A note on a problem of Henning and Yeo about the transversal number of uniform linear systems whose 2-packing number is fixed

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Abstract

A linear system is a pair \((P, \mathcal{L})\) where \(\mathcal{L}\) is a family of subsets of a ground finite set \(P\) such that \(|l \cap l'| \leq 1\), for every pair of different lines \(l, l' \in \mathcal{L}\). If every member of \(\mathcal{L}\) has \(r\) elements, then the linear system \((P, \mathcal{L})\) is called \(r\)-uniform linear system. The transversal number \(\tau(P, \mathcal{L})\) of a linear system \((P, \mathcal{L})\) is the minimum cardinality of a subset \(l \cap l' \neq \emptyset\), for every \(l \in \mathcal{L}\). The 2-packing number \(\nu_2(P, \mathcal{L})\) of a linear system \((P, \mathcal{L})\) is the maximum cardinality of a subset \(R \subseteq \mathcal{L}\) such that every triplet of different elements of \(R\) do not have a common point. Henning and Yeo [Discrete Math. 313 (2013) 959–966] state the following question: is it true that if \((P, \mathcal{L})\) is an \(r\)-uniform linear system then \(\tau(P, \mathcal{L}) \leq (|P|+|\mathcal{L}|)/(r+1)\) holds for every \(r \geq 2\)? In this note, we prove that the mentioned inequality holds for several classes of \(r\)-uniform linear systems having a fixed 2-packing number.

Keywords: linear systems; 2-packing number; transversal number; finite projective plane.

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1. Introduction

A linear system is a pair \((P, \mathcal{L})\) where \(\mathcal{L}\) is a family of subsets of a ground finite set \(P\) such that \(|l \cap l'| \leq 1\), for every pair of different lines \(l, l' \in \mathcal{L}\). The linear system \((P, \mathcal{L})\) is intersecting if \(|l \cap l'| = 1\) for every pair of different lines \(l, l' \in \mathcal{L}\). The elements of \(P\) and \(\mathcal{L}\) are called points and lines, respectively. An \(r\)-line is a line containing exactly \(r\) points. An \(r\)-uniform linear system \((P, \mathcal{L})\) is a linear system such that all lines of \(\mathcal{L}\) are \(r\)-lines. In this context, a simple graph is a 2-uniform linear system. Throughout this paper, we will consider linear systems of rank \(r \geq 2\).

Let \((P, \mathcal{L})\) be a linear system and consider a point \(p \in P\). The set of lines incident to \(p\) is denoted by \(\mathcal{L}_p\). The degree of \(p\) is defined as \(\deg(p) = |\mathcal{L}_p|\) and the maximum degree over all points of the linear system \((P, \mathcal{L})\) is denoted by \(\Delta = \Delta(P, \mathcal{L})\). Two points \(p, q \in P\) are adjacent if there is a line \(l \in \mathcal{L}\) such that \(|p, q| \subseteq l\).

A linear subsystem \((P', \mathcal{L}')\) of a linear system \((P, \mathcal{L})\) is a linear system such that for every line \(l' \in \mathcal{L}'\) there exists a line \(l \in \mathcal{L}\) satisfying \(l' = l \cap P'\). The linear subsystem induced by a set of lines \(\mathcal{L}' \subseteq \mathcal{L}\) is the linear system \((P', \mathcal{L}')\), where \(P' = \bigcup_{l \in \mathcal{L}'} l\). The linear subsystem \((P', \mathcal{L}')\) of \((P, \mathcal{L})\) is called spanning linear subsystem if \(P' = P\). Given a linear system \((P, \mathcal{L})\) and a point \(p \in P\), the linear system obtained from \((P, \mathcal{L})\) by deleting point \(p\) is the linear subsystem \((P', \mathcal{L}')\) induced by \(\mathcal{L}' = \mathcal{L} \setminus \{l\}\). On the other hand, given a linear system \((P, \mathcal{L})\) and a line \(l \in \mathcal{L}\), the linear system obtained from \((P, \mathcal{L})\) by deleting the line \(l\) is the linear subsystem \((P', \mathcal{L}')\) induced by \(\mathcal{L}' = \mathcal{L} \setminus \{l\}\). Finally, let \((P', \mathcal{L}')\) and \((P, \mathcal{L})\) be two linear systems. The linear systems \((P', \mathcal{L}')\) and \((P, \mathcal{L})\) are isomorphic, \((P', \mathcal{L}') \simeq (P, \mathcal{L})\), if after of deleting points of degree 1 or 0 from both, the linear systems \((P', \mathcal{L}')\) and \((P, \mathcal{L})\) are isomorphic as hypergraphs, see [5].

