Parameter-Covariance Maximum Likelihood Estimation

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1 Introduction

Complicated time series often contain different regimes of dynamical evolution that are largely neglected in the practice of system identification. Employing a regime labelling/segmentation can greatly simplify data analysis, but this introduces a heavy dependence of analysis quality on the regime labelling process. This segmentation process is embarrassingly often done by eye, leading to label disagreements that are hard to resolve, especially when the regime-labelling criteria are imprecisely specified. Many different techniques have been developed to tackle these challenges (1), but they tend to be highly data and discipline dependent. A popular theoretically flexible and practically useful framework incorporates “model switching” to account for these shifts in dynamical behavior. The resulting problem is to then simultaneously assign regime labels and find parameters characterizing the behavior in these regimes.

This scheme is an extension of the celebrated Gaussian Mixture Model (GMM):

\[
p(x_l, z_l | \mu, \Sigma, \pi) = \prod_{k=1}^{m} \left( \pi_k N(x_l | \mu_k, \Sigma_k) \right)^{z_{lk}} \implies p(x_l | \mu, \Sigma, \pi) = \sum_{k=1}^{m} \pi_k N(x_l | \mu_k, \Sigma_k)
\]

which has seen widespread success in unsupervised learning for encoding complicated probability distributions. A common scheme for switched system identification can be summarized as (17):

\[
p(x_l, z_l | \Phi, \Sigma, \pi) = \prod_{k=1}^{m} \left( \pi_k N(x_{l+1} | \Phi_k x_l + \mu_k, \Sigma_k) \right)^{z_{lk}} \implies p(x_l | \Phi, \Sigma, \pi) = \ldots
\]

where \( \theta \) is a parameterization of the state-transition matrix. This is the simplest possible scheme and is adequate in many situations. When it doesn’t work—say because regime labels contradict knowledge about the underlying system—the inherent nonconvexity in the likelihood severely complicates mitigation strategies. These tend to resemble:

- Restricting the class of covariance matrices in the search space (22)
- Using informed priors to encourage particular switching patterns (21) and (20)
- Seeding better starting points using other algorithms, especially hierarchical clustering (11)

and never relax the noise whiteness assumptions. This begets limitations akin to those faced by the vanilla K-means algorithm; just as K-means treats all clusters as spherical (8), vector autoregressive mixtures treat all injected noise as purely white. We surmise that more general noise processes can better suit time-series data in the same way that considering non-spherical clusters can better suit anisotropic mixtures. Accommodating this situation requires simultaneously parameterizing the noise filter and the dynamical system it is driving. This additional sophistication requires more precision, but overall, we will show the adjustments (which show up as boundary conditions) do not lead to problem intractability.

Precisely speaking, consider the vector-ARMA\((p, q)\) model of form:

\[
x_{l+1} = \sum_{\tau=0}^{p-1} \Phi_{\tau} x_{l-\tau} + \sum_{\tau=0}^{q} \Psi_{\tau} w_{l-\tau} \quad \text{and} \quad w_{l} \sim \mathcal{N}(0, I)
\]

\(^1\)We will use the notation introduced in Bishop here and throughout.
and mixtures of these models. When we say the moving-average component is commonly neglected, we refer to the case when \( q = 0 \). This form is less expressive, but these concerns are somewhat assuaged by raising the autoregressive order because ARMA\((p, q)\) models often have an AR\((\infty)\) representation. This problem has been studied by (3 and 4), who give consistent estimators and asymptotic results on the autoregressive parameter values. It is plausible that an ARMA\((p, q)\) model may admit both a better fit and more parsimony (10), a line of reasoning almost completely ignored in mixture modelling.

There is one notable exception (14) that considers more general noise structure, but this methodology is ill suited for our purposes. Its focus on the inverse-covariance matrix (rather than the covariance matrix) and reliance on the Markov random field framework (rather than time series analysis) induces network structure more specialized than what we assume here. It is possible to directly identify the covariance matrices via semi-definite programming.

