Expansion of polynomial Lie group integrals in terms of certain maps on surfaces, and factorizations of permutations

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Abstract
Using the diagrammatic approach to integrals over Gaussian random matrices, we find a representation for polynomial Lie group integrals as infinite sums over certain maps on surfaces. The maps involved satisfy a specific condition: they have some marked vertices, and no closed walks that avoid these vertices. We also formulate our results in terms of permutations, arriving at new kinds of factorization problems.

Keywords: Lie groups, quantum chaos, maps on surfaces, factorizations of permutations

(Some figures may appear in colour only in the online journal)

1. Introduction

We are interested in integrals over the orthogonal group $O(N)$ of $N$-dimensional real matrices $O$ satisfying $OO^T = 1$, where $T$ means transpose, and over the unitary group $U(N)$ of $N$-dimensional complex matrices $U$ satisfying $UU^\dagger = 1$, where $\dagger$ means transpose conjugate. These groups have a unique invariant probability measure, known as Haar measure, and integrals over them may be seen as averages over ensembles of random matrices.

We will consider averages of functions that are polynomial in the matrix elements, i.e. quantities like

$$\int_{O(N)} dO \prod_{k=1}^n U_{u_k} U_{p_k}^\dagger$$

for the orthogonal group and

$$\int_{U(N)} dU \prod_{k=1}^n U_{u_k} U_{p_k}^\dagger$$

for the unitary group.
\[
\int_{\mathcal{O}(N)} dO \prod_{k=1}^{n} O_{k,0} O_{i,j}^{k}
\]  
(2)

for the orthogonal one (results for the unitary symplectic group \(Sp(N)\) are very close to those for \(\mathcal{O}(N)\), so we do not consider this group in detail). From the statistical point of view, these are joint moments of the matrix elements, considered as correlated random variables. Their study started in physics [1, 2], with applications to quantum chaos [3–5], and found its way into mathematics, initially for the unitary group [6] and soon after for the orthogonal and symplectic ones [7, 8]. Since then, they have been extensively explored [9, 10], related to Jucys–Murphy elements [11, 12] and generalized to symmetric spaces [13]. Unsurprisingly, after these developments some new applications have been found in physics [14–19], especially in quantum physics when some ‘complex’ or ‘random’ system—represented by a unitary or orthogonal matrix—must be ‘averaged out’ by being integrated over. In scattering problems, for example, the scattering matrix may behave as a random unitary matrix in some circumstances.

For the unitary group average, equation (1), the result is different from zero only if the \(q\)-labels are a permutation of the \(i\)-labels, and the \(p\)-labels are a permutation of the \(j\)-labels. The basic building blocks of the calculation, usually called Weingarten functions, are of the kind

\[
W_{\mathcal{U}}^U(\pi) = \int_{\mathcal{U}(N)} dU \prod_{k=1}^{n} U_{k,0} U_{i,j}^{k}
\]  
(3)

where \(\pi\) is an element of the permutation group, \(\pi \in S_n\). In general, if there is more than one permutation relating the sets of labels (due to repeated indices, e.g. \(|U_{1,2}|\)), the result is a sum of Weingarten functions.

The cycletype of a permutation in \(S_n\) is a partition of \(n\), \(\alpha \vdash n\), whose parts are the lengths of its cycles; the function \(W_{\mathcal{U}}^U(\pi)\) depends only on the cycletype of \(\pi\) [6, 7]. An explicit formula for it can be found in terms of irreducible characters \(\chi_\lambda(\pi)\) of \(S_n\) (for a very short derivation of the result below, see [20]). Namely, it is given by

\[
W_{\mathcal{U}}^U(\pi) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{d_{\lambda}}{[N^n_\lambda]} \chi_\lambda(\pi),
\]  
(4)

where \(d_\lambda\) is the dimension of a irreducible representation of \(S_n\) and

\[
[N^n_\lambda] = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (\alpha(j - 1) + N - i + 1).
\]  
(5)

The result of the orthogonal group average, equation (2), is different from zero only if the \(i\)-labels satisfy some matching (see section 2.1 for matchings) and the \(j\)-labels also satisfy some matching [7–9]. For concreteness, we may choose the \(i\)’s to satisfy only the trivial matching, and the \(j\)’s to satisfy only some matching \(m\) of coset type \(\beta\). The basic building blocks of polynomial integrals over the orthogonal group are

\[
W_{\mathcal{O}}^O(\beta) = \int_{\mathcal{O}(N)} dOO \cdots O_{a,b} O_{a,b} \cdots O_{a,b} \cdots
\]  
(6)

As happens for the unitary group, an explicit formula can be found involving quantities related to representation theory. In this case it is

\[
W_{\mathcal{O}}^O(\beta) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{d_{2\lambda}}{[N^n_\lambda]} \omega_\lambda(\beta),
\]  
(7)

where \(\omega_\lambda\) are some zonal spherical functions.
As examples of unitary and orthogonal Weingarten functions, let us mention

\[ W_{g_2}^{U}(2) = \int_{H(2)} dU U_1 U_2^* U_2 U_1^* = \frac{-1}{(N-1)N(N+1)}, \]  

(8)

corresponding to the permutation \( \pi = (12) \), which has cycletype \((2)\), and

\[ W_{g_2}^{O}(2) = \int_{O(N)} dO O_1 O_2 O_2 O_1 = \frac{-1}{(N-1)N(N+2)}, \]  

(9)

corresponding to the matching \( \{1, 2\}, \{\hat{1}, \hat{2}\} \) for the \( j \)'s, which has coset type \((2)\).

It is well known that small blocks of large random unitary/orthogonal matrices behave like Gaussian random variables. Therefore, it is natural to ask about the \( N \gg 1 \) regime of the Weingarten functions. The explicit formulas above are not very transparent, and different treatments have been developed [4, 6, 11, 21, 22] which relate the problem to combinatorial questions involving maps and factorizations of permutations.

This work is a contribution to this line of investigation. Both in the unitary and orthogonal cases, our approach is different from the previous ones and the maps/factorizations we encounter are new. In particular, the factorizations must satisfy some very restrictive conditions on the positions of some elements of the factors, which apparently have never been considered. We were originally motivated by applications in physics, in the semiclassical approximation to the quantum mechanics of chaotic systems. The problem in question was to construct correlated sets of chaotic trajectories that are responsible for the relevant quantum effects in the semiclassical limit [40]. Connections between this topic and factorizations of permutations had already been noted [39, 41–44]. In [18] we suggested that such sets of trajectories could be obtained from the diagrammatic expansion (in terms of the maps used in the present work) of a certain matrix integral, provided the dimension of the matrices be set to zero after the calculation. When we realized the connection to truncated random unitary matrices, this became our theorem 1. Initially restricted to systems without time-reversal symmetry, this method was later extended to systems that have this symmetry [19]; the connection with truncated random orthogonal matrices then gave rise to our theorem 3.

Our calculation will proceed by considering Weingarten functions in the context of ensembles of random truncated matrices; then relating those to Gaussian ensembles, and finally using the rich combinatorial structure which is already known for these.

In the next section we review some basic concepts and facts related to permutations, maps and Gaussian random matrices. In section 3 we collect some previous results about the combinatorics of large-\( N \) expansion of Weingarten functions. This helps put our work in perspective and also serves to group in a single place results that were scattered through the literature. In section 4 we present a sketch of our results. Details of calculations are shown in section 5 for the unitary group and in section 6 for the orthogonal group.

The connection between our previous work [18] and truncated unitary matrices was first suggested by Yan Fyodorov.

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2. Basic concepts

In this section we review some basic concepts and facts that are needed in order to understand the combinatorics associated with the large-\( N \) expansion of Weingarten functions.
2.1. Partitions, permutations and matchings

By $\alpha \vdash n$ we mean $\alpha$ is a partition of $n$, i.e. a weakly decreasing sequence of positive integers such that $\sum \alpha_i = |\alpha| = n$. The number of non-zero parts is called the length of the partition and denoted by $\ell(\alpha)$.

The group of permutations of $n$ elements is $S_n$. The cycletype of $\pi \in S_n$ is the partition $\alpha \vdash n$ whose parts are the lengths of the cycles of $\pi$. The length of a permutation is the number of cycles it has, $\ell(\pi) = \ell(\alpha)$, while its rank is $r(\pi) = n - \ell(\pi) = r(\alpha)$. We multiply permutations from right to left, e.g. $(13)(12) = (123)$. The conjugacy class $C_\lambda$ is the set of all permutations with cycletype $\lambda$. Its size is $|C_\lambda| = n!/\prod j^v_j$, with $v_j$ the number of times part $j$ occurs in $\lambda$.

