ON SIMULTANEOUS PALINDROMES

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Abstract. A palindrome in base $g$ is an integer $N$ that remains the same when its digit expansion in base $g$ is reversed. Let $g$ and $h$ be given distinct integers $>1$. In this paper we discuss how many integers are palindromes in base $g$ and simultaneously palindromes in base $h$.

1. Introduction

Let $a, g \in \mathbb{Z}$ with $a \geq 0$ and $g \geq 2$. If $a$ has a symmetric digit expansion in base $g$, i.e. $a$ read from left to right is the same as read from right to left, then we call $a$ a palindrome in base $g$. In particular, we will use the following definition

Definition 1. Let $a$ be a positive integer with $g$-adic digit expansion

$$a = \sum_{i=0}^{k} a_i g^i, \quad \text{with } a_i \in \{0, 1, \ldots, g-1\}, \text{ and } a_k \neq 0$$

then we write

$$(a)_g = \sum_{i=0}^{k} a_i g^{k-i}$$

for the digit reversed companion to $a$. We call $a$ a palindrome in base $g$, if we have $a = (a)_g$.

There is a rich literature on integers that are as well palindromes for some fixed base $g$ as well have some other property like being a square [8], a $k$-th power [7, 4], almost a $k$-th power [13, 11], member of a recurrence sequence [10] or some other sequences (in case of arithmetic sequence see [5]), a prime [2] and many other properties. Also some authors considered the case of palindromes that are palindromes in two or more bases simultaneously. In particular, Goins [6] proved that there are only finitely many palindromes in base 10 with $d \geq 2$ digits and $N$ is at the same time a palindrome with...
d digits in a base \( b \neq 10 \) (for a similar result see also [3]). On the other hand Luca and Togbé [11] prove that there are only finitely many binary palindromes which are decimal palindromes of the form \( 10^n \pm 1 \).

In this paper we consider the following problem

**Problem 1.** For which pairs of bases \((g, h) \in \mathbb{Z}^2 \) with \( 2 \leq h < g \) are there only finitely many positive integers that are simultaneously palindromes in base \( g \) and \( h \).

Note that the answer to this problem is negative if \( g = h^k \) for some \( k \geq 2 \), since all integers of the form \( g^n \pm 1 \) are palindromes in base \( g \) as well as in base \( h \). Therefore we consider Problem 1 only for bases \( g, h \) such that \( g \) is not a perfect power of \( h \).

Unfortunately we cannot give an answer to this problem yet, but using ideas from Luca and Togbé [11], who proved the finiteness of binary palindromes of the form \( 10^n \pm 1 \), we were able to prove the following theorem.

**Theorem 1.** Let \( 2 \leq h < g \) be integers and assume that \( h \mid g \) and that \( h \) and \( g \) are multiplicatively independent. If \( N = ag^n + (a)g \) is a palindrome in base \( h \), then

\[
\begin{align*}
n \leq & \max \left\{ \frac{\log ga}{\log h}, \frac{\log g(\log(agh))^2}{(\log 2)^3}, 1.91 \cdot 10^7 \log a(\log \log a)^3, \\
& 5.11 \cdot 10^{12} \log g(\log(agh))(\log(\log g \log(agh)))^2 \right\}. \end{align*}
\]

Note that the result of Luca and Togbé [11] can be derived from our Theorem 1 together with an extensive computer search. In particular, if we put \( h = 2, g = 10 \) and \( a \) some integer smaller than \( 10^6 \) then Theorem 1 implies that \( N = a10^n + (a)10 \) can be a binary palindrome only if \( n \leq 2.65 \cdot 10^{15} \). Our second result is:

**Theorem 2.** Let \( 2 \leq h < g \) be fixed integers and assume that \( h \mid g \) and that \( h \) and \( g \) are multiplicatively independent. For all \( \varepsilon > 0 \) there are at least \( \Omega_{\varepsilon,g,h}(x^{1/2-\varepsilon}) \) palindromes \( N \leq x \) in base \( g \) that are not palindromes in base \( h \). Moreover the constants involved in the \( \Omega \)-term are explicitly computable.

This theorem means that for \( h \mid g \) most numbers which are palindromes in base \( g \) are not palindromes in base \( h \).

