Black Holes and the Penrose Inequality in General Relativity

Hubert L. Bray *

Abstract

In a paper [23] in 1973, R. Penrose made a physical argument that the total mass of a spacetime which contains black holes with event horizons of total area $A$ should be at least $\sqrt{A/16\pi}$. An important special case of this physical statement translates into a very beautiful mathematical inequality in Riemannian geometry known as the Riemannian Penrose inequality. One particularly geometric aspect of this problem is the fact that apparent horizons of black holes in this setting correspond to minimal surfaces in Riemannian 3-manifolds. The Riemannian Penrose inequality was first proved by G. Huisken and T. Ilmanen in 1997 for a single black hole [17] and then by the author in 1999 for any number of black holes [6]. The two approaches use two different geometric flow techniques. The most general version of the Penrose inequality is still open.

In this talk we will sketch the author’s proof by flowing Riemannian manifolds inside the class of asymptotically flat 3-manifolds (asymptotic to $\mathbb{R}^3$ at infinity) which have nonnegative scalar curvature and contain minimal spheres. This new flow of metrics has very special properties and simulates an initial physical situation in which all of the matter falls into the black holes which merge into a single, spherically symmetric black hole given by the Schwarzschild metric. Since the Schwarzschild metric gives equality in the Penrose inequality and the flow decreases the total mass while preserving the area of the horizons of the black holes, the Penrose inequality follows. We will also discuss how these techniques can be generalized in higher dimensions.

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1. Introduction

A natural interpretation of the Penrose inequality is that the mass contributed by a collection of black holes is (at least) $\sqrt{A/16\pi}$, where $A$ is the total area of the event horizons of the black holes. More generally, the question “How much matter

*Mathematics Department, 2-179, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA. E-mail: bray@math.mit.edu
is in a given region of a spacetime?" is still very much an open problem [12]. In this paper, we will discuss some of the qualitative aspects of mass in general relativity, look at examples which are informative, and sketch a proof of the Riemannian Penrose inequality.

1.1. Total mass in general relativity

Two notions of mass which are well understood in general relativity are local energy density at a point and the total mass of an asymptotically flat spacetime. However, defining the mass of a region larger than a point but smaller than the entire universe is not very well understood at all.

Suppose \((M^3, g)\) is a Riemannian 3-manifold isometrically embedded in a \((3+1)\) dimensional Lorentzian spacetime. Suppose that \(M^3\) has zero second fundamental form in the spacetime. This is a simplifying assumption which allows us to think of \((M^3, g)\) as a \("t = 0"\) slice of the spacetime. The Penrose inequality (which allows for \(M^3\) to have general second fundamental form) is known as the Riemannian Penrose inequality when the second fundamental form is set to zero.

We also want to only consider \((M^3, g)\) that are asymptotically flat at infinity, which means that for some compact set \(K\), the "end" \(M^3 \setminus K\) is diffeomorphic to \(\mathbb{R}^3 \setminus B_1(0)\), where the metric \(g\) is asymptotically approaching (with certain decay conditions) the standard flat metric \(\delta_{ij}\) on \(\mathbb{R}^3\) at infinity. The simplest example of an asymptotically flat manifold is \((\mathbb{R}^3, \delta_{ij})\) itself. Other good examples are the conformal metrics \((\mathbb{R}^3, u(x)^4 \delta_{ij})\), where \(u(x)\) approaches a constant sufficiently rapidly at infinity. (Also, sometimes it is convenient to allow \((M^3, g)\) to have multiple asymptotically flat ends, in which case each connected component of \(M^3 \setminus K\) must have the property described above.)

The purpose of these assumptions on the asymptotic behavior of \((M^3, g)\) at infinity is that they imply the existence of the limit

\[
m = \frac{1}{16\pi} \lim_{\sigma \to \infty} \int_{S_\sigma} \sum_{i,j} (g_{ij,i} \nu^j - g_{ii,j} \nu^j) \, d\mu,
\]

where \(S_\sigma\) is the coordinate sphere of radius \(\sigma\), \(\nu\) is the unit normal to \(S_\sigma\), and \(d\mu\) is the area element of \(S_\sigma\) in the coordinate chart. The quantity \(m\) is called the total mass (or ADM mass) of \((M^3, g)\) (see [1], [2], [24], and [27]).

