THE CENTER OF THE MODULAR SUPER YANGIAN $Y_{m|n}$

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ABSTRACT. The present paper is devoted to studying the super Yangian $Y_{m|n}$ associated to the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ over a field of positive characteristic. We extend Drinfeld-type presentations of $Y_{m|n}$ and the special super Yangian $SY_{m|n}$ to positive characteristic. Moreover, the center $Z(Y_{m|n})$ of $Y_{m|n}$ is described: it is generated by its Harish-Chandra center together with a large $p$-center. We also study the $p$-center of $SY_{m|n}$ and provide another description of the $p$-center of $Y_{m|n}$ in terms of the RTT generators.

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1. INTRODUCTION

For each simple finite-dimensional Lie algebra $\mathfrak{g}$ over the field of complex numbers, the corresponding Yangian $Y(\mathfrak{g})$ was defined by Drinfeld in [D1] as a canonical deformation of the universal enveloping algebra $U(\mathfrak{g}[x])$ for the current Lie algebra $\mathfrak{g}[x]$. In [D2], Drinfeld gave a new presentation for Yangians. The Yangians form a remarkable family of quantum groups related to rational solutions of the classical Yang-Baxter equation. The Yangian $Y_n = Y(\mathfrak{gl}_n)$ of the Lie algebra $\mathfrak{gl}_n$ was earlier considered in the works of mathematical physicists from St.-Petersburg; see for instance [TF]. It is an associative algebra whose defining relations can be written in a specific matrix form, which is called the RTT relation; see e.g. [MNO].

The super Yangian $Y_{m|n}$ associated to the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ over the complex field was defined by Nazarov [Na] in terms of the RTT presentation as a super analogue of $Y_n$. It has been studied by several authors. The Drinfeld-type presentation was found by Gow in [Gow2]. In [Peng1, Peng3], Peng obtained the superalgebra generalization of the parabolic presentations of [BK]. Recently, Tsymbaliuk [Isy] provided a generalization of $Y_{m|n}$ to arbitrary parity sequences.

In characteristic zero, the center $Z(Y_n)$ is generated by the coefficients of the quantum determinant, see [MNO, Theorem 2.13]. In his article [Na], Nazarov defined the quantum Berezinian which plays a similar role in the study of the super Yangian $Y_{m|n}$ as the quantum determinant does in the case of $Y_n$. Later, Gow used the Drinfeld presentation to determine the generators of the center of $Y_{m|n}$. More precisely, the center $Z(Y_{m|n})$ can be generated freely by the elements $\{c^{(r)}; r > 0\}$, where the elements $c^{(r)}$ are the coefficients of the quantum Berezinian ([Na], [Gow1, Theorem 2], [Gow2, Theorem 4]).

In [BT], Brundan and Topley developed the theory of the Yangian $Y_n$ over a field of positive characteristic. In particular, they gave a description of the center $Z(Y_n)$ of $Y_n$. One of the key features which differs from characteristic zero is the existence of a large central subalgebra $Z_p(Y_n)$, called the $p$-center. Also, these results give important applications to the theory of modular finite $W$-algebras (see [GT1, GT2]).

The main goal of this article is to obtain the superalgebra generalization of the $p$-center of [BT] for the modular super Yangian of type $A$. We define the super Yangian $Y_{m|n}$ over an algebraically closed field $\mathbb{k}$ of positive characteristic to be the associative superalgebra by the usual RTT presentation from [Na]. To describe the center $Z(Y_{m|n})$, we found that it was easier to work initially with the Drinfeld-type presentation. Thus the first step is to establish the modular version of Drinfeld-type presentation for $Y_{m|n}$. Since we are in characteristic $p := \text{char } \mathbb{k} > 0$, the modular phenomenon could happen everywhere. One needs extra care when treating the issues arising from odd elements. Actually, we have more relations than [Gow2] (see Theorem 3.8). Meanwhile, comparing with the Yangian $Y_n$, we need to insert the necessary sign factors in almost every formula.

In characteristic zero, it is well-known ([Gow2, Corollary 1]) that $Y_{m|n}$ is a deformation of the universal enveloping algebra $U(\mathfrak{gl}_{m|n}[x])$ for the current Lie superalgebra $\mathfrak{gl}_{m|n}[x]$. It is clear that the element $z_r := (e_{1,1} + \cdots + e_{m+n,m+n}) \otimes x^r$ belongs to the center $Z(U(\mathfrak{gl}_{m|n}[x]) := Z(U(\mathfrak{gl}_{m|n}[x]))$. Moreover, the coefficient of quantum Berezinian $c^{(r+1)}$ is a lift of $z_r$. These definitions make sense when $\text{char } \mathbb{k} > 0$. The algebra generated by the coefficients $\{c^{(r)}; r > 0\}$ will be denoted by $Z_{HC}(Y_{m|n})$ which is a subalgebra of the center $Z(Y_{m|n})$. We call it the
Harish-Chandra center of $Y_{m|n}$. Suppose that $p = \text{char } \mathbb{k} > 0$. The current Lie superalgebra $\mathfrak{gl}_{m|n}[x]$ admits a natural structure of restricted Lie superalgebra. That is, the even subalgebra $(\mathfrak{gl}_{m|n}[x])_0$ is a restricted Lie algebra with the $p$-map $(\mathfrak{gl}_{m|n}[x])_0 \rightarrow (\mathfrak{gl}_{m|n}[x])_0$ sending $a \mapsto a^{[p]}$, and the odd part $(\mathfrak{gl}_{m|n}[x])_1$ is a restricted module by the adjoint action of the even subalgebra. Then for each even element $a \in (\mathfrak{gl}_{m|n}[x])_0$, the element $a^p - a^{[p]} \in U(\mathfrak{gl}_{m|n}[x])$ is central. Denote by $Z_p(\mathfrak{gl}_{m|n}[x])$ the subalgebra of $Z(\mathfrak{gl}_{m|n}[x])$ generated by all $a^p - a^{[p]}$ with $a \in (\mathfrak{gl}_{m|n}[x])_0$. This subalgebra is often called the $p$-center of $U(\mathfrak{gl}_{m|n}[x])$. It is natural to look for lifts of the $p$-central elements in $Z(Y_{m|n})$. In Section 4, we will give a description of the $p$-center $Z_p(Y_{m|n})$ and show that the generators of $Z_p(Y_{m|n})$ provide the lifts of generators for $Z_p(\mathfrak{gl}_{m|n}[x])$. With this information in hand, we show in particular that the center $Z(Y_{m|n})$ is generated by $Z_{HC}(Y_{m|n})$ and $Z_p(Y_{m|n})$.

We organize this article in the following manner. In Section 2, we introduce the current Lie superalgebra and determine the center of its universal enveloping algebra. Following Nazarov’s definition, we define the modular super Yangian $Y_{m|n}$ in the RTT realization in Section 3. After recalling some basic properties of $Y_{m|n}$, we extend the Drinfeld-type presentation from characteristic zero to positive characteristic. Section 4 is concerned with the center of $Y_{m|n}$. We investigate various $p$-central elements by employing the Drinfeld presentation. Moreover, we give a description of the center of $Y_{m|n}$ and obtain the precise formulas for the generators. Section 5 is devoted to the study about the special super Yangian $SY_{m|n}$, which may be viewed as the modular version of the super Yangian for the Lie superalgebra $\mathfrak{sl}_{m|n}$. In particular, we obtain another description of the $p$-center of $Y_{m|n}$ in terms of the RTT generators.

Throughout this paper, $\mathbb{k}$ denotes an algebraically closed field of characteristic $\text{char}(\mathbb{k}) =: p > 0$.

2. The current superalgebra

2.1. The current superalgebra. Let $\mathfrak{gl}_{m|n}[x]$ denote the current superalgebra is defined to be the Lie superalgebra $\mathfrak{gl}_{m|n}[x] := \mathfrak{gl}_{m|n} \otimes \mathbb{k}[x]$. We will always denote the Lie algebra by $\mathfrak{g}$ and write $U(\mathfrak{g})$ for its enveloping algebra and $S(\mathfrak{g})$ for the symmetric superalgebra. Now let the indices $i, j$ run through $1, \ldots, m + n$. Set $|i| = 0$ if $1 \leq i \leq m$ and $|i| = 1$ if $m < i \leq m + n$. The elements $e_{i,j}x^r := e_{i,j} \otimes x^r$ with $r = 0, 1, 2, \ldots$ and $i, j = 1, \ldots, m + n$ make a basis of $\mathfrak{g}$. The $\mathbb{Z}_2$-grading on $\mathfrak{g}$ is defined by $\deg e_{i,j}x^r = |i| + |j|$. The supercommutation relations with $r, s \geq 0$ is given by

$$[e_{i,j}x^r, e_{k,l}x^s] = \delta_{k,j}e_{i,l}x^{r+s} - (-1)^{|i|+|j|(|k|+|l|)}\delta_{l,i}e_{k,j}x^{r+s}. \tag{2.1}$$

The adjoint action of $\mathfrak{g}$ on itself extends uniquely to actions of $\mathfrak{g}$ on $U(\mathfrak{g})$ and $S(\mathfrak{g})$ by derivations. The corresponding invariant subalgebras are denoted $U(\mathfrak{g})^{\mathfrak{g}}$ and $S(\mathfrak{g})^{\mathfrak{g}}$. In particular, the center $Z(\mathfrak{g}) = Z(\mathfrak{g})_0 \oplus Z(\mathfrak{g})_1 = U(\mathfrak{g})^{\mathfrak{g}}$, where

$$Z(\mathfrak{g})_i = \{ z \in U(\mathfrak{g}); za = (-1)^{ij}az, \forall a \in (\mathfrak{g})_j \text{ for } j \in \mathbb{Z}_2 \}, \quad i \in \mathbb{Z}_2.$$

There is one obvious family of even central elements in $U(\mathfrak{g})$. For any $r \in \mathbb{N}$, we set

$$z_r := e_{1,1}x^r + \cdots + e_{m+n,m+n}x^r \in \mathfrak{g}. \tag{2.2}$$

Using (2.1) one can show by direct computation that the set $\{ z_r; \ r \geq 0 \}$ forms a basis for the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$, so that $\mathbb{k}[z_r; \ r \geq 0]$ is a subalgebra of $Z(\mathfrak{g})$. 
2.2. Symmetric invariants. There is a filtration
\[ U(g) = \bigcup_{r \geq 0} F_r U(g) \]
(2.3)
of the enveloping algebra \( U(g) \), which is defined by placing \( e_{i,j}x^r \) in degree \( r + 1 \), i.e., \( F_r U(g) \) is the span of all monomials of the form \( e_{i_1,j_1}x^{r_1} \cdots e_{i_s,j_s}x^{r_s} \) with total degree \( (r_1 + 1) + \cdots + (r_s + 1) \leq r \). The associated graded algebra \( \text{gr}U(g) \) is isomorphic (both as a graded algebra and as a graded \( g \)-module) to \( S(g) \). It follows that
\[ \text{gr}Z(g) \subseteq S(g)^g. \]
(2.4)

Lemma 2.1. The invariant algebra \( S(g)^g \) is generated by \( \{ z_r; \ r \geq 0 \} \) together with \( ((g)_0)^p := \{ a^p; \ a \in (g)_0 \} \subseteq S(g) \). In fact, \( S(g)^g \) is freely generated by
\[ \{ z_r; \ r \geq 0 \} \cup \{(e_{i,j}x^r)^p; \ 1 \leq i, j \leq m + n \ with \ (i, j) \neq (1, 1), r \geq 0, e_{i,j}x^r \in (g)_0 \}. \]
(2.5)

Proof. The proof is essentially the same as [BT, Lemma 3.2], except that we need consider the odd elements. If \( a \in (g)_0 \), then the Leibniz rule implies \( a^p \in S(g)^g \). Let \( I(g) \) be the subalgebra of \( S(g)^g \) generated by \( \{ z_r; \ r \geq 0 \} \) and \( ((g)_0)^p \). Let
\[ B_0 := \{(i, j, r); \ 1 \leq i, j \leq m + n \ with \ (i, j) \neq (1, 1), r \geq 0, e_{i,j}x^r \in (g)_0 \}, \]
and \( B := B_0 \cup B_1 \) for short. Since the elements \( \{ z_r; \ r \geq 0 \} \cup \{(e_{i,j}x^r); \ (i, j, r) \in B_0 \cup B_1 \} \) give a basis of \( g \), it follows that
\[ S(g) = \mathbb{K}[z_r; \ r \geq 0][e_{i,j}x^r; \ (i, j, r) \in B_0] \otimes \Lambda[e_{i,j}x^r; \ (i, j, r) \in B_1], \]
\[ I(g) = \mathbb{K}[z_r; \ r \geq 0][(e_{i,j}x^r)^p; \ (i, j, r) \in B_0], \]
where \( \Lambda[e_{i,j}x^r; \ (i, j, r) \in B_1] \) is the exterior algebra (cf. [CW, (1.5.1)]). Hence, \( S(g) \) is free as an \( I(g) \)-module with basis \( \{ \prod_{(i,j,r) \in B_0 \cup B_1} (e_{i,j}x^r)^{w(i,j,r)}; \ \omega \in \Omega \} \), where
\[ \Omega := \left\{ \omega : B \rightarrow \mathbb{N}; \ 0 \leq \omega(i, j, r) < p, \ \forall(i, j, r) \in B_0 \ and \ \omega(i, j, r) \in \{0, 1\}, \ \forall(i, j, r) \in B_1 \right\}. \]
Now, we must show that \( S(g)^g \subseteq I(g) \). Given \( f \in S(g)^g \), we thus write
\[ f = \sum_{\omega \in \Omega} c_{\omega} \prod_{(i,j,r) \in B} (e_{i,j}x^r)^{w(i,j,r)} \]
for \( c_{\omega} \in I(g) \), all but finitely many of which are zero. Also fix a non-zero function \( \omega \), we have to prove that \( c_{\omega} = 0 \).

Suppose first that \( \omega(i, j, r) > 0 \) for some \( (i, j, r) \in B \) with \( i \neq j \). Choose an integer \( s \) that it is bigger than all \( r' \) such that \( \omega(i', j', r') > 0 \) for \( (i', j', r') \in B \), we have
\[ \text{ad}(e_{i,x^s})(f) = \sum_{\omega \in \Omega} c_{\omega} \sum_{(i', j', r', r') \in B} (-1)^{\text{sgn}(i', j', r', r')} \omega(i', j', r')(e_{i', j'}x^{r'})^{\omega(i', j', r')-1} \left[ e_{i,x^s}, e_{i', j'}x^{r'} \right] \]
\[ \times \prod_{(i'', j'', r'') \in B} (e_{i'', j''}x^{r''})^{\omega(i'', j'', r'')}, \]
where \( \text{sgn}(\omega, i', j', r') \in \{0, 1\} \) depends on \( \omega, i', j', r' \). Thanks to the choice of \( s \), the coefficient of

\[
(e_{i,j}x^r)^{\omega(i,j,r)-1}e_{i,j}x^{s+r} \prod_{(i'',j'',r'') \in B} (e_{i'',j'',r''})^{\omega(i'',j'',r'')}\]

in this expression is \((-1)^{\text{sgn}(\omega,i,j,r)}c_{\omega}(i,j,r)\). It must be zero since \( f \in S(\mathfrak{g})^p \). As \( \omega(i,j,r) \) is non-zero in \( \mathbb{k} \), we conclude that \( c_{\omega} = 0 \) as required.

