Research Article
Hybrid Iterations for the Fixed Point Problem and Variational Inequalities

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A hybrid iterative algorithm with Meir-Keeler contraction is presented for solving the fixed point problem of the pseudocontractive mappings and the variational inequalities. Strong convergence analysis is given as \( \lim_{n \to \infty} d(S^T x_n, T S x_n) \).

1. Introduction
Throughout, we assume that \( H \) is a real Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \) and the norm \( \| \cdot \| \) and \( C \subset H \) is a nonempty closed convex set.

Definition 1. A mapping \( T : C \to C \) is said to be pseudocontractive if

\[
\langle T u - T u^*, u - u^* \rangle \leq \| u - u^* \|^2
\]

(1)

for all \( u, u^* \in C \).

We use \( \text{Fix}(T) \) to denote the set of fixed points of \( T \).

Remark 2. It is easily seen that (1) is equivalent to the following:

\[
\| T u - T u^* \|^2 \leq \| u - u^* \|^2 + \| (I - T) u - (I - T) u^* \|^2
\]

(2)

for all \( u, u^* \in C \).

Definition 3. A mapping \( T : C \to C \) is said to be \( L \)-Lipschitzian if

\[
\| T u - T u^* \| \leq L \| u - u^* \|
\]

(3)

for all \( u, u^* \in C \), where \( L > 0 \) is a constant.

If \( L = 1 \), \( T \) is said to be nonexpansive.

One of our purposes of this paper is to find the fixed points of the pseudocontractive mappings in Hilbert spaces. In the literature, there are a large number references associated with the fixed point algorithms for the pseudocontractive mappings. See, for instance, [1–9]. The first interesting algorithm for finding the fixed points of the Lipschitz pseudocontractive mappings in Hilbert spaces was presented by Ishikawa [4] in 1974.

Ishikawa’s Algorithm. For any \( x_0 \in C \), define the sequence \( \{ x_n \} \) iteratively by

\[
y_n = (1 - \alpha_n) x_n + \alpha_n T x_n,
\]

\[
x_{n+1} = (1 - \epsilon_n) x_n + \epsilon_n T y_n
\]

(4)
for all $n \in \mathbb{N}$, where $\{a_n\} \subset [0, 1]$ and $\{b_n\} \subset [0, 1]$ satisfy the following conditions:

(a) $\lim_{n \to \infty} a_n = 0$;

(b) $\sum_{n=1}^{\infty} a_n b_n = \infty$.

Ishikawa proved that the sequence $\{x_n\}$ generated by (4) converges strongly to a fixed point of $T$ provided $C$ is a compact set.

Recently, Zhou [9] suggested the following algorithm.

**Zhou’s Algorithm.** For any $x_0 \in C$, define the sequence $\{x_n\}$ iteratively by

$$
\begin{align*}
y_n &= (1 - b_n) x_n + b_n Tx_n, \\
z_n &= (1 - a_n) x_n + a_n T y_n, \\
c_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 \\
&\quad - a_n b_n \left(1 - 2a_n + 2a_n^2\right) \|z - T x_n\|^2\}, \\
q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= \text{proj}_{c_n \cap q_n}(x_0), \quad n \in \mathbb{N},
\end{align*}
$$

where $\{a_n\}$ and $\{b_n\}$ are two real sequences in $(0, 1)$ satisfying the following conditions:

(a) $a_n \leq a_0$ for all $n \in \mathbb{N}$;

(b) $0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n \leq 1/(\sqrt{1 + L^2} + 1)$.

Zhou proved that the sequence $\{x_n\}$ generated by (5) converges strongly to $\text{proj}_{\text{Fix}(T)}(x_0)$ without the compactness assumption.

**Definition 4.** A mapping $A : C \to \mathbb{H}$ is said to be in**verse strongly monotone if there exists $\xi > 0$ such that

$$
\langle u - v, Au - Av \rangle \geq \xi \|Au - Av\|^2
$$

for all $u, v \in C$.

The variational inequality problem is to find $u \in C$ such that

$$
\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.
$$

2. **Preliminaries**

Recall that the metric projection $\text{proj}_C : \mathbb{H} \to C$ satisfies

$$
\|u - \text{proj}_C(u)\| = \inf \left\{\|u - u^*\| : u^* \in C\right\}.
$$

The metric projection $\text{proj}_C$ is a typically firmly nonexpansive mapping, that is,

$$
\|\text{proj}_C(u) - \text{proj}_C(u')\|^2 \\
\leq \langle \text{proj}_C(u) - \text{proj}_C(u'), u - u' \rangle
$$

for all $u, u' \in \mathbb{H}$.