Let $[n] = \{1, \ldots, n\}$, $\hat{[n]} = \{\hat{1}, \ldots, \hat{n}\}$ and let $[n] \cup [\hat{n}]$ be a set with $2n$ elements. A matching on this set is a collection of $n$ disjoint subsets with two elements each. The set of all such matchings is $M_n$. The trivial matching is defined as $t = \{(1, \hat{1}), (2, \hat{2}), \ldots, (n, \hat{n})\}$. A permutation $\pi$ acts on matchings by replacing blocks like $\{a, b\}$ by $\{\pi(a), \pi(b)\}$. If $\pi(t) = m$ we say that $\pi$ produces the matching $m$.

Given a matching $m$, let $G_m$ be a graph with $2n$ vertices having labels in $[n] \cup [\hat{n}]$, two vertices being connected by an edge if they belong to the same block in either $m$ or $t$. Since each vertex belongs to two edges, all connected components of $G_m$ are cycles of even length. The coset type of $m$ is the partition of $n$ whose parts are half the number of edges in the connected components of $G_m$. See examples in figure 1. A permutation has the same coset type as the matching it produces.

We realize the group $S_{2n}$ as the group of all permutations acting on the set $[n] \cup [\hat{n}]$. It has a subgroup called the hyperoctahedral, $H_n$, with $|H_n| = 2^n n!$ elements, which leaves invariant the trivial matching and is generated by permutations of the form $(a \ a)\hat{b}$ or of the form $(a b)(\ a)\hat{b}$. The cosets $S_{2n}/H_n \sim M_n$ can therefore be represented by matchings. The trivial matching identifies the coset of the identity permutation. We may inject $M_n$ into $S_{2n}$ by using fixed-point free involutions. This is done by the simple identification $m = \{(a_1, a_2), (a_3, a_4), \ldots\} \mapsto \sigma_m = (a_1 a_2)(a_3 a_4) \cdots$.

The double cosets $H_n \setminus S_{2n}/H_n$, on the other hand, are indexed by partitions of $n$: two permutations belong to the same double coset if and only if they have the same coset type [45].
(hence this terminology). We denote by $K_{\lambda}$ the double coset of all permutations with coset type $\lambda$.

Given a sequence of $2n$ numbers, $(i_1, ..., i_{2n})$ we say that it satisfies the matching $m$ if the elements coincide when paired according to $m$. This is quantified by the function $\Delta_m(t) = \prod_{b \in m} \delta_{b_1, b_2}$, where the product runs over the blocks of $m$ and $b_1, b_2$ are the elements of block $b$.

### 2.2. Maps

Maps are graphs drawn on a surface, with a well defined sense of rotation around each vertex. We show in figure 2 two different maps that correspond to the same graph. We shall represent the edges of a map by fat ribbons. These ribbons merge at the vertices, which are represented by disks, and the region of a vertex between two ribbons is called a corner.

Each one of our ribbons has two boundaries. It is possible to go from one vertex to another by walking along any one of those boundaries. Upon arriving at a vertex, a walker may move around the boundary of the disk to a boundary of the next ribbon (i.e. around a corner), and then depart again. Such a walk we call a boundary walk. A boundary walk that eventually retraces itself is called closed and delimits a face of the map.

Let $V$, $E$ and $F$ be respectively the numbers of vertices, edges and faces of a map. For instance, both examples in figure 2 have $V = 4$, $E = 4$ and $F = 2$. The Euler characteristic of a map is $\chi = V - E + F$, and it is additive in the connected components (we do not require maps to be connected, and each connected component is understood to be drawn on a different surface). For example, it is very well known that $\chi = 2$ for every convex polyhedron.

In all of the maps used in this work, ribbons have boundaries of two different colors. For convenience, we represent those colors by simply drawing these boundaries with two types os lines: dashed lines and solid lines. Ribbons are attached to vertices in such a way that all corners and all faces have a well defined color, i.e. our maps are face-bicolored. Examples are shown in figures 5 and 6.

### 2.3. Non-hermitian Gaussian random matrices

We consider $N$-dimensional matrices for which the elements are independent and identically-distributed Gaussian random variables, the so-called Ginibre ensembles [23]. For real matrices we will use the notation $M$, for complex ones we use $Z$.

Normalization constants for these ensembles are defined as

$$Z_R = \int dMe^{-\frac{1}{2} \text{Tr}(M^2)}, \quad Z_C = \int dZe^{-\text{Tr}(Z^2)}.$$  

(10)
They can be computed (as we do in sections 5 and 6) using singular value decomposition. Average values are denoted by

\[ \langle f(M) \rangle = \frac{1}{Z} \int dMe^{-\frac{1}{2}Tr(MM^\dagger)}f(M), \tag{11} \]

and

\[ \langle f(Z) \rangle = \frac{1}{Z} \int dZe^{-\frac{1}{2}Tr(ZZ^\dagger)}f(Z), \tag{12} \]

the meaning of \( \langle \cdot \rangle \) being clear from context. We have the simple covariances

\[ \langle M_{ab}M_{cd} \rangle = \frac{1}{\Omega} \delta_{ac}\delta_{bd}, \tag{13} \]

and

\[ \langle Z_{ab}Z_{cd} \rangle = \langle Z_{ab}Z_{cd}^\dagger \rangle = 0, \quad \langle Z_{ab}Z_{cd}^* \rangle = \frac{1}{\Omega} \delta_{ac}\delta_{bd}. \tag{14} \]

Polynomial integrals may be computed using Wick’s rule, which is a combinatorial prescription for combining covariances. It simply states that, since the elements are independent, the average of a product can be decomposed in terms of products of covariances. In the complex case, we may consider the elements of \( Z \) fixed and then permute the elements of \( Z^\dagger \) in all possible ways,

\[ \left\langle \prod_{k=1}^{n} Z_{ab} Z_{cd}^k \right\rangle = \frac{1}{\Omega^n} \sum_{\pi \in S_n} \prod_{k=1}^{n} \delta_{a_{\pi(k)} a_{\pi(k)}}, \delta_{b_{\pi(k)} c_{\pi(k)}}, \tag{15} \]

For example,

\[ \langle Z_{ab} Z_{cd}^* Z_{ab} Z_{cd}^* \rangle = \frac{1}{\Omega^2} \delta_{a_{a} a_{a}} \delta_{b_{b} c_{c}} + \frac{1}{\Omega^2} \delta_{a_{a} d_{d}} \delta_{b_{b} c_{c}} \delta_{a_{a} d_{d}} \delta_{b_{b} c_{c}}. \tag{16} \]

In the real case, we must consider all possible matchings among the elements,

\[ \left\langle \prod_{k=1}^{2n} M_{ab} \right\rangle = \frac{1}{\Omega^n} \sum_{m \in M_n} \Delta_m(a)\Delta_m(b). \tag{17} \]

For example,

\[ \langle M_{ab} M_{ab} M_{ab} M_{ab} \rangle = \frac{1}{\Omega^4} \delta_{a_{a} a_{a}} \delta_{a_{a} a_{a}} \delta_{b_{b} b_{b}} \delta_{b_{b} b_{b}} + \frac{1}{\Omega^4} \delta_{a_{a} a_{a}} \delta_{a_{a} a_{a}} \delta_{b_{b} b_{b}} \delta_{b_{b} b_{b}} + \frac{1}{\Omega^4} \delta_{a_{a} a_{a}} \delta_{a_{a} a_{a}} \delta_{b_{b} b_{b}} \delta_{b_{b} b_{b}}. \tag{18} \]

These Wick’s rules have a well known diagrammatic implementation (see, e.g. [24–28]). In the complex case, matrix elements are represented by ribbons having borders oriented in the same direction but with different colors. Ribbons from elements of \( Z \) have a marked head, while ribbons from elements of \( Z^\dagger \) have a marked tail. Ribbons coming from traces are arranged around vertices, so that all marked ends are on the outside and all corners have a well defined color. Wick’s rule consists in making all possible connections between ribbons, using marked ends, respecting orientation. This produces a map (not necessarily connected). According to equation (14), each edge leads to a factor \( \Omega^{-1} \). In the real cases, the boundaries of the ribbons
are not oriented and the maps need not be orientable: the edges may contain a ‘twist’. We show an example for the complex case in figure 3, and an example for the real case in figure 4.

3. Previous results

In this section we review some previous results regarding the large-$N$ expansion of Weingarten functions.

