DIMENSIONAL REDUCTION AND QUIVER BUNDLES

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ABSTRACT. The so-called Hitchin–Kobayashi correspondence, proved by Donaldson, Uhlenbeck and Yau, establishes that an indecomposable holomorphic vector bundle over a compact Kähler manifold admits a Hermitian–Einstein metric if and only if the bundle satisfies the Mumford–Takemoto stability condition. In this paper we consider a variant of this correspondence for $G$-equivariant vector bundles on the product of a compact Kähler manifold $X$ by a flag manifold $G/P$, where $G$ is a complex semisimple Lie group and $P$ is a parabolic subgroup. The modification that we consider is determined by a filtration of the vector bundle which is naturally defined by the equivariance of the bundle. The study of invariant solutions to the modified Hermitian–Einstein equation over $X \times G/P$ leads, via dimensional reduction techniques, to gauge-theoretic equations on $X$. These are equations for hermitian metrics on a set of holomorphic bundles on $X$ linked by morphisms, defining what we call a quiver bundle for a quiver with relations whose structure is entirely determined by the parabolic subgroup $P$. Similarly, the corresponding stability condition for the invariant filtration over $X \times G/P$ gives rise to a stability condition for the quiver bundle on $X$, and hence to a Hitchin–Kobayashi correspondence. In the simplest case, when the flag manifold is the complex projective line, one recovers the theory of vortices, stable triples and stable chains, as studied by Bradlow, the authors, and others.

INTRODUCTION

Let $M$ be a compact Kähler manifold and let $\mathcal{F}$ be a holomorphic vector bundle over $M$. It is well-known that a natural differential equation to consider for a Hermitian metric $h$ on $\mathcal{F}$ is the Hermitian–Einstein equation, also referred sometimes as the Hermitian–Yang–Mills equation. This says that $F_h$, the curvature of the Chern connection of $h$ must satisfy

$$\sqrt{-1} \Lambda F_h = \lambda I,$$

were $\Lambda$ is the contraction with the Kähler form of $M$, $\lambda$ is a real number determined by the topology and $I$ is the identity endomorphism of $\mathcal{F}$. A theorem of Donaldson, Uhlenbeck and Yau [D1, D2, UY], also known as the Hitchin–Kobayashi correspondence establishes that the existence of such metric is equivalent to the stability of $\mathcal{F}$ in the sense of Mumford–Takemoto.

Suppose now that a compact Lie group $K$ acts on $M$ by isometries so that $M/K$ is a smooth Kähler manifold and the action on $M$ can be lifted to an action on $\mathcal{F}$. One can then apply dimensional reduction techniques to study $K$-invariant solutions to the Hermitian–Einstein equation on $\mathcal{F}$ and the corresponding stability condition to obtain a theory expressed entirely in terms of the orbit space $M/K$. Many important equations in gauge theory arise in this way (cf. e.g. [AH, FM, H, W]). In this paper we carry out this programme for the manifold $M = X \times G/P$, where $X$ is a compact Kähler manifold, $G$ is a connected simply connected semisimple complex Lie group and $P \subset G$ is a parabolic subgroup, i.e. $G/P$ is a flag manifold. The group $G$ (and hence, its maximal compact subgroup $K \subset G$) act trivially on $X$ and in the standard way on $G/P$. The Kähler structure on $X$ together with a $K$-invariant Kähler structure on $G/P$ define a product Kähler structure on $X \times G/P$. In [AG1] we studied the case in which $G/P = \mathbb{P}^1$, the complex projective line, which is obtained as the quotient of $G = \text{SL}(2, \mathbb{C})$ by the subgroup of lower triangular matrices

$$P = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$
generalizing previous work by \cite{G1, G2, BG}.

Already in the study of the dimensional reduction on \( X \times \mathbb{P}^1 \) \cite{AG1} one realizes that the Hermitian–Einstein is not quite the appropriate equation to consider. It turns out that every \( SL(2, \mathbb{C}) \)-equivariant holomorphic vector bundle on \( X \times \mathbb{P}^1 \) admits an equivariant holomorphic filtration

\[
\mathcal{F} : 0 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_m = \mathcal{F},
\]

which, in turn, is in one-to-one correspondence with a chain

\[
\mathcal{E}_m \xrightarrow{\phi_m} \mathcal{E}_{m-1} \xrightarrow{\phi_{m-1}} \cdots \xrightarrow{\phi_1} \mathcal{E}_0,
\]

consisting of holomorphic vector bundles \( \mathcal{E}_i \) on \( X \) and morphisms \( \phi_i : \mathcal{E}_i \to \mathcal{E}_{i-1} \). If one considers \( SU(2) \)-invariant solutions to the Hermitian–Einstein equation on \( \mathcal{F} \) one obtains certain equations of vortex-type for Hermitian metrics on \( \mathcal{E}_i \). These involve, of course the homomorphisms \( \phi_i \), which in this context are referred as Higgs fields. The key point is that these equations naturally have as many real parameters as morphisms in the chain. By weighting the Kähler structure on \( X \times \mathbb{P}^1 \) one can accommodate one parameter but not all, unless the chains are just one step —the so-called triples— \cite{G2, BG}. In the general case the filtration on \( \mathcal{F} \) has \( m \) steps, and a natural equation to consider for a metric \( h \) on \( \mathcal{F} \) is

\[
\sqrt{-1} \Lambda F_h = \begin{pmatrix} \tau_0 I_0 & \tau_1 I_1 & \cdots & \tau_m I_m \end{pmatrix},
\]

where the RHS is a diagonal matrix, with constants \( \tau_0, \tau_1, \ldots, \tau_m \in \mathbb{R} \), written in blocks corresponding to the splitting which a hermitian metric \( h \) defines in the filtration \( \mathcal{F} \). This equation reduces of course to the Hermitian–Einstein equation when \( \tau_0 = \tau_1 = \cdots = \tau_m = \lambda \). In \cite{AG1} we proved a Hitchin–Kobayashi correspondence relating this equation to a stability criterion for the filtration \( \mathcal{F} \) that depends on the parameters, of which only \( m \) are actually independent.

Coming back to the general situation that we will deal with in this paper, the first thing we do to study dimensional reduction on \( X \times G/P \) is to analyse the structure \( G \)-equivariant holomorphic vector bundles on \( X \times G/P \). As in the case of \( X \times \mathbb{P}^1 \), the basic tool we use is the study of the holomorphic representations of \( P \). This is simply because \( G \)-equivariant bundles on \( X \times G/P \) are in one-to-one correspondence with \( P \)-equivariant bundles on \( X \), where the action of \( P \) on \( X \) is trivial. Each fibre of such a bundle is hence a representation of \( P \). This is carried out in Section \[1\]. A key fact in our study is the existence of a quiver with relations \((Q, \mathcal{K})\) naturally associated to the subgroup \( P \). A quiver is a pair \( Q = (Q_0, Q_1) \) formed by two sets, where \( Q_0 \) is the set of vertices and \( Q_1 \) is the set of arrows, together with two maps \( t, h : Q_1 \to Q_0 \), which to any arrow \( a : \lambda \to \mu \) associate the tail \( ta = \lambda \) of the arrow and the head \( ha = \mu \) of the arrow. A relation of the quiver is a formal complex linear combination of paths of the quiver. The set of vertices in this case coincides with the set of irreducible representations of \( P \). The description of the arrows and relations involves studying certain isotopical decompositions related to the nilradical of the Lie algebra of \( P \). This construction was initially studied by Bondal and Kapranov in \cite{BK} and later by Hille (cf. e.g. \cite{Hi2}). The quiver we obtain is the same as in \cite{BK}, but different from \cite{Hi2}, while the relations differ from those of the previous authors. This is due to the possibility of defining different quivers with relations whose categories of representations are equivalent (the set of vertices is always the same). The quiver and relations that we obtain seem to be arising more naturally from the point of view of dimensional reduction (that is, when studying the relation between the moment maps for actions on the spaces of unitary connections induced by actions on homogeneous bundles, and the moment maps for actions on quiver representations). In Section \[1\] we also provide examples of quivers and relations associated to \( P \), for some parabolic subgroups.

A representation of a quiver with relation \((Q, \mathcal{K})\) consists of a collection of complex vector spaces \( V_\lambda \) indexed by the vertices \( \lambda \in Q_0 \), and a collection of linear maps \( \phi_a : V_{ta} \to V_{ha} \) indexed by the
arrows $a \in Q_1$, satisfying the relations $\mathcal{K}$. The crucial fact is that holomorphic representations of $P$ are in one-to-one correspondence with representations of $(Q, \mathcal{K})$ which, in turn, are in one-to-one correspondence with holomorphic homogeneous vector bundles on $G/P$. In Section 2, we extend this result to $G$-equivariant bundles on $X \times G/P$. To do this, we first introduce the notion of quiver bundle (term due to Alastair King), more precisely $(Q, \mathcal{K})$-bundle on $X$. This is a collection of holomorphic vector bundles $\mathcal{E}_\lambda$ on $X$ indexed by $\lambda \in Q_0$ and a collection of homomorphisms $\phi_a : \mathcal{E}_{ta} \to \mathcal{E}_{ha}$ indexed by $a \in Q_1$, that satisfy the relations $\mathcal{K}$. One has that there is an equivalence of categories between $G$-equivariant bundles on $X \times G/P$ and $(Q, \mathcal{K})$-bundles on $X$. By standard techniques in representation theory of complex Lie groups on Fréchet spaces of sections of equivariant coherent sheaves (with a holomorphic action of the complex Lie group), this equivalence can be extended to coherent sheaves.

