An \((s, S)\) Inventory System with Unreliable Service and Repeated Calls in a Random Environment

James Cordeiro*, Ying-Ju Chen, Andrés Larraín-Hubach

University of Dayton, 300 College Park, Dayton, Ohio, 45458 USA

Abstract

Queueing systems are often used as the foundation for industrial models of all types, to include production lines and inventory systems. Here, we consider an inventory system with repeated calls and failures of service. The inventory system features a single product that experiences Markovian demand and service intervals with random service interruptions. Lead time for replenishment is instantaneous. The Markov-modulated duration of service and inter-demand times permit time-dependent variation of the exponential parameters of these processes, which thus imparts a greater degree of realism to the model. Due to the resulting complexity of the underlying Markov chain, a traffic intensity condition is difficult to obtain via conventional means. We thus employ a novel technique that utilizes a matrix-analytic drift approach to obtain a closed-form drift condition for the stability of the process. Estimates of the steady-state distribution of such models are then used to compute various system performance measures.

Keywords: Queueing, \((s, S)\) inventory, Lyapunov drift, retrial queue

*Corresponding author

Email address: cordeirj@yahoo.com (James Cordeiro)
1. Introduction

The \((s, S)\) inventory model, which was first investigated in \cite{Arrow1951}, has appeared frequently in studies of working inventory systems of all types. In such a system, an order is placed for enough units of inventory to restore the level to its maximum capacity of \(S > 0\) when the inventory drops to a trigger level \(s > 0\). As with the Arrow, et al. model, a stochastic delay between replenishment orders and deliveries is often specified whenever the threshold level \(s\) is attained. Alternatively, this delay may be deterministic, and perhaps even instantaneous, depending upon the nature of the agency that performs the restocking service. Instantaneous review models are appropriate in particular situations in which little to no delay is expected during the replenishment process, such as when items are produced on-site. The issue of replenishment leads to another important characteristic of inventory models, which is to determine when and how observations of the product level in inventory are performed. As with the original \((s, S)\) models, the level of restocking is monitored constantly, hence the designation continuous review. Such monitoring would be appropriate for situations in which autonomous capability is present or there is very little additional cost that results from delays in replenishment. Otherwise, and particularly when optimal control decisions are to be made, inventory level monitoring might take place at stochastic or deterministic time epochs.

The \((s, S)\) inventory models aptly describe a variety of inventory systems, to include those that pertain to brick-and-mortar as well as e-commerce retail locations. In particular, e-commerce websites commonly direct customers to single items via a hyperlink, which admits them to a queue that services incoming demands for the featured item. In support of this demand stream is an inventory of the featured item that is physically located at the servicing distribution center. Problems serving incoming demands inevitably arise in such systems, and may lead to a reduction in capacity (and hence revenue) if these problems are not addressed. For instance, in the event that the physical server becomes unavailable due to lack of capacity or maintenance activities, demands may leave the system before they are fulfilled. A simple way to mitigate this difficulty is to maintain a holding area for blocked demands. In a working service center, patience will limit the amount of time demands can be kept in limbo, and so a timely replenishment system becomes a matter of necessity.

Such emphasis on retaining transactions in the face of electronic blocking issues and inventory depletion has suggested the utility of including repeated calls in an inventory system. In particular, retrial queueing systems and the vast body of literature for the performance assessment of such
models may be leveraged to study working inventory systems. In such queues, customers whose demands are not immediately satisfied may conveniently be placed into a special buffer called a retrial orbit, in which they remain until they successfully regain service and are subsequently discharged from the system. If customer demand is not satisfied due to unavailability of service processing, demand is retained in the orbit until the system comes back on line at some time in the future.

The imposition of such a queueing framework to inventory models allows the modeler to leverage the suite of powerful techniques developed for the performance analysis of queues in steady-state operation. Matrix-analytic queueing theory, which was developed by M. Neuts in the 1960s and 1970s, has facilitated an algorithmic approach to eliciting the steady-state measures of queueing systems that fall into the mold of certain generalized birth-death systems such as the quasi-birth-and-death (QBD) system, among others Neuts (1978a,b 1981). Bright & Taylor (1995) were the first to develop algorithms used to compute the matrix-geometric distribution of steady-state probabilities for the level-dependent QBD (LDQBD), which is a state-dependent generalization of the QBD that will be featured later in this paper. More recently, Cordeiro et al. (2019) elicited analytic ergodicity criteria for LDQBDs using a matrix-analytic drift approach. Because of these developments, it has become possible at last to describe, not only estimated steady-state distributions for complex inventory systems, but also closed-form traffic intensity formulas that provide parameter thresholds that guarantee the stability of such systems.

2. Literature review

An \((s, S)\) queueing inventory system based upon an \(M/M/1\) retrial queue with an infinite-capacity retrial orbit and exponential lead times was first studied by Artalejo et al. (2006). A similar retrial queueing inventory model was independently studied by Ushakumari (2006). Although these earlier systems were described in terms of Markov chains, they were not identified as LDQBDs, though they clearly could be classified as such. Later, Krishnamoorthy et al. (2012) does frame the \((s, S)\) retrial inventory system as a LDQBD and capitalizes on its structure to determine the steady-state distribution of the model. Ko (2020) describes a \((s, S)\) perishable inventory system (i.e., a model for which a lifetime distribution on inventory items is imposed) as a retrial queueing system. The long-run behavior of the model is analyzed using a generalized drift approach, whereupon a closed-form ergodicity criterion is obtained. The underlying LDQBD structure is applied in this case, not to the theoretical investigation of the asymptotic behavior of the model, but rather to
leverage existing numerical techniques for the estimation of its steady-state distribution.

In order to incorporate a greater degree of realism, \((s, S)\) inventory systems featuring imperfect demand processing may be considered. Here, we distinguish stochastic shutdowns of demand fulfillment from simple depletion of the inventory. Inventory systems with repeated calls are useful for capturing demand lost from such breakdowns. Prior to the publication of Artalejo et al. (2006), which considers demand blocking due to consumption of the inventory, Özekici & Parlar (1999) considers a model in which the server fails when the random environment transitions into a certain set of what are called trigger states. Later, Krishnamoorthy et al. (2012) incorporates breakdowns of demand processing that proceed according to a dedicated Poisson process, which is the protocol employed in this paper.

Queues in random environments in the context of level-independent QBDs were first studied by Neuts (1971, 1978a,b). However, this pedigree for inventory models incorporating random environments extends somewhat further back in time. Inventory systems that incorporate nonstationary demand distributions go back at least as far as the publication of the articles Karlin (1960); Iglehart & Karlin (1962). The first to study a \((s, S)\) inventory system with a compound-Poisson demand process modulated by a finite-state Markovian random environment is Feldman (1978). Models with unreliable demand processes modulated by a random environment include Song & Zipkin (1993); Özekici & Parlar (1999); Yadavalli & Schoor (2012). The assumption of imperfect demand processing likewise provides a greater degree of fidelity to a working inventory system. More recently, Perry & Posner (2002) features a production inventory with a demand process server that functions according to a two-state random environment that undergoes successive ON and OFF periods.

Although inventory models with one or more of the features included in this paper, such as Markov modulation, imperfect service, and repeated calls, are ubiquitous in the literature, there are no such models that provide for the modulation of every model process, to include those of arrival, service, breakdowns, repairs, and retrials. In addition, this study leverages a new result (c.f. Cordeiro et al. (2019)) that allows for a procedural determination of an analytic traffic intensity for a class of models with underlying Markov chains described as LDQBDs, regardless of complexity. This analytic condition, moreover, is expressed in a form that elegantly and conveniently incorporates information about every environmental state. Further extensions of this model and to other, more complex models, is anticipated from the example presented in this paper.

In the following sections, we will define the LDQBD and introduce the average drift method of Cordeiro et al. (2019) for a class of LDQBDs that is convergent over infinite rows of the infinitesimal
generator in order to discern necessary and sufficient conditions for stability and non-stability of the system. After defining the instantaneous replenishment model and constructing the LDQBD representation of the system, an analytic traffic intensity formula for the system will be derived based upon the method of Cordeiro et al. (2019). Performance measures will be derived for two positive-recurrent inventory systems by utilizing an algorithm of Bright & Taylor (1995) to compute the stationary probabilities of the system states.