By the same token, we can treat the case that \( \omega(j,j,r) > 0 \) for some \( (j,j,r) \in B \). \( \square \)

2.3. Restricted Lie superalgebra. A Lie superalgebra \( \mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1 \) is called a restricted Lie superalgebra if \( (\mathfrak{l}_0, [p]) \) is a restricted Lie algebra and \( \mathfrak{l}_1 \) is a restricted \( \mathfrak{l}_0 \)-module. By definition, for each \( x \in \mathfrak{l}_0 \), the element \( x^p - x^{[p]} \in U(\mathfrak{l}) \) is central and the map \( \xi : \mathfrak{l} \to U(\mathfrak{l}) ; x \mapsto x^p - x^{[p]} \) is \( p \)-semilinear.

For any associative \( \mathbb{k} \)-superalgebra \( A = A_0 \oplus A_1 \), there is a natural way to define a Lie bracket \([,] \) in \( A \), i.e., by the equality,

\[
[a, b] := ab - (-1)^{|a||b|}ba. \tag{2.6}
\]

The Lie superalgebra \((A,[,])\) will be denoted \( A^- \). Since we are in characteristic \( p > 0 \), the mapping \( a \to a^p; a \in A_0 \) endows \( A^- \) with the restricted structure.

**Lemma 2.2.** The current superalgebra \( \mathfrak{g} \) is a restricted Lie superalgebra with \( p \)-map defined on the basis by the rule \((ax^r)^{[p]} := a^p x^{rp} \), where \( a^p \) denotes the \( p \)-th matrix power of \( a \in \mathfrak{gl}_m \oplus \mathfrak{gl}_n \).

**Proof.** Let \( \text{Mat}_{m|n} \) be the matrix superalgebra. By definition, we have \( \mathfrak{gl}_{m|n} = (\text{Mat}_{m|n})^- \), so that \( \mathfrak{gl}_{m|n} \) is restricted with the \( p \)-map given by the \( p \)-th power of matrices. Then the claim follows immediately from the rules of Lie bracket (2.1) and \( p \)-map. \( \square \)

2.4. The center of \( U(\mathfrak{g}) \). We refer to \( Z_p(\mathfrak{g}) := \mathbb{k} \langle x^p - x^{[p]} ; x \in (\mathfrak{g})_0 \rangle \) as the \( p \)-center of \( U(\mathfrak{g}) \). Since the \( p \)-map is \( p \)-semilinear, we have that

\[
Z_p(\mathfrak{g}) = \mathbb{k} \left[ (e_{i,j}x^r)^p - \delta_{i,j}e_{i,j}x^{rp} ; 1 \leq i, j \leq m + n, r \geq 0, |i| + |j| = 0 \right] \tag{2.7}
\]

as a free polynomial algebra.

**Theorem 2.3.** The center \( Z(\mathfrak{g}) \) of \( U(\mathfrak{g}) \) is generated by \( \{z_r ; r \geq 0\} \) and \( Z_p(\mathfrak{g}) \). In fact, \( Z(\mathfrak{g}) \) is freely generated by

\[
\{z_r ; r \geq 0\} \cup \{(e_{i,j}x^r)^p - \delta_{i,j}e_{i,j}x^{rp} ; (i,j) \neq (1,1), r \geq 0, |i| + |j| = 0\}. \tag{2.8}
\]

**Proof.** The proof is similar to proof of [BT, Theorem 3.4], and will be skipped here. \( \square \)

**Remark 1.** In view of Theorem 2.3, the center \( Z(\mathfrak{g}) \) consists of only even elements, i.e.,

\( Z(\mathfrak{g}) = Z(\mathfrak{g})_0 \). This fact can be proved directly by the triangular decomposition of basic classical Lie superalgebras (see [CW, (2.2.2)]).

3. **Modular super Yangian \( Y_{m|n} \) and Drinfeld-type presentation**

In this section, we study the Yangian \( Y_{m|n} \) in positive characteristic.
3.1. RTT presentation of \( Y_{m|n} \). Following [Na], the super Yangian associated to the general linear Lie superalgebra \( \mathfrak{gl}_{m|n} \), denoted by \( Y_{m|n} \), is the associated superalgebra over \( \mathbb{k} \) with the RTT generators \( \{ t^{(r)}_{i,j}; 1 \leq i, j \leq m + n, r \geq 1 \} \) subject to the following relations:

\[
\left[ t^{(r)}_{i,j}, t^{(s)}_{k,l} \right] = (-1)^{|i||j|+|i||k|+|j||k|} \sum_{t=0}^{\min(r,s)-1} \left( \sum_{m=0}^{\min(r,s)-t} \left( t^{(t)}_{k,j} (r+s-1-t) t^{(s)}_{i,l} - t^{(r+s-1-t)}_{k,j} t^{(t)}_{i,l} \right) \right),
\]

(3.1)

where the parity of \( t^{(r)}_{i,j} \) is defined by \( |i| + |j| \pmod{2} \), and the bracket is understood as the supercommutator. By convention, we set \( t^{(0)}_{i,j} := \delta_{i,j} \).

The element \( t^{(r)}_{i,j}; r > 0 \) is called an even (odd, respectively) element if its parity is 0 (1, respectively). We define the formal power series

\[
t_{i,j}(u) := \sum_{r \geq 0} t^{(r)}_{i,j} u^{-r} \in Y_{m|n}[[u^{-1}]],
\]

and a matrix \( T(u) := \{ t_{i,j}(u) \}_{1 \leq i,j \leq m+n} \). It is easily seen that, in terms of the generating series, the initial defining relation (3.1) may be rewritten as follows:

\[
[t_{i,j}(u), t_{k,l}(v)] = \frac{(-1)^{|i||j|+|i||k|+|j||k|}}{(u-v)}(t_{k,j}(u)t_{i,l}(v) - t_{k,j}(v)t_{i,l}(v)).
\]

(3.2)

Note that the matrix \( T(u) \) is invertible, we observe the following notation for the entries of the inverse of the matrix \( T(u) \):

\[
T(u)^{-1} := \{ t'_{i,j}(u) \}_{i,j=1}^{m+n},
\]

and we also have another relation (see [Gow2, (5)], [Peng1, (2.4)]):

\[
[t_{i,j}(u), t'_{k,l}(v)] = \frac{(-1)^{|i||j|+|i||k|+|j||k|}}{(u-v)}(\delta_{k,j} \sum_{s=1}^{m-n} t_{i,s}(u)t'_{s,l}(v) - \delta_{i,l} \sum_{s=1}^{m+n} t'_{k,s}(u)t_{s,j}(v)).
\]

(3.3)

For homogeneous elements \( x_1, \ldots, x_s \) in a superalgebra \( A \), a supermonomial in \( x_1, \ldots, x_s \) means a monomial of the form \( x_1^{i_1} \cdots x_s^{i_s} \) for some \( i_1, \ldots, j_s \in \mathbb{Z}_{>0} \) and \( i_j \leq 1 \) if \( x_j \) is odd. The following proposition is a PBW theorem for \( Y_{m|n} \), where the proof in [Gow2, Theorem 1] works perfectly in positive characteristic.

**Theorem 3.1.** Ordered supermonomial in the generators \( \{ t^{(r)}_{i,j}; 1 \leq i, j \leq m + n, r \geq 1 \} \) taken in some fixed order forms a linear basis for \( Y_{m|n} \).

3.2. Loop filtration. Define the loop filtration on \( Y_{m|n} \)

\[
Y_{m|n} = \bigcup_{r \geq 0} F_r Y_{m|n}
\]

(3.4)

by setting \( \deg t^{(r)}_{i,j} = r - 1 \), i.e., \( F_r Y_{m|n} \) is the span of all supermonomials of the form \( t^{(r_1)}_{i_{1,j_1}} \cdots t^{(r_m)}_{i_{m,j_m}} \) with \( (r_1 - 1) + \cdots + (r_m - 1) \leq r \). To describe the associated graded superalgebra \( \text{gr} Y_{m|n} \), We recall that \( U(\mathfrak{g}) \) has the natural filtration and grading with \( \deg e_{i,j} x^r = r \).
Lemma 3.2. The assignment
\[ t_{i,j}^{(r)} \mapsto (-1)^{|i|} e_{i,j} x^{r-1} \]
gives rise to an isomorphism \( \text{gr} Y_{m|n} \cong U(\mathfrak{g}) \) of graded superalgebras.

Proof. Relation (3.1) implies that
\[
[\text{gr}_r t_{i,j}^{(r)} : \text{gr}_s t_{k,l}^{(s)}] = [t_{i,j}^{(r)} : t_{k,l}^{(s)}] + F_{r+s-3} Y_{m|n}
\]
\[
= (-1)^{|i|+|j|} \left( \delta_{k,j} t_{i,j}^{r+s-1} - \delta_{i,j} t_{k,l}^{r+s-1} \right) + F_{r+s-3} Y_{m|n}
\]
Comparing with (2.1), we deduce that the map in the statement of the lemma is well defined. To see that it is an isomorphism, one uses the PBW basis from Theorem 3.1 to see that a basis for \( \text{gr} Y_{m|n} \) is sent to a basis for \( U(\mathfrak{g}) \).

3.3. Gauss decomposition and quasideterminants. Note that the leading minors of the matrix \( T(u) \) are always invertible and hence the matrix \( T(u) \) possesses a Gauss decomposition
\[
T(u) = F(u) D(u) E(u)
\]
for unique matrices
\[
D(u) = \begin{pmatrix}
    d_1(u) & 0 & \cdots & 0 \\
    0 & d_2(u) & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & d_{m+n}(u)
\end{pmatrix},
\]
\[
E(u) = \begin{pmatrix}
    1 & e_{1,2}(u) & \cdots & e_{1,m+n}(u) \\
    0 & 1 & \cdots & e_{2,m+n}(u) \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1
\end{pmatrix},
\quad F(u) = \begin{pmatrix}
    1 & 0 & \cdots & 0 \\
    f_{2,1}(u) & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{m+n,1}(u) & f_{m+n,2}(u) & \cdots & 1
\end{pmatrix}.
\]
In terms of quasideterminants of [GR], we have the following descriptions (cf. [Gow2, Section 3]):
\[
d_i(u) = \begin{vmatrix}
    t_{1,1}(u) & \cdots & t_{1,i-1}(u) & t_{1,i}(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,i}(u) \\
    t_{i,1}(u) & \cdots & t_{i,i-1}(u) & t_{i,i}(u)
\end{vmatrix},
\]
\[
e_{i,j}(u) = d_i(u)^{-1} \begin{vmatrix}
    t_{1,1}(u) & \cdots & t_{1,i-1}(u) & t_{1,i}(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,i}(u) \\
    t_{i,1}(u) & \cdots & t_{i,i-1}(u) & t_{i,i}(u)
\end{vmatrix},
\]
\[
f_{j,i}(u) = \begin{vmatrix}
    t_{1,1}(u) & \cdots & t_{1,i-1}(u) & t_{1,i}(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,i}(u) \\
    t_{j,1}(u) & \cdots & t_{j,i-1}(u) & t_{j,i}(u)
\end{vmatrix} d_i(u)^{-1}.
\]
We use the following notation for the coefficients:

\[ d_i(u) = \sum_{r \geq 0} d_i^{(r)} u^{-r}; \quad (d_i(u))^{-1} = \sum_{r \geq 0} d_i^{(r)} u^{-r}; \]

\[ e_{i,j}(v) = \sum_{r \geq 1} e_{i,j}^{(r)} v^{-r}; \quad f_{j,i}(v) = \sum_{r \geq 1} f_{j,i}^{(r)} v^{-r}. \]

Let \( e_j(u) := e_{j,j+1}(u), f_j(v) := f_{j+1,j}(v) \) for short. By the above, we immediately have \( e_{j-1}^{(1)} = e_{j-1,j}^{(1)} \) and \( f_{j-1}^{(1)} = e_{j-1,j-1}^{(1)} \). By induction, one may show that for each pair \( i, j \) such that \( 1 < i + 1 < j \leq m + n - 1 \), we have

\[ e_{i,j}^{(r)} = (-1)^{j-1}[e_{i,j-1}^{(r)}, e_{j-1}^{(1)}]; \quad f_{j,i}^{(r)} = (-1)^{j-1}[f_{j-1}^{(1)}, e_{j-1,i}^{(r)}]. \quad (3.9) \]

By multiplying out the matrix product (3.5), we see that each \( t_{i,j}^{(r)} \) can be expressed as a sum of monomials in \( d_i^{(r)}, e_{i,j}^{(r)} \) and \( f_{j,i}^{(r)} \), appearing in certain order that all \( f \)'s before \( d \)'s and all \( d \)'s before \( e \)'s. By (3.9), it is enough to use \( d_i^{(r)}, e_j^{(r)} \) and \( f_j^{(r)} \) only, so that we have the following theorem.

**Theorem 3.3.** The super Yangian \( Y_{m|n} \) is generated as an algebra by the following elements:

\[ \{d_i^{(r)}; 1 \leq i \leq m + n, r \geq 0\} \cup \{e_j^{(r)}, f_j^{(r)}; 1 \leq j \leq m + n - 1, r \geq 1\}. \]

### 3.4. Maps between Super Yangians

To explicitly write down the relations among the Drinfeld generators, we start with the special cases when \( m \) and \( n \) are either 1 or 2, that are relatively less complicated, and then to apply the maps in this section to obtain the relations in the general case.

**Proposition 3.4.** (1) The map \( \rho_{m|n} : Y_{m|n} \to Y_{n|m} \) defined by

\[ \rho_{m|n}(t_{i,j}(u)) = t_{m+n+1-i,m+n+1-j}(-u) \]

is an algebra isomorphism.

(2) The map \( \omega_{m|n} : Y_{m|n} \to Y_{m|n} \) defined by

\[ \omega_{m|n}(T(u)) = (T(-u))^{-1} \]

is an algebra isomorphism.

(3) For any \( k \in \mathbb{Z}_{\geq 0} \), the map \( \psi_k : Y_{m|n} \to Y_{m+k|n} \) defined by

\[ \psi_k = \omega_{m+k|n} \circ \varphi_{m|n} \circ \omega_{m|n} \]

where \( \varphi_{m|n} : Y_{m|n} \to Y_{m+k|n} \) is the injective algebra homomorphism which sends each \( t_{i,j}^{(r)} \in Y_{m|n} \) to \( t_{k+i,k+j}^{(r)} \in Y_{m+k|n} \).

(4) The map \( \zeta_{m|n} : Y_{m|n} \to Y_{n|m} \) defined by

\[ \zeta_{m|n} = \rho_{m|n} \circ \omega_{m|n} \]

is an algebra isomorphism.

**Proof.** The proof follows easily from the defining relations, see also [Gow2, Section 4]. \( \square \)

We call \( \psi_k \) the **shift map** and \( \zeta_{m|n} \) the **swap map**. The following Proposition is due to Gow and the proof given in characteristic zero in [Gow2, Proposition 1, Lemma 2] works as well in positive characteristic.
Lemma 3.6. For \(k, l \geq 1\), we have
\[
\psi_k(d_l(u)) = d_k+l(u),
\]
\[
\psi_k(e_l(u)) = e_k+l(u),
\]
\[
\psi_k(f_l(u)) = f_k+l(u).
\]

The subalgebras \(Y_k\) and \(\psi_k(Y_{m|n})\) in \(Y_{m+k|n}\) supercommute with each other.

