GLOBAL CURVATURE ESTIMATE OF THE $k$-HESSIAN EQUATION FOR $k \geq \frac{n}{2}$

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Abstract. In this paper, we establish the curvature estimate of $k$-convex hypersurface satisfying the Weingarten curvature equation $\sigma_k(\kappa(X)) = f(X, \nu(X))$ for $k \geq \frac{n}{2}$, and discuss some applications.

1. Introduction

Let $M \subset \mathbb{R}^{n+1}$ be a closed hypersurface. In this paper we consider the following curvature equation in a general form:

$$\sigma_k(\kappa(X)) = f(X, \nu(X)) \text{ for } X \in M,$$

where $\sigma_k$ is the $k$-th elementary symmetric function, $\kappa(X)$ and $\nu(X)$ are principal curvatures and outer normal vector at $X \in M$. $\sigma_k(\kappa)$ ($k = 1, 2, \cdots, n$) are the Weingarten curvatures of $M$. In particular, $\sigma_1(\kappa)$, $\sigma_2(\kappa)$ and $\sigma_1(\kappa)$ are the mean curvature, scalar curvature and Gauss curvature, respectively.

The curvature equation (1.1) plays a significant role in geometry. Many important geometric problem can be transformed into (1.1) with a special form of $f$, including the Minkowski problem ([28, 29, 30, 11]), the problem of prescribing general Weingarten curvature on outer normals by Alexandrov ([1, 19]), the problem of prescribing curvature measures in convex geometry ([2, 29, 21, 20]) and the prescribing curvature problem considered in [3, 36, 9].

The curvature equation (1.1) has been studied extensively. When $k = 1$, equation (1.1) is quasi-linear, so the curvature estimate follows from the classical theory of quasi-linear PDEs. When $k = n$, equation (1.1) is of Monge-Ampère type. The desired estimate was established by Caffarelli-Nirenberg-Spruck [7].

When $1 < k < n$, if $f$ is independent of $\nu$, Caffarelli-Nirenberg-Spruck [9] established the curvature estimate; if $f$ depends only on $\nu$, the curvature estimate was proved by Guan-Guan [19]. In [23, 24], Ivochkina studied the Dirichlet problem of equation (1.1) on domains in $\mathbb{R}^n$ and obtained the curvature estimate under some additional assumptions on the dependence of $f$ on $\nu$. For the prescribing curvature measures problem, Guan-Lin-Ma [21] and Guan-Li-Li [20] proved the curvature estimate for $f(X, \nu) = \langle X, \nu \rangle \tilde{f}(X, \nu)$. 

1
For general right-hand side \( f(X, \nu) \), establishing the curvature estimate for equation (1.1) is very important and interesting problem in both geometry and PDEs. In [22], Guan-Ren-Wang solved the case \( k = 2 \) (in [35], Spruck-Xiao gave a simplified proof), and obtained the curvature estimate for convex hypersurface when \( k \geq 3 \). In [31, 32], Ren-Wang removed the convexity assumption when \( k = n - 1 \) and \( k = n - 2 \), so they solved this problem in these two cases. The other cases \( 2 < k < n - 2 \) are still open. In this paper, our main result solves the cases \( k \geq \frac{n}{2} \).

**Theorem 1.1.** Let \( M \) be a closed \( k \)-convex hypersurface satisfying curvature equation (1.1) for some positive function \( f \in C^2(\Gamma) \), where \( \Gamma \) is an open neighborhood of the unit normal bundle of \( M \) in \( \mathbb{R}^{n+1} \times S^n \). If \( k \geq \frac{n}{2} \), then there exists a constant \( C \) depending only \( n, k, \|M\|_{C^1}, \inf f \) and \( \|f\|_{C^2} \) such that

\[
\max_{X \in M, i=1, \ldots, n} |\kappa_i(X)| \leq C.
\]

Here a \( C^2 \) regular hypersurface \( M \subset \mathbb{R}^{n+1} \) is called \( k \)-convex if \( \kappa(X) \in \Gamma_k \) for all \( X \in M \), where \( \Gamma_k \) is the Gårding cone defined by

\[
\Gamma_k = \{ \kappa \mid \sigma_m(\kappa) > 0, \ m = 1, 2, \ldots, k \}
\]

and

\[
\sigma_m(\kappa) = \sum_{i_1 < \cdots < i_m} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_m}.
\]

Similarly, for a domain \( \Omega \subset \mathbb{R}^n \), a function \( u \in C^2(\Omega) \) is said to be \( k \)-convex if the eigenvalues of \( D^2u(x) \) is in \( \Gamma_k \) for all \( x \in \Omega \).

**Remark 1.2.** By the similar argument, the curvature estimate in Theorem 1.1 can be extended to hypersurfaces in space forms, or even warped product manifolds. We refer the reader to [35, 10].

**Remark 1.3.** If \( M \) is \((k+1)\)-convex, then the assumption \( k \geq \frac{n}{2} \) can be removed (see Remark 3.4). Thus we prove the curvature estimate of convex hypersurface satisfying equation (1.1) for any \( k \), which gives an alternative proof of the main result in [22] proved by Guan-Ren-Wang.

Next we discuss some applications of Theorem 1.1. As in [22, 31, 32], the first application is the existence of solution to the curvature equation (1.1). To establish \( C^0 \) and \( C^1 \) estimates, we need to impose some technical conditions on the prescribed function \( f \) as in [31, 36, 9]. For convenience, we write \( \rho(X) = |X| \).