3. Level-Dependent Quasi-Birth-and-Death Processes (LDQBD)

A brief description of the general LDQBD model will be required in order to define the Markov representation of the inventory model. A continuous-time level-dependent quasi-birth-and-death (LDQBD) process is a bivariate CTMC \( \Phi = \{(X(t), Y(t)) : t \geq 0\} \) with state space

\[
S_\Phi = \{(i, j) : i \in \mathbb{Z}_+, j \in \{1, \ldots, m\}\},
\]

where \( m < \infty \) is some positive integer value. The \( x \)-coordinate of \( S_\Phi \) is denoted as the level of the process while the \( y \)-coordinate is the phase. The infinitesimal generator \( Q^* \) of a LDQBD is of the distinctive tridiagonal form given by

\[
Q^* = [q^*_{ij}] =
\begin{bmatrix}
A_1^{(0)} & A_0^{(0)} & 0 & 0 & 0 & \cdots \\
A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & 0 & 0 & \cdots \\
0 & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & 0 & \cdots \\
0 & 0 & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} & \cdots \\
0 & 0 & 0 & A_2^{(4)} & A_1^{(4)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

for which the corresponding entries of each \( K \times K \) block matrix \( A_j^{(i)} \) vary according to the level \( i \). The process is called \textit{skip-free} in deference to the characteristic that no transition of the process may exceed one level in either the positive or negative direction.

Next, we consider the discrete-time Markov chain (DTMC)

\[
\tilde{\Phi} = \{(X_n, Y_n) : n \in \mathbb{Z}_+\}
\]

with state space \( S^* \) that is embedded at transitions of the CTMC \( \Phi \); this is known as the \textit{jump}
process of $\Phi$. Its transition probability matrix $P$ has the same tridiagonal block structure

$$
\tilde{P} = \begin{bmatrix}
\tilde{A}_1^{(0)} & \tilde{A}_0^{(0)} & 0 & 0 & 0 & \cdots \\
\tilde{A}_2^{(1)} & \tilde{A}_1^{(1)} & \tilde{A}_0^{(1)} & 0 & 0 & \cdots \\
0 & \tilde{A}_2^{(2)} & \tilde{A}_1^{(2)} & \tilde{A}_0^{(2)} & 0 & \cdots \\
0 & 0 & \tilde{A}_2^{(3)} & \tilde{A}_1^{(3)} & \tilde{A}_0^{(3)} & \cdots \\
0 & 0 & 0 & \tilde{A}_2^{(4)} & \tilde{A}_1^{(4)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

(2)

with block entries

$$
\begin{align*}
\tilde{A}_2^{(i)}_{kk'} &= \mathbb{P}(X_{n+1} = i - 1, Y_{n+1} = k' \mid X_n = i, Y_n = k) \\
\tilde{A}_1^{(i)}_{kk'} &= \mathbb{P}(X_{n+1} = i, Y_{n+1} = k' \mid X_n = i, Y_n = k) \\
\tilde{A}_0^{(i)}_{kk'} &= \mathbb{P}(X_{n+1} = i + 1, Y_{n+1} = k' \mid X_n = i, Y_n = k).
\end{align*}
$$

(3)

It is thus seen that the entries of $\tilde{A}_2^{(R)}$ contain the probabilities of transitions that result in the level decreasing by 1, those of $\tilde{A}_1^{(R)}$ the probabilities of remaining at the same level, and lastly those of $\tilde{A}_0^{(R)}$ the probabilities of the level increasing by 1.

For the purpose of determining system stability, it is useful here to restrict consideration to the class of LDQBD jump processes that satisfy the assumption

**Assumption 1.** For the discrete-time LDQBD process $\tilde{\Phi}$, the matrix

$$
\tilde{A}^* = \tilde{A}_2^* + \tilde{A}_1^* + \tilde{A}_0^*,
$$

where the sub-stochastic matrices $\tilde{A}_j^*$ for $j = 0, 1, 2$ are defined by the element-wise limits

$$
\tilde{A}_j^* = \lim_{i \to \infty} \tilde{A}_j^{(i)}, \quad k = 0, 1, 2,
$$

exists and is stochastic.

We will show that that queueing inventory model that is the subject of this paper is an LDQBD whose jump process satisfies Assumption 1.

### 4. Model Description

We assume that the model, which is depicted in Figure 1, is a continuous-review $(s, S)$ inventory system which consists of a single-product storage facility and a single server that processes incoming demands. There is no waiting area for demands. We denote $S \in \mathbb{N}$ to be the fixed inventory storage...
capacity and $0 \leq s < S$ to be the threshold level at which a replenishment of the inventory is triggered. If the level of product in the inventory drops to the threshold level of $s$, an instantaneous replenishment of $S - s$ items occurs. Such a replenishment policy maintains the inventory level in the range $[s + 1, S]$, which enforces the requirement that only one replenishment takes place at any instant of time. Consequently, no further constraints need be placed on the relative values of the capacity $S$ and threshold replenishment level $s$ of the inventory.

Before any incoming demand is satisfied, it must be processed by the system server. The server is assumed at all times to be in one of three states, namely idle and operational, busy and operational, or failed. A server that is failed may not satisfy a demand for product. The server remains operational for an exponential duration with average length $1/\xi$, after which it is considered to be in a failed state. Repair of the server commences immediately for an exponential duration of average length $1/\alpha$, after which the server is returned to a fully-operational and idle state.

At time $t = 0$, it is assumed that the server is idle and operational, the inventory is at its maximum level $I(0) = S$, and there are no demands in the system. Thereafter, single demands arrive to the serving station according to a simple Poisson process with average interarrival rate of $1/\lambda$. If the server is idle, processing of the incoming demand is commenced, upon which the server is considered to be in a busy state. Satisfaction of the demand is completed at the end of an exponential duration with average length $1/\mu$, after which the inventory is decremented by one item and the demand leaves the system.

Figure 1: Model Illustration
On the other hand, if a demand encounters a busy or a failed server, it will proceed to the retrial orbit, which is an uncapacitated buffer for unprocessed demands in the system. Likewise, if the server fails during a busy period, the demand being processed will immediately proceed to the retrial orbit. Once in orbit, a demand will reattempt processing at an exponential rate of $\theta$ in a manner independent of all other demands currently in the orbit. The reattempt will be successful only if the server is idle, otherwise, it remains in orbit. As with a normal incoming demand, it will leave the system upon successfully completing its processing.

The various exponential rates of the system are each determined via the evolution of an external random environment, which is a finite-state CTMC $\{Z(t) : t \in \mathbb{R}_+\}$ with state space $S = \{1, \ldots, m\}$ and infinitesimal generator $Q = [q_{zz'}]_{z,z' \in S}$. In the standard way, we use the following notation to define the total rate out of state $z \in S$:

$$q_z = -q_{zz} = \sum_{z' \neq z} q_{zz'}, \quad z \in S.$$  

If $Z(t) = z$ at a time instant $t \geq 0$, then the exponential parameters of each process are given as follows:

| Process Name | Exponential Rate |
|--------------|-----------------|
| Arrival      | $\lambda_z$     |
| Service      | $\mu_z$         |
| Uptime       | $\xi_z$         |
| Downtime     | $\alpha_z$      |
| Retrial      | $\theta_z$      |

For convenience, the parameters are expressed as entries of the respective $m$-vectors $\lambda, \mu, \xi, \alpha, \theta,$ and $q$.

