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THE CONVOLUTION IN ANISOTROPIC BESOV SPACES

We study the boundedness of the convolution operator in Nikol’skii-Besov anisotropic spaces $B^{\alpha q p \tau}_{\mu \nu \rho}$. These spaces are constructed on the basis of anisotropic Lorentz spaces $L^p_\tau$, where $p$ и $\tau$ are vector parameters. The properties of anisotropic Nikol’skii-Besov spaces are investigated. The main goal of the paper is to solve the following problem: let $f$ and $g$ be functions from some classes of the Nikol’skii-Besov space scale. It is necessary to determine which space belongs to their convolution $f \ast g$. We proved the inequality of different Nikol’skii metrics for trigonometric polynomials with spectrum in binary blocks in anisotropic Lorentz spaces $L^p_\tau$. Conditions are obtained in terms of the corresponding vector parameters $\alpha, p, q, \tau, \mu, \beta, \eta, h, \nu, \gamma, \xi$, which are necessary and sufficient conditions for embeddings

$$B^{\delta \eta}_{\mu \nu} \ast B^{\gamma \xi}_{\mu \nu} \hookrightarrow B^{\alpha q p \tau}_{\mu \nu \rho}.$$  

This statement is an analogue of O’Nei inequality for Lorentz spaces. In particular, the classical O’Neil inequality follows from the proved results. The obtained criterion is generalized by the results of Burenkov and Batyrov, who considered this problem in Besov spaces with scalar parameters.

Key words: Young-O’Neil inequality, anisotropic Besov spaces, convolution operator.

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Свертка в анизотропных пространствах Бесова

В работе исследуется ограниченность оператора свертки в анизотропных пространствах Никольского-Бесова $B^p_{\alpha q \tau}$. Данные пространства построены на основе анизотропных пространств Лоренца $L^p_{\tau r}$. Докажено неравенство разных метрик Никольского для тригонометрических полиномов со спектром в двойных пачках в анизотропных пространствах Лоренца $L_{pr}$. Получены условия в терминах соответствующих векторных параметров $\alpha, p, q, \tau, r, \mu, \eta, h, \nu, \gamma, \xi$, являющихся необходимыми и достаточными условиями для вложений $B^p_{\alpha q} \rightarrow B^q_{\beta \eta} h^r \mu^* B^q_{\gamma \xi} h^\nu \eta^* B^p_{\beta \eta}$.

Данное утверждение является аналогом неравенства О’Нейла для пространств Лоренца. В частности, из доказанных результатов следует классическое неравенство О’Нейла. Полученный критерий обобщает результаты Буренкова и Батырова, которые рассмотрели данную задачу в пространствах Бесова со скалярными параметрами.

**Ключевые слова:** неравенство Юнга-О’Нейла, анизотропные пространства Бесова, оператор свертки.

1 Introduction and review of literature

Let $I$ be either a $n$-dimensional torus $\mathbb{T}^n = [0, 1)^n$, or a Euclidean space $\mathbb{R}^n$. Let $f(x)$ and $g(x)$ be determined and measurable functions on $I$ with respect to the $n$-dimensional Lebesgue measure such that for almost all $x \in I$ there exists an integral

$$
\int_I f(x - y)g(y)dy.
$$

In this case, it is said that the convolution of these functions is defined

$$(f \ast g)(x) = \int_I f(x - y)g(y)dy. \tag{1}
$$

The classical Young’s inequality [1, 199] has the form: suppose

$$
1 \leq p, r, q \leq \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}. \tag{2}
$$

If $f \in L_p(I)$, $g \in L_r(I)$, then almost everywhere in $I$ there exists a convolution $f \ast g$, belonging to the space $L_q(I)$ and the following inequality holds

$$
\|f \ast g\|_{L_q(I)} \leq \|f\|_{L_p(I)}\|g\|_{L_r(I)}. \tag{3}
$$

We write this statement in the form of a relation

$L_p(I) \ast L_r(I) \hookrightarrow L_q(I)$. 

These inequalities play an important role in harmonic analysis and in the theory of partial differential equations \([1–3]\).

If

\[ 1 < p, r, q < \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}, \]

then for \(g_0(x) = \frac{1}{|x|^r}\) the inequality holds

\[ \|f * g_0\|_{L_q(I)} \leq C\|f\|_{L_p(I)}. \]

This inequality is called the Hardy-Littlewood-Sobolev inequality. It does not follow from Young’s inequality, since \(\|g_0\|_{L_r(I)} = \infty\). Generalization of inequality (3) obtained by O’Neil \([4]\) (see also \([5, 6]\)).

If (4) is true and \(0 < s_1, s_2, s \leq \infty, \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}\), then

\[ L_{ps_1} * L_{rs_2} \hookrightarrow L_s \]

and in particular

\[ L_p * L_{r\infty} \hookrightarrow L_q, \]

where \(L_{ps}\) is Lorentz space.

Note that in relation (5), condition (4) is essential. The limiting cases of the O’Neil inequality with condition (2) were considered in \([7]\).

