A Computer Verification of a Conjecture About The Erdős-Mordell Curve

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Abstract — In this paper we consider Erdős-Mordell inequality and its extension in the plane of triangle to the Erdős-Mordell curve. Algebraic equation of this curve is derived, and using modern computer tools in mathematics, we verified one conjecture that relates to Erdős-Mordell curve.

Keywords — algebraic equation, Erdős-Mordell curve, computer verification, conjecture.

I. INTRODUCTION

In theory of geometrical inequalities [1]-[4], a special place holds the Erdős-Mordell inequality for triangle:

\[ R_A + R_B + R_C \geq 2(r_a + r_b + r_c) \]  \hspace{1cm} (1)

where \( R_A, R_B \) and \( R_C \) are the distances from the arbitrary point \( M \) in the interior of \( \Delta ABC \) to the vertices \( A, B \) and \( C \) respectively, and \( r_a, r_b \) and \( r_c \) are the distances from the point \( M \) to the sides \( BC, CA \) and \( AB \) respectively (Fig. 1).

The importance of the Erdős-Mordell inequality can be observed in a recent result of V. Pambuccian which proved that this inequality is an equivalent of negative curvature of absolute plane [5]. In the paper [6], is given an extension of the Erdős-Mordell inequality from the interior of triangle to the interior of the Erdős-Mordell curve. Also, the Erdős-Mordell inequality has its interpretation in the following geometric problem: "Which set of paths/communication lines is the fastest/shortest?" [7].

In this paper we give the algebraic equation of the Erdős-Mordell curve, and in addition, in the case of equilateral triangle, we give a conjecture about the Erdős-Mordell curve brought out from [6], which is verified on the large number of triangles. Value of constant \( \varepsilon_0 \) from the conjecture is determined by use of symbolic-numerical functions of MatLAB and the conjecture is numerically tested in programming language Java with the use of visual representation of equidistant region in \( \varepsilon \geq \varepsilon_0 \).

II. ERDŐS-MORDELL CURVE

In paper [6], a set of points \( E \) was introduced, which fulfills:

\[ R_A + R_B + R_C \geq \left( \frac{c}{b} + \frac{b}{c} \right) r_a + \left( \frac{a}{c} + \frac{c}{a} \right) r_b + \left( \frac{a}{b} + \frac{b}{a} \right) r_c \]  \hspace{1cm} (2)

where \( a = |BC|, b = |CA| \) and \( c = |AB| \). The set \( E \) is determined in the intersection of areas:

\[ E_A = \left\{ (x,y) \mid R_A \geq \frac{c}{a} r_a + \frac{b}{a} r_c \right\} \]  \hspace{1cm} (3)

\[ E_B = \left\{ (x,y) \mid R_B \geq \frac{c}{b} r_a + \frac{a}{b} r_c \right\} \]  \hspace{1cm} (4)

\[ E_C = \left\{ (x,y) \mid R_C \geq \frac{b}{c} r_a + \frac{a}{c} r_c \right\} \]  \hspace{1cm} (5)

The previous sets which correspond to the points \( A, B \) and \( C \) are corner areas, see [6]. If (2) is true, then the Erdős-Mordell inequality is also fulfilled. In the paper [6], the Erdős-Mordell curve is defined by the following equation:

\[ R_A + R_B + R_C = 2(r_a + r_b + r_c) \]  \hspace{1cm} (6)

where

\[ R_A = \sqrt{x^2 + (y - r)^2}, \quad r_a = \frac{|y(q - p)|}{\sqrt{(q - p)^2 + 1}} = |y| \]

\[ R_B = \sqrt{(x - p)^2 + y^2}, \quad r_b = \frac{-q(y - r) - rx}{\sqrt{r^2 + q^2}} \]

\[ R_C = \sqrt{(x - q)^2 + y^2}, \quad r_c = \frac{-p(y - r) - rx}{\sqrt{r^2 + p^2}} \]

where \( A = (0,r), B = (p,0) \) and \( C = (q,0) \). Let us denote \( E' \) as interior of the Erdős-Mordell curve; then the set \( E' \) is maximal set of points in the plane where the Erdős-Mordell inequality is true. In the paper [6] it is proven that the set \( E' \) contains the set \( E \), and the set \( E \) contains initial triangle \( \Delta ABC \). The curve (6) is a union of parts of algebraic curves of eight order (Fig. 2), which we are proving in this section.
We start from the Erdös-Mordell curve defined by equality (6). Let us denote expressions:
\[ Q_1 = x^2 + (y - r)^2, \]
\[ Q_2 = (x - p)^2 + y^2, \]
\[ Q_3 = (x - q)^2 + y^2 \]
and denote a sum of absolute values
\[ S = 2(r_x + r_y + r_z). \]

