Protocol Coding through Reordering of User Resources, Part I: Capacity Results

Petar Popovski* and Zoran Utkovski†

* Department of Electronic Systems, Aalborg University, Denmark
† Institute of Information Technology, University of Ulm, Germany

Email: petarp@es.aau.dk, zoran.utkovski@uni-ulm.de

Abstract

The vast existing wireless infrastructure features a variety of systems and standards. It is of significant practical value to introduce new features and devices without changing the physical layer/hardware infrastructure, but upgrade it only in software. A way to achieve it is to apply protocol coding: encode information in the actions taken by a certain (existing) communication protocol. In this work we investigate strategies for protocol coding via combinatorial ordering of the labelled user resources (packets, channels) in an existing, primary system. Such a protocol coding introduces a new secondary communication channel in the existing system, which has been considered in the prior work exclusively in a steganographic context. Instead, we focus on the use of secondary channel for reliable communication with newly introduced secondary devices, that are low-complexity versions of the primary devices, capable only to decode the robustly encoded header information in the primary signals. We introduce a suitable communication model, capable to capture the constraints that the primary system operation puts on protocol coding. We have derived the capacity of the secondary channel under arbitrary error models. The insights from the information–theoretic analysis are used in Part II of this work to design practical error–correcting mechanisms for secondary channels with protocol coding.

I. INTRODUCTION

A. Motivation and Initial Observations

After two decades of explosive growth, the starting point for wireless innovation is changed. With the vast amount of deployed infrastructure and variety of existing systems, it is of significant practical value to introduce new features without changing the physical layer/hardware of the
infrastructure, but only upgrade it in software. This can be achieved by a suitable, backward–compatible upgrade of the communication protocols. We use the term protocol coding to refer to techniques that convey information by modulating the actions of a communication protocol.

Consider the example on Fig. 1 where a cellular base station (BS) a group of primary terminals in its range. It is assumed that the cellular system is frame–based (WiMax [1], LTE [2], etc.). The metadata contained in the frame header informs the terminals how to receive/interpret the actual data that follows. The frame header is commonly encoded more robustly compared to the data, such that it can be reliably received in an area that is larger than the nominal coverage area, as depicted on Fig. 1. In such a context, while still using the same infrastructure, we can introduce new secondary devices, which are able to operate in the extended coverage area. These can be e. g. machine-type devices [3], such as sensors or actuators, that are controlled by the cellular BS. The secondary devices are simple and have a limited functionality, capable to decode only the frame header, but not the complex high–rate codebooks used for data. The main idea is that BS can send information to the secondary devices in the frame header. However, one could immediately object that the frame header carries important metadata that cannot be changed arbitrarily. The BS decides how to schedule the primary users based on certain QoS criterion. Nevertheless, there could be still freedom to rearrange the headers and thereby send information to the secondary devices. To illustrate this point, assume that there are two OFDMA channels, 1 and 2, defined in a diversity mode [1], such that if a user Alice is scheduled in a given frame, it is irrelevant whether it is assigned to channel 1 or 2. Hence, if BS schedules Alice and Bob in a given frame, then it can encode 1–bit secondary information as follows: allocating Alice to channel 1 and Bob to channel 2 is a bit value 0, otherwise it is a bit value 1. Taking this simple example further, let there be three OFDMA channels, but still only two users, Alice and Bob. In a given frame, each of them can get from 0 up to 3 channels assigned, which is decided by the primary scheduling criterion; the secondary transmitter can encode information by assigning these channels to Alice/Bob in a particular way. If there are 2(1) packets for Alice (Bob), they can be assigned in 3 possible ways and in that particular frame, \( \log_2(3) \) secondary bits can be sent. However, if all 3 packets are addressed to Alice, no secondary information can be sent in that frame. This variable amount of information due to the primary operation is the crux of the communication model considered in this work.

The objective of this and the companion paper [4] is to investigate the fundamental properties
of communication systems that use protocol coding to send information, under restrictions imposed by a primary system. The secondary information is encoded in the ordering of labelled resources (packets, channels) of the primary (legacy) users. In this paper we introduce a suitable communication model that can capture the restrictions imposed by the primary system. The model captures the key feature of a secondary communication: in a given scheduling epoch, the primary system decides which packets/users to send data to, while secondary information can be sent by only rearranging these packets. Each primary packet is subject to an error (e.g. erasure), which induces a corresponding error model for secondary communication. In this paper we analyze the model using information-theoretic tools and obtain capacity-achieving communication strategies, which we then apply in Part II of the work to obtain practical encoding strategies.

B. Related Work and Contributions

Protocol coding can appear in many flavors. An early work that mentions the possibility to send data by modulating the random access protocol is [5], but in a rather “negative” context, since the model used explicitly prohibits to decide the protocol actions based on user data. The seminal work [6] uses a form of protocol coding: the information is modulated in the arrival times of data packets. More recent works on possible encoding of information in relaying scenarios through protocol-level choice of whether to transmit or receive is presented in [7] [8] and [9]. At a conceptual level, protocol coding bridges information theory and networking [10]. The idea of communication based on packet reordering is not new per se and has been presented in the context of covert channels [11] [12] [13]. However, the big difference with our work is that our objective is not steganographic, but rather what kind of communication strategies can be used when the degrees of freedom for secondary communication are limited by a certain (random) process in the primary system. The practical coding strategies are related to the frequency permutation arrays for power line communications [14], [15].

Preliminary results of this work have appeared in [16] and [17]. In [16] we have introduced the notion of a secondary channel and sketched of the communication strategies when the primary packets are subject to an erasure channel, while in [17] we treated the case when the error model for the primary packets is represented by a Z-channel. In this paper we devise capacity-achieving strategies for arbitrary error model incurred on the primary packets and provide the detailed proofs. We first show that our communication model is related to the model of Shannon...
for channels with causal side information at the transmitter (CSIT) \[18\]. We then develop a new framework for computing the secondary capacity, which leads us to explicit specification of the communication strategies that are applied to convolutional codes in Part II \[4\].

II. SYSTEM MODEL

A. Communication Scenario

The communication model is depicted on Fig. 2. A Base Station (BS) transmits downlink data to a set of two users, addressed 0 and 1, respectively. The BS serves the users in scheduling frames with Time Division Multiple Access (TDMA). Each frame has a fixed number of $F$ packets. Each packet carries the address of a user to whom the packet is destined, as well as data for that user. This is called primary data, destined to either user 0 or user 1. There is a third receiving device, termed secondary device, that listens the TDMA frames sent by the BS. This device only records the address of each packet and ignores the packet data. Since this work is focused on the secondary communication, the notions “transmitter” and “receiver” will be used to refer to secondary transmitter and receiver, respectively. By addressing the packets in a given frame in a particular order, the BS sends secondary information. Thus, an input symbol for the secondary channel is an $F$–dimensional binary vector $x \in X = \{0, 1\}^F$.

The model with only two primary is limiting, but extension to $K$ primary addresses entails complexity that is outside the scope of this initial paper on the topic. Yet, the results with binary secondary inputs provide novel insights for the communication strategies and set the basis for generalizations to $K > 2$. Furthermore, the binary input captures the following practical setup. Consider the case in which the arrival of packets in the primary system is random and in a certain frame the BS has only $F' < F$ packets to send, then $(F - F')$ of the slots will be empty. In this case we can still use the binary input model. We assign address 0 to a the empty packet slots, such that these empty slots can be actually treated as valid secondary input symbols. On the other hand, the presence of a packet in a given slot is treated as a secondary symbol 1. The secondary receiver only needs to detect packet presence/absence, without decoding its header.