3.1. Unitary case

A monotone factorization of $\pi$ is a sequence $(\tau_1, \ldots, \tau_k)$ of transpositions $\tau_i = (s_i, t_i)$, $t_i > s_i$ and $t_i \geq t_{i-1}$, such that $\pi = \tau_1 \cdots \tau_k$. The number of transpositions, $k$, is the length of the factorization. Let $M^k_{\alpha}$ be the number of length $k$ monotone factorizations of a permutation $\pi$ with
cycletype $\alpha$. Using the theory of Jucys–Murphy elements, Matsumoto and Novak showed that

$$W_{\alpha}(\pi) = (-1)^{\pi + \ell(\alpha)} \sum_{k=0}^{\infty} M_{\alpha}^{k} N^{-n-k}.$$  \hspace{1cm} (19)

A proper factorization of $\pi$ is a sequence of permutations $(\tau_1, \ldots, \tau_k)$, in which no one is the identity, such that $\pi = \tau_1 \cdots \tau_k$. The depth of a proper factorization is the sum of the ranks of the factors, $\sum_{j=1}^{k} r(\tau_j)$. Let $P_{\alpha}^{k,d}$ be the number of proper factorizations of $\pi$, with cycletype $\alpha$, having length $k$ and depth $d$. It is known that \[56\]

$$P_{\alpha}^{k,d} = \frac{1}{n!} \sum_{\lambda \vdash n} |C_{\lambda}| \chi_{\lambda}(\alpha) \sum_{\mu \vdash k} \left( \prod_{j=1}^{k} |C_{\mu_j}| \frac{\chi_{\lambda}(\mu_j)}{\chi_{\lambda}(\mu)} \right)^{\delta_{\sum_{j=1}^{k} r(\tau_j),d}},$$ \hspace{1cm} (20)

where all partitions $\mu$ are different from $1_n$. Starting from the character expansion of the Weingarten function,

$$W_{\alpha}(\pi) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{\chi_{\lambda}(\pi)^2}{s_{\lambda}(\pi)} \chi_{\lambda}(\alpha),$$ \hspace{1cm} (21)

which is in fact the same as equation (4), Collins used [6] the Schur function expansion

$$s_{\lambda}(\pi^n) = \frac{1}{n!} \sum_{\mu \vdash n} |C_{\mu}| \chi_{\lambda}(\mu) N^{r(\mu)} = \chi_{\lambda}(\pi) N^{\ell(\pi)} \left( 1 + \sum_{\rho \vdash n, \rho \neq \pi} \frac{|C_{\rho}|}{\chi_{\lambda}(\rho)} \chi_{\lambda}(\pi) N^{-r(\rho)} \right),$$ \hspace{1cm} (22)

to arrive at the expression

$$W_{\alpha}(\pi) = \frac{1}{n!} \sum_{\lambda \vdash n} \chi_{\lambda}(\pi) \chi_{\lambda}(\alpha) \sum_{k=1}^{\infty} (-1)^k \sum_{\rho_1 \cdots \rho_k} \left( \prod_{j=1}^{k} |C_{\rho_j}| \frac{\chi_{\lambda}(\rho_j)}{\chi_{\lambda}(\rho)} \right)^{\delta_{\sum_{j=1}^{k} r(\tau_j),d}}.$$ \hspace{1cm} (23)

Comparing with (20) one concludes that

$$W_{\alpha}(\pi) = \sum_{d=0}^{d} \sum_{k=1}^{d} (-1)^k P_{\alpha}^{k,d} N^{-n-d}.$$ \hspace{1cm} (24)

A cycle factorization of $\pi$ is a sequence of permutations $(\tau_1, \ldots, \tau_k)$, in which all factors have only one cycle besides singletons, i.e. their cycletypes are hook partitions. Inequivalent cycle factorizations are equivalence classes of cycle factorizations, two factorizations being equivalent if they differ by the swapping of adjacent commuting factors. Berkolaiko and Irving show [32] that the number of such factorizations of $\pi$, with cycletype $\alpha$, having length $k$ and depth $d$, denoted by $I_{\alpha}^{k,d}$, satisfy

$$\sum_{k} (-1)^k I_{\alpha}^{k,d} = \sum_{k} (-1)^k P_{\alpha}^{k,d} = (-1)^{\pi + \ell(\alpha)} M_{\alpha}^{d}.$$ \hspace{1cm} (25)

These results are indexed by depth, but one can use Euler characteristic instead, by resorting to the equality $\chi = n + \ell(\alpha) - d$.

3.2. Orthogonal case

Consider permutations from $S_{2n}$ acting on the set $[n] \cup \overline{[n]}$. Let $h$ be the operation of ‘forgetting’ the hat, i.e. for all $a \in [n]$ we define $h(\hat{a}) = h(a) = a$.
Matsumoto defined the following analogue of monotone factorizations [38]. Let \( m \) be a matching and let \( (\tau_1, \ldots, \tau_k) \) be a sequence of transpositions \( \tau = (s_1 t_i) \), in which all \( t_i \in [n] \) with \( t_i \geq t_{i-1} \) and \( t_i > h(s_i) \), such that \( m = \tau \cdots \tau_2 \tau_1 \), where \( t \) is the trivial matching. Let \( \tilde{M}_\mu \) be the number of length \( k \) such factorizations of some \( m \) with coset type \( \beta \). Then,

\[
W_{\delta_\beta}(\gamma) = \sum_{k=0}^{\infty} (-1)^k \tilde{M}^k_\beta N^{-n-k}.
\]  

(26)

The analogue of proper factorizations in this context is, for a permutation \( \sigma \) with coset type \( \beta \), a sequence of permutations \( (\tau_1, \ldots, \tau_k) \), no one having the same coset type as the identity, such that \( \sigma = \tau_1 \cdots \tau_k \). Let \( \tilde{P}^{k,d}_\beta \) be the number of such factorizations having length \( k \) and depth \( d \). We know from [35–37] that (actually these works only consider \( k = 2 \), but the extension to higher values of \( k \) is straightforward)

\[
\tilde{P}^{k,d}_\beta = \frac{1}{(2n)!^d} \sum_{\lambda \vdash n} \chi_{\lambda}(\tilde{P}^d) \omega_\lambda(\beta) \sum_{\mu_1 \cdot \mu_2} \left\{ \prod_{j=1}^{k} |K_{\mu_j}| \omega_\lambda(\mu_j) \right\} \delta_{\sum_{j=1}^{d} r(\mu_1, \mu_2), d}.
\]  

(27)

where

\[
\omega_\lambda(\tau) = \frac{1}{2^n n!} \sum_{\xi \in \mathcal{U}_n} \chi_{\lambda}(\tau \xi)
\]  

(28)

are the zonal spherical function of the Gelfand pair \((S_{2n}, H_n)\) (they depend only on the coset type of \( \tau \); see [45]).

The relation between the above factorizations and orthogonal Weingarten functions comes as follows. The character-theoretic expression for the orthogonal Weingarten function is [8]

\[
W_{\delta_\beta}(\gamma) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \chi_{\lambda}(\tilde{P}) \omega_\lambda(\beta),
\]  

(29)

where \( Z_\lambda \) are zonal polynomials (this is in fact the same as equation (7)). Following the same procedure used for the unitary group, we expand \( Z_\lambda(\tilde{P}) = \frac{1}{2^n n!} \sum_{\mu \vdash n} |K_{\mu}| \omega_\lambda(\mu) N^{\mu} \) to arrive at

\[
W_{\delta_\beta}(\gamma) = \frac{2^n n!}{(2n)!} \sum_{\mu_1 \cdot \mu_2} \chi_{\mu_1}(\tilde{P}) \omega_\lambda(\beta) \omega_{\lambda}(\mu_1) \omega_{\lambda}(\mu_2) N^{-n-\mu_1-\mu_2}.
\]  

(30)

Comparing with (27), we see that

\[
W_{\delta_\beta}(\gamma) = \sum_{d=0}^{\infty} \sum_{k=0}^{d} \frac{(-1)^k}{(2^n n!)^d} \tilde{P}^{k,d}_\beta N^{-n-d}.
\]  

(31)

Finally, Berkolaiko and Kuiers have provided a combinatorial description of the coefficients in the \( 1/N \) expansion of the function \( W_{\delta_{N+1}} \) [39] (they actually worked with the so-called Circular Orthogonal Ensemble of unitary symmetric matrices, but the Weingarten function of that ensemble coincides [13] with \( W_{\delta_{N+1}} \)). A palindromic monotone factorization is a sequence \( (\tau_1, \ldots, \tau_k) \) of transpositions \( \tau = (s_1 t_i) \), with \( t_i > s_i \) and \( t_i > t_{i-1} \), such that \( \pi \equiv \tau \cdots \tau_2 \tau_1 \). Let \( \tilde{M}^k_\beta \) be the number of length \( k \) palindromic monotone factorizations of \( \pi \), with \( \pi \) a permutation of cycle type \( \beta \). Then,
\[ W_{\beta}^{0}(\beta) = \sum_{k=0}^{\infty} (-1)^{k} \beta^{k} A^{n-k}. \] (32)

An appropriate analogue of inequivalent cycle factorizations is currently missing, but we conjecture that, whatever they are, their counting function will be related to coefficients in $1/N$ expansions of orthogonal Weingarten functions.

