Combinatorics of flag simplicial 3-polytopes

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We shall consider convex simplicial 3-polytopes (polyhedra). Their boundaries are simplicial 2-spheres, and the converse also holds by a theorem of Steinitz. Hence it is sufficient to investigate the combinatorics of simplicial 2-spheres, leaving aside their polyhedral realizations.

**Theorem** (Steinitz, 1922; see [3]). A graph $\Gamma$ is the graph of a convex 3-polytope if and only if $\Gamma$ is simple, 3-connected, and planar.

**Definition 1.** A set of vertices of a complex such that it does not form a face itself but its proper subsets do form faces is called a missing face of the complex. If all missing faces of a complex contain two vertices, then this is called a flag complex. A simplicial polytope is called a flag polytope if its boundary is a flag simplicial complex.

**Definition 2.** The contraction of an edge $e = \{v_1, v_2\}$ of a complex $K$ is the operation of replacing a union of two stars $\text{st}_K v_1 \cup \text{st}_K v_2$ by the star $\text{st}_K v$ of a new vertex $v$ as follows: the vertices $v_1$ and $v_2$ and all the simplexes containing them are removed, after which the vertex $v$ and all the simplexes $v \sqcup \sigma$ such that $v_1 \sqcup \sigma \in K$ or $v_2 \sqcup \sigma \in K$ are added. The resulting complex is denoted by $K/e$.

**Theorem 3.** Each flag simplicial 3-polytope can be reduced to an octahedron by a sequence of contractions of edges so that at each step a flag simplicial 3-polytope is produced.

It was proved in [1] that a complex of dimension at most 2 can be obtained from any geometric subdivision of it by a sequence of edge contractions. This generalizes a result due to Steinitz and Rademacher [2] that a tetrahedron is the unique minimal (not admitting edge contractions) triangulation of the 2-sphere. However, it will not suffice to show that the boundary of a flag simplicial polyhedron is a geometric subdivision of an octahedron, because the polyhedra obtained in the process are not necessarily flag polyhedra. Therefore, we give an explicit proof.

**Proposition 4.** A simplicial 2-sphere $K$ is a flag complex if and only if for each edge $e$ of it the complex $K/e$ is a simplicial 2-sphere.

**Proof.** Assume that $K$ has a missing face $V$ with $|V| > 2$. If $|V| = 4$, then $K$ is the boundary of a tetrahedron, and the contraction of any edge gives a triangle. If $|V| = 3$, then let $V = \{v_1, v_2, v_3\}$. Contracting the edge $\{v_1, v_2\}$ to a vertex $v$, we obtain a complex which is not homeomorphic to a 2-sphere, since its 1-skeleton is not 3-connected (the set obtained by removing $v$ and $v_3$ is disconnected).

Assume that $K/e$ is not a polyhedral complex. By the Steinitz theorem there exist vertices $v$ and $w$ such that $(K/e) \setminus \{v, w\}$ is disconnected. Since the original complex is polyhedral, it cannot become disconnected upon removal of two vertices, so we can assume that $v$ is the image of an edge $e = \{v_1, v_2\}$ under a contraction. Hence $K \setminus \{v_1, v_2, w\}$ is disconnected. Now we look at a polyhedral realization of $K$ with vertices in general position. Drawing a plane $H$ through $v_1$, $v_2$, and $w$, we divide the remaining vertices of $K$
into the ones lying in the lower and the upper half-spaces. It can easily be shown that any two vertices in the lower (upper) half-space can always be joined by a path lying entirely in the same half-space. Hence \( K \setminus \{v_1, v_2, w\} \) consists of two connected components, formed by the vertices lying in the lower and the upper half-space. The plane \( H \) does not intersect an edge joining vertices in the different half-spaces. All this means that \( v_1, v_2, \) and \( w \) are connected by edges and form a missing face of \( K \).

We introduce a partial ordering in the set of flag 2-spheres by setting \( P \preceq Q \) if \( P \) can be obtained from \( Q \) by a sequence of edge contractions. In this case minimal spheres are ones which fail to be flag complexes after the contraction of any edge.

**Definition 5.** We say that a set of vertices \( \{v_1, v_2, v_3, v_4\} \) of a simplicial complex forms a **belt** \( \square \) if the simplicial subcomplex generated by this set is simplicially isomorphic to the boundary of a square.