The key assumption in the model is that the packets that are scheduled in a frame are decided by the primary communication system: the primary system decides that $s$ packets in a frame will be addressed to user 1 and $(F - s)$ packets will be addressed to user 0, where $0 \leq s \leq F$. This assumption captures the essence of protocol coding: secondary communication is realized
by modulating the degrees of freedom left over from the operation of the original, primary communication system. In other words, it is assumed that the operational requirements of the primary system are contained in the set of packets that the BS decides to send in a given frame.

The number of packets $s$ addressed to user 1 in a given frame is called state of the frame. We assume that the primary system selects packets in a memoryless fashion: in each frame, a packet is addressed $1(0)$ with probability $a(1-a)$, independently of the other packets and the previous frames. Hence, the probability that a frame is in state $s$ is binomial $P_S(s) = \binom{F}{s} a^s (1-a)^{F-s}$.

With the state $s$ decided by the primary system, the secondary transmitter is only allowed to rearrange the packets in the frame. Since $s$ is a random variable over which the secondary transmitter has no control, a frame carries a variable amount of secondary information. For example, if $F = 4$ and the primary system decides $s = 3$, then the possible secondary symbols for the frame are $1110$, $1101$, $1011$, $0111$. But, if $s = F = 4$, then in that frame the secondary transmitter cannot send any information.

B. Error Models for the Secondary Channel

From the perspective of a secondary transmitter/receiver, each packet is sent over a memoryless channel with binary inputs. Several suitable error models can be inferred from the physical setup. In erasure channel, the receiver either correctly decodes the packet address 0 or 1 or the header checksum in incorrect, leading to erasure $\epsilon$. In a binary symmetric channel, the receiver uses error-correction decoding to decide whether it is more likely that address 0 or 1 is received. This results in only two possible outputs and symmetric error events. Finally, the Z-channel is suitable if 0/1 corresponds to packet absence/presence, respectively. The probability that, in absence of a packet, the noise produces a valid packet detection sequence, is practically 0, while the probability that packet transmission is not detected is $p_e > 0$.

In the general case of a channel with binary inputs, there can be $J$ possible outputs from the set $\mathcal{J}$. The special cases above have $\mathcal{J} = \{0, 1, \epsilon\}$ and $\mathcal{J} = \{0, 1\}$. When $i = 0, 1$ is sent, there are $J$ transition probabilities, represented by a vector:

$$q_i = (q_{i1}, q_{i2}, \ldots q_{iJ})$$

where $q_{ij} = P(y = j|x = i)$ and some $q_{ij}$ can be equal to 0. A secondary output symbol is $y \in \mathcal{Y} = \mathcal{J}^F$. The input/output variables of the secondary channel are denoted by $X$ and $Y$,
respectively. By denoting \( x = (x_1, x_2, \cdots x_F) \) with \( x_f \in \{0, 1\} \) and \( y = (y_1, y_2, \cdots y_F) \) with \( y_f \in \mathcal{J} \), we can define the channel \( X - Y \) through the transition probabilities:

\[
P_{Y|X}(y|x) = \prod_{f=1}^{F} q_{x_f y_f}
\]

When there is no risk for confusion, we simply write \( P(y|x) \). Thus, the channel \( X - Y \) is specified by the memoryless binary channel through which each packet is passed.

The following notation will be used. \( S = \{0, 1, \ldots F\} \) to denote the set of possible states. The set of input and output symbols of the secondary channel is denoted by \( \mathcal{X} \) and \( \mathcal{Y} \), respectively. The set of input symbols is partitioned into \( F + 1 \) subsets \( \mathcal{X}_s \) defined as follows:

\[
x \in \mathcal{X}_s \iff \sum_{i=1}^{F} x_i = s
\]

When the frame state is \( S = s \), then only \( x \in \mathcal{X}_s \) can be sent over the secondary channel.

### III. Framework for Analyzing the Capacity of a Secondary Channel

#### A. Relation to the Shannon’s Model with Causal State Information at the Transmitter (CSIT)

The secondary channel can be represented by the framework of Shannon for channels with causal state information at the transmitter (CSIT) [18]. Shannon showed that instead of considering the original channel with CSIT, one can consider an ordinary, discrete memoryless channel with equivalent capacity that has a larger input alphabet. The input variable of the equivalent channel is \( T \) and each possible input letter \( t \), termed strategy [19], represents a mapping from the state alphabet \( S \) to the input alphabet \( \mathcal{X} \) of the original channel. A particular strategy \( t \in T \) is defined by the vector of size \(|S|\): \((t(1), \ldots t(|S|))\), where \( t(s) \in \mathcal{X} \). Therefore, if each \( s \in S \) can be mapped map to any \( x \in \mathcal{X} \), then the total number of possible strategies is \(|\mathcal{X}|^{|S|}\) and therefore \(|T| \leq |\mathcal{X}|^{|S|}\). The capacity of the equivalent channel can be found as:

\[
C = \max_{P_T(\cdot)} I(T; Y)
\]

where \( P_T(\cdot) \) is a probability distribution defined over the set \( T \) which is independent of the state \( S \). The maximization is performed across all the joint distributions that satisfy [19]:

\[
P_{S,T,X,Y}(s,t,x,y) = P_S(s)P_T(t)\delta(x,t(s))P_{Y|X,S}(y|x,s)
\]

where \( \delta(x,t(s)) = 1 \) if \( x = t(s) \) and \( \delta(x,t(s)) = 0 \) otherwise. Following the properties of mutual information ([20], Section 8.3), the required cardinality of \( T \) is not more than \(|\mathcal{Y}|\).
However, Shannon’s result is for the general case of channels with causal CSIT. The secondary channel considered here has a specific structure that permits more explicit characterization of the communication strategies. As noted in relation to (3), for a given state $S = s$ only a subset $X_s \in \mathcal{X}$ of symbols $x$ may be produced. For example, when $F = 4$ and $s = 2$, it is not possible to send the symbol $x = 1011$. Nevertheless, in the model with causal CSIT the distribution $P_{Y|X,S}(y|x, s)$ needs to be defined for all pairs $(x, s)$, irrespective of the fact that in the original model some $x$ are incompatible with $s$, i.e. when the state is $S = s$, the symbols $x \notin \mathcal{X}_s$ cannot be sent. In order to deal with this situation, we need to extend the model. Given $P_{Y|X}(y|x)$, we define $P_{Y|X,S}(y|x, s)$ in the following way: For each $x_u \notin \mathcal{X}_s$ we take one $x_v \in \mathcal{X}_s$ and define:

$$P_{Y|X,S}(y|x_u, s) \equiv P_{Y|X,S}(y|x_v, s) \quad \forall y \in \mathcal{Y}. \quad (6)$$

The idea behind this approach is the following. For example, let us assume $F = 4$ and the erasure model. When $s = 0$ only $x = 0000$ can be sent. But we can look at it in another way: when $s = 0$ only $y = 0000$ are the versions of 0000 with erasures can occur. Hence, we can equivalently say that when $s = 0$, any $x$ can be sent, but, in absence of errors, the output is always 0000. Picking a strategy $t''$ in which $t''(s) = x_u$ is equivalent to picking $t'$ in which $t'(s) = x_v$. In short, for given $s$, we define $P_{Y|X,S}$ in order to discourage selection of symbols $x$ for which $x \neq y$ in absence of channel errors.

As pointed out in [19], expressing the capacity in terms of strategies might pose some conceptual and practical problems for code construction and implementation when $F$ is large. On the other hand, our objective is to use the specific way in which the set of states partitions the possible set of transmitted symbols $\mathcal{X}$ in order to provide insights in the capacity–achieving communication strategies. Therefore, a different framework for capacity analysis will be used. A practical dividend of such a framework is presented in the companion paper [4], where the capacity–achieving strategies are converted into convolutional code designs.

