Optimal linear Glauber model

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Abstract. For a generic Ising system, the nonlinear glauber model (NLGM) is not exactly solvable although the detailed balance condition at equilibrium is exactly satisfied by the nonlinear choice of transition rate \( w_j \). On the contrary, the linear Glauber model (LGM), where the choice of \( w_j \) is linear, is exactly solvable although the detailed balance condition is not exactly satisfied. In this work we address the issue of writing \( w_j \) in a best possible linear form such that the mean squared error in satisfying the detailed balance condition is least. For a generic system where a spin is coupled to \( z \) neighbors with different coupling constants, our linear choice of \( w_j \) involves \( z \) independent parameters whose optimal values are derived by a linear regression process. The Moore-Penrose pseudoinverse matrix involved in the regression process depends solely on the configuration matrix and takes a simple form of dimension \( z \times 2^z \).

The work presented in this paper will help us anticipate how the kinetic properties of an arbitrary Ising system depend on the temperature and the coupling constants. We analyze the critical and dynamic properties of the linear model in the present context. We will also show in this paper that the effect of magnetic field \( (H) \) can easily be studied within our approach; in particular, our study finds that the critical temperatures \( (T_C) \) for two and three dimensional systems decrease quadratically with (weak) magnetic field \( (\delta T_C/T_C \propto -H^2) \).

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1. Introduction

It will not be probably too much of a generalization to say that everything in the universe is evolving with some time scale. The notion of equilibrium is a limiting concept which is well understood within the realms of the equilibrium thermodynamics and statistical mechanics. Though a widely accepted formalism exists for studying systems in equilibrium, we are yet to develop a general framework to study some irreversible process. The non-equilibrium statistical mechanics is a very active field where new exciting results are appearing regularly. To get better understanding of how systems
evolve, it is important to study dynamics of simple physical models. The Ising model is probably the simplest non-trivial model in physics. Studying dynamics of this model may give us some new insights into the general feature of a non-equilibrium process.

Even for this simple Ising model, studying dynamics is not generally an easy task. Roy J. Glauber showed in his classic original work how one can study dynamics in a simple (ferromagnetic) Ising chain [1]. Since then this work has been extended to study, both analytically and numerically, different systems in numerous physical situations with varying degrees of success [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In Glauber model, the form of transition rate (between two possible orientations of a spin) in studying dynamics is not unique; it can be chosen according to convenience provided that it satisfies the equation of detailed balance at equilibrium for all possible configurations of neighbors. It is possible to get a non-linear form of the transition rate for which the detailed balance condition is exactly satisfied for all configurations. Unfortunately, with this non-linear form of transition rate, analytical calculations become intractable. It has been a real challenge to develop a kinetic model which will be solvable for a generic system without compromising on the detailed balance condition. In this regard, there exists some popular approaches with varying degrees of success, e.g., the mean field approaches [12], the effective field approach [13], etc.

A widely studied kinetic model is the linear Glauber model (or the voter model with noise [14, 15, 16]). In this model a linear form of the transition rate is taken; a parameter is introduced here which determines the strength of the noise. This model is exactly solvable in all dimensions; its properties have been extensively studied as a function of the parameter. For one dimensional system (coordination number $z = 2$), it is possible to relate this parameter to the temperature and the coupling constant in such a way that the detailed balance condition is exactly satisfied [1, 14, 15].

We now notice that, on the one hand, the actual nonlinear Glauber model (NLGM) is not exactly solvable for a generic Ising system although the detailed balance condition is exactly satisfied, and on the other hand, the linear Glauber model (LGM) is exactly solvable but the detailed balance condition is not exactly satisfied. In this context we pose the following question: for a generic Ising system, what is the best value of the parameter of the linear model for which the mean squared error in satisfying the detailed balance condition is least? or in other words, what is the best way of relating the parameter of the LGM to the temperature and the coupling constant such that the results obtained from the model are as close as possible to the results obtained from the nonlinear Glauber model?

In this paper we address the above issue in a more general set-up. For a generic Ising system where a spin is coupled to $z$ neighbors with different coupling constants, we show how the nonlinear exact form of the transition rate can be linearized in an optimal way. Here the optimization is done by a linear regression process; this process ensures that, the mean squared error of a relevant quantity is minimum. For the obvious reason, we call the present model the optimal linear Glauber model (OLGM). The Moore-Penrose pseudoinverse matrix involved in the regression process is obtained solely from
the configuration matrix and takes a simple form of dimension $z \times 2^z$. This pseudoinverse matrix does not depend on the Hamiltonian parameters or dimensionality of system, this makes our approach very appealing. For one dimensional system ($z = 2$), our method can reproduce results of Glauber’s original work; this shows generality of our approach. Further more, at criticality it is shown that our model reduces to the voter model (without noise).

The work presented in this paper will help us anticipate how the kinetic properties of an arbitrary Ising system depend on the temperature and the coupling constants. In the present context, we study here both the static and dynamic scaling properties of Ising systems in different dimensions. It is easy to study the effect of magnetic field within our approach. In this paper we demonstrate that our transition rate in the presence of magnetic field works more efficiently than the commonly used one. In particular, using our approach, we estimate the critical temperatures of two and three dimensional Ising systems and show that the critical temperature decreases quadratically with weak magnetic field.

Regarding the nature of the steady state that we obtain from the OLGM, we will later see that (section 2.5), although the local probability currents are non-zero in the steady state, its average over all possible configurations of the neighbors is zero; in addition, our approach ensures that the individual opposite currents are on average as small in strength as possible. This facts allow us to safely say that, within a linearization approach, the steady state that we get here is as close to the equilibrium state as possible.

Our paper is organized in the following way. In section 2, we give a detailed description of our approach. In the next section (sec 3), we apply our method to study different Ising systems. We conclude our work in section 4.

2. General theory

Let us consider a system of $N$ interacting Ising spins ($\sigma_i = \pm 1$). These spins can be arranged in any spatial dimension where each spin is assumed to interact with $z$ neighbors ($z$ is called coordination number). In our approach, nature and strength of the coupling constants can be different. We here assume that these neighbors are not directly interacting with each other, i.e., there is no next-nearest neighbor interactions. Let us now consider that $p(\{\sigma\};t)$ be the probability that the spins take the values $\sigma_1, \ldots, \sigma_N$ at time $t$. We note that there can be $2^N$ possibilities of spin configurations, and sum of the probabilities corresponding to all these possibilities is 1. We now assume that $w_j(\sigma_j)$ is the transition rate of $j$th spin, i.e., the probability per unit time that the $j$th spin will flip from the state $\sigma_j$ to $-\sigma_j$ while the neighboring spins are momentarily remain fixed. This rate should intuitively depend on the states of neighbors. We will later discuss in detail the form of $w_j$’s. Following Glauber, we can now write the master equation which gives the time derivative of the probability:

$$\frac{d}{dt} p(\{\sigma\};t) = - \sum_j w_j(\sigma_j)p(\{\sigma\};t) + \sum_j w_j(-\sigma_j)p(R_j\{\sigma\};t),$$  \hspace{1cm} (1)
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where $R_j\{\sigma\}$ represents the same spin configuration as $\{\sigma\}$ with $j$th spin flipped.

By considering $\sigma_i(t)$ as stochastic function of time, we can define two important quantities, namely, a time dependent average spin value $q_i(t)$ and a time dependent correlation function $r_{i,j}(t)$. These are given below,

$$q_i(t) = \langle \sigma_i(t) \rangle = \sum_{C(N)} \sigma_i p(\{\sigma\};t) \quad (2)$$

$$r_{i,j}(t) = \langle \sigma_i(t)\sigma_j(t) \rangle = \sum_{C(N)} \sigma_i\sigma_j p(\{\sigma\};t). \quad (3)$$

Here sum is over all possible ($2^N$ in number) spin configurations, $C(N)$. It may be noted that $r_{i,i} = 1$.

