SCHUR-WEYL DUALITY FOR DELIGNE CATEGORIES II: THE LIMIT CASE

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ABSTRACT. This paper is a continuation of a previous paper of the author ([3]), which gave an analogue to the classical Schur-Weyl duality in the setting of Deligne categories.

Given a finite-dimensional unital vector space $V$ (a vector space $V$ with a chosen non-zero vector $1$), we constructed in [2] a complex tensor power of $V$: an $Ind$-object of the Deligne category $Rep(S_t)$ which is a Harish-Chandra module for the pair $(\mathfrak{g}(V), \Phi_1)$, where $\Phi_1 \subset GL(V)$ is the mirabolic subgroup preserving the vector $1$.

This construction allowed us to obtain an exact contravariant functor $\overline{SW}_{t,V}$ from the category $Rep^{ab}(S_t)$ (the abelian envelope of the category $Rep(S_t)$) to a certain localization of the parabolic category $O$ associated with the pair $(\mathfrak{g}(V), \Phi_1)$.

In this paper, we consider the case when $V = \mathbb{C}^\infty$. We define the appropriate version of the parabolic category $O$ and its localization, and show that the latter is equivalent to a "restricted" inverse limit of categories $O^p_{t,C,N}$ with $N$ tending to infinity. The Schur-Weyl functors $\overline{SW}_{t,C,N}$ then give an anti-equivalence between this category and the category $Rep^{ab}(S_t)$.

This duality provides an unexpected tensor structure on the category $O^p_{t,C,\infty}$.

1. Introduction

1.1. The Karoubian rigid symmetric monoidal categories $Rep(S_t)$, $t \in \mathbb{C}$, were defined by P. Deligne in [4] as a polynomial family of categories interpolating the categories of finite-dimensional representations of the symmetric groups; namely, at points $n = t \in \mathbb{Z}_+$, the category $Rep(S_{t=n})$ allows an essentially surjective additive symmetric monoidal functor onto the standard category $Rep(S_n)$. The categories $Rep(S_t)$ were subsequently studied by P. Deligne and others (e.g. by V. Ostrik, J. Comes in [1, 2]).

In [5], we gave an analogue to the classical Schur-Weyl duality in the setting of Deligne categories. In order to do this, we defined the “complex tensor power” of a finite-dimensional unital complex vector space (i.e. a vector space $V$ with a distinguished non-zero vector $1$). This “complex tensor power” of $V$, denoted by $V^{\otimes t}$, is an $Ind$-object in the category $Rep(S_t)$, and comes with an action of $\mathfrak{g}(V)$ on it; moreover, this $Ind$-object is a Harish-Chandra module for the pair $(\mathfrak{g}(V), \Phi_1)$, where $\Phi_1 \subset GL(V)$ is the mirabolic subgroup preserving the vector $1$.

The “$t$-th tensor power” of $V$ is defined for any $t \in \mathbb{C}$; for $n = t \in \mathbb{Z}_+$, the functor $Rep(S_{t=n}) \to Rep(S_n)$ takes this $Ind$-object of $Rep(S_{t=n})$ to the usual tensor power $V^{\otimes n}$ in $Rep(S_n)$. Moreover, the action of $\mathfrak{g}(V)$ on the former object corresponds to the action of $\mathfrak{g}(V)$ on $V^{\otimes n}$.

This allowed us to define an additive contravariant functor, called the Schur-Weyl functor,

$$SW_{t,V} : Rep^{ab}(S_t) \to O^p_V, \quad SW_{t,V} := \text{Hom}_{Rep^{ab}(S_t)}(\cdot, V^{\otimes t})$$

Here $Rep^{ab}(S_t)$ is the abelian envelope of the category $Rep(S_t)$ (this envelope was described in [2, 4 Chapter 8]). The category $O^p_V$ is a version of the parabolic category $O$ for $\mathfrak{g}(V)$ associated with the pair $(V, 1)$, which is defined as follows.

We define $O^p_V$ to be the category of Harish-Chandra modules for the pair $(\mathfrak{g}(V), \Phi_1)$ on which the group $GL(V/\mathbb{C}1)$ acts by polynomial maps, and which satisfy some additional finiteness conditions (similar to the ones in the definition of the usual BGG category $O$).

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We now consider the localization of \(O^p_t\) obtained by taking the full subcategory of \(O^p_t\) consisting of modules of degree \(t\) (i.e., modules on which \(\text{Id}_V \in \text{End}(V)\) acts by the scalar \(t\)), and localizing by the Serre subcategory of \(\mathfrak{gl}(V)\)-polynomial modules. This quotient is denoted by \(\hat{O}^p_{t,V}\).

It turns out that for any unital finite-dimensional space \((V, \mathbb{1})\) and any \(t \in \mathbb{C}\), the contravariant functor \(\hat{SW}_{t,V}\) makes \(\hat{O}^p_{t,V}\) a Serre quotient of \(\text{Rep}^{ab}(S_t)^{op}\).

In this paper, we will consider the categories \(\hat{O}^p_{t,C^N}\) for \(N \in \mathbb{Z}_+\) and for \(N = \infty\).

Defining appropriate restriction functors
\[
\hat{\text{Res}}_{n-1,n} : \hat{O}^p_{t,C^n} \longrightarrow \hat{O}^p_{t,C^{n-1}}
\]
allows us to consider the inverse limit of the system \((\hat{O}^p_{t,C^n})_{n \geq 0}, (\hat{\text{Res}}_{n-1,n})_{n \geq 1}\). Inside this inverse limit we consider a full subcategory which is equivalent to \(\hat{O}^p_{t,C^\infty}\). This subcategory is called the “the restricted inverse limit” of the system \((\hat{O}^p_{t,C^n})_{n \geq 0}, (\hat{\text{Res}}_{n-1,n})_{n \geq 1}\), and will be denoted by \(\lim_{\leftarrow}^{n \geq 1, \text{restr}} \hat{O}^p_{t,C^n}\). This category has an intrinsic description, which we give in this paper (intuitively, this is the inverse limit among finite-length categories).

Similarly to [5], we define the complex tensor power of the unital vector space \((\mathbb{C}^\infty, \mathbb{1} := e_1)\), and the corresponding Schur-Weyl contravariant functor \(\hat{SW}_{t,C^\infty}\). As in the finite-dimensional case, this functor induces an exact contravariant functor \(\hat{SW}_{t,C^\infty}\), and we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Rep}^{ab}(S_t)^{op} & \xrightarrow{\lim_{\leftarrow}^{n \geq 1, \text{restr}}} & \hat{O}^p_{t,C^n} \\
\hat{SW}_{t,lim} & \downarrow & \uparrow \\
\hat{SW}_{t,C^\infty} & \longrightarrow & \hat{O}^p_{t,C^\infty}
\end{array}
\]

The contravariant functors \(\hat{SW}_{t,C^\infty}, \hat{SW}_{t,lim}\) turn out to be anti-equivalences induced by the Schur-Weyl functors \(SW_{t,C^\infty}\).

The anti-equivalences \(\hat{SW}_{t,C^\infty}, \hat{SW}_{t,lim}\) induce an unexpected structure of a rigid symmetric monoidal category on
\[
\hat{O}^p_{t,C^\infty} \cong \lim_{n \geq 1, \text{restr}} \hat{O}^p_{t,C^n}
\]
We obtain an interesting corollary: the duality in this category given by the tensor structure will coincide with the one arising from the usual notion of duality in BGG category \(O\).

1.2. Notation and structure of the paper. The base field throughout the paper will be \(\mathbb{C}\). The notation and definitions used thoughtout this paper can be found in [5, Section 2].

Sections [2] and [3] contain preliminaries on the Deligne category \(\text{Rep}(S_t)\) (throughout the paper, we use the parameter \(\nu\) instead of \(t\), the categories of polynomial representations of \(\mathfrak{gl}_N\) (\(N \in \mathbb{Z}_+ \cup \{\infty\}\)) and the parabolic category \(O\) for \(\mathfrak{gl}_N\). These sections are based on [5], [6] and [12].

In Section [5], we give a description of the parabolic category \(O\) for \(\mathfrak{gl}_\infty\) as a restricted inverse limit of the parabolic categories \(O\) for \(\mathfrak{gl}_n\) as \(n\) tends to infinity.

In Sections [6] and [7] we recall the definition of the complex tensor power \((\mathbb{C}^N)^{\otimes\nu}\), and define the functors \(SW_{\nu,V} : \text{Rep}^{ab}(S_\nu)^{op} \rightarrow \hat{O}^p_{\nu,V}\) and \(\hat{SW}_{\nu,V} : \text{Rep}^{ab}(S_\nu)^{op} \rightarrow \hat{O}^p_{\nu,V}\) for a unital vector space \((V, \mathbb{1})\) (finite or infinite-dimensional). In Subsection [8,2] we recall the finite-dimensional case (studied in [5]).

Section [9] discusses the restricted inverse limit construction in the case of the classical Schur-Weyl duality, which motivates our construction for the Deligne categories. Sections [9] and [10] prove the main results of the paper. Section [14] discusses the relation between the rigidity (duality) in \(\text{Rep}^{ab}(S_\nu)\) and the duality in the parabolic category \(O\) for \(\mathfrak{gl}_\infty\).
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2. Deligne category $\text{Rep}(S_\nu)$

A detailed description of the Deligne category $\text{Rep}(S_t)$ and its abelian envelope can be found in [1] [2] [3] [4] as well as [5]. Throughout the paper, we will use the parameter $\nu$ instead of the parameter $t$ used in the introduction.

2.1. General description. For any $\nu \in \mathbb{C}$, the category $\text{Rep}(S_\nu)$ is generated, as a $\mathbb{C}$-linear Karoubian tensor category, by one object, denoted $\mathfrak{h}$. This object is the analogue of the permutation representation of $S_n$, and any object in $\text{Rep}(S_\nu)$ is a direct summand in a direct sum of tensor powers of $\mathfrak{h}$.

For $\nu \notin \mathbb{Z}_+$, $\text{Rep}(S_\nu)$ is a semisimple abelian category.

If $\nu$ is a non-negative integer, then the category $\text{Rep}(S_\nu)$ has a tensor ideal $\mathcal{I}_\nu$, called the ideal of negligible morphisms (this is the ideal of morphisms $f : X \rightarrow Y$ such that $tr(fu) = 0$ for any morphism $u : Y \rightarrow X$). In that case, the classical category $\text{Rep}(S_n)$ of finite-dimensional representations of the symmetric group for $n := \nu$ is equivalent to $\text{Rep}(S_{\nu=n})/\mathcal{I}_\nu$ (equivalent as Karoubian rigid symmetric monoidal categories).

The full, essentially surjective functor $\text{Rep}(S_{\nu=n}) \rightarrow \text{Rep}(S_n)$ defining this equivalence will be denoted by $S_n$.

Note that $S_n$ sends $\mathfrak{h}$ to the permutation representation of $S_n$.

The indecomposable objects of $\text{Rep}(S_\nu)$, regardless of the value of $\nu$, are parametrized (up to isomorphism) by all Young diagrams (of arbitrary size). We will denote the indecomposable object in $\text{Rep}(S_\nu)$ corresponding to the Young diagram $\tau$ by $X_\tau$.

For non-negative integer $\nu =: n$, we have: the partitions $\lambda$ for which $X_\lambda$ has a non-zero image in the quotient $\text{Rep}(S_{\nu=n})/\mathcal{I}_{\nu=n} \cong \text{Rep}(S_n)$ are exactly the $\lambda$ for which $\lambda_1 + |\lambda| \leq n$.

If $\lambda_1 + |\lambda| \leq n$, then the image of $\lambda$ in $\text{Rep}(S_n)$ is the irreducible representation of $S_n$ corresponding to the Young diagram $\lambda(n)$: the Young diagram obtained by adding a row of length $n - |\lambda|$ on top of $\lambda$.

For each $\nu$, we define an equivalence relation $\sim$ on the set of all Young diagrams: we say that $\lambda \sim \lambda'$ if the sequence $(\nu - |\lambda|, \lambda_1 - 1, \lambda_2 - 2, \ldots)$ can be obtained from the sequence $(\nu - |\lambda'|, \lambda'_1 - 1, \lambda'_2 - 2, \ldots)$ by permuting a finite number of entries.

The equivalence classes thus obtained are in one-to-one correspondence with the blocks of the category $\text{Rep}(S_\nu)$ (see [1]).

We say that a block is trivial if the corresponding equivalence class is trivial, i.e. has only one element (in that case, the block is a semisimple category).

The non-trivial equivalence classes (respectively, blocks) are parametrized by all Young diagrams of size $\nu$; in particular, this happens only if $\nu \in \mathbb{Z}_+$. These classes are always of the form $\{\lambda^{(i)}\}i$, with

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \ldots$$

(each $\lambda^{(i)}$ can be explicitly described based on the Young diagram of size $\nu$ corresponding to this class).

2.2. Abelian envelope. As it was mentioned before, the category $\text{Rep}(S_\nu)$ is defined as a Karoubian category. For $\nu \notin \mathbb{Z}_+$, it is semisimple and thus abelian, but for $\nu \in \mathbb{Z}_+$, it is not abelian. Fortunately, it has been shown that $\text{Rep}(S_\nu)$ possesses an “abelian envelope”, that is, that it can be embedded (as a full monoidal subcategory) into an abelian rigid symmetric monoidal category, and this abelian envelope has a universal mapping property (see [2] Theorem 1.2], 3 [8.21.2]).

We will denote the abelian envelope of the Deligne category $\text{Rep}(S_\nu)$ by $\text{Rep}^ab(S_\nu)$ (with $\text{Rep}^ab(S_\nu) := \text{Rep}(S_\nu)$ for $\nu \notin \mathbb{Z}_+$).
An explicit construction of the category $\text{Rep}^{ab}(S_{\nu=\infty})$ is given in [2], and a detailed description of its structure can be found in [3].

It turns out that the category $\text{Rep}^{ab}(S_{\nu})$ is a highest weight category (with infinitely many weights) corresponding to the partially ordered set ($\{\text{Young diagrams}\}, \geq$), where

$$\lambda \geq \mu \iff \lambda \sim \mu, \lambda \subset \mu$$

(namely, in a non-trivial $\sim$-class, $\lambda(i) \geq \lambda(j)$ if $i \leq j$).

Thus the isomorphism classes of simple objects in $\text{Rep}^{ab}(S_{\nu})$ are parametrized by the set of Young diagrams of arbitrary sizes. We will denote the simple object corresponding to a Young diagram of arbitrary sizes. We will denote the simple object corresponding to $\nu$ by $L(\lambda)$.

