An Upper Bound on the number of distinct Composition Series in a finite group

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Abstract

In this paper we prove that among all finite groups of order ≤ n (where n ≥ 4 be a natural number) the number of distinct composition series is bounded above by \( \prod_{i=1}^{\lfloor \log_2 n \rfloor} (2^i - 1) \) and it is attained if and only if \( G \) is the elementary abelian 2-group of order \( 2^\alpha \), where \( \alpha = \lfloor \log_2 n \rfloor \). This bound is non trivial upper bound of composition series and so far best possible.

Keywords: Finite group, composition series, Sylow subgroup, elementary abelian group
Mathematics Subject Classifications(2010): 20D30, 20D15

1 Introduction

A composition series, that is a series of subgroups each normal in the previous such that corresponding factor groups are simple. Any finite group has a composition series. The concept of a composition series in a group is due to Evariste Galois(1831)[Theorem 5.9, [1]]. The famous Jordan-Holder theorem which says that, the composition factors in a composition series are unique up to isomorphism was proved in nineteenth century [5, 6]. Sometimes a group of small order has a huge number of distinct composition series. For example an elementary abelian group of order 64 has 615195 distinct composition series. To find an upper bound for composition series in a finite group is a natural question. In [7], there is an algorithm in GAP to find the distinct composition series of any group of finite order. The aim of this paper is to provide an upper bound of the number of distinct composition series of any group of finite order. The approach of this is combinatorial and the method is elementary. All the groups considered in this paper are of finite order.

There are three sections in the paper namely section 2, section 3 and section 4. In section 2 we prove Theorem 2.1. which is the main theorem of section 2. All the theorems and results in section 2 are well known. In section 3 we prove Theorem 3.7. which is a new result and it is the main theorem of section 3. In section 4 we prove Theorem 4.3. by proving an inequality in number theory. This theorem is the main theorem of this paper.

2 Preliminary Results

Theorem 2.1. Among all finite groups of order \( n \), the abelian group with elementary abelian Sylow subgroups, has the highest number of distinct composition series.

Before proving theorem 2.1 we recall some useful theorems in group theory at first.

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Theorem 2.2 (Theorem 2.28, [1]). (Correspondence Theorem) Given \( G \triangleright N \), denote the the family of subgroups of \( G \) that contain \( N \) by \( \text{sub}(G;N) \) and the family of subgroups of \( G/N \) by \( \text{sub}(G/N) \). Then there is a bijection
\[
\phi : \text{sub}(G;N) \rightarrow \text{sub}(G/N),
\]
\[
H \mapsto H/N
\]
which preserves all subgroup lattice and normality relationships.

Theorem 2.3 (Exercise 14, Section 5.2, [8]). A finite abelian group is self dual.

Lemma 2.4 (Lemma 4.2.7, [9]). Suppose \( G \) be a finite group and let \( \prod(G) = \{ p_1, p_2, \ldots, p_r \} \) where \( \prod(G) \) denotes the distinct prime factors in the order of \( G \). Let \( P_i \in \text{Syl}_{p_i}(G) \). If \( G = P_1 \times P_2 \times \cdots \times P_r \) then
\[
|m(G)| = \sum_{i=1}^{r} |m(P_i)|,
\]
where \( m(G) \) denotes the distinct maximal subgroups in \( G \).

Corollary 2.5. If \( G \) be the abelian group with elementary abelian Sylow subgroups of order \( n \) where \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) then \( m(G) = \sum_{i=1}^{r} \left( \frac{p_i^{\alpha_i} - 1}{p_i - 1} \right) \).

Proof. Since \( G \) is a finite abelian group then it is self dual and direct products of its Sylow subgroups. Therefore number of maximal subgroups in \( G \) is equal to the sum of all the maximal subgroups in it’s Sylow subgroups. Now elementary abelian group is self dual. So number of maximal subgroups in an elementary abelian subgroup is exactly same with the number of minimal subgroups. Therefore number of subgroups of index \( p_i \) is number of distinct subgroups of order \( p_i \) in an elementary abelian \( p_i \) group of order \( p_i^{\alpha_i} \) which is \( \frac{p_i^{\alpha_i} - 1}{p_i - 1} \) and the result follows immediately.

Example 2.6. If \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) then \( |G| = 3600 \) and
\[
m(G) = \frac{2^4 - 1}{2 - 1} + \frac{3^2 - 1}{3 - 1} + \frac{5^2 - 1}{5 - 1} = 15 + 4 + 6 = 25.
\]

Lemma 2.7. If \( G = S_1 \times S_2 \), where \( S_1 \), \( S_2 \) are two finite simple groups. If \( G_1 \) is a proper non-trivial normal subgroup of \( G \) then \( G_1 \) is isomorphic to either \( S_1 \) or \( S_2 \).