We now write time derivative of these quantities as first step to obtain them as function of time. It is easy to get them by multiplying $\sigma_k$ and $\sigma_j\sigma_k$ respectively to the Eq. (1) and then sum them over all possible spin configurations. Following Glauber, these time derivatives can be written as,

$$\frac{d}{dt}q_k(t) = -2 \sum_{C(N)} \sigma_kw_k p(\{\sigma\};t) \quad (4)$$

$$\frac{d}{dt}r_{j,k}(t) = -2 \sum_{C(N)} \sigma_j\sigma_k \{w_j(\sigma_j) + w_k(\sigma_k)\} p(\{\sigma\};t). \quad (5)$$

To solve these equations, a choice of $w_j(\sigma_j)$ has to be made. As we may expect, the tendency of the $j$th spin to be up or down should depend on the states of the neighboring spins as well on the nature and strength of the coupling constants between $j$th spin and the neighboring spins. For example, if the $j$th and $k$th spins are coupled by a ferromagnetic interaction, then the $j$th spin will try to align itself parallel to the $k$th spin. There can be many ways to choose $w_j(\sigma_j)$ to obey this tendency, although the option is constrained by the fact that it has to satisfy the equation of detailed balance at equilibrium for all possible configurations of neighbors. Now we will introduce a general mathematical approach by which one will be able to find the optimal linear form of the transition rate for an arbitrary Ising system.

When the system reaches equilibrium at temperature $T$, the probability $p(\sigma_j)$ that the $j$th spin will be in the state $\sigma_j$ as opposed to $-\sigma_j$ (for a given configuration of neighbors), is just proportional to the Maxwell-Boltzmann factor $e^{-\beta h_j(\sigma_j)}$. Here $\beta = 1/k_B T$ with $k_B$ being the Boltzmann constant. In the factor, $h_j(\sigma_j)$ is the interaction energy associated with the $j$th spin (when in the state $\sigma_j$) with its neighbors, and this is given by

$$h_j(\sigma_j) = \sum_{k=1}^z J_k \sigma_j \cdot \sigma_k, \quad (6)$$

where $J_k$ is the coupling constant between the $j$th and $k$th spins. In the equilibrium, for a momentarily fixed configuration of other spins, the $j$th spin should satisfy the equation of detailed balance,

$$\frac{p(\sigma_j)}{p(-\sigma_j)} = \frac{w_j(-\sigma_j)}{w_j(\sigma_j)}. \quad (7)$$
To proceed further, let us now write the probability factor in the following way:

\[
e^{-\beta h_j(\sigma_j)} = \cosh \left[ \beta \sum_{k=1}^{z} J_k \sigma_k \right] - \sigma_j \sinh \left[ \beta \sum_{k=1}^{z} J_k \sigma_k \right]
\]

\[
= \cosh \left[ \beta \sum_{k=1}^{z} J_k \sigma_k \right] \left\{ 1 - \sigma_j \tanh \left[ \beta \sum_{k=1}^{z} J_k \sigma_k \right] \right\}
\]

(8)

If we use the above expression of the probability factor in the equation of detailed balance, Eq. (7), we immediately get an exact form of \( w_j \),

\[
w_j(\sigma_j) = \frac{\alpha}{2} \left( 1 + \sigma_j \tanh \left[ \beta \sum_{k=1}^{z} J_k \sigma_k \right] \right),
\]

(9)

where \( \alpha/2 \) is the transition rate for non-interacting case. This nonlinear form of \( w_j \) not only fulfills the orientational tendencies of \( j \)th spin mentioned earlier, it exactly satisfies the equation of detailed balance (Eq. (7)) at equilibrium for all possible configurations. Unfortunately, this nonlinear form is intractable for the analytical study of dynamics. A linear form of \( w_j \) is easy to handle, but, except for a few special cases, it does not exactly satisfy the detailed balance condition. It will be shown here how this nonlinear \( w_j \) can be linearized in an optimal way such that mean squared error in satisfying detailed balance condition is least. It will be also clear in the process why we have chosen this particular nonlinear form of \( w_j \) while one has other options.

Noting the series \( \tanh x = x - \frac{x^3}{3} + \cdots \), we can attempt to linearize \( w_j \) by considering,

\[
\tanh \left[ \beta \sum_{k=1}^{z} J_k \sigma_k \right] \approx \sum_{k=1}^{z} \gamma_k \sigma_k.
\]

(10)

Here the coefficients \( \gamma_k \)'s are not just \( \beta J_k \)'s that appear in the first order term of the hyperbolic-tan series. These coefficients also have contributions from the higher order terms of the series (this will be clear by noting that, \( \sigma_k^n = 1 \) if \( n \) is even and \( \sigma_k^n = \sigma_k \) if \( n \) is odd). Although by analyzing the series it is possible to find out the exact values of \( \gamma_k \)'s, it is best to take the optimized values for the \( \gamma_k \)'s which can be obtained by a linear regression process. By taking the optimized values, we ensure that the mean squared error in satisfying the detailed balance condition is least. The optimization process somewhat compensates the absence of the nonlinear terms in our desired linear form of \( w_j \) (nonlinear terms are typically product of different \( \sigma \)'s).

To do a linear regression, we will consider \( \gamma_k \)'s in Eq. (10) as the parameters of the regression process. We may note that, Eq. (10) actually represents \( 2^z \) linear equations in \( z \) parameters. Each of these linear equations corresponds to the one of the \( 2^z \) configurations of the \( z \) neighbors. Obviously, no set of values for the \( \gamma \)'s can simultaneously satisfy the overdetermined set of \( 2^z \) linear equations (except for a special case discussed later). We will now see how the best possible values for \( \gamma \)'s, for which mean squared error is minimum, can be obtained.
Before discussing the linear regression process, it may be worth mentioning here that, the function \( \tanh x \) is linear about the origin \((x = 0)\). Since the term \( \frac{\beta}{2} \sum_{k=1}^{z} J_k \sigma_k \) is zero or close to zero for a good fraction of the total number of configurations (at least for isotropic case when \( J_k \)'s are equal), we expect our linearization to work reasonably good in a normal situation.

### 2.1. Theory of linear regression (LR)

Before we use it, let us first briefly present the linear regression theory [17] necessary for our present work. We consider an overdetermined system of \( m \) linear equations in \( n \) \((< m)\) unknown coefficients \( \mu_1, \mu_2, \ldots, \mu_n \):  

\[
\sum_{j=1}^{n} a_{ij} \mu_j = y_i \quad (i = 1, 2, \ldots, m)
\]

Here \( y_i \)'s are regressands or dependent variables while \( a_{i1}, a_{i2}, \ldots, a_{in} \) are regressors or independent variables. This can be written in the matrix form as,

\[
A \mu = Y, \quad (11)
\]

where \( A \) is an \( m \times n \) matrix with \( a_{ij} \) being the \( ij \)th element, \( \mu \) is a column vector (dimension \( n \times 1 \)) with \( \mu_j \) being the \( j \)th element and \( Y \) is again a column vector (dimension \( m \times 1 \)) with \( y_i \) being its \( i \)th element. For a particular set of values of \( \mu_i \)'s, we can define an error function \( S(\mu) = \sum_{i=1}^{m} |y_i - \sum_{j=1}^{n} a_{ij} \mu_j|^2 = ||Y - A \mu||^2 \); this error function can be minimized with respect to \( \mu_i \)'s to obtain the best possible values for the parameters. It can be easily shown that the minimization problem reduced to finding solution of the following equation,

\[
A^T A \mu = A^T Y. \quad (12)
\]

There are many ways to solve this equation; if \( A^+ \) be the pseudoinverse matrix of \( A \), called the Moore-Penrose pseudoinverse, then the solution can be written as,

\[
\mu = A^+ Y. \quad (13)
\]

If the \( n \) columns of the \( A \) matrix are linearly independent, then \( A^T A \) is invertible and the pseudoinverse matrix can simply be obtained as,

\[
A^+ = (A^T A)^{-1} A^T. \quad (14)
\]

If \((A^T A)^{-1}\) does not exist, then there are ways to get the pseudoinverse matrix \( A^+ \); for example by doing Tikhonov regularization or by doing a singular value decomposition (SVD) of matrix \( A \) [17]. We may note that if \( U \Sigma V^T \) is the SVD of \( A \), then \( A^+ = V \Sigma^+ U^T \), where \( \Sigma^+ \) is just obtained by taking the reciprocal of each non-zero element on the diagonal of matrix \( \Sigma \). For our present problem, the \( n \) column vectors of the \( A \) matrix are linearly independent; this is because they are generated by \( n \) independent Ising variables \( (\sigma_i)'s \).

