HOMOGENEOUS 4-DIMENSIONAL
KÄHLER–WEYL STRUCTURES

M. BROZOS-VÁZQUEZ, E. GARCÍA-RÍO, P. GILKEY,
AND R. VÁZQUEZ-LORENZO

Abstract. Any pseudo-Hermitian or para-Hermitian manifold of dimension 4 admits a unique Kähler–Weyl structure; this structure is locally conformally Kähler if and only if the alternating Ricci tensor \( \rho_a \) vanishes. The tensor \( \rho_a \) takes values in a certain representation space. In this paper, we show that any algebraic possibility \( \Xi \) in this representation space can in fact be geometrically realized by a left-invariant Kähler–Weyl structure on a 4-dimensional Lie group in either the pseudo-Hermitian or the para-Hermitian setting. MSC 2010: 53A15, 53C15, 15A72.

1. Introduction

Let \( M \) be a smooth manifold of dimension \( m = 2\bar{m} \geq 4 \) with \( H^1(M; \mathbb{R}) = 0 \); we are only really interested in local theory so this cohomology vanishing condition poses no real restriction. Let \( \nabla \) be a torsion free connection on the tangent bundle of \( M \), and let \( g \) be a pseudo-Riemannian metric on \( M \) of signature \((p, q)\).

1.1. Weyl structures. The triple \((M, g, \nabla)\) is said to be a Weyl structure and \( \nabla \) is said to be a Weyl connection if there exists a smooth 1-form \( \phi \) on \( M \) so that

\[
\nabla g = -2\phi \otimes g.
\]

Weyl [34] used these geometries in an attempt to unify gravity with electromagnetism – although this approach failed for physical reasons, the resulting geometries are still of importance and there is a vast literature on the subject. See, for example, [1, 10, 14, 15, 20, 21]; note that the indefinite signature setting is of particular importance [3, 11, 20, 22] as is the complex setting [18, 19, 23]. The field is a vast one and we only cite a few representative recent examples. We introduce the following notational conventions and follow the treatment of [2] (Section 6.5) which is based on work of [6, 13, 28, 29]. Let \( R \) be the curvature operator of a Weyl connection \( \nabla \) and let \( R \) be the associated curvature tensor:

\[
R(x, y)z := (\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]} )z \text{ and } R(x, y, z, w) := g(R(x,y)z, w).
\]

Let \( \rho \) be the Ricci tensor, \( \rho_a \) be the alternating Ricci tensor, and let \( \rho_s \) be the symmetric Ricci tensor:

\[
\rho(x, y) := \text{Tr}\{z \to R(z, x) y\}, \quad \rho_a(x, y) = \frac{1}{2}(\rho(x, y) - \rho(y, x)), \quad \rho_s(x, y) = \frac{1}{2}(\rho(x, y) + \rho(y, x)).
\]

We have the symmetries:

\[
R(x, y, z, w) + R(y, z, x, w) = 0, \\
R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0, \\
R(x, y, z, w) + R(x, y, w, z) = -\frac{4}{m}\rho_a(x, y)g(z, w).
\]

This is a conformal theory; if \( \tilde{g} = e^{2f}g \) is a conformally equivalent metric, then \((M, \tilde{g}, \nabla)\) is again a Weyl structure with associated 1-form \( \tilde{\phi} := \phi - df \). The
Weyl structure is said to be trivial (or integrable) if \( \nabla \) is the Levi-Civita connection of a conformally equivalent metric or, equivalently (given our assumption that \( H^1(M; \mathbb{R}) = 0 \)), if \( d\phi = 0 \). The following formula shows that the Weyl structure is trivial if and only if \( \rho_a = 0 \):

\[
d\phi = -\frac{2}{m}\rho_a. \tag{1.b}
\]

1.2. Complex manifolds. Let \( J_- \) be an almost complex structure on \( M \), i.e. an automorphism of the tangent bundle \( TM \) so that \( J_-^2 = -\text{Id} \). We say that \( J_- \) is integrable if there exist local coordinates \((x_1, \ldots, x_m, y_1, \ldots, y_m)\) on a neighborhood of any point of \( M \) so that

\[
J_- \partial_{x_i} = \partial_{y_i}, \quad \text{and} \quad J_- \partial_{y_i} = -\partial_{x_i}.
\]

We define the Nijenhuis tensor by setting:

\[
N_{J_-}(x, y) := [x, y] + J_-[J_- x, y] + J_-[x, J_- y] - [J_- x, J_- y].
\]

Then \( J_- \) is integrable if and only if \( N_{J_-} \) vanishes. Let \( T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C} \) be the complexified tangent bundle. Let \( \mathcal{W}_\pm \) be the \( \pm\sqrt{-1} \) eigenbundles of \( J_- \):

\[
\mathcal{W}_\pm := \{ Z \in T_{\mathbb{C}}M : J_- Z = \pm \sqrt{-1} Z \} = \mathbb{C} \cdot \{ E \mp \sqrt{-1} J_- E \}_{E \in TM}.
\]