**Lemma 6.** Let \( K \) be a flag simplicial 2-sphere with edge \( e \). Then the simplicial sphere \( K/e \) is a flag complex if and only if \( e \) lies in no belt in \( K \).

**Proof.** We shall show that if a belt in \( K \) contains \( e \), then it is the inverse image of a missing face of cardinality 3 in \( K/e \) under the map \( K \to K/e \).

If \( e = \{v_1, v_2\} \) lies in a belt \( \{v_1, v_2, v_3, v_4\} \) in \( K \), then \( K/e \) contains the missing face \( \{v, v_3, v_4\} \), where \( v \) is the image of \( e \) under the contraction.

Assume that the complex \( K/e \) obtained by contracting the edge \( e = \{v_1, v_2\} \) of the flag complex \( K \) into the vertex \( v \) is not a flag complex. Consider a missing face \( V \) of \( K/e \) with \( |V| > 2 \). Since \( K \) is a flag complex, it follows that \( v \in V \). On the other hand, \( K/e \) cannot be the boundary of a tetrahedron, so \( |V| = 3 \). Let \( V = \{v, v_3, v_4\} \). Then \( v_1, v_2 \in \text{lk}_K v_3 \cup \text{lk}_K v_4 \), but since \( V \) is not a simplex in \( K/e \), one of \( v_1 \) and \( v_2 \) is contained in \( \text{lk}_K v_3 \setminus \text{lk}_K v_4 \) while the other lies in \( \text{lk}_K v_4 \setminus \text{lk}_K v_3 \). Hence \( \{v_1, v_2, v_3, v_4\} \) is a belt containing \( e \).

**Lemma 7.** Let \( K \) be a minimal flag simplicial 2-sphere. Then it has a vertex \( w \in K \) whose link is a belt in \( K \).

**Proof.** Take an arbitrary vertex \( v \in K \). By Lemma 6 each edge lies in a belt. Let \( W_i^1 \) and \( W_i^2 \) denote the simplicial balls into which a belt \( \square_i \) divides \( K \).

There exists a belt \( \square_0 \) containing \( v \) such that any belt \( \square' \) containing \( v \) and intersecting the interior of \( W_0^1 \) also intersects the interior of \( W_0^2 \). Indeed, consider a belt \( \square_1 \) containing \( v \). If there exists another belt \( \square_2 \) which lies in \( W_1^1 \), then \( W_2^1 \subset W_1^1 \). We go on by selecting \( \square_i \) so that \( W_i^1 \subset W_{i-1}^1 \). At some step it will be impossible to find the next belt, so the last belt selected will satisfy the above requirement.

Let \( v' \in \square_0 \) and assume that \( \{v, v'\} \) is not an edge. Each edge \( \{v, w\} \), where \( w \) is an interior vertex of \( W_0^1 \), lies in a belt \( \square_{v, w} \) intersecting the interior of \( W_0^j \). Then \( \square_0 \cap \square_{v, w} = \{v, v'\} \), since \( \square_{v, w} \) is 2-connected. Hence all the vertices in \( W_0^1 \setminus \text{lk}_K v \) are joined to \( v' \) by an edge. It follows from the choice of \( W_0^1 \) that \( w \) is the only interior vertex in this set. Hence \( \text{lk}_K w = \square_0 \).

**Proof of Theorem 3.** Let \( K \) be a minimal flag simplicial 2-sphere. We shall show that \( K \) is the boundary of an octahedron. Let \( w \) be the vertex from Lemma 7. We take a belt \( \square_1 \) containing \( w \) and a belt \( \square_2 \) containing \( w \) together with vertices in \( \text{lk}_K w \setminus \square_1 \). Then \( \square_1 \) and \( \square_2 \) have the common vertex \( w \). Since \( K \setminus \square_1 \) is disconnected and \( \square_2 \) is 2-connected, they must have another common vertex \( v \). Then all 4 vertices in the subcomplex \( \text{lk}_K w \) are joined by edges to \( v \) and \( w \). Since \( K \) is a flag sphere, it has no other vertices and thus is the boundary of an octahedron.
Bibliography

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