Geometric Spinors, Generalized Dirac Equation and Mirror Particles

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Abstract

It is shown that since the geometric spinors are elements of Clifford algebras, they must have the same transformation properties as any other Clifford number. In general, a Clifford number $\Phi$ transforms into a new Clifford number $\Phi'$ according to $\Phi \rightarrow \Phi' = R \Phi S$, i.e., by the multiplication from the left and from the right by two Clifford numbers $R$ and $S$. We study the case of $Cl(1,3)$, which is the Clifford algebra of the Minkowski spacetime. Depending on choice of $R$ and $S$, there are various possibilities, including the transformations of vectors into 3-vectors, and the transformations of the spinors of one minimal left ideal of $Cl(1,3)$ into another minimal left ideal. This, among others, has implications for understanding the observed non-conservation of parity.

1 Introduction

We will follow the approach [1]–[4] in which spinors are constructed in terms of nilpotents formed from the spacetime basis vectors represented as generators of the Clifford algebra

$$\gamma_a \cdot \gamma_b \equiv \frac{1}{2} (\gamma_a \gamma_b + \gamma_b \gamma_a) = \eta_{ab}$$

$$\gamma_a \wedge \gamma_b \equiv \frac{1}{2} (\gamma_a \gamma_b - \gamma_b \gamma_a).$$

(1)

The inner, symmetric, product of basis vectors gives the metric. The outer, antisymmetric, product of basis vectors gives a basis bivector.

The generic Clifford number is

$$\Phi = \varphi^A \gamma_A.$$  

(2)

where $\gamma_A \equiv \gamma_{a_1 a_2 ... a_r} \equiv \gamma_{a_1} \wedge \gamma_{a_2} \wedge ... \wedge \gamma_{a_r}$, $r = 0, 1, 2, 3, 4$.

Spinors are particular Clifford numbers, $\Psi = \psi^\alpha \xi_\alpha$, where $\xi_\alpha$ are spinor basis elements, composed from $\gamma_A$. We will consider transformation properties of Clifford numbers.

In general, a Clifford number transforms according to

$$\Phi \rightarrow \Phi' = R \Phi S.$$  

(3)

Here $R$ and $S$ are Clifford numbers, e.g., $R = e^{a A} \gamma_A$, $S = e^{b A} \gamma_A$.

In particular, if $S = 1$, we have

$$\Phi \rightarrow \Phi' = R \Phi.$$  

(4)
As an example, let us consider the case

\[ R = e^{\frac{i}{2} \gamma_1 \gamma_2} = \cos \frac{\alpha}{2} + \gamma_1 \gamma_2 \sin \frac{\alpha}{2}, \quad S = e^{\frac{1}{2} \gamma_1 \gamma_2} = \cos \frac{\beta}{2} + \gamma_1 \gamma_2 \sin \frac{\beta}{2} \]  

(5)

and examine [5], how various Clifford numbers, 

\[ X = X C \gamma C, \]  

(6)

transform under (3), which now reads:

\[ X \rightarrow X' = R X S. \]  

(7)

(i) If \( X = X^1 \gamma_1 + X^2 \gamma_2 \) then

\[ X' = X^1 \left( \gamma_1 \cos \frac{\alpha - \beta}{2} + \gamma_2 \sin \frac{\alpha - \beta}{2} \right) + X^2 \left( -\gamma_1 \sin \frac{\alpha - \beta}{2} + \gamma_2 \cos \frac{\alpha - \beta}{2} \right). \]  

(8)

(ii) If \( X = X^3 \gamma_3 + X^{123} \gamma_{123} \) then

\[ X' = X^3 \left( \gamma_3 \cos \frac{\alpha + \beta}{2} + \gamma_{123} \sin \frac{\alpha + \beta}{2} \right) + X^{123} \left( -\gamma_3 \sin \frac{\alpha + \beta}{2} + \gamma_{123} \cos \frac{\alpha + \beta}{2} \right). \]  

(9)

(iii) If \( X = s \frac{1}{2} + X^{12} \gamma_{12} \), then

\[ X' = s \left( \frac{1}{2} \cos \frac{\alpha + \beta}{2} + \gamma_{12} \sin \frac{\alpha + \beta}{2} \right) + X^{12} \left( -\frac{1}{2} \sin \frac{\alpha + \beta}{2} + \gamma_{12} \cos \frac{\alpha + \beta}{2} \right). \]  