4. Our results

In this section we state our results without proof. Definitions and detailed calculations are shown in the following sections.

4.1. Unitary group

4.1.1. Maps. For the unitary group, we represent the Weingarten function as an infinite sum over orientable maps.

**Theorem 1.** If $\alpha$ is a partition with $\ell(\alpha)$ parts, then

\[ W_{\beta}^{U}(\alpha) = \frac{(-1)^{\ell(\alpha)}}{N^{2|\alpha|} + \ell(\alpha)} \sum_{\chi} N^{\chi} \sum_{w \in B(\alpha, \chi)} (-1)^{V(w)}, \] (33)

where the first sum is over Euler characteristic, and $V(w)$ is the number of vertices in the map $w$.

The (finite) set $B(\alpha, \chi)$ contains all maps, not necessarily connected, with the following properties: (i) they are orientable with Euler characteristic $\chi$; (ii) they have $\ell(\alpha)$ marked vertices with valencies $(2, 2, \ldots, 2)$; (iii) all other vertices have even valence larger than 2; (iv) all closed walks along the boundaries of the edges/ribbons visit the marked vertices in exactly one corner; (v) they are face-bicolored and have $|\alpha|$ faces of each color.

As an example, the $1/N$ expansion for the function in equation (8), for which $\alpha = (2)$, starts

\[ -\frac{1}{N^{3}} - \frac{1}{N^{5}} - \frac{1}{N^{7}} - \cdots. \] (34)

The first two orders come from maps like the ones shown in figure 5. The map in panel (a) is the only leading order contribution. It has $\chi = 2$ and $V = 2$, so its value is indeed $-1/N^{3}$. The next order comes from elements of $B(2, 0)$. There are in total 21 different maps in that set having four vertices of valence 4; one of them is shown in panel (b). There are 28 different maps having two vertices of valence 4 and one vertex of valence 6; one of them is shown in panel (c). Finally, there are 8 different maps having one vertex of valence 4 and one vertex of valence 8; one of them is shown in panel (d). They all have $\chi = 0$, and their combined contribution is $(-21 + 28 - 8)/N^{5} = -1/N^{5}$.

4.1.2. Factorizations. By a factorization of $\Pi$ we mean an ordered pair, $f \equiv (\tau_1, \tau_2)$, such that $\Pi = \tau_1 \tau_2$. We call $\Pi$ the ‘target’ of the factorization $f$. If $\Pi \in S_k$, then the Euler characteristic of $f$ is given by $\chi = \ell(\Pi) - k + \ell(\tau_1) + \ell(\tau_2)$.

The number of factorizations of a permutation depends only on its cycletype, so it makes sense to restrict attention to specific representatives. Call a permutation ‘standard’ if each of its cycles contains only adjacent numbers, and the cycles have weakly decreasing length.
when ordered with respect to least element. For example, \((123)(45)\) is standard. Since \(\pi \in S_n\) depends only on the cycletype of \(\pi \in S_n\), we may take \(\pi\) to be standard.

The relevant factorizations for \(W_{\theta N}^{U}(\pi)\) are those whose target is of the kind \(\pi \rho\Pi = \cdots\), where the ‘complement’ \(\rho\) is a standard permutation acting on the set \(\{n + 1, \ldots, n + m\}\), for some \(m \geq 0\). They satisfy the following properties: (i) they have Euler characteristic \(\chi\); (ii) the complement \(\rho\) has no fixed points; (iii) every cycle of the factors \(\tau_1, \tau_2\) must have exactly one element in \(\{1, \ldots, n\}\). Notice that the last condition implies \(\ell(\pi_1) = \ell(\pi_2) = n\). Let the (finite) set of all such factorizations be denoted \(F(\pi, \chi)\). Then, we have

**Theorem 2.** Let \(\pi \in S_n\) be a standard permutation, then

\[
W_{\theta N}^{U}(\pi) = \frac{(-1)^{\ell(\pi)}}{N^{2n+\alpha+\gamma}} \sum_{\chi} N^\chi \sum_{f \in F(\pi, \chi)} \frac{(-1)^{\ell(\Pi)}}{z_\rho},
\]

where \(z_\rho = \prod_j j^v_j v_j!\), with \(v_j\) the number of times part \(j\) occurs in the cycletype of the complement \(\rho\).

Theorem 2 follows from theorem 1 by a simple procedure for associating factorizations to maps in \(\mathcal{B}(\alpha, \chi)\), discussed in section 5.3. Associations of this kind are well known [29–31].

For example, the leading order in equation (34), for which \(\pi = (12)\), has a contribution from the factorization \((12)(34) = (14)(23)(13)(24)\), which has \(\rho = (34)\) and is one of two factorizations that can be associated with the map in figure 5(a). Several factorizations can be associated with the other maps in figure 5. We mention one possibility for each of them: \((12)(34)(56)(78) = (148)(25763)(13)(285746)\) for the map in figure 5(b); \((12)(345)(67) = (17)(23546)(16)(27435)\) for the map in figure 5(c); \((12)(3456) = (164)(253)(1365)(24)\) for the map in figure 5(d). Notice how they have different complements.
In the previous section, we mentioned other factorization problems also related to the coefficients in the $1/N$ expansion for $\mathcal{W}_g^{(\pi)}(\pi)$. In particular, they satisfy some curious sum rules, equation (25). Interestingly, our factorizations satisfy a very similar sum rule, namely

$$\sum f_{\pi,\chi}^{\alpha} (-1)^{\ell(\alpha)} \tilde{z}_p = (-1)^{\rho + \ell(\alpha)} M_{\alpha, \chi},$$  \hspace{1cm} (36)

An important difference between (36) and (25) is that the factorizations in (36) take place in $S_{n+m}$ for some $m \geq 0$, while all those in (25) take place in $S_n$. Notice that our factorizations must satisfy condition (iii), which is related to the distribution of the elements from the set $\{1, \ldots, n\}$ among the cycles of the factors $\tau_1, \tau_2$. This is close in spirit to the kind of questions studied by Bóna, Stanley and others [33, 34], which count factorizations satisfying some placement conditions on the elements of the target.

4.2. Orthogonal group

4.2.1. Maps. For the orthogonal group, we represent the Weingarten function as an infinite sum over maps (orientable and non-orientable).

**Theorem 3.** Let $\beta$ be a partition with $\ell(\beta)$ parts, then

$$\mathcal{W}_{g_{N+1}}^{O}(\beta) = \frac{(-2)^{\ell(\beta)}}{N^{2\ell(\beta) + \ell(\beta)}} \sum_{\chi} N^x \sum_{w \in \mathcal{NB}(\beta, \chi)} \left(-\frac{1}{2}\right)^{V(w)},$$  \hspace{1cm} (37)

where the first sum is over Euler characteristic, and $V(w)$ is the number of vertices in the map $w$.

The (finite) set $\mathcal{NB}(\beta, \chi)$ contains all maps, not necessarily connected or orientable, with the following properties: (i) they have Euler characteristic $\chi$; (ii) they have $\ell(\beta)$ marked vertices with valencies $(2, \beta_1, 2, \beta_2, \ldots)$; (iii) all other vertices have even valence larger than 2; (iv) all closed walks along the boundaries of the edges visit the marked vertices in exactly one corner; (v) they are face-bicolored and have $|\beta|$ faces of each color.

Notice that the expansion is in powers of $N^{-1}$ for the group $O(N + 1)$ and not for $O(N)$. For example, the first terms in the expansion of the function in equation (9) at dimension $N + 1$ are

$$\frac{-1}{N(N + 1)(N + 3)} = -\frac{1}{N^3} + \frac{4}{N^4} - \frac{13}{N^5} - \ldots$$  \hspace{1cm} (38)

The leading order comes from the map shown in figure 5(a), which is orientable. The next order comes from non-orientable maps in the set $\mathcal{NB}(2, 1)$. There are in total 8 different maps having two vertices of valence 2; one of them is shown in figure 6(a). There are in total 4 different maps having one vertex of valence 2 and one vertex of valence 3; one of them is shown in figure 6(b). Their combined contribution is $(8 - 4)/N^4 = 4/N^4$.