Let \((P, \mathcal{L})\) be a linear system. A subset of points \(T\) of \(P\) is a transversal of \((P, \mathcal{L})\) (also called vertex cover or hitting set) if \(T \cap l \neq \emptyset\), for every line \(l \in \mathcal{L}\). The minimum cardinality of a transversal of a linear system \((P, \mathcal{L})\), \(\tau = \tau(P, \mathcal{L})\), is called transversal number of \((P, \mathcal{L})\). On the other hand, a subset of lines \(R\) of \(\mathcal{L}\) is called a 2-packing of \((P, \mathcal{L})\) if every triplet of different elements of \(R\) do not have a common point. The maximum cardinality of a 2-packing of \((P, \mathcal{L})\), \(\nu_2 = \nu_2(P, \mathcal{L})\), is called 2-packing number of \((P, \mathcal{L})\). This new parameter has been studied in some papers, see for example [3–5, 18–20].

Araujo-Pardo et al. in [5] proved a relationship between the transversal and the 2-packing numbers

\[
\frac{\nu_2}{2} \leq \tau \leq \frac{\nu_2(\nu_2 - 1)}{2}.
\]

Hence, the transversal number of any linear system is upper bounded by a quadratic function of their 2-packing number. For some linear systems the transversal number is bounded above by a linear function of their 2-packing number, see [3–5].

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Eustis and Verstraete in [13] proved, using probabilistic methods, the existence of \( k \)-uniform linear systems \((P, \mathcal{L})\) for infinitely many \( k \)'s and \( n = |P| \) large enough, which transversal number is \( \tau = n - o(n) \). This \( k \)-uniform linear systems has 2-packing number upper bounded by \( \frac{2n}{3} \).

There are works which the transversal number of an \( r \)-uniform linear system is bounded above by a function of their number points and lines, see for example [12,14]. Henning and Yeo in [14] stated the following question: is it true that if \((P, \mathcal{L})\) is an \( r \)-uniform linear system then

\[
\tau \leq \frac{|P| + |\mathcal{L}|}{r + 1},
\]

for every \( r \geq 2 \). Chvátal and McDiarmid in [10] proved (2) when \( r \in \{2, 3\} \). Dorfling and Henning in [12] proved (2) when \( \Delta \leq 2 \) and there are only two families of \( r \)-uniform linear systems that achieve equality in the bound. Also, Dorfling and Henning in [12] gave a better upper bound for the transversal number in terms of the number of points and the number of lines, namely, they proved that if \((P, \mathcal{L})\) is an \( r \)-uniform linear system with \( \Delta \leq 2 \) and if \( r \geq 3 \) is an odd integer, then \( r(r^2 - 3)\tau \leq (r - 2)(r + 1)n + (r - 1)^2 m + r - 1 \); similar bounds were also proved when \( r \geq 2 \) is an even integer.

This note is organized as follows. In Section 2, we present an infinite family of \( r \)-uniform linear systems \((P, \mathcal{L})\) for which equality in (2) holds, where \( r \geq 3 \) is an odd integer. This family of linear systems was defined in [3]. In Section 3, we prove that if \((P, \mathcal{L})\) is an intersecting \( r \)-uniform linear system with \( \tau = r \), then (2) holds. In Section 4, we prove if \((P, \mathcal{L})\) is an \( r \)-uniform linear system with \( \nu_2 \in \{2, 3, 4\} \), then (2) holds. Finally, in Section 5, we prove that if \((P, \mathcal{L})\) is an \( r \)-uniform linear system with \( \Delta = 2 \), then (2) holds, and equality in (2) holds if and only if \((P, \mathcal{L})\) is an \( (\nu_2 - 1) \)-uniform linear system with \( \nu_2 \geq 2 \) an even integer. This result was obtained first by Dorfling and Henning in [12].