We will also use the fact that blocks of the category $\text{Rep}^{ab}(S_{\nu})$, just like the blocks of $\text{Rep}(S_{\nu})$, are parametrized by $\sim$-equivalence classes.

For each $\sim$-equivalence class, the corresponding block of $\text{Rep}(S_{\nu})$ is the full subcategory of tilting objects in the corresponding block of $\text{Rep}^{ab}(S_{\nu})$ (see [2] Proposition 2.9, Section 4).

3. $\mathfrak{gl}_\infty$ and the Restricted Inverse Limit of Representations of $\mathfrak{gl}_n$

In this section, we discuss the category of polynomial representations of the Lie algebra $\mathfrak{gl}_\infty$ and its relation to the categories of polynomial representations of $\mathfrak{gl}_N$ for $N \geq 0$. The representations of the Lie algebra $\mathfrak{gl}_\infty$ are discussed in detail in [11], [3], as well as [12, Section 3].

Most of the constructions and the proofs of the statements appearing in this section can be found in [6, Section 7].

3.1. The Lie algebra $\mathfrak{gl}_\infty$. Let $\mathbb{C}^\infty$ be a complex vector space with a countable basis $e_1, e_2, e_3, \ldots$.

Consider the Lie algebra $\mathfrak{gl}_\infty$ of infinite matrices $A = (a_{ij})_{i,j \geq 1}$ with finitely many non-zero entries. We have a natural action of $\mathfrak{gl}_\infty$ on $\mathbb{C}^\infty$ and on the restricted dual $\mathbb{C}^{\infty}_* = \text{span}_\mathbb{C}(e_1^*, e_2^*, e_3^*, \ldots)$ (here $e_i^*$ is the linear functional dual to $e_i$: $e_i^*(e_j) = \delta_{ij}$).

Let $N \in \mathbb{Z}_+ \cup \{\infty\}$, and let $m \geq 1$. We will consider the Lie subalgebra $\mathfrak{gl}_m \subset \mathfrak{gl}_N$ which consists of matrices $A = (a_{ij})_{1 \leq i,j \leq N}$ for which $a_{ij} = 0$ whenever $i > m$ or $j > m$. We will also denote by $\mathfrak{gl}_m$ the Lie subalgebra of $\mathfrak{gl}_N$ consisting of matrices $A = (a_{ij})_{1 \leq i,j \leq N}$ for which $a_{ij} = 0$ whenever $i \leq m$ or $j \leq m$.

Remark 3.1.1. Note that $\mathfrak{gl}_m \cong \mathfrak{gl}_{N-m}$ for any $N, m$.

3.2. Categories of Polynomial Representations of $\mathfrak{gl}_N$. In this subsection, $N \in \mathbb{Z}_+ \cup \{\infty\}$. The notation $\mathbb{C}^*_N$ will stand for $(\mathbb{C}^N)^*$ whenever $N \in \mathbb{Z}_+$, and for $\mathbb{C}^{\infty}_*$ when $N = \infty$.

Consider the category $\text{Rep}(\mathfrak{gl}_N)_{\text{poly}}$ of polynomial representations of $\mathfrak{gl}_N$: this is the category of the representations of $\mathfrak{gl}_N$ which can be obtained as summands of a direct sum of tensor powers of the tautological representation $\mathbb{C}^N$ of $\mathfrak{gl}_N$.

It is easy to see that this is a semisimple abelian category, whose simple objects are parametrized (up to isomorphism) by all Young diagrams of arbitrary sizes whose length does not exceed $N$: the simple object corresponding to $\lambda$ is $S^\lambda \mathbb{C}^N$.

Remark 3.2.1. Note that $\text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}}$ is the free abelian symmetric monoidal category generated by one object (see [12 (2.2.11)]). It has an equivalent definition as the category of polynomial functors of bounded degree, which can be found in [9] and in [12].

Next, we define a natural $\mathbb{Z}_+$-grading on objects in $\text{Ind} - \text{Rep}(\mathfrak{gl}_N)_{\text{poly}}$ (c.f. [12 (2.2.2)]):

Definition 3.2.2. The objects in $\text{Ind} - \text{Rep}(\mathfrak{gl}_N)_{\text{poly}}$ have a natural $\mathbb{Z}_+$-grading. Namely, given $M \in \text{Ind} - \text{Rep}(\mathfrak{gl}_N)_{\text{poly}}$, we consider the decomposition $M = \bigoplus_{\lambda} S^\lambda \mathbb{C}^N \otimes \text{mult}_\lambda$ (here $\text{mult}_\lambda$ is the multiplicity space of $S^\lambda \mathbb{C}^N$ in $M$), we define

$$gr_k(M) := \bigoplus_{\lambda:|\lambda|=k} S^\lambda \mathbb{C}^N \otimes \text{mult}_\lambda$$
Of course, the morphisms in $\text{Ind} - \text{Rep}(\mathfrak{gl}_n)_{\text{poly}}$ respect this grading.

3.3. Specialization and restriction functors. We now define specialization functors from the category of representations of $\mathfrak{gl}_\infty$ to the categories of representations of $\mathfrak{gl}_n$ (c.f. [12 Section 3]):

**Definition 3.3.1.**

$$\Gamma_n : \text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}} \to \text{Rep}(\mathfrak{gl}_n)_{\text{poly}}, \Gamma_n := (\cdot)^{\mathfrak{gl}_n}_d$$

One can easily check (see [6 Section 7]) that the functor $\Gamma_n$ is well-defined. The following Lemma is proved in [11], [12 Section 3]:

**Lemma 3.3.2.** The functors $\Gamma_n$ are additive symmetric monoidal functors between semisimple symmetric monoidal categories. Their effect on the simple objects is described as follows: for any Young diagram $\lambda$, $\Gamma_n(S^\lambda \mathbb{C}^\infty) \cong S^\lambda \mathbb{C}^n$.

Next, we define the restriction functors we will use:

**Definition 3.3.3.** Let $n \geq 1$. We define the functor

$$\text{Res}_{n-1,n} : \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \to \text{Rep}(\mathfrak{gl}_{n-1})_{\text{poly}}, \text{Res}_{n-1,n} := (\cdot)^{\mathfrak{gl}_{n-1}}_d$$

Again, one can easily show that these functors are well-defined.

**Remark 3.3.4.** There is an alternative definition of the functors $\text{Res}_{n-1,n}$. One can think of the functor $\text{Res}_{n-1,n}$ acting on a $\mathfrak{gl}_n$-module $M$ as taking the restriction of $M$ to $\mathfrak{gl}_{n-1}$ and then considering only the vectors corresponding to “appropriate” central characters.

More specifically, we say that a $\mathfrak{gl}_n$-module $M$ is of degree $d$ if $\text{Id}_{\mathbb{C}^n} \in \mathfrak{gl}_n$ acts by $d \cdot \text{Id}_M$ on $M$. Also, given any $\mathfrak{gl}_n$-module $M$, we may consider the maximal submodule of $M$ of degree $d$, and denote it by $\text{deg}_d(M)$. This defines an endo-functor $\text{deg}_d$ of $\text{Rep}(\mathfrak{gl}_n)_{\text{poly}}$.

Note that a simple module $S^\lambda \mathbb{C}^n$ is of degree $|\lambda|$.

The notion of degree gives a decomposition

$$\text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \cong \bigoplus_{d \in \mathbb{Z}_+} \text{Rep}(\mathfrak{gl}_n)_{\text{poly},d}$$

where $\text{Rep}(\mathfrak{gl}_n)_{\text{poly},d}$ is the full subcategory of $\text{Rep}(\mathfrak{gl}_n)_{\text{poly}}$ consisting of all polynomial $\mathfrak{gl}_n$-modules of degree $d$.

Then

$$\text{Res}_{n-1,n} = \bigoplus_{d \in \mathbb{Z}_+} \text{Res}_{d,n-1,n} : \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \to \text{Rep}(\mathfrak{gl}_{n-1})_{\text{poly}}$$

where

$$\text{Res}_{d,n-1,n} : \text{Rep}(\mathfrak{gl}_n)_{\text{poly},d} \to \text{Rep}(\mathfrak{gl}_{n-1})_{\text{poly},d}, \text{Res}_{d,n-1,n} := \text{deg}_d \circ \text{Res}_{\mathfrak{gl}_n,n-1}^{\mathfrak{gl}_{n-1}}$$

where $\text{Res}_{\mathfrak{gl}_n,n-1}^{\mathfrak{gl}_{n-1}}$ is the usual restriction functor for the pair $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$.

Once again, the functors $\text{Res}_{n-1,n}$ are additive functors between semisimple categories, and satisfy:

**Lemma 3.3.5.** $\text{Res}_{n-1,n}(S^\lambda \mathbb{C}^n) \cong S^\lambda \mathbb{C}^{n-1}$ for any Young diagram $\lambda$.

Moreover, these functors are compatible with the functors $\Gamma_n$ defined before.

**Lemma 3.3.6.** For any $n \geq 1$, we have a commutative diagram:

$$
\begin{array}{ccc}
\text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}} & \xrightarrow{\Gamma_n} & \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \\
\downarrow \Gamma_{n-1} & & \downarrow \text{Res}_{n-1,n} \\
\text{Rep}(\mathfrak{gl}_{n-1})_{\text{poly}} & \xrightarrow{\Gamma_n} & \text{Rep}(\mathfrak{gl}_{n-1})_{\text{poly}}
\end{array}
$$

That is, there is a natural isomorphism $\Gamma_{n-1} \cong \text{Res}_{n-1,n} \circ \Gamma_n$. 


In particular, this implies:

**Corollary 3.3.7.** The functors \( \text{Res}_{n-1,n} : \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \to \text{Rep}(\mathfrak{gl}_{n-1})_{\text{poly}} \) are symmetric monoidal functors.

### 3.4. The restricted inverse limit of categories \( \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \)

This subsection gives a description of the category \( \text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}} \) as a “restricted” inverse limit of categories \( \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \) (see \([6]\) for details).

We will use the framework developed in \([6]\) for the inverse limits of categories with \( \mathbb{Z}_+ \)-filtrations on objects, and the restricted inverse limits of finite-length categories (abelian categories in which every object admits a Jordan-Holder filtration). The notions of \( \mathbb{Z}_+ \)-filtered functors and shortening functors are defined loc. cit.

We define a \( \mathbb{Z}_+ \)-filtration on the objects of \( \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \) for each \( n \in \mathbb{Z}_+ \):

**Notation 3.4.1.** For each \( k \in \mathbb{Z}_+ \), let \( \text{Fil}_k(\text{Rep}(\mathfrak{gl}_n)_{\text{poly}}) \) be the full additive subcategory of \( \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \) generated by \( S^\lambda \mathbb{C}^n \) such that \( \ell(\lambda) \leq k \).

Clearly the subcategories \( \text{Fil}_k(\text{Rep}(\mathfrak{gl}_n)_{\text{poly}}) \) give us a \( \mathbb{Z}_+ \)-filtration on the objects of the category \( \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \). Furthermore, by Lemma 3.3.5 the functors \( \text{Res}_{n-1,n} \) are \( \mathbb{Z}_+ \)-filtered functors, i.e. they induce functors

\[
\text{Res}_{n-1,n}^k : \text{Fil}_k(\text{Rep}(\mathfrak{gl}_n)_{\text{poly}}) \to \text{Fil}_k(\text{Rep}(\mathfrak{gl}_{n-1})_{\text{poly}})
\]

This allows us to consider the inverse limit

\[
\varprojlim_{n \in \mathbb{Z}_+, \text{Z}_+\text{-filt}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \cong \varprojlim_{k \in \mathbb{Z}_+, n \in \mathbb{Z}_+} \text{Fil}_k(\text{Rep}(\mathfrak{gl}_n)_{\text{poly}})
\]

This is an abelian category (with a natural \( \mathbb{Z}_+ \)-filtration on objects).

Note that by Lemma 3.3.5 the functors \( \text{Res}_{n-1,n} \) are shortening functors; furthermore, the system \( \left( \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \right)_{n \in \mathbb{Z}_+}, (\text{Res}_{n-1,n})_{n \geq 1} \) satisfies the conditions listed in \([6]\) Section 6], and therefore the category \( \varprojlim_{n \in \mathbb{Z}_+, \text{Z}_+\text{-filt}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \) is also equivalent to the restricted inverse limit of this system, \( \varprojlim_{n \in \mathbb{Z}_+, \text{restr}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \).

**Remark 3.4.2.** The functors \( \text{Res}_{n-1,n} \) are symmetric monoidal functors, so the category \( \varprojlim_{n \in \mathbb{Z}_+, \text{restr}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \) is a symmetric monoidal category.

**Proposition 3.4.3.** We have an equivalence of symmetric monoidal Karoubian categories

\[
\Gamma_{\lim} : \text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}} \to \varprojlim_{n \in \mathbb{Z}_+, \text{restr}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}}
\]

induced by the symmetric monoidal functors

\[
\Gamma_n = (\cdot)^{\mathfrak{gl}_n^+} : \text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}} \to \text{Rep}(\mathfrak{gl}_n)_{\text{poly}}
\]

### 4. Parabolic category \( O \)

In this section, we describe a version of the parabolic category \( O \) for \( \mathfrak{gl}_N \) which we are going to work with. We will give a definition which will describe both the relevant category for \( \mathfrak{gl}_n \), and for \( \mathfrak{gl}_\infty \).

#### 4.1. For the benefit of the reader, we will start by giving a definition for \( \mathfrak{gl}_N \) when \( N \) is a positive integer; this definition is analogous to the usual definition of the category \( O \). The generic definition will then be just a slight modification of the first to accomodate the case \( N = \infty \).

This version of the parabolic category \( O \) is attached to a pair: a vector space \( V \) and a fixed non-zero vector \( 1 \) in it. Such a pair is called a *unital vector space*. In our case, we will just consider \( V = \mathbb{C}^N \), with the standard basis \( e_1, e_2, \ldots \), and the chosen vector \( 1 := e_1 \).

Fix \( N \in \mathbb{Z}, N \geq 1 \).

The following notation will be used throughout the paper:
Notation 4.1.1.

- We denote by $p_N \subset \mathfrak{gl}_N$ the parabolic Lie subalgebra which consists of all the endomorphisms $\phi : \mathbb{C}^N \to \mathbb{C}^N$ for which $\phi(1) \in \mathbb{C}1$. In terms of matrices this is $span\{E_{1,j} | j > 1\}$.
- $u^+_N \subset p_N$ denotes the algebra of endomorphisms $\phi : \mathbb{C}^N \to \mathbb{C}^N$ for which $Im(\phi) \subset \mathbb{C}1 \subset Ker(\phi)$. In terms of matrices, $u^+_N = span\{E_{1,j} | j > 1\}$.

