ON GROTHENDIECK’S STANDARD CONJECTURES OF TYPE C+ AND D IN POSITIVE CHARACTERISTIC

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Abstract. Making use of topological periodic cyclic homology, we extend Grothendieck’s standard conjectures of type C+ and D (with respect to crystalline cohomology theory) from smooth projective schemes to smooth proper dg categories in the sense of Kontsevich. As a first application, we prove Grothendieck’s original conjectures in the new cases of linear sections of determinantal varieties. As a second application, we prove Grothendieck’s (generalized) conjectures in the new cases of “low-dimensional” orbifolds. Finally, as a third application, we establish a far-reaching noncommutative generalization of Berthelot’s cohomological interpretation of the classical zeta function and of Grothendieck’s conditional approach to “half” of the Riemann hypothesis. Along the way, following Scholze, we prove that the topological periodic cyclic homology of a smooth proper scheme X agrees with the crystalline cohomology theory of X (after inverting the characteristic of the base field).

1. Introduction

Let k be a perfect base field of positive characteristic p > 0, W(k) the associated ring of p-typical Witt vectors, and K := W(k)[1/p] the fraction field of W(k). Given a smooth projective k-scheme X, let $H^*_\text{crys}(X) := H^*_\text{crys}(X/W(k)) \otimes_{W(k)} K$ be the crystalline cohomology of X, $\pi^+_X$ the $i^{th}$ Künneth projector of $H^*_\text{crys}(X)$, $Z^*(X)_Q$ the Q-vector space of algebraic cycles on X, and $Z^*(X)_{Q/\sim_{\text{hom}}}$ and $Z^*(X)_{Q/\sim_{\text{num}}}$ the quotients of $Z^*(X)_Q$ with respect to the homological and numerical equivalence relations, respectively. Following Grothendieck [9] (see also Kleiman [14, 15]), the standard conjecture of type C+, denoted by $C^+(X)$, asserts that the even Künneth projector $\pi^+_X := \sum i \pi^{2i}_X$ is algebraic, and the standard conjecture of type D, denoted by $D(X)$, asserts that $Z^*(X)_{Q/\sim_{\text{hom}}} = Z^*(X)_{Q/\sim_{\text{num}}}$. Both these conjectures hold whenever $\dim(X) \leq 2$. Moreover, the standard conjecture of type C+ holds for abelian varieties (see Kleiman [15, 2. Appendix]) and also whenever the base field k is finite (see Katz-Messing [12]). In addition to these cases (and to some other scattered cases), the aforementioned important conjectures remain wide open.

A differential graded (dg) category $\mathcal{A}$ is a category enriched over complexes of k-vector spaces; see §4.1. Every (dg) k-algebra $A$ gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes (or, more generally, by algebraic stacks) since the category of perfect complexes $\text{perf}(X)$ of every quasi-compact quasi-separated k-scheme X (or algebraic stack $\mathcal{X}$) admits a canonical dg enhancement $\text{perf}_{\text{dg}}(X)$. When X is quasi-projective this dg enhancement is moreover unique; see Lunts-Orlov [21, Thm. 2.12].

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As explained in §7 below, given a smooth proper dg category $\mathcal{A}$ in the sense of Kontsevich, Grothendieck’s standard conjectures of type $C^+$ and $D$ admit noncommutative analogues $C^+_{\text{nc}}(\mathcal{A})$ and $D_{\text{nc}}(\mathcal{A})$, respectively.

**Theorem 1.1.** Given a smooth projective $k$-scheme $X$, we have the equivalences of conjectures $C^+(X) \equiv C^+_{\text{nc}}(\text{perf}_d(X))$ and $D(X) \equiv D_{\text{nc}}(\text{perf}_d(X))$.

Intuitively speaking, Theorem 1.1 shows that Grothendieck’s standard conjectures of type $C^+$ and $D$ belong not only to the realm of algebraic geometry but also to the broad setting of smooth proper dg categories. In what follows, we describe two of the manyfold applications\footnote{For example, Theorem 1.1 implies immediately that if two smooth projective $k$-schemes $X$ and $Y$ have (Fourier-Mukai) equivalent derived categories, then $C^+(X) \equiv C^+(Y)$ and $D(X) \equiv D(Y)$.} of this noncommutative viewpoint; consult also §11 below for a third application of this noncommutative viewpoint.

## 2. Application I: HPD-invariance

For a survey on Homological Projective Duality (=HPD), we invite the reader to consult Kuznetsov [19] and/or Thomas [33]. Let $X$ be a smooth projective $k$-scheme equipped with a line bundle $\mathcal{L}_X(1)$; we write $V := H^0(X, \mathcal{L}_X(1))^*$. Assume that the triangulated category $\text{perf}(X)$ admits a Lefschetz decomposition $\langle a_0, a_1(1), \ldots, a_{i-1}(i-1) \rangle$ with respect to $\mathcal{L}_X(1)$ in the sense of [20, Def. 4.1]. Following [20, Def. 6.1], let $Y$ be the HP-dual of $X$, $\mathcal{L}_Y(1)$ the HP-dual line bundle, and $Y \to \mathbb{P}(V^*)$ the morphism associated to $\mathcal{L}_Y(1)$. Given a linear subspace $L \subset V^*$, consider the linear sections $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^*)$ and $Y_L := Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$.

**Theorem 2.1** (HPD-invariance). Let $X$ and $Y$ be as above. Assume that $X_L$ and $Y_L$ are smooth, that $\dim(X_L) = \dim(X) - \dim(L)$ and $\dim(Y_L) = \dim(Y) - \dim(L^*)$, and that the conjecture $C^+_{\text{nc}}(\mathcal{A}^d_{0})$, resp. $D_{\text{nc}}(\mathcal{A}^d_{0})$, holds, where $\mathcal{A}^d_{0}$ stands for the dg enhancement of $A_0$ induced from $\text{perf}_d(X)$. Under these assumptions, we have the equivalence $C^+(X_L) \equiv C^+(Y_L)$, resp. $D(X_L) \equiv D(Y_L)$.

**Remark 2.2.** The linear section $X_L$ is smooth if and only if $Y_L$ is smooth; see [19, page 9]. Moreover, given any general linear subspace $L \subset V^*$, $X_L$ and $Y_L$ are smooth, and $\dim(X_L) = \dim(X) - \dim(L)$ and $\dim(Y_L) = \dim(Y) - \dim(L^*)$.

Making use of Theorem 2.1, we are now able to prove Grothendieck’s standard conjectures of type $C^+$ and $D$ in new cases. Here is one family of examples:

**Determinantal duality.** Let $U_1$ and $U_2$ be two $k$-vector spaces of dimensions $d_1$ and $d_2$, respectively, $V := U_1 \otimes U_2$, and $0 < r < \min\{d_1, d_2\}$ an integer.

Consider the determinantal variety $Z^r_{d_1, d_2} \subset \mathbb{P}(V)$ defined as the locus of those matrices $U_2 \to U_1^*$ with rank $\leq r$; recall that the condition (rank $\leq r$) can be described as the vanishing of the $(r+1)$-minors of the matrix of indeterminates:

\[
\begin{pmatrix}
x_{1,1} & \cdots & x_{1,d_2} \\
\vdots & \ddots & \vdots \\
x_{d_1,1} & \cdots & x_{d_1,d_2}
\end{pmatrix}.
\]

**Example 2.3** (Segre varieties). In the particular case where $r = 1$, the determinantal varieties agree with the classical Segre varieties. Concretely, $Z^1_{d_1, d_2}$ agrees with...
with Remark 7.5, the linear section below, this implies

$$\{(x_{1,2} : x_{1,2} : x_{2,1} : x_{2,2}) \mid \det \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = 0\} \subset \mathbb{P}^3.$$  

In contrast with the Segre varieties, the determinantal varieties $Z_{d_1,d_2}^r$ are not smooth. The singular locus of $Z_{d_1,d_2}^r$ consists of those matrices $U_2 \to U_1^*$ with rank $< r$, i.e. it agrees with the closed subvariety $Z_{d_1,d_2}^{r-1}$. Nevertheless, it is well-known that $Z_{d_1,d_2}^r$ admits a canonical Springer resolution of singularities given by the (incidence) projective bundle $X_{d_1,d_2}^r := \mathbb{P}(U_2 \otimes \mathbb{Q}) \to Z_{d_1,d_2}^r$, where $\mathbb{Q}$ stands for the tautological quotient vector bundle of the Grassmannian $Gr(r,U_1)$.

