Lectures on F-theory compactifications and model building

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Abstract
These lecture notes are devoted to formal and phenomenological aspects of F-theory. We begin with a pedagogical introduction to the general concepts of F-theory, covering classic topics such as the connection to type IIB orientifolds, the geometry of elliptic fibrations and the emergence of gauge groups, matter and Yukawa couplings. As a suitable framework for the construction of compact F-theory vacua we describe a special class of Weierstrass models called Tate models, whose local properties are captured by the spectral cover construction. Armed with this technology we proceed with a survey of F-theory grand unified theory (GUT) models, aiming at an overview of basic conceptual and phenomenological aspects, in particular in connection with GUT breaking via hypercharge flux.

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1. Introduction

String phenomenology is the branch of string theory that takes the theory seriously as a consistent, fundamental description of gravitational and particle interactions in four dimensions. Its primary goal is to understand the solutions of this theory and their implications for physics in four dimensions. This includes the investigation of both the mathematical structure of the space of string vacua and of their concrete particle phenomenological and cosmological properties.

In the perturbative region of the string landscape, two corners exhibit particularly attractive four-dimensional solutions. The heterotic string heavily exploits the availability of exceptional gauge symmetry, into which the observed gauge group of the standard model can be embedded in elegant ways realizing the idea of a grand unified theory (GUT). Many different techniques have been developed to study these solutions in various regions of the moduli space, ranging from heterotic orbifolds and free-fermionic constructions to smooth Calabi–Yau compactifications with vector bundles, see e.g. the review [1] and references
therein. Since both gravity and gauge dynamics descend from the closed string sector, all aspects of four-dimensional physics are sensitive to the global structure of the compactification space.

Perturbative type II orientifolds with D-branes, on the other hand, are based on the classical gauge groups $U(N)$, $SO(N)$ and $Sp(2N)$. This type of constructions can directly accommodate the standard model gauge group, while GUT physics is not immediate. The huge literature on type IIA orientifolds with intersecting D6-branes and on type IIB orientifolds with D7/D3-branes (reviewed e.g. in [2–4]) exploits furthermore the localization of the gauge degrees of freedom along the D-branes, which are therefore of a different physical origin than the gravity sector in the bulk. This opens up the possibility of exploring a certain subclass of phenomenological questions in the context of local models, while other aspects of the associated four-dimensional physics remain sensitive to the full compactification details.

The strict separation between the phenomenological properties of the heterotic and brane constructions ceases to exist at finite values of the string coupling $g_s$. In fact, in all but a minute class of brane constructions $g_s$ is dynamical and varies over the compactification space in such a way as to leave the perturbative regime somewhere. The correct way to describe type IIB compactifications with 7-branes in this generic situation is called F-theory [5]. F-theory is a fascinating subject by itself because it geometrizes the backreaction of the 7-branes on the ambient space and is therefore, in some sense, the way to think about 7-branes. It incorporates certain strong coupling effects with breath-taking elegance. The rich and by now classic literature on the more formal aspects of F-theory reflects the amount to which these non-perturbative phenomena have mesmerized string theorists.

More recently it has been exploited that the four-dimensional solutions of F-theory are also interesting from a phenomenological viewpoint [6–9]. This is because at strong coupling new degrees of freedom—$(p, q)$ strings—become light and realize exceptional gauge symmetries even in a theory based on branes. The perturbative dichotomy between the localization of gauge degrees of freedom and exceptional gauge dynamics is therefore resolved. This bears exciting prospects for GUT model building and has been the subject of fruitful and intensive investigations in recent times. Most importantly, F-theory compactifications also inherit the favourable properties of M-theory and of type IIB orientifolds with respect to moduli stabilization. The combination of 3-brane instanton effects and background fluxes allows in principle for the stabilization of Kähler and complex structure moduli within the framework of warped Calabi–Yau compactifications. This is motivation enough to take F-theory seriously as a promising corner for string phenomenology.

When dealing with F-theory compactifications one must be aware that to date there exists no description of F-theory as a fundamental theory. In this respect F-theory has a status very different from M-theory, which can—at least conjecturally—be conceived as the theory reducing in its long-wavelength limit to 11-dimensional supergravity coupled to membranes. Early ideas [5, 10, 11] to define an analogous 12-dimensional theory whose fundamental objects are 3-branes have not led to a consistent picture. Rather F-theory should be thought of as a genuinely non-perturbative description of a class of string vacua which, in certain limits, is accessible by string dualities from three different corners of the M-theory star. These are

- F-theory as (strongly coupled) type IIB theory with 7-branes and varying dilaton,
- F-theory as dual to $E_8 \times E_8$ heterotic theory,
- F-theory as dual to M-theory on a vanishing $T^2$.

Of these three, the F/M-theory duality probably captures the dynamics in the most general way. While particularly fruitful for concrete applications, the first two dualities can sometimes
be misleading because they describe only certain aspects of the dynamics of a typical F-theory compactification.

These lectures intend to provide a pedagogical introduction to some of the technology and the phenomenological applications of F-theory model building. A complimentary set of lecture notes covering general aspects of F-theory is [12], while the specifics of F-theory GUT model building are surveyed in the review [13].

Section 2 aims at a pedagogical introduction to the main concepts of F-theory. In section 2.1 we begin with a definition of F-theory from the perspective of type IIB orientifolds with 7-branes. The F/M-theory duality is briefly sketched in section 2.2. Given the geometric nature of F-theory, it is crucial to understand the geometry of elliptic fibrations, which we approach in an elementary manner in section 2.3. The connection to the type II orientifold picture is made via Sen’s orientifold limit in section 2.4. A closer investigation of the appearance of gauge degrees of freedom from the singularities of the elliptic fibration concludes our first encounter with F-theory in section 2.5.

In section 3 we introduce some of the more advanced technology of F-theory compactifications. The basic algorithm to read off the gauge groups from a given model is presented in section 3.1. The corresponding Tate models represent a convenient framework for F-theory compactifications. Besides the pure geometry, the gauge flux is an essential ingredient in F-theory models (see section 3.2). Note that it is this aspect that is currently the least understood. An account of charged matter and Yukawa couplings follows in section 3.3. In section 3.4 we recall basic aspects of F-theory/heterotic duality; an important ingredient is the construction of vector bundles on elliptically fibred Calabi–Yau 3-folds via the spectral cover construction. Section 3.5 concludes our tour through the model-building rules of F-theory with a review of the recent application of the spectral cover construction in general F-theory models with no heterotic dual.

In the final part of these lecture notes we apply what we have learned in sections 2 and 3 to the construction of $SU(5)$ GUT vacua. This is now a vast and dynamical field, and rather than aiming at completeness we focus on some fundamental aspects. After briefly describing the decoupling idea underlying local models in section 4.1, we investigate various options for GUT symmetry breaking in section 4.2. GUT breaking by hypercharge flux is critically assessed in section 4.3. The experimentum crucis in GUT model building is proton decay, and we explain some of the challenges within the F-theory context in section 4.4. An incomplete list of further phenomenological topics can be found in section 4.5.

2. A first encounter with F-theory

2.1. The need for a non-perturbative formulation of type IIB with 7-branes

2.1.1. Backreaction from 7-branes. In this section we approach F-theory as the strong coupling limit of type IIB orientifolds with O7/O3-planes and D7/D3-branes. For a background on type II orientifolds we refer to existing reviews such as [2–4]. The usual philosophy in the description of perturbative type II orientifolds with D-branes is to neglect the backreaction of the branes and the orientifold planes on the background geometry in the spirit of a probe approximation. This is justified as long as asymptotically away from the brane the backreaction becomes negligible. In this case, one can consider a large volume limit in which the knowledge of the detailed form of the solution is not required at least to understand the main properties of the string vacuum.

To see when this approximation is justified, we consider a $p$-brane in ten dimensions. It represents a source term in the normal $n = 9 − p$ spatial directions. At a heuristic level, this
leads to a Poisson equation for the background fields sourced by the brane. Schematically denoting these sourced fields as $\Phi_1$, one can write this as

$$\Delta \Phi(r) \simeq \delta(r) \implies \Phi(r) \simeq \frac{1}{r^{n-2}} \quad n > 2.$$  \hfill (1)

More precisely, within type II supergravity the BPS solution for a stack of $N_p$-branes along directions $\mu = 0, 1, \ldots, p, p < 7$, takes the form [14]

$$d s^2 = H^{-1/2} \eta_{\mu \nu} \, d x^\mu \, d x^\nu + H^{1/2} \sum_i d x^i \, d x^i,$$  \hfill (2)

$$e^{2\phi} = e^{2\phi_0} H^{1-p \over p}, \quad C_{p+1} = H^{p-1 \over e^{\phi_0}} \, d x^0 \wedge \cdots \wedge d x^p,$$  \hfill (3)

$$H_p = 1 + \left(\frac{r_p}{r}\right)^{(7-p)} \quad r_p^{(7-p)} = \text{const} \, e^{\phi_0} N.$$  \hfill (4)

Here $\phi$ denotes the ten-dimensional dilaton and the Ramond–Ramond (RR) $(p+1)$-form potential $C_{p+1}$ couples electrically to the $p$-brane. The backreaction is governed by the harmonic function $H_p$, which asymptotes to unity away from the brane. In particular, $e^{\phi_0}$ denotes the asymptotic value of $e^\phi$ and can be taken as the value of the string coupling $g_s$ relevant in the large volume limit.

However, the above logic goes through only if the codimension of the brane $n > 2$. The critical case $n = 2$ corresponds precisely to D7-branes in type IIB theory. A D7-brane along, say, dimensions $0, \ldots, 7$ looks like a charged point particle localized in the two normal directions 8, 9—a cosmic string [15]. Solutions to the two-dimensional Poisson equation scale logarithmically with the distance to the source. Such a logarithmic profile is in sharp contrast with the favourable asymptotics for lower dimensional branes encountered above.

Let us see how this heuristic argument applies in more detail. Recall, e.g., from [4] that the string frame type IIB effective action in the democratic formulation is given by

$$S^{(S)}_{\text{IIB}} = \frac{2\pi}{\ell_s^2} \left( \int d^{10} x \, e^{-2\phi} \left( -\frac{1}{2} \sqrt{-g} R + 4 \partial_\mu \phi \partial^\mu \phi \right) - \frac{1}{2} e^{-2\phi} \int H_3 \wedge * H_3 
- \frac{1}{4} \sum_{p=0}^4 \int F_{2p+1} \wedge * F_{2p+1} - \frac{1}{2} \int C_4 \wedge H_3 \wedge F_3 \right).$$  \hfill (5)

Here $\ell_s = 2\pi \sqrt{\alpha'}$ and the field strengths are defined as

$$H_3 = d B_2, \quad F_1 = d C_0, \quad F_3 = d C_2 - C_0 d B_2,$$

$$F_5 = d C_4 - \frac{1}{2} C_2 \wedge d B_2 + \frac{1}{2} B_2 \wedge d C_2,$$  \hfill (6)

supplemented by the duality relations $F_6 = * F_1, F_7 = - * F_3, F_5 = * F_5$ at the level of equations of motion.

The RR-field sourced electrically by a D7-brane is $C_8$ and is dual to the axion $C_0$ that combines with the string coupling $g_s = e^\phi$ into the complex axio-dilaton

$$\tau = C_0 + \frac{i}{g_s}.$$  \hfill (7)

The D7-brane action is the sum of the two terms

$$S_{\text{DBI}} = -\frac{2\pi}{\ell_s^8} \int d^7 \xi \, e^{-\phi} \sqrt{\det(g + 2\pi \alpha' F/2\pi)}.$$  \hfill (8)
where $2\pi\alpha' F = 2\pi\alpha' F + B_2$ in terms of the Yang–Mills field strength $F$ and the last factor denotes curvature contribution in terms of the A-roof-genus (see e.g. [16]) of the tangent and normal bundle to the D7-brane. Define the complex coordinate $z = x^8 + i x^9$ for the dimensions perpendicular to the 7-brane. Taking into account constraints from supersymmetry, it turns out that the axio-dilaton must be a holomorphic function in $z$. Therefore, the Poisson equation for $C_8$ in the presence of a 7-brane at $z = z_0$ takes the form

$$d \ast F_9 = \delta(2\pi)(z - z_0).$$

(10)

The integrated form of (10) is

$$1 = \int_C d \ast F_9 = \oint_{S^1} F_1 = \oint_{S^1} dC_0.$$ 

(11)

A simple solution can be found that is valid close to the brane at $z = z_0$,

$$\tau(z) = \tau_0 + \frac{1}{2\pi i} \ln(z - z_0) + \cdots,$$

(12)

where we have omitted possible regular terms in $z$. Note in particular that $g_s \to 0$ at the position of the brane $z = z_0$.

Far away from the brane, the simple solution of $\tau$ must be modified. Nonetheless we see already that the background exhibits a logarithmic branch cut in the complex plane normal to the D7-brane. A careful analysis of the Einstein equations reveals [15] that asymptotically away from the brane, spacetime becomes locally flat, but suffers a deficit angle. Contrary to the backreaction of lower dimensional branes, this effect does not asymptote away and the probe philosophy is, strictly speaking, not applicable in this region.

Despite the non-trivial deficit angle at infinity, one can identify a region where the backreaction of the brane on the geometry is negligible. Following the lucid discussion in [17] one can, ignoring the higher terms, rewrite the solution (12) as

$$\tau(z) = \frac{1}{2\pi i} \ln\frac{z - z_0}{\lambda} \quad \Rightarrow \quad e^{-\phi} = -\frac{1}{2\pi} \ln \left| \frac{z - z_0}{\lambda} \right|$$

(13)

in the vicinity of the brane, where $\lambda$ is related to $\tau_0$. At the point $z = z_0 = \lambda$ one encounters $g_s = e^{\phi} \to \infty$. The presence of this special point breaks the naively expected rotational invariance around the source, a clear sign of backreaction. However, for $|z - z_0| \ll |\lambda|$ the geometry is approximately flat. It is this region that is the analogue of the asymptotic large distance limit where the backreaction of $p$-branes with $p < 7$ discussed above is negligible. The limit of weak coupling is the one where this region (and distances therein) is large enough to trust effective supergravity, i.e. where $\lambda \to \infty$. That is, in this limit the inevitable increase of $g_s$ away from the brane occurs at larger and larger distances, and one remains at weak coupling as long as one focuses on a suitable radius around the brane.

Generically, however, the fact that $g_s$ develops a strongly varying non-trivial profile obscures an interpretation of the background in terms of perturbation theory. As just discussed, in generic situations $\tau$ inevitably takes values of order 1 and larger in some regions of the compactification manifold. Besides, the expression for $\tau$ must receive strong stringy

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1 This does not mean that the gauge theory on the 7-brane is trivial. The relevant frame is the Einstein frame, where the 7-brane gauge coupling is independent of $g_s$ and given to leading order by the volume of the internal 4-cycle wrapped by the brane (for the case of compactification to four dimensions).

2 In compactifications, while the profile of $\tau$ is set by the brane configurations, the integration constant $\tau_0$ or overall scale $\lambda$ remains a modulus of the low-energy effective theory.
corrections in the vicinity of an orientifold plane, which carries $-4$ units of $C_8$-charge. A naive solution $\tau(z) \simeq -\frac{4}{2\pi i} \ln(z - z_0)$ which would be suggested by the above arguments is unphysical as the dilaton would become negative close to the O-plane.

An exception to the inevitable variation of $\tau$ in the presence of 7-branes and O-planes is the very special situation where all 7-brane charges cancel locally because a suitable number of D7-branes and O7-planes lie on top of each other. In this case, $g_s$ is constant on the entire compactification manifold and in the absence of stabilizing effects such as fluxes it can freely be chosen to lie in the perturbative window $g_s \ll 1$ everywhere. In fact, one should reserve the term ‘perturbative’ for this non-generic configuration. This establishes that a proper treatment of backgrounds with varying axio-dilaton is not an option, but rather required for a satisfactory definition of generic configurations with 7-branes.

2.1.2. Monodromies and $SL(2, \mathbb{Z})$ invariance. A peculiar feature of the 7-brane backreaction is the appearance of monodromies. The monodromy associated with the simple solution (12) implies that as one encircles the position of the D7-brane in the $z$-plane, the axio-dilaton transforms as

$$\tau \rightarrow \tau + 1.$$ (14)

At first sight this might come as a shock as it seems to make a consistent interpretation of the background solution impossible. The deus ex machina approaching to our rescue is the fact that type IIB is invariant under $SL(2, \mathbb{Z})$ transformations, of which (14) is the simplest example. As is most readily verified after transforming (5) into the Einstein frame (see e.g. [4]), the classical type IIB action enjoys the $SL(2, \mathbb{R})$ invariance

$$\tau \rightarrow a\tau + b, \quad \tau \rightarrow c\tau + d, \quad \det M = 1.$$ (15)

Note that $C_4$ is invariant under these transformations. This classical symmetry is broken at the non-perturbative level to $SL(2, \mathbb{Z})$. The reason is that $D(-1)$ instanton effects involve a factor $e^{2\pi i \tau}$. Invariance of such quantum effects under transformations of the type

$$M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : \quad C_0 \rightarrow C_0 + b$$

restricts $b \in \mathbb{Z}$, thus reducing $SL(2, \mathbb{R})$ to $SL(2, \mathbb{Z})$.