The above Theorems 1 and 2 can both be deduced from the following lemma

**Lemma 1.** Let \( 2 \leq h < g \) be integers and assume \( h \mid g \) and \( h \) and \( g \) are multiplicatively independent. Moreover, let \( N \) be a palindrome in bases \( g \) of the form

\[
ag^n + \sum_{i=0}^{n-m-1} a_i g^i, \quad a_i \in \{0, 1, \ldots, g-1\},
\]
where \( n \geq m + 1 \) and \( a = (a_{n-1} \ldots a_0)_g \) is an \((n - m)\)-digit number in base \( g \). This means that \( N \) is a palindrome in base \( g \) starting with the digits of \( a \) (in base \( g \)) followed by \( m \) zeros. If

\[
m > C(a, g, h, n) := \max \left\{ \frac{\log ga}{\log h}, \frac{\log g(\log(agh))^2}{(\log 2)^3}, 142(\log n)^2, 2.022 \cdot 10^{10} \log g \log(agh) \log n \right\}
\]

then \( N \) cannot be a palindrome in base \( h \).

More precisely assume that \( m > \frac{\log ga}{\log h} \) and write \( \alpha = \frac{a}{(a)_g}_h \). If \( g, h \) and \( \alpha \) are multiplicatively independent then \( N \) can be a palindrome in base \( h \) only if

\[
|\log \alpha - k \log h + n \log g| < \frac{11}{9h^{-m}}.
\]

If \( \alpha^r = g^s h^{-t} \) for some integers \( r, s, t \) not all zero, then \( N \) is a palindrome in base \( h \) only if

\[
|(n + s) \log g - (k + t) \log h| < \frac{11r}{9h^{-m}}.
\]

Note that for proving Theorems 1 and 2 only the first part of the lemma is essential. However the second part of the lemma is useful if one wants to find all simultaneous palindromes of a special form, e.g. finding all binary palindromes of the form \( a10^n + (a)_{10} \) for some fixed \( a \).

In the next section we will give a proof of the fundamental Lemma 1 and in Section 3 we deduce Theorems 1 and 2 from that lemma. In Section 4 we present some numeric results on simultaneous palindromes in bases 2 and 10. In the last section we present some numeric results for other bases.

2. PROOF OF THE MAIN LEMMA

The purpose of this section is to prove Lemma 1. Therefore assume that \( N \) is a palindrome in base \( g \) as well as in base \( h \). Let \( n_a = \left\lfloor \frac{\log a}{\log g} \right\rfloor + 1 \) be the number of digits of \( a \) in base \( g \). Since \( N \) is a palindrome in bases \( g \) as well as in base \( h \) and since \( h|g \) we have

\[
N \equiv (a)_g \mod h^{m+n_a}.
\]

Therefore we know the last \( m + n_a \) digits of \( N \) in base \( h \), provided that \( (a)_g < h^{m+n_a} \).

**Lemma 2.** \( (a)_g < h^{m+n_a} \) if \( m > \frac{\log ga}{\log h} \).

**Proof.** Note that \( a \) and \( (a)_g \) have the same number of \( g \)-adic digits, i.e. \( (a)_g < ag \). Using the formula for \( n_a \) we see that \( (a)_g < h^{m+n_a} \) if \( ga < h^m h^{\log a/\log g + 1} \), hence

\[
\frac{\log ga}{\log h} - \frac{\log a}{\log g} - 1 < \frac{\log ga}{\log h} < m.
\]
implies \((a)_g < h^{n+n_a}\).

Therefore we assume from now on \(m > \log_{ga} h\). Because \(N\) is a palindrome in base \(h\) we also know now the first \(m + n_a\) digits of \(N\) in base \(h\). In particular, we have

\[
N = \left(\frac{(a)_g}{h}\right)_h h^k + \sum_{i=0}^{k-m-1} b_i h^i,
\]

where \(\bar{m} = m + n_a - \tilde{n}_a\) with \(\tilde{n}_a\) denoting the number of digits of \((a)_g\) in base \(h\), and where

\[
k = \left\lfloor \frac{\log N - \log (a)_g}{\log h} \right\rfloor.
\]

Since

\[
m + n_a = \bar{m} + \left\lfloor \frac{\log (a)_g}{\log h} \right\rfloor + 1
\]

we have

\[
\bar{m} = m + \left\lfloor \frac{\log (a)_g}{\log h} \right\rfloor - \left\lfloor \frac{\log (a)_g}{\log h} \right\rfloor.
\]

