addition, if $h : V^* \to A^+$ is a Hamiltonian section then we define an affine isomorphism $\flat h$ (over the identity of $V^*$) between the affine bundles $\rho_A^*(TV^*) \to V^*$ and $(\mathcal{L}_{\tau_V} V)^* \to V^*$. Here, $\rho_A^*(TV^*)$ is the pull-back of the vector bundle $T\tau_V^* : TV^* \to TM$ over $\rho_A$ and $\mathcal{L}_{\tau_V} V$ is the prolongation of $V$ over $\tau_V^*$. The map $\flat h$, along with a canonical vector bundle isomorphism $A_A : \rho_A^*(TV^*) \to (\mathcal{L}_{\tau_A} V)^*$ (related with $\sigma_A$) gives us the Tulczyjew’s triple associated with $A$ and $h$.

In Section 6, we introduce the notion of a symplectic Lie affgebroid as a Lie affgebroid modelled on a symplectic Lie algebroid (that is, a Lie algebroid $\tau_V : V \to M$ which admits a non-degenerate 2-cocycle). Moreover, we prove that if $\tau_A : A \to M$ is a symplectic Lie affgebroid then its prolongation $\mathcal{J}^A A$ is also a symplectic Lie affgebroid. The notion of a Lagrangian Lie subaffgebroid of a symplectic Lie affgebroid is introduced, in a natural way, in Section 7. Examples and properties of this type of objects are discussed in this section.

In Section 8, using the results of Section 7, we introduce the definition of a Lagrangian submanifold of a symplectic Lie affgebroid. Then, if $h : V^* \to A^+$ is a Hamiltonian section we deduce that $S_h = R_h(V^*)$ is a Lagrangian submanifold of the symplectic Lie affgebroid $\rho_A^*(TV^*)$ and, in addition, there exists a bijection between admissible curves in $S_h$ and solutions of the Hamilton equations for $h$. Similarly, given a Lagrangian function $L : A \to \mathbb{R}$, we prove that $S_L = (A_A^{-1} \circ d\mathcal{L}_{\tau_A} V L)(A)$ is a Lagrangian submanifold of $\rho_A^*(TV^*)$ and that there exists a bijection between admissible curves in $S_L$ and solutions of the Euler-Lagrange equations for $L$. When $L$ is hyperregular and $h$ is the corresponding Hamiltonian section, we deduce that $S_L = S_h$.

Finally, we describe some applications in Section 9. In fact, in Section 9.1, we prove that if the Lie affgebroid $A$ is a Lie algebroid then we recover the results obtained in [10] about the relation between Lagrangian submanifolds and dynamics on Lie algebroids. In addition, in Section 9.2 we apply the results of the paper to the particular case when the Lie affgebroid $A$ is the 1-jet bundle $\tau_{1,0} : J^1 \tau \to M$ of local sections of a fibration $\tau : M \to \mathbb{R}$. As a consequence, we deduce that the classical Euler-Lagrange (Hamilton) equations of time-dependent Mechanics are just the local equations defining Lagrangian submanifolds of a symplectic Lie affgebroid. On the other hand, in Section 9.3 we consider a principal $G$-bundle $p : Q \to M$ such that the base space $M$ is fibred on $\mathbb{R}$, that is, there exists a fibration $\nu : M \to \mathbb{R}$. Then, if $\tau = \nu \circ p$, we have that the quotient affine bundle $\tau_{1,0} | G : J^1 G / G \to M$ admits a Lie affgebroid structure in such way that the bidual Lie algebroid is just the Atiyah algebroid $\pi_Q | G : TQ / G \to M$ associated with the principal $G$-bundle $p : Q \to M$ (see [10]). For this reason, the affine bundle $\tau_{1,0} | G : J^1 G / G \to M$ is called an Atiyah affgebroid. We obtain that, in this case, the solutions of the Euler-Lagrange (Hamilton) equations for a Lagrangian (resp., a Hamiltonian section) are the solutions of the classical nonautonomous Lagrange-Poincaré (resp. Hamilton-Poincaré) equations for the corresponding $G$-invariant Lagrangian (resp. $G$-invariant Hamiltonian section). Moreover, all
these equations are reinterpreted as those defining the corresponding Lagrangian submanifolds of an Atiyah symplectic Lie affgebroid. Manifolds are real, paracompact and $C^\infty$. Maps are $C^\infty$. Sum over crossed repeated indices is understood.

2. Lie algebroids and Lie affgebroids

2.1. Lie algebroids. Let $E$ be a vector bundle of rank $n$ over the manifold $M$ of dimension $m$ and $\tau : E \to M$ be the vector bundle projection. Denote by $\Gamma(\tau)$ the $C^\infty(M)$-module of sections of $\tau : E \to M$. A Lie algebroid structure $([\cdot, \cdot], \rho)$ on $E$ is a Lie bracket $[\cdot, \cdot]$ on the space $\Gamma(\tau)$ and a bundle map $\rho : E \to TM$, called the anchor map, such that if we also denote by $\mu : \Gamma(\tau) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then $[X, fY] = f[X, Y] + \rho(X)(f)Y$, for $X, Y \in \Gamma(\tau)$ and $f \in C^\infty(M)$. The triple $(E, [\cdot, \cdot], \rho)$ is called a Lie algebroid over $M$ (see [11]). In such a case, the anchor map $\rho : \Gamma(\tau) \to \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(\tau), [\cdot, \cdot])$ and $(\mathfrak{X}(M), [\cdot, \cdot])$.

If $(E, [\cdot, \cdot], \rho)$ is a Lie algebroid, one may define a cohomology operator, which is called the differential of $E$, $d^E : \Gamma(\wedge^k \tau^*) \to \Gamma(\wedge^{k+1} \tau^*)$, as follows

$$(d^E \mu)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \rho(X_i)(\mu(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) + \sum_{i<j} (-1)^{i+j} \mu([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k),$$

for $\mu \in \Gamma(\wedge^k \tau^*)$ and $X_0, \ldots, X_k \in \Gamma(\tau)$. Moreover, if $X \in \Gamma(\tau)$ one may introduce, in a natural way, the Lie derivative with respect to $X$, as the operator $\mathcal{L}_X^E : \Gamma(\wedge^k \tau^*) \to \Gamma(\wedge^k \tau^*)$ given by $\mathcal{L}_X^E = i_X \circ d^E + d^E \circ i_X$.

If $E$ is the standard Lie algebroid $TM$ then the differential $d^E = dTM$ is the usual exterior differential associated with $M$, which we will denote by $d_0$.

Now, suppose that $(E, [\cdot, \cdot], \rho)$ and $(E', [\cdot, \cdot]', \rho')$ are Lie algebroids over $M$ and $M'$, respectively, and that $F : E \to E'$ is a vector bundle morphism over the map $f : M \to M'$. Then $(F, f)$ is said to be a Lie algebroid morphism if

$$d^E((F, f)^* \phi') = (F, f)^*(d^{E'} \phi'), \text{ for } \phi' \in \Gamma(\wedge^k (\tau')^*) \text{ and for all } k.$$ 

Note that $(F, f)^* \phi'$ is the section of the vector bundle $\wedge^k E^* \to M$ defined by

$$( (F, f)^* \phi')(x)(a_1, \ldots, a_k) = \phi'_f(x)(F(a_1), \ldots, F(a_k)),$$

for $x \in M$ and $a_1, \ldots, a_k \in E_x$. If $(F, f)$ is a Lie algebroid morphism, $f$ is an injective immersion and $F_{E_x} : E_x \to E'_{f(x)}$ is injective, for all $x \in M$, then $(E, [\cdot, \cdot], \rho)$ is said to be a Lie subalgebroid of $(E', [\cdot, \cdot]', \rho')$.

2.1.1. The prolongation of a Lie algebroid over a smooth map. In this section, we will recall the definition of the Lie algebroid structure on the prolongation of a Lie algebroid over a smooth map (see [8, 11]).

Let $(E, [\cdot, \cdot], \rho)$ be a Lie algebroid of rank $n$ over a manifold $M$ of dimension $m$ with vector bundle projection $\tau : E \to M$ and $f : M' \to M$ be a smooth map.
We consider the subset $\mathcal{L}^f E$ of $E \times TM'$ and the map $\tau^f : \mathcal{L}^f E \to M'$ defined by

$$\mathcal{L}^f E = \{(b, v') \in E \times TM' : \rho(b) = (Tf)(v')\}, \quad \tau^f(b, v') = \pi_M(v'),$$

where $Tf : TM' \to TM$ is the tangent map to $f$ and $\pi_M : TM' \to M'$ is the canonical projection.

Now, assume that $\rho(E_f(x')) + (T_x f)(T_x M') = T_f(x') M$ for all $x' \in M'$. Then, $\tau^f : \mathcal{L}^f E \to M'$ is a vector bundle over $M'$ of rank $n + \dim M' - m$ which admits a Lie algebroid structure $([\cdot, \cdot]^f, \rho^f)$ characterized by

$$[[X \circ f, U'], (Y \circ f, V')]^f = ([X, Y] \circ f, [U', V']), \quad \rho^f(X \circ f, U') = U',$$

for all $X, Y \in \Gamma(\tau)$ and $U', V'$ $f$-projectable vector fields to $\rho(X)$ and $\rho(Y)$, respectively. $([\cdot, \cdot]^f, \rho^f)$ is called the prolongation of the Lie algebroid $E$ over the map $f$ (for more details, see [3 III]).

Next, we consider a particular case of the above construction. Let $E$ be a Lie algebroid over a manifold $M$ with vector bundle projection $\tau : E \to M$ and $\mathcal{L}^\tau E$ be the prolongation of $E$ over the projection $\tau^* : E^* \to M$. $\mathcal{L}^\tau E$ is a Lie algebroid over $E^*$ and we can define a canonical section $\lambda_E$ of the vector bundle $(\mathcal{L}^\tau E)^* \to E^*$ as follows. If $a^* \in E^*$ and $(b, v) \in (\mathcal{L}^\tau E)_a^*$ then

$$(2.1) \quad \lambda_E(a^*)(b, v) = a^*(b).$$

$\lambda_E$ is called the Liouville section associated with the Lie algebroid $E$.

Now, one may consider the nondegenerate section $\Omega_E = -d\mathcal{L}^\tau E \lambda_E$ of $\wedge^2 (\mathcal{L}^\tau E)^* \to E^*$. It is clear that $d\mathcal{L}^\tau E \Omega_E = 0$. In other words, $\Omega_E$ is a symplectic section. $\Omega_E$ is called the canonical symplectic section associated with the Lie algebroid $E$. Using the symplectic section $\Omega_E$ one may introduce a linear Poisson structure $\Pi_{E^*}$ on $E^*$, with linear Poisson bracket $\{\cdot, \cdot\}_{E^*}$ given by

$$\{F, G\}_{E^*} = -\Omega_E(X_F, Y_G), \quad \text{for } F, G \in C^\infty(E^*),$$

where $X_F$ and $X_G$ are the Hamiltonian sections associated with $F$ and $G$, that is, $i_{X_F} \Omega_E = d\mathcal{L}^\tau E F$ and $i_{X_G} \Omega_E = d\mathcal{L}^\tau E G$.

Suppose that $(x^i)$ are local coordinates on an open subset $U$ of $M$ and that $\{e_\alpha\}$ is a local basis of sections of the vector bundle $\tau^{-1}(U) \to U$ such that

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}, \quad \{e_\alpha, e_\beta\} = C^\gamma_{\alpha\beta} e_\gamma.$$

Then, $\{e_\alpha, \tilde{e}_\alpha\}$ is a local basis of sections of the vector bundle $(\tau^*)^{-1}((\tau^*)^{-1}(U)) \to (\tau^*)^{-1}(U)$, where $\tau^* : \mathcal{L}^\tau E \to E^*$ is the vector bundle projection and

$$\tilde{e}_\alpha(a^*) = (e_\alpha(\tau^*(a^*)), \rho^i_\alpha \frac{\partial}{\partial x^i}|_{a^*}), \quad \tilde{e}_\alpha(a^*) = (0, \frac{\partial}{\partial y_\alpha}|_{a^*}).$$

Here, $(x^i, y_\alpha)$ are the local coordinates on $E^*$ induced by the local coordinates $(x^i)$ and the dual basis $\{e^\alpha\}$ of $\{e_\alpha\}$. Moreover, we have that

$$(2.2) \quad [\tilde{e}_\alpha, \tilde{e}_\beta]^* = C^\gamma_{\alpha\beta} \tilde{e}_\gamma, \quad [\tilde{e}_\alpha, \bar{e}_\beta]^* = [\tilde{e}_\alpha, e_\beta]^* = 0, \quad \rho^* \tilde{e}_\alpha = \rho^i_\alpha \frac{\partial}{\partial x^i}, \quad \rho^* \tilde{e}_\alpha = \frac{\partial}{\partial y_\alpha},$$

where $\rho = (\rho^i_\alpha)$.
and
\[ \lambda_E(x^i, y_\alpha) = y_\alpha \epsilon^\alpha, \quad \Omega_E(x^i, y_\alpha) = \epsilon_\alpha \wedge \epsilon^\alpha + \frac{1}{2} C_{\alpha\beta}^\gamma y_\gamma \epsilon^\alpha \wedge \epsilon^\beta, \]

(2.4) \[ \Pi_E = \frac{1}{2} C_{\alpha\beta}^\gamma y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta} + \rho_\alpha^i \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial x^i}, \]

for more details, see [10, 13].

2.1.2. Action Lie algebroids. In this section, we will recall the definition of the Lie algebroid structure of an action Lie algebroid (see [8, 10]).

Let \((E, [[\cdot, \cdot], \rho])\) be a Lie algebroid over a manifold \(M\) and \(f : M' \to M\) be a smooth map. Then, the pull-back of \(E\) over \(f\), \(f^*E = \{(x', a) \in M' \times E / f(x') = \tau(a)\}\), is a vector bundle over \(M'\) whose vector bundle projection is the restriction to \(f^*E\) of the first canonical projection \(pr_1 : M' \times E \to M'\). However, \(f^*E\) is not, in general, a Lie algebroid.

Now, suppose that \(\Psi : \Gamma(\tau) \to \mathcal{X}(M')\) is an action of \(E\) on \(f\), that is, \(\Psi\) is a \(\mathbb{R}\)-linear map which satisfies the following conditions
\[ \Psi(hX) = (h \circ f)\Psi X, \quad \Psi[X, Y] = [\Psi X, \Psi Y], \quad \Psi X(h \circ f) = \rho(X)(h) \circ f, \]

for \(X, Y \in \Gamma(\tau)\) and \(h \in C^\infty(M)\). Then, one may introduce a Lie algebroid structure \([[], \rho\Psi]\) on the vector bundle \(f^*E \to M'\) which is characterized by the following conditions
\[ [[X \circ f, Y \circ f]]\Psi = [X, Y] \circ f, \quad \rho\Psi(X \circ f) = \Psi(X), \quad \text{for } X, Y \in \Gamma(\tau). \]

The resultant Lie algebroid is denoted by \(E \ltimes f\) and we call it an action Lie algebroid.

Next, we will apply the above construction to a particular case. First of all, we recall that there is a one-to-one correspondence between linear functions on a vector bundle \(E\) and sections of \(E^*\). In this paper, we don’t distinguish between a section of \(E^*\) and its associated linear function on \(E\). Let \((E, [[\cdot, \cdot], \rho])\) be a Lie algebroid with vector bundle projection \(\tau : E \to M\) and \(X\) be a section of \(\tau : E \to M\). Then, we define the vertical lift of \(X\) as the vector field on \(E\) given by \(X^v(\alpha) = X(\tau(\alpha)))_\alpha^\alpha\), for \(\alpha \in E\), where \(\alpha^\alpha : E_{\tau(\alpha)} \to T_{\alpha}(E_{\tau(\alpha)})\) is the canonical vector space isomorphism. In addition, there exists a unique vector field \(X^c\) on \(E\), the complete lift of \(X\), which satisfies the following conditions: i) \(X^c\) is \(\tau\)-projectable on \(\rho(X)\) and ii) \(X^c(\alpha) = \mathcal{L}_X^E \alpha\), for all \(\alpha : E \to \mathbb{R}\) section of \(E^*\) (for more details, see [3, 7, 10]).

On the other hand, it is well-known (see, for instance, [3]) that the tangent bundle to \(E, TE\), is a vector bundle over \(TM\) with vector bundle projection the tangent map to \(\tau, T\tau : TE \to TM\). Moreover, the tangent map to \(X, TX : TM \to TE\), is a section of the vector bundle \(T\tau : TE \to TM\). We may also consider the section \(\dot{X} : TM \to TE\) of \(T\tau : TE \to TM\) given by
\[ \dot{X}(u) = (T_x0)(u) + X(x)_0^0(x), \]

for \(u \in T_xM\), where \(0 : M \to E\) is the zero section of \(E\) and \(\alpha^0_{0(x)} : E_x \to T_{0(x)}(E_x)\) is the canonical isomorphism between \(E_x\) and \(T_{0(x)}(E_x)\).

If \(\{e_\alpha\}\) is a local basis of \(\Gamma(\tau)\) then \(\{Te_\alpha, \bar{e}_\alpha\}\) is a local basis of \(\Gamma(T\tau)\).

The vector bundle \(T\tau : TE \to TM\) admits a Lie algebroid structure with anchor map \(\rho^T\) given by \(\rho^T = \sigma_T \circ T\rho, \sigma_T : TM \to T(TM)\) being the canonical involution of the double
tangent bundle. The Lie bracket \([\cdot, \cdot]^T\) on the space \(\Gamma(T\tau)\) is characterized by the following equalities
\[
[TX, TY]^T = T[X, Y], \quad [TX, \hat{Y}]^T = [\hat{X}, Y], \quad [\hat{X}, \hat{Y}]^T = 0,
\]
for \(X, Y \in \Gamma(\tau)\) (see [10, 13]).

Furthermore, there exists a unique action \(\Psi : \Gamma(T\tau) \to \mathfrak{x}(E)\) of the Lie algebroid \((TE, [\cdot, \cdot]^T, \rho^T)\) over the anchor map \(\rho : E \to TM\) such that \(\Psi(TX) = X^e\) and \(\Psi(\hat{X}) = \hat{X}^e\), for \(X \in \Gamma(\tau)\) (see [10]). Thus, the vector bundle \(\rho^*(TE)\) is a Lie algebroid with Lie algebroid structure \(([\cdot, \cdot]^T, \rho^T)\), which is characterized as in (2.3).

2.2. Lie affgebroids. Let \(\tau_A : A \to M\) be an affine bundle with associated vector bundle \(\tau_V : V \to M\). Denote by \(\tau_{A^+} : A^+ = Aff(A, \mathbb{R}) \to M\) the dual bundle whose fibre over \(x \in M\) consists of affine functions on the fibre \(A_x\). Note that this bundle has a distinguished section \(1_A \in \Gamma(\tau_{A^+})\) corresponding to the constant function \(1\) on \(A\). We also consider the bidual bundle \(\tau_{\hat{A}} : \hat{A} \to M\) whose fibre at \(x \in M\) is the vector space \(A_x^\perp = (A_x^\perp)^*\). Then, \(A\) may be identified with an affine subbundle of \(\hat{A}\) via the inclusion \(i_A : A \to \hat{A}\) given by \(i_A(a)(\varphi) = \varphi(a)\), which is injective affine map whose associated linear map is denoted by \(i_V : V \to A\). Thus, \(V\) may be identified with a vector subbundle of \(\hat{A}\). Using these facts, one can prove that there is a one-to-one correspondence between affine functions on \(A\) and linear functions on \(\hat{A}\). On the other hand, there is an obvious one-to-one correspondence between affine functions on \(A\) and sections of \(A^+\).

A Lie affgebroid structure on \(A\) consists of a Lie algebra structure \(([\cdot, \cdot]_V)\) on the space \(\Gamma(\tau_V)\) of the sections of \(\tau_V : V \to M\), a \(\mathbb{R}\)-linear action \(D : \Gamma(\tau_A) \times \Gamma(\tau_V) \to \Gamma(\tau_V)\) of the sections of \(A\) on \(\Gamma(\tau_V)\) and an affine map \(\rho_A : A \to TM\), the anchor map, satisfying the following conditions:
- \(D_X[Y, Z]_V = [D_XY, Z]_V + [Y, D_XZ]_V\),
- \(D_X + \rho_A(X)f Y = f D_XY + \rho_A(X)(f)\hat{Y}\),

for \(X \in \Gamma(\tau_A), Y, Z \in \Gamma(\tau_V)\) and \(f \in C^\infty(M)\) (see [11, 12]).

If \(([\cdot, \cdot]_V, D, \rho_A)\) is a Lie affgebroid structure on an affine bundle \(A\) then \((V, [\cdot, \cdot]_V, \rho_V)\) is a Lie algebroid, where \(\rho_V : V \to TM\) is the vector bundle map associated with the affine morphism \(\rho_A : A \to TM\).

A Lie affgebroid structure on an affine bundle \(\tau_A : A \to M\) induces a Lie algebroid structure \(([\cdot, \cdot]_\hat{A}, \rho_{\hat{A}})\) on the bidual bundle \(\hat{A}\) such that \(1_A \in \Gamma(\tau_{A^+})\) is a 1-cocycle in the corresponding Lie algebroid cohomology, that is, \(d^31_A = 0\). Indeed, if \(X_0 \in \Gamma(\tau_A)\) then for every section \(\hat{X}\) of \(\hat{A}\) there exists a function \(f \in C^\infty(M)\) and a section \(\hat{X} = fX_0 + \hat{X}\) such that \(\hat{X} = fX_0 + \hat{X}\) and
\[
\rho_{\hat{A}}(fX_0 + \hat{X}) = f\rho_A(X_0) + \rho_V(\hat{X}),
\]
\[
[fX_0 + \hat{X}, gX_0 + \hat{Y}]_{\hat{A}} = (\rho_V(\hat{X})(g) - \rho_V(\hat{Y})(f) + f\rho_A(X_0)(g) - g\rho_A(X_0)(f))X_0 + [\hat{X}, \hat{Y}]_V + fDX_0\hat{Y} - gDX_0\hat{X}.
\]

Conversely, let \((U, [\cdot, \cdot]_U, \rho_U)\) be a Lie algebroid over \(M\) and \(\phi : U \to \mathbb{R}\) be a 1-cocycle of \((U, [\cdot, \cdot]_U, \rho_U)\) such that \(\phi|_{U_x} \neq 0\), for all \(x \in M\). Then, \(A = \phi^{-1}\{1\}\) is an affine bundle over \(M\) which admits a Lie affgebroid structure in such a way that \((U, [\cdot, \cdot]_U, \rho_U)\) may be identified with the bidual Lie algebroid \((\hat{A}, [\cdot, \cdot]_{\hat{A}}, \rho_{\hat{A}})\) to \(A\) and, under this identification, the 1-cocycle
$1_A : \tilde{A} \to \mathbb{R}$ is just $\phi$. The affine bundle $\tau_A : A \to M$ is modelled on the vector bundle $\tau_V : V = \phi^{-1}\{0\} \to M$. In fact, if $i_V : V \to U$ and $i_A : A \to U$ are the canonical inclusions, then

$$i_V \circ [X, Y] U = [i_V \circ X, i_V \circ Y] U, \quad \rho_A(X) = \rho_U(i_A \circ X),$$

for $X, Y \in \Gamma(\tau_V)$ and $X \in \Gamma(\tau_A)$.

A trivial example of a Lie algebroid may be constructed as follows. Let $\tau : M \to \mathbb{R}$ be a fibration and $\tau_{1,0} : J^1\tau \to M$ be the 1-jet bundle of local sections of $\tau : M \to \mathbb{R}$. It is well known that $\tau_{1,0} : J^1\tau \to M$ is an affine bundle modelled on the vector bundle $\pi = (\pi_M)_{\mid \tau} : V\tau \to M$, where $V\tau$ is the vertical bundle of $\tau : M \to \mathbb{R}$. Moreover, if $t$ is the usual coordinate on $\mathbb{R}$ and $\eta$ is the closed 1-form on $M$ given by $\eta = \tau^*(dt)$ then we have the following identification

$$J^1\tau \cong \{ v \in TM/\eta(v) = 1 \}$$

(see, for instance, [20]). Note that $V\tau = \{ v \in TM/\eta(v) = 0 \}$. Thus, the bidual bundle $\tilde{J^1\tau}$ to the affine bundle $\tau_{1,0} : J^1\tau \to M$ may be identified with the tangent bundle $TM \to M$ and, under this identification, the Lie algebroid structure on $\pi_M : TM \to M$ is the standard Lie algebroid structure and the 1-cocycle $1_{\tau}$ on $\pi_M : TM \to M$ is just the 1-form $\eta$.

Let $\tau_A : A \to M$ be a Lie algebroid modelled on the Lie algebroid $\tau_V : V \to M$. Suppose that $(x^i)$ are local coordinates on an open subset $U$ of $M$ and that $\{ e_0, e_\alpha \}$ is a local basis of sections of $\tau_A : \tilde{A} \to M$ in $U$ which is adapted to the 1-cocycle $1_A$, i.e., such that $1_A(e_0) = 1$ and $1_A(e_\alpha) = 0$, for all $\alpha$. Note that if $\{ e_0^\prime, e_\alpha^\prime \}$ is the dual basis of $\{ e_0, e_\alpha \}$ then $e_0^\prime = 1_A$. Denote by $(x^i, y^0, y^\alpha)$ the corresponding local coordinates on $\tilde{A}$. Then, the local equation defining the affine subbundle $A$ (respectively, the vector subbundle $V$) of $\tilde{A}$ is $y^0 = 1$ (respectively, $y^0 = 0$).

Thus, $(x^i, y^\alpha)$ may be considered as local coordinates on $A$ and $V$.

Now, let $\tau_A : A \to M$ (respectively, $\tau_A^0 : A^0 \to M^0$) be an affine bundle with associated vector bundle $\tau_V : V \to M$ (respectively, $\tau_V^0 : V^0 \to M^0$) and $F : A \to A^0$ be an affine bundle morphism over the map $f : M \to M'$ with associated morphism $(F^i, f)$ between the vector bundles $\tau_V : V \to M$ and $\tau_V^0 : V^0 \to M^0$.

Then, a direct computation proves that the map $\tilde{F} : \tilde{A} \to \tilde{A'}$ given by

$$\tilde{F}(\tilde{a})(\varphi') = \tilde{a}(\varphi' \circ F), \quad \text{for } \tilde{a} \in \tilde{A}_+ \text{ and } \varphi' \in (A')^+_f(x)^i, \text{ with } x \in M,$$

defines a morphism between the vector bundles $\tilde{A}$ and $\tilde{A'}$ over $f$ and, moreover, $(\tilde{F}, f)^{-1}1_{A'} = 1_A$.