2 Vector ARMA Modelling

2.a Preliminaries

It is possible to identify vector-ARMA models (18), but this is almost never done in practice, (especially outside of econometrics) for implementation reasons (10). Our first main result poses this vector-ARMA parameter identification problem as a convex semi-definite program, which requires some notation to express sensibly. For ease of exposition, we largely following the derivations in 16 and 15, sans stationarity assumption. This situation is commonplace outside of econometrics, and requires some specific initial conditions, which we denote as \( \{G_{-1}, G_{-2}, \ldots, G_{-?}\} \). Consider the vector time sequence \( \{x_0, x_1, \ldots, x_{T-1}\} \) concatenated into one long vector:

\[
X_{0:T-1} = [x_0^T, x_1^T, \ldots, x_{T-1}^T]^T \in \mathbb{R}^{nT}
\]

Its first moment can be derived by simply using the deterministic update rule specified by the AR parameters, \( \Phi \), and recursive substitution, that is,

\[
E[X_{0:T-1}] = \left[ \left( \sum_{\tau=0}^{p-1} \Phi_{\tau} x_{-1-\tau} \right)^T, \left( \sum_{\tau=0}^{p-1} \Phi_{\tau} x_{0-\tau} \right)^T, \ldots, \left( \sum_{\tau=0}^{p-1} \Phi_{\tau} x_{(T-1)-\tau} \right)^T \right]^T
\]

where the first term can be determined entirely by the initial conditions, and subsequent ones by recursive substitution. The second moment, \( \Sigma \), is the symmetric positive semidefinite block-toeplitz matrix filled with autocovariance parameters:

\[
\Gamma^{(k)} = \sum_{\tau} \Psi_{\tau} \Psi_{\tau + k}^T, \quad \Gamma^{(-k)} = \Gamma^{(k)^T} \quad \Sigma = \begin{bmatrix} \Gamma^{(0)} & \Gamma^{(1)} & \cdots \\ \Gamma^{(-1)} & \Gamma^{(0)} & \cdots \\ \vdots & \ddots & \ddots \end{bmatrix} \in \mathbb{R}^{nT \times nT}
\]

Knowing both the first and second moments allows us to evaluate the likelihood of observing the entire time sequence. There are additional important features of \( \Sigma \) we mention; proofs/derivations can be found in the references cited, especially in (16):

1. \( \Sigma \) is of finite block bandwidth \( q \), that is, when \( k \notin [-q + 1, q - 1] \), \( \Gamma^{(k)} \) is the zero matrix.

2. The inverse of \( \Sigma \) is not block-Toeplitz. In fact, the inverse of a Toeplitz matrix is not Toeplitz at all—it’s persymmetric (22).

3. \( \Sigma \) is sparse with block-Toeplitz structure when we specify \( q \ll T \), which automatically grants parsimony. This is the primary motivation for techniques like graphical-LASSO (12), which instead promotes sparsity in the inverse-covariance matrix.

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\footnote{We choose a causal model in this situation for the sake of familiarity to the reader. There is no requirement that this model is causal!}

\footnote{A discussion on omitting stationarity is relegated to the appendix.}
Employing vector ARMA models requires that we search for the autocovariance parameters describing $\Sigma$, whose feasible set can be encoded as an LMI cone. However, its size is hard to deal with, especially since time series data frequently contain thousands of samples. This can easily overwhelm a computer’s memory, serving as another deterrent for practical usage.

