Improved Approximations for Free-Order Prophets and Second-Price Auctions

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Abstract

We study the fundamental problem of selling a single indivisible item to one of \( n \) buyers with independent and potentially nonidentical value distributions. We focus on two simple and widely used selling mechanisms: the second price auction with eager personalized reserve prices and the sequential posted price mechanism. Using a new approach, we improve the best-known performance guarantees for these mechanisms. We show that for every value of the number of buyers \( n \), the eager second price (ESP) auction and sequential posted price mechanisms respectively earn at least 0.6620 and 0.6543 fractions of the optimal revenue. We also provide improved performance guarantees for these mechanisms when the number of buyers is small, which is the more relevant regime for many applications of interest. This in particular implies an improved bound of 0.6543 for free-order prophet inequalities.

Motivated by our improved revenue bounds, we further study the problem of optimizing reserve prices in the ESP auctions when the sorted order of personalized reserve prices among bidders is exogenous. We show that this problem can be solved polynomially. In addition, by analyzing a real auction dataset from Google’s advertising exchange, we demonstrate the effectiveness of order-based pricing.

Introduction

One of the most fundamental problems in auction/economic theory is selling a single indivisible item to one of \( n \) buyers with independent and possibly heterogeneous value distributions. Optimizing the performance of this auction has remarkable theoretical and practical impact. In online advertising markets alone, such problems need to be solved every fraction of a second to decide what ads to show to Internet users (WSJ 2017). An overwhelming majority of these decisions are made via running eager second price (ESP) auctions (Chawla et al. 2014, Dhangwatnotai et al. 2015, Paes Leme et al. 2016). Motivated by this, we seek to answer the following key question: What fraction of optimal revenue can the ESP auction obtain?

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1In “eager” second price auctions, first the buyers that have not cleared their reserve prices are filtered. Subsequently, the winner is determined among non-filtered buyers. In contrast, in “lazy” second price auctions, first the
The first non-trivial performance guarantee for this auction was given by Chawla et al. (2010a), who prove a $1 - \frac{1}{e}$ approximation to the optimal revenue. They do so by proving the same approximation for the \textit{sequential posted price mechanism} (SPM) and making the important connection that the same approximation factor carries over to the ESP auction that uses the posted prices as personalized reserve prices. The sequential posted price problem is another fundamental selling mechanism studied in the mechanism design literature and is interesting in its own right. The mechanism computes $n$ prices, one per buyer (without soliciting any bids), and approaches each buyer in descending order of prices to make a take-it-or-leave-it offer while the supply lasts. As stated in Chawla et al. (2010a), there are several advantages to SPMs compared to a traditional auction, explaining their ubiquitous presence (Holahan 2008): (i) SPMs have trivial game dynamics: responding truthfully is a dominant strategy, i.e., a buyer purchases the item when his value exceeds the posted price. (ii) These mechanisms satisfy strong notions of collusion resistance, like group strategy-proofness: a buyer interested in helping another buyer has to decline an item with a price below his value, thereby hurting his own utility. Considering these properties, characterizing the revenue bounds of these robust mechanisms is independently worthy of study.

Since the result of Chawla et al. (2010a), the $1 - \frac{1}{e}$ approximation (or, more generally, the $1 - (1 - \frac{1}{n})^n$ approximation when the number of buyers is $n$) for the SPM was found to be obtainable with various techniques: pipage rounding (Calinescu et al. 2011), correlation gap (Agrawal et al. (2012) and Yan (2011)), prophet secretary (Esfandiari et al. 2017) and Bernoulli selection lemma (Correa et al. 2017a). The first improvement over this bound was due to the work of Azar et al. (2017), who show how to improve $1 - (1 - \frac{1}{n})^n$ to $1 - \frac{1}{e} + 1/400$ for $n \geq 74$. For smaller values of $n$, $1 - (1 - \frac{1}{n})^n$ has remained the best known bound until our current paper.

We now summarize our main contributions.

\textbf{Improved Revenue Bounds.} We provide an improved universal bound over $1 - (1 - \frac{1}{n})^n$ (i.e., valid for any number of buyers $n$) for both SPMs and ESP auctions. In particular, for SPMs and ESP auctions, we prove improved approximations of 0.6543 and 0.6620 to the optimal revenue, respectively. While the universal bounds of 0.6543 and 0.6620 are pessimistic (they are obtained when $n$ goes to infinity), they are useful in settings where either there is significant uncertainty in the number of buyers, or the number of buyers is rather large.

The approximation factors are noticeably larger for smaller values of $n$. Table 1 presents our improved bounds for $n = 1, 2, \ldots, 10$ along with the value of $1 - (1 - \frac{1}{n})^n$ for comparison. We stress that, thanks to our novel proof techniques, our bounds are valid even if the distributions of buyers’ value are irregular. The table shows that for SPMs and ESP auctions, our approach outperforms the bound of $1 - (1 - \frac{1}{n})^n$ by up to 3% and 4%, respectively. We note that for the motivating application of display advertising markets, the number of buyers in each auction is typically small, and therefore, improvement for small $n$ is particularly relevant.

We want to highlight that our ESP auction results are strictly better than our SPM results, i.e., we do not just invoke the result of Chawla et al. (2010a) to reduce ESP auctions to SPM, but also use some properties specific to ESP auctions to get these better bounds. Prior to this work, the SPM approximation ratio always matched the ESP approximation ratio (including in the work of Azar et al. (2017)), as they all invoke Chawla et al. (2010a) for ESP auction to SPM reduction.

\textbf{Novel Proof Techniques.} To establish our revenue bound for both SPMs and ESP auctions, we consider the best of the two pricing rules: \textit{Uniform} and \textit{Myersonian}. In the uniform pricing buyer with the highest submitted bid is chosen as the potential winner. Then, this buyer is announced as the winner if his bid exceeds his reserve price.
rule, the mechanisms post the same price for every buyer. In the Myersonian pricing rule, the mechanisms aim to imitate the optimal mechanism proposed by Myerson (1981). We apply the simple concept of the taxation principle, which enables one to view any deterministic incentive-compatible mechanism as a posted price mechanism for each buyer, with the price being dependent on the bids of other buyers. In particular, we use the fact that any deterministic incentive-compatible mechanism, including the optimal Myerson’s mechanism, can be seen as posting a price for each buyer, where the price is a function of other buyers’ bids.

Taking inspiration from this fact, the Myersonian pricing rule computes a price for each buyer $i$ by sampling the bids of other buyers $j \neq i$ from their respective distributions and then computes the posted price that Myerson’s mechanism would have come up with given these bids from other buyers. We perform fresh and independent sampling while computing the prices for different buyers. Thus, while the prices from Myerson’s mechanism are highly correlated, the prices that we compute are independent of each other, easing our analysis. Having considered the best of uniform and Myersonian pricing rules, we then show our improved revenue bound by constructing novel factor revealing linear programs (LPs).

**Optimal Order-Based Eager Reserve Prices.** While the discussion so far focused on obtaining improved revenue bounds via simple reserve pricing schemes, the question of computing the optimal vector of reserve prices for independent value distributions is still open. In this paper, we make some progress towards answering this important question. We develop a dynamic programming method to show that if we are given a sorted ordering over the personalized reserve prices among the buyers, the problem of computing the optimal vector of personalized reserve prices obeying that order can be done in polynomial time. Besides being a relevant subproblem to the problem of computing optimal personalized reserve prices, order-based optimal reserves can be interesting in their own right. For example, in display advertising markets, web publishers may be willing to treat certain buyers more preferentially than other buyers due to long-term relationships. In ESP auctions, this would amount to giving smaller reserve prices for the more preferred buyers. Another motivation for this problem is to develop a new method to improve prior pricing heuristics such as Myerson’s monopoly reserve prices or those proposed by Paes Leme et al. (2016), Ronen (2001), and Roughgarden and Wang (2016). In particular, ESP auctions can obtain the order over reserve prices from one of these several heuristics and optimize its revenue subject to this

| $n$       | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 8     | 10    |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 − (1 − 1/n)$^2$ | 1.0000 | 0.7500 | 0.7037 | 0.6836 | 0.6723 | 0.6651 | 0.6601 | 0.6564 | 0.6536 | 0.6513 |
| SPM       | 1.0000 | 0.7586 | 0.7168 | 0.6990 | 0.6891 | 0.6828 | 0.6785 | 0.6753 | 0.6728 | 0.6709 |
| ESP       | 1.0000 | 0.7611 | 0.7210 | 0.7040 | 0.6946 | 0.6887 | 0.6846 | 0.6815 | 0.6792 | 0.6774 |

Table 1: Revenue bounds of our mechanisms for different number of buyers. All numbers are rounded up.

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$^2$In our proof, we do not invoke virtual values at all.

$^3$Myerson’s mechanism, even for irregular distributions, where it is usually thought to be randomized, can be implemented as a deterministic mechanism by breaking ties consistently (e.g. lexicographically) between buyers. See Chawla and Sivan (2014).

$^4$For the correlated value distribution, Paes Leme et al. (2016) show that the problem of optimizing reserve prices in ESP auctions is NP-complete.

$^5$For example, for small values of $n$, enumerating over all the $n!$ orderings of reserve prices to find the truly optimal vector of reserve prices is possible.
order. This is the first work that compares the performance of various prominent reserve pricing heuristics in a very data rich real world environment, besides proposing a practical technique to improve them.

**Empirical and Numerical Studies.** We evaluate the performance of order-based ESP auctions on synthetic and fully-anonymized real auction datasets from Google’s advertising exchange. We present results for synthetic data so that our results are replicable, and the real auction dataset enables us to assess our order-based ESP auctions in a realistic environment that does not necessarily satisfy our independency assumption. Specifically, we show that although submitted bids in our dataset are not independent across buyers, the order-based ESP auctions perform really well.

We investigate the performance of the ESP auctions that obey the order of prices in the four main pricing heuristics from the literature: Myerson’s monopoly reserve prices and the respective pricing heuristics proposed by Paes Leme et al. (2016), Ronen (2001), and Roughgarden and Wang (2016). We note that to the best of our knowledge, the present paper is the first work that compares these heuristics in a realistic environment\(^6\) and provides a practical technique to improve them. We further study the performance of the order-based ESP auctions that follow the order of the average and coefficient of variation (CV) of the submitted bids. These orderings are motivated by Golrezaei et al. (2017), who show that to extract more revenue from buyers, the mechanism should favor buyers with smaller CVs and average bids. We show that by obeying the order suggested by the prior pricing heuristics and optimizing reserve prices subject to the order, the revenue of ESP auctions increases by up to 8.75\%. In addition, we show that ordering based on the average and CV of submitted bids performs well.

The rest of the paper is organized as follows. In Section 1, we review the related literature. In Section 2, we present our model, and we formally define SPMs and ESP auctions. Sections 3 and 4 present our universal revenue bounds for SPMs and ESP auctions. In Section 5, we provide our revenue bounds that take into account the number of buyers \(n\). Sections 6 and 7 are respectively dedicated to the order-based reserve price optimization problem and our empirical studies. For the sake of brevity, we only include proofs of selected results in the main text; the detailed proofs of the rest of the statements are deferred to the appendix.

### 1 Related Work

Our work relates and contributes to the literature on optimal auction design. The seminal work of Myerson (1981) shows that when buyers’ value distributions are regular and homogeneous, the optimal mechanism can be implemented via a second price auction with reserve. However, the structure of the optimal mechanism can be complex when the value distributions are heterogeneous and irregular (Myerson 1981). Because of this, several papers have studied simpler auction formats, such as second price auctions with (personalized) reserve prices (Hartline and Roughgarden (2009), Paes Leme et al. (2016), Roughgarden and Wang (2016), and Allouah and Besbes (2018)), boosted second price auctions (Golrezaei et al. 2017), the BIN-TAC mechanism (Celis et al. 2014), and first price auctions (Bhalgat et al. 2012), to name a few.

Hartline and Roughgarden (2009) study the question of approximating the optimal revenue via a Vickrey auction with personalized reserve prices, and show that for regular distributions, the second price auction with so-called monopoly reserve prices yields a 2-approximation for regular revenue.\(^6\) We observe that the pricing heuristic of Paes Leme et al. (2016) outperforms the other aforementioned heuristics.
distributions, and that for irregular distributions, no constant factor approximation is possible with the monopoly reserves. Paes Leme et al. (2016) consider second price auctions and study the question of computing the optimal personalized reserve prices in a correlated distribution setting, and they show that the problem is NP-complete. Roughgarden and Wang (2016) show that this problem is APX-hard for correlated distributions and give a $\frac{1}{2}$-approximation. Finally, Golrezaei et al. (2017)—using empirical and theoretical analyses—show that when buyers are heterogeneous, their proposed mechanism, called boosted second price auction, gets a high fraction (more than $\frac{1}{2}$) of the optimal revenue and outperforms the second price auction with reserve. In the current work, we provide an improved approximation factor for second price auctions and show that this auction format, despite its simple structure, performs well even when the distributions are heterogeneous and irregular.

Our work is also related to the literature on prophet inequalities (Krengel and Sucheston (1977, 1978) and Kennedy (1987)). Specifically, studying posted price mechanisms has been intimately connected to the work on prophet inequalities. In the classic prophet inequality setting, $n$ independent (but not necessarily identical) random variables arrive in an adversarial sequence, and after each random variable arrives, the gambler faces two choices: accept the random variable and stop, or reject and continue. The objective is to maximize the expected value of the random variable selected by the gambler. Performance is measured based on the ratio of the gambler’s choice to the expected value of the maximum of $n$ random variables (the objective that a prophet with complete foresight can obtain).

**Adversarial Prophets.** Hill and Kertz (1981) show that when variables are independent but not identical and their orders are chosen adversarially, the gambler can obtain at least $\frac{1}{2}$ of the expected value obtained by a prophet; see also Samuel-Cahn (1984).7 This $\frac{1}{2}$-approximation was later used by Chawla et al. (2010a) to give a $\frac{1}{2}$-approximation for the posted prices mechanism when the buyers arrive in an adversarial order. When the random variables are i.i.d. and their orders are adversarial, Hill and Kertz (1982) show that the gambler can obtain at least $1 - \frac{1}{e}$ of the prophet’s value and also show examples that prove that one cannot obtain a factor beyond $\frac{1}{1.342} \sim 0.745$. Kertz (1986) later conjecture that $\frac{1}{1.342} \sim 0.745$ is the best possible approximation. The first formal proof that one can go beyond $1 - \frac{1}{e}$ was given by Abolhassani et al. (2017), who give 0.738 for all $n$ beyond a certain constant $n_0$. Simultaneously and independently, Correa et al. (2017a) show a 0.745 approximation for this problem, thereby completely closing the gap. We highlight that the 0.745 approximation factor proved by Correa et al. (2017a) is not applicable to our setting, as in our work, the values of buyers are not i.i.d.

