Semiclassical parametrix for the Maxwell equation and applications to the electromagnetic transmission eigenvalues

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Abstract

We introduce an analog of the Dirichlet-to-Neumann map for the Maxwell equation in a bounded domain. We show that it can be approximated by a pseudodifferential operator on the boundary with a matrix-valued symbol and we compute the principal symbol. As an application, we obtain a parabolic region free of the transmission eigenvalues associated with the Maxwell equation.

Keywords: Maxwell equation, Semiclassical parametrix, Transmission eigenvalues

1 Introduction

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded, connected domain with a \( C^\infty \) smooth boundary \( \Gamma = \partial \Omega \), and consider the Maxwell equation

\[
\begin{cases}
\nabla \times E = i\lambda \mu(x)H & \text{in } \Omega, \\
\nabla \times H = -i\lambda \varepsilon(x)E & \text{in } \Omega, \\
v \times E = f & \text{on } \Gamma,
\end{cases}
\]

(1.1)

where \( \lambda \in \mathbb{C}, |\lambda| \gg 1, v = (v_1, v_2, v_3) \) denotes the Euclidean unit normal to \( \Gamma \), and \( \mu, \varepsilon \in C^\infty(\overline{\Omega}) \) are scalar-valued strictly positive functions. The functions \( E = (E_1, E_2, E_3) \in \mathbb{C}^3 \) and \( B = (B_1, B_2, B_3) \in \mathbb{C}^3 \) denote the electric and magnetic fields, respectively. Equation (1.1) describes the propagation of electromagnetic waves in \( \Omega \) with a frequency \( \lambda \) moving with a speed \((\varepsilon \mu)^{-1/2}\). Recall that given two vectors \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \), \( a \times b \) denotes the vector \((a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)\) and it is perpendicular to both \( a \) and \( b \). Thus, we have

\[
\nabla \times E = (\partial_{x_2}E_3 - \partial_{x_3}E_2, \partial_{x_3}E_1 - \partial_{x_1}E_3, \partial_{x_1}E_2 - \partial_{x_2}E_1)
\]

and similarly for \( \nabla \times H \). Throughout this paper, given \( s \in \mathbb{R} \) we will denote by \( \mathcal{H}_s \) the Sobolev space \( H^s(\Gamma; \mathbb{C}^3) \). Introduce the spaces

\[
\mathcal{H}^s_s := \{ f \in \mathcal{H}_s : \langle \nu(x), f(x) \rangle = 0 \}, \quad s = 0, 1,
\]

where \( \langle \nu, f \rangle := v_1f_1 + v_2f_2 + v_3f_3 \). In view of Theorem 3.1, we can introduce the operator

\[
\mathcal{N}(\lambda) : \mathcal{H}^1_1 \rightarrow \mathcal{H}^0_0
\]

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defined by
\[ N(\lambda)f = v \times H|_{\Gamma}, \]
which can be considered as an analog of the Dirichlet-to-Neumann map. Set \( h = |\operatorname{Re}\lambda|^{-1} \) if \( |\operatorname{Re}\lambda| \geq |\operatorname{Im}\lambda| \) and \( h = |\operatorname{Im}\lambda|^{-1} \) if \( |\operatorname{Im}\lambda| \geq |\operatorname{Re}\lambda| \). Then for every \( f \in \mathcal{H}, \) we have the estimate (1.2) provides a good approximation of the operator \( N(\lambda) \) as long as \( \theta \geq h^{1/3} \).

Our main result is the following:

**Theorem 1.1** Let \( \theta \geq h^{2/5} \) and \( 0 < h < 1 \), where \( 0 < \epsilon \ll 1 \) is arbitrary. Then for every \( f \in \mathcal{H}_1 \), we have the estimate
\[
\left\| N(\lambda)f - \text{Op}_h(m + h\bar{m})(v \times f) \right\|_{\mathcal{H}_0} \lesssim h^{\theta - 5/2} \| f \|_{\mathcal{H}_1} \tag{1.2}
\]
where \( \bar{m} = C^\infty(T^*\Gamma) \) is a matrix-valued function independent of \( h \), belonging to the space \( \operatorname{C}^0_{\Gamma} \) uniformly in \( z \) and such that \( \mu_0 \bar{m} \) is independent of \( \epsilon \) and \( \mu \).

Hereafter the Sobolev spaces are equipped with the \( h \)-semiclassical norm. Clearly, the estimate (1.2) provides a good approximation of the operator \( N(\lambda) \) as long as \( \theta \geq h^{2/5} \).

**Corollary 1.2** Let \( \theta \geq h^{2/5} \) and \( 0 < h < 1 \). Then for every \( f \in \mathcal{H}_1 \), we have the estimate
\[
\left\| N(\lambda)f \right\|_{\mathcal{H}_0} \lesssim \theta^{-1/2} \| f \|_{\mathcal{H}_1}. \tag{1.3}
\]

Note that analogous estimates for the Dirichlet-to-Neumann operator associated with the Helmholtz equation are proved in [9, 11] for \( \theta \geq h^{1/2} \), in [13] for \( \theta \geq h^{2/3} \) and in [10] for \( \theta \geq h^{2/3}, 0 < \epsilon \ll 1 \) being arbitrary. In the last case, it is assumed that the boundary is strictly concave. In all these papers, the approximation of the Dirichlet-to-Neumann map is used to get parabolic regions free of transmission eigenvalues.

To prove Theorem 1.1, we build in Sect. 4 a semiclassical parametrix near the boundary for the solutions to Eq. (1.1). It takes the form of oscillatory integrals with a complex-valued phase function \( \psi \) satisfying the eikonal equation modulo \( \mathcal{O}(x_1^N) \) (see (4.5)), where \( N \gg 1 \) is arbitrary and \( 0 < x_1 \ll 1 \) denotes the normal variable near the boundary, that is, the distance to \( \Gamma \). The amplitudes satisfy some kind of transport equations modulo \( \mathcal{O}(x_1^N) \) (see (4.2)). Thus, the parametrix satisfies the Maxwell equation modulo an error term which is
given by oscillatory integrals with amplitudes of the form $O(x_1^N) + O(h^N)$. To estimate the difference between the exact solution to Eq. (1.1) and its parametrix, we use the a priori estimate (3.5). Note that there exists a different approach suggested in [3] which could probably lead to (1.2) as well. It consists of using the results in [9,11] to approximate the normal derivatives $-ih\partial_r E|_{\Gamma}$ and $-ih\partial_n H|_{\Gamma}$ by $\text{Op}_h(\rho) E|_{\Gamma}$ and $\text{Op}_h(\rho) H|_{\Gamma}$. Thus, Eq. (1.1) can be reduced to a system of $h - \Psi$DOs on $\Gamma$ by restricting the equations in (1.1) on the boundary.

In analogy with the Helmholtz equation, Theorem 1.1 can be used to study the location on the complex plane of the transmission eigenvalues associated with the Maxwell equation (see Sect. 5). It can also be used to study the complex eigenvalues associated with the Maxwell equation with dissipative boundary conditions like that one considered in [3].

2 Preliminaries

We will first introduce the spaces of symbols which will play an important role in our analysis and will recall some basic properties of the $h - \Psi$DOs. Given $k \in \mathbb{R}, \delta_1, \delta_2 \geq 0$, we denote by $S^k_{\delta_1, \delta_2}$ the space of all functions $a \in C^\infty(T^*\Gamma)$, which may depend on the semiclassical parameter $h$, satisfying

$$\left| \partial_\xi^\alpha \partial_{\xi'}^\beta a(x', \xi', h) \right| \leq C_{\alpha, \beta} \langle \xi' \rangle^{k-\delta_1|\alpha|-\delta_2|\beta|}$$

for all multi-indices $\alpha$ and $\beta$, with constants $C_{\alpha, \beta}$ independent of $h$. More generally, given a function $\omega > 0$ on $T^*\Gamma$, we denote by $S^k_{\delta_1, \delta_2}(\omega)$ the space of all functions $a \in C^\infty(T^*\Gamma)$, which may depend on the semiclassical parameter $h$, satisfying

$$\left| \partial_\xi^\alpha \partial_{\xi'}^\beta a(x', \xi', h) \right| \leq C_{\alpha, \beta} \omega^{k-\delta_1|\alpha|-\delta_2|\beta|}$$

for all multi-indices $\alpha$ and $\beta$, with constants $C_{\alpha, \beta}$ independent of $h$ and $\omega$. Thus, $S^k_{\delta_1, \delta_2} = S^k_{\delta_1, \delta_2}(\langle \xi' \rangle)$. Given a matrix-valued symbol $a$, we will say that $a \in S^k_{\delta_1, \delta_2}$ if all entries of $a$ belong to $S^k_{\delta_1, \delta_2}$.

Also, given $k \in \mathbb{R}, 0 \leq \delta < 1/2$, we denote by $S^k$ the space of all functions $a \in C^\infty(T^*\Gamma)$, which may depend on the semiclassical parameter $h$, satisfying

$$\left| \partial_\xi^\alpha \partial_{\xi'}^\beta a(x', \xi', h) \right| \leq C_{\alpha, \beta} h^{-\delta(|\alpha|+|\beta|)} \langle \xi' \rangle^{k-|\beta|}$$

for all multi-indices $\alpha$ and $\beta$, with constants $C_{\alpha, \beta}$ independent of $h$. Again, given a matrix-valued symbol $a$, we will say that $a \in S^k$ if all entries of $a$ belong to $S^k$.

The $h - \Psi$DO with a symbol $a$ is defined by

$$(\text{Op}_h(a)f)(x') = (2\pi h)^{-2} \int e^{-\frac{i}{h} [x'-y', \xi']} a(x', \xi', h) f(y') d\xi' dy'.$$

If $a \in S^k_{\delta_1, \delta_2}$, then the operator $\text{Op}_h(a) : H^k_h(\Gamma) \to L^2(\Gamma)$ is bounded uniformly in $h$, where

$$\|u\|_{H^k_h(\Gamma)} := \|\text{Op}_h(\langle \xi' \rangle^k)u\|_{L^2(\Gamma)}.$$

It is also well known (e.g., see Section 7 of [4]) that, if $a \in S^0, 0 \leq \delta < 1/2$, then $\text{Op}_h(a) : H^k_h(\Gamma) \to H^k_h(\Gamma)$ is bounded uniformly in $h$. More generally, we have the following (see Section 2 of [9]):

**Proposition 2.1** Let $h^{\ell_+} a^{\pm} \in S^{\pm k}$, $0 \leq \delta < 1/2$, where $\ell_+ \geq 0$ are some numbers. Assume in addition that the functions $a^{\pm}$ satisfy

$$\left| \partial_\xi^{\alpha_1} \partial_{\xi'}^{\beta_1} a^+(x', \xi') \partial_\xi^{\alpha_2} \partial_{\xi'}^{\beta_2} a^-(x', \xi') \right| \leq \kappa C_{\alpha_1, \beta_1, \alpha_2, \beta_2} h^{-([\alpha_1]+|\beta_1|+|\alpha_2|+|\beta_2|)/2}$$

(2.1)
for all multi-indices $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $|\alpha_j| + |\beta_j| \geq 1$, $j = 1, 2$, with constants $C_{\alpha_1, \beta_1, \alpha_2, \beta_2} > 0$ independent of $h$ and $\kappa$. Then we have

$$\| \text{Op}_h(a^+)\text{Op}_h(a^-) - \text{Op}_h(a^+ a^-)\|_{L^2(\Gamma)} \lesssim h + \kappa. \quad (2.2)$$

Let $\eta \in C^\infty(T^*\Gamma)$ be such that $\eta = 1$ for $r_0 \leq C_0$, $\eta = 0$ for $r_0 \geq 2C_0$, where $C_0 > 0$ does not depend on $h$. It is easy to see (e.g., see Lemma 3.1 of [9]) that taking $C_0$ big enough we can arrange

$$C_1 \theta^{1/2} \leq |\rho| \leq C_2, \quad \text{Im} \ \rho \geq C_3|\theta||\rho|^{-1} \geq C_4|\theta|$$
on supp $\eta$, and

$$|\rho| \geq \text{Im} \ \rho \geq C_5|\xi'|$$
on supp$(1 - \eta)$ with some constants $C_j > 0$. We will say that a function $a \in C^\infty(T^*\Gamma)$ belongs to $S^k_{\delta_1, \delta_2}(\omega_1) + S^k_{\delta_3, \delta_4}(\omega_2)$ if $\eta a \in S^k_{\delta_1, \delta_2}(\omega_1)$ and $(1 - \eta)a \in S^k_{\delta_3, \delta_4}(\omega_2)$. It is shown in Section 3 of [9] (see Lemma 3.2 of [9]) that

$$\rho^k \in S^k_{\delta_2,22}(\rho) + S^k_{\delta_0,1}(\rho) \subset S^{k,\bar{k}}_{L,1}(\theta) + S^0_{\delta,1} \subset \theta^{2N}S^0_{1/2-\epsilon} + S^0_{\delta,1} \subset \theta^{2N}S^k_{1/2-\epsilon} \quad (2.3)$$
as long as $\theta \geq h^{1/2-\epsilon}$, uniformly in $\theta$ and $h$, where $\bar{k} = 0$ if $k \geq 0$, $\bar{k} = -k$ if $k \leq 0$ and $N \gg 1$ is arbitrary. Proposition 2.1 implies the following:

**Proposition 2.2** Let $h^{1/2-\epsilon} \leq \theta \leq 1$, $\ell_{\pm} \geq 0$, and let

$$a^\pm \in S^0_{\delta,1}(\theta_{\pm}) + S^k_{\delta,1} \subset \theta^{2N}S^k_{1/2-\epsilon}.$$  

Then we have

$$\left\| \text{Op}_h(a^+)\text{Op}_h(a^-) - \text{Op}_h(a^+a^-) \right\|_{H^\ell(\Gamma) \rightarrow L^2(\Gamma)} \lesssim \frac{h\theta^{1-\ell,\epsilon}}{\theta^{1-\ell,\epsilon}} \quad (2.4)$$

where $k = k_+ + k_- - 1$.

