On Bound States in Quantum Field Theory

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Abstract
In this paper, a new method to describe the energy spectrums of bound states in Quantum Field Theory is presented. We point out that the fundamental field and its dual soliton combine together to form bound states and the soliton corresponds to the ghost particle in our regularization scheme which takes advantage of dimensional regularization and Pauli-Villars regularization. Based on this point of view, we discuss the bound states of massive Thirring model, the positronium \((e^+ e^-)\) in QED and the vector meson in QCD. We also give a new way to obtain the mass of soliton (quantum soliton) from the stationary condition (gap equation). Our results agree with experimental data to high precision. We argue that the hypothetic \(X_{17}\) particle in decay of \(^8\text{Be}\) and \(^4\text{He}\) is a soliton. For vector meson, we find the squared masses of \(\rho\) resonances are \(m^2(n) \sim (an^{1/3} - b)^2\) \((n \in N)\) which coincide well with experiments.

Keywords Quantum field theory · Bound state · Soliton

1 Introduction

The correlation function and S-matrix are the basic building blocks of Quantum Field Theory (QFT). These quantities can be calculated perturbatively at weak coupling. The perturbative QFT have achieved great successes [1], but these are insufficient to solve the non-perturbative aspects of QFT. We need new methods to study the non-perturbative problems especially the bound states which are important objects in quantum theory and have been widely studied. There are many methods aim to solve it completely, e.g. the Bethe-Salpeter equation [2] and Lattice QCD [3]. The Bethe-Salpeter equation is difficult to solve. Lattice calculations give us useful numerical results, but we still need to understand many important phenomena from analytical aspects. In this paper we study the bound states from the analytical structures of the correlation function. The general representations of the vacuum expectation value of two Heisenberg operators are given by the Källén – Lehmann spectral representation [4]. We give the Källén – Lehmann form for the vacuum expectation

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value of the time-ordered product of two vector fields which can be obtained from that of
the ordinary product of the two operators, the Wightman function:

$$\langle \Omega | A_\mu(x) A_\nu(y) | \Omega \rangle = \sum_n \langle \Omega | A_\mu(x) | n \rangle \langle n | A_\nu(y) | \Omega \rangle,$$

where the complete set of states \{ | n \rangle \} has been used. Based on general assumptions about
invariance and the spectrum, the expression of the Wightman function is

$$\langle \Omega | A_\mu(x) A_\nu(y) | \Omega \rangle = i \int_0^\infty dm^2 \rho_{\mu\nu}(m^2) \Delta_1(\Delta^+(x - y; m^2)).$$

The theory is determined by the spectral density function \( \rho_{\mu\nu} \). The exact Feynman prop-
gagator for the gauge field in the Källén – Lehmann spectral representation is given by

$$\langle \Omega | T(A_\mu(x) A_\nu(y)) | \Omega \rangle = \frac{1}{2} \int_0^\infty dm^2 \rho_{\mu\nu}(m^2) \Delta_F(x - y; m^2). \quad (1)$$

If a bound state exists in the theory, the pole of the Fourier transform of (1) gives the mass
of the bound state. We fail to obtain the mass spectrums of bound states from the vacuum
expectation value of two Heisenberg operators by the perturbative methods. We find that the
spectrums of bound states can be obtained by taking into account the soliton contributions.
The soliton plays an important role in QFT. Dirac [5] argued that in quantum mechanics
the unobservability of phase permits singularities which manifest themselves as sources of
magnetic fields and the electric charge \( q \) and magnetic charge \( g \) satisfies the quantization
condition

$$q g = \frac{n}{2} \quad (2)$$

with \( n \) an arbitrary integer. The Dirac quantization condition is a topological property.
The magnetic monopoles necessarily arise as solitons in gauge theories, e. g. the ‘t Hooft-
Polyakov monopole [6]. The free monopole has not yet been discovered due to the mass
\( M \sim \frac{M_w}{\alpha} \sim 137 M_w \), where \( M_w \) is the mass of the intermediate vector boson of the weak
interaction. There is another particle known as the dyon carrying electric and magnetic
charge. The mass of a dyon of magnetic number \( n_e \) and electric number \( n_m \) is [7]

$$M \geq \sqrt{2} |Z| \quad (3)$$

with complex charge

$$Z = v(n_e + i \frac{1}{\alpha} n_m). \quad (4)$$

Where \( v \) is the Higgs expectation value and \( \alpha = \frac{g^2}{4\pi} \). In bound states, we find that the
mass of fundamental field and its dual soliton satisfies a new relation similar to the Dirac
quantization condition (2). From this relation, we can study the energy eigenvalues of the
bound states and the masses of solitons. This also indicates that the fundamental field and its
dual soliton combine together to form bound states. Besides this, we give a clue for physical
regularization in QFT.

The paper is organized as follows. In Section 2, we give a general framework of our
method. Based on a new regularization scheme, we propose a equation for energy eigen-
values of the bound states. We argue that the mass of the fictitious particle in Pauli-Villars
regularization is the same as the dual soliton mass. In Section 3, we discuss the bound states
of massive Thirring model to show our method can obtain the right results. In Section 4,
we discuss the positronium \((e^+ e^-)\) in QED. We give a new way to obtain the mass of soli-
ton from the stationary condition (gap equation) by analytic continuation. In Section 5, we
consider vector meson in QCD. We end with the conclusions.
2 A General Framework

