Research Article

Oluwatson T. Mewomo* and Olawale K. Oyewole

An iterative approximation of common solutions of split generalized vector mixed equilibrium problem and some certain optimization problems

https://doi.org/10.1515/dema-2021-0019
received October 9, 2020; accepted May 19, 2021

Abstract: In this paper, we study the problem of finding a common solution of split generalized vector equilibrium problem (SGVMEP), fixed point problem (FPP) and variational inequality problem (VIP). We propose an inertial-type iterative algorithm, which uses a projection onto a feasible set and a linesearch, which can be easily calculated. We prove a strong convergence of the sequence generated by the proposed algorithm to a common solution of SGVMEP, fixed point of a quasi-$\phi$-nonexpansive mapping and VIP for a general class of monotone mapping in 2-uniformly convex and uniformly smooth Banach space $E_1$ and a smooth, strictly convex and reflexive Banach space $E_2$. Some numerical examples are presented to illustrate the performance of our method. Our result improves some existing results in the literature.

Keywords: split feasibility problem, vector equilibrium problem, generalized mixed equilibrium problem, quasi-monotone mapping, strong convergence, Banach space

MSC 2020: 47H06, 47H09, 46N10

1 Introduction

Let $C$ denote a nonempty, closed and convex subset of a real Banach space $E$ with norm $\|\cdot\|$ and $f : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f\|^2, \ \forall x \in E\},$$

where $E^*$ is the dual space of $E$ and $\langle \cdot, \cdot \rangle$ is the duality pairing between the elements of $E$ and $E^*$.

Let $C$ and $X$ be nonempty, closed and convex subsets of real Banach spaces $E_1$ and $E_2$, respectively, then the split feasibility problem (SFP) (see [1,2]), consists of finding a point

$$x \in C \text{ such that } Lx \in X,$$

where $L : E_1 \to E_2$ is a bounded linear operator. The SFP is a variant of the inverse problem and finds real life applications in image processing, radiation therapy and remote sensing [3–6]. For approximating the solutions of (2) and related optimization problems in Hilbert, Banach, Hadamard and p-uniformly convex metric spaces, researchers have developed several iterative methods that adopt a fixed point approach
(see [3,7–9] and references therein). It has been shown that a point $x \in C$ is a solution of (2) if and only if $x$ is a fixed point of the operator $P_L (I - y L (I - P y)) L$ (see [10]), where $P_C$, $P_X$ are metric projections onto $C$ and $X$, respectively, $L$ is a bounded linear operator with adjoint $L^*$ and $y$ is a positive parameter. Let $T : C \to C$ be a mapping. A point $x \in C$ is called a fixed point of $T$, if $x = Tx$. We shall denote the set of fixed points of $T$ by $\text{Fix}(T)$, that is $\text{Fix}(T) = \{ x \in C : x = Tx \}$.

Let $X$ be a nonempty, convex subset of a real Banach space $E$. Assume that $P$ is a proper, pointed, closed, convex cone of a real Hausdorff $Y$ and $e \in \text{int} P$. Let $f : X \times X \to Y$ be a bifunction, $\psi : X \to E$ be a nonlinear mapping and $\phi : X \to Y$ be a function. The generalized vector mixed equilibrium problem (GVMEP) is the problem of finding a point $x \in X$ such that

$$f(x, y) + \langle y - x, \psi(x) \rangle + \phi(y) - \phi(x) \in P, \quad \forall y \in X.$$  \hfill (3)

Problem (3) so defined is called strong GVMEP. The problem is however said to be weak if the notation $e \in P$ in (3) is replaced by the notation $e \notin \text{int} P$. Now, suppose in (3) we let $\psi = 0$, then problem reduces to the generalized vector equilibrium problem (GVEP) studied by Kazmi and Farid (see [11]). Also, if we set $\psi = \phi = 0$, then (3) reduces to a vector equilibrium problem (VEP), where VEP consists of finding $x \in X$ such that

$$f(x, y) \in P, \quad \forall y \in X.$$  \hfill (4)

Moreover, if we set $Y = \mathbb{R}$ and $e = 1$, then Problem (3) reduces to the generalized mixed equilibrium problem considered by Peng and Yao [12]. Furthermore, problem (4) reduces to the classical equilibrium due to Blum and Oettli [13]. Vector equilibrium represents a unified framework for studying several problems, including vector optimization, vector variational inequality, vector complementarity problems and so on [14,15]. In recent years, iterative algorithms for obtaining the equilibrium problems, zero points problems and related optimization problems have been studied in the literature (see [14–20] and references therein).

Let $C$ be a nonempty, closed and convex subset of a real Banach space $E$ with dual $E^*$ and $F : C \to E^*$ be a mapping. The variational inequality problem (VIP) is to find a point $x \in C$ such that

$$\langle y - x, F(x) \rangle \geq 0, \quad \forall y \in C.$$  \hfill (5)

We shall denote the solution set of (5) by $\text{VIP}(C, F)$. Closely related to (5) (see [21]) is the problem of finding $y \in C$ such that

$$\langle y - x, F(y) \rangle \geq 0, \quad \forall y \in C.$$  \hfill (6)

Following [21], we shall refer to (6) as the dual variational inequality problem (DVIP) of (5). The VIP is one of the central problems in nonlinear analysis (see [22–24]) with monotonicity playing a major role in its study. For instance, monotone operators are important tools in the study of several problems in the domain of optimization, nonlinear analysis, differential equation and other related fields. However, there have also been studies of variational inequalities with weaker monotonicity conditions such as pseudomonotone, quasimonotone, strictly quasimonotone, etc. In 2019, Chang et al. [25] studied an iterative approximation of solution of VIP for a semistrictly quasimonotone operator in the framework of infinite-dimensional Hilbert spaces (see [26] and references therein). It is known that extragradient methods for solving VIP require projections onto a set which are difficult to evaluate especially when the structure of the set is not simple. He et al. [27] introduced a totally relaxed self subgradient extragradient method (TRSSEM) involving feasible sets which are easily defined for solving the VIP. Let $U = \{ z \in E : h_j(z) \leq 0 \}$, where $h_j : E \to \mathbb{R}$ for all $j = 1, 2, \ldots, m$ are convex and differentiable functions. For the TRSSEM, the feasible set is defined as $C = \bigcap_{j=1}^{m} C_j$.

Furthermore, obtaining a common element in the solution set of a fixed point problem (FPP), VIP and EP has recently been considered by authors in the literature due to its various applications, see [28–30]. In 2012, Shan and Huang [31] introduced the concept of generalized mixed vector equilibrium problem (GMVEP). They obtain the existence result in the framework of Hilbert space for this problem. They further
proposed an iterative algorithm for obtaining a common element in the solution set of GMVEP, VIP and fixed point of a nonexpansive mapping. Very recently, Farid and Kazmi [32] introduced and studied a general iterative algorithm for approximating a common solution of split generalized equilibrium problem (SGEP), VIP and FPP. They proved a strong convergence theorem for the sequences generated by the proposed algorithm. For more references on this see [26].

In this paper, motivated by Shan and Huang [31], Chang et al. [25] and He et al. [27], we study an iterative approximation of a common solution of split generalized vector mixed equilibrium problem (SGVMEP), VIP and fixed point of quasi-$\phi$-nonexpansive mapping. We proposed an inertial-type iterative algorithm which uses projection onto a feasible set and a linesearch with Halpern method. We prove a strong convergence theorem for the sequence generated by this algorithm to a common solution of these problems in the frame work of 2-uniformly convex and uniformly smooth Banach space $E_1$ and a smooth, strictly convex and reflexive Banach space $E_2$. Finally, some numerical examples are presented to illustrate the performance of our method.

The rest of the paper is organized as follows: We first recall some basic definitions, required assumptions and results in Section 2. We give an explicit statement of the problem and show that its solution set is well defined, and we also propose an iterative process and prove a strong convergence of the method to a solution of the problem in Section 3. Some numerical experiments of our results are given in Section 4. We give concluding remarks in Section 5.

2 Preliminaries

In this section, we give some important definitions, results and restrictions which are useful in establishing our main results. Throughout this paper, we denote the weak and strong convergence of a sequence $\{x_k\}$ in a real Banach space $E$ to a point $x \in E$ by $x_k \rightharpoonup x$ and $x_k \to x$, respectively.

Let $E$ be a real Banach space, a function $h : E \to \mathbb{R}$ is said to be:

(a) Gâteaux differentiable at a point $x \in E$, if there exists an element in $E$ denoted by $h'(x)$ or $\nabla h(x)$ such that

$$\lim_{t \to 0} \frac{h(x + ty) - h(x)}{t} = \langle y, h'(x) \rangle, \quad \forall y \in E,$$

where $h'(x)$ or $\nabla h(x)$ is called Gâteaux differential or gradient of $h$ at $x$. $h$ is said to be Gâteaux differentiable on $E$ if it is Gâteaux differentiable on every $x \in E$;

(b) weakly lower semicontinuous at $x \in E$ if $\{x_k\} \subset E$, $x_k \to x$ implies $h(x) \leq \liminf_{k \to \infty} h(x_k)$. Also, $h$ is weakly lower semicontinuous on $E$ if $h$ is weakly lower semicontinuous for each $x \in E$;

(c) if $h$ is such that $h((1 - \lambda)x + \lambda y) \leq (1 - \lambda)h(x) + \lambda h(y)$, for each $x, y \in E$ and $\lambda \in (0, 1)$, then $h$ is said to be a convex function. $h$ is said to be differentiable if the set

$$\partial h(x) = \{w \in E^* : h(y) - h(x) \geq \langle w, y - x \rangle, \forall y \in E\} \neq \emptyset. \quad (7)$$

Each element $\partial h(x)$ is called a subgradient of $h$ at $x$ or the subdifferential of $h$ and inequality (7) is said to be the subdifferential inequality of $h$ at $x$. We say that the function $h$ is subdifferentiable at $E$ if it is subdifferentiable at every point of $E$. It is known that if $h$ is Gâteaux differentiable at $x$, then $h$ is subdifferentiable at $x$ and $\partial h(x) = \partial h(x)$, which implies $\partial h(x)$ is singleton (see [33]). For more details on these, see [4,34,35] and references therein.

Following [36], Albert introduced a generalized projection operator $\Pi_C : E \to C$ defined by

$$\Pi_C(x) = \arg \min_{y \in C} \phi(y, x), \quad \forall x \in E,$$

where $\phi : E \times E \to \mathbb{R}$ is the Lyapunov functional defined by
\[
\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.
\]

The functional \( \phi \) is known to satisfy the following properties:

- (P1) \( (\|x\|^2 - \|y\|^2) \leq \phi(x, y) \leq (\|x\|^2 + \|y\|^2); \)
- (P2) \( \phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle; \)
- (P3) \( \phi(z, Ax + (1 - \lambda)y) \leq \lambda \phi(z, x) + (1 - \lambda)\phi(z, y), \) for all \( x, y, z \in E \) and \( \lambda \in (0, 1); \)
- (P4) \( \phi(x, y) + \phi(y, x) = \langle x - y, Jx - Jy \rangle, \quad \forall x, y \in E; \)
- (P5) \( \phi(x, y) \leq \|x\| \|Jx - Jy\| + \|y\| \|x - y\|. \)

In Hilbert space, \( \Pi_c = P_c \), the metric projection and \( \phi(x, y) = \|x - y\|^2 \), see [37] for details on \( P_c \).

**Remark 2.1.** [38, 39] If \( E \) is a reflexive, strictly convex and smooth Banach space, then for \( x, y \in E \), \( \phi(x, y) = 0 \) if and only if \( x = y \).