**Proof** Let $\eta_0, \eta_1, \eta_2 \in C^\infty_0(T^*\Gamma)$ be such that $\eta_1 = 1$ on supp $\eta$, $\eta_2 = 1$ on supp $\eta_1$, $\eta = 1$ on supp $\eta_0$. Then we have

$$\text{Op}_h(a^+ a^-) - \text{Op}_h(\eta a^+ \eta a^-)\text{Op}_h(\eta_2) - \text{Op}_h((1 - \eta)a^+(1 - \eta_0)a^-)$$

$$= \text{Op}_h(\eta a^+ \eta a^-)\text{Op}_h(1 - \eta_2) = O(h\infty) : H^\ell(\Gamma) \rightarrow L^2(\Gamma),$$

$$\text{Op}_h(a^+)\text{Op}_h(a^-) - \text{Op}_h((1 - \eta)a^+ \eta a^-)\text{Op}_h(1 - \eta_2) - \text{Op}_h((1 - \eta)a^+)\text{Op}_h((1 - \eta_0)a^-)$$

$$= \text{Op}_h(\eta a^+)\text{Op}_h((1 - \eta_1)a^-) + \text{Op}_h((1 - \eta)a^+)\text{Op}_h(\eta_0 a^-)$$

$$+ \text{Op}_h(\eta a^+)\text{Op}_h(\eta_1 a^-)\text{Op}_h(1 - \eta_2) = O(h\infty) : H^\ell(\Gamma) \rightarrow L^2(\Gamma).$$

By assumption, $\eta a^+ \in S^{-\ell^+}_{\ell^+,1}(\theta_+)$, $\eta a^- \in S^{-\ell^-}_{\ell^-,1}(\theta_-)$, which implies that the functions $\eta a^+$ and $\eta_1 a^-$ satisfy the condition (2.1) with $\kappa = h\theta^{1-\ell,\epsilon}_{\ell^+_+} \theta^{1-\ell,\epsilon}_{\ell^-_-}$. Therefore, by (2.2) we have

$$\left\| \left( \text{Op}_h(\eta a^+ \eta a^-)\text{Op}_h(\eta a^-) \right)\text{Op}_h(\eta_2)\text{Op}_h(f) \right\|_{L^2} \lesssim h\theta^{1-\ell,\epsilon}_{\ell^+_+} \theta^{1-\ell,\epsilon}_{\ell^-_-} \left\| \text{Op}_h(\eta_2)\text{Op}_h(f) \right\|_{L^2} \lesssim h\theta^{1-\ell,\epsilon}_{\ell^+_+} \theta^{1-\ell,\epsilon}_{\ell^-_-} \left\| f \right\|_{H^\ell(\Gamma)}.$$ 

On the other hand, $(1 - \eta)a^+ \in S^k_{\ell^+,1}(1 - \eta_0)a^- \in S^k_{\ell^-,1}$. The standard pseudodifferential calculus gives that, mod $O(h\infty)$, the operator

$$\text{Op}_h((1 - \eta)a^+ (1 - \eta_0)a^-) - \text{Op}_h((1 - \eta)a^+)\text{Op}_h((1 - \eta_0)a^-)$$


is an $h - \Psi$DO with symbol $\omega \in S^k_{0,1}$ uniformly in $h$, where $k = k_+ + k_- - 1$. Therefore,
\[
\| \text{Op}_h((1 - \eta)a^+(1 - \eta_0)a^-)f - \text{Op}_h((1 - \eta_0)a^-)\text{Op}_h((1 - \eta)a^+)f \|_{L^2} \lesssim h\| f \|_{H_0^1}.
\]
Clearly, (2.4) follows from the above estimates. □

Proposition 2.1 also implies the following:

**Proposition 2.3** Let $h^{1/2-\epsilon} \leq \theta \leq 1$, $\ell \geq 0$, and let
\[
a \in S^{-\ell}(\theta) + S^k_{0,1} \subset \theta^{-\ell}S^k_{1/2-\epsilon}.
\]

Then we have
\[
\| \text{Op}_h(a) \|_{H_0^k(\Gamma) \rightarrow L^2(\Gamma)} \lesssim \theta^{-\ell}.
\] (2.5)

Note that these propositions remain valid for matrix-valued symbols.

We will next write the gradient $\nabla$ in the local normal geodesic coordinates near the boundary (see also Section 2 of [3]). Fix a point $y^0 \in \Gamma$ and let $\mathcal{U} \subset \mathbb{R}^3$ be a small neighbourhood of $y^0$. Let $\mathcal{U}_0$ be a small neighbourhood of $x' = 0$ in $\mathbb{R}^2$ and let $x' = (x_2, x_3)$ be local coordinates in $\mathcal{U}_0$. Then there exists a diffeomorphism $s : \mathcal{U}_0 \rightarrow \mathcal{U} \cap \Gamma$. Let $y = (y_1, y_2, y_3) \in \mathcal{U} \cap \Omega$, denote by $y' \in \Gamma$ the closest point from $y$ to $\Gamma$ and let $v(y')$ be the unit inner normal to $\Gamma$ at $y'$. Set $s_1 = \text{dist}(y, \Gamma)$, $x' = s^{-1}(y')$ and $v(x') = v(s(x')) = (v_1(x'), v_2(x'), v_3(x'))$. We have
\[
y = s(x') + x_1 v(x'),
\]
and hence,
\[
\frac{\partial}{\partial y_j} = v_j(x') \frac{\partial}{\partial x_1} + \sum_{k=2}^3 a_{j,k}(x) \frac{\partial}{\partial x_k},
\]
where $a_{j,k} = \frac{\partial x_k}{\partial y_j}$, provided $x_1$ is small enough. Note that the matrix $\left( \frac{\partial x_k}{\partial y_j} \right)$, $1 \leq k, j \leq 3$, is the inverse of $\left( \frac{\partial y_k}{\partial x_j} \right)$, $1 \leq k, j \leq 3$. In particular, this implies the identities
\[
\sum_{j=1}^3 v_j(x') a_{j,k}(x) = 0, \quad k = 2, 3.
\]
Set $\zeta_1 = (1, 0, 0)$, $\zeta_2 = (0, 1, 0)$, $\zeta_3 = (0, 0, 1)$. Clearly, we can write the Euclidean gradient $\nabla = (\partial_{y_1}, \partial_{y_2}, \partial_{y_3})$ in the coordinates $x = (x_1, x')$ as
\[
\nabla = \gamma(x)\nabla_x = v(x') \frac{\partial}{\partial x_1} + \sum_{k=2}^3 \gamma(x) \zeta_k \frac{\partial}{\partial x_k},
\]
where $\gamma$ is a smooth matrix-valued function such that $\gamma(x)\zeta_1 = v(x')$, $\gamma(x)\zeta_k = (\alpha_{1,k}, \alpha_{2,k}, \alpha_{3,k})$, $k = 2, 3$. Notice that the above identities can be rewritten in the form
\[
(v(x'), \gamma(x)\zeta_k) = 0, \quad k = 2, 3.
\] (2.6)
Let $(\xi_1, \xi') = (\xi_2, \xi_3)$, be the dual variable of $(x_1, x')$. Then the symbol of the operator $-i\nabla|_{x_1=0}$ in the coordinates $(x, \xi)$ takes the form $\xi_1 v(x') + \beta(x', \xi')$, where
\[
\beta(x', \xi') = \sum_{k=2}^3 \xi_k \gamma(0, x') \zeta_k.
\]
Thus, we get that the principal symbol of $-\Delta|_{x_1=0}$ is equal to 
\[ \xi_1^2 + (\beta(x', \xi'), \beta(x', \xi')). \]
This implies that the principal symbol, $r_0(x', \xi')$, of the positive Laplace–Beltrami operator on $\Gamma$ is equal to 
\[ \langle \beta(x', \xi'), \beta(x', \xi') \rangle. \]
Note also that (2.6) implies the identity 
\[ \langle v(x'), \beta(x', \xi') \rangle = 0 \] (2.7)
for all $(x', \xi')$.

In what follows, in this section we will solve the linear system 
\[
\begin{align*}
\psi_0 \times a - z \mu_0 b &= a^\flat, \\
\psi_0 \times b + z \varepsilon_0 a &= b^\flat, \\
v \times a &= g,
\end{align*}
\] (2.8)
where $\psi_0 = \rho v - \beta$ and $\langle g, v \rangle = 0$. To this end, we rewrite it in the form 
\[
\begin{align*}
\beta \times a + z \mu_0 b &= \rho g - a^\flat, \\
\rho v \times b - \beta \times b + z \varepsilon_0 a &= b^\flat, \\
v \times a &= g.
\end{align*}
\] (2.9)
Using the identity $-\beta \times (\beta \times a) = \langle \beta, \beta \rangle a - \langle \beta, a \rangle \beta$, we obtain 
\[
z \rho \mu_0 v \times b = z \mu_0 \beta \times b - z^2 \varepsilon_0 \mu_0 a + z \mu_0 b^\flat
\]
\[
= -\beta \times (\beta \times a) - z^2 \varepsilon_0 \mu_0 a + \beta \times (\rho g - a^\flat) + z \mu_0 b^\flat
\]
\[
= (\langle \beta, \beta \rangle - z^2 \varepsilon_0 \mu_0) a - (\beta, a) \beta + \beta \times (\rho g - a^\flat) + z \mu_0 b^\flat
\]
\[
= (r_0 - z^2 \varepsilon_0 \mu_0) a - (\beta, a) \beta + \beta \times (\rho g - a^\flat) + z \mu_0 b^\flat
\]
\[
= -\rho^2 a - (\beta, a) \beta + \beta \times (\rho g - a^\flat) + z \mu_0 b^\flat.
\]
Taking the scalar product of this identity with $v$ and using that $\langle v, \beta \rangle = 0$ and $\langle v, v \times b \rangle = 0$, we get 
\[ \langle v, a \rangle = \rho^{-1} \langle v, \beta \times g \rangle - \rho^{-2} \langle \beta \times a^\flat, v \rangle + z \mu_0 \rho^{-2} \langle b^\flat, v \rangle. \]
On the other hand, $a_t = a - \langle v, a \rangle v$ satisfies $v \times a_t = v \times a = g$. Hence, 
\[ v \times g = v \times (v \times a_t) = -\langle v, v \rangle a_t + \langle v, a_t \rangle v = -a_t. \]
Thus, we find 
\[ a = -v \times g + \rho^{-1} \langle v, \beta \times g \rangle v - \rho^{-2} \langle \beta \times a^\flat, v \rangle v + z \mu_0 \rho^{-2} \langle b^\flat, v \rangle v, \]
\[ z \mu_0 b = \rho g + \beta \times (v \times g) - \rho^{-1} (v, \beta \times g) \beta \times v\]
\[ -a^\flat + \rho^{-2} \langle \beta \times a^\flat, v \rangle \beta \times v - z \mu_0 \rho^{-2} \langle b^\flat, v \rangle \beta \times v, \]
\[ z \mu_0 v \times b = -\rho a + \beta \times g + \rho^{-1} \langle \beta, v \times g \rangle \beta - \rho^{-1} \beta \times a^\flat + z \mu_0 b^\flat
\]
\[ = \rho v \times g + \beta \times g - (v, \beta \times g) \beta - \rho^{-1} (v, \beta \times g) \beta\]
\[ = -\rho^{-1} \beta \times a^\flat + \rho^{-1} \langle \beta \times a^\flat, v \rangle v + z \mu_0 b^\flat - z \mu_0 \rho^{-1} (b^\flat, v). \]
Since $\langle v, g \rangle = 0$ and $\langle v, \beta \rangle = 0$, we have 
\[ \beta \times g - \langle v, \beta \times g \rangle v = 0. \]
Thus, we obtain
\[ z\mu_0 v \times b = \rho v \times g + \rho^{-1} \langle \beta, v \times g \rangle \beta - \rho^{-1} \beta \times a^2 + \rho^{-1} (\beta \times a^2, v) v + z\rho^{-1} \mu_0 b^2 - z\rho^{-1} \mu_0 (b^2, v) v. \]