**Condition (1):** There are two positive constants \( r_1 < 1 < r_2 \) such that

\[
\begin{cases}
  f\left( X, \frac{X}{|X|} \right) \geq \frac{\sigma_k(1, \ldots, 1)}{r_1^*} & \text{for } |X| = r_1, \\
  f\left( X, \frac{X}{|X|} \right) \leq \frac{\sigma_k(1, \ldots, 1)}{r_2^*} & \text{for } |X| = r_2.
\end{cases}
\]
Condition (2): For any fixed unit vector $\nu$,
\[
\frac{\partial}{\partial \rho}(\rho^k f(X, \nu)) \leq 0, \quad \text{where } |X| = \rho.
\]

**Theorem 1.4.** If $k \geq \frac{n}{2}$ and positive function $f \in C^2((B_{r_2} \setminus B_{r_1}) \times S^n)$ satisfies Conditions (1) and (2), then the equation (1.1) has a unique $C^{3,\alpha}$ star-shaped solution $M$ in $\{r_1 \leq |X| \leq r_2\}$ for any $\alpha \in (0, 1)$.

Here a hypersurface is said to be star-shaped if it can be viewed as a radial graph of $S^n$ for some positive radial function.

The second application is to solve the prescribed $k$-curvature problem of the $n$-dimensional spacelike graphic hypersurface in the Minkowski space $\mathbb{R}^{n,1}$. Here “spacelike” means the tangent space of hypersurface lies outside the light cone. A function is called spacelike if its graph defined by $(x, u(x))$ in $\mathbb{R}^{n,1}$ is spacelike.

**Theorem 1.5.** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let $\phi \in C^4(\overline{\Omega})$ be a spacelike function and $f \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be a positive function with $f_u \geq 0$. We consider the following Dirichlet problem
\[
\begin{aligned}
\sigma_k(\kappa(x, u(x))) &= f(x, u, Du) \quad \text{in } \Omega, \\
u = \phi &\quad \text{on } \partial \Omega,
\end{aligned}
\]
where $\kappa(x, u(x))$ denotes the principal curvatures of the spacelike graphic hypersurface defined by $(x, u(x))$ in $\mathbb{R}^{n,1}$. If $k \geq \frac{n}{2}$ and this problem admits a subsolution, then there exists a unique spacelike solution $u$ in $\Gamma_k$ belonging to $C^{3,\alpha}(\overline{\Omega})$ for any $\alpha \in (0, 1)$.

The prescribing curvature problem of the spacelike graphic hypersurface over bounded domain in the Minkowski space $\mathbb{R}^{n,1}$ has received considerable attention. Bartnik-Simon [4] proposed and solved the prescribed mean curvature problem ($k = 1$). Delanoe [13] solved the prescribed Gauss curvature problem ($k = n$). In [5] [6], Bayard proposed the prescribing $k$-curvature problems for $2 \leq k < n$ and studied them under some additional assumptions. The prescribed scalar curvature problem ($k = 2$) was solved by Urbas [38]. In [31] [32], Ren-Wang solved the cases $k = n - 1$ and $k = n - 2$. Hypersurfaces with prescribed curvatures in Lorentzian manifolds has also been studied by Gerhardt [15] [16] and Schnürer [33].

The third application is the $C^2$ estimate for the Dirichlet problem of the $k$-Hessian equation.

**Theorem 1.6.** Let $u$ be a $k$-convex solution of the following Dirichlet problem
\[
\begin{aligned}
\sigma_k(D^2u) &= f(x, u, Du) \quad \text{in } \Omega, \\
u = \phi &\quad \text{on } \partial \Omega,
\end{aligned}
\]
where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $f$ is a positive function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and $\phi$ is a function defined on $\overline{\Omega}$. If $k \geq \frac{n}{2}$, then there exists a
constant $C$ depending only on $n$, $k$, $\|u\|_{C^1(\Omega)}$, $f$ and $\Omega$ such that

$$\|u\|_{C^2(\Omega)} \leq C + C \max_{\partial \Omega} |D^2 u|.$$ 

When $f$ is independent of $Du$, the $C^2$ estimate was studied by Caffarelli-Nirenberg-Spruck [8], Li [27], Guan [17] and Trudinger [37] etc. When $k = n$ and $f$ depends on $Du$, the $C^2$ estimate was established by Caffarelli-Nirenberg-Spruck [7]. For general $k$, Guan [18] proved the $C^2$ estimate when $f(x, \xi, p)$ defined on $(x, \xi, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ is convex in $p$.

The fourth application is the interior $C^2$ estimate for the Dirichlet problem of the $k$-Hessian equation.

**Theorem 1.7.** Let $u$ be a $k$-convex solution of the following Dirichlet problem:

$$\begin{aligned}
\sigma_k (D^2 u) &= f(x, u, Du) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}$$

If $k \geq \frac{n}{2}$, then we have

$$\sup_{\Omega} \left[ (-u)^{\beta} \Delta u \right] \leq C,$$

where $\beta$ and $C$ are constants depending only on $n$, $k$, $\|u\|_{C^1(\Omega)}$, $f$ and $\Omega$.

When $f$ is independent of $Du$, the interior $C^2$ estimate was first initiated by Pogorelov [30] for Monge-Ampère equation ($k = n$). In [12], Chou-Wang generalized Pogorelov’s estimate to the $k$-Hessian equation. When $f$ depends on $Du$, Li-Ren-Wang [26] established the interior $C^2$ estimate for $2$-convex solution to the $2$-Hessian equation and $(k + 1)$-convex solution to the $k$-Hessian equation when $k \geq 3$.

We now discuss the proof of Theorem 1.1. Compared to the previous works in [22, 31, 32], we take a different approach. To establish the curvature estimate, the main difficulty is how to deal with the third order terms. First, we apply the maximum principle to a quantity involving the largest principal curvature $\kappa_1$, instead of the symmetric function of $\kappa$. This gives us some “good” third order terms. Second, we make use of concavity of the operator log $\sigma_k$, instead of $\sigma_k^\frac{1}{k}$ and $\left( \frac{\sigma_k}{\sigma_l} \right)^{\frac{1}{k-l}}$ for $l < k$. Then more “good” third order terms appear when we differentiate the equation (1.1) twice. After some delicate calculation and estimates, the “good” third order terms are just barely enough to control the “bad” third order terms, and we obtain the desired estimate.