3.5. Gauss decomposition of \(Y_{m|n}\). We first consider the case, where \(m = 2\) and \(n = 1\). The following lemma is a generalization and modular analogue of [Gow2, Lemma 3].

Lemma 3.6. The following identities hold in \(Y_{2|1}[[u^{-1}, v^{-1}, w^{-1}]]\):
\[
(u-v)[d_i(u), e_j(v)] = \begin{cases} 
(\delta_{ij} - \delta_{i,j+1})d_i(u)(e_j(v) - e_j(u)), & \text{if } j = 1; \\
(\delta_{ij} + \delta_{i,j+1})d_i(u)(e_j(v) - e_j(u)), & \text{if } j = 2;
\end{cases}
\]
\[
(u-v)[d_i(u), f_j(v)] = \begin{cases} 
-(\delta_{ij} - \delta_{i,j+1})(f_j(v) - f_j(u)d_i(u), & \text{if } j = 1; \\
-(\delta_{ij} + \delta_{i,j+1})(f_j(v) - f_j(u)d_i(u), & \text{if } j = 2;
\end{cases}
\]
\[
(u-v)[e_j(u), f_k(v)] = (-1)^{j+1}\delta_{jk} \left( d_j(u)^{-1}d_{j+1}(u) - d_j(v)^{-1}d_{j+1}(v) \right);
\]
\[
(u-v)[e_j(u), e_j(v)] = (-1)^{j+1}(e_j(u) - e_j(v))^2;
\]
\[
(u-v)[f_j(u), f_j(v)] = (-1)^{j+1}(f_j(u) - f_j(v))^2;
\]
\[
(u-v)[e_1(u), e_2(v)] = e_1(u)e_2(v) - e_1(v)e_2(v) - e_{1,3}(u) + e_{1,3}(v);
\]
\[
(u-v)[f_1(u), f_2(v)] = -f_2(v)f_1(u) + f_2(v)f_1(v) + f_{3,1}(u) - f_{3,1}(v);
\]
\[
[e_{13}(u), e_2(v)] = e_2(v) [e_1(u), e_2(v)];
\]
\[
[e_1(u), e_{13}(v) - e_1(v)e_2(v)] = -[e_1(u), e_2(v)] e_1(u);
\]
\[
[[e_i(u), e_j(v)], e_j(v)] = 0 \text{ if } |i - j| = 1;
\]
\[
[[f_i(u), f_j(v)], f_j(v)] = 0 \text{ if } |i - j| = 1;
\]
\[
[[e_i(u), e_j(v)], e_j(w)] + [[e_i(u), e_j(w)], e_j(v)] = 0, \text{ if } |i - j| = 1;
\]
\[
[[f_i(u), f_j(v)], f_j(w)] + [[f_i(u), f_j(w)], f_j(v)] = 0, \text{ if } |i - j| = 1.
\]
Proof. Equations (3.13)-(3.15) were proved over $\mathbb{C}$ in [Gow2, Lemma 3], the same proof works here.

To establish (3.16), we first consider the Yangian $Y_2$. According to [BT, (4.30)] (see also [BK, Lemma 5.4(iv)]), we have $(u - v)[e_1(u), e_1(v)] = (e_1(u) - e_1(v))^2$. The standard embedding $Y_2 \hookrightarrow Y_{2|1}$ yields (3.18) for $j = 1$. Using exactly the same degree argument as in [Peng1, (5.4)], we see that $(u - v)[e_1(u), e_1(v)] = -(e_1(u) - e_1(v))^2$ holds in the Yangian $Y_{1|1}$. By applying the shift map $\psi_1 : Y_{1|1} \to Y_{2|1}$ (see (3.11)), we obtain (3.16) for $j = 2$. The relation (3.17) follows from a consecutive application of [BK, (5.6)] and the swap map $\xi_{1|1}$ to (3.16) with suitable choices of indices.

For the remaining relations, we just give a brief account. This can be proven in the same manner as [Peng1, Section 6]. For example, for (3.24), it suffices to verify that

$$(u - v)(u - w)(v - w)[[e_1(u), e_2(v)], e_2(w)]$$

is symmetric in $v$ and $w$. Actually, this follows from (3.18)-(3.23). To establish (3.18), we use (3.3) to get

$$(u - v)[t_{1,2}(u), l_{1,3}^2(v)] = t_{1,1}(u)t_{1,3}^2(v) + t_{1,2}(u)t_{1,3}^2(v) + t_{1,3}(u)t_{1,3}^2(v).$$

Substituting by the Drinfeld’s generators (see for example [Gow2, Page 807]) in the above identity, we have

$$[u - v][d_1(u)e_1(u), -e_2(v)d_3(v)^{-1}] = d_1(u)(e_1(v)e_2(v) - e_{1,3}(v) - e_1(u)e_2(v) + e_{1,3}(u)d_3(v)^{-1}.$$}

Then we multiply both sides on the left by $d_1(u)^{-1}$ and on the right by $d_3(v)$ to get (3.18). For (3.20), we have by (3.3) and (3.13) that

$$0 = [t_{1,3}(u), -l_{1,3}^2(v)] = [d_1(u)e_{1,3}(u), e_2(v)d_3(v)^{-1}] = d_1(u)[e_{1,3}(u), e_2(v)d_3(v)^{-1}].$$

It follows that $[e_{1,3}(u), e_2(v)d_3(v)^{-1}] = 0$, this readily yields

$$(*): [e_{1,3}(u), e_2(v)]d_3(v)^{-1} = e_2(v)[e_{1,3}(u), d_3(v)^{-1}].$$

Note that (3.13) gives the following identity:

$$(u - v)[d_3(v)^{-1}, e_2(u)] = (e_2(u) - e_2(v))d_3(v)^{-1}.$$}

Taking the coefficient of $u^0$ and using the Super-Jacobi identity, we have

$$[e_{1,3}(u), d_3(v)^{-1}] = [e_1 u, e_2(v)]d_3(v)^{-1}.$$}

Now substituting this into (*) gives (3.20). The proof of (3.21) is similar to (3.20), and we need to consider the relation $[t_{1,2}(u), l_{1,3}^2(v)] = 0$ using (3.3). For (3.22) in the case $i = 1, j = 2$, we have by (3.18) and (3.20) to see that

$$(u - v + 1)[[e_1(u), e_2(v)], e_2(v)] = [[[e_1(v), e_2(v)], e_2(v)].$$}

Now let $u = v - 1$ to deduce that the right-hand side is zero, then divide by $(u - v + 1)$ to complete the proof. \qed

For general $m, n$, the relations among $d's, e's$ and $f's$ are given in the following lemma, which is a generalization and modular analogue of [Gow2, Lemma 4].
Lemma 3.7. The following relations hold in $Y_{m|n}[[u^{-1}, v^{-1}, w^{-1}]]$:

$$[d_i(u), d_j(v)] = 0 \quad \text{for all } 1 \leq i, j \leq m + n;$$ \hfill (3.26)

$$[e_i(u), e_j(v)] = [f_i(u), f_j(v)] = 0 \quad \text{if } |i - j| > 1;$$ \hfill (3.27)

$$(u - v)[d_i(u), e_j(v)] = (-1)^{|i|}((\delta_{ij} - \delta_{i,j+1})d_i(u)(e_j(v) - e_j(u)));$$ \hfill (3.28)

$$(u - v)[d_i(u), f_j(v)] = (-1)^{|i|}(-\delta_{ij} + \delta_{i,j+1})(f_j(v) - f_j(u))d_i(u);$$ \hfill (3.29)

$$(u - v)[e_i(u), f_j(v)] = (-1)^{|j+1|}\delta_{ij}(d_i(u)^{-1}d_{i+1}(u) - d_i(v)^{-1}d_{i+1}(v));$$ \hfill (3.30)

$$(u - v)[e_j(u), e_j(v)] = (-1)^{|j|+1}(e_j(u)e_j(v) - e_j(v)e_{j+1}(v) - e_{j+2}(u) + e_{j+2}(v));$$ \hfill (3.33)

$$(u - v)[f_j(u), f_j(v)] = -(-1)^{|j+1|}(f_j(u)f_j(v) - f_{j+1}(v)f_j(v) - f_{j+2}(u) + f_{j+2}(v));$$ \hfill (3.34)

$$[[e_i(u), e_j(v)], e_j(v)] = 0 \quad \text{if } |i - j| = 1;$$ \hfill (3.35)

$$[[f_i(u), f_j(v)], f_j(v)] = 0 \quad \text{if } |i - j| = 1;$$ \hfill (3.36)

$$[[e_i(u), e_j(v)], e_j(w)] + [[e_i(u), e_j(w)], e_j(v)] = 0, \quad \text{if } |i - j| = 1;$$ \hfill (3.37)

$$[[f_i(u), f_j(v)], f_j(w)] + [[f_i(u), f_j(w)], f_j(v)] = 0, \quad \text{if } |i - j| = 1;$$ \hfill (3.38)

$$[[e_i, e_i^{(1)}], [e_i, e_i^{(s)}]] = 0;$$ \hfill (3.39)

$$[[f_i^{(r)}, f_i^{(1)}], [f_i^{(1)}, f_i^{(s)}]] = 0.$$ \hfill (3.40)

Proof. The relations (3.26) and (3.27) follow directly from Proposition 3.5(3).

For the relations (3.28)-(3.38), we use Lemma 3.6, [BT, Theorem 4.3], together with the shift maps and swap maps. We just go through (3.35) in the case $j = i + 1$, i.e.,

$$[[e_i(u), e_i+1(v)], e_i+1(v)] = 0,$$

since the others are similar. There are four cases: (a) $1 \leq i \leq m - 2$; (b) $i = m - 1$; (c) $i = m$; (d) $m + 1 \leq i \leq m + n - 2$. For these, (a) and (d) are immediate from [BT, (4.25)]. We consider the composite map:

$$\eta : Y_{m|n} \xrightarrow{\psi_{m-1}} Y_{m|n-1} \xrightarrow{\zeta_{m-1}} Y_{m|m} \xrightarrow{\psi_{m-1}} Y_{m|m} \xrightarrow{\zeta_{n|m}} Y_{m|n}.$$

Now Proposition 3.5 implies $\eta(e_1) = e_{m-1}$ and $\eta(e_2) = e_m$, so that (3.22) yields (b). By using (3.23) instead of (3.22), (c) is a similar argument to (b).
The proof of (3.40) is similar to (3.39), so we just prove (3.39). Taking its coefficient of $u^{-r}v^{-s}w^{-t}$ in (3.37), we have
\[
\left[e^{(r)}_{i}, e^{(s)}_{j}, e^{(t)}_{j}\right] + \left[e^{(r)}_{i}, e^{(t)}_{j}, e^{(s)}_{j}\right] = 0, \quad \text{if } |i - j| = 1. \tag{3.41}
\]
Then taking the $u^{-r}v^{-2t}$-coefficient in (3.35) in conjunction with (3.41) gives
\[
\left[e^{(r)}_{i}, e^{(t)}_{j}\right] = 0 \quad \text{if } |i - j| = 1. \tag{3.42}
\]
To show (3.39), we consider the following three cases: (i) $1 \leq i \leq m - 1$; (ii) $i = m$; (iii) $m + 1 \leq i \leq m + n - 2$. As before, (iii) is a similar argument to (i), so we just prove (i) and (ii). Suppose that $1 \leq i \leq m - 1$. By (3.42), (3.27) and super-Jacobi identity, we have:
\[
[[e^{(r)}_{i-1}, e^{(1)}_{i}], e^{(s)}_{i+1}] = [e^{(1)}_{i}, e^{(r)}_{i-1}, e^{(1)}_{i}] = e^{(1)}_{i}, e^{(1)}_{i}, e^{(r)}_{i-1}]
\]
\[
= [e^{(s)}_{i-1}, e^{(1)}_{i}, e^{(r)}_{i-1}] + [e^{(r)}_{i-1}, e^{(s)}_{i}, e^{(1)}_{i}] = -[e^{(r)}_{i-1}, e^{(s)}_{i}, e^{(1)}_{i}].
\]
Hence we have proved (i) in case $\text{char } k = p \neq 2$. The proof of (ii) is the same argument as in the proof of [Gow2, Lemma 5]. Now we assume that $p = 2$ and consider the composite map
\[
\xi : Y_{2|i} \xrightarrow{\psi_{i-2}} Y_{2|i} \xrightarrow{\zeta_{i-2}} Y_{2|i} \xrightarrow{\psi_{m+n-i-2}} Y_{m+n-1+i} \xrightarrow{\zeta_{m+n-1+i}} Y_{m+n+i}.
\]
Note that $Y_{m+n+i} = Y_{m+i}$. Then Gow’s argument ([Gow2, Lemma 5]) in conjunction with the map $\xi$ yields (i) for $p = 2$.

Remark 2. (1) Note that (3.31) and (3.32) are different from the ones in [Gow2, (30)-(31)]. Suppose that $p \neq 2$. We may switch $u$ and $v$. Since both $e_{m}(u)$ and $e_{m}(v)$ are odd, this forces $(u - v)[e_{m}(u), e_{m}(v)] = 0$. If $p = 2$, then $Y_{m+i} \cong Y_{m+n+i}$, so that $(u - v)[e_{m}(u), e_{m}(v)] = -(e_{m}(u) - e_{m}(v))^{2} = (e_{m}(u) - e_{m}(v))^{2}$ (cf. [BT, (4.30)]).

(2) For the quartic Serre relations (3.39, 3.40), we need consider all admissible $i$. If $p \neq 2$, then the proof of Lemma 3.7 implies that the quartic Serre relations for $j \neq m$ already follow from the quadratic and cubic relations, however we cannot derive the quartic Serre relations from others in the case $p = 2$ (cf. [BT, Theorem 4.3]).

3.6. Drinfeld-type presentation. Recall that the fact in Section 3.3 that $Y_{m+i}$ is generated as an algebra by the set $\{d^{(r)}_{i}, d^{(r)}_{i}, e^{(s)}_{j}, f^{(r)}_{j}\}$. The following theorem describes the relations among these generators, and the relations are enough as defining relations of the super Yangian $Y_{m+i}$ (cf. [Gow2, Theorem 3]).