Therefore, as one encircles a D7-brane, the full background transforms by the $SL(2, \mathbb{Z})$ action $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and the monodromy action is merely a symmetry of the theory.

2.1.3. $[p, q]$-branes. Once we have accepted that a consistent interpretation of D7-branes forces us to take the $SL(2, \mathbb{Z})$ symmetry at the face value, we are led to more exotic objects than D7-branes which have no interpretation in terms of perturbation theory. The new objects we must include are $[p, q]$-branes and corresponding $(p q)$ strings [18]. To see this, recall from the Polyakov worldsheet action that the fundamental superstring is charged electrically under the NS–NS $B_2$-field. Given the mixing of $B_2$ with the RR 2-form $C_2$ under general $SL(2, \mathbb{Z})$ transformations as in (15) there must also exist an analogous object charged electrically under $C_2$. This, of course, is nothing other than the D1-string, and its corresponding coupling to $C_2$ is the Chern–Simons action. It is therefore appropriate to combine the F1- and the D1-string into an $SL(2, \mathbb{Z})$ doublet. An F1-string is represented as the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and a D1-string is a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ object. A general $\begin{pmatrix} p \\ q \end{pmatrix}$ string carries $p$ units of electric $B_2$-charge and $q$ units of electric $C_2$-charge. Such objects exist as supersymmetric bound states for $p, q$ co-prime [19]. Note
that in perturbative type IIB theory only \((1,0)\) strings are present as the fundamental objects, while the D1-string enters as a solitonic, non-perturbative object.

In perturbative type IIB theory, a D7-brane is by definition a hypersurface on which fundamental strings can end. This motivates the definition, in strongly coupled type IIB theory, of a \([p,q]\)-7-brane as the hypersurface on which \((p,q)\) strings can end. Two \([p,q]\)-7-branes of different \(p,q\) are called mutually non-local.

A general \([p,q]\)-brane induces an \(SL(2,\mathbb{Z})\) monodromy \(M_{p,q}\) on the background fields as one encircles the location of the brane. This monodromy generalizes the perturbative action \((14)\) induced by \(T = M_{1,0}\) and can be shown to take the form
\[
M_{p,q} = g_{p,q} M_{1,0} g_{p,q}^{-1} = \begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix}.
\]

Here \(g_{p,q} = \binom{p & q}{q & -p}\) is the \(SL(2,\mathbb{Z})\) matrix that transforms a \((1,0)\) string into a \((p,q)\) string\(^3\). The only eigenvector of \(M_{p,q}\) is a \((p,q)\)-string itself; any other type of string gets transformed as transported around the location of the \([p,q]\)-7-brane.

Every \([p,q]\)-brane as such can be mapped into a \([1,0]\)-brane, i.e. a conventional D7-brane, by an \(SL(2,\mathbb{Z})\) transformation. Locally around each single 7-brane the geometry therefore is indistinguishable from the one backreacted by a D7-brane. However, in the presence of mutually non-local \([p,q]\)-branes in the above sense, new phenomena arise because the various branes cannot be simultaneously transformed into a D7-brane. And, most importantly, a consistent compactification necessarily includes 7-branes of different types.

The simplest example where this becomes apparent starts from a perturbative type IIB orientifold on \(T^2\) modded out by \(\Omega(-1)^F\sigma\). The geometric orientifold action \(\sigma\) transforms the complex coordinate of the torus as \(z \rightarrow -z\). Its four fixed points are the location of O7-branes. Local tadpole cancellation in perturbative models (in the sense introduced above) requires four D7-branes (plus their image branes) on top of each O7-plane, resulting in a famous \(SO(8)^4\) gauge group. The \([p,q]\)-brane interpretation of this configuration was given by Sen [20], who showed that eight-dimensional type II compactifications require a set of three different \([p,q]\)-branes with \([p,q]\)-labels
\[
A : [1, 0], \quad B : [3, -1], \quad C : [1, -1].
\]

In this notation \(A\)-type branes correspond to perturbative D7-branes. In fact the branes \(B\) and \(C\) are chosen such that the combined monodromy \(M_{BC} = M_{3,-1} M_{1,-1}\) acts on a \((1,0)\) string by orientation reversal,
\[
M_{3,-1} M_{1,-1} \binom{1}{0} = -\binom{1}{0},
\]
as in the context of an orientifold theory. This identifies the \(BC\)-system as a perturbative O7-plane.

In generic configurations, however, objects with general \(SL(2,\mathbb{Z})\) monodromies will play a role, simply because the action of \(M_{1,0}\) and \(M_{BC}\) does not generate the full \(SL(2,\mathbb{Z})\). This raises a puzzle: What are the consistency conditions for theories with mutually non-local \([p,q]\)-branes? Is any configuration of coincident branes like the \(BC\) system allowed? Given the dyonic nature of \([p,q]\)-branes, perturbative methods are bound to fail. For compactifications to eight dimensions one might try to keep track of the \(SL(2,\mathbb{Z})\) monodromies by hand and search for intrinsically consistent configurations, but already in six dimensions this becomes intractable.

\(^3\) Note that \(r\) and \(s\) are not uniquely determined. This ambiguity drops out in all physically relevant quantities.
It is therefore quite remarkable that there exists a reformulation of the problem that allows us to almost blindly read off the consistent configurations of 7-branes. This formulation is the much-sought-after F-theory.

2.1.4. Towards a geometric description. The crucial insight [5] that underlies the formulation of such a non-perturbative theory is the identification of the $SL(2, \mathbb{Z})$ symmetry of ten-dimensional type IIB supergravity with the geometric $SL(2, \mathbb{Z})$ action on the complex structure of a two-torus $T^2$. Inspired by the $SL(2, \mathbb{Z})$ transformation of $\tau$ one interprets the axio-dilaton as the complex structure of a fictitious elliptic curve. The variation of $\tau$ in the presence of a set of 7-branes is therefore modelled as the variation of the complex structure of an elliptic curve transverse to the location of the 7-branes. Such a structure defines an elliptic fibration, and the non-triviality of the fibration is a measure for how strongly the axio-dilaton varies as a consequence of the backreaction of the branes.

The F-theory conjecture states that the physics of type IIB orientifold compactifications with 7-branes on the complex $n$-fold $B_n$ is encoded in the geometry of an $(n+1)$-fold $Y_{n+1}$ which is elliptically fibred over $B_n$. The elliptic fibre itself is not part of the physical spacetime but merely a book-keeping device that accounts for the variation of $\tau$. In particular, at the location of 7-branes the axio-dilaton $\tau$ diverges as seen for the solution (12). If the complex structure of an elliptic curves diverges, this indicates the shrinking of a 1-cycle and thus the degeneration of the elliptic curve. Thus, the degeneration locus of the elliptic curve describes the presence of 7-branes.

In fact, duality with M-theory yields additional restrictions on the relevant elliptic fibrations. First, for $N = 1$ supersymmetry to be conserved, the space $Y_{n+1}$ has to be Calabi–Yau. Further, only the limit of vanishing elliptic fibre is to be considered. Both these facts will become apparent in the context of F/M-theory duality reviewed in the next section.

To return to the simple example alluded to before, understanding compactifications of type IIB orientifolds to eight dimensions involves F-theory on a Calabi–Yau 2-fold $Y_2$. In this case there is only one such Ricci flat 2-fold, the famous $K3$. The physically most interesting situation of course corresponds to F-theory on an elliptically fibred 4-fold $Y_4$, which is supposed to capture type IIB orientifolds compactified to four dimensions on the 3-fold $B_3$.

2.2. F/M-theory duality and Calabi–Yau 4-folds

So far we have motivated F-theory as a clever way to think about type IIB orientifolds with 7-branes. To determine the details of the effective action and of the physical degrees of freedom though, it is useful to approach F-theory via duality with 11-dimensional M-theory, taken as its long-wavelength limit of 11-dimensional supergravity coupled to M2/M5-branes. This duality also provides a more direct way to uncover the appearance of the elliptic curve in F-theory.

The starting point is therefore compactification of 11-dimensional supergravity on $\mathbb{R}^{1,9} \times T^2$. Let $\tau$ be the complex structure of the torus $T^2 = S^1_A \times S^1_B$. For example, for the special case of a rectangular torus, we have $\tau = i \frac{R_A}{R_B}$ in terms of the radii $R_A$ and $R_B$ of $S^1_A$ and $S^1_B$. The M/F-theory duality consists in taking the limit of vanishing torus volume in the following two-step procedure [21].

- The circle $S^1_A$ with radius $R_A$ is identified as the M-theory circle in the reduction from M-theory to type IIA theory in ten dimensions so that $g_{\text{IIA}} \simeq \frac{R_A}{\tau}$. The limit $R_A \to 0$ therefore corresponds to taking the perturbative IIA limit of M-theory, and we recover weakly coupled type IIA theory on $\mathbb{R}^{1,9} \times S^1_B$. 
Now perform a T-duality along the remaining circle $S^1_\ell$ with radius $R_B$. This yields type IIB theory on $\mathbb{R}^{1,9} \times S^1_\ell$ with dual radius $\tilde{R}_B = \frac{i\ell s}{R_B}$. The limit $R_B \to 0$ results in ten-dimensional type IIB theory.

Application of the T-duality transformation identifies the IIB coupling as $g_{\text{IIB}} \simeq g_{\text{IIA}} \frac{\ell s}{R_B} \simeq \frac{\ell s}{R_B} \simeq \text{Im}(\tau)$. The last step applies to rectangular tori but has simple generalizations. A more precise analysis as worked out in detail e.g. in [12] allows one to trace back the type IIB RR complex structure of the M-theory torus.

What we have sketched is a description of type IIB theory in ten dimensions in terms of M-theory on $\mathbb{R}^{1,9} \times T^2$, where the IIB axio-dilaton $\tau = C_0 + \frac{i\ell s}{R_B}$ is identified with the complex structure of the M-theory $T^2$. F-theory on $T^2$ can therefore be defined as the ten-dimensional IIB theory which is dual, in the above sense, to M-theory on $\mathbb{R}^{1,9} \times T^2$. Note that for the duality to work the volume of the M-theory $T^2$ has to vanish. This is the physical reason why only the complex structure $\tau$, but not the volume of the elliptic curve, appears as a physical field in type IIB/F-theory.

The above logic extends adiabatically to the more general case of a non-trivial elliptic fibration rather than a direct product $\mathbb{R}^{1,9} \times T^2$. The physically most interesting situation is of course that of M-theory compactification on $\mathbb{R}^{1,2} \times Y_4$, where $Y_4 : T^2 \to B_3$ is a complex 4-fold elliptically fibred over a 3-complex dimensional base $B_3$. If the 4-fold $Y_4$ is Calabi–Yau, this yields a three-dimensional effective theory with four supercharges [22–24]. In the limit of vanishing fibre volume the fourth dimension grows large and this setup is dual to the four-dimensional effective theory obtained by type IIB compactification on $B_3$ or, by definition, F-theory on $Y_4$. The four supercharges lead to an $\mathcal{N} = 1$ effective theory in four dimensions. Thus, the Calabi–Yau property of $Y_4$ is an obvious requirement from the M-theory perspective.

The Kaluza–Klein reduction of the M-theory 3-form allows one to also recover the higher type IIB RR-forms. We denote by $\alpha$ and $\beta$ the periodic coordinates along the M-theory torus $T^2 = S^1_\ell \times S^2_\ell$ and focus on a generic non-singular fibre for the time being. Reduction of $C_3$ yields

$$C_3 = \hat{C}_3 + B_2 \wedge d\alpha + C_2 \wedge d\beta + B_1 \wedge d\alpha \wedge d\beta. \quad (19)$$

After T-duality along $\beta$ and decompactification to four dimensions, $\hat{C}_3$ furnishes the degrees of freedom of the RR 4-form $C_4 = \hat{C}_3 \wedge d\beta$, while $B_1$ becomes part of the four-dimensional metric $g_{\beta\beta}$. $B_2$ and $C_2$, on the other hand, translate into the NS–NS and R–R 2-forms in type IIB. Their transformation properties as a doublet under the $SL(2, \mathbb{Z})$ group are derived immediately from the geometric $SL(2, \mathbb{Z})$ transformation of the $A$ and $B$ cycle of the M-theory torus.

It is beyond the scope of these lectures to enter a detailed discussion of the F-theory effective action. An in-depth derivation of the four-dimensional F-theory effective action via M/F-theory duality has been provided in [25] (see also [12]), which we recommend for more details. An important ingredient in the Kaluza–Klein reduction, e.g. to determine the precise massless supergravity spectrum, is the topological properties of Calabi–Yau 4-folds. Many aspects of the geometry and topology of (elliptic) Calabi–Yau 4-folds can be found e.g. in [26]. Suffice it here to recall for completeness that the Hodge diagram of Calabi–Yau 4-folds is characterized by three independent Hodge numbers $h^{1,1}$, $h^{2,1}$, $h^{3,1}$ to which $h^{2,2}$ is related via

$$h^{2,2} = 2(22 + 2h^{1,1} + 2h^{3,1} - h^{2,1}). \quad (20)$$
An important quantity for model building, which will appear prominently in what follows, is the Euler characteristic

\[ \chi(Y_4) = \int_{Y_4} c_4(T_{Y_4}) = 6(8 + h^{1,1} + h^{2,1}). \]  

(21)

2.3. The geometry of elliptic fibrations

The above considerations show that understanding the non-perturbative F-theoretic region of the string landscape requires familiarity with the concept of elliptic fibrations, to which we now turn in some detail. The problem splits into two parts. We first need to understand elliptic curves as such, and then find a way to describe their fibration over the base \( B_n \) such as to form an elliptically fibred Calabi–Yau manifold \( Y_{n+1} \):

\[ \pi : Y_{n+1} \rightarrow B_n. \]  

(22)

2.3.1. Elliptic curves as \( P_{2,3,1} \)[6]. There are different ways to describe an elliptic curve, the simplest being as a hypersurface or, more generally, as a complete intersection of some weighted projective space. As an example consider the weighted projective space \( P_{2,3,1} \) spanned by the homogeneous coordinates \((x, y, z) \simeq (\lambda^2 x, \lambda^3 y, \lambda z)\), where \((x, y, z)\) are complex coordinates and \(\lambda \subset \mathbb{C}^*\). Due to the scaling relation this space is 2-complex dimensional. An elliptic curve is a flat 1-complex dimensional space, i.e. a Calabi–Yau 1-fold. It can thus be described by the hypersurface cut out by the vanishing locus of a homogenous polynomial in \((x, y, z)\) of degree 6 under rescaling by \(\lambda\). Here we used the general fact that a hypersurface in weighted projective space is Ricci-flat whenever the degree of its defining polynomial equals the sum of degrees of the homogenous coordinates. A degree-6 polynomial in \( P_{2,3,1} \) defines the space \( P_{2,3,1} \)[6]. One can show that after suitable coordinate redefinitions such a polynomial can always be brought into the so-called Weierstrass form

\[ P_W = y^2 - x^3 - f x z^4 - g z^6 = 0, \]  

(23)

where \(f, g \in \mathbb{C}\) specify the shape of the elliptic curve as detailed below. Note at this stage that alternative representations of elliptic curves along these lines are the hypersurfaces \( P_{1,1,1} \)[4] or \( P_{1,1,1} \)[3].

As argued at the end of section 2.1 the physics of the 7-branes is encoded in the degenerations of the elliptic fibre. A general mathematical fact is that a hypersurface described by equation \( P_W = 0 \) becomes singular whenever

\[ P_W = 0 \quad \text{and} \quad dP_W = 0, \]  

(24)

where the last condition indicates a degeneration of the tangent space. It is easy to work out these conditions for the Weierstrass curve (23). First use the scaling relations to go to inhomogeneous coordinates by setting \(z = 1\). Then the above conditions are equivalent to

\[ y = 0, \]
\[ -x^3 - f x - g = (x - a_1)(x - a_2)(x - a_3) = 0, \]  

\[ (x - a_1)(x - a_2) + (x - a_2)(x - a_3) + (x - a_1)(x - a_3) = 0, \]  

(25)

where \(a_1, a_2, a_3\) are the three complex roots of the cubic polynomial \(-x^3 - f x - g\). Clearly the above requires that two or more of these roots coincide. As always, the structure of the roots of this cubic polynomial is encoded in the discriminant \(\Delta\), which vanishes if and only if at least
two roots coincide. For the Weierstrass model the discriminant of the cubic \(-x^3 - fx - g\) takes the simple form

\[
\Delta = 27g^2 + 4f^3, \tag{26}
\]

which shows that the structure of the elliptic curve indeed depends on the parameters \(f\) and \(g\).