The O’Neil inequality for anisotropic Lorentz spaces was studied in \([8–10]\). In the case of \(n \geq 2\) these results are extended the inequality (6). In the one-dimensional case, the O’Neil inequality was extended in \([11, 12]\).

There are generalizations of the Young and O’Neil inequalities for various functional spaces: weighted \(L_p\) spaces, classical and Lorentz weighted spaces, Hardy spaces, Wiener spaces, Orlicz spaces; see \([5, 6, 8, 13–18]\), and references therein.

Convolution operators were studied in spaces of smooth functions in \([19–24]\).

V.I. Burenkov and B.E. Batyrov in \([21]\) received the following statement: Let \(-\infty < l_1, l_2, l_3 < \infty, 0 < p_1, p_2, p_3 \leq \infty, 0 < \theta_1, \theta_2, \theta_3 \leq \infty\). In order for there to exist a number \(c_3 > 0\) such that for any \(f_1 \in B_{p_1, \theta_1}^{l_1}(\mathbb{R}^n), f_2 \in B_{p_2, \theta_2}^{l_2}(\mathbb{R}^n)\) such that \(Ff_1\) and \(Ff_2\) are regular generalized functions and their (pointwise) product \(Ff_1 \cdot Ff_2 \in S(\mathbb{R}^n)\), the inequality

\[ \|f_1 * f_2\|_{B_{p_3, \theta_3}^{l_3}(\mathbb{R}^n)} \leq c_3 \|f_1\|_{B_{p_1, \theta_1}^{l_1}(\mathbb{R}^n)} \|f_2\|_{B_{p_2, \theta_2}^{l_2}(\mathbb{R}^n)} \]

it is necessary and sufficient that the following conditions be fulfilled:

1) \(p_3 \geq p_1, p_3 \geq p_2\);
2) \(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1 \geq 0\);

and one of the conditions

3a) \(l_3 < l_1 + l_2 - n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1\right)\)

or
3b) \( l_3 = l_1 + l_2 - n \left( \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1 \right) \) \( \frac{1}{\theta_3} \leq \frac{1}{\theta_1} + \frac{1}{\theta_2} \),

where \( Ff \) is the Fourier transform of the function \( f \):

\[
(Ff)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(\xi) d\xi.
\]

For \( p_2 = p_3, \theta_2 = \theta_3, 0 < l_2 < l_3 < \infty \) inequality (7) and some of its generalizations follow from the results obtained in the works of K.K. Golovkin and V.A. Solonnikov [19, 20] and [23].

In [24] we investigated the boundedness of the norm of the convolution operator in Sobolev spaces, with the dominant mixed derivative and anisotropic Nikol’skii-Besov spaces. For Sobolev spaces with the dominant mixed derivative, an analogue of Young’s inequality is obtained, namely, relations of the form

\[
W_p^r * W_q^s \hookrightarrow W_\alpha^t
\]

are proved when the corresponding conditions on the parameters are satisfied. Using relation (8) and the Nursultanov interpolation theorem for anisotropic spaces, an analogue of the O’Neil theorem was obtained for the Nikol’skii-Besov space scale \( B_{pq}^\alpha \), where \( \alpha, p, q \) are vector parameters. Relations of the form \( B_{p_1}^\alpha * B_{q_2}^\beta \hookrightarrow B_{q_3}^\gamma \) are obtained, with the corresponding ratios of vector parameters.

The theorems obtained in [24] complement the results of Batyrov and Burenkov, where similar problems were considered in isotropic Nikol’skii-Besov spaces, that is, in spaces where the parameters are scalars.

2 Material and methods

Let \( \alpha \in \mathbb{R}^2, 0 < q = (q_1, q_2), \tau = (\tau_1, \tau_2) \leq \infty, 1 \leq p = (p_1, p_2) < \infty, \mathbb{T}^2 = [0, 1)^2. \)

We denote the space \( B_{pr}^{\alpha}(\mathbb{T}^2) \) as the set of all trigonometric series \( f = \sum_{m \in \mathbb{Z}^2} a_m(f) e^{2\pi i (m \cdot x)} \)

(generally speaking, divergent) for which

\[
\|f\|_{B_{pr}^{\alpha}(\mathbb{T}^2)} = \left( \sum_{k_2=0}^{\infty} \left( \sum_{k_1=0}^{\infty} (2^{\alpha_1 k_1 + \alpha_2 k_2} \|\Delta_k f\|_{L_{p_1, p_2}^{\tau_1, \tau_2}(\mathbb{T}^2)})^{q_1} \right)^{q_2/q_1} \right)^{1/q_2}
\]

is finite, are called as Besov type spaces \( B_{pr}^{\alpha}(\mathbb{T}^2) \), where

\[
\Delta_k f(x_1, x_2) = \sum_{2^{k_2-1} \leq |m_2| < 2^{k_2}} \sum_{2^{k_1-1} \leq |m_1| < 2^{k_1}} a_{m_1, m_2}(f) e^{2\pi i (m_1 x_1 + m_2 x_2)},
\]

\( k \in \mathbb{N} \). In the isotropic case, these spaces were investigated in [25], where the interpolation properties were studied.

We define the concept of convolution for the elements of this spaces.