Proof. We consider the projection map \( p : G \rightarrow S_1 \) and the projection map is onto. Therefore the image of a normal subgroup under the projection map \( p \) will be a normal subgroup in \( S_1 \). Since \( S_1 \) is a simple group we have the following two choices:

Case 1 : If \( p(G_1) = \{ e \} \). Then \( G_1 \) is a subgroup of \( S_2 \) and it is a non trivial normal subgroup of the simple group \( S_2 \) then \( G_1 = S_2 \) i.e. \( G_1 \cong S_2 \).

Case 2 : If \( p(G_1) = S_1 \). Now we will take the kernel \( K = S_2 \cap G_1 \) of the projection map. Here \( K \) will become a normal subgroup of the simple group \( S_2 \). So either \( K = \{ e \} \) or \( S_2 \). If \( K = \{ e \} \) then \( G_1 \cong S_1 \). If \( K = S_2 \) then \( G_1 \cong S_1 \times S_2 \), which is a contradiction since \( G_1 \) is a proper subgroup of \( G \).

Corollary 2.8. If \( G = S_1 \times S_2 \times \cdots \times S_k \) be the direct product of \( k \) non-abelian simple groups then \( G \) has \( 2^k \) normal subgroups namely \( N_1 \times N_2 \times \cdots \times N_k \) where either \( N_i = \{ e \} \) or \( N_i = S_i \), for \( 1 \leq i \leq k \). Also \( G \) has only \( k \) maximal normal subgroups namely \( M_1, M_2, \ldots, M_k \) where \( M_i = S_1 \times S_2 \times \cdots \times S_{i-1} \times e \times S_{i+1} \times \cdots \times S_k \) and \( G/M_i \ncong S_i \), for \( 1 \leq i \leq k \).

Example 2.9. Let \( G = A_5 \times A_5 \) then \( |G| = 3600 \) and \( G \) has only \( 2^2 = 4 \) normal subgroups namely \( e \times e, e \times A_5, A_5 \times e, G \) and it has only 2 maximal normal subgroups namely \( e \times A_5 \) and \( A_5 \times e \).

Theorem 2.10 (Theorem 8.6, [3]). (Birkhoff) Every algebra \( A \) is isomorphic to subdirect product of subdirectly irreducible algebras.

Now we present the proof of theorem 2.1.
Proof. Let $G$ be the abelian group with elementary abelian Sylow subgroups such that $|G| = n$. Now by Lemma 2.4, the number of maximal subgroups in a finite nil-potent group of order $n$ is exactly same of the sum of distinct maximal subgroups in each Sylow subgroup. But we know that among all finite $p$ groups of same order the elementary abelian group has highest number of maximal subgroups. This proves that among all nil-potent groups of order $n$, $G$ has the highest number of maximal normal subgroups and hence the highest number of composition series. Let $K$ be a solvable group of order $n$ with $m$ maximal normal subgroups namely $M_1, M_2, \ldots, M_m$. Then index of $M_i$ in $K$ is a prime and $K/M_i \cong \mathbb{Z}_p$ for some prime $p$. Let $C'$ be the commutator subgroup of $K$. Then $C' \leq M_i$. Let $M = \cap_{i=1}^{m} M_i$ be the intersection of all the maximal subgroups of $K$. $M$ is a normal subgroup of $K$ such that $C' \leq M$ and therefore $K/M$ is abelian. Now by Correspondence Theorem there exists a bijection

$$\phi : \text{sub}(K; M) \longrightarrow \text{sub}(K/M),$$

which preserves subgroup lattice and normality relationship. Therefore $K/M$ is an abelian group with exactly $m$ maximal normal subgroups. But it is already proved that among all abelian groups of same order the abelian group with elementary abelian Sylow subgroups, has the highest number of maximal normal subgroups. For non-solvable non-simple group we take the Jacobson radical $J(G)$ which is the intersection of all the maximal normal subgroups of $G$ and we will use the subdirect irreducible theorem in universal algebra which says any algebra can be expressed as a subdirect product of subdirectly irreducible algebras. Then $J(G)$ is a characteristic subgroup of $G$ and also $J(G/J(G)) = 1$ (By [3], page 4). So $J(G)$ is a normal subgroup of $G$ such that the quotient $G/J(G)$ is the direct product of simple groups. (By [2], Lemma 4.1). Now if $J(G)$ is trivial then $G$ is direct product of simple groups. Otherwise, let $J(G)$ be a non trivial then we can get a minimal normal subgroup $P$ such that $P$ is a subgroup of $J(G)$. Then we take the quotient $G/P$ and get a normal covering. Proceeding this way it reduces to the case when $G$ is a direct products of simple groups as $G$ embeds in $G/N_1 \times G/N_2 \times \cdots \times G/N_k$ such that $1 = \cap_{i=1}^{k} N_i$ where each $N_i$ is a maximal normal subgroup and where $k$ is a natural number. Now if $H$ is a non-solvable non-simple group of order $n$ then $H$ is direct product of isomorphic non-abelian simple groups by Corollary 2.8, direct product of $k$ copies of non-abelian simple groups has only $k$ maximal normal subgroups, which proves the theorem. \hfill $\square$

