The content of $f(R)$ gravity

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Abstract

We analyze the propagating degrees of freedom in gravity models where the scalar curvature in the action is replaced by a generic function $f(R)$ of the curvature. That these gravity models are equivalent to Einstein’s gravity with an extra scalar field had previously been shown by applying a conformal transformation. We confirm this result by calculating the particle propagators. This provides further evidence of the inability of these models to explain the accelerated expansion of the Universe without contradicting solar system experiments.

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A suitable modification of gravity at large distances could explain the current accelerated expansion of the Universe observed in measurements of type 1a supernovae [1]. This idea had been suggested in the work [2], the authors of which developed it further in the context of the proposed in [3] model. On the other hand, cosmologically motivated theories that explain the small acceleration rate of the Universe via modifications of gravity at very large scales can be tested in solar system experiments [4], making these infrared modifications an even more attractive subject for studying.

In this article we address the question of what other than DGP [2], [3] kinds of Einstein’s gravity modifications would make a pathology free theory. Bearing in mind the acceleration problem, we will concentrate on the attractively simple modifications of Einstein’s equation through changing the action by replacing the scalar curvature with a function of it [5]. Specifically, we start from a sufficiently generic theory with the action

\[ S = \int d^4x \sqrt{-g} f(R) + S_{\text{matter}}, \]  

where \( f(R) \) is some function of the scalar curvature and \( S_{\text{matter}} \) is the action for matter fields. It was shown some time ago that, by use of a conformal transformation, these kind of theories reduce essentially to the theory of Einstein’s gravity plus that of an extra scalar field (see, for example, [6], [7]). The decoupling of this scalar mode would make these theories no different from quintessence. However, the decoupling of the scalar mode would violate general covariance thus leading to an unsatisfactory result. Moreover, there exists an argument about the existence of a preferred frame, questioning thus the legitimacy of such conformal transformation [8]. The presence of a coupled to matter scalar mode would mean that we are dealing with a scalar tensor theory, which would have a valid perturbation expansion but an incorrect Newtonian limit.

Here we will consider the modification (1) from the point of view of propagating degrees of freedom by deriving the particle propagators of this theory. The variation of the action with respect to the metric leads to the equations of motion

\[ f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\nabla_\lambda \nabla^\lambda - \nabla_\mu \nabla_\nu) f'(R) = T_{\mu\nu}. \]  

One can see that for \( f(R) = R \) they give the standard Einstein equation. These equations allow a constant curvature \( R = R_0 = \text{const} \) solution, \( R_0 \)
being defined by the following condition

\[ f'(R_0)R_0 = 2f(R_0). \]  

(3)

Since this is a maximally symmetric solution, it implies

\[ R_{\lambda\mu\nu\sigma} = \frac{R_0}{12} (g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\sigma}g_{\mu\nu}) \]  

(4)

and

\[ R_{\mu\nu} = \frac{1}{4} R_0 g_{\mu\nu}. \]  

(5)

We linearize now the equations of motion on the constant curvature solution background. We take the metric in the form

\[ g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \]

where \( g^{(0)}_{\mu\nu} \) is the solution of (2) corresponding to our constant scalar curvature \( R_0 \), next expand the equations of motion till the linear order terms. Inverting the operator acting on \( h_{\mu\nu} \), we can find the propagator. In order to write down this propagator, it is convenient to use spin projectors \( P^2, P^1_m, P^1_e, P^1_b, P^1_{me}, P^1_{em}, P^0_s, P^0_w, P^0_{sw}, P^0_{ws} \) \([2]\) (the explicit form of the projectors is given in the Appendix):

\[ \begin{align*}
- \frac{P^2}{(\nabla^2 + R/2)f'(R)} & - \frac{P^1_m}{(R/2)f'(R)} - \\
- \frac{f'(R)/2 + R/4f''(R)}{(\nabla^2 + R/2)f'(R) (f''(R)(3\nabla^2 + R) - f'(R))} P^0_s - \\
- \frac{\sqrt{3}(f'(R)/2 - (\nabla^2 + R/4)f''(R))}{(\nabla^2 + R/2)f'(R) (f''(R)(3\nabla^2 + R) - f'(R))} (P^0_{sw} + P^0_{ws}) + \\
+ \frac{((4\nabla^2 + R/2)f'(R) - 3(2\nabla^2 + R/2)f''(R))}{(\nabla^2 + R/2)f'(R) (f''(R)(3\nabla^2 + R) - f'(R))} P^0_w.
\end{align*} \]  