The energy eigenvalues of bound states are the poles of propagators. The general form of off-shell propagator $G_2(p^2)$ by the chain approximation in momentum space has a factor [8]

$$G_2(q^2) \propto \frac{1}{1 - \Pi(q^2, m)},$$

(5)

where the $\Pi(q^2, m)$ is obtained from loop diagram and contains ultraviolet (UV) divergence. The UV divergence can be regulated by the Pauli-Villars (PV) [9] regularization and dimensional regularization [10]. The PV regularization requires that for each particle of mass $m$ a new unphysical ghost particle of mass $\tilde{m}$ is added with either the wrong statistics or wrong-sign kinetic term. The PV regularization is complicated because it involves introducing, in a gauge invariant manner, sets of heavy fields. The dimensional regularization respects gauge invariance and can regulate IR or UV divergences, but it has the disadvantage of being unphysical. We use a new regularization scheme which takes advantage of dimensional regularization and Pauli-Villars regularization. We first calculate the Feynman diagrams in $d = n - 2\varepsilon$ dimensions, then we add a ghost particle to cancel the UV divergence, where the ghost particle is determined by the bound state. The new regularization scheme will be physical one without any fictitious particle. Applying the new regularization scheme in $G_2(p^2)$, it modifies the propagator $G_2(p^2)$ as follows:

$$G_2(q^2) \propto \frac{1}{1 - \Pi(q^2, m)} \rightarrow \tilde{G}_2(q^2) \propto \frac{1}{1 - (\Pi(q^2, m) - \Pi(q^2, \tilde{m}))},$$

(6)

where the $\Pi(q^2, m)$ is calculated in $d = n - 2\varepsilon$ dimensions. Then the energy eigenvalues of the bound states are the solutions of the equation

$$\lim_{\varepsilon \to 0} [1 - (\Pi(q^2, m) - \Pi(q^2, \tilde{m}))] = 0.$$  

(7)

To get the excited spectrum of bound states, we need to know the fictitious particle mass $\tilde{m}$. We find that the mass of the fictitious particle is the same as the dual soliton mass:

$$\text{ghost particle} \iff \text{soliton solution},$$

(8)

then we have the relation

$$\Pi^{\text{soliton}}(q^2, \tilde{m}) = -\Pi(q^2, \tilde{m}).$$

(9)

From this point, we argue that the soliton contributions present a natural physical UV subtraction. According to the (7), the masses of solitons (quantum solitons) can be obtained by the energy eigenvalues of bound states.

In order to study the bound states, we define the integral of a complex function $f(z)$ along a smooth contour $C[a, b]$ in the complex plane. Suppose the function $f(z)$ has poles or branch cuts (Fig. 1), then the integral of $f(z)$ along the contour $C[a, b]$ can be expressed as

$$\int_{C[a, b]} f(z) dz = P \int_a^b f(z) dz + \sum n_i \oint_{C_i} f(z) dz.$$  

(10)

Where $C_i$ is a closed curve circling the pole or branch cut. The $P \int_a^b f(z) dz$ denotes the principal value which takes value in the main single-valued branch of multi-valued function. The winding number $n_i \in \mathbb{Z}$ denotes the contour circling $n_i$ times around the pole or branch cut.
Fig. 1 A smooth contour $C[a, b]$ in complex plane starting from $a$ to $b$. The dots and wave line denote the poles and branch cut of function $f(z)$ separately.

For example, we obtain the following integral (Fig. 2)

\[
\int_{C[-1,1]} \frac{1}{z} dz = P \int_{-1}^{1} \frac{1}{z} dz + 2\pi ni = \int_{-\delta}^{-1} \frac{1}{z} dz + \int_{C_\delta} \frac{1}{z} dz + \int_{1}^{\delta} \frac{1}{z} dz + 2\pi ni = \pi i + 2\pi ni,
\]

where $n \in \mathbb{Z}$ and $\delta$ is the infinitely small radius of semi-circle $C_\delta$.

We then consider the function $f(z) = \ln[(z-a)(z-b)]$ ($a < b$), which has a branch cut between $z = a$ and $z = b$, as shown in Fig. 3. Considering a closed counter-clockwise contour $C$, the contour integral $\oint_C f(z) dz$ is

\[
\oint_C f(z) dz = \oint_C \ln[(z-a)(z-b)] dz = 2\pi i (b-a).
\]
In our related work [11], we studied the axial (ABJ) anomaly [12, 13] by the formulæ (10) and obtained the non-perturbative masses of neutral pseudoscalar mesons. The divergence of the axial current is

\[
q^\mu M_{\rho\mu} = \frac{m^2}{2\pi^2} \epsilon^{\mu\nu} \rho \kappa_1^\gamma \kappa_2^\sigma \left[ \frac{1}{q^2} P \int_0^1 \frac{1}{x} \ln \frac{1}{1 - x(1 - x) \frac{q^2}{m^2}} dx + \frac{1}{q^2} 2\pi i k \ln \frac{1 + \sqrt{1 - \frac{4m^2}{q^2}}}{1 - \sqrt{1 - \frac{4m^2}{q^2}}} + \frac{1}{q^2} (2\pi i)^2 k l \right] - \frac{1}{4\pi^2} \epsilon^{\mu\nu} \rho \kappa_1^\gamma \kappa_2^\sigma \left( \frac{\mu}{k_1} \leftrightarrow \frac{\nu}{k_2} \right) .
\]

Where the \( k \) and \( l \) are \( k, l \in \mathbb{Z} \). The anomaly free condition is then

\[
q^2 = 8k l \pi^2 m^2 = 8n \pi^2 m^2 = m^2_P(n), \quad n = kl \in \mathbb{N} .
\]

Where the \( m_P(n) \) is the neutral pseudoscalar meson mass and the \( m \) is the quark mass in meson.

