GLOBAL WELL POSEDNESS FOR THE DRIFT-DIFFUSION-MAXWELL SYSTEM IN 2D

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Abstract. We prove global existence of strong solutions to the drift-diffusion-Maxwell system in two space dimension. We also provide an exponential growth estimate for the $H^1$ norm of the solution.

1. Introduction

We consider a coupled system of equations consisting of the equation of the charge and current density and Maxwell’s equations of electromagnetism, the coupling comes from the Lorentz force.

1.1. The model. We consider the Drift-Diffusion-Maxwell system (DD-M) for short, namely:

\[
\begin{aligned}
\partial_t \rho + \text{div}_x j &= 0, & \text{in } (0, T) \times \mathbb{R}^2, \\
\partial_t E - \text{curl}_x B &= -j, & \text{in } (0, T) \times \mathbb{R}^2, \\
\partial_t B + \text{curl}_x E &= 0, & \text{in } (0, T) \times \mathbb{R}^2, \\
\text{div}_x E &= \rho, & \text{in } (0, T) \times \mathbb{R}^2, \\
\text{div}_x B &= 0, & \text{in } (0, T) \times \mathbb{R}^2, \\
j &= \rho E - \nabla_x \rho,
\end{aligned}
\]

(1)

where $E$, $B$ are the electric and magnetic fields and $\rho$, $j$ are respectively the charge and current densities. We also supplement (1) with the following initial data

\[
\rho(t = 0) = \rho_0, \quad B(t = 0) = B_0, \quad E(t = 0) = E_0.
\]

(2)

Here, $\rho$, $B$, $E$ are defined on $\mathbb{R}^2$ and take their values in $\mathbb{R}^3$, i.e., $E = (E_1(t, x), E_2(t, x), E_3(t, x))$, $B = (B_1(t, x), B_2(t, x), B_3(t, x))$, $\rho = \rho(t, x)$ for any $(t, x) \in (0, T) \times \mathbb{R}^2$. The notation $\text{curl}_x B$ corresponds to

\[
\nabla \wedge B = \begin{pmatrix}
\partial_1 \\
\partial_2 \\
0
\end{pmatrix} \wedge \begin{pmatrix}
B_1 \\
B_2 \\
B_3
\end{pmatrix}
\]

(3)

The system (1) has the following energy identity:

\[
\frac{1}{2} \partial_t (\|\rho\|^2_{L^2} + \|E\|^2_{L^2} + \|B\|^2_{L^2}) + \|\nabla \rho\|^2_{L^2} \leq 0
\]

(4)

which is similar to the energy identity for the Maxwell-Navier-Stokes system used in [15].

Before stating our main result, let us mention that the Drift-Diffusion-Maxwell model (1) is derived from a Vlasov-Maxwell-Fokker-Planck system [7] which is motivated from...
plasma physics and Drift-Diffusion models can also be derived from other singular limits. We refer for instance to [8] where the Drift-Diffusion-Poisson model is derived from a Vlasov-Poisson-Fokker-Planck system. Let us recall that the Drift-Diffusion model is a standard model for semiconductors physics and suited for numerical computations we refer to [2, 4, 17] for a discussion about this model. Here, we would like to explain a little bit the relevance of the model. The first equation in (1) is the mass conservation equation (Continuity Equation). The second equation is the Ampere-Maxwell equation which includes here the displacement current $\partial_t E$. The third equation of (1) is the Faraday’s law and finally, the forth and fifth equation are the Gauss’s law (electric and magnetic).

1.2. Statement of the result. We want to prove in 2 space dimension, the global existence of solutions such that $\rho_0 \in L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ and $B_0$, $E_0 \in H^1(\mathbb{R}^2)$. The proof uses the conservation of the energy as well as a logarithmic estimate to bound the $L^\infty$ norm of $\rho$ in terms of the $H^1$ norm of $\rho$.

Our main result is the following:

**Theorem 1.1.** Take $\rho_0 \in L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ and $B_0$, $E_0 \in H^1(\mathbb{R}^2)$. Then, there exists a unique global solution $(\rho, E, B)$ of (1) such that for all $T > 0$, $\rho \in C([0, T); L^2) \cap L^2(0, T; H^1)$ and $E, B \in C([0, T); H^1)$. Moreover, $j \in L^2(0, T; L^2) \cap L^2(0, T; H^1)$ and $\rho \in L^1(0, T; H^2)$. In addition, the energy identity (4) holds and we have the following exponential growth estimate for all $t > 0$:

$$
\|\rho\|_{L^1(0, t; H^1)} + \|(E, B)(t)\|_{H^1} \leq (1 + \|(E_0, B_0)\|_{H^1}) e^{C_0(t+1)},
$$

where $C_0 = C[\|\rho_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + 1]$ for some constant $C$.

In the next Section 1.3, we give preliminaries about some regularity estimates. In Section 2, we prove some a priori estimate and derive the growth bound (5). In Section 3, we prove Theorem 1.1 by using a Galerkin approximation.