This yields the following inequality for \(N\):

\[
\left(\frac{(a)_g}{h}\right)_h h^k + h^{k-\bar{m}} > N > \left(\frac{(a)_g}{h}\right)_h h^k.
\]

Dividing this inequality through \(\left(\frac{(a)_g}{h}\right)_h h^k\) yields

\[
(3)\quad \left| \frac{a g^n}{(a)_g} h^k - 1 \right| < \frac{h^{-\bar{m}}}{h - \bar{m}}
\]

On the other hand

\[
\frac{h^{-\bar{m}}}{(a)_g} = h^{-m - \left\lfloor \frac{\log (a)_g}{\log g} \right\rfloor + \left\lfloor \frac{\log (a)_g}{\log h} \right\rfloor - \left\lfloor \frac{\log (a)_g}{\log h} \right\rfloor} \leq h^{-m} \leq \frac{1}{4}
\]

provided \(m \geq 2\). Writing \(\alpha := \frac{a}{(a)_g}\) and using that the inequality \(\log |1 - x| \leq \frac{11x}{9}\) holds, provided \(x \leq \frac{1}{4}\), which can easily be proved by a Taylor expansion with Cauchy’s remainder term from equation (3) we obtain

\[
(4)\quad |\log \alpha - k \log h + n \log g| \leq \frac{11}{9h^{-m}}.
\]
which is inequality (1) in Lemma 1. Inequality (2) is deduced from (1) by multiplying it by $r$ and noting that $r \log \alpha = s \log g - t \log h$.

We distinguish now between two cases. The first case is that $\alpha$ is multiplicatively independent of $g$ and $h$ and the second is that $\alpha$ is multiplicatively dependent of $g$ and $h$. The first case requires lower bounds for linear forms in three logarithms (we will use a result due to Matveev [12]) and in the second case our inequality will reduce to an inequality in linear forms in two logarithms, where sharper bounds are known (we will use a result due to Laurent et. al. [9]). Unfortunately using this result will involve the prime decompositions of $g$, $h$ and $\alpha$.

We start with the first case. Let us state Matveev’s theorem [12]:

**Theorem 3.** Denote by $\alpha_1, \ldots, \alpha_n$ algebraic numbers, not 0 nor 1, by $\log \alpha_1, \ldots, \log \alpha_n$ determinations of their logarithms, by $D$ the degree over $\mathbb{Q}$ of the number field $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, and by $b_1, \ldots, b_n$ rational integers. Furthermore let $\kappa = 1$ if $K$ is real and $\kappa = 2$ otherwise. Choose

$$A_i \geq \max \{ Dh(\alpha_i), |\log \alpha_i| \} \quad (1 \leq i \leq n),$$

where $h(\alpha)$ denotes the absolute logarithmic Weil height of $\alpha$ and

$$B = \max \{ 1, \max \{|b_j|A_j/A_n : 1 \leq j \leq n\} \}.$$ 

Assume $b_n \neq 0$ and $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Z}$; then

$$\log |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| \geq -C(n)C_0W_0D^2\Omega,$$

with

$$\Omega = A_1 \cdots A_n,$$

$$C(n) = C(n, \kappa) = \frac{16}{n! \kappa} e^n (2n + 1 + 2\kappa)(n + 2)(4(n + 1))^{n+1} \left( \frac{1}{2} en \right)^\kappa,$$

$$C_0 = \log \left( e^{4.4n+7}n^{5.5}D^2 \log(eD) \right), \quad W_0 = \log(1.5eBD \log(eD)).$$

We will apply Theorem 3 directly to (4). Obviously $\kappa = 1$ and $D = 1$, and we may choose $A_1 = \log(agh)$ since $a, \left( (a)\big)_g < agh$, $A_2 = \log h$ and $A_3 = \log g$. Next we have to estimate $B$:

**Lemma 3.** $B < 2n$ if $m \geq 2 > \frac{\log h}{\log g} + 1$.

**Proof.** First note that $2n > \frac{\log \alpha}{\log g} = |b_1|A_1^{-1}$, since

$$m \geq 1 > \frac{\log g + \log h}{2 \log g} > \frac{\log \alpha}{2 \log g}$$

and $n \geq m$. 