The distribution \( \mathcal{W}_- \) (or, equivalently, \( \mathcal{W}_+ = \mathcal{W}_- \)) determines the almost complex structure \( J_- \). Furthermore, \( J_- \) is integrable if and only if the complex Frobenius condition is satisfied:

\[
[C^\infty(\mathcal{W}_+), C^\infty(\mathcal{W}_-)] \subset C^\infty(\mathcal{W}_-).
\]

1.3. Para-complex manifolds. Let \( J_+ \) be an almost para-complex structure on \( M \), i.e. an automorphism \( J_+ \) of \( TM \) so that \( J_+^2 = \text{Id} \) and so that \( \text{Tr}(J_+) = 0 \). We say that \( J_+ \) is integrable if there exist local coordinates \((x_1, \ldots, x_m, y_1, \ldots, y_m)\) on a neighborhood of any point of \( M \) so that

\[
J_+ \partial_{x_i} = \partial_{y_i}, \quad \text{and} \quad J_+ \partial_{y_i} = \partial_{x_i}, \quad \text{for} \quad 1 \leq i \leq m.
\]

We form the real eigenbundles

\[
\mathcal{W}_\pm := \{ X \in C^\infty(TM) : J_+ X = \pm X \} = \{ E \pm J_+ E \}_{E \in TM}.
\]

Then \( J_+ \) is integrable if and only if \([C^\infty(\mathcal{W}_+), C^\infty(\mathcal{W}_-)] \subset C^\infty(\mathcal{W}_-)\); in contrast to the complex setting, both conditions are required. The Nijenhuis tensor in this context is defined by

\[
N_{J_+}(x, y) := [x, y] - J_+[J_+ x, y] - J_+[x, J_+ y] + [J_+ x, J_+ y];
\]

the two distributions \( \mathcal{W}_+, \mathcal{W}_- \) determine \( J_+ \) and the para-complex structure is integrable if and only if \( N_{J_+} = 0 \).

1.4. (Para)-Kähler–Weyl structures. Let \( g \) be a pseudo-Riemannian metric on \( M \) of signature \((p, q)\). In the complex setting, we assume that \( J_- \) is almost pseudo-Hermitian (i.e. \( J_- g = g \)) and in the para-complex setting, we assume that \( J_+ \) is almost para-Hermitian (i.e. \( J_+ g = -g \)); here, we extend \( J_\pm \) to act naturally on tensors of all types. We shall use the notation \( J_\pm \) as it is a convenient formalism for discussing both geometries in a parallel format; we shall never, however, be considering both the complex setting (-) and the para-complex setting (+) at the same moment.

Let \((M, g, \nabla)\) be a Weyl structure and let \((M, g, J_\pm)\) be a pseudo-Hermitian (-) or a para-Hermitian (+) structure. We say that the quadruple \((M, g, \nabla, J_\pm)\) is a Kähler–Weyl structure if \( \nabla J_\pm = 0 \); this necessarily implies that \( J_\pm \) is integrable.
so we restrict to this setting henceforth. We then have the additional curvature symmetry:
\[
R(x, y, J_{±} z, J_{±} w) = \mp R(x, y, z, w). \tag{1.d}
\]
We say that the Kähler–Weyl structure is trivial (or integrable) if \( \nabla \) is the Levi-Civita connection of some conformally equivalent Kähler metric. As the Kähler–Weyl structure is trivial if and only if the alternating Ricci tensor \( \rho_0 = 0 \), attention is focused on this tensor. Pedersen, Poon, and Swann \cite{27} used work of Vaisman \cite{32, 33} to establish the following result in the positive definite setting; the extension to the pseudo-Hermitian or to the para-Hermitian setting is immediate (see, for example, the discussion in \cite{2});

**Theorem 1.1.** If \( m > 4 \), then any Kähler–Weyl structure is trivial.

Thus only dimension \( m = 4 \) is interesting in this theory. Let
\[
\Omega(x, y) := g(x, J_{±} y)
\]
denote the Kähler form. Let \( \delta \) be the co-derivative. If \( * \) is the Hodge operator and if \( \Omega_{ij,k} \) are the components of \( \nabla \Omega \), then the Lee form \( \delta \Omega \) is given by:
\[
\delta \Omega = -*d*\Omega = g^{ij}\Omega_{ij,k}dx^k.
\]
Note that \( J_{±} \delta \Omega \) is called the anti-Lee form. We refer to Section 3.1 for further details. The following result was established \cite{10} in the Riemannian setting; the proof extends without change to this more general context – we also refer to \cite{8, 9} for another treatment and to \cite{12} for related material.