Denote $U_N := span\{e_2, e_3, ..., e_N\}$.

We have a splitting $\mathfrak{gl}_N \cong p_N \oplus u^+_N$, where $u^-_N \cong U_N = span\{E_{i,j} | i > 1\}$. This gives us an analogue of the triangular decomposition:

$$\mathfrak{gl}_N \cong \mathbb{C}Id_{\mathbb{C}^N} \oplus u^-_N \oplus u^+_N \oplus \mathfrak{gl}(U_N)$$

We can now give a precise definition of the parabolic category $O$ which we will use:

**Definition 4.1.2.** We define the category $O^{p^N}_{\mathbb{C}^N}$ to be the full subcategory of $Mod_{\mathcal{U}(\mathfrak{gl}_N)}$ whose objects $M$ satisfy the following conditions:

- Viewed as a $\mathcal{U}(\mathfrak{gl}(U_N))$-module, $M$ is a direct sum of polynomial $\mathcal{U}(\mathfrak{gl}(U_N))$-modules (that is, $M$ belongs to $Ind - Rep(\mathfrak{gl}(U_N))_{poly}$).
- $M$ is locally finite over $u^+_N$.
- $M$ is a finitely generated $\mathcal{U}(\mathfrak{gl}_N)$-module.

**Remark 4.1.3.** One can replace the requirement that $u^+_N$ act locally finitely on $M$ by the requirement that $\mathcal{U}(u^+_N)$ act locally nilpotently on $M$.

**Remark 4.1.4.** One can, in fact, give an equivalent definition of the category $O^p_N$ corresponding to a finite-dimensional unitary vector $(V, \mathbb{1})$ without choosing a splitting(c.f. [5, Section 5] and Introduction [1]).

We now recall the following definition:

**Definition 4.1.5.** A module $M$ over the Lie algebra $\mathfrak{gl}_N$ will be said to be of degree $K \in \mathbb{C}$ if $Id_{\mathbb{C}^N} \in \mathfrak{gl}_N$ acts by $K \ Id_M$ on $M$.

We will denote by $O^{p^N}_{\nu, \mathbb{C}^N}$ the full subcategory of $O^{p^N}_{\mathbb{C}^N}$ whose objects are modules of degree $\nu$.

Note that for a module $M$ of $O^{p^N}_{\mathbb{C}^N}$ to be of degree $\nu$ is the same as to require that $E_{1,j}$ acts on each subspace $S^\lambda U_N$ of $M$ by the scalar $\nu - |\lambda|$.

**Definition 4.1.6.** Let $\nu \in \mathbb{C}$. Define the functor $deg_\nu : Mod_{\mathcal{U}(\mathfrak{gl}_N)} \to Mod_{\mathcal{U}(\mathfrak{gl}_N)}$ by putting $deg_\nu(E)$ to be the maximal submodule of $E$ of degree $\nu$ (see Definition [13.4]). For a morphism $f : E \to E'$ of $\mathfrak{gl}_N$-modules, we put $deg_\nu(f) := f|_{deg_\nu(E)}$.

Let $E \in Mod_{\mathcal{U}(\mathfrak{gl}_N)}$. The maximal submodule of $E$ of degree $\nu$ is well-defined: it is the subspace of $E$ consisting of all vectors on which $Id_{\mathbb{C}^N}$ acts by the scalar $\nu$, and it is a $\mathfrak{gl}_N$-submodule since $Id_{\mathbb{C}^N}$ lies in the center of $\mathfrak{gl}_N$.

One can show that the functor $deg_\nu : Mod_{\mathcal{U}(\mathfrak{gl}_N)} \to Mod_{\mathcal{U}(\mathfrak{gl}_N)}$ is left-exact. Moreover, it is easy to show that the category $O^{p^N}_{\nu, \mathbb{C}^N}$ is a direct summand of $O^{p^N}_{\mathbb{C}^N}$, and the functor $deg_\nu : O^{p^N}_{\mathbb{C}^N} \to O^{p^N}_{\nu, \mathbb{C}^N}$ is exact.

**4.2. Parabolic category $O$ for $\mathfrak{gl}_N$.** We now give a definition of the parabolic category $O$ which for $\mathfrak{gl}_N$, $N$ being either a positive integer or infinity.

Again, we let $N \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$.

Consider a unital vector space $(\mathbb{C}^N, \mathbb{1})$, where $\mathbb{1} := e_1$. Put $U_N := span_{\mathbb{C}}(e_2, e_3, ..., e_N) \subset \mathbb{C}^N$, so that we have a splitting $\mathbb{C}^N = Ce_1 \oplus U_N$. We will also denote $U_{N,+} := span(e_2^+, e_3^+, ...)$ (so $U_{N,+} = U^+_N$ whenever $N \in \mathbb{Z}$).

We have a decomposition

$$\mathfrak{gl}_N \cong \mathfrak{gl}(U_N) \oplus \mathfrak{gl}_1 \oplus u^+_N \oplus u^-_N$$
Of course, for any \( N, u_{p_N}^- \cong U_N \); moreover, \( u_{p_N}^+ \cong U_{N,s} \).

We will also use the isomorphisms \( \mathfrak{gl}(U_N) \cong \mathfrak{gl}_1^+ \cong \mathfrak{gl}_{N-1} \).

**Definition 4.2.1.**

- Define the category \( \text{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N) - \text{poly}} \) to be the category of \( \mathfrak{gl}_N \)-modules whose restriction to \( \mathfrak{gl}(U_N) \) lies in \( \text{Ind} - \text{Rep}(\mathfrak{gl}_{U_N})_{\text{poly}} \); that is, \( \mathfrak{gl}_N \)-modules whose restriction to \( \mathfrak{gl}(U_N) \) is a (perhaps infinite) direct sum of Schur functors applied to \( U_N \).

The morphisms would be \( \mathfrak{gl}_N \)-equivariant maps.

- We say that an object \( M \in \text{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N) - \text{poly}} \) is of degree \( \nu \) (\( \nu \in \mathbb{C} \)) if on every summand \( S^\lambda U_N \subset M \), the element \( E_{1,1} \in \mathfrak{gl}_N \) acts by \( (\nu - |\lambda|) \text{Id}_{S^\lambda U_N} \).

- Let \( M \in \text{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N) - \text{poly}} \). We have a commutative algebra \( \text{Sym}(U_N) \cong U(u_{p_N}^-) \) (the enveloping algebra of \( u_{p_N}^- \subset \mathfrak{gl}_N \)). The action of \( \mathfrak{gl}_N \) on \( M \) gives \( M \) a structure of a \( \text{Sym}(U_N) \)-module.

We say that \( M \) is finitely generated over \( \text{Sym}(U_N) \) if \( M \) is a quotient of a “free finitely-generated \( \text{Sym}(U_N) \)-module”; that is, as a \( \text{Sym}(U_N) \)-module, \( M \) is a quotient (in \( \text{Ind} - \text{Rep}(\mathfrak{gl}_{U_N})_{\text{poly}} \) of \( \text{Sym}(U_N) \otimes E \) for some \( E \in \text{Rep}(\mathfrak{gl}(U_N))_{\text{poly}} \).

- Let \( M \in \text{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N) - \text{poly}} \). We have a commutative algebra \( \text{Sym}(U_N, s) \cong U(u_{p_N}^+) \) (the enveloping algebra of \( u_{p_N}^+ \subset \mathfrak{gl}_N \)). The action of \( \mathfrak{gl}_N \) on \( M \) gives \( M \) a structure of a \( \text{Sym}(U_N, s) \)-module.

We say that \( M \) is locally nilpotent over the algebra \( U(u_{p_N}^+) \) if for any \( \nu \in M \), there exists \( m \geq 0 \) such that for any \( A \in \text{Sym}^m(U_N, s) \) we have: \( A.v = 0 \).

Recall the natural \( \mathbb{Z}_+ \)-grading on the object of \( \text{Ind} - \text{Rep}(\mathfrak{gl}_{U_N})_{\text{poly}} \).

For each \( M \in \text{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N) - \text{poly}} \), the above definition implies: \( \mathfrak{gl}(U_N) \) acts by operators act by operators of degree zero, \( U_{N,s} \) acts by operators of degree 1. We now define the parabolic category \( O \) for \( \mathfrak{gl}_N \) which we will use throughout the paper:

**Definition 4.2.2.** We define the category \( \mathcal{O}_{\mu, \mathbb{C}^N} \) to be the full subcategory of \( \text{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N) - \text{poly}} \) whose objects \( M \) satisfy the following requirements:

- \( M \) is of degree \( \nu \).
- \( M \) is finitely generated over \( \text{Sym}(U_N) \).
- \( M \) is locally nilpotent over the algebra \( U(u_{p_N}^+) \).

Of course, for a positive integer \( N \), this is just the category \( \mathcal{O}_{\mu, \mathbb{C}^N} \) we defined in the beginning of this section.

We will also consider the localization of the category \( \mathcal{O}_{\mu, \mathbb{C}^N} \) by its Serre subcategory of polynomial \( \mathfrak{gl}_N \)-modules of degree \( \nu \); such modules exist iff \( \nu \in \mathbb{Z}_+ \). This localization will be denoted by

\[
\widehat{\mathcal{O}}_{\mu, \mathbb{C}^N} : \mathcal{O}_{\mu, \mathbb{C}^N} \rightarrow \widehat{\mathcal{O}}_{\mu, \mathbb{C}^N}
\]

and will play an important role when we consider the Schur-Weyl duality in complex rank.

#### 4.3. Duality in category \( O \).

Recall that in the category \( O \) for \( \mathfrak{gl}_n \), we have the notion of a duality (c.f. [10] Section 3.2); namely, given a \( \mathfrak{gl}_n \)-module \( M \) with finite-dimensional weight spaces, we can consider the twisted action of \( \mathfrak{gl}_n \) on the dual space \( M^* \), given by \( A.f := f \circ A^T \), where \( A^T \) means the transpose of \( A \in \mathfrak{gl}_n \). This makes \( M^* \) a \( \mathfrak{gl}_n \)-module. We then take \( M' \) to be the maximal submodule of \( M^* \) lying in category \( O \).

More explicitly, considering \( M \) as a direct sum of its finite-dimensional weight spaces

\[
M = \bigoplus \lambda M_{\lambda}
\]


we can consider the restricted twisted dual

$$M^\vee := \bigoplus_\lambda M_\lambda^\vee$$

(that is, we take the dual to each weight space separately). The action of $\mathfrak{gl}_n$ is given by $A \cdot f := f \circ A^T$ for any $A \in \mathfrak{gl}_n$.

The module $M^\vee$ is called the dual of $M$, and we get an exact functor $(\cdot)^\vee : O^{op} \to O$.

**Proposition 4.3.1.** The category $O^{p_n}_{\nu, \mathbb{C}_n}$ is closed under taking duals, and the duality functor $(\cdot)^\vee : (O^{p_n}_{\nu, \mathbb{C}_n})^{op} \to O^{p_n}_{\nu, \mathbb{C}_n}$ is an equivalence of categories.

In fact, a similar construction can be made for $O^{p_{\infty}}_{\nu, \mathbb{C}_n}$. All modules $M$ in $O^{p_{\infty}}_{\nu, \mathbb{C}_n}$ are weight modules with respect to the subalgebra of diagonal matrices in $\mathfrak{gl}_{\infty}$, and the weight spaces are finite-dimensional (due to the polynomiality condition in the definition of $O^{p_{\infty}}_{\nu, \mathbb{C}_n}$). This allows one to construct the restricted twisted dual $M^\vee$ in the same way as before, and obtain an exact functor

$$(\cdot)^\vee : (O^{p_{\infty}}_{\nu, \mathbb{C}_n})^{op} \to O^{p_{\infty}}_{\nu, \mathbb{C}_n}$$

**Remark 4.3.2.** It is obvious that for $n \in \mathbb{Z}_+$, the functor $(\cdot)^\vee : (O^{p_n}_{\nu, \mathbb{C}_n})^{op} \to O^{p_n}_{\nu, \mathbb{C}_n}$ takes finite-dimensional (polynomial) modules to finite-dimensional (polynomial) modules.

In fact, one can easily check that the functor $(\cdot)^\vee : (O^{p_{\infty}}_{\nu, \mathbb{C}_n})^{op} \to O^{p_{\infty}}_{\nu, \mathbb{C}_n}$ takes polynomial modules to polynomial modules as well.

### 4.4. Structure of the category $O^{p_n}_{\nu, \mathbb{C}_n}$

In this subsection, we present some facts about the category $O^{p_n}_{\nu, \mathbb{C}_n}$ which will be used later on. The material of this section is discussed in more detail in [1] Section 5 and is mostly based on [10] Chapter 9.

Fix $\nu \in \mathbb{C}$, and fix $n \in \mathbb{Z}_+$. We denote by $e_1, e_2, ..., e_n$ the standard basis of $\mathbb{C}^n$, and denote $1 := e_1, U_n := \text{span}\{e_2, e_3, ..., e_n\}$.

We will consider the category $O^{p_n}_{\nu, \mathbb{C}_n}$ for the unital vector space $(\mathbb{C}^n, 1)$ and the splitting $\mathbb{C}^n = \mathbb{C}1 \oplus U_n$.

**Proposition 4.4.1.** The category $O^{p_n}_{\nu, \mathbb{C}_n}$ (resp. $\text{Ind} - O^{p_n}_{\nu, \mathbb{C}_n}$) is closed under taking duals, direct sums, submodules, quotients and extensions in $O_{\mathfrak{gl}_n}$, as well as tensoring with finite dimensional $\mathfrak{gl}_n$-modules.

The category $O^{p_n}_{\nu, \mathbb{C}_n}$ decomposes into blocks (each of the blocks is an abelian category in its own right). To each $\nu$-class of Young diagrams corresponds a block of $O^{p_n}_{\nu, \mathbb{C}_n}$. If all Young diagrams $\lambda$ in this $\nu$-class have length at least $n$, then the corresponding block is zero. To each non-zero block of $O^{p_n}_{\nu, \mathbb{C}_n}$ corresponds a unique $\nu$-class.

Moreover, the blocks corresponding to trivial $\nu$-classes are either semisimple (i.e. equivalent to the category $\text{Vect}_\mathbb{C}$), or zero.

