SPECIAL HOMOMORPHISMS BETWEEN PROBABILISTIC GENE REGULATORY NETWORKS

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ABSTRACT. In this paper we study finite dynamical systems with $n$ functions acting on the same set $X$, and probabilities assigned to these functions, that it is called Probabilistic Regulatory Gene Networks (PRN) in [3]. This concept is the same or a natural generalization of the concept Probabilistic Boolean Networks (PBN), introduced by I. Shmulevich, E. Dougherty, and W. Zhang in [9], particularly the model PBN has been used to describe genetic networks and has therapeutic applications, see [10]. In PRNs the most important question is to describe the steady states of the systems, so in this paper we pay attention to the idea of transforming a network to another without lost all the properties, in particular the probability distribution. Following this objective we develop the concepts of homomorphism and $\epsilon$-homomorphism of probabilistic regulatory networks, since these concepts bring the properties from one networks to another. Projections are special homomorphisms, and hey always induce invariant subnetworks that contain all cycles and steady states in the network.

INTRODUCTION

Genes can be understanding in their complexity behavior using models according with their discrete or continuous action. Developing computational tools permits describe gene functions and understand the mechanism of regulation [4, 5]. This understanding will have a significant impact on the development of techniques for drugs testing and therapeutic intervention for treating human diseases [3, 8, 10].

We focus our attention in the discrete structure of genetic regulatory networks, instead of, its dual moving continuo-discrete. Probabilistic Gene Regulatory Network (PRgN) is a natural generalizations of the model Probabilistic Boolean Network (PBN), introduced by I. Shmulevich, E. Dougherty, and W. Zhang in [9]. The mathematical background of the model PRgN, is introduced here, for simplicity we work with functions defined over a set $X$ to itself, with probabilities assigned to these functions. $X$ is a set of states of genes, for example $X = \{0, 1\}^n$, if our network is a Boolean network. Working in this way, we can observe the dynamic of the network indeed focus our attention in the description of functions. The set

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\(X\) can be a subset of \(\{0,1\}^n\), and we can extend some classical ideas to regulatory network, such as invariant subnetworks, automorphisms group, etc. In particular if \(X\) is a vector space over a finite field, the functions are lineal functions, then we can use linear Algebra to describe the state space. Mapping are important in the study of networks, because they permit to recognize subnetworks, in particular determine when two networks are similar or equivalent. Special mappings are homomorphisms and \(\epsilon\)-homomorphisms, we use both to describe subnetworks and similar networks. An homomorphism transform a network to another in such a way the discrete structure giving by the first network can lives in part of the other one, or these two networks are very similar but no equals, in particular in the probabilistic way. An \(\epsilon\)-homomorphism is the same but with the condition that the probability distributions of the networks are close, and we use a preestablishes \(0 < \epsilon < 1\) as a distance between the probabilities.

1. Finite dynamical systems and probabilistic Boolean networks

Two finite dynamical systems \((X, f)\) and \((Y, g)\) are isomorphic (or equivalents) if there exists a bijection \(\phi : X \rightarrow Y\) such that \(\phi \circ f = g \circ \phi\), ( or \(f = \phi^{-1} \circ g \circ \phi\)). If \(\phi\) is not a bijection map then \(\phi\) is an homomorphism.

If \(Y \subset X\) is such that \(f(Y) \subset Y\) then \((Y, f|_Y)\) is a sub-FDS of \((X, f)\), where \(f|_Y\) is the map restricted to \(Y\). There exists naturally an injective morphism from \(Y\) to \(X\) called inclusion and denoted by \(i\). The state space of a FDS \((X, f)\), is a digraph whit vertices the set \(X\), and with an arrow from \(u\) to \(v\) if \(f(u) = v\).