**Theorem 1.2.** Every para-Hermitian (+) or pseudo-Hermitian (−) manifold of dimension 4 admits a unique Kähler–Weyl structure where we have that \( \phi = \pm J_{±} \delta \Omega \), \( \nabla_{±} y := \nabla^g y + \phi(x)y + \phi(y)x - g(x, y)\phi^* \) (here \( \phi^* \) is the dual vector field), and \( \rho_0 = -dJ_{±}\delta \Omega = -2d\phi \).

1.5. The (para)-unitary group. We study the quadruple \( (TP, gp, J_{±}, R_P) \) where \( P \) is a point of a Kähler–Weyl manifold. We shall eventually be interested in the homogeneous setting and thus the point \( P \) will be inessential. We pass to the algebraic setting and work abstractly. Let \( (V, \langle \cdot, \cdot \rangle, J_{±}) \) be a pseudo-Hermitian (−) or a para-Hermitian (+) vector space. Introduce the following structure groups:
\[
\mathcal{O} = \mathcal{O}(V, \langle \cdot, \cdot \rangle) := \{ T \in GL(V) : T^*\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \},
\]
\[
\mathcal{U} = \mathcal{U}(V, \langle \cdot, \cdot \rangle, J_{±}) := \{ T \in \mathcal{O} : TJ_{±} = J_{±}T \},
\]
\[
\mathcal{U}^* = \mathcal{U}^*(V, \langle \cdot, \cdot \rangle, J_{±}) := \{ T \in \mathcal{O} : TJ_{±} = J_{±}T \text{ or } TJ_{±} = -J_{±}T \}.
\]
There is a natural \( \mathbb{Z}_2 \) valued character \( \chi \) of \( \mathcal{U}^* \) so that
\[
TJ_{±} = \chi(T)J_{±}T \quad \text{for } T \in \mathcal{U}^*.
\]
We let \( \mathcal{O}_0 \) and \( \mathcal{U}_0 \) denote the connected component of the identity.

These groups act on tensors of all types. Let \( \mathfrak{g} \) be the trivial \( \mathcal{U}^* \) module and, by an abuse of notation, let \( \chi \) be the linear representation space corresponding to the character given above. We define the following modules:
\[
S^2_{0,±} = \{ \theta \in S^2 : J_{±}\theta = \mp \theta \} \quad S^2_{±} = \{ \theta \in S^2 : J_{±}\theta = \pm \theta \},
\]
\[
\Lambda^2_{0,±} = \{ \theta \in \Lambda^2 : J_{±}\theta = \mp \theta \} \quad \Lambda^2_{±} = \{ \theta \in \Lambda^2 : J_{±}\theta = \pm \theta \}.
\]
The \( \mathcal{O} \) module decomposition \( V^* \otimes V^* = \Lambda^2 \oplus S^2 \) is an orthogonal direct sum of the alternating and symmetric 2-tensors. We may further decompose:
\[
\Lambda^2 = \chi \otimes \Lambda^2_{0,±} \oplus \Lambda^2_{±}, \quad S^2 = \mathfrak{g} \oplus S^2_{0,±} \oplus S^2_{±} \quad \text{(complex setting)},
\]
\[
\Lambda^2 = \chi \otimes \Lambda^2_{0,±} \oplus \Lambda^2_{±}, \quad S^2 = \mathfrak{g} \oplus S^2_{0,±} \oplus S^2_{±} \quad \text{(para-complex setting)}.
\]
The decompositions of $\Lambda^2$ and $S^2$ given above are into irreducible and inequivalent $U^*$ modules. In the para-Hermitian setting, the modules are not irreducible if we replace $U^*$ by $U$. In the Hermitian setting, these modules are still irreducible but not inequivalent if we replace $U^*$ by $U$. Thus $U^*$ is the appropriate structure group for our purposes. The space of algebraic Kähler–Weyl curvature tensors is given by $S_{2n} \subset \otimes^2 V^*$ if $\sigma$ is defined by imposing the symmetries of Equation (1.1a) and Equation (1.1a). There is $\mathbb{R}_{2n}$ an orthogonal direct sum decomposition of $\mathbb{R}_{2n}$ into

$$\mathbb{R}_{2n} = \left\{ \begin{array}{l} W_1 \oplus W_2 \oplus W_3 \oplus L^2_{0,+} \oplus L^2_+ \text{ (complex setting)} \\ W_1 \oplus W_2 \oplus W_3 \oplus L^2_{0,-} \oplus L^2_- \text{ (para-complex setting)} \end{array} \right\}.$$ 