### B. Capacity Analysis through a Cascade of Channels

Recall that $T$ is an auxiliary random variable defined over the set of possible strategies $\mathcal{T}$. For given $T = t$ and each $s \in \mathcal{S}$ there is a single representative of $t$ in $s$ $x = t(s) \in \mathcal{X}_s$. In the text that follows we use “strategies” and “input symbols” interchangeably. Hence, $\mathcal{T}$ consists of...
the input symbols \(\{1, 2, \ldots, |T|\}\). The set of \(F + 1\) representatives \(\{x_s(t)\}\) for given \(t\) will be called a \textit{multisymbol} of \(t\).

Due to the randomized state change, each \(t \in T\) induces a distribution on \(\mathcal{X}\). For example, if \(F = 2\) and the strategy is defined as \(t(0) = 00, t(1) = 01, t(2) = 11\), then we can define
\[
P_{X|T}(x = 00|t) = (1 - a)^2 = P_S(0), \quad P_{X|T}(x = 11|t) = a^2 = P_S(2), \quad P_{X|T}(x = 01|t) = 2a(1 - a) = P_S(1), \quad \text{and} \quad P_{X|T}(x = 10|t) = 0.
\]
In general, \(P_{X|T}(\cdot)\) should satisfy that for each \(s \in S\) there is a single \(x \in \mathcal{X}_s\) such that \(P_{X|T}(x|t) = P_S(s)\). The set of such distributions is:
\[
\mathcal{P}_{X|T} = \{P_{X|T}(\cdot)\forall t \in T, \forall s \in S, \exists! x \in \mathcal{X}_s \text{ such that } P_{X|T}(x|t) = P_S(s)\}\]

(7)

In this way, we do not need to explicitly consider state in the capacity analysis, but instead we model the secondary communication channel by using a cascade of two channels \(T - X - Y\) and the primary constraints are reflected in the definition of \(\mathcal{P}_{X|T}\). In order to express the mutual information \(I(T; Y)\), we write \(I(T, X; Y) = I(T; Y) + I(X; Y|T) = I(X; Y) + I(T; Y|X)\).

Using the Markov property for the cascade we get \(I(T; Y|X) = 0\), which implies:
\[
I(T; Y) = I(X; Y) - I(X; Y|T)
\]

(8)

Let \(\mathcal{P}_T\) denote the set of all distributions \(P_T(\cdot)\). Our objective is to find the pair of distributions \((P_T(\cdot), P_{X|T}(\cdot))\) that maximizes \(I(T; Y)\). Thus, the capacity of the secondary channel is:
\[
C = \max_{P_T(\cdot), P_{X|T}(\cdot)} I(T; Y)
\]

(9)

We will always that \(P_{X|T}(\cdot) \in \mathcal{P}_{X|T}\) always. The expression \((9)\) can be upper–bounded:
\[
C \leq \max_{P_T(\cdot), P_{X|T}(\cdot)} I(X; Y) - \min_{P_T(\cdot), P_{X|T}(\cdot)} I(X; Y|T)
\]

(10)

where the equality is achieved if and only if there is a pair of distributions \((P_T(\cdot), P_{X|T}(\cdot))\) that simultaneously attains the max/min in the first/second term, respectively. We will decompose the problem \((9)\) into two sub–problems, maximization of \(I(X; Y)\) and minimization of \(I(X; Y|T)\).

Fig. 3 illustrates the cascade of channels where \(F = 2\) and erasure model for \(X - Y\) with \(\mathcal{J} = \{0, 1, \epsilon\}\) and \(q_{00} = q_{11} = 1 - p\), while \(q_{0\epsilon} = q_{1\epsilon} = p\). Let us assume that the primary constraint uses \(a = \frac{1}{2}\). The two multisymbols, corresponding to \(t = 1\) and \(t = 2\) are \(\{00, 01, 11\}\) and \(\{00, 10, 11\}\), respectively. It is seen that uniform \(P_T(\cdot)\) induces uniform \(P_X(\cdot)\). On the other hand, the capacity of the vector channel with erasures \(X - Y\) is achieved when \(P_X(\cdot)\) is uniform. The reader can check that uniform \(P_T(\cdot)\) and the choice of \(P_{X|T}(\cdot)\) according to Fig. 3 simultaneously maximizes \(I(X; Y)\) and minimizes \(I(X; Y|T)\).
IV. Maximization of $I(X;Y)$

Each pair of distributions $(P_T(\cdot), P_{X|T}(\cdot))$ induces a distribution $P_X$ on $\mathcal{X}$. Let $\mathcal{P}_X$ denote the set of all possible distributions $P_X(\cdot)$, while $\mathcal{P}_X^T \subset \mathcal{P}_X$ containing the distributions $P_X(\cdot)$ that can be induced by all possible pairs $(P_T(\cdot), P_{X|T}(\cdot))$. Then the following holds:

**Proposition 1:** The set of distributions $\mathcal{P}_X^T$ is a subset of $\mathcal{P}_{X,S}$, where $\mathcal{P}_{X,S} \subset \mathcal{P}_X$ and:

$$\mathcal{P}_{X,S} = \left\{ P_X(\cdot) \mid \sum_{x \in \mathcal{X}_s} P_X(x) = P_S(s), \forall s = 0, 1, \ldots F \right\} \quad (11)$$

**Proof:** We need to show that if $P_X(\cdot) \in \mathcal{P}_X^T$, then $P_X(\cdot) \in \mathcal{P}_{X,S}$. Let $P_X(\cdot) \in \mathcal{P}_X^T$, then:

$$\sum_{x \in \mathcal{X}_s} P_X(x) = \sum_{x \in \mathcal{X}_s} \sum_{t \in T} P_T(t) P_{X|T}(x|t) = \sum_{t \in T} P_T(t) \sum_{x \in \mathcal{X}_s} P_{X|T}(x|t) \stackrel{(a)}{=} P_S(s) \sum_{t \in T} P_T(t) \stackrel{(b)}{=} P_S(s)$$

where (a) follows from the definition and (b) from $\sum_{t \in T} P_T(t) = 1$. □

The previous proposition implies $\max_{P_T(\cdot), P_{X|T}(\cdot)} I(X;Y) \leq \max_{P_X(\cdot) \in \mathcal{P}_{X,S}} I(X;Y)$. We will first look for the distribution $P_{X^*}(\cdot) \in \mathcal{P}_{X,S}$ that maximizes $I(X;Y)$. Once $P_{X^*}(\cdot)$ is known, we choose $(P_T(\cdot), P_{X|T}(\cdot))$ in order to induce the desired $P_{X^*}(\cdot)$. Let us define:

$$C_{XY} = \max_{P_X(\cdot) \in \mathcal{P}_{X,S}} I(X;Y) \quad (12)$$

which is never larger than the capacity of $X - Y$, achieved by selecting over all $P_X(\cdot) \in \mathcal{P}_X$. For example, if the probability $a \neq \frac{1}{2}$ and there are erasure–type errors, then $C_{XY} < F(1 - p)$, where $F(1 - p)$ is the capacity of $F$ erasure channel uses. This is because the achieving the capacity of the erasure channel requires uniform distribution $P_{U\cdot X}(x) = 2^{-F}$, which induces the necessary condition $\sum_{x \in \mathcal{X}_s} P_{U\cdot X}(x) = \left(\frac{F}{s}\right)2^{-F}$, but this is not equal to $P_S(s)$ if $a \neq \frac{1}{2}$.

In this text we are interested in channels $X - Y$ where each single channel use $x$ consists of $F$ uses of a more elementary, identical channels, leading to the following symmetry: the set of transition probabilities $\{P_Y|X(y|x)\}$ is identical for all $x \in \mathcal{X}_s$, as they are all permutations of a vector with $s$ 1s and $F - s$ 0s. This is valid irrespective of the the type of elementary channel used for a single primary packet. Such a symmetry is instrumental for making statements about $C_{XY}$. The following lemma is proved in Appendix A.