Instead of thinking of total mass as given by equation 1, it is better to consider the following example. Going back to the example \((\mathbb{R}^3, u(x)^4 \delta_{ij})\), if we suppose that \(u(x) > 0\) has the asymptotics at infinity

\[
u(x) = a + b/|x| + \mathcal{O}(1/|x|^2)
\]

(and derivatives of the \(\mathcal{O}(1/|x|^2)\) term are \(\mathcal{O}(1/|x|^3)\)), then the total mass of \((M^3, g)\) is

\[m = 2ab.\]

Furthermore, suppose \((M^3, g)\) is any metric whose "end" is isometric to \((\mathbb{R}^3 \setminus K, u(x)^4 \delta_{ij})\), where \(u(x)\) is harmonic in the coordinate chart of the end \((\mathbb{R}^3 \setminus K, \delta_{ij})\)
and goes to a constant at infinity. Then expanding $u(x)$ in terms of spherical harmonics demonstrates that $u(x)$ satisfies condition 2. We will call these Riemannian manifolds $(M^3, g)$ **harmonically flat at infinity**, and we note that the total mass of these manifolds is also given by equation 3.

A very nice lemma by Schoen and Yau is that, given any $\epsilon > 0$, it is always possible to perturb an asymptotically flat manifold to become harmonically flat at infinity such that the total mass changes less than $\epsilon$ and the metric changes less than $\epsilon$ pointwise, all while maintaining nonnegative scalar curvature (discussed in a moment). Hence, it happens that to prove the theorems in this paper, we only need to consider harmonically flat manifolds! Thus, we can use equation 3 as our definition of total mass. As an example, note that $(R^3, \delta_{ij})$ has zero total mass. Also, note that, qualitatively, the total mass of an asymptotically flat or harmonically flat manifold is the $1/r$ rate at which the metric becomes flat at infinity.

### 1.2. Local energy density

Another quantification of mass which is well understood is local energy density. In fact, in this setting, the local energy density at each point is

$$\mu = \frac{1}{16\pi} R,$$

where $R$ is the scalar curvature of the 3-manifold (which has zero second fundamental form in the spacetime) at each point. Thus, we note that $(R^3, \delta_{ij})$ has zero energy density at each point as well as zero total mass. This is appropriate since $(R^3, \delta_{ij})$ is in fact a “$t = 0$” slice of Minkowski spacetime, which represents a vacuum. Classically, physicists consider $\mu \geq 0$ to be a physical assumption. Hence, from this point on, we will not only assume that $(M^3, g)$ is asymptotically flat, but also that it has nonnegative scalar curvature,

$$R \geq 0. \quad (5)$$

This notion of energy density also helps us understand total mass better. After all, we can take any asymptotically flat manifold and then change the metric to be perfectly flat outside a large compact set, thereby giving the new metric zero total mass. However, if we introduce the physical condition that both metrics have nonnegative scalar curvature, then it is a beautiful theorem that this is in fact not possible, unless the original metric was already $(R^3, \delta_{ij})$! (This theorem is actually a corollary to the positive mass theorem discussed in a moment.) Thus, the curvature obstruction of having nonnegative scalar curvature at each point is a very interesting condition.

Also, notice the indirect connection between the total mass and local energy density. At this point, there does not seem to be much of a connection at all. Total mass is the $1/r$ rate at which the metric becomes flat at infinity, and local energy density is the scalar curvature at each point. Furthermore, if a metric is changed in a compact set, local energy density is changed, but the total mass is unaffected.
The reason for this is that the total mass is not the integral of the local energy density over the manifold. In fact, this integral fails to take potential energy into account (which would be expected to contribute a negative energy) as well as gravitational energy (discussed in a moment). Hence, it is not initially clear what we should expect the relationship between total mass and local energy density to be, so let us begin with an example.

1.3. Example using superharmonic functions in $\mathbb{R}^3$

Once again, let us return to the $(\mathbb{R}^3, u(x)^4 \delta_{ij})$ example. The formula for the scalar curvature is

$$R = -8u(x)^{-5} \Delta u(x).$$

Hence, since the physical assumption of nonnegative energy density implies non-negative scalar curvature, we see that $u(x) > 0$ must be superharmonic ($\Delta u \leq 0$). For simplicity, let’s also assume that $u(x)$ is harmonic outside a bounded set so that we can expand $u(x)$ at infinity using spherical harmonics. Hence, $u(x)$ has the asymptotics of equation 2. By the maximum principle, it follows that the minimum value for $u(x)$ must be $a$, referring to equation 2. Hence, $b \geq 0$, which implies that $m \geq 0$! Thus we see that the assumption of nonnegative energy density at each point of $(\mathbb{R}^3, u(x)^4 \delta_{ij})$ implies that the total mass is also nonnegative, which is what one would hope.

1.4. The positive mass theorem

More generally, suppose we have any asymptotically flat manifold with non-negative scalar curvature, is it true that the total mass is also nonnegative? The answer is yes, and this fact is known as the positive mass theorem, first proved by Schoen and Yau [25] in 1979 using minimal surface techniques and then by Witten [30] in 1981 using spinors.