We have $W_3 = \ker(\rho) \cap \mathbb{R}_{2n}$ and, furthermore, that $\rho_s$ and $\rho_a$ define $U^*$ module isomorphisms $[2] \mathbb{R}_{2n}$:

$$\text{Tr}(\rho) : W_1 \xrightarrow{\approx} \mathbb{R}, \quad \rho_s : W_2 \xrightarrow{\approx} S^2_{0, \pm}, \quad \rho_a : L^2_{0,+} \xrightarrow{\approx} \Lambda^2_{0,+}, \quad \rho_a : L^2_\pm \xrightarrow{\approx} \Lambda^2_{\pm}.$$ 

We note for further reference that in dimension 4 we have:

$$\dim\{W_1\} = 1, \quad \dim\{W_2\} = 3, \quad \dim\{W_3\} = 5, \quad \dim\{\Lambda^2_{0,\pm}\} = 3, \quad \dim\{L^2_\pm\} = 2.$$ 

1.6. Lie groups. Let $G$ be a 4-dimensional Lie group which is equipped with an integrable left invariant complex structure (resp. para-complex structure) and a left invariant pseudo-Hermitian metric (resp. para-Hermitian metric). Then the associated Lie algebra $\mathfrak{g}$ is equipped with an almost complex structure (resp. para-complex structure) with vanishing Nijenhuis tensor and a pseudo-Hermitian (resp. para-Hermitian) inner product $\langle \cdot, \cdot \rangle$. Conversely, given $(\mathfrak{g}, \langle \cdot, \cdot \rangle, J_\pm)$ where $\mathfrak{g} = (\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ is a 4-dimensional Lie algebra equipped with an integrable pseudo-Hermitian ($-$) or para-Hermitian ($+$) structure, there is a unique simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$ so $J_{\pm}$ induces a left invariant integrable complex (resp. para-complex) structure on $G$ and $\langle \cdot, \cdot \rangle$ induces a left-invariant pseudo-Hermitian metric (resp. para-Hermitian metric) on $G$. Thus we can work in the algebraic context henceforth. Fix a pseudo-Hermitian ($-$) or para-Hermitian ($+$) vector space $(V, \langle \cdot, \cdot \rangle, J_{\pm})$. Given $\Xi \in \Lambda^2_{0,+} \oplus \Lambda^2_-$ or $\Xi \in \Lambda^2_{0,-} \oplus \Lambda^2_+$, we shall try to define a bracket $[\cdot, \cdot]$ so that $J_\pm$ is integrable and so that if $\nabla$ is the associated Kähler–Weyl connection, then $\rho_a = \Xi$; by Equation (1.1a) and Theorem (1.2) one has:

$$\rho_a = -2d\delta_\omega = \left\{ \begin{array}{l} dJ_- \delta\Omega \text{ if } J_- \text{ is complex} \\ -dJ_+ \delta\Omega \text{ if } J_+ \text{ is para-complex} \end{array} \right\}.$$ 

The following theorem is the fundamental result of this paper (we refer to [2] for a survey of other results concerning geometric realizability):

**Theorem 1.3.**

(1) Let $(V, \langle \cdot, \cdot \rangle, J_-)$ be a 4-dimensional Hermitian vector space of signature $(0, 4)$. Then every element of $\Lambda^2_{0,+} \oplus \Lambda^2_-$ is realizable by a 4-dimensional Hermitian Lie group.

(2) Let $(V, \langle \cdot, \cdot \rangle, J_+)$ be a 4-dimensional para-Hermitian vector space of signature $(2, 2)$. Then every element of $\Lambda^2_{0,-} \oplus \Lambda^2_+$ is realizable by a 4-dimensional para-Hermitian Lie group.

**Remark 1.4.** The corresponding geometrical realization question without the assumption of homogeneity was established previously in [7, 8]; the question at hand of providing homogeneous examples realizing all such tensors $\Xi$ was posed to us by Prof. Alekseevsky and we are grateful to him for the suggestion. Related questions have been examined previously. See, for example, the discussion in [17] of homogeneous Einstein–Weyl structures on symmetric spaces, the discussion in [4] of (complex) 3-dimensional homogeneous metrics which admit Einstein–Weyl connections,
and the discussion in [3] dealing with (among other matters) the 4-dimensional Einstein–Weyl equations in the homogeneous setting.

Here is a brief outline of the remainder of this paper. In Section 2 we recall some of the geometry of complex and para-complex Lie algebras and in Section 3 we establish Theorem 1.3.

2. A review of complex and para-complex geometry

2.1. The action of the unitary group on $\Lambda_{0,2}^\pm \oplus \Lambda_2^\pm$. If $G$ is a Lie group, let $G_0$ be the connected component of the identity. The following is a useful fact; we omit the proof as it is an entirely elementary computation:

**Lemma 2.1.** Let $(V, \langle \cdot, \cdot \rangle, J_\pm)$ be a pseudo-Hermitian (−) or a para-Hermitian (+) vector space. The natural action of the unitary group $U$ on $\Lambda_{0,2}^\pm \oplus \Lambda_2^\pm$ defines a surjective group homomorphism $\pi$ from $U_0$ to $O(\Lambda_{0,2}^\pm) \oplus O(\Lambda_2^\pm)$.