**Example of FDS-homomorphism** The FDSs \(X = (\{0,1\}^2, f_1(x, y) = (xy, y))\), and \(Y = (\{0,1\}^2, f_2(x, y) = (x, (x+1)y))\) are isomorphics, because their state spaces are isomorphics.

\[
\begin{align*}
(1,0) & \rightarrow^{f_1} (0,0) \quad \circ \quad (0,1) \quad \circ \rightarrow^{f_1} (1,1) \\
(1,1) & \rightarrow^{f_2} (1,0) \quad \circ \quad (0,0) \quad \circ \rightarrow^{f_2} (0,1)
\end{align*}
\]

In fact, the isomorphism \(\phi : \{0,1\}^2 \rightarrow \{0,1\}^2\) is the bijection \(\phi(1,0) = (1,1)\), \(\phi(0,0) = (1,0)\), \(\phi(0,1) = (0,0)\), and \(\phi(1,1) = (0,1)\). The following is an example of homomorphism (inclusion) with \(Z = \{(0,0), (1,0)\}, f_1\).

\[
\begin{align*}
(1,0) & \rightarrow f_1 (0,0) \\
(0,0) & \rightarrow f_1 (1,0)
\end{align*}
\]

A Probabilistic Boolean Network \(A = (V, F, C)\) is defined by the following sort (type) of objects \(\mathbb{B}\): a set of nodes (genes) \(V = \{x_1, \ldots, x_n\}, x_i \in \{0,1\}\), for all \(i\); a family \(F = \{F_1, F_2, \ldots, F_n\}\) of ordered sets \(F_i = \{f_1^{(i)}, f_2^{(i)}, \ldots, f_{\ell(i)}^{(i)}\}\) of Boolean functions \(f_j^{(i)} : \{0,1\}^n \rightarrow \{0,1\}\), for all \(j\) called predictors; and a list \(C = (C_1, \ldots, C_n)\), \(C_i = \{c_1^{(i)}, \ldots, c_{\ell(i)}^{(i)}\}\), of selection probabilities. The selection probability that the function \(f_j^{(i)}\) is used for the vertex \(i\) is \(c_j^{(i)} = Pr(f^{(i)} = f_j^{(i)})\). The dynamic of the PBN is given by a vector of functions \(f_k = (f_{k_1}^{(1)}, f_{k_2}^{(2)}, \ldots, f_{k_n}^{(n)})\).
for \(1 \leq k_i \leq l(i)\), and \(f_{k_i}^{(i)} \in F_i\), where \(k = [k_1, \ldots, k_n]\), \(1 \leq k_i \leq l(i)\). The map \(f_k : \{0,1\}^n \to \{0,1\}^n\) acts as a transition function. Each variable \(x_i \in \{0,1\}^n\) represents the state of the vertex \(i\). All functions are updated synchronously. At every time step, one of the functions is selected randomly from the set \(F_i\) according to a predefined probability distribution. The selection probability that the transition function \(f_k = (f_{k_1}^{(1)}, f_{k_2}^{(2)}, \ldots, f_{k_n}^{(n)})\) is used to go from the state \(u \in \{0,1\}\) to another state \(f_k(u) = v \in \{0,1\}\) is given by

\[
c_{k_i} = \prod_{i=1}^{n} c_{k_i}^{(i)}.
\]

The dynamical transition structure of a PBN can be described by a Markov chain with fixed transition probabilities. There are two digraph structures associated with a PBN: the low-level digraph \(\Gamma\), consisting of genes functions essentiality relations; and the high-level digraph which consists of the states of the system and the transitions between states. The matrix \(T\) associated to the high level digraph formed by placing \(p(u,v)\) in row \(u\) and column \(v\), where \(u, v \in \{0,1\}\) is called the transition probability matrix or chain matrix, \(p(u,v) = \sum_{k_i \in \Gamma} \chi_{k_i}^{(i)}\).

2. Probabilistic Regulatory Gene Networks

A Probabilistic Gene Regulatory Network (PRN) is a triple \(\mathcal{X} = (X,F,C)\) where \(X\) is a finite set and \(F = \{f_1, \ldots, f_n\}\) is a set of functions from \(X\) into itself, with a list \(C = (c_1, \ldots, c_n)\) of selection probabilities, where \(c_i = p(f_i)\). We associate with each PRN a weighted digraph, whose vertices are the elements of \(X\), and if \(u, v \in X\), there is an arrow going from \(u\) to \(v\) for each function \(f_i \) such that \(f_i(u) = v\), and the probability \(c_i\) is assigned to this arrow. This weighted digraph will be called the state space of \(\mathcal{X}\). In this paper, we use the notation PRN for one or more networks.

Example 2.1.