**Free-Order Prophets.** In another variation of prophet inequalities, the gambler can pick the random variables in her desired order. This variation is known as free-order prophets. We first note, as recently shown by Correa et al. (2017b), that any approximation for SPMs implies the same approximation for free-order prophets. The bound on the best possible approximation by Kertz (1986) also holds for this setting. Chawla et al. (2010a) give a $1 - \frac{1}{e}$ approximation for the SPM problem. Recall that under SPMs, the seller approaches the buyers in decreasing order of their prices. This bound of $1 - \frac{1}{e}$ was recently surpassed by Azar et al. (2017) to $1 - \frac{1}{e} + 0.0025$.

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7This constant cannot be improved, even if the prophet were allowed to use adaptive strategies and even if we make large market assumptions, like each distribution occurring at least $m$ times for any arbitrarily large $m$ (Abolhassani et al. 2017).
and their bound also extends to random order prophets, in which the random variables arrive in a uniformly random order. Our bound of $0.6543 \sim 1 - \frac{1}{e} + 0.022$ for SPMs, which implies the same bound for free-order prophets by the result of Correa et al. (2017b), is the best known bound for free-order prophets so far. More precisely, just like for SPMs and ESP auctions, ours is the first paper to go beyond the $1 - (1 - \frac{1}{n})^n$ bound for every $n$ in the free-order prophets setting.

**Posted Prices, Prophet Inequalities and Generalizations.** The connection between prophet inequalities and mechanism design was initiated by Hajiaghayi et al. (2007), who interpret the prophet inequality algorithms as truthful mechanisms for online auctions. Chawla et al. (2008) give a 4-approximation to the single-agent $n$-items unit-demand pricing problem, via upper bounding the revenue of the multi-parameter setting by that of the single-item $n$-buyers single-parameter problem. Chawla et al. (2010a,b) further this connection, and develop constant fraction approximations for several multi-parameter unit-demand settings, by establishing constant factor approximations to the corresponding single-parameter posted price settings through connections to prophet inequalities. Yan (2011) makes a connection between the revenue of SPMs and correlation gap for submodular functions (Agrawal et al. 2012). Chakraborty et al. (2010) develop a PTAS for computing the optimal adaptive and non-adaptive SPMs in a $k$-item single-parameter setting. Kleinberg and Weinberg (2012) generalize the prophet inequalities result to the matroidal environment, and use this to give improved approximations for the multi-parameter unit-demand mechanism design problem. For other generalizations of posted prices and prophet inequalities to combinatorial settings, see Dütting and Kleinberg (2015), Ehsani et al. (2018), Feldman et al. (2015), Rubinstein and Singla (2017), and Dütting et al. (2017).

## 2 Model

There is a single indivisible item to be sold to one of $n$ buyers indexed by $i \in [n]$, where $[n] = \{1, 2, \ldots, n\}$. The value of buyer $i$ for the item, denoted by $v_i$, is drawn independently from distribution $F_i$ with density function $f_i$. For each $i \in [n]$, distribution $F_i$ is public information, while value $v_i$ is buyer $i$'s private information.

In the following, we formally define the sequential posted price mechanisms, denoted by SPM($p$), and we then proceed to define the eager second price auctions, denoted by ESP($p$). Here, $p = (p_1, p_2, \ldots, p_n)$ is a vector of posted prices in SPM($p$) and a vector of reserve prices in ESP($p$).

Before presenting these mechanisms, we note that in this work, we focus on the eager second price auctions. The lazy second price auctions are incomparable to the eager second price auctions for general correlated distributions but are within a factor of 2 of each other (Paes Leme et al. 2016). Further, Paes Leme et al. (2016) show that the optimal revenue from the eager auction dominates the optimal revenue from the lazy auction when the value distributions are independent. Motivated by this, we study ESP auctions in the current work. We note that it is known from an example in Ronen (2001) that it is impossible to obtain a better than $1/2$-approximation for the lazy auctions with respect to the optimal revenue.

**Sequential Posted Price Mechanisms** SPM($p$)

- Buyers are sorted in decreasing order of their posted prices $p_i$, $i \in [n]$. Without loss of generality, we assume that $p_1 \geq p_2 \geq \ldots \geq p_n$. 

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- The mechanism approaches buyers in decreasing order of their posted prices. Precisely, for $i \in [n]$, the mechanism offers price $p_i$ to buyer $i$. If the buyer accepts the offer, i.e., $v_i \geq p_i$, the item will be allocated to buyer $i$ at a price of $p_i$, and the mechanism will stop. Otherwise, the mechanism proceeds to the next buyer with the highest posted price, i.e., buyer $i + 1$.

**Eager Second Price Auctions ESP($p$)**

- Each buyer $i \in [n]$ submits his bid/value $v_i$.

- All the buyers with value $v_i < p_i$ are first eliminated. Let $S = \{i : v_i \geq p_i\}$ be the set of all the buyers who clear their reserve prices.

- The item is then allocated to buyer $i^* = \arg \max_{i \in S} v_i$, who has the highest value among all buyers in set $S$, and he pays $\max(p_{i^*}, \max_{i \in S, i \neq i^*} v_i)$. For other buyers, their payment is zero.

The following lemma is an important observation about the revenue of ESP($p$) and SPM($p$) mechanisms made by Chawla et al. (2010a). Note that the revenue of a mechanism is the total (expected) payment that it charges the buyers, where the expectation is taken with respect to the randomness in buyers’ value.

**Lemma 1** (ESP Dominates SPM). For any vector of prices $p = (p_1, \ldots, p_n)$ and any value distributions, the revenue of ESP($p$) is at least the revenue of SPM($p$).

**Proof.** Proof of Lemma 1: For any realization of buyers’ value, if SPM($p$) has a winner, then ESP($p$) must also have a winner. If the winner in the two auctions is the same, then the revenue of ESP($p$) must be at least as large as that of SPM($p$). If the winner is different, then the winner of ESP($p$) is paying at least the bid of the winner in SPM($p$), and the latter is at least the posted price in SPM($p$). □

Throughout the proof, with a slight abuse of notation, we respectively denote the expected revenue of ESP($p$) and SPM($p$) by ESP($p$) and SPM($p$), where the expectation is taken with respect to the randomness in the buyers’ value.

### 2.1 Optimal Revenue Benchmark

Our benchmark, which we refer to as Opt, is an incentive-compatible (truthful) and individually rational revenue-optimal auction in the independent value setting. This mechanism was designed by Myerson (1981). For most of our results, the specific form of the optimal auction is irrelevant. We will use only the fact that it is a deterministic truthful auction, and hence, the taxation principle (Hammond 1979) gives a simple equivalent form of expressing such a mechanism. As mentioned earlier, even when the distributions are not regular, Myerson’s mechanism can be implemented as a deterministic mechanism; see Chawla and Sivan (2014).

The following lemma describes the taxation principle in any deterministic truthful mechanism. We do not prove it here: its proof can be derived from any standard auction theory text.

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8Since ESP auctions are truthful, buyers’ bids are equal to their values.
Lemma 2 (Taxation Principle). Given a single-item deterministic truthful mechanism \( M \), there are threshold functions \( t_i(v_{-i}) \) for each buyer \( i \), depending only on the bids of the other buyers \( v_{-i} = (v_j)_{j \neq i} \), such that the allocation and payment of mechanism \( M \) can be described as follows:

- if \( v_i > t_i(v_{-i}) \), then the item is allocated to buyer \( i \), and he is charged the threshold \( t_i(v_{-i}) \).
- if \( v_i = t_i(v_{-i}) \), then either the item is allocated to buyer \( i \), and he is charged his value \( v_i = t_i(v_{-i}) \), or the item is not allocated to him, and he is not charged.
- if \( v_i < t_i(v_{-i}) \), then the item is not allocated to buyer \( i \), and he is not charged.

We note that the threshold functions \( t_i \)’s can be computed for any deterministic incentive-compatible mechanism. In such a mechanism, each buyer \( i \) has a critical value \( v_{\min} \)—which only depends on other buyers’ values—such that he gets allocated and pays \( v_{\min} \) if his value \( v_i > v_{\min} \). When \( v_i < v_{\min} \), he does not get allocated and pays 0. When \( v_i = v_{\min} \), the mechanism can either allocate the item to buyer \( i \) and charge him \( v_{\min} \), or not allocate the item to him and charge 0. The threshold function is defined as \( t_i(v_{-i}) = v_{\min} \). We give two examples to illustrate how these thresholds are computed.

Example 3. Second price auction with no reserve: The critical value for buyer \( i \) is \( t_i(v_{-i}) = \max_{j \neq i} v_j \).

Example 4. Optimal auction with uniform distributions: Suppose that there are two buyers with values \( v_1 \) and \( v_2 \) drawn independently from the uniform distributions in \([0,1]\) and \([0,2]\), respectively. In the optimal auction, the item is allocated to the buyer with the highest non-negative virtual value.\(^9\) The virtual values of buyers 1 and 2 are respectively \( 2v_1 - 1 \) and \( 2v_2 - 2 \). Thus, buyer 1 is allocated when \( v_1 \geq \max(0.5, v_2 - 0.5) \), and buyer 2 is allocated when \( v_2 \geq \max(1, v_1 + 0.5) \). Therefore, the threshold functions are given by \( t_1(v_2) = \max(0.5, v_2 - 0.5) \) and \( t_2(v_1) = \max(1, v_1 + 0.5) \).

Thresholds, \( t_i(v_{-i}) \), are constructed in such a way that at most one buyer is strictly above his threshold. However, it is possible that more than one buyer is at the threshold, in which case a tie-breaking rule needs to be determined to exactly describe the allocation rule of the mechanism.

There are two important cases in which the issues of tie breaking can be ignored. The first case is when the value distributions are independent and continuous, with no atoms: in this case, the thresholds for any deterministic auction can be constructed in such a way that there will be at most one buyer who meets/exceeds the threshold, i.e., \( v_i \geq t_i(v_{-i}) \), in which case that buyer gets allocated and pays his threshold. To avoid issues with ties, throughout the paper, we assume that distributions are discrete, and in such a case, thresholds are set in a manner such that there is at most one buyer who meets/exceeds the threshold in the auction. The continuous case can be handled by discretizing the distributions and taking the limit of the discretization.

2.2 Definitions and Notations

Thresholds: In the rest of the paper, \( t_i(v_{-i}) \) refers to the threshold of buyer \( i \) corresponding to the optimal auction (see Lemma 2). Whenever it is clear from context, we abbreviate \( t_i(v_{-i}) \) or the function \( t_i(\cdot) \) by \( t_i \).

\(^9\)The virtual value of buyer \( i \) with value \( v \) is given by \( v - \frac{1 - F_i(v)}{f_i(v)} \).
Re-sampled thresholds: We will often refer to the thresholds computed from independently re-sampled values: namely, for each buyer $i$, sample $v'_{j,i} \sim F_j$ for all $j \neq i$, and denote by $t_i(v_{-i}')$ the re-sampled threshold where $v_{-i}' = (v'_{1,i}, \ldots, v'_{i-1,i}, v'_{i+1,i}, \ldots, v'_{n,i})$. Observe that we do not reuse samples: for each buyer $i$, we freshly re-sample the values of all other buyers. We abbreviate $t_i(v_{-i}')$ by $t'_i$ whenever it is clear from the context. Note that although for each $i$, the distribution of $t_i$ is the same as the distribution of $t'_i$, $t'_i$'s are independent across $i$'s, while $t_i$'s are correlated.

Myersonian posted prices: This refers to the tuple of $n$ posted prices, one per buyer, namely, the re-sampled threshold $t'_i$ for buyer $i$.

Uniform posted price: This refers to the highest revenue yielding uniform posted price, namely, $p^\star = \arg \max_p p \cdot \mathbb{P}[\max_i v_i \geq p]$.

Revenue lower bound probabilities: Let $s_i(\tau) = \mathbb{P}[v_i \geq t_i(v_{-i}) \geq \tau]$, $i \in [n]$, be the probability that buyer $i$ wins and pays at least $\tau$ in the optimal auction, where the expectation is taken w.r.t. $v_{-i}$ and $v_i$. Further, let $s(\tau) = \sum_{i \in [n]} s_i(\tau)$ be the probability that the winner pays at least $\tau$. It follows immediately that $s(\tau)$ is a weakly decreasing function whose integral defines the optimal auction’s revenue:

$$\text{Opt} = \int_0^\infty s(\tau) d\tau. \quad (1)$$

3 Universal Revenue Bound for Sequential Posted Price Mechanisms

In this section, we provide a universal approximation factor for the SPMs that hold for any value of $n$. We note that our approximation factor is valid even if value distributions are not regular.

Theorem 5 (Revenue Bound of SPM). There exists a vector of prices $p = (p_1, p_2, \ldots, p_n)$ such that $\text{SPM}(p) \geq 0.6543 \cdot \text{Opt}$.

Relation to free-order prophets: We note that Theorem 5 also implies that an improved bound of 0.6543 for the free-order prophet problem Correa et al. (2017b). In this problem, there are $n$ independent random variables $X_i$ with known distributions $G_i$. A decision maker knows the distributions but not the realizations and he chooses an order to inspect the variables. Upon inspecting a variable, he learns its realized value and needs to choose between stopping and obtaining its value as a reward or abandoning that variable forever and continuing to inspect other variables. An algorithm for this problem is a policy that determines an order to inspect the variables and for each variable inspected decides whether to stop and obtain that reward or continue inspecting. If $I$ is the index of the random variable chosen by the decision maker, the performance of the algorithm is $\mathbb{E}[X_I]$. The goal is to compare the performance of the decision maker with a prophet that knows all the values in advance and therefore can obtain $\mathbb{E}[\max(X_1, \ldots, X_n)]$. Correa et al. (2017b) recently showed that this problem is equivalent to the sequential posted prices problem. In particular, each variable $X_i$ in the free-order prophet problem can be mapped to a buyer $i$ with value $v_i$ such that the virtual value associated with $v_i$ has distribution $G_i$. Then, they showed
that one can solve the sequential posted prices problem for \( v_i \)’s and then map the policy back to a free-order prophet policy. Using this mapping, any algorithm for sequential posted prices can be transformed (in a black-box manner) to an algorithm for the free-order prophets problem with the same approximation ratio.

