Anomalous Amplitudes in a Thermal Bath

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I review the implications of the axial anomaly in a thermal bath. I assume that the Adler-Bardeen theorem applies at nonzero temperature, so that the divergence of the axial current remains independent of temperature. Nevertheless, I argue that while the anomaly does not change with temperature, "anomalous" mesonic couplings do. This is verified by explicit calculations in a low temperature expansion, and near the chiral phase transition.

§1. Introduction

In this paper I provide a pedagogical review of recent works on the nature of anomalous interactions in a thermal bath.1) To forestall any possible confusion, at the outset I stress that I assume that the Adler-Bardeen theorem remains valid in a thermal bath. With the proper regularization scheme, the Adler-Bardeen theorem states that the divergence of the axial current is given identically by its value at one loop order.2) Diagramatically, this is because the axial anomaly is due to the ultraviolet behavior of one loop graphs. Since a thermal bath only affects the infrared, it is natural that the Adler-Bardeen theorem will still hold at nonzero temperature. This is confirmed by explicit calculations.3), 4)

Nevertheless, in this note I show that while the anomaly itself remains unchanged, anomalous amplitudes — such as multi-point amplitudes between axial vector and vector currents — do change in a thermal bath. This is due directly to the loss of lorentz covariance in a thermal bath, and is not special to anomalous currents. Consider, for example, the two point function between two conserved, vector currents. At zero temperature, lorentz invariance and current conservation implies that this two point function involves only one scalar function. At nonzero temperature, the lack of explicit lorentz covariance implies that there are four scalar functions; imposing current conservation leaves three independent functions. Similar considerations enter for anomalous currents; there are some slight changes because the currents are anomalous instead of conserved, but this is really secondary.

This understanding originated in work by Itoyama and Mueller;4) it was then demonstrated by explicit calculations in various specific models.1) Unlike our presentation in the literature, in this note I begin by reviewing the general analysis,1) and then summarize how these results are realized about zero temperature, and then about the chiral phase transition. While it is opposite to how things were understood historically, the presentation is more logically coherent.
§2. General analysis

Consider a vector current, $J_\alpha$, and an axial current, $J_{5,\gamma}$. I assume that the vector current is conserved,

$$\partial^\alpha J_\alpha = 0 ,$$

while the axial current is anomalous,

$$\partial^\alpha J_{5,\alpha} = -\frac{e^2 N_c}{4\pi^2} F_{\alpha\beta} F^{\alpha\beta} .$$

The Adler-Bardeen theorem is the statement that the coefficient of the right hand side, computed to one loop order, is exact to any loop order; this coefficient is also expected to be independent of temperature and density.

The classic quantity to compute is the three point Green's function between one axial vector current and two vector currents:

$$T_{\alpha\beta\gamma}(P_1, P_2; T) = -ie^2 \int d^4X_1 d^4X_2 e^{i(P_1 \cdot X_1 + P_2 \cdot X_2)}
\times \frac{\operatorname{Tr}(e^{-H/T} J_\alpha(X_1) J_\beta(X_2) J_{5,\gamma}(0))}{\operatorname{Tr}(e^{-H/T})} .$$

This is a true Green's function in a thermal bath, with $H$ the hamiltonian. Given the abelian anomaly, this $AVV$ correlation function is the simplest Green's function in which the anomaly enters. For the nonabelian anomaly, besides $AVV$, there are also box diagrams, such as $AVVV$, and pentagon diagrams, such as $AVVVV$ and $AAAVV$. All the other Green's functions can be analyzed by similar means, although because of a proliferation of independent functions, the details become increasingly complicated.

Since the vector currents are conserved, $T_{\alpha\beta\gamma}$ satisfies

$$P_1^{\alpha} T_{\alpha\beta\gamma} = P_2^{\beta} T_{\alpha\beta\gamma} = 0 ;$$

similarly, the divergence of the axial vector current is anomalous,

$$Q^\gamma T_{\alpha\beta\gamma} = -\frac{e^2 N_c}{12\pi^2} \epsilon_{\alpha\beta\gamma\delta} P_1^{\gamma} P_2^{\delta} ,$$

where $Q = P_1 + P_2$.