It is well known that, in a real Hilbert space $\mathbb{H}$, the following equality holds:

$$
\|\xi u + (1 - \xi) u'\|^2 \\
= \xi \|u\|^2 + (1 - \xi) \|u'\|^2 - (1 - \xi) \|u - u'\|^2
$$

for all $u, u' \in \mathbb{H}$ and $\xi \in [0, 1]$.

**Lemma 5 (see [9]).** Let $\mathbb{H}$ be a real Hilbert space and let $C$ be a closed convex subset of $\mathbb{H}$. Let $T : C \to C$ be a continuous pseudocontractive mapping. Then,

(i) $\text{Fix}(T) \subset C$ is a closed convex set;

(ii) $(1 - T)$ is demiclosed at zero.

Let $\{C_n\} \subset \mathbb{H}$ be a sequence of nonempty closed convex sets. We define the symbols $s$-$\text{Li}_n C_n$ and $w$-$\text{Li}_n C_n$ as follows.

(1) $x^* \in s$-$\text{Li}_n C_n$ if there exists $\{x_n\} \subset C_n$ such that $x_n \to x^*$ strongly.

(2) $x^* \in w$-$\text{Li}_n C_n$ if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{x_{n_i}\}$ in $C_{n_i}$ such that $x_{n_i} \rightharpoonup x^*$ weakly.

If $C_0$ satisfies the following:

$$
C_0 = s$-$\text{Li}_n C_n = w$-$\text{Li}_n C_n, 
$$

then we say that $\{C_n\}$ converges to $C_0$ in the sense of Mosco [19] and we write $C_0 = M$-$\lim_{n \to \infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco.

Tsukada [20] proved the following theorem for the metric projection.

**Lemma 6 (see [20]).** Let $\mathbb{H}$ be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of $\mathbb{H}$. If $C_0 = M$-$\lim_{n \to \infty} C_n$ exists and is nonempty, then, for each $x \in \mathbb{H}$, $\{\text{proj}_{C_n}(x)\}$ converges strongly to $\text{proj}_{C_0}(x)$, where $\text{proj}_{C_n}$ and $\text{proj}_{C_0}$ are the metric projections of $\mathbb{H}$ onto $C_n$ and $C_0$, respectively.

Let $(E, d)$ be a complete metric space. A mapping $\psi : E \to E$ is called a Meir-Keeler contraction [21] if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$
d(u, v) < \varepsilon + \delta \implies d(\psi(u), \psi(v)) < \varepsilon
$$

(12)
for all \( u, u^† \in E \). It is well known that the Meir-Keeler contraction is a generalization of the contraction.

Lemma 7 (see [21]). A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.

Lemma 8 (see [22]). Let \( \psi \) be a Meir-Keeler contraction on a convex subset \( C \) of a Banach space \( E \). Then, for any \( \varepsilon > 0 \), there exists \( \sigma \in (0, 1) \) such that

\[
\| u - u^† \| \geq \varepsilon \implies \| \psi(u) - \psi(u^†) \| \leq \sigma \| u - u^† \| 
\]

for all \( u, u^† \in E \).

Lemma 9 (see [22]). Let \( C \) be a convex subset of a Banach space \( E \). Let \( T \) be a nonexpansive mapping on \( C \) and let \( \psi \) be a Meir-Keeler contraction on \( C \). Then the following holds.

(i) \( T\psi \) is a Meir-Keeler contraction on \( C \).

(ii) For each \( \zeta \in (0, 1) \), \( (1 - \zeta)T + \alpha\psi \) is a Meir-Keeler contraction on \( C \).

3. Main Results

In this section, we firstly introduce a hybrid iterative algorithm for finding the common element of the fixed point problem and the variational inequality problem.