We now proceed to discuss standard objects in $O^{p_n}_{\nu, \mathbb{C}_n}$.

**Definition 4.4.2.** Let $\lambda$ be a Young diagram. The generalized Verma module $M_{p_n}(\nu - |\lambda|, \lambda)$ is defined to be the $\mathfrak{gl}_n$-module

$$\mathcal{U}(\mathfrak{gl}_n) \otimes_{\mathcal{U}(p_n)} S^\lambda U_n$$

where $\mathfrak{gl}(U_n)$ acts naturally on $S^\lambda U_n$, $\text{Id}_{\mathbb{C}^n} \in p_n$ acts on $S^\lambda U_n$ by scalar $\nu$, and $u^+_{p_n}$ acts on $S^\lambda U_n$ by zero.

Thus $M_{p_n}(\nu - |\lambda|, \lambda)$ is the parabolic Verma module for $(\mathfrak{gl}_n, p_n)$ with highest weight $(\nu - |\lambda|, \lambda)$ iff $n - 1 \geq \ell(\lambda)$, and zero otherwise.

**Definition 4.4.3.** $L(\nu - |\lambda|, \lambda)$ is defined to be zero (if $n \geq \ell(\lambda)$), or the simple module for $\mathfrak{gl}_n$ of highest weight $(\nu - |\lambda|, \lambda)$ otherwise.
The following basic lemma will be very helpful:

**Lemma 4.4.4.** Let $\lambda$ be a Young diagram such that $\ell(\lambda) < n$. We then have an isomorphism of $\mathfrak{gl}(U_n)$-modules:

$$M_{p_n}(\nu - |\lambda|, \lambda) \cong \text{Sym}(U_n) \otimes S^\lambda U_n$$

We will also use the following lemma.

**Lemma 4.4.5.** Let $\{\lambda^{(i)}\}_i$ be a non-trivial $\sim$-class, and $i \geq 0$ be such that $\ell(\lambda^{(i)}) < n$.

Then there is a short exact sequence

$$0 \to L(\nu - |\lambda^{(i+1)}|, \lambda^{(i+1)}) \to M_{p_n}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to L(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to 0$$

**Corollary 4.4.6.** The isomorphism classes of the generalized Verma modules and the simple polynomial modules in $O_{\nu,C^n}^{p_n}$ form a basis for the Grothendieck group of $O_{\nu,C^n}^{p_n}$.

5. **Stable inverse limit of parabolic categories $O$**

### 5.1. Restriction functors.

**Definition 5.1.1.** Let $n \geq 1$. Define the functor

$$\mathcal{R}es_{n-1,n} : O_{\nu,C^n}^{p_n} \to O_{\nu,C^{n-1}}^{p_{n-1}}, \quad \mathcal{R}es_{n-1,n} := (\cdot)^{\mathfrak{gl}_{n-1}}$$

Again, the subalgebras $\mathfrak{gl}_{n-1}, \mathfrak{gl}_{n-1}^\perp \subset \mathfrak{gl}_n$ commute, and therefore the subspace of $\mathfrak{gl}_{n-1}$-invariants of a $\mathfrak{gl}_n$-module automatically carries an action of $\mathfrak{gl}_{n-1}$.

We need to check that this functor is well-defined. In order to do so, consider the functor $\mathcal{R}es_{n-1,n} : O_{\nu,C^n}^{p_n} \to \text{Mod}_{\mathfrak{gl}(\mathfrak{gl}_{n-1})}$. This functor is well-defined, and we will show that the objects in the image lie in the full subcategory $O_{\nu,C^{n-1}}^{p_{n-1}}$ of $\text{Mod}_{\mathfrak{gl}(\mathfrak{gl}_{n-1})}$.

Note that the functor $\mathcal{R}es_{n-1,n}$ can alternatively be defined as follows: for a module $M$ in $O_{\nu,C^n}^{p_n}$, we restrict the action of $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n-1}$, and then only take the vectors in $M$ attached to specific central characters. More specifically, we have:

**Lemma 5.1.2.** The functor $\mathcal{R}es_{n-1,n}$ is naturally isomorphic to the composition $\text{deg}_{\nu} \circ \mathcal{R}es_{\mathfrak{gl}_n}^{\mathfrak{gl}_{n-1}}$ (the functor $\text{deg}_{\nu}$ was defined in Definition 4.4.2).

**Proof.** Let $M \in O_{\nu,C^n}^{p_n}$. For any vector $m \in M$, we know that $\text{Id}_{C^n} \cdot m = (E_{1,1} + E_{2,2} + \ldots + E_{n,n}) \cdot m = \nu m$. Then the requirement that $\text{Id}_{C^{n-1}} \cdot m = (E_{1,1} + E_{2,2} + \ldots + E_{n-1,n-1}) \cdot m = \nu m$ is equivalent to requiring that $E_{n,n} \cdot m = 0$, namely that $m \in M^{\mathfrak{gl}_{n-1}}$. $\square$

We will now use this information to prove the following lemma:

**Lemma 5.1.3.** The functor $\mathcal{R}es_{n-1,n} : O_{\nu,C^n}^{p_n} \to O_{\nu,C^{n-1}}^{p_{n-1}}$ is well-defined.

**Proof.** Let $M \in O_{\nu,C^n}^{p_n}$, and consider the $\mathfrak{gl}_{n-1}$-module $\mathcal{R}es_{n-1,n}(M)$. By definition, this is a module of degree $\nu$. We will show that it lies in $O_{\nu,C^{n-1}}^{p_{n-1}}$.

First of all, consider the inclusion $\mathfrak{gl}(U_{n-1})^\perp \subset \mathfrak{gl}(U_{n-1}) \subset \mathfrak{gl}(U_n)$. This inclusion gives us the restriction functor (see Definition 3.3.3)

$$\mathcal{R}es_{U_{n-1},U_{n-1}} : \text{Rep}(\mathfrak{gl}(U_{n-1}))_{\text{poly}} \to \text{Rep}(\mathfrak{gl}(U_n))_{\text{poly}}, \quad \mathcal{R}es_{U_{n-1},U_{n-1}} := (\cdot)^{\mathfrak{gl}(U_{n-1})^\perp}$$

The latter is an additive functor between semisimple categories, and takes polynomial representations of $\mathfrak{gl}(U_{n-1})$ to polynomial representations of $\mathfrak{gl}(U_n)$.

Now, the restriction to $\mathcal{R}es_{U_{n-1},U_{n-1}}$ of the $\mathfrak{gl}_{n-1}$-module $\mathcal{R}es_{n-1,n}(M)$ is isomorphic to $\mathcal{R}es_{U_{n-1},U_{n-1}}(M^{\mathfrak{gl}(U_{n-1})})$, and thus is a polynomial representation of $\mathfrak{gl}(U_{n-1})$.

Secondly, $\mathcal{R}es_{n-1,n}(M)$ is locally nilpotent over $U(\mathfrak{u}_{p_n}^+)$, since $M$ is locally nilpotent over $U(\mathfrak{u}_{p_n}^+)$ and $U(\mathfrak{u}_{p_n}^+) \subset U(\mathfrak{u}_{p_n}^+)$. 

10
It remains to check that given $M \in O^p_{\nu, C_n}$, the module $\mathfrak{Res}_{n-1,n}(M)$ is finitely generated over $\text{Sym}(U_{n-1})$. Indeed, we know that there exists a polynomial $\mathfrak{gl}(U_n)$-module $E$ and a surjective $\mathfrak{gl}(U_n)$-equivariant morphism of $\text{Sym}(U_n)$-modules $\text{Sym}(U_n) \otimes E \rightarrow M$. Taking the $\mathfrak{gl}(U_{n-1})^+$-invariants and using Lemma 3.3.7, we conclude that there is a surjective $\mathfrak{gl}(U_{n-1})$-equivariant morphism of $\text{Sym}(U_{n-1})$-modules

$$\text{Sym}(U_{n-1}) \otimes E^{\mathfrak{gl}(U_{n-1})^+} \rightarrow \mathfrak{Res}_{n-1,n}(M)$$

Thus $\mathfrak{Res}_{n-1,n}(M)$ is finitely generated over $\text{Sym}(U_{n-1})$.

\[\square\]

**Lemma 5.1.4.** The functor $\mathfrak{Res}_{n-1,n} : O^p_{\nu, C_n} \rightarrow O^p_{\nu, C_{n-1}}$ is exact.

**Proof.** We use Lemma 5.1.2. The functor $\mathfrak{deg}_\nu : O_{C_{n-1}} \rightarrow O^p_{\nu, C_{n-1}}$ is exact, so the functor $\mathfrak{Res}_{n-1,n}$ is obviously exact as well. \[\square\]

**Lemma 5.1.5.** The functor $\mathfrak{Res}_{n-1,n}$ takes parabolic Verma modules to either parabolic Verma modules or to zero:

$$\mathfrak{Res}_{n-1,n}(M_{p_n}(\nu - |\lambda|, \lambda)) \cong M_{p_{n-1}}(\nu - |\lambda|, \lambda)$$

(recall that the latter is a parabolic Verma module for $\mathfrak{gl}_{n-1}$ iff $\ell(\lambda) \leq n-2$, and zero otherwise).

**Proof.** Consider the parabolic Verma module $M_{p_n}(\nu - |\lambda|, \lambda)$, where the Young diagram $\lambda$ has length at most $n - 1$.

By definition of the parabolic Verma module $M_{p_n}(\nu - |\lambda|, \lambda)$, we have:

$$M_{p_n}(\nu - |\lambda|, \lambda) = U(\mathfrak{gl}_n) \otimes u_{p_n} S^\lambda U_n$$

The branching rule for $\mathfrak{gl}(U_{n-1}) \subset \mathfrak{gl}(U_n)$ tells us that

$$S^\lambda U_n |_{\mathfrak{gl}(U_{n-1})} \cong \bigoplus_{\lambda'} S^{\lambda'} U_{n-1}$$

the sum taken over the set of all Young diagrams obtained from $\lambda$ by removing several boxes, no two in the same column.

So

$$\mathfrak{Res}_{\mathfrak{gl}_n}(M_{p_n}(\nu - |\lambda|, \lambda)) \cong \left( \bigoplus_{\lambda' \subset \lambda} M_{p_{n-1}}(\nu - |\lambda|, \lambda') \right) \otimes U \left( \frac{u_{p_n}}{u_{p_{n-1}}} \right)$$

Here

- $M_{p_{n-1}}(\nu - |\lambda|, \lambda')$ is either a parabolic Verma module for $\mathfrak{gl}_{n-1}$ of highest weight $\nu - |\lambda| + |\lambda'|$ (note that it is of degree $\nu - |\lambda| + |\lambda'|$) or zero.
- $\mathfrak{gl}(U_{n-1})$ acts trivially on the space $U \left( \frac{u_{p_n}}{u_{p_{n-1}}} \right)$. This space is isomorphic, as a $\mathbb{Z}_+$-graded vector space, to $\mathbb{C}[t]$ ($t$ standing for $E_{n,1} \in \mathfrak{gl}_n$) and $E_{1,1}$ acts on it by derivations $\frac{t d}{dt}$.

Thus $\mathbb{I}_{d_{n-1}} \in \mathfrak{gl}_n$ acts on the subspace $M_{p_{n-1}}(\nu - |\lambda|, \lambda') \otimes t^k \subset M_{p_n}(\nu - |\lambda|, \lambda)$ by the scalar $\nu - |\lambda| + |\lambda'| - k$.

We now apply the functor $\mathfrak{deg}_\nu$ to the module $\mathfrak{Res}_{\mathfrak{gl}_n}(M_{p_n}(\nu - |\lambda|, \lambda))$.

To see which subspaces $M_{p_{n-1}}(\nu - |\lambda'|, \lambda') \otimes t^k$ of $M_{p_n}(\nu - |\lambda|, \lambda)$ will survive after applying $\mathfrak{deg}_\nu$, we require that $|\lambda| - |\lambda'| + k = 0$. But we are only considering Young diagrams $\lambda'$ such that $\lambda' \subset \lambda$, and non-negative integers $k$, which means that the only relevant case is $\lambda' = \lambda$, $k = 0$.

We conclude that

$$\mathfrak{Res}_{n-1,n}(M_{p_n}(\nu - |\lambda|, \lambda)) \cong M_{p_{n-1}}(\nu - |\lambda|, \lambda)$$

\[\square\]
Lemma 5.1.6. Given a simple $\mathfrak{gl}_n$-module $L_n(\nu - |\lambda|, \lambda)$,
\[ \text{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong L_{n-1}(\nu - |\lambda|, \lambda) \]
(recall that the latter is a simple $\mathfrak{gl}_{n-1}$-module iff $\ell(\lambda) \leq n - 2$, and zero otherwise).

Proof. Note that the statement follows immediately from Lemma 5.1.5 when $\lambda$ lies in a trivial $\lambda$-class; for a non-trivial $\lambda$-class $\{\lambda^{(i)}\}_i$, we have short exact sequences (see Lemma 1.4.5):
\[ 0 \to L_n(\nu - \lambda^{(i+1)}, \lambda) \to M_{\rho_n}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to L_n(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to 0 \]
Using the exactness of $\text{Res}_{n-1,n}$, we can prove the required statement for $L_n(\nu - |\lambda^{(i)}|, \lambda^{(i)})$ by induction on $i$, provided the statement is true for $i = 0$.

So it remains to check that
\[ \text{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong L_{n-1}(\nu - |\lambda|, \lambda) \]
for the minimal Young diagram $\lambda$ in any non-trivial $\lambda$-class.

Recall that in that case, $L_n(\nu - |\lambda|, \lambda) = S^{\lambda(\nu)}C^n$ is a finite-dimensional simple representation of $\mathfrak{gl}_n$.

The branching rule for $\mathfrak{gl}_n, \mathfrak{gl}_{n-1}$ implies that
\[ \text{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(S^{\lambda(\nu)}C^n) \cong \bigoplus_{\mu} S^{\mu}C^{n-1} \]
the sum taken over the set of all Young diagrams obtained from $\lambda(\nu)$ by removing several boxes, no two in the same column.