It turns out that the parabolicity of $P$ implies that the quiver has no oriented cycles. An important consequence of this is that a $G$-equivariant holomorphic vector bundle admits a natural $G$-equivariant filtration by holomorphic subbundles. This suggests that the natural equation for a Hermitian metric to consider is again the deformed Hermitian–Einstein equation, like in the case of $X \times \mathbb{P}^1$. To study this equation and its dimensional reduction we first study in Section 2 the correspondence established in Section 2 by means of Dolbeault operators. We reinterpret in these terms many of the ingredients studied in the previous section. In particular, we see that the relations of the quiver correspond to the integrability of the Dolbeault operator on the corresponding homogeneous vector bundle on $G/P$. We go on to consider the dimensional reduction of a $K$-invariant solution of the filtered Hermitian–Einstein equation on a $G$-equivariant vector bundle $\mathcal{F}$ over $X \times G/P$. We obtain that such solutions are in correspondence with a collection of metrics satisfying a set of vortex-type equations on the bundles $\mathcal{E}_\lambda$ in the $(Q, \mathcal{K})$-bundle defined by $\mathcal{F}$. We also show that the stability of $\mathcal{F}$ as an equivariant filtration is equivalent to an appropriate stability condition for the $(Q, \mathcal{K})$-bundle on $X$. As a corollary we obtain a Hitchin–Kobayashi correspondence for $(Q, \mathcal{K})$-bundles. We finish this section by observing that the quiver vortex equations that we obtain make actually perfect sense for arbitrary quivers with relations (i.e. not necessarily related to a parabolic group). One can indeed prove a Hitchin–Kobayashi correspondence in this more general situation — this is a subject of the paper [AG2].

In the last section, we use the examples provided in Section 1 to illustrate the general theory developed throughout this paper. We compute explicitly the dimensional reduction of the gauge equations and stability conditions. In particular, when $G/P$ is the projective line we recover the results of [AG1, BG] from our general theory.

A dimensional reduction problem closely related to the one treated in this paper has been studied by Steven Bradlow, Jim Glazebrook and Franz Kamber [BGK]. It would be very interesting to combine the two points of view to pursue further research in the subject.

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1. Homogeneous Bundles and Quiver Representations

One goal of this paper is to investigate the dimensional reduction of natural gauge equations and stability conditions for equivariant vector bundles on the product of a compact Kähler manifold $X$ by a flag variety $G/P$, where $G$ is a connected semisimple complex Lie group, and $P \subset G$ is a parabolic subgroup. As we shall see, the process of dimensional reduction leads to new objects defined on $X$, called quiver bundles, which we shall define in §2, where we shall also prove that there is a one-to-one
relation between equivariant vector bundles on $X \times G/P$ and quiver bundles on $X$. Preparing and leading up to this correspondence, this section is concerned with the case where $X$ is a single point. This section is organised as follows. After introducing notation and some preliminaries in §§1.1 and 1.2, in §1.3 we define the quiver with relations associated to any complex Lie group. In §1.4 we prove the equivalence between the category of holomorphic homogeneous vector bundles on the flag variety $G/P$ and the category of representations of the quiver with relations associated to $P$. Then, in §1.5 we prove that the quiver associated to a parabolic subgroup $P \subset G$ has no oriented cycles, and as a result, the holomorphic homogeneous vector bundles on $G/P$ admit certain filtrations by homogeneous subbundles.

1.1. Preliminaries and notation. Throughout this paper, $G$ is a connected simply connected complex semisimple Lie group, $P$ is a parabolic subgroup of $G$, and $X$ is a compact Kähler manifold. Given a complex manifold $M$ with a holomorphic action of $G$, a $G$-equivariant holomorphic vector bundle on $M$ is a holomorphic vector bundle $\pi : F \to M$ on $M$, together with a holomorphic $G$-action $\rho : G \times F \to F$ on its total space $F$ which commutes with $\pi$, and such that for all $(g, p) \in G \times M$, the map $\rho_{g,p} : F_p \to F_{g \cdot p}$, from $F_p = \pi^{-1}(p)$ into $F_{g \cdot p} = \pi^{-1}(g \cdot p)$, induced by this action, is an isomorphism of vector spaces. In this paper we are concerned with the $G$-equivariant holomorphic vector bundles on the $G$-manifold $M = X \times G/P$. The group $G$ acts on $X \times G/P$ by the trivial action on $X$ and the (left) standard action on the flag variety $G/P$.

1.1.1. Induction and reduction. It is a standard result that there is a one-to-one correspondence between the isomorphism classes of $G$-equivariant holomorphic vector bundles on $X \times G/P$ and the isomorphism classes of $P$-equivariant holomorphic vector bundles on $X$, where $P$ acts trivially on $X$. The correspondence is defined as follows. A $G$-equivariant holomorphic vector bundle $F$ on $X \times G/P$ defines a $P$-equivariant holomorphic vector bundle $i^*F$ on $X$ by restriction to the slice $i : X \cong X \times P/P \hookrightarrow X \times G/P$. Conversely, a $P$-equivariant holomorphic vector bundle $E$ on $X$ defines by induction a $G$-equivariant holomorphic vector bundle $G \times_P E$ on $X \times G/P$. This holomorphic vector bundle is by definition the quotient of $G \times E$ by the action of $P$, defined by $p \cdot (g, e) = (gp^{-1}, p \cdot e)$ for $p \in P$ and $(g, e) \in G \times E$. The bundle projection is

$$G \times_P E \to X \times G/P, \quad [g, e] \mapsto (x, gp)$$

for $e \in E_x, x \in X$. The vector bundle $G \times_P E \to X \times G/P$ has an equivariant action of $G$ given by

$$G \times (G \times_P E) \to G \times_P E, \quad (g', [g, e]) \mapsto [g'g, e].$$

This construction defines equivalence functors between the categories of $G$-equivariant holomorphic vector bundle on $X \times G/P$ and $P$-equivariant holomorphic vector bundles on $X$.

1.1.2. Homogeneous vector bundles. A holomorphic homogeneous vector bundle on $G/P$ is a $G$-equivariant holomorphic vector bundle on $G/P$. If we take $X$ to be a point in the previous correspondence, we get an equivalence between the category of holomorphic homogeneous vector bundles on $G/P$ and the category of holomorphic representations of $P$. Now, $X \times G/P$ is a ‘family’ of flag varieties $\{x\} \times G/P \cong G/P$, parametrised by the points $x \in X$. Let $F$ be a $G$-equivariant holomorphic vector bundle on $X \times G/P$, and $E = i^*F$ the corresponding holomorphic $P$-equivariant vector bundle on $X$. Given $x \in X$, the restriction $F_x := i^*_x F$ of $F$ to the flag variety $i^*_x : G/P \cong \{x\} \times G/P \hookrightarrow X \times G/P$ is a holomorphic homogeneous vector bundle on $G/P$. So $F$ is also a ‘family’ of holomorphic homogeneous vector bundles $F_x$ on $G/P$. The holomorphic homogeneous vector bundle $F_x$ is in correspondence with the holomorphic representation $E_x$ of $P$.

1.1.3. Irreducible representations of $P$. It is apparent from the constructions above that a detailed study of the holomorphic representations of $P$ is important in classifying the $G$-equivariant holomorphic vector bundles on $X \times G/P$. To study the holomorphic representations of $P$, it is convenient
to start with the irreducible ones. Throughout this paper, \( U \) is the unipotent radical of \( P \), and \( L \) is a (reductive) Levi subgroup of \( P \), so there is a semidirect decomposition \( P = U \ltimes L \), coming from a short exact sequence

\[
1 \longrightarrow U \longrightarrow P \longrightarrow L \longrightarrow 1.
\]

It is an immediate consequence of Engels theorem that a holomorphic representation \( V \) of \( P \) is irreducible if and only if the action of \( U \) on \( V \) is trivial and \( V \) is irreducible when considered as a holomorphic representation of \( L \). Therefore there is a one-to-one correspondence between irreducible representations of \( P \) and irreducible representations of its Levi subgroup \( L \).

1.1.4. **Notation.** Throughout this paper, \( H \) is a Cartan subgroup of \( G \) such that \( H \subset L \), and \( p, u, l, h \) are the Lie algebras of \( P, U, L, H \), respectively. The lattice of integral weights of \( H \), which parametrises the (isomorphism classes of) irreducible representations of \( H \), is denoted \( \Lambda \subset h^* \). We fix a fundamental chamber \( \Lambda^\circ \) in \( \Lambda \) for the reductive Lie group \( L \), so its elements are called integral dominant weights of \( P \). The fundamental chamber parametrises the (isomorphism classes of) irreducible representations of \( L \) (or \( P \), cf. §1.1.3). (The fundamental chamber \( \Lambda^\circ \) only depends on \( P \), since any two Levi subgroups are conjugate, by Malcev’s theorem, cf. e.g. [OV].) Thus, given an integral dominant weight for \( P \), \( \lambda \in \Lambda^+_P \), we fix an irreducible representation \( M_\lambda \) of \( P \) (or \( L \) ) (cf. §1.1.3), with highest weight \( \lambda \).

A holomorphic representation \( \rho : P \to GL(V) \) of \( P \) on \( V \) restricts to holomorphic representations \( \sigma := \rho|_L : L \to GL(V), \tau := \rho|_U : U \to GL(V) \) of \( L \) and \( U \) on \( V \). Let \( \sigma : L \to GL(V) \) and \( \tau : U \to GL(V) \) be holomorphic representations of \( L \) and \( U \) on a vector space \( V \). To determine when there exists a holomorphic representation \( \rho \) of \( P \) whose restrictions to \( L \) and \( U \) are \( \sigma \) and \( \tau \), it is convenient to use the representation theory of quivers. Good references for details on quivers and their representations are [ARS] [GR].