In this paper, we omit the proofs of Theorem 1.4, 1.5 and 1.6. In fact, we just need to consider the test function

$$Q = \log \kappa_1 + \text{lower order terms}$$

or

$$Q = \log \lambda_1(D^2 u) + \text{lower order terms},$$
where $\lambda_1(D^2u)$ denotes the largest eigenvalue of $D^2u$. Applying the similar argument of Theorem 1.1 to deal with the third order terms, we obtain the curvature estimate or $C^2$ estimate. The remaining proofs are almost identical to that of [31, Corollary 3, Theorem 4, Theorem 5].

2. Preliminary

In [25], Korevaar proved that the Gårding cone $\Gamma_k$ can be characterized as follows

$$\Gamma_k = \left\{ \kappa \mid \sigma_k(\kappa) > 0, \frac{\partial \sigma_k(\kappa)}{\partial \kappa_{i_1}} > 0, \cdots, \frac{\partial^k \sigma_k(\kappa)}{\partial \kappa_{i_1} \cdots \partial \kappa_{i_k}} > 0, \right. $$

for all $1 \leq i_1 < \cdots < i_k \leq n \left\} \right.$.

Some properties of the Gårding cone $\Gamma_k$ are listed below.

Lemma 2.1. If $\kappa \in \Gamma_k$, then we have

1. $(\kappa|i) \in \Gamma_{k-1}$ for any $1 \leq i \leq n$, where $(\kappa|i) = (\kappa_1, \cdots, \kappa_{i-1}, \kappa_{i-1} \cdots \kappa_n)$.
2. $\kappa_1 \sigma_{k-1}(\kappa|1) \geq \theta \sigma_k(\kappa)$, where $\theta$ is a positive constant depending only on $k$ and $n$.
3. $-\kappa_i < \frac{n-k}{k} \kappa_1$ for $1 \leq i \leq n$.
4. $\kappa_1 \cdots \kappa_s \leq \sigma_s(\kappa)$ for any $s < k$.

Proof. (1) is a corollary of (2.1). For (2), (3) and (4), We refer the reader to [12, p.1037] and [32, Lemma 10, Lemma 11].

For convenience, we use the following notations:

$$F = \log \sigma_k(\kappa), \quad F^{ij} = \frac{\partial F}{\partial h_{ij}}, \quad F^{ijkl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}.$$ 

For any point $X_0 \in M$, let $\{e_i\}_{i=1}^n$ be a local orthonormal frame near $X_0$ such that $h_{ij} = \delta_{ij} \kappa_i$, $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$.

Then at $X_0$, we have (see e.g. [14, 34])

$$F^{ij} = \frac{\sigma_{k-1}(\kappa|i)}{\sigma_k(\kappa)}$$

and

$$F^{ij,pq} = \begin{cases} F^{ii,pp} & \text{if } i = j, p = q; \\ F^{ip,pi} & \text{if } i = q, p = j, i \neq p; \\ 0 & \text{otherwise}, \end{cases}$$

where

$$F^{ii,pp} = \frac{(1 - \delta_{ip}) \sigma_{k-2}(\kappa|ip)}{\sigma_k(\kappa)} - \frac{\sigma_{k-1}(\kappa|i) \sigma_{k-1}(\kappa|p)}{[\sigma_k(\kappa)]^2}$$

and

$$F^{ip,pi} = -\frac{\sigma_{k-2}(\kappa|ip)}{\sigma_k(\kappa)}.$$
where $\sigma_s(\kappa_1 \cdot \cdot \cdot \kappa_r)$ denotes $s$-th elementary symmetric function with $\kappa_{i_1} = \cdot \cdot \cdot = \kappa_{i_r} = 0$.

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. First we define a function $u$ by

$$u = \langle X, \nu(X) \rangle - \inf_{X \in M} \langle X, \nu(X) \rangle + 1.$$ 

By assumptions, there exists a uniform constant $C$ such that $1 \leq u \leq C$ for $X \in M$.

Let $\kappa_1$ be the largest principal curvature. From $\kappa \in \Gamma_k \subset \Gamma_1$, we see that the mean curvature is positive. To prove Theorem 1.1, it suffices to prove $\kappa_1$ is uniformly bounded from above. Without loss of generality, we may assume that the set $\Omega = \{ \kappa_1 > 0 \}$ is not empty. On $\Omega$, we consider the following function $Q = \log \kappa_1 - Au$, where $A > 1$ is a constant to be determined later. Note that $Q$ is continuous on $\Omega$, and goes to $-\infty$ on $\partial \Omega$. Hence $Q$ achieves a maximum at a point $X_0$ with $\kappa_1(X_0) > 0$. However, the function $Q$ may be not smooth at $X_0$ when the eigenspace of $\kappa_1$ has dimension strictly larger than 1, i.e., $\kappa_1(X_0) = \kappa_2(X_0)$. To deal with this case, we apply the standard perturbation argument. Let $g$ be the first fundamental form of $M$ and $D$ be the corresponding Levi-Civita connection. We choose a local orthonormal frame $\{ e_i \}_{i=1}^n$ near $X_0$ such that

$$D e_i e_j = 0, \quad h_{ij} = \delta_{ij} \kappa_i, \quad \kappa_1 \geq \kappa_2 \geq \cdot \cdot \cdot \geq \kappa_n \quad \text{at } X_0.$$

Now we apply a perturbation argument. Near $X_0$, we define a new tensor $B$ by

$$B(V_1, V_2) = g(V_1, V_2) - g(V_1, e_1)g(V_2, e_1),$$

for tangent vectors $V_1$ and $V_2$. Let $B_{ij} = B(e_i, e_j)$. It is clear that

$$B_{ij} = \delta_{ij} B_{ii}, \quad B_{11} = 0, \quad B_{ii} = 1 \quad \text{for } i > 1,$$