If \(X = \{0,1\}^2\), \(F = \{f_1(x,y) = (x,y), f_2(x,y) = (x,0), f_3(x,y) = (1,y), f_4(x,y) = (1,0)\}\); and \(C = \{.46, .21, .22, .11\}\), the state space of \(\mathcal{X} = (X,F,C)\) is the following:

\[
\begin{array}{ccc}
.67 & .33 \\
(0,0) & (0,1) & .46 \\
\text{.33} & \uparrow \text{.21 \ .46 \ .11 \ .22} & T = \\
\end{array}
\]

3. Homomorphisms and \(\epsilon\)-homomorphisms of PRN

If \(C\) is a set of selection probabilities we denote by \(\chi\) the characteristic function over \(C\). That is \(\chi : \mathcal{C} \cup \{0\} \to \{0,1\}\) such that \(\chi(c) = 1\), if \(c \neq 0\) and \(\chi(0) = 0\). Let \(\mathcal{X}_1 = (X_1, F = (f_i)^n_{i=1}, C)\) and \(\mathcal{X}_2 = (X_2, G = (g_j)^m_{j=1}, D)\) be two PRN.

Definition 3.1 (Homomorphisms of PRN). A map \(\phi : \mathcal{X}_1 \to \mathcal{X}_2\) is an homomorphism from \(\mathcal{X}_1\) to \(\mathcal{X}_2\), if for all \(f_i\) there exists a \(g_j\), such that for all \(u, v\) in \(\mathcal{X}_1\),

\[
(1) \phi \circ f_i = g_j \circ \phi; \text{ and } (2) \chi(d_{g_j}(\phi(u),\phi(v))) \geq \chi(e_{f_i}(u,v)).
\]
If φ : X₁ → X₂ is a bijective map, then φ is an isomorphism.

Example 3.2 (PRN-Homomorphism).

If \( \mathcal{T} = (X; F; C) \) is the PRN in Example 1, and \( \mathcal{X}_1 = (X; F' = \{f_1, f_2, f_3\}; C' = \{.47, .28, .25\}) \) is a new PRN over the same set \( X \) with different probabilities and only three functions.

The homomorphism \( \phi : \mathcal{X}_1 \rightarrow \mathcal{T} \) is a bijective map, \( \phi(x) = x \), over the set of states, but an inclusion over the set of arrows, because the arrow going from \((0, 1)\) to \((1, 1)\) in \( \mathcal{T} \) doesn’t appear in \( \mathcal{X}_1 \). The first condition for homomorphism is obvious. The condition (2) holds, because the inclusion of arrows. The two transition matrices are connected by this inclusion, since if the place \( ij \) in the first matrix \( \neq 0 \) then this place is \( \neq 0 \) in the second network too. The two PRN are not isomorphics because the probabilities are not equals. Since, there are no specific condition about the probability distribution in both PRN, we include a third condition, obtaining in this way a new concept that we will call \( \epsilon \)-homomorphism of PRN.

Condition (3) for \( \epsilon \)-Homomorphism The distributions of probabilities following the homomorphism are enough close. An \( \epsilon \)- homomorphism is an homomorphism that satisfies the condition, for all \( i, j, \max|p(u_i, u_j) - p(\phi(u_i), \phi(u_j))| \leq \epsilon \), where \( \epsilon > 0 \) is a real number that we previously determine for the applications.

As a consequence of this condition, if we use a test, as Kolmogorov-Smirnov test, the differences between the two distributions are \( \leq \epsilon \) again. In order to determine \( \epsilon \) for the homomorphism, we use the transition matrices. In the above example \( \epsilon = .11 \).

\[
T_1 - \mathcal{T} = \begin{bmatrix}
.08 & 0 & -.08 & 0 \\
.07 & .01 & -.11 & .03 \\
0 & 0 & 0 & 0 \\
0 & 0 & -.04 & .04
\end{bmatrix}
\]

Conclusion

If the homomorphism is a bijective map like here, the transition matrices \( T_1 \) and \( T_2 \) have the same order, and \( \sum_{i=1}^{n}(T_1 - T_2)_{ij} = 0 \), for \( j = 1, n \)

Theorem 3.3. If \( \phi : \mathcal{X}_1 \rightarrow \mathcal{X}_2 \) is an \( \epsilon \)-homomorphism, then the transition matrices \( T_1 \) and \( T_\phi \) satisfy the condition:
Example 4.1.