I now need to relate this $AVV$ correlation function to the amplitude for pion decay; to do so, I follow Shore and Veneziano. At low temperature the pion couples to the axial current as

$$\langle 0 | J_{5,\alpha}^a | \pi^b(Q) \rangle = iQ_\alpha f_\pi \delta^{ab} .$$

I work in the chiral limit, in which the pions are massless. Strictly speaking, (2.6) is valid only to lowest nontrivial order about zero temperature. In a nonlinear sigma model with pion decay constant $f_\pi$, that is $\sim T^2/f_\pi^2$, to this order, pions obtain a finite, temperature dependent renormalization constant, but otherwise they propagate without damping. To higher order, however, such as $\sim T^4/f_\pi^4$, pions
are damped,\textsuperscript{6} and their self energy acquires an imaginary part even on mass shell. This damping in turn implies that there is a nontrivial matrix element not just between the axial vector current and one pion, but also between the axial vector current and three pions. This complicates, but does not invalidate, the following analysis. Implicitly, I ignore pion damping, because even then I shall see that the direct connection between the axial anomaly and the electromagnetic decay of the $\pi^0$ is already lost.

To obtain the amplitude for $\pi^0 \rightarrow \gamma\gamma$, I introduce $Q^2$ times the matrix element between two vector currents and a pion,

$$\mathcal{T}_{\alpha\beta} = e^2 Q^2 \int d^4 X_1 d^4 X_2 e^{i(P_1 \cdot X_1 + P_2 \cdot X_2)} \frac{\text{Tr} \left( e^{-H/T} J_\alpha(X_1) J_\beta(X_2) \pi(0) \right)}{\text{Tr} (e^{-H/T})}.$$  \hspace{1cm} (2.7)

This is related to the pion decay amplitude as

$$\mathcal{M} = \lim_{Q^2 \rightarrow 0} \epsilon^\alpha_1 \epsilon^\beta_2 \mathcal{T}_{\alpha\beta},$$  \hspace{1cm} (2.8)

where $\epsilon^\alpha_1$ and $\epsilon^\beta_2$ are the polarization tensors for the two photons.

The original amplitude contains terms which are one particle irreducible; in addition, it also contains terms which are one particle reducible. Of the latter, I pick out those which are one pion reducible. I then subtract the one pion pole term from (2.3) to define $\hat{T}_{\alpha\beta\gamma},$

$$\hat{T}_{\alpha\beta\gamma} = \mathcal{T}_{\alpha\beta\gamma} + f_\pi \frac{1}{Q^2} \mathcal{T}_{\alpha\beta},$$  \hspace{1cm} (2.9)

$\hat{T}_{\alpha\beta\gamma}$, which by definition is one pion irreducible, satisfies Ward identities similar to those for $\mathcal{T}_{\alpha\beta\gamma}$. The condition for current conservation is identical,

$$P^\alpha_1 \hat{T}_{\alpha\beta\gamma} = P^\beta_2 \hat{T}_{\alpha\beta\gamma} = 0.$$  \hspace{1cm} (2.10)

The anomalous Ward identity differs, receiving a contribution from the one pion pole,

$$Q^\gamma \hat{T}_{\alpha\beta\gamma} = f_\pi T_{\alpha\beta} - \frac{e^2 N_c}{12\pi^2} \epsilon_{\alpha\beta\gamma\delta} P^\gamma_1 P^\delta_2.$$  \hspace{1cm} (2.11)

I now see what general relations can be deduced from these relations, using Bose symmetry between the two photons, $P_1, \alpha \rightleftarrows P_2, \beta.$