Algorithm 10. Let \( H \) be a real Hilbert space and \( C \subset H \) a nonempty closed convex set. Let \( \psi : C \to C \) be a Meir-Keeler contractive mapping. Let \( \lambda : C \to \Lambda \) be an inverse strongly monotone mapping. Let \( T : C \to C \) be a \( \kappa \)-Lipschitz pseudocontractive mapping with \( \kappa > 1 \). For \( x_0 \in C_0 = C \) arbitrarily, define a sequence \( \{x_n\} \) iteratively by

\[
u_n = \left( 1 - \gamma_n \right) u_n + \gamma_n T\mu_n, \\
\mu_n = \left( 1 - \omega_n \right) u_n + \omega_n T\nu_n, \\
C_{n+1} = \{ \mu \in C_n : \| \mu_n - \mu \| \leq \| x_n - \mu \| \}, \\
x_{n+1} = \text{proj}_{C_{n+1}} \psi(x_n), \quad n \in \mathbb{N},
\]

where \( \alpha \in (0, 2\lambda) \) is a constant and \( \{\omega_n\} \) and \( \{\gamma_n\} \) are two real number sequences in \( (0, 1) \) satisfying \( 0 < \gamma_1 < \omega_n \leq \omega_n \leq c_2 < 1/(\sqrt{2} + \kappa^2 + 1) \).

Next, we show the strong convergence of (14).

Theorem 11. Suppose that \( \Lambda = \text{VI}(C, \lambda) \cap \text{Fix}(T) \neq \emptyset \). Then the sequence \( \{x_n\} \) defined by (14) converges strongly to \( x^† = \text{proj}_\Lambda \psi(x^†) \).

Remark 12. Note that \( \Lambda \) is a closed convex subset of \( C \). Thus \( \text{proj}_\Lambda \) is well defined. Since \( \psi \) is a Meir-Keeler contraction of \( C \), it follows that \( \text{proj}_\Lambda \psi \) is a Meir-Keeler contraction of \( C \) by Lemma 9. According to Lemma 7, there exists a unique fixed point \( x^† \in \Lambda \) such that \( x^† = \text{proj}_\Lambda \psi(x^†) \).

Proof. The outline of our proof is as follows.

Step 1. \( \Lambda \subset C_n \) for all \( n \in \mathbb{N} \);

Step 2. \( C_n \) is closed and convex for all \( n \in \mathbb{N} \);

Step 3. \( \lim_{n \to \infty} \| x_n - y \| = 0 \) where \( y = \text{proj}_{C_n} \psi(y) \);

Step 4. \( y \in \text{Fix}(T) \);

Step 5. \( y \in \text{VI}(C, \lambda) \);

Step 6. \( y = \text{proj}_\Lambda \psi(y) = x^† \).

Proof of Step 1. We prove this step by induction. (i) \( \Lambda \subset C_0 \) is obvious. (ii) Suppose that \( \Lambda \subset C_k \) for some \( k \in \mathbb{N} \). Pick up \( x^* \in C_{ \Lambda \subset C_k} \). Then, we have

\[
\| u_n - x^* \| = \| \text{proj}_C [x_n - \alpha \lambda x_n] - \text{proj}_C [x^* - \alpha \lambda x^*] \| \\
\leq \| (x_n - \alpha \lambda x_n) - (x^* - \alpha \lambda x^*) \| \\
\leq \| x_n - x^* \|. 
\]

By (2), we have

\[
\| Tu_n - x^* \| \leq \| u_n - x^* \| + \| Tu_n - u_n \|, \\
\| Tv_n - x^* \| \leq \| T((1 - \epsilon_n) u_n + \epsilon_n Tu_n) - x^* \|^2 \\
\leq \| (1 - \epsilon_n) (u_n - x^*) + \epsilon_n (Tu_n - x^*) \|^2 \\
+ \| (1 - \epsilon_n) u_n + \epsilon_n Tu_n - Tv_n \|^2.
\]

From (10), we obtain

\[
\| (1 - \epsilon_n) u_n + \epsilon_n Tu_n - Tv_n \|^2 \\
= \| (1 - \epsilon_n) (u_n - Tv_n) + \epsilon_n (Tu_n - Tv_n) \|^2 \\
= (1 - \epsilon_n) \| u_n - Tv_n \|^2 + \epsilon_n \| Tu_n - Tv_n \|^2 \\
- \epsilon_n (1 - \epsilon_n) \| u_n - Tu_n \|^2. 
\]