Conversely, suppose that $\tau_U : U \to M$ and $\tau_{U'} : U' \to M'$ are vector bundles and that $\phi$ and $\phi'$ are sections of the vector bundles $\tau_U^* : U^* \to M$ and $\tau_{U'}^* : (U')^* \to M'$ such that $\phi(x) \neq 0$, for all $x \in M$, and $\phi'(x') \neq 0$, for all $x' \in M'$. Assume also that the pair $(\tilde{F}, f)$ is a morphism between the vector bundles $\tau_U : U \to M$ and $\tau_{U'} : U' \to M'$ such that $(\tilde{F}, f)^{-1}\phi' = \phi$ and denote by $A$ and $V$ (respectively, $A'$ and $V'$) the subsets of $U$ (respectively, $U'$) defined by $A = \phi^{-1}\{1\}$ and $V = \phi^{-1}\{0\}$ (respectively, $A' = (\phi')^{-1}\{1\}$ and $V' = (\phi')^{-1}\{0\}$). Then, it is easy to prove that $\tilde{F}(A) \subseteq A'$ and $\tilde{F}(V) \subseteq V'$. Thus, the pair $(F, f)$ is a morphism between the affine bundles $\tau_A = (\tau_U)_{\mid A} : A \to M$ and $\tau_{A'} = (\tau_{U'})_{\mid A'} : A' \to M'$, where $F : A \to A'$ is the restriction of $\tilde{F}$ to $A$. The corresponding morphism between the vector bundles $\tau_V = (\tau_U)_{\mid V} : V \to M$ and $\tau_{V'} = (\tau_{U'})_{\mid V'} : V' \to M'$ is the pair $(F^i, f), F^i : V \to V'$ being the restriction of $\tilde{F}$ to $V$. 
Now, suppose that \((\tau_A : A \to M, \tau_V : V \to M, ([\cdot, \cdot]_V, D, \rho_A))\) and \((\tau_{A'} : A' \to M', \tau_{V'} : V' \to M', ([\cdot, \cdot]_{V'}, D', \rho_{A'})\) are two Lie affgebroids and that \(((F, f), (F', f'))\) is a morphism between the affine bundles \((\tau_A : A \to M, \tau_V : V \to M)\) and \((\tau_{A'} : A' \to M', \tau_{V'} : V' \to M')\). Then, the pair \(((F, f), (F', f'))\) is said to be a Lie affgebroid morphism if:

(i) The pair \((F^t, f)\) is a morphism between the Lie algebroids \((V, [\cdot, \cdot]_V, \rho_V)\) and \((V', [\cdot, \cdot]_{V'}, \rho_{V'})\),

(ii) \(Tf \circ \rho_A = \rho_{A'} \circ F\) and

(iii) If \(X\) (respectively, \(X'\)) is a section of \(\tau_A : A \to M\) (respectively, \(\tau_{A'} : A' \to M'\)) and \(\tilde{Y}\) (respectively, \(\tilde{Y}'\)) is a section of \(\tau_{V'} : V' \to M'\) such that \(X' \circ f = F \circ X\) and \(\tilde{Y}' \circ f = F' \circ \tilde{Y}\) then \(F' \circ D_X \tilde{Y} = (D_X \tilde{Y}) \circ f\).

Now, using \((\ref{14})\), one may deduce the following result.

**Proposition 2.1.** Suppose that \((\tau_A : A \to M, \tau_V : V \to M)\) and \((\tau_{A'} : A' \to M', \tau_{V'} : V' \to M')\) are Lie affgebroids. If \(((F, f), (F', f'))\) is a Lie affgebroid morphism and \(\tilde{A} : \tilde{A} \to \tilde{A}'\) is the corresponding morphism between the bidual vector bundles \(\tilde{A}\) and \(\tilde{A}'\), then the pair \((\tilde{F}, f)\) is a morphism between the Lie algebroids \(\tilde{A}\) and \(\tilde{A}'\).

Finally, using \((\ref{14})\), one may prove

**Proposition 2.2.** Suppose that \(\tau_U : U \to M\) and \(\tau_{U'} : U' \to M'\) are Lie algebroids and that \(\phi \in \Gamma(\tau_U^*)\) and \(\phi' \in \Gamma(\tau_{U'}^*)\) are 1-cocycles of \(\tau_U : U \to M\) and \(\tau_{U'} : U' \to M'\), respectively, such that \(\phi(x) \neq 0\), for all \(x \in M\), and \(\phi'(x') \neq 0\), for all \(x' \in M'\). Then, if the pair \((\tilde{F}, f)\) is a Lie algebroid morphism between the Lie algebroids \(\tau_U : U \to M\) and \(\tau_{U'} : U' \to M'\) satisfying \((\tilde{F}, f)^* \phi' = \phi\), we have that the corresponding morphism \(((F, f), (F', f'))\) between the Lie algebroids \(\tau_A = (\tau_U)|_A : A = \phi^{-1}(1) \to M\), \(\tau_V = (\tau_U)|_V : V = \phi^{-1}(0) \to M\) and \((\tau_{A'} = (\tau_{U'})|_{A'} : A' = (\phi')^{-1}(1) \to M'\), \(\tau_{V'} = (\tau_{U'})|_{V'} : V' = (\phi')^{-1}(0) \to M'\) is a Lie affgebroid morphism.

### 3. Hamiltonian and Lagrangian formalism on Lie affgebroids

#### 3.1. The Hamiltonian formalism.

In this section, we will develop a geometric framework, which allows to write the Hamilton equations associated with a Hamiltonian section on a Lie affgebroid (see \(\text{(14)}\)).

Suppose that \((\tau_A : A \to M, \tau_V : V \to M, ([\cdot, \cdot]_V, D, \rho_A))\) is a Lie affgebroid. Now, let \((L^\tau A, [\cdot, \cdot]_A, \rho_A)\) be the prolongation of the bidual Lie algebroid \((\tilde{A}, [\cdot, \cdot]_{\tilde{A}}, \rho_{\tilde{A}})\) over the fibration \(\tau_V^*: V* \to M\).

Let \((x^i)\) be local coordinates on an open subset \(U\) of \(M\) and \(\{e_0, e_\alpha\}\) be a local basis of sections of the vector bundle \(\tau_{A}^{-1}(U) \to U\) adapted to \(A\) and

\[
[e_0, e_\alpha]_{\tilde{A}} = C_{\alpha \gamma}^\tau e_\gamma, \quad [e_\alpha, e_\beta]_{\tilde{A}} = C_{\alpha \beta}^\tau e_\gamma, \quad \rho_{\tilde{A}}(e_0) = \rho_0^\tau \frac{\partial}{\partial x^i}, \quad \rho_{\tilde{A}}(e_\alpha) = \rho_\alpha^\tau \frac{\partial}{\partial x^i}.
\]

Denote by \((x^i, y^\alpha, y^\alpha)\) the corresponding local coordinates on \(\tilde{A}\) and by \((x^i, y_0, y_\alpha)\) the dual coordinates on the dual vector bundle \(\tau_{A^+} : A^+ \to \tilde{A}\) to \(\tilde{A}\). Then, \((x^i, y_\alpha)\) are local coordinates
on $V^*$ and $\{\tilde{e}_0, \tilde{e}_\alpha, \tilde{e}_\alpha\}$ is a local basis of sections of the vector bundle $\tau_{\tilde{A}}^V: \mathcal{L}^\ast \tilde{A} \to V^*$, where

$$\tilde{e}_0(\psi) = (e_0(\tau_V^V(\psi)), \rho_0^i \frac{\partial}{\partial x^i} |_V), \quad \tilde{e}_\alpha(\psi) = (e_\alpha(\tau_V^V(\psi)), \rho_\alpha^i \frac{\partial}{\partial x^i} |_V), \quad \tilde{e}_\alpha(\psi) = (0, \frac{\partial}{\partial y^\alpha} |_V).$$

Using this local basis one may introduce local coordinates $(x^i, y^\alpha, z^0, z^\alpha, v_\alpha)$ on $\mathcal{L}^\ast \tilde{A}$.

Let $\mu : A^+ \to V^*$ be the canonical projection given by $\mu(\phi) = \phi^l$, for $\phi \in A^+_X$, with $x \in M$, where $\phi^l \in V^*_x$ is the linear map associated with the affine map $\phi$ and $h : V^* \to A^+$ be a Hamiltonian section of $\mu$.

Now, we consider the Lie algebroid prolongation $\mathcal{L}^\gamma A^+$ of the Lie algebroid $\tilde{A}$ over $\tau_A^+: A^+ \to M$ with vector bundle projection $\tau_{\tilde{A}}^+ : \mathcal{L}^\gamma A^+ \to A^+$ (see Section 2.1). Then, we may introduce the map $\mathcal{L}h : \mathcal{L}^\gamma A \to \mathcal{L}^\gamma A^+$ defined by $\mathcal{L}h(\tilde{a}, X_\alpha) = (\tilde{a}, (T_a h)(X_\alpha))$, for $(\tilde{a}, X_\alpha) \in (\mathcal{L}^\gamma A)_\alpha$, with $\alpha \in V^*$. It is easy to prove that the pair $(\mathcal{L}h, h)$ is a Lie algebroid morphism between the Lie algebroids $\tau_{\tilde{A}}^+ : \mathcal{L}^\gamma A \to V^*$ and $\tau_{A}^+ : \mathcal{L}^\gamma A^+ \to A^+$.

Next, denote by $\lambda_h$ and $\Omega_h$ the sections of the vector bundles $(\mathcal{L}^\ast \mathcal{L}^\gamma A^+)^* \to V^*$ and $\Lambda^2(\mathcal{L}^\ast \mathcal{L}^\gamma A^+)^* \to V^*$ given by

$$\lambda_h = (\mathcal{L}h, h)^* (\lambda_{\tilde{A}}), \quad \Omega_h = (\mathcal{L}h, h)^* (\Omega_{\tilde{A}}),$$

where $\lambda_{\tilde{A}}$ and $\Omega_{\tilde{A}}$ are the Liouville section and the canonical symplectic section, respectively, associated with the Lie algebroid $\tilde{A}$. Note that $\Omega_h = -d\mathcal{L}^\gamma A^+_h \lambda_h$.

On the other hand, let $\eta : \mathcal{L}^\gamma A \to \mathbb{R}$ be the section of $(\mathcal{L}^\gamma A^+)^* \to V^*$ defined by

$$\eta(\tilde{a}, X_\alpha) = 1_A(\tilde{a}),$$

for $(\tilde{a}, X_\alpha) \in (\mathcal{L}^\gamma A)_\alpha$, with $\alpha \in V^*$. Note that if $pr_1 : \mathcal{L}^\gamma A \to \tilde{A}$ is the canonical projection on the first factor then $(pr_1, \tau_{\tilde{A}}^+)$ is a morphism between the Lie algebroids $\tau_{\tilde{A}}^+ : \mathcal{L}^\gamma A \to V^*$ and $\tau_A^+ : \mathcal{L}^\gamma A \to M$ and $(pr_1, \tau_{\tilde{A}}^+)^* (1_A) = \eta$. Thus, since $1_A$ is a 1-cocycle of $\tau_{\tilde{A}}^+ : \mathcal{L}^\gamma A \to M$, we deduce that $\eta$ is a 1-cocycle of the Lie algebroid $\tau_{\tilde{A}}^+ : \mathcal{L}^\gamma A \to V^*$.

Suppose that $h(x^i, y^\alpha) = (x^i, -H(x^i, y^\beta), y^\alpha)$ and that $\{\tilde{e}^0, \tilde{e}^\alpha, \tilde{e}_\alpha\}$ is the dual basis of $\{\tilde{e}_0, \tilde{e}_\alpha, \tilde{e}_\alpha\}$. Then $\eta = e^0$ and, from (2.1), and the definition of the map $\mathcal{L}h$, it follows that

$$\Omega_h = \tilde{e}^\gamma \wedge e^\gamma + \frac{1}{2} C_{\gamma\beta \gamma}^\alpha y_\alpha \tilde{e}^\gamma \wedge e^\beta + (\rho_\gamma^i \frac{\partial H}{\partial x^i} - C_0^\alpha_{\gamma \gamma} y_\alpha) \tilde{e}^\gamma \wedge \tilde{e}_\alpha + \frac{\partial H}{\partial y^\gamma} \tilde{e}^\gamma \wedge \tilde{e}_0.$$

Thus, it is easy to prove that the pair $(\Omega_h, \eta)$ is a cosymplectic structure on the Lie algebroid $\tau_{\tilde{A}}^+ : \mathcal{L}^\gamma A \to V^*$, that is,

$$\langle \eta \wedge \Omega_h \wedge \ldots \wedge \Omega_h \rangle (\alpha) \neq 0, \text{ for all } \alpha \in V^*, \quad d\mathcal{L}^\gamma A^+_h \eta = 0, \quad d\mathcal{L}^\gamma A^+_h \Omega_h = 0.$$

**Remark 3.1.** Let $\mathcal{L}^\gamma V$ be the prolongation of the Lie algebroid $V$ over the projection $\tau_V^V : V^* \to M$. Denote by $\lambda_V$ and $\Omega_V$ the Liouville section and the canonical symplectic section, respectively, of $V$ and by $(iv, Id) : \mathcal{L}^\gamma V \to \mathcal{L}^\gamma \tilde{A}$ the canonical inclusion. Then, using (2.1), (3.1), (3.2) and the fact that $\mu \circ h = Id$, we obtain that

$$\langle (iv, Id)^* (\lambda_h) \rangle = \lambda_V, \quad \langle (iv, Id)^* (\eta) \rangle = 0.$$
Thus, since \((i_V, Id)\) is a Lie algebroid morphism, we also deduce that

\[
(i_V, Id)^*(\Omega_h) = \Omega_V. \tag{3.5}
\]

Now, let \(R_h \in \Gamma(\tau^{\tau_V}_{\tilde{\alpha}})\) be the Reeb section of the cosymplectic structure \((\Omega_h, \eta)\) characterized by the following conditions

\[
i_{R_h} \Omega_h = 0 \quad \text{and} \quad i_{R_h} \eta = 1. \tag{3.6}
\]

With respect to the basis \(\{\tilde{e}_0, \tilde{e}_\alpha, \tilde{e}_\alpha\}\) of \(\Gamma(\tau^{\tau_V}_{\tilde{\alpha}})\), \(R_h\) is locally expressed as follows:

\[
R_h = \tilde{e}_0 + \frac{\partial H}{\partial y_\alpha} \tilde{e}_\alpha - (C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i} - C^\gamma_{0\alpha} y_\gamma) \tilde{e}_\alpha. \tag{3.7}
\]

Thus, the vector field \(\rho^{\tau_V}_{\tilde{\alpha}}(R_h)\) is locally given by

\[
\rho^{\tau_V}_{\tilde{\alpha}}(R_h) = (\rho^0_\alpha + \frac{\partial H}{\partial y_\alpha} \rho^i_\alpha) \frac{\partial}{\partial x^i} + (- \rho^i_\alpha \frac{\partial H}{\partial x^i} + y_\gamma (C^\gamma_{0\alpha} + C^\gamma_{\beta\alpha} \frac{\partial H}{\partial y_\beta})) \frac{\partial}{\partial y_\alpha}, \tag{3.8}
\]

and the integral curves of \(R_h\) (i.e., the integral curves of \(\rho^{\tau_V}_{\tilde{\alpha}}(R_h)\)) are just the solutions of the Hamilton equations for \(h\),

\[
\frac{dx^i}{dt} = \rho^i_0 + \frac{\partial H}{\partial y_\alpha} \rho^i_\alpha, \quad \frac{dy_\alpha}{dt} = -\rho^i_\alpha \frac{\partial H}{\partial x^i} + y_\gamma (C^\gamma_{0\alpha} + C^\gamma_{\beta\alpha} \frac{\partial H}{\partial y_\beta}), \tag{3.9}
\]

for \(i \in \{1, \ldots, m\}\) and \(\alpha \in \{1, \ldots, n\}\).

Next, we will present an alternative approach in order to obtain the Hamilton equations. For this purpose, we will use the notion of an aff-Poisson structure on an AV-bundle which was introduced in [4] (see also [5]).

Let \(\tau_Z : Z \to M\) be an affine bundle of rank 1 modelled on the trivial vector bundle \(\tau_{M \times \mathbb{R}} : M \times \mathbb{R} \to M\), that is, \(\tau_Z : Z \to M\) is an AV-bundle in the terminology of [5].

Then, we have an action of \(\mathbb{R}\) on the fibers of \(Z\). This action induces a vector field \(X_Z\) on \(Z\) which is vertical with respect to the projection \(\tau_Z : Z \to M\).

On the other hand, there exists a one-to-one correspondence between the space of sections of \(\tau_Z : Z \to M\), \(\Gamma(\tau_Z)\), and the set

\[
\{F_h \in C^\infty(Z)/X_Z(F_h) = -1\}. \tag{3.10}
\]

In fact, if \(h \in \Gamma(\tau_Z)\) and \((x^i, s)\) are local fibred coordinates on \(Z\) such that \(X_Z = \frac{\partial}{\partial s}\) then \(h\) may be considered as a local function \(H\) on \(M\), \(x^i \to H(x^i)\), and the function \(F_h\) on \(Z\) is locally given by

\[
F_h(x^i, s) = -H(x^i) - s, \tag{3.10}
\]

(for more details, see [5]).

Now, an aff-Poisson structure on the affine bundle \(\tau_Z : Z \to M\) is a bi-affine map

\[
\{,\} : \Gamma(\tau_Z) \times \Gamma(\tau_Z) \to C^\infty(M)
\]

which satisfies the following properties:

i) Skew-symmetric: \(\{h_1, h_2\} = -\{h_2, h_1\}\).
ii) Jacobi identity:
\[ \{h_1, \{h_2, h_3\}\}_2 + \{h_2, \{h_3, h_1\}\}_2 + \{h_3, \{h_1, h_2\}\}_2 = 0, \]
where \(\{\cdot, \cdot\}_2\) is the affine-linear part of the bi-affine bracket.

iii) If \(h \in \Gamma(\tau_Z)\) then
\[ \{h, \cdot\} : \Gamma(\tau_Z) \to C^\infty(M), \quad h' \mapsto \{h, h'\}, \]
is an affine derivation.

Condition iii) implies that the linear part \(\{h, \cdot\}_V : C^\infty(M) \to C^\infty(M)\) of the affine map
\(\{h, \cdot\} : \Gamma(\tau_Z) \to C^\infty(M)\) defines a vector field on \(M\), which is called the Hamiltonian vector field of \(h\) (see [5]).

In [5], the authors proved that there is a one-to-one correspondence between aff-Poisson brackets \(\{\cdot, \cdot\}_V\) on \(\tau_Z : Z \to M\) and Poisson brackets \(\{\cdot, \cdot\}_\Pi\) on \(Z\) which are \(X_Z\)-invariant, i.e., which are associated with Poisson 2-vectors \(\Pi\) on \(Z\) such that \(\mathcal{L}_{X_Z} \Pi = 0\). This correspondence is determined by
\[ \{h, h'\} \circ \tau_Z = \{F_h, F_{h'}\}_\Pi, \quad \text{for } h, h' \in \Gamma(\tau_Z). \]

Note that the function \(\{F_h, F_{h'}\}_\Pi\) on \(Z\) is \(\tau_Z\)-projectable, i.e., \(\mathcal{L}_{X_Z} \{F_h, F_{h'}\}_\Pi = 0\) (because the Poisson 2-vector \(\Pi\) is \(X_Z\)-invariant).

Using this correspondence we will prove the following result.

**Theorem 3.2.** Let \(\tau_A : A \to M\) be a Lie algebroid modelled on the vector bundle \(\tau_V : V \to M\). Denote by \(\tau_{A^+} : A^+ \to M\) (resp., \(\tau_V^* : V^* \to M\)) the dual vector bundle to \(A\) (resp., to \(V\)) and by \(\mu : A^+ \to V^*\) the canonical projection. Then:

i) \(\mu : A^+ \to V^*\) is an AV-bundle which admits an aff-Poisson structure.

ii) If \(h : V^* \to A^+\) is a Hamiltonian section (that is, \(h \in \Gamma(\mu)\)) then the Hamiltonian vector field of \(h\) with respect to the aff-Poisson structure is a vector field on \(V^*\) whose integral curves are just the solutions of the Hamilton equations for \(h\).

**Proof.** i) It is clear that \(\mu : A^+ \to V^*\) is an AV-bundle. In fact, if \(a^+ \in A^+_x\), with \(x \in M\), and \(t \in \mathbb{R}\) then
\[ a^+ + t = a^+ + t1_A(x). \]

Thus, the \(\mu\)-vertical vector field \(X_{A^+}\) on \(A^+\) is just the vertical lift \(1^V_A\) of the section \(1_A \in \Gamma(\tau_A^+)\). Moreover, one may consider the Lie algebroid \(\tau_A^* : A^* \to M\) and the corresponding linear Poisson 2-vector \(\Pi_{A^+}\) on \(A^+\). Then, using the fact that \(1_A^*\) is a 1-cocycle of \(\tau_A^* : A^* \to M\), it follows that the Poisson 2-vector \(\Pi_{A^+}\) is \(X_{A^+}\)-invariant. Therefore, \(\Pi_{A^+}\) induces an aff-Poisson structure \(\{\cdot, \cdot\}_\mu\) on \(\mu : A^+ \to V^*\) which is characterized by the condition
\[ (3.11) \quad \{h, h'\} \circ \mu = \{F_h, F_{h'}\}_{\Pi_{A^+}}, \quad \text{for } h, h' \in \Gamma(\mu). \]

One may also prove this first part of the theorem using the relation between special Lie algebroid structures on an affine bundle \(A\) and aff-Poisson structures on the AV-bundle \(AV(\{A'\})\) (see Theorem 23 in [5]).

ii) From (3.10) and (3.11), we deduce that the linear map \(\{h, \cdot\}_V : C^\infty(V^*) \to C^\infty(V^*)\) associated with the affine map \(\{h, \cdot\} : \Gamma(\mu) \to C^\infty(V^*)\) (that is, the Hamiltonian vector field of
(3.12) \( \{h, \cdot \}_V(\varphi) \circ \mu = \{F_h, \varphi \circ \mu \}_{\Pi_A^+}, \) for \( \varphi \in C^\infty(V^*). \)

Now, suppose that the local expression of \( h \) is
\begin{equation}
(3.13) \quad h(x^i, y^\alpha) = (x^i, -H(x^j, y^\alpha)).
\end{equation}

On the other hand, using (2.4), we have that
\begin{equation}
(3.14) \quad \Pi_{A^+} = \frac{1}{2} C^{\alpha\beta\gamma}_{\alpha\beta\gamma} \frac{\partial}{\partial y^\alpha} \wedge \frac{\partial}{\partial y^\beta} + C^{\alpha\beta\gamma}_{\alpha\beta\gamma} \frac{\partial}{\partial y^\gamma} \wedge \frac{\partial}{\partial y^\alpha} + \rho^i_0 \frac{\partial}{\partial y^0} \wedge \frac{\partial}{\partial x^i} + \rho^i_\alpha \frac{\partial}{\partial y^\alpha} \wedge \frac{\partial}{\partial x^i}.
\end{equation}

Thus, from (3.22), (3.23) and (3.24), we conclude that the Hamiltonian vector field of \( h \) is locally given by
\begin{equation}
(\rho^i_0 + \frac{\partial H}{\partial y^0} \rho^i_0) \frac{\partial}{\partial x^i} + (-\rho^i_\alpha \frac{\partial H}{\partial y^\alpha} + y^\gamma (C^0_{\alpha\beta} + C^0_{\beta\gamma} \frac{\partial H}{\partial y^\beta})) \frac{\partial}{\partial y^\gamma}
\end{equation}
which proves our result (see (3.8)).

3.2. The Lagrangian formalism. In this section, we will develop a geometric framework, which allows to write the Euler-Lagrange equations associated with a Lagrangian function \( L \) on a Lie algebroid \( A \) in an intrinsic way (see [40]).

Suppose that \((\tau_A : A \to M, \tau_V : V \to M, ([\cdot, \cdot]_V, D, \rho_A))\) is a Lie algebroid on \( M \). Then, the bidual bundle \( \tau_{A^+} : A \to M \) to \( A \) admits a Lie algebroid structure \(([\cdot, \cdot]_{\tau_A}, \rho_{\tau_A})\) such that the section \( 1_A \) of the dual bundle \( A^+ \) is a 1-cocycle.

Now, we consider the Lie algebroid prolongation \((\mathcal{L}^\tau_{\tau_A} \tilde{A}, [\cdot, \cdot]_{\tau_A}^\tau, \rho_{\tau_A}^\tau)\) of the Lie algebroid \((\tilde{A}, [\cdot, \cdot]_{\tilde{A}}, \rho_{\tilde{A}})\) over the fibration \( \tau_A : A \to M \) with vector bundle projection \( \tau_A^\tau : \mathcal{L}^\tau_{\tau_A} \tilde{A} \to A \).

If \((x^i)\) are local coordinates on an open subset \( U \) of \( M \) and \( \{e_0, e_\alpha\} \) is a local basis of sections of the vector bundle \( \tau_{\tilde{A}}^{-1}(U) \to U \) adapted to \( 1_A \), then \( \{\tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_\alpha\} \) is a local basis of sections of the vector bundle \( (\tau_{\tau_A}^\tau)^{-1}(\tau_{\tilde{A}}^{-1}(U)) \to \tau_{\tilde{A}}^{-1}(U) \), where
\begin{equation}
(3.15) \quad \tilde{T}_0(a) = (e_0(\tau_A(a)), \rho^i_0 \frac{\partial}{\partial x^i}|_a), \quad \tilde{T}_\alpha(a) = (e_\alpha(\tau_A(a)), \rho^i_\alpha \frac{\partial}{\partial x^i}|_a), \quad \tilde{V}_\alpha(a) = (0, \frac{\partial}{\partial y^\alpha}|_a),
\end{equation}
\((x^i, y^\alpha)\) are the local coordinates on \( A \) induced by the local coordinates \((x^i)\) and the basis \( \{e_\alpha\} \) and \( \rho^i_0, \rho^i_\alpha \) are the components of the anchor map \( \rho_{\tilde{A}} \). Therefore, we have that
\begin{equation}
(3.16) \quad \tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_\alpha = C^\gamma_{\alpha\beta} \tilde{T}_\beta, \quad \tilde{T}_0, \tilde{T}_\alpha = C^\gamma_{\alpha\beta} \tilde{T}_\beta, \quad \tilde{T}_0, \tilde{V}_\alpha = 0, \quad \tilde{T}_\alpha, \tilde{V}_\beta = 0,
\end{equation}
\(\rho^\tau_{\tau_A} \tilde{T}_0 = \rho^i_0 \frac{\partial}{\partial x^i}, \quad \rho^\tau_{\tau_A} \tilde{T}_\alpha = \rho^i_\alpha \frac{\partial}{\partial x^i}, \quad \rho^\tau_{\tau_A} \tilde{V}_\alpha = \frac{\partial}{\partial y^\alpha},\)
where \(C^\gamma_{\alpha\beta}\) and \(C^\gamma_{\alpha\beta}\) are the structure functions of the Lie algebroid \([\cdot, \cdot]_{\tilde{A}}\) with respect to the basis \(\{e_0, e_\alpha\}\). Note that, if \(\tilde{T}_0, \tilde{T}_\alpha, \tilde{V}^\alpha\) is the dual basis of \(\{\tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_\alpha\}\), then \(\tilde{T}_0\) is globally defined and it is a 1-cocycle. We will denote by \(\phi_0\) the 1-cocycle \(T_0\). Thus, we have that
\begin{equation}
(3.17) \quad \phi_0(a)(\tilde{b}, X_a) = 1_A(\tilde{b}), \quad \text{for} \ (\tilde{b}, X_a) \in (\mathcal{L}^\tau_{\tau_A} \tilde{A})_a.
\end{equation}
One may also consider the vertical endomorphism \( S: \mathcal{L}^\gamma \tilde{A} \to \mathcal{L}^\gamma \tilde{A} \), as a section of the vector bundle \( \mathcal{L}^\gamma \tilde{A} \otimes (\mathcal{L}^\gamma \tilde{A})^* \to A \), whose local expression is (see [10])

\[
S = (\tilde{T}^\alpha - y^\alpha \tilde{T}^0) \otimes \tilde{V}_\alpha.
\]

Now, a curve \( \gamma: I \subseteq \mathbb{R} \to A \) in \( A \) is said to be admissible if \( \rho^\gamma_0 \circ i_A \circ \gamma = (\tau^\gamma_A \circ \gamma) \) or, equivalently, \((i_A(\gamma(t)), \dot{\gamma}(t)) \in (\mathcal{L}^\gamma \tilde{A})_{\gamma(t)} \) for all \( t \in I \), \( i_A: A \to \tilde{A} \) being the canonical inclusion. Thus, if \( \gamma(t) = (x^i(t), y^a(t)) \), for all \( t \in I \), then \( \gamma \) is an admissible curve if and only if

\[
\frac{dx^i}{dt} = \rho_0^i + \rho_a^i y^a, \quad \text{for} \quad i \in \{1, \ldots, m\}.
\]

A section \( \xi \) of \( \tau^\gamma_A: \mathcal{L}^\gamma \tilde{A} \to A \) is said to be a second order differential equation (SODE) on \( A \) if the integral curves of \( \xi \), that is, the integral curves of the vector field \( \rho^\gamma_0(\xi) \), are admissible. If \( \xi \in \Gamma(\tau^\gamma_A) \) is a SODE then \( \xi = \tilde{T}_0 + y^\alpha \tilde{T}_\alpha + \xi^a \tilde{V}_a \), where \( \xi^a \) are arbitrary local functions on \( A \), and

\[
\rho^\gamma_0(\xi) = (\rho_0^i + y^\alpha \rho_a^i) \frac{\partial}{\partial x^i} + \xi^a \frac{\partial}{\partial y^a}.
\]

On the other hand, let \( L: A \to \mathbb{R} \) be a Lagrangian function. Then, we introduce the Poincaré-Cartan 1-section \( \Theta_L \in \Gamma((\tau^\gamma_A)^*) \) and the Poincaré-Cartan 2-section \( \Omega_L \in \Gamma(\wedge^2(\tau^\gamma_A)^*) \) associated with \( L \) defined by

\[
\Theta_L = L\phi_0 + (d\xi^\gamma \tilde{A}L) \circ S, \quad \Omega_L = -d\xi^\gamma \tilde{A}\Theta_L.
\]

From [10, 11] and (5.19) we obtain that

\[
\Theta_L = (L - y^\alpha \frac{\partial L}{\partial y^\alpha}) \tilde{T}^0 + \frac{\partial L}{\partial y^\alpha} \tilde{T}^\alpha,
\]

\[
\Omega_L = (i_{\xi_\gamma}(d\xi^\gamma \tilde{A}L) - \frac{\partial L}{\partial \gamma^\alpha}(C^\gamma_{\alpha \beta} + C^{\gamma \beta}_{\alpha \gamma} - \rho_a^i \frac{\partial L}{\partial x^i}) \theta^\alpha \wedge \tilde{T}^0
\]

\[
+ \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \theta^\alpha \wedge \psi^\beta + \frac{1}{2}(\rho_0^i \frac{\partial^2 L}{\partial x^i \partial y^\alpha} - \rho_a^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} + \frac{\partial L}{\partial y^\gamma} C^{\gamma \beta}_{\alpha \gamma} \theta^\alpha \wedge \psi^\beta),
\]

where \( \theta^\alpha = \tilde{T}^\alpha - y^\alpha \tilde{T}^0 \), \( \psi^\alpha = \tilde{V}_\alpha - \xi_0^a \tilde{T}^0 \) and \( \xi_0 = \tilde{T}_0 + y^\alpha \tilde{T}_\alpha + \xi_0^a \tilde{V}_a \) is an arbitrary SODE.