I first discuss zero temperature, where euclidean invariance can be invoked. The most general pseudo-tensor $\hat{T}_{\alpha\beta\gamma}$ which satisfies all of our conditions can be shown to involve only three terms:

$$\hat{T}_{\alpha\beta\gamma} = T_1 \epsilon_{\alpha\beta\gamma\delta} (P^\delta_1 - P^\delta_2) + T_2 (\epsilon_{\alpha\beta\gamma\delta} P^\delta_2 - \epsilon_{\beta\gamma\delta\alpha} P^\delta_1) P^\kappa_1 P^\kappa_2 + T_3 (\epsilon_{\alpha\gamma\delta\kappa} P^\delta_1 - \epsilon_{\beta\gamma\delta\kappa} P^\delta_2) P^\kappa_1 P^\kappa_2.$$  \hspace{1cm} (2.12)
Current conservation, (2·10), gives
\[ T_1 + P_1^2 T_2 + P_1 \cdot P_2 T_3 = 0, \]  
while from the anomalous Ward identity, (2·11),
\[ -2T_1 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}. \]  
Combining these two relations,
\[ 2P_1^2 T_2 + 2P_1 \cdot P_2 T_3 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}. \]  
Putting the photons on their mass shell \( P^2 = P_1^2 = P_2^2 \), the left-hand side on (2·15) reduces to \( Q^2 T_3 \). Since by definition I constructed \( \widehat{T} \) to be one pion irreducible, this must vanish on the pion mass shell, \( Q^2 \to 0 \); there is no possibility for poles in \( 1/Q^2 \) to enter into \( T_3 \). Hence the left-hand side of (2·15) vanishes, and I obtain a relation between \( g_{\pi\gamma\gamma} \) and the coefficient of the axial anomaly,
\[ 0 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}. \]  
This analysis, and especially the tensor decomposition of (2·12), is identical to the derivation of the Sutherland-Veltman theorem. Historically, this theorem predated the anomaly, and was used originally to conclude that \( g_{\pi\gamma\gamma} = 0 \); that is, that the electromagnetic decay of the neutral pion was chirally suppressed. By adding the axial anomaly through the anomalous Ward identity (2·11), however, I obtain \( g_{\pi\gamma\gamma} \sim e^2 N_c/f_\pi \), (2·16). This was first derived by Adler, and is reasonably accurate. From a modern perspective, then, it is precisely the anomaly plus the Sutherland-Veltman theorem which allows us to relate the amplitude for \( \pi^0 \to \gamma\gamma \) to the coefficient of the axial anomaly. If, for example, \( Q^2 T_3 \) did not vanish, then while there would be a condition from the anomalous Ward identity, it would not uniquely predict the amplitude for \( \pi^0 \to \gamma\gamma \).

This is something like what happens at nonzero temperature. I follow Itoyama and Mueller, and write the most general tensor decomposition for \( \widehat{T}_{\alpha\beta\gamma} \). The crucial point is obvious: in a thermal bath, euclidean symmetry is lost, so that I can introduce a new vector, \( n_\mu = (1, \vec{0}) \), which denotes the rest frame of the bath. There are now much more tensors which can enter. A partial list of the new tensors includes
\[ \widehat{T}_{\alpha\beta\gamma} = T_1 \varepsilon_{\alpha\beta\gamma \delta} (P_1^\delta - P_2^\delta) + T_2 (\varepsilon_{\alpha\gamma\delta\kappa} P_2^\beta - \varepsilon_{\beta\gamma\delta\kappa} P_1^\beta) P_1^\delta P_2^\kappa + T_3 (\varepsilon_{\alpha\gamma\delta\kappa} P_1^\beta - \varepsilon_{\beta\gamma\delta\kappa} P_2^\beta) P_1^\delta P_2^\kappa + T_4 n \cdot Q \varepsilon_{\alpha\beta\gamma\delta\kappa} P_1^\delta P_2^\kappa n^\gamma + T_5 (n \cdot P_2 \varepsilon_{\alpha\gamma\delta\kappa} n^\beta - n \cdot P_1 \varepsilon_{\beta\gamma\delta\kappa} n^\alpha) P_1^\delta P_2^\kappa + \cdots . \]  
(2·17)
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I have only included the terms in $\hat{T}$ which contribute to the Ward identities (2.10) and (2.11); what other tensors enter will not matter for our considerations. Current conservation gives

$$T_1 + P_1^2 T_2 + P_1 \cdot P_2 T_3 + (n \cdot P_1)^2 T_5 = 0,$$

while the anomalous Ward identity fixes

$$-2T_1 + (n \cdot Q)^2 T_4 = f_\pi(T) g_{\pi\gamma\gamma}(T) - \frac{e^2 N_c}{12\pi^2}.$$  

