ALGEBRAIC DELOCALIZATION FOR THE
SCHRÖDINGER EQUATION ON LARGE TORI

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ABSTRACT. Let $\mathcal{L}$ be a fixed $d$-dimensional lattice. We study the localization properties of solutions of the stationary Schrödinger equation with a positive $L^\infty$ potential on tori $\mathbb{R}^d/\mathcal{L}$ in the limit, as $L \to \infty$, for dimension $d \leq 3$.

We show that the probability measures associated with $L^2$-normalized solutions, with eigenvalue $E$ near the bottom of the spectrum, satisfy an algebraic delocalization theorem which states that these probability measures cannot be localized inside a ball of radius $r = o(E^{-1/4+\epsilon})$, unless localization occurs with a sufficiently slow algebraic decay.

In particular, we apply our result to Schrödinger operators modeling disordered systems, such as the $d$-dimensional continuous Anderson-Bernoulli model, where almost sure exponential localization of eigenfunctions, in the limit as $E \to 0$, was proved by Bourgain-Kenig in dimension $d \geq 2$, and show that our theorem implies an algebraic blow-up of localization length in this limit.

1. Introduction

We are motivated by the study of the localization properties of eigenfunctions of Schrödinger operators which describe disordered systems. For instance, Bourgain-Kenig [3] considered the Anderson-Bernoulli model on $\mathbb{R}^d$, $d \geq 2$, (the case $d = 1$ had previously been treated in [4])

\begin{equation}
H_{AB} = -\Delta + \sum_{\xi \in \mathbb{Z}^d} \alpha_\xi \varphi(\cdot - \xi)
\end{equation}

where $\varphi \in C^\infty_c(\mathbb{R}^d)$, $0 \leq \varphi \leq 1$, supp $\varphi \subset B(0,1/10)$ with i.i.d. Bernoulli couplings $\alpha_\xi \in \{0, 1\}$. One notes that $\inf \sigma(H_\alpha) = 0$ almost surely.

Bourgain-Kenig found that solutions of the stationary Schrödinger equation $H_{AB} \Psi = E \Psi$ were almost surely exponentially localized in the limit $E \to 0$, therefore proving Anderson localization [2] in the spectral sense. A natural way to study this phenomenon is to consider the operator restricted to a box $[0,L]^d \subset \mathbb{R}^d$, $d \geq 2$, with appropriate boundary conditions and investigate the localization properties of the eigenfunctions in the thermodynamic limit, as $L \to +\infty$.

In this article, instead of considering random operators we will study deterministic operators for any given configuration of scatterer positions and

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coupling constants. For example, in the case of the continuous Anderson-Bernoulli model, fix a function $\alpha : \mathbb{Z}^d \to \{0, 1\}$ and associate with it the Schrödinger operator
\begin{equation}
H_\alpha = -\Delta + \sum_{\xi \in \mathbb{Z}^d} \alpha(\xi) \varphi(\cdot - \xi).
\end{equation}

We will then study the localization properties of the eigenfunctions of $H_\alpha$ for any given Boolean function $\alpha$ on the lattice $\mathbb{Z}^d$.

A key application of the algebraic delocalization theorem which is proved in this article will be the divergence of the localization length associated with an eigenfunction of $H_\alpha$, as $E \to 0$. We stress for the case $d = 2$ that this type of delocalization does not constitute a contradiction to the scaling theory of localization [1]: in fact, a similar type of divergence of localization length has been observed in the physics literature for certain models [5, 6]. In this context, the authors found multifractal properties of the eigenfunctions, but no full transition to a regime with extended states.

2. Statement of results

We consider Schrödinger operators on boxes with periodic boundary conditions (our method should work for other self-adjoint b.c.s and more general domains). We study $L^\infty$ potentials which remain bounded in the thermodynamic limit, with $\inf V = 0$ and $\sup V = V_1 > 0$, a condition which is, in particular, satisfied by the Anderson-Bernoulli operator (1.2) for any configuration of coupling constants.

