Abstract

This is the second of the two related papers analysing origins and possible explanations of a paradoxical phenomenon of the quantum potential (QP). It arises in quantum mechanics (QM) of a particle in the Riemannian $n$-dimensional configurational space obtained by various procedures of quantization of the non-relativistic natural Hamilton systems. Now, the two questions are investigated: 1) Does QP appear in the non-relativistic QM generated by the quantum theory of scalar field (QFT) non-minimally coupled to the space-time metric? 2) To which extent is it in accord with quantization of the natural systems? To this end, the asymptotic non-relativistic equation for the particle-interpretable wave functions and operators of canonical observables are obtained from the primary QFT objects. It is shown that, in the globally-static space-time, the Hamilton operators coincide at the origin of the quasi-Euclidean space coordinates in the both alternative approaches for any constant of non-minimality $\tilde{\xi}$, but a certain requirement of the Principle of Equivalence to the quantum field propagator distinguishes the unique value $\tilde{\xi} = 1/6$. Just the same value had the constant $\xi$ in the quantum Hamiltonians arising from the traditional quantizations of the natural systems: the DeWitt canonical, Pauli-DeWitt quasiclassical, geometrical and Feynman ones, as well as in the revised Schrödinger variational quantization. Thus, QP generated by mechanics is tightly related to non-minimality of the quantum scalar field. Meanwhile, an essential discrepancy exists between the non-relativistic QMs derived from the two alternative approaches: QFT generate a scalar QP, whereas various quantizations of natural mechanics, lead to PQs depending on choice of space coordinates as physical observables and non-vanishing even in the flat space if the coordinates are curvilinear.
Unfinished History and Paradoxes of Quantum Potential.
II. Relativistic Point of View

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1 Introduction

In the accompanying paper under the same general title and the subtitle "I. Non-Relativistic Origin, History and Paradoxes.,” to which I shall refer as (I), the main formalisms of quantization of the natural Hamilton systems were analyzed with interesting and sometimes paradoxical conclusions. The natural systems are those whose Hamilton functions are non-uniform quadratic forms in momenta \( p_a \) with coefficients \( \omega^{ab}(q) \) depending on coordinates \( q^{(a)}, \ a, b, \cdots = 1, \ldots, n \) of configurational space \( V_n \):

\[
H^{(\text{nat})}(q, p; \omega) = \frac{1}{2m} \omega^{ab}(q)p_ap_b + V^{(\text{ext})}(q). \tag{1}
\]

Here and further, the notation is used, which is standard in General Relativity (GR). An important physical representative of this class of systems is the particle moving in an external static gravitational field defined general-relativistically as the metric form of an \( n + 1 \)-dimensional (Lorentzian) space-time \( V_{1,n} \) in the normal Gaussian system of coordinates \( \{x^0 \equiv ct, q^{(a)}\} :\)

\[
ds_{(g)}^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta = c^2 dt^2 - \omega_{ab}(q) dq^a dq^b, \quad \alpha, \beta, \cdots = 0, 1, \ldots n; \tag{2}
\]

Then, this construction is a foliation of \( V_{1,n} \) (frame of reference) by the normal geodesic translations of any space-like hypersurface

\[
\Sigma = \{x \in V_{1,n}, \Sigma(x) = \text{const}, \partial^\alpha \Sigma(x) \partial_{\alpha} \Sigma(x) > 0\} \tag{3}
\]

the interior geometry of which is that of \( V_n \). If a metric tensor \( \omega_{ab} \) does not depend on \( t \), \( V_{1,n} \) is a globally static space-time.

Analysis of various quantization procedures of the generic natural system in [I] has shown that the resulting non-relativistic QMs of a particle do not reconcile with the basic principles
of GR, namely, the Principles of General Covariance and of Equivalence, owing to inevitable appearance of QPs in the Hamilton operators or propagators. In the formers, these QPs are not invariant (not scalars) with respect to general transformations of coordinates $q^a = q'^a(q)$ and they single out persistently the potential term:

$$V^{(qm)}(y) = -\frac{\hbar^2}{2m} \cdot \frac{1}{6} R_{(\omega)}(0) + O(y), \quad (4)$$

at the origin of the quasi-Euclidean (normal Riemannian) coordinates $y^a$, where $R_{(\omega)}(q)$ is the scalar curvature of $V_n$. It contradicts formally to the Principle of Equivalence (PE) in S. Weinberg’s formulation [9] quoted also in [1], Section 3.

In view of this paradoxes, we shall consider now an alternative approach to construction the non-relativistic QM in the globally static $V_{1,n}$, which starts from the general-relativistic quantum theory of a neutral scalar field and produces a non-relativistic QM as the limit for $c^{-1} \to 0$ of the one-quasi-particle sector of an appropriate Fock representation. The initial theory is general-covariant and extraction of QM from it is covariant with respect to transformations of the spatial Gaussian coordinates $q^a$. As concerns PE in quantum theory, the field-theoretical approach shows, in which sense it is satisfied on the relativistic level, and originates the term (4) in the non-relativistic QM.

The paper is organized as follows. In Section 2, a brief exposition of the classical theory of scalar field in $V_{1,n}$ non-minimally coupled to the metric is given. In Section 3 and relation of the energy-momentum tensor in the conformal covariant version of the theory to the Dirac scalar-tensor theory of gravitation is shown. In Sections 4 - 6, the Fock representations of quantum theory of the field is constructed and and relation to PE of the structure on the light conoid of the propagator is considered. Restriction to the time-independent (globally static) case, which is necessary for comparison with conclusions of [1], is considered in Sections 7-8. A logical chain of conclusions of the both papers is given in Section 9.