We can now use the pseudoinverse matrix to get the minimum of the error function \( S(\mu) \). Using Eq. \((13)\) in the error function, we get the following expression for the minimum:

\[
S_{\text{min}} = ||(I - AA^+) Y||^2, \quad (15)
\]

where \( I \) is the \( m \times m \) Identity matrix.
2.2. Application of LR Theory to Glauber dynamics

To apply this linear regression theory to our problem in hand, we first note that, Eq. (10) will give a linear equation in $\gamma$'s corresponding to each of the $2^z$ spin configurations. For us $n = z$ and $m = 2^z$. In Eq. (11), the elements of the column matrix $\mu$ are the parameters $\gamma_i$'s; we will denote this matrix as $\Gamma$. Each of the rows of the $A$ matrix will represent one of the $2^z$ spin configurations of the $z$ neighbors; we will call this matrix as configuration matrix $C$. For $z = 4$, the form of the matrix $C$ can be seen in Eq. (16).

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \end{pmatrix} \quad \text{(for } z = 4) \quad (16)$$

The $i$th element of the column matrix $Y$ is just the value of $\tanh [\beta \sum_{k=1}^{z} J_k \sigma_k]$ for the spin configuration as appears in the $i$th row of the matrix $A$; we will denote this column matrix as $\Omega$. With the present set of relevant notations, Eq. (11) is rewritten as $C \Gamma = \Omega$ for our regression problem. The best possible values of $\gamma_i$'s can be obtained by

$$\Gamma = C^+ \Omega, \quad (17)$$

where $C^+$ is the Moore-Penrose pseudoinverse of the configuration matrix $C$.

We will now determine the form of the pseudoinverse matrix $C^+$ for any coordination number $z$. We first note that $\sum_{i=1}^{2^z} C^2_{i,j} = 2^z$ for any $j$. We also note that, along a row of any two columns $i$ and $j$ ($i \neq j$), there can appear only four configurational states, namely, $1 1, 1 -1, -1 1, -1 -1$. Each of the states appears exactly $2^{z-2}$ times; this is due to the fact that, when two Ising spins are in one of the four states, rest of the spins ($z - 2$ in number) can assume any of the $2^{z-2}$ possible configurations. This implies that, $\sum_{k=1}^{2^z} C_{k,i} C_{k,j} = 0$, i.e., all the $z$ columns of the $C$ matrix are orthogonal. From these facts we can easily conclude that $C^T C$ is a diagonal matrix ($z \times z$ in dimension) with $(C^T C)_{i,j} = 2^z \delta_{i,j}$. This result helps us to write the pseudoinverse matrix $C^+$ in the
following form,

\[ C^+ = (C^T C)^{-1} C^T = 2^{-z} C^T \]  

(18)

By using this pseudoinverse matrix we can get the best possible values of the parameters (\( \gamma \)'s) from the equation \( \Gamma = C^+ \Omega \). This set of parameters can then be used in Eq. (10) to obtain a best possible linearized version of \( w_j(\sigma_j) \),

\[ w_j(\sigma_j) = \frac{\alpha}{2}(1 + \sum_{k=1}^{z} \gamma_k \sigma_j \cdot \sigma_k). \]  

(19)

If we use this linearized form of \( w_j \), the net local probability current between two spin configurations of a site will not be zero in the steady state for the different configurations of neighbors (except for a special case). As a result, this steady state does not qualify for an equilibrium state. However, our approach makes sure that the average local current is zero and the opposite currents are individually as small as possible on the average (cf. section 2.5). This allows us to say that the steady state that we get here is as close to the equilibrium state as possible within a linearization approach.

Practical advantage of this linearized form of \( w_j \) should be clear now. If we use it in Eqs. (4) and (5), we will get two decoupled sets of equations for \( q_k \)'s and \( r_{i,j} \)'s respectively. Solving these sets of equations is much easier as equations for \( q_k \)'s will not contain any correlation term \( r_{i,j} \) and vice versa.

2.3. Reduction of problem by the use of symmetries

The problem of regression can be reduced if \( Z_2 \) symmetry is available, i.e., the external magnetic field is absent. In this case we only need to consider half of the configurations which are not transformed to each other by \( Z_2 \) symmetry. The configuration matrix in this case will be of dimension \( 2^{z-1} \times z \); we denote this matrix as \( \tilde{C} \). We see that the configurations in the first half of the \( C \) matrix (1 to \( 2^{z-1} \) rows) are just the spin flipped version of the configurations in the second half. This simply implies that, the columns of \( \tilde{C} \) matrix are also orthogonal, and consequently \( \tilde{C}^T \tilde{C} \) is a diagonal matrix with \( (\tilde{C}^T \tilde{C})_{i,j} = 2^{z-1} \delta_{i,j} \). In this case,

\[ \tilde{C}^+ = (\tilde{C}^T \tilde{C})^{-1} \tilde{C}^T = 2^{-z+1} \tilde{C}^T \]  

(20)

Now we will discuss another way of reducing the regression problem. We see that in the expression of \( w_j(\sigma_j) \) (see Eq. (19)), \( \gamma_l \) is expected to be same as \( \gamma_m \) if \( J_l = J_m \) in Eq. (6), i.e., if \( j \)th spin is coupled to the \( l \)th and \( m \)th spin by the same coupling constant. In this case, we can work with \( z - 1 \) number of parameters (\( \gamma_i \)'s) instead of considering all \( z \) parameters. We also note that, with this consideration, the matrix (which we denote by \( \tilde{C} \)) whose pseudoinverse we seek in the regression process, will have \( z - 1 \) columns. It should be clear that, one of the columns of the \( \tilde{C} \) matrix will be just the sum of the \( l \)th and \( m \)th columns of the actual configuration matrix \( C \) matrix, and rest of its columns will be same as those of the \( C \) matrix. Properties of the \( \tilde{C} \) matrix can be directly derived from the \( C \) matrix; in fact it can be seen that \( \tilde{C}^T \tilde{C} \) is
a \((z - 1) \times (z - 1)\) diagonal matrix with the diagonal elements being \(2^z\) except for the row/column which corresponds to the sum of two columns of the \(C\) matrix. The value of this particular element is \(2 \cdot 2^z\) or \(2^z + 1\).

Now we will consider in little more detail the important special case when all the coupling constants are same (the isotropic case). In this case it is enough to consider only one parameter \(\gamma\) in the expression of \(w_j(\sigma_j)\) (see Eq. (19)). Clearly, here \(\bar{C}\) will be just a column vector and consequently \(\bar{C}^T \bar{C}\) is just a number whose value is \(z \cdot 2^z\). For this isotropic case, the pseudoinverse matrix (which is a row vector) is, \(\bar{C}^+ = \frac{2^z}{z} \bar{C}^T\). We may further note that the first element of the column vector \(\bar{C}\) is \(z\), next \(z\) elements are all \((z - 2)\), and so on till we get the last element \((2^z)\) as \(-z\). On the other hand, the first element of the column vector \(\Omega\) is \(\tanh(\beta z J)\), next \(z\) elements are all \(\tanh(\beta(z - 2) J)\), and so on till we get the last element \((2^z)\) as \(\tanh(\beta(-z) J)\). We may note that, in this special case, \(\Gamma\) will have only one element which is the parameter \(\gamma\). This parameter \(\gamma\) can be calculated using Eq. (17); its explicit form is given by the following formula:

\[
\gamma = \frac{2^{-z+1}}{z} \sum_{i=1}^{L} Z_i \cdot C_{i-1} \tanh(\beta Z_i J),
\]

where \(Z_i = z - 2i + 2\) and \(L\) is \(\frac{z}{2}\) or \(\frac{z+1}{2}\) depending on whether \(z\) is even or odd respectively.

