Black holes and entropy in loop quantum gravity: An overview

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Abstract

Black holes in equilibrium and the counting of their entropy within Loop Quantum Gravity are reviewed. In particular, we focus on the conceptual setting of the formalism, briefly summarizing the main results of the classical formalism and its quantization. We then focus on recent results for small, Planck scale, black holes, where new structures have been shown to arise, in particular an effective quantization of the entropy. We discuss recent results that employ in a very effective manner results from number theory, providing a complete solution to the counting of black hole entropy. We end with some comments on other approaches that are motivated by loop quantum gravity.

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I. INTRODUCTION

Black holes (BH) have become rather prominent in fundamental physics ever since the fundamental results in the early 70’s showing that black holes satisfy some ‘thermodynamic-like laws’, summarized in the celebrated laws of black hole mechanics [1],

\[ \delta M = \frac{\kappa}{8\pi G} \delta A, \]  

From which one can formally relate,

\[ M \leftrightarrow E, \quad \kappa \leftrightarrow T, \quad A \leftrightarrow S, \]

where the relation between geometrical variables on Eq. (1) can be seen as the analogue of the first law of thermodynamics if the above association between geometric and thermodynamical objects is made. This analogy is further motivated by the fact that the surface gravity \( \kappa \) of a Killing horizon is constant and the area of an event horizon always grows. This observation, together with the proposal by Bekenstein and Hawking that BH possess a physical entropy and temperature, as confirmed by the computation of particle creation on black hole background, gave raise to a true identification between geometrical quantities and thermodynamical variables as follows [2]:

\[ E = M \quad T = \frac{\kappa}{2\pi} \quad \text{and} \quad S = \frac{A}{4G\hbar}. \]

It is not unnatural to interpret that black holes must behave as thermodynamic systems, and in particular possess a non-zero temperature (that vanishes in the classical limit) and an entropy (that blows up). Quantum theory was needed in order to identify temperature and entropy with geometrical objects, by means of Planck’s constant \( \hbar \), suggesting that these identifications are quantum in nature. But, in order to have a full analogy, the question of what are the underlying degrees of freedom responsible for entropy became a pressing one. In other words, how can we account for the (huge) entropy associated to the black hole horizons?

The standard wisdom is that only with a full marriage of Gravity and the Quantum will we be able to understand this issue. This is one of the main challenges that faces any candidate quantum theory of gravity.

During the past 20 years there have been several attempts to identify those degrees of freedom. In particular one has to mention the success of string theory in explaining the entropy of extremal and near-extremal BH in several dimensions [3]. There have also been some proposals based on causal sets [4] and on the use of entanglement entropy of matter fields [5]. Within loop quantum gravity [6, 7], a leading candidate for a quantum theory of gravity, there has been some progress in describing black holes ‘in equilibrium’. In particular this implies that the objects to be studied are assumed to be isolated, in such a way that a study of its properties will guarantee that one can separate their description from that of the rest of the environment (as one normally does in thermodynamics). The resulting quantum picture is that the interaction between ‘bulk states’ as described by spin networks as they puncture the horizon, create horizon degrees of freedom that can (and do) fluctuate. These degrees of freedom are, on the one hand, independent of the bulk degrees of freedom, and on the other hand, fluctuate ‘in tandem’ with their bulk counterparts, as dictated by specific quantum conditions warranting the existence of the quantum horizon.

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The original program was developed in a series of papers [8, 9, 10] and has been further studied [11, 13, 14, 15, 16, 17, 18, 21, 22] and also widely reviewed in [25, 26]. The purpose of this contribution is to provide a bird’s eye view into the field, briefly summarizing the progress made in the past 12 years, including some recent results. This contribution can also be seen as a starting point and as a reading guide for those interested in more details.

In what follows, we shall in particular try to answer the following questions: How do we characterize black holes in equilibrium? That is, what are the quantum horizon states? How do we know which states we should count? Can we learn how entropy behaves? Can we make contact, for large black holes, with the Bekenstein-Hawking entropy? Can we extend the formalism and consider small, Planck scale BH’s? How small is small? That is, where does the transition from the Planck scale to the ‘large area limit’ occurs?

We shall not include topics such as the possibility of treating Hawking radiation [27, 28] or the criteria for dealing with black holes in thermal equilibrium [29].