Example 4.1. Let \( X \) be the digraph is:

\[
X = \begin{pmatrix}
X_0 & 0 & 0 \\
0 & X_1 & 0 \\
0 & 0 & X_2
\end{pmatrix}
\]

Then our aim holds.

4. Algebra of Probabilistic Regulatory Networks

Sum of two PRN

Let \( X_1 = (X_1, F = (f_i)_{i=1}^n, C) \) and \( X_2 = (X_2, G = (g_j)_{j=1}^m, D) \) be two PRN. The sum \( X_1 \oplus X_2 = (X_1 \cup X_2, F \cup G, C \cup D) \) is a PRN where

1. \( X_1 \cup X_2 \) is the disjoint union of \( X_1 \) and \( X_2 \).
2. The function \( h_{ij} = (f_i \lor g_j) \) is defined by \( h_{ij}(x) = f_i(x) \) if \( x \in X_1 \) and \( h_{ij}(x) = g_j(x) \) if \( x \in X_2 \).
3. The probability \( p(h_{ij}) = c_i \lor d_j \), that is \( p(h_{ij}) = c_i \) if \( h_{ij} = f_i \) or \( p(h_{ij}) = d_j \) if \( h_{ij} = g_j \).

If \( T_1 \) and \( T_2 \) are the transition matrices of \( X_1 \) and \( X_2 \) respectively, Then \( T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \) is the transition matrix of \( X_1 \oplus X_2 \).

Example 4.1.

An example of sum is the PRN obtained by summing the same PRN twice, \( X \oplus X \). To make the disjoint union, we subindicate \( X \) with 0 for the first \( X \) and with 1 for the second \( X \). That is, the new set is

\[
X_0 \cup X_1 = \{(0,0,0), (0,1,0), (0,1,0), (1,1,0)\}
\cup \{(0,0,1), (0,1,1), (0,1,1), (1,1,1)\}.
\]

The digraph is:

\[
\begin{array}{cccc}
(1,1,0) & \rightarrow^4 & (1,0,0) & \rightarrow^6 & (0,0,0) \\
& & & \uparrow^1 & \\
(0,1,0) \\
& & & \uparrow^1 & \\
(1,1,1) & \rightarrow^4 & (1,0,1) & \rightarrow^6 & (0,0,1) \\
& & & \uparrow^1 & \\
(0,1,1)
\end{array}
\]

This is a way to construct a PRN over \( \{0,1\}^n \) using either one or two PRN over \( \{0,1\}^{n-1} \), since \( 2^{n-1} + 2^{n-1} = 2^n \).

Superposition

It is clear that a PRN is the superposition of several Finite dynamical Systems (FDS) over the same set \( X \) with probabilities assigned to each FDS. Since each
functions defined over a finite field can be wrote as a polynomial function, we will use this notation for functions over a finite field, \([2]\). If \(X = \{0, 1\} = \mathbb{Z}_2\), the finite field of two elements, all the FDSs over \(X\) have one of the following state space, where \(f_1(x) = x; f_2(x) = 1; f_3(x) = 0; f_4(x) = x + 1, \forall x \in X:\)

\[
\begin{array}{cccc}
L_1 & L_2 & L_3 & L_4 \\
0 \odot & 0 & 0 \odot & 0 \\
\downarrow & \uparrow & \uparrow & \downarrow \\
1 \odot & 1 \odot & 1 & 1 \\
\end{array}
\]

If \(p_i\) denotes the probability assigned to \(L_i\), and \(T_i\) denotes its transition matrix, then the set of all PRN is described as follows.

\[
\left\{(X, F, C) \mid T = \sum_{i=1}^{4} p_i T_i = \begin{pmatrix}
    p_1 + p_3 & p_2 + p_4 \\
    p_3 + p_4 & p_1 + p_2
\end{pmatrix}, \sum_{i=1}^{4} p_i = 1 \right\}
\]