Now, a curve \( \gamma: I = (-\epsilon, \epsilon) \subseteq \mathbb{R} \to A \) in \( A \) is a solution of the Euler-Lagrange equations associated with \( L \) if and only if \( \gamma \) is admissible and \( i_{\xi_{\gamma(\gamma(t))}}\Omega_L(\gamma(t)) = 0 \), for all \( t \).

If \( \gamma(t) = (x^i(t), y^a(t)) \) then \( \gamma \) is a solution of the Euler-Lagrange equations if and only if

\[
\frac{dx^i}{dt} = \rho_0^i + \rho_a^i y^a, \quad \frac{d}{dt}(\frac{\partial L}{\partial y^\alpha}) = \rho_a^i \frac{\partial L}{\partial x^i} + (C^\gamma_{\alpha \beta} + C^{\gamma \beta}_{\alpha \gamma}) \frac{\partial L}{\partial y^\gamma},
\]

for \( i \in \{1, \ldots, m\} \) and \( \alpha \in \{1, \ldots, n\} \).

The Lagrangian \( L \) is regular if and only if the matrix \((W_{\alpha \beta}) = (\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta})\) is regular or, in other words, the pair \((\Omega_L, \phi_0)\) is a cosymplectic structure on \( \mathcal{L}^\gamma \tilde{A} \).

If the Lagrangian \( L \) is regular, then the Reeb section \( R_L \) of \((\Omega_L, \phi_0)\) is the unique Lagrangian SODE associated with \( L \), that is, the integral curves of the vector field \( \rho^\gamma_0(R_L) \) are solutions of
the Euler-Lagrange equations associated with $L$. In such a case, $R_L$ is called the Euler-Lagrange section associated with $L$ and its local expression is

$$R_L = \dot{T}_0 + y^a \dot{T}_a + \omega^{\alpha\beta}(\rho^\beta_0 \frac{\partial L}{\partial x^\alpha} - (\rho^\alpha_0 + y^\alpha \rho^\alpha_1)) \frac{\partial^2 L}{\partial x^\alpha \partial y^\beta} + (C^\mu_{0\beta} + y^\mu C^\mu_{1\beta}) \frac{\partial L}{\partial y^\beta} \dot{V}_\alpha.$$  

where $(\omega^{\alpha\beta})$ is the inverse matrix of $(\omega^{\alpha\beta})$.

3.3. The Legendre transformation and the equivalence between the Lagrangian and Hamiltonian formalisms. Let $L : A \to \mathbb{R}$ be a Lagrangian function and $\Theta_L \in \Gamma((\tau^A)_A)$ be the Poincaré-Cartan 1-section associated with $L$. We introduce the extended Legendre transformation associated with $L$ as the smooth map $Leg_L : A \to \mathbb{R}^+$ defined by $Leg_L(a)(b) = \Theta_L(a)(z)$, for $a, b \in A$, where $z$ is a point in the fibre of $\mathcal{L}^A \mathbb{A}$ over the point $a$ such that $pr_1(z) = i_A(b)$, $pr_1 : \mathcal{L}^A \mathbb{A} \to \mathbb{A}$ being the restriction to $\mathcal{L}^A \mathbb{A}$ of the first canonical projection $pr_1 : A \times TA \to A$. The map $Leg_L$ is well-defined and its local expression in fibre coordinates on $A$ and $A^+$ is

$$Leg_L(x^i, y^\alpha) = (x^i, L - \frac{\partial L}{\partial y^\alpha} y^\alpha, \frac{\partial L}{\partial y^\alpha}).$$

Thus, we can define the Legendre transformation associated with $L$, $leg_L : A \to V^*$, by $leg_L = \mu \circ Leg_L$. From (3.23) and since $\mu(x^i, y^\alpha, v^\alpha) = (x^i, y^\alpha)$, we have that

$$leg_L(x^i, y^\alpha) = (x^i, \frac{\partial L}{\partial y^\alpha}).$$

The maps $Leg_L$ and $leg_L$ induce the maps $\mathcal{L}Leg_L : \mathcal{L}^A \mathbb{A} \to \mathcal{L}^A \mathbb{A}$ and $\mathcal{L}eg_L : \mathcal{L}^A \mathbb{A} \to \mathcal{L}^A \mathbb{A}$ defined by

$$\mathcal{L}Leg_L(\tilde{b}, X_a) = (\tilde{b}, (T_0 Leg_L)(X_a)), \quad (\mathcal{L}eg_L)(\tilde{b}, X_a) = (\tilde{b}, (T_0 leg_L)(X_a)),$$

for $a \in A$ and $(\tilde{b}, X_a) \in (\mathcal{L}^A \mathbb{A})_a$.

Now, let $\{\tilde{T}_0, \tilde{T}_a, \tilde{V}_a\}$ (respectively, $\{\tilde{e}_0, \tilde{e}_a, \tilde{e}_0, \tilde{e}_a\}$) be a local basis of $\Gamma((\tau^A)_A)$ as in Section 3.2 (respectively, of $\Gamma((\tau^A)_A)$ as in Section 2.1.1) and denote by $(x^i, y^\alpha; z^0, z^\alpha, v^\alpha)$ (respectively, $(x^i, y^\alpha; z^0, z^\alpha, v^\alpha)$) the corresponding local coordinates on $\mathcal{L}^A \mathbb{A}$ (respectively, $\mathcal{L}^A \mathbb{A}$). In addition, suppose that $(x^i, y^\alpha; z^0, z^\alpha, v^\alpha)$ are local coordinates on $\mathcal{L}^{V\mathbb{A}} \mathbb{A}$ as in Section 3.1.

Then, using (3.22), (3.23) and (3.24), we deduce that the local expression of the maps $\mathcal{L}Leg_L$ and $\mathcal{L}eg_L$ is

$$\mathcal{L}Leg_L(x^i, y^\alpha; z^0, z^\alpha, v^\alpha) = (x^i, L - \frac{\partial L}{\partial y^\alpha} y^\alpha, \frac{\partial L}{\partial y^\alpha}; z^0, z^\alpha, z^\alpha, z^0 \rho^0_0 \frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial x^i \partial y^\alpha} y^\alpha)$$

$$+ z^\alpha \rho^i_0 \frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial x^i \partial y^\alpha} y^\beta - \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} y^\beta, z^\alpha \rho^i_0 \frac{\partial^2 L}{\partial x^i \partial y^\alpha} + \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} y^\beta, z^\alpha \rho^i_0 \frac{\partial^2 L}{\partial x^i \partial y^\alpha} + \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} y^\beta),$$

$$\mathcal{L}eg_L(x^i, y^\alpha; z^0, z^\alpha, v^\alpha) = (x^i, \frac{\partial L}{\partial y^\alpha}; z^0, z^\alpha, z^0 \rho^0_0 \frac{\partial L}{\partial x^i} + z^\alpha \rho^i_0 \frac{\partial^2 L}{\partial x^i \partial y^\alpha} + \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} y^\beta).$$

Thus, using (3.26), (3.27) and (3.28), we can prove the following result.
Theorem 3.3. The pair \((\mathcal{L}Leg_L, Leg_L)\) is a morphism between the Lie algebroids \((\mathcal{L}_L, P, \rho_L)\) and \((\mathcal{L}^*L, \bar{\rho}_L)\). Moreover, if \(\Theta_L\) and \(\Omega_L\) (respectively, \(\Theta_A\) and \(\Omega_A\)) are the Poincaré-Cartan 1-section and 2-section associated with \(L\) (respectively, the Liouville 1-section and the canonical symplectic section associated with \(A\)) then
\[
(\mathcal{L}Leg_L, Leg_L)^*(\lambda_A) = \Theta_L, \quad (\mathcal{L}Leg_L, Leg_L)^*(\Omega_A) = \Omega_L.
\]

From (3.23), it follows

Proposition 3.4. The Lagrangian \(L\) is regular if and only if the Legendre transformation \(\text{leg}_L : A \to V^*\) is a local diffeomorphism.

Next, we will assume that \(L\) is hyperregular, that is, \(\text{leg}_L\) is a global diffeomorphism. Then, from (3.25) and Theorem 3.3, we conclude that the pair \((\mathcal{L}Leg_L, leg_L)\) is a Lie algebroid isomorphism. Moreover, if \(\gamma : I \to A\) is a solution of the Hamilton equations for \(h_L\), then \(\gamma = \text{leg}_L^{-1} \circ \tilde{\gamma}\) is a solution of the Euler-Lagrange equations for \(L\).

Theorem 3.5. If the Lagrangian \(L\) is hyperregular then the Euler-Lagrange section \(R_L\) associated with \(L\) and the Hamiltonian section \(h_L\) associated with \(L\) satisfy the following relation
\[
(R_L \circ \text{leg}_L) = \mathcal{L}Leg_L \circ R_L.
\]

Moreover, if \(\gamma : I \to A\) is a solution of the Euler-Lagrange equations associated with \(L\), then \(\text{leg}_L \circ \gamma : I \to V^*\) is a solution of the Hamilton equations associated with \(h_L\) and, conversely, if \(\tilde{\gamma} : I \to V^*\) is a solution of the Hamilton equations for \(h_L\) then \(\gamma = \text{leg}_L^{-1} \circ \tilde{\gamma}\) is a solution of the Euler-Lagrange equations for \(L\).

Now, we will analyze the local expression of the transformation \(\text{leg}_L^{-1} : V^* \to A\). Suppose that
\[
\text{leg}_L^{-1}(x^i, y_\alpha) = (x^i, y^\alpha(x^j, y_\beta))
\]
and \(h_L(x^i, y_\alpha) = (x^i, -H_L(x^j, y_\beta), y_\alpha)\). Then, from (3.26) and (3.27), it follows that
\[
H_L(x^i, y_\alpha) = y^\alpha(x^i, y_\beta)y_\alpha - L(x^i, y^\alpha(x^j, y_\beta)).
\]

Thus, we obtain that
\[
\frac{\partial H_L}{\partial y_\alpha} = y^\alpha \frac{\partial y^\beta}{\partial y_\alpha} y_\beta - \frac{\partial L}{\partial y^\beta} \frac{\partial y^\beta}{\partial y_\alpha}
\]
and, using (3.26) and (3.27), we deduce that \(\frac{\partial H_L}{\partial y_\alpha} = y^\alpha\). Therefore, we conclude that
\[
\text{leg}_L^{-1}(x^i, y_\alpha) = (x^i, \frac{\partial H_L}{\partial y_\alpha}).
\]

Note that, since \(\text{leg}_L^{-1} : V^* \to A\) is a diffeomorphism, it follows that the matrix \(\left(\frac{\partial^2 H_L}{\partial y_\alpha \partial y_\beta}\right)\) is regular.
Next, we will introduce the notion of a hyperregular Hamiltonian section and we will prove that given a hyperregular Hamiltonian section \( h : V^* \to A^+ \) then one may construct a hyperregular Lagrangian function \( L : A \to \mathbb{R} \) and \( h_L = h \).

Let \( h : V^* \to A^+ \) be a Hamiltonian section. If \( R \) is an arbitrary section of \( A \), we may consider the real \( C^\infty \)-function \( H_R : V^* \to \mathbb{R} \) on \( V^* \) given by

\[
H_R(\alpha) = h(\alpha)(R(\tau_{V^*}(\alpha))), \quad \text{for} \quad \alpha \in V^*.
\]

Using the function \( H_R \) we may define the map \((\mathcal{F}h)_R : V^* \to V\) as follows

\[
\alpha \in V^*_x, \quad \text{with} \quad x \in M \Rightarrow (\mathcal{F}h)_R(\alpha) \in V_x
\]

and

\[
\beta((\mathcal{F}h)_R(\alpha)) = \frac{d}{dt}|_{t=0} H_R(\alpha + t\beta), \quad \text{for} \quad \beta \in V^*_x.
\]

Now, we introduce the map \( \mathcal{F}h : V^* \to A \) given by

\[
(\mathcal{F}h)(\alpha) = R(\tau_{V^*}(\alpha)) + (\mathcal{F}h)_R(\alpha).
\]

If the local expressions of \( h \) and \( R \) are

\[
h(x^i, y_{\alpha}) = (x^i, -H(x^j, y_{\beta}), y_{\alpha}), \quad \mathcal{R}(x^i) = (x^i, R^\alpha(x^i)),
\]

then, from (3.33), (3.34) and (3.35), we obtain that

\[
H_R(x^i, y_{\alpha}) = -H(x^j, y_{\beta}) + \mathcal{R}^\alpha(x^i)y_{\alpha},
\]

\[
(\mathcal{F}h)_R(x^i, y_{\alpha}) = (x^i, -\frac{\partial H}{\partial y_{\alpha}}(x^j, y_{\beta}) + \mathcal{R}^\alpha(x^i)),
\]

\[
(\mathcal{F}h)(x^i, y_{\alpha}) = (x^i, \frac{\partial H}{\partial y_{\alpha}}).
\]

Thus, the map \( \mathcal{F}h \) doesn’t depend on the chosen section \( R \).

The Hamiltonian section \( h \) is said to be \textit{regular} if the map \( \mathcal{F}h : V^* \to A \) is a local diffeomorphism or equivalently if the matrix \( \left( \frac{\partial^2 H}{\partial y_{\alpha} \partial y_{\beta}} \right) \) is regular. \( h \) is said to be \textit{hyperregular} if \( \mathcal{F}h : V^* \to A \) is a global diffeomorphism.

It is clear that if \( L : A \to \mathbb{R} \) is a hyperregular Lagrangian function and \( h_L : V^* \to A^+ \) is the Hamiltonian section associated with \( L \) then \( h_L \) is hyperregular. In fact, from (3.32) and (3.35), it follows that \( \mathcal{F}h_L \) is a diffeomorphism and \( \mathcal{F}h_L = leg_L^{-1} \).

Now, we will prove that the converse is also true.

\textbf{Theorem 3.6.} If \( h : V^* \to A^+ \) is a hyperregular Hamiltonian section then there exists a hyperregular Lagrangian function \( L : A \to \mathbb{R} \) such that the Hamiltonian section associated with \( L \) is just \( h \). In other words, \( h_L = h \).
Proof. We define the Lagrangian function \( L : A \to \mathbb{R} \) by
\[
L(a) = (\mathcal{F}h)^{-1}(a)(a - \mathcal{R}(\tau_A(a))) + H_{\mathcal{R}}((\mathcal{F}h)^{-1}(a)), \quad \text{for } a \in A.
\]
The function \( L \) doesn’t depend on the chosen section \( \mathcal{R} \). In fact, if \((\mathcal{F}h)^{-1}(x^i, y^\alpha) = (x^i, y_\alpha(x^j, y^\beta))\) then, using (3.36) and (3.38), we have that
\[
\tau(a) = \frac{\partial L}{\partial y^\alpha} = y_\alpha(x^j, y^\beta) - H(x^i, y_\alpha(x^j, y^\beta)).
\]
Therefore, we deduce that
\[
\frac{\partial L}{\partial y^\alpha} = y_\alpha.
\]
and using that \( \frac{\partial H}{\partial y^\alpha} = y^\beta \) (see (3.38)), it follows that
\[
\frac{\partial L}{\partial y^\alpha} = y^\beta.
\]
This implies that (see (3.24)) \( \text{Leg}_L = (\mathcal{F}h)^{-1} \) and consequently, from (4.25), (4.28), (4.29) and since \( h_L = \text{Leg}_L \circ \text{Leg}_{L}^{-1} \), we conclude that \( h_L = h \).

4. The canonical involution associated with a Lie affgebroid

Let \((\tau_A : A \to M, \tau_V : V \to M)\) be a Lie affgebroid. Denote by \( \rho_V : V \to TM \) the anchor map of the Lie algebroid \( \tau_V : V \to M \), by \(([,\cdot]_A, \rho_A)\) the Lie algebroid structure on the bidual bundle \( \tau_A^{-1} : \tilde{A} \to M \) to \( A \) and by \( 1_A : \tilde{A} \to \mathbb{R} \) the distinguished 1-cocycle on \( \tilde{A} \).

We consider the subset \( \mathcal{J}^A A \) of the product manifold \( A \times TA \) defined by
\[
\mathcal{J}^A A = \{ (a, v) \in A \times TA / \rho_A(a) = (T\tau_A)(v) \}.
\]

Next, we will see that \( \mathcal{J}^A A \) admits two Lie affgebroid structures. We will also see that these Lie affgebroid structures are isomorphic under the so-called canonical involution associated with \( A \).

i) The first structure:

Let \((\mathcal{L}^A_\tau \tilde{A}, [\cdot,\cdot]_A^\tau, \rho_A^\tau)\) be the prolongation of the Lie algebroid \((\tilde{A}, [\cdot,\cdot]_A, \rho_A)\) over the fibration \( \tau_A : A \to M \) and \( \phi_0 : \mathcal{L}^A_\tau \tilde{A} \to \mathbb{R} \) be the 1-cocycle of the Lie algebroid cohomology complex of \((\mathcal{L}^A_\tau \tilde{A}, [\cdot,\cdot]_A^\tau, \rho_A^\tau)\) given by (3.17). Using (3.17) and the fact that \( (1_A|_{A_x}) \neq 0 \), for all \( x \in M \), we deduce that \( (\phi_0)|_{(\mathcal{L}^A_\tau \tilde{A})_a} \neq 0 \), for all \( a \in A \).

Moreover, we have that
\[
(\phi_0)^{-1}\{1\} = \{ (\tilde{a}, v) \in \tilde{A} \times TA / \rho_A^\tau(\tilde{a}) = (T\tau_A)(v), 1_A(\tilde{a}) = 1 \} = \mathcal{J}^A A.
\]

In addition, if \( \mathcal{L}^V_\tau V \) is the prolongation of the Lie algebroid \((V, [\cdot,\cdot]_V, \rho_V)\) over the fibration \( \tau_A : A \to M \) then, it is easy to prove that
\[
(\phi_0)^{-1}\{0\} = \mathcal{L}^V_\tau V.
\]

We will denote by \(([\cdot,\cdot]_V^\tau, \rho_V^\tau)\) the Lie algebroid structure on \( \tau^A_\tau : \mathcal{L}^V_\tau V \to A \).

From (13.1), we conclude that \( \mathcal{J}^A A \) is an affine bundle over \( \tilde{A} \) with affine bundle projection \( \tau^A_\tau : \mathcal{J}^A A \to A \) defined by
\[
\tau^A_\tau(a, v) = \pi_A(v),
\]
and, moreover, the affine bundle $\tau_{\tilde{A}}^A : J^A A \to A$ admits a Lie affgebroid structure in such a way that the bidual Lie algebroid to $\tau_{\tilde{A}}^A : J^A A \to A$ is just $(L^{\tau_{\tilde{A}}}A, \{\cdot, \cdot\}_A^A, \rho_{\tilde{A}}^A)$. Finally, using (1.2), it follows that the Lie affgebroid $\tau_{\tilde{A}}^A : J^A A \to A$ is modelled on the Lie algebroid $(L^{\tau_{\tilde{A}}}V, \{\cdot, \cdot\}_V^A, \rho_V^A)$.

**Remark 4.1.** Let $L^{\tau_{\tilde{A}}} \tilde{A}$ be the prolongation of the Lie algebroid $(\tilde{A}, \{\cdot, \cdot\}_A^A, \rho_{\tilde{A}})$ over the fibration $\tau_{\tilde{A}} : \tilde{A} \to M$. Denote by $(Id,Ti_A) : L^{\tau_{\tilde{A}}} \tilde{A} \to L^{\tau_{\tilde{A}}} \tilde{A}$ the inclusion defined by $$ (Id,Ti_A)(\tilde{a},v) = (\tilde{a}, Ti_A(v_b)), $$
for $(\tilde{a},v_b) \in (L^{\tau_{\tilde{A}}} \tilde{A})_b$, with $b \in A$. Then, it is easy to prove that the pair $((Id,Ti_A), i_A)$ is a Lie algebroid morphism. Thus, $(L^{\tau_{\tilde{A}}} \tilde{A}, \{\cdot, \cdot\}_A^A, \rho_{\tilde{A}}^A)$ is a Lie subalgebroid of $(L^{\tau_{\tilde{A}}} \tilde{A}, \{\cdot, \cdot\}_A^A, \rho_{\tilde{A}})$. \hfill $\diamond$

**ii) The second structure:**
As we know, the tangent bundle to $\tilde{A}$, $T\tilde{A}$, is a Lie algebroid over $TM$ with vector bundle projection $T\tau_{\tilde{A}} : T\tilde{A} \to TM$. Now, we consider the subset $d_0(1_A)^0$ of $T\tilde{A}$ given by
\begin{equation}
(4.3) \quad d_0(1_A)^0 = \{ \tilde{v} \in T\tilde{A}/d_0(1_A)(\tilde{v}) = 0 \} = \{ \tilde{v} \in T\tilde{A}/\tilde{v}(1_A) = 0 \}.
\end{equation}
$d_0(1_A)^0$ is the total space of a vector subbundle of $T\tau_{\tilde{A}} : T\tilde{A} \to TM$. More precisely, suppose that $\tilde{X} \in \Gamma(T\tilde{A})$ and denote by $T\tilde{X} : TM \to T\tilde{A}$ the tangent map to $\tilde{X}$ and by $\tilde{X} : TM \to T\tilde{A}$ the section of $T\tau_{\tilde{A}} : T\tilde{A} \to TM$ defined by (2.7). Then, using (4.3), we deduce the following facts:

i) If $1_A(\tilde{X}) = c$, with $c \in \mathbb{R}$, we have that $T\tilde{X}(TM) \subseteq d_0(1_A)^0$ and, thus, $T\tilde{X} : TM \to d_0(1_A)^0$ is a section of the vector bundle $(T\tau_{\tilde{A}})|_{d_0(1_A)^0} : d_0(1_A)^0 \to TM$.

ii) If $1_A(\tilde{X}) = 0$ it follows that $\tilde{X}(TM) \subseteq d_0(1_A)^0$ and, therefore, $\tilde{X} : TM \to d_0(1_A)^0$ is a section of the vector bundle $(T\tau_{\tilde{A}})|_{d_0(1_A)^0} : d_0(1_A)^0 \to TM$.

In fact, if $\{e_0, e_\alpha\}$ is a local basis of $\Gamma(T\tilde{A})$ adapted to $1_A$, then $\{Te_0, Te_\alpha, \tilde{e}_\alpha\}$ is a local basis of $\Gamma((T\tau_{\tilde{A}})|_{d_0(1_A)^0})$. Consequently, the canonical inclusion $i : d_0(1_A)^0 \to T\tilde{A}$ is a monomorphism (over the identity of $TM$) between the vector bundles $(T\tau_{\tilde{A}})|_{d_0(1_A)^0} : d_0(1_A)^0 \to TM$ and $T\tau_{\tilde{A}} : T\tilde{A} \to TM$. Moreover, using (2.7) and the fact that $1_A : \tilde{A} \to \mathbb{R}$ is a 1-cocycle of the Lie algebroid $(\tilde{A}, \{\cdot, \cdot\}_A^A, \rho_{\tilde{A}})$, we deduce that the Lie bracket $\{\cdot, \cdot\}_A^T$ on $\Gamma(T\tau_{\tilde{A}})$ restricts to a Lie bracket on the space $\Gamma((T\tau_{\tilde{A}})|_{d_0(1_A)^0})$. Therefore, we have proved the following result.

**Proposition 4.2.** The vector bundle $(T\tau_{\tilde{A}})|_{d_0(1_A)^0} : d_0(1_A)^0 \to TM$ is a Lie algebroid and the canonical inclusion $i : d_0(1_A)^0 \to T\tilde{A}$ is a monomorphism between the Lie algebroids $(T\tau_{\tilde{A}})|_{d_0(1_A)^0} : d_0(1_A)^0 \to TM$ and $T\tau_{\tilde{A}} : T\tilde{A} \to TM$. Thus, $(T\tau_{\tilde{A}})|_{d_0(1_A)^0} : d_0(1_A)^0 \to TM$ is a Lie subalgebroid of $T\tau_{\tilde{A}} : T\tilde{A} \to TM$.

Next, we consider the pull-back of the vector bundle $(T\tau_{\tilde{A}})|_{d_0(1_A)^0} : d_0(1_A)^0 \to TM$ over the anchor map $\rho_A : A \to TM$, that is,$$
\rho_A^\ast(d_0(1_A)^0) = \{(a, \tilde{v}) \in A \times d_0(1_A)^0/\rho_A(a) = (T\tau_{\tilde{A}})|_{d_0(1_A)^0}(\tilde{v})\}.
$$
$\rho_A^\ast(d_0(1_A)^0)$ is a vector bundle over $A$ with vector bundle projection$$
pr_1 : \rho_A^\ast(d_0(1_A)^0) \to A, \ (a, \tilde{v}) \mapsto a.
$$
On the other hand, we will denote by \((i_A, i) : \rho^*_A(d_0(1_A)^0) \to \rho^*_A(T\tilde{A})\) the monomorphism (over the canonical inclusion \(i_A : A \to \tilde{A}\)) between the vector bundles \(\rho^*_A(d_0(1_A)^0) \to A\) and \(\rho^*_A(T\tilde{A}) \to \tilde{A}\) defined by

\[
(i_A, i)(a, \tilde{v}) = (i_A(a), \tilde{v}), \quad \text{for} \quad (a, \tilde{v}) \in \rho_A^*(d_0(1_A)^0).
\]

We recall that the vector bundle \(\rho^*_A(T\tilde{A}) \to \tilde{A}\) is an action Lie algebroid (see Section 2.1.2). Furthermore, we have

**Proposition 4.3.** i) The vector bundle \(\rho^*_A(d_0(1_A)^0) \to A\) is a Lie algebroid over \(A\) and the pair \(((i_A, i), i_A)\) is a monomorphism between the Lie algebroids \(\rho^*_A(d_0(1_A)^0) \to A\) and \(\rho^*_A(T\tilde{A}) \to \tilde{A}\).

ii) If \(\pi_{\tilde{A}} : T\tilde{A} \to \tilde{A}\) is the canonical projection and \(\varphi_0 : \rho_A^*(d_0(1_A)^0) \to \mathbb{R}\) is the linear map given by

\[
\varphi_0(a, \tilde{v}) = 1_A(\pi_{\tilde{A}}(\tilde{v})),
\]

then \(\varphi_0\) is a 1-cocycle of the Lie algebroid \(\rho^*_A(d_0(1_A)^0) \to A\) and \(\varphi_0|_{\rho^*_A(d_0(1_A)^0)} \neq 0\), for all \(a \in A\).