(10)

(iv) If \( X = \tilde{X}^1 \gamma_5 \gamma_1 + \tilde{X}^2 \gamma_5 \gamma_2 \), then

\[ X' = \tilde{X}^1 \left( \gamma_5 \gamma_1 \cos \frac{\alpha - \beta}{2} + \gamma_5 \gamma_2 \sin \frac{\alpha - \beta}{2} \right) + \tilde{X}^2 \left( -\gamma_5 \gamma_1 \sin \frac{\alpha - \beta}{2} + \gamma_5 \gamma_2 \cos \frac{\alpha - \beta}{2} \right). \]  

(11)

Usual rotations of vectors or pseudovectors are reproduced, if the angle \( \beta \) for the right transformation is equal to minus angle for the left transformation, i.e., if \( \beta = -\alpha \). Then all other transformations which mix the grade vanish. But in general, if \( \beta \neq \alpha \), the transformation (7) mixes the grade.

2 Clifford algebra and spinors in Minkowski space

Let us introduce a new basis, called the Witt basis,

\[ \theta_1 = \frac{1}{2} (\gamma_0 + \gamma_3), \quad \theta_2 = \frac{1}{2} (\gamma_1 + i \gamma_2), \]

\[ \bar{\theta}_1 = \frac{1}{2} (\gamma_0 - \gamma_3), \quad \bar{\theta}_2 = \frac{1}{2} (\gamma_1 - i \gamma_2), \]

(12)

where

\[ \gamma_a = (\gamma_0, \gamma_1, \gamma_2, \gamma_3). \]  

(13)
The new basis vectors satisfy
\[ \{\theta_a, \bar{\theta}_b\} = \eta_{ab}, \quad \{\theta_a, \theta_b\} = 0, \quad \{\bar{\theta}_a, \bar{\theta}_b\} = 0, \] (14)
which are fermionic anticommutation relations. We now observe that the product
\[ f = \bar{\theta}_1 \bar{\theta}_2 \] (15)
satisfies
\[ \bar{\theta}_a f = 0, \quad a = 1, 2. \] (16)
Here \( f \) can be interpreted as a 'vacuum, and \( \bar{\theta}_a \) can be interpreted as operators that annihilate \( f \).

An object constructed as a superposition
\[ \Psi = (\psi^0 + \psi^1 \theta_1 + \psi^2 \theta_2 + \psi^{12} \theta_1 \theta_2) f \] (17)
is a 4-component spinor. It is convenient to change the notation:
\[ \Psi = (\psi^1 + \psi^2 \theta_1 \theta_2 + \psi^3 \theta_1 + \psi^4 \theta_2) f = \psi^\alpha \xi_\alpha, \quad \alpha = 1, 2, 3, 4 \] (18)
where \( \xi_\alpha \) is the spinor basis.

The even part of the above expression is a left handed spinor
\[ \Psi_L = (\psi^1 + \psi^2 \theta_1 \theta_2) \bar{\theta}_1 \bar{\theta}_2, \] (19)
whereas the odd part is a right handed spinor
\[ \Psi_R = (\psi^3 \theta_1 + \psi^4 \theta_2) \bar{\theta}_1 \bar{\theta}_2. \] (20)

We can verify that the following relations are satisfied:
\[ i\gamma_5 \Psi_L = -\Psi_L, \quad i\gamma_5 \Psi_R = \Psi_R \] (21)
Under the transformations
\[ \Psi \to \Psi' = R \Psi, \] (22)
where
\[ R = \exp\left[ \frac{1}{2} \gamma_{a_1} \gamma_{a_2} \varphi \right], \] (23)
the Clifford number \( \Psi \) transforms as a spinor.