Let us consider following transformations of equality (6):
\[ S_1 = Q_1 + Q_2 + Q_3 = S - \sqrt{Q_1}, \]
\[ S_2 = 2\sqrt{Q_1} + 2\sqrt{Q_2} + 2\sqrt{Q_3} = S^2 - 2S\sqrt{Q_1} + Q_1, \]
\[ S_3 = 8S\sqrt{Q_1}\sqrt{Q_2}\sqrt{Q_3} = S^4 + Q_1 - Q_2 - Q_3, \]
\[ S_4 = 64S^2Q_1Q_2Q_3 = (S^2 + Q_1 - Q_2 - Q_3)^5 - 4Q_1Q_2Q_3 - 4S^2Q_1^3. \]

If we analyze single cases of absolute values in the last equality, in subexpression S given by (7), then we get algebraic equalities of eight order, which fulfill the Erdös-Mordell curve, piece by piece. By application of previous transformation on different ordering of terms \( Q_1, Q_2, \) and \( Q_3, \) there follows a conclusion that the Erdös-Mordell curve represents the union of parts of algebraic curves of eight order.

The Erdös-Mordell curve gives us a set of telecommunication paths of length \( 2(r_x + r_y + r_z) \) in a sense of telecommunication problem in [7]. Let us remark that the Erdös-Mordell curve is always in the exterior of the given triangle, and in the special case of equilateral triangle, it additionally contains the center of the triangle as a single isolated point.

III. NUMERICAL DETERMINING OF \( \varepsilon_0 \)

The open problem given in [6] is proposed as: to prove or disprove that there exists a positive number \( \varepsilon_0 \) such that the area of \( E' \) is larger than \( 1 + \varepsilon_0 \) times the area of the triangle for every triangle in the plane. In [6], there was a conjecture stated: for the finite area of \( E' \), the value \( \varepsilon_0 \) is determined in the case of equilateral triangle \( \triangle ABC \). Let us denote:
\[ P_{\triangle ABC} - \text{area of triangle } \triangle ABC \]
\[ P_{E'} - \text{area of interior parts of the Erdös-Mordell curve.} \]

In this section we describe procedure of numerical determining of constant
\[ \varepsilon_0 = \frac{(P_{E'} - P_{\triangle ABC})}{P_{\triangle ABC}} \]
in case of equilateral triangle \( \triangle ABC \) with vertices \( A=A(0,\sqrt{3}), B=B(-1,0), C=C(1,0) \) and area \( P_{\triangle ABC} = \sqrt{3} \). Let us form intersection points \( P_i = P_i(x_i, y_i), i=1..6, \) of the Erdös-Mordell curve with straight lines set through the sides of triangle (Fig. 3):
\[ x_i = -x_i = -\frac{\sqrt{3}}{\sqrt{15} - 8\sqrt{3}} = -1.61966 \ldots \]
\[ x_2 = -x_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2\sqrt{15} - 8\sqrt{3}} = -1.30983 \ldots \]
\[ x_3 = -x_a = -\frac{1}{2} + \frac{\sqrt{3}}{2\sqrt{15} - 8\sqrt{3}} = 0.30983 \ldots \]
\[ y_1 = y_a = 0 \]
\[ y_2 = y_3 = \frac{3}{2} - \frac{\sqrt{3}}{2\sqrt{15} - 8\sqrt{3}} = -0.53664 \ldots \]
\[ y_3 = y_a = \frac{3}{2} + \frac{3}{2\sqrt{15} - 8\sqrt{3}} = 2.26869 \ldots \]

The previous points are obtained by using the symbolic command solve of MatLAB.
Let us define branch $C_1 = \text{arch } P_1P_2$ with the appropriate function $y=f_1(x)$ and branch $C_2 = \text{arch } P_2P_3$ which is defined by the function $y=f_2(x)$. Expressions for $y=f_1(x)$ and $y=f_2(x)$ are determined by using the symbolic command solve of MatLAB, thus solving by $y$ the appropriate algebraic equations of eight order by $x$, piece by piece. Numerical value of constant $\varepsilon_0$ is given by integration:

$$
\varepsilon_0 = \sqrt{3} \left[ \int_{\varepsilon_1}^{\varepsilon_2} \left(-f_1(x)\right) dx + \int_{\varepsilon_1}^{\varepsilon_2} \left(-f_2(x)\right) dx \right] + \int_{\varepsilon_1}^{\varepsilon_2} \left(-\sqrt{3}x + \sqrt{3} - f_3(x)\right) dx \right],
$$

(10)

Using numerical integration by MatLAB we get:

$$
\varepsilon_0 = 0.8140420779 \ldots
$$

(11)

Let us note that the previous calculation can be executed in any program that allows solving the polynomial equalities of higher order and numerical integration.