In fact, the dependence of the complex structure \(\tau\) on \(f\) and \(g\) can be made more precise. Since \(\tau\) undergoes \(SL(2, \mathbb{Z})\) monodromies it is more convenient to work with the \(SL(2, \mathbb{Z})\) invariant Jacobi \(j\)-function, which yields an isomorphism from the fundamental domain of \(SL(2, \mathbb{Z})\) to the Riemann sphere. Instead of introducing the formal definition of \(j(z)\) in terms of theta-functions it suffices here to note the expansion

\[
j(z) = \exp(-2\pi iz) + 744 + (\exp(2\pi iz)) + \cdots. \tag{27}
\]

A classical mathematical result is that the complex structure \(\tau\) of a Weierstrass elliptic curve is determined by \(f, g\) via the relation

\[
j(\tau) = \frac{4(24f)^3}{\Delta}. \tag{28}
\]

### 2.3.2. From elliptic curves to elliptic fibrations

Armed with this representation of a single elliptic curve, we can proceed to elliptic fibrations. Suppose we have some \(n\)-complex-dimensional manifold \(B_n\), covered by local coordinates \(u_i\). Then a fibration of the Weierstrass curve (23) over \(B_n\) is obtained by promoting the constants \(f\) and \(g\) in (23) to suitable polynomials in the coordinates \(u_i\) of \(B_n\), \(f = f(u_i), g = g(u_i)\). This defines the corresponding Weierstrass model, also known as an \(E_8\)-fibration. Note that in principle one can also consider fibrations of the elliptic curves \(P^1_1, 1, 2\) or \(P^1_1, 1, 1\) (called \(E_7\)- and, respectively, \(E_6\)-fibrations) and their generalizations. The crucial point, however, is the following mathematical theorem (see e.g. [27]). Every elliptic fibration with a section can be represented by a Weierstrass model defined in terms of equation (23) with varying \(f\) and \(g\). All other representations such as the \(E_7\) and \(E_6\)-fibrations are birationally equivalent (equivalent up to a flop transition).

Via the relation (28) the complex structure \(\tau\) is now dependent on the base coordinates \(u_i\). In particular, the elliptic fibre degenerates on a codimension-1 sublocus on \(B_n\) determined by the vanishing of the likewise \(u_i\)-dependent discriminant \(\Delta = 27g^2 + 4f^3\). In view of what we said before, this vanishing locus must be interpreted as a divisor wrapped by a stack of 7-branes. Indeed, suppose \(\Delta\) vanishes to order \(N\) along the divisor \(S\) on \(B_n\) defined by \(w = 0\). Then equations (27) and (28) imply that in the directions normal to this divisor

\[
\tau \simeq \frac{N}{2\pi i} \ln(w) \tag{29}
\]

up to regular terms, which is exactly the behaviour expected for \(N\) coincident 7-branes. The connection between this geometric description of 7-branes as the degeneration locus of the elliptic fibration and the concept of \([p, q]\)-7-branes comes about as follows. A single \([p, q]\)-7-brane is the locus on the base \(B_n\) along which a \((p, q)\)-cycle on the elliptic fibre degenerates. This can be shown by matching the geometric \(SL(2, \mathbb{Z})\) monodromies with the action (16) of \(M_{p,q}\). Locally around each 7-brane we are free to declare the degenerate 1-cycle in the fibre to correspond to the \((1, 0)\) fibre by a choice of basis for \(H_2(T^2, \mathbb{Z})\). This corresponds to the assertion that locally each brane looks like a D7-brane by an \(SL(2, \mathbb{Z})\) transformation. In IIB theory, a stack of \(N\) coincident 7-branes in the generic position with respect to the O7-plane gives rise to the gauge group \(U(N)\). In the F-theory picture the gauge group along a divisor is described by the details of the degeneration of the elliptic fibre as we will see below.
First, however, we need to discuss the Calabi–Yau property in greater detail. The first Chern class of an elliptic fibration $Y_{n+1}$ is related to the first Chern class of the base space $B_n$ and the degeneration locus of the fibration as [28]

$$c_1(T_{Y_{n+1}}) \simeq \pi^* \left( c_1(T_{B_n}) - \sum_i \frac{\delta_i}{\Gamma_1} \right), \quad \delta_i = O(\Delta)_{[\Gamma_i]}.$$  

(30)

Here

$$\pi : Y_{n+1} \to B_n$$  

(31)

denotes the projection from the fibration to the base, and its pullback $\pi^*$ maps $H^2(B_n, \mathbb{Z}) \to H^2(Y_{n+1}, \mathbb{Z})$. Furthermore, the discriminant is supposed to vanish to order $\delta_i$ along the divisors $[\Gamma_i]$ on the base, with dual 2-form $[\Gamma_i]$. Strictly speaking this formula is correct only for elliptic K3-manifolds, but the extra complications due to higher codimension degenerations on the base do not affect the present argument [29]. We see that the Calabi–Yau property of $Y_{n+1}$ forces $B_n$ to acquire positive curvature such as to compensate for the degenerations of the fibre. In particular, $B_n$ itself is not Calabi–Yau. Since the discriminant locus describes the positions of the 7-branes, we find that the total amount of 7-branes leading to a consistent compactification is thus constrained by the curvature of the base space via

$$\sum_i \delta_i [\Gamma_i] = 12c_1(T_{B_n}).$$  

(32)

This equation is reminiscent of the 7-brane tadpole cancellation condition in perturbative type IIB theory

$$\sum_i N_i [\Gamma_i] = 4[O_7].$$  

(33)

This demonstrates the very different nature of string model building in perturbative type IIB orientifolds and F-theory in that the 7-brane consistency conditions are automatically incorporated in a well-defined geometry.

From a mathematical perspective, the relation (32) constrains the degree of the polynomials $f$ and $g$ that appear in the Weierstrass model as follows. The class of the left-hand side is just the class of the full discriminant $\Delta$, and (32) states that $[\Delta] = 12c_1(T_{B_n})$. In view of $\Delta = 27g^2 + 4f^3$ this determines the class of $f, g$. In general $f$ and $g$ are not globally defined functions of the base space, but rather sections of a line bundle $L$. In terms of the canonical bundle $K_{B_n}$, which has the first Chern class $c_1(K_{B_n}) = -c_1(T_{B_n})$, it follows that $f$ and $g$ are sections of $K_{B_n}^{-4}$ and $K_{B_n}^{-6}$, respectively. Homogeneity of the defining Weierstrass polynomial then also forces $x$ and $y$ to transform as sections of the base. In all, we find

$$x \in H^0(B_n, K_{B_n}^{-2}), \quad y \in H^0(B_n, K_{B_n}^{-3}), \quad z \in H^0(B_n, \mathcal{O}),$$  

(34)

$$f \in H^0(B_n, K_{B_n}^{-4}), \quad g \in H^0(B_n, K_{B_n}^{-6}).$$  

(35)

2.3.3. Example: F-theory on K3. As a simple example consider K3 on the locus of its moduli space where it arises as an elliptic fibration over $B_1 = \mathbb{P}^1$. In terms of the normalized volume form $i$ of $\mathbb{P}^1$, the first Chern class of $K_{\mathbb{P}^1}$ is simply $c_1(K_{\mathbb{P}^1}) = -2i$. This follows from the Hirzebruch–Riemann–Roch theorem, whereby

$$\chi = 2 - 2g = \int_{\mathbb{P}^1} c_1(T_{\mathbb{P}^1}).$$  

(36)

The standard notation is of course $K_{\mathbb{P}^1} = \mathcal{O}(-2)$. More generally a section of the line bundle $\mathcal{O}(-n)$ is a homogenous polynomial of degree $n$ in the homogeneous coordinates $[u_0, u_1]$ of
Together with the original scaling relation of the Weierstrass model the elliptically fibred \( K3 \) is spanned by the coordinates

\[
(u_0, u_1; x, y, z) \simeq (u_0, \mu u_1; \mu^4 x, \mu^6 y, z).
\]

Famously, the discriminant \( \Delta \) is thus a polynomial of degree 24 on the base, with 24 zeros known as the position of the 24 7-branes of compactification of IIB theory on \( \mathbb{P}^1 \). Not all of these 24 branes describe perturbative D7-branes. As discussed already around equation (18), in the strict perturbative limit the tadpoles are cancelled locally by grouping the 24 branes into four stacks of six branes such that each stack corresponds to four D7-branes, i.e. \( A \)-type branes, on top of a system of \( BC \) branes. Recall that the latter furnish the F-theoretic description of the IIB \( O7 \)-plane.

At a more advanced level, the F-theory solution teaches us several exciting lessons. In [20] Sen showed how the \( O7 \)-brane splits as one removes one or more \( A \)-branes from the \( A^4 BC \) system. Recall from the discussion after (12) that in such a situation where tadpoles are no longer cancelled locally the naive supergravity solution must break down close to the \( O7 \)-plane. In F-theory, the geometry adjusts itself via this non-perturbative brane split so as to render the configuration self-consistent. This not only reveals that the \( O7 \)-plane is a dynamical object per se but it also furnishes a highly non-trivial check that F-theory correctly captures the non-perturbative degrees of the theory. Indeed it is quite remarkable that a comparatively straightforward analysis of the geometry of \( K3 \) alone dictates all consistent configurations, \([p, q] \)-brane configurations. Along these lines, the authors of [30] found more general solutions with constant, but non-perturbatively large axio-dilaton for which the 24 branes group e.g. into three bunches of the type \( A^5 BC^2 \) and others. This illustrates that the number of branes of a given \([p, q] \)-type on a given compactification space changes from configuration to configuration due to the appearance of monodromies as we start moving the branes around.

2.4. Sen’s orientifold limit

After this first encounter with the geometric description of F-theory, let us go back and analyse more generally how to recover the weakly coupled type IIB orientifold picture. This is accomplished by a procedure due to Sen [31]. Recall from the discussion after equation (12) that in the IIB limit one considers the 7-branes as probe objects and takes the axio-dilaton as non-varying and perturbatively small. We already noted that strictly speaking this can only be realized if all 7-brane tadpoles are cancelled locally by placing a suitable amount of 7-branes on top of the orientifold planes. More generally, there can however exist a limit in which one can approximately take \( \tau \) small and non-varying even though not all tadpoles are cancelled locally. As argued after equation (13) this corresponds to the limit \( \lambda \to \infty \). Since the profile of \( \tau \) is encoded in the Jacobi function (28) a constant \( \tau \) can be achieved as long as \( j(\tau) \) is non-varying over the base \( B_3 \) of the F-theory 4-fold. To arrange for this one makes the general ansatz

\[
f = -3h^2 + \epsilon \eta, \quad g = -2h^3 + \epsilon h \eta - \frac{\epsilon^2}{12} \chi,
\]

where \( \epsilon \) is an arbitrary constant. Sen’s orientifold limit corresponds to the limit \( \epsilon \to 0 \) with \( h, \eta \) and \( \chi \) generically non-vanishing. In this case one finds that indeed the string coupling becomes arbitrarily weak everywhere except at the location \( h = 0 \), where \( g_s \to \infty \). It can thus be considered as a perturbative parameter of the theory. The discriminant locus factorizes to leading order in \( \epsilon \) as

\[
\Delta_\epsilon = -9\epsilon^2 h^2 (\eta^2 - h \chi) + \mathcal{O}(\epsilon^3)
\]
so that the fibration degenerates at \( h = 0 \) and at \( \eta^2 - h \chi \). The latter may factorize further depending on the particular form of the polynomials \( \eta \) and \( \chi \). By closer inspection of the monodromies around these loci it can be shown that \( h = 0 \) corresponds to the location of an O7-plane, while the latter locus is that of an ordinary D7-brane. The inclusion of higher order terms in \( \epsilon \), on the other hand, destroys this factorization of \( \Delta \) into the orientifold and brane piece. This process can be thought of as a recombination of the O7-plane and the D7-brane. Since in perturbation theory there are no light strings stretching between the O7-plane and a D7-brane, this recombination is truly non-perturbative. The involved recombination modes are expected to be \( (p,q) \) strings.

The type IIB theory is defined on the Calabi–Yau 3-fold \( X \) which is a double cover of the base \( B \) branched over \( h = 0 \). Concretely, suppose the base \( B_3 \) is embedded into some projective space with projective coordinates \( u_i \). Then one can define \( X \) as the complete intersection in the projective space spanned by the coordinates \( u_i \) and \( \xi \) which is defined by the original hypersurface polynomial and the additional constraint

\[
\xi^2 - h(u_i) = 0. \tag{40}
\]

Conversely, given a type IIB orientifold on \( X \) together with a \( \mathbb{Z}_2 \) action \( \sigma \), one can construct the F-theory uplift by reversing Sen’s limit above. This has been extended and applied in the more recent literature [32–34].

Note from the form of the 7-brane locus \( \eta^2 - \xi^2 \chi = 0 \) (where we replaced \( h \) in \( \eta^2 - h \chi = 0 \) by \( \xi^2 \) according to (40)) that for the defining equation for a single D7-brane configuration with a well-defined F-theory uplift is non-generic [17, 35]. Brane/image-brane configurations require a specialization of the polynomials \( \eta \) and \( \chi \), e.g. by factorization of \( \chi \). The relation of such even more non-generic brane splits to elliptic fibrations of \( E_7 \)- and \( E_6 \)-type has been investigated in [36, 37].

Finally we would like to point out that the limit \( \epsilon \rightarrow 0 \) with all other polynomials generic is not the only way to make \( \tau \) non-varying along \( B_3 \). There exist in addition the two possibilities of taking either \( f \equiv 0 \) or \( g \equiv 0 \) [30]. In these cases, \( \tau \) is constant and fixed at non-perturbatively large values, in agreement with the appearance of exceptional gauge groups. These configurations correspond to generalizations of the \( \mathbb{Z}_2 \) orientifold action of the strongly coupled type IIB string to \( \mathbb{Z}_n \) actions.

2.5. Gauge symmetry from degenerations

From a physics perspective the most essential data of an elliptic fibration are the locus and the type of fibre degenerations because these allow us to deduce the nature of the 7-branes wrapping the corresponding divisor. The different ways how the complex structure of the elliptic fibre can degenerate have been classified by Kodaira [28] for the case of a Weierstrass model of \( K3 \). The relevant criterion to be imposed here is that the singularities can be resolved without destroying the Calabi–Yau property. These results have subsequently been generalized to higher Calabi–Yau spaces (see [38] and references therein for a physics discussion).

Consider a Weierstrass model \( Y_{n+1} \). One distinguishes degenerations that merely render the fibration singular without inducing an actual singularity in \( Y_{n+1} \) versus loci where the Calabi–Yau \((n + 1)\)-fold becomes singular itself. The simplest kind of degeneration of the first type corresponds to a so-called \( I_1 \) singularity, which arises in the fibre over just a single

\footnote{Note that this reference is devoted to an analysis of elliptic Calabi–Yau 3-folds. For higher dimensional manifolds additional effects due to a more complicated monodromy structure can occur, and as of this writing no complete classification exists in the literature. Also the results of [38] apply to situations where the discriminant locus itself is non-singular as a divisor of \( Y_3 \).}
7-brane. For a \((p, q)\)-7-brane the \((p, q)\) 1-cycle of the elliptic fibre pinches off and the elliptic fibre forms a sausage-type \(\mathbb{P}^1\) whose north and south pole touch each other. Following Kodaira, \(I_1\) singularities arise in a Weierstrass model whenever \(\Delta\) develops a simple zero while \(f\) and \(g\) are non-zero. In this case two of the roots \(a_i\) in the cubic (25) coincide. From equations (27), (28) the simple zero in \(\Delta\) leads to the logarithmic profile (12) of \(\tau\) characteristic of a single 7-brane.

When \(\Delta\) vanishes to higher order, generically also \(Y_{n+1}\) develops an actual singularity as a manifold. For the simplest case of \(Y_2 = K_3\) the possible singularities that can occur are precisely the ones listed in the famous ADE classification \([28]\). For higher dimensional \(Y_{n+1}\) with \(n > 2\) also the action of monodromies along the brane have to be taken into account. This can create more general singularities including non-simply laced examples of \(B\) and \(C\) type and such exceptional cases as \(G_2\) or \(F_4\).

Let us focus for definiteness on Calabi–Yau 4-folds \(Y_4\). One intuitive way to understand the connection between a singularity over the divisor \(S \subset B_3\) and a Lie algebra \(G\) is as follows. The singularity in \(Y_4\) can be resolved by standard methods of algebraic geometry. For our purposes it suffices to consider cases that allow for a so-called split simultaneous resolution.

