Abstract

In this work we provide alternative formulations of the concepts of lambda theory and extensional theory without introducing the notion of substitution and the sets of all, free and bound variables occurring in a term. We also clarify the actual role of \( \alpha \)-renaming and \( \eta \)-extensionality in the lambda calculus: both of them can be described as properties of extensionality for certain classes of terms.

1 Introduction

In the early thirties, Church [11] introduced a formal system intended to provide a common foundation for logic and mathematics. However, after the discovery of some paradoxes, a proper portion of the original system was extracted. This part constitutes what is known as the lambda calculus today; see Church [13], Curry and Feys [19], Curry, Hindley and Seldin [20], Stenlund [48], Stoy [50], Barendregt [5], Révész [42], Krivine [30], Hankin [24], Sørensen and Urzyczyn [47], Hindley and Seldin [26], Cardone and Hindley [10] and Seldin [45] for good expositions.

In a nutshell, the lambda calculus can be described as a general theory of computable functions: it provides a formalism for dealing with functions-as-terms in the context of the foundations of mathematics: In axiomatic set theories, functions from a given set to itself are defined by their graphs as — typically infinite — sets of ordered pairs. By contrast, in the lambda calculus functions are simply implemented by means of some — always finite — formal expressions called terms. Furthermore, in set theory it follows from the axioms that a function cannot be a member of its domain. In the lambda calculus, every term is in the domain of a given function; in particular, every function can take itself as input.

The most interesting feature of this calculus is that it constitutes a model of computation and for this reason this powerful language is the theoretical core of many modern functional programming languages.

To be more precise, the lambda calculus in itself has no expressive power at all. It becomes an expressive language only when equipped with a suitable relation which allows us to determine when two terms are intended to be the same. While in set theory two functions from a given set to itself are the same if they have identical input-output behaviour, in the lambda calculus the situation is rather different; there is a huge collection of relations called lambda theories and each of these relation is intended to capture some notion of sameness.

A little more technically, lambda theories are congruences on terms which satisfy the conditions of \( \alpha \)-renaming and \( \beta \)-rule. The two most important lambda theories are lambda conversion — the least lambda theory — and extensional conversion — the least lambda theory which also satisfies the condition of \( \eta \)-extensionality. Here we adopt the Church’s terminology [13]; other names for lambda conversion and extensional conversion are \( \alpha \beta \eta \)-equality and \( \alpha \beta \eta \)-equality, respectively. In order to avoid misunderstanding, in this paper we call \( \alpha \)-renaming, \( \beta \)-rule and \( \eta \)-extensionality the conditions that in the literature are usually denoted by \( (\alpha) \), \( (\beta) \) and \( (\eta) \). Since several versions of \( (\alpha) \), \( (\beta) \) and \( (\eta) \) have been proposed, for the sake of clarity we use our terminology when we do not refer to a specific formulation. We reserve the names \( (\alpha) \), \( (\beta) \) and \( (\eta) \) to the particular conditions that will be introduced in Section 7.

In general, each lambda theory has to be seen as a reasonable relation of sameness for terms. We refer to Barendregt [5], Berline [6, 7] and Manzonetto and Salibra [32] for a survey of results on lambda theories.

In this paper we focus our attention on the syntactical formalizations of the notions of lambda theory and extensional theory — we call extensional theory any lambda theory which also satisfy \( \eta \)-extensionality. The usual way to proceed is to introduce some auxiliary notions first:

- the sets of all, free and bound variables occurring in a term,
- a meta-theoretic operation of substitution.

Henceforth, we refer to these auxiliary notions as ancillary concepts. Note that they usually occur, explicitly or implicitly, in the traditional formulations of the conditions of \( \alpha \)-renaming, \( \beta \)-rule and \( \eta \)-extensionality. One of the aims of this paper is to show that

in order to syntactically define lambda and extensional theories it is not necessary to introduce any ancillary concept.

Regarding our motivations, we mention that it is well-known that the usual definition of substitution is intrinsically difficult and error-prone. The same problems also arise in logics with quantifiers and more generally in languages with binding operators. As for the lambda calculus, we believe that the actual complications do not lie in the operation of substitution itself but rather in the formulation of the \( \beta \)-rule. In fact, it is not so well-known that it is possible to consider a simple definition of substitution at the price of having some restrictions in the \( \beta \)-rule. This is the approach of Barendregt’s dissertation [4], which is actually a more elegant variant of the original approach developed by Church [13]. By contrast, following Curry and Feys [19], the vast majority of the presentations of the lambda calculus prefers to consider a complicated notion of substitution
which allows a simple formulation of the $\beta$–rule. In Section 7 we shall compare the two options in more detail. But in either case, we believe it is definitely more convenient to not consider substitution at all and replace the $\beta$–rule with the more elementary conditions we shall introduce later on.

Historically, these complications were well–known to the fathers of modern formal logic. In fact, approximately in the same period the lambda calculus was conceived, a way to circumvent the problems caused by substitution was proposed by Shönfinkel [46] and Curry [18, 15, 20] under the name of combinatory logic. However, we think that in such a framework the elimination of concepts is too drastic: the ancillary concepts are eliminated by means of the rejection of the notion of (bound) variable. As a negative side effect, in that setting the “functional intuitions” coming from the lambda notation are completely lost. This is not an accident, since this elimination of variables was one of the objectives of Shönfinkel (see Cardone and Hindley [10] for a survey of Shönfinkel’s work). By contrast, in this paper we want to maintain the syntax of the lambda calculus unaltered and perform a more precise surgery which allows us to eliminate only the ancillary concepts.

Regarding the sets of all, free and bound variables occurring in a term, we see these concepts as mere “syntactic bureaucracy” without any real mathematical substance. As we shall see, these sets can be eliminated from the picture with almost no effort and hence in this way more elegant formulations of the notions of lambda theory and extensional theory can be obtained.

In addition to the theoretical and pedagogical interest of our results, we also see a more concrete advantage as we now explain. In the model theory of the lambda calculus, in order to prove that a given mathematical structure is a model of, say, lambda conversion, all we need to do is to check that the binary relation obtained by relating terms which have the same interpretation in the given structure — the relation usually called theory of a model — is a lambda theory. In equivalent words, we have to prove a soundness theorem. To show this theorem, a typical approach is to prove the substitution lemma, see Wadsworth [54] and Meyer [33], as well as Stoy [50, pp. 161–166] for a detailed proof of this lemma. We believe that a direct verification of the substitution–free conditions which define (our formulation of) lambda theories is not only easier than the lemma, but also more convenient inasmuch as the proof — and the exportability — of the lemma largely depends on the specific definition of substitution. (In particular, the substitution lemma cannot be simply invoked by authors who use a different notion of substitution. Unfortunately, we also believe that this delicate point has been overlooked by many authors.) To convince the reader, a direct verification of some of our substitution–free conditions will be given in Section 9.

In order to avoid terminological misunderstandings, in the rest of the paper the relations of sameness on terms defined by using our formalizations are called lambda and extensional congruences, while we exclusively reserve the terminology lambda and extensional theory for the traditional axiomatizations of these relations. Then, one of our main results can be stated as follows:

Lambda congruences and lambda theories coincide, as well as extensional congruences and extensional theories.

Another aspect we want to analyze in this paper is the precise role of $\alpha$–renaming and $\eta$–extensionality in the lambda calculus. As for the latter, it is well–known that this condition allows us to consider terms which have the same input–output behaviour as the same. Thus, expressed in this form $\eta$–extensionality has a very intuitive and satisfactory mathematical significance: it reproduces in the lambda calculus a form of extensionality familiar from ordinary set theory. As for the former, the situation is not so clear. To the best of our knowledge, the most intuitive informal description of this concept is this: terms which only differs in the names of bound variables should be considered as the same. We think that this explanation is too syntactical — it even mentions an ancillary concept! — and we do not see any clear mathematical content in it. One of the aim of this paper is to clarify the meaning of $\alpha$–renaming in intuitive and mathematically reasonable terms. It turns out that $\alpha$–renaming admits the following simple and satisfactory equivalent description:

abstractions which have the same input–output behaviour should be considered as the same,

as we shall show in Section 5. Here “abstractions” are, roughly speaking, terms which begin with a $\lambda$. Thus, $\alpha$–renaming can be regarded as a form of extensionality for abstractions. We believe that when expressed in this form the real mathematical significance of this condition emerges.

To obtain the aforementioned results we have to consider a new factorization of the conditions which define lambda and extensional theories, in the sense we now explain. Consider, for instance, lambda theories. Following Church [13], a way of introducing this concept is to put $\alpha$–renaming and the $\beta$–rule together in the definition of these theories, like in the original formulation of lambda conversion. Following Barendregt [5], another approach is to define an auxiliary equivalence relation on terms called alpha conversion. Terms which belongs to the same equivalence class are then identified. After this identification, substitution and the $\beta$–rule are finally introduced. We refer to Crole [17] and the references therein for more on alpha conversion.

Our factorization is completely different and, to the best of our knowledge, it seems to be new. Roughly speaking, we obtain it by proceeding as follows. First, we only consider our substitution–free version of the $\beta$–rule. We obtain a family of relations that we call prelambda congruences. These relations need to satisfy neither $\alpha$–renaming nor $\eta$–extensionality; in fact, we shall show is that there exists a prelambda congruence in which both conditions do not hold. Only at this stage we add $\alpha$–renaming to prelambda congruences. We then obtain our version of lambda theories: lambda congruences. We shall prove that

$\alpha$–renaming and the property of extensionality for abstractions mentioned above are equivalent in every prelambda congruence.

Also, we add $\eta$–extensionality to prelambda congruences
and get our version of extensional theories: extensional congruences. Similarly, we shall prove that

\[ \eta - \text{extensionality and the property of extensionality (for all terms) mentioned above are equivalent in every prelambda congruence.} \]

Notice that these equivalences are significant precisely because neither \( \alpha \)-renaming nor \( \eta \)-extensionality is valid in every prelambda congruence.

Regarding our methodology, our analysis of \( \alpha \)-renaming and \( \eta \)-extensionality in the lambda calculus is somehow inspired by the usual treatment of classical logic from a constructive point of view. In that context, intuitionistic logic is first introduced and developed. Only at a second stage, the law of excluded middle is motivated, discussed and added, and some equivalent formulations are established — such as double negation elimination. Equivalents of excluded middle are of interest precisely because this law is not a theorem of intuitionistic logic. Of course, \( \alpha \)-renaming and \( \eta \)-extensionality are by no means “controversial” in the lambda calculus as excluded middle is in constructive mathematics, but we do believe they deserve a better understanding as they are too often taken “for granted” in the literature, especially \( \alpha \)-renaming. One of the aims of this paper is to provide a more intuitive and clear comprehension of these conditions. To sum up, the result of our analysis is that

\[ \text{in every prelambda congruence both } \alpha \text{-renaming and } \eta \text{-extensionality can be both described in a natural way as properties of extensionality.} \]

This completes the description of this paper.

We now discuss some related work. As far as we know, there are three lines of apparently independent research which implicitly or explicitly try to eliminate the ancillary concepts without modifying the syntax of the lambda calculus.

In the sixties, following some ideas of Church [12], Henkin introduced a theory of propositional types based on the simply typed lambda calculus [25]. In order to develop a deductive system for his theory, Henkin decomposed the typed \( \beta \)-rule in several substitution-free clauses which are strikingly similar to the conditions that we use in this paper. The reason for this decomposition was the simplification of the proof of the soundness theorem of his deductive system with respect to the set-theoretic semantics of his type theory. Again, here we see strong similarities between his and our motivations. Following Henkin, Andrews employed similar clauses in his work on type theory [2,3]. However, some ancillary concepts are present in all these works.

In the early eighties, Révész [40,41,42] proposed a substitution-free formalization of lambda conversion. His and our motivations are rather similar, the main difference is that he was mainly interested in developing concrete computer implementation while we also think that substitution-free formalizations are also useful to simplify some practical aspects of the model theory of the lambda calculus — namely, the elimination of the substitution lemma. Révész’s conditions are very similar to (untyped versions) of Henkin’s ones and in particular some ancillary concepts are still present in his formalization. From an historical perspective, the main novelty of his work was the first substitution-free formulation of \( \alpha \)-renaming.

Some years later, with the aim of providing a general algebraic setting for the lambda calculus, Pigolzi and Salibra [36, 37, 38] introduced the theory of lambda abstraction algebras. Despite the aims of this line of work are completely different from ours, from a purely syntactical standpoint the improvement emerging from the introduction of these algebras is the complete elimination the ancillary concepts. In fact, the conditions defining lambda abstraction algebras are very similar to those we introduce in this paper, as we shall see.

We finally point out that the conditions we use in this paper to define lambda and extensional congruences do not form just a selection of clauses taken from these works. On the contrary, the crucial conditions (\( \beta_e \), (\( \alpha_e \)) and (\( \eta_e \)) that we use in this paper are simplified — for our purposes — reworked versions of some similar clauses present in the aforementioned literature.

The paper is organized as follows. In Section 2 we shall recall the syntax of the terms of the lambda calculus and introduce some notation and terminology. In Section 3 we shall introduce the concept of prelambda congruence and in Section 4 we shall study the basic properties of this notion. In Section 5 we shall discuss the real — for us — reason why \( \alpha \)-renaming is so important in the lambda calculus and introduce the concept of lambda congruence. We shall also prove that in every prelambda congruence \( \alpha \)-renaming is equivalent to a suitable principle of extensionality. Similarly, in Section 6 we shall define the notion of extensional congruence and prove that in every prelambda congruence \( \eta \)-extensionality is equivalent to a principle of extensionality. In Section 7 we shall recall the concepts of lambda and extensional theory and in Section 8 we shall prove the equivalence between lambda congruences and theories as well as the one between extensional congruences and theories. In Section 9 we shall provide a construction of a model whose theory forms a prelambda congruence which do not satisfy a simple instance of \( \alpha \)-renaming. Finally, in Section 10 we shall conclude the paper.

## 2 The Lambda Calculus

In this section we recall the syntax of the terms of the lambda calculus and introduce some terminology.

**Definition 2.1 (Variables).** Let \( \mathbb{V} \) be an infinite set. We call its member variables and we denote them by \( x, y, z, \ldots, \Delta \)

Some comments concerning the set \( \mathbb{V} \) are given at the end of this section.

The basic elements of the lambda calculus are called terms. They are special words built from the infinite alphabet consisting of all variables together with the following auxiliary symbols: \( \lambda \) (lambda–abstractor), [ (left lambda–bracket) and ] (right lambda–bracket). In this paper, we write words by juxtaposition of symbols; in particular, we do not notationally distinguish between members of the alphabet and unary words.
**Definition 2.2** (Term). The terms of the lambda calculus are the elements of the set $T$ which is inductively defined as follows:

$$(T_1) \ x \in V \implies x \in T;$$
$$(T_2) \ A \in T \text{ and } B \in T \implies [AB] \in T;$$
$$(T_3) \ x \in V \text{ and } A \in T \implies [\lambda x A] \in T.$$  

In the sequel, we use $A$, $B$, $C$, $\ldots$ to denote arbitrary terms.

