Refined Belief-Propagation Decoding of Quantum Codes with Scalar Messages

Kao-Yueh Kuo and Ching-Yi Lai
Institute of Communications Engineering, National Chiao Tung University
Hsinchu 30010, Taiwan
kykuo@nctu.edu.tw and cylai@nctu.edu.tw

Abstract—Codes based on sparse matrices have good performance and can be efficiently decoded by belief-propagation (BP). Decoding binary stabilizer codes needs a quaternary BP for (additive) codes over GF(4), which has a higher check-node complexity compared to a binary BP for codes over GF(2). Moreover, BP decoding of stabilizer codes suffers a performance loss from the short cycles in the underlying Tanner graph. In this paper, we propose a refined BP algorithm for decoding quantum codes by passing scalar messages. For a given error syndrome, this algorithm decodes to the same output as the conventional quaternary BP but with a check-node complexity the same as binary BP. As every message is a scalar, the message normalization can be naturally applied to improve the decoding performance and error-floor in computer simulation.

Index Terms—Quantum stabilizer codes, LDPC codes, sparse matrices, belief-propagation, sum-product algorithm, message-update schedule, message normalization.

I. INTRODUCTION

In the 1960s, Gallager proposed the low-density parity-check (LDPC) codes (or called sparse codes) and sum-product (decoding) algorithm to have a near Shannon-capacity performance for many channels, including the binary symmetric channel (BSC) and additive white Gaussian noise (AWGN) channel [1]–[5]. The sum-product algorithm runs an iterative message-passing on the Tanner graph [6] defined by a parity-check matrix of the code [7]–[9], where the procedure is usually understood as a realization of Pearl’s belief-propagation (BP)—a widely-used algorithm in machine-learning and artificial-intelligence [10]–[12].

The parity-check matrix of a classical code is used to do syndrome measurements for (syndrome-based) decoding. In the 1990s, quantum error-correction was proposed to be done in a similar way by using stabilizer codes [13]–[17]. There will be a check matrix defined for a stabilizer code to do the projective (syndrome) measurements for decoding. Many sparse stabilizer codes have been proposed, such as topological codes [18], [19], random sparse codes (in particular, bicycle codes) [20], and hypergraph-product codes [21].

Decoding a binary stabilizer code is equivalent to decoding an additive code over GF(q = 4) (representing Pauli errors I, X, Y, Z) [17]. That is, a quaternary BP (BP4) is required, with every message conventionally a length-four vector to represent a distribution over GF(4). In contrast, decoding a binary classical code only needs a binary BP (BP2), with every message a scalar (e.g., as a log-likelihood ratio (LLR) [2] or likelihood difference (LD) [5]). As a result, BP2 has a check-node efficiency \( q^2 = 16 \) times better than the conventional BP4 [22]. To reduce the complexity, it often treats a binary stabilizer code as a binary classical code with doubled length so that BP2 can be used [20], followed by additional processes to handle the X/Z correlations [25], [26].

A stabilizer code inevitably has many four-cycles in its Tanner graph, which degrade the performance of BP [20], [27]. To improve, additional pre/post-processes are usually used, such as heuristic flipping from nonzero syndrome bits [27], (modified) enhanced-feedback [28], [29], training a neural BP [30], augmented decoder (adding redundant rows to the check matrix) [26], and ordered statistics decoding (OSD) [31].

In this paper, we first simplify the conventional BP4. An important observation is that, though a binary stabilizer code is considered as a quaternary code, its error syndrome is binary. Inspired by this observation and MacKay’s LD-based computation rule (\( \delta \)-rule) (cf. (47)–(53) in [5]), we derive a \( \delta \)-rule based BP4 that is equivalent to the conventional BP4 but passing every message as a scalar, which improves the check-node efficiency 16 times and does not need additional processes to handle the X/Z correlations.

Then we improve BP4 by introducing classical BP techniques. First, the messages can be updated in different orders (schedules) [32]–[34]. Second, as we have scalar messages, the message normalization can be applied [35]–[37]. With these techniques, our (scalar-based) BP4 shows a much improved empirical performance in computer simulation.