**Theorem 1** (Schoen-Yau) Let $(M^3, g)$ be any asymptotically flat, complete Riemannian manifold with nonnegative scalar curvature. Then the total mass $m \geq 0$, with equality if and only if $(M^3, g)$ is isometric to $(\mathbb{R}^3, \delta)$.

1.5. Black holes

Another very interesting and natural phenomenon in general relativity is the existence of black holes. Instead of thinking of black holes as singularities in a spacetime, we will think of black holes in terms of their horizons. Given a surface in a spacetime, suppose that it admits an outward shell of light. If the surface area of this shell of light is decreasing everywhere on the surface, then this is called a trapped surface. The outermost boundary of these trapped surfaces is called the apparent horizon of the black hole. Apparent horizons can be computed based on their local geometry, and an apparent horizon always implies the existence of an event horizon outside of it [15].

Now let us return to the case we are considering in this paper where $(M^3, g)$ is a “t = 0” slice of a spacetime with zero second fundamental form. Then it is a
very nice geometric fact that apparent horizons of black holes intersected with $M^3$ correspond to the connected components of the outermost minimal surface $\Sigma_0$ of $(M^3, g)$.

All of the surfaces we are considering in this paper will be required to be smooth boundaries of open bounded regions, so that outermost is well-defined with respect to a chosen end of the manifold [6]. A minimal surface in $(M^3, g)$ is a surface which is a critical point of the area function with respect to any smooth variation of the surface. The first variational calculation implies that minimal surfaces have zero mean curvature. The surface $\Sigma_0$ of $(M^3, g)$ is defined as the boundary of the union of the open regions bounded by all of the minimal surfaces in $(M^3, g)$. It turns out that $\Sigma_0$ also has to be a minimal surface, so we call $\Sigma_0$ the outermost minimal surface.

We will also define a surface to be (strictly) outer minimizing if every surface which encloses it has (strictly) greater area. Note that outermost minimal surfaces are strictly outer minimizing. Also, we define a horizon in our context to be any minimal surface which is the boundary of a bounded open region.

It also follows from a stability argument (using the Gauss-Bonnet theorem interestingly) that each component of a stable minimal surface (in a 3-manifold with nonnegative scalar curvature) must have the topology of a sphere. Furthermore, there is a physical argument, based on [23], which suggests that the mass contributed by the black holes (thought of as the connected components of $\Sigma_0$) should be defined to be $\sqrt{A_0/16\pi}$, where $A_0$ is the area of $\Sigma_0$. Hence, the physical argument that the total mass should be greater than or equal to the mass contributed by the black holes yields that following geometric statement.

**The Riemannian Penrose Inequality**

Let $(M^3, g)$ be a complete, smooth, 3-manifold with nonnegative scalar curvature which is harmonically flat at infinity with total mass $m$ and which has an outermost minimal surface $\Sigma_0$ of area $A_0$. Then

$$m \geq \sqrt{\frac{A_0}{16\pi}},$$

with equality if and only if $(M^3, g)$ is isometric to the Schwarzschild metric $(\mathbb{R}^3\setminus\{0\}, (1 + \frac{m^2}{2|x|})^4\delta_{ij})$ outside their respective outermost minimal surfaces.

The above statement has been proved by the author [6], and by Huisken and Ilmanen [17] where $A_0$ is defined instead to be the area of the largest connected component of $\Sigma_0$. We will discuss both approaches in this paper, which are very different, although they both involve flowing surfaces and/or metrics.

We also clarify that the above statement is with respect to a chosen end of $(M^3, g)$, since both the total mass and the definition of outermost refer to a particular end. In fact, nothing very important is gained by considering manifolds with more than one end, since extra ends can always be compactified by connect summing them (around a neighborhood of infinity) with large spheres while still preserving nonnegative scalar curvature, for example. Hence, we will typically consider manifolds with just one end. In the case that the manifold has multiple ends,
we will require every surface (which could have multiple connected components) in this paper to enclose all of the ends of the manifold except the chosen end.

Other contributions on the Penrose Conjecture have also been made by Herzlich [16] using the Dirac operator which Witten [30] used to prove the positive mass theorem, by Gibbons [14] in the special case of collapsing shells, by Tod [29], by Bartnik [4] for quasi-spherical metrics, and by the author [7] using isoperimetric surfaces. There is also some interesting work of Ludvigsen and Vickers [21] using spinors and Bergqvist [5], both concerning the Penrose inequality for null slices of a space-time.