3 Some New Results

In this section we prove Theorem 3.7, which is the main result in this section.

Definition 3.1. We define $C_n$ as the set of all distinct composition series of the group $G$.

Theorem 3.2. Let $n \geq 2$ be a positive integer where $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}$. Then

$$|C_{Z_n}| = \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}.$$

Proof. We know that for each positive divisor $d$ of $n$ there exists a unique subgroup of $\mathbb{Z}_n$ of order $n$. Let $S$ be the set of sequence of primes from the set $\{p_1, \ldots, p_r\}$ of length $\alpha_1 + \alpha_2 + \cdots + \alpha_r$, where $p_i$ occurs $\alpha_i$ times. Then

$$|S| = \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}.$$

Now $C_{Z_n}$ is the set of distinct composition series of $\mathbb{Z}_n$. Now define a function $f : S \to C_{Z_n}$ by

$$f(\beta_1\beta_2\ldots \beta_{\alpha_1+\alpha_2+\cdots+\alpha_r}, \beta_{\alpha_1+\alpha_2+\cdots+\alpha_r}) = \{e\} \leq Z_{\beta_1} \leq Z_{\beta_1\beta_2} \leq \cdots \leq Z_{\beta_1\beta_2\ldots \beta_{\alpha_1+\alpha_2+\cdots+\alpha_r}} = Z_n,$$

where $\beta_1, \beta_2, \ldots, \beta_{\alpha_1+\alpha_2+\cdots+\alpha_r}$ are primes such that $\beta_i = p_i$ has exactly $\alpha_i$ solutions for $1 \leq i \leq \alpha_1 + \alpha_2 + \cdots + \alpha_r$. So $\beta_1\beta_2\ldots \beta_{\alpha_1+\alpha_2+\cdots+\alpha_r} \in S$. Now $\{e\} \leq Z_{\beta_1} \leq Z_{\beta_1\beta_2} \leq \cdots \leq Z_{\beta_1\beta_2\ldots \beta_{\alpha_1+\alpha_2+\cdots+\alpha_r}} = Z_n$ is a composition series of $Z_n$ where $\beta = \prod_{i=1}^{r} \beta_i$, $1 \leq i \leq \alpha_1 + \alpha_2 + \cdots + \alpha_r$. Then $f : S \to C_{Z_n}$ is an injective mapping by its construction. Now we will prove that it is surjective also. Let $\{e\} = G_0 \leq G_1 \leq \cdots \leq G_{\alpha_1+\alpha_2+\cdots+\alpha_r} = Z_n$ be a composition series of $Z_n$. Since for each divisor $d$ of $n$, there exists a unique subgroup of order $d$ of $\mathbb{Z}_n$ and any subgroup of $\mathbb{Z}_n$ is cyclic then $|G_1| \div |G_{i-1}|$ is a prime number.

3
Define $g : C_{Z_n} \rightarrow S$ by
\[ g(C_{Z_n}) = q_1 q_2 \cdots q_{\alpha_1 + \alpha_2 + \cdots + \alpha_r}, \]
where $q_i = \frac{|G_i|}{|G_{i-1}|}$, for $1 \leq i \leq \alpha_1 + \alpha_2 + \cdots + \alpha_r$. Then each $q_i$ is a prime for $1 \leq i \leq \alpha + \alpha_2 + \cdots + \alpha_r$.

So $q_1 q_2 \cdots q_{\alpha_1 + \alpha_2 + \cdots + \alpha_r}$ is a sequence of primes such that $q_i = p_i$ has exactly $\alpha_i$ solutions for $1 \leq i \leq \alpha_1 + \alpha_2 + \cdots + \alpha_r$. So, $q_1 q_2 \cdots q_{\alpha_1 + \alpha_2 + \cdots + \alpha_r} \in S$ and therefore $f^{-1} = g$ and hence $f$ is surjective and hence bijective also. Therefore,
\[ |S| = |C_{Z_n}| = \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!} \]
i.e. $Z_n$ has $\frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ distinct composition series.