As an example let us consider the case
\[ R = e^{\frac{i}{2} \gamma_1 \gamma_2 \varphi} = \cos \frac{\varphi}{2} + \gamma_1 \gamma_2 \sin \frac{\varphi}{2}. \] (24)

Then we have
\[ \Psi \to \Psi' = R \Psi = \left( e^{\frac{i \varphi}{2}} \psi^1 + e^{-\frac{i \varphi}{2}} \psi^2 \theta_1 \theta_2 + e^{i \frac{\varphi}{2}} \psi^3 \theta_1 + e^{-\frac{i \varphi}{2}} \psi^4 \theta_2 \right) f. \] (25)
This is the well-known transformation of a 4-component spinor.
2.1 Four independent spinors

There exist four different possible vacua \([1, 6, 3]\):

\[ f_1 = \bar{\theta}_1 \theta_2, \quad f_2 = \theta_1 \theta_2, \quad f_3 = \theta_1 \bar{\theta}_2, \quad f_3 = \bar{\theta}_1 \theta_2 \]  

(26)

to which there correspond four different kinds of spinors:

\[
\begin{align*}
\Psi^1 &= (\psi^{11} + \psi^{21} \theta_1 \theta_2 + \psi^{31} \theta_1 + \psi^{41} \theta_2) f_1 \\
\Psi^2 &= (\psi^{12} + \psi^{22} \theta_1 \theta_2 + \psi^{32} \theta_1 + \psi^{42} \theta_2) f_2 \\
\Psi^3 &= (\psi^{13} \theta_1 + \psi^{23} \theta_2 + \psi^{33} + \psi^{43} \bar{\theta}_1 \theta_2) f_3 \\
\Psi^4 &= (\psi^{14} \theta_1 + \psi^{24} \bar{\theta}_2 + \psi^{34} + \psi^{44} \theta_1 \theta_2) f_4.
\end{align*}
\]  

(27)

Each of those spinors lives in a different minimal left ideal of \(Cl(1, 3)\), or in general, of its complexified version if we assume complex \(\psi^{\alpha i}\).

An arbitrary element of \(Cl(1, 3)\) is the sum:

\[ \Phi = \Psi^1 + \Psi^2 + \Psi^3 + \Psi^4 = \psi^{\alpha i} \xi_{\alpha i} \equiv \psi^{\hat{A}} \xi_{\hat{A}}, \]  

(28)

where

\[ \xi_{\hat{A}} \equiv \xi_{\alpha i} = \{ f_1, \theta_1 \theta_2 f_1, ..., \theta_1 f_4, \bar{\theta}_2 f_4, f_4, \bar{\theta}_1 \theta_2 f_4 \}, \]  

(29)

is a spinor basis of \(Cl(1, 3)\). Here \(\Phi\) is a generalized spinor.

In matrix notation we have

\[
\begin{pmatrix}
\psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\
\psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\
\psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\
\psi^{41} & \psi^{42} & \psi^{43} & \psi^{44}
\end{pmatrix}
\]

(30)

Here, for instance, the second column in the left matrix contains the components of the spinor of the second left ideal. Similarly, the second column in the right matrix contains the basis elements of the second left ideal.

A general transformation is

\[ \Phi = \psi^{\hat{A}} \xi_{\hat{A}} \rightarrow \Phi' = R \Phi S = \psi^{\hat{A}} \xi'_{\hat{A}} = \psi^{\hat{A}} L_{\hat{A}} \bar{\theta} \xi_{\bar{B}} = \psi^{\hat{B}} \xi_{\bar{B}} \]  

(31)

where

\[ \xi'_{\hat{A}} = R \xi_{\hat{A}} S = L_{\hat{A}} \bar{\theta} \xi_{\bar{B}}, \quad \psi^{\hat{B}} = \psi^{\hat{A}} L_{\hat{A}} \bar{\theta}. \]  

(32)

This is an active transformation, because it changes an object \(\Phi\) into another object \(\Phi'\).

The transformation from the left,

\[ \Phi' = R \Phi, \]  

(33)

reshuffles the components within each left ideal, whereas the transformation from the right,

\[ \Phi' = \Phi S, \]  

(34)

reshuffles the left ideals.
3 Behavior of spinors under Lorentz transformations

Let us consider the following transformation of the basis vectors
\[ \gamma_a \rightarrow \gamma'_a = R \gamma_a R^{-1}, \quad a = 0, 1, 2, 3, \] (35)
where \( R \) is a proper or improper Lorentz transformation. A generalized spinor, \( \Phi \in Cl(1,3) \), composed of \( \gamma_a \), then transforms according to
\[ \Phi = \psi^\hat{A} \xi_{\hat{A}} \rightarrow \Phi' = \psi'^{\hat{A}} \xi'_{\hat{A}} = \psi^{\hat{A}} R \xi_B R^{-1} = R \Phi R^{-1}. \] (36)
The transformation (35) of the basis vectors has for a consequence that the object \( \Phi \) does not transform only from the right, but also from the left. This had led Piazzese to the conclusion that spinors cannot be interpreted as the minimal ideals of Clifford algebras [7].