Note that at nonzero temperature, I allow $f_\pi$ and $g_{\pi\gamma\gamma}$ to depend upon temperature; explicit calculation shows that they can and do. Combining these two relations, I find

$$2P_1^2 T_2 + 2P_1 \cdot P_2 T_3 + (n \cdot Q)^2 T_4 + 2(n \cdot P_1)^2 T_5$$

$$= f_\pi(T) g_{\pi\gamma\gamma}(T) - \frac{e^2 N_c}{12\pi^2}.$$  

This relation is valid for arbitrary momenta. I then put the photons on their mass shell, $P_1^2 = P_2^2 = 0$, as well as the pion, $Q^2 \to 0$. Since by construction $T_3$ is one pion irreducible, it cannot have a pole in $\sim 1/Q^2$, and so I find

$$(n \cdot Q)^2 T_4 + 2(n \cdot P_1)^2 T_5 = f_\pi(T) g_{\pi\gamma\gamma}(T) - \frac{e^2 N_c}{12\pi^2}.$$  

The terms on the right-hand side are the same as at zero temperature. But now I find two terms on the left-hand side, which involve the energy squared for the pion, $(n \cdot Q)^2$, and the same for one photon, $n \cdot P_1$. Even letting all fields go on their mass shell, I can do so without letting the energies vanish. Further, there is no reason why the amplitudes $T_4$ and $T_5$ should vanish at these points. Thus I see that at nonzero temperature, while there is a condition from the axial anomaly, it cannot be used to uniquely relate $g_{\pi\gamma\gamma}(T)$, $f_\pi(T)$, and the coefficient of the axial anomaly; I also need the values of $T_4$ and $T_5$.

In general terms, I have assumed that the Adler-Bardeen theorem applies. What failed was the Sutherland-Veltman theorem: the terms on the left-hand side of (2.21) do not need to, and in general do not, vanish.

What happens beyond lowest order in an expansion about zero temperature? When the effects of pion damping are included, one will have to deal with a pion mass shell which is not only off the light cone, but also has an imaginary part. Since Goldstone’s theorem remains valid in a thermal bath, this is not a problem in principle. More states contribute to the anomalous Ward identity, but it remains valid.

Initially, one might well wonder why the Sutherland-Veltman theorem should apply at zero, but not at any nonzero, temperature. Even at zero temperature, though, it should be remembered that the Sutherland-Veltman theorem only applies in a very strictly defined regime: in the chiral limit, with all fields, the pion and both photons, on their mass shell. For example, consider the case in which one is in the chiral limit, but only one photon is on its mass shell. Then the left-hand side of (2.20)
does not vanish, and \( g_{\pi \gamma \gamma} \) is not simply related to the anomaly. In fact, consider the limit in which \( P_1^2 \) is large; then even in the chiral limit, \( Q^2 \to 0, g_{\pi \gamma \gamma} \sim e^2 (f_\pi / P_1^2) \). This agrees with a simple power counting of the underlying quark diagrams which contribute to \( g_{\pi \gamma \gamma} \); any coupling falls off like powers of momenta at large momenta.

§3. Low temperature

About zero temperature, it is most convenient to use a nonlinear sigma model. This is a nonrenormalizable theory, but one can still treat it with a cutoff. Furthermore, for the temperature dependent effects which I am interested in, everything is obviously ultraviolet finite.

The technical details of computing with a nonlinear sigma model are involved. To include anomalous couplings, one adds a Wess-Zumino-Witten term to the (gauged) nonlinear sigma model lagrangian, and then compute loop effects with that lagrangian. For definiteness, all calculations are for two massless flavors of quarks.