It is now worth noting that, for a dimerized linear Ising chain, the coordination number \(z = 2\). In the absence of magnetic field, we can use the matrix \(\bar{C}\) for the regression process. The dimension of this matrix is \(2^{z-1} \times z\), i.e., \(2 \times 2\). This implies that, we can get exact values of the two parameters \((\gamma_1\) and \(\gamma_2)\) appearing in the expressions of \(w_j\)’s. This can be also understood from the Eq. (10), which in this particular case will give two independent equations in two unknown parameters \((\gamma\)’s). When dimerization is zero, these two parameters will turn out to be same. This is the special case studied by Glauber in his original paper. This shows that our approach to the dynamics is truly general; in principle this same approach can be used in studying dynamics of any type of Ising system. From this point of view, our approach can be seen as natural extension of what Glauber did in his work.

2.4. Presence of magnetic field

In the presence of external magnetic field \((H)\), the interaction energy associated with the \(j\)th spin will have now an extra term \(-H\sigma_j\), and therefore Eq. (6) will be modified accordingly,

\[
h'_j(\sigma_j) = \sum_{k=1}^{z} J_k \sigma_j \cdot \sigma_k - H \sigma_j.
\]

It is common to write the modified transition rate \((w'_j)\) in terms of \(w_j\) in the following way [11]:

\[
w'_j(\sigma_j) = w_j(\sigma_j)(1 - \sigma_j \tanh \beta H).
\]
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Main problem with this form of \( w'_{j} \) is that, when used in Eq. (4), the equations for \( q_k \)'s get coupled with the correlation functions \( r_{i,j} \)'s. On the other side, when this form of \( w'_{j} \) is used in Eq. (5), the equations for \( r_{i,j} \)'s not only get coupled with \( q_k \)'s, they now get coupled with more complex three point correlation functions. Here we will present another way of handling this issue of applied magnetic field. Here transition rates (\( w'_{j} \)'s) will have one more independent parameter than the number of neighbors (\( z \)). If we use this form of \( w'_{j} \)'s, we will get a decoupled set of equations for \( q_k \)'s (\( r_{i,j} \)'s do not appear in them) and the equations for \( r_{i,j} \)'s will not contain any three point correlation terms (though will get coupled with \( q_k \)'s).

To present our approach, we first note that, \( w'_{j} \) will have following exact but nonlinear form (cf. Eq. (9)):

\[
w'_{j}(\sigma_{j}) = \frac{\alpha}{2} \left( 1 + \sigma_{j} \tanh \left( \beta \left( \sum_{k=1}^{z} J_{k} \sigma_{k} - H \right) \right) \right),
\]

(24)

As before (see Eq. (10)), we can attempt to linearize \( w'_{j} \) by considering,

\[
\tanh \left( \beta \left( \sum_{k=1}^{z} J_{k} \sigma_{k} - H \right) \right) \approx \sum_{k=1}^{z} \gamma_{k} \sigma_{k} + \gamma_{H}.
\]

(25)

A linear regression will give the best possible values for the parameters \( \gamma \)'s. We here note that, Eq. (25) actually represents \( 2^z \) linear equations in \( z + 1 \) parameters. Each of these equations correspond to one of the \( 2^z \) configurations of \( z \) neighbors.

To get the optimal values for the parameters \( \gamma \)'s, the matrix equation to be solved in this case is \( C_H \Gamma_H = \Omega_H \) (cf. Eq. (11)). Here \( \Gamma_H \) is a column matrix containing \( z + 1 \) parameters (\( \gamma_1, \cdots, \gamma_z, \gamma_H \)). The \( 2^z \times (z + 1) \) matrix \( C_H \) is just the configuration matrix \( C \) with an additional column whose elements are all 1. The column matrix \( \Omega_H \) has \( 2^z \) elements, with ith element being the value of \( \tanh \left( \beta \left( \sum_{k=1}^{z} J_{k} \sigma_{k} - H \right) \right) \) for the ith configuration (as appear in the ith row of the \( C \) matrix). Now a linear regression can be done to obtain the best possible values of the parameters (for which mean squared error is minimum). The best possible values can be formally written using the Moore-Penrose pseudoinverse matrix: \( \Gamma_H = C_H^+ \Omega_H \), here the pseudoinverse matrix \( C_H^+ \) is \( (C_H^T C_H)^{-1} C_H^T \).

We know that the columns of the configuration matrix \( C \) are orthogonal, and they are also orthogonal to the extra column of the \( C_H \) matrix as each column of the \( C \) matrix has same number of +1 and -1. This implies that the \( (z + 1) \times (z + 1) \) matrix \( (C_H^T C_H) \) is diagonal with the elements being \( (C_H^T C_H)_{i,j} = 2^z \delta_{i,j} \). Therefore the pseudoinverse matrix in this case is given by, \( C_H^+ = 2^{-z} C_H^T \). As we mentioned, this matrix will help us get best possible values of the parameters (\( \gamma \)'s) from the equation \( \Gamma_H = C_H^+ \Omega_H \). We may here note that all these parameters will be the functions of the magnetic field \( H \).

This optimal values of the set of parameters can now be used in Eq. (25) to obtain a best possible linearized version of \( w'_{j}(\sigma_{j}) \),

\[
w'_{j}(\sigma_{j}) = \frac{\alpha}{2} (1 + \sum_{k=1}^{z} \gamma_{k} \sigma_{j} \cdot \sigma_{k} + \gamma_{H} \sigma_{j}).
\]

(26)
For the important special case when all the coupling constants are same, say $J$, it is possible to give explicit formulae for the parameters (in the absence of magnetic field, it is given in Eq. (21)). For this isotropic case, there will be only two parameters, $\gamma$ and $\gamma_H$, which are respectively the first and second elements of the column vector $\Gamma_H$. Here the $C_H$ matrix (whose pseudoinverse has to be found) has just two columns. All the elements of the second column are 1. The first element of the first column is $z$, next $z^2C_1$ elements of the column is $(z - 2)$, and so on till we get the last element $(2z)^{th}$ of the column as $-z$. On the other hand, the first element of the column vector $\Omega_H$ is $tanh[\beta\{zJ - H\}]$, next $z^2C_1$ elements are all $tanh[\beta\{(z - 2)J - H\}]$, and so on till we get the last element $(2z)^{th}$ of the column as $tanh[\beta\{-zJ - H\}]$. Since two columns of this reduced $C_H$ matrix are orthogonal, $C_H^T C_H$ is a diagonal matrix with first diagonal element being $z^2$ and second one being $2^z$. This implies that, first row of the pseudoinverse matrix $C_H^+$ is just the transpose of the first column of $C_H$ multiplied by $\frac{2^z}{z}$. On the other hand, second row of $C_H^+$ is the transpose of second column of $C_H$ multiplied by $2^{-z}$. Now we can write explicit formulae for the parameters $\gamma$ and $\gamma_H$ by using $\Gamma_H = C_H^+ \Omega_H$:

$$\gamma = \frac{2^{-z}}{z} \sum_{i=1}^{z+1} Z_i^z C_{i-1} \tanh[\beta(Z_i J - H)] \quad \text{and} \quad (27)$$

$$\gamma_H = 2^{-z} \sum_{i=1}^{z+1} z C_{i-1} \tanh[\beta(Z_i J - H)], \quad (28)$$

where $Z_i = z - 2i + 2$.

It is easy to verify that, when the magnetic field $H$ becomes vanishingly small, the parameter $\gamma_H$ approaches zero. In this limit, the parameter $\gamma$ given in Eq. (27) reduces to the parameter obtained for the case when the magnetic field was absent (see Eq. (21)).