**Lemma 1:** The distribution $P_X(\cdot) \in \mathcal{P}_{X,S}$ that achieves $C_{XY}$ is, for all $s$ and each $x \in \mathcal{X}_s$:

$$P_X(x) = \frac{P_S(s)}{\left(\frac{F}{s}\right)} \quad (13)$$
Having found $P_X(\cdot)$ that attains $C_{XY}$, it remains to find $T$, $P_T(\cdot)$ and $P_{X|T}(\cdot)$ (i.e. the representatives of each $T = t$) such that (13) is satisfied. For example, let $F = 4$ and $|\mathcal{X}_s| = 1, 4, 6, 4, 1$ for $s = 0, 1, 2, 3, 4$, respectively. Let at first take $|T| = 4m$ and uniform $P_T(t) = \frac{1}{4m}$. Then each $x \in \mathcal{X}_1$ can be a representative of exactly $m$ different elements of $\mathcal{T}$, such that $P_X(X = x) = P_S(1) \cdot m \cdot \frac{1}{4m} = P_S(1)/\left(\begin{array}{c} 4 \\ 1 \end{array}\right)$. In general, if $|T| = \left(\begin{array}{c} F \\ s \end{array}\right) \cdot m$ and uniform $P_T(t)$, we can choose $x \in \mathcal{X}_s$ to be a representative of exactly $m$ elements from $\mathcal{T}$; i.e. $P_{X|T}(x|t) = P_S(s)$ for $m$ different values $t$ and zero otherwise. The resulting $P_X(\cdot)$ satisfies (13). To satisfy this condition for all $s$ simultaneously, $|T|$ should be divisible with $\left(\begin{array}{c} F \\ s \end{array}\right)$ for all $s = 0 \cdots F$, leading to the following lemma, stated without proof (lcm stands for “least common multiplier”):

**Lemma 2:** The distribution $P_X(\cdot)$ that satisfies (13) can be achieved by choosing uniform $P_T(\cdot)$ over a set with a minimal cardinality of $|T| = \text{lcm} \left(\begin{array}{c} F \\ 0 \end{array}, \begin{array}{c} F \\ 1 \end{array}, \ldots, \begin{array}{c} F \\ F \end{array}\right)$.

V. MINIMIZATION OF $I(X; Y|T)$

A. Definition of Minimal Multisymbols

The multisymbol $\mathcal{M}_t = \{x_0(t), \cdots x_F(t)\}$ corresponding to $t$ has one representative in each $x_s(t) = \mathcal{X}_s$, such that $P_{X|T}(x_s(t)|t) = P_S(s)$ and is zero for the other $x$. Since $I(X; Y|T = t)$ depends on the choice of representatives in $\mathcal{M}_t$, we will denote it by $I(X; Y|\mathcal{M}_t)$, such that:

$$I(X; Y|T) = \sum_{t \in T} I(X; Y|\mathcal{M}_t) \quad (14)$$

For example, let $F = 5$ with $\mathcal{M}_1 = \{00000, 00001, 00011, 00111, 01111, 11111\}$ and $\mathcal{M}_2 = \{00000, 00001, 00110, 11100, 10111, 11111\}$. Assuming a binary symmetric channel with $q_{00} = q_{11} = 0.8, q_{01} = q_{10} = 0.2$ it can be seen that $I(X; Y|\mathcal{M}_1) < I(X; Y|\mathcal{M}_2)$. For intuitive explanation, consider two representatives $x_{s_i} \in \mathcal{X}_{s_i}, i = 1, 2$. From (3) the Hamming weight of $x_{s_i}$ is $s_i$ and, without loss of generality, assume $s_1 > s_2$. For the multisymbol $\mathcal{M}_1$, the Hamming distance between any two representatives is given by:

$$d_H(x_{s_1}, x_{s_2}) = s_2 - s_1 \quad (15)$$

and is minimal possible. Informally, any two representatives from $\mathcal{M}_1$ are as similar to each other as possible since they represent the same input $T = 1$, which is not the case for $\mathcal{M}_2$.

The multisymbols satisfying (15) are of special interest and will be termed *minimal* multisymbols. Among them, there is one termed *basic multisymbol* $\mathcal{M}_b$ with a particular structure:
the representative in $X_s$ is $00\cdots011\cdots1$ starts with $F - s$ consecutive zeros and $s$ consecutive ones. It can be shown that any minimal multisymbol can be obtained from the basic one via permutation, such that there are $F!$ different minimal multisymbols. For example, let $M^b = \{000,001,011,111\}$ and we apply the permutation $\pi = 321$: the components of each $x \in M^b$ are permuted according to $\pi$ to obtain $M^m = \{000,100,110,111\}$. In general, for a given permutation $\pi$ we define $\gamma_\pi(\cdot)$:

$$M' = \gamma_\pi(M)$$

(16)

such that each $x'_s \in M'$ is obtained from the corresponding $x_s \in M$ by permuting the packets according to $\pi$ and the Hamming distance between any two representatives is preserved $d_H(x_{s_1}, x_{s_2}) = d_H(x'_{s_1}, x'_{s_2}) = s_2 - s_1$.

B. Analysis of $I(X; Y|T = t) = I(X; Y|\mathcal{M}_t)$

We write the mutual information $I(X; Y|\mathcal{M}_t) = H(Y|\mathcal{M}_t) - H(Y|X, \mathcal{M}_t)$ and first consider:

$$H(Y|X, \mathcal{M}_t) = \sum_{s=0}^{F} P_s(s) H(Y|x_s(t))$$

(17)

Since each component of $x_s$ uses identical memoryless channel, $H(Y|x_s(t))$ depends only on the Hamming weight $s$, but not on how the 0s and 1s are arranged in $x_s$. This is stated through:

**Lemma 3**: The conditional entropy for $x_s \in X_s$, having a Hamming weight of $s$, is given by:

$$H(Y|X = x_s) = sH(q_1) + (F - s)H(q_0) = H_s$$

(18)

where $H(q_i) = -\sum_{j=1}^{J} q_{ij} \log_2 q_{ij}$ for $i = 0, 1$ and $q_{i}$ is given by (1).

**Proof**: In order to determine $H(Y|X = x) = -\sum_{y \in J} P(y|x) \log_2 P(y|x)$, we use the fact that $P(y|x) = \prod_{f=1}^{F} q_{xfy_f}$ is a product distribution, such that we can write $H(Y|X = x)$ as:

$$-\sum_{y \in J} \prod_{i=1}^{F} q_{x_iy_i} \prod_{j=1}^{F} \log_2 q_{x_jy_j} = -\sum_{j=1}^{F} \sum_{y_1 \in J} \prod_{y_f \in J} q_{x_1y_1} \log_2 q_{x_jy_j} \prod_{i=1}^{F} q_{x_iy_i}$$

(19)

where (a) follows from changing the order of summation. If we consider the component $j = 1$:

$$-\sum_{y_1 \in J} \prod_{y_f \in J} q_{x_1y_1} \prod_{i=2}^{F} q_{x_iy_i} = -\sum_{y_1 \in J} q_{x_1y_1} \log_2 q_{x_1y_1} \sum_{y_2 \in J} \prod_{y_f \in J} q_{x_2y_2} \prod_{i=2}^{F} q_{x_iy_i}$$

(19)

$$= -\sum_{y_1 \in J} \log_2 q_{x_1y_1} \cdot q_{x_1y_1} = H(q_{x_1})$$

February 3, 2012 DRAFT
where (b) follows from $\sum_{y_2 \in \mathcal{J}} \cdots \sum_{y_F \in \mathcal{J}} \prod_{i=2}^{F} q_{x_i y_i} = 1$. Doing the same for $j = 2 \ldots F$ shows that each $x_j = i$, $i = 0, 1$, contributes $H(q_i)$ to $H(Y|X = x)$, which proves the lemma.