**Proof.** i) Let \(\tilde{X}\) be a section of \(\tau_{\tilde{A}} : \tilde{A} \to M\) and \(\tilde{X}^c\) (respectively, \(\tilde{X}^v\)) be the complete lift (respectively, the vertical lift) of \(\tilde{X}\). If \(1_A(\tilde{X}) = c\), with \(c \in \mathbb{R}\), it follows that \(\tilde{X}^c|_{1_A(\tilde{A})} = 0\) and, thus, the restriction of \(\tilde{X}^c\) to \(A\) is tangent to \(A\). In addition, if \(1_A(\tilde{X}) = 0\) we obtain that \(\tilde{X}^v|_{1_A(\tilde{A})} = 0\) and, therefore, the restriction of \(\tilde{X}^v\) to \(A\) is tangent to \(A\).

Now, proceeding as in the proof of Theorem 4.4 in [10], we deduce that there exists a unique action \(\Psi_0\) of the Lie algebroid \((T\tau_{\tilde{A}})|_{d_0(1_A)^0} : d_0(1_A)^0 \to TM\) over the anchor map \(\rho_A : A \to TM\) such that \(\Psi_0(T\tilde{X}) = \tilde{X}^c|_{1_A(\tilde{A})}\), for \(\tilde{X} \in \Gamma(\tau_{\tilde{A}})\) with \(1_A(\tilde{X}) = c\) and \(c \in \mathbb{R}\), and \(\Psi_0(\tilde{X}) = \tilde{X}^v|_{1_A(\tilde{A})}\), for \(\tilde{X} \in \Gamma(\tau_{\tilde{A}})\) such that \(1_A(\tilde{X}) = 0\).

ii) Let \(\{e_0, e_\alpha\}\) be a local basis of \(\Gamma(\tau_{\tilde{A}})\) adapted to \(1_A\). Then,

\[
\{T^{\rho_A}e_0 = Te_0 \circ \rho_A, T^{\rho_A}e_\alpha = Te_\alpha \circ \rho_A, e_0^\rho = e_\alpha \circ \rho_A\}
\]

is a local basis of sections of the vector bundle \(\rho_A^*(d_0(1_A)^0) \to A\). Moreover, if \(((\cdot, \cdot), \Psi_0), (\rho_A^*|_{\Psi_0})\) is the Lie algebroid structure on \(\rho_A^*(d_0(1_A)^0) \to A\), we have that

\[
(T^{\rho_A}e_0, T^{\rho_A}e_\alpha)|_{\Psi_0} = T[e_0, e_\alpha]_A \circ \rho_A,
\]

\[
(T^{\rho_A}e_\alpha, T^{\rho_A}e_\beta)|_{\Psi_0} = T[e_\alpha, e_\beta]_A \circ \rho_A,
\]

\[
(T^{\rho_A}e_0, e_0^\rho)|_{\Psi_0} = ((T^{\rho_A}e_\alpha, e_0^\rho)|_{\Psi_0} = (e_\alpha^\rho)|_{\Psi_0} = 0,
\]

and

\[
(\rho_A|_{\Psi_0})^T e_0 = (e_0)|_{\Psi_0}, \quad (\rho_A|_{\Psi_0})^T e_\alpha = (e_\alpha)|_{\Psi_0}, \quad (\rho_A|_{\Psi_0})^T (e_\alpha^\rho) = (e_\alpha)|_{\Psi_0}.
\]
On the other hand, if $\tilde{X} \in \Gamma(\tau_{\tilde{A}})$ then, from \eqref{4.3} and \eqref{4.4}, we obtain that
\begin{equation}
1_A(\tilde{X}) = c \in \mathbb{R} \Rightarrow \varphi_0(T\tilde{X} \circ \rho_A) = c,
\end{equation}
\begin{equation}
1_A(\tilde{X}) = 0 \Rightarrow \varphi_0(T\tilde{X} \circ \rho_A) = 0.
\end{equation}
Thus, using \eqref{4.5}, \eqref{4.6} and the fact that $1_A$ is a 1-cocycle of $(\tilde{A}, \cdot, \cdot_{\tilde{A}}, \rho_{\tilde{A}})$, we conclude that
$\varphi_0$ is a 1-cocycle of the Lie algebroid $\rho_A^*d_0(1_A^0) \rightarrow A$.

Now, from \eqref{4.4}, it follows that
\begin{equation}
\varphi_0^{-1}\{1\} = \{(a, v) \in A \times TA/\rho_A(a) = (T\tau_A)(v)\} = \mathcal{J}^A A.
\end{equation}
Therefore, we deduce that $\mathcal{J}^A A$ is an affine bundle over $A$ with affine bundle projection $pr_1 : \mathcal{J}^A A \rightarrow A$ defined by $pr_1(a, v) = a$ and, moreover, the affine bundle $pr_1 : \mathcal{J}^A A \rightarrow A$ admits a Lie algebroid structure in such a way that the bidual Lie algebroid to $pr_1 : \mathcal{J}^A A \rightarrow A$ is just $(\rho_A^*d_0(1_A^0), \{\cdot, \cdot\}_\varphi, (\rho_{\tilde{A}})^T\varphi_{\tilde{A}})$.

On the other hand, using \eqref{4.3}, we obtain that
\begin{equation}
\varphi_0^{-1}\{0\} = \{(a, v) \in A \times TV/\rho_A(a) = (T\tau_V)(u)\} = \rho_A^*(TV).
\end{equation}
Consequently, the affine bundle $pr_1 : \mathcal{J}^A A \rightarrow A$ is modelled on the vector bundle $pr_1 : \rho_A^*(TV) \rightarrow A$. Furthermore, using \eqref{4.3}, we deduce that the corresponding Lie algebroid structure is induced by an action $\Psi_V$ of the Lie algebroid $(TV, \{\cdot, \cdot\}_V, \rho_V^T)$ over the anchor map $\rho_A : A \rightarrow TM$. For this action, we have that
\begin{equation}
\Psi_V(T\tilde{X}) = (iv \circ \tilde{X})^v|_A, \quad \Psi_V(\tilde{X}) = (iv \circ \tilde{X})^v|_A, \quad \text{for } \tilde{X} \in \Gamma(\tau_V).
\end{equation}

The canonical involution:
Let $L^\tau \tilde{A}$ be the prolongation of the Lie algebroid $(\tilde{A}, \cdot, \cdot_{\tilde{A}}, \rho_{\tilde{A}})$ over the fibration $\tau_{\tilde{A}} : \tilde{A} \rightarrow M$ and $\rho^*_{\tilde{A}}(T\tilde{A}) \equiv L^\tau \tilde{A}$ be the pull-back of the Lie algebroid $\tau_{\tilde{A}} : T\tilde{A} \rightarrow TM$ over the anchor map $\rho_{\tilde{A}} : \tilde{A} \rightarrow TM$. If $(\tilde{a}, \tilde{v}) \in (L^\tau \tilde{A})_b$, with $\tilde{b} \in \tilde{A}_x$ and $x \in M$, then there exists a unique tangent vector $\tilde{u}_a \in T_{\tilde{a}}\tilde{A}$ such that:
\begin{equation}
\tilde{u}_a(f \circ \tau_{\tilde{A}}) = (d^\tau f)(\tilde{b}), \quad \tilde{u}_a(\theta) = \tilde{v}_\theta(\tilde{b}) + (d^\tau \theta)(\tilde{b}, \tilde{u}_a),
\end{equation}
for $f \in C^\infty(M)$ and $\theta : \tilde{A} \rightarrow \mathbb{R} \in \Gamma(\tau_{\tilde{A}}^+)$. Thus, one may define the map $\sigma_{\tilde{A}} : L^\tau \tilde{A} \rightarrow \rho^*_{\tilde{A}}(T\tilde{A})$ as follows
\begin{equation}
\sigma_{\tilde{A}}(\tilde{a}, \tilde{v}) = (\tilde{b}, \tilde{u}_a), \quad \text{for } (\tilde{a}, \tilde{v}) \in (L^\tau \tilde{A})_b.
\end{equation}
$\sigma_{\tilde{A}}$ is an isomorphism (over the identity $Id : \tilde{A} \rightarrow \tilde{A}$) between the Lie algebroids $(L^\tau \tilde{A}, \cdot, \cdot_{\tilde{A}}, \rho^*_{\tilde{A}})$ and $(\rho^*_{\tilde{A}}(T\tilde{A}), \{\cdot, \cdot\}_{\rho^*_{\tilde{A}}}, (\rho_{\tilde{A}})^T\varphi_{\tilde{A}})$ and, moreover, $\sigma_{\tilde{A}}^2 = Id$. $\sigma_{\tilde{A}}$ is called the canonical involution associated with the Lie algebroid $(\tilde{A}, \cdot, \cdot_{\tilde{A}}, \rho_{\tilde{A}})$ (for more details, see \cite{10}).

\textbf{Theorem 4.4.} The restriction of $\sigma_{\tilde{A}}$ to $\mathcal{J}^A A$ induces an isomorphism $\sigma_A : \mathcal{J}^A A \rightarrow \mathcal{J}^A A$ between the Lie algebroids $\sigma^\tau_{\tilde{A}} : \mathcal{J}^A A \rightarrow A$ and $pr_1 : \mathcal{J}^A A \rightarrow A$ and, moreover, $\sigma_A^2 = Id$. The corresponding Lie algebroid isomorphism $\sigma^\tau_A : L^\tau A \rightarrow \rho_A^*(TV)$ between the Lie algebroids $\tau^\tau_A : L^\tau A \rightarrow A$ and $pr_1 : \rho_{\tilde{A}}^*(TV) \rightarrow A$ is the restriction of $\sigma_{\tilde{A}}$ to $L^\tau A$, that is, $\sigma_A^\tau = (\sigma_{\tilde{A}})|_{L^\tau A}$. 

Finally, the local equations defining the vector subbundles \( \sigma_z \) and \( \sigma_{\rho} \) coordinates is (see [10]) \( \rho^* (d_0 (1_A)^0) \), then, using again (4.8) and (4.9), we deduce that \( \sigma_A (b, \tilde{u}_a) \in (L^T \tilde{A})_b \). Thus, since \( \sigma_A \) is an involution, we conclude that the restriction of \( \sigma_A \) to the prolongation \( L^T \tilde{A} \) induces an isomorphism \( \tilde{\sigma}_A : L^T \tilde{A} \to \rho^*_A (d_0 (1_A)^0) \) (over the identity \( Id : A \to A \)) between the vector bundles \( \tau^T_A : L^T \tilde{A} \to A \) and \( pr_1 : \rho^*_A (d_0 (1_A)^0) \to A \).

On the other hand, as we know, \( L^T \tilde{A} \) (respectively, \( \rho^*_A (d_0 (1_A)^0) \)) is a Lie subalgebroid of \( L^T \tilde{A} \) (respectively, \( \rho^*_A (T \tilde{A}) \)). Therefore, using that \( \sigma_A : L^T \tilde{A} \to \rho^*_A (T \tilde{A}) \) is a Lie algebra isomorphism, we obtain that \( \tilde{\sigma}_A : L^T \tilde{A} \to \rho^*_A (d_0 (1_A)^0) \) is also a Lie algebra isomorphism.

Now, denote by \( \phi_0 \) (respectively, \( \varphi_0 \)) the 1-cocycle of the Lie algebroid \( \tau^T_A : L^T \tilde{A} \to A \) (respectively, \( pr_1 : \rho^*_A (d_0 (1_A)^0) \to A \)) given by (3.17) (respectively, 4.4). From (3.17), (4.4), (4.8) and (4.9), it follows that \( \varphi_0 \circ \tilde{\sigma}_A = \phi_0 \). Consequently, using (3.17), (4.4), Proposition 2.2 and the fact that the bidual Lie algebroid to the Lie algebroid \( \tau^T_A : \tilde{A} \to A \) is \( \tilde{A} \), we prove the result.

\[ \Box \]

**Definition 4.5.** The map \( \sigma_A : \tilde{A} \to \tilde{A} \) is called the canonical involution associated with the Lie algebroid \( A \).

Suppose that \( (x^i) \) are local coordinates on an open subset \( U \) of \( M \) and that \( \{ e_0, e_\alpha \} \) is a local basis of sections of \( \tau^T_A : \tilde{A} \to M \) in \( U \). Denote by \( (x^i, y^\gamma, \alpha^\beta) \) the corresponding local coordinates on \( \tilde{A} \) and by \( \rho^0_\alpha \) and \( \rho^i_\alpha \) the components of the anchor map \( \rho_A \). Then, \( \{ \tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_0, \tilde{V}_\alpha \} \) is a local basis of sections of \( \tau^T_A : L^T \tilde{A} \to \tilde{A} \), where

\[
\begin{align*}
\tilde{T}_0 (\tilde{a}) &= (e_0 (\tau^T_A (\tilde{a})), \rho^0_\alpha \frac{\partial}{\partial x^i | \tilde{a}}), \\
\tilde{T}_\alpha (\tilde{a}) &= (e_\alpha (\tau^T_A (\tilde{a})), \rho^i_\alpha \frac{\partial}{\partial x^i | \tilde{a}}), \\
\tilde{V}_0 (\tilde{a}) &= (0, \frac{\partial}{\partial y^\gamma | \tilde{a}}), \\
\tilde{V}_\alpha (\tilde{a}) &= (0, \frac{\partial}{\partial y^\alpha | \tilde{a}}).
\end{align*}
\]

This local basis induces a system of local coordinates \( (x^i, y^\gamma, \alpha^\beta, z^0, \alpha^\gamma, v^0, \alpha^\alpha) \) on \( L^T \tilde{A} \equiv \rho^*_A (T \tilde{A}) \). The local expression of the canonical involution \( \sigma_A : L^T \tilde{A} \to \rho^*_A (T \tilde{A}) \) in these coordinates is (see (3.11))

\[
\sigma_A (x^i, y^\gamma, \alpha^\beta, z^0, \alpha^\gamma, v^0, \alpha^\alpha) = (x^i, z^0, \alpha^\gamma, y^\gamma, y^\gamma, v^0, v^\alpha + C_{\gamma \beta} (z^0 y^\gamma - z^\gamma y^0) + C_{\beta \gamma} z^\beta y^\gamma).
\]

Here, \( C_{\alpha \gamma} \) and \( C_{\alpha \beta} \) are the structure functions of the Lie bracket \([,]_A \) with respect to the basis \( \{ e_0, e_\alpha \} \).

On the other hand, the local equations defining the affine subbundle \( \tilde{J}^A A \) of \( L^T \tilde{A} \) are \( y^0 = 1, \) \( z^0 = 1, \) \( v^0 = 0 \). Thus, \( (x^i, y^\gamma, \alpha^\beta, v^\alpha) \) may be considered as local coordinates on \( \tilde{J}^A A \). Using these coordinates, we deduce that the local expression of \( \sigma_A : \tilde{J}^A A \to \tilde{J}^A A \) is

\[
\sigma_A (x^i, y^\gamma, \alpha^\beta, z^0, v^\alpha) = (x^i, z^0, \alpha^\gamma, y^\gamma, v^\alpha + C_{\gamma \beta} (y^\gamma - z^\gamma) + C_{\beta \gamma} z^\beta y^\gamma).
\]

Finally, the local equations defining the vector subbundles \( L^T V \) and \( \rho^*_A (TV) \) of \( L^T \tilde{A} \) and \( \rho^*_A (T \tilde{A}) \), respectively, are \( y^0 = 1, \) \( z^0 = 0, \) \( v^0 = 0 \), and \( z^0 = 1, \) \( y^0 = 0, \) \( v^0 = 0 \). Therefore, \( (x^i, y^\gamma, \alpha^\beta, v^\alpha) \) may be considered as local coordinates on \( L^T V \) and \( \rho^*_A (TV) \). Using these coordinates, we obtain that the local expression of \( \sigma_A^I : L^T V \to \rho^*_A (TV) \) is

\[
\sigma_A^I (x^i, y^\gamma, \alpha^\beta, z^0, v^\alpha) = (x^i, z^0, y^\gamma, v^\alpha - C_{\gamma \beta} z^\gamma + C_{\beta \gamma} z^\beta y^\gamma).
\]
5. TULCZYJEW’S TRIPLE ASSOCIATED WITH A LIE AFFGEBROID AND A HAMILTONIAN SECTION

Let $\tau_A : A \to M$ be a Lie affgebroid modelled on the Lie algebroid $\tau_V : V \to M$. Denote by $([\cdot, \cdot]_V, D, \rho_A)$ the Lie affgebroid structure on $A$, by $\rho_A^*(TV)$ (respectively, $\rho_A^*(TV^*)$) the pull-back of the vector bundle $T\tau_V : TV \to TM$ (respectively, $T\tau_V^* : TV^* \to TM$) over the anchor map $\rho_A : A \to TM$ and by $\mathcal{L}^\tau V$ (respectively, $\mathcal{L}^\tau A$ and $\mathcal{L}^\tau V$) the prolongation of the Lie algebroid $\tau_A^* : \bar{A} \to M$ (respectively, $\tau_V : V \to M$) over the projection $\tau_V^* : V^* \to M$ (respectively, $\tau_A : A \to M$ and $\tau_A^* : V^* \to M$).

Suppose that $(x^i)$ are local coordinates on an open subset $U$ of $M$ and that $\{e_0, e_\alpha\}$ is a local basis of $\Gamma(\tau_A)$ adapted to $1_A$. Denote by $(x^i, y^\alpha; z^\alpha, v^\alpha)$ (respectively, $(x^i, y^\alpha)$) the corresponding local coordinates on $\bar{A}$ (respectively, $V$ and $A$). Then, we may consider local coordinates $(x^i, y^\alpha; z^\alpha, v^\alpha)$ of $\rho_A^*(TV)$ and $\mathcal{L}^\tau A$ as in Section 4 and the corresponding dual coordinates $(x^i, y^\alpha; z_\alpha, v_\alpha)$ may be considered as a system of local coordinates on $\rho_A^*(TV^*)$ and $\mathcal{L}^\tau V$.

Next, we will introduce the so-called Tulczyjew’s triple associated with the Lie affgebroid $A$ and a Hamiltonian section.

First step: In this first step, we will introduce a canonical isomorphism $A_A : \rho_A^*(TV^*) \to (\mathcal{L}^\tau V)^*$, over the identity of $A$, between the vector bundles $\rho_A^*(TV^*) \to A$ and $(\mathcal{L}^\tau V)^* \to A$.

Let $(\cdot, \cdot) : V \times_M V^* \to \mathbb{R}$ be the natural pairing given by

\[ \langle u, \alpha \rangle = \alpha(u), \quad \text{for} \quad (u, \alpha) \in V_x \times V_x^*, \]

with $x \in M$. If $b \in A$, $(b, X_u) \in \rho_A^*(TV)_b$ and $(b, X_\alpha) \in \rho_A^*(TV^*)_b$ then

\[ (X_u, X_\alpha) \in T_{(u, \alpha)}(V \times_M V^*) = \{ (X'_u, X'_\alpha) \in T_u V \times T_\alpha V^*/(T_u \tau_V)(X'_u) = (T_\alpha \tau_V^*)(X'_\alpha) \}, \]

and we may consider the map $\widehat{T}(\cdot, \cdot) : \rho_A^*(TV) \times A \rho_A^*(TV^*) \to \mathbb{R}$ defined by

\[ \widehat{T}(\cdot, \cdot)((b, X_u), (b, X_\alpha)) = dt_{(u, \alpha)}(\langle T_{(u, \alpha)}(\cdot, \cdot)(X_u, X_\alpha) \rangle) \]

where $t$ is the usual coordinate on $\mathbb{R}$. The local expression of the map $\widehat{T}(\cdot, \cdot)$ is

\[ \widehat{T}(\cdot, \cdot)((x^i, y^\alpha; z^\alpha, v^\alpha), (x^i, y^\alpha; z_\alpha, v_\alpha)) = y^\alpha v_\alpha + v^\alpha y_\alpha. \]

Thus, $\widehat{T}(\cdot, \cdot)$ is also a non-singular pairing and it induces an isomorphism (over the identity of $A$) between the vector bundles $\rho_A^*(TV) \to A$ and $\rho_A^*(TV^*) \to A$ which we also denote by $\widehat{T}(\cdot, \cdot)$, that is, $\widehat{T}(\cdot, \cdot) : \rho_A^*(TV) \to \rho_A^*(TV^*)$. Note that

\[ (5.1) \quad \widehat{T}(\cdot, \cdot)(x^i, y^\alpha; z^\alpha, v^\alpha) = (x^i, y^\alpha; z^\alpha, y^\alpha). \]

Next, we consider the isomorphism of vector bundles $A_A^* : \mathcal{L}^\tau V \to \rho_A^*(TV^*)$ given by

\[ (5.2) \quad A_A^* = \widehat{T}(\cdot, \cdot) \circ \sigma^I_A, \]
From (4.10), (5.1) and (5.2), it follows that

\[ A(x^i, y^\alpha; z^\alpha, v^\alpha) = (x^i, v^\gamma - C^\gamma_{\alpha\beta}z^\alpha v^\beta; y^\gamma, z^\gamma), \]

and therefore

\[ A(x^i, y^\alpha; z^\alpha, v^\alpha) = (x^i, y^\gamma - C^\beta_{\alpha\gamma}z^\alpha y^\beta; y^\gamma, z^\gamma). \]

**Second step**: Let \( \tau_A : A^+ \to M \) be the dual vector bundle to the affine bundle \( \tau_A : A \to M \), \( \mu : A^+ \to V^* \) be the canonical projection and \( h : V^* \to A^+ \) be a Hamiltonian section, that is, \( h \) is a section of \( \mu : A^+ \to V^* \).

Denote by \( (\Omega_A, \eta) \) the cosymplectic structure on \( \mathcal{L}^* \mathcal{V} \mathcal{A} \) given by (3.1) and (3.2). Then, a direct computation, using (3.2), proves that

\[ \eta^{-1}(1) = \rho_A^*(V^*) \quad \text{and} \quad \eta^{-1}(0) = \mathcal{L}^* \mathcal{V} \mathcal{A}. \]

Thus, \( \rho_A^*(TV^*) \) is an affine bundle over \( V^* \) with affine bundle projection \( \tilde{\pi}_V : \rho_A^*(TV^*) \to V^* \), defined by \( \tilde{\pi}_V(a, X) = \pi_{TV^*}(X) \) and, furthermore, this affine bundle admits a Lie affgebroid structure in such a way that the bidual Lie algebroid is \( \tau_A^*: \mathcal{L}^* \mathcal{V} \mathcal{A} \to V^* \).

In this step, we will introduce an affine isomorphism \( \tilde{\pi}_V : \rho_A^*(TV^*) \to (\mathcal{L}^* \mathcal{V} \mathcal{A})^* \), over the identity of \( V^* \), between the affine bundle \( \tilde{\pi}_V : \rho_A^*(TV^*) \to V^* \) and the vector bundle \( (\tau_A^*)^*: (\mathcal{L}^* \mathcal{V} \mathcal{A})^* \to V^* \).

The map \( \tilde{\pi}_V \) is defined as follows. If \( \alpha \in V^* \) and \( (a, X_\alpha) \in \rho_A^*(TV^*) \), then

\[ \{\tilde{\pi}_V(\alpha)(a, X_\alpha)(u, Y_\alpha) = \Omega_A(\alpha)(i_A(a), X_\alpha, i_V(u), Y_\alpha), \]

for \( (u, Y_\alpha) \in (\mathcal{L}^* \mathcal{V} \mathcal{A})^* \).

On the other hand, let \( \tilde{\pi}_V : (\mathcal{L}^* \mathcal{V} \mathcal{A})^* \to V^* \) be the canonical isomorphism over the identity of \( V^* \) induce the canonical symplectic section \( \Omega_V \) associated with the Lie algebroid \( \tau_V : V \to M \), that is,

\[ \{\tilde{\pi}_V(\alpha)(u, Y_V)(v, Z_\alpha) = \Omega_V(\alpha)((u, Y_V), (v, Z_\alpha)), \]

for \( (u, Y_V), (v, Z_\alpha) \in (\mathcal{L}^* \mathcal{V} \mathcal{A})^* \), \( \alpha \in V^* \). Then, using (3.1), (3.2) and (5.5), it follows that \( \tilde{\pi}_V \) is an affine isomorphism over the identity of \( V^* \) and the corresponding linear isomorphism between the vector bundles \( \tau_V^*: (\mathcal{L}^* \mathcal{V} \mathcal{A})^* \to V^* \) and \( (\tau_V^*)^*: (\mathcal{L}^* \mathcal{V} \mathcal{A})^* \to V^* \) is just the map \( \tilde{\pi}_V \).

In conclusion, we have the following commutative diagram:

\[ (\mathcal{L}^* \mathcal{V} \mathcal{A})^* \]

\[ A \]

\[ \rho_A^*(TV^*) \]

\[ \tilde{\pi}_V \]

\[ (\tau_V^*)^* \]

\[ \Omega_A \]

\[ (\mathcal{L}^* \mathcal{V} \mathcal{A})^* \]

\[ V^* \]

\[ (\tau_V^*)^* \]
Moreover, using the properties of the complete and vertical lifts (see [10, 12]), we deduce that the vector fields (6.4) 
\[ (6.2) \quad (fg)^c = f^c g^v + f^v g^c, \quad (fg)^v = f^v g^v, \quad \text{for } f, g \in C^\infty(M). \]

A direct computation proves that
\[ (6.1) \quad f^c(a) = \rho_A(a)(f), \quad f^v(a) = f(\tau_A(a)), \quad \text{for } a \in A. \]

Now, if \( \tilde{X} \) is a section of \( \tau_A : \tilde{A} \to M, \) we may consider the vertical and complete lift \( \tilde{X}^v \) and \( \tilde{X}^c \) as the vector fields on \( \tilde{A} \) defined in Section 2.1.2. Using these constructions, we may introduce the sections \( \tilde{X}^v \) and \( \tilde{X}^c \) of the prolongation \( \tau_A^\infty : \mathcal{L}^\infty \tilde{A} \to \tilde{A} \) given by
\[ \tilde{X}^v(\tilde{a}) = (0(\tau_A(\tilde{a})), \tilde{X}^v(\tilde{a})), \quad \tilde{X}^c(\tilde{a}) = (\tilde{X}(\tau_A(\tilde{a})), \tilde{X}^c(\tilde{a})), \]
for \( \tilde{a} \in \tilde{A}. \) If \( \{e_0^c, e_\alpha^c\} \) is a local basis of \( \Gamma(\tau_A^\infty) \) adapted to \( 1_A : \tilde{A} \to \mathbb{R}, \) then \( \{e_0^c, e_{\alpha}^c, e_{\alpha}^v, e_{\alpha}^v\} \) is a local basis of \( \Gamma(\tau_A^\infty) \) (see [10, 12]).

Next, denote by \( i_A : A \to \tilde{A}, \) \( i_V : V \to \tilde{A} \) the canonical inclusions and by \( \tau^\infty_A : \mathcal{L}^\infty A \to A \) the prolongation of the Lie algebroid \( \tau_V : V \to M \) over the projection \( \tau_A : \tilde{A} \to M. \)

If \( X \) is a section of \( \tau_V : V \to M \) then \( i_V \circ X \in \Gamma(\tau_A) \) and we have that the restriction to \( A \) of the vector fields \( (i_V \circ X)^v \) and \( (i_V \circ X)^c \) are tangent to \( A. \) Thus,
\[ X^v = (i_V \circ X)^v|_A \in \Gamma(\tau^\infty_V), \quad X^c = (i_V \circ X)^c|_A \in \Gamma(\tau^\infty_V). \]

Moreover, using the properties of the complete and vertical lifts (see [10, 12]), we deduce that
\[ (f X)^v = f^v X^v + f^c X^c, \quad (f X)^c = f^c X^v, \]
\[ [X^c, Y^c]^\tau_A = [X, Y]^\epsilon, \quad [X^c, Y^v]^\tau_A = [X, Y]^\epsilon, \quad [X^v, Y^v]^\tau_A = 0, \]
\[ \rho^\tau_A(X^v) = (i_V \circ X)^v|_A, \quad \rho^\tau_A(X^c) = (i_V \circ X)^c|_A, \]
for \( f \in C^\infty(M) \) and \( X, Y \in \Gamma(\tau_V), \) where \( ([\cdot, \cdot]_V, \rho_V) \) and \( ([\cdot, \cdot]^\tau_A, \rho^\tau_A) \) are the Lie algebroid structures on \( V \) and \( \mathcal{L}^\tau A \), respectively. In addition, if \( \{e_\alpha\} \) is a local basis of \( \Gamma(\tau_V) \) then \( \{e_{\alpha}^v, e_{\alpha}^c\} \) is a local basis of \( \Gamma(\tau^\infty_V) \) (for more details, see [10]).