1.6. The Schwarzschild metric

The Schwarzschild metric \((\mathbb{R}^3\setminus\{0\}, (1 + \frac{m}{w^2})^4 \delta_{ij})\), referred to in the above statement of the Riemannian Penrose Inequality, is a particularly important example to consider, and corresponds to a zero-second fundamental form, space-like slice of the usual \((3+1)\)-dimensional Schwarzschild metric (which represents a spherically symmetric static black hole in vacuum). The 3-dimensional Schwarzschild metrics have total mass \(m > 0\) and are characterized by being the only spherically symmetric, geodesically complete, zero scalar curvature 3-metrics, other than \((\mathbb{R}^3, \delta_{ij})\). They can also be embedded in 4-dimensional Euclidean space \((x, y, z, w)\) as the set of points satisfying \(|(x, y, z)| = \frac{w^2}{8m} + 2m\), which is a parabola rotated around an \(S^2\). This last picture allows us to see that the Schwarzschild metric, which has two ends, has a \(Z_2\) symmetry which fixes the sphere with \(w = 0\) and \(|(x, y, z)| = 2m\), which is clearly minimal. Furthermore, the area of this sphere is \(4\pi(2m)^2\), giving equality in the Riemannian Penrose Inequality.

2. The conformal flow of metrics

Given any initial Riemannian manifold \((M^3, g_0)\) which has nonnegative scalar curvature and which is harmonically flat at infinity, we will define a continuous, one parameter family of metrics \((M^3, g_t), 0 \leq t < \infty\). This family of metrics will converge to a 3-dimensional Schwarzschild metric and will have other special properties which will allow us to prove the Riemannian Penrose Inequality for the original metric \((M^3, g_0)\).

In particular, let \(\Sigma_0\) be the outermost minimal surface of \((M^3, g_0)\) with area \(A_0\). Then we will also define a family of surfaces \(\Sigma(t)\) with \(\Sigma(0) = \Sigma_0\) such that \(\Sigma(t)\) is minimal in \((M^3, g_t)\). This is natural since as the metric \(g_t\) changes, we expect that the location of the horizon \(\Sigma(t)\) will also change. Then the interesting quantities to keep track of in this flow are \(A(t)\), the total area of the horizon \(\Sigma(t)\) in \((M^3, g_t)\), and \(m(t)\), the total mass of \((M^3, g_t)\) in the chosen end.

In addition to all of the metrics \(g_t\) having nonnegative scalar curvature, we will also have the very nice properties that

\[
A'(t) = 0, \quad m'(t) \leq 0
\]
for all \( t \geq 0 \). Then since \((M^3, g_t)\) converges to a Schwarzschild metric (in an appropriate sense) which gives equality in the Riemannian Penrose Inequality as described in the introduction,

\[
m(0) \geq m(\infty) = \sqrt{\frac{A(\infty)}{16\pi}} = \sqrt{\frac{A(0)}{16\pi}}
\]

which proves the Riemannian Penrose Inequality for the original metric \((M^3, g_0)\).

The hard part, then, is to find a flow of metrics which preserves nonnegative scalar curvature and the area of the horizon, decreases total mass, and converges to a Schwarzschild metric as \( t \) goes to infinity.

### 2.1. The definition of the flow

In fact, the metrics \( g_t \) will all be conformal to \( g_0 \). This conformal flow of metrics can be thought of as the solution to a first order o.d.e. in \( t \) defined by equations 11, 12, 13, and 14. Let

\[
g_t = u_t(x)^4 g_0
\]

and \( u_0(x) \equiv 1 \). Given the metric \( g_t \), define

\[
\Sigma(t) = \text{the outermost minimal area enclosure of } \Sigma_0 \text{ in } (M^3, g_t)
\]

where \( \Sigma_0 \) is the original outer minimizing horizon in \((M^3, g_0)\). In the cases in which we are interested, \( \Sigma(t) \) will not touch \( \Sigma_0 \), from which it follows that \( \Sigma(t) \) is actually a strictly outer minimizing horizon of \((M^3, g_t)\). Then given the horizon \( \Sigma(t) \), define \( v_t(x) \) such that

\[
\begin{cases}
\Delta_g v_t(x) &\equiv 0 \quad \text{outside } \Sigma(t) \\
v_t(x) &\equiv 0 \quad \text{on } \Sigma(t) \\
\lim_{x \to \infty} v_t(x) &\equiv -e^{-t}
\end{cases}
\]

and \( v_t(x) \equiv 0 \) inside \( \Sigma(t) \). Finally, given \( v_t(x) \), define

\[
u_t(x) = 1 + \int_0^t v_s(x) ds
\]

so that \( u_t(x) \) is continuous in \( t \) and has \( u_0(x) \equiv 1 \).

Note that equation 14 implies that the first order rate of change of \( u_t(x) \) is given by \( v_t(x) \). Hence, the first order rate of change of \( g_t \) is a function of itself, \( g_0 \), and \( v_t(x) \) which is a function of \( g_0, t, \) and \( \Sigma(t) \) which is in turn a function of \( g_t \) and \( \Sigma_0 \). Thus, the first order rate of change of \( g_t \) is a function of \( t, g_t, g_0, \) and \( \Sigma_0 \).