Dually, consider the variety $W_{d_1,d_2}^r \subset \mathbb{P}(V^*)$, defined as the locus of those matrices $U_2^* \to U_1$ with corank $\geq r$, and the associated Springer resolution of singularities $Y_{d_1,d_2}^r := \mathbb{P}(U_2^* \otimes U^*) \to W_{d_1,d_2}^r$, where $U$ stands for the tautological sub-vector bundle of $Gr(r,U_1)$. As explained in [27, §1], work of Bernardara-Bolognesi-Faenzi [3] and Buchweitz-Leuschke-Vanden Bergh [7] implies that $X := X_{d_1,d_2}^r$ and $Y := Y_{d_1,d_2}^r$ are HP-dual to each other with respect to a certain Lefschetz decomposition $\text{perf}(X) = \langle A_0, A_1(1), \ldots, A_{d_2,r-1}(d_2r - 1) \rangle$. Moreover, the dg category $A_0^\text{dg}$ is Morita equivalent to a finite-dimensional $k$-algebra of finite global dimension $A$; consult the proof of [27, Prop. 1.5]. Thanks to Proposition 7.5 below, this implies that the conjectures $C_{+}^\text{uc}(A_0^\text{dg})$ and $D_{\text{uc}}(A_0^\text{dg})$ hold. Consequently, by combining Theorem 2.1 with Remark 2.2, we obtain the following result:

**Corollary 2.4.** Given any general linear subspace $L \subset V^*$, we have the equivalences of equivalences $C^+(X_L) \leftrightarrow C^+(Y_L)$ and $D(X_L) \leftrightarrow D(Y_L)$.

By construction, $\dim(X) = r(d_1 + d_2 - r) - 1$ and $\dim(Y) = r(d_1 - d_2 - r) + d_1d_2 - 1$. Consequently, the associated linear sections have the following dimensions:

$$\dim(X_L) = r(d_1 + d_2 - r) - 1 - \dim(L) \quad \text{and} \quad \dim(Y_L) = r(d_1 - d_2 - r) - 1 + \dim(L).$$

Since Grothendieck’s standard conjectures of type C and D hold in dimensions $\leq 2$, we hence obtain from Corollary 2.4 the following result(s):

**Theorem 2.5** (Linear sections of determinantal varieties). Let $X_L$ and $Y_L$ be smooth linear sections of determinantal varieties as in Corollary 2.4.

(i) When $r(d_1 + d_2 - r) - 1 - \dim(L) \leq 2$, the conjectures $C^+(Y_L)$ and $D(Y_L)$ hold.

(ii) When $r(d_1 - d_2 - r) + \dim(L) \leq 2$, the conjectures $C^+(X_L)$ and $D(X_L)$ hold.

**Corollary 2.6** (Square matrices). Let $d_1 = d_2$. Given any general linear subspace $L \subset V^*$ of dimension $r^2 + i$, $1 \leq i \leq 3$, the conjectures $C^+(X_L)$ and $D(X_L)$ hold.

To the best of the author’s knowledge, Theorem 2.5 (and Corollary 2.6) is new in the literature. In particular, it proves Grothendieck’s standard conjecture of type D in several new cases; consult Remarks 2.7-2.8 below. For example, note that in Corollary 2.6 the linear section $X_L$ is of dimension $r((2d - r) - r) - 1 - i$, $1 \leq i \leq 3$, with $d := d_1 = d_2$. Therefore, by letting $d \to \infty$, we obtain infinitely many new examples of smooth projective $k$-schemes $X_L$, of arbitrary high dimension, satisfying Grothendieck’s standard conjecture of type D.

**Remark 2.7** (Standard conjecture of type B). Recall from [9] the definition of Grothendieck’s standard conjecture of type B (a.k.a. the standard conjecture of
Lefschetz type). This conjecture holds for the projective bundles $X$ and $Y$, is stable under hyperplane sections, and implies the standard conjecture of type $C^+$; see [14, §4]. Consequently, the standard conjecture of Lefschetz type yields an alternative “geometric” proof of Theorem 2.5 for the standard conjecture of type $C^+$.

Remark 2.8 (Standard conjecture of type I). Recall from [9] the definition of Grothendieck’s standard conjecture of type I (a.k.a. the standard conjecture of Hodge type). As explained in loc. cit., given any smooth projective $k$-scheme $X$, we have the implication $B(X) + I(X) \Rightarrow D(X)$. On the one hand, when the base field $k$ is of characteristic zero$^2$, the conjecture $I(X)$ holds (thanks to the Hodge index theorem). On the other hand, in positive characteristic, the conjecture $I(X)$ is only known to hold when $\dim(X) \leq 3$. Consequently, in contrast with Remark 2.7, the standard conjecture of Lefschetz type does not yield an alternative “geometric” proof of Theorem 2.5 for the standard conjecture of type D.

Finally, note that whenever $\mathbb{P}(L^\perp) \subset \mathbb{P}(V)$ does not intersects the singular locus of $Z_{d_1,d_2}^r$, we have $X_L = \mathbb{P}(L^\perp) \cap Z_{d_1,d_2}^r$. In other words, $X_L$ is a linear section of a determinantal variety. Here are some examples:

Example 2.9 (Segre varieties). Let $r = 1$. In this case, as mentioned in Example 2.3, the determinantal variety $Z_{d_1,d_2}^1 \subset \mathbb{P}^{d_1 d_2 - 1}$ agrees with the smooth Segre variety of dimension $d_1 + d_2 - 2$. Therefore, thanks to Theorem 2.5(ii), given any general linear subspace $L \subset V^*$ of dimension $(d_2 - d_1) + i, 2 \leq i \leq 4$, the associated smooth linear section $X_L \subset Z_{d_1,d_2}^1$ has dimension $2d_1 - 2 - i, 2 \leq i \leq 4$, and satisfies Grothendieck’s standard conjectures of type $C^+$ and $D$.

Example 2.10 (Rational normal scrolls). Let $r = 1$ and $d_2 = 2$. In this case, the Segre variety $Z_{d_1,2}^2 \subset \mathbb{P}^{2d_1 - 1}$ agrees with the rational normal $d_1$-fold scroll $S_{1,1,1,1,1}$; see [10, Ex. 8.27]. Take $d_1 = 4$, resp. $d_1 = 5$, and choose a linear subspace $L \subset V^*$ of dimension 1 for which the associated hyperplane $\mathbb{P}(L^\perp) \subset \mathbb{P}^7$, resp. $\mathbb{P}(L^\perp) \subset \mathbb{P}^9$, does not contains any 3-plane, resp. 4-plane, of the rulling of $S_{1,1,1,1,1}$, resp. $S_{1,1,1,1,1}$. Note that this is a general condition on $L$. By combining Example 2.9 with [8, Prop. 2.5], we hence conclude that the rational normal 3-fold scroll $X_L = S_{1,1,2}$, resp. 4-fold scroll $X_L = S_{1,1,1,2}$, satisfies Grothendieck’s standard conjectures of type $C^+$ and $D$.

Example 2.11 (Square matrices). Let $d_1 = d_2 = 4$ and $r = 2$. In this case, the determinantal variety $Z_{4,4}^2 \subset \mathbb{P}^{15}$ has dimension 11 and its singular locus is the 6-dimensional Segre variety $Z_{4,4}^1$. Given any general linear subspace $L \subset V^*$ of dimension 7, the associated smooth linear section $X_L$ is 4-dimensional and, thanks to Corollary 2.6, it satisfies Grothendieck’s standard conjectures of type $C^+$ and $D$. Note that since $\text{codim}(L^\perp) = 7 > 6 = \dim(Z_{4,4}^1)$, the subspace $\mathbb{P}(L^\perp) \subset \mathbb{P}^{15}$ does not intersects the singular locus $Z_{4,4}^1$ of $Z_{4,4}^2$. Therefore, in all these cases, the 4-fold $X_L$ is a linear section of the determinantal variety $Z_{4,4}^2$.

