Vertex models, TASEP and Grothendieck polynomials

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Abstract
We examine the wavefunctions and their scalar products of a one-parameter family of integrable five-vertex models. At a special point of the parameter, the model investigated is related to an irreversible interacting stochastic particle system—the so-called totally asymmetric simple exclusion process (TASEP). By combining the quantum inverse scattering method with a matrix product representation of the wavefunctions, the on-/off-shell wavefunctions of the five-vertex models are represented as a certain determinant form. Up to some normalization factors, we find that the wavefunctions are given by Grothendieck polynomials, which are a one-parameter deformation of Schur polynomials. Introducing a dual version of the Grothendieck polynomials, and utilizing the determinant representation for the scalar products of the wavefunctions, we derive a generalized Cauchy identity satisfied by the Grothendieck polynomials and their duals. Several representation theoretical formulae for the Grothendieck polynomials are also presented. As a byproduct, the relaxation dynamics such as Green functions for the periodic TASEP are found to be described in terms of the Grothendieck polynomials.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Symmetric polynomials [1] are ubiquitous objects in mathematics and mathematical physics. One of the most basic and important symmetric polynomials is the Schur polynomial

\[ s_\lambda(z) = \frac{\det_N(z_j^{\lambda_i+N-i})}{\prod_{1 \leq j < k \leq N}(z_j - z_k)}, \tag{1.1} \]

where \( z = \{z_1, \ldots, z_N\} \) is a set of variables and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \) is a sequence of weakly decreasing non-negative integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0 \). A sequence \( \lambda \) can be represented as a Young diagram whose \( i \)th row has \( \lambda_i \) boxes.
Schur polynomials appear not only in representation theory but also in many contexts in mathematical physics, especially in integrable systems. For example, the tau functions of the KP hierarchy have Schur polynomial expansions [2]. Schur polynomials also appear as the singular vectors in conformal field theory [3, 4], the Green function of the vicious walkers [5, 6], the domain wall boundary partition function of the six-vertex model [7, 8] and the Schur processes as one of the most fundamental examples of determinantal processes [9], to list a few. The relation between integrable models and symmetric polynomials can be extended from Schur polynomials to Jack, Hall–Littlewood, Macdonald polynomials and so on.

In this paper, we develop a novel relation between integrable models and symmetric polynomials. We consider a family of integrable five-vertex models. At a special point of the parameter, the model we investigate is related to an irreversible interacting stochastic particle system called the totally asymmetric simple exclusion process (TASEP) [10–15]. At another point, the vertex models reduce to the four-vertex model describing the one-dimensional quantum Ising model [16].

We show that up to normalization factors, the wavefunctions of the five-vertex model are given by the Grothendieck polynomials [17–19]

$$G_\lambda(z; \beta) = \frac{\det_N \left( (z_j + N - k) z_j \right)^{1 - k}}{\prod_{1 \leq j < k \leq N} (z_j - z_k)},$$  \hspace{1cm} (1.2)

which are a one-parameter extension of the Schur polynomials. The Grothendieck polynomial was originally introduced [17] in the context of the intersection between geometry and representation theory as a \( K \)-theoretical extension of the Schubert polynomials, i.e. as polynomial representatives of Schubert classes in the Grothendieck ring of the flag manifold. The formal parameter \( \beta \) corresponds to the \( K \)-theoretical extension. For flag varieties of type \( A \), the Schubert polynomial is the Schur polynomial itself, and it was shown recently [18, 19] that Grothendieck polynomials for flag varieties of type \( A \) are expressed in the determinant form (1.2). We show the equivalence between the wavefunctions and the Grothendieck polynomials by combining the quantum inverse scattering method with a matrix product representation of the wavefunctions. We find that the wavefunctions correspond to the Grothendieck polynomials (1.2), and the dual wavefunctions to the following ‘dual’ Grothendieck polynomials:

$$\overline{G}_\lambda(z; \beta) = \frac{\det_N \left( (z_j + N - k) z_j^{-1} \right)^{1 - k}}{\prod_{1 \leq j < k \leq N} (z_j - z_k)}.$$  \hspace{1cm} (1.3)

From this relation between integrable models and symmetric polynomials, we note that studying integrable five-vertex models leads us to representation theoretical results of the Grothendieck polynomials. We present several results for the Grothendieck polynomials in this way, i.e. by studying the integrable five-vertex models. We find the following Cauchy identity for the Grothendieck and dual Grothendieck polynomials:

$$\sum_{\lambda \subseteq (M-N)^N} G_\lambda(z; \beta) \overline{G}_\lambda(y; \beta) = \prod_{1 \leq j < k \leq N} \frac{1}{(z_j - z_k)(y_j - y_k)} \det_N \left[ \frac{(z_j y_k)^M - \left\{ (1 + \beta z_j)/(1 + \beta y_k^{-1}) \right\}^{N-1}}{z_j y_k - 1} \right].$$  \hspace{1cm} (1.4)

which generalizes the one for the Schur polynomials. We show this identity by evaluating the scalar products of the wavefunctions in two ways. We can evaluate the determinant representation of the scalar products directly in a way by the use of the recursive relations called the Izergin–Korepin approach. The scalar products can also be evaluated by inserting the completeness relation and the determinant forms of the wavefunctions and dual wavefunctions.
The two ways of the evaluation of the scalar products turn out to yield the Cauchy identity for the Grothendieck polynomials (1.4). In short, we find a generalization of the celebrated Cauchy identity by analyzing a family of integrable five-vertex models with the recently developed techniques to analyze integrable models (see [20, 21] for the six-vertex model, [22–26, 28] for the XXZ chain and [27] for the TASEP). As a special case of the Cauchy identity, we also obtain the summation formula for the Grothendieck polynomials. There are several results on the generalizations of the Cauchy identity for symmetric polynomials related to geometry in the past (see [28, 29] for example). The one for the Grothendieck polynomials presented in this paper has advantages in that the connection with the Schur polynomials is explicit and easily understandable, which seems not to be known before.

As a byproduct of the determinant forms of the wavefunctions, we formulate the exact relaxation dynamics of the periodic TASEP for arbitrary initial condition, generalizing the case for the step and alternating initial conditions [30, 31].

This paper is organized as follows. In the next section, we introduce a one-parameter family of integrable five-vertex models by solving the RLL-relation, a version of the Yang–Baxter relation which guarantees the integrability of the model. In section 3, we evaluate the scalar products by the Izergin–Korepin approach to find its determinant form. In section 4, by combining the quantum inverse scattering method with a matrix product representation of the wavefunctions, the determinant representation of the wavefunctions is obtained. In section 5, we establish the relation between the wavefunctions and the Grothendieck polynomials. Combining the results in sections 3 and 4, we derive the Cauchy identity and the summation formula for the Grothendieck polynomials. We give a formulation of the exact relaxation dynamics of the periodic TASEP for arbitrary initial condition in section 6. Section 7 is devoted to the conclusion of this paper.

2. One-parameter family of five-vertex models

A key ingredient in constructing quantum integrable models is to find a solution of the relation (RLL-relation)

\[ R_{\mu \nu}(u, v)L_{\mu j}(u)L_{\nu j}(v) = L_{\nu j}(v)L_{\mu j}(u)R_{\mu \nu}(u, v) \]  

(2.1)

holding in \( \text{End}(W_\mu \otimes W_\nu \otimes V_j) \) for arbitrary \( u, v \in \mathbb{C} \). Here the matrix \( R_{\mu \nu}(u, v) \in \text{End}(W_\mu \otimes W_\nu) \) satisfies the Yang–Baxter equation

\[ R_{\mu \nu}(u, v)R_{\mu \gamma}(u, w)R_{\nu \gamma}(v, w) = R_{\nu \gamma}(v, w)R_{\mu \gamma}(u, w)R_{\mu \nu}(u, v) \]  

(2.2)

and \( L_{\mu j}(u) \) is an operator acting on the space \( W_\mu \otimes V_j \). By convention, we call \( W \) and \( V \) the auxiliary space and the quantum space, respectively.

In the following, we shall take both of the spaces \( W \) and \( V \) to be the two-dimensional vector spaces \( W = V = \mathbb{C}^2 \) spanned by the ‘empty state’ \( |0\rangle = (1, 0) \) and the ‘particle occupied state’ \( |1\rangle = (0, 1) \). (Note that \( W_\mu \) (resp. \( V_j \)) denotes a copy of \( \mathbb{C}^2 \) spanned by the \( \mu \)th (resp. \( j \)th) states \( |0\rangle_\mu \) and \( |1\rangle_\mu \) (resp. \( |0\rangle_j \) and \( |1\rangle_j \)).) One solution to the Yang–Baxter equation (2.2) is the following \( R \)-matrix whose elements are the Boltzmann weights associated with a five-vertex model:

\[ R(u, v) = \begin{pmatrix}
 f(v, u) & 0 & 0 & 0 \\
 0 & 0 & g(v, u) & 0 \\
 0 & g(v, u) & 1 & 0 \\
 0 & 0 & 0 & f(v, u)
\end{pmatrix} \]  

(2.3)
with
\[ f(v, u) = \frac{u^2}{u^2 - v^2}, \quad g(v, u) = \frac{uv}{u^2 - v^2}. \] (2.4)

As a solution of the RLL-relation (2.1) with the R-matrix (2.3), we find the following L-operator

\[ L(\mu)(u) = u s_\mu s_j + \sigma^-_\mu \sigma^+_j + \sigma^+_\mu \sigma^-_j + (\alpha u - u^{-1}) n_\mu s_j + \alpha u n_\mu n_j, \] (2.5)

where \( \sigma^\pm, \sigma^z \) are the spin-1/2 Pauli matrices; \( s = (1 + \sigma^z)/2 \) and \( n = (1 - \sigma^z)/2 \) are the projection operators onto the states \(|0\rangle\) and \(|1\rangle\), respectively. Note that the operators with subscript \( \mu \) (resp. \( j \)) act on the auxiliary (resp. quantum) space \( W_\mu \) (resp. \( V_j \)). See also figure 1 for a pictorial description of the L-operator (2.5), which allows for an intuitive understanding of the subsequent calculations.