Let \( f = \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} a_{k_1, k_2} e^{2\pi i (k_1 x_1 + k_2 x_2)} \) and \( g = \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} b_{k_1, k_2} e^{2\pi i (k_1 x_1 + k_2 x_2)} \) be trigonometric series. By convolution of these series we mean the series

\[
(f * g)(y_1, y_2) = \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} a_{k_1, k_2}(f) b_{k_1, k_2}(g) e^{2\pi i (k_1 y_1 + k_2 y_2)}.
\]
Note that for the "good" functions \( f \) and \( g \), the convolution defined by equality (9) coincides with the classical definition (1). If the functions \( f \) and \( g \) from the corresponding spaces in (3), then \( f(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)e^{2\pi i(k,x)} \) and \( g(x) = \sum_{k \in \mathbb{Z}^n} \hat{g}(k)e^{2\pi i(k,x)} \) and

\[
(f * g)(x) = \int_{\mathbb{T}^n} f(x-y)g(y)dy = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)\hat{g}(k)e^{2\pi i(k,x)}.
\]

Here, equalities are understood in the sense of the corresponding metrics.

Lemma 1

\[
\Delta_k(f * g)(y_1, y_2) = \int_0^1 \int_0^1 \Delta_k f(x_1, x_2)\Delta_k g(y_1 - x_1, y_2 - x_2)dx_1dx_2.
\]

Lemma 2 (\cite{10}) Let \( 1 < q, p, r < \infty, 1 \leq h, \xi, \eta < \infty \), and \( 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} \), \( \frac{1}{h} = \frac{1}{\xi} + \frac{1}{\eta} \). Suppose that \( f \) and \( K \) are respectively measurable on \([0,1]^2\) and \([-1,1]^2\) functions such that \( f \in L_{pq}(\mathbb{R}^2) \) and \( K \in L_{rh}\mathbb{R}^2 \). Then \( f * K \in L_{qh}\mathbb{R}^2 \) and

\[
\|f * K\|_{L_{qh}} \leq 4(q'_2g'_2)^2\|f\|_{L_{pq}}\|K\|_{L_{rh}}.
\]

Lemma 3 Let \( 0 < h, \xi, \eta \leq \infty, 1 < p, r, q < \infty \)

1. If \( \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r_1} \) and \( \frac{1}{h_1} = \frac{1}{\xi} + \frac{1}{\eta_1} \), then the following inequality

\[
\|f *_1 g\|_{L_{(q_1, q_2), (h_1, h_2)}} \leq C\|f\|_{L_{(p_1, q_2), (\xi_1, h_2)}}\|g\|_{L_{(q_1, r_2), (\eta_1, h_2)}}
\]

holds for the transformation

\[
(f *_1 g)(x_1, x_2) = \int_{-\infty}^{\infty} f(y_1, x_2)g(x_1 - y_1, x_2)dy_1.
\]

2. If \( \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r_2} \) and \( \frac{1}{h_2} = \frac{1}{\xi} + \frac{1}{\eta_2} \), then the following inequality

\[
\|f *_2 g\|_{L_{(q_1, q_2), (h_1, h_2)}} \leq C\|f\|_{L_{(p_1, r_2), (\xi_1, h_2)}}\|g\|_{L_{(q_1, r_2), (\eta_1, h_2)}}
\]

holds for the transformation

\[
(f *_2 g)(x_1, x_2) = \int_{-\infty}^{\infty} f(x_1, y_2)g(x_1, x_2 - y_2)dy_2.
\]
Proof. The proof of the lemma is similar to the proof of Theorem 3.1 from [10]. We note here that the application of the classical O’Neil inequality in one variable does not give the statement we need.

Lemma 4 Let

\[ T_k(x_1, x_2) = \sum_{2^{k_2-1} \leq |m_2| < 2^{k_2}} \sum_{2^{k_1-1} \leq |m_1| < 2^{k_1}} a_{m_1, m_2} e^{2\pi i (m_1 x_1 + m_2 x_2)}. \]

Let \( 1 \leq p, q < \infty, 0 < \tau \leq \infty, \theta_i = \frac{1}{p_i} - \frac{1}{q_i} \geq 0, \frac{1}{\tau_i} = \frac{1 - \text{sgn} \theta_i}{\tau_i}, \quad i = 1, 2, \)
then

\[ \|T_k\|_{L_{qr}} \leq C 2^{k_1 \theta_1 + k_2 \theta_2} \|T_k\|_{L_{pq}}. \]

Proof. Let \( \theta_i > 0, i = 1, 2. \) Note that for \( T_k \) there is a representation

\[ T_k(x) = T_k * D_k, \]

where \( D_k(x_1, x_2) = \sum_{2^{k_2-1} \leq |m_2| < 2^{k_2}} \sum_{2^{k_1-1} \leq |m_1| < 2^{k_1}} e^{2\pi i (m_1 x_1 + m_2 x_2)}. \)

Using Lemma 2, we have

\[ \|T_k\|_{L_{qr}} \leq C \|T_k\|_{L_{pq}} \|D_k\|_{L_{rr}}, \]

where \( \frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}. \)