Since \( T \) is \( \kappa \)-Lipschitzian and \( u_n - v_n = \epsilon_n (u_n - Tu_n) \), by (18), we get

\[
\| (1 - \epsilon_n) u_n + \epsilon_n Tu_n - Tv_n \|^2 \\
\leq (1 - \epsilon_n) \| u_n - Tu_n \|^2 + \epsilon_n \kappa^2 \| u_n - Tu_n \|^2 \\
- \epsilon_n (1 - \epsilon_n) \| u_n - Tu_n \|^2 \\
= (1 - \epsilon_n) \| u_n - Tu_n \|^2 \\
+ (\epsilon_n \kappa^2 + \epsilon_n^2 - \epsilon_n) \| u_n - Tu_n \|^2.
\]
By (10) and (16), we have
\[
\| (1 - \varrho_n) (u_n - x^*) + \varrho_n (T u_n - x^*) \|^2 \\
= \| (1 - \varrho_n) (u_n - x^*) + \varrho_n (T u_n - x^*) \|^2 \\
= (1 - \varrho_n) \| u_n - x^* \|^2 + \varrho_n \| T u_n - x^* \|^2 \\
- \varrho_n (1 - \varrho_n) \| u_n - T u_n \|^2 \\
\leq (1 - \varrho_n) \| u_n - x^* \|^2 + \varrho_n \left( \| u_n - x^* \|^2 + \| u_n - T u_n \|^2 \right) \\
- \varrho_n (1 - \varrho_n) \| u_n - T u_n \|^2 \\
= \| u_n - x^* \|^2 + \varrho_n^2 \| u_n - T u_n \|^2.
\]
(20)

From (17), (19), and (20), we deduce
\[
\| T u_n - x^* \|^2 \leq \| u_n - x^* \|^2 + (1 - \varrho_n) \| u_n - T u_n \|^2 \\
- \varrho_n \left( 1 - 2 \varrho_n - \varrho_n^2 \kappa^2 \right) \| u_n - T u_n \|^2.
\]
(21)

Since \( \varrho_n < \zeta < 1/(\sqrt{1 + \kappa^2} + 1) \), we have
\[
1 - 2 \varrho_n - \varrho_n^2 \kappa^2 > 0
\]
(22)
for all \( n \in \mathbb{N} \). This together with (21) implies that
\[
\| T u_n - x^* \|^2 \leq \| u_n - x^* \|^2 + (1 - \varrho_n) \| u_n - T u_n \|^2 \\
\leq \| u_n - x^* \|^2 + \varrho_n \| u_n - T u_n \|^2
\]
(23)

By (10), (15), and (23) and noting that \( \omega_n \leq \varrho_n \), we have
\[
\| u_n - x^* \|^2 = \| (1 - \omega_n) u_n + \omega_n T u_n - x^* \|^2 \\
= (1 - \omega_n) \| u_n - x^* \|^2 + \omega_n \| T u_n - x^* \|^2 \\
- \omega_n (1 - \omega_n) \| u_n - T u_n \|^2 \\
\leq \| u_n - x^* \|^2 + \omega_n \| u_n - T u_n \|^2 \\
\leq \| x_n - x^* \|^2
\]
(24)
and hence \( x^* \in C_{k+1} \). This indicates that \( \Lambda \subset C_n \) for all \( n \in \mathbb{N} \).

**Proof of Step 2.** In fact, it is obvious from the assumption that \( C_0 = C \) is closed convex. Suppose that \( C_k \) is closed and convex for some \( k \in \mathbb{N} \). For any \( \mu \in C_k \), we know that \( \| y_k - \mu \| \leq \| x_k - \mu \| \) equivalent to
\[
\| y_k - x_k \|^2 + 2 \langle y_k - x_k, x_k - \mu \rangle \leq 0.
\]
(25)
So \( C_{k+1} \) is closed and convex. By induction, we deduce that \( C_n \) is closed and convex for all \( n \in \mathbb{N} \).