**Example 3.3.** Let us try to understand the above theorem for $Z_{360}$. Note that, $360 = 2^3 3^2 5^1$. Now $\prod_{i=1}^{(3+2+1)} = 60$ distinct sequence of primes can be formed of length 6 such that each sequence contains 3 times prime 2, 2 times prime 3 and 1 time prime 5. Some of them are 232253, 323522, 523322. Now we will construct a composition series corresponding to these numbers.

1. The composition series corresponding to 232253 will be
\[ \{e\} \trianglelefteq Z_2 \trianglelefteq Z_6 \trianglelefteq Z_{12} \trianglelefteq Z_{24} \trianglelefteq Z_{120} \trianglelefteq Z_{360}. \]

2. The composition series corresponding to 323522 will be
\[ \{e\} \trianglelefteq Z_3 \trianglelefteq Z_6 \trianglelefteq Z_{18} \trianglelefteq Z_{90} \trianglelefteq Z_{180} \trianglelefteq Z_{360}. \]

3. The composition series corresponding to 523322 will be
\[ \{e\} \trianglelefteq Z_5 \trianglelefteq Z_{10} \trianglelefteq Z_{30} \trianglelefteq Z_{90} \trianglelefteq Z_{180} \trianglelefteq Z_{360}. \]

**Theorem 3.4.** Let $G$ be an abelian group of order $n$ where $n \geq 2$ be a natural number such that $n = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$. Then $|C_G| = \prod_{i=1}^{r} t_i (\frac{\sum_{i=1}^{r} \alpha_i!}{\prod_{i=1}^{r} \alpha_i!})$, where $t_i$ are the numbers of distinct composition series of the Sylow $p_i$-subgroups of $G$.

**Proof.** Note that in finite abelian group there exists at least one subgroup of order $h$ for each divisor $h$ of the order of the group $n$ and also each composition factor of an abelian group is of prime order. The order of composition factors of an finite abelian group is a prime number and using them we can make a sequence of primes which belongs to $S$ (which is described in the Theorem 3.2).

Therefore the number of distinct composition series of $G$ is a multiple of $\prod_{i=1}^{r} t_i (\frac{\sum_{i=1}^{r} \alpha_i!}{\prod_{i=1}^{r} \alpha_i!})$. Now let $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G_{m+1} \trianglelefteq \cdots \trianglelefteq G_{\alpha_1 + \alpha_2 + \cdots + \alpha_r} = G$ be a composition series of $G$ where $\alpha_i$ be the least positive integer such that $p_1^{\alpha_1}$ divide $G_j$ and $k$ be the least positive integer such that $p_1^{\alpha_1 + k}$ divide $|G_k|$ when
\[ 1 \leq j < k \leq \alpha_1 + \alpha_2 + \cdots + \alpha_r, 1 \leq s_1 < s_2 + 1 \leq \alpha_1. \]

Therefore $|G_j| = p_1^{s_1} q$ where $p_1^{s_1}, q \in \mathbb{N}$ with $\gcd(p_1^{s_1}, q) = 1$. $|G_k| = p_1^{s_1 + k} u$, where $p_1^{s_1 + k}, u \in \mathbb{N}$ with $\gcd(p_1^{s_1 + k}, u) = 1$.

Let $G_j = A_j B_j$ where $A_j$ and $B_j$ are two subgroups of $G$ of order $p_1^{s_1}$ and $q$ respectively. Let $G_k = C_k D_k$ where $C_k$ and $D_k$ are two subgroups of $G$ of order $p_1^{s_1 + k}$ and $u$ respectively. We will prove that $A_j \subseteq C_k$; if not then, $\exists$ some $x \in A_j$ such that $x \notin C_k$, $x \in A_j \implies x = p_1^{s_1}$. Now $x \in A_j \implies x \in G_{j+1}$ as $G_j$ is a normal subgroup of $G_{j+1}$. $x \in G_{j+1} \implies x \in G_{j+2}$ as $G_{j+1}$ is a normal subgroup of $G_{j+2}$. So proceeding in this way we can prove that $x \in G_k$. Now $G_k = C_k D_k$ where $C_k \cap D_k = \{e\}$, $x \in G_k \implies x \in C_k D_k$, i.e. $x = cd$ where $c \in C_k, d \in D_k$. Therefore $x = p_1^{s_1} d, p_1^{s_1} \in C_k$.