We define the matrix by $\tilde{h}_{ij} = h_{ij} - B_{ij}$, and denote its eigenvalues by $\tilde{\kappa}_1 \geq \tilde{\kappa}_2 \geq \cdot \cdot \cdot \geq \tilde{\kappa}_n$. It then follows that $\kappa_1 \geq \tilde{\kappa}_1$ near $X_0$ and

$$\tilde{\kappa}_i = \begin{cases} \kappa_i & \text{if } i = 1, \\ \kappa_i - 1 & \text{if } i > 1, \end{cases} \quad \text{at } X_0.$$

Thus $\tilde{\kappa}_1 > \tilde{\kappa}_2$ at $X_0$. We can now consider the perturbed quantity $\hat{Q}$ defined by

$$\hat{Q} = \log \tilde{\kappa}_1 + Au,$$

which still obtains a local maximum at $X_0$. From now on, all the calculations will be carried out at $X_0$. For any $1 \leq i \leq n$, since $\tilde{\kappa}_1 = \kappa_1$, we have

$$0 = \hat{Q}_i = \frac{\tilde{\kappa}_1}{\kappa_1} - A u_i = \frac{\tilde{\kappa}_1}{\kappa_1} - A u_i$$

(3.1)
and
\[
0 \geq F^{ii} \tilde{Q}_{ii} = F^{ii}(\log \tilde{\kappa}_1)_{ii} - AF^{ii} u_{ii}.
\]

Before we prove Theorem 1.1, we list some well-known formulas which we will need later.

**Guass formula:**
\[X_{ij} = -h_{ij} \nu,\]

**Weingarten equation:**
\[\nu_i = h_{ij} e_j,\]

**Codazzi formula:**
\[h_{ijk} = h_{ikj},\]

**Guass equation:**
\[R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk},\]

where \(R_{ijkl}\) is the curvature tensor of \(M\). We also have
\[h_{ijkl} = h_{ijlk} + h_{mk} R_{imlk} - h_{mi} R_{jmlk} = h_{klij} + (h_{mj} h_{il} - h_{ml} h_{ij}) h_{mk} + (h_{mj} h_{kl} - h_{ml} h_{kj}) h_{mi}.\]

In the following lemma, we estimate each term in (3.2) and obtain a lower bound of \(F^{ii} \tilde{Q}_{ii}\).

**Lemma 3.1.** We have
\[0 \geq F^{ii} \tilde{Q}_{ii} \geq 2 \sum_{i>1} F_{11} h_{1i}^2 - 2 \sum_{i>1} F_{i1} h_{1i}^2 \frac{h_{11i}}{\kappa_1} - \frac{F_{ii} h_{11i}}{\kappa_1^2} + (A - C) F^{ii} h_{ii}^2 - CA.
\]

**Proof.** First, let us recall the first and second derivatives of \(\tilde{\kappa}_1\) at \(X_0\) (see e.g. [34]):
\[\tilde{\kappa}_1^{pq} := \frac{\partial \tilde{\kappa}_1}{\partial h_{pq}} = \delta_{1p} \delta_{1q},\]
\[\tilde{\kappa}_1^{pq,kl} := \frac{\partial^2 \tilde{\kappa}_1}{\partial h_{pq} \partial h_{kl}} = (1 - \delta_{1p}) \frac{\delta_{1q} \delta_{1k} \delta_{1l}}{\tilde{\kappa}_1 - \tilde{\kappa}_p} + (1 - \delta_{1k}) \frac{\delta_{1l} \delta_{1p} \delta_{1q}}{\tilde{\kappa}_1 - \tilde{\kappa}_k}.
\]

We compute
\[\tilde{\kappa}_{1,i} = \tilde{\kappa}_1^{pq} \tilde{h}_{pq} = \tilde{h}_{11i},\]
\[\tilde{\kappa}_{1,ii} = \tilde{\kappa}_1^{pq} \tilde{h}_{pqii} + \tilde{\kappa}_1^{pq,kl} \tilde{h}_{kl1} \tilde{h}_{pq} = \tilde{h}_{11ii} + 2 \sum_{p>1} \frac{\tilde{h}_{1p}^2}{\kappa_1 - \kappa_p},\]
where we used \(\tilde{\kappa}_1 = \kappa_1\). Using the definition of tensor \(B\) and \((D e_i e_j)(X_0) = 0\), we see that
\[B_{ij,k} = 0, \quad B_{11,ii} = 0 \quad at \ X_0,\]
which implies
\[\tilde{h}_{ijk} = h_{ijk}, \quad \tilde{h}_{11ii} = h_{11ii} \quad at \ X_0.\]
It then follows that

\begin{equation}
\tilde{\kappa}_{1,i} = h_{11i}, \quad \tilde{\kappa}_{1,ii} = h_{11ii} + 2 \sum_{p > 1} \frac{h_{1p}^2}{\kappa_1 - \tilde{\kappa}_p}.
\end{equation}

For the term \( F^{ii}(\log \tilde{\kappa}_1)_{ii} \) in (3.2), using Codazzi formula, we compute

\begin{equation}
F^{ii}(\log \tilde{\kappa}_1)_{ii} = \frac{F^{ii} \tilde{\kappa}_{1,ii}}{h_{11}^2} - \frac{F^{ii} \tilde{\kappa}_{1,ii}^2}{h_{11}^2} \geq F^{ii} h_{11ii} + 2 \sum_{p > 1} \frac{F^{ii} h_{1pi}^2}{\kappa_1 - \tilde{\kappa}_p} - \frac{F^{ii} h_{11ii}^2}{\kappa_1^2}.
\end{equation}