As described e.g. in \([39]\) this means that there exists a new, non-singular Calabi–Yau \(\overline{Y}_4\) defined by replacing the singular fibre over \(S\) by a tree of \(\mathbb{P}^1\)'s which we call in the following \(\Gamma_i, i = 1, \ldots, \rk(G)\). Note that in this process \(h^{1,1}\) increases by \(\rk(G)\). These \(\mathbb{P}^1\)'s have the property that they intersect one another like the simple roots of the Lie algebra \(G\) in the following sense. The resolution 2-cycles \(\Gamma_i\) are fibred over the 7-brane \(S \subset B_3\). Denote the resulting divisors of \(\overline{Y}_4\) by \(\hat{D}_i : \Gamma_i \to S\).

\[
\hat{D}_i : \Gamma_i \to S.
\] (41)

By construction these are \(\mathbb{P}^1\)-fibrations. Consider furthermore the linear combination \(\hat{D}_0 = \hat{S} - \sum_i a_i \hat{D}_i\), where \(\hat{S}\) is the elliptic fibration over \(S\) and \(a_i\) denote the Dynkin labels for the Lie algebra \(G\) associated with the singularity over \(S\). Then \(\hat{D}_0, \hat{D}_i\) encode the extended Dynkin diagram of \(G\) in the sense that their Poincaré dual 2-forms \([\hat{D}_i] \in H^2(\overline{Y}_4)\) satisfy

\[
\int_{\Gamma_i} [\hat{D}_i] \wedge [\hat{D}_j] \wedge \tilde{\omega} = -C_{ij} \int_{\hat{S}} \tilde{\omega}, \quad i, j = 0, 1, \ldots, \rk(G)
\] (42)

for \(\tilde{\omega} \in H^2(B_3)\) and \(C_{ij}\) the Cartan matrix of \(G\). Conversely, the singular F-theory limit can be understood as the limit in which the \(\Gamma_i\) shrink to zero volume. The singularity is the result of a collision of \(\rk(G)\) zero size \(\mathbb{P}^1\)'s.

The appearance of the corresponding gauge groups along the 7-brane wrapping the singular locus can be understood by F/M-theory duality. As sketched in (19), Kaluza–Klein reduction of \(C_3\) along non-singular fibres results in the type IIB closed string 2-forms \(B_2\) and \(C_2\). As we have just seen, in the fibre above the discriminant locus extra 2-cycles appear along which \(C_3\) can be reduced. For simplicity consider an \(A_{n-1}\) singularity associated with the gauge group \(SU(n)\). The resolution \(\mathbb{P}^1\)'s \(\Gamma_i, i = 1, \ldots, (n - 1)\), correspond to the nodes of the Dynkin diagram of \(A_{n-1}\). Massless vector states arise from two sources.

- The M-theory 3-form \(C_3\) can be reduced along \(\Gamma_i\), leading to 1-forms \(A_i = \int_{\Gamma_i} C_3\) along the 7-brane. These are interpreted as the gauge potentials for the Abelian group factors in the Cartan subalgebra of \(SU(n)\).
- The M-theory M2-brane can wrap chains of 2-cycles \(S_{ij} = \Gamma_i \cup \Gamma_{i+1} \cup \cdots \cup \Gamma_j\) for any \(i \leq j\). Taking into account the two possible orientations this gives rise to \(2^{n^2-n}\) states along the brane. In the singular limit of vanishing volume of the \(\Gamma_i\) these states become massless and lead to the \(W\)-bosons of \(SU(n)\).

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In all this yields the \( n^2 - 1 \) generators of \( SU(n) \). This logic can be generalized to other simple groups.

In addition there can be extra Abelian gauge factors which do not arise as the Cartan generators of a non-Abelian gauge group and which are strictly speaking not localized along individual divisors. An unambiguous way to determine the total rank of the gauge group of a singular Weierstrass model is to note that all \( U(1) \) generators are derived from the reduction of \( C_3 \) as \( C_3 = A_1 \wedge \omega_k + \cdots \), where the 2-forms \( \omega_k \) have one leg along the \( \mathbb{T}^2 \) fibre and one leg along the base \([29, 40]\). Therefore,

\[
r = h^{1,1}(\overline{Y_{n+1}}) - h^{1,1}(B_n) - 1
\]

(43)
gives the total rank of the F-theory gauge group including possible Abelian gauge factors\(^5\). The physics of these extra \( U(1) \)s in compactifications to four dimensions has been discussed in \([37]\).

Let us stress that for model-building purposes it is essential to have control over the resolution \( \overline{Y}_d \) of the singular fibration \( Y_d \), e.g. to determine the complete rank of the gauge group or in the context of 3-brane tadpole cancellation, see section 3.2. In fact, for the class of Calabi–Yau 4-folds constructed as hypersurfaces or complete intersections of toric spaces powerful tools have been developed to perform this resolution in terms of the divisors \( D_i \) explicitly \([41, 42]\). These have recently been exploited in the context of F-theory model building in \([37, 43–45]\).

To conclude this section, we point out that the appearance of exceptional and more general gauge groups has a beautiful interpretation in terms of \( (p,q) \)-strings \([46, 47]\). This can be understood already for F-theory on \( K3 \), where the possible gauge groups are exhausted by the ADE classification. By definition, only \( (p,q) \)-strings can end on \([p,q]\)-7-branes. Due to the appearance of monodromies, however, \( (p,q) \)-strings stretched between two branes along different paths can lead to inequivalent states. A mild version of this phenomenon was encountered already in the perturbative limit. A \((p,q)\)-string encircling a \( BC \)-system gets orientation reversed, see equation (18). The resulting unoriented strings correspond to strings between a stack of branes and their orientifold images. More generally, one can classify allowed paths that lead to BPS-strings and determine their interactions via consistent splitting and joining. Suffice it here to highlight from \([46]\) that the various \( (p,q) \)-strings occurring for the \([p,q]\)-7-brane system \( A^nBC^2 \) indeed span the adjoint of \( E_{5+n}, n = 1, 2, 3 \), in agreement with the monodromies for the generalized orientifolds of \([30]\). The underlying reason for this new richness is that \( (p,q) \)-strings of different types can form string junctions with more than just two endpoints. This overcomes the limitation to two-index representations as carried by the Chan–Paton factors of perturbative \( (p,q) \)-strings. The latter allow only for the construction of \( U(N), Sp(2N) \) and \( SO(N) \) groups.

### 3. Technology for F-theory compactifications

#### 3.1. Tate models

Given the importance of the singularity structure for an analysis of the F-theory landscape, we find it useful to present the methods to detect the singularities in slightly greater detail. There exists an algorithm due to Tate that allows one to read off the singularity structure of an elliptic

\(^5\) In addition, there are \( h^{2,1}(B_n) \) bulk \( U(1) \) fields, corresponding to the reduction of the RR 4-form \( C_4 \) along 3-cycles on \( B_n \).
fibration, say \( Y : T^2 \to B_3 \). Tate’s formalism consists in a local coordinate redefinition that brings the Weierstrass constraint \( P_W = 0 \) in (23) into the Tate form

\[
P_W = x^3 - y^2 + xyx_1 + x^2 z^2 a_2 + yz^3 a_3 + xz^3 a_4 + z^6 a_6 = 0.
\] (44)

For a general Weierstrass model this can be achieved \textit{locally} via the so-called Tate algorithm described in detail in [38]. In what follows we will only be working with the inhomogeneous Tate form by setting \( z = 1 \). The \( a_i(u_i) \) depend on the complex coordinates \( u_i \) of the base \( B \).

They encode the discriminant locus of the elliptic fibration. To recover the Weierstrass model one first introduces the combinations

\[
\beta_2 = a_1^2 + 4a_2, \quad \beta_4 = a_1 a_3 + 2a_4, \quad \beta_6 = a_3^2 + 4a_6.
\] (45)

One can check by completing the square and the cube in (44) that the Weierstrass sections \( f \) and \( g \) are related to these via

\[
f = -\frac{1}{38} (\beta_2^2 - 24 \beta_4), \quad g = -\frac{1}{864} (-\beta_2^3 + 36 \beta_2 \beta_4 - 216 \beta_6).
\] (46)

The discriminant \( \Delta = 27g^2 + 4f^3 \) can then be expressed as

\[
\Delta = -\frac{1}{4} \beta_2^4 (\beta_2 \beta_6 - \beta_2^2) - 8 \beta_2^3 - 27 \beta_6^2 + 9 \beta_2 \beta_4 \beta_6.
\] (47)

In general, the discriminant \( \Delta \) will factorize with each factor describing the location of a 7-brane on a divisor in \( B_3 \). The precise group is encoded by the vanishing degree of the \( a_i \) and \( \Delta \). This has been classified in table 2 of [38].

For example, an \( SU(5) \) gauge group along the divisor

\[
S : \quad w = 0
\] (48)

corresponds to

\[
a_1 = b_5, \quad a_2 = b_4 w, \quad a_3 = b_3 w^2, \quad a_4 = b_2 w^3, \quad a_6 = b_0 w^5,
\] (49)

where the \( b_i \) generically depend on all coordinates \( u_i \) of the base \( B \) but do not contain an overall factor of \( w \). It is straightforward to evaluate the discriminant \( \Delta \) of the elliptic fibration in terms of the new sections \( b_i \) as

\[
\Delta = -w^2 \left( b_2^4 P + w b_2^2 (8b_4 P + b_3 R) + w^2 (16b_3 b_2^2 + b_5 O) + O(w^3) \right)
\] (50)

with

\[
P = b_5^2 b_4 - b_2 b_3 b_5 + b_0 b_2^2, \quad R = 4b_0 b_2 b_5 - b_3^2 - b_2^2 b_5.
\] (51)

Generically the expression in brackets in (50), denoted as \( S_1 \), does not factorize further and thus constitutes the single-component locus of an \( I_1 \) singularity. Cohomologically, one thus finds that the class \([\Delta]\) splits as \([\Delta] = [S_5] + [S_1]\).

As stressed in section 2.3, the most general elliptic fibration with a section can always be written as a Weierstrass model. A special subclass of such models, however, can be brought into the Tate form not just locally around the discriminant locus, but globally. That is, for those models the defining constraint is (44) in terms of globally defined sections \( a_i \in H^0(B, K^+_B) \). Clearly each such global Tate model defines also a Weierstrass model in that each set of \( a_i \) defines the Weierstrass sections \( f \) and \( g \) as in (46). The converse, however, is not true globally as the transformation from \( f \) and \( g \) to \( a_i \) involves branch cuts.

Models which are globally of the Tate form are very convenient because one can immediately read off the gauge group from table 2 in [38] without going through the algorithm to bring the Weierstrass polynomial locally into the form (44). It is important to keep in mind, though, that they do not give rise to the most general singularity structure. In particular, there

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6 Note that this table was derived for the case of elliptic Calabi–Yau 3-folds, see footnote 4.
is a sense in which such models are based on an underlying gauge group $E_8$. Consider the situation that the gauge group $G$ realized along the divisor $S$ is contained within $E_8$. Then, in the global Tate model the dynamics of the gauge sector can be understood via breaking an original $E_8$ symmetry down to the commutant of a complimentary gauge group $H$ such that $E_8 \rightarrow G \times H$. This can be seen by organizing the sections of the Tate model as a sum of two terms, the first of which corresponds to singularity enhancement $E_8$, while the second encodes an $H$-bundle responsible for the breaking down to $G$. For more technical information on this point we refer to [37] and references therein.

### 3.2. Fluxes and 3-brane tadpoles

While the entire information of 7-brane charge is contained in the geometric data of the F-theory Calabi–Yau 4-fold, the induced 5- and 3-brane charge is extra data encoded in the background flux. In the type IIB orientifold picture one distinguishes closed string fluxes—most notably the 3-form flux $G_3 = F_3 - \tau H_3$—and brane fluxes $F$. The latter are the background value of the Yang–Mills field strength on the 7-branes. In F-theory, both RR 3-form fluxes and brane fluxes enjoy a uniform description in terms of the M-theory 4-form flux $G_4$. In principle, brane fluxes arise by the reduction

$$G_4 = \sum_i F^{(i)} \wedge \omega_i + \cdots,$$

where $\omega_i$ denote those normalizable harmonic 2-forms of $Y_{n+1}$ that are neither elements of $H^{1,1}(B_n)$ nor represent the fibre class itself. For example, this expression straightforwardly describes the Abelian gauge flux associated with a Cartan $U(1)$ generator contained in the gauge group $G$ along $S$. In this case $\omega_i \equiv \omega_i^G$ are the 2-forms Poincaré dual to the resolution divisors $D_i$ introduced in (41). Such flux breaks the gauge group $G$ to the commutant of the Cartan generator, a mechanism that will be heavily exploited in the context of GUT symmetry breaking in section 4.2.

More general fluxes, however, are extremely difficult to describe in detail because they encode the information of a non-Abelian gauge bundle. This is best understood in the context of the global Tate model introduced before. The flux we have in mind is associated with the orthogonal gauge group $H \subset E_8$ and therefore does not affect the gauge group $G$. An auxiliary tool to study such fluxes is given by the spectral cover construction, which is the subject of section 3.5. Note that the gauge flux is essential to achieve a chiral matter spectrum, see the discussion in the next section.

The total amount of F-theory fluxes is constrained by the 3-brane tadpole cancellation condition, which is dual to M2-brane charge cancellation in M-theory.

Here we simply quote the result [48]

$$\chi(Y_4) = N_{M2} + \frac{1}{2} \int_{\mathcal{T}_4} G_4 \wedge G_4,$$

where $N_{M2}$ is the number of M2-branes filling the three non-compact dimensions. By F/M-duality $N_{M2} = N_{D3}$, the number of spacetime-filling D3-branes.

The object $\chi(Y_4)$ is the Euler characteristic, introduced in equation (21), of the resolved Calabi–Yau 4-fold. It encodes all curvature-induced 3-form charge of the 7-branes. Indeed in models with a weakly coupled type IIB description one can match $\chi(Y_4)$ with the curvature-dependent terms in the Chern–Simons action of the O7-plane and D7-branes [35]. Special
care has to be taken in the computation of $\chi(Y_4)$. As long as the 4-fold is non-singular, $\chi(Y_4) = \int c_4(Y)$, and for a Weierstrass model over $B_3$ a simple formula exists [26, 48]:

$$\chi(Y_4) = 12 \int_{B_3} c_1(B_3)c_2(B_3) + 360 \int_{B_3} c_3^3(B_3). \quad (54)$$

For the phenomenologically relevant case of F-theory compactification with non-Abelian gauge groups, however, $Y_4$ is singular and the computation of its Euler characteristic requires the resolution of $Y_4$ to $\overline{Y}_4$. For general singular Calabi–Yau 4-folds the computation of $\chi/24$ can become quite involved [49, 50]. However, as stressed above, for the class of 4-folds that can be described with the help of toric methods a well-defined algorithm exists that allows one to deduce $\chi/24$, see [43–45] for examples of this type in the context of global F-theory GUT models. Furthermore, for global Tate models with non-Abelian singularities solely along a single divisor $S$, a simple closed formula for $\chi/24$ has been conjectured [43], see the discussion around (100).

Note that the value of $\chi$ for a singular 4-fold $Y_4$ is usually significantly lower than the expression (54) for the corresponding smooth Weierstrass model. This is of direct relevance for the construction of four-dimensional F-theory vacua because moduli stabilization and the engineering of a realistic particle spectrum require the inclusion of 4-form flux. In consistent models the flux is not allowed to overshoot $\chi/24$ as this would necessitate the inclusion of anti 3-branes, which would lead to instabilities\(^7\). Typically, F-theory vacua with high-rank non-Abelian gauge groups are therefore much more constrained than vacua with lower rank.

### 3.3. Matter curves and Yukawa points

In previous sections we have described the localization of the non-Abelian gauge degrees of freedom in F-theory along divisors $D_a$ of the base space $B_n$ of the elliptic Calabi–Yau. Each such divisor carries a $G_a$ gauge theory with four supercharges, corresponding to $N = 1$ super-Yang–Mills in four dimensions. Besides the vector multiplet in the adjoint representation of $G_a$ there arises extra massless charged matter.

In type IIB orientifolds on a Calabi–Yau 3-fold $X$, chiral multiplets in the adjoint representation of $G_a$ correspond to D7-brane moduli [51]. These include brane deformation moduli and Wilson line moduli, counted respectively by $H^{(3)}(D_a) \simeq H^0(S, K_S)$ and $H^{(0,1)}(S)$. The subscripts indicate that only the odd eigenspaces of the cohomology groups under the orientifold action $\sigma$ are relevant. In F-theory there is no clear separation between open and closed string moduli. In fact the brane position moduli become part of the complex structure moduli $H^3(Y_4)$. For a precise account of these fields and the Wilson line moduli in the effective action derived by the dimensional reduction of the dual M-theory we refer to [25].

Apart from these non-chiral matter fields, there exist two types of potentially chiral charged matter: the so-called bulk states that propagate along the whole divisor and matter at the intersection locus of two divisors $D_a$ and $D_b$. The appearance of these chiral multiplets parallels the situation in type IIB orientifolds, which is therefore worthwhile recalling in this context. For this we will follow mostly the discussion in [52].

