FULLY-DISCRETE FINITE ELEMENT APPROXIMATIONS FOR
A FOURTH-ORDER LINEAR STOCHASTIC PARABOLIC EQUATION
WITH ADDITIVE SPACE-TIME WHITE NOISE
II. 2D AND 3D CASE

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Abstract. We consider an initial- and Dirichlet boundary-value problem for a fourth-order linear
stochastic parabolic equation, in two or three space dimensions, forced by an additive space-time white
noise. Discretizing the space-time white noise a modeling error is introduced and a regularized fourth-
order linear stochastic parabolic problem is obtained. Fully-discrete approximations to the solution
of the regularized problem are constructed by using, for discretization in space, a standard Galerkin
finite element method based on $C^1$ piecewise polynomials, and, for time-stepping, the Backward Euler
method. We derive strong a priori estimates for the modeling error and for the approximation error to
the solution of the regularized problem.

1. Introduction

1.1. The main problem. Let $d = 2$ or $3$, $T > 0$, $D = (0, 1)^d \subset \mathbb{R}^d$ and $(\Omega, \mathcal{F}, P)$ be a complete
probability space. Then we consider an initial- and Dirichlet boundary-value problem for a fourth-
order linear stochastic parabolic equation formulated, typically, as follows: find a stochastic function
$u : [0, T] \times \overline{D} \to \mathbb{R}$ such that
\begin{align}
\partial_t u + \Delta^2 u &= \dot{W}(t, x) \quad \forall (t, x) \in (0, T] \times D, \\
\Delta^m u(t, \cdot) \bigg|_{\partial D} &= 0 \quad \forall t \in (0, T], \ m = 0, 1, \\
u(0, x) &= 0 \quad \forall x \in D,
\end{align}
a.s. in $\Omega$, where $\dot{W}$ denotes a space-time white noise on $[0, T] \times D$ (see, e.g., [26], [15]). The mild solution
of the problem above (cf. [5], [10]), known as ‘stochastic convolution’, is given by
\begin{equation}
\begin{aligned}
u(t, x) &= \int_0^t \int_D G(t - s; x, y) dW(s, y).
\end{aligned}
\end{equation}
Here, $G(t; x, y)$ is the space-time Green kernel of the corresponding deterministic parabolic problem: find
a deterministic function $w : [0, T] \times \overline{D} \to \mathbb{R}$ such that
\begin{align}
\partial_t w + \Delta^2 w &= 0 \quad \forall (t, x) \in (0, T] \times D, \\
\Delta^m w(t, \cdot) \bigg|_{\partial D} &= 0 \quad \forall t \in (0, T], \ m = 0, 1, \\
w(0, x) &= w_0(x) \quad \forall x \in D,
\end{align}
where $w_0$ is a deterministic initial condition. In particular, we have
\begin{equation}
w(t, x) = \int_D G(t; x, y) w_0(y) dy \quad \forall (t, x) \in (0, T] \times \overline{D}
\end{equation}
1.2. The regularized problem. Following the approach for a second order one dimensional stochastic parabolic equation with additive space-time white noise proposed in [1], we construct below an approximate initial and boundary value problem:

For $N_*, J_* \in \mathbb{N}$, define the mesh-lengths $\Delta t := \frac{T}{N_*}$, $\Delta x := \frac{1}{J_*}$, and the nodes $t_n := n \Delta t$ for $n = 0, \ldots, N_*$ and $x_j := j \Delta x$ for $j = 0, \ldots, J_*$. Then, we define the sets $N_* := \{1, \ldots, N_* \}$, $J_* := \{1, \ldots, J_* \}$, $T_n := (t_n, t_n)$ for $n \in N_*$, $D_j := (x_j, x_j)$ for $j \in J_*$, $D_{n, \mu} := \prod_{j=1}^{d} D_j$, for $\mu \in J_*^d$, and $S_{n, \mu} := T_n \times D_{\mu}$ for $n \in N_*$ and $\mu \in J_*^d$. Next, consider the fourth-order linear stochastic parabolic problem:

$$
\begin{align*}
&\partial_t \hat{u} + \Delta^2 \hat{u} = \hat{W} \quad \text{in } (0, T) \times D, \\
&\Delta^m \hat{u}(t, \cdot)|_{\partial D} = 0 \quad \forall \ t \in (0, T], \ m = 0, 1, \\
&\hat{u}(0, x) = 0 \quad \forall \ x \in D,
\end{align*}
$$

(1.6)
a.e. in $\Omega$, where

$$
\hat{W}(t, x) := \frac{1}{\Delta t(\Delta x)^d} \sum_{n \in N_*} \sum_{\mu \in J_*} X_{S_{n, \mu}}(t, x) R^{n, \mu} \quad \forall \ (t, x) \in [0, T] \times \overline{D},
$$

and $X_S$ is the index function of $S \subset [0, T] \times \overline{D}$.

The solution of the problem (1.6), according to the standard theory for parabolic problems (see, e.g., [19]), has the integral representation

$$
\hat{u}(t, x) = \int_0^t \int_D G(t-s; x, y) \hat{W}(s, y) \, ds dy \quad \forall \ (t, x) \in [0, T] \times \overline{D}.
$$

(1.7)

Remark 1.1. The properties of the stochastic integral (see, e.g., [26]), yield that $R^{n, \mu} \sim \mathcal{N}(0, \Delta t(\Delta x)^d)$ for all $(n, \mu) \in N_* \times J_*^d$. Also, we observe that $\mathbb{E}[R^{n, \mu} R^{n', \mu'}] = 0$ for $(n, \mu) \neq (n', \mu')$. Thus, the random variables $(R^{n, \mu})_{(n, \mu) \in N_* \times J_*^d}$ are independent.

1.3. Main results of the paper. The problem (1.1) is a linearized formulation of the stochastic Cahn-Hilliard equation (cf. [5], [10]) which was introduced by Cook [6] for the investigation of phase separation in spinodal decomposition (see, e.g., [10], [12]). For the convergence analysis of approximation methods for fourth-order stochastic parabolic problems driven by a space-time white noise, we refer the reader to [4] which considers a finite difference method for the stochastic Cahn-Hilliard equation, and to [22], [13] and [14] which consider time-stepping methods for a wide family of evolution problems that includes (1.1), while the finite element method is not among the space-discretization techniques considered in [13] and [14]. Also, we refer the reader to [8], [1], [17], [25], [27] and [2] for the analysis of the finite element method for second order stochastic parabolic problems.

In the paper at hand, we extend some of our results for the 1D case, given in [18], to the 2D and 3D case. In particular, we consider approximations of the solution $\hat{u}$ of (1.6) produced by the Backward Euler time-stepping combined with a $C^1$–finite element method, and analyze its convergence to the mild solution of (1.1). This error splits in two parts: the modeling error that appears by approximating $u$ by $\hat{u}$ and the numerical approximation error for $\hat{u}$. The estimation of the modeling error is achieved, in Theorem 3.1, by obtaining the inequality

$$
\begin{align*}
&\max_{t \in [0, T]} \left\{ \int_D \left( \int_D |u(t, x) - \hat{u}(t, x)|^2 \, dx \right) \, dP \right\}^{\frac{1}{2}} \leq C \left[ \epsilon^{-\frac{1}{2}} \Delta x^{\frac{4-d}{2}} \epsilon + \Delta t^{\frac{4-d}{4}} \right],
\end{align*}
$$

(1.8)
Moving to the direction of building approximations of \( \hat{u} \), we let \( M \in \mathbb{N} \), \((\tau_m)_{m=0}^M\) the nodes of a partition of \([0,T]\), i.e. \( \tau_0 = 0 \), \( \tau_M = T \) and \( \tau_{m-1} < \tau_m \) for \( m = 1, \ldots, M \), and define \( \Delta_m := (\tau_{m-1}, \tau_m) \) and \( k_m := \tau_m - \tau_{m-1} \) for \( m = 1, \ldots, M \), and set \( k_{\text{max}} := \max_{1 \leq m \leq M} k_m \); also, we let \( M_h \subset H^1_0(D) \cap H^2(D) \) be a finite element space consisting of functions which are piecewise polynomials over a partition of \( D \) in triangles or rectangulars with maximum diameter \( h \), and define a discrete biharmonic operator \( B_h : M_h \rightarrow M_h \) by

\[
(B_h\varphi, \chi)_{o,D} = (\Delta \varphi, \Delta \chi)_{o,D} \quad \forall \varphi, \chi \in M_h,
\]

and the usual \( L^2(D) \)-projection operator \( P_h : L^2(D) \rightarrow M_h \) by

\[
(P_h f, \chi)_{o,D} = (f, \chi)_{o,D} \quad \forall \chi \in M_h, \quad \forall f \in L^2(D).
\]

To construct approximations to \( \hat{u} \), we employ the Backward Euler finite element method which begins by setting

\[
\hat{U}_h^0 := 0,
\]

and, then for \( m = 1, \ldots, M \), finds \( \hat{U}_h^m \in M_h \) such that

\[
\hat{U}_h^m - \hat{U}_h^{m-1} + k_m B_h \hat{U}_h^m = \int_{\tau_m}^{\tau_{m+1}} P_h \hat{W} \, ds.
\]

Estimating the numerical approximation error for \( \hat{u} \), we derive first, in Theorem 6.2, the discrete in time \( L^2_t(L^2_p(L^2_v)) \) error estimate:

\[
\left\{ \sum_{m=1}^M k_m \int_\Omega \left( \int_D \left| \hat{U}_h^m(x) - \hat{u}(\tau_m, x) \right|^2 \, dx \right) \, dP \right\}^{\frac{1}{2}} \leq C \left[ (k_{\text{max}})^{\frac{1}{2}} + \epsilon_2^{-\frac{1}{2} h^{d-\epsilon}} \right],
\]

where: \( \nu = \nu(r, d) \) is given in (5.13) and depends on the space dimension \( d \) and a parameter \( r \in \{2, 3, 4\} \) which is related to the approximation properties of the finite element spaces \( M_h \) (see (2.19)). Only for the needs of the proof of (1.12), we introduce a space-discrete approximation \( \hat{u}_h \) of \( \hat{u} \) and analyze its convergence in the \( L^\infty_t(L^2_p(L^2_v)) \) norm (see Theorem 5.2). Also, assuming that the nodes \((\tau_m)_{m=0}^M\) are equidistributed with \( \Delta \tau = k_m \) for \( m = 1, \ldots, M \), we arrive at the discrete in time \( L^\infty_t(L^2_p(L^2_v)) \) error estimate (see Theorem 6.5):

\[
\max_{0 \leq m \leq M} \left\{ \int_\Omega \left( \int_D \left| \hat{u}_h^m(x) - \hat{u}(\tau_m, x) \right|^2 \, dx \right) \, dP \right\}^{\frac{1}{2}} \leq C \left[ \epsilon_1^{-\frac{1}{2} \Delta \tau^{\frac{d-\epsilon}{4}}} + \epsilon_2^{-\frac{1}{2} h^{d-\epsilon}} \right].
\]

To get the estimate above, first we consider the Backward Euler time-discrete approximations of \( \hat{u} \) and analyze their convergence in the discrete in time \( L^\infty_t(L^2_p(L^2_v)) \) norm above (see Theorem 1.2), and then, we compare the Backward Euler fully-discrete with the Backward Euler time-discrete approximations of \( \hat{u} \) (see Proposition 6.4). This procedure gives the possibility to estimate separately the space and the time discretization error in contrast to the technique used in [23] and [2] for second order problems. The reason of including the error estimate (1.12) in addition to the stronger norm error estimate (1.13), is that the order of convergence is slightly better and it allows a nonuniform partition of the time interval which is non standard among references (cf. [27], [13], [14], [23], [24]).

We close the section by an overview of the paper. Section 2 introduces notation, and recalls or proves several results often used in the paper. Section 3 is dedicated to the estimation of the modeling error. Section 4 defines the Backward Euler time-discrete approximations of \( \hat{u} \) and analyzes its convergence. Section 5 defines a finite element space-discrete approximation of \( \hat{u} \) and estimates its approximation error. Section 6 contains the error analysis for the Backward Euler fully-discrete approximations of \( \hat{u} \).

2. Notation and Preliminaries

2.1. Function spaces and operators. We denote by \( L^2(D) \) the space of the Lebesgue measurable functions which are square integrable on \( D \) with respect to Lebesgue’s measure \( dx \), provided with the standard norm \( \|g\|_{o,D} := \left\{ \int_D |g(x)|^2 \, dx \right\}^{\frac{1}{2}} \) for \( g \in L^2(D) \). The standard inner product in \( L^2(D) \) that produces the norm \( \| \cdot \|_{o,D} \) is written as \( (\cdot, \cdot)_{o,D} \), i.e., \( (g_1, g_2)_{o,D} := \int_D g_1(x)g_2(x) \, dx \) for \( g_1, g_2 \in L^2(D) \).
For $s \in \mathbb{N}_0$, $H^s(D)$ will be the Sobolev space of functions having generalized derivatives up to order $s$ in the space $L^2(D)$, and by $\| \cdot \|_{s,D}$ its usual norm, i.e. $\|g\|_{s,D} := \left\{ \sum_{|\alpha| \leq s} \| \partial_x^\alpha g \|_{0,D}^2 \right\}^{\frac{1}{2}}$ for $g \in H^s(D)$. Also, by $H^1_0(D)$ we denote the subspace of $H^1(D)$ consisting of functions which vanish at the boundary $\partial D$ of $D$ in the sense of trace. We note that in $H^1_0(D)$ the, well-known, Poincaré-Friedrichs inequality holds, i.e.,

\[
\|g\|_{0,D} \leq C_{PF} \|\nabla g\|_{0,D} \quad \forall g \in H^1_0(D),
\]

where $\|\nabla v\|_{0,D} := \left\{ \sum_{|\alpha| = 1} \|\partial_x^\alpha v\|_{0,D}^2 \right\}^{\frac{1}{2}}$ for $v \in H^1(D)$.