On the other hand, if \( pr_V : V \times TA \to V \) is the canonical projection on the first factor then the pair \( (pr_V|_\mathcal{L}^\tau A V, \tau_A) \) is a morphism between the Lie algebroids \( (\mathcal{L}^\tau A V, [\cdot, \cdot]^\tau_A, \rho^\tau_A) \) and \( (V, [], V, \rho_V) \). Thus, if \( \alpha \in \Gamma(\wedge^k \tau_V^\infty) \) we may consider the section \( \alpha^v \) of the vector bundle \( \wedge^k(\mathcal{L}^\tau A V)^* \to A \) defined by
\[ \alpha^v = (pr_V|_\mathcal{L}^\tau A V, \tau_A)^*(\alpha). \]
\( \alpha^v \) is called the vertical lift to \( \mathcal{L}^\tau A V \) of \( \alpha \) and it is clear that
\[ d^\mathcal{L}^\tau A V \alpha^v = (d^V \alpha)^v. \]
Furthermore, we have that \( (\alpha_1 \wedge \cdots \wedge \alpha_k)^v = \alpha_1^v \wedge \cdots \wedge \alpha_k^v, \quad \text{for } \alpha_i \in \Gamma(\wedge^k V^*). \)
Now, we define the complete lift of $\alpha$ as follows.

**Proposition 6.1.** If $\alpha$ is a section of the vector bundle $\wedge^k V^* \rightarrow M$, then there exists a unique section $\alpha^c$ of the vector bundle $\wedge^k (\mathcal{L}^* A)^* \rightarrow A$ such that

$$
\alpha^c(X_1^\tau, \ldots, X_k^\tau) = \alpha(X_1, \ldots, X_k),
\alpha^c(X_1^\tau, X_2^\tau, \ldots, X_k^\tau) = \alpha(X_1, X_2, \ldots, X_k),
\alpha^c(X_1^\tau, \ldots, X_s^\tau, X_{s+1}^\tau, \ldots, X_k^\tau) = 0, \quad \text{if} \quad 2 \leq s \leq k,
$$

for $X_1, \ldots, X_k \in \Gamma(\tau_V)$. Moreover, $d\wedge^k V \alpha^c = (dV \alpha)^c$.

**Proof.** Using (6.2), (6.5), (6.6), (6.7) and proceeding as in the proof of Proposition 6.3 in [10], we deduce the result. □

The section $\alpha^c$ of the vector bundle $\wedge^k (\mathcal{L}^* A)^* \rightarrow A$ is called the complete lift of $\alpha$.

**Remark 6.2.**

i) If $\{e_\alpha\}$ is a basis of $\Gamma(\tau_V)$ and $\{e^\alpha\}$ is the dual basis to $\{e_\alpha\}$ then

$$(e^\alpha)^c((e^\beta)^c) = (e^\alpha)^c((e^\beta)^c) = \delta_{\alpha\beta}, \quad (e^\alpha)^c((e^\beta)^c) = (e^\alpha)^c((e^\beta)^c) = 0.$$

Therefore, $\{(e^\alpha)^c, (e^\alpha)^c\}$ is the dual basis to the local basis $\{(e_\alpha)^c, (e_\alpha)^c\}$ of $\Gamma(\tau_V^\tau)$.

ii) If $\alpha_i \in \Gamma(\wedge^k \tau_V)$, $i = 1, \ldots, k$, then

$$
(\alpha_1 \wedge \cdots \wedge \alpha_k)^c = \sum_{i=1}^{k} \alpha_1^c \wedge \cdots \wedge \alpha_i^c \wedge \cdots \wedge \alpha_k^c.
$$

Next, we will introduce the notion of a symplectic Lie affgebroid.

**Definition 6.3.** A Lie affgebroid $\tau_A : A \rightarrow M$ modelled on the Lie algebroid $\tau_V : V \rightarrow M$ is said to be symplectic if $\tau_V : V \rightarrow M$ admits a symplectic section $\Omega$, that is, $\Omega$ is a section of the vector bundle $\wedge^2 V^* \rightarrow M$ such that:

(i) For all $x \in M$, the 2-form $\Omega(x) : V_x \times V_x \rightarrow \mathbb{R}$ on the vector space $V_x$ is non-degenerate and

(ii) $\Omega$ is a 2-cocycle, i.e., $dV \Omega = 0$.

**Examples 6.4.**

(i) Let $\tau : M \rightarrow \mathbb{R}$ be a fibration. Then, as we know (see Section 2.2), the 1-jet bundle $\tau_{1,0} : J^1 \tau \rightarrow M$ is a Lie affgebroid modelled on the Lie algebroid $(\pi_M)|_{\tau_V} : V \tau \rightarrow M$. Now, suppose that $M$ has odd dimension $2n+1$ and that $(\Omega, \eta)$ is a symplectic structure on $M$, with $\eta = \tau^*(dt)$, $t$ being the usual coordinate on $\mathbb{R}$. This means that $\Omega$ is a closed 2-form and that $\eta \wedge \Omega = \eta \wedge \Omega \wedge \cdots \wedge \Omega$ is a volume element on $M$. Thus, it is easy to prove that the restriction to $V \tau$ of $\Omega$ is a symplectic section of $(\pi_M)|_{\tau_V} : V \tau \rightarrow M$ and, therefore, the Lie affgebroid $\tau_{1,0} : J^1 \tau \rightarrow M$ is symplectic.

(ii) Let $\tau_A : A \rightarrow M$ be a Lie affgebroid modelled on the Lie algebroid $\tau_V : V \rightarrow M$. Denote by $\rho_A^*(TV^*)$ the pull-back of the vector bundle $T\tau^*_V : TV^* \rightarrow TM$ over the anchor map $\rho_A : A \rightarrow TM$. Then, $\rho_A^*(TV^*)$ is a Lie affgebroid over $V^*$ with affine bundle projection $\pi_V^* : \rho_A^*(TV^*) \rightarrow V^*$ given by $\pi_V^*(a, X) = \pi_V^*(X)$, for $(a, X) \in \rho_A^*(TV^*)$ (see Section 5). Moreover, the Lie affgebroid $\pi_V^* : \rho_A^*(TV^*) \rightarrow V^*$ is modelled on the Lie algebroid $\tau_V^*: V^* \rightarrow M$. 


\( \mathcal{L}^\tau V \to V^* \) which admits a canonical symplectic section \( \Omega_V \) (see Section 2.1.1). Therefore, the Lie affgebroid \( \tilde{\pi}_V^\tau : \rho_A^*(TV^*) \to V^* \) is symplectic. \( \triangle \)

Now, we will see that the prolongation of a symplectic Lie affgebroid over the affine bundle projection is also a symplectic Lie affgebroid. We recall that if \( \tau_A : A \to M \) is a Lie affgebroid modelled on the Lie algebroid \( \tau_V : V \to M \) then the prolongation

\[
\mathcal{J}^A A = \{(a, v) \in A \times TA/\rho_A(a) = (T\tau_A)(v)\}
\]

is a Lie affgebroid modelled on the Lie algebroid \( \tau^\tau_V^A : \mathcal{L}^\tau V \to A \) (see Section 4). Furthermore, we have that

**Theorem 6.5.** Let \( \tau_A : A \to M \) be a symplectic Lie affgebroid modelled on the Lie algebroid \( \tau_V : V \to M \) and \( \Omega \) be a symplectic section of \( \tau_V : V \to M \). Then, the prolongation \( \mathcal{J}^A A \) of the Lie affgebroid \( \tau_A \) over the projection \( \tau_A : A \to M \) is a symplectic Lie affgebroid and the complete lift \( \Omega^c \) of \( \Omega \) to the prolongation \( \mathcal{L}^\tau A \) is a symplectic section of \( \tau^\tau_V^A \).

**Proof.** Using (6.3) and proceeding as in the proof of the Theorem 6.5 in [10], we deduce the result. \( \square \)

**Example 6.6.** Let \( \tau_A : A \to M \) be a Lie affgebroid modelled on the Lie algebroid \( \tau_V : V \to M \).

Then, as we know, the pull-back \( \tilde{\pi}_V^\tau : \rho_A^*(TV^*) \to V^* \) of the vector bundle \( T\tau_V^* : TV^* \to TM \) over the anchor map \( \rho_A : A \to TM \) is a Lie affgebroid modelled on the symplectic Lie algebroid \( \tau^\tau_V^\tau : \mathcal{L}^\tau V \to V^* \).

Now, suppose that \( \{x^i\} \) are local coordinates on \( M \) and that \( \{e_\alpha\} \) is a local basis of \( \Gamma(\tau_V) \). Then, we may consider the corresponding local coordinates \( \{x^i, y_\alpha\} \) of \( V^* \) and the corresponding local basis \( \{\tilde{e}_\alpha, \tilde{e}_\alpha\} \) of \( \Gamma(\tau^\tau_V^\tau) \) (see Section 2.1.1). This local basis induces a system of local coordinates \( \{x^i, y_\alpha; z^\alpha, v_\alpha\} \) on \( \mathcal{L}^\tau V \). Moreover, if \( \mathcal{L}^\tau V \) is the prolongation of \( \mathcal{L}^\tau V \) over the projection \( \tilde{\pi}_V^\tau : \rho_A^*(TV^*) \to V^* \) then \( \{\tilde{e}_\alpha, \tilde{e}_\alpha, \tilde{e}_\alpha, \tilde{e}_\alpha\} \) is a local basis of \( \Gamma((\tau^\tau_V^\tau)^\tau^\tau V^* \) and if \( \{\tilde{e}_\alpha, \tilde{e}_\alpha\} \) is the dual basis to \( \{\tilde{e}_\alpha, \tilde{e}_\alpha\} \) then

\[
\{(\tilde{e}_\alpha)^\gamma, (\tilde{e}_\alpha)^c, (\tilde{e}_\alpha)^c, (\tilde{e}_\alpha)^c\}
\]

is the dual basis of \( \{\tilde{e}_\alpha, \tilde{e}_\alpha, \tilde{e}_\alpha, \tilde{e}_\alpha\} \). Note that if \( \rho_A, C^\gamma_{\alpha\beta} \) and \( C^\gamma_{\alpha\beta} \) are the structure functions of the Lie algebroid \((\tilde{A}, [\cdot, \cdot], \tilde{\rho}_A, \tilde{\rho}_A)\) with respect to the coordinates \( \{x^i\} \) and to the basis \( \{e_0, e_\alpha\} \) then, from (2.22), (6.8) and (6.9), we deduce that

\[
\begin{align*}
\tilde{e}_\alpha^c(x^i, y_\beta; z^\gamma, v_\gamma) &= (\tilde{e}_\alpha(x^i, y_\beta), \rho_A(x^j) \frac{\partial}{\partial x^i} - (C^\gamma_{\alpha\beta}(x^j) - C^\gamma_{\alpha\beta}(z^\beta) \frac{\partial}{\partial z^\gamma}), \\
(6.9)~\tilde{e}_\alpha^c(x^i, y_\beta; z^\gamma, v_\gamma) &= (e_\alpha(x^i, y_\beta), \frac{\partial}{\partial y_\alpha}), \\
\tilde{e}_\alpha^c(x^i, y_\beta; z^\gamma, v_\gamma) &= (0, \frac{\partial}{\partial z^\alpha}), \quad \tilde{e}_\alpha^c(x^i, y_\beta; z^\gamma, v_\gamma) = (0, \frac{\partial}{\partial v_\alpha}).
\end{align*}
\]
In addition, using (5.5) and (5.8), we deduce that the local expression of the complete lifts $\lambda^c_V$ and $\Omega^c_V$ are
\[
\lambda^c_V(x^i, y^a, z^a, \alpha) = v^a(\tilde{e}^a)^\gamma + y^a(\tilde{e}^a)^\beta,
\]
\[
(6.10) \quad \Omega^c_V(x^i, y^a, z^a, \alpha) = (\tilde{e}^a)^\epsilon \wedge (\tilde{e}^a)^\gamma + (\tilde{e}^a)^\gamma \wedge (\tilde{e}^a)^\epsilon + C^\gamma_{\alpha\beta} y_\gamma (\tilde{e}^a)^\epsilon \wedge (\tilde{e}^a)^\beta + \frac{1}{2}(\rho^i + \rho^i_{\mu} z^\mu) \frac{\partial C^\gamma_{\alpha\beta}}{\partial x^i} y_\gamma + C^\gamma_{\alpha\beta} v_\gamma (\tilde{e}^a)^\epsilon \wedge (\tilde{e}^a)^\beta.
\]

\[\triangle\]

7. LAGRANGIAN LIE SUBAFFGBROIDS IN SYMPLECTIC LIE AFFGBROIDS

First of all, we will introduce the notion of a Lie subaffgebroid of a Lie affgebroid.

**Definition 7.1.** Let $\tau_A : A \to M$ be a Lie affgebroid modelled on the Lie algebroid $\tau_V : V \to M$. A Lie subaffgebroid of $A$ is a Lie affgebroid morphism $((j : A' \to A, i : M' \to M), (j^i : V' \to V, i : M' \to M))$ such that $i$ is an injective immersion and $j : A' \to A$ is also injective.

**Examples 7.2.** Let $\tau_A : A \to M$ be a Lie affgebroid modelled on the Lie algebroid $\tau_V : V \to M$.

i) Denote by $((\cdot, [\cdot \cdot, \cdot], \rho^V_A))$ the Lie algebroid structure on the bidual bundle $\tau^\vee_A : \tilde{A} \to A$.

Now, suppose that $X$ is a section of $\tau_A : A \to M$ and consider the map $(\text{Id}, TX \circ \rho^V_A) : \tilde{A} \to \mathcal{L}^\tau_A \tilde{A}$ given by
\[
(\text{Id}, TX \circ \rho^V_A)(\tilde{a}) = (\tilde{a}, (TX)(\rho^V_A(\tilde{a}))), \quad \text{for} \quad \tilde{a} \in \tilde{A}.
\]

Using the definition of the anchor map of the Lie algebroid $\tau^A_\tilde{A} : \mathcal{L}^\tau_A \tilde{A} \to A$ we find the fact that $\rho^V_A : \Gamma(\tau^\vee_A) \to X(M)$ is a Lie algebra morphism, we deduce that the pair $((\text{Id}, TX \circ \rho^V_A), X)$ is a Lie algebroid morphism. Moreover, it follows that $((\text{Id}, TX \circ \rho^V_A), X)^* (\phi_0) = 1_A$, where $\phi_0$ is the 1-cocycle on $\tau^A_\tilde{A} : \mathcal{L}^\tau_A \tilde{A} \to A$ defined by (3.17). Thus, the pair $((\text{Id}, TX \circ \rho^V_A), X)$ defines a Lie subaffgebroid of $(\tau^A_\tilde{A} : \mathcal{J}^A_A \to A, \tau^V_\tilde{A} : \mathcal{L}^\tau_A V \to A)$

\[
\begin{array}{cccc}
A & \xrightarrow{(\text{Id}, TX \circ \rho^V_A)} & \mathcal{J}^A_A & \xrightarrow{\tau^A_\tilde{A}} & \mathcal{L}^\tau_A V \\
\tau_A & \xrightarrow{X} & A \\
\end{array}
\]

ii) Denote by $\rho^V_A(TV^*)$ the pull-back of the vector bundle $T\tau^A_\tilde{A} : TV^* \to TM$ over the anchor map $\rho^V_A : A \to TM$. $\rho^V_A(TV^*)$ is a vector bundle over $A$ with vector bundle projection $\text{pr}_{1|\rho^V_A(TV^*)} : \rho^V_A(TV^*) \to A$ defined by $\text{pr}_{1|\rho^V_A(TV^*)}(a, Y) = a$, for $(a, Y) \in \rho^V_A(TV^*)$.

Now, suppose that $\hat{X} : A \to \rho^V_A(TV^*)$ is a section of $\text{pr}_{1|\rho^V_A(TV^*)} : \rho^V_A(TV^*) \to A$ and let $\mathcal{L}^\tau_A \tilde{A}$ be the prolongation of the Lie algebroid $\tau_A : \tilde{A} \to M$, over the map $\tau^\vee_\tilde{A} : V^* \to M$ and $\pi^\vee_A : \rho^V_A(TV^*) \to V^*$ be the canonical projection. Then, we may consider the map $(\text{Id}, T(\pi^\vee_A \circ \hat{X})) : \mathcal{L}^\tau_A \tilde{A} \to \mathcal{L}^\tau_A \tilde{A}$ given by
\[
(\text{Id}, T(\pi^\vee_A \circ \hat{X}))(\tilde{a}, Y) = (\tilde{a}, T(\pi^\vee_A \circ \hat{X})(Y)), \quad \text{for} \quad (\tilde{a}, Y) \in \mathcal{L}^\tau_A \tilde{A}.
\]

It is easy to prove that the pair $((\text{Id}, T(\pi^\vee_A \circ \hat{X})), \pi^\vee_A \circ \hat{X})$
Consequently, the pair $((\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^* \to \mathcal{L}^\tau \tilde{A})$ defines a Lie subaffgebroid of the Lie affgebroid $(\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^* \to \mathcal{L}^\tau \tilde{A}$ given by

$$((\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^* \to \mathcal{L}^\tau \tilde{A})$$

Next, denote by $\mathcal{L}^\tau \tilde{A} : \mathcal{L}^\tau \tilde{A} \to V^*$ over the projection $\tilde{\pi} V^*$ and by $((\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to \mathcal{L}^\tau \tilde{A})$ the map given by

$$((\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^*)$$

for $(\tilde{a}, Y) \in \mathcal{L}^\tau \tilde{A}$. Then, using the above facts, we conclude that

$$\mathcal{L}^\tau \tilde{A} : ((\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^*)$$

is a Lie algebroid morphism.

On the other hand, as we know (see Section 5), the affine bundle $\tilde{\pi} V^* : \rho_A^*(TV^*) \to V^*$ is a Lie affgebroid and the bidual bundle to $\tilde{\pi} V^* : \rho_A^*(TV^*) \to V^*$ may be identified with the Lie algebroid $\mathcal{L}^\tau \tilde{A} : \mathcal{L}^\tau \tilde{A} \to V^*$. Therefore, one may consider the 1-cocycle $\tilde{\phi}_0 : \mathcal{L}^\tau \tilde{A} \to \mathcal{L}^\tau \tilde{A}$ given by $\tilde{\phi}_0((\tilde{a}, Y), Z) = 1_A(\tilde{a})$, for $(\tilde{a}, Y, Z) \in \mathcal{L}^\tau \tilde{A}$ (see 8.17). Moreover, it follows that

$$((\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^*)$$

where $\phi_0 : \mathcal{L}^\tau \tilde{A} \to \mathcal{L}^\tau \tilde{A}$ is the 1-cocycle of $\mathcal{L}^\tau A : \mathcal{L}^\tau \tilde{A} \to A$ defined by 8.17. Note that

$$\tilde{\phi}_0^{-1}(1) = \mathcal{J} \rho_A^*(TV^*) \rho_A^*(TV^*)$$

$$\phi_0^{-1}(1) = \mathcal{J} A \mathcal{A}, \quad \phi_0^{-1}(0) = \mathcal{L}^\tau V.$$

Consequently, the pair $((\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^*)$ defines a Lie subaffgebroid of the Lie affgebroid $(\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^* \to \mathcal{L}^\tau \tilde{A}$ given by

$$\mathcal{L}^\tau \tilde{A} : ((\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^*)$$

for $(\tilde{a}, Y) \in \mathcal{L}^\tau \tilde{A}$. Then, using the above facts, we conclude that

$$\mathcal{L}^\tau \tilde{A} : ((\pi V^*, \tilde{\pi} V^*) : \mathcal{L}^\tau A \to V^*)$$

is a Lie algebroid morphism.
Next, we will introduce the notion of a Lagrangian Lie subaffgebroid of a symplectic Lie affgebroid.

**Definition 7.3.** Let $\tau_A : A \to M$ be a symplectic Lie affgebroid modelled on the Lie algebroid $\tau_V : V \to M$ with symplectic section $\Omega$ and

\[
\begin{array}{ccc}
A' & \xrightarrow{j} & A \\
\tau_{A'} & & \tau_A \\
M' & \xleftarrow{i} & M
\end{array}
\]

be a Lie subaffgebroid. Then, the Lie subaffgebroid is said to be Lagrangian if $j^l(V'_{x'})$ is a Lagrangian subspace of the symplectic vector space $(V_{i(x')}, \Omega_{i(x')})$, for all $x' \in M'$. In other words, we have that:

(i) $\text{rank } V' = \frac{1}{2}\text{rank } V$ and
(ii) $(\Omega_{i(x')})_{j^l(V'_{x'})} = 0$, for all $x' \in M'$.

Now, let $\tau_A : A \to M$ be a symplectic Lie affgebroid modelled on the Lie algebroid $\tau_V : V \to M$ with symplectic section $\Omega$. Then, as we know (see Theorem 6.5), the Lie affgebroid $\tau_A^\star : J^A A \to A, \tau_A^\ast : \mathcal{L}^A V \to A$ is symplectic and the complete lift $\Omega^c$ of $\Omega$ is a symplectic section of $\tau_A^\ast : \mathcal{L}^A V \to A$.

Denote by $(\mathcal{L}^A V, D, \rho_A)$ the Lie affgebroid structure of $A$ and suppose that $X : M \to A$ is a section of $\tau_A : A \to M$. The section $X$ allows us to define the Lie subaffgebroid of $(\tau_A^\star : J^A A \to A, \tau_V^\ast : \mathcal{L}^A V \to A)$ considered in Examples 7.2 (i) and, in addition, we will obtain a necessary and sufficient condition for such a Lie subaffgebroid to be Lagrangian.

For this purpose, we will introduce the operator $D_X : \Gamma(\wedge^k \tau_V^\ast) \to \Gamma(\wedge^k \tau_V^\ast)$ defined as follows. If $\alpha \in \Gamma(\wedge^k \tau_V^\ast)$ then

\[
(7.1) \quad (D_X \alpha)(X_1, \ldots, X_k) = \rho_A(X)(\alpha(X_1, \ldots, X_k)) - \sum_{i=1}^k \alpha(X_1, \ldots, D_X X_i, \ldots, X_k)
\]

for $X_1, \ldots, X_k \in \Gamma(\tau_V)$. Note that

\[
D_X (fX_i) = \rho_A(X)(f)X_i + fD_X X_i, \quad \text{for } f \in C^\infty(M),
\]

and, thus, $D_X \alpha \in \Gamma(\wedge^k V^\ast)$. 

\[
\begin{array}{ccc}
L^\tau_A V & \xrightarrow{\mathcal{L}\tau_V^\ast} (L\tau_V^\ast)(L\tau^\ast V) \\
\tau^\tau_A & \xrightarrow{\tau\tau_V} \tilde{X} \\
A & \xrightarrow{\rho_A(TV^\ast)} \rho_A(TV^\ast)
\end{array}
\]
Remark 7.4. The canonical inclusion \( i_V : V \rightarrow \tilde{A} \) is a Lie algebroid monomorphism over the identity of \( M \) (see (2.31)) and, moreover, one may choose \( \tilde{\alpha} \in \Gamma(\wedge \kappa \tau_A^+) \) such that \( \alpha = i_V^* \tilde{\alpha} \). Then, from (2.49) and (2.51), we deduce that

\[
D_x \alpha = i_V^*(\mathcal{L}^\tilde{A} M_{(A \circ X)} \tilde{\alpha}),
\]

where \( i_A : A \rightarrow \tilde{A} \) is the canonical inclusion and \( \mathcal{L}^\tilde{A} M_{(A \circ X)} \) is the Lie derivative in the Lie algebroid \( \tau_A : \tilde{A} \rightarrow M \) with respect to the section \( i_A \circ X : M \rightarrow \tilde{A} \).

Using the operator \( D_x \), we have that

\[
\text{Proposition 7.5. Let } (\tau_A : A \rightarrow M, \tau_V : V \rightarrow M) \text{ be a symplectic Lie affgebroid with symplectic section } \Omega \text{ and } X : M \rightarrow A \text{ be a section of } A. \text{ Then, the Lie subaffgebroid of } (\tau_A^*: J^A A \rightarrow A, \tau_V^*: \mathcal{L}^\tau_A V \rightarrow A) \text{ considered in Examples 7.3 (i)} \text{ is Lagrangian if and only if } D_X \Omega = 0.
\]

\text{Proof. Let } Y \text{ be a section of } \tau_V : V \rightarrow M \text{ and denote by } \tilde{X} \text{ and } \tilde{Y} \text{ the sections of } \tau_A : \tilde{A} \rightarrow M \text{ given by } \tilde{X} = i_A \circ X \text{ and } \tilde{Y} = i_V \circ Y. \text{ Then, using some results in [10] (see Examples 7.5 in [10]), we obtain that}

\[
(\tau \tilde{X})(\rho_A(\tilde{Y})) = (Y^c - [\tilde{X}, \tilde{Y}]^v_A) \circ \tilde{X},
\]

where \( \tilde{Y}^c \in \mathfrak{X}(\tilde{A}) \) and \([\tilde{X}, \tilde{Y}]^v_A \in \mathfrak{X}(\tilde{A}) \) are the complete and vertical lift of \( \tilde{Y} \in \Gamma(\tau_A) \) and \([\tilde{X}, \tilde{Y}]_A \in \Gamma(\tau_A) \), respectively. Since the restriction to \( A \) of \( \tilde{Y}^c \) and \([\tilde{X}, \tilde{Y}]^v_A \) are tangent to \( A \) and \( \rho_A(\tilde{Y}) = \rho_V(Y) \), it follows that \( TX(\rho_V(Y)) = (Y^c)_A - ([\tilde{X}, \tilde{Y}]^v_A)_A \circ X \), and thus, using (1.53) and (1.54), we deduce that

\[
(1, TX \circ \rho_V) \circ Y = (Y^c - (D_X Y)^c) \circ X,
\]

where \( Y^c \in \Gamma(\tau_V^\tau) \) and \( (D_X Y)^c \in \Gamma(\tau_V^\tau) \) are the complete and vertical lift of \( Y \in \Gamma(\tau_V) \) and \( D_X Y \in \Gamma(\tau_V) \) (see Section 6).

On the other hand, if \( Y, Z \in \Gamma(\tau_V) \) then, from Proposition 6.1, we have that

\[
\Omega^c(Y^c - (D_X Y)^c, Z^c - (D_X Z)^c) = \Omega(Y, Z)^c - \Omega(D_X Y, Z)^v - \Omega(Y, D_X Z)^v.
\]

Therefore, using (6.31), it follows that

\[
\Omega^c(Y^c - (D_X Y)^c, Z^c - (D_X Z)^c) \circ X = \rho_A(X)(\Omega(Y, Z)) - \Omega(D_X Y, Z) - \Omega(Y, D_X Z) = (D_X \Omega)(Y, Z)
\]

and \( \Omega^c(Y^c - (D_X Y)^c, Z^c - (D_X Z)^c) \circ X = 0 \), for all \( Y, Z \in \Gamma(\tau_V) \), if and only if \( D_X \Omega = 0 \). Consequently, taking into account that the rank of \( V \) is \( \frac{1}{2} \text{rank}(\mathcal{L}^\tau_A V) \), we deduce the result.

In the particular case when the symplectic section \( \Omega \) is a 1-coboundary, we obtain the following corollary.