We also require the functional \( V : E \times E^* \to \mathbb{R} \) defined by

\[
V(u, v) = \|u\|^2 - 2\langle u, v \rangle + \|v\|^2, \quad \text{for each } u \in E \text{ and } v \in E^*.
\]

It is easy to see that \( V(u, v) = \phi(u, J^{-1}v) \) for all \( u \in E \) and \( v \in E^* \). It is well known (see [40]) that if \( E \) is a reflexive, strictly convex and smooth Banach space, then

\[
V(u, v) \leq V(u, v + w) - 2\langle J^{-1}v - u, w \rangle,
\]

for all \( u \in E \) and \( v, w \in E^* \).

Let \( C \) be a nonempty, closed and convex subset of a real Banach space and \( T : C \to C \) be a mapping. A point \( \bar{x} \in C \) is called an asymptotic fixed point of \( T \) if \( C \) contains a sequence \( \{x_k\} \) such that \( x_k \to \bar{x} \) and \( \|x_k - Tx_k\| \to 0 \) as \( k \to \infty \). We denote by \( \text{Fix}(T) \) the asymptotic fixed point set of \( T \). A mapping \( T \) from \( C \) into itself is said to be relatively nonexpansive \([41, 42]\), if \( \text{Fix}(T) = \text{Fix}(T) \) and \( \phi(p, Tx) \leq \phi(p, x) \) for all \( x \in C \) and \( p \in \text{Fix}(T) \). Furthermore, \( T \) is said to be \( \phi \)-nonexpansive, if \( \phi(Tx, Ty) \leq \phi(x, y) \) for \( x, y \in C \). It is said to be quasi-\( \phi \)-nonexpansive if \( \text{Fix}(T) \neq \emptyset \) and \( \phi(p, Tx) \leq \phi(p, x) \) for all \( x \in C \) and \( p \in \text{Fix}(T) \). The class of quasi-\( \phi \)-nonexpansive mappings is more general than the class of relatively nonexpansive mappings \([41, 43]\) as the latter requires the strong restriction \( \text{Fix}(T) = \text{Fix}(T) \).

Let \( E \) a real Banach space. The modulus of smoothness of \( E \) is the function \( \rho_E : [0, \infty) \to [0, \infty) \) defined by

\[
\rho_E(t) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.
\]

\( E \) is uniformly smooth if and only if

\[
\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0.
\]

Let \( E \) be a real Banach space with a dimension greater or equal to 2, the modulus of convexity of \( E \) is the function \( \delta_E : [0, 2] \to [0, 1] \) defined by

\[
\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| = 1 = \|y\|; \|x - y\| = \varepsilon \right\}.
\]

\( E \) is uniformly convex if for any \( \varepsilon \in (0, 2] \), there exists a \( \delta = \delta(\varepsilon) > 0 \) such that if \( x, y \in E \) with \( \|x\| \leq 1, \|y\| \leq 1 \) and \( \|x - y\| \geq \varepsilon \), then \( \frac{1}{2} \|x + y\| \leq 1 - \delta \). Equivalently, \( E \) is uniformly convex if and only if \( \delta_E(\varepsilon) > 0 \) for all \( \varepsilon \in (0, 2] \). \( E \) is said to be 2-uniformly convex if there exists a constant \( c > 0 \) such that \( \delta_E(\varepsilon) > c\varepsilon^2 \) for every \( \varepsilon \in (0, 2] \), where \( c \) is the 2-uniformly convexity constant of \( E \). It is known [38,44,45] that every 2-uniformly convex Banach space is uniformly convex and reflexive. The space \( E \) is strictly convex, if for each \( x \neq y \in E \) and \( \|x\| = \|y\| = 1 \), we have \( \|\lambda x + (1 - \lambda)y\| < 1 \) for all \( \lambda \in (0, 1) \).
Lemma 2.2. [46] Given a number $s > 0$. A real Banach space $E$ is uniformly convex if and only if there exists a continuous strictly increasing function $g : (0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

for all $x, y \in E$, $\lambda \in [0, 1]$, with $\|x\| < s$ and $\|y\| < s$.

Lemma 2.3. [36] Let $C$ be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space $E$. If $x \in E$ and $q \in C$, then

$$q = \Pi_C x \iff \langle y - q, Jx - Jq \rangle \leq 0, \quad \forall y \in C$$

and

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C, \quad x \in E.$$  

Lemma 2.4. [47] Let $E$ be a smooth and uniformly convex real Banach space and let $\{x_k\}$ and $\{y_k\}$ be two sequences in $E$. If either $\{x_k\}$ or $\{y_k\}$ is bounded and $\phi(x_k, y_k) \to 0$ as $k \to \infty$, then $\|x_k - y_k\| \to 0$ as $k \to \infty$.

Remark 2.5. [48] Using P5, it is easy to see that converse of Lemma 2.4 is also true whenever both $\{x_k\}$ and $\{y_k\}$ are bounded.

Lemma 2.6. [49] Let $E$ be a 2-uniformly convex and smooth Banach space. Then for every $x, y \in E$

$$\phi(x, y) \geq \nu \|x - y\|^2,$$

where $\nu > 0$ is the 2-uniformly convexity constant of $E$.

Lemma 2.7. [46] Let $E$ be a 2-uniformly smooth Banach space with the best smoothness constant $d > 0$. Then, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + \langle y, Jx \rangle + 2d\|y\|^2, \quad \forall x, y \in E.$$

Definition 2.8. [20,50] Let $X$ and $Y$ be two Hausdorff topological spaces and let $D$ be a nonempty, convex subset of $X$ and $P$ be a pointed, proper, closed and convex cone of $Y$ with int$P \neq \emptyset$. Let $0$ be the zero point of $Y$, $N(0)$ the neighbourhood set of $0$, $N(x_0)$ be the neighbourhood set of $x_0$ and $f : D \to Y$ be a mapping.  

1) If, for any $V \in N(0) \subset Y$, there exists $U \in N(x_0)$ such that

$$f(x) \in f(x_0) + V + P, \quad \forall x \in U \cap D,$$

then $f$ is called upper $P$-continuous on $x_0$. If $f$ is upper $P$-continuous for all $x \in D$, then $f$ is called upper $P$-continuous on $D$;

2) If, for any $V \in N(0) \subset Y$, there exists $B \in N(x_0)$ such that

$$f(x) \in f(x_0) + V - P, \quad \forall x \in U \cap D,$$

then $f$ is called lower $P$-continuous on $x_0$. If $f$ is lower $P$-continuous for all $x \in D$, then $f$ is called lower $P$-continuous on $D$;

3) If, for any $x, y \in D$ and $\lambda \in [0, 1]$, the mapping $f$ satisfies

$$f(x) \in f(\lambda x + (1 - \lambda)y) + P \quad \text{or} \quad f(y) \in f(\lambda x + (1 - \lambda)y) + P,$$

then $f$ is called proper $P$-quasiconvex;

4) If, for any $x, y \in D$ and $\lambda \in [0, 1]$, the mapping $f$ satisfies

$$\lambda f(x) + (1 - \lambda)f(y) \in f(\lambda x + (1 - \lambda)y) + P,$$

then $f$ is called $P$-convex.

Lemma 2.9. [51] Let $X$ and $Y$ be two real Hausdorff topological spaces, $D$ is a nonempty, compact, convex subset of $X$ and $P$ is a pointed, proper, closed and convex cone of $Y$ with int$P \neq \emptyset$. Assume $f : D \times D \to Y$ and $\Phi : D \to Y$ are two vector mappings. Suppose $f$ and $\Phi$ satisfy
Then, there exists a point \( x \in D \) that satisfies \( G(x, y) \in P \setminus \{0\}, \ \forall y \in D \) where

\[
G(x, y) = f(x, y) + \Phi(y) - \Phi(x), \quad \forall x, y \in D.
\]

For solving the GMVEP, we give the following assumptions: Let \( X \subset E_2 \) be a nonempty, compact, convex subset of real Banach space \( E_2 \) and \( Y \) a real Hausdorff topological space, \( P \subset Y \) be a proper, closed and convex cone. Let \( f : X \times X \rightarrow Y \) and \( \psi : X \rightarrow Y \) be two mappings. For any \( x \in E_2 \), define a mapping \( \Psi : X \times X \rightarrow Y \) as follows:

\[
\Psi(y, z) = f(z, y) + \psi(z) - \psi(y) + \frac{e}{r}(y - z, Jz - Jx),
\]

where \( r \) is a positive number in \( \mathbb{R} \) and \( e \in \text{int}P \). Let \( \Psi, f \) and \( \psi \) satisfy the following conditions:

(R1) For all \( x \in X \), \( f(x, x) = 0 \);

(R2) \( f \) is monotone, that is \( f(x, y) + f(y, x) \in -P, \ \forall x, y \in X \);

(R3) \( f(\cdot, \cdot) \) is continuous, \( \forall y \in X \);

(R4) \( f(x, \cdot) \) is weakly continuous and \( P \)-convex, that is,

\[
\lambda f(x, y) + (1 - \lambda)f(x, z) \in f(x, \lambda y + (1 - \lambda)z) + P, \quad \forall x, y, z \in X \quad \text{and} \quad \forall \lambda \in [0, 1];
\]

(R5) \( \Psi(\cdot, y) \) is lower \( P \)-continuous for all \( y \in X \) and \( z \in E_2 \);

(R6) \( \psi(\cdot) \) is \( P \)-continuous;

(R7) \( \psi(\cdot, \cdot) \) is proper \( P \)-quasiconvex for all \( x \in X \) and \( z \in E_2 \).

The following result was proved in [31] in the framework of Hilbert space, but can easily be adapted for this study.

**Lemma 2.10.** [31] Let \( f \) and \( \psi \) satisfy restrictions (R1)–(R7). Define a mapping \( K_r : E_2 \rightarrow X \) as follows:

\[
K_r(x) = \{ z \in E_2 : f(z, y) + \psi(z) - \psi(y) + \frac{e}{r}(y - z, Jz - Jx) \in P, \ \forall y \in X \}.
\]

Then,

(i) \( K_r(x) \neq \emptyset \) for all \( x \in E_2 \);

(ii) \( K_r \) is single valued;

(iii) \( K_r \) is firmly nonexpansive-type mapping, that is for all \( x, y \in E_2 \),

\[
\langle K_r(x) - K_r(y), JK_r(x) - JK_r(y) \rangle \leq \langle K_r(x - K_r(y), Jx - Jy) \rangle;
\]

(iv) \( \text{Fix}(K_r) = \text{Sol}(\text{GV EP}) \);

(v) \( \text{Sol}(\text{GV EP}) \) is closed and convex.

Let \( E \) be a real Banach space. Given \( x, y \in E \), define the open segment

\[
(x, y) = \{ tx + (1 - t)y : 0 < t < 1 \}.
\]

The segments \( (x, y), [x, y] \) and \([x, y]\) are defined analogously.

**Definition 2.11.** [25] A mapping \( F : E \rightarrow E^* \) is said to be

(a) weakly hemicontinuous if \( F \) is upper semicontinuous from line segments to the weak topology of \( E \);

(b) sequentially weakly semicontinuous if for each sequence \( \{x_k\} \) in \( E \) with \( x_k \rightharpoonup q \), then \( Fx_k \rightharpoonup Fq \).

It is easy to check that (b) implies (a).
Lemma 2.12. [21] A solution of DVIP is always a solution of VIP, if the operator $F$ is weakly hemicontinuous.

Remark 2.13. It is well known that $p \in C$ is a solution of (5) if and only if $p$ is a fixed point of the operator $P_{\lambda}(I - AF)$ for all $\lambda > 0$.

