Tropical $F$-polynomials and general presentations

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Abstract
We introduce the tropical $F$-polynomial $f_M$ of a quiver representation $M$. We study its interplay with the general presentation for any finite-dimensional basic algebra. We give an interpretation of evaluating $f_M$ at a weight vector. As a consequence, we give a presentation of the Newton polytope $\mathcal{N}(M)$ of $M$. We study the dual fan and 1-skeleton of $\mathcal{N}(M)$. We propose an algorithm to determine the generic Newton polytopes, and show that it works for path algebras. As an application, we give a representation-theoretic interpretation of Fock–Goncharov’s duality pairing. We give an explicit construction of dual clusters, which consists of real Schur representations. We specialize the above general results to the cluster-finite algebras and the preprojective algebras of Dynkin type.

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1 | INTRODUCTION

For fixed projective representations \( P_-, P_+ \) of a given path algebra with relations \( A = kQ/I \), a general presentation \( d : P_- \to P_+ \) is a general element in the vector space \( \text{Hom}(P_-, P_+) \). The study of general presentations of algebras was initiated in [11]. The theory was developed in parallel with that of general representations of quivers (without relations) (e.g., [30, 40]). The dimension vectors in our setting is replaced by the weight vectors \( \delta \in \mathbb{Z}^Q_0 \). The presentation space of weight \( \delta \) is the space

\[
\text{PHom}(\delta) := \text{Hom}(P([-\delta]_+, P([\delta]_+))
\]

where we denote \([\delta]_+ := \text{max}(\delta, 0)\), \( P(\beta) := \bigoplus_{i \in Q_0} \beta(i)P_i \), and \( P_i \) is the indecomposable projective representation corresponding to the vertex \( i \). For two presentations \( d_1, d_2 \), we defined a finite-dimensional space \( E(d_1, d_2) \) that plays the role of \( \text{Ext}^1 \) for path algebras (without relations). We denote by \( e(\delta_1, \delta_2) \) the minimal value of \( \text{dim} E(-, -) \) on \( \text{PHom}(\delta_1) \times \text{PHom}(\delta_2) \). We found many analogous results about general representations for general presentations. For example, the following analog of Kac’s canonical decomposition.

**Definition** [11]. A weight vector \( \delta \in \mathbb{Z}^Q_0 \) is called **indecomposable** if a general presentation in \( \text{PHom}(\delta) \) is indecomposable. We call \( \delta = \delta_1 \oplus \delta_2 \oplus \cdots \oplus \delta_s \) the **canonical decomposition** of \( \delta \) if a general element in \( \text{PHom}(\delta) \) decompose into (indecomposable) ones in each \( \text{PHom}(\delta_i) \).
**Theorem 1.1** [11, Theorem 4.4]. \( \delta = \delta_1 \oplus \delta_2 \oplus \cdots \oplus \delta_s \) is the canonical decomposition of \( \delta \) if and only if \( \delta_1, \ldots, \delta_s \) are indecomposable, and \( e(\delta_i, \delta_j) = 0 \) for \( i \neq j \).

However, an analog of the following interesting result in [40] is missing.

**Theorem 1.2** [40, Theorem 5.4]. Let \( \alpha \) and \( \beta \) be dimension vectors for the quiver \( Q \). Then

\[
\text{ext}(\alpha, \beta) = \max_{\beta \rightarrow \beta'} \{-\langle \alpha, \beta' \rangle\} = \max_{\alpha' \hookrightarrow \alpha} \{-\langle \alpha', \beta \rangle\}.
\]

Here \( \text{ext}(\alpha, \beta) := \min \{\dim \text{Ext}(M, N) | M \in \text{rep}_\alpha(Q), N \in \text{rep}_\beta(Q)\} \) and \( \langle -, - \rangle \) is the Euler form of \( Q \).

It seems reasonable to find an analog for \( \dim E(d_1, d_2) \) where \( d_1 \) and \( d_2 \) are general presentations. Recall that the space \( E(d_1, d_2) \) only depends on the cokernel of \( d_2 \), so it makes sense to define \( E(d, M) \) as \( E(d, d_M) \) where \( d_M \) is any presentation of \( M \). In fact, we find something more general, a formula for \( \dim E(d, M) \) where \( d \) is a general presentation, while \( M \) is an arbitrary representation. However, the statement is not as neat as Schofield’s.

To state this result, we need to introduce the other main character of this paper, the tropical \( F \)-polynomials of representations, which interplay with the general presentations. Let \( M \) be a finite-dimensional representation of \( A \).

**Definition.** The tropical \( F \)-polynomial \( f_M \) of a representation \( M \) is the function \( (\mathbb{Z}^Q_0)^* \rightarrow \mathbb{Z}_{\geq 0} \) defined by

\[
\delta \mapsto \max_{L \hookrightarrow M} \delta(\dim L).
\]

The dual tropical \( F \)-polynomial \( \check{f}_M \) of a representation \( M \) is the function \( (\mathbb{Z}^Q_0)^* \rightarrow \mathbb{Z}_{\geq 0} \) defined by

\[
\delta \mapsto \max_{M \twoheadrightarrow N} \delta(\dim N).
\]

We denote by \( \text{hom}(\delta, M) \) and \( e(\delta, M) \) the dimension of the kernel and cokernel of

\[
\text{Hom}(P_+, M) \rightarrow \text{Hom}(P_-, M),
\]

which is induced from a general presentation \( P_- \rightarrow P_+ \) in \( \text{PHom}(\delta) \). Similarly, we can define \( \text{hom}(M, \check{\delta}) \) and \( \check{e}(M, \check{\delta}) \) using a general injective presentation of weight \( \check{\delta} \). Here is our analog of Theorem 1.2.

**Theorem 1.3** (Theorem 3.6). For any representation \( M \) and any \( \delta \in \mathbb{Z}^Q_0 \), there is some \( n \in \mathbb{N} \) such that

\[
f_M(n\delta) = \text{hom}(n\delta, M), \quad \check{f}_M(-n\delta) = e(n\delta, M).
\]

Similarly, for any representation \( M \) and any \( \check{\delta} \in \mathbb{Z}^Q_0 \), there is some \( \check{n} \in \mathbb{N} \) such that

\[
\check{f}_M(\check{n}\delta) = \text{hom}(M, \check{n}\delta), \quad f_M(-\check{n}\delta) = \check{e}(M, \check{n}\delta).
\]
Moreover, \( n \) can be replaced by \( kn \) for any \( k \in \mathbb{N} \). If \( m \) is the minimum of all such \( n \), then \( m\delta \) cannot be decomposed as \( m\delta = k\delta \oplus (m-k)\delta \) for any \( k \). In particular, if \( e(\delta, \delta) = 0 \), then \( m = 1 \).

When \( A \) is the Jacobian algebra of a quiver with potential, we show that \( m = 1 \) in the following two cases:

1. \( M \) is negative reachable (Theorem 3.22);
2. the quiver is mutation-acyclic and \( M \) is the cokernel of a general presentation (Corollary 3.26).

The following direct consequence says that the evaluation of \( f_M \) computes the asymptotic \( \hom(a\delta, M) \) as \( a \) increases, which generalizes a result of W. Crawley–Boevey on quivers without relations [10].

**Corollary 1.4** (Corollary 3.10). The following limits exist and we have the equalities:

\[
\lim_{a \to \infty} \frac{1}{a} \hom(a\delta, M) = f_M(\delta) \\
\lim_{a \to \infty} \frac{1}{a} e(a\delta, M) = f_M(-\delta);
\]

\[
\lim_{a \to \infty} \frac{1}{a} \hom(M, a\delta) = f_M(\delta) \\
\lim_{a \to \infty} \frac{1}{a} e(M, a\delta) = f_M(-\delta).
\]

The tropical \( F \)-polynomial \( f_M \) is completely determined by the Newton polytope of \( M \).

**Definition.** The Newton polytope \( N(M) \) of a representation \( M \) is the convex hull of 

\[ \{ \dim L \mid L \hookrightarrow M \} \]

in \( \mathbb{R}^{Q_0} \). The dual Newton polytope \( \check{N}(M) \) of a representation \( M \) is the convex hull of 

\[ \{ \dim N \mid M \twoheadrightarrow N \} \]

in \( \mathbb{R}^{Q_0} \).

As an easy consequence of Theorem 1.3, we get a presentation of \( N(M) \).

**Theorem 1.5** (Theorem 5.1). The Newton polytope \( N(M) \) is defined by 

\[ \{ \gamma \in \mathbb{R}^{Q_0} \mid \delta(\gamma) \leq \hom(\delta, M), \forall \delta \in \mathbb{Z}^{Q_0} \} \].

The dual Newton polytope \( \check{N}(M) \) is defined by 

\[ \{ \gamma \in \mathbb{R}^{Q_0} \mid \delta(\gamma) \leq \hom(M, \delta), \forall \delta \in \mathbb{Z}^{Q_0} \} \].

It is then natural to study the Newton polytope of a representation. The vertices and facets were studied in detail in [19]. In this paper, we focus on their duals—the normal vectors and normal cones. Recall that

**Definition** [11]. A weight vector \( \delta \in \mathbb{Z}^{Q_0} \) is called real if \( e(d, d) = 0 \) for some \( d \in \text{PHom}(\delta) \). A maximal set of real indecomposable weight vectors \( \{\delta_1, \ldots, \delta_n\} \) such that \( e(\delta_i, \delta_j) = 0 \) for \( i \neq j \) is called a cluster.
**Definition.** For a fixed algebra $A$, a weight vector $\delta$ is called **normal** if it is an outer normal vector of the Newton polytope of some $M \in \text{rep} A$.

We show in Corollary 5.3 that any indivisible outer normal vector of $N(M)$ must be indecomposable. It is natural to ask if the converse is true.

**Question.** Is any indecomposable $\delta$-vector normal?

The answer is positive if $\delta$ is real. We give an equivalent condition for $\delta$ being normal (Proposition 5.7).

In our setting, the **normal cone** $F_\gamma(M)$ of a vertex $\gamma \in N(M)$ is the cone spanned by $\delta$ satisfying $\delta(\gamma) = f_M(\delta)$.

The two most important normal cones are the ones corresponding to the vertices $0$ and $M$. The lattice points inside the cones are precisely

\[
\{\delta \in \mathbb{Z}^Q_0 \mid \text{hom}(n\delta, M) = 0 \text{ for some } n \in \mathbb{N}\};
\]

\[
\{\delta \in \mathbb{Z}^Q_0 \mid e(n\delta, M) = 0 \text{ for some } n \in \mathbb{N}\}.
\]

Clearly, $F_0(M)$ always contains the **negative cluster** $(-e_1, \ldots, -e_n)$ and $F_M(M)$ always contains the **positive cluster** $(e_1, \ldots, e_n)$. In particular, Theorem 1.3 provides us a presentation for them (Corollary 5.11).

Our most important result about the normal cones is the following.

**Theorem 1.6** (Theorem 5.17). Let $\delta_1, \ldots, \delta_m$ be finitely many clusters. Then there is some representation $M$ such that each $\delta_i$ spans a normal cone of $N(M)$.

The normal cones of $N(M)$ fit together into a complete fan $F(M)$, the **normal fan** of $N(M)$. The generalized cluster fan defined below refines the cluster fan introduced in [11].

**Definition.** Let $F(\text{rep} A)$ be the set of all cones spanned by $\{\delta_1, \ldots, \delta_P\}$ such that each $\delta_i$ is normal and $e(\delta_i, \delta_j) = 0$ for $i \neq j$. It turns out that $F(\text{rep} A)$ forms a simplicial fan. We call it **generalized cluster fan**.

**Proposition 1.7** (Proposition 8.4). The fan $F(M)$ is a coarsening of the generalized cluster fan $F(\text{rep} A)$.

To study the dual picture, namely, the 1-skeleton of $N(M)$, we need the Schur representations, especially the real ones.

**Definition.** A representation $V$ is called **Schur** if $\text{Hom}(V, V) = k$. It is called **real Schur** if, in addition, we require $\text{Ext}^1(V, V) = 0$.

Suppose that $\{\delta_\_\} \cup \delta_0$ and $\{\delta_+\} \cup \delta_0$ are two **adjacent** clusters. We assume that $(\delta_\_, \delta_+)$ is a **regular exchange pair**, that is, $e(\delta_\_, \delta_+) = 1$. In this case, we define the sign of $\delta_\_$ in the cluster...
{\delta_\pm} \cup \delta_0 to be negative, and the sign of \delta_+ in the cluster {\delta_\pm} \cup \delta_0 to be positive. Let \(d_-\) and \(d_+\) be general presentations of weight \(\delta_-\) and \(\delta_+.\) Let \(L = \text{Coker}(d_+)\) and \(N = \text{Coker}(d_-),\) then \(\text{hom}(L, \tau N) = e(d_-, L) = 1.\) We consider the exact sequence

\[0 \to K \to L \to \tau N \to C \to 0.\]

Let \(I\) be the image of \(L \to \tau N.\) It is not hard to show that \(I\) is a real Schur representation (Lemma 7.2).

Let \(\delta = \{\delta_1, \delta_2, \ldots, \delta_n\}\) be a cluster, and \(\delta'_j = (\delta \setminus \{\delta_j\}) \cup \{\delta'_j\}\) be the adjacent cluster. Let \(I_j\) be defined as above for each (unordered) exchange pair \(\{\delta_j, \delta'_j\},\) and \(\epsilon_j\) be the sign of \(\delta_j\) in \(\delta.\) We say \(\delta\) is a regular cluster if each exchange pair is regular. Below we use the upright \(\delta\) to denote the usual delta-function. We write \(\delta^\perp\) for the abelian subcategory

\[\delta^\perp := \{M \in \text{rep} A \mid \text{hom}(\delta, M) = e(\delta, M) = 0\}.\]

**Theorem 1.8** (Theorem 7.6). Let \(\{\delta_i\}_i\) be a regular cluster and \(I_j\) be defined as above. Then

\[\text{hom}(\delta_i, I_j) = [\epsilon_j]_+ \delta(i, j) \text{ and } e(\delta_i, I_j) = [-\epsilon_j]_+ \delta(i, j).\]

Moreover, the simple objects in the category \(\delta^\perp_i := \bigcap_{i \in I} \delta^\perp_i\) are precisely \(I_j (j \not\in I).\)

Now we state the results about the 1-skeleton of \(N(M).\)

**Proposition 1.9** (Proposition 8.6). If \(L_- L_+\) is an edge in \(N(M),\) then either \(L_- \subset L_+\) or \(L_+ \subset L_-\). Say \(L_- \subset L_+\), then

\[L_- = t_\delta(M) \text{ and } L_+ = \tilde{t}_\delta(M) \text{ for any } \delta \text{ in the interior of } F_{L_- L_+}(M).\]

Here, \(t_\delta\) and \(\tilde{t}_\delta\) are two functors introduced in [19], and \(F_{L_- L_+}(M) = F_{L_-}(M) \cap F_{L_+}(M).\) Moreover, we have the following.

1. \(\delta_+(L_+/L_-) \geq 0\) for any \(\delta_+ \in F_{L_+}(M)\) and \(\delta_-(L_+/L_-) \leq 0\) for any \(\delta_- \in F_{L_-}(M)\) with the equality holding only when \(\delta_\pm \in F_{L_- L_+}(M).\)
2. If \(F_{L_-}(M)\) is spanned by a regular cluster, then \(L_+/L_-\) is a direct sum of isomorphic real Schur representations.

**Definition.** We assign the orientation \(L_0 \to L_1\) for each edge \(L_0 L_1\) with \(L_0 \subset L_1.\) We call the resulting oriented graph the edge quiver of \(N(M),\) denoted by \(N_1(M).\)

The above results, especially Theorem 1.6 and Proposition 1.7 and 1.9, when being applied to some special cases, already produce new and nontrivial results. For example,

**Proposition 1.10** (Proposition 9.5). Suppose that \(A\) is cluster-finite. Let \(M\) be the direct sum of all \(E\)-rigid representations. Then the normal fan \(F(M)\) is the cluster fan of \(A,\) and the edge quiver \(N_1(M)\) is the exchange quiver of \(A.\)

In view of Propositions 1.7 and 1.10, the generalized cluster fan \(F(\text{rep} A)\) can be viewed heuristically as the normal fan of the infinite-dimensional representation \(\bigoplus_{M \in \text{rep} A} M.\)
Proposition 1.11 (Proposition 9.9). Suppose that $A$ is a preprojective algebra of Dynkin type. The vertices of $N(A)$ are labeled by the ideals $I_w$, and $F_w(A)$ is the cluster corresponding to $I_w$. So, $F(A)$ is the cluster fan $F(\text{rep} A)$, which is a Weyl fan.

Finally, let us come back to the generic setting as in Schofield’s original paper. We are interested in determining the Newton polytopes of general representations.

Theorem 1.12 (Theorem 6.4). Let $\alpha$ be any dimension vector of $Q$. Each normal cone $F_{\gamma}(\alpha)$ of $N(\alpha)$ contains a cluster. Hence, the Newton polytope $N(\alpha)$ is completely determined by Newton polytopes of real Schur representations.

“Determine” here means that we can explicitly compute all vertices of $N(\alpha)$ by what we observed in Observation 6.1. More generally, we are interested in the Newton polytope of the cokernel of a general presentation, especially for the Jacobian algebras. In some optimistic situation (e.g., Question 6.2 is positive), the method would work for such generic Newton polytopes (see Observation 6.9). We will explain below why this is an important problem in the cluster algebra theory. This approach also gives an alternative proof of Schofield’s Theorem 1.2.

