A FORMULA FOR THE EULER CHARACTERISTIC OF $\overline{M}_{2,n}$

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Abstract. In this paper we compute the generating function for the Euler characteristic of the Deligne-Mumford compactification of the moduli space of smooth $n$-pointed genus 2 curves. The proof relies on quite elementary methods, such as the enumeration of the graphs involved in a suitable stratification of $\overline{M}_{2,n}$.

1. Introduction

Let $\overline{M}_{g,n}$, $2g + n - 2 > 0$, denote the Deligne-Mumford compactification of the moduli space of smooth $n$-pointed genus $g$ curves. As explained in [AC], there exists a stratification $\overline{M}_{g,n} \supseteq \partial \overline{M}_{g,n} \supseteq \partial^2 \overline{M}_{g,n} \supseteq \ldots \supseteq \partial^{3g-3+n} \overline{M}_{g,n}$, whose codimension $k$ strata $\partial^k \overline{M}_{g,n}$, $1 \leq k \leq 3g-3+n$, can be thoroughly described by genus $g$ graphs with a finite set $V$ of vertices, a finite set of edges $E$ and characterised in the following way:

- each vertex can have labelled half-edges, which we shall refer to as leaves, with labels in the set $\{1, \ldots, n\}$;
- there exists a map $\gamma : V \to \{0, \ldots, g\}$;
- for each vertex $v$, $2\gamma(v) + h(v) + l(v) - 2 > 0$, where $l(v)$ is the number of edges issuing from $v$ and $h(v)$ is the number of leaves stemming from $v$;
- $g = \sum_{v \in V} \gamma(v) +$ the topological genus of the graph.

Observe that a graph with $\gamma(v) = 0$, for every $v$, and without loops is simply a tree.

Let us now consider $X_j := \partial^j \overline{M}_{g,n} - \partial^{j+1} \overline{M}_{g,n}$, $0 \leq j \leq 3g-4+n$, $\partial^i \overline{M}_{g,n} := \overline{M}_{g,n}$, which is easily seen to be a union of quasi-projective subvarieties $\{X_i\}_{i \in I(j)}$. With each $X_i$ we can associate a graph which describes all the elements contained in a codimension $j$ stratum, but not in its closure. We will make use of this stratification to calculate generating functions for the Euler characteristic of $\overline{M}_{1,n}$, $\overline{M}_{2,n}$. More precisely, given a graph $\Gamma_i$ associated with $X_i$ there exists a morphism $\xi_{\Gamma_i} : \Pi_{v \in V_{\Gamma_i}} \overline{M}_{g(v), h(v)+l(v)} \to \overline{M}_{g,n}$ which is described as follows. Each vertex $v$ of the graph corresponds to a smooth $(h(v)+l(v))$-pointed genus $g(v)$ curve $[C; x_1, \ldots, x_{h(v)}, x_{h(v)+1}, \ldots, x_{l(v)}]$ which is attached to another curve, corresponding to another vertex, through one of the marked points $x_j$, $h(v) + 1 \leq j \leq l(v)$. By the multiplicativity of the Euler characteristic, this means that $\chi(X_i)$ is generically given by the product of Euler characteristics of each moduli space corresponding to vertices of $\Gamma_i$. In some cases, depending on the symmetries of the graph, we should take into account a symmetric group action.
By the additivity of the Euler characteristic, we just have to find out how to enumerate all the graphs involved in the complete description of boundary strata. To this end, we introduce the generating function for trees, which satisfies the following recursive relation:

$$D(t) := t + \sum_{n \geq 2} \chi(M_{0,n+1}) \frac{D(t)^n}{n!},$$

and observe that our graphs are obtained by some particular configurations to which trees are attached (see sections 5 and 7 for these configurations and the definition of graph-type).

We can finally state our main results. Consider the generating function

$$K_g(t) := \sum_{2g+n-2 > 0} \chi(M_{g,n}) \frac{t^n}{n!}. $$

Then the following theorems hold.

**Theorem 1.1.**

$$K_1(t) = \frac{19}{12} D + \frac{23}{24} D^2 + \frac{5}{18} D^3 + \frac{D^4}{24} - \frac{E}{12} - \frac{1}{2} \log (1 - \log (1 + D)).$$

**Theorem 1.2.**

$$K_2(t) = \frac{1}{1440 (1 + D)^2 (E - 1)^3} \left[ -2 D^8 (E - 1)^2 - 24 D^7 (E - 1)^2 (17 E - 7) \
- 3 D^6 (E - 1)^2 (201 E + 259) + 30 D^5 (E - 1)^2 (61 E - 221) \
+ 15 D^4 (631 E^3 - 2640 E^2 + 3395 E - 1386) \
+ 60 D^3 (341 E^3 - 1322 E^2 + 1633 E - 652) \
+ 180 D^2 (138 E^3 - 519 E^2 + 635 E - 254) \
+ 360 D (45 E^3 - 167 E^2 + 206 E - 84) \
+ 60 (73 E^3 - 270 E^2 + 336 E - 144) \right],$$

where we have set $E := \log (1 + D)$.

Although a generating function for $\chi(M_{1,n})$ has already been found by Getzler in [G1], we propose an elementary and simplified algorithm to determine it. This will allow the reader to become more familiar with calculations for the genus 2 case. While preparing this work, we came to know that a generating function for $\chi(M_{g,n})$ has been found by J. Harer; even if our computation works in theory for every genus $g$, at the moment it turns out to be effective only for small genera, but in these cases it provides very direct algebraic formulas for the sought-for generating functions. We also notice that for the computation we need to know the characteristic of the open sets $M_{g,n} \subseteq M_{g,n}$ (when $g = 0, 1, 2$): although a general method to compute $\chi(M_{g,n})$ can be found in [HZ], our paper is self-contained, since we shall make explicit calculations for $\chi(M_{0,n})$, $\chi(M_{1,n})$ and $\chi(M_{2,n})$ on the basis of a convenient stratification as suggested in [AC]. We also checked that our results, for low $n$, coincide with the ones obtained by E. Getzler in [G2], and with some recent computations of him, which he kindly informed us about.
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Many of our explicit computations were performed using Mathematica\(^1\); we are grateful to D. Finocchiaro for his patience in teaching us to use it.

2. Quotients of products of \(M_{0,n}\) and \(M_{1,n}\)

As remarked in the Introduction, we shall compute \(\chi(M_{1,n})\) and \(\chi(M_{2,n})\) summing the contributions provided by the Euler characteristic of the subvarieties \(X_i\) represented by graphs. When computing the characteristic associated with each graph, we have to deal with the characteristic of \(M_{0,n}\) and some of its quotients with respect to the symmetric group action which permutes the marked points. Furthermore, we will need quotients of products of \(M_{0,n}\) with respect to suitable actions of the symmetric group. In addition, these quotients are also useful to describe strata in the stratification introduced in sections 3 and 4 in order to compute directly the characteristic of the open sets \(M_{1,n}\) and \(M_{2,n}\).

This section is devoted to these computations, which are of independent interest and will be carried out using two different techniques. On one side, one can study the \(S_n\)-invariants in the rational cohomology ring of \(M_{0,n}\) (subsection 2.1), and on the other side one can employ geometric arguments involving branched covering (subsection 2.2). In this last subsection, we are going to use some results of section 4, namely the formula for the Euler characteristic of \(M_{1,n}\); nevertheless this formula will be proved without assuming any of the previous results of this paper.

Obviously, we start by recalling the Euler characteristic of \(M_{0,n}\). This is provided by the following

**Theorem 2.1.** For \(n \geq 3\)

\[
\chi(M_{0,n}) = (-1)^{n-3}(n-3)!.
\]

**Proof.** Consider the fibration \(\pi : M_{0,n+1} \to M_{0,n}\) with fiber \(\mathbb{P}^1 - \{n \text{ points}\}\).

This gives the recursive formula

\[
\chi(M_{0,n+1}) = (2-n)\chi(M_{0,n}),
\]

with initial data \(\chi(M_{0,3}) = 1\).

\(\Box\)

Let us now consider the case of the symmetric group \(S_j\) acting on \(M_{0,n}\) (here, for \(j \leq n\), we identify \(S_j\) with a subgroup of \(S_n\)). We notice that when \(n-j \geq 3\) the quotient map \(q : M_{0,n} \to M_{0,n}/S_j\) is unramified, since any automorphism of \(\mathbb{P}^1\) fixing three or more points is the identity. This implies that

\[
\chi(M_{0,n}/S_j) = \frac{\chi(M_{0,n})}{j!} = \frac{(-1)^{n-3}(n-3)!}{j!}.
\]

The following subsection provides a description of the \(S_n\) module \(H^*(M_{0,n}; \mathbb{Q})\) which allows us to compute the Euler characteristic of the quotient spaces \(M_{0,n}/S_n, M_{0,n}/S_{n-1}, M_{0,n}/S_{n-2}\).

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2.1. $S_n$-invariants of $H^*(\mathcal{M}_{0,n}, \mathbb{Q})$. The following well known theorem of Invariant Theory points out the relations between the cohomology ring of $\mathcal{M}_{0,n}/S_j$ ($1 \leq j \leq n$) and the $S_j$-invariants in $H^*(\mathcal{M}_{0,n}, \mathbb{Q})$.

**Theorem 2.2.** Let $X$ be a variety and $G$ a finite group which acts on $X$. Then

$$H^*(X/G, \mathbb{Q}) \cong (H^*(X, \mathbb{Q}))^G.$$  

Let us first recall some results about the symmetric group action on $H^*(\mathcal{M}_{0,n}, \mathbb{Q})$. For every $n \geq 3$ and every $j \leq n$ we will denote by $Ch_j(\mathcal{M}_{0,n})$ (resp. $Ch_j^1(\mathcal{M}_{0,n})$) the character of the $S_j$ representation on $H^*(\mathcal{M}_{0,n}, \mathbb{Q})$ (resp. on $H^i(\mathcal{M}_{0,n}, \mathbb{Q})$). Furthermore, we will denote by $I_j$ and $P_j$ respectively the characters of the trivial and standard representations of $S_j$. Then we have

**Theorem 2.3.** (see [Ga], [Ma]) For every $n \geq 3$,

$$Ch_{n-1}^i(\mathcal{M}_{0,n}) = Ch_{n-1}^1(\mathcal{M}_{0,n-1}) + P_{n-1}Ch_{n-1}^i(\mathcal{M}_{0,n-1}). \quad (2.1)$$

We will also refer to the following theorem which was first obtained by Lehrer in [L] and that can be derived by formula (2.1).