The sequence of pairs $\{ (\lambda_\alpha, \varepsilon_\alpha) \}_{\alpha \in \mathbb{N}^d}$ is a solution to the eigenvalue/eigenfunction problem: find nonzero $\varphi \in H^2(D) \cap H^1_0(D)$ and $\sigma \in \mathbb{R}$ such that $-\Delta \varphi = \sigma \varphi$ in $D$. Since $(\epsilon_\alpha)_{\alpha \in \mathbb{N}^d}$ is a complete $(\cdot,\cdot)_D$-orthonormal system in $L^2(D)$, for $s \in \mathbb{N}$, a subspace $\hat{H}^s(D)$ of $L^2(D)$ (see [23]) is defined by

\[
\hat{H}^s(D) := \left\{ v \in L^2(D) : \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha^s (v, \varepsilon_\alpha)^2_{0,D} < \infty \right\}
\]

and provided with the norm $\|v\|_{\hat{H}^s} := \left( \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha^s (v, \varepsilon_\alpha)^2_{0,D} \right)^{\frac{1}{2}} \forall v \in \hat{H}^s(D)$. Let $m \in \mathbb{N}_0$. It is well-known (see [23]) that

\[
\hat{H}^m(D) = \left\{ v \in H^m(D) : \Delta^i v \big|_{\partial D} = 0 \quad \text{if} \quad 0 \leq i \leq \frac{m}{2} \right\}
\]

and there exist constants $C_{m,A}$ and $C_{m,B}$ such that

\[
C_{m,A} \|v\|_{m,D} \leq \|v\|_{\hat{H}^m} \leq C_{m,B} \|v\|_{m,D} \quad \forall v \in \hat{H}^m(D).
\]

Also, we define on $L^2(D)$ the negative norm $\| \cdot \|_{-m,D}$ by

\[
\|v\|_{-m,D} := \sup \left\{ \frac{(v, \varphi)_{0,D}}{\|\varphi\|_{-(m-1),D}} : \varphi \in \hat{H}^m(D) \quad \text{and} \quad \varphi \neq 0 \right\} \quad \forall v \in L^2(D),
\]

for which, using (2.3), it is easy to conclude that there exists a constant $C_{-m} > 0$ such that

\[
\|v\|_{-m,D} \leq C_{-m} \|v\|_{\hat{H}^{-m}} \quad \forall v \in L^2(D).
\]

Let $\mathbb{L}_2 = (L^2(D), (\cdot, \cdot)_{0,D})$ and $\mathcal{L}(\mathbb{L}_2)$ be the space of linear, bounded operators from $\mathbb{L}_2$ to $\mathbb{L}_2$. We say that, an operator $\Gamma \in \mathcal{L}(\mathbb{L}_2)$ is Hilbert-Schmidt, when $\|\Gamma\|_{HS} := \left\{ \sum_{k=1}^{\infty} \|\Gamma \varepsilon_k\|_{0,D}^2 \right\}^{\frac{1}{2}} < \infty$, where $\|\Gamma\|_{HS}$ is the so-called Hilbert-Schmidt norm of $\Gamma$. We note that the quantity $\|\Gamma\|_{HS}$ does not change when we replace $\{ \varepsilon_k \}_{k=1}^{\infty}$ by another complete orthonormal system of $\mathbb{L}_2$. It is well known (see, e.g., [11]) that an operator $\Gamma \in \mathcal{L}(\mathbb{L}_2)$ is Hilbert-Schmidt iff there exists a measurable function $g : D \times D \to \mathbb{R}$ such that $\Gamma[v](\cdot) = \int_D g(\cdot, y) v(y) dy$ for $v \in L^2(D)$, and then, it holds that

\[
\|\Gamma\|_{HS} = \left( \int_D \int_D g^2(x, y) dxdy \right)^{\frac{1}{2}}.
\]

Let $\mathcal{L}_{HS}(\mathbb{L}_2)$ be the set of Hilbert Schmidt operators of $\mathcal{L}(\mathbb{L}_2)$ and $\Phi : [0, T] \to \mathcal{L}_{HS}(\mathbb{L}_2)$. Also, for a random variable $X$, let $\mathbb{E}[X]$ be its expected value, i.e., $\mathbb{E}[X] := \int_\mathbb{N} X dP$. Then, the Itô isometry property for stochastic integrals, which we will use often in the paper, reads

\[
\mathbb{E} \left[ \left\| \int_0^T \Phi dW \right\|_{0,D}^2 \right] = \int_0^T \|\Phi(t)\|_{HS}^2 dt.
\]

For later use, we introduce the projection operator $\hat{\Pi} : L^2((0, T) \times D) \to L^2((0, T) \times D)$ defined by

\[
\hat{\Pi}(g)(\cdot) \big|_{s_{n,\mu}} := \frac{1}{2\Delta t} \int_{s_{n,\mu}} g(t, x) dt dx \quad \forall n \in \mathcal{N}^*, \quad \forall \mu \in \mathcal{J}^d
\]

for $g \in L^2((0, T) \times D)$, which has the following property:
Lemma 2.1. For \( g \in L^2((0, T) \times D) \), it holds that

\[
\int_0^T \int_D \widehat{\Pi}(g; s, y) \, dW(s, y) = \int_0^T \int_D \widehat{W}(t, x) \, g(t, x) \, dt \, dx.
\]

Proof. To obtain (2.8) we work, using (2.7) and the properties of \( W \), as follows:

\[
\int_0^T \int_D \widehat{\Pi}(g; s, y) \, dW(s, y) = \frac{1}{\Delta t(\Delta x)^2} \sum_{n \in \mathbb{N}, \mu \in \mathbb{J}_d} \left( \int_{S_{n,\mu}} g \, dt \, dx \right) \left( \int_0^T \int_D \chi_{S_{n,\mu}}(s, y) \, dW(s, y) \right)
= \frac{1}{\Delta t(\Delta x)^2} \sum_{n \in \mathbb{N}, \mu \in \mathbb{J}_d} \left( \int_{S_{n,\mu}} g(t, x) \, dt \, dx \right) R^m,\mu
= \frac{1}{\Delta t(\Delta x)^2} \sum_{n \in \mathbb{N}, \mu \in \mathbb{J}_d} \sum_{\iota = 1}^T \int_0^T g(t, x) \, \chi_{S_{n,\mu}}(t, x) \, R^m,\mu \, dt \, dx
= \int_0^T \int_D g(t, x) \, \widehat{W}(t, x) \, dt \, dx.
\]

We close this section, by stating some asymptotic bounds for series that will often appear in the rest of the paper and for a proof of them we refer the reader to Appendix A and Appendix B.

Lemma 2.2. Let \( d \in \{1, 2, 3\} \) and \( c_* > 0 \). Then, there exists a constant \( C > 0 \) that depends on \( c_* \) and \( d \), such that

\[
\sum_{\alpha \in \mathbb{N}^d} |\alpha|^{(d+c_*)} \leq C \epsilon^{-1} \quad \forall \epsilon \in (0, 2].
\]

Lemma 2.3. Let \( d \in \{2, 3\} \) and \( \delta > 0 \). Then there exists a constant \( C > 0 \) which is independent of \( \delta \), such that

\[
\sum_{\alpha \in \mathbb{N}^d} \frac{1}{\alpha!} \chi_{\mathbb{N}^d}^\alpha \leq C \quad \forall \delta \in (0, 1),
\]

where \( p_d(\delta) := 1 + \sum_{i=1}^d s_i \).

2.2. Linear elliptic and parabolic operators. For given \( f \in L^2(D) \) let \( v_E \in H^2(D) \cap H^1_0(D) \) be the solution of the boundary value problem

\[
\Delta v_E = f \quad \text{in} \quad D,
\]

and \( T_E : L^2(D) \to H^2(D) \cap H^1_0(D) \) be its solution operator, i.e. \( T_E f := v_E \), which has the property

\[
\|T_E f\|_{m, D} \leq C_E \|f\|_{m-2, D}, \quad \forall f \in H^{\text{max}(0, m-2)}(D), \quad \forall m \in \mathbb{N}_0.
\]

Also, for \( f \in L^2(D) \) let \( v_B \in H^4(D) \) be the solution of the following biharmonic boundary value problem

\[
\Delta^2 v_B = f \quad \text{in} \quad D,
\]

\[
\Delta^m v_B|_{\partial D} = 0, \quad m = 0, 1,
\]

and \( T_B : L^2(D) \to \dot{H}^4(D) \) be the solution operator of (2.13), i.e. \( T_B f := v_B \), which satisfies

\[
\|T_B f\|_{m, D} \leq C \|f\|_{m-4, D}, \quad \forall f \in H^{\text{max}(0, m-4)}(D), \quad \forall m \in \mathbb{N}_0.
\]

Due to the type of boundary conditions of (2.13), we conclude that

\[
T_B f = T_E^2 f, \quad \forall f \in L^2(D),
\]

which, easily, yields

\[
(T_B v_1, v_2)_{0,D} = (T_E v_1, T_E v_2)_{0,D} \quad \forall v_1, v_2 \in L^2(D).
\]
Proof. Let \( (S(t)w_0)_{t \in [0,T]} \) be the standard semigroup notation for the solution \( w \) of (1.3), we can easily establish the following property (see, e.g., [23, 21]): for \( \ell \in \mathbb{N}_0, \beta, p \in \mathbb{R}_0^+ \) and \( q \in [0,p+4\ell] \) there exists a constant \( C > 0 \) such that:
\[
(2.24) \quad \int_{t_a}^{t_b} (r - t_a)^\beta \| \partial_t^{\ell} S(t)w_0 \|_{\dot{H}^r}^2 \, dt \leq C \| w_0 \|_{\dot{H}^{p+4\ell-2\beta-2}}^2 \quad \forall t_b > t_a \geq 0, \quad \forall w_0 \in \dot{H}^{p+4\ell-2\beta-2}(D).
\]

2.3. Discrete spaces and operators. For \( r \in \{2,3,4\} \), we consider a finite element space \( M_h \subset H^1_0(D) \cap H^2(D) \) consisting of functions which are piecewise polynomials over a partition of \( D \) in triangles or rectangulars with maximum mesh-length \( h \). We assume that the space \( M_h \) has the following approximation property
\[
(2.19) \quad \inf_{\chi \in M_h} \| v - \chi \|_{2,D} \leq C h^{r-1} \| v \|_{r+1,D} \quad \forall v \in H^{r+1}(D) \cap H^1_0(D),
\]
which covers several classes of \( C^1 \) finite element spaces, for example the tensor products of \( C^1 \) splines, the Argyris triangle elements, the Hsieh-Clough-Tocher triangle elements and the Bell triangle (cf. [7, 3]).

A finite element approximation \( v_{0,h} \in M_h \) of the solution \( v_B \) of (2.13) is defined by the requirement
\[
(2.20) \quad B_h v_{0,h} = P_h f,
\]
and we denote by \( T_{B,h} : L^2(D) \to M_h \) the solution operator of (2.20), i.e. \( T_{B,h} := v_{0,h} = B_h^{-1} P_h f \) for \( f \in L^2(D) \). It is easy to verify that \( T_{B,h} \) is selfadjoint, i.e.,
\[
(2.21) \quad (T_{B,h} f, g)_{0,D} = (f, T_{B,h} g)_{0,D} \quad \forall f, g \in L^2(D).
\]
Also, using (2.20), (2.19) and (2.15) we conclude that
\[
(2.22) \quad \| \Delta T_{B,h} f \|_{0,D} \leq C \| f \|_{2,D} \quad \forall f \in L^2(D).
\]
Applying the standard theory of the finite element method (see, e.g., [7, 3]) and using (2.15), we get
\[
(2.23) \quad \| \Delta (T_B f - T_{B,h} f) \|_{0,D} \leq C h^{r-1} \| f \|_{r-3,D}, \quad \forall f \in H^{r+1}(D),
\]
while error estimates in the \( L^2(D) \) norm are obtained in the proposition below.