1.1 Motivation and relation to cluster algebras

The tropical $F$-polynomials and general presentations discussed in this paper are originated from the theory of cluster algebras [24]. We know from [25] that for cluster algebras of geometric type, any cluster variable can be written as

$$X(\delta) := x^{-\delta}F_\delta(y),$$

where $y$ is a certain monomial change of the initial cluster variables $x$. Here we use $x^a$ to denote the monomial $\prod_i x_i^{a(i)}$.

If the cluster algebra is skew-symmetric, we have a nondegenerate quiver with potential $(Q, \mathcal{P})$ to model this algebra [16]. Let $A$ be the Jacobian algebra associated to $(Q, \mathcal{P})$. The above polynomial $F_\delta$ is the $F$-polynomial of some E-rigid representation $M$ of $A$ [16]. Moreover, the minimal injective presentation of $M$ has weight exactly $\delta$. Since the coefficients of $F$ are all positive, we can tropicalize it in the usual sense. The tropicalization is precisely the tropical $F$-polynomial of $M$.

Moreover, if $\{X(\delta_1), \ldots, X(\delta_n)\}$ forms a cluster in the cluster algebra $C(Q)$, then $\{\delta_1, \ldots, \delta_n\}$ is a cluster in $\text{rep} A$ [11, 16]. So, the cluster fan $F'(\text{rep} A)$ is the original cluster fan for $C(Q)$. In this setting, the signed dimension vector $\epsilon_j \dim I_j$ of $I_j$ in Theorem 1.8 is the corresponding $c$-vectors of the cluster.

If the Jacobian algebra is cluster-finite, then we get an easy consequence of Proposition 1.10. In this case, the Newton polytope is the so-called generalized associahedron [23].

Corollary 1.13 (Corollary 9.7). Suppose that $A$ is a cluster-finite Jacobian algebra. Let $M$ be the direct sum of all E-rigid representations of $A$. Then the dual fan $F(M)$ is the cluster fan of $C(Q)$, and the edge quiver $N_1(M)$ is the exchange quiver of $C(Q)$. Moreover, the signed dimension vectors of the real Schur representations attached to the arrows from/to a fixed vertex $L$ are the signed $c$-vectors dual to the cluster $F_L(M)$.
The formula (1.1) has a naive generalization where we consider the \( F \)-polynomial \( F_{\delta} \) of the kernel of a general injective presentation of any weight \( \delta \in \mathbb{Z}^{Q_0} \) [18]. In many cases, they are turned out to be a basis of the upper cluster algebra \( \tilde{C}(Q) \) [38]. The Newton polytope of this \( F_{\delta} \) is exactly given by the generic Newton polytope \( N(\delta) \).

In the meanwhile, a remarkable positive basis consisting of theta functions for all cluster algebras was introduced in [26]. For each \( \delta \)-vector, there is a theta function \( \varphi_{\delta} \), which is of the form

\[
\varphi_{\delta} = x^{-\delta} \varphi_{\delta}(y).
\]

In general, the theta function can be a Laurent series, but let us assume that it is a Laurent polynomial, so \( \varphi_{\delta} \) is a polynomial with positive coefficients. Another very interesting positive (quantum) basis called triangular basis was introduced in [39]. It has a similar form

\[
T_{\delta, q} = x^{-\delta} T_{\delta, q}(y).
\]

In particular, \( \varphi_{\delta} \) and \( T_{\delta, q} \) can be tropicalized and the tropicalization is determined by its Newton polytope. We have the following conjecture.

**Conjecture 1.14.** The Newton polytopes of \( \varphi_{\delta} \) and \( T_{\delta, q} \) are the same as the generic Newton polytope \( N(\delta) \).

Another related problem is the Fock–Goncharov duality conjecture [22, Conjecture 4.1]. Recall that a skew-symmetrizable matrix \( B \) gives rise to a pair of cluster varieties \(( \mathcal{A}, \mathcal{X} )\), and their Langlands dual \(( \mathcal{A}^\vee, \mathcal{X}^\vee )\). The conjecture says that the tropical points \( \mathcal{X}^\vee(\mathbb{Z}^l) \) of \( \mathcal{X}^\vee \) parametrize a basis of ring of regular functions \( \mathcal{O}(\mathcal{A}) \) of \( \mathcal{A} \), and we can interchange the roles of \( \mathcal{A} \) and \( \mathcal{X} \). The duality conjecture fails in general, but can hold with some mild assumption, or if replaced with a certain formal version (see [26] for detail). Let us assume that the parametrizations exist and we denote them by

\[
I_{\mathcal{A}} : \mathcal{A}(\mathbb{Z}^l) \leftrightarrow \mathcal{O}(\mathcal{A}^\vee) \quad \text{and} \quad I_{\mathcal{X}^\vee} : \mathcal{X}^\vee(\mathbb{Z}^l) \leftrightarrow \mathcal{O}(\mathcal{A}).
\]

The duality conjecture further asserts that we can require the parametrized bases to be universally positive and satisfy several interesting properties. One of them concerns the pairing

\[
\mathcal{A}(\mathbb{Z}^l) \times \mathcal{X}^\vee(\mathbb{Z}^l) \to \mathbb{Z}.
\]

There are two canonical (conjecturally equal) ways to define this pairing:

\[
I_{\mathcal{A}}(a)^{\text{trop}}(x) \quad \text{and} \quad I_{\mathcal{X}^\vee}(x)^{\text{trop}}(a)
\]

for \( a \in \mathcal{A}(\mathbb{Z}^l), \ x \in \mathcal{X}^\vee(\mathbb{Z}^l) \).

We give a representation-theoretic interpretation of the pairing in some special cases.

**Theorem 1.15** (Fock–Goncharov duality pairing). Suppose that \( B \) is skew-symmetric. The pairings \( \mathcal{A}(\mathbb{Z}^l) \times \mathcal{X}^\vee(\mathbb{Z}^l) \to \mathbb{Z} \) given by \( I_{\mathcal{A}}(a)^{\text{trop}}(\delta) \) and \( I_{\mathcal{X}^\vee}(\delta)^{\text{trop}}(a) \) are both equal to \( \text{hom}(aB^T, \delta) - a \cdot \delta \) in the following two situations.

1. The quiver of \( B \) is mutation-equivalent to an acyclic quiver.
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(2) Either $I_{X^\vee}(\delta)$ or $I_X(ab^T)$ is a cluster variable, or equivalently either $\delta$ or $ab^T$ is negative reachable.

Although the main part of Fock–Goncharov duality conjecture was intensively studied, the meaning of the duality pairing is only known in few cases (e.g., [21, Proposition 12.1]). The verification of the equality in this generality is new.

1.2 | Organization

In Section 2, we review the theory of general presentations developed in [11]. In Section 3, we introduce the tropical $F$-polynomial of a representation and its Newton polytope. We prove our first main result—Theorem 3.6. Then we improve this result in the case of quivers with potentials (Theorem 3.22 and Corollary 3.26). In Section 4, we review the two pairs of functors considered in [19]. In Section 5, we give a presentation of the Newton polytope—Theorem 5.1. We study the normal vectors and the normal cones, and prove another main result—Theorem 5.17. In Section 6, we propose an algorithm to determine the generic Newton polytopes. We show in Theorem 6.4 that the algorithm works for path algebras. Observation 6.9 explains why we speculate the algorithm should work more generally. We make some connection to the cluster algebra theory, including an interpretation of the Fock–Goncharov duality pairing (Theorem 6.12). In Section 7, we give an explicit construction of dual clusters consisting of real Schur representations in Theorem 7.6. In Section 8, we study the normal fan and edge quiver of the Newton polytope. For the general case, the two main results here are Propositions 8.4 and 8.6. For the quiver case, we state an interesting bijection in Conjecture 8.11. In Section 9, we apply the above results to two special cases. One is the cluster-finite algebra (Proposition 9.5) and the other is the preprojective algebra of Dynkin type (Proposition 9.9).

1.3 | Notation and conventions

Throughout we only deal with finite-dimensional basic algebras over an algebraically closed field $k$ of characteristic 0. If we write an algebra $A = kQ$, we assume implicitly that $Q$ is finite and has no oriented cycles. For general $A = kQ/I$, we allow $Q$ to have oriented cycles. We denote by $Q_0$ the set of vertices of $Q$.

Unless otherwise stated, unadorned Hom and other functors are all over the algebra $A$, and the superscript $*$ is the trivial dual for vector spaces. For direct sum of $n$ copies of $M$, we write $nM$ instead of the traditional $M^{\oplus n}$. We write hom, ext and $e$ for dim Hom, dim Ext, and dim E.

- $\text{rep } A$ the category of finite-dimensional representations of $A$
- $\text{rep}_\alpha(A)$ the space of $\alpha$-dimensional representations of $A$
- $S_i$ the simple representation supported on the vertex $i$
- $P_i$ the projective cover of $S_i$
- $I_i$ the injective envelope of $S_i$
- $\dim M$ the dimension vector of $M$
2 | REVIEW ON GENERAL PRESENTATIONS

2.1 | The E-invariant of presentations

Let $A$ be a finite-dimensional basic algebra over an algebraically closed field $k$ of characteristic 0. Then $A$ can be presented as a path algebra modulo an ideal generated by admissible relations: $A = kQ/I$ [2]. We denote by $P_v$ (respectively, $I_v$) the indecomposable projective (respectively, injective) representation of $A$ corresponding the vertex $v$ of $Q$. For $\beta \in \mathbb{Z}^{Q_0}_{\geq 0}$, we write $P(\beta)$ for $\bigoplus_{v \in Q_0} \beta(v)P_v$; similarly write $I(\beta)$ for $\bigoplus_{v \in Q_0} \beta(v)I_v$. Following [11], we call a homomorphism between two projective representations, a projective presentation (or presentation in short). As a full subcategory of the category of complexes in $\text{rep} A$, the category of projective presentations is Krull–Schmidt as well.

**Definition 2.1.** The $\delta$-vector (or reduced weight vector) of a presentation $d : P(\beta_-) \to P(\beta_+)$ is the difference $\beta_+ - \beta_- \in \mathbb{Z}^{Q_0}$. When working with injective presentations $\hat{d} : I(\hat{\beta}+) \to I(\hat{\beta}-)$, we call the vector $\hat{\beta}_+ - \hat{\beta}_-$ the $\hat{\delta}$-vector (or reduced weight vector) of $\hat{d}$.

Let $\nu$ be the Nakayama functor $\text{Hom}(-, A)^\ast$. There is a map still denoted by $\nu$ sending a projective presentation to an injective one

$$P_- \to P_+ \mapsto \nu(P_-) \to \nu(P_+).$$

We say a presentation $d$ nonnegative if $d$ has no direct summands of form $P_- \to 0$. If $d$ is nonnegative, then $\text{Ker}(\nu d) = \tau \text{Coker}(d)$ where $\tau$ is the classical Auslander–Reiten translation [2].

**Definition 2.2** [11, 16]. Given any projective presentation $d : P_- \to P_+$, we define $\text{Hom}(d, M)$ and $E(d, M)$ to be the kernel and cokernel of the induced map:

$$0 \to \text{Hom}(d, M) \to \text{Hom}(P_+, M) \xrightarrow{C(d, M)} \text{Hom}(P_-, M) \to \text{E}(d, M) \to 0.$$ (2.1)

Similarly, for an injective presentation $\hat{d} : I_+ \to I_-$, we define $\text{Hom}(M, \hat{d})$ and $\hat{E}(M, \hat{d})$ to be the kernel and cokernel of the induced map $\text{Hom}(M, I_+) \to \text{Hom}(M, I_-)$. It is clear that

$$\text{Hom}(d, M) = \text{Hom}(\text{Coker}(d), M) \quad \text{and} \quad \text{Hom}(M, \hat{d}) = \text{Hom}(M, \text{Ker}(\hat{d})).$$

In this paper, we never use $\text{Hom}(d, M)$ for the above $k$-linear map $C(d, M)$ induced by $d$.

**Lemma 2.3** [11]. We have the following properties.

---

†The $\delta$-vector is the same one defined in [11], but is the negative of the g-vector defined in [16].
(1) Any exact sequence \(0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0\) in \(\text{rep } A\) gives the long exact sequence:

\[
0 \rightarrow \text{Hom}(d, L) \rightarrow \text{Hom}(d, M) \rightarrow \text{Hom}(d, N) \rightarrow \text{E}(d, L) \rightarrow \text{E}(d, M) \rightarrow \text{E}(d, N) \rightarrow 0.
\]

(2) \(\text{E}(d, M) \supseteq \text{Ext}^1(\text{Coker}(d), M)\) for any \(d\) and \(M\).

(3) \(\text{E}(d, M) \cong \text{Hom}(M, \text{Ker}(\nu d))^\ast\) for any \(d\) and \(M\).

Readers can easily formulate the analogous statements for \(\hat{E}\).

Sometimes, it is convenient to view presentations as elements in the homotopy category \(K^b(\text{proj } - A)\) of bounded complexes of projective representations of \(A\). Our convention is that \(P_{-}\) sits in degree \(-1\) and \(P_{+}\) sits in degree 0. Then the \(\delta\)-vector of a presentation is just the corresponding element in the Grothendieck group of \(K^b(\text{proj } - A)\). Given any two presentation \(d_1\) and \(d_2\), we also define

\[
\text{E}(d_1, d_2) = \text{Hom}_{K^b(\text{proj } - A)}(d_1, d_2[1]).
\]

It turns out [11] that

\[
\text{E}(d_1, d_2) = \text{E}(d_1, \text{Coker}(d_2)) \quad \text{and} \quad \hat{E}(d_1, d_2) = \hat{E}(\text{Ker}(d_1), d_2).
\]

For any representation \(M\), we denote by \(d_M\) (respectively, \(\hat{d}_M\)) its minimal projective (respectively, injective) presentation. Given any two representation \(M\) and \(N\), we define

\[
\text{E}(M, N) := \text{E}(d_M, N) \quad \text{and} \quad \hat{E}(M, N) := \hat{E}(M, \hat{d}_N).
\]

### 2.2 General presentations

By a general presentation in \(\text{Hom}(P_{-}, P_{+})\), we mean a presentation in some open (and thus dense) subset of \(\text{Hom}(P_{-}, P_{+})\). Any \(\delta \in \mathbb{Z}^{Q_0}\) can be written as \(\delta = \delta_+ - \delta_-\) where \(\delta_+ = \max(\delta, 0)\) and \(\delta_- = \max(-\delta, 0)\). We put

\[
\text{PHom}(\delta) := \text{Hom}(P(\delta_-), P(\delta_+)).
\]

It is well known that a general presentation in \(\text{Hom}(P(\beta_-), P(\beta_+))\) is homotopy equivalent to a general presentation in \(\text{PHom}(\beta_+ - \beta_-)\) for any \(\beta_-, \beta_+ \in \mathbb{Z}^{Q_0}_{\geq 0}\).

There is some open subset \(U\) of \(\text{PHom}(\delta)\) such that for any \(d \in U\), we have

(1) \(\text{Hom}(d, M)\) has constant dimension for a fixed \(M \in \text{rep } A\),

(2) \(\text{Coker}(d)\) has constant subrepresentation lattice.

Note that (1) implies that \(\text{E}(d, M)\) has constant dimension as well. It follows from (1) or (2) that \(\text{Coker}(d)\) has a constant dimension vector \(\alpha\). In fact, we can ask \(\text{Coker}(d)\) lie in a fixed irreducible component of \(\text{rep}_\alpha(A)\) (see [1, Section 2]). We denote by \(\text{Coker}(\delta)\) the cokernel of a general presentation in \(\text{PHom}(\delta)\). Similarly, we can define the injective presentation space \(\text{IHom}(\delta)\), and denote by \(\text{Ker}(\delta)\) the kernel of a general element there.
Definition 2.4. We denote by \( \text{hom}(\delta, M) \) and \( e(\delta, M) \) the value of \( \text{hom}(d, M) \) and \( e(d, M) \) for a general presentation \( d \in \text{PHom}(\delta) \). \( \text{hom}(M, \delta) \) and \( \tilde{e}(M, \delta) \) are defined analogously.

Recall the isomorphism \( \text{Hom}(P_i, P_j) \cong \text{Hom}(I_i, I_j) = \text{Hom}(vP_i, vP_j) \). If \( d \) is general in \( \text{PHom}(\delta) \), then \( \nu d \) is general in \( \text{IHom}(-\delta) \). We obtain the obvious relations

\[
\text{hom}(\delta, M) = \tilde{e}(M, -\delta) \quad \text{and} \quad e(\delta, M) = \text{hom}(M, -\delta).
\]

(2.2)

Definition 2.5 [1]. A weight vector \( \delta \in \mathbb{Z} Q_0 \) is called indecomposable if a general presentation in \( \text{PHom}(\delta) \) is indecomposable. We call \( \delta = \bigoplus_{i=1}^{s} \delta_i \) a decomposition of \( \delta \) if a general element \( d \) in \( \text{PHom}(\delta) \) decomposes into \( \bigoplus_{i=1}^{s} d_i \) with each \( d_i \in \text{PHom}(\delta_i) \). It is called the canonical decomposition of \( \delta \) if each \( d_i \) is indecomposable.

The function \( \dim E(-, -) \) is upper semicontinuous on \( \text{PHom}(\delta_1) \times \text{PHom}(\delta_2) \). We denote by \( e(\delta_1, \delta_2) \) the minimal value of \( \dim E(-, -) \) on \( \text{PHom}(\delta_1) \times \text{PHom}(\delta_2) \). One of the motivation of introducing the space \( E \) is the following theorem.