2.2. Hermitian signature $(0, 4)$. Let $(V, \langle \cdot, \cdot \rangle, J_\pm)$ be a 4-dimensional Hermitian vector space of signature $(0, 4)$. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for $V$ and let $\{e^1, e^2, e^3, e^4\}$ be the associated dual basis for $V^*$. We normalize the choice so that the complex structure $J_-$ and diagonal inner products are given by:

\[
J_- e_1 = e_2, \quad J_- e_2 = -e_1, \quad J_- e_3 = e_4, \quad J_- e_4 = -e_3,
\]

\[
\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = 1,
\]

\[
\langle e^1, e^1 \rangle = \langle e^2, e^2 \rangle = \langle e^3, e^3 \rangle = \langle e^4, e^4 \rangle = \frac{1}{2}.
\]

We define a complex basis $\{Z_1, Z_2, \bar{Z}_1, \bar{Z}_2\}$ for $V_C := V \otimes \mathbb{C}$ and the corresponding complex dual basis $\{Z^1, Z^2, \bar{Z}^1, \bar{Z}^2\}$ for the complex dual space $V_C^*$ by setting:

\[
Z_1 := \frac{1}{2}(e_1 - \sqrt{-1}e_2), \quad Z_2 := \frac{1}{2}(e_3 - \sqrt{-1}e_4),
\]

\[
\bar{Z}_1 := \frac{1}{2}(e_1 + \sqrt{-1}e_2), \quad \bar{Z}_2 := \frac{1}{2}(e_3 + \sqrt{-1}e_4),
\]

\[
Z^1 := (e^1 + \sqrt{-1}e^2), \quad Z^2 := (e^3 + \sqrt{-1}e^4),
\]

\[
\bar{Z}^1 := (e^1 - \sqrt{-1}e^2), \quad \bar{Z}^2 := (e^3 - \sqrt{-1}e^4).
\]

Then we have:

\[
\langle Z_1, Z_1 \rangle = 1, \quad \langle Z^1, Z^1 \rangle = 1, \quad \langle Z_2, \bar{Z}_2 \rangle = 1, \quad \langle Z^2, \bar{Z}^2 \rangle = 1,
\]

\[
J_- Z_1 = \sqrt{-1}Z_1, \quad J_- \bar{Z}_1 = -\sqrt{-1}Z_1,
\]

\[
J_- Z_2 = \sqrt{-1}Z_2, \quad J_- \bar{Z}_2 = -\sqrt{-1}Z_2,
\]

\[
J_- Z^1 = -\sqrt{-1}Z^1, \quad J_- \bar{Z}^1 = -\sqrt{-1}Z^1,
\]

\[
J_- Z^2 = -\sqrt{-1}Z^2, \quad J_- \bar{Z}^2 = -\sqrt{-1}Z^2.
\]

Because $\Omega(Z_j, \bar{Z}_j) = \langle Z_j, J_- \bar{Z}_j \rangle = -\sqrt{-1}$, the Kähler form is given by:

\[
\Omega = -\sqrt{-1}(Z^1 \wedge \bar{Z}^1 + Z^2 \wedge \bar{Z}^2).
\]

Let $[\cdot, \cdot]$ be a Lie bracket on $V_C$. Then $[\cdot, \cdot]$ is integrable if and only if

\[
[Z_1, Z_2] \in \text{Span}\{Z_1, Z_2\}
\]

and $[\cdot, \cdot]$ arises from an underlying real bracket on $V$ if and only if

\[
[x, y] = \langle [\cdot, \cdot] \rangle \text{ for all } x, y \in V_C.
\]

We define a basis $\{\theta_1, \theta_2, \theta_3\}$ for $\Lambda_{0,4}^\pm$ and a basis $\{\theta_4, \theta_5\}$ for $\Lambda_2^\pm$ by setting:

\[
\theta_1 := \sqrt{-1}(Z^1 \wedge \bar{Z}^1 + Z^2 \wedge \bar{Z}^2), \quad \theta_2 := Z^1 \wedge \bar{Z}^1 + Z^2 \wedge \bar{Z}^2,
\]

\[
\theta_3 := \sqrt{-1}(Z^1 \wedge \bar{Z}^1 - Z^2 \wedge \bar{Z}^2), \quad \theta_4 := Z^1 \wedge \bar{Z}^1 - Z^2 \wedge \bar{Z}^2,
\]