1.1.5. **Representations of quivers.** A quiver, or directed graph, is a pair of sets \( Q = (Q_0, Q_1) \) together with two maps \( h, t : Q_1 \to Q_0 \). The elements of \( Q_0 \) (resp. \( Q_1 \)) are called the vertices (resp. arrows) of the quiver. For each arrow \( a \in Q_1 \), the vertex \( ta \) (resp. \( ha \) ) is called the tail (resp. head) of the arrow \( a \), and the arrow \( a \) is sometimes represented by \( a : v \to v' \), with \( v = ta, v' = ha \). We will not require \( Q_0 \) to be finite, but will require \( Q \) to be locally finite, i.e. \( h^{-1}(v), t^{-1}(v) \) should be finite, for \( v \in Q_0 \). A (non-trivial) path in \( Q \) is a sequence \( p = a_0 \cdots a_m \) of arrows \( a_i \in Q_1 \) which compose, i.e. with \( ta_{i-1} = ha_i \) for \( 1 \leq i \leq m \):

\[
p : \bullet \quad \bullet \quad \bullet \to \cdots \to \bullet \quad \bullet \quad \bullet
\]

The vertices \( tp := ta_m \) and \( hp := ha_0 \) are called the tail and the head of the path \( p \). The trivial path \( e_v \) at \( v \in Q_0 \) consists of the vertex \( v \) with no arrows. A (complex) relation of a quiver \( Q \) is a formal finite sum \( r = c_1 p_1 + \cdots + c_p p_l \) of paths \( p_1, \ldots, p_l \) with coefficients \( c_i \in \mathbb{C} \). A quiver with relations is a pair \( (Q, K) \), where \( Q \) is a quiver and \( K \) is a set of relations of \( Q \). A linear representation \( R = (V, \varphi) \) of \( Q \) is given by a collection \( V \) of complex vector spaces \( V_v \), for each vertex \( v \in Q_0 \), together with a collection \( \varphi \) of linear maps \( \varphi_a : V_{ta} \to V_{ha} \), for each arrow \( a \in Q_1 \). We also require that \( V_v = 0 \) for all but finitely many \( v \). A morphism \( f : R \to R' \) between two representations \( R = (V, \varphi) \) and \( R' = (V', \varphi') \) is given by morphisms \( f_v : V_v \to V'_v \), for each \( v \in Q_0 \), such that \( \varphi' \circ f_a = f_h \circ \varphi_a \) for all \( a \in Q_1 \). Given a quiver representation \( R = (V, \varphi) \), a (non-trivial) path \( \{1\} \) induces a linear map

\[
\varphi(p) := \varphi_{a_0} \circ \cdots \circ \varphi_{a_m} : V_{tp} \longrightarrow V_{hp}.
\]

The linear map induced by the trivial path \( e_v \) at \( v \in Q_0 \) is \( \varphi(e_v) = id : V_v \to V_v \). A linear representation \( R = (V, \varphi) \) of \( Q \) is said to satisfy the relation \( r = \sum c_i p_i \) if \( \sum_i c_i \varphi(p_i) = 0 \). Given a set \( K \) of relations of \( Q \), a \( (Q, K) \)-module is a linear representation of \( Q \) satisfying the relations in \( K \). The category of \( (Q, K) \)-modules is clearly abelian.
1.2. The twisted nilradical representations. This subsection provides the key technical point of the understanding of representations of a complex Lie group \( P \) as quiver representations. The basic ingredients are vector spaces \( A_{\mu \lambda}, B_{\mu \lambda} \) and linear maps \( \psi_{\mu \nu \lambda}, \psi_{\mu \lambda}, \) for \( \lambda, \mu, \nu \) integral dominant weights of \( P \).

As a motivation for the definitions below, note that, if \( V \) is a \( P \)-module, with action of its Lie algebra \( \mathfrak{p} \) given by \( \rho : \mathfrak{p} \rightarrow \text{Aut}(V) \), then the action of \( L \) is specified by the decomposition of \( V \), as an \( L \)-module, into isotopic components (since \( L \) is reductive), and the rest of the \( \rho \) is determined by a linear map \( \tau = \rho | \mathfrak{p} : \mathfrak{p} \rightarrow \text{End}(V) \). If \( \sigma : l \rightarrow \text{End}(V) \) is the representation of \( L \) induced by \( \rho \), it is clear that \( \tau \) satisfies the following commutation relation \( \tau([f, e]) = [\sigma(f), \tau(e)] \) for \( f \in l \) and \( e \in \mathfrak{p} \); and \( \tau([e, e']) = [\tau(e), \tau(e')] \) for \( e, e' \in \mathfrak{p} \). The former condition means that \( \tau \) is \( L \)-equivariant, and motivates the definition of the linear spaces \( A_{\mu \lambda} \), while the latter is more delicate, and motivates the definition of the linear spaces \( B_{\mu \lambda} \) and the linear maps \( \psi_{\mu \nu \lambda}, \psi_{\mu \lambda} \).

1.2.1. Isotopical decompositions. Consider the nilradical \( \mathfrak{u} \subset \mathfrak{p} \) as a representation of \( L \). Decompose the \( L \)-modules obtained by twisting the exterior powers of the dual nilradical representation with the irreducible \( P \)-modules, \( \bigwedge^2 \mathfrak{u}^* \otimes M_{\mu}, \bigwedge^2 \mathfrak{u}^* \otimes M_{\mu} \), for \( \mu \in \Lambda^+_p \), into irreducible components, as a representation of \( L \):

\[
\mathfrak{u}^* \otimes M_{\mu} = \bigoplus_{\lambda \in \Lambda^+_p} A_{\mu \lambda} \otimes M_{\lambda}, \quad A_{\mu \lambda} := (\mathfrak{u}^* \otimes \text{Hom}(M_{\lambda}, M_{\mu}))^L,
\]

\[
\bigwedge^2 \mathfrak{u}^* \otimes M_{\mu} = \bigoplus_{\lambda \in \Lambda^+_p} B_{\mu \lambda} \otimes M_{\lambda}, \quad B_{\mu \lambda} := (\bigwedge^2 \mathfrak{u}^* \otimes \text{Hom}(M_{\lambda}, M_{\mu}))^L.
\]

1.2.2. Linear maps. (a) For any \( \lambda, \mu, \nu \in \Lambda^+_p \), define the linear map

\[
\psi_{\mu \nu \lambda} : (\mathfrak{u}^* \otimes \text{Hom}(M_{\mu}, M_{\nu})) \otimes (\mathfrak{u}^* \otimes \text{Hom}(M_{\nu}, M_{\lambda})) \rightarrow \bigwedge^2 \mathfrak{u}^* \otimes \text{Hom}(M_{\lambda}, M_{\mu})
\]

where

\[
\psi_{\mu \lambda}(a^\mu)(e, e') = -a([e, e']) \quad \text{for } e, e' \in \mathfrak{u}.
\]

By the exterior product \( a' \wedge a \) of part (a), of course we mean \( \psi_{\mu \lambda}(a' \otimes a) = a' \wedge a = (s' \wedge s) \otimes (f' \circ f) \) for \( a = s \otimes f, a' = s' \otimes f' \), with \( s, s' \) \( \in \mathfrak{u}^* \) and \( f \in \text{Hom}(M_{\mu}, M_{\nu}), f' \in \text{Hom}(M_{\nu}, M_{\lambda}) \).

The following lemma is straightforward. We shall need it in the definition \( \mathcal{K} \) associated to the complex Lie group \( P \).

Lemma 1.2. If \( \lambda, \mu, \nu \in \Lambda^+_p \), then \( \psi_{\mu \nu \lambda}(A_{\mu \nu} \otimes A_{\nu \lambda}) \subset B_{\mu \lambda} \) and \( \psi_{\mu \lambda}(A_{\mu \lambda}) \subset B_{\mu \lambda} \).

1.3. Quiver and relations associated to the Lie group \( P \).

1.3.1. Quiver. Given \( \lambda, \mu \in \Lambda^+_p \), let \( \{ a^{(i)}_{\mu \lambda} | i = 1, \ldots, n_{\mu \lambda} \} \) be a basis of \( A_{\mu \lambda} \), with \( n_{\mu \lambda} := \dim A_{\mu \lambda} \). The quiver \( Q \) associated to \( P \) has as vertex set, \( Q_0 = \Lambda^+_p \), i.e. the set of irreducible representations of \( P \), and as arrow set \( Q_1 = \{ a^{(i)}_{\mu \lambda} | \lambda, \mu \in Q_0, 1 \leq i \leq n_{\mu \lambda} \} \), the set of all basis elements of the vector spaces \( A_{\mu \lambda} \). The tail and head maps \( t, h : Q_1 \rightarrow Q_0 \) are defined by

\[
t(a^{(i)}_{\mu \lambda}) = \lambda, \quad h(a^{(i)}_{\mu \lambda}) = \mu.
\]
1.3.2. Relations. Let \( Q = (Q_0, Q_1) \) be as above. Let \( \{b_{\mu\lambda}^{(p)}|1 \leq p \leq m_{\mu\lambda}\} \) be a basis of \( B_{\mu\lambda} \), with \( m_{\mu\lambda} := \dim B_{\mu\lambda} \), for \( \lambda, \mu \in Q_0 \). Expand \( \psi_{\mu\nu\lambda}(a_{\mu\nu}^{(j)} \otimes a_{\nu\lambda}^{(i)}) \in B_{\mu\lambda} \) and \( \psi_{\mu\lambda}(a_{\mu\lambda}^{(k)}) \in B_{\mu\lambda} \) in this basis (cf. Lemma 1.2), for \( \lambda, \mu, \nu \in Q_0 \):

\[
\psi_{\mu\nu\lambda}(a_{\mu\nu}^{(j)} \otimes a_{\nu\lambda}^{(i)}) = \sum_{p=1}^{m_{\mu\lambda}} c_{\mu\nu\lambda}^{(j,i,p)} b_{\mu\lambda}^{(p)}, \quad \psi_{\mu\lambda}(a_{\mu\lambda}^{(k)}) = \sum_{p=1}^{m_{\mu\lambda}} c_{\mu\lambda}^{(k,p)} b_{\mu\lambda}^{(p)}.
\]

The set of relations of the quiver \( Q \) associated to \( P \) is \( K = \{r_{\mu\lambda}^{(p)}|\lambda, \mu \in Q_0, 1 \leq p \leq m_{\mu\lambda}\} \), where

\[
r_{\mu\lambda}^{(p)} = \sum_{\nu\in Q_0} \sum_{i=1}^{n_{\mu\nu}} \sum_{j=1}^{n_{\nu\lambda}} c_{\mu\nu\lambda}^{(j,i,p)} a_{\mu\nu}^{(j)} a_{\nu\lambda}^{(i)} + \sum_{k=1}^{n_{\mu\lambda}} c_{\mu\lambda}^{(k,p)} a_{\mu\lambda}^{(k)}.
\]

Note that in this definition \( a_{\mu\nu}^{(j)} a_{\nu\lambda}^{(i)} \) does not mean the composition of \( a_{\nu\lambda}^{(i)} \in u^* \otimes \Hom(M_\lambda, M_\nu) \) with \( a_{\mu\nu}^{(j)} \in u^* \otimes \Hom(M_\mu, M_\lambda) \), but the path that these two arrows define.