The proof of Theorem 5 is based on a novel technique that uses the the best of two posted price mechanisms (and hence, the ultimate mechanism is also a posted price mechanism). The first one, which we call the Myersonian posted price mechanism, posts prices that mimic the threshold functions in the optimal auction, and the second one, called the uniform posted price mechanism, posts the same price for every buyer. We show that choosing the best of these two mechanisms gives us the desired bound.

**Myersonian Posted Price Mechanism:** Approach the buyers in decreasing order of their Myersonian posted prices, i.e., the re-sampled thresholds \( t'_i \) (defined in Section 2.2), and allocate to the first buyer \( i \) whose value \( v_i \) exceeds his threshold \( t'_i \). Recall that \( t'_i \)'s are independent random variables across \( i \)'s, and each \( t'_i \) is distributed identically to \( t_i \). Let \( MP \) denote the expected revenue of this mechanism, where the expectation is taken w.r.t. randomness in both the re-sampled posted prices and the buyers’ values.

**Uniform Posted Price Mechanism:** Approach buyers in an arbitrary order, and allocate to the first buyer whose value exceeds the price \( p^* = \arg \max \{ p : \Pr \left[ \max_i v_i \geq p \right] \} \). Let \( UP \) be the expected revenue of this mechanism, where the expectation is taken with respect to the buyers’ values.

In Theorem 5, we show that \( \max(UP, MP) \) is at least a 0.6543 fraction of the optimal revenue, denoted by \( \text{Opt} \). This implies that there exist thresholds \( p = (p_1, p_2, \ldots, p_n) \) such that \( \text{SPM}(p) \geq (1 - \frac{1}{e} + 0.022)\text{Opt} \approx 0.6543 \cdot \text{Opt} \).

We focus on the best of these two mechanisms because they complement each other, i.e., in the worst instances of Myersonian pricing, uniform pricing has a good performance.\(^{10}\) We are now ready to prove Theorem 5 by constructing a factor revealing LP: the proof immediately follows from Claim 6.

**Claim 6** (Detailed Statement of Theorem 5). *The maximum of Myersonian Posted Price Mechanism’s and Uniform Posted Price Mechanism’s revenue is at least \( \frac{\text{Opt}}{\text{LP-SPM}} = 0.6543 \cdot \text{Opt} \), where \( \text{LP-SPM} \) is defined as:

\[
\text{LP-SPM} = \max_{\{s(\tau) : \tau \geq 0, s(\tau) \geq 0\}} \int_0^\infty s(\tau)d\tau \\
\text{s.t.} \quad 0 \leq s(\tau) \leq \min(1, 1/\tau) \quad \forall \quad \tau \geq 0 \\
\int_0^\infty s(\tau)f(\tau)d\tau \leq 1,
\]

where \( f(\tau) = \frac{1 - e^{-\min(1,1/\tau)}}{\min(1,1/\tau)} \).

\(^{10}\)To illustrate that \( UP \) complements \( MP \), consider the setting where there are \( n \) buyers whose values are independently drawn from the uniform distribution in \([1, 1 + \epsilon]\) for a tiny \( \epsilon \). Then, the optimal auction is simply the second price auction with a uniform reserve of 1, and the uniform pricing scheme that just posts a price of 1 gets very close to this optimal auction. However, in the Myersonian posted price mechanism, each buyer is offered a random threshold that is the maximum of \( n - 1 \) uniform variables. Thus, each buyer is above such a threshold with probability \( 1/n \). Since they are all independent, with probability \( (1 - 1/n)^n \rightarrow 1/e \), no buyer is above the threshold. Thus, \( MP \) just makes a \( 1 - 1/e \) approximation for this particular choice of prices.
Proof. Proof of Claim 6: We first show that max(MP, UP) ≥ \frac{1}{LP-SPM} Opt. Subsequently we prove that \frac{1}{LP-SPM} = 0.6543.

Without loss of generality, assume that the revenue of the posted price mechanism that chooses the best of uniform and Myersonian prices is normalized to 1: i.e., max(MP, UP) = 1 (this can be done by scaling all values). Note that the objective function of LP-SPM is the optimal revenue (see Equation (1)). Thus, proving that the constraints of Problem LP-SPM follow from upper/lower bounds on UP, MP along with max(MP, UP) = 1 will imply that max(MP, UP) ≥ \frac{1}{LP-SPM} Opt. We now show that the constraints follow from upper and lower bounds on UP, MP.

**Upper Bounds on UP (First Set of Constraints):** Consider the posted price mechanism that posts a price of τ for every buyer. The revenue of this mechanism is equal to \( \tau \cdot \mathbb{P}[\max_i v_i \geq \tau] \), which is at least \( \tau s(\tau) \). Therefore, \( \text{UP} \geq \tau s(\tau) \) for every \( \tau \geq 0 \); that is,

\[
\max_{\tau \geq 0} \tau s(\tau) \leq \text{UP} \leq 1, \tag{2}
\]

where the second inequality follows from max(MP, UP) = 1. Equation (2) leads to

\[
0 \leq s(\tau) \leq \min(1, 1/\tau) \quad \forall \tau \geq 0. \tag{3}
\]

In the inequality, we also used the fact that \( s(\tau) \) is a probability and is at most 1. Note that Equation (3) is the first set of constraints in LP-SPM.

**Lower Bounds on MP (Second Constraint):** Let \( m(\tau) \) be the probability that the Myersonian posted price mechanism sells with price at least \( \tau \), which is the probability that there is at least one buyer with \( v_i \geq t'_i \geq \tau \), where \( t'_i \) is the posted-price for buyer \( i \) in MP. Then, we have:

\[
\text{MP} = \int_0^\infty m(\tau)d\tau \leq 1, \tag{4}
\]

where the inequality follows from max(MP, UP) = 1. Next, we present a lower bound on MP by connecting \( m(\tau) \) to probability \( s(\tau) \) that describes the optimal auction.

By construction of \( t'_i \), the probability \( m_i(\tau) \) that \( v_i \geq t'_i \geq \tau \), is the same probability as buyer \( i \) wins in the optimal auction and is charged at least \( \tau \), which is \( s_i(\tau) \). Since the buyers’ values are all independent, and the prices \( t'_i \) are all computed independently with fresh re-sampling for each buyer \( i \), we have:

\[
\begin{align*}
    m(\tau) &= 1 - \prod_{i=1}^n (1 - s_i(\tau)) \\
    &\geq 1 - e^{-\sum_{i=1}^n s_i(\tau)} = 1 - e^{-s(\tau)} = s(\tau) \frac{1 - e^{-s(\tau)}}{s(\tau)} \geq s(\tau) f(\tau),
\end{align*}
\tag{5}
\]

where \( f(\tau) = \frac{(1 - e^{-\min(1, 1/\tau)})}{\min(1, 1/\tau)} \). The first inequality follows from \( 1 - x \leq e^{-x} \) and the last inequality follows from \( \frac{1 - e^{-x}}{x} \) being decreasing in \( x \) and that \( s(\tau) \leq \min(1, 1/\tau) \) (see Equation (3)). Invoking Equations (4) and (5) leads to the second constraint in LP-SPM.

Next, we compute the objective value LP-SPM. It is not difficult to guess the optimal solution of Problem LP-SPM. Since \( f(\tau) \) is increasing in \( \tau \), an optimal solution must satisfy that whenever \( s(\tau) > 0 \), it must be that \( s(\tau') = \min(1, 1/\tau') \) for every \( \tau' < \tau \). Hence, the optimal solution of Problem LP-SPM has the following form:

\[
s(\tau) = \begin{cases} 
    \min(1, 1/\tau) & \text{if } 0 \leq \tau \leq \tau^*; \\
    0 & \text{if } \tau > \tau^*,
\end{cases}
\]
where $\tau^* > 1$ is the unique threshold for which $\int_0^\infty s(\tau)f(\tau)d\tau = 1$. This leads to

$$\int_0^\infty s(\tau)f(\tau)d\tau = \int_0^1 s(\tau)f(\tau)d\tau + \int_1^{\tau^*} s(\tau)f(\tau)d\tau = (1 - e^{-1}) + \int_1^{\tau^*} (1 - e^{-1/\tau})d\tau = 1.$$  

By solving the above equation numerically, we get $\tau^* = 1.696$, and the optimal solution of Problem LP-SPM is given by $1 + \ln(\tau^*) = 1.5283$. To see why note that

$$\int_0^\infty s(\tau)d\tau = \int_0^1 d\tau + \int_1^{\tau^*} \frac{1}{\tau}d\tau = 1 + \ln(\tau^*) = 1.5283.$$  

Hence, the SPM that chooses the best of uniform and Myersonian pricing rule yields at least $1/1.5283 \approx 0.6543$ of Opt, which is the bound in Theorem 5. ■

4 Universal Revenue Bound for Eager Second Price Auctions

The bound presented in Theorem 5 is also a valid bound for the ESP auctions; see Lemma 1. However, the ESP auctions can potentially earn higher revenue than SPM by leveraging the second highest bid. We now show how to exploit the second highest bid to obtain an improved bound for the ESP auction.

The following is the main result of this section.

**Theorem 7** (Revenue Bound of ESP). *There exist reserve prices $p = (p_1, p_2, \ldots, p_n)$ such that $\text{ESP}(p) \geq 0.6620 \cdot \text{Opt}$.*

The proof, in spirit, is similar to that of Theorem 5. We consider an ESP auction that chooses the best of Myersonian and Uniform reserve prices. We construct a factor revealing LP to bound its performance.

**Proof of Theorem 7**

Similar to the proof of Theorem 5, we define the following two ESP auctions.

**Myersonian ESP Auction:** We run the ESP auction with personalized reserve prices for each buyer, with buyer $i$ facing the re-sampled threshold $t'_i$ as his reserve price (see the definition in Section 2.2). Let $\text{ME}$ denote the expected revenue of this auction.

**Uniform ESP Auction:** We run the ESP auction with a uniform reserve price of $p^*_E = \arg\max_p \mathbb{E}_{v \sim \prod_i F_i} \left[ \max(p, v(2)) \cdot \mathbb{I}[\max_i v_i \geq p] \right]$, where $v(2)$ is the second highest bid (which is also equal to the second highest value in a truthful auction). We denote the revenue of this auction by $\text{UE}$.

In the following claim, we show that the best of two aforementioned ESP auctions has the approximation factor given in Theorem 7, concluding its proof.

**Claim 8** (Detailed Statement of Theorem 7). *For every positive integer $k$, the maximum of Myersonian ESP Auction’s and Uniform ESP Auction’s revenue is at least $\frac{1}{\text{LP-ESP}(k)} \cdot \text{Opt}$, where $s_i = i/k$,
for $i \in [k]$, and

$$\text{LP-ESP}(k) = \max_w \sum_{i \in [k]} w_i$$

s.t.  

$$\sum_{i=1}^{j} w_i \frac{2(1 - e^{-s_i}) - s_i e^{-s_i}}{s_i} + \sum_{i=j+1}^{k} w_i \frac{s_j + (1 - e^{-s_i})}{s_i} \leq 2, \ \forall j \in [k]$$

$$\sum_{i \in [k]} w_i \frac{1 - e^{-s_i}}{s_i} \leq 1, \ \forall i \in [k]$$

$$w_i \geq 0, \ \forall i \in [k]$$

(LP-ESP)

In particular, setting $k = 3200$, the approximation factor is $\frac{1}{\text{LP-ESP}(3200)} = 0.6620$.

The factor revealing LP, given in Claim 8, does not have a closed form solution. In the following table, we present the value of $\text{LP-ESP}(k)$ for different values of $k$. Since $\frac{1}{\text{LP-ESP}(3200)}$ is a valid approximation factor for every $k$, it follows that $\frac{1}{\text{LP-ESP}(3200)} = 0.6620$ is a valid approximation factor. As it becomes clear in the proof, parameter $k$ determines the precision of our discretization. Larger values of $k$ imply more granular discretization.

| $k$  | 50  | 100 | 200 | 400 | 800 | 1600 | 3200 |
|------|-----|-----|-----|-----|-----|------|------|
| $\frac{1}{\text{LP-ESP}(k)}$ | 0.6606 | 0.6613 | 0.6617 | 0.6618 | 0.6619 | 0.6620 | 0.6620 |

Table 2: $\frac{1}{\text{LP-ESP}(k)}$ for different values of $k$.

Before presenting the proof of Claim 8, we briefly discuss LP-ESP. The objective of the problem is the optimal revenue. Recall that by Equation (1), the optimal revenue is $\int_0^\infty s(\tau) d\tau$ where $s(\tau)$ is the probability that the optimal auction sells at a price at least $\tau$. We define $0 = \tau_k \leq \tau_{k-1} \leq \ldots \leq \tau_1 \leq \tau_0 = \infty$ such that $\tau_j = \inf\{ \tau : s(\tau) \leq j/k \}, \ j \in [k-1]$. We then have

$$\text{Opt} = \sum_{i \in [k]} w_i, \ \text{where} \ \ w_i = \int_{\tau_{i-1}}^{\tau_i} s(\tau) d\tau. \ \ (6)$$

Equation (6) verifies that the objective function of Problem LP-ESP is Opt. Next, we explain the constraints of the LP. The first set of constraints follows from lower and upper bounds on $\text{UE} + \text{ME}$. The second constraint follows from bounding ME.

We now present the proof of Claim 8.

Proof. Proof of Claim 8: The goal is to show that $\max(\text{UE}, \text{ME}) \geq \frac{1}{\text{LP-ESP}(k)} \cdot \text{Opt}$. Similar to the proof of Theorem 5, without loss of generality, let $\max(\text{UE}, \text{ME}) = 1$. By the definition of $w_i$’s, the objective function of Problem LP-ESP is equal to Opt. This implies that to show Claim 8, it suffices to prove that the constraints of Problem LP-ESP follow from upper/lower bounds on $\text{UE}, \text{ME}$ and $\max(\text{UE}, \text{ME}) = 1$.