Considering only the summands of degree $\nu$, we see that
\[ \text{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong S^{\lambda(\nu)}C^{n-1} \cong L_{n-1}(\nu - |\lambda|, \lambda) \]
\[ \square \]

The functor $\text{Res}_{n-1,n} : \hat{O}^p_{\nu, C^n} \to \hat{O}^p_{\nu, C^{n-1}}$ clearly takes polynomial modules to polynomial modules; together with Lemma 5.1.6, this means that $\text{Res}_{n-1,n}$ factors through an exact functor
\[ \hat{\text{Res}}_{n-1,n} : \hat{O}^p_{\nu, C^n} \to \hat{O}^p_{\nu, C^{n-1}} \]
i.e. we have a commutative diagram
\[ \begin{array}{ccc}
\hat{O}^p_{\nu, C^n} & \xrightarrow{\text{Res}_{n-1,n}} & \hat{O}^p_{\nu, C^{n-1}} \\
\hat{\pi}_n \downarrow & & \downarrow \hat{\pi}_{n-1} \\
\hat{O}^p_{\nu, C^n} & \xrightarrow{\text{Res}_{n-1,n}} & \hat{O}^p_{\nu, C^{n-1}}
\end{array} \]
(see Subsection 4.2 for the definition of the localizations $\hat{\pi}_n$).

5.2. Specialization functors.

Definition 5.2.1. Let $n \geq 1$. Define the functor
\[ \Gamma_n : O_{\nu, C^n}^p \to O_{\nu, C^n}^p, \quad \Gamma_n := (\cdot)^{\mathfrak{gl}_n^+} \]
As before, the subalgebras $\mathfrak{gl}_n, \mathfrak{gl}_n^+ \subset \mathfrak{gl}_\infty$ commute, and therefore the subspace of $\mathfrak{gl}_n^+$-invariants of a $\mathfrak{gl}_\infty$-module automatically carries an action of $\mathfrak{gl}_n$.

Lemma 5.2.2. The functor $\Gamma_n : O_{\nu, C^n}^p \to O_{\nu, C^n}^p$ is well-defined.

Proof. The proof is essentially the same as in Lemma 5.1.3
Next, we check that the functor $\Gamma_n$ is exact:

**Lemma 5.2.3.** Let $\nu_{\infty} \to \nu_{n}$ be the restriction functors described in Subsection 5.1 and $\nu_{\infty} \to \nu_{n}$ be the restriction functors. By [6, Section 6], these limits are equivalent to the respective restricted inverse limits on objects (see 5.1 for the relevant definitions).

**Proof.** These statements follow directly from Lemma 5.1.6, which tells us that $\nu_{\infty}(L_n(\nu - |\lambda|, \lambda)) \cong L_n(\nu - |\lambda|, \lambda) \cong \lim_{k \in \mathbb{Z}_+} \nu_{\infty}(L_n(\nu - |\lambda|, \lambda))$.

We can now consider the inverse limits of the $\mathbb{Z}_+$-filtered systems $((O_{\nu_{\infty}}^n)_{n \geq 1}, (\nu_{\infty})_{n \geq 1})$ and $((\nu_{\infty})_{n \geq 1}, (\nu_{\infty})_{n \geq 1})$. By [6], Section 6, these limits are equivalent to the respective restricted inverse limits $\nu_{\infty}((O_{\nu_{\infty}}^n)_{n \geq 1}, (\nu_{\infty})_{n \geq 1})$ and $((\nu_{\infty})_{n \geq 1}, (\nu_{\infty})_{n \geq 1})$.

5.3. Stable inverse limit of categories $O_{\nu_{n}}^{\nu_{\infty}}$ and the category $O_{\nu_{n}}^{\nu_{\infty}}$. The restriction functors $\nu_{\infty} \to \nu_{n}$ described in Subsection 5.1 allow us to consider the inverse limit of the system $((O_{\nu_{n}}^n)_{n \geq 1}, (\nu_{\infty}^{\nu_{n}})_{n \geq 1})$.

Similarly, we can consider the inverse limit of the system $((\nu_{\infty}^{\nu_{n}})_{n \geq 1}, (\nu_{\infty}^{\nu_{n}})_{n \geq 1})$.

Let $n \geq 1$.

**Notation 5.3.1.** For each $k \in \mathbb{Z}_+$, let $\text{Fil}_k(O_{\nu_{n}}^{\nu_{n}})$ (respectively, $\text{Fil}_k(\nu_{\infty}^{\nu_{n}})$) be the Serre sub-category of $O_{\nu_{n}}^{\nu_{n}}$ (respectively, $\nu_{\infty}^{\nu_{n}}$) generated by simple modules $L_n(\nu - |\lambda|, \lambda)$ (respectively, $\nu_{n}(L_n(\nu - |\lambda|, \lambda))$, with $|\lambda| \leq k$.

This defines $\mathbb{Z}_+$-filtrations on the objects of $O_{\nu_{n}}^{\nu_{n}}$, $\nu_{\infty}^{\nu_{n}}$, i.e.

$O_{\nu_{n}}^{\nu_{n}} \cong \lim_{k \in \mathbb{Z}_+} \text{Fil}_k(O_{\nu_{n}}^{\nu_{n}})$, $\nu_{\infty}^{\nu_{n}} \cong \lim_{k \in \mathbb{Z}_+} \text{Fil}_k(\nu_{\infty}^{\nu_{n}})$

**Lemma 5.3.2.** Let $n \geq 1$. The functors $\nu_{\infty} \to \nu_{n}$ are both shortening and $\mathbb{Z}_+$-filtered functors between finite-length categories with $\mathbb{Z}_+$-filtrations on objects (see [6] for the relevant definitions).

**Proof.** These statements follow directly from Lemma 5.1.6, which tells us that $\nu_{\infty}(L_n(\nu - |\lambda|, \lambda)) \cong L_n(\nu - |\lambda|, \lambda)$. 

We can now consider the inverse limits of the $\mathbb{Z}_+$-filtered systems $((O_{\nu_{n}}^n)_{n \geq 1}, (\nu_{\infty}^{\nu_{n}})_{n \geq 1})$ and $((\nu_{\infty}^{\nu_{n}})_{n \geq 1}, (\nu_{\infty}^{\nu_{n}})_{n \geq 1})$. By [6], Section 6, these limits are equivalent to the respective restricted inverse limits $\nu_{\infty}((O_{\nu_{n}}^n)_{n \geq 1}, (\nu_{\infty}^{\nu_{n}})_{n \geq 1})$ and $((\nu_{\infty}^{\nu_{n}})_{n \geq 1}, (\nu_{\infty}^{\nu_{n}})_{n \geq 1})$. 

...
The functors $\Gamma_n$ described above induce exact functors

$$\Gamma_{\lim} : \mathcal{O}_{\nu, \mathbb{C}}^{\infty} \longrightarrow \varprojlim_{n \geq 1} \mathcal{O}_{\nu, \mathbb{C}}^{p_n}$$

and

$$\hat{\Gamma}_{\lim} : \hat{\mathcal{O}}_{\nu, \mathbb{C}}^{\infty} \longrightarrow \varprojlim_{n \geq 1} \hat{\mathcal{O}}_{\nu, \mathbb{C}}^{p_n}$$

We would like to show that this functor is an equivalence of categories:

**Proposition 5.3.3.** The functors $\Gamma_n$ induce an equivalence

$$\Gamma_{\lim} : \mathcal{O}_{\nu, \mathbb{C}}^{\infty} \longrightarrow \varprojlim_{n \geq 1, \text{ restr}} \mathcal{O}_{\nu, \mathbb{C}}^{p_n}$$

**Proof.** First of all, we need to check that this functor is well-defined. Namely, we need to show that for any $M \in \mathcal{O}_{\nu, \mathbb{C}}^{\infty}$, the sequence $\{\ell_U(\mathfrak{gl}_{n+1})(\Gamma_{n+1}(M))\}_n$ stabilizes. In fact, it is enough to show that this sequence is bounded (since it is obviously increasing).

Recall that we have a surjective map of $\text{Sym}(u_{\nu, \mathbb{C}}^{\infty})$-modules $\text{Sym}(u_{\nu, \mathbb{C}}^{\infty}) \otimes E \to M$ for some $E \in \text{Rep}(\mathfrak{gl}(U_\infty))_{\text{poly}}$. Since $\Gamma_{n+1}$ is exact, it gives us a surjective map $\text{Sym}(u_{\nu, \mathbb{C}}^{\infty}) \otimes \Gamma_{n+1}(E) \to \Gamma_{n+1}(M)$ for any $n \geq 0$, with $\Gamma_{n+1}(E)$ being a polynomial $\mathfrak{gl}(U_{n+1})$-module.

Now,

$$\ell_U(\mathfrak{gl}_{n+1})(\Gamma_{n+1}(M)) \leq \ell_U(\mathfrak{gl}_{n+1})(\Gamma_{n+1}(M)) \leq \ell_U(\mathfrak{gl}(U_{n+1}))(\Gamma_{n+1}(E))$$

The sequence $\{\ell_U(\mathfrak{gl}(U_{n+1}))(\Gamma_{n+1}(E)))\}_n$ is bounded by Proposition 3.4.3 and thus the sequence $\{\ell_U(\mathfrak{gl}_{n+1})(\Gamma_{n+1}(M)))\}_n$ is bounded as well.

We now show that $\Gamma_{\lim}$ is an equivalence.

A construction similar to the one appearing in [6, Section 7.5] gives a left-adjoint to the functor $\Gamma_{\lim}$; that is, we will define a functor

$$\Gamma_{\star} : \varprojlim_{n \geq 1, \text{ restr}} \mathcal{O}_{\nu, \mathbb{C}}^{p_n} \longrightarrow \mathcal{O}_{\nu, \mathbb{C}}^{\infty}$$

Let $((M_n)_{n \geq 1}, (\phi_{n-1,n})_{n \geq 2})$ be an object of $\varprojlim_{n \geq 1, \text{ restr}} \mathcal{O}_{\nu, \mathbb{C}}^{p_n}$.

The isomorphisms $\phi_{n-1,n} : \mathfrak{Res}_{n-1,n}(M_n) \rightarrow M_{n-1}$ define $\mathfrak{gl}_{n-1}$-equivariant inclusions $M_{n-1} \rightarrow M_n$. Consider the vector space

$$M := \bigcup_{n \geq 1} M_n$$

which has a natural action of $\mathfrak{gl}_{\infty}$ on it.

It is easy to see that the obtained $\mathfrak{gl}_{\infty}$-module $M$ is a direct sum of polynomial $\mathfrak{gl}(U_\infty)$-modules, and is locally nilpotent over the algebra

$$\mathcal{U}(u_{\nu, \mathbb{C}}^{\infty}) \cong \text{Sym}(U_{\infty, \ast}) \cong \bigcup_{n \geq 1} \text{Sym}(U_n^\ast)$$

We now prove the following sublemma:

**Sublemma 5.3.4.** Let $((M_n)_{n \geq 1}, (\phi_{n-1,n})_{n \geq 2})$ be an object of $\varprojlim_{n \geq 1, \text{ restr}} \mathcal{O}_{\nu, \mathbb{C}}^{p_n}$. Then $M := \bigcup_{n \geq 1} M_n$ is a finitely generated module over $\text{Sym}(U_\infty) \cong \mathcal{U}(u_{\nu, \mathbb{C}}^{\infty})$.

**Proof.** Recall from [6, Section 4] that all the objects in the abelian category $\varprojlim_{n \geq 1, \text{ restr}} \mathcal{O}_{\nu, \mathbb{C}}^{p_n}$ have finite length, and that the simple objects in this category are exactly those of the form $((L_n(\nu - |\lambda|, \lambda))_{n \geq 1}, (\phi_{n-1,n})_{n \geq 2})$ for a fixed Young diagram $\lambda$. So we only need to check that applying the above construction to these simple objects gives rise to finitely generated modules over $\text{Sym}(U_\infty) \cong \mathcal{U}(u_{\nu, \mathbb{C}}^{\infty})$.

Using Corollary 4.4.1, we now reduce the proof of the sublemma to proving the following two statements:
• Let $\lambda$ be a fixed Young diagram and let $((L_n(\nu - |\lambda|), \phi))_{n \geq 1}$ be a simple object in $\lim_{\leftarrow n \geq 1, \text{restr}} O_{\nu, C_n}$ in which $L_n(\nu - |\lambda|, \lambda)$ is polynomial for every $n$ (i.e. $\lambda$ is minimal in its non-trivial $\mathcal{S}$-class).

Then $L := \bigcup_{n \geq 1} L_n(\nu - |\lambda|, \lambda)$ is a polynomial $\mathfrak{gl}_\infty$-module (in particular, a finitely generated module over $\text{Sym}(U_{\infty}) \cong \mathcal{U}(\mathfrak{u}_{\infty}^-)$).

• Let $\lambda$ be a fixed Young diagram and let $((M_n(\nu - |\lambda|), \phi(n-1,n))_{n \geq 2})$ be an object of $\lim_{\leftarrow n \geq 1, \text{restr}} O_{\nu, C_n}$ (this is a sequence of “compatible” parabolic Verma modules). Then

$$M := \bigcup_n M_n(\nu - |\lambda|, \lambda)$$

is a finitely generated module over $\text{Sym}(U_{\infty}) \cong \mathcal{U}(\mathfrak{u}_{\infty}^-)$.

The first statement follows immediately from Proposition 3.4.3 (cf. [6, Section 7.5]).

To prove the second statement, recall that

$$M_n(\nu - |\lambda|, \lambda) \cong \text{Sym}(U_n) \otimes S^\lambda U_n$$

(Lemma 4.4.4).

$$M := \bigcup_n M_n(\nu - |\lambda|, \lambda) \cong \bigcup_n \text{Sym}(U_n) \otimes S^\lambda U_n \cong \text{Sym}(U_{\infty}) \otimes S^\lambda U_{\infty}$$

which is clearly a finitely generated module over $\text{Sym}(U_{\infty}) \cong \mathcal{U}(\mathfrak{u}_{\infty}^-)$. □

This allows us to define the functor $\Gamma^*_\lim$ by setting

$$\Gamma^*_\lim((M_n)_{n \geq 1}, (\phi(n-1,n))_{n \geq 2}) := \bigcup_{n \geq 1} M_n$$

and requiring that it act on morphisms accordingly.

The definition of $\Gamma^*_\lim$ gives us a natural transformation

$$\Gamma^*_\lim \circ \Gamma^*_\lim \cong \text{Id}_{O_{\nu, C_n}}$$

Restricting the action of $\mathfrak{gl}_\infty$ to $\mathfrak{gl}(U_{\infty})$ and using Proposition 3.4.3, we conclude that this natural transformation is an isomorphism.

Notice that the definition of $\Gamma^*_\lim$ implies that this functor is faithful. Thus we conclude that the functor $\Gamma^*_\lim$ is an equivalence of categories, and so is $\Gamma^*_\lim$.

