DIFFERENTIAL OPERATOR APPROACH TO $ı$QUANTUM GROUPS AND THEIR OSCILLATOR REPRESENTATIONS

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Abstract. For a quasi-split Satake diagram, we define a modified $q$-Weyl algebra, and show that there is an algebra homomorphism between it and the corresponding $ı$quantum group. In other words, we provide a differential operator approach to $ı$quantum groups. Meanwhile, the oscillator representations of $ı$quantum groups are obtained. The crystal basis of the irreducible subrepresentations of these oscillator representations are constructed.

1. Introduction

Quantum groups were introduced independently by Drinfeld and Jimbo in [7, 21]. To date, the theory of quantum groups has played important roles in representation theory, low dimensional topology and mathematical physics etc. From the perspective of the development of quantum groups, mathematicians have extended various theories of quantum groups to more and more generalized cartan data, such as from semi-simple cases to Kac-Moody cases, and then to Borcherds cases. Another direction to develop quantum groups is the so-called $ı$quantum groups, which are the main objects studied in the current paper.

Let $(\mathfrak{g}, \mathfrak{g}^\theta)$ be a symmetric pair, where $\mathfrak{g}$ is a Kac-Moody algebra, $\theta$ is an involution of $\mathfrak{g}$ and $\mathfrak{g}^\theta$ is the $\theta$-stable subalgebra of $\mathfrak{g}$. The $ı$quantum groups are coideal subalgebras of $U_q(\mathfrak{g})$, which specializes to $U(\mathfrak{g}^\theta)$ at $q \to 1$ and denoted by $U'_q(\mathfrak{g}^\theta)$. In general, the algebra $U_q(\mathfrak{g}^\theta)$ is not a Hopf algebra. The $ı$quantum groups are classified by Satake diagrams. Moreover, $ı$quantum groups are isomorphic to ordinary quantum groups if the Satake diagram is quasi-split of diagonal type.

In 2013, Bao and Wang initiated the $ı$-program in [3]. The purpose of this program is to generalize various results for ordinary quantum groups to $ı$quantum groups. From geometric side, in [1], Bao, Kujawa, Li and Wang gave a geometric realization of $U^j$ and $U^i$, the $ı$quantum groups of type $B/C$, by using equivariant function on double flag varieties. This generalizes the classic work of Beilinson, Lusztig and MacPherson for ordinary quantum group in [4]. Later on, the first author and his collaborators further gave a geometric realization of the $ı$quantum groups of affine type $C$ in [13] by using a similar approach. Moreover, the first author, Ma and Xiao provided another geometric realization of the $ı$quantum groups $U^j$ in [16] by using equivariant $K$-group of corresponding Steinberg variety. This fills Ginzburg and Vasserot’s work in [17] into $ı$-program.

From algebraic side, $q$-Schur duality, Hall algebra and categorification are important theory related to quantum groups. In the framework of $ı$-program, these theories have been generalized to certain $ı$quantum groups. The Hecke-algebraic approach to the $ı$quantum groups was given in [14], which was a generalization of affine type $A$ construction in [8]. The three-parameter $q$-Schur duality between $ı$quantum groups and Hecke algebras was studied in [15], which can be degenerated to $q$-Schur dualities of affine types $B, C$ and $D$ by various specialization of parameters. In [28, 5], Ringel and Bridgeland gave a realization of the half part and entire of quantum groups by using Hall algebras, respectively. The $ı$Hall algebra approach

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toward quantum groups was developed by Lu and Wang in series papers [23, 24, 25, 26]. The highest
weight theory and crystal (global) basis theory of quantum groups \( U_q \) was studied in [29] and
[30]. The categorification of the modified quantum group \( \hat{U}_q \) was developed in
[2]. In [10, 11], Du and Wu presented a new realization of the quantum groups \( U_q \) and \( U^l \) in
terms of a BLM type basis.

In [18], Hayashi used differential operators on polynomial ring to construct spinor and
oscillator representations for quantum groups of types \( A_{N-1} \), \( B_N \), \( C_N \), \( D_N \) and \( A_{N-1}^{(1)} \). In
[20, 31, 12, 9], Hu and Du further studied the differential operator realization for quantum
groups \( U_q(\mathfrak{sl}_n) \), \( U_q(\mathfrak{sp}_{2n}) \), \( U_q(\mathfrak{gl}_{m|n}) \) and \( U_q(q_n) \) respectively.

In this paper, we provide a differential operator realization of quasi-split finite and affine
quantum groups \( ^tU(\mathcal{S}) \), where \( \mathcal{S} \) is the Satake diagram of types \( A_r \) or \( A_r^{(1)} \) with nontrivial
diagram involutions.

This realization can be regarded as an application of Hayashi’s approach [18] to quantum
groups. To achieve this goal, we introduce the modified \( q \)-Weyl algebra \( A_q(\mathcal{S}) \) associated
to the Satake diagrams \( \mathcal{S} \), generated by certain linear operators on the polynomial ring
\( \mathbb{Q}(q)[X_0, X_1, \cdots, X_{r+1}] \), and show that there is an algebra homomorphism from \( ^tU(\mathcal{S}) \) to
\( A_q(\mathcal{S}) \). Thus we obtain naturally a differential operator realization of \( ^tU(\mathcal{S}) \) on the \( q \)-Weyl
algebra \( A_q(A_{r+1}) \) and representations of \( ^tU(\mathcal{S}) \) on the polynomial ring which are called the
oscillator representations. Following the framework of crystal basis theory in [22] and [19, Chapter 4],
we construct the crystal basis of irreducible modules arising from the oscillator representations
of quantum groups \( ^tU(A_{2r+1}) \), \( ^tU(A_{2r+1}^{(1)}) \) and \( ^tU(A_1^{(1)}) \).

The paper is organized as follows. In Section 2, we recall the differential operator
realization of the quantum group \( U_q(\mathfrak{sl}_{r+2}) \) and its oscillator representation. In Section 3, we
define the modified \( q \)-Weyl algebras \( A_q(\mathcal{S}) \) associated with Satake diagrams and study their
representations. In Section 4, we construct the differential operator realizations of quantum
groups \( ^tU(\mathcal{S}) \). In Section 5, we study the oscillator representations of \( ^tU(\mathcal{S}) \) and the crystal
basis of irreducible modules over quantum groups \( ^tU(A_{2r+1}) \), \( ^tU(A_{2r+1}^{(1)}) \) and \( ^tU(A_1^{(1)}) \).