**Theorem 2.4.** For every $n \geq 3$,

$$Ch_{n-1}(\mathcal{M}_{0,n}) = Ind_{S_2}^{S_{n-1}}(I_2).$$

Let us now denote by $(\cdot , \cdot)_{S_j}$ the inner product in the space of class functions on $S_j$ (in the sequel we may omit the subscript $S_j$ if it is clear to which group we are referring to).

**Lemma 2.5.** For $n \geq 3$,

$$(Ch_n(\mathcal{M}_{0,n}), I_n)_{S_n} = 1 \quad (2.2)$$

$$(Ch_{n-1}(\mathcal{M}_{0,n}), I_{n-1})_{S_{n-1}} = 1 \quad (2.3)$$

$$(Ch_{n-2}(\mathcal{M}_{0,n}), I_{n-2})_{S_{n-2}} = n - 2. \quad (2.4)$$

**Proof.** This is a consequence of Theorem 2.4. In fact we can write

$$(Ch_{n-1}(\mathcal{M}_{0,n}), I_{n-1})_{S_{n-1}} = (Ind_{S_2}^{S_{n-1}}(I_2), I_{n-1})_{S_{n-1}}$$

which, by Frobenius reciprocity law, is equal to

$$(I_2, Res_{S_2}^{S_{n-1}}(I_{n-1}))_{S_2} = (I_2, I_2)_{S_2} = 1$$

This gives relation (2.3). As for relation (2.2) we note that, since $Res_{S_{n-1}}^{S_n} (Ch_n(\mathcal{M}_{0,n}))$ is equal to $Ch_{n-1}(\mathcal{M}_{0,n})$, then

$$1 = (Ch_n(\mathcal{M}_{0,n}), I_{n-1})_{S_n} =$$

$$=(Ch_n(\mathcal{M}_{0,n}), Ind_{S_{n-1}}^{S_n}(I_{n-1}))_{S_n} = (Ch_n(\mathcal{M}_{0,n}), I_n + P_n)_{S_n}$$

Since $\dim H^0(\mathcal{M}_{0,n}, \mathbb{Q}) = 1$ we know that $(Ch_n(\mathcal{M}_{0,n}), I_n)_{S_n} \geq 1$. This implies that $(Ch_n(\mathcal{M}_{0,n}), I_n)_{S_n} = 1$ and $(Ch_n(\mathcal{M}_{0,n}), P_n)_{S_n} = 0$.

It remains to prove the last assertion, which can be formulated as

$$(Res_{S_{n-2}}^{S_{n-1}}(Ch_{n-1}(\mathcal{M}_{0,n})), I_{n-2})_{S_{n-2}} = n - 2.$$
Therefore, in the equation (2.5) we find the product its decomposition into irreducibles (see [FH] Chap. 4) is equal to two other irreducible characters.

Ch hypothesis, (\(H\) space of \(Ch\) is at least 1. In fact we then observe that the top cohomology of \(M\) is \(\geq 3\) and we can apply Lemma 2.5. For \(i = 0\) our assertion is trivial. Let us then suppose \(i \geq 1\). From the relation of Theorem 2.1 we deduce

\[
Ch^i_{n-2}(M_{0,n}) = Ch^i_{n-2}(M_{0,n-1}) + \text{Res}^{S_{n-1}}_{S_{n-2}}(P_{n-1})Ch^{i-1}_{n-2}(M_{0,n-1}),
\]

that is to say,

\[
Ch^i_{n-2}(M_{0,n}) = Ch^i_{n-2}(M_{0,n-1}) + (P_{n-2} + I_{n-2})Ch^{i-1}_{n-2}(M_{0,n-1})
\]

(2.5)

If \(i = 1\), we have \(Ch^{i-1}_{n-2}(M_{0,n-1}) = I_{n-2}\), therefore \(I_{n-2}\) appears in the irreducible decomposition of \(Ch^i_{n-2}(M_{0,n})\). If \(i \geq 2\), we observe that, by the inductive hypothesis, \((Ch^{i-1}_{n-2}(M_{0,n-1}), I_{n-3}) = 1\). But by Frobenius reciprocity law

\[
1 = (Ch^{i-1}_{n-3}(M_{0,n-1}), I_{n-3}) = (Ch^{i-1}_{n-2}(M_{0,n-1}), I_{n-2} + P_{n-2}).
\]

Now \((Ch^{i-1}_{n-2}(M_{0,n-1}), I_{n-2}) = 0\) since we have already proven that the only subspace of \(H^*(M_{0,n-1}, \mathbb{Q})\) which affords the trivial representation \(I_{n-2}\) is \(H^0(M_{0,n-1}, \mathbb{Q})\). Then we have

\[
1 = (Ch^{i-1}_{n-2}(M_{0,n-1}), P_{n-2}).
\]

Therefore, in the equation (2.3) we find the product \(P_{n-2}P_{n-2}\) as an addendum: its decomposition into irreducibles (see [FH] Chap. 4) is equal to \(I_{n-2} + P_{n-2}\) plus two other irreducible characters. □
2.2. The geometric method. Let us now apply the fibration method to other quotients of \(\mathcal{M}_{0,n}\). A first case is given by the action of the Klein group \(S_2 \times S_2\) which intertwines two pairs of marked points, i.e.

\[
[\mathbb{P}^1; x_1, \ldots, x_{n-1}, x_n] \to [\mathbb{P}^1; x_1, \ldots, x_n, x_{n-1}].
\]

**Proposition 2.8.** For \(n \geq 4\)

\[
\chi(\mathcal{M}_{0,n}/(S_2 \times S_2)) = \begin{cases} 
0, & n = 4, \\
0, & n = 5, \\
-2, & n = 6, \\
\frac{\chi(\mathcal{M}_{0,n})}{4}, & n \geq 7.
\end{cases}
\]

**Proof.** For \(n \geq 7\), the claim follows easily, since there are at least three fixed points. The remaining cases can be treated in the same way and we show \(\chi(\mathcal{M}_{0,6}/(S_2 \times S_2))\) as a sample case. The quotient map \(\varphi : \mathcal{M}_{0,6} \to \mathcal{M}_{0,6}/(S_2 \times S_2)\) is a \(4-1\) covering ramified over the branch locus

\[\mathcal{B} := \{[\mathbb{P}^1; 0, \infty, 1, -1, b, -b] : b \neq 0, \infty, 1, -1\}\].

In fact, the fiber of a generic point \([\mathbb{P}^1; 0, \infty, 1, a, b, c]\) of \(\mathcal{M}_{0,6}/(S_2 \times S_2)\) contains the points

\[\mathbb{P}^1; 0, \infty, 1, a, b, c,]; [\mathbb{P}^1; 0, \infty, 1, a, c, b], [\mathbb{P}^1; 0, \infty, 1, a, b, c/a], \text{ and}\]

\[\mathbb{P}^1; 0, \infty, 1/a, c/a, b/a].\]

These are distinct points except when \(a = -1, b = -c\) in which case the fiber is made up of two points. We now have the following relation

\(\chi(\mathcal{M}_{0,6}) = 4\chi(\mathcal{M}_{0,6}/(S_2 \times S_2)) - 2\chi(\mathcal{B}).\)

Since the Euler characteristic of the branch locus is \(-1\) (as being isomorphic to \((\mathbb{P}^1 - \{0, \infty, 1, -1\})/S_2\) ), we deduce that \(\chi(\mathcal{M}_{0,6}/(S_2 \times S_2)) = -2.\)

Another important example is given by the action of the dihedral group \(D_4\) on \(\mathcal{M}_{0,n}\). Here we identify \(D_4\) with the subgroup of \(S_4\) generated by \(\sigma\) and \(\tau\) such that

\[\sigma : [\mathbb{P}^1; x_1, x_2, x_3, x_4, \ldots, x_n] \to [\mathbb{P}^1; x_2, x_1, x_3, x_4, \ldots, x_n],\]

\[\tau : [\mathbb{P}^1; x_1, x_2, x_3, x_4, \ldots, x_n] \to [\mathbb{P}^1; x_3, x_4, x_1, x_2, \ldots, x_n].\]

**Proposition 2.9.** For \(n \geq 4\)

\[
\chi(\mathcal{M}_{0,n}/D_4) = \begin{cases} 
0, & n = 4, \\
0, & n = 5, \\
-1, & n = 6, \\
\frac{\chi(\mathcal{M}_{0,n})}{8}, & n \geq 7.
\end{cases}
\]
Proof. The proof proceeds analogously as in Proposition 2.8. In the case \( n = 6 \), the branch locus is exactly the same and so the result follows.

Let us now come to a detailed analysis of those quotients of products we shall use. Take \( n_1, n_2 \geq 3 \) positive integers such that \( n_1 \leq n_2 \) and consider the product \( \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \) endowed with the action of \( S_2 \) which permutes a pair of marked points in each element of both factors of the product.