We also note that

\[ D_{k_1, k_2}^{s_1, s_2}(t_1, t_2) \leq C \min \left( 2^{k_1}, \frac{1}{t_1} \right) \min \left( 2^{k_2}, \frac{1}{t_2} \right) \]

and therefore

\[ \|D_k\|_{L_{rr}} \asymp 2^{k_1 \left( \frac{1}{p_1} - \frac{1}{r_1} \right) + k_2 \left( \frac{1}{p_2} - \frac{1}{r_2} \right)}. \]

Let now \( \theta_1 > 0, \theta_2 = 0. \) Then

\[ T_k(x_1, x_2) = \int_{-\infty}^{\infty} T_k(y_1, x_2) D_{k_1}(x_1 - y_1) dy_1, \]

where \( D_{k_1}(x_1) = \sum_{2^{k_1-1} \leq |m_1| < 2^{k_1}} e^{2\pi i x_1 m_1}. \)

Further, applying the Lemma 3, we derive

\[ \|T_k\|_{L_{(q_1, q_2), (r_1, r_2)}} \leq C \|T_k\|_{L_{(p_1, q_2), (\infty, r_2)}} \|D_{k_1}\|_{L_{r_1, r_2}} \asymp 2^{k_1 \theta_1} \|T_k\|_{L_{(p_1, q_2), (\infty, r_2)}}, \]

where \( \frac{1}{r_1} = 1 + \frac{1}{q_1} - \frac{1}{p_1}. \)

The case \( \theta_2 > 0, \theta_1 = 0 \) is considered similarly.

The case \( \theta_1 = \theta_2 = 0 \) is obvious.
Lemma 5 Let \( \alpha, \beta \in \mathbb{R}^2 \), \( 1 \leq p, q \leq \infty \), \( 0 < \tau \leq \infty \). Let \( \alpha - \frac{1}{q} = \beta - \frac{1}{p} \), \( \alpha - \beta = \theta \geq 0 \), then

\[
B^\alpha_{qt} \hookrightarrow B^\beta_{pt},
\]

where \( \frac{1}{t} = \frac{1 - \text{sgn} \theta}{\tau} \).

Proof. Let \( \theta_i > 0, i = 1, 2 \). Let \( f \in B^\alpha_{qt} \). Using Lemma 4, we have

\[
\|f\|_{B^\alpha_{qt}} = \left( \sum_{k=0}^{\infty} \left( \sum_{k_1=0}^{\infty} \left( 2^{\beta_1 k_1 + \beta_2 k_2} \|\Delta_k f\|_{L^p_{\tau}} \right)^{s_2/s_1} \right)^{1/s_2} \right)^{1/s_1} \leq C \sum_{k=0}^{\infty} \left( \sum_{k_1=0}^{\infty} \left( 2^{\beta_1 k_1 + \beta_2 k_2} \|\Delta_k f\|_{L^p_{\tau}} \right)^{s_2/s_1} \right)^{1/s_2} = C\|f\|_{B^\alpha_{qt}}.
\]

Other cases are checked similarly.

Lemma 6 Let \( \alpha, \tilde{\alpha} \in \mathbb{R}^2 \), \( 1 \leq p, \tilde{p} < \infty \), \( \theta' = \tilde{\alpha} - \alpha \geq 0 \), \( \theta'' = \tilde{p} - p \geq 0 \), \( 0 < q, \tau \leq \infty \). Then

\[
B^{\tilde{\alpha}q}_{p^\tau} \hookrightarrow B^{\alpha q}_{p^\tau},
\]

where \( \frac{1}{q} = \frac{1 - \text{sgn} \theta'}{q}, \frac{1}{\tau} = \frac{1 - \text{sgn} \theta''}{\tau} \).

Proof. The proof follows from the embeddings \( L^{\tilde{\alpha}q}_{p^\tau} \hookrightarrow L^{\alpha q}_{p^\tau} \) and \( L^p_{\tau} \hookrightarrow L^p_{\tau} \).

Lemma 7 Let \( \alpha, \tilde{\alpha} \in \mathbb{R}^2 \), \( 1 \leq p, \tilde{p} < \infty \), \( 0 < q, \tau \leq \infty \), \( \delta' = \left( \tilde{\alpha} - \frac{1}{\tilde{p}} \right) - \left( \alpha - \frac{1}{p} \right) \geq 0 \), \( \theta' = \tilde{\alpha} - \alpha \geq 0 \). Then

\[
B^{\tilde{\alpha}q}_{p^\tau} \hookrightarrow B^{\alpha q}_{p^\tau},
\]

where \( \frac{1}{\tau} = \frac{(1 - \text{sgn} \delta')(1 - \text{sgn} \theta')}{t}, \frac{1}{q} = \frac{1 - \text{sgn} \delta' \text{sgn} \theta'}{q} \).