### 3 A priori estimates

Let \( \tilde{f} \in \mathcal{H} \) and let the functions \( U_1, U_2 \in L^2(\Omega; \mathbb{C}^3) \) be such that \( \text{div} \ U_1, \text{div} \ U_2 \in L^2(\Omega) \), \( u_1 := \langle v, U_1 |_\Gamma \rangle \in L^2(\Gamma) \). In this section, we will prove a priori estimates for the restrictions on the boundary of the solutions \( E \) and \( H \) to the Maxwell equation
\begin{equation}
\begin{cases}
\mathbf{h} \nabla \times E = iz\mu(x)H + U_1 & \text{in } \Omega, \\
\mathbf{h} \nabla \times H = -iz\varepsilon(x)E + U_2 & \text{in } \Omega, \\
v \times E = \tilde{f} & \text{on } \Gamma.
\end{cases}
\end{equation}

Since \( \langle \nabla, \nabla \times E \rangle = \langle \nabla, \nabla \times H \rangle = 0 \), the solutions to (3.1) must satisfy the equation
\begin{equation}
\begin{cases}
\langle \nabla, E \rangle = (iz\varepsilon)^{-1} \langle \nabla, U_2 \rangle - \varepsilon^{-1} \langle \nabla \varepsilon, E \rangle & \text{in } \Omega, \\
\langle \nabla, H \rangle = -(iz\mu)^{-1} \langle \nabla, U_1 \rangle - \mu^{-1} \langle \nabla \mu, H \rangle & \text{in } \Omega.
\end{cases}
\end{equation}

To simplify the notations, in what follows we will denote by \( \| \cdot \| \) (resp. \( \| \cdot \|_0 \)) the norm on \( L^2(\Omega; \mathbb{C}^3) \) (resp. \( L^2(\Gamma; \mathbb{C}^3) \)) or on \( L^2(\Omega) \) (resp. \( L^2(\Gamma) \)). We also set \( Y = (E, H), U = (U_1, U_2) \), and define the norms \( \| Y \|, \| U \| \) and \( \| \text{div} \ U \| \) by
\begin{align*}
\| Y \|^2 &= \| E \|^2 + \| H \|^2, \\
\| U \|^2 &= \| U_1 \|^2 + \| U_2 \|^2, \\
\| \text{div} \ U \|^2 &= \| \text{div} \ U_1 \|^2 + \| \text{div} \ U_2 \|^2.
\end{align*}

By the Gauss divergence theorem, we have the identity
\begin{equation}
\int_{\Omega} \langle E, \nabla \times \overline{H} \rangle - \int_{\Omega} \langle \overline{H}, \nabla \times E \rangle = \int_{\Gamma} \langle \overline{H} \times E, v \rangle.
\end{equation}

We will use (3.3) to prove the following:

**Theorem 3.1** Let \( \theta > 0 \) and \( 0 < \eta \ll 1 \). Suppose that \( E \) and \( H \) satisfy Eq. (3.1) with \( U_1 = U_2 = 0 \). Then the functions \( f = E|_\Gamma, g = H|_\Gamma \) satisfy the estimate
\begin{equation}
\| f \|_{\mathcal{T}_0} + \| g \|_{\mathcal{T}_0} \lesssim \theta^{-1}\| \tilde{f} \|_{\mathcal{T}_1}.
\end{equation}

Suppose that \( E \) and \( H \) satisfy Eq. (3.1) with \( \tilde{f} = 0 \). Then the functions \( f = E|_\Gamma, g = H|_\Gamma \) satisfy the estimate
\begin{equation}
\| f \|_{\mathcal{T}_0} + \| g \|_{\mathcal{T}_0} \lesssim \| u_1 \|_0 + \eta\theta^{-1}\| U \| + \eta^{1/2}\| \text{div} \ U \|.
\end{equation}

**Proof** We decompose the vector-valued functions \( f \) and \( g \) as \( f = f_t + f_n, g = g_t + g_n \), where \( f_t = \langle v, f \rangle v, g_n = \langle v, g \rangle v \). Clearly, we have the identities \( \langle f_t, f_n \rangle = \langle g_t, g_n \rangle = 0 \) and \( v \times f = \nabla \times f_t, v \times g = \nabla \times g_t, f_t = -v \times (v \times f), g_t = -v \times (v \times g) \). Applying (3.3) to the solutions of Eq. (3.1) leads to the identity
\begin{equation}
\int_{\mathcal{E}} \langle \varepsilon | E \rangle^2 - iz \int_{\Omega} \mu | H |^2 = \int_{\Omega} \langle \overline{H}, U_1 \rangle - \int_{\Omega} \langle E, \overline{U}_2 \rangle + \eta \int_{\Gamma} \langle \overline{H} \times f_t, v \rangle.
\end{equation}

Taking the real part yields the estimate
\begin{equation}
\| Y \|^2 \lesssim \theta^{-2}\| U \|^2 + \theta^{-1}\| g_t \|_0 \| f_t \|_0.
\end{equation}
By Eq. (3.2), we also have
\[
|\langle \nabla, E \rangle | + |\langle \nabla, H \rangle | \lesssim |\text{div } U| + |Y|.
\] (3.7)

Restricting the first equation of (3.1) on $\Gamma$ and taking the scalar product with $v$ leads to the estimate
\[
\|g_\nu\|_0 = \| (v, g) \|_0 \lesssim \| (v, h\nabla \times E) \|_0 + \| u_1 \|_0.
\] (3.8)

In the normal coordinates $(x_1, x')$, $x' \in \Gamma$, the gradient takes the form $\nabla = \gamma \nabla_{x_1} + \gamma \nabla_{x'}$, where $\overline{v} = (1, 0, 0)$ and $\overline{\nabla}_{x'} = (0, \nabla_{x'})$. So, we have
\[
\nabla |_{x_1=0} = \gamma_0 \nabla_{x_1} + \gamma_0 \nabla_{x'}, \quad \gamma_0(x') = \gamma(0, x').
\]

Hence,
\[
\langle v, h\nabla \times E \rangle |_{\Gamma} = h \langle v, v \times \partial_{x_1} E |_{x_1=0} \rangle + \langle v, h\gamma_0 \nabla_{x'} \times E |_{x_1=0} \rangle
\]
\[
= \langle v, h\gamma_0 \nabla_{x'} \times f \rangle = \langle v, h\gamma_0 \nabla_{x'} \times f_\nu \rangle + h \langle v, \gamma_0 \nabla_{x'} \times f_\nu \rangle.
\]

On the other hand,
\[
\langle v, \gamma_0 \nabla_{x'} \times f_\nu \rangle = \langle v, f \rangle \langle v, \gamma_0 \nabla_{x'} \times v \rangle + \langle v, \gamma_0 \nabla_{x'} (\langle v, f \rangle) \times v \rangle
\]
\[
= \langle v, f \rangle \langle v, \gamma_0 \nabla_{x'} \times v \rangle.
\]

Therefore, (3.8) gives
\[
\|g_\nu\|_0 \lesssim \|f\|_{H_1} + \|u_1\|_0 + \|h\|f\|_0.
\] (3.9)

We will now bound the norms of $f_\nu$ and $g_\nu$. Let the function $\phi_0 \in C_0^\infty(\mathbb{R})$ be such that $\phi_0(\sigma) = 1$ for $|\sigma| \leq 1$, $\phi_0(\sigma) = 0$ for $|\sigma| \geq 2$, and set $\phi(\sigma) = \phi_0(\sigma/\delta)$, where $0 < \delta \ll 1$.

Then the functions $Y^\nu := (E^\nu, H^\nu) = (\phi(x_1)E, \phi(x_1)H)$ satisfy equation
\[
\begin{cases}
  h(\gamma \nabla_{x_1} + \gamma \nabla_{x'}) \times E^\nu = i\epsilon_0 \mu H^\nu + U_1^\nu & \text{in } \Omega, \\
  h(\gamma \nabla_{x_1} + \gamma \nabla_{x'}) \times H^\nu = -i\epsilon \mu E^\nu + U_2^\nu & \text{in } \Omega,
\end{cases}
\]
(3.10)

where $U^\nu := (U_1^\nu, U_2^\nu)$ satisfy $\|U^\nu\|_0 \lesssim \|U\|_0 + \|h\|Y\|_0$. By (3.7), the functions
\[
p = \langle \gamma \overline{v}, \partial_{x_1} E^\nu \rangle + \langle \gamma \overline{\nabla}_{x'} E^\nu \rangle, \quad q = \langle \gamma \overline{v}, \partial_{x_1} H^\nu \rangle + \langle \gamma \overline{\nabla}_{x'} H^\nu \rangle,
\]

satisfy
\[
|p| + |q| \lesssim |\text{div } U| + |Y|.
\] (3.11)

Denote by $\langle \cdot, \cdot \rangle_0$ the scalar product in $L^2(\Gamma; \mathbb{C}^3)$ or in $L^2(\Gamma)$, that is,
\[
\langle a, b \rangle_0 = \int_\Gamma \langle a, \overline{b} \rangle \quad \text{if } a, b \in L^2(\Gamma; \mathbb{C}^3),
\]
\[
\langle a, b \rangle_0 = \int_\Gamma a \overline{b} \quad \text{if } a, b \in L^2(\Gamma).
\]

Introduce the functions
\[
F_1(x_1) = \|\gamma \overline{v} \times E^\nu\|_0^2 - \|\langle \gamma \overline{v}, E^\nu \rangle\|_0^2,
\]
\[
F_2(x_1) = \|\gamma \overline{v} \times H^\nu\|_0^2 - \|\langle \gamma \overline{v}, H^\nu \rangle\|_0^2.
\]
Since \( v = \gamma_0 \tilde{v} = \gamma \tilde{v}_{x_1=0} \), we have

\[
F_1(0) = \|f_1\|_0^2 - \|f_0\|_0^2, \quad F_2(0) = \|g_1\|_0^2 - \|g_0\|_0^2.
\]

Using Eq. (3.10), we will calculate the first derivatives \( F'_j(x_1) \). In view of (3.11), we get

\[
F'_j(x_1) = 2 \text{Re} \left\{ \gamma \tilde{v} \times \partial_{x_1} E^j, \gamma \tilde{v} \times E^j \right\}_0 + 2 \text{Re} \left\{ \gamma \tilde{v} \times E^j, \gamma \tilde{v} \times E^j \right\}_0 - 2 \text{Re} \left\{ \left( \gamma \tilde{v}, \partial_{x_1} E^j \right), \left( \gamma \tilde{v}, E^j \right) \right\}_0 - 2 \text{Re} \left\{ \left( \gamma \tilde{v}, E^j \right), \left( \gamma \tilde{v}, E^j \right) \right\}_0
\]

\[
= -2 \text{Re} \left\{ \gamma \tilde{v} \times E^j, \gamma \tilde{v} \times E^j \right\}_0 + 2 h^{-1} \text{Re} \left\{ \left( i z H^j + U_1^j, \gamma \tilde{v} \times E^j \right) \right\}_0 + 2 \text{Re} \left\{ \left( \gamma \tilde{v} \times E^j, \gamma \tilde{v}, E^j \right) \right\}_0 - 2 \text{Re} \left\{ \left( \gamma \tilde{v}, \gamma \tilde{v}, E^j \right) \right\}_0 + R
\]

with a remainder term \( R \) satisfying the estimate

\[
|R| \lesssim h^{-1} \| Y \|_0^2 + h^{-1} \| U \|_0^2 + \| E \|_0^2 \| \text{div} U_1 \|_0.
\]