**Proposition 2.10.**

\[
\chi((\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2})/S_2) = \begin{cases} 
\chi((\mathcal{M}_{0,n_2}/S_2), & n_1 = 3, \\
\chi((\mathcal{M}_{0,n_2}/S_2) - \chi(\mathcal{M}_{0,n_2}), & n_1 = 4, \\
\frac{\chi(\mathcal{M}_{0,n_2})\chi(\mathcal{M}_{0,n_2})}{2}, & n_1 \geq 5.
\end{cases}
\]

**Proof.** Since \( \mathcal{M}_{0,3} \) is a point, the first statement follows easily. On the other hand, when \( n_1 \geq 5 \), the Euler characteristic of the product is simply given by the relation

\[
\chi(\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}) = 2\chi((\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2})/S_2),
\]

since in this case the projection of \( (\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}) \) onto the quotient does not have any branch points. For the second case, we recall that \( \mathcal{M}_{0,4} \) maps onto \( \mathcal{M}_{0,4}/S_2 \) with a branch point, namely \([\mathbb{P}^1; 0, \infty, 1, -1]\). Therefore, because of the action of \( S_2 \) on \( \mathcal{M}_{0,4} \times \mathcal{M}_{0,n_2} \), we have

\[
\chi(\mathcal{M}_{0,4} \times \mathcal{M}_{0,n_2}) = 2\chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,n_2})/S_2) - \chi(U_{n_2}),
\]

where \( U_{n_2} \) is the branch locus of the map \( \mathcal{M}_{0,n_2} \to \mathcal{M}_{0,n_2}/S_2 \). Since

\[
\chi(\mathcal{M}_{0,n_2}) = 2\chi((\mathcal{M}_{0,n_2})/S_2) - \chi(U_{n_2}),
\]

we finally deduce that

\[
\chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,n_2})/S_2) = \chi((\mathcal{M}_{0,n_2}/S_2) - \chi(\mathcal{M}_{0,n_2}).
\]

Let us consider the action of \( S_3 \) on the product \( \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \) by permuting triples of marked points in each element of the product. Then we can prove the following

**Lemma 2.11.** i) \( \chi((\mathcal{M}_{0,3} \times \mathcal{M}_{0,n_2})/S_3) = \chi((\mathcal{M}_{0,n_2}/S_3)) \),

ii) \( \chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,4})/S_3) = 2\chi((\mathcal{M}_{0,5} \times \mathcal{M}_{0,4})/S_3) = 1\chi((\mathcal{M}_{0,5} \times \mathcal{M}_{0,5})/S_3) = 2 \),

iii) \( \chi((\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2})/S_3) = \frac{1}{6}\chi(\mathcal{M}_{0,n_1})\chi(\mathcal{M}_{0,n_2}), \) when at least one of the \( n_i \)'s is greater or equal to 6.

**Proof.** Statements i) and iii) are obvious.

For the proof of ii), let us examine directly what happens when \( (n_1, n_2) = (4, 4) \), \( (n_1, n_2) = (4, 5) \), and \( (n_1, n_2) = (5, 5) \). We first observe that the map

\[
\mathcal{M}_{0,4} \to \mathcal{M}_{0,4}/S_3,
\]

is ramified along the fiber with two points \( p_1 = [\mathbb{P}^1; 0, \infty, 1, \alpha, \alpha^2] \) and \( p_2 = [\mathbb{P}^1; 0, \infty, 1, \alpha^2, \alpha] \), \( \alpha \) a primitive third root of unity, and the fiber with three
points \( q_1 = [\mathbb{P}^1; 0, \infty, 1, -1] \), \( q_2 = [\mathbb{P}^1; 0, \infty, 1, 1/2] \), \( q_3 = [\mathbb{P}^1; 0, \infty, 1, 2] \). This allows to compute directly branch points of the map

\[
\mathcal{M}_{0,4} \times \mathcal{M}_{0,4} \to (\mathcal{M}_{0,4} \times \mathcal{M}_{0,4})/S_3,
\]

yielding that

\[
\chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,4})) = 6\chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,4})/S_2) - 11,
\]

i.e. \( \chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,4})/S_2) = 2 \).

For the other cases, we recall that the map

\[
\mathcal{M}_{0,5} \to \mathcal{M}_{0,5}/S_3
\]

has a fiber with two points, instead of six, namely \( v_1 = [\mathbb{P}^1; 0, \infty, 1, \alpha, \alpha^2] \) and \( v_2 = [\mathbb{P}^1; 0, \infty, 1, \alpha^2, \alpha] \). So when \( (n_1, n_2) = (4, 5) \) branch points are given by \( (p_1, v_1) \), \( (p_2, v_2) \), \( (p_1, v_2) \), \( (p_2, v_1) \); hence it is easy to compute \( \chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,5})/S_3) = 2 \).

In the same way, one can treat the remaining case. Notice that these results could also have been obtained via representation theory of the symmetric group.

We next consider the action on \( \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \) of the group \( S_2 \times S_2 \) generated by the following involutions:

\[
\sigma_1 \left( \left[ \mathbb{P}^1; x_1, \ldots, x_{n_1-1}, x_{n_1} \right], \left[ \mathbb{P}^1; x_1, \ldots, x_{n_2-3}, x_{n_2-2}, x_{n_2-1}, x_{n_2} \right] \right) \rightarrow \left( \left[ \mathbb{P}^1; x_1, \ldots, x_{n_1-1}, x_{n_1} \right], \left[ \mathbb{P}^1; x_1, \ldots, x_{n_2-3}, x_{n_2-2}, x_{n_2-1}, x_{n_2} \right] \right),
\]

\[
\sigma_2 \left( \left[ \mathbb{P}^1; x_1, \ldots, x_{n_1-1}, x_{n_1} \right], \left[ \mathbb{P}^1; x_1, \ldots, x_{n_2-3}, x_{n_2-2}, x_{n_2-1} x_{n_2} \right] \right) \rightarrow \left( \left[ \mathbb{P}^1; x_1, \ldots, x_{n_1-1}, x_{n_1} \right], \left[ \mathbb{P}^1; x_1, \ldots, x_{n_2-3}, x_{n_2-2}, x_{n_2}, x_{n_2-1} \right] \right).
\]

**Proposition 2.12.** Take integers \( n_1, n_2 \) such that \( 3 \leq n_1 < n_2 \). Then

i) \( \chi((\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2})/(S_2 \times S_2)) = \chi(\mathcal{M}_{0,n_2}/(S_2 \times S_2)) \), for \( n_1 = 3 \),

ii) \( \chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,n_2})/(S_2 \times S_2)) = \begin{cases} 
0, & n_2 = 4, \\
1, & n_2 = 6,
\end{cases} 
\]

iii) \( \chi((\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2})/(S_2 \times S_2)) = \frac{1}{2} \chi(\mathcal{M}_{0,n_1}) \chi((\mathcal{M}_{0,n_2})/S_2) \), for \( n_1 \geq 5 \), and \( n_2 \geq 4 \).

**Proof.** i) is obvious. For ii) one needs to recall that the only ramification point in the quotient map \( \mathcal{M}_{0,4} \to \mathcal{M}_{0,4}/(S_2 \times S_2) \) is \([\mathbb{P}^1; 0, \infty, 1, -1]\). In the other cases, ramification points are given by maps introduced in Proposition 2.8. Finally, iii) follows by the fact that a rational point with more than three marked points is automorphism free.

\[ \square \]
Eventually, let us now consider the group $S_2 \times S_2$ acting on $\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \times \mathcal{M}_{0,n_3}$, generated by:

$$
\left( [\mathbb{P}^1; x_1, \ldots, x_{n_1-1}, x_{n_1}], [\mathbb{P}^1; x_1, \ldots, x_{n_2-1}, x_{n_2}], [\mathbb{P}^1; x_1, \ldots, x_{n_3-3}, x_{n_3-2}, x_{n_3-1}, x_{n_3}] \right) \to
\left( [\mathbb{P}^1; x_1, \ldots, x_{n_1}, x_{n_1-1}], [\mathbb{P}^1; x_1, \ldots, x_{n_2-1}, x_{n_2}], [\mathbb{P}^1; x_1, \ldots, x_{n_3-2}, x_{n_3-3}, x_{n_3-1}, x_{n_3}] \right),
$$

$$
\left( [\mathbb{P}^1; x_1, \ldots, x_{n_1-1}, x_{n_1}], [\mathbb{P}^1; x_1, \ldots, x_{n_2-1}, x_{n_2}], [\mathbb{P}^1; x_1, \ldots, x_{n_3-2}, x_{n_3-3}, x_{n_3-1}, x_{n_3}] \right) \to
\left( [\mathbb{P}^1; x_1, \ldots, x_{n_1-1}, x_{n_1}], [\mathbb{P}^1; x_1, \ldots, x_{n_2-1}, x_{n_2}], [\mathbb{P}^1; x_1, \ldots, x_{n_3-3}, x_{n_3-2}, x_{n_3-1}, x_{n_3}] \right),
$$

with $n_1 \geq 3$, $n_3 \geq 4$, $n_2 \geq 3$, $n_1 \leq n_2$. In this case we have

**Proposition 2.13.**

i) $\chi((\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \times \mathcal{M}_{0,n_3})/(S_2 \times S_2)) = \chi((\mathcal{M}_{0,n_2} \times \mathcal{M}_{0,n_3})/(S_2 \times S_2))$, for $n_1 = 3$.

ii) $\chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,n_3})/(S_2 \times S_2)) =
\begin{cases}
-1, & n_3 = 4, \\
0, & n_3 = 5, \\
-2, & n_3 = 6, \\
\frac{1}{2} \chi(\mathcal{M}_{0,n_3}), & n_3 \geq 7.
\end{cases}$

iii) $\chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,n_2} \times \mathcal{M}_{0,n_3})/(S_2 \times S_2)) = \frac{1}{2} \chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,n_3})/S_2) \chi(\mathcal{M}_{0,n_2})$, for $n_2 \geq 5$.

iv) $\chi((\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \times \mathcal{M}_{0,n_3})/(S_2 \times S_2)) = \frac{1}{2} \chi((\mathcal{M}_{0,n_2} \times \mathcal{M}_{0,n_3})/S_2) \chi(\mathcal{M}_{0,n_1})$, for $n_1 \geq 5$.

**Proof.** The only statement to prove is ii). Let us work out the case $n_3 = 5$ as a sample case. The quotient map $\mathcal{M}_{0,4} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,5} \xrightarrow{\varphi} (\mathcal{M}_{0,4} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,5})/(S_2 \times S_2)$ is a degree four map with branch locus the image under $\varphi$ of the following set of points

$$
\{ ([\mathbb{P}^1; \infty, 0, 1, -1], [\mathbb{P}^1; \infty, 0, 1, b, 1 - b], [\mathbb{P}^1; \infty, 0, 1, -1]) : b \neq 0, \infty, 1, 1/2 \}.
$$

Therefore

$$
2 = \chi(\mathcal{M}_{0,4} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,5}) = 4 \chi((\mathcal{M}_{0,4} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,5})/S_2 \times S_2) + 2
$$

and the claim follows.