2.b Model Identification (Main Result 1)

Some authors have tackled the problem of identification ([18, 19, 16]), but we find them extraordinarily challenging to generalize to the mixture setting. Further, they neglect to mention one of the most important properties of the optimization problem at hand: it can be expressed as a convex program on an LMI cone. While the unmodified maximum likelihood gaussian identification problem is nonconvex ([9]):

\[
\begin{align*}
\min_{\mu, \Sigma} & \quad \log \det(\Sigma) + \frac{1}{2} \text{tr} \left( \Sigma^{-1} \sum_{t=1}^T (x_t - \mu)(x_t - \mu)^T \right) \\
\text{subject to:} & \quad \Sigma \succ 0
\end{align*}
\]

a popular mitigation strategy restores convexity by instead dealing with the inverse covariance matrix ([12]). This will not work in our case because we will need to constrain one of the optimization variables, $\Sigma$, to reside in an LMI cone encoding the block Toeplitz structure. Translating this constraint to the inverse covariance matrix is unfruitful, so our first main result deals with the matrix inversion.

**Lemma 2.1 (LMI Inverse Hull Lemma).** Consider the sets:

\[ \mathcal{F} = \text{span}(S_1, \ldots, S_n) \cap \mathbb{S}_++ \quad \mathcal{G} = \text{hull} \circ \text{image}(\text{inv}(\mathcal{F})) \]

that is, $\mathcal{G}$ is the convex hull of the image under matrix inversion of the set $\mathcal{F}$. Next consider the set:

\[ \mathcal{G}' = \left\{ M_2 \in \mathbb{S}_n \left| \begin{array}{cc} M_1 & I \\ I & M_2 \end{array} \right| \succeq 0, \quad M_1 \in \mathcal{F} \right\} \]

$\mathcal{G}$ is equivalent to $\mathcal{G}'$.

**Proof.** First we prove the forward direction $\mathcal{G} \subset \mathcal{G}'$. Let $A$ and $B$ belong to $\mathcal{G}$, i.e., there are corresponding $x^{(A)}$ and $x^{(B)}$ such that:

\[
A^{-1} = \sum_{k=1}^n x^{(A)}_k S_k > 0, \quad B^{-1} = \sum_{k=1}^n x^{(B)}_k S_k > 0
\]

Since $\mathcal{G}$ is a convex set, $C = \theta A + (1-\theta) B \in \mathcal{G}$. We now want to show that $C \in \mathcal{G}'$, i.e. there exists an $x$ such that:

\[
\begin{bmatrix}
\sum_{k=1}^n x_k S_k & I \\
I & C
\end{bmatrix} \succeq 0
\]

We obtain this using the Schur-Complement. Since the inverse function is matrix convex [7], we have

\[
\sum_{k=1}^n x_k S_k - \text{inv}(\theta A + (1-\theta) B) \succeq \sum_{k=1}^n x_k S_k - \left( \theta A^{-1} + (1-\theta) B^{-1} \right) = \sum_{k=1}^n \left( x_k - \theta x_k^{(A)} + (1-\theta) x_k^{(B)} \right) S_k
\]

With choice $x = \theta x^{(A)} + (1-\theta) x^{(B)}$, we obtain the desired inequality in the Schur Complement which implies our desired positive definiteness. Next, we prove the reverse direction, through the contrapositive ($M \notin \mathcal{G} \Rightarrow M \notin \mathcal{G}'$). If $M \notin \mathcal{G}$, then there is a separating hyperplane $\mathcal{H}$ (parameterized by $V_H \in \mathbb{R}^n$ and $c_H \in \mathbb{R}$) such that:

\[
\text{tr}(V_H \Sigma^{-1}) - c_H > 0 \text{ for all } \Sigma^{-1} \in \mathcal{G}, \quad \text{AND} \quad \text{tr}(V_H M) - c_H \leq 0
\]

*The astute reader may immediately see how to construct such a cone, which is elementary, but takes a lot of space to describe, so we relegate this to the appendix.*
Instead of finding a hyperplane between \( M \) and \( \mathcal{F} \), we find one separating \( \mathcal{F} \) and \( \Pi_{L \cap S_+^n}(M) \), i.e., the projection of \( M \) onto \( \text{closure}(K_{\mathcal{F}}) \).