Theorem 2.6 [11, Theorem 4.4]. \( \delta = \delta_1 \oplus \delta_2 \oplus \cdots \oplus \delta_s \) is the canonical decomposition of \( \delta \) if and only if \( \delta_1, \ldots, \delta_s \) are indecomposable, and \( e(\delta_i, \delta_j) = 0 \) for \( i \neq j \).

Definition 2.7 [11]. A weight vector \( \delta \in \mathbb{Z} Q_0 \) is called real if \( e(d, d) = 0 \) for some \( d \in \text{PHom}(\delta) \); is called tame if it is not real but \( e(\delta, \delta) = 0 \); is called wild if \( e(\delta, \delta) > 0 \).

If an indecomposable \( \delta \) is real or tame, then by Theorem 2.6, the canonical decomposition of \( m\delta \) is a sum of \( m \) copies of \( \delta \) for any \( m \in \mathbb{N} \). In particular, \( \delta \) is indivisible.

2.3 \ E-rigid presentations

The group \( \text{Aut}(P_-) \times \text{Aut}(P_+) \) acts on \( \text{Hom}(P_-, P_+) \) by \((g_-, g_+)d = g_+d g_-^{-1} \). The space \( E(d, d) \) can be interpreted as the normal space to the orbit of \( d \) in \( \text{Hom}(P_-, P_+) \).

Definition 2.8. A presentation \( d \) is called rigid \(^1\) if \( E(d, d) = 0 \) (\( \tilde{E}(\tilde{d}, \tilde{d}) = 0 \) for an injective presentation \( \tilde{d} \)). A representation \( M \) is called \( E \)-rigid \(^2\) (respectively, \( \tilde{E} \)-rigid) if \( E(M, M) = 0 \) (respectively, \( \tilde{E}(M, M) = 0 \)).

So, the orbit of such a presentation is dense in its ambient space. In this case, the weight vector of \( \tilde{d} \) must be real. The dual of Lemma 2.3.(3) says that \( \tilde{E}(M, \tilde{d}) \cong \text{Hom} (\text{Coker}(\nu^{-1} \tilde{d}), M)^* \). So, we have that

\[
E(d, d) \cong \text{Hom}(\text{Coker}(d), \text{Ker}(\nu d))^* \cong \tilde{E}(\nu d, \nu d).
\]

(2.3)

This implies that \( d \) is rigid if and only if \( \nu d \) is rigid.

\(^1\) In [32], this is called presilting, which is defined for any complex in \( K^b(\text{proj} - A) \).

\(^2\) Due to the equation \( E(M, M) = \text{Hom}(M, \tau M)^* \), it is also called \( \tau \)-rigid in [1].
One can always complete a rigid presentation $d$ to a maximal rigid one $\tilde{d}$, in the sense that $E(\tilde{d} \oplus d', \tilde{d} \oplus d') \neq 0$ for any indecomposable $d' \not\in \text{ind}(d)$. Here we denote by $\text{ind}(d)$ the set of nonisomorphic indecomposable direct summands of $d$. The maximal rigid presentation can be characterized as follows.

**Theorem 2.9** [11, Theorem 5.4], [1]. The following are equivalent for a rigid presentation $d$.

1. $d$ is maximal rigid.
2. $|\text{ind}(d)| = |Q_0|.
3. $\text{ind}(d)$ generates $K^b(\text{proj } A)$.

**Definition 2.10.** If $d$ is maximal rigid, then we call $\text{ind}(d)$ a cluster of presentations. We also call the weight vectors of presentations in $\text{ind}(d)$ a cluster of $\delta$-vectors. A maximal set of presentations $\{d_1, \ldots, d_r\}$ satisfying $e(d_i, d_j) = 0$ for $i \neq j$ is called a generalized cluster of presentations. Their weights $\{\delta_1, \ldots, \delta_r\}$ are also called a generalized cluster of $\delta$-vectors.

**Proposition 2.11** [11, Proposition 5.7]. If a rigid presentation $d$ is almost complete, that is, $|\text{ind}(d)| = |Q_0| - 1$, then it has exactly two complements $d_-$ and $d_+$. They are related by the triangle $d_+ \rightarrow d' \rightarrow d'' \rightarrow d_- \rightarrow d'_+ [1]$, where $e = \dim E(d_-, d_+)$. Moreover, both $d' \oplus d_-$ and $d'' \oplus d_+$ are rigid and $E(d_+, d_-) = E(d_-, d'_+) = E(d_-, d_-) = 0$. In particular, $e = 1$ if and only if $d' = d''$ belongs to the subcategory generated by $\text{ind}(d)$.

**Definition 2.12.** We call the above pair $(d_-, d_+)$ an exchange pair of presentations. If $e = 1$, the exchange pair is called regular. The two clusters $\{d_\pm\} \cup \text{ind}(d)$ and $\{d_\pm\} \cup \text{ind}(d)$ are called adjacent to each other. A cluster $\{d_1, \ldots, d_n\}$ is called regular if each $\{d_i, d'_i\}$ can be ordered to be a regular exchange pair, where $d'_i$ appears in the adjacent cluster $(d_1, \ldots, d'_i, \ldots, d_n)$.

An open problem posed in [11] is how to characterize algebras for which all clusters are regular.

### 3 TROPICAL $F$-POLYNOMIALS AND GENERAL PRESENTATIONS

#### 3.1 Tropical $F$-polynomials

We keep assuming that $A = kQ/I$. Throughout we identify the Grothendieck group $K_0(\text{rep } A)$ with $\mathbb{Z}^{Q_0}$. Let $M$ be a finite-dimensional representation of $A$.

**Definition 3.1.** The tropical $F$-polynomial $f_M$ of a representation $M$ is the function $(\mathbb{Z}^{Q_0})^* \rightarrow \mathbb{Z}_{\geq 0}$ defined by

$$\delta \mapsto \max_{L \preceq M} \delta(\text{dim}L).$$

The dual tropical $F$-polynomial $\tilde{f}_M$ of a representation $M$ is the function $(\mathbb{Z}^{Q_0})^* \rightarrow \mathbb{Z}_{\geq 0}$ defined by

$$\delta \mapsto \max_{M \rightarrow N} \delta(\text{dim}N).$$
Clearly, $f_M$ and $\hat{f}_M$ are related by $f_M(\delta) - \hat{f}_M(-\delta) = \delta(\dim M)$. The definition of $f_M$ is motivated by the $F$-polynomial of $M$ defined in [16]

$$F_M(y) = \sum_{\gamma} \chi(\text{Gr}_\gamma(M)) y^\gamma,$$

where $\text{Gr}_\gamma(M)$ is the variety parametrizing the $\gamma$-dimensional subrepresentations of $M$, and $\chi(-)$ is the topological Euler characteristic. In general, $\chi(\text{Gr}_\gamma(M))$ may not be a positive number. If the $F$-polynomial $F_M$ has nonnegative coefficients, then the tropical $F$-polynomial $f_M$ is the usual tropicalization of $F_M$.

**Definition 3.2.** The *Newton polytope* $N(M)$ of a representation $M$ is the convex hull of

$$\{ \dim L \mid L \hookrightarrow M \}$$

in $\mathbb{R}^{Q_0}$. The *dual* Newton polytope $\hat{N}(M)$ of a representation $M$ is the convex hull of

$$\{ \dim N \mid M \twoheadrightarrow N \}$$

in $\mathbb{R}^{Q_0}$.

**Remark 3.3.** We have two remarks.
(1) The tropical $F$-polynomial $f_M$ is completed determined by the Newton polytope of $M$.
(2) It is shown in [19] that the Newton polytope of $M$ is the same as the usual Newton polytope of the polynomial $F_M$.

**Lemma 3.4** [16, Proposition 3.2]. $F_{M \oplus N} = F_M F_N$ for any two representations $M$ and $N$. In particular, we have that $f_{M \oplus N} = f_M + f_N$.

When paired with a dimension vector or evaluated by some $f_M$, a weight $\delta$ is viewed as an element in $(\mathbb{Z}^{Q_0})^*$ via the usual dot product. It follows from (2.1) that for any presentation $d$ of weight $\delta$,

$$\delta(\dim M) = \text{hom}(d, M) - \epsilon(d, M);$$

(3.1)

$$\hat{\delta}(\dim M) = \text{hom}(M, \hat{d}) - \hat{\epsilon}(M, \hat{d}).$$

(3.2)

Let $M \rightarrow N$ be a homomorphism. We fix some general presentation $d$ of weight $\delta$. Throughout we use the notation $\text{Hom}(\delta, M) \rightarrow \text{Hom}(\delta, N)$ for the induced map $\text{Hom}(d, M) \rightarrow \text{Hom}(d, N)$. The notation $\text{E}(\delta, M) \rightarrow \text{E}(\delta, N)$ has the similar meaning.

**Lemma 3.5.** We have the following inequalities for any representation $M$ and any $\delta \in \mathbb{Z}^{Q_0}$:

$$f_M(\delta) \leq \text{hom}(\delta, M), \quad \hat{f}_M(-\delta) \leq \epsilon(\delta, M);$$

(3.1)

$$\hat{f}_M(\hat{\delta}) \leq \text{hom}(M, \hat{\delta}), \quad f_M(-\hat{\delta}) \leq \hat{\epsilon}(M, \hat{\delta}).$$

(3.2)
Proof. Since $\text{Hom}(\delta, L) \hookrightarrow \text{Hom}(\delta, M)$ for any subrepresentation $L$ of $M$, we have that $\delta(\text{dim}L) \leq \text{hom}(\delta, L) \leq \text{hom}(\delta, M)$. Hence, $f_M(\delta) \leq \text{hom}(\delta, M)$. Then $\tilde{f}_M(-\delta) \leq e(\delta, M)$ follows from (3.1). The other half is proved similarly.

Here is the main result of this section.

**Theorem 3.6.** For any representation $M$ and any $\delta \in \mathbb{Z}^{Q_0}$, there is some $N \in \mathbb{N}$ such that

$$f_M(n\delta) = \text{hom}(n\delta, M), \quad \tilde{f}_M(-n\delta) = e(n\delta, M)$$

for any $n \geq N$. Similarly, for any representation $M$ and any $\delta \in \mathbb{Z}^{Q_0}$, there is some $\tilde{N} \in \mathbb{N}$ such that

$$\tilde{f}_M(n\delta) = \text{hom}(n\delta, M), \quad f_M(-n\delta) = \tilde{e}(n\delta, M)$$

for any $n \geq \tilde{N}$. Moreover, $n$ can be replaced by $kn$ for any $k \in \mathbb{N}$. If $m$ is the minimum of all such $n$, then $m\delta$ cannot be decomposed as $m\delta = k\delta \oplus (m-k)\delta$ for any $k$. In particular, if $\delta$ is not wild, then $m = 1$.

**Example 3.7.** We remark that $N$ or $\tilde{N}$ may not always chosen to be 1. Let $Q$ be the three-arrow Kronecker quiver $\bullet \xrightarrow{1} \xrightarrow{0} \xleftarrow{1} \bullet$. Consider $\delta = (1, -1)$ and $M \in \text{rep}_{(3,3)}(Q)$ given by

$$M(a) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M(b) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M(c) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$ 

Then one can easily check that $f_M(\delta) = \text{hom}(n\delta, M) = 0$ for any $n \geq 2$ but $\text{hom}(\delta, M) = 1$.

Before giving a proof, we hasten to mention an interesting corollary (Corollary 3.10). It says that the evaluation of $f_M$ is related to the asymptotic $\text{hom}(a\delta, M)$ as $a$ increases, generalizing a result of W. Crawley–Boevey on quivers without relations [10].

**Lemma 3.8.** If $\delta = \delta_1 + \delta_2$, then $\text{hom}(\delta, M) \leq \text{hom}(\delta_1, M) + \text{hom}(\delta_2, M)$ for any $M$. If $\delta = \delta_1 \oplus \delta_2$, then $\text{hom}(\delta, M) = \text{hom}(\delta_1, M) + \text{hom}(\delta_2, M)$ for any $M$. Moreover, all $\text{hom}$ can be replaced by $e$.

**Proof.** Let $d_i$ be a general presentation of weight $\delta_i$ ($i = 1, 2$). Then the weight of $d_1 \oplus d_2$ is $\delta$ and $\text{hom}(d_1 \oplus d_2, M) = \text{hom}(d_1, M) + \text{hom}(d_2, M) = \text{hom}(\delta_1, M) + \text{hom}(\delta_2, M)$. By the lower semicontinuity, we have that $\text{hom}(\delta, M) \leq \text{hom}(d_1 \oplus d_2, M)$. If $\delta = \delta_1 \oplus \delta_2$, then we can assume $d_1 \oplus d_2$ is general, so $\text{hom}(\delta, M) = \text{hom}(\delta_1, M) + \text{hom}(\delta_2, M)$.

**Question 3.9.** Is it true that if $\text{hom}(\delta, M) = \text{hom}(\delta_1, M) + \text{hom}(\delta_2, M)$ for any $M$, then $\delta = \delta_1 \oplus \delta_2$?

**Corollary 3.10.** The following limits exist and we have the equalities:

$$\lim_{a \to \infty} \frac{1}{a} \text{hom}(a\delta, M) = f_M(\delta) \quad \lim_{a \to \infty} \frac{1}{a} e(a\delta, M) = \tilde{f}_M(-\delta);$$

$$\lim_{a \to \infty} \frac{1}{a} \text{hom}(M, a\delta) = \tilde{f}_M(\delta) \quad \lim_{a \to \infty} \frac{1}{a} \tilde{e}(M, a\delta) = f_M(-\delta).$$
Proof. Let $n$ be the number such that $f_M(n\delta) = \text{hom}(n\delta, M)$ as in Theorem 3.6. For any $a \in \mathbb{N}$, we can write $a = qn + r$ with $q, r \in \mathbb{Z}_{\geq 0}$ and $r < n$. Then by Lemma 3.8,

$$\frac{1}{a} \text{hom}(a\delta, M) \leq \frac{1}{qn} (q \text{hom}(n\delta, M) + r \text{hom}(\delta, M))$$

$$\leq f_M(\delta) + \frac{1}{q} \text{hom}(\delta, M),$$

which tends to $f_M(\delta)$ as $a \to \infty$. On the other hand, we have by Lemma 3.5 that

$$\frac{1}{a} \text{hom}(a\delta, M) \geq \frac{1}{a} f_M(a\delta) = f_M(\delta).$$

Hence, the limit exists and equals to $f_M(\delta)$. The others can be proved similarly. □

The proof of Theorem 3.6 requires some preparation.

### 3.2 Stability and semi-invariants

A. King introduced Mumford’s GIT into the setting of quiver representation theory [33]. Recall that any weight $\delta \in \mathbb{Z}^{Q_0}$ gives a multiplicative characters $\chi_{\delta}$ of $\text{GL}_\alpha := \prod_{v \in Q_0} \text{GL}_{\alpha(v)}$:

$$(g(v))_{v \in Q_0} \mapsto \prod_{v \in Q_0} (\det g(v))^{\delta(v)}.$$

A semi-invariant function of weight $\delta$ is an element in

$$\text{SI}_{\alpha}(A)_{\delta} := \{ s \in k[\text{rep}_\alpha(A)] \mid g(s) = \chi_{\delta}(g)s, \forall g \in \text{GL}_\alpha \}.$$

The graded semi-invariant algebra associated to $\delta$ is

$$\text{SI}_\delta^\alpha(A) := \bigoplus_{n \geq 0} \text{SI}_{\alpha}(A)_{n\delta}.$$

A representation $M$ is called $\delta$-semistable if there is some $s \in \text{SI}_{\alpha}(A)$ such that $s(M) = 0$.

**Lemma 3.11** [33, Proposition 3.1]. A representation $M$ is $\delta$-semistable (respectively, $\delta$-stable) if and only if $\delta(\dim M) = 0$ and $\delta(\dim L) \leq 0$ (respectively, $\delta(\dim L) < 0$) for any nontrivial subrepresentation $L$ of $M$.

For any projective presentation $d$ of weight $\delta$ such that $\delta(\alpha) = 0$, Schofield constructed the following semi-invariant function of weight $\delta$ on $\text{rep}_\alpha(A)$. We apply the functor $\text{Hom}(\cdot, M)$ to $d$ for $M \in \text{rep}_\alpha(A)$

$$\text{Hom}(P_+, M) \xrightarrow{C(d, M)} \text{Hom}_Q(P_-, M).$$

Since $\delta(\alpha) = 0$, $C(d, M)$ is a square matrix. We define

$$c_d(M) := \det C(d, M).$$
**Theorem 3.12** [12, Theorem 1],[13, 17, 41]. The space $\text{SI}_\alpha(A)_\delta$ is spanned by semi-invariants of the form $c_d$ where $d$ has weight $\delta$.

**Lemma 3.13.** A representation $M \in \text{rep}_\alpha(A)$ is $\delta$-semistable if and only if

$$\text{hom}(n\delta, M) = \delta(\alpha) = 0 \text{ for some } n \in \mathbb{N}.$$  

Moreover, $n$ can be replaced by $kn$ for any $k \in \mathbb{N}$. If $m$ is the minimum of all such $n$, then $m\delta$ cannot be decomposed as $m\delta = k\delta \oplus (m-k)\delta$ for any $k$. In particular, if $\delta$ is not wild, then $m = 1$.