So $p_1^{s_1} = p_1^{s_1} d$ i.e. $p_1^{s_1 - c_1} = d$. As $\gcd(p_1^{s_1}, u) = 1$ so $P_1 \cap D_k = \{e\}$ but $p_1^{s_1 - c_1} \in P_1 \cap D_k$ i.e. $p_1^{s_1 - c_1} = c, p_1^{s_1 - c_1} = p_1^{s_1}$ hence $x (= p_1^{s_1}) \in C_k$, a contradiction.

Therefore in the composition series of Sylow $p_1$-group, $A_j$ is a subgroup of $C_k$. So if the Sylow $p_1$-group has $t_1$ distinct composition series then we have $t_1 (\frac{\sum_{i=1}^{r} \alpha_i!}{\prod_{i=1}^{r} \alpha_i!})$ such possible composition series due to Sylow $p_1$-subgroup in $G$. In a similar way for the other Sylow $p_2$-group, . . . , Sylow $p_r$-group we have $\prod_{i=1}^{r} t_i (\frac{\sum_{i=1}^{r} \alpha_i!}{\prod_{i=1}^{r} \alpha_i!})$ distinct composition series for $G$. \[\square\]
Theorem 3.5. In an elementary abelian group of order \( p^n \), there are \( \prod_{k=1}^{n} \left( \frac{p^k - 1}{p - 1} \right) \) distinct composition series.

Proof. Let \( \{c\} \leq p \leq p^2 \leq \cdots \leq p^k \leq \cdots \leq p^n \) be a composition series for the elementary abelian group of order \( p^n \). To find the number of distinct composition series, we will find the possible choice from \( p^{k-1} \) to \( p^k \) at first.

Let there be \( C_{k-1} \) choice available for \( p^{k-1} \) to \( p^k \). Now in an elementary abelian group of order \( p^n \) there are

\[
\prod_{i=0}^{k-1} \left( \frac{(p^n - 1) - (p^i - 1)}{p^k - p^i} \right) = A \text{ (say)}
\]
distinct subgroups of order \( p^k \) each of which is an elementary abelian (By [10], page - 65). In an elementary abelian group of order \( p^n \), also there are

\[
\prod_{i=0}^{k-2} \left( \frac{(p^n - 1) - (p^i - 1)}{p^k - p^i} \right) = B \text{ (say)}
\]
distinct subgroups of order \( p^{k-1} \) each of which is an elementary abelian (By [10], page-65). Again in an elementary abelian group of order \( p^k \) there are

\[
\prod_{i=0}^{k-2} \left( \frac{(p^k - 1) - (p^i - 1)}{p^k - p^i} \right) = D \text{ (say)}
\]
distinct subgroups of order \( p^{k-1} \) each of which is elementary abelian (By [10], page-65). Now there is an identical algebraic structure of any two subgroups of same order in an elementary abelian group i.e. in elementary abelian group of order \( p^n \), any two distinct subgroups of order \( p^k \) have the same number of subgroup order \( p^{k-1} \). Then

\[
C_{k-1} = \frac{A-B}{D} = \frac{p^{n+1-k} - 1}{p - 1}.
\]

As a result we have \( \prod_{i=1}^{n-1} \left( \frac{p^{n+1-i} - 1}{p - 1} \right) \) distinct possible choice for the composition series of an elementary abelian group of order \( p^n \) (as \( k \geq 1 \)), which is same as

\[
\prod_{k=1}^{n} \left( \frac{p^{n+1-k} - 1}{p - 1} \right) = \prod_{k=1}^{n} \left( \frac{p^k - 1}{p - 1} \right) = \prod_{k=1}^{n} \left( \frac{p^k - 1}{(p - 1)^n} \right).
\]

\[\square\]

Theorem 3.6. Let \( G \) be an abelian group with elementary abelian Sylow subgroups of order \( n \geq 2 \) where \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_r^{\alpha_r} \), then

\[
|G| = \prod_{i=1}^{r} \left( \prod_{j=1}^{\alpha_i} \frac{p_j^i - 1}{p_j - 1} \right) \left( \sum_{j=1}^{\alpha_i} \alpha_i \right)! / \prod_{i=1}^{\alpha_i} \alpha_i !.
\]

Proof. This followed from theorem 3.4 and theorem 3.5 \[\square\]

Theorem 3.7. Let \( G \) be a finite group of order \( n \) where \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_r^{\alpha_r} \), then

\[
|G| \leq \prod_{i=1}^{r} \left( \prod_{j=1}^{\alpha_i} \frac{p_j^i - 1}{p_j - 1} \right) \left( \sum_{j=1}^{\alpha_i} \alpha_i \right)! / \prod_{i=1}^{\alpha_i} \alpha_i !.
\]

and the equality is hold if and only if \( G \) is the abelian group with elementary abelian sylow subgroups.