Proof. Let the conditions of the lemma be satisfied. Then there are \( \tilde{\alpha} \) and \( \tilde{p} \) such that

\[
\tilde{\alpha} - \frac{1}{\tilde{p}} = \tilde{\alpha} - \frac{1}{p}, \quad \tilde{\alpha} \geq \alpha \geq \alpha, \quad \tilde{p} \geq p.
\]

Moreover, for \( \delta_i > 0, \theta_i > 0, \tilde{\alpha}_i > \alpha_i > \alpha, \tilde{p}_i > p \) Lemma 6 implies the embedding

\[
B^{\tilde{\alpha}q}_{p^\tau} \hookrightarrow B^{\alpha q}_{p^\tau}.
\]
where
\[ \frac{1}{q} = \frac{1 - \text{sgn}(\tilde{\alpha} - \alpha)}{q}, \quad \frac{1}{\tau} = \frac{1 - \text{sgn}(\tilde{p} - p)}{\tau}. \]

Applying Lemma 5, we have
\[ B_{p\tau}^{\tilde{\alpha} \tilde{q} \tilde{\tau}} \hookrightarrow B_{p\tau}^{\alpha q \tau}, \]
where
\[ \frac{\tilde{\alpha} - 1}{q} = \frac{\tilde{\alpha} - 1}{q} \geq \frac{\alpha - 1}{p}. \]
\[ \frac{1}{\tilde{\tau}} = \frac{(1 - \text{sgn}(\tilde{\alpha} - \alpha))}{\tau} = \frac{(1 - \text{sgn}(\tilde{\alpha} - \alpha))(1 - \text{sgn}(\tilde{p} - p))}{\tau}, \]
\[ \frac{1}{q} = \frac{1 - \text{sgn}(\tilde{\alpha} - \alpha)}{q}. \]

We note that \((\tilde{\alpha}_i - \alpha_i) > 0\) if and only if \(\delta'_i > 0\) и \(\theta'_i > 0\), i.e.
\[ \text{sgn}(\tilde{\alpha}_i - \alpha_i) = \text{sgn} \delta'_i \cdot \text{sgn} \theta'_i. \]

And \(\tilde{\alpha}_i - \alpha_i = 0, \tilde{p}_i - p_i = 0\) if and only if \(\delta_i = 0, \theta_i = 0\), which means
\[ (1 - \text{sgn}(\tilde{\alpha} - \alpha))(1 - \text{sgn}(\tilde{p} - p)) = (1 - \text{sgn} \delta')(1 - \text{sgn} \theta'). \]

**Theorem 1** Let \(\alpha, \beta, \gamma \in \mathbb{R}^2\), \(1 < p, r, h < \infty\), \(0 < \tau, \mu, \nu, q, \xi, \eta \leq \infty\). In order for the inequality
\[ \|f^* g\|_{B_{p\tau}^{\alpha q \tau}(0,1)^2} \leq C \|f\|_{B_{p\tau}^{\beta \eta}(0,1)^2} \|g\|_{B_{h \nu}^{\gamma \xi}(0,1)^2} \] (11)
to hold for \(f \in B_{p\tau}^{\beta \eta}(0,1)^2\) and \(g \in B_{h \nu}^{\gamma \xi}(0,1)^2\) it is necessary and sufficient that the following conditions are met:
\[ \theta = \beta + \gamma - \alpha \geq 0; \] (12)
\[ \delta = \beta + \gamma - \alpha + 1 + \frac{1}{p} - \frac{1}{r} - \frac{1}{h} \geq 0; \] (13)
\[ \frac{(1 - \text{sgn} \delta)(1 - \text{sgn} \theta)}{\tau} \leq \frac{1}{\mu} + \frac{1}{\nu}; \] (14)
\[ \frac{(1 - \text{sgn} \delta \text{sgn} \theta)}{q} \leq \frac{1}{\xi} + \frac{1}{\eta}. \] (15)

**Proof.** Let conditions (12)-(15) be satisfied. In this case, there are \(\tilde{p}, \tilde{\alpha}\) such that \(\beta + \gamma \geq \tilde{\alpha} \geq \alpha, \tilde{p} \geq p\) и \(\beta + \gamma - \tilde{\alpha} + 1 + \frac{1}{\tilde{p}} - \frac{1}{r} - \frac{1}{h} = 0\). Moreover, if \(\delta_i > 0, \theta_i > 0\), then \(\beta_i + \gamma_i > \tilde{\alpha}_i > \alpha_i, \tilde{p}_i > p\). Applying Lemma 7, we have
\[ B_{p\tau}^{\tilde{\alpha} \tilde{q} \tilde{\tau}} \hookrightarrow B_{p\tau}^{\alpha q \tau}, \]
where
\[
\frac{1}{\tau} = 1 - \text{sgn}\left(\frac{\tilde{\alpha} - \frac{1}{p}}{p} - \frac{\alpha - \frac{1}{p}}{p}\right) (1 - \text{sgn}(\tilde{\alpha} - \alpha))
\]
\[
\frac{1}{\tilde{q}} = 1 - \text{sgn}\left(\frac{\tilde{\alpha} - \frac{1}{p}}{p} - \frac{\alpha - \frac{1}{p}}{p}\right) \text{sgn}(\tilde{\alpha} - \alpha)
\]
We note, that
\[
\left(1 - \text{sgn}\left(\frac{\tilde{\alpha} - \frac{1}{p}}{p} - \frac{\alpha - \frac{1}{p}}{p}\right)\right) (1 - \text{sgn}(\tilde{\alpha} - \alpha)) = (1 - \text{sgn} \delta)(1 - \text{sgn} \theta),
\]
\[
1 - \text{sgn}\left(\frac{\tilde{\alpha} - \frac{1}{p}}{p} - \frac{\alpha - \frac{1}{p}}{p}\right) \text{sgn}(\tilde{\alpha} - \alpha) = 1 - \text{sgn} \delta \text{sgn} \theta,
\]
that is
\[
\frac{1}{\tau} = \frac{(1 - \text{sgn} \delta)(1 - \text{sgn} \theta)}{\tau}, \quad \frac{1}{\tilde{q}} = \frac{1 - \text{sgn} \delta \text{sgn} \theta}{q}.
\]
Therefore
\[
\|f * g\|_{B_{p,r}^{\alpha,q}} \leq C \|f * g\|_{B_{p,r}^{\tilde{\alpha},\tilde{q}}} =
\]
\[
= C \left( \sum_{k_2=0}^{\infty} \left( \sum_{k_1=0}^{\infty} \left( 2^{\tilde{\alpha}_1 k_1 + \tilde{\alpha}_2 k_2} \|\Delta_k (f * g)\|_{L_{p,r}} \right) \right)^{\tilde{q}_2/\tilde{q}_1} \right)^{1/\tilde{q}_2}
\]