(6)

In particular, for the flat background Einstein equation \( (f(R) \equiv R, R = 0) \), we get the standard result for the graviton propagator

\[ - \frac{P^2}{\nabla^2} + \frac{1}{2} \frac{P^0_s}{\nabla^2}, \]  

(7)

where the seemed to be ghost \( P^0_s \) is just a cancelation term for the mass degree of freedom of the massive spin-2 projector \( P^2 \).
To analyze the particle content of the theory, we have to look at the $P^2$ and $P^0_s$ terms. The corresponding part of the propagator can be rewritten as

$$ -\frac{(P^2 - \frac{1}{2} P^0_s)}{(\nabla^2 + R/2) f'(R)} - \frac{P^0_s}{(\nabla^2 + R/3 - f'(R)/(3f''(R))) f'(R)}, $$

thus we have a massless graviton in a curved background, $-P^2 + 1/2 P^0_s$, and a scalar particle, which becomes a ghost only for $f'(R) < 0$. This condition, however, cannot be satisfied since it would make graviton itself a ghost. Our result agrees with that obtained by the conformal transformation, according to which the modification of gravity by use of some arbitrary enough function of the scalar curvature is equivalent to the addition of a scalar field with a specific type of potential. We also would like to notice that the recently suggested model of ghost inflation [10] doesn’t reduce to the above modification even ignoring higher derivatives terms.

The modified gravity model should give us a reasonably flat observable Universe, which implies a tiny mass for the scalar field. However, light scalar gravity is in contradiction with solar system experiments. It seems that it is impossible to decouple the scalar mode even by fine tuning the function $f(R)$. As one can see, its derivative defines the coupling of both tensor and scalar modes, the $f(R)$ model is essentially a scalar tensor gravity. Thus we have to conclude that such modifications of gravity are ruled out. On the other hand it seems that discussions of a preferred physical frame are irrelevant, at least in the linear approximation.

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**Appendix**

Below we give the expressions for the ten operators which span the space of solutions to the linearized field equations [9]:

\[
\begin{align*}
P^2 & = \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho}) - \frac{1}{3} \theta_{\mu\nu} \theta_{\rho\sigma}, \\
P^1_m & = \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}), \\
P^1_e & = \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} \theta_{\mu\sigma} \omega_{\nu\rho} - \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}),
\end{align*}
\]

\[
\begin{align*}
P^1_b & = \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} - \theta_{\mu\sigma} \theta_{\nu\rho}).
\end{align*}
\]
\[ P^1_{me} = \frac{1}{2} (\theta_{\mu\nu} \omega_{\nu\sigma} - \theta_{\nu\sigma} \omega_{\mu\nu} + \theta_{\nu\rho} \omega_{\mu\sigma} - \theta_{\mu\sigma} \omega_{\nu\rho}), \]

\[ P^1_{em} = \frac{1}{2} (\theta_{\mu\nu} \omega_{\nu\sigma} + \theta_{\nu\sigma} \omega_{\mu\nu} - \theta_{\nu\rho} \omega_{\mu\sigma} - \theta_{\mu\sigma} \omega_{\nu\rho}), \]

\[ P^0_s = \frac{1}{3} \theta_{\mu\nu} \theta_{\rho\sigma}, \]

\[ P^0_w = \omega_{\mu\nu} \omega_{\rho\sigma}, \]

\[ P^0_{sw} = \frac{1}{\sqrt{3}} \theta_{\mu\nu} \omega_{\rho\sigma}, \]

\[ P^0_{ws} = \frac{1}{\sqrt{3}} \omega_{\mu\nu} \theta_{\rho\sigma}. \]

where the transversal and longitudinal projectors in the momentum space are respectively

\[ \theta_{\mu\nu} = \delta_{\mu\nu} - \frac{\nabla_{\mu} \nabla_{\nu}}{\nabla^2}, \quad \omega_{\mu\nu} = \frac{\nabla_{\mu} \nabla_{\nu}}{\nabla^2}. \]

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