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Notations. Let

\[
[a] = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad [a]^k = [ka][k(a - 1)] \cdots [2k][k], \quad X^{(a)_k} = \frac{X^a}{[a]^k!} (a, k \in \mathbb{Z}),
\]

\((a; x)_0 = 1, \quad (a; x)_n = (1 - a)(1 - ax) \cdots (1 - ax^{n-1}), \quad n \geq 1,\]

\[
[n]_d = \begin{cases} \frac{[n][n-1] \cdots [n-d+1]}{[d]^n}, & \text{if } d > 0, \\ 1, & \text{if } d = 0, \\ 0, & \text{if } d < 0. \end{cases}
\]

2. Differential operator realization of \( U_q(\mathfrak{sl}_{r+2}) \)

Let \( r \geq 0 \) and \( \mathbb{I} = \{0, 1, \cdots, r + 1\} \). The \( q \)-Weyl algebra \( A_q(A_{r+1}) \) associated to Dynkin
diagram of type \( A_{r+1} \) is generated by \( \mathcal{D}_i, \mathcal{X}_i, \mathcal{M}_i^{\pm 1} (i \in \mathbb{I}) \) over \( \mathbb{Q}(q) \) subject to the following
relations (see [18]):

\begin{align}
(2.1) & \quad M_i^{-1}M_i = M_iM_i^{-1} = 1, \quad M_iM_j = M_jM_i, \\
(2.2) & \quad D_iM_j = M_jD_i, \quad X_iM_j = M_jX_i, \quad D_iX_j = X_jD_i, \quad \text{if } i \neq j, \\
(2.3) & \quad D_iD_j = D_jD_i, \quad X_iX_j = X_jX_i, \\
(2.4) & \quad D_iM_i = qM_iD_i, \quad X_iM_i = q^{-1}M_iX_i, \\
(2.5) & \quad D_iX_i = \frac{qM_i - q^{-1}M_i^{-1}}{q - q^{-1}}, \quad X_iD_i = \frac{M_i - M_i^{-1}}{q - q^{-1}}.
\end{align}

Let \( P = \mathbb{Q}(q)[X_0, X_1, \ldots, X_{r+1}] \) be the polynomial ring over \( \mathbb{Q}(q) \). We define \( A_q(A_{r+1}) \) acting on \( P \), for any \( f \in P \), as follows

\begin{align}
(2.6) \quad D_i f(X_0, \ldots, X_{r+1}) &= \frac{f(X_0, \ldots, qX_i, \ldots, X_{r+1}) - f(X_0, \ldots, q^{-1}X_i, \ldots, X_{r+1})}{qX_i - q^{-1}X_i}, \\
X_i f(X_0, \ldots, X_i, \ldots, X_{r+1}) &= X_i f(X_0, \ldots, X_i, \ldots, X_{r+1}), \\
M_i^{\pm 1} f(X_0, \ldots, X_i, \ldots, X_{r+1}) &= f(X_0, \ldots, \pm X_i, \ldots, X_{r+1}).
\end{align}

**Theorem 2.1** ([18, Proposition 2.1]). The polynomial ring \( P \) is an irreducible \( A_q(A_{r+1}) \)-module.

The generators \( D_i, X_i, M_i \) \((i \in \mathbb{N})\) of the algebra \( A_q(A_{r+1}) \) can be viewed as linear operators on the polynomial ring \( P \). Moreover, the operator \( D_i \) is called \( q \)-differentiation operator, which satisfies the \( q \)-analog of the Leibniz rule as follows:

\begin{align}
D_i(f(X_0, \ldots, X_i, \ldots, X_{r+1})g(X_0, \ldots, X_i, \ldots, X_{r+1})) \\
= D_i f(X_0, \ldots, X_i, \ldots, X_{r+1})g(X_0, \ldots, q^{-1}X_i, \ldots, X_{r+1}) \\
+ f(X_0, \ldots, qX_i, \ldots, X_{r+1})D_ig(X_0, \ldots, X_i, \ldots, X_{r+1}).
\end{align}

Let \( c_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j} \) for \( i, j \in \mathbb{Z} \). Recall that the quantum group \( U_q(sl_{r+2}) \) is generated by \( E_i, F_i, K_i^\pm(0 \leq i \leq r) \) satisfying the following relations:

\begin{align}
K_iK_i^{-1} &= K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i, \\
K_iE_jK_i^{-1} &= q^{\epsilon_{ij}}E_j, \quad K_iF_jK_i^{-1} = q^{-\epsilon_{ij}}F_j, \\
E_iF_j - F_jE_i &= \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
E_iE_j &= E_jE_i, \quad F_iF_j = F_jF_i, \quad \text{if } |i - j| > 1, \\
E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 &= 0, \quad \text{if } |i - j| = 1, \\
F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 &= 0, \quad \text{if } |i - j| = 1.
\end{align}

**Theorem 2.2** ([18, Theorem 3.2]). There is a \( \mathbb{Q}(q) \)-algebra homomorphism \( \chi_r : U_q(sl_{r+2}) \rightarrow A_q(A_{r+1}) \) such that

\begin{align}
E_i \mapsto X_iD_{i+1}, \quad F_i \mapsto X_{i+1}D_i, \quad K_i \mapsto M_iM_i^{-1}.
\end{align}

By Theorems 2.1 and 2.2, the polynomial ring \( P \) is a \( U_q(sl_{r+2}) \)-module via pull-back map. The polynomial ring \( P \) is called the oscillator representation of \( U_q(sl_{r+2}) \). Moreover, we have

**Theorem 2.3** ([18, Theorem 4.1]). Let \( s \geq 0 \). Then the subspace of \( P \) spanned by the set

\begin{align}
\{X_0^a X_1^{a_1} \cdots X_{r+1}^{a_r} | a_0 + a_1 + \cdots + a_{r+1} = s\}
\end{align}

is an irreducible \( U_q(sl_{r+2}) \)-module.
3. Modified $q$-Weyl algebra

In this section, we shall introduce the modified $q$-Weyl algebra associated to a given Satake diagram. Let $I$ be a finite set and assume $|I| \geq r + 2$. Denote by $D$ the Dynkin diagram of type $A_{|I|-1}$ or $A_{|I|-1}^{(1)}$. The nodes in $D$ are labeled by the elements in $I$. For $i, j \in I$, we define the Cartan datum $(I, \cdot)$ on $D$ as follows:

\[
i \cdot j = \begin{cases} 
2, & \text{if } i = j, \\
-k, & \text{if there are } k \text{ lines connecting } i \text{ and } j, \\
0, & \text{otherwise.}
\end{cases}
\]

Let $(Y, X, \langle \cdot, \cdot \rangle)$ be a root datum of type $(I, \cdot)$; cf. [27, Section 2.1]. Let $\tau$ be an involution of the Cartan datum $(I, \cdot)$ and $W$ be the Weyl group generated by simple reflections $s_i$ for $i \in I$. Given a subset $I_\circ \subset I$, let $W_{I_\circ} = \langle s_i \mid i \in I_\circ \rangle$ be the parabolic subgroup of $W$ with longest element $w_\circ$, and let $\rho_\circ$ (resp. $\rho^\vee_\circ$) be the half sum of all positive roots (resp. coroots) in the root system $R_{I_\circ}$ (resp. $R_{I_\circ}^\vee$).

