NOTES ON PLANAR SEMIMODULAR LATTICES. IX.
\[ \mathcal{C}_1\text{-DIAGRAMS} \]

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Abstract. A planar semimodular lattice \( L \) is slim if \( M_3 \) is not a sublattice of \( L \). In a recent paper, G. Czédli introduced a very powerful diagram type for slim, planar, semimodular lattices, the \( \mathcal{C}_1 \)-diagrams. This short note proves the existence of such diagrams.

Background. The basic concepts and notation not defined in this note are available in Part I of the book [10], see arXiv:2104.06539 it is freely available. We will reference it, for instance, as [CFL2, p. 4]. In particular, a planar semimodular lattice \( L \) is slim if \( M_3 \) is not a sublattice of \( L \) and a grid \( G \) is a direct product of two nontrivial chains. For the lattice \( S_7 \), see Figure 1 and [10, pages xxi, 34]. Following my paper [15] with E. Knapp, a semimodular lattice \( L \) is rectangular if the left and right boundary chains have exactly one doubly-irreducible element each and these elements are complementary.

In my paper [10] with H. Lakser, and E. T. Schmidt, we prove that every finite distributive lattice \( D \) can be represented as the congruence lattice of a (planar) semimodular lattice \( L \). Since \( M_3 \) sublattices play a crucial role in the construction of \( L \), it was natural to raise the question what can be said about congruence lattices of slim, planar, semimodular (SPS) lattices (see [CFL2, Problem 24.1], originally raised in my paper [11]). The papers in the References list some contributions to this topic. In particular, my presentation [13] gently reviews the background of this topic.

\( \mathcal{C}_1 \)-diagrams. This research tool played an important role in some recent papers, see G. Czédli [3] and [4], G. Czédli and G. Grätzer [6], and G. Grätzer [13]; for the definition, see G. Czédli [3 Definition 5.3], G. Czédli [4 Definition 2.1], and G. Czédli and G. Grätzer [6 Definition 3.1].

In the diagram of an SPS lattice \( K \), a normal edge (line) has a slope of 45° or 135°. If it is the first, we call the edge (line) normal-up, otherwise, normal-down. Any edge (line) of slope strictly between 45° and 135° is steep.

A cover-preserving \( S_7 \) of a lattice \( L \) is a sublattice isomorphic to \( S_7 \) such that the covers in the sublattice are covers in the lattice \( L \).

Definition 1. A diagram of an SPS lattice \( L \) is a \( \mathcal{C}_1 \)-diagram if the middle edge of any cover-preserving \( S_7 \) is steep and all other edges are normal.

G. Czédli [3 Definition 5.11] also defines the much smaller class of \( \mathcal{C}_2 \)-diagrams.

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This note presents a short and direct proof of the existence theorem of $C_1$-diagrams, see G. Czédli [3, Theorem 5.5], utilizing only Theorem 3 the Structure Theorem of Slim Rectangular Lattices.

**Theorem 2.** Every slim, planar, semimodular lattice $L$ has a $C_1$-diagram.

For an SPS lattice $K$ and 4-cell $C$ in $K$, we denote the fork extension of $K$ at $C$ by $K[C]$, see G. Czédli and E. T. Schmidt [7] (see also [CFL2, Section 4.2]), illustrated by Figure 2.

**Theorem 3** (Structure Theorem of Slim Rectangular Lattices). For every slim rectangular lattice $K$, there is a grid $G$ and sequences

(1) \[ G = K_1, K_2, \ldots, K_{n-1}, K_n = K \]

of slim rectangular lattices and

(2) \[ C_1 = \{o_1, c_1, d_1, i_1\}, C_2 = \{o_2, c_2, d_2, i_2\}, \ldots, C_{n-1} = \{o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1}\} \]

of 4-cells in the appropriate lattices such that

(3) \[ G = K_1^K[1] = K_2, \ldots, K_{n−1}[C_{n−1}] = K_n = K. \]

Moreover, the principal ideals $\downarrow c_{n−1}$ and $\downarrow d_{n−1}$ are distributive.

**Proof of Theorem 3 for rectangular lattices.** Let the rectangular lattice $K$ be represented as in (3). We prove the Theorem by induction on $n$. For $n = 1$, the statement is trivial. Let us assume that the statement holds for $n-1$ and so $K_{n-1}$ has $C_1$-diagrams; we fix one. By the induction hypothesis, the 4-cell $C = C_{n-1}$ with $0_C = o$ and $1_C = i$ has (at least) two normal edges: $[o, c]$ and $[o, d]$, see Figure 2(i) and by the last clause of Theorem 3 the principal ideals $\downarrow c$ and $\downarrow d$ are distributive.

Utilizing that $\downarrow c$ is distributive, we place the element $a$ inside the edge $[o, c]$ so that the area bounded by the (dotted) normal-up line through $a$ and the normal-up
line through \( o \) contains no element below \( a \); we place the element \( b \) symmetrically on the other side, as in Figure 2(ii). The two dotted lines meet inside \( C \) since the two lower edges of \( C \) are normal and the upper edges are normal or steep. We place the third element of the fork at their intersection and connect it with a steep edge to the element \( i \). We add more elements to the lower left and lower right of \( C \) as part of the fork construction, see Figure 2(iii). We can use normal edges for this because of the way \( a \) and \( b \) were placed. The diagram we obtain is a \( C_1 \)-diagram of \( K \). □

Now let \( K \) be an SPS lattice. G. Czédli and E. T. Schmidt define in [7] a corner element \( a \) of \( K \) as a doubly irreducible element on the boundary of \( K \) such that \( a^* \) is meet-reducible, \( a^* \) is join-reducible, and \( a^* \) has exactly two lower covers.

By G. Czédli and E. T. Schmidt [7], \( K \) is obtained from a slim rectangular lattice \( K \) with a fixed \( C_1 \)-diagram by removing corners. In a cover-preserving sublattice \( S_7 \) of \( K \), there are only two doubly irreducible elements but neither is a corner (since the upper cover of a corner has at most two lower covers). Hence, when \( S_7 \) is a cover-preserving sublattice (of \( K \) or any other SPS lattice), then this \( S_7 \) contains no corner of \( K \). So the \( S_7 \)'s remain \( S_7 \)'s, the steep edges remain the “legitimately” steep edges of these remaining \( S_7 \)'s. All other edges that are left after removing corners remain of normal slopes. Thus, \( K \) is a \( C_1 \)-diagram, as required.

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