Let us consider $\mathcal{M}_{1,n}$, $n \geq 2$ and let the group $S_2$ act on it by permuting the last two marked points in each genus 1 $n$-pointed curve.
Proposition 2.14. i) 

\[
\chi(M_{1,n}/S_2) = \begin{cases} 
1, & n = 2, \\
1, & n = 3, \\
1, & n = 4, \\
0, & n = 5, \\
6, & n = 6.
\end{cases}
\]

ii) \(\chi(M_{1,n}/S_2) = \frac{1}{2}\chi(M_{1,n})\), when \(n \geq 7\).

Proof. Since an \(n\)-pointed genus 1 curve has non-trivial automorphisms when \(n \geq 5\), ii) follows easily. On the other hand, each case in i) must be analyzed separately. For \(n = 2\), the action of \(S_2\) is free, since the two pointed elliptic curves \([C; 0, p]\) and \([C; p, 0]\), \(p \neq 0\), are the same because of the structure of \(C\) as \(\mathbb{C}\) modulo a lattice.

We now turn to the case \(n = 3\). Suppose \([C; 0]\) is elliptic and \(p, q\) nonzero distinct points on \(C\). Clearly, if \([C; 0]\) is general, the \(-1\) involution around zero exchanges \(p\) and \(q\). Now write \(C = C/\Lambda\), and let \(p\) and \(q\) be the classes of the two complex numbers \(z\) and \(w\). Suppose first \(\Lambda = \mathbb{Z} + i\mathbb{Z}\). If the automorphism given by multiplication by \(i\) interchanges \(p\) and \(q\), then \(iz \equiv w \mod \Lambda\) and \(iw \equiv z \mod \Lambda\). Thus \(-z \equiv z \mod \Lambda\), i.e. \(p\) is a 2-torsion point. Since \(p\) and \(q\) are distinct, they must be the classes of \(1/2\) and \(i/2\). Suppose next that \(\Lambda = \mathbb{Z} + \omega\mathbb{Z}\), where \(\omega\) is a primitive third root of unity, and let \(\varphi\) be an order 6 automorphism. If \(\varphi\) interchanges \(p\) and \(q\), then \(-p = \varphi^3(p) = q\). In conclusion, the branch locus of \(M_{1,3} \to M_{1,3}/S_2\), consists of an isolated point plus the set \(U\) of isomorphism classes of curves \([C; 0, p, -p]\) such that \(p\) is not a torsion 2-point. Since \(U\) is isomorphic to \(M_{0,5}/S_3\), we finally deduce that

\[
0 = \chi(M_{1,3}) = 2\chi(M_{1,3}/S_2) - 2.
\]

For the other cases, we proceed analogously. When \(n = 4\), the branch locus of the map \(M_{1,4} \to M_{1,4}/S_2\), consists of the isolated point \([C; 0, i/2, 1/2, 1/2 + i/2]\) and the set of points \(U'\) of isomorphism classes of curves \([C; 0, v, p, -p]\), where \(v\) is a torsion 2-point and \(p\) is not a torsion 2-point. Since this stratum is isomorphic to \(M_{0,5}/S_2\), we have

\[
0 = \chi(M_{1,4}) = 2\chi(M_{1,4}/S_2) - 1 - 1.
\]

Other computations are similar and can be worked out by the reader.

We end this subsection with a mixed product. Let \(S_2\) act on \(M_{1,n_1} \times M_{0,n_2}\), \(n_1 \geq 1\), \(n_2 \geq 3\), as follows:

\[
([C; x_1, \ldots, x_{n_1-1}, x_{n_1}], [\mathbb{P}^1; y_1, \ldots, y_{n_2-1}, y_{n_2}]) \mapsto
\]
where $C$ is an elliptic curve.

Similarly to Proposition 2.10 one can prove:

**Proposition 2.15.**

$$
\chi((\mathcal{M}_{1,n_1} \times \mathcal{M}_{0,n_2})/S_2) = \begin{cases} 
\chi(\mathcal{M}_{1,n_1}/S_2), & n_2 = 3, \\
\chi(\mathcal{M}_{1,n_1}/S_2) - \chi(\mathcal{M}_{1,n_1}), & n_2 = 4, \\
\frac{1}{2}\chi(\mathcal{M}_{0,n_2})\chi(\mathcal{M}_{1,n_2}), & n_2 \geq 5.
\end{cases}
$$

3. The Euler characteristic of $\mathcal{M}_{2,n}$

For every $n$-pointed curve $C$ of genus 2, we denote by $\tau$ the hyperelliptic involution. We can stratify $\mathcal{M}_{2,n}$ according to the action of $\tau$ on the marked points. In fact, we are going to decompose $\mathcal{M}_{2,n}$ into a disjoint union of quasi projective subvarieties, which will be doubly indexed, up to isomorphism: the first index counts the number of $\tau$-fixed marked points and the other one counts the couples of points in involution.

In order to fix notation, we say that $\mathcal{M}_{2,n}$ is the union of $a_{j,r}$ subvarieties of type $\{j,r\}$, isomorphic to $U_{j,r}$, hence

$$
\chi(\mathcal{M}_{2,n}) = \sum_{j=0}^{\min(n,6)} \sum_{r=0}^{[n-j]} a_{j,r} \chi(U_{j,r}).
$$

Set

$$
U_{j,r} := \{ [C,p_1,...,p_n] : \tau(p_i) = p_i, i = 1,...,j, \tau(p_{j+2i}) = p_{j+2i-1}, i = 1,...,r, \tau(p_j) \neq p_k \text{ otherwise} \}.
$$

Since the subvarieties isomorphic to $U_{j,r}$ are obtained by permuting the markings, it is easy to see by combinatorial arguments, that

$$
a_{j,r} = \binom{n}{j} \frac{(n-j)!}{2^r(n-j-2r)!r!}.
$$

Let us consider the covering map

$$
f_{j,r} : U_{j,r} \to \frac{\mathcal{M}_{0,n+6-r-j}}{S_{6-j}}
$$

sending the class $[C,p_1,...,p_n]$ to the class

$$
[C/\tau, [p_1],..., [p_j], [p_{j+1}] = [p_{j+2}],..., [p_{j+2r-1}] = [p_{j+2r}], q_1,...,q_{6-j}],
$$

where $\{q_1,...,q_{6-j}\}$ are the ramification points of $\tau$ other than the images of the marked points, and the group $S_{6-j}$ acts permuting exactly these points.

Conversely, given such data, a $n$ pointed genus 2 curve is determined up to the choice of the branch of the covering where the marked points are, and we can say that the map $f_{j,r}$ is a covering of degree $2^{n-j-r-1}$, except for the cases where $j = n$, and $r = 0$, when it is an isomorphism.

We claim that this map is unramified unless $j = 0$, and $n - r = 2$, where the target space is $\frac{\mathcal{M}_{0,8}}{S_6}$. A ramification point of this map implies the existence of an
isomorphism (different from the hyperelliptic involution $\sigma$) between two genus 2, $n$-pointed curves, which represent the same equivalence class modulo $\sigma$; this induces an automorphism of the rational curve $C/\tau$, fixing the classes of the markings, and permuting the classes of the $\tau$-fixed points. Thus one can see that if $j > 0$ or $n - r > 2$, this automorphism is the identity, and the isomorphism between the genus 2 curves is the identity too, since it fixes at least 6 points and is not the hyperelliptic involution.

But if $j = 0$ and $n - r = 2$, our claim is that the ramification locus of $f_{j,r}$ is isomorphic to $\frac{M_{0,4,2}}{S_3}$; observe that there are only three such cases, namely $U_{0,0} \subset M_{2,2}$, $U_{0,1} \subset M_{2,3}$, $U_{0,2} \subset M_{2,4}$. Let us explain the simpler case, namely $f_{0,0} : U_{0,0} \to \frac{M_{0,4,2}}{S_3}$; we need to find out for which curves $(C, p_1, p_2) \in U_{0,0}$ there exists an isomorphism

$$\sigma : (C, p_1, p_2) \to (C, p_1, \tau(p_2)).$$

By Riemann-Hurwitz formula, and uniqueness of the hyperelliptic involution, the curve $C/\sigma$ is elliptic, and the quotient map is ramified exactly over the images of $p_1$ and $\tau(p_1)$. Observe that the two automorphism $\sigma$ and $\tau$ commute, and $\tau$ (resp. $\sigma$) induces an automorphism of $C/\sigma$ (resp. $C/\tau$). Everything matches in the following commutative diagram:

$$\begin{array}{ccc}
C & \xrightarrow{\phi} & C/\tau \\
\psi \downarrow & & \overline{\psi} \downarrow \\
C/\sigma & \xrightarrow{\overline{\sigma}} & C/\langle \sigma, \tau \rangle
\end{array}$$

By the conditions on $\sigma$ and $\tau$, the map $\overline{\sigma}$ is ramified over $\overline{\psi}(p_2)$, and over the images of the ramification points of $\phi$; since the map $\overline{\psi}$ has degree 2, these six points form exactly three fibers of it; the last point we have to mark on $C/\langle \sigma, \tau \rangle$ is $\overline{\psi}(p_1)$; hence a genus 2, 2-pointed curve satisfying our requirements determines a genus 0 curve, with 5 marked points, three of which are indistinguishable. Conversely, given a point in $\frac{M_{0,4,2}}{S_3}$, a point $(C, p_1, p_2) \in U_{0,0} \subset M_{2,2}$ matching our conditions is uniquely determined by building $C/\tau$ and $C/\sigma$ as ramified coverings over the marked points of the rational curve, namely, by reversing the construction. This proves our claim.