1.4. Correspondence between group and quiver representations.

Theorem 1.4. Let \( Q \) and \( K \) be the quiver and the set of relations associated to the group \( P \). There is an equivalence of categories

\[
\begin{align*}
\{ \text{finite dimensional} & \quad \text{holomorphic representations} \\
& \quad \text{of the Lie group } P \} \\
\leftrightarrow & \quad \{ \text{finite dimensional} & \quad \text{representations of the quiver } Q \}
\end{align*}
\]

This equivalence was first proved by Bondal and Kapranov [BK] when \( P \) is a Borel subgroup or the simple components of \( G \) are in the series \( A, D, E \). They were actually only interested in homogeneous bundles over projective spaces, so these cases were enough for their purposes. Bondal and Kapranov also gave a very simple and explicit description of the quiver and the relations in these cases (we collect their results about the quivers in Propositions 1.21 and 1.23). Unfortunately, their theorem is not always true with the simple definition of the relations given in [BK], as shown by a counterexample found by Hille [Hl2]. Our definition of the quiver is precisely as in [BK], but the relations have been appropriately corrected. Hille also defined a quiver with relations for any parabolic subgroup, and proved the corresponding theorem of equivalence of categories in [Hl2]. Hille uses a different definition of the quiver associated to \( P \), obtained by removing certain arrows from the quiver in [BK] (although they coincide when \( U \) is abelian so the relations are quadratic). As mentioned in the introduction, the quiver and relations that we obtain seem to arise more naturally from the point of view of dimensional reduction (cf. Theorem 4.13).

Proof. To prove the theorem, we shall define an equivalence functor from the category of representations of \( P \) into the category of \( (Q, K) \)-modules. Given a representation \( \rho : \tilde{P} \rightarrow \Aut(V) \) of \( P \) on a (finite-dimensional) complex vector space \( V \), we obtain by restriction a representation \( \sigma = \rho|_L : L \rightarrow \Aut(V) \) of \( L \subset P \) on \( V \). Since \( L \) is reductive, \( \sigma \) decomposes into a direct sum

\[
V = \bigoplus_{\lambda \in Q_0} V_\lambda \otimes M_\lambda, \quad V_\lambda = \Hom_L(M_\lambda, V).
\]

The vector spaces \( V_\lambda \) are the multiplicity spaces, and have trivial \( L \)-action. Let us fix a representation \( \sigma : L \rightarrow \Aut(V) \) of \( L \) on \( V \) as in (1.5). We consider the set \( \text{Rep}_P(\sigma) \) of (isomorphism classes of) representations \( \rho : P \rightarrow \Aut(V) \) of \( P \) on \( V \) whose restriction to \( L \) is \( \rho|_L = \sigma \). Apart from \( L \), the rest of the \( P \)-module structure of \( V \) is given by the action \( \tau = \rho|_U : U \rightarrow GL(V) \) of \( U \) on \( V \). Now, for the unipotent complex Lie group \( U \), the map \( \exp : u \rightarrow U \) is a \( P \)-equivariant isomorphism (of
algebraic varieties, cf. e.g. [DV, §3.3.6, Theorem 7]), and for any element \( u = \exp e \) of \( U \), with \( e \in u \), \( \tau(e) \) is a nilpotent operator, so its exponential is a finite sum

\[
\tau(u) = \exp(\tau(e)) = \sum_{i} \frac{1}{i!} \tau(e)^i,
\]

hence a polynomial in \( u \). (We use the same symbol \( \tau \) for the representation \( d\tau : u \to \text{End}(V) \) of the Lie algebra \( u \) of \( U \).) The conditions for a linear map \( \tau : u \to \text{End}(V) \) to define a representation \( \rho \in \text{Rep}_P(\sigma) \) consist of two kinds of commutation relations:

1. (CR1) \( \tau([f, e]) = [\sigma(f), \tau(e)] \) for \( f \in \mathfrak{L} \) and \( e \in u \);
2. (CR2) \( \tau([e, e']) = [\tau(e), \tau(e')] \) for \( e, e' \in u \).

If we consider \( u \) as an \( L \)-module, and \( \text{End}(V) \) as the \( L \)-module obtained from the isotopical decomposition \([1,2] \), then condition (CR1) simply means that \( \tau : u \to \text{End}(V) \) is a morphism of \( L \)-modules. In other words, (CR1) is satisfied if and only if \( \tau \) belongs to the \( L \)-invariant part \( W_1^L \) of the \( L \)-module

\[
W_1 = u^* \otimes \text{End}(V).
\]

(The subindex ‘1’ accounts for condition (CR1).) The isotopical decomposition \([1,2] \) applies to give

\[
W_1 \cong \bigoplus_{\lambda, \mu, \nu \in Q_0} A_{\mu \lambda} \otimes \text{Hom}(V_\lambda, V_\mu) \otimes \text{Hom}(M_\mu, M_\mu),
\]

while Schur’s lemma implies

\[
W_1^L \cong \bigoplus_{\lambda, \mu \in Q_0} A_{\mu \lambda} \otimes \text{Hom}(V_\lambda, V_\mu).
\]

Let \( V \) be a collection of linear spaces \( V_\lambda \), for each \( \lambda \in Q_0 \). There is a linear isomorphism between \( W_1^L \) and the space of quiver representations into \( V \),

\[
\mathcal{R}(Q, V) = \bigoplus_{a \in Q_1} \text{Hom}(V_{i a}, V_{h a}),
\]

i.e. the space of representations \( \mathbf{R} = (V, \varphi) \) with fixed \( V \). This isomorphism takes any

\[
\tau = \sum_{\lambda, \mu \in Q_0} \sum_{i=1}^{n_{\mu \lambda}} a^{(i)}_{\mu \lambda} \otimes \varphi^{(i)}_{\mu \lambda} \in \bigoplus_{\lambda, \mu \in Q_0} A_{\mu \lambda} \otimes \text{Hom}(V_\lambda, V_\mu),
\]

into the representation \( \mathbf{R} = (V, \varphi) \) given by the morphisms \( \varphi_a = \varphi^{(i)}_{\mu \lambda} : V_\lambda \to V_\mu \), for \( a = a^{(i)}_{\mu \lambda} \).

Next we consider how condition (CR2) translates into \( \mathcal{R}(Q, V) \). Define the \( L \)-module

\[
W_2 := \wedge^2 u^* \otimes \text{End}(V)
\]

(the subindex ‘2’ accounts for condition (CR2)) and the (non-linear) map

\[
\psi : W_1 \to W_2
\]

by

\[
\psi(\tau)(e, e') = [\tau(e), \tau(e')] - \tau([e, e']), \quad \text{for } \tau \in W_1, \ e, e' \in u.
\]

Then \( \tau \) satisfies condition (CR2) if and only if \( \psi(\tau) = 0 \), so the space of representations \( \rho \) of \( P \) with \( \rho|_L \cong \sigma \) is \( \text{Rep}_P(\sigma) \cong \psi^{-1}(0) \cap W_1^L \); note that this is contained in \( W_1^L \cong \mathcal{R}(Q, V) \). In order to express this condition in terms of the relations \( \varphi^{(i)}_{\mu \lambda} \), we rewrite \( \psi \) using the following linear maps \( \psi_1, \psi_2 \). The first linear map is

\[
\psi_1 : W_1 \otimes W_1 \to W_2 \quad \tau' \otimes \tau \quad \mapsto \tau' \wedge \tau
\]
where by the exterior product $\tau' \wedge \tau$, we mean $\psi_1(\tau' \otimes \tau) = \tau' \wedge \tau = (s' \wedge s) \otimes (f' \circ f)$ for $\tau = s \otimes f$, $\tau' = s' \otimes f'$, with $s, s' \in u^*$ and $f, f' \in \text{End}(V)$ (this map should be compared with [1.2.2](a)). The second linear map

$$\psi_2 : W_1 \longrightarrow W_2,$$

is given by

$$\psi_2(\tau)(e, e') = -\tau([e, e']), \quad \text{for } \tau \in W_1, \ e, e' \in u^*$$

(compare with [1.2.2](b)). Obviously

$$\psi(\tau) = \psi_1(\tau \otimes \tau) + \psi_2(\tau), \quad \text{for } \tau \in W_1.$$ 