\[
\theta_5 := \sqrt{-1}(Z^1 \wedge \bar{Z}^1 - Z^2 \wedge \bar{Z}^2).
\]
The collection \{\theta_1, ..., \theta_5\} is an orthogonal set with the diagonal inner products given by:
\[\langle \theta_1, \theta_1 \rangle = 2, \quad \langle \theta_2, \theta_2 \rangle = 2, \quad \langle \theta_3, \theta_3 \rangle = 2, \quad \langle \theta_4, \theta_4 \rangle = 2, \quad \langle \theta_5, \theta_5 \rangle = 2.\]
Consequently, \(\Lambda^2_{0, +}\) has signature \((0, 3)\) and \(\Lambda^2_{+}\) has signature \((0, 2)\). Let
\[\Theta^{x, y} := \{\Xi \in \Lambda^2_{0, +} \oplus \Lambda^2_+ : |\Xi_0|^2 = x \text{ and } |\Xi_+|^2 = y\}.\]

**Lemma 2.2.** If \((V, \langle \cdot, \cdot \rangle, J_-)\) is a Hermitian inner product space of signature \((0, 4)\), then \(\{\Theta^{x, y}\}_{x \geq 0, y \geq 0}\) are the orbits of \(U\) acting on \(\Lambda^2_{0, +} \oplus \Lambda^2_+\).

**Proof.** The sets \(\Theta^{x, y}\) are the product of a sphere of radius \(x\) in \(\Lambda^2_{0, +}\) and a sphere of radius \(y\) in \(\Lambda^2_+\) and thus represent the orbits of \(O(\Lambda^2_{0, +}) \oplus O(\Lambda^2_+)\) acting on \(\Lambda^2_{0, +} \oplus \Lambda^2_+\). Consequently, by Lemma 2.1 \(U\) acts transitively on these sets. On the other hand, since \(U\) acts orthogonally, \(U\) preserves these sets. \(\square\)

### 2.3. Para-Hermitian signature \((2, 2)\)

Let \((V, \langle \cdot, \cdot \rangle, J_+)\) be a para-Hermitian vector space of signature \((2, 2)\). Choose a basis \(\{e_1, e_2, e_3, e_4\}\) for \(V\) so that the basis is hyperbolic and so that \(J_+\) is diagonalized:
\[\langle e_1, e_3 \rangle = \langle e_2, e_4 \rangle = 1, \quad J_+ e_1 = e_1, \quad J_+ e_2 = e_2, \quad J_+ e_3 = -e_3, \quad J_+ e_4 = -e_4.\]
We then have that
\[\Omega = -e^1 \wedge e^3 - e^2 \wedge e^4.\]
A Lie bracket on \(V\) is integrable if and only if
\[\{e_1, e_2\} \in \text{Span}\{e_1, e_2\} \quad \text{and} \quad \{e_3, e_4\} \in \text{Span}\{e_3, e_4\}.\]

We define an orthogonal basis \(\{\theta_1, \theta_2, \theta_3\}\) for \(\Lambda^2_{0, -}\) and an orthogonal basis \(\{\theta_4, \theta_5\}\) for \(\Lambda^2_+\) by setting:
\[
\begin{align*}
\theta_1 &:= e^1 \wedge e^3 - e^2 \wedge e^4, \quad \theta_2 := e^1 \wedge e^4 + e^2 \wedge e^3, \quad \theta_3 := e^1 \wedge e^3 - e^2 \wedge e^4, \\
\theta_4 &:= e^1 \wedge e^2 + e^3 \wedge e^4, \quad \theta_5 := e^1 \wedge e^2 - e^3 \wedge e^4.
\end{align*}
\]
The diagonal inner products are given by:
\[
\begin{align*}
\langle \theta_1, \theta_1 \rangle = -2, \quad \langle \theta_2, \theta_2 \rangle = -2, \quad \langle \theta_3, \theta_3 \rangle = 2, \\
\langle \theta_4, \theta_4 \rangle = 2, \quad \langle \theta_5, \theta_5 \rangle = -2.
\end{align*}
\]
Thus \(\Lambda^2_{0, -}\) has signature \((2, 1)\) and \(\Lambda^2_+\) has signature \((1, 1)\).

**Lemma 2.3.** Every orbit of the action of \(U\) on \(\Lambda^2_{0, -} \oplus \Lambda^2_+\) contains a representative perpendicular to \(\theta_1\).