The parameter \( \alpha \) can be taken arbitrary. In fact, the models at special points of \( \alpha \) are related to some physically interesting models. To see this, let us construct the monodromy matrix \( T(\mu)(u) \) by a product of L-operators:

\[ T_\mu(u) = \prod_{i=1}^{M} L_{\mu j}(u), \] (2.6)

which acts on \( W_\mu \otimes (V_1 \otimes \cdots \otimes V_M) \). Tracing out the auxiliary space, one defines the transfer matrix \( \tau(\mu)(u) \in \text{End}(V \otimes M) \):

\[ \tau(\mu)(u) = \text{Tr}_{W_\mu} T_\mu(u). \] (2.7)

Thanks to the RLL-relation, the transfer matrix \( \tau(\mu)(u) \) mutually commutes, i.e.

\[ [\tau(\mu)(u), \tau(\nu)(v)] = 0. \] (2.8)

After taking the logarithmic derivative of the transfer matrix with respect to the spectral parameter, one obtains the quantum Hamiltonian which is, in general, non-Hermitian,

\[ \mathcal{H} := \sum_{j=1}^{M} \left\{ \alpha \sigma^+_j \sigma^-_{j+1} + \frac{1}{4} (\sigma^+_j \sigma^z_{j+1} - 1) \right\} = \frac{1}{2\sqrt{\alpha}} \frac{\partial}{\partial u} \log \left\{ (\sqrt{\alpha u})^{-M} \tau(\mu)(u) \right\} \bigg|_{u=\frac{1}{\sqrt{\alpha}}}. \] (2.9)

3 Note that the parameter \( \alpha \) is different from the parameter \( q \) of a quantum group \( U_q(sl_2) \).
At \( \alpha = 1 \), the \( L \)-operator is essentially the same with the \( R \)-matrix (2.3). In this case, the quantum Hamiltonian \( \mathcal{H} \) (2.9) can be interpreted as a stochastic matrix describing an irreversible interacting stochastic particle system called the TASEP \([15, 27]\) (see section 6 for details). On the other hand, in the limit \( \alpha \rightarrow \infty \), this model is related to an irreversible noncollisional diffusion process (i.e. a vicious random walker model): \( (\mathcal{H} / \alpha - 1)^{\otimes N}|_{\alpha \rightarrow \infty} \) is nothing but a transition matrix describing the process of \( N \) vicious random walkers. Finally, the \( L \)-operator at \( \alpha = 0 \) reduces to the four-vertex model \([16]\), and through relation (2.9) it is related to the well-known Ising model.

The quantum integrability of the model (2.9) is easily understood by the commutativity of the transfer matrix (2.8) and its Hamiltonian limit (2.9): the transfer matrix \( \tau (u) \) is just a generator of nontrivial conserved quantities.

In the next section, by means of the quantum inverse scattering method, we construct state vectors of the Hamiltonian (2.9) (or equivalently, of the transfer matrix (2.7)) and calculate their scalar products.

3. Scalar products of state vectors

Here we construct a state vector of the integrable models defined in the preceding section by using the quantum inverse scattering method (i.e. the algebraic Bethe ansatz). The resultant \( \mathcal{N} \)-particle state \( |\psi ((u)_{\mathcal{N}})\rangle \) is characterized by \( \mathcal{N} \) unknown numbers \( u_{j} \in \mathbb{C} \) (\( 1 \leq j \leq \mathcal{N} \)), which becomes an eigenstate of (2.9) (or (2.7)) if we choose the parameters \( |u|_{\mathcal{N}} \) as an arbitrary set of solutions of certain algebraic equation (i.e. the Bethe ansatz equation, see (3.7)). Hereafter, we call the eigenstates the on-shell states, while we call the states with arbitrary complex values of \( |u|_{\mathcal{N}} \) the off-shell states. In this section, we construct the arbitrary off-shell states and show how their scalar products can be expressed as a determinant form.

First let us consider the monodromy matrix:

\[
T_{\mu}(u, \{ w \}) = \prod_{j=1}^{M} L_{\mu j}(u/w_{j}) = \begin{pmatrix}
(A(u, \{ w \}) & B(u, \{ w \}) \\
(C(u, \{ w \}) & D(u, \{ w \})
\end{pmatrix}_{\mu}.
\]

(3.1)

Here, for later convenience, we introduced the inhomogeneous parameters \( w_{1}, \ldots, w_{M} \in \mathbb{C} \). Taking the homogeneous limit \( w_{j} \rightarrow 1 \) (\( 1 \leq j \leq M \)), (2.6) is recovered:

\[
T(u, \{ w \})|_{w_{1}=1, \ldots, w_{M}=1} = T(u).
\]

(3.2)

As in the above equation, hereafter we will omit \( \{ w \} \) for the quantities in the homogeneous limit (e.g. \( A(u) \leftarrow A(u, \{ w \})|_{w_{1}=1, \ldots, w_{M}=1} \)). The four elements of the monodromy matrix \( A(u, \{ w \}) \), etc are the operators acting on the quantum space \( V_{1} \otimes \cdots \otimes V_{M} \). The diagrammatic representations of these four elements are given in figure 2.

Applying the RLL-relation (2.1) repeatedly, the below intertwining relation follows:

\[
R_{\mu \nu}(u, v)T_{\mu}(u, \{ w \})T_{\nu}(v, \{ w \}) = T_{\nu}(v, \{ w \})T_{\mu}(u, \{ w \})R_{\mu \nu}(u, v).
\]

(3.3)

The relations listed below are obtained by the above equation, which play a key role in the following calculations:

\[
C(u, \{ w \})B(v, \{ w \}) = g(u, v) [A(u, \{ w \})D(v, \{ w \}) - A(v, \{ w \})D(u, \{ w \})],
\]

(3.4)

\[
A(u, \{ w \})B(v, \{ w \}) = f(v, u)B(v, \{ w \})A(u, \{ w \}) + g(v, u)B(u, \{ w \})A(v, \{ w \}),
\]

\[
D(u, \{ w \})B(v, \{ w \}) = f(v, u)B(v, \{ w \})D(u, \{ w \}) + g(u, v)B(u, \{ w \})D(v, \{ w \}),
\]

\[
[B(u, \{ w \}), B(v, \{ w \})] = [C(u, \{ w \}), C(v, \{ w \})] = 0.
\]

(3.4)

The transfer matrix \( \tau (u, \{ w \}) \) is then expressed as elements of the monodromy matrix:

\[
\tau (u, \{ w \}) = \text{Tr}_{w_{\mu}} T_{\mu}(u, \{ w \}) = A(u, \{ w \}) + D(u, \{ w \}).
\]

(3.5)
The arbitrary $N$-particle state $|\psi([u]_N, [w])\rangle$ (resp. its dual $\langle \psi([u]_N, [w])|$) (not normalized) with $N$ spectral parameters $[u]_N = \{u_1, u_2, \ldots, u_N\}$ is constructed by a multiple action of the $B$ (resp. $C$) operator on the vacuum state $|\Omega\rangle := |0^M\rangle := |0\rangle_1 \otimes \cdots \otimes |0\rangle_M$ (resp. $\langle \Omega | := \langle 0^M | := \langle 0 | \otimes \cdots \otimes \langle 0 |_M$):

$$
|\psi([u]_N, [w])\rangle = \prod_{j=1}^{N} B(u_j, [w])) \Omega, \quad \langle \psi([u]_N, [w])| = \langle \Omega | \prod_{j=1}^{N} C(u_j, [w]).
$$

Due to the commutativity of the operators $B$ or $C$ (3.4), the states defined above (and also their scalar products) do not depend on the order of the product of $B$ or $C$.

By the standard procedure of the algebraic Bethe ansatz, we have the following.

**Proposition 3.1.** The $N$-particle state $|\psi([u]_N, [w])\rangle$ and its dual $\langle \psi([u]_N, [w])|$ become an eigenstate (on-shell states) of the transfer matrix (3.5) when the set of parameters $[u]_N$ satisfies the Bethe ansatz equation:

$$
a(u, [w]) = \prod_{j=1}^{M} \frac{u}{w_j}, \quad d(u, [w]) = \prod_{j=1}^{M} \left( \frac{au}{w_j} - \frac{w_j}{u} \right)
$$

(3.7)

where

$$
a(u, [w]) = \prod_{j=1}^{N} f(u_j, [w]), \quad d(u, [w]) = \prod_{j=1}^{N} f(u_j, [w]).
$$

(3.8)

Then the eigenvalue of the transfer matrix is given by

$$
\tau(u, [w]) = a(u, [w]_{j=1}^{N} f(u, [u], [w]) + d(u, [w]) \prod_{j=1}^{N} f(u_j, [w]).
$$

(3.9)

The scalar product between the arbitrary off-shell state vectors, which is mainly considered in this section, is defined as

$$
\langle \psi([u]_N, [w])| \psi([v]_N, [w]) \rangle = \langle \Omega | \prod_{j=1}^{N} C(u_j, [w]) \prod_{k=1}^{N} B(v_k, [w]) |\Omega\rangle
$$

(3.10)

with $u_j, v_k \in \mathbb{C}$. In the homogeneous limit $w_j \to 1$ (1 $\leq j \leq N$), the following theorem is known [15, 22, 41].