**Theorem 2** Taken together, equations 11, 12, 13, and 14 define a first order o.d.e. in \( t \) for \( u_t(x) \) which has a solution which is Lipschitz in the \( t \) variable, \( C^1 \) in the \( x \) variable everywhere, and smooth in the \( x \) variable outside \( \Sigma(t) \). Furthermore, \( \Sigma(t) \) is a smooth, strictly outer minimizing horizon in \((M^3, g_t)\) for all \( t \geq 0 \), and \( \Sigma(t_2) \) encloses but does not touch \( \Sigma(t_1) \) for all \( t_2 > t_1 \geq 0 \).
Since \( v_t(x) \) is a superharmonic function in \((M^3, g_0)\) (harmonic everywhere except on \( \Sigma(t) \), where it is weakly superharmonic), it follows that \( u_t(x) \) is superharmonic as well. Thus, from equation 14 we see that \( \lim_{x \to \infty} u_t(x) = e^{-t} \) and consequently that \( u_t(x) > 0 \) for all \( t \) by the maximum principle. Then since
\[
R(g_t) = u_t(x)^{-5}(-8\Delta g_0 + R(g_0))u_t(x), \tag{15}
\]
it follows that \((M^3, g_t)\) is an asymptotically flat manifold with nonnegative scalar curvature.

Even so, it still may not seem like \( g_t \) is particularly naturally defined since the rate of change of \( g_t \) appears to depend on \( t \) and the original metric \( g_0 \) in equation 13. We would prefer a flow where the rate of change of \( g_t \) can be defined purely as a function of \( g_t \) (and \( \Sigma_0 \) perhaps), and interestingly enough this actually does turn out to be the case. In section 2.4, we prove this very important fact and define a new equivalence class of metrics called the harmonic conformal class. Then once we decide to find a flow of metrics which stays inside the harmonic conformal class of the original metric (outside the horizon) and keeps the area of the horizon \( \Sigma(t) \) constant, then we are basically forced to choose the particular conformal flow of metrics defined above.

**Theorem 3** The function \( A(t) \) is constant in \( t \) and \( m(t) \) is non-increasing in \( t \), for all \( t \geq 0 \).

The fact that \( A'(t) = 0 \) follows from the fact that to first order the metric is not changing on \( \Sigma(t) \) (since \( v_t(x) = 0 \) there) and from the fact that to first order the area of \( \Sigma(t) \) does not change as it moves outward since \( \Sigma(t) \) is a critical point for area in \((M^3, g_t)\). Hence, the interesting part of theorem 3 is proving that \( m'(t) \leq 0 \). Curiously, this follows from a nice trick using the Riemannian positive mass theorem, which we describe in section 2.3.

Another important aspect of this conformal flow of the metric is that outside the horizon \( \Sigma(t) \), the manifold \((M^3, g_t)\) becomes more and more spherically symmetric and “approaches” a Schwarzschild manifold \((\mathbb{R}^3 \setminus \{0\}, s)\) in the limit as \( t \) goes to \( \infty \). More precisely,

**Theorem 4** For sufficiently large \( t \), there exists a diffeomorphism \( \phi_t \) between \((M^3, g_t)\) outside the horizon \( \Sigma(t) \) and a fixed Schwarzschild manifold \((\mathbb{R}^3 \setminus \{0\}, s)\) outside its horizon. Furthermore, for all \( \epsilon > 0 \), there exists a \( T \) such that for all \( t > T \), the metrics \( g_t \) and \( \phi_t^*(s) \) (when determining the lengths of unit vectors of \((M^3, g_t)\)) are within \( \epsilon \) of each other and the total masses of the two manifolds are within \( \epsilon \) of each other. Hence,
\[
\lim_{t \to \infty} \frac{m(t)}{\sqrt{A(t)}} = \sqrt{\frac{1}{16\pi}}. \tag{16}
\]

Theorem 4 is not that surprising really although a careful proof is reasonably long. However, if one is willing to believe that the flow of metrics converges to a spherically symmetric metric outside the horizon, then theorem 4 follows from two
facts. The first fact is that the scalar curvature of \((M^3, g_t)\) eventually becomes identically zero outside the horizon \(\Sigma(t)\) (assuming \((M^3, g_0)\) is harmonically flat). This follows from the facts that \(\Sigma(t)\) encloses any compact set in a finite amount of time, that harmonically flat manifolds have zero scalar curvature outside a compact set, that \(u_t(x)\) is harmonic outside \(\Sigma(t)\), and equation 15. The second fact is that the Schwarzschild metrics are the only complete, spherically symmetric 3-manifolds with zero scalar curvature (except for the flat metric on \(R^3\)).