First Set of Constraints: To derive the first set of constraints, we use lower and upper bounds on $\text{ME} + \text{UE}$. Let $\mathcal{T}_x = \inf\{ \tau : s(\tau) \leq x \}, \ x \in [0, 1]$ (note that $\mathcal{T}_x$ is different from the
\( \tau_j = \inf\{ \tau : s(\tau) \leq j/k \}, j \in [k-1] \) defined earlier). In addition, with a slight abuse of notation, let \( \text{UE}_x \) be the revenue of the ESP auction that posts a uniform price of \( \hat{\tau}_x \) for all buyers. By definition of the uniform ESP auction, we have \( \text{UE} = \max_{x \in [0,1]} \text{UE}_x \). We now bound \( \text{ME} + \text{UE}_x \) for any \( x \in [0,1] \).

We start by bounding \( \text{UE}_x \). Define \( u_x(\tau) \) as the probability that ESP auction with uniform price \( \hat{\tau}_x \) sells with price at least \( \tau \). Then, \( \text{UE}_x = \int_{\tau=0}^{\infty} u_x(\tau) d\tau \). We next bound \( \text{UE}_x \) via bounding \( u_x(\tau) \).

For \( \tau \leq \hat{\tau}_x \), we bound \( u_x(\tau) \) via

\[
u_x(\tau) \geq s(\hat{\tau}_x) \geq x, \quad \tau \leq \hat{\tau}_x.
\]  

This bound holds because (i) while the ESP auction with uniform price \( \hat{\tau}_x \) can sell the item with price at least \( \tau \) if there exists at least one buyer \( i \) with value \( v_i \geq \hat{\tau}_x \), the optimal auction can sell at price at least \( \hat{\tau}_x \) only if there is at least one buyer \( i \) with \( v_i \geq t_i(v_{-i}) \geq \hat{\tau}_x \), and as a result, \( u_x(\tau) \geq s(\hat{\tau}_x) \), and (ii) by definition of \( \hat{\tau}_x \), we have \( s(\hat{\tau}_x) \geq x \); to see this recall that \( \hat{\tau}_x = \inf\{ \tau : s(\tau) \leq x \} \). So when \( \tau_x \in \{ \tau : s(\tau) \leq x \} \), by monotonicity of \( s(\tau) \), it must be the case that \( s(\hat{\tau}_x) = x \). Further, if \( \tau_x \notin \{ \tau : s(\tau) \leq x \} \), we have \( s(\hat{\tau}_x) > x \). Thus, \( s(\hat{\tau}_x) \geq x \).

For \( \tau > \hat{\tau}_x \) we bound \( u_x(\tau) \) by noting that the ESP auction with uniform price \( \hat{\tau}_x \) can sell at price at least \( \tau \) only if there are at least two buyers bidding above \( \tau \). Let \( \tilde{Z}_\tau \) be the cardinality of set \( \{ i \in [n], v_i \geq \tau \} \) and \( Z_\tau \) be the cardinality of set \( \{ i \in [n], v_i \geq t_i' \geq \tau \} \).

Then, we have:

\[
u_x(\tau) = \mathbb{P}[\tilde{Z}_\tau \geq 2] \geq \mathbb{P}[Z_\tau \geq 2] = 1 - \mathbb{P}[Z_\tau = 0] - \mathbb{P}[Z_\tau = 1], \quad \tau > \hat{\tau}_x.
\]  

Combining the bounds in (7) and (8), we get

\[
\text{UE}_x = \int_{\tau=0}^{\infty} u_x(\tau) d\tau \geq \int_{\tau=0}^{\hat{\tau}_x} x d\tau + \int_{\hat{\tau}_x}^{\infty} (1 - \mathbb{P}[Z_\tau = 0] - \mathbb{P}[Z_\tau = 1]) d\tau.
\]  

We next bound \( \text{ME} \). With a slight abuse of notation, let \( m(\tau) \) be the probability that the Myersonian ESP auction sells at price greater than or equal to \( \tau \). Then, by the definition of \( Z_\tau \) we have

\[
m(\tau) = \mathbb{P}[Z_\tau \geq 1] \geq 1 - \mathbb{P}[Z_\tau = 0].
\]  

Then, considering that \( \text{ME} = \int_{\tau=0}^{\infty} m(\tau) d\tau \) and by using Equations (9) and (10), we get

\[
2 \geq \text{UE}_x + \text{ME} \geq \int_{0}^{\hat{\tau}_x} (x + 1 - \mathbb{P}[Z_\tau = 0]) d\tau + \int_{\hat{\tau}_x}^{\infty} (2 - 2\mathbb{P}[Z_\tau = 0] - \mathbb{P}[Z_\tau = 1]) d\tau,
\]  

where the first inequality follows from our assumption that \( \max(\text{ME}, \text{UE}) = 1 \). To simplify the r.h.s. of (11), we make use of Lemma 9, stated at the end of this section, that says \( \mathbb{P}[Z_\tau = 0] \leq e^{-s(\tau)} \), and \( \mathbb{P}[Z_\tau = 0] + \mathbb{P}[Z_\tau = 1] \leq (2 + s(\tau))e^{-s(\tau)} \).

By Equation (11) and Lemma 9, we get

\[
\int_{0}^{\hat{\tau}_x} (x + (1 - e^{-s(\tau)})) d\tau + \int_{\hat{\tau}_x}^{\infty} (2 - 2e^{-s(\tau)} - s(\tau)e^{-s(\tau)}) d\tau \leq 2.
\]  

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The above equation holds for any \( x \in [0,1] \). Set \( x = j/k \). Then, we have

\[
2 \geq \int_0^{\tau_x} (x + (1 - e^{-s(\tau)}))d\tau + \int_{\tau_x}^{\infty} (2 - 2e^{-s(\tau)} - s(\tau)e^{-s(\tau)})d\tau
\]

\[
= \sum_{i=j+1}^{k} \int_{\tau_i}^{\tau_{i-1}} \frac{(x + (1 - e^{-s(\tau)}))}{s(\tau)}d\tau + \sum_{i=1}^{j} \int_{\tau_i}^{\tau_{i-1}} \frac{(2 - 2e^{-s(\tau)} - s(\tau)e^{-s(\tau)})}{s(\tau)}d\tau
\]

\[
\geq \sum_{i=j+1}^{k} w_i \frac{x + (1 - e^{-s_i})}{s_i} + \sum_{i=1}^{j} w_i \left( (2 - 2e^{-s_i} - s_i e^{-s_i}) \right), \tag{13}
\]

where the equality follows from the definition of \( \tau_j \)'s and \( \tau_x \), and the fact that at \( x = j/k, \tau_x = \tau_j \).

The second inequality follows from definition of \( w_i \)'s and \( s_i \)'s, and the fact that \( y \mapsto \frac{1}{y}(2 - (2 + y)e^{-y}) \) and \( y \mapsto \frac{1}{y}(x + 1 - e^{-y}) \) are decreasing in \( y \in [0,1] \) (for proof, see Lemma 10, which is stated at the end of this section) and that \( s(\tau) \) itself is a decreasing function. Then, since \( x = s_j = j/k \), Equation (13) gives us the first set of constraints in Problem LP-ESP.

**The Second Constraint:** We now apply the same procedure to bound \( ME = \int_{\tau=0}^{\infty} m(\tau)d\tau \).

By Equation (10), Lemma 9, and the fact that \( \max(ME,UE) = 1 \), we have

\[
1 \geq ME = \int_{\tau=0}^{\infty} m(\tau)d\tau \geq \int_{\tau=0}^{\infty} \frac{1 - e^{-s(\tau)}}{s(\tau)}s(\tau)d\tau \geq \sum_{i=1}^{k} w_i \frac{1 - e^{-s_i}}{s_i},
\]

which results in the second constraint of Problem LP-ESP. \( \square \)

**Lemma 9.** Let \( Z_\tau \) be the number of buyers with \( v_i \geq t'_i \geq \tau \); that is, \( Z_\tau = |\{i \in [n], v_i \geq t'_i \geq \tau\}| \).

Then,

\[
P[ Z_\tau = 0 ] \leq q_n(s(\tau)) \leq \lim_{n \to \infty} q_n(s(\tau)) = e^{-s(\tau)} \tag{14}
\]

\[
2P[ Z_\tau = 0 ] + P[ Z_\tau = 1 ] \leq r_n(s(\tau)) \leq \lim_{n \to \infty} r_n(s(\tau)) = (2 + s(\tau))e^{-s(\tau)}, \tag{15}
\]

where \( q_n(y) = (1 - \frac{y}{n})^n \) and \( r_n(y) = 2(1 - \frac{y}{n})^n + y(1 - \frac{y}{n})^{n-1} \).

**Lemma 10.** Functions \( y \mapsto \frac{1}{y}(2 - r_n(y)) \) and \( y \mapsto \frac{1}{y}(x + 1 - q_n(y)) \) are decreasing in \( y \in [0,1] \) for every positive integer \( n \) and every \( x \geq 0 \), where \( q_n(y) = (1 - \frac{y}{n})^n \) and \( r_n(y) = 2(1 - \frac{y}{n})^n + y(1 - \frac{y}{n})^{n-1} \).

In addition, functions \( y \mapsto \frac{1}{y}(2 - (2 + y)e^{-y}) \) and \( y \mapsto \frac{1}{y}(x + 1 - e^{-y}) \) are decreasing in \( y \in [0,1] \) and \( y \in [x, 1] \).\footnote{Note that \( \lim_{n \to \infty} \frac{1}{y}(2 - r_n(y)) = \frac{1}{y}(2 - (2 + y)e^{-y}) \) and \( \lim_{n \to \infty} \frac{1}{y}(x + 1 - q_n(y)) = \frac{1}{y}(x + 1 - e^{-y}) \).
}

Proofs of Lemmas 9, 10 are given in the appendix.

### 5 Revenue Bounds for a Finite Number of Buyers

The bounds presented in Theorems 5 and 7 hold for any number of buyers \( n \). As stated earlier, in online advertising markets, due to targeting and heterogeneous preference of buyers (advertisers), the number of buyers is rather small. Inspired by this fact, in this section, we obtain improved
bounds for the SPMs and ESP auctions when the number of buyers \( n \) is small. We use the same ideas as in Theorems 5 and 7 to get these improvements, with the main new ingredient being the usage of \( n \)-dependent bounds of Lemma 9, namely, \( q_n(\tau) \) and \( r_n(\tau) \), rather than the \( n \)-independent limiting versions of these quantities used in Theorems 7.

In Figure 1, we illustrate our improved bounds for the SPMs and ESP auctions. We also depict the best bound known for these mechanisms prior to this work, i.e., \( 1 - (1 - 1/n)^n \). As stated earlier, this bound is due to Chawla et al. (2010a). We observe that our bound for the SPMs and ESP auctions improves the prior bound by up to 3% and 4%, respectively. Further, the revenue bounds increase as the number of buyers decreases. This justifies the widespread use of the ESP auctions in advertising exchanges where the number of buyers in each auction is rather small.

In this section, to highlight the dependency of our bounds on the number of buyers \( n \), we denote the revenue of SPMs and ESP auctions with vector of prices \( p = (p_1, p_2, \ldots, p_n) \) by \( \text{SPM}_n(p) \) and \( \text{ESP}_n(p) \), respectively. We further denote the optimal revenue by \( \text{Opt}_n \).

### 5.1 Posted Price Mechanism

The following theorem is the main result of this section.

**Theorem 11 (SPM with a Finite Number of Buyers).** There exist reserve prices \( p = (p_1, p_2, \ldots, p_n) \) such that \( \text{SPM}_n(p) \geq \text{Opt}_n \cdot \max_k \left( \frac{1}{\text{LP-SPM}(n,k)} \right) \) for every positive integer \( k \), where

\[
\text{LP-SPM}(n,k) = \max_w \sum_{i=1}^k w_i \quad \text{s.t.} \quad \sum_{i=j+1}^k w_i \frac{s_i}{s_j} \leq 1, \quad \forall j \in [k-1] \\
\sum_{i=1}^k w_i \frac{1 - q_n(s_i)}{s_i} \leq 1 \\
w_i \geq 0, \quad \forall i \in [k]
\]

Here, \( s_i = i/k, i \in [k] \), and \( q_n(y) = (1 - \frac{y}{n})^n \).
The proof of Theorem 11 is similar to that of Theorem 7, and is provided in the appendix.

Table 3 shows \( \frac{1}{\text{LP-SPM}(n,k)} \) for \( n \in \{10\} \) and \( k = 200, 400, 800, \) and 1600. Here, again, \( k \) determines the granularity of our discretization.

| n  | k   | \( \frac{1}{\text{LP-SPM}(n,k)} \) | n  | \( \frac{1}{\text{LP-SPM}(n,k)} \) | n  | \( \frac{1}{\text{LP-SPM}(n,k)} \) | n  | \( \frac{1}{\text{LP-SPM}(n,k)} \) |
|----|-----|-----------------------------------|----|-----------------------------------|----|-----------------------------------|----|-----------------------------------|
| 1  | 200 | 1.0000                            | 2  | 0.7585                            | 3  | 0.7167                            | 4  | 0.6988                            | 5  | 0.6889                            |
|    | 400 | 1.0000                            | 2  | 0.7586                            | 3  | 0.7168                            | 4  | 0.6989                            | 5  | 0.6890                            |
|    | 800 | 1.0000                            | 2  | 0.7586                            | 3  | 0.7168                            | 4  | 0.6990                            | 5  | 0.6891                            |
|    | 1600| 1.0000                            | 2   | 0.7586                           | 3   | 0.7168                           | 4   | 0.6990                           | 5   | 0.6891                           |
| 6  | 200 | 0.6826                            | 7  | 0.6782                            | 8  | 0.6750                            | 9  | 0.6726                            | 10 | 0.6706                            |
|    | 400 | 0.6827                            | 7  | 0.6784                            | 8  | 0.6752                            | 9  | 0.6727                            | 10 | 0.6708                            |
|    | 800 | 0.6828                            | 7  | 0.6784                            | 8  | 0.6752                            | 9  | 0.6728                            | 10 | 0.6708                            |
|    | 1600| 0.6828                            | 7   | 0.6785                           | 8   | 0.6753                           | 9   | 0.6728                           | 10  | 0.6709                           |

Table 3: \( \frac{1}{\text{LP-SPM}(n,k)} \) for \( n \in \{10\} \) and \( k = 200, 400, 800, \) and 1600. For each \( n \in \{10\} \), the maximum value of \( \frac{1}{\text{LP-SPM}(n,k)} \) (among the \( k \)'s considered) is boldfaced.