2.5. Nature of steady state and its closeness to the equilibrium state

It may be noted that, any arbitrary choice of the set of $\gamma$’s would make the system evolve to some steady state (which is the solution of the master equation at large time), but in general this steady state will not be the actual equilibrium state that the given system would relax to. That steady state will be the actual equilibrium state of the given system only when the equation of detailed balance (Eq. (7)) is satisfied for all possible spin configurations of the neighbors. Unfortunately, except for a special case, no choice of $\gamma$’s would obey this condition exactly, as we have less number of parameters than the number of configurations (see Eq. (10) and discussion there). Our method makes sure that the steady state comes as close to the actual equilibrium state as possible within the linearization approach. Let us see more physically how this is done. First define for the $j$th site the net local probability current flowing between two configurations, $I_j = w_j(\sigma_j)p(\sigma_j) - w_j(-\sigma_j)p(-\sigma_j)$. Here $p(\sigma_j)$ is the the Maxwell-Boltzmann probability
factor defined for the given system while $w_j(\sigma_j)$ is the transition rate with an arbitrary set of $\gamma$’s. Clearly the current $I_j$ will be positive for some configurations of the neighbors and negative for other configurations. If we could choose a set of $\gamma$’s for which the equation of detailed balance was exactly satisfied for all configurations, then the current $I_j$ would have been identically zero for each and every configuration. In this context, our method does the following: it makes sure that average current $< I_j >$ (average over all possible configurations of neighbors) is zero in the absence of magnetic field and small, if not zero, when magnetic field is present. In addition, our method ensures that two opposite tendencies (forward current and backward current depending on the sign of $I_j$) are individually as low as possible on the average.

To prove that $< I_j >= 0$ when magnetic field is absent, we first note that, this current can be written as, $I_j = \frac{2}{\sqrt{1-X^2}}[Y-X]$; here $X = tanh[\beta \sum_{k=1}^{z} J_k \sigma_k]$ and $Y = \sum_{k=1}^{z} \gamma_k \sigma_k$. We may here note that, in the expression of $I_j$, we have missed out a constant prefactor which does not change for different configurations. We have also taken $\sigma_j = 1$ to arrive at the expression; this implies that when $I_j > 0$, there will be a net current flowing from the up state to the down state of the $j$th spin (in other words, there will be a net tendency for the spin to flip if it’s in the up state) and similarly, when $I_j < 0$, there will be a net opposite current flowing from the down state to the up state of the $j$th spin. Now let us consider two configurations (say, $p$ and $p'$) connected by $Z_2$ symmetry. Clearly, if $X_k$ and $Y_k$ denotes respectively the values of the $X$ and $Y$ for the $k$th configuration, then $X_p = -X_{p'}$ and $Y_p = -Y_{p'}$. This implies that the current $I_j$ is exactly opposite for two configurations of neighbors related by $Z_2$ symmetry. This proves that $< I_j >= 0$ when magnetic field is absent.

In case when magnetic field is present, we again can write the current in the form $I_j = \frac{2}{\sqrt{1-X^2}}[Y_H - X_H]$, where now $X_H = tanh[\beta \sum_{k=1}^{z} J_k \sigma_k - H]$ and $Y_H = \sum_{k=1}^{z} \gamma_k \sigma_k + \gamma_H$. Here for two configurations, $p$ and $p'$, connected by $Z_2$ symmetry, $(X_H)^p \neq -(X_H)^{p'}$ and $(Y_H)^p \neq -(Y_H)^{p'}$. Therefore, unlike when magnetic field is absent, the current $I_j$ is now not exactly opposite for two configurations connected by $Z_2$ symmetry. But we note that, $\sum_{i=1}^{2z}(Y_H - X_H)i = 0$. This can be seen in the following way. We can write, $\sum_{i=1}^{2z}(Y_H - X_H)i = \sum_{i=1}^{2z}(C_H \Gamma_H - \Omega_H)i$, where $(C_H \Gamma_H - \Omega_H)$ is the $i$th element of column vector $(C_H \Gamma_H - \Omega_H)$ (see section 2.4). We now have, $\sum_{i=1}^{2z}(C_H \Gamma_H - \Omega_H)i = \sum_{i=1}^{2z} \sum_{k=1}^{z+1}(C_H)i,k(\Gamma_H)k - \sum_{i=1}^{2z}(\Omega_H)i$. But $\sum_{i=1}^{2z}(C_H)i,k = 0$ for all $k$’s except when $k = z + 1$, for which $\sum_{i=1}^{2z}(C_H)i,z+1(\Gamma_H)z+1 = 2^2 \gamma_H$. We have seen that the value of $\gamma_H$ that we get by regression (or by solving $\Gamma_H = C_H^+ \Omega_H$) is $2^{-z} \sum_{i=1}^{2z}(\Omega_H)i$. This implies that $\sum_{i=1}^{2z}(C_H \Gamma_H - \Omega_H)i = 0$. Noticing that, $|X_H| < 1$ for any possible configuration, we have the following series, $\frac{1}{\sqrt{1-X_H^2}} = 1 + \frac{1}{2}X_H^2 + \frac{3}{8}X_H^4 + \cdots$.

In fact when magnetic field and Hamiltonian parameters take reasonable values, we expect $|X_H|$ to be well below one for most of the configurations. We note that there is no linear term in the above series, and variation of existing higher order terms are expected to be very small. Therefore as first approximation we take a constant value (say, $K$) for $\frac{1}{\sqrt{1-X_H^2}}$. This gives, $< I_j >\approx K \sum_{i=1}^{z}(Y_H - X_H)i$, i.e., $< I_j >\approx 0$ when
magnetic field is present.

We now see that for a linear model, \( < J_j > \) is small, if not zero, even when the parameters \( \gamma_1, \gamma_2, \cdots, \gamma_z \) (not \( \gamma_H \)) are chosen arbitrarily. But arbitrary choice of parameters does not ensure whether the forward currents and the backward currents are individually as weak as possible. This is then done by choosing the values of the parameters as obtained by a linear regression process (see sections 2.1 and 2.2). This facts allow as to safely say that, even though our steady state is generally not an equilibrium state, it is as close to the equilibrium state as possible within a linearization approach.

It is important to study how close we reach to the actual equilibrium state. \( S_{\text{min}} \), as given in Eq. (15), can be taken as the measure for this closeness. Lower the value of \( S_{\text{min}} \) implies that we are closer to the actual equilibrium state.

In general this minimum of the error function (i.e., \( S_{\text{min}} \)) is not zero and its value can be obtained using the pseudoinverse matrix. Taking \( C^+ \) as given in Eq. (18), we get directly from Eq. (15): 
\[
S_{\text{min}} = ||(I - 2^{-z}CC^T)\Omega||^2.
\]
Since rows of the \( C \) matrix are not orthogonal in general, analyzing behavior of \( S_{\text{min}} \) as function of coordination number \( z \) and temperature \( (\beta^{-1}) \) is not always easy. For the special cases we can get simple form of \( S_{\text{min}} \) which allows us to analyze its behavior analytically.

In the absence of magnetic field, \( \tilde{C}\tilde{C}^+ \) is an Identity matrix for a system with \( z = 2 \). This implies that, in this special case, \( S_{\text{min}} = 0 \) (see Eq. (15)). This result is not unexpected, as we noted earlier, one can have exact solution for the \( \gamma \)'s in this particular situation (or in other words, here one does not need to do regression).

Now we will analyze an important special case when all the coupling constants are same (say, \( J \)). In the absence of magnetic field, the matrix involved in the regression is a column vector \( (\tilde{C}) \) with \( 2^{z-1} \) elements. In this isotropic case the column vector is just the sum of \( z \) column vectors of \( \tilde{C} \) matrix (configuration matrix representing those configurations which are not transformed to each other by \( Z_2 \) symmetry). This implies that, the first element of the column vector \( \tilde{C} \) is \( z \), then there are \( zC_1 \) number of elements each equals to \( (z-2) \), and so on. We note that, if \( z \) is even then the last \( \frac{1}{2} z^2 C_1 \) elements of the column vector are all zero. On the other hand, if \( z \) is odd then the last \( zC_1 \) elements of the column vector are all one. Following the arguments given in section 2.3, it is not difficult to see that, \( \tilde{C}^T\tilde{C} = z \cdot 2^{z-1} \). This implies that, the pseudoinverse matrix \( (\tilde{C}^+) \) is a row vector with \( 2^{z-1} \) elements, and is given by \( \tilde{C}^+ = 2^{z+1} \tilde{C}^T \).