II. PRELIMINARIES

This section has two parts. In the first one we review the motivation for the need of a notion of horizon that is local and not teleological as is the case of the traditional event horizon. In the second part we briefly review the main ideas behind the isolated horizons formalism.

A. Motivation

Physically, one is interested in describing black holes in equilibrium. That is, equilibrium of the horizon, not the exterior. Just as in the standard analysis of physical systems subject to thermodynamics considerations, one requires \textit{the system} and not the whole universe to be in equilibrium. The use of globally stationary solutions to Einstein’s equations to study the thermodynamics of horizons is very restrictive since one is requiring the whole universe to be stationary and not just the system, i.e. the horizon. Can one capture that notion via quasi-local boundary conditions? Yes! And the answer is provided by the isolated horizons (IH) formalism [9, 25].

The main idea is that \textit{some} boundary conditions are imposed on an inner boundary of the spacetime region under consideration. The interior region of the horizon is cut out, since the isolated horizon is regarded as a boundary. Is this a physical boundary? No! but one can ask whether one can make sense of it, namely whether there is a consistent prescription for incorporating this hypothesis, and a consistent variational principle is possible. A second question pertains to the physical interpretation of the boundary. If ‘physics’ does not end there, in the sense that in a realistic spacetime, matter and observers can fall into the interior region with a well defined evolution, what is then the justification for ‘arbitrarily’ cutting this region out?

The justification is that, being null surfaces, the exterior region (say in an asymptotic region) will not have access to any events inside the horizon (even if the isolated horizon does not coincide with a possible event horizon, it will lie \textit{inside} it), the information of what happens inside is not needed for describing the physical processes in the outside region. One can then interpret the horizon, and the degrees of freedom on it, as a ‘screen’ that keeps track of those aspects of the degrees of freedom that fell in but that can still interact with
the outside region. For instance, the mass of the isolated horizon has certain information of the energy of the matter that fell in, and is responsible for the gravitational field outside the horizon. The same is true for other quantities such as charge, angular momentum, etc. These horizon charges (multipole moments) will carry this information, and is the input needed in formulating the theory.

Let us summarize the main features of IH and their quantum treatment:

i) The boundary $\Delta$, the 3-D isolated horizon, provides an effective description of the degrees of freedom of the inside region, that is cut out in the formalism.

ii) The boundary conditions are such that they capture the intuitive description of a horizon in classical equilibrium and allow for a consistent variational principle.

iii) The quantum geometry of the horizon has independent degrees of freedom that fluctuate ‘in tandem’ with the bulk quantum geometry.

iv) The quantum boundary degrees of freedom are then responsible for the entropy.

v) The entropy thus found can be interpreted as the entropy assigned by an ‘outside observer’ to the (2-dim) horizon $S = \Sigma \cap \Delta$.

![Diagram](image)

Fig.1 Left: The physical situation one expects to describe. The collapse of a stellar object creates an event horizon that settles down (rather quickly) and in the asymptotic future is non-expanding, giving rise to an Isolated Horizon $\Delta$. Right: even if there is more matter falling in the future, there will be portions of the horizon that will be isolated.

Just as for other approaches to black hole entropy, the LQG treatment is not free from some interpretational issues. For instance, is the entropy to be regarded as the entropy contained by the horizon? Is there some ‘holographic principle’ in action? Can the result be associated to entanglement entropy between the interior and the exterior?, etc. Some of these questions have been clarified but there are still some for which we have no answer yet (see for instance the discussion in [24]).
B. Isolated Horizons

In this part we will provide the main ideas in the definition of isolated horizons. For full details see [25]. An isolated horizon is a null, non-expanding 3D-surface $\Delta$, equipped with some notion of translational symmetry along its generators (it is assumed to have a congruence of null vectors generating it). There are three main consequences of these boundary conditions:

i) The gravitational degrees of freedom induced on the horizon are captured by a $U(1)$ connection,

$$W_a = -\frac{1}{2} \Gamma^i_a r_i$$

where $\Gamma^i_a$ is the spin connection of the canonical theory on $\Sigma$. Thus, there is an effective reduction of the gauge symmetry from $SU(2)$ to $U(1)$.