Furthermore, we have the inequality
\[
2n = \frac{2 \log N - 2 \log a}{\log g} > \log N - \log \left( \frac{(a)g}{h} \right) - \log h - \log g + \log N - \log a
\]
\[
= k \frac{\log h}{\log g} + n - \frac{\log h}{\log g} - 1 > k \frac{\log h}{\log g} = |b_2| \frac{A_2}{A_3}.
\]

Therefore we obtain
\[
W = 1.152 \log n \text{ provided } n > m \geq 10^6.
\]
Now Theorem 3 together with inequality (4) yields
\[
2.022 \cdot 10^{10} \log g (\log agh) \log n > m,
\]
which proves the first case. Note that the bound 2.022-10^{10} \log g (\log agh) \log n contains the bound m \leq 10^6 in any case.

Now we consider the second case. Since by assumption \alpha, g and h are multiplicatively dependent, but g and h are multiplicatively independent thus there exist integers r, s, t with greatest common divisor 1 such that
\[
\alpha^r = g^s h^t \text{ with } r \neq 0.
\]

Lemma 4.
\[
|r| \leq \frac{\log g \log h (\log agh)}{(\log 2)^3}, \quad |s| \leq \frac{\log h (\log agh)^2}{(\log 2)^3}, \quad |t| \leq \frac{\log g (\log agh)^2}{(\log 2)^3}
\]

Proof. Now let \( p_1, p_2 \) be primes that divide gh, let \( e_{\beta,i} = v_{p_i}(\beta) \) for \( i = 1, 2 \) and \( \beta \in \{ g, h, \alpha \} \). Here \( v_p(x) \) denotes the p-adic valuation of x. Further, assume that the vectors \( (e_{g,1}, e_{h,1}) \) and \( (e_{g,2}, e_{h,2}) \) are linearly independent over \( \mathbb{Z}^2 \). Note that this pair of primes exists since g and h are multiplicatively independent. A technical but easy computation shows that
\[
s = (e_{h,1} e_{\alpha,2} - e_{h,2} e_{\alpha,1}) e_{\alpha,2};
\]
\[
t = (e_{g,1} e_{\alpha,2} - e_{g,2} e_{\alpha,1}) e_{\alpha,2};
\]
\[
r = (e_{g,2} e_{h,1} - e_{g,1} e_{h,2}) e_{\alpha,2}
\]
is a solution and since \( (e_{g,1}, e_{h,1}) \) and \( (e_{g,2}, e_{h,2}) \) are linearly independent \( r \neq 0 \). The statement of the lemma now follows from the simple estimates
\[
e_{h,i} \leq \frac{\log h}{\log 2}; \quad e_{g,i} \leq \frac{\log g}{\log 2}; \quad |e_{\alpha,i}| \leq \frac{\log (agh)}{\log 2};
\]
for \( i = 1, 2 \).

We multiply inequality (4) by \( r \) and obtain
\[
|(n + s) \log g - (k + t) \log h| < \frac{r 11}{9h-m}
\]
This time we apply the following result due to Laurent et. al. [9]:

\[
\begin{align*}
|\gamma(\log x) - \gamma(\log y)| &< \frac{1}{\log x} \frac{1}{\log y} \max(x,y) \log x \log y \\
&< \frac{1}{\log x} \frac{1}{\log y} \max(x,y) \log x \log y \\
&< \frac{1}{\log x} \frac{1}{\log y} \max(x,y) \log x \log y
\end{align*}
\]
Theorem 4. Let $\alpha_1$ and $\alpha_2$ be two positive, real, multiplicatively independent elements in a number field of degree $D$ over $\mathbb{Q}$. For $i = 1, 2$, let $\log \alpha_i$ be any determination of the logarithm of $\alpha_i$, and let $A_i > 1$ be a real number satisfying
\[
\log A_i \geq \max\{h(\alpha_i), |\log \alpha_i|/D, 1/D\}.
\]
Further, let $b_1$ and $b_2$ be two positive integers. Define
\[
b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1} \quad \text{and} \quad \log b = \max\left\{\log b' + 0.14, 21/D, \frac{1}{2}\right\}.
\]
Then
\[
|b_2 \log \alpha_2 - b_1 \log \alpha_1| \geq \exp\left(-24.34D^4(\log b)^2 \log A_1 \log A_2\right).
\]
We choose $\alpha_1 = g$ and $\alpha_2 = h$, thus we have $D = 1$. Put $\log A_1 = \log g$ and $\log A_2 = \frac{\log h}{\log 2} \geq 1$ and start with estimating $b'$:
\[
b' = \frac{(n + s) \log 2}{\log h} + \frac{k + t}{\log g}
\]
\[
< \frac{2n}{\log h} + \frac{1}{\log g} + \frac{2}{\log h} + \frac{1}{(\log 2)^2} + \frac{1}{(\log 2)^3} (\log agh)^2
\]
\[
< \frac{2n}{\log h} + 6.3(\log agh)^2 < \frac{4n}{\log h} < 6n
\]
provided that $n > m > 3.15 \log h(\log agh)^2$. Note that the first inequality is true because of
\[
k = \left\lfloor \frac{\log N - \log (a^g)}{\log h} \right\rfloor
\]
\[
< \frac{\log a + (n + 1) \log g - \log a + \log g + \log h}{\log h} = n \frac{\log g}{\log h} + 2 \frac{\log g}{\log h} + 1.
\]
The inequality for $b'$ now implies that we may choose
\[
\log b = 2 \log n > \max\{\log n + \log 6 + 0.14, 21\}
\]
provided that $n > m > 37000$. Now Theorem 4 yields
\[
141(\log n)^2 \log g \log h < m \log h + \log(9/11) - \log \left(\frac{\log g(\log agh)^2}{(\log 2)^3}\right).
\]
Let us assume $n > m > \frac{\log g(\log agh)^2}{(\log 2)^3}$ and since we also assume $n > m > 37000$ this last inequality turns into
\[
142(\log n)^2 \log g > m.
\]
Therefore we have completely proved Lemma 1.