3. Application II: Grothendieck’s Standard Conjectures for Orbifolds

Theorem 1.1 allows us to easily extend Grothendieck’s standard conjectures of type $C^+$ and $D$ from smooth projective $k$-schemes $X$ to smooth proper algebraic $k$-stacks $\mathcal{X}$ by setting $C^+(\mathcal{X}) := C^+_{nc}(\text{perf}_{dg}(\mathcal{X}))$ and $D(\mathcal{X}) := D_{nc}(\text{perf}_{dg}(\mathcal{X}))$.

$^2$The characteristic zero analogue of Theorem 2.5 was proved in [25, Thm. 1.11].
Theorem 3.1 (Orbifolds). Let $G$ be a finite group of order $n$, $X$ a smooth projective $k$-scheme equipped with a $G$-action, and $\mathcal{X} := [X/G]$ the associated global orbifold. If $p \nmid n$, then we have the following implications of conjectures

$$\sum_{\sigma \in G} \mathbb{C}^+(X^\sigma) \Rightarrow \mathbb{C}^+(\mathcal{X}) \quad \sum_{\sigma \in G} D(X^\sigma \times \text{Spec}(k[\sigma])) \Rightarrow D(\mathcal{X}),$$

where $\sigma$ is a cyclic subgroup of $G$. Moreover, whenever $k$ contains the $n^{th}$ roots of unity, the conjecture $D(X^\sigma \times \text{Spec}(k[\sigma]))$ can be replaced by $D(X^\sigma)$.

Theorem 3.1 leads automatically to a proof of Grothendieck’s standard conjectures of type $\mathbb{C}^+$ and $D$ for (global) orbifolds in the following cases:

Corollary 3.2. Assume that $p \nmid n$ and let $\mathcal{X} := [X/G]$ be as in Theorem 3.1.

(i) If the base field $k$ is finite, then the conjecture $\mathbb{C}^+(\mathcal{X})$ holds.

(ii) If $\dim(X) \leq 2$, if $\dim(X) = 3$ and $\mathbb{C}^+(X)$ holds, or if $X$ is an abelian variety and $G$ acts by group homomorphisms, then the conjecture $\mathbb{C}^+(\mathcal{X})$ holds.

(iii) If $\dim(X) \leq 2$, then the conjecture $D(\mathcal{X})$ holds.

(iv) Assume moreover that $k$ contains the $n^{th}$ roots of unity. If $\dim(X) = 3$ and $D(X)$ holds, then the conjecture $D(\mathcal{X})$ holds.

4. Preliminaries

4.1. Dg categories. For a survey on dg categories, we invite the reader to consult Keller’s ICM address [13]. Let $(\mathcal{C}(k), \otimes, k)$ be the category of dg $k$-vector spaces. A differential graded (=dg) category $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ and a dg functor $F : \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$. In what follows, we write dgcat$(k)$ for the category of (essentially small) dg categories and dg functors.

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\text{op}}$ has the same objects and $\mathcal{A}^{\text{op}}(x,y) := \mathcal{A}(y,x)$. A right dg $\mathcal{A}$-module is a dg functor $\mathcal{A}^{\text{op}} \to \mathcal{C}_{\text{dg}}(k)$ with values in the dg category $\mathcal{C}_{\text{dg}}(k)$ of dg $k$-vector spaces. Following [13, §3.2], the derived category $D(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of the category of right dg $\mathcal{A}$-modules with respect to the objectwise quasi-isomorphisms. In what follows, we write $D_c(\mathcal{A})$ for the triangulated subcategory of compact objects. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a Morita equivalence if it induces an equivalence on derived categories $D(\mathcal{A}) \simeq D(\mathcal{B})$; see [13, §4.6]. The tensor product $\mathcal{A} \otimes \mathcal{B}$ of dg categories is defined as follows: the set of objects is $\text{obj}(\mathcal{A}) \times \text{obj}(\mathcal{B})$ and $(\mathcal{A} \otimes \mathcal{B})((x,w),(y,z)) := \mathcal{A}(x,y) \otimes \mathcal{B}(w,z)$. A dg $\mathcal{A}-\mathcal{B}$-bimodule is a dg functor $\mathcal{A} \otimes \mathcal{B}^{\text{op}} \to \mathcal{C}_{\text{dg}}(k)$. An example is the dg $\mathcal{A}-\mathcal{B}$-bimodule $f_* \mathcal{B} : \mathcal{A} \otimes \mathcal{B}^{\text{op}} \to \mathcal{C}_{\text{dg}}(k)$, $(x,z) \mapsto \mathcal{B}(z,F(x))$ associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$. Following Kontsevich [16, 17, 18], a dg category $\mathcal{A}$ is called smooth if the dg $\mathcal{A}-\mathcal{A}$-bimodule $\text{id}_\mathcal{A}$ belongs to $D_c(\mathcal{A}^{\text{op}} \otimes \mathcal{A})$ and proper if $\sum_i \dim H^i \mathcal{A}(x,y) < \infty$ for any ordered pair of objects $(x,y)$. Examples include finite-dimensional $k$-algebras of finite global dimension $A$ as well as the dg categories of perfect complexes perf$_{\mathcal{A}_{\text{dg}}}(X)$ associated to smooth proper $k$-schemes $X$. In what follows, we write dgcat$_{\text{per}}(k)$ for the full subcategory of smooth proper dg categories.

4.2. Orbit categories. Let $(\mathcal{C}, \otimes, 1)$ be a $\mathbb{Q}$-linear, additive, symmetric monoidal category and $\mathcal{O} \in \mathcal{C}$ a $\otimes$-invertible object. The associated orbit category $\mathcal{C}/_{-\otimes \mathcal{O}}$ has the same objects as $\mathcal{C}$ and morphisms

$$\text{Hom}_{\mathcal{C}/_{-\otimes \mathcal{O}}}(a,b) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(a, b \otimes \mathcal{O}^{\otimes n}).$$
Given objects $a, b, c$ and composable morphisms $f = \{f_n\}_{n \in \mathbb{Z}}$ and $g = \{g_n\}_{n \in \mathbb{Z}}$, the $i$th-component of $g \circ f$ is defined as $\sum_n (g_{i-n} \otimes \mathcal{O}^{\otimes n}) \circ f_n$. The canonical functor

$$\iota: \mathcal{C} \to \mathcal{C}/-\otimes \mathcal{O}, \quad a \mapsto a \quad f \mapsto f = \{f_n\}_{n \in \mathbb{Z}},$$

where $f_0 = f$ and $f_n = 0$ if $n \neq 0$, is endowed with an isomorphism $\iota \circ (- \otimes \mathcal{O}) \Rightarrow \iota$ and is 2-universal among all such functors. Moreover, the category $\mathcal{C}/-\otimes \mathcal{O}$ is $\mathbb{Q}$-linear, additive, and inherits from $\mathcal{C}$ a symmetric monoidal structure making $\iota$ into a symmetric monoidal functor.

5. Topological Periodic Cyclic Homology

Thanks to the work of Hesselholt [11, §4] and Blumberg-Mandell\(^3\) [6, Thm. A], topological periodic cyclic homology\(^4\) yields a symmetric monoidal functor

$$TP_{\pm}(-)_{1/p}: \mathrm{dgc}(k) \to \mathrm{vec}_{\mathbb{Z}/2}(K)$$

with values in the category of finite-dimensional $\mathbb{Z}/2$-graded $K$-vector spaces; consult also [28, §4]. The following result, which is of independent interest, will be used below in the proof of Theorem 1.1 and Corollary 11.5.