Definition 2.14. Let $C$ be a nonempty closed and subset of a real Banach space $E$ with dual $E^\ast$. The mapping $F : C \to E^\ast$ is said to be:

(i) $m$-inverse strongly monotone on $C$ with constant $m > 0$ if for each $u, v \in C$, there holds

$$\langle u - v, F(u) - F(v) \rangle \geq m\|F(u) - F(v)\|^2;$$

(ii) strongly monotone on $C$ with constant $\tau > 0$ if for each $u, v \in C$, there holds

$$\langle u - v, F(u) - F(v) \rangle \geq \tau\|u - v\|^2;$$

(iii) strictly monotone on $C$ if for all distinct $u, v \in C$, there holds

$$\langle u - v, F(u) - F(v) \rangle > 0;$$

(iv) monotone on $C$ if for each $u, v \in C$, there holds

$$\langle u - v, F(u) - F(v) \rangle \geq 0;$$

(v) pseudo-monotone on $C$ if for each $u, v \in C$, there holds

$$\langle u - v, F(v) \rangle \geq 0 \Rightarrow \langle u - v, F(u) \rangle \geq 0;$$

(vi) quasi-monotone on $C$ if for each $u, v \in C$, there holds

$$\langle u - v, F(v) \rangle > 0 \Rightarrow \langle u - v, F(u) \rangle \geq 0;$$

(vii) (see [52]) semistrictly quasi-monotone on $C$ if $F$ is quasi-monotone on $C$ and for all distinct points $u, v \in C$, we have that

$$\langle u - v, F(v) \rangle > 0 \Rightarrow \langle u - v, F(w) \rangle > 0,$$

for some $w \in \left(\frac{1}{2}(u + v), u\right)$.

Lemma 2.15. [53] Let $C$ be a nonempty, closed and convex subset of $E$ and $F : C \to E$ be a weakly hemi-continuous and semistrictly quasi-monotone mapping. Then $DVI(C, F)$ at least has one solution.

Lemma 2.16. [54] Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{k+1} \leq (1 - a_k)a_k + a_k \alpha_k + \gamma_k, \quad k \geq 0,$$

where

(a) $\{\alpha_k\} \subset [0, 1], \lim_{k \to \infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(b) $\limsup_{k \to \infty} \alpha_k \leq 0$;

(c) $\gamma_k \geq 0, (k \geq 1)$ and $\sum_{k=1}^{\infty} \gamma_k < \infty$.

Then, $\lim_{k \to \infty} a_k = 0$.

Lemma 2.17. [9,42] Let $\{a_k\}$ be a sequence of real numbers such that there exists a subsequence $\{k_j\}$ of $\{k\}$ such that $a_{k_j} < a_{k_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing subsequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$: $a_{m_k} < a_{m_{k+1}}$ and $a_k < a_{m_{k+1}}$.

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.
3 Main result

In this section, we prove our main result. First, we explicitly state the problem considered in this paper, then we introduce a linesearch algorithm for obtaining the solution of this problem and finally discuss its convergence analysis.

Let $C$ be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space $E_1$, $X$ be a nonempty, compact and convex subset of a smooth, strictly convex and reflexive Banach space $E_2$, $L : E_1 \to E_2$ be a bounded linear operator with $L^*: E_2^* \to E_1^*$ its adjoint. Let $P$ be a pointed, proper, closed and convex cone of a real Hausdorff topological space $Y$. Let $f : X \times X \to Y$ and $A : X \to Y$ functions satisfying assumptions $(R_1)$–$(R_7)$. Let $\psi : X \to E_2$ be an $m$-inverse strongly monotone mapping and $T : C \to C$ be a quasi-$\phi$-nonexpansive mapping. Also, let $F : C \to E_2$ be a semistrictly quasi-monotone, sequentially weakly continuous mapping. We consider the problem of finding a point $p \in C$ such that

$$p \in \text{VIP}(C, F) \cap \text{Fix}(T) \quad \text{such that} \quad Lp \in \text{GVMEP}(f, A, \psi, X). \quad (13)$$

Assume $\Gamma \neq \emptyset$, where $\Gamma$ denotes the solution set of problem (13). We note that $\Gamma$ is closed and convex. Indeed, following [41], the fixed point of quasi-$\phi$-nonexpansive mapping $T$ is closed and convex. Also by Lemma 2.10(v), we have that $\text{GVMEP}(f, A, \psi, X)$ is closed and convex and finally by Lemma 2.15, $\text{VIP}(C, F)$. Hence, the solution of (13) is well defined. To obtain the solution of (13), we consider the following iterative algorithm:

**Algorithm 3.1.** Pick $x_0, x_0, u \in C$ and choose the parameters $\rho \in (0, 1), \eta \in (0, \nu)$, where $\nu$ is the 2-uniform convexity constant of $E_1$ and suppose that $[a_k] \subseteq [a, e]$ for some $0 < a < e < 1$, $[\beta_k] \subseteq [d, b]$ for some $0 < d < b < 1$, $[\eta_k]$ be a sequence of positive real numbers and $\gamma \in (0, \frac{1}{2d^2\nu^2})$ and $d$ is the best smoothness constant of $E_2$. For $j = 1, 2, \ldots, m$, let $h_j : E \to R$ be a family of convex, weakly lower semicontinuous and Gâteaux differentiable functions such that $h_j^\prime(\cdot) = K_{f_j}$-Lipschitz continuous with $K = \max_{1 \leq j \leq m} K_j$. The feasible set $C$ is defined by

$$C := \bigcap_{i=1}^m C^i,$$

where

$$C^i := \{z \in E : h_j(z) \leq 0\}.$$

We also assume the following conditions are satisfied:

(B1) $\lim_{k \to \infty} \beta_k = 0, \sum_{k=0}^{\infty} \beta_k = \infty$;

(B2) $\liminf_{k \to \infty} a_k (1 - a_k) > 0$;

(B3) $\sum_{k=1}^{\infty} \phi(x_k, x_{k-1}) < \infty$;

(B4) $\lim_{k \to \infty} a_k \beta_k = 0.$

For each $k > 0$, having the $k$-iterate $\{x_k\}$, compute the following steps:

**Step I:** For $j = 1, 2, \ldots, m$ and given the current iterate, construct the family of half spaces

$$C^j_k := \{w \in E_1 : h_j(w_k) + \langle h_j^\prime(w_k), w - w_k \rangle \leq 0\}$$

and set

$$C_k := \bigcap_{j=1}^m C^j_k.$$

Compute

$$w_k = J_{l_j}^{-1}(j_l x_k + \theta_k(j_l x_{k-1} - j_l x_k)),
\quad z_k = \Pi_{C_k} J_{l_j}^{-1}(j_l w_k - A_k F w_k), \quad (14)$$
where $\lambda_k = \rho^k$ and $l_k$ is the smallest nonnegative integer such that
\[ \lambda_k \| Fw_k - Fz_k \| \leq \eta \| w_k - z_k \|. \]  
(15)

**Step II:** If $w_k = z_k$ ($w_k \in \text{VIP}(C, F)$), then $w_{k+1} = y_k$ and go to step III. Otherwise, compute the next iterate by
\[ y_k = \Pi_{Q_k} J^{-1}(Jw_k - \lambda_k Fz_k), \]  
(16)

where $Q_k = \{ w \in E_1 : \langle w - z_k, Jw_k - \lambda_k Fw_k - Jz_k \rangle \leq 0 \}$.

**Step III:** Compute
\[
\begin{align*}
    u_k &= J^{-1}(a_k Jy_k + (1 - a_k) J y_k), \\
    v_k &= J^{-1}(J u_k + \rho L Jz_k \left( (K_y(I - rA) - I) L u_k ) ) ,
    x_{k+1} &= J^{-1}(B J u_k + (1 - \beta_k) I y_k).
\end{align*}
\]  
(17)

**Step IV:** Set $k = k + 1$ and go to step I.

**Lemma 3.2.** If $w_k = z_k$, then $w_k \in \text{VIP}(C, F)$.

**Proof.** Suppose $w_k = z_k$, then by the characteristics of $\Pi_{Q_k}$ and (15), we have
\[ \langle Jw_k - \lambda_k F(w_k) - Jw_k, y - z_k \rangle \leq 0, \quad \forall y \in C_k \]
or equivalently
\[ \lambda_k \langle F(w_k), y - w_k \rangle \geq 0, \quad \forall y \in C_k. \]  
(18)

Since $\lambda_k > 0$, we obtain
\[ \langle F(w_k), y - w_k \rangle \geq 0, \quad \forall y \in C_k. \]  
(19)

Hence, $w_k \in \text{VIP}(C_k, F)$. We conclude therefore from this, $w_k \in C$ and $C \subseteq C_k$ that $w_k \in \text{VIP}(C, F)$. 

In what follows, we shall show that the Armijo Linesearch rule (15) is well defined.

**Lemma 3.3.** There exists a nonnegative integer $l_k$ satisfying (15) of Algorithm 3.1.

**Proof.** If $w_k \in \text{VIP}(C, F)$, then $w_k = \Pi_{C_k} J^{-1}(Jw_k - \lambda_k F w_k)$ and $l_k = 0$. Hence, we consider the case where $w_k \notin \text{VIP}(C, F)$ and assume the contrary, that is for $l > 0$
\[ \lambda_k \| Fw_k - Fz_k \| > \eta \| w_k - z_k \|. \]

This implies
\[ \| Fw_k - Fz_k \| > \frac{\eta \| w_k - z_k \|}{\lambda_k}. \]  
(20)

Next, we consider the following possibilities, $w_k \in C_k$ and $w_k \notin C_k$. First, let $w_k \in C$, then $w_k \in C_k$ and $w_k = \Pi_{C_k}(w_k)$. Using the continuity of $F$ and $\Pi_{Q_k}$, then $\lim_{k \to \infty} \| w_k - z_k \| = 0$. Consequently,
\[ \lim_{k \to \infty} \| Fw_k - Fz_k \| = 0. \]  
(21)

We have from (20) and (21) that
\[ \lim_{k \to \infty} \frac{\| w_k - z_k \|}{\lambda_k} = 0. \]

Using the continuity of $J$ on bounded subsets of $E_i$, we get
\[ \lim_{k \to \infty} \frac{\| Jw_k - Jz_k \|}{\lambda_k} = 0. \]  
(22)
By (9), we obtain
\[ \langle J_z k - J_{w k} + \lambda_k F_{w k}, y - z_k \rangle \geq 0, \quad \forall y \in C, \]
this implies
\[ \langle y - z_k, F_{w k} \rangle \geq \frac{\langle J_{w k} - J_z k, y - z_k \rangle}{\lambda_k}, \quad \forall y \in C. \]

Letting \( k \to \infty \) and using (22), we get
\[ \langle y - w_k, F_{w k} \rangle \geq 0, \quad \forall y \in C, \]
which implies \( w_k \in \text{VIP}(C, F) \). It is easy to see from \( w_k \in C \) and \( C \subset C_k \) that \( w_k \in \text{VIP}(C, F) \), a contradiction. On the other hand, suppose \( w_k \in C_k \), then
\[ \|y - w_k\| = \|y - w_k\| > \lim_{k \to \infty} \Pi_{C_k} w_k > 0 \]
and
\[ \lim_{k \to \infty} \lambda_k \|F_{w k} - F_{z_k}\| = 0. \]

By using (20), (25) and (26), we obtain a contradiction. Therefore, the linesearch (15) is well defined.

For our convergence analysis, we will assume that \( \sup_{k \geq 1} \lambda_k < \infty \), which implies \( \inf_{k \geq 1} \lambda_k > 0 \).

In the following result, we prove the boundedness of the sequence generated by our proposed method.

**Lemma 3.4.** Let \( \{x_k\} \) be the sequence given by Algorithm 3.1 and \( x^* \in \Gamma \). Then, \( \{\phi(x^*, x_k)\} \) is bounded. Consequently, the sequences \( \{x_k\}, \{y_k\}, \{w_k\}, \{u_k\}, \{v_k\} \) and \( \{L u_k\} \) are bounded.