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\(^7\) One might argue that the anti 3-brane should be stabilized in a warped region as in KKLT and thus do no further harm, but this assumes a sufficient amount of 3-form flux allowed by the 3-brane tadpole to create a warped throat in the first place.
3.3.1. Bulk matter. Consider two parallel stacks of type IIB D7-branes along the same divisor $S$, each carrying internal gauge flux $2\pi a' F_a$ and $2\pi a' F_b$, respectively. This flux is the curvature of two vector bundles $V_a$ and $V_b$ with the Chern character

$$\text{tr}[e^{2\pi a' F}]=\text{ch}(V).$$  \hfill (55)

In the following we restrict ourselves to line bundles $L_a, L_b$ characterized entirely by their first class $c_1(L_i) = \frac{1}{2\pi} \text{tr} \tilde{c}_i^2 F_i$. For simplicity we assume that $S$ is at a general position with respect to the O7-plane so that the gauge group is $U(N_a) \times U(N_b)$. The bulk matter describes chiral multiplets in the bifundamental representation propagating along the whole divisor $S$. Such matter is in general counted by the so-called extension groups

$$\text{Ext}^n(\iota_* L_a, \iota_* L_b), \quad n = 0, \ldots, 3,$$  \hfill (56)

where $\iota: S \to X$ denotes the inclusion of the divisor $S$ into the Calabi–Yau 3-fold $X$. This was derived in detail from an analysis of B-model vertex operators in \cite{53}. The value $n=1$ refers to anti-chiral multiplets transforming as $(\overline{N}_a, N_b)$, while $n=2$ corresponds to chiral multiplets in the same representation. For consistency, the states counted by the groups corresponding to $n=0$ and $n=3$ must be absent. These states do not describe matter fields but rather refer to gauge multiplets. One can show that the sheaf extension groups (equation (56)) translate into certain cohomology groups for the line bundles on the divisor $S$, concretely:

$$\begin{align*}
\text{Ext}^0(\iota_* L_a, \iota_* L_b) &= H^0(S, L_a \otimes L_b^\vee), \\
\text{Ext}^1(\iota_* L_a, \iota_* L_b) &= H^1(S, L_a \otimes L_b^\vee) + H^2(S, L_a^\vee \otimes L_b), \\
\text{Ext}^2(\iota_* L_a, \iota_* L_b) &= H^2(S, L_a \otimes L_b^\vee) + H^1(S, L_a^\vee \otimes L_b), \\
\text{Ext}^3(\iota_* L_a, \iota_* L_b) &= H^3(S, L_a \otimes L_b^\vee).
\end{align*}$$  \hfill (57)

The net chirality follows as \cite{52}

$$I^\text{bulk}_{ab} = \sum_{n=0}^{3} (-1)^n \dim \text{Ext}^n(\iota_* L_a, \iota_* L_b) = -\int_X [S] \wedge [S] \wedge (c_1(L_a) - c_1(L_b)).$$  \hfill (58)

As a variant of this setup one can also consider a single brane stack with gauge group $G$ along $S$ and consider the gauge flux in terms of a non-trivial embedding of a vector bundle with the structure group $G_2 \subset G$. This breaks $G$ to the commutant $G_1$ of $G_2$. In this case chiral multiplets charged under $G_1$ arise, which originate from the adjoint representation of $G$. More precisely, the group theoretic decomposition

$$G \to G_1 \times G_2$$  \hfill (59)

$$\text{ad}_G \to (\text{ad}_{G_1}, 1) \oplus (1, \text{ad}_{G_2}) \oplus \sum (R_x, U_x)$$  \hfill (60)

leads to bulk matter in suitable representations $R_x$ under the visible gauge group $G_1$. The individual states are counted by appropriate cohomology groups with values in the bundle representation $U_x$. For instance, a non-trivial gauge line bundle $L$ associated with a Cartan $U(1)$ of $G$ breaks $G \to \tilde{G} \times U(1)$, and if the representations of $\tilde{G}$ carry $U(1)$ charge $q$, the relevant bundle is $L^q$. This generalizes the appearance of $L_a \otimes L_b^\vee$ in (57) for bifundamental states.

Even though derived originally in the perturbative type IIB context, these expressions continue to hold for general 7-branes in F-theory. This can be derived e.g. with the help of the eight-dimensional twisted field theory introduced in \cite{7}.
3.3.2. Localized matter. From the intersecting brane perspective it is natural to expect that additional massless matter appears at the intersection of two 7-branes. What happens in F-theory is that as two singular loci collide transversely, the singularity type of the elliptic fibration enhances over the intersection locus [54]. This intersection occurs in complex codimension one along the base. In F-theory compactifications on Calabi–Yau 4-folds, two divisors $D_a$ and $D_b$ intersect along a complex curve $C_{ab} = D_a \cap D_b$. The appearance of extra matter can be understood in terms of the tree of zero-size $\mathbb{P}^1$s over each singular locus. Along the intersection locus the two sets of $\mathbb{P}^1$s unite to form the (affine) Dynkin diagram of a new gauge group $G_{ab}$. Its rank is the sum of the gauge groups $G_a$ and $G_b$ along the two divisors. Note that group theoretically there may be several types of enhancements possible, each leading to different types of matter states. By abuse of notation one calls the group $G_{ab}$ the enhanced gauge group along the intersection locus even though in actuality there exists no $N = 1$ SYM theory with a corresponding vector multiplet. However, from the reduction of the M-theory 3-form and from wrapped M2-branes along the various $\mathbb{P}^1$s in the fibre one does find as many massless states along $C_{ab}$ as is necessary to form the adjoint of $G_{ab}$. These states not only include the states propagating along $D_a$ and $D_b$ but also contain extra matter localized at $C_{ab}$. Under the group theoretic decomposition

$G_{ab} \rightarrow G_a \times G_b$ \hspace{1cm} (61)

$\text{ad}_{G_{ab}} \rightarrow (\text{ad}_{G_a}, 1) \oplus (1, \text{ad}_{G_b}) \oplus \sum (R_x, U_x)$ \hspace{1cm} (62)

the localized states transform in the representation $(R_x, U_x)$. These states descend from M2-branes wrapping chains of $\mathbb{P}^1$s associated with the Dynkin diagram of both $G_a$ and $G_b$, in analogy to the chains $\delta_\beta$ defined in the paragraph after (42). Away from the intersection locus $C_{ab}$, such chains are not of zero size and the extra matter becomes massive.

Note that for enhancements to non-classical groups $G_{ab}$ the irreps $(R_x, U_x)$ need not correspond to two-index representations, as would be the case for perturbative type IIB models. The larger set of possibilities is again due to the multiple endpoints of multi-pronged $(p|q)$ strings, which furnish a genuine strong coupling effect. While the Chan–Paton factors of a fundamental $(1|0)$ string with two endpoints always give rise to two-index representations, a multi-pronged string can accommodate more general charges.

An analysis of the twisted defect SYM theory along the matter curves [7] confirms that this massless matter is counted by the same expressions familiar from perturbative type IIB models [53]:

$H^i(C_{ab}, F_x \otimes K_{C_{ab}}\frac{1}{2})$, \hspace{1cm} $i = 0, 1$. \hspace{1cm} (63)

Here $i = 0$ and $i = 1$ respectively count chiral and anti-chiral multiplets in the representation $(R_x, U_x)$, and $F_x$ is schematic for suitable combinations of the gauge flux restricted onto $C_{ab}$. The chiral index follows via Hirzebruch–Riemann–Roch as

$\chi = \dim H^0(C_{ab}, F_x \otimes K_{C_{ab}}\frac{1}{2}) - \dim H^1(C_{ab}, F_x \otimes K_{C_{ab}}\frac{1}{2}) = \int_{C_{ab}} F_x$. \hspace{1cm} (64)

We will be more specific about the concrete chirality formulae in the context of the spectral cover construction of the gauge flux in section 3.5.

The type of singularity enhancement can be read off from the discriminant locus with the help of Tate’s algorithm. As a non-trivial example we consider again the $SU(5)$ gauge theory along the divisor $S$ introduced above. In this case the localized matter states arise at the intersection of the $SU(5)$ brane $S$ with the $I_1$-locus $S_1$. Since the latter carries no non-Abelian gauge group, the singularity gets enhanced by rank 1. Two types of rank-1 enhancements are possible for $SU(5)$ [6, 7].
• Enhancement $A_4 \rightarrow D_5$ corresponding to $SU(5) \rightarrow SO(10)$. The extra matter transforms in the $10$ representation of $SU(5)$ as can be determined from the branching rule
\[ 45 \rightarrow (24)_0 + (1)_0 + 10_2 + \overline{10}_{-2}. \]  
(65)
The subscripts denote the (formal) $U(1)$ charges under the decomposition $SO(10) \rightarrow SU(5) \times U(1)$.

From Tate’s algorithm this $D_5$ enhancement occurs along the curve
\[ P_{10} : \quad w = 0 \quad \cap \quad b_5 = 0, \]  
(66)
along which the discriminant (50) scales like $w^7$, as required for a $D_5$ singularity.

• Enhancement $A_4 \rightarrow A_5$ corresponding to $SU(5) \rightarrow SU(6)$. The $SU(6)$ locus hosts matter in the $5$ arising via
\[ 35 \rightarrow (24)_0 + (1)_0 + 5_1 + \overline{5}_{-1}. \]  
(67)
This enhancement occurs whenever
\[ P_3 : \quad w = 0 \quad \cap \quad P = b_4^2 b_4 - b_2 b_3 b_5 + b_0 b_5^2 = 0, \]  
(68)
consistent with the scaling $\Delta \propto w^6$ in (50).

In addition there can appear localized GUT singlets $1$ on matter curves away from (but possibly intersecting) the $SU(5)$ brane $S$. These localized states appear at self-intersections of the $I_1$-part $S_1$ of the discriminant, along which the gauge group enhances to $A_1$ [37].

3.3.3. Yukawa points. In compactifications on Calabi–Yau 4-folds, two or more matter curves can meet in points. Here the singularity type of the fibre enhances even further by the same mechanism that leads to the enhancement along matter curves. Consider for example the intersection of three matter curves $C_{ab}, C_{bc}, C_{ac}$ in one point. Group theoretically the representations of the matter states along the three curves combine to form the adjoint $ad_{abc}$ of an enhanced gauge group $G_{abc}$. Even though there exists no actual gauge theory associated with $G_{abc}$, the cubic interaction term for the adjoint of this (hypothetical) enhanced gauge group leads to Yukawa interactions for the massless matter [6, 7]. This can be argued by a closer analysis of the local gauge theory description of the geometry responsible for the enhancements. The resulting cubic interaction in the product theory $G_a \times G_b \times G_c$ can be determined by decomposing the triple product $ad_{abc}^3$ into gauge invariant triple products for the irreducible representations of $G_a, G_b$ and $G_c$. This picture is in perfect agreement with expectations from weakly coupled type IIB theory, where Yukawa couplings are known to be localized at the intersection of matter curves.

To illustrate this picture in the context of our $SU(5)$ gauge theory, we note that there exist three possible rank-2 enhancements of $A_4$ to $E_6$, $D_6$ and $A_7$ with the following Yukawa structure [6, 7].

• The $10 \overline{10} 5$ Yukawa is localized at a point of $E_6$ enhancement $b_5 = 0 = b_4$.
• The $10 5 \overline{5}$ is localized at a $D_6$ point $b_5 = 0 = b_3$.

Indeed $b_4 = 0 = b_5$ and $b_2 = 0 = b_3$ correspond to a single and double zero of $P_5$ in agreement with the number of $5$ representations appearing in the coupling.

• At $P = 0 = R$ but $(b_4, b_5) \neq (0, 0)$ the singularity type enhances to $A_6$. This realizes the coupling $5 \overline{5} 1$. The state $1$ represents a (possibly localized) GUT singlet.
3.4. F-theory-heterotic duality

3.4.1. Generalities on the duality. Thus far we have approached F-theory via type IIB orientifolds and via duality with M-theory. In this section we highlight some aspects of the duality with the heterotic string. This duality is a particularly fruitful source of inspiration for many applications to concrete model building.

The basic assertion is that F-theory on an elliptic $K3 : T^2 \rightarrow P^1$ is dual to the heterotic string on $T^2$ [5]. This can be argued by comparing the moduli spaces of both theories. In particular, the heterotic string coupling $\exp(2\phi)$ is dual to the volume of the base $P^1$ of the F-theory $K3$.

This basic duality can be extended ‘adiabatically’ by fibring an elliptic $K3$-surface and, respectively, an elliptic curve over a common $n$-complex-dimensional base $B_n$ in such a way as to produce Calabi–Yau manifolds on either side of the duality. Thus, F-theory on a $K3$-fibred Calabi–Yau $(n+2)$-fold $Y_{n+2} : K3 \rightarrow B_n$ is dual to the heterotic string on the elliptic Calabi–Yau $(n+1)$-fold $Z_{n+1} : T^2 \rightarrow B_n$. Since $Y_{n+2}$ exhibits a double fibration structure as a $K3$-fibration over $B_n$ and, as always, as an elliptic fibration $Y_{n+2} : T^2 \rightarrow B_{n+1}$, the base $B_{n+1}$ is by itself $P^1$-fibred, $B_{n+1} : P^1 \rightarrow B_n$. Therefore, only a very special subclass of F-theory compactifications possesses a straightforward heterotic dual in terms of an elliptic fibration $Z_{n+1}$.

On the heterotic side, a perturbative $SO(32)$ or $E_8 \times E_8$ gauge group descends from the respective ten-dimensional theory by compactification on a smooth Calabi–Yau. If present, singularities on the heterotic compactification space lead to extra gauge group factors not related to this perturbative gauge group which can be of potentially huge rank and which are due to massless non-perturbative states. In the following we will focus on $E_8 \times E_8$ heterotic models with only the perturbative group, i.e. on models for which the elliptic fibration $Z_{n+1}$ is smooth. Heterotic models on smooth Calabi–Yau spaces are defined in terms of a holomorphic vector bundle $V_1 \oplus V_2$ with structure group $H_1 \times H_2$ embedded into $E_8 \times E_8$. This breaks the gauge group down to the commutant $G_1 \times G_2$. The vector bundle data must therefore map into the singularity structure of the elliptic fibration $Y_{n+2}$ on the dual F-theory side.

Heterotic/F-theory duality is particularly powerful in compactifications to six dimensions, where it relates the heterotic string on a $K3$ which is elliptically fibred over $P^1$ to F-theory on a $K3$-fibration over that same $P^1$. This implies that the base $B_2$ of the F-theory elliptic fibration is itself a $P^1$-fibration over $P^1$. Such a fibration is called a rationally ruled or Hirzebruch surface $F_k$, where $k = 0, 1, \ldots$ determines the structure of the fibration. The geometry of elliptic fibrations over $F_k$ and applications to F-theory/heterotic duality in six dimensions have been discussed in detail in the pioneering [29] (more general elliptic 3-folds are analysed in [40]). The value $k$ of the F-theory base $F_k$ determines the second Chern class or instanton number of the heterotic bundle $V_1 \times V_2$ embedded into $E_8 \times E_8$ as $(12 - k, 12 + k)$, $k = 0, 1, \ldots, 12$. The heterotic dilaton is now related to the volume of the fibre and of the base $P^1$ of the F-theory $B_2$ as

$$\exp(2\phi) = \frac{\text{vol}_f}{\text{vol}_b}. \tag{69}$$

For compactifications to four dimensions, the set of possible F-theory $K3$-fibrations is constrained by the fact that only a small number of complex base spaces $B_2$ allow for elliptic Calabi–Yau spaces $Z_3 : T^2 \rightarrow B_2$ as required on the heterotic side. In fact, for smooth heterotic compactifications with $\mathcal{N} = 1$ supersymmetry $Z_3$ must have $SU(3)$ holonomy and $B_2$ can only be a (blow-up in $r$ points of) $P_2$, a (blow-up of) $F_k$ or the Enriques surface $K3/Z_2$ [55] (see [40] for more explanations). The base $B_1$ of the dual F-theory elliptic 4-fold is then a $P^1$-fibration over these complex surfaces. Such a fibration is characterized by a
line bundle $T$ over $B_2$ with first Chern class $c_1(T) = t$ as follows. Consider the rank 2 bundle $O \oplus T$ over $B_2$. Its fibre consists of two copies of $C$, the fibres of the trivial line bundle $O$ and of $T$. In the same manner as one forms a $\mathbb{P}^1$ by projectivizing two complex coordinates $(0,0) \neq (z_1,z_2) \sim \lambda (z_1,z_2)$, a $\mathbb{P}^1$-fibration can be obtained as the projectivization $\mathbb{P}(O \oplus T) = B_3$. That is, one projectivizes each fibre $C \oplus C$ and then fibres over $B_2$. Let $r = c_1(O(1))$, where by abuse of notation $O(1)$ is the line bundle over $B_3$ that reduces, along each $\mathbb{P}^1$ fibre, to $O_{\mathbb{P}^1}(1)$. Then one can show that [56]

$$r(r + t) = 0, \quad c_1(B_3) = c_1(B_2) + 2r + t. \quad (70)$$

The upshot is that the class $t$ generalizes the integer $k$ characterizing the Hirzebruch surface $\mathbb{F}_1$ for $\text{F}$-theory in six dimensions. In order to understand the generalization of the relation between $k$ and the instanton number $(12 - k, 12 + k)$ of the heterotic bundle, we need to familiarize ourselves with the construction of vector bundles on elliptic 3-folds $Z_3$. Thanks to the work of Donagi [57] and Friedman, Morgan and Witten [56] a large class of holomorphic vector bundles on elliptic fibrations are known in terms of the spectral cover construction. In the following we give a brief summary of the basic idea and quantities characterizing a spectral cover bundle.