5.2 Eager Second Price Mechanism

The following theorem presents our improved bounds for the ESP auction when the number of buyers is \( n \).

**Theorem 12** (ESP with a Finite Number of Buyers). There exists a vector of reserve prices \( \mathbf{p} = (p_1, p_2, \ldots, p_n) \) such that \( \text{ESP}(n, \mathbf{p}) \geq \text{Opt}_n \cdot \max_k (\frac{1}{\text{LP-ESP}(n,k)}) \) for every positive integer \( k \), where

\[
\text{LP-ESP}(n,k) = \max_w \sum_{i \in [k]} w_i \\
\text{s.t.} \quad \sum_{i=1}^{j} w_i \frac{2 - r_n(s_i)}{s_i} + \sum_{i=j+1}^{k} w_i \frac{s_j + (1 - q_n(s_i))}{s_i} \leq 2, \quad \forall j \in [k] \\
\sum_{i=1}^{k} w_i \frac{1 - q_n(s_i)}{s_i} \leq 1, \quad \forall i \in [k] \\
w_i \geq 0
\]

Here, \( s_i = i/k, \) \( i \in [k], \) \( q_n(y) = (1 - \frac{y}{n})^n, \) and \( r_n(y) = 2 \left(1 - \frac{y}{n}\right)^n + y \left(1 - \frac{2y}{n}\right)^{n-1}. \)

The proof of Theorem 12 is similar to the proof of Theorem 7; thus it is omitted. The only difference between the proofs is that here we provide tighter lower bounds for the revenue of the Myersonian and uniform ESP auctions using Lemma 9. This lemma provides \( n \)-dependent and \( n \)-independent bounds. The \( n \)-independent bounds, i.e., \( 2e^{-s_i} + s_i e^{-s_i} \) and \( e^{-s_i} \), were used in Theorem 7 while the \( n \)-dependent bounds, i.e., \( r_n(s_i) \) and \( q_n(s_i) \), are used in Theorem 12 to obtain an improved approximation factor. Observe that if in Problem \( \text{LP-ESP}(n,k) \), we replace \( r_n(s_i) \) and \( q_n(s_i) \) respectively with \( 2e^{-s_i} + s_i e^{-s_i} \) and \( e^{-s_i} \), we can recover Problem \( \text{LP-ESP}. \)
To better understand Theorem 12, in Table 4, we present \( \frac{1}{\text{LP-ESP}(n,k)} \) for \( n \in [10] \) and \( k = 200, 400, 800, \) and 1600.

| \( n \) | \( k \) | \( \frac{1}{\text{LP-ESP}(n,k)} \) | \( n \) | \( \frac{1}{\text{LP-ESP}(n,k)} \) | \( n \) | \( \frac{1}{\text{LP-ESP}(n,k)} \) | \( n \) | \( \frac{1}{\text{LP-ESP}(n,k)} \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 200 | 1.0000 | 2 | 0.7610 | 3 | 0.7207 | 4 | 0.7038 |
| 400 | 1.0000 | 0.7611 | 709 | 0.7209 | 0.7039 | 5 | 0.6944 |
| 800 | 1.0000 | 0.7611 | 0.7209 | 0.7040 | 0.6945 |
| 1600 | 1.0000 | 0.7611 | 0.7210 | 0.7040 | 0.6946 |

| 6 | 200 | 0.6884 | 7 | 0.6843 | 8 | 0.6813 | 9 | 0.6790 |
| 400 | 0.6886 | 0.6844 | 0.6814 | 0.6791 | 0.6773 |
| 800 | 0.6886 | 0.6845 | 0.6815 | 0.6792 | 0.6774 |
| 1600 | 0.6887 | 0.6846 | 0.6815 | 0.6792 | 0.6774 |

Table 4: \( \frac{1}{\text{LP-ESP}(n,k)} \) for \( n \in [10] \) and \( k = 200, 400, 800. \)

## 6 Reserve Prices for Ordered Buyers

We continue the study of the problem of setting reserve prices in ESP auctions with an extra constraint: we assume that the order among the reserves is exogenous. In other words, given an ordering \( \pi : [n] \to [n] \) and distributions \( F_1, \ldots, F_n \) for the buyers’ value, we want to choose reserve prices \( p = (p_1, p_2, \ldots, p_n) \) to maximize \( \text{ESP}(p) \) while ensuring that \( p_{\pi(1)} \leq \ldots \leq p_{\pi(n)} \). This problem can be written as follows

\[
\text{ESP}^*(\pi) = \max_p \text{ESP}(p) \quad \text{subject to} \quad p_{\pi(1)} \leq \ldots \leq p_{\pi(n)}. \tag{Ordered-ESP}
\]

In what follows, without loss of generality, we assume that \( \pi(i) = i, i \in [n] \). That is, we would like to have \( p_1 \leq p_2 \leq \ldots \leq p_n \). We note that as shown by Paes Leme et al. (2016) and Roughgarden and Wang (2016), the problem of optimizing reserve prices in the ESP auctions is NP-complete. Here, we show that this problem becomes tractable when we fix the order over reserve prices.

The motivation for studying this problem is threefold. First, there are often good heuristics to choose an ordering among buyers, such as ordering by the mean of the distribution, the monopoly price, the monopoly revenue, or the coefficient of variation. Second, the sellers might have reasons to set lower reserves for some buyers, for example, as a consequence of the deals and contracts that they might have signed. Finally, one can see order-based reserve prices as a technique to enhance the performance of the prior pricing heuristics (like, for example, Ronen (2001), Roughgarden and Wang (2016), and Paes Leme et al. (2016)). That is, one can enhance the revenue of a pricing heuristic by finding the best reserve prices that respect the order suggested by the heuristic. In Section 7, we show empirically that such an approach improves the revenue of these heuristics significantly.

We now focus on Problem Ordered-ESP. We show that this problem can be solved effectively. I.e., given a fixed ordering of the reserves, the optimal reserves can be computed in polynomial time. To be able to reason about computation, it is useful to assume that the value distributions have support in a finite discrete set. The result generalizes to continuous distributions by discretizing the distributions and applying the algorithm.
Our approach is based on dynamic programming (DP) method. We define \( \text{DP}_i(p) \) as the expected revenue of the ESP auction when only buyers 1, 2, \ldots, \( i \) participate in the auction and their reserve prices satisfy the following constraint: \( p_1 \leq p_2 \leq \ldots \leq p_i \leq p \). Let \( \tilde{F}_i(y) = \mathbb{P}[v_i < y] \) (this is slightly different from \( F_i(y) = \mathbb{P}[v_i \leq y] \)). For \( i \geq 1 \), \( \text{DP}_i(p) \) can be computed recursively as follows:

\[
\forall i \geq 0, \forall p \in V:\quad \text{DP}_{i+1}(p) = \max_{y \leq p} \{ \tilde{F}_{i+1}(y)\text{DP}_i(y) + (1 - \tilde{F}_{i+1}(y))R_{i+1}^\geq(y) \}, \quad (16)
\]

\[
\forall p \in V:\quad \text{DP}_0(p) = 0
\]

where \( R_{i+1}^\geq(y) \) is the revenue of the ESP auction when

1. only buyers 1, 2, \ldots, \( i+1 \) participate in the auction,
2. reserve price of all these \( i + 1 \) buyers is equal to \( y \),
3. the value distribution of buyer \( i + 1 \) is \( F_{i+1}^\geq \) where \( F_{i+1}^\geq \) is the distribution of \( F_{i+1} \) conditioned on the value being at least \( y \), and
4. the value distributions of buyers \( j \in [i] \) is \( F_j \).

To understand the DP, observe that in the first term of the r.h.s. of Equation (16), buyer \( i + 1 \) cannot clear his reserve price \( y \), i.e., \( v_{i+1} < y \). Then, considering the fact that \( p_1 \leq \ldots \leq p_{i+1} = y \), the revenue of the ESP auction in that case is equal to \( \text{DP}_i(y) \). In the second term of the r.h.s., buyer \( i + 1 \) clears his reserve price, i.e., \( v_{i+1} \geq y \). In that case, the revenue of the ESP auction is equal to \( R_{i+1}^\geq(y) \), as any buyer \( j \in [i] \) cannot win the item if his value is less than \( y \). That is, the effective reserve price for any buyer \( j \in [i] \) is \( y \).

In the following, we show that how to use the DP in (16) to compute the optimal reserve prices of Problem Ordered-ESP. To present our result, we need a few more definitions. Let \( V_i \) be the support of buyer \( i \)'s value distribution and let \( V = \bigcup_{i \in [n]} V_i \) be the support of buyers’ value. Define \( p^*_i, i \in [n] \), as the optimal reserve price of buyer \( i \) in Problem Ordered-ESP. Further, for any \( i \in [n - 1] \), let

\[
\text{OP}_{i+1}(p) = \arg \max_{y \leq p} \{ \tilde{F}_{i+1}(y)\text{DP}_i(y) + (1 - \tilde{F}_{i+1}(y))R_{i+1}^\geq(y) \}. \quad (17)
\]

That is, \( \text{OP}_{i+1}(p) \) is the value of \( y \leq p \) that maximizes the expression in \( \text{DP}_{i+1}(p) \); see Equation (16).

We are now ready to present our result.

**Theorem 13** (Optimal Solution of Problem Ordered-ESP). Without loss of generality, assume that \( \pi(i) = i \) for any \( i \in [n] \). Let \( \bar{p} = \max\{y : y \in V\} \), and let \( p^*_i \) be the optimal reserve price for buyer \( i \) in the Problem Ordered-ESP. Then,

\[
\begin{align*}
    p^*_n & = \text{OP}_n(\bar{p}) \\
    p^*_i & = \text{OP}_i(p^*_{i+1}) \quad \forall i \in [n - 1] \\
    \text{ESP}^*(\pi) & = \text{DP}_n(\bar{p}).
\end{align*}
\]

Here, \( \text{DP}_n(\bar{p}) \) and \( \text{OP}_i(\cdot) \) are respectively defined in Equations (16) and (17), \( \text{ESP}^*(\pi) \) is the optimal value of Problem Ordered-ESP. The optimal reserve prices \( \{p^*_i\}_{i \in [n]} \) can be computed in time \( O(n^4 |V|^2) \), where \( V \) is the support of buyers’ value.
Theorem 13 shows a simple polynomial time algorithm for computing the optimal order-based reserve prices recursively.

The proof of Theorem 13 is presented in the appendix. The proof at a high level is as follows. Except for the runtime, the rest of the theorem follows from the definition of \( \text{OP}_i(\cdot), \text{DP}_i(\cdot), \text{R}^i_{\geq}(\cdot) \), and the recursive equations given in (16) and (17). The runtime is derived by characterizing the complexity of computing \( \text{R}^i_{\geq}(y) \) and using the DP in Equation (16). We show that for each \( i \in [n] \) and \( y \in V \), \( \text{R}^i_{\geq}(y) \) can be computed in \( O(n^3 |V|) \). Thus, we construct a look-up table that includes \( \text{R}^i_{\geq}(y) \) for any \( i \in [n] \) and \( y \in V \) in \( O(n^4 |V|^2) \) time. With the look-up table and the DP presented in (16), the runtime of the dynamic programming algorithm is \( O(n^4 |V|^2) \).

7 Empirical and Numerical Studies

In this section, we evaluate our order-based reserve price optimization method in two settings. In the first setting we use synthetic data and in the second setting, we use the real auction dataset from Google’s advertising exchange. The dataset is fully anonymized and subsampled. We begin by describing the orderings over reserve prices that we consider for this study.

Order over Reserve Prices: We consider six orderings for reserve prices. The first four orderings are based on four important pricing heuristics for ESP auctions that have been proposed in the literature: monopoly reserve prices and heuristics proposed by Ronen (2001), Roughgarden and Wang (2016), and Paes Leme et al. (2016). We later describe these heuristics in detail. In these four orders, ESP auctions follow the order of reserve prices computed by these heuristics. The last two orderings are based on average and coefficient of variation (CV) of submitted bids of the buyers, where CV of a buyer is the standard deviation of his submitted bids divided by his average bids. Precisely, in the orderings based on the average (respectively CV) of bids, buyers with higher average bids (respectively CVs) are assigned higher reserve prices. As stated earlier, these last two orderings are motivated by Golrezaei et al. (2017).

We now explain these pricing heuristics.

Heuristics. Since the goal is to evaluate ordered-based ESP in datasets, it is best to explain the pricing heuristics by describing how to compute the reserve prices in them given a data set of \( K \) ESP auctions, with a vector of submitted bids for each auction. Let \( b^k_i \) be the submitted bid of buyer \( i \) in auction \( k \in [K] \), and, let \( b^{(2)}_k \) denote the second highest submitted bid in auction \( k \). Finally, let \( W_i \) be the subset of these \( K \) auctions in which buyer \( i \) submits the highest bid.

- Monopoly Reserve Prices (M): The M reserve price of buyer \( i \) is given by

\[
    p^M_i = \arg \max_y \sum_{k \in [K]} y \cdot \mathbb{1}(b^k_i \geq y). \tag{18}
\]

Monopoly reserve prices are inspired by their role in the design of the optimal auction (Myerson 1981). In particular, the monopoly price is the optimal reserve price when there is only a single buyer in the auction.

\footnote{For instance, if ESP follows the order of the monopoly reserve prices, and if the monopoly reserves are \( p^M_1 < p^M_2 < \ldots < p^M_n \), in Problem Ordered-ESP, we let \( \pi(i) = i \) for \( i \in [n] \) to ensure that \( p_1 \leq p_2 \leq \ldots \leq p_n \).}
- **Roughgarden and Wang’s Reserve Prices (RW):** The RW reserve price for buyer $i$ is computed as:

$$p_{RW}^i = \arg \max_y \sum_{k \in W_i} (y - b_k^{(2)})^+ \cdot I(b_k^i > y),$$

where $(y)^+$ is $y$ when $y \geq 0$, and zero otherwise. Roughgarden and Wang (2016) show that the best of $p_{RW}^i$ and zero reserve price obtains at least half of the optimal revenue.