**Remark 1.** It is known [13] that the Weingarten function of the unitary symplectic group $Sp(N)$ is proportional to $\mathcal{W}_{g_{2N+1}}^{O}$. Therefore the appropriate dimension for our map expansion in $Sp(N)$ is also different from $N$: it has to be $Sp(N - 1/2)$ (a non-integer dimension is to be understood by keeping in mind that Weingarten functions are rational functions of $N$). Interestingly, if we assign a parameter $\alpha = 2, 1, 1/2$ for orthogonal, unitary and symplectic groups, respectively (sometimes called Jack parameter), then the appropriate dimensions for the map expansion in powers of $N^{-1}$ can be written as $N + \alpha - 1$. 

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4.2.2. Factorizations. Let \( n = \{1, \ldots, n\} \) and \( \tilde{n} = \{\tilde{1}, \ldots, \tilde{n}\} \). We consider the action of \( S_\tilde{n} \) on \( [n] \cup [\tilde{n}] \). Define the ‘hat’ involution on permutations as \( \hat{\pi}(a) = \pi^{-1}(\tilde{a}) \) (assuming \( \tilde{a} = a \)). We call permutations that are invariant under this transformation ‘palindromic’, e.g., \( 12 \hat{\circledcirc} 21 \) and \( \hat{\circledcirc} 12 21 \) are palindromic.

Given a partition \( \beta \), define \( \pi \in S_n \) to be the standard permutation that has cycletype \( \beta \) and define \( \pi \rho \Pi = \circledcirc \), where the ‘complement’ \( \rho \) is a standard permutation acting on the set \( \{n + 1, \ldots, n + m\} \) for some \( m \geq 0 \). Define the fixed-point free involutions \( p_1 \), whose cycles are \( (a \tilde{a}) \) for \( 1 \leq a \leq n + m \), and \( p_2 \), whose cycles are of the type \( (\tilde{a} a + 1) \), but with the additions computed modulo the cycle lengths of \( \Pi \), i.e.,

\[
p_2 = (\hat{1} 2)(\hat{2} 3) \cdots (\hat{\beta}_1 1)(\tilde{\beta}_1 + 1 \beta_1 + 2) \cdots (\tilde{\beta}_1 + \beta_1 \beta_1 + 1) \cdots.
\]  

(39)

By construction, they provide a factorization of the palindromic version of the target, \( p_2 p_1 = \Pi \tilde{\Pi} \).

(40)

The problem we need to solve is to find all factorizations \( \Pi \tilde{\Pi} = f_1 f_2 \), that satisfy the following properties: (i) their Euler characteristic, defined as \( \ell(\Pi - m - n + \ell(f_1) + \ell(f_2)) \), is \( \chi \); (ii) the complement \( \rho \) has no fixed points; (iii) the factors may be written as \( f_1 = \theta \rho_1 \) and \( f_2 = p_2 \theta \) for some fixed-point free involution \( \theta \); (iv) \( f_1 \) is palindromic; (v) every cycle of the factors \( f_1, f_2 \) contains exactly one element in \( [n] \cup [\tilde{n}] \). Clearly, the crucial quantity is actually \( \theta \). Let the (finite) set of all pairs \( (\Pi, \theta) \) satisfying these conditions be denoted \( N\mathcal{F}(\beta, \chi) \). Then, we have

**Theorem 4.** For a given partition \( \beta \) of length \( \ell(\beta) \),

\[
W^\theta_{N+1}(\beta) = \frac{(-2)^{\ell(\beta)}}{(N^2)^{\ell(\beta)}} \sum_{\Pi, \theta \in N\mathcal{F}(\beta, \chi)} \frac{1}{z_{\rho}} \left( -\frac{1}{2} \right)^{\ell(\Pi)},
\]

where \( z_{\rho} = \prod_j j^{v_j} v_j! \), with \( v_j \) the number of times part \( j \) occurs in the cycle type of the complement \( \rho \).

Theorem 4 follows from theorem 3 by a simple procedure for describing combinatorially the maps in \( N\mathcal{B}(\beta, \chi) \), discussed in section 6.3. Such kind of descriptions are well known [35–37].

For example, the leading order in equation (38) has a contribution from the pair \( \Pi = (12)(34), \theta = (13)(24)(42)(31) \). The next order comes from factorizations associated with the maps in figure 6. We mention one for each of them: \( \Pi = (12)(34)(56), \theta = (13)(13)(25)(24)(46)(56) \) for (a) and \( \Pi = (12)(345), \theta = (13)(13)(25)(24)(45) \) for (b).
5. Unitary group

In this section, we present the derivation of our results for the unitary group.

5.1. Truncations

Let $U$ be a random matrix uniformly distributed in $\mathcal{U}(N)$ with the appropriate normalized Haar measure. Let $A$ be the $M_1 \times M_2$ upper left corner of $U$, with $N \geq M_1 + M_2$ and $M_1 \leq M_2$. It is known [46–50] that $A$, which satisfies $AA^\dagger < 1_{M_2}$, becomes distributed with probability density given by

$$P(A) = \frac{1}{\mathcal{Y}_1} \det(1 - AA^\dagger)^{N_0}, \quad (42)$$

where

$$N_0 = N - M_1 - M_2 \quad (43)$$

and $\mathcal{Y}_1$ is a normalization constant.

The value of $\mathcal{Y}_1$ can be computed using the singular value decomposition $A = WDV$, where $W$ and $V$ are matrices from $\mathcal{U}(M_1)$ and $\mathcal{U}(M_2)$, respectively. Matrix $D$ is real, diagonal and non-negative. Let $\mathcal{W}(t_1, t_2, \ldots) = \prod_{i=1}^{M_1} \prod_{j=i+1}^{M_2} (t_j - t_i)$. Then [24, 51, 52],

$$\mathcal{Y}_1 = \int_{(\mathcal{U}(M_1))} dW \int_{(\mathcal{U}(M_2))} dV \int_0^1 \prod_{i=1}^{M_1} \prod_{j=1}^{M_2} (1 - t_i)(1 - t_j)^{N_0} |D_i - M_j| \Delta(t_i)^2, \quad (44)$$

where

$$\Delta(t_i) = \prod_{i=1}^{M_1} \prod_{j=1}^{M_2} (t_j - t_i). \quad (45)$$

If we denote the angular integrals by

$$\int_{(\mathcal{U}(M_1))} dW \int_{(\mathcal{U}(M_2))} dV = \mathcal{V}_1, \quad (46)$$

then Selberg’s integral tells us that [53, 54]

$$\mathcal{Y}_1 = \mathcal{V}_1 \prod_{j=1}^{M_1} \frac{\Gamma(j + 1)\Gamma(M_2 + 1 - j)\Gamma(N - M_2 - j + 1)}{\Gamma(N - j + 1)}, \quad (47)$$

Consider now an even smaller subblock of $U$, which is contained in $A$. Namely, let $\tilde{U}$ be the $N_1 \times N_2$ upper left corner of $U$, with $N_1 \leq M_1$ and $N_2 \leq M_2$. We shall make use of the obvious fact that integrals involving matrix elements of $\tilde{U}$ can be computed either by integrating over $U$ or over $A$. In particular, the quantity

$$W_{\tilde{U}}^{(\pi)}(\nu) = \int_{(\mathcal{U}(N))} dU \prod_{k=1}^{n} \tilde{U}_{k,\pi(k)}^{\nu} \tilde{U}_{k,\pi(k)}^\dagger \quad (48)$$

with $n \leq N_1, N_2$ and $\pi \in S_n$, can also be written as

$$W_{\tilde{U}}^{(\pi)}(\nu) = \frac{1}{\mathcal{Y}_1} \int_{AA^\dagger < 1_{M_2}} dA \det(1 - AA^\dagger)^{N_0} \prod_{k=1}^{n} \tilde{A}_{k,\pi(k)}, \quad (49)$$
Notice that, although this may not be evident at first sight, the right-hand-side of equation (49) is actually independent of $M_1$ and $M_2$.