We now introduce some useful terminology and notation.

As is standard, two terms $A$ and $B$ are said to be equal, in symbols $A = B$, if and only if $A$ and $B$ are exactly the same word. In particular, we have $x = y$ as elements of $T$ if and only if $x = y$ as elements of $V$. Our notion of equality corresponds to the relation which is often called syntactical equality in the literature.

In order to save space, we always write $AB$ and $\lambda x A$ for $[AB]$ and $[\lambda x A]$, respectively. Henceforth, we also refer to a term of the form $AB$ as an application and to a term of the form $\lambda x A$ as an abstraction.

As usual, we call a subset of $T^2$ (the Cartesian product of $T$ with itself) a binary relation on terms and we use the symbol $\sim$ to denote binary relations on terms. We also write $A \sim B$ for $(A,B) \in \sim$ and $A \not\sim B$ for $(A,B) \notin \sim$. Furthermore, if we have $A \sim B$ and $B \sim C$, then sometimes we simply write $A \sim B \sim C$ to express this fact. Analogously, we may write $A \sim B \sim C \sim D$ in case we have $A \sim B$, $B \sim C$ and $C \sim D$, and so on.

An important class of binary relations on terms is that of congruences, that we now formally define and discuss.

**Definition 2.3** (Congruence, structural conditions). Let $\sim$ be a binary relation on terms. We say that $\sim$ is a congruence if it satisfies the following conditions:

$$(r) \ A \sim A;$$
$$(s) \ A \sim B \implies B \sim A;$$
$$(t) \ A \sim B \text{ and } B \sim C \implies A \sim C;$$
$$(t) \ A \sim B \implies \lambda x A \sim \lambda x B;$$
$$(a) \ A \sim B \text{ and } C \sim D \implies AC \sim BD;$$

where $A$, $B$, $C$ and $D$ are arbitrary terms, and $x$ is an arbitrary variable. We collectively refer to conditions $(r)$ to $(a)$ above as structural conditions.

Conditions $(r)$, $(s)$ and $(t)$ ensure that any congruence is an equivalence relation between terms, while conditions $(t)$ and $(a)$ express the fact that any congruence is a relation compatible with the operations of term formation $(T_2)$ and $(T_3)$ of Definition 2.2. In particular, a congruence defined as above is a congruence in the usual algebraic sense. In other words, structural conditions are nothing more than a formalization of the algebraic principle usually called replacement of equals by equals. This principle — which is ubiquitous in algebra — is often used in an implicit and tacit fashion. In this paper, for the sake of clarity — but in contrast to the algebraic tradition — we shall explicitly mention when and where structural conditions are used in formal proofs of our results.

While important, congruences alone cannot be considered as candidates for a good formalization of the notion of sameness for terms: in fact, in order to develop the theory of functions using the lambda calculus more conditions are needed. This situation is similar to what happens for the traditional presentations of sequent calculi for classical logic in proof theory: there, in addition to structural rules, we also need logical rules to be able to derive classical tautologies in a syntactical way. In our context, “structural rules” are, unsurprisingly, what we are calling structural conditions and the “logical rules” specific to this work that we shall introduce and discuss in detail in later sections are $(\beta_1)$, $(\beta_2)$, $(\beta_3)$, $(\beta_4)$, $(\alpha_e)$ and $(\eta_e)$.

We now give an informal presentation of our logical rules. In the lambda calculus, the functional intuition on terms is this: an abstraction of the form $\lambda x A$ should be thought as a function depending on the variable $x$. If, for a moment, we think of $A$ as the polynomial $2x + y$, then $\lambda x A$ represents the function given by $f(x) = 2x + y$. Also, an application of the form $[\lambda x A]D$ should be intuitively regarded as the function $\lambda x A$ applied to the specific input $D$. Thus, if $D$ represent the number 7, then $[\lambda x A]D$ represents $f(7)$. It follows from elementary rules of algebra that this expression can be simplified to $14 + y$. Now, in our formalizations of lambda and extensional theories, the role of conditions $(\beta_1)$, $(\beta_2)$, $(\beta_3)$, $(\beta_4)$ and $(\beta_5)$ is in some sense algebraic: they serve to simplify expressions like $[\lambda x A]D$ by performing some symbolic manipulations. The remaining conditions, namely $(\alpha_e)$ and $(\eta_e)$, have a different task; in this work, both of them are seen as conditions of extensionality in the sense that they allow us to infer that two terms (abstractions, in case of $(\alpha_e))$ are the same (i.e., related by $\sim$) if they have the same input–output behaviour, as we shall see in detail later on.

As promised, some comments on the set of variables $V$ are now in order. Firstly, we make a rather trivial — but essential — remark: we observe that the property of being infinite is crucial. To see this, let us call a set of variables $S$ cofinite if its complement with respect to $V$ is finite, in other words if $S = V \setminus F$ for some finite set of variables $F$. Now, the reason for the infinity of $V$ is that in order to develop even the most elementary part of the lambda calculus we need the following facts to hold:

- If $S$ and $T$ are cofinite, then so is $S \cap T$;
- if $S$ is cofinite and $S \subseteq T$, then $T$ is cofinite;
- every cofinite set is non–empty.

These properties immediately follow from the definition of cofinite set and from the infinity of $V$. In the sequel, in order to make the exposition less pedantic, we shall use these simple properties without explicitly mentioning them. Our choice is perfectly in line with what it is done in the literature, where we often find statements like “let $z$ be a variable which occurs neither free in $A$ nor bound in $B$” without the explicit proof of the existence of this variable (which simply goes as follows: First, observe that the sets of variables which occur free or bound in a term are finite, so that their complements are cofinite. Then, the result follows, as the intersection of cofinite sets is cofinite and cofinite sets cannot be empty).
Secondly, we note that it is fairly common in the literature of the lambda calculus to assume that the set of variables is countably infinite, see for instance Hindley and Seldin [26, Def. 1.1]. Of course, in order to make the above properties of cofinite sets true, countable infinity suffices. In this paper we do not need assume that the set of variables is countable and the reason is plain: we never invoke the existence of a choice point of the definition we need to invoke the existence of a function $f$ from cofinite sets of variables to variables such that $f(S) \in S$ for every $S$. (Note that cofinite sets must be non-empty for this discussion to make sense and this is indeed the case!) We refer to Vestergaard [52] for his approach to infinitary lambda theories and extensional theories without assuming any ordering on variables. To sum up, in the present work we do not need to impose any ordering on the set of variables.

Finally — and most importantly — we observe that it is also common to assume that the set of variables comes equipped with an ordering; again, see [26, Def. 1.1]. Here, the motivation is ultimately related to the definition of substitution: if this notion is formalized as in [26], then at some point of the definition we need to invoke the existence of a choice function $f$ from cofinite sets of variables to variables such that $f(S) \in S$ for every $S$. (Note that cofinite sets must be non-empty for this discussion to make sense and this is indeed the case!) We refer to Vestergaard [52] for more on this aspect, where the axiom of cofinite choice is explicitly discussed. In passing, we also mention that choice functions are specifically introduced by Stoughton [49] in his approach to simultaneous substitution, see also Copello, Szasz and Tasistro [13]. The crucial point of the present discussion is this: if the set of variables is appropriately ordered, then a choice function $f$ naturally arises. Indeed, for each cofinite set $S$ we can simply define $f(S)$ as the first, in the given order, variable which is a member of $S$ (see also Remark 7.1). In this paper we do not define substitution as in [26]; in Section 7 we follow an elegant approach due to Barendregt [4] which allows us to define substitution, lambda theories and extensional theories without assuming any ordering on variables. To sum up, in the present work we do not need to impose any ordering on the set of variables. Even more generally, we do not need to appeal to the existence of some choice function either.

### 3 Prelambda Congruences

In this section, we introduce the notion of prelambda congruence and discuss the conditions which define this concept. Basic properties of prelambda congruences will be studied in the next section.

Prelambda congruences are our first step towards our formalizations of the concepts of lambda theory and extensional theory. Even though it is technically incorrect, it may be helpful to think of prelambda congruences as lambda theories which do not necessarily satisfy the condition of $\alpha$–renaming — but, of course, they have to satisfy the $\beta$–rule. (In a similar manner, preorders can be seen as partial orders which do not necessarily satisfy anti-symmetry but they do satisfy reflexivity and transitivity.) As a matter of fact, we show that there exists a prelambda congruence $\sim$ with the following property: we have $\lambda xx \not\sim \lambda yy$ for all variables $x$ and $y$ such that $x \neq y$. Since $\lambda xx \sim \lambda xx$ holds in every lambda theory $\sim$, in order to develop our formalization of lambda theory it is actually necessary to add $\alpha$–renaming at some stage.

Prelambda congruences which satisfy a suitable condition of $\alpha$–renaming will be considered in Section 5 under the name of lambda congruences. Similarly, in Section 6 we shall introduce extensional congruences, which are prelambda congruences which satisfy an appropriate condition of $\eta$–extensionality.

There are essentially two motivations for introducing the concept of prelambda congruence.

Firstly, the results that we prove in the next section on the concept of independence of variables do not depend on $\alpha$–renaming and $\eta$–extensionality.

Secondly, the factorization of lambda and extensional congruences respectively as “prelambda congruences plus $\alpha$–renaming” and “prelambda congruences plus $\eta$–extensionality” will provide us with a mathematically reasonable understanding of these two syntactical conditions in terms of extensionality.

It is time for us to formally introduce prelambda congruences.

**Definition 3.1** (Prelambda congruence). Let $\sim$ be a congruence. We say that $\sim$ is a prelambda congruence if it also satisfies the following conditions:

1. $(\beta_1)$ $\lambda x A \equiv D$;
2. $(\beta_2)$ $\lambda x y D \sim y$, provided $x \neq y$;
3. $(\beta_3)$ $\lambda x [A B] D \sim [\lambda x A] D [\lambda x B] D$;
4. $(\beta_4)$ $\lambda x [\lambda x A] D \sim \lambda x A$;
5. $(\beta_5)$ $\lambda y D x \sim D$ implies $\lambda x [\lambda y A] D \sim \lambda y [\lambda x A] D$, provided $x \neq y$;

where $A$, $B$ and $D$ are arbitrary terms, and $x$ and $y$ are arbitrary variables.

We collectively refer to conditions $(\beta_1)$ to $(\beta_5)$ above as beta conditions.

Thus, in plain words a prelambda congruence is any binary relation on terms which simultaneously satisfies all structural and beta conditions.

A simple example of prelambda congruence is the whole set $\equiv$. A more interesting example is given in the following definition.

**Definition 3.2** (Prelambda conversion). We call prelambda conversion the prelambda congruence inductively defined by structural and beta conditions.

In Corollary 3.4 below it is shown that $\equiv^2$ and prelambda conversion are in fact different relations. Hence, if we regard prelambda conversion as an equational theory where equations are expressions of the form $A \sim B$, then prelambda conversion is consistent in the sense of Hilbert and Post.

Let us now discuss the conditions of Definition 3.1 in some detail.

With possibly some minor differences, conditions $(\beta_1)$, $(\beta_3)$, $(\beta_4)$ and $(\beta_5)$ also appear in some work by Henkin [25], Andrews [3], Révész [40, 41, 42], and Pigolotti and Salibra [36, 37, 38]. For the sake of completeness, we also mention that conditions $(\beta_1)$ and $(\beta_2)$ are also considered by Andrews [2], but there conditions $(\beta_2)$ of $(\beta_4)$ are condensed into the following clause:

1. $(\gamma_1)$ $\lambda x A \equiv D$, provided $x$ does not occur free in $A$. 

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Among beta conditions, the more interesting one is certainly ($\beta_3$). This condition constitutes a point of divergence in the relevant literature. Actually, ($\beta_3$) as formulated in this paper seems to be new. Consider the following conditions:

\[(\gamma_2) \quad [\lambda x[\lambda y A]]D \sim \lambda y[[\lambda x A]D],\]

provided $x \neq y$ and $y$ does not occur free in $D$;

\[(\gamma_3) \quad [\lambda x[\lambda y A]]D \sim \lambda y[[\lambda x A]D],\]

provided $y$ is distinct from $x$ and all variables occurring in $D$;

\[(\gamma_4) \quad [\lambda x[\lambda y A]]D \sim \lambda y[[\lambda x A]D],\]

provided $x \neq y$ and at least one of these two conditions hold: $x$ does not occur free in $A$, $y$ does not occur free in $D$;

\[(\gamma_5) \quad [\lambda y D]z \sim D \implies [\lambda x[\lambda y A]][\lambda y D]z \sim [\lambda y [[\lambda x A][\lambda y D]z]],\]

provided $x \neq y$ and $y \neq z$;

\[(\gamma_6) \quad [\lambda x[\lambda y A]][\lambda y D]z \sim [\lambda y [[\lambda x A][\lambda y D]z]],\]

provided $x \neq y$ and $y \neq z$.

Again, up to minor details, ($\gamma_2$) is the version of ($\beta_3$) considered by Henkin [25, Axiom 7.5 p. 331] and Andrews [2, 4 p. 3], ($\gamma_3$) is another version considered by Andrews [3, 4 p. 164] and finally ($\gamma_4$) is the one taken into account by Révész [42, p. 29]. As we can see, all these conditions explicitly mention some ancillary concept.

The condition which is most similar to our ($\beta_3$) is the one given by Pigozzi and Salibra in the context of lambda abstraction algebras ($\gamma_3$) [37, $\beta_6$ p. 12]; in fact, our condition ($\beta_3$) is actually a simplification of ($\gamma_3$) — in the sense that our condition requires two variables and one inequality to be expressed, while ($\gamma_3$) needs three variables and two inequalities. Our ($\gamma_6$) is a shortened version of ($\gamma_2$) due to Pigozzi and Salibra [37, Prop. 1.5]. Conditions ($\gamma_5$) and ($\gamma_6$) are equivalent in every lambda abstraction algebra. Finally, it is worth mentioning that Pigozzi and Salibra also consider another “beta condition” in their setting, namely $[\lambda x A]x \sim A$ (see [37, $\beta$ p. 12]). But for our purposes, we do not need to consider this condition at all.

As we have just seen, (suitable versions of) beta conditions have already been considered in the literature. They perhaps seem a bit complicated at first, but a closer look reveals that they are in fact quite natural and even easy to memorize. To see this, notice that they are all of the form $[\lambda x M]D \sim G$. Now, if we replace each term of the form $[\lambda x M]D$ by an expression of the form $[D/x](M)$ and write $=\sim$ instead of $\sim$, then by a slight rearrangement we obtain

\[(\beta_1) \quad [D/x](x) = D;\]

\[(\beta_2) \quad [D/x](y) = y, \text{ if } x \neq y;\]

\[(\beta_3) \quad [D/x](AB) = [D/x](A)[D/x](B);\]

\[(\beta_4) \quad [D/x](\lambda x A) = \lambda y [D/x](A),\]

if $x \neq y$ (provided $[x/y](D) = D$).