**Proof.** Fix \( p \in \Gamma \), then from Lemma 2.3, we have that
\[
\phi(p, y_k) = \phi(p, \Pi_{Q_k} J^{-1}(J_{w k} - \lambda_k F_{z_k}))
\]
\[
\leq \phi(p, J^{-1}(J_{w k} - \lambda_k F_{z_k})) - \phi(y_k, J^{-1}(J_{w k} - \lambda_k F_{z_k}))
\]
\[
= \|p\| - 2(p, J_{w k} - \lambda_k F_{z k}) + \|J_{w k} - \lambda_k F_{z k}\|^2 - \|y_k\|^2 - 2(p, J_{w k} - \lambda_k F_{z k}) + \|J_{w k} - \lambda_k F_{z k}\|^2
\]
\[
= \|p\| - 2(p, J_{w k} - \lambda_k F_{z k}) - \|y_k\|^2 + 2(y_k, J_{w k} - \lambda_k F_{z k})
\]
\[
= \|p\| - 2(p, J_{w k}) + \|p\| - \|y_k\|^2 - 2(p, J_{w k}) + \|y_k\|^2 + 2\lambda_k (p - y_k, F_{z k})
\]
\[
= \phi(p, w_k) - \phi(y_k, w_k) + 2\lambda_k (p - y_k, F_{z k}),
\]
which by (P2) implies
\[
\phi(p, y_k) = \phi(p, w_k) - (\phi(y_k, z_k) + \phi(z_k, w_k) + 2(y_k - z_k, J_{w k} - J_{w k})) + 2\lambda_k (p - z_k, F_{z k}) + \langle z_k - y_k, F_{z k}\rangle
\]
\[
= \phi(p, w_k) - \phi(y_k, z_k) - \phi(z_k, w_k) - 2(y_k - z_k, J_{w k} - J_{w k}) + 2\lambda_k (p - z_k, F_{z k}) - 2\lambda_k (y_k - z_k, F_{z k})
\]
\[
= \phi(p, w_k) - \phi(y_k, z_k) - \phi(z_k, w_k) + 2(y_k - z_k, J_{w k} - \lambda_k F_{z k} - J_{w k}) + 2\lambda_k (p - z_k, F_{z k})
\]
\[
\leq \phi(p, w_k) - \phi(y_k, z_k) - \phi(z_k, w_k) + 2\lambda_k (y_k - z_k, J_{w k} - \lambda_k F_{z k} - J_{w k}).
\]

We obtained the last inequality by using the fact that \( p \in \Gamma \) and the definition of \( F \). Indeed, for any \( p \in \Gamma \), we have \( \langle y - p, F(p) \rangle \geq 0, \quad \forall y \in C \). Replacing \( y \) by \( z_k \), we get \( \langle z_k - p, F(p) \rangle \geq 0. \) Since \( F \) is semi-quasi strictly monotone, we obtain \( \langle z_k - p, F(z_k) \rangle > 0 \Rightarrow \langle z_k - p, F(z_k) \rangle > 0. \) Thus, \( \langle p - z_k, F(z_k) \rangle \leq 0. \)

By the definition of \( Q_k \) and Cauchy-Schwarz inequality, we obtain
\[ 2\langle y_k - z_k, J_{w k} - \lambda_k F_{z k} - J_{z_k} \rangle = 2\langle y_k - z_k, J_{w k} - \lambda_k F_{w k} - J_{z_k} \rangle + 2\lambda_k (y_k - z_k, F_{w k} - F_{z k}) \]
\[
\leq 2\lambda_k (y_k - z_k, F_{w k} - F_{z k})
\]
and by (15), we get
\[ 2\langle y_k - z_k, J_{w k} - \lambda_k F_{z k} - J_{z_k} \rangle \leq 2\eta \|y_k - z_k\| \|w_k - z_k\|. \]
Using Lemma 2.6, one gets
\[ 2\eta\|y_k - z_k\| \leq 2\eta \sqrt{\frac{\phi(y_k, z_k)}{v}} \times \frac{\phi(z_k, w_k)}{v} \leq \frac{\eta}{v} (\phi(y_k, z_k) + \phi(z_k, w_k)), \]
that is
\[ 2(y_k - z_k, J_w z_k - J_z z_k) \leq \frac{\eta}{v} (\phi(y_k, z_k) + \phi(z_k, w_k)). \] (28)

Therefore, from (27) and (28), we obtain that
\[ \phi(p, y_k) \leq \phi(p, w_k) - \left(1 - \frac{\eta}{v}\right) (\phi(y_k, z_k) + \phi(z_k, w_k)). \] (29)

Now, from (17) and Lemma 2.7, we have
\[
\phi(p, u_k) = \phi(p, J^{-1}_I(a_k y_k + (1 - a_k)J_T y_k))
\]
\[ = \|p\|^2 - 2\langle p, a_k y_k + (1 - a_k)J_T z_k \rangle + \|a_k J_T y_k + (1 - a_k)J_T z_k\|^2 \]
\[ = \|p\|^2 - 2a_k \langle p, J_T y_k \rangle - 2(1 - a_k)\langle p, J_T z_k \rangle + a_k \|\|J_T y_k\| + (1 - a_k)\|J_T z_k\|^2 \]
\[ - a_k(1 - a_k)g(||J_T y_k - J_T z_k||) \leq a_k \phi(p, y_k) + (1 - a_k)\phi(p, z_k) - a_k(1 - a_k)g(||J_T y_k - J_T z_k||) \]
\[ \leq \phi(p, y_k) - a_k(1 - a_k)g(||J_T y_k - J_T z_k||) \leq \phi(p, y_k). \] (30)

Again by using (17) and Lemma 2.7, we have
\[
\phi(p, v_k) = \phi(p, J^{-1}_I(J_L u_k + yL^2 ((K_n(I - nA) - I)L_u_k)))
\]
\[ = \|p\|^2 - 2\langle p, J_L u_k + yL^2 ((K_n(I - nA) - I)L_u_k) \rangle + \|\|J_L u_k + yL^2 ((K_n(I - nA) - I)L_u_k)\|\|^2 \]
\[ + 2\langle yL^2 ((K_n(I - nA) - I)L_u_k) \rangle + \|\|yL^2 ((K_n(I - nA) - I)L_u_k)\|\|^2 \]
\[ = \|p\|^2 - 2\langle p, J_L u_k \rangle - 2y\langle Lp, J_L ((K_n(I - nA) - I)L_u_k) \rangle + \|u_k\|^2 \]
\[ + 2d^2y^2\|L_u\|^2\|K_n(I - nA) - I\|L_u_k\|^2 \]
\[ = \phi(p, u_k) - 2\chi(Lp - L_u_k, J_L ((K_n(I - nA) - I)L_u_k)) + 2d^2y\|L_u\|^2\|K_n(I - nA) - I\|L_u_k\|^2. \] (31)

Since \((K_n(I - nA) - I)\) is firmly nonexpansive type, we have
\[
\langle Lp - L_u_k, \langle J_L ((K_n(I - nA) - I)L_u_k) \rangle \rangle
\]
\[ = \langle Lp - K_n(I - nA)L_u_k, J_L ((K_n(I - nA) - I)L_u_k) \rangle + \|\|K_n(I - nA) - I\|L_u_k\|^2 \]
\[ \geq \|\|K_n(I - nA) - I\|L_u_k\|^2, \]
using this in (31), we get that
\[
\phi(p, v_k) \leq \phi(p, u_k) - 2y\|\|K_n(I - nA) - I\|L_u_k\|^2 + 2d^2y^2\|L_u\|^2\|K_n(I - nA) - I\|L_u_k\|^2
\]
\[ = \phi(p, u_k) - 2y(1 - yd^2\|L_u\|^2)\|\|K_n(I - nA) - I\|L_u_k\|^2 \]
\[ \leq \phi(p, u_k). \] (32)

Furthermore, from (17), \(0 < \eta < v\) and (P3), we have
\[
\phi(p, x_{k+}) = \phi(p, J^{-1}_I(\beta_k J_u + (1 - \beta_k)J_T y_k))
\]
\[ \leq \beta_k \phi(p, u) + (1 - \beta_k)\phi(p, v_k) \]
\[ \leq \beta_k \phi(p, u) + (1 - \beta_k)\phi(p, v_k) \] (33)
\[ \leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, y_k) \]
\[ = \beta_k \phi(p, u) + (1 - \beta_k) \left[ \phi(p, w_k) - \left(1 - \frac{\eta}{v} \right) (\phi(y_k, z_k) + \phi(w_k, z_k)) \right] \]
\[ \leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, w_k) - (1 - \beta_k) \left(1 - \frac{\eta}{v} \right) \phi(y_k, z_k) + \phi(w_k, z_k) \]
\[ \leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, w_k). \]

Again from (P3) and (14), we have
\[ \phi(p, w_k) = \phi(p, j^{-1}(j(x_k + \theta_k(j(x_{k-1} - j(x_k))))) \]
\[ = \phi(p, j^{-1}(j(x_{k-1} + (1 - \theta_k))j(x_k))) \]
\[ \leq (1 - \theta_k) \phi(p, x_k) + \theta_k \phi(p, x_{k-1}), \]

hence
\[ \phi(p, x_{k+1}) \leq \beta_k \phi(p, u) + (1 - \beta_k) \left[ (1 - \theta_k) \phi(p, x_k) + \theta_k \phi(p, x_{k-1}) \right] \]
\[ \leq \beta_k \phi(p, u) + (1 - \beta_k) \left[ \phi(p, x_k) + \phi(p, x_{k-1}) \right] \]
\[ \leq \max \{ \phi(p, u), (\phi(p, x_k) + \phi(p, x_{k-1})) \} \]
\[ \leq \max \{ \phi(p, u), (\phi(p, x_k) + \phi(p, x_0)) \}, \quad \forall k \geq 1. \]

Thus, \( \{ \phi(p, x_k) \} \) is bounded. Therefore, \( \{ x_k \} \) is bounded and the conclusion of the lemma holds. \( \square \)

In what follows, we obtain a result, which is a consequence of the boundedness of \( \{ x_k \} \).

**Lemma 3.5.** Let \( \{ x_k \} \) be a subsequence of the sequence \( \{ x_k \} \) defined by Algorithm 3.1 such that \( x_k \to q \). Let \( \| w_k - z_k \| \to 0 \) and \( \| w_k - x_k \| \to 0 \) as \( i \to \infty \) hold. Then \( q \in \text{VIC}(C, F) \).

**Proof.** First, we show that \( q \in C \). Indeed, it follows from \( z_k \in C_k \) that
\[ h_j(w_k) + \langle h'_j(w_k), z_k - w_k \rangle \leq 0. \]
By using the Cauchy-Schwartz inequality, we have
\[ h_j(w_k) \leq \langle h'_j(w_k), w_k - z_k \rangle \leq \| h'_j(w_k) \| \cdot \| w_k - z_k \|. \]

Since \( h'_j \) is Lipschitz continuous and \( \{ w_k \} \) is bounded, we have that \( \{ h'_j(w_k) \} \) is bounded. Thus, there exists \( K_j > 0 \) such that \( \| h'_j(w_k) \| \leq K_j \) for each \( i \). Therefore, we obtain
\[ h_j(w_k) \leq K \cdot \| w_k - z_k \|, \]
where \( K = \max_{1 \leq j \leq m} \{ K_j \} \). Hence, by the weakly continuity of \( h_j \), we have
\[ h_j(q) \leq \liminf_{k \to \infty} h_j(w_k) \leq \lim_{k \to \infty} K \cdot \| w_k - z_k \| = 0. \]

Thus, \( q \in C \).