The Riemannian Penrose inequality, inequality 7, then follows from equation 10 using theorems 2, 3 and 4, for harmonically flat manifolds [6]. Since asymptotically flat manifolds can be approximated arbitrarily well by harmonically flat manifolds while changing the relevant quantities arbitrarily little, the asymptotically flat case also follows. Finally, the case of equality of the Penrose inequality follows from a more careful analysis of these same arguments.

2.2. Qualitative discussion

The diagrams above and below are meant to help illustrate some of the properties of the conformal flow of the metric. The above picture is the original metric which has a strictly outer minimizing horizon \(\Sigma_0\). As \(t\) increases, \(\Sigma(t)\) moves outwards, but never inwards. In the diagram below, we can observe one of the consequences of the fact that \(A(t) = A_0\) is constant in \(t\). Since the metric is not changing inside \(\Sigma(t)\), all of the horizons \(\Sigma(s), 0 \leq s \leq t\) have area \(A_0\) in \((M^3, g_t)\). Hence, inside \(\Sigma(t)\), the manifold \((M^3, g_t)\) becomes cylinder-like in the sense that it is laminated (meaning foliated but with some gaps allowed) by all of the previous horizons which all have the same area \(A_0\) with respect to the metric \(g_t\).
Now let us suppose that the original horizon $\Sigma_0$ of $(M^3, g)$ had two components, for example. Then each of the components of the horizon will move outwards as $t$ increases, and at some point before they touch they will suddenly jump outwards to form a horizon with a single component enclosing the previous horizon with two components. Even horizons with only one component will sometimes jump outwards, but no more than a countable number of times. It is interesting that this phenomenon of surfaces jumping is also found in the Huisken-Ilmanen approach to the Penrose conjecture using their generalized $1/H$ flow.

2.3. Proof that $m'(t) \leq 0$

The most surprising aspect of the flow defined in section 2.1. is that $m'(t) \leq 0$. As mentioned in that section, this important fact follows from a nice trick using the Riemannian positive mass theorem.

The first step is to realize that while the rate of change of $g_t$ appears to depend on $t$ and $g_0$, this is in fact an illusion. As is described in detail in section 2.4., the rate of change of $g_t$ can be described purely in terms of $g_t$ (and $\Sigma_0$). It is also true that the rate of change of $g_t$ depends only on $g_t$ and $\Sigma(t)$. Hence, there is no special value of $t$, so proving $m'(t) \leq 0$ is equivalent to proving $m'(0) \leq 0$. Thus, without loss of generality, we take $t = 0$ for convenience.

Now expand the harmonic function $v_0(x)$, defined in equation 13, using spherical harmonics at infinity, to get

$$v_0(x) = -1 + \frac{c}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad (17)$$

for some constant $c$. Since the rate of change of the metric $g_t$ at $t = 0$ is given by $v_0(x)$ and since the total mass $m(t)$ depends on the $1/r$ rate at which the metric $g_t$ becomes flat at infinity (see equation 3), it is not surprising that direct calculation gives us that

$$m'(0) = 2(c - m(0)). \quad (18)$$

Hence, to show that $m'(0) \leq 0$, we need to show that

$$c \leq m(0). \quad (19)$$

In fact, counterexamples to equation 19 can be found if we remove either of the requirements that $\Sigma(0)$ (which is used in the definition of $v_0(x)$) be a minimal
surface or that \((M^3, g_0)\) have nonnegative scalar curvature. Hence, we quickly see that equation 19 is a fairly deep conjecture which says something quite interesting about manifolds with nonnegative scalar curvature. Well, the Riemannian positive mass theorem is also a deep conjecture which says something quite interesting about manifolds with nonnegative scalar curvature. Hence, it is natural to try to use the Riemannian positive mass theorem to prove equation 19.

Thus, we want to create a manifold whose total mass depends on \(c\) from equation 17. The idea is to use a reflection trick similar to one used by Bunting and Masood-ul-Alam for another purpose in [11]. First, remove the region of \(M^3\) inside \(\Sigma(0)\) and then reflect the remainder of \((M^3, g_0)\) through \(\Sigma(0)\). Define the resulting Riemannian manifold to be \((\bar{M}^3, \bar{g}_0)\) which has two asymptotically flat ends since \((M^3, g_0)\) has exactly one asymptotically flat end not contained by \(\Sigma(0)\). Note that \((\bar{M}^3, \bar{g}_0)\) has nonnegative scalar curvature everywhere except on \(\Sigma(0)\) where the metric has corners. In fact, the fact that \(\Sigma(0)\) has zero mean curvature (since it is a minimal surface) implies that \((\bar{M}^3, \bar{g}_0)\) has distributional nonnegative scalar curvature everywhere, even on \(\Sigma(0)\). This notion is made rigorous in [6]. Thus we have used the fact that \(\Sigma(0)\) is minimal in a critical way.