Note that

$$W_2^L = (\wedge^2 u^* \otimes \text{End}(V))^L \cong \bigoplus_{\lambda, \mu \in Q_0} B_{\mu \lambda} \otimes \text{Hom}(V_{\lambda'}, V_{\mu}).$$

When $\tau \in W_1^L$ is given by (1.7), linearity of $\psi_1, \psi_2$ allows one to obtain $\psi(\tau)$:

$$\psi_1(\tau \otimes \tau) = \sum_{\lambda, \nu, \mu, \nu' \in Q_0} \sum_{i=1}^{n_{\lambda \mu \nu \nu'}} \psi_1((a_{\mu \nu}^{(j)}) \otimes (\varphi_{\mu \nu}^{(j)})) \otimes (a_{\nu \lambda}^{(i)} \otimes \varphi_{\nu \lambda}^{(i)}))$$

$$= \sum_{\lambda, \nu, \mu \in Q_0} \sum_{i=1}^{n_{\lambda \mu \nu \nu'}} \psi_{\mu \nu \lambda} (a_{\mu \nu}^{(j)}) \otimes (\varphi_{\mu \nu}^{(j)} \circ \varphi_{\nu \lambda}^{(i)}),$$

and

$$\psi_2(\tau) = \sum_{\lambda, \mu \in Q_0} \sum_{i=1}^{n_{\mu \lambda}} \psi_2(a_{\mu \lambda}^{(i)} \otimes \varphi_{\mu \lambda}^{(i)}),$$

$$= \sum_{\lambda, \mu \in Q_0} \sum_{i=1}^{n_{\mu \lambda}} \psi_{\mu \lambda} (a_{\mu \lambda}^{(i)}) \otimes \varphi_{\mu \lambda}^{(i)} = \sum_{\lambda, \mu \in Q_0} \sum_{p=1}^{n_{\mu \lambda}} b_{\mu \lambda}^{(p)} \otimes \left( \sum_{k=1}^{n_{\mu \lambda}} c_{\mu \lambda}^{(k)} \varphi_{\mu \lambda}^{(k)} \right),$$

so that (1.8), (1.9), and (1.10) imply

$$\psi(\tau) = \sum_{\lambda, \mu \in Q_0} \sum_{p=1}^{n_{\mu \lambda}} b_{\mu \lambda}^{(p)} \otimes \left( \sum_{\nu \in Q_0} \sum_{i=1}^{n_{\nu \mu \lambda}} \sum_{j=1}^{n_{\mu \nu \lambda}} c_{\mu \lambda}^{(j,i,p)} \varphi_{\mu \lambda}^{(i)} \circ \varphi_{\mu \lambda}^{(j)} + \sum_{k=1}^{n_{\nu \mu \lambda}} c_{\mu \lambda}^{(k,p)} \varphi_{\mu \lambda}^{(k)} \right).$$

This equation gives the relations of the quiver $Q$ which realise condition (CR2).

It is worth remarking that the inclusion $\wedge^2 u^* \subset u^* \otimes u^*$ gives

$$B_{\mu \lambda} \subset (M_{\mu} \otimes u^* \otimes u^* \otimes M_{\lambda})^L \cong \bigoplus_{\nu, \nu' \in Q_0} A_{\mu \nu} \otimes M_{\nu} \otimes A_{\nu' \lambda} \otimes M_{\nu'}^L \cong \bigoplus_{\nu \in Q_0} A_{\mu \nu} \otimes A_{\nu \lambda}.$$
that the category of representations of \((Q, \mathcal{K})\) is equivalent to the category of rational representations of \(P\).

As a result of \([1,1.2]\) and Theorem \([1.4]\), it follows that:

**Corollary 1.13.** Let \(G\) be a connected complex Lie group and \(P \subset G\) a parabolic subgroup. Let \(Q\) and \(\mathcal{K}\) be the quiver and the set of relations associated to \(P\). There is an equivalence of categories

\[
\begin{align*}
\{ \text{holomorphic homogeneous vector bundles on } G/P \} & \rightarrow \{ \text{finite dimensional representations of the quiver } Q \} \\
\text{satisfying the relations in } \mathcal{K} & .
\end{align*}
\]

1.5. **Holomorphic filtrations of homogeneous bundles.** Let \(Q\) be the quiver associated to \(P\). This quiver has no oriented cycles, i.e. there are no paths of length \(>0\) in \(Q\) whose tail and head coincide. To prove this, we first introduce the notion of \(\Sigma\)-height, adapting \([Hu, \S 10.1]\) (see also \([CSS]\)). An essential ingredient will be that \(P\) is a parabolic subgroup of \(G\). We should mention that Hille \([H12]\) has defined a function similar to our \(\Sigma\)-height, that he calls a level function, for the quivers that he associates to the parabolic subgroups.

1.5.1. **Notation.** Choose a system \(\mathcal{S}\) of simple roots of \(\mathfrak{g}\) with respect to \(\mathfrak{h}\), such that all the negative roots of \(\mathfrak{g}\) with respect to \(\mathfrak{h}\) are roots of \(\mathfrak{p}\). Let \(\Sigma\) be the set of non-parabolic simple roots, i.e. it consists of those simple roots of \(\mathfrak{g}\) which are not roots of \(\mathfrak{p}\) (in terms of \(I\), this means that \(\mathcal{S} \setminus \Sigma\) is a system of simple roots of \(I\), while the roots in \(\Sigma\) are not roots of \(I\)). Given a \(P\)-module \(V\), let \(\Delta(V)\) be its set of weights with respect to \(\mathfrak{h}\), so

\[
V = \bigoplus_{\lambda \in \Delta(V)} V^\lambda,
\]

where \(V^\lambda := \{ v \in V | h \cdot v = \lambda(h)v \text{ for } h \in \mathfrak{h} \}\) for \(\lambda \in \Lambda\), and \(\Delta(V) := \{ \lambda \in \Lambda | V^\lambda \neq 0 \}\).

1.5.2. **\(\Sigma\)-height.** Any integral weight \(\mu \in \Lambda\) admits a decomposition \(\mu = \sum_{\alpha \in \mathcal{S}} n_\alpha \alpha\), with \(n_\alpha \in \mathbb{Q}\). We define the \(\Sigma\)-height of \(\mu\) as the rational number

\[
ht_\Sigma(\mu) = \sum_{\alpha \in \Sigma} n_\alpha.
\]

**Lemma 1.15.** If the quiver \(Q\) has an arrow \(\lambda \rightarrow \mu\), then \(ht_\Sigma(\lambda) > ht_\Sigma(\mu)\). Therefore, the quiver \(Q\) is directed.

**Proof.** If the irreducible \(L\)-modules \(M^\lambda, M_\mu, u^\ast\) have weight space decompositions

\[
M_\mu = \bigoplus_{\mu' \in \Delta(M_\mu)} M^\mu_{\mu'}, \quad M^\lambda = \bigoplus_{\lambda' \in \Delta(M^\lambda)} (M^\lambda)^{\lambda'}, \quad u^\ast \cong \bigoplus_{\gamma \in \Delta(u)} \mathfrak{g}^{-\gamma},
\]

(see \([1.14]\) above), then \(H \subset L\) implies

\[
A_{\mu\lambda} = (M_\mu \otimes u^\ast \otimes M^\lambda)^L \subset (M_\mu \otimes u^\ast \otimes M^\lambda)^H \cong \bigoplus_{(\mu', \lambda') \in \Delta(M_\mu, M^\lambda)} M^{\mu'}_{\mu} \otimes \mathfrak{g}^{-(\mu' - \lambda')} \otimes (M^\lambda)^{-\lambda'},
\]

where \(\Delta(M_\mu, M^\lambda)\) is the set of pairs \((\mu', \lambda')\), with \(\mu' \in \Delta(M_\mu), \lambda' \in \Delta(M^\lambda)\) such that \(\mu' - \lambda' \in \Delta(u)\). If \(A_{\mu\lambda} \neq 0\), there are \(\gamma \in \Delta(u), \lambda' \in \Delta(M^\lambda), \mu' \in \Delta(M^\lambda)\) with \(\mu' - \lambda' = \gamma\). But \(ht_\Sigma(\lambda') = ht_\Sigma(\lambda), ht_\Sigma(\mu') = ht_\Sigma(\mu)\), so \(ht_\Sigma(\mu) - ht_\Sigma(\lambda) = ht_\Sigma(\mu - \lambda) = ht_\Sigma(\gamma) < 0\). □
1.5.3. A total order in $Q_0$. By Lemma 1.15, defining $\lambda > \mu$ if $ht_{\Sigma}(\lambda) > ht_{\Sigma}(\mu)$, for each $\lambda, \mu \in Q_0$ provides a partial order in $Q_0$. We now enlarge this partial order to get a total order in $Q_0$: For each $q \in \mathbb{Q}$ with $ht_{\Sigma}^{-1}(q) \neq \emptyset$, we choose a total order $(\prec)_q$ on $ht_{\Sigma}^{-1}(q)$, and define a total order in $Q_0$ by saying that $\mu \prec \lambda$, for each $\lambda, \mu \in Q_0$, if either $ht_{\Sigma}(\mu) < ht_{\Sigma}(\lambda)$, or $\lambda, \mu \in ht_{\Sigma}^{-1}(q)$ for some $q \in \mathbb{Q}$ and $\mu(\prec)_q \lambda$. Thus, if there is an arrow $\lambda \rightarrow \mu$, then $\lambda > \mu$.