## 3 The Bound States of Massive Thirring Model

We first discuss the ground state of massive Thirring model [14] which is a theory of a fermion field \( \psi \) in (1+1)-dimensional space-time, with the Lagrangian density

\[
\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - M) \psi - \frac{g}{2} (\bar{\psi} \gamma^\mu \psi)^2 .
\]

Coleman [15] proved that the zero-charge sector of the massive Thirring model is equivalent to the sine-Gordon theory with the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^4}{\lambda} \left[ \cos \left( \sqrt{\frac{\lambda}{m}} \phi \right) - 1 \right] .
\]

The sine-Gordon kink mass and the breather mass are given by [16]

\[
M_{kink} = \frac{8m}{\gamma'} + O(\lambda/m), \quad M_n = \frac{16m}{\gamma'} \sin \left( \frac{n \gamma'}{16} \right), \quad n = 1, 2, \ldots, < \frac{8\pi}{\gamma'} ,
\]
where \( y' = \frac{\lambda}{m^2} (1 - \frac{\lambda}{8\pi m^2})^{-1} \). To examine the spectrum for \( \lambda \) just below \( 4\pi m^2 \), we write
\[
\frac{\lambda}{4\pi m^2} = \frac{1}{1 + g/\pi}.
\] (15)

The energy of the lowest breather mode can be written as
\[
M_1 = M_{kink}[2 - g^2 + \frac{4g^3}{\pi} + O[g^4]].
\] (16)

The weak-coupling \( \frac{\lambda}{m^2} \to 0 \) limit of the sine-Gordon theory would correspond to the infinity strong-coupling \( g \to \infty \) limit of the massive Thirring model, while the \( g \to 0 \) weak-coupling limit of the latter would correspond to the strong coupling limit \( \frac{\lambda}{m^2} \to 4\pi \).

To check the correspondences, we study the bound states of massive Thirring model in the weak-coupling limit \( g \to 0 \). We review some results on the bound states of massive Thirring model which have been discussed in [17] by the chain approximation. The Fourier component of the propagator for a scalar state is
\[
\Delta(k) = \frac{1}{i(2\pi)^2} \int d^2x <0 | T \bar{\psi}(x)\psi(x) \cdot \bar{\psi}(y)\psi(y) | 0 > e^{-ik(x-y)} = \frac{\Pi_s(k^2)}{1 + \frac{g_0^2}{\pi} \Pi_s(k^2)}. \] (17)

Where the \( \Pi_s(k^2) \) is given by
\[
\Pi_s(k^2) = \frac{1}{i} \int \frac{d^2q}{(2\pi)^2} \text{Tr} \left( \frac{1}{(q + k/2 - M)(q - k/2 - M)} \right). \] (18)

The poles of the propagator represent the energy eigenvalues of a scalar bound states
\[
1 + \frac{g_0^2}{2\pi} \Pi_s(k^2) = 0. \] (19)

The \( \Pi_s(k^2) \) has ultraviolet divergence.

To remove the divergence, we need renormalization. One method is to make the replacement \( \Pi_s(k^2) \to \hat{\Pi}_s(k^2) = \Pi_s(k^2) - \Pi_s(0) \) and obtain
\[
\hat{\Pi}_s(k^2) = \frac{1}{2\pi} \int_0^1 dx \ln(1 - x(1-x)^2)
\]
\[
= \frac{1}{\pi} \sqrt{\frac{4}{t^2} - 1} \arctan\left( \frac{t}{\sqrt{4 - t^2}} \right) - 1], \] (20)

where we have defined the new variable \( t = \sqrt{\frac{k^2}{M^2}} \). If we only consider the fermion contributions, the energy eigenvalue (19) has no solutions of bound states [17] in the weak coupling limit. The reason is that
\[
1 + \frac{g_0^2}{2\pi} \hat{\Pi}_s(k^2) > 0 \text{ for } g_0 \to 0. \] (21)

The contour integral along the semi-infinite branch cut of the function \( f(x) = \ln(1 - x(1-x)t^2) \) is infinite for \( t < 2 \) (Fig. 4), we need to take into account the soliton contribution and apply the formulae (10) to obtain the energy eigenvalues of bound states (Fig. 5).

Then the energy eigenvalues of the bound states are solutions of the equation
\[
1 + \frac{g_0^2}{2\pi} \Pi_s(k^2) + \frac{g_0}{2\pi} \Pi_{s,\text{soliton}}(k^2) = 1 + \frac{g_0}{4\pi} \int_0^1 dx \ln(1 - x(1-x)t^2) - \frac{g_0}{4\pi} \int_0^1 dx \ln(1 - x(1-x)t^2) + \frac{g_0}{4\pi} \ln \frac{M^2}{\tilde{M}^2} = 0.
\] (22)
Fig. 4 The branch cuts of function $f(x) = \ln(1 - x(1 - x)t^2)$ with different $t$. The contour integral along the semi-infinite branch cut is infinite for $t < 2$.