2.1. Algebraic delocalization. Let $d \leq 3$ and $T_L^d = \mathbb{R}^d / L\mathbb{L}_0$, where $L > 0$ is a large parameter, and $\mathcal{L}_0 = \mathbb{Z}(1, 0, \cdots, 0) \oplus \mathbb{Z}(0, \gamma_1, \cdots, 0) \oplus \mathbb{Z}(0, \cdots, 0, \gamma_{d-1})$, $\gamma_1, \cdots, \gamma_{d-1} \geq 1$. Let $V \in L^\infty(T_L^d)$ with $V \geq 0$ and $\inf V = 0$. We consider a solution of the Schrödinger equation
\begin{equation}
(-\Delta + V) \Psi = E \Psi, \quad \|\Psi\|_{L^2(T_L^d)} = 1, \quad 0 < E < 1.
\end{equation}

We show that the $L^2$-density $d\mu_\Psi = |\Psi|^2d\mu$ associated with a solution $\Psi$ of (2.1) can only vary subject to certain algebraic constraints. In particular we choose scales $\ell \asymp E^{-1/4+\eta(1-d/4)}$, $\eta \in (0, 1/(4-d))$, and $r = \ell E^{-\eta}$. We show that for any $x_0 \in T_L^d$ the ball $B(x_0, \ell)$ contains either less than half of the $L^2$-mass of $\Psi$: $\mu_\Psi(B(x_0, \ell)) < 1/2$, or the larger ball $B(x_0, r)$ contains less than $1 - E$ of the $L^2$-mass of $\Psi$: $\mu_\Psi(B(x_0, r)) < 1 - E$.

**Theorem 2.1.** There exist absolute constants $c_1, c_2 > 0$ such that the following holds. Let $\eta \in (0, 1/(d-4))$. Denote
\[c_{V} = c_1 (1 + \|V\|_\infty^{1/2})^{-1/2}.
\]

Let $\Psi$ be a solution of (2.1) with eigenvalue
\begin{equation}
E \in [(2c_{V}/L)^{4/(1+dn)}, \min(c_{V}^{4/(1+dn)}, c_2^{2/dn})],
\end{equation}
where we assume that $L$ is sufficiently large.
Then, we have for any $x_0 \in T^d_L$

$$\int_{B(x_0, cV^E - 1/4 + \eta(1 - d/4))} |\Psi|^2 d\mu < \frac{1}{2} \quad (2.3)$$

or

$$\int_{T^d_L \setminus B(x_0, cV^E - 1/4 - d\eta/4)} |\Psi|^2 d\mu > E. \quad (2.4)$$

2.2. Blow-up of localization length. We will illustrate the meaning of Theorem 2.1 by demonstrating that, if we suppose that a solution of the Schrödinger equation (2.1) is localized with respect to a localization center, then our theorem implies that the localization length admits an algebraic singularity, as $E \to 0$, unless the decay of the localization is no faster than algebraic with exponent $d - 4$.

Let $\delta : [1, +\infty) \mapsto [0, \frac{1}{2}]$ be a continuous, strictly decreasing function. We say that $\Psi$ is localized with respect to $x_0$ with localization length $\ell_{loc}$ and decay $\delta : [1, +\infty) \mapsto [0, \frac{1}{2}]$, if we have for any $r \geq \ell_{loc}$

$$\int_{T^d_L \setminus B(x_0, r)} |\Psi|^2 d\mu \leq \delta \left( \frac{r}{\ell_{loc}} \right). \quad (2.5)$$

One may now use the result above to deduce lower bounds for the localization length.

Take $\ell_1 = cV^E - 1/4 + d\eta/4$, where $cV = c_1 (1 + \|V\|_{L^\infty}^{1/2})^{-1/2}$. Suppose that $\ell_1 \geq \ell_{loc}$ (if $\ell_1 < \ell_{loc}$, then we have our lower bound). Using the result above we know that for each $x_0 \in T^d_L$ either (2.3) or (2.4) must hold. We will show that both inequalities (2.3) and (2.4) imply a blow-up of localization length, as $E \to 0$.