Note that all our assumptions on $m$ made during the proof together with the bounds for $m$ cumulate in the lower bound of Lemma 1.
3. Proof of Theorems 1 and 2

We start with the proof of Theorem 1. In this case $N = ag^n + \overline{(a)g}$ and therefore

$$m = n - \left\lfloor \frac{\log a}{\log g} \right\rfloor > n - \frac{\log a}{\log g} - 1.$$ 

In view of Lemma 1 this implies: If $n > C\left(a, g, h, n\right) + \frac{\log a}{\log g} + 1$, then $N$ is not a palindrome in base $h$. Let us consider the two inequalities $n > 142(\log n)^2 + \frac{\log a}{\log g}$ and $n > 2.022 \cdot 10^{10} \log g \log(agh) \log n + \frac{\log a}{\log g}$. We note that the largest solution to $n = A\log n + B$ is smaller than the largest solution to $\frac{n}{\log n} = A + \frac{B}{\log A}$ and the largest solution to $\frac{n}{\log n} = C$ is smaller than $x(\log x)^2$ provided $x > e^2$. Now let $A = 2.022 \cdot 10^{10} \log g \log(agh)$ and $B = \frac{\log a}{\log g} + 1$, then

$$A + \frac{B}{\log A} = 2.022 \cdot 10^{10} \log g \log(agh) + \frac{\log a}{\log g} + 1 > 23.73 + \log(\log g \log(agh)) > 2.023 \cdot 10^{10} \log g \log(agh).$$

Therefore the inequality $n > 2.022 \cdot 10^{10} \log g \log(agh) \log n + \frac{\log a}{\log g} + 1$ is fulfilled whenever

$$n > 5.11 \cdot 10^{12} \log g \log(agh)(\log(\log g \log(agh)))^2.$$ 

Similarly the largest solution to $n = A(\log n)^2 + B$ is smaller than the largest solution to $\frac{n}{(\log n)^2} = A + \frac{B}{(\log A)^2}$ and the largest solution to $\frac{n}{(\log n)^2} = C$ is smaller than $x(\log x)^3$ provided $x > 62$. This time we put $A = 142$ and $B = \frac{\log a}{\log g} + 1$ and

$$A + \frac{B}{(\log A)^2} < 142 + \frac{\log a}{24.56 \log g} + 0.041.$$ 

If $a \leq 2$, then the lower bound for $n$ will be 17295 which is absorbed by the much larger bound found in the paragraph above. Therefore we may assume that $a \geq 3$ and obtain

$$A + \frac{B}{(\log A)^2} = 142 + \frac{\log a}{24.56 \log g} < 130 \log a$$

and therefore the inequality $n > 142(\log n)^2 + \frac{\log a}{\log g}$ is fulfilled if

$$n > 1.91 \cdot 10^7 \log a(\log a)^3.$$ 

Therefore the proof of Theorem 1 is complete.