1.3. Preliminaries. The system (1) has the following energy identity:

$$
\frac{1}{2} \partial_t [\|\rho\|_{L^2}^2 + \|E\|_{L^2}^2 + \|B\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2] = \int \rho \nabla . j dx + \int (\text{curl}_x(B).E - \text{curl}_x(E).B) dx + \int \nabla \rho. E dx - \int \rho |E|^2 dx.
$$

Indeed, multiplying the first equation of (1) by $\rho$, the second one by $E$ and the third one by $B$ and integrating by parts, the energy estimate reads

$$
\frac{1}{2} \partial_t [\|\rho\|_{L^2}^2 + \|E\|_{L^2}^2 + \|B\|_{L^2}^2] = - \int \rho \nabla . j dx + \int (\text{curl}_x(B).E - \text{curl}_x(E).B) dx + \int \nabla \rho. E dx - \int \rho |E|^2 dx.
$$

Since, $j = \rho E - \nabla \rho$ and $\nabla . j = \rho \text{div} E + \nabla \rho \cdot E - \Delta \rho$ Then, $\int \rho \nabla . E dx = \int \rho^2 \text{div} E dx + \int \rho \nabla . E dx - \int \rho \Delta \rho dx$ Moreover, $\int (\text{curl}_x(B).E - \text{curl}_x(E).B) dx = 0$ We obtain,

$$
\frac{1}{2} \partial_t [\|\rho\|_{L^2}^2 + \|E\|_{L^2}^2 + \|B\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2] = - \int \rho E \nabla \rho dx - \int \rho^3 dx - \int \rho E^2 dx + \int E \nabla \rho dx.
$$
We have
\[ I_1 = - \int E \nabla (\rho^2) dx = \int \text{div} E \rho^2 dx = \frac{1}{2} \int \rho^3 dx. \]
So, \( I_1 + I_2 = - \frac{1}{2} \int \rho^3 dx \leq 0, \) \( I_3 \leq 0 \) and
\[ I_4 = - \int \text{div} E \rho dx = - \int \rho^2 dx \leq 0. \]
which yields the desired energy estimate.

**Remark 1.2.** The energy identity given here is not sufficient to deduce that \( E, B \in L^\infty(0, T; H^1) \)

We will use the following lemma giving regularity result for the Maxwell equation:

**Lemma 1.3.** [15] If \((E, B)\) solves
\[
\begin{align*}
\partial_t E - \text{curl}_x B &= -j, \\
\partial_t B + \text{curl}_x E &= 0, \\
E(t = 0) &= E_0, \quad B(t = 0) = B_0
\end{align*}
\]
On some time interval \((0, T)\) then, we have
\[
\| (E, B) \|_{C([0, T); H^1)} \leq \| (E_0, B_0) \|_{H^1} + \| j \|_{L^1(0, T; H^1)}
\]
We can use for the proof of this lemma the Duhamel formula and write
\[
F = e^{tL}F_0 + \int_0^t e^{(t-s)L} f(s) ds
\]
where \(F = (E, B), \) \(L\) is the operator define by \(L(E, B) = (\text{curl} B, -\text{curl} E)\) and \(f(s) = (j(s), 0).\) It is then clear that \(e^{tL}\) defines an isometry on \(H^s\) and hence the claim follows.

We refer to [8] for the proof of the following result which give a regularity of the density:

**Lemma 1.4.** Let \( \rho \) be a positive function such that \( \rho \in L^\infty(0, T; L^1(\mathbb{R}^2)) \), satisfying
\[
\begin{align*}
\nabla \sqrt{\rho} - \frac{1}{2} E \sqrt{\rho} &= H \in L^2(0, T; L^2(\mathbb{R}^2)), \\
\text{div}_x E &= \rho, \\
E &\in L^\infty(0, T; L^2(\mathbb{R}^d)),
\end{align*}
\]
then
\[
\rho \in L^2(0, T; L^2(\mathbb{R}^2)), \quad E \sqrt{\rho} \in L^2(0, T; L^2(\mathbb{R}^2)),
\]
and
\[
\sqrt{\rho} \in L^2(0, T; H^1(\mathbb{R}^2)).
\]

We recall here the Littlewood-Paley decomposition of a function. We define \( C \) to be the ring of center 0, of small radius \( 1/2 \) and great radius 2. There exist two nonnegative radial functions \( \chi \) and \( \varphi \) belonging respectively to \( \mathcal{D}(B(0, 1)) \) and to \( \mathcal{D}(C) \) so that
\[
\chi(\xi) + \sum_{q \geq 0} \varphi(\xi) = 0
\]
\[ |p - q| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-q}) \cap \text{Supp } \chi(2^{-p}) = \emptyset. \]
For instance, one can take \( \chi \in \mathcal{D}(B(0, 1)) \) such that \( \chi \equiv 1 \) on \( B(0, 1/2) \) and take,
\[
\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi).
\]
Then, we are able to define the Littlewood-Paley decomposition. Let us denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^d$. Let $h$, $\widetilde{h}$, $\triangle_q$, $S_q$ ($q \in \mathbb{Z}$) be defined as follows:

$$h = \mathcal{F}^{-1} \varphi$$

and

$$\triangle_q u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u) = 2^{qd} \int h(2^q y) u(x-y) dy,$$

$$S_q u = \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u) = 2^{qd} \int \widetilde{h}(2^q y) u(x-y) dy.$$

We point out that $S_q u = \sum_{q' \leq q-1, q' \in \mathbb{Z}} \triangle_q u$.

**Lemma 1.5.** If $\rho \in L^2(0, T; H^2(\mathbb{R}^2))$, then we have:

$$\|\rho\|_{L^1_t L^\infty_x} \leq C_0^{1/2} T^{1/2} + C T^{1/2} \|\nabla \rho\|_{L^2_{t,x}} \log(e + \frac{\|\nabla^2 \rho\|_{L^2_{t,x}}}{\|\nabla \rho\|_{L^2_{t,x}}})$$

**Proof.** For the proof of this lemma, we use the Littlewood-Paley decomposition of a function. When dealing with functions which depend on $t$ and $x$, the Littlewood-Paley decomposition will only apply to the $x$ variable.