We next define the random variables that reflect the state of the system at time $t \geq 0$. Let

$$R(t) = \text{the number of demands in orbit at time } t$$
$$I(t) = \text{the number of items in the inventory at time } t$$
$$X(t) = \text{the status of the server at time } t,$$ where

$$X(t) = \iota \text{ if the server is idle}$$
$$X(t) = \beta \text{ if the server is busy}$$
$$X(t) = \gamma \text{ if the server is broken}$$

$$Z(t) = \text{the state of the random environment at time } t.$$
Due to the fact that all of the time duration of the process are exponentially distributed, the Markov property holds. Consequently, we may define the system as the Markov chain

\[ \Phi = \{(R(t), I(t), X(t), Z(t)) : t \in \mathbb{Z}_+\} \]

with the state space

\[ S_\Phi = \{(R, I, X, Z) : R \in \mathbb{Z}_+, I \in [s+1, S] \cap \mathbb{Z}_+, X \in \{\iota, \beta, \gamma\}, Z \in S\} \]

For convenience, we define the phase state partition of \( S_\Phi \) as the set

\[ S_{\Phi}^{ph} = \{(I, X, Z) : I \in [s, S] \cap \mathbb{Z}_+, X \in \{\iota, \beta, \gamma\}, Z \in S\} \]

If the elements of this set are enumerated in lexicographic order as

\[ \mathcal{L} = \{1, 2, \ldots, (S - s) \cdot (3m)\} \]

we may then rewrite the state space as

\[ S_\Phi = \{(R, k) : R \in \mathbb{Z}_+, k \in \mathcal{L}\} \]

Moreover, the process \( \Phi \) possesses an infinitesimal generator matrix \( Q^* = [q_{yy'}] \), where \( y = (R, I, \iota, Z) \in S_\Phi \) and \( y' = (R', I', \iota', Z') \in S_\Phi \). The rows and columns of the matrix are arranged according in the lexicographic order of ascending orbit size \( R \) (level) and the order given in \( \mathcal{L} \) at each level \( R \). The matrix consequently appears as in (1).

We will next describe transitions of the process by specifying elements of the generator matrix \( Q^* \) of \( \Phi \). Let \( \Delta(x) \) denote the \( m \times m \) matrix whose diagonal entries are the corresponding entries of the \( m \)-vector \( x \). Due to the skip-free nature of the transitions of \( \Phi \), the entries of \( Q^* \) are naturally organized according to the \( (S - s) \cdot (3m) \)-dimensional square block matrices \( A_2^{(R)} \), \( A_1^{(R)} \), and \( A_0^{(R)} \), which are arranged as follows:

\[
A_2^{(R)} = \begin{bmatrix}
\Theta_R & 0 & 0 & 0 & \cdots & 0 \\
0 & \Theta_R & 0 & 0 & \cdots & 0 \\
0 & 0 & \Theta_R & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \Theta_R & 0 \\
0 & 0 & 0 & \cdots & 0 & \Theta_R
\end{bmatrix}, \quad R = 1, 2, \ldots
\]
\[
A_1^{(R)} = \begin{bmatrix}
\Gamma_R & 0 & 0 & 0 & \cdots & M \\
M & \Gamma_R & 0 & 0 & \cdots & 0 \\
0 & M & \Gamma_R & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & M & \Gamma_R \\
0 & 0 & \cdots & 0 & M & \Gamma_R
\end{bmatrix}, \quad A_0^{(R)} = \begin{bmatrix}
\Lambda & 0 & 0 & 0 & \cdots & 0 \\
0 & \Lambda & 0 & 0 & \cdots & 0 \\
0 & 0 & \Lambda & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \Lambda & 0 \\
0 & 0 & \cdots & 0 & \Lambda
\end{bmatrix},
\]

\[R = 0, 1, \cdots,\]

where the \((3m \times 3m)\) matrices \(\Theta_R, \Gamma_R, \Lambda,\) and \(M\) are given by

\[
\Theta_R = \begin{bmatrix}
0 & \Delta(R\theta) & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \Gamma_R = \begin{bmatrix}
Q_i & \Delta(\lambda) & \Delta(\xi) \\
0 & Q_\beta & 0 \\
\Delta(\alpha) & 0 & Q_\gamma
\end{bmatrix},
\]

\[
\Lambda = \begin{bmatrix}
0 & 0 & 0 \\
0 & \Delta(\lambda) & \Delta(\xi) \\
0 & 0 & \Delta(\lambda)
\end{bmatrix}, \quad M = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \Delta(\mu) & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

and \(\Delta(u)\) is the diagonal matrix whose diagonal entries consist of the entries of the vector \(u\).

Additionally, for each \(R \in \mathbb{Z}_+\), the \(m \times m\) matrices \(Q_i, Q_\beta,\) and \(Q_\gamma\) are defined as

\[
Q_i(R) = Q - \Delta(\lambda + \xi + R\theta), \quad R \in \mathbb{Z}_+ \\
Q_\beta = Q - \Delta(\lambda + \mu + \xi) \\
Q_\gamma = Q - \Delta(\lambda + \alpha).
\]

Further define for each \(R \in \mathbb{Z}_+\) the scalar values \(d_i(z, R), d_\beta(z),\) and \(d_\gamma(z),\) which are the absolute values of the \(z\)th diagonal elements of \(Q_i(R), Q_\beta,\) and \(Q_\gamma,\) respectively. This gives

\[
d_i(z, R) = q_z + \lambda_z + \xi_z + R\theta_z \\
d_\beta(z) = q_z + \lambda_z + \mu_z + \xi_z \\
d_\gamma(z) = q_z + \lambda_z + \alpha_z.
\]

Table I provides the nonzero scalar entries of the generator block matrix terms \(A_2^{(R)}, A_1^{(R)},\) and \(A_0^{(R)}\).
As a result of the preceding construction, we obtain the following:

**Theorem 1.** The process $\Phi$ is an irreducible continuous-time LDQBD with infinitesimal generator $Q^*$ whose nonzero entries are given in Table 1. Furthermore, the jump process $\tilde{\Phi}$ of $\Phi$ satisfies Assumption [1].

**Proof.** It remains to show that $\tilde{\Phi}$ satisfies Assumption [1]. This may be done by constructing the block matrices $\tilde{A}^{(R)}_k$ for each $k = 0, 1, 2$, followed by the computation of the element-wise limit (if the limit exists)

$$\tilde{A}^* = \lim_{R \to \infty} \tilde{A}^{(R)}_k,$$

where $\tilde{A}^{(R)} = \tilde{A}^{(R)}_0 + \tilde{A}^{(R)}_1 + \tilde{A}^{(R)}_2$.

For convenience, we will define the $m$- (row) vectors $d_{\iota}(R)$, $d_{\beta}$, and $d_{\gamma}$, whose entries consist of terms $d_{\iota}(z, R)$, $d_{\beta}(z)$, and $d_{\gamma}(z)$, for each $z = 1, \ldots, m$. Further define for each $R \in \mathbb{Z}_+$ the

| $k$ | $R$ | Initial $y$ | Terminal $y'$ | Description |
|-----|-----|--------------|---------------|-------------|
| 0   | ≥ 0 | $(R, I, \iota, Z)$ | $(R + 1, I, \gamma, Z)$ | $\xi_z$ Server fails when busy |
| 1   | ≥ 0 | $(R, I, \iota, Z)$ | $(R, I, \iota, Z')$ | $q_{ZZ'}$ Environment (idle) |
|     |     | $(R, I, \iota, Z)$ | $(R, I, \beta, Z)$ | $\lambda_z$ Arrival while idle |
|     |     | $(R, I, \iota, Z)$ | $(R, I, \gamma, Z)$ | $\xi_z$ Server fails while idle |
|     |     | $(R, I, \iota, Z)$ | $(R, I, \iota, Z)$ | $-d_{\iota}(z, R)$ Diagonal entry (idle) |
|     |     | $(R, I, \beta, Z)$ | $(R, I, \iota, Z)$ | $\mu_z$ Demand, $I - 1 > s$ |
|     |     | $(R, s + 1, \beta, Z)$ | $(R, S, \iota, Z)$ | $\mu_z$ Demand, restocked |
|     |     | $(R, I, \beta, Z)$ | $(R, I, \beta, Z)$ | $-d_{\beta}(z)$ Diagonal entry (busy) |
|     |     | $(R, I, \gamma, Z)$ | $(R, I, \gamma, Z')$ | $q_{ZZ'}$ Environment (failed) |
|     |     | $(R, I, \gamma, Z)$ | $(R, I, \iota, Z)$ | $\alpha_z$ Server repaired |
|     |     | $(R, I, \gamma, Z)$ | $(R, I, \gamma, Z)$ | $-d_{\gamma}(z)$ Diagonal entry (failed) |

Table 1: Nonzero entries of infinitesimal generator $Q^*$ of $\Phi$. 