**Theorem 5.2 (Scholze [24]).** Given a smooth proper $k$-scheme $X$, we have a natural isomorphism of $\mathbb{Z}/2$-graded $K$-vector spaces:

$$TP_{\pm}(\mathrm{perf}_{dg}(X)) \otimes \mathbb{Q} \cong \bigoplus_{i \in \mathrm{odd}} H_{\mathrm{crys}}^i(X) + \bigoplus_{i \in \mathrm{even}} H_{\mathrm{crys}}^i(X).$$

**Proof.** In order to simplify the exposition, we will write $TP(X)$ for the spectrum $TP(\mathrm{perf}_{dg}(X))$. Following Bhatt-Morrow-Scholze [5, §9.4], let us choose a prime number $l \neq p$ and consider the associated Adams operation $\psi_l$; since we are working over a perfect base field $k$ of characteristic $p > 0$, the spectrum $TP(X)$ is already $p$-complete. As proved in [5, Thm. 1.12(2)], the spectrum $TP(X)$ admits a “motivic” exhaustive decreasing $\mathbb{Z}$-indexed filtration $\{\mathrm{fil}^n TP(X)\}_{n \in \mathbb{Z}}$. After inverting $p$, this leads to an induced filtration $\{\mathrm{fil}^n TP(X)[1/p]\}_{n \in \mathbb{Z}}$ of $TP(X)[1/p]$. Since the Adams operation $\psi_l$ preserves this filtration, we hence obtain the $K$-linear homomorphisms

\begin{equation}
(\pi_*(\mathrm{fil}^n TP(X)[1/p]))^{\psi_l = l^n} \to (\pi_*(TP(X)[1/p]))^{\psi_l = l^n} \cong TP_\ast(\mathrm{fil}^n TP(X)[1/p])^{\psi_l = l^n}
\end{equation}

\begin{equation}
(\pi_*(\mathrm{fil}^n TP(X)[1/p]))^{\psi_l = l^n} \to (\pi_*(\mathrm{gr}^n TP(X)[1/p]))^{\psi_l = l^n},
\end{equation}

where $(-)^{\psi_l = l^n}$ stands for the $K$-linear subspace of those elements $v$ such that $\psi_l(v) = l^n \cdot v$. We claim that the above homomorphisms (5.4)-(5.5) are invertible. Consider the following cofiber sequence of spectra

\begin{equation}
\mathrm{fil}^n TP(X)[1/p] \to TP(X)[1/p] \to \frac{TP(X)[1/p]}{\mathrm{fil}^n TP(X)[1/p]} =: \text{cofiber}
\end{equation}

and the endomorphism of the associated long exact sequence of $K$-vector spaces:

$$\cdots \pi_{i+1}(\text{cofiber}) \to \pi_i(\mathrm{fil}^n TP(X)[1/p]) \to \pi_i(TP(X)[1/p]) \to \pi_i(\text{cofiber}) \to \cdots$$

$$\psi^l - l^n \quad \quad \psi^l - l^n \quad \quad \psi^l - l^n$$

\begin{equation}
\cdots \pi_{i+1}(\text{cofiber}) \to \pi_i(\mathrm{fil}^n TP(X)[1/p]) \to \pi_i(TP(X)[1/p]) \to \pi_i(\text{cofiber}) \to \cdots
\end{equation}

\(^3\)See also the work of Antieau-Mathew-Nikolaus [1].

\(^4\)Recall that topological periodic cyclic homology is defined as the Tate cohomology of the circle group acting on topological Hochschild homology.
Since $X$ is a smooth proper $k$-scheme, the dg category $\text{perf}^b_{K^2}(X)$ is smooth and proper. This implies that the $K$-vector spaces $\pi_*(TP(X)[1/p]) \simeq TP_*(X)[1/p]$ (and hence $\pi_*(\text{fil}^nTP(X)[1/p])$) are finite dimensional. Therefore, thanks to the general Lemma 5.10 below, in order to prove that the above homomorphisms (5.4) are invertible, it suffices to show that the following endomorphisms are invertible:

$$\psi^l - l^n : \pi_*(\text{cofiber}) \rightarrow \pi_*(\text{cofiber}).$$

Note that the spectrum $\text{TP}(X)[1/p]$ comes naturally equipped with the following exhaustive decreasing filtration $\left\{ \text{TP}(X)[1/p] \right\}_{m<n}$, whose graded pieces are equal to $\{ \text{gr}^mTP(X)[1/p] \}_{m<n}$. As proved in [5, Prop. 9.14], the induced endomorphism $\psi^l$ of $\pi_*(\text{gr}^mTP(X)[1/p])$ acts by multiplication with $l^m$. Since $m < n$ and $K$ is of characteristic zero, this implies that the following endomorphisms are invertible:

$$\psi^l - l^n = l^m - l^n : \pi_*(\text{gr}^mTP(X)[1/p]) \rightarrow \pi_*(\text{gr}^mTP(X)[1/p]) \quad m < n.$$ 

Now, a standard inductive argument using the isomorphisms (5.8) and the 5-lemma allows us to conclude that the above homomorphisms (5.4) are invertible. The proof of the invertibility of the homomorphisms (5.5) is similar: simply replace the above cofiber sequence (5.6) by the following fiber sequence of spectra:

$$\text{fil}^{n+1}TP(X)[1/p] \rightarrow \text{fil}^nTP(X)[1/p] \rightarrow \text{gr}^nTP(X)[1/p].$$

This concludes the proof of our claim.

As mentioned above, the induced endomorphism $\psi^l$ of $\pi_*(\text{gr}^nTP(X)[1/p])$ acts by multiplication with $l^n$. Thanks to (5.4)-(5.5), this leads to natural isomorphisms:

$$TP_*(X)_{1/p}^{\psi^l=l^n} \simeq (\pi_*(\text{gr}^nTP(X)[1/p]))^{\psi^l=l^n} = \pi_*(\text{gr}^nTP(X)[1/p]).$$

Hence, using the fact that the filtration $\{ \text{fil}^nTP(X)[1/p] \}_{n \in \mathbb{Z}}$ of $TP(X)[1/p]$ is exhaustive, we obtain the following natural isomorphisms of $K$-vector spaces:

$$TP_*(X)_{1/p} \simeq \bigoplus_{n \in \mathbb{Z}} TP_*(X)^{\psi^l=l^n}_{1/p} \simeq \bigoplus_{n \in \mathbb{Z}} \pi_*(\text{gr}^nTP(X)[1/p]).$$

Making use of the natural isomorphisms $\pi_*(\text{gr}^nTP(X)[1/p]) \simeq H^{2n-2n}_{\text{crys}}(X)$ constructed by Bhattach-Morrow-Scholze in [5, Thms. 1.10 and 1.12(4)], we hence conclude that $TP_*(X)_{1/p} \simeq \bigoplus_{n \in \mathbb{Z}} H^{2n-2n}_{\text{crys}}(X)$. This automatically yields the natural isomorphism of $\mathbb{Z}/2$-graded $K$-vector spaces (5.3), and so the proof is finished. □

**Lemma 5.10.** Consider the following commutative diagram of vector spaces:

$$\begin{array}{cccc}
V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & V_4 \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\
V_1' & \longrightarrow & V_2' & \longrightarrow & V_3' & \longrightarrow & V_4'.
\end{array}$$

Assume that the rows are exact, that $V_2$ and $V_3$ are finite dimensional, that $f_1$ is surjective, and that $f_4$ is injective. Under these assumptions, the induced homomorphism $\text{Ker}(f_2) \rightarrow \text{Ker}(f_3)$ is invertible.

**Proof.** On the one hand, a simple diagram chasing argument implies that the induced homomorphism $\text{Ker}(f_2) \rightarrow \text{Ker}(f_3)$ is surjective. On the other hand, a dual diagram chasing argument implies that the induced homomorphism $\text{coKer}(f_2) \rightarrow \text{coKer}(f_3)$ is injective. Making use of the equalities $\dim \text{Ker}(f_2) = \dim \text{coKer}(f_2)$ and $\dim \text{Ker}(f_3) = \dim \text{coKer}(f_3)$ and of the finite dimensionality of the vector
spaces \( \ker(f_2) \) and \( \ker(f_3) \), we hence conclude that the induced homomorphism \( \ker(f_2) \to \ker(f_3) \) is moreover injective. \( \square \)

6. Noncommutative motives

For a book, resp. survey, on noncommutative motives, we invite the reader to consult [26], resp. [29]. Recall from [26, §4.1] the definition of the category of noncommutative Chow motives \( \text{NChow}(k) \). By construction, this \( \Q \)-linear category is additive, rigid symmetric monoidal, and comes equipped with a symmetric monoidal functor \( U(-) \colon \text{dgcat} \to \text{NChow}(k) \). Moreover, we have

\[
\text{Hom}_{\text{NChow}(k)}(U(A), U(B)) \simeq K_0(D_c(A^{op} \otimes B))_\Q =: K_0(A^{op} \otimes B)_\Q.
\]

Recall from [28, Prop. 4.2] that the above functor (5.1) yields a \( \Q \)-linear symmetric monoidal functor \( TP_\pm(-)_{1/p} \colon \text{NChow}(k)_\Q \to \text{vect}_{\Z/2}(K) \). Under these notations, the category of noncommutative homological motives \( \text{NHom}(k)_\Q \) is defined as the idempotent completion of the quotient \( \text{NChow}(k)_\Q / \ker(TP_\pm(-)_{1/p}) \).