We turn now to the proof of Theorem 2. We consider palindromes described in Lemma 1 with $a < g$ and $m = (\log n)^3$. Then by Lemma 1 we know that for some constant $C_{g,h}$ depending only on $g$ and $h$ we have $m > C(a,g,h,n)$ (see Lemma 1) for all $n > C_{g,h}$, i.e. these palindromes cannot be palindromes in base $h$ if $n > C_{g,h}$. On the other hand there are
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\[ x^{1/2-\epsilon} \] palindromes of the described form provided \( \frac{1}{(\log n)^x} > \epsilon \), which proves Theorem 2.

4. Numerical Considerations

The purpose of this section is to consider the case \( g = 10 \) and \( h = 2 \) more closely and think of it as a model case. The aim is to find decimal palindromes that are also binary palindromes, however we did not find many such palindromes.

**Proposition 1.** Let \( N < 10^{20} \) be a palindrome in base 10 which is also a binary palindrome, then \( N \) is one of the following 53 palindromes:

1, 3, 5, 7, 9, 33, 99, 313, 585, 717, 7447, 9009, 15351, 32223, 39993, 53235, 53835, 73737, 585585, 1758571, 1934391, 1979791, 3129213, 5071705, 5259525, 581485, 13500531, 719848917, 910373019, 939474939, 1290880921, 7451111547, 10050905001, 18462126481, 32479297423, 75015151057, 11094884911, 136525525631, 1234104014321, 1413899983141, 147492229471, 1792704072971, 179409904971, 1999925299991, 565262226565, 7227526257227, 7284717174827, 9484874784849, 34141388314143, 552212535212255, 1793770770773971, 3148955775598413, 933138363831339.

**Proof.** For all \( a < 10^{10} \) we construct decimal palindromes \( N < 10^{20} \) with an even number of digits by reversing the digits of \( a \) and appending the reversed string of digits at the string of digits of \( a \), i.e. if \( a = \sum_{i=0}^{n} a_i 10^i \) we compute the palindrome

\[
N = \sum_{i=0}^{n} a_i 10^{n+1+i} + \sum_{i=0}^{n} a_{n-i} 10^i.
\]

Similarly we construct for all \( a < 10^{10} \) palindromes \( N < 10^{20} \) with an odd number of digits by

\[
N = \sum_{i=0}^{n} a_i 10^{n+i} + \sum_{i=0}^{n-1} a_{n-i} 10^i.
\]

With this procedure we have a complete list of all decimal palindromes \( N < 10^{20} \).

Now we test for each decimal palindrome \( N \) whether it is a binary palindrome by the following algorithm. First we compute the number of binary digits

\[
k = \left\lfloor \frac{\log N}{\log 2} \right\rfloor + 1.
\]
Let us put \( n_k = n_0 = N \) and compute subsequently for all \( 0 \leq i \leq \lfloor k/2 \rfloor \) the \( i \)-th highest and \( i \)-th lowest binary digits of \( N \)

\[
d_{k-i} = \left\lfloor \frac{n_{k-i}}{2^{k-i}} \right\rfloor, \quad d_i = n_i \mod 2
\]

and

\[
n_{k-i-1} = \frac{n_{k-i} - 2^{k-i}d_{k-i}}{2}, \quad n_{i+1} = \frac{n_i - d_i}{2}.
\]

If \( d_{k-i} \neq d_i \) for some \( i \) then \( N \) is not a binary palindrome. If \( d_{k-i} = d_i \) for all \( 0 \leq i \leq \lfloor k/2 \rfloor \) then \( N \) is also a binary palindrome.

Note that if \( N \) is not a binary palindrome we do not have to compute all binary digits of \( N \) and in many cases after computing a few digits of \( N \) will yield a result. Indeed implementing this algorithm in sage [14] and running it on a Silicon Graphics International SGI Altix ICE8400EX supercomputer with Intel Xeon X5680 (Westmere EP) processors (each containing 6 cores) we used about 5821 hours of CPU time. In principle we used 240 cores simultaneously during about 60 hours of wall-clock time. □

**Proposition 2.** Let \( N = a10^n + \overline{\underbrace{a}_{\text{a digits}}} \) be a binary palindrome with \( 10 \nmid a \) and \( a < \min\{10^6, 10^n\} \), then it is already contained in the list of palindromes in Proposition 1.