2 Scalar Field in Riemannian space-time, conformal covariance and Principle of Equivalence

Thus, we start with the (classical) real scalar field $\varphi(x), \quad x \in V_{1,n}$, which satisfies to the so called non-minimal generalization of the standard Klein–Gordon–Fock equation:

$$\Box \varphi + \xi R_{(g)}(x) + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0, \quad \Box \overset{\text{def}}{=} g^{\alpha\beta} \nabla_\alpha \nabla_\beta \equiv (-g)^{-\frac{1}{2}} \partial_\gamma \left((-g)^{\frac{1}{2}} g^{\alpha\beta} \partial_\beta \right). \quad (5)$$

Notation here and in sequel is

- $\nabla_\alpha$ is the covariant derivative in $V_{1,n}$;
- $R_{(g)} = g^{\alpha\beta} R_{(g)\alpha\gamma\beta}$ is the scalar curvature of $V_{1,n}$ and the Riemann-Christoffel curvature tensor is determined so that $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) f_\gamma = R_{(g)\gamma\alpha\beta} f_\delta$ for any twice differentiable 1-form $f_\gamma(x)$;
\[ \eta \equiv \text{const} \] is a (dimensionless) parameter of non-minimality of the coupling of \( \varphi(x) \) to the external gravitation represented by the metric tensor \( g_{\alpha\beta}(x) \); the value \( \eta = 0 \) corresponds to the minimal coupling traditionally adopted in theoretical physics up to the end of 1960s.

Among the arbitrary values of \( \eta \), there is a distinguished value

\[ \eta = \eta_{(\text{conf})}(n) \overset{\text{def}}{=} \frac{n - 1}{4n} \]  

for which eq.(5) is asymptotically conformal covariant for \( m \to 0 \), that is, if \( \varphi(x), x \in V_{1,n} \) is a solution of eq.(5) with \( m = 0 \), then \( \tilde{\varphi}(x) = \frac{\Omega^{1/n}}{\Omega} \varphi(x), x \in \tilde{V}_{1,n} \), is a solution of the same equation in \( \tilde{V}_{1,n} \) whose metric tensor is \( \tilde{g}_{\alpha\beta}(x) = \Omega^2(x) g_{\alpha\beta}(x) \) and \( \Omega(x) \) is an arbitrary sufficiently smooth function. Conformal covariance ensures conformal invariance of eq.(5) and corresponding conservation laws if \( V_{1,n} \) under consideration admits a group of conformal isometries (motions).

The term \( \tilde{\eta} R(g)(x) \) in eq.(5) again, as in the Schrödinger equation with QP, causes the question on PE (see the formulation by S. Weinberg [9] reproduced also in [I]) since the term does not disappear in the quasi-Cartesian coordinates with the origin at \( x \) if \( R(g)(x) \neq 0 \). Some answer on the question gives an investigation of structure of singularities of the Green functions for the field equation (5). First, in 1974, S. Il’in and the present author [11] had shown that for

\[ \lim_{x \to x'} \left\{ G_{V_{1,3}}(x, x'; \tilde{\eta}) - G_{E_{1,3}}(\Gamma(x, x')) \right\} = \frac{\theta(\Gamma(x, x'))}{8\pi}(\tilde{\eta} - \frac{1}{6})R(g)(x') \]  

where \( G_{V_{1,3}}(x, x'; \tilde{\eta}) \) is the classical Green function in \( V_{1,3} \) and \( \Gamma(x, x') \) is the geodesic interval between \( x, x' \). Thus, singularities of \( G_{V_{1,3}}(x, x'; \tilde{\eta}) \) on the light conoid \( \Gamma(x, x') = 0 \) (the locus of isotropic geodesics, emanated from \( x' \) ) are the same as in the tangent space \( E_{1,3} \), ”a locally inertial coordinate system” in Weinberg’s formulation of PE, see [I]. Thus, PE is satisfied in this sense in the classical field theory with \( \tilde{\eta} = \frac{1}{6} \) and \( n = 3 \) (The direct recalculation in \( V_{1,n} \) shows that the same property takes place also for arbitrary \( n \) ). Unfortunately, the authors of [11] had not recognized sufficiently the significance of their result for justification of PE for eq.(5). Therefore, it is not surprising that much later, Sonego and Faraoni [12] have reproduced, in fact, the same result but as a verification of PE.

Generalization of this verification to the quantum theory given by A. A. Grib and E. A. Poberii [19] will be noted in Section 6 after quantization of field \( \varphi \).

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1This property of eq.(5) was first pointed out by R. Penrose [3] but only for \( n = 3 \) and with no consideration of physical consequences. Detailed study of properties and quantization of \( \varphi(x) \) satisfying eq.(5) with \( \tilde{\eta} = \tilde{\eta}_{(\text{conf})}(n) \) for arbitrary dimension \( n \) was done in [II] (see also [5]). Soon after that, eq.(5) attracted a serious attention in theoretical physics and cosmology in its asymptotically conformal covariant as well general non-minimal forms; see the analytic review [6] by V. Faraoni on the role of different values of \( \tilde{\eta} \) in the generalized inflation models in cosmology.
3 Energy-momentum tensor and Dirac scalar-tensor theory

Eq. (5) is the unique linear covariant scalar field equation if one introduces no new dimensional constant into the theory [8]. It follows from variation of \( \varphi \) in the functional of action

\[
A\{g(\cdot), \varphi(\cdot); \tilde{\xi}\} \overset{\text{def}}{=} \int L(x)(-g)^{1/2} \text{d}^{n+1}x; \quad L \overset{\text{def}}{=} \frac{1}{2} \partial^\alpha \varphi \partial_\alpha \varphi - \frac{1}{2} \left( \frac{(mc)^2}{\hbar^2} + \tilde{\xi} R(g) \right) \varphi^2.
\]

Its variation by \( g_{\alpha\beta}(x) \) gives the (metric) energy-momentum tensor

\[
T_{\alpha\beta}(x; \tilde{\xi}) \overset{\text{def}}{=} \frac{\delta A\{g(\cdot), \varphi(\cdot); \tilde{\xi}\}}{\delta g^{\alpha\beta}(x)} = \varphi_\alpha \varphi_\beta - Lg_{\alpha\beta} - \tilde{\xi}(R(g)_{\alpha\beta} - \frac{1}{2} R(g)g_{\alpha\beta} + \nabla_\alpha \partial_\beta - g_{\alpha\beta} \square) \varphi^2,
\]

For solutions of eq.(5), one has

\[
T(x; \tilde{\xi}) \overset{\text{def}}{=} g^{\alpha\beta} T_{\alpha\beta}(x; \tilde{\xi}) = \left( \frac{mc}{\hbar} \right)^2 \varphi^2 + n \left( \tilde{\xi} - \tilde{\xi}_{(\text{conf})}(n) \right) \left( \varphi_\alpha \varphi_\alpha - 2 \left( \tilde{\xi} R(g) + \left( \frac{mc}{\hbar} \right)^2 \right) \varphi^2 \right),
\]

and consequently

\[
T(x; \tilde{\xi}_{(\text{conf})}(n)) = \left( \frac{mc}{\hbar} \right)^2 \varphi^2,
\]

i.e., it has the property which is inherent also for fields with spin 1/2 and 1 and which provides all these fields with the asymptotic conservation laws corresponding to conformal isometries (if any) when \( m \to 0 \). Note also, that \( T_{\alpha\beta}(x; \tilde{\xi}) \neq T_{\alpha\beta}(x; 0) \) even in \( E_{1,n} \) if \( \tilde{\xi} \neq 0 \).