\textbf{Corollary 7.6. Let } (\tau_A : A \rightarrow M, \tau_V : V \rightarrow M) \text{ be a symplectic Lie affgebroid with symplectic section } \Omega = -d^V \lambda, \lambda \text{ being a section of the vector bundle } \tau_V^* : V^* \rightarrow M \text{ and suppose that } X : M \rightarrow A \text{ is a section of } \tau_A : A \rightarrow M. \text{ Then, the Lie subaffgebroid of } (\tau_A^*: J^A A \rightarrow A, \tau_V^*: \mathcal{L}^\tau_A V \rightarrow A) \text{ considered in Examples 7.2 (i)} \text{ is Lagrangian if and only if the section } D_X \lambda \text{ of } \tau_V : V^* \rightarrow M \text{ is a 1-cocycle of } \tau_V : V \rightarrow M.
Proof. If $\alpha \in \Gamma(A^\wedge \tau_V)$ then, from Remark 1, and since $i_V : V \to \tilde{A}$ is a Lie algebroid morphism, it follows that $D_X (d^V \alpha) = d^V (D_X \alpha)$. In particular, this implies that $D_X \Omega = -d^V (D_X \lambda)$. Thus, using this fact and Proposition 7.5, we deduce the result. \qed

Let $\tau_A : A \to M$ be a Lie algebroid modelled on the Lie algebroid $\tau_V : V \to M$ and $\rho_A^\ast (TV^\ast)$ be the pull-back of the vector bundle $\tau_V^\ast : TV^\ast \to TM$ over the anchor map $\rho_A : A \to TM$. Then, $\rho_A^\ast (TV^\ast)$ is a Lie algebroid over $V^\ast$ modelled on the vector bundle $\tau_V^\ast : \mathcal{L}^{\tau_V} V \to V^\ast$ and with affine bundle projection $\tilde{\pi}_V : \rho_A^\ast (TV^\ast) \to V^\ast$. Denote by $\Omega_V$ the canonical symplectic section of $\mathcal{L}^{\tau_V} V$. As we know (see Example 6.3), the Lie algebroid $(\tilde{\pi}_V^\ast) : \mathcal{L}^{\tau_V} (\mathcal{L}^{\tau_V} V) \to \mathcal{L}^{\tau_V} (TV^\ast) \to \rho_A^\ast (TV^\ast)$ is symplectic and the complete lift $\Omega_V$ to $\mathcal{L}^{\tau_V} (TV^\ast)$ is a symplectic section of $(\tilde{\pi}_V^\ast) : \mathcal{L}^{\tau_V} (\mathcal{L}^{\tau_V} V) \to \rho_A^\ast (TV^\ast)$. Now, suppose that $\tilde{X} : A \to \rho_A^\ast (TV^\ast)$ is a section of the vector bundle $pr_{1|\rho_A^\ast (TV^\ast)} : \rho_A^\ast (TV^\ast) \to A$. Then, $\tilde{X}$ allows us to define the Lie subaffgebroid of $(\tilde{\pi}_V^\ast) : \mathcal{L}^{\tau_V} (\mathcal{L}^{\tau_V} V) \to \rho_A^\ast (TV^\ast)$ considered in Examples 7.2 (ii).

Next, we will obtain a necessary and sufficient condition for such a Lie subaffgebroid to be Lagrangian. For this purpose, we will introduce a section of $(\mathcal{L}^{\tau_A} V)^\ast \to A$ as follows.

Let $A_A : \rho_A^\ast (TV^\ast) \to (\mathcal{L}^{\tau_A} V)^\ast$ be the canonical isomorphism, over the identity of $A$, between the vector bundles $\rho_A^\ast (TV^\ast) \to A$ and $(\mathcal{L}^{\tau_A} V)^\ast \to A$ considered in Section 5 (see 5.3) and $\alpha_{\tilde{X}}$ be the section of $(\mathcal{L}^{\tau_A} V)^\ast \to A$ given by

\begin{equation}
\alpha_{\tilde{X}} = A_A \circ \tilde{X}.
\end{equation}

Then, we have the following result.

Proposition 7.7. Let $\tau_A : A \to M$ be a Lie affgebroid modelled on the Lie algebroid $\tau_V : V \to M$ and $\tilde{X} : A \to \rho_A^\ast (TV^\ast)$ be a section of $pr_{1|\rho_A^\ast (TV^\ast)} : \rho_A^\ast (TV^\ast) \to A$. Then, the Lie subaffgebroid of $(\tilde{\pi}_V^\ast) : \mathcal{L}^{\tilde{\pi}_V} (\mathcal{L}^{\tau_V} V) \to \rho_A^\ast (TV^\ast), (\tilde{\pi}_V^\ast) : \mathcal{L}^{\tilde{\pi}_V} (\mathcal{L}^{\tau_V} V) \to \rho_A^\ast (TV^\ast)$ considered in Examples 7.2 (ii) is Lagrangian if and only if $\alpha_{\tilde{X}}$ is a 1-cocycle of the Lie algebroid $\tau_{\tilde{X}}^\ast : \mathcal{L}^{\tau_A} V \to A$.

Proof. Let $(\Psi_{\tilde{X}}, \tilde{X})$ be the monomorphism between the Lie algebroids $\tau_{\tilde{X}}^\ast : \mathcal{L}^{\tau_A} V \to A$ and $(\tau_{\tilde{X}}^\ast) : \mathcal{L}^{\tilde{\pi}_V} (\mathcal{L}^{\tau_V} V) \to \rho_A^\ast (TV^\ast)$ considered in Examples 7.2 (ii).

Suppose that $(x^i)$ are local coordinates on $M$ and that $\{e_0, e_\alpha\}$ is a local basis of $\Gamma(\tau_A)$ adapted to $A$. Denote by $(x^i, y^\alpha)$ (respectively, $(x^i, y^\alpha, z^\alpha, v_\alpha)$) the corresponding coordinates on $A$ (respectively, $\rho_A^\ast (TV^\ast)$) and by $\{\tilde{T}_\alpha, \tilde{V}_\alpha\}$ the corresponding local basis of $\Gamma(\tau_{\tilde{X}}^\ast)$ (see 3.15).

If the local expression of $\tilde{X} : A \to \rho_A^\ast (TV^\ast)$ is

\begin{equation}
\tilde{X}(x^i, y^\alpha) = (x^i, \tilde{X}_\alpha; y^\alpha, \tilde{X}_\alpha^\prime)
\end{equation}

then, we have the following result.
then, using (6.11) and (7.5), we deduce that

\[ \Psi_{\dot{T}}(\dot{T}_\alpha(x^i, y^\gamma)) = \dot{e}_\alpha^c(\dot{X}(x^i, y^\gamma)) + \rho^{\beta}_\alpha(x^i) \frac{\partial \dot{X}_\beta}{\partial y^\gamma}(x^i, y^\gamma) \dot{e}_\gamma^c(\dot{X}(x^i, y^\gamma)) - C^\gamma_{\alpha\beta}(x^i) \dot{e}_\gamma^c(\dot{X}(x^i, y^\gamma)) + C^\gamma_{\alpha\beta}(x^i) y^\beta \dot{e}_\gamma^c(\dot{X}(x^i, y^\gamma)) \]

(7.5)

\[ \Psi_{\dot{V}}(\dot{V}_\alpha(x^i, y^\gamma)) = \frac{\partial \dot{X}_\beta}{\partial y^\gamma}(x^i, y^\gamma) \dot{e}_\gamma^c(\dot{X}(x^i, y^\gamma)) + \frac{\partial \dot{X}_\beta}{\partial y^\gamma}(x^i, y^\gamma) \dot{e}_\gamma^c(\dot{X}(x^i, y^\gamma)), \]

where \( \dot{e}_\alpha \) and \( \dot{e}_\alpha^c \) (respectively, \( \dot{e}_\alpha^v \) and \( \dot{e}_\alpha^v^c \)) are the complete lifts (respectively, vertical lifts) of \( \dot{e}_\alpha \) and \( \dot{e}_\alpha^c \) to \( \Gamma(\tau^{\pi V}) \) and \( \rho^{\beta}_\alpha \), \( C^\gamma_{\alpha\beta} \) and \( \tilde{C}^\gamma_{\alpha\beta} \) are the structure functions of the Lie algebroid \( \tau_A : A \to M \) with respect to the local coordinates \( (x^i) \) and to the basis \{\( e_0, e_\alpha \}\).

Thus, if \( \lambda_V \) is the Liouville section of \( \mathcal{L}^{\pi V} V \) then, from (6.11) and (7.5), we obtain that

\[ (\Psi_{\dot{T}}, \dot{X})^*(\lambda_V^c) = (\dot{X}^\alpha + \dot{X}_\gamma C^\gamma_{\alpha\beta} y^\beta + \dot{X}_\gamma C^\gamma_{\alpha0} \dot{T}^\alpha + \dot{X}_\alpha \dot{V}^\alpha), \]

\( \{\dot{T}^\alpha, \dot{V}^\alpha\} \) being the dual basis of \( \{\dot{T}_\alpha, \dot{V}_\alpha\} \).

Therefore, using (6.10), (7.3) and (7.4), it follows that

\[ (\Psi_{\dot{T}}, \dot{X})^*(\lambda_V^c) = \alpha_{\dot{X}}. \]

Now, since \( \Omega_V^c = -d\mathcal{L}^{\pi V} V \lambda_V \) (see Proposition 6.11), we have that

(7.6)

\[ (\Psi_{\dot{X}}, \dot{X})^*(\Omega_V^c) = -d\mathcal{L}^{\tau_A V} \alpha_{\dot{X}}. \]

Consequently, using (6.10) and the fact that \( \text{rank}(\mathcal{L}^{\tau_A V}) = \frac{1}{2} \text{rank}(\mathcal{L}^{\pi V} V \mathcal{L}^{\tau_A V}) \), we deduce the result. \( \square \)

8. Lagrangian submanifolds, Tulczyjew’s triple and Euler-Lagrange (Hamilton) equations

Let \( (\tau_A : A \to M, \tau_V : M, (\tau_A, D, \rho_A)) \) be a symplectic Lie affgebroid with symplectic section \( \Omega \). Then, as we know (see Theorem 6.6), the Lie affgebroid \( (\tau^{\tau_A} : \mathcal{J}^A A \to A, \tau^{\tau_A}_V : \mathcal{L}^{\tau_A V} \to A) \) is symplectic and the complete lift \( \Omega^c \) of \( \Omega \) is a symplectic section of \( \tau^{\tau_A} \) : \( \mathcal{L}^{\tau_A V} \to A \).

**Definition 8.1.** Let \( S \) be a submanifold of the symplectic Lie affgebroid \( A \) and \( i : S \to A \) be the canonical inclusion. Denote by \( \tau^{\tau_A}_S : S \to M \) the map given by \( \tau^{\tau_A}_S = \tau_A \circ i \) and suppose that \( \rho_v(V_{\tau^{\tau_A}_S}(a)) \tau^{\tau_A}_S(T_a S) = T_{\tau^{\tau_A}_S(a)} M \), for all \( a \in S \). Then, the submanifold \( S \) is said to be Lagrangian if the corresponding Lie subaffgebroid \( (\tau^{\tau_A}_A, \rho_A(T S) \to S, \tau^{\tau_A}_V : \mathcal{L}^{\tau_A V} \to S) \) of the symplectic Lie affgebroid \( (\tau^{\tau_A}_A : \mathcal{J}^A A \to A, \tau^{\tau_A}_V : \mathcal{L}^{\tau_A V} \to A) \) is Lagrangian.

Now, we have the following result.
Thus, using Proposition 7.7, we deduce that the submanifold \( S = X(M) \) of \( A \) is Lagrangian if and only if the section \( D_X \lambda \) of \( \tau_V^* : V^* \to M \) is a 1-cocycle of \( \tau_V : V \to M \).

**Proof.** Let \((\text{Id},TX \circ \rho_A) : A \to \mathcal{J}^A A\) (respectively, \((\text{Id},TX \circ \rho_V) : V \to \mathcal{L}^* V\)) be the morphism defined as in Examples [7.2](#). Then, a direct computation proves that \((\text{Id},TX \circ \rho_A)(A) = \rho_A^*(TS)\) and \((\text{Id},TX \circ \rho_V)(V \circ \lambda) = \mathcal{L}_V^* \lambda \). Thus, using Corollary [7.6](#) we deduce that \( S \) is Lagrangian if and only if \( D_X \lambda \) is a 1-cocycle.

If \( \tau_A : A \to M \) is a Lie affgebroid modelled on the Lie algebroid \( \tau_V : V \to M \), we will denote by \( \rho_A^*(TV^*) \) the pull-back of the vector bundle \( T\tau_V^* : TV^* \to TM \) over the anchor map \( \rho_A : A \to TM \) and by \( A_A : \rho_A^*(TV^*) \to (\mathcal{L}^* V)^* \) the canonical isomorphism, over the identity of \( A \), between the vector bundles \( \rho_A^*(TV^*) \to A \) and \( (\mathcal{L}^* V)^* \to A \). Considered in Section 5 (see [5.3](#)).

**Corollary 8.3.** Let \( \tau_A : A \to M \) be a Lie affgebroid modelled on the Lie algebroid \( \tau_V : V \to M \) and \( \bar{X} : A \to \rho_A^*(TV^*) \) be a section of \( \rho_A^*(TV^*) \). Denote by \( S \) the submanifold \( \bar{X} A \) of \( \rho_A^*(TV^*) \) and by \( \alpha_{\bar{X}} \) the section of \( (\mathcal{L}^* V)^* \) given by \( \alpha_{\bar{X}} = \bar{X} \circ \bar{X} \). Then, \( S \) is a Lagrangian submanifold of the symplectic Lie affgebroid \( (\bar{X} \rho_A^*(TV^*), \bar{X} \mathcal{L}_V^* \lambda) \) if and only if \( \alpha_{\bar{X}} \) is a 1-cocycle of the Lie algebroid \( \mathcal{L}^* V \to A \).

**Proof.** Let \((\text{Id},T(\bar{X} \circ \bar{X})), T\bar{X} \) : \( \mathcal{J}^A A \to \mathcal{J}^A(\mathcal{L}^* V) \) (respectively, \((\text{Id},T(\bar{X} \circ \bar{X})), T\bar{X} \) : \( \mathcal{L}^* V \to \mathcal{L}^* V \)) be the morphism defined as in Examples [7.3](#). Then, it is easy to prove that

\[
((\text{Id},T(\bar{X} \circ \bar{X})), T\bar{X}((\mathcal{J}^A A) = \rho_A^*(TV^*)) T(\mathcal{L}^* V)) = \mathcal{L}^* \rho_A^*(TV^*) \lambda \mathcal{L}^* V.
\]

Thus, using Proposition 7.7, we deduce that \( S \) is Lagrangian if and only if \( \alpha_{\bar{X}} \) is a 1-cocycle.
is said to be *admissible* if the curve \( \gamma_2 : I \to TV^* \) is a tangent lift, that is, \( \gamma_2(t) = \dot{c}(t) \), where \( c : I \to V^* \) is the curve in \( V^* \) given by \( \pi_{V^*} \circ \gamma_2 \), \( \pi_{V^*} : TV^* \to V^* \) being the canonical projection.

**Theorem 8.4.** Under the bijection \( \Psi_h \), the admissible curves in the Lagrangian submanifold \( S_h \) correspond with the solutions of the Hamilton equations for \( h \).

**Proof.** Suppose that \( \gamma : I \to S_h \subseteq \mathcal{L}^* \tilde{A} \subseteq \tilde{A} \times TV^* \), \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \) is an admissible curve in \( S_h \). Then, \( \gamma_2(t) = \dot{c}(t) \), for all \( t \), where \( c : I \to V^* \) is the curve in \( V^* \) given by \( c = \pi_{V^*} \circ \gamma_2 \). Now, since \( R_h \) is a section of the vector bundle \( \tau^*_A : \mathcal{L}^* \tilde{A} \to V^* \) and \( \gamma(I) \subseteq S_h = R_h(V^*) \), it follows that

\[
(8.2) \quad R_h(c(t)) = \gamma(t), \quad \text{for all } t,
\]

that is, \( c = \Psi_h(\gamma) \). Thus, from \( \text{(3.2)} \), we obtain that \( \rho^*_A(R_h) \circ c = \gamma_2 = \dot{c} \). In other words, \( c \) is an integral curve of the vector field \( \rho^*_A(R_h) \) and, therefore, \( c \) is a solution of the Hamilton equations associated with \( h \) (see Section \( \text{3.3} \)). Conversely, assume that \( c : I \to V^* \) is a solution of the Hamilton equations associated with \( h \), that is, \( c \) is an integral curve of the vector field \( \rho^*_A(R_h) \) or, equivalently,

\[
(8.3) \quad \rho^*_A(R_h) \circ c = \dot{c}.
\]

Then, \( \gamma = R_h \circ c \) is a curve in \( S_h \) and, from \( \text{(3.3)} \), we deduce that \( \gamma \) is admissible. \( \square \)

Next, suppose that \( L : A \to \mathbb{R} \) is a Lagrangian function. Then, from Corollary \( \text{3.3} \), we obtain that \( S_L = (A^*_A \circ d\mathcal{L}^*A^*L)(A) \) is a Lagrangian submanifold of the symplectic Lie affgebroid \( \rho^*_A(TV^*) \).

On the other hand, we have a bijective correspondence \( \Psi_L \) between the set of curves in \( S_L \) and the set of curves in \( A \). In fact, if \( \gamma : I \to S_L \) is a curve in \( S_L \) then there exists a unique curve \( c : I \to A \) in \( A \) such that \( A_A(\gamma(t)) = (d\mathcal{L}^*A^*L)(c(t)) \), for all \( t \). Note that

\[
pr_1(\gamma(t)) = (\tau^*_A)^*(A_A(\gamma(t))) = (\tau^*_A)^*((d\mathcal{L}^*A^*L)(c(t))) = c(t), \quad \text{for all } t,
\]

where \( pr_1 : \rho^*_A(TV^*) \subseteq A \times TV^* \to A \) is the canonical projection on the first factor and \( (\tau^*_A)^* : (\mathcal{L}^*A^*)^* \to A \) is the vector bundle projection. Thus,

\[
\gamma(t) = (c(t), \gamma_2(t)) \in \rho^*_A(TV^*) \subseteq \mathcal{L}^* A \subseteq A \times TV^*, \quad \text{for all } t.
\]

A curve \( \gamma \) in \( S_L \)

\[
\gamma : I \to S_L \subseteq \rho^*_A(TV^*) \subseteq A \times TV^*, \quad t \mapsto (c(t), \gamma_2(t)),
\]

is said to be *admissible* if the curve \( \gamma_2 : I \to TV^* \) is a tangent lift, that is, \( \gamma_2(t) = \dot{c}^*(t) \), where \( c^* : I \to V^* \) is the curve in \( V^* \) given by \( c^* = \pi_{V^*} \circ \gamma_2 \).

**Theorem 8.5.** Under the bijection \( \Psi_L \), the admissible curves in the Lagrangian submanifold \( S_L \) correspond with the solutions of the Euler-Lagrange equations for \( L \).

**Proof.** Suppose that \( (x^i) \) are local coordinates on \( M \) and that \( \{e_0, e_\alpha\} \) is a local basis of \( \Gamma(\tau_A) \) adapted to \( \text{1}_A \). Denote by \( (x^i, y^\alpha) \) (respectively, \( (x^i, y_\alpha) \) and \( (x^i, y_\alpha; z^\alpha, v_\alpha) \)) the corresponding coordinates on \( A \) (respectively, \( V^* \) and \( \rho^*_A(TV^*) \)). Then, using \( \text{(3.3)} \), it follows that the
submanifold $S_L$ is characterized by the following equations

$$y_\alpha = \frac{\partial L}{\partial y^\alpha}, \quad z^\alpha = y^\alpha, \quad v_\alpha = \rho_\alpha i \frac{\partial L}{\partial x^i} + (C^\gamma_{0\alpha} + C^\gamma_{\beta\alpha} y^\beta) \frac{\partial L}{\partial y^\gamma}, \quad \text{for all } \alpha.$$  \hspace{1cm} (8.4)

Now, let $\gamma : I \to S_L$ be an admissible curve in $S_L$

$$\gamma(t) = (c(t), \gamma_2(t)) \in S_L \subseteq \rho_A^* (TV^*) \subseteq A \times TV^*, \quad \text{for all } t,$$

and denote by $c^* : I \to V^*$ the curve in $V^*$ satisfying

$$\gamma_2(t) = \dot{c}^*(t), \quad \text{for all } t,$$

i.e.,

$$c^*(t) = \pi_{V^*} (\gamma_2(t)), \quad \text{for all } t.$$  \hspace{1cm} (8.5)

If the local expressions of $\gamma$ and $c$ are

$$\gamma(t) = (x^i(t), y_\alpha(t); z^\alpha(t), v_\alpha(t)), \quad c(t) = (x^i(t), y^\alpha(t)),$$

then we have that

$$y^\alpha(t) = z^\alpha(t), \quad \text{for all } \alpha.$$  \hspace{1cm} (8.6)

Moreover, from $S_5$ and $S_6$, we deduce that

$$c^*(t) = (x^i(t), y_\alpha(t)), \quad \gamma_2(t) = \frac{dx^i}{dt} \frac{\partial}{\partial x^i | c^*(t)} + \frac{dy_\alpha}{dt} \frac{\partial}{\partial y_\alpha | c^*(t)}.$$  \hspace{1cm} (8.8)

Thus,

$$v_\alpha(t) = \frac{dy_\alpha}{dt}, \quad \text{for all } \alpha.$$  \hspace{1cm} (8.9)

Therefore, using $S_8$, $S_9$, $S_{10}$, $S_{11}$, $S_{12}$ and the fact that $\rho_A (c(t)) = (T \tau^*_V)(\gamma_2(t))$, it follows that

$$\frac{dx^i}{dt} = \rho^i_0 + \rho^i_\alpha y^\alpha, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial y^\alpha} \right) = \rho_\alpha \frac{\partial L}{\partial x^i} + (C^\gamma_{0\alpha} + C^\gamma_{\beta\alpha} y^\beta) \frac{\partial L}{\partial y^\gamma},$$

for all $i$ and $\alpha$, that is, $c$ is a solution of the Euler-Lagrange equations for $L$.

Conversely, let $c : I \to A$ be a solution of the Euler-Lagrange equations for $L$ and $\gamma : I \to S_L$ be the corresponding curve in $S_L$, $c = \Psi_L (\gamma)$. Suppose that

$$\gamma(t) = (c(t), \gamma_2(t)) \in S_L \subseteq \rho_A^* (TV^*) \subseteq A \times TV^*, \quad \text{for all } t,$$

and denote by $c^* : I \to V^*$ the curve in $V^*$ given by $c^* = \pi_{V^*} \circ \gamma_2$. If the local expressions of $\gamma$ and $c$ are

$$\gamma(t) = (x^i(t), y_\alpha(t); z^\alpha(t), v_\alpha(t)), \quad c(t) = (x^i(t), y^\alpha(t)),$$

then $y^\alpha(t) = z^\alpha(t)$, for all $\alpha$, and the local expressions of $c^*$ and $\gamma_2$ are

$$c^*(t) = (x^i(t), y_\alpha(t)), \quad \gamma_2(t) = \left( \rho^i_0 + z^\alpha(t) \rho_\alpha (x^i(t)) \right) \frac{\partial}{\partial x^i | c^*(t)} + v_\alpha(t) \frac{\partial}{\partial y_\alpha | c^*(t)}.$$  \hspace{1cm} (8.10)

Thus, using $S_{14}$ and the fact that $c$ is a solution of the Euler-Lagrange equations for $L$, we deduce that $\gamma_2(t) = \dot{c}^*(t)$, for all $t$, which implies that $\gamma$ is admissible. \hspace{1cm} $\square$
Now, assume that the Lagrangian function $L : A \to \mathbb{R}$ is hyperregular and denote by $\Theta_L$ and $\Omega_L = -d^{\mathcal{L}^TA}\Theta_L$ the Poincaré-Cartan sections associated with $L$. We consider the map $(i_V, \text{Id}) : \mathcal{L}^TA \to \mathcal{L}^TA$ given by

$$(i_V, \text{Id})(v, X_a) = (i_V(v), X_a), \text{ for all } (v, X_a) \in (\mathcal{L}^TA)_a, \text{ with } a \in A,$$

$i_V : V \to \tilde{A}$ being the canonical inclusion. We have that $(i_V, \text{Id})$ is a Lie algebroid morphism over the identity of $A$. Furthermore, if $\phi_0$ is the section of the dual bundle to $\mathcal{L}^TA$ defined by $3.17$, it follows that the pair $(\Omega_L, \phi_0)$ is a cosymplectic structure on $\mathcal{L}^TA \tilde{A}$ (see $3.20$) and it is easy to prove that $(i_V, \text{Id})^*\phi_0 = 0$. This implies that $(i_V, \text{Id})^*\Omega_L$ is a symplectic section of the Lie algebroid $\tau_V^A : \mathcal{L}^TA \to A$ and, thus, the Lie algebroid $(\tau_V^A : \mathcal{J}^AA \to A, \tau_V^A : \mathcal{L}^TA \to A)$ is symplectic. Note that $(i_V, \text{Id})^*\Omega_L = -d^{\mathcal{L}^TV}((i_V, \text{Id})^*(\Theta_L))$.

Next, denote by $R_L$ the Reeb section of the cosymplectic structure $(\Omega_L, \phi_0)$. Since $\phi_0(R_L) = 1$, we deduce that $R_L$ is a section of $\tau_V^A : \mathcal{J}^AA \to A$. Moreover, we have that $\mathcal{L}^A_{R_L} : \mathcal{L}^A \to \mathcal{L}^{\mathcal{J}AA}$ is a cosymplectic structure on $\mathcal{L}^{\mathcal{J}AA} \tilde{A}$. Therefore, from $\tau^2$ and Corollary $8.2$, we deduce that $S_{R_L} = R_L(A)$ is a Lagrangian submanifold of the symplectic Lie algebroid $J^AA$.

On the other hand, it is clear that there exists a bijective correspondence $\Psi_{S_{R_L}}$ between the set of curves in $S_{R_L}$ and the set of curves in $A$.

A curve $\gamma$ in $S_{R_L}$

$$\gamma : I \to S_{R_L} \subseteq \mathcal{J}^AA \subseteq A \times TA, \quad t \mapsto (\gamma_1(t), \gamma_2(t)),$$

is said to be admissible if the curve $\gamma_2 : I \to TA$ is a tangent lift, that is, $\gamma_2(t) = \dot{c}(t)$, for all $t$, where $c : I \to A$ the curve in $A$ defined by $c = \pi_A \circ \gamma_2, \pi_A : TA \to A$ being the canonical projection.

**Theorem 8.6.** If the Lagrangian $L$ is hyperregular then under the bijection $\Psi_{S_{R_L}}$ the admissible curves in the Lagrangian submanifold $S_{R_L}$ correspond with the solutions of the Euler-Lagrange equations for $L$.

**Proof.** Let $\gamma : I \to S_{R_L} \subseteq \mathcal{J}^AA \subseteq A \times TA$ be an admissible curve in $S_{R_L}$, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, for all $t$. Then, $\gamma_2(t) = \dot{c}(t)$, for all $t$, where $c : I \to A$ is the curve in $A$ given by $c = \pi_A \circ \gamma_2$.

Now, since $R_L$ is a section of the vector bundle $\tau_V^A : \mathcal{J}^AA \to A$ and $\gamma(I) \subseteq S_{R_L} = R_L(A)$, it follows that $R_L(c(t)) = \gamma(t)$, for all $t$, that is, $c = \Psi_{S_{R_L}}(\gamma)$. Thus, we obtain that $\rho_A^{\tau_A}(R_L) \circ c = \gamma_2 = \dot{c}$, that is, $c$ is an integral curve of the vector field $\rho_A^{\tau_A}(R_L)$. Therefore, $c$ is a solution of the Euler-Lagrange equations associated with $L$ (see Section $3.2$).