Inequality (2.3) together with the localization hypothesis implies

$$\delta \left( \frac{\ell_1}{\ell_{loc}} \right) \geq \int_{T^d_L \setminus B(x_0, \ell_1)} |\Psi|^2 d\mu > \frac{1}{2}$$

which is a contradiction, because $\delta(\ell_1/\ell_{loc}) \leq \frac{1}{2}$. So, we have the lower bound

$$\ell_{loc} > \ell_1 = cV^E - 1/4 + \eta(1 - d/4).$$

Let us suppose that inequality (2.4) holds and deduce a blow-up of localization length in this case. Denote $\ell_2 = \ell_1 E^{-\eta}$. We have $\ell_2 \geq \ell_1 \geq \ell_{loc}$ by our assumption $\ell_1 \geq \ell_{loc}$. Therefore,

$$\delta \left( \frac{\ell_2}{\ell_{loc}} \right) \geq \int_{T^d_L \setminus B(x_0, \ell_1)} |\Psi|^2 d\mu > E.$$ 

Because $\delta$ is strictly decreasing, we have

$$\frac{\ell_2}{\ell_{loc}} < \delta^{-1}(E)$$

\footnote{Note that (2.2) ensures that $1 \leq cV^E - 1/4 + d\eta/4 \leq \frac{1}{2}L$}
which yields the lower bound
\[ \ell_{\text{loc}} > \frac{\ell_2}{\delta^{-1}(E)} = c_V E^{-1/4 - d\eta/4} \delta^{-1}(E). \]

In conclusion, we must have the following lower bound for the localization length
\[ (2.6) \quad \ell_{\text{loc}} > c_V E^{-1/4 + \eta(1-d/4)} \min \left( 1, \frac{E^{-\eta}}{\delta^{-1}(E)} \right) \]

2.2.1. Delocalization for exponential decay. In the case of an exponential decay with respect to a localization center \( x_0 \), we have for \( C, \beta > 0 \)
\[ \int_{T^d_0 \setminus B(x_0, r)} |\Psi|^2 d\mu \leq C e^{-\beta r} = \delta \left( \frac{r}{\ell_{\text{loc}}} \right) \]
with
\[ \delta(r) = C e^{-\log(2C)r}, \quad \ell_{\text{loc}} = \frac{\log(2C)}{\beta}. \]
Since \( \delta^{-1}(E) = -\log(E/C)/\log(2C) \), we obtain from (2.6) the lower bound (recall \( \eta \in (0, 1/(4-d)) \))
\[ \ell_{\text{loc}} > c_V E^{-1/4 + \eta(1-d/4)} \min \left( 1, \log(2C) \frac{E^{-\eta}}{\log(C/E)} \right) \]

2.2.2. Delocalization for sufficiently fast algebraic decay. In particular, we see that for any sufficiently fast algebraic decay we may deduce \( \ell_{\text{loc}} \to +\infty \) as \( E \to 0^+ \). To see this take \( \delta(r) = C r^{-\alpha} \), which gives
\[ \ell_{\text{loc}} > c_V E^{-1/4 + \eta(1-d/4)} \min \left( 1, C^{-1/\alpha} E^{-\eta + 1/\alpha} \right). \]
So, for \( \alpha \geq 1/\eta \), we have
\[ \ell_{\text{loc}} \lesssim E^{-1/4 + \eta(1-d/4)}, \]
because for small \( E \), \( E^{-\eta + 1/\alpha} \) is large. We then have a blow-up, since \( \eta \in (0, 1/(4-d)) \).
Moreover, if \( \alpha < 1/\eta \), then \( E^{-\eta + 1/\alpha} \) is small if \( E \to 0 \). So, the lower bound is of order
\[ \ell_{\text{loc}} \lesssim E^{-1/4 - d\eta/4 + 1/\alpha}, \]
and we have a blow-up provided \( 1/\alpha < 1/4 + d\eta/4 \). This yields the general condition \( \alpha > 4/(1 + d\eta) \) to ensure a blow-up of localization length. Since \( \eta \in (0, 1/(4-d)) \), we can rule out any decay faster than exponent \( 4 - d \).
2.3. Generalization to spectral projectors. In the context of Anderson localization it is natural to study the localization properties of all eigenfunctions in a given spectral window \( I = [E, 2E] \), \( 0 < E < 1 \). Let \( h : \mathbb{R} \to \mathbb{R} \) be a continuous, compactly supported function with \( \text{supp} \ h \subset I \).