**Proposition 2.1.** Let \( r \in \{2,3,4\} \). Then, it holds that:
\[
(2.24) \quad \| T_B f - T_{B,h} f \|_{0,D} \leq C \begin{cases} \begin{align*}
h^5 & \quad r = 4 \\
h^4 & \quad r = 3 \\
h^2 & \quad r = 2 \end{align*} \end{cases} \quad \forall f \in H^{\max\{r-3,0\}}(D).
\]

**Proof.** Let \( f \in H^{\max\{0,r-3\}}(D) \) and \( e = T_B f - T_{B,h} f \). Also, we define a bilinear form \( \gamma : H^2(D) \times H^2(D) \to \mathbb{R} \) by \( \gamma(v_1, v_2) := (\Delta v_1, \Delta v_2)_{0,D} \) for \( v_1, v_2 \in H^2(D) \). Now, let \( w_A, w_B \in H^4(D) \) be defined by \( T_B \Delta e = w_A \) and \( T_B e = w_B \). Then, using Galerkin orthogonality, we have:
\[
(2.25) \quad \| \nabla e \|_{0,D}^2 = - \gamma(w_A, e)_{0,D} \leq \| \Delta e \|_{0,D} \inf_{\chi \in M_h} \| w_A - \chi \|_{2,D}
\]
and
\[
(2.26) \quad \| e \|_{0,D}^2 = - \gamma(w_A, e)_{0,D} \leq \| \Delta e \|_{0,D} \inf_{\chi \in M_h} \| w_B - \chi \|_{2,D}.
\]

**Case 1:** Let \( r \in \{2,3\} \). Then, using (2.20), (2.23), (2.19) and (2.22), we obtain
\[
\| e \|_{0,D}^2 \leq C h^{r-1} \| f \|_{r-3,D} h^{r-1} \| w_B \|_{r+1,D} \leq C h^{2(r-1)} \| f \|_{r-3,D} \| e \|_{r-3,D}
\]
which, obviously, yields (2.24).
Case 2: Let \( r = 4 \). Then, combining, (2.26), (2.19), (2.15) and (2.1), we get
\[
\|e\|^2_{o,D} \leq C \|\Delta e\|_{o,D} h^3 \|T_0 e\|_{s,D}
\]
\[
\leq C \|\Delta e\|_{o,D} h^3 \|\nabla e\|_{o,D}.
\]
Also, we observe that (2.25) and (2.15) yield
\[
\|\nabla e\|_{o,D} \leq \|\Delta e\|_{o,D} T_0 \Delta e\|_{s,D}
\]
\[
\leq C \|\Delta e\|_{o,D} h^3 \|e\|_{s,D}.
\]
Now, we combine (2.27), (2.28) and (2.29) to have
\[
\|e\|^2_{o,D} \leq C h^3 \|\Delta e\|_{o,D}^2
\]
\[
\leq C h^{15} \|f\|_{s,D}^2,
\]
which obviously leads to (2.24) for \( r = 4 \). \( \square \)

3. An Estimate for the Modeling Error

Here, we derive an \( L^\infty_t (L^2_x) \) bound for the modeling error \( u - \hat{u} \), in terms of \( \Delta t \) and \( \Delta x \).

**Theorem 3.1.** Let \( u \) and \( \hat{u} \) be defined, respectively, by (1.2) and (1.7). Then, there exists a real constant \( C > 0 \), independent of \( T \), \( \Delta t \) and \( \Delta x \), such that
\[
(3.1) \quad \max_{[0,T]} \{ E \left[ \|u - \hat{u}\|^2_{o,D} \right] \}^{1/2} \leq C \left\{ \left( p_d(\Delta t) \Delta t^{1-\epsilon} + \epsilon^{1/2} \Delta x^{4-d-\epsilon} \right) \forall \epsilon \in (0,\frac{4-d}{2}) \right\}.
\]

**Proof.** Using (1.2) and (1.7), we conclude that
\[
(3.2) \quad u(t,x) - \hat{u}(t,x) = \int_0^T \int_D \left[ \mathcal{X}_{(0,t)}(s) G(t-s;x,y) - \tilde{G}(t,x,s,y) \right] dW(s,y) \quad \forall (t,x) \in [0,T] \times D,
\]
where \( \tilde{G} : (0,T) \times D \to L^2((0,T) \times D) \) given by
\[
\tilde{G}(t,x;\cdot) \bigg|_{s_{n,\mu}} = \frac{1}{\Delta t(\Delta x)^d} \int_{s_{n,\mu}} \mathcal{X}_{(0,t)}(s') G(t-s';x,y') \ ds' dy', \quad \forall n \in N_*, \ \forall \mu \in J^d_*.
\]
Let \( \Theta := \{ E \left[ \|u_L - \hat{u}_L\|^2_{o,D} \right] \}^{1/2} \) and \( t \in (0,T) \). Using (3.2) and Itô isometry (2.6), we obtain
\[
\Theta(t) = \frac{1}{\Delta t(\Delta x)^d} \left\{ \sum_{n \in N_*} \sum_{\mu \in J^d_*} \int_D \left\{ \int_{s_{n,\mu}} \left[ \mathcal{X}_{(0,t)}(s) G(t-s;x,y)
\right.ight.ight.
\]
\[
- \left. \mathcal{X}_{(0,t)}(s') G(t-s';x,y') \right] ds' dy' \right\} dsdy\right\}^{1/2}
\]
Now, we introduce the splitting
\[
(3.3) \quad \Theta(t) \leq \Theta_A(t) + \Theta_B(t),
\]
where
\[
\Theta_A(t) := \frac{1}{\Delta t(\Delta x)^d} \left\{ \sum_{n \in N_*} \sum_{\mu \in J^d_*} \int_D \left\{ \int_{s_{n,\mu}} \left[ \mathcal{X}_{(0,t)}(s) \right] G(t-s;x,y)
\right.
\]
\[
- \left. G(t-s;x,y') \right] ds' dy' \right\}^{1/2} dsdy\right\}^{1/2}
\]
\[
\Theta_B(t) := \frac{1}{\Delta t(\Delta x)^d} \left\{ \sum_{n \in N_*} \sum_{\mu \in J^d_*} \int_D \left\{ \int_{s_{n,\mu}} \left[ \mathcal{X}_{(0,t)}(s) \right] \right.
\]
\[
\left. - \mathcal{X}_{(0,t)}(s') \right] ds' dy' \right\}^{1/2} dsdy\right\}^{1/2}
\]
and
\[ \Theta_\mu(t) = \frac{1}{\Delta t (\Delta x)^d} \left\{ \sum_{n \in \mathbb{N}} \sum_{\mu \in \mathcal{J}_d^2} \int_D \left\{ \int_{S_{n,\mu}} \left[ X_{0,t}(s) G(t-s;x,y') - X_{0,t}(s') G(t-s';x,y') \right] ds' dy' \right\} ds dy \right\}^{\frac{1}{2}}. \]

**Estimation of \( \Theta_\mu(t) \):** Using \( (\ref{1.3}) \) and the \( (\cdot, \cdot)_{0, D} \)-orthogonality of \( (\varepsilon_\alpha)_{\alpha \in \mathbb{N}^d} \), we have
\[
\Theta^2_\mu(t) = \frac{1}{(\Delta x)^d} \sum_{n \in \mathbb{N}} \sum_{\mu \in \mathcal{J}_d^2} \int_D \left\{ \int_{S_{n,\mu}} \left[ \sum_{\alpha \in \mathbb{N}^d} X_{0,t}(s) e^{-2\lambda_\alpha^2(t-s)} \left( \int_{D_\mu} (\varepsilon_\alpha(y) - \varepsilon_\alpha(y')) dy' \right)^2 \right] ds dy \right\} dx
\]
\[
= \frac{1}{(\Delta x)^d} \sum_{n \in \mathbb{N}} \sum_{\mu \in \mathcal{J}_d^2} \left\{ \sum_{\alpha \in \mathbb{N}^d} \int_{T_n} X_{0,t}(s) e^{-2\lambda_\alpha^2(t-s)} ds \right\} \left\{ \sum_{\mu \in \mathcal{J}_d^2} \int_{D_\mu} \left( \int_{D_\mu} (\varepsilon_\alpha(y) - \varepsilon_\alpha(y')) dy' \right)^2 dy \right\}
\]
from which, using the Cauchy-Schwarz inequality, follows that
\[
(\ref{3.4}) \quad \Theta^2_\mu(t) \leq \sum_{\alpha \in \mathbb{N}^d} \left( \int_0^t e^{-2\lambda_\alpha^2(t-s)} ds \right) \left[ \frac{1}{(\Delta x)^d} \sum_{\mu \in \mathcal{J}_d^2} \int_{D_\mu} |\varepsilon_\alpha(y) - \varepsilon_\alpha(y')|^2 dy' dy \right].
\]
Observing that \( \int_0^t e^{-2\lambda_\alpha^2(t-s)} ds \leq \frac{1}{2} \lambda_\alpha^{-2} \) for \( \alpha \in \mathbb{N}^d \), and that
\[
\sup_{y, y' \in D_\mu} |\varepsilon_\alpha(y) - \varepsilon_\alpha(y')| \leq 2^{d+1} \min \left\{ 1, \frac{d}{2} \Delta x |\alpha|_\infty \right\}
\]
\[
\leq 2^{d+1-\gamma} \pi^{\gamma} d^\gamma \Delta x^{\gamma} |\alpha|_\infty^{\gamma}, \quad \forall \gamma \in [0, 1], \quad \forall \alpha \in \mathbb{N}^d, \quad \forall \mu \in \mathcal{J}_d^2,
\]
\( (\ref{3.3}) \) yields
\[
(\ref{3.5}) \quad \Theta^2_\mu(t) \leq 2^{d+1-\gamma} d^\gamma \pi^{2\gamma-4} (\Delta x)^{2\gamma} \sum_{\alpha \in \mathbb{N}^d} |\alpha|_\infty^{\gamma d-2\gamma}.
\]
The series in \( (\ref{3.3}) \) converges when \( 2(2 - \gamma) > d \) or equivalently \( \gamma < \frac{d}{2} \). Thus, combining \( (\ref{3.5}) \) and \( (\ref{2.3}) \), we, finally, conclude that
\[
(\ref{3.6}) \quad \Theta_\mu(t) \leq C e^{- \frac{1}{2} \Delta x^{\frac{4d}{2d-4} - \varepsilon}} \quad \forall \varepsilon \in \left( 0, \frac{4-d}{2} \right].
\]
**Estimation of \( \Theta_\mu(t) \):** For \( t \in (0, T] \), let \( \tilde{N}(t) := \min \{ \ell \in \mathbb{N} : 1 \leq \ell \leq N, \text{ and } t \leq t_\ell \} \) and
\[
\tilde{T}_n(t) := T_n \cap (0, t) = \begin{cases} T_n, & \text{if } n < \tilde{N}(t) \\ (t_{\tilde{N}(t)-1}, t), & \text{if } n = \tilde{N}(t) \end{cases}, \quad n = 1, \ldots, \tilde{N}(t).
\]
Now, we use \( (\ref{1.3}) \) and the \( (\cdot, \cdot)_{0, D} \)-orthogonality of \( (\varepsilon_\alpha)_{\alpha \in \mathbb{N}^d} \) as follows
\[
\Theta^2_\mu(t) = \frac{(\Delta x)^d}{(\Delta t (\Delta x)^d)^2} \sum_{n \in \mathbb{N}} \sum_{\mu \in \mathcal{J}_d^2} \int_D \left\{ \int_{T_n} \left[ X_{0,t}(s) G(t-s;x,y') - X_{0,t}(s') G(t-s';x,y') \right] ds' dy' \right\} ds dy \right\} dx
\]
\[
= \frac{(\Delta x)^d}{(\Delta t (\Delta x)^d)^2} \sum_{\alpha \in \mathbb{N}^d} \left( \sum_{\mu \in \mathcal{J}_d^2} \left( \int_{T_n} \varepsilon_\alpha(y') dy' \right)^2 \right) \left( \int_{T_n} \left( \sum_{n=1}^{\tilde{N}(t)} \int_{T_n} \left( X_{0,t}(s) e^{-\lambda_\alpha^2(t-s)} - X_{0,t}(s') e^{-\lambda_\alpha^2(t-s')} \right) ds' \right)^2 ds \right)
which yields that

\[(3.7) \quad \Theta_n^2(t) \leq 2^d \sum_{\alpha \in \mathbb{N}^d} \left( \frac{1}{(\Delta t)^2} \sum_{n=1}^{\mathring{N}(t)} \Psi_n^\alpha(t) \right),\]

where

\[\Psi_n^\alpha(t) := \int_{\tau_n} \left( \int_{\tau_n} \left( X_{(0,t)}(s) e^{-\lambda_n^2(t-s)} - X_{(0,t)}(s') e^{-\lambda_n^2(t-s')} \right) ds' \right)^2 ds.\]