We have

$$\rho = S_1 \rho + \sum_{q=1}^{q=N} \triangle_q \rho + \sum_{q \geq N+1} \triangle_q \rho$$

when $N$ is the integer. Hence, we have

$$\|\rho\|_{L^\infty} \leq \|S_1 \rho\|_{L^\infty} + \sum_{q=1}^{q=N} \|\triangle_q \rho\|_{L^\infty} + \sum_{q \geq N} \|\triangle_q \rho\|_{L^\infty}$$

$$\leq \|S_1 \rho\|_{L^\infty} + \sum_{q=1}^{q=N} 2^{-q} \|\nabla (\triangle_q \rho)\|_{L^\infty} + \sum_{q \geq N} 2^{-2q} \|\nabla^2 (\triangle_q \rho)\|_{L^\infty}$$

$$\leq C \|\rho\|_{L^2} + C \sum_{q=1}^{q=N} \|\nabla (\triangle_q \rho)\|_{L^2} + C \sum_{q \geq N} 2^{-q} \|\nabla^2 (\triangle_q \rho)\|_{L^2}$$

$$\leq C \|\rho\|_{L^2} + C \sum_{q=1}^{q=N} \|\nabla \rho\|_{L^2} + C \sum_{q \geq N} 2^{-q} \|\nabla^2 \rho\|_{L^2}$$

$$\leq C \|\rho\|_{L^2} + C N \|\nabla \rho\|_{L^2} + C 2^{-N} \|\nabla^2 \rho\|_{L^2}$$

Then, by using the Cauchy-Schwartz inequality, we get

$$\int_0^T \|\rho\|_{L^\infty} dt \leq C \|\rho\|_{L^1_t L^2_x} + C N \int_0^T \|\nabla \rho\|_{L^2_x} dt + C 2^{-N} \int_0^T \|\nabla^2 \rho\|_{L^2_x} dt$$

we optimize in $N$, by taking $N$ of the order $\frac{1}{\log(2)} \log(e + \frac{\|\nabla^2 \rho\|_{L^2_x}}{\|\nabla \rho\|_{L^2_x}})$. Hence,

$$\|\rho\|_{L^1_t L^\infty_x} \leq C_0^{1/2} T^{1/2} + C T^{1/2} \|\nabla \rho\|_{L^2_{t,x}} \log(e + \frac{\|\nabla^2 \rho\|_{L^2_{t,x}}}{\|\nabla \rho\|_{L^2_{t,x}}})$$
Lemma 1.6. There exists a constant $C$ and $C_0$, such that for all $T > 0$, we have
\[ \|\rho\|_{L_1^1 H^1} \leq C(e + \|(E_0, B_0)\|_{H^1})e^{C_0 T}. \] (8)

Proof. We have
\[ \|\rho\|_{L_2^2 H^1} \leq T^{1/2}\|\rho\|_{L_\infty^\infty L^2} + \|\nabla \rho\|_{L_2^2 L^2} \]
\[ \leq C_0^{1/2}(T^{1/2} + 1) \]
So, we get
\[ \|\rho\|_{L_1^1 H^1} \leq T^{1/2}\|\rho\|_{L_2^2 H^1} \]
\[ \leq C_0^{1/2}T^{1/2}(T^{1/2} + 1) \]
\[ \leq Ce^{C_0 T}. \]
\[ \Box \]

Finally, we recall a 2D Gagliardo-Nirenberg estimate and some classical inequalities.

Lemma 1.7. For any $u \in H^1(\mathbb{R}^2)$, we have
\[ \|u\|_{L^4} \leq \|u\|_{L_2^2} \|\nabla u\|_{L_2^2}. \] (9)

Lemma 1.8. Let $a \geq 1$ and $b \geq 0$ be two real numbers. Then
\[ \sqrt{a + b} \leq \sqrt{a} + b, \] (10)
and
\[ \log (a + b) \leq \log (a) + b. \] (11)

2. A priori estimates

The system (1) has the following energy identity
\[ \frac{1}{2}\partial_t (\|\rho\|_{L_2^2}^2 + \|E\|_{L_2^2}^2 + \|B\|_{L_2^2}^2) + \|\nabla \rho\|_{L_2^2}^2 \leq 0 \]
and hence
\[ \frac{1}{2}\|\rho\|_{L_2^2}^2 + \|E\|_{L_2^2}^2 + \|B\|_{L_2^2}^2] + \int_0^t \|\nabla \rho\|_{L_2^2}^2 \leq \frac{1}{2}\|\rho_0\|_{L_2^2}^2 + \|E_0\|_{L_2^2}^2 + \|B_0\|_{L_2^2}^2 \leq C_0. \] (12)

This formally yields the bounds
\[ \rho \in L_\infty(0, T; L^2) \cap L^2(0, T; H^1), E, B \in L_\infty(0, T; L^2). \]

Here and below $C_0$ will denote any constant of the form $C[\|\rho_0\|_{L_2^2}^2 + \|E_0\|_{L_2^2}^2 + \|B_0\|_{L_2^2}^2 + 1]$ where $C$ may change from one line to the other.