ii) The total symplectic structure of the theory (and this is true even when matter is present) gets split as,

$$\Omega_{tot} = \Omega_{bulk} + \Omega_{hor}$$

with

$$\Omega_{hor} = \frac{a_0}{8\pi G} \oint_S dW \wedge dW'$$

This is precisely the symplectic structure one would get if we were considering a Chern-Simons theory for the $U(1)$ connection $W_a$ on the three dimensional manifold $\Delta$ with $S$ a spatial section (Recall that Chern Simons does not require a metric, so the fact that $\Delta$ is null is irrelevant).

iii) Finally, the 'connection part' and the 'triad part' at the horizon must satisfy the condition,

$$F_{ab} = -\frac{2\pi \gamma}{a_0} F^i_{ab} r_i,$$

the so called 'horizon constraint'. Here $F_{ab}$ is the curvature of the $U(1)$ connection $W_a$.

C. Constraints

It is interesting to explore the consequences of the boundary conditions in the Hamiltonian framework. A detailed study of the canonical theory [9, 25] reveals an interesting structure. In particular, the formalism tells us what is gauge and what not. To be precise, with respect to the constraints that appear in the canonical formalism, we know that:

a) The relation between curvature and triad, the horizon constraint (4), is equivalent to Gauss’ law.

b) Diffeomorphisms that leave $S$ invariant (i.e. that, when restricted to the horizon map $S$ to itself) are gauge (i.e. the vector fields generating infinitesimal diffeomorphisms are tangent to $S$).

c) The scalar constraint must have a vanishing lapse $N|_{hor} = 0$ at the horizon. Thus, the gauge transformations generated by the scalar constraint (that depend of the lapse), leave the horizon untouched. In particular, this implies that any gauge and diff-invariant observable is a full Dirac observable. This list includes all multipole moments of the horizon.
This last point is the reason behind the fact that one can sensibly talk about the quantum theory of black holes in LQG even when we have not solved the quantum dynamics in the bulk. That is, since in the quantum theory one has to implement the constraints, the fact that the lapse vanishes at the horizon implies that, from the horizon perspective, any quantum state that satisfies Gauss’ law and is diffeomorphism invariant will by a physical state, given that the Hamiltonian constraint imposes no further condition. Of course, one has to make sure that the quantum horizon states ‘interact’ properly with the bulk states for which the dynamics is still not fully understood. This represents one of the current challenges.

III. QUANTUM THEORY: THE BULK

Loop quantum gravity [6] is based on a canonical formulation of general relativity in terms of connections and triads (For a brief introduction see [7]).

The basic canonical variables are:

\[ A^i_a \text{ a } SU(2) \text{ connection } ; \quad E^a_i \text{ a densitized triad} \]  

(5)

with \( A^i_a = \Gamma^i_a - \gamma K^i_a \), and \( \gamma \) real the Barbero-Immirzi parameter (BI). Loop Quantum gravity defined on a manifold without boundary is based on two fundamental observables of the basic variables:

\[ h_e(A) := \mathcal{P} \exp(\int_e A) \]  

(6)

and

\[ E(f, S) := \int_S dS^{ab} E_{iab} f^i. \]  

(7)

where \( E^e_i = \tilde{\eta}^{abc} E_{ab}^i \). The main assumption of Loop Quantum Gravity is that these quantities become well defined operators in the quantum theory. Thus, the starting point for LQG is the so called Holonomy-Flux algebra \( \mathcal{HF} \) [31]. An important question is how many consistent representations of the \( \mathcal{HF} \)-algebra there are. In recent years, the LOST collaboration proved the following result: There is a unique representation of the Holonomy-Flux algebra on a Hilbert space that is diffeomorphism invariant [32]. This representation corresponds precisely to the construction of Ashtekar and Lewandowski [6]. Let us now give a brief description of this resulting Hilbert space. First we can characterize it in terms of Spin Networks:

\[ \mathcal{H}_{AL} = \bigoplus_{\text{graphs}} \mathcal{H}_\Upsilon = \text{Span of all Spin Networks } |\Upsilon, \vec{j}, \vec{m}\rangle \]  

(8)

Fig 2. A Spin network \((\Upsilon, \vec{j}, \vec{m})\) consists of a graph \(\Upsilon\) together with labels \(j_i\) for the edges and \(m_i\) for the vertices.
A Spin Network $|\Upsilon, \vec{j}, \vec{m}\rangle$, represents a particularly convenient basis for the theory. It is a state labelled by a graph $\Upsilon$, and some colorings $(\vec{j}, \vec{m})$ associated to edges and vertices.