**Theorem 3.2.** The scalar product (3.10) in the homogeneous limit $w_j \to 1$ (1 $\leq j \leq N$) is given by a determinant form:

$$
\langle \psi([u]_N)| \psi([v]_N) \rangle = \prod_{1 \leq j < k \leq N} \frac{1}{(u_j^2 - u_k^2)(v_j^2 - v_k^2)} \det_N Q([u]_N|[v]_N).
$$

(3.11)
where \{u\}_N and \{v\}_N are arbitrary sets of complex values (i.e. off-shell conditions), and \(Q\) is an \(N \times N\) matrix with matrix elements

\[
Q(\{u\}_N|\{v\}_N|\{w\})_{jk} = \frac{a(u_j)d(v_k)v_j^{2N-1} - a(v_j)d(u_j)u_j^{2N-1}}{v_j/u_j - u_j/v_j}.
\]

Here we will show the above determinant formula by utilizing a method recently developed by Wheeler in the calculation of the scalar product of the spin-1/2 XXZ chain [25]. This technique is based on the Izergin–Korepin procedure [20, 21], which is originally a method to calculate the domain wall boundary partition function of the six-vertex model [20, 21]. In contrast to the spin-1/2 XXZ chain, in our case there is no need to impose the Bethe ansatz equation (i.e. on-shell condition) to show the determinant formula. In other words, the determinant formula (3.11) is valid for arbitrary off-shell states.

The following intermediate scalar products play a fundamental role in this method (see also figure 3 for a diagrammatic representation):

\[
S(\{u\}_n|\{v\}_N|\{w\})\bigg|_{un=\pm\alpha^{-(M-1)/2}} = \alpha^{M-2N-1}S([u]_n|\{v\}_N|\{w\})\bigg|_{un=\pm\alpha^{-(M-1)/2}}.
\]

The term ‘intermediate’ stems from the fact that (3.13) interpolates the scalar product \((n = N)\) and the domain wall boundary partition function \((n = 0)\).

We have the following lemma regarding the properties of the intermediate scalar product.

**Lemma 3.3.** The intermediate scalar product (3.13) \(S([u]_n|\{v\}_N|\{w\})\) satisfies the following properties.

1. \(S([u]_n|\{v\}_N|\{w\})\) is symmetric with respect to the variables \(\{w\}_1, \ldots, \{w\}_{M-N+n}\).
2. \(\prod_{n=1}^{M-2N-1}S([u]_n|\{v\}_N|\{w\})\) is a polynomial of degree \(M - N + n - 1\) in \(u_j^n\).
3. The following recursive relations between the intermediate scalar products hold:

\[
S([u]_n|\{v\}_N|\{w\})\bigg|_{u_j=\pm\alpha^{-(M-1)/2}} = \alpha^{N-n-(M-1)/2}(\pm 1)^{M-1}\frac{u_j^{M+n-N}}{\prod_{j=1}^{M}u_j^n}S([u]_{n-1}|\{v\}_N|\{w\}).
\]

\[\]
\[
0 \ 0 \ \cdots \ 0 \ 1 \ \cdots \ 1 \\
\vdots \ 0 \ 0 \ \cdots \ 0 \ 1 \ \cdots \ 1 \\
0 \ 0 \ \cdots \ 0 \ 1 \ \cdots \ 1 \\
\vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\
0 \ 0 \ \cdots \ 0 \ 1 \ \cdots \ 1 \\
\vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\
0 \ 0 \ \cdots \ 0 \ 1 \ \cdots \ 1 \\
\vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\
0 \ 0 \ \cdots \ 0 \ 1 \ \cdots \ 1 \\
\vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\
0 \ 0 \ \cdots \ 0 \ 1 \ \cdots \ 1 \\
\vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\
0 \ 0 \ \cdots \ 0 \ 1 \ \cdots \ 1
\]

Figure 4. The intermediate scalar products where the completeness relation is inserted (3.18). Note the parameter \(u_n\) comes only from the top row.

(4) The case \(n = 0\) of the intermediate scalar products has the following form:

\[
S([u]_0|v_N|w) = \alpha^{N(N-1)/2} \prod_{j=1}^{N-M} \prod_{k=1}^{M-N} \left( \frac{u_j}{w_k} - \frac{w_k}{v_j} \right) \prod_{j=1}^{N-N-1} w_{j}^{N-1},
\]

(3.15)

Proof. Property 1 follows from the RLL-relation

\[
\tilde{R}_{jk}(w_j/w_k) L_{jk}(u/w_j) L_{jk}(u/w_k) = L_{jk}(u/w_j) L_{jk}(u/w_k) \tilde{R}_{jk}(w_j/w_k)
\]

(3.16)

holding in \(\text{End}(W_\mu \otimes V_j \otimes V_k)\). Here, \(\tilde{R}\) is given by

\[
\tilde{R}(u) = \begin{pmatrix}
u & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \alpha(u - u^{-1}) & 0 \\
0 & 0 & 0 & u
\end{pmatrix},
\]

(3.17)

which intertwines the L-operators acting on a common auxiliary space (but acting on different quantum spaces). Note that the usual RLL-relation (3.16) intertwines the L-operators acting on the same quantum space but different auxiliary spaces. The above RLL-relation (3.17) allows one to construct the monodromy matrix as a product of the L-operators acting on the same quantum space (see also the next section), and rewriting the intermediate scalar products in terms of the resultant monodromy matrices makes one see that property 1 holds.

Property 2 can be shown by inserting the completeness relation into the intermediate scalar products (see figure 4 for a graphical interpretation):

\[
S([u]_n|v|w) = (0^{M-N+n} 1^{N-n}) \prod_{j=1}^{n} C(u_j, \{w\}) \prod_{k=1}^{N} B(v_k, \{w\}) \Omega
\]

\[
= \sum_{k=1}^{M-N+n} (0^{M-N+n} 1^{N-n}) C(u_n, \{w\}) (0^{k-1} 10^{M-N+n-k} 1^{N-n})
\]

\[
\times (0^{k-1} 10^{M-N+n-k} 1^{N-n}) \prod_{j=1}^{n-1} C(u_j, \{w\}) \prod_{k=1}^{N} B(v_k, \{w\}) \Omega.
\]

(3.18)
Figure 5. The graphical representation of the recursive relation (3.14). We can see that the top row is frozen by setting the spectral parameter \( u_n \) to \( u_n = \pm \alpha^{-1/2} w_{M-N+n} \).

Figure 6. The intermediate scalar products (3.15) for \( n = 0 \), which corresponds to the domain wall boundary partition function. One sees all the internal states are frozen when the boundary states are fixed to the configuration in the figure.

and noting that the factor containing \( u_n \) is calculated as

\[
\langle 0^{M-N+n} 1^{N-n} | C(u_n, \{ w \}) | 0^{M-N+n-k} 1^{N-n} \rangle = \alpha^{N-n} w_{M-N+n+1} \prod_{j=1}^{n-1} w_j \prod_{j=M-N+n+1}^{M-N+n} \left( \frac{\alpha u_n}{w_j} - \frac{w_j}{u_n} \right). \tag{3.19}
\]

Property 3 can be obtained by setting \( u_n = \pm \alpha^{-1/2} w_{M-N+n} \) in (3.18), or can be directly observed by its graphical representation (figure 5) that the top row is completely frozen.

Property 4 can be shown by noting that all the internal states are frozen (figure 6), and reading out and multiplying all the weights of the \( L \)-operators to find (3.15).

\[\square\]

**Lemma 3.4.** The properties in lemma 3.3 uniquely determine the intermediate scalar product (3.13).

**Proof.** The proof is by induction on \( n \). For \( n = 0 \), by property 4, the assertion is trivial. Assume by induction that the assertion holds for \( n - 1 \). Taking into account property 1, one finds that
property 3 gives values of $\prod_{j=1}^{n} u_j^{M+2n-2N-1} S(\{u_n\}|\{v\}|\{w\})$ at $M - N + n$ distinct points of $u_j^n$. By this, together with property 2, $S(\{u_n\}|\{v\}|\{w\})$ is uniquely determined. Thus, the assertion holds for $n$.

Due to lemma 3.4, the following determinant representation for the intermediate scalar product is valid.

**Theorem 3.5.** The intermediate scalar product $S(\{u\}|\{v\}|\{w\})$ (3.13) has the following determinant form:

$$S(\{u\}|\{v\}|\{w\}) = \prod_{M-N+n+1 \leq j < k \leq M} \frac{1}{u_j^2 - u_k^2} \prod_{1 \leq j < k \leq n} \frac{1}{u_j^2 - u_k^2} \prod_{1 \leq j < k \leq N} \frac{1}{v_j^2 - v_k^2} \times \det_N Q(\{u\}|\{v\}|\{w\})$$

(3.20)

with an $N \times N$ matrix $Q(\{u\}|\{v\}|\{w\})$ whose matrix elements are given by

$$Q(\{u\}|\{v\}|\{w\})_{jk} = \begin{cases} a(u_j, \{w\})d(v_k, \{v\})v_k^{2(N-1)} - a(v_k, \{v\})d(u_j, \{w\})u_j^{2(N-1)} & (1 \leq j \leq n) \\ (v_k/u_j - u_j/v_k) \prod_{l=M-N+n+1}^{M} (u_j^{2} - \alpha^{-1}u_l^{2}) & (n+1 \leq j \leq N) \\ v_j^{2(N-1)} \prod_{l=1}^{M} \left( \frac{\alpha v_k}{w_j} - \frac{w_j}{\alpha v_k} \right) & (l \neq M-N+j) \end{cases}$$

(3.21)

**Proof.** We can directly see that the determinant formula (3.20) satisfies all the properties in lemma 3.3. To show property 2, we just use the fact that the singularities $u_j^n = u_j^1 (1 \leq j \leq n-1)$ in the prefactor, and $u_j^n = \alpha^{-1}w_j^2 (M - N + n + 1 \leq j \leq M)$ and $u_j^n = v_j^2 (1 \leq j \leq n)$ in elements of the determinant, are removed. For property 4, we utilize the Cauchy determinant formula to obtain

$$\det_N \left( \frac{\alpha^{1/2}v_k}{w_{M-N+j}} - \frac{w_{M-N+j}}{\alpha^{1/2}v_k} \right)^{-1} = \prod_{j=1}^{M-N+1} \frac{w_j}{u_j} \prod_{1 \leq j < k \leq N} \frac{v_j}{v_k} \prod_{k=M-N+1}^{M} \left( \frac{\alpha^{1/2}v_j}{w_k} - \frac{w_k}{\alpha^{1/2}v_j} \right)$$

(3.22)

Finally due to lemma 3.4, the determinant formula (3.20) holds.