**Proof.** \(\{\theta_1, \theta_2, \theta_3\}\) is an orthogonal basis for \(\Lambda^2_{0, -}\) where \(\{\theta_1, \theta_2\}\) are timelike and \(\theta_3\) is spacelike. Lemma 2.3 follows from Lemma 2.4 since \(\pi(U_0)\) contains \(O_0(\Lambda^2_{0, -})\) and the corresponding assertion holds for this group. \(\square\)

### 3. The proof of Theorem 1.3

In Section 3.1 we discuss the Hodge operator. In Section 3.2 we discuss a specific Lie algebra which will be used in Section 3.3 to prove Theorem 1.3 (2) and which will be used in Section 3.4 to prove Theorem 1.3 (1).
3.1. The Hodge $*$ operator. Let $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$ be a basis for $\mathbb{C}^4$ and let $\{\Psi^1, \Psi^2, \Psi^3, \Psi^4\}$ be the corresponding dual basis for the dual vector space. Take a hyperbolic metric whose non-zero components are defined by:

$$\langle \Psi_1, \Psi_3 \rangle = \langle \Psi_2, \Psi_4 \rangle = \langle \Psi^1, \Psi^3 \rangle = \langle \Psi^2, \Psi^4 \rangle = 1.$$ 

This is a convenient notation as it is consistent with previous sections:

1. For Section 2.2, set $\Psi_1 = Z_1$, $\Psi_2 = Z_2$, $\Psi_3 = \bar{Z}_1$, and $\Psi_4 = \bar{Z}_2$.
2. For Section 2.3, set $\Psi_1 = \epsilon_1$, $\Psi_2 = \epsilon_2$, $\Psi_3 = \epsilon_3$, and $\Psi_4 = \epsilon_4$.

Let $*$ be the Hodge operator, let $d\nu = \Psi^1 \wedge \Psi^3 \wedge \Psi^2 \wedge \Psi^4$ be the volume form, and let $\delta$ be the co-derivative. We use the identity

$$\omega_1 \wedge * \omega_2 = g(\omega_1, \omega_2) d\nu$$

to compute:

$$* \Psi^1 = - \Psi^1 \wedge \Psi^2 \wedge \Psi^4,$$
$$* \Psi^2 = \Psi^1 \wedge \Psi^2 \wedge \Psi^3,$$
$$* \Psi^3 = - \Psi^2 \wedge \Psi^3 \wedge \Psi^4,$$
$$* \Psi^4 = - \Psi^1 \wedge \Psi^3 \wedge \Psi^4.$$ 

3.2. An example: We define a complex Lie algebra by setting:

$$[\Psi_1, \Psi_2] = \epsilon_1 \Psi_1, \quad [\Psi_1, \Psi_4] = \alpha_2 \Psi_1, \quad [\Psi_2, \Psi_3] = - \bar{\alpha}_3 \Psi_3,$$

$$(\Psi_2, \Psi_4) = \alpha_2 \Psi_4 - \bar{\alpha}_2 \Psi_3,$$

$$[\Psi_3, \Psi_4] = \bar{\epsilon}_1 \Psi_3.$$ 

We verify that the Jacobi identity is satisfied:

$$[[\Psi_1, \Psi_2], \Psi_3] + [[\Psi_2, \Psi_3], \Psi_1] + [[\Psi_3, \Psi_1], \Psi_2] = 0.$$ 

We define a para-complex structure $J_+$ setting:

$$J_+ \Psi^1 = \Psi^1, \quad J_+ \Psi^2 = \Psi^2, \quad J_+ \Psi^3 = - \Psi^3, \quad J_+ \Psi^4 = - \Psi^4.$$ 

We then have

$$\Omega_+ := -(\Psi^1 \wedge \Psi^3 \wedge \Psi^2 \wedge \Psi^4).$$

We use the formula $d\Psi^i(J_+ \Psi^j, \Psi^k) = -\Psi^i([\Psi^j, \Psi^k])$ to compute:

$$d \Psi^1 = - \epsilon_1 \Psi^1 \wedge \Psi^2 - \alpha_3 \Psi^2 \wedge \Psi^4 + 0, \quad d \Psi^2 = 0,$$
$$d \Psi^3 = - \bar{\alpha}_3 \Psi^2 \wedge \Psi^4 + 0, \quad d \Psi^4 = 0.$$ 

Since $\delta = - * d *$ and $* \Omega = - \Omega$, we have:

$$\delta \Omega_+ = - * d * \Omega_+ = - * d (\Psi^1 \wedge \Psi^3 + \Psi^2 \wedge \Psi^4)$$

$$= * \{ (\epsilon_1 \Psi^1 \wedge \Psi^2 + \alpha_3 \Psi^1 \wedge \Psi^4 + \alpha_2 \Psi^2 \wedge \Psi^4) \wedge \Psi^3 \}$$

$$+ * \{ \Psi^1 \wedge (\bar{\alpha}_3 \Psi^2 \wedge \Psi^3 + \bar{\alpha}_2 \Psi^2 \wedge \Psi^4 - \bar{\epsilon}_1 \Psi^3 \wedge \Psi^4) \}$$