Given a \( \Q \)-linear, additive, rigid symmetric monoidal category \( (\mathcal{C}, \otimes, 1) \), its \( \mathcal{N} \)-ideal is defined as follows (\( \text{tr}(g \circ f) \) stands for the categorical trace of \( g \circ f \)):

\[
\mathcal{N}(a, b) := \{ f \in \text{Hom}_\mathcal{C}(a, b) \mid \forall g \in \text{Hom}_\mathcal{C}(b, a) \text{ we have } \text{tr}(g \circ f) = 0 \}.
\]

This is the largest \( \otimes \)-ideal of \( \mathcal{C} \) distinct from the entire category. Under these notations, the category of noncommutative numerical motives \( \text{NNum}(k)_\Q \) is defined as the idempotent completion of the quotient \( \text{NChow}(k)_\Q / \mathcal{N} \).

7. Noncommutative standard conjectures of type \( C^+ \) and \( D \)

Given a smooth proper dg category \( \mathcal{A} \), consider the even Künneth projector \( \pi^A_+ \) of the \( \Z/2 \)-graded \( K \)-vector space \( TP_\pm(A)_{1/p} \), as well as the following \( \Q \)-vector spaces\(^5\):

\[
K_0(\mathcal{A})_{\Q} / \sim_{\text{hom}} := \text{Hom}_{\text{NHom}(k)_\Q}(U(k)_\Q, U(\mathcal{A})_\Q)
\]

\[
K_0(\mathcal{A})_{\Q} / \sim_{\text{num}} := \text{Hom}_{\text{NNum}(k)_\Q}(U(k)_\Q, U(\mathcal{A})_\Q).
\]

Under these notations, Grothendieck’s standard conjectures of type \( C^+ \) and \( D \) admit the following noncommutative counterparts:

**Conjecture** \( C^+_\nc(A) \): The even Künneth projector \( \pi^A_+ \) is algebraic, i.e. there exists an endomorphism \( \pi^A_+ \) of \( U(\mathcal{A})_\Q \) such that \( TP_\pm(\pi^A_+)^{1/p}_\Q = \pi^A_+ \).

**Conjecture** \( D^+_\nc(A) \): The equality \( K_0(\mathcal{A})_{\Q} / \sim_{\text{hom}} = K_0(\mathcal{A})_{\Q} / \sim_{\text{num}} \) holds.

**Remark** 7.1 (Morita invariance). Let \( \mathcal{A} \) and \( \mathcal{B} \) be two smooth proper dg categories. By construction, the functor \( U(-)^{-}_{\Q} \colon \text{dgcat} \to \text{NChow}(k)_\Q \) sends Morita equivalences to isomorphisms. Therefore, whenever \( \mathcal{A} \) and \( \mathcal{B} \) are Morita equivalent, we have \( C^+_\nc(\mathcal{A}) \Leftrightarrow C^+_\nc(\mathcal{B}) \) and \( D^+_\nc(\mathcal{A}) \Leftrightarrow D^+_\nc(\mathcal{B}) \).

**Remark** 7.2 (Odd Künneth projector). Let \( \pi^A_- \) be the odd Künneth projector of the \( \Z/2 \)-graded \( K \)-vector space \( TP_\pm(A)_{1/p} \). Note that if the even Künneth projector \( \pi^A_+ \) is algebraic, then the odd Künneth projector \( \pi^A_- \) is also algebraic: simply take for \( \pi^A_- \) the difference \( \text{id}_{U(\mathcal{A})_\Q} - \pi^A_+ \).

---

\(^5\)As explained in [28, §6], the \( \Q \)-vector space \( K_0(\mathcal{A})_{\Q} / \sim_{\text{num}} \) can be alternatively defined as the \( \Q \)-linearization of the quotient of \( K_0(\mathcal{A}) \) by the (left=right) kernel of the classical Euler pairing.
Remark 7.3 (Stability under tensor products). Given smooth proper dg categories \( A \) and \( B \), we have the equality \( \pi^A \otimes B = \pi^A \otimes \pi^B \otimes \pi^B \). Consequently, since \( TP_k(-)_{1/p} \) is an additive symmetric monoidal functor, we obtain the implication:

\[
C^+_\text{nc}(A) + C^+_\text{nc}(B) \Rightarrow C^+_\text{nc}(A \otimes B)
\]

Given smooth projective \( k \)-schemes \( X \) and \( Y \), the dg categories \( \text{perf}_{dg}(X \times Y) \) and \( \text{perf}_{dg}(X) \otimes \text{perf}_{dg}(Y) \) are Morita equivalent; see [30, Lem. 4.26]. Therefore, by combining (7.4) with Theorem 1.1, we obtain \( C^+(X) + C^+(Y) \Rightarrow C^+(X \times Y) \).

Proposition 7.5. Given a finite-dimensional \( k \)-algebra of finite global dimension \( A \), the conjectures \( C^+_\text{nc}(A) \) and \( D_{\text{nc}}(A) \) hold.

Proof. The proof is similar for both cases. Hence, we will address solely conjecture \( D_{\text{nc}}(A) \). Thanks to [31, Thm. 3.15], we have \( U(A)_Q \simeq U(A/J(A))_Q \), where \( J(A) \) stands for the Jacobson radical of \( A \). Let us write \( S_1, \ldots, S_m \) for the simple (right) \( A/J(A) \)-modules and \( D_1 := \text{End}_A(J(A)(S_1), \ldots, D_m := \text{End}_A(J(A)(S_m) \) for the associated division \( k \)-algebras. The Artin-Wedderburn theorem implies that the semi-simple quotient \( A/J(A) \) is Morita equivalent to \( D_1 \times \cdots \times D_m \). Moreover, the center \( Z_i \) of \( D_i \) is a finite field extension of \( k \) and \( D_i \) is a central simple \( Z_i \)-algebra. Making use of [31, Thm. 2.1], we hence conclude that \( U(D_i)_Q \simeq U(Z_i)_Q \). Consequently, thanks to Theorem 1.1, we obtain the equivalences of conjectures:

\[
D_{\text{nc}}(A) \Leftrightarrow D_{\text{nc}}(A/J(A)) \Leftrightarrow D_{\text{nc}}(Z_1) + \cdots + D_{\text{nc}}(Z_m) \Leftrightarrow D(Z_1) + \cdots + D(Z_m) .
\]

The proof follows now from the fact that \( \dim(\text{Spec}(Z_i)) = 0 \) for every \( i \). \( \square \)

8. Proof of Theorem 1.1

Type \( C^+ \). Let \( \text{Chow}(k)_Q \) be the classical category of Chow motives; see Manin [22]. By construction, this \( \mathbb{Q} \)-linear category is additive, rigid symmetric monoidal, and comes equipped with a symmetric monoidal functor \( h(-)_Q : \text{SmProj}(k)^{op} \to \text{Chow}(k)_Q \) defined on smooth projective \( k \)-schemes. Crystalline cohomology gives rise to a symmetric monoidal functor \( H^\text{crys}_i : \text{Chow}(k)_Q \to \text{ect}_{Z}(K) \) with values in the category of finite-dimensional \( \mathbb{Z} \)-graded \( K \)-vector spaces. By composing it with the functor \( \text{vect}_{Z}(K) \to \text{vect}_{Z/2}(K) \) that sends \( \{ V_i \}_{i \in \mathbb{Z}} \) to \( (\oplus_{i \text{even}} V_i, \oplus_{i \text{odd}} V_i) \), we hence obtain the following \( \mathbb{Q} \)-linear symmetric monoidal functor:

\[
(8.1) \quad \text{Chow}(k)_Q \to \text{vect}_{Z/2}(K) \quad h(X)_Q \to (\bigoplus_{i \text{even}} H^i_{\text{crys}}(X), \bigoplus_{i \text{odd}} H^i_{\text{crys}}(X)) .
\]