Conditions ($\beta_1$), ($\beta_2$), ($\beta_3$) and ($\beta_4$) above really look like a kind of inductive definition of substitution which is not too dissimilar to the ones we can find in the literature (cf. Definition 2.2). In passing, it is worth observing that these clauses are very similar to some of the conditions which axiomatize lambda substitution algebras, algebraic structures introduced by Diskin and Beylin [21] with the purposes of algebraizing the lambda calculus — in fact, these algebras share several similarities with lambda abstraction algebras.

At this point, we hope that we have almost convinced the reader that beta conditions are quite natural and intuitive. However, if there is some reader which still feels uncomfortable with the precondition $[\lambda y D]x \sim D$ in condition ($\beta_3$) (that is to say, the proviso $[x/y](D) = D$ in ($\beta_3^*$) above), then we would like to reassure the reader by saying that the whole next section is dedicated to the study of the “consequences” of $[\lambda y D]x \sim D$.

We now show that $\alpha$-renaming is not available in every prelambda congruence. For, let $x$ and $y$ be two distinct variables, and let $\sim$ be an arbitrary prelambda congruence. If $\sim$ were a lambda theory, we would certainly have $\lambda x x \sim \lambda y y$ by using $\alpha$-renaming. However, here nothing ensures that we have $\lambda x x \sim \lambda y y$. In fact, the following theorem shows that any attempt of deriving $\lambda x x \sim \lambda y y$ by means of our structural and beta conditions is doomed to failure.

**Theorem 3.3.** Let $x$ and $y$ be two variables such that $x \neq y$. Then, there exists a prelambda congruence $\sim_1$ such that $\lambda x x \not\sim_1 \lambda y y$.

The proof of Theorem 3.3 is quite long and technical. Furthermore, in order to produce it we are forced to introduce further notions and methods that we do not need in the rest of this paper. For these reasons, it is postponed to Section 9. Nevertheless, the interested reader can directly go there without any hesitation, as no further material is needed to read that part of this work.

As a consequence of the previous result, a very simple proof of the fact that $\mathbb{T}^2$ and prelambda conversion are distinct relations is at our disposal.

**Corollary 3.4.** Let $\sim$ be prelambda conversion, and let $x$ and $y$ be two distinct variables. Then, we have $\lambda x x \not\sim \lambda y y$. In particular, $\sim$ is consistent.

**Proof.** Since $\sim$ is the inductively defined prelambda congruence, it is the intersection of all prelambda congruences. In particular, we have $\sim \subseteq \sim_1$. From this and Theorem 3.3 it follows that $\lambda x x \not\sim_1 \lambda y y$. In particular, we have $\sim \neq \mathbb{T}^2$. □

### 4 Basic Properties of Prelambda Congruences

In the previous section, we did not discussed the significance of the precondition $[\lambda y D]x \sim D$ in condition ($\beta_3$). In this section, we aim to fill this gap. Since this requirement turns out to be very important in our setting, it is convenient to introduce some terminology to describe this situation.

**Definition 4.1** (Independence of variables). Let $\sim$ be a prelambda congruence and let $A$ be a term. We define the set of variables $\sim(A)$ as follows:

$\sim(A) \equiv \{ x \mid [\lambda x A]z \sim A \text{ for some variable } z \text{ such that } x \neq z \} \,$.

Let $x$ be a variable. We say that $A$ is independent of $x$ in $\sim$ if $x \in \sim(A)$.

□
Thus, since in $(\beta_3)$ we have $x \neq y$, in the new terminology the precondition $[\lambda yD]x \sim D$ means that $D$ is independent of $y$ in $\sim$. There is a clear analogy with the proviso “$y$ does not occur free in $D$” of Henkin’s version of $(\beta_3)$ — the condition we called $(\gamma_1)$ in the previous section. A precise relationship between independent variables and free variables is given in Theorem 4.8.

The notion of independence of variables in the lambda calculus has been introduced and studied by Piggozzi and Salibra in the context of lambda abstraction algebras [36, 37, 38]. Even though prelambda congruences are not lambda abstraction algebras, facts which look quite similar to “sharpened versions” of Proposition 4.2 and Lemma 4.3 below have been already established, see 37, Lem 1.6 and Prop. 1.7. With a little effort, we could generalize our facts in the same vein, but we find more instructive to present them in the actual form because we do not need anything more elaborated to prove our main results.

Technically speaking, the peculiar use of the concept of independence of variables we make in this paper allows us to avoid the introduction of the sets of all, free and bound variables occurring in a term in the definitions of our notions of prelambda, lambda and extensional congruences.

To begin the study of the concept of independence of variables in prelambda congruences, we show that a property similar to the condition called $(\gamma_1)$ that we discussed in the previous section also holds in our setting.

**Proposition 4.2.** Let $\sim$ be a prelambda congruence. Let $x$ be a variable and let $A$ be a term. Suppose that $x \in \sim(A)$. Then, we have $[\lambda xA]D \sim D$ for every term $D$.

**Proof.** Let $D$ be a term. Since $x \in \sim(A)$, there exists a variable $z$ such that $x \neq z$ and $[\lambda xA]z \sim A$. Let $C \equiv [\lambda x(\lambda xA)]zD$, $E \equiv [\lambda x(\lambda xA)]D$ and $F \equiv [\lambda xz]D$. Since $[\lambda xA]z \sim A$, we obtain $\lambda x([\lambda xA]z) \sim \lambda xA$ by (i). We have $D \sim D$ by (r), and we get $C \sim [\lambda xA]D$ by (a). Now, by $(\beta_3)$ we deduce $C \sim EF$ and we have $E \sim \lambda xA$ by $(\beta_3)$. As $x \neq z$, we obtain $F \sim z$ by $(\beta_3)$ and it follows from (a) that $EF \sim [\lambda xA]z$. By using (s) we have $[\lambda xA]D \sim C \sim EF \sim [\lambda xA]z \sim A$ and we finally get $[\lambda xA]D \sim A$ from (t).

In regard to the syntactical category a term belongs to, the following lemma gives a partial description of sets of independent variables.

**Lemma 4.3.** Let $\sim$ be a prelambda congruence. Let $x$ be a variable, and let $B$ and $C$ be terms. Then, the following statements hold:

(i) $\forall y \in \sim(x)$;
(ii) $\sim(B) \cap \sim(C) \subseteq \sim(B \cap C)$;
(iii) $\sim(B \cup C) \subseteq \sim(B \cup C)$ (in particular, $\sim(B) \subseteq \sim(\lambda xB)$).

**Proof.** (i) Let $y \in \forall y \in \sim(x)$. Since $x \neq y$, we have $[\lambda yx]x \sim x$ by $(\beta_3)$. Hence, we obtain $y \in \sim(x)$.

(ii) Let $y \in \sim(B) \cap \sim(C)$ and let $z$ be a variable such that $y \neq z$. By Proposition 4.2 it follows that $[\lambda yB]z \sim B$ and $[\lambda yC]z \sim C$, and we obtain $[\lambda yB][\lambda yC]z \sim BC$ by (a). From $(\beta_3)$, we have $[\lambda yB][\lambda yC]z \sim [\lambda yB][\lambda yC]z$ and we get $[\lambda yB][\lambda yC]z \sim BC$ from (t). Therefore, we conclude that $y \in \sim(BC)$.

(iii) Let $z$ be a variable such that $x \neq z$. Then, we have $[\lambda x][\lambda B]z \sim \lambda xB$ by $(\beta_4)$ and hence it follows that $x \in \sim(\lambda xA)$. Now, let $y \in \sim(B \setminus \{x\}$ and let $z$ be a variable such that $x \neq z$ and $y \neq z$. By Proposition 4.2 we get $[\lambda yB]z \sim B$. Since $x \neq y$ and $x \neq z$, we have $[\lambda xz]y \sim z$ by $(\beta_3)$ and hence $[\lambda y][\lambda xB]z \sim \lambda x([\lambda xB]z) \sim \lambda xB$ by $(\beta_3)$. Since $[\lambda yB]z \sim B$, we also have $[\lambda x][\lambda yB]z \sim \lambda xB$ from (t). Finally, we obtain $[\lambda y][\lambda xB]z \sim \lambda xB$ from (t). As $y \neq z$, we conclude that $y \in \sim(\lambda xB)$.

Our next step is to relate the notion of independence of variables in prelambda congruences to the usual concept of free variable. Up to now, we dealt with them informally and appealed to the reader’s previous knowledge of this concept. Since at this point we need to prove some formal facts about free variables, we now recall the formal definition. Nevertheless, for the sake of better readability, in informal discussions below we shall often express the concept of free variable in words, rather than employing the symbolism of Definition 4.4.

As already noticed by Welch [55, Rem. 0.3.4], for the development of the lambda calculus it is actually more clear and convenient to formalize the concept of variable which does not occur free in a given term. Here, we follow the same idea because this approach makes the connection between non–free and independent variables tighter. Non–free variables are sometimes called *fresh* variables in the literature, for instance in Copello, Szasz and Tasistro [15].

**Definition 4.4** (Non–free variable). Let $A$ be a term. We define the cofinite set of variables $\forall(A)$ by induction on the structure of $A$ as follows:

- $\forall(x) \equiv \forall x \setminus \{x\}$;
- $\forall(BC) \equiv \forall(B) \cap \forall(C)$;
- $\forall(\lambda xB) \equiv \forall(B) \cup \{x\}$.

Finally, we say that a variable $x$ does not occur free in $A$ if $x \in \forall(A)$.

The next theorem is fundamental for the main results of this paper.

**Theorem 4.5.** Let $\sim$ be a prelambda congruence and let $A$ be a term. Then, we have $\forall(A) \subseteq \sim(A)$.

**Proof.** We proceed by induction on the structure of $A$.

Suppose that $A = x$. By Lemma 4.3(i), we have $\forall(x) = \forall x \setminus \{x\}$.

Suppose that $A = BC$. By inductive hypothesis, we have $\forall(B) \subseteq \sim(B)$ and $\forall(C) \subseteq \sim(C)$. Thus, we obtain $\forall(BC) = \forall(B) \cap \forall(C) \subseteq \sim(B) \cap \sim(C) \subseteq \sim(BC)$ by using Lemma 4.3(ii).

Finally, suppose that $A = \lambda xB$. By inductive hypothesis, we have $\forall(B) \subseteq \sim(B)$. By Lemma 4.3(iii), it follows that $\forall(\lambda xB) = \forall(B) \cup \{x\} \subseteq \sim(B \cup \{x\} \subseteq \sim(\lambda xB)$.

An immediate consequence of Theorem 4.5 is that in every prelambda congruence $\sim$ all sets of variables of the form $\sim(A)$ are cofinite. A more remarkable consequence is that condition $(\gamma_2)$, discussed in the previous section, always holds in our setting as we show in the second part of the following proposition.
**Proposition 4.6.** Let \( \sim \) be a prelambda congruence. Let \( A \) and \( D \) be terms. Let \( x \) and \( y \) be variables such that \( x \neq y \). Then, the following statements hold:

(i) Suppose that \( y \in \sim(D) \). Then, we have \([\lambda y ] (\lambda y A) \] \( \sim \lambda y (\lambda x A) D \].

(ii) Suppose that \( y \in \{ (D) \). Then, we have \([\lambda x ] (\lambda y A) \] \( \sim \lambda y (\lambda x A) D \].

**Proof.** (i) We have \([\lambda y ] D x \sim D \) by Proposition 4.2. As \( x \neq y \), we obtain \([\lambda x ] (\lambda y A) \] \( \sim \lambda y (\lambda x A) D \) by \((\beta_3)\).

(ii) It follows from (i) above, as we have \( y \in \sim(D) \) by Theorem 4.5.

We now further analyze the relationship between independent and non–free variables, though the facts we are now going to establish are not strictly needed to prove our main results. To begin with, we show that independence and non–freedom are not equivalent notions. In the following example we show that the converse of Theorem 4.5 does not hold.

**Example 4.7.** Let us consider the case \( \sim = T^2 \). Since all terms are related, we have \( \sim(A) = \forall \) for every term \( A \). However, it is not true that \( \forall(A) = \forall \) for every term \( A \); for instance, we have \( \forall(x) = \forall \setminus \{ x \} \) for every variable \( x \). \( \triangle \)

The previous example does not exclude the existence of a prelambda congruence \( \sim \) in which it is possible to have \( \sim(A) = \forall(A) \) for every term \( A \). However, we can show that such a \( \sim \) cannot exist. Actually, we prove an even stronger version of this fact in Corollary 4.9 below, which is a consequence of the theorem that we now show.

**Theorem 4.8** (Independence and non–free variables). \( \sim \) be a prelambda congruence and let \( A \) be a term. We have \( \sim(A) = \{ v \mid v \in \forall(B) \text{ for some term } B \text{ such that } B \sim A \} \).

**Proof.** Let \( S = \{ v \mid v \in \forall(B) \text{ for some term } B \text{ such that } B \sim A \} \). Suppose that \( x \in \sim(A) \). Then, there exists a variable \( z \) such that \( x \neq z \) and \([\lambda x ] z \sim A \). Let \( B = [\lambda x ] z \). Since \( x \neq z \), we have \( x \in \{ \forall(A) \setminus \{ x \} \} \cap (\forall \setminus \{ z \} ) = \forall(B) \). As \( B \sim A \), we conclude that \( x \in S \).

Suppose now that \( x \in S \). Let \( B \) be a term such that \( x \in \forall(B) \) and \( B \sim A \). Since \( x \in \forall(B) \), we have \( x \in \sim(B) \) by Theorem 4.5. In particular, there exists a variable \( z \) such that \( x \neq z \) and \([\lambda x ] z \sim B \). Now, since \( B \sim A \), we have \( \lambda x B \sim \lambda x A \) by \((\iota)\). From \((\iota)\) it follows that \( z \sim A \). So, we obtain \([\lambda x ] z \sim \forall(B) \) by \((\alpha)\). By using \((s)\), we have \([\lambda x ] z \sim \lambda x B \sim A \) and we obtain \([\lambda x ] z \sim \lambda x A \) from \((\iota)\). As \( x \neq z \), we conclude that \( x \in \sim(A) \).

**Corollary 4.9.** There exists a term \( A \) such that in every prelambda congruence \( \sim \) we have \( \sim(A) \subseteq \forall(A) \).

**Proof.** Let \( x \) and \( y \) be variables such that \( x \neq y \), and let \( A = [\lambda x y] \). Let \( \sim \) be a prelambda congruence. Since \( \forall(A) = \{ \forall(x) \} \cap \forall(y) \cap \forall(\setminus \{ x \} ) \}, \) we have \( x \notin \forall(A) \). Since \( x \neq y \), we have \( A \sim y \) from \((\beta_3)\) and \( y \sim A \) from \((s)\). Also, we have \( x \in \forall \setminus \{ y \} = \forall(y) \) and so it follows that \( x \in \forall(B) \) for some term \( B \) such that \( B \sim A \). Thus, we obtain \( x \in \sim(A) \) by means of Theorem 4.8.