□

**Proposition 5.3.5.** The functors $\hat{\Gamma}_n$ induce an equivalence

$$\hat{\Gamma}_\lim : \widehat{O}_{\nu, C_n} \to \lim_{\leftarrow n \geq 1, \text{restr}} \widehat{O}_{\nu, C_n}$$

**Proof.** Let $M \in O_{\nu, C_n}$. First of all, we need to check that the functor $\hat{\Gamma}_\lim$ is well-defined; that is, we need to show that the sequence $\{\ell_{\widehat{O}_{\nu, C_n}}((\hat{\pi}_n(\Gamma_n(M))))\}_{n \geq 1}$ is bounded from above.

Indeed,

$$\ell_{\widehat{O}_{\nu, C_n}}((\hat{\pi}_n(\Gamma_n(M)))) \leq \ell_{O_{\nu, C_n}}((\Gamma_n(M)))$$

But the sequence $\{\ell_{O_{\nu, C_n}}((\Gamma_n(M))))\}_{n \geq 1}$ is bounded from above by Lemma 5.3.3 so the original sequence is bounded from above as well.

Thus we obtain a commutative diagram

$$\begin{array}{ccc}
\text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}, \nu} & \longrightarrow & O_{\nu, C_n} \\
\Gamma^*_\lim \downarrow & & \hat{\Gamma}_\lim \downarrow \\
\lim_{\leftarrow n \geq 1, \text{restr}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}, \nu} & \longrightarrow & \lim_{\leftarrow n \geq 1, \text{restr}} O_{\nu, C_n} \\
\end{array}$$

$$\begin{array}{ccc}
& \xrightarrow{\hat{\pi}_n} & \\
& \xrightarrow{\hat{\pi}_n} & \\
\hat{\Gamma}_\lim & \longrightarrow & \hat{\Gamma}_\lim \\
\end{array}$$

$$\begin{array}{ccc}
\xrightarrow{\ell_{\widehat{O}_{\nu, C_n}}} & & \xrightarrow{\ell_{\widehat{O}_{\nu, C_n}}} \\
\xrightarrow{\ell_{O_{\nu, C_n}}} & & \xrightarrow{\ell_{O_{\nu, C_n}}} \\
\text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}, \nu} & \longrightarrow & \lim_{\leftarrow n \geq 1, \text{restr}} O_{\nu, C_n} \\
\end{array}$$
where \( \text{Rep}(\mathfrak{gl}_N)_{\text{poly}, \nu} \) is the Serre subcategory of \( \hat{O}^p_{\nu, \mathbb{C}} \) consisting of all polynomial modules of degree \( \nu \). The rows of this commutative diagram are "exact" (in the sense that \( \hat{O}^p_{\nu, \mathbb{C}} \)) is the Serre quotient of the category \( O^p_{\nu, \mathbb{C}} \) by the Serre subcategory \( \text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}, \nu} \), and similarly for the bottom row.

The functors
\[
\Gamma_{\lim} : \text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}, \nu} \to \lim_{n \geq 1, \text{restr}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}, \nu}
\]
and
\[
\Gamma_{\lim} : O^p_{\nu, \mathbb{C}} \to \lim_{n \geq 1, \text{restr}} O^p_{\nu, \mathbb{C}_n}
\]
are equivalences of categories (by Propositions 3.4.3 and 5.3.3), and thus the functor \( \Gamma_{\lim} \) is an equivalence as well.

6. Complex tensor powers of a unital vector space

In this section we describe the construction of a complex tensor power of the unital vector space \( \mathbb{C}^N \) with the chosen vector \( 1 := e_1 \) (again, \( N \in \mathbb{Z}_+ \cup \{ \infty \} \)). A general construction of the complex tensor power of a unital vector space is given in [5, Section 6].

Again, we denote \( U_N := \text{span}\{e_2, e_3, \ldots\} \), and \( U_{N*} := \text{span}\{e^*_2, e^*_3, \ldots\} \subset \mathbb{C}_n^N \). As before, we have a decomposition:
\[
\mathfrak{gl}_N \cong \mathbb{C} \text{Id}_{\mathbb{C}^N} \oplus u^+_p N \oplus \mathfrak{gl}(U_N)
\]
such that \( U_N \cong u^+_p N \), \( U_{N*} \cong u^-_p N \), and if \( N \) is finite, we have \( U^*_{N*} \cong U_{N*} \).

Fix \( \nu \in \mathbb{C} \).

Recall from [3, Section 4] that for any \( \nu \in \mathbb{C} \), in the Deligne category \( \text{Rep}(S_\nu) \) we have the objects \( \Delta_k \) \((k \in \mathbb{Z}_+)\). These objects interpolate the representations \( \mathcal{C}Inj(\{1, \ldots, k\}, \{1, \ldots, n\}) \cong \text{Ind}_{S_{n-k} \times S_k \times S_k}(\mathbb{C}) \) of the symmetric groups \( S_n \); in fact, for any \( n \in \mathbb{Z}_+ \) we have:
\[
S_n(\Delta_k) \cong \mathcal{C}Inj(\{1, \ldots, k\}, \{1, \ldots, n\})
\]
where \( S_n : \text{Rep}(S_{\nu = n}) \to \text{Rep}(S_n) \) is the monoidal functor discussed in Subsection 2.1.

**Definition 6.0.6** (Complex tensor power). Define the object \( (\mathbb{C}^N)^{\otimes \nu} \) of \( \text{Ind} - (\text{Rep}^{ab}(S_\nu) \boxtimes O^p_{\nu, \mathbb{C}}) \) by setting
\[
(\mathbb{C}^N)^{\otimes \nu} := \bigoplus_{k \geq 0} (U^\otimes N_k \otimes \Delta_k)^{S_k}
\]
The action on \( \mathfrak{gl}_N \) on \( (\mathbb{C}^N)^{\otimes \nu} \) is given as follows:

- \( E_{1,1} \in \mathfrak{gl}_N \) acts by scalar \( \nu - k \) on each summand \( (U^\otimes N_k \otimes \Delta_k)^{S_k} \).
- \( A \in \mathfrak{gl}(U_N) \subset \mathfrak{gl}_N \) acts on \( (U^\otimes N_k \otimes \Delta_k)^{S_k} \) by
  \[
  \sum_{1 \leq i \leq k} A^{(i)}|_{U^\otimes N_k} \otimes \text{Id}_{\Delta_k} : (U^\otimes N_k \otimes \Delta_k)^{S_k} \to (U^\otimes N_k \otimes \Delta_k)^{S_k}
  \]
- \( u \in U_N \cong u^+_p N \) acts by morphisms of degree 1, which are given explicitly in [5, Section 6.2].
- \( f \in U_{N*} \cong u^-_p N \) acts by morphisms of degree \(-1\), which are given explicitly in [5, Section 6.2].
Proposition 6.0.10. Let inclusion following statement (when $N$ \text{ and } $\Gamma$\text{C}) \text{ with respect to the Lie subalgebra } \mathcal{CE}_{1,1} \oplus \mathfrak{gl}(U_N)$, the above fact implies that this isomorphism is also $\mathfrak{g}l_N$-equivariant.

In other words, if there exists a way to define an action of $\mathfrak{g}l_N$ whose restriction to the the Lie subalgebra $\mathcal{CE}_{1,1} \oplus \mathfrak{gl}(U_N)$ is given by the formulas above, then such an action of $\mathfrak{g}l_N$ is unique.

Remark 6.0.7. The actions of the elements of $u_+^+$, $u_-^-$, though not written here explicitly, are in fact uniquely determined by the actions of $E_{1,1}$ and $\mathfrak{gl}(U_N)$.

To see this, note that the ideal in the Lie algebra $\mathfrak{g}l_N$ generated by the Lie subalgebra $\mathcal{CE}_{1,1} \oplus \mathfrak{gl}(U_N)$ is the entire $\mathfrak{g}l_N$. Given two $\mathfrak{g}l_N$-modules $M_1$, $M_2$ and an isomorphism $M_1 \rightarrow M_2$ which is equivariant with respect to the Lie subalgebra $\mathcal{CE}_{1,1} \oplus \mathfrak{gl}(U_N)$, the above fact implies that this isomorphism is also $\mathfrak{g}l_N$-equivariant.

Remark 6.0.8. The proof that the object $(\mathcal{C}^N)_{\mathbb{Z}_+}$ lies in the category $\text{Ind} - (\text{Rep}(S_\nu) \boxtimes \mathcal{O}^{pN}_{N,\mathbb{C}^N})$ is an easy check, and can be found in [5]. In particular, it means that the action of the mirabolic subalgebra $\mathfrak{g}l_\mathbb{C}$ on the complex tensor power $(\mathcal{C}^N)_{\mathbb{Z}_+}$ integrates to an action of the mirabolic subgroup $\mathfrak{p}_1$, thus making $(\mathcal{C}^N)_{\mathbb{Z}_+}$ a Harish-Chandra module in $\text{Ind} - \text{Rep}^{ab}(S_\nu)$ for the pair $(\mathfrak{g}l_N, \mathfrak{p}_1)$.

The definition of the complex tensor power is compatible with the usual notion of a tensor power of a unital vector space (see [5] Section 6):

Proposition 6.0.9. Let $d \in \mathbb{Z}_+$. Consider the functor

$$\hat{S}_d : \text{Ind} - (\text{Rep}(S_\nu = d) \boxtimes \mathcal{O}^{pN}_{d,\mathbb{C}^N}) \rightarrow \text{Ind} - (\text{Rep}(S_d) \boxtimes \mathcal{O}^{pN}_{d,\mathbb{C}^N})$$

induced by the functor

$$S_d : \text{Ind} - (\text{Rep}(S_\nu=d) \rightarrow \text{Rep}(S_n))$$

described in Subsection 6.2. Then $\hat{S}_d((\mathcal{C}^N)_{\mathbb{Z}_+}) \cong (\mathcal{C}^N)_{\mathbb{Z}_+}$.

The construction of the complex tensor product is also compatible with the functors $\mathfrak{R} \mathfrak{e} \mathfrak{s}_{n,n+1}$ and $\Gamma_n$, defined in Definitions 6.1.1, 5.2.1. These properties can be seen as special cases of the following statement (when $N = n+1$ and $N = \infty$, respectively):

Proposition 6.0.10. Let $n \geq 1$, and let $N \geq n$, $N \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Recall that we have an inclusion $\mathfrak{g}l_n \oplus \mathfrak{g}l_n^\perp \subset \mathfrak{g}l_N$, and consider the functor

$$(\cdot)^{\mathfrak{g}l_n^\perp} : \text{Ind} - (\text{Rep}^{ab}(S_\nu) \boxtimes \mathcal{O}^{pN}_{\nu,\mathbb{C}^N}) \rightarrow \text{Ind} - (\text{Rep}^{ab}(S_\nu) \boxtimes \mathcal{O}^{pN}_{\nu,\mathbb{C}^N})$$

induced by the functor $(\cdot)^{\mathfrak{g}l_n^\perp} : \mathcal{O}^{pN}_{\nu,\mathbb{C}^N} \rightarrow \mathcal{O}^{pN}_{\nu,\mathbb{C}^N}$. The functor $(\cdot)^{\mathfrak{g}l_n^\perp}$ then takes $(\mathcal{C}^N)_{\mathbb{Z}_+}$ to $(\mathcal{C}^N)_{\mathbb{Z}_+}$.

Proof. The functor $(\cdot)^{\mathfrak{g}l_n^\perp} : \mathcal{O}^{pN}_{\nu,\mathbb{C}^N} \rightarrow \mathcal{O}^{pN}_{\nu,\mathbb{C}^N}$ induces an endofunctor of $\text{Ind} - \text{Rep}^{ab}(S_\nu)$. We would like to say that we have an isomorphism of $\text{Ind} - \text{Rep}^{ab}(S_\nu)$-objects

$$((\mathcal{C}^N)_{\mathbb{Z}_+})^{\mathfrak{g}l_n^\perp} \cong (\mathcal{C}^N)_{\mathbb{Z}_+},$$

and that the action of $\mathfrak{g}l_n \subset \mathfrak{g}l_N$ on $((\mathcal{C}^N)_{\mathbb{Z}_+})$ corresponds to the action of $\mathfrak{g}l_n$ on $((\mathcal{C}^N)_{\mathbb{Z}_+})$.

In order to do this, we first consider $(\mathcal{C}^N)_{\mathbb{Z}_+}$ as an object in $\text{Ind} - \text{Rep}^{ab}(S_\nu)$ with an action of $\mathfrak{g}l(U_N)$:

$$(\mathcal{C}^N)_{\mathbb{Z}_+} \cong \bigoplus_{k \geq 0} (\Delta_k \otimes U_N^{\otimes k}) S_k$$

If we consider only the actions of $\mathfrak{g}l(U_N), \mathfrak{g}l(U_n)$, the functor $\Gamma_n$ is induced by the additive monoidal functor $(\cdot)^{\mathfrak{g}l(U_n)^\perp} : \text{Ind} - \text{Rep}(\mathfrak{g}l(U_N))_{\text{poly}} \rightarrow \text{Ind} - \text{Rep}(\mathfrak{g}l(U_N))_{\text{poly}}$. This shows that we have an isomorphism of $\text{Ind} - \text{Rep}^{ab}(S_\nu)$-objects

$$((\mathcal{C}^N)_{\mathbb{Z}_+})^{\mathfrak{g}l_n^\perp} \cong \bigoplus_{k \geq 0} (\Delta_k \otimes U_n^{\otimes k}) S_k \cong (\mathcal{C}^N)_{\mathbb{Z}_+}$$

and the actions of $\mathfrak{g}l(U_n)$ on both sides are compatible. From the definition of the complex tensor power (Definition 6.0.6) one immediately sees that the actions of $E_{1,1}$ on both sides are compatible as well. Remark 6.0.7 now completes the proof. \qed
7. Schur-Weyl duality in complex rank: the Schur-Weyl functor and the finite-dimensional case

We fix $\nu \in \mathbb{C}$, and $N \in \mathbb{Z}_+ \cup \{\infty\}$. Again, we consider the unital vector space $\mathbb{C}^N$ with the chosen vector $\mathbf{1} := e_1$ and the complement $U_N := \text{span}\{e_2, e_3, \ldots\}$.

7.1. Schur-Weyl functor.

**Definition 7.1.1.** Define the Schur-Weyl contravariant functor

$$SW_{\nu} : \text{Rep}^{ab}(S_{\nu}) \longrightarrow \text{Mod}_{U}(\mathfrak{gl}_N)$$

by

$$SW_{\nu} := \text{Hom}_{\text{Rep}^{ab}(S_{\nu})}(\mathbf{1}, (\mathbb{C}^N)^{\otimes \nu})$$

**Remark 7.1.2.** The functor $SW_{\nu} : \text{Rep}^{ab}(S_{\nu}) \longrightarrow \text{Mod}_{U}(\mathfrak{gl}_N)$ is a contravariant $\mathbb{C}$-linear additive left-exact functor.