**Corollary 3.6.** Taking $n = N$ in (3.20) yields the determinant representation of the scalar product for the five-vertex model with inhomogeneous parameters (3.10):

$$\langle \psi(\{u\}, \{w\}) | \psi(\{v\}, \{w\}) \rangle = \prod_{1 \leq j < k \leq n} \frac{1}{u_j^2 - u_k^2} \frac{1}{v_j^2 - v_k^2} \det_N Q(\{u\}|\{v\}|\{w\})$$

(3.23)

with

$$Q(\{u\}|\{v\}|\{w\})_{jk} = \frac{a(u_j, \{w\})d(v_k, \{v\})v_k^{2(N-1)} - a(v_k, \{v\})d(u_j, \{w\})u_j^{2(N-1)}}{v_k/u_j - u_j/v_k}$$

(3.24)

Further taking the homogeneous limit $w_j \to 1 (1 \leq j \leq n)$ yields (3.11) in theorem 3.5.
The state vectors $|\psi (\{u\}_N)\rangle$ and $\langle \psi (\{u\}_N)|$ become the energy eigenstates of (6.3), when an arbitrary set of solutions $\{u\}_N$ to the Bethe ansatz equation (3.7) in the homogeneous limit $w_j \to 1$ ($1 \leq j \leq N$) is substituted into the state vectors. Then we have the following corollary regarding the norm of the eigenstates.

**Corollary 3.7.** The norm of the eigenstates in the homogeneous limit $w_j \to 1$ ($1 \leq j \leq n$) is given by

$$
\langle \psi (\{u\}_N)|\psi (\{u\}_N)\rangle = \prod_{j=1}^{N} u_j^{2(M-N-1)} \prod_{j,k=1}^{N} \frac{1}{u_j^2 - u_k^2} \det_{\{u\}_N} Q(\{u\}_N) \quad (3.25)
$$

with

$$
\tilde{Q}_{jk}(\{u\}_N) = -1 + \frac{\alpha N + (M - N)u_j^{-2}}{\alpha - u_j^{-2}} \delta_{jk}.
$$

By use of Sylvester’s determinant theorem, the determinant in the above further reduces to

$$
\det_{\{u\}_N} Q = \prod_{j=1}^{N} \frac{\alpha N + (M - N)u_j^{-2}}{\alpha - u_j^{-2}} \left( 1 - \sum_{j=1}^{N} \frac{\alpha - u_j^{-2}}{\alpha N + (M - N)u_j^{-2}} \right). \quad (3.27)
$$

4. Wavefunctions

In this section, we compute the overlap between an arbitrary off-shell $N$-particle state $|\psi (\{u\}_N)\rangle$ and the (normalized) state with an arbitrary particle configuration $|\{x_1 \cdots x_N\}\rangle$ ($x_1 < \cdots < x_N$), where $x_j$ denotes the positions of the particles. Namely, here we evaluate the wavefunction $\langle \{x_1 \cdots x_N|\psi (\{u\}_N)\rangle$ and its dual $\langle \psi (\{u\}_N)|\{x_1 \cdots x_N\}\rangle$. One finds that these quantities are crucial to describe physically interesting phenomena such as the relaxation dynamics as in section 6, because the state $|\psi (\{u\}_N)\rangle$ becomes an eigenstate of the Hamiltonian (2.9) (correspondingly $\langle \{x_1 \cdots x_N|\psi (\{u\}_N)\rangle$ becomes an energy eigenfunction). If we choose $\{u\}_N$ as an arbitrary set of solutions of the Bethe ansatz equation (see proposition 3.1 and section 6 for details). Here and in what follows, we consider the homogeneous case $w_1 = 1, \ldots, w_M = 1$, and as noted in the previous section, we omit $\{w\}$ as in (3.2).

The main results in this section are summarized in the following theorem.

**Theorem 4.1.** The wavefunctions can be written as the following determinant formulae:

$$
\langle \{x_1 \cdots x_N|\psi (\{v\}_N)\rangle = \prod_{j=1}^{N} v_j^{M-1}(\alpha v_j^2 - 1)^{-1} \det_{\{v\}_N} (v_j^{2k}(\alpha - v_j^{-2})^{\sigma_{x_j}}), \quad (4.1)
$$

$$
\langle \psi (\{u\}_N)|\{x_1 \cdots x_N\}\rangle = \prod_{j=1}^{N} (\alpha u_j - u_j^{-1})^{M} u_j^{-N-1} \det_{\{u\}_N} (u_j^{2k}(\alpha - u_j^{-2})^{-\sigma_{x_j}}), \quad (4.2)
$$

where

$$
\langle \{x_1 \cdots x_N| = \langle \Omega| \prod_{j=1}^{N} \sigma_{x_j}^+, \quad |\{x_1 \cdots x_N\} = \prod_{j=1}^{N} \sigma_{x_j}^-|\Omega\rangle, \quad (4.3)
$$

and $\{v\}_N$ and $\{u\}_N$ are sets of arbitrary complex parameters.
The strategy to show theorem 4.1 is as follows. We first rewrite the wavefunctions into a matrix product form following [27]. The matrix product form can be expressed as a determinant with some overall factor which remains to be calculated. The information of the particle configuration \{x_1, x_2, \ldots, x_N\} is encoded in the determinant. On the other hand, the overall factor is independent of the particle positions, and therefore we can determine this factor by considering the specific configuration: we explicitly calculate it with the help of the result for the overlap of the consecutive configuration (i.e. \(x_j = j\)) obtained in [30, 31].

Let us begin to compute the wavefunctions. We consider (4.2) first. The proof of (4.1) can be shown in a similar way. First we shall rewrite the wavefunction \(\psi(\{u\}_N|x_1 \cdots x_N\) into the matrix product representation. With the help of graphical description, one finds that the wavefunction can be written as

\[
\langle \Omega | \prod_{j=1}^N C(u_j)|x_1 \cdots x_N\rangle = Tr_{W^\otimes N} \left[ \langle \Omega | \prod_{\mu=1}^M T_\mu(\{u\}_N)|x_1 \cdots x_N\rangle \right], \tag{4.4}
\]

where \(P = |0^N\rangle\langle 1^N|\) is an operator acting on the tensor product of auxiliary spaces \(W_1 \otimes \cdots \otimes W_N\). The trace here is also over the auxiliary spaces. Due to the commutativity of the operators \(B\) or \(C\) (3.4), the wavefunctions do not depend on the order of the product of \(B\) or \(C\). In other words, the wavefunctions are symmetric with respect to the parameters \(\{u\}_N\) or \(\{v\}_N\). Changing the viewpoint of the products of the monodromy matrices, we have

\[
\prod_{\mu=1}^M T_\mu(\{u\}_N) = \prod_{j=1}^M T_j(\{u\}_N), \tag{4.5}
\]

where \(T_j(\{u\}_N) := \prod_{\mu=1}^N L_{uj}(u_\mu) \in \text{End}(W^\otimes N \otimes V_j)\) can be regarded as a monodromy matrix consisting of \(L\)-operators acting on the same quantum space \(V_j\) (but acting on different auxiliary spaces). The monodromy matrix is decomposed as

\[
T_j(\{u\}_N) := \begin{pmatrix}
A_N(\{u\}_N) & B_N(\{u\}_N) \\
C_N(\{u\}_N) & D_N(\{u\}_N)
\end{pmatrix}, \tag{4.6}
\]

where the elements \((A_N, \text{ etc})\) act on \(W_1 \otimes \cdots \otimes W_N\). The wavefunction (4.4) can then be rewritten by \(T_j(\{u\}_N)\) as

\[
\langle \psi(\{u\}_N)|x_1 \cdots x_N\rangle = Tr_{W^\otimes N} \left[ \langle \Omega | \prod_{j=1}^M T_j(\{u\}_N)|x_1 \cdots x_N\rangle \right], \tag{4.7}
\]

In figure 7, we depict the elements \(A_n(\{u\}_n)\) and \(B_n(\{u\}_n)\) of the monodromy matrix \(T_j(\{u\}_n)\), which explicitly appear in (4.7).

For these operators, one finds the following recursive relations:

\[
A_{n+1}(\{u\}_{n+1}) = A_n(\{u\}_n) \otimes \begin{pmatrix} u_{n+1} & 0 \\ 0 & \alpha u_{n+1} - u_{n+1} \end{pmatrix} + B_n(\{u\}_n) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.8}
\]

\[
B_{n+1}(\{u\}_{n+1}) = A_n(\{u\}_n) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + B_n(\{u\}_n) \otimes \begin{pmatrix} 0 & 0 \\ 0 & \alpha u_{n+1} \end{pmatrix}, \tag{4.9}
\]

with the initial condition

\[
A_1 = \begin{pmatrix} u_1 & 0 \\ 0 & \alpha u_1 - u_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.10}
\]

See figure 8 for a graphical description of the recursion relation for the operator \(A_n(\{u\}_n)\). By using the recursive relations (4.8) and (4.9), one sees that these operators satisfy the following simple algebra.
\[ A_n((u)_n) = \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·
where the $2n \times 2n$ matrix $H_n$ remains to be determined. Using the induction hypothesis for $n$, one obtains

$$G_{n+1}^{-1} A_{n+1} G_{n+1} = \begin{pmatrix} u_{n+1} - u_0 & \alpha \nu \nu + \alpha \nu \nu - (\alpha u_{n+1} - u_{n+1}^{-1}) H_n \nu \nu \\ 0 & (\alpha u_{n+1} - u_{n+1}^{-1}) \nu \nu \end{pmatrix}.$$  \hspace{1cm} (4.15)