11
We divide each of the rows of \( A^{(R)} \) by the corresponding diagonal (nonzero) entries of \( \Delta \Gamma_R \) to obtain the \((S-s)(3m)\)-dimensional square matrix \( \tilde{A}^{(R)} \) of the jump process:

\[
\tilde{A}^{(R)} = \begin{bmatrix}
\Delta \Gamma^{-1}_R S_R & 0 & 0 & 0 & \cdots & \Delta \Gamma^{-1}_R M \\
\Delta \Gamma^{-1}_R M & \Delta \Gamma^{-1}_R S_R & 0 & 0 & \cdots & 0 \\
0 & \Delta \Gamma^{-1}_R M & \Delta \Gamma^{-1}_R S_R & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \Delta \Gamma^{-1}_R M & \Delta \Gamma^{-1}_R S_R & 0 \\
0 & 0 & \cdots & 0 & \Delta \Gamma^{-1}_R M & \Delta \Gamma^{-1}_R S_R \\
\end{bmatrix},
\]

where the \(3m \times 3m\) block matrix \( S_R = \Lambda + \Gamma_R + \Theta_R \) and \( M \) are as defined in Section 4.

The subsequent computation of the limiting matrix \( \tilde{A}^* = \lim_{R \to \infty} \tilde{A}^{(R)} \) will be accomplished in an element-wise fashion. Its expression will require the \(3m \times 3m\) limiting matrix

\[
\tilde{S}^* = \lim_{R \to \infty} \Delta \Gamma^{-1}_R S_R = \tilde{\Lambda} + \tilde{\Gamma}^* + \tilde{\Theta}^*,
\]

where the terms

\[
\tilde{M} = \Delta \Gamma^{-1}_R M, \quad \tilde{\Lambda} = \Delta \Gamma^{-1}_R \Lambda, \\
\tilde{\Gamma}^* = \lim_{R \to \infty} \Delta \Gamma^{-1}_R \Gamma_R, \quad \tilde{\Theta}^* = \lim_{R \to \infty} \Delta \Gamma^{-1}_R \Theta_R,
\]

are evaluated in an element-wise manner. Using the shorthand

\[
\frac{A}{B} = B^{-1}A
\]

for two square matrices \( A \) and \( B \), we obtain the limiting matrix

\[
\tilde{A}^* = \lim_{R \to \infty} \tilde{A}^{(R)} = \begin{bmatrix}
\tilde{S}^* & 0 & 0 & 0 & \cdots & \tilde{M} \\
\tilde{M} & \tilde{S}^* & 0 & 0 & \cdots & 0 \\
0 & \tilde{M} & \tilde{S}^* & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \tilde{M} & \tilde{S}^* & 0 \\
0 & 0 & \cdots & 0 & \tilde{M} & \tilde{S}^* \\
\end{bmatrix},
\]
with $3m \times 3m$ block elements given by

$$\tilde{S}^* = \begin{bmatrix} 0 & I_3 & 0 \\ 0 & \frac{\Delta(\lambda+q)+Q}{\Delta(d_\gamma)} & 0 \\ \frac{\Delta(\alpha)}{\Delta(d_\gamma)} & 0 & \frac{\Delta(\lambda+q)+Q}{\Delta(d_\gamma)} \end{bmatrix}$$

and

$$\tilde{M} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\Delta(\mu)}{\Delta(d_\gamma)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have thus demonstrated that $\tilde{\Phi}$ satisfies Assumption 1.

5. System Stability

An analytic traffic intensity formula will now be derived using the methods of Cordeiro et al. (2019), which utilizes a Foster-Lyapunov drift approach to obtain a criterion based on the average drift. It is a well-known fact that, for an irreducible LDQBD $\Phi$, its ergodicity or non-ergodicity is equivalent to that of its jump chain $\tilde{\Phi}$. Hence, we will derive a traffic intensity condition for the inventory system based upon the analytic ergodicity criterion that will be derived for $\tilde{\Phi}$ using Proposition 3 of Cordeiro et al. (2019).

Let $\tilde{P}$, which is depicted in (2), be the transition probability matrix of jump chain $\tilde{\Phi}$ that follows Assumption 1. We subsequently define the average drift $D^*$ of process $\tilde{\Phi}$ to be the scalar quantity

$$D^* = \pi^* \left( \bar{A}_0^* - \bar{A}_2^* \right) e$$

where $\pi^*$ solves the linear system

$$\pi^* \tilde{A}^* = \pi^*, \quad \pi^* \cdot e = 1.$$  

(7)

Under Assumption 1, it is shown in Cordeiro et al. (2019) that a unique solution to (7) exists. Further results that establish the condition $D^* < 0$ to the positive recurrence of $\tilde{\Phi}$ lead eventually to the following stability criterion.

**Theorem 2.** The continuous-time LDQBD process $\Phi$ is positive recurrent if and only if $D < 0$, where

$$D = e^\top (\Delta(\alpha)\Delta(\lambda - \mu) + \Delta(\lambda)\Delta(\xi)) e.$$  

(8)

**Proof.** It is necessary to compute the limiting average drift term $D^*$ defined in equation (6) in order prove the result. To do this, the vector solution

$$\pi^* = \begin{bmatrix} \pi_{ijk} \end{bmatrix}_{i=\bar{s}+1,...,S, j=\bar{\alpha},\beta,\gamma, k=1,...,m}$$

is required.
of $[7]$ is required. Note that the vector is written in partitioned form according to the states $(i, j, k) \in S_{\Phi}$. For example, the notation $\pi_{i*,*}$ denotes the $(3 \times m)$-dimensional partition of $\pi^*$ for which $i$ is held constant and $\pi_{i,j,*}$ the $m$-dimensional partition for which both $i$ and $j$ are fixed.

When expanded, system of equations $[7]$ becomes

$$
\pi_{s+1,*} S^* + \pi_{s+2,*} \tilde{M} = \pi_{s+1,*}
$$

$$
\pi_{s+2,*} S^* + \pi_{s+3,*} \tilde{M} = \pi_{s+2,*}
$$

$$
\vdots
$$

$$
\pi_{S,*} S^* + \pi_{s+1,*} \tilde{M} = \pi_{S,*}
$$

$$
\pi^* e = 1.
$$

We will proceed by induction on the inventory difference term $(S - s)$. To this end, consider an inventory system in which $S - s = 2$. Then system $[9]$ may be written in vector-matrix form as

$$
\begin{bmatrix}
\pi_{s+1,*} & \pi_{s+2,*}
\end{bmatrix}
\begin{bmatrix}
\tilde{S}^* \\
\tilde{M}
\end{bmatrix}
\begin{bmatrix}
e_{3m}^T
\end{bmatrix}
= \begin{bmatrix}
\pi_{s+1,*} & \pi_{s+2,*} & 1
\end{bmatrix},
$$

where $e_n$ is a column vector consisting entirely of $n \geq 1$ ones and

$$
\pi^*_2 = \begin{bmatrix}
\pi_{s+1,*} & \pi_{s+2,*}
\end{bmatrix}
$$

is its solution. For convenience, we can write this as the transpose system

$$
\begin{bmatrix}
(\tilde{S}^*)^T & (\tilde{M})^T \\
(\tilde{M})^T & (\tilde{S}^*)^T \\
e_{3m}^T & e_{3m}^T
\end{bmatrix}
\begin{bmatrix}
\pi_{(s+1)*} \\
\pi_{(s+2)*}
\end{bmatrix}
= \begin{bmatrix}
\pi_{(s+1)*}^T \\
\pi_{(s+2)*}^T & 1
\end{bmatrix}.
$$

When expanded, the system becomes

$$
\begin{bmatrix}
0 & 0 & \frac{\Delta(\alpha)}{\Delta(d_f)} & 0 & \frac{\Delta(\mu)}{\Delta(d_f)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\Delta(\xi)}{\Delta(d_f)} & \frac{\Delta(\lambda + \gamma)}{\Delta(d_f)} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\Delta(\alpha)}{\Delta(d_f)} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\Delta(\xi)}{\Delta(d_f)} & \frac{\Delta(\lambda + \gamma)}{\Delta(d_f)} \\
e_{m}^T & e_{m}^T & e_{m}^T & e_{m}^T & e_{m}^T & e_{m}^T
\end{bmatrix}
\begin{bmatrix}
\pi_{(s+1),*} \\
\pi_{(s+1),*} \\
\pi_{(s+2),*} \\
\pi_{(s+2),*} \\
\pi_{(s+2),*} \\
\pi_{(s+2),*}
\end{bmatrix}
$$

14
\[
\begin{bmatrix}
\pi_{(s+1)\iota} & \pi_{(s+1)\beta} & \pi_{(s+1)\gamma} & \pi_{(s+2)\iota} & \pi_{(s+2)\beta} & \pi_{(s+2)\gamma} & 1
\end{bmatrix}^\top,
\]

where \(0\) denotes the \(m \times m\) matrix of zeros.