It turns out that the image of the functor $SW_{\nu} : \text{Rep}^{ab}(S_{\nu}) \longrightarrow \text{Mod}_{U}(\mathfrak{gl}_N)$ lies in $\mathcal{O}^{\mathfrak{gl}_N}_{\nu \oplus \mathbb{C}^N}$ (cf. Remark 6.0.8).

We can now define another Schur-Weyl functor which we will consider: it is the contravariant functor $\hat{SW}_{\nu}^{ab} : \text{Rep}^{ab}(S_{\nu}) \longrightarrow \hat{\mathcal{O}}^{\mathfrak{gl}_N}_{\nu \oplus \mathbb{C}^N}$. Recall from Section 4.2 that $\hat{\pi}_N : \mathcal{O}^{\mathfrak{gl}_N}_{\nu \oplus \mathbb{C}^N} \longrightarrow \hat{\mathcal{O}}^{\mathfrak{gl}_N}_{\nu \oplus \mathbb{C}^N} := \mathcal{O}^{\mathfrak{gl}_N}_{\nu \oplus \mathbb{C}^N} / \text{Rep}(\mathfrak{gl}_N)_{\text{poly}, \nu}$ is the Serre quotient of $\mathcal{O}^{\mathfrak{gl}_N}_{\nu \oplus \mathbb{C}^N}$ by the Serre subcategory of polynomial $\mathfrak{gl}_N$-modules of degree $\nu$.

We then define

$$\hat{SW}_{\nu \oplus \mathbb{C}^N} := \hat{\pi}_N \circ SW_{\nu \oplus \mathbb{C}^N}$$

7.2. The finite-dimensional case. Let $n \in \mathbb{Z}_+$. We then have the following theorem, which can be found in [5, Section 7]:

**Theorem 7.2.1.** The contravariant functor $\widehat{SW}_{\nu \oplus \mathbb{C}^N} : \text{Rep}^{ab}(S_{\nu}) \longrightarrow \hat{\mathcal{O}}^{\mathfrak{gl}_N}_{\nu \oplus \mathbb{C}^N}$ is exact and essentially surjective. Moreover, the induced contravariant functor

$$\text{Rep}^{ab}(S_{\nu}) / \text{Ker}(\widehat{SW}_{\nu \oplus \mathbb{C}^N}) \longrightarrow \hat{\mathcal{O}}^{\mathfrak{gl}_N}_{\nu \oplus \mathbb{C}^N}$$

is an anti-equivalence of abelian categories, thus making $\hat{\mathcal{O}}^{\mathfrak{gl}_N}_{\nu \oplus \mathbb{C}^N}$ a Serre quotient of $\text{Rep}^{ab}(S_{\nu})^{\text{op}}$.

We will show that a similar result holds in the infinite-dimensional case, when the contravariant functor $\widehat{SW}_{\nu \oplus \mathbb{C}^N}$ is in fact an anti-equivalence of categories.

In the proof of Theorem 7.2.1 we established the following fact (see [5, Theorem 7.2.3]):

**Lemma 7.2.2.** The functor $\widehat{SW}_{\nu \oplus \mathbb{C}^N}$ takes a simple object to either a simple object, or zero. More specifically, we have:

- Let $\lambda$ be a Young diagram lying in a trivial $\nu$-class. Then
  $$\widehat{SW}_{\nu \oplus \mathbb{C}^N}(L(\lambda)) \cong \hat{\pi}(L_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda))$$

- Consider a non-trivial $\nu$-class $\{\lambda^{(i)}\}_{i \geq 0}$. Then
  $$\widehat{SW}_{\nu \oplus \mathbb{C}^N}(L(\lambda^{(i)})) \cong \hat{\pi}(L_{\mathfrak{p}_n}(\nu - |\lambda^{(i+1)}|, \lambda^{(i+1)}))$$

whenever $i \geq 0$.

**Remark 7.2.3.** Recall that $L_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ is zero if $\ell(\lambda) \geq n$. 

18
8. CLASSICAL SCHUR-WEYL DUALITY AND THE RESTRICTED INVERSE LIMIT

8.1. A short overview of the classical Schur-Weyl duality. Let $V$ be a vector space over $\mathbb{C}$, and let $d \in \mathbb{Z}_+$. The symmetric group $S_d$ acts on $V^\otimes d$ by permuting the factors of the tensor product (the action is semisimple, by Mashke’s theorem):

$$\sigma(v_1 \otimes v_2 \otimes ... \otimes v_d) := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes ... \otimes v_{\sigma^{-1}(d)}$$

The actions of $\mathfrak{gl}(V)$, $S_d$ on $V^\otimes d$ commute, which allows us to consider a contravariant functor

$$\mathcal{SW}_{d,V} : \text{Rep}(S_d) \rightarrow \text{Rep}(\mathfrak{gl}(V))_{\text{poly}}, \mathcal{SW}_{d,V} := \text{Hom}_{S_d}(\cdot, V^\otimes d)$$

The contravariant functor $\mathcal{SW}_{d,V}$ is $\mathbb{C}$-linear and additive, and sends a simple representation $\lambda$ of $S_d$ to the $\mathfrak{gl}(V)$-module $S^\lambda V$.

Next, consider the contravariant functor

$$\mathcal{SW}_V : \bigoplus_{d \in \mathbb{Z}_+} \text{Rep}(S_d) \rightarrow \text{Rep}(\mathfrak{gl}(V))_{\text{poly}}, \mathcal{SW}_V := \oplus_d \mathcal{SW}_{d,V}$$

This functor $\mathcal{SW}_V$ is clearly essentially surjective and full (this is easy to see, since $\text{Rep}(\mathfrak{gl}(V))_{\text{poly}}$ is a semisimple category with simple objects $S^\lambda V \cong \mathcal{SW}(\lambda)$).

The kernel of the functor $\mathcal{SW}_V$ is the full additive subcategory (direct factor) of $\bigoplus_{d \in \mathbb{Z}_+} \text{Rep}(S_d)$ generated by simple objects $\lambda$ such that $\ell(\lambda) > \dim V$.

8.2. Classical Schur-Weyl duality: inverse limit. In this subsection, we prove that the classical Schur-Weyl functors $\mathcal{SW}_{\mathbb{C}^n}$ make the category $\bigoplus_{d \in \mathbb{Z}_+} \text{Rep}(S_d)$ dual (anti-equivalent) to the category

$$\text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}} \cong \lim_{n \in \mathbb{Z}_+ \cup \{\infty\}, \text{restr}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}}$$

The contravariant functor $\mathcal{SW}_{\mathbb{C}N}$ sends the Young diagram $\lambda$ to the $\mathfrak{gl}_N$-module $S^\lambda \mathbb{C}^N$.

Let $n \in \mathbb{Z}_+$. We start by noticing that the functors $\mathfrak{Res}_{n,n+1}$ and the functors $\Gamma_n$ (defined in Section 3) are compatible with the classical Schur-Weyl functors $\mathcal{SW}_{\mathbb{C}^n}$:

**Lemma 8.2.1.** We have natural isomorphisms

$$\mathfrak{Res}_{n,n+1} \circ \mathcal{SW}_{\mathbb{C}^{n+1}} \cong \mathcal{SW}_{\mathbb{C}^n}$$

and

$$\Gamma_n \circ \mathcal{SW}_{\mathbb{C}\infty} \cong \mathcal{SW}_{\mathbb{C}^n}$$

for any $n \geq 0$.

**Proof.** It is enough to check this on simple objects in $\bigoplus_{d \in \mathbb{Z}_+} \text{Rep}(S_d)$, in which case the statement follows directly from the definitions of $\mathfrak{Res}_{n,n+1}$, $\Gamma_n$ together with the fact that $\mathcal{SW}_{\mathbb{C}^n}(\lambda) \cong S^\lambda \mathbb{C}^N$ for any $N \in \mathbb{Z}_+ \cup \{\infty\}$. \qed

The above Lemma implies that we have a commutative diagram

```
\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$\bigoplus_{d \in \mathbb{Z}_+} \text{Rep}(S_d)^{\text{op}}$};
  \node (B) at (3,0) {$\text{Rep}(\mathfrak{gl}_n)_{\text{poly}}$};
  \node (C) at (0,-3) {$\mathcal{SW}_{\mathbb{C}_n}$};
  \node (D) at (3,-3) {$\text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}}$};
  \node (E) at (0,-6) {$\mathcal{SW}_{\mathbb{C}\infty}$};
  \node (F) at (3,-6) {$\Gamma_n \circ \mathcal{SW}_{\mathbb{C}\infty}$};
  \node (G) at (3,-2) {$\lim_{n \geq 1, \text{restr}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}}$};

  \draw[->] (A) to node {$\mathcal{SW}_{\mathbb{C}_n}$} (C);
  \draw[->] (C) to node {$\mathcal{SW}_{\mathbb{C}_n}$} (E);
  \draw[->] (E) to node {$\mathcal{SW}_{\mathbb{C}\infty}$} (F);
  \draw[->] (A) to node {$\text{Pr}_n$} (B);
  \draw[->] (B) to node {$\Gamma_n$} (G);
  \draw[->] (C) to node {$\mathcal{SW}_{\mathbb{C}_n}$} (D);
  \draw[->] (D) to node {$\Gamma_n$} (F);
  \end{tikzpicture}
\end{center}
```

the functor $\Gamma_{\text{lim}}$ being an equivalence of categories (by Proposition 3.3.3), and $\text{Pr}_n$ being the canonical projection functor.
Proposition 8.2.2. The contravariant functors
\[ \text{SW}_\infty : \bigoplus_{d \in \mathbb{Z}_+} \text{Rep}(S_d) \rightarrow \text{Rep}(\mathfrak{gl}_\infty)_{\text{poly}} \]
and
\[ \text{SW}_{\lim} : \bigoplus_{d \in \mathbb{Z}_+} \text{Rep}(S_d) \rightarrow \lim_{n \in \mathbb{Z}_+, \text{restr}} \text{Rep}(\mathfrak{gl}_n)_{\text{poly}} \]
are anti-equivalences of semisimple categories.

Proof. As it was said in Subsection 8.1, the functor \( \text{SW}_N \) is full and essentially surjective for any \( N \). In this case, the functor \( \text{SW}_\infty \) is also faithful, since the simple object \( \lambda \) in \( \bigoplus_{d \in \mathbb{Z}_+} \text{Rep}(S_d) \) is taken by the functor \( \text{SW}_\infty \) to the simple object \( S^\lambda \mathbb{C}^\infty \neq 0 \). This proves that the contravariant functor \( \text{SW}_\infty \) is an anti-equivalence of categories. The commutative diagram above then implies that the contravariant functor \( \text{SW}_{\lim} \) is an anti-equivalence as well.

9. \( \text{Rep}^{ab}(S_\nu) \) AND THE INVERSE LIMIT OF CATEGORIES \( \widehat{\text{O}}^N_{\nu, \mathbb{C}^N} \)

9.1. In this section we are going to prove that the Schur-Weyl functors defined in Section 7.1 give us an equivalence of categories between \( \text{Rep}^{ab}(S_\nu) \) and the restr inverse limit \( \lim_{N \in \mathbb{Z}_+, \text{restr}} \widehat{\text{O}}^N_{\nu, \mathbb{C}^N} \).

We fix \( \nu \in \mathbb{C} \).

Proposition 9.1.1. The functor \( \text{Res}_{n-1,n} \) satisfies: \( \text{Res}_{n-1,n} \circ \text{SW}_{\nu, \mathbb{C}^n} \cong \text{SW}_{\nu, \mathbb{C}^{n-1}} \), i.e. there exists a natural isomorphism \( \eta_n : \text{Res}_{n-1,n} \circ \text{SW}_{\nu, \mathbb{C}^n} \rightarrow \text{SW}_{\nu, \mathbb{C}^{n-1}} \).

Proof. Follows directly from Proposition 6.0.10.

Corollary 9.1.2. We have \( \text{Res}_{n-1,n} \circ \text{SW}_{\nu, \mathbb{C}^n} \cong \text{SW}_{\nu, \mathbb{C}^{n-1}} \), i.e. there exists a natural isomorphism \( \eta_n : \text{Res}_{n-1,n} \circ \text{SW}_{\nu, \mathbb{C}^n} \rightarrow \text{SW}_{\nu, \mathbb{C}^{n-1}} \).

Proof. By definition of \( \text{Res}_{n-1,n}, \text{SW}_{\nu, \mathbb{C}^n} \), together with Proposition 9.1.1, we have a commutative diagram

\[
\begin{array}{cccccc}
\text{Rep}^{ab}(S_\nu)^{\text{opp}} & \rightarrow & \text{O}^N_{\nu, \mathbb{C}^n} & \rightarrow & \text{Res}_{n-1,n} & \rightarrow & \text{O}^N_{\nu, \mathbb{C}^{n-1}} \\
\text{SW}_{\nu, \mathbb{C}^n} & \downarrow & \eta_n & \downarrow & \eta_{n-1} & \downarrow & \text{Res}_{n-1,n} \\
\text{SW}_{\nu, \mathbb{C}^{n-1}} & & & & & & \text{O}^N_{\nu, \mathbb{C}^{n-1}} \\
\end{array}
\]

Since \( \eta_{n-1} \circ \text{SW}_{\nu, \mathbb{C}^{n-1}} = \text{SW}_{\nu, \mathbb{C}^{n-1}} \), we get \( \text{Res}_{n-1,n} \circ \text{SW}_{\nu, \mathbb{C}^n} \cong \text{SW}_{\nu, \mathbb{C}^{n-1}} \).

Notation 9.1.3. For each \( k \in \mathbb{Z}_+ \), \( \text{Fil}_k(\text{Rep}^{ab}(S_\nu)) \) is defined to be the Serre subcategory of \( \text{Rep}^{ab}(S_\nu) \) generated by the simple objects \( L(\lambda) \) such that the Young diagram \( \lambda \) satisfies either of the following conditions:

- \( \lambda \) belongs to a trivial \( \zeta \)-class, and \( \ell(\lambda) \leq k \).
- \( \lambda \) belongs to a non-trivial \( \zeta \)-class \( \{\lambda^{(i)}\}_{i \geq 0}, \lambda = \lambda^{(i)} \), and \( \ell(\lambda^{(i+1)}) \leq k \).