\[
V_H = \Pi_{L \cap S_+^n}(M) - M \quad \text{and} \quad c_H = \text{tr}(\Pi_{L \cap S_+^n}(M) - M, \Pi_{L \cap S_+^n}(M))
\]

And we observe that this separating hyperplane also supports \( \mathcal{F} \). This hyperplane will be useful shortly. Assume for a contradiction that \( M \notin \mathcal{G} \) but \( M \in \mathcal{G}' \). This implies the existence of a corresponding \( \Sigma \in \mathcal{F} \) such that:

\[
\begin{bmatrix}
\Sigma & I \\
I & M
\end{bmatrix} \succ 0
\]

The Schur complement characterization of positive definiteness implies:

\[
(M - \Sigma^{-1}) \succ 0 \iff \text{tr}(V_H(M - \Sigma^{-1})) \geq 0
\]

However,

\[
\text{tr}(V_H(M - \Sigma^{-1})) = \text{tr}(V_HM) - \text{tr}(V_H\Sigma^{-1}) \leq c_H - \text{tr}(V_HM) < c_H - c_H = 0
\]

But this is a contradiction, thus \( S_2 \not\subseteq S_1 \). \( \square \)

This lemma, which is heavily inspired by the matrix geometric mean \((7)\), plays a huge role in our first main result:

**Theorem 2.1.** *The non-convex problem:*

\[
\begin{aligned}
\text{minimize} & \quad \log \det(\Sigma) + \text{tr}(\Sigma^{-1}C) \\
\text{subject to:} & \quad \Sigma > 0, \Sigma = x_1S_1 + \ldots + x_{n_y}S_{n_y}
\end{aligned}
\]

*is equivalent to the convex problem:*

\[
\begin{aligned}
\text{minimize} & \quad -\log \det(M_2) + \text{tr}(M_2C) \\
\text{subject to:} & \quad \begin{bmatrix}
M_1 & I \\
I & M_2
\end{bmatrix} \succeq 0, \quad M_1 = x_1S_1 + \ldots + x_{n_y}S_{n_y} > 0
\end{aligned}
\]

*when \( C \) is positive semi-definite.*

**Proof.** Consider first minimizing over \( M_2 \), then over \( M_1 \) and \( x \). The block matrix linear matrix inequality we use here is the convex hull we constructed earlier. Assume for a contradiction that there exists a \( \Lambda \in S_+^n \) such that:

\[
\begin{bmatrix}
\hat{M}_1 & I \\
I & \hat{M}_2
\end{bmatrix} \succeq 0 \iff \hat{M}_2 \succeq \hat{M}_1^{-1} \iff \hat{M}_2 = \hat{M}_1^{-1} + \Delta
\]

where \( \hat{M}_2 \) is optimal given some \( \hat{M}_1 \). Consider the point \( (\hat{M}_1, \hat{M}_1^{-1}) \); since \( (\hat{M}_1, \hat{M}_2) \) is optimal in \( M_2 \), we have:

\[
-\log \det(\hat{M}_2) + \text{tr}(\hat{M}_2C) \leq -\log \det(\hat{M}_1^{-1}) + \text{tr}((\hat{M}_1)^{-1}C)
\]

We can rewrite our expression as:

\[
-\log \det(\hat{M}_1^{-1} + \Delta) + \text{tr}(\Delta C) + \text{tr}((\hat{M}_1)^{-1}C) \leq -\log \det(\hat{M}_1^{-1}) + \text{tr}((\hat{M}_1)^{-1}C)
\]

Cancelling our the trace terms and combining the determinant terms, we get:

\[
-\log \det(I + \Delta \hat{M}_1) + \text{tr}(\Delta C) \leq 0
\]

\[
-\log \det(I + \hat{M}_1^{1/2} \Delta \hat{M}_1^{1/2}) + \text{tr}(\Delta C) \leq 0
\]