Recall from [26, Thm. 4.3] that there exists a \( \mathbb{Q} \)-linear, fully-faithful, symmetric monoidal functor \( \Phi \) making the following diagram commute

\[
(8.2) \quad \begin{array}{ccc}
\text{SmProj}(k)^{op} & \xrightarrow{X \mapsto \text{perf}_{dg}(X)} & \text{dgcat}^{sp}_k(k) \\
\downarrow h(-)_Q & & \downarrow U(-)_Q \\
\text{Chow}(k)_Q & \xrightarrow{i} & \text{NChow}(k)_Q \\
\downarrow \Phi & & \\
\text{Chow}(k)_Q / - \otimes Q(1) & \rightarrow & \end{array}
\]
there exists an endomorphism $\pi$ of the Chow motive $\mathfrak{h}(X)_Q$ such that $H^*_\text{cryst}(\pi_X^+) = \pi_X^+$. Thanks to the natural isomorphism (5.3) and to the commutative diagram (8.2), the composition (8.3) is naturally isomorphic to the above functor (8.1). Consequently, by taking the image of $\pi_X^+$ under the composition $\Phi \circ \iota$, we conclude that the conjecture $\text{C}^+_\text{nc}(\text{perf}_{dg}(X))$ also holds.

Assume now that the conjecture $\text{C}^+_\text{nc}(\text{perf}_{dg}(X))$ holds, i.e. that there exists an endomorphism $\pi^+_{X}$ of $U(\text{perf}_{dg}(X))_{\otimes}$ such that $TP_{\pm}(\pi^+_{X})_{1/p} = \pi^+_{\text{perf}_{dg}(X)}$. Thanks to the commutativity of the diagram (8.2) and to the fully-faithfulness of the functor $\Phi$, the endomorphism $\pi^+_{X}$ corresponds to an endomorphism $\{\pi^+_{n}\}_{n \in \mathbb{Z}}$ of the object $\mathfrak{h}(X)_Q$ in the orbit category $\text{Chow}(k)_{Q/\otimes Q(1)}$. Moreover, since the composition (8.3) is naturally isomorphic to (8.1), the image of $\{\pi^+_{n}\}_{n \in \mathbb{Z}}$ under the composition $TP_{\pm}(-)_{1/p} \circ \Phi$ agrees with the endomorphism $(id,0)$ of the $\mathbb{Z}/2\mathbb{Z}$-graded $K$-vector space $(\bigoplus_{i \text{even}} H^i_{\text{cryst}}(X), \bigoplus_{i \text{odd}} H^i_{\text{cryst}}(X))$. Note that the following morphism

$$\pi^+: \mathfrak{h}(X)_Q \to \mathfrak{h}(X)_Q(n)$$

in $\text{Chow}(k)_{Q}$, where $\mathfrak{h}(X)_Q(n)$ stands for $\mathfrak{h}(X)_Q \otimes Q(1)^{\otimes n}$, induces an homomorphism of degree $-2n$ in crystalline cohomology theory:

$$H^*_\text{cryst}(\pi^+_n): H^*_\text{cryst}(X) \to H^{*-2n}_\text{cryst}(X).$$

Since the image of $\{\pi^+_{n}\}_{n \in \mathbb{Z}}$ under the composition $TP_{\pm}(-)_{1/p} \circ \Phi$ is given by $\sum_n H^*_\text{cryst}(\pi^+_{n})$, this implies that all the homomorphisms $H^*_\text{cryst}(\pi^+_{n})$, with $n \neq 0$, are necessarily equal to zero. Consequently, $\pi^+_0$ is an endomorphism of the Chow motive $\mathfrak{h}(X)_Q$ whose image under the functor (8.1) agrees with the above endomorphism $(id,0)$ of the $\mathbb{Z}/2\mathbb{Z}$-graded $K$-vector space $(\bigoplus_{i \text{even}} H^i_{\text{cryst}}(X), \bigoplus_{i \text{odd}} H^i_{\text{cryst}}(X))$. By construction of the functor (8.1), we hence conclude finally that the image of $\pi^+_0$ under the functor $H^*_\text{cryst}: \text{Chow}(k)_{Q} \to \text{vect}(K)$ agrees with the even Künneth projector $\pi^+_X := \sum_i \pi^{2i}_X$. This proves the conjecture $\text{C}^+(X)$.

**Type D.** Let $\text{Hom}(k)_{Q}$ be the classical category of homological motives (with respect to crystalline cohomology theory) and $\text{Num}(k)_{Q}$ the classical category of numerical motives. Recall from [26, §4.6] that there exists a $Q$-linear, fully-faithful, symmetric monoidal functor $\Phi_X$ making the following diagram commute:

$$(8.4) \quad \text{Chow}(k)_{Q} \xrightarrow{\iota} \text{Chow}(k)_{Q/\otimes Q(1)} \xrightarrow{\Phi} \text{NChow}(k)_{Q} \xrightarrow{\Phi_N} \text{NNum}(k)_{Q}.$$

By construction, the kernel of crystalline cohomology $H^*_\text{cryst}: \text{Chow}(k)_{Q} \to \text{vect}(K)$ agrees with the kernel of the above symmetric monoidal functor (8.1). Therefore, since the composition (8.1) is naturally isomorphic to (8.3) and (6.1) is the largest
The ⊗-ideal of the categories $\text{Chow}(k)_Q$ and $\text{NChow}(k)_Q$, the preceding diagram (8.4) admits the following “factorization”

\[
\begin{array}{ccc}
\text{Chow}(k)_Q & \xrightarrow{\iota} & \text{Chow}(k)_Q/\otimes Q(1) \\
\downarrow & & \downarrow \\
\text{Hom}(k)_Q & \xrightarrow{\iota} & \text{Hom}(k)_Q/\otimes Q(1) \\
\downarrow & & \downarrow \\
\text{Num}(k)_Q & \xrightarrow{\iota} & \text{Num}(k)_Q/\otimes Q(1)
\end{array}
\xrightarrow{\Phi} \begin{array}{c}
\text{NChow}(k)_Q \\
\text{NHom}(k)_Q \\
\text{NNum}(k)_Q,
\end{array}
\]

where $\Phi_H$ stands for the functor induced by the universal property of the orbit category $\text{Hom}(k)_Q/\otimes Q(1)$.

**Lemma 8.6.** The induced functor $\Phi_H$ is full.

**Proof.** Let us write $(\text{Chow}(k)_Q/\otimes Q(1))/\text{Ker}$ for the idempotent completion of the quotient of the orbit category $\text{Chow}(k)_Q/\otimes Q(1)$ by the kernel of the composition $TP_h(-)_1 \circ \Phi$. Under this notation, we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Chow}(k)_Q/\otimes Q(1) & \xrightarrow{\iota} & \text{Chow}(k)_Q/\otimes Q(1) \\
\downarrow & & \downarrow \\
\text{Hom}(k)_Q/\otimes Q(1) & \xrightarrow{\iota} & (\text{Chow}(k)_Q/\otimes Q(1))/\text{Ker} \\
\downarrow & & \downarrow \Phi_H' \\
\text{Num}(k)_Q/\otimes Q(1) & \xrightarrow{\iota} & \text{NNum}(k)_Q,
\end{array}
\]

where $\theta$, resp. $\Phi_H'$, stands for the canonical, resp. induced, functor. The proof follows now from the fact that the functor $\theta$, resp. $\Phi_H'$, is full (see [23, Lem. 4.7]), resp. fully-faithful, and that $\Phi_H = \Phi_H' \circ \theta$. $\square$