3.4.2. The heterotic spectral cover construction. The construction of a rank $n$ spectral cover bundle involves two concepts: that of a spectral surface $C^{(n)}$ and of a spectral line bundle $N$, both describable in terms of cohomological data on $B_2$. Under $\text{F}$-theory/heterotic duality these data map to the geometry of the singular $Y_3$ and to the gauge flux $G_4$.

The basic idea of the spectral cover construction is to first construct a stable $(S) U(n)$ bundle on the elliptic fibre over each point of the base and then extend it over the whole manifold $Z_3$ by gluing the data together suitably. Recall that in general, an $(S)U(n)$ bundle defines a rank $n$ complex vector bundle. Its restriction to the elliptic fibre $E_b$ over $b \in B_2$ can be shown to be isomorphic to the direct sum of $n$ complex line bundles

$$V|_{E_b} = N_1 \oplus \cdots \oplus N_n, \quad (71)$$

each of which has to be of zero degree. If $G = SU(n)$ as opposed to $U(n)$, $V|_{E_b}$ must in addition be of trivial determinant, i.e. $\bigotimes_{i=1}^{n} N_i = O_{E_b}$. The zero degree condition on $N_i$ implies that there exists for each $N_i$ a meromorphic section with precisely one zero at some $Q_i$ and a pole at $p$, the zero of the elliptic curve. That is, $N_i = O_{E_b}(Q_i - p)$. Consequently, stable $(S) U(n)$ bundles on an elliptic curve are in one-to-one correspondence with the unordered $n$-tuple of points $Q_i$ and the reduction of $U(n)$ to $SU(n)$ is encoded in the additional requirement that $\sum_i (Q_i - p) = 0$ in the group law of the elliptic curve.

Having understood the restriction of a rank $n$ bundle $V$ to each elliptic fibre, [56] proceeds to construct the whole of $V$. In intuitive terms, the above implies that over an elliptically fibred manifold a $U(n)$ vector bundle determines a set of $n$ points, varying over the base. More precisely, the bundle $V$ over $Z_3$ with the property [56]

$$V|_{E_b} = \bigoplus_{i=1}^{n} O(Q_i - p) \quad (72)$$

uniquely defines an $n$-fold (ramified) cover $C^{(n)}$ of $B_2$, the spectral cover. It is defined by a projection

$$\pi_n : C^{(n)} \to B_2 \quad \text{such that} \quad C^{(n)} \cap E_b = \bigcup_i Q_i. \quad (73)$$

That is, the intersection points of $C^{(n)}$ with the elliptic fibre are the $n$ points characterizing the restriction of the bundle $V$ to the fibre. $C^{(n)}$ is conveniently described, as a hypersurface in $Z_3$,
by its Poincaré 2-form. The Weierstrass model $Z_3$ possesses a section $\sigma$ which identifies the base $B_2$ as an element of $H^2(B_2, \mathbb{Z})$. This section has the important property

$$\sigma \cdot \sigma = -\sigma c_1(S).$$

(74)

The class of $C^{(n)}$ can then be written as

$$[c^{(n)}] = n\sigma + \pi^* \eta(\eta) \in H^2(Z_3, \mathbb{Z})$$

(75)

for $\eta$ some effective class in $H^2(B_2, \mathbb{Z})$. In particular the first piece $n\sigma$ shows that the spectral surface is an $n$-fold cover of the base $B_2$.

Several distinct bundles over $Z_3$ may well give rise to the same spectral cover $C^{(n)}$ since the latter only encodes the information about the restriction of $V$ to the fibre $E_b$. To recover $V$ from the spectral data we need to specify in addition how it varies over the base, i.e. $V|_{B_2}$. As discussed in [56] this is uniquely accomplished by the so-called spectral line bundle $\mathcal{N}$ on $C^{(n)}$ with the property

$$\pi_{n*}\mathcal{N} = V|_{B_2}.$$  

(76)

For the first Chern class $c_1(\mathcal{N}) \in H^2(C^{(n)}; \mathbb{Z})$ of the spectral bundles one can make the general decomposition ansatz [56, 58]

$$c_1(\mathcal{N}) = \frac{r}{2} + \gamma.$$ 

(77)

Here we have abbreviated

$$r = -c_1(c^{(n)}) + \pi^*_n c_1(S), \quad \gamma = \frac{1}{n} \pi^*_n c_1(V) + \gamma_n,$$

(78)

where $\gamma_n$ is chosen such that it satisfies $\pi_{n*}\gamma_n = 0$. This yields

$$\gamma_n = \frac{\lambda}{n} (n\sigma - \pi^*_n \eta + n\pi^*_n c_1(S)),$$

(79)

for a number $\lambda \in \mathbb{Q}$ subject to certain constraints to be discussed shortly. Let us further parametrize $c_1(V)$ by some element $\zeta \in H^2(S; \mathbb{Z})$ [58]:

$$\zeta = c_1(V).$$

(80)

Note that this bundle exists for a generic complex structure since it only involves $\sigma$ and the pullback of classes from $B_2$.

The parameter $\lambda \in \mathbb{Q}$ has to be chosen such that $c_1(\mathcal{N})$ defines an integer class in $H^2(C^{(n)}; \mathbb{Z})$. On the non-Calabi–Yau space $X$ the adjunction formula leads to

$$-c_1(C^{(n)}) = (n-2)\sigma + \pi^*_n(\eta - 2c_1(B_2)).$$

(81)

Putting everything together, we have

$$c_1(\mathcal{N}) = -\sigma + n\left(\frac{1}{2} + \lambda\right) \sigma + \left(\frac{1}{2} - \lambda\right) \pi^*_n \eta$$

$$+ \left(\frac{-1}{2} + n\lambda\right) \pi^*_n c_1(B_2) + \frac{1}{n} \pi^*_n \zeta.$$ 

(82)

$$+ \left(\frac{-1}{2} + n\lambda\right) \pi^*_n c_1(B_2) + \frac{1}{n} \pi^*_n \zeta.$$ 

(83)

For example, for an $SU(5)$ bundle, integrality of $c_1(\mathcal{N})$ puts the value of $\lambda \in \mathbb{Q}$ subject to the constraints

$$5 \left(\frac{1}{2} + \lambda\right) \in \mathbb{Z}, \quad \left(\frac{1}{2} - \lambda\right) \eta + (5\lambda - \frac{1}{2}) c_1(S) \in H^2(S; \mathbb{Z}).$$

(84)

From these data one can compute the higher Chern classes. For an $SU(n)$ bundle these are [56, 59]

$$\int_{B_2} c_2(V) = \int_{B_2} \eta \sigma - \frac{1}{24} \chi_{SU(n)} - \frac{1}{2} \int_{B_2} \pi_{n*}(\eta^2),$$

$$\int_{B_2} c_3(V) = \lambda \eta (\eta - nc_1(B_2)).$$

(85)

$$\int_{B_2} c_2(V) = \int_{B_2} \eta \sigma - \frac{1}{24} \chi_{SU(n)} - \frac{1}{2} \int_{B_2} \pi_{n*}(\eta^2),$$

$$\int_{B_2} c_3(V) = \lambda \eta (\eta - nc_1(B_2)).$$

where $\chi_{SU(n)} = \int_{B_2} c_1^2(B_2)(n^3 - n) + 3n \eta (\eta - nc_1(B_2)).$
To summarize, a $U(n)$ spectral cover bundle is characterized by the following topological data:

- the class of the spectral surface $[\mathcal{C}^{(n)}] = n \sigma + \pi^* \eta$, $\eta \in H^2(B_2, \mathbb{Z})$;
- the first Chern class of the spectral line bundle $c_1(N)$ as in (77).

Under F-theory-heterotic duality the bundle data map partly into the singular geometry of $Y_4$ and partly into the gauge flux $G_4$. The gauge groups $G_1$ and $G_2$ are localized on the divisors given by the base $B_2$ located at the north and south pole of the $\mathbb{P}^1$ that constitutes the fibre of $B_3 : \mathbb{P}^1 \to B_2$.

- The classes $\eta_i$ for the two bundles embedded into $E_8^{(1)}$ and $E_8^{(2)}$ correspond on the F-theory side to

$$\eta_1 = 6c_1(B_2) - t, \quad \eta_2 = 6c_1(B_2) + t, \quad (86)$$

where $t = c_1(T)$, see the discussion around (70).

- The quantity $\gamma_u$ appearing in (79) maps into $G_4$ flux; in particular

$$\int_{\mathcal{Y}_4} G_4 \wedge G_4 = - \int_{B_3} \pi_{n*}(\gamma_1^2 + \gamma_2^2). \quad (87)$$

In [56], also methods for the construction of gauge bundles with a more general structure group including $E_6, E_7, E_8$ are developed. If one embeds a bundle with the structure group $H_1 \times E_8$ into $E_8 \times E_8$, the visible heterotic gauge group is $G_1$. On the F-theory side this maps into a single gauge group $G_1$ along the base of the $\mathbb{P}^1$-fibration $B_3$. Note that this is precisely the structure encountered for global Tate models.

### 3.5. The spectral cover construction for F-theory models

Let us come back to general F-theory models on an elliptic fibration $Y_4$. As we have reviewed, the global structure of the Weierstrass model $Y_4$ is specified by the sections $f \in H^0(B_3, K_{B_3}^{-4})$ and $g \in H^0(B_3, K_{B_3}^{-6})$. Locally this can be brought into the form (44) of a Tate model. A special class of fibrations is even globally of the Tate form and thus based on an underlying $E_8$ gauge symmetry broken to the gauge group $G$ along a single divisor $S$.

If one is interested not in the full details of the global 4-fold geometry, but merely in the physics on the divisor $S$, one can restrict the Tate model to the neighbourhood of $S \subset B_3$. This restriction of the Tate constraint to the neighbourhood of $S$ likewise goes under the name spectral cover construction, whose application we just described in the context of heterotic model building. By heterotic/F-theory duality it is clear that spectral covers have a natural appearance also for F-theory compactifications with a heterotic dual. More recently, however, it has been appreciated [61, 62] that spectral covers are useful to describe the geometry and gauge flux of F-theory compactifications even without (simple) heterotic duals—at least in and to some amount even beyond a local picture.

#### 3.5.1. The general philosophy

Before discussing the technicalities, let us try and gain an intuitive understanding of the appearance of the spectral cover construction. For general F-theory models, the essence of the spectral cover idea is to zoom into the local neighbourhood of the divisor $S : w = 0$ within $B_1$ by discarding all terms of higher power in the normal coordinate $w$ that appear in the sections $b_n$, defined in (49) for the case of $SU(5)$. The restrictions of $b_n$ to the divisor $S$,

$$b_n = b_n|_{w=0}, \quad (88)$$
are therefore sections entirely on $S$. In this local picture the brane $S$ is described as the base of the bundle $K_S \rightarrow S$, given by $s = 0$. The neighbourhood of $S$ is then modelled by a spectral surface viewed as a divisor of the total space of $K_S$. In the following we will concentrate on the Tate model for an $SU(5)$ GUT symmetry along $S$ with associated spectral surface

$$C^{(5)} : b_0s^5 + b_2s^3 + b_3s^2 + b_4s + b_5 = 0.$$  \hspace{1cm} (89)$$

One can think of $C^{(5)}$ as encoding the information about the discriminant locus in the local vicinity of $S$. In particular, as we will see the intersections of $C^{(5)}$ and $S$ determine the matter curves (66) on $S$. It is also clear from relation (88), though, that all the information in $b_i$ contained in terms higher in $w$ is lost in the spectral cover approach.

The spectral cover approach to F-theory model building serves in particular as an auxiliary construction to construct the gauge flux required for chirality of the model. Given the local nature of the spectral covers, it is reasonable to suspect that this yields a correct description of the gauge flux near the brane $S$ and in particular along the matter curves on $S$. This is good news as it is the restriction of the fluxes onto these curves which governs the chirality of a model, but more work is needed to fully understand the continuation of the fluxes in a global construction.

Recall from section 3.4 that in models with a heterotic dual, the elliptic 4-fold $Y_4$ also has the structure of a $K3$-fibration $K3 \rightarrow B_2$ over a complex surface $B_2$, i.e. the base space $B_3$ of the elliptic fibration $Y$ is itself globally $\mathbb{P}^1$-fibred over $B_2$. If in addition on the heterotic side the second $E_8$ factor is broken by an $E_8$ bundle, then the only non-Abelian gauge group is localized along $B_2$, which in the $SU(5)$ example we would identify with the GUT divisor $S$. The GUT divisor is therefore the base of a globally defined fibration in models with heterotic dual. In general F-theory models, we have seen that this is not the case. However, from the discussion of the split resolution in section 2.5 we know that one can locally view $S$ as the basis of an $ALE$ fibration which describes the singularity structure along $S$. The $ALE$ fibre contains a distinguished set of 2-cycles $\Gamma_{E_8}$ whose intersection form is related to the Cartan matrix of $E_8$. For a generic non-zero size of these 2-cycles the $E_8$ symmetry is broken. If the divisor $S$ exhibits enhanced gauge symmetry $G$ this is because some of the $\Gamma_{E_8}$, called $\Gamma_G$, in the following, in the fibre shrink to zero size. The intersection matrix of the 2-cycles $\Gamma_H$ with non-zero size is related to the Cartan matrix of the commutant $H \subset E_8$ of $G$.

This picture is of course very reminiscent of the breaking of $E_8$ to $G$ in the heterotic string by means of a gauge bundle with the structure group $H$. Furthermore, as seen at the end of section 3.4, in F-theory some of the degrees of freedom of the heterotic vector bundle are encoded purely geometrically, while others map to the gauge flux. The geometric part is interpreted in the local field theory of [7] as encoding the vacuum expectation value of the Higgs field $\varphi \in H^0(S, K_S)$ associated with the normal fluctuations of the 7-brane. The spectral cover now is designed to describe the size of the non-zero 2-cycles responsible for the breaking of $E_8$ to $G$ along $S$.

Finally we stress once again that for the special case of global Tate models the underlying $E_8$ structure is exact. It is then clear that the spectral cover construction, which geometrizes the breaking of $E_8$ to $G$ via the Higgs bundle of the structure group $H$, has a chance to capture more than just the very local neighbourhood of the divisor $S$.

3.5.2. Technical details of spectral covers for an $SU(5)$ model. In what follows we restrict ourselves to the spectral cover description of a $G = SU(5)_{\text{GUT}}$ singularity along a divisor $S \subset B_3$. The complement of $SU(5)_{\text{GUT}}$ in $B_3$ is denoted by $H = SU(5)_{\perp}$. For more background and for details on more general configurations we refer to [61, 62].

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The starting point is to construct an auxiliary non-Calabi–Yau 3-fold $\mathcal{W}$ as a fibration over $S$ which encodes the singular geometry of $S$ in $B_5$. This space $\mathcal{W}$ serves as a compactification of the total space of the bundle $K_S \to S$ introduced before. We will therefore think of $S$ either as a divisor on $B_5$ or as the base of a fictitious 3-fold $\mathcal{W}$. The definition of $\mathcal{W}$ is as the projectivized bundle over the GUT divisor $S$:

$$\mathcal{W} = \mathbb{P}(O_S \oplus K_S), \quad p_\mathcal{W} : \mathcal{W} \to S,$$

(90)

where $p_\mathcal{W}\mathcal{W}$ is the projection to the base of the bundle. The base $S$ is viewed as the vanishing locus of the section $\sigma$ in $\mathcal{W}$. This section satisfies the important relation

$$\sigma \cdot \sigma = -\sigma c_1(S).$$

(91)

This should ring a bell and remind us of relation (74) encountered for the heterotic spectral cover construction on the physical elliptic Calabi–Yau space $Z_3$. Indeed from now on the construction formally proceeds in the same fashion as described in section 3.4.2, replacing the base $B_5$ of the total space of the bundle $K_S \to S$, which encodes the singular geometry of $S$, by $\mathcal{W}$.

The starting point is to construct an auxiliary non-Calabi–Yau 3-fold $W$ as a fibration over $S$, where $p_W : W \to S$. The spectral cover is constructed as a 5-fold cover of $S$. As for the flux on the matter branes, the spectral cover

$$c_1(\mathcal{W}) = 2\sigma + 2c_1(S).$$

(92)

The spectral cover is constructed as a 5-fold cover of $S$ within $\mathcal{W}$,

$$\pi_5 : C^{(5)} \to S.$$

(93)

While in the heterotic context the intersection points of $C^{(5)}$ with the elliptic fibre encode the information about the vector bundle $V$ restricted to the fibre, see equation (73), here the five intersection points with the $\mathbb{P}^1$-fibre denote the five eigenvalues of the $SU(5)_H$-valued Higgs field $\varphi$ along $S$ [61, 62], which in turn specifies the local neighbourhood of $S$ inside $B_5$ [7, 62].