Using the results of section 2, we have that

$$\chi \left( \frac{M_{0,n+6-r-j}}{S_6-j} \right) = \begin{cases} 
(-1)^{n+3-j-r} \frac{(n+3-j-r)!}{(6-j)!}, & \text{for } n - r \geq 3, \\
0, & \text{for } n - r = 2, j \text{ even}, \\
1, & \text{for } n - r = 2, j \text{ odd}, \\
1, & \text{for } n - r = 1, \\
1, & \text{for } n - r = 0.
\end{cases}$$

If $n \geq 7$, then the fiber of the universal curve

$$M_{2,n+1} \to M_{2,n}$$
is a genus 2 curve without \( n \) points, hence has Euler characteristic \(-(n + 2)\); for \( n \geq 8 \),

\[
\chi (\mathcal{M}_{2,n}) = \prod_{h=7}^{n-1} -(2 + h)\chi (\mathcal{M}_{2,7}) = \\
= (-1)^{n-1} - (n - 1 + 2)!\frac{\chi (\mathcal{M}_{2,7})}{8!} = \\
= (-1)^{n+1} (n + 1)!\frac{\chi (\mathcal{M}_{2,7})}{8!}.
\]

Moreover, for the case \( n = 7 \), the fiber of the universal curve

\[\mathcal{M}_{2,7} \rightarrow \mathcal{M}_{2,6}\]

over \( \mathcal{M}_{2,6}\backslash U_{6,0} \) is a genus 2 curve without 6 points, of characteristic \(-8\), and over \( U_{6,0} \) is a genus 2 curve without the 6 points fixed by the involution, modulo the involution itself, hence a genus 0 curve without 6 points, of characteristic \(-4\); therefore, since \( U_{6,0} \cong \mathcal{M}_{0,6} \), we get

\[
\chi (\mathcal{M}_{2,7}) = -8 (\chi (\mathcal{M}_{2,6}) - \chi (\mathcal{M}_{0,6})) - 4\chi (\mathcal{M}_{0,6}) + \\
= -8\chi (\mathcal{M}_{2,6}) - 24.
\]

Since we have all the ingredients, we begin computing directly the cases \( n = 1, \ldots, 6 \).

\[
\chi (\mathcal{M}_{2,0}) = \chi \left( \frac{\mathcal{M}_{0,6}}{S_6} \right) = 1.
\]

\[
\chi (\mathcal{M}_{2,1}) = \sum_{j=0}^{1} 2^{-j} \chi \left( \frac{\mathcal{M}_{0,7-j}}{S_{6-j}} \right) + \frac{1}{2} \chi \left( \frac{\mathcal{M}_{0,6}}{S_5} \right) \\
= \chi \left( \frac{\mathcal{M}_{0,7}}{S_6} \right) + \chi \left( \frac{\mathcal{M}_{0,6}}{S_5} \right) = 2.
\]

\[
\chi (\mathcal{M}_{2,2}) = 2\chi \left( \frac{\mathcal{M}_{0,8}}{S_6} \right) - \chi \left( \frac{\mathcal{M}_{0,5}}{S_3} \right) + \chi \left( \frac{\mathcal{M}_{0,7}}{S_6} \right) \\
+ 2\chi \left( \frac{\mathcal{M}_{0,7}}{S_5} \right) + \chi \left( \frac{\mathcal{M}_{0,6}}{S_4} \right) = 2.
\]

\[
\chi (\mathcal{M}_{2,3}) = 4 \cdot \chi \left( \frac{\mathcal{M}_{0,9}}{S_6} \right) + 6 \cdot \chi \left( \frac{\mathcal{M}_{0,8}}{S_6} \right) - 3\chi \left( \frac{\mathcal{M}_{0,5}}{S_3} \right) \\
+ 6\chi \left( \frac{\mathcal{M}_{0,8}}{S_5} \right) + 3\chi \left( \frac{\mathcal{M}_{0,7}}{S_3} \right) + 3\chi \left( \frac{\mathcal{M}_{0,7}}{S_4} \right) + \chi \left( \frac{\mathcal{M}_{0,6}}{S_3} \right) = 0.
\]

With the same kind of computations, we get that \( \chi (\mathcal{M}_{2,4}) = -4 \), \( \chi (\mathcal{M}_{2,5}) = 0 \), and \( \chi (\mathcal{M}_{2,6}) = -24 \). Finally,

\[
\chi (\mathcal{M}_{2,7}) = -8\chi (\mathcal{M}_{2,6}) - 24 = 168.
\]
From this we can conclude that, for \( n \geq 7 \),
\[
\chi(M_{2,n}) = (-1)^{n+1} (n+1)! \frac{\chi(M_{2,7})}{8!} = (-1)^{n+1} \frac{(n+1)!}{240}.
\]

In the following table we list our results:

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | \( \geq 7 \) |
|---------|---|---|---|---|---|---|---|---------|
| \( \chi(M_{2,n}) \) | 1 | 2 | 2 | 0 | -4 | 0 | -24 | \( (-1)^{n+1} \frac{(n+1)!}{240} \) |

4. The Euler characteristic of \( M_{1,n} \)

A similar calculation can be done for the Euler characteristic of \( M_{1,n} \) (see [AC]). In this case, \( \tau \) denotes the hyperelliptic involution around the last marked point, and the subvarieties \( U_{j,r} \) have the same definition, assuming that \( j \) counts the \( \tau \)-fixed marked points except the last one. In this case, for \( n \geq 5 \), the curves have no automorphisms, and we get

\[
\chi(M_{1,n}) = \prod_{h=5}^{n-1} -h \chi(M_{1,5}) = (-1)^{n-1} \frac{(n-1)! \chi(M_{1,5})}{4!};
\]

moreover, the fiber of the universal curve
\( M_{1,5} \to M_{1,4} \),
over \( M_{1,4} \setminus U_{3,0} \) is a genus 1 curve without 4 points, of characteristic \(-4\), and over \( U_{3,0} \) a genus 0 curve without 4 points, of characteristic \(-2\); we then get

\[
\chi(M_{1,5}) = -4 (\chi(M_{1,4}) - \chi(M_{0,4})) - 2 \chi(M_{0,4})
= -4 \chi(M_{1,4}) - 2.
\]

The formula we get for \( n \leq 3 \) is

\[
\chi(M_{1,n+1}) = \sum_{j=0}^{n} \sum_{r=0}^{\left\lfloor \frac{n-j}{2} \right\rfloor} a_{j,r} \chi(U_{j,r})
= \sum_{j=0}^{n} \sum_{r=0}^{\left\lfloor \frac{n-j}{2} \right\rfloor} \binom{n}{j} \frac{(n-j)!}{(n-j-2r)! r!} 2^{n-j-2r-1} \chi\left( \frac{M_{0,n-r+4-j}}{S_{3-j}} \right) + \frac{1}{2} \chi\left( \frac{M_{0,4}}{S_{3-n}} \right).
\]

Since we know that \( M_{1,1} \cong \mathbb{C} \), then \( \chi(M_{1,1}) = 1 \); from the formula we calculate:

\[
\chi(M_{1,2}) = \sum_{j=0}^{1} 2^{-j} \chi\left( \frac{M_{0,5-j}}{S_{3-j}} \right) + \frac{1}{2} \chi\left( \frac{M_{0,4}}{S_{2}} \right) = 1;
\]
A FORMULA FOR THE EULER CHARACTERISTIC OF $\overline{M}_{2,n}$

\[ \chi(M_{1,3}) = 2 \sum_{j=0}^{2} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \binom{2}{j} \frac{(2-j)!}{(2-j-2r)!} 2^{j-2r} \chi \left( \frac{\mathcal{M}_{0,6-r-j}}{S_{3-j}} \right) + \frac{1}{2} \chi(M_{0,4}) \]

\[ = 2 \chi \left( \frac{\mathcal{M}_{0,6}}{S_{3}} \right) + 3 \chi \left( \frac{\mathcal{M}_{0,5}}{S_{2}} \right) + \chi(M_{0,4}) = 0; \]

\[ \chi(M_{1,4}) = 3 \sum_{j=0}^{3} \sum_{r=0}^{\lfloor \frac{j}{3} \rfloor} \binom{3}{j} \frac{(3-j)!}{(3-j-2r)!} 2^{j-2r} \chi \left( \frac{\mathcal{M}_{0,7-r-j}}{S_{3-j}} \right) + \frac{1}{2} \chi(M_{0,4}) \]

\[ = 3 \sum_{j=0}^{3} \sum_{r=0}^{\lfloor \frac{j}{3} \rfloor} \binom{3}{j} \frac{(3-j)!2^{j-2r}(-1)^{r+j}(4-r-j)!}{(3-j-2r)!r!(3-j)!} \frac{1}{2} = 0. \]

Finally, \( \chi(M_{1,5}) = -2 \) and, for \( n \geq 5 \), \( \chi(M_{1,n}) = (-1)^n \frac{(n-1)!}{12} \); in the following table we summarize the results:

| \( n \) | 1 | 2 | 3 | 4 | \( \geq 5 \) |
|---------|---|---|---|---|----------|
| \( \chi(M_{1,n}) \) | 1 | 1 | 0 | 0 | \( (-1)^n \frac{(n-1)!}{12} \) |

5. Generating functions and graphs

In this section we briefly describe our combinatorial strategy in computing the Euler characteristics of \( \overline{M}_{1,n} \) and \( \overline{M}_{2,n} \).

As mentioned in the Introduction, the quasi-projective subvarieties \( X_i \) of \( \overline{M}_{g,n} \) are in correspondence with a collection of genus \( g \) graphs with \( n \) leaves. Considering a graph \( \Gamma \) of this collection, its contribution to the Euler characteristic is provided by the product of the Euler characteristics of the moduli spaces associated to its vertices.

Now it turns out that all the genus \( g \) graphs (with any number of leaves) can be obtained by attaching trees to some loops or to some vertex representing a curve of genus greater than 1. Therefore, if we let \( D \) to be a generating function which counts the contributions of trees to the Euler characteristic, the computation of the series \( K_g \) reduces to the sum of the contributions of all the possible combinations of loops and vertices with genus greater than 1, multiplied by a suitable power of \( D \).