The above matrix is guaranteed to be diagonal when

$$\nu \nu = \left(\alpha u_{n+1} - u_{n+1}^{-1}\right) H_n \nu \nu = u_{n+1} \nu \nu H_n.$$  \hspace{1cm} (4.16)

Utilizing the above relation and recalling that $\nu \nu$ and $\nu^{(j)}$ satisfy the same relation as that in (4.11), one finds

$$H_n = \nu \nu^{-1} \sum_{j=1}^{n} \left(\alpha u_j - u_j^{-1}\right) \nu \nu^{(j)}.$$  \hspace{1cm} (4.17)

One thus obtains the diagonal matrix $\nu \nu + 1$:

$$\nu \nu + 1 = \begin{pmatrix} u_{n+1} - u_0 & 0 \\ 0 & (\alpha u_{n+1} - u_{n+1}^{-1}) \nu \nu \end{pmatrix}.$$  \hspace{1cm} (4.18)

The remaining task is to derive $\nu^{(j)}$ and to prove that relations (4.11)–(4.13) hold for $n + 1$. Combining (4.9), (4.14) and (4.17), and also inserting relations (4.12) and (4.13), one arrives at $\nu^{(j)} = \sum_{j+1}^{n} \nu^{(j)}$, where

$$\nu^{(j)} = \begin{pmatrix} 1 \\ \alpha u_{n+1} - u_n^{-1} \nu \nu^{(j)} \\ 0 \\ 0 \end{pmatrix}$$  \hspace{1cm} (4.19)

Finally, recalling that $\nu \nu$ and $\nu^{(j)}$ are supposed to satisfy relations (4.11)–(4.13) and using the explicit form of $\nu \nu + 1$ (4.18) and $\nu^{(j)}$ (4.19), one sees that they satisfy the same algebraic relations as those in (4.11)–(4.13) for $n + 1$.  \hspace{1cm} \text{\blacksquare}

Due to the algebraic relations (4.11) and (4.12) in lemma 4.2, the matrix product form for the wavefunction (4.7) can be rewritten as

$$\langle \psi | [u] \rangle x \epsilon \cdots x_N \rangle = \sum_{\sigma \in \Omega_N} \prod_{j=1}^{N} u_{\sigma(j)}^{-2(M-N)} \left(\alpha u_{\sigma(j)}^2 - 1\right) M-N+j \nu \nu^{(j)} (\alpha - u_{\sigma(j)}^{-2})^{-x_i}$$  \hspace{1cm} \times Tr_{W^{[\nu \nu+1]}} [B_1^{(N)} \cdots B_1^{(N)} \nu \nu A_{M-N} P],$$  \hspace{1cm} (4.20)

where $\Omega_N$ is the symmetric group of order $N$. Using (4.13) to arrange the order of the matrix product $B_1^{(N)} \cdots B_1^{(N)}$ in the canonical order $B_1^{(N)} \cdots B_1^{(N)}$ yields the following determinant form:

$$\langle \psi | [u] \rangle x \epsilon \cdots x_N \rangle = K \sum_{\sigma \in \Omega_N} (-1)^\sigma \prod_{j=1}^{N} u^{-2(j)} (\alpha - u_{\sigma(j)}^{-2})^{-x_i}$$  \hspace{1cm} = K \det_N \left[ u_j^{-2} (\alpha - u_j^{-2})^{-x_i} \right].$$  \hspace{1cm} (4.21)

where the prefactor $K$ given below remains to be determined:

$$K = \prod_{j=1}^{N} (\alpha u_j^2 - 1) M-N+j u_j^{-2(M-N)} \nu \nu A_{M-N} P.$$  \hspace{1cm} (4.22)

In (4.21), we note that the information of the particle configuration $\{x_1, x_2, \ldots, x_N\}$ is encoded in the determinant, while the overall factor $K$ is independent of the configuration. This
fact allows us to determine the factor $K$ by evaluating the overlap for a particular particle configuration. In fact, the overlaps for some particular cases can be directly evaluated as in \[30, 31\]. For instance, we find the following explicit expression for the case $x_j = j (1 \leq j \leq n)$:

$$\langle \psi (|u|_N) | 12 \cdots N \rangle = a^{N(N-1)/2} \prod_{j=1}^{N} u_j^{N-1} (au_j - u_j^{-1})^{M-N},$$  \tag{4.23}

which can be evaluated with the help of its graphical description, in just the same way as with the $n=0$ case in the intermediate scalar products (3.15). A comparison of (4.23) with (4.21) for $x_j = j (1 \leq j \leq N)$ determines the desired prefactor $K$:

$$K = \frac{\prod_{j=1}^{N} (au_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1 \leq j < k \leq N} (u_j^2 - u_k^2)},$$  \tag{4.24}

where we have used the Vandermonde determinant $\det_{\nu} (\lambda_{\nu}^{N-j}) = \prod_{1 \leq j < k \leq N} (x_j - x_k)$ to evaluate the determinant in (4.21) for the case $x_j = j (1 \leq j \leq N)$. The insertion of the result of $K$ into (4.21) yields (4.2).

We can also evaluate the dual expression (4.1) in a similar manner. In this case, the corresponding matrix product representation is given by

$$\langle 1 \cdots x_N \psi (|u|_N) \rangle = Tr_{\mathcal{W}} \left[ \alpha_1^{M-x_1} C_1 \alpha_2^{M-x_2} C_2 \cdots \alpha_N^{M-x_N} C_N \right],$$  \tag{4.25}

where $C_N$ is an element of the monodromy matrix defined in (4.6) and $Q$ is a projection operator $Q = |1\rangle \langle 0^N |$ acting on $W_1 \otimes \cdots \otimes W_N$ (cf (4.4) and (4.7)). The algebraic relations satisfied by the operators $\mathcal{A}$ and $C$ are summarized in the following lemma.

\textbf{Lemma 4.3.} There exists a decomposition of $C_n$: $C_n = \sum_{j=1}^{n} c_n^{(j)}$ such that the following algebraic relations hold for $\mathcal{A}_n$ and $C_n^{(j)}$:

$$\mathcal{A}_n C_n^{(j)} = \frac{u_j}{au_j - u_j} c_n^{(j)} \mathcal{A}_n,$$  \tag{4.26}

$$\left( C_n^{(j)} \right)^2 = 0,$$  \tag{4.27}

$$\left( au_j^2 - 1 \right) C_n^{(j)} c_n^{(k)} = -\left( au_j^2 - 1 \right) c_n^{(k)} C_n^{(j)} \quad (j \neq k).$$  \tag{4.28}

According to lemma 4.3 and the explicit expression for the wavefunction

$$\langle 12 \cdots N | \psi (|u|_N) \rangle = a^{N(N-1)/2} \prod_{j=1}^{N} u_j^{M-1},$$  \tag{4.29}

the matrix product representation of the overlap (4.25) reduces to the determinant expression given in (4.1).

\textbf{Example 4.4.} The wavefunction (4.2) for the configuration $x_j = 2j - 1 (1 \leq j \leq N)$ is obtained as follows:

$$\langle \Omega | \prod_{j=1}^{N} C(u_j) | x_1 \cdots x_N \rangle = \prod_{j=1}^{N} \left( au_j - u_j^{-1} \right)^{M-2j+1} \det_{\nu} \left[ u_j^{-2k} (au_j - u_j^{-2})^{2(k-1)} \right]$$

$$= \prod_{j=1}^{N} \left( au_j - u_j^{-1} \right)^{M-2N+1} \det_{\nu} \left[ (au_j - u_j^{-1})^{2(N-j)} \right]$$

$$= \prod_{j=1}^{N} \left( au_j - u_j^{-1} \right)^{M-2N+1} \prod_{1 \leq j < k \leq N} \left( au_j - u_j^{-1} \right)^2 (u_j^2 - u_k^2).$$  \tag{4.30}
Theorem 4.5. The off-shell wavefunction \( \langle x_1 \cdots x_N | \psi ((v|N)) \rangle \) (4.1) satisfies the following summation formula:

\[
\sum_{1 \leq x_1, \ldots, x_N \leq M} \alpha^{MN-\sum_{i=1}^{N} j_i} \langle x_1 \cdots x_N | \psi ((v|N)) \rangle = \prod_{j=1}^{N} u_j^{M+1} \prod_{1 \leq j < k \leq N} \frac{1}{v_k^j - v_k^j} \text{det}_N V, \tag{4.31}
\]

where \( V \) is an \( N \times N \) matrix with the elements

\[
V_{jk} = \sum_{m=0}^{M} (-1)^m \alpha^{M-m} \left( \frac{M}{m} \right) v_k^{2(m-j+1)} \quad (1 \leq j \leq N - 1),
\]

\[
V_{Nk} = - \sum_{m=\max(N-1,1)}^{M} (-1)^m \alpha^{M-m} \left( \frac{M}{m} \right) v_k^{2(m-N+1)}, \tag{4.32}
\]

while the dual off-shell wavefunction \( \langle \psi ((u|N)) | x_1 \cdots x_N \rangle \) (4.2) satisfies the following:

\[
\sum_{1 \leq x_1, \ldots, x_N \leq N} \sum_{\gamma=1}^{V} \alpha^{\gamma j - x_N} \langle \psi ((u|N)) | x_1 \cdots x_N \rangle = \prod_{j=1}^{N} u_j^{M+1} \prod_{1 \leq j < k \leq N} \frac{1}{u_j^k - u_j^k} \text{det}_N \tilde{V}, \tag{4.33}
\]

with an \( N \times N \) matrix \( \tilde{V} \) whose elements are given by

\[
\tilde{V}_{jk} = \sum_{m=0}^{M} (-1)^m \alpha^{M-m} \left( \frac{M}{m} \right) u_k^{2(m-j-N)} \quad (2 \leq j \leq N),
\]

\[
\tilde{V}_{Nk} = - \sum_{m=\max(N-1,1)}^{M} (-1)^m \alpha^{M-m} \left( \frac{M}{m} \right) u_k^{2(m-N+1)}, \tag{4.34}
\]