But if the reference frame transforms as
\[ \gamma_a \rightarrow \gamma'_a = R \gamma_a, \] (37)
then
\[ \Phi = \psi^\hat{A} \xi_{\hat{A}} \rightarrow \Phi' = \psi'^{\hat{A}} \xi'_{\hat{A}} = \psi^{\hat{A}} R \xi_B = R \Phi. \] (38)
This is a transformation of a spinor. Therefore, the description of spinors in terms of ideals is consistent.

As we have seen in Sec.1, the transformation (37) is also a possible transformation within a Clifford algebra. It is a transformation that changes the grade of a basis element. Usually, we do not consider such transformations of basis vectors. Usually reference frames are “rotated” (Lorentz rotated) according to
\[ \gamma_a \rightarrow \gamma'_a = R \gamma_a R^{-1} = L_a^b \gamma_b, \] (39)
where \( L_a^b \) is a proper or improper Lorentz transformation. Therefore, a “rotated observer sees (generalized) spinors transformed according to
\[ \Phi \rightarrow \Phi' = R \Phi R^{-1}. \] (40)
With respect to a new reference frame, the object \( \Phi = \psi^\hat{A} \xi_{\hat{A}} \) is expanded according to
\[ \Phi = \psi'^{\hat{A}} \xi'_{\hat{A}}, \] (41)
where
\[ \psi'^{\hat{A}} = \psi^{\hat{B}} (L^{-1})_{\hat{B}}^\hat{A}. \] (42)
Recall that \( \alpha, \beta = 1, 2, 3, 4 \), and \( i, j = 1, 2, 3, 4 \). The corresponding matrix \( \psi'^{\alpha i} \) transforms from the left and from the right.

If the observer, together with the reference frame, starts to rotate, then after having exhibited the \( \varphi = 2\pi \) turn, he observes the same spinor \( \Psi \), as he did at \( \varphi = 0 \). The sign of the spinor did not change, because this was just a passive transformation, so that the same (unchanged) objects was observed from the transformed (rotated) references frames at different angles \( \varphi \). In the new reference frame the object was observed to be transformed according to \( \Psi' = R \Psi R^{-1} \). There must also exist the corresponding active transformation such that in a fixed reference frame the spinor transforms as \( \Psi' = R \Psi R^{-1} \).
3.1 Examples

3.1.1 Rotation

Let us consider the following rotation:

\[
\begin{align*}
\gamma_0 & \rightarrow \gamma_0, & & \gamma_1 & \rightarrow \gamma_1, & & \gamma_2 & \rightarrow \gamma_2 \cos \vartheta + \gamma_3 \sin \vartheta, \\
\gamma_3 & \rightarrow -\gamma_2 \sin \vartheta + \gamma_3 \cos \vartheta.
\end{align*}
\] (43)

In the case \(\vartheta = \pi\), we have

\[
\begin{align*}
\gamma_0 & \rightarrow \gamma_0, & & \gamma_1 & \rightarrow \gamma_1, & & \gamma_2 & \rightarrow -\gamma_2, & & \gamma_3 & \rightarrow -\gamma_3.
\end{align*}
\] (44)

The Witt basis then transforms as

\[
\begin{align*}
\theta_1 & \rightarrow \bar{\theta}_1, & & \theta_2 & \rightarrow \bar{\theta}_2, & & \bar{\theta}_1 & \rightarrow \theta_1, & & \bar{\theta}_2 & \rightarrow \theta_2.
\end{align*}
\] (45)

A consequence is that, e.g., a spinor of the first left ideal transforms as

\[
(\psi^{11} + \psi^{21} \theta_1 \theta_2 + \psi^{31} \theta_1 + \psi^{41} \theta_2) \bar{\theta}_1 \bar{\theta}_2 \rightarrow (\psi^{11} + \psi^{21} \bar{\theta}_1 \bar{\theta}_2 + \psi^{31} \bar{\theta}_1 + \psi^{41} \bar{\theta}_2) \theta_1 \theta_2.
\] (46)

By inspecting the latter relation and taking into account Eqs. (27), (19), (20), we see that a left handed spinor of the first ideal transforms into a right handed spinor of the second ideal. Similarly, a right handed spinor of the first ideal transforms into a right handed spinor of the second ideal.