Moreover, the first equation of (1) can be written as
\[ \partial_t \rho - \Delta \rho = -E.\nabla \rho - \rho^2. \]
Applying the operator $\nabla$ to this equation
\[ \partial_t \nabla \rho - \Delta \nabla \rho = -\nabla E.\nabla \rho - E.\nabla^2 \rho - 2\rho \nabla \rho. \]
Multiplying the result by $\nabla \rho$, we obtain
\[
\partial_t \frac{\nabla \rho^2}{2} - \nabla \rho \nabla \rho = - (\nabla E \cdot \nabla \rho) \nabla \rho - (E \cdot \nabla \nabla \rho) \nabla \rho - 2 \rho |\nabla \rho|^2
\]
Now, integrating it with respect to $x$
\[
\frac{1}{2} \partial_t \int |\nabla \rho|^2 dx + \int |\nabla^2 \rho|^2 dx = \int (\nabla E \cdot \nabla \rho) \nabla \rho dx
\tag{13}
- \int (E \cdot \nabla^2 \rho) \nabla \rho dx - 2 \int \rho |\nabla \rho|^2 dx.
\]
Applying the operator $\nabla$ to the second equation of (1)
\[
\partial_t \nabla E - \text{curl}_x \nabla B = \nabla^2 \rho - \nabla \rho E - \rho \nabla E.
\]
Multiplying the result by $\nabla E$ and integrating it with respect to $x$
\[
\frac{1}{2} \partial_t \int |\nabla E|^2 dx - \int \nabla E \text{curl}_x \nabla B dx = \int \nabla^2 \rho \nabla E dx
\tag{14}
- \int \nabla \rho \nabla \nabla E dx - \int \rho |\nabla E|^2 dx.
\]
Applying the operator $\nabla$ to the third equation of (1)
\[
\partial_t \nabla B + \text{curl}_x \nabla E = 0
\]
Multiplying the result by $\nabla B$ and integrating it with respect to $x$
\[
\frac{1}{2} \partial_t \int |\nabla B|^2 dx + \int \nabla B \text{curl}_x \nabla E dx = 0.
\tag{15}
\]
Combining (13),(14) and (15), we deduce that
\[
\frac{1}{2} \partial_t \left[ \left( |\nabla \rho|^2 + |\nabla E|^2 + |\nabla B|^2 \right) dx + \int |\nabla^2 \rho|^2 dx \right] = \left[ \int \nabla^2 \rho \nabla E dx \right]_{t_1}
- \left[ \int \nabla \rho \nabla E dx \right]_{t_2} - \left[ \int \rho |\nabla E|^2 dx \right]_{t_3} - \left[ \int (\nabla E \cdot \nabla \rho) \nabla \rho dx \right]_{t_4}
- \left[ \int (E \cdot \nabla^2 \rho) \nabla \rho dx \right]_{t_5} - 2 \int \rho |\nabla \rho|^2 dx_{t_6}.
\]
Now, using the Young’s inequality ($ab \leq \alpha a^2 + \frac{1}{4\alpha} b^2, \forall \alpha > 0$), we can see that
\[
I_1 = \int \nabla^2 \rho \nabla E dx \leq \|\nabla E\|_{L^2} \|\nabla^2 \rho\|_{L^2}
\leq 2 \|\nabla F\|^2_{L^2} + \frac{1}{8} \|\nabla^2 \rho\|^2_{L^2}
\]
The Gagliardo-Nirenberg inequality (9), gives

\[
I_2 = -\int \nabla \rho E \nabla E \, dx = -\int \nabla \rho \nabla \frac{E^2}{2} \, dx
\]

\[
= \int \nabla^2 \rho \frac{E^2}{2} \, dx
\]

\[
\leq \frac{1}{2} \|\nabla^2 \rho\|_{L^2} \|E\|_{L^4}^2
\]

\[
\leq \frac{1}{2} \|\nabla^2 \rho\|_{L^2}^2 \|E\|_{L^2} \|\nabla E\|_{L^2}
\]

\[
\leq \frac{1}{8} \|\nabla^2 \rho\|_{L^2}^2 + \frac{1}{2} \|E\|_{L^2} \|\nabla F\|_{L^2}^2
\]

\[I_4 = -\int (\nabla E \cdot \nabla \rho) \nabla \rho \, dx \leq \|\nabla E\|_{L^2} \|\nabla \rho\|_{L^4}^2
\]

\[
\leq \|\nabla E\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla^2 \rho\|_{L^2}
\]

\[
\leq \frac{1}{8} \|\nabla^2 \rho\|_{L^2}^2 + 2\|\nabla \rho\|_{L^2} \|\nabla F\|_{L^2}^2
\]

\[I_5 = -\int (E \nabla \nabla \rho) \nabla \rho \, dx = -\int E \nabla \left(\frac{|\nabla \rho|^2}{2}\right) \, dx
\]

\[
= \int \rho \frac{|\nabla \rho|^2}{2} \, dx
\]

\[
\leq \frac{1}{2} \|\rho\|_{L^2} \|\nabla \rho\|_{L^4}^2
\]

\[
\leq \frac{1}{2} \|\rho\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla^2 \rho\|_{L^2}
\]

\[
\leq \frac{1}{8} \|\nabla^2 \rho\|_{L^2}^2 + \frac{1}{2} \|\rho\|_{L^2} \|\nabla \rho\|_{L^2}^2
\]

Combining \(I_1, I_2, I_3, I_4, I_5\) and \(I_6\), we deduce that

\[
\frac{1}{2} \partial_t \|\nabla F\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2 \leq \frac{1}{2} \|\nabla^2 \rho\|_{L^2}^2 + \frac{1}{2} \|\rho\|_{L^2} \|\nabla \rho\|_{L^2}^2
\]

\[
+ \left[2 + \frac{1}{2} \|E\|_{L^2}^2 + 2\|\nabla \rho\|_{L^2}^2\right] \|\nabla F\|_{L^2}^2
\]

Then,

\[
\partial_t \|\nabla F\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2 \leq \|\rho\|_{L^2} \|\nabla \rho\|_{L^2}^2
\]

\[
+ \left[4 + \|E\|_{L^2}^2 + 4\|\nabla \rho\|_{L^2}^2\right] \|\nabla F\|_{L^2}^2
\]