The spin networks have a very nice interpretation in terms of the quantum geometry they generate. They are the eigenstates of the quantized geometry, such as the area operator,

$$\hat{A}[S] \cdot |\Upsilon, \vec{j}, \vec{m}\rangle = 8\pi\ell_P^2 \gamma \sum_{\text{edges}} \sqrt{j_I(j_I+2)} |\Upsilon, \vec{j}, \vec{m}\rangle$$

where the sum is over all the intersection points $p_I$ of the edges $e_I$ with the surface $S$. The standard interpretation is that the edges of the graph excite the quantum geometry of the surface $S$ at the intersection points between $S$ and $\Upsilon$. The edges $e_I$ can be seen as quantum fluxes of area.

Fig. 3. An artist impression of a black hole in LQC. The edges of the state on the bulk puncture the horizon $S = \Sigma \cap \Delta$ endowing it with area through the labels $j$'s and with intrinsic curvature through the $m$'s.

IV. QUANTUM THEORY OF THE HORIZON

Just as in the classical description of the gravitational field with an IH, the phase space could be decomposed in a bulk part and a horizon part, a basic assumption is that the total Hilbert Space is a tensor product of the form:

$$\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$$

where $\mathcal{H}_S$, the surface Hilbert Space, can be built from Chern Simons Hilbert spaces for a sphere with punctures. This represents the ‘kinematical Hilbert space’.
In order to go to the physical theory, the conditions on $H$ that we need to impose are:

Invariance under diffeomorphisms of $S$ and the quantum condition on $\Psi$, the quantum equivalence of Eq. (4):

$$
\left( \text{Id} \otimes \hat{F}_{ab} + \frac{2\pi \gamma}{a_0} \hat{E}_{ab} r_i \otimes \text{Id} \right) \cdot \Psi = 0.
$$

(11)

Then, the theory we are considering is a quantum gravity theory, with an isolated horizon of fixed area $a_0$ (and multiple-moments). A Physical state would be such that, in the bulk satisfy the ordinary constraints and, at the horizon, the quantum horizon condition.

**Entropy.** We shall consider the simplest case of pure gravity with a non-rotating horizon. In this case, from the outset, we are given a black hole of area $a_0$. The question then is: what entropy can we assign to it? Let us take the microcanonical viewpoint. To compute the entropy we shall count the number of states $\mathcal{N}$ such that they satisfy:

- The area eigenvalue is in the interval $\langle \hat{A} \rangle \in [a_0 - \delta, a_0 + \delta]$
- The quantum horizon condition (11) is satisfied.

The entropy $S$ will be then given by,

$$
S = \ln \mathcal{N}.'
$$

(12)

The challenge now is to identify those states that satisfy the two conditions, and count them.

**Characterization of the States.** There is a convenient way of characterizing the states by means of the spin network basis. If an edge of a spin network with label $j_I$ ends at the horizon $S$, it creates a puncture, with label $j_I$. The area of the horizon will be the area that the operator on the bulk assigns to it: $A = 8\pi \gamma \ell_P^2 \sum_i \sqrt{j_I(j_I + 1)}$.

Is there any other quantum number associated to the punctures $p_I$? Yes! They are given by eigenstates of $\hat{E}_{ab}$ that are also half integers $m_I$, such that $-j_I \leq m_I \leq j_I$. The quantum horizon condition relates these eigenstates to those of the Chern-Simons theory. The requirement that the horizon is a (topological) sphere then imposes a ‘total projection condition’ on $m$’s:

$$
\sum_I m_I = 0
$$

(13)

that has to be taken into account as well.

A quantum horizon state can be conveniently characterized by a set of punctures $p_I$ and to each one a pair of half integer $(j_I, m_I)$, where the three previous conditions impose some restrictions on the possible values of the labels.

If we are given $N$ punctures and two assignments of labels $(j_I, m_I)$ and $(j'_I, m'_I)$. Are they physically distinguishable? or a there some ‘permutations’ of the labels that give indistinguishable states? That is, what is the statistics of the punctures?