By (3.3), we have

\[ F^{ii} h_{11ii} = \sum_i F^{ii} h_{ii11} + \sum_i F^{ii} h_{im} R_{imi1} + \sum_i F^{ii} h_{mi} R_{imi1} \]
\[ = F^{ii} h_{ii11} + F^{ii} (h_{ii}^2 - h_{ii} h_{11}) h_{ii} + F^{ii} (h_{ii} h_{11} - h_{ii}^2) h_{11} \]
\[ = F^{ii} h_{ii11} - F^{ii} h_{ii}^2 h_{11} + F^{ii} h_{ii} h_{11}^2 \]
\[ = F^{ii} h_{ii11} - F^{ii} h_{ii}^2 h_{11} + kh_{11}^2. \]

where we used

\begin{equation}
\sum_i F^{ii} h_{ii} = \sum_i \frac{\kappa_i \sigma_{k-1}(\kappa|i)}{\sigma_k(\kappa)} = k.
\end{equation}

On the other hand, the curvature equation (1.1) can be written as

\begin{equation}
F = \log \sigma_k(\kappa) = \tilde{f},
\end{equation}

where \( \tilde{f} = \log f \). Differentiating (3.7) twice, we obtain

\[ F^{ii} h_{ii11} \geq -F^{ij,k} h_{ij} h_{kl1} + \sum_k h_{k11}(d_v \tilde{f})(e_k) - Ch_{11}^2 - C. \]

Thanks to the concavity of \( F \) and Codazzi formula, we have

\[ -F^{ij,k} h_{ij} h_{kl1} \geq -2 \sum_{i > 1} F^{ii,i1} h_{11i}^2 - F^{11,11} h_{111}^2 \]
\[ = -2 \sum_{i > 1} F^{ii,i1} h_{11i}^2 - F^{11,11} h_{11i}^2. \]

Hence,

\[ F^{ii} h_{11ii} \geq -2 \sum_{i > 1} F^{ii,i1} h_{11i}^2 - F^{11,11} h_{11i}^2 \]
\[ + \sum_k h_{k11}(d_v \tilde{f})(e_k) - F^{ii} h_{ii}^2 h_{11} - Ch_{11}^2 - C, \]
Substituting this into (3.5),

\[
F^{ii}(\log \tilde{\kappa})_{ii} \geq 2 \sum_{i>1} \frac{F^{11i}h_{11i}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_i)} - 2 \sum_{i>1} \frac{F^{1i}h_{11i}^2}{\kappa_1} - \frac{F^{1i}h_{11i}^2}{\kappa_1} + 2 \sum_{i>1} \frac{F^{ii}h_{11i}^2}{\kappa_1} + \frac{1}{\kappa_1} \sum_k h_{k11}(d_{\nu} \hat{f})(e_k)
\]

- \frac{F^{ii}h_{ii}^2}{\kappa_1^2} - Ch_{11} - C.

(3.8)

By Guass formula, Weingarten equation and Codazzi formula, we see that

\[
u_{ii} = \sum_k h_{iik} \langle e_k, X \rangle - u_{ii} h_{ii}^2 + h_{ii}.
\]

For the term \(-AF^{ii}u_{ii}\) in (3.2), we compute

\[
-AF^{ii}u_{ii} = -A \sum_k F^{ii}h_{iik} \langle e_k, X \rangle + AuF^{ii}h_{ii}^2 - AF^{ii}h_{ii}^2 - A \sum_k F^{ii}h_{ii}^2 \langle e_k, X \rangle - Ak,
\]

(3.9)

where we used \(u \geq 1\) and (3.6). Differentiating (3.7), we obtain

\[
F^{ii}h_{iik} = (d_{X} \hat{f})(e_k) + h_{kk}(d_{\nu} \hat{f})(e_k).
\]

Substituting this into (3.9), we have

(3.10)

\[-AF^{ii}u_{ii} \geq -A \sum_k h_{kk}(d_{\nu} \hat{f})(e_k) \langle e_k, X \rangle + AF^{ii}h_{ii}^2 - CA.
\]

Combining (3.2), (3.8) and (3.10), we obtain

\[
0 \geq F^{ii}Q_{ii} \geq 2 \sum_{i>1} \frac{F^{11i}h_{11i}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_i)} - 2 \sum_{i>1} \frac{F^{1i}h_{11i}^2}{\kappa_1} - \frac{F^{1i}h_{11i}^2}{\kappa_1} + 2 \sum_{i>1} \frac{F^{ii}h_{11i}^2}{\kappa_1} + \frac{1}{\kappa_1} \sum_k h_{k11}(d_{\nu} \hat{f})(e_k) - A \sum_k h_{kk}(d_{\nu} \hat{f})(e_k) \langle e_k, X \rangle + (A - 1)F^{ii}h_{ii}^2 - Ch_{11} - CA.
\]

Using (3.1), (3.4) and \(u_k = h_{kk} \langle e_k, X \rangle\), for any \(k\), we have

\[
\frac{h_{11k}}{\kappa_1} - Ah_{kk} \langle e_k, X \rangle = 0.
\]

Combining this with Codazzi formula, it is clear that

\[
\frac{1}{\kappa_1} \sum_k h_{k11}(d_{\nu} \hat{f})(e_k) - A \sum_k h_{kk}(d_{\nu} \hat{f})(e_k) \langle e_k, X \rangle = 0.
\]

By Lemma 2.1 (2), we have

\[
Ch_{11} \leq CF^{ii}h_{ii}^2.
\]
\[
0 \geq F^{ii} \hat{Q}_{ii} \\
\geq 2 \sum_{i>1} \frac{F^{11} h_{11i}^2}{\kappa (\kappa_1 - \hat{\kappa}_i)} - 2 \sum_{i>1} \frac{F^{1i,ii} h_{11i}^2}{\kappa_1} - \frac{F^{11,11} h_{11i}^2}{\kappa_1} - \frac{F^{ii} h_{11i}^2}{\kappa_1^2} \\
+ (A - C) F^{ii} h_{ii}^2 - CA.
\]

as required. \(\square\)

**Lemma 3.2.** For any \(\delta \in (0, 1)\), we have

\[
(1 - \delta) \sum_{i \in I} \frac{F^{ii} h_{11i}^2}{\kappa_i^2} \leq 2 \sum_{i \in I} \frac{F^{11} h_{11i}^2}{\kappa_1 (\kappa_1 - \hat{\kappa}_i)} - 2 \sum_{i \in I} \frac{F^{1i,ii} h_{11i}^2}{\kappa_1},
\]

assuming without loss of generality that \(\kappa_1 \geq C \delta^{-1}\) for some uniform constant \(C\).