Let \(\alpha \in \mathbb{N}^d\) and \(n \in \{1, \ldots, \mathring{N}(t) - 1\}\). Then, we have

\[\Psi_n^\alpha(t) = \int_{\tau_n} \left( \int_{\tau_n}^{\tau_{n+1}} \lambda_n^2 e^{-\lambda_n^2(t-s)} ds' \right)^2 \]
\[\leq \int_{\tau_n} \left( \int_{\tau_n}^{\tau_{n+1}} \lambda_n^2 e^{-\lambda_n^2(t-s)} ds' \right)^2 ds \]
\[\leq 2 \int_{\tau_n} \left( \int_{\tau_n}^{\tau_{n+1}} \lambda_n^2 e^{-\lambda_n^2(t-s')} ds' \right)^2 ds + 2 \int_{\tau_n} \left( \int_{\tau_n}^{\tau_{n+1}} \lambda_n^2 e^{-\lambda_n^2(t-s)} ds \right)^2 ds \]
\[\leq 2 \Delta t \left( \int_{\tau_n}^{\tau_{n+1}} \lambda_n^2 e^{-\lambda_n^2(t-s')} ds' \right)^2 + 2 (\Delta t)^2 \int_{\tau_n} \left( \int_{\tau_n}^{\tau_{n+1}} \lambda_n^2 e^{-\lambda_n^2(t-s)} ds \right)^2 ds,\]

from which, using the Cauchy-Schwarz inequality and integrating by parts, we obtain

\[\Psi_n^\alpha(t) \leq 4 (\Delta t)^2 \int_{\tau_n} \left( e^{-\lambda_n^2(t-s)} - e^{-\lambda_n^2(t-s')} \right)^2 ds \]
\[\leq 4 (\Delta t)^2 (1 - e^{-\lambda_n^2 \Delta t})^2 \int_{\tau_n} e^{-2\lambda_n^2(s-t)} ds \]
\[\leq 2 (\Delta t)^2 (1 - e^{-\lambda_n^2 \Delta t})^2 e^{-\lambda_n^2(t-s') - \lambda_n^2(t-s')} ds.\]

Thus, by summing with respect to \(n\), we obtain

\[(3.8) \quad \frac{1}{(\Delta t)^2} \sum_{n=1}^{\mathring{N}(t)-1} \Psi_n^\alpha(t) \leq 2 \frac{(1 - e^{-\lambda_n^2 \Delta t})^2}{\lambda_n^2}.\]

Considering, now, the case \(n = \mathring{N}(t)\), we have

\[(3.9) \quad \Psi_{\mathring{N}(t)}^\alpha(t) = \Psi_A^\alpha(t) + \Psi_B^\alpha(t)\]

with

\[\Psi_A^\alpha(t) := \int_t^{t_{\mathring{N}(t)-1}} \left( \int_t^{t_{\mathring{N}(t)-1}} \lambda_n^2 e^{-\lambda_n^2(t-s)} ds' \right)^2 ds \]
\[\Psi_B^\alpha(t) := \int_t^{t_{\mathring{N}(t)}} \left( \int_{t_{\mathring{N}(t)-1}}^{t_{\mathring{N}(t)}} e^{-\lambda_n^2(t-s')} ds' \right)^2 ds.\]

Then, we have

\[\Psi_B^\alpha(t) \leq \frac{\Delta t}{\lambda_n^2} \left[ 1 - e^{-\lambda_n^2 \Delta t} \right]^2 \]
\[\leq \frac{\Delta t}{\lambda_n^2 (1 - e^{-\lambda_n^2 \Delta t})^2}.\]
and
\[
\Psi^0_A(t) - \int_{t_{S(t)-1}}^{t} \left[ \int_{t_{S(t)-1}}^{s} \lambda^2_a e^{-\lambda_a^2 (t-r)} d\tau ds' + \Delta t e^{-\lambda_a^2 (t-s)} \right]^2 ds \\
\leq 2 \int_{t_{S(t)-1}}^{t} \left[ \int_{t_{S(t)-1}}^{s} \lambda^2_a e^{-\lambda_a^2 (t-r)} d\tau ds' \right]^2 ds + \frac{(\Delta t)^2}{\lambda_a^2} \left[ 1 - e^{-2\lambda_a^2 (t-t_{S(t)-1})} \right] \\
\leq 2 \int_{t_{S(t)-1}}^{t} \left[ \int_{t_{S(t)-1}}^{s} \lambda^2_a e^{-\lambda_a^2 (t-r)} d\tau ds' \right]^2 ds + \frac{(\Delta t)^2}{\lambda_a^2} \left( 1 - e^{-2\lambda_a^2 \Delta t} \right) \\
\leq 8 (\Delta t)^2 \int_{t_{S(t)-1}}^{t} \left[ \int_{t_{S(t)-1}}^{s} \lambda^2_a e^{-\lambda_a^2 (t-r)} d\tau \right]^2 ds + \frac{(\Delta t)^2}{\lambda_a^2} \left( 1 - e^{-2\lambda_a^2 \Delta t} \right) \\
\leq 8 (\Delta t)^2 \int_{t_{S(t)-1}}^{t} \left[ e^{-\lambda_a^2 (t-s)} - e^{-\lambda_a^2 (t-t_{S(t)-1})} \right]^2 ds + \frac{(\Delta t)^2}{\lambda_a^2} \left( 1 - e^{-2\lambda_a^2 \Delta t} \right),
\]
which, along with (3.9), gives
\[
\Psi^0_{\hat{x}(t)} \leq 5 \frac{(\Delta t)^2}{\lambda_a^2} \left( 1 - e^{-2\lambda_a^2 \Delta t} \right) + \frac{2 \Delta t}{\lambda_a^2} \left( 1 - e^{-2\lambda_a^2 \Delta t} \right)^2.
\]
Since the mean value theorem yields: \(1 - e^{-2\lambda_a^2 \Delta t} \leq \lambda_a^2 \Delta t\), the above inequality takes the form
\[
(\Delta t)^2 \Psi^0_{\hat{x}(t)} \leq 6 \frac{1-e^{-2\lambda_a^2 \Delta t}}{\lambda_a^2}.
\]
Combining (3.11), (8.7) and (8.10) we obtain
\[
\Theta^0_{\theta}(t) \leq 8 \sum_{\alpha \in \mathbb{N}^d} \frac{1-e^{-2\lambda_a^2 \Delta t}}{\lambda_a^2}.
\]
Now, combine (3.11) and (2.10) to arrive at
\[
\Theta_{\theta}(t) \leq C (p_d(\Delta t^{+}))^{+} \Delta t^{\frac{d-2}{2}}.
\]
The error bound (3.11) follows by observing that \(\Theta(0) = 0\) and combining the bounds (6.6), (8.6) and (3.12).

4. Time-Discrete Approximations

The Backward Euler time-discrete approximations to the solution \(\hat{u}(\tau_m, \cdot)\) of the problem (1.6) are defined as follows: first, set
\[
\hat{U}^0 := 0,
\]
and then, for \(m = 1, \ldots, M\), find \(\hat{U}^m \in \dot{H}^4(D)\) such that
\[
\hat{U}^m - \hat{U}^{m-1} + k_m \Delta^2 \hat{U}^m = \int_{\Delta_m} \hat{W} \, ds \quad a.s.
\]
To develop an error estimate in a discrete in time \(L^\infty_t(L^2_p(L^2_x))\) norm for the above time-discrete approximations, we need an error estimate for the Backward Euler time-discrete approximations, \((W^m)_{m=0}^M\), of the solution \(w\) to the deterministic problem (1.3), given below: First, set
\[
W^0 := w_0.
\]
Then, for \(m = 1, \ldots, M\), find \(W^m \in \dot{H}^4(D)\) such that
\[
W^m - W^{m-1} + k_m \Delta^2 W^m = 0.
\]
Proposition 4.1. Let \((W^m)^{m=0}_{m=M}\) be the Backward Euler time-discrete approximations of the solution \(w\) of the problem \((1.3)\) defined in \((4.3)\) - \((4.4)\). If \(w_0 \in \dot{H}^2(D)\), then, there exists a constant \(C > 0\), independent of \(T, \Delta t, \Delta x, M\) and \((k_m)^{m=0}_{m=M}\), such that

\[
\left( \sum_{m=1}^{M} k_m \|W^m - w(\tau_m, \cdot)\|_{0,D}^2 \right)^{\frac{1}{2}} \leq C (k_{\max})^{\theta} \|w_0\|_{H^{4-\theta}} \quad \forall \theta \in [0,1].
\]

Proof. The proof is omitted since it is moving along the lines of the proof of the one dimensional case which is exposed in Proposition 5.1 of [18].

Next theorem proves a discrete in time \(L^\infty(T)\) convergence estimate for the Backward Euler time discrete approximations of \(\tilde{u}\), over a uniform partition of \([0,T]\).

Theorem 4.2. Let \(\tilde{u}\) be the solution of \((1.0)\) and \((\tilde{U}^m)^{m=0}_{m=M}\) be the Backward Euler time-discrete approximations specified in \((4.1)\) - \((4.2)\). If \(k_m = \Delta \tau\) for \(m = 1, \ldots, M\), then there exists constant \(C > 0\), independent of \(T, \Delta t, \Delta x\) and \(\Delta \tau\), such that

\[
\max_{1 \leq m \leq M} \left\{ \mathbb{E}\left[ \|\tilde{U}^m - \tilde{u}(\tau_m, \cdot)\|_{0,D}^2 \right] \right\}^{\frac{1}{2}} \leq C \tilde{\omega}(\Delta \tau, \epsilon) \Delta \tau^{\frac{4-d}{8}} - \epsilon, \quad \forall \epsilon \in (0,\frac{4-d}{8}]
\]

where \(\tilde{\omega}(\Delta \tau, \epsilon) := [\epsilon^{-\frac{1}{2}} + (\Delta \tau)^{\epsilon} (p_d(\Delta \tau^{\frac{1}{2}}))]^{\frac{1}{2}}\).

Proof. Let \(I : L^2(D) \to L^2(D)\) be the identity operator, \(\Lambda : L^2(D) \to \dot{H}^4(D)\) be the inverse elliptic operator \(\Lambda := (I + \Delta \tau \Delta \tau^{-1})^{-1}\) which has Green function \(G(x, y) = \sum_{\alpha \in \mathbb{N}^d} \frac{\varepsilon_{\alpha}(x) \varepsilon_{\alpha}(y)}{1 + \alpha^2 \lambda_{\alpha}}\), i.e. \(\Lambda f(x) = \int_D G(x, y) f(y) dy\) for \(x \in D\) and \(f \in L^2(D)\). Obviously, \(G(x, y) = G(x, y)\) for \(x, y \in D\), and \(G \in L^2(D \times D)\). Also, for \(m \in \mathbb{N}\), we denote by \(G_{\lambda,m}\) the Green function of \(\Lambda^m\). Thus, from \((4.2)\), using an induction argument, we conclude that \(\tilde{U}^m = \sum_{j=1}^{m} \int_{\Delta_j} \Lambda^{m-j+1} \tilde{W}(\tau, \cdot) d\tau\) for \(m = 1, \ldots, M\), which is written, equivalently, as follows:

\[
\tilde{U}^m(x) = \int_0^{\tau_m} \int_D \tilde{K}_m(\tau; x, y) \tilde{W}(\tau, y) dy d\tau \quad \forall x \in D, \ m = 1, \ldots, M,
\]

where \(\tilde{K}_m(\tau; x, y) := \sum_{j=1}^{m} X_j(\tau) G_{\lambda,m-j+1}(x, y) \quad \forall \tau \in [0,T], \quad \forall x, y \in D.\)

Let \(m \in \{1, \ldots, M\}\) and \(\mathcal{E}^m := \mathbb{E}\left[ \|\tilde{U}^m - \tilde{u}(\tau, \cdot)\|_{0,D}^2 \right]\). First, we use \((4.7), (1.7), (2.6)\) and \((2.5)\), to obtain

\[
\mathcal{E}^m = \mathbb{E}\left[ \int_D \left( \int_0^{\tau_m} \int_D X_{(0,\tau_m)}(\tau) \left[ \tilde{K}_m(\tau; x, y) - G(\tau_m, \tau; x, y) \right] \tilde{W}(\tau, y) dy d\tau \right)^2 dx \right]
\]

Then, we apply the Cauchy-Schwarz inequality and \((2.5)\) to arrive at

\[
\mathcal{E}^m \leq \int_0^{\tau_m} \left( \int_D \left( \int_D \left[ \tilde{K}_m(\tau; x, y) - G(\tau_m, \tau; x, y) \right]^2 dy dx \right) d\tau \right)
\]

\[
\leq \sum_{j=1}^{m} \int_{\Delta_j} \left( \int_D \left[ \tilde{K}_m(\tau; x, y) - G(\tau_m, \tau; x, y) \right]^2 dy dx \right) d\tau.
\]

Now, we introduce the splitting

\[
\mathcal{E}^m \leq \mathcal{E}_1^m + \mathcal{E}_2^m,
\]
where

\[ B_1^m := 2 \sum_{\ell=1}^{m} \int_{\Delta_{\ell}} \| \Lambda^{m-\ell+1} - S(\tau_m - \tau_{\ell-1}) \|_{HS}^2 d\tau, \]

\[ B_2^m := 2 \sum_{\ell=1}^{m} \int_{\Delta_{\ell}} \| S(\tau_m - \tau_{\ell-1}) - S(\tau_m - \tau) \|_{HS}^2 d\tau. \]

**Estimation of** \( B_1^m \): By the definition of the Hilbert-Schmidt norm, we have

\[
B_1^m \leq 2 \Delta \tau \sum_{\ell=1}^{m} \left( \sum_{\alpha \in \mathbb{N}^d} \| \Lambda^{m-\ell+1} \varepsilon_\alpha - S(\tau_m - \tau_{\ell-1}) \varepsilon_\alpha \|_{0,D}^2 \right)
\leq 2 \sum_{\alpha \in \mathbb{N}^d} \left( \sum_{\ell=1}^{m} \Delta \tau \| \Lambda^{m-\ell+1} \varepsilon_k - S(\tau_m - \tau_{\ell-1}) \varepsilon_k \|_{0,D}^2 \right)
\leq 2 \sum_{\alpha \in \mathbb{N}^d} \left( \sum_{\ell=1}^{m} \Delta \tau \| \Lambda^\ell \varepsilon_k - S(\tau_\ell) \varepsilon_k \|_{0,D}^2 \right).
\]

Let \( \theta \in [0, \frac{4-d}{8}) \). Using the deterministic error estimate (4.5), we obtain

\[
B_1^m \leq C \Delta \tau^{2\theta} \sum_{\alpha \in \mathbb{N}^d} \| \varepsilon_\alpha \|_{HS}^2 (1 + 2^{-2}) \leq C \Delta \tau^{2\theta} \sum_{\alpha \in \mathbb{N}^d} \frac{1}{m^{\frac{4-d}{8}}}. \tag{4.9}
\]

The convergence of the series is ensured because \( 4(1 - 2\theta) > d \).