Hence,

\[
\|\nabla F\|_{L^2}^2(t) + \int_0^t \|\nabla^2 \rho\|_{L^2}^2 \leq \|\nabla F_0\|_{L^2}^2 + \|\rho\|_{L^2}^2 \int_0^t \|\nabla \rho\|_{L^2}^2 + \int_0^t \left(4 + \|E\|_{L^2}^2 + 4\|\nabla \rho\|_{L^2}^2\right) \|\nabla F\|_{L^2}^2(t).
\]

So, we have

\[
\|\nabla F\|_{L^2}^2(t) \leq \|\nabla F_0\|_{L^2}^2 + \|\rho\|_{L^2}^2 \int_0^t \|\nabla \rho\|_{L^2}^2 + \int_0^t \left(4 + \|E\|_{L^2}^2 + 4\|\nabla \rho\|_{L^2}^2\right) \|\nabla F\|_{L^2}^2(t).
\]
We deduce from Gronwall lemma that
\[ \| \nabla F \|_{L^2}^2(t) \leq \left( \| \nabla F_0 \|_{L^2}^2 + \| \rho \|_{L^2}^2 \int_0^t \| \nabla \rho \|_{L^2}^2 \right) e^{\int_0^t \left( 4 + \| E \|_{L^2}^2 + 4 \| \nabla \rho \|_{L^2}^2 \right)} \]
for \( 0 < t < T \). This, formally, yields the bound \( \nabla F \in L^\infty(0, T; L^2(\mathbb{R}^2)) \).

Moreover, if we return to (16) we get
\[ \int_0^t \| \nabla^2 \rho \|_{L^2}^2 \leq \| \nabla F_0 \|_{L^2}^2 + \| \rho \|_{L^2}^2 \int_0^t \| \nabla \rho \|_{L^2}^2 + \int_0^t \left( [4 + \| E \|_{L^2}^2 + 4 \| \nabla \rho \|_{L^2}^2] \| \nabla F \|_{L^2}^2 \right) \]
We obtain that \( \nabla^2 \rho \in L^2(0, T; L^2) \). Then, we deduce that \( \nabla \rho \in L^\infty(0, T; L^2 \cap L^2(0, T; H^1)) \) and \( \nabla E, \nabla B \in L^\infty(0, T; L^2 \cap L^2(0, T; H^1)) \).

Now, by using the previous section, we have
\( \rho \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \) and \( E, B \in L^\infty(0, T; H^1) \).

If we denote \( F = (E, B) \) then we get from Lemma 1.3 that for \( t > 0 \):
\[ \| F(t) \|_{H^1} \leq \| F_0 \|_{H^1} + \int_0^t \| j(s) \|_{H^1} ds \]
Using that
\[ j = \rho E - \nabla \rho \text{ and } \nabla j = \nabla \rho E + \rho \nabla E - \nabla^2 \rho, \]
we obtain
\[ \| F(t) \|_{H^1} \leq \| F_0 \|_{H^1} + Z_1(t) + Z_2(t) + Z_3(t) + Z_4(t). \]

Where
\[ Z_1(t) = \| \rho E - \nabla \rho \|_{L^1 L^2}, \ Z_2(t) = \| \nabla \rho E \|_{L^1 L^2}, \]
\[ Z_3(t) = \| \rho \nabla E \|_{L^1 L^2} \text{ and } Z_4(t) = \| \nabla^2 \rho \|_{L^1 L^2}. \]