**Theorem 3.8.** The Yangian $Y_{m+i}$ is generated by the elements $\{d^{(r)}_{i}, d^{(r)}_{i}; 1 \leq i \leq m + n, r \geq 1\}$ and $\{e^{(s)}_{j}, f^{(r)}_{j}; 1 \leq j \leq m + n - 1, r \geq 1\}$ subject only to the following relations:

\[
d^{(0)}_{i} = 1, \sum_{t=0}^{r} d^{(t)}_{i} d^{(r-t)}_{i} = \delta_{r,0}; \tag{3.43}
\]
\[
[d^{(r)}_{i}, d^{(s)}_{j}] = 0; \tag{3.44}
\]
\[
[e^{(r)}_{i}, e^{(s)}_{j}] = 0. \tag{3.45}
\]
\[
[d_i^{(r)}, e_j^{(s)}] = (-1)^{|i|} (\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-1-t)};
\]
\[
[d_i^{(r)}, f_j^{(s)}] = (-1)^{|i|} (\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} f_j^{(r+s-1-t)} d_i^{(t)};
\]
\[
[e_i^{(r)}, f_j^{(s)}] = -(-1)^{|i|} \delta_{ij} \sum_{t=0}^{r+s-1} d_i^{(t)} d_{i+1}^{(r+s-1-t)};
\]
\[
[e_j^{(r)}, e_j^{(s)}] = (-1)^{|j+1|} \sum_{t=1}^{s-1} e_j^{(t)} e_j^{r+s-1-t} - \sum_{t=1}^{r-1} e_j^{(t)} e_j^{(r+s-1-t)};
\]
\[
[f_j^{(r)}, f_j^{(s)}] = (-1)^{|j+1|} \sum_{t=1}^{r-1} f_j^{(t)} f_j^{r+s-1-t} - \sum_{t=1}^{s-1} f_j^{(t)} f_j^{(r+s-1-t)};
\]
\[
[e_j^{(r+1)}, e_j^{(s+1)}] = (1+j+1) e_j^{(r+1)} e_j^{(s+1)};
\]
\[
[f_j^{(r+1)}, f_j^{(s+1)}] = (1+j+1) f_j^{(r+1)} f_j^{(s+1)};
\]
\[
[e_i^{(r)}, e_j^{(s)}] = 0 = [f_i^{(r)}, f_j^{(s)}] \text{ if } |i-j| > 1;
\]
\[
[[e_i^{(r)}, e_j^{(s)}], e_j^{(t)}] + [[e_i^{(r)}, e_j^{(t)}], e_j^{(s)}] = 0, \text{ if } |i-j| = 1;
\]
\[
[[f_i^{(r)}, f_j^{(s)}], f_j^{(t)}] + [[f_i^{(r)}, f_j^{(t)}], f_j^{(s)}] = 0, \text{ if } |i-j| = 1;
\]
\[
[[e_i^{(r)}, e_j^{(t)}], e_j^{(s)}] = 0 \text{ if } |i-j| = 1;
\]
\[
[[f_i^{(r)}, f_j^{(t)}], f_j^{(s)}] = 0, \text{ if } |i-j| = 1;
\]
\[
[[e_i^{(r)}, e_i^{(1)}], e_i^{(1)}, e_i^{(s+1)}] = 0;
\]
\[
[[f_i^{(r)}, f_i^{(1)}], f_i^{(1)}, f_i^{(s+1)}] = 0.
\]

**Remark 3.** (1) There are several differences between Theorem 3.8 and [Gow2, Theorem 3]. The relations (41)-(43) of *loc. cit.* are expressed here as the two relations (3.48)-(3.49). Also relations (47)-(48) of *loc. cit.* are expressed here as the four relations (3.53)-(3.56). This is essential in characteristic 2 (see Remark 2).

(2) We note that the relations (39), (44)-(45) in [Gow2, Theorem 3] contain some typos.

(3) Relations (3.48) and (3.49) are equivalent to the following relations:
\[
[e_j^{(r+1)}, e_j^{(s+1)}] = -(-1)^{|j+1|} (e_j^{(s)} e_j^{(r)} + e_j^{(r)} e_j^{(s)}),
\]
\[
[f_j^{(r+1)}, f_j^{(s+1)}] = -(-1)^{|j+1|} (f_j^{(s)} f_j^{(r)} + f_j^{(r)} f_j^{(s)}).
Proof. We have before proved (3.55)-(3.57) (see (3.39)-(3.40) and (3.42)) and (3.58) is a similar argument to (3.57), while the others come from Lemma 3.7 and the identity
\[
\frac{g(v) - g(u)}{u - v} = \sum_{r,s \geq 1} g^{(r+s-1)}u^{-r}v^{-s}
\]
for any formal series \(g(u) = \sum_{r \geq 0} g^{(r)}u^{-r}\).

Now we consider the second part of the proof. Let \(\hat{Y}_{m|n}\) be the algebra generated by elements and relations as in the theorem. We may further define all the other \(e^{(r)}_{i,j}\) and \(f^{(r)}_{j,i}\) in \(\hat{Y}_{m|n}\) by setting \(e^{(r)}_{i,i+1} := e^{(r)}_{i}\) and \(f^{(r)}_{j+1,j} := f^{(r)}_{j}\), then using the formula (3.9) inductively when \(|i - j| > 1\). Let \(\theta : \hat{Y}_{m|n} \rightarrow Y_{m|n}\) be the map sending every element in \(\hat{Y}_{m|n}\) into the element in \(Y_{m|n}\) with the same name. The previous paragraph implies that \(\theta\) is a well-defined surjective homomorphism. Therefore, it remains to prove that \(\theta\) is also injective. The injectivity will be proved in Proposition 3.10.

\(\square\)

Let \(\hat{Y}^+_{m|n}\) denote the subalgebra of \(\hat{Y}_{m|n}\) generated by the elements \(\{e^{(r)}_{i,j} : 1 \leq i < j \leq m + n, r > 0\}\). Define a filtration on \(\hat{Y}^+_{m|n}\) by declaring that the elements \(e^{(r)}_{i,j}\) are of filtered degree \(r - 1\), and denote by \(\text{gr}\hat{Y}^+_{m|n}\) the corresponding graded algebra. Let \(\bar{e}^{(r)}_{i,j} := \text{gr}_{r-1}e^{(r)}_{i,j}\) be the image of \(e^{(r)}_{i,j}\) in the graded algebra \(\text{gr}_{r-1}\hat{Y}^+_{m|n}\).

The following equations were proven over \(\mathbb{C}\) in [Gow2, (53)-(55)], Using (3.50), (3.53) and (3.9), this can be obtained by exactly the same proof.

Lemma 3.9. The following identities hold in \(\text{gr}\hat{Y}^+_{m|n}\):
\[
[\bar{e}^{(r)}_{i,i+1}, \bar{e}^{(s)}_{k,k+1}] = 0, \quad \text{if } |i - k| \neq 1,
\]
\[
[\bar{e}^{(r+1)}_{i,i+1}, \bar{e}^{(s)}_{k,k+1}] = [\bar{e}^{(r)}_{i,i+1}, \bar{e}^{(s+1)}_{k,k+1}], \quad \text{if } |i - k| \neq 1,
\]
\[
[\bar{e}^{(r)}_{i,i+1}, [\bar{e}^{(s)}_{i,i+1}, \bar{e}^{(t)}_{k,k+1}]] = -[\bar{e}^{(s)}_{i,i+1}, [\bar{e}^{(r)}_{i,i+1}, \bar{e}^{(t)}_{k,k+1}]], \quad \text{if } |i - k| \neq 1,
\]
\[
\bar{e}^{(r)}_{i,j} = (-1)^{|i-j|-1}[\bar{e}^{(r)}_{i,j-1}, \bar{e}^{(1)}_{j-1,j}], \quad \text{for } j > i + 1.
\]

Proposition 3.10. The map \(\theta\) in Theorem 3.8 is injective.

Proof. By the same argument as in [Gow2, pp. 814] (see also [Peng1, Section 8]), in order to proving the injectivity of \(\theta\), it suffices to show that the following relation
\[
[\bar{e}^{(r)}_{i,j}, \bar{e}^{(s)}_{k,l}] = (-1)^{|j|\delta_{k,l}\bar{e}^{(r+s+1)}_{i,j} - (-1)^{|i||j|+|i||k|+|j||l|}\delta_{i,k}\bar{e}^{(r+s+1)}_{i,j}].
\]

The proof of (3.64) is the same argument as in the proof of [Gow2, Theorem 3] except the following four relations (cf. [Gow2, (56)-(59)]):
\[
[\bar{e}^{(r)}_{i,i+2}, \bar{e}^{(s)}_{i+1,i+2}] = 0, \quad \text{for } 1 \leq i \leq m + n - 2,
\]
\[
[\bar{e}^{(r)}_{i,i+1}, \bar{e}^{(s)}_{i+1,i+2}] = 0, \quad \text{for } 1 \leq i \leq m + n - 2,
\]
\[
[\bar{e}^{(r)}_{i,i+2}, \bar{e}^{(s)}_{i+1,i+3}] = 0, \quad \text{for } 1 \leq i \leq m + n - 3,
\]
\[
[\bar{e}^{(r)}_{i,j}, \bar{e}^{(s)}_{k,k+1}] = 0, \quad \text{for } 1 \leq i \leq k < j \leq m + n.
\]
By Lemma 3.9, we have

\[ (-1)^{|i|+1} [\tilde{e}_{i,i+2}^{(r)}, \tilde{e}_{i+1,i}^{(s)}] = \left\{ \begin{array}{ll} 0 & \text{if } i < k, \\ \tilde{e}_{i,i+1}^{(r)}, [\tilde{e}_{i,i+1}^{(s)}, \tilde{e}_{i+1,i}^{(s)}] & \text{if } i = k. \end{array} \right. \]

and the last term is zero by (3.55), this proves (3.65). For (3.66), the same method in (3.65) works, except that we apply (3.63) on the term \( \tilde{e}_{i,i+2}^{(s)} \). By applying (3.63) on the left side of (3.67), we obtain

\[ [\tilde{e}_{i,i+2}^{(r)}, \tilde{e}_{i+1,i+3}^{(s)}] = (-1)^{|i|+1+|i+2|} [\tilde{e}_{i,i+1}^{(r)}, [\tilde{e}_{i,i+1}^{(s)}, \tilde{e}_{i+1,i+2}^{(s)}]]. \]

which is zero by (3.57), hence (3.67) is true.

To establish (3.68), we first consider the case \( i = k \). We proceed by induction on \( j-i \). When \( j-i = 1 \) (resp. \( j-i = 2 \)), this follows from (3.60) (resp. (3.66)). The super-Jacobi identity in conjunction with (3.63) gives

\[ (-1)^{|i|} [\tilde{e}_{i,i+1}^{(s)}, \tilde{e}_{i,i}^{(r)}] = [\tilde{e}_{i,i+1}^{(s)}, [\tilde{e}_{i+1,j-1}^{(r)}, \tilde{e}_{j-1,j}^{(s)}]] + (-1)^{|i|+|i+1|+|j-1|} [\tilde{e}_{i,j-1}^{(r)}, [\tilde{e}_{i+1,j}^{(s)}, \tilde{e}_{j,j-1}^{(s)}]] = 0, \]

where the last equality follows from the induction hypothesis and (3.60). For the case \( i < k \) in (3.68), we use (3.63) to reduce the problem to showing

\[ [\tilde{e}_{i,i+1}^{(r)}, \tilde{e}_{i,i}^{(s)}] = 0 = [\tilde{e}_{i,i+1}^{(r)}, \tilde{e}_{k,k}^{(s)}]. \]

By using again (3.63)-(3.67), this follows from the induction on \( k-i \).

Recall that by Lemma 3.2, we may identify \( e_{i,j} x^r \) with \( (-1)^{|i|} \text{gr}_r e_{i,j}^{(r+1)} \) via the identification \( U(\mathfrak{g}) \) and \( \text{gr}Y_{m,n} \). Using (3.6)-(3.8), one sees that \( d_{i}^{(r+1)}, e_{i,j}^{(r+1)} \) and \( f_{j,i}^{(r+1)} \) all belong to \( F_rY_{m,n} \). Combining with [Peng1, Proposition 8.1, 8.4], and under our identification we have that

\[ e_{i,j} x^r = \left\{ \begin{array}{ll} (-1)^{|i|} \text{gr}_r d_{i}^{(r+1)} & \text{if } i = j, \\ (-1)^{|i|} \text{gr}_r e_{i,j}^{(r+1)} & \text{if } i < j, \\ (-1)^{|i|} \text{gr}_r f_{j,i}^{(r+1)} & \text{if } i > j. \end{array} \right. \]

Using PBW theorem for \( U(\mathfrak{g}) \), we obtain the PBW basis for \( Y_{m,n} \) (cf. [Peng1, Corollary 8.5]).

**Theorem 3.11.** Ordered supermonomials in the elements

\[ \{d_{i}^{(r)} \mid 1 \leq i \leq m+n, r > 0\} \cup \{e_{i,j}^{(r)} \mid 1 \leq i < j \leq m+n, r > 0\} \]

taken in any fixed ordering form a basis for \( Y_{m,n} \).
Lemma 3.12. The following relations hold in $Y_{m|n}[[u^{-1}, v^{-1}]]$ for all $l \geq 0$:

\[(u - v)[e_i(u), (e_i(v) - e_i(u))^l] = (-1)^{|i|+1}l(e_i(v) - e_i(u))^{l+1}
= (-1)^{|i|}l(e_i(v) - e_i(u))^{l+1},\]  
(3.70)

\[(u - v)[e_i(u), d_i(v)(e_i(v) - e_i(u))^l] = (-1)^{|i|}(l - 1)d_i(v)(e_i(v) - e_i(u))^{l+1},\]  
(3.71)

\[(u - v)[e_i(u), d_{i+1}(v)(e_i(v) - e_i(u))^l] = (-1)^{|i|+1}(l + 1)d_{i+1}(v)(e_i(v) - e_i(u))^{l+1},\]  
(3.72)

\[(u - v)[e_i(u), d_{i+1}(v)(e_i(v) - e_i(u))^{l-1}] = (-1)^{|i|+1}(l + 2)d_{i+1}(v)(e_i(v) - e_i(u))^{l+1}d_i(v)^{-1}.\]  
(3.73)

Proof. We first check (3.70). If $i \neq m$, then $|i| = |i + 1|$ and $e_i(u)$ is even. In this case, the relation (3.70) follows from (3.31) and the Leibniz rule. It remains to treat the case $i = m$. There is nothing to prove, if $p = 2$. So assume $p \neq 2$. Now Remark 2 readily yields

\[(u - v)[e_m(u), e_m(v)] = -(e_m(v) - e_m(u))^2 = 0,
so that all terms of (3.70) are equal to 0.

In view of (3.28), we have

\[(u - v)[e_i(u), d_i(v)] = (-1)^{|i|}d_i(v)(e_i(u) - e_i(v)),\]  
(3.74)

\[(u - v)[e_i(u), d_i(v)^{-1}] = (-1)^{|i|}(e_i(v) - e_i(u))d_i(v)^{-1},\]  
(3.75)

\[(u - v)[e_i(u), d_{i+1}(v)] = (-1)^{|i|+1}d_{i+1}(v)(e_i(v) - e_i(u)),\]  
(3.76)

\[(u - v)[e_i(u), d_{i+1}(v)^{-1}] = (-1)^{|i|+1}(e_i(u) - e_i(v))d_{i+1}(v)^{-1}.\]  
(3.77)

Then (3.71)-(3.73) follows from (3.70), (3.74)-(3.76) using Leibniz again. $\square$

The following result follows from (3.74) and (3.76) by specializing $v$.