We denote by \(L_1 L_2\) the superposition of \(L_1\) and \(L_2\), and similarly \(L_1 L_3\) is the superposition of \(L_1\) and \(L_3\). The state spaces are the following:

\[
\begin{array}{cccc}
L_1 L_2 & L_1 L_3 & L_1 L_4 \\
0 \odot & 0 \odot & 0 \odot \\
\uparrow & \uparrow & \uparrow \\
1 \odot & 1 \odot & 1 \odot \\
\end{array}
\]

\[
\begin{array}{cccc}
L_2 L_3 & L_2 L_4 \\
0 \odot & 0 \\
\uparrow & \uparrow \\
1 \odot & 1 \odot \\
\end{array}
\]

\[
\begin{array}{cccc}
L_3 L_4 \\
0 \odot \\
\uparrow \\
1 \\
\end{array}
\]

For example, with transition matrices

\[
T_{12} = T_1 + T_2 = \begin{pmatrix}
p_1 & p_2 \\
0 & 1
\end{pmatrix}, \quad T_{13} = T_1 + T_3 = \begin{pmatrix}
1 & 0 \\
p_3 & p_1
\end{pmatrix}
\]

**Product of two PRN**

Let \(X_1 = (X_1, F = (f_i)_{i=1}^n, C)\) and \(X_2 = (X_2, G = (g_j)_{j=1}^m, D)\) be two PRN. The product \(X_1 \times X_2 = (X_1 \times X_2, F \times G, C \wedge D)\) is a PRN where

1. \(X_1 \times X_2\) is the cartesian product of \(X_1\) and \(X_2\).
2. the function \(h_{ij} = (f_i, g_j)\) is defined by
   \[
   h_{ij}(x_1, x_2) = (f_i(x_1), g_j(x_2))
   \]
   for \(x_1 \in X_1\), and \(x_2 \in X_2\).
3. the probability \(p(h_{ij})\) is a function of \(c_i\) and \(d_j\), for example \(p(h_{ij}) = \frac{c_i + d_j}{2}\).

**Example 4.2.**
The product $L_1 L_2 \times L_1 L_3$ is the PRN with four states \{(0, 0), (0, 1), (1, 0), (1, 1)\} and four functions 

\[ f_{11}(x, y) = (x, y), f_{13}(x, y) = (x, 0), \]
\[ f_{21}(x, y) = (1, y), f_{23}(x, y) = (1, 0). \]

The state space is the following:

\[ L_1 L_2 \times L_1 L_3 \]
\[ \begin{array}{ccc}
  p_{11} + p_{13} & 0 & p_{23} + p_{21} \\
p_{13} & p_{11} & p_{23} \\
 0 & 0 & 1 \\
 0 & 0 & p_{13} + p_{23}
\end{array} \]

The transition matrix is the following

\[ T = \begin{pmatrix}
p_{11} + p_{13} & 0 & p_{23} + p_{21} \\
p_{13} & p_{11} & p_{23} \\
 0 & 0 & 1 \\
 0 & 0 & p_{13} + p_{23}
\end{pmatrix} \]

4.1. Linear Probabilistic Regulatory Networks. A linear PRN is a superposition of linear FDS. A linear FDS is a pair $(X, f)$ where $f$ is a linear function, and $X$ is a vector space over a finite field. So, a linear PRN is a triple $(X, (f_i)_{i=1}^n, C)$, where $X$ is a finite vector space, the functions $f_i : X \rightarrow X$ are linear functions, and $C = \{c_i = p(f_i)\}$. The set $X$ has cardinality a power of a prime number and each linear function is determined by its characteristic polynomial and the companion matrix.

If $X = \mathbb{Z}_3 = \{0, 1, 2\}$ is the field of integer modulo 3, then the linear functions are: $f_1(x) = x$, $f_2(x) = 2x$, and $f_3(x) = 0$ for all $x \in \mathbb{Z}_3$. So, the linear PRN are the following:

\[ \begin{array}{cc}
  \{f_1, f_2\} & \{f_1, f_3\} \\
  \bigcirc^1 & \bigcirc^1 \\
 0 & 0
\end{array} \]

If $X = \mathbb{Z}_2 \times \mathbb{Z}_2$ is the vector space with 4 elements over the field $\mathbb{Z}_2$, then there are 4 linear FDS not isomorphics. In fact, using matrix, the possible characteristics polynomials $p_f(\lambda)$ are: $\lambda^2$, $\lambda^3 + 1$, $\lambda^2 + \lambda + 1$. The companion matrices of these linear functions are:

\[ A_1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} A_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} A_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A_4 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \]