As usual, we should let the theory tell us. One does not postulate any statistics. If one treats in a careful way the action of the diffeomorphisms on the punctures one learns that when one has a pair of punctures with the same labels $j$’s and $m$’s, then the punctures are indistinguishable and one should not count them twice. In all other cases the states are distinguishable.

**The counting.** We start with an isolated horizon, with area $a_0$ (assumed to be of the order of several Planck areas) and ask how many states are there compatible with the two conditions,
and taking into account the distinguishability of the states. One can approach the problem in a two step process.

First step: Count just the different configurations and forget about $\sum I m_j = 0$. Thus, given \( \{n_j\}_{j=1}^{s_{\text{max}}} = (n_{1/2}, n_1, n_{3/2}, \ldots, n_{s_{\text{max}}/2}) \), where \( n_j \) means the number of punctures with label \( j \), we count the number of states:

\[
N = \frac{N!}{\prod_j (n_j!)} \prod_j (2j + 1)^{n_j} \tag{14}
\]

with \( N = \sum_j n_j \). Taking the large area approximation \( A >> \ell_{\text{Pl}} \), and using the Sterling approximation, one gets as the dominant term:

\[
S = \frac{A}{4\ell_{\text{Pl}}^2} \frac{\gamma_0}{\gamma} \tag{15}
\]

with \( \gamma_0 \) the solution\(^1\) to \( \sum_j (2j + 1) e^{2\pi \gamma_0 \sqrt{j(j+1)}} = 1 \).

As a second step one introduces the projection constraint. This has the effect of introducing a correction to the entropy area relation as an infinite series, where the first correction is logarithmic \([13, 15, 16]\):

\[
S = \frac{A}{4\ell_{\text{Pl}}^2} \frac{\gamma_0}{\gamma} - \frac{1}{2} \ln(A) + \ldots \tag{16}
\]

Note that one gets, in the complete counting, the asymptotic linear dependence on area. If we want to make contact with the Bekenstein-Hawking formula we have to make use of the freedom in LQG provided by the BI parameter and choose \( \gamma = \gamma_0 \).\(^2\) The coefficient of the logarithmic correction seems to be universal and independent of the particular counting (for other topologies of the horizon, it might change \([33]\)). An important observation is that the formalism can be generalized to more general situations, the combinatorial problem is the same and therefore the result is that \textit{the same value of} \( \gamma \) \textit{will yield the BH entropy}. These more general horizons include arbitrary distortion and rotation in vacuum gravity \([11]\) as well as coupling to electromagnetic, dilatonic, Yang-Mills, cosmological constant \([9, 10]\), and non-minimally coupled scalar fields \([12]\).

In the following sections we shall review new developments that have occurred since 2006. These include a new phenomena found when computing directly the number of states for small black holes, and an exact counting of states by use of methods from number theory.

V. DIRECT COUNTINGS AND ENTROPY QUANTIZATION

In this section we will describe the results found when considering small Planck size horizons for which the counting of states is possible. For that one tells a computer how to count for a range of area \( a_0 \) at the Planck scale \([17]\). With the availability of having an exact algorithm under control, one can ask, for instance, what is the effect of considering

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\(^1\) This counting was first done in detail in \([16]\). There is a slightly different counting (sometimes denoted as the DLM counting) that does not distinguish configurations with different \( j \)'s if the \( m \)'s are the same. In that case we get a different linear dependence with $\sum j e^{2\pi \gamma_0 \sqrt{j(j+1)}} = 1 \ ([14, 15])$.

\(^2\) As noted before, the value of \( \gamma_0 \) depends on the counting.
or not the ‘projection constraint’. In the large area approximation it is responsible for the first, logarithmic, correction term. One could also ask when is the linear dependence with area first observed. That is, when do we see a transition from deep quantum effects to ‘large areas’? Let us briefly summarize the results reported in [17]. What was found is that, without the projection constraint, the entropy approaches very fast a ‘smooth’ function of area with the slope found in the analytical calculations. When including the constraint, the relation between entropy and area became oscillatory, with a well identified period \( \delta A_0 \), that on average, introduced the expected logarithmic corrections. This was already identified for horizons that are as small as \( 100 \ell_{Pl}^2 \). For details see [17].