Given the implicit underlying $E_8$ structure of the ALE fibration, the massless matter representations of $G = SU(5)$ can be understood as the irreducible representations $\mathbf{R}_i$ in the decomposition $248 \to \sum \text{ad}_i + \sum (\mathbf{R}_i, U_i)$:

$$248 \mapsto (24, 1) + (1, 24) + \{(10, 5) + (\mathbf{5}, \mathbf{10}) + \text{h.c.}\}.$$  

(94)

The matter curve $P_{10}$ is the locus $b_5 = 0$ on $S$. It is associated with the spectral cover in the fundamental representation of $H = SU(5)_H$ because the $\mathbf{10}$ appears as $(10, 5)$ in (94). Let us also define the object $P_{10}$ viewed as a curve in $\mathcal{W}$ [62]:

$$P_{10} = C^{(5)} \cap \sigma \subset \mathcal{W}.$$  

(95)

Then the matter curve on $S$ is related to $P_{10}$ as

$$[P_{10}] = [P_{10}]_\sigma = (5\sigma + \pi_5(\eta))_\sigma = \eta - 5c_1(S).$$  

(96)

with the help of (91), i.e. the restriction of $P_{10}$ is cohomologically equivalent to the matter curve $P_{10}$.

The matter curve for the $\mathbf{5}$ on $X$ is more complicated and was analysed in detail in the context of the heterotic string in [63–65], to which we refer for more details.

### 3.5.3. Gauge flux in $SU(5)$ GUT models

So far all that the spectral cover approach has done for us is to rewrite the geometric data in a seemingly more complicated manner. Its actual power, however, becomes apparent once one includes the gauge flux into the compactification. In the context of $SU(5)$ GUT theories one distinguishes between the gauge flux along the GUT divisor $S$ itself and the flux along the matter branes which constitute the $I_1$ locus of the discriminant. The first type of gauge flux takes values in $G = SU(5)_\text{GUT}$ and will thus further break the $SU(5)$ GUT symmetry. As for the flux on the matter branes, the spectral cover...
approach makes use of the fact that $C^{(5)}$ describes the geometry in the vicinity of $S$ and in particular the local geometry of the $I_1$ component. In the machinery of the spectral cover, the gauge flux on $I_1$ is therefore given in terms of the so-called spectral line bundle $\mathcal{N}$ along $C^{(5)}$ defined by its first Chern class

$$c_1(\mathcal{N}) \in H^{(1,1)}(C^{(5)}; \mathbb{Z}).$$

(97)

Since $C^{(5)}$ is a 5-fold cover of $S$ in $\mathcal{W}$ one can push this line bundle forward to $S$ via $\pi_5$. This defines a rank-5 vector bundle

$$V = \pi_5^* \mathcal{N}$$

(98)
on $S$. Following the general logic of the ALE fibration over $S$ the structure group of this bundle $V$ is identified with the commutant $SU(5)_\perp \subset E_8$ of the GUT $SU(5)$ along $S$ and therefore indeed associated with the flux on $I_1$.

Therefore the construction of the gauge flux on the matter branes is reduced to the problem of constructing an $SU(5)_H$ bundle $V$ on the basis of the $\mathbb{P}^1$-fibration $\mathcal{W}$. At this point we can again refer to the discussion of the spectral line bundle and its associated spectral cover bundle in the heterotic context, which proceeds, mutatis mutandis, in the same manner. The spectral cover approach thus provides a concrete dictionary to describe a certain subclass of gauge fluxes in terms of a few parameters. Moreover, it allows for the computation of the chiral index for the various matter representations localized along the matter curves in terms of these input parameters and is therefore of considerable practical use for model-building purposes. It would lead too far to derive these formulae for the chirality in full detail here. We simply state without proof that the chiral index of states in the $10$ representation localized on the curve $P_{10}$ on $S$ is given by

$$\chi_{10} = \sigma \cdot C^{(5)} \cdot \gamma_u = -\lambda \int_S \eta \left( \eta - 5c_1(S) \right),$$

(99)

with $\gamma_u$ as in equation (79) for $SU(5)_H$ bundles. Consistently, this is also the number of chiral families in the $\overline{5}$ representation.

Let us conclude this general presentation of the spectral cover construction by taking up our initial concerns about its global validity. We have stressed several times that as of this writing it yet remains to find a general description of the gauge flux in terms of global $G_4$ flux. Nonetheless it is remarkable that the spectral cover construction does capture some of the global aspects of the geometry correctly. More precisely this is the case for the special subclass of elliptic fibrations that can globally be described as a Tate model with the non-Abelian gauge group $G \subset E_8$ solely along a divisor $S$. If we take the spectral cover construction at face value for a second one can conjecture a simple closed expression for the Euler characteristic of the resolution $\overline{Y}_G$ of the singular 4-fold given by

$$\chi(\overline{Y}_G) = \chi^*(Y) + \chi_H - \chi_{E_8}.$$  

(100)

Here $\chi^*(Y) = 12 \int_B c_1(B_3)c_2(B_3) + 360 \int_B c_1^3(B_3)$ denotes the expression valid for a smooth elliptic fibration over the base $B_3$, and the remaining two terms are listed in table 1. They involve data solely on $S$ and reflect the underlying $E_8$ structure of the spectral cover construction, in which we think of first enhancing the singularity over $S$ to $E_8$ and then breaking it via the $H$-bundle down to $G$. This formula can be derived for models with a heterotic dual by F-theory-heterotic duality. The point is now that for global Tate models the underlying $E_8$ structure is correct globally. Indeed for global Tate models in which an explicit resolution of the singularities is available, the result of (100) can be compared to the value of $\chi(\overline{Y}_G)$ computed explicitly via resolution. This has been performed in [43, 44] for a number of cases
using the machinery of toric geometry, finding perfect agreement. For such geometries it is not unreasonable that also the gauge flux constructed via the spectral covers has a global extension. It is this expectation that underlies the global F-theory models as existent in the literature as of this writing, but a more complete understanding of the gauge flux without relying on the spectral cover construction is desirable. For example [66] proposes a global extension of spectral cover fluxes in terms of a so-called spectral divisor designed so as to reproduce the expressions for the chirality of states along the GUT branes, i.e. the local aspects associated with non-Abelian gauge symmetry. It remains to be seen how to capture also genuinely global aspects of the flux such as the integral $\int_Y G_4 \wedge G_4$ or the chiral index of singlets under the non-Abelian group $G$.

4. Phenomenological applications to GUT model building

As stressed in the introduction, F-theory combines two characteristic aspects of type II orientifolds with D-branes on the one hand and of heterotic string vacua on the other which are of general phenomenological interest and of significant use in the context of realistic GUT model building. These are

- the appearance of exceptional gauge groups as in $E_8 \times E_8$ heterotic string constructions and
- the localization of gauge degrees of freedom, matter states and Yukawa interactions as in perturbative D-brane models.

Beginning with [6–9] the prospects of F-theory compactifications for GUT phenomenology have recently been under intense investigation. While in principle applicability of F-theory is by no means restricted to GUT models, the relevance of exceptional gauge groups in the context of unification singles out this class of constructions as the one with the most distinctive F-theoretic features as compared with intersecting brane models in perturbative orientifolds. Promising starting points for the construction of GUT models are the gauge groups $SO(10)$ and $SU(5)$. Most efforts in the F-theory literature have focused on the latter, mainly due to complications with GUT breaking via internal fluxes.

Given the localization of gauge degrees of freedom in F-theory, a considerable number of phenomenological questions can be discussed already at the level of local models. Among these are the structure of the GUT matter curves and the details of GUT matter Yukawa couplings. Several other issues, by contrast, can only be addressed in a satisfactory manner in the context of a globally defined compactification. This is true in particular for all aspects of $U(1)$ symmetries—including $SU(5)$ GUT breaking via hypercharge flux and Abelian selection rules—and the physics of GUT singlets (e.g. certain aspects of neutrino physics), which are localized away from the GUT divisor.

| $G = \frac{E_8}{H}$ | $H$ | $\chi_H$ |
|----------------------|-----|---------|
| $E_8$, $n \leq 5$   | $SU(n)$ | $\int_S c_1^2(S)(n^3 - n) + 3n \eta(n - nc_1(S))$ |
| $SU(3)$             | $E_6$  | 72 $\int_S (\eta)^2 - 7\eta c_1(S) + 13c_1^2(S)$ |
| $SU(2)$             | $E_7$  | 18 $\int_S (8\eta^2 - 64\eta c_1(S) + 133c_1^2(S))$ |
|                     | $E_8$  | 120 $\int_S (3\eta^2 - 27\eta c_1(S) + 62c_1^2(S))$ |
It is beyond the scope of these lectures to survey all exciting aspects of F-theory GUT model building that have emerged recently; rather we will outline some of the general philosophy behind the construction of SU(5) GUT models. Armed with this background the interested reader can easily delve into more advanced topics.

4.1. SU(5) GUT models and the principle of decoupling

SU(5) is the mother of all GUT groups. In Georgi–Glashow SU(5) models [67], the embedding of the MSSM gauge group SU(3) × SU(2) × U(1)Y rests on the identification of the U(1)Y generator with the Cartan generator \( T = \text{diag}(2, 2, 2, -3, -3) \) within SU(5). The MSSM matter is organized into SU(5) multiplets as

\[
\begin{align*}
10 &\leftrightarrow (Q_L, u_R^c, e_R^c), \\
\bar{5}_m &\leftrightarrow (d_R^c, L), \\
1 &\leftrightarrow \nu_R^c, \\
\bar{5}_H &\leftrightarrow (T_u, H_u), \\
\bar{3}_H &\leftrightarrow (T_d, H_d).
\end{align*}
\]

The triplets \( T_u, T_d \), which are not present in the MSSM, must receive high-scale masses via doublet–triplet splitting. An alternative SU(5) GUT scenario called flipped SU(5) [68, 69] starts from the gauge group SU(5) × U(1)X and the MSSM matter is related to the identifications (101) by ‘flipping’ \( e_R^c \leftrightarrow \nu_R^c \) and \( d_R^c \leftrightarrow u_R^c \). Since U(1)Y arises as a combination of U(1)X and the Cartan generator \( T = \text{diag}(2, 2, 2, -3, -3) \) of SU(5), flipped SU(5) is a unified model only if SU(5) × U(1)X is itself embedded into a higher group such as SO(10). For definiteness we focus for now on Georgi–Glashow SU(5) models.

The geometric origin [6–9] of the SU(5) GUT gauge group, the matter representations and the Yukawa couplings in terms of codimension 1, 2 and 3 singularities of the elliptic fibration has been discussed in sections 3.1 and 3.3. Matter charged under SU(5) is a unified model only if SU(5) × U(1)X is itself embedded into a higher group such as SO(10). For definiteness we focus for now on Georgi–Glashow SU(5) models.

The MSSM Yukawa couplings follow from the SU(5) GUT interactions by decomposition of the SU(5) representations as

\[
\begin{align*}
10 &\rightarrow Q_L u_R^c H_u, \\
10 &\rightarrow Q_L u_R^c H_u, \\
10 &\rightarrow Q_L u_R^c H_u + Q_L d_R^c H_d.
\end{align*}
\]

The 10 10 5H Yukawa coupling localizes at a point of \( E_6 \) singularity, which is a strong coupling phenomenon. It is for the sake of this coupling that exceptional symmetry is essential. In perturbative type II orientifolds this interaction is forbidden by global U(1) selection rules and can only be generated by D-brane instantons [70]. The natural presence of this crucial coupling, from which the up-quark masses descend, is the prime motivation to pursue F-theory as the framework for GUT models with 7-branes; in perturbative models, the volume of the D-brane instanton generating the 10 10 5H Yukawa coupling would have to be tuned so as to prevent too large suppression, see [52] for recent examples. On the other hand, in flipped SU(5) models, where the 10 10 5H coupling gives rise to down-quark masses, such a non-perturbative suppression can be a welcome rationale to argue for the hierarchy between the top and bottom quark mass.

An important question concerns the nature of the GUT brane \( S \), which must be a Kähler surface embedded into the base space \( B \) as a holomorphic divisor. In full generality the only unavoidable restriction on the topology of the GUT cycle \( S \) is that it must support one of the GUT breaking mechanisms which will be discussed in section 4.2. A reasonable, though not
strictly necessary organizing principle, however, is to require the existence of a well-defined decoupling limit for gravity [8]. This paradigm is inspired by the separation of the four-dimensional GUT scale $M_{\text{GUT}} = 10^{16}$ GeV and the Planck scale $M_{\text{Pl}} = 10^{19}$ GeV together with UV completeness of GUT models.

The four-dimensional Planck scale arises from the dimensional reduction of the Einstein–Hilbert term in the Einstein frame

$$S_{\text{EH}} = M^8 \int_{\mathbb{R}^{1,3} \times B} \sqrt{-g} R \quad \Rightarrow \quad M_{\text{Pl}}^2 = M^8 \text{Vol}(B).$$

(103)

Here $M_s$ is the M-theory fundamental length scale inherited via M/F-theory duality and can be viewed, in the IIB limit, as the value of $\ell_s^{-1}$ in the Einstein frame. On the other hand, the GUT scale, determined by the breaking scale of $SU(5)$ down to the standard model gauge group, is parametrically given by the volume of the GUT divisor $S$:

$$M_{\text{GUT}}^2 \simeq \text{Vol}^{-1}(S).$$

(104)

For example in the context of GUT breaking via hypercharge flux as discussed in section 4.2 this approximate relation arises because the volume of $S$ sets the flux-induced mass of the $X$–$Y$ gauge bosons within $SU(5)$; after all these states propagate along the whole divisor $S$. The observed hierarchy of about $10^{-3}$ between $M_{\text{GUT}}$ and $M_{\text{Pl}}$ translates into a small hierarchy between the typical radii of the GUT brane and the six-dimensional F-theory base of approximately

$$\ell_s \ll R_S \ll R_B, \quad x = 10^{-16} \text{ GeV}^{-1}.$$  

(105)

The relation to $\ell_s$ has been chosen in such a way that in addition

$$\alpha_{\text{GUT}}^{-1} = M_s^4 \text{Vol}(S) \simeq 24.$$

(106)

This equality in turn follows parametrically by reduction of the eight-dimensional Yang–Mills action $S_{YM} = M_s^4 \int_{\mathbb{R}^{1,3} \times S} F^2$.

Note that for phenomenological viability of a model it is sufficient to stabilize the moduli in agreement with the crude estimate of (105). A fortiori, it is often postulated [8, 9, 71] that the GUT brane $S$ allow for a limit

$$\text{Vol}(B) \to \infty, \quad \text{Vol}(S) \text{ finite}$$

(107)

such as to decouple gravity completely—at least in principle. Alternatively to this physical decoupling limit one can consider the mathematical decoupling limit [71]

$$\text{Vol}(S) \to 0, \quad \text{Vol}(B) \text{ finite},$$

(108)

even though the two are not completely equivalent [44]. This mathematical decoupling limit can be taken if the surface $S$ is Fano, which amounts to requiring that $\int_C K_S^{-1} > 0$ for every holomorphic curve $C$ in $S$. The list of such Fano surfaces is very restrictive and consists of the del Pezzo surfaces $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ and $dP_r, r = 1, \ldots, 8$. The latter are defined as $\mathbb{P}^2$ with $r$ points in the generic position blown up to $\mathbb{P}^1$. Since these surfaces are used extensively in the F-theory GUT literature, we briefly collect some of their basic topological properties. The second homology $H_2(dP_r, \mathbb{Z})$ is spanned by the elements $l, E_1, \ldots, E_r$, where $E_i$ denote the $i$th blow-up $\mathbb{P}^1$ and $l$ the hyperplane class inherited from $\mathbb{P}^2$. The non-vanishing intersection numbers are $l \cdot l = 1, E_i \cdot E_j = -\delta_{ij}$. The first Chern class of the anti-canonical bundle is given by $c_1(K_{dP_r}) = c_1(dP_r) = 3l + \sum_{i=1}^{r} E_i$ with $c_1^2(dP_r) = 9 - r$ and $c_2(dP_r) = 3 + r$.

Generic Fano surfaces are shrinkable in the sense of (108). At closer inspection, however, the existence of suitable matter curves and possible global embeddings in $SU(5)$ models poses certain restrictions on the shrinkability of the GUT divisor and forbids shrinkability to a point
in generic situations. Rather the GUT brane can only shrink to a curve or to a point in such a way that another divisor shrinks simultaneously. For more information on these restrictions we refer to [44, 71].

4.2. Options for GUT breaking

Let us now discuss possible ways to break $SU(5)$ to the observed $SU(3) \times SU(2) \times U(1)_Y$ MSSM gauge group. As in all brane constructions there exist three options to accomplish this.