Proof. By the graphical description, it can be easily shown that

\[
\langle \Omega \rangle \lim_{u_j \to \infty} \prod_{j=1}^{N} u_j^{M+1} C(u_j) = \sum_{1 \leq x_1, \ldots, x_N \leq M} \alpha^{MN-\sum_{i=1}^{N} j_i} \langle x_1 \cdots x_N | \Omega \rangle,
\]

\[
\lim_{v_j \to \infty} \prod_{j=1}^{N} v_j^{M+1} B(v_j) \rangle = \sum_{1 \leq x_1, \ldots, x_N \leq M} \alpha^{MN-\sum_{i=1}^{N} j_i} \langle x_1 \cdots x_M | \Omega \rangle. \tag{4.35}
\]

By substituting them into the determinant representation of the scalar product (3.11), one sees that the resultant expressions coincide with (4.31) and (4.33). Here, the limiting procedure \( u_j \to \infty \) and \( v_j \to \infty \) for \( 1 \leq j \leq N \) in the scalar product (3.11) can be taken by expanding \( d(u_j) = (u_j - au_j^{-1})^M \) and \( d(v_k) = (v_k - av_k^{-1})^M \) in the numerator of the elements for the determinant (3.12), dividing the numerator by the denominator and then making use of the formula

\[
\lim_{u_j \to u} \frac{\text{det}_N [\Phi(u_j, v_k)]}{\prod_{1 \leq j < k \leq N} (u_k - u_j)} = \text{det}_N \left[ \frac{1}{(j-1)!} \left( \frac{\partial}{\partial u} \right)^{j-1} \Phi(u, v_k) \right]. \tag{4.36}
\]

where \( \Phi(u, v) \) denotes \((N - 1)\)-times differentiable functions of \( u \). \qed

Note that there exists a misprint in the index of the summation of the elements of the determinant corresponding to (4.34).
5. Grothendieck polynomials and the Cauchy identity

The wavefunctions (4.1) and (4.2) play a key role in analyzing physically interesting quantities such as Green functions. Because the operators $B$ (or $C$) in (3.4) mutually commute, the wavefunctions (and the corresponding Green functions) can be described by some symmetric polynomials of $\{v\}_N$ (or $\{u\}_N$). In this section, we show that the wavefunctions for generic value of $\alpha$ are written as the Grothendieck polynomial which is a one-parameter deformation of the Schur polynomial. Combining the completeness relation and the determinant form of the scalar product (3.11), one obtains the Cauchy identity of the Grothendieck polynomials.

Let us first define the Grothendieck polynomials.

**Definition 5.1.** The Grothendieck polynomial is defined to be the following determinant [18]:

$$G_\lambda(z; \beta) = \frac{\det_N(z_j^{x_j+N-k} (1 + \beta z_j)^{k-1})}{\prod_{1 \leq j < k \leq N}(z_j - z_k)}, \quad (5.1)$$

where $z = \{z_1, \ldots, z_N\}$ is a set of variables and $\lambda$ denotes a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ with weakly decreasing non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. For our purpose, we further define the ‘dual’ Grothendieck polynomial (we discuss the orthogonality of the original and the dual Grothendieck polynomials later):

$$\tilde{G}_\lambda(z; \beta) = \frac{\det_N(z_j^{x_j+N-k} (1 + \beta z_j^{-1})^{1-k})}{\prod_{1 \leq j < k \leq N}(z_j - z_k)}. \quad (5.2)$$

The Grothendieck polynomial (5.1) and its dual version (5.2) can be regarded as a one-parameter deformation of the Schur polynomial, since they reduce to the Schur polynomial $\psi(\{x\}_N)$ and the corresponding Green functions) can be described by some symmetric polynomials (5.1) and its dual version (5.2).

**Lemma 5.2.** By inserting the relation $\lambda_j = x_{N-j+1} - N + j - 1$ and setting

$$z_j = \alpha - v_j^{-2}, \quad y_j^{-1} = \alpha - u_j^{-2}, \quad \beta = -1/\alpha, \quad (5.4)$$

the wavefunctions (4.1) and (4.2) can, respectively, be expressed as the Grothendieck polynomials (5.1) and its dual version (5.2):

$$\langle x_1 \cdots x_N | \psi(\{v\}_N) \rangle = \alpha^{N(N-1)/2} \prod_{j=1}^N y_j^{-M-1} G_\lambda(z; \beta),$$

$$\langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle = \alpha^{N(N-1)/2} \prod_{j=1}^N v_j^{M-1} y_j^{-M+N} (1 + \beta y_j^{-1})^{N-1} \tilde{G}_\lambda(y; \beta). \quad (5.5)$$
The Cauchy identity holding for the Schur polynomials can be extended to that for the Grothendieck polynomials.

Theorem 5.3. The following identity holds true for the Grothendieck polynomials (5.1) and (5.2):

\[
\sum_{\lambda \subseteq (M-N)^N} G_{\lambda}(z; \beta) G_{\lambda}(y; \beta) = \prod_{1 \leq j < k \leq N} \frac{1}{(z_j - z_k)(y_j - y_k)} \det_N \left[ \frac{(z_j y_k)^N - \left\{ (1 + \beta z_j)/(1 + \beta y_k^{-1}) \right\}^{N-1}}{z_j y_k - 1} \right].
\] (5.6)

The usual Cauchy identity holding for the Schur polynomials is recovered by taking \( \beta = 0 \).

Proof. First, substituting the completeness relation, one decomposes the scalar product as

\[
\langle \psi(\{u\}_N) | \psi(\{v\}_N) \rangle = \sum_{1 \leq x_1 < \cdots < x_N \leq M} \langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle \langle x_1 \cdots x_N | \psi(\{v\}_N) \rangle.
\] (5.7)

Then substituting the determinant representation for the scalar product (3.11) into the rhs of the above and utilizing the relations in lemma 5.2 yields the one-parameter deformation of the Cauchy identity (5.6).

Taking \( M \to \infty \), one has the following identity.

Corollary 5.4. \[
\sum_{\lambda \subseteq (M-N)^N} G_{\lambda}(z; \beta) G_{\lambda}(y; \beta) = \prod_{j=1}^{N} \left( \frac{1 + \beta z_j}{1 + \beta y_j^{-1}} \right)^{N-1} \prod_{j,k=1}^{N} \frac{1}{1 - z_j y_k},
\] (5.8)

where the sum is over all Young diagrams of shape \( \lambda = (\lambda_1, \ldots, \lambda_N) \). Taking \( \beta = 0 \), the well-known Cauchy identity for the Schur functions is recovered:

\[
\sum_{\lambda} s_{\lambda}(z) s_{\lambda}(y) = \prod_{j,k=1}^{N} \frac{1}{1 - z_j y_k}.
\] (5.9)

We also list the summation formulae for the Grothendieck polynomials, which are obtained by inserting (5.5) into (4.31) and (4.33).

Theorem 5.5. The following summation formula is valid for the Grothendieck polynomials (5.1):

\[
\sum_{\lambda \subseteq (M-N)^N} (-\beta)^{\sum_{j=1}^{N-1} \lambda_j} G_{\lambda}(z; \beta) = \prod_{1 \leq j < k \leq N} \frac{1}{z_k - z_j} \det_N V^{(M)}(M),
\] (5.10)

with an \( N \times N \) matrix \( V^{(M)}(M) \) whose matrix elements are

\[
V_{jk}^{(M)} = \sum_{m=0}^{j-1} (-1)^m (-\beta)^{j-N} \binom{M}{m} (1 + \beta z_k)^{m-j+N-1} (1 \leq j \leq N-1),
\]

\[
V_{NK}^{(M)} = - \sum_{m=\max(N-1,1)}^{M} (-1)^m \binom{M}{m} (1 + \beta z_k)^{m-1},
\] (5.11)

while the dual Grothendieck polynomials (5.2) satisfy

\[
\sum_{\lambda \subseteq (M-N)^N} (-\beta)^{\sum_{j=1}^{N-1} \lambda_j} \overline{G}_{\lambda}(y; \beta) = \prod_{j=1}^{N} \prod_{1 \leq j < k \leq N} \frac{1}{y_k - y_j} \det_N \overline{V}^{(M)},
\] (5.12)
where $\tilde{V}^{(M)}$ is an $N \times N$ matrix whose elements are given by

$$
\tilde{V}_{jk}^{(M)} = \sum_{m=0}^{N-j} (-1)^{m}(-\beta)^{-j+1-M+N} \binom{M}{m}(1+\beta y_k^{-1})^{m+j-N-1} \quad (2 \leq j \leq N),
$$

$$
\tilde{V}_{jk}^{(M)} = -\sum_{m=\max(N-1,1)}^{M} (-1)^{m}(-\beta)^{-M+N} \binom{M}{m}(1+\beta y_k^{-1})^{m-N}.
$$

(5.13)

Finally, we discuss the orthogonality of the Grothendieck polynomials and the dual Grothendieck polynomials. We now impose the periodic boundary condition on the model, i.e., suppose that the spectral parameters $[z]_N$ satisfy the Bethe ansatz equations

$$(1 + \beta z_k)^N + (-1)^N z_k^M \prod_{j=1}^{N}(1 + \beta z_j) = 0 \quad (1 \leq k \leq N).$$

(5.14)

We insert into $\{x_1 \cdots x_N \mid x'_1 \cdots x'_N\} = \prod_{j=1}^{N} \delta_{x_j x'_j}$ the completeness of the Bethe states

$$
I = \sum_{[u]_N} \frac{|\psi([u]_N)\rangle \langle \psi([u]_N)|}{\langle \psi([u]_N)\rangle \langle \psi([u]_N)|},
$$

(5.15)

where the summation is over all of the solutions of the Bethe ansatz equations. We have

$$
\sum_{[u]_N} \frac{\{x_1 \cdots x_N \mid [u]_N\} \langle \psi([u]_N)|[\psi([u]_N)]\rangle}{\langle \psi([u]_N)|\psi([u]_N)\rangle} = \prod_{j=1}^{N} \delta_{x_j x'_j},
$$

(5.16)

which, with the use of the expressions (3.25) and (5.5), can be translated to the following orthogonality relation between the Grothendieck polynomials and the dual Grothendieck polynomials.