Corollary 3.13. The following hold in $Y_{m|n}[[u^{-1}]]$:

\[e_i(u - (-1)^{|i|})d_i(u) = d_i(u)e_i(u), \quad d_i(u)^{-1}e_i(u - (-1)^{|i|}) = e_i(u)d_i(u)^{-1},\]  
(3.78)

\[e_i(u + (-1)^{|i|+1})d_{i+1}(u) = d_{i+1}(u)e_i(u), \quad d_{i+1}(u)^{-1}e_i(u + (-1)^{|i|+1}) = e_i(u)d_{i+1}(u)^{-1}.\]  
(3.79)

Lemma 3.14. For any $i = 1, \ldots, m + n - 1$, $l \geq 0$ and $r, s > 0$, we have that

\[\begin{bmatrix}
    e_i^{(r)}
    \sum_{s_1, \ldots, s_l \geq r} e_i^{(s_1)} \cdots e_i^{(s_l)}
\end{bmatrix} = (-1)^{|i|}l \sum_{s_1, \ldots, s_{l+1} \geq r} e_i^{(s_1)} \cdots e_i^{(s_{l+1})},\]  
(3.80)

\[\begin{bmatrix}
    e_i^{(r)}
    \sum_{s_1, \ldots, s_l \leq r - 1} e_i^{(s_1)} \cdots e_i^{(s_l)}
\end{bmatrix} = (-1)^{|i|}l \sum_{s_1, \ldots, s_{l+1} \leq r - 1} e_i^{(s_1)} \cdots e_i^{(s_{l+1})},\]  
(3.81)

\[\begin{bmatrix}
    e_i^{(r)}
    \sum_{s_1, \ldots, s_l \geq r, t \geq 0} d_i^{(t)} e_i^{(s_1)} \cdots e_i^{(s_l)}
\end{bmatrix} = (-1)^{|i|}(l - 1) \sum_{s_1, \ldots, s_{l+1} \geq r, t \geq 0} d_i^{(t)} e_i^{(s_1)} \cdots e_i^{(s_{l+1})},\]  
(3.82)
\[\left[ e_i^{(r)}, \sum_{s_1, \ldots, s_l \geq r, t \geq 0} d_{i+1}^{(t)} e_i^{(s_1)} \cdots e_i^{(s_l)} \right] = (-1)^{|i|+1}(l+1) \sum_{s_1, \ldots, s_{l+1} \geq r, t \geq 0} d_{i+1}^{(t)} e_i^{(s_1)} \cdots e_i^{(s_{l+1})}, \quad (3.83)\]

\[\left[ e_i^{(r)}, \sum_{s_1, \ldots, s_l \geq r, u, t \geq 0} d_{i}^{(t)} e_i^{(s_1)} \cdots e_i^{(s_l)} d_i^{(u)} \right] = (-1)^{|i|+1}(l+2) \sum_{s_1, \ldots, s_{l+1} \geq r, u, t \geq 0} d_{i+1}^{(t)} e_i^{(s_1)} \cdots e_i^{(s_{l+1})} d_i^{(u)}. \quad (3.84)\]

**Proof.** Using \((3.48)\) and the same argument as in the proof of Lemma 3.12 one obtains

\[\left[ e_i^{(r)}, e_i^{(s_j)} \right] = (-1)^{|i|} \sum_{s_j', e_i^{(r)}} e_i^{(s_j')} e_i^{(s_j')} = (-1)^{|i|+1} \sum_{s_j', e_i^{(r)}} e_i^{(s_j')} e_i^{(s_j')} \quad (3.85)\]

for \(0 < r \leq s_j\), and \((3.75)\) in conjunction with \((3.59)\) gives

\[\left[ e_i^{(r)}, d_i^{(s)} \right] = (-1)^{|i|} \sum_{t=0}^{l-1} e_i^{(r+s-1-t)} d_i^{(t)}. \quad (3.86)\]

If \(p = 2\) or \(i \neq m\), then the lemma can be proved in the same method as in [BT, Lemma 4.9] using the relations \((3.85)\) and \((3.86)\), and hence we skip the detail.

So we assume \(i = m\) and \(p > 2\). For \((3.80)\), we shall show

\[(\dagger) \quad \left[ e_i^{(r)}, \sum_{s_1, \ldots, s_l \geq r} e_i^{(s_1)} \cdots e_i^{(s_l)} \right] = 0 = (-1)^{|i|+1} \sum_{s_1, \ldots, s_{l+1} \geq r} e_i^{(s_1)} \cdots e_i^{(s_{l+1})}. \]

It is clear that the above equality holds if \(l = 0\). Since \(e_i^{(r)}\) is odd, \((3.48)\) implies that \([e_m^{(r)}, e_m^{(s)}] = 0\) for all \(r, s > 0\), so that in particular \((e_m^{(r)})^2 = 0\). This yields

\[\sum_{s_1, \ldots, s_l \geq r} e_i^{(s_1)} \cdots e_i^{(s_l)} = 0\]

whenever \(l \geq 2\). Hence, \((\dagger)\) holds for all \(l \geq 1\). The other parts are proved similarly. \(\square\)

We define

\[d_{i;k}(u) := d_i(u)d_i(u-1) \cdots d_i(u-k+1),\]
\[d_{i;k}(u) := d_i(u)d_i(u+1) \cdots d_i(u+k-1).\]

**Lemma 3.15.** The following relations hold for all \(k \geq 1:\)

\[(u-v)[d_{i;k}(u), e_i(v)] = kd_{i;k}(u)(e_i(v) - e_i(u)), \quad \text{for} \ 1 \leq i \leq m, \quad (3.87)\]
\[(u-v)[d_{i;k}(u), e_i(v)] = -kd_{i;k}(u)(e_i(v) - e_i(u)), \quad \text{for} \ m + 1 \leq i \leq m + n - 1, \quad (3.88)\]
\[(u-v)[d_{i;k}(u), e_{i-1}(v)] = kd_{i;k}(u)(e_{i-1}(u) - e_{i-1}(v)), \quad \text{for} \ 2 \leq i \leq m, \quad (3.89)\]
\[(u-v)[d_{i;k}(u), e_{i-1}(v)] = -kd_{i;k}(u)(e_{i-1}(u) - e_{i-1}(v)), \quad \text{for} \ m + 1 \leq i \leq m + n. \quad (3.90)\]

**Proof.** We only prove \((3.88)\), as others can be treated similarly. Actually, we will prove it in the following equivalent form:

\[(\ast) \quad (u-v)[d_{i;k}(u), e_i(v)] = (u-v)e_i(v)d_{i;k}(u) + kd_{i;k}(u)e_i(u).\]
This follows when \( k = 1 \) from (3.74). To prove (*) in general, we proceed by induction on \( k \). Given (*) for some \( k \geq 1 \), multiply both sides on the left by \((u - v + k + 1)d_k(u + k)\) to deduce that:

\[
(u - v + k + 1)(u - v + k)d_{tk+1}(u)e_i(v) = (u - v)(u - v + k + 1)d_k(u + k)e_i(v)d_{tk}(u) + k(u - v + k + 1)d_{tk+1}(u)e_i(v). \tag{3.91}
\]

Using the case of \( k = 1 \) in (*) and replacing \( u \) by \( u + k \) give that

\[
(u - v + k + 1)d_k(u + k)e_i(v) = (u - v + k)e_i(v)d_k(u + k) + d_k(u + k)e_i(u + k).
\]

Then substituting the above identity into (3.91) and using (3.78) we obtain (*) with \( k \) replaced by \( k + 1 \), as required.

We shall consider more diagonal elements, we let

\[
h_i(u) = \sum_{r \geq 0} h_i^{(r)} u^{-r} := (-1)^{|i|} d_{i+1}(u) d_i(u)^{-1}
\]

assuming \( 1 \leq i \leq m + n - 1 \). According to (3.43), we have \( d_i^{(0)} = 1 \) and \( d_i^{(r)} = -\sum_{l=0}^r d_i^{(l)} d_i^{(r-l)} \), so that in particular \( h_i^{(r+1)} \in F_r Y_{m|n}, h_i^{(0)} = (-1)^{|i|} \) and \( gr_r d_i^{(r+1)} = -gr_r d_i^{(r+1)} \). Moreover, the identification (3.69) yields:

\[
gr_r h_i^{(r+1)} = e_{i,i} x^r - (-1)^{|i|+|i+1|} e_{i+1,i+1} x^r. \tag{3.93}
\]

Note also by Corollary 3.13 that

\[
h_i(u)e_i(u - (-1)^{|i|}) = e_i(u + (-1)^{|i+1|}) h_i(u). \tag{3.94}
\]

**Lemma 3.16.** The following relations hold in \( Y_{m|n}[[u^{-1}, v^{-1}]] \):

\[
(u - v - (-1)^{|i|})[h_i(u), e_i(v)] = (-1)^{|i|+1}2 h_i(u)(e_i(u - (-1)^{|i|}) - e_i(v)), \tag{3.95}
\]

\[
(u - v + (-1)^{|i+1|})[h_i(u), e_i(v)] = (-1)^{|i|+1}2(e_i(u + (-1)^{|i+1|}) - e_i(v)) h_i(u), \tag{3.96}
\]

\[
(u - v)[h_{i-1}(u), e_i(v)] = (-1)^{|i|} h_{i-1}(u)(e_i(v) - e_i(u)), \tag{3.97}
\]

\[
(u - v - (-1)^{|i|})[h_{i-1}(u), e_i(v)] = (-1)^{|i|}(e_i(v) - e_i(u - (-1)^{|i|})) h_{i-1}(u), \tag{3.98}
\]

\[
(u - v)[h_{i+1}(u), e_i(v)] = (-1)^{|i|+1}(e_i(v) - e_i(u)) h_{i+1}(u), \tag{3.99}
\]

\[
(u - v + (-1)^{|i+1|})[h_{i+1}(u), e_i(v)] = (-1)^{|i|+1} h_{i+1}(u)(e_i(v) - e_i(u + (-1)^{|i+1|})). \tag{3.100}
\]

***Proof.*** We prove (3.96), (3.98) and (3.99) in detail here, while the others can be proved in a similar fashion. To establish (3.96), we have by (3.73) that

\[
(u - v)[e_i(u), d_{i+1}(v)d_i(v)^{-1}] = (-1)^{|i|+1}2d_{i+1}(v)(e_i(v) - e_i(u))d_{i+1}(v)^{-1}. \tag{3.101}
\]

From (3.76), we have that

\[
(v - u + (-1)^{|i+1|})d_{i+1}(v)e_i(u) = (v - u)e_i(u)d_{i+1}(v) + (-1)^{|i|+1}d_{i+1}(v)e_i(v).
\]

Then we multiply (3.101) by \((v - u + (-1)^{|i+1|})\) and use these identities in combination with (3.79) to obtain

\[
(u - v)(v - u + (-1)^{|i+1|})[e_i(u), h_i(v)] = (-1)^{|i|+1}2(v - u)(e_i(u + (-1)^{|i+1|}) - e_i(u)) h_i(v).
\]
Dividing through by \((u - v)\) and interchanging \(u\) and \(v\) give the result. For (3.98), the Leibniz rule for \(\text{ad} \, e_i(v)\) in conjunction with (3.28) implies that
\[
(u - v - (-1)^{|i|})[d_i(u)d_{i-1}(u)^{-1}, e_i(v)] = ((v - u + (-1)^{|i|})[e_i(v), d_i(u)]d_{i-1}(u)^{-1}.
\]
Substituting the last bracket by (3.74) and simplifying the result, we have
\[
(u - v - (-1)^{|i|})[d_i(u)d_{i-1}(u)^{-1}, e_i(v)] = (-1)^{|i|}(e_i(v)d_i(u) - d_i(u)e_i(v))d_{i-1}(u)^{-1}.
\]
Then the assertion follows from applying (3.78) to the above equality. Finally, for (3.99), this follows easily from (3.77) and the Leibniz rule, using that \(d_{i+2}(u)\) commutes with \(e_i(v)\) by (3.28).

**Corollary 3.17.** The following hold in \(Y_{m|n}[[u^{-1}, v^{-1}]]\):
\[
(u - v)[h_i(u), e_i(v)] = \begin{cases} (-1)^{|i|+1}(2h_i(u)e_i(u - (-1)^{|i|}) - h_i(u)e_i(v) - e_i(v)h_i(u)), & i \neq m; \\ 0, & i = m; \end{cases}
\]
(3.102)
\[
(u - v)[h_{i-1}(u + \frac{(-1)^{|i|}}{2}), e_i(v)] = \frac{(-1)^{|i|}}{2} \left( h_{i-1}(u + \frac{(-1)^{|i|}}{2})e_i(v) + e_i(v)h_{i-1}(u + \frac{(-1)^{|i|}}{2}) \right)
- \frac{(-1)^{|i|}}{2} \left( h_{i-1}(u + \frac{(-1)^{|i|}}{2})e_i(u + \frac{(-1)^{|i|}}{2}) + e_i(u - \frac{(-1)^{|i|}}{2})h_{i-1}(u + \frac{(-1)^{|i|}}{2}) \right),
\]
(3.103)
\[
(u - v)[h_{i+1}(u - \frac{(-1)^{|i|+1}}{2}), e_i(v)] = \frac{(-1)^{|i|+1}}{2} \left( h_{i+1}(u - \frac{(-1)^{|i|+1}}{2})e_i(v) + e_i(v)h_{i+1}(u - \frac{(-1)^{|i|+1}}{2}) \right)
- \frac{(-1)^{|i|+1}}{2} \left( h_{i+1}(u - \frac{(-1)^{|i|+1}}{2})e_i(u + \frac{(-1)^{|i|+1}}{2}) + e_i(u - \frac{(-1)^{|i|+1}}{2})h_{i+1}(u - \frac{(-1)^{|i|+1}}{2}) \right),
\]
(3.104)
assuming that \(\text{char } k \neq 2\) for the last two.

**Proof.** Suppose that \(\text{char } k \neq 2\). The relations (3.103) and (3.104) follow by averaging the corresponding pairs identities from Lemma 3.16, e.g. (3.103) is (3.97)\(+(3.98)\)/2 with \(u\) replaced by \(u + \frac{(-1)^{|i|}}{2}\). For (3.102), we note that \((-1)^{|i|} = (-1)^{|i|+1}\) when \(i \neq m\). When combined with (3.94), the first equality follows by averaging (3.95) and (3.96). If \(i = m\), then the right hand side of (3.95) and (3.96) are equal. In conjunction with (3.94) this implies \([h_i(u), e_i(v)] = 0\). To establish (3.102) when \(\text{char } k = 2\), we observe by (3.95) that \([h_i(u), e_i(v)] = 0\), which easily implies the desired identity. \(\square\)

### 3.7. Automorphisms

We list the following (anti)automorphisms of \(Y_{m|n}\) which are needed in the next section; see [MNO, Proposition 1.12] and [Peng4, (5.22)].

1. (“Multiplication by a power series”) For any power series \(f(u) \in 1 + u^{-1}k[[u^{-1}]]\), there is an automorphism \(\mu_f : Y_{m|n} \to Y_{m|n}\) defined from \(\mu_f(T(u)) = f(u)T(u)\). On Drinfeld generators, it is easy to show by induction and (3.5) that \(\mu_f(d_i(u)) = f(u)d_i(u), \mu_f(e_j(u)) = e_j(u)\) and \(\mu_f(f_j(u)) = f_j(u)\).

2. (“Transposition”) By the presentation of \(Y_{m|n}\), there is an anti-automorphism \(\tau : Y_{m|n} \to Y_{m|n}\) of order 2 defined by \(\tau(d_i^{(r)}) = d_i^{(r)}, \tau(e_j^{(r)}) = f_j^{(r)}, \tau(f_j^{(r)}) = e_j^{(r)}\).