This paper is organized as follows. In Sec. II we define the notation and review BP2. In Sec. III, we show that BP4 can be efficiently done by passing scalar messages. In Sec. IV we introduce the message normalization and provide simulation results. Finally, we conclude in Sec. V.

II. CLASSICAL BINARY BELIEF-PROPAGATION DECODING

A. The notation and parallel/serial BP2

We consider the syndrome-based decoding. Consider an \([N,K]\) classical binary linear code defined by an \( M \times N \)

1The complexity can be \( O(q \log q) \) for the fast Fourier transform (FFT) method [33], [24]. But a comparison based on this is more suitable for a larger \( q > 4 \), since there is additional cost running FFT and inverse FFT.
parity-check matrix \( H \in \{0,1\}^{M \times N} \) (not necessarily of full rank) with \( M \geq N - K \). Upon the reception of a noisy codeword disturbed by an unknown error \( E = (E_1, E_2, \ldots, E_N) \in \{0,1\}^N \), syndrome measurements (defined by rows of \( H \)) are performed to generate a syndrome vector \( z \in \{0,1\}^M \). The decoder is given \( H, z \), and an a priori distribution \( p_n = (p_n^0, p_n^1) \) for each \( E_n \) to infer an \( \hat{E} \) such that the probability \( P(E = E) \) is as high as possible.\(^2\) BP decoding is done by updating each \( p_n \) as an a posteriori distribution \( q_n = (q_n^0, q_n^1) \) and inferring \( \hat{E} = (\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_N) \) by \( \hat{E}_n = \arg \max_{\hat{E}_n} \hat{p}_n \). This is done by doing an iterative message-passing on the Tanner graph defined by \( H \).

The Tanner graph corresponding to \( H \) is a bipartite graph consisting of \( N \) variable nodes and \( M \) check nodes, and there is an edge \((m,n)\) connecting check node \( m \) and variable node \( n \) if entry \( H_{mn} = 1 \). Let \( H_m \) be row \( m \) of \( H \). An example Tanner graph of \( H = [1 1 1 1] \) is shown in Fig. 1.

![Fig. 1. The Tanner graph of an example \( H = [1 1 1 1] \). The three circles are variable nodes and the two squares are check nodes.](image)

For the iterative message-passing, there will be a variable-to-check message \( d_{n \rightarrow m} \) and a check-to-variable message \( \delta_{m \rightarrow n} \) passed on edge \((m,n)\). Let \( N(m) = \{ n : H_{mn} = 1 \} \) and \( M(n) = \{ m : H_{mn} = 1 \} \). (And we will write things like \( N(m) \setminus \{ n \} \) as \( M(n) \setminus m \) to simplify the notation.) Assume that every \( \delta_{m \rightarrow n} = 0 \) initially. An iteration is done by:

- \( d_{n \rightarrow m} \) is computed by \( \{ n' \in N(m) \setminus \{ n \} \} \) with \( p_n \).
- \( \delta_{m \rightarrow n} \) is computed by \( \{ n' \in M(n) \setminus n \} \) with \( z_m \).
- \( q_n \) is computed by \( \{ m \in M(n) \} \) with \( p_m \).

The messages \( d_{n \rightarrow m} \) and \( \delta_{m \rightarrow n} \) will be denoted, respectively, by \( d_{mn} \) and \( \delta_m \) for simplicity. The message-passing is usually done with a parallel schedule, referred to as parallel BP2 (cf. [39], Algorithm 1), which has \( d_{mn} \) and \( \delta_m \) as likelihood differences (LDs) and is due to MacKay and Neal \([3,4,5]\).

The message-passing can also be done with a serial schedule, referred to as serial BP2 (Algorithm 2). We compared parallel/serial BP2 by decoding a [13298, 3296] code in [39] Figs. 4 and 5], showing that serial BP2 has a better convergence within limited iterations (as expected in \([32,33,34]\).