Furthermore, it was seen that, by analyzing the ‘black hole spectrum’ (i.e. the degeneracy of states as function of area), both the oscillations found with a large value of \( \delta \) as well as these structures in the ‘spectrum’ possess the same periodicity \( \delta A_0 \approx 2.41 \ell_{Pl}^2 \). A natural question is whether there is any physical significance to this periodicity. It turns out that, if one chooses the interval: \( 2 \delta = \Delta A_0 \), the plot of the entropy vs area becomes stair-like [18], as can be seen in Fig.4:

![Fig. 4. The entropy as function of area shows a step-like behavior when the interval \( \delta \) is chosen to coincide with the periodicity.](image)

What one notes is that the entropy has a completely different behavior for this particular choice of interval: Instead of oscillations, the entropy seems to increase in discrete steps. Furthermore, the height of the steps seems to approach a constant value as the area of the horizon grows, thus implementing in a rather subtle way the conjecture by Bekenstein that entropy should be equidistant for large black holes. Quite remarkably, this result is robust, namely, it is independent of the counting.

While the constant number in which the entropy of large black holes ‘jumps’ seems to approach [18]:

\[ 10 \]
\[ \Delta S \mapsto 2 \gamma_0 \ln (3) \] 

(17)

Fig. 5. The black hole spectrum shows some peaks of higher degeneracy together with some valleys. This is the origin of the step-like behavior of entropy.

Some recent proposals have provided a heuristic understanding of the origin of these peaks and valleys [19, 21]. To summarize these results, the model there proposed allows one to think of the states as organized in bands, labelled by certain combination of the total number of punctures and ‘spin’. By employing the analytic \( n_j \) distribution that maximizes degeneracy, as originally introduced in [16], one can find the ‘average area’, for each band associated with this maximum degeneracy configuration, from which one can compute the change in area from peak to peak as,

\[
\Delta A = \frac{8 \pi \gamma}{3} \sum_s \sqrt{s(s+2)} (s+1) e^{-2 \pi \gamma_0 \sqrt{s(s+2)}} + 2
\] 

(18)

An interesting observation is that if one parametrizes this number as \( \Delta A = \chi \gamma \), one can see that \( \chi \) must be a constant, independent of the counting (since it only depends on the degeneracy of the states as functions of \( j \)'s and \( m \)'s). From the observed periodicity in the direct counting, it was conjectured in [18] that the value of \( \chi \) is \( 8 \ln 3 \). Interestingly, the approximate formula found in [19, 21] for both countings, yield a slightly different approximate values \( \chi \) for the parameter \( \chi \), with \( \chi_{DLM} < 8 \ln 3 < \chi_{GM} \), and the relative difference of the order of \( 10^{-4} \). This shows that the approximation is not exact and one needs a better analytical understanding of the combinatorial problem.
VI. EXACT COUNTING: NUMBER THEORY

Recent progress using number theoretical considerations has turned out to be useful for the purpose of understanding the emergence of the discrete structures [22]. In this study, a reformulation of the counting of states and an exact counting of the number of states has been achieved recently. In this part I shall briefly summarize these results. There are two main steps involved in the counting. In the first one, one finds a complete characterization of the area spectrum, that is, of the eigenvalues of the area operator in terms of so-called ‘square-free numbers’. Then, given an allowed area-eigenvalue, one computes the number of possible ‘sets’ of labels (be them \( j' \)s and/or \( m' \)s) that are compatible with that values. This sets contain also the number of punctures that have a given label assigned. In the second part of the counting process one assigns a degeneracy to each ‘set’ coming from the possible ‘permutation’ of labels. At the end, one obtains an exact number of consistent states for the given value of area. The final step, that is, the computation of the entropy can then be computed either by considering an interval as previously defined, or by summing over all values of area up until the prescribed value \( A_0 \).

Let us now describe how one achieves the first step in the counting process. First, one notes that the area eigenvalues (when measured in units of \( 8\pi G\gamma \)) can be written as:

\[
A = \sum_{I=1}^{N} \sqrt{(k_I + 1)^2 - 1} = \sum_{k=1}^{k_{\text{max}}} n_k \sqrt{(k + 1)^2 - 1} \tag{19}
\]

where \( k_I = 2J_I \) are integers labelling the punctures and we have recast the sum by rearranging the punctures by their label \( k \) (\( n_k \) is the number of punctures with label \( k \)) and summing over labels. The idea here is to employ the square free numbers as a basis for the area eigenvalues (the numbers are ‘linearly independent’ under arbitrary linear combinations with integer coefficients). Each of the terms in the sum can be recast as an integer \( q_i \) times a ‘square free number’ \( \sqrt{p_i} \) (by means of the prime decomposition of the quantities inside the square root). Thus the sum becomes \( \sum_{i=1}^{r} q_i \sqrt{p_i} \), where \( \sqrt{(k + 1)^2 - 1} = y \sqrt{p_i} \), for some integer \( y \).