$$= \bar{\alpha}_2 \Psi^1 \wedge (\epsilon_1 + \bar{\alpha}_3) \Psi^2 - \alpha_2 \Psi^3 + (\bar{\epsilon}_1 + \alpha_3) \Psi^4.$$
which by Theorem 1.2 yields
\[ \rho_\alpha = \tilde{a}_2 \varepsilon_1 \Psi^1 \wedge \Psi^2 + \tilde{a}_2 \alpha_1 \Psi^1 \wedge \Psi^4 - \alpha_2 \tilde{a}_1 \Psi^2 \wedge \Psi^4 + \alpha_2 \tilde{a}_1 \Psi^3 \wedge \Psi^4. \] (3.a)

3.3. The proof of Theorem 1.3 (2). We now deal with the para-Hermitian setting. Let \( \Xi \in \mathbb{A}_+ \). By Lemma 2.3 we may assume that the coefficient of \( \theta = e^1 \wedge e^3 - e^2 \wedge e^4 \) in \( \Xi \) vanishes, i.e.
\[ \Xi = \mu_{12} e^1 \wedge e^2 + \mu_{14} e^1 \wedge e^4 + \mu_{23} e^2 \wedge e^3 + \mu_{34} e^3 \wedge e^4. \]

We must show that \( \Xi \) is geometrically realizable by a 4-dimensional para-Hermitian Lie group. We consider the Lie algebra of Section 3.2 where the parameters \( \{ \varepsilon_1, \tilde{\varepsilon}_1, \alpha_2, \alpha_3, \tilde{\alpha}_3 \} \) are real and where we set \( \Psi_1 = e_1; \) this Lie algebra is modeled on \( A_{2.2} \) in the classification of [26] for generic values of the parameters. We apply Equation (3.a). We set \( \alpha_2 = \tilde{\alpha}_2 = 1 \). The remaining parameters are then determined; we complete the proof by taking:
\[ \varepsilon_1 = \mu_{12}, \ \alpha_3 = \mu_{14}, \ \tilde{\alpha}_3 = -\mu_{23}, \ \tilde{\varepsilon}_1 = \mu_{34}. \]

\[ \square \]

3.4. The proof of Theorem 1.3 (1). The question of realizability is invariant under the action of the structure group \( \mathbb{U} \). Thus by Lemma 2.2 only the norms of \( |\Xi_0,+) \) and \( |\Xi_-) \) are relevant in establishing Theorem 1.3 (1). Again, we use the Lie algebra of Section 3.2. We set \( \Psi_1 = Z_1, \ \Psi_2 = Z_2, \ \Psi_3 = \tilde{Z}_1, \ \Psi_4 = \tilde{Z}_2, \) and take \( \tilde{\varepsilon}_1 = \varepsilon_1, \ \tilde{\alpha}_2 = \alpha_2, \) and \( \tilde{\alpha}_3 = \alpha_3 \) to define an underlying real Algebra which is modeled on \( A_{4.12} \) in the classification of [26] for generic values of the parameters; see also related work in [24] [25] [30] [31]. We set \( J_- = \sqrt{-1} J_+ \). We then have \( \Omega_- = \sqrt{-1} \Omega_+ \) so
\[ \phi_- = \frac{-1}{2} J_- \delta \ast \Omega_- = \frac{1}{2} J_+ \delta \ast \Omega_+ = \phi_+, \]
\[ \rho_\alpha = \tilde{a}_2 \varepsilon_1 Z^1 \wedge Z^2 + \tilde{a}_2 \alpha_1 Z^1 \wedge Z^2 - \alpha_2 \tilde{a}_1 Z^2 \wedge \tilde{Z}^1 + \alpha_2 \tilde{a}_1 \tilde{Z}^1 \wedge \bar{Z}^2, \]
\[ |\Xi_0,+)^2 = |\alpha_2|^2 |\alpha_3|^2, \quad |\Xi_-) = |\alpha_2|^2 |\varepsilon_1|^2. \]
If we set \( \alpha_2 = 1, \ \alpha_3 = \sqrt{\varepsilon_1^2}, \) and \( \varepsilon_1 = \sqrt{\varepsilon_1^2}, \) then \( \rho_\alpha \in \Theta^{x,y} \) as desired. \[ \square \]

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MB: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF A CORUÑA, SPAIN
E-mail address: mbrozos@udc.es

EG and RV: FACULTY OF MATHEMATICS, UNIVERSITY OF SANTIAGO DE COMPOSTELA, SPAIN
E-mail address: eduardo.garcia.rio@usc.es and ravazlor@edu.xunta.es

PG: MATHEMATICS DEPARTMENT, UNIVERSITY OF OREGON, EUGENE OR 97403, USA
E-mail address: gilkey@uoregon.edu