### 2. Examples of linear systems \((P, \mathcal{L})\) with \( \tau = \frac{|P| + |\mathcal{L}|}{r + 1} \)

Let \((\Gamma, +)\) be an additive Abelian group, with neutral element \( e \), satisfying \( \sum_{g \in \Gamma} g = e \) and \( 2g \neq e \), for all \( g \in \Gamma \setminus \{e\} \). An example of this groups is \((\mathbb{Z}_n, +)\), with \( n \geq 3 \) an odd integer.

Let \( n = 2k + 1 \), with \( k \) a positive integer; and let \((\Gamma, +)\) be an additive Abelian group of order \( n \) as above. Alfaro et al. in [3] defined the following linear system \( C_{n,n+1} = (P_n, \mathcal{L}_n) \), where

\[
P_n = (\Gamma \times \Gamma \setminus \{e\}) \cup \{p, q\} \quad \text{and} \quad \mathcal{L}_n = \mathcal{L} \cup \mathcal{L}_p \cup \mathcal{L}_q,
\]

with

\[
\mathcal{L} = \{L_g : g \in \Gamma \setminus \{e\}\}, \quad \text{and} \quad L_g = \{(h, g) : h \in \Gamma\},
\]

for \( g \in \Gamma \setminus \{e\} \), and

\[
\mathcal{L}_p = \{l_{p_g} : g \in \Gamma\}, \quad \text{with} \quad l_{p_g} = \{(g, h) : h \in \Gamma \setminus \{e\}\} \cup \{p\},
\]

for \( g \in \Gamma \), and

\[
\mathcal{L}_q = \{l_{q_g} : g \in \Gamma\}, \quad \text{and} \quad l_{q_g} = \{(h, f_g(h)) : h \in \Gamma, f_g(h) = h + g \text{ with } f_g(h) \neq e\} \cup \{q\},
\]

for \( g \in \Gamma \).

The set of lines \( \mathcal{L} \) is a set of pairwise disjoint lines with \( |\mathcal{L}| = n - 1 \) and each line of \( \mathcal{L} \) has \( n \) points. The set of lines \( \mathcal{L}_p \) and \( \mathcal{L}_q \) are lines incidents to \( p \) and \( q \), respectively, with \( |\mathcal{L}_p| = |\mathcal{L}_q| = n \), and each line of \( \mathcal{L}_p \cup \mathcal{L}_q \) has \( n \) points. This linear system is an \( n \)-uniform linear system with \( n(n - 1) + 2 \) points and \( 3n - 1 \) lines. Moreover, this linear system has 2 points of degree \( n \) (points \( p \) and \( q \)) and \( n(n - 1) \) points of degree 3.

Alfaro et al. in [3] proved the following:

**Theorem 2.1.** [3] The linear system \( C_{n,n+1} \) satisfies \( \tau(C_{n,n+1}) = \nu_2(C_{n,n+1}) = n + 1 \).

A consequence of Theorem 2.1 is the following corollary.

**Corollary 2.1.** Let \((P, \mathcal{L})\) be an \( r \)-uniform linear system such that \((P, \mathcal{L}) \simeq C_{n,n+1} \), where \( r \geq n \), then \( \tau \leq \frac{|P| + |\mathcal{L}|}{r + 1} \). Moreover, the equality holds if and only if \((P, \mathcal{L}) = C_{n,n+1} \).