Then the FDS associated to these matrices are:

\[ \begin{array}{ccc}
  A_1 & A_2 \\
  \bigcirc (0, 0) & \bigcirc (0, 0) \\
  \bigcirc (1, 0) & \bigcirc (1, 0)
\end{array} \]

\[ \begin{array}{ccc}
  \bigcirc (0, 1) & \bigcirc (1, 1) \\
  \bigcirc (1, 1) & \bigcirc (0, 1)
\end{array} \]
The linear PRN with two functions are the following:

\[
\begin{align*}
A_1, A_2 &\quad A_1, A_3 \\
\odot (0, 0) &\quad \odot (1, 0) \\
&\quad \odot (0, 0) \searrow \nearrow \\
&\quad (1, 0) \\
&\quad \odot (0, 1) \\
&\quad \odot (1, 1) \quad (0, 1) \rightarrow (1, 1)
\end{align*}
\]

5. Invariant Subnetworks and Projections

A subnetwork \(Y \subseteq X\) of \(X = (X, F, C)\) is an invariant subnetwork or a sub-PRN of \(X\) if \(f_i(u) \in Y\) for all \(u \in Y\), and \(f_i \in F\). Sub-PRNs are sections of a PRN, where there aren’t arrows going out. The complete network \(X\), and any cyclic state with probability 1, are sub-PRNs. An invariant subnetwork is irreducible if doesn’t have a proper invariant subnetwork. An endomorphism is a projection if \(\pi^2 = \pi\).

Theorem 5.1. If there exists a projection from \(X\) to a subnetwork \(Y\) then \(Y\) is an invariant subnetwork of \(X\).

Proof. Suppose that there exists a projection \(\pi : X \rightarrow Y\). If \(y \in Y\), by definition of projection \(\pi(y) = y\), and \(f_i(\pi(y)) = \pi(g_j(y))\). Therefore all arrows in the subnetwork \(Y\) are going inside \(Y\), and the network is invariant. \(\square\)

Example 5.2.

The PRN \(\widehat{X}\) has two invariant subnetworks with projections \(\pi_1(x, y) = (x, 0)\) and \(\pi_2(x, y) = (1, y)\).
Checking the probabilities for $\pi_1$ and $\pi_2$, we have $\epsilon_1 = .68$; and $\epsilon_2 = .67$.

We can observe that $X \cong S_1 \times S_2$.

**Example 5.3.**

The subnetwork $X_1 = \{(x, y, 1)\}, F, C$ is an invariant subnetwork of $X = (\{0, 1\}^3, F, C)$.

$X_1 \cong X_1$

Using the projection $\pi : X \to X_1$, $\pi(x, y, z) = (x, y, 1)$; and the isomorphism $\rho(x, y, 1) = (x, y)$, the network $X$ is projected over the network $X_1$. Checking the arrows the projection $\pi$ is a $\epsilon$-homomorphism.

**5.1. Mathematical background.**

**Theorem 5.4.** If $\phi_1 : X_1 \to X_2$ is an $\epsilon_1$-homomorphism, and $\phi_2 : X_2 \to X_3$ is another $\epsilon_2$-homomorphism. Then $\phi = \phi_2 \circ \phi_1 : X_1 \to X_3$ is an $\epsilon$-homomorphism. Therefore the Probabilistic Regulatory Networks with the homomorphisms of PRN form the category PRN.

**Proof.** The Probabilistic Regulatory Networks with the PRN homomorphisms is a category if: the composition is an homomorphism, and satisfy the associativity law; and there exists an identity homomorphism for each PRN.