**Proposition 1.16.** Let $V$ be a $P$-module with isotopical decomposition, as an $L$-module,

\[(1.17)\quad V = \bigoplus_{\lambda \in Q_0(V)} V_\lambda \otimes M_\lambda, \quad V_\lambda = \text{Hom}_L(M_\lambda, V),\]

where $Q_0(V) \subset Q_0$ is a finite set. Let us list the set of vertices in $Q_0(V)$ in ascending order as $Q_0(V) = \{\lambda_0, \lambda_1, \ldots, \lambda_m\}$, $\lambda_0 < \lambda_1 < \cdots < \lambda_m$. Then $V$ admits a flag of $P$-submodules with completely reducible quotients:

\[(1.18)\quad V_{(\leq 0)} : 0 \subset V_{(\leq 0)} \subset V_{(\leq 1)} \subset \cdots \subset V_{(\leq m)} = V, \quad V_{(\leq s)}/V_{(\leq s-1)} \cong V_{\lambda_s} \otimes M_{\lambda_s}\]

(In fact, one can prove that if $V$ is indecomposable as a $P$-module, then $ht_{\Sigma}(\lambda_s) - ht_{\Sigma}(\lambda_{s-1})$ is one or zero for all $1 \leq s \leq m$.)

**Proof.** From the proof of Theorem 1.4, we see that the $L$-module structure of $V$ is given by an isotopical decomposition, as an $L$-module, $V = \bigoplus_{i=0}^m V_{\lambda_i} \otimes M_{\lambda_i}$, where $V_{\lambda_i} \neq 0$ for $0 \leq s \leq m$, while the $u$-structure is given by an $L$-invariant morphism

$$
\tau = \sum_{0 \leq s, s' \leq m} n_{\lambda_{s'}, \lambda_s} \sum_{i=1}^{m} a_{\lambda_{s'}, \lambda_s}^{(i)} \otimes \varphi_{\lambda_{s'}, \lambda_s}^{(i)}
$$

of $W = u^* \otimes \text{End}(V)$. Define the flag $V_{(\leq s)}$ of vector spaces as in (1.18), where $V_{(\leq s)} = \bigoplus_{j=0}^s V_{\lambda_{s'}} \otimes M_{\lambda_{s'}}$. This is obviously a flag of $L$-modules. To prove that it is a flag of $P$-modules as in (1.18), it is enough to see that $u$ takes $V_{(\leq s)}$ into $V_{(\leq s-1)}$, for $1 \leq s \leq m$. Let $e \in u$. The action of $e$ on $V$ is given by $\tau(e) \in \text{End}(V)$, where

$$
\tau(e) = \sum_{0 \leq s, s' \leq m} n_{\lambda_{s'}, \lambda_s} \sum_{i=1}^{m} a_{\lambda_{s'}, \lambda_s}^{(i)} (e) \otimes \varphi_{\lambda_{s'}, \lambda_s}^{(i)}
$$

with $a_{\lambda_{s'}, \lambda_s}^{(i)} (e) \in \text{Hom}(M_{\lambda_{s'}}, M_{\lambda_s})$ for $0 \leq s, s' \leq m$. By part (c) of Lemma 1.15, this is zero unless $s' < s$, so that $\tau(e)$ takes $V_{\lambda_s} \otimes M_{\lambda_s}$ into $\bigoplus_{s' < s} V_{\lambda_{s'}} \otimes M_{\lambda_{s'}}$. \hfill \Box

1.5.4. Note. Given an irreducible representation $M_\lambda$ of $P$, corresponding to an integral dominant weight $\lambda \in \Lambda_0^+$, $\mathcal{O}_\lambda := G \times P M_\lambda$ is the induced irreducible holomorphic homogeneous vector bundle on $G/P$.

**Corollary 1.19.** Any holomorphic homogeneous vector bundle $\mathcal{F}$ on $G/P$ admits a filtration of holomorphic homogeneous vector subbundles $\mathcal{F}_s$, with completely reducible quotients with respect to the $G$-action,

\[(1.20)\quad \mathcal{F} : 0 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_m = \mathcal{F}, \quad \mathcal{F}_s/\mathcal{F}_{s-1} \cong V_{\lambda_s} \otimes \mathcal{O}_{\lambda_s}, \quad 1 \leq s \leq m,\]

for some dominant integral weights $\lambda_s$ with $\lambda_0 < \lambda_1 < \cdots < \lambda_m$, where $V_{\lambda_s}$ are vector spaces determined by $\mathcal{F}$. 
Proof. This result follows from §1.1.3 and Proposition 1.16. First, \( F \cong G \times_P V \), where \( V_o = F_o \), the fibre at the base point \( o = P \in G/P \), is a representation of \( P \). Then \( V \) admits the isotopical decomposition (1.17) and the flag of \( P \)-submodules (1.18). Let \( F_s = G \times_P V_{(\leq s)} \) be the homogeneous holomorphic vector subbundle of \( F \) induced by \( V_{(\leq s)} \), for \( 0 \leq s \leq m \). Then \( V_{(\leq s)}/V_{(\leq s-1)} \cong V_{\lambda_s} \otimes M_{\lambda_s} \), implies \( F_s/F_{s-1} \cong V_{\lambda_s} \otimes O_{\lambda_s} \). \( \square \)

1.6. Examples. In this subsection we shall present explicit expressions for the quiver and the relations corresponding to some flag varieties. We would like to remark that the problem of classification of the quivers with relations associated to all parabolic subgroups is a subtle one. At present a complete classification is not known, and it would require a deep study which is out of the scope of this paper. To obtain the quiver one only has to evaluate dimension formulas for the vector spaces \( A_{\mu\lambda} \). However, to obtain the relations, as given in §1.3, is more difficult, since one has to choose bases of the vector spaces \( A_{\mu\lambda} \) and \( B_{\mu\lambda} \), and express the linear maps \( \psi_{\mu\lambda}, \psi_{\mu\lambda} \) in these bases, as in §1.3.2.

1.6.1. Quiver and relations for Borel subgroups. Let us assume that \( B = P \) is a Borel subgroup of \( G \). The set of integral dominant weights of \( B \) is precisely the weight lattice \( \Lambda \cong \mathbb{Z}^{\text{rank}(G)} \) of integral weights (cf. §1.1.4), for the Cartan subgroup \( H \) is a Levi subgroup of \( B \). Let \( \Delta \) be the set of roots of \( (\mathfrak{g}, \mathfrak{h}) \), and for \( \alpha \in \Delta \), let \( \mathfrak{g}^{\alpha} \) be the root subspace of \( \mathfrak{g} \) corresponding to \( \alpha \). We choose the sets \( \Delta_+, \Delta_- \subset \Delta \) of positive and negative roots, so that the Lie algebra of \( B \) is \( \mathfrak{b} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha} \). For each \( \alpha \in \Delta \), let \( e_\alpha \in \mathfrak{g}^{\alpha} \) be the corresponding Chevalley generator (see e.g. [Hu, §25.21]). Let \( N_{\alpha\beta} \in \mathbb{Z} \), for \( \alpha, \beta \in \Delta \), be the coefficients defined by the condition \( [e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta} \) if \( \alpha + \beta \in \Delta \), and \( N_{\alpha\beta} = 0 \) if \( \alpha + \beta \notin \Delta \).

Proposition 1.21. The quiver with relations \((Q, \mathcal{K})\) associated to \( B \) is given as follows.

1. The vertex set \( Q_0 \) is the weight lattice \( \Lambda \cong \mathbb{Z}^{\text{rank}(G)} \).
2. The arrow set \( Q_1 \) consists of the arrows \( a_{\mu\lambda}^{(\gamma)} : \lambda \to \mu \), for \( \lambda, \mu \in \Lambda \), with \( \gamma = \mu - \lambda \in \Delta_- \).
3. The set \( \mathcal{K} \) consists of the relations \( r_{\Lambda_{\mu\lambda}}^{(\gamma')}(\gamma, \gamma') = a_{\mu\lambda}^{(\gamma)} a_{\mu\lambda}^{(\gamma')} - a_{\mu\lambda}^{(\gamma)} a_{\mu\lambda}^{(\gamma')} - N_{\gamma\gamma'} a_{\mu\lambda}^{(\gamma+\gamma')} \), for \( \lambda, \mu \in \Lambda \) and \( \gamma, \gamma' \in \Delta_- \), with \( \gamma + \gamma' = \mu - \lambda \) and \( \gamma \neq \gamma' \), where \( \nu := \lambda + \gamma \) and \( \nu' := \lambda + \gamma' \).

(The term \( a_{\mu\lambda}^{(\gamma')} a_{\mu\lambda}^{(\gamma)} \) (resp. \( a_{\mu\lambda}^{(\gamma')} a_{\mu\lambda}^{(\gamma)} \); \( N_{\gamma\gamma'} a_{\mu\lambda}^{(\gamma+\gamma')} \)) is ignored in the definition of \( r_{\Lambda_{\mu\lambda}}^{(\gamma, \gamma')} \), whenever the basis vectors \( a_{\mu\lambda}^{(\gamma')} \) or \( a_{\mu\lambda}^{(\gamma)} \) (resp. \( a_{\mu\lambda}^{(\gamma')} \) or \( a_{\mu\lambda}^{(\gamma)} \); \( a_{\mu\lambda}^{(\gamma+\gamma')} \)) do not make sense.)