At zero temperature, one can perform a nontrivial check of both the Sutherland-Veltman and Adler-Bardeen theorems. At tree level, the Wess-Zumino-Witten term defines a coupling between the pion and two photons, which satisfies

\[
f_\pi g_{\pi \gamma \gamma} = \frac{e^2}{4\pi^2}.
\]  

(3.1)

This is a relationship between bare quantities in the tree lagrangian. One can then compute to one loop order. Since the nonlinear sigma model is nonrenormalizable, it is unremarkable to find that quadratic divergences arise. For example, the relationship between the bare and renormalized pion decay constants is

\[
f_\pi^{\text{ren}} = \left(1 - \frac{1}{f_\pi^2} \int \frac{d^4K}{(2\pi)^4} \frac{1}{K^2}\right) f_\pi.
\]  

(3.2)

The renormalized coupling between a pion and two photons is similarly found to be

\[
g_{\pi \gamma \gamma}^{\text{ren}} = \left(1 + \frac{1}{f_\pi^2} \int \frac{d^4K}{(2\pi)^4} \frac{1}{K^2}\right) g_{\pi \gamma \gamma}.
\]  

(3.3)

Now of course to be well defined, I should introduce some (chirally invariant) regularization scheme, such as dimensional regularization. But in fact no matter how one regulates the quadratic divergence in these expressions, it is clear that to one loop order, \( \sim 1/f_\pi^2 \), the product of renormalized quantities satisfies the same relationship as in (3.1),

\[
f_\pi^{\text{ren}} g_{\pi \gamma \gamma}^{\text{ren}} = f_\pi g_{\pi \gamma \gamma} = \frac{e^2}{4\pi^2}.
\]  

(3.4)

In terms of the previous section, no diagrams contribute to \( T_2 \) or \( T_3 \). Two diagrams contribute to \( T_1 \), but cancel against each other. Because \( T_1 = T_2 = T_3 = 0 \), at zero temperature both the Sutherland-Veltman and Adler-Bardeen theorems apply.
At nonzero temperature, at first one might reason\(^\text{11}\) that topology should similarly constrain the couplings as in (3·4). Indeed, the diagrams are absolutely identical to those at zero temperature. Thus one would expect that I simply extract the temperature dependent piece from the (quadratic) divergences in \(f_\pi\) and \(g_{\pi\gamma\gamma}\), as

\[
\int \frac{d^4K}{(2\pi)^4} \frac{1}{K^2} = (T = 0) + \frac{T^2}{12}. \tag{3·5}
\]

However the divergence at zero temperature is regulated, the temperature dependent piece in the integral is perfectly well defined.

This works for the pion decay constant; to \(\sim T^2/f_\pi^2\), it decreases as this naive argument and (3·2) would suggest

\[
f_\pi(T) = \left(1 - \frac{1}{12} \frac{T^2}{f_\pi^2}\right) f_\pi. \tag{3·6}
\]

The only diagrams which contribute to \(f_\pi\), however, are wave function renormalization for the pion, and a renormalization of the axial vector current. Both of these diagrams are "tadpole" type diagrams, and are clearly independent of the external momentum.

This is not true for the coupling between a pion and two photons. Most of the diagrams which contribute are tadpole type diagrams, but one, in which a single photon couples to a pion loop, is not. For this last diagram, the momentum dependence must be treated with care. Doing so, one finds the following. To lowest order about zero temperature, \(f_\pi(T)\) decreases, (3·6). If the result at nonzero temperature were like that at zero temperature, (3·3), then (3·5) would predict that \(g_{\pi\gamma\gamma}\) increases with temperature. Instead, I find that it decreases

\[
g_{\pi\gamma\gamma}(T) = \left(1 - \frac{1}{12} \frac{T^2}{f_\pi^2}\right) g_{\pi\gamma\gamma}. \tag{3·7}
\]

In terms of the anomalous Ward identity, calculation shows that at nonzero temperature, while no diagrams contribute to \(T_2 = T_3 = 0\), \(T_1\) is nonzero:

\[
T_1 = \frac{T^2}{12 f_\pi^2} \frac{\epsilon^2}{4\pi^2}. \tag{3·8}
\]