By the definition of \( z_k \) and characterization of \( \Pi_{C_k} \), we have
\[ \langle j_jw_k - \lambda_k Fw_k - j_jz_k, w - z_k \rangle \leq 0, \quad \forall w \in C_k \]
or
\[ \langle j_jw_k - j_jz_k, w - z_k \rangle \leq \lambda_k \langle Fw_k, w - z_k \rangle, \quad \forall w \in C_k. \]
This implies that
\[ \langle j_jw_k - j_jz_k, w - z_k \rangle + \lambda_k \langle Fw_k, z_k - w_k \rangle \leq \lambda_k \langle Fw_k, w - w_k \rangle, \quad \forall w \in C_k. \]
Fix $w \in C_k$ and let $i \to \infty$ in (37), by hypothesis, $\lambda_k > 0$ and uniform continuity of $J_1$ on bounded subsets of $E_1$, we have

$$\liminf_{i \to \infty} \langle w - w_{k_i}, Fw_{k_i} \rangle \geq 0, \quad \forall w \in C_k.$$  \hfill (38)

Thus, we have from (38), the fact that $w_{k_i} \in C$ and $C \subset C_k$, that

$$\liminf_{i \to \infty} \langle w - w_{k_i}, Fw_{k_i} \rangle \geq 0, \quad \forall w \in C.$$  \hfill (39)

Let $|\epsilon_i|$ be a sequence of positive numbers such that $\epsilon_i \to 0$ as $i \to \infty$ From (39), we can find $N$ large enough such that

$$\langle w - w_{k_i}, Fw_{k_i} \rangle + \epsilon_i > 0, \quad \forall k \geq N.$$  \hfill (40)

For some $q_{k_i} \in E_1$ satisfying $\langle Fw_{k_i}, q_{k_i} \rangle = 1$, we can write (40) as

$$\langle w + \epsilon_i q_{k_i} - w_{k_i}, Fw_{k_i} \rangle > 0, \quad \forall k \geq N.$$  \hfill (41)

Since $F$ is semistrictly quasi-monotone, we have that

$$\langle w + \epsilon_i q_{k_i} - w_{k_i}, F(w + \epsilon_i q_{k_i}) \rangle > 0$$  \hfill (42)

from which we get

$$\langle w - w_{k_i}, Fw \rangle > \langle w + \epsilon_i q_{k_i} - w_{k_i}, Fw - F(w + \epsilon_i q_{k_i}) \rangle - \langle \epsilon_i q_{k_i}, Fw \rangle, \quad \forall k \geq N.$$  \hfill (43)

Letting $i \to \infty$, we obtain

$$\liminf_{i \to \infty} \langle w - w_{k_i}, Fw \rangle \geq 0.$$  \hfill (44)

Hence, for all $w \in C$

$$\langle w - p, Fw \rangle = \lim_{i \to \infty} \langle w - w_{k_i}, Fw \rangle = \liminf_{i \to \infty} \langle w - w_{k_i}, Fw \rangle \geq 0.$$  \hfill (45)

It follows from Lemma 2.12 and Remark 2.13 that $q \in \text{VIC}(C, F)$.

**Theorem 3.6.** Let $\{x_k\}$ be the sequence generated by Algorithm 3.1 such that assumptions (B1)–(B4) are satisfied. Then $\{x_k\}$ converges strongly to a point $\Pi \Gamma u = p \in \Gamma$, where $\Pi \Gamma$ is the generalized projection of $E_1$ onto $\Gamma$.

**Proof.** As in Lemma 3.4, let $p \in \Gamma$. Then from (8) and (17), we have

$$\phi(p, x_{k+1}) = \phi(p, J_1^{-1}(\beta_k J_1 u + (1 - \beta_k) J_1 v_k))$$

$$= V(p, \beta_k J_1 u + (1 - \beta_k) J_1 v_k) + 2\langle -\beta_k (J_1 u - J_1 p), J_1^{-1}(\beta_k J_1 u + (1 - \beta_k) J_1 v_k) - p \rangle$$

$$\leq V(p, \beta_k J_1 u + (1 - \beta_k) J_1 v_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle$$

$$\leq \beta_k V(p, J_1 p) + (1 - \beta_k) V(p, J_1 v_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle$$

$$= \beta_k \phi(p, p) + (1 - \beta_k) \phi(p, v_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle$$

$$= (1 - \beta_k) \phi(p, v_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle$$

$$\leq (1 - \beta_k) \phi(p, v_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle$$

$$\leq (1 - \beta_k) \langle \beta_k \phi(p, x_k) + \theta_k \phi(p, x_{k-1}) \rangle + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle$$

$$\leq (1 - \beta_k) \langle \beta_k \phi(p, x_k) + \beta_k \left( \frac{\theta_k}{\beta_k} \phi(p, x_{k-1}) + 2(J_1 u - J_1 p, x_{k+1} - p) \right)$$

for each $k \in N$. 


Now, consider the following two possible cases:

**Case 1.** Suppose that there exists \( n_0 \in \mathbb{N} \) such that \( \{\phi(p, x_k)\}_{k=n_0} \) is either non-increasing or non-decreasing. Then, by the boundedness of \( \{\phi(p, x_0)\} \), it follows that \( \{\phi(p, x_k)\} \) is convergent and 

\[
\phi(p, x_k) - \phi(p, x_{k+1}) \to 0 \quad \text{as} \quad k \to \infty.
\]

From (30), (33) and (35), we have

\[
\phi(p, x_{k+1}) \leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, u_k)
\]

\[
\leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, y_k) - \alpha_k (1 - \alpha_k) g(||Jy_k - J_T y_k||)
\]

\[
\leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, w_k) - \alpha_k (1 - \alpha_k) g(||Jy_k - J_T y_k||)
\]

\[
\leq \beta_k \phi(p, u) + (1 - \beta_k) [\theta_k \phi(p, x_{k-1}) + \theta_k \phi(p, x_k) - \alpha_k (1 - \alpha_k) g(||Jy_k - J_T y_k||)],
\]

which implies

\[
a_k (1 - \alpha_k) (1 - \beta_k) g(||Jy_k - J_T y_k||) \leq \beta_k \left[ \frac{\theta_k}{\beta_k} \phi(p, x_{k-1}) + \phi(p, u) \right] + (1 - \beta_k) \phi(p, x_k) - \phi(p, x_{k+1}).
\]

By letting \( k \to \infty \), we get \( a_k (1 - \alpha_k) (1 - \beta_k) g(||Jy_k - J_T y_k||) \to 0 \), since \( a_k (1 - \alpha_k) (1 - \beta_k) > 0 \) we obtain

\[
g(||Jy_k - J_T y_k||) \to 0 \quad \text{as} \quad k \to \infty.
\]

Using the property of \( g \) and the uniform continuity of \( J^{-1} \) on bounded subsets of \( E_1^* \), we have

\[
\lim_{k \to \infty} \|y_k - T y_k\| = 0 = \phi(y_k, T y_k),
\]

(46)

from Remark 2.5. Observe from (17) that

\[
\phi(y_k, u_k) = \phi(y_k, J^{-1}(a_k Jy_k + (1 - a_k) J_T y_k))
\]

\[
= \|y_k\|^2 - 2 \langle y_k, a_k Jy_k + (1 - a_k) J_T y_k \rangle + \|a_k Jy_k + (1 - a_k) J_T y_k\|^2
\]

\[
= \|y_k\|^2 - 2 a_k \langle y_k, Jy_k \rangle - 2(1 - a_k) \langle y_k, J_T y_k \rangle + a_k \|y_k\|^2 + (1 - a_k) \|J_T y_k\|^2 - a_k (1 - \alpha_k) g(||Jy_k - J_T y_k||)
\]

\[
= a_k \phi(y_k, y_k) + (1 - a_k) \phi(y_k, T y_k) - a_k (1 - \alpha_k) g(||Jy_k - J_T y_k||),
\]

which implies by using (46) that \( \phi(y_k, u_k) \to 0 \) as \( k \to \infty \). By invoking Lemma 2.4, we get

\[
\lim_{k \to \infty} \|y_k - u_k\| = 0.
\]

(47)

Also, from (34), we have

\[
(1 - \beta_k) \left[ 1 - \frac{\eta}{\nu} \right] \phi(y_k, z_k) + \phi(z_k, w_k) \leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, w_k) - \phi(p, x_{k+1})
\]

\[
\leq \beta_k \left[ \frac{\theta_k}{\beta_k} \phi(p, x_{k-1}) + \phi(p, u) \right] + (1 - \beta_k) \phi(p, x_k) - \phi(p, x_{k+1}),
\]

thus by condition (B2), we get \( \phi(y_k, z_k) + \phi(z_k, w_k) \to 0 \) as \( k \to \infty \), that is

\[
\lim_{k \to \infty} \phi(y_k, z_k) = \lim_{k \to \infty} \phi(y_k, z_k) = 0,
\]

Lemma 2.4 ensures

\[
\lim_{k \to \infty} \|y_k - z_k\| = \lim_{k \to \infty} \|y_k - z_k\| = 0.
\]

(48)

Next, we show that \( \|(K_r(I - nA) - I) L u_k\| \to 0 \) as \( k \to \infty \). Indeed, we have from (32) and (33) that

\[
\phi(p, x_{k+1}) \leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, v_k)
\]

\[
\leq \beta_k \phi(p, u) + (1 - \beta_k) [\phi(p, u_k) - 2y(1 - y^2) L_2^2 (K_r(I - nA) - I) L u_k^2],
\]

\[
\leq \beta_k \phi(p, u) + (1 - \beta_k) ||\phi(p, u_k) - 2y(1 - y^2) L_2^2 (K_r(I - nA) - I) L u_k^2||
\]

\[
\leq \beta_k \phi(p, u) + (1 - \beta_k) [\phi(p, u_k) - 2y(1 - y^2) L_2^2 (K_r(I - nA) - I) L u_k^2],
\]

\[
\leq \beta_k \phi(p, u) + (1 - \beta_k) ||\phi(p, u_k) - 2y(1 - y^2) L_2^2 (K_r(I - nA) - I) L u_k^2||
\]

\[
\leq \beta_k \phi(p, u) + (1 - \beta_k) [\phi(p, u_k) - 2y(1 - y^2) L_2^2 (K_r(I - nA) - I) L u_k^2],
\]
by further calculations, we get
\[ 2\gamma(1 - yd^2\|L\|^2)((K_n(I - \eta A) - I)Lu_k)^2 \leq \beta_k\left(\frac{1}{\beta_k}\phi(p, x_{k-1}) + \phi(p, u)\right) + (1 - \beta_k)\phi(p, x_k) - \phi(p, x_{k+1}). \]

Thus, we obtain
\[
\lim_{k \to \infty} \|(K_n(I - \eta A) - I)Lu_k\| = 0. \tag{49}
\]

Observe from (17) that
\[
\phi(u_k, v_k) = \phi(u_k, J^{-1}_u(\eta u_k + yL^*J_z((K_n(I - \eta A) - I)Lu_k)))
= \|u_k\|^2 - 2\langle u_k, J(u_k + yL^*J_z((K_n(I - \eta A) - I)Lu_k))\rangle + \|u_k\|^2 + yL^*J_z((K_n(I - \eta A) - I)Lu_k)^2
+ 2y\langle u_k, L^*J_z((K_n(I - \eta A) - I)Lu_k)\rangle + 2d^2\gamma\|L\|^2\|(K_n(I - \eta A) - I)Lu_k\|^2,
\]
which by (49) and Lemma 2.4, that
\[
\lim_{k \to \infty} \phi(v_k, u_k) = \lim_{k \to \infty} \|v_k - u_k\| = 0. \tag{50}
\]

Furthermore, we obtain the limit of \(|x_k - x\| as k \to \infty. Observe first from (14) and condition (B3) that
\[
\lim_{k \to \infty} \phi(w_k, x_k) = \lim_{k \to \infty} \theta_k\phi(x_k, x_{k-1}) = 0, \tag{51}
\]
that is, by Lemma 2.4, we have
\[
\|w_k - x_k\| \to 0 \quad \text{as} \quad k \to \infty. \tag{52}
\]

It is also easy to see that
\[
\begin{align*}
\lim_{k \to \infty} \|v_k - y_k\| &= \lim_{k \to \infty} \|(v_k - u_k) + (u_k - y_k)\| = 0, \\
\lim_{k \to \infty} \|y_k - x_k\| &= \lim_{k \to \infty} \|(y_k - w_k) + (w_k - x_k)\| = 0,
\end{align*} \tag{53}
\]

Now,
\[
\phi(x_k, x_{k+1}) = \phi(x_k, J^{-1}(\beta_k J u + (1 - \beta_k)J v_k))
= \|x_k\|^2 - 2\langle x_k, \beta_k J u + (1 - \beta_k)J v_k\rangle + \|\beta_k J u + (1 - \beta_k)J v_k\|^2
= \beta_k\phi(x_k, u) + (1 - \beta_k)\phi(x_k, v_k) - \beta_k(1 - \beta_k)\phi(||J u - J v_k||),
\]
thus, we obtain
\[
\lim_{k \to \infty} \phi(x_k, x_{k+1}) = \lim_{k \to \infty} \|x_{k+1} - x_k\| = 0, \tag{54}
\]
where we have used Lemma 2.4.