**Proof.** We define

\[I = \{ i \in \{2, \cdots, n\} \mid \kappa_i = \kappa_1 \}.\]

For \(i \in I\), we have \(F^{ii} = F^{11}\). Recalling that \(\hat{\kappa}_i = \kappa_i - 1\), we see that

\[
2 \sum_{i \in I} \frac{F^{11} h_{11i}^2}{\kappa_1 (\kappa_1 - \hat{\kappa}_i)} \geq 2 \sum_{i \in I} \frac{F^{11} h_{11i}^2}{\kappa_1} \geq \sum_{i \in I} \frac{F^{ii} h_{11i}^2}{\kappa_1^2},
\]

as long as \(\kappa_1 \geq \frac{1}{2}\).

For \(i \notin I\), by Lemma 2.1 (3) and \(k \geq \frac{n}{2}\), we have

\[\kappa_1 - \hat{\kappa}_i \leq 2 \kappa_1 + 1.\]

Thus

\[
2 \sum_{i \notin I} \frac{F^{11} h_{11i}^2}{\kappa_1 (\kappa_1 - \hat{\kappa}_i)} \geq (1 - \delta) \sum_{i \notin I} \frac{F^{11} h_{11i}^2}{\kappa_1^2},
\]

as long as \(\kappa_1 \geq \delta^{-1}\). On the other hand, using \(\kappa_i < \kappa_1\) and Lemma 2.1 (3), we have

\[-F^{1i,i} = \frac{\sigma_{k-2}(\kappa_1^1 i)}{\sigma_k(\kappa)} = \frac{\sigma_{k-1}(\kappa_1^1 i) - \sigma_{k-1}(\kappa_1^1)}{\sigma_k(\kappa)(\kappa_1 - \hat{\kappa}_i)} = \frac{F^{ii} - F^{11}}{\kappa_1 - \hat{\kappa}_i} \geq \frac{F^{ii} - F^{11}}{2 \kappa_1}.\]

It then follows that

\[
-2 \sum_{i \notin I} \frac{F^{1i,ii} h_{11i}^2}{\kappa_1} \geq \sum_{i \notin I} \frac{(F^{ii} - F^{11}) h_{11i}^2}{\kappa_1^2}.
\]

Combining (3.12) and (3.13), we obtain

\[
2 \sum_{i \notin I} \frac{F^{11} h_{11i}^2}{\kappa_1 (\kappa_1 - \hat{\kappa}_i)} - 2 \sum_{i \notin I} \frac{F^{1i,ii} h_{11i}^2}{\kappa_1} \geq (1 - \delta) \sum_{i \notin I} \frac{F^{ii} h_{11i}^2}{\kappa_1^2}.
\]

Then Lemma 3.2 follows from (3.11) and (3.14). \(\square\)
Lemma 3.3. For any $\delta \in (0, 1)$, there exists a uniform constant $C_\delta$ depending on $\delta$ such that if

$$F^{11}_{11} \left(1 - \frac{1}{\kappa_1^2}\right) > -\frac{F^{11,11}}{\kappa_1},$$

then $\kappa_1 \leq C_\delta$.

Proof. Since

$$-F^{11,11} = \frac{[\sigma_{k-1}(\kappa|1)]^2}{[\sigma_k(\kappa)]^2} = (F^{11})^2,$$

then (3.15) gives

$$\kappa_1 \sigma_{k-1}(\kappa|1) < (1 - \delta) \sigma_k(\kappa).$$

Recalling that $\sigma_k(\kappa) = \kappa_1 \sigma_{k-1}(\kappa|1) + \sigma_k(\kappa|1)$, we obtain

$$\kappa_1 \sigma_{k-1}(\kappa|1) < \delta^{-1} (1 - \delta) \sigma_k(\kappa|1).$$

This shows $\sigma_k(\kappa|1) > 0$. Combining this with Lemma 2.1 (1), we obtain $(\kappa|1) \in \Gamma_k$. By Lemma 2.1 (4), we obtain

$$\kappa_2 \cdots \kappa_k \leq \sigma_{k-1}(\kappa|1),$$

which implies

$$\kappa_1 \cdots \kappa_k \leq \kappa_1 \sigma_{k-1}(\kappa|1) \leq \delta^{-1} \sigma_k(\kappa|1).$$

If $\kappa_n \geq 0$, then $\kappa_i \geq 0$ for any $i$, so

$$\kappa_1 \cdots \kappa_k \leq \delta^{-1} \sigma_k(\kappa|1) \leq C \delta^{-1} \kappa_2 \cdots \kappa_{k+1}.$$

It then follows that

$$\kappa_{k+1} \geq C_\delta \kappa_1.$$

On the other hand, by (3.16) and $\sigma_k(\kappa) = f$, we have

$$\kappa_1 \cdots \kappa_k \leq \delta^{-1} \sigma_k(\kappa|1) \leq \delta^{-1} \sigma_k(\kappa) \leq C \delta^{-1}.$$