For the terms special to nonzero temperature, \(T_4\) vanishes, while

\[
T_5 = -\frac{1}{(n \cdot P_1)^2} \frac{T^2}{12 f_\pi^2} \frac{\epsilon^2}{4\pi^2}. \tag{3·9}
\]

Comparing to the left-hand side of (2·21), because \(T_5\) is nonzero, the Sutherland-Veltman theorem does not apply even to leading order at nonzero temperature, \(\sim T^2/f_\pi^2\). In contrast, to this order the anomalous Ward identity, and so the Adler-Bardeen theorem, are satisfied. In fact, \(T_5\) and the anomalous Ward identity provides a nice check of our results for \(f_\pi(T)\) and \(g_{\pi\gamma\gamma}(T)\) in (3·6) and (3·7).
As the Sutherland-Veltman theorem fails even when it pion damping can be neglected, there is no point in considering it to higher order in an expansion about zero temperature. Surely the Adler-Bardeen theorem remains valid, although how in detail it is manifested is presumably involved and a question of interest.

Diagrammatically, while the same diagrams contribute to $g_{\pi\gamma\gamma}(T)$ to one loop order at $T = 0$ and to $\sim T^2/f_\pi^2$, it is the delicate momentum dependence of one diagram at nonzero temperature which gives rise to the unexpected result in (3.7). This comes about because of a surprising analogy. The momentum dependent diagram which contributes to $g_{\pi\gamma\gamma}$ is proportional to

$$T \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \frac{K^\alpha K^\beta}{K^2(K-P)^2}, \quad (3.10)$$

where $P = (p^0, \vec{p})$ is an external momentum for the gauge field.

Exactly the same function enters into what appears to be a very different problem: the self energy for a gauge field, coupled to massless fields, in the limit of high, as opposed to low, temperature. It is well known for gauge fields that the self energy is an involved function of the external momenta, and that different results are obtained depending upon how the zero momentum limit is reached: in particular, the static limit, $p^0 = 0$, then $\vec{p} \rightarrow 0$, differs from the limit on the light one, $p^0 = i\omega$, $\omega = p \rightarrow 0$. Technically, this is why the guess for $g_{\pi\gamma\gamma}(T)$ failed: since $P$ is the momentum of the external photon, the zero momentum limit which enters is not the static one, but that on the light cone.

This dependence on the external momentum is more easily understood when one constructs an effective lagrangian for $\pi^0 \rightarrow \gamma\gamma$:

$$L_{\pi^0\gamma\gamma}(T) = \left( \frac{e^2 N_c}{4\pi^2} \right) \frac{1}{f_\pi(T)} \pi^0 F_{\alpha\beta} \tilde{F}^{\alpha\beta} - \frac{T^2}{12f_\pi^2} \left( \frac{e^2 N_c}{4\pi^2} \right) \int d\Omega_{\hat{k}} \frac{1}{4\pi} H_{\gamma\alpha} \frac{\hat{K}^\alpha \hat{K}^\beta}{-(\partial \cdot \hat{K})^2} F_{\gamma\beta}, \quad (3.11)$$

where $\tilde{F}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}/2$, $H_{\alpha\beta} = \partial_{\alpha} H_{\beta} - \partial_{\beta} H_{\alpha}$ and

$$H_{\alpha} = \frac{1}{f_\pi} \epsilon_{\alpha\beta\gamma\delta} F_{\beta\gamma} \partial_{\delta} p^0. \quad (3.12)$$

The vector $\hat{K} = (i, \hat{k})$; one then integrates over all angles $\hat{k}$. This integration represents the hard, massless field in the one loop integral.