The solution \(\pi^\star_2\) of (10) is most expediently obtained if one first solves the system in terms of the composite block entries of the corresponding coefficient matrix, from whence we will obtain a solution in terms of block matrices. This solution may then be easily converted to actual scalar vector solution of (10).

Some modifications to (10) must be made. First, we replace each occurrence of the \(m\)-dimensional vector variables \(\pi_{IX}\) with the symbolic \(m\)-vectors \(\pi_{IX}\) for each \(I \in \mathbb{Z}_+\) and for each \(x = \iota, \beta, \gamma\). To convert from the symbolic block vector to the corresponding \(3m(S-s)\)-entry vector, the relationship

\[\pi_{IX} = \pi_{IX} \cdot e_{3m(S-s)}\]  

may be applied. More specifically, \(\pi_{IX}\) is a vector partition consisting of \(m\) scalar entries whilst \(\pi_{IX}\) may be expressed as a \(m \times m\) block matrix.

Next, we replace the entries of the last row of the coefficient matrix with the identity matrix \(I_m\). For the moment, the 1 on the right-hand side will be replaced with an indeterminate \(m\)-dimensional ‘vector’ \(u\) until the appropriate value can be determined. Relationship (11) also applies to \(u\) in the sense that \(u \cdot e_{3m(S-s)}\) is a vector partition consisting of \(m\) scalar entries whilst \(u\) is symbolic \(m \times m\) block matrix term.

We may then rewrite system (10) as

\[
\begin{bmatrix}
0 & 0 & \frac{\Delta(\alpha)}{\Delta(d_{\iota})} & 0 & \frac{\Delta(\mu)}{\Delta(d_{\beta})} & 0 \\
I_m & \frac{\Delta(\lambda+q)+Q^T}{\Delta(d_{\gamma})} & 0 & 0 & 0 \\
0 & \frac{\Delta(\xi)}{\Delta(d_{\iota})} & \frac{\Delta(\lambda+q)+Q^T}{\Delta(d_{\iota})} & 0 & 0 & 0 \\
0 & \frac{\Delta(\mu)}{\Delta(d_{\beta})} & 0 & 0 & 0 & \frac{\Delta(\alpha)}{\Delta(d_{\beta})} \\
0 & 0 & 0 & I_m & \frac{\Delta(\lambda+q)+Q^T}{\Delta(d_{\gamma})} & 0 \\
0 & 0 & 0 & 0 & \frac{\Delta(\xi)}{\Delta(d_{\iota})} & \frac{\Delta(\lambda+q)+Q^T}{\Delta(d_{\gamma})} \\
I_m & I_m & I_m & I_m & I_m & I_m
\end{bmatrix}
\begin{bmatrix}
\pi_{(s+1)\iota}^\top \\
\pi_{(s+1)\beta}^\top \\
\pi_{(s+1)\gamma}^\top \\
\pi_{(s+2)\iota}^\top \\
\pi_{(s+2)\beta}^\top \\
\pi_{(s+2)\gamma}^\top \\
u
\end{bmatrix}.
\]

We shall denote the solution of base system (12) as the symbolic (row) vector \(\pi^\star_6\), again, for which the relationship (11) applies.
In order to solve (12) using conventional methods for linear systems with scalar unknowns, it would be necessary for all elements of the coefficient matrix to be (symbolic) diagonal matrices. However, the generator matrix $Q$ of the random environment is not diagonal, which makes the system exceedingly difficult to solve. In fact, since row sums of $A^*$ are not multiples of $I_m$, it is not ‘stochastic’ in this block-matrix sense, which, in effect, causes the system to become inconsistent for any value of $u$. To circumvent this difficulty, we will first solve for what will be termed a $Q$-homogeneous solution $\pi^*_h$ of (12) by setting $Q = 0$. When this is done, the matrix $A^*$ becomes ‘stochastic’ in the sense that the row sums are $I_m$. Hence, the following system with $u = I_m$ is consistent:

$$
\begin{bmatrix}
0 & 0 & \frac{\Delta(\alpha)}{\Delta(d)} & 0 & \frac{\Delta(\mu)}{\Delta(d)} & 0 \\
I_m & \frac{\Delta(\lambda + q)}{\Delta(d)} & 0 & 0 & 0 & 0 \\
0 & \frac{\Delta(\xi)}{\Delta(d)} & \frac{\Delta(\lambda + q)}{\Delta(d)} & 0 & 0 & 0 \\
0 & \frac{\Delta(\mu)}{\Delta(d)} & 0 & 0 & 0 & \frac{\Delta(\alpha)}{\Delta(d)} \\
0 & 0 & 0 & I_m & \frac{\Delta(\lambda + q)}{\Delta(d)} & 0 \\
0 & 0 & 0 & 0 & \frac{\Delta(\xi)}{\Delta(d)} & \frac{\Delta(\lambda + q)}{\Delta(d)} \\
I_m & I_m & I_m & I_m & I_m & I_m
\end{bmatrix}
\begin{bmatrix}
\pi_{h(s+1)}^* \\
\pi_{h(s+1)\beta}^* \\
\pi_{h(s+1)\gamma}^* \\
\pi_{h(s+2)}^* \\
\pi_{h(s+2)\beta}^* \\
\pi_{h(s+2)\gamma}^*
\end{bmatrix}
= 
\begin{bmatrix}
\pi_{h(s+1)}^* \\
\pi_{h(s+1)\beta}^* \\
\pi_{h(s+1)\gamma}^* \\
\pi_{h(s+2)}^* \\
\pi_{h(s+2)\beta}^* \\
\pi_{h(s+2)\gamma}^*
\end{bmatrix}
\begin{bmatrix}
u = I_m
\end{bmatrix}.
$$

(13)

The resulting symbolic block solution is

$$
(\pi^*_h)^T = \frac{1}{2} \cdot \frac{I_m}{\Delta(\alpha)\Delta(\lambda + 2\mu + q) + \Delta(\xi)\Delta(3\alpha + \lambda + q)}.
$$

where action of the binary operator ‘·’ is clear from the context.

Next, the non-$Q$-homogeneous system (12) will be solved. In order to do this, we first define the $m \times m$ matrix

$$
\Pi = \begin{bmatrix} p^T & p^T & \ldots & p^T \end{bmatrix}^T,
$$

where $p$ is the stationary probability (row) vector of $Q$. 

16
Claim 1: The block matrix solution to system (12) with \( u = \Pi \) is

\[
(\pi_b^*)^\top = \frac{\Pi^\top}{2} \cdot \frac{I_m}{\Delta(\alpha)\Delta(\lambda + 2\mu + q) + \Delta(\xi)\Delta(3\alpha + \lambda + q)} \cdot \begin{bmatrix}
\Delta(\alpha)\Delta(\mu + \xi) \\
\Delta(\alpha)\Delta(\mu + \xi + q) \\
\Delta(\xi)\Delta(\alpha + \lambda + q) \\
\Delta(\alpha)\Delta(\mu + \xi) \\
\Delta(\alpha)\Delta(\mu + \xi + q) \\
\Delta(\xi)\Delta(\alpha + \lambda + q)
\end{bmatrix}.
\]

Proof of Claim 1: This is easily demonstrated by evaluating system (12) with the given value of \( \pi_b^* \) and subsequently applying the identity

\[
Q^\top \cdot p^\top = 0.
\]

We may use Claim 1 to construct a vector solution \( \pi_r^* \) with scalar entries to system (10), which is detailed in the following result:

Claim 2: The \( 3m(S - s) \)-vector solution \( \pi_2^* \) to system (10) is given by

\[
\pi_2^* = \frac{1}{m} \pi_b^* \cdot e_{3m(S-s)}.
\]