- **Ronen’s Reserve Prices (R):** The R reserve price of buyer $i$ is computed as:

$$p_R^i = \arg \max_y \sum_{k \in W_i} y \cdot I(b_k^i \geq y).$$

This heuristic is proposed by Ronen (2001). The difference between Ronen’s and monopoly reserve prices is the set of auctions that each considers. To compute $p_R^i$, we only consider the set of auctions $W_i$ in which buyer $i$ has the highest submitted bid. To compute $p_M^i$, we consider all the auctions.

- **Paes Leme et al.’s Reserve Prices (PPV):** The PPV reserve price of buyer $i$ is given by:

$$p_{PPV}^i = \arg \max_y \sum_{k \in W_i} \max(y, b_k^i) \cdot I(b_k^i \geq y).$$

The PPV reserve prices are optimal when one is constrained to run a lazy second price auction.

### 7.1 First Setting: Synthetic Dataset

We assume that there are 10 buyers attending the ESP auction, where the buyer $i$’s value is drawn independently from a lognormal distribution $\text{Logn}(\mu_i, \sigma_i)$. Here, $\mu_i$ and $\sigma_i$ are independently drawn from a uniform distribution in $[0, 1]$. We note that lognormal distributions have proved to be a good fit for the distribution of valuations of advertisers in online advertising markets; see, for example, Edelman et al. (2007), Edelman and Schwarz (2006), Xiao et al. (2009), Balseiro et al. (2014), and Golrezaei et al. (2017).

As discussed earlier, we consider six orderings over reserve prices. The first four orderings are based on the four pricing heuristics, and the last two orders are based on the Coefficient of Variation (CV) and average of the submitted bids of buyers. Table 5 shows the expected revenue of the ESP auctions as a percentage of the optimal revenue. To compute these numbers, we consider 200 problem instances, where each problem instance is associated with a random set of parameters $\mu = (\mu_1, \ldots, \mu_{10})$ and $\sigma = (\sigma_1, \ldots, \sigma_{10})$. For each instance, we run 5000 auctions, i.e., draw 5000 independent values of $v_i \sim \text{Logn}(\mu_i, \sigma_i)$, and compute the expected revenue of the ESP auction across these auctions. This results in 200 values, one for each problem instance. The reported numbers in the table are obtained by taking the average over these 200 values.

We now discuss the results in the table.

---

Given that random variable $X \sim \text{Logn}(\mu, \sigma)$, then the mean and standard deviation of the variable’s natural logarithm are respectively $\mu_i$ and $\sigma_i$. 
• ESP: The second row of the table contains the expected revenue of ESP auctions as a percentage of the optimal revenue $\text{Opt}$, where the reserve prices come from M’s, RW’s, R’s and PPV’s heuristics. The columns corresponding to CV and Average (AVG) of bids are not well defined here because these only define orderings, and not any reserve prices: so ESP auctions with OR-CV or OR-AVG as reserve prices is not well defined.

• OR-ESP: The third row of the table contains the percentage of optimal revenue obtained by the expected revenue of the optimal order-based ESP. The columns refer to the percentage obtained when using the ordering from M, RW, R, PPV, OR-CV and OR-AVG. Note that the expected revenue of the optimal order-based ESP, namely $\text{ESP}^*(\pi)$ defined in Problem Ordered-ESP, is computed using our DP formulation.

• Rev Gain: The fourth row is just the percentage increase in the third row when compared to the second row. Precisely, for each pricing heuristic, the Rev Gain is equal to $100 \cdot \frac{\text{OR-ESP} - \text{ESP}}{\text{ESP}}$.

We observe that the OR-ESP auctions can improve the revenue of these prominent pricing heuristics by up to 5.89%, where the highest gain happens for RW’s reserve prices. The revenue gain of OR-ESP auctions is shown in the last row of the table. In addition, regardless of the order that we imposed, the performance of OR-ESP is better than that of the ESP auctions under the four pricing heuristics. As another observation, OR-AVG and OR-CV respectively obtain 96.8% and 98.4% of the optimal revenue, confirming the good performance of the order-based price optimization for ESP auctions. Finally, these results show that the ESP auctions can perform much better than our theoretical revenue bounds.

| Heuristics | M  | RW | R  | PPV | OR-CV | OR-AVG |
|------------|----|----|----|-----|-------|--------|
| ESP        | 93.2 | 93.3 | 95.7 | 93.3 | -     | -      |
| OR-ESP     | 96.2 | 98.8 | 98.8 | 97.8 | 98.4  | 96.8   |
| Rev Gain in % | 3.22 | **5.89** | 3.24 | 4.82 | -     | -      |

Table 5: The expected revenue of ESP and OR-ESP on synthetic data as a percentage of the optimal revenue. The standard error of all the numbers is less than 0.1%.

### 7.2 Second Setting: Real Auction Dataset

While using synthetic data makes our study replicable, it only allows us to evaluate the performance of the OR-ESP auction in a stylized environment where all the buyers participate in all the auctions and their valuations are independent of each other. To overcome this issue, in this section, we assess the performance of the OR-ESP auction in a more realistic environment where these assumptions do not necessarily hold. In particular, we use a dataset from Google’s advertising exchange that runs second price auctions with reserve, and we show that the OR-ESP auctions have a good performance despite the fact that submitted bids in our dataset are correlated across buyers. We work with a dataset that has impression level granularity (i.e., data for each auction). We perform our evaluation on four ad slots that have the highest traffic volume.

We randomly divide the auctions for each ad slot into two sets with equal size, creating training
and test dataset. We use the training dataset to optimize the reserve prices.\footnote{We only optimize the reserve prices of the top 10 buyers with the highest winning rate. This is due to the thinness of the auctions: i.e., the participation rate of other buyers is rather small. For these small buyers, we replace their bids with zero. Retaining the small buyers yields similar results too.} We then evaluate the optimized reserve prices on the test dataset. Precisely, we randomly sample 20\% of the test dataset 200 times.\footnote{The entire dataset contains a sample from over 100,000 real world auctions. Each sample of 20\% of the test dataset accounts for more than 20000 auctions.} We then compute the revenue of the ESP and OR-ESP auctions on each of the sampled dataset, and take the average.

To compute the reserve prices in OR-ESP auctions, we use our DP formulation, provided in Equation (16). To do so, we need to compute $R_i^\geq(y)$ for any $i \in [n]$ and $y \in V$; see the precise definition of $R_i^\geq(y)$ right after Equation (16). (As stated earlier, without loss of generality, we assume that $p_1 \leq p_2 \leq \ldots \leq p_n$.) To compute $R_i^\geq(y)$, we consider all the auctions in the training dataset in which buyer $i$ submits a bid greater than $y$, and set the bids of buyers $i+1, i+2, \ldots, n$ in these auctions to zero. Then, considering the submitted bids of buyers 1 to $i$ in these auctions, we compute the revenue of the ESP, assuming that the reserve price of buyers 1, 2, \ldots, $i$ is set to $y$. The average revenue of ESP in these auctions is $R_i^\geq(y)$.

Table 6 shows revenue obtained by ESP and OR-ESP auctions, for each of the top four ad slots, as the percentage of revenue of the second price auction with no reserve.\footnote{On real auction data, evaluating the optimal revenue is very challenging, as buyers’ values are correlated and value distributions may be irregular.} The meaning of the rows in Table 6 are similar to what we have in Table 5, with the only differences being

1. Table 6 has four different subtables, one for each ad slot. If one is looking to compare synthetic and real datasets’ results, we should compare Table 5 with each of these 4 subtables in Table 6.

2. The percentages in Table 6 are w.r.t. the second price auction with no reserves, instead of being w.r.t. the optimal auction, and hence the entries in each of the subtables of Table 6 are larger than 100.

We note here that our paper is the first work that evaluates and compares the performance of these prominent pricing heuristics in a real environment. Even before getting into how our DP formulation improves these heuristics significantly, there are already some interesting takeaways. For example, while it is clear that introducing reserve prices to second price auctions increases its revenue, Table 6 shows that in real data, the increase can be quite significant. For instance, in ad slot 1, the revenue of the ESP auction under all four pricing heuristics is more than 2 times the revenue of the second price auction with no reserve. For other ad slots, revenue of the ESP auction under the pricing heuristics is 28-86 percentage more than that of second price auction with no reserve. As another observation, PPV’s prices outperforms other pricing heuristics in most of the ad slots.

The main take-away is that in each ad slot, and for every pricing heuristic, the OR-ESP auction (obtained from our DP formulation) outperforms the ESP auction. This is particularly interesting as it empirically establishes that OR-ESP outperforms ESP even when the bidders’ distributions are correlated in real world setting. Furthermore, the revenue gains are often significant, reaching up to 8.75\%.
Table 6: The expected revenue of ESP and OR-ESP on real auction data as a percentage of the revenue of second price auction with no reserve. The standard error of all the numbers is less than 0.1%.

8 Conclusion

We improve the best-known performance guarantee on the revenue of two simple and commonly used auction formats, ESP auctions and SPMs. For each auction format, we present two revenue bounds. Our first revenue bound is universal: it is valid for any number of buyers \( n \). Our second revenue bound improves our universal bound via taking into account the number of buyers in the auctions: we show that when the number of buyers is small, ESP auctions and SPMs get a higher fraction of the optimal revenue. This justifies the wide-spread use of the ESP auctions in advertising markets where the number of buyers in each auction is rather small.

We then show that in the ESP auctions, reserve prices can be optimized in polynomial time once we fix their order. Such an order could be obtained from the buyer’s bidding behavior and/or the deals/contracts that the seller has with the buyer, or be obtained from existing heuristics for reserve price optimization. Via both empirical and numerical analysis, we show that the ESP auctions with exogenous orders perform well and can improve the performance of prominent pricing heuristics by up to 8.75%.

References

Abolhassani M, Ehsani S, Esfandiari H, Hajijaghayi M, Kleinberg RD, Lucier B (2017) Beating 1-1/e for ordered prophets. Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017, 61–71.

Agrawal S, Ding Y, Saberi A, Ye Y (2012) Price of correlations in stochastic optimization. Operations Research 60(1):150–162.

Allouah A, Besbes O (2018) Prior-independent optimal auctions .

Azar Y, Chiplunkar A, Kaplan H (2017) Prophet secretary: Surpassing the 1 – 1/e barrier. arXiv preprint arXiv:1711.01834 .

Balseiro SR, Feldman J, Mirokni V, Muthukrishnan S (2014) Yield optimization of display advertising with ad exchange. Management Science 60(12):2886–2907.

Bhalgat A, Feldman J, Mirokni V (2012) Online allocation of display ads with smooth delivery. Proceedings of the 18th ACM SIGKDD international conference on Knowledge discovery and data mining, 1213–1221 (ACM).

Calinescu G, Chekuri C, Pál M, Vondrák J (2011) Maximizing a monotone submodular function subject to a matroid constraint. SIAM Journal on Computing 40(6):1740–1766.
Celis LE, Lewis G, Mobius M, Nazerzadeh H (2014) Buy-it-now or take-a-chance: Price discrimination through randomized auctions. *Management Science* 60(12):2927–2948.

Chakraborty T, Even-Dar E, Guha S, Mansour Y, Muthukrishnan S (2010) Approximation schemes for sequential posted pricing in multi-unit auctions. *Internet and Network Economics - 6th International Workshop, WINE 2010, Stanford, CA, USA, December 13-17, 2010. Proceedings*, 158–169.

Chawla S, Fu H, Karlin AR (2014) Approximate revenue maximization in interdependent value settings. *ACM Conference on Economics and Computation, EC '14, Stanford , CA, USA, June 8-12, 2014, 277–294*.

Chawla S, Hartline JD, Kleinberg R (2008) Algorithmic pricing via virtual valuations. *CoRR* abs/0808.1671, URL http://arxiv.org/abs/0808.1671.

Chawla S, Hartline JD, Malec DL, Sivan B (2010a) Multi-parameter mechanism design and sequential posted pricing. *Proceedings of the forty-second ACM symposium on Theory of computing*, 311–320 (ACM).

Chawla S, Malec DL, Sivan B (2010b) The power of randomness in bayesian optimal mechanism design. *Proceedings 11th ACM Conference on Electronic Commerce (EC-2010), Cambridge, Massachusetts, USA, June 7-11, 2010, 149–158*, URL http://dx.doi.org/10.1145/1807342.1807366.

Chawla S, Sivan B (2014) Bayesian algorithmic mechanism design. *ACM SIGecom Exchanges* 13(1):5–49.

Correa J, Foncea P, Hoeksma R, Oosterwijk T, Vredeveld T (2017a) Posted price mechanisms for a random stream of customers. *Proceedings of the 2017 ACM Conference on Economics and Computation*, 169–186 (ACM).

Correa J, Foncea P, Pizarro D, Verdugo V (2017b) From pricing to prophets, and back. *Dynamic Pricing Workshop, Santiago de Chile, Chile, Dec 11-15, 2017*.

Dhangwatnotai P, Roughgarden T, Yan Q (2015) Revenue maximization with a single sample. *Games and Economic Behavior* 91:318–333.

Dütting P, Feldman M, Kesselheim T, Lucier B (2017) Prophet inequalities made easy: Stochastic optimization by pricing non-stochastic inputs. *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, 540–551.

Dütting P, Kleinberg R (2015) Polymatroid prophet inequalities. *Algorithms - ESA 2015 - 23rd Annual European Symposium, Patras, Greece, September 14-16, 2015, Proceedings*, 437–449.

Edelman B, Ostrovsky M, Schwarz M (2007) Internet advertising and the generalized second-price auction: Selling billions of dollars worth of keywords. *American economic review* 97(1):242–259.

Edelman B, Schwarz M (2006) Optimal auction design in a multi-unit environment: The case of sponsored search auctions. *Unpublished manuscript, Harvard Business School* .

Ehsani S, Hajiaghayi M, Kesselheim T, Singla S (2018) Prophet secretary for combinatorial auctions and matroids. *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, 700–714.

Esfandiari H, Hajiaghayi M, Liaghat V, Monemizadeh M (2017) Prophet secretary. *SIAM Journal on Discrete Mathematics* 31(3):1685–1701.