**Proof of Step 3.** Firstly, from Step 2, we note that \( \{ x_n \} \) is well defined. Since \( \bigcap_{n=1}^{\infty} C_n \) is closed convex, we also have that \( \text{proj}_{\bigcap_{n=1}^{\infty} C_n} \) is well defined and so \( \text{proj}_{\bigcap_{n=1}^{\infty} C_n} \psi \) is a Meir-Keeler contraction on \( C \). By Lemma 7, there exists a unique fixed point \( v \in \bigcap_{n=1}^{\infty} C_n \) of \( \text{proj}_{\bigcap_{n=1}^{\infty} C_n} \psi \). Since \( C_n \) is a nonincreasing sequence of nonempty closed convex subsets of \( H \) with respect to inclusion, it follows that
\[
\emptyset \neq A \subset \bigcap_{n=1}^{\infty} C = M - \lim_{n \to \infty} C_n.
\]
(26)

Setting \( s_n := \text{proj}_{C_n} \psi (v) \) and applying Lemma 6, we can conclude that
\[
\lim_{n \to \infty} s_n = \text{proj}_{\bigcap_{n=1}^{\infty} C_n} \psi (v) = v.
\]
(27)

Now, we show that \( \lim_{n \to \infty} \| x_n - v \| = 0 \). Assume that \( M = \lim_{n \to \infty} \| x_n - v \| > 0 \). Then, for any \( \epsilon \) with \( 0 < \epsilon < M \), we can choose \( \delta_1 > 0 \) such that
\[
\lim_{n \to \infty} \| x_n - v \| > \epsilon + \delta_1.
\]
(28)
Since \( \psi \) is a Meir-Keeler contraction, for the positive \( \epsilon \), there exists another \( \delta_2 > 0 \) such that
\[
\| x - y \| < \epsilon + \delta_2 \Rightarrow \| \psi (x) - \psi (y) \| < \epsilon
\]
(29)
for all \( x, y \in C \).

In fact, we can choose a common \( \delta > 0 \) such that (28) and (29) hold. If \( \delta_1 > \delta_2 \), then
\[
\lim_{n \to \infty} \| x_n - v \| > \epsilon + \delta_1 > \epsilon + \delta_2.
\]
(30)
If \( \delta_1 \leq \delta_2 \), then, from (29), it follows that
\[
\| x - y \| < \epsilon + \delta_1 \Rightarrow \| \psi (x) - \psi (y) \| < \epsilon
\]
(31)
for all \( x, y \in C \). Thus, we have
\[
\lim_{n \to \infty} \| x_n - v \| > \epsilon + \delta,
\]
(32)
\[
\| x - y \| < \epsilon + \delta \Rightarrow \| \psi (x) - \psi (y) \| < \epsilon
\]
(33)
for all \( x, y \in C \). Since \( s_n \rightarrow v \), there exists \( n_0 \in \mathbb{N} \) such that
\[
\| s_n - v \| < \delta
\]
(34)
for all \( n \geq n_0 \).

Now, we consider two possible cases.

**Case I.** There exists \( n_1 \geq n_0 \) such that
\[
\| x_n - v \| \leq \epsilon + \delta.
\]
(35)
By (33) and (34), we get
\[
\| x_{n+1} - v \| \leq \| x_{n+1} - s_{n+1} \| + \| s_{n+1} - v \|
\]
(36)
\[
= \| \text{proj}_{C_{n+1}} \psi (x_{n+1}) - \text{proj}_{C_{n+1}} \psi (v) \| \\
+ \| s_{n+1} - v \|
\]
\[
\leq \| \psi (x_{n+1}) - \psi (v) \| + \| s_{n+1} - v \|
\]
\[
\leq \epsilon + \delta.
\]
By induction, we can obtain that
\[\|x_{n+1} - y\| \leq \epsilon + \delta\] (37)
for all \(m \geq 1\), which implies that
\[\lim_{n \to \infty} \|x_n - y\| \leq \epsilon + \delta,\] (38)
which contradicts (32). Therefore, we conclude that \(\|x_n - y\| \to 0\) as \(n \to \infty\).

Case 2 (\(\|x_n - y\| > \epsilon + \delta\) for all \(n \geq n_0\)). Now, we prove that Case 2 is impossible. Suppose that Case 2 is true. By Lemma 8, there exists \(\sigma \in (0,1)\) such that
\[\|\psi(x_n) - \psi(y)\| \leq \sigma \|x_n - y\|\] (39)
for all \(n \geq n_0\). Thus we have
\[\|x_{n+1} - s_{n+1}\| = \|\text{proj}_{C_n} \psi(x_n) - \text{proj}_{C_n} \psi(y)\|\]
\[\leq \|\psi(x_n) - \psi(y)\|\]
\[\leq \sigma \|x_n - y\|\] (40)
for all \(n \geq n_0\). It follows that
\[\lim_{n \to \infty} \|x_{n+1} - y\| = \lim_{n \to \infty} \|x_{n+1} - s_{n+1}\|\]
\[\leq \sigma \lim_{n \to \infty} \|x_n - y\|\]
\[< \lim_{n \to \infty} \|x_n - y\|,\] (41)
which gives a contradiction. Hence we obtain
\[\lim_{n \to \infty} \|x_n - y\| = 0.\] (42)