Now, if \( \sim(A) \subseteq \forall(A) \) were true, then we would get \( x \in \forall(A) \) but this would contradict \( x \notin \forall(A) \). Therefore, we have \( \sim(A) \not\subseteq \forall(A) \).

## 5 Lambda Congruences

We now turn our attention to the intuitive interpretation of terms as functions in prelambda congruences. Let \( \sim \) be an arbitrary prelambda congruence.

In Section 2 we observed that a term of the form \( \lambda x A \) should be thought as a function whose domain is the whole set of terms \( T \) (including \( \lambda x A \) itself). Under this perspective, when the function \( \lambda x A \) is applied to an input \( D \), any term \( B \) such that \( B \sim [\lambda x ] D \) should be thought as the output of \( \lambda x A \) on input \( D \). This view is justified by the fact that the terms \( B \) and \( [\lambda x ] D \) are meant to be same in \( \sim \) when \( B \sim [\lambda x ] D \) holds, so that syntactically different terms related to \( [\lambda x ] D \) are just different representations of the same output. Depending on the practical aims, better and more informative representations of the output can be computed using structural and beta conditions. Clearly, different prelambda congruences can produce different outcomes; for instance, if \( \sim = T^2 \) then any term is the output of any function on any input.

Notice that we do not require that every term has to be thought as a function. So, variables and applications can be simply thought as constituents for forming functions — of course, they can be inputs and outputs of functions. (In a similar fashion, in set theory with urelements it is not required elements to be sets, but they can be used to build sets.) In Section 6 the possibility of regarding every term as a function will be investigated.

So far so good. However, if we seriously want to develop a reasonable theory of functions inside the lambda calculus we should expect the following property of extensionality for functions to be true: two functions should be considered as the same if they have identical input–output behaviour. This property clearly holds in set theory: given a set \( Z \) and two functions \( f \) and \( g \) from \( Z \) to itself we always have \( f(d) = g(d) \) for every \( d \in Z \) implies \( f = g \).

Thus, if we want to think abstractions as functions then we should expect the following property of extensionality for abstractions to hold:

\([\lambda x ] A \sim [\lambda y ] B \] for every term \( D \) implies \( \lambda x A \sim \lambda y B \).

Quite suggestively, we observe that our formulation of extensionality for abstractions strikingly resembles the axiom of extensionality of von Neumann’s set theory based on functions [53, p. 397 and Axiom I4 p. 399].

A serious problem of prelambda congruences is that it is not always the case that extensionality for abstractions holds. For instance, if \( \sim \) is prelambda conversion (see Definition 3.2), then for distinct variables \( x \) and \( y \) and any term \( D \) we have \([\lambda x ] D \sim D \) and \([\lambda y ] D \sim D \) by \((\beta_1)\). So, by using \((s)\) we obtain \([\lambda x ] D \sim D \sim [\lambda y ] D \) and by \((\iota)\) we get \([\lambda x ] D \sim [\lambda y ] D \) which shows that the two abstractions \( \lambda x x \) and \( \lambda y y \) have the same input–output behaviour. But
\(\lambda x \sim \lambda y y\) does not hold, by Corollary [3.4]. Hence, in the section we focus our attention on prelambda congruences which satisfy extensionality of abstraction.

Nevertheless, we do not regard prelambda congruences that do not satisfy such a form of extensionality — such as prelambda conversion — as uninteresting. We strongly believe that these congruences may serve as a foundation for the development of a truly intensional (as opposed to extensional) theory of algorithms... but this is another story.

Another intriguing motivation for considering extensionality for abstractions is the fact that this condition has been present in the lambda calculus from the very beginning — though under a completely different guise. In fact, in every prelambda congruence extensionality for abstractions and \(\alpha\)-renaming are equivalent conditions, as we show in Theorem 5.5.

We believe that the usual informal explanations of \(\alpha\)-renaming, as "the names of bound variables are immaterial" and the like, provide neither a good justification of what is really going on nor a good motivation for accepting this condition as natural. Moreover, in the literature we are not aware of any line of work which attempts to clarify the real significance of this condition. In this paper, we can give a mathematically significant answer: \(\alpha\)-renaming is just a compact reformulation of the property of extensionality for abstractions.

We now formally introduce the concept of lambda congruence: prelambda congruences which satisfy our condition of \(\alpha\)-renaming \((\alpha_e)\).

**Definition 5.1 (Lambda congruence).** Let \(\sim\) be a prelambda congruence. We say that \(\sim\) is a **lambda congruence** if it also satisfies the following condition:

\[ (\alpha_e) \quad [\lambda y A] x \sim A \text{ implies } \lambda x A \sim \lambda y[[\lambda x A] y]; \]

where \(x\) and \(y\) are arbitrary variables and \(A\) is an arbitrary term.

In other words, a lambda congruence is any binary relation on terms which simultaneously satisfies all structural and beta conditions together with condition \((\alpha_e)\). In particular, every lambda congruence is also a prelambda congruence and so the general results on independence of variables we proved in the previous section can be applied to lambda congruences as well.

Since our formulation of \(\alpha\)-renaming as \((\alpha_e)\) seems to be new, we now discuss our condition in relation to other clauses of \(\alpha\)-renaming proposed in the relevant literature, as we did in Section 3 for \((\beta_2)\). Other conditions of \(\alpha\)-renaming which are perhaps more standard are considered by Crole [17, Sec. 3]; however, they do not fit the aims of the present work because their formulations require either substitution or new operations such as atom swapping. Consider the following conditions:

\[ (\delta_1) \quad \lambda x A \sim \lambda y[[\lambda x A] y], \text{ provided } y \text{ does not occur free in } A; \]
\[ (\delta_2) \quad [\lambda y A] x \sim A \text{ implies } \lambda x A \sim \lambda y[[\lambda x A] y], \text{ provided } y \neq x; \]
\[ (\delta_3) \quad \lambda x [[\lambda y A] z] \sim \lambda y[[\lambda x [\lambda y A] z] y], \text{ provided } y \neq z. \]

Up to minor differences, \((\delta_1)\) is the condition of \(\alpha\)-renaming considered by Révész [42, p. 29], while conditions \((\delta_2)\) and \((\delta_3)\) have been introduced by Pigozzi and Salibra in the theory of lambda abstraction algebras 57, 6 p. 12 and Prop. 1.5]. In their setting, \((\delta_2)\) and \((\delta_3)\) are equivalent. Our condition \((\alpha_e)\) is actually a simplified version of \((\delta_2)\) inasmuch as \((\alpha_e)\) requires two variables to be expressed, while \((\delta_2)\) three variables and one inequality.

In the type theories developed by Henkin [25] and Andrews [2, 3], the situation is rather different; in their setting it is present a condition which is similar to our extensionality for abstractions. Roughly speaking, they consider a condition of extensionality for terms which have the same "arrow type"; this condition in turn implies \(\alpha\)- renaming for some classes of terms, see for instance [25, Axiom Schema 6 p. 330 and Thm. 7.21]. A crucial difference is that their condition of extensionality can be simply expressed as a term of their language. By contrast, in our setting extensionality for abstractions is a first–order sentence of the meta–language.

Regarding our formulation of \(\alpha\)-renaming we also observe the following fact.

**Remark 5.2.** Let \(\sim\) be a prelambda congruence. Let \(A\) be a term and let \(x\) be a variable. Suppose that \([\lambda x A] x \sim A\). Then, from (\(\ell\)) we have \(\lambda x[[\lambda x A] x] \sim \lambda x A\) and from (\(s\)) we obtain \(\lambda x A \sim \lambda x[[\lambda x A] x]\). This show that condition \((\alpha_e)\) is already present in any prelambda congruence in the special case \(x = y\). Therefore, the real content of \((\alpha_e)\) is actually \((\delta_4)\) \([\lambda y A] x \sim A \text{ implies } \lambda x A \sim \lambda y[[\lambda x A] y]\), provided \(x \neq y\).

However, we prefer to express our condition of \(\alpha\)-renaming by means of \((\alpha_e)\) rather than \((\delta_4)\) because the former admits a shorter formulation.

In order to improve its intuitive understanding, we may rewrite, in the notation employed in Section 3 condition \((\alpha_e)\) in the following form:

\[ (\alpha_e') \quad \lambda x A = \lambda y[[y/x](A)] \quad (\text{provided } [x/y](A) = A). \]

In this shape, it appears to be very similar to the usual conditions of \(\alpha\)-renaming which one finds in the literature.

Contrarily to our treatment of prelambda congruences, it is not worth giving examples of specific lambda congruences. The reason is that in Section 3 we shall show that lambda congruences and lambda theories are actually the same concept, and in the literature of the lambda calculus examples of lambda theories abound. But it is worth giving an example of a prelambda congruence which is not a lambda congruence.

**Proposition 5.3.** Let \(\sim\) be a lambda congruence. Let \(x\) and \(y\) be two variables such that \(x \neq y\). Then, we have \(\lambda x x \sim \lambda y y\). In particular, prelambda conversion is not a lambda congruence.

**Proof.** We have \([\lambda y x] x \sim x\) by (\(\beta_2\)). Thus, we obtain \(\lambda x x \sim \lambda y y[[\lambda x x y]]\) by \((\alpha_e)\). By (\(\beta_1\)) it follows that \([\lambda x x y] \sim y\) and thus we get \(\lambda y[[\lambda x x y]] \sim \lambda x x y\) from (\(\ell\)). Hence, we obtain \(\lambda x x \sim \lambda x x y\) by (\(\ell\)). In particular, by Corollary 3.4 prelambda conversion cannot be a lambda congruence.

We now show, in the second part of the following proposition, that condition \((\delta_1)\) above holds in every lambda congruence.
Proposition 5.4. Let \( \sim \) be a lambda congruence. Let \( A \) be a term, and let \( x \) and \( y \) be variables. Then, the following statements hold:

(i) Suppose that \( y \in \sim(A) \). Then, we have \( \lambda x A \sim \lambda y [\lambda x A] y \).

(ii) Suppose that \( y \in \Pi(A) \). Then, we have \( \lambda x A \sim \lambda y [\lambda x A] y \).

Proof. (i) We have \( [\lambda y A] x \sim A \) by Proposition 4.2 and we obtain \( \lambda x A \sim [\lambda y [\lambda x A] y] \) from (\( \alpha_e \)).

(ii) It follows from (i) above, as we have \( y \in \sim(A) \) by Theorem 5.5.

Our next step is to show that in every prelambda congruence conditions (\( \alpha_e \)) and extensionality for abstractions are equivalent.

Theorem 5.5. Let \( \sim \) be a prelambda congruence. Then, \( \sim \) is a lambda congruence if and only if it satisfies the condition of extensionality for abstractions:

\[(\alpha_e) \quad [\lambda x A] D \sim [\lambda y B] D \text{ for every term } D \text{ implies } \lambda x A \sim \lambda y B,\]

for all terms \( A \) and \( B \), and all variables \( x \) and \( y \).

Proof. Suppose that \( \sim \) is a lambda congruence. Let \( A \) and \( B \) be terms, and let \( x \) and \( y \) be variables. Let \( E \equiv \lambda x A \) and \( F \equiv \lambda y B \). Assume \( E \sim F \) for every term \( D \). We now show that \( E \sim F \). Let \( z \in \sim(A) \cap \sim(B) \). Then, we have

\[Ez \sim Fz \text{ and } \lambda z [Ez] \sim \lambda z [Fz] \text{ from } (t).\]

As \( x \neq y \), it follows that \( [\lambda x y z] z \sim [\lambda y x z] z \) by \( (\beta_e) \). Hence, we get \( M \sim [\lambda x [\lambda y A] z] \) by \( (\beta_e) \). Since \( x \neq y \), we have \( y \in \sim(A) \) from the assumption \( [\lambda y A] x \sim A \). Hence, by Proposition 4.2 we obtain \( [\lambda y A] z \sim A \) and we get \( \lambda x [\lambda y A] z \sim E \) from \( (t) \). Thus, we have \( M \sim E \) from \( (f) \). Now, by \( (\beta_e) \) we obtain \( N \sim z \) and we get \( MN \sim Ez \) from \( (a) \). By \( (t) \), we obtain \( Fz \sim Ez \) and so \( \lambda z [Ez] \sim \lambda z [Fz] \) from \( (s) \) and \( (t) \). Let \( D \) be an arbitrary term. From \( (r) \) we have \( D \sim D \) and by \( (a) \) we obtain \( [\lambda z [Ez] D] \sim [\lambda z [Fz] D] \). Let \( G \in \{E, F\} \). We have \( [\lambda z [Gz] D] \sim [\lambda z [Gz] D] \) by the \( \beta_e \). Since \( z \in \sim(G) \), we have \( [\lambda z [Gz] D] \sim G \) by Proposition 4.2. By \( (\beta_e) \) we obtain \( [\lambda z [Gz] D] \sim D \). Thus, by \( (a) \) it follows that \( [\lambda z [Gz] D] [\lambda z [Gz] D] \sim \) GD and by \( (t) \) we obtain \( [\lambda z [Gz] D] \sim GD \). This shows that we have \( [\lambda z [Gz] D] \sim ED \) and \( [\lambda z [Fz] D] \sim FD \). By using \( (s) \) we obtain \( E \sim [\lambda z [Ez] D] \sim [\lambda z [Fz] D] \sim FD \) and \( ED \sim FD \) from \( (t) \). Since the term \( D \) is arbitrary, we have \( ED \sim FD \) for every term \( D \). As \( E \) and \( F \) are both abstractions, we obtain \( E \sim F \) by using \( (\alpha_e) \).

Thus, by the previous theorem our formulation of \( \alpha \)-renaming \( (\alpha_e) \) and extensionality for abstractions \( (\alpha_e) \) are equivalent. This equivalence is significant precisely because, by Proposition 5.3 \( (\alpha_e) \) does not hold in every prelambda congruence. We decided to axiomatize lambda congruences using \( (\alpha_e) \) because this condition is more compact and also because this approach is more akin to traditional presentations of lambda theories. But for us, the actual meaning of \( \alpha \)-renaming is precisely the property of extensionality for abstractions.

6 Extensional Congruences

In the previous section, we followed the idea that only abstractions should be thought as functions. In section we expand our vision and explore the possibility of thinking every term as a function.

In order to implement this idea, the most natural way is to consider lambda congruences \( \sim \) where each term is related to an abstraction. The typical condition of \( \eta \)-extensionality — that we shall discuss in the next section —

\[A \sim \lambda y \, [Ay], \quad \text{provided } y \text{ does not occur free in } A\]

clearly does the required job. In our setting, to achieve the same result without introducing any ancillary concept, it suffices to consider prelambda congruences which satisfy a simpler condition of \( \eta \)-extensionality, our condition \( (\eta_e) \).

This lead us to the formalization of the concept of extensional congruence.

Definition 6.1 (Extensional congruence). Let \( \sim \) be a prelambda congruence. We say that \( \sim \) is an extensional congruence if \( \sim \) also satisfies the following condition:

\[(\eta_e) \quad y \sim \lambda x [yx], \quad \text{provided } x \neq y;\]

where \( x \) and \( y \) are arbitrary variables.