Proof. Part (1) is obvious. The nilpotent radical of \( \mathfrak{b} \) is \( \mathfrak{u} = \oplus_{\gamma \in \Delta_-} \mathfrak{g}_\gamma \), with basis \( \{ e_\gamma | \gamma \in \Delta_- \} \). Let \( \{ e_\gamma | \gamma \in \Delta_- \} \) be its dual basis. Since the Levi subgroup \( H \subset B \) is abelian, its irreducible representations \( M_\lambda \), for \( \lambda \in \Lambda \), are one dimensional. Let \( v_\lambda \) be a basis vector of \( M_\lambda \), and \( v^\lambda \in M^\lambda_\lambda \) be its dual basis vector. Let \( A_{\mu\lambda} = (u^* \otimes \text{Hom}(M_\lambda, M_\mu))^H \), for \( \lambda, \mu \in \Lambda \), as in §1.2.1. The weight of \( e_\gamma \) with respect to \( \mathfrak{h} \) is \( -\gamma \). Thus, if \( \mu - \lambda \notin \Delta_- \), then \( A_{\mu\lambda} = 0 \), while if \( \gamma := \mu - \lambda \in \Delta_- \), then \( A_{\mu\lambda} \) is one dimensional, with basis vector

\[
a_{\mu\lambda}^{(\gamma)} := e_\gamma \otimes v_\mu \otimes v^\lambda.
\]

This proves part (2). To get part (3), let \( B_{\mu\lambda} = (\Lambda^2 u^* \otimes \text{Hom}(M_\lambda, M_\mu))^H \), as in §1.2.1. Let \( \lambda \) be a total order for the set \( \Delta_- \). A basis of \( \Lambda^2 u^* \) is \( \{ e_\gamma \wedge e_\gamma' | \gamma, \gamma' \in \Delta_- , \gamma < \gamma' \} \). The weight of \( e_\gamma \wedge e_\gamma' \) with respect to \( \mathfrak{h} \) is \( -\gamma - \gamma' \). Thus, if \( \mu - \lambda \neq \gamma + \gamma' \) for any \( \gamma, \gamma' \in \Delta_- \), then \( B_{\mu\lambda} = 0 \), while if \( \mu - \lambda = \gamma + \gamma' \) for some \( \gamma, \gamma' \in \Delta_- \) with \( \gamma < \gamma' \), then a basis of \( B_{\mu\lambda} \) is \( \{ b_{\mu\lambda}^{(\gamma, \gamma')} | \gamma, \gamma' \in \Delta_- , \gamma < \gamma', \gamma + \gamma' = \mu - \lambda \} \), where

\[
b_{\mu\lambda}^{(\gamma, \gamma')} := (e_\gamma \wedge e_\gamma') \otimes v_\mu \otimes v^\lambda.
\]
To express $\psi_{\mu\nu}(a_{\mu\nu}^{(\gamma)})$ and $\psi_{\mu\nu}(a_{\mu\nu}^{(\gamma')})$ in the bases $\{a_{\mu\nu}^{(\gamma)}\}$ and $\{a_{\mu\nu}^{(\gamma')}\}$, let $\lambda, \mu \in \Lambda$ be such that $B_{\mu\lambda} \neq 0$. Hence, there are $\gamma, \gamma' \in \Delta_+$, with $\gamma < \gamma', \mu - \lambda = \gamma + \gamma'$. Let $\nu = \lambda + \gamma, \nu' = \lambda + \gamma'$. Then

$$
\psi_{\mu\nu\lambda}(a_{\mu\nu\lambda}^{(\gamma)}) \otimes a_{\nu'\lambda}^{(\gamma')} = b_{\mu\nu\lambda}^{(\gamma,\gamma')}, \quad \psi_{\mu\nu'\lambda}(a_{\mu\nu'\lambda}^{(\gamma)}) \otimes a_{\nu\lambda}^{(\gamma')} = -b_{\mu\nu'\lambda}^{(\gamma,\gamma')}.
$$

Let $\lambda, \mu \in \Lambda$ with $A_{\mu\lambda} \neq 0$. Then $a_{\mu\lambda}^{(\phi)} = e^{\phi} \otimes v_{\mu} \otimes v_{\lambda}$, where $\phi := \mu - \lambda \in \Delta_-$, so

$$
(1.22) \quad \psi_{\mu\lambda}(a_{\mu\lambda}^{(\phi)})(e_\epsilon, e_\epsilon') = -a_{\mu\lambda}^{(\phi)}(N_{e_\epsilon e_\epsilon' + e_\epsilon'} = -\delta_{\epsilon+\epsilon'} \delta_{\gamma'\gamma} v_{\epsilon} \otimes v_{\lambda}, \quad \text{for } \epsilon, \epsilon' \in \Delta_-
$$

($\delta_{\epsilon+\epsilon'}$ is Kronecker’s delta). If $\gamma, \gamma' \in \Delta_+$, with $\gamma < \gamma'$, are such that $\mu - \lambda = \gamma + \gamma'$, then $-N_{\gamma\gamma'}b_{\mu\lambda}^{(\gamma,\gamma')}(e_\epsilon, e_\epsilon') = -N_{e_\epsilon} \delta_{\epsilon'\epsilon} \delta_{\gamma'\gamma} v_{\epsilon} \otimes v_{\lambda}$. Comparing this with (1.22), we see that

$$
\psi_{\mu\lambda}(a_{\mu\lambda}^{(\phi)}) = -\sum_{(\gamma, \gamma') \in \Delta_{\mu\lambda}} N_{\gamma\gamma'} b_{\mu\lambda}^{(\gamma,\gamma')},
$$

where $\Delta_{\mu\lambda} := \{ (\gamma, \gamma') | \gamma, \gamma' \in \Delta_+, \gamma + \gamma = \mu - \lambda, \gamma < \gamma' \}$. It follows from §1.3.2 that the relations $r_{\mu\lambda}^{(\gamma,\gamma')}$ in $\mathcal{K}$ are as given in part (3), for $\lambda, \mu \in \Lambda$ and $(\gamma', \gamma') \in \Delta_{\mu\lambda}$. For $\gamma > \gamma'$ we get the negative of these $r_{\mu\lambda}^{(\gamma,\gamma')} = -r_{\mu\lambda}^{(\gamma',\gamma)}$, since $N_{\gamma\gamma'} = -N_{\gamma'\gamma}$. So the relations corresponding to $\gamma > \gamma'$ do not provide more constraints on the representations. This proves part (3).

The previous proposition has also been proved in [BK]. Since different authors describe generally the quiver with relations in different ways, we include a proof based on our definitions.

### 1.6.2. Homogeneous bundles on products of complex projective lines.

The product of $N$ complex projective lines $\mathbb{P}^1$ can be written as a quotient $(\mathbb{P}^1)^N = G/P$ of groups

$$
G = \prod_{\alpha=1}^{N} G_{\alpha}, \quad P = \prod_{\alpha=1}^{N} P_{\alpha}, \quad \text{where } G_{\alpha} = SL(2, \mathbb{C}), \quad P_{\alpha} = \left( \begin{array}{cc} * & 0 \\ * & * \end{array} \right) \subset G_{\alpha}.
$$

Since $P \subset G$ is a Borel subgroup, its associated quiver with relations $(Q, \mathcal{K})$ is given by Proposition [1.21]. Hence, the vertex set is $Q_0 = \mathbb{Z}^N$. We easily see that the roots of $u$ are $\gamma_k = -2L_k$, for $1 \leq k \leq N$, where $\{L_1, \ldots, L_N\}$ is the dual of the standard basis of $\mathfrak{h} \cong \mathbb{C}^N$. Thus, the arrow set is $Q_1 = \{a_{\lambda}^{(i)} | \lambda \in \mathbb{Z}^N, 1 \leq i \leq N\}$, with $a_{\lambda}^{(i)} : \lambda \rightarrow \lambda - 2L_i$. For any arrow $a_{\lambda}^{(i)} : \lambda \rightarrow \mu = \lambda - 2L_i$, we have $\sum_{i=1}^{N} \mu_i = \sum_{i=1}^{N} \lambda_i - 2$. Thus, the quiver decomposes into two full subquivers $Q^{(h)}$, for $h = 0, 1$, whose vertices $\lambda$ satisfy $\sum_{i=1}^{N} \lambda_i \equiv h \mod 2$. For the example $G/P = \mathbb{P}^1 \times \mathbb{P}^1$, the picture of any connected component is given in Fig. 1 below. Since $u$ is abelian, the coefficients $N_{\gamma',\gamma}$, in Proposition [1.21](3), are zero. Thus, the relations $r_{\lambda}^{(j,i)}$ of $P$ are parametrised by a vertex $\lambda$ and a basis vector of $\wedge^2 \mathbb{C}^N$. They are given by

$$
r_{\lambda}^{(j,i)} = a_{\lambda - 2L_i}^{(j)} a_{\lambda - 2L_j}^{(i)} - a_{\lambda - 2L_j}^{(i)} a_{\lambda}^{(j)}, \quad \text{for } 1 \leq i < j \leq N.
$$
Summarizing, the category of $G$-equivariant holomorphic vector bundles on $G/P = (\mathbb{P}^1)^N$ is equivalent to the category of commutative diagrams on the quiver $Q$.

**Proposition 1.23.** Let $\Delta(u)$ be the set of weights of $u$ with respect to $\mathfrak{h}$, $Q$ be the quiver associated to $P$, and $\lambda, \mu \in Q_0$. If all the simple components of $G$ are in one of the series $A, D, E$, then the number of arrows from $\lambda$ to $\mu$ is

$$n_{\mu \lambda} = \begin{cases} 1 & \text{if } \mu - \lambda \in \Delta(u); \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** The proof is as in [BK, Proposition 2], where an isotropical decomposition of $u \otimes M_\mu$, as an $L$-module, is obtained by applying Weyl’s character formula, and the fact that, for the series $A, D, E$, the off-diagonal elements of the Cartan matrix are equal to $0$ or $-1$. We only have to adapt the proof to get an isotropical decomposition of $u^* \otimes M_\mu$ instead.