**Proof of Step 4.** By Step 3, we deduce immediately that \(\{x_n\}\) is bounded. Observe that
\[\|x_{n+1} - x_n\| \leq \|x_n - y\| + \|y - s_{n+1}\| + \|s_{n+1} - x_n\|\]
\[= \|x_n - y\| + \|y - s_{n+1}\| + \|s_{n+1} - s_{n+1}\| + \|x_n - y\|\]
\[+ \|\text{proj}_{C_n} \psi(x_n) - \text{proj}_{C_n} \psi(y)\|\]
\[\leq \|x_n - y\| + \|y - s_{n+1}\| + \|\psi(x_n) - \psi(y)\|.\] (43)
Therefore, we have
\[\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.\] (44)
Since \(x_{n+1} \in C_{n+1}\), we have
\[\|u_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|.\] (45)
This together with (44) implies that
\[\lim_{n \to \infty} \|u_n - x_{n+1}\| = \lim_{n \to \infty} \|u_n - x_n\| = 0.\] (46)
From (15) and (24), we have
\[\|u_n - x^*\|^2 \leq \|u_n - x^*\|^2\]
\[\leq \|(x_n - \alpha A x_n) - (x^* - \alpha A x^*)\|^2\]
\[= \|x_n - x^*\|^2 + \alpha^2 \|A x_n - A x^*\|^2\]
\[\leq \|x_n - x^*\|^2 + 2 \langle A x_n - A x^*, x_n - x^* \rangle\]
\[\leq \|x_n - x^*\|^2 + \alpha (\alpha - 2 \lambda) \|A x_n - A x^*\|^2.\] (47)
Then we have
\[(2\lambda - \alpha) \|A x_n - A x^*\|^2\]
\[\leq \|x_n - x^*\|^2 - \|u_n - x^*\|^2\] (48)
\[\leq \|x_n - u_n\| \left(\|x_n - x^*\| + \|u_n - x^*\|\right).\]
By (46) and (48), we obtain
\[\lim_{n \to \infty} \|A x_n - A x^*\| = 0.\] (49)
Since \(\text{proj}_{C}\) is firmly nonexpansive, we have
\[\|u_n - x^*\|^2 = \|\text{proj}_{C} [x_n - \alpha A x_n] - \text{proj}_{C} [x^* - \alpha A x^*]\|^2\]
\[\leq \langle (x_n - \alpha A x_n) - (x^* - \alpha A x^*), u_n - x^* \rangle\]
\[= \frac{1}{2} \left(\|x_n - \alpha A x_n\|^2 + \|u_n - x^*\|^2 - \|x_n - x^*\|^2 - \|\alpha A x_n - \alpha x^*\|^2\right)\]
\[\leq \frac{1}{2} \left(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2\right)\]
\[+ \frac{1}{2} \left(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2\right)\]
\[+ 2 \alpha \langle x_n - u_n, \alpha A x_n - \alpha x^* \rangle - \alpha^2 \|A x_n - A x^*\|^2.\] (50)
It follows that
\[\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|u_n - u_n\|^2\]
\[+ 2 \alpha \langle x_n - u_n, \alpha A x_n - \alpha x^* \rangle\]
\[\leq \alpha^2 \|A x_n - A x^*\|^2.\] (51)
From (24) and (51), we get
\[
\|w_n - x^*\|^2 \\
\leq \|u_n - x^*\|^2 \\
\leq \|x_n - x^*\|^2 - \|w_n - x_n\|^2 \\
+ 2\alpha \langle x_n - u_n, \beta x_n - x^* \rangle - \alpha^2 \|\beta x_n - x^*\|^2 \\
\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \\
+ 2\alpha \|x_n - u_n\| \|\beta x_n - x^*\|.
\]
and so
\[
\|x_n - u_n\|^2 \leq \|x_n - x^*\|^2 - \|w_n - x_n\|^2 \\
+ 2\alpha \|x_n - u_n\| \|\beta x_n - x^*\| \\
\leq \|x_n - w_n\| (\|x_n - x^*\| + \|w_n - x_n\|) \\
+ 2\alpha \|x_n - u_n\| \|\beta x_n - x^*\|.
\]
This together with (46) and (49) implies that
\[
\lim_{n \to \infty} \|x_n - u_n\| = 0.
\] (54)