It is now easy to find the elements of the \( 2^{z-1} \times 2^{z-1} \) matrix \( (I - \tilde{C}\tilde{C}^+) \). We see that the elements of the first row of the \( CC^+ \) is larger than the corresponding elements of the other rows. In fact, the elements of the first row are \( z\tilde{C}^+ \), the elements of the next \( zC_1 \) rows are \( (z-2)\tilde{C}^+ \), and so on. Now we note that, the first element of \( \Omega \) is \( tanh [\beta zJ] \), next \( zC_1 \) elements are all \( tanh [\beta (z-2)J] \), and so on. It is now not difficult to see that the absolute value of the first element of the column matrix \( (I - \tilde{C}\tilde{C}^+)\Omega \) is the largest. This allows us to get the following upper bound: 
\[
S_{\text{min}} \leq 2^{z-1} ||(I - z\tilde{C}^+)\Omega||^2 ,
\]
where \( I \) is now a row vector whose first element is one and rest of the elements are zero. A more appropriate quantity to study here is the minimum of the root mean square error (\( \tilde{S}_{\text{min}} \)),
which is basically the minimum of the average error per configuration. This quantity is given by:  \( \bar{S}_{\text{min}} = \sqrt{\frac{S_{\text{min}}}{z-1}} \). Now using the bound for \( S_{\text{min}} \), we get the following bound for the quantity:

\[
\bar{S}_{\text{min}} \leq |(I - zC^+)\Omega|.
\] (29)

We now notice that the absolute value of the first element of the row vector \( (I - zC^+) \) is larger than the absolute value of any other element. This fact is also true for the column vector \( \Omega \). This implies that, \(|(I - zC^+)\Omega| \leq 2^{z-1}|(1 - \frac{2^{z-1} + z^2}{z})\tanh[\beta z J]|\). If we use this inequality in Eq. (29), we get the following upper bound for \( \bar{S}_{\text{min}} \),

\[
\bar{S}_{\text{min}} \leq (2^{z-1} - z)\tanh[\beta z |J|].
\] (30)

We may here note that, if we had not used \( Z_2 \) symmetry, we would have found twice of what we have got as the upper bound for \( S_{\text{min}} \); the bound for \( \bar{S}_{\text{min}} \) will though remain same even if we work with the full configuration matrix.

Although the upper bound given in Eq. (30) is not a tight one, by noting that \( 0 \leq \tanh[\beta z |J|] \leq 1 \), it makes some sense to infer the followings. \( \bar{S}_{\text{min}} \) is a weak function of temperature \( (\beta^{-1}) \) and coupling strength \( |J| \); in fact, \( \bar{S}_{\text{min}} \) increases slowly with the parameter \( \beta |J| \). Though Eq. (30) suggests a strong dependence of \( \bar{S}_{\text{min}} \) on the coordination number \( z \), we will see in the following sections that the result (critical temperature) obtained for three dimensional system is somewhat better than that for two dimensional system.

A similar bound can be found in case of non-zero external magnetic field. Without going into details of calculation, we can safely say that \( \bar{S}_{\text{min}} \) is a weak function of magnetic field \( H \), as it is of the parameter \( \beta |J| \). This is due to the fact that \( H \) appears in \( \bar{S}_{\text{min}} \) only through the argument of hyperbolic-tan function.

3. Application to different Ising systems

In this section we will study different Ising spin systems using the method we developed in the preceding section. In particular we study the relaxation time both in the presence and absence of magnetic field. From the divergence of this relaxation time, it is possible to estimate critical temperatures of different systems. We also discuss how the critical temperature changes with applied magnetic field. We may here note that, the analysis of our optimal linear Glauber model (OLGM) and the linear Glauber model (LGM) are essentially same; the advantage of our present work is, we will now get to know how the static and dynamic properties of an Ising model depend on temperature and coupling constants. We have added a subsection (3.3) to discuss different scaling properties of a linear model in the present context.

3.1. Relaxation time for a generic system

For a general lattice, the transition rate \( w_j \) in Eq. (19) is rewritten as,

\[
w_{\vec{r}}(\sigma_{\vec{r}}) = \frac{\alpha}{2} (1 + \sum_{\vec{R}} \gamma_{\vec{R}} \sigma_{\vec{r}} \cdot \sigma_{\vec{R}}).
\] (31)
Here $\vec{r}$ is the position vector of a site while $\vec{R}$ is the separation vector identifying neighbors connected to the site. In this notation, the equation for $q_k$ as given in Eq. (4) is recasted as,

$$\frac{d}{dt} q_{\vec{r}}(t) = -2 \sum_{C(N)} \sigma_{\vec{r}} w_{\vec{r}}(\sigma_{\vec{r}}) p(\{\sigma\}; t)$$

(32)

Now if we use Eq. (31) in Eq. (32), we will get the following equation for $q_{\vec{r}},$

$$\frac{d}{dt} q_{\vec{r}}(t) = -\alpha \left[ q_{\vec{r}}(t) + \sum_{\vec{R}} \gamma_{\vec{R}} q_{\vec{R}}(t) \right].$$

(33)

If we denote the total magnetization, $\sum_{\vec{r}} q_{\vec{r}}(t)$, by $M(t)$, then it is not difficult to see that, $\sum_{\vec{r}} q_{\vec{R}}(t) = M(t)$ for a particular $\vec{R}$. To get the equation for $M(t)$, we sum both sides of the Eq. (33) over all sites; this gives,

$$\frac{d}{dt} M(t) = -\alpha \left[ 1 + \sum_{\vec{R}} \gamma_{\vec{R}} \right] M(t).$$

(34)

Solution of this equation gives us,

$$M(t) = M(0)e^{-t/\tau},$$

(35)

where $M(0)$ is the magnetization of the system at $t = 0$, and

$$\tau = \frac{1}{\alpha} \left[ 1 + \sum_{\vec{R}} \gamma_{\vec{R}}(T) \right]^{-1}$$

(36)

is the relaxation time of the system at temperature $T$. In the above expression it is explicitly shown that the $\gamma$’s are all functions of $T$.

When magnetic field is present, the transition rate $w_{\vec{r}}$ has to be replaced by appropriate $w'_{\vec{r}}$ (see Eq. (26) where $w'_j$ is given). A simple calculation now leads us to the following equation for the magnetization,

$$\frac{d}{dt} M(t) = -\alpha \left[ 1 + \sum_{\vec{R}} \gamma_{\vec{R}} \right] M(t) - \alpha N \gamma_H,$$

(37)

where, $N$ is the total number of sites in the system. Solution of this equation gives us,

$$M(t) = M(0)e^{-t/\tau_H} - N \tau_H \gamma_H (1 - e^{-t/\tau_H}),$$

(38)

where $M(0)$ is the magnetization of the system at $t = 0$, and

$$\tau_H = \frac{1}{\alpha} \left[ 1 + \sum_{\vec{R}} \gamma_{\vec{R}}(T, H) \right]^{-1}$$

(39)

is the relaxation time of the system at temperature $T$. In the above expression it is explicitly shown that the $\gamma$’s are all functions of $T$ and $H$.

During second order phase transition, the relaxation time of a system is expected to diverge. Using this fact it is possible to estimate the critical temperature $T_C$ from
Optimal linear Glauber model

Eq. (36). Using Eq. (39), it is also possible to study how this $T_C$ changes with magnetic field $H$.

In the next section we study the criticality of the isotropic Ising systems in different dimensions.

3.2. Study of criticality for some systems

In this section, we study how one, two and three dimensional ferromagnetic isotropic Ising systems behave close to criticality. In particular, we calculate the critical temperatures ($T_C$) for the systems in different dimensions and show how $T_C$ changes with the strength of magnetic field $H$ for two and three dimensional systems.