Using Lemma 1, Lemma 2, and Hölder’s inequality, we derive
\[
\|f * g\|_{B_{p,r}^{\alpha,q}} \leq C \left( \sum_{k_2=0}^{\infty} \left( \sum_{k_1=0}^{\infty} \left( 2^{\alpha_1 k_1 + \alpha_2 k_2} \|\Delta_k (f * g)\|_{L_{p,r}} \right) \right)^{\tilde{q}_2/\tilde{q}_1} \right)^{1/\tilde{q}_2}
\]
\[
\leq C \left( \sum_{k_2=0}^{\infty} \left( \sum_{k_1=0}^{\infty} \left( 2^{\beta_1 k_1 + \beta_2 k_2} \|\Delta_k (f * g)\|_{L_{p,r}} \right) \right)^{\tilde{q}_2/\tilde{q}_1} \right)^{1/\tilde{q}_2}
\]
\[
\leq C \left( \sum_{k_2=0}^{\infty} \left( \sum_{k_1=0}^{\infty} \left( 2^{\gamma_1 k_1 + \gamma_2 k_2} \|\Delta_k (f * g)\|_{L_{p,r}} \right) \right)^{\tilde{q}_2/\tilde{q}_1} \right)^{1/\tilde{q}_2}
\]
\[
\times \left( \sum_{k_2=0}^{\infty} \left( \sum_{k_1=0}^{\infty} \left( 2^{\eta_1 k_1 + \eta_2 k_2} \|\Delta_k (g)\|_{L_{p,r}} \right)^{\tilde{q}_2/\tilde{q}_1} \right)^{1/\tilde{q}_2}
\right)
\]
\[
= \|f\|_{B_{p,r}^{\alpha,q}} \|g\|_{B_{p,r}^{\gamma,q}},
\]
where \(\tilde{\mu} \geq \mu, \tilde{\nu} \geq \nu, \tilde{\eta} \geq \eta, \tilde{\xi} \geq \xi\) and
\[
\frac{1}{\tau} = \frac{1}{\tilde{\mu}} + \frac{1}{\tilde{\nu}} \leq \frac{1}{\mu} + \frac{1}{\nu}.
\]
\[
\frac{1}{q} = \frac{1}{\eta} + \frac{1}{\xi} \leq \frac{1}{\eta} + \frac{1}{\nu}.
\]

And therefore, considering that
\[
B^{\beta \eta}_{\nu \mu} \leftarrow B^{\beta \eta}_{\nu \mu} , \quad B^{\gamma \xi}_{\nu \mu} \leftarrow B^{\gamma \xi}_{\nu \mu},
\]
we obtain the inequality (11).

Conversely, we show that the conditions (12)-(15) are necessary.

Let (11) hold.

Let \( k \in \mathbb{N}^2 \). We consider the functions
\[
f_1(x_1, x_2) = e^{2\pi i(2k_1x_1 + 2k_2x_2)}, \quad g_1(x_1, x_2) = e^{2\pi i(2k_1x_1 + 2k_2x_2)}.
\]
Then
\[
(f_1 * g_1)(x_1, x_2) = e^{2\pi i(2k_1x_1 + 2k_2x_2)}
\]

From the inequality (11) we have
\[
2^{\alpha_1 k_1 + \alpha_2 k_2} \leq C 2^{\beta_1 k_1 + \beta_2 k_2 + \gamma_1 k_1 + \gamma_2 k_2}.
\]

Given the correct choice of \( k \in \mathbb{N}^2 \), we derive
\[
\alpha_i \leq \beta_i + \gamma_i. \quad (16)
\]