**Estimation of** \( B_2^m \): Using, again, the definition of the Hilbert-Schmidt norm we have

\[
B_2^m = 2 \sum_{\alpha \in \mathbb{N}^d} \left( \sum_{\ell=1}^{m} \int_{\Delta_{\ell}} \| S(\tau_m - \tau_{\ell-1}) \varepsilon_\alpha - S(\tau_m - \tau) \varepsilon_\alpha \|_{0,D}^2 d\tau \right).
\]

Observing that \( S(t) \varepsilon_\alpha = e^{-\lambda_\alpha^2 t} \varepsilon_\alpha \) for \( t \geq 0 \), (4.10) yields

\[
B_2^m = 2 \sum_{\alpha \in \mathbb{N}^d} \left[ \sum_{\ell=1}^{m} \int_{\Delta_{\ell}} \left( \int_{D} \left[ e^{-\lambda_\alpha^2 (\tau_m - \tau_{\ell-1})} - e^{-\lambda_\alpha^2 (\tau_m - \tau)} \right]^2 e_\alpha^2(x) dx \right) d\tau \right]
\leq 2 \sum_{\alpha \in \mathbb{N}^d} \left[ \sum_{\ell=1}^{m} \int_{\Delta_{\ell}} e^{-2\lambda_\alpha^2 (\tau_m - \tau)} \left(1 - e^{-\lambda_\alpha^2 (\tau - \tau_{\ell-1})} \right)^2 d\tau \right]
\leq 2 \sum_{\alpha \in \mathbb{N}^d} \left(1 - e^{-\lambda_\alpha^2 \Delta \tau} \right)^2 \int_{0}^{\tau_m} e^{-2\lambda_\alpha^2 (\tau_m - \tau)} d\tau
\leq \sum_{\alpha \in \mathbb{N}^d} \frac{1 - e^{-2\lambda_\alpha^2 \Delta \tau}}{\lambda_\alpha^2 \Delta \tau},
\]

from which, applying (2.10), we obtain

\[
B_2^m \leq C \ p_d(\Delta \tau^{1/4}) \ \Delta \tau^{\frac{4-d}{8}}. \tag{4.11}
\]

Thus, we obtain the estimate (4.6) as a conclusion of (4.8), (4.9), (4.11) and (2.9). \( \square \)
5. Space-Discrete Approximations

Let \( r \in \{2, 3, 4\} \). The space-discrete approximation of the solution \( \hat{u} \) of (1.6) is a stochastic function \( \hat{u}_h : [0, T] \rightarrow M_h \) such that

\[
\partial_t \hat{u}_h + B_h \hat{u}_h = P_h \hat{W} \quad \text{on } (0, T], \\
\hat{u}_h(0) = 0 \quad \text{a.s.}
\]

(5.1)

To develop an \( L_t^\infty(L_x^2(L_y^2)) \) convergence estimate for the space-discrete approximation \( \hat{u}_h \), we will derive first an \( L_t^2(L_x^2) \) error estimate for the corresponding space-discrete approximation \( w_h \) of the solution \( w \) of (1.3) (cf. [24] and [2]), which is a function \( w_h : [0, T] \rightarrow M_h \) such that

\[
\partial_t w_h + B_h w_h = 0 \quad \text{on } (0, T], \\
w_h(0) = P_h w_0.
\]

(5.2)

Since \( w_h \) can be considered as the value of a linear operator of the initial condition \( w_0 \), we will write it as \( w_h(t, \cdot) = [S_h(t)w_0](\cdot) \) for \( t \in [0, T] \). Thus, by Duhamel’s principle (cf. [24]), we have

\[
\hat{u}_h(t, x) = \int_0^t [S_h(t-s)\hat{W}(s, \cdot)](x) \, ds \quad \text{a.s..}
\]

(5.3)

Proposition 5.1. Let \( r \in \{2, 3, 4\} \), \( w \) be the solution of (1.3) and \( w_h \) be its space-discrete approximation given in (5.2). If \( w_0 \in H^r(D) \), then, there exists a constant \( C > 0 \), independent of \( T \) and \( h \), such that

\[
\left( \int_0^T \left\| w - w_h \right\|_{0,D}^2 \, dt \right)^{\frac{1}{2}} \leq C h^{\tilde{\nu}(r, \theta)} \left\| w_0 \right\|_{H^\theta(D)} \quad \forall \theta \in [0, 1],
\]

where

\[
\tilde{\nu}(r, \theta) := \begin{cases} 
2 \theta & \text{if } r = 2 \\
4 \theta & \text{if } r = 3 \\
5 \theta & \text{if } r = 4
\end{cases} \quad \text{and} \quad \tilde{\xi}(r, \theta) := \begin{cases} 
3 \theta - 2 & \text{if } r = 2 \\
4 \theta - 2 & \text{if } r = 3 \\
5 \theta - 2 & \text{if } r = 4
\end{cases}.
\]

(5.5)

Proof. Let \( e := w - w_h \) and \( \rho := (T_{\theta, x} - T_{\theta})\Delta^2 w \). We will derive (5.4) by interpolation, after showing that it holds for \( \theta = 1 \) and \( \theta = 0 \).

Observing that \( T_{\theta, x} e_t + e = \rho \) on \([0, T]\), and then taking the \((\cdot, \cdot)_{0,D}\) inner product with \( e \), we easily arrive at

\[
\int_0^T \| e \|_{0,D}^2 \, dt \leq \int_0^T \| \rho \|_{0,D}^2 \, dt.
\]

(5.6)

For \( r = 2 \), using (1.6), (2.24), (2.3) and (2.18), we have

\[
\left( \int_0^T \| e \|_{0,D}^2 \, dt \right)^{\frac{1}{2}} \leq C h^2 \left( \int_0^T \| w \|_{H^3}^2 \, dt \right)^{\frac{1}{2}} \leq C h^2 \| w_0 \|_{H^1}.
\]

(5.7)

Also, for \( r = 3, 4 \), combining (5.6), (2.24), (2.3) and (2.18) we get

\[
\left( \int_0^T \| e \|_{0,D}^2 \, dt \right)^{\frac{1}{2}} \leq C h^{r+1} \left( \int_0^T \| w \|_{H^{r+1}}^2 \, dt \right)^{\frac{1}{2}} \leq C h^{r+1} \| w_0 \|_{H^{r-1}}.
\]

(5.8)

Thus, relations (5.7) and (5.8) yield (5.4) for \( \theta = 1 \).

Since \( T_{\theta, x} w_t + w = 0 \) on \([0, T]\), we obtain \( (T_{\theta, x} w_t, w)_{0,D} + \| w \|_{0,D}^2 = 0 \) on \([0, T]\), which, along with (2.17), yields \( \frac{d}{dt}\| w_t \|_{0,D}^2 + \| w \|_{0,D}^2 = 0 \) on \([0, T]\). Then, integrating over \([0, T]\) and using (2.12), we get

\[
\left( \int_0^T \| w \|_{0,D}^2 \, dt \right)^{\frac{1}{2}} \leq C \| w_0 \|_{-2,D}.
\]

(5.9)
Since $T_{h,x}\partial_x w_h + w_h = 0$ on $[0, T]$, we obtain $(T_{h,x}\partial_x w_h, w_h)_{o,D} + \|w_h\|_{o,D}^2 = 0$ on $[0, T]$. Then, integrating over $[0, T]$ and using (2.22), we have

$$
\left( \int_0^T \|w_h\|_{o,D}^2 \, dt \right)^{\frac{1}{2}} \leq \|\Delta T_{h,x} P_h w_0\|_{o,D}^2 \\
\leq \|\Delta T_{h,x} w_0\|_{o,D} \\
\leq C \|w_0\|_{H^{-2}}.
$$

(5.10)

Hence, from (5.9), (5.10) and (2.4), we obtain \( \left( \int_0^T \|v\|_{o,D}^2 \, dt \right)^{\frac{1}{2}} \leq C \|v_0\|_{H^{-2}} \), which yields (5.4) with \( \theta = 0 \).

Next lemma shows that a discrete analogue of (1.4) holds.

**Lemma 5.1.** Let \( r \in \{2, 3, 4\} \) and \( w_h \) be the space-discrete approximation of the solution \( w \) of (1.3) defined in (5.2). Then, there exists a map \( G_h : [0, T] \to C(\overline{D} \times \overline{D}) \) such that

$$
(5.11) \quad w_h(t; x) = \int_D G_h(t; x, y) \, w_0(y) \, dy \quad \forall t \in [0, T], \quad \forall x \in \overline{D},
$$

and \( G_h(t; x, y) = G_h(t; y, x) \) for \( x, y \in \overline{D} \) and \( t \in [0, T] \).

**Proof.** Let \( \dim(M_h) = n_h \) and \( \gamma_h : M_h \times M_h \to \mathbb{R} \) be an inner product on \( M_h \) given by \( \gamma_h(\chi_A, \chi_B) := \langle \Delta\chi_A, \Delta\chi_B \rangle_{o,D} \) for \( \chi_A, \chi_B \in M_h \). We can construct a basis \( (\xi_j)_{j=1}^{n_h} \) of \( M_h \) which is \( L^2(D) \)-orthonormal, i.e., \( \langle \xi_i, \xi_j \rangle_{o,D} = \delta_{ij} \) for \( i, j = 1, \ldots, n_h \), and \( \gamma_h \)-orthogonal, i.e., there are \( (\lambda_h\ell)_{\ell=1}^{n_h} \subset (0, +\infty) \) such that \( \gamma_h(\xi_i, \xi_j) = \lambda_h\ell \delta_{ij} \) for \( i, j = 1, \ldots, n_h \) (see Section 8.7 in [9]). Thus, there exists a map \( \omega : [0, T] \to \mathbb{R}^{n_h} \) such that \( w_h(t; x) = \sum_{j=1}^{n_h} \omega_j(t) \chi_j(x) \). Since \( w_h(0) = P_h w_0 \), it follows that \( \omega_j(0) = (w_0, \chi_j)_{o,D} \) for \( j = 1, \ldots, n_h \). Now, (5.2) yields that \( \frac{d}{dt} \omega(t) = B \omega(t) \) for \( t \in [0, T] \), where \( B \in \mathbb{R}^{n_h \times n_h} \) with \( B_{ij} := -\gamma_h(\xi_i, \xi_j) = -\lambda_h\ell \delta_{ij} \) for \( i, j = 1, \ldots, n_h \). Hence, it follows that \( \omega(t) = e^{-\lambda_h\ell t} (w_0, \chi_\ell)_{o,D} \) for \( t \in [0, T] \) and \( \ell = 1, \ldots, n_h \), which yields (5.11) with \( G_h(t; x, y) = \sum_{\ell=1}^{n_h} e^{-\lambda_h\ell t} \chi_\ell(x) \chi_\ell(y) \).

We are ready to derive a convergence estimate, in an \( L^\infty(t) \theta L^2(\theta L^2) \) norm, for the space-discrete approximation \( \tilde{u}_h \) to the solution \( \tilde{u} \) of the regularized problem.