Tensor \( T_{\alpha\beta}(x; \tilde{\xi}_{(\text{conf})}(3)) \) has been re-discovered later and called "a new energy-momentum tensor" by Callan, Coleman and Jackiv [14]. They had postulated \( T_{\alpha\beta}(x; \tilde{\xi}_{(\text{conf})}(3)) \) in the form of eq.(9) for the particular case of \( E_{1,3} \) and generalized it afterwards for \( V_{1,3} \). Their reasoning is evidently an inversion of the straightforward general-relativistic approach with the requirement of the conformal symmetry in [4].

More interesting is that, in 1973, Dirac [13] formulated a scalar-tensor theory of gravitation in relation with his famous hypothesis on large numbers. For \( n = 3 \) and \( \tilde{\xi} = 1/6 \), the integral \( A \) is just the gravitational (geometrical) part of the action integral of the Dirac theory [13], formula (5.2) there. (The full Dirac action integral includes also the electromagnetic \( F_{\mu\nu} F^{\mu\nu} \) and non-linear \( \text{const} \cdot \varphi^4 \) terms.) Therefore, our \( T_{\alpha\beta}(x; 1/6) \) is just the left-hand side of the scalar-tensor Dirac equation. In fact, Dirac had been motivated by simplicity of the trace \( T(x; \tilde{\xi}) \), eq[11], when \( \tilde{\xi} = \tilde{\xi}_{(\text{conf})}(3) \equiv 1/6 \). However, we see that the same reasoning is correct for any \( n \) and, thus, the Dirac theory can be generalized to any \( V_{1,n} \) as a conformal-covariant one. In fact, the theory based on the action integral \( A\{g(\cdot), \varphi(\cdot); \tilde{\xi}_{(\text{conf})}(3)\} \) is used for construction of so called conformal cosmology, an alternative to the standard model, and applied to fit recent data on distant supernovae taken as standard candles, [14] and references therein. Thus, determination of value of \( \tilde{\xi} \) acquires a "practical" interest.
4 Quantization of the scalar field in the general Riemannian space-time

Now, the quantum theory of the field $\varphi(x)$, $x \in V_{1,n}$ (denoted as QFT in sequel) will be formulated to extract from it a structure similar to the non-relativistic QM considered in [4]. The program of construction of a particle-interpreted Fock representation for quantum field $\tilde{\varphi}(x)$, $x \in V_{1,n}$, has been fulfilled in [15] with use of formulations from [17], Chapter 2, and [18], Chapter 3, ("check" over symbols will denote operators in the Fock spaces $\mathcal{F}$). Here, the main points of that program with some improvements including a consideration of PE in QFT will be reproduced in the following four sections for a consecutive statement of the problem and conclusions.

The program starts with complexification $\Phi_c = \Phi \otimes \mathbb{C}$, of the space $\Phi$ of solutions to eq.(5) and a subspace $\Phi'_c \subset \Phi_c$ such that

$$\Phi'_c = \Phi^- \oplus \Phi^+$$

where $\Phi^\pm$ are supposed to be mutually complex conjugate spaces. They are selected so that the conserved (i.e. independent on choice of $\Sigma$) Hermitian sesquilinear form

$$\{\varphi_1, \varphi_2\}_\Sigma \overset{\text{def}}{=} i \int_\Sigma d\sigma^\alpha (\bar{\varphi}_1(x) \partial_\alpha \varphi_2(x) - \partial_\alpha \bar{\varphi}_1(x) \varphi_2(x)),$$

be positive (negative) definite in $\Phi^+(\Phi^-)$, where $d\sigma^\alpha$ is the normal volume element of a Cauchy hypersurface $\Sigma$ induced by the metric of $V_{1,n}$ and determined for an arbitrary vector field $f^\alpha(x)$ and arbitrary interior coordinates $q^a$ on $\Sigma$ by relation

$$f_\alpha d\sigma^\alpha = (-g)^{1/2} \begin{vmatrix} f^0 & f^1 & \cdots & f^n \\ \frac{\partial x^0}{\partial q^0} dx^0 & \frac{\partial x^1}{\partial q^0} dx^1 & \cdots & \frac{\partial x^n}{\partial q^0} dx^n \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial x^n}{\partial q^n} dx^0 & \frac{\partial x^0}{\partial q^n} dx^1 & \cdots & \frac{\partial x^n}{\partial q^n} dx^n \end{vmatrix}$$

The form (13) can thus be considered as an inner product in $\Phi^-$ providing the last with a pre-Hilbert structure.

It is clear that bi-partition (12) of $\Phi_c$ can be done by an infinite set of ways. In $E_{1,n}$ and the globally static $V_{1,n}$, there is a discriminated bi-partition by the positive- and negative-frequency solution owing to existence of the conserved positive definite observable of energy. However, for a time being, the generically time-dependent $V_{1,n}$ makes sense to be considered.