Where the $\Pi^\text{soliton}(\tilde{t}^2)$ is the soliton contribution and the new variable $\tilde{t}$ is defined as $\tilde{t} = \sqrt{g_0^2 \ln \frac{M^2}{M^2}}$. The term $\frac{g_0}{4\pi} \ln \frac{M^2}{M^2}$ has the property that is $\frac{g_0}{4\pi} \ln \frac{M^2}{M^2} \to 0$ for small $g_0$. We defined the renormalized coupling constant as following
\begin{equation}
\frac{g_0}{4\pi} \ln \frac{M^2}{M^2} = \frac{g_0}{1 + \frac{g_0}{4\pi} \ln \frac{M^2}{M^2}}. \tag{23}
\end{equation}

The (22) becomes
\begin{equation}
1 + \frac{g}{4\pi} \int_{C[0,1]} dx \ln(1 - x(1 - x)t^2) - \frac{g}{4\pi} \int_{C[0,1]} dx \ln(1 - x(1 - x)\tilde{t}^2) = 0. \tag{24}
\end{equation}

The soliton mass $\tilde{M}$ of massive Thirring model can be obtained by the inverse scattering method [18]. From the Lagrangian density (12), we can obtain the equation of motion
\begin{equation}(i\gamma^\mu \partial_\mu - M)\psi - g(\bar{\psi}\gamma^\mu \psi)\gamma_\mu \psi = 0. \tag{25}\end{equation}

The equation has a classical solution which behaves as an elementary particle with mass $\tilde{M} = \frac{2M}{g}$. The quantum corrections to the soliton mass can be obtained similar to the sine-Gordon kink mass which is [18]
\begin{equation}
\frac{\tilde{M}}{M} = \frac{2}{g} - \frac{1}{\pi} + \frac{3g}{4} + \frac{17g^2}{24\pi} + O[g^3]. \tag{26}
\end{equation}

Fig. 5 The bound states have two main contributions. The soliton mass $\tilde{M}$ is $\tilde{M} \approx \frac{2M}{g}$.
According to the formulae (10) and the contour integral along the branch cuts (Fig. 6), the (24) becomes

\[
1 + \frac{g}{2\pi} \left[ \sqrt{\frac{4}{t^2} - 1} \arctan\left( \frac{t}{\sqrt{4 - t^2}} \right) - 1 \right] - \frac{g}{2\pi} \left[ \sqrt{\frac{4}{\tilde{t}^2} - 1} \arctan\left( \frac{\tilde{t}}{\sqrt{4 - \tilde{t}^2}} \right) - 1 \right]
- \frac{g}{2\sqrt{t^2}} - 1 + \frac{g}{2\sqrt{\tilde{t}^2}} - 1 = 0.
\] (27)

From the (26) and (27), we obtain the ground state mass

\[
M_{\text{bound}} = M \left[ 2 - g^2 + \frac{4g^3}{\pi} + O[g^4] \right].
\] (28)

Which is consistent with the lowest breather mass of sine-Gordon model given by (16).

4 Positronium \((e^+e^-)\) Systems in QED

Then we study the positronium \((e^+e^-)\) bound states. We review some results of QED which can be found in famous papers and text books [8, 9, 19, 20]. We define \(i \Pi^{\mu\nu}(q)\) to be the sum of all 1-particle-irreducible (1PI) insertions into the photon propagator.

The \(\Pi^{\mu\nu}(q)\) has the tensor structure:

\[
\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi(q^2).
\]

Fig. 6 After taking into account the soliton contribution, the contour integrals along the branch cuts become finite.
The $\alpha (\alpha = \frac{e^2}{4\pi})$ contribution to $i \Pi^{\mu \nu}(q) = i(P_2^{\mu \nu}(q) = i(q^2 g^{\mu \nu} - q^{\mu} q^{\nu}) \Pi_2(q^2)$ coming from the electron loop (Fig. 7), where $\Pi_2(q^2)$ is

$$\Pi_2(q^2) = -8e^2 \int_0^1 dx (1 - x) \int d^4l_E \frac{1}{(2\pi)^4 (l_E^2 + \Delta)^2}$$

$$= -\frac{2\alpha}{\pi} \int_0^1 dx (1 - x) \int_0^\infty \frac{ydy}{(y + \Delta)^2}. \quad (29)$$

The $\Delta$ is $\Delta = m_e^2 - x(1 - x)q^2$. The excited spectrums are the poles of the two-point function

$$G_{\mu \nu}(q^2) = \frac{-i g_{\mu \nu}}{q^2 (1 - \Pi(q^2))}.$$  

The two-point function has ultraviolet divergence. We choose our new scheme to regulate the ultraviolet divergence

$$\tilde{G}_{\mu \nu}(q^2) = \frac{-i g_{\mu \nu}}{q^2 (1 - [\Pi(q^2, m_e) - \Pi(q^2, \tilde{m}_e)])}. \quad (30)$$

Where the $\tilde{m}_e$ is the mass of the soliton in QED that is $\Pi_2^{\text{soliton}}(q^2, \tilde{m}_e) = -\Pi(q^2, \tilde{m}_e)$. We consider the positronium ($e^+e^-$) bound states which correspond to the case $t < 2$. To simplify our discussion, we define the new variable $t = \sqrt{\frac{q^2}{m_e^2}}$. Similar to the massive Thirring model, the two-point function (30) hasn’t the pole of bound state in the weak coupling limit. Applying the formula (10), we also can’t find the pole of bound state. The reason is that the contour integral along the semi-infinite branch cut of the function $f(x) = \ln(1 - x(1 - x)t^2)$ is infinite for $t < 2$ (Fig. 4). We need to take into account the soliton contribution $\Pi_2^{\text{soliton}}(q^2, \tilde{m}_e)$ (Fig. 8).