Let $I_o = I \setminus I_\circ$. A Satake diagram $S$ is the pair $(I = I_o \cup I_\circ, \tau)$ such that $\tau(I_o) = I_\circ$, the actions of $\tau$ and $-w_\circ$ on $I_\circ$ coincide, and $\langle \rho_\circ ' , j' \rangle \in \mathbb{Z}$ if $\tau j = j \in I_o$. By abuse of notations, denote by $I$ the set of $\tau$-orbits of $I$.

In this paper, we only consider the case $I = I_o$. We use natural numbers \{0, 1, 2, \cdots\} to label the elements in $I$.

**Definition 3.1.** Given a Satake diagram $S$, the associated modified $q$-Weyl algebra, $A_q(S)$, is generated by $d_i, r_i, m_i \ (i \in I)$ over $\mathbb{Q}(q)$ subject to the following relations:

\[
\begin{align*}
(3.1) & \quad m_i m_i^{-1} = m_i^{-1} m_i = 1, \quad m_i m_j = m_j m_i, \\
(3.2) & \quad d_i m_j = m_j d_i, \quad r_i m_j = m_j r_i, \quad d_i r_j = r_j d_i, \quad \text{if } i \neq j, \\
(3.3) & \quad d_i d_j = d_j d_i, \quad r_i r_j = r_j r_i, \\
(3.4) & \quad d_i m_i = q^{\xi_i} m_i d_i, \quad r_i m_i = q^{-\xi_i} m_i r_i, \\
(3.5) & \quad d_i r_i = \frac{q^{\xi_i} m_i - q^{-\xi_i} m_i^{-1}}{q - q^{-1}}, \quad r_i d_i = \frac{m_i - m_i^{-1}}{q - q^{-1}},
\end{align*}
\]

where $\xi_i = 1 - i \cdot \tau i$.

In the rest of the paper, we shall consider the following Satake diagrams.

**Diagram I**: Satake diagram of type $A_{2r+3}$.

```
0 ─── 1 ─── \cdots ─── r ─── r + 1
\big\{ \big\} \big\{ \big\} \big\} \big\} \big\}
0 ─── 1 ─── \cdots ─── r + 1 ─── r + 2
\big\{ \big\} \big\} \big\} \big\} \big\}
2r + 3 2r + 2 \cdots r + 3 r + 2
```

**Diagram II**: Satake diagram of type $A_{2r+2,1}$.

```
0 ─── 1 ─── \cdots ─── r ─── r + 1
\big\{ \big\} \big\} \big\} \big\}
0 ─── 1 ─── \cdots ─── r + 2
\big\{ \big\} \big\} \big\} \big\}
2r + 2 2r + 1 \cdots r + 2
```
Diagram III: Satake diagram of type $A^{(1)}_{2r+3}$.

Diagram IV: Satake diagram of type $A^{(1)}_{2r+2,1}$.

Diagram V: Satake diagram of type $A^{(1)}_{1,2r+2}$.

Diagram VI: Satake diagram of type $A^{(1)}_{2r+1,2}$.

**Proposition 3.1.** Let $S$ be the Satake diagram in Diagram I-VI. Then there is a $Q(q)$-algebra homomorphism $\iota: A_q(S) \to A_q(A_{r+1})$ which sends

| $d_i$ | $x_i$ | $m_i$ | $d_i$ | $x_i$ | $m_i$ |
|-------|-------|-------|-------|-------|-------|
| $D_i$ | $X_i$ | $M_i^\pm$ | $-D_i$ | $X_i$ | $M_i^\pm$ |

if $\xi_i > 0$, then

$$\iota(d_i x_i) = D_i \left( \sum_{k=0}^{\xi_i-1} M_i^{\xi_i-1-2k} \right) X_i = D_i X_i \left( \sum_{k=0}^{\xi_i-1} q^{\xi_i-1-2k} M_i^{\xi_i-1-2k} \right) = \frac{q M_i - q^{-1} M_i^{-1}}{q - q^{-1}} \left( \sum_{k=0}^{\xi_i-1} q^{\xi_i-1-2k} M_i^{\xi_i-1-2k} \right) = \frac{q^{\xi_i} M_i^{\xi_i} - q^{-\xi_i} M_i^{-\xi_i}}{q - q^{-1}}$$

**Proof.** Since $d_i$, $x_i$, and $m_i$ are generators of $A_q(S)$, it’s enough to show that the map $\iota$ is well defined. The verification of relations (3.1)-(3.4) is straightforward. We shall only verify the relations in (3.5). If $\xi_i > 0$, by using the relations (2.4) and (2.5), we have
and Proposition 3.1.

\[
\frac{g^{\xi_i}(m_i) - q^{-\xi_i}(m_i^{-1})}{q - q^{-1}},
\]

and

\[
\iota(x,\partial_i) = \sum_{k=0}^{\xi_i-1} \frac{M_i^k - M_i^{-k}}{q - q^{-1}} = \frac{\xi_i - 1}{q - q^{-1}}.
\]

The case for \(\xi_i < 0\) can be shown similarly. \qed

**Remark 3.1.** The diagram constructed from Diagram I by neglecting the edge between \(r + 1\) and \(r + 2\) is called the Satake diagram of diagonal type. In this case, the modified q-Weyl algebra \(A_q(S)\) is exactly the q-Weyl algebra \(A_q(A_{r+1})\).

Let \(X = (X_0, \ldots, X_{r+1})\) and \(a = (a_i)_{i \in \mathbb{E}} \in \mathbb{Z}^{r+2}_{\geq 0}\). We denote \(X^a = X_0^{a_0} X_1^{a_1} \cdots X_{r+1}^{a_{r+1}}\). Let \(e_i\) be the \((r + 2)\)-tuple such that the \(i\)-th entry is 1 and 0 otherwise. We define a lexicographic order \(\triangleright_{\text{lex}}\) on the set \(\mathbb{Z}^{r+2}_{\geq 0}\) as follows.