Let \( k \in \mathbb{N}^2 \). We consider the functions
\[
f_2(x_1, x_2) = \sum_{m_2=2^{k_2-1}}^{2^{k_2-1}} \sum_{m_1=2^{k_1-1}}^{2^{k_1-1}} e^{2\pi i(2m_1x_1 + 2m_2x_2)} = g_2(x_1, x_2).
\]

Then \( f_2 * g_2 = f_2 = g_2 \). Using Hardy-Littlewood theorem, we have
\[
\|f_2 * g_2\|_{B^{\beta \eta}_{\nu \mu}} = 2^{\alpha_1 k_1 + \alpha_2 k_2} \|\Delta_k f_2\|_{L^{\nu \mu}} \approx 2^{(\alpha_1 + \frac{1}{\mu}) k_1 + (\alpha_2 + \frac{1}{\mu}) k_2}
\]
\[
\|f\|_{B^{\beta \eta}_{\nu \mu}} \approx 2^{(\beta_1 + \frac{1}{\nu}) k_1 + (\beta_2 + \frac{1}{\nu}) k_2},
\]
\[
\|g\|_{B^{\gamma \xi}_{\nu \mu}} \approx 2^{(\gamma_1 + \frac{1}{\nu}) k_1 + (\gamma_2 + \frac{1}{\nu}) k_2}.
\]

From the inequality (11), since \( k \) is arbitrary, we have
\[
\alpha_i - \beta_i - \gamma_i \leq 1 + \frac{1}{p_i} - \frac{1}{r_i} - \frac{1}{h_i}, \quad i = 1, 2,
\]
that is, the condition (13) is necessary.

From (13) and (16) follows (12).

The condition (15) makes sense when \( \delta_i = 0 \) \((i = 1, 2)\). We consider the functions
\[
f_3(x_1, x_2) = \sum_{k=0}^{N} \sum_{m=2^{k-1}}^{2^{k-1}} 2^{-\left(\frac{1}{r_i} + \beta_i\right)k} e^{2\pi imx_i},
\]
The convolution in anisotropic Besov spaces

\[ g_3(x_1, x_2) = \sum_{k=0}^{N} \sum_{m=2^k-1} 2^{k-1} \sum_{m=2^k-1} 2^{-\left(\frac{1}{\tau_i} + \gamma_i\right)k} e^{2\pi i m x_i}, \]

then

\[ (f_3 * g_3)(x_1, x_2) = \sum_{k=0}^{N} \sum_{m=2^k-1} 2^{-\left(\frac{1}{\tau_i} + \frac{1}{\mu_i} + (\beta_i + \gamma_i)\right)k} e^{2\pi i m x_i}. \]

Then we have

\[ \| f * g \|_{B^{\alpha q}_{p\tau}} \asymp \left( \sum_{k=0}^{N} \left( 2^{\left(\alpha_i + \frac{1}{\mu_i} - \frac{1}{\tau_i} - \beta_i - \gamma_i\right)k} \right)^{\tau_i} \right)^{1/\tau_i} = N^{\frac{1}{\tau_i}}, \]

\[ \| f \|_{B^{\beta \eta}_{p\mu}} \asymp N^{\frac{1}{\eta_i}}, \]

\[ \| g \|_{B^{\gamma \xi}_{h\nu}} \asymp N^{\frac{1}{\xi_i}}, \]

Therefore, \( \frac{1}{\tau_i} \leq \frac{1}{\eta_i} + \frac{1}{\xi_i} \) follows from (11).

The condition (14) makes sense when \( \delta_i = 0, \theta_i = 0 \). This means that \( \alpha_i = \beta_i + \gamma_i, \)

\[ 1 + \frac{1}{p_i} = 1 + \frac{1}{r_i}. \]

Let \( k \in \mathbb{N} \).

\[ f_4(x_1, x_2) = 2^{-\beta_i k_i} \sum_{m=2^k-1} (m - 2^{k-1}) \frac{1}{\tau_i} e^{2\pi i m x_i}, \]

\[ g_4(x_1, x_2) = 2^{-\gamma_i k_i} \sum_{m=2^k-1} (m - 2^{k-1}) \frac{1}{\gamma_i} e^{2\pi i m x_i}. \]

Then

\[ \| f \|_{B^{\beta \eta}_{p\mu}} \asymp \left( \sum_{m=1}^{2^{k-1}} (m - 1)^{\frac{1}{\tau_i}} \right)^{\frac{1}{\tau_i}} \asymp k^{\frac{1}{\tau_i}}, \]

\[ \| g \|_{B^{\gamma \xi}_{h\nu}} \asymp k^{\frac{1}{\xi_i}}, \]

\[ \| f * g \|_{B^{\alpha q}_{p\tau}} \asymp k^{\frac{1}{\tau_i}}. \]

Therefore \( \frac{1}{\tau_i} \leq \frac{1}{m_i} + \frac{1}{\nu_i} \).
3 Conclusion

In conclusion, we note that in the article we investigate the boundedness of the norm of the convolution operator in anisotropic Besov spaces. We proved a criterion for the fulfillment of the inequality

$$\| f * g \|_{B^{p,q}_{\tau}(0,1)^2} \leq C \| f \|_{B^{r,q}_{\tau}(0,1)^2} \| g \|_{B^{r,s}_{\tau}(0,1)^2}$$

in terms of the corresponding parameters. The resulting theorem:
1) summarizes the result of Burenkov and Batyrov [21];
2) it implies the classical O’Neil inequalities.