3. (“Permutation”) Let \(S_{m+n}\) be the Symmetric group on \(m + n\) objects and let \(S_m \times S_n \subseteq S_{m+n}\) denote its Young subgroup associated to \((m, n)\). Suppose that \(p \neq 2\). For each \(w \in S_m \times S_n \subseteq S_{m+n}\), there is an automorphism \(w : Y_{m|n} \to Y_{m|n}\) sending \(\lambda_{ij}^{(r)} \mapsto \rho_{w(i),w(j)}^{(r)}\). This is clear from the RTT relation (3.1) (see also [Peng2, Section
2] and [Tsy, Lemma 2.24]). If \( p = 2 \), then each element \( w \in S_{m+n} \) gives rise to an automorphism ([BT, Section 4.5(5)]).

(4) (“Translation”) For \( c \in \mathbb{K} \), there is an automorphism \( \eta_c : Y_{m|n} \to Y_{m|n} \) defined from \( \eta_c(t_{i,j}(u)) = t_{i,j}(u - c) \), i.e. \( \eta_c(t_{i,j}^{-1}) = \sum_{s=1}^{r} (r_{-s}^{s}) c^{r-s} t_{i,j}^{s} \). In terms of Drinfeld generators, \( \eta_c \) sends \( d_i(u) \to d_i(u - c) \), \( e_{i,j}(u) \to e_{i,j}(u - c) \) and \( f_{j,i}(u) \to f_{j,i}(u - c) \).

This can be easily checked by the relations (3.2) and (3.6)-(3.8).

**Lemma 3.18.** [BT, Lemma 4.16] Suppose that \( p > 2 \). For \( 1 \leq i < j \leq m + n \) and \( |i| + |j| = 0 \), the permutation automorphism of \( Y_{m|n} \) defined by the transposition \( (i+1, j) \) maps \( e_i(u) \to e_{i,j}(u) \) and \( f_i(u) \to f_{j,i}(u) \).

**Proof.** This follows from (3.6)-(3.8). \( \square \)

4. The center of \( Y_{m|n} \)

In this section, we will describe the center of the modular super Yangian \( Y_{m|n} \), and give precise formulas for the generators.

4.1. Harish-Chandra center. Following [Na], we define the quantum Berezinian of the matrix \( T(u) \) as the following power series:

\[
c(u) := \sum_{\rho \in S_m} \text{sgn}(\rho) t_{\rho(1),1}(u) t_{\rho(2),2}(u - 1) \cdots t_{\rho(m),m}(u - m + 1) \\
\times \sum_{\sigma \in S_n} \text{sgn}(\sigma) t'_{m+1,m+\sigma(1)}(u - m + 1) \cdots t'_{m+n,m+\sigma(n)}(u - m + n).
\]

Thanks to [Gow1, Theorem 1], we may also write the quantum Berezinian as follows:

\[
c(u) = d_1(u) d_2(u - 1) \cdots d_m(u - m + 1) \times d_{m+1}(u - m + 1)^{-1} \cdots d_{m+n}(u - m + n)^{-1} \\
=: 1 + \sum_{r \geq 1} c^{(r)} u^{-r}.
\]  

(4.1)

The algebra generated by the coefficients \( \{c^{(r)}; r \geq 1\} \) will be denoted by \( Z_{HC}(Y_{m|n}) \). We call it the Harish-Chandra center of \( Y_{m|n} \).

**Proposition 4.1.** The elements \( \{c^{(r)}; r \geq 1\} \) are central. Furthermore, we have that \( c^{(r)} \) has degree \( r - 1 \) with respect to the loop filtration and \( \text{gr}_{r-1} c^{(r)} = z_{r-1} \in U(\mathfrak{g}) \). Hence, \( c^{(1)}, c^{(2)}, \ldots \) are algebraically independent.

**Proof.** Using (3.44) and the anti-automorphism \( \tau \), we just need to check \( [c(u), e_i(v)] = 0 \) for all \( i \). This can be proven in the same manner as [BK, Theorem 7.2] and [Gow1, Theorem 2] using the relation (3.28). By the proof of [Gow2, Theorem 4], we have

\[
c^{(r)} = t_{1,1}^{(r)} + \cdots + t_{m,m}^{(r)} - t_{m+1,m+1}^{(r)} - \cdots - t_{m+n,m+n}^{(r)} + \text{terms of lower degree}
\]

Consequently, \( \text{gr}_{r-1} c^{(r)} = z_{r-1} \) (see Lemma 3.2). The final assertion follows because \( z_0, z_1, \ldots \) are algebraically independent in \( U(\mathfrak{g}) \). \( \square \)
4.2. Off-diagonal $p$-central elements. This subsection is a super generalization of [BT, Section 5.2]. We may assume $p > 2$ because the case $p = 2$ has been considered in loc. cit. In this subsection, we investigate the $p$-central elements that lie in the “even root subalgebras” $Y_{i,j}^+, Y_{j,i}^- \subseteq Y_{m|n}$ for $1 \leq i < j \leq m+n$ and $|i| + |j| = 0$, that is, the subalgebras generated by $\{e_{i,j}^{(r)}; r > 0, |i| + |j| = 0\}$ and $\{f_{j,i}^{(r)}; r > 0, |i| + |j| = 0\}$, respectively.

**Lemma 4.2.** For $1 \leq i < j \leq m+n$ and $|i| + |j| = 0$, all coefficients in the power series $(e_{i,j}(u))^p$ and $(f_{j,i}(u))^p$ belong to $Z(Y_{m|n})$.

**Proof.** Using Lemma 3.18 and the anti-automorphism $\tau$ it only needs to be proved that the coefficients of $(e_{i}(u))^p$ are central in $Y_{m|n}$ for each $i = 1, \ldots , m-1, m+1, \ldots , m+n-1$.

Since we are in characteristic $p$, Theorem 3.3 implies that it is enough to show the following identities in $Y_{m|n}[[u^{-1}, v^{-1}]]$ for all admissible $j, k$:

\[
(\text{ad } e_{i}(u))^p(e_{j}(v)) = 0, \quad (4.2)
\]

\[
(\text{ad } e_{i}(u))^p(d_{k}(v)) = 0, \quad (4.3)
\]

\[
(\text{ad } e_{i}(u))^p(f_{j}(v)) = 0. \quad (4.4)
\]

A consecutive application of the swap map and the anti-automorphism $\tau$ implies that we may assume $1 \leq i \leq m-1$. Note that there is a standard embedding $Y_{m} \hookrightarrow Y_{m|n}$. Then [BT, Lemma 5.2] implies that equations (4.2)-(4.4) hold for all $1 \leq j \leq m-1$ and $1 \leq k \leq m$. Due to (3.27), (3.28) and (3.30) it remains to prove that $(\text{ad } e_{m-1}(u))^p(e_{m}(v)) = 0$. This follows immediately from (3.35). \hfill \Box

**Lemma 4.3.** For $1 \leq i < j \leq m+n, |i| + |j| = 0$ and $r > 0$, we have that $(e_{i,j}^{(r)})^p, (f_{j,i}^{(r)})^p \in Z(Y_{m|n})$.

**Proof.** Similar to the proof of Lemma 4.2, it is enough to show that $(e_{i}^{(r)})^p \in Z(Y_{m|n})$ for each $1 \leq i \leq m-1$. This reduces to checking

\[
(\text{ad } e_{i}^{(r)})^p(e_{j}^{(s)}) = 0, \quad (4.5)
\]

\[
(\text{ad } e_{i}^{(r)})^p(d_{k}^{(s)}) = 0, \quad (4.6)
\]

\[
(\text{ad } e_{i}^{(r)})^p(f_{j}^{(s)}) = 0. \quad (4.7)
\]

Owing to [BT, Lemma 5.3], the equations (4.5)-(4.7) hold for all $1 \leq j \leq m-1$ and $1 \leq k \leq m$. Thanks to (3.45), (3.46) and (3.52), we only have to show that $(\text{ad } e_{m-1}^{(r)})^p(e_{m}^{(s)}) = 0$. But this follows from (3.55). \hfill \Box

**Remark 4.** Lemmas 4.2 and 4.3 can also be deduced using Lemmas 3.12 and 3.14, respectively.

Also, we put

\[
p_{i,j}(u) = \sum_{r \geq p} p_{i,j}^{(r)} u^{-r} := e_{i,j}(u)^p, \quad q_{j,i}(u) = \sum_{r \geq p} q_{j,i}^{(r)} u^{-r} := f_{j,i}(u)^p. \quad (4.8)
\]

**Theorem 4.4.** For $1 \leq i < j \leq m+n, |i| + |j| = 0$, the algebras $Z(Y_{m|n}) \cap Y_{i,j}^+$ and $Z(Y_{m|n}) \cap Y_{j,i}^-$ are infinite rank polynomial algebras freely generated by the central elements $\{e_{i,j}^{(r)}; r > 0\}$ and
\{(f^{(r)}_{j,i})^p; \ r > 0\}, respectively. We have that \((e^{(r)}_{i,j})^p, (f^{(r)}_{j,i})^p \in F_{rp-p}Y_{m|n}\) and
\[
\text{gr}_{rp-p}(e^{(r)}_{i,j})^p = (-1)^{|i|}(e_{i,j}x^{r-1})^p, \quad \text{gr}_{rp-p}(f^{(r)}_{j,i})^p = (-1)^{|j|}(e_{j,i}x^{r-1})^p.
\]
(4.9)

For \(r \geq p\) we have that
\[
P_{i,j}^{(r)} = \begin{cases} 
(-1)^{|i|}(e^{(r/p)}_{i,j})^p + (\ast) & \text{if } p \mid r, \\
(\ast) & \text{if } p \nmid r,
\end{cases}
\]
(4.10)
where \((\ast) \in F_{r-p-1}Y_n\) is a polynomial in the elements \(e^{(s)}_{i,j}\) for \(1 \leq s < [r/p]\). Hence, the central elements \(\{p_{i,j}^{(r)}; \ r > 0\}\) give another algebraically independent set of generators for \(Z(Y_{m|n}) \cap Y^+_i\)
lifting the central elements \(\{(e_{i,j}x^{r-1})^p; \ r > 0\}\) of \(\text{gr}Y_{m|n}\). Analogous statements with \(Y^+_i, e, p\) and \(e_{i,j}x^{r-1}\) replaced by \(Y^-_i, f, q\) and \(e_{i,j}x^{r-1}\) also hold.

**Proof.** The proof, which uses Theorem 2.3, Lemma 4.2 and Lemma 4.3, is similar to the proof of [BT, Theorem 5.4]. \(\square\)

### 4.3. Diagonal \(p\)-central elements.
This subsection we introduce the \(p\)-central elements that belong to the diagonal subalgebras

\[Y^0_i := \mathbb{K}[d^{(r)}_i; \ r > 0]\]
of \(Y_{m|n}\) for \(1 \leq i \leq m + n\). Note that \(Y^0_i = \mathbb{K}[d^{(r)}_i; \ r > 0]\) by (3.43).

We define
\[
b_i(u) := \sum_{r \geq 0} b_i^{(r)}u^{-r} := \begin{cases} 
d_{i,p}(u) = d_i(u)d_i(u-1) \cdots d_i(u-p+1) & \text{if } |i| = 0, \\
d_{i,p}(u)^{-1} = d_i(u)^{-1}d_i(u-1)^{-1} \cdots d_i(u-p+1)^{-1} & \text{if } |i| = 1.
\end{cases}
\]
(4.11)

**Lemma 4.5.** For all \(i = 1, \ldots, m + n\) and \(r > 0\), the elements \(b_i^{(r)}\) belongs to \(Z(Y_{m|n})\).

**Proof.** In view of Theorem 3.3 and (3.44), it suffices to check that
\[
[b_i(u), e_j(v)] = 0 = [b_i(u), f_j(v)]
\]
for all \(1 \leq j \leq m + n - 1\). By applying the anti-automorphism \(\tau\) (Section 3.7(2)), it suffices to check just the first equality. By (3.28), the first equality obviously holds when \(j \notin \{i - 1, i\}\). Consider first the case that \(j = i - 1\). When \(|i| = 0\), then (3.89) implies that
\[
[b_i(u), e_{i-1}(v)] = [d_{i,p}(u), e_{i-1}(v)] = [d_{i,p}(u - p + 1), e_{i-1}(v)] = 0.
\]
Now let \(|i| = 1\), then (3.90) yields \([d_{i,p}(u), e_{i-1}(v)] = 0\), so that
\[
[b_i(u), e_{i-1}(v)] = [d_{i,p}(u)^{-1}, e_{i-1}(v)] = 0.
\]
One argues similarly for the case \(j = i\) using (3.87), (3.88) instead of (3.89), (3.90). \(\square\)

**Theorem 4.6.** For \(1 \leq i \leq m + n\), the algebra \(Z(Y_{m|n}) \cap Y^0_i\) is an infinite rank polynomial algebra freely generated by the central elements \(\{b_i^{(r)}; \ r > 0\}\). We have that \(b_i^{(r)} \in F_{rp-p}Y_{m|n}\) and
\[
\text{gr}_{rp-p} b_i^{(r)} = (e_{i,i}x^{r-1})^p - e_{i,i}x^{rp-p}.
\]
(4.12)

For \(0 < r < p\), we have that \(b_i^{(r)} = 0\). For \(r > p\) with \(p \nmid r\), we have that \(b_i^{(r)} \in F_{r-p-1}Y_{m|n}\) and it is a polynomial in the elements \(\{b_i^{(s)}; \ 0 < s \leq [r/p]\}\).
Proof. Recall that $\text{gr}_r d_i^{(r+1)} = -\text{gr}_r d_i^{(r-1)}$, so that the identification (3.69) implies that $\text{gr}_r d_i^{(r+1)} = e_i x^r$ when $|i| = 1$. Then the proof, which uses Theorem 2.3, Lemma 4.5, is similar to the proof of [BT, Theorem 5.8].

4.4. The center $Z(Y_{m|n})$. We define the $p$-center $Z_p(Y_{m|n})$ of $Y_{m|n}$ to be the subalgebra generated by

$$\{y_i^{(rp)}; 1 \leq i \leq m+n, r > 0\} \cup \left\{(e_i^{(r)})^p, (f_{j,i}^{(r)})^p; 1 \leq i < j \leq m+n, r > 0, |i| + |j| = 0\right\}.$$  

(4.13)

According to Proposition 4.1, Lemma 4.3 and Lemma 4.5, we know that both $Z_{HC}(Y_{m|n})$ and $Z_p(Y_{m|n})$ are subalgebras of $Z(Y_{m|n})$. Note also by (3.69) and (4.12) that $\text{gr} Z_p(Y_{m|n})$ may be identified with the $p$-center $Z_p(g)$ of $U(g)$ from (2.7).

We need one more family of elements. Recalling (4.1) and (4.11), we let

$$bc(u) := \sum_{r \geq 0} bc^{(r)} u^{-r}$$

$$:= b_1(u)b_2(u-1) \cdots b_m(u-m+1)b_{m+1}(u-m+1) \cdots b_{m+n}(u-m+n)$$

$$= c(u)c(u-1) \cdots c(u-p+1).$$  

(4.14)

By definition each $bc^{(r)}$ can be expressed as a polynomial in the elements $\{c^{(s)}; s > 0\}$, so that it belongs to $Z_{HC}(Y_{m|n})$. It is also a polynomial in the elements $\{b_i^{(s)}; 1 \leq i \leq m+n, s > 0\}$, so that it belongs to $Z_p(Y_{m|n})$ by Theorem 4.6. Consequently, $bc^{(r)} \in Z_{HC}(Y_{m|n}) \cap Z_p(Y_{m|n})$.