**B. The simplified rules for BP2**

The computation in parallel/serial BP2 can be substantially simplified and represented by simple update rules, corresponding to weight-2 and weight-3 rows. These simple rules can in turn be extended to define the general rules \([7, Sec. V-E]\). Consider \( H = [1 1 1 1] \) as in Fig. 1. Recall that we denote \( P(E = 0) = p_0 \) and \( P(E = 1) = p_1 \). Let \( E_2 + E_3 \) be a super bit indexed by \{2,3\}. Then we have the representation \( P(E_2 + E_3 = 0) = p_{(2,3)}^0 \) and \( P(E_2 + E_3 = 1) = p_{(2,3)} \). By Bayes rule and some derivations, we can update the (conditional) distribution of \( E_n \), say \( n = 1 \), by:

**LD-rule (δ-rule):** The distribution of \( E_1 \) can be updated by (conditions) \( H_1 \) and \( H_2 \), respectively, by:

\[
H_1 = [1,1,0] : \begin{bmatrix} p_1^0 \ p_1^1 \end{bmatrix} \to \begin{bmatrix} q_1^0 \ q_1^1 \end{bmatrix} \propto \begin{bmatrix} p_1^0 p_2^0 \ p_1^1 p_2^1 \end{bmatrix},
\]

\[
H_2 = [1,1,1] : \begin{bmatrix} p_1^0 \ p_1^1 \end{bmatrix} \to \begin{bmatrix} q_1^0 \ q_1^1 \end{bmatrix} \propto \begin{bmatrix} p_1^0 p_{(2,3)}^0 \ p_1^1 p_{(2,3)} \end{bmatrix},
\]

where

\[
p_{(2,3)}^0 = p_2^0 p_3^0 + p_1^1 p_3^1 = (1 + d_2 d_3)/2, \tag{2}
\]

\[
p_{(2,3)}^1 = p_2^0 p_3^1 + p_1^1 p_3^0 = (1 - d_2 d_3)/2,
\]

where \( d_n \) is the likelihood difference (LD) of \( E_n \),

\[
d_n = p_n^0 - p_n^1. \tag{3}
\]

We describe the δ-rule following [5] rather than [9]. This allows us to generalize the computation to the quantum case later. The δ-rule is equivalent to the LLR rule as follows.

**LLR-rule (Λ-rule):** Let \( \Lambda_n = \ln \frac{p_n^0}{p_n^1} \) be the log-likelihood ratio (LLR) of \( E_n \). Define \( f(x) = \ln \left( \frac{e^x + 1}{e^x - 1} \right) = \ln \left( \coth \left( \frac{x}{2} \right) \right) \) like what Gallager did [11], where \( x \geq 0 \), \( f(0) \equiv \infty \), and \( f(\infty) \equiv 0 \). (Note that \( f^{-1} = f \).) The LLR of \( E_1 \) can be updated by (conditions) \( H_1 \) and \( H_2 \), respectively, by:

\[
H_1 = [1,1,0] : \Lambda_1 \to \Lambda_1 + \Lambda_2,
\]

\[
H_2 = [1,1,1] : \Lambda_1 \to \Lambda_1 + \Lambda_2 + \Lambda_{(2,3)} = \Lambda_1 + (\Lambda_2 \boxplus \Lambda_3),
\]

where \( \Lambda_{(2,3)} = \Lambda_2 \boxplus \Lambda_3 \) is defined by

\[
\Lambda_2 \boxplus \Lambda_3 = \text{sign}(\Lambda_2 \Lambda_3) f(f(\Lambda_2) + f(\Lambda_3)). \tag{4}
\]

It only needs an addition after a transform by \( f \). To prevent handling the signs separately, it can use a transform by \( \tanh \) but needs a multiplication \([40]\) by

\[
\Lambda_2 \boxplus \Lambda_3 = 2 \tanh^{-1}(\tanh(\Lambda_2/2) \tanh(\Lambda_3/2)). \tag{5}
\]

Using simple algebra can confirm that the \( \Lambda_2 \boxplus \Lambda_3 \) in (4) and (5) are equivalent, and \( \Lambda_2 \boxplus \Lambda_3 = \ln \left( \frac{p_{(2,3)}^0}{p_{(2,3)}^1} \right) \) for the \( p_{(2,3)}^0 \) and \( p_{(2,3)}^1 \) defined in (2). More equivalent rules can be found in [9, Sec. V-E].