Let us then give the following suitable definition of \( D \):

**Definition 5.1.**

\[ D(t) = t + \sum_{n=2}^{\infty} \sum_{\Gamma} \chi(\Gamma) \frac{t^n}{n!} \]

Here the second sum ranges over the collection of “admissible trees”, which are all the oriented rooted trees on \( n \) numbered leaves with the following two further properties

1. There is an unlabelled half-edge going into the root.
2. The tree is stable, that is to say, for every vertex of the tree the number of outgoing edges (including the leaves), plus 1, is greater than or equal to 3.

Furthermore, \( \chi(\Gamma) \) is the Euler characteristic of the stratum of \( \mathcal{M}_{0,n} \) which corresponds to \( \Gamma \).
As an example of the way in which we use $D$, the contribution to $K_2$ of all the graphs of the following kind

![Graph Example](image)

can be computed by means of the series

$$\sum_{n \geq 1} \chi(M_{2,n}) \frac{D^n}{n!}$$

Let us now focus on the series $D$: we can easily find a recursive relation for it noticing that, if we cut an edge which stems from the root of a genus 0 admissible tree, the cut part is again a genus 0 admissible tree. Therefore we can write the following recursive relation for $D$ (which was taken as a definition in the Introduction):

$$D = t + \sum_{n=2}^{\infty} \chi(M_{0,n+1}) \frac{D^n}{n!}.$$  

Substituting the values for $\chi(M_{0,n+1})$ we obtain

$$D = t + \sum_{n=2}^{\infty} (-1)^{n-1} (n-2)! \frac{D^n}{n!},$$

which, after differentiating with respect to $t$ gives

$$D' = 1 + D' \sum_{n=1}^{\infty} (-1)^n \frac{D^n}{n}$$

$$D'(1 - \log(1 + D)) = 1.$$  

This is a differential equation that allows us to compute recursively all the coefficients of $D$:

$$D(t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{7t^4}{24} + \frac{17t^5}{60} + \frac{71t^6}{240} + \frac{163t^7}{504} + o(t^8).$$

We notice that, in computing Euler characteristics, we will often come across the series $E = \log(1 + D)$, which can be equivalently written in the following way

$$E = \sum_{n=1}^{\infty} \chi(M_{0,n+2}) \frac{D^n}{n!}$$

and summarizes the contribution provided by trees that stem from a vertex of a polygon.

Let us now pass to motivate the introduction of two different operations on generating series, namely the derivative with respect to $D$ and the derivative with respect to $t$. Let us consider a graph $\Delta$ of the following kind
or more generally a graph which is made by two components of genus 1 attached by a genus 0 path.

When computing the contribution of such graphs to the series $K_2$, we can imagine to cut the picture in the following way

What remains on the left is a graph $\Delta_1$ of genus 1 with the artificial leaf $L_1$, while on the right there is another graph $\Delta_2$ of genus 1 with the artificial leaf $L_2$ instead of a tree.

A first approximation to the contribution of such graphs is provided by $\frac{1}{2} \frac{\partial K_1}{\partial t} \frac{\partial K_1}{\partial D}$.

In fact

1. The factor $\frac{\partial K_1}{\partial t}$ takes into account the contribution of the left part of the graph. The derivative with respect to $t$ cancels the mistake due to the presence in $\Delta_1$ of the leaf $L_1$ which is not a leaf of $\Delta$.

2. The factor $\frac{\partial K_1}{\partial D}$ takes into account the contribution of the right part of the graph. The derivative with respect to $D$ is the translation in terms of generating series of the presence, in $\Delta_2$, of the artificial leaf $L_2$ instead of a tree.

3. The coefficient $\frac{1}{2}$ is needed since in general there are two possible ways to cut the graph $\Delta$.

There are some exceptional cases; for instance one is provided by the following configuration, which should be carefully looked at:

Here and from now on, when we draw a dotted line outgoing from a vertex we mean that any number of trees can stem from the vertex itself.

In this case, the contribution to $K_2$ turns out to be equal to

$$\sum_{n \geq 1} \chi(M_{0,n}/S_2) \frac{D^n}{n!}$$

instead of

$$\frac{1}{2} \sum_{n \geq 1} \chi(M_{0,n}) \frac{D^n}{n!}$$
which was implicit in the expression \( \frac{1}{2} \frac{\partial K_1}{\partial t} \frac{\partial K_1}{\partial D} \).

6. The generating function for genus 1

In this section we obtain with our elementary and direct methods the generating function for the genus one case, which was already calculated by Getzler in [34].

There are two different contributions which should be considered: the first one comes from graphs of this kind

\[
\begin{align*}
\sum_{n \geq 1} \chi(M_{1,n}) \frac{D^n}{n!} &= D + \frac{D^2}{2} + \frac{1}{12} \left( \sum_{n \geq 1} (-1)^n \frac{D^n}{n} + D - D^2 + \frac{D^3}{3} - \frac{D^4}{4} \right) \\
&= \frac{13}{12} D + \frac{11}{24} D^2 + \frac{D^3}{36} - \frac{D^4}{48} - \frac{1}{12} \log(1 + D) \\
&= \frac{13}{12} D + \frac{11}{24} D^2 + \frac{D^3}{36} - \frac{D^4}{48} - \frac{E}{12};
\end{align*}
\]

the second contribution comes from the graphs containing a loop, and is

\[
\frac{1}{2} \sum_{l \geq 3} E^l + \sum_{n \geq 1} \chi\left( \frac{M_{0,n+2}}{S_2} \right) \frac{D^n}{n!} + \frac{1}{2} \sum_{n,m \geq 1} \chi\left( \frac{M_{0,n+2} \times M_{0,m+2}}{S_2} \right) \frac{D^{n+m}}{n!m!},
\]

where the action of \( S_2 \) exchanges the last two markings, simultaneously in the second case.

It can be written as

\[
\begin{align*}
&= \frac{1}{2} \left( -\log(1 - E) - E - \frac{E^2}{2} \right) + \frac{1}{2} \left( E - D + \frac{D^2}{2} \right) + \frac{1}{4} \left( E^2 - D^2 + D^3 - \frac{D^4}{4} \right) \\
&+ \sum_{1 \leq n \leq 2} \chi\left( \frac{M_{0,n+2}}{S_2} \right) \frac{D^n}{n!} + \frac{1}{2} \sum_{1 \leq n,m \leq 2} \chi\left( \frac{M_{0,n+2} \times M_{0,m+2}}{S_2} \right) \frac{D^{n+m}}{n!m!} \\
&= \frac{1}{2} \left( -\log(1 - E) - E - \frac{E^2}{2} \right) + \frac{1}{2} \left( E - D + \frac{D^2}{2} \right) + \\
&+ \frac{1}{4} \left( E^2 - D^2 + D^3 - \frac{D^4}{4} \right) + D + \frac{D^2}{2} + \frac{D^3}{4} + \frac{D^4}{16} \\
&= -\frac{1}{2} \log(1 - E) + \frac{D}{2} + \frac{D^2}{2} + \frac{D^3}{4} + \frac{D^4}{16}.
\end{align*}
\]

Summing up we get the generating function for genus 1.

\[
K_1 = \frac{13}{12} D + \frac{11}{24} D^2 + \frac{D^3}{36} - \frac{E}{12} - \frac{1}{2} \log(1 - E) + \frac{D}{2} + \frac{D^2}{2} + \frac{D^3}{4} + \frac{D^4}{16}.
\]
7. Graphs and components

Let us consider the set $\mathcal{F}$ of graphs representing boundary components of the moduli space of genus 2 pointed curves, including the one representing the open part. We define the graph-type of each graph as the graph obtained by deleting the markings, contracting all trees, and smoothing each vertex of valence 2.

There are seven different graph-types, and we denote by $\mathcal{F}_1, ..., \mathcal{F}_7$ the collections of graphs of each type. We count separately the contribution these sets give to the generating function for the characteristic of $\overline{\mathcal{M}}_{2,n}$.

![Graph-types](image)

**Figure 1.** Graph-types

The group of automorphisms $G_i$ of the graph-type acts on $\mathcal{F}_i$, and generically the contribution should have coefficient $\frac{1}{|G_i|}$; in this general formula we should “correct” the contribution of the components corresponding to graphs with non trivial stabilizer. In this case, the stabilizer of the graph acts generically non-trivially on the boundary component, since it acts on the added markings of the irreducible components of the curve. If there are enough markings on this component, the action is free, and nothing changes in the coefficient $\frac{1}{|G_i|}$; but, for a low number of markings, we should analyze the contribution this component gives independently.

As an example, graphs of this kind

![Graph](image)

(which are of type 5) give the contribution $\frac{1}{2} \sum_{n \geq 0} \chi(\mathcal{M}_{1,n+2}) \frac{B^n}{n!}$ in the generic formula, since the graph-type has automorphism group of order 2. In fact, what we should really put in the formula is $\sum_{n \geq 0} \chi \left( \frac{\mathcal{M}_{1,n+2}}{S_2} \right) \frac{B^n}{n!}$, where $S_2$ permutes the two added markings. The two formulas coincide for $n \geq 5$, when the action becomes free, but the initial terms should be corrected.

Now let us analyze separately each contribution.

7.1. Graphs of type 1. Since there are no graph-type automorphisms, the contribution is $\sum_{n \geq 0} \chi(\mathcal{M}_{2,n}) \frac{B^n}{n!}$.
From our formulas, we get
\[
\sum_{n \geq 0} \chi(\mathcal{M}_{2,n}) \frac{D^n}{n!} = 1 + 2D + D^2 - \frac{D^4}{6} - \frac{D^6}{30} + \frac{1}{240} \sum_{n \geq 7} (-1)^{n+1} (n+1)D^n = \frac{1}{240 (1 + D)^2} + \frac{241}{240} + \frac{239D}{120} + \frac{81D^2}{80} - \frac{D^3}{60} - \frac{7D^4}{48} - \frac{D^5}{40} - \frac{D^6}{240}.
\]

7.2. Graphs of type 2,3,4. These graph-types could be considered together, as they all could be seen in exactly two different ways as the union of a genus 1 graph with one cut leaf, and a genus 1 graph with one cut tree.