Since \(|x_k| is bounded by Lemma 3.4, there exists a subsequence \(|x_k| of \{|x_k| such that \(x_k \to q. It is easy to see that there exists \{|y_k|, \{w_k|, \{u_k| and \{y_k| all converge weakly to \(q. Hence, by (48) and Lemma 3.5, we have that \(q \in VIP(C, F). Also, by (46) and demiclosedness of \((I - T)\), we get that \(q \in Fix(T). On the other hand, by the linearity of \(L\) we obtain that \((L) = Lq. We now show that
\[
Lq \in GVMEP(f, \psi, \phi, X). \quad \text{Indeed, suppose} \quad v_k = K_n(I - \eta A)Lu_k, \text{then by}
\]
\[
f(v_k, z) + \psi(z) - \psi(v_k) + e\langle z - v_k, ALu_k \rangle + \frac{e}{r_k}\langle z - v_k, v_k - Lu_k \rangle \in P, \quad \forall z \in X, \tag{55}
\]
which implies that
\[
0 \in f(z, v_k) - \left\{\psi(z) - \psi(v_k) + e\langle z - v_k, ALu_k \rangle + \frac{e}{r_k}\langle z - v_k, v_k - Lu_k \rangle\right\} + P, \quad \forall z \in X. \tag{56}
\]
Let $z_t = (1 - \lambda)Lq + \lambda z$ for all $\lambda \in (0, 1)$, then $z_t \in E_2$. By (55), we have
\[
e(z_t - v_{k_t}, Az_t) \leq f(z_t, v_{k_t}) - (\psi(z_t) - \psi(v_{k_t})) + e(z_t - v_{k_t}, Az_t) - e(z_t - v_{k_t}, ALu_{k_t}) - \frac{\psi}{n_t} \langle z_t - v_{k_t}, v_{k_t} - Lu_{k_t} \rangle + P
\]
\[= f(z_t, v_{k_t}) + e(z_t - v_{k_t}, Az_t - Av_{k_t}) + e(z_t - v_{k_t}, Av_{k_t} - ALu_{k_t}) - e \left( z_t - v_{k_t}, \frac{v_{k_t} - Lu_{k_t}}{n_t} \right) \tag{57}\]
\[- (\psi(z_t) - \psi(v_{k_t})) + P.
\]
Using (49), the properties of $A$ and $f$, we have
\[
\|Av_{k_t} - ALu_{k_t}\| \to 0, \quad \|v_{k_t} - Lu_{k_t}\| \to 0, \quad \text{and} \quad \langle z_t - v_{k_t}, Av_{k_t} - ALu_{k_t} \rangle \to 0.
\]
Let $i \to \infty$, we get
\[
e(z_t - Lq, Az_t) \leq f(z_t, Lq) - (\psi(z_t) - \psi(Lq)) + P. \tag{58}
\]
It follows from (R1), (R2) and (R6) that
\[
\lambda(f(z_t, z) + (1 - \lambda)f(z_t, Lq) + \lambda \psi(z) + (1 - \lambda)\psi(Lq) - \psi(z_t) - \psi(z) + P \in P, \tag{59}
\]
that is
\[
-\lambda[f(z_t, z) + \psi(z) - \psi(z_t)] - (1 - \lambda)[f(z_t, Lq) + \psi(Lq) - \psi(z_t)] \in -P. \tag{60}
\]
Using this and (58), we have
\[-\lambda[f(z_t, z) + \psi(z) - \psi(z_t)] - e(1 - \lambda)\lambda(z - Lq, Az_t) \in -P, \tag{61}
\]
and so
\[-\lambda[f(z_t, z) + \psi(z) - \psi(z_t)] - e(1 - \lambda)(z - Lq, Az_t) \in -P, \tag{62}
\]
Letting $\lambda \to 0$, we get
\[f(Lq, z) + e(z - Lq, ALq) + \psi(z) - \psi(Lq) \in P, \quad \forall z \in E_2
\]
and so $Lq \in GVMEP(f, A, \psi, X)$.

To conclude Case 1, we show that $x_k \to \Pi Tu = p$. Let $\{x_{k_t}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_t} \to q$ and
\[
\limsup_{k \to \infty} \langle J_{k_t}u - J_{k_t}p, x_{k_t+1} - p \rangle = \lim_{k \to \infty} \langle J_{k_t}u - J_{k_t}p, x_{k+1} - p \rangle. \tag{63}
\]
Since $\|x_{k_t} - x_d\| \to 0$ as $k \to \infty$, it follows that $x_{k_t} \to q$. By using Lemma 2.3 (9), we obtain
\[
\limsup_{k \to \infty} \langle J_{k_t}u - J_{k_t}p, x_{k_t+1} - p \rangle = \lim_{k \to \infty} \langle J_{k_t}u - J_{k_t}p, x_{k+1} - p \rangle = \langle J_{k_t}u - J_{k_t}p, q - p \rangle \leq 0. \tag{64}
\]
It follows from (45) and (64) that $\phi(p, x_k) \to q$ as $k \to \infty$. Therefore, by Lemma 2.4, we obtain $|x_k - p| \to 0$ as $k \to \infty$. Hence, $x_k \to p$.

**Case 2.** Suppose that $|\phi(p, x_k)|$ is not a monotone sequence. Then there exists a subsequence $\{k_i\}$ of $\{k\}$ such that
\[
\phi(p, x_{k_i}) < \phi(p, x_{k_{i+1}}), \quad \forall i \in \mathbb{N}.
\]

For some $N$ large enough, let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $k \geq N$ by
\[
\tau(k) = \max\{i \in \mathbb{N} : i \leq k, \phi(p, x_k) \leq \phi(p, x_{k+1})\}.
By Lemma 2.17, \( \tau(k) \) is non-decreasing with \( \tau(k) \to \infty \) as \( k \to \infty \) and
\[
\phi(p, x_{r(k)}) \leq \phi(p, x_{r(k)+1}) \quad \text{and} \quad \phi(p, x_{k}) \leq \phi(p, x_{r(k)+1}).
\]

Following similar argument as above in Case 1, we can conclude that
\[
\lim_{k \to \infty} \| ( K_{\tau(k)}(I - r_{\tau(k)}A) - I) L U_{\tau(k)} \| = 0,
\]
\[
\lim_{k \to \infty} \| y_{\tau(k)} - z_{\tau(k)} \| = \lim_{k \to \infty} \| w_{\tau(k)} - z_{\tau(k)} \| = 0,
\]
\[
\lim_{k \to \infty} \| y_{\tau(k)} - T y_{\tau(k)} \| = 0,
\]
\[
\lim_{k \to \infty} \| x_{\tau(k)+1} - x_{\tau(k)} \| = 0,
\]
and
\[
\lim_{k \to \infty} \langle J u - J p, x_{\tau(k)+1} - p \rangle \leq 0.
\]

Since \( \{x_{\tau(k)}\} \) is bounded, we can find a subsequence of \( \{x_{\tau(k)}\} \), still denoted \( \{x_{\tau(k)}\} \) such that \( x_{\tau(k)} \to q \in \text{Fix}(T) \cap \text{VIP}(C, F) \) and \( q \in \text{GVMEP}(f, A, \psi, X) \). It follows from (45) that
\[
\phi(p, x_{\tau(k)+1}) \leq (1 - \beta_{\tau(k)}) \phi(p, x_{\tau(k)}) + \beta_{\tau(k)} \left( \theta_{\tau(k)} \phi(p, x_{\tau(k)-1}) + 2 \langle J u - J p, x_{\tau(k)+1} - p \rangle \right) \quad (67)
\]
for each \( \tau(k) \in \mathbb{N} \). Since \( \phi(p, x_{\tau(k)}) \leq \phi(p, x_{\tau(k)+1}) \), we have that \( \phi(p, x_{\tau(k)}) - \phi(p, x_{\tau(k)+1}) \leq 0 \). Thus, we obtain from (67) that
\[
\beta_{\tau(k)} \phi(p, x_{\tau(k)}) \leq \phi(p, x_{\tau(k)}) - \phi(p, x_{\tau(k)+1}) + \beta_{\tau(k)} \left( \frac{\theta_{\tau(k)}}{\beta_{\tau(k)}} \phi(p, x_{\tau(k)-1}) + 2 \langle J u - J p, x_{\tau(k)+1} - p \rangle \right) \leq \beta_{\tau(k)} \frac{\theta_{\tau(k)}}{\beta_{\tau(k)}} \phi(p, x_{\tau(k)-1}) + 2 \langle J u - J p, x_{\tau(k)+1} - p \rangle \quad (68)
\]
that is
\[
\phi(p, x_{\tau(k)}) \leq \left( \frac{\theta_{\tau(k)}}{\beta_{\tau(k)}} \phi(p, x_{\tau(k)-1}) + 2 \langle J u - J p, x_{\tau(k)+1} - p \rangle \right),
\]
since \( \beta_{\tau(k)} > 0 \).

Using condition (B4) and (66), we get that \( \phi(p, x_{\tau(k)}) \to 0 \) as \( k \to \infty \). This together with (67) implies that \( \phi(p, x_{\tau(k)-1}) \to 0 \) as \( k \to \infty \). But \( \phi(p, x_{k}) \leq \phi(p, x_{\tau(k)+1}) \) for all \( k \in \mathbb{N} \), we get that \( \phi(p, x_{k}) \to 0 \). By Lemma 2.4, we obtain \( x_{k} \to p \) as \( k \to \infty \). Therefore, from the above two cases, we can conclude that \( \{x_{k}\} \) converges strongly to a point \( p = \Pi_{T} u \). Thus, completing the proof. \( \square \)

The following is a direct consequence of our main result: Suppose we set \( Y = \mathbb{R} \) and \( e = 1 \), then we have the following result for obtaining a common solution of SFP for variational inequality, FPP and GMEP.

**Corollary 3.7.** Let \( C \) and \( Q \) be nonempty, closed and convex subsets of a 2-uniformly convex and uniformly smooth Banach space \( E_{1} \), and smooth, strictly convex and reflexive Banach space \( E_{2} \) with dual space \( E_{2}^{*} \) and \( E_{1}^{*} \), respectively. Let \( L : E_{1} \to E_{2} \) be a bounded linear operator with \( L^* : E_{2}^{*} \to E_{1}^{*} \) its adjoint. Let \( f : Q \times Q \to \mathbb{R} \) satisfying assumptions (A1)-(A4) (see [55]) and \( A : Q \to \mathbb{R} \cup \{+\infty\} \) be a proper, convex and lower semi-continuous function. Let \( \psi : Q \to E_{2}^{*} \) be an \( m \)-inverse strongly monotone mapping and \( T : C \to C \) be a quasi-\( \phi \)-nonexpansive mapping such that Fix\((T) \neq \emptyset \). Let \( F : C \to E^{*} \) be a semistrictly quasi-monotone, sequentially weakly continuous mapping. Let \( \Gamma = \{p \in \text{VIP}(C, F) \cap \text{Fix}(T) : L p \in \text{GMEP}(f, \phi, \psi) \} \neq \emptyset \). Then, the sequence \( \{x_{k}\} \) generated by Algorithm 3.1 converges strongly to \( x^{*} \in \Gamma \).
4 Numerical examples

In this section, we give some numerical examples to illustrate the performance of our method.