Recall from equation 13 that \(v_0(x)\) was defined to be the harmonic function equal to zero on \(\Sigma(0)\) which goes to \(-1\) at infinity. We want to reflect \(v_0(x)\) to be defined on all of \((\bar{M}^3, \bar{g}_0)\). The trick here is to define \(v_0(x)\) on \((\bar{M}^3, \bar{g}_0)\) to be the harmonic function which goes to \(-1\) at infinity in the original end and goes to \(1\) at infinity in the reflect end. By symmetry, \(v_0(x)\) equals 0 on \(\Sigma(0)\) and so agrees with its original definition on \((M^3, g_0)\).

The next step is to compactify one end of \((\bar{M}^3, \bar{g}_0)\). By the maximum principle, we know that \(v_0(x) > -1\) and \(c > 0\), so the new Riemannian manifold \((\bar{M}^3, (v_0(x) + 1)^4\bar{g}_0)\) does the job quite nicely and compactifies the original end to a point. In fact, the compactified point at infinity and the metric there can be filled in smoothly (using the fact that \((M^3, g_0)\) is harmonically flat). It then follows from equation 15 that this new compactified manifold has nonnegative scalar curvature since \(v_0(x) + 1\) is harmonic.

The last step is simply to apply the Riemannian positive mass theorem to \((\bar{M}^3, (v_0(x) + 1)^4\bar{g}_0)\). It is not surprising that the total mass \(\bar{m}(0)\) of this manifold involves \(c\), but it is quite lucky that direct calculation yields

\[
\bar{m}(0) = -4(c - m(0)), \tag{20}
\]

which must be positive by the Riemannian positive mass theorem. Thus, we have that

\[
m'(0) = 2(c - m(0)) = -\frac{1}{2} \bar{m}(0) \leq 0. \tag{21}
\]

2.4. The harmonic conformal class of a metric

As a final topic which is also of independent interest, we define a new equivalence class and partial ordering of conformal metrics. These new objects provide a natural motivation for studying conformal flows of metrics to try to prove the
Riemannian Penrose inequality. Let
\[ g_2 = u(x)^{\frac{4}{n-2}} g_1, \tag{22} \]
where \( g_2 \) and \( g_1 \) are metrics on an \( n \)-dimensional manifold \( M^n, n \geq 3 \). Then we get the surprisingly simple identity that
\[ \Delta_{g_1} (u \phi) = u^{\frac{n+2}{n-2}} \Delta_{g_2} (\phi) + \phi \Delta_{g_1} (u) \tag{23} \]
for any smooth function \( \phi \). This motivates us to define the following relation.

**Definition 1** Define
\[ g_2 \sim g_1 \]
if and only if equation 22 is satisfied with \( \Delta_{g_1} (u) = 0 \) and \( u(x) > 0 \).

Then from equation 23 we get the following lemma.

**Lemma 1** The relation \( \sim \) is reflexive, symmetric, and transitive, and hence is an equivalence relation.

Thus, we can define the following equivalence class of metrics.

**Definition 2** Define
\[ [g]_H = \{ \bar{g} \mid \bar{g} \sim g \} \]
to be the harmonic conformal class of the metric \( g \).

Of course, this definition is most interesting when \( (M^n, g) \) has nonconstant positive harmonic functions, which happens for example when \( (M^n, g) \) has a boundary.

Also, we can modify the relation \( \sim \) to get another relation \( \succeq \).

**Definition 3** Define
\[ g_2 \succeq g_1 \]
if and only if equation 22 is satisfied with \( -\Delta_{g_1} (u) \geq 0 \) and \( u(x) > 0 \).

Then from equation 23 we get the following lemma.

**Lemma 2** The relation \( \succeq \) is reflexive and transitive, and hence is a partial ordering.

Since \( \succeq \) is defined in terms of superharmonic functions, we will call it the superharmonic partial ordering of metrics on \( M^n \). Then it is natural to define the following set of metrics.

**Definition 4** Define
\[ [g]_S = \{ \bar{g} \mid \bar{g} \succeq g \} \]
This set of metrics has the property that if \( g \in [g]_S \), then \([g]_S \subset [g]_S\)

Also, the scalar curvature transforms nicely under a conformal change of the metric. In fact, assuming equation 22 again,
\[ R(g_2) = u(x)^{-\frac{4(n-2)}{n-2}} \left( -c_n \Delta_{g_1} + R(g_1) \right) u(x) \tag{24} \]
where \( c_n = \frac{4(n-1)}{n-2} \). This gives us the following lemma.
**Lemma 3** The sign of the scalar curvature is preserved pointwise by ∼. That is, if \( g_2 \sim g_1 \), then \( \text{sgn}(R(g_2)(x)) = \text{sgn}(R(g_1)(x)) \) for all \( x \in \mathbb{M}^m \). Also, if \( g_2 \geq g_1 \), and \( g_1 \) has non-negative scalar curvature, then \( g_2 \) has non-negative scalar curvature.