We define the $\hat{\Pi}_2^{\text{soliton}}(q^2)$ as

$$\hat{\Pi}_2^{\text{soliton}}(q^2) \equiv -(\Pi_2(q^2, \tilde{m}_e) - \Pi_2(0, \tilde{m}_e))$$

$$= \frac{2\alpha}{\pi} \int_{C[0,1]} dxx (1 - x) \log \left( \frac{1}{1 - x(1 - x)} \right).$$

**Fig. 7** The order $\alpha$ contribution to the photon propagator.
The bound states have two main contributions. The mass of soliton in QED is difficult to find. We need a new way to calculate it. The photon in bound state becomes massive, so we consider the 4-Fermi effective theory (Fig. 9) with Lagrangian density

$$\mathcal{L}(\psi, \bar{\psi}) = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi + \frac{G_F}{2} (\bar{\psi} \psi)^2,$$

(31)

which is known as the massive Gross-Neveu (GN) model [21]. In QED, the mass $m$ is $m_e$ and the coupling constant $G_F$ is $G_F \approx \frac{e^2}{2m_e^2}$. The Lagrangian of Gross-Neveu (GN) model (31) can be re-written by using a scalar auxiliary field $\sigma(x)$:

$$\mathcal{L}(\psi, \bar{\psi}, \sigma) = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi - \sigma \bar{\psi} \psi - \frac{1}{2} G_F \sigma^2. \quad (32)$$

The (32) is classically equivalent to (31) by the equation of motion for $\sigma(x)$. The Lagrangian (32) is purely quadratic in the fermion field we can integrate directly

$$\int D\bar{\psi} D\psi D\sigma \exp[i \int d^d x \mathcal{L}(\psi, \bar{\psi}, \sigma)] = \int D\sigma \exp[i \int d^d x (\frac{1}{2 G_F} \sigma^2) + \text{Tr} (\log (i \gamma^\mu \partial_\mu - m - \sigma))].$$

(33)

Then the effective theory can be calculated by the method of steepest descents. The vacuum expectation value $M$ of the field $\sigma(x)$ is the solution of the stationary condition (gap equation)

$$- \frac{M}{G_F} + \int \frac{d^d p}{(2\pi)^d} \text{Tr} (\frac{i}{p - m - M}) = 0. \quad (34)$$

We find a new method to deal with the gap (34) by using the formulae (10). From the gap (34), we can obtain the mass of soliton.

Let us briefly recall the results of GN model with $N$ Dirac fermions in $d = 1 + 1$ dimensions. The solitons in the massless ($m = 0$) GN model are Callan-Coleman- Gross-
Zee (CCGZ) kink and DHN saddle point configurations (fermion bag solitons) [22] in the large \( N \) limit. The ground state is fixed by the gap equation

\[
- \frac{M_0}{G_F} + N \int \frac{d^d p}{(2\pi)^d} \text{Tr}(\frac{i}{p - M_0}) = 0, \tag{35}
\]

in which \( M = M_0 \) is constant. The dynamically generated mass of small fluctuations of the Dirac fields around this vacuum is

\[
m = M_0 = \tilde{\mu} \exp\left\{ -\frac{\pi}{\alpha_F(p^2)} \right\}, \tag{36}
\]

where the \( \tilde{\mu} \) is the renormalization scale. The mass of CCGZ kink is

\[
M_{\text{kink}} = \frac{m N}{N G_F} = \frac{m}{G_F} = \frac{m N}{\pi}, \tag{37}
\]

where the \( N G_F = \pi \) by choosing the renormalization point at \( \tilde{\mu} = m \). The mass of fermion bag soliton is

\[
M_{\text{soliton}}(n) = \frac{2m N}{\pi} \sin\left( \frac{\pi n}{2N} \right), \tag{38}
\]

which is the bound state of \( n \) (\( n \in N \)) kinks. The solitons in the 1+1 dimensional massive generalization of the GN model were carried in[23] with masses

\[
M_{\text{soliton}}(v) = Nm\left( \frac{2}{\pi} \sin \theta + \gamma \log \frac{1 + \sin \theta}{1 - \sin \theta} \right). \tag{39}
\]

The \( \theta \) is a function of the filling fraction \( \nu = \frac{N_f}{N} \) and \( \gamma \) is the renormalization group invariant ratio \( \gamma = \frac{m_0}{m} \ln \frac{\Lambda}{m} \). In massive (\( m \neq 0 \)) GN model, the CCGZ kink becomes infinitely massive and disappears by classical analysis. We argue that there is quantum effect to allow the CCGZ kink exist, which is quantum soliton [24].

We now use a new method to find mass of soliton from gap (34). In our approach, we adopt the new regularization scheme. We let \( d = 2 - 2\varepsilon \) and introduce a parameter \( \mu \) with dimension of mass. Unlike the \( \tilde{\mu} \) in (36) which is an renormalization scale, the \( \mu \) is determined by the gap (34). If we consider mass \( m \) to be the physical mass of fermion, then the \( M \) is pure imaginary. Similar to the imaginary part of complex charge \( Z \) (4), the \( M \) connects with the soliton mass. We assume that the \( M \) is \( M = i\tilde{m} \), then the (34) becomes

\[
- \frac{i\tilde{m}}{G_F} = -2(m + i\tilde{m}) i \mu^{2\varepsilon} \int \frac{d^d p}{(2\pi)^d} \left( \frac{1}{p^2 - (m + i\tilde{m})^2} \right) = \frac{1}{2\pi}(m + i\tilde{m})(\frac{\Gamma(1 - \frac{\varepsilon}{2})}{((\frac{m + i\tilde{m}}{\mu})^2)^{1 - \frac{\varepsilon}{2}}} - 2\pi\tilde{n}i). \tag{40}
\]