We have from Cauchy-Schwartz inequality that
\[ Z_1(t) \leq \int_0^t \| \rho(s) \|_{L^\infty} \| F(s) \|_{H^1} ds + \int_0^t \| \nabla \rho(s) \|_{L^2} ds \]
\[ \leq \int_0^t \| \rho(s) \|_{L^\infty} \| F(s) \|_{H^1} ds + t^\frac{1}{2} \left( \int_0^t \| \nabla \rho(s) \|_{L^2}^2 ds \right)^{\frac{1}{2}} \]
\[ \leq \int_0^t \| \rho(s) \|_{L^\infty} \| F(s) \|_{H^1} ds + t^\frac{1}{2} C_0^{\frac{1}{2}}. \]
and we have from Gagliardo-Nirenberg inequality (9) that
\[
Z_2(t) \leq \int_0^t \|\nabla \rho(s)\|_{L^1} \|E(s)\|_{L^4} ds \\
\leq \int_0^t \|\nabla \rho(s)\|_{L^2} \|\nabla^2 \rho(s)\|_{L^2} \|E(s)\|_{L^2} \|\nabla E(s)\|_{L^2} ds \\
\leq \int_0^t \|\nabla^2 \rho(s)\|_{L^2} ds + \int_0^t \|\nabla \rho(s)\|_{L^2} \|E(s)\|_{L^2} \|\nabla E(s)\|_{L^2} ds \\
\leq \int_0^t \|\nabla^2 \rho(s)\|_{L^2} ds + \|E\|_{L^\infty(L^2)} \int_0^t \|\nabla \rho(s)\|_{L^2} \|F(s)\|_{H^1} ds \\
\leq \int_0^t \|\nabla^2 \rho(s)\|_{L^2} ds + C_0 \int_0^t \|\nabla \rho(s)\|_{L^2} \|F(s)\|_{H^1} ds.
\]
Moreover,
\[
Z_3(t) \leq \int_0^t \|\rho(s)\|_{L^\infty} \|F(s)\|_{H^1} ds.
\]
Combining \(Z_1, Z_2, Z_3\) and \(Z_4\), we get
\[
\|F(t)\|_{H^1} \leq \|F_0\|_{H^1} + t^{\frac{1}{2}} C_0^\frac{3}{2} + 2 \int_0^t \|\rho(s)\|_{L^\infty} \|F(s)\|_{H^1} ds \\
+ 2 \int_0^t \|\nabla^2 \rho(s)\|_{L^2} ds + C_0 \int_0^t \|\nabla \rho(s)\|_{L^2} \|F(s)\|_{H^1} ds \\
\leq \|F_0\|_{H^1} + C_0 t^{\frac{1}{2}} \\
+ \int_0^t (2 \|\rho(s)\|_{L^\infty} + C_0 \|\nabla \rho(s)\|_{L^2}) \|F(s)\|_{H^1} ds.
\]
We deduce from Gronwall lemma that
\[
\|F(t)\|_{H^1} \leq \left(\|F_0\|_{H^1} + C_0 t^{\frac{1}{2}}\right) e^{\int_0^t (2 \|\rho(s)\|_{L^\infty} + C_0 \|\nabla \rho(s)\|_{L^2}) ds}.
\]
Then, using the inequality \((\log(e + a e^b)) \leq \log(e + a) + b, a \geq 0, b \geq 0\), we obtain
\[
\log(e + \|F(t)\|_{H^1}) \leq \log(e + \|F_0\|_{H^1}) + C_0 \frac{3}{2} t^{\frac{1}{2}} + C \|\rho\|_{L^1 L^\infty}.
\]
By using inequality (11) and Lemma 1.6, we obtain
\[
\log(e + \|F(t)\|_{H^1}) \leq \log(e + \|F_0\|_{H^1}) + C_0 \frac{3}{2} t^{\frac{1}{2}} \\
+ C t^{1/2} \|\nabla \rho\|_{L^2_{t,x}} \log(e + \frac{\|\nabla^2 \rho\|_{L^2_{t,x}}}{\|\nabla \rho\|_{L^2_{t,x}}}).
\]
Then, we use that the function \(x \to x \log(e + \frac{c_0}{x})\) is increasing in \(x\) to deduce that there exists a \(C_0\) such that for all \(T > 0\), we have
\[
\sup_{0 \leq t \leq T} \log^{\frac{1}{2}}(e + \|F(t)\|_{H^1}) \leq \log^{\frac{1}{2}}(e + \|F_0\|_{H^1}) + C_0 \frac{3}{2} T^{\frac{1}{2}}.
\]
Therefore, there exists a constant \(C\) such that for all \(T > 0\), we have
\[
\log(e + \|F(T)\|_{L^\infty H^1}) \leq C \log(e + \|F_0\|_{H^1}) + C_0 T.
\]
Then, there exists a constant \( D_0 \) depending on \( \|F_0\|_{H^1} \) such that for all \( T > 0 \), we have
\[
\|F(T)\|_{H^1} \leq D_0(e + \|F_0\|_{H^1})e^{C_0T}.
\]

3. Proof of Theorem 1.1

3.1. Existence of solutions. The existence of a solution \((\rho, E, B)\) which solves (1) follows from the a priori estimates proved in the last section. We shall use the very classical Friedrichs method (also called Galerkin method in the periodic case) which consists in approximating the system (1) by a cutoff in the frequency system. For this, let us define the regularization operator \( J_n \) by:
\[
\forall n \in \mathbb{N}, \quad J_n u := \mathcal{F}^{-1}(1_{B(0,n)}(\xi)\hat{u}(\xi)), \tag{17}
\]
where the Fourier transform \( \mathcal{F} \) in the space variables defined by
\[
\mathcal{F}(u)(\xi) := \hat{u}(\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} u(x) dx.
\]

This operator has a regularizing effect since the Plancherel equality allows to write that, for all \( s \geq 0 \):
\[
\exists C > 0, \forall n \in \mathbb{N}, \quad \|J_n u\|_{H^s(\mathbb{R}^d)} \leq C(1 + n)^s \|u\|_{L^2(\mathbb{R}^d)}.
\]
On the other hand, we have by the Lebesgue theorem that for any \( u \in \dot{H}^s(\mathbb{R}^d) \),
\[
\lim_{n \to \infty} \|J_n u - u\|_{\dot{H}^s(\mathbb{R}^d)} = 0,
\]
where
\[
\dot{H}^s(\mathbb{R}^d) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^d) \mid \|u\|_{\dot{H}^s} < \infty \}
\]
and
\[
\|u\|_{\dot{H}^s} = (\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2)^{\frac{1}{2}}.
\]