In general, under the \(\vartheta = \pi\) rotation in the \((\gamma_2, \gamma_3)\) plane, a generalized spinor

\[
\Phi = (\psi^{11} + \psi^{21} \theta_1 \theta_2 + \psi^{31} \theta_1 + \psi^{41} \theta_2) \bar{\theta}_1 \bar{\theta}_2 + (\psi^{12} + \psi^{22} \theta_1 \theta_2 + \psi^{32} \theta_1 + \psi^{42} \theta_2) \theta_1 \theta_2 + (\psi^{13} \bar{\theta}_1 + \psi^{23} \bar{\theta}_2 + \psi^{33} + \psi^{43} \bar{\theta}_1 \theta_2) \theta_1 \theta_2 + (\psi^{14} \theta_1 + \psi^{24} \theta_2 + \psi^{34} + \psi^{44} \theta_1 \theta_2) \bar{\theta}_1 \bar{\theta}_2
\] (47)

transforms into

\[
\Phi' = (\psi^{11} + \psi^{21} \bar{\theta}_1 \bar{\theta}_2 + \psi^{31} \bar{\theta}_1 + \psi^{41} \bar{\theta}_2) \theta_1 \theta_2 + (\psi^{12} + \psi^{22} \bar{\theta}_1 \bar{\theta}_2 + \psi^{32} \bar{\theta}_1 + \psi^{42} \bar{\theta}_2) \bar{\theta}_1 \bar{\theta}_2 + (\psi^{13} \theta_1 + \psi^{23} \theta_2 + \psi^{33} + \psi^{43} \theta_1 \bar{\theta}_2) \theta_1 \theta_2 + (\psi^{14} \bar{\theta}_1 + \psi^{24} \bar{\theta}_2 + \psi^{34} + \psi^{44} \bar{\theta}_1 \bar{\theta}_2) \bar{\theta}_1 \bar{\theta}_2.
\] (48)

The matrix of components

\[
\psi^{\alpha i} = \begin{pmatrix}
\psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\
\psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\
\psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\
\psi^{41} & \psi^{42} & \psi^{43} & \psi^{44}
\end{pmatrix}
\] transforms into \(\psi'^{\alpha i} = \begin{pmatrix}
\psi^{12} & \psi^{11} & \psi^{14} & \psi^{13} \\
\psi^{22} & \psi^{21} & \psi^{24} & \psi^{23} \\
\psi^{32} & \psi^{31} & \psi^{34} & \psi^{33} \\
\psi^{42} & \psi^{41} & \psi^{44} & \psi^{43}
\end{pmatrix}\). (49)

We see that in the transformed matrix, the first and the second column are interchanged. Similarly, also the third and forth column are interchanged. Different columns represent different left minimal ideals of \(Cl(1, 3)\), and thus different spinors.

Let us now focus our attention on the spinor basis states of the first and second ideal:

\[
\xi_{11} = \bar{\theta}_1 \bar{\theta}_2, \quad \xi_{21} = \theta_1 \theta_2 \bar{\theta}_1 \bar{\theta}_2, \quad \xi_{12} = \theta_1 \theta_2, \quad \xi_{22} = \bar{\theta}_1 \bar{\theta}_2 \theta_1 \theta_2.
\] (50)
which span the left handed part of the 4-component spinor (see Eqs. (19, 20)).

Under the \( \vartheta = \pi \) rotation (44), (45), we have

\[
\begin{align*}
\xi_{11} & \rightarrow \xi_{12}, & \xi_{21} & \rightarrow \xi_{22}, & \xi_{12} & \rightarrow \xi_{11}, & \xi_{22} & \rightarrow \xi_{21}, \\
-\frac{i}{2} \gamma_1 \gamma_2 \xi_{11} & = \frac{1}{2} \xi_{11}, & -\frac{i}{2} \gamma_1 \gamma_2 \xi_{21} & = -\frac{1}{2} \xi_{21}, \\
-\frac{i}{2} \gamma_1 \gamma_2 \xi_{12} & = -\frac{1}{2} \xi_{12}, & -\frac{i}{2} \gamma_1 \gamma_2 \xi_{22} & = \frac{1}{2} \xi_{22}.
\end{align*}
\]