The function \( \frac{\Gamma(1 - \frac{\varepsilon}{2})}{((\frac{m + i\tilde{m}}{\mu})^2)^{1 - \frac{\varepsilon}{2}}} \) has a \( \log((\frac{m + i\tilde{m}}{\mu})^2) \) term which is multi-valued, so we have the \( 2\pi\tilde{n}i \) term with \( \tilde{n} \in \mathbb{Z} \) in the expression (40). The soliton mass satisfies \( \tilde{m} \gg m \) in the weak coupling limit, then the (40) is equivalent to

\[
\begin{align*}
\begin{cases}
m\tau + \tilde{m}\tilde{n} = 0, & \text{for } (\mu^2 < 0) \quad \text{and} \quad \frac{\tilde{m}}{G_F} = \tilde{m} - \tau \tilde{m}, \\
m\tau + \tilde{m}(\tilde{n} - 1/2) = 0, & \text{for } (\mu^2 > 0).
\end{cases}
\end{align*}
\tag{41}
\]
Where the \( \tau \) is defined as \( \tau = \text{Re} \left[ \frac{1}{2\pi} \frac{\Gamma(1-\frac{d}{2})}{(\frac{\alpha}{\mu}+imn)^{1-\frac{d}{2}}} \right] \). The extra term \( 1/2 \) comes from the relation \( \log(\mu^2) = \log \mu^2 + i\pi \). The solutions of the (40) have two branches:

\[
m^2 = \begin{cases} \frac{m}{nG_F} & \text{for } \mu^2 < 0; \\ \frac{m}{(n-1/2)G_F} & \text{for } \mu^2 > 0. 
\end{cases}
\]

(42)

Where the \( n = |\tilde{\mu}| \in \mathbb{N} \). Then we obtain \( \tilde{m} \gg m \) in the weak coupling limit

\[
\tilde{m} \approx \begin{cases} \frac{m}{nG_F} & \text{CCGZ kink (quantum soliton)}; \\ \frac{m}{(n-1/2)G_F} & \text{fermion bag soliton}. 
\end{cases}
\]

(43)

The solutions (43) are the quantum solitons [24] which also include the classical configurations. When \( n = 1 \), it leads to the mass of CCGZ kink (37) and fermion bag soliton (39) at kink-antikink threshold in leading order term.

We now consider the soliton mass of massive GN model in four dimensions. As the same as massive GN model in the \( 1+1 \) dimensions, we let \( d = 4 - 2\varepsilon \) and \( M = i\tilde{m} \). Then the gap (34) becomes

\[
-\frac{i\tilde{m}}{G_F} = -4(m + i\tilde{m})i\mu \epsilon \int \frac{d^dp}{(2\pi)^d} \left( \frac{1}{p^2 - (m + i\tilde{m})^2} \right) = \frac{(m + i\tilde{m})^3}{4\pi^2} \left( \frac{\Gamma(1-\frac{d}{2})}{(m + i\tilde{m})^2 - \frac{d}{2}} + 2\pi\tilde{m}\epsilon \right).
\]

(44)

Similar to the \( 1+1 \) dimensions GN model, the solutions of (44) have two branch solutions

\[
\tilde{m} \approx \begin{cases} \sqrt[3]{\frac{6\pi m}{nG_F}} \approx m \sqrt[3]{\frac{3}{n\alpha}} & \text{for } \mu^2 < 0; \\ \sqrt[3]{\frac{6\pi m}{(n-1/2)G_F}} \approx m \sqrt[3]{\frac{3}{(n-1/2)\alpha}} & \text{for } \mu^2 > 0. 
\end{cases}
\]

(45)

Where the \( n = |\tilde{\mu}| \in \mathbb{N} \). The soliton solutions (45) with mass \( m \sqrt[3]{\frac{3}{n\alpha}} \) are similar to the CCGZ kinks in two dimensions. Then the leading term of the soliton mass \( \tilde{m}_e \) in QED is \( \tilde{m}_e \sim m_e \sqrt[3]{\frac{3}{n\alpha}} \). To obtain the correct bound states, the other contributions of \( \tilde{m}_e \) are needed. We find

\[
\frac{\tilde{m}_e^2(\alpha, n, a)}{m_e^2} = \Gamma(\alpha, n, a) = -\frac{(\alpha^2 - 8n^2)^2}{n^4} + \frac{(\alpha^3 - 8an^2)^4}{n^4 (\alpha, n, a)^{4/3}} + \frac{\gamma(\alpha, n, a)^{1/3}}{n^4 \alpha^4} = -\frac{\left( \frac{3}{\alpha} \right)^{2/3} \sqrt{n^2}}{\alpha^{2/3}} - 1 + \frac{\left( \frac{2}{\alpha} \right)^{2/3} \alpha^{2/3}}{\sqrt[3]{\frac{3}{n^2}}} + O[\alpha]^{4/3} = \Omega(n\alpha, a) + O[\alpha]^{4/3}.
\]

(46)

Where the \( \gamma(\alpha, n, a) \) is defined as

\[
\gamma(\alpha, n, a) \equiv \alpha^{12} \left( \alpha^2 - 8n^2 \right)^6 - 2a^2\alpha^{10} \left( \alpha^2 - 16n^2 \right) \left( \alpha^4 + 96n^4 - 16\alpha^2 n^2 \right)^2 + 4096n^{12} \\
\sqrt[4]{\alpha^{16} \left( -2a^2 \left( \frac{\alpha^2}{4n^2} \right)^6 + 2a^2 \left( \frac{\alpha^2}{4n^2} \right)^2 + 2\alpha^2 + \alpha^4 \left( \frac{\alpha^2}{4n^2} \right)^6 \right)^2 - \alpha^8 \left( \frac{\alpha^2}{4n^2} \right)^{12}}.
\]

(47)
The function $\Omega(x, a)$ is defined as

$$\Omega(x, a) = \left( \frac{3a^2}{2} \right)^{2/3} + \left( \frac{2x}{3a} \right)^{2/3} - 1. \quad (48)$$