We introduce
\[
\Pi_h(x, y) = \sum_{\lambda} h(\lambda) \Psi_{\lambda}(x) \overline{\Psi_{\lambda}(y)}.
\]

We note that, while individual eigenfunctions are expected to be exponentially localized with respect to a localization center, these centers may be different for each eigenfunction. Therefore one considers a spectral projector \( \Pi_h(x, y) \) which ought to satisfy an exponential decay with respect to the distance \( |x - y| \).

Let us fix \( x_0 \in L^d \) and let \( F = \Pi_h(\cdot, x_0) / \| \Pi_h(\cdot, x_0) \|_{L^2(T^d)} \). Let \( \mu_h \) be the probability measure associated with the probability density \( d\mu_h = |F|^2 d\mu \).

We have the following algebraic delocalization theorem which is a generalization of Theorem 2.1.

**Theorem 2.2.** Assume that \( E \) satisfies the conditions of Theorem 2.1. Let \( B_\ell \subset T^d_L \) be any ball of radius \( \ell = c_1 E^{-1/4 + \eta(1-d/4)} \), and \( B_r \subset T^d_L \) be a ball with the same center and radius \( r = \ell E^{-\eta} \). We have either \( \mu_h(B_\ell) < 1/2 \) or \( \mu_h(B_r) < 1 - E \).

Arguing as in the previous section one may now deduce a blow-up of localization length for any decay faster than exponent \( d - 4 \).

3. Proof of Theorem 2.1

We argue by contradiction. Recall \( \ell = c_1 E^{-1/4 + \eta(1-d/4)}(1 + \| V \|_{L^\infty}^{1/2})^{-1/2} \). Let us suppose there exists \( x_0 \in T^d_L \) such that

\[
(3.1) \quad \int_{B(x_0, \ell)} |\Psi|^2 d\mu \geq \frac{1}{2}
\]

and

\[
(3.2) \quad \int_{T^d_L \setminus B(x_0, \ell E^{-\eta})} |\Psi|^2 d\mu \leq E.
\]

3.1. A variation bound. Let \( r = \ell E^{-\eta} \) for \( \eta \in (0, 1/(4 - d)) \). We recall that our assumption (2.2) implies \( 1 \leq r \leq \frac{1}{4} L \). Let \( \Lambda_{2r} \subset T^d_L \) be a box of side length \( 2r \) centered on \( x_0 \) inside the torus such that \( B(x_0, r) \subset \Lambda_{2r} \). Let \( \chi \in C^\infty(T^d_L) \) be a smoothed indicator in the sense that \( \Lambda_{2r} \subset \text{supp} \chi \subset \Lambda_{2r+1} \), and \( \chi|_{\Lambda_{2r}} = 1 \). Moreover we may choose \( \chi \) in such a way that \( \sup \| \nabla \chi \| \leq k_1 \) and \( \sup |\Delta \chi| \leq k_2 \) for absolute constants \( k_1, k_2 \).

Let \( x_1, x_2 \in \Lambda_{2r} \). Let \( r' = 2r + 1 \). We recall that the kernel \( G_\lambda \) of the resolvent \( (-\Delta - \lambda)^{-1} \) on \( \Lambda_{r'} \) can be expanded with respect to the o. n. b.
of exponentials $e(\xi \cdot x/r')r'^{-d/2}$, $\xi \in \mathbb{Z}^d$, as
\[
G_{\lambda}(x, y) = -\frac{1}{r^{d\lambda}} + \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{4\pi^2 |\xi/r'|^2 - \lambda} \frac{e(\xi \cdot (x - y)/r')}{r^{d\lambda}},
\]
where $\lambda \notin \sigma(-\Delta)$ and we denote $e(\tau) = e^{2\pi i \tau}$.