For the isotropic case (when all coupling constant are same) all the parameters ($\gamma$’s) take the same value and is given by Eq. (21) in the absence of magnetic field. Here the relaxation time of Eq. (36) will take the following form:

$$\tau = \frac{1}{\alpha \left[ 1 + z\gamma \right]^{-1}}.$$ (40)

For one dimensional isotropic system, $z = 2$ and $\gamma = -\frac{1}{2} \tanh (2\beta|J|)$ (note for ferromagnetic systems, $J = -|J|$). In this case the Eq. (40) takes the following form: $\tanh (2\beta|J|) = 1$. This will be only satisfied when $\beta \to \infty$. Therefore in this case $T_C = 0$, in accordance with the fact that the one dimensional Ising system behaves critically only near to absolute zero temperature.

For two dimensional system (square lattice), $z = 4$ and $\gamma = -\frac{1}{8} \left[ \tanh (4\beta|J|) + 2 \tanh (2\beta|J|) \right]$. In this case the Eq. (40) takes the following form: $\tanh (4\beta|J|) + 2 \tanh (2\beta|J|) = 2$. Solution of this equation gives $T_C = 3.089 |J|/k_B$, whereas its exact value is know to be $T_C = 2.269 |J|/k_B$ [18].

For three dimensional system (simple cubic lattice), $z = 6$ and $\gamma = -\frac{1}{32} \left[ \tanh (6\beta|J|) + 4 \tanh (4\beta|J|) + 5 \tanh (2\beta|J|) \right]$. In this case the Eq. (40) takes the following form: $\tanh (6\beta|J|) + 4 \tanh (4\beta|J|) + 5 \tanh (2\beta|J|) = 16/3$. Solution of this equation gives $T_C = 5.071 |J|/k_B$, whereas its actual value is expected to be about $T_C = 4.511 |J|/k_B$ [19, 20, 21].

The values for the $T_C$ obtained for our optimal linear model are somewhat better in comparison with the mean field values (where $T_C = z|J|/k_B$). Deviation of our result from the actual values of $T_C$ may not be surprising, as our linearized transition rate does not exactly satisfy the detailed balance condition. It is though encouraging to notice here that our approach correctly captures the basic physics of the Ising model in different dimensions, viz., while criticality exists only at absolute zero for one dimensional system, for two and three dimensional systems, criticality exists at finite temperatures.

Now we will discuss how for an isotropic system, $T_C$ changes in the presence of magnetic field. If $\beta_c$ and $\beta'_c$ are the inverse critical temperatures of the system in the absence and presence of magnetic field respectively, then for a weak field $H$,

$$1 + z\gamma(\beta_c', H) = 1 + z\gamma(\beta_c, 0) + z \frac{\partial \gamma(\beta_c, 0)}{\partial \beta} \delta \beta_c + \frac{z}{2} \frac{\partial^2 \gamma(\beta_c, 0)}{\partial H^2} H^2.$$ (41)
Here expression for $\gamma(\beta, H)$ is given in Eq. (27). All the partial derivatives are evaluated at $\beta = \beta_c$ and $H = 0$. We may note that $\frac{\partial \gamma(\beta_c, 0)}{\partial H} = 0$, so we took second order term in $H$ in order to study change in critical temperature due to presence of magnetic field. We also note that, $\frac{\partial^2 \gamma(\beta, 0)}{\partial \beta^2} = 0$. Now since, $1 + \frac{z \gamma(\beta', H)}{1 + \frac{z \gamma(\beta_c, 0)}{0}}$, we get,

$$\delta \beta_c = -\frac{1}{2} \left( \frac{\partial^2 \gamma(\beta, 0)}{\partial H^2} \right) H^2. \quad (42)$$

Since, $\delta \beta_c = -\frac{\delta T_c}{T_c} \beta_c$, we get from Eq. (42),

$$\frac{\delta T_c}{T_c} = \frac{1}{2 \beta_c} \left( \frac{\partial^2 \gamma(\beta, 0)}{\partial H^2} \right) H^2. \quad (43)$$

For square lattice ($z = 4$), Eq. (43) gives, $\frac{\delta T_c}{T_c} = -0.0859 \frac{H^2}{T}$. For simple cubic lattice ($z = 6$) Eq. (43) gives, $\frac{\delta T_c}{T_c} = -0.0339 \frac{H^2}{T}$. Therefore we see that, the critical temperature decreases quadratically with magnetic field (for weak $H$). This behavior was also found numerically [22], although the constant prefactors were somewhat different (-0.0544 and -0.0097 respectively) [23].

### 3.3. Static and dynamic scaling properties

Main advantage of the linear Glauber model is that it is exactly solvable. This model has been extensively studied in the past [14, 15, 16]. Analysis of our present optimal linear model essentially remains same as the linear Glauber model. We now discuss some of the important scaling properties of the linear model in the present context.

For a linear Glauber model, following transition rate is considered, $w_r(\sigma_r) = \frac{\lambda}{2} \left( 1 - \frac{\lambda}{2d} \sum_{\vec{R}} \sigma_{\vec{R}} \cdot \sigma_{\vec{R}} \right)$, where $\lambda$ is a parameter ($0 < \lambda \leq 1$) which quantifies strength of ‘noise’ while $d$ is the dimensionality of the system (meaning of the other symbols are already explained before). It was shown that [15], with this transition rate, the susceptibility $\chi \sim \epsilon^{-1}/|\ln \epsilon|$ for $d = 2$ and $\chi \sim \epsilon^{-1}$ for $d \geq 3$. Here $\epsilon = \frac{1}{N^c}$. Now identifying $\lambda = 2d|\gamma|$ (cf. Eq. (19)), where $\gamma$ is given by Eq. (21), it is not difficult to see that, $\chi \sim \frac{T - T_c}{T_c} - 1$ for all dimensions (with logarithmic correction for $d = 2$). (Note: Usual symbol for the critical exponent related to susceptibility is $\gamma$, which is 1 in the present case; we do not use this symbol here as it has been already used as an optimization parameter.) This result can also be derived from Eq. (38). When $t \to \infty$, Eq. (38) gives, $\chi = \frac{\partial M(t \to \infty)}{\partial H} |_{H=0} = -N \tau \frac{\partial \gamma(\beta, 0)}{\partial \beta} |_{H=0}$. Here $\tau$ is given by Eq. (36) and $\gamma_H$ is given by Eq. (21). Using the relevant expressions for $\tau$ and $\gamma_H$, it is now not difficult to show that, near criticality, $\chi \sim \frac{T - T_c}{T_c} - 1$ for all dimensions ($d \geq 2$). This result is the direct consequence of the fact that the relaxation time itself diverges as $\tau \sim \frac{T - T_c}{T_c}$.

It may be worth mentioning here that, at criticality, $|\gamma| = \frac{1}{2d}$ (cf. Eq. (10)), i.e., $\lambda = 1$. This shows that our optimal linear model reduces to the voter model (without noise) at criticality ($T = T_c$).

To analyze the dynamic properties of the model, we now shift to the continuum limit. In this limit, the correlation function for two spins separated by the position
vector $\vec{r}$ satisfies the following diffusion-decay equation (this can be easily derived once we use the linear form of $w_k$'s in the equation of two-point correlation $r_{i,j}$; cf. Eq. (5) and Eq. (19)):

$$\frac{\partial G(\vec{r}, t)}{\partial t} = D \nabla^2 G(\vec{r}, t) - \kappa G(\vec{r}, t),$$

(44)

where $D = 2\alpha|\gamma|$ and $\kappa = 2\alpha(1 - 2d|\gamma|)$. At criticality when $|\gamma| = \frac{1}{2d}$, we have $\kappa = 0$ and $D = \alpha/d$; this reduces Eq. (44) to the diffusion equation for the voter model \[24\]. We may note that, the solution of Eq. (44) can be written in the following form,

$$G(\vec{r}, t) = P(\vec{r}) - Q(\vec{r}, t)e^{-\kappa t},$$

(45)

where $P(\vec{r})$ and $Q(\vec{r}, t)$ respectively satisfy the following equations:

$$\nabla^2 P(\vec{r}) = \frac{\kappa}{D} P(\vec{r}) \quad \text{and}$$

(46)

$$\frac{\partial Q(\vec{r}, t)}{\partial t} = D \nabla^2 Q(\vec{r}, t).$$

(47)