Proof. This followed from theorem 2.1 and theorem 3.6. \[\square\]
4 Main Result

In this section we prove theorem 4.3 which is the main theorem of the paper. This section is more elementary and computational.

Lemma 4.1. Let \( X = \prod_{i=0}^{a_1+k}(2^i - 1) \alpha_1! \alpha_r! \) and \( Y = \prod_{i=1}^{\alpha_r}(p - 1)(\alpha_1 + \alpha_r)! \), where \( p \geq 5 \) is a prime and \( \alpha_1 \geq 0 \) is an integer, \( \alpha_r \in \mathbb{N} \), such that \( t = p^{\alpha_r} \) and \( \lfloor \log_2 t \rfloor = k \). Also \( \alpha_r \geq 2 \) whenever \( p = 3 \). Then \( X \not| Y \) is a monotone increasing function of \( \alpha_1 \).

Proof. Let

\[
f(\alpha_1) = \frac{\prod_{i=1}^{a_1+k}(2^i - 1)}{\prod_{i=1}^{\alpha_1}(2^i - 1) \prod_{i=1}^{\alpha_r}(p - 1)(\alpha_1 + \alpha_r)!}.
\]

Then

\[
f(\alpha_1 + 1) = \frac{\prod_{i=1}^{a_1+k+1}(2^i - 1)}{\prod_{i=1}^{\alpha_1+1}(2^i - 1) \prod_{i=1}^{\alpha_r}(p - 1)(\alpha_1 + \alpha_r)!}.
\]

Therefore

\[
\frac{f(\alpha_1 + 1)}{f(\alpha_1)} = \frac{2^{a_1+k+1} - 1}{2^{a_1+1} - 1} \frac{\alpha_1 + 1}{\alpha_1 + \alpha_r + 1}.
\]

In order to prove \( f(\alpha_1 + 1) > f(\alpha_1) \) we have to prove that

\[
\frac{2^{a_1+k+1} - 1}{2^{a_1+1} - 1} \frac{\alpha_1 + 1}{\alpha_1 + \alpha_r + 1} > 2^k \quad \text{and} \quad \alpha_r + 1 \geq \frac{\alpha_1 + 1}{\alpha_1 + 1}.
\]

As \( \frac{2^{a_1+k+1} - 1}{2^{a_1+1} - 1} > 2^k \) and \( \alpha_r + 1 \geq \frac{\alpha_1 + 1}{\alpha_1 + 1} \). So it is sufficient enough to prove

\[
2^k > \alpha_r + 1 \iff k > \log_2(\alpha_r + 1).
\]

Now \( \log_2 t = k + f \), where \( 0 < f < 1 \). So \( k = \log_2 t - f \). Therefore, we have to prove that

\[
\log_2 t - f > \log_2(\alpha_r + 1) \iff \log_2 \left( \frac{t}{\alpha_r + 1} \right) > f.
\]

As \( 0 < f < 1 \) it is sufficient enough to prove \( \log_2 \left( \frac{t}{\alpha_r + 1} \right) > 1 \) since \( \log x \) is a monotone increasing function for \( x > 0 \), i.e.

\[
\frac{t}{\alpha_r + 1} > 2 \iff p^{\alpha_r} > 2\alpha_r + 2
\]

for all prime \( p \geq 5 \) with \( \alpha_r \geq 1 \) and \( p = 3 \) with \( \alpha_r \geq 2 \), which can easily be proved using mathematical induction. Hence \( \frac{t}{\alpha_r + 1} \) is a monotone increasing function of \( \alpha_1 \).

Theorem 4.2. Among all \( p \)-groups of order \( \leq n \), where \( n \geq 4 \) is a positive integer; the elementary abelian group of order \( 2^n \) where \( \alpha = \lfloor \log_2 n \rfloor \), has the highest number of composition series.

Proof. We know that among all finite \( p \)-groups of equal order the elementary abelian group has the highest number of composition series. Let \( G_2 \) be an elementary abelian group of order \( 2^n \), where \( \alpha = \lfloor \log_2 n \rfloor \) and let \( G_p \) be an elementary abelian \( p \)-group of order \( p^\beta \), where \( \beta = \lfloor \log_p n \rfloor \) with \( p \geq 3 \) be a prime and \( n \geq 4 \).