\( ^{\text{Probably should prove this; the first inequality in } M \text{ is satisfied by construction, but the one for } \Sigma^{-1} \in \mathcal{G} \text{ is annoying and seems to be a consequence of the supporting hyperplane theorem.}} \)}
Since \(-\log \det(X) \geq \text{tr}(X - I)\) for any positive definite matrix \(X\), if we replace \(X\) with \(I + \hat{M}_1^{1/2} \Delta \hat{M}_1^{1/2}\), we get \(-\log \det(I + \hat{M}_1^{1/2} \Delta \hat{M}_1^{1/2}) \geq \text{tr}(\hat{M}_1^{1/2} \Delta \hat{M}_1^{1/2})\). Applying this to our expression, we obtain:

\[
\text{tr} \left( \hat{M}_1^{1/2} \Delta \hat{M}_1^{1/2} \right) + \text{tr}(\Delta C) \leq 0
\]

\[
\text{tr}(\Delta \hat{M}_1) + \text{tr}(\Delta C) \leq 0
\]

The first term strictly positive because \(M_2\) is positive definite which leads to a contradiction (a positive number is not smaller than zero). Since this implies that \(M_2 = M_1^{-1}\), if we apply this substitution on our second optimization problem, we recover the first.

It is unlikely that we are the first to show the equivalence between these two problems due to the ubiquity of Gaussian models and popularity of semidefinite programming. However, this equivalent problem is so poorly publicized that we contend the reader is likely unfamiliar with it despite its utility:

1. We can substitute \(M_1 = M_2^{-1}\) for the covariance matrix, and/or \(M_2 = M_1^{-1}\) for the inverse covariance matrix with *impunity* even though some combinations are nonconvex. This recognition was missed by [3]

2. This objective function resembles a semidefinite objective with an added log-barrier term, and is easy to manipulate algebraically. A consequence of this formulation (and strong duality) is that the sample-covariance matrix is the solution to the maximum likelihood covariance problem under i.i.d. sampling. Typical proofs of this statement are far more opaque.

3. In time series analysis, we frequently have a singular observed sequence. Clearly, \(XX^\top\) can be well accomodated by this programme. In fact, any scenario where we perform partial/complete concatenation is amenable since \(C\) only needs to be positive semidefinite.

We will discuss particulars of solving this problem (and its adaptation to mixture models) but there are some degeneracies we first address. These are purely of theoretical concern

### 2.3 Regularity Conditions

The presence of the log-determinant term in the objective is potentially problematic due to its *coercive* nature. We specifically wish to avoid situations with degenerate gaussian distributions, and we present a simple condition guaranteeing this.

**Definition 2.1** (Autocovariances). It is standard to call the \(\Gamma_k\) terms the autocovariance(s) of lag \(k\), and to call the top left \((q_2 + q_1)n \times (q_2 + q_1)n\) block the autocovariance matrix:

\[
\Sigma \left( \left\{ \Psi_t \right\}_{q_2 - q_1} \right) = \begin{bmatrix}
\Gamma_0 & \Gamma_1 & \cdots & \Gamma_{q_2 + q_1 - 1} \\
\Gamma_1 & \Gamma_0 & \cdots & \Gamma_{q_2 + q_1 - 2} \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma_{q_2 + q_1 - 1} & \gamma_{-(q_2 + q_1 - 2)} & \cdots & \Gamma_0 \\
\end{bmatrix}
\]

Furthermore, each auto-covariance sum has the equivalent definition:

\[
\Gamma_k = \Gamma_k^\top = \sum_{t = 0}^{q_2 + q_1 - 1 - k} \Psi_t^\top \Psi_{t+k}^\top = \begin{cases}
\sum_{t = 0}^{q_2 + q_1 - 1 - k} \Psi_t^\top \Psi_{t+k}^\top & \text{for } 0 \leq k \leq q_2 + q_1 - 1 \\
\sum_{t = -k}^{q_2 + q_1 - 1} \Psi_t^\top \Psi_{t+k}^\top & \text{for } -(q_2 + q_1 - 1) \leq k < 0 \\
0 & \text{otherwise}
\end{cases}
\]

This is why we only work with the causal model; dealing with the non-causal model leads to additional notation for very little gain, especially since converting between the two is simple.