Thus, an extensional congruence is any binary relation on terms which simultaneously satisfies all structural and beta conditions together with our condition of \( \eta \)-extensionality \( (\eta_e) \). Of course, since every extensional congruence is also a prelambda congruence, the general facts on independence of variables we observed in Section 4 can be applied to extensional congruences as well.

Regarding our formalization, our exact formulation of condition \( (\eta_e) \) seems to be new, even though we recognize a strong similarity with a condition introduced by Hindley and Longo for studying models of the lambda calculus [27, Eq. 7 Lem. 4.2]. Their condition is, essentially, one instance of our condition \( (\eta_e) \). The main difference is that in their setting \( \alpha \)-renaming is available; so it is possible to use \( \alpha \)-renaming to generate more instances. On the contrary, here we are working with arbitrary prelambda congruences and we are somehow forced to consider every combination of variables in our \( (\eta_e) \). But the actual power of \( (\eta_e) \) is quite remarkable: it allows us to derive our condition of \( \alpha \)-renaming \( (\alpha_e) \) in every extensional congruence, as we show in Theorem 6.6.

We now survey some conditions of \( \eta \)-extensionality considered in the literature which do not mention any ancillary concept:

\[(r_1) \quad \lambda x [[\lambda x A] yx] \sim [\lambda x A] y, \quad \text{provided } x \neq y;\]
(ε₂) \( y \sim λxA \) for some term A, provided \( x \neq y \);
(ε₃) \( i \sim 1 \).

Up to inessential differences, conditions (ε₁) and (ε₂) are due to Salbria; the former is specifically employed in the axiomatization of the concept of extensional lambda abstraction algebra, while the latter is provably equivalent to (ε₁) [44, Def. 57 and Prop. 58]. Our condition (ηₑ) clearly resembles (ε₂).

Condition (ε₃) is often employed in model theoretical studies of the lambda calculus, again see [27, Lem. 4.2] and [44, Prop. 58]. Here i and 1 respectively denote the terms \( λy \lambda y \) and \( λy[λx][xy] \) for \( x \neq y \). Again, (ε₃) is specifically designed to work in settings where α-renaming is available.

As already said, in Theorem 6.3 we show that every extensional congruence is a lambda congruence. Thus, there is no reason to explicitly include condition (αₑ) in the axiomatization. The property that α-renaming can be derived from η-extensionality is not so well-known but it has been already noticed for the simply typed lambda calculus by Došen and Petrić [22, Sec. 3]. Indeed, we have taken account of their idea in our formalization of extensional congruence.

Since in Section 5 we shall show that extensional congruences precisely correspond to extensional theories, in the present section we do not give concrete examples of extensional congruences; rather, we concentrate on establishing the technical facts we need to show the aforementioned equivalence and on explaining the real significance of η-extensionality in our setting.

The first result we present is given in the second part of following proposition, where we show that the traditional formulation of η-extensionality holds every extensional congruence.

**Proposition 6.2.** Let \( \sim \) be an extensional congruence. Let A be a term and let \( y \) be a variable. Then, the following statements hold:

(i) Suppose that \( y \in \sim(A) \). Then, we have \( A \sim λy[Ay] \).

(ii) Suppose that \( y \notin \sim(A) \). Then, we have \( A \sim λy[Ay] \).

**Proof.** (i) Let \( x \) be a variable such that \( x \neq y \). Let \( B ≜ [λxx][λxy][Ay] \), \( C ≜ [λxy][Ay] \) and \( D ≜ λx[λx][xy] \). From (ε₂), we have \( C \sim B \). From (ε₁) and (ε₂) we obtain \( [λxx][λxy][Ay] \sim [λxy][Ay] \) from (i). Now, as \( x \neq y \), we have \( [λxy][Ay] \sim [λxy][Ay] \) from (ηₑ) and \( [λx][λx][xy] \sim [λx][λx][xy] \) from (ε₁). By (r), we get \( A \sim Ay \). So, by (a) we have \( [λxx][λxy][Ay] \sim DA \). Since \( x \neq y \) and \( y \in \sim(A) \), we obtain \( DA \sim λC \) by Proposition 4.6(i). Using (s), we have \( A \sim [λxx][λxy][Ay] \sim λC \sim λ[Ay] \) and we get \( A \sim λC[λAy] \) from (i).

(ii) It follows from (i) above, as we have \( y \in \sim(A) \) by Theorem 6.3.

In order to achieve all the goals we set in the beginning of this section, it only remains to prove that (ε₃) holds in every extensional congruence.

**Theorem 6.3.** Every extensional congruence is a lambda congruence.

**Proof.** Let \( \sim \) be an extensional congruence. Let \( A \) be a term, and let \( x \) and \( y \) be variables. Let \( B ≜ λx[A] \). We have to show that \( [λyxy][Ay] \sim A \) implies \( B \sim λy[By] \). Assume \( [λyxy][Ay] \sim A \). If \( x = y \), then we proceed as in Remark 5.2. If \( x \neq y \), then we have \( y \in \sim(A) \) and we obtain \( y \in \sim(B) \) by Lemma 4.3(iii). By Proposition 6.2(ii) we obtain \( B \sim λy[By] \).

In the following corollary we show that the set of all extensional congruences is properly included in the set of all prelambda congruences.

**Corollary 6.4.** There exists a prelambda congruence which is not an extensional congruence.

**Proof.** By Proposition 5.3 a prelambda congruence is not a lambda congruence and hence it cannot be an extensional congruence by Proposition 6.3.

Regarding the real significance of η-extensionality in our setting we now consider the following condition that we call extensionality for terms:

\( AD \sim BD \) for every term \( D \) implies \( A \sim B \).

This form of extensionality has been considered by many authors: for instance, in combinatory logic by Rosenbloom [43, p. 112] and in the lambda calculus by Wadsworth [54, p. 493], Hindley and Longo [27, p. 297] and Hindley and Seldin [26, p. 77]. While each condition of η-extensionality so far considered has a strong syntactical and artificial flavour and no apparent connection with any property of extensionality, the property of extensionality for terms above has a clear mathematical significance: two terms should be regarded as the same if they have the same input–output behaviour. We now prove that in every prelambda congruence extensionality for terms is equivalent to (ηₑ).

**Theorem 6.5.** Let \( \sim \) be a prelambda congruence. Then, \( \sim \) is an extensional congruence if and only if it satisfies the condition of extensionality for terms:

(ε₄) \( AD \sim BD \) for every term \( D \) implies \( A \sim B \),
for all terms \( A \) and \( B \).

**Proof.** Suppose that \( \sim \) is an extensional congruence. We now show that \( \sim \) satisfies (ε₄). For, we assume \( AD \sim BD \) for every term \( D \) and prove that \( A \sim B \).

Let \( z \in \sim(A) \cap \sim(B) \). We have \( Az \sim Bz \) and we get \( λz[Az] \sim λz[Bz] \) by (l). By Proposition 6.2(i), we get \( A \sim λz[Az] \) and \( B \sim λz[Bz] \). By (s) we have \( A \sim λz[Az] \sim λz[Bz] \sim B \) and we get \( A \sim B \) from (l).

Suppose now that \( \sim \) is a prelambda congruence which satisfies (ε₄). Let \( x \) and \( y \) be two variables such that \( x \neq y \). We now prove that \( y \sim λx[xy] \). Let \( D \) be an arbitrary term. We have \( [λxy][xy]D \sim [λxy][xy][Ay] \) by (β₁). By (β₂) and (β₃) we get \( [λxy][xy]D \sim yD \) and \( [λxy][xy]D \sim [λxy][xy]D \) by and using (i) we obtain \( yD \sim [λxy][xy]D \) for every term \( D \). Hence, we can apply (ε₄) to obtain \( y \sim λx[xy] \).
Again, observe that this equivalence is significant because, by Corollary 6.4, \( \sim_e \) does not hold in every prelambda congruence.

We can now give a proper conclusion to our informal discussion of terms as functions. Recall that for us a class of terms intended to play the role of functions must satisfy the condition of extensionality discussed in the Section 5, where we examined the case of abstractions as functions. By Theorem 6.5, in every extensional congruence we can think every term as a function and consider the condition \( \sim_e \) as a compact and equivalent way to express the property of extensionality for terms — this also gives a proper justification for the terminology “\( \eta \)-extensionality” employed for our condition \( \sim_e \).

### 7 Lambda and Extensional Theories

Our next step is to prove that lambda and extensional congruences precisely correspond to lambda and extensional theories, respectively. In this section, we recall the traditional definitions of these theories and prove some preliminary properties that we need to prove our results.

Our definitions of lambda and extensional theories are perfectly in line with the ones which can be found in the literature; see, e.g., Meyer [33, p. 92].

In order to provide the axiomatizations, we need to formalize the notion of substitution first. To this aim, we point out that there exist several different definitions of substitution in the literature. A typical approach is to proceed as in the following remark.

**Remark 7.1.** The definition of substitution which is often found in the literature is the one given by Hindley and Seldin [26, Def. 1.12]. It is actually a variation of the one proposed by Curry and Feys [19] and it is commonly called capture–free substitution.

We now recall its definition in order to discuss and emphasize some of its aspects. Expressed in our notation, it can be formulated as follows:

1. \( \{D/x\}(x) \equiv D; \)
2. \( \{D/x\}(y) \equiv y, \text{ if } x \neq y; \)
3. \( \{D/x\}(AB) \equiv \{D/x\}(A)\{D/x\}(B); \)
4. \( \{D/x\}(\lambda yA) \equiv \lambda y\{D/x\}(A); \)
5. \( \{D/x\}(\lambda yA) \equiv \lambda y\{D/x\}(A), \) if \( x \neq y \) and \( x \) does not occur free in \( A; \)
6. \( \{D/x\}(\lambda yA) \equiv \lambda y\{D/x\}(A), \) if \( x \neq y \) and \( x \) occurs free in \( A \) and \( y \) does not occur free in \( D; \)
7. \( \{D/x\}(\lambda yA) \equiv \lambda z\{D/x\}((z/y)(A)), \) if \( x \neq y \), \( x \) occurs free in \( A \) and \( y \) occurs free in \( D, \) where \( z \) is the first variable in which does not occur free in both \( A \) and \( D. \)

Despite its complicated axiomatization, substitution as defined above has the following pleasant property: the usual \( \beta \)-rule of the lambda calculus can be simply expressed as \( [\lambda xA]D \sim \{D/x\}(A), \) without any restriction. However, there are also some disadvantages in considering the above formalization.

Firstly, the definition above is not given by induction on the structure of terms. The reason is that in condition (7) above there is \( (z/y)(A) \) and not \( A \) in the right hand side \( \lambda z\{D/x\}((z/y)(A)) \). In fact, the previous definition is by induction on the size of terms. So, in order to define substitution in this way, we need first a definition of the concept of size of a term. Furthermore, we need to prove that the previous construction is well–defined and to do so, it is also necessary to prove that \( (z/y)(A) \) and \( A \) have the same size.

Secondly, in condition (7) above, it is assumed that the set of variables is equipped with an appropriate ordering. As already said in Section 2, we do not want to assume this inasmuch as there are other ways to define substitution. △

In this article, we follow the simpler and more elegant approach to substitution developed by Barendregt in his dissertation [4]. (It should not be confused with the more familiar one developed by Barendregt in his classic book [5].)

**Definition 7.2 (Substitution).** Let \( A \) and \( D \) be terms, and let \( x \) be a variable. By induction on the structure of \( A \), we define the term \( \langle D/x\rangle(A) \) as follows:

1. \( \langle D/x\rangle(x) \equiv D; \)
2. \( \langle D/x\rangle(y) \equiv y, \text{ if } x \neq y; \)
3. \( \langle D/x\rangle(BC) \equiv \langle D/x\rangle(B)\langle D/x\rangle(C); \)
4. \( \langle D/x\rangle(\lambda yB) \equiv \lambda y\langle D/x\rangle(B), \text{ if } x \neq y. \)

We call the term \( \langle D/x\rangle(A) \) the result of the substitution of \( x \) for \( D \) in \( A. \)

In order to define the concepts of lambda and extensional theory we now formally define the notion of variable occurring bound in a term. However, for the sake of conformity with our formal treatment of free variables, we prefer to define the set of variables which do not occur bound in a given term instead.

**Definition 7.3 (Non–bound variable).** Let \( A \) be a term. We define the cofinite set of variables \( \mathcal{B}(A) \) by induction on the structure of \( A \) as follows:

- \( \mathcal{B}(x) \equiv \forall; \)
- \( \mathcal{B}(BC) \equiv \mathcal{B}(B) \cap \mathcal{B}(C); \)
- \( \mathcal{B}(\lambda yB) \equiv \mathcal{B}(B) \setminus \{x\}. \)

We also say that a variable \( x \) does not occur bound in \( A \) if \( x \in \mathcal{B}(A). \)

Having now properly defined the necessary ancillary concepts, we are finally ready to define the notions of lambda and extensional theory.

**Definition 7.4 (Lambda and extensional theory).** Let \( \sim \) be a congruence. We say that \( \sim \) is a lambda theory if it also satisfies the following conditions:

1. \( [\lambda xA]D \sim \{D/x\}(A), \) provided \( \mathcal{B}(A) \cup \mathcal{B}(D) = \forall; \)
2. \( \lambda yA \sim \lambda y(y/x)(A), \) provided \( y \in \mathcal{B}(A) \cap \mathcal{B}(A); \)
where $A$ and $D$ are arbitrary terms, and $x$ and $y$ are arbitrary variables.

Let $\sim$ be a lambda theory. We say that $\sim$ is an **extensional theory** if it also satisfies the following condition:

1. $A \sim \lambda y[Ay]$, provided $y \in \mathcal{I}(A)$;

where $A$ is an arbitrary term and $y$ is an arbitrary variable.

△

Thus, a lambda theory is any binary relation on terms which simultaneously satisfies all structural conditions and conditions $(\beta)$ and $(\alpha)$ given above. If, in addition, the relation is also closed under $(\eta)$, then it is an extensional theory.

The most important examples of lambda and extensional theories are **lambda conversion** and **extensional conversion**.

**Definition 7.5** (Lambda and extensional conversion). We call **lambda conversion** the lambda theory inductively defined by structural conditions, $(\beta)$ and $(\alpha)$. We also define **extensional conversion** as the extensional theory inductively defined by structural conditions, $(\beta)$, $(\alpha)$ and $(\eta)$.

△

Let us now discuss the conditions of Definition [7.4] in some detail.

Conditions $(\beta)$ and $(\alpha)$ are as in Barendregt [4, p. 4]. Note that $(\beta)$ presents a restriction on its applicability, namely $\mathcal{I}(A) \cup \mathcal{I}(D) = \mathcal{V}$. In the original formulation, this restriction is equivalently expressed follows: bound variables of $A$ and free variables of $D$ are disjoint sets. In this paper, instead of putting complications directly inside the definition of substitution (cf. the capture–free substitution of Remark[7.1]), we prefer to have a simple notion of substitution at the price of this restriction. In the next example we can see how the proviso $\mathcal{I}(A) \cup \mathcal{I}(D) = \mathcal{V}$ works.