Proof of Claim 2: In order to show that the assertion of Claim 2 is true, we must first apply the identity (11), which produces a \( 3m(S - s) \) vector term. By doing so, and by simultaneously applying the identity

\[
\Pi \cdot e_m = e_m,
\]

we obtain the chain of equalities

\[
\pi_b^* \cdot e_{3m(S-s)} = e_{3m(S-s)}^\top \cdot \pi_b^* \\
= \begin{bmatrix}
e_m^\top & e_m^\top & \ldots & e_m^\top
\end{bmatrix} \cdot \begin{bmatrix}
\Delta(\alpha)\Delta(\mu + \xi) \\
\Delta(\alpha)\Delta(\mu + \xi + q) \\
\Delta(\xi)\Delta(\alpha + \lambda + q) \\
\Delta(\alpha)\Delta(\mu + \xi) \\
\Delta(\alpha)\Delta(\mu + \xi + q) \\
\Delta(\xi)\Delta(\alpha + \lambda + q)
\end{bmatrix}.
\]
We thus arrive at an expression relating the $Q$- and the non-$Q$-homogeneous solutions. Because this $Q$-homogeneous solution $\pi^*_h$ must satisfy the original scalar-coefficient system (10), the relationship

$$e_{3m(S-s)}^T \cdot \pi^*_h \cdot e_{3m(S-s)} = 1$$  \hspace{1cm} (14)$$

must hold. However, the fact that $\pi^*_h$ solves system (13) implies that the following equalities likewise hold:

$$e_{3m(S-s)}^T \cdot \pi^*_h \cdot e_{3m(S-s)} = e_m^T \cdot I_m \cdot e_m = e_m^T \cdot e_m = n$$

Thus, by Equation (14), we must have

$$\pi^*_2 = \frac{1}{m} e_{3m(S-s)}^T \cdot \pi^*_h$$

which proves the Claim 2.

Now that a limiting stationary vector $\pi^*_2$ is in hand, we proceed to compute the corresponding limiting drift expression. First, we observe that $\pi^*_2$ is composed of repeating blocks of $3m$-dimensional vectors $\pi_r$, where

$$\pi_r^T = \begin{bmatrix} \Delta(\alpha) \Delta(\mu + \xi) \\ \Delta(\alpha) \Delta(\lambda + 2\mu + q) + \Delta(\xi) \Delta(3\alpha + \lambda + q) \\ \Delta(\alpha) \Delta(\mu + \xi + q) \\ \Delta(\alpha) \Delta(\lambda + 2\mu + q) + \Delta(\xi) \Delta(3\alpha + \lambda + q) \\ \Delta(\alpha) \Delta(\mu + \lambda + q) \\ \Delta(\alpha) \Delta(\lambda + 2\mu + q) + \Delta(\xi) \Delta(3\alpha + \lambda + q) \end{bmatrix},$$

which permits us to write

$$\pi_2^* = \frac{1}{2m} \begin{bmatrix} \pi_r e_m & \pi_r e_m \end{bmatrix}.$$
By (6), we compute

\[
D^* = \pi_2^* (\tilde{A}_0^* - \tilde{A}_2^*) e
\]

\[
= \frac{1}{2m} e_{3m}^T \left[ \begin{array}{cc} \tilde{\Lambda} - \tilde{\Theta}^* & 0 \\ 0 & \tilde{\Lambda} - \tilde{\Theta}^* \end{array} \right] \left[ \begin{array}{c} e_{3m}^T \\ e_{3m}^T \end{array} \right]
\]

\[
= \frac{1}{m} e_{3m}^T \pi_r (\tilde{\Lambda} - \tilde{\Theta}^*) e_{3m}
\]

\[
= \frac{1}{m} e_{3m}^T \pi_r \left[ \begin{array}{cccc} 0 & -I_m & 0 \\ 0 & \frac{\Delta(\lambda)}{\Delta(\lambda+\mu+\xi+q)} & \frac{\Delta(\xi)}{\Delta(\lambda+\mu+\xi+q)} \\ 0 & 0 & \frac{\Delta(\lambda)}{\Delta(\alpha+\lambda+q)} \end{array} \right] \left[ \begin{array}{c} e_m^T \\ e_m^T \\ e_m^T \end{array} \right]
\]

\[
= \frac{1}{m} e_m^T \frac{\Delta(\alpha)\Delta(\lambda - \mu) + \Delta(\lambda)\Delta(\xi)}{\Delta(\alpha)\Delta(\lambda + 2\mu + q) + \Delta(\xi)\Delta(3\alpha + \lambda + q)} e_m.
\]  \hspace{1cm} (15)

For the induction step, we assume that the drift expression (15) holds for \((S - s - 1)\). For this model, we obtain the stationary vector

\[
\pi_2^* = \frac{1}{(S - s - 1)m} \left[ \pi_r e_m \pi_r e_m \pi_r e_m \ldots \pi_r e_m \right].
\]

For an \((S - s)\) model, the matrix \(\tilde{A}^*\) gains an additional repeated block matrix row, from which we deduce the new value of the stationary probability vector to be

\[
\pi_{(S-s)}^* = \frac{1}{(S - s)m} \left[ \pi_r e_m \pi_r e_m \pi_r e_m \ldots \pi_r e_m \right].
\]

We now repeat the previous computation of drift \(D^*\) as

\[
D^* = \pi_{(S-s)}^* (\tilde{A}_0^* - \tilde{A}_2^*) e
\]

\[
= \frac{1}{(S - s)m} e_{3m}^T \left[ \begin{array}{cccc} \tilde{\Lambda} - \tilde{\Theta}^* & 0 & 0 & 0 \ldots & 0 \\ 0 & \tilde{\Lambda} - \tilde{\Theta}^* & 0 & 0 & 0 \ldots & 0 \\ : & 0 & \tilde{\Lambda} - \tilde{\Theta}^* & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & \tilde{\Lambda} - \tilde{\Theta}^* \end{array} \right] \left[ \begin{array}{c} e_{3m}^T \\ e_{3m}^T \\ e_{3m}^T \\ e_{3m}^T \end{array} \right]
\]

\[
= \frac{1}{m} e_{3m}^T \pi_r (\tilde{\Lambda} - \tilde{\Theta}^*) e_{3m}
\]
\[
\frac{1}{m} e_m^\top \frac{\Delta(\alpha)\Delta(\lambda - \mu) + \Delta(\lambda)\Delta(\xi)}{\Delta(\alpha)\Delta(\lambda + 2\mu + q) + \Delta(\xi)\Delta(3\alpha + \lambda + q)} e_m. \quad (16)
\]

Hence, the expression (16) is the average drift \( D^* \) of the jump chain \( \tilde{\Phi} \) for all values of \( S - s \geq 2 \).

Since, by Proposition 3 of Cordeiro et al. (2019), \( D^* < 0 \) indicates positive recurrence of the Markov chain, we may discard positive factors of (16) to obtain

\[
D = e_m^\top (\Delta(\alpha)\Delta(\lambda - \mu) + \Delta(\lambda)\Delta(\xi)) e_m,
\]

which is the drift function (8) in the statement of the Theorem.

It is customary to restate the drift condition in Theorem 2 as a traffic intensity, which we shall denote as \( \rho \). By rearranging terms in the average drift (8), we arrive at the following Corollary to Theorem 2.

**Corollary 1.** The traffic intensity \( \rho \) of the process \( \Phi \) may be written as

\[
\rho = \frac{e^\top (\Delta(\lambda)\Delta(\alpha + \xi)) e}{e^\top (\Delta(\mu)\Delta(\alpha)) e}. \tag{17}
\]

Subsequently, the continuous-time LDQBD process \( \Phi \) is positive recurrent if and only if \( \rho < 1 \).

A non-Markov-modulated queueing system tends to become more unstable (closer to null-recurrent) as the traffic intensity approaches 1 from below. However, as we shall see later, this positive correlation of traffic intensity to average orbit size is degraded somewhat when considering this quantity over multiple environment states.