Now we know that if \( \lfloor \log_q r \rfloor = t \) then \( \lfloor \log_q qr \rfloor = t + 1 \) for all prime \( q \geq 2 \) and \( r, t \in \mathbb{N} \) with \( q \leq r \). Obviously \( 2 \) divides \( p - 1 \). Therefore we get

\[
2^{\lfloor \log_2 n \rfloor - 1} > \frac{p^{\lfloor \log_p n \rfloor} - 1}{p - 1}, \quad \text{whenever} \ n \geq 4.
\]

\[
| C_{G_2} | = \prod_{i=1}^{\lfloor \log_2 n \rfloor} (2^i - 1),
\]

\[
| C_{G_p} | = \prod_{i=1}^{\lfloor \log_p n \rfloor} \left( \frac{p^i - 1}{p - 1} \right).
\]
From (1) it implies that
\[ \prod_{i=1}^{[\log_2 n]} (2^i - 1) > \prod_{i=1}^{[\log_2 n]} \left( \frac{p^i - 1}{p - 1} \right) \]
i.e. \(|C_{G_2}| > |C_{G_r}|\). Therefore among all \(p\) groups of order \(\leq n\), the elementary abelian group of order \(2^n\) has the highest number of composition series where \(\alpha = [\log_2 n]\).

**Theorem 4.3.** Let \(n \geq 4\) be a positive integer. Among all finite groups of order \(\leq n\), the elementary abelian group of order \(2^n\) has the highest number of composition series where \(\alpha = [\log_2 n]\).

**Proof.** \(n \geq 4\) is a natural number and \(\alpha = [\log_2 n]\) and let \(G\) be the elementary abelian group of order \(2^n\) then
\[ |C_G| = \prod_{i=1}^{[\log_2 n]} (2^i - 1) . \]
As we know that among all finite groups of order \(n \geq 2\), the abelian group with elementary abelian Sylow subgroups has the highest number of composition series. So in order to prove our claim we have to prove that
\[ \prod_{i=1}^{[\log_2 n]} (2^i - 1) \geq \prod_{i=1}^{q} \left( \frac{\beta_i}{p_i - 1} \right) \prod_{i=1}^{r} \left( \frac{\sum_{j=1}^{q} \beta_j!}{\prod_{j=1}^{r} \beta_j!} \right) \]
where each \(p_i\) is a prime and \(\beta_i, q \in \mathbb{N}\) such that \(\prod_{i=1}^{q} \beta_i! \leq n\) and the equality is hold if and only if \(q = 1, p_1 = 2\) and \(\beta_1 = [\log_2 n]\). The inequality will be proved in the following steps.

Step 1: Let \(H\) be an abelian group of order \(m\) with elementary abelian Sylow subgroups such that \(4 \leq m \leq n\) where \(m = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_r^{\alpha_r}\) also \(|C_H|\) denotes the number of distinct composition series of \(H\). Let \(p_r\) be a prime such that \(p_r \neq 2\) and \(p_r^{\alpha_r} = t\), \([\log_2 t] = k\). Then \(2^k < p_r^{\alpha_r}\). Let \(H'\) be the abelian group of order \(\frac{m}{p_r^{\alpha_r}}\) with elementary abelian Sylow subgroups. Then \(|H'| < |H|\). We will prove that \(|C_{H'}| > |C_H|\).

Step 2: If \(|H'| = 2^k p_1^{\alpha_1} \cdots p_r^{\alpha_r-1}\) i.e. 2 is not a divisor of \(m\) then we have \(k > \alpha_r\) and
\[ \prod_{i=1}^{k} (2^k - 1) > \prod_{i=1}^{\alpha_r} \left( \frac{p_i^k - 1}{p_i - 1} \right) . \]
Therefore, obviously \(|C_{H'}| > |C_H|\).