**Proof.** By Theorem 3.12, $M$ is $\delta$-semistable if and only if $c_d(M) \neq 0$ for some $d$ of weight $n\delta$ for some $n \in \mathbb{N}$. This happens if and only if the matrix $C(d, M)$ is invertible, which is equivalent to that $\text{hom}(d, M) = e(d, M) = 0$. The condition that $\text{hom}(d, M) = e(d, M) = 0$ for some $d$ is clearly equivalent to that $\text{hom}(n\delta, M) = \delta(\alpha) = 0$ for some $n \in \mathbb{N}$.

The moreover part follows easily from Lemma 3.8. If $\delta$ is not wild, then $n\delta = \delta \oplus \cdots \oplus \delta$ by Theorem 2.6. □

We remark that the first statement of [8, Theorem 1.1] follows from a special case of this lemma when the $\delta$ is real or equivalently the general presentation of weight $\delta$ is rigid.

### 3.3 Proof of Theorem 3.6

We need to review the notion of one-point extension of $A$. Let $M$ be a right $A$-module. Treating $M$ as a $k$-$A$-bimodule, the triangular algebra $A[M] := \begin{pmatrix} A & 0 \\ M & k \end{pmatrix}$ is called (trivial) one-point extension of $A$ by $M$. There is an obvious dual notion of one-point coextension $A[M^*] := \begin{pmatrix} k & 0 \\ M^* & A \end{pmatrix}$.

Suppose that $M \in \text{rep} A$ is presented by $P(\beta_-) \xrightarrow{d} P(\beta_+) \rightarrow M \rightarrow 0$. Then the algebra $\widetilde{A} = A[M]$ can be presented by a new quiver $Q(M)$, which is obtained from $Q$ by adjoining a new vertex $\square$ and $\beta_+(v)$ new arrows from $\square$ to the vertex $v \in Q$. The relations are clearly given by the presentation $d$. In reality, the presentation is always chosen to be minimal. The one-point coextension $A[M^*]$ can be similarly described using injective presentation of $M$. By convention, the newly adjoined vertex is denoted by $\square$.† By construction, we have the following exact sequences:

\[
\begin{align*}
0 \rightarrow (M, 0) \rightarrow P_{\square} \rightarrow S_{\square} \rightarrow 0 & \quad \text{for one-point extensions}, \quad (3.3) \\
0 \rightarrow S_{\square} \rightarrow I_{\square} \rightarrow (0, M) \rightarrow 0 & \quad \text{for one-point coextensions.} \quad (3.4)
\end{align*}
\]

Let $B$ either be the algebra $A[M]$ or the algebra $A[M^*]$. We have a restriction functor $\text{res}_A : \text{rep} B \rightarrow \text{rep} A$ sending $M$ to $M e$ where $e = 1 - e_\pm$ and $e_\pm$ is the idempotent corresponding to the vertex $\square$ or $\square$. The restriction functor has two induction functors $T_B := - \otimes_A e B$ and $L_B := \text{Hom}_A(B e, -)$.

**Lemma 3.14** [2, Theorem I.6.8]. $T_B$ (respectively, $L_B$) is left (respectively, right) adjoint to $\text{res}_A$. Moreover, they satisfy $\text{res}_A T_B \cong \text{Id}_{\text{rep} A} \cong \text{res}_A L_B$.

†The notation suggests that $\square$ and $\square$ should be visualized as the frozen vertices with label $-$ and $+$.  

---
Corollary 3.15. If $d$ is a general presentation of weight $\delta$, then $T_B(d)$ is a general presentation of weight $(\delta,0)$ or $(0,\delta)$ (depending on $B = A[M]$ or $A[M^*]$). Moreover, $\text{Coker}(T_B(d)) = T_B(\text{Coker}(d))$.

Proof. The first statement follows from the equality:

$$\text{Hom}_B(T_B(V), T_B(W)) \cong \text{Hom}_A(V, \text{res}_A T_B(W)) \cong \text{Hom}_A(V, W)$$

for any $V, W \in \text{rep} A$. The second statement is due to the right exactness of $T_B$. □

Let $M$ be a representation of $A$. We extend $A$ by $M$ and obtain the algebra $A^- := A[M]$. Then we coextend $A[M]$ by the indecomposable projective representation $P_\square$ of $A[M]$, and obtain the algebra $(A[M])[P^\square_\square]$. We denote $A^\pm := (A[M])[P^\square_\square] = A^-[P^\square_\square]$. Note that

$$A^\pm = \left[ \begin{array}{ccc} k & 0 & 0 \\ M^* & A & 0 \\ k & M & k \end{array} \right].$$

Throughout we use $P_\square$ to denote the above indecomposable projective representation of $A[M]$ rather than the indecomposable projective representation of $A^\pm$.

Lemma 3.16. We have that

$$\text{Hom}_{A^\pm}(T_{A^\pm}(T_{A^-}(V)), I_\square) \cong \text{Hom}_A(V, M).$$

Proof. We have that $\text{res}_A^-(I_\square) = P_\square$ and $\text{res}_A^-(P_\square) = M$. So, apply Lemma 3.14 twice, and we get

$$\text{Hom}_{A^\pm}(T_{A^\pm}(T_{A^-}(V)), I_\square) \cong \text{Hom}_{A^-}(T_{A^-}(V), P_\square) \cong \text{Hom}_A(V, M).$$ □

Definition 3.17. A vertex $v$ is called \textit{maximal} in a representation $M$ if $\dim M(v) = 1$ and all strict subrepresentations of $M$ are not supported on $v$.

Let $f_\square$ (respectively, $f_\square$) be the tropical $F$-polynomial of $I_\square$ (respectively, $P_\square$).

Lemma 3.18. $\square$ is a maximal vertex of $I_\square \in \text{rep} A^\pm$. Moreover, we have that

$$f_\square((\delta, \delta_-)) = \max(f_M(\delta), \delta(\dim M) + \delta_-),$$

$$f_\square((\delta_+, \delta_-)) = \max(0, f_\square((\delta, \delta_-)) + \delta_+).$$

Proof. Recall that we have two exact sequences

$$0 \to (M, 0) \to P_\square \to S_\square \to 0,$$

$$0 \to S_\square \to I_\square \to (0, P_\square) \to 0.$$

Since the 1-dimensional subspace of $P_\square$ at vertex $\square$ generates $P_\square$, we see that a subrepresentation of $P_\square$ is either a subrepresentation of $M$ or $P_\square$ itself. Hence,

$$f_\square((\delta, \delta_-)) = \max(f_M(\delta), \delta(\dim M) + \delta_-).$$
Next, whenever there is a subrepresentation $S$ of $P_{\square}$, we have a subrepresentation $(k, S)$ of $I_{\square}$. Conversely, any nonzero subrepresentation of $I_{\square}$ must be supported on $\square$. Hence,

$$f_{\square}(\delta_+ \delta \delta_-) = \max(0, f_{\square}(\delta, \delta) + \delta_+).$$

\begin{lemma}
Suppose that a representation $M$ contains a maximal vertex $v$. Then $f_M(\delta) = 0$ if and only if $\text{hom}(n\delta, M) = 0$ for some $n \in \mathbb{N}$.
\end{lemma}

\begin{proof}
If $\text{hom}(n\delta, M) = 0$, then $\text{hom}(n\delta, L) = 0$, and thus, $\delta(\dim L) \leq 0$ for all subrepresentations $L$ of $M$. Conversely, suppose that $\text{PHom}(\delta) = \text{Hom}(P_-, P_+)$. We add $c = -\delta(\dim M) \geq 0$ copies of $P_v$'s to $P_+$ so that a presentation in $\text{Hom}(P_-, P_+ \oplus cP_v)$ has weight $\delta' = \delta + ce_v$. It satisfies that $\delta'(\dim M) = 0$ and $\delta'(\dim L) = \delta(\dim L) \leq 0$ for all subrepresentations $L \subsetneq M$. By King’s criterion (Lemma 3.11), we see that $M$ is $\delta'$-semistable, and thus, $\text{hom}(n\delta', M) = 0$ for some $n \in \mathbb{N}$ by Lemma 3.13. Now a general presentation $nP_- \xrightarrow{(d, d')} nP_+ \oplus ncP_v$ must have $d$ general in $\text{Hom}(nP_-,nP_+)$. Hence, $\text{hom}(n\delta', M) = 0$ implies $\text{hom}(n\delta, M) = 0$.
\end{proof}

\begin{remark}
This lemma was proved in [20, Lemma 6.6] for $A$ being a Jacobian algebra. The argument actually works for any finite-dimensional algebras. Unfortunately, there is a gap in the proof where we assume that $n$ can always be 1. However, each representation $T_{i,j}$ in [20, Section 6] is negative reachable. So, this would not affect the correctness of the main results there due to Theorem 3.22 below.
\end{remark}

\begin{proof}[Proof of Theorem 3.6 (1)] Let $d$ be general presentation of any weight $\delta$ in $\text{rep} A$ and $V = \text{Coker}(d)$. By Corollary 3.15, $T_{A^+}(T_{A^+}(V)) = \text{Coker}(T_{A^+}(T_{A^+}(d)))$ and $T_{A^+}(T_{A^-}(d))$ is a general presentation of weight $\tilde{\delta} = (0, \delta, 0)$. By Lemma 3.16, we have that

$$\text{hom}(\tilde{\delta}, I_{\square}) = \text{hom}(\delta, M).$$

Coker($\tilde{\delta}$) may not be supported on the original quiver $Q$. We are going to put an appropriate negative weight $\delta_-$ on the vertex $\square$. By Lemma 3.18, we have that

$$f_{\square}(\delta_+ \delta) = \max(0, f_{\square}(\delta, 0) + \delta_+) = \max(0, f_M(\delta) + \delta_+).$$

Let $\delta_+ = -f_M(\delta)$, then $f_{\square}(\delta_+ \delta) = 0$. By Lemma 3.19, $\text{hom}(n(\delta_+, \delta), I_{\square}) = 0$ for some $n \in \mathbb{N}$. But $\text{hom}(n\delta_+, \delta, 0, I_{\square}) = 0$ implies that $\text{hom}(0, n\delta, 0, I_{\square}) \leq -n\delta_+$. Hence, we have

$$f_M(n\delta) \geq \text{hom}(0, n\delta, 0, I_{\square}) = \text{hom}(n\delta, M).$$

We get the equality by Lemma 3.5. Then the equality $\tilde{j}_M(\gamma) = e(n\delta, M)$ follows from the relation (3.1).

The other half can be proved by the dual argument. The moreover part follows from the corresponding part in Lemma 3.13.
\end{proof}

\section{The case of quivers with potentials}

We refer the readers to the original papers [15, 16] for the theory of the quivers with potential. In this subsection, $(Q, P)$ is a quiver with potential such that its Jacobian algebra $A = J(Q, P)$ is
finite-dimensional. The key notion introduced in [15] is the \textit{mutation} of a quiver of potential and its decorated representations.

A vertex is called \textit{admissible} if it is not involved in any oriented cycle of length \( \leq 2 \). For each admissible vertex \( u \in Q_0 \), there is an operation \( \mu_u \), which yields a new quiver with potential \( \mu_u(Q, P) \). A \textit{decorated} representation \( \mathcal{M} = (M, V) \) consists of two parts: \( M \) is a usual representation and the decorated part \( V \) is a \( k^{Q_0} \)-module. A usual representation \( M \) can be regarded as a decorated representation \( (M, 0) \). For each (decorated) representation \( \mathcal{M} \), there is a mutated representation \( \mu_u(\mathcal{M}) \) of \( \mu_u(Q, P) \). For any weight vector \( \delta \in \mathbb{Z}^{Q_0} \), there is a mutated weight vector \( \mu_u(\delta) \) defined by [16, (2.11)]. For a decorated representation \( \mathcal{M} = (M, V) \), its tropical \( F \)-polynomial and related functors, such as \( \text{Hom}(\cdot, M) \) and \( \text{E}(\cdot, M) \), are all defined to be those of \( M \).

**Lemma 3.21.** Let \( \mu \) be a sequence of mutations (at admissible vertices). We denote \( M' := \mu(M) \) and \( \delta' := \mu(\delta) \). We have the following relation for any representation \( M \) and \( \delta \in \mathbb{Z}^{Q_0} \):

\[
\hat{f}_{M'}(\delta') - f_M(\delta) = \text{hom}(\delta', M') - \text{hom}(\delta, M);
\]
\[
\hat{f}_{M'}(-\delta') - \hat{f}_M(-\delta) = e(\delta', M') - e(\delta, M).
\]

There are similar relations for \( \hat{f}_{M}(\hat{\delta}) \), \( \text{hom}(M, \hat{\delta}) \), and \( \hat{e}(M, \hat{\delta}) \).

**Proof.** By induction, it suffices to show for any one-step mutation \( \mu_u \). We knew from [16, Proposition 6.1] that

\[
\text{hom}(\delta', M') - \text{hom}(\delta, M) = [\delta'(u)]_+ [\hat{\delta}_{M'}(u)]_+ - [\delta(u)]_+ [\hat{\delta}_M(u)]_+.
\]

Recall from [16, Lemma 5.2] that

\[
(1 + y_u^{h_u}) F_M(y) = (1 + y_u^{h_u'}) F_{M'}(y'),
\]
where \( h_u = -[\hat{\delta}_M(u)]_+ \) and \( h_u' = -[\hat{\delta}_{M'}(u)]_+ \). Taking the Newton polytope (see the remark after Definition 3.1),\(^\dagger\) we get

\[
- [\delta(u)]_+ [\hat{\delta}_M(u)]_+ + f_M(\delta) = -[\delta'(u)]_+ [\hat{\delta}_{M'}(u)]_+ + f_{M'}(\delta'),
\]
\[
\Rightarrow f_{M'}(\delta') - f_M(\delta) = [\delta'(u)]_+ [\hat{\delta}_{M'}(u)]_+ - [\delta(u)]_+ [\hat{\delta}_M(u)]_+.
\]

Compare with (3.5), and we obtain the first relation. The other relation follows easily from (3.1). \( \square \)

We say a representation \( M \) of \((Q, P)\) \textit{negative reachable} if there is a sequence of mutations \( \mu \) such that \( \mu(M) \) is \textit{negative}, that is, \( \mu(M) \) has only the decorated part.

**Theorem 3.22.** If \( M \) is negative reachable, then for any \( \delta, \delta' \in \mathbb{Z}^{Q_0} \), we have that

\[
f_M(\delta) = \text{hom}(\delta, M), \quad \hat{f}_M(-\delta) = e(\delta, M);
\]
\[
\hat{f}_M(\hat{\delta}) = \text{hom}(M, \hat{\delta}), \quad f_M(-\hat{\delta}) = \hat{e}(M, \hat{\delta}).
\]

\(^\dagger\) Equivalently, we can take tropicalization here but we need the nontrivial positivity results.
Proof. By Lemma 3.21, it is enough to notice that if $M$ is negative, then $f_M(\delta) = \text{hom}(\delta, M) = 0$ and $\tilde{f}_M(\delta) = \text{hom}(M, \delta) = 0$ for any $\delta$ and $\tilde{\delta}$.

**Corollary 3.23.** If $I_i$ is negative reachable, then the dimension vector $\alpha$ of $\text{Coker}(\delta)$ can be computed by

$$\alpha(i) = f_{I_i}(\delta).$$

If $P_i$ is negative reachable, then the dimension vector $\tilde{\alpha}$ of $\text{Ker}(\tilde{\delta})$ can be computed by

$$\tilde{\alpha}(i) = \tilde{f}_{P_i}(\delta).$$

**Example 3.24.** If $A$ is not a Jacobian algebra, then Corollary 3.23 may fail. We plug $M$ and $A = kQ$ in Example 3.7 into the construction of Section 3.3. Then from the proof of Theorem 3.6, we see that $f_{I_i}(\tilde{\delta}) = f_M(\delta) = 0$ but $\text{hom}(\tilde{\delta}, I_i) = \text{hom}(\delta, M) = 1$. We also note that $\tilde{f}_{\text{Coker} \delta}(e_i) = \text{hom}(\delta, I_i) = 1$.

**Question 3.25.** Does the conclusion of Theorem 3.22 still hold if $M = \text{Ker}(\tilde{\delta})$ in (3.6) and $M = \text{Coker}(\delta)$ in (3.7)?

This is certainly true for acyclic quivers due to Schofield’s result (Theorem 1.2). By Lemma 3.21, this is also true for mutation-acyclic QPs. Moreover, in this case, $M = \text{Ker}(\tilde{\delta})$ for some $\tilde{\delta}$ iff $M = \text{Coker}(\delta)$ for some $\delta$.

**Corollary 3.26.** If $(Q, P)$ is mutation-equivalent to an acyclic quiver, then the conclusion of Theorem 3.22 holds for $M = \text{Ker}(\tilde{\delta})$ or $M = \text{Coker}(\delta)$.

### 4 Functors Associated To $\delta$

In this section, we briefly review the two pairs of functors considered in [19].

**Lemma 4.1** [19, Lemma 3.3]. Let $L$ be any subrepresentation of $M$. Then $\delta(\text{dim}L) = \text{hom}(\delta, M)$ if and only if $\text{hom}(\delta, M/L) = e(\delta, L) = 0$. Moreover, if $L'$ is another such subrepresentation, that is, $\delta(\text{dim}L') = \text{hom}(\delta, M)$, then both $L \cap L'$ and $L + L'$ are such subrepresentations.

Let $\mathcal{L}(\delta, M)$ be the set of all subrepresentations $L$ of $M$ such that $\delta(\text{dim}L) = f_M(\delta)$.

**Theorem 4.2** [19, Theorem 3.4]. The set $\mathcal{L}(\delta, M)$ contains a unique minimal element $L_{\text{min}}$ and a unique maximal element $L_{\text{max}}$. Moreover, $L_1/L_0$ is $\delta$-semistable for any $L_0 \subset L_1$ in $\mathcal{L}(\delta, M)$.