5.2. Sum over maps

The key to the diagrammatic formulation of our integral is the identity

$$\det(1 - AA')^{N_0} = e^{N_0 \text{Tr} \log(1 - AA')} = e^{-N_0 \sum_{i=1}^{\infty} \frac{1}{i} \text{Tr}(AA')^i}.$$  \hfill (50)

We shall consider the first term in the series separately from the rest, and incorporate it into the measure, i.e. we will write

$$dAe^{-N_0 \sum_{i=1}^{\infty} \frac{1}{i} \text{Tr}(AA')^i} = d_G(A)e^{-N_0 \sum_{i=2}^{\infty} \frac{1}{i} \text{Tr}(AA')^i},$$  \hfill (51)

where $d_G(A)$ is a Gaussian measure,

$$d_G(A) = dAe^{-N_0 \text{Tr}(AA')}.$$  \hfill (52)

This leads to

$$W_{\chi}_{N_0}(\pi) = \frac{1}{\mathcal{Z}_1} \int_{\text{AA'} < 1_{N_0}} d_G(A)e^{-N_0 \sum_{i=2}^{\infty} \frac{1}{i} \text{Tr}(AA')^i} \prod_{k=1}^{n} A_{k,k} A_{k,\pi(k)}^\dagger.$$  \hfill (53)

Taking into account that the series in the exponent diverges for $AA' \geq 1_{M}$ and that $e^{-\infty} = 0$, we extend the integration to general matrices $A$,

$$W_{\chi}_{N_0}(\pi) = \frac{1}{\mathcal{Z}_1} \int d_G(A)e^{-N_0 \sum_{i=2}^{\infty} \frac{1}{i} \text{Tr}(AA')^i} \prod_{k=1}^{n} A_{k,k} A_{k,\pi(k)}^\dagger.$$  \hfill (54)

Now we are in the realm of Gaussian integrals, and may apply Wick’s rule. For each cycle of $\pi$, the elements of $A$ and $A'$ in the last product can be arranged in counterclockwise order around vertices. This produces what we call ‘marked’ vertices, in number of $\ell(\pi)$. Formally expanding the exponential in equation (54) as a Taylor series in $N_0$ will produce other vertices, let us call them ‘internal’, all of them of even valence larger than 2. This leads to an infinite sum over maps with arbitrary numbers of internal vertices and edges. The contribution of a map will be proportional to $(-1)^{N_0 - \hat{v}}$, if it has $v$ internal vertices and $E$ edges, which can be written as

$$(-1)^{N_0 - E} = \left(-1\right)^{\hat{v}}.$$  \hfill (55)

However, the application of Wick’s rule may lead to closed boundary walks that visit only the internal vertices and avoid the marked ones. If a map has $r_1$ closed boundary walks of one color and $r_2$ closed boundary walks of the other color that avoid the marked vertices, its contribution will be proportional to $M_1^{r_1} M_2^{r_2}$. Crucially, since we know that $W_{\chi}(\pi)$ is actually independent of $M_1, M_2$, we are free to consider only the cases $r_1 = r_2 = 0$, or equivalently to take the simplifying limits $M_1 \to 0$ and $M_2 \to 0$. We will therefore be left only with maps in which all closed boundary walks visit the marked vertices.

Another point we must address is that the normalization constant $\mathcal{Z}_1$ is not equal to the corresponding Gaussian one, $\mathcal{Z}_G = \int d_G(A)$, so that the diagrammatic expression for $W_{\chi}(\pi)$ will be multiplied by $\mathcal{Z}_G \mathcal{Z}_1$. Using again the singular value decomposition of $A$, and
\[ \int_0^\infty \prod_{j=1}^M dt e^{-N_0 t} \mathcal{M}_{i_1 \ldots i_j} \Delta(t)^2 = \left( \frac{1}{N_0} \right)^{M_0} \prod_{j=1}^M \Gamma(j+1) \Gamma(M_2 + 1 - j). \] (56)

which can be obtained as a limiting case of the Selberg integral, we arrive at

\[ \frac{\mathcal{Z}_C}{\mathcal{Y}_1} = \frac{1}{N_0^{M_0}} \prod_{j=1}^M (N-j)! (N-M_2-j)! . \] (57)

and, therefore,

\[ \frac{\mathcal{Z}_C}{\mathcal{Y}_1} = \frac{1}{N_0^{M_0}} \prod_{j=1}^M \frac{(N-j)!}{(N-M_2-j)!} . \] (58)

In the end, we have the simplification

\[ \lim_{M_0 \to 0} \frac{\mathcal{Z}_C}{\mathcal{Y}_1} = 1. \] (59)

The limit $M_0 \to 0$ is trivial. Notice that the limit of $N_0$ is simply $N$.

Because of the way we arranged the elements around the marked edges, our maps will always have $2n$ faces, being $n$ of each color. We have therefore produced exactly the maps in the set $\mathcal{B}(\alpha, \chi)$. This proves our theorem 1.

5.3. Sum over factorizations

Suppose a standard permutation $\pi \in S_n$ with cycletype $\alpha \vdash n$. We shall associate factorizations of $\pi$ with the maps in $\mathcal{B}(\alpha, \chi)$. In order to do that, we must label the corners of the maps. We proceed as follows.

Consider first the marked vertex with largest valency, and pick a corner delimited by solid line. Label this corner and the corner following it in counterclockwise order with number 1. Proceed to another marked vertex in weakly decreasing order of valency, and repeat the above with integers from $\alpha_1 + 1$ to $\alpha_2 + 1$. Repeat until all marked vertices have been labelled, producing thus the cycles of $\pi$. The same procedure is then applied to the internal vertices, producing the cycles of another standard permutation $\rho$, acting on the set \{n + 1, ..., $E$\}, where $E = n + m$ is the number of edges in the map. Notice that $\rho$ has no fixed points, since all internal vertices of maps in $\mathcal{B}(\alpha, \chi)$ have even valence larger than 2.

Let $\Pi = \pi \rho$. See figure 7, where $\Pi = (12)(34)$ for panel (a), $\Pi = (12)(34)(56)(78)$ for panel (b), $\Pi = (12)(345)(67)$ for panel (c), and $\Pi = (12)(3456)$ for panel (d).

Define the permutation $\omega_1$ to be such that its cycles have the integers in the order they are visited by the arrows in solid line. In the example of figure 7, this would be $\omega_1 = (14)(23)$ for panel (a), $\omega_1 = (184)(23675)$ for panel (b), $\omega_1 = (17)(26453)$ for panel (c) and $\omega_1 = (146)(235)$ for panel (d). The cycles of $\omega_1$ correspond to the closed boundary walks in solid line.

Permutation $\gamma_2$ is defined analogously in terms of the arrows in dashed lines. In figure 7, this would be $\gamma_2 = (13)(24)$ for panel (a), $\gamma_2 = (13)(285746)$ for panel (b), $\gamma_2 = (16)(27435)$ for panel (c) and $\gamma_2 = (1365)(24)$ for panel (d). The cycles of $\gamma_2$ correspond to the closed boundary walks in dashed line.

Suppose an initial integer, $i$. The arrow in dashed line which departs from the corner labelled by $i$ arrives at the corner labelled by $\gamma_2(i)$. On the other hand, the image of $i$ under the permutation $\Pi$ corresponds to the label of an outgoing arrow in solid line which, following $\omega_1$,
also arrives at \( \tau_2(i) \). Therefore, we have, by construction, \( \omega_1 \Pi = \tau_2 \) or, equivalently, writing \( \tau \omega = -1 \), we have the factorization

\[
\Pi = \tau \omega.
\]

(60)

For the maps in \( \mathcal{B}(\alpha, \chi) \), all boundary walks visit the marked vertices, which means that all cycles of \( \tau \) ad of \( \tau_2 \) must have exactly one element in the set \( \{1, ..., n\} \). Therefore, the permutations satisfy the conditions we listed in section 4.1.2.

When we label the vertices of the map to produce a factorization, there are two kinds of ambiguities. First, for a vertex of valency \( 2j \) there are \( j \) possible choices for the first corner to be labelled. Second, if there are \( m_j \) vertices of valency \( j \), there are \( m_j! \) ways to order them. Hence, to a map for which the complement is \( \rho \) there correspond \( z_\rho = \prod_j j^{m_j} m_j! \) factorizations, where \( m_j \) is the multiplicity of part \( j \) in the cycletype of \( \rho \). The sum in (33) can indeed be written as

\[
W_{\delta}(\pi) = \frac{1}{N^{2n+1}(\pi)} \sum_{\chi} N! \sum_{f \in \mathcal{F}(\pi, \chi)} (-1)^{\ell(\rho)} z_\rho,
\]

(61)

where \( \mathcal{F}(\pi, \chi) \) is the set of factorizations of the kind we have described for given \( \alpha \) and \( \chi \). This proves our theorem 2.

6. Orthogonal group

In this section, we present the derivation of our results for the orthogonal group.

6.1. Truncations

Let \( O \) be a random matrix uniformly distributed in \( \mathcal{O}(N+1) \) with the appropriate normalized Haar measure. Let \( A \) be the \( M_1 \times M_2 \) upper left corner of \( O \), with \( N \geq M_1 + M_2 \) and \( M_1 \leq M_2 \).
It is known [49, 55] that $A$, which satisfies $AA^T < 1_{M_2}$, becomes distributed with probability density given by

$$P(A) = \frac{1}{\mathcal{Y}_2} \det(I - AA^T)^{N_0/2},$$

where

$$N_0 = N - M_1 - M_2$$

and $\mathcal{Y}_2$ is a normalization constant. Notice that we start with $O(N + 1)$ and not $O(N)$.