Feldman M, Gravin N, Lucier B (2015) Combinatorial auctions via posted prices. *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015*, 123–135.

Golrezaei N, Lin M, Mirrokni V, Nazerzadeh H (2017) Boosted second-price auctions for heterogeneous bidders .

Hajiaghayi MT, Kleinberg RD, Sandholm T (2007) Automated online mechanism design and prophet inequalities. *Proceedings of the Twenty-Second AAAI Conference on Artificial Intelligence, July 22-26, 2007, Vancouver, British Columbia, Canada, 58–65*, URL http://www.aaai.org/Library/AAAI/2007/aaai07-009.php.

Hammond PJ (1979) Straightforward individual incentive compatibility in large economies. *The Review of Economic Studies* 46(2):263–282.
Hartline JD, Roughgarden T (2009) Simple versus optimal mechanisms. Proceedings 10th ACM Conference on Electronic Commerce (EC-2009), Stanford, California, USA, July 6–10, 2009, 225–234.

Hill TP, Kertz RP (1981) Ratio comparisons of supremum and stop rule expectation. Z. Wahrsch. verw. Gebiete 56:283–285.

Hill TP, Kertz RP (1982) Comparisons of stop rules and supremum expectations of i.i.d. random variables. Ann. Probab. 10:336–345.

Holahan C (2008) Auctions on eBay: A Dying Breed. http://www.businessweek.com/technology/content/jun2008/tc2008062112762.htm.

Kennedy D (1987) Prophet-type inequalities for multi-choice optimal stopping. Stochastic Processes and their Applications 24(1):77 – 88, ISSN 0304-4149, URL http://dx.doi.org/https://doi.org/10.1016/0304-4149(87)90029-9.

Kertz RP (1986) Stop rule and supremum expectations of i.i.d. random variables: A complete comparison by conjugate duality. Journal of Multivariate Analysis 19(1):88 – 112, ISSN 0047-259X, URL http://dx.doi.org/https://doi.org/10.1016/0047-259X(86)90095-3.

Krengel U, Sucheston L (1977) Semiamarts and finite values. Bull. Amer. Math. Soc. 83(4):745–747, URL https://projecteuclid.org:443/euclid.bams/1183538915.

Krengel U, Sucheston L (1978) On semiamarts, amarts and processes with finite values. Adv. in Probab. 4:197–266.

Myerson RB (1981) Optimal auction design. Mathematics of operations research 6(1):58–73.

Paes Leme R, Pál M, Vassilvitskii S (2016) A field guide to personalized reserve prices. Proceedings of the 25th International Conference on World Wide Web, 1093–1102 (International World Wide Web Conferences Steering Committee).

Ronen A (2001) On approximating optimal auctions. Proceedings of the 3rd ACM conference on Electronic Commerce, 11–17 (ACM).

Roughgarden T, Wang JR (2016) Minimizing regret with multiple reserves. Proceedings of the 2016 ACM Conference on Economics and Computation, EC ‘16, Maastricht, The Netherlands, July 24–28, 2016, 601–616.

Rubinstein A, Singla S (2017) Combinatorial prophet inequalities. Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, 1671–1687.

Samuel-Cahn E (1984) Comparison of threshold stop rules and maximum for independent nonnegative random variables. Ann. Probab. 12(4):1213–1216, URL http://dx.doi.org/10.1214/aop/1176993150.

WSJ (2017) How google?s ad auctions work. https://www.wsj.com/articles/how-gogles-ad-auctions-work-1484827203.

Xiao B, Yang W, Li J (2009) Optimal reserve price for the generalized second-price auction in sponsored search advertising. Journal of Electronic Commerce Research 10(3):114.

Yan Q (2011) Mechanism design via correlation gap. Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms, 710–719 (Society for Industrial and Applied Mathematics).
A Other Proofs

A.1 Proof of Lemma 9

Lemma 9 (restated). Let $Z_r$ be the number of buyers with $v_i \geq t'_i \geq \tau$; that is, $Z_r = \{i \in [n], v_i \geq t'_i \geq \tau\}$. Then,
\[
P[Z_r = 0] \leq q_n(s(\tau)) \leq e^{-s(\tau)}
\]
\[
2P[Z_r = 0] + P[Z_r = 1] \leq r_n(s(\tau)) \leq (2 + s(\tau))e^{-s(\tau)},
\]
where $q_n(y) = (1 - \frac{y}{n})^n$ and $r_n(y) = 2 \left(1 - \frac{y}{n}\right)^n + y \left(1 - \frac{y}{n}\right)^{n-1}$.

Proof. Define $z_i = I(v_i \geq t'_i \geq \tau)$, where $I(\cdot)$ is an indicator function. Then, $Z_r$ can be written as follows
\[
Z_r = \sum_{i \in [n]} z_i,
\]
where $z_i$'s are independent 0/1 Bernoulli random variables with $E[z_i] = s(\tau)$. This implies that $E[Z_r] = \sum_{i \in [n]} E[z_i] = \sum_{i \in [n]} s_i(\tau) = s(\tau)$. Then,
\[
P[Z_r = 0] = \prod_{i \in [n]} P[z_i = 0] = \prod_{i \in [n]} (1 - s_i(\tau)) \leq \left(1 - \frac{\sum_{i \in [n]} s_i(\tau)}{n}\right)^n \leq e^{-\sum_{i \in [n]} s_i(\tau)} = e^{-s(\tau)},
\]
where the first inequality follows from the fact that for any sequence $a_1, a_2, \ldots, a_n$, we have $\prod_{i \in [n]} a_i \leq \left(\frac{\sum_{i \in [n]} a_i}{n}\right)^n$. By definition of $q_n(\cdot)$, the above equation leads to Equation (22), which is the first desired result.

Next, we will show Equation (23). We start by observing that the l.h.s. of this equation can be written as a symmetric polynomial in $s_1(\tau), \ldots, s_n(\tau)$, namely,
\[
2P[Z_r = 0] + P[Z_r = 1] = 2 \prod_{i \in [n]} (1 - s_i(\tau)) + \sum_{i \in [n]} s_i(\tau) \prod_{j \neq i} (1 - s_j(\tau)).
\]
In the rest of the proof, to ease the notation, we denote $s_i(\tau), i \in [n]$, by $s_i$. Define polynomial
\[
P_n(s_1, \ldots, s_n) := 2 \prod_{i \in [n]} (1 - s_i) + \sum_{i \in [n]} s_i \prod_{j \neq i} (1 - s_j).
\]
To provide an upper bound on $2P[Z_r = 0] + P[Z_r = 1]$, we show that subject to the constraint $\sum_{i \in [n]} s_i = s(\tau)$, the value of the polynomial $P_n$ is maximized when $s_1 = s_2 = \cdots = s_n = s(\tau)/n$.

To prove this, consider a point $s = (s_1, \ldots, s_n)$ such that $\sum_{i \in [n]} s_i = s(\tau)$. Pick any pair of coordinates (without loss of generality, 1 and 2) and consider increasing one and decreasing the other. Now, note that $P(s_1 + \delta, s_2 - \delta, s_3, \ldots, s_n)$ is a quadratic function of $\delta$. It is not difficult to verify that the quadratic coefficient is negative. Then, considering the fact that $P$ is symmetric, it follows that the maximum in this direction is achieved at $\delta$ such that $s_1 + \delta = s_2 - \delta$, i.e. when the two coordinates are equal, for every profile of values for the remaining coordinates. Since this argument holds for any pair of coordinates, it follows that $P(s)$ is maximized when all coordinates are equal, i.e., $s_i = s(\tau)/n$ for $i \in [n]$.

So far, we have established that
\[
2P[Z_r = 0] + P[Z_r = 1] \leq P_n\left(\frac{s(\tau)}{n}, \ldots, \frac{s(\tau)}{n}\right) = r_n(s(\tau)),
\]
where the equality follows from definition of $P_n$ and $r_n$. The above equation gives us the first desired inequality in (23). For the second inequality, observe that:
\[
P_n\left(\frac{s(\tau)}{n}, \ldots, \frac{s(\tau)}{n}\right) = P_{n+1}\left(\frac{s(\tau)}{n}, \ldots, \frac{s(\tau)}{n}, 0\right) \leq P_{n+1}\left(\frac{s(\tau)}{n+1}, \ldots, \frac{s(\tau)}{n+1}, \frac{s(\tau)}{n+1}\right).
\]
In particular,
\[
P_n\left(\frac{s(\tau)}{n}, \ldots, \frac{s(\tau)}{n}\right) \leq \lim_{k \to \infty} P_k\left(\frac{s(\tau)}{k}, \ldots, \frac{s(\tau)}{k}\right) = (2 + s(\tau))e^{-s(\tau)}.
\]
Lemma 10 (restated). Functions \( y \mapsto \frac{1}{y}(2 - r_n(y)) \) and \( y \mapsto \frac{1}{y}(x + 1 - q_n(y)) \) are decreasing in \( y \in [0, 1] \) for every positive integer \( n \) and every \( x \geq 0 \), where \( q_n(y) = (1 - \frac{2}{n})^n \) and \( r_n(y) = 2(1 - \frac{2}{n})^n + y(1 - \frac{2}{n})^{n-1} \). Further, the limiting functions are also decreasing, i.e., functions \( y \mapsto \frac{1}{y}(2 - (2 + y)e^{-y}) \) and \( y \mapsto \frac{1}{y}(x + 1 - e^{-y}) \) are decreasing in \( y \in [0, 1] \) and \( y \in [x, 1] \).

Proof. The proof follows from taking derivatives and showing that they are negative in the desired ranges. We first show that functions \( y \mapsto \frac{1}{y}(2 - (2 + y)e^{-y}) \) and \( y \mapsto \frac{1}{y}(x + 1 - e^{-y}) \) are decreasing respectively in \( y \in [0, 1] \) and \( y \in [x, 1] \). For the first function, note that \( \frac{d}{dy}(\frac{1}{y}(2 - (2 + y)e^{-y})) = \frac{1}{y^2}(2ye^{-y} + y^2e^{-y} + 2e^{-y} - 2) \leq 0 \) due to the inequality \( e^y \geq 1 + y + \frac{y^2}{2} \). For the second function,

\[
\frac{\partial}{\partial y} \left( \frac{1}{y}(x + 1 - e^{-y}) \right) = \frac{ye^{-y} - x - 1 + e^{-y}}{y^2} \leq 0,
\]

where the inequality holds because \( 1 + y \leq e^y \) and \( x \geq 0 \).

We next show that functions \( y \mapsto \frac{1}{y}(2 - r_n(y)) \) and \( y \mapsto \frac{1}{y}(x + 1 - q_n(y)) \) are decreasing in \( y \in [0, 1] \) and \( y \in [x, 1] \) respectively for every positive integer \( n \) and every \( x \geq 0 \), where \( q_n(\cdot) \) and \( r_n(\cdot) \) are defined in Lemma 9. While taking derivatives below, we assume \( n \geq 2 \). It is straightforward to verify for the case of \( n = 1 \).

By definition of \( r_n(\cdot) \), the derivative of the first function is given by

\[
\frac{d}{dy} \left( \frac{1}{y}(2 - r_n(y)) \right) = \frac{(1 - \frac{2}{n})^n - 2y(1 - \frac{2}{n}) + (n - 2)\frac{2}{n^2}}{y^2}.
\]

Note that the derivative is non-positive if

\[
y^2\frac{n - 1}{2n} + y(1 - \frac{2}{n}) + (1 - \frac{2}{n})^2 = 1 + \frac{n - 2}{n}y + \frac{n - 1}{2}\frac{y^2}{n^2} \leq \frac{1}{(1 - \frac{2}{n})^{n-2}},
\]

where the equality follows from a simple algebra. In the following, we will verify the inequality. This shows that \( \frac{1}{y}(2 - r_n(y)) \) is decreasing in \( y \). For any \( y \in [0, 1) \), we have

\[
\frac{1}{(1 - \frac{2}{n})^{n-2}} \geq \left( 1 + \frac{y}{n} + \frac{y^2}{n^2} \right)^{n-2} \geq 1 + (n - 2)\frac{2}{n} + \frac{(n - 2)(n - 3)}{2} \frac{y^2}{n^2} = 1 + \frac{n - 2}{n}y + \frac{n - 1}{2} \frac{(n - 2)}{n} \frac{y^2}{n^2}.
\]

The last equation is the desired result.

We next show that function \( \frac{1}{y}(x + 1 - q_n(y)) \) is decreasing in \( y \). The derivative of this function w.r.t. \( y \) is given by

\[
\frac{\partial}{\partial y} \left( \frac{1}{y}(x + 1 - q_n(y)) \right) = -\frac{x - 1 + y(1 - \frac{2}{n})^{n-1} + (1 - \frac{2}{n})^n}{y^2} = \frac{(1 - \frac{2}{n})^n - \frac{x + 1}{y} + y + 1 - \frac{2}{n}}{y^2}.
\]

To show that \( \frac{d}{dy}(\frac{1}{y}(x + 1 - q_n(y))) \leq 0 \), we verify that \( -\frac{x + 1}{(1 - \frac{2}{n})^{n-1}} + y + 1 - \frac{2}{n} \leq 0 \). For \( y < 1 \) and \( x \geq 0 \), we have

\[
\frac{x + 1}{(1 - \frac{2}{n})^{n-1}} \geq (x + 1) \left( 1 + \frac{y}{n} \right)^{n-1} \geq (x + 1) \left( 1 + \frac{n - 1}{n} \frac{y}{n} \right) \geq 1 + \frac{n - 1}{n} y.
\]

The last inequality implies that \( \frac{1}{y}(x + 1 - q_n(y)) \) is decreasing in \( y \).
\section*{A.3 Proof of Theorem 11}

\textbf{Theorem 11 (SPM with a Finite Number of Buyers, restated).} There exist reserve prices $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ such that $\text{SPM}_n(p) \geq \text{Opt}_n \cdot \max_k \left( \frac{1}{\text{LP-SPM}_n(k)} \right)$ for every positive integer $k$, where 

$$\text{LP-SPM}_n(k) = \max_{w} \sum_{i \in [k]} w_i$$

subject to 

$$\sum_{i=j+1}^{k} w_i \frac{s_j}{s_i} \leq 1, \quad \forall j \in [k-1] \quad \text{(LP-SPM-n)}$$

$$\sum_{i=1}^{k} w_i \left( 1 - \frac{q_a(s_i)}{s_i} \right) \leq 1,$$

$$w_i \geq 0, \quad \forall i \in [k]$$

Here, $s_i = i/k$, $i \in [k]$, and $q_a(y) = \left(1 - \frac{k}{y}\right)^a$.