5
Lemma 2.2 (Triangular Toeplitz Decomposition). Let the covariance matrix $\Sigma$ generated by the matrix sequence $\{\Psi_i\}_{i=0}^{q-1}$ be defined as:

$$\Sigma \left( \Gamma_{i-j} \right) = \begin{bmatrix}
\Gamma_0 & \Gamma_1 & \cdots & \Gamma_{q-1} \\
\Gamma_1 & \Gamma_0 & \cdots & \Gamma_{q-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma_{q-1} & \Gamma_{q-2} & \cdots & \Gamma_0
\end{bmatrix}$$

$\Gamma_k = \sum_{t=0}^{q-1-k} \Psi_t \Psi_t^T$ (when $k > 0$)

$\Sigma \left( \Psi_i \right)_{i=0}^{T-1}$ can be decomposed as:

$$\Sigma \left( \Psi_i \right)_{i=0}^{T-1} = \begin{bmatrix}
Q_1^T & Q_2^T
\end{bmatrix}
\begin{bmatrix}
Q_1 & Q_2
\end{bmatrix}$$

Proof. We simply verify this statement block index-wise.

$$(e_i^{(q)} \otimes I_n) \Gamma(\cdot)(e_j^{(q)} \otimes I_n) = (e_i^{(q)} \otimes I_n) \Gamma_1 Q_1 Q_1^T (e_j^{(q)} \otimes I_n) + (e_i^{(q)} \otimes I_n) \Gamma_2 Q_2 Q_2^T (e_j^{(q)} \otimes I_n)$$

Block indexing can be used to find which terms should be summed.

$$\begin{bmatrix}
0 & 0 & \cdots & 0 \\
\Psi_0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\Psi_{q-2} & \cdots & 0 & \Psi_{q-1}
\end{bmatrix}$$

We choose to sum from right to left, so the total number of summed terms is $\min(q-i, q-j) + 1 = q + 1 - \max(i, j)$ when $i, j \in \{1, \cdots, q\}$. This matrix product is therefore:

$$\begin{bmatrix}
B_{0} & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}$$

The second term can be evaluated in a similar fashion:

$$(e_i^{(q)} \otimes I_n) \Gamma_2 Q_2 Q_2^T (e_j^{(q)} \otimes I_n) = \begin{bmatrix}
B_{j-2} & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}$$

We implicitly assume here that $i > 1$ AND $j > 1$—otherwise, the matrix product evaluates to zero. There are $\min(i-2, j-2) + 1$ terms, so summing from left to right:

$$\sum_{t=0}^{\min(i-2,j-2)} B_{j-2-t} B_{i-2-t}^T$$
First, we apply a sum-reversal to obtain:

\[ \sum_{\ell=0}^{\min(i,j)-2} B_{i-\min(i,j)+\ell} B_{j-\min(i,j)+\ell} \]

We notice that the two sums can be combined because their indices are adjacent and non-overlapping:

\[
(\epsilon_i \otimes I_n)^T Q_1Q_1^T(\epsilon_j \otimes I_n) = \sum_{\ell=0}^{q-1+\min(i,j)-\max(i,j)} B_{i-\min(i,j)+\ell} B_{j-\min(i,j)+\ell}
\]

We shift all indices up by \( i - \min(i,j) \):

\[
= \sum_{\ell=i-\min(i,j)}^{q-1+i-\max(i,j)} B_{\ell} B_{\ell+i-j} = \sum_{\ell=\min(j-i,0)}^{\min(q-1,q-1-(j-i))} B_{\ell} B_{\ell+i-j}
\]

We see that when \( k = j - i \), then we get a desired form for the autocovariance. □

Next, we consider when this expression is positive definite.