For any \(a = (a_0, \ldots, a_{r+1})\), \(b = (b_0, \ldots, b_{r+1}) \in \mathbb{Z}^{r+2}_{\geq 0}\), \(a \triangleright_{\text{lex}} b\) if the leftmost nonzero entry of \(a - b \in \mathbb{Z}^{r+2}\) is positive.

**Theorem 3.2.** The polynomial ring \(\mathbb{P}\) is a \(A_q(S)\)-module which is irreducible if \(q\) is not a root of unity.

**Proof.** By Theorem 2.1 and Proposition 3.1, \(\mathbb{P}\) is a \(A_q(S)\)-module via pull-back. More precisely, the generators \(\partial_i, \tau_i\) and \(m_i\) in \(A_q(S)\) act on \(\mathbb{P}\) as follows:

\[
\partial_i X^a = [\xi_i a_i] X^{a-e_i}, \quad \tau_i X^a = X^{a+e_i}, \quad m_i X^a = q^{\xi_i a_i} X^a,
\]

where \(\partial_i X^a = 0\) if \(a_i = 0\).

Let \(\mathbb{P}'\) be a nonzero submodule of \(\mathbb{P}\). We choose a non-zero vector \(w = \sum c_a X^a \in \mathbb{P}'\) with \(c_a \in \mathbb{Q}(q)\). Let \((k_i)_{0 \leq i \leq r+1}\) be the maximum element in the set \(\{ a \in \mathbb{Z}_{\geq 0}^{r+2} | c_a \neq 0 \}\) with respect to the order \(\triangleright_{\text{lex}}\). It follows that

\[
\partial_0^{k_0} \partial_1^{k_1} \cdots \partial_{r+1}^{k_{r+1}} w = c_k \prod_{i=0}^{r+1} [k_i]^{\xi_i} X^0.
\]

Since the coefficient of \(X^0\) is not zero, we have \(X^0 \in \mathbb{P}'\). For any element \(v = X^a \in \mathbb{P}\) with \(a = (a_i)_{0 \leq i \leq r+1}\), we have \(t_0^{a_0} t_1^{a_1} \cdots t_{r+1}^{a_{r+1}} X^0 = v \in \mathbb{P}'\), i.e., \(\mathbb{P} \subseteq \mathbb{P}'\). Hence \(\mathbb{P} = \mathbb{P}'\). Since \(\mathbb{P}'\) is arbitrary, \(\mathbb{P}\) is irreducible. \qed

4. Differential operator approach to \(\mathfrak{g}\)-quantum groups

In this section, we shall show that there is an algebra homomorphism from \(\mathfrak{g}U(S)\) to \(A_q(S)\) for a quasi-split Satake diagram S. In other words, this provides a differential operator approach to \(\mathfrak{g}U(S)\). Let us first recall the definition of \(\mathfrak{g}U(S)\) from [6].

**Definition 4.1.** Given a Satake diagram \(S\), the associated \(\mathfrak{g}\)-quantum group \(\mathfrak{g}U(S)\) is the \(\mathbb{Q}(q)\)-algebra generated by \(B_i, H_i\) (\(i \in I\)), subjecting to the following relations

\[
H_i H_{ri} = 1, \quad H_i H_j = H_j H_i,
\]

\[
H_j B_i - q^{i \cdot \tau_j - i} B_i H_j = 0,
\]

\[
(4.1)
\]

\[
H_j B_i - q^{i \cdot \tau_j - i} B_i H_j = 0,
\]

\[
(4.2)
\]
For each case of $\Phi$, we shall give an explicit construction of $\Phi_S$, and show that it is an algebra homomorphism.

(1) If $S$ is $A_{2r+1}$ ($r \geq 0$) in Diagram I, the map $\Phi_S$ is given by

$$e_i \mapsto \tau_i \delta_{i+1}, \quad f_i \mapsto \tau_{i+1} \delta_i, \quad k_i \mapsto m_i m_{i+1}^{-1}.$$  

(2) If $S$ is $A_{2r+2,1}$ ($r \geq 0$) in Diagram II, the map $\Phi_S$ is given by

$$e_i \mapsto \tau_i \delta_{i+1}, \quad f_i \mapsto \tau_{i+1} \delta_i, \quad k_i \mapsto (-1)^{\delta_i \cdot r} m_i m_{i+1}^{-1} \delta_{i+1}, \quad t_{r+1} \mapsto \tau_{r+1} \delta_{r+1}.$$  

(3) If $S$ is $A_{2r+1}^{(1)}$ ($r \geq 1$) in Diagram III, the map $\Phi_S$ is given by

$$e_i \mapsto \tau_i \delta_{i+1}, \quad f_i \mapsto \tau_{i+1} \delta_i, \quad k_i \mapsto q^{-2\delta_i \cdot 0} m_i m_{i+1}^{-1}.$$  

(4) If $S$ is $A_{1}^{(1)}$ in Diagram III, the map $\Phi_S$ is given by

$$e_0 \mapsto \tau_0 \delta_1, \quad f_0 \mapsto \tau_1 \delta_0, \quad k_0 \mapsto q^{-1} m_0 m_1^{-1}.$$  

(5) If $S$ is $A_{2r+2,1}^{(1)}$ ($r \geq 0$) in Diagram IV, the map $\Phi_S$ is given by

$$e_i \mapsto \tau_i \delta_{i+1}, \quad f_i \mapsto \tau_{i+1} \delta_i, \quad k_i \mapsto (-1)^{\delta_i \cdot r} q^{-2\delta_i \cdot 0} m_i m_{i+1}^{-1} \delta_{i+1}, \quad t_{r+1} \mapsto \tau_{r+1} \delta_{r+1}.$$  

(6) If $S$ is $A_{1,2r+2}^{(1)}$ ($r \geq 0$) in Diagram V, the map $\Phi_S$ is given by

$$e_i \mapsto \tau_{i-1} \delta_i, \quad f_i \mapsto \tau_i \delta_{i-1}, \quad k_i \mapsto (-1)^{\delta_i \cdot 1} m_{i-1}^{-1} m_{i-1}^{-1}, \quad t_0 \mapsto \tau_0 \delta_0.$$
(7) If $S$ is $A_{2r+1,2}^{(1)}$ ($r \geq 1$) in Diagram VI, the map $\Phi_S$ is given by

\[
e_i \mapsto t_i \partial_i, \quad f_i \mapsto t_{i+1} \partial_i, \quad k_i \mapsto (-1)^{\delta_i} m_i m_{i+1}^{-(1)} t_i, \quad 0 \mapsto t_1 \partial_1, \quad t_{r+1} \mapsto t_r \partial_{r+1}.
\]

We shall only prove the cases (1)-(4), the other cases can be shown similarly.