Let, further, $\{\varphi(x; A) \subset \Phi^+ \}$ be a basis enumerated by a multi-index $A$, which has values on a set $\{A\}$ with a measure $\mu(A)$, and orthonormalized with respect to the inner product (13). Then,

$$\tilde{\varphi}(x) = \int_{\{A\}} d\mu(A) \left( \hat{c}^+(A) \varphi(x; A) + \hat{c}^-(A) \varphi(x; A) \right) \equiv \tilde{\varphi}^+(x) + \tilde{\varphi}^-(x),$$

(14)
with the operators \( \hat{c}^+(A) \) and \( \hat{c}^-(A) \) of creation and annihilation of the field modes \( \varphi^-(x; A) \in \Phi^- \) (or, of the quasi-particles), which satisfy the canonical commutation relations

\[
[\hat{c}^+(A), \hat{c}^+(A')] = [\hat{c}^-(A), \hat{c}^-(A')] = 0, \quad \int_{\{A\}} d\mu(A) f(A) [\hat{c}^-(A), \hat{c}^+(A')] = f(A')
\]

for any appropriate function \( f(A) \). They act in the Fock space \( \mathcal{F} \) with the cyclic vector \( \vert 0 \rangle \) (the quasi-vacuum) defined by equations

\[
\hat{c}^-(A) \vert 0 \rangle = 0.
\]

The conservation property of the “scalar product” (13) allows to consider the basis as defined in the present paper is restricted by the traditional conjecture in theoretical physics that only local manifestations of the curvature are taken into account.) Then, it is sufficient to introduce the operators \( \hat{\mathcal{N}} \{ \hat{\varphi}; \Sigma \} \)

correspondingly, \( \vert 0 \rangle = \vert 0; \Sigma \rangle \) and \( \mathcal{F} \equiv \mathcal{F} \{ \Sigma \} \). Then, operators of the basic observables in \( \mathcal{F} \{ \Sigma \} \) can be defined as follows.

The operator of number of quasi-particles

\[
\hat{\mathcal{N}} \{ \hat{\varphi}; \Sigma \} \overset{\text{def}}{=} i \int_{\Sigma} d\sigma^\alpha (\hat{\varphi}^+ \partial_\alpha \hat{\varphi}^- - \partial_\alpha \hat{\varphi}^+ \hat{\varphi}^-) \overset{\text{def}}{=} \int_{\Sigma} d\sigma(x) \hat{N}(x), \quad d\sigma \overset{\text{def}}{=} \frac{\partial_\alpha \Sigma d\sigma^\alpha}{(\partial^\alpha \Sigma \partial_\alpha \Sigma)^{\frac{1}{2}}}
\]

The operator of projection of momentum of field \( \hat{\varphi}(x) \) on a given vector field \( K^\alpha(x) \):

\[
\hat{\mathcal{P}}_K \{ \hat{\varphi}; \Sigma \} \overset{\text{def}}{=} \int_{\Sigma} d\sigma^\alpha K^\beta T_{\alpha\beta}(\hat{\varphi}) \overset{\text{def}}{=}
\]

where and in sequel the colons denote the normal product of operators \( \hat{c}_{\Sigma}^\pm \).

To define a QFT- prototype \( \hat{Q}^{(a)} \{ \hat{\varphi}; \Sigma \} \), \( a, b, \ldots = 1, \ldots n \) of non-relativistic QM position operators \( \hat{q}^a \) which played a basic role in [L], introduce first \( n \) position-type functions \( q^{(a)}(x) \), \( x \in V_{1,n} \) which are defined in [13], Section 2, in terms of fibre bundles. Consideration in the present paper is restricted by the traditional conjecture in theoretical physics that \( V_{1,n} \) is a trivial manifold. (It is equivalent in physics to assumption that only local manifestations of the curvature are taken into account.) Then, it is sufficient to introduce \( q^{(i)}_{\Sigma}(x) \) are scalar functions of \( x^\alpha \) w.r.t. general transformations \( \hat{x}^\alpha = \tilde{x}^\alpha(x) \), which satisfy the conditions

\[
\partial^\alpha \hat{x} \partial_\alpha q^{(i)}_{\Sigma} \bigg|_{\Sigma} = 0, \quad \text{rank} \{ \partial_\alpha q^{(i)}_{\Sigma} \bigg|_{\Sigma} \} = 3.
\]

So, they define a point on the Cauchy hypersurface \( \Sigma = \{ x \in V_{1,3} \mid \Sigma(x) = \text{const} \} \). Their restrictions on \( \Sigma \) can serve as internal coordinates on it.

Assuming that the corresponding QFT–operators \( \hat{Q}^{(i)} \{ \hat{\varphi}; \Sigma \} \) have the same structure as the operators \( \hat{\mathcal{N}} \) and \( \hat{\mathcal{P}}_K \) introduced above, let us impose the following conditions on them:

1. \( \hat{Q}^{(i)} \{ \hat{\varphi}; \Sigma \} \) should be local sesquilinear Hermitean forms in the operators \( \hat{\varphi}^\pm(x) \), and linear functionals of \( q^{(a)}_{\Sigma}(x) \) expressed as invariant integrals over \( \Sigma \).
2. $\mathcal{Q}^{(i)}\{\varphi; \Sigma\}$ should not contain derivatives of $q_{\Sigma}^{(i)}(x)$.

3. $\mathcal{Q}^{(i)}\{\varphi; \Sigma\}$ should lead to the operator of multiplication by $q_{\Sigma}^{(i)}(x)$ in the configuration space of the standard non-relativistic QM, i.e. for $c^{-1} = 0$.

These conditions lead apparently to the following unique set of $n$ operators $\mathcal{Q}^{(i)}$ on $\mathcal{F}$:

$$\mathcal{Q}^{(i)}\{\varphi; \Sigma\} \overset{def}{=} i \int_{\Sigma} d\sigma^a(x) \: q_{\Sigma}^{(i)}(x) \left( \varphi^+(x) \partial^a \varphi^- - \partial^a \varphi^+ \varphi^- - \partial^a \varphi^- \varphi^+ \right)$$

$$\equiv \int_{\Sigma} d\sigma(x) q_{\Sigma}^{(i)}(x) \mathcal{N}(x). \quad (19)$$

This definition, in a certain sense, leads to a generalization for $V_{1,3}$ of the known Newton–Wigner operator of the Cartesian coordinate operators as it is shown in [15], Section 6.