Let us summarize. If we specify a square free number \( p_i \), we want to know for which values of integers \( k \) and \( y \) is the previous equation satisfied, which would tell us (for each possible solution) the allowed values of the labels \( k \). This equation is known as the Pell equation and has an infinite number of solutions (labelled by \( m \)). We can then use these solutions to rewrite (19) as

\[
A = \sum_{i=1}^{r} \sum_{m=1}^{\infty} n_{k_i m} y_i^m \sqrt{p_i}
\]

If we use the fact that the numbers \( \sqrt{p_i} \) are linearly independent, we can split the equation in a series of different equations of the form \( \sum_{m=1}^{\infty} n_{k_i m} y_i^m = q_i \), where the \( y \)’s and the \( q \)’s are known, as solutions to the Pell equation, and the unknowns here are the numbers \( n_{k_i m} \). If these Diophantine equations admit solutions \( \sum_{i=1}^{r} q_i \sqrt{p_i} \), then \( A \) belongs to the relevant part of the spectrum of the area operator, the numbers \( k \)’s give the spins involved, and the numbers \( n \)’s count the number of times that edges labelled by the spin \( k_i m / 2 \) pierce the horizon.

Thus, given a linear combination of square free number as the area eigenvalue, the procedure here described provides an answer to the ‘degeneracy’ associated to the different pairs
\{k_i^m, n_{k_i}^m\}$ defining the different spin configurations. The next step in order to obtain the total number of states is to count the ‘$m$-degeneracy’, namely the different ways of accommodating the $m$’s on a given spin configuration. It is at this point that the two different countings (GM and DLM) provide different answers. Both cases can be treated in terms of fusion numbers and fusion matrices employed in CFT. For details see [22]. Just as an illustration, for the DLM counting the answer can be exactly written as

\[
\frac{2^N M^{-1}}{M} \sum_{s=0}^{N} \prod_{I=1}^{N} \cos\left(2\pi s K_I / M\right)
\]

with $M = 1 + \sum_{I=1}^{N} k_I$, allowing to have exact expressions for the degeneracy of states. Of course, this strategy confirms the results of [18], but also allows to compute the spectrum of larger black holes with the same computational capacity. These results represent a starting point for more refined asymptotic analysis, by means of generating functions for the combinatorial problem, that will shed more light on the behavior of macroscopic black holes [23]. For instance, an important question to be addressed is whether the oscillatory behavior on entropy, the entropy quantization, together with its possible implications for Hawking radiation, is still present for large black holes.

VII. OTHER APPROACHES

Let us now discuss some open questions regarding quantum black holes and the progress that has been made within LQG. By the mere fact that in the isolated horizon framework one is considering only the outside region of a spacetime containing a back hole, one is not addressing the issue of the singularity. The possible singularity resolution has been analyzed in a series of papers using loop quantization techniques [34].

The starting point of such treatments is the minisuperspace of homogeneous cosmologies on a spatial manifold with topology $S^2 \times \mathbb{R}$. These ‘Kantowski-Sachs models’ are important given that the interior region of the global Schwarzschild solution belongs to this class. It is thus natural to attempt to employ the same techniques that have been extremely useful in the treatment of (minisuperspaces corresponding to) homogeneous and isotropic models in cosmology (See, for instance [35] for a recent summary of such methods).

Those results suggest that the classical singularity inside the horizon, just as in the case of isotropic cosmologies, is avoided, and the quantum evolution continues past it, but more work is needed to reach a definite conclusion. In particular, none of the presently available models [34] is able to overcome consistency requirements that select a unique quantization in the isotropic sector [35].

An important open issue is how to specify black hole/horizon states from the full theory. That is, without assuming that there was a classical horizon to begin with. Some progress in this direction has been made in two fronts, but still at some preliminary stage. A proposal for defining coherent states that approximate a spacetime with a black hole, have been used to count the number of black hole states [38]. Even when potentially important, this approach is still in its early stages given that the coherent states are kinematical, and there is not a full control on the dynamical sector of the theory.