Depending on the application (the channel model, decoder hardware, etc.), a suitable rule can be chosen. For example, the Λ-rule is suitable for the AWGN channel (since the received symbol’s magnitude is linearly mapped to \( |\Lambda_n| \) [2]), and the operation \( \boxplus \) can be efficiently computed/approximated (e.g., by table lookup [36]). On the other hand, the δ-rule is suitable for the BSC, and the computation can be efficiently programmed/computed by a general-purpose computer.

The δ-rule is very suitable for our purpose of decoding quantum codes in the next section.
III. QUATERNARY BP DECODING FOR QUANTUM CODES

A. The simplified rule for BP₄

We will use the notation based on Pauli operators \( \{ I = [1, 0, 0], X = [0, 1, 0], Y = [i, 0, 0], Z = [0, 0, 1] \} \) [41], and consider \( N \)-fold Pauli operators with an inner product

\[
\langle \cdot, \cdot \rangle : \{ I, X, Y, Z \}^{\otimes N} \times \{ I, X, Y, Z \}^{\otimes N} \to \{ 0, 1 \},
\]

which has an output 0 if the inputs commute and an output 1 if the inputs anticommute. (This is mathematically equivalent to mapping \( \{ I, X, Y, Z \}^{\otimes N} \to GF(4)^N \) with a Hermitian trace inner-product to the ground field GF(2) [17].) It suffices to omit \( \otimes \) in the following discussion. An unknown error will be denoted by \( E = E_1 E_2 \cdots E_N \in \{ I, X, Y, Z \}^N \).

There will be a check matrix \( S = \{ S_{11}, S_{12}, S_{21}, S_{22}, S_{23} \} \), where there are five non-identity entries. For simplicity, fix it as \( S = [\frac{1}{2} \frac{1}{2} \frac{1}{2}] \) and the other cases can be handled similarly. The Tanner graph (with generalized edge types \( X, Y, Z \)) of the example \( S \) is shown in Fig.2.

For decoding quantum codes, we are to update an initial \( \delta \)-rule and some derivations, we have:

\[
\delta = (p_1^X + p_2^X) - (p_1^Y + p_2^Y),
\]

\[
\delta = (p_1^X + p_3^X) - (p_2^X + p_3^X).
\]

Observing the similarity and difference in (1)-(3) and (7)-(9). This even suggests a way to generalize the above strategy to a case of higher-order GF(q = 2^l) with binary syndromes.

This pre-add strategy is like a partial FFT-method since for \( GF(q = 2^l) \), the basic operation of the ground field GF(2) is addition (subtraction) [24]. However, the FFT-method needs \( O(q \log q) \) additions and \( O(q) \) multiplications to combine two distributions [23]. The above strategy, by generalizing (8) and (9), would only need \( O(q) \) additions and \( O(1) \) multiplication to combine two distributions if the syndrome is binary.

Another point is that the FFT-method still treats a message as a vector over GF(q), but we treat the message as a scalar and this opens new possibilities for improvement (e.g., using message normalization (shown later) or allowing a neural BP₄, where (since training needs scalar messages) a previous neural BP is only considered with BP₄). For decoding quantum codes, with LD-rule and some derivations, we have:

**LD-rule (δ-rule) for BP₄:** The distribution of \( E \) can be updated by (conditions) \( S_1 \) and \( S_2 \), respectively, by:

\[
S_1 = XYI : \begin{bmatrix} p_1^I \\ p_1^X \\ p_1^Y \\ p_1^Z \end{bmatrix} \times \begin{bmatrix} q_1^I \\ q_1^X \\ q_1^Y \\ q_1^Z \end{bmatrix} \times \begin{bmatrix} p_1^I (p_2^X + p_2^Y) \\ p_1^X (p_2^Y + p_2^Z) \\ p_1^Y (p_2^Z + p_2^X) \end{bmatrix},
\]

\[
S_2 = ZZY : \begin{bmatrix} p_3^I \\ p_3^X \\ p_3^Y \\ p_3^Z \end{bmatrix} \times \begin{bmatrix} q_3^I \\ q_3^X \\ q_3^Y \\ q_3^Z \end{bmatrix} \times \begin{bmatrix} p_3^I (p_2^I + p_2^2) \\ p_3^X (p_2^I + p_2^2) \\ p_3^Y (p_2^I + p_2^2) \end{bmatrix},
\]