Clearly, we have a similar expression for \( F'_j(x_1) \) as well. Let us see now that

\[
\text{Re} \left\{ \gamma \tilde{v} \times E^j, \gamma \tilde{v} \times E^j \right\}_0 - \text{Re} \left\{ \left( \gamma \tilde{v} \times E^j, \gamma \tilde{v}, E^j \right) \right\}_0 = O \left( \| E \|_0^2 \right).
\]

(3.12)

It suffices to check (3.12) at a symbol level. Let \( \tilde{\xi}' = (0, \xi') \) denote the symbol of \(-i \tilde{v} \times \vec{E} \). We have the identity

\[
\left\{ \gamma \tilde{v} \times E^j, \gamma \tilde{v} \times E^j \right\}_0 = \left\{ \gamma \tilde{v}, \gamma \tilde{v} \right\}_0 \left\{ E^j, E^j \right\}_0 - \left\{ E^j, \gamma \tilde{v} \right\}_0 \left\{ \gamma \tilde{v}, E^j \right\}_0 = - \left\{ E^j, \gamma \tilde{v} \right\}_0 \left\{ \gamma \tilde{v}, E^j \right\}_0,
\]

where we have used that \( \left\{ \gamma \tilde{v}, \gamma \tilde{v} \right\}_0 = 0 \) [see (2.6)]. Hence,

\[
\text{Im} \left\{ \gamma \tilde{v} \times E^j, \gamma \tilde{v} \times E^j \right\}_0 - \text{Im} \left\{ \left( \gamma \tilde{v} \times E^j, \gamma \tilde{v}, E^j \right) \right\}_0 = 0,
\]

which clearly implies (3.12). Thus, we conclude

\[
|F'_1(x_1)| + |F'_2(x_1)| \lesssim h^{-1} \| Y \|_0^2 + h^{-1} \| U \|_0^2 + h \| \text{div} U \|_0^2.
\]

(3.13)

Since

\[
F_j(0) = - \int_0^{25} F'_j(x_1) dx_1,
\]

we deduce from (3.13),

\[
|F_1(0)| + |F_2(0)| \lesssim h^{-1} \| Y \|_0^2 + h^{-1} \| U \|_0^2 + h \| \text{div} U \|_0^2.
\]

(3.14)

By (3.6) and (3.14),

\[
\|f_0\|_0^2 + \|g_0\|_0^2 \lesssim \|f_1\|_0^2 + \|g_1\|_0^2 + \|f_2\|_0 \|g_2\|_0 + h^{-1} \theta^{-2} \| U \|_0^2 + h \| \text{div} U \|_0^2,
\]

which implies

\[
\|f_0\|_0^2 + \|g_0\|_0^2 \lesssim \theta^{-2} \|f_1\|_0^2 + \|g_1\|_0^2 + h^{-1} \theta^{-2} \| U \|_0^2 + h \| \text{div} U \|_0^2.
\]

(3.15)

Clearly, the estimates (3.4) and (3.5) follow from (3.9) and (3.15) by taking \( h \) small enough.
4 Parametrix construction

We keep the notations from the previous sections and will suppose that $\theta \geq h^{2/5-\epsilon}$, $0 < \epsilon \ll 1$. Let $(x_1, x')$ be the local normal geodesic coordinates introduced in Sect. 2. Clearly, it suffices to build the parametrix locally. Then the global parametrix is obtained by using a suitable partition of the unity on $\Gamma$ and summing up the corresponding local parametrices. We will be looking for a local parametrix of the solution to Eq. (1.1) in the form

$$\tilde{E} = (2\pi h)^{-2} \int \int e^{i \frac{1}{h}(y', \xi') + \psi(x, \xi')/h)} \chi(x, \xi') a(x, y', \xi', z, h) \rho \rho \text{d}y \text{d}y',$$

$$\tilde{H} = (2\pi h)^{-2} \int \int e^{i \frac{1}{h}(y', \xi') + \psi(x, \xi')/h)} \chi(x, \xi') b(x, y', \xi', z, h) \rho \rho \text{d}y \text{d}y',$$

where

$$\chi = \phi_0(x_1/\delta) \phi_0(x_1/\rho)^2 \delta,$$

the function $\phi_0$ being as in the previous section (see the proof of Theorem 3.1) and $0 < \delta \ll 1$ is a parameter independent of $h$ and $\theta$, which is fixed in Lemma 4.1. We require that $\tilde{E}$ satisfies the boundary condition $\nu \times \tilde{E} = f$ on $x_1 = 0$, where $f \in \mathcal{H}^1_1$. The phase function is of the form

$$\varphi = \sum_{k=0}^{N-1} x_k^k \varphi_k, \quad \varphi_0 = -(x', \xi'), \quad \varphi_1 = \rho,$$

where $N \gg 1$ is an arbitrary integer and the functions $\varphi_k, k \geq 2$, are determined from the eikonal Eq. (4.5). The amplitudes are of the form

$$a = \sum_{j=0}^{N-1} h^j a_j, \quad b = \sum_{j=0}^{N-1} h^j b_j.$$

In what follows, we will determine the functions $a_j$ and $b_j$ in terms of $f$ so that $(\tilde{E}, \tilde{H})$ satisfy the Maxwell equation modulo an error term. We have

$$e^{-i\varphi/h} \left( h \nabla \times (e^{i\varphi/h} a) - iz \mu e^{i\varphi/h} b \right) = i(\nu \nabla \varphi) \times a - iz \mu b + h (\nu \nabla \varphi) \times a$$

$$= \sum_{j=0}^{N-1} h^j \left( i(\nu \nabla \varphi) \times a_j - iz \mu b_j + (\nu \nabla \varphi) \times a_{j-1} \right)$$

$$+ h^N (\nu \nabla \varphi) \times a_{N-1},$$

$$e^{-i\varphi/h} \left( h \nabla \times (e^{i\varphi/h} b) + iz \epsilon e^{i\varphi/h} a \right) = i(\nu \nabla \varphi) \times b + iz \epsilon a + h (\nu \nabla \varphi) \times b$$

$$= \sum_{j=0}^{N-1} h^j \left( i(\nu \nabla \varphi) \times b_j + iz \epsilon a_j + (\nu \nabla \varphi) \times b_{j-1} \right)$$

$$+ h^N (\nu \nabla \varphi) \times b_{N-1},$$

where $a_{-1} = b_{-1} = 0$. We let now the functions $a_j$ and $b_j$ satisfy the equations

$$\begin{cases}
(\nu \nabla \varphi) \times a_0 - z \mu b_0 = x_1 \tilde{\Psi}_0, \\
(\nu \nabla \varphi) \times b_0 + z \epsilon a_0 = x_1 \tilde{\Psi}_0 \\
v \times a_0 = g \quad \text{on} \quad x_1 = 0,
\end{cases}$$

where $v = v(x') = (v_1(x'), v_2(x'), v_3(x'))$ is the unit normal vector at $x' \in \Gamma$,

$$g = -v(x') \times (v(y') \times f(y')) = f(y') - (v(x') - v(y')) \times (v(y') \times f(y')),$$
for $1 \leq j \leq N - 1$. We will be looking for solutions of the form
\[
a_j = \sum_{k=0}^{N-1} x_k^j a_{jk}, \quad b_j = \sum_{k=0}^{N-1} x_k^j b_{jk}.
\]

Let us expand the functions $\mu$, $\varepsilon$ and $\gamma$ as
\[
\mu(x) = \sum_{k=0}^{N-1} x_k^j \mu_k(x') + x_1^N \mathcal{M}(x),
\]
\[
\varepsilon(x) = \sum_{k=0}^{N-1} x_k^j \varepsilon_k(x') + x_1^N \varepsilon(x),
\]
\[
\gamma(x) = \sum_{k=0}^{N-1} x_k^j \gamma_k(x') + x_1^N \Theta(x).
\]

Observe that
\[
\nabla_x \varphi = \sum_{k=0}^{N-1} x_k^j e_k
\]
with $e_0 = (\rho, -\xi_2, -\xi_3)$, $e_k = (\varphi_{k+1}, \partial_x \varphi_k, \partial_x \gamma_k)$, $k \geq 1$. Hence,
\[
\nabla_x \varphi = \sum_{k=0}^{N-1} x_k^j \psi_k + x_1^N \widetilde{\Theta},
\]
where $\psi_0 = \gamma_0 \varphi_0 = \rho \nu - \beta$,
\[
\psi_k = \sum_{\ell=0}^{k} \gamma_k e_{k-\ell}, \quad k \geq 1,
\]
\[
\widetilde{\Theta} = \sum_{k=N}^{2N-2} x_k^{k-N} \psi_k + \Theta(\nabla_x \varphi).
\]

We will first solve Eq. (4.1). We let the functions $a_{0,0}$ and $b_{0,0}$ satisfy the equation
\[
\begin{cases}
\psi_0 \times a_{0,0} - z \mu_0 b_{0,0} = 0, \\
\psi_0 \times b_{0,0} + \varepsilon_0 a_{0,0} = 0, \\
v \times a_{0,0} = g.
\end{cases}
\] (4.3)

Equation (4.3) is solved in Section 2 and we have
\[
a_{0,0} = -v \times g + \rho^{-1} (v, \beta \times g) v,
\]
\[
b_{0,0} = \rho (z \mu_0)^{-1} g + (z \mu_0)^{-1} \beta \times (v \times g) - (z \mu_0)^{-1} \rho^{-1} (v, \beta \times g) \beta \times v,
\]
\[
z \mu_0 v \times b_{0,0} = \rho \nu \times g + \rho^{-1} (\beta, v \times g) \beta.
\] (4.4)

Next we let $\varphi$ satisfy the eikonal equation mod $\mathcal{O}(x_1^N)$:
\[
(\gamma \nabla_x \varphi, \gamma \nabla_x \varphi) - z^2 \varepsilon_0 \mu_0 = x_1^N \Phi.
\] (4.5)

This equation is solved in Section 4 of [9]. The functions $\varphi_k$, $k \geq 2$, are determined uniquely and have the following properties (see Lemma 4.1 of [9]):
Lemma 4.1 We have
\[ \varphi_k \in S_{2,2}^{-3k}(\rho) + S_{0,1}^1(|\rho|), \quad k \geq 1, \] (4.6)
\[ \partial_{x_i}^k \phi \in S_{2,2}^{-2N-3k}(\rho) + S_{0,1}^2(|\rho|), \quad k \geq 0, \] (4.7)
uniformly in \( z \) and \( 0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\} \). Moreover, if \( \delta > 0 \) is small enough, independent of \( \rho \), we have
\[ \text{Im} \varphi \geq x_1 \text{Im} \rho / 2 \quad \text{for} \quad 0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}. \] (4.8)
Furthermore, there are functions \( \varphi^\beta_k \in S_{0,1}^1 \), independent of \( \epsilon \) and \( \mu \), such that
\[ (1 - \eta)\varphi_k - \varphi^\beta_k \in S_{0,1}^{-1}. \]

Set
\[ \tilde{\varphi} = \sum_{k=1}^{N-1} x_1^k \varphi_k. \]

Let \( \eta \) be the function introduced in Sect. 2. Using the above lemma, we will prove the following:

Lemma 4.2 There exists a constant \( C > 0 \) such that we have the estimates
\[ \left| \partial_x^\alpha \partial_\xi^\beta \left( e^{\beta / h} \right) \right| \leq \begin{cases} C_{a,\beta} \theta^{-|\alpha|} |e^{-C_\beta h / \rho}| & \text{on supp} \eta, \\ C_{a,\beta} |\xi|^{-|\beta|} |e^{-C_\beta |\xi| / h}| & \text{on supp}(1 - \eta), \end{cases} \] (4.9)
for \( 0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\} \) and all multi-indices \( \alpha \) and \( \beta \) with constants \( C_{a,\beta} > 0 \) independent of \( x_1, \theta, z \) and \( h \).