**6. Steady-State Distribution and Performance Measures**

If \( D^* < 0 \), then by Theorem 1, \( \Phi \) is positive recurrent. In this case, the steady state probabilities defined as

\[
P(R,i,x,z) = \lim_{t \to \infty} P((R(t),I(t),X(t),Z(t)) = (R,i,x,z)),
\]

\( R \in \mathbb{Z}_+, (i,x,z) \in \mathcal{L} \)

exist. Since \( \mathcal{L} \) is a finite set, we may enumerate the elements of this set as \( \mathcal{L} = \{1, 2, \ldots, M\} \), where we define the \( k \)th element of \( \mathcal{L} \) as \( (i_k,x_k,z_k) \) and \( M = (S-s) \cdot (3m) \). The steady-state probabilities may then be expressed more concisely as

\[
p_{R,k} = P(R,i_k,x_k,z_k),
\]

20
whereupon we may define the $M$-dimensional row vectors

$$p_R = (p_{R,1}, p_{R,2}, \ldots, p_{R,M}), \quad R \in \mathbb{Z}_+$$

of steady-state probabilities of $\Phi$ grouped according to orbit size $R$. Assuming the positive recurrence of $\Phi$, one may infer the presence of the matrix-geometric relationship between terms of $p_R$, which is given in this article as

$$p_R = \mathbf{p}_0 \sum_{R=0}^{\infty} \left[ \prod_{\ell=0}^{R-1} R_\ell \right] e, \quad (18)$$

where the rate matrices $\{R_\ell : \ell \in \mathbb{Z}_+\}$ are the minimal non-negative solutions to the system of equations

$$A_\ell^{(0)} + R_\ell A_\ell^{(1)} + R_\ell \left[ R_{\ell+1} A_{\ell+2}^{(1)} \right] = 0, \quad \ell \in \mathbb{Z}_+, \quad (19)$$

and the level 0 steady state probability $p_0$ is the minimal vector solution to

$$p_0 \left[ A_{1}^{(0)} + R_0 A_{2}^{(1)} \right] = 0. \quad (20)$$

Since it is unlikely that (19) and (20) have closed-form solutions, however, it is more expedient to produce estimated measures of performance. To this end, one may apply one of several established algorithms that were developed for the purpose of estimating the steady-state distribution of an LDQBD, such as that of Bright & Taylor (1995). The method, via Algorithm 1, produces estimated stationary probabilities $p_{R,k}(R^*) \sim p_{R,k}$ of a truncated system $\Phi(R^*)$, say, at some level (orbit size) $R^*$ that is sufficiently large. The term ‘sufficiently large’ is used in the context of the fact that

$$p_R = \lim_{R^* \to \infty} p_R(R^*), \quad k \in \mathbb{Z}_+. \quad (21)$$

In other words, the estimates become progressively more accurate as the system is truncated at larger levels $R^*$. Because Algorithm 1 produces successive estimates of $p_R$ by means of the matrix geometric recurrence relation (18), there is a need to efficiently compute the rate matrices $R_\ell$, a task for which Algorithm 2 is utilized. Algorithm 3 provides iterative intermediate computations that are used during the execution of Algorithm 2.

With the steady-state distribution in hand, one may compute the asymptotic performance measures of the queuing inventory system $\Phi$. We begin with the marginal steady-state probabilities $p_i, p_b, p_f$ of the server status, which are

Idle Probability: \hspace{1cm} $p_i = \sum_{R=0}^{\infty} \sum_{i=s+1}^{S} \sum_{k=1}^{m} \mathbb{P}(R, i, \ell, k)$
Busy Probability: \[ p_\beta = \sum_{R=0}^{\infty} \sum_{i=s+1}^{S} \sum_{k=1}^{m} \mathbb{P}(R, i, \beta, k) \]

Failure Probability: \[ p_\gamma = \sum_{R=0}^{\infty} \sum_{i=s+1}^{S} \sum_{k=1}^{m} \mathbb{P}(R, i, \gamma, k) \]

Likewise, the steady-state probability \( p_R \) of the number in orbit is the marginal probability
\[
p_R = \sum_{i=s+1}^{S} \sum_{k=1}^{m} \left[ \mathbb{P}(R, i, \iota, k) + \mathbb{P}(R, i, \beta, k) + \mathbb{P}(R, i, \gamma, k) \right]
\]

The long-run expected numbers in orbit \( (L_R) \) and the system \( (L) \) may then be computed in the usual way as
\[
L_R = \sum_{R=0}^{\infty} R \cdot p_R \quad \text{and} \quad L = L_R + p_\beta.
\]

For the temporal measures of queueing performance, we will employ Little’s Law, which in turn will require the computation of the long-run average exponential rate of input over environment states, which is
\[
\bar{\lambda} = \sum_{z=1}^{m} \lambda_z \cdot p_z = \lambda \cdot p_e,
\]
where \( p_e = [p_z]_{z \in S} \) is the stationary vector of probabilities of being in each state \( z \in S \) for the random environment, and as such, must solve the linear system
\[
p_e \cdot Q = 0, \quad p_e \cdot e = 1.
\]

We may then apply Little’s Law to compute the long-run expected times in orbit \( (W_R) \) and in the system \( (W) \):
\[
W_R = L_R / \bar{\lambda} \quad \text{and} \quad W = L / \bar{\lambda}.
\]

For performance measures having to do with the amount in inventory, we must first have the steady-state distribution of the number in inventory. The marginal long-run probability of there being \( I \in [s + 1, S] \) in inventory is given by
\[
p_I = \sum_{R=0}^{\infty} \sum_{k=1}^{m} \left[ \mathbb{P}(R, i, \iota, k) + \mathbb{P}(R, i, \beta, k) + \mathbb{P}(R, i, \gamma, k) \right].
\]

From this distribution, we may obtain the long-run expected inventory level
\[
B_{inv} = \sum_{I=s+1}^{S} I \cdot p_I.
\]

Lastly, we consider the asymptotic expected time \( D_S \) to deplete (or replenish) the inventory from the maximum level \( S \). Since the inventory decrements by one just before a customer leaves
the system, depletion from the maximum level of \( S \) items occurs whenever \( S - s \) customers are successfully processed. Thus

\[
D_S = (S - s)W.
\]

7. Numerical Examples

In this section, a pair of multi-environment inventory systems is studied. First, estimates of the steady-state distributions are computed via the Bright and Taylor algorithms. From these distributions, the various system performance measures are derived and then utilized in a comparison of both systems. One of the striking features of a Markov-modulated queueing system (c.f. Neuts (1981)) is that a system that is unstable for one or more fixed environment states may still be observed to be stable in the sense of Corollary 1. It is possible to evaluate such a fixed-environment inventory system simply by computing the traffic intensity \( \rho_z \) for a system with one state \( z \in S \), which, by (17), results in

\[
\rho_z = \frac{\lambda_z}{\mu_z} \left( 1 + \frac{\xi_z}{\alpha_z} \right).
\]

The comparison study will be based upon the examination of stable low- and high-traffic seven-state systems with varying levels of the the threshold \( s \) for a set upper limit \( S = 100 \) of the inventory level. The determination of traffic severity is founded upon a comparison of the average orbit sizes \( L_R \), with lower-traffic models demonstrating lower values of \( L_R \) as compared to their high-traffic counterparts. Exponential parameters for each system appear in Tables 2 and 3, respectively, along with the overall system traffic intensity \( \rho \) in (17). Observe that each example system exhibits one or more state-dependent traffic intensities \( \rho_z \) for which \( \rho_z \geq 1 \). Nevertheless, both systems demonstrate overall stability as measured by the traffic intensity \( \rho \).

As mentioned in the postscript to Corollary 1 the relative values of the traffic intensity \( \rho \) do not positively correlate to the ‘busyness’ of each system, at least with regard to average orbit size. The reason for the low-traffic system (see Table 2) having \( \rho = 0.45 \), as opposed to that of the high-traffic system (see Table 3) having \( \rho = 0.40 \), is that the low-traffic system is highly unstable in environment states 2 and 7. This degree of instability, in comparison, only occurs in state 7 for the other system. However, since the traffic intensity \( \rho \) does not contain \( q \), the measure is insensitive to the frequency of transitions into states 1 and 7. Consequently, \( \rho \) is conservative in its representation of the degree to which a system is considered stable.