(1) By the relations (4.2) and (4.5), we have

\[
\begin{align*}
\Phi_S&(k_i e_j k_i^{-1} = q e_i^{\delta_i} e_j, \quad k_i f_j k_i^{-1} = q^{-e_i^{\delta_i}} f_j, \\
e_i^2 f_r + f_r e_i^2 &= (q + q^{-1})(e_r f_r e_r - e_r (qk_r + q^{-1} k_r^{-1})), \\
f_r^2 e_r + e_r f_r^2 &= (q + q^{-1})(f_r e_r f_r - (qk_r + q^{-1} k_r^{-1}) f_r).
\end{align*}
\]

By the involution $\tau$ in Diagram I, it follows that $\xi_i = 2^{\delta_i}$. According to the definition of homomorphism $\Phi_S$ and relations (3.1)-(3.5), for $i = j = r$ in the relations (4.7), we have

\[
\begin{align*}
\Phi_S(k_r e_r k_r^{-1}) &= m_r m_{r+1}^{-1} t_r \partial_{r+1} m_{r+1}^{-1} t_{r+1} = q^3 \partial_r \partial_{r+1} = q^3 \Phi_S(e_r), \\
\Phi_S(k_r f_r k_r^{-1}) &= m_r m_{r+1}^{-1} t_r \partial_{r+1} m_{r+1}^{-1} t_{r+1} = q^3 \partial_r \partial_{r+1} = q^3 \Phi_S(f_r),
\end{align*}
\]

and the verification of other cases in the relations (4.7) is similar.

For the relation (4.8), we have

\[
\Phi_S(e_r^2 f_r) + \Phi_S(f_r e_r^2) = t_r \partial_r t_r^{-1} \partial_{r+1} t_{r+1} \partial_r t_{r+1} \partial_{r+1} t_{r+1}
\]

\[
= (q - q^{-1})^2 t_r \partial_{r+1}((m_r - m_{r+1}^{-1})(q^2 m_{r+1} - q^{-2} m_{r+1}^{-1}) + (q^2 m_r - q^{-2} m_r^{-1})(q^{-2} m_{r+1} - q^2 m_{r+1}^{-1}))
\]

\[
= (q - q^{-1})^2 t_r \partial_{r+1}((q^2 + 1)m_r m_{r+1}^{-1} + (q^2 - 1)m_{r+1}^{-1} m_r^{-1})
\]

\[
- (q^2 - q^4)m_r m_{r+1}^{-1} - (q^2 + q^4)m_{r+1}^{-1}
\]

\[
= (q + q^{-1})(q - q^{-1})^2 t_r \partial_{r+1}((q m_r m_{r+1} + q^{-1} m_r^{-1} m_{r+1}^{-1})
\]

\[
- (q + q(q - q^{-1})^2 m_r m_{r+1}^{-1} - q^{-1} + q^{-1}(q - q^{-1})^2 m_{r+1}^{-1} m_r^{-1})
\]

\[
= (q - q^{-1})^2 t_r \partial_{r+1}((q m_r m_{r+1}^{-1} - m_{r+1}^{-1})(q m_r - q^{-1} m_r^{-1}) - q m_r m_{r+1}^{-1} - q^{-1} m_{r+1}^{-1} m_r^{-1})
\]

\[
= (q + q^{-1})(q m_r m_{r+1}^{-1} t_r \partial_{r+1} m_{r+1}^{-1} t_{r+1} m_{r+1}^{-1} m_{r+1}^{-1} m_{r+1}^{-1}
\]

\[
= (q + q^{-1})(q m_r m_{r+1}^{-1} t_r \partial_{r+1} m_{r+1}^{-1} t_{r+1} m_{r+1}^{-1} m_{r+1}^{-1} m_{r+1}^{-1})
\]

\[
= (q + q^{-1})(q m_r m_{r+1}^{-1} t_r \partial_{r+1} m_{r+1}^{-1} t_{r+1} m_{r+1}^{-1} m_{r+1}^{-1} m_{r+1}^{-1})
\]

\[
= (q + q^{-1})(q m_r m_{r+1}^{-1} t_r \partial_{r+1} m_{r+1}^{-1} t_{r+1} m_{r+1}^{-1} m_{r+1}^{-1} m_{r+1}^{-1})
\]

(2) For $i, j \in \{r, r + 1\}$, the relations (4.4) and (4.6) can be converted to

\[
\begin{align*}
e_i^2 t_{r+1} + t_{r+1} e_i^2 &= [2] e_r t_{r+1} e_r, \\
f_i^2 t_{r+1} + t_{r+1} f_i^2 &= [2] f_r t_{r+1} f_r \\
e_i t_{r+1} &= t_{r+1} e_i, \quad f_i t_{r+1} = t_{r+1} f_i, \quad \text{if } i \neq r, \\
t_{r+1} e_i + e_i t_{r+1}^2 &= [2] t_{r+1} e_r t_{r+1} + e_r, \\
t_{r+1}^2 f_i + f_i t_{r+1}^2 &= [2] t_{r+1} f_r t_{r+1} + f_r.
\end{align*}
\]

By the involution $\tau$ in Diagram II, we have $\xi_i = (-1)^{\delta_i}$. For the relations (4.10) and (4.13), we have

\[
\Phi_S(e_i^2 t_{r+1} + f_i^2 t_{r+1})
\]

\[
= [2] \frac{m_{r+1} - m_{r+1}^{-1}}{q - q^{-1}} + [2] \frac{q^2 m_{r+1} - q^{-2} m_{r+1}^{-1}}{q - q^{-1}}
\]

\[
= (q + q^{-1}) \frac{2 q^2}{q - q^{-1}} \frac{m_{r+1} - m_{r+1}^{-1}}{q - q^{-1}} = [2] \Phi_S(e_r t_{r+1} e_r),
\]
\[ \Phi_S(t_{r+1}^2e_r) + \Phi_S(e_r t_{r+1}^2) \]
\[ = (q - q^{-1})^{-2}((m_{r+1} - m_{r+1})^2 + (q^{-1}m_{r+1} - qm_{r+1})^2)_{r,0} \partial_{r+1} \]
\[ = (q - q^{-1})^{-2}(1 + q^{-2})m_{r+1}^2 + (1 + q^2)m_{r+1}^2 - 4)_{r,0} \partial_{r+1} \]
\[ = (q - q^{-1})^{-2}(q + q^{-1})(m_{r+1} - m_{r+1})(q^{-1}m_{r+1} - qm_{r+1})r,0 \partial_{r+1} + \varepsilon,0 \partial_{r+1} \]
\[ = [2] \Phi_S(t_{r+1}e_r t_{r+1}) + \Phi_S(e_r). \]