Proof Let us see that the functions
\[ c_{a,\beta} = e^{-\beta / h} \partial_x^\alpha \partial_\xi^\beta \left( e^{\beta / h} \right), \quad |\alpha| + |\beta| \geq 1, \]
satisfy the bounds
\[ \left| \partial_x^\alpha \partial_\xi^\beta c_{a,\beta} \right| \lesssim \sum_{j=1}^{\max(|\alpha| + |\beta| + |\alpha'| + |\beta'|)} \left( \frac{x_1}{h |\rho|} \right)^j |\rho|^{-2(|\alpha| + |\beta| + |\alpha'| + |\beta'| - j)} \] (4.10)
on supp \( \eta \), and
\[ \left| \partial_x^\alpha \partial_\xi^\beta c_{a,\beta} \right| \lesssim \sum_{j=1}^{\max(|\alpha| + |\beta| + |\alpha'| + |\beta'|)} \left( \frac{x_1}{h |\xi|} \right)^j |\xi|^{-(|\beta| + |\beta'| - j)} \] (4.11)
on supp \( (1 - \eta) \), for all multi-indices \( \alpha' \) and \( \beta' \). We will proceed by induction in \( |\alpha| + |\beta| \).

Let \( \alpha_1 \) and \( \beta_1 \) be multi-indices such that \( |\alpha_1| + |\beta_1| = 1 \) and observe that
\[ c_{a + \alpha_1,\beta + \beta_1} = \partial_x^\alpha \partial_\xi^\beta c_{a,\beta} + i h^{-1} c_{a,\beta} \partial_x^{\alpha_1} \partial_\xi^{\beta_1} \tilde{\varphi}. \]

More generally, we have
\[ \partial_x^\alpha \partial_\xi^\beta c_{a + \alpha_1,\beta + \beta_1} = \partial_x^{\alpha + \alpha'} \partial_\xi^{\beta + \beta'} c_{a,\beta} + i h^{-1} \partial_x^{\alpha'} \partial_\xi^{\beta'} \left( c_{a,\beta} \partial_x^{\alpha_1} \partial_\xi^{\beta_1} \tilde{\varphi} \right). \] (4.12)

By Lemma 4.1, we have
\[ x_1^{-1} \tilde{\varphi} \in S_{2,2}^1(|\rho|) + S_{0,1}^1(|\rho|) \] (4.13)
for \(0 \leq x_1 \leq 2 \delta \min(1, |\rho|^3)\). By (4.12) and (4.13), it is easy to see that if (4.10) and (4.11) hold for \(c_{a,\beta}\), they hold for \(c_{a+a_1,\beta+\beta_1}\) as well. Using (4.10) together with (4.8), we obtain

\[
|e^{\hat{c}/h}c_{a,\beta}| \lesssim \sum_{j=1}^{[\delta]} \left( x_1 \over h|\rho| \right)^j |\rho|^{-2[(\delta|+|\beta|)-j]} e^{-2C\hat{c}/h|\rho|} \lesssim \sum_{j=1}^{[\delta]} \theta^{-j} |\rho|^{-2[(\delta|+|\beta|)-j]} e^{-C_1 \theta/h} \lesssim \theta^{-|\beta|} e^{-C_1 \theta/h}.
\]

Similarly, by (4.11) we obtain

\[
|e^{\hat{c}/h}c_{a,\beta}| \lesssim \sum_{j=1}^{[\delta]} \left( x_1 \over h|\rho| \right)^j |\rho|^{-2[(\delta|+|\beta|)-j]} e^{-2C_1 h|\rho|} \lesssim \sum_{j=1}^{[\delta]} \theta^{-j} |\rho|^{-2[(\delta|+|\beta|)-j]} e^{-C_1 h|\rho|} \lesssim \theta^{-|\beta|} e^{-C_1 h|\rho|}.
\]

\[\square\]

We take \(a_{0,k} = \tilde{a}_{0,k}v\) for \(k \geq 1\), where \(\tilde{a}_{0,k}\) are scalar functions to be determined such that

\[
\langle \gamma \nabla_x \psi, a_0 \rangle = x_1^{N} \tilde{\Phi}.
\]

(4.14)

Using that \(\langle \psi_0, a_{0,0} \rangle = 0\), we can expand the left-hand side as

\[
\sum_{k=1}^{2N-2} x_1^k \left( \sum_{\ell=0}^{k-1} \langle \psi_\ell, v \rangle \tilde{a}_{0,k-\ell} + \langle \psi_k, a_{0,0} \rangle \right) + x_1^N \langle \tilde{\Theta}, a_0 \rangle.
\]

Therefore, if

\[
\langle \psi_k, a_{0,0} \rangle + \sum_{\ell=0}^{k-1} \langle \psi_\ell, v \rangle \tilde{a}_{0,k-\ell} = 0, \quad 1 \leq k \leq N-1,
\]

then (4.14) is satisfied with

\[
\tilde{\Phi} = \sum_{k=N}^{2N-2} x_1^k \sum_{\ell=0}^{k-1} \langle \psi_\ell, v \rangle \tilde{a}_{0,k-\ell} + \langle \tilde{\Theta}, a_0 \rangle.
\]

Since \(\langle \psi_0, v \rangle = \rho\), we arrive at the relations

\[
\tilde{a}_{0,k} = -\rho^{-1} \langle \psi_k, a_{0,0} \rangle - \rho^{-1} \sum_{\ell=0}^{k-1} \langle \psi_\ell, v \rangle \tilde{a}_{0,k-\ell}
\]

(4.15)

which allow us to find all \(\tilde{a}_{0,k}\) and hence to find \(a_0\). To find \(b_0\), we will use the expansion

\[
(\gamma \nabla_x \psi) \times a_j = \sum_{k=0}^{N-1} x_1^k \psi_k \times \sum_{k=0}^{N-1} x_1^k a_{j,k} + x_1^N \tilde{\Theta} \times a_j
\]

\[
= \sum_{k=0}^{N-1} x_1^k \sum_{\ell=0}^{k} \psi_{k-\ell} \times a_{j,\ell} + x_1^N \sum_{k=N}^{2N-2} x_1^k \sum_{\ell=0}^{k} \psi_{k-\ell} \times a_{j,\ell} + x_1^N \tilde{\Theta} \times a_j
\]

with \(j = 0\). We take

\[
b_{0,k} = (z\mu_0)^{-1} \sum_{\ell=0}^{k} \psi_{k-\ell} \times a_{0,\ell}, \quad 0 \leq k \leq N-1.
\]

(4.16)

Then the first equation of (4.1) is satisfied with

\[
\psi_0 = \sum_{k=N}^{2N-2} x_1^{k-N} \sum_{\ell=0}^{k} \psi_{k-\ell} \times a_{0,\ell} + \tilde{\Theta} \times a_0.
\]
On the other hand, we have the identity

\[(\gamma \nabla_x \psi) \times ((\gamma \nabla_x \psi) \times a_0) = -(\gamma \nabla_x \psi, \gamma \nabla_x \psi) a_0 + (\gamma \nabla_x \psi, a_0) \gamma \nabla_x \psi.\]

Therefore, in view of (4.5) and (4.14), the second equation of (4.1) is satisfied with

\[\tilde{\Psi}_0 = (z \mu)^{-1} \left( -\Phi a_0 + \Phi \gamma \nabla_x \psi \right).\]

To solve Eq. (4.2), we will use the expansion

\[(\gamma \nabla_x) \times a_j = \sum_{k=0}^{N-1} x_1^k (\gamma_k \nabla_x) \times \sum_{k=0}^{N-1} x_1^k a_{j,k} + x_1^N (\Theta \nabla_x) \times a_j \]

\[= \sum_{k=0}^{N-1} x_1^k \sum_{\ell=0}^{k} \left( (\gamma_{k-\ell} \nabla_{x}) \times a_{j,\ell} + (\ell + 1) \gamma_{k-\ell} \nabla \times a_{j,\ell+1} \right) \]

\[+ x_1^N \sum_{k=N}^{2N-2} x_1^{k-N} \sum_{\ell=0}^{k} \left( (\gamma_{k-\ell} \nabla_{x}) \times a_{j,\ell} + (\ell + 1) \gamma_{k-\ell} \nabla \times a_{j,\ell+1} \right) \]

\[+ x_1^N (\Theta \nabla_x) \times a_j,\]

where \(\tilde{\nu} = (1, 0, 0)\) and \(\nabla_{x} = (0, \nabla_x)\). Clearly, we have similar expansions with \(a_j\) replaced by \(b_j\). We let the functions \(a_{j,k}\) and \(b_{j,k}\) satisfy the equations

\[\psi_0 \times a_{j,k} - z \mu_0 b_{j,k} = - \sum_{\ell=0}^{k-1} (\psi_{k-\ell} \times a_{j,\ell} - z \mu_{k-\ell} b_{j,\ell}) \]

\[+ \sum_{\ell=0}^{k-1} i (\gamma_{k-\ell} \nabla_{x}) \times a_{j-1,\ell} + (\ell + 1) \gamma_{k-\ell} \nabla \times a_{j-1,\ell+1} =: a_{j,k}^\nu\]

\[\psi_0 \times b_{j,k} + z \varepsilon_0 a_{j,k} = - \sum_{\ell=0}^{k-1} (\psi_{k-\ell} \times b_{j,\ell} + z \varepsilon_{k-\ell} a_{j,\ell}) \]

\[+ \sum_{\ell=0}^{k-1} i (\gamma_{k-\ell} \nabla_{x}) \times b_{j-1,\ell} + (\ell + 1) \gamma_{k-\ell} \nabla \times b_{j-1,\ell+1} =: b_{j,k}^\nu,\]

\(v \times a_{j,k} = 0\), for \(1 \leq j \leq N - 1\) and \(0 \leq k \leq N - 1\). Then Eq. (4.2) is satisfied with

\[\Psi_j = \sum_{k=N}^{2N-2} x_1^{k-N} \sum_{\ell=0}^{k} (\psi_{k-\ell} \times a_{j,\ell} - z \mu_{k-\ell} b_{j,\ell}) + \tilde{\Theta} \times a_j - z M b_j \]

\[+ \sum_{k=N}^{2N-2} x_1^{k-N} \sum_{\ell=0}^{k} (\gamma_{k-\ell} \nabla_{x}) \times a_{j-1,\ell} + (\ell + 1) \gamma_{k-\ell} \nabla \times a_{j-1,\ell+1} + (\Theta \nabla_x) \times a_{j-1},\]

\[\tilde{\Psi}_j = \sum_{k=N}^{2N-2} x_1^{k-N} \sum_{\ell=0}^{k} (\psi_{k-\ell} \times b_{j,\ell} + z \varepsilon_{k-\ell} a_{j,\ell}) + \tilde{\Theta} \times b_j + z D a_j \]

\[+ \sum_{k=N}^{2N-2} x_1^{k-N} \sum_{\ell=0}^{k} (\gamma_{k-\ell} \nabla_{x}) \times b_{j-1,\ell} + (\ell + 1) \gamma_{k-\ell} \nabla \times b_{j-1,\ell+1} + (\Theta \nabla_x) \times b_{j-1},\]
where \(a_{-1,\ell} = b_{-1,\ell} = 0\). The above equations are solved in Sect. 2 and we have the formulas

\[
a_{j,k} = \rho^{-2} (\beta \times a^z_{j,k}, v) v + z \mu_0 \rho^{-2} (b^z_{j,k}, v) v,
\]
\[
b_{j,k} = (z \mu_0)^{-1} a^z_{j,k} - (z \mu_0)^{-1} \rho^{-2} (\beta \times a^z_{j,k}, v) \beta \times v - \rho^{-2} (b^z_{j,k}, v) \beta \times v,
\]
\[
z \mu_0 \rho v \times b_{j,k} = \beta \times a^z_{j,k} - (\beta \times a^z_{j,k}, v) v + z \mu_0 b^z_{j,k} - z \mu_0 (b^z_{j,k}, v) v.
\]

(4.17)

Thus, we can express all functions \(a_{j,k}, b_{j,k}\) in terms of \(g\). More precisely, they are of the form

\[
a_{j,k} = A_{j,k}(x', \xi', \bar{\psi}(y') \bar{\beta}, b_{j,k} = B_{j,k}(x', \xi', \bar{\psi}(y') \bar{\beta}),
\]

where \(\bar{\psi}(y') = v(y') \times f(y') = v(y') \bar{\beta}(y') \bar{\psi}(y')\), \(v\) being a \(3 \times 3\) matrix, and \(A_{j,k}, B_{j,k}\) are smooth matrix-valued functions whose main properties are given in Lemma 4.3 below. In what follows, given a vector-valued function \(a\) of the form \(A(x', \xi') \bar{\psi}(y')\), we will write \(a \in S^k_{\ell_1, \ell_2} \bar{\psi}\) if all entries of \(A\) belong to \(S^k_{\ell_1, \ell_2}\).