(51)

which means that the spin 1/2 state of the 1st ideal transforms into the spin state of the 2nd ideal, and vice versa. The above states are eigenvalues of the spin operator, \(-\frac{1}{2} \gamma_1 \gamma_2\),

\[
\begin{align*}
-\frac{i}{2} \gamma_1 \gamma_2 \xi_{11} & = \frac{1}{2} \xi_{11}, & -\frac{i}{2} \gamma_1 \gamma_2 \xi_{21} & = -\frac{1}{2} \xi_{21}, \\
-\frac{i}{2} \gamma_1 \gamma_2 \xi_{12} & = -\frac{1}{2} \xi_{12}, & -\frac{i}{2} \gamma_1 \gamma_2 \xi_{22} & = \frac{1}{2} \xi_{22}.
\end{align*}
\]

(52)

(53)

Let us now introduce the new basis states

\[
\begin{align*}
\xi_{1/2}^1 & = \frac{1}{\sqrt{2}} (\xi_{11} + \xi_{22}), & \xi_{1/2}^2 & = \frac{1}{\sqrt{2}} (\xi_{11} - \xi_{22}), \\
\xi_{-1/2}^1 & = \frac{1}{\sqrt{2}} (\xi_{21} + \xi_{12}), & \xi_{-1/2}^2 & = \frac{1}{\sqrt{2}} (\xi_{21} - \xi_{12}).
\end{align*}
\]

(54)

which are superpositions of the states of the 1st and the 2nd ideal. Under the rotation (44), (45) we have

\[
\begin{align*}
\xi_{1/2}^1 & \rightarrow \frac{1}{\sqrt{2}} (\xi_{12} + \xi_{21}) = \xi_{-1/2}^1, \\
\xi_{1/2}^2 & \rightarrow \frac{1}{\sqrt{2}} (\xi_{22} + \xi_{11}) = \xi_{1/2}^1, \\
\xi_{-1/2}^1 & \rightarrow \frac{1}{\sqrt{2}} (\xi_{12} - \xi_{21}) = -\xi_{-1/2}^2, \\
\xi_{-1/2}^2 & \rightarrow \frac{1}{\sqrt{2}} (\xi_{22} - \xi_{11}) = -\xi_{1/2}^2.
\end{align*}
\]

(55)

These states also have definite spin projection:

\[
\begin{align*}
-\frac{i}{2} \gamma_1 \gamma_2 \xi_{\pm 1/2}^1 & = \pm \frac{1}{2} \xi_{\pm 1/2}^1, \\
-\frac{i}{2} \gamma_1 \gamma_2 \xi_{\pm 1/2}^2 & = \pm \frac{1}{2} \xi_{\pm 1/2}^2.
\end{align*}
\]

(56)

(57)

(58)

The states (55) have the property that under the \( \vartheta = \pi \) rotation, the spin 1/2 state \( \xi_{1/2}^1 \) transforms into the spin \(-1/2\) state \( \xi_{-1/2}^1 \), and vice versa. Analogous hold for the other set of states, \( \xi_{1/2}^2, \xi_{-1/2}^2 \).

Let us stress again that the transformation in the above example is of the type \( \Phi' = R \Phi R^{-1} \). This is a reason that, under such a transformation, a spinor of one ideal is transformed into the spinor of a different ideal. A transformation \( R^{-1} \), acting from the right, mixes the ideals. Another kind of transformation is \( \Phi' = R \Phi \), in which case there is no mixing of ideals. Such are the usual transformations of spinors. By considering the objects of the entire Clifford algebra and possible transformations among them, we find out that spinors are not a sort of objects that transform differently than vectors under rotations. They can transform under rotations in the same way as vectors, i.e., according to \( \Phi' = R \Phi R^{-1} \). Here \( \Phi \) can be a vector, spinor or any other object of Clifford algebra. In addition to this kind of transformations, there exist also the other kind of transformations, namely, \( \Phi' = R \Phi \), where again \( \Phi \) can be any object of \( Cl(1,3) \), including a vector or a spinor. These are particular cases of the more general transformations, \( \Phi' = R \Phi S \), considered in Sec. 1.
3.2 Space inversion