Using the lemma, (17) can be rewritten as $H(Y|X, \mathcal{M}_t) = \sum_{s=0}^{F} P_S(s) H_s$ and is not affected by the actual choice of $\mathcal{M}_t$, as long as there is a representative in each $X_s$.

C. Analysis of $H(Y|\mathcal{M}_t)$

To gain intuition, we first consider a special type of $P_S(\cdot)$, in which only two states $s_1, s_2 \in S$ occur with non-zero probability $P_S(s_1) = \lambda$ and $P_S(s_2) = 1 - \lambda$, such that $\mathcal{M}_t = \{x_{s_1}, x_{s_2}\}$. Due to the symmetry implied by Lemma 3 without losing generality, we first pick an arbitrary $x_{s_1} \in X_{s_1}$. Then, how to select $x_{s_2} \in X_{s_2}$ in order to minimize the $H(Y|\mathcal{M}_t)$? Slightly abusing the notation from (15), we use $d_H(x)$ to denote the Hamming weight of $x$. Recall that $d_H(x) = s$ for $x \in X_s$. Let $g_{uv}(x_{s_1}, x_{s_2})$, where $u, v \in \{0, 1\}$ denote the number of positions $f$ at which $x_{s_1 f} = u$ and $x_{s_2 f} = v$. For example, if $x_{s_1} = 00110$, $x_{s_2} = 11011$, then $g_{00} = 0$, $g_{01} = 3$, $g_{10} = 1$, and $g_{11} = 1$ (we write $g_{uv}$ for brevity). Using similar arithmetics as in Lemma 3,

$$H(Y|T = t) = g_{00} H(q_0) + g_{11} H(q_1) + g_{01} H(\lambda q_0 + (1-\lambda)q_1) + g_{10} H((1-\lambda)q_0 + \lambda q_1)$$  \hspace{1cm} (20)$$

The Hamming distance is $d_H(x_{s_1}, x_{s_2}) = g_{01} + g_{10}$. The following lemma formalizes the intuition that $H(Y|\mathcal{M}_t)$ is minimized when any two representatives are as similar to each other as possible.

**Lemma 4:** When $\mathcal{M}_t$ consists of only two representatives $x_{s_1}, x_{s_2}$, $H(Y|\mathcal{M}_t)$ is minimized when the Hamming distance $d_H(x_{s_1}, x_{s_2}) = |s_2 - s_1|$ is minimal possible.

**Proof:** Without loss of generality, assume that $s_2 > s_1$. Then $g_{10}(x_{s_1}, x_{s_2}) < g_{01}(x_{s_1}, x_{s_2})$ since $d_H(x_{s_1}) < d_H(x_{s_2})$. Assume that $g_{10}(x_{s_1}, x_{s_2}) > 0$ and let there be $f_1, f_2$ such that:

$$(x_{s_1, f_1}, x_{s_2, f_1}) = (1, 0) \quad (x_{s_1, f_2}, x_{s_2, f_2}) = (0, 1)$$  \hspace{1cm} (21)$$

Let $z_{s_2}$ be another representative from $X_{s_2}$, obtained by swapping the positions $f_1, f_2$ in $x_{s_2}$, but keeping the other values of $x_{s_2}$, such that $z_{s_2, f_1} = 1$ and $z_{s_2, f_2} = 0$. Then:

$$g_{00}(x_{s_1}, x_{s_2}) + 1 = g_{00}(z_{s_1}, z_{s_2}) \quad g_{11}(x_{s_1}, x_{s_2}) + 1 = g_{11}(z_{s_1}, z_{s_2})$$

$$g_{01}(x_{s_1}, x_{s_2}) - 1 = g_{01}(z_{s_1}, z_{s_2}) \quad g_{10}(x_{s_1}, x_{s_2}) - 1 = g_{10}(z_{s_1}, z_{s_2})$$  \hspace{1cm} (22)$$

Using the concavity of the entropy function, we can write:

$$H(\lambda q_0 + (1-\lambda)q_1) + H((1-\lambda)q_0 + \lambda q_1) \geq \lambda H(q_0) + (1-\lambda)H(q_1) + (1-\lambda)H(q_0) + \lambda H(q_1) = H(q_0) + H(q_1)$$  \hspace{1cm} (23)$$
where \(Q\) is empty. Without losing generality, we will "empty" the set \(Q\) and \(u\) such that \(u, v\) are created, such that:

\[ u = (Q_{00} + Q_{01})q_0 + (Q_{10} + Q_{11})q_1 \quad u = (Q_{00} + Q_{01})q_0 + (Q_{10} + Q_{11})q_1 \]

\[ g_{00}H(q_0) + g_{11}H(q_1) + (g_{01} - 1)H(\lambda q_0 + (1 - \lambda)q_1) + (g_{10} - 1)H((1 - \lambda)q_0 + \lambda q_1) = H_{x_{s1}, x_{s2}} \]

where \(g_{uv} = g_{uv}(x_{s1}, x_{s2})\) and \(H_{x_{s1}, x_{s2}} = H(Y|M_t = \{x_{s1}, x_{s2}\})\). We can analogously continue the swap the positions in \(x_{s2}\) until getting \(g_{10} = 0\). Each swap does not increase \(H(Y|M_t)\), which means that when \(g_{10} = 0\), \(H(Y|M_t)\) is minimal.

We now consider a general \(P_S(\cdot)\). As indicated above, \(H(Y|M_t)\) can be written as:

\[ H(Y|M_t) = \sum_{f=1}^{F} H(u_f) \] (24)

where \(u_f\) is the probability distribution that corresponds to the \(f\)-th position, defined as:

\[ u_f = \sum_{s=0}^{F} P_s [(1 - x_{s,f})q_0 + x_{s,f}q_1] \quad \text{where } x_{s,f} \in \{0, 1\} \] (25)

Without losing generality, let us take the first value \(x_{s1}\) of each of the representatives \(x_s\) can create \((F + 1)\)-dimensional vector \(z_1\). In a similar way \(z_2\) is created, such that:

\[ z_1 = (x_{01}, x_{11}, \cdots x_{F1}) \quad z_2 = (x_{02}, x_{12}, \cdots x_{F2}) \] (26)

The probability distribution vectors \(u_1\) and \(u_2\) can be written as:

\[ u_1 = (Q_{00} + Q_{01})q_0 + (Q_{10} + Q_{11})q_1 \quad u_2 = (Q_{00} + Q_{10})q_0 + (Q_{01} + Q_{11})q_1 \] (27)

where \(Q_{uv} = \sum_{s \in G_{uv}(z_1, z_2)} P_s\) and the sets \(G_{uv}(z_1, z_2) = \{s|x_{s,1} = u, x_{s,2} = v\}\) for \(u, v \in \{0, 1\}\).

**Lemma 5:** The contribution of the positions 1 and 2 to the entropy \(H(Y|M_t)\) is minimized when one of the sets \(G_{01}, G_{10}\) is empty.

**Proof:** Let us start with a multisymbol \(\{x_s\}\) in which none of the sets \(G_{01}(z_1, z_2), G_{10}(z_1, z_2)\) is empty. Without losing generality, we will “empty” the set \(G_{01}(z_1, z_2)\) as follows: If there is \(s \in S\) such that \(x_{s,1} = 0, x_{s,2} = 1\), these two positions in the representative \(x_s\) are swapped. That is, if there is a representative \(x = 01 \cdots\), it is changed to 10 \cdots. Using the concavity of the entropy, we can show that these swapping operations can decrease the contribution of the positions \(f = 1, 2\) to the entropy (24). Note that after swapping (27), the new distributions are:

\[ u'_1 = (Q_{00} + Q_{01})q_0 + (Q_{01} + Q_{10} + Q_{11})q_1 \quad u'_2 = (Q_{00} + Q_{01} + Q_{10})q_0 + Q_{11}q_1 \] (28)
Using the concavity property, it can be shown that
\[ H(u_1) + H(u_2) \geq H(u'_1) + H(u'_2) \quad (29) \]
where \( u_1, u_2 \) and \( u'_1, u'_2 \) are given by (27) and (28), respectively. Analogously, the contribution from the two positions will decrease to the value (29) if the set \( G(z_1, z_2) \) is emptied.