(1) Let $\phi_1 : X_1 \to X_2$ be an $\epsilon_1$-homomorphism, and let $\phi_2 : X_2 \to X_3$ be an $\epsilon_2$-homomorphism. If $q_l, g_k$ and $f_j$ are functions in each PRN, and such that $\phi_1 \circ f_j = g_k \circ \phi_1$ and $\phi_2 \circ g_k = q_l \circ \phi_2$, then we will prove that: $\phi \circ f_j = q_l \circ \phi$. In fact,

$$(\phi_2 \circ \phi_1) \circ f_j = \phi_2 \circ (\phi_1 \circ f_j) =$$

$$(\phi_2 \circ g_k \circ \phi_1) = (\phi_2 \circ g_k) \circ \phi_1 =$$
(2) We want to prove that
\[
\chi(t_k(\phi(u), \phi(v))) \geq \chi(c_i(u, v)).
\]
Suppose that \(\chi(c_i(u, v)) = 1\). Then, since \(\phi_1\) is an homomorphism of PRN, we have that
\[
\chi(d_j(\phi_1(u), \phi_1(v))) \geq \chi(c_i(u, v))
\]
which is 1. Since \(\phi_2\) is an homomorphism of PRN, we obtain that
\[
\chi(t_k(\phi(u), \phi(v))) = \chi(t_k(\phi_2(\phi_1(u)), \phi_2(\phi_1(v))))
\]
\[
\geq \chi(c_j(\phi_1(u), (\phi_1(v))) = 1.
\]
Therefore we obtain that
\[
\chi(t_k(\phi_2(\phi_1(u)), \phi_2(\phi_1(v)))) = 1.
\]

Then the composition of two PRN-homomorphisms is an homomorphism.

(3) To verify the third condition for \(\epsilon\)-homomorphism, we do the following. If \(p(\phi(u_1), \phi(u_2)) > 1\), with \(u_1, u_2 \in X_1\), then we need to prove that there exists an \(\epsilon\) such that
\[
|p(u_1, u_2) - p(\phi(u_1), \phi(u_2))| < \epsilon.
\]
In fact:
\[
|p(u_1, u_2) - p(\phi(u_1), \phi(u_2))| = |p(u_1, u_2) - p(\phi_1(u_1), \phi_1(u_2)) + p(\phi_1(u_1), \phi_1(u_2)) - p(\phi_2(\phi_1(u_1)), \phi_2(\phi_1(u_2)))|< |p(u_1, u_2) - p(\phi_1(u_1), \phi_1(u_2))| + |p(\phi_1(u_1), \phi_1(u_2)) - p(\phi_2(\phi_1(u_1)), \phi_2(\phi_1(u_2)))| \leq \epsilon_1 + \epsilon_2
\]
because \(\phi_1\) and \(\phi_2\) are \(\epsilon\)-homomorphisms.

The associativity and identity laws are easily checked, therefore our claim holds, and PRN is a category.

It is clear that, the PRN with the homomorphism between them form a category that we will denote \(\mathcal{PRN}\). The category PRN is a subcategory of \(\mathcal{PRN}\), since an homomorphism is not always an homomorphism for some \(\epsilon \in \mathbb{R}\) enough small. But, if we don’t include the condition for \(\epsilon\) to be enough small, the two categories are the same, because always an homomorphism is an \(\epsilon\)-homomorphism for some \(\epsilon \in \mathbb{R}\).

**Theorem 5.5.** Let \(X_1 \times X_2 = (X_1 \times X_2, H, E)\) be a product of PRN \(X_1 = (X_1, F, C)\) and \(X_2 = (X_2, G, D)\). If \(\delta_i : X \rightarrow X_i\) are two PRN-homomorphisms, then there exists an homomorphism \(\delta : X \rightarrow X_1 \times X_2\), such that \(\phi_i \circ \delta = \delta_i\) for \(i = 1, 2\). That is, the following diagram commutes

\[
\begin{array}{ccc}
X_1 \times X_2 & \xrightarrow{\phi_1 \times \phi_2} & X_1 \\
\downarrow \phi_1 \times \phi_2 & & \downarrow \phi_i \\
X_1 & \xrightarrow{\delta_i} & X_2
\end{array}
\]

This homomorphism is unique.
Proof. The function $\delta : X \to X_1 \times X_2$ is defined as follows $\delta(x) = (\delta_1(x), \delta_2(x))$, $x \in X$. $\delta$ is an homomorphism, in fact:

(1) Let $X = (X, L, P)$ be a PRN. Since $\delta_1$ and $\delta_2$ are homomorphism, for all function $l_t \in L$ there exist two functions $f_i \in F$ and $g_j \in G$, such that $\delta_1 \circ l_t = f_i \circ \delta_1$, and $\delta_2 \circ l_t = g_j \circ \delta_2$. Then for the function $l_t$ there exists the function $(f_i, g_j)$ that satisfies $\delta \circ l_t = (f_i, g_j) \circ \delta$.