We then have
\[
G_{\lambda}(x, x_1) - G_{\lambda}(x, x_2) = \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} \frac{e(\xi \cdot (x_1)/r') - e(\xi \cdot (x_2)/r') e(\xi \cdot x/r')}{4\pi^2 |\xi/r'|^2 - \lambda}.
\]

We estimate the integral term by term. We start with the third term
\[
(3.3) \quad \Psi(x_1) - \Psi(x_2) = \int_{\Lambda_{\nu'}} (-\Delta b) \chi \Psi \, d\mu = \int_{\Lambda_{\nu'}} b (-\Delta (\chi \Psi)) \, d\mu
\]

We may now expand
\[-\Delta (\chi \Psi) = -\Psi \Delta \chi - 2\nabla \chi \cdot \nabla \Psi - \chi \Delta \Psi.
\]
We estimate the integral term by term. We start with the third term
\[
(3.4) \quad \int_{\Lambda_{\nu'}} b \chi (-\Delta \Psi) \, d\mu = E \int_{\Lambda_{\nu'}} b \chi \Psi \, d\mu - \int_{\Lambda_{\nu'}} b \chi V \Psi \, d\mu
\]
where we used $(-\Delta + V)\chi = E\chi$.

Moreover, the identity
\[0 \leq \int_{\mathbb{T}^d_L} |V\Psi|^2 \, d\mu = \int_{\mathbb{T}^d_L} -\Delta \Psi \cdot \nabla \Psi \, d\mu = \int_{\mathbb{T}^d_L} (E - V)|\Psi|^2 \, d\mu
\]
yields
\[\int_{\Lambda_{\nu'}} V |\Psi|^2 \, d\mu \leq \int_{\mathbb{T}^d_L} V |\Psi|^2 \, d\mu \leq E,
\]
where we used $\|\Psi\|_{L^2(\mathbb{T}^d_L)} = 1$.

Hence, Cauchy-Schwarz yields
\[
(3.5) \quad \left| \int_{\Lambda_{\nu'}} b \chi V \Psi \, d\mu \right| \leq \|b\chi\|_{L^2(\mathbb{T}^d_L)} \|V\|_{L^\infty}^{1/2} \left( \int_{\mathbb{T}^d_L} V |\Psi|^2 \right)^{1/2}
\]
\[\leq \|b\|_{L^2(\mathbb{T}^d_L)} \|V\|_{L^\infty}^{1/2} E^{1/2}.
\]
ALGEBRAIC DELOCALIZATION

\[ \frac{1}{2\pi^2} \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{r^d|\xi/r|^4} \] 

\[ \leq \frac{r^{(4-d)/2}L_d(2)^{1/2}}{2\pi^2} \]

where we denote the Dirichlet series

\[ L_d(s) = \sum_{n=1}^{\infty} \frac{r_d(n)}{n^s} \]

associated with \( r_d(n) \), the number of ways the integer \( n \) can be represented as a number of \( d \) squares. Note that the series converges for \( s = 2 \), since \( d \leq 3 \).

Moreover,

\[ E \left| \int_{\Lambda_r} b\chi \Psi d\mu \right| \leq E \|b\chi\|_{L^2(\Lambda_r)} \]

where we used \( \|\Psi\|_{L^2(T^d_L)} = 1 \) and \( \Lambda_r \subset T^d_L \).

Let us continue with the second term. We have

\[ \left| \int_{\Lambda_r} b\nabla \chi \cdot \nabla \Psi d\mu \right| \leq \|b\|_{L^2(\Lambda_r)} \|\nabla \chi \cdot \nabla \Psi\|_{L^2(\Lambda_r)} \]

and

\[ \|\nabla \chi \cdot \nabla \Psi\|_{L^2(\Lambda_r)} \leq \left( \int_{\Lambda_r} |\nabla \chi|^2 |\nabla \Psi|^2 d\mu \right)^{1/2} \leq k_1 \|\nabla \Psi\|_{L^2(\Lambda_r)} \]

where we recall \( |\nabla \chi| \leq k_1 \), and, as we saw above,

\[ \|\nabla \Psi\|_{L^2(\Lambda_r)} \]

\[ \leq \left( \int_{\Lambda_r} |\nabla \chi|^2 d\mu \right)^{1/2} \]

\[ \leq k_2 \|\nabla \Psi\|_{L^2(T^d_L)} \]

where we used \( \|\Psi\|_{L^2(T^d_L)} = 1 \) and \( V \geq 0 \).