Hence, the harmonic conformal equivalence relation ∼ and the superharmonic partial ordering ≥ are useful for studying questions about scalar curvature. In particular, these notions are useful for studying the Riemannian Penrose inequality which concerns asymptotically flat 3-manifolds \((M^3, g)\) with non-negative scalar curvature. Given such a manifold, define \( m(g) \) to be the total mass of \((M^3, g)\) and \( A(g) \) to be the area of the outermost horizon (which could have multiple components) of \((M^3, g)\). Define \( P(g) = \frac{m(g)}{\sqrt{A(g)}} \) to be the Penrose quotient of \((M^3, g)\).

Then an interesting question is to ask which metric in \([g]_S\) minimizes \( P(g) \).

Section 2. of this paper can be viewed as an answer to the above question. We showed that there exists a conformal flow of metrics (starting with \( g_0 \)) for which the Penrose quotient was non-increasing, and in fact this conformal flow stays inside \([g_0]_S\). Furthermore, \( g_{t_2} \in [g_0]_S \) for all \( t_2 \geq t_1 \geq 0 \). We showed that no matter which metric we start with, the metric converges to a Schwarzschild metric outside its horizon. Hence, the minimum value of \( P(g) \) in \([g]_S\) is achieved in the limit by metrics converging to a Schwarzschild metric (outside their respective horizons).

In the case that \( g \) is harmonically flat at infinity, a Schwarzschild metric (outside the horizon) is contained in \([g]_S\). More generally, given any asymptotically flat manifold \((M^3, g)\), we can use \( \mathbb{R}^3 \setminus B_r(0) \) as a coordinate chart for the asymptotically flat end of \((M^3, g)\) which we are interested in, where the metric \( g_{ij} \) approaches \( \delta_{ij} \) at infinity in this coordinate chart. Then we can consider the conformal metric

\[
g_C = \left(1 + \frac{C}{|x|}\right)^4 g
\]  

in this end. In the limit as \( C \) goes to infinity, the horizon will approach the coordinate sphere of radius \( C \). Then outside this horizon in the limit as \( C \) goes to infinity, the function \( (1 + \frac{C}{|x|}) \) will be close to a superharmonic function on \((M^3, g)\) and the metric \( g_C \) will approach a Schwarzschild metric (since the metric \( g \) is approaching the standard metric on \( \mathbb{R}^3 \)). Hence, the Penrose quotient of \( g_C \) will approach \((16\pi)^{-1/2}\), which is the Penrose quotient of a Schwarzschild metric.

As a final note, we prove that the first order o.d.e. for \( \{g_t\} \) defined in equations 11, 12, 13, and 14 is naturally defined in the sense that the rate of change of \( g_t \) is a function only of \( g_t \) and not of \( g_{0} \) or \( t \). To see this, given any solution \( g_t = u_t(x)^4 g_0 \) to equations 11, 12, 13, and 14, choose any \( s > 0 \) and define \( \tilde{u}_t(x) = u_t(x)/u_s(x) \) so that

\[
g_t = \tilde{u}_t(x)^4 g_s
\]  

and \( \tilde{u}_s(x) \equiv 1 \). Then define \( \tilde{v}_t(x) \) such that

\[
\begin{align*}
\Delta_{g_t} \tilde{v}_t(x) &\equiv 0 & \text{ outside } \Sigma(t) \\
\tilde{v}_t(x) &\equiv 0 & \text{ on } \Sigma(t) \\
\lim_{x \to \infty} \tilde{v}_t(x) &\equiv -e^{-(t-s)}
\end{align*}
\]  

(27)
and $\bar{v}_t(x) \equiv 0$ inside $\Sigma(t)$. Then what we want to show is

$$\bar{u}_t(x) = 1 + \int_s^t \bar{v}_r(x)dr$$  \hspace{1cm} (28)

To prove the above equation, we observe that from equations 23, 27, and 13 it follows that

$$v_t(x) = \bar{v}_t(x) u_s(x)$$  \hspace{1cm} (29)

since $\lim_{x \to \infty} u_s(x) = e^{-s}$. Hence, since

$$u_t(x) = u_s(x) + \int_s^t v_r(x)dr$$  \hspace{1cm} (30)

by equation 14, dividing through by $u_s(x)$ yields equation 28 as desired. Thus, we see that the rate of change of $g_t(x)$ at $t = s$ is a function of $\bar{v}_s(x)$ which in turn is just a function of $g_s(x)$ and the horizon $\Sigma(s)$. Hence, to understand properties of the flow we need only analyze the behavior of the flow for $t$ close to zero, since any metric in the flow may be chosen to be the base metric.

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