The important aspect of (3.11) is that it is nonlocal. The nonlocality is familiar from hard thermal loops in gauge theories, and is responsible for the sensitivity to how the zero momentum limit is reached. The complete expression for the temperature dependent terms in the Wess-Zumino-Witten lagrangian was derived by Manuel.1)
§4. Near the chiral phase transition

In this section I explain what was historically the first example of how the coupling of $\pi^0 \to \gamma\gamma$ changes with temperature.\(^1\) I work near the chiral phase transition, which is assumed to be of second order, and show that in this limit, $g_{\pi\gamma\gamma}$ vanishes as the point of phase transition is approached. I emphasize the simplicity of the phenomenon; for technical reasons which I will discuss, the analysis is not as complete as about zero temperature.

I employ a constituent quark model. At the outset I confess that I do not think that this model is at all a realistic model to calculate detailed properties of the chiral phase transition. I do think it is good enough to explain the qualitative physics, in essence at the level of a type of mean field theory.

The coupling between the mesons and quark fields is take to be

$$\mathcal{L} = \bar{\psi} \left( D + 2\tilde{g} \left( \sigma t_0 + i \pi \cdot \vec{r}^5 \right) \right) \psi. \quad (4.1)$$

I take two flavors, with $t_0 = 1/2$ and $tr(t^a t^b) = \delta^{ab}/2$. This lagrangian is invariant under the standard chiral symmetry of $SU(2)_L \times SU(2)_R$. The meson fields include the $\sigma$ meson and pions.

At zero temperature I assume that the $\sigma$ field acquires a vacuum expectation value; for two flavors, at tree level the pion decay constant is identically this v.e.v, $f_\pi = \sigma_0$. (The ratio of $f_\pi/\sigma_0 \neq 1$ for three or more flavors.) At nonzero temperature, $f_\pi$ is no longer strictly equal to $\sigma_0$; this can be seen by an expansion about zero temperature, as the terms $\sim T^2/f_\pi^2$ differ. If $T_{ch}$ is the temperature for the chiral phase transition, which is assumed to be of second order, then both should vanish as $T \to T_{ch}$ in the same manner, $\sigma_0 \sim f_\pi \sim (T_{ch} - T)^\beta$. In mean field theory, $\beta = 1/2$.

The coupling between a pion and two photons is given by a triangle diagram, similar to that which contributes to the axial anomaly. For the axial anomaly, however, one computes the divergence of the axial current; in the fermion loop, the vertex brings in one factor of the momentum, since it is the divergence I am computing, and one factor of the Dirac matrix $\gamma^5$, as an axial current. For the coupling between a pion and two photons, the pion vertex brings in one $\gamma^5$, but no factor of the momentum, just the coupling $\tilde{g}$. For the divergence of the axial anomaly, the factor of the momentum means that the triangle diagram is sensitive to the ultraviolet regime, and completely insensitive to the infrared regime. For the decay of a pion, without the power of momentum upstairs, the associated triangle diagram becomes completely insensitive to the ultraviolet, but sensitive to the infrared.

For the axial anomaly, the diagram involves the following trace over Dirac matrices:

$$\text{tr}(\gamma^5 \Delta(K) \gamma^\alpha \Delta(K - P_1) \gamma^\beta \Delta(K - P_2)) \quad (4.2)$$

with $\Delta$ the fermion propagator. The detailed momentum dependence does not matter for my arguments. As stated, for the axial anomaly only the ultraviolet behavior matters, so I can take the propagators to be massless. Then I am left with the trace of $\gamma^5$ times six Dirac matrices; since the trace of $\gamma^5$ times four, six, etc., Dirac matrices is nonzero, I find a nontrivial result.
In contrast, for pion decay the trace over Dirac matrices is

\[ \text{tr}(\gamma^5 \Delta(K) \gamma^\alpha \Delta(K - P_1) \gamma^\beta \Delta(K - P_2)) \]. \tag{4.3} \]

If I take each fermion propagator to be massless, the integral vanishes identically, since I have the trace of \( \gamma^5 \) times five Dirac matrices. To have any nonzero result, there must be one power of the mass from a fermion propagator. This is the essential origin of the suppression of pion decay near the chiral phase transition.