**Lemma 4.3** We have

\[
A_{j,k} \in S^1_{-1-3k-5j}(|\rho|) + S^1_{0,j}(|\rho|), \quad j \geq 0, k \geq 0,
\]

(4.18)

\[
B_{j,k} \in S^1_{-1-3k-5j}(|\rho|) + S^1_{0,j}(|\rho|), \quad j \geq 0, k \geq 0,
\]

(4.19)

\[
u v(x') B_{j,k} \in S^1_{-1-3k-5j}(|\rho|) + S^1_{0,j}(|\rho|), \quad j \geq 1, k \geq 0,
\]

(4.20)

\[
\delta^k_{x_1} \Psi_j \in S^1_{-1-3(N+k)-5j}(|\rho|) \bar{\psi} + S^1_{0,j}(|\rho|) \bar{\psi}, \quad j \geq 0,
\]

(4.21)

\[
\delta^k_{x_1} \bar{\psi} \in S^1_{-1-3(N+k)-5j}(|\rho|) \bar{\psi} + S^1_{0,j}(|\rho|) \bar{\psi}, \quad j \geq 0.
\]

(4.22)

uniformly in \(z\) and \(0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}\).

**Proof** By Lemma 4.1,

\[
\psi_k \in S^1_{-1-3k}(|\rho|) + S^1_{0,1}(|\rho|), \quad \bar{\psi} \in S^1_{-1-3N}(|\rho|) + S^1_{0,1}(|\rho|).
\]

(4.23)

It is easy to see from (4.15) and (4.16) by induction in \(k\) that (4.23) implies (4.18) and (4.19) for \(j = 0\) and all \(k \geq 0\). To prove the assertion for all \(j \geq 1\) and \(k \geq 0\) we will proceed by induction in \(j + k\). Suppose it is fulfilled for all \(0 \leq j \leq J, k \geq 0\), as well as for \(j = J + 1\) and \(k \leq K\), where \(J \geq 0, K \geq -1\) are integers. This implies

\[
a^x_{j+1,K+1} \in S^1_{-1-3k-5j}(|\rho|) \bar{\psi} + S^1_{0,j}(|\rho|) \bar{\psi},
\]

(4.24)

\[
b^x_{j+1,K+1} \in S^1_{-1-3k-5j}(|\rho|) \bar{\psi} + S^1_{0,j}(|\rho|) \bar{\psi}.
\]

(4.25)

Recall that \(a^x_{0,0} = b^x_{0,0} = 0\). Using (2.3) with \(k = -2\) and the formulas for \(a_{j,k}\) and \(b_{j,k}\) in terms of \(a^x_{j,k}\) and \(b^x_{j,k}\), we get from (4.24) and (4.25) that (4.18) and (4.19) hold with \(j = J + 1\) and \(k = K + 1\), as desired. It is also clear that (4.20) follows from (4.24) and (4.25) used with \(K = k - 1, J = J - 1\) and (4.17) together with (2.3) with \(k = -1\). Since the functions \(\Psi_j\) and \(\bar{\psi}\) are expressed in terms of \(A_{j,k}, B_{j,k}, \psi_k\) and \(\bar{\psi}\), one can derive (4.21) and (4.22) from (4.18),(4.19) and (4.23). One just needs the following simple observation: if

\[
a \in S^1_{\ell_1}(|\rho|) + S^1_{\ell_2}(|\rho|),
\]
Thus, by (4.17) we obtain
\[ x^k a \in S^{\ell_1 + 3k}_{2,2}(|\rho|) + S^{\ell_2}_{0,1}(|\rho|), \quad k \geq 0. \]
\[ \square \]

Clearly, we have \( v \times \vec{E}|_{x_1=0} = f \) and
\[ v \times \vec{H}|_{x_1=0} = i_v(x') \vec{H}|_{x_1=0} = \sum_{j=0}^{N-1} h^j \text{Op}_h (i_v B_{j,0}) \tilde{f} \]
\[ = \text{Op}_h (i_v B_{0,0} + h(1-\eta)i_v B_{1,0}) \tilde{f} + K_1 \tilde{f}, \]
where
\[ K_1 = h \text{Op}_h (\eta i_v B_{1,0}) + \sum_{j=2}^{N-1} h^j \text{Op}_h (i_v B_{j,0}). \]

**Lemma 4.4** There exists a matrix-valued function \( B^\phi_{1,0} \in S^0_{0,1} \) such that
\[ (1-\eta)i_v B_{1,0} - B^\phi_{1,0} \in S^{-1}_{0,1} \]
(4.26)
and \( \mu_0 B^\phi_{1,0} \) is independent of \( \epsilon \) and \( \mu \).

**Proof** In view of (4.15) and (4.16), we have
\[ -ia^x_{1,0} = y_0 \tilde{\nabla} \times a_{0,0} + \nu \times a_{0,1} = y_0 \tilde{\nabla} \times a_{0,0}, \]
\[ -ib^x_{1,0} = y_0 \tilde{\nabla} \times b_{0,0} + \nu \times b_{0,1} \]
\[ = y_0 \tilde{\nabla} \times b_{0,0} + (z\mu_0)^{-1} (\psi_1 \times a_{0,0} + \psi_0 \times a_{0,1}) \]
\[ = y_0 \tilde{\nabla} \times b_{0,0} + (z\mu_0)^{-1} (\psi_1 \times a_{0,0} - \rho^{-1} (\psi_1, a_{0,0}) \psi_0 \times \nu) \]
\[ = y_0 \tilde{\nabla} \times b_{0,0} + (z\mu_0)^{-1} (\psi_1 \times a_{0,0} + \rho^{-1} (\psi_1, a_{0,0}) \beta \times \nu). \]

Thus, by (4.17) we obtain
\[ -iz \mu_0 t_v B_{1,0} \tilde{f} = -iz \mu_0 v \times b_{1,0} \]
\[ = \rho^{-1} \beta \times (y_0 \tilde{\nabla} \times a_{0,0}) - \rho^{-1} (\beta \times (y_0 \tilde{\nabla} \times a_{0,0}) \nu) + z \mu_0^{-1} y_0 \tilde{\nabla} \times b_{0,0} - z \mu_0 \rho^{-1} (y_0 \tilde{\nabla} \times b_{0,0}) \nu + \rho^{-1} \psi_1 \times a_{0,0} - \rho^{-1} (\psi_1 \times a_{0,0} \nu) + \rho^{-2} (\psi_1, a_{0,0}) \beta \times \nu. \]

Observe now that
\[ \rho = i_0 (1 + O(r_0^{-1})) = i_0 + O \left( \frac{1}{r_0} \right) \quad \text{as} \quad r_0 \to \infty. \]

More generally, we have
\[ (1-\eta)(\rho - i_0) \in S^{-1}_{0,1}, \]
\[ (1-\eta)(\rho^{-k} - (i_0)^{-k}) \in S_{0,1}^{-k-2}, \quad k = 1, 2. \]

Define \( a^\phi_{0,0} \) and \( b^\phi_{0,0} \) by replacing in the formulas for \( a_{0,0} \) and \( b_{0,0} \) above the function \( \rho \) by \( i_0 \). Clearly, \( a^\phi_{0,0} \) and \( \mu_0 b^\phi_{0,0} \) are independent of \( \epsilon \) and \( \mu \). Moreover, we have
\[ (1-\eta)(a_{0,0} - a^\phi_{0,0}) \in S^{-1}_{0,1} \tilde{f}, \quad (1-\eta)(b_{0,0} - b^\phi_{0,0}) \in S^0_{0,1} \tilde{f}. \]
Define \( \psi_1^0 \in S^1_{0,1} \) by replacing in the definition of \( \psi_1 \) the function \( \psi_2 \) by \( \psi_2^0 \) and \( \rho \) by \( i \sqrt{r_0} \). We also define \( B_{1,0}^\nu \) by replacing in the formula for \( i \nu B_{1,0} \) above the function \( \rho \) by \( i \sqrt{r_0} \), \( \psi_1 \) by \( \psi_1^0 \), \( a_{0,0} \) and \( b_{0,0} \) by \( a_{0,0}^0 \) and \( b_{0,0}^0 \). Set \( B_{1,0}^\nu = (1 - \eta)B_{1,0}^\nu \). With this choice, one can easily check that the conclusions of the lemma hold.

Clearly, we can write the matrix \( i \nu \) in the form \( \sum_{j=1}^3 v_j I_j \), where \( I_j \) are constant matrices. In view of (4.4), we have

\[
\begin{align*}
\psi_1\left(\frac{\tau}{[0,1]}\right) &= \frac{1}{i} \sum_{j=1}^3 \left( \psi_1 \right)_j \left( \frac{\tau}{[0,1]} \right) \
\end{align*}
\]

Define

\[
\begin{align*}
\tilde{\psi}^\nu &= \psi_1^0 \tilde{\tau} = \psi_1^0 \nu \\
\tilde{m} &= m(\nu \times g) = m\tilde{f} + m\nu \sum_{j=1}^3 (v_j(y') - v_j(x')) I_j \tilde{f},
\end{align*}
\]

where \( m = (\nu \mu)^{-1}(\nu I + \rho^{-1} B) \). Set \( m_0 = i(\nu \mu)^{-1} \sqrt{r_0}(I - r_0^{-1} B) \). We have

\[
\begin{align*}
\Op_h(i\nu B_{0,0})\tilde{f} &= \Op_h(m\tilde{f} + \sum_{j=1}^3 \left[ \Op_h(m I_j), v_j \right] \tilde{f} \\
&= \Op_h(m\tilde{f} + \sum_{j=1}^3 \left[ \Op_h((1 - \eta)m_0 I_j), v_j \right] \tilde{f} \\
&\quad + \sum_{j=1}^3 \left[ \Op_h((\eta m + (1 - \eta)(m - m_0))I_j), v_j \right] \tilde{f} \\
&= \Op_h(m + \nu n)\tilde{f} + \sum_{j=1}^3 \left[ \left[ \Op_h((1 - \eta)m_0 I_j), v_j \right] - \Op_h(hn_j) \right] \tilde{f} \\
&\quad + \sum_{j=1}^3 \left[ \Op_h((\eta m + (1 - \eta)(m - m_0))I_j), v_j \right] \tilde{f},
\end{align*}
\]

where \( n = \sum_{j=1}^3 n_j \) with

\[
n_j = -i \sum_{|\alpha| = 1} \partial^\alpha_{\nu} v_j \partial^\alpha_{\xi} ((1 - \eta)m_0) I_j.
\]

Thus, we obtain

\[
\begin{align*}
\nu \times \tilde{H}|_{x_1 = 0} &= \Op_h(m + \nu m)\tilde{f} + \mathcal{K}\tilde{f},
\end{align*}
\]

where we have put \( \tilde{m} = n + B_{1,0}^\nu \) and \( \mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 \) with

\[
\begin{align*}
\mathcal{K}_2 &= \h \Op_h \left( (1 - \eta)I B_{1,0} - B_{1,0}^\nu \right), \\
\mathcal{K}_3 &= \sum_{j=1}^3 \left[ \left[ \Op_h((1 - \eta)m_0 I_j), v_j \right] - \Op_h(hn_j) \right] \\
&\quad + \sum_{j=1}^3 \left[ \Op_h((\eta m + (1 - \eta)(m - m_0))I_j), v_j \right].
\end{align*}
\]