The Bright and Taylor algorithm is then applied to each of the low- and high-traffic systems, which are each truncated to a maximum orbit size of \( R^* = 75 \). Using a common random environment
Table 2: Parameter values of the low-traffic seven-state system, $S = 35$ and $s = 10$.

| Environment ($z$) | $\lambda_z$ | $\mu_z$ | $\xi_z$ | $\alpha_z$ | $\theta_z$ | $\rho_z$ | $\rho$ |
|-------------------|-------------|---------|---------|------------|-----------|---------|-------|
| 1                 | 1.0         | 13.0    | 0.05    | 7.0        | 1.00      | 0.0775  | 0.4546|
| 2                 | 15.0        | 1.2     | 3.80    | 0.8        | 0.10      | 71.8750 |       |
| 3                 | 0.3         | 17.0    | 0.02    | 15.0       | 4.00      | 0.0177  |       |
| 4                 | 2.0         | 12.0    | 0.30    | 12.0       | 2.00      | 0.1708  |       |
| 5                 | 0.5         | 18.7    | 1.00    | 5.0        | 5.00      | 0.0321  |       |
| 6                 | 1.0         | 15.0    | 1.20    | 2.8        | 0.10      | 0.0952  |       |
| 7                 | 12.0        | 6.0     | 14.00   | 0.5        | 0.05      | 58.0000 |       |

Table 3: Parameter values of the high-traffic seven-state system, $S = 35$ and $s = 10$.

| Environment ($z$) | $\lambda_z$ | $\mu_z$ | $\xi_z$ | $\alpha_z$ | $\theta_z$ | $\rho_z$ | $\rho$ |
|-------------------|-------------|---------|---------|------------|-----------|---------|-------|
| 1                 | 10.0        | 7.0     | 5.0     | 2.0        | 1.0       | 5.0000  | 0.4041|
| 2                 | 0.1         | 4.2     | 0.8     | 0.1        | 1.7       | 0.2143  |       |
| 3                 | 1.0         | 8.0     | 1.0     | 15.0       | 2.0       | 0.1333  |       |
| 4                 | 0.8         | 10.0    | 0.3     | 12.0       | 2.0       | 0.0820  |       |
| 5                 | 2.0         | 4.5     | 1.0     | 5.0        | 5.0       | 0.5333  |       |
| 6                 | 0.5         | 2.0     | 0.7     | 13.0       | 1.0       | 0.2635  |       |
| 7                 | 11.0        | 0.3     | 0.2     | 0.5        | 0.5       | 51.3333 |       |

with infinitesimal generator $Q$ given by

$$Q = \begin{bmatrix}
-17.5 & 4.5 & 2.6 & 1.1 & 0.0 & 6.1 & 3.2 \\
5.8 & -32.3 & 3.2 & 7.8 & 4.4 & 8.2 & 2.9 \\
2.2 & 9.6 & -40.4 & 0.8 & 8.8 & 7.4 & 11.6 \\
0.1 & 1.7 & 5.1 & -19.8 & 0.0 & 12.9 & 0.0 \\
6.5 & 0.0 & 8.2 & 8.1 & -27.4 & 3.7 & 0.9 \\
6.6 & 8.9 & 16.2 & 3.9 & 8.2 & -45.9 & 2.1 \\
1.8 & 2.8 & 9.5 & 0.8 & 7.9 & 0.0 & -22.8
\end{bmatrix},$$

steady-state distributions for the low- and high-traffic systems are obtained. Although the exact distributions are omitted for the sake of brevity, the marginal steady-state distributions of orbit size for each example are depicted in Figures 2 and 3, respectively.

From the steady-state probabilities, we compute the marginal probabilities of server status and the performance measures described in Section 6 for each of the low- and high-traffic models. The results appear in Table 4.
As expected, the marginal busy probability, average orbit and system sizes, waiting times, and times $D_s$ to deplete the inventory are less in value overall for the light-traffic system as compared to their counterparts in the high-traffic system. This is true despite the tendency of the light-traffic system to more often experience a failed server. However, the average inventory content $B_{inv}$ is constant over both models due to the fact that there is no waiting time distribution for replenishment.

8. Conclusion

Due to the novel approach enabled by the results contained in Cordeiro et al. (2019), it was possible to derive a closed-form traffic intensity condition for a complex inventory system with every exponential rate parameter modulated by a random environment. Of note is the fact that the
expression is independent of the number of states in the environment, which facilitates the easy computation of the intensity matrix-vector operations. Using this information, it becomes possible to easily distinguish stable versus unstable systems, and thus allow the investigator to construct stable systems, estimate the steady-state distribution using existing algorithmic approaches, and subsequently obtain the corresponding long-run performance measures.

In effect, the method described herein to deriving a closed-form traffic intensity may potentially be used for more complex models whose underlying Markov chains exhibit the behavior described by Assumption 1. Models based upon a retrial queue with infinite capacity are particularly good candidates for this approach, and these are very important in describing real-world processes ranging from telecommunications networks to retail services, among many others. Furthermore, the analysis of systems subject to stochastic replenishment times, as well as those featuring the many variations of service interruptions and retrial regimes, is amenable to the drift-based methods demonstrated here.

Another important avenue for future investigation is an optimal control study based upon a long-run average cost. This would potentially require the employment of a version of the \((s, S)\) inventory system that allows for periodic, rather than continuous, reviews of the inventory level in a
Markov decision framework. Regardless, the current study opens up boundless possibilities for the range of models accessible to analytical as well as numerical investigation. Even more importantly, it represents the validation of the matrix-analytic approach to queueing analysis in studying very complex models in every field of application.

References

Arrow, K. J., Harris, T., & Marschak, J. (1951). Optimal Inventory Policy. *Econometrica, 19*, 250–272.

Artalejo, J., Krishnamoorthy, A., & Lopez-Herrero, M. (2006). Numerical analysis of (s, S) inventory systems with repeated attempts. *Annals of Operations Research, 141*, 67–83.

Bright, L., & Taylor, P. (1995). Calculating the equilibrium distribution in level dependent quasi-birth-and-death processes. *Communications in Statistics: Stochastic Models, 11*, 497–525.

Cordeiro, J., Kharoufeh, J., & Oxley, M. (2019). On the ergodicity of a class of level-dependent quasi-birth-and-death processes. *Advances in Applied Probability, 51*, 1109–1128.

Feldman, R. M. (1978). Continuous review (s, S) inventory system in a random environment. *J Appl Probab, 15*, 654–659.

Iglehart, D., & Karlin, S. (1962). Optimal policy for dynamic inventory process with nonstationary stochastic demands. In K. Arrow, S. Karlin, & H. Scarf (Eds.), *Studies in Applied Probability and Management Science* (pp. 127–147). Stanford University Press.

Karlin, S. (1960). Dynamic Inventory Policy with Varying Stochastic Demands. *Management Science, 6*, 231–258.

Ko, S.-S. (2020). A nonhomogeneous quasi-birth-death process approach for an (s, S) policy for a perishable inventory system with retrial demands. *Journal of Industrial & Management Optimization, 16*, 1415–1433.

Krishnamoorthy, A., Nair, S., & Narayanan, V. C. (2012). An inventory model with server interruptions and retrials. *Operational Research, 12*, 151–171.

Neuts, M. F. (1971). A queue subject to extraneous phase changes. *Adv. Appl. Prob., 3*, 78–119.
Neuts, M. F. (1978a). The $M/M/1$ queue with randomly varying arrival and service rates. *OPSEARCH, 15*, 139–157.

Neuts, M. F. (1978b). Further results on the $M/M/1$ queue with randomly varying rates. *OPSEARCH, 15*, 158–168.

Neuts, M. F. (1981). *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*. Dover Books on Advanced Mathematics. New York, NY: Dover Publications.

Özekici, S., & Parlar, M. (1999). Inventory models with unreliable suppliers in a random environment. *Annals of Operations Research*, .

Perry, D., & Posner, M. (2002). Production-inventory models with an unreliable facility operating in a two-state random environment. *Probability in the Engineering and Informational Sciences*, .

Song, J. S., & Zipkin, P. (1993). Inventory control in a fluctuating demand environment. *Operations Research, 41*, 351–370.

Ushakumari, P. V. (2006). On (s, S) inventory system with random lead time and repeated demands. *Journal of Applied Mathematics & Stochastic Analysis, Volume 2006. Article ID 81508*, 1 – 22.

Yadavalli, V. S. S., & Schoor, C. d. W. V. (2012). A perishable product inventory system operating in a random environment. *South African Journal of Industrial Engineering, 15*. 