So, in summary, we bound the second term as follows:

\[ \left| \int_{\Lambda_r} b\nabla \chi \cdot \nabla \Psi d\mu \right| \leq k_1 \frac{L_d(2)^{1/2}}{2\pi^2} r^{(4-d)/2} E^{1/2}. \]

For the first term, we use [3.2]:

\[ \left| \int_{\Lambda_r} b\Psi \Delta \chi d\mu \right| \leq \|b\|_{L^2(\Lambda_r)} \sup |\Delta \chi| \left( \int_{\text{supp}(\Delta \chi)} |\Psi|^2 d\mu \right)^{1/2} \]

\[ \leq k_2 \frac{L_d(2)^{1/2}}{2\pi^2} r^{(4-d)/2} E^{1/2} \]

where we recall \( |\Delta \chi| \leq k_2 \), as well as \( \|b\|_{L^2(\Lambda_r)} \leq r^{(4-d)/2} L_d(2)^{1/2}/2\pi^2 \).
In the estimate of the integral over \( \text{supp}(\Delta \chi) \), we note that by construction of the cutoff function \( \chi \) we have \( \text{supp}(\Delta \chi) \subset \Lambda_{r'} \setminus \Lambda_2 \) (recall \( r' = 2r + 1 \)) and, therefore,

\[
\text{supp}(\Delta \chi) \subset \mathbb{T}^d_L \setminus \Lambda_{2r} \subset \mathbb{T}^d_L \setminus B(x_0, r)
\]

and, thus, using (3.2) (recall \( r = \ell E^{-\gamma} \)), we obtain

\[
\int_{\text{supp}(\Delta \chi)} |\Psi|^2 d\mu \leq \int_{\mathbb{T}^d_L \setminus B(x_0, r)} |\Psi|^2 d\mu \leq E.
\]

Combining all of the above estimates, we obtain

(3.9) \[ |\Psi(x_1) - \Psi(x_2)| \leq c \ell^{(4-d)/2} (1 + \|V\|_{L^\infty}^{1/2}) \ell^{1/2} \]

where \( c > 0 \) denotes an absolute constant and \( r' = 2r + 1 \leq 3r \), as \( r \geq 1 \).

3.2. Proof. In order to prove the result we will first of all show that our assumption

\[
\int_{B(x_0, \ell)} |\Psi|^2 d\mu \geq \frac{1}{2}
\]

implies that there exists \( x' \in B(x_0, \ell) \) s. t. \( |\Psi(x')| \geq \sqrt{\frac{1}{2\pi}} \ell^{-d/2} \).

To see this, we argue by contradiction. Suppose we have \( |\Psi(x)| < \sqrt{\frac{1}{2\pi}} \ell^{-d/2} \) for all \( x \in B \). Then,

\[
\int_B |\Psi|^2 d\mu < \frac{1}{2\pi} \ell^{-d} \text{vol}(B) = \frac{1}{2}.
\]

For any \( x \in \Lambda_{2r} \) we have

\[
|\Psi(x)| \geq |\Psi(x')| - |\Psi(x) - \Psi(x')| \\
\geq \frac{1}{\sqrt{2\pi}} \ell^{-d/2} - c(1 + ||V||_{L^\infty}^{1/2}) \ell^{(4-d)/2} E^{1/2}.
\]

In order to have a lower bound of order \( \ell^{-d/2} \), we then need

\[
c' \ell^{-d/2} \geq (1 + ||V||_{L^\infty}^{1/2}) \ell^{(4-d)/2} E^{1/2} = (1 + ||V||_{L^\infty}^{1/2}) \ell^{(4-d)/2} E^{1/2 - \eta(4-d)/2}
\]

for some absolute constant \( c' > 0 \), which, in turn, is equivalent to

\[
\ell \leq \left( \frac{c'}{1 + ||V||_{L^\infty}^{1/2}} \right)^{1/2} E^{-1/4 + \eta(1-d/4)}
\]

and we recall that this holds with \( c_1 = c'^{1/2} \).