where (since \( S_{22}S_{23} = ZY \))

\[
p_{(0)}^{(1)} = \begin{cases} p_3^I p_3^X + p_2^X p_2^Y & \text{if } n = 1 \text{ and } \beta = 1 \\ p_3^I p_3^X + p_2^X p_2^Y + p_3^X p_3^Y & \text{if } n = 1 \text{ and } \beta = 0 \end{cases},
\]

\[
p_{(1)}^{(2)} = \begin{cases} p_3^I p_3^X + p_2^X p_2^Y + p_2^X p_2^Y & \text{if } n = 1 \text{ and } \beta = 1 \\ p_3^I p_3^X + p_2^X p_2^Y + p_3^X p_3^Y & \text{if } n = 1 \text{ and } \beta = 0 \end{cases}.
\]

The last two probabilities (with 16 multiplications as above) can be efficiently computed (with one multiplication) by

\[
p^{(0)}_{(2,3)} = \frac{1 + \delta_3}{2} \quad \text{and} \quad p^{(1)}_{(2,3)} = \frac{1 - \delta_3}{2},
\]

where (since \( S_{22} = Z \) and \( S_{23} = Y \))

\[
\delta_3 = (p_1^X + p_2^X) - (p_1^Y + p_2^Y),
\]

\[
\delta_3 = (p_3^X + p_3^Y) - (p_3^X + p_3^Y).
\]

B. The conventional BP₄ with vector messages

An \([N, K]\) stabilizer code is a \(2^K\)-dimensional subspace of \( C^{2^N}\) fixed by the operators (not necessarily all independent) defined by a check matrix \( S \in \{ I, X, Y, Z \}^{M \times N} \) with \( M \geq N - K \). Each row \( S_m \) of \( S \) corresponds to an \( N \)-fold Pauli operator that stabilizes the code space. The matrix \( S \) is self-orthogonal with respect to the inner product [6], i.e., \( \langle S_m, S_m' \rangle = 0 \) for any two rows \( S_m \) and \( S_m' \). The code space is the joint-(+1) eigenspace of the rows of \( S \). The vectors in the (multiplicative) rowspace of \( S \) are called stabilizers [41].

It suffices to consider discrete errors like Pauli errors per the error discretization theorem [41]. Upon the reception of a noisy coded state suffered from an unknown \( N \)-qubit error \( E \in \{ I, X, Y, Z \}^N \), \( M \) stabilizers \( \{ S_m \}_{m=1}^M \) are measured to determine a binary error syndrome \( z = (z_1, z_2, \ldots, z_M) \in \{ 0, 1 \}^M \) using (6), where

\[
z_m = \langle E, S_m \rangle \in \{ 0, 1 \}.
\]

Given \( z, s \), and an a priori distribution \( p_n = (p_n^I, p_n^X, p_n^Y, p_n^Z) \) for each \( E_n \) (e.g., let \( p_n = (1 - \epsilon, \epsilon/3, \epsilon/3, \epsilon/3) \) for a depolarizing channel with error rate \( \epsilon \)), a decoder has to estimate an \( \hat{E} \in \{ I, X, Y, Z \}^N \) such that \( \hat{E} \) is equivalent to \( E \), up to a stabilizer, with a probability as high as possible. (The solution is not unique due to the degeneracy [27, 42].)

For decoding quantum codes, the neighboring sets are \( \mathcal{N}(m) = \{ n : S_{mn} \neq I \} \) and \( \mathcal{M}(n) = \{ m : S_{mn} \neq I \} \). Let \( E_{\mathcal{N}(m)} \) be the restriction of \( E = E_1 E_2 \cdots E_N \) to \( \mathcal{N}(m) \). Then \( \{ E, S_m \} = \{ E_{\mathcal{N}(m)}, S_m \}_{m \in \mathcal{N}(m)} \) for any \( E \) and \( S_m \).