Let us now consider space inversion, under which the basis vectors of a reference frame transform according to

$$\gamma_0 \rightarrow \gamma'_0 = \gamma_0, \quad \gamma_r \rightarrow \gamma'_r = -\gamma_r, \quad r = 1, 2, 3.$$  \hspace{1cm} (59)

The vectors of the Witt basis (12) then transform as

$$\begin{align*}
\theta_1 &\rightarrow \frac{1}{2}(\gamma_0 - \gamma_3) = \tilde{\theta}_1, \\
\theta_2 &\rightarrow \frac{1}{2}(\gamma_1 - i\gamma_2) = -\theta_2, \\
\tilde{\theta}_1 &\rightarrow \frac{1}{2}(\gamma_0 + \gamma_3) = \theta_1, \\
\tilde{\theta}_2 &\rightarrow \frac{1}{2}(\gamma_1 + i\gamma_2) = -\theta_2.
\end{align*}$$  \hspace{1cm} (60)

A spinor of the first left ideal transforms as\cite{3}

$$(\psi_{11}^1 + \psi^{21} \theta_1 \theta_2 + \psi^{31} \theta_1 + \psi^{41} \theta_2) \tilde{\theta}_1 \tilde{\theta}_2 \rightarrow (-\psi_{11}^1 + \psi^{21} \tilde{\theta}_1 \theta_2 - \psi^{31} \tilde{\theta}_1 + \psi^{41} \theta_2) \theta_1 \theta_2.$$  \hspace{1cm} (61)

The latter equation shows that a left handed spinor of the first ideal transforms into a right handed spinor of the third ideal.

In general, under space inversion, the matrix of the spinor basis elements

$$\xi_{ai} = \begin{pmatrix}
\begin{array}{cccc}
\theta_1 f_1 & \theta_2 f_1 & \tilde{\theta}_1 f_3 & \theta_1 f_4 \\
\theta_1 \theta_2 f_1 & \bar{\theta}_1 \theta_2 f_2 & \bar{\theta}_3 f_3 & \bar{\theta}_2 f_4 \\
\theta_1 f_1 & \tilde{\theta}_1 f_2 & f_3 & \bar{\theta}_2 f_4 \\
\theta_2 f_1 & \tilde{\theta}_2 f_2 & \bar{\theta}_1 \theta_2 f_3 & \theta_1 \theta_2 f_4
\end{array}
\end{pmatrix},$$  \hspace{1cm} (62)

transforms into

$$\xi'_{ai} = \begin{pmatrix}
\begin{array}{cccc}
-f_3 & -f_4 & -\theta_1 f_1 & -\tilde{\theta}_1 f_2 \\
-\theta_1 \theta_2 f_3 & -\theta_1 \theta_2 f_4 & -\theta_2 f_1 & -\tilde{\theta}_2 f_2 \\
-\theta_1 f_3 & -\theta_1 f_4 & -f_1 & -f_2 \\
-\theta_2 f_3 & -\theta_1 f_2 & -\theta_1 f_1 & \tilde{\theta}_1 \theta_2 f_2
\end{array}
\end{pmatrix}. $$  \hspace{1cm} (63)

The matrix of components

$$\psi_{ai} = \begin{pmatrix}
\begin{array}{cccc}
\psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\
\psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} \\
\psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \\
\psi_{41} & \psi_{42} & \psi_{43} & \psi_{44}
\end{array}
\end{pmatrix}$$

transform into

$$\psi'^{ai} = \begin{pmatrix}
\begin{array}{cccc}
-\psi_{33} & -\psi_{34} & -\psi_{31} & -\psi_{32} \\
-\psi_{43} & \psi_{44} & \psi_{41} & \psi_{42} \\
-\psi_{13} & -\psi_{14} & -\psi_{11} & -\psi_{12} \\
\psi_{23} & \psi_{24} & \psi_{21} & \psi_{22}
\end{array}
\end{pmatrix}. $$  \hspace{1cm} (64)

By comparing (62) and (63), or by inspecting (64), we find that the spinor of the 1st ideal transforms into the spinor of the 3rd ideal, and the spinor of the 2nd ideal transforms into the spinor of the 4th ideal.