First, we give an example in $\ell_p$ ($1 \leq p < \infty$) with $p \neq 2$, which is not a Hilbert space. It is well known that the dual space $(\ell_p)^*$ is isomorphic to $\ell_q$ provided $\frac{1}{p} + \frac{1}{q} = 1$ (see for instance [56, Lemma 2.2]).

Example 4.1. Let $E_1 = E_2 = X = Y = \ell_2(\mathbb{R})$ where $\ell_2(\mathbb{R}) = \{x = (x_1, x_2, x_3, \ldots) \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ with norm $\|x\|_{\ell_2} = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}$ and inner product $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$ for all $x = (x_1, x_2, x_3, \ldots), y = (y_1, y_2, y_3, \ldots) \in E$.

Let $L : \ell_2 \to \ell_2$ be defined by $L(x) = \left(\frac{2y_1}{5}, \frac{2y_2}{5}, \frac{2y_3}{5}, \ldots\right)$ for every $x \in \ell_2$. Then, $L^*(y) = \left(\frac{x_1}{1+x_1}, \frac{x_2}{1+x_2}, \frac{x_3}{1+x_3}, \ldots\right)$ for each $y \in \ell_2$. Let $C = \{x = (x_1, x_2, x_3, \ldots) \in \ell_2 : \|x\| \leq 4\}$, define the mapping $T : C \to C$ by $T(x) = \left(\frac{11y_1}{1+y_1}, \frac{11y_2}{1+y_2}, \frac{11y_3}{1+y_3}, \ldots\right)$ for all $x \in C$. Define the function $F : C \to E_1^*$ by $F(x) = (6 - \|x\|)x$. Then, $F$ is semistrictly quasi-monotone and weakly sequentially continuous on $C$, see [25].

Also, define the mappings $f : X \times X \to Y$ by $f(x) = -x^2 + y^2, \forall x, y \in \ell_2, A(x) = \left(\frac{x_1}{3}, \frac{x_2}{3}, \frac{x_3}{3}, \ldots\right), \forall x \in \ell_2$ and $\psi(x) = 0, \forall x \in \ell_2$. It is easy to see that the mappings $f, A$ and $\psi$ satisfy conditions (R1)–(R7) of Theorem 3.6. Let $n_k = 1$ for all $k$, by Lemma 2.10, we can easily find $z \in E_2$ such that

$$f(z, y) + \langle Az, y - z \rangle \psi(y) - \psi(x) + \frac{e}{\|x\|} \|y - z, Jz - Jx\| \in P, \quad \forall y \in X.$$ 

It can be easily checked that

$$(K_n(I - \lambda A) - I)u_k = \left(\frac{5L_{u_1}^2}{16}, \frac{5L_{u_2}^2}{16}, \frac{5L_{u_3}^2}{16}, \ldots\right).$$

It follows therefore that $\Gamma = \{0\} \neq \emptyset$. Choose the sequences $\alpha_k = \frac{3^k}{5^k + 7}, \beta_k = \frac{1}{k+10}, \theta_k = \frac{1}{3} - \frac{2^k}{5^k + 3k}, \rho = 0.9, y = 0.25/14, \lambda_k = \rho^2$ and $u = [-0.2345, 0.8943, 0, \ldots]^T$. Take $\epsilon = 1 \times 10^{-5}$ and choose the following initial values:

(Case 1) $x_0 = [0.15, 0.25, 0.75, \ldots]^T, x_1 = [0.75, 0.5, 0.25, \ldots]^T, u = [0.1, 0.9, 0.3, \ldots]^T$;

(Case 2) $x_0 = [1.5, 2.5, 3.5, \ldots]^T, x_1 = [7.5, 5, 2.5, \ldots]^T, u = [10, 5, 3, \ldots]^T$;

(Case 3) $x_0 = [-5, 0, 5, \ldots]^T, x_1 = [-10, -20, 10, \ldots]^T, u = [10, 10, 10, \ldots]^T$.

We plot the graphs of errors against the number of iterations in each case. The numerical results can be found in Figure 1.

Example 4.2. Let $E_1 = E_2 = X = Y = \ell_2(\mathbb{R})$ be the space of square-summable sequences of real numbers $\{x_i\}_{i=1}^{\infty}$, that is

$\ell_2(\mathbb{R}) = \left\{x = (x_1, x_2, \ldots, x_i, \ldots) : x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty\right\},$

with inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ and $\| \cdot \| : \ell_2 \to \mathbb{R}$ defined by $\langle u, v \rangle = \sum_{i=1}^{\infty} u_i v_i$ and $\|u\| = \left(\sum_{i=1}^{\infty} |u_i|^2\right)^{1/2}$, respectively, for all $u, v \in \ell_2$.

Let $L : \ell_2 \to \ell_2$ be defined by $L(x) = \left(\frac{2y_1}{5}, \frac{2y_2}{5}, \frac{2y_3}{5}, \ldots\right)$ for every $x \in \ell_2$. Then, $L^*(y) = \left(\frac{x_1}{1+x_1}, \frac{x_2}{1+x_2}, \frac{x_3}{1+x_3}, \ldots\right)$ for each $y \in \ell_2$. Let $C = \{x \in E_1 : \|x\| \leq 4\}$, define the mapping $T : C \to C$ by $T(x) = \left(\frac{2y_1}{7}, \frac{2y_2}{7}, \frac{2y_3}{7}, \ldots\right)$ for all $x \in C$. Define the function $F : C \to E_1^*$ by $F(x) = (6 - \|x\|)x$. Then, $F$ is semistrictly quasi-monotone and weakly sequentially continuous on $C$, see [25].

Also, define the mappings $f : X \times X \to Y$ by $f(x) = -x^2 + y^2, \forall x, y \in \ell_2, A(x) = \left(\frac{x_1}{5}, \frac{x_2}{5}, \frac{x_3}{5}, \ldots\right), \forall x \in \ell_2$ and $\psi(x) = 0, \forall x \in \ell_2$. It is easy to see that the mappings $f, A$ and $\psi$ satisfy conditions (R1)–(R7) of Theorem 3.6 and $0 \in \text{GVMEP}(f, A, \psi, X)$. Let $n_k = 1$ for all $k$, by Lemma 2.10, we can easily find $z \in E_2$ such that

$$f(z, y) + \langle Az, y - z \rangle \psi(y) - \psi(x) + \frac{e}{\|x\|} \|y - z, Jz - Jx\| \in P, \quad \forall y \in X.$$
It can be easily checked that
\[
(K_\gamma(I - r_k A) - I)Lu_k = \left( \frac{5Lu_k^i}{16}, \frac{5Lu_k^i}{16}, \frac{5Lu_k^i}{16}, \ldots \right).
\]

It follows therefore that $\Gamma = \{0\} \neq \emptyset$. Choose the sequences $\alpha_k = \frac{3k}{4k^2 + 7}$, $\beta_k = \frac{1}{1000(k+1)}$, $\theta_k = \frac{1}{3} - \frac{2k}{k^2 + 3k}$, $\rho = 0.07$, $\lambda_k = \rho^k$ and $u = [-0.2345, 0.8943, 0, \ldots, 0, \ldots]^T$. By taking $\varepsilon = 1 \times 10^{-6}$, we choose the following initial values:

(Case 1) $x_0 = [4.1285, -2.9018, 0, \ldots, 0, \ldots]^T$ and $x_1 = [3.2158, -5.8091, 0, \ldots, 0, \ldots]^T$,

(Case 2) $x_0 = [3.9015, 2.0345, 0, \ldots, 0, \ldots]^T$ and $x_1 = [1.2456, 2.3125, 0, \ldots, 0, \ldots]^T$,

(Case 3) $x_0 = [0.4563, 1.2098, 0, \ldots, 0, \ldots]^T$ and $x_1 = [-0.8924, 1.3521, 0, \ldots, 0, \ldots]^T$.

We plot the graphs of errors against the number of iterations in each case. The numerical results can be found in Figure 2.
Example 4.3. Let $E_1 = E_2 = X = Y = \mathbb{R}$ and $L : \mathbb{R} \to \mathbb{R}$ be defined by $L(x) = \frac{2x}{5}$ for every $x \in \mathbb{R}$. Then, $L'(y) = \frac{2y}{5}$ for each $y \in E_1$. Let $C = \{x \in \mathbb{R} : |x| \leq 4\}$, define the mapping $T : C \to C$ by $T(x) = \frac{2x}{7}$ for all $x \in \mathbb{R}$. Define the function $F : C \to \mathbb{R}$ by $F(x) = (6 - |x|)x$. Then, $F$ is semistrictly quasi-monotone and weakly sequentially continuous on $C$, see [25].

Also, define the mappings $f : X \times X \to Y$ by $f(x) = -x^2 + y^2$, $\forall x, y \in \mathbb{R}$, $A(x) = \frac{x}{5}$, $\forall x \in \mathbb{R}$ and $\psi(x) = 0$, $\forall x \in \mathbb{R}$. It is easy to see that the mappings $f, A$ and $\psi$ satisfy conditions (R1)–(R7) of Theorem 3.6 and $0 \in \text{GVMEP}(f, A, \psi, X)$. Let $\{\eta_k\} = 1$ for all $k$, by Lemma 2.10, we can easily find $z \in \mathbb{R}$ such that

$$f(z, y) + \langle Az, y - z \rangle \psi(y) - \psi(x) + \frac{\sigma}{\eta_k}(y - z, Jz - Jx) \in P, \quad \forall y \in X.$$  

It can be easily checked that

$$(K_\sigma (I - \eta A) - I) L u_k = \frac{5 L u_k}{16}.$$
It follows therefore that $\Gamma = \{0\} \neq \emptyset$. Choose the sequences $a_k = \frac{3k}{k^2 + 7}$, $\beta_k = \frac{1}{1000(k+1)}$, $\eta = \frac{0.17}{13}$, $\theta_k = \frac{1}{3} - \frac{2k}{k^2 + 3}$ and $\rho = 0.07$. Take $\varepsilon = 1 \times 10^{-6}$ and choose the following initial values:

(Case 1) $x_0 = 1.5$, $x_1 = 0.5$, $u = -0.5$ and $\lambda_k = \rho$;
(Case 2) $x_0 = 2.5$, $x_1 = -0.5$, $u = -0.5$ and $\lambda_k = \rho^2$;
(Case 3) $x_0 = 10$, $x_1 = 15$, $u = 1$ and $\lambda_k = \rho^4$.

We plot the graphs of errors against the number of iterations in each case. The numerical results can be found in Figure 3.

Figure 3: Example 4.3, top left: Case (1); top right: Case (2); and bottom: Case (3).
5 Conclusion

In this paper, we introduced an iterative algorithm of inertial form for approximating an element in the solution set of SGVMEP, which is also a fixed point of a quasi-ϕ-nonexpansive mapping and solves a VIP for a weakly sequentially continuous and quasi-monotone mapping in Banach spaces. The result obtained in this sequel extends and unifies the works of Chang et al. [25], Kazmi and Farid [11], Shan and Huang [31] and others in the literature. Using numerical example, we showed the efficacy of our method for arriving at an element in the solution set.

Remark 5.1.

(i) If we take \( F = 0 \), then Theorem 3.6 reduces to the theorems for finding a common element in the solution of SGVMEP and fixed point of quasi-ϕ-nonexpansive mapping.