From (3.17) and (3.18), we obtain $\kappa_1 \leq C_\delta$. Next we assume without loss of generality that $\kappa_n < 0$. Since $(\kappa|1) \in \Gamma_k$, we have $\sigma_{k-1}(\kappa|1n) > 0$. Thus (3.16) implies

$$\kappa_1 \cdots \kappa_k \leq \delta^{-1} \sigma_k(\kappa|1) = \delta^{-1} \sigma_k(\kappa|1n) + \delta^{-1} \kappa_n \sigma_{k-1}(\kappa|1n) < \delta^{-1} \sigma_k(\kappa|1n),$$

so $\sigma_k(\kappa|1n) > 0$. Combining this with $(\kappa|1) \in \Gamma_k$ and Lemma 2.1 (1), we obtain $(\kappa|1n) \in \Gamma_k$. If $\kappa_{n-1} \geq 0$, then $\kappa_i \geq 0$ for any $1 \leq i \leq n-1$, so

$$\kappa_1 \cdots \kappa_k \leq \delta^{-1} \sigma_k(\kappa|1n) \leq C \delta^{-1} \kappa_2 \cdots \kappa_{k+1},$$

which implies $\kappa_{k+1} \geq C_\delta \kappa_1$. Combining this with (3.18), we get $\kappa_1 \leq C_\delta$. We may assume that $\kappa_{n-1} < 0$. Since $(\kappa|1n) \in \Gamma_k$, we have $\sigma_{k-1}(\kappa|1n-1) > 0$, so

$$\kappa_1 \cdots \kappa_k \leq \delta^{-1} \sigma_k(\kappa|1n) \leq \delta^{-1} \sigma_k(\kappa|1n-1).$$

Repeating the above argument, we obtain $\kappa_1 \leq C_\delta$ or

$$\kappa_1 \cdots \kappa_k \leq \delta^{-1} \sigma_k(\kappa|1 k + 2 \cdots n).$$
In the latter case, it is clear that
\[ \kappa_1 \cdots \kappa_k \leq C \delta^{-1} \kappa_2 \cdots \kappa_{k-1}, \]
which implies \( \kappa_{k+1} \geq C \delta \kappa_1 \). Combining this with (3.15), we obtain \( \kappa_1 \leq C \delta \).

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.3, we assume without loss of generality that
\[ \frac{F^{11,11}}{\kappa_1} \geq (1 - \delta) \frac{F^{11}}{\kappa_1^2}. \]
This implies
\[ \frac{F^{11,11} h_{11}^2}{\kappa_1} \geq (1 - \delta) \frac{F^{11} h_{11}^2}{\kappa_1^2}. \]
Combining this with Lemma 3.1 and 3.2, we have
\[ 0 \geq -\delta \frac{F^{ii} h_{11i}^2}{\kappa_1^2} + (A - C) F^{ii} h_{ii}^2 - CA, \]
Using (3.1), (3.4) and \( u_i = h_{ii} \langle e_i, X \rangle \), we see that
\[ -\delta \frac{F^{ii} h_{11i}^2}{\kappa_1^2} = \delta A^2 F^{ii} u_i^2 \leq C \delta A^2 F^{ii} h_{ii}^2. \]
It then follows that
\[ 0 \geq (A - C_0 - C_0 \delta A^2) F^{ii} h_{ii}^2 - C_0 A, \]
for some uniform constant \( C_0 \). Choosing \( A = 2C_0 + 1 \) and \( \delta = \frac{1}{A^2} \), we have
\[ 0 \geq F^{ii} h_{ii}^2 - C \geq F^{11} \kappa_1^2 - C. \]
Thanks to Lemma 2.1 (2), we obtain \( \kappa_1 \leq C \), as required. \( \square \)

Remark 3.4. In the proof of Theorem 1.1, the assumption \( \kappa \geq \frac{n}{2} \) is only used in the proof of Lemma 3.3. If \( M \) is \((k+1)\)-convex, then \( \kappa \in \Gamma_{k+1} \). By [26] Lemma 7, we have \( \kappa_i \geq -C \) for some uniform constant, which implies \( \kappa_1 - \kappa_i \leq 2 \kappa_k + 1 \). Thus Lemma 3.3 can be proved by the same argument.

4. PROOF OF THEOREM 1.5

In this section, we give the proof of Theorem 1.5. Since it is very similar to that of Theorem 1.1, we only give a sketch here.

Proof of Theorem 1.5. For convenience, we denote the eigenvalues of \( D^2 u \) by \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). We consider the following quantity
\[ Q = \log \lambda_1 + \frac{A}{2} |Du|^2 + \beta \log(-u), \]
where \( A, \beta > 1 \) are constants to be determined later. Let \( x_0 \) be the maximum point of \( Q \). Without loss of generality, we assume that \( Q \) is smooth at \( x_0 \),
i.e., $\lambda_1 > \lambda_2$ at $x_0$. Otherwise, we apply the perturbation argument as in
Section 3. Near $x_0$, we choose the coordinate system such that

$$u_{ij} = \delta_{ij} u_{ii}, \quad u_{11} \geq u_{22} \geq \cdots \geq u_{nn} \text{ at } x_0.$$ 

By the similar calculation of Lemma 3.1 at $x_0$, we have

$$0 \geq F_{ii} Q_{ii}$$

(4.1)

$$\geq 2 \sum_{i>1} \frac{F_{ii}^2 u_{11i}^2}{\lambda_1 (\lambda_1 - \lambda_i)} - 2 \sum_{i>1} \frac{F_{ii}^2 u_{11i}^2}{\lambda_1} - \frac{F_{ii}^1 u_{111i}^2}{\lambda_1} - \frac{F_{ii}^1 u_{111i}^2}{\lambda_1^2}
- \beta \frac{F_{ii}^1 u_{1i}^2}{u^2} + C \beta \frac{u_{1i}^2}{u} + (A - C) F_{ii}^2 u_{ii}^2 - CA.$$ 

The proof splits into two cases:

**Case 1:** $\lambda_n < -\frac{1}{2} \lambda_1$.

In this case, applying the same argument of Theorem 1.1, we have

$$0 \geq (1 - \delta) \frac{F_{ii}^1 u_{11i}^2}{\lambda_1^2} \leq 2 \sum_{i>1} \frac{F_{ii}^2 u_{11i}^2}{\lambda_1 (\lambda_1 - \lambda_i)} - 2 \sum_{i>1} \frac{F_{ii}^2 u_{11i}^2}{\lambda_1} - \frac{F_{ii}^1 u_{111i}^2}{\lambda_1} - \frac{F_{ii}^1 u_{111i}^2}{\lambda_1^2},$$

assuming without loss of generality that $\lambda_1 \geq C_\delta$. Combining this with (4.1), we obtain

$$0 \geq -\delta \frac{F_{ii}^1 u_{11i}^2}{\lambda_1^2} - \beta \frac{F_{ii}^1 u_{1i}^2}{u^2} + C \beta \frac{u_{1i}^2}{u} + (A - C) F_{ii}^2 u_{ii}^2 - CA.$$ 

Using $Q_{ii}(x_0) = 0$, it is clear that

$$\frac{u_{11i}}{\lambda_1} + A u_{ii} u_i + \frac{\beta u_i}{u} = 0,$$

which implies

$$\delta \frac{F_{ii}^1 u_{11i}^2}{\lambda_1^2} \leq C \delta A^2 F_{ii}^2 u_{ii}^2 + \frac{C \delta \beta^2}{u^2} \sum_i F_{ii}.$$ 

(4.4)

On the other hand, we have

$$\beta \frac{F_{ii}^1 u_{1i}^2}{u^2} \leq \frac{C \beta}{u^2} \sum_i F_{ii}.$$ 

(4.5)

Substituting (4.4) and (4.5) into (4.2), we obtain

$$0 \geq (A - C_0 - C_0 \delta A^2) F_{ii}^2 u_{ii}^2 - \frac{C_0 (\delta \beta^2 + \beta)}{u^2} \sum_i F_{ii}^2 + \frac{C_0 \beta}{u} - C_0 A,$$

for some uniform constant $C_0$. Choosing $A = 2C_0 + 1$, $\beta = 2A^2$ and $\delta = \frac{1}{2C}$, we obtain

$$0 \geq F_{i11} u_{i11} - \frac{C}{u^2} \sum_i F_{ii} + \frac{C}{u} - C.$$
Recalling that $\lambda_n < -\frac{1}{2}$ and $F_{ii} \leq F_{nn}$ for any $i$, it is clear that

$$0 \geq \frac{\lambda_1^2}{C} \sum_{i} F_{ii} - \frac{C}{u^2} \sum_{i} F_{ii} + \frac{C}{u} - C.$$ 

This shows $(-u)^{\beta} \lambda_1 \leq C$, as required.

**Case 2:** $\lambda_n \geq -\frac{1}{2} \lambda_1$.

In this case, for any $i$, we have

$$\lambda_1 - \lambda_i \leq \frac{3}{2} \lambda_1.$$

Combining this and the argument of Lemma 3.2, we get

$$\frac{4}{3} \sum_{i > 1} \frac{F_{ii}^2 u_{1ii}^2}{\lambda_1^2} \leq 2 \sum_{i > 1} \frac{F_{11}^2 u_{1ii}^2}{\lambda_1 (\lambda_1 - \lambda_i)} - 2 \sum_{i > 1} \frac{F_{11,ii}^2 u_{1ii}^2}{\lambda_1}.$$

Thanks to Lemma 3.3, we may assume that

$$-\frac{F_{11,11}^2 u_{111}^2}{\lambda_1} \geq (1 - \delta) \frac{F_{11}^2 u_{111}^2}{\lambda_1^2}.$$ 

Hence,

$$0 \geq \frac{1}{3} \frac{F_{ii}^2 u_{1ii}^2}{\lambda_1^2} - \frac{\delta F_{11}^2 u_{111}^2}{\lambda_1^2} - \frac{\beta F_{ii}^2 u_{1ii}^2}{u^2} + \frac{C \beta}{u} + (A - C) F_{ii}^2 u_{1ii}^2 - CA.$$ 

Using (4.3), we see that

$$\delta \frac{F_{11}^2 u_{111}^2}{\lambda_1^2} \leq C \delta A^2 F_{11}^2 u_{111}^2 + C \delta \beta^2 \frac{F_{11}^2}{u^2}.$$ 

and

$$\beta \sum_{i > 1} \frac{F_{ii}^2 u_{1ii}^2}{u^2} \leq \frac{2}{\beta} \sum_{i > 1} \frac{F_{ii}^2 u_{11i}^2}{\lambda_1^2} + \frac{2 A^2}{\beta} \sum_{i > 1} F_{ii}^2 u_{ii}^2.$$ 

Substituting (4.7) and (4.8) into (4.6), we have

$$0 \geq \left( \frac{1}{3} - \frac{2}{\beta} \right) \sum_{i > 1} \frac{F_{ii} u_{11i}^2}{\lambda_1^2} - C_0 (\delta \beta^2 + \beta) \frac{F_{11}^2}{u^2}$$

$$+ \frac{C_0 \beta}{u} + \left( A - C_0 - \frac{2 A^2}{\beta} - C_0 \delta A^2 \right) F_{ii}^2 u_{ii}^2 - C_0 A.$$ 

Choosing $A = 2C_0 + 2$, $\beta = 2A^2$ and $\delta = \frac{1}{4A}$, we obtain

$$0 \geq F_{11}^2 u_{111}^2 - \frac{C F_{11}^2}{u^2} + \frac{C}{u} - C,$$

which implies $(-u)^{\beta} \lambda_1 \leq C$, as required. $\square$
GLOBAL CURVATURE ESTIMATE

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