**Proof.** Let \((P, \mathcal{L})\) be an \( r \)-uniform linear system such that \((P, \mathcal{L}) \simeq C_{n,n+1} \). Then \( |P| = n(n - 1) + 2 + k|\mathcal{L}| \), where \( n + k = r \) with \( k \geq 0 \), and \( |\mathcal{L}| = 3n - 1 \). Hence

\[
\frac{|P| + |\mathcal{L}|}{r + 1} = \frac{n(n - 1) + 2 + (3n - 1)(k + 1)}{n + k + 1} = \frac{(n - 1)(n + k + 1) + 2(n(k + 1) + 1)}{n + k + 1} = (n + 1) + \frac{2k(n - 1)}{n + k + 1} \geq n + 1.
\]
Hence, by Theorem 2.1
\[ \tau \leq \frac{|P| + |\mathcal{L}|}{n + 1}. \]

The equality holds if and only if \( k = 0 \), that is, if and only if \((P, \mathcal{L}) = \mathcal{C}_{n,n+1}\).

\[ \Box \]

**Theorem 2.2.** If \( r \geq 2 \) is an positive integer and \((P, \mathcal{L})\) is an \( r \)-uniform linear system with \( \Delta \geq \nu_2 - 1 \), \( |\mathcal{L}| \geq \nu_2 + \Delta - 2 \) and \( \Delta \geq 3 \), then
\[ \nu_2 - 1 \leq \frac{|P| + |\mathcal{L}|}{r + 1}. \]

**Proof.** Since \( |P| \geq \Delta(r - 1) + 1 \), we have
\[ |P| + |\mathcal{L}| \geq \frac{\Delta(r - 1) + 1 + \nu_2 + \Delta - 2}{r + 1} = \frac{r\Delta + \nu_2 - 1}{r + 1} \geq \frac{(\nu_2 - 1)(r + 1)}{r + 1} = \nu_2 - 1. \]

\[ \Box \]

**Corollary 2.2.** If \( r \geq 2 \) is an positive integer and \((P, \mathcal{L})\) is an \( r \)-uniform linear system with \( \Delta \geq \nu_2 \), \( |\mathcal{L}| \geq \nu_2 + \Delta - 1 \) and \( \Delta \geq 3 \), then
\[ \nu_2 \leq \frac{|P| + |\mathcal{L}|}{r + 1}. \]

**Proof.** The proof of this corollary is analogous to the proof of Theorem 2.2.

\[ \Box \]

3. Intersecting \( r \)-uniform linear systems

Through this paper, all linear systems \((P, \mathcal{L})\) satisfy \( |\mathcal{L}| > \nu_2 \), due to the fact that \( |\mathcal{L}| = \nu_2 \) if and only if \( \Delta \leq 2 \).

The proofs of Lemma 3.1 and Lemma 3.2 are analogous to the proofs of Lemma 2.4 and Lemma 2.5 of [11], respectively.

**Lemma 3.1.** [11] Let \((P, \mathcal{L})\) be an intersecting \( r \)-uniform linear system, with \( r \geq 3 \). If \( \tau = r \), then every line of \((P, \mathcal{L})\) has at most one point of degree two and \( \Delta = r \).

**Lemma 3.2.** [11] Let \((P, \mathcal{L})\) be an intersecting \( r \)-uniform linear system, with \( r \geq 3 \). If \( \tau = r \), then \( 3(r - 1)|L| \leq |\mathcal{L}| \leq r^2 - r + 1 \) and \( |P| = r^2 - r + 1 \).

The proof of Lemma 3.3 is analogous to the proof of Lemma 4.1 of [18].

**Lemma 3.3.** [18] Let \((P, \mathcal{L})\) be an intersecting \( r \)-uniform linear system, with \( r \geq 3 \) be an odd integer. If \( \tau = r \), then \( \nu_2 = r + 1 \).

**Corollary 3.1.** Let \( r \geq 3 \) be an odd integer. If \((P, \mathcal{L})\) is an intersecting \( r \)-uniform linear system with \( \tau = r \), then
\[ \tau \leq \frac{|P| + |\mathcal{L}|}{r + 1}. \]

**Proof.** By Lemma 3.1 and Lemma 3.3 then \( \Delta = \nu_2 - 1 \). On the other hand, by Theorem 2.2 and \( \nu_2 = r + 1 \) which implies
\[ \tau \leq \frac{|P| + |\mathcal{L}|}{r + 1}. \]

Let us consider the case when \( r \) is an even integer. If \((P, \mathcal{L})\) is an intersecting \( r \)-uniform linear system, then \( \nu_2 \leq r + 1 \). However, if \( r \) is an even integer, then by the next lemma, it holds that \( \nu_2 \leq r \).