$(\delta \circ l_t)(x) = \delta(l_t(x)) = (\delta_1(l_t(x)), \delta_2(l_t(x))) = (f_i(\delta_1(x)), g_j(\delta_2(x))) = ((f_i, g_j) \circ \delta)(x)$

(2) In order to prove $\chi(e_{ij}(\delta(x), \delta(x'))) \geq \chi(p_t(x, x'))$, suppose $\chi(p_t(x, x')) = 1$. Then $l_t(x) = x'$, and $\delta(x') = \delta(l_t(x)) = (f_i, g_j)(\delta(x))$ by part (1). Therefore $\chi(e_{ij}(\delta(x), \delta(x'))) = 1$, and our claim holds.

It is easy to check that $\phi_i \circ \delta = \delta_i$, in fact

$\phi_1(\delta(x)) = \phi_1(\delta_1(x), \delta_2(x)) = \delta_1(x)$,

for all $x \in X$.

If $\delta_i$, $i = 1, 2$, are $\epsilon_i$-homomorphism then

$max|p(x, x') - p(\phi_1(\delta(x)), \phi_1(\delta(x')))| \leq \epsilon_1.$

But

$|p(x, x') - p(\delta(x), \delta(x')) + p(\delta(x), \delta(x')) - p(\phi_1(\delta(x)), \phi_1(\delta(x')))| \leq$

$|p(x, x') - p(\delta(x), \delta(x'))| +$

$|p(\delta(x), \delta(x')) - p(\phi_1(\delta(x)), \phi_1(\delta(x')))| \leq \epsilon_1.$

Therefore

$|p(x, x') - p(\delta(x), \delta(x'))| \leq \epsilon_1 - |p(\delta(x), \delta(x')) - p(\phi_1(\delta(x)), \phi_1(\delta(x')))|$

$|p(x, x') - p(\delta(x), \delta(x'))| \leq \epsilon_1 - \epsilon_1 = \epsilon_1.$

Therefore $\delta$ is an $\epsilon$-homomorphism. So, the theorem holds for $\epsilon$-homomorphism.

It is an immediate consequence the following result, also is true for $\epsilon$-homomorphisms.

**Theorem 5.6.** Let $X_1 \oplus X_2 = (X_1 \times X_2, H, E)$ be a product of PRN $X_1 = (X_1, F, C)$ and $X_2 = (X_2, G, D)$. If $\gamma_i : X_i \to X$ are two PRN-homomorphisms, then there exists an homomorphism $\gamma : X_1 \oplus X_2 \to X$, such that $\gamma \circ \iota_i = \gamma_i$ for $i = 1, 2$. That is, the following diagram commutes

$$
\begin{array}{ccc}
X_1 \oplus X_2 & \xrightarrow{\iota_1 \oplus \iota_2} & X_1 \\
\gamma \downarrow & & \downarrow \gamma_1 \\
X_1 \oplus X_2 & \xrightarrow{\gamma_2} & X_2
\end{array}
$$

This homomorphism is unique.

**Theorem 5.7** (Fundamental Theorem). All reducible PRN is either a product of its non trivial sub-PRN or a subnetwork of this product.

Proof. It is trivial by definition of Product and sub-PRN. \qed
6. Conclusions

The intersection, and the union of two sub-PRN is a sub-PRN, therefore the class of sub-PRN of a particular PRN is a lattice. Reduction mappings described in [7] and defined for PBN using the influence of a gene, for example \( x_n \), on the predictor function \( f_j^{(1)} \) to determine the selected predictor, can be extended to PRN. In order to extend this procedure to more than boolean functions, we use the polynomial description of genetic functions given in [2], the partial derivative is the usual in calculus and all the concepts in [7] can be using for PRN. Similarly our definition of projection, the reduction mappings are \( \varepsilon \)-homomorphisms, and we can use for genes with more than two quantization, since this extension is not a trivial work we develop the theory and methods in [1].

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