Furthermore, it is easy to see that

\[
\nabla \times (\chi a) = \chi \nabla \times a + \tilde{\chi} a,
\]

where \( \tilde{\chi} \) is a smooth matrix-valued function, which is a linear combinations of \( \partial_{\xi} \chi \). Therefore, \( \tilde{\chi} \) is supported in \( \delta \min\{1, |\rho|^3\} \leq x_1 \leq 2\delta \min\{1, |\rho|^3\} \). We have

\[
\begin{align*}
\h \nabla \times \tilde{E} - iz\mu \phi \tilde{H} = (2\pi \hbar)^{-2} \int \int e^{i\xi'(y',\xi') + \psi} V_1(x, y', \xi', h, z) \, d\xi' \, dy' = U_1,
\end{align*}
\]
\[ h \nabla \times \vec{H} + i e \phi \vec{E} = (2\pi h)^{-2} \int \int e^{i(x', \xi') \cdot (y', \eta') \cdot V_2(x, y', \xi', h, z) \, d\xi' \, dy' =: U_2, \]

where
\[
V_1 = h\vec{\nabla} a + h\vec{\nabla} \chi \gamma \nabla x \times a_N + \sum_{j=0}^{N-1} h^j \chi \Psi_j, \\
V_2 = h\vec{\nabla} b + h\vec{\nabla} \chi \gamma \nabla x \times b_N + \sum_{j=0}^{N-1} h^j \chi \tilde{\Psi}_j.
\]

Let \( \alpha \) be a multi-index such that \( |\alpha| \leq 1 \). Then we can write
\[
((h \partial_x)^\alpha U_2)(x, \cdot) = \text{Op}_h (e^{\phi/\theta} V_j(\alpha)) \tilde{f},
\]
where \( V_j^{(0)} = V_j \) and
\[
V_j^{(\alpha)} = i \partial_x^\alpha \phi V_j + (h \partial_x)^\alpha V_j
\]
if \( |\alpha| = 1 \). Since \( (E - \vec{F}, H - \vec{H}) \) satisfy Eq. (3.1) with \( \tilde{f} = 0 \), by (3.5) together with (4.27) we get the estimate
\[
\left\| \nabla (\lambda f - \text{Op}_h (m + h\tilde{m}) (v \times f)) \right\|_{\mathcal{H}_0} \\
\lesssim h^{-1/2} \theta^{-1} \| U \| + h^{1/2} \| \nabla U \| + \| u_1 \|_0 + \| \mathcal{K} \tilde{f} \|_0.
\]

We need now the following:

**Lemma 4.5** We have the estimates
\[
\| \mathcal{K} \tilde{f} \|_0 \lesssim h \theta^{-1/2} \| f \|_{\mathcal{H}_{-1}},
\]
\[
\| u_1 \|_0 + \| U \| + \| \nabla U \| \lesssim h^{5N/2 - \ell} \| f \|_{\mathcal{H}_{-1}},
\]
with some constant \( \ell > 0 \) independent of \( N \).

**Proof** By (4.20),
\[
\eta_i B_{j,0} t_v \in S_{2,2}^{-1} (|\rho|) \subset S_{1,1}^{-1} (\theta), \quad j \geq 1, \\
(1 - \eta) t_v B_{j,k} t_v \in S_{0,1}^{1-1} (|\rho|) \subset S_{0,1}^{-1}, \quad j \geq 2.
\]

Therefore, Proposition 2.3 yields
\[
\| \mathcal{K} \tilde{f} \|_0 \lesssim \sum_{j=1}^{N-1} h^j \| \text{Op}_h (\eta_i B_{j,0} t_v f) \|_0 + \sum_{j=2}^{N-1} h^j \| \text{Op}_h ((1 - \eta) t_v B_{j,k} t_v f) \|_0 \\
\lesssim \sum_{j=1}^{N-1} h^j \theta^{-1/2} \| f \|_{\mathcal{H}_{-1}} + \sum_{j=2}^{N-1} h^j \| f \|_{\mathcal{H}_{-1}} \lesssim h \theta^{-5/2} \| f \|_{\mathcal{H}_{-1}}.
\]

Furthermore, (4.26) clearly implies \( \mathcal{K}_2 = \mathcal{O}(h) : \mathcal{H}_{-1} \to \mathcal{H}_0 \). To bound the norm of \( \mathcal{K}_3 \), we will use Proposition 2.2 twice—with
\[
a^+ = (\eta m + (1 - \eta)(m - m_0)) i, j, \quad \theta^+ = \theta, \quad a^- = v_j, \quad \theta^- = 1,
\]
and with
\[
a^+ = v_j, \quad \theta^+ = 1, \quad a^- = (\eta m + (1 - \eta)(m - m_0)) i, j, \quad \theta^- = \theta.
\]
Since 
\[(\eta m + (1 - \eta)(m - m_0))c_i I_j \in S_{-2,2}^{-1}(\rho) + S_{0,1}^{-1}(\rho) \subset S_{-1,1}^{-1/2}(\theta) + S_{0,1}^{-1},\]
by Proposition 2.2,
\[\left\|\left[\text{Op}_h((\eta m + (1 - \eta)(m - m_0))c_i I_j), v_j\right]\right\|_{\mathcal{H}_{-1} \to \mathcal{H}_0} \lesssim h^{\theta - 3/2}.
\]
On the other hand, the standard pseudodifferential calculus gives that, mod \(O(h^\infty)\), the operator \(\left[\text{Op}_h((1 - \eta)m_0 c_i I_j), v_j\right]\) is an \(h - \Psi \text{DO}\) with a principal symbol \(h n_j, n_j \in S_{0,1}\)
being as above. This implies that 
\[\left[\text{Op}_h((1 - \eta)m_0 c_i I_j), v_j\right] - \text{Op}_h(h n_j)\]
is an \(h - \Psi \text{DO}\) with a symbol \(h^2\omega\), with \(\omega \in S_{0,1}^{-1}\). Hence,
\[\left\|\left[\text{Op}_h((1 - \eta)m_0 c_i I_j), v_j\right] - \text{Op}_h(h n_j)\right\|_{\mathcal{H}_{-1} \to \mathcal{H}_0} \lesssim h^2,
\]
which completes the proof of (4.29). Furthermore, since 
\[\frac{x_1 N}{1} h^{-c_0 x_1} / h \lesssim h^N \theta^{-N}, \quad \frac{x_1 N}{1} h^{-c_0 x_1} / h \lesssim h^N |\xi|^{-N},\]
we deduce from Lemma 4.2 that 
\[h^{-N} x_1 N h^{-c_0 / h} \in S_{1,1}^{-N}(\theta) + S_{0,1}^{-N}\]
uniformly in \(x_1\) and \(h\). On supp \(\tilde{x}\), we have the bounds 
\[e^{-c_0 x_1 / h} \leq e^{-c_0 |\alpha| / h} \leq e^{-c_0 x_1 / h} \lesssim h^N \theta^{-N}, \quad e^{-c_0 x_1 / h} \lesssim h^N |\xi|^{-N}.
\]
Therefore, by Lemma 4.2 we have
\[h^{-N} x_1 N h^{-c_0 / h} \in S_{1,1}^{-N/2}(\theta) + S_{0,1}^{-N}.\]
Notice that \(h^N \theta^{-5j/2} \leq 1\) for \(j \geq 1\) as long as \(\theta \geq h^{2/3-\epsilon}\). Taking this into account, one can easily check that (4.31) and (4.32) together with Lemma 4.3 imply
\[h^{-N} x_1 N h^{-c_0 / h} \in S_{1,1}^{-N/2}(\theta) + S_{0,1}^{-N} + \tilde{\ell}_a \:
\]
with some \(\tilde{\ell}_a > 0\) independent of \(N\), whose exact values are not important in the analysis that follows. Let \(N > \tilde{\ell}_a + 1\). By (4.33) and Proposition 2.3, we get
\[\left\|\left((h\partial_\alpha^0 U_j)(x_1,\cdot)\right)\right\|_{\mathcal{H}_{-1}} \lesssim h^N \theta^{-N/2 - \tilde{\ell}_a / 5} \left\|f\right\|_{\mathcal{H}_{-1}} \lesssim h^{5N/2 - 2\tilde{\ell}_a / 5} \left\|f\right\|_{\mathcal{H}_{-1}}
\]
as long as \(\theta \geq h^{2/3-\epsilon}\), uniformly in \(x_1\). Observe also that 
\[h^{-N} V_1 |x_1| = \left(\gamma \nabla x\right) \times a_{N-1} |x_1| = \left(\gamma_0 \nabla x\right) \times a_{N-1,0} + v \times a_{N-1,1}
\]
\[= (\gamma_0 \nabla x) \times (A_{N-1,0}) + v \times (A_{N-1,1}) =: \omega\tilde{f}.
\]
By Lemma 4.3,
\[\omega \in S_{1,1}^{-N/2}(\theta) + S_{0,1}^{-N+1},\]
which together with Proposition 2.3 yield
\[\text{Op}_h(\omega) = O(h^{-N/2}) : \mathcal{H}_{-1} \to \mathcal{H}_{0}.
\]
Since \(U_1 |x_1| = h^N \text{Op}_h(\omega)\tilde{f}\), we get
\[\left\|U_1 |x_1| \right\|_{\mathcal{H}_{-1}} \lesssim h^N \theta^{-N/2} \left\|f\right\|_{\mathcal{H}_{-1}} \lesssim h^{5N/2} \left\|f\right\|_{\mathcal{H}_{-1}}.
\]
Clearly, (4.30) follows from (4.34) and (4.35). 
\[\square
\]
Taking \(N\) big enough depending on \(\epsilon\), it is easy to see that the estimate (1.2) follows from (4.28) and Lemma 4.5.
5 Electromagnetic transmission eigenvalues

A complex number $\lambda$ is said to be an electromagnetic transmission eigenvalue if the following boundary value problem has a nontrivial solution:

$$\begin{align*}
\nabla \times E_1 &= i \lambda \mu_1(x) H_1 & \text{in } \Omega, \\
\nabla \times H_1 &= -i \lambda \varepsilon_1(x) E_1 & \text{in } \Omega, \\
\nabla \times E_2 &= i \lambda \mu_2(x) H_2 & \text{in } \Omega, \\
\nabla \times H_2 &= -i \lambda \varepsilon_2(x) E_2 & \text{in } \Omega, \\
\nu \times (E_1 - E_2) &= 0 & \text{on } \Gamma, \\
\nu \times (c_1 H_1 - c_2 H_2) &= 0 & \text{on } \Gamma,
\end{align*}$$

where $\mu_j, \varepsilon_j \in C^\infty(\Omega), c_j \in C^\infty(\Gamma), j = 1, 2,$ are scalar-valued strictly positive functions.

The most important question that arises in the theory of the transmission eigenvalues is to know the conditions on the coefficients under which they form a discrete set on the complex plane. This question has been largely investigated in the context of the acoustic transmission eigenvalues, that is, those associated with the Helmholtz equation. Several sufficient conditions have been found that guarantee not only the discreteness, but also Weyl asymptotics for the counting function of the acoustic transmission eigenvalues (see [6–8]). In particular, it was proved in [7] that the existence of parabolic eigenvalue-free regions implies the Weyl asymptotics. On the other hand, such regions were obtained in [9–13] under various conditions, by approximating appropriately the Dirichlet-to-Neumann operator associated with the Helmholtz equation with smooth refraction index.

It was proved in [12] that, under quite general conditions on the coefficients on the boundary, all transmission eigenvalues are located in a strip $|\text{Im} \lambda| \leq C$, which turns out to be optimal. The situation, however, is very different as far as the electromagnetic transmission eigenvalues are concerned. In this context, there are few results and they are mainly concerned with the question of discreteness (e.g. see [1,5]). The most general one is in [1], where the authors considered the case $c_1 \equiv c_2 \equiv 1$ and proved the discreteness under the condition

$$\varepsilon_1 \neq \varepsilon_2, \quad \mu_1 \neq \mu_2, \quad \frac{\varepsilon_1}{\mu_1} \neq \frac{\varepsilon_2}{\mu_2} \quad \text{on } \Gamma.$$  (5.2)

They also proved that given any $\gamma > 0$ there is $C_\gamma > 0$ such that there are no electromagnetic transmission eigenvalues in the region $|\text{Im} \lambda| \geq \gamma |\text{Re} \lambda|, |\lambda| \geq C_\gamma$. The question now is whether we have larger eigenvalue-free regions similar to those existing for the acoustic transmission eigenvalues. In particular, one can ask whether there are coefficients $c_j, \varepsilon_j, \mu_j, j = 1, 2$, for which the corresponding electromagnetic transmission eigenvalues are located in a strip $|\text{Im} \lambda| \leq C$ (see [2]). Note that this problem is still open and most probably the answer would be negative.