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References

[1] Bennett C., Sharpley R., "Interpolation of Operators", Pure and Applied Mathematics 129, Boston, MA, Academic Press, INC (1988): 469.
[2] Brézis H., Wainger S., "A note on limiting cases of Sobolev embeddings and convolution inequalities", Comm. Partial Differential Equations vol. 5, no. 7 (1980): 773-789.
[3] Hörmander L., "The analysis of linear partial differential operators I", Distribution theory and Fourier analysis Reprint of the second edition / Berlin: Classics in Mathematics, Springer-Verlag (2013): 440.
[4] O’Neil R., "Convolution operators and $L(p, q)$ spaces", Duke Math. J. 30 (1963): 129-142.
[5] Yap L.Y.H., "Some remarks on convolution operators and $l(p, q)$ spaces", Duke Math. J. 36 (1969): 647-658.
[6] Hunt R.A., "On $L(p, q)$ spaces", Enseignement Math. vol. 12, no. 2 (1966): 249-276.
[7] Nursultanov E., Tikhonov S., "Convolution inequalities in Lorentz spaces", J. Fourier Anal. Appl. 17 (2011): 486-505.
[8] Blozinski A.P., "On a convolution theorem for $L(p, q)$ spaces", Trans. Amer. Math. Soc. 164 (1972): 255-265.
[9] Nursultanov E.D., Tleukhanova N.T., "O multiplikatorah kratnyh ryadov Fur’e [Multipliers of Multiple Fourier Series]", Proc. Steklov Inst. Math. 227 (1999): 231-236.
[10] Tleukhanova N.T., Sadykova K.K. "O’Neil-type inequalities for convolutions in anisotropic Lorentz spaces", Eurasian Mathematical Journal vol. 10, no. 3 (2019): 68-83.
[11] Nursultanov E., Tikhonov S., Tleukhanova N., "Norm inequalities for convolution operators", C. R. Acad. Sci. Paris vol. I, no. 347 (2009): 1385-1388.
[12] Nursultanov E., Tikhonov S., Tleukhanova N., "Norm convolution inequalities in Lebesgue spaces", Rev. Mat. Iberoam vol. 34, no. 2 (2018): 811-838.
[13] Heil C., "An introduction to weighted Wiener amalgams. In Wavelets and their applications", Allied Publishers, New Delhi (2003): 183-216.
[14] Kamińska A., "On convolution operator in Orlicz spaces", Rev. Mat. Univ. Complutense 2 (1989): 157-178.
[15] Kerman R.A., "Convolution theorems with weights", Trans. Amer. Math. Soc. vol. 280, no. 1 (1983): 207-219.
[16] Kerman R., Sawyer E., "Convolution algebras with weighted rearrangement-invariant norm", *Studia Math.* vol. 108, no. 2 (1994): 103-126.

[17] Nursultanov E., Tikhonov S., "Weighted norm inequalities for convolution and Riesz potential", *Potential Analysis* vol. 42, no. 2 (2015): 435-456.

[18] Sampson G., Naparstek A., Drobot V., "*(L_p, L_q)* mapping properties of convolution transforms", *Studia Math.* vol. 55, no. 1 (1976): 41-70.

[19] Golovkin K.K., Solonnikov V.A., "Ocenki integral'nyh operatorov v translyacionno-invariantnyh normah [Estimates of integral operators in translation-invariant norms]", *Tr. MIAN* 70 (1964): 47-58.

[20] Golovkin K.K., Solonnikov V.A., "Ocenki integral'nyh operatorov v translyacionno-invariantnyh normah. II [Estimates of integral operators in translation-invariant norms. II]", *Tr. MIAN* 92 (1966): 5-30.

[21] Batyrov B.E., Burenkov V.I., "Estimates for convolutions in Nikol'skii-Besov spaces", *Dokl. Akad. Nauk* vol. 330, no. 1 (1993): 9-11.

[22] Bui H., "Weighted Young's inequality and convolution theorems on weighted Besov spaces", *Math. Nachr.* 170 (1994): 25-37.

[23] Golovkin K.K., Solonnikov V.A., "Ob ocenkah operatorov svertki [Estimates of convolution operators]", *Zap. Naučn. Sem. Leningrud. Otdel. Mat. Inst. Steklov. (LOMI)* 7 (1968): 6-86.

[24] Sadykova K.K., Tleukhanova N.T., "Estimates of the norm of the convolution operator in anisotropic Besov spaces with the dominated mixed derivative", *Bulletin of the Karaganda University-Mathematics*, vol. 95, no. 3 (2019): 51-59.

[25] Bekmaganbetov K., Nursultanov E., "Interpolation of Besov *B*_p^σ space and Lizorkin-Triebel *F*^σ^p τ space", *Analysis Mathematica*, 35 (2009): 169-188.