This analysis leads us to the following theorem (proof in Appendix B) and corollary:

**Theorem 1:** When each individual packet in a frame is sent over an identical channel with binary inputs and general outputs, the minimal multisymbol minimizes \( H(Y|M_t) \).

**Corollary 1:** The following mutual information is constant for all minimal multisymbols \( M^m \):
\[
I(X;Y|M^m) = H(Y|M^m) - H(Y|X, M^m) = I_m \quad (30)
\]

**VI. Achieving the Capacity of the Secondary Channel**

Here we analyze (10) and find \( \mathcal{T} \) and \( \{M_t\} \) (i.e. \( P_T(\cdot) \) and \( P_{X|T}(\cdot) \), respectively) that simultaneously maximizes \( I(X;Y) \) according to Lemma \( \square \) and minimizes \( I(X;Y|T) = I_m \) according to (30). Recall that uniform \( T \) with \( |T| = \lcm((F_0), (F_1), \ldots, (F_l)) = L \) can achieve \( C_{XY} \). Since there are \( F! \geq L \) multisymbols, then in principle it should be possible to select \( L \) minimal multisymbols in order to have \( I(X;Y|T) = I_m \) and maximize \( I(X;Y) \).

In order to show that it is always possible to select \( \{M_t\} \) with \( \{|M_t|\} = L \) and uniform \( T \), we first take an example with \( F = 4 \). The set of \( L = 12 \) multisymbols can be selected as on Fig. 4(a). Multisymbols can be represented by a directed graph, see Fig. 4(b). Each node in the graph represents a particular \( x \in X \). An edge exists between \( x_s \in X_s \) and \( x_{s+1} \in X_{s+1} \) if and only if the Hamming distance is \( d_H(x_s, x_{s+1}) = 1 \). The directed edge from \( x_s \) to \( x_{s+1} \) exists if they can both belong to a same minimal multisymbol \( M_t \). A multisymbol is represented by a path of length \( F \) that starts at 00...0 and ends at 11...1. To each edge we can assign a nonnegative integer, which denotes the number of multisymbols (paths) that contain that edge. On Fig. 4(b), each edge that starts from 0000 has a weight 3, each edge between an element of \( X_1 \) and \( X_2 \) has a weight 1, etc. The weight of each edge between \( x_s \) and \( x_{s+1} \) can be treated as an outgoing weight for \( x_s \) and incoming weight for \( x_{s+1} \). Using this framework, we need to prove that, for each \( s = 0 \ldots F - 1 \), it is possible to match all outgoing weights from \( X_s \) to all incoming weights from \( X_{s+1} \). This is stated with the following theorem (proof in Appendix C):
Theorem 2: If \( L = \text{lcm} \left( \binom{F}{0}, \binom{F}{1}, \ldots, \binom{F}{F} \right) \) and the distribution over \( \mathcal{T} \) is uniform, then the multisymbols can be chosen such as to achieve the capacity of the secondary channel.

If \( F = 4 \) it turns out that \( \frac{m_s}{F-s} \) is always an integer, such that all the outgoing/incoming weights to the same node are identical. This is not the case if, e. g., \( F = 7 \), then \( L = 105 \), \( m_1 = 15 \) and \( \frac{m_1}{7-1} = \frac{15}{6} \), such that each node from \( \mathcal{X}_1 \) has 3 outgoing edges of weight 3 and 3 of weight 2.

VII. FURTHER CONSIDERATIONS AND NUMERICAL ILLUSTRATIONS

In absence of errors \( Y = X \), such that \( I(T; Y) = I(T; X) \) and the capacity is

\[
C_{F,0} = \sum_{s=0}^{F} P_S(s) \log_2 \left( \frac{F}{s} \right) \tag{31}
\]

When there are no errors, the state \( s \) is always known also at the receiver and the communication strategy is different, see [9]. Each state \( s \) is seen as a different subchannel, also denoted \( s \), and both the transmitter \( X \) and the receiver \( Y \) know which subchannel is used in a frame. Let \( r(F, s) = \log_2 \left( \frac{F}{s} \right) \) denote the number of bits that are sent in a single use of the subchannel \( s \). Considering a large number of channel uses \( n \to \infty \), then the realization of the sequence of frame states becomes typical [20] and the state \( s \) occurs approximately \( nP_S(s) \) times. The sender segments the message into submessages and each submessage is sent over a separate subchannel. The submessage sent over the subchannel \( s \) contains approximately \( nP(s)r(F, s) \) bits. If during the \( i \)-th channel use the sender observes that the state \( s \), then it takes the next \( r(F, s) \) bits from the corresponding submessage. Thus, the whole message is sent by time–interleaving of all the available subchannels and the time–interleaved sequence is perfectly observed by the receiver.

We now consider the model with erasures. An upper bound on the secondary capacity is simply taking \( C_{XY} \), as defined in [12]. If \( a = \frac{1}{2} \), then \( C_{XY} = F(1 - p) \), the capacity of the erasure channel with \( F \) uses. Consider now the asymptotic case \( F \to \infty \) and observe a single frame (one single channel use). The state becomes typical and, with high probability, \( s \in \left( \frac{F(1-\epsilon)}{2}, \frac{F(1+\epsilon)}{2} \right) \), where \( \epsilon \to 0 \) as \( F \to \infty \). We sketch how the capacity can be achieved in this case. First note that it suffices that the \( \mathcal{T} \) is \( \left( \frac{F}{2}, \frac{F(1-\epsilon)}{2} \right) \), where the latter is assumed to be integer. Then a multisymbol for each \( T = t \) has representatives in the sets \( \mathcal{X}_s \), where \( s \in \left[ \frac{F(1-\epsilon)}{2}, \frac{F(1+\epsilon)}{2} \right] \). If a state \( s \) outside of that interval occurs, then an arbitrary \( x \) is sent. With this strategy, there are some \( x \in \mathcal{X}_s \) with \( s > \frac{F(1-\epsilon)}{2} \) that are unused, but this is asymptotically negligible, and it can be shown that

\[
\lim_{F \to \infty} \frac{C_F}{F} = (1 - p) \tag{32}
\]
where $C_F$ is the capacity when the frame size is $F$. In other words, the normalized capacity approaches the capacity of a binary erasure channel, which is expected. A numerical illustration for the erasure channel is given on Fig. 5 and it can be seen that for relatively small $F$, the gap between the capacity of the secondary channel and $F$–uses erasure channel is substantial. Note that the equation (32) does not state that the gap will disappear, but only that it is of a type $o(F)$, i. e. becomes asymptotically zero compared to $F$.

We finally consider the case of a Z-channel, introduced in Section II-B. Recall that this is suitable when address 0 is an “empty” user, while address 1 means that there is a packet transmission (irrespective to which user it is addressed). The capacity of a binary Z-channel with crossover probability $p$ is given by $C_Z(p) = \log_2 \left(1 + (1 - p)p^{(1-p)}\right)$. The capacity–achieving distribution for the Z–channel requires nonuniform input distribution $P_{U,X}(x) \neq 2^{-F}$. As a simple outer bound on the capacity of the secondary channel, we again take $C_{XY}$, which for given input probability $a$ is given by $C_{XY} = C_{F,out} = FC_{Z,a}(p)$, where $C_{Z,a}(p)$ is the capacity of the binary Z-channel under a fixed value of the input probability $a$, given by $C_{Z,a}(p) = -(1 - a(1 - p)) \log_2(1 - a(1 - p) + a(1 - p) \log_2 a(1 - p)) + a (p \log_2(p) + (1 - p) \log_2(1 - p))$. Some illustrative results for the Z–channel model are provided on Fig. 6. The channel capacity is compared to the outer bound in dependency of the frame length $F$, for a fixed crossover probability $p = 0.2$ and $a = 0.5$.