5 One-quasi-particle subspace of Fock space

A normalized one-quasi-particle state vector in $\mathcal{F}\{\varphi; \Sigma\}$ is

$$|\varphi\rangle \overset{def}{=} \{\varphi, \varphi\}_{\Sigma}^{-1/2} \int_{\{A\}} d\mu(A) \{\varphi(., A), \varphi(.)\}_{\Sigma} \hat{c}^+(A) |0; \Sigma\rangle. \quad (20)$$

It determines the field configuration

$$\Phi^- \ni \varphi(x) = \int_{\{A\}} d\mu(A) \{\varphi(., A), \varphi(.)\}_{\Sigma} \varphi(x; A).$$

Obviously $\langle \varphi|\varphi \rangle = 1$.

Consider matrix elements of operators $\mathcal{N}(\varphi; \Sigma)$, $\mathcal{P}_K(\varphi; \Sigma)$ and $\mathcal{Q}^a\{\varphi; \Sigma\}$ between two such states $|\varphi_1\rangle$ and $|\varphi_2\rangle$. Simple calculations with use of Eqs.(16), (9), (19) and (20) give:

$$\langle \varphi_1|\mathcal{N}(\varphi; \Sigma)|\varphi_2 \rangle = \frac{\{\varphi_1, \varphi_2\}_{\Sigma}}{\{\varphi_1, \varphi_1\}_{\Sigma}^{1/2} \{\varphi_2, \varphi_2\}_{\Sigma}^{1/2}}, \quad (21)$$

$$\langle \varphi_1|\mathcal{P}_K(\varphi; \Sigma)|\varphi_2 \rangle = \frac{P_K(\varphi_1, \varphi_2; \Sigma)}{\{\varphi_1, \varphi_1\}_{\Sigma}^{1/2} \{\varphi_2, \varphi_2\}_{\Sigma}^{1/2}} \quad (22)$$

where

$$P_K(\varphi_1, \varphi_2; \Sigma) = \hbar \int_{\Sigma} d\sigma^a \left( \partial^a \varphi_1 K^\beta \partial^\beta \varphi_2 + K^\beta \partial^\beta \varphi_1 \partial^a \varphi_2 \right)$$

$$- K_\alpha \left( \partial^\beta \varphi_1 \partial^\beta \varphi_2 - \left( \frac{mc}{\hbar} \right)^2 \hat{\xi} R(\beta) \right) \varphi_1 \varphi_2$$

$$- \tilde{\xi} \int_{\Sigma} d\sigma^a (\tilde{K}_{\alpha \beta} \partial^a - \nabla^\beta \tilde{K}_{\alpha \beta})(\varphi_1 \varphi_2) \quad (23)$$

where

$$\tilde{K}_{\alpha \beta} \overset{def}{=} \nabla_\alpha K_\beta + \nabla_\beta K_\alpha - \nabla K g_{\alpha \beta}, \quad (24)$$
and

\[
\langle \varphi_1 | \hat{Q}^a \{ \tilde{\varphi}; \Sigma \} | \varphi_2 \rangle = \frac{\{ \varphi_1, q^{(a)}_\Sigma \varphi_2 \} \Sigma}{\{ \varphi_1, \varphi_1 \}^{1/2} \{ \varphi_2, \varphi_2 \}^{1/2}} \tag{25}
\]

These matrix elements are sesquilinear functionals of two functions \( \varphi_1(x), \varphi_2(x) \in \Phi^- \) which are obviously Hermitean in the sense that, given a functional \( Z(\varphi_1, \varphi_2; \Sigma) \), the following equality takes place:

\[
Z(\varphi_1, \varphi_2; \Sigma) = \overline{Z(\varphi_2, \varphi_1; \Sigma)}.
\tag{26}
\]

### 6 Principle of Equivalence in quantum field theory

Representation (14) of the quantum field \( \hat{\phi} \) allows to obtain the causal Green function (or, the propagator of the quasi-particle).

\[
G^{(\text{causal})}_{V_1,n}(x, x'; \tilde{\xi}) \overset{\text{def}}{=} \frac{1}{i} < 0; \Sigma | T(\hat{\phi}(x)\hat{\phi}(x')) | 0; \Sigma >
\]

\[
= G_{V_1,n}(x, x'; \tilde{\xi}) + \frac{i}{2} G^{(1)}_{V_1,n}(x, x'; \tilde{\xi}),
\tag{27}
\]

where \( T \) denotes the chronological product and \( G^{(1)} \) is the Hadamard elementary solution for the field equation (5) which is determined up to a regular solution of (5) \( w(x, x') \) satisfying the initial condition \( w(x, x') \to 0 \) for \( x \to x' \). Since, in general, the definition of quasi-particles and quasi-vacuum depend on choice of the initial Cauchy hypersurface \( \Sigma_0 \), the bi-scalar \( w(x, x') \) does, too, according to definition (27), and determines creation and annihilation of the newly determined quasi-particles when \( \Sigma_0 \) (system of reference) is changed.

Contrary to [11], A. A. Grib and E. A. Poberii [19] studied both terms in eq.(28) together and have obtained that

\[
\lim_{x \to x'} \left\{ G^{(1)}_{V_1,3}(x, x'; \tilde{\xi}) - G^{(1)}_{E_1,3}(\Gamma(x, x')) \right\}
\]

\[
= \lim_{x \to x'} \left\{ \frac{1}{8\pi^2} (2\gamma + \ln |m^2 \Gamma(x, x')|) (\tilde{\xi} - \frac{1}{6}) R(x') + w(x, x') \right\}.
\]

Thus, they have shown directly that the quantum Green function supports PE if \( \xi = 1/6 \). All the works mentioned above are restricted by the case of \( n = 3 \) but re-calculation for arbitrary \( n \) leads to the same result and therefore we come to an important conclusion that the (asymptotic) conformal covariance and PE are in accord only for \( n=3 \) and thus the dimensionality of our real space is distinguished by that.