**Lemma 3.4.** [19] Let \((P, \mathcal{L})\) be an \( r \)-uniform intersecting linear system where \( r \geq 2 \) is an even integer. If \( \nu_2 = r + 1 \), then \( \tau = \frac{r + 2}{2} \).

**Corollary 3.2.** Let \((P, \mathcal{L})\) be an \( r \)-uniform intersecting linear system with \( r \geq 2 \) be an even integer. If \( \nu_2 = r + 1 \), then
\[ \tau \leq \frac{|P| + |\mathcal{L}|}{r + 1}. \]

**Proof.** It is not difficult to prove \( \Delta = 2 \), see [19]. Hence, by Corollary 5.4 (see in Section 5), \( \tau \leq \frac{|P| + |\mathcal{L}|}{r + 1}. \)

Therefore, if \( \tau > \frac{r + 2}{2} \), then \( \nu_2 \leq r \) and \( r \geq 4 \) being an even integer. The proof of Lemma 3.5 is analogous to the proof of Lemma 6 of [19].

**Lemma 3.5.** [19] Let \((P, \mathcal{L})\) be an intersecting \( r \)-uniform linear system, with \( r \geq 4 \) be an even integer. If \( \tau = r \), then \( \nu_2 = r \).

**Corollary 3.3.** Let \( r \geq 4 \) be an even integer. If \((P, \mathcal{L})\) is an intersecting \( r \)-uniform linear system with \( \tau = r \), then \( \tau \leq \frac{|P| + |\mathcal{L}|}{r + 1}. \)
Proof. By Lemma 3.1 and Lemma 3.5, \( \Delta = \nu_2 = r \). Hence, by Corollary 2.2, it holds that \( \tau \leq \frac{|P| + |L|}{r + 1} \).

By Corollary 3.1 and Corollary 3.3, we have

**Theorem 3.1.** Let \( r \geq 3 \) be an integer. If \((P, L)\) is an intersecting \( r \)-uniform linear system with \( \tau = r \), then \( \tau \leq \frac{|P| + |L|}{r + 1} \).

A finite projective plane, or merely projective plane, is an intersecting linear system satisfying the following conditions:

- any pair of points have a common line,
- any pair of lines have a common point, and
- there exist four points in general position (there are not three collinear points).

It is well known that if \((P, L)\) is a projective plane then there exists a number \( q \in \mathbb{N} \), called order of projective plane, such that every point (line, respectively) of \((P, L)\) is incident to exactly \( q + 1 \) lines (points, respectively), and \((P, L)\) contains exactly \( q^2 + q + 1 \) points (lines, respectively). Also, it is well known that projective planes of order \( q \), denoted by \( \Pi_q \), exist when \( q \) is a power prime. For more information about the existence and the unicity of projective planes see, for instance, [7, 8].

In relation to the transversal number of projective planes, it is well known that every line in \( P \), \( q \) of odd order \( \Pi_q \) = \( q + 1 \). On the other hand, related to the 2-packing number of a projective planes, since projective planes are dual systems, this parameter coincides with the cardinality of an oval, which is the maximum number of points in a general position (no three of them collinear), and it is equal to \( q + 1 \) when \( q \) is odd integer, and it is equal to \( q + 2 \) when \( q \) is even integer (see for example [8]).

Consequently, for the projective planes \( \Pi_q \) of odd order \( q \) we have \( \tau(\Pi_q) = \nu_2(\Pi_q) = q + 1 \), and for the projective planes \( \Pi_q \) of even order \( q \) we have \( \tau(\Pi_q) = \nu_2(\Pi_q) - 1 = q + 1 \), see [5].

**Corollary 3.4.** Let \( q \) be a prime power, and let \((P, L)\) be an \((q + 1)\)-uniform linear system such that \((P, L) \simeq \Pi_q\), then \( \tau \leq \frac{|P| + |L|}{q + 2} \).

Proof. The proof is a simple consequence of Theorem 2.2 and Corollary 2.2, since \( \Delta = q + 1 \) and \( \nu_2 = q + 2 \) if \( q \) is an even integer, and \( \nu_2 = q + 1 \) if \( q \) is an odd integer.