Similar to the discussion for the erasure channel, for the Z–channel we also consider the asymptotic case $F \to \infty$ and observe a single frame (channel use). Using similar arguments as for the erasure channel, for the asymptotic case with a Z–channel model it can be shown that

$$\lim_{F \to \infty} \frac{C_F}{F} = C_{Z,a}(p)$$

(33)

VIII. CONCLUSION

We have introduced a class of communication channels with protocol coding, i. e. the information is modulated in the actions taken by the communication protocol of an existing, primary system. In particular, we have considered strategies in which protocol coding is done by combinatorial ordering of the labelled user resources (packets, channels) in the primary system. Differently from the previous works, our focus here is not on the steganographic usage of this type of protocol coding. Our aim is rather on its ability to introduce a new secondary communication channel, intended for reliable communication with newly introduced secondary devices, that are
low-complexity versions of the primary devices, capable only to decode the robustly encoded header information in the primary signals. The key feature of the communication model is that it captures the constraints that the primary system operation puts on protocol coding i.e. the secondary information can only be sent by rearranging the set of packets made available by the primary system. The challenge is that the amount of information that can be sent in this way is not controllable by the secondary - e.g. if all the primary packets in a given scheduling epoch carry the same label, then all re-arrangements look equivalent to a secondary receiver and no secondary information can be sent. Since the main application of the secondary channels introduced here is reliable communication, we have focused on investigating the communication strategies that can be used under various error models. We have derived the capacity of the secondary channel under arbitrary error models. The insights obtained from the capacity-achieving communication strategies are used in Part II of this work to design practical error-correcting mechanisms for secondary channels with protocol coding.

ACKNOWLEDGMENT

The authors would like to thank Prof. Osvaldo Simeone (New Jersey Institute of Technology) for useful discussions on the channels with causal channel state information at the transmitter.

APPENDIX

A. Proof of Lemma 7

Proof: We generalize the Theorem 4.5.1 from [21] to reflect the fact that the maximization is over $P_{X,S}$ rather than $P_X$. Let us denote $P_X(x_{s,k}) = \alpha_{s,k}$ where $x_{s,k}$ is the $k-$th element (e.g. in a lexicographic order) within the set $X_s$. Let $\alpha = (\alpha_{0,1}, \alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{F,F})$ be the $2^F$-dimensional probability vector. Then $I(X, Y) = f(\alpha)$ and the maximization problem is:

$$\max f(\alpha) \quad \text{such that} \quad \sum_{k=1}^{K_s} \alpha_{s,k} = p_s, \quad \forall s \in S$$

(34)

where $p_s = P_S(s)$ and $K_s = |X_s| = (F_s)$. The constraint $\sum_{s,k} \alpha_{s,k} = 1$ is redundant, since $\sum_s p_s = 1$. We need to use $(F+1)$ Lagrangian multipliers and maximize $f(\alpha) - \sum_s \lambda_s (\sum_k \alpha_{s,k} - p_s)$. For each $s,k$ we have $\frac{\partial f}{\partial \alpha_{s,k}} = \lambda_s$ when $\alpha_{s,k} > 0$ and $\frac{\partial f}{\partial \alpha_{s,k}} \leq \lambda_s$ when $\alpha_{s,k} = 0$. With these conditions, Theorem 4.5.1 in [21] is generalized as follows. We define:

$$I(X = x_{s,k}; Y) = \sum_{y \in Y} p(y|x_{s,k}) \log \frac{p(y|x_{s,k})}{\sum_{s,k} \alpha_{s,k} p(y|x_{s,k})}$$

(35)
The necessary and sufficient conditions for an input probability vector \( \alpha \in \mathcal{P}_{X,S} \) to maximize this mutual information are stated as follows. For some set of numbers \( \{ C_s \} \), where \( s \in S \):

If \( \alpha_{s,k} > 0 \) then \( I(X = x_{s,k}; Y) = C_s \); otherwise, if \( \alpha_{s,k} = 0 \) then \( I(X = x_{s,k}; Y) \leq C_s \).

Let \( Y_A \) be the set of all \( y \) whose elements are permutations of a certain \( y_A \). The \( K_s \times |Y_A| \) sub–matrix that contains \( p(y|x_{s,k}) \) which correspond to the inputs from the state \( S = s \) and the outputs from the subset \( Y_A \) exhibits a symmetry: each row of this sub–matrix is a permutation of each other row. Using the definition of symmetric channel from [21] and setting all the inputs \( x \in X \) equiprobable with \( \alpha_{s,k} = p_s K_s \). Then \( p(y) = \sum_s p_s K_s \sum_k p(y|x_{s,k}) \), one can check that \( I(X = x_{s,k}; Y) = C_s \) is constant for all inputs that belong to the same state \( s \).

**Proof**: The members on the left-handed side of (??) can be written as:

\[
H((Q_1 + Q_2)q_0 + (Q_3 + Q_4)q_1) = H(\lambda v_1 + (1-\lambda)v_2) \quad H((Q_1 + Q_3)q_0 + (Q_2 + Q_4)q_1) = H((1-\lambda)v_1 + \lambda v_2)
\]

where \( v_1 = Q_1q_0 + (Q_2 + Q_3 + Q_4)q_1 \), \( v_2 = (Q_1 + Q_2 + Q_3)q_0 + Q_4q_1 \), and \( \lambda = \frac{Q_3}{Q_2 + Q_4} \). Since \( H(\cdot) \) is concave, we finalize the proof by writing:

\[
H(\lambda v_1 + (1-\lambda)v_2) + H((1-\lambda)v_1 + \lambda v_2) \geq \lambda H(v_1) + (1-\lambda)H(v_2) + (1-\lambda)H(v_1) + \lambda H(v_2) = H(v_1) + H(v_2)
\]

B. **Proof of Theorem 7**

**Proof**: Let the basic multisymbol associated with \( T = t \) be represented by a matrix:

\[
M = \begin{pmatrix}
  z_1 & z_2 & z_3 & \cdots & z_{F-1} & z_F \\
  x_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
  x_1 & 0 & 0 & 0 & \cdots & 0 & 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  x_{F-1} & 1 & 1 & 1 & \cdots & 1 & 0 \\
  x_F & 1 & 1 & 1 & \cdots & 1 & 1 
\end{pmatrix}
\]

(36)

It can be easily checked that for any pair \( z_{f_1}, z_{f_2} \) either the set \( \mathcal{G}_{10}(z_1, z_2) \) or the set \( \mathcal{G}_{01}(z_1, z_2) \) is empty. According to Lemma 5, that permutation (swapping) of the values within one or more \( x_s \) cannot further decrease the entropy contribution of the positions that are swapped. Hence, the basic multisymbol (36) results in the minimal possible value of \( H(Y|M_t) \). The same observation can be made whenever \( M_t \) is a minimal multisymbol, which proves the theorem. •
C. Proof of Theorem 2