(ii) Let \( Y = \mathbb{R} \) and \( P = [0, +\infty) \), then the result presented in this paper reduces to finding a common element in the solution set of SGMEP considered in [12,55] and fixed point of quasi-ϕ-nonexpansive mapping which is also a solution of a VIP.

(iii) If we let \( E_1 = E_2 = H \) a real Hilbert space, the result presented in this sequel is a unification of the result presented in [25] and [31].

Acknowledgment: The authors sincerely thank the reviewers for their careful reading, constructive comments and fruitful suggestions that improved the manuscript.

Funding information: Olawale K. Oyewole acknowledges with thanks the bursary and financial support from Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Doctoral Bursary. Oluwatosin T. Mewomo is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS and NRF.

Conflict of interest: The authors declare that they have no competing interests.

References

[1] H. A. Abass, K. O. Aremu, L. O. Jolaoso, and O. T. Mewomo, An inertial forward-backward splitting method for approximating solutions of certain optimization problems, J. Nonlinear Funct. Anal. 2020 (2020), 6, DOI: https://doi.org/10.23952/jnfa.2020.06.
[2] C. Izuchukwu, G. N. Ogwo, and O. T. Mewomo, An inertial method for solving generalized split feasibility problems over the solution set of monotone variational inclusions, Optimization (2020), DOI: https://doi.org/10.1080/02331934.2020.1808648.
[3] C. Byrne, Iterative oblique projection onto convex sets and split feasibility problem, Inverse Problems 18 (2002), 441–453.
[4] Y. Censor and A. Lent, An iterative row-action method for interval complex programming, J. Optim. Theory Appl. 34 (1981), 321–353.
[5] T. O. Alakoya, A. Taiwo, O. T. Mewomo, and Y. J. Cho, An iterative algorithm for solving variational inequality, generalized mixed equilibrium, convex minimization and zeros problems for a class of nonexpansive-type mappings, Ann. Univ. Ferrara Sez. VII Sci. Mat. 67 (2021), 1–21.
[6] E. C. Godwin, C. Izuchukwu, and O. T. Mewomo, An inertial extrapolation method for solving generalized split feasibility problems in real Hilbert spaces, Bol. Univ. Mat. Ital. 14 (2021), 379–401.
[7] T. O. Alakoya, L. O. Jolaoso, and O. T. Mewomo, A self adaptive inertial algorithm for solving split variational inclusion and fixed point problems with applications, J. Ind. Manag. Optim. 17 (2021), 2733–2759, DOI: http://dx.doi.org/10.3934/jimo.2020152.
[8] K. O. Aremu, C. Izuchukwu, G. N. Ogwo, and O. T. Mewomo, Multi-step Iterative algorithm for minimization and fixed point problems in p-uniformly convex metric spaces, J. Ind. Manag. Optim. 17 (2021), 2161–2180.
[9] O. K. Oyewole, H. A. Abass, and O. T. Mewomo, A strong convergence algorithm for a fixed point constrained split null point problem, Rend. Circ. Mat. Palermo II 70 (2021), 389–408.

[10] A. Latif, D. R. Sahu, and Q. H. Ansari, Variable KM-like algorithms for fixed point problems and split feasibility problems, Fixed Point Theory Appl. 211 (2014), 211, DOI: https://doi.org/10.1186/1687-1812-2014-211.

[11] K. R. Kazmi and M. Farid, Some iterative schemes for generalized vector equilibrium problems and relatively nonexpansive mappings in Banach spaces, Math. Sci. 7 (2013), 19.

[12] J. W. Peng and J. C. Yao, A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems and variational inequality problems, Taiwan J. Math. 12 (2008), 1401–1432.

[13] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994), 123–145.

[14] G. Chen, X. Huang, and X. Yang, Vector optimization: Set-valued and variational analysis, Lecture Notes in Economics and Mathematical Systems, Vol. 541, Springer, Berlin, Germany, 2005.

[15] F. Giannessi, Vector Variational Inequalities and Vector Equilibrium, Vol. 38, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[16] T. O. Alakoya, L. O. Jolaoso, and O. T. Mewomo, Strong convergence theorems for finite families of pseudomonotone equilibrium and fixed point problems in Banach spaces, Afr. Mat. 32 (2021), 897–923, DOI: https://doi.org/10.1007/s13370-020-00869-z.

[17] T. O. Alakoya, L. O. Jolaoso, A. Taiwo, and O. T. Mewomo, Inertial algorithm with self-adaptive stepsize for split common null point and common fixed point problems for multivalued mappings in Banach spaces, Optimization (2021), DOI: https://doi.org/10.1080/02331934.2021.1895154.

[18] O. K. Oyewole and O. T. Mewomo, A subgradient extragradient algorithm for solving split equilibrium and fixed point problems in reflexive Banach spaces, J. Nonlinear Funct. Anal. 2020 (2020), 2.

[19] O. K. Oyewole and O. T. Mewomo, A strong convergence theorem for split null point problem and generalized mixed equilibrium problem in real Hilbert spaces, Axioms 10 (2020), 16.

[20] D. T. Luc, Theory of vector optimization, Lecture Notes in Economics and Mathematical Systems, Vol. 319, Springer, Berlin, 1989.

[21] A. Danilidis and N. Hadjisavvas, Characterization of nonsmooth semistrictly quasiconvex and strictly quasiconvex functions, J. Optim. Theory Appl. 102 (1999), 525–536.

[22] L. Ćirić, A. Rafiq, S. Radenović, M. Rajović, and J. S. Ume, Common fixed point theorems for non-self-mappings in metric spaces of hyperbolic type, J. Comput. Appl. Math. 233 (2010), 2966–2974.

[23] L. Ćirić, A. Rafiq, S. Radenović, M. Rajović, and J. S. Ume, On Mann implicit iterations for strongly accretive and strongly pseudo-contractive mappings, Appl. Math. Comput. 198 (2008), 128–137.

[24] T. O. Alakoya, L. O. Jolaoso, and O. T. Mewomo, Modified inertial subgradient extragradient method with self-adaptive stepsize for solving monotone variational inequality and fixed point problems, Optimization 70 (2020), no. 3, 545–574, DOI: https://doi.org/10.1080/02331934.2020.1723586.

[25] S. S. Chang, S. Salahuddin, L. Wang, and M. Liu, On the weak convergence for solving semistrictly quasi-monotone variational inequality problems, J. Inequal. Appl. 2019 (2019), 74.

[26] S. S. Chang, H. W. JosephLee, and C. K. Chan, A new method for solving equilibrium problem, fixed point problem and variational inequality problem with application to optimization, Nonlinear Anal. 70 (2009), 3307–3319.

[27] S. He, T. Wu, A. Gibali, and Q.-L. Dong, Totally relaxed, self-adaptive algorithm for solving variational inequalities over the intersection of sub-level sets, Optimization 67 (2018), no. 9, 1487–1504.

[28] M. A. Olona, T. O. Alakoya, A. O.-E. Owolabi, and O. T. Mewomo, Inertial shrinking projection algorithm with self-adaptive step size for split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings, Demonstr. Math. 54 (2021), 47–67.

[29] M. A. Olona, T. O. Alakoya, A. O.-E. Owolabi, and O. T. Mewomo, Inertial algorithm for solving equilibrium, variational inclusion and fixed point problems for an infinite family of strictly pseudocontractive mappings, J. Nonlinear Funct. Anal. 2021 (2021), 10.

[30] A. O.-E. Owolabi, T. O. Alakoya, A. Taiwo, and O. T. Mewomo, A new inertial-projection algorithm for approximating common solution of variational inequality and fixed point problems of multivalued mappings, Numer. Algebra Control Optim. (2021), DOI: http://dx.doi.org/10.1016/j.naco.2021004.

[31] S.-Q. Shan and N.-J. Huang, An iterative method for generalized mixed vector equilibrium problems and fixed point of nonexpansive mappings and variational inequalities, Taiwan J. Math. 16 (2012), 1681–1705.

[32] M. Farid and K. R. Kazmi, A new mapping for finding a common solution of split generalized equilibrium problem, variational inequality problem and fixed point problem, Korean. J. Math. 27 (2019), 297–327.

[33] J. B. Hiriart-Urruty and C. Lemarchal, Fundamentals of Convex Analysis, Springer-Verlag, Berlin, 2001.

[34] A. Taiwo, T. O. Alakoya, and O. T. Mewomo, Strong convergence theorem for solving equilibrium problem and fixed point of relatively nonexpansive multi-valued mappings in a Banach space with applications, Asian-Eur. J. Math. 14 (2021), no. 8, 2150137, DOI: https://doi.org/10.1142/S1793557121501370.

[35] A. Taiwo, L. O. Jolaoso, and O. T. Mewomo, Inertial-type algorithm for solving split common fixed-point problem in Banach spaces, J. Sci. Comput. 86 (2021), 12.
[36] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in: A. G. Kartsatos (ed.), Theory and Applications of Nonlinear Operators and Accretive and Monotone Type, Lecture Notes in Pure and Applied Mathematics, Vol. 178, Dekker, New York, 1996, pp. 15–50.

[37] G. N. Ogwo, C. Izuchukwu, and O. T. Mewomo, *Inertial methods for finding minimum-norm solutions of the split variational inequality problem beyond monotonicity*, Numer. Algorithms 88 (2021), 1419–1456, DOI: https://doi.org/10.1007/s11075-021-01081-1.

[38] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear*, Kluwer, Dordrecht, 1990.

[39] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, Japan, 2009.

[40] R. T. Rockfellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. 16 (1977), 877–808.

[41] D. Butnariu, S. Reich, and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal. 7 (2001), 151–174.

[42] A. Taiwo, T. O. Alakoya, and O. T. Mewomo, *Halpern-type iterative process for solving split common fixed point and monotone variational inclusion problem between Banach spaces*, Numer. Algorithms 86 (2021), 1359–1389.

[43] S. Y. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory 134 (2005), 257–266.

[44] K. Ball, E. A. Carlen, and E. H. Lieb, *Sharp uniform convexity and smoothness inequalities for trace norms*, Invent. Math. 115 (1994), 463–482.

[45] C. E. Chidume, *Geometric properties of Banach spaces and nonlinear iteration*, Appl. Math. Comput. 271 (2015), 251–258.

[46] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. 16 (1991), 1127–1138.

[47] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. 13 (2002), 938–945.

[48] C. E. Chidume, S. I. Ikechukwu, and A. Adamu, *Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps*, Fixed Point Theory Appl. 2018 (2018), 9.

[49] K. Nakajo, *Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces*, Appl. Math. Comput. 271 (2015), 251–258.

[50] N. X. Tan, *On the existence of solution of quasivariational inclusion problems*, J. Optim. Theory Appl. 123 (2004), 619–638.

[51] X. H. Gong and H. M. Yue, *Existence of efficient solutions and strong solutions for vector equilibrium problems*, J. Nanchang Univ. 32 (2008), 1–5.

[52] N. Hadjisavvas and S. Chaible, *On strong pseudomonotonicity and (semi)strict quasimonotonicity*, J. Optim. Theory Appl. 79 (1993), 139–155.

[53] I. V. Konnov, *On quasimonotone variational inequalities I*, J. Optim. Theory Appl. 99 (1998), 165–181.

[54] G. N. Ogwo, C. Izuchukwu, and O. T. Mewomo, *A modified extragradient algorithm for a certain class of split pseudomonotone variational inequality problem*, Numer. Algebra Control Optim. (2021), DOI: http://dx.doi.org/10.3934/naco.2021011.

[55] S. Zhang, *Generalized mixed equilibrium problem in Banach spaces*, Appl. Math. Mech.-Engl. Ed. 30 (2009), 1105–1112.

[56] A. Bowers and N. J. Kalton, *An Introductory Course in Functional Analysis*, Springer, New York, 2014.