(3) By the relation (4.5), we have

(4.15) \[ e_s f_j - f_j e_s = \delta_{ij} k_i - k_i^{-1} \quad \text{if} \quad (i, j) \neq (0, 0), (r, r), \]
(4.16) \[ e_0^2 f_0 + f_0 e_0^2 = (q + q^{-1})(e_0 f_0 e_0 - (q^0 + q^{-1})e_0), \]
(4.17) \[ e_r^2 f_r + f_r e_r^2 = (q + q^{-1})(e_r f_r e_r - (q^r + q^{-1})e_r), \]
(4.18) \[ f_0^2 e_0 + e_0 f_0^2 = (q + q^{-1})(f_0 e_0 f_0 - f_0(q^0 + q^{-1})), \]
(4.19) \[ f_r^2 e_r + e_r f_r^2 = (q + q^{-1})(f_r e_r f_r - (q^r + q^{-1})). \]

We provide a detailed argument for the relation (4.16). By the involution \( \tau \) in Diagram III, we have \( \xi_2 = 2 \xi_{0,0} + \xi_{r,r+1} \). Then

\[ \Phi_S(e_0^2 f_0) + \Phi_S(f_0 e_0^2) = \varepsilon_0 \partial_{0} \partial_{0} \partial_{1,0} \partial_{1,0} \partial_{0} + \varepsilon_1 \partial_{0} \partial_{0} \partial_{1,0} \partial_{1,0} \partial_{0} \]
\[ = (q - q^{-1})^{-2} \varepsilon_0 \partial_{0} ((m_0 - m_0^{-1})(q^0 m_1^{-1} - q^{-1}m_1^{-1}))(q^{-1}m_0 - q^0 m_0^{-1}) \]
\[ = (q - q^{-1})^{-2} \varepsilon_0 \partial_{0} ((q + q^{-1})m_0 m_1 - (q^{-1} + q^0) m_0^{-1} m_1 + (q + q^{-1}) m_0^{-1} m_1^{-1}) \]
\[ = (q - q^{-1})^{-2} (q + q^{-1}) \varepsilon_0 \partial_{0} (q^2 m_0 m_1 + q^{-2} m_0^{-1} m_1^{-1}) \]
\[ - (q^2 + (q - q^{-1})^2 q^2) m_0 m_1^{-1} - (q^2 + (q - q^{-1})^2 q^{-2}) m_0^{-1} m_1 \]
\[ = (q + q^{-1}) \varepsilon_0 \partial_{0} ((q + q^{-1})^{-2} (m_0 - m_0^{-1})(q^2 m_0 - q^{-2} m_0^{-1}))(q^2 m_0 m_1^{-1} + q^2 m_0^{-1} m_1) \]
\[ = (q + q^{-1}) \varepsilon_0 \partial_{0} (q^2 m_0 m_1 - q^2 m_0^{-1} m_1) \]
\[ = (q + q^{-1}) (\Phi_S(e_0 f_0 e_0) - q \Phi_S(k_0 e_0) - q^{-1} \Phi_S(k_0^{-1} e_0)). \]

(4) It follows by the relations (4.1), (4.2) and (4.5) that

(4.20) \[ k_0 k_0^{-1} = 1, \quad k_0 e_0 = q^4 e_0 k_0, \quad k_0 f_0 = q^4 f_0 k_0, \]
(4.21) \[ e_0^3 f_0 - [3] e_0^2 f_0 e_0 + [3] e_0 f_0 e_0^2 - f_0 e_0^3 = [3]!(q - q^{-1})e_0(k_0 - k_0^{-1})e_0, \]
(4.22) \[ f_0^3 e_0 - [3] f_0^2 e_0 f_0 + [3] f_0 e_0 f_0^2 - e_0 f_0^3 = [-3]!(q - q^{-1})f_0(k_0 - k_0^{-1})f_0. \]

We need to make slight modifications for the algebra \( \mathbf{A}_q(S) \) defined in Definition 3.1. Let \( r = 0 \) and \( \xi_0 = 3.1 \). We only verify the relation (4.21) and we have

\[ \Phi_S(e_0^3 f_0) - [3] \Phi_S(e_0^2 f_0 e_0) + [3] \Phi_S(e_0 f_0 e_0^2) - \Phi_S(f_0 e_0^3) \]
\[ = \varepsilon_0 \partial_{0} \partial_{0} \partial_{1} \partial_{0} \partial_{1} \partial_{0} \partial_{1} \partial_{0} + [3] \varepsilon_0 \partial_{1} \partial_{0} \partial_{1} \partial_{0} \partial_{1} \partial_{0} \partial_{1} \partial_{0} - \varepsilon_1 \partial_{0} \partial_{0} \partial_{1} \partial_{0} \partial_{1} \partial_{0} \partial_{1} \partial_{0} \]
\[ = (q - q^{-1})^{-2} \varepsilon_0 \partial_{0}^2 ((m_0 - m_0^{-1})(q^2 m_1 - q^{-3} m_1^{-1}) + [3](q^2 m_0 - q^{-2} m_0^{-1}))(q^2 m_0 - q^{-3} m_0^{-1}) \]
\[ + [3](q^2 m_0 - q^{-2} m_0^{-1})(q^{-3} m_1 - q^3 m_1^{-1}) - (q^6 m_1 - q^6 m_1^{-1})(q^{-3} m_0 - q^{-3} m_0^{-1})) \]
\[ = (q - q^{-1})^{-2} (q^2 m_0 m_1 - q^{-3} m_1^{-1}) - ((q^3 - q^{-3}) + [3](q^3 - q^{-3}) m_0 m_1^{-1}) \]
\[ - ((q^3 - q^{-3}) + [3](q^{-3} - q^3)) m_0 m_1^{-1} - (q^{-3} - q^3) + [3](q - q^{-1})) m_0 m_1^{-1} \]
\[ = [3]!(q - q^{-1}) \varepsilon_0 \partial_{0} (q^{-1} m_0 m_1^{-1} - q^0 m_0 m_1^{-1} \varepsilon_0 \partial_{1} \]
\[ = \ldots \]

= [3]!(q - q^{-1})\Phi_S(e_0(k_0 - k_0^{-1})e_0).

The other relations in $U(S)$ can be verified in a similar way. □

5. CRYSTAL BASIS OF THE OSCILLATOR REPRESENTATIONS

In this section, we study the irreducible modules of quantum groups and construct the crystal basis of these irreducible modules over quantum groups $U(A_{2r+1})$, $U(A_{2r+1}^{(1)})$ and $U(A_1^{(1)})$.