**Theorem 5.2.** Let \( r \in \{2, 3, 4\} \), \( \tilde{u} \) be the solution of (1.6) and \( \tilde{u}_h \) be its space-discrete approximation defined in (5.1). Then, there exists a constant \( C > 0 \), independent of \( T, \Delta t, \Delta x \) and \( h \), such that

$$
(5.12) \quad \max_{[0,T]} \left\{ \mathbb{E} \left[ \|\tilde{u}_h - \tilde{u}\|_{o,D}^2 \right] \right\}^{\frac{1}{2}} \leq C e^{-\frac{1}{2} h^{\nu(r,d)}} \quad \forall \epsilon \in (0, \nu(r,d)],
$$

where

$$
(5.13) \quad \nu(r,d) := \begin{cases} \frac{4-r}{d} & \text{if } r = 2 \\ \frac{4-r}{2} & \text{if } r = 3, 4 \end{cases}.
$$

**Proof.** Let \( \tilde{e} := \tilde{u}_h - \tilde{u} \) and \( t \in (0, T) \). Then, (5.3), (5.11) and (1.7) yield

$$
\tilde{e}(t, x) = \int_0^t \int_D \left[ G_h(t - s; x, y) - G(t - s; x, y) \right] \tilde{W}(s, y) \, ds \, dy \quad \forall x \in \overline{D}, \quad \text{a.s.}
$$
Thus, using the Itô isometry property of the stochastic integral and the Cauchy-Schwarz inequality, we have

\[
\mathbb{E} \left[ \| e(t, \cdot)^2 \|_{\infty, D}^2 \right] = \mathbb{E} \left[ \int_D \left( \int_0^T \mathcal{X}_{(0,t)}(s) \left[ G_h(t-s,x,y) - G(t-s,x,y) \right] \, \tilde{W}(s,y) \, ds \right) dy \right]^2 dx
\]

\[
= \frac{1}{\sigma^2(t)} \int_{\mathbb{R}^d} \sum_{n \in \mathbb{N}^d} \sum_{\mu \in \mathcal{F}_n^d} \left( \int_{\mathbb{R}^d} \mathcal{X}_{(0,t)}(s') \left[ G_h(t-s';x,y') - G(t-s';x,y') \right] \, ds' \, dy' \right)^2 \, dx
\]

\[
\leq \int_0^t \left( \int_D \int_D \left[ G_h(t-s;x,y) - G(t-s;x,y) \right]^2 \, dy dx \right) ds
\]

which, along with \( (5.14) \), yields

\[
(5.14) \quad \mathbb{E} \left[ \| e(t, \cdot)^2 \|_{\infty, D}^2 \right] \leq \int_0^t \| S(s) - \mathcal{S}_h(s) \|^2_{\infty, D} \, ds.
\]

Since \( e(0, \cdot) = 0 \), we use \( (5.14) \), the definition of the Hilbert-Schmidt norm and \( (5.4) \), to obtain

\[
\max_{[0,T]} \mathbb{E} \left[ \| e \|^2_{\infty, D} \right] \leq \int_0^T \left( \sum_{\alpha \in \mathbb{N}^d} \| S(s) \varepsilon_\alpha - \mathcal{S}_h(s) \varepsilon_\alpha \|_{\infty, D}^2 \right) \, ds
\]

\[
\leq \sum_{\alpha \in \mathbb{N}^d} \left( \int_0^T \| S(s) \varepsilon_k - \mathcal{S}_h(s) \varepsilon_k \|_{\infty, D}^2 \, ds \right)
\]

\[
\leq C \tilde{h}^{2\dot{r}(r, \theta)} \sum_{\alpha \in \mathbb{N}^d} \| \varepsilon \|^{2}_{H\dot{r}(r, \theta)}
\]

\[
\leq C \tilde{h}^{2\dot{r}(r, \theta)} \pi^{2\dot{r}(r, \theta)} \sum_{\alpha \in \mathbb{N}^d} | \alpha |^{2\dot{r}(r, \theta)}.
\]

The series in the right hand side of \( (5.15) \) converges if and only if \(-2\tilde{\xi}(r, \theta) > d\). Thus, in view of \( (2.9) \), we arrive at \( (5.12) \) and \( (5.13) \). \( \Box \)

6. Convergence of the Fully-Discrete Approximations

6.1. Consistency estimates. First, we derive some Hölder-type bounds for \( \tilde{u} \).

Lemma 6.1. Let \( \tilde{u} \) be the solution of \( (1.10) \). Then, there exist a real positive constant \( C \), which is independent of \( T, \Delta t \) and \( \Delta x \), such that

\[
(6.1) \quad \left\{ \mathbb{E} \left[ \left\| \int_{\tau_a}^{\tau_b} [\tilde{u}(\tau_b, \cdot) - \tilde{u}(\tau_a, \cdot)] \, d\tau \right\|_{\infty, D}^2 \right] \right\}^{\frac{1}{2}} \leq C \left( p_d((\tau_b - \tau_a)^{\frac{1}{4}}) \right)^{\frac{1}{2}} | \tau_b - \tau_a |^{1 + \frac{1}{4p_d}}
\]

and

\[
(6.2) \quad \left\{ \mathbb{E} \left[ \left\| \tilde{u}(\tau_a, \cdot) - \tilde{u}(\tau_a, \cdot) \right\|_{\infty, D}^2 \right] \right\}^{\frac{1}{2}} \leq C \left( p_d((\tau_b - \tau_a)^{\frac{1}{4}}) \right)^{\frac{1}{2}} | \tau_b - \tau_a |^{1 - \frac{1}{4p_d}}
\]

for \( \tau_a, \tau_b \in [0, T] \) with \( \tau_a \leq \tau_b \).

Proof. We will omit the proof of \( (6.2) \) because it is similar to that of \( (6.1) \) which follows.

Let \( m \in \{ 1, \ldots, M \} \), \( \tau_a \in [0, T] \) and \( \tau_a \in [0, T] \) with \( \tau_a < \tau_b \) and \( \mu(\cdot) \) := \( \int_{\tau_a}^{\tau_b} \tilde{u}(\tau_a, \cdot) \, d\tau \). First we assume that \( \tau_a > 0 \). Then, we use \( (1.7) \), \( (2.8) \), the Itô-isometry property of the stochastic
integral, (1.5) and the \( L^2(D) \)-orthogonality of \( (\varepsilon_\alpha)_{\alpha \in \mathbb{N}^d} \), to obtain

\[
\mathbb{E} [ \| \mu \|_{0,D}^2 ] = \frac{1}{\Delta(t) \Delta x} \int_D \left\{ \sum_{n \in \mathbb{N}^d} \int \left( \int_{\tau_n}^{\tau_{n+1}} \left[ \mathcal{X}_{(\tau_n)}(s') G(\tau - s'; x, y') - \mathcal{X}_{(\tau)}(s') G(\tau - s'; x, y') \right] d\tau ds' dy' \right)^2 \right\} dx
\]

\[
= \frac{1}{\Delta(t) \Delta x} \sum_{n \in \mathbb{N}^d} \left\{ \sum_{m \in \mathbb{N}^d} \left( \int_{\tau_n}^{\tau_{n+1}} \left[ \mathcal{X}_{(\tau_n)}(s') e^{-\lambda_n^2(\tau_n - s')} - \mathcal{X}_{(\tau)}(s') e^{-\lambda_n^2(\tau_n - s')} \right] d\tau ds' \right)^2 \left( \int_{\Delta^m} \varepsilon_\alpha(y') dy' \right)^2 \right\}
\]

which, along with the use of the Cauchy-Schwarz inequality, yields

\[
\mathbb{E} [ \| \mu \|_{0,D}^2 ] \leq \sum_{n \in \mathbb{N}^d} \left\{ \int_0^\tau \left[ \int_{\tau_n}^{\tau_{n+1}} \left[ \mathcal{X}_{(\tau_n)}(s') e^{-\lambda_n^2(\tau_n - s')} - \mathcal{X}_{(\tau)}(s') e^{-\lambda_n^2(\tau_n - s')} \right] d\tau ds' \right]^2 ds' \right\}
\]

\[
\leq (\tau_b - \tau_n) \sum_{n \in \mathbb{N}^d} \left( \int_{\tau_n}^{\tau_{n+1}} \left[ e^{-\lambda_n^2(\tau_n - s')} - e^{-\lambda_n^2(\tau_n - s')} \right]^2 ds' d\tau \right.
\]

\[\left. + \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{\tau_{n+1}} e^{-2\lambda_n^2(\tau_n - s')} ds'd\tau \right) \leq (\tau_b - \tau_n)^2 \sum_{n \in \mathbb{N}^d} \frac{1-e^{-2\lambda_n^2(\tau_b - \tau_n)}}{\lambda_n^2}. \tag{6.3} \]

Finally, we combine (6.3) and (2.10) to arrive at (6.1). The case \( \tau_a = 0 \) follows by moving along the lines of the proof above using that \( \hat{u}(0, x) = 0 \). \( \square \)

Next, we show a consistency result for the Backward Euler time-discrete approximations of \( \hat{u} \), which is based on the result of Lemma 6.1.

**Proposition 6.1.** Let \( \hat{u} \) be the solution of (1.6) and \( (\hat{\sigma}_m)_{m=1} \) be stochastic functions defined by

\[
\hat{u}(\tau_m, \cdot) - \hat{u}(\tau_{m-1}, \cdot) + k_m \Delta^2 \hat{u}(\tau_m, \cdot) = \int_{\Delta_m} \hat{W} d\tau + \hat{\sigma}_m \quad \text{a.s.,} \quad m = 1, \ldots, M.
\]

Then it holds that

\[
(\mathbb{E} [ \| T_\theta \hat{\sigma}_m \|_{0,D}^2 ] )^{\frac{1}{2}} \leq C \left( \frac{p_d(k_m)}{2} \right)^{\frac{1}{2}} \left( k_m \right)^{1 + \frac{4-d}{8}}, \quad m = 1, \ldots, M. \tag{6.5} \]

**Proof.** Let \( m \in \{1, \ldots, M\} \). Integrating the equation in (1.6) over \( \Delta_m \) and subtracting it from (6.4), we conclude that \( T_\theta \hat{\sigma}_m(\cdot) = \int_{\Delta_m} [\hat{u}(\tau_m, \cdot) - \hat{u}(\tau_m, \cdot)] d\tau \) a.s.. Thus, to get the bound (6.5), we apply the result (6.1) on the latter equality. \( \square \)

**6.2. Discrete in time \( L^2_t(L^2_x((L^2_x)^m)) \) error estimate.** We first obtain a discrete in time \( L^2_t(L^2_x((L^2_x)^m)) \) error estimate for the Backward Euler fully-discrete approximations of \( \hat{u} \), by connecting it to the error estimate of Theorem 5.2 for the space-discrete approximation of \( \hat{u} \).

**Theorem 6.2.** Let \( r \in \{2, 3, 4\} \), \( \hat{u} \) be the solution of (1.9) and \( (\hat{U}_m)_{m=0}^M \subset M_h \) be the Backward Euler fully-discrete approximations of \( \hat{u} \) defined in (1.10)-(1.11). Then there exists a constant \( C > 0 \), independent of \( T, r, \Delta t, \Delta x, h, M \) and \( (k_m)_{m=1}^M \), such that:

\[
\sum_{m=1}^M k_m \mathbb{E} [ \| \hat{U}_m - \hat{u}(\tau_m, \cdot) \|_{0,D}^2 ] \leq C \sqrt{T} \left[ \varepsilon^{-\frac{1}{2}} h^{\nu(r,d) - \epsilon} + \hat{\omega}(k_{max}) (k_{max})^{\frac{4-d}{8}} \right]
\]

for \( \epsilon \in (0, \nu(r,d)) \), where \( \hat{\omega}(k_{max}) := \left( p_d(k_{max}) \right)^{\frac{1}{2}} \) and \( \nu(r,d) \) is defined in (5.13).
Proof. Let \( \hat{u}_h \) be the space-discrete approximation of \( \hat{u} \) defined in (6.1). \( \hat{e} = u - \hat{u}_h \), \( z^m_h := \hat{U}^m_h - \hat{u}_h(\tau_m) \in S^r_h \) for \( m = 0, \ldots, M \), and \( V_h := \{ \sum_{m=1}^M k_m \mathbb{E} \left[ \| z_h^m \|^2_{0,D} \right] \}^{1/2} \). First, we observe that

\[
\left\{ \sum_{m=1}^M k_m \mathbb{E} \left[ \| \hat{U}^m_h - \hat{u}(\tau_m, \cdot) \|^2_{0,D} \right] \right\}^{1/2} \leq V_h + \sqrt{T} \max_{[0,T]} \left\{ \mathbb{E} \left[ \| \hat{e} \|^2_{0,D} \right] \right\}^{1/2}.
\]

Integrating (6.1) over \( \Delta_m \) and subtracting the obtained relation from (1.11), we arrive at

\[
T_{B,h}(z^m_h - z^{m-1}_h) + k_m z^m_h = \rho_{h,m} \text{ a.s., } m = 1, \ldots, M,
\]

where \( \rho_{h,m} := \int_{\Delta_m} \left[ \hat{u}_h(\tau, \cdot) - \hat{u}_h(\tau_m, \cdot) \right] d\tau \). Take the \( \langle \cdot, \cdot \rangle_{0,D} \)-inner product of both sides of (6.8) with \( z^m_h \), sum with respect to \( m \) from 1 up to \( M \), to obtain

\[
\sum_{m=1}^M (\Delta T_{B,h} z^m_h - \Delta T_{B,h} z^{m-1}_h, \Delta T_{B,h} z^m_h)_{0,D} + \sum_{m=1}^M k_m \| z^m_h \|^2_{0,D} = \sum_{m=1}^M (\rho_{h,m}, z^m_h)_{0,D} \text{ a.s.}
\]

Since \( z^0_h = 0 \), we conclude that \( \sum_{m=1}^M (\Delta T_{B,h} z^m_h - \Delta T_{B,h} z^{m-1}_h, \Delta T_{B,h} z^m_h)_{0,D} \geq \frac{1}{2} \| \Delta T_{B,h} z^m_h \|^2_{0,D} \) a.s.. Thus, taking expected values in (6.9) and using the Cauchy-Schwarz inequality we get

\[
(V_h)^2 \leq \mathbb{E} \left[ \sum_{m=1}^M k_m^{-1} \| \rho_{h,m} \|^2_{0,D} \right] \leq \mathbb{E} \left[ \sum_{m=1}^M \int_D \int_{\Delta_m} \left[ \hat{u}_h(\tau, x) - \hat{u}_h(\tau_m, x) \right]^2 d\tau dx \right] \leq \sum_{m=1}^M \int_{\Delta_m} \mathbb{E} \left[ \| \hat{u}_h(\tau, \cdot) - \hat{u}_h(\tau_m, \cdot) \|^2_{0,D} \right] d\tau.
\]

Using (6.10) and (6.2), we conclude that

\[
V_h \leq \left( \sum_{m=1}^M \int_{\Delta_m} \mathbb{E} \left[ \| \hat{e}(\tau, \cdot) - \hat{e}(\tau_m, \cdot) \|^2_{0,D} \right] d\tau \right)^{1/2} + \left\{ \sum_{m=1}^M \int_{\Delta_m} \mathbb{E} \left[ \| \hat{e}(\tau, \cdot) - \hat{e}(\tau_m, \cdot) \|^2_{0,D} \right] d\tau \right\}^{1/2} \leq C \sqrt{T} \left( \max_{[0,T]} \left\{ \mathbb{E} \left[ \| \hat{e} \|^2_{0,D} \right] \right\}^{1/2} + (p_d((k_{max})^{1/2})^{1/2} (k_{max})^{1/2}) \right).
\]

Thus, (6.6) follows from (6.7), (6.11) and (6.12). \( \square \)

### 6.3. Discrete in time \( L^\infty(L^2(D)) \) error estimate

To get a discrete in time \( L^\infty(L^2(D)) \) error estimate for the Backward Euler fully-discrete approximations of \( \hat{u} \), we compare them to the Backward Euler time-discrete approximations of \( \hat{u} \) defined in (4.1)–(4.2).