To solve Eq. (44), we must now set the initial condition(s) for $G(\vec{r}, t)$ and discuss the asymptotic behavior of the function. We note that, $r_{i,i} = 1$ for all time $t$, also, $r_{i,j} = 0$ for $t = 0$ ($i \neq j$). In the continuum limit, it is convenient to set a lower cutoff $a > 0$, such that condition of self-correlation becomes, $G(|\vec{r}| = a, t = 1) = 1$. Here the second condition becomes, $G(|\vec{r}| > a, t = 0) = 0$. Physically, for large $|\vec{r}|$, the correlation between two spins will first increase and then saturate to its steady state value after a long time. The steady state value of the correlation function is given by $P(\vec{r})$ which is a solution of Eq. (46). It is here natural to set $P(a) = 1$. With the stated conditions on $G(\vec{r}, t)$ and $P(\vec{r})$, we expect $Q(\vec{r}, t)$ to behave in such a way that, $Q(a, t) = 0$ and $Q(|\vec{r}| > a, 0) = P(\vec{r})$ (cf. Eq. (45)). We now notice that, the solution for $Q(\vec{r}, t)$ is the solution of a problem where we have an absorbing sphere of radius $a$ surrounded by moving particles. Initially, the concentration of the particles is high near the surface of the sphere and it decreases with the distance from the sphere according to the functional form of $P(\vec{r})$. Now following the same line of arguments as given in Ref. [24], it is possible to get the following asymptotic solution for $Q(r, t)$ (where $r = |\vec{r}|$; it should not be confused with the symbol for two-point correlation):

$$Q(r, t) \simeq P(\sqrt{Dt}) \times \begin{cases} \frac{r}{\sqrt{Dt}}, & d = 1 \ & 0 < r < \frac{a}{r} \\ \frac{\ln(r/a)}{\ln(\sqrt{Dt}/a)}, & d = 2 \ & a < r \\ 1 - \frac{a}{r}, & d = 3 \ & a < r \end{cases}$$

(48)

It should be mentioned here that the functional form of $P(\sqrt{Dt})$ will be different in different dimensions. In Eq. (48), we have assumed $r < \sqrt{Dt}$ for $d = 1$ and 2. Accuracy of $Q(r, t)$ also depends on whether $\sqrt{Dt}$ is less than the correlation length ($\xi$). If $\sqrt{Dt} \gtrsim \xi$, then $Q(r, t)$ is expected to decay faster than what we get from Eq. (48). This is because, the concentration of particles far away ($\gtrsim \xi$) from the absorbing sphere will be very low. In this situation, when $r \sim \xi \lesssim \sqrt{Dt}$, the particles at position $r$ will not only diffuse into the absorbing sphere, now they will also diffuse away towards the outer low
concentration zone. For this reason, one expects Eq. (48) to work good near criticality where correlation length is very large. We are here, though, not much interested in the case when the correlation length ($\xi$) is short; in such case, the correlation function $G(r, t)$ for large $r$ anyway always remains close to zero at all times.

Now we turn our attention to find the steady state solution $P(r)$ from Eq. (46). A trial solution of the form $e^{-r/\xi}$ can be taken to find the desired solution for $P(r)$. Here $\xi = \sqrt{D/\kappa}$ is the correlation length and $k$ is a constant to be determined. We find that for $d = 1$ and 3, $k = 0$ and 1 respectively. For $d = 2$, the above trial form does not yield any solution of Eq. (46). For this special case, we take the following trial form:

$$P(r) = \sum_{n=0}^{\infty} a_n r^n.$$  

If we put this in Eq. (46), we get the following equation for $S(r)$:

$$\frac{d^2 S(r)}{dr^2} + (\xi - 2r) \frac{dS(r)}{dr} - S(r) = 0. \quad (49)$$

The solution of this equation can be most easily found by a trial series of the form

$$S(r) = \sum_{n=0}^{\infty} a_n r^n.$$  

After some calculations, we find that,

$$a_n = \frac{(2n-1)!!}{n! \xi^{2n}} a_0.$$  

This gives,

$$S(r) = a_0 \left[ 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{n! \xi^{2n}} (r/\xi)^n \right].$$

By ratio test it can be verified that this series is convergent for any finite $r$. With this in hand, we now write solution for the $P(r)$:

$$P(r) \simeq a_0 e^{-r/\xi} \begin{cases} 1, & d = 1 \\ 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(n!)^2} (r/\xi)^n, & d = 2 \\ \frac{1}{r}, & d = 3. \end{cases} \quad (50)$$

The value of $a_0$ can be determined by the normalization condition $P(a) = 1$. For $d = 1$ and 3, $a_0$ is respectively $e^{a/\xi}$ and $ae^{a/\xi}$.

As we might expect, at criticality, the correlation function $G(r, t)$ reduces to the one for the voter model (cf. Ref. [24]). This can be easily checked by taking $\xi \to \infty$ in Eqs. (48) and (50). In this limiting case, the interface density $\rho$ (or domain wall density) scales with time as $t^{-1/2}$, $(\ln t)^{-1}$ and $O(1)$ respectively for $d = 1, 2$ and 3. This temporal behavior is supposed to continue even when $T \gtrsim T_C$. This can be understood from the fact that $\rho$ is defined as $\frac{1}{2} (1 - G(r, t))$, where $r$ is the distance between nearest neighbors, i.e., technically, $r$ is here approximately equal to $a$ (but not exactly). If we now use Eqs. (48) and (50) to calculate $\rho$, we will get back the same asymptotic temporal behavior as we mentioned before.

To calculate the dynamic exponent, we do a Fourier analysis of the Eq. (44). If we insert $G(r, t) = \sum_{k} G(\vec{k}, t) e^{i \vec{k} \cdot \vec{r}}$ in the equation, we get,

$$\frac{\partial G(\vec{k}, t)}{\partial t} = -(Dk^2 + \kappa) G(\vec{k}, t). \quad (51)$$

The solution of this equation gives,

$$G(\vec{k}, t) = G(\vec{k}, 0) e^{-t/\tau(k)}, \quad (52)$$

where $\tau(k)$ is the relaxation time $(Dk^2 + \kappa)^{-1}$ for the $\vec{k}$ mode. This relaxation time can be rewritten as,

$$\tau(k) = \frac{1}{D(k^2 + \xi^{-2})}. \quad (53)$$
where $\xi$ is, as mentioned earlier, the correlation length: $\xi = \sqrt{D/\kappa} = \sqrt{|\gamma| \frac{1-2d/\gamma}{|\gamma|}}$. Here it may be briefly mentioned that, near criticality the correlation length diverges as, $\xi \sim |T-T_C|^{-1/2}$.

The dynamic exponent (denoted by $z$; not to be confused with coordination number) is defined by how the maximum possible value of the relaxation time ($\tau_{\text{max}}$) scales with the system’s relevant length scale. For a thermodynamic system (size $L \to \infty$) where lowest possible value of $k$ is zero, we get from Eq. (53), $\tau_{\text{max}} \sim \xi^2$, i.e., the exponent $z = 2$. On the other hand, for a finite system of size $L$, near criticality, $\tau_{\text{max}} \sim k_{\text{min}}^2$, i.e., $\tau_{\text{max}} \sim L^2$. Here again the exponent $z = 2$.

4. Conclusion

Contrary to the actual nonlinear Glauber model, the linear Glauber model is exactly solvable, although the detailed balance condition is not generally satisfied. In this work we have addressed the issue of writing transition rate ($w_j$) in a best possible linear form such that the mean squared error in satisfying the detailed balance condition is least. For a generic system, we have shown how this can be done using a simple Moore-Penrose pseudoinverse matrix. Our work in this paper will help us anticipate how the kinetic properties of an arbitrary Ising system depend on the temperature and the coupling constants. Both the static (steady state) and dynamic properties of the Ising systems (in different dimensions) are analyzed using our optimal linearization approach. We also demonstrated in our paper that the effect of the magnetic field can be easily treated within our approach; our transition rate in the presence of magnetic field works more efficiently than the commonly used one.

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