Our goal is to obtain a parabolic eigenvalue-free region under the condition

$$\frac{c_1}{\mu_1} = \frac{c_2}{\mu_2}, \quad \varepsilon_1 \mu_1 \neq \varepsilon_2 \mu_2 \quad \text{on } \Gamma.$$  (5.3)

Indeed, using Theorem 1.1 we will prove the following:

**Theorem 5.1** Under the condition (5.3), there exists a constant $C > 0$ such that there are no electromagnetic transmission eigenvalues in the region

$$|\text{Im} \lambda| \geq C(|\text{Re} \lambda| + 1)^{\frac{1}{2}}.$$  (5.4)
Proof Denote by $N_j(\lambda)$, $j = 1, 2$, the operator introduced in Sect. 1 corresponding to $(\varepsilon_j \mu_j)$, and set $T(\lambda) = c_1 N_1(\lambda) - c_2 N_2(\lambda)$. We define the functions $\rho_j$ by replacing in the definition of $\rho$ the function $\varepsilon \mu |r|$ by $\varepsilon_j \mu_j |r|$. Set $f = v \times E_1 = v \times E_2 \in \mathcal{H}_1'$. Then $\lambda$ is an electromagnetic transmission eigenvalue if $f \neq 0$ and $T(\lambda) f = 0$. Therefore, to get the free region (5.4) we need to show that the operator $T(\lambda)$ is invertible there. By Theorem 1.1, we have

$$
\| \text{Op}_h(T)(v \times f) \|_{\mathcal{H}_0} = \| T(\lambda) f - \text{Op}_h(T)(v \times f) \|_{\mathcal{H}_0} \lesssim h^{d/2-\epsilon} \| f \|_{\mathcal{H}_-1} \tag{5.5}
$$
for $\theta \geq h^{2/\epsilon} - \epsilon$, where

$$
T = \frac{c_1 \rho_1 I + c_1 \rho_2 I - c_2 \rho_2 I}{\mu_1 \mu_2} - \frac{c_2 \rho_2 I}{\mu_2 \mu_2}
= \frac{c_1}{\mu_1} (\rho_1 - \rho_2) (I - (\rho_1 \rho_2)^{-1} B).
$$

Since

$$(\rho_1 - \rho_2)(\rho_1 + \rho_2) = \rho_1^2 - \rho_2^2 = r^2 \varepsilon_1 \mu_1 - r^2 \varepsilon_2 \mu_2,$$

we have $T = w \tilde{T}$, where

$$
w = \frac{c_1}{\mu_1} (\varepsilon_1 \mu_1 - \varepsilon_2 \mu_2) \neq 0,
\tilde{T} = (\rho_1 + \rho_2)^{-1} (I - (\rho_1 \rho_2)^{-1} B).
$$

Using that $B^2 = n_0 B$, one can easily check the identity

$$(I + (\rho_1 \rho_2 - r_0) B)(I - (\rho_1 \rho_2)^{-1} B) = I \tag{5.6}$$

Lemma 5.2 For all integers $k \geq 1$ and all multi-indices $\alpha$ and $\beta$, we have the estimates

$$
\left| \partial_{x}^{\alpha} \partial_{\xi}^{\beta} (r_0 - \rho_1 \rho_2)^{-k} \right| \leq \begin{cases} \frac{C_{k, \alpha, \beta} \theta^{2-k-|\alpha| - |\beta|}}{|\xi|} & \text{on supp } \eta, \\ \frac{C_{k, \alpha, \beta} \theta^{2-k-|\alpha| - |\beta|}}{|\xi|} & \text{on supp}(1 - \eta), \end{cases} \tag{5.7}
$$

$$
\left| \partial_{x}^{\alpha} \partial_{\xi}^{\beta} (\rho_1 + \rho_2)^{-k} \right| \leq \begin{cases} \frac{C_{k, \alpha, \beta} \theta^{-|\alpha| - |\beta|}}{|\xi|} & \text{on supp } \eta, \\ \frac{C_{k, \alpha, \beta} \theta^{-|\alpha| - |\beta|}}{|\xi|} & \text{on supp}(1 - \eta), \end{cases} \tag{5.8}
$$

$$
\left| \partial_{x}^{\alpha} \partial_{\xi}^{\beta} ((\rho_1 \rho_2)^{-1} (\rho_1 + \rho_2)^{-1}) \right| \leq \begin{cases} \frac{C_{k, \alpha, \beta} \theta^{1/2 - |\alpha| - |\beta|}}{|\xi|} & \text{on supp } \eta, \\ \frac{C_{k, \alpha, \beta} \theta^{1/2 - |\alpha| - |\beta|}}{|\xi|} & \text{on supp}(1 - \eta). \end{cases} \tag{5.9}
$$

Proof We will first prove the estimates on supp$(1 - \eta)$. Since $\rho_j = i \sqrt{r_0} \left(1 + O(r_0^{-1}) \right)$ as $r_0 \to \infty$, we have

$$
r_0 - \rho_1 \rho_2 = 2r_0 \left(1 + O(r_0^{-1}) \right), \quad \rho_1 + \rho_2 = 2i \sqrt{r_0} \left(1 + O(r_0^{-1}) \right).
$$

Therefore, $|r_0 - \rho_1 \rho_2| \geq r_0$ and $|\rho_1 + \rho_2| \geq \sqrt{r_0}$ on supp$(1 - \eta)$, provided the constant $C_0$ in the definition of $\eta$ is taken large enough (what we can do without loss of generality).

To prove (5.7) for all $\alpha$ and $\beta$ we will proceed by induction in $|\alpha| + |\beta|$. Suppose that (5.7) holds on supp$1 - \eta)$ for $\alpha$, $\beta$ such that $|\alpha| + |\beta| \leq K$ and all integers $k \geq 1$. We will show that it holds for all $\alpha$, $\beta$ such that $|\alpha| + |\beta| = K + 1$ and all integers $k \geq 1$. Let $\alpha_1$ and $\beta_1$ be multi-indices such that $|\alpha_1| + |\beta_1| = 1$. We have

$$
\partial_{x}^{\alpha_1} \partial_{\xi}^{\beta_1} (r_0 - \rho_1 \rho_2)^{-k} = -k (r_0 - \rho_1 \rho_2)^{-k-1} \partial_{x}^{\alpha_1} \partial_{\xi}^{\beta_1} (r_0 - \rho_1 \rho_2),
$$
and more generally, if $\alpha$, $\beta$ are such that $|\alpha| + |\beta| = K$, we have

$$
\partial_{x}^{\alpha + \alpha_1} \partial_{\xi}^{\beta + \beta_1} (r_0 - \rho_1 \rho_2)^{-k} = -k \partial_{x}^{\alpha_1} \partial_{\xi}^{\beta_1 - k} \left( (r_0 - \rho_1 \rho_2)^{-k-1} \partial_{x}^{\alpha_1} \partial_{\xi}^{\beta_1} (r_0 - \rho_1 \rho_2) \right).
$$
Recall now that \(u_0\) is a homogeneous polynomial of order two in \(\xi\). Hence, \(\partial_{\xi}^\alpha \partial_{\eta}^\beta u_0 = O((|\xi|)^2 - |\beta|)\). Furthermore, by (2.3) we have \(\partial_{\xi}^\alpha \partial_{\eta}^\beta (\rho_1 \rho_2) = O((|\xi|)^2 - |\beta|)\) on \(\text{supp}(1 - \eta)\). Using this, one can easily deduce from the above identity that \((5.7)\) holds on \(\text{supp}(1 - \eta)\) for \(\alpha + \alpha_1, \beta + \beta_1\) and all integers \(k \geq 1\). Clearly, the same argument also works for \((5.8)\).

The estimate \((5.9)\) on \(\text{supp}(1 - \eta)\) follows from \((5.8)\).

To prove \((5.7)\) on \(\text{supp} \eta\), we will use the identity

\[
(r_0 + \rho_1 \rho_2)(r_0 - \rho_1 \rho_2) = r_0^2 - \rho_1^2 \rho_2^2 = z^2 (r_0 (\varepsilon_1 \mu_1 + \varepsilon_2 \mu_2) - z^2 \varepsilon_1 \mu_1 \varepsilon_2 \mu_2) =: w_1 (w_2 r_0 - z^2)
\]

which we rewrite in the form

\[
(r_0 - \rho_1 \rho_2)^{-k} = w_1^{-k} (r_0 + \rho_1 \rho_2)^k (w_2 r_0 - z^2)^{-k}.
\]

By induction, in the same way as above, one can easily prove the estimates

\[
\left| \partial_{\xi}^\alpha \partial_{\eta}^\beta (w_2 r_0 - z^2)^{-k} \right| \leq C_{k,\alpha, \beta} \theta^{-k - |\alpha| - |\beta|}
\]

on \(\text{supp} \eta\). On the other hand, by (2.3) we have \(\partial_{\xi}^\alpha \partial_{\eta}^\beta (r_0 + \rho_1 \rho_2)^k = O(\theta^{-|\alpha| - |\beta|})\) on \(\text{supp} \eta\). Therefore, \((5.7)\) on \(\text{supp} \eta\) follows from the above estimates. The estimates \((5.8)\) and \((5.9)\) on \(\text{supp} \eta\) can be obtained in the same way, using (2.3) and the identities

\[
(r_1 + \rho_2)^{-k} = w_3^{-k} (r_1 - \rho_2)^k, \quad w_3 := z^2 (\varepsilon_1 \mu_1 - \varepsilon_2 \mu_2),
\]

\[
(r_1 \rho_2)^{-1} (r_1 + \rho_2)^{-1} = w_3^{-1} (\rho_2^{-1} - \rho_1^{-1} - 1).
\]

We rewrite the identity \((5.6)\) in the form

\[
T_1 \tilde{T} = (\xi')^{-1} I,
\]

where

\[
T_1 = (\xi')^{-1} (r_1 + \rho_2) (I + (\rho_1 \rho_2 - r_0)^{-1} B).
\]

It follows from Lemma 5.2 together with (2.3) that

\[
T_1 \in S_{-1,1}^{-1}(\theta) + S_{0,1}^0 \subset \theta^{-1} S_{1/2, -\epsilon}^1,
\]

\[
\tilde{T} \in S_{-1,1}^{-1/2}(\theta) + S_{0,1}^{-1} \subset \theta^{-1/2} S_{1/2, -\epsilon},
\]

as long as \(\theta \geq h^{1/2, -\epsilon}\). Therefore, by Proposition 2.3 we get

\[
\| Op_{\theta}(T_1) \|_{\tau_0 \to \tau_0} \lesssim \theta^{-1},
\]

(5.11)

while Proposition 2.2 yields

\[
\| Op_{\theta}(T_1 \tilde{T}) - Op_{\theta}(T_1) Op_{\theta}(\tilde{T}) \|_{\tau_{-1} \to \tau_0} \lesssim h^{\theta^{-7/2}},
\]

(5.12)

Combining (5.10), (5.11) and (5.12) leads to

\[
\| Op_{\theta}((\xi')^{-1} \tilde{f}) \|_{\tau_0} \lesssim h^{\theta^{-7/2}} \| \tilde{f} \|_{\tau_{-1}} + \| Op_{\theta}(T_1) Op_{\theta}(\tilde{T}) \tilde{f} \|_{\tau_0} \lesssim h^{\theta^{-7/2}} \| \tilde{f} \|_{\tau_{-1}} + \theta^{-1} \| Op_{\theta}(\tilde{T}) \tilde{f} \|_{\tau_0}
\]

(5.13)

where \(\tilde{f} = v \times f\). Since the norms \(\| Op_{\theta}((\xi')^{-1} \tilde{f}) \|_{\tau_0}, \| \tilde{f} \|_{\tau_{-1}}\) and \(\| f \|_{\tau_{-1}}\) are equivalent, by (5.5) and (5.13) we obtain

\[
\| f \|_{\tau_{-1}} \lesssim h^{\theta^{-7/2}} \| f \|_{\tau_{-1}}.
\]

(5.14)
Thus, if $\hbar \theta^{-7/2} \ll 1$ we deduce from (5.14) that $f = 0$. In other words, the region $\hbar \theta^{-7/2} \ll 1$ is free of transmission eigenvalues. It is easy to see that this region is equivalent to (5.4) on the $\lambda$-plane. □

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