The overall behavior of the diagrams can be estimated on the basis of power counting and gauge invariance. Gauge invariance tells us that the gauge fields have to enter through the form \( F_{\alpha\beta} \tilde{F}^{\alpha\beta} \). Thus the divergence of the axial current, and this operator, have each dimension four, so the coefficient is a pure number. The Dirac trace in (4.2) should just give us that part of the one loop result.

For the coupling between a pion and two gauge fields, since the latter enter as \( F_{\alpha\beta} \tilde{F}^{\alpha\beta} \), the coupling must have dimensions of inverse mass. One can read off the relevant factors without direct computation: there is one factor of \( \tilde{g} \) from the coupling, and one factor of \( m \) from the Dirac trace in (4.3). That leaves an integral with dimensions of mass squared; about zero momentum, the only natural mass scale is \( 1/m^2 \). Thus at zero temperature,

\[ \sim e^2 \frac{\tilde{g} m}{m^2} \pi^0 F_{\alpha\beta} F^{\alpha\beta} = \frac{e^2}{12\pi^2} \frac{1}{f_\pi} \pi^0 F_{\alpha\beta} F^{\alpha\beta}. \tag{4.4} \]

Here I have used the fact that as can be read off from (4.1), the constituent quark mass \( m = \tilde{g} m_0 = \tilde{g} f_\pi \). The result in (4.4) is the first term in the Wess-Zumino-Witten lagrangian. The full lagrangian is complicated, although the overall coefficient is dictated by topology, which indirectly reflects the topology underlying the Adler-Bardeen theorem.

To estimate the coupling at nonzero temperature, I only need recognize that while the factors of \( \tilde{g} \) and \( m = \tilde{g} f_\pi \) remain the same, the integral, being sensitive to the infrared, changes. The diagram involves fermions at nonzero temperature. In the limit in which the constituent quark mass is much less than the temperature, the only mass scale in the loop integral is the temperature \( T \). Due to Fermi-Dirac statistics, there are no infrared divergences, and the temperature just acts as an infrared cutoff, as the coupling between \( \pi \rightarrow \gamma \gamma \) becomes

\[ \sim e^2 \frac{\tilde{g} m(T)}{T^2} \pi^0 F_{\alpha\beta} F^{\alpha\beta} = \frac{7 \zeta(3) e^2 \tilde{g}^2}{8\pi^4 T^2} f_\pi(T) \pi^0 F_{\alpha\beta} F^{\alpha\beta}. \tag{4.5} \]

This result is confirmed by direct calculation, which also gives the coefficient of (4.5) to one loop order. I have assumed that the constituent quark mass, \( m(T) = \tilde{g} f_\pi(T) \), changes with temperature, but neglected the dependence of the coupling constant \( \tilde{g} \) with temperature. This is because \( f_\pi(T) \) changes like a power of \( T \), while as a typical dimensionless coupling constant, \( \tilde{g} \) should only change logarithmically. Even if the coupling \( \tilde{g} \) does flow to a fixed point, there is every reason to believe that it will be a nontrivial fixed point; then I just replace the "bare" \( \tilde{g} \) with that value.

There is a technical qualification: the expression in (4.5) is correct only at zero momentum, approached in the static limit. It is rather more difficult to compute
the analogous amplitudes away from the static limit, which is the limit of interest of compute the anomalous Ward identity, with photons and pions which are on their mass shell. Thus I cannot directly verify how the anomalous Ward identity is satisfied. Even with explicit calculation, though, I can assume that like $g_{\pi\gamma\gamma}$, $T_1$, $T_2$ and $T_3$ vanish at $T = T_{ch}$. Nevertheless, the anomalous Ward identity can easily be satisfied if $T_4$ and/or $T_5$ are nonzero.

Besides verifying the anomalous Ward identity, there is another reason for computing anomalous processes near the chiral phase transition. The processes in this section represent one term in what the Wess-Zumino-Witten lagrangian becomes near $T_{ch}$. Like the terms about zero temperature, (3-12), it is undoubtedly nonlocal. The nonlocality will not be identical to those of hard thermal loops, though, they represent a new, nonlocal lagrangian which governs anomalous processes about the chiral phase transition.

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