We recall that the generating function for the genus 1 case is
\[
K_1(D) = \frac{19D}{12} + \frac{23D^2}{24} + \frac{5D^3}{18} + \frac{D^4}{24} - \frac{\log (1 + D)}{12} - \frac{\log (1 - \log (1 + D))}{2},
\]
the generic contribution is:
\[
\frac{1}{2} \left( \frac{\partial K_1(D)}{\partial D} \right)^2 D' = \frac{1}{2} \left( \frac{19}{12} + \frac{23D}{12} + \frac{5D^2}{6} + \frac{D^3}{6} - \frac{1}{12 (1 + D)} + \frac{1}{2 (1 - E) (1 + D)} \right)^2 \frac{1}{1 - E}.
\]
\[
= \frac{288 (1 - E)}{(361 + 874D + 909D^2 + 536D^3 + 192D^4 + 40D^5 + 4D^6)} - \frac{19 + 23D + 10D^2 + 2D^3}{144 (1 - E) (1 + D)} + \frac{19 + 23D + 10D^2 + 2D^3}{24 (1 - E)^2 (1 + D)} - \frac{1}{24 (1 - E)^2 (1 + D)^2} + \frac{1}{288 (1 - E) (1 + D)^2} + \frac{1}{8 (1 - E)^3 (1 + D)^2};
\]

moreover, the group exchanging the two sides of the graph type fixes the following graphs of type 2 and 4:

\[
\begin{align*}
1) & \quad \begin{array}{c}
\text{1} \\
\text{1}
\end{array} \\
2) & \quad \begin{array}{c}
\text{1} \\
\text{1}
\end{array} \\
3) & \quad \begin{array}{c}
\text{1} \\
\text{1}
\end{array} \\
4) & \quad \begin{array}{c}
\text{1} \\
\text{1}
\end{array}
\end{align*}
\]

in each of the first two cases, the contribution is 1 instead of \( \frac{1}{2} \), while in the third one we should replace \( \frac{1}{2} \left( D - \frac{D^2}{2} \right) \) with
\[
\sum_{1 \leq n \leq 2} \chi(\mathcal{M}_{0,n+2}) \frac{D^n}{n!} = D.
\]
The fourth graph requires a little more work: in fact we have to analyze the action of $S_2$ on 
\[ \mathcal{M}_{0,n+2} \times \mathcal{M}_{1,1} \times \mathcal{M}_{1,1}, \]
which is free for $n > 2$. For the case $n = 1$, the action is trivial, whereas for the case $n = 2$, the quotient map is generically $2 : 1$, and it is ramified on 
\[ [\mathbb{P}^1 : \infty, 0, 1, -1] \times \Delta, \]
where $\Delta$ is the diagonal in $\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}$. Thus we get 
\[ \chi (\mathcal{M}_{0,n+2} \times \mathcal{M}_{1,1} \times \mathcal{M}_{1,1}/S_2) = 0. \]
Once more we should replace \( \frac{1}{2} \left( D - \frac{D^2}{2} \right) \) with $D$.

Finally we have:
\[
\begin{align*}
1 + D + \frac{D^2}{2} &+ \frac{(361 + 874D + 909D^2 + 536D^3 + 192D^4 + 40D^5 + 4D^6)}{288 (1 - E)} \\
&- \frac{19 + 23D + 10D^2 + 2D^3}{144 (1 - E) (1 + D)} + \frac{19 + 23D + 10D^2 + 2D^3}{24 (1 - E)^2 (1 + D)} - \frac{1}{24 (1 - E)^2 (1 + D)^2} \\
&+ \frac{1}{288 (1 - E) (1 + D)^2} + \frac{1}{8 (1 - E)^3 (1 + D)^2}.
\end{align*}
\]

### 7.3. Graphs of type 5

$G_5 \cong S_2$ acts on the graph-type. The formula to be corrected is
\[
\begin{align*}
\frac{1}{2} \left( \sum_{n \geq 0} \chi (\mathcal{M}_{1,n+2}) \frac{D^n}{n!} \right) D' \\
= \frac{1}{2(1 - E)} \left( \sum_{n \geq 3} (-1)^n \frac{(n + 1)! \cdot D^n}{12 n!} + 1 \right) \\
= \frac{1}{2(1 - E)} \left( \frac{1}{12} \left( \sum_{n \geq 0} (-1)^n (n + 1) D^n \right) - \frac{1}{12} + \frac{D}{6} - \frac{D^2}{4} + 1 \right) \\
= \frac{1}{2(1 - E)} \left( \frac{1}{12 (1 + D)^2} + \frac{11}{12} + \frac{D}{6} - \frac{D^2}{4} \right),
\end{align*}
\]
the graphs stabilized by $G_5$ are

\[
\begin{aligned}
\cdots \circ 1
\end{aligned}
\]
and
which, taking into account only the cases where the elliptic curve has non trivial automorphisms, give contribution

\[
\sum_{0 \leq n \leq 4} \chi \left( \frac{\mathcal{M}_{1,n+2}}{S_2} \right) \frac{D^n}{n!}
\]

\[
= \chi \left( \frac{\mathcal{M}_{1,2}}{S_2} \right) + \chi \left( \frac{\mathcal{M}_{1,3}}{S_2} \right) D + \chi \left( \frac{\mathcal{M}_{1,4}}{S_2} \right) \frac{D^2}{2}
\]

\[
+ \chi \left( \frac{\mathcal{M}_{1,5}}{S_2} \right) \frac{D^3}{6} + \chi \left( \frac{\mathcal{M}_{1,6}}{S_2} \right) \frac{D^4}{24}
\]

\[
= 1 + D + \frac{D^2}{2} + \frac{D^4}{4}
\]

instead of

\[
\frac{1}{2} \sum_{0 \leq n \leq 4} \chi (\mathcal{M}_{1,n+2}) \frac{D^n}{n!} = \frac{1}{2} \left( 1 - \frac{D^3}{3} + \frac{5D^4}{12} \right),
\]

and

\[
\sum_{0 \leq n \leq 4} \chi \left( \frac{\mathcal{M}_{1,n+2} \times \mathcal{M}_{0,m+2}}{S_2} \right) \frac{D^{n+m}}{n!m!} = D + D^2 + D^3 + \frac{D^4}{4} + \frac{D^5}{6} - \frac{D^6}{12}
\]

instead of

\[
\frac{1}{2} \sum_{0 \leq n \leq 4} \chi (\mathcal{M}_{1,n+2}) \frac{D^n}{n!} \left( D - \frac{D^2}{2} \right)
\]

\[
= \frac{1}{2} \left( 1 - \frac{D^3}{3} + \frac{5D^4}{12} \right) \left( D - \frac{D^2}{2} \right)
\]

\[
= \frac{D}{2} - \frac{D^2}{4} - \frac{D^4}{6} + \frac{7D^5}{24} - \frac{5D^6}{48}
\]

Therefore the contribution of graphs of type 5 is

\[
= \frac{1}{24 (1 - E) (1 + D)^2} + \frac{11 + 2D - 3D^2}{24 (1 - E)}
\]

\[
+ \frac{1}{2} + \frac{3}{2} D + \frac{7}{4} D^2 + \frac{7}{6} D^3 + \frac{11}{24} D^4 - \frac{1}{8} D^5 + \frac{1}{48} D^6.
\]

7.4. Graphs of type 6. \( G_6 = H_7 \times L_7 \times R_7 \cong S_2 \times S_2 \times S_2 \) acts on the graph type: \( H_7 \) exchanges the two loops, \( L_7 \) and \( R_7 \) reverse the orientation on the left and on the right loop; the starting formula is

\[
\frac{1}{8} \sum_{n \geq 0} \chi (\mathcal{M}_{0,n+4}) \frac{D^n}{n!} (D')^2 = -\frac{1}{8 (1 + D)^2 (1 - E)^2}.
\]
A formula for the Euler characteristic of $\mathcal{M}_{2,n}$ stabilizes the graph, and the contribution is

\[
\sum_{0 \leq n \leq 3} \chi \left( \frac{\mathcal{M}_{0,n+4}}{D_4} \right) \frac{D^n}{n!} = \chi \left( \frac{\mathcal{M}_{0,4}}{D_4} \right) + \chi \left( \frac{\mathcal{M}_{0,5}}{D_4} \right) D + \chi \left( \frac{\mathcal{M}_{0,6}}{D_4} \right) \frac{D^2}{2} = -\frac{D^2}{2},
\]

which should replace $\frac{1}{8} \left( -1 + 2D - 3D^2 \right)$ in the formula.