Conversely, assume that $c : I \to A$ is a solution of the Euler-Lagrange equations associated with $L$, that is, $c : I \to A$ is an integral curve of the vector field $\rho_A^{\tau_A}(R_L)$ or, equivalently,

$$(8.10) \quad \rho_A^{\tau_A}(R_L) \circ c = \dot{c}.$$ 

Then, $\gamma = R_L \circ c$ is a curve in $S_{R_L}$, and, from $8.10$, we deduce that $\gamma$ is admissible. \hfill \Box

If $L : A \to \mathbb{R}$ is hyperregular then the Legendre transformation $leg_L : A \to V^*$ associated with $L$ is a global diffeomorphism. So, we may consider the Hamiltonian section $h_L : V^* \to A^+$ defined by $h_L = Leg_L \circ leg_L^{-1}$, $Leg_L : A \to A^+$ being the extended Legendre transformation, and the Reeb section $R_L$ (respectively, $R_{h_L}$) of the cosymplectic structure $(\Omega_L, \phi_0)$ (respectively, $(\Omega_{h_L}, \eta)$) on $\mathcal{L}^AA$ (respectively, $\mathcal{L}^TA$).
Thus, we have:
- The Lagrangian submanifolds $S_L$ and $S_{hL}$ of the symplectic Lie affgebroid $\rho_A(TV^*)$.
- The Lagrangian submanifold $S_{RL}$ of the symplectic Lie affgebroid $J^A \hat{A}$.

Denote by $(\mathcal{L}eg_L, leg_L)$ the Lie algebroid isomorphism between the Lie algebroids $\mathcal{L}^A \tilde{A}$ and $\mathcal{L}^{\tau} \tilde{A}$ induced by the transformation $leg_L : A \to V^*$.

**Theorem 8.7.** If the Lagrangian function $L : A \to \mathbb{R}$ is hyperregular and $h_L : V^* \to A^+$ is the corresponding Hamiltonian section then the Lagrangian submanifolds $S_L$ and $S_{hL}$ are equal and

\begin{equation}
\mathcal{L}eg_L(S_{RL}) = S_L = S_{hL}.
\end{equation}

**Proof.** Using (3.30), we obtain that

\begin{equation}
A_A \circ R_{hL} \circ leg_L = A_A \circ \mathcal{L}eg_L \circ R_L.
\end{equation}

Now, suppose that $(x^i)$ are local coordinates in $M$ and that $\{e_0, e_\alpha\}$ is a local basis of $\Gamma(\tau_A)$ adapted to $1_A$. Denote by $(x^i, y^\alpha)$ the corresponding coordinates on $A$ and by $(x^i, y^\alpha; z^\alpha, v^\alpha)$ (respectively, $(x^i, y^\alpha; z^\alpha, v^\alpha)$) the corresponding ones on $J^A \hat{A}$ (respectively, $\rho_A(TV^*)$ and $(\mathcal{L}^A V)^*$). Then, from (8.12), we deduce that

\begin{equation}
(A_A \circ \mathcal{L}eg_L \circ R_L)(x^i, y^\alpha) = (x^i, y^\alpha; \rho_\alpha \frac{\partial L}{\partial x^i}; \frac{\partial L}{\partial y^\alpha}).
\end{equation}

Thus, it follows that $(A_A \circ \mathcal{L}eg_L \circ R_L)(x^i, y^\alpha) = d\mathcal{L}^A V L (x^i, y^\alpha)$, that is (see (8.12)), $A_A \circ R_{hL} \circ leg_L = d\mathcal{L}^A V L$. Therefore, $S_L = S_{hL}$.

On the other hand, using (3.30), we obtain that (8.11) holds.

\[\square\]

9. Applications

9.1. The particular case of a Lie algebroid. In this first example, we will show that when we consider a Lie algebroid as a Lie affgebroid and we apply the results obtained in this paper, we recover the constructions made in (10).

Let $(E, [\cdot, \cdot], \rho)$ be a Lie algebroid on $M$ with vector bundle projection $\tau_E : E \to M$. In this case, $\tau_E : E \to M$ can be considered as an affine bundle with associated vector bundle $\tau_E : E \to M$. Then, its dual bundle $E^+$ is $E^* \times \mathbb{R}$ and therefore, its bidual bundle $\tilde{E}$ is just $E \times \mathbb{R}$. Moreover, we can identify the distinguished section $1_E \in \Gamma(\tau_{E^+}) = \Gamma(\tau_{\tilde{E}}) \times C^\infty(M)$ with the section $(0, 1) \in \Gamma(\tau_{E^+}) \times C^\infty(M)$ induced by the constant function 1 on $M$.

On the other hand, a local basis $\{e_0, e_\alpha\}$ of sections of $\tilde{E} = \tilde{E} \times \mathbb{R}$ adapted to the 1-cocycle $1_E = (0, 1)$ can be constructed as follows

$$e_0 = (0, 1) \in \Gamma(\tau_{E^+}) = \Gamma(\tau_E) \times C^\infty(M), \quad e_\alpha = (e'_\alpha, 0) \in \Gamma(\tau_{E^+}) = \Gamma(\tau_E) \times C^\infty(M),$$

where $\{e'_\alpha\}$ is a local basis of sections of $E$.

Now, suppose that $(x^i)$ are local coordinates on an open subset $U$ of $M$. Denote by $(x^i, y^\alpha, y^0)$ the local coordinates on $\tilde{E} = \tilde{E} \times \mathbb{R}$ induced by $(e_0, e_\alpha)$. The local equation defining $E$ as affine subbundle (respectively, as vector subbundle) of $\tilde{E}$ is $y^0 = 1$ (respectively, $y^0 = 0$). Thus, $(x^i, y^\alpha)$ may be considered as local coordinates on $E$. 
In such a case, the local expressions of the Lie bracket and the anchor map on $\tilde{E} = E \times \mathbb{R}$ are the following:

$$\left[ e_0, e_\alpha \right]_{E \times \mathbb{R}} = 0, \quad \left[ e_\alpha, e_\beta \right]_{E \times \mathbb{R}} = C_{\alpha \beta \gamma} e_\gamma,$$

$$\rho_{E \times \mathbb{R}}(e_0) = 0, \quad \rho_{E \times \mathbb{R}}(e_\alpha) = \rho^E_{\alpha} \frac{\partial}{\partial x^\alpha}.$$

On the other hand, the prolongation, $\mathcal{L}^{\tau_{\tilde{E}}} \tilde{E}$, of the bidual Lie algebroid $\tilde{E}$ over the dual projection of the vector bundle $\tau_E : E \to M$ is just the product vector bundle $\mathcal{L}^{\tau_{\tilde{E}}} E \times \mathbb{R} \to E^*$.

So, the space of sections of $\mathcal{L}^{\tau_{\tilde{E}}} \tilde{E}$ can be identified with $\Gamma(\tau_{\tilde{E}}^\ast) \times C^\infty(E^*)$.

Since the map $\mu : E^+ = E^* \times \mathbb{R} \to E^*$ is the canonical projection on the first factor, a Hamiltonian section $h : E^* \to E^* \times \mathbb{R}$ may be identified with a Hamiltonian function $H : E^* \to \mathbb{R}$ is such a way that $h(\beta) = (\beta, -H(\beta))$, for $\beta \in E^*$. Moreover, the cosymplectic structure $(\Omega_h, \eta)$ on the Lie algebroid $\mathcal{L}^{\tau_{\tilde{E}}} \tilde{E}$ can be expressed in terms of the canonical symplectic section of $\mathcal{L}^{\tau_{\tilde{E}}} \tilde{E}$, $\Omega_E$, and the section $(0, 1) \in \Gamma(\tau_{\tilde{E}}^\ast) \times C^\infty(E^*)$ as follows

\begin{equation}
\Omega_h = \Omega_E + d\mathcal{L}^{\tau_{\tilde{E}}} \tilde{E} H \wedge (0, 1), \quad \eta = (0, 1).
\end{equation}

Thus, the local expression of the Reeb section of $(\Omega_h, \eta)$, $R_h \in \Gamma(\tau_{\tilde{E}}^\ast) \cong \Gamma(\tau_{\tilde{E}}^\ast) \times C^\infty(E^*)$, is

\begin{equation}
R_h = (\xi_H, 1),
\end{equation}

$\xi_H$ being the unique section of $\Gamma(\tau_{\tilde{E}}^\ast)$ satisfying $\iota_{\xi_H} \Omega_E = d\mathcal{L}^{\tau_{\tilde{E}}} \tilde{E} H$. This implies that the vector fields $\rho_{E \times \mathbb{R}}(R_h)$ and $\rho^\ast(\xi_H)$ on $E^*$ coincide. Therefore, one deduces that the Hamilton equations associated with $h$ on $E$ (as a Lie affgebroid) are just the Hamilton equations associated with $H$ considering on $E$ the structure of Lie algebroid (see Section 3.3 in [10]).

Now, let $L : E \to \mathbb{R}$ be a Lagrangian function. Then, if we write the Euler-Lagrange equations associated with $L$ (see (9.24)), we obtain the Euler-Lagrange equations associated with $L$ considering $E$ as a Lie affgebroid (see (2.40) in [10]).

On the other hand, if the Lagrangian function $L$ is hyperregular, we can consider the corresponding Hamiltonian section $h_L : E^* \to E^+ = E^* \times \mathbb{R}$ defined by $h_L = \text{Leg}_L \circ \text{leg}_L^{-1}$. Thus, $h_L(\alpha) = (\alpha, -H_L(\alpha))$, where $H_L : E^* \to \mathbb{R}$ is a Hamiltonian function on $E^*$. One may prove that $H_L = E_L \circ \text{leg}_L^{-1}$, where $E_L : E \to \mathbb{R}$ is the Lagrangian energy associated with $L$ (for the definition of $E_L$, see [10]). Thus, $H_L$ is the Hamiltonian function associated with $L$ considered in [10].

Next, we are going to describe the Tulczyjew’s triple associated with $E$ (as a Lie affgebroid) and the Hamilton section $h : E^* \to E^* \times \mathbb{R}$:

\begin{equation}
(\mathcal{L}^{\tau_{\tilde{E}}} E)^\ast \xrightarrow{\rho^\ast(TE^\ast)} \mathcal{L}^{\tau_{\tilde{E}}} \tilde{E} \xrightarrow{b_{\Omega_h}} (\mathcal{L}^{\tau_{\tilde{E}}} E)^\ast \xrightarrow{\pi_{E^\ast}} (\tau_{\tilde{E}}^\ast)^\ast
\end{equation}

where $\rho^\ast(TE^\ast) \cong \mathcal{L}^{\tau_{\tilde{E}}} E$, $\pi_{E^\ast} : (\tau_{\tilde{E}}^\ast)^\ast \to E^\ast$.
Note that the space \( \mathcal{J}^{E}E \) is the prolongation \( \mathcal{L}^{E}E \) of \( E \) over \( \tau_{E} : E \to M \). Moreover, the canonical involution \( \sigma: \mathcal{J}^{E}E \to \mathcal{J}^{E}E \) associated with the Lie affgebroid \( E \) is just the canonical involution associated with the Lie algebroid \( E \) which was introduced in [10] (see Section 4 in [10]). Thus, it is easy to prove that the map \( \eta_{E} : \rho_{E}^{*}(TE) \equiv \mathcal{L}^{E}E \to (\mathcal{L}^{E}E)^{*} \) coincides with the isomorphism of vector bundles also introduced in [10] (see Section 5 in [10]).

On the other hand, let \( \gamma_{E} : \mathcal{L}^{E}E \to (\mathcal{L}^{E}E)^{*} \) be the canonical isomorphism between the vector bundles \( \gamma_{E} : \mathcal{L}^{E}E \to E^{*} \) and \( (\mathcal{L}^{E}E)^{*} \). Then, one can check that (see (9.1))

\[
\gamma_{\Omega_{h}} = \gamma_{E} \circ (d\mathcal{L}^{E}E H) \circ \gamma_{E}.
\]

Finally, we will analyze the Lagrangian submanifolds which describe the dynamics on the Lie affgebroid \( E \).

If \( L : E \to \mathbb{R} \) is a Lagrangian function then it is clear that \( S_{L} = (A_{E}^{-1} \circ d\mathcal{L}^{E}E L)(E) \) is just the Lagrangian submanifold of the symplectic Lie algebroid \( \mathcal{E}^{E}E \colon \mathcal{L}^{E}E \to E^{*} \) which was considered in [10] in order to describe the Lagrangian dynamics (see Section 8 in [10]). On the other hand, suppose that \( h : E^{*} \to E^{*} \) is a Hamiltonian section and that \( H : E^{*} \to \mathbb{R} \) is the corresponding Hamiltonian function. Then, the vector bundle \( \pi_{E}^{*} : \mathcal{L}^{E}E \to E^{*} \) may be identified with the affine subbundle \( \mathcal{E}^{E}E \times \{1\} \to E^{*} \) of \( \mathcal{L}^{E}E \times \mathbb{R} = \mathcal{L}^{E}E \to E^{*} \) and, under this identification, the Reeb section \( \rho_{h} \) is the Hamiltonian section \( \xi_{H} \) of \( \pi_{E}^{*} : \mathcal{L}^{E}E \to E^{*} \) (see (9.2)). Therefore, \( \mathcal{S}_{h} = \mathcal{R}_{h}(E) \) may be identified with the Lagrangian submanifold \( \mathcal{S}_{H} = \xi_{H}(E^{*}) \) of the symplectic Lie algebroid \( \pi_{E}^{*} : \mathcal{L}^{E}E \to E^{*} \) associated with the Hamiltonian function \( H \). This submanifold was considered in [10] in order to describe the Hamiltonian dynamics.

9.2. Lie affgebroids and time-dependent Mechanics. Let \( \tau : M \to \mathbb{R} \) be a fibration and \( \tau_{1,0} : J^{1}\tau \to M \) be the associated Lie affgebroid modelled on the vector bundle \( \pi = (\tau_{M})_{V\tau} : V_{\tau} \to M \). As we know, the bidual vector bundle \( J^{1}\tau \) to the affine bundle \( \tau_{1,0} : J^{1}\tau \to M \) may be identified with the tangent bundle \( TM \) to \( M \) and, under this identification, the Lie algebroid structure on \( \tau_{M} : TM \to M \) is the standard Lie algebroid structure and the 1-cocycle \( J_{t} \) on \( \tau_{M} : TM \to M \) is just the 1-form \( \eta = \tau^{*}(dt) \), \( t \) being the coordinate on \( \mathbb{R} \) (see Section 2.2). If \( (t, q^{i}) \) are local fibrad coordinates on \( M \) then \( \{\frac{\partial}{\partial \tau^{i}}\} \) (respectively, \( \{\frac{\partial}{\partial \tau^{i}}, \frac{\partial}{\partial q^{j}}\} \)) is a local basis of sections of \( \pi : V_{\tau} \to M \) (respectively, \( \pi_{M} : TM \to M \)). Denote by \( (t, q^{i}, \dot{t}, \dot{q}^{i}) \) (respectively, \( (t, q^{i}, \dot{t}, \dot{q}^{i}) \)) the corresponding local coordinates on \( V_{\tau} \) (respectively, \( TM \)). Then, the (local) structure functions of \( TM \) with respect to this local trivialization are given by

\[
C_{ij}^{k} = 0 \quad \text{and} \quad \rho_{i}^{j} = \delta_{ij}, \quad \text{for} \quad i, j, k \in \{0, 1, \ldots, n\}.
\]

Now, let \( \pi^{*} : V^{*}\tau \to M \) be the dual vector bundle to \( \pi : V_{\tau} \to M \) and suppose that \( \pi_{1}^{*} : V^{*}\tau \to \mathbb{R} \) is the fibration defined by \( \pi_{1}^{*} = \tau \circ \pi^{*} \), that \( (\pi_{1}^{*})_{1,0} : J^{1}\pi_{1}^{*} \to V^{*}\tau \) is the 1-jet bundle of local sections of \( \pi_{1}^{*} : V^{*}\tau \to \mathbb{R} \) and that \( \rho_{J^{1}\tau} : J^{1}\tau \to TM \) is the anchor map of \( \tau_{1,0} : J^{1}\tau \to M \) (\( \rho_{J^{1}\tau} \) is the canonical inclusion of \( J^{1}\tau \) on \( TM \)). Then, one may introduce an isomorphism \( F \) (over the identity of \( V^{*}\tau \)) between the Lie affgebroids \( (\pi_{1}^{*})_{1,0} : J^{1}\pi_{1}^{*} \to V^{*}\tau \) and \( \pi_{V^{*}\tau} : \rho_{J^{1}\tau}(T(V^{*}\tau)) \to V^{*}\tau \) as follows. If \( j_{t}^{1} \psi \in J^{1}\pi_{1}^{*} \), with \( t \in I \) and \( \psi : I \subset \mathbb{R} \to V^{*}\tau \) is a local section of \( \pi_{1}^{*} : V^{*}\tau \to \mathbb{R} \), then there exists a unique \( z \in J^{1}\tau \) such that \( \rho_{J^{1}\tau}(z) = \)
\[(T_{\psi(t)}\pi^*)(\dot{\psi}(t))\) and we define
\[F(j_1^*\psi) = (z, (T_{\psi(t)}\pi^*)(\dot{\psi}(t))).\]

On the other hand, since the anchor map of \(\pi_M : TM \to M\) is the identity of \(TM\), it follows that the Lie algebroids \(\pi^*_M : \mathcal{L}^\pi(TM) \to V^*\tau\) and \(\pi_{V^*\tau} : T(V^*\tau) \to V^*\tau\) are isomorphic. Note that if \((t, q^i, p_i; \dot{q}^i, \dot{p}_i)\) are the local coordinates on \(T(V^*\tau)\) induced by \((t, q^i, p_i)\) then the local equation defining \(J^1\pi^*_1\) as an affine subbundle of \(\pi_{V^*\tau} : T(V^*\tau) \to V^*\tau\) is \(i = 1\). Therefore, \((t, q^i, p_i; \dot{q}^i, \dot{p}_i)\) is a system of local coordinates on \(J^1\pi^*_1\).

Next, let \(h\) be a Hamiltonian section, that is, \(h : V^*\tau \to (J^1\tau^1)^+ \cong T^*M\) is a section of the canonical projection of \(\mu : (J^1\tau^1)^+ \cong T^*M \to V^*\tau\). \(h\) is locally given by
\[h(t, q^i, p_i) = (t, q^i, -H(t, q^i, p_i), p_i).\]
Moreover, the cosymplectic structure \((\Omega, \eta)\) on the Lie algebroid \(\pi^*_M : \mathcal{L}^\pi(TM) \cong T(V^*\tau) \to V^*\tau\) is, in this case, the standard cosymplectic structure \((\Omega, \eta)\) on the manifold \(V^*\tau\) locally given by
\[\Omega_h = dq^i \wedge dp_i + \frac{\partial H}{\partial q^i} dq^i \wedge dt + \frac{\partial H}{\partial p_i} dp_i \wedge dt, \quad \eta = dt.\]
Thus, the Reeb section of \((\Omega_h, \eta)\) is the vector field \(R_h\) on \(V^*\tau\) defined by
\[R_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}.\]
It is clear that the integral sections of \(R_h\)
\[t \mapsto (t, q^i(t), p_i(t))\]
give just the solutions of the classical non-autonomous Hamilton equations
\[\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.\]

Moreover, if \(S_h\) is the Lagrangian submanifold of the symplectic Lie algebroid \(\pi_{V^*\tau} : \rho^*_J, (V^*\tau) \cong J^1\pi^*_1 \to V^*\tau\) given by \(S_h = R_h(V^*\tau)\) then the local equations defining \(S_h\) are
\[\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}\]
that is, the Hamilton equations for \(h\).

**Remark 9.1.** As we know (see Section 3.3), \((J^1\tau^1)^+ \cong T^*M\) is an affine bundle over \(V^*\tau\) of rank 1 modelled on the trivial vector bundle \(\tau V^*\tau \times \mathbb{R} \to V^*\tau\) and the affine bundle projection is the map \(\mu : (J^1\tau^1)^+ \cong T^*M \to V^*\tau\). Furthermore, the Hamiltonian section \(h\) induces an affine function \(F_h\) on \(T^*M\). In the particular case when the fibration \(\tau\) is trivial, that is, \(M = \mathbb{R} \times Q\) and \(\tau\) is the canonical projection on the first factor then \(T^*M \cong T^*\left(\mathbb{R} \times Q\right) \cong (\mathbb{R} \times \mathbb{R}) \times T^*Q, V^*\tau \cong \mathbb{R} \times T^*Q\) and under these identifications \(\mu\) is the projection given by
\[\mu(t, p, \alpha_q) = (t, \alpha_q), \quad \text{for} \quad \alpha_q \in T^*_q Q \quad \text{and} \quad (t, p) \in \mathbb{R} \times \mathbb{R}.\]
Thus, \(h\) may be considered as the Hamiltonian function \(H\) on \(\mathbb{R} \times T^*Q\). In addition, the affine function \(F_h \equiv F_H\) on \(T^*M \cong (\mathbb{R} \times \mathbb{R}) \times T^*Q\) is given by
\[F_h(t, p, \alpha_q) \equiv F_H(t, p, \alpha_q) = -H(t, \alpha_q) - p.\]
Note that $F_h \equiv F_H$ is, up to the sign, the classical extension of $H$ to a Hamiltonian function $H^\tau$ on the extended phase space $T^\tau(\mathbb{R} \times Q)$ (see [10] and the references therein).

Finally, if $L : J^1\tau \to \mathbb{R}$ is a time-dependent Lagrangian function then the local equations defining the corresponding Lagrangian submanifold $S_L$ are (see [11])
\begin{equation}
 p_i = \frac{\partial L}{\partial q^i},
\end{equation}
\begin{equation}
 p^j = \frac{\partial L}{\partial q^i}.
\end{equation}

Note that Eqs. (9.4) give the definition of the momenta and Eqs. (9.5) are just the Euler-Lagrange equations for $L$.

9.3. Atiyah affgebroids and nonautonomous Hamilton(Lagrange)-Poincaré equations.

9.3.1. Atiyah affgebroids. Let $p : Q \to M$ be a principal $G$-bundle. Denote by $\Phi : G \times Q \to Q$ the free action of $G$ on $Q$ and by $T\Phi : G \times TQ \to TQ$ the tangent action of $G$ on $TQ$. Then, one may consider the quotient vector bundle $\pi_Q|G : TQ/G \to M = Q/G$ and the sections of this vector bundle may be identified with the vector fields on $Q$ which are invariant under the action $\Phi$. Using that every $G$-invariant vector field on $Q$ is $p$-projectable and that the usual Lie bracket on vector fields is closed with respect to $G$-invariant vector fields, we can induce a Lie algebroid structure on $TQ/G$. This Lie algebroid is called the Atiyah algebroid associated with the principal $G$-bundle $p : Q \to M$ (see [10]).

Now, we suppose that $\nu : M \to \mathbb{R}$ is a fibration of $M$ on $\mathbb{R}$. Denote by $\tau : Q \to \mathbb{R}$ the composition $\tau = \nu \circ p$. Then, $\Phi$ induces an action $J^1\Phi : G \times J^1\tau \to J^1\tau$ of $G$ on $J^1\tau$ such that
\begin{equation}
 J^1\Phi(g, j^1_1\gamma) = j^1_1(\Phi_g \circ \gamma),
\end{equation}
for $g \in G$ and $\gamma : I \subset \mathbb{R} \to Q$ a local section of $\tau$, with $t \in I$. Moreover, the projection
\begin{equation}
 \tau_1,0|G : J^1\tau/G \to M, \quad [j^1_1\gamma] \mapsto p(\tau_1,0(j^1_1\gamma)) = p(\gamma(t))
\end{equation}
defines an affine bundle on $M$ which is modelled on the quotient vector bundle
\begin{equation}
 \pi|G : V\tau/G \to M, \quad [u_q] \mapsto p(q), \text{ for } u_q \in V_q\tau,
\end{equation}
$\pi : V\tau \to Q$ being the vertical bundle of the fibration $\tau : Q \to \mathbb{R}$. Here, the action of $G$ on $V\tau$ is the restriction to $V\tau$ of the tangent action $T\Phi$ of $G$ on $TQ$.

In addition, the bidual vector bundle of $J^1\tau/G \to M$ is $\pi_Q|G : TQ/G \to M$.

On the other hand, if $t$ is the usual coordinate on $\mathbb{R}$, the 1-form $\tau^*(dt)$ is $G$-invariant and defines a non-zero 1-cocycle $\phi : TQ/G \to \mathbb{R}$ on the Atiyah algebroid $TQ/G$. Note that $\phi^{-1}\{1\} \cong J^1\tau/G$ and therefore, one may consider the corresponding Lie affgebroid structure on $J^1\tau/G$ (see [10]). $J^1\tau/G$ endowed with this structure is called the Atiyah affgebroid associated with the principal $G$-bundle $p : Q \to M$ and the fibration $\nu : M \to \mathbb{R}$.