The value of $\mathcal{Y}_2$ can be computed using the singular value decomposition $A = WDV$, where $W$ and $V$ are matrices from $O(M_1)$ and $O(M_2)$, respectively. Matrix $D$ is real, diagonal and non-negative. Let $T = D^2 = \text{diag}(t_1, t_2, \ldots)$. Then [51, 52],

$$\mathcal{Y}_2 = \int_{(O(M_1))} dW \int_{(O(M_2))} dV \int_0^1 \prod_{i=1}^M dt_i (1 - t_i)^{N_0/2} |M_2 - M_1 - 1/2| \Delta(t_i).$$

If we denote the angular integrals by $\int_{(O(M))} dW \int_{(O(M))} dV = \mathcal{V}_2$, then we have again from Selberg’s integral that

$$\mathcal{Y}_2 = \mathcal{V}_2 \prod_{j=1}^M \frac{\Gamma(j/2 + 1) \Gamma((M_2 + 1 - j)/2) \Gamma((N - M_2 - j)/2 + 1)}{\Gamma((N - j)/2 + 1) \Gamma(3/2)}.$$

Consider now an even smaller subblock of $O$, which is contained in $A$. Namely, let $\hat{O}$ be the $N_1 \times N_2$ upper left corner of $O$, with $N_1 \leq M_1$ and $N_2 \leq M_2$. The average value of any function of matrix elements of $\hat{O}$ can be computed either by integrating over $O$ or over $A$. In particular, the quantity

$$W_{\hat{O}}^{O_{N+1}}(\beta) = \int_{(O(N+1))} dO \prod_{k=1}^n \hat{O}_{k,\beta} \hat{O}_{k,\beta},$$

where $k \leq N_1, N_2$ and the $j$’s only satisfy some matching of cosettype $\beta$, can also be written as

$$W_{\hat{O}}^{O_{N+1}}(\beta) = \frac{1}{\mathcal{Y}_2} \int_{AA^T < 1_{M_2}} dA \det(1 - AA^T)^{N_0/2} \prod_{k=1}^n A_{k,\beta} A_{k,\beta}.$$

Notice that the right-hand-side of equation (67) is actually independent of $M_1$ and $M_2$.

6.2. Sum over maps

Analogously to the unitary case, we have

$$W_{\hat{O}}^{O_{N+1}}(\beta) = \frac{1}{\mathcal{Y}_2} \int dG(A) e^{-\frac{N_0}{2} \sum_{\alpha=2} \frac{1}{T_{\alpha} AA^T,\alpha} \prod_{k=1}^n A_{k,\beta} A_{k,\beta}},$$

where now $dG(A) = e^{-\frac{N_0}{2} T_{\alpha} AA^T,\alpha}$. The diagrammatical considerations proceed as previously, except that we use the Wick’s rule of the real case and the resulting maps need not be orientable. Also, a map now contributes $(-N_0/2)$ for each internal vertex and $1/N_0$ for each edge. This gives a total contribution which is proportional to

$$N_0^{-\ell} \left( -\frac{1}{2} \right)^V = \frac{N_0^{N_0}}{N_0^{(N_0)}} \left( -\frac{1}{2} \right)^V (-2)^{\ell(\beta)},$$
where \( v \) is the number of internal vertices, \( V = v + \ell \) is the total number of vertices, \( E \) is the number of edges and \( \chi = F - E + V \) is the Euler characteristic, where \( F \) is the number of faces. When we take \( M_2 \to 0 \), and then \( M_1 \to 0 \), we arrive at maps with no closed boundary walks that avoid the marked vertices, having \( 2n \) faces, \( n \) of each color. We thus arrive at the maps in the set \( \mathcal{NB}(\beta, \chi) \).

The Gaussian normalization constant is

\[
Z_R = \int dG(A) = V_2 \int_0^\infty M_1 \prod_{j=1}^M dt_j e^{-\frac{N_0}{2}t_j(M_j - M_j - 1)}/2 \prod_{j=1}^M |t_j - t_j| \tag{70}
\]

and we have

\[
\lim_{M_1 \to 0} \frac{Z_R}{V_2} = \lim_{M_1 \to 0} \left( \frac{2}{N_0} \right)^{M_1 \cdot M_2} \prod_{j=1}^M \frac{\Gamma(1 + j/2)\Gamma((M_j + 1 - j)/2)}{\Gamma(3/2)} \tag{71}
\]

and

\[
\lim_{M_1 \to 0} \frac{Z_R}{V_2} = \lim_{M_1 \to 0} \left( \frac{2}{N_0} \right)^{M_1 \cdot M_2} \prod_{j=1}^M \frac{\Gamma((N + 2 - j)/2)}{\Gamma((N - M_j + 2 - j)/2)} = 1. \tag{72}
\]

Once again, the limit \( M_1 \to 0 \) is trivial. This reduces \( N_0 \) to \( N \). Taking into account the contribution of the maps, already discussed, we arrive at our theorem 3.

### 6.3. Sum over factorizations

We now label the maps in \( \mathcal{NB}(\beta, \chi) \) in order to relate them to permutations. We only need to change slightly the labelling procedure we used for the maps in \( \mathcal{B}(\alpha, \chi) \) in section 5.3. First, we replace the labels of the corners in dashed line by ’hatted’ versions. Second, instead of labelling corners, we now label half-edges, by rotating the previous labels counterclockwise. This is shown in figure 8 (where the hatted labels are enclosed by boxes, while the normal ones are enclosed by circles).

The unhatted labels, read in anti-clockwise order around vertices, produce a permutation \( \Pi \) which is standard. This can be written as \( \Pi = \pi \rho \), where \( \pi \in S_n \) has cycle type \( \beta \) and the complement \( \rho \) acts on the set \( \{n + 1, \ldots, E\} \) where \( E = n + m \) is the number of edges. As before, \( \rho \) has no fixed points, since all internal vertices of maps in \( \mathcal{NB}(\beta, \chi) \) have even valence larger than 2.

A fixed-point free involution \( \theta \) can be constructed from the labels that appear at the ends of each edge. Namely, in the examples shown in figure 8 it is given by \( \theta = (1 \hat{3})(1 \hat{3})(2 \hat{4}) \) for (a), \( \theta = (1 \hat{3})(1 \hat{3})(2 \hat{5})(2 \hat{4})(4 \hat{6})(5 \hat{6}) \) for (b) and \( \theta = (1 \hat{3})(1 \hat{3})(2 \hat{5})(2 \hat{4})(4 \hat{6}) \) for (c).

We also define the hatted version of any permutation \( \pi \) by the equation \( \hat{\pi}(a) = \pi^{-1}(a) \), assuming \( \hat{\alpha} = a \). This is clearly an involution. Permutations that are invariant under this operation are called ’palindromic’, such as \((12 \hat{2} 1)\) or \((12 \hat{2} 1)\). Any permutation that can be written as \( \pi \hat{\pi} \) where \( \pi \) is another permutation is automatically palindromic.

Define two special fixed-point free involutions,

\[
p_1 = (1 \hat{1})(2 \hat{2}) \cdots, \tag{73}
\]

and

\[
p_2 = (1 \hat{2})(2 \hat{3}) \cdots (\hat{\beta}_1 1)(\hat{\beta}_1 + 1 \hat{\beta}_1 + 2) \cdots (\hat{\beta}_1 + \hat{\beta}_2 \hat{\beta}_1 + 1) \cdots. \tag{74}
\]
Notice that the cycles of $p_1$ contain labels which delimit corners of dashed line, while that the cycles of $p_2$ contain labels which delimit corners of solid line. Notice also that they factor the palindromic version of the vertex permutation, $p_2p_1 = \pi \bar{\pi}$.

By construction, the permutation $f_1 = \theta p_1$ contains every other label encountered along boundary walks around the faces delimited by boundaries in dashed line. For example, in figure 8 it would be $f_1 = (13)(24)(\bar{4}\bar{2})(\bar{3}\bar{1})$ for (a), $f_1 = (13)(2465)(5\bar{6}\bar{4}\bar{2})(31)$ for (b) and $f_1 = (13)(24\bar{5})(5\bar{4}\bar{2})(31)$ for (c). In particular, $f_1$ is always palindromic.

Conversely, permutation $f_2 = p_2\theta$ contains every other label encountered along boundary walks around the faces delimited by boundaries in solid line. In figure 8 it would be $f_2 = (14)(23)(\bar{3}\bar{2})(\bar{4}\bar{1})$ for (a), $f_2 = (14)(26\bar{6}\bar{3})(\bar{3}\bar{2})(\bar{4}\bar{5}\bar{5})$ for (b) and $f_2 = (14)(2\bar{4}\bar{3})(\bar{3}\bar{2})(\bar{1}\bar{5}\bar{5})$ for (c). Permutation $f_2$ needs not be palindromic. Notice that $f_1$ and $f_2$ are factors for the palindromic version of the vertex permutation, $\pi \bar{\pi} = f_2f_1$.