Note that
\[
\|u_n - T u_n\| \leq \|u_n - w_n\| + \|w_n - T u_n\| \\
\leq \|u_n - w_n\| + (1 - \omega_n) \|u_n - T u_n\| \\
+ \omega_n \|T v_n - T u_n\| \\
\leq \|u_n - w_n\| + (1 - \omega_n) \|u_n - T u_n\| \\
+ \omega_n \|v_n - u_n\| \\
\leq \|u_n - w_n\| + (1 - \omega_n) \|u_n - T u_n\| \\
+ \omega_n c \kappa \|u_n - T u_n\|.
\] (55)

It follows that
\[
\|u_n - T u_n\| \leq \frac{1}{\omega_n (1 - \omega_n \kappa)} \|u_n - w_n\| \\
\leq \frac{1}{c_1 (1 - \omega_n \kappa)} \|u_n - w_n\| \to 0.
\] (56)

Since \(x_n \to v\), we have \(u_n \to v\) by (54). So, from (56) and Lemma 5, we deduce that \(v \in \text{Fix}(T)\).

**Proof of Step 5.** Define a mapping \(\phi\) by
\[
\phi(v) = \begin{cases} 
\beta v + N \psi, & v \in C, \\
0, & v \notin C.
\end{cases}
\] (57)

Then \(\phi\) is maximal monotone (see [15]). Let \((v, w) \in \text{G}(\phi)\).
Since \(w - \beta v \in N \psi - v\) and \(u_n \in C\), we have \(\langle v - u_n, w - \beta v \rangle \geq 0\).
On the other hand, from \(u_n = \text{proj}_C[x_n - \alpha \beta x_n]\), we have
\[
\langle v - u_n, u_n - (x_n - \alpha \beta x_n) \rangle \geq 0,
\] (58)

that is,
\[
\langle v - u_n, \frac{u_n - x_n}{\alpha} + \beta x_n \rangle \geq 0.
\] (59)

Therefore, we have
\[
\langle v - u_n, \omega \rangle \\
\geq \langle v - u_n, \beta \omega \rangle \\
\geq \langle v - u_n, \beta \omega \rangle - \langle v - u_n, \frac{u_n - x_n}{\alpha} + \beta x_n \rangle \\
= \langle v - u_n, \beta \omega - \beta \beta x_n - \frac{u_n - x_n}{\alpha} \rangle \\
= \langle v - u_n, \beta \omega - \beta \beta x_n \rangle + \langle v - u_n, \beta \beta x_n - \beta x_n \rangle \\
- \langle v - u_n, \frac{u_n - x_n}{\alpha} \rangle \\
\geq \langle v - u_n, \beta \beta x_n - \beta x_n \rangle - \langle v - u_n, \frac{u_n - x_n}{\alpha} \rangle.
\] (60)

Noting that \(\|u_n - x_n\| \to 0\) and \(\beta\) is Lipschitz continuous, we obtain \(\langle v - v, \omega \rangle \geq 0\). Since \(\phi\) is maximal monotone, we have \(v \in \phi^{-1}(0)\) and hence \(v \in \text{VI}(C, A)\).

**Proof of Step 6.** Since \(x_{n+1} = \text{proj}_{C_{n+1}} \psi(x_n)\), we have
\[
\langle \psi(x_n) - x_{n+1}, x_{n+1} - y \rangle \geq 0
\] (61)
for all \(y \in C_{n+1}\). Since \(A \subset C_{n+1}\), we get
\[
\langle \psi(x_n) - x_{n+1}, x_{n+1} - y \rangle \geq 0
\] (62)
for all \(y \in A\). Noting that \(x_n \to v \in A\), we deduce
\[
\langle \psi(v) - v, v - y \rangle \geq 0
\] (63)
for all \(y \in A\). Thus \(v = \text{proj}_A \psi(v) = x^*\). This completes the proof. \(\square\)

**Conflicts of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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