In order to prove this proposition we have to consider the Diophantine inequalities (4) and (5). An upper bound for \( m \) is given by Lemma 1 but this bound is very huge. Therefore we will use continued fractions in case of (2) and a method due to Baker and Davenport [1] to reduce the upper bound in case of (1). Let us state a variant of this reduction method:

**Lemma 5.** Given a Diophantine inequality of the form

\[
|n_1 + n_2 \epsilon + \delta| < c_1 \exp(-n_2c_2)
\]

Assume \( n_2 < X \) and assume that there is a real number \( \kappa > 1 \) and also assume there exists a convergent \( p/q \) to \( \epsilon \) with \( \frac{X}{q} < \frac{1}{2\kappa} \) such that

\[
\|q\epsilon\| < \frac{1}{2\kappa X} \quad \text{and} \quad \|q\delta\| > \frac{1}{\kappa},
\]

\( \|q\delta\| > \frac{1}{\kappa} \), where \( \| \cdot \| \) denotes the distance to the nearest integer. Then we have

\[
n_2 \leq \frac{\log(2\kappa q c_1)}{c_2}.
\]

**Proof.** We consider equation (6) and we multiply it by \( q \). Then under our assumptions we obtain

\[
c_1 q \exp(-n_2c_2) > |qn_1 + n_2 q \epsilon + q \delta| \geq \|\|q\epsilon\| - \|q\delta\|| > \frac{1}{2\kappa}.
\]

The last inequality holds since \( p/q \) is a convergent to \( \epsilon \) and therefore \( |q \epsilon - p| < \frac{1}{q} \). Solving this inequality for \( n_2 \) we obtain the lemma. □
Proof of Proposition 2. Now let us apply the second part of Lemma 1 to the present situation. Since we assume \( a < 10^6 \) we have \( m \geq n - \frac{\log a}{\log 10} > n - 6 \). And therefore either \( n > 30 \) or one of the two inequalities (1) and (2) are fulfilled with \( n \leq 2.65 \cdot 10^{15} := X \) - since the upper bound for \( n \) obtained in Theorem 1. We have to distinguish between the two cases \( \alpha^r = 10^s \cdot 2^t \) for some integers \( r, s, t \) with \( r^2 + s^2 + t^2 \neq 0 \) and \( \alpha \), 2 and 10 are multiplicatively independent.

In the second case we divide by \( \log 2 \) and obtain inequality (6), with \( n_1 = k, n_2 = n, \epsilon = \frac{\log 10}{\log 2}, \delta = \frac{\log \alpha}{\log 2}, c_1 = \frac{11 \cdot 2^6}{9 \log 2} \) and \( c_2 = \log 2 \). With this choice we apply Lemma 5. Since \( \epsilon \) is independent of \( a \) we can precompute suitable pairs \((q, \kappa)\) which may be applied to our situation. Therefore we compute the first 50 convergents to \( \frac{\log 10}{\log 2} \). For each convergent \( p/q \) with \( q > 1.06 \cdot 10^{16} \) we form the pair \((q, \kappa)\) with \( \kappa := 2X/q \) and get a list of 16 potential pairs applicable to Lemma 5. Now let us fix \( a \). We subsequently test whether in our list is a pair \((q, \kappa)\) such that \( \|q\delta\| > \frac{1}{\kappa} \), hence by Lemma 5 we get a new bound that should be rather small and indeed in all cases our new bound yields \( n \leq 81 \). Further we want to emphasize that it is highly improbable that for a given \( a \) no pair of our list of candidates yields an application of Lemma 5 and therefore no new upper bound for \( n \). Therefore we are left to test all remaining \( n \) for our fixed \( a \), which can be done by a quick computer search.

In case of \( \alpha^r = 10^s \cdot 2^t \) for some integers \( r, s, t \) with \( r^2 + s^2 + t^2 \neq 0 \) we know that we can choose \( r = 1 \). Indeed the free \( \mathbb{Z}\)-Module generated by \( \{\log 10, \log 2\} \) is the same as the free \( \mathbb{Z}\)-Module generated by \( \{\log 5, \log 2\} \) and \( \log \alpha \) is contained in the later one and therefore also in the first \( \mathbb{Z}\)-Module. Now we obtain by Lemma 1 and in particular by inequality (2)

\[
\left| \frac{\log 10}{\log 2} - \frac{k + t}{n + s} \right| < \frac{11 \cdot 2^6}{9 \cdot 2^{-n}(n + s) \log 2}.
\]