**Example 7.6.** Let $\sim$ be a lambda theory. Let $x$ and $y$ be variables such that $x \neq y$. We have $(y/x)\lambda x y) = \lambda y y/x)(x) = \lambda y y$. However, we cannot infer $\lambda x \lambda x y y) \sim \lambda y y$ directly from $(\beta)$, as we have $\mathcal{I}(\lambda x) \cup \mathcal{I}(y) = (\mathcal{I}(x) \setminus \{y\}) \cup (\mathcal{V} \setminus \{y\}) = (\mathcal{V} \setminus \{y\}) \cup \mathcal{V} \setminus \{y\} = \mathcal{V} \setminus \{y\} \neq \mathcal{V}$. △

As for $(\alpha)$, when compared with other formalizations of $\alpha$–renaming that one usually finds in the literature, the main difference lies in the restriction $y \in \mathcal{I}(A) \cap \mathcal{I}(A)$ which is not just the “usual” proviso $y \in \mathcal{I}(A)$ (that is, $y$ does not occur free in $A$). The next example explains the situation.

**Example 7.7.** Let $\sim$ be a lambda theory. Let $x$ and $y$ be variables such that $x \neq y$. As before, we have $(y/x)\lambda x y) = \lambda y y/x)(x) = \lambda y y$. From this, it follows that $\lambda y y/x)(x) \sim \lambda y y$. Since $x \neq y$, we have $y \in \mathcal{V} \setminus \{x\} = (\mathcal{V} \setminus \{x\}) \cup \{y\} = \mathcal{I}(y)$. However, we cannot infer $\lambda x \lambda y y) \sim \lambda y y$ directly from $(\alpha)$, as we have $y \notin \mathcal{V} \setminus \{y\} = \mathcal{I}(x) \setminus \{y\} = \mathcal{I}(y)$ and in particular $y \notin \mathcal{I}(y) \cap \mathcal{I}(y)$.

Notice that if we use the capture–free substitution of Remark[7.1], then, in order to calculate $(y/x)(\lambda x)(y)$, we are forced to use condition $(\eta)$ as we have $x \neq y$, $x \notin \mathcal{I}(y)$ and $y \notin \mathcal{I}(y)$. In this case, we obtain $(y/x)(\lambda x)(y) = \lambda z \lambda x)(z/y)(y) = \lambda z \lambda x)(z/x)(x) = \lambda z y$ where $z$ is the first variable in $\mathcal{V} \setminus \{y\} = \mathcal{I}(x) \cap \mathcal{I}(y)$. From this, it follows that $\lambda y y/x)(x) \sim \lambda y y$ and we can infer $\lambda x \lambda y y) \sim \lambda y y$ by using $(\alpha)$ with the “usual” restriction $y \notin \mathcal{I}(A)$.

△

Finally, we note that condition $(\eta)$ is exactly as in the literature.

As already pointed out, Došen and Petrić [22] observed that it is not necessary to include $\alpha$–renaming in the axiomatization extensional theories. Indeed, we followed their idea in our formalization of extensional congruences. However, we do not take their observation into account in the present context because traditional formulations do include an explicit condition of $\alpha$–renaming. Also, even if condition $(\alpha)$ were removed from the definition, other ancillary concepts would still be present.

In order to reach our goals, we now prove a series of preliminary results.

First, we show that every lambda theory satisfies the condition of extensionality for abstractions discussed in Section [6].

**Proposition 7.8.** Let $\sim$ be a lambda theory. Let $\Lambda A$ and $\lambda y B$ be terms and suppose that $[\Lambda x A]D \sim [\lambda y B]D$ for every term $D$. Then, we have $\lambda x A \sim \lambda y B$.

**Proof.** Let $z \in \mathcal{I}(\Lambda x A) \cap \mathcal{I}(\lambda y B) \cap \mathcal{I}(\Lambda x A) \cap \mathcal{I}(\lambda y B)$. Then, we have $[\Lambda x A]z \sim [\lambda y B]z$. Since $\mathcal{I}(z) = \mathcal{V} \setminus \{z\}$, $z \in \mathcal{I}(\Lambda x A)$ and $z \in \mathcal{I}(\lambda y B)$, we have $\mathcal{I}(\Lambda x A) \cup \mathcal{I}(z) = \mathcal{V}$ and $\mathcal{I}(\lambda y B) \cup \mathcal{I}(z) = \mathcal{V}$. So, we can apply $(\beta)$ and we obtain $\mathcal{I}(\Lambda x A) \sim (z/x)(\Lambda x A)$ and $\mathcal{I}(\lambda y B) \sim (z/y)(\lambda y B)$. By using $(\alpha)$ we obtain $(z/x)(\Lambda x A) \sim (\Lambda x A)z \sim (z/y)(\lambda y B)$ and so $\lambda z (z/x)(\Lambda x A) \sim \lambda z (z/y)(\lambda y B)$ from $(\alpha)$ and $(\beta)$. Now, since $z \in \mathcal{I}(\Lambda x A) = \mathcal{I}(A) \setminus \{x\}$ we obtain $z \in \mathcal{I}(A)$ and $x \neq z$. Similarly, as $z \in \mathcal{I}(\lambda y B) = \mathcal{I}(B) \setminus \{y\}$ we have $z \in \mathcal{I}(B)$ and $y \neq z$. As $z \in \mathcal{I}(\Lambda x A) = \mathcal{I}(A) \setminus \{x\}$ and $x \neq z$, it follows that $z \in \mathcal{I}(A)$. Analogously, as $z \in \mathcal{I}(\lambda y B) = \mathcal{I}(B) \setminus \{y\}$ and $y \neq z$, we obtain $z \in \mathcal{I}(B)$. Hence, we can now apply $(\alpha)$ to obtain $\lambda z (z/x)(\Lambda x A) \sim \lambda z (z/y)(\lambda y B)$ and finally $\lambda z A \sim \lambda z B$ from $(\alpha)$.

We now establish other technical properties.

**Lemma 7.9.** Let $A$ be a term, and let $y$ and $w$ be variables. Then, we have $\mathcal{I}(w/y)(A) \sim \mathcal{I}(A)$.

**Proof.** We reason by induction on the structure of $A$.

Suppose that $A = z$. If $y = z$, then we have $\mathcal{I}(w/y)(A) = \mathcal{I}(w) = \mathcal{V} = \mathcal{I}(A)$. If $y \neq z$, then we have $\mathcal{I}(w/y)(z) = \mathcal{I}(z)$.

Suppose that $A = BC$. By inductive hypothesis, we have $\mathcal{I}(w/y)(B) = \mathcal{I}(B)$ and $\mathcal{I}(w/y)(C) = \mathcal{I}(C)$. Then, it follows that $\mathcal{I}(w/y)(BC) = \mathcal{I}(w/y)(B/C) = \mathcal{I}(w/y)(B) \cap \mathcal{I}(w/y)(C) = \mathcal{I}(B) \cap \mathcal{I}(C) = \mathcal{I}(BC)$.

Finally, suppose that $A = \lambda z B$. If $y = z$, then we have $\mathcal{I}(w/y)(\lambda z B) = \mathcal{I}(w/y)(B) = \mathcal{I}(B) = \mathcal{I}(\lambda z B)$. If $y \neq z$, then, by inductive hypothesis, we have $\mathcal{I}(w/y)(B) = \mathcal{I}(B)$. Hence, it follows that $\mathcal{I}(w/y)(\lambda z B) = \mathcal{I}(w/y)(B) = \mathcal{I}(w/y)(B) \setminus \{z\} = \mathcal{I}(B) \setminus \{z\} = \mathcal{I}(\lambda z B)$.

**Proposition 7.10.** Let $\sim$ be a lambda theory, and let $A$ and $B$ be terms. Then, there exists a term $G$ such that $G \sim A$, and $\mathcal{I}(G) \cup \mathcal{I}(D) = \mathcal{V}$.

**Proof.** The proof is by induction on the structure of $A$.

Suppose that $A = x$. Let $G = \lambda x x$. Then, we have $G \sim x$ by $(\eta)$. Since $\mathcal{I}(G) = \mathcal{V}$, we have $\mathcal{I}(G) \cup \mathcal{I}(D) = \mathcal{V}$. (r)
Suppose that $A = BC$. By inductive hypothesis there exist $M$ and $N$ such that $M \sim B$ and $\theta(M) \cup \eta(D) = \emptyset$, as well as $N \sim C$ and $\theta(N) \cup \eta(D) = \emptyset$. Let $G \equiv MN$. Then, we have $G \sim BC$ by ($\alpha$) and $\theta(G) \cup \eta(D) = (\theta(M) \cup \eta(N)) \cup \eta(D) = (\theta(M) \cup \eta(D)) \cap (\theta(N) \cup \eta(D)) = \emptyset \lor \emptyset = \emptyset$.

Finally, suppose that $A = \lambda z B$. By inductive hypothesis, there exists a term $M$ such that $M \sim B$ and $\theta(M) \cup \eta(D) = \emptyset$. Let $w \in \theta(M) \cap \eta(D)$ and let $G \equiv \lambda w (w/y)(M)$. We have $\lambda y M \sim \lambda y B$ from ($\theta$) and from ($\alpha$) it follows that $\lambda y M \sim G$, as $w \in \eta(M) \cap \eta(M)$. By using ($\alpha$), we obtain $G \sim \lambda y M \sim \lambda y B$. Hence, we have $G \sim \lambda y B$ from ($\theta$). In order to finish the proof, we now show that $\theta(G) \cup \eta(D) = \emptyset$. For, we have $\theta(G) = \theta((w/y)(M)) \setminus \{w\}$. By Lemma 7.9, it follows that $\theta((w/y)(M)) = \theta(M)$. Hence, we get $\theta(G) \cup \eta(D) = (\theta(M) \setminus \{w\}) \cup \eta(D) = \emptyset \cup \eta(D)$. Now, observe that $(\theta(M) \setminus \{w\}) \cup \eta(D) = \emptyset (M \cup \eta(D))$, as $w \in \eta(D)$. From this, we obtain $\theta(G) \cup \eta(D) = \emptyset$ by the inductive hypothesis.

**Lemma 7.11.** Let $A$ and $D$ be terms, and let $x$ and $y$ be variables such that $x \neq y$. Then, we have $\langle D/x \rangle (\langle y/x \rangle (A)) = (\langle y/x \rangle (A))$.

**Proof.** We proceed by induction on the structure of $A$.

Suppose that $A = z$. If $x = z$, then we have $\langle D/x \rangle (\langle y/x \rangle (y)) = \langle D/x \rangle (y) = (y/x)(x)$, as $x \neq y$. If $x \neq z$, then it follows that $\langle D/x \rangle (\langle y/x \rangle (z)) = \langle D/x \rangle (z) = z$.

Suppose that $A = \lambda z B$. By inductive hypothesis, we have $\langle D/x \rangle (\langle y/x \rangle (B)) = (y/x)(B)$ and $\langle D/x \rangle (\langle y/x \rangle (C)) = (y/x)(C)$. Then, we get $\langle D/x \rangle (\langle y/x \rangle (BC)) = \langle D/x \rangle (\langle y/x \rangle (B))(\langle y/x \rangle (C)) = \langle D/x \rangle (\langle y/x \rangle (B))(\langle y/x \rangle (C)) = (y/x)(B)(y/x)(C) = (y/x)(BC)$.

Finally, suppose that $A = \lambda z B$. If $x = z$, then we have $\langle D/x \rangle (\langle y/x \rangle (\lambda z B)) = \langle D/x \rangle (\lambda z B) = (y/x)(\lambda z B)$. If $x \neq z$, then, by inductive hypothesis, we have $\langle D/x \rangle (\langle y/x \rangle (\lambda z B)) = (y/x)(B)$. Then, we obtain $\langle D/x \rangle (\langle y/x \rangle (\lambda z B)) = \langle D/x \rangle (\lambda z (y/x)(B)) = \lambda z (y/x)(B) = (y/x)(\lambda z B)$.

**8 Lambda and Extensional Theories Revisited**

In this section, we finally prove that the notions of lambda and extensional congruence are respectively equivalent to the concepts of lambda and extensional theory. To begin with, we show in the next proposition that prelambda congruences behave well with respect to condition ($\beta$).

**Proposition 8.1.** Let $\sim$ be a prelambda congruence. Let $x$ be a variable, and let $A$ and $D$ be terms such that $\theta(A) \cup \eta(D) = \emptyset$. Then, we have $[\lambda x A]D \sim (D/x)(A)$.

**Proof.** We reason by induction on the structure of $A$.

Suppose that $A = y$. If $x = y$, then we have $\langle D/x \rangle (y) = D$ and it follows that $\lambda x x D \sim D$ from ($\beta$). If $x \neq y$, then we have $\langle D/x \rangle (y) = y$ and we obtain $[\lambda x y]D \sim y$ by ($\beta$).

Suppose that $A = BC$. Since $\theta(A) \subseteq \theta(B)$, $\theta(A) \subseteq \theta(C)$ and $\theta(A) \cup \eta(D) = \emptyset$, we have $\theta(B) \cup \eta(D) = \emptyset$ and $\theta(C) \cup \eta(D) = \emptyset$. Hence, by inductive hypothesis, it follows that $[\lambda x B]D \sim (D/x)(B)$ and $[\lambda x C]D \sim (D/x)(C)$. From this, we get $[[\lambda x B]D][[\lambda x C]D] \sim (D/x)(B)(D/x)(C)$ from ($\alpha$). Now, by ($\beta$), we have $[\lambda x B]D \sim [[\lambda x B]D][[\lambda x C]D]$, and we obtain $[\lambda x B]D \sim (D/x)(B)(D/x)(C)$ from ($\beta$). As $x \neq y$, we obtain $[\lambda y B]D \sim (D/x)(B)$ by Proposition 4.6(ii). Since $\lambda x A \sim \lambda y B$ by ($\beta$), we have $\lambda x A \sim \lambda y B$ by Propostion 4.6(ii). Since $\lambda x A \sim \lambda y B$ by ($\beta$), we have $\lambda y (D/x)(B)$.

Finally, suppose that $A = \lambda z B$. If $x = z$, then we have $\langle D/x \rangle (\langle y/x \rangle (\lambda z B)) = \langle D/x \rangle (\lambda z B) = \lambda z (y/x)(B)$. If $x \neq z$, then, by inductive hypothesis, we have $\langle D/x \rangle (\langle y/x \rangle (\lambda z B)) = (y/x)(B)$. Then, we obtain $\langle D/x \rangle (\langle y/x \rangle (\lambda z B)) = \langle D/x \rangle (\lambda z (y/x)(B)) = \lambda z (y/x)(B) = (y/x)(\lambda z B)$.

**8.3 Every lambda theory is a lambda congruence.**

**Proof.** Let $\sim$ be a lambda congruence. In order to prove that $\sim$ is a lambda theory we only have to show that $\sim$ satisfies conditions ($\beta$) and ($\alpha$).