Let us consider the approximate system:
\[
\begin{align*}
\partial_t \rho_n + J_n \text{div}_x (J_n \rho_n J_n E_n) - \triangle J_n \rho_n &= 0, \quad \text{in } (0, T) \times \mathbb{R}^2, \\
\partial_t E_n - \text{curl}_x J_n B_n &= -j_n, \quad \text{in } (0, T) \times \mathbb{R}^2, \\
\partial_t B_n + \text{curl}_x J_n E_n &= 0, \quad \text{in } (0, T) \times \mathbb{R}^2, \\
\text{div}_x (J_n E_n) &= \partial_n, \quad \text{in } (0, T) \times \mathbb{R}^2, \\
\text{div}_x B_n &= 0, \quad \text{in } (0, T) \times \mathbb{R}^2, \\
j_n = J_n (J_n \rho_n J_n E_n - \nabla_x J_n \rho_n),
\end{align*}
\]
with the following initial data
\[
\rho_n(t = 0) = J_n(\rho_0), \quad B_n(t = 0) = J_n(B_0), \quad E_n(t = 0) = J_n(E_0).
\]
The above system appears as a system of ordinary differential equations on
\[
L^2_n = \{ u \in L^2(\mathbb{R}^2) \mid J_n u = u \}.
\]
Indeed, by using (18) we can transform (20) as a system of ordinary differential equations
\[
\partial_t u_n = F(u_n)
\]
where
\[
u_n = (\rho_n, E_n, B_n)
\]
and
\[
F(u_n) = (J_n \triangle J_n \rho_n - J_n \text{div}_x (J_n \rho_n J_n E_n), \text{curl}_x J_n B_n - j_n, -\text{curl}_x J_n E_n)
\]
So we remark that
\[
\|J_n \triangle J_n \rho_n\|_{L^2(\mathbb{R}^2)} \leq C(1 + n)^2 \|\rho_n\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq C(1 + n)^2 \|u_n\|_{L^2(\mathbb{R}^2)}
\]
by using (18) and the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $s > d/2$, we get
\[
\|J_n \text{div}_x (J_n \rho_n J_n E_n)\|_{L^2(\mathbb{R}^2)} \leq C(1 + n) \|J_n \rho_n J_n E_n\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq C(1 + n) \|J_n \rho_n\|_{L^\infty(\mathbb{R}^2)} \|J_n E_n\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq C_n \|J_n \rho_n\|_{L^1(\mathbb{R}^2)} \|E_n\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq C_n \|\rho_n\|_{L^2(\mathbb{R}^2)} \|E_n\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq C_n \|u_n\|_{L^2(\mathbb{R}^2)}^2
\]
\[
\|\text{curl}_x J_n B_n\|_{L^2(\mathbb{R}^2)} \leq C(1 + n) \sum_{i,j} \|\partial_i J_n B_n^j\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq C(1 + n) \|J_n B_n\|_{H^1(\mathbb{R}^2)}
\]
\[
\leq C_n \|B_n\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq C_n \|u_n\|_{L^2(\mathbb{R}^2)}
\]
\[
\|J_n (J_n \rho_n J_n E_n - J_n \nabla J_n \rho_n)\|_{L^2(\mathbb{R}^2)} \leq C \|J_n \rho_n J_n E_n\|_{L^2(\mathbb{R}^2)} + C \|\nabla J_n \rho_n\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq C_n \|J_n \rho_n\|_{L^\infty(\mathbb{R}^2)} \|J_n E_n\|_{L^2(\mathbb{R}^2)} + C_n \|J_n \rho_n\|_{H^1(\mathbb{R}^2)}
\]
\[
\leq C_n \|\rho_n\|_{L^2(\mathbb{R}^2)} \|E_n\|_{L^2(\mathbb{R}^2)} + C_n \|\rho_n\|_{L^2(\mathbb{R}^2)} + 1
\]
\[
\leq C_n \|\rho_n\|_{L^2(\mathbb{R}^2)} \|\rho_n\|_{L^2(\mathbb{R}^2)} + 1
\]

So, the usual Cauchy-Lipschitz theorem implies the existence of a unique solution $u_n$ of (20) which is in $C^1([0, T_n]; L^2)$ and a strictly positive maximal time $T_n$ which verify:

\[
T_n < \infty \quad \Rightarrow \quad \|u_n(T_n)\|_{L^2} = \infty \quad (21)
\]

But, as $J_n^2 = J_n$, we claim that $J_n u_n$ is also a solution, so uniqueness implies that $J_n u_n = u_n$ and hence, one can remove all the $J_n$ in front of $\rho_n$, $B_n$ and $E_n$ keeping only those in front of nonlinear terms:

\[
\begin{align*}
\partial_t \rho_n + J_n \text{div}_x (\rho_n E_n) - J_n \triangle \rho_n &= 0, & \text{in } (0, T) \times \mathbb{R}^2, \\
\partial_t E_n - \text{curl}_x B_n &= -J_n, & \text{in } (0, T) \times \mathbb{R}^2, \\
\partial_t B_n + \text{curl}_x E_n &= 0, & \text{in } (0, T) \times \mathbb{R}^2, \\
\text{div}_x E_n &= \rho_n, & \text{in } (0, T) \times \mathbb{R}^2, \\
\text{div}_x B_n &= 0, & \text{in } (0, T) \times \mathbb{R}^2, \\
j_n = J_n (\rho_n E_n - \nabla_x \rho_n),
\end{align*}
\]

(22)

The main goal is to prove that $T_n$ can be taken to be equal to $+\infty$ and that we have some local in time estimate which are uniform in $n$. Then, one can pass to the limit and recover a solution of the initial system (1).
As \(J_n\) is a Fourier multiplier, it commutes with constant coefficient differentiations and hence, the energy estimate (12) still holds:

\[
\frac{1}{2}\|\rho_n\|_{L^2}^2 + \|E_n\|_{L^2}^2 + \|B_n\|_{L^2}^2(t) + \int_0^t \|\nabla \rho_n\|_{L^2}^2 \geq \frac{1}{2}\|J_n(\rho_0)\|_{L^2}^2 + \|J_n(E_0)\|_{L^2}^2 \leq \frac{1}{2}\|\rho_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 \leq C_0.
\]

This implies that by (21) the \(L^2\) norm of \((\rho_n, B_n, E_n)\) is controlled and hence, \(T_n = +\infty\) for all \(n \in \mathbb{N}\). Moreover, the estimates performed in the previous section apply in the same way to the system (22) and hence the a priori estimates derived there still hold (with bounds which are independent of \(n\)), namely we have:

\[
\|F_n(t)\|_{H^1} \leq C(e + \|F_0\|_{H^1})e^{C_0 t}
\]

and

\[
\|\rho_n\|_{L^1; H^1} \leq C(e + \|F_0\|_{H^1})e^{C_0 T}.
\]