- **Via a GUT Higgs** (scalar field) in the adjoint representation $24$ of $SU(5)$. This effectively realizes the GUT breaking mechanism of conventional field-theoretic GUT models. The string theoretic origin of the Higgs would be a brane deformation modulus counted by $H^0(S, K_S)$. An obvious challenge associated with this approach is the generation of a suitable potential for the GUT Higgs field which leads to dynamical symmetry breaking. In principle 7-brane deformation moduli are stabilized by background fluxes, but no concrete setup has been described so far that incorporates dynamical GUT breaking.

- **Via Wilson line moduli**. These correspond to elements of $H^1(S)$ and likewise transform in the adjoint of $SU(5)$. The same remarks concerning the generation of a potential for dynamical symmetry breaking apply with the added complication that Wilson line moduli are not fixed by background fluxes. Alternatively one can consider GUT breaking by discrete Wilson lines. These are available whenever the brane $S$ has a discrete, but non-trivial first homotopy group $\pi_1(S)$. The VEV of the discrete Wilson line is now topological and part of the defining data of the compactification. Note that this is exactly the same strategy as pursued in Calabi–Yau compactifications of the $E_8 \times E_8$ heterotic string with $SU(N)$ gauge bundles. In some sense this is the cleanest approach to GUT symmetry breaking. The implementation of F-theory models along GUT divisors with non-trivial $\pi_1(S)$ has not been achieved in the literature so far, but seems a promising avenue for future research.

- **Via hypercharge flux**. This corresponds to turning on the non-trivial gauge flux $F_Y$ associated with the hypercharge generator $T_Y = \text{diag}(2, 2, 2, -3, -3)$. As for Wilson lines the flux is part of the defining topological data of the compactification and circumvents the quest for the dynamical generation of a symmetry-breaking scalar potential. This approach does not rest on any strong coupling effect and is equally possible for perturbative type IIB models.

Note that the last two GUT breaking mechanisms are not available in conventional, four-dimensional field theoretic GUTs. Studying their possible consequences for the more detailed phenomenology of $SU(5)$ GUTs is therefore particularly interesting.

4.3. Some constraints from hypercharge flux

For definiteness we focus on the last GUT breaking mechanism. This is also the only possibility for del Pezzo surfaces, which have no geometric deformations and no Wilson line moduli. The use of hypercharge flux was suggested in the context of F-theory model building in [8, 9] and first realized in compact models with 7-branes within perturbative type IIB orientifolds in [52]. In IIB language one simply embeds a non-trivial line bundle $L_Y$ along the GUT brane into $SU(5)$ by identifying its structure group with $U(1)_Y$. In F/M-theory language this corresponds to $G_4$ flux of the form $G_4 = F_Y \wedge \omega_Y$. Here $\omega_Y$ is the 2-form dual to the zero-size $\mathbb{P}^1$ in the fibre over $S$ that corresponds to the node in the $SU(5)$ Dynkin diagram associated with the Cartan generator $T_Y$. 

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In the presence of the $U(1)_Y$ gauge flux the GUT matter decomposes into representations of $SU(3) \times SU(2) \times U(1)_Y$ as
\begin{equation}
\begin{aligned}
24 & \mapsto (8, 1)_{0_Y} + (1, 3)_{0_Y} + (1, 1)_{0_Y} + (3, 2)_{5_Y} + (\bar{3}, 2)_{-5_Y}, \\
\bar{5} & \mapsto (\bar{3}, 1)_{2_Y} + (1, 2)_{-3_Y}, \\
10 & \mapsto (3, 2)_{1_Y} + (\bar{3}, 1)_{-4_Y} + (1, 1)_{0_Y}, \\
\bar{5}_H & \mapsto (3, 1)_{-2_Y} + (1, 2)_{3_Y}, \\
5_{\text{H}} & \mapsto (\bar{3}, 1)_{2_Y} + (1, 2)_{-3_Y}.
\end{aligned}
\end{equation}

The cohomology classes counting the MSSM matter contain factors of $L^q_Y$ with $q$ the $U(1)_Y$ charge of the state. In order to guarantee the same number of states within each $SU(5)$ family, the net $U(1)_Y$ flux through each matter curve must therefore vanish [8, 9]. As we will discuss in section 4.4, the absence of dimension 4 proton decay operators requires that the $5$ matter curve $P_5$ splits into a $\bar{5}_m$ matter curve and a Higgs curve. Then what we need for the absence of exotics in incomplete multiplets is
\begin{equation}
c_1(L_Y) \cdot P_{10} = 0 = c_1(L_Y) \cdot P_{5_m}.
\end{equation}

On the other hand, if in addition the Higgs curve splits into two curves $P_{H_1}$ and $P_{H_2}$, the non-zero hypercharge flux through these Higgs curves allows for an elegant solution to the doublet–triplet splitting problem if it is chosen such as to project out the massless $(3, 1)_{2_Y}$ within $\bar{5}_H$ [8]. Having outlined the general approach we next discuss three of the challenges which must be met for constructions with hypercharge flux breaking.

4.3.1. Massless $U(1)_Y$. $U(1)_Y$ must remain massless after GUT symmetry breaking. In the language of IIB 7-branes with the gauge flux, it is well-known that the Chern–Simons coupling (8) leads to a St¨uckelberg mass for $U(1)_Y$ by dimensional reduction of the RR 4-form $C_4$ along 2-cycles on $S$:
\begin{equation}
S_{\text{Stückelberg}} \simeq \int_{S^{4,1}} F_{4_Y}^\text{LD} \wedge c_2^{(1)\text{tr}} T_Y^2 \int_S c_1(L_Y) \wedge t^* \omega_i.
\end{equation}

Here $\omega_i$ denotes a basis of $H^2(B, \mathbb{Z})$ and we have decomposed $C_4 = c_2^{(1)\text{tr}} \wedge \omega_i$ with $c_2^{(1)\text{tr}}$ denoting 2-forms in four dimensions. A mass term for $U(1)_Y$ can only be avoided if the gauge flux $F_Y$ is switched on exclusively along 2-cycles in $S$ which are homologically trivial as 2-cycles in the ambient geometry [51, 72]; in this case $c_1(L_Y)$ is orthogonal to $t^* H^2(B, \mathbb{Z})$. Cycles of this type are said to lie in the relative cohomology of $S$ with respect to $B$. Note that to ensure this topological constraint one needs full control of the global compactification geometry. This is because the question of triviality of a 2-cycle on $S$ can only be answered by studying an implicit embedding of $S$ into a compact geometry.

4.3.2. Absence of massless bulk exotics. The decomposition of the $SU(5)$ GUT matter displayed in (109) contains the fields $(3, 2)_{5_Y} + (\bar{3}, 2)_{-5_Y}$. From the point of view of the MSSM these are exotic matter states which must be absent at the massless level for phenomenological viability of the model. Since they descend from the adjoint of $SU(5)$ they correspond to modes propagating along the entire GUT divisor $S$ and are therefore ‘bulk’ states. From the discussion of bulk matter states in section 3.3 we recall that these states are counted by the cohomology groups of the hypercharge flux $L_Y^q$ along $S$, where $q$ is the $U(1)_Y$ charge. The absence of exotic states $(3, 2)_{5_Y} + (\bar{3}, 2)_{-5_Y}$ therefore requires that $H^i(S, L_Y^{15}) = 0$.

Vanishing cohomology for such a high power of line bundles is difficult to engineer. A way out is to admit a suitably fractional line bundle $L_Y$ instead of the integer quantized bundle.
A clean way to define this is by a twisting procedure that works both in perturbative type IIB [52] and in the $E_8$-based F-theory models [33] discussed in these lectures. For such embeddings, the potential exotics are eventually counted by $H'(S, L_{Y}^{± 1})$, which therefore has to vanish. Here $L_{Y}$ denotes the hypercharge bundle in the twisted embedding. This constraint poses certain restrictions on the type of hypercharge flux switched on along the GUT brane. For example, for a del Pezzo surface it can be achieved if and only if $c_1(L_{Y}) = E_i - E_j$ for $i \neq j$ [8].

As a final remark we note that it is this absence of bulk exotics that cannot be achieved for an analogous breaking of the GUT group $SO(10)$ by internal fluxes [8]. This is the technical reason why $SO(10)$ GUT models have received less attention in the F-theory literature. Alternatively, $SO(10)$ can be broken to flipped $SU(5) \times U(1)_X$ via suitable fluxes, and subsequently the standard field theoretic GUT Higgs mechanism [69] can be invoked to break $SU(5) \times U(1)_X$ to the MSSM gauge group. For recent realizations of flipped $SU(5)$ beyond strictly local models see [45, 73, 74]; an assessment of the phenomenological prospects of flipped $SU(5)$ and its realizations within F-theory has appeared in [75] and references therein.

4.3.3. Gauge coupling unification. The most subtle and controversial challenge of the hypercharge GUT breaking scenario arises in the context of gauge coupling unification [9, 76]. The problem is that—at least from a type IIB perspective—hypercharge flux along the GUT divisor seems to spoil the tree-level equality of the MSSM gauge couplings at the GUT scale. To see this one must obtain the gauge kinetic function $f$ of the four-dimensional GUT gauge theory, defined as

$$S^{(4D)}_{YM} = \frac{1}{2} \text{Re}(f) \int_{\mathbb{R}^4} \text{tr} F \wedge \star F + \frac{1}{2} \text{Im}(f) \int_{\mathbb{R}^4} \text{tr} F \wedge F,$$

from the 7-brane effective action by dimensional reduction. In type IIB 7-brane language, the kinetic Yang–Mills and the topological Chern–Simons term follow from the reduction of $S_{DBI}$ and $S_{CS}$ in equation (8), respectively. The flux-induced corrections to the leading order gauge kinetic function

$$f_S = \frac{1}{g_s} \frac{\text{Vol}_S}{\epsilon_s^4} + i \int_S C_4$$

can be deduced from the contribution to Im($f$) encoded in the Cherns–Simons term proportional to $\int C_0 \wedge \text{tr} F^4$. To this end one takes into account holomorphicity of $f$ together with the fact that $C_0$ and $g_s$ combine into the holomorphic field $\tau = C_0 + \frac{1}{g_s}$. What is important is that the contribution from hypercharge flux differs for the three gauge couplings $\alpha_s, \alpha_w, \alpha_Y$ of the MSSM gauge groups and distorts the gauge coupling relations at the Kaluza–Klein scale. The exact relation depends on the precise group theoretic embedding, see [9] and [76] for two different types of embedding. Final agreement on the interpretation of the physical consequences for unification has not yet been achieved in the literature. A conceptual difficulty is that a derivation of the critical $F^4$-term purely in F/M-theory and without reference to the weakly coupled type IIB language has not been provided so far. In addition, threshold corrections from Kaluza–Klein states have to be taken into account [9]. The conclusions of [76] are that extra thresholds below the GUT scale are required to reconcile the flux-induced splitting of the gauge couplings with one-loop GUT unification. The minimal such threshold could be played by the unavoidable massive Higgs triplets [76], while more radical approaches consider incomplete multiplets of massive exotic matter [77, 78] (see also [79]).
4.4. Proton decay

A classic topic in GUT model building is proton stability. In particular, avoiding dimension 4 and dimension 5 proton decay operators is the key to the phenomenological viability of SU(5) models [80, 81]. We therefore face the question if string compactifications—here those of F-theory—add new ingredients to achieve these requirements. On the one hand, we can seek for stringy realizations of known field theoretic mechanisms such as favourable symmetries. In a more ambitious vein, new types of selection rules might become available in string theory that have no obvious four-dimensional counterpart.

Let us begin with dimension 4 proton decay. Dangerous MSSM operators of the type $u^c R d^c_R d^c R L e^c R, Q L d^c_R$ descend from a potential coupling $10 \mathcal{F}_m \mathcal{F}_m$, which must therefore be prevented. The same holds for its unwanted cousin $10 \mathcal{F}_H \mathcal{F}_H$. It was already noted in [8] that a necessary condition for the absence of $10 \mathcal{F}_m \mathcal{F}_m$ while allowing at the same time for the Yukawa couplings $10 \mathcal{F}_m \mathcal{F}_m$ and $10 \mathcal{F}_H \mathcal{F}_H$ is that the $5$ matter curve splits into (at least) two curves $P_m$ for $\mathcal{F}_m$ and $P_H$ for $\mathcal{F}_H$. From the perspective of the Weierstrass model, the splitting of matter curves corresponds to a non-generic situation that requires the restriction of some of the complex structure moduli, more on that in a second. If this splitting were sufficient to prevent dimension 4 proton decay, it would furnish an example of a geometric selection rule.

As found in [61], however, typically $P_m$ and $P_H$ intersect at points without further singularity enhancements. If this happens, the wavefunction of $\mathcal{F}_m$ and $\mathcal{F}_H + \mathcal{F}_H$ obey the same boundary conditions and dangerous couplings of the type $10 \mathcal{F}_m \mathcal{F}_m$ are re-introduced. To be on the safe side, explicit field theoretic selection rules have to be implemented. The minimal such selection rule—R-parity—could descend from a geometric discrete $Z_2$ symmetry that acts appropriately on the massless modes [61], but no concrete realizations of this idea have been constructed as of this writing. More radically, a (massive) $U(1)$ selection symmetry can forbid unwanted Yukawa couplings [61]. The maybe simplest example of such a $U(1)$ symmetry is $U(1)_X$ with charge assignments [77]

$$10, \quad (\mathcal{F}_m)_{-3}, \quad (\mathcal{F}_H)_{-2} + (\mathcal{F}_H)_2,$$ (114)

but other examples such as a Peccei–Quinn-type $U(1)_{PQ}$ [78] with different charge assignments for $H_u$ and $H_d$ or models with several $U(1)$s [82] have also been considered.

At the level of model building the implementation of such Abelian symmetries requires a further specification of the complex structure moduli of the Weierstrass model. For global Tate models this leads to the so-called $U(1)$ restricted Tate models [37]. In particular the engineering of, say, a $U(1)_X$ symmetry automatically leads to a split of the $\mathcal{F}_m$ and the Higgs curve. Note that as remarked before all questions associated with $U(1)$ symmetries defy a local treatment and require knowledge of the full compactification data of the 4-fold. This is intuitively clear because a VEV of a $U(1)$ charged GUT singlet localized away from the GUT brane $S$ can Higgs the Abelian symmetry. Previously, the implementation of $U(1)$ symmetries had been studied via so-called split spectral covers [77]. Split spectral covers can be regarded as the restriction of $U(1)$ restricted Tate models to the neighbourhood of the GUT brane. As such they are not sufficient to guarantee the presence of $U(1)$ symmetries [37, 83].

Dimension 5 proton decay can in principle be prevented via a missing partner mechanism [8] if also $\mathcal{F}_H$ and $\mathcal{F}_H$ localize on separate curves. One possibility of achieving this is in the context of $U(1)_{PQ}$ extended models [78], where the Abelian selection rule forbids dimension 5 proton decay. For the present-day realizations of this scenario, exotics in incomplete GUT multiplets result as a side-effect because of a tension of condition (110) with the precise structure of the matter curves. These exotics in turn affect gauge coupling unification as outlined in the previous section. Whether or not such incomplete multiplets are strictly
unavoidable within the hypercharge GUT breaking framework if dimension 5 proton decay is to be prevented is still under investigation.

4.5. Further developments

Our presentation of the phenomenological properties of F-theory GUTs has only covered some of the crudest aspects, and many more advanced phenomenological topics have been studied in the literature. For a review devoted specifically to the phenomenology of F-theory constructions and a more complete list of references we recommend [13]. Topics worth highlighting include the following.

- The local nature of brane models offers the possibility of studying the structure of Yukawa couplings without referring to the details of the global geometry. Investigations of the Yukawa and flavour structure of SU(5) GUT models in this context and various phenomenological scenarios have appeared in [61, 84–93].
- Possible connections with neutrino physics are the subject of [94, 95].
- On a more formal level, instanton effects in F-theory have been reconsidered in the recent literature [96–99] with special attention to the generation of phenomenologically viable matter couplings known from the weakly coupled type II limit (see e.g. [100] for a review and references).

Another focus in the recent literature is the realization of these and other model-building ideas in concrete compact examples. The motivation behind this is, as was stressed already, that certain questions of phenomenological relevance cannot be disentangled from the global geometric structure. Compact F-theory GUT vacua have been constructed in [43–45, 77, 78, 101]. Methods of toric geometry, as applied in [43–45], allow one to explicitly construct fully fledged singular Calabi–Yau 4-folds and their explicit resolution in a way that keeps full control of the singularities of the Tate model. In particular, out of the GUT vacua constructed so far, the Euler characteristic, which enters the 3-brane tadpole cancellation condition, has been computed from first principles for the examples of [43–45]. As far as the construction of the gauge flux is concerned, the models [43–45, 77, 78, 101] rely on the spectral cover approach outlined in section 3.5. Modulo the caveats pointed out there, 3-generation GUT models have been achieved.

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