($\beta$) Suppose that $\theta(A) \cup \eta(D) = \emptyset$. Since every lambda congruence is also a prelambda congruence, we obtain $[\lambda x A]D \sim (D/x)(A)$ by Proposition 8.1.

($\alpha$) Suppose that $y \in \eta(A) \cap \eta(B)$. As $y \in \eta(A)$, we have $\lambda x A \sim \lambda y (\lambda x A)$ by Proposition 5.5(ii). As $y \in \eta(A)$ and $\theta(A) = \emptyset$, we have $\lambda x A \cup \eta(A) = \emptyset$. By Proposition 8.1, it follows that $\lambda x A \sim \lambda y (\lambda x A)$, and we get $[\lambda x A]B \sim \lambda y (\lambda x A)(B)$ from ($\beta$). Then, we conclude that $\lambda x A \sim \lambda y (\lambda x A)(B)$ from ($\beta$).
Every extensional theory is an extensional congruence and so we get \( \lambda x \lambda y G \sim \lambda y \lambda x G \) by \((\eta)\). Hence, we conclude that \( \lambda x \lambda y G \sim \lambda y \lambda x \lambda A D \) from \((I)\).

\((\epsilon_e)\) This condition holds by Proposition 7.8. \(\square\)

**Proposition 8.4.** Every extensional congruence is an extensional theory.

**Proof.** Let \( \sim \) be an extensional congruence. In order to prove that \( \sim \) is an extensional theory we have to show that \( \sim \) satisfies conditions \((\beta)\), \((\alpha)\) and \((\eta)\). By Theorem 8.6 we know that \( \sim \) is a lambda congruence and so by Proposition 8.2 it satisfies \((\beta)\) and \((\alpha)\). As for \((\eta)\), it holds by Proposition 6.2 (ii). \(\square\)

**Proposition 8.5.** Every extensional theory is an extensional congruence.

**Proof.** Let \( \sim \) be an extensional theory. In order to prove that \( \sim \) is an extensional congruence we have to show that \( \sim \) satisfies all beta conditions and \((\eta)\). Since, by definition, \( \sim \) is also a lambda theory, we know by Proposition 8.3 that \( \sim \) satisfies all beta conditions. As for \((\eta)\), suppose that \( x \not\sim y \). Then, as \( x \in \{y \mid y = \eta(y)\} \), we obtain \( y \sim \lambda x \lambda y \) by \((\eta)\). \(\square\)

**Theorem 8.6.** The concepts of lambda and extensional congruences are respectively equivalent to the notions of lambda and extensional theory.

**Proof.** By the previous four propositions. \(\square\)

By using the above theorem, it is also possible to give simplified formalizations of lambda conversion and extensional conversion as we now explain.

Recall that lambda conversion is the lambda conversion inductively defined by using structural conditions, \((\beta)\) and \((\alpha)\). In other words, lambda conversion is the intersection of all lambda theories. By Theorem 8.6 the set of all lambda theories is equal to the set of all lambda congruences. Thus, lambda conversion is also the intersection of all lambda congruences. In particular, lambda conversion can be equivalently characterized as the relation inductively defined by using all structural and beta conditions together with \((\alpha)\).

By replacing \((\alpha)\) by \((\eta)\) in the discussion above, a simplified axiomatization of extensional conversion is similarly given.

### 9 The Prelambda Congruence \(\sim_I\)

The aim of this section is to show that there exists a prelambda congruence \(\sim_I\) such that for all distinct variables \(x\) and \(y\) we have \(\lambda x x \not\sim_I \lambda y y\). In other words, we now prove Theorem 5.3 whose proof was omitted in Section 5.

This is not an easy task, though. The reason is that the defining conditions of prelambda congruence only deal with “provability” while our aim is to show the “unprovability” of \(\lambda x x \sim_I \lambda y y\), that is \(\lambda x x \not\sim_I \lambda y y\).

A reassuring fact is that similar situations frequently happen in logic and in that setting well–known methods to solve these kinds of problems are available. Consider for instance intuitionistic theories and the law of excluded middle — for the purposes of this discussion an intuitionistic theory is just a set of formulas which is closed under the rules of intuitionistic logic. We say that an intuitionistic theory is classical if it contain every instance of excluded middle. Of course, there are several intuitionistic theories which are classical: classical logic is the primary example.

Suppose that we want to show that there exists an intuitionistic theory which is not classical. In order to do this, it suffices to construct a model \(M\) and prove that the set of formulas which are valid in \(M\) forms a non–classical intuitionistic theory. Note that \(M\) cannot be a standard model of classical logic — excluded middle never fails there — but it has to be a model specifically designed to this aim, such as a topological model or a Kripke model; see, e.g., Sørensen and Urzyczyn [47, Ch. 2].

Now, if we think of prelambda congruences as intuitionistic theories and “\(\lambda x x \sim_I \lambda y y\) for all \(x\) and \(y\)” as the law of excluded middle, then the situation is strikingly similar to the one described above. Hence, in order to solve our
problem we now build a kind of model with the required properties. Since all models considered in the literature of the lambda calculus have been conceived for the purpose of \textit{validating} (not \textit{refuting}) \(^\lambda x x \sim \lambda y y\) for all \(x\) and \(y\), our construction, to the best of our knowledge, seems to be new.

We now introduce some notation. Let \(A\) and \(B\) be sets. We write \(B \subseteq_f A\) to express the fact that \(B\) is a finite subset of \(A\). We also write \(P(A)\) and \(P_f(A)\) for the set of all subsets and finite subsets of \(A\), respectively. Furthermore, we write \(A \times B\) to denote the Cartesian product of \(A\) and \(B\).

In order to construct our model we now introduce the notion of \textit{formula}.

\textbf{Definition 9.1} (Formula). Let \(\mathfrak{F}\) be the set inductively defined as follows:

\begin{enumerate}[(\(\mathfrak{F}_1\))]
\item \(\mathfrak{F}_1\) \(x \in \mathcal{V}\) implies \(x \in \mathfrak{F}\);
\item \(F \subseteq_f \mathfrak{F}\) and \(\ell \in \mathfrak{F}\) imply \((F, \ell) \in \mathfrak{F}\).
\end{enumerate}

We call \textit{formula} any element of the set \(\mathfrak{F}\). Henceforth, we use \(F, G, H,\ldots\) and \(\ell, m, n,\ldots\) to denote finite subsets and elements of \(\mathfrak{F}\), respectively. We also use the expression \(F \vdash \ell\) as an alternative notation for the ordered pair \((F, \ell)\).

In particular, we write \(G \vdash F \vdash \ell\) for \((G, (F, \ell))\).

Let \(\ell\) be a formula. We say that \(\ell\) is a \textit{atomic formula} if \(\ell \in \mathcal{V}\); equivalently, if it is of the form \(F \vdash m\).

We now observe that our set \(\mathfrak{F}\) belongs to the class of structures called \textit{graph algebras}, in the terminology of Engeler [23]. In particular, \(\mathfrak{F}\) corresponds to the full graph algebra \(G(\mathcal{V})\) built on the set of variables \(\mathcal{V}\). For our aims the fact that \(\mathfrak{F}\) is specifically constructed out of the set of variables \(\mathcal{V}\) turns out to be essential, as we make clear later.

Graph algebras are also known as Engeler models and PSE–algebras in the literature; see, e.g., Meyer [33], Longo [31], Barendregt [5], Krivine [30], Plotkin [39], Berline [3, 7] and Hindley and Seldin [26]. Regarding our terminology, we refer to elements \(\mathfrak{F}\) as \textit{formulas} because a similar terminology is employed in [30] for similar structures.

Our aim is to interpret terms using the structure we have just defined. In order to do so, it is necessary to introduce the concepts of environment and update first.

\textbf{Definition 9.2} (Environment, update). We call \textit{environment} any function \(\rho\) from \(\mathcal{V}\) to \(P_f(\mathfrak{F})\). In the sequel, we use \(\sigma, \rho, \tau, \ldots\) to denote environments. We also denote the set of all environments by \(\mathcal{E}\).

Let \(\sigma\) be an environment. Let \(x \in \mathcal{V}\) and let \(F \subseteq_f \mathfrak{F}\). We denote by \(\{F/x\}\sigma\) the environment given by:

\[
\{F/x\}\sigma(y) = \begin{cases} 
F & \text{if } x = y \\
\sigma(y) & \text{if } x \neq y,
\end{cases}
\]

for \(y \in \mathcal{V}\).

We call \(\{F/x\}\sigma\) an \textit{update of} \(\sigma\).

Let \(x\) and \(y\) be variables, and let \(F\) and \(G\) in \(P_f(\mathfrak{F})\). Let \(\sigma\) be an environment. If \(\rho = \{F/x\}\sigma\) and \(\tau = \{G/y\}\rho\), then we also write \(\tau\) as \(\{G/y\}\{F/x\}\sigma\).

Updates obey the algebraic laws that we show in the next lemma.

\textbf{Lemma 9.3.} Let \(\sigma\) be an environment. Let \(x\) and \(y\) be variables such that \(x \neq y\), and let \(F\) and \(G\) be finite subsets of \(\mathfrak{F}\). Then, the following properties hold:

\begin{enumerate}[(i)]
\item \(\{G/x\}\{F/x\}\sigma = \{G/x\}\sigma\);
\item \(\{G/y\}\{F/x\}\sigma = \{F/x\}\{G/y\}\sigma\).
\end{enumerate}

\textbf{Proof.} (i) We have \(\{G/x\}\{F/x\}\sigma(z) = \{G/x\}\sigma(z)\). Let \(z\) be a variable such that \(x \neq z\). Then, we have \(\{G/x\}\{F/x\}\sigma(z) = \{F/x\}\sigma(z)\).

(ii) Since \(x \neq y\), we have \(\{G/y\}\{F/x\}\sigma(x) = \{F/x\}\sigma(x)\) and \(\{G/y\}\{F/x\}\sigma(y) = \{G/y\}\sigma(y)\) if \(x \neq z\) and \(y \neq z\). Then, we have \(\{G/y\}\{F/x\}\sigma(z) = \{F/x\}\sigma(z)\).

We now define our interpretation of terms. As is standard for graph algebras, each term is interpreted via an environment as a subset of formulas.

\textbf{Definition 9.4} (Interpretation of terms in \(P(\mathfrak{F})\)). We define the function \(I\) from \(\mathcal{T} \times \mathcal{E}\) to \(P(\mathfrak{F})\) by induction on the structure of terms as follows:

\begin{enumerate}[(I_1)]
\item \(I(x, \sigma) \equiv \sigma\{x\}\);
\item \(I(AB, \sigma) \equiv \{F \mid \text{there exists } F \subseteq_f I(B, \sigma) \text{ such that } F \vdash \ell \in I(A, \sigma)\};
\item \(I(\lambda x A, \sigma) \equiv \{F \vdash m \mid m \in I(A, \{F/x\}\sigma) \cup \{x\}\}\).
\end{enumerate}

Let \(A\) and \(B\) be terms. We write \(A \sim B\) and say that \(A\) and \(B\) have the same \textit{interpretation} in \(P(\mathfrak{F})\) if \(I(A, \sigma) = I(B, \sigma)\) for every environment \(\sigma\). We also call the relation \(\sim\) the \textit{theory} of \(P(\mathfrak{F})\).

While our set \(\mathfrak{F}\) is just an example of graph algebra, our interpretation of terms in \(P(\mathfrak{F})\) is definitely non-standard, as we now explain.

Firstly, environments are usually taken as function from \(\mathcal{V}\) to \(P(\mathfrak{F})\) and not as functions from \(\mathcal{V}\) to \(P_f(\mathfrak{F})\) as we do in this paper. The reason is practical: to show our results we found it is not necessary consider environments having infinite sets of formulas in their range.

Secondly, following the standard interpretation of terms in graph algebras \(I(\lambda x A, \sigma)\) should be the set of formulas \(\{F \vdash m \mid m \in I(A, \{F/x\}\sigma)\}\) and not our set \(\{F \vdash m \mid m \in I(A, \{F/x\}\sigma) \cup \{x\}\}\). As a consequence of this fact, in our setting each set of the form \(I(\lambda x A, \sigma)\) contains compound formulas of the form \(F \vdash m\) and \textit{exactly one} atomic formula, namely \(x\). This formula \(x\) gives us a “tag” for the “\(x\)” in the interpretation of \(\lambda x A\) and the presence of this unique atomic formula turns out to be crucial for our purposes. For this reason, the fact that variables are also formulas is very important in our setting.

We now show two lemmas that we need during the proof of Theorem 9.7.

\textbf{Lemma 9.5.} Let \(x\) be a variable and let \(A\) be a term. Let \(F\) and \(G\) finite subsets of formulas such that \(F \subseteq G\). Then, for every environment \(\sigma\) we have \(I(A, \{F/x\}\sigma) \subseteq I(A, \{G/x\}\sigma)\).

\textbf{Proof.} We reason by induction on the structure of \(A\). Let \(\sigma\) be an environment and let \(\ell\) be a formula.
Suppose that $A = y$. If $x = y$, then we have $I(x, \{F/x\} \sigma) = \{F/x\} \sigma(x) = F \subseteq G = \{G/x\} \sigma(x) = I(x, \{G/x\} \sigma)$. If $x \neq y$, then we have $I(y, \{F/x\} \sigma) = \{F/x\} \sigma(y) = \sigma(y) = I(y, \{G/x\} \sigma)$.

Suppose that $A = BC$. By inductive hypothesis, for every environment $\rho$ and every $D \subseteq \{B, C\}$ we have $I(D, \{F/x\} \rho) \subseteq I(D, \{G \rho\} \sigma)$. Assume $\ell \in I(BC, \{F/x\} \sigma)$. Then, there exists $H \subseteq I(C, \{F/x\} \sigma)$ such that $H \vdash \ell \in I(B, \{F/x\} \sigma)$. By inductive hypothesis, it follows that $H \subseteq I(C, \{G/x\} \sigma)$ and $H \vdash \ell \in I(B, \{G/x\} \sigma)$. From this, we conclude that $\ell \in I(BC, \{G/x\} \sigma)$.

Suppose that $A = \lambda y B$. Assume $\ell \in I(\lambda y B, \{F/x\} \sigma)$. Suppose that $x = y$. If $\ell = x$, then we obviously have $x \in I(\lambda x B, \{G/x\} \sigma)$. Otherwise, it follows that $\ell = H + m$ for some $H$ and $m$ such that $m \in I(B, \{H/x\} \{F/x\} \sigma)$. By using Lemma 9.3(i) it follows that $m \in I(B, \{H/x\} \{G/x\} \sigma)$ and we obtain $H + m \in I(\lambda x B, \{G/x\} \sigma)$. Suppose now that $x \neq y$. If $\ell = y$, then we clearly have $y \in I(\lambda y B, \{G/x\} \sigma)$. Otherwise, it follows that $\ell = H + m$ for some $H$ and $m$ such that $m \in I(B, \{H/y\} \{F/x\} \sigma)$. By inductive hypothesis, for every environment $\rho$ we have $I(B, \{F/x\} \rho) \subseteq I(B, \{G/x\} \rho)$. From this and Lemma 9.3(ii) we obtain $m \in I(B, \{H/y\} \{F/x\} \sigma) = I(B, \{F/x\} \{H/y\} \sigma) \subseteq I(B, \{G/x\} \{H/y\} \sigma) = I(B, \{H/y\} \{G/x\} \sigma)$, as $x \neq y$. Therefore, we get $H + m \in I(\lambda x B, \{G/x\} \sigma)$.