**Lemma 2.3.** The matrix sum-of-squares:

\[ P = \sum_{k=1}^{M} Q^{(k)}Q^{(k)^T} \]

is positive definite if and only if

\[ 0 = \bigcap_{k=1}^{M} \ker(Q^{(k)^T}) \]

**Proof.** Trivial, use the quadratic form definition of positive definiteness. □

We can conclude that when \( \Sigma(\{\Psi_t\}^q_{t=0}) \) is positive definite, i.e.,

\[ \Sigma(\{\Psi_t\}^q_{t=0}) = Q^{(1)}Q^{(1)^T} + Q^{(2)}Q^{(2)^T} > 0 \]

\( Q^{(1)^T}, Q^{(2)^T} \) only share the zero vector as within their nullspaces. This characterization is useful because it succinctly summarizes the definiteness of the covariance matrix, or as we wish to interpret it, an autocovariance matrix generated by a zero-padded parameter matrix sequence \( \{\tilde{\Psi}_\ell\}_{\ell \geq q} \), where \( \tilde{\Psi}_\ell = 0 \) when \( \ell \geq q \). This is an identifiability property we will need to ensure uniqueness of our solution. After zero-padding, we see that this assumption needs to be strengthened.

**Theorem 2.2 (Time Structured Covariance Positive Definiteness).** The covariance matrix generated by the zero-padded parameters \( \Sigma(\{\Psi_t\}^{T-1}_{t=0}) \), where

\[ \{\Psi_t\}^{T-1}_{0} = \{\Psi_0, \ldots, \Psi_{q-1}, 0, \ldots, 0\} \]

is positive definite if and only if:

\[
\ker\left(\begin{bmatrix} \Psi_{q-1} & \Psi_{q-2} & \cdots & \Psi_0 \\ \Psi_{q-1} & \cdots & \Psi_1 \\ \vdots & \ddots & \vdots \\ \Psi_0 & \cdots & \Psi_{q-1} \end{bmatrix}\right) = 0 \text{ and } \ker\left(\begin{bmatrix} \Psi_0 \\ \vdots \\ \Psi_{q-2} & \cdots & \Psi_0 \end{bmatrix}\right) = 0
\]
Proof. $Q_1(\{\tilde{B}\})$ and $Q_2(\{\tilde{B}\})$ are upper/lower triangular matrices with zeros on the diagonal, so all of their eigenvalues are zero. We will arrive at our criterion constructively, so first we consider the matrices $Q_1(\{\tilde{B}\})$ and $Q_2(\{\tilde{B}\})$. We can easily construct the associated null-space vectors of $Q_1(\{\tilde{B}\})$ and $Q_2(\{\tilde{B}\})$

$$Q_1(\{\tilde{B}\}) = \begin{bmatrix} 0 & \cdots & 0 & B_{q-1} & \cdots & B_0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & B_{q-1} \\ \cdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \vdots & \ddots & \vdots \\ \end{bmatrix}$$

$(T-q)n$ zeros

$$V(Q_1(\{\tilde{B}\})) = \begin{bmatrix} e_1 & \cdots & e_{(T-q)n} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 \\ \eta_1^{(Q_1)} & \cdots & \eta_{|N(Q_1)|}^{(Q_1)} \end{bmatrix}$$

Where $\eta_1^{(Q_1)}, \ldots, \eta_{|N(Q_1)|}^{(Q_1)}$ are the nullspace vectors of $Q_1(\{\tilde{B}\})$. Similarly,

$$Q_2(\{\tilde{B}\}) = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & B_0 \\ \vdots & \ddots & \vdots & \vdots \\ B_{q-2} & \cdots & B_0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$qn$ zeros

$$V(Q_2(\{\tilde{B}\})) = \begin{bmatrix} \eta_1 \end{bmatrix} \begin{bmatrix} \eta_{|N(Q_2)|}^{(Q_2)} \end{bmatrix} \begin{bmatrix} e_{(T-q)n+1} & \cdots & e_T \end{bmatrix}$$

$\eta_1^{(Q_2)}, \ldots, \eta_{|N(Q_2)|}^{(Q_2)}$ are the nullspace vectors of the left lower block of $Q_2$ (the portion of concern in our theorem). We observe that when our theorem statement holds, these “extra” null-space vectors disappear and the remaining null-space eigenvectors partition the space $\mathbb{R}^{nT}$ thus (by the preceding lemma), $\Gamma(\{\tilde{B}\})$ is positive definite.