For the maps in $\mathcal{NB}(\beta,\chi)$ all boundary walks visit the marked vertices, which means that all cycles of $f_1$ and of $f_2$ must have exactly one element in the set $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$. Therefore, the permutations satisfy the conditions we listed in section 4.2.2. When we label the vertices of the map to produce a factorization, the same ambiguities arise as for the unitary group, which are accounted for by division by the factor $z_{\rho}$. We have therefore arrived at factorizations in $\mathcal{NF}(\beta,\chi)$ and proved our theorem 4.
References

[1] Weingarten D 1978 Asymptotic behavior of group integrals in the limit of infinite rank J. Math. Phys. 19 999
[2] Samuel S 1980 U(N) integrals, 1/N, and de Wit–’t Hooft anomalies J. Math. Phys. 21 2695
[3] Mello P A 1990 Averages on the unitary group and applications to the problem of disordered conductors J. Phys. A: Math. Gen. 23 4061
[4] Brouwer P W and Beenakker C W J 1996 Diagrammatic method of integration over the unitary group, with applications to quantum transport in mesoscopic systems J. Math. Phys. 37 4904
[5] Degli Esposti M and Knaut A 2004 On the form factor for the unitary group J. Math. Phys. 45 4957
[6] Collins B 2003 Moments and cumulants of polynomial random variables on unitary groups, the Itzykson–Zuber integral, and free probability Int. Math. Res. Not. 17 953
[7] Collins B and Sniady P 2006 Integration with respect to the Haar measure on unitary, orthogonal and symplectic group Commun. Math. Phys. 264 773
[8] Brouwer P W and Beenakker C W J 1996 Diagrammatic method of integration over the unitary group, with applications to quantum transport in mesoscopic systems J. Math. Phys. 37 4904
[9] Zuber J-B 2008 The large-N limit of matrix integrals over the orthogonal group J. Phys. A: Math. Theor. 41 382001
[10] Banica T, Collins B and Schlenker J-M 2011 On polynomial integrals over the orthogonal group J. Comb. Theory A 118 78
[11] Matsumoto S and Novak J 2013 Jucys–Murphy elements and unitary matrix integrals Int. Math. Res. Not. 2 362
[12] Zinn-Justin P 2010 Jucys–Murphy elements and Weingarten matrices Lett. Math. Phys. 91 119
[13] Matsumoto S 2013 Weingarten calculus for matrix ensembles associated with compact symmetric spaces Random Matrices: Theory Appl. 2 1350001
[14] Scott A J 2008 Optimizing quantum process tomography with unitary 2-designs J. Phys. A: Math. Theor. 41 055308
[15] Žnidarič M, Pineda C and García-Mata I 2011 Non-Markovian behavior of small and large complex quantum systems Phys. Rev. Lett. 107 080404
[16] Cramer M 2012 Thermalization under randomized local Hamiltonians New J. Phys. 14 053051
[17] Vinayak and Žnidarič M 2012 Subsystem’s dynamics under random Hamiltonian evolution J. Phys. A: Math. Theor. 45 125204
[18] Novaes M 2013 A semiclassical matrix model for quantum chaotic transport J. Phys. A: Math. Theor. 46 502002
[19] Novaes M 2015 Semiclassical matrix model for quantum chaotic transport with time-reversal symmetry Ann. Phys. 361 51
[20] Novaes M 2014 Elementary derivation of Weingarten functions of classical Lie groups (arXiv:1406.2182v2 [math-ph])
[21] Zinn-Justin P and Zuber J-B 2003 On some integrals over the U(N) unitary group and their large N limit J. Phys. A: Math. Gen. 36 3173
[22] Collins B, Guionnet A and Maurel-Segala E 2009 Asymptotics of unitary and orthogonal matrix integrals Adv. Math. 222 172
[23] Ginibre J 1965 Statistical ensembles of complex, quaternion, and real matrices J. Math. Phys. 6 440
[24] Morris T R 1991 Chequered surfaces and complex matrices Nucl. Phys. B 356 703
[25] ‘t Hooft G 1974 A planar diagram theory for strong interactions Nucl. Phys. B 72 461
[26] Bessis D, Itzykson C and Zuber J B 1980 Quantum field theory techniques in graphical enumeration Adv. Appl. Math. 1 109
[27] Di Francesco P 2003 Rectangular matrix models and combinatorics of colored graphs Nucl. Phys. B 648 461
[28] Novaes M 2015 Statistics of time delay and scattering correlation functions in chaotic systems. II. Semiclassical approximation J. Math. Phys. 56 062109
[29] Bousquet-Mélon M and Schaeffer G 2000 Enumeration of planar constellations Adv. Appl. Math. 24 337
[30] Irving J 2009 Minimal transitive factorizations of permutations into cycles Can. J. Math. 61 1092
[31] Bouttier J 2011 Matrix integrals and enumeration of maps The Oxford Handbook of Random Matrix Theory ed G Akemann et al (Oxford: Oxford University Press) ch 26
[32] Berkolaiko G and Irving J 2016 Inequivalent factorizations of permutations J. Combin. Theory A 140 1–37
[33] Bóna M and Pittel B 2016 On the cycle structure of the product of random maximal cycles (arXiv:1601.00319v1)
[34] Bernardi O, Morales A, Stanley R and Du R 2014 Separation probabilities for products of permutations Comb. Probab. Comput. 23 201
[35] Morales A H and Vassilieva E A 2010 Bijective evaluation of the connection coefficients of the double coset algebra (arXiv:1011.5001v1)
[36] Hanlon P J, Stanley R P and Stembridge J R 1992 Some combinatorial aspects of the spectra of normally distributed random matrices Contemp. Math. 138 151
[37] Goulden I P and Jackson D M 1996 Maps in locally orientable surfaces, the double coset algebra, and zonal polynomials Can. J. Math. 48 569
[38] Matsumoto S 2011 Jucys–Murphy elements, orthogonal matrix integrals, and Jack measures Ramanujan J. 26 69
[39] Berkolaiko G and Kuipers J 2013 Combinatorial theory of the semiclassical evaluation of transport moments I: equivalence with the random matrix approach J. Phys. A: Math. Theor. 54 112103
[40] Müller S, Heusler S, Braun P and Haake F 2007 Semiclassical approach to chaotic quantum transport New J. Phys. 9 12
[41] Müller S, Heusler S, Braun P, Haake F and Altland A 2005 Periodic-orbit theory of universality in quantum chaos Phys. Rev. E 72 046207
[42] Berkolaiko G and Kuipers J 2013 Combinatorial theory of the semiclassical evaluation of transport moments II: algorithmic approach for moment generating functions J. Math. Phys. 54 123505
[43] Novaes M 2012 Semiclassical approach to universality in quantum chaotic transport Europhys. Lett. 98 20006
[44] Novaes M 2013 Combinatorial problems in the semiclassical approach to quantum chaotic transport J. Phys. A: Math. Theor. 46 095101
[45] Macdonald I G 1995 Symmetric Functions and Hall Polynomials 2nd edn (Oxford: Oxford University Press)
[46] Friedman W A and Mello P A 1985 Marginal distribution of an arbitrary square submatrix of the S-matrix for Dyson’s measure J. Phys. A: Math. Gen. 18 425
[47] Życzkowski K and Sommers H-J 2000 Truncations of random unitary matrices J. Phys. A: Math. Gen. 33 2045
[48] Neretin Y A 2002 Hua-type integrals over unitary groups and over projective limits of unitary groups Duke Math. J. 114 239
[49] Forrester P J 2006 Quantum conductance problems and the Jacobi ensemble J. Phys. A: Math. Gen. 39 6861
[50] Fyodorov Y V and Khoruzhenko B A 2007 A few remarks on colour-flavour transformations, truncations of random unitary matrices, Berezin reproducing kernels and Selberg-type integrals J. Phys. A: Math. Theor. 40 669
[51] Shen J 2001 On the singular values of Gaussian random matrices Linear Algebra: Appl. 326 1
[52] Edelman A and Rao N R 2005 Random matrix theory Acta Numer. 14 233
[53] Mehta M L 2004 Random Matrices (New York: Academic) ch 17
[54] Forrester P J and Warnaar S O 2008 The importance of the Selberg integral Bull. Am. Math. Soc. 45 489
[55] Khoruzhenko B A, Sommers H-J and Życzkowski K 2010 Truncations of random orthogonal matrices Phys. Rev. E 82 040106
[56] Stanley R P 1999 Enumerative Combinatorics vol 2 (Cambridge: Cambridge University Press)