\textit{Proof.} Similar to the proof of Theorem 5, we consider the posted price mechanism that chooses the best of Myersonian and uniform pricing rules. We then show that the revenue of this mechanism is at least $\max_k \left( \frac{1}{\text{LP-SPM}_n(k)} \right)$ fraction of the optimal revenue where $\text{LP-SPM}_n(k)$ is defined in Theorem 11.

By definition of $w_i$’s in \eqref{eq:wi}, the objective function of Problem LP-SPM-n is the optimal revenue. In addition, without loss of generality, we normalize the revenue of the posted price mechanism that chooses the best of Myersonian and uniform pricing rules to one; that is, $\max(\text{MP, UP}) = 1$. Thus, to show the result, it suffices to verify the constraints.

\textbf{The First Set of Constraints:} Define $\mathcal{F}_x = \inf\{\tau : s(\tau) \leq x\}$, $x \in [0, 1]$. In addition, with a slight abuse of notation, let $\text{UP}_x$ be the revenue of the posted price mechanism that posts a uniform price of $\mathcal{F}_x$ for all buyers. By definition of the uniform posted price mechanism, we have $\text{UP} \geq \max_{x \in [0, 1]} \text{UP}_x$. Then, by Equation \eqref{eq:9}, and our assumption that $\max(\text{MP, UP}) = 1$, we have

$$1 \geq \text{UP}_x \geq \int_0^{\mathcal{F}_x} x d\tau = \int_0^{\mathcal{F}_x} \frac{x}{s(\tau)} s(\tau) d\tau.$$ 

In the following, we set $x$ to $s_j = j/k$. Then, we get

$$1 \geq \int_0^{\mathcal{F}_x} \frac{s_j}{s(\tau)} s(\tau) d\tau \geq \sum_{i=j+1}^{k} \int_{\tau_i}^{\tau_{i-1}} \frac{s_j}{s(\tau)} s(\tau) d\tau \geq \sum_{i=j+1}^{k} \frac{s_j}{s_i} \frac{w_i}{s_i},$$

where the first inequality follows from the monotonicity of $s(\cdot)$ and definition of $\tau_i$’s and $\mathcal{F}_x$. The second inequality follows from definition of $w_i$ and the fact that $s(\cdot)$ is weakly decreasing. Note that the above equation verifies the first set of constraints.

\textbf{Second Constraint:} Let $m(\tau)$ be the probability that the Myersonian posted price mechanism sells with price at least $\tau$. Then, by construction of the prices, $t_i$’s, in this mechanism, we have $m(\tau) = 1 - \mathbb{P}[Z_\tau = 0]$, where $Z_\tau$ is the number of buyers that satisfy $s_i \geq t_i \geq \tau$. This implies that 

$$1 \geq \text{MP} = \int_0^{\infty} (1 - \mathbb{P}[Z_\tau = 0]) d\tau \geq \int_0^{\infty} (1 - \frac{q_a(s(\tau))}{s(\tau)}) d\tau$$

$$= \int_0^{\infty} \frac{1 - q_a(s(\tau))}{s(\tau)} s(\tau) d\tau \geq \sum_{i=1}^{k} \frac{w_i}{s_i} \frac{1 - q_a(s_i)}{s_i},$$

where the second inequality follows from Lemma 9 and third inequality follows from Lemma 10 where we show $\frac{1 - q_a(y)}{y}$ is decreasing in $y$. The above equation gives the second constraint, and completes the proof.

\section*{A.4 Proof of Theorem 13}

\textbf{Theorem 13 (Optimal Solution of Problem Ordered-ESP, restated).} Without loss of generality, assume that $\pi(i) = i$ for any $i \in [n]$. Let $\bar{p} = \max \{ y : y \in V \}$, and let $p^*_i$ be the optimal reserve price for buyer $i$ in the
Problem **Ordered-ESP**. Then,

\[
\begin{align*}
p_n^* &= \text{OP}_n(\bar{p}) \\
p_i^* &= \text{OP}_i(p_{i+1}^*) \quad \forall i \in [n-1] \\
\text{ESP}^*(\pi) &= \text{DP}_n(\bar{p}),
\end{align*}
\]

where

\[
\begin{align*}
\forall i \geq 0, \forall p \in V : \quad \text{DP}_{i+1}(p) &= \max_{y \leq p} \{ \tilde{F}_{i+1}(y) \text{DP}_i(y) + (1 - \tilde{F}_{i+1}(y)) \text{R}_n^*(y) \}, \\
\forall i \geq 0, \forall p \in V : \quad \text{OP}_{i+1}(p) &= \arg \max_{y \leq p} \{ \tilde{F}_{i+1}(y) \text{DP}_i(y) + (1 - \tilde{F}_{i+1}(y)) \text{R}_n^*(y) \}, \\
\forall p \in V : \quad \text{DP}_0(p) &= 0, \\
\forall i > 0, \forall y \in V : \quad \tilde{F}_i(y) &= \mathbb{P}[v_i < y],
\end{align*}
\]

and \(\text{ESP}^*(\pi)\) is the optimal value of Problem **Ordered-ESP**. The optimal reserve prices \(\{p_i^*\}_{i \in [n]}\) can be computed in time \(O(n^4 |V|^2)\), where \(V\) is the support of buyers’ value.

**Proof.** By definition, \(\text{DP}_n(\bar{p})\) is the expected revenue of the ESP auction when buyers 1, 2, \ldots, \(n\) participate in the auction and their reserve prices satisfy the following equation: \(p_1 \leq p_2 \leq \ldots \leq p_n \leq \bar{p}\). This implies that \(\text{ESP}^*(\pi) = \text{DP}_n(\bar{p})\) and reserve price of buyer \(n\) is \(p_n^* = \text{OP}_n(\bar{p})\). Similarly, the reserve price of buyer \(n-1\) is equal to \(\text{OP}_{n-1}(p_n^*)\) because by constraints of Problem **Ordered-ESP**, \(p_1 \leq p_2 \leq \ldots \leq p_n\). Likewise, it follows that \(p_i^* = \text{OP}_i(p_{i+1}^*)\) for any \(i \in [n-1]\).

To establish the runtime, note that the optimal value of this problem, \(\text{ESP}^*(\pi)\), is equal to \(\text{DP}_n(\bar{p})\). To compute \(\text{DP}_n(\bar{p})\), we have to evaluate \(\text{R}_n^*(y)\) for any \(i \in [n]\) and \(y \in V\). Lemma 14 establishes the complexity of computing \(\text{R}_n^*(y)\) for any given \(i\), \(y\) to be \(O(n^4 |V|)\). Using this we are able to construct a look-up table that contains \(\text{R}_n^*(y)\) for all values of \(i \in [n]\) and \(y \in V\) in \(O(n^4 |V|^2)\). Having access to this look-up table, we next show that the complexity of computing \(\text{DP}_n(\bar{p})\) is \(O(n^4 |V|^2)\). To compute \(\text{DP}_n(\bar{p})\), we need to evaluate \(\text{DP}_i(y)\) for any \(i \in [n]\) and \(y \in V\). Further, computing \(\text{DP}_i(y)\) involves evaluating the maximum over \(O(|V|)\) values. Thus, by having access to the look-up table, we can compute \(\text{DP}_n(\bar{p})\) in \(O(n^4 |V|^2)\). Considering the fact that the look-up table can be calculated in \(O(n^4 |V|^2)\), the overall complexity of computing \(\text{DP}_n(\bar{p})\) is \(O(n^4 |V|^2) + O(n^4 |V|^2) = O(n^4 |V|^2)\).

**Lemma 14.** For any \(y \in V\) and \(i \in [n]\), the quantity \(\text{R}_i^*(y)\) can be computed in \(O(n^4 |V|)\).

**Proof.** **Proof of Lemma 14:** Observe that the complexity of \(\text{R}_i^*(y)\) is increasing in \(i\). Thus, in the following, we present the complexity of computing \(\text{R}_i^*(y)\).

Decompose the probability space into events of the following types. Throughout the proof, when more than one buyer wins, i.e., when more than one buyer has the highest submitted bid, we choose the lexicographically largest as the winner. Followed by this, if more than one buyer acts as the price-setter, we declare the lexicographically largest as the price-setter.

- **Type-1:** Winner is priced at \(y\).
- **Type-2:** Winner is priced at \(p > y\), where \(p\) is the second highest buyer’s bid (which also equals his value).

In a Type-1 event, the revenue is exactly \(y\). So, the total revenue contribution from this event is \(y \cdot \mathbb{P}[\text{Type-1 event}]\).

Type-1 event occurs when either no bidder \(i\) \((i \neq n)\) has a value strictly larger than \(y\), in which case bidder \(n\) is the winner (even when there is a tie, due to our lexicographically largest tie-breaking), or when exactly one bidder \(i\) \((i \neq n)\) has a value strictly larger than \(y\), and bidder \(n\) has value equal to \(y\). The probability of these two events is expressed below:

\[
\mathbb{P}[\text{Type-1 event}] = \sum_{i \in [n-1]} \left[ (1 - F_i(y)) \cdot \left( \prod_{j \in (1,i)} F_j(y) \right) \cdot \frac{f_n(y)}{1 - F_n(y)} \right] + \prod_{j \notin (n)} F_j(y).
\]

In a Type-2\(_p\) event, the revenue is \(p \cdot \mathbb{P}[\text{Type-2\(_p\) event}]\). So, the total revenue contribution from all the Type-2\(_p\) events, \(p \in V\), is \(\sum_{p \in V} p \cdot \mathbb{P}[\text{Type-2\(_p\) event}]\). To compute the probability of a Type 2\(_p\) event: note that it can happen when

- there is either exactly one buyer with value strictly larger than \(p\) and at least one buyer with value at \(p\), or,
- there are no buyers with value strictly above \(p\) and at least two buyers with value equal to \(p\).
The event of having exactly one buyer above \( p \) and at least one buyer with value at \( p \) can be split further as follows. Let \( i \) be the winner, and let \( j \) be the price-setter.

1. \( i \neq n, j \neq n \). In this case, given our reverse lexicographic tie-breaking, for \( j \) to be price-setter, we need: \( v_i > p, \ v_j = p, \ v_k \leq p \) for \( k < j \ (k \neq i) \) and \( v_k < p \) for \( k > j \ (k \neq i) \). The probability of this event is:

\[
P_{11} = \sum_{i \in [n-1]} \left[ (1 - F_i(p)) \cdot \frac{\tilde{F}_u(p)}{1 - \tilde{F}_u(y)} \cdot \sum_{j \notin \{i, n\}} \left\{ f_j(p) \cdot \left( \prod_{k < j, k \notin \{i\}} F_k(p) \right) \cdot \left( \prod_{k > j, k \notin \{i, n\}} \tilde{F}_k(p) \right) \right\} \right].
\]

2. \( i \neq n, j = n \). We need \( v_i > p, \ v_j = p, \ v_k \leq p \) for \( k < n \ (k \neq i) \). The probability of this event is:

\[
P_{12} = \sum_{i \in [n-1]} \left[ (1 - F_i(p)) \cdot \frac{f_u(p)}{1 - F_u(y)} \cdot \left( \prod_{k < n, k \notin \{i\}} F_k(p) \right) \right].
\]

3. \( i = n, j \neq n \). We need \( v_n > p, \ v_j = p, \ v_k \leq p \) for \( k < j \ (k \neq n) \) and \( v_k < p \) for \( k > j \ (k \neq n) \). The probability of this event is:

\[
P_{13} = \frac{1 - F_n(p)}{1 - F_n(y)} \sum_{j \notin \{n\}} \left[ f_j(p) \cdot \left( \prod_{k < j, k \notin \{i\}} F_k(p) \right) \cdot \left( \prod_{k > j, k \notin \{n\}} \tilde{F}_k(p) \right) \right].
\]

The event of there being no buyers with value strictly above \( p \) and at least two buyers with a value equal to \( p \) can be split further as follows. Let \( i \) be the winner and \( j \) be the price-setter (and therefore both have value at \( p \)).

1. \( i \neq n \) (note that \( j < i \), because otherwise by tie-breaking we would have declared \( j \) as the winner). We need \( v_i = p, \ v_j = p, \ v_k \leq p \) for \( k < j \) and \( v_k < p \) for \( k > j \ (k \neq i) \). The probability of this event is:

\[
P_{21} = \sum_{i \in [n-1]} \left[ f_i(p) \cdot \sum_{j < i} \left\{ f_j(p) \cdot \left( \prod_{k < j} F_k(p) \right) \cdot \left( \prod_{k > j, k \notin \{i\}} \tilde{F}_k(p) \right) \cdot \frac{\tilde{F}_u(p)}{1 - \tilde{F}_u(y)} \right\} \right].
\]

2. \( i = n \). We need \( v_n = p, \ v_j = p, \ v_k \leq p \) for \( k < j \) and \( v_k < p \) for \( k > j \ (k \neq n) \). The probability of this event is:

\[
P_{22} = \frac{f_n(p)}{1 - F_n(y)} \sum_{j < n} \left[ f_j(p) \cdot \left( \prod_{k < j} F_k(p) \right) \cdot \left( \prod_{k > j, k \notin \{n\}} \tilde{F}_k(p) \right) \right].
\]

For each of these five probabilities \( \{P_{11}, P_{12}, P_{13}, P_{21}, P_{22}\} \), it is straightforward to observe, given the double summation over \( i, j \), that we are adding over at most \( n^2 \) terms, and that each of these terms involves computing a product of at most \( n \) terms. Thus, computing the probability for any Type-2 event, namely \( P_{11} + P_{12} + P_{13} + P_{21} + P_{22} \), for any given \( p \) takes \( O(n^3) \) time. Thus, computing the type-2 \( p \) probability for every \( p \in V \) takes \( O(n^3 |V|) \) time, and as a result, \( R^n \) (\( y \)) can be computed in \( O(n^3 |V|) \) time.

\[\Box\]