### 7 From quasi-particles to a quantum point-like particle

Now, our main aim is to extract a counterpart to non-relativistic QM of the natural mechanical systems, that had been considered in [1], from the ambiguous relativistic one-quasi-particle
structure just described, and to compare these two QMs. The space $\Phi^-$ so discriminated could be interpreted on a sufficient physical basis as the space of wave functions of particles instead of the ambiguous notion of a quasi-particle. In $E_{1,3}$ and globally static space-times, there exists an unique decomposition \cite{ worried } such that an irreducible representation of the space-time symmetry is realized on $\Phi^- \Sigma$ but, even in these exceptional cases, one should restore the quantum-mechanical operators on $L^2(V_\alpha; C; d\sigma)$ of canonical observables of coordinates $q^a$ and of momenta $p^a$ conjugate to them; this is not a completely evident task. In sequel the operators in $L^2(V_\alpha; C; d\sigma)$ and its analogs are denoted by "hat" on top; and the superscript "(ft)" denotes objects of the field-theoretical origin. All "hatted" operators act along the hypersurface $\Sigma \ni x$ or its normal geodesic translations $S_\Sigma = \text{const}$ are expressed in terms of projections of covariant derivatives $\nabla_a$ onto these hypersurfaces:

$$D_\alpha \overset{\text{def}}{=} h_\alpha^\beta \nabla_\beta, \quad h_\alpha^\beta \overset{\text{def}}{=} \delta_\alpha^\beta - \partial_\alpha S_\Sigma \partial^\beta S_\Sigma,$$

(i.e. $h_{\alpha \beta}$ is the tensor of projection on $S_\Sigma$). I recall that, up for a time being, we consider non-static $V_{1,n}$ for generality.

Our first task is to construct a map

$$\Phi^- \ni \varphi \rightarrow \psi(x) \in L^2(S_\Sigma; C; \omega^{1/2} d^n q)$$

so that eq.\cite{ worried } would generate Schrödinger -DeWitt-type equation, eq.\cite{ worried } in terms of $\psi(x) \in L^2(\Sigma; C; \omega^{1/2} d^n q)$ so that the inner product in the latter were induced by the scalar product \cite{ worried }. In the generic $V_{1,n}$, map \cite{ worried } can be constructed only as the quasi-non-relativistic asymptotic(i.e. for $c^{-2} \rightarrow 0$). In \cite{ worried }, the space $\Phi^- \Sigma \{S_\Sigma\}$ of the following asymptotic in $c^{-2}$ solutions of eq.\cite{ worried } is taken as $\Phi^-$:

$$\varphi(x; N) = \sqrt{\frac{\hbar}{2mc}} \exp \left(-i \frac{mc}{\hbar} S_\Sigma(x)\right) \hat{V}(x; N) \psi(x; N), \quad N = 0, 1, \ldots.$$  

The objects $\Sigma$, $S_\Sigma$, $\psi$, and $\hat{V}(x)$ are:

- $\Sigma$ is a given Cauchy hypersurface in $V_{1,n}$ as defined by eq. \cite{ worried };
- $S_\Sigma$ is a solution of the Hamilton–Jacobi equation $\partial_n S_\Sigma \partial^a S_\Sigma = 1$, with the initial conditions $S_\Sigma(x)|_{\Sigma} = 0$; any hypersurface $S_\Sigma(x) = \text{const}$ forms a level surface of the normal geodesic flow through $\Sigma$ which plays the role of proper frame of reference for the quantum particle under consideration;
- $\psi(x; N)$ is a solution of the Schrödinger equation

$$i \hbar c(\partial^a S \partial_\alpha + \frac{1}{2} \square S) \psi(x; N) = \left(\hat{H}^{(t)}_{N^-}(x) + O(c^{-2(N+1)})\right) \psi(x; N),$$

$$\hat{H}^{(t)}_{N^-}(x) \overset{\text{def}}{=} \hat{H}^{(t)}_0(x) + \sum_{n=1}^{N} \hat{h}_n(x),$$

$$\hat{H}^{(t)}_0(x) \overset{\text{def}}{=} -\frac{\hbar^2}{2m} \left(\Delta_S(x) - \xi R_g(x) + \frac{1}{2}(\partial^a S \partial_\alpha \square S) + \frac{1}{4}(\square S)^2\right);$$

- $S \text{ } \equiv \text{ } S_\Sigma \text{ } (\text{here and in sequel for simplicity});$
the superscript \((\text{ft})\) denotes the field-theoretical origin of the object. Operators \(\hat{h}_n(x)\) are determined by recurrent relations starting with \(\hat{h}_0 = \hat{H}_0^{(\text{ft})}\); their concrete form is not essential for purposes of the present paper because, finally, it will be concentrated on exactly non-relativistic case of \(N = 0\). Wave functions \(\psi(x; N) \in L^2(S_\Sigma; \mathbb{C}; \text{d}\sigma_S)\) (\(\text{d}\sigma_S\) being defined as in eq.\((16)\) with \(\Sigma \sim S_\Sigma\)) in the following asymptotic sense:

\[\{\varphi_1, \varphi_2\}_S = (\psi_1, \psi_2)_S \overset{\text{def}}{=} \int_S \text{d}\sigma_S \overline{\psi}_1 \psi_2 + O\left(e^{-2(N+1)}\right), \quad \varphi_1, \varphi_2 \in \Phi_N^{-1}\{S\}; \quad (36)\]

- \(\hat{V}(x; N)\) is a differential operator on \(L^2(S_\Sigma; \mathbb{C}; \text{d}\sigma)\) the particular form of which is not important in sequel except that \(\hat{V}(x; N) = \hat{1} + O(e^{-2})\)

All "hatted" operators act along the hypersurface \(S \ni x\) that is they are differential operators containing only the covariant derivatives \(D_a\) along \(S\).

Eq.\((36)\) provides \(\Phi_N^{-1}\{S\}\) with the structure of \(L^2(S_\Sigma; \mathbb{C}; \text{d}\sigma_S)\) and \(\psi\) by the standard Born probabilistic interpretation in each configurational space \(S = \text{const}\), i.e. \(|\psi(x)|^2\) is the probability density to observe the field configuration which may be called "a particle" at the point \(x \in S\). At least, this field configuration satisfies an intuitive idea of the quantum particle as a localizable object.