Lemma 4.7. For $r > 0$, we have that $bc^{(rp)} \in F_{rp-p}Y_{m|n}$ and

$$\text{gr}_{rp-p} bc^{(rp)} = z_{r-1}^p - z_{rp-p}.$$  

(4.15)

Proof. By definition of $bc(u)$ (4.14), we have

$$bc(u) = \prod_{i=1}^p \left( \sum_{r \geq 0} c^{(r)}(u-i+1)^{-r} \right).$$

Since $c^{(0)} = 1$ and the set $\{c^{(r)}; r > 0\}$ are commuting elements by Proposition 4.1, it follows from [BT, Lemma 2.9] that $bc^{(rp)} \in F_{rp-p}Y_{m|n}$ and

$$bc^{(rp)} \equiv (c^{(r)})^p - c^{(rp-p+1)} \mod F_{rp-p-1}Y_{m|n}.$$  

Our assertion now follows from the fact that $\text{gr}_{r-1} c^{(r)} = z_{r-1}$. □

Now we can state the main result of this section. The foregoing observations in conjunction with (2.7), (3.69) and Theorem 2.3 yield the following Theorem. The proof is similar to the proof of [BT, Theorem 5.11].

Theorem 4.8. The centre $Z(Y_{m|n})$ is generated by $Z_{HC}(Y_{m|n})$ and $Z_p(Y_{m|n})$. Moreover:

1. $Z_{HC}(Y_{m|n})$ is the free polynomial algebra generated by $\{c^{(r)}; r > 0\}$;
2. $Z_p(Y_{m|n})$ is the free polynomial algebra generated by

$$\{y_i^{(rp)}; 1 \leq i \leq m+n, r > 0\} \cup \left\{(e_i^{(r)})^p, (f_{j,i}^{(r)})^p; 1 \leq i < j \leq m+n, r > 0, |i| + |j| = 0\right\};$$  

(4.16)
(3) $Z(Y_{m|n})$ is the free polynomial algebra generated by
\[ \{ b_i^{(r)}, c(r); \ 2 \leq i \leq m + n, r > 0 \} \cup \{ e_i^{(r)}, f_{i,j}^{(r)}; \ 1 \leq i < j \leq m + n, r > 0, |i| + |j| = 0 \}; \] 
(4.17)

(4) $Z_{HC}(Y_{m|n}) \cap Z_p(Y_{m|n})$ is the free polynomial algebra generated by $\{ b c^{(r)}; \ r > 0 \}$. 

**Corollary 4.9.** The super Yangian $Y_{m|n}$ is free as a module over its center, with basis given by the ordered supermonomials in
\[ \{ d_i^{(r)}; \ 2 \leq i \leq m + n, r > 0 \} \cup \{ e_i^{(r)}, f_{i,j}^{(r)}; \ 1 \leq i < j \leq m + n, r > 0 \} \] 
in which no exponent is $p$ or more for $d_i^{(r)}$ and $e_i^{(r)}, f_{i,j}^{(r)}$ with $|i| + |j| = 0$. 

**Proof.** Let $Y^0_{m|n}$ denote the subalgebra of $Y_{m|n}$ generated by the elements $\{ d_i^{(r)} \}$. We consider the sets
\[ \{ b_i^{(r)}, c(r); \ 2 \leq i \leq m + n, r > 0 \} \] 
(4.19)
and
\[ \{ d_i^{(r)}; \ 2 \leq i \leq m + n, r > 0 \}. \] 
(4.20)

It suffices to show that the set consisting of ordered monomials in (4.19) multiplied by ordered monomials in (4.20) with exponents $< p$ gives a basis for $Y^0_{m|n}$. To see this, we pass to the associated graded algebra using (3.69), (4.12) and Proposition 4.1 to reduce to showing that the monomials
\[ \prod_{r \geq 0} z_r^{a_{i,r}} \prod_{2 \leq i \leq m+n, \ r \geq 0} (e_{i,i} t^{r})^{p} - e_{i,i} t^{rp})^{a_{i,r}} \prod_{2 \leq i \leq m+n, \ r \geq 0} (e_{i,i} t^{r})^{b_{i,r}} \]
for $a_{i,r} \geq 0$ and $0 \leq b_{i,r} < p$ form a basis for $\text{gr} Y^0_{m|n}$. This is quite straightforward: these monomials are related to a PBW basis of $\text{gr} Y^0_{m|n}$ by a uni-triangular transition matrix. \( \square \)

Similarly, we have:

**Corollary 4.10.** The super Yangian $Y_{m|n}$ is free as a module over $Z_p(Y_{m|n})$ with basis given by the ordered supermonomials in
\[ \{ d_i^{(r)}; \ 1 \leq i \leq m + n, r > 0 \} \cup \{ e_i^{(r)}, f_{i,j}^{(r)}; \ 1 \leq i < j \leq m + n, r > 0 \} \] 
in which no exponent is $p$ or more for $d_i^{(r)}$ and $e_i^{(r)}, f_{i,j}^{(r)}$ with $|i| + |j| = 0$. 

5. **MODULAR SUPER YANGIAN OF $\mathfrak{sl}_{m|n}$**

In this section, we define the modular version of the super Yangian of $\mathfrak{sl}_{m|n}$, which is a subalgebra $SY_{m|n}$ of $Y_{m|n}$. We give a presentation for $SY_{m|n}$ valid in any characteristic by using the diagonal elements defined in (3.92). We will show that this presentation is equivalent to the usual Drinfeld-type presentation (see [Gow2, Proposition], [Tsy, Section 2.5]) whenever $\text{char } \mathbb{K} \neq 2$. 
5.1. The special super Yangian. We define the special super Yangian associated to the special linear Lie superalgebra \( \mathfrak{sl}_m|_n \) as the following subalgebra of \( Y_{m|n} \):

\[
SY_{m|n} := \{ x \in Y_{m|n}; \; \mu_f(x) = x \text{ for all } f(u) \in 1 + u^{-1}k[[u^{-1}]] \},
\]

(5.1)

where we take \( \mu_f \) as defined as in Section 3.7(1).

Let \( g' := \mathfrak{sl}_m[x] \) be the current superalgebra associated to \( \mathfrak{sl}_m|_n \). The following theorem is a generalization and modular analogue of [Gow2, Proposition 3, Lemma 7].

**Theorem 5.1.** The algebra \( SY_{m|n} \) has a basis consisting of ordered supermonomials in

\[
\{ h_i^{(r)}; \; 1 \leq i < m+n, r > 0 \} \cup \{ e_{i,j}^{(r)}, f_{j,i}^{(r)}; \; 1 \leq i < j < m+n, r > 0 \}
\]

(5.2)

taken in any fixed order. Hence, \( \text{gr} \; SY_{m|n} = U(g') \), and multiplication defines a vector space isomorphism

\[
SY_{m|n} \otimes Y_1 \xrightarrow{\sim} Y_{m|n}
\]

(5.3)

where \( Y_1 \) is identified with the subalgebra of \( Y_{m|n} \) generated by the elements \( d_i^{(r)} \) in the obvious way. If we assume in addition that \( p \nmid (m-n) \) then multiplication defines an algebra isomorphism

\[
SY_{m|n} \otimes Z_{HC}(Y_{m|n}) \xrightarrow{\sim} Y_{m|n}.
\]

(5.4)

**Proof.** The proof is the same as in the non-super case (see [BT, Theorem 6.1]). We just give a brief indication.

First note that all \( h_i^{(r)} \) belong to \( SY_{m|n} \) by the definitions of \( h_i(u) \) (3.92) and the automorphism \( \mu_f \) (Section 3.7(1)), and of course all \( e_{i,j}^{(r)} \) and \( f_{j,i}^{(r)} \) belong to \( SY_{m|n} \) too. Let \( SY_{m|n} \) be the subspace of \( SY_{m|n} \) spanned by the ordered supermonomials in the elements (5.2). Passing to the associated graded space induced by the filtration of \( Y_{m|n} \) and using (3.69) and (3.93), it follows that \( \text{gr} \; SY_{m|n} = U(g') \) and thus the ordered supermonomials that span \( SY_{m|n} \) are in particular linearly independent. Then, multiplying them by ordered monomials in \( \{ d_i^{(r)}; \; r > 0 \} \) gives the following isomorphism:

\[
\overline{SY_{m|n}} \otimes k[d_i^{(r)}; \; r > 0] \cong Y_{m|n}.
\]

Furthermore, we can use the above isomorphism to show that the inclusion \( SY_{m|n} \subseteq \overline{SY_{m|n}} \). This proves the isomorphism (5.3). Finally, the assumption \( p \nmid (m-n) \) ensures that the elements

\[
\{ e_{i,j}x^r, (-1)^{|i|+|i+1|}e_{i+1,j,i+1}x^r; \; 1 \leq i < m+n \} \cup \{ z_r; \; r \geq 0 \} \cup \{ e_{i,j}x^r; \; 1 \leq i \neq j \leq m+n, r \geq 0 \}
\]

form a basis of \( g' \). By considering the associated graded algebra, the isomorphism (5.4) can be proven similarly.

In view of (3.9), Theorem 5.1 implies that \( SY_{m|n} \) can be generated by the elements \( \{ h_i^{(r)}, e_i^{(r)}, f_i^{(r)} \} \). Then we have the following presentation for the subalgebra \( SY_{m|n} \).

**Theorem 5.2.** The algebra \( SY_{m|n} \) is generated by the elements

\[
\{ h_i^{(r)}, e_i^{(r)}, f_i^{(r)}; \; 1 \leq i < m+n, r > 0 \}
\]

(5.5)
subject only to the relations (3.48)-(3.58) plus the following:

\[
[h_i^{(r)}, h_j^{(s)}] = 0, \quad \text{if } |i - j| > 1 \tag{5.6}
\]

\[
[e_i^{(r)}, f_j^{(s)}] = (-1)^{|i|+|i+1|} \delta_{i,j} h_i^{(r+s-1)}, \quad \text{if } |i - j| > 1 \tag{5.7}
\]

\[
[h_i^{(r)}, e_j^{(s)}] = 0 \quad \text{if } |i - j| > 1 \tag{5.8}
\]

\[
[h_i^{(r)}, f_j^{(s)}] = 0 \quad \text{if } |i - j| > 1 \tag{5.9}
\]

\[
[h_{i-1}^{(r+1)}, e_i^{(s)}] - [h_{i-1}^{(r)}, e_i^{(s+1)}] = (-1)^{|i|} h_{i-1}^{(r)} e_i^{(s)}, \quad \text{if } i \neq m, \tag{5.10}
\]

\[
[h_{i}^{(r+1)}, f_i^{(s+1)}] - [h_{i}^{(r+1)}, f_i^{(s)}] = (-1)^{|i|} h_{i}^{(r)} f_i^{(s)}, \quad \text{if } i \neq m, \tag{5.11}
\]

\[
[h_{i}^{(r+1)}, e_i^{(s)}] - [h_{i}^{(r)}, e_i^{(s+1)}] = \begin{cases} \begin{aligned}
-(-1)^{|i|+1} & \left(h_i^{(r)} e_i^{(s)} + e_i^{(s)} h_i^{(r)}\right), & \text{if } i \neq m, \\
0 & , & \text{if } i = m,
\end{aligned} \end{cases} \tag{5.12}
\]

\[
[h_{i}^{(r+1)}, f_i^{(s+1)}] - [h_{i}^{(r+1)}, f_i^{(s)}] = \begin{cases} \begin{aligned}
-(-1)^{|i|+1} & \left(f_i^{(s)} h_i^{(r)} + h_i^{(r)} f_i^{(s)}\right), & \text{if } i \neq m, \\
0 & , & \text{if } i = m,
\end{aligned} \end{cases} \tag{5.13}
\]

\[
[h_{i+1}^{(r+1)}, e_i^{(s)}] - [h_{i+1}^{(r+1)}, e_i^{(s+1)}] = (-1)^{|i|+1} e_i^{(s)} h_{i+1}^{(r)}, \tag{5.14}
\]

\[
[h_{i+1}^{(r+1)}, f_i^{(s+1)}] - [h_{i+1}^{(r+1)}, f_i^{(s)}] = (-1)^{|i|+1} h_{i+1}^{(r)} f_i^{(s)}, \tag{5.15}
\]

for all admissible $i, j, r, s$ including $r = 0$ in (5.10)-(5.15); remember also $h_0^{(0)} = (-1)^{|i|}.$

**Proof.** Clearly, (3.48)-(3.58) hold, and the relations (5.6), (5.8)-(5.9) follow from (3.44)-(3.46).

Referring to the definition (3.92), we have

\[
h_i^{(r+s-1)} = (-1)^{|i|} \sum_{t=0}^{r+s-1} d_i^{(t)} d_{i+1}^{(r+s-1-t)}. \tag{5.26}
\]

Now the relation (5.7) follows directly from (3.47). For the remaining relations, (5.10), (5.12) and (5.14) follow by taking coefficients of $u^{-r}v^{-s}$ in (3.97), (3.102) and (3.99), respectively. Then (5.12), (5.13) and (5.15) follow by applying the anti-automorphism $\tau.$ For the rest of the proof, one can show by the same argument as [BT, Theorem 6.3]. \qed

Suppose that $\text{char } \mathbb{k} = p \neq 2.$ We have the Drinfeld presentation for the super Yangian $\text{SY}_{m,n}$ (cf. [Stu, Definition 2], [Gow2, Proposition 5] and [Ity, Section 2.5]). We shall use the “opposite” presentation (See [BK, Remark 5.12] for the Yangian $\mathcal{Y}(\mathfrak{s}\mathfrak{l}_n))$. In more detail, we define $\kappa_i, \xi_i$ for $i = 1, \ldots, m + n - 1$ and $s \geq 0$ from the equations

\[
\kappa_i(u) = \sum_{s \geq 0} \kappa_{i,s} u^{-s-1} := (-1)^{|i|} + \eta(-1)^{|i|} h_i(u),
\]

\[
\xi_i^+(u) = \sum_{s \geq 0} \xi_{i,s}^+ u^{-s-1} := \eta(-1)^{|i|} e_i(u),
\]

\[
\xi_i^-(u) = \sum_{s \geq 0} \xi_{i,s}^- u^{-s-1} := \eta(-1)^{|i|} f_i(u).
\]