This defines a \( \mathbb{Z}_+ \)-filtration on the objects of the category \( \text{Rep}^{ab}(S_\nu) \). That is, we have:

\[ \text{Rep}^{ab}(S_\nu) \cong \lim_{k \in \mathbb{Z}_+} \text{Fil}_k(\text{Rep}^{ab}(S_\nu)) \]
Lemma 9.1.4. The functors \( \check{SW}_{\nu, C^n} \) are \( \mathbb{Z}_+ \)-filtered shortening functors (see [6] for the relevant definitions).

Proof. Follows from the fact that \( \check{SW}_{\nu, C^n} \) are exact, together with Lemma 7.2.2. \( \Box \)

This Lemma, together with Corollary 9.1.2, gives us a contravariant \( (\mathbb{Z}_+\text{-filtered shortening}) \) functor

\[
\check{SW}_{\nu, \text{lim}} : \text{Rep}^{ab}(S_\nu) \rightarrow \lim_{n \geq 1, \text{restr}} \hat{O}^{p_n}_{\nu, C^n}
\]

\[
X \mapsto \left\{ \left\{ \check{SW}_{\nu, C^n}(X) \right\}_{n \geq 1}, \left\{ \eta_n(X) \right\}_{n \geq 2} \right\}
\]

\[
(f : X \rightarrow Y) \mapsto \left\{ \check{SW}_{\nu, C^n}(f) : \check{SW}_{\nu, C^n}(Y) \rightarrow \check{SW}_{\nu, C^n}(X) \right\}_{n \geq 1}
\]

This functor is given by the universal property of the restricted inverse limit described in [6], and makes the diagram below commutative:

\[
\check{SW}_{\nu, C^n} \Rightarrow \hat{O}^{p_n}_{\nu, C^n}
\]

\[
\text{Rep}^{ab}(S_\nu)^{\text{op}} \xrightarrow{\text{Pr}_n} \lim_{n \geq 1, \text{restr}} \hat{O}^{p_n}_{\nu, C^n}
\]

(here \( \text{Pr}_n \) is the canonical projection functor).

We now show that there is an equivalence of categories \( \text{Rep}^{ab}(S_\nu)^{\text{op}} \) and \( \lim_{n \geq 1, \text{restr}} \hat{O}^{p_n}_{\nu, C^n} \).

Theorem 9.1.5. The Schur-Weyl contravariant functors \( \check{SW}_{\nu, C^n} \) induce an anti-equivalence of abelian categories, given by the (exact) contravariant functor

\[
\check{SW}_{\nu, \text{lim}} : \text{Rep}^{ab}(S_\nu) \rightarrow \lim_{n \geq 1, \text{restr}} \hat{O}^{p_n}_{\nu, C^n}
\]

Proof. The functors \( \check{SW}_{\nu, C^n} \) are exact for each \( n \geq 1 \), which means (see [6] Section 3.2) that the functor \( \check{SW}_{\nu, \text{lim}} \) is exact as well.

To see that it is an anti-equivalence, we will use [6] Proposition 5.1.10. All we need to check is that the functors \( \check{SW}_{\nu, C^n} \) satisfy the “stabilization condition” ([6] Condition 5.1.9): that is, for each \( k \in \mathbb{Z}_+ \), there exists \( n_k \in \mathbb{Z}_+ \) such that

\[
\check{SW}_{\nu, C^n} : \text{Fil}_k(\text{Rep}^{ab}(S_\nu)) \rightarrow \text{Fil}_k(\hat{O}^{p_n}_{\nu, C^n})
\]

is an anti-equivalence of categories for any \( n \geq n_k \).

Indeed, let \( k \in \mathbb{Z}_+ \), and let \( n \geq k+1 \).

The category \( \text{Fil}_k(\text{Rep}^{ab}(S_\nu)) \) decomposes into blocks (corresponding to the blocks of \( \text{Rep}^{ab}(S_\nu) \)), and the category \( \text{Fil}_k(\hat{O}^{p_n}_{\nu, C^n}) \) decomposes into blocks corresponding to the blocks of \( \hat{O}^{p_n}_{\nu, C^n} \).

The requirement \( n \geq k+1 \) together with Lemma 7.2.2 means that for any semisimple block of \( \text{Fil}_k(\text{Rep}^{ab}(S_\nu)) \), the simple object \( L(\lambda) \) corresponding to this block is not sent to zero under \( \check{SW}_{\nu, C^n} \). This, in turn, implies that \( \check{SW}_{\nu, C^n} \) induces an anti-equivalence between each semisimple block of \( \text{Fil}_k(\text{Rep}^{ab}(S_\nu)) \) and the corresponding semisimple block of \( \text{Fil}_k(\hat{O}^{p_n}_{\nu, C^n}) \).

Now, fix a non-semisimple block \( B_\lambda \) of \( \text{Rep}^{ab}(S_\nu) \), and denote by \( \text{Fil}_k(B_\lambda) \) the corresponding non-semisimple block of \( \text{Fil}_k(\text{Rep}^{ab}(S_\nu)) \). We denote by \( B_{\lambda, n} \) the corresponding block in \( \hat{O}^{p_n}_{\nu, C^n} \). The corresponding block of \( \text{Fil}_k(\hat{O}^{p_n}_{\nu, C^n}) \) will then be \( \hat{\pi}(\text{Fil}_k(B_{\lambda, n})) \).

We now check that the contravariant functor

\[
\check{SW}_{\nu, C^n}|_{\text{Fil}_k(B_\lambda)} : \text{Fil}_k(B_\lambda) \rightarrow \hat{\pi}(\text{Fil}_k(B_{\lambda, n}))
\]

is an anti-equivalence of categories when \( n \geq k+1 \).
Since $n \geq k + 1$, the Serre subcategories $Fil_k(B_\lambda)$ and $Ker(SW_{\nu,\mathbb{C}^n})$ of $Rep^{ab}(S_\nu)$ have trivial intersection (see Lemma 7.2.2), which means that the restriction of $SW_{\nu,\mathbb{C}^n}$ to the Serre subcategory $Fil_k(B_\lambda)$ is both faithful and full (the latter follows from Theorem 7.2.1).

It remains to establish that the functor $SW_{\nu,\mathbb{C}^n}|_{Fil_k(B_\lambda)}$ is essentially surjective when $n \geq k + 1$. This can be done by checking that this functor induces a bijection between the sets of isomorphism classes of indecomposable projective objects in $Fil_k(B_\lambda)$, $\hat{\pi}(Fil_k(\mathbb{B}_{\lambda,n}))$ respectively (see [5] Proof of Theorem 7.2.7 where we use a similar technique). The latter fact follows from the proof of [5] Theorem 7.2.7.

Thus $SW_{\nu,\mathbb{C}^n} : Fil_k(B_\lambda) \to Fil_k(\hat{\pi}(\mathbb{B}_{\lambda,n}))$ is an anti-equivalence of categories for $n \geq k + 1$, and

$$\hat{SW}_{\nu,\mathbb{C}^n} : Fil_k(Rep^{ab}(S_\nu)) \to Fil_k(\hat{O}^{\nu}_{\nu,\mathbb{C}^n})$$

is an anti-equivalence of categories for $n \geq k + 1$, which completes the proof. \hfill \Box

10. Schur-Weyl Duality for $Rep^{ab}(S_\nu)$ and $\mathfrak{gl}_\infty$

Let $\mathbb{C}^\infty$ be a complex vector space with a countable basis $e_1, e_2, e_3, \ldots$. Fix $\mathbf{1} := e_1$ and $U_\infty := span_\mathbb{C}(e_2, e_3, \ldots)$.

**Lemma 10.1.1.** We have a commutative diagram

$$\begin{array}{ccc}
Rep^{ab}(S_\nu)^{op} & \xrightarrow{SW_{\nu,\lim}} & \lim_{n \geq 1, \text{ restr}} \hat{O}^{\nu}_{\nu,\mathbb{C}^n} \\
\hat{SW}_{\nu,\mathbb{C}^n} & \xrightarrow{\hat{\pi}_{\nu,\mathbb{C}^n}} & \hat{O}^{\nu}_{\nu,\mathbb{C}^n} \\
SW_{\nu,\mathbb{C}^\infty} & \xrightarrow{\hat{\Gamma}_{\nu,\mathbb{C}^\infty}} & \hat{O}^{\nu}_{\nu,\mathbb{C}^\infty}
\end{array}$$

Namely, there is a natural isomorphism $\hat{\eta} : \hat{\Gamma}_{\nu,\mathbb{C}^\infty} \circ \hat{SW}_{\nu,\mathbb{C}^\infty} \to \hat{SW}_{\nu,\lim}$.

**Proof.** In order to prove this statement, we will show that for any $n \geq 1$, the following diagram is commutative:

$$\begin{array}{ccc}
Rep^{ab}(S_\nu)^{op} & \xrightarrow{SW_{\nu,\mathbb{C}^\infty}} & O^{\nu}_{\nu,\mathbb{C}^\infty} \\
\hat{SW}_{\nu,\mathbb{C}^\infty} & \xrightarrow{\hat{\pi}_{\nu,\mathbb{C}^\infty}} & \hat{O}^{\nu}_{\nu,\mathbb{C}^\infty} \\
SW_{\nu,\mathbb{C}^\infty} & \xrightarrow{\hat{\Gamma}_{\nu,\mathbb{C}^\infty}} & \hat{O}^{\nu}_{\nu,\mathbb{C}^\infty}
\end{array}$$

which will prove the required statement. The commutativity of this diagram follows from the existence of a natural isomorphism $\hat{\Gamma}_{n} \circ SW_{\nu,\mathbb{C}^\infty} \cong SW_{\nu,\mathbb{C}^n}$ (due to Proposition 6.0.10) and a natural isomorphism $\hat{\Gamma}_{n} \circ \hat{\pi}_{\infty} \cong \hat{\pi}_{n} \circ \Gamma_{n}$ (see proof of Proposition 5.3.5).

\hfill \Box

Thus we obtain a commutative diagram

$$\begin{array}{ccc}
Rep^{ab}(S_\nu)^{op} & \xrightarrow{\lim_{n \geq 1, \text{ restr}} \hat{O}^{\nu}_{\nu,\mathbb{C}^n}} & \hat{O}^{\nu}_{\nu,\mathbb{C}^n} \\
\hat{SW}_{\nu,\mathbb{C}^\infty} & \xrightarrow{\hat{\Gamma}_{\nu,\mathbb{C}^\infty}} & \hat{O}^{\nu}_{\nu,\mathbb{C}^\infty} \\
SW_{\nu,\mathbb{C}^\infty} & \xrightarrow{\hat{\Gamma}_{\nu,\mathbb{C}^\infty}} & \hat{O}^{\nu}_{\nu,\mathbb{C}^\infty}
\end{array}$$
Theorem 10.1.2. The contravariant functor \( \hat{SW}_{\nu,C} : \text{Rep}^{ab}(S_\nu) \to \hat{O}^p_{\nu,C} \) is an anti-equivalence of abelian categories.

Proof. The functor \( \hat{\Gamma}_{\text{lim}} \) is an equivalence of categories (see Lemma 5.3.5), and the functor \( \hat{SW}_{\nu,\text{lim}} \) is an anti-equivalence of categories (see Theorem 9.1.5). The commutative diagram above implies that the contravariant functor \( \hat{SW}_{\nu,C} \) is an anti-equivalence of categories as well. \( \square \)

11. Schur-Weyl functors and duality structures

11.1. Let \( n \in \mathbb{Z}_+ \).

Recall the contravariant duality functors
\[
(\cdot)^\vee_n : \left( O^p_{\nu,C^n} \right)^{\text{op}} \to O^p_{\nu,C^n}
\]
discussed in Subsection 4.3. This functor takes polynomial modules to polynomial modules, and therefore descends to a duality functor
\[
(\cdot)^\vee_n : \left( \hat{O}^p_{\nu,C^n} \right)^{\text{op}} \to \hat{O}^p_{\nu,C^n}
\]

Next, the definition of duality functor in \( O^p_{\nu,C^n} \) that the duality functors commute with the restriction functors \( \text{Res}_{n-1,n} \), namely, that for any \( n \geq 2 \), we have:
\[
(\cdot)^\vee_{n-1} \circ \text{Res}_{n-1,n}^{op} \cong \text{Res}_{n-1,n}^{op} \circ (\cdot)^\vee_n
\]

This allows us to define duality functors
\[
(\cdot)^\vee_{\text{lim}} : \left( \lim_{n \geq 1, \text{ restr}} O^p_{\nu,C^n} \right)^{\text{op}} \to \lim_{n \geq 1, \text{ restr}} O^p_{\nu,C^n}
\]
and
\[
(\cdot)^\vee_{\lim} : \left( \lim_{n \geq 1, \text{ restr}} \hat{O}^p_{\nu,C^n} \right)^{\text{op}} \to \lim_{n \geq 1, \text{ restr}} \hat{O}^p_{\nu,C^n}
\]

Under the equivalence \( O^p_{\nu,C^\infty} \cong \lim_{n \geq 1, \text{ restr}} O^p_{\nu,C^n} \) established in Subsection 5.3, the functor \( (\cdot)^\vee_{\text{lim}} \) corresponds to the duality functor
\[
(\cdot)^\vee_{\infty} : \left( O^p_{\nu,C^\infty} \right)^{\text{op}} \to O^p_{\nu,C^\infty}
\]
discussed in Subsection 4.3. Again, this functor descends to a contravariant duality functor
\[
(\cdot)^\vee_{\infty} : \left( \hat{O}^p_{\nu,C^\infty} \right)^{\text{op}} \to \hat{O}^p_{\nu,C^\infty}
\]

As a consequence of Theorem 7.2.1, a connection was established between the notions of duality in the Deligne category \( \text{Rep}^{ab}(S_\nu) \) and the duality in the category \( \hat{O}^p_{\nu,C^N} \) for \( N \in \mathbb{Z}_+ \) (see [5] Section 7.3). The above construction allows us to extend this connection to the case when \( N = \infty \). Namely, Theorems 9.1.3 and 10.1.2 together with [5] Section 7.3, imply the following statement:

**Proposition 11.1.1.** Let \( N \in \mathbb{Z}_+ \cup \{\infty\} \), \( \nu \in \mathbb{C} \). There is an isomorphism of (covariant) functors
\[
\hat{SW}_{\nu,\infty} \circ (\cdot)^* \to (\cdot)^\vee_{\infty} \circ SW_{\nu,C^N}
\]

23
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