For that we derive first a discrete in time \( L^2(L^2) \) error estimate between the Backward Euler time-discrete and the Backward Euler fully discrete approximations of the solution \( w \) of (1.3) given below:

First set

\[
W_h^0 := P_h w_0.
\]

Then, for \( m = 1, \ldots, M \), find \( W_h^m \in M_h \) such that

\[
W_h^m - W_h^{m-1} + k_m B_h W_h^m = 0.
\]

**Proposition 6.3.** Let \( \hat{r} \in \{2, 3, 4\} \), \( w \) be the solution of the problem (1.3), \( (W_h^m)_{m=0}^M \) be the Backward Euler time-discrete approximations of \( w \) defined in (4.1)–(4.2), and \( (W_h^m)_{m=0}^M \) be the Backward Euler fully-discrete approximations of \( w \) specified in (6.12)–(6.13). If \( w_0 \in H^\hat{r}(D) \), then, there exists a constant \( C > 0 \), independent of \( T, h, M \) and \( (k_m)_{m=1}^M \), such that

\[
\left( \sum_{m=1}^M k_m \| W_h^m - W_h^{m-1} \|^2_{0,D} \right)^{1/2} \leq C h^{\hat{r}(r, \theta)} \| w_0 \|_{H^\hat{r}(r, \theta)} \quad \forall \theta \in [0, 1],
\]
where $\tilde{\nu}(r, \theta)$ and $\tilde{\xi}(r, \theta)$ are defined in (5.5).

Proof. Let $E^m := W^m - W_h^m$ for $m = 0, \ldots, M$. We will get (6.14) by interpolation, showing it for $\theta = 0$ and $\theta = 1$.

We use (4.1) and (6.13), to obtain: $T_{B,h}(E^m - E^{m-1}) + k_m E^m = k_m (T_B - T_{B,h}) \Delta^2 W^m$ for $m = 1, \ldots, M$. Since $T_{B,h} E^0 = 0$, proceeding as in the proof of Theorem 6.2, it follows that

$$\sum_{m=1}^{M} k_m \|E^m\|_{0,D}^2 \leq \sum_{m=1}^{M} k_m \|(T_B - T_{B,h}) \Delta^2 W^m\|_{0,D}^2. \hspace{1cm} (6.15)$$

Let $r = 3$. Then, by (2.21) and (6.14), we obtain

$$\sum_{m=1}^{M} k_m \|E^m\|_{0,D}^2 \leq C h^8 \sum_{m=1}^{M} k_m \|\Delta^2 W^m\|_{0,D}^2. \hspace{1cm} (6.16)$$

Taking the $(\cdot, \cdot)_{0,D}$–inner product of (4.4) with $\Delta^2 W^m$, and then integrating by parts and summing with respect to $m$ from 1 up to $M$, it follows that

$$\sum_{m=1}^{M} (\Delta W^m - \Delta W^{m-1}, \Delta W^m)_{0,D} + \sum_{m=1}^{M} k_m \|\Delta^2 W^m\|_{0,D}^2 = 0. \hspace{1cm} (6.17)$$

Since $\sum_{m=1}^{M} (\Delta W^m - \Delta W^{m-1}, \Delta W^m)_{0,D} \geq \frac{1}{2} \left( \|\Delta W^M\|_{0,D}^2 - \|\Delta W^0\|_{0,D}^2 \right)$, (6.17) yields

$$\sum_{m=1}^{M} k_m \|\Delta^2 W^m\|_{0,D}^2 \leq \frac{1}{2} \|w_0\|_{1,D}^2. \hspace{1cm} (6.18)$$

Combining, now, (6.16), (6.18) and (6.15), we obtain

$$\left( \sum_{m=1}^{M} k_m \|E^m\|_{0,D}^2 \right)^\frac{1}{2} \leq C h^4 \|w_0\|_{H^2}. \hspace{1cm} (6.19)$$

Let $r = 2$. Then, by (2.21), (2.21) and (6.15), we obtain

$$\sum_{m=1}^{M} k_m \|E^m\|_{0,D}^2 \leq C h^4 \sum_{m=1}^{M} k_m \|\Delta^2 W^m\|_{H^{-1}}^2 \leq C h^4 \left[ - \sum_{m=1}^{M} k_m \left( T_E \Delta^2 W^m, \Delta^2 W^m \right)_{0,D} \right]. \hspace{1cm} (6.20)$$

Taking the $(\cdot, \cdot)_{0,D}$–inner product of (4.3) with $\Delta W^m$, integrating by parts and summing with respect to $m$ from 1 up to $M$, it follows that

$$\sum_{m=1}^{M} (\nabla W^m - \nabla W^{m-1}, \nabla W^m)_{0,D} - \sum_{m=1}^{M} k_m (\Delta^2 W^m, \Delta W^m)_{0,D} = 0. \hspace{1cm} (6.21)$$

Since $\sum_{m=1}^{M} (\nabla W^m - \nabla W^{m-1}, \nabla W^m)_{0,D} \geq \frac{1}{2} \left[ \|\nabla W^M\|_{0,D}^2 - \|\nabla W^0\|_{0,D}^2 \right]$, (6.21) yields

$$\sum_{m=1}^{M} k_m (\Delta^2 W^m, \Delta W^m)_{0,D} \leq \frac{1}{2} \|w_0\|_{1,D}^2. \hspace{1cm} (6.22)$$

Combining (6.20), (6.22) and (2.3) we get

$$\left( \sum_{m=1}^{M} k_m \|E^m\|_{0,D}^2 \right)^\frac{1}{2} \leq C h^2 \|w_0\|_{H^4}. \hspace{1cm} (6.23)$$
Let $r = 4$. Then, observing that $\Delta^2 W^m \in \dot{H}^1(D)$ and using the relations (6.24), (2.4) and (6.19), we obtain
\[
\sum_{m=1}^{M} k_m \| E^m \|_{0,D}^2 \leq C h^{rac{5}{4}} \sum_{m=1}^{M} k_m \| \Delta^2 W^m \|_{H^1}^2
\]
\[
\leq C h^{rac{5}{4}} \sum_{m=1}^{M} k_m \| \Delta^3 W^m \|_{H^1}^2
\]
(6.24)
\[
\leq C h^{rac{5}{4}} \left[ \sum_{m=1}^{M} k_m (T_k \Delta^3 W^m, \Delta^3 W^m)_{0,D} \right]
\]
\[
\leq C h^{rac{5}{4}} \left[ \sum_{m=1}^{M} k_m (\Delta^2 W^m, \Delta^3 W^m)_{0,D} \right].
\]
After, Applying the operator $\Delta$ on (4.4), take the $(\cdot, \cdot)_{0,D}$ inner product of the obtained relation with $\Delta^2 W^m$, integrate by parts and sum with respect to $m$ from 1 up to $M$, to get
\[
- \sum_{m=1}^{M} (\Delta W^m - \Delta W^{m-1}, \Delta^2 W^m)_{0,D} - \sum_{m=1}^{M} k_m (\Delta^3 W^m, \Delta^2 W^m)_{0,D} = 0.
\]
Also, we have
\[
- \sum_{m=1}^{M} (\Delta W^m - \Delta W^{m-1}, \Delta^2 W^m)_{0,D} \geq \sum_{m=1}^{M} \left( \| \Delta W^m \|_{H^1}^2 - \| \Delta W^m \|_{H^1} \| \Delta W^{m-1} \|_{H^1} \right)
\]
\[
\geq \frac{1}{2} \left( \| \Delta W^m \|_{H^1}^2 - \| \Delta W^0 \|_{H^1} \right).
\]
Thus, (6.25) and (6.26) yield
\[
- \sum_{m=1}^{M} k_m (\Delta^3 W^m, \Delta^2 W^m)_{0,D} \leq \frac{1}{2} \| w_0 \|_{H^3}^2.
\]
Combining (6.24) and (6.27) we get
\[
\left( \sum_{m=1}^{M} k_m \| E^m \|_{0,D}^2 \right)^{\frac{1}{2}} \leq C h^5 \| w_0 \|_{H^3}.
\]
Thus, the relations (6.19), (6.23) and (6.28) yield (6.14) for $\theta = 1$.
Since $T_{\theta,h}(W^m_h - W^{m-1}_h) + k_m W^m_h = 0$ for $m = 1, \ldots, M$, we obtain
\[
(\Delta T_{\theta,h} W^m - \Delta T_{\theta,h} W^{m-1}_{W^m_h})_{0,D} + k_m \| W^m_h \|_{0,D}^2, \quad m = 1, \ldots, M,
\]
which, along with (2.22) and (2.4), yields
\[
\sum_{m=1}^{M} k_m \| W^m_h \|_{0,D}^2 \leq \frac{1}{2} \| \Delta T_{\theta,h} w_0 \|_{0,D}^2
\]
(6.29)
\[
\leq C \| w_0 \|_{H^2}.
\]
Now, using (1.4) and (2.17), we obtain $(T_{\theta} W^m - T_{\theta} W^{m-1}, T_{\theta} W^m)_{0,D} + k_m \| W^m \|_{0,D}^2 = 0$ for $m = 1, \ldots, M$, which yields $\| T_{\theta} W^m \|_{0,D}^2 - \| T_{\theta} W^{m-1} \|_{0,D}^2 + 2 k_m \| W^m \|_{0,D}^2 \leq 0$ for $m = 1, \ldots, M$. Then, summing with respect to $m$ from 1 up to $M$, and using (2.13) and (2.4) we obtain
\[
\sum_{k=1}^{M} k_m \| W^m \|_{0,D}^2 \leq \frac{1}{2} \| T_{\theta} w_0 \|_{0,D}^2
\]
(6.30)
\[
\leq C \| w_0 \|_{H^2}.
\]
Finally, combine (6.29) with (6.30) to get $\left( \sum_{m=1}^{M} k_m \| E^m \|_{0,D}^2 \right)^{\frac{1}{2}} \leq C \| w_0 \|_{H^2}$, which is equivalent to (6.14) for $\theta = 0$. □
The following lemma ensures the existence of a continuous Green function for the solution operator of a discrete elliptic problem.

**Lemma 6.2.** Let \( r \in \{2, 3, 4\} \), \( \epsilon > 0 \), \( f \in L^2(D) \) and \( \psi_h \in M_h \) such that
\[
\epsilon B_h \psi_h + \psi_h = P_h f.
\]
(6.31)
Then there exists a function \( G_{h,\epsilon} \in C(\overline{D} \times \overline{D}) \) such that
\[
\psi_h(x) = \int_D G_{h,\epsilon}(x, y) f(y) \, dy \quad \forall \ x \in \overline{D}
\]
and \( G_{h,\epsilon}(x, y) = G_{h,\epsilon}(y, x) \) for \( x, y \in \overline{D} \).

**Proof.** Keeping the notation and the constructions of the proof of Lemma 5.1, we conclude that there are \( (\mu_j)_{j=1}^{n_h} \subset \mathbb{R} \) such that \( \psi_h = \sum_{j=1}^{n_h} \mu_j \chi_j \). Thus, (6.31) is equivalent to \( \mu_i = \frac{1}{1 + \epsilon \lambda_i} \langle f, \chi_i \rangle_{S,D} \) for \( i = 1, \ldots, n_h \). Finally, we obtain (6.32) with \( G_{h,\epsilon}(x, y) = \sum_{j=1}^{n_h} \frac{\chi(x)\chi(y)}{1 + \epsilon \lambda_{h,j}} \).