**Theorem 5.6.** The following orthogonality relation between the Grothendieck polynomials and the dual Grothendieck polynomials holds:

$$
\sum_{[z]_N} w([z]_N) \tilde{G}_\lambda(z^{-1}; \beta) \tilde{G}_\mu(z; \beta) = \delta_{\lambda \mu},
$$

(5.17)

where the summation is over all of the solutions of the Bethe ansatz equation (5.14), and the weight $w([z]_N)$ given by

$$
w([z]_N) = \left(1 + \sum_{j=1}^{N} \frac{\beta z_j}{M + (M-N)\beta z_j} \right)^{-1} \prod_{j,k=1}^{N} (z_j - z_k) \prod_{j=1}^{N} \frac{z_j^{1-N}(1 + \beta z_j)}{M + (M-N)\beta z_j}.
$$

(5.18)

**Corollary 5.7.** Setting $\beta = 0$ and taking the limit $M \to \infty$ yields the well-known orthogonality relation for the Schur polynomials (see [39] for example):

$$
\frac{1}{(2\pi i)^N N!} \oint_{|z| = 1} \cdots \oint_{|z| = 1} \prod_{j=1}^{N} dz_j s_\lambda(z^{-1}) s_\mu(z) \prod_{j,k=1}^{N} (z_j - z_k) \prod_{j=1}^{N} z_j^{-N} = \delta_{\lambda \mu}.
$$

(5.19)
Proof. Setting $\beta = 0$ and using relation (5.3), one finds that (5.17) reduces to

$$\sum_{\{z\}} s_i(z^{-1}) s_k(z) \prod_{j=1}^{N} \frac{z_j^I}{M} \prod_{j,k=1 \atop j \neq k}^{N} (z_j - z_k) \prod_{j=1}^{N} e^{-I_j} = \delta_{i,k}. \quad (5.20)$$

From the Bethe ansatz equation (5.16) for $\beta = 0$, one observes that the roots are located on the unit circle in the complex plane: $z_j = \exp(2\pi i I_j/M)$, where $I_j \in \mathbb{Z}$ ($I_j \in (2\mathbb{Z} + 1)/2$) for $N \in 2\mathbb{Z} + 1$ ($N \in 2\mathbb{Z}$) and $0 \leq I_1 < I_2 < \cdots < I_N \leq M - 1$. Recalling that the sum in the above is taken over all the sets of the solutions, and ignoring the order of $\{z_j\}$, we can rewrite the sum as the multiple integrals

$$\lim_{M \to \infty} \sum_{\{z\}} \prod_{j=1}^{N} \frac{z_j^I}{M} = \lim_{M \to \infty} \sum_{\{z\}} \prod_{j=1}^{N} e^{2\pi i I_j/M} = \frac{1}{(2\pi i)^N N!} \oint_{|z|=1} \cdots \oint_{|z|=1} \prod_{j=1}^{N} dz_j. \quad (5.21)$$

Inserting this limiting procedure into (5.20), one arrives at (5.19).

6. Totally asymmetric simple exclusion process

In the previous sections, we have evaluated the arbitrary off-shell wavefunctions for the one-parameter family of the five-vertex model by making use of the matrix product representations. The most significant is that the resultant determinant representation of the wavefunctions can be expressed by the Grothendieck polynomials which are a one-parameter deformation of the Schur polynomials. As mentioned in section 2, the five-vertex model includes several physically interesting models. As an application of the results obtained in the previous sections, we consider the TASEP and formulate the relaxation dynamics.

The TASEP is a stochastic interacting particle system consisting of biased random walkers obeying the exclusion principle, whose dynamics can be formulated as follows. We consider the $N$-particle system on the periodic lattice with $M$ sites. By the exclusion rule, each site can be occupied by at most one particle. The dynamical rule of the TASEP is: during the time interval $dt$, a particle at a site $j$ jumps to the $(j + 1)$th site with probability $d \delta$, if the $(j + 1)$th site is vacant. The probability of being in the (normalized) state $\{|x_1 \cdots x_N\}$ is denoted as $P_t(x_1, \ldots, x_N)$. Then the arbitrary states can be written as

$$|\psi(t)\rangle = \sum_{1 \leq x_1 < \cdots < x_N \leq M} P_t(x_1, \ldots, x_N) |x_1 \cdots x_N\rangle. \quad (6.1)$$

Note that the probability is given as the amplitude of each state, which is in contrast to the quantum mechanics where the probability is given by the squared magnitude of the amplitude. The time evolution of the state vector is subject to the master equation

$$\frac{d}{dt} |\psi(t)\rangle = \mathcal{H} |\psi(t)\rangle. \quad (6.2)$$

Here the stochastic matrix $\mathcal{H}$ of the TASEP is given by (2.9) for the case $\alpha = 1$:

$$\mathcal{H} = \sum_{j=1}^{M} \left\{ \sigma_j^+ \sigma_{j+1}^- + \frac{1}{4} (\sigma_j^+ \sigma_{j+1}^- - 1) \right\}. \quad (6.3)$$

The eigenvalue spectrum of the stochastic matrix (6.3) can be calculated by the Bethe ansatz method [14, 15, 22, 38] as formulated in section 3. Namely, taking the logarithmic derivative of
the eigenvalue of the transfer matrix (3.9) according to (2.9), and setting \( w_j = 1 \) for \( 1 \leq j \leq M \), \( \alpha = 1 \) and \( z_j = 1 - u_j^{-2} \) for \( 1 \leq j \leq N \), one obtains

\[
\mathcal{H}(z) = -N + \sum_{j=1}^{N} z_j^{-1},
\]

(6.4)

where the parameters \( \{z_j\}_N \) must satisfy the Bethe ansatz equation (3.7). Explicitly, it reads

\[
z_k^{-M}(1 - z_k)^N = (-1)^{N-1} \prod_{j=1}^{N}(1 - z_j) \quad (1 \leq k \leq N).
\]

(6.5)

The state vector \( |\psi(z)\rangle \) (resp. \( |\psi(\mu)\rangle \)) defined by setting \( u_j^{-2} = 1 - z_j \) in \( |\psi((\mu)\rangle \) (resp. \( \langle \psi((\mu)|\rangle \)) becomes an energy eigenstate of (6.3), when we choose the set of parameters \( z \) as an arbitrary set of solutions of (6.5). Then the norm of the eigenstate is given by (3.25) after setting \( \alpha = 1 \) and \( u_j^{-2} = 1 - z_j \).

The Green function \( G(x|x) \) which is the probability that the particles starting at initial positions \( x = \{x_1, \ldots, x_N\} \) \( (1 \leq x_1 < \cdots < x_N \leq M) \) arrive at positions \( x' = \{x'_1, \ldots, x'_N\} \) \( (1 \leq x'_1 < \cdots < x'_N \leq M) \) at time \( t \) is given by solving the master equation (6.2):

\[
G_t(x'|x) = \langle x'_1 \cdots x'_N \rangle e^{\mathcal{H}t} |x_1 \cdots x_N\rangle.
\]

(6.6)

Utilizing the results in the previous section, one finds that the Green function can be written in terms of the Grothendieck polynomials.

**Proposition 6.1.** The Green function \( G_t(x'|x) \) of the TASEP whose stochastic matrix is given by (6.3) is expressed as the Grothendieck polynomials (5.1) and (5.2) with \( \beta = -1/\alpha = -1 \):

\[
G_t(x'|x) = \sum_{\lambda} \frac{\prod_{\gamma \in (M-N)^n} G_\gamma(z^{-1}; -1)}{\prod_{\nu \in (N)^n} G_\nu(\mu^{-1}; -1)} e^{\mathcal{H}t},
\]

(6.7)

where \( \lambda = (\lambda_1, \ldots, \lambda_N) \) and \( \mu = (\mu_1, \ldots, \mu_N) \) denote the Young diagram characterized by the initial and final positions \( \lambda_j = x_{j-1} - N + j - 1 \) and \( \mu_j = x'_j - N + j - 1 \), respectively. The arguments of the Grothendieck polynomials \( z = \{z_1, \ldots, z_N\} \) and \( z^{-1} = \{z_1^{-1}, \ldots, z_N^{-1}\} \) are expressed as the solutions to the Bethe ansatz equation (6.5). The summation is over all the sets of the solutions to the Bethe ansatz equation.

**Proof.** Substituting the resolution of the identity operator into (6.6), we have

\[
G_t(x'|x) = \sum_{\zeta} \frac{\langle x_1 \cdots x_N | \psi(z)| \rangle \langle \psi(z)| x_1 \cdots x_N \rangle}{\langle \psi(z)| \psi(z) \rangle} e^{\mathcal{H}t},
\]

(6.8)

where the parameters \( \zeta = \{z_1, \ldots, z_N\} \) are the solutions to the Bethe ansatz equation (6.5) and the summation is over all the sets of the solutions. Finally, utilizing the expression of the wavefunctions (5.5) and the deformed Cauchy identity (5.6), one arrives at (6.7). \( \square \)

Let us check the validity of (6.8) for the steady state. After infinite time, the system will relax to the steady state \( |S_N\rangle \):

\[
|S_N\rangle = \binom{M}{N}^{-1} \sum_{1 \leq x_1 < \cdots < x_N \leq M} |x_1 \cdots x_N\rangle.
\]

(6.9)

Up to some overall factor, the steady state corresponds to the zero-energy state \( |\psi((\mu)|\rangle \) with \( u_j = \infty \) for \( 1 \leq j \leq N \). Due to the Perron–Frobenius theorem, all the energy spectrum
On the other hand, one finds that the Grothendieck polynomials $G_\lambda(z, -1)$ and $\overline{G}_\mu(z^{-1}, -1)$ do not depend on the shapes $\lambda$ and $\mu$ in the limit $z_j \to 1$ ($1 \leq j \leq N$):

$$G_\lambda(z, -1)|_{z \to 1} = 1, \quad \prod_{j=1}^{N}(1 - z_j)^{N-1}\overline{G}_\mu(z^{-1}, -1)|_{z \to 1} = 1,$$

which follows from the formula (4.36). Thus the rhs of (6.7) reduces to $1/\sum_{\lambda \subseteq (M-N)^n} 1 = \binom{M}{N}^{-1}$, which is nothing but the rhs of (6.10). The following is a consequence of proposition 6.1 and the conservation law of the total probability: $\sum_{1 \leq x_1 < \ldots < x_N \leq M} P_t(x_1, \ldots, x_N) = 1$.