When the moving average order is much smaller than the number of samples, that is, $q \ll T$, we can expect $\Sigma$ to be positive definite when it satisfies the conditions of this theorem. This ensures the coercive nature of the log-determinant is avoided and the optimization problem returns a reasonable result. Finally, we remark that this expression is eerily similar to the Gohberg-Semencul formula for the inverse covariance matrix ([13]), which expresses the inverse using its impulse response. Further maps between covariance and inverse covariance matrices can easily be derived using these formulas, but we do not investigate this further here. This lemma leads to the a straightforward regularity condition on the covariance matrix.

2.d A convex optimization formulation of parameter identification (Main Result 2)

We end this section on vector-ARMA modelling by writing down the optimization problem that identifies the parameters. We assume the regularity condition just described on the model generating the data, $X_{0:T-1}$. With a slight abuse of notation, let:
1. $\Phi$ be the convolution operator used to define $E[X]$

2. $S_1, \ldots, S_{n_B}$ be the symmetric matrices that form a basis for the block-Toeplitz matrices

3. $M_1$ and $M_2$ play the roles of the covariance matrix ($\Sigma$) and inverse-covariance matrix ($\Sigma^{-1}$) respectively

$$\begin{align*}
\text{minimize} & \quad - \log \det(M_2) + \text{tr} \left( M_2 \left( X_{0:T-1} - \Phi X_{-p:T-1-p} \right) \left( X_{0:T-1} - \Phi X_{-p:T-1-p} \right)^T \right) \\
\text{subject to:} & \quad \begin{bmatrix} M_1 & I \\ I & M_2 \end{bmatrix} \succeq 0, \quad M_1 = c_1 S_1 + \ldots + c_{n_B} S_{n_B} > 0
\end{align*}$$

By inspection, we see this is a convex problem on the semidefinite cone so there are many options for solving this problem. One fairly simple way to solve this would be through block alternating coordinate descent ([5]). Employing this technique with block coordinates $\{\Phi\}$, and $\{M_1, M_2, c\}$ leads to two well known problems; generalized least squares and a semidefinite programme (some trivial manipulations to make this exact). Of course, we can also employ Newton’s method—and its analogues—as ([16], [18]) do.

3 Regime Switched vector-ARMA

One of the key ideas in the analysis of the EM algorithm derivation is to analyze the posterior probability:

$$p(Z|X) = \frac{p(X|Z)p(Z)}{p(X)}$$

4 Expectation Maximization

Finally, we present the forms of “expectation” and “maximization” steps commonly used for mixture model parameter identification, specifically tailored to vector-ARMA mixtures. From this point forward, we simply use the shorthand:

$$P + I.C. := \Phi, \Sigma, \pi, Z, X_{-p:-1}$$

Lemma 4.1 (Expectation Step). $P(Z|X, P + I.C.)$

The maximization step follows the exact same principle in ([8]), where we maximize the expected likelihood of under the posterior distribution. We write this down simply to emphasize that this step is still convex, albeit hard to solve.

Lemma 4.2 (Maximization Step).

$$(\Phi^+, \Sigma^+, \pi^+) = \maximize_{\Phi, \Sigma, \pi} \sum_{k=1}^{n} \log(p_k) + \log \det(\Sigma_k) - \frac{1}{2} \text{tr} ()$$

subject to: $\Sigma > 0, \Sigma \in \text{Block-Toeplitz}$

We follow the exact same model form as from

5 Numerical Experiments

6 Conclusion

The
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