Further, let \(\hat{O}\) signifies any of the QFT-operators of observables in the Fock representation determined by the space \(\Phi_N^{-1}\{S_\Sigma\}\), which have been introduced above in Section 3. Then, owing to relation \((26)\), the corresponding asymptotically Hermitean quasi-non-relativistic QM-operator \(\hat{O}\) is determined up to an asymptotic unitary transformation by the following general relation:

\[<\varphi_1|\hat{O}|\varphi_2> = (\psi_1\hat{O}_N\psi_2)_S \overset{\text{def}}{=} \int_S \text{d}\sigma_S \overline{\psi}_1 \hat{O}_N \psi_2 + O\left(e^{-2(N+1)}\right), \varphi_1, \varphi_2 \in \Phi_N^{-1}\{S\} \quad (37)\]

\[\hat{O}_N \overset{\text{def}}{=} \hat{O}_0 + \sum_{l=1}^{N} \frac{\hat{a}_l}{(2mc^2)^l}; \quad (38)\]

again, \(\hat{a}_n\) are differential QM-operators along \(S\) determined by recurrence relations starting with \((\hat{O})_0\). The simplest example of the relation is

\[<\varphi_1|\hat{N}(\hat{\varphi}; \Sigma)|\varphi_2> = \frac{(\psi_1, \psi_2 S_\Sigma)(\psi_1, \psi_1 S_\Sigma^1/2, (\psi_2, \psi_2 S_\Sigma^1/2) + O\left(e^{-2(N+1)}\right)}, \quad (39)\]

and hence the operator of the number of particles \(\hat{N}(\hat{\varphi}; \Sigma)\) is represented in the space \(\Psi_N\{S_\Sigma\} \sim L^2(S_\Sigma; \mathbb{C}; \text{d}\sigma_S)\) by the unity operator as it should be in quantum mechanics of a single stable particle.

In the same way, one could determine the asymptotic QM-operators of particle position \(\hat{q}^{a}(x)\) and of projection of momentum on a vector field \(K^{a}(x)\) acting on \(\Psi_N\{S_\Sigma\}\) and along the hypersurface \(S_\Sigma(x) = \text{const}\). The formulae in their generality are somewhat lengthy and I refer for them to \([15]\). Instead, having in view as the main aim, comparison of the present
asymptotic structure of the field-theoretic origin with QM in [I] obtained by quantization of the conservative natural mechanics, I give here a summary of the operators for the case when $V_{1,n}$ is a globally static space-time. In this case, coordinates $x$ can be chosen as $\{x^a\} \sim \{t, q^a\}$ so that the metric of $V_{1,n}$ acquires the form (2), $S = ct$ and

$$R(q)(x) \equiv R(\omega)(x).$$

Then, the asymptotic expansions of the QM-operators of observables can be represented as the formal closed expressions [15]:

$$\hat{H}^{(n)}_\infty = mc^2 \left( \hat{1} + \frac{2\hat{H}_0^{(n)}}{mc^2} \right)^{1/2} - \hat{1}; \quad \hat{H}_0^{(n)} = -\frac{\hbar^2}{2m}(\Delta_S - \tilde{\xi} R(\omega)); \quad \text{(41)}$$

$$\hat{V}_\infty = \left( \hat{1} + \frac{2\hat{H}_0^{(n)}}{mc^2} \right)^{-1/4}; \quad \text{(42)}$$

$$\hat{p}_K(x) = -\frac{i\hbar}{2} \hat{V}_\infty^{-1} \cdot (K^a D_a) \cdot \hat{V}_\infty + \frac{i\hbar}{2} \hat{V}_\infty \cdot (K^a D_a)^{\dagger} \cdot \hat{V}_\infty^{-1}, \quad (K^a \partial_a S = 0); \quad \text{(43)}$$

$$c (\hat{p}_S)^{(i)}(x) = mc^2 \left( \hat{1} + \frac{2\hat{H}_0^{(n)}}{mc^2} \right)^{1/2}, \quad \text{(the energy operator);} \quad \text{(44)}$$

$$\hat{q}_S^{(i)}(x) = q_S^{(i)}(x) \cdot \hat{1} + \frac{1}{2} \left[ \hat{V}_\infty, q_S^{(i)}(x) \right], \hat{V}_\infty^{-1}. \quad \text{(45)}$$

These formulae are of interest for separate investigation when $c^{-1} > 0$. For example, it is seen that operators of coordinates $\hat{q}_S^{(i)}(x)$ do not commute except the case of $S \sim E_n$ and $\hat{q}_S^{(i)}(x) \equiv y^a$, the Cartesian coordinates. However, I shall not dwell on these interesting questions here and pass directly to the non-relativistic QM resulting from this asymptotic structure in the limit $c^{-1} = 0$.

8 Non-relativistic Quantum Mechanics generated by Quantum Field Theory

It is seen that the expressions (41 – 45) are invariant as w.r.t. the point transformations $x^a \rightarrow \tilde{x}^a(x)$ as well as w.r.t. the choice of classical position-type observables $q_S^{(i)}(x) \rightarrow \tilde{q}_S^{(i)}(x)$ generated by the chosen initial $\Sigma$.

The expressions for quantum observables for $c^{-1} = 0$ in terms of arbitrary coordinates $q^a$ on foliums $S$ of $V_{1,n}$ are the following differential operators acting on $\psi(t, q) \in L^2(V_n; \mathbb{C}; \omega^{1/2} d^n q)$:

- the Hamilton operator for Schrödinger equation

$$\hat{H}_0^{(\ell)}(q) = -\frac{\hbar^2}{2m}(\Delta_S(q) - \tilde{\xi} R(\omega)(q) \cdot \hat{1}) \quad \text{(46)}$$

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the operator of projection of momentum on the vector field $K^a(q)$ on $V_n$

$$\hat{p}_K(q) = -i\hbar \left( K^a \nabla_a^{(\omega)} + \frac{1}{2} (\nabla_a^{(\omega)} K^a) \right) \cdot \hat{1} \equiv -i\hbar \frac{1}{\omega^{1/4}} \frac{\partial}{\partial q^a} \cdot \left( \omega^{1/4} K^a \right) \cdot \hat{1}. \quad (47)$$

where $\hat{O}_1 \cdot \hat{O}_2$ denotes the operator product of these operators, $\nabla_a^{(\omega)}$ is the covariant derivative in $S \sim V_n$ (i.e., i.e. w.r.t. the metric tensor $\omega_{bc}$) and $\omega \defeq \det \| \omega_{bc} \|$;