The constant $a$ is $a \approx 2.00408$ for positronium ($e^+e^-$) bound states. The soliton mass $\tilde{m}_e$ corresponding to the ground state is $\tilde{m}_e(1) \approx 7.4m_e$. Similar to the above discussion, we present a soliton interpretation to the hypothetic $X_{17}$ particle which has mass $m_{X_{17}} \approx 32.681m_e$ [25]. Anomaly in decay of $^8$Be and $^4$He is interpreted as a signature of a decay via emission of neutral boson $X_{17}$. To discuss the $X_{17}$, we need to study the photon interacting with up quarks. The effective coupling constant $\alpha_u$ is $\alpha_u = N_c Q_f^2 \alpha$, where $N_c = 3$ is color factor and $Q_f = 2/3$ is the electric charge of the up quark. Using the same formulae as (46), we obtain the soliton mass $\tilde{m}_u \approx 31.4093m_e \approx m_{X_{17}}$. The almost same mass indicates that the hypothetic $X_{17}$ particle might be a soliton (quantum soliton).

According to the formulae (10) and the contour integral along the branch cuts (Figs. 6 and 10), the energy spectrum of positronium is the solution of the equation

$$1 - \Pi_2(t^2) - \Pi_2^{\text{soliton}}(t^2) \approx 1 - \Pi_1(t^2) - \Pi_1^{\text{soliton}}(t^2)$$

$$\approx 1 + \frac{\alpha(t\sqrt{4 - t^2}(12 + 5t^2) + 6(-8 - 2t^2 + t^4) \arctan \left( \frac{t}{\sqrt{4 - t^2}} \right))}{9\pi t^5 \sqrt{4 - t^2}} - 4\pi \int_{x_1}^{\tilde{x}_2} dxx(1 - x)$$

$$= 1 + \frac{\alpha(t\sqrt{4 - t^2}(12 + 5t^2) + 6(-8 - 2t^2 + t^4) \arctan \left( \frac{t}{\sqrt{4 - t^2}} \right))}{9\pi t^3 \sqrt{4 - t^2}} - \frac{2an}{3\alpha} \sqrt{\frac{4}{t^2} - 1}(1 + \frac{2}{t^2}) = 0. \quad (49)$$

The $x_{1,2} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{4\Gamma(\alpha,n,a)}{t^2}}$ are the solutions of the equation $\Delta = \tilde{m}_e^2 - x(1 - x)q^2 = 0$. The solution is consistent with the known result of positronium to high precision. To

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig10.png}
\caption{Two branch cuts can be approximated as one single branch cut in weak-coupling region.}
\end{figure}
### Table 1 Comparing with the known results

| n  | $t - 2$            | $-\frac{\alpha^2}{4n^2}$ |
|-----|--------------------|----------------------------|
| 1   | $-1.33128 \times 10^{-5}$ | $-1.33128 \times 10^{-5}$ |
| 2   | $-3.32830 \times 10^{-6}$ | $-3.32821 \times 10^{-6}$ |
| 3   | $-1.47926 \times 10^{-6}$ | $-1.47920 \times 10^{-6}$ |
| 4   | $-8.32084 \times 10^{-7}$ | $-8.32052 \times 10^{-7}$ |
| 5   | $-5.32534 \times 10^{-7}$ | $-5.32514 \times 10^{-7}$ |

To illustrate this, we compare the $t-2$ of the solution of (49) with the result of (50) in Table 1 with $\alpha^{-1} = 137.03599913$. The excited spectrum of positronium to order $\alpha^2$ is [20]

$$E_n = 2m_e - m_e \frac{\alpha^2}{4n^2}$$

(50)

We discuss the physical meaning of non-perturbative term. The leading term of (49) is

$$1 - 4n\alpha \int_{\bar{x}_1}^{\bar{x}_2} dx x(1-x) = 1 - \frac{2n\alpha}{3} \sqrt{\frac{4\Gamma(\alpha, n, a)}{t^2}} - 1(1 + \frac{2\Gamma(\alpha, n, a)}{t^2})$$

$$\approx 1 - \frac{2n\alpha}{3} \sqrt{\frac{4\Omega(n\alpha, a)}{t^2}} - 1(1 + \frac{2\Omega(n\alpha, a)}{t^2}) = 1 - \frac{2an}{3\alpha} \sqrt{\frac{4}{t^2}} - 1(1 + \frac{2}{t^2}) = 0.$$

We can view the $n\alpha$ as the effective coupling constant $\bar{\alpha}$. The leading term of $\frac{m_e^2(\alpha,n,a)}{\Omega(\alpha,n,a)}$ (46) that is $\Omega(n\alpha, a) = \Omega(\bar{\alpha}, a)$ depends on the effective coupling constant $\bar{\alpha}$. The soliton contribution of positronium ($e^+e^-$) bound state in weak-coupling is equivalent to the system of ($e^+e^-$) in strong-coupling (Fig. 11). This is also a kind of weak-strong transformation similar to electric-magnetic duality [5, 26].

### 5 Vector Meson Spectrum in QCD

Finally, we discuss the vector meson spectrum in QCD. The gluon-gluon interactions in QCD have no analogue in QED, and it can be shown that they lead to properties of the strong interaction that differ significantly from those of the electromagnetic interaction. These properties are colour confinement and asymptotic freedom. The colour confinement

![strong coupling](e+)

![weak coupling](s)

**duality**

Fig. 11 The soliton contribution of positronium ($e^+e^-$) bound state in weak-coupling is equivalent to the system of ($e^+e^-$) in strong-coupling
have many theories, e.g. the dual superconductor picture of confinement [27]. We discuss the vector meson spectrum which is similar to the positronium.