Another proposal for identifying black hole states from the full set of states was made within the context of symmetry reduced models in Ref. [39]. Here, the idea is to specify operators on a kinematical Hilbert space such that they project the kinematical states onto...
a ‘black hole sector’. Just as in the previous case, there are still several consistence criteria that this approach must satisfy before one can make concrete predictions. In particular both approaches lack a description for dynamical processes that the horizon might undergo.

If the singularity resolution were also generic, and there existed a spacetime interpretation beyond the ‘would be singularity’, one would be lead to the Ashtekar-Bojowald paradigm for evaporation and (lack of) information loss \[30\]. This picture includes the description of dynamical processes that are no longer described by the Isolated Horizon formalism. One needs then to consider the more general framework of dynamical horizons \[25\]. If this picture is physically correct, there is no classical singularity and no event horizon ever forms. Still, there is a horizon that forms, grows and then shrinks due to Hawking radiation. Information is not lost, even when, for certain observers, Hawking radiation appears to be thermal (for more details within a simple model see \[37\]). These results are certainly intriguing and suggest a resolution of the ‘information loss problem’ due to true quantum geometric effects. Needless to say, more work is needed to unravel this mystery.

VIII. CONCLUSIONS AND OUTLOOK

Let us summarize what we have learned from the merger of isolated horizons and loop quantum gravity. First, as we have shown, isolated horizons provide a consistent framework to incorporate black holes that are physically in equilibrium, as classical objects. As we have argued, one can consistently quantize the theory, as described by the IH phase space, employing both the methods of quantum geometry that are useful in the bulk, together with techniques from $U(1)$ Chern-Simons theory on a sphere. A detailed study of the action of the constraints allows us to give a full characterization of the quantum horizon degrees of freedom that contribute to the entropy. It is found that the entropy is finite, without the need of a regulator nor a cut-off, and that its dominant term is linear in area for large horizons in Planck units. Furthermore, the formalism allows us to translate the entropy counting into a purely combinatorial problem for which one can attempt algorithmic brute force computations \[17\], as well as number-theoretic treatments \[22\].

A very important feature of this formalism is that one can incorporate and count the entropy of a whole class of different black holes, where one can include arbitrary distortion and rotation in vacuum gravity \[11\] as well as coupling to electromagnetic, dilatonic, Yang-Mills \[9, 10\] and non-minimally coupled scalar fields \[12\]. In all these cases the combinatorial problem to be solved is the same (even when its translation into physically relevant quantities might vary) and therefore, entropy is always proportional to area in the large area limit (or with the expected contribution from the scalar field in the non-minimally coupled case).

As we have explored, when one considers the problem of a direct counting of the number of states, and thus being forced to consider small horizons, several unexpected features appear for these Planck size black holes. While one recovers the asymptotic linear dependence on area and the logarithmic correction (with the right coefficient), from which we can say something about BI parameter, a new behavior is observed for small horizons. It is found that there are oscillations in entropy with a constant periodicity. Furthermore, when properly interpreted, this points to an effective quantization of the entropy in equidistant steps \[18\]. This observed behavior suggests that loop quantum gravity can make contact, in a rather subtle manner, with both Bekenstein’s heuristic model \[18\], and the Mukhanov-Bekenstein effect \[20, 40\]. Whether this scenario is realized or not remains an intriguing question.
Recently, attempts to understand the origin of the ‘black hole spectrum’ responsible for the entropy quantization have been put forward [19, 21, 22], which have been able to give some intuitive understanding of the effect. In particular, there has been some progress to understand, from a heuristic perspective, the origin of the ‘bands’ in the spectrum and their equidistant nature. A pressing question here is whether the discrete structures found at the Planck scale are still present for macroscopic black holes. One would also like to understand whether the constant $\chi$ actually is equal to $8 \ln 3$ as the numerical computations and the heuristic considerations seem to suggest. If this were the case, one would need to understand its origin.

As we have here tried to convey, in the past couple years there has been exciting progress in our understanding of quantum black holes within Loop Quantum Gravity, but there are still important questions that remain open regarding the detailed relation between gravity, entropy and the quantum.

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