**Definition 4.3** [19]. Let $(t_{\overline{\delta}}, f_{\overline{\delta}})$ and $(I_{\overline{\delta}}, f_{I_{\overline{\delta}}})$ be the pairs of functors associated to the torsion pair $(T(\overline{\delta}), F(\overline{\delta}))$ and $(\tilde{T}(\overline{\delta}), \tilde{F}(\overline{\delta}))$, where

$$F(\overline{\delta}) = \{ N \in \text{rep}(A) \mid \text{hom}(n\delta, N) = 0 \text{ for some } n \in \mathbb{N} \},$$

$$\tilde{F}(\overline{\delta}) = \{ L \in \text{rep}(A) \mid e(n\delta, L) = 0 \text{ for some } n \in \mathbb{N} \}.$$
If $\delta$ is not wild, by Lemma 3.8, we can let $n = 1$ in the definition of $F(\delta)$ and $\tilde{F}(\delta)$. In this case, the functors will be denoted by $(t_\delta, f_\delta)$ and $(\tilde{t}_\delta, \tilde{f}_\delta)$.

**Theorem 4.4** [19, Theorem 3.10]. We have that for any representation $M$ and any $\delta \in \mathbb{Z}^{Q_0}$,

- $t_\delta(M) = L_{\min}$ and $f_\delta(M) = M/L_{\min}$;
- $\tilde{t}_\delta(M) = L_{\max}$ and $\tilde{f}_\delta(M) = M/L_{\max}$.

In particular, we have for any $L \in \mathcal{L}(\delta, M)$ that

$\text{Hom}(t_\delta(M), M/L) = 0$ and $\text{Hom}(L, \tilde{f}_\delta(M)) = 0$.

Suppose that $\text{hom}(M, N) = h$. We choose a basis of $\text{Hom}(M, N)$ and take $hM \to N$ to be the canonical map with respect to this basis. We call this map a *universal homomorphism* from $\text{add}(M)$ to $N$.

**Corollary 4.5** [19, Corollary 3.13]. Suppose that $d$ is a rigid presentation with weight $\delta$. Then $t_\delta(M)$ is the image of the universal homomorphism $h \text{Coker}(d) \to M$, while $\tilde{t}_\delta(M)$ is the kernel of the universal homomorphism $M \to e \text{Ker}(\nu d)$, where $h = \text{hom}(\delta, M)$ and $e = e(\delta, M)$.

## 5 | NEUTON POLYTOPES OF REPRESENTATIONS

### 5.1 | A presentation of $N(M)$

In this subsection, we mostly follow [3, Section 4.2]. Let $V$ be a $\mathbb{R}$-vector space. To a nonempty compact convex subset $P$ of $V$, we associate its support function $\psi_P : V^* \to \mathbb{R}$, which maps a linear function $f \in V^*$ to the maximal value $f$ takes on $P$. Then $\psi_P$ is a sublinear function on $V^*$. One can recover $P$ from the datum of $\psi_P$ by the Hahn–Banach theorem [28]

$P = \{v \in V \mid \alpha(v) \leq \psi_P(\alpha), \ \forall \alpha \in V^*\},$

and the map $P \mapsto \psi_P$ is a bijection from the set of all nonempty compact convex subsets of $V$ onto the set of all sublinear functions on $V^*$.

**Theorem 5.1.** The Newton polytope $N(M)$ is defined by

$\{\gamma \in \mathbb{R}^{Q_0} \mid \delta(\gamma) \leq \text{hom}(\delta, M), \ \forall \delta \in \mathbb{Z}^{Q_0}\}$.

The dual Newton polytope $\tilde{N}(M)$ is defined by

$\{\gamma \in \mathbb{R}^{Q_0} \mid \tilde{\delta}(\gamma) \leq \text{hom}(M, \tilde{\delta}), \ \forall \tilde{\delta} \in \mathbb{Z}^{Q_0}\}$.

**Proof.** In our setting of $P = N(M)$, the support function is given by $\delta \mapsto f_M(\delta)$. So, $N(M)$ is defined by

$\{\gamma \in \mathbb{R}^{Q_0} \mid \delta(\gamma) \leq f_M(\delta), \ \forall \delta \in \mathbb{R}^{Q_0}\}$.

We know *in priori* that $N(M)$ have integral vertices, so its normal vector can be chosen as integral as well. It is enough to consider all $\delta \in \mathbb{Z}^{Q_0}$. In general, we have that $f_M(\delta) \leq \text{hom}(\delta, M)$. But
We know *a priori* that the Newton polytope has a (finite) hyperplane representation. In fact, we only need those δ-vectors that are outer normal vectors of N(M). It is an interesting problem to find a finite set of δ-vectors determining the Newton polytope. This is achieved for general representations of any acyclic quiver in [19].

5.2 | Facets and normals

Recall that a δ-vector is called indecomposable if a general presentation in PHom(δ) is indecomposable.

**Lemma 5.2.** Suppose that \{δ_1, ..., δ_r\} satisfies \(e(δ_i, δ_j) = 0\) for \(i \neq j\). Then

\[
f_M \left( \sum_i c_i δ_i \right) = \sum_i c_i f_M(δ_i).
\]

**Proof.** By Lemma 3.8, we have that \(e(aδ_i, bδ_j) = 0\) for \(i \neq j\) and any \(a, b \in \mathbb{N}\). We set \(δ := \sum_i c_i δ_i\), then δ decomposes as \(δ = \bigoplus_i c_i δ_i\) (\(c_i δ_i\) may be decomposable). By Theorem 3.6, there is some \(γ \in N(M)\) and \(n \in \mathbb{N}\) such that \(nδ(γ) = \text{hom}(nδ, M)\) for some \(n \in \mathbb{N}\). Note that \(nδ\) decomposes as \(nδ = \bigoplus_i n_c c_i δ_i\). So,

\[
\sum_i n_c c_i δ_i(γ) = nδ(γ) = \text{hom}(nδ, M) = \sum_i \text{hom}(n_c c_i δ_i, M),
\]

but each \(n_c c_i δ_i(γ) \leq \text{hom}(n_c c_i δ_i, M)\). Hence, \(n_c c_i δ_i(γ) = \text{hom}(n_c c_i δ_i, M) = f_M(n_c c_i δ_i)\) for each \(i\), so \(δ_i(γ) = f_M(δ_i)\). Then

\[
\sum_i c_i f_M(δ_i) = \sum_i c_i δ_i(γ) = δ(γ) \leq f_M(δ).
\]

Finally, the equality follows from the sublinearity of \(f_M\). □

**Corollary 5.3.** Let δ be an indivisible outer normal vector of N(M). Then in any decomposition \(nδ = δ_1 \oplus δ_2\), δ_1 must be a multiple of δ. In particular, δ is indecomposable.

**Proof.** Suppose that none of δ_1 and δ_2 is a multiple of δ. For any γ \(\in N(M)\) on this facet, we have that

\[
δ_1(γ) + δ_2(γ) = nδ(γ) = f_M(nδ) = f_M(δ_1) + f_M(δ_2).
\]

Since \(δ_i(γ) \leq f_M(δ_i)\), we must have that \(δ_i(γ) = f_M(δ_i)\) for \(i = 1, 2\). This implies that both δ_1 and δ_2 are out normal vectors of this facet. A contradiction. □

**Definition 5.4.** For a fixed algebra \(A\), a weight vector δ is called normal if it is an outer normal vector of the Newton polytope of some \(M \in \text{rep} \ A\).

**Question 5.5.** Is any indecomposable δ-vector normal?
Later, we shall see that each real indecomposable $\delta$-vector is normal. Moreover, if $A$ has no relations, then each indecomposable $\delta$-vector is normal.

**Definition 5.6.** Suppose that $\delta = \bigoplus_i \delta_i$ is the canonical decomposition of $\delta$, and the dimension of the subspace spanned by $\{\delta_i\}_i$ is $r$. We say that $\text{rep} A$ has enough $\delta$-stable representations if there are $|Q_0| - r$ $\delta$-stable representations with linearly independent dimension vectors.

This is equivalent to say that the dimension vectors of $\delta$-semistable representations span a codimension $r$ subspace in $K_0(\text{rep} A)$.

**Proposition 5.7.** An indecomposable $\delta$ is normal if and only if $\text{rep} A$ has enough $\delta$-stable representations.

**Proof.** If $\delta$ is a normal vector of $N(M)$, then the convex hull of dimension vectors in $\mathcal{L}(\delta, M)$ has codimension 1. By Theorem 4.2, $L/L_{\text{min}}$ is $\delta$-semistable for any $L \in \mathcal{L}(\delta, M)$, and the dimension vectors of $L/L_{\text{min}}$ span a codimension 1 subspace.

Conversely, if $\text{rep} A$ has $|Q_0| - 1$ $\delta$-stable representations $\{L_i\}_i$ with linearly independent dimension vectors, then let $M = \bigoplus_i L_i$. We claim that $\delta$ is a normal vector of $N(M)$. Since $M$ is $\delta$-semistable, we have that $\text{hom}(n\delta, M) = 0$ for some $n \in \mathbb{N}$. So, $\{\gamma \mid \delta(\gamma) = 0\}$ supports a face of $N(M)$. Since each $L_i$ lies on this face, its codimension is exactly 1. □

One of the main results in [19] gives an explicit formula for the restriction of the $F$-polynomial $F_M$ to a facet of its Newton polytope. In particular, this result specializes to the tropical setting. Roughly speaking, any facet of Newton polytope $N(M)$ is a shifted Newton polytope $N(M')$ for a representation $M'$ of another algebra. We refer the readers to [19, Section 6] for more details.

## 5.3 Vertices and dual cones

If $P$ is a polytope, then its support function is piecewise linear. The maximal regions of linearity of $\psi_P$ are exactly the dual cones of the vertices of $P$: for each vertex $v$ of $P$, the support function $\psi_P$ is linear on $\{\alpha \in V^* \mid \alpha(v) = \psi_P(\alpha)\}$. The extremal rays of the dual cone are precisely the normal vectors of all facets of $P$ containing $v$. For this reason, it is also called the normal cone of $v$. In our setting, the dual cone $F_\gamma(M)$ of a vertex $\gamma \in N(M)$ is the cone spanned by $\delta$ satisfying

$$\delta(\gamma) = f_M(\delta).$$

Similarly, the dual cone $\hat{F}_\gamma(M)$ of a vertex $\gamma \in \hat{N}(M)$ is the cone spanned by $\hat{\delta}$ satisfying

$$\hat{\delta}(\gamma) = \hat{f}_M(\hat{\delta}).$$

Let $V(M)$ and $\hat{V}(M)$ be the set of vertices in $N(M)$ and $\hat{N}(M)$. We first recall some results in [19].

**Proposition 5.8** [19, Lemma 4.1, Proposition 4.7]. $\gamma \in V(M)$ if and only if it is the dimension vector of $t_\gamma(M)$ or $\hat{t}_\gamma(M)$ for some weight $\delta \in \mathbb{Z}^Q_0$. In particular, there is a unique subrepresentation $L$ of $M$ of dimension $\gamma$, and it satisfies $\text{Hom}(L, M/L) = 0$.

It is quite clear that $\delta$ can be any weight in the interior of $F_\gamma(M)$. The converse of the last statement is not true.
**Definition 5.9.** For any $\gamma \in \mathcal{V}(M)$, we call the unique subrepresentation $L$ with $\dim L = \gamma$ a **vertex subrepresentation** of $M$. The vertex quotient representation is defined analogously.

In particular, we can label the vertices of $N(M)$ by the vertex subrepresentations of $M$.

**Corollary 5.10.** Suppose that $M = \bigoplus_i M_i$. Then each vertex subrepresentation $L$ of $M$ is of the form $L = \bigoplus_i L_i$ where each $L_i$ is a vertex subrepresentation of $M_i$. In particular, $N(M) = \sum_i N(M_i)$ where the sum on the right side is the Minkowski sum.

Consider the sets
\[
\Delta_0(M) = \{ \delta \in \mathbb{Z}^{Q_0} \mid \hom(n\delta, M) = 0 \text{ for some } n \in \mathbb{N} \},
\]
\[
\Delta_1(M) = \{ \delta \in \mathbb{Z}^{Q_0} \mid e(n\delta, M) = 0 \text{ for some } n \in \mathbb{N} \}.
\]

They span the two most important dual cones, namely, $F_0(M)$ and $F_M(M)$. We call them the major cones of $N(M)$. Clearly, $F_0(M)$ always contains the negative cluster $(-e_1, \ldots, -e_n)$ and $F_M(M)$ always contains the positive cluster $(e_1, \ldots, e_n)$. Moreover, $\Delta(M) := \Delta_0(M) \cap \Delta_1(M)$ consists of all weights $\delta$ such that $M$ is $\delta$-semistable. Due to the relation $f_M(\delta) - \hat{f}_M(-\delta) = \delta(\dim M)$, we have the obvious duality
\[
F_L(M) = -\hat{F}_{M/L}(M).
\]
It follows from Theorem 3.6 that

**Corollary 5.11.** $\Delta_0(M)$ (respectively, $\Delta_1(M)$) are precisely the lattice points in the polyhedral cone defined by $\delta(v) \leq 0$ for all $v \in \mathcal{V}(M)$ (respectively, $\delta(v) \geq 0$ for all $v \in \bar{\mathcal{V}}(M)$).

One interesting result in [19] says that if $M$ is a general representation of an acyclic quiver, then the normal vectors of $N(M)$ are precisely given by the extremal rays in $F_0(M)$ and $F_M(M)$.

The following proposition says that other dual cones are intersections of the major cones.

**Proposition 5.12.** Suppose that $L$ is a vertex subrepresentation of $M$. We have that
\[
F_L(M) = F_0(M/L) \cap F_L(L).
\]

**Proof.** If $\delta \in F_L(M)$, then $\delta(\dim L) = f_M(\delta)$. Since every subrepresentation of $L$ is a subrepresentation of $M$, $\delta(\dim L) \leq f_L(\delta) \leq f_M(\delta)$. So $\delta \in F_L(L)$. Similarly, we can show that $\delta \in F_0(M/L)$. Conversely, if $\delta \in F_0(M/L) \cap F_L(L)$, then by Theorem 3.6, there is some $n \in \mathbb{N}$ such that $e(n\delta, L) = 0$ and $\hom(n\delta, M/L) = 0$. By Lemma 4.1, we have that $n\delta(\dim L) = \hom(n\delta, M) = f_M(n\delta)$, that is, $\delta \in F_L(M)$. \qed

**Lemma 5.13.** Suppose that $M = \bigoplus_i M_i$, and $\delta \in F_L(M)$. Then $\delta \in F_{L_i}(M_i)$ for each $i$ where $L_i = L \cap M_i$. So each $F_{L_i}(M_i)$ is a union of dual cones of $N(M)$.

**Proof.** By Corollary 5.10, each $L_i$ is a vertex subrepresentation. $\delta \in F_L(M)$ implies that $\delta(\dim L) = f_M(\delta)$. So, we have that
\[
\sum_i \delta(\dim L_i) = \delta(\dim L) = f_M(\delta) = \sum_i f_{M_i}(\delta).
\]
But \( \delta(\dim L_i) \leq f_{M_i}(\delta) \) for each \( i \). We must have that \( \delta(\dim L_i) = f_{M_i}(\delta) \) for each \( i \). Hence \( \delta \in F_{L_i}(M_i) \). Conversely, suppose that \( \delta \in F_{L_i}(M_i) \) for each \( i \). Then \( \delta \in F_{g_i}(M) \). It follows that each \( F_{L_i}(M_i) \) is a union of \( F_{L_j}(M) \) where \( L \cap M_i = L_i \).

**Lemma 5.14.** Suppose that \( \{\delta_1, \ldots, \delta_r\} \) satisfies \( e(\delta_i, \delta_j) = 0 \) for \( i \neq j \). Then all \( \delta_i \)'s are contained in some dual cone of \( N(M) \).

**Proof.** This is just a reformulation of Lemma 5.2.

**Lemma 5.15.** Let \( M \) be an \( E \)-rigid representation with weight vector \( \delta \), and \( N \) is a quotient representation of \( M \). Then \( \delta(\dim N) = 0 \) if and only if \( N = 0 \).

**Proof.** Suppose that \( N \neq 0 \), then \( \text{hom}(M, N) > 0 \). Since \( M \) is \( E \)-rigid, \( e(M, N) \leq e(M, M) = 0 \). We have that \( \delta(\dim N) = \text{hom}(M, N) - e(M, N) > 0 \).

**Lemma 5.16.** Suppose that \( \delta_- \) and \( \delta_+ \) are real, and \( e(\delta_-, \delta_+) > 0 \). Then \( \delta_- \) and \( \delta_+ \) cannot lie in the same dual cone of \( F(M) \) where \( M = \text{Coker}(\delta_+) \).

**Proof.** Since \( \delta_+ \) is real, by Lemma 5.15, \( \delta_+ \in F_M(M) \) and \( \delta_+ \notin F_L(M) \) if \( L \neq M \). But \( \delta_-(\dim M) < \text{hom}(\delta_-, M) \) because \( e(\delta_-, M) > 0 \). Since \( \delta_- \) is real, \( f_M(\delta_-) = \text{hom}(\delta_-, M) \). Hence \( \delta_- \notin F_M(M) \).

**Theorem 5.17.** Let \( \delta_1, \ldots, \delta_m \) be finitely many clusters. Then there is some representation \( M \) such that each \( \delta_i \) spans a dual cone of \( N(M) \).