**Corollary 6.2.** The following sum rule holds for the Grothendieck polynomials:

$$\sum_{\lambda} \sum_{\mu \subseteq (M-N)^n} G_{\mu}(z; -1) G_{\lambda}(z^{-1}; -1) e^{H(z)} = 1,$$

where the summation is over all the sets of the solutions to the Bethe ansatz equation (6.5).

Finally, we comment on the relaxation dynamics of a physical quantity $\mathcal{A}$. The time evolution of the expectation value for $\mathcal{A}$ starting from an initial state $|x_1 \cdots x_N\rangle$ is defined as

$$\langle \mathcal{A} \rangle_t = \langle S_N | e^{tM} | x_1 \cdots x_N \rangle,$$

where $|S_N\rangle$ is the left steady state vector.

**Proposition 6.3.**

$$\langle \mathcal{A} \rangle_t = \sum_{\lambda} \left[ \sum_{\nu \subseteq (M-N)^n} \sum_{\mu \subseteq (M-N)^n} A_\mu^\nu G_{\mu}(z; -1) \overline{G}_\lambda(z^{-1}; -1) e^{H(z)} \right],$$

where the matrix element $A_\mu^\nu$ is given by $A_\mu^\nu = \langle y_1 \cdots y_N | \mathcal{A} | x_1 \cdots x_N \rangle$ with $x_j = \lambda_{N-j+1} + j$ ($M - N \geq \lambda_1 \geq \cdots \lambda_N \geq 0$) and $y_j = \mu_{N-j+1} + j$ ($M - N \geq \mu_1 \geq \cdots \mu_N \geq 0$), and the summation is over all the sets of the solutions to the Bethe ansatz equation (6.5).

For instance, the relaxation dynamics of the local densities $\mathcal{A} = n_j = 1 - s_j$ and currents $\mathcal{A} = j_i = (1 - s_i)s_{i+1}$ can be explicitly evaluated by applying the following theorem.

**Theorem 6.4.** Let $\mathcal{A} = s_l \cdots s_{l+n-1}$ ($-l + 1 \leq n \leq M$; $l \in \mathbb{Z}$). Then the following formula holds for arbitrary complex values $z_j \in \mathbb{C}$ ($1 \leq j \leq N$):

$$\sum_{\nu \subseteq (M-N)^n} \sum_{\mu \subseteq (M-N)^n} A_\mu^\nu G_{\mu}(z; -1) = \prod_{j=1}^{N} s_j^{J_{l-j}^{l+n-1}} \prod_{1 \leq j < k \leq N} \frac{1}{z_k - z_j} \det_{\nu} V(M-n),$$
where $A = s_1 \cdots s_{l+n-1}$, and the $N \times N$ matrix $V$ is written as

$$V_{jk}^{(M-n)} = \sum_{m=0}^{j-1} (-1)^m \frac{(M-n)!}{m!(M-m-n)!} (1 - z_k)^{m-j+N-1} \quad (1 \leq j \leq N-1),$$

$$V_{Nk}^{(M-n)} = -\sum_{m=\max(N-1,1)}^{M-n} (-1)^m \frac{(M-n)!}{m!(M-m-n)!} (1 - z_k)^{m-1}.$$

(6.17)

**Proof.** The formula directly follows from the determinant representation of the form factor for $\langle S_N | s_1 \cdots s_{l+n-1} | \psi(z) \rangle$ obtained in [30, 31].

Setting $n = 0$ and $l = 1$ in the above formula, we find that $A_{\nu \mu} = \delta_{\nu \mu}$, and then the above formula reduces to (5.10) for $\beta = -1$.

### 7. Conclusion

In this paper, we studied the determinant structures of a one-parameter family of integrable five-vertex models. By use of the algebraic Bethe ansatz and the matrix product representation of the wavefunctions, the on-/off-shell wavefunctions are expressed in terms of determinant forms. We found that the resultant wavefunctions are given by the Grothendieck polynomials which are a one-parameter deformation of the Schur polynomials. By use of the properties satisfied by the wavefunctions, we derived several important formulae such as the Cauchy identity, summation formulae and so on for the Grothendieck polynomials.

The Grothendieck polynomial was originally introduced in the context of the Schubert calculus. This paper investigates the objects of the (geometric) representation theory from the perspectives of integrable models. See also [40–47] for the integrable model approach to the (geometric) representation theory or the classical integrable interpretation of integrable models. It is interesting to study the geometric and classical integrable interpretation of the Cauchy identity, or to examine other representation theoretical objects from the integrable model side, the Littlewood–Richardson coefficient for example. The Cauchy identity also seems to have potential applications to boxed plane partitions and determinantal processes, which we would like to pursue in the near future.

From the physics side, the evaluation of the wavefunctions by means of the matrix product representation allows us to formulate the exact relaxation dynamics of the periodic TASEP for arbitrary initial condition, beyond the step and alternating initial conditions studied in our former works [30, 31]. We can now extract the asymptotics, fluctuations and so on from the formulation. Moreover, since we started from the one-parameter extension of the $L$-operator which corresponds to the TASEP with an effective long-range potential [48–50], we are in a position to make an extensive study of them. One of the continuations of this paper is to study the properties of the model.

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Appendix

Let us derive (2.5) as a solution to the RLL-relations (2.1) with the R-matrix (2.3), by making the ansatz on the L-operator

\[ L_{n,j}(u) = d_1(u)s_j + d_2(u)\sigma_j^+\sigma_j^- + d_3(u)n_j, \]

where \( d_j(u) \) are the functions to be determined. In this paper, we consider the case \( d_2(u) \) is not identically equal to zero\(^5\) (\( d_2(u) \neq 0 \)). The equations to be solved are listed as

\[ vd_1(u)d_2(v) - ud_1(v)d_2(u) = 0, \]
\[ d_2(u)d_3(v) - d_2(v)d_3(u) = 0, \]
\[ vd_3(u)d_2(v) - ud_3(v)d_2(u) = 0, \]
\[ vd_1(u)d_3(v) - ud_1(v)d_3(u) = 0, \]
\[ vd_3(u)d_3(v) - ud_3(v)d_3(u) = 0, \]
\[ (u^2 - v^2)d_2(u)d_3(v) + uv(d_1(u)d_3(v) - d_1(v)d_3(u)) = 0, \]
\[ (u^2 - v^2)d_3(u)d_2(v) + uv(d_1(u)d_2(v) - d_1(v)d_2(u)) = 0, \]
\[ (u^2 - v^2)d_3(u)d_3(v) + uv(d_3(u)d_4(v) - d_3(v)d_4(u)) = 0. \]

From (A.2)–(A.4), we have the relations between \( d_1(u), d_3(u), d_5(u) \) and \( d_2(u) \) with the use of arbitrary constants \( A, B \) and \( C \) as

\[ d_1(u) = Aud_2(u), \]
\[ d_3(u) = Bd_3(u), \]
\[ d_5(u) = Cud_2(u). \]

Substituting the above relations into the remaining equations, we find that (A.5) and (A.6) are automatically satisfied, and we are left with (A.7)–(A.9) which now read

\[ (Bd_2(u) + Aud_4(u))u^2d_2(v) - (Bd_2(v) + Aud_4(v))u^2d_2(u) = 0, \]
\[ u(Cud_2(u) - d_4(u))d_2(v) - u(Cud_2(v) - d_4(v))d_2(u) = 0. \]

These equations lead to following relations between \( d_4(u) \) and \( d_2(u) \):

\[ Bd_2(u) + Aud_4(u) = Eu^2d_2(u), \]
\[ Cu^2d_2(u) - uAd_4(u) = Fd_2(u), \]

with constants \( E \) and \( F \). Assuming \( A \neq 0 \), the compatibility between the two relations (A.15) and (A.16) leads to \( E = AC, F = A^{-1}B \). Considering also the case \( A = 0 \), we finally find the elements of the L-operator satisfying the RLL-relation under the ansatz (A.1) to be

\(^5\) For \( d_2(u) = 0 \), one sees from (A.2)–(A.9) that the model reduces to the four-vertex model: \( d_1(u) = Au, d_3(u) = f(u), d_4(u) = d_5(u) = Bu, \) where \( A \) and \( B \) are some constants, and \( f(u) \) is a rational function not identically equal to zero.
\[ d_1(u) = Af(u), \]  
\[ d_2(u) = f(u), \]  
\[ d_3(u) = Bf(u), \]  
\[ d_4(u) = (Cu - Du^{-1})f(u), \]  
\[ d_5(u) = Cu f(u), \]

where \( f(u) \neq 0 \) is a rational function of \( u \) and \( A, B, C, D \) are constants satisfying the constraints \( B - AD = 0 \). Taking \( A = B = D = 1 \) and \( C = \alpha \), we have the desired \( L \)-operator (2.5) up to the overall factor \( f(u) \). (See [51] for example for a brute force search of more complicated integrable models of higher ranks or higher spins.)

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