3For example, if \( S_m = IXZ \) and \( E = E_1 E_2 E_3 \), then \( E_{\mathcal{N}(m)} = (E_1, E_2, E_3)_{\mathcal{N}(m)} = E_2 E_3 \) and \( S_m \in \mathcal{M}(m) = XZ \). Then

\[
\langle E, S_m \rangle = \langle E_1 E_2 E_3, IXZ \rangle = \langle E_2 E_3, XZ \rangle = \langle \mathcal{N}(m), S_m \rangle_{\mathcal{N}(m)}.
\]
A conventional BP₄ for decoding binary stabilizer codes is done as follows [27]. In the initialization step, every variable node \( n \) passes the message \( q_{mn} = (q_{mn}, q_{mn}, q_{mn}, q_{mn}) = p_n \) to every neighboring check node \( m \in \mathcal{M}(n) \).

- **Horizontal Step:** At check node \( m \), compute \( r_{mn} = (r_{mn}^l, r_{mn}^c, r_{mn}^y, r_{mn}^z) \) and pass \( r_{mn} \) as the message \( m \rightarrow n \) for every \( n \in \mathcal{N}(m) \), where \( \forall W \in \{I, Y, Z\} \):
  \[
  r_{mn}^W = E_n \left( \sum_{n' \in \mathcal{N}(m) \setminus n} \prod_{n'' \in \mathcal{M}(n)} q_{n''}^{E_{n''}} \right).
  \]

- **Vertical Step:** At variable node \( n \), compute \( q_{mn} = (q_{mn}^l, q_{mn}^c, q_{mn}^y, q_{mn}^z) \) and pass \( q_{mn} \) as the message \( n \rightarrow m \) for every \( m \in \mathcal{M}(n) \), where
  \[
  q_{mn}^W = a_{mn} p_m^W \prod_{m' \in \mathcal{M}(n) \setminus m} q_{mn'}^W
  \]
  with \( a_{mn} \) a chosen scalar to let \( \sum_{W \in \{I, Y, Z\}} q_{mn}^W = 1 \).

At variable node \( n \), it also computes \( E_n = p_n^W \prod_{m \in \mathcal{M}(n)} q_{mn}^W \) to infer \( \hat{E}_n = \arg \max_{W \in \{I, Y, Z\}} q_{mn}^W \). The horizontal and vertical steps are iterated until an estimated \( \hat{E} = \hat{E}_1 \hat{E}_2 \cdots \hat{E}_N \) is valid or a maximum number of iterations is reached.

### C. The refined BP₄ with scalar messages

For a variable node, say variable node 1, and its neighboring check node \( m \), we know from [10] that
\[
(E_1, S_{m1}) = z_m + \sum_{n=2}^{N} (E_n, S_{mn}) \mod 2.
\]

Given the value \( (E_n, S_{mn}) \) of (a possible) \( E_n \in \{I, X, Y, Z\} \) and some \( S_{mn} \in \{X, Y, Z\} \) and \( S_{mn} \in \{X, Y, Z\} \) and \( S_{mn} \in \{X, Y, Z\} \) \( \setminus S_{mn} \). Consequently, the passed message should indicate more likely whether \( E_n \in \{I, S_{mn}\} \) or \( E_n \in \{X, Y, Z\} \) \( \setminus S_{mn} \). In other words, the message from a neighboring check will tell us more likely whether \( E_1 \) commutes or anticommutes with \( S_{m1} \). This suggests that a BP decoding of stabilizer codes with scalar messages is possible and we provide such an algorithm in Algorithm 1, referred to as parallel BP₄ (the same as [39] Algorithm 3). Note that the notation \( r_{mn}((W(S), S)) \) is simplified as \( r_{mn}((W(S)) \).

It can be shown that Algorithm 1 has exactly the same output as the conventional BP₄ (as outlined in Sec. III-A). In particular, Algorithm 1 has a check-node efficiency 16-fold improved from the conventional BP₄.

Similar to the classical case, we can consider running Algorithm 1 with a serial schedule, referred to as serial BP₄ (cf. [39] Algorithm 4). Using serial BP₄ may prevent the decoding oscillation (as an example based on the [5,1,3] code in [39] Fig. 7) and improve the empirical performance (e.g., for decoding a hypergraph-product code [39] Fig. 8).