**Proof.** By Lemmas 5.13 and 5.14, it suffices to show for a single cluster, say \( \delta = (\delta_1, \ldots, \delta_r) \). Let \( (\delta_-, \delta_+) \) be an exchange pair. In particular, we have that \( e(\delta_-, \delta_+) > 0 \). By Lemma 5.16, there is a representation \( N \) separating \( (\delta_-, \delta_+) \) in the sense that they lie in two different dual cones of \( N(N) \). Let \( N_i \) be the representation separating the (unordered) exchange pair \( \{\delta_i, \delta'_i\} \) with respect to \( \delta \setminus \{\delta_i\} \). By Lemmas 5.13 and 5.14, \( M = \bigoplus_i N_i \) is the desired representation.

**Remark 5.18.** The proof shows that \( M \) can be chosen to be a direct sum of \( E \)-rigid representations. Later, we will see in Corollary 7.9 that \( M \) can be a direct sum of real Schur representations in the dual clusters under some mild assumption. This theorem also implies that in particular each real indecomposable weight vector is normal.

### 6 | GENERIC NEWTON POLYTOPES

#### 6.1 | Generic Newton polytopes of \( \delta \)

We first extend the notation \( \text{hom}(\delta, M) \) and \( \text{hom}(M, \delta) \) in Definition 2.4 in an obvious manner. We write

\[
\text{hom}(\delta, \delta) := \text{hom}(\delta, \text{Ker}(\delta)) = \text{hom}(\text{Coker}(\delta), \delta).
\]
Similarly, we write
\[ f_\delta(\delta) := f_{\text{Ker}(\delta)}(\delta) \text{ and } f_\delta(\hat{\delta}) = f_{\text{Coker}(\delta)}(\delta). \]

As we have seen in Example 3.24 that \( f_\delta(\delta) \neq f_\delta(\hat{\delta}) \) in general even if one of \( \delta \) and \( \hat{\delta} \) is real. We denote by \( N(\hat{\delta}) \) the Newton polytope of the kernel of a general presentation in \( \text{IHom}(\hat{\delta}) \). We hope to determine \( N(\hat{\delta}) \) when \( A \) is the Jacobian algebra of a quiver with potential. The idea is based on the following observation. In the rest of this section, we assume that \( A \) is a Jacobian algebra of some QP.

**Observation 6.1.** According to Lemma 5.14, any cluster \( \{\delta_i\} \) lies in some dual cone \( \mathcal{C}L(M) \) of \( \mathcal{N}(M) \). Such a cluster determines the vertex \( \dim L \) by the formula \( \delta_i(\dim L) = \text{hom}(\delta_i, M) \) for each \( i \). The vertex can be explicitly computed as
\[ \dim L = h^{-1} \delta, \]
where \( \delta \) is the matrix \( (\delta_1^T, \delta_2^T, ..., \delta_n^T) \) and \( h(i) = \text{hom}(\delta_i, M) \). In general, computing \( \text{hom}(\delta_i, M) \) is not easy. However, when the cluster is negative reachable and \( M \) is a generic kernel of \( \text{IHom}(\hat{\delta}) \), Theorem 3.22 implies that
\[ \text{hom}(\delta_i, M) = \text{hom}(\delta_i, \hat{\delta}) = f_{\delta_i}^*(\delta). \]

Moreover, the tropical \( F \)-polynomial \( f_{\delta_i}^* \) or equivalently \( \mathcal{N}(\delta_i) \), the dual Newton polytope of \( \text{Coker}(\delta_i) \), may be computed by the mutation algorithm [16, 25].

Now the question is whether each dual cone of \( \mathcal{N}(\hat{\delta}) \) contains a cluster. In fact, according to Theorem 2.9 and Lemma 5.14, the question is equivalent to whether each dual cone of \( \mathcal{N}(\hat{\delta}) \) contains a real \( \delta \)-vector.

**Question 6.2.** Does each dual cone of \( \mathcal{N}(\hat{\delta}) \) contain a real \( \delta \)-vector?

We shall give a positive answer for acyclic quivers. This is based on the following lemma.

Recall that for any acyclic quiver \( Q \), we can associate each dimension vector \( \alpha \) a weight \( \delta_{\alpha} := -\langle -, \alpha \rangle \in (\mathbb{Z}^{\Delta_0})^* \) where \( \langle -, - \rangle \) is the Euler form of \( Q \). We denote by \( N(\alpha) \) the Newton polytope of a general \( \alpha \)-dimensional representation. Since there is an open set of \( \text{rep}_\alpha(Q) \) in which the representations have minimal injective presentations of weight \( \delta_{\alpha} \), we have that \( N(\delta_{\alpha}) = N(\alpha) \).

**Lemma 6.3** [19, Lemma 6.17]. Let \( M \) be a general representation in \( \text{rep}_\alpha(Q) \). If \( \delta \) corresponds to an imaginary root and \( \text{hom}(\delta, M) > 0 \), then the convex hull of the dimension vectors in \( L(\delta, M) \) has codimension at least 2. In particular, such a \( \delta \)-vector cannot be a normal vector of \( N(M) \).

**Theorem 6.4.** Let \( \alpha \) be any dimension vector of \( Q \). Each normal cone \( \text{F}_{\gamma}(\alpha) \) of \( N(\alpha) \) contains a cluster. Hence, the Newton polytope \( N(\alpha) \) is completely determined by the Newton polytopes of real Schur representations.

\[ {^*} \text{It is not so easy to confuse this notation with the functors introduced in Section 4 under appropriate context.} \]
Proof. If $\gamma = 0$, $F_\gamma(\alpha)$ contains the negative cluster. If $\gamma \neq 0$, it must be contained in some facet supported by $\{ \gamma \in \mathbb{R}_Q^0 \mid \delta(\gamma) = h > 0 \}$. Its normal vector $\delta$ cannot be imaginary by Lemma 6.3. So, one ray of the cone is real. By the above remark, it must contain a cluster. \qed

Algorithm 6.5. For a fixed dimension vector $\alpha$ of $Q$, the (primitive) normal vectors of $N(\alpha)$ are bounded. It may be hard to give a sharp bound, but it is easy to estimate some rough bound. Let $\Delta_\alpha$ be the set of all real $\delta$-vectors within this bound. We can use the mutation algorithm ([25, Proposition 5.1]) to find the tropical $F$-polynomial of any $\delta \in \Delta_\alpha$. In general, using the mutation algorithm to compute the $F$-polynomial of $\delta$ is very expensive, but it is much cheaper to find the tropical one by the tropical version of [25, Proposition 5.1]. Since the exchange graph of acyclic quivers are connected [27], searching for all $\delta$ in $\Delta_\alpha$ can be terminated in finite steps. Finally, according to the above theorem, the generic Newton polytope $N(\alpha)$ are determined by these tropical $F$-polynomials.

Example 6.6. Let $Q$ be the quiver $\xrightarrow{1} 2 \xrightarrow{a} 3$, and $\alpha$ be the dimension vector $(3,5,2)$. Except for zero and itself, the Newton polytope $N(\alpha)$ has four vertices, which are listed in the left column. The middle column is one of the clusters determining the vertex, and the right column is the sequence of mutations to reach this cluster.

$\begin{array}{ccc}
(0, 3, 0) & (-e_1, e_2 - e_3, -e_3) & 2 \\
(0, 0, 2) & (-e_1, -e_2, e_3) & 3 \\
(0, 5, 2) & (-e_1, e_2 - e_3, e_2) & (2, 3) \\
(2, 3, 2) & (3e_1 - 2e_2, 2e_1 - e_2, e_3) & (3, 2, 1, 2, 1)
\end{array}$

Let us test Question 6.2 in a very simple example.

Example 6.7. Consider the quiver

$\begin{array}{ccc}
1 & \xrightarrow{a} & 4 \\
\updownarrow & & \downarrow b \\
2 & \xrightarrow{b} & 3
\end{array}$

with potential $abc$. Let $\delta = (0, -3, 1, 1)$. One can check that $\delta$ is not real, and $M = \text{Ker}(\delta)$ has dimension vector $(1, 1, 1, 2)$. Except for zero and itself, the Newton polytope $N(\delta)$ has six vertices.

$\begin{array}{ccc}
(0, 0, 0, 1) & (-e_1, -e_2, -e_3, e_4 - e_3) & 4 \\
(0, 0, 1, 0) & (-e_1, -e_2, e_3 - e_1, -e_4) & 3 \\
(0, 0, 1, 2) & (-e_1, -e_2, e_4, e_4 - e_3) & (4, 3) \\
(1, 0, 0, 1) & (e_1, -e_2, -e_3, e_4 - e_3) & (4, 1) \\
(1, 0, 1, 1) & (e_1, -e_2, e_3, e_1 - e_4) & (1, 4, 3) \\
(1, 0, 1, 2) & (e_1, -e_2, e_4, e_4 - e_3) & (4, 1, 3)
\end{array}$

Conjecture 6.8. We have that $f_\delta(\delta) = \tilde{f}_\delta(\tilde{\delta})$ for any $\delta$ and $\tilde{\delta}$.

A more optimistic conjecture is that $f_\delta(\delta) = \tilde{f}_\delta(\tilde{\delta}) = \text{hom}(\delta, \tilde{\delta})$ (see Question 3.25).
**Observation 6.9.** The positive answer to Question 6.2 implies Conjecture 6.8. If this is the case, we can determine each vertex of \( \mathcal{N}(\tilde{\delta}) \) using the method described in Observation 6.1.

**Proof.** Let \( M = \ker(\delta) \). Suppose that \( \delta \in F_1(M) \) and \( \{\delta_i\}_i \) is a cluster in \( F_1(M) \). By Theorem 2.9, we can write \( \delta \) as an integral linear combination of \( \delta_i \)'s: \( \delta = \sum_i c_i\delta_i \). Then we have the following equalities, where the second one and the last one are due to Theorems 3.6 and 3.22, respectively,

\[
\delta_i(y) = f_\delta(\delta_i) = \hom(\delta_i, \delta) = \tilde{f}_\delta(\delta).
\]

Then we have the following equalities, where the fourth one is due to Lemmas 2.6 and 3.4.

\[
f_\delta(\delta) = \delta(y) = \sum_i c_i\delta_i(y) = \sum_i c_i\tilde{f}_\delta(\delta) = \tilde{f}\sum_i c_i(\delta) = \tilde{f}_\delta(\delta).
\]

In fact, they both equal to \( \hom(\delta, \delta) \) by Lemma 3.8. \( \square \)

Due to Theorem 6.4, we have Schofield’s Theorem 1.2 as a corollary of our Theorem 3.6 and Observation 6.9.

**Remark 6.10 (Relation to the cluster algebras).** Determine \( \mathcal{N}(\tilde{\delta}) \) when \( A \) is a Jacobian algebra is an important problem in the cluster algebra theory. Let \( Q \) be a 2-acyclic quiver, and \( B \) be its associated skew-symmetric matrix given by

\[
B(u, v) = |\text{arrows } u \to v| - |\text{arrows } v \to u|.
\]

We denote by \( \overline{C}(Q) \) the associated upper cluster algebra [6]. Let \((Q, P)\) be a nondegenerate quiver with potential. We still keep the assumption that \( A = J(Q, P) \) is finite-dimensional.

In [18] the author introduced a set of elements \( \{X_{\delta}\} \) indexed by the \( \tilde{\delta} \)-vectors (or \( \delta \)-vectors), of the form

\[
X_{\delta} = x^{-\tilde{\delta}}F_{\delta}(y),
\]

where \( F_{\delta} \) is the \( F \)-polynomial of \( \ker(\tilde{\delta}) \) [16] and \( y \) is a monomial change of variables from \( x \): \( y_u = \prod_v x_{uv}^{B(u,v)} \). In many cases, they are turned out to be a basis of \( \overline{C}(Q) \) [38]. The Newton polytope of this polynomial \( F_{\delta} \) is exactly given by the generic Newton polytope \( \mathcal{N}(\tilde{\delta}) \).

In the meanwhile, a remarkable positive basis consisting of theta functions for all cluster algebras was introduced in [26]. For each \( \tilde{\delta} \)-vector, there is a theta function \( \varphi_{\delta} \), which is of the form

\[
\varphi_{\delta} = x^{-\delta}\varphi_{\delta}(y).
\]

In general, the theta function can be a Laurent series, but let us assume that it is a Laurent polynomial, so \( \varphi_{\delta} \) is a polynomial with positive coefficients. Another very interesting positive (quantum) basis called triangular basis was introduced in [39] as a far-reaching generalization of [7]. It has a similar form

\[
T_{\delta,q} = x^{-\delta}\psi_{\delta,q}(y).
\]

In particular, \( \varphi_{\delta} \) and \( \psi_{\delta,q} \) can be tropicalized and the tropicalization is determined by its Newton polytope. We expect that all “interesting” bases of cluster algebras should contain the cluster
monomials and their tropicalizations satisfy the Conjecture 6.8. In light of Theorem 6.4, we have the following conjecture.

**Conjecture 6.11.** The Newton polytopes of $\varphi_\delta$ and $\psi_{\delta,q}$ are the same as the generic Newton polytope $N(\delta)$. Moreover, the coefficients in $F_\delta$, $\varphi_\delta$, and $\psi_{\delta,q}$ corresponding to the vertices of $N(\delta)$ are all 1’s (the statement for $F_\delta$ has been settled in [19]).

### 6.2 Application to the Fock–Goncharov duality pairing

We first briefly recall the Fock–Goncharov’s duality pairing [22]. Recall that a skew-symmetrizable matrix $B$ gives rise to a pair of cluster varieties $(\mathcal{A}, \mathcal{X})$, and their Langlands dual $(\mathcal{A}^\vee, \mathcal{X}^\vee)$. Fock–Goncharov duality conjecture [22, Conjecture 4.1] says that the tropical points $\mathcal{X}^\vee(\mathbb{Z}_t)$ of $\mathcal{X}^\vee$ parametrize a basis of ring of regular functions $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$, and we can interchange the roles of $\mathcal{A}$ and $\mathcal{X}$. The duality conjecture fails in general, but can hold with some mild assumption, or if we replace it with a formal version (see [26] for detail). From now on let us assume that the duality conjecture holds, and denote the parametrizations by

$$I_{\mathcal{A}} : \mathcal{A}(\mathbb{Z}_t) \hookrightarrow \mathcal{O}(\mathcal{X}^\vee) \quad \text{and} \quad I_{\mathcal{X}^\vee} : \mathcal{X}^\vee(\mathbb{Z}_t) \hookrightarrow \mathcal{O}(\mathcal{A}).$$

The duality conjecture further asserts that we can require the parametrized bases to be universally positive and satisfy several interesting properties. One of them concerns the pairing

$$\mathcal{A}(\mathbb{Z}_t) \times \mathcal{X}^\vee(\mathbb{Z}_t) \to \mathbb{Z}.$$

There are two canonical ways to define this pairing:

$$I_{\mathcal{A}}(a)^{\text{trop}}(x) \quad \text{and} \quad I_{\mathcal{X}^\vee}(x)^{\text{trop}}(a) \quad \text{for } a \in \mathcal{A}(\mathbb{Z}_t), \ x \in \mathcal{X}^\vee(\mathbb{Z}_t).$$

The conjecture says that they are equal. We are going to give a representation-theoretic interpretation of the above pairings in some special cases. As a consequence, we shall see that the two ways of pairings are equal. Recall that there is a canonical map $\tilde{p} : \mathcal{A}^\vee \to \mathcal{X}^\vee$ given by $\tilde{p}^*(y_u) = \prod_u x_v^{B(u,v)}$, where $x$ and $y$ are the coordinates of $\mathcal{A}^\vee$ and $\mathcal{X}^\vee$. At the level of tropical points, this is given by $\mathcal{A}^\vee(\mathbb{Z}_t) \to \mathcal{X}^\vee(\mathbb{Z}_t)$, $a \mapsto aB^T$. Note that if $B$ is invertible, then $\tilde{p}^*$ is injective.

As one can see immediately, the two pairings depend on the map $I_{\mathcal{A}}$ and $I_{\mathcal{X}^\vee}$. According to Conjecture 6.11, this may not be an issue for the known interesting bases. At this stage, let us first resolve this issue by letting $I_{\mathcal{A}}$ and $I_{\mathcal{X}^\vee}$ be the generic basis map. More precisely, $I_{\mathcal{X}^\vee}$ and $I_{\mathcal{A}}$ are given by

$$I_{\mathcal{X}^\vee}(\delta) = x^{-\delta}F_\delta(x) \quad \text{and} \quad I_{\mathcal{A}}(a) = y^{-a\tilde{F}_\delta(a)}(y),$$

where $F_\delta$ is the $F$-polynomial of $\text{Ker}(\delta)$ in the $x$-coordinate, and $\tilde{F}_\delta$ is the dual $F$-polynomial of $\text{Coker}(\delta)$ in the $y$-coordinate. The reason why we switch to the dual $F$-polynomial is due to the transposition of $B$ in the Langlands dual.

It is known that the $F$-polynomials may have negative coefficients, so the usual tropicalization is not well defined. However, we can modify the usual tropicalization by considering the tropical $F$-polynomials. Besides Remark 6.10, this approach is further justified in [19, Remark 1.4]. At least
when the $F$-polynomial has positive coefficients, the two notions agree. So, let us define

$$I_{\chi^\vee}(\delta)^{\text{trop}} = f_\delta \circ B^T - \delta \quad \text{and} \quad I_A(a)^{\text{trop}} = \tilde{f}_a - a,$$

where $B^T$ is the map of multiplication by the matrix $B^T$.