• the position operator

$$\hat{q}^{(i)}(q) \defeq q^{(i)}(q) \cdot \hat{1}. \quad (48)$$

Recall that $q^{(i)}$ are scalar functions of $x^a$ and, thus, of $q^a$, which are subordinated to conditions (18). Thus, operators $\hat{q}^{(i)}(q), \hat{p}_K(q), \hat{H}_0^{(1f)}(q)$, are independent on choice of $q^a$ but depend on choice of scalars $q(i)$ and the vector field $K^a$. In particular, $n$ vectors

$$K^{(i)a}(q) \defeq \omega^{ab} \frac{\partial q^{(i)}}{\partial q^b}. \quad (49)$$

form a basis in the tangent spaces of $V_n$ determined by $q^{(i)}$. Then, if the values of the latter scalars are taken as coordinates $q^a$, i.e.

$$q^a \equiv q^{(a)}(q). \quad (50)$$

Then, $K^{(i)a} = \delta^{(i)a}$ and the brackets in the superscript $(i)$ may be omitted. Finally, we come Pauli’s expression (12) in [1]:

$$\hat{p}_a = -i\hbar \frac{1}{\omega^{1/4}} \frac{\partial}{\partial q^a} \cdot \omega^{1/4}. \quad (51)$$

Though it looks as non-invariant operator w.r.t. transformations of $q^a$, actually it is tightly related to choice of canonically conjugate $q^a$ the values of which are fixed by the scalar functions $q^{(i)}(x)|_{\Sigma}$ and cannot be transformed. Thus, there is no sense to ask, is it an 1-form or not. Actually, it is a form-invariant: if we take another set of scalars $\tilde{q}^{(i)}$ that formalizes measurement of position in the configurational space $\Sigma$ by a complete set of operators $\tilde{\hat{q}}^a$, then other set of momentum operators $\tilde{\hat{p}}_a$ should be taken in the form of eq.(51). Consequently, returning to canonical quantization as in Section 3 of [1] with these changed basic observables gives different QP related to the the canged scalars $q^{(i)}(x)$ formalising observation of a particle position on a folium $S$.

9 Conclusion

Summarizing the main results of the both papers we come to the following logical chain.

1. If the Schrödinger variational quantization procedure [21] is revised so that the canonically conjugate primary quantum observables $\hat{q}^a, \hat{p}_b$ were Hermitean operators (condition of observability), then QP appears in the Hamilton operator, which paradoxically depend on choice of coordinates $q^a$, (see [1], Section 3).
2. Then, it was natural to review and investigate other popular quantization procedures in application to the natural systems. Actually, QP was discovered by DeWitt (1952) in a particular version of canonical quantization and it remarkably coincides with the revised version of Schrödinger quantization. Another versions of canonical quantization, as well as quasi-classical, geometrical and Feynman (path integration) quantizations also generate different QPs with the common property that at the origin of quasi-Euclidean coordinates $y^a$ all these quantization generate QP of the form

$$V^{(qm)}(y) = -\frac{\hbar^2}{2m} \cdot \tilde{\xi} R_{(\omega)}(y) + O(y).$$

(52)

Moreover, the mentioned latter three quantizations as well as the (revised) Schrödinger variational and canonical DeWitt quantizations give

$$\xi = \frac{1}{6},$$

(53)

that is the formula (I)

Generalization of the canonical quantization general (I, Section 4) can give any value of $\xi$ and some form of non-invariant QP persists to appear.

3. If QM of a natural system considered as QM of a particle in an external static gravitational ($n$-dimensional) field presented general-relativistically as $V_{1,n} \sim R \times V_n$, then the term $[52]$ in the Hamiltonian may be considered as a violation of PE in Weinberg’s formulation, see [I], Section 3, if Schrödinger equation may be considered as ”a law of nature” assumed by Weinberg.

4. In view of this discouraging features of QP in the non-relativistic QM of natural systems, an alternative approach to construction of QM of a particle in the generic Riemannian space-time $V_{1,n}$ has been considered. It starts with quantum theory of linear scalar field non-minimally coupled to the metric with the arbitrary constant $\tilde{\xi}$ of non-minimality.

5. Despite that there are a continuum of the Fock representations of the quantum field, the condition of accord with PE of the structure of singularities of the causal Green functions (propagators) fixes uniquely the value of $\tilde{\xi}$ just by eq.(53) for any space dimension $n$. This value coincides with the constant of conformal coupling $\tilde{\xi}(\text{conf}) \equiv (n - 1)/(4n)$ is just for $n = 3$ and our real space-time $V_{1,3}$ is exceptional in this sense.

6. Relation between $\tilde{\xi}$ and $\xi$ from (I) is ascertained by extraction of the non-relativistic QM in $V_n$ from QFT in our alternative approach. It is done by determination of the unique Fock representation the one-quasi-particle sector of which simulate the structure of QMs generated in (I) by quantization of the generic natural system. The result is

$$\tilde{\xi} = \xi,$$

(54)
though $\hat{H}_0^{(r)}(q)$ differs from hamiltonians $\hat{H}(q)$ in (I) obtained by quantization of the natural mechanics by that the former does not contain the part of QP depending on choice of coordinates $q$, that is the terms that are hid in the residual term $O(y)$ in eq.(53).

7. That $\tilde{\xi} = 1/6$ required by PE and Eq.(54) together mean that QP is not an artefact or a mistake and inevitable in the frameworks of the traditional (non-relativistic) quantization formalisms and the canonical quantization of general-relativistic non-minimal scalar field.

Meanwhile, there is a difference between QMs in $V_{1,n}$ in that quantization of the natural systems generates a more complicate QP which does not vanish even in the Euclidean space-time $E_{1,n}$ if curvilinear coordinates are taken as the position observables $q^a$. I have attempted in [I] to interpret this phenomenon as intervention of information on the (speculative) classical position detecting device. into the quantum Hamiltonian. The relativistic theory cannot include information on such a device in principle and takes into account only the local QP in (46). The difference between the two approaches is not a discrepancy, in my opinion, but different particular manifestations of a more deep quantum physics still unknown for us completely but apparently related to the problem of measurement. Recall also that some essential considerations related to the problem are given in the last section of [I].

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