### Algorithm 1: δ-rule based quaternary BP decoding for binary stabilizer codes with a parallel schedule (parallel BP₄)

**Input:** \( S \subseteq \{I, X, Y, Z\}^{M \times N}, z \in \{0, 1\}^M \), and initial \( \{(p_n, p_n^X, p_n^Y, p_n^Z)\}_{n=1}^N \).

**Initialization.** For \( n = 1, 2, \ldots, N \) and \( m \in \mathcal{M}(n) \), let
\[
d_{mn} = q_{mn}^{(0)} - q_{mn}^{(1)},
\]
where \( q_{mn}^{(0)} = p_n^I + p_m^{S_{mn}} \) and \( q_{mn}^{(1)} = 1 - q_{mn}^{(0)} \).

**Horizontal Step.** For \( m = 1, \ldots, M \) and \( n \in \mathcal{N}(m) \), compute
\[
\delta_{mn} = (-1)^{z_m} \prod_{n' \in \mathcal{N}(m) \setminus n} \prod_{m' \in \mathcal{M}(n')} d_{mn'}.
\]

**Vertical Step.** For \( n = 1, 2, \ldots, N \) and \( m \in \mathcal{M}(n) \), do:

- Compute \( r_{mn}^{(0)} = (1 + \delta_{mn})/2 \)
\[
r_{mn}^{(0)} = p_n^I \prod_{m' \in \mathcal{M}(n) \setminus m} r_{mn'}^{(0)}.
\]

- Compute \( r_{mn}^{(1)} = (1 - \delta_{mn})/2 \)
\[
r_{mn}^{(1)} = p_n^I \prod_{m' \in \mathcal{M}(n) \setminus m} r_{mn'}^{(1)}.
\]

- Let \( q_{mn}^{(0)} = a_{mn}(q_{mn}^{(0)} + q_{mn}^{(1)}) \) and \( q_{mn}^{(1)} = a_{mn}(\sum_{W \in \{I, X, Y, Z\}} S_m q_{mn}^{(W)}) \),
where \( a_{mn} \) is a chosen scalar such that \( q_{mn}^{(0)} + q_{mn}^{(1)} = 1 \).

- Update: \( d_{mn} = q_{mn}^{(0)} - q_{mn}^{(1)} \).

**Hard Decision.** For \( n = 1, 2, \ldots, N \), compute
\[
\hat{q}_n^{(0)} = p_n^I \prod_{m \in \mathcal{M}(n)} r_{mn}^{(0)}
\]
\[
\hat{q}_n^{(1)} = p_n^I \prod_{m \in \mathcal{M}(n)} r_{mn}^{(1)}.
\]

Let \( \hat{E}_n = \arg \max_{W \in \{I, X, Y, Z\}} q_{mn}^{(W)} \).

- If \( \langle \hat{E}, S_m \rangle = z_m \forall m \), halt and return “SUCCESS”;
- otherwise, if a maximum number of iterations is reached, halt and return “FAIL”;
- otherwise, repeat from the horizontal step.

### Algorithm 2: Normalized BP₄ with a given parameter \( \alpha \)

Identical to Algorithm 1 except that \( q_{mn}^{(0)} \) and \( q_{mn}^{(1)} \) are respectively replaced by \( q_{mn}^{(0)} = a_{mn}(q_{mn}^{(0)} + q_{mn}^{(1)})^{1/\alpha} \) and \( q_{mn}^{(1)} = a_{mn}(\sum_{W \in \{I, X, Y, Z\}} S_m q_{mn}^{(W)})^{1/\alpha} \) for some \( \alpha > 0 \), where \( a_{mn} \) is a chosen scalar such that \( q_{mn}^{(0)} + q_{mn}^{(1)} = 1 \).

### IV. Simulation Results

Having scalar messages allows us to apply the message normalization [35]–[37]. This suppresses the wrong belief looped in the short cycles (due to sub-matrices like \( \begin{bmatrix} Z & Z \end{bmatrix} \) in Fig. 2).