The stabilizer is $L_7 \times R_7$; we put

\[
\frac{1}{2} \chi \left( \frac{\mathcal{M}_{0,4}}{R_7} \right) \left( E - D + \frac{D^2}{2} \right) + D \sum_{0 \leq n \leq 2} \chi \left( \frac{\mathcal{M}_{0,n+4} \times \mathcal{M}_{0,3}}{L_7 \times R_7} \right) \frac{D^n}{n!} + \frac{D^2}{2} \sum_{0 \leq n \leq 2} \chi \left( \frac{\mathcal{M}_{0,n+4} \times \mathcal{M}_{0,4}}{L_7 \times R_7} \right) \frac{D^n}{n!} = -\frac{3}{2} D^3 + \frac{D^4}{4}
\]

instead of

\[
\left( \frac{D}{4} - \frac{D^2}{8} \right) \left( -1 + 2D - 3D^2 \right) - \frac{1}{4} \left( E - D + \frac{D^2}{2} \right)
\]

\[
= -\frac{E}{4} + \frac{D^2}{2} - D^3 + \frac{3}{8} D^4.
\]

This set of graphs gives

\[
\frac{1}{2} \sum_{0 \leq n \leq 2} \sum_{1 \leq p,q \leq 2} \chi \left( \frac{\mathcal{M}_{0,n+4} \times \mathcal{M}_{0,p+2} \times \mathcal{M}_{0,q+2}}{L_7 \times R_7} \right) \frac{D^{n+p+q}}{n!p!q!} = \frac{1}{2} \left( \frac{D^4}{4} - D^4 - D^4 + \frac{D^5}{2} - \frac{D^6}{4} \right) = -\frac{9}{8} D^4 + \frac{D^5}{4} - \frac{D^6}{8}
\]
replacing

$$\frac{1}{8} \left( -1 + 2D - 3D^2 \right) \left( D - \frac{D^2}{2} \right)^2$$

and

$$\frac{1}{2} \sum_{1 \leq p \leq 2} \chi \left( \frac{M_{0.4} \times M_{0.0+2}}{S_2} \right) \frac{D^p}{p!} \left( E - D + \frac{D^2}{2} \right)$$

replacing

$$\frac{-1}{4} \left( E - D + \frac{D^2}{2} \right) \left( D - \frac{D^2}{2} \right)$$

$$= E \left( -\frac{D}{4} + \frac{D^2}{8} \right) + \frac{D^2}{4} - \frac{D^3}{4} + \frac{D^4}{16}.$$

The contribution of these graphs (here and from now in the pictures a pentagon stands for any polygon with more than three edges) is

$$\frac{1}{2} \chi \left( \frac{M_{0.4}}{S_2} \right) \frac{E^2}{1 - E} = 0$$

where in the formula we have \(-\frac{1}{4} \frac{E^2}{1 - E}\), and finally we have

which give contribution

$$\frac{E^2}{2(1 - E)} \left( \chi \left( \frac{M_{0.4} \times M_{0.3}}{S_2} \right) D + \chi \left( \frac{M_{0.4} \times M_{0.4}}{S_2} \right) \frac{D^2}{2} \right)$$

$$= \frac{1}{4} \frac{D^2 E^2}{1 - E}$$
and not $\frac{E^2}{4E} \left(-D + \frac{D^2}{2}\right)$.

From graph type 6:

\[
-\frac{1}{8(1 + D)^2(1 - E)^2} = \frac{D^2}{2} + \frac{1}{8} - \frac{D}{4} + \frac{3}{8}D^2
\]

\[
-\frac{3}{2}D^3 + \frac{D^4}{4} + \frac{E}{4} - \frac{D^2}{2} + D^3 - \frac{3}{8}D^4
\]

\[
-\frac{9}{8}D^4 + \frac{D^5}{4} - \frac{D^6}{8} + \frac{D^2}{8} - \frac{3D^3}{8} + \frac{21}{32}D^4 - \frac{7}{16}D^5 + \frac{3}{32}D^6
\]

\[
+ED^2 - \frac{D^3}{4} + \frac{D^4}{8} - E\left(-\frac{D}{4} + \frac{D^2}{8}\right) - \frac{D^2}{4} + \frac{D^3}{4} - \frac{D^4}{16}
\]

\[
+\frac{1}{41 - E} + \frac{1}{4}E^2D^2 + \frac{1}{42 - E} - \frac{1}{81 - E}E^2D^2
\]

\[
= -\frac{1}{8(1 + D)^2(1 - E)^2} + \frac{E^2}{4(1 - E)} \left(\frac{D^2}{2} + D + 1\right)
\]

\[
+E\left(\frac{D^2}{8} + \frac{D}{4} + \frac{1}{4}\right) + \frac{1}{8} - \frac{D}{4} - \frac{D^2}{2} - \frac{7}{8}D^3 - \frac{11}{32}D^4 - \frac{3}{16}D^5 - \frac{D^6}{32}.
\]

7.5. **Graphs of type 7.** The group acting is $G_7 = H_7 \times K_7$, where $H_7 \cong S_2$ interchanges the two vertices, and $K_7 \cong S_3$ permutes the three edges.

The generic contribution is

\[
\frac{1}{12}\left(\sum_{n \geq 0} \chi(\mathcal{M}_{0,n+3}) \frac{D^n}{n!}\right)^2 (D')^3 = \frac{1}{12} \frac{1}{(1 + D)^2} \frac{1}{(1 - E)^2};
\]

we need to correct the following contributions:

![Graph](image)

The stabilizer is the whole group $G_7$; it is clear that there exists only such a curve, so that the real contribution is 1; our formula gives $\frac{1}{12}$.
The stabilizer is $H_7 \times K'_7$, where $K'_7$ is a subgroup of order 2 of $K_7$; the contribution is
\[
\sum_{n \geq 1} \chi \left( \frac{M_{0,n+2}}{S_2} \right) \frac{D^n}{n!} = D + \frac{E}{2} - \frac{D}{2} + \frac{D^2}{4}
\]
\[
= \frac{E}{2} + \frac{D}{2} + \frac{D^2}{4}
\]
and the formula gives $\frac{E}{4}$.

\[
\begin{array}{c}
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\bullet \\
\bullet \\
\end{array}
\]

The stabilizer is $H_7$, and the contribution is
\[
\frac{1}{2} \sum_{1 \leq n, m \leq 2} \chi \left( \frac{M_{0,n+2} \times M_{0,m+2}}{S_2} \right) \frac{D^{n+m}}{n!m!} = \frac{D^2}{2} + \frac{D^4}{8},
\]
where in the formula we get $\frac{1}{4} \left( D^2 - D^3 + \frac{D^4}{4} \right) = \frac{D^2}{4} - \frac{D^3}{4} + \frac{D^4}{16}$.

\[
\begin{array}{c}
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\bullet \\
\end{array}
\]

Still $H_7$ stabilizes all the graphs of this kind, and the real contribution is the following:
\[
\frac{1}{6} \sum_{1 \leq n, m, p \leq 2} \chi \left( \frac{M_{0,n+2} \times M_{0,m+2} \times M_{0,p+2}}{S_2} \right) \frac{D^{n+m+p}}{n!m!p!} = \frac{D^3}{6} + \frac{D^5}{8}
\]
The formula gives instead $\frac{1}{12} \left( D^3 - \frac{3}{2} D^4 + \frac{3}{4} D^5 - \frac{D^6}{8} \right) = \frac{D^3}{12} - \frac{D^4}{8} + \frac{D^5}{16} - \frac{D^6}{96}$.

\[
\begin{array}{c}
\bullet \\
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\bullet \\
\end{array}
\]

(we assume that at least one vertex has valence $\geq 4$). The stabilizer is $K_7$, and the contribution is
\[
\frac{1}{2} \sum_{0 \leq n, m \leq 2} \chi \left( \frac{M_{0,n+3} \times M_{0,m+3}}{S_3} \right) \frac{D^{n+m}}{n!m!} = D + \frac{3}{2} D^2 + \frac{3}{2} D^3 + \frac{D^4}{4},
\]
whereas the formula gives $\frac{1}{12} \left( -2D + 3D^2 - 2D^3 + D^4 \right) = -\frac{D}{6} + \frac{D^2}{4} - \frac{D^3}{12} + \frac{D^4}{12}$.

The stabilizer is a subgroup of order 2 of $K_7$, and we should put
\[
\frac{1}{2} \sum_{0 \leq n, m \leq 1} \chi \left( \frac{M_{0,n+3} \times M_{0,m+3}}{S_2} \right) \frac{D^{n+m}}{n!m!} \cdot \frac{ED'}{2(1-E)}
\]

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]
A FORMULA FOR THE EULER CHARACTERISTIC OF $\mathcal{M}_{2,n}$

instead of $\frac{E(-2D + D^2)}{4(1 - E)}$, and, for the case $n = m = 0$, since we already corrected part of this term, we substitute $\frac{E^2}{2(1 - E)}$ to $\frac{E^2}{1 - E}$.

The final contribution of type 7 graphs is

$$
\frac{1}{12} \left( \frac{1}{(1 + D)^2} \right)^3 + \frac{1}{12} \left( \frac{1}{(1 - E)^3} \right)^3 + \frac{11}{12} + \frac{E}{2} + \frac{D}{2} + \frac{D^2}{4} - \frac{E}{4} + \frac{D^2}{8} + \frac{D^4}{16} - \frac{D^4}{4} + \frac{D^3}{12} + \frac{D^5}{16} - \frac{D^6}{96}
$$

$$
+ D + \frac{3}{2} D^2 + \frac{D^3}{2} + \frac{D^4}{4} + \frac{D^5}{8} - \frac{D^6}{16} + \frac{ED^3}{4(1 - E)} - \frac{2DE + ED^2}{4(1 - E)} + \frac{E^2}{4(1 - E)}
$$

$$
= \frac{1}{12} \left( \frac{1}{(1 + D)^2} \right)^3 + \frac{1}{4(1 - E)^3} + \frac{E}{4} \left( \frac{2D + D^2}{4(1 - E)} \right) + \frac{3}{8} D + \frac{7}{16} D^2 + \frac{D^3}{16} + \frac{17}{48} D^4 + \frac{D^5}{96} + \frac{D^6}{96}.
$$

Now we are able to write down the complete generating function for genus 2:

$$
K_2 = \frac{1}{1440}(1 + D)^2(E - 1)^3(-2D^6(E - 1)^2(7 + 3E) - 2AD^7(E - 1)^2(-7 + 17E) + 30D^5(E - 1)^2(61E - 221) - 3D^6(E - 1)^2(259 + 201E) + 360D(45E^3 - 167E^2 + 206E - 84) + 60(73E^3 - 270E^2 + 336E - 144) + 180D^2(138E^3 - 519E^2 + 635E - 254) + 60D^3(341E^3 - 1322E^2 + 1633E - 652) + 15D^4(631E^3 - 2640E^2 + 3395E - 1386)].
$$

By developing in power series, we get:

$$
K_2(t) = 6 + 131 + 21t^2 + \frac{181}{6} t^3 + \frac{251}{6} t^4 + \frac{6893}{120} t^5 + \frac{27971}{720} t^6 + \frac{177673}{1680} t^7 + o(t^8);
$$

from this we read

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $\chi(\mathcal{M}_{2,n})$ | 6 | 13 | 42 | 181 | 1004 | 6853 | 55942 | 533019 |
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