**Theorem 6.12** (Fock–Goncharov duality pairing). Suppose that $B$ is skew-symmetric. The pairings $\mathcal{A}(\mathbb{Z}^l) \times \mathcal{X}^\vee(\mathbb{Z}^l) \to \mathbb{Z}$ given by $I_A(a)^{\text{trop}}(\delta)$ and $I_{\chi^\vee}(\delta)^{\text{trop}}(a)$ are both equal to $\text{hom}(aB^T, \delta) - a \cdot \delta$ in the following two situations.

1. The quiver of $B$ is mutation-equivalent to an acyclic quiver.
2. Either $I_{\chi^\vee}(\delta)$ or $I_A(aB^T)$ is a cluster variable, or equivalently either $\delta$ or $aB^T$ is negative reachable.

**Proof.** Due to Corollary 3.26 for (1) and Theorems 3.6 and 3.22 for (2), we have that

$$I_{\chi^\vee}(\delta)^{\text{trop}}(a) = f_\delta(aB^T) - a \cdot \delta = \text{hom}(aB^T, \delta) - a \cdot \delta; \quad (6.1)$$

$$I_A(a)^{\text{trop}}(\delta) = \tilde{f}_{aB^T}(\delta) - a \cdot \delta = \text{hom}(aB^T, \delta) - a \cdot \delta. \quad (6.2)$$

□

**Remark 6.13.** It is clear that Conjecture 6.8 implies the equality of the two pairings in all skew-symmetric cases. If $B$ is invertible, we can set $\delta = aB^T$ and write $\text{hom}(aB^T, \delta) - a \cdot \delta$ in a more symmetric form

$$\text{hom}(\delta, \tilde{\delta}) + \delta B^{-1} \tilde{\delta}^T.$$

One can check that this pairing is mutation-invariant using [16, Proposition 6.1 and (2.11)].

Although the main part of Fock–Goncharov duality conjecture was intensively studied, the meaning of the duality pairing is only known in few cases. For the moduli space of the $\text{PGL}_2 / \text{SL}_2$-local systems of surfaces, the duality pairing can be interpreted as the intersection pairing of laminations [21, Proposition 12.1]. The verification of the equality in this generality is new.

## 7 SCHUR REPRESENTATIONS AND DUAL CLUSTERS

**Definition 7.1.** A representation $V$ is called Schur if $\text{Hom}(V, V) = k$. It is called real Schur if in addition we require $\text{Ext}^1(V, V) = 0$.

Here is a method to produce such $V$. We start with any representation $M$.

**Lemma 7.2.** Suppose that $\delta$ is $E$-rigid such that $\text{hom}(\delta, M) = 1$. Then $t_\delta(M)$ is Schur. Dually, suppose that $\tilde{\delta}$ is $\tilde{E}$-rigid such that $\text{hom}(M, \tilde{\delta}) = 1$. Then $\tilde{f}_\delta(M)$ is Schur.

Moreover, if $M$ is $\tilde{E}$-rigid (respectively, $E$-rigid), then $t_\delta(M)$ (respectively, $\tilde{f}_\delta(M)$) is real Schur.
Proof. By Corollary 4.5 $L = t_\delta(M)$ is the image of the nonzero homomorphism $C \to M$ where $C = \text{Coker}(\delta)$. Since $L$ is a quotient of $C$, we have that
\[ k \subseteq \text{Hom}(L, L) \subseteq \text{Hom}(C, L) = k. \]
Hence $\text{Hom}(L, L) = k$. If $M$ is $\mathbb{E}$-rigid, then by [19, Proposition 4.8], we have $\text{Ext}^1(L, L) = 0$. □

Let $d_0$ be an $E$-rigid presentation with $\text{ind}(d_0) = |Q_0| - 1$. Then by Proposition 2.11, there are two complements $d_-$ and $d_+$ of $d_0$ satisfying $e(d_-, d_+) = e > 0$ and $e(d_+, d_-) = 0$. In this case, we define the sign of $d_\pm$ in the cluster \{d_\pm\} $\cup \text{ind}(d_0)$ to be $\pm$. Throughout this section, we will always assume that $e = 1$. In other words, $(d_-, d_+)$ is a regular exchange pair.† In this case, $d_-$ and $d_+$ fit into the triangle in $K^b(\text{proj} - A)$
\[ d_+ \to d \to d_- \to d_+[1], \]
where $d \in \text{add}(d_0)$. Let $\delta_0$ and $\delta_\pm$ be the weight vectors of $d_0$ and $d_\pm$ respectively.

Let $L = \text{Coker}(d_+)$, $N = \text{Coker}(d_-)$, and $N^\tau = \text{Ker}(\nu d_-)$. Note that if $d_-$ is nonnegative (i.e., $\neq (P_i \to 0)$), then $N^\tau = \tau N$. We have that $\text{hom}(L, N^\tau) = e(d_-, L) = 1$. We consider the exact sequence
\[ 0 \to K \to L \to N^\tau \to C \to 0, \]
where $L \to N^\tau$ spans $\text{Hom}(L, N^\tau)$. Let $I$ be the image of $L \to N^\tau$.

Definition 7.3. We called $\dim I$, the $c$-vector of the exchange pair $(d_-, d_+)$. We also called $\pm \dim I$ the signed $c$-vector of $d_\pm$ for the cluster \{d_\pm\} $\cup \text{ind}(d_0)$.

According to Corollary 4.5, we have that \[ K = \tilde{t}_\delta(L), \quad C = f_{\delta_+}(N^\tau), \quad \text{and} \quad I = \tilde{f}_{\delta_-}(L) = t_{\delta_+}(N^\tau). \]

Lemma 7.4. We have that
\[ \text{Hom}(d_0, I) = 0, \quad E(d_0, I) = 0; \]
\[ \text{Hom}(d_+, I) = k, \quad E(d_+, I) = 0; \]
\[ \text{Hom}(d_-, I) = 0, \quad E(d_-, I) = k. \]
Moreover, $I$ is real Schur.

Proof. Since $\text{Hom}(d_0 \oplus d_+, N^\tau) = E(d_-, d_0 \oplus d_-) = 0$ and $I$ is a subrepresentation of $N^\tau$, we get $\text{Hom}(d_0 \oplus d_-, I) = 0$. On the other hand, $E(d_0 \oplus d_+, L) = 0$ and $I$ is a quotient of $L$, so we have that $E(d_0 \oplus d_+, I) = 0$. By Lemma 4.1, we have that $E(d_+, I) = E(d_+, N^\tau) = k$; $E(d_-, I) = E(d_-, L) = k$.

Moreover, $I$ is real Schur and follows from Lemma 7.2. □

† It is known that this assumption is always satisfied if the algebra is the Jacobian algebra of some quiver with generic potential and the cluster is (negative or positive) reachable.
We remark that Lemma 4.1 also tells us \( \text{Hom}(L, C) = 0 \) and \( \text{E}(L, \tau N) \cong \text{E}(L, C) \). Dually we have that \( \text{E}(N, K) = 0 \) and \( \text{Hom}(N, K) \cong \text{Hom}(N, L) \).

Let \( d = \{d_1, d_2, \ldots, d_n\} \) be a cluster of presentations, and \( d'_j = (d \setminus \{d_j\}) \cup \{d'_j\} \) be the adjacent cluster. Let \( I_j \) be defined as above for each (unordered) exchange pair \( \{d_j, d'_j\} \), and \( \varepsilon_j \) be the sign of \( d_j \) in \( d \).

**Definition 7.5.** For \( I \in \rep A \) and a sign \( \varepsilon = \pm \), we denote \( I^\varepsilon := \left\{ \begin{array}{ll} I & \text{if } \varepsilon = + \\ I[1] & \text{if } \varepsilon = - \end{array} \right. \) as an element in the bounded derived category \( D^b(\rep A) \). We define the dual cluster of \( d \) as the ordered elements \( (\varepsilon_1 I_1, \ldots, \varepsilon_n I_n) \) in \( D^b(\rep A) \).

In this notation, we can rephrase Lemma 7.4 as

\[
\text{Hom}(d_0, \pm I) = \text{Hom}(d_+, I) = \text{Hom}(d_-, -I) = 0 \quad \text{and} \quad \text{Hom}(d_-, -I) = \text{Hom}(d_+, I) = k.
\]

We use the upright \( \delta \) to denote the usual delta-function. We write \( \delta^\perp \) for the abelian subcategory of \( \rep A \)

\[
\delta^\perp := \{ M \in \rep A \mid \text{hom}(\delta, M) = e(\delta, M) = 0 \}.
\]

**Theorem 7.6** (cf. [34, Lemma 5.3]). Let \( \{\delta_i\} \) be a regular cluster and \( I_j \) be defined as above. Then

\[
\text{hom}(\delta_i, \varepsilon_j I_j) = \delta(i, j) \quad \text{and} \quad \text{hom}(\delta_i, -\varepsilon_j I_j) = 0.
\]

Moreover, the simple objects in category \( \delta^\perp_i := \bigcap_{i \in I} \delta^\perp_i \) are precisely \( I_j \) for \( j \notin I \).

**Proof.** The first statement is a direct consequence of Lemma 7.4. For the second statement, we already have that \( I_j \) (\( j \notin I \)) are the simple objects in the category \( \delta^\perp_i \). [29, Theorem 3.8] says that the category \( \delta^\perp_i \) is equivalent to the module category of some (basic) algebra whose quiver has \( |I| \) vertices less than \( Q_0 \). In particular, there are exactly \( |Q_0| - |I| \) simple objects in \( \delta^\perp_i \). \( \Box \)

**Remark 7.7.** If we embed \( K^b(\proj -A) \) canonically into \( D^b(\rep A) \), then the Euler form

\[
\langle d, C \rangle = \sum (-1)^p \text{Hom}_{D^b(\rep A)}(d, C[p]) \quad \text{on} \quad K^b(\proj -A) \times D^b(\rep A)
\]

gives us a nondegenerate pairing. This theorem shows in particular that the classes dual to the basis \( \{\delta_i\} \) are given by \( \{[\varepsilon_i I_i]\} = \{\varepsilon_i \text{dim} I_i\} \).

When \( A \) is a finite-dimensional Jacobian algebra associated to a nondegenerate QP \( (Q, P) \), the \( c \)-vectors in [36] are defined as such dual basis. For those reachable clusters, the \( c \)-vectors defined this way agree with the \( c \)-vectors of the corresponding clusters in the cluster algebra \( C(Q) \). This duality was further studied in skew-symmetrizable cases in [37]. Here we gave an explicit construction of the real Schur representations corresponding to the \( c \)-vectors for any regular cluster. The sign coherence of the \( c \)-vectors is thus obvious from our construction.

**Corollary 7.8.** Suppose that we have the exchange triangle

\[
d^+_i \rightarrow \bigoplus_j b_{ij} d_j \rightarrow d^-_i \rightarrow d^+_i[1].
\]
Let \( \{\epsilon_j I_j\}_j \) (respectively, \( \{\epsilon'_j I'_j\}_j \)) be the dual cluster of \( \{d_1, \ldots, d^+_i, \ldots, d^n\} \) (respectively, \( \{d_1, \ldots, d^-_i, \ldots, d^n\} \)). Then \( b_{ij} = \delta^-_i ([\epsilon_j I_j]) = \hom(\delta^-_i, \epsilon_j I_j) = \delta^+_i ([\epsilon'_j I'_j]) = \hom(\delta^+_i, \epsilon'_j I'_j), \) and \( \delta^\pm_i ([I_i]) = \pm 1. \)

**Proof.** We pair the triangle with the dual basis \( [\epsilon_j I_j] \), and we obtain

\[
b_{ij} = \delta^-_i ([\epsilon_j I_j]) \quad \text{and} \quad \delta^-_i ([I_i]) = -1.
\]

If \( \epsilon_j \) is positive, then \( I_j \) is a quotient of \( \text{Coker}(\delta_j) \). We have that \( \epsilon(\delta^-_i, I_j) = 0 \) so \( b_{ij} = \hom(\delta^-_i, I_j) \).

If \( \epsilon_j \) is negative, then \( I_j \) is a subrepresentation of \( \text{Ker}(\nu\delta_j) \). We have that \( \hom(\delta^-_i, I_j) = 0 \) so \( b_{ij} = e(\delta^-_i, I_j) = \hom(\delta^-_i, -I_j) \). The rest can be proved similarly using the dual basis \( [\epsilon'_j I'_j] \). \( \Box \)

**Corollary 7.9.** Let \( \{\epsilon, I_i\}_I \) be the dual cluster of \( \{\delta_i\}_i \), and \( M \) be the direct sum \( \bigoplus_i I_i \). Then one of dual cone of \( \mathcal{N}(M) \) is precisely spanned by this cluster.

**Proof.** Consider \( M_\pm = \bigoplus_{q_i = \pm} I_{ij} \), then \( M_\pm \) is a vertex subrepresentation of \( M \). We claim that \( F_{M_+}(M) = \langle \delta_i \rangle_i \). We first show that each \( \delta_i \in F_{M_+}(M) \), or equivalently

\[
\delta_i (\dim M_+) = \hom(\delta_i, M).
\]

But this is rather clear from (7.1).

Next, we show that each adjacent \( \delta'_i \not\in F_{M_+}(M) \), or equivalently

\[
\delta'_i (\dim M_+) < \hom(\delta'_i, M),
\]

which is equivalent to

\[
-e(\delta'_i, M_+) < \hom(\delta'_i, M_-).
\]

But it is clear from Corollary 7.8 that if \( \epsilon_i > 0 \), then \( \hom(\delta'_i, M_-) \geq 0 \) and \( e(\delta'_i, M_+) \geq 1 \); if \( \epsilon_i < 0 \), then \( \hom(\delta'_i, M_-) \geq 1 \) and \( e(\delta'_i, M_+) \geq 0 \). \( \Box \)

Finally, we pose some questions. Consider the following three sets consisting of

1. all real Schur representations;
2. all real Schur representations constructed from Lemma 7.2;
3. all real Schur representations constructed from exchange pairs (Lemma 7.4).

It is clear that (1) contains (2), and (2) contains (3).

**Conjecture 7.10.** For the finite-dimensional Jacobian algebras, the three sets are equal.

**Problem 7.11.** We say that a set of real Schur representations is **compatible** if they are a part of some dual cluster. Find some reasonable conditions without referring to the original cluster that can verify the compatibility.
8 | THE DUAL FAN AND THE EDGE QUIVER

**Definition 8.1.** A fan $\mathcal{F}$ in a real vector space $V$ is a finite collection of nonempty polyhedral cones in $V$ such that

1. every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$;
2. the intersection of any two cones in $\mathcal{F}$ is a face of both.

A fan is called complete if the union of all the cones in $\mathcal{F}$ is $V$.

The dual cones of a polytope $\mathcal{P}$ fit together into a complete fan, the dual fan of $\mathcal{P}$. It is also called the normal fan of $\mathcal{P}$. To pedantically stick to the definition, we need the cones dual to faces (not just vertices) of $\mathcal{P}$. Let $L$ be a face of $N(M)$. The dual cone $F_L(M)$ of $L$ is spanned by

\[ \{ \delta \in \mathbb{Z}_+^Q \mid \delta(\gamma) = f_M(\delta), \forall \gamma \in L \}, \]

which is the intersection $\bigcap_\gamma F_{\gamma}(M)$ over all vertices $\gamma \in L$. The dual cones of vertices are the maximal cones of the dual fan. We denote the dual fan of $N(M)$ by $F(M)$.

A fan $\mathcal{F}_1$ is said to be a coarsening of a fan $\mathcal{F}_2$ if every cone of $\mathcal{F}_2$ is contained in some cone of $\mathcal{F}_1$. A fan $\mathcal{F}_2$ is said to be a refinement of a fan $\mathcal{F}_1$ if every cone of $\mathcal{F}_1$ is a union of cones of $\mathcal{F}_2$. If $\mathcal{F}_1$ is complete, then it is clear that $\mathcal{F}_2$ is a refinement of $\mathcal{F}_1$, then $\mathcal{F}_1$ is a coarsening of $\mathcal{F}_2$, but not vice versa. It follows from Lemma 5.13 that

**Lemma 8.2** (cf. [42, Proposition 7.12]). Let $M_1$ and $M_2$ be any two representations of $A$. Then $F(M_1 \oplus M_2)$ is the common refinement of $F(M_1)$ and $F(M_2)$.

Let us recall the cluster fan $F'(rep A)$ of $A$ introduced in [11]. The cones of $F'(rep A)$ are spanned by $\{\delta_1, ..., \delta_p\}$ such that each $\delta_i$ is real indecomposable and $e(\delta_i, \delta_j) = 0$ for $i \neq j$. Note that the maximal cones of $F'(rep A)$ are precisely those spanned by the clusters.

**Definition 8.3.** Let $F(rep A)$ be the set of all cones spanned by $\{\delta_1, ..., \delta_p\}$ such that each $\delta_i$ is normal and $e(\delta_i, \delta_j) = 0$ for $i \neq j$. By Theorem 2.6 and Corollary 5.3, $F(rep A)$ forms a simplicial fan as well. We call it generalized cluster fan.

It follows from Lemma 5.14 that

**Proposition 8.4.** The fan $F(M)$ is a coarsening of the generalized cluster fan $F(rep A)$.

**Remark 8.5.** In view of Lemma 8.2, Propositions 8.4 and 9.5, $F(rep A)$ can be viewed heuristically as the normal fan of the infinite-dimensional representation $\bigoplus_{M \in rep A} M$.