Now, using that \(E_n, B_n \in L^\infty(0, T; L^2)\) we can see that \(\partial_t B_n = \text{curl} E_n\) is bounded in \(L^\infty(0, T; H^{-1})\) and \(\partial_t E_n = \text{curl} B_n - j_n\) is bounded in \(L^\infty(0, T; H^{-1}) + L^1 H^1\). We also have, \(\partial_t \rho_n = J_n \Delta \rho_n - J_n \text{div}(\rho_n E_n)\). Since we know that \(\rho_n\) is bounded in \(L^2(0, T; H^1)\) then \(\Delta \rho_n\) is bounded in \(L^2(0, T; H^{-1})\) and recall that \(\rho_n\) is bounded in \(L^2(0, T; L^\infty)\) and \(E_n\) is bounded in \(L^\infty(0, T; L^2)\) so due to lemma 1.6 we get that for all \(T > 0\), there exists a constant \(C_T\) such that

\[
\|\partial_t \rho_n\|_{L^2(0, T; H^{-1})} \leq C_T \text{ and } \|\partial_t E_n\|_{L^2(0, T; H^{-1})} \leq C_T.
\]

Hence, extracting a subsequence, standard compactness arguments allow us to pass to the limit in (22). This yields the existence of a solution \((\rho, E, B)\) to (1) (see for instance [15]) with the initial data (2).

### 3.2. Uniqueness of solutions.

Here, we prove the uniqueness of solutions to (1) in \(L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \times L^\infty(0, T; H^1) \times L^\infty(0, T; H^1)\). Actually, we prove here a uniqueness result slightly stronger than the one stated in the theorem since we do not require the continuity in time. This actually is a very small improvement since one can get the continuity just from the fact that \((\rho_i, E_i, B_i)\) solves the system. Take \((\rho_1, E_1, B_1)\) and \((\rho_2, E_2, B_2)\) two solutions of (1) with the same initial condition (2) and such that for \(i = 1, 2\), we have \(\rho_i \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)\) and \(E_i, B_i \in L^\infty(0, T; H^1)\). We denote \(\rho = \rho_2 - \rho_1, E = E_2 - E_1\), \(j = j_2 - j_1\) and \(B = B_2 - B_1\). We have:

\[
\begin{cases}
\partial_t \rho + \nabla_x \rho E + \rho^2 - \Delta \rho = 0, & \text{in } (0, T) \times \mathbb{R}^2, \\
\partial_t E - \text{curl}_x B = -j, & \text{in } (0, T) \times \mathbb{R}^2, \\
\partial_t B + \text{curl}_x E = 0, & \text{in } (0, T) \times \mathbb{R}^2, \\
\text{div}_x E = \rho, & \text{in } (0, T) \times \mathbb{R}^2, \\
\text{div}_x B = 0, & \text{in } (0, T) \times \mathbb{R}^2, \\
j = \rho E - \nabla_x \rho, & \text{in } (0, T) \times \mathbb{R}^2.
\end{cases}
\]

We denote \(X = L^\infty(0, T; L^2) \cap L^2(0, T; H^1)\). We also denote \(Y = X \times L^\infty(0, T; H^1) \times L^\infty(0, T; H^1)\) and use that \(X \subset L^2(0, T; H^1)\).
Multiplying the first equation of (27) by $\rho$ and integrating by parts,
\[
\frac{1}{2} \partial_t \|\rho\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 = -\int \rho \nabla_x \rho \, Edx - \int \rho^3 \, dx
\]
\[
= -\int \nabla_x (\frac{\rho^2}{2}) \, Edx - \int \rho^3 \, dx
\]
\[
= \frac{1}{2} \int \rho^3 \, dx - \int \rho^3 \, dx
\]
\[
= \frac{1}{2} \|\rho\|_{L^\infty} \|\rho\|_{L^2}^2.
\]
Now, integrating the result with respect to $t$. Thus we obtain
\[
\|\rho(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla \rho\|_{L^2}^2 \, d\tau \leq \|\rho_0\|_{L^2}^2 + \int_0^t \|\rho\|_{L^\infty} \|\rho\|_{L^2}^2 \, d\tau.
\] (28)
Hence, by Gronwall lemma, we obtain
\[
\|\rho(t)\|_{L^2}^2 \leq \|\rho_0\|_{L^2}^2 \exp(\int_0^t \|\rho\|_{L^\infty} \, d\tau)
\] (29)
that $\|\rho(t)\|_{L^2} = 0$ since $\rho_0 = 0$. Therefore, we deduce from (28), we get $\|\rho\|_X = 0$. On the other hand, we have,
\[
\|F\|_{L^\infty H^1} \leq C \|j\|_{L^1 H^1}
\]
\[
\leq C \|\rho E\|_{L^1 H^1} + C \|\nabla_x \rho\|_{L^1 H^1}
\]
\[
\leq C \|\rho\|_{L^1 L^\infty} \|E\|_{L^\infty H^1} + C \|\nabla_x \rho\|_{L^3 L^2} + C \|\nabla_x^2 \rho\|_{L^3 L^2}
\]
\[
\leq C \|\rho\|_{L^1 L^\infty} \|F\|_{L^\infty H^1} + C \|\rho\|_{L^1 H^1} + CT^{1/2} \|\nabla_x^2 \rho\|_{L^2 L^2}
\]
\[
\leq CT^{1/2} \|F\|_{L^\infty H^1} + CT^{1/2} \|\rho\|_{L^2 H^1} + CT^{1/2} \|\nabla_x^2 \rho\|_{L^2 L^2}
\]
Choosing $T$ small enough, we get $F = 0$, which yields the uniqueness of the solution on a small time interval. One can then repeat the argument and get the uniqueness on the whole real line.

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