The message normalization can be applied to (Algorithm 1, Algorithm 5) or (Algorithm 1, Algorithm 5) and (Algorithm 6). Many simulation results were provided in [39], and normalizing \((q_{mn}^{(0)}, q_{mn}^{(1)})\) has a lower error-floor. We restate [39] Algorithm 6 as Algorithm 2 here (where \((\cdot)^{1/\alpha}\) can be efficiently done by one multiplication and two additions [43]). Some more results will be provided. We try to collect at least 100 logical errors for every data-point; or otherwise an error bar between two crosses shows a 95% confidence interval (omitted in Fig. 5).
In the simulation, we do not restrict the measurements to be local. We consider bicycle codes (a kind of random sparse codes) [20] since they have small ancilla overhead and good performance, benchmarked by code rate vs. physical error rate [20, Fig. 1]. But BP decoding of this kind of codes have some error-floor subject to improve [20, Fig. 6].

First, a random bicycle code is easy to construct [27] but could have a high error-floor [39]. For convenience, we replotted [39, Fig. 16] as Fig. 3 here. Observe the high error-floor before using message normalization. Using $\alpha_v$ improves the error-floor a lot. We implemented an overflow (underflow) protection in BP as in [5] and confirmed that the original error-floor is not due to numerical overflow/underflow. It should be due to the random construction (especially that a row-deletion step is performed randomly). MacKay, Mitchison, and McFadden suggested to use a heuristic approach to do the construction (to have more uniform column-weights in the check matrix after the row-deletion step) [20].

An $[[800, 400]]$ bicycle code is constructed by the heuristic approach, with the other parameters the same as the code in Fig. 3. The performance of this new code is shown in Fig. 4 where the BP$^4$ error-floor (without message normalization) is improved compare to Fig. 3. However, using $\alpha_v = 1.5$ further lowers the error-floor to a level like the improved case in Fig. 3. The two figures show that running (scalar-based) BP$^4$ with message normalization has a more robust performance. This provides more flexibility in code construction (for if some rows (measurements) must be kept or deleted).

Next we construct a $[[3786, 946]]$ bicycle code with row-weight 24, using the heuristic approach. A code with such parameters has some error-floor for a block error rate $< 10^{-4}$, when decoded by BP$^2$ in the bit-flip channel [20, Fig. 6]. We decode the constructed code by BP$^4$ in the depolarizing channel, and plot the decoding performance in Fig. 5. There is a similar error-floor for a logical error rate $< 10^{-4}$ before using message normalization. After using $\alpha_v$, the error-floor performance is much improved. A smaller $\epsilon$ usually needs a larger $\alpha_v$ for suppression (since $p_p$ is quite biased with a large $p_p = 1 - \epsilon$). As a reference, we derive Fig. 6 from Fig. 5 by using different $\alpha_v$ for different $\epsilon$ to see the improvement.

V. Conclusion

We proposed a $\delta$-rule based BP$^4$ for decoding quantum stabilizer codes by passing scalar messages. This is achieved by

\begin{itemize}
  \item Using random sparse codes needs long qubit-connectivity, which is possibly supported by trapped-ions or nuclear-spins [44]. If the measurements must be local, then topological codes are usually considered (with another benchmark called threshold usually used). Our proposed algorithm can also decode topological codes but needs some extension to have a good performance. This will be addressed in another manuscript under preparation.
  \item Except for special irregular designs, BP converges more well when the column-weights are more uniform [2, 5, 20]. We consider to either minimize the column-weight variance (MIN-VAR approach) or minimize the difference of the largest and smallest column-weights (MIN-MAX approach).
\end{itemize}

$\delta$-rule based BP$^4$ is equivalent to the conventional BP$^4$ but with a check-node efficiency 16-fold improved. The message normalization can be naturally applied to the scalar messages. We performed the simulation by decoding several bicycle codes. Different decoding schedules were also considered. Using the proposed BP$^4$ with message normalization and a suitable schedule results in a significantly improved performance and error-floor.

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Fig. 5. Performance of decoding the $[[3786, 946]]$ bicycle code.

Fig. 6. Performance of decoding the $[[3786, 946]]$ bicycle code with different $\alpha$, used for different $\epsilon$. This figure is derived from the data in Fig. 5.

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