Next, we discuss the 1-skeleton of $N(M)$. We will represent an edge of $N(M)$, that is, an 1-dimensional face of $N(M)$, by $L_0L_1$ where $L_0$ and $L_1$ are vertex subrepresentations. Recall the functors $t_\delta$ and $\tilde{t}_\delta$ in Section 4.

**Proposition 8.6.** If $L_-L_+$ is an edge in $N(M)$, then either $L_- \subset L_+$ or $L_+ \subset L_-$. Say $L_- \subset L_+$, then $L_- = t_\delta(M)$ and $L_+ = \tilde{t}_\delta(M)$ for any $\delta$ in the interior of $F_{L_-L_+}(M)$. 
Moreover, we have the following.

1. \( \delta_+(L_+/L_-) \geq 0 \) for any \( \delta_+ \in F_{L_+}(M) \) and \( \delta_-(L_+/L_-) \leq 0 \) for any \( \delta_- \in F_{L_-}(M) \) with the equality holding only when \( \delta_\pm \in F_{L_-L_+}(M) \).

2. If \( F_{L_-}(M) \) is spanned by a regular cluster, then \( L_+/L_- \) is a direct sum of isomorphic real Schur representations.

Proof. The convex hull of \( \mathcal{L}(\delta, M) \) contains \( L_-L_+ \) for \( \delta \in F_{L_-L_+}(M) = F_{L_-}(M) \cap F_{L_+}(M) \). If \( \delta \) is in the interior of \( F_{L_-L_+}(M) \), then \( \delta \notin F_{L_-}(M) \) for any other vertex \( L \), so the convex hull of \( \mathcal{L}(\delta, M) \) is exactly \( L_-L_+ \). By Theorem 4.2, we have either \( L_- \subset L_+ \) or \( L_+ \subset L_- \).

For (1), \( L_+/L_- = i_\delta(M) / i_\gamma(M) \) is \( \delta \)-semistable by Theorem 4.2. If \( \delta_+ \in F_{L_+}(M) \setminus F_{L_-}(M) \), then \( \delta_+(L_-) < \delta_+(L_+) = f_M(\delta_+) \). Hence \( \delta_+(L_+ / L_-) > 0 \). Similarly, we get \( \delta_-(L_+ / L_-) < 0 \).

For (2), suppose that \( F_{L_-}(M) \) is spanned by a regular cluster \( \delta \). Then \( L_+/L_- \) is \( \delta \)-semistable for any \( \delta \in F_{L_-L_+}(M) \). There is only one element in \( \delta \) lying outside \( F_{L_-L_+}(M) \), so by Theorem 7.6, \( L_+/L_- \) must be an iterated extension of a real Schur representation \( E \). But \( \text{Ext}^1(E, E) = 0 \), so it has to be a direct sum of \( E \).

\[ \square \]

Definition 8.7. We assign the orientation \( L_0 \to L_1 \) for each edge \( L_0L_1 \) with \( L_0 \subset L_1 \). We call the resulting oriented graph the edge quiver of \( \mathbb{N}(M) \), denoted by \( \mathbb{N}_1(M) \). We call \( L_1 / L_0 \) an edge factor of \( M \).

For any two consecutive arrows \( L_0 \to L_1 \to L_2 \), we have that \( \text{Hom}(L_1/L_0, L_2/L_1) = 0 \). Indeed, by Proposition 5.8, we have that \( \text{Hom}(L_1, M/L_1) = 0 \). Since \( L_1/L_0 \) is a quotient of \( L_1 \) and \( L_2/L_1 \) is a subrepresentation of \( M/L_1 \), we have that \( \text{Hom}(L_1/L_0, L_2/L_1) = 0 \). However, there could be some homomorphism if the two arrows are not consecutive as shown in the following example.

Example 8.8. Consider the same quiver with potential as in Example 6.7. Let \( M = \text{Coker}(-1, 1, 1, 0) \). The Newton polytope of \( M \) was computed in [19, Example 6.10]. There is a path \( 0 \to S_3 \to S_34 \to L \to M \) in \( \mathbb{N}_1(M) \), where \( L \) is the vertex subrepresentation such that \( M/L_1 = S_3 \).

We see that \( \text{Hom}(S_3/0, M/L) = k \).

The point of this example is that the filtration of \( M \) given by a path from 0 to \( M \) in \( \mathbb{N}_1(M) \) may not be the Harder–Narasimhan filtration associated to any stability condition.

Definition 8.9. The exchange quiver of \( A \) is the dual graph of \( \mathbb{F}^r(\text{rep } A) \) with orientation given by \( \{ \delta_- \} \cup \delta_0 \to \{ \delta_+ \} \cup \delta_0 \) if \( \text{e}(\delta_-, \delta_+) > 0 \).

In the end of this section, we state a conjecture relating the maximal paths in a general representation of quiver to the Schur sequences introduced in [14]. Recall from [30] that if \( \text{rep}_a(Q) \) contains a Schur representation, then \( a \) is called a Schur root. \( a \) is called real if \( \langle a, a \rangle = 1 \); otherwise, it is called imaginary. It is also called isotropic if \( \langle a, a \rangle = 0 \). We denote \( L \perp N \) if \( \text{hom}(L, N) = \text{ext}(L, N) = 0 \), and denote \( y \perp \beta \) if \( \text{hom}(y, \beta) = \text{ext}(y, \beta) = 0 \). If \( y \perp \beta \), then the number of \( y \)-dimensional subrepresentations of a general \( (\beta + y) \)-dimensional representation is finite. We denote this number by \( y \circ \beta \).
Definition 8.10 [14]. We call two dimension vectors $\gamma$ and $\beta$ strongly perpendicular if $\gamma \circ \beta = 1$. We denote this by $\gamma \perp \perp \beta$. A sequence $(\beta_1, \beta_2, \ldots, \beta_s)$ of Schur root is called a Schur sequence if $\beta_i \perp \perp \beta_j$ for all $i < j$.

The Schur sequence was introduced as a simultaneous generalization of exceptional sequences, sequences arising from the canonical decomposition and the stable decomposition in the quiver setting. It has interesting applications in the Schubert calculus. We refer readers to [14] for more background and motivation.

Let $S(\alpha)$ be the set of all Schur sequences $(\beta_1, \beta_2, \ldots, \beta_r)$ (of any length) such that $\alpha$ is a positive integral combination $\alpha = \sum_{i=1}^r c_i \beta_i$ and $c_i = 1$ whenever $\beta_i$ is not real or isotropic.

Conjecture 8.11. There is a bijection between $S(\alpha)$ and the maximal paths in $N_1(\alpha)$.

Example 8.12. Let $Q$ be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, and $\alpha$ be the dimension vector $(3,5,2)$. Except for zero and itself, $N(M)$ has four vertex subrepresentations $L_1, L_2, L_3, L_4$ of dimension $(0,3,0), (0,0,2), (2,3,2), \text{and} (0,5,2)$, respectively (see Example 6.6). We list all paths from 0 to $M$ in $N_1(M)$ together with the corresponding Schur sequences.

$0 \to M$ (3, 5, 2)
$0 \to L_1 \to M$ $e_2, (3, 2, 2)$
$0 \to L_3 \to M$ (2, 3, 2), (1, 2, 0)
$0 \to L_2 \to L_3 \to M$ $e_3, (2, 3, 0), (1, 2, 0)$
$0 \to L_1 \to L_4 \to M$ $e_2, (0, 1, 1), e_1$
$0 \to L_2 \to L_4 \to M$ $e_3, e_2, e_1$

9 | EXAMPLES

In this section, we give some concrete examples of $F(M)$ and $N_1(M)$. There are at least two parameters that we can vary. One is the representation $M$, and the other is the algebra $A$.

9.1 The case when $M = A$

Lemma 9.1. A vertex subrepresentation $I$ of $A$ is a two-sided ideal of $A$. Dually, for a vertex quotient representation $I^*$ of $A^*$, $I$ is a two-sided ideal of $A$.

Proof. Recall our convention that all modules are right. We need to show that $AI \subseteq I$. If not, there is some $a$ such that $aI \nsubseteq I$. By the Wedderburn–Malcev theorem, we can assume that $a$ is in the radical of $A$. Then $(1 + a)I \nsubseteq I$ as well. But $1 + a$ is invertible, so $(1 + a)I$ has the same dimension vector as $I$. This contradicts the fact that $I$ is a vertex subrepresentation (see Proposition 5.8). □

Proposition 9.2. Assume that $\tilde{f}_A(\tilde{\delta}) = \hom(A, \tilde{\delta})$. If $\tilde{\delta} \in \tilde{F}_{A/I}(A)$ and $\tilde{d} \in \Pi \hom(\tilde{\delta})$, then $\tilde{d}$ is surjective with the same kernel after tensoring with $A/I$. Dually, assume that $f_{A^*}(\delta) = \hom(\delta, A^*)$. If $\delta \in \tilde{F}_{A^*/I^*}(A^*)$ and $d \in \Pi \hom(\delta)$, then $d$ becomes injective with the same cokernel when tensoring with $A/I$. 


Proof. $\tilde{\delta} \in \tilde{T}_{A/I}(A)$ implies that $\tilde{\delta}(A/I) = \text{hom}(A, \tilde{\delta})$. By the dual of Lemma 4.1, we have that $\text{hom}(A/I, \tilde{\delta}) = \text{hom}(A, \tilde{\delta})$. This implies that the kernel of $\tilde{d}$ does not change after tensoring with $A/I$. Then $\text{dim Ker}(\tilde{\delta}) = \text{hom}(A/I, \tilde{\delta}) = \tilde{\delta}(A/I) = \tilde{\delta}((A/I)^*)$, which implies that $\tilde{\delta}$ is surjective after tensoring with $A/I$. □

Example 9.3. Consider the quiver

![Quiver diagram](image)

with relation $ab$. Except for the two trivial ones, $N(A)$ has seven vertex subrepresentations as listed in the left column. The middle and right columns are the corresponding ideals and dual cones.

| (P₁, S₃, S₃) | ⟨e₁, e₃⟩ | (e₁, −e₂, −e₃) |
| S₂ ⊕ S₃, P₂, S₃ | ⟨e₂, e₃⟩ | (−e₁, e₂ − e₁, −e₃) |
| (S₂, S₃, S₃) | ⟨e₃⟩ | (e₁, e₂ − e₁, −e₃) |
| (S₂, P₂, 0) | ⟨e₂⟩ | (−e₁, −e₂, e₃ − e₂ − e₁) |
| (P₁, P₂, 0) | ⟨e₁, e₂⟩ | (e₁, −e₂, e₃ − e₂ − e₁) |
| (P₁, 0, 0) | ⟨e₁⟩ | (−e₁, e₂ − e₁, e₃ − e₁, e₃ − e₂ − e₁) |
| (S₂, 0, 0) | ⟨a⟩ | (e₁, e₃ − e₂ − e₁, e₃ − e₁) |

9.2 | Cluster-finite algebras

Definition 9.4. We call an algebra cluster-finite if it has only finitely many indecomposable $E$-rigid representations.

A cluster-finite algebra may not be representation-finite. For example, the preprojective algebra of Dynkin type (other than $A_i$, $i < 5$). We showed that the cluster fan of a cluster-finite algebra is complete ([11, Proposition 6.1]). In particular, the cluster fan is the same as the generalized cluster fan.

Proposition 9.5. Suppose that $A$ is cluster-finite. Let $M$ be the direct sum of all $E$-rigid representations. Then the dual fan $F(M)$ is the cluster fan of $A$, and the edge quiver $N₁(M)$ is the exchange quiver of $A$.

Proof. The claim about the dual fan is a direct consequence of Theorem 5.17 and the completeness of the cluster fan. So, if $(\delta_−, \delta₊)$ is an exchange pair and $\delta_± \in F_{L_±}(M)$, then there is an arrow between $L_-$ and $L_+$. We need to show that the arrow has the correct direction. By Proposition 8.6, it suffices to show that $\delta_−(\text{dim } L₀) < 0$ and $\delta_+(\text{dim } L₀) > 0$ where $L₀ = \overline{\delta₀}(M)$ for $\delta₀ \in F_{\overline{L₀}}(M)$. Apply Hom$(-, L₀)$ to the exchange triangle $d_+ \rightarrow d₀ \rightarrow d_− \rightarrow d_+[1]$ of Proposition 2.11, and we get the exact sequence

$$0 \rightarrow \text{Hom}(eδ_−, L₀) \rightarrow 0 \rightarrow \text{Hom}(δ₊, L₀) \rightarrow \text{E}(eδ_−, L₀) \rightarrow 0 \rightarrow \text{E}(δ₊, L₀) \rightarrow 0.$$  

Hence $\delta_−(\text{dim } L₀) < 0$ and $\delta_+(\text{dim } L₀) > 0$. □
Example 9.6. We continue with Example 9.3. There are nine indecomposable representations of $A$. Except for indecomposable projective, injective, and simple representations, they are $R = \text{Coker}(1, -1, 0)$ and $T = \text{Coker}(1, 1, -1)$. They are either $E$-rigid or $\tilde{E}$-rigid. It turns out that to get the cluster fan of $A$, we do not need all of them as in Proposition 9.5. We have two minimal choices. One is $P_2, P_3, I_1, I_2, R, T$, and the other is $P_1, P_2, P_3, I_1, I_2, I_3$. Here is the polytope for the first choice. We also display the edge quiver and the edge factors. The 18 vertices correspond to the 18 clusters.

The statement for the dual cluster in Corollary 9.7 also holds here because we can check that each cluster is regular.

Let $(Q, P)$ be a nondegenerate quiver with potential such that its Jacobian algebra $A$ is finite-dimensional and cluster-finite. Let $C(Q)$ be the cluster algebra associated to the quiver $Q$. Then the results in [16] imply that the map sending each $\delta$ to the corresponding cluster variables induces an isomorphism from the cluster fan of $A$ to the ordinary cluster fan of $C(Q)$.

Corollary 9.7. Let $M$ be the direct sum of all $E$-rigid representations of $A$. Then the dual fan $F(M)$ is the cluster fan of $C(Q)$, and the edge quiver $N_1(M)$ is the exchange quiver of $C(Q)$. Moreover, the signed dimension vectors of the real Schur representations attached to the arrows from/to a fixed vertex $L$ are the signed $c$-vectors dual to the $g$-vectors in the corresponding cluster.

Remark 9.8. We recover and generalize the main result in [5], where the authors obtain the similar result for Dynkin quivers (without potentials). In such cases, the Newton polytope is the so-called generalized associahedron [23].

We conjecture that any strict subset of $\text{ind}(M)$ cannot do the job. More precisely, let $N$ be a direct sum of elements in any strict subset, then $F(N)$ is not the cluster fan of $C(Q)$. We are able to prove this conjecture for the Dynkin quivers. By contrast, we will see that for the preprojective algebras of Dynkin type, we need very few $E$-rigid representations, namely, projective ones only.
9.3 | Preprojective algebras

In this subsection, we let \( A \) be the preprojective algebra of a Dynkin diagram. In [4], three authors showed that if \( M \) is a general representation in some irreducible component of \( \text{rep}_\nu(A) \), then \( \text{N}(M) \) is the \( MV \) polytope of certain basis element of \( k[U] \) associated to \( M \), where \( U \) is the maximal unipotent group of the simple, connected, simply connected Lie group of the same Dynkin type. This is also part of our motivation for studying the Newton polytope of a representation.

An interesting result in [35] says that the maximal rigid presentations \( d_w \) can be labeled by the elements \( w \) in the Weyl group of the same Dynkin type. The cokernel of \( d_w \) is the ideal \( I_w \) of \( A \) introduced in [9].

**Proposition 9.9.** The vertices of \( \text{N}(A) \) are labeled by the ideals \( I_w \), and \( F_{I_w}(A) \) is the cluster corresponding to \( d_w \). So \( F(A) \) is the cluster fan \( F(\text{rep } A) \), which is a Weyl fan.

**Proof.** Let \( \delta_w \) be the weight vector of the maximal rigid presentation \( d_w \). We claim that \( t_{\delta_w}(A) = I_w \), which implies that \( I_w \) is the vertex subrepresentation of \( A \) such that \( F_{I_w}(A) \) is the cluster corresponding to \( d_w \). It is known (e.g., [4]) that the \( I_w \) determines a torsion pair

\[
T(I_w) = \{ M \in \text{rep } A | \text{Ext}^1(I_w, M) = 0 \} \quad \text{and} \quad F(I_w) = \{ M \in \text{rep } A | \text{Hom}(I_w, M) = 0 \}.
\]

On the other hand, recall from Definition 4.3 that the torsion-free class \( F(\delta_w) \) associated to \( \delta_w \) is \( F(I_w) \) as well. So, its associated torsion class is \( T(I_w) \). Now the claim follows from the exact sequence \( 0 \to I_w \to A \to A/I_w \to 0 \). Indeed, from \( I_w \in T(I_w) \) and \( A/I_w \in F(I_w) \) [4], we conclude that \( t_{\delta_w}(A) = I_w \).

Since \( F(A) \) is a coarsening of \( F(\text{rep } A) \), we must have the equality \( F(A) = F(\text{rep } A) \), and thus, there are no more vertices other than \( I_w \).  

**Example 9.10.** Let \( T_{ij} \) be the indecomposable representation with socle \( S_i \) and top \( S_j \), and \( R_2 \) (respectively, \( R_2^2 \)) be the \((1,1,1)\)-dimensional indecomposable representation with socle \( S_2 \) (respectively, top \( S_2 \)). We display the Newton polytope of \( A \) for Dynkin type \( A_3 \). The vertices are labeled by the 24 permutations of the symmetric group \( \mathfrak{S}_4 \).
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