Lemma 9.6. Let $x$ and $z$ be variables such that $x \neq z$, and let $A$ be a term. Then, $I([\lambda x A]z \sim A, \{\}).$ Suppose that $\lambda x A[z \sim A, \{\}$ and $I(F/x) \sigma = I(A, \{F/x\} \sigma)$ for every environment $\sigma$.

Proof. Let $\sigma$ be an environment and let $\ell$ be a formula. As $\lambda x A[z \sim A, \sigma = I(\lambda x A, \sigma) = I(\lambda x A, \{F/x\} \sigma) = I(A, \{F/x\} \sigma)$. Assume $\ell \in I(\lambda x A, \{\}$. Then, there exists $G \subseteq I(\lambda x A, \sigma)$ such that $G \vdash \ell \in I(\lambda x A, \sigma)$ and $\ell \in I(\lambda x A, \{F/x\} \sigma)$. Since $x \neq z$, it follows that $G \subseteq I(\lambda x A, \{F/x\} \sigma)$ and hence $G \vdash \ell \in I(\lambda x A, \{F/x\} \sigma)$. Since $G \subseteq I(\lambda x A, \{F/x\} \sigma)$, we obtain $\ell \in I(\lambda x A, \{F/x\} \sigma) = I(\lambda x A, \{F/x\} \sigma).$ This shows that $\lambda x A[z \sim A, \{\}$. To show the opposite inclusion, assume $\ell \in I(\lambda x A, \{F/x\} \sigma)$. Then, there exists some $G \subseteq I(\lambda x A, \{F/x\} \sigma)$ such that $G \vdash \ell \in I(\lambda x A, \{F/x\} \sigma)$ and $\ell \in I(\lambda x A, \{F/x\} \sigma)$. By using Lemma 9.3(i), we get $\ell \in I(\lambda x A, \{F/x\} \sigma)$.

We can now prove the consistency of our model is a prelambda congruence.

Theorem 9.7. The relation $\sim_1$ is a prelambda congruence.

Proof. All we have to do is to check that $\sim_1$ satisfies all structural and beta conditions. Let $\sigma$ be an environment and let $\ell$ be a formula.

As for $(\iota)$, $(\iota)$ and $(\iota)$, let $\rho$ be an environment. Since $I(D, \rho)$ is a set for each term $D$, the following conditions hold: $I(A, \rho) = I(A, \rho)$, $I(A, \rho) = I(B, \rho)$ implies $I(B, \rho) = I(A, \rho) = I(B, \rho) = I(C, \rho)$ imply $I(A, \rho) = I(B, \rho) = I(C, \rho)$. From this observation, the fact that $\sim_1$ satisfies $(\iota)$, $(\iota)$ and $(\iota)$ follows.

(i) Suppose that $I(A, \rho) = I(B, \rho)$ for every environment $\rho$. We have to show that $I(\lambda x A, \sigma) = I(\lambda x B, \sigma)$. For $M \in \{A, B\}$, let

$$N \equiv \begin{cases} B & \text{if } M = A \\ A & \text{if } M = B. \end{cases}$$

Suppose that $\ell \in I(\lambda x M, \sigma)$. If $\ell = x$, then we clearly have $x \in I(\lambda x N, \sigma)$. Otherwise, it follows that $\ell = F + m$ for some $F$ and $m$ such that $m \in I(M, \{F/x\} \sigma)$. Since $I(A, \{F/x\} \sigma) = I(B, \{F/x\} \sigma)$, we have $I(M, \{F/x\} \sigma) = I(N, \{F/x\} \sigma)$ and hence $m \in I(N, \{F/x\} \sigma)$.

(a) Suppose that $I(A, \rho) = I(B, \rho)$ for every environment $\rho$ and $I(C, \tau) = I(D, \tau)$ for every environment $\tau$. We have to show that $I(\lambda x A, \sigma) = I(\lambda x B, \sigma)$.

(B) We have to show that $I(\lambda x x D, \sigma) = I(\lambda x x D, \sigma)$. Suppose that $\ell \in I(\lambda x x D, \sigma)$. Then, there exists $F \subseteq I(\lambda x x D, \sigma)$ such that $\ell = \ell \in I(M, \sigma)$.

Suppose that $\ell \in I(\lambda x x D, \sigma)$. Let $\ell \in I(\lambda x x D, \sigma)$. Then, there exists $F \subseteq I(\lambda x x D, \sigma)$ such that $\ell = \ell \in I(M, \sigma)$. Then, there exists $F \subseteq I(\lambda x x D, \sigma)$ such that $\ell = \ell \in I(M, \sigma)$ and hence $\ell \in I(\lambda x x D, \sigma)$. Since $F \subseteq I(\lambda x x D, \sigma)$, we obtain $\ell \in I(\lambda x x D, \sigma) = I(\lambda x x D, \sigma)$. This shows that $\ell \subseteq I(\lambda x x D, \sigma)$. To show the opposite inclusion, assume $\ell \in I(\lambda x x D, \sigma)$. Then, there exists some $G \subseteq I(\lambda x x D, \sigma)$ such that $G \vdash \ell \in I(\lambda x x D, \sigma)$ and $\ell \in I(\lambda x x D, \sigma)$. By using Lemma 9.3(i), we get $\ell \in I(\lambda x x D, \sigma)$.

We have to show that $I(\lambda x x D, \sigma) = I(\lambda x x D, \sigma)$. Since $x \neq y$, we have $\ell \in I(y, \{F/x\} \sigma) = \{F/x\} \sigma(y) = \sigma(y) = I(y, \{x\} \sigma)$. As for the opposite inclusion, assume now $\ell \in I(\lambda x x D, \sigma)$. Then, there exists some $G \subseteq I(\lambda x x D, \sigma)$ such that $G \vdash \ell \in I(\lambda x x D, \sigma)$ and $\ell \in I(\lambda x x D, \sigma)$. By using Lemma 9.3(i), we get $\ell \in I(\lambda x x D, \sigma)$.
We can now prove Theorem 9.3 as follows. By Theorem 9.7, the relation $\sim_1$ is a prelambda congruence. Now, let $x$ and $y$ be two variables such that $x \neq y$. Let $\sigma$ be an environment. We have $x \in \ell(\lambda x, \sigma)$ and $x \notin \ell(\lambda y, \sigma)$, as $x \neq y$ and $y$ is the only atomic formula in $I(\lambda y, \sigma)$. This shows that $\lambda x \not\sim_1 \lambda y$.

### 10 Conclusions and Some Directions for Future Work

In this work we have obtained alternative and simplified formulations of the concepts of lambda theory and extensional theory without introducing the meta-theoretic notion of substitution and the conceptually inelegant sets of all, free and bound variables occurring in a term. We have also clarified the actual role of $\alpha$-renaming and $\eta$-extensionality in the lambda calculus: from a convenient point of view — our prelambda congruences — both of them can be equivalently described as properties of extensionality for certain classes of terms.

Our proof of the elimination of the ancillary concepts is rather technical, but conceptually speaking very simple. We also point out that in the relevant literature discussed in Section 1 we could not find complete and detailed proofs of results similar to those we proved in Section 9 and Section 9. For this reason — and also because we want to make the article readable by a wider audience — we decided to make our exposition self-contained as much as possible.

As for future work, we plan to apply the ideas we followed in this paper to other contexts. For instance, we would like to provide similar formalizations of various notions of derivability (formulas and sequents) for several first- and higher-order logics and theories. As already observed by Révész [41], some results for derivability in classical first-order logic with equality which are in line with our motivations — the elimination of some meta-theoretic notions included in our ancillary concepts — have been already established by Tarski [51], Kalish and Montague [28] and Monk [34]. Similar results for second-order classical logic have been established by Cocchiarella [14]. But in our opinion, the most satisfactory axiomatization of first-order classical logic with equality, due to Németi, is reported in the book of Blok and Pigozzi [9, App. C]. In contrast to the aforementioned work, the formulation described in this book has the pleasant property of being completely free of ancillary concepts.

We believe that there is a more uniform way to tackle the problem of eliminating the ancillary concepts from several logics, as we now intuitively explain. It is well-known that substitution — which is usually introduced as a tool for developing the proof-theory of the quantifiers — can be completely handled by the lambda notation, as in Church’s theory of simple types [12]. This theory can be seen as (an extension of) classical higher-order logic and we refer to Coquand [16, Sec. 1] for a very elegant presentation of its purely intensional part in an intuitionistic setting. We think that the ideas and the approach employed in the present article can be exploited to remove the ancillary concepts — which have no mathematical and logical substance — from logic.
and hence simplify the presentations of intuitionistic and classical higher-order logics.

We also plan to export our ideas to give new presentations of some frameworks based on the lambda calculus (without types) whose aim is to formalize (considerable parts of) mathematics such as the type–free systems introduced by Myhill and Flagg [35] and map theory, a setting originally introduced by Grue and recently simplified by Berline and Grue [8].

Finally, we also believe that some ideas we have introduced in this paper can be useful to develop another approach to lambda calculi with explicit substitutions, see Abadi, Cardelli, Curien and Lévi [1] and also Kesner [29] for a brief survey. Our suggestion is to internalize substitutions as in the theory of lambda substitution algebras, the algebraic framework introduced by Diskin and Beylin [21] discussed in Section 8. The expected advantage should be the following: no ancillary concept other than substitution — which is not a meta–theoretic notion in that context — would appear in the formalizations of lambda and extensional theories in systems with explicit substitutions.

More generally, we believe that the algebraic approach to binding operations, see Cardone and Hindley [10, p. 736] for a brief survey, has been relatively overlooked by the computer science community — exceptions are, of course, lambda abstraction and lambda substitution algebras. While the primary goals of those lines of work, namely representation theorems, are perhaps more palatable to algebraists than computer scientists, we think that the study of the algebraic approach to quantification can be very useful for the developments of better syntactical formalizations of theories in structures with binding operations.

References

[1] M. Abadi, L. Cardelli, P.-L. Curien, and J.-J. Lévy. Explicit substitutions. Journal of Functional Programming, 1(4):375–416, 1991. doi:10.1017/S0956796800000186

[2] P. B. Andrews. A Transfinite Type Theory with Type Variables. North–Holland, 1965. doi:10.1016/0049-237X(65)71167-4

[3] P. B. Andrews. Introduction to Mathematical Logic and Type Theory: To Truth Through Proof. Academic Press, 1986.

[4] H. P. Barendregt. Some extended term models for combinatory logics and λ–calculus. PhD thesis, University of Utrecht, 1971. URL http://hdl.handle.net/2066/27329 Revised in 2006.

[5] H. P. Barendregt. The Lambda Calculus. Its Syntax and Semantics. North–Holland, 1984. doi:10.1016/0049-237X(84)90001-0 Revised Edition.

[6] C. Berline. From computation to foundations via functions and application: The λ-calculus and its webbed models. Theoretical Computer Science, 249(1):81–161, 2000. doi:10.1016/S0304-3975(00)00057-8

[7] C. Berline. Graph models of λ-calculus at work, and variations. Mathematical Structures in Computer Science, 16(2):185–221, 2006. doi:10.1017/S0960129506005123

[8] C. Berline and K. Grue. A synthetic axiomatization of Map Theory. Theoretical Computer Science, 614:1–62, 2016. doi:10.1016/j.tcs.2015.11.028
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[30] J.-L. Krivine. Lambda–Calculus, Types and Models. Ellis Horwood and Prentice–Hall, 1993.

[31] G. Longo. Set-theoretical models of $\lambda$–calculus: theories, expansions, isomorphisms. *Annals of Pure and Applied Logic*, 24(2):153–188, 1983. doi:10.1016/0168-0023(83)90030-1.

[32] G. Manzonetto and A. Salibra. Applying Universal Algebra to Lambda Calculus. *Journal of Logic and Computation*, 20(4):877–915, 2010. doi:10.1093/logcom/exn085.

[33] A. R. Meyer. What is a model of the lambda calculus? *Information and Control*, 52(1):87–122, 1982. doi:10.1016/S0019-9958(82)90087-9.

[34] D. Monk. Substitutionless predicate logic with identity. *Archiv für mathematische Logik und Grundlagenforschung*, 7(3):102–121, 1965. doi:10.1007/BF01969435.

[35] J. Myhill and B. Flagg. A type–free system extending (ZFC). *Annals of Pure and Applied Logic*, 43(1):79–97, 1989. doi:10.1016/0168-0072(89)90026-2.

[36] D. Pigozzi and A. Salibra. Applying Universal Algebra to Lambda Calculus. *Journal of Logic and Computation*, 20(4):877–915, 2010. doi:10.1093/logcom/exn085.

[37] D. Pigozzi and A. Salibra. Lambda abstraction algebras: representation theorems. *Theoretical Computer Science*, 140(1):5–52, 1995. doi:10.1016/0304-3975(94)00203-U.

[38] D. Pigozzi and A. Salibra. Lambda Abstraction Algebras: Coordinating Models of Lambda Calculus. *Fundamenta Informaticae*, 33(2):149–200, 1998. doi:10.3233/FI-1998-33203.

[39] G. D. Plotkin. Set-theoretical and other elementary models of the $\lambda$–calculus. *Theoretical Computer Science*, 121(1):351–409, 1993. doi:10.1016/0304-3975(93)90094-A.

[40] G. E. Révész. Lambda–calculus without substitution. *Bulletin of the European Association for Theoretical Computer Science*, 11:140, 1980.

[41] G. E. Révész. Lambda–Calculus, Combinators, and Functional Programming. Cambridge University Press, 1988.

[42] P. C. Rosenbloom. The Relation between Computational and Denotational Properties for Scott’s $D_\infty$–Models of the Lambda–Calculus. *SIAM Journal on Computing*, 5(3):488–521, 1976. doi:10.1137/0205036.

[43] J. von Neumann. An Axiomatization of Set Theory. In J. van Heijenoort, editor, From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931, pages 393–413. Harvard University Press, 1967.

[44] C. P. Wadsworth. The Relation between Computational and Denotational Properties for Scott’s $D_\infty$–Models of the Lambda–Calculus. *SIAM Journal on Computing*, 5(3):488–521, 1976. doi:10.1137/0205036.

[45] P. H. Welch. The Minimal Continuous Semantics of the Lambda–Calculus. PhD thesis, University of Warwick, 1974. URL http://vrap.warwick.ac.uk/72113/.

[46] J. E. Stoy. Denotational Semantics: The Scott–Strachey Approach to Programming Language Theory. MIT Press, 1977.