The $i \Pi_2^{\mu,\nu}(q)$ has the following structure

$$i \Pi_2^{\mu,\nu}(q) = i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab} \Pi_2(q^2).$$

Where $\Pi_2(q^2)$ is calculated from Fig. 12 [8]

$$\Pi_2(q^2) = \Pi_2^{f}(q^2) + \Pi_2^{g}(q^2).$$

Where the $\Pi_2^{f}(q^2)$ is the fermion loop diagram contribution and $\Pi_2^{g}(q^2)$ is the three diagrams of Fig. 12 from the gluon sector.

$$\Pi_2^{f}(q^2) = \frac{-2C(r)n_f \alpha_g}{\pi} \int_{C[0,1]} dxx(1-x)\frac{\Gamma(2 - \frac{d}{2})}{(m_q^2 - x(1-x)q^2)^{2-\frac{d}{2}}}$$

$$\Pi_2^{g}(q^2) = \frac{C_2(G)\alpha_g}{4\pi} \int_{C[0,1]} dx\frac{\Gamma(2 - \frac{d}{2})}{(-x(1-x)q^2)^{2-\frac{d}{2}}}[(1 - \frac{d}{2})(1-2x)^2 + 2].$$

Here $\alpha_g = \frac{g^2}{4\pi}$, the $n_f$ is the number of fermion species, $C(r)$ is $C(r) = \frac{1}{2}$ for fundamental representations, $C_2(G)$ is $C_2(G) = N$ for $SU(N)$ and the $m_q$ is the free quark mass. We then consider the soliton contribution. Similar to QED, we first present soliton contributions $\Pi_2^{f,s}(q^2)$

$$\Pi_2^{f,s}(q^2) = \frac{2C(r)n_f \alpha_g}{\pi} \int_{C[0,1]} dxx(1-x)\frac{\Gamma(2 - \frac{d}{2})}{(\tilde{m}_q^2 - x(1-x)q^2)^{2-\frac{d}{2}}}$$

which comes from the dual field of the fermion. When come to the dual soliton of the gluon, we find the different case. In non-Abelian gauge theories, the Pauli-Villars regularization does not work, which is a gauge invariant regularization but not a gauge covariant regularization. To preserve the gauge symmetry, we don’t have the soliton contribution corresponding to gluon (or the mass of soliton being infinite). This leads to the colour confinement. Then the energy eigenvalues of the bound states are solutions of equation

$$1 - \Pi_2^{f}(q^2) - \Pi_2^{g}(q^2) - \Pi_2^{f,s}(q^2) = 0.$$  (53)

The leading contribution of (53) is

$$C(r)n_f \frac{2nf(\alpha_g)}{3} \sqrt{\frac{4}{t^2} - 1 + \frac{2}{t^2}} - b_q = 0.$$  (54)

Fig. 12  The gluon propagator in order $g^2$
Table 2 The theoretical and experimental values for vector meson $\rho$ masses. The mass $m_{th}(n) \sim (1631.5578086364482 n^{1/3} - 856.0578086364481)(GeV)$

| $\rho$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|--------|-----|-----|-----|-----|-----|-----|-----|
| $m_{ex}(GeV)$ | 0.7755 | 1.282 | 1.465 | 1.720 | 1.909 | 2.149 | 2.265 |
| $m_{th}(GeV)$ | 0.7755 | 1.200 | 1.497 | 1.733 | 1.934 | 2.109 | 2.265 |
| error  | 0%  | 6%  | 2%  | 0.7% | 1.3% | 1.9% | 0%  |

The concrete expression of the function $f(\alpha_g)$ which is obtained from the (53) is unimportant for our discussion. The constant $a_q$ and $b_q$ are defined as $a_q = C(r) \frac{2f(\alpha_g)}{3}$ and $b_q = \Pi_2(q^2) \propto \Gamma(2 - \frac{d}{2}) \to \infty$. We obtain the solution of the (54) which is

$$t^2(n) \approx \left(\frac{4na_q}{b_q}\right)^{2/3} \Rightarrow m(n) \approx m_q\left(\frac{4na_q}{b_q}\right)^{1/3}.$$  \hspace{1cm} (55)

The observed physical mass $m_{th}(n)$ is $m_{th}(n) = m(n) - c_q = m_q\left(\frac{4na_q}{b_q}\right)^{1/3} - c_q$, where the $c_q$ and $m_q\left(\frac{4na_q}{b_q}\right)^{1/3}$ (the free quark mass $m_q \to \infty$) can determined by the experimental data fitting. We obtain $c_q \approx 856.1$(GeV) and $m_q\left(\frac{4na_q}{b_q}\right)^{1/3} \approx 1631.6$(GeV) which is consistent with the experimental data [28] (Table 2). It’s difficult to find free quark because $t$ is very small, which leads to the colour confinement. Our new experimental data fitting is as good as Regge theory [29].

6 Conclusions and Discussions

In this paper, we have given a new method to calculate the the excited spectrums of bound states in Quantum Field Theory. There are no correctly bound state poles in the two-point functions at one loop in traditional methods. We need to take into account the soliton contribution which can regulate the ultraviolet divergence. We have discussed the bound state of massive Thirring model, the positronium ($e^+e^-$) in QED and the vector meson in QCD separately. We also gave a new way to obtain the mass of soliton from the stationary condition (gap equation). Our results agree with experimental data to high precision. We argued that the hypothetic $X_{17}$ particle in decay of $^8$Be and $^4$He is a soliton. Especially for vector meson, we found squared masses of $\rho$ resonances are $m^2(n) \sim (an^{1/3} - b)^2$. The new experimental data fitting is as good as Regge theory. To make these results accurate, we need to calculate the higher order loops.

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