Note that therefore \( \frac{k + t}{n + s} \) has to be a convergent to \( \frac{\log 10}{\log 2} \) unless \( n \leq 30 \) or

\[
\frac{1}{2(n + s)^2} < \frac{11 \cdot 2^8}{9 \cdot 2^{-n}(n + s) \log 2}.
\]

Let us note that this inequality does not hold for large \( n \), in particular in all cases that we consider we can choose the bound \( n \geq 30 \). Therefore we know that \( n + s \) has to be a multiple of \( q \), where \( p/q \) is a convergent to \( \frac{\log 10}{\log 2} \), i.e. \( n = kq - s \) for some positive integer \( k \). But, already for rather small \( k \) and fixed convergent \( p/q \) this choice will contradict the inequality

(7)

\[
\left| \frac{\log 10}{\log 2} - \frac{p}{q} \right| < \frac{11 \cdot 2^6}{9 \cdot 2^{kq-s}(kq) \log 2}.
\]

We claim that inequality (7) is never fulfilled for \( n \geq 30 \). Since Lemma 4 we know \( s \leq 34 \) and therefore we may assume \( q < 2.66 \cdot 10^{15} \). In particular we have to prove inequality (7) for 32 convergents. If we replace in (7) the quantities \( kq - s = n \) and \( kq = n + s \) by 30 - we may do so since we assume
For the remaining 24 cases we replace in (7) \( kq - s = n \) by \( q - 34 \) and \( kq \) by \( q \). If this new inequality still holds also (6) holds. A quick computation using a computer algebra system like sage [14] resolves this case.

Therefore we also have in this case a very efficient method to find all simultaneous palindromes, i.e. we only have to test all \( n \leq 30 \).

We implemented the idea above in sage [14] and computed for all \( 2 < a < 10^6 \) with \( 10 \nmid a \) and \( (a)_{10} \) is odd all \( n \) with \( a < \min\{10^6, 10^n\} \) such that \( N = a10^n + (a)_{10} \) is a binary palindrome. Note that in case of \( (a)_{10} \) is even then the last binary digit of \( N \) would be 0 and \( N \) would not be a binary palindrome. The computer search took on a single PC about 80 minutes.

5. Other bases

In this section we want to discuss Problem 1 for further base pairs \((h, g)\). In case of \((h, g) = (2, 10)\) the preceding sections show that there are only few integers that are simultaneously palindromes in bases 2 and 10. Looking at our results we even guess that there are only finitely many simultaneous palindromes for the bases 2 and 10. In this last section we want to present shortly our numeric considerationes for other base pairs. In particular, we considered the pairs \((2, 3), (6, 15), (5, 7), (11, 13)\) and \((7, 29)\) and counted the number \( N \) of simultaneous palindromes smaller than some bound \( B \). Our results are listed in Table 1 below. The algorithms were implemented in sage [14] and except the case \((h, g) = (2, 3)\) were run on a single PC.

The case \((h, g) = (2, 3)\) required more than 4648 days CPU time on a Silicon Graphics International SGI Altix ICE8400EX supercomputer with Intel Xeon X5680 (Westmere EP) processors (each containing 6 cores). In principle we used 120-420 cores simultaneously, depending on availability, and so the computation has been finished in less than 18 days.

Table 1. Number of simultaneous palindromes

| \( g \) | \( h \) | \( B \) | \( N \) | Time |
|-----|-----|-----|-----|-----|
| 2   | 3   | \( 3^{37} \approx 1.22 \cdot 10^{37} \) | 5   | 4648d 10h 55m |
| 6   | 15  | \( 6^{20} \approx 3.66 \cdot 10^{15} \) | 58  | 1d 0h 49m |
| 5   | 7   | \( 5^{24} \approx 5.96 \cdot 10^{16} \) | 60  | 4d 22h 27m |
| 11  | 13  | \( 13^{14} \approx 3.94 \cdot 10^{15} \) | 58  | 19h 26m |
| 7   | 29  | \( 7^{20} \approx 7.98 \cdot 10^{16} \) | 72  | 4d 21h 29m |
| 2   | 10  | \( 10^{16} \approx 10^{16} \) | 53  | 1d 13h 9m |

Looking at Table 1 the number of simultaneous palindromes to the bases 2 and 3 is very small and indeed we are led by our numeric computations to the following conjecture

**Conjecture 1.** The only positive integers that are a palindrome in bases 2 and 3 simultaneously are 1, 6643, 1422773, 5415589 and 90396755477.
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