We are ready to compare, in the discrete in time \( L^\infty(\ell^2(L^2_\ell)) \) norm, the time-discrete with the fully-discrete Backward Euler approximations of \( \hat{u} \).

**Proposition 6.4.** Let \( r \in \{2, 3, 4\} \), \( \hat{u} \) be the solution of the problem (1.10), \( (\hat{U}_h^m)^M_{m=0} \) be the Backward Euler fully-discrete approximations of \( \hat{u} \) specified in (1.10)–(1.11), and \( (\hat{U}^m)^M_{m=0} \) be the Backward Euler time-discrete approximations of \( \hat{u} \) specified in (1.11)–(1.12). If the partition \( (\tau_m)^M_{m=0} \) is uniform, i.e. \( k_m = \Delta \tau \) for \( m = 1, \ldots, M \), then, there exists a constant \( C > 0 \), independent of \( \Delta x, \Delta t, h, M \) and \( \Delta \tau \), such that
\[
\max_{1 \leq m \leq M} \mathbb{E} \left[ \left\| \hat{U}_h^m - \hat{U}^m \right\|_{o,D}^2 \right]^{1/2} \leq C \epsilon^{-\frac{1}{4}} h^{\nu(r,d) - \epsilon}, \quad \forall \ \epsilon \in (0, \nu(r,d)]
\]
where \( \nu(r,d) \) has been defined in (5.13).

**Proof.** Let \( I : L^2(D) \to L^2(D) \) be the identity operator and \( \Lambda_h : L^2(D) \to S'_r \) be the inverse discrete elliptic operator given by \( \Lambda_h := (I + \Delta \tau B_h)^{-1} P_h \) and having a Green function \( G_{h,\Delta} \) (cf. Lemma 6.2). Also, for \( \ell \in \mathbb{N} \), we denote by \( G_{h,\Delta,\ell} \) the Green function of \( \Lambda_h^{\ell} \). Using, now, an induction argument, from (1.11) we conclude that \( \hat{U}_h^m = \sum_{j=1}^{m} \int_{\Delta_j} \Lambda_h^{m-j+1} \hat{W}(\tau,\cdot) \, d\tau \), \( m = 1, \ldots, M \), which is written, equivalently, as follows:
\[
\hat{U}_h^m(x) = \int_0^{\tau_m} \int_D \hat{D}_{h,m}(\tau, x, y) \hat{W}(\tau, y) \, dy \, d\tau \quad \forall \ x \in \overline{D}, \ m = 1, \ldots, M,
\]
where
\[
\hat{D}_{h,m}(\tau, x, y) := \sum_{j=1}^{m} \Lambda_h(\tau) G_{h,\Delta,\ell,j+1}(x, y) \quad \forall \ \tau \in [0, T], \ \forall \ x, y \in D.
\]
Using (1.1), (6.34), the Itô-isometry property of the stochastic integral, (2.5) and the Cauchy-Schwarz inequality, we get
\[
\mathbb{E} \left[ \left\| \hat{U}^m - \hat{U}_h^m \right\|_{o,D}^2 \right] \leq \int_0^{\tau_m} \left( \int_D \int_D \left| \hat{K}_m(\tau, x, y) - \hat{D}_{h,m}(\tau, x, y) \right|^2 \, dy \, dx \right) \, d\tau \leq \sum_{j=1}^{m} \int_{\Delta_j} \left\| \Lambda^{m-j+1} - \Lambda_h^{m-j+1} \right\|_{\text{HS}}^2 \, d\tau, \ m = 1, \ldots, M,
where \( \Lambda \) is the inverse elliptic operator defined in the proof of Theorem 4.2. Now, we use the definition of the Hilbert-Schmidt norm and the deterministic error estimate (6.14), to have

\[
E \left[ \| \hat{U}^m - \hat{U}_h^m \|_{0,D}^2 \right] \leq \sum_{j=1}^{m} \Delta \tau \sum_{\alpha \in \mathbb{N}^d} \| \Lambda^{m-j+1} \epsilon_x - \Lambda_h^{m-j+1} \epsilon_x \|_{0,D}^2
\]

\[
\leq \sum_{\alpha \in \mathbb{N}^d} \left( \sum_{j=1}^{m} \Delta \tau \| \Lambda^j \epsilon_x - \Lambda_h^j \epsilon_x \|_{0,D}^2 \right)
\]

\[
\leq C \Delta \tau \| \epsilon_x \|_{H^2(\bar{\Omega},r)}^2, \quad m = 1, \ldots, M, \quad \forall \theta \in [0,1].
\]

Thus, we arrive at

\[
\max_{1 \leq m \leq M} E \left[ \| \hat{U}^m - \hat{U}_h^m \|_{0,D}^2 \right] \leq C \Delta \tau \| \epsilon_x \|_{H^2(\bar{\Omega},r)}^2, \quad \forall \theta \in [0,1].
\]

from which, requiring \(-2 \tilde{\epsilon}(r,\theta) > d\), (6.33) easily, follows (cf. Theorem 5.2).

The available error estimates allow us to conclude a discrete in time \( L^\infty(L^2(L^2)) \) convergence of the Backward Euler fully-discrete approximations of \( \hat{u} \), over a uniform partition of \([0,T]\).

**Theorem 6.5.** Let \( r \in \{2,3,4\} \), \( \nu(r,d) \) be defined by (5.13), \( \hat{u} \) be the solution of problem (1.6), and \((\hat{U}_h^m)_{m=0}^M\) be the Backward Euler fully-discrete approximations of \( \hat{u} \) constructed by (1.10)-(1.11). If the partition \((\tau_m)_{m=0}^M\) is uniform, i.e., \( \Delta \tau = \Delta t \) for \( m = 1, \ldots, M \), then, there exists a constant \( C > 0 \), independent of \( T, \Delta \tau, \Delta t \) and \( \Delta x \), such that

\[
\max_{0 \leq m \leq M} \left\{ E \left[ \| \hat{U}^m - \hat{u}(\tau_m,\cdot) \|_{0,D}^2 \right] \right\}^{\frac{1}{2}} \leq C \left[ \tilde{\omega}(\Delta \tau, \epsilon_1) \Delta \tau^{\frac{4-d}{4-d} - \epsilon_1} + \epsilon_2^{\frac{2}{4+d}} \Delta \tau^{\nu(r,d)-\epsilon_2} \right],
\]

for \( \epsilon_1 \in (0,\frac{4-d}{4-d}] \) and \( \epsilon_2 \in (0,\nu(r,d)) \) where \( \tilde{\omega}(\Delta \tau, \epsilon_1) := \epsilon_1^{\frac{4-d}{4-d} - \epsilon_1} + (\Delta \tau)^{\nu(r,d)}(p_d(\Delta \tau)^{\frac{4}{4+d}})^{\frac{4}{4+d}} \).

**Proof.** The estimate is a simple consequence of the error bounds (6.33) and (4.6). \(\square\)

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Proof of (2.9). Let $\epsilon \in (0, 2]$. First, we observe that
\[
\sum_{n=1}^{\infty} \frac{1}{n^{1+c_{\star} \epsilon}} \leq 1 + \int_{1}^{+\infty} \frac{1}{x^{1+c_{\star} \epsilon}} \, dx \\
\leq \left( 2 + \frac{1}{c_{\star}} \right) \frac{1}{\epsilon},
\]
which easily yields (2.9) for $d = 1$. For $d = 2$, we have
\[
\sum_{\alpha \in \mathbb{N}^2} |\alpha|_{\mathbb{N}^2}^{-(2+c_{\epsilon})} \leq 2 \sum_{n=1}^{\infty} \frac{1}{(1+n^2)^{1+c_{\epsilon}}} + \int_{(1, +\infty)^2} |x|^{-(2+c_{\epsilon})} \, dx \\
\leq 2 \sum_{n=1}^{\infty} \frac{1}{n^{1+c_{\epsilon}}} + \int_{1}^{+\infty} \int_{0}^{\pi} \int_{0}^{\pi} r^{-(1+c_{\epsilon})} \, dr \, d\theta \, d\phi \\
\leq C \epsilon^{-1}.
\]
For $d = 3$, using (2.9) for $d = 2$, we proceed similarly as follows
\[
\sum_{\alpha \in \mathbb{N}^3} |\alpha|_{\mathbb{N}^3}^{-(3+c_{\epsilon})} \leq 3 \sum_{\beta \in \mathbb{N}^2} \left( 1 + |\beta|_{\mathbb{N}^2}^{2} \right)^{\frac{3+c_{\epsilon}}{2}} + \int_{(1, +\infty)^3} |x|^{-(3+c_{\epsilon})} \, dx \\
\leq 3 \sum_{\beta \in \mathbb{N}^2} |\beta|_{\mathbb{N}^2}^{-(2+c_{\epsilon})} + \int_{1}^{+\infty} \int_{0}^{\pi} \int_{0}^{\pi} \sin(\theta) r^{-(1+c_{\epsilon})} \, dr \, d\theta \, d\phi \\
\leq C \epsilon^{-1}.
\]
Appendix B.

Proof of (2.10). First, we recall from [18] that \(\sum_{k=1}^{\infty} \frac{1-e^{-\delta t^{2}}}{k^{4} \pi^{4}} \leq C (1 + \delta^{\frac{1}{2}}) \delta^{\frac{3}{2}}. \) For \(d = 2\), using the latter inequality, we have
\[
\sum_{\alpha \in \mathbb{N}^{2}} \frac{1-e^{-\lambda_{\alpha}^{2} t^{4}}}{\lambda_{\alpha}^{4}} \leq 2 \sum_{n=1}^{\infty} \frac{1-e^{-\pi^{4}(1+k^{2})^{2} t^{4}}}{\pi^{4}(1+k^{2})^{2} t^{4}} + \int_{(1, +\infty)^{2}} \frac{1-e^{-\pi^{4} |x|^{4} t^{4}}}{\pi^{4} |x|^{4} t^{4}} \, dx
\]
\[
\leq 2 \sum_{n=1}^{\infty} \frac{1-e^{-\pi^{4} k^{4} t^{4}}}{\pi^{4} k^{4} t^{4}} + \int_{0}^{+\infty} \int_{1}^{+\infty} \frac{1-e^{-\pi^{4} s^{4} t^{4}}}{\pi^{4} s^{4} t^{4}} \, dr \, d\theta
\]
\[
\leq C (1 + \delta^{\frac{1}{2}}) \delta^{\frac{3}{2}} + \frac{1}{2\pi} \int_{0}^{+\infty} \frac{1-e^{-\pi^{4} r^{4} d^{4}}}{\pi^{4} r^{4} d^{4}} \, dz
\]
\[
\leq C (1 + \delta^{\frac{1}{2}}) \delta^{\frac{3}{2}} + \frac{1}{2\pi} \int_{0}^{+\infty} e^{-z^{3}} \, dz,
\]
which yields \(\sum_{\alpha \in \mathbb{N}^{2}} \frac{1-e^{-\lambda_{\alpha}^{2} t^{4}}}{\lambda_{\alpha}^{4}} \leq C (1 + \delta^{\frac{1}{2}} + \delta^{\frac{3}{2}}) \delta^{\frac{1}{2}}. \) Finally, when \(d = 3\), using (2.10) for \(d = 2\), we obtain
\[
\sum_{\beta \in \mathbb{N}^{2}} \frac{1-e^{-\lambda_{\beta}^{2} t^{4}}}{\lambda_{\beta}^{4}} \leq 3 \sum_{\beta \in \mathbb{N}^{3}} \frac{1-e^{-\pi^{4} |\beta|^{2} s^{4}}}{\pi^{4} |\beta|^{2} s^{4}} + \int_{(1, +\infty)^{3}} \frac{1-e^{-\pi^{4} |x|^{4} s^{4}}}{\pi^{4} |x|^{4} s^{4}} \, dx
\]
\[
\leq 3 \sum_{\beta \in \mathbb{N}^{2}} \frac{1-e^{-\lambda_{\beta}^{4} s^{4}}}{\lambda_{\beta}^{4}} + \int_{0}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} \sin(\phi) \frac{1-e^{-\pi^{4} s^{4}}}{\pi^{4} s^{4}} \, dr \, d\theta \, d\phi
\]
\[
\leq C (1 + \delta^{\frac{1}{2}} + \delta^{\frac{3}{2}}) \delta^{\frac{1}{2}} + \frac{1}{2\pi} \int_{0}^{+\infty} \frac{1-e^{-z^{4}}}{z^{4}} \, dz
\]
\[
\leq C (1 + \delta^{\frac{1}{2}} + \delta^{\frac{3}{2}}) \delta^{\frac{1}{2}} + \frac{1}{2\pi} \int_{0}^{+\infty} e^{-z^{2}} \, dz,
\]
which yields \(\sum_{\alpha \in \mathbb{N}^{3}} \frac{1-e^{-\lambda_{\alpha}^{2} t^{4}}}{\lambda_{\alpha}^{4}} \leq C (1 + \delta^{\frac{1}{2}} + \delta^{\frac{3}{2}} + \delta^{\frac{1}{2}}) \delta^{\frac{1}{2}}. \) \(\Box\)