Then we have for any \( x \in \Lambda_{2r} \)

\[ |\Psi(x)| \geq c'' \ell^{-d/2}, \]

for an absolute constant \( c'' > 0 \). This then yields

\[
\int_{\Lambda_{2r}} |\Psi(x)|^2 d\mu \geq c''^2 \frac{(2r)^d}{\ell^d} = 2^d c''^2 E^{-\eta d}
\]
and, thus, for an absolute constant $0 < c_2 < c''$ we have $E \leq c_2^{2/d} < (2c'^{2/d})^{1/n}$ and we arrive at a contradiction to the normalization $\|\Psi\|_{L^2(T^d_L)} = 1$.

4. Proof of Theorem 2.2

For given $x_0 \in T^d_L$, we introduce
\[ F = \sum_{\lambda \in [E,2E]} h(\lambda) \overline{\Psi_\lambda(x_0)} \Psi_\lambda \]
We sketch the argument which is very similar to the one above for individual eigenfunctions. In particular, we have
\[ -\Delta F = \tilde{F} -VF \]
where
\[ \tilde{F} = \sum_{\lambda \in [E,2E]} \frac{\lambda h(\lambda) \overline{\Psi_\lambda(x_0)} \Psi_\lambda}{(\sum_{\lambda \in [E,2E]} |\Psi_\lambda(x_0)|^2)^{1/2}} \]
and
\[ \|\tilde{F}\|_{L^2}^2 = \sum_{\lambda \in [E,2E]} \frac{\lambda^2 |h(\lambda)|^2 |\Psi_\lambda(x_0)|^2}{(\sum_{\lambda \in [E,2E]} |\Psi_\lambda(x_0)|^2)^{1/2}} \leq 4 \|h\|_{L^\infty}^2 E^2 \]
Following the argument above we get for $x_1, x_2 \in \Lambda_r$
\[ |F(x_1) - F(x_2)| \leq \left| \int_{T^d_L} b \Delta(\chi F) \, d\mu \right| \]
Let us estimate the integral with the Laplace term
\[ \int_{T^d_L} b \chi \Delta F \, d\mu = \int_{T^d_L} b \chi \tilde{F} \, d\mu - \int_{T^d_L} b \chi V F \, d\mu \]
For the first term, we find an estimate of the same order as above
\[ \left| \int_{T^d_L} b \chi \tilde{F} \, d\mu \right| \leq 2E\|b\|_2 \lesssim r^{(4-d)/2}E^{1/2} \]
For the second term, we have
\[ \left| \int_{T^d_L} b \chi V F \, d\mu \right| \leq \|b\|_{L^2(\Lambda_r)} \|V\|_{L^\infty} \left( \int_{T^d_L} V |F|^2 \, d\mu \right)^{1/2} \]
and
\[ 0 \leq \int_{T^d_L} |\nabla F|^2 \, d\mu = \int_{T^d_L} (-\Delta F) F \, d\mu = \int_{T^d_L} \tilde{F} F \, d\mu - \int_{T^d_L} V |F|^2 \, d\mu \]
which yields
\[ \int_{T^d_L} V |F|^2 \, d\mu \leq \int_{T^d_L} \tilde{F} F \, d\mu \leq \|\tilde{F}\|_2 \|F\|_2 \leq 2 \|h\|_{L^\infty} E \]
And again, we get an estimate of the same order as above
\[ \left| \int_{T^d_L} b \chi V F \, d\mu \right| \lesssim \|V\|_{L^\infty} r^{(4-d)/2} E^{1/2} \]
Moreover, for the gradient term we get the same estimate as above due to

\[
\int_{T_L^d} |\nabla F|^2 d\mu = \int_{T_L^d} \tilde{F} F d\mu - \int_{T_L^d} V|F|^2 d\mu \leq \int_{T_L^d} \tilde{F} F d\mu \leq 2\|h\|_{L^\infty} E.
\]

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