5.1. Irreducible modules of quantum groups. By Proposition 3.1 and Theorem 4.1, we have

Corollary 5.1. There exists a $\mathbb{Q}(q)$-algebra homomorphism $\Phi : U(S) \rightarrow A_q(A_{r+1})$.

From Theorems 3.2 and 4.1, it’s easy to see that $\mathbb{P}$ is a $U(S)$-module. The $U(S)$-action is given as follows.

1) For the quantum group $U(A_{2r+1})$ $(r \geq 0)$, the action is given by
\[ e_iX^a := [2\delta_i, r, a_{i+1}]X^{a+e_i - e_i+1}, \quad f_iX^a := [a_i]X^{a-e_i + e_i+1}, \quad k_iX^a := q^{\delta_i, r, a_{i+1}}X^a. \]

2) For the quantum group $U(A_{2r+2})$ $(r \geq 0)$, the action is given by
\[ e_iX^a := [(-1)\delta_i, r, a_{i+1}]X^{a+e_i - e_i+1}, \quad f_iX^a := [a_i]X^{a-e_i + e_i+1}, \quad k_iX^a := (-1)\delta_i, r, q^{a_{i+1}}X^a, \quad t_{r+1}X^a := -[a_{r+1}]X^a. \]

3) For the quantum group $U(A_{2r+1}^{(1)})$ $(r \geq 1)$, the action is given by
\[ e_iX^a := [2\delta_i, r, a_{i+1}]X^{a+e_i - e_i+1}, \quad f_iX^a := [2\delta_i, 0, a_i]X^{a-e_i + e_i+1}, \quad k_iX^a := q^{2\delta_i, r, a_{i+1} - 2\delta_i, 0}X^a, \]
\[ k_iX^a := q^{2\delta_i, r, a_{i+1} - 2\delta_i, 0}X^a, \quad k_iX^a := q^{2\delta_i, r, a_{i+1} - 2\delta_i, 0}X^a, \]
\[ k_iX^a := q^{2\delta_i, r, a_{i+1} - 2\delta_i, 0}X^a, \quad t_{r+1}X^a := -[a_{r+1}]X^a. \]

4) For the quantum group $U(A_1^{(1)})$, the action is given by
\[ e_iX^a := [3a_i]X^{a+e_0 - e_i}, \quad f_iX^a := [a_0]X^{a-e_0 + e_i}, \quad k_0X^a := q^{a_0 - 3a_1}X^a. \]

5) For the quantum group $U(A_{2r+1}^{(1)})$ $(r \geq 0)$, the action is given by
\[ e_iX^a := (-1)\delta_i, r, [a_{i+1}]X^{a+e_i - e_i+1}, \quad f_iX^a := [2\delta_i, 0, a_i]X^{a-e_i + e_i+1}, \quad k_iX^a := (-1)\delta_i, r, q^{2\delta_i, 0, a_{i+1} - 2\delta_i, 0}X^a, \quad t_{r+1}X^a := -[a_{r+1}]X^a. \]

6) For the quantum group $U(A_1^{(1)})$ $(r \geq 0)$, the action is given by
\[ e_iX^a := [2\delta_i, r, a_{i+1}]X^{a+e_i - e_i+1}, \quad f_iX^a := [(-1)\delta_i, 1, a_{i+1}]X^{a-e_i + e_i+1}, \quad k_iX^a := (-1)\delta_i, r, q^{a_{i+1} - 2\delta_i, 1}X^a, \quad t_0X^a := -[a_0]X^a. \]

7) For the quantum group $U(A_{2r+1}^{(1)})$ $(r \geq 1)$, the action is given by
\[ e_iX^a := (-1)\delta_i, r, [a_{i+1}]X^{a+e_i - e_i+1}, \quad f_iX^a := [a_i]X^{a-e_i + e_i+1}, \quad k_iX^a := (-1)\delta_i, r, q^{a_{i+1} - 2\delta_i, 0}X^a, \quad t_0X^a := [a_1]X^a, \quad t_{r+1}X^a := -[a_{r+1}]X^a. \]
The \( \mathcal{U}(S) \)-module \( P \) is called the oscillator representation of \( \mathcal{U}(S) \). The following commutative diagram explains the relations between \( \mathcal{U}(S) \), \( A_q(S) \), \( A_q(A_{r+1}) \) and \( P \).

\[
\mathcal{U}(S) \xrightarrow{\Phi_S} A_q(S) \xrightarrow{\iota} A_q(A_{r+1}) \xrightarrow{\cdot} \text{End}(P)
\]

For \( s \geq 0 \), let \( \Lambda_s = \{ a \in \mathbb{Z}_{\geq 0}^r \mid \sum_{i=0}^{r-1} a_i = s \} \). The polynomial ring \( P \) has the following direct sum decomposition

\[
P = \bigoplus_{s=0}^{\infty} P_s, \quad P_s = \bigoplus_{a \in \Lambda_s} \mathbb{K}X^a.
\]

Since the actions of \( e_i, f_i, k_i \) and \( t_j \) on \( P \) preserve the degree of arbitrary monomial in \( P_s \), the subspace \( P_s \) is a \( \mathcal{U}(S) \)-module.

**Theorem 5.1.** The \( \mathcal{U}(S) \)-module \( P_s \) is irreducible if \( q \) is not a root of unity.

**Proof.** Let \( P'_s \) be a non-zero submodule of \( P_s \). We fix a non-zero element \( X^a \in P'_s \). It follows that

\[
\left( \prod_{i=0}^{r} e_i^{\sum_{j=0}^{i} a_j} \right) X^a = \left( \prod_{i=0}^{r} e_i^{a_i} \right) X^a = \left( \prod_{i=1}^{r+1} a_i + a_{i+1} + \cdots + a_{r+1} \right) X^a \in P'_s.
\]

Since the coefficient is not zero, we get \( X^a \in P'_s \). Let \( X^b \) be any monomial in \( P \), where \( b = (b_i)_{0 \leq i \leq r+1} \). It follows that

\[
\left( \prod_{i=0}^{r} f_i^{s-(\sum_{j=0}^{i} b_j)} \right) X^a = \left( \prod_{i=0}^{r} f_i^{a_i} \right) X^a = \left( \prod_{i=1}^{r+1} s - (\sum_{j=0}^{i-1} b_j) \right) X^a \in P'_s.
\]

The relation \( P_s = P'_s \) holds automatically due to the non-zero coefficient of \( X^b \). Hence the \( \mathcal{U}(S) \)-module \( P_s \) is irreducible. \( \square \)

### 5.2. Crystal basis of irreducible module.

In this subsection, we only consider the quantum groups \( \mathcal{U}(A_{2r+1}), \mathcal{U}(A_{2r+1}^{(1)}) \) and \( \mathcal{U}(A_{r+1}^{(1)}) \).