4. The \( r \)-uniform linear systems with \( \nu_2 \in \{2, 3, 4\} \)

Let \((P, L)\) be an \( r \)-uniform linear system with \( \nu_2 \in \{2, 3\} \). It is not difficult to prove (see [5]) that \( \nu_2 = 2 \) if and only if \( \tau = 1 \).

Also, if \( \nu_2 = 3 \), then \( \tau = 2 \), see [5].

**Lemma 4.1.** [5] Any linear system \((P, L)\) with \( \nu_2 = 4 \) and \( \Delta \geq 5 \) satisfies \( \tau \leq \nu_2 - 1 \).

**Corollary 4.1.** Any linear system \((P, L)\) with \( \nu_2 = 4 \) and \( \Delta \geq 5 \) satisfies \( \tau \leq \frac{|P| + |L|}{r + 1} \).

Proof. The required result follows from Theorem 2.2.

**Lemma 4.2.** [5] Let \((P, L)\) be a linear system with \( \nu_2 = 4 \) and \( \Delta = 3 \). If \((P, L) \cong C_{3,4}\), then \( \tau = \nu_2 \), otherwise \( \tau \leq \nu_2 - 1 \).

**Corollary 4.2.** Let \( r \geq 2 \) be an integer and let \((P, L)\) be an \( r \)-uniform linear system. If \( \nu_2 = 4 \) and \( \Delta = 3 \), then

\[ \tau \leq \frac{|P| + |L|}{r + 1} \]

The equality holds if and only if \((P, L) = C_{3,4}\).

Proof. Let \((P, L)\) be an \( r \)-uniform linear system with \( \nu_2 = 4 \) and \( \Delta = 3 \) such that \((P, L) \not\cong C_{3,4}\). By Theorem 2.2 and Lemma 4.2, we have

\[ \tau \leq \frac{|P| + |L|}{r + 1} \]

On the other hand, if \((P, L) \cong C_{3,4}\), then \( |P| = 8 + 8k \) and \( |L| = 8 \), where \( k + 3 = r \) and \( k \geq 0 \). Hence

\[ \frac{|P| + |L|}{r + 1} = \frac{8(k + 2)}{k + 4} \geq \frac{16}{4} = 4 = \tau. \]

Therefore, \( \tau \leq \frac{|P| + |L|}{r + 1} \), where the equality holds if and only if \( k = 0 \), that is, if and only if \((P, L) = C_{3,4}\).
Lemma 4.3. [5] If $(P, L)$ is a linear system with $\nu_2 = 4$ and $\Delta = 4$, then $\tau \leq \nu_2$.

Corollary 4.3. Let $r \geq 2$ be an integer and $(P, L)$ be an $r$-uniform linear system. If $\nu_2 = 4$ and $\Delta = 4$, then $\tau \leq \frac{|P| + |L|}{r + 1}$.

**Proof.** The required result follows from Corollary 2.2 and Lemma 4.3.

Therefore, the main result of this section states as:

Theorem 4.1. Let $r \geq 2$ be an integer and $(P, L)$ be an $r$-uniform linear system with $|L| > \nu_2$. If $\nu_2 = 4$, then $\tau \leq \frac{|P| + |L|}{r + 1}$.

**Proof.** The required result follows from Corollary 4.1, Corollary 4.2 and Corollary 4.3.

5. The $r$-uniform linear systems with $\Delta = 2$

In this section, we present some results regarding $r$-uniform linear systems $(P, L)$ with $\Delta = 2$ satisfying $\tau \leq \frac{|P| + |L|}{r + 1}$.

**Proposition 5.1.** If $(P, L)$ is a linear system with $\Delta = 2$, then $\tau \leq \nu_2 - 1$.

**Proof.** Let $A$ be a maximum subset of $P$ such that every $p \in A$ satisfies $\deg(p) = 2$, and $\{p, q\} \not\subseteq I$, for every $p, q \in A$. Since $\Delta = 2$, then $A \neq \emptyset$. Let $L_A = \bigcup_{p \in A} L_p$ and $L' = L \setminus L_A$. Hence, if $L' \neq \emptyset$, then the set of lines of $L'$ is pairwise disjoint. Therefore, the following set $T = A \cup B$, where $B = \{p_l : l \in L' \land p_l \in I\}$, is a transversal of $(P, L)$. Hence, $\tau \leq |T| = |A| + |B| \leq |L| - 1 = \nu_2 - 1$.