To ease notation we let \((a_{i,j})_{m+n-1}\) be the Cartan matrix associated to the Lie superalgebra $\mathfrak{sl}_{m,n},$ that is, $a_{i,j} = (-1)^{|i|} (\delta_{i,j} - \delta_{i,j+1}) - (-1)^{|i+1|} (\delta_{i+1,j} - \delta_{i+1,j+1}).$
Proposition 5.3. The Yangian $\text{SY}_{m,n}$ is generated by the elements $\{\kappa_{i, s}, \xi_{i, s}^\pm; 1 \leq i < m + n, s \geq 0\}$ subject only to the following relations:

\[
[k_{i, r}, \kappa_{j, s}] = 0, \quad (5.16)
\]
\[
[e_{i, r}^+, \kappa_{j, s}] = (-1)^{|i|+|i+1|} \delta_{i, j} \kappa_{i, r+s}, \quad (5.17)
\]
\[
[k_{i, 0}, \xi_{j, s}^\pm] = \pm (-1)^{|i|} a_{i, j} \kappa_{j, s}^\pm, \quad (5.18)
\]
\[
[k_{i, r}, \kappa_{j, s}^\pm] - [k_{i, s+1}, \xi_{j, r}^\pm] = \pm \frac{a_{i, j}}{2} (\kappa_{i, r} \xi_{j, s}^\pm + \xi_{j, r}^\pm \kappa_{i, s}), \text{ for } i, j \not\equiv 0 \mod{m}, \quad (5.19)
\]
\[
[k_{m, r+1}, \xi_{j, s}^\pm] = 0, \quad (5.20)
\]
\[
[k_{i, r}, \xi_{j, s+1}^\pm] - [k_{i, s+1}, \xi_{j, r}^\pm] = \pm \frac{a_{i, j}}{2} (\xi_{i, r}^\pm \xi_{j, s}^\pm + \xi_{j, r}^\pm \xi_{i, s}^\pm), \text{ for } i, j \not\equiv 0 \mod{m}, \quad (5.21)
\]
\[
[k_{i, r}, \xi_{j, s}^\pm] = 0, \quad (5.22)
\]
\[
[k_{i, r}, [\xi_{j, s}^\pm, \xi_{j, t}^\pm]] + [\xi_{i, s}^\pm, [\xi_{i, r}^\pm, \xi_{j, t}^\pm]] = 0 \text{ if } |i - j| = 1, \quad (5.23)
\]
\[
[k_{i, r}, [\xi_{i, s}^\pm, \kappa_{j, s}^\pm]] = 0. \quad (5.24)
\]

Proof. The proof is just a rephrasing of Theorem 5.2 for these generators by taking some special coefficients in the power series. For example, the relation (5.17) follows from (3.30) in conjunction with the identities

\[
\frac{\kappa_{i}(u) - \kappa_{i}(v)}{v - u} = \sum_{r, s \geq 0} \kappa_{i, r+s-1} u^{-r-1} v^{-s-1};
\]

To check (5.18) in the case $i = j + 1$ and the sign is $+$, we set $u' := u - \frac{(-1)^{|i|+|i+1|}}{2} (m - i)$ and $v' := v - \frac{(-1)^{|i|}}{2} (m - i)$ in (3.104). Then taking the $v'^{-s-1}$-coefficient yields

\[
[k_{i+1, 0}, \xi_{i, s}^\pm] = -1 = (-1)^{|i|+1} a_{i+1, i} \kappa_{i, s}^\pm;
\]

The relations (5.19)-(5.20) follow from (3.102)-(3.104) and (5.21) follows from (3.31)-(3.34). As $p \neq 2$ and both $e_m$ and $f_m$ are odd, the relations (3.31)-(3.32) yield

\[
(u - v) [e_m(u), e_m(v)] = 0 = (u - v) [f_m(u), f_m(v)],
\]

so the relation (5.22) follows; the relation (5.23) can be deduced from (3.27); Finally, (5.24)-(5.25) follow from (3.37)-(3.40).

Remark 5. (1) Note that $p \neq 2$. We may set $r = s$ in (5.24) to see that

\[
[e_{i, r}^\pm, [\xi_{i, r}^\pm, \xi_{j, t}^\pm]] = 0 \text{ if } |i - j| = 1.
\]

For the quartic Serre relations (5.25), we just consider the case for $i = m$ (See Remark 2).

(2) The case $i = j$ in (5.21) and (5.22) are equivalent to the following relations:

\[
[e_{i, r}^+, \xi_{i, s}^+] = (-1)^{|i|+1} \left( \sum_{t=0}^{r-1} \xi_{i, t+r}^+ \xi_{i, s-t-1}^+ - \sum_{t=0}^{s-1} \xi_{i, t+s}^+ \xi_{i, r-t-1}^+ \right);
\]
\[ [\xi_{i,r}, \xi_{i,s}] = (-1)^{|i+1|} \left( \sum_{t=0}^{r-1} \xi_{i,t+s} \xi_{i,r-t-1} - \sum_{t=0}^{s-1} \xi_{i,t+r} \xi_{i,s-t-1} \right). \]

### 5.2. The $p$-centre of $SY_{m|n}$

Let
\[
a_i(u) = \sum_{r \geq 0} a_i^{(r)} u^{-r} := h_i(u) h_i(u - 1) \cdots h_i(u - p + 1) = \begin{cases} 
-b_{i+1}(u) b_i(u)^{-1} & \text{if } i < m, \\
-b_{i+1}(u) b_i(u) b_i(u)^{-1} & \text{if } i = m, \\
b_{i+1}(u) b_i(u)^{-1} & \text{if } i > m,
\end{cases}
\]
(5.26)

where the last equality follows from the definition ((3.92) and (4.11)). According to Lemma 4.5, each $a_i^{(r)}$ belongs to the center of $SY_{m|n}$. We define the $p$-center of $SY_{m|n}$ to be the sub-algebra $Z_p(SY_{m|n})$ of $Z(SY_{m|n})$ generated by
\[
\{ a_i^{(rp)}; 1 \leq i < m + n, r > 0 \} \cup \left\{ (e_{i,j})^p, (f_{i,j})^p; 1 \leq i < j \leq m + n, r > 0, |i| + |j| = 0 \right\}.
\]
(5.27)

We also let
\[
Z_p(\mathfrak{g}') := \mathbb{k}\langle x^p - x^{[p]}; x \in (\mathfrak{g}')_0 \rangle
\]
be the $p$-center of $U(\mathfrak{g}')$.

**Theorem 5.4.** The generators (5.27) of $Z_p(SY_{m|n})$ are algebraically independent, and we have that $gr Z_p(SY_{m|n}) = Z_p(\mathfrak{g}')$. Moreover, $SY_{m|n}$ is free as a module over $Z_p(SY_{m|n})$ with basis given by the ordered supermonomials in
\[
\{ h_i^{(r)}; 1 \leq i < m + n, r > 0 \} \cup \{ e_{i,j}^{(r)}, f_{i,j}^{(r)}; 1 \leq i < j \leq m + n, r > 0 \}
\]
in which no exponent is $p$ or more for $h_i^{(r)}$ and $e_{i,j}^{(r)}, f_{i,j}^{(r)}$ with $|i| + |j| = 0$.

**Proof.** We put
\[
\tilde{a}_i(u) := -(-1)^{|i|} a_i(u) = \sum_{r \geq 0} \tilde{a}_i^{(r)} u^{-r}
\]
and
\[
\tilde{h}_i(u) := -(-1)^{|i|} h_i(u) = \sum_{r \geq 0} \tilde{h}_i^{(r)} u^{-r}.
\]

By (5.26), we have
\[
\tilde{a}_i(u) = \prod_{j=1}^{p} \left( \sum_{r \geq 0} \tilde{h}_i^{(r)}(u - j + 1)^{-r} \right).
\]

Note that $\tilde{h}_i^{(0)} = 1$ and the set $\{ \tilde{h}_i^{(r)}; r > 0 \}$ are commuting elements by (3.44). Then [BT, Lemma 2.9] implies that $\tilde{a}_i^{(rp)} \in F_{rp-p} Y_{m|n}$ and
\[
\tilde{a}_i^{(rp)} \equiv (\tilde{h}_i^{(r)})^p - \tilde{h}_i^{(rp-p+1)} (mod F_{rp-p} Y_{m|n}).
\]

Now (3.93) readily yields
\[
\n_{rp-p} a_i^{(rp)} = (e_i x^{r-1} - (-1)^{|i|+|i+1|} e_{i+1,i+1} x^{r-1})^p - (e_i x^{rp-p} - (-1)^{|i|+|i+1|} e_{i+1,i+1} x^{rp-p}).
\]
In conjunction with (4.9) this implies that the generators (5.27) of \( Z_p(SY_{m|n}) \) are lifts of generators for \( Z_p(g') \) coming from a basis for \( g' \). This establishes the algebraic independence and that \( gr Z_p(SY_{m|n}) = Z_p(g') \). The final assertion follows by similar argument to the proof of Corollary 4.9 using the PBW basis for \( SY_{m|n} \) from Theorem 5.1. □

**Proposition 5.5.** If \( p \nmid (m - n) \), then \( Z_p(SY_{m|n}) = Z(SY_{m|n}) \).

**Proof.** Obviously, \( Z_p(SY_{m|n}) \subseteq Z(SY_{m|n}) \). When combined with Theorem 5.4, this implies
\[ Z_p(g') = gr Z_p(SY_{m|n}) \subseteq gr Z(SY_{m|n}) \subseteq Z(g') . \]
It suffices to verify that \( Z_p(g') = Z(g') \). Our assumption on \( p \) ensures that \( g = g' \oplus \mathfrak{z}(g) \). Hence, \( Z(g) \cong Z(g') \otimes \mathbb{K}[z_r; \ r \geq 0] \). Moreover, Theorem 2.3 in combination with the assumption \( p \nmid (m - n) \) shows that the elements \( \{ x^p - x^r^{|p|}; x \in g' \} \cup \{ z_r; r \geq 0 \} \) generate \( Z(g) \). This implies the assertion. □

### 5.3. Another description of the \( p-center of Y_{m|n} \)

Given \(|i| + |j| = 0\), we consider the even elements \( \{ t_{i,j}^{(r)}; r \geq 0 \} \). Using the defining relation (3.1) and induction on \( r + s \), it is easy to see that
\[ t_{i,j}^{(r)} t_{i,j}^{(s)} = t_{i,j}^{(s)} t_{i,j}^{(r)} \]
for all \( r, s \geq 0 \). For \(|i| + |j| = 0\), we define
\[ s_{i,j}(u) = \sum_{r \geq 0} s_{i,j}^{(r)} u^{-r} = \begin{cases} t_{i,j}(u) t_{i,j}(u - 1) \cdots t_{i,j}(u - p + 1) & \text{if } |i| = 0, \\ t_{i,j}(u) t_{i,j}(u - 1) \cdots t_{i,j}(u - p + 1) & \text{if } |i| = 1. \end{cases} \quad (5.28) \]
The foregoing observations in conjunction with Proposition 3.4(4) imply that the order of the product on the right hand side here is irrelevant.

**Lemma 5.6.** All of the elements \( s_{i,j}^{(r)} \) belong to the \( p-center Z_p(Y_{m|n}) \).

**Proof.** First we show that each \( s_{i,j}^{(r)} \) belongs to \( Z(Y_{m|n}) \). The general assumption \(|i| + |j| = 0\) then implies that we need to consider the two cases: (a) \( 1 \leq i, j \leq m \); (b) \( m + 1 \leq i, j \leq m + n \). For (a), one uses the permutation automorphism from Section 3.7(3) to reduce the problem to showing that
\[ (\dagger) \quad \text{all coefficients of } s_{1,1}(u) \text{ and of } s_{1,2}(u) \text{ are central.} \]

Using the swap map (Proposition 3.4) one can show by direct computation that
\[ \zeta_{m|n}(t_{i,j}^{(r)}(u)) = t_{m+n+1-i,m+1-j}(u). \]
If \( m+1 \leq i, j \leq m+n \), then \( \zeta_{m|n}(s_{i,j}(u)) = s_{m+n+1-i,m+1-j}(u) \in Y_{n|m} \). We can also use the permutation automorphism to reduce case (b) to the problem of showing (\dagger). Since the coefficients of \( b_i(u) \) and \( p_{i,j} \) are contained in the \( p-center of Y_{m|n} \), the (\dagger) follows because the following claim.
\[ s_{1,1}(u) = b_1(u), \quad s_{1,2}(u) = b_1(u)p_{1,2}(u). \quad (5.29) \]

The definition of \( b_i(u) \) (4.11) in combination with (3.6) gives \( t_{1,1}(u) = d_1(u) \), so that the first identity (5.29) follows. For (5.30), we set \( i = 1 \) and \( v = u - k \) in (3.87) to deduce that
\[ e_1(u - k)d_1(u - k + 1) \cdots d_1(u - 1)d_1(u) = d_1(u - k + 1) \cdots d_1(u - 1)d_1(u)e_1(u) \]
for each \( k = 1, \ldots, p - 1 \), while the Gauss decomposition (3.5) yields \( t_{1,2}(u) = d_1(u)e_1(u) \). Consequently,
\[
\begin{align*}
  s_{1,2}(u) &= t_{1,2}(u - p + 1) \cdots t_{1,2}(u - 1)t_{1,2}(u) \\
  &= d_1(u - p + 1)e_1(u - p + 1) \cdots d_1(u - 1)d_1(u)e_1(u) \\
  &= d_1(u - p + 1) \cdots d_1(u - 1)d_1(u)e_1(u)^p = b_1(u)p_{1,2}(u).
\end{align*}
\]
This establishes (5.30).

By Theorem 4.8(2), we have that \( Z_p(Y_{m|n}) = Y_{m|n} \cap Z_p(Y_{m|n+1}) \), where we are using the natural embedding \( Y_{m|n} \hookrightarrow Y_{m|n+1} \); \( t_{i,j}^{(p)} \mapsto t_{i,j}^{(p)} \). In order to prove that \( s_{i,j}^{(p)} \in Z_p(Y_{m|n}) \), we may thus assume that that \( p \not| (m-n) \). Equivalently, we show that \( s_{i,j}(u) \in Z_p(Y_{m|n})[[u^{-1}]] \). This is immediate by (5.29) in case \( i = j = 1 \). In general, we will show that \( s_{i,j}^{(u)}s_{1,1}^{(u)-1} \in Z_p(Y_{m|n})[[u^{-1}]] \). Using the definition (5.1), we get that \( s_{i,j}(u)s_{1,1}^{(u)-1} \in SY_{m|n}[[u^{-1}]] \). Since we have shown its coefficients are central already, it therefore lies in \( Z_p(SY_{m|n})[[u^{-1}]] \), which by Proposition 5.5. However, the definition of \( Z_p(SY_{m|n}) \) immediately implies \( Z_p(SY_{m|n})[[u^{-1}]] \subset Z_p(Y_{m|n})[[u^{-1}]] \), as desired. \( \square \)

**Theorem 5.7.** The \( p \)-center \( Z_p(Y_{m|n}) \) is freely generated by \( \{s_{i,j}^{(r)}; 1 \leq i, j \leq m+n, \ r > 0, |i| + |j| = 0 \} \). We have that \( s_{i,j}^{(r)} \in F_{rp-p}Y_{m|n} \) and
\[
\text{gr}_{rp-p} s_{i,j}^{(r)} = (e_{i,j}t^{r-1}p - \delta_{i,j}e_{i,j}t^{r-p}p).
\]
For \( 0 < r < p \), we have that \( s_{i,j}^{(r)} = 0 \). For \( r \geq p \) with \( p \not| r \), the central element \( s_{i,j}^{(r)} \) belongs to \( F_{rp-p-1}Y_{m|n} \) and it may be expressed as a polynomial in the elements \( \{s_{i,j}^{(r)}; 0 < s \leq |r/p| \} \).

**Proof.** Let \( t_{i,j}^{(r)}(u) := \sum_{r \geq 0} t_{i,j}^{(r)}u^{-r} \). By multiplying out the matrix products \( T(u) = F(u)D(u) \) and \( T(u)^{-1} = E^{-1}(u)^{-1}D(u)^{-1}F(u)^{-1} \), one obtain that \( t_{i,j}^{(r+1)} \in F_rY_{m|n} \) and \( \text{gr}_r t_{i,j}^{(r+1)} = (-1)^{|i|}e_{i,j}u^r \). By using Lemma 5.6 and passing to the associated graded algebra, the rest of the proof is the same as in the non-super case [BT, Theorem 6.9], and will be skipped. \( \square \)

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