IV. JAVA APPLET FOR TESTING CONJECTURE AND VISUAL REPRESENTATION OF TRIANGLES FOR GIVEN $\varepsilon$

On the internet at Wolfram Demonstrations Project for The Erdös-Mordell Inequality [9] are available applications that present inner points of Erdös-Mordell curve of the selected triangle. In this paper, the special Java application for visual verification of conjecture from paper [6] will be presented. Let us denote the vertices $A=\text{A}(u,v)$, $B=\text{B}(-100,0)$ and $C=\text{C}(100,0)$ of the triangle $ABC$. Then, for fixed $A$ and arbitrary $M=M(x,y)$ we calculate values:

$$
r_u = \left| y \right|,
$$

$$
r_v = \frac{\left| x+100 \right|}{\sqrt{v^2+(x+100)^2}},
$$

$$
r_r = \frac{\left| x-100 \right|}{\sqrt{v^2+(x-100)^2}},
$$

$$
R_u = \sqrt{(x-u)^2 + (y-v)^2},
$$

$$
R_v = \sqrt{(x+100)^2 + y^2},
$$

$$
R_r = \sqrt{(x-100)^2 + y^2},
$$

$$
g = R_u + R_v + R_r - 2(r_u + r_v + r_r)
$$

(12)

For values $-600 \leq u, v \leq 600$ the vertex $A=\text{A}(u,v)$ with fixed $B$ and $C$, determines triangle $ABC$ in the visible area of the plane. For every such triangle $ABC$, the equation $g(x,y)=0$ determines the Erdös-Mordell curve and the numerical value of constant $\varepsilon = (P_{\text{abc}} - P_{\text{ABC}})/P_{\text{ABC}}$ when Erdös-Mordell curve is closed in domain $-1000 \leq x, y \leq 1000$. The condition for testing whether the Erdös-Mordell curve is closed is reduced to checking the following inequality $g(x,y) < 0$ for values $x = \pm1000$, $y = \pm1000$. Let us note that inequality $g(x,y) \geq 0$ determines all points in domain which are inside of the Erdös-Mordell curve. Since only the integer values with increment of one were used for the points, for each point with positive value of the function $g(x,y)$, value of 1 had been added to area of the Erdös-Mordell curve in square pixels. The integer values were used to improve speed of calculations. As the conjecture was tested on triangles with vertex $A=(u,v)$ where $-600 \leq u, v \leq 600$, exactly $1200^2 = 1.44 \cdot 10^6$ triangles were tested. Domain of testing points is wider than domain in which triangles are formed, as the Erdös-Mordell curve has at least 81% larger area than the triangle for which it is formed, and in many case much larger. For each triangle, if $-1000 \leq x, y \leq 1000$, values of $2000^2 = 4 \cdot 10^6$ points were tested for inequality which brings to exactly $1.44 \cdot 10^6 \cdot 4 \cdot 10^6 = 5.76 \cdot 10^{12}$ points processed.

One of the first limitations for application used to prove the conjecture lies in the fact that computers currently lack enough processing power to calculate data, and visually represent them in real time. As a way to circumvent this limitation part of data that would require hours of calculation was done with use of separate application and stored in form that is usable by applications for visualization. To improve the time of calculation, graphical processor unit was used, which allowed massive multithread computation.

The application for visualization of data consists of several areas. The most important area is the one that maps values of $\varepsilon$ as equidistant lines. As a visual verification of the conjecture, the color assigned to the value of constant $\varepsilon_0$ appears in the points where the equilateral triangle is formed with two other fixed points, and for $0 \leq \varepsilon < \varepsilon_0$ application shows empty screen, as application finds no values that fall in this range. Another area allows visual representation of the Erdös-Mordell curve for user-selected vertexes of triangle. This was achieved by methods of visualization of implicitly expressed surfaces. One of the limitations in visual representation is also the fact that the Erdös-Mordell curve has to be closed. The additional data, such for the area of the Erdös-Mordell curve, are also provided for further analysis in this area. Developed Java application will be available at site http://symbolicalgebra.esf.bg.ac.rs/Java-Applications/.

The conjecture processed in this paper, appertains to experimental mathematics [10]-[12]. Verification of conjecture in one part of visual domain can be expanded and verified also in much wider visual domain.

![Fig. 4. Color mapping for $\varepsilon$ from $\varepsilon_0$ to 0.825.](image-url)
V. CONCLUSION

In the presented paper it is proved that the Erdős-Mordell curve is a union of algebraic curves of eight order and numerical value of constant $\varepsilon_0$ is determined. In addition, Java applet was formed, for visual testing of conjecture that will be the subject of further researches.

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