Since $L$ divides each $\binom{F}{s}$, the number of multisymbols that contain $x_s \in \mathcal{X}_s$ is an integer $m_s = \frac{L}{\binom{F}{s}}$. The number of outgoing edges from $x_s$ is $(F-s)$, while the number of incoming edges to $x_s$ is $s$. The sum of incoming weights and the sum of outgoing weights for $x_s$ is equal to $m_s$. Note that the average outgoing weight for $x_s$ is $\frac{m_s}{F-s}$, while the average incoming weight for any $x_{s+1} \in \mathcal{X}_{s+1}$ is $\frac{m_{s+1}}{s+1}$. However, the following holds

\[ \frac{m_s}{F-s} \binom{F}{s} = \frac{L}{\binom{F}{s}(F-s)} = \frac{L}{\binom{F}{s}(s+1)} = \frac{m_{s+1}}{s+1} \]

i.e., the average outgoing weight from $\mathcal{X}_s$ is equal to the average incoming weight at $\mathcal{X}_{s+1}$, which is a necessary condition for the multisymbols that achieve the secondary capacity. We now prove that for each outgoing weight from $\mathcal{X}_s$ there is a matched incoming weight at $\mathcal{X}_{s+1}$. We choose the weight of each edge to be either $w_1 = \lfloor \frac{m_s}{F-s} \rfloor$ or $w_2 = \lceil \frac{m_s}{F-s} \rceil$. Then $b$ weights have to be chosen to be equal to $w_2 = \lceil \frac{m_s}{F-s} \rceil$, where $b$ is given by

\[ m_s = a(F-s) + b, \quad a \in \{\mathbb{N} \cup 0\}, \quad 0 \leq b \leq F-s-1. \tag{37} \]

There are $s+1$ incoming edges at $x_{s+1}$. The weight of each incoming edge is also either $w_1$ or $w_2$, since $\frac{m_s}{F-s} = \frac{m_{s+1}}{s+1}$. In order to satisfy the condition that the total incoming weight of $x_{s+1}$ is $m_{s+1}$, $d$ weights should be chosen to be equal to $w_2$, where $d$ is given by

\[ m_{s+1} = c(s+1) + d, \quad c \in \{\mathbb{N} \cup 0\}, \quad 0 \leq d \leq s. \tag{38} \]

If (37) and (38) are satisfied, then $b\binom{F}{s} = d\binom{F}{s+1}$ needs to be fulfilled, which follows from $\binom{F}{s} = \binom{F}{s+1}$ and the equality of average incoming/outgoing weights. For each outgoing weight from $\mathcal{X}_s$ there is a matched incoming weight at $\mathcal{X}_{s+1}$. Since $L \leq F!$, it will be always possible to select $L$ different paths.

REFERENCES

[1] J. G. Andrews, A. Ghosh, and R. Muhamed, Fundamentals of WiMAX. Prentice-Hall, 2007.
[2] 3GPP, “LTE–Advanced,” http://www.3gpp.org/article/lte-advanced.
[3] S.-Y. Lien, K.-C. Chen, and Y. Lin, “Toward Ubiquitous Massive Accesses in 3GPP Machine-to-Machine Communications,” IEEE Communications Magazine, vol. 49, no. 4, pp. 66–74, Apr. 2011.
[4] P. Popovski, Z. Utkovski, and K. F. Trillingsgaard, “Protocol Coding through Reordering of User Resources, Part II: Practical Coding Strategies,” submitted to IEEE Trans. Communications, 2012.
[5] J. L. Massey, Channel Models for Random–Access Systems, ser. Performance Limits in Communication Theory and Practice, NATO Advances Studies Institutes Series E142. Kluwer Academic, 1988, pp. 391–402.
[6] V. Anantharam and S. Verdu, “Bits through Queues,” IEEE Trans. Inform. Theory, vol. 44, pp. 4–18, Jan. 1996.
[7] G. Kramer, “Models and Theory for Relay Channels with Receive Constraints,” in *Proc. 42 Annual Allerton Conference on Communications, Control and Computing*, Urbana-Champaign, IL, USA, Sep. 2004.

[8] T. Lutz, C. Hausl, and R. Kötter, “Bits Through Relay Cascades with Half-Duplex Constraint,” 2009, submitted, (arXiv:0906.1599).

[9] P. Popovski and O. Simeone, “Protocol Coding for Two-Way Communications with Half-Duplex Constraints,” in *IEEE GLOBECOM*, Miami, FL, USA, Dec. 2010.

[10] A. Ephremides and B. Hajek, “Information Theory and Communication Networks: An Unconsummated Union,” *IEEE Trans. Inform. Theory*, vol. 44, no. 6, pp. 2416–2434, Oct. 1998.

[11] K. Ashan, *Covert Channel Analysis and Data Hiding in TCP/IP*. M. Sc. thesis, Dept. of Electrical and Computer Engineering, University of Toronto, August 2002.

[12] R. C. Chakinala, A. Kumarasubramanian, R. Manokaran, G. Noubir, C. P. Rangan, and R. Sundaram, “Steganographic communication in ordered channels,” in *Information Hiding, Lecture Notes in Computer Science*, vol. 4437. Springer-Verlag, 2009.

[13] A. El-Atawy and E. Al-Shaer, “Building covert channels over the packet reordering phenomenon,” in *Proc. of IEEE INFOCOM*, Apr. 2009.

[14] A. J. H. Vinck, “Coded Modulation for Power Line Communications,” *AEÜ Journal*, pp. 45–49, Jan. 2000.

[15] W. Chu, C. J. Colbourn, and P. Dukes, “Constructions for Permutation Codes in Powerline Communications,” *Designs, Codes and Cryptography, Kluwer Academic Publishers*, vol. 32, pp. 51–64, 2004.

[16] P. Popovski and Z. Utkovski, “On the Secondary Capacity of the Communication Protocols,” in *IEEE GLOBECOM*, Honolulu, HI, USA, Dec. 2009.

[17] Z. Utkovski and P. Popovski, “Protocol Coding with Reordering of User Resources: Capacity Results for the Z-Channel,” in *49th Annual Allerton Conference on Communication, Control, and Computing*, Monticello, Illinois, USA, Sep. 2011.

[18] C. E. Shannon, “Channels with side information at the transmitter,” *IBM Journal of Research and Development*, vol. 2, pp. 289–293, Oct. 1958.

[19] G. Keshet, Y. Steinberg, and N. Merhav, *Channel Coding in the Presence of Side Information*, ser. Foundations and Trends in Communications and Information Theory, 2007, vol. 4, no. 6.

[20] T. Cover and J. Thomas, *Elements of Information Theory*. Wiley-Interscience, 2nd Edition, 2006.

[21] R. Gallager, *Information Theory and Reliable Communication*. John Wiley and Sons, 1968.
Fig. 1. Illustration of a secondary communication through protocol coding in cellular systems. A primary device can decode any information sent by the base station, while the secondary device has a limited functionality can only decode the information sent by protocol coding. The range of the primary communication system (white circle) is smaller than the range of the secondary information (shaded circle).
Fig. 2. The primary system consists of a Base Station (BS) and two primary devices. Each primary packet has a header that contains address $a_i \in \{0, 1\}$. The BS selects the orders of the packets in a frame in order to send information to the secondary device.
Fig. 3. Example choice of the probability distribution $P_{X|T}$ with $F = 2$ and $T = \{1, 2\}$. The transition probabilities on the channel $X - Y$ are not marked, but it is assumed that each packet 0 or 1 can become erased $\epsilon$ independently with probability $p$. 
Fig. 4. Selection of the representative sets for $F = 4$ that achieve the capacity. (a) Multisymbols for the 12 inputs (b) Graph representation of the process for selecting the multisymbols $x_s(t)$. 
Fig. 5. Comparison between the secondary capacity with the reference capacity (erasure channel with $F$ uses). The probability $a = \frac{1}{2}$. 

Packet erasure probability $p=0.2$

Capacity [bits/frame]

Frame size $F$

Secondary capacity

Reference capacity
Fig. 6. Comparison between the capacity of the combinatorial model and the outer bound for $p = 0.2$ and $a = 0.5$. 

Packet erasure probability $p=0.2$

Capacity [bits/frame]

Frame size $F$

Secondary capacity

Reference capacity