Let \( X^a = \prod_{i=0}^{r+1} X_{i}^{(a_i)} \) for a sequence \( a \in \Lambda_s \). It's easy to verify:

\[
X^a = f^{(a_{i+1})}_{i+1} X^{(a_{i+1}(a_i-e_i+1))}.
\]

Let \( P_s = \bigoplus_{a \in \Lambda_s} P_a \), where \( P_a = \mathbb{Q}(q)X^a \). We define Kashiwara operators \( \bar{e}_i \) and \( \bar{f}_i \) \((0 \leq i \leq r)\) on \( P_s \) as follows:

\[
\bar{e}_i X^a = f_{i}^{(a_{i+1}-1)} X^{(a_{i+1}(a_i-e_i-e_{i+1}))}, \quad \bar{f}_i X^a = f_{i}^{(a_{i+1})} X^{(a_{i+1}(a_i-e_i-e_{i+1}))}.
\]

Let \( \Lambda_0 = \{ f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0 \} \). The crystal lattice and crystal basis of the irreducible module \( P_s \) are defined as follows.

**Definition 5.2.** Let \( L(s) \) be an \( \Lambda_0 \)-submodule of \( P_s \). We say that \( L(s) \) is a crystal lattice of \( P_s \) if

1. \( L(s) \) is a free \( \Lambda_0 \)-module of rank \( \dim_{\mathbb{Q}(q)} P_s \), and \( \mathbb{Q}(q) \otimes_{\Lambda_0} L(s) = P_s \),
2. \( \mathcal{L}(s) = \bigoplus_{a \in \Lambda_s} L(s)_a \), where \( L(s)_a := L(s) \cap P_a \),
3. \( \bar{f}_i(L(s)) \subset L(s) \) and \( \bar{e}_i(L(s)) \subset L(s) \).

**Definition 5.3.** A crystal basis of the module \( P_s \) is a pair \( (L(s), \mathcal{B}(s)) \) such that
(B1) \( \mathcal{L}(s) \) is a crystal lattice of \( \mathbb{P}_s \),

(B2) \( \mathcal{B}(s) \) is a \( \mathbb{Q} \)-basis of \( \mathcal{L}(s)/q\mathcal{L}(s) \),

(B3) \( \mathcal{B}(s) = \bigcup_{\lambda \in \Lambda_s} \mathcal{B}(s)_{\lambda} \), where \( \mathcal{B}(s)_{\lambda} := \mathcal{B}(s) \cap (\mathcal{L}(s)_{\lambda}/q\mathcal{L}(s)_{\lambda}) \),

(B4) \( \tilde{f}_i(\mathcal{B}(s)) \subset \mathcal{B}(s) \cup \{0\} \) and \( \tilde{e}_i(\mathcal{B}(s)) \subset \mathcal{B}(s) \cup \{0\} \),

(B5) for each \( b, b' \in \mathcal{B}(s) \), one has \( \tilde{f}_i(b) = b' \) if and only if \( b = \tilde{e}_i(b') \).

From Theorem 5.1, \( \mathbb{P}_s \) is an irreducible \( 'U(S) \)-module with \( \mathbb{Q}(q) \)-basis \( \{ X^\sigma | a \in \Lambda_s \} \). We define

\[
\mathcal{L}^*(s) = \bigoplus_{a \in \Lambda_s} \mathcal{L}^*(s)_a, \quad \mathcal{B}^*(s) = \{ X^\sigma + q\mathcal{L}^*(s) | a \in \Lambda_s \},
\]

where \( \mathcal{L}^*(s)_a = A_0 X^\sigma \).

**Theorem 5.2.** The pair \( (\mathcal{L}^*(s), \mathcal{B}^*(s)) \) is a crystal basis of \( \mathbb{P}_s \).

**Proof.** It’s straightforward to verify that the conditions (L1), (L2) and (L3) hold in Definition 5.2 due to the definition of Kashiwara operators in (5.1). Hence \( \mathcal{L}^*(s) \) is a crystal lattice of \( \mathbb{P}_s \). From the definition of \( \mathcal{B}^*(s) \) in (5.2), it’s easy to see \( \mathcal{B}^*(s) \) is a \( \mathbb{Q} \)-basis of \( \mathcal{L}^*(s)/q\mathcal{L}^*(s) \cong \mathbb{Q} \otimes_{A_0} \mathcal{L}^*(s) \). The conditions (B3) and (B4) are easy to verify. Now let us verify (B5):

Let \( b = X^\sigma \) and \( b' = f_i^{(a_{i+1}+1)_{i+1}} X^{(a_{i+1}+1)(a_{i-1}-e_{i+1})} \). Then we have \( \tilde{f}_i b = b' \).

\[
\tilde{e}_i b' = \tilde{e}_i f_i^{(a_{i+1}+1)_{i+1}} X^{(a_{i+1}+1)(a_{i-1}-e_{i+1})} = \tilde{e}_i f_i^{(a_{i+1}+1)_{i+1}} X^{(a_{i+1}+1)(a_{i-1}-e_{i+1})} = f_i^{(a_{i+1}+1)_{i+1}} X^{(a_{i+1}+1)(a_{i-1}-e_{i+1})}
\]

\[
= f_i^{(a_{i+1}+1)_{i+1}} X^{(a_{i+1}+1)(a_{i-1}-e_{i+1})} = X^\sigma = b.
\]

Hence \( (\mathcal{L}^*(s), \mathcal{B}^*(s)) \) is a crystal basis of \( \mathbb{P}_s \). \qed

**Example 5.3.** Let \( r = 1 \) and \( s = 3 \), we use \( (a_0 a_1 a_2) \) to represent \( X_0^{(a_0)_{i_0}} X_1^{(a_1)_{i_1}} X_2^{(a_2)_{i_2}} \). We use the red arrow to represent the action of \( \tilde{f}_0 \) and the blue arrow to represent the action of \( \tilde{f}_1 \). The crystal graph is described as follows:
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\[(300) \quad \rightarrow \quad (210)\]
\[(201) \quad \rightarrow \quad (201) \quad \rightarrow \quad (030)\]
\[(111) \quad \rightarrow \quad (102) \quad \rightarrow \quad (021)\]
\[(012) \quad \rightarrow \quad (012) \quad \rightarrow \quad (003)\]

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