**Corollary 5.1.** Let $(P, L)$ be a linear system with $\Delta = 2$ and let $L'$ as above. If $|L'| \leq 1$, then $\tau = \lfloor \nu_2 / 2 \rfloor$. Moreover, if $|L'| = \nu_2 - 2$, then $\tau = \nu_2 - 1$.

**Corollary 5.2.** If $(P, L)$ is an $r$-uniform linear system with $\Delta = 2$ and $\nu_2 \geq 4$, then $\lfloor \nu_2 / 2 \rfloor \leq \frac{|P| + |L|}{r + 1} \leq \nu_2 - 1$.

**Proof.** Let $A$ as in the proof of Proposition 5.1. If $|A| = k$, where $1 \leq k \leq \nu_2(\nu_2 - 1)/2$, then $r\nu_2 - k \leq |P| \leq r\nu_2 - 1$. Hence

$$\frac{|P| + |L|}{r + 1} \leq \frac{\nu_2 - \frac{1}{r + 1}}{r + 1} = \nu_2 - 1.$$

On the other hand, since $|P| \geq r\nu_2 - k$ then

$$\frac{|P| + |L|}{r + 1} \geq \nu_2 - \frac{k}{r + 1} \geq \nu_2 - \frac{\nu_2(\nu_2 - 1)/2}{r + 1} \geq \nu_2/2,$$

and the statement holds.

In [12], the following result was proved.

**Theorem 5.1.** [12] If $(P, L)$ is a linear system with $\Delta = 2$, then $\tau \leq \frac{|P| + |L|}{r + 1}$.

As a simple consequence, since $\tau \in \mathbb{N}$, we have $\tau \leq \left\lfloor \frac{|P| + |L|}{r + 1} \right\rfloor$.

**Theorem 5.2.** If $(P, L)$ is an $r$-uniform linear system with $\nu_2 - 1 \leq r$, then

$$\left\lfloor \nu_2 / 2 \right\rfloor \leq \frac{|P| + |L|}{r + 1}.$$

**Proof.** Since $|P| \geq r\nu_2 - \frac{\nu_2(\nu_2 - 1)}{2}$ and $|L| \geq \Delta + \nu_2 - 2$, then

$$\frac{|P| + |L|}{r + 1} \geq \frac{r\nu_2 - \frac{\nu_2(\nu_2 - 1)}{2} + \nu_2 + \Delta - 2}{r + 1} = \nu_2 \left[1 - \frac{\nu_2 - 1}{2(r + 1)} + \frac{\Delta - 2}{r + 1}\right].$$

Since $\nu_2 - 1 \leq r$ then

$$\frac{|P| + |L|}{r + 1} \geq \nu_2 \left[1 - \frac{\nu_2 - 1}{2\nu_2} + \frac{\Delta - 2}{r + 1}\right] = \frac{\nu_2 + 1}{2} + \frac{\Delta - 2}{r + 1} \geq \nu_2/2.$$

Hence, the theorem holds.

**Corollary 5.3.** Let $(P, L)$ be an $r$-uniform linear system with $\nu_2 - 1 \leq r$ and $\tau = \lfloor \nu_2 / 2 \rfloor$, then $\tau \leq \frac{|P| + |L|}{r + 1}$.

**Corollary 5.4.** Let $(P, L)$ be an $r$-uniform intersecting linear system with $\Delta = 2$, then $\tau \leq \frac{|P| + |L|}{r + 1}$.

**Corollary 5.5.** Let $(P, L)$ be an $r$-uniform intersecting linear system with $\Delta = 2$ and $r \geq 2$ an even integer. Then

$$\tau = \frac{|P| + |L|}{r + 1}$$

if and only if $r = \nu_2 - 1$. 
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