As an example, in the rest of §1.6.3, we apply Proposition 1.23 to describe the quiver corresponding to the complex projective plane $\mathbb{P}^2$, slightly modifying the treatment in [BK]. Let $U = U' \oplus U''$ be a 3-dimensional vector space, where $U'$ is 2-dimensional and $U''$ is 1-dimensional. Thus, $\mathbb{P}^2 = \mathbb{P}(U) = G/P$, where $G$ is $SL(3, \mathbb{C}) = SL(U)$ and $P$ is its parabolic subgroup

$$P = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

of block lower triangular matrices, i.e. automorphisms preserving $U' \subset U$. A Levi subgroup $L$ is the subgroup of determinant one matrices in $GL(U') \times GL(U'')$. Let $\{u_1, u_2, u_3\}$ be a basis of $U'$, with $u_1, u_2 \in U'$, and $u_3 \in U''$. Let $\{E_{i,j}\}_{1 \leq i, j \leq 3}$ be the basis of $End U$ given by $E_{i,j} : u_k = \delta_{ij} u_i$. Let $\mathfrak{h}_* \subset End U$ be the subspace with basis vectors $H_{*,i} := E_{i,i}$, and let $L_{*,1}, L_{*,2}, L_{*,3}$ be its dual basis. The subspace $\mathfrak{h} \subset \mathfrak{h}_*$ of traceless vectors $c_1 H_{*,1} + c_2 H_{*,2} + c_3 H_{*,3}$, i.e. with $c_1 + c_2 + c_3 = 0$, is a Cartan subalgebra of $\mathfrak{g}$, with dual $\mathfrak{h}^* = \mathfrak{h}_*^\perp / \mathfrak{d}$, where $\mathfrak{d} = \mathbb{C} \cdot (L_{*,1} + L_{*,2} + L_{*,3})$. Let $L_1$ be the image of $L_{*,1}$ under the projection $\mathfrak{h}_* \to \mathfrak{h}^*$. Then $p = u \rtimes I$, where $u = CE_{1} \oplus CE_{2}$, $I = \mathfrak{h} \oplus CE_{1,2} \oplus CE_{2,1}$, with $E_1 := E_{3,1}, E_2 := E_{3,2}$. The sets of roots of $\mathfrak{g}$, $I$ and $u$ are $\Delta = \{\alpha_{i,j}\}_{0 \leq i \neq j \leq 3}$, $\Delta(I) = \{\alpha_{1,2}, \alpha_{2,1}\}$, $\Delta(u) = \{\gamma_1, \gamma_2\}$, resp., with $\alpha_{i,j} = L_i - L_j$, $\gamma_k = \alpha_{3,k}$. We choose $\alpha_i := \alpha_{i,i+1}$, for $i = 1, 2$, as simple roots of $(\mathfrak{g}, \mathfrak{h})$, so $\Sigma = \{\alpha_2\}$ (cf. §1.3.1). The fundamental weights of $\mathfrak{g}$ (resp. $I$) are $\lambda_{\alpha_1} = L_1, \lambda_{\alpha_2} = L_1 + L_2$, (resp. $\lambda_{\alpha_1} = L_1$). Thus, $\Lambda_\mu$ is the set of weights $\lambda \in \mathfrak{h}^*$ which are integral for $\mathfrak{g}$ and dominant for $I$, i.e. $\lambda = l_1 \lambda_{\alpha_1} + l_2 \lambda_{\alpha_2}$, with $l_1, l_2 \in \mathbb{Z}$, and $l_1 \geq 0$. Expressing them in the basis $\{L_1, L_2\}$, $\lambda = \lambda_1 L_1 + \lambda_2 L_2$ with $\lambda_1 = l_1 + l_2$, $\lambda_2 = l_2$, so the condition $l_1 \geq 0$ is equivalent to $\lambda_1 \geq \lambda_2$. To
get a nice picture of this quiver, we use the vectors \( \epsilon_1 = -\frac{1}{3}L_1 - \frac{2}{3}L_2, \epsilon_2 = -\frac{2}{3}L_1 - \frac{1}{3}L_2 \) as the standard basis of \( \mathfrak{h}^* \), so \( L_1 = (1, -2), L_2 = (-2, 1) \). Let \( \Lambda \) be the lattice generated by \( L_1, L_2 \). Any \( \lambda = \lambda_1L_1 + \lambda_2L_2 \) can be written as \( \lambda = x_1\epsilon_1 + x_2\epsilon_2 \), with \( x_1 = \lambda_1 - 2\lambda_2, x_2 = -2\lambda_1 + \lambda_2 \). The condition \( \lambda_1 \geq \lambda_2 \) is equivalent to \( x_1 \geq x_2 \), for \( x_1 - x_2 = 3(\lambda_1 - \lambda_2) \). So the vertex set is

\[
Q_0 = \{ (x_1, x_2) \in \Lambda | x_1 \geq x_2 \}.
\]

To get the arrows, we see that \( L_3 = -L_1 - L_2 \), so \( \gamma_1 = L_3 - L_1 = -2L_1 - L_2 = 3\epsilon_2, \gamma_2 = L_3 - L_2 = -L_1 - 2L_2 = 3\epsilon_1 \). Applying Proposition 1.23, the arrows are

\[
a_x^{(1)}: x = (x_1, x_2) \rightarrow (x_1, x_2 + 3), \quad a_x^{(2)}: x = (x_1, x_2) \rightarrow (x_1 + 3, x_2),
\]

where \( x \in Q_0 \). Given an arrow \( a_x^{(i)}: (x_1, x_2) \rightarrow (y_1, y_2) \), with \( i = 1, 2 \), we see that \( y_1 + y_2 = x_1 + x_2 + 3 \); thus, the quiver decomposes into three connected components, say \( Q^{(0)}, Q^{(1)}, Q^{(2)} \), with \( Q_0^{(h)} = \{ (x_1, x_2) \in Q_0 | x_1 + x_2 \equiv -h \mod 3 \} \). The picture of any connected component is given in Fig. 2 above. Although we shall not obtain the relations associated to \( P \), we can easily prove that they are quadratic. In fact, \( u \) is an abelian algebra, for \( [E_1, E_2] = 0 \), so the linear maps \( \psi_{\mu\lambda} \) of \( \{1, 2, 3\} \) are zero. Now, when the relations are quadratic, our definition of the quiver and the relations associated to \( P \) coincide with those given by Hille, who has worked out explicitly the case \( n = 2 \) (cf. [HI1, HI3]), showing that the relations are \( r_x = a_{x-3\epsilon_1}^{(2)}a_x^{(1)} - a_{x-3\epsilon_2}^{(2)}a_x^{(1)} \), for \( x \in Q_0 \), i.e. commutative diagrams.

2. Equivariant bundles, equivariant sheaves and quivers

The object of this section is to generalize the results of §3 to equivariant vector bundles and sheaves on \( X \times G/P \). To do this, in §§2.1 and 2.2, we define the categories of holomorphic equivariant bundles and \( G \)-equivariant holomorphic filtrations, as well as the corresponding categories of sheaves. The definition of a quiver bundle applies to any quiver, while the definition of a \( G \)-equivariant holomorphic filtration applies to any complex \( G \)-manifold. In §2.3, we extend the results of §§2.1 and 2.2 to equivariant vector bundles and sheaves on \( X \times G/P \). Thus, we prove that there is an equivalence between the category of holomorphic equivariant vector bundles (resp. coherent equivariant sheaves) on \( X \times G/P \) and the category of holomorphic quiver bundles (resp. quiver sheaves) on \( X \). We also show that such an equivariant holomorphic vector bundle or sheaf admits a natural equivariant filtration.

2.1. Quiver bundles and quiver sheaves. The notion of quiver bundle generalises previous concepts of vector bundles with additional structure. In this subsection, \( Q \) is a (locally finite) quiver.

Definition 2.1. A \( Q \)-sheaf \( \mathcal{R} = (\mathcal{E}, \phi) \) on \( X \) is given by a collection \( \mathcal{E} \) of coherent sheaves \( \mathcal{E}_v \) for each vertex \( v \in Q_0 \), together with a collection \( \phi \) of morphisms \( \phi_a: \mathcal{E}_m \rightarrow \mathcal{E}_{tp} \), for each arrow \( a \in Q_1 \), such that \( \mathcal{E}_v = 0 \) for all but finitely many \( v \in Q_0 \).

Given a \( Q \)-sheaf \( \mathcal{R} = (\mathcal{E}, \phi) \) on \( X \), every (non-trivial) path \( p = a_0 \cdots a_m \) in \( Q \) induces a morphism of sheaves \( \phi(p) := \phi_{a_0} \circ \cdots \circ \phi_{a_m}: \mathcal{E}_{tp} \rightarrow \mathcal{E}_{hp} \). The trivial path \( \epsilon_v \) at a vertex \( v \) induces \( \phi(\epsilon_v) = id: \mathcal{E}_v \rightarrow \mathcal{E}_v \). A \( Q \)-sheaf \( \mathcal{R} = (\mathcal{E}, \phi) \) satisfies a relation \( r = \sum c_i p_i \) if \( \sum c_i \phi(p_i) = 0 \). Let \( \mathcal{K} \) be a set of relations of \( Q \). A \( Q \)-sheaf with relations \( \mathcal{K} \), or a \( (Q, \mathcal{K}) \)-sheaf, is a \( Q \)-sheaf satisfying the relations in \( \mathcal{K} \). A holomorphic \( Q \)-bundle is a \( Q \)-sheaf \( \mathcal{R} = (\mathcal{E}, \phi) \) such that all the sheaves \( \mathcal{E}_v \) are holomorphic vector bundles, (i.e. locally free sheaves). A morphism \( f: \mathcal{R} \rightarrow \mathcal{S} \) between two \( Q \)-sheaves \( \mathcal{R} = (\mathcal{E}, \phi), \mathcal{S} = (\mathcal{F}, \psi) \), is given by morphisms \( f_v: \mathcal{E}_v \rightarrow \mathcal{F}_v \), for each \( v \in Q_0 \), such that \( \psi_a \circ f_{ta} = f_{ha} \circ \phi_a \), for each \( a \in Q_1 \). It is immediate that \( (Q, \mathcal{K}) \)-sheaves form an abelian category. Important concepts in relation