Thanks to the commutative diagram (8.2), the bottom right-hand side square in (8.5) yields the following commutative square of $\mathbb{Q}$-vector spaces:

\[
\begin{array}{ccc}
\text{Hom}_{\text{Hom}(k)_Q/\otimes Q(1)}(h(k)_Q, h(X)_Q) & \rightarrow & \text{Hom}_{\text{NHom}(k)_Q}(U(k)_Q, U(\text{perf}_{dg}(X))_Q) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{Num}(k)_Q/\otimes Q(1)}(h(k)_Q, h(X)_Q) & \xrightarrow{\sim} & \text{Hom}_{\text{NNum}(k)_Q}(U(k)_Q, U(\text{perf}_{dg}(X))_Q).
\end{array}
\]

Note that thanks to Lemma 8.6, the upper horizontal homomorphism is surjective. Note also that by construction of the categories of (noncommutative) homological and numerical motives, the preceding commutative square identifies with

\[
\begin{array}{ccc}
Z^*(X)_Q/\sim_{\text{hom}} & \rightarrow & K_0(\text{perf}_{dg}(X))_Q/\sim_{\text{hom}} \\
\downarrow & & \downarrow \\
Z^*(X)_Q/\sim_{\text{num}} & \xrightarrow{\sim} & K_0(\text{perf}_{dg}(X))_Q/\sim_{\text{num}}.
\end{array}
\]

We now have all the ingredients necessary to prove the equivalence of conjectures $D(X) \Leftrightarrow D_{nc}(\text{perf}_{dg}(X))$. Assume first that the conjecture $D(X)$ holds, i.e. that the vertical left-hand side homomorphism in (8.7) is injective. From the commutativity of (8.7), we conclude that the vertical right-hand side homomorphism in (8.7) is also injective, i.e. that the conjecture $D_{nc}(\text{perf}_{dg}(X))$ also holds.
Assume now that the conjecture $D_{nc}(\text{perf}_{dg}(X))$ holds, i.e. that vertical right-hand side homomorphism in (8.7) is injective. Note that the vertical left-hand side homomorphism in (8.7) is the diagonal quotient homomorphism from the direct sum $\bigoplus_{i=0}^{\dim(X)} Z^i(X)_{Q/\sim\text{hom}}$ to the direct sum $\bigoplus_{i=0}^{\dim(X)} Z^i(X)_{Q/\sim\text{num}}$. Therefore, thanks to the commutativity of (8.7), in order to prove the conjecture $D(X)$ it suffices to show that the following homomorphisms are injective:

\[(8.8)\quad Z^i(X)_{Q/\sim\text{hom}} \longrightarrow K_0(\text{perf}_{dg}(X))_{Q/\sim\text{hom}} \quad 0 \leq i \leq \dim(X).\]

Note that the composed functor $\Phi_H \circ \iota$ in (8.7) is faithful. In particular, for every $0 \leq i \leq \dim(X)$, the induced homomorphism is injective:

\[(8.9)\quad \text{Hom}_{\text{Hom}(k)_{Q}}(h(k)_{Q}, h(X)_{Q}(i)) \longrightarrow \text{Hom}_{\text{Hom}(k)_{Q/\sim\mathbb{Q}(1)}}(U(k)_{Q}, U(\text{perf}_{dg}(X))_{Q}).\]

By construction of the category of (noncommutative) homological motives and of the orbit category $\text{Hom}(k)_{Q/\sim\mathbb{Q}(1)}$, the preceding homomorphisms (8.9) (induced by the functor $\Phi_H \circ \iota$) correspond to the above homomorphisms (8.8) (induced by the functor $\Phi_H$). This implies that the homomorphisms (8.8) are injective, and hence proves the conjecture $D(X)$.

9. Proof of Theorem 2.1

Thanks to Theorem 1.1, the proof of Theorem 2.1 is similar to the proof of [25, Thm. 1.4]. Simply replace $\mathcal{O}_X(r)$ by $\mathcal{L}_X(r)$, $\text{perf}(Y; \mathcal{F})$ by $\text{perf}(Y)$, and the references [20, Thm. 6.3] and [19, §2.4] (in characteristic zero) by the reference [2, Thm. 2.3.4] (in arbitrary characteristic).

10. Proof of Theorem 3.1

We start by proving the first claim. Since by assumption $p \nmid n$, i.e. since $1/n \in k$, it follows from [32, Thm. 1.1 and Rk. 1.4] that the noncommutative Chow motive $U(\text{perf}_{dg}(X))_{Q}$ is a direct summand of $\bigoplus_{\sigma \leq G} U(\text{perf}_{dg}(X^\sigma \times \text{Spec}(k[\sigma])))_{Q}$. By definition, the noncommutative standard conjectures of type $C^+$ and $D$ are stable under direct sums and direct summands. Therefore, we obtain the implications:

$$\sum_{\sigma \leq G} C^+_{nc}(\text{perf}_{dg}(X^\sigma \times \text{Spec}(k[\sigma]))) \Rightarrow C^+_{nc}(\text{perf}_{dg}(X)) =: C^+(X)$$

$$\sum_{\sigma \leq G} D_{nc}(\text{perf}_{dg}(X^\sigma \times \text{Spec}(k[\sigma]))) \Rightarrow D_{nc}(\text{perf}_{dg}(X)) =: D(X).$$

The proof is now a consequence of Theorem 1.1 and of the following implications

$$C^+(X^\sigma) \Rightarrow (a) C^+(X^\sigma) + C^+(\text{Spec}(k[\sigma])) \Rightarrow (b) C^+(X^\sigma \times \text{Spec}(k[\sigma])),$$

where (a) follows from the fact that $\dim(\text{Spec}(k[\sigma])) = 0$ and (b) from Remark 7.3.

Let us now prove the second claim. If $k$ contains moreover the $n$th roots of unity of $k$, then it follows from [32, Cor. 1.6(i)] that the noncommutative Chow motive $U(\text{perf}_{dg}(X))_{Q}$ is a direct summand of $\bigoplus_{\sigma \leq G} U(\text{perf}_{dg}(X^\sigma))_{Q}$. Therefore, since the noncommutative standard conjecture of type $D$ is stable under direct sums and direct summands, the proof is a consequence of the following implications

$$\sum_{\sigma \leq G} D(X^\sigma) \Rightarrow (a) \sum_{\sigma \leq G} D_{nc}(\text{perf}_{dg}(X^\sigma)) \Rightarrow D_{nc}(\text{perf}_{dg}(X)) =: D(X),$$

where (a) follows from Theorem 1.1.
11. Application III: Zeta functions of endomorphisms

Let \( \mathcal{A} \) be a smooth proper dg category and \( f \) an endomorphism of the noncommutative Chow motive \( U(\mathcal{A})_\mathbb{Q} \); see §6. Following [28, §5], the zeta function of \( f \) is defined as the following formal power series

\[
Z(f; t) := \exp \left( \sum_{n \geq 1} \frac{\text{tr}(f^{on}) t^n}{n} \right) \in \mathbb{Q}[t],
\]

where \( f^{on} \) stands for the composition of \( f \) with itself \( n \)-times, \( \text{tr}(f^{on}) \) stands for the categorical trace of \( f^{on} \), and \( \exp(t) := \sum_{m \geq 0} \frac{t^m}{m!} \in \mathbb{Q}[t] \). Recall from [28, Rk. 5.2] that when \( f = [B]_\mathbb{Q} \) with \( B \in D_c(\mathcal{A}^{op} \otimes \mathcal{A}) \), we have the computation

\[
(11.1) \quad \text{tr}(f^{on}) = [HH(\mathcal{A}; B \otimes \mathcal{L}_A^1 \cdots \otimes \mathcal{L}_A^m B)] \in K_0(k) \simeq \mathbb{Z},
\]

where \( HH(\mathcal{A}; B \otimes \mathcal{L}_A^1 \cdots \otimes \mathcal{L}_A^m B) \) stands for the Hochschild homology of \( \mathcal{A} \) with coefficients in the dg \( \mathcal{A} \)-\( \mathcal{A} \) bimodule \( B \otimes \mathcal{L}_A^1 \cdots \otimes \mathcal{L}_A^m B \).