4 Generalized Dirac equation (Dirac-Kähler equation)

Let us now consider the Clifford algebra valued fields, $\Phi(x)$, that depend on position $x \equiv x^\mu$ in spacetime. We will assume that a field $\Phi$ satisfies the following equation\cite{3} (see also refs. [4, 6]):

$$(i \gamma^\mu \partial_\mu - m)\Phi = 0, \quad \Phi = \phi^A \gamma_A = \psi^{\tilde{A}} \xi_{\tilde{A}} = \psi^{ai} \xi_{ai}.$$  \hspace{1cm} (65)
where $\gamma_A$ is a multivector basis of $Cl(1,3)$, and $\xi_A \equiv \xi_{\alpha i}$ is a spinor basis of $Cl(1,3)$, or more precisely, of its complexified version if $\psi^{\alpha i}$ are complex-valued. Here $\alpha$ is the spinor index of a left minimal ideal, whereas the $i$ runs over four left ideals of $Cl(1,3)$.

Multiplying Eq. (65) from the left by $(\xi_A)^\dagger$, where $\dagger$ is the operation of reversion that reverses the order of vectors in a product, and using the relation

\[
\langle (\xi_A)^\dagger \gamma^\mu \xi_B \rangle_S \equiv (\gamma^\mu)^A_B ,
\]

and where $\langle \rangle_S$ is the (properly normalized [9]) scalar part of an expression, we obtain the following matrix form of the equation (65):

\[
\left( i (\gamma^\mu)^A_B \partial_\mu - m \delta^A_B \right) \psi^B = 0 .
\] (67)

The $16 \times 16$ matrices can be factorized according to

\[
(\gamma^\mu)^A_B = (\gamma^\mu)^\alpha_\beta \delta^{ij} ,
\] (68)

where $(\gamma^\mu)^\alpha_\beta$ are $4 \times 4$ Dirac matrices. Using the latter relation (68), we can write Eq. (67) as

\[
\left( i (\gamma^\mu)^\alpha_\beta \partial_\mu - m \delta^\alpha_\beta \right) \psi^{\beta i} = 0 ,
\] (69)

or more simply,

\[
(i \gamma^\mu \partial_\mu - m) \psi^i = 0 .
\] (70)

In the last equation we have omitted the spinor index $\alpha$.

The action that leads to the generalized Dirac equation (65) is

\[
I = \int d^4x \bar{\psi}^i (i \gamma^\mu D_\mu - m) \psi^j z_{ij} .
\] (71)

This is an action that describes four spinors $\psi^i$, belonging to the four minimal left ideals of $Cl(1,3)$. Here $z_{ij}$ is the metric in the space of ideals. It is a part of the metric

\[
(\xi_A)^\dagger \xi_B = z_{AB} = z_{(\alpha i)(\beta j)} = z_{\alpha \beta} z_{ij}
\] (72)

of the Clifford algebra $Cl(1,3)$, represented in the basis $\xi_A$:

\[
z_{ij} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad z_{\alpha \beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} .
\] (73)

Gauge covariant action is

\[
I = \int d^4x \bar{\psi}^i (i \gamma^\mu D_\mu - m) \psi^j z_{ij} , \quad D_\mu \psi^i = \partial_\mu \psi^i + G^i_\mu \psi^j .
\] (74)

This action contains the ordinary particles and mirror particles. The first and the second columns of the matrix $\psi^{\alpha i}$, written explicitly in eq. (30), describe the ordinary particles, whereas the third and the forth column in (30) describe mirror particles.
The SU(2) gauge group acting within the 1st and 2nd ideal can be interpreted as the weak interaction gauge group for ordinary particles. The SU(2) gauge group acting within the 3rd and 4th ideal can be interpreted as the weak interaction gauge group for mirror particles. The corresponding two kinds of weak interaction gauge fields that can be transformed into each other by space inversion are contained in $G_{\mu}^i j$, which is a generalized gauge field occurring in the covariant action [74].

Mirror particles were first proposed by Lee and Yang [10]. Subsequently, the idea of mirror particles has been pursued by Kobzarev et al. [11], and in Refs. [12]–[17]. The possibility that mirror particles are responsible for dark matter has been explored in many works, e.g., in [18]–[25]. A demonstration that mirror particles can be explained in terms of algebraic spinors (elements of Clifford algebras) was presented in Ref. [4].

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