Now, let $K : TQ \to g$ be a connection in the principal bundle $p : Q \to M$. This connection will allow us to determine an isomorphism between the vector bundles $TQ/G \to M$ and $TM \oplus \mathfrak{g} \to M$. \[\text{D. IGLESIAS, J.C. MARRERO, E. PADRÓN, D. SOSA}\]}
Moreover, suppose that between the Lie affgebroids $J$ and $\phi$ denote by $G$ bundle and that as follows. Let $U$ be and let $TM$ be the principal bundle $g$ given by $\varphi$ that is, $\xi$ is a Lie algebroid isomorphism. In addition, the 1-cocycle $K$ $(\partial t, \partial x^e)$, that is, $\xi$ for $\{1\}$ may be identified with the affine bundle (over $M$) $J^1 \nu + \tilde{g}$. Thus, the affine bundle $J^1 \nu + \tilde{g} \rightarrow M$ is a Lie affgebroid and, from Proposition $I_K$ induces an isomorphism between the Lie affgeboids $J^1 \tau / G$ and $J^1 \nu + \tilde{g}$. Denote by $B: TQ \oplus TQ \rightarrow g$ the curvature of $K$. Then, we will obtain a local basis of $\Gamma(\pi_Q(G))$ as follows. Let $U \times G$ be a local trivialization of $p: Q \rightarrow M$, where $U$ is an open subset of $M$ and let $e$ be the identity element of $G$. Assume that there are local fibred coordinates $(t, x^i)$ in $U$ and that $\{\xi_a\}$ is a basis of $g$. Denote by $\{\xi^b_a\}$ the corresponding left-invariant vector fields on $G$, that is, $\xi^b_a(g) = (T_e L_g) (\xi_a)$, for $g \in G$, where $L_g: G \rightarrow G$ is the left translation by $g$. Moreover, suppose that

$$K\left(\frac{\partial}{\partial t}(x,e)\right) = K^i_0(x)\xi_a, \quad K\left(\frac{\partial}{\partial x^i}(x,e)\right) = K^i_0(x)\xi_a, \quad i \in \{1, \ldots, m\},$$

$$B^e_0 = \frac{\partial K^e_0}{\partial t} - \frac{\partial K^e_0}{\partial x^i} - K^a_0 K^b_i c^e_{ab}, \quad B^e_i = \frac{\partial K^e_i}{\partial t} - \frac{\partial K^e_i}{\partial x^j} - c^e_{ab} K^a_j K^b_i.$$

for $x \in U$. Note that if $\{c^e_{ab}\}$ are the structure constants of $g$ with respect to the basis $\{\xi_a\}$ then

Now, the horizontal lift of the vector fields $\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\right\}$ to $p^{-1}(U) \cong U \times G$ is given by

$$\left(\frac{\partial}{\partial t}\right)^h = \frac{\partial}{\partial t} - K_0^a \xi^L_a, \quad \left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - K_0^a \xi^L_a.$$

Therefore, the vector fields on $U \times G$

$$\left(9.6\right) \left\{e_0 = \frac{\partial}{\partial t} - K_0^a \xi^L_a, e_i = \frac{\partial}{\partial x^i} - K_0^a \xi^L_a, e_b = \xi^L_b\right\}$$

are $G$-invariant and they define a local basis $\left\{e_0, e_i, e_b\right\}$ of $\Gamma(\pi_Q(G))$. We will denote by $(t, x^i, \dot{t}, \dot{x}^i, \ddot{v}^a)$ (respectively, $(t, x^i, \dot{x}^i, \ddot{v}^a)$) the corresponding fibred coordinates on $TQ/G$ (respectively, $J^1 \tau / G$). Then, if $(\cdot, \cdot)_{TQ/G, \rho TQ/G}$ is the Lie algebra structure on $TQ/G$, we deduce that

$$\left[e'_0, e'_j\right]_{TQ/G} = -B^b_0 e'_c, \quad \left[e'_i, e'_j\right]_{TQ/G} = -B^b_i e'_c, \quad \left[e'_0, e'_a\right]_{TQ/G} = c^b_{ab} K^b_0 e'_c,$$

$$\left[e'_i, e'_a\right]_{TQ/G} = c^b_{ab} K^b_i e'_c, \quad \left[e'_i, e'_j\right]_{TQ/G} = c^b_{ab} K^b_i e'_c,$$

$$\rho_{TQ/G}(e'_0) = \frac{\partial}{\partial t}, \quad \rho_{TQ/G}(e'_i) = \frac{\partial}{\partial x^i}, \quad \rho_{TQ/G}(e'_a) = 0.$$
Thus, the (local) structure functions of the Lie algebroid $\pi_Q|G : TQ/G \to M$ with respect to a local trivialization are zero except the following

\begin{equation}
\begin{aligned}
C^a_{0j} &= -B^a_{0j}, & C^c_{0a} &= -C^c_{ab} = c^c_{ab} K^b_0, \\
C^a_{ij} &= -B^a_{ij}, & C^c_{ia} &= -C^c_{ai} = c^c_{ab} K^b_i, & C^c_{ab} &= c^c_{ab}, \\
\rho^0_j &= 1, & \rho^i_j &= \delta_{ij}.
\end{aligned}
\end{equation}

9.3.2. Nonautonomous Hamilton-Poincaré equations. Let $\pi^* : V^* \tau \to Q$ be the dual vector bundle to the vertical bundle $\pi : V\tau \to Q$. Then, the Lie group $G$ acts on $V^* \tau$ and one may consider the corresponding quotient vector bundle $\pi^*|G : V^* \tau/G \to M = Q/G$. It is easy to prove that this vector bundle is isomorphic to the dual vector bundle to $\pi|G : V\tau/G \to M$. Moreover, using that the action of $G$ on $V^* \tau$ is free, it follows that $V^* \tau$ is a principal $G$-bundle over $V^* \tau/G$ with bundle projection $p_{V^* \tau} : V^* \tau \to V^* \tau/G$. Thus, we have the corresponding Atiyah algebroid $\pi_V^*|G : J(V^* \tau)/G \to V^* \tau/G$ and, in addition, the exact 1-form $(\pi^*_1)(dt)$ on $V^* \tau$, which is $G$-invariant, induces a 1-cocycle $\tilde{\eta} : T(V^* \tau)/G \to \mathbb{R}$ on $\pi_V^*|G : T(V^* \tau)/G \to V^* \tau/G$. In fact, $\tilde{\eta}$ is given by

\begin{equation}
\tilde{\eta}([X_{\alpha_q}]) = X_{\alpha_q}(\pi^*_1),
\end{equation}

for $X_{\alpha_q} \in T_{\alpha_q}(V^* \tau)$ and $\alpha_q \in V^*_q$. Now, denote by $\mathcal{L}^{\pi^*|G}(TQ/G)$ the prolongation of the Atiyah algebroid $\pi_Q|G : TQ/G \to M$ over the fibration $\pi^*|G : V^* \tau/G \to M$ and by $\eta : \mathcal{L}^{\pi^*|G}(TQ/G) \to \mathbb{R}$ the 1-cocycle defined by (9.2). Then, we introduce the map

\begin{equation}
(p_{TQ} \circ T\pi^*, Tp_{V^* \tau}) : T(V^* \tau) \to \mathcal{L}^{\pi^*|G}(TQ/G) \subseteq TQ/G \times T(V^* \tau/G)
\end{equation}

given by

\begin{equation}
(p_{TQ} \circ T\pi^*, Tp_{V^* \tau})(X_{\alpha_q}) = (p_{TQ}((T_{\alpha_q}\pi^*))(X_{\alpha_q})), (T_{\alpha_q}p_{V^* \tau})(X_{\alpha_q}),
\end{equation}

for $X_{\alpha_q} \in T_{\alpha_q}(V^* \tau)$ and $\alpha_q \in V^*_q$, where $p_{TQ} : TQ \to TQ/G$ is the canonical projection. Next, we will consider the fibration $\nu \circ \pi^*|G : V^* \tau/G \to \mathbb{R}$ and the Atiyah algebroid associated with the principal $G$-bundle $p_{V^* \tau} : V^* \tau \to (V^* \tau)/G$ and the projection $\nu \circ \pi^*|G$. Since $\nu \circ \pi^*|G \circ p_{V^* \tau} = \pi^*_1$, it follows that the Atiyah algebroid is the quotient affine bundle $(\pi^*_1)_{1,0}|G : J^1\pi^*_1/G \to V^* \tau/G$, where $(\pi^*_1)_{1,0}|G$ is defined by

\begin{equation}
((\pi^*_1)_{1,0}|G)([j^1_t \gamma]) = [(\pi^*_1)_{1,0}(j^1_t \gamma)] = [\gamma(t)],
\end{equation}

for $\gamma : I \subseteq \mathbb{R} \to V^* \tau$ a local section of $\pi^*_1$ and $t \in I$. Note that the Lie algebroids $\tilde{\eta}^{-1}\{1\}$ and $(\pi^*_1)_{1,0}|G : J^1\pi^*_1/G \to V^* \tau/G$ may be identified in such a way that the bidual Lie algebroid to $(\pi^*_1)_{1,0}|G : J^1\pi^*_1/G \to V^* \tau/G$ also may be identified with $\pi_{V^* \tau}|G : T(V^* \tau)/G \to V^* \tau/G$. Under the above identifications, the 1-cocycle $1_{(J^1\pi^*_1/G)}$ is just $\tilde{\eta}$.

**Theorem 9.2.** (i) The map $(p_{TQ} \circ T\pi^*, Tp_{V^* \tau})$ induces an isomorphism $(p_{TQ} \circ T\pi^*, Tp_{V^* \tau})$, over the identity of $V^* \tau/G$, between the Lie algebroids $\pi_{V^* \tau}|G : T(V^* \tau)/G \to V^* \tau/G$ and $\tau_{TQ/G}^\pi : \mathcal{L}^{\pi^*|G}(TQ/G) \to V^* \tau/G$. Furthermore,

\begin{equation}
((p_{TQ} \circ T\pi^*, Tp_{V^* \tau}), Id)^* \eta = \tilde{\eta}.
\end{equation}
(ii) The restriction of the map \((p_TQ \circ T \pi^*, T \rho_{V^*})\) to \(J^1 \pi^*_1/G\) induces an isomorphism, over the identity of \(V^* \tau/G\), between the Atiyah affgebroid \((\pi_1)_{1,0}^*: J^1 \pi^*_1/G \to V^* \tau/G\) and the Lie affgebroid \(\pi_{V^*}(V^* \tau/G): \rho_{J^1 \tau/G}(T(V^* \tau/G)) \to V^* \tau/G\).

**Proof.** The cotangent lift of \(\Phi\) defines an action of \(G\) on \(T^*Q\) and it is clear that the canonical projection \(\mu: T^*Q \to V^* \tau\) is \(G\)-equivariant. Thus, \(\mu\) induces an epimorphism \(\mu: T^*Q/G \to V^* \tau/G\) between the quotient vector bundles \(T^*Q/G\) and \(V^* \tau/G\). Therefore, if \((\pi_Q|G)*: T^*Q/G \to M\) is the dual vector bundle to the Atiyah algebroid \(\pi_Q|G: TQ/G \to M\) and \(L^{(\pi_Q|G)*}(TQ/G)\) is the prolongation of \(\pi_Q|G: TQ/G \to M\) over the fibration \((\pi_Q|G)*: T^*Q/G \to M\), one may introduce the epimorphism

\[
(Id, T(\mu|G)) : L^{(\pi_Q|G)*}(TQ/G) \to L^{\pi_*|G}(TQ/G)
\]

between the Lie algebroids \(L^{(\pi_Q|G)*}(TQ/G) \subseteq TQ/G \times T(T^*Q/G)\) and \(L^{(\pi_*|G)}(TQ/G) \subseteq TQ/G \times T(V^* \tau/G)\) defined by

\[
(Id, T(\mu|G))((u_q, X_{[\alpha_q]})) = ([u_q], (T_{[\alpha_q]}(\mu|G))(X_{[\alpha_q]})),
\]

for \(u_q \in T_qQ\) and \(X_{[\alpha_q]} \in T_q(T^*Q/G)\), with \(\alpha_q \in T^*_qQ\).

The tangent map to \(\mu\), \(T\mu: T(T^*Q) \to T(V^* \tau)\), is also \(G\)-invariant with respect to the tangent actions of \(G\) on \(T(T^*Q)\) and \(T(V^* \tau)\). This implies that \(T\mu\) induces an epimorphism between the vector bundles \(\pi_{T^*Q}|G: T(T^*Q)/G \to T^*Q/G\) and \(\pi_{V^* \tau}|G: T(V^* \tau)/G \to V^* \tau/G\). In addition, since \(T\mu\) is an epimorphism over \(\mu\) between the Lie algebroids \(\pi_{T^*Q}: T(T^*Q) \to T^*Q\) and \(\pi_{V^* \tau}: T(V^* \tau) \to V^* \tau\), we deduce that the map \(T\mu|G: T(T^*Q)/G \to T(V^* \tau)/G\) is also an epimorphism over \(\mu|G: T^*Q/G \to V^* \tau/G\) between the Lie algebroids \(T(T^*Q)/G\) and \(T(V^* \tau)/G\).

Now, denote by \(\pi_Q^*: T^*Q \rightarrow Q\) the bundle projection, by \(p_TQ: T^*Q \rightarrow T^*Q/G\) the canonical projection and by \((p_TQ \circ T \pi^*_Q, T \rho_{V^*})\) \(T(T^*Q) \rightarrow L^{(\pi_Q|G)*}(TQ/G)\) the map defined by

\[
(p_TQ \circ T \pi^*_Q, T \rho_{V^*})(X_{\alpha_q}) = (p_TQ((T_{\alpha_q} \pi^*_Q)(X_{\alpha_q})), (T_{\alpha_q} p_TQ))(X_{\alpha_q}),
\]

for \(X_{\alpha_q} \in T_{\alpha_q}(T^*Q)\), with \(\alpha_q \in T^*_qQ\).

Then, this map induces an isomorphism \((p_TQ \circ T \pi^*_Q, T \rho_{V^*})\), over the identity of \(T^*Q/G\), between the Lie algebroids \(T(T^*Q)/G \rightarrow T^*Q/G\) and \(L^{(\pi_Q|G)*}(TQ/G) \rightarrow T^*Q/G\) (see Theorem 9.3 in [10]).

On the other hand, using \([99]\), it follows that the map \((p_TQ \circ T \pi^*, T \rho_{V^*}) : T(V^* \tau) \rightarrow L^{\pi_*|G}(TQ/G)\) induces a map \((p_TQ \circ T \pi^*, T \rho_{V^*}) : T(V^* \tau)/G \rightarrow L^{\pi_*|G}(TQ/G)\) between the spaces \(T(V^* \tau)/G\) and \(L^{\pi_*|G}(TQ/G)\). Furthermore, it is easy to prove that the following diagram is commutative

\[
\begin{array}{ccc}
T(T^*Q)/G & \xrightarrow{T\mu|G} & T(V^* \tau)/G \\
(p_TQ \circ T \pi^*_Q, T \rho_{V^*}) \downarrow & & \downarrow (p_TQ \circ T \pi^*, T \rho_{V^*}) \\
L^{(\pi_Q|G)*}(TQ/G) & \xrightarrow{(Id, T(\mu|G))} & L^{\pi_*|G}(TQ/G)
\end{array}
\]
Thus, the map \( (p_{TQ} \circ T \pi^*, T \pi_{\nu^*}) \) is an epimorphism over the identity of \( V^* \gamma / G \) between the Lie algebroids \( T(V^* \gamma) / G \to V^* \gamma / G \) and \( \mathcal{L}^{*} [G](TQ / G) \to V^* \gamma / G \). Therefore, using that the ranks of these Lie algebroids are equal, we deduce that the map \( (p_{TQ} \circ T \pi^*, T \pi_{\nu^*}) \) is an isomorphism, over the identity of \( V^* \gamma / G \), between the Lie algebroids. Moreover, from (3.2), (3.3) and (3.4), it follows (9.10). This proves (i).

On the other hand, using (i) and since \( \eta^{-1} \{1\} \) is the total space of the Lie affgebroid \( \pi \tau_{1,0}(T(V^* \gamma / G)) \to V^* \gamma / G \) (see Section 5), we deduce (ii).

As we know, the bidual Lie algebroid to \( \tau_{1,0} | G \) may be identified with the Atiyah algebroid \( \pi_Q | G : TQ / G \to M \). Thus, the vector bundles \( (J^1 \tau / G)^+ \to M \) and \( T^* Q / G \to M \) are isomorphic.

Now, suppose that \( h : V^* \gamma / G \to (J^1 \tau / G)^+ \to T^* Q / G \) is a Hamiltonian section. Then, we have that \( h \) induces a Hamiltonian section \( \tilde{h} : V^* \gamma \to T^* Q \) with respect to the Lie algebroid \( \tau_{1,0} : J^1 \tau \to Q \) (note that \( (T^* Q / G)_{p(q)} \to T^* Q, \) for all \( q \in Q \)). Moreover, using the \( G \)-equivariant character of \( \tilde{h} \), we deduce that the standard symplectic structure \( (\Omega_{\tilde{h}}, \tilde{\eta}) \) on \( V^* \gamma \) (see Section 4.2) is \( G \)-invariant. Thus, it induces a symplectic structure on the Lie algebroid \( \mathcal{L}^{*} [G](TQ / G) \cong T(V^* \gamma) / G \to V^* \gamma / G \). This symplectic structure is just \( (\Omega_{\tilde{h}}, \tilde{\eta}) \).

In addition, it is clear that the Reeb vector field \( R_{\tilde{h}} \) of \( (\Omega_{\tilde{h}}, \tilde{\eta}) \) is also \( G \)-invariant and, therefore, it induces a section of the Lie algebroid \( \mathcal{L}^{*} [G](TQ / G) \cong T(V^* \gamma) / G \to V^* \gamma / G \) which is just the Reeb section \( R_{\tilde{h}} \) of \( (\Omega_{\tilde{h}}, \tilde{\eta}) \). Consequently, the solutions of the Hamilton equations for \( h \) (that is, the integral curves on \( V^* \gamma / G \) of the vector field \( \pi \tau_{1,0}(R_{\tilde{h}}) \)) are just the solutions of the nonautonomous Hamilton-Poincaré equations for \( \tilde{h} \).

Next, we will obtain the expression of these equations. Let \( \{e'_0, e'_1, e'_a\} \) be the local basis of \( \Gamma(\pi_Q | G) \) considered in Section 3.6. Then, \( \{e'_0, e'_1, e'_a\} \) is a local basis of \( \Gamma(\pi | G) \) and may consider the corresponding local coordinates \( (t, x^i, p_i, \bar{p}_a) \) (respectively, \( (t, x^i, p_i, \bar{p}_a) \)) on \( V^* \gamma / G \) (respectively, \( T^* Q / G \)). Suppose that \( h \) is locally given by

\[ h(t, x^i, p_i, \bar{p}_a) = (t, x^i, -H(t, x^i, p_i, \bar{p}_a), p_i, \bar{p}_a). \]

Using (3.9) and (3.7), we get the Hamilton equations for \( h \) (the Hamilton-Poincaré equations for \( \tilde{h} \)),

\[
\begin{align*}
\frac{dx^i}{dt} &= \frac{\partial H}{\partial p_i} - \bar{p}_b \left( B_{0i}^{b} + B_{ki}^{b} \frac{\partial H}{\partial p_k} + c_{ac}^{b} K_{ac}^{b} \frac{\partial H}{\partial p_a} \right), \\
\frac{dp_i}{dt} &= \bar{p}_b \left( c_{ak}^{b} K_{ak}^{b} + c_{bk}^{a} K_{bk}^{a} \frac{\partial H}{\partial p_b} - c_{ab}^{a} \frac{\partial H}{\partial x^a} \right).
\end{align*}
\]

Now, we will obtain the local equations defining the Lagrangian submanifold \( S_{\tilde{h}} = R_{\tilde{h}}(V^* \gamma / G) \) of the symplectic Lie affgebroid \( \rho_{J^1 \gamma / G}(T(V^* \gamma / G)) \cong J^1 \gamma / G \). For this purpose, we consider local coordinates \( (t, x^i, p_i, \bar{p}_a, \bar{x}^i, \bar{p}_i, \bar{\bar{p}}_a) \) on \( \rho_{J^1 \gamma / G}(T(V^* \gamma / G)) \cong J^1 \gamma / G \).

Using (3.7) and (3.6), we get that the Reeb section \( R_{\tilde{h}} \) is given by

\[
R_{\tilde{h}} = \tilde{c}_0 + \frac{\partial H}{\partial p_i} \tilde{c}_i + \frac{\partial H}{\partial p_a} \tilde{c}_a - \left( \frac{\partial H}{\partial x^i} + B_{0i}^{b} \bar{p}_b + B_{ki}^{b} \bar{p}_b + c_{ac}^{b} K_{ac}^{b} \frac{\partial H}{\partial p_a} \right) \tilde{e}_i + \left( e_{ab}^{b} K_{0i}^{b} \bar{p}_c + e_{ab}^{b} K_{ak}^{b} \frac{\partial H}{\partial p_b} - c_{ab}^{a} \frac{\partial H}{\partial x^a} \right) \tilde{e}_a,
\]
where \( \{\tilde{e}_0, \tilde{e}_1, \tilde{e}_a, \tilde{e}_i, \tilde{e}_a\} \) is the local basis of sections of \( \mathcal{L}^{\tau^*_1}(TQ/G) \to V^*\tau/G \) induced by \( \{e^0, e^1, e^a\} \). Thus, the local equations defining the submanifold \( S_h \) of \( \rho_{J^1\tau/G}(T(V^*\tau/G)) \cong J^1\tau^*_1/G \)

\[
\tilde{v}^a = \frac{\partial H}{\partial p_a}, \quad \tilde{x}^i = \frac{\partial H}{\partial p_i}, \quad \tilde{p}_a = \frac{\partial H}{\partial x^a} \left( B_{10} + B_{1k} \frac{\partial H}{\partial p_k} + c_{ac}^{\tau} K_c^b \frac{\partial H}{\partial p_b} \right),
\]

In other words,

\[
\tilde{v}^a = \frac{\partial H}{\partial p_a}, \quad \tilde{x}^i = \frac{\partial H}{\partial p_i}, \quad \tilde{p}_a = \frac{\partial H}{\partial x^a} \left( B_{10} + B_{1k} \frac{\partial H}{\partial p_k} + c_{ac}^{\tau} K_c^b \frac{\partial H}{\partial p_b} \right).
\]

Note that Eqs. (9.11) are just the Hamilton-Poincaré equations for \( \tilde{h} \).

9.3.3. Nonautonomous Lagrange-Poincaré equations. The action \( J^1\Phi \) of \( G \) on \( J^1\tau \) is free and \( J^1\tau \) is the total space of a principal \( G \)-bundle over \( J^1\tau/G \) with bundle projection \( p_{J^1\tau} : J^1\tau \to J^1\tau/G \). Therefore, we have the corresponding Atiyah algebroid \( \pi_{J^1\tau}: G : T(J^1\tau)/G \to J^1\tau/G \) and, in addition, the exact 1-form \( \tau^*_1(dt) \) on \( J^1\tau \), which is \( G \)-invariant, induces a 1-cocycle \( \phi_0 : T(J^1\tau)/G \to \mathbb{R} \) on \( \pi_{J^1\tau} \). Here, \( \tau_1 : J^1\tau \to \mathbb{R} \) is the map \( \tau \circ \tau_{1,0} \). Thus, \( \tau \) is free and \( J^1\tau/G \) is \( G \)-invariant, induces a 1-cocycle \( \phi_0 \).

Now, denote by \( \mathcal{L}^{\tau^*_1}(TQ/G) \) the prolongation of the Atiyah algebroid \( \pi_Q : G : TQ/G \to M \) over the fibration \( \tau_{1,0} : J^1\tau/G \to \mathbb{R} \) and by \( \phi_0 : \mathcal{L}^{\tau^*_1}(TQ/G) \to \mathbb{R} \) the 1-cocycle defined by \( \phi_0 \). We recall that \( \phi_0^{-1}\{1\} \) is the Lie algebroid \( (\tau_1,0,G)^{(\tau_1,0,G)} : J^1\tau/G \to J^1\tau/G \) (see Section 4). Moreover, we may introduce the map

\[
(p_{TQ} \circ Tp_{J^1\tau}, Tp_{J^1\tau}) : T(J^1\tau) \to \mathcal{L}^{\tau^*_1}(TQ/G) \subseteq TQ/G \times T(J^1\tau/G)
\]

given by

\[
(p_{TQ} \circ Tp_{J^1\tau}, Tp_{J^1\tau})(X_{j^1\gamma}) = (p_{TQ}((T_{j^1\gamma} p_{J^1\tau})(X_{j^1\gamma})), (T_{j^1\gamma} p_{J^1\tau})(X_{j^1\gamma}))
\]

for \( X_{j^1 \gamma} \in \tau_{j^1 \gamma}(J^1\tau) \) and \( j^1 \gamma \in J^1\tau \), where \( p_{J^1\tau} : J^1\tau \to TQ \) is the anchor map of the Lie algebroid \( \tau_{1,0} : J^1\tau \to Q \).

On the other hand, we consider the fibration \( \nu \circ \tau_{1,0} : G : J^1\tau/G \to \mathbb{R} \) and the Atiyah algebroid associated with the principal \( G \)-bundle \( p_{J^1\tau} : J^1\tau \to J^1\tau/G \) and the projection \( \nu \circ \tau_{1,0} \). Since \( \nu \circ \tau_{1,0} \) is free and \( J^1\tau/G \) is \( G \)-invariant, we may identify \( \tau_{1,0} \) with the Atiyah algebroid \( J^1\tau/G \) by \( \tau_1 \).

Since \( \nu \circ \tau_{1,0} \) is free and \( J^1\tau/G \) is \( G \)-invariant, we may identify \( \tau_{1,0} \) with the Atiyah algebroid \( J^1\tau/G \) by \( \tau_1 \).

The above identifications, the 1-cocycle \( 1_{(J^1\tau)/G} \) is just \( \tilde{\phi}_0 \).
Theorem 9.3. (i) The map \((p_{TQ} \circ T \rho_{J^1\tau}, T p_{J^1\tau}) : T(J^1\tau) \to \mathcal{L}^{\tau,0}(T Q / G)\) induces an isomorphism \((p_{TQ} \circ T \rho_{J^1\tau}, T p_{J^1\tau})\), over the identity of \(J^1\tau / G\), between the Lie algebroids \(\pi_{J^1\tau} : T(J^1\tau) / G \to J^1\tau / G\) and \((\pi_Q (\tau / G)^{\tau,0}) : \mathcal{L}^{\tau,0}(T Q / G) \to J^1\tau / G\). Furthermore, 
\[ ((p_{TQ} \circ T \rho_{J^1\tau}, T p_{J^1\tau}), Id)^* \phi_0 = \tilde{\phi}_0. \]

(ii) The restriction of the map \((p_{TQ} \circ T \rho_{J^1\tau}, T p_{J^1\tau})\) to \((J_1\tau_1) / G\) induces an isomorphism, over the identity of \((J^1\tau) / G\), between the Atiyah algebroid \((\tau_1)_{1,0} : (J^1\tau_1) / G \to J^1\tau / G\) and the Lie algebroid \((\tau_1,0) : \mathcal{J}(\tau / G)^{\tau,0} : J^1\tau / G \to J^1\tau / G\).

Proof. (i) From \cite{IGLESIAS} it follows that \((p_{TQ} \circ T \rho_{J^1\tau}, T p_{J^1\tau})\) induces a morphism, over the identity of \(J^1\tau / G\), between the Lie algebroids \(\pi_{J^1\tau} : T(J^1\tau) / G \to J^1\tau / G\) and \((\pi_Q (\tau / G)^{\tau,0}) : \mathcal{L}^{\tau,0}(T Q / G) \to J^1\tau / G\). We denote this morphism by \((p_{TQ} \circ T \rho_{J^1\tau}, T p_{J^1\tau})\). Moreover, proceeding as in the proof of Theorem 9.1 in \cite{IGLESIAS} and using the G-equivariant character of \(\rho_{J^1\tau}\), we deduce (i).

(ii) It follows using (i). \(\square\)

Now, suppose that \(l : J^1\tau / G \to \mathbb{R}\) is a Lagrangian function. Then, we will denote by \(L : J^1\tau \to \mathbb{R}\) the function given by \(L = l \circ p_{J^1\tau}\). Since \(L\) is a G-invariant Lagrangian function, we deduce that the Poincaré-Cartan 2-form \(\Omega_l\) on \(J^1\tau\) is also G-invariant. Thus, it induces a section of the vector bundle \(\wedge^2 (L^{\tau,0}(T Q / G))^* \cong \wedge^2 (T^* (J^1\tau) / G) \to J^1\tau / G\). This section is just the Poincaré-Cartan 2-section \(\Omega_l\) associated with \(l\). Therefore, the solutions of the Euler-Lagrange equations for \(l\) are just the solutions of the nonautonomous Lagrange-Poincaré equations for \(L\).

Next, we will obtain the expression of these equations. Let \(\{e_0, e_i, e_a\}\) be the local basis of G-invariant vector fields on \(Q\) defined as in \cite{JCG} and \((t, x^i, \dot{x}^i, \dot{v}^a)\) (respectively, \((t, x^i, \dot{t}, \dot{x}^i, \dot{v}^a)\)) be the corresponding local fibred coordinates on \(J^1\tau / G\) (respectively, \(J^1\tau / G = T Q / G\)). Using \cite{IGLESIAS} and \cite{MARRERO}, we obtain the Euler-Lagrange equations for \(l\) (that is, the Lagrange-Poincaré equations for \(L\)),

\[
\frac{\partial l}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial l}{\partial x^i} \right) = B^b_{ij} \frac{\partial l}{\partial \dot{v}^b} + B^b_{ij} \frac{\partial l}{\partial \dot{v}^b} + c_{ac} K^c_{ij} \dot{x}^j - c_{ad} \dot{v}^d \right),
\]

\[
\dot{p}_a = \dot{p}_b \left( c_{ac} K^c_{ij} + c_{ac} K^c_{ij} \dot{x}^j - c_{ad} \dot{v}^d \right).
\]

On the other hand, we consider the local coordinates \((t, x^i, \dot{p}_i, \dot{x}^i, \dot{v}^a, \dot{p}_i, \dot{v}_a)\) on \(J^1\pi^*_1 / G \cong \rho_{J^1\tau} / G(T V^* \tau / G))\). Then, the local equations defining the Lagrangian submanifold \(S_t\) are (see \cite{MARRERO})

\[
p_i = \frac{\partial l}{\partial \dot{x}^i}, \quad \dot{p}_a = \frac{\partial l}{\partial \dot{v}^a}, \quad \dot{p}_i = \frac{\partial l}{\partial \dot{x}^i} + B^b_{ij} \frac{\partial l}{\partial \dot{v}^b} + B^b_{ij} \dot{x}^j - c_{ac} K^c_{ij} \dot{x}^j - c_{ad} \dot{v}^d \right),
\]

\[
\dot{p}_a = \frac{\partial l}{\partial \dot{v}^a} \left( c_{ac} K^c_{ij} + c_{ac} K^c_{ij} \dot{x}^j - c_{ad} \dot{v}^d \right).
\]
or, in other words,

\begin{equation}
\label{9.13}
p_i = \frac{\partial l}{\partial \dot{x}^i}, \quad \bar{p}_a = \frac{\partial l}{\partial \bar{v}^a},
\end{equation}

\begin{equation}
\label{9.14}
\frac{\partial l}{\partial x^i} - \frac{dp_i}{dt} = \bar{p}_b \left( B^b_{0i} + B^b_{j} \dot{x}^j + c^b_{dl} K^{c}_{d} \bar{v}^{d} \right),
\end{equation}

\begin{equation}
\dot{\bar{p}}_a = \bar{p}_b \left( c^b_{ac} K^{c}_{0} + c^b_{ac} K^{c}_{j} \dot{x}^j - c^b_{ad} \bar{v}^{d} \right).
\end{equation}

Eqs. \(9.13\) give the definition of the momenta and Eqs. \(9.14\) are just the Lagrange-Poincaré equations for \(L\).
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