**Example 11.2** (Classical zeta function). Let \( k = \mathbb{F}_q \) be a finite field of characteristic \( p \), \( X \) a smooth projective \( k \)-scheme, and \( Fr \) the Frobenius of \( X \). Recall from [28, Example 5.4] that when \( \mathcal{A} = \text{perf}_{dg}(X) \) and \( B \) is the dg bimodule associated to the pull-back dg functor \( Fr^* : \text{perf}_{dg}(X) \to \text{perf}_{dg}(X) \), the above integer \((11.1)\) agrees with \(|X(\mathbb{F}_q)|\). Consequently, in this particular case, the zeta function of \( f = [B]_\mathbb{Q} \) reduces to the classical zeta function \( Z_X(t) := \exp(\sum_{n \geq 1} |X(\mathbb{F}_q)| t^n) \) of \( X \).

As proved in [28, Thm. 5.8], the formal power series \( Z(f; t) \) is rational and satisfies a functional equation. In the particular case of Example 11.2, these results yield an alternative proof of “half” of the Weil conjectures; consult [28, Cor. 5.12] for details. In loc. cit., we established moreover the following equality:

\[
(11.3) \quad Z(f; t) = \frac{\det(\text{id} - t\text{TP}_-(f)_{1/p} | \text{TP}_-(\mathcal{A})_{1/p})}{\det(\text{id} - t\text{TP}_+(f)_{1/p} | \text{TP}_+(\mathcal{A})_{1/p})} \in K(t).
\]

**Theorem 11.4.** If the conjecture \( C_{nc}^+(\mathcal{A}) \) holds, then the numerator and denominator of \((11.3)\) are polynomials with \( \mathbb{Q} \)-coefficients. Moreover, when \( f = [B]_\mathbb{Q} \) with \( B \in D_c(\mathcal{A}^{op} \otimes \mathcal{A}) \), the same holds with \( \mathbb{Z} \)-coefficients.

Note that thanks to Corollary 3.2, resp. Proposition 7.5, the preceding Theorem 11.4 can be applied, for example, to any “low-dimensional” orbifold, resp. to any finite dimensional \( k \)-algebra of finite global dimension.

**Corollary 11.5.** Let \( X \) be a smooth projective \( \mathbb{F}_q \)-scheme.

(i) We have the following equality:

\[
(11.6) \quad Z_X(t) = \frac{\prod_{\text{odd}} \det(\text{id} - tH_{\text{cryst}}(Fr) | H_{\text{cryst}}^i(X))}{\prod_{\text{even}} \det(\text{id} - tH_{\text{cryst}}(Fr) | H_{\text{cryst}}^i(X))} \in K(t).
\]

(ii) If the conjecture \( C^+(\mathcal{X}) \) holds, then the numerator and denominator of \((11.6)\) are polynomials with \( \mathbb{Z} \)-coefficients.

**Proof.** On the one hand, item (i) follows from the combination of \((11.3)\) with Example 11.2 and Theorem 5.2. On the other hand, item (ii) follows from the combination of Theorems 1.1, 5.2, and 11.4, with Example 11.2. \(\square\)
On the one hand, item (i) is Berthelot’s cohomological interpretation of the classical zeta function in terms of crystalline cohomology theory; see [4, page 583]. On the other hand, item (ii) is Grothendieck’s conditional approach to “half” of the Riemann hypothesis; see [9, §1-2] and [15, 4.1 Theorem]. Corollary 11.5 provides us with an alternative proof of these important results. Moreover, the above equality (11.3), resp. Theorem 11.4, establishes a far-reaching noncommutative generalization of Berthelot’s cohomological interpretation of the classical zeta function, resp. of Grothendieck’s conditional approach to “half” of the Riemann hypothesis.

Proof of Theorem 11.4. If the conjecture $C_{\text{ur}}^+(\mathcal{A})$ holds, then there exists an endomorphism $\pi^+_A$ of $U(\mathcal{A})_\mathbb{Q}$ such that $TP_A^+(\pi^+_A) = \pi^+_A$. In what follows, we write $f_+$ for the composition $\pi^+_A \circ f$. Note that $TP_A^+(f_+|_{1/p}) = TP_A^+(f|_{1/p}).$

We start by proving the first claim. Thanks to the classical Newton identities, the coefficients of the characteristic polynomial $\det(\text{id} - t TP_A^+(f|_{1/p}) | TP_A^+ (\mathcal{A})|_{1/p})$ can be written as polynomials with $\mathbb{Q}$-coefficients in the power sums $\alpha_1^n + \cdots + \alpha_r^n$, $n \geq 1$, where $\alpha_1, \ldots, \alpha_r$ are the eigenvalues (with multiplicities) of the $K$-linear homomorphism $TP_A^+(f|_{1/p})$. Therefore, it suffices to show that these power sums are rational numbers. This follows from the following equalities

\begin{equation}
\alpha_1^n + \cdots + \alpha_r^n = \text{tr}(TP_A^+(f^{on}|_{1/p}) = \text{tr}(TP_A^+(f^{on}|_{1/p}) = \text{tr}(f^{on})
\end{equation}

and from the fact that $\text{tr}(f^{on}) \in \mathbb{Q}$.

Let us now prove the second claim. By construction, the category of noncommutative Chow motives with $\mathbb{Z}$-coefficients $N\text{Chow}(k|\mathbb{Z})$ (see [20, Rk. 4.2]) is additive, rigid symmetric monoidal, and comes equipped with a symmetric monoidal functor $U(\mathcal{A})_\mathbb{Z} : \text{dgcat}^\text{sp}_k(k) \to N\text{Chow}(k|\mathbb{Z})$ as well as with a $\mathbb{Q}$-linearization symmetric monoidal functor $(\cdot)_\mathbb{Q} : N\text{Chow}(k|\mathbb{Z}) \to N\text{Chow}(k|\mathbb{Q})$. Therefore, if $f = [B]_\mathbb{Q}$ with $B \in D_c(\mathcal{A}^{op} \otimes \mathcal{A})$, i.e. if $f$ is the $\mathbb{Q}$-linearization of an endomorphism $[B]^\circ$ of $U(\mathcal{A})_\mathbb{Z}$, then $\text{tr}(f^{on}) = \text{tr}([B]^{on}) \in \mathbb{Z}$ for every $n \geq 1$. The endomorphism $\pi^+_A$ of $U(\mathcal{A})_\mathbb{Q}$ is not necessarily the $\mathbb{Q}$-linearization of an endomorphism of $U(\mathcal{A})_\mathbb{Z}$. Nevertheless, by removing denominators, there exists an integer $\lambda > 0$ such that $\lambda \cdot \pi^+_A = [B']_\mathbb{Q}$ for some $\mathcal{A} \otimes \mathcal{A}$ bimodule $B' \in D_c(\mathcal{A}^{op} \otimes \mathcal{A})$. Consequently, making use of the following equalities

\[ \lambda \cdot \text{tr}(f^{on}) = \lambda \cdot \text{tr}(\pi^+_A \circ f^{on}) = \text{tr}((\lambda \cdot \pi^+_A) \circ f^{on}) = \text{tr}([B']^\circ) = \text{tr}([B]^{on}) , \]

we conclude that $\lambda \cdot \text{tr}(f^{on}) \in \mathbb{Z}$ for every $n \geq 1$. Thanks to the above equalities (11.7), [15, 2.8 Lemma] hence implies that the coefficients of the characteristic polynomial $\det(\text{id} - t TP_A^+(f|_{1/p}) | TP_A^+ (\mathcal{A})|_{1/p})$ are algebraic integers. Since these numbers are also rational, we conclude that they are necessarily integers.

Finally, note that the proof concerning the coefficients of the characteristic polynomial $\det(\text{id} - t TP_A^-(f|_{1/p}) | TP_A^- (\mathcal{A})|_{1/p})$ is similar: simply replace $\pi^+_A$ by $\pi^-_A$.

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6The other “half” of the Riemann hypothesis asserts that the roots of the characteristic polynomial $\det(\text{id} - t H^{crys}_{crys}(\mathcal{F}r) | H^{crys}_{crys}(\mathcal{X}))$ have absolute value $q^{1/2}$. 

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