Step 3: If \(|H| = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}\) then \(|H'| = 2^{\alpha_1+k} p_2^{\alpha_2} \cdots p_r^{\alpha_r-1}\). In this case we will prove that \(|C_{H'}| > |C_H|\).
\[ |C_H| = \prod_{i=1}^{\alpha_1} (2^{i-1}) \prod_{i=1}^{r} \left( \frac{p_i^{\alpha_i} - 1}{p_i - 1} \right) \prod_{i=2}^{r-1} \left( \frac{\prod_{j=1}^{\alpha_i} p_j^i - 1}{\prod_{j=1}^{r-1} \alpha_i!} \right) , \]
\[ |C_{H'}| = \prod_{i=1}^{\alpha_1+k} (2^{i-1}) \prod_{i=2}^{r-1} \left( \frac{\prod_{j=1}^{\alpha_i} p_j^i - 1}{\prod_{j=1}^{r-1} \alpha_i!} \right) \frac{\prod_{j=1}^{r} \alpha_i + k}{\prod_{j=1}^{r-1} \alpha_i!} , \]
Therefore,
\[ \frac{|C_{H'}|}{|C_H|} = \prod_{i=\alpha_1+1}^{\alpha_1+k} (2^{i-1}) \frac{(2^i - 1) (k + s)! \alpha_1! \alpha_r!}{\prod_{j=1}^{\alpha_r} \frac{p_j^i - 1}{p_j - 1} (\alpha_1 + k)! (\alpha_r + s)!} . \]
where \(s = \sum_{i=1}^{r-1} \alpha_i\). So in order to prove our claim we have to prove that
\[ \prod_{i=\alpha_1+1}^{\alpha_1+k} (2^{i-1}) (k + s)! \alpha_1! \alpha_r! > \prod_{j=1}^{\alpha_r} \left( \frac{p_j^i - 1}{p_j - 1} \right) (\alpha_1 + k)! (\alpha_r + s)! \]
(2)
Note that \( s = \sum_{i=1}^{r-1} \alpha_i \geq \alpha_1 \), \( \Rightarrow s = \alpha_1 + a \), where \( a \geq 0 \), \( k > \alpha_r \Rightarrow k = \alpha_r + b \), \( b > 0 \); \( a, b \in \mathbb{Z} \).

Then inequality (3) reduces to

\[
\prod_{i=\alpha_1+1}^{\alpha_1+k} (2^i - 1) (\alpha_1 + \alpha_r + a + b)! \alpha_1! \alpha_r! > \prod_{j=1}^{\alpha_r} \left( \frac{p_j^k - 1}{p_j - 1} \right) (\alpha_1 + \alpha_r + a)! (\alpha_1 + \alpha_r + b)!
\]  

(3)

[Here we are ignoring the case \( p_r = 3 \) with \( \alpha_r = 1 \), as \( m \geq 4 \) at least one other case must exist.] Note

\[
\frac{(\alpha_1 + \alpha_r + a + b)! \alpha_1! \alpha_r!}{(\alpha_1 + \alpha_r + a)! (\alpha_1 + \alpha_r + b)!}
\]

is a monotone increasing function of \( a \) for fixed \( \alpha_1, \alpha_r, b \). So it is sufficient enough to prove the inequality for \( a = 0 \) i.e.

\[
\prod_{i=\alpha_1+1}^{\alpha_1+k} (2^i - 1) \alpha_1! \alpha_r! > \prod_{j=1}^{\alpha_r} \left( \frac{p_j^k - 1}{p_j - 1} \right) (\alpha_1 + \alpha_r)!
\]  

(4)

Let \( X = \prod_{i=\alpha_1+1}^{\alpha_1+k} (2^i - 1) \alpha_1! \alpha_r! \) and \( Y = \prod_{j=1}^{\alpha_r} \left( \frac{p_j^k - 1}{p_j - 1} \right) (\alpha_1 + \alpha_r)! \). Then according to lemma 4.1 \( X \) is a monotone increasing function of \( \alpha_1 \) whenever \( p_r \) and \( \alpha_r \) are fixed. So it is sufficiently enough to prove the inequality (4) for \( \alpha_1 = 0 \) i.e.

\[
\prod_{i=1}^{k} (2^i - 1) > \prod_{j=1}^{\alpha_r} \left( \frac{p_j^k - 1}{p_j - 1} \right),
\]

which is following from theorem 4.2. As a result we prove that \(| C_{H'} | > | C_H |\). Now we will repeat the same process in the group \( H' \) and get a group \( H'' \) such that \(| C_{H''} | > | C_{H'} |\). Since \(| H' |\) is finite then \(| H' |\) has only finitely many prime divisors. So after a finite number of steps we get an elementary abelian group of order \( 2^n \), where \( n = \left\lfloor \log_2 n \right\rfloor \).

Step 4 : Let \(| H | = 2^{\alpha_1+3} \) for any \( \alpha_1 \in \mathbb{N} \). Then \(| C_H | = \prod_{i=1}^{\alpha_1} (2^i - 1) (\alpha_1 + 1) \). Then we can find an elementary abelian group \( H' \) such that \(| H' | = 2^{\alpha_1+1} \) and \(| C_{H'} | = \prod_{i=1}^{\alpha_1} (2^i - 1) \). Obviously \(| C_{H'} | > | C_H |\) as \( 2^{\alpha_1+1} > \alpha_1 + 2 \) for each \( \alpha_1 \in \mathbb{N} \). Among all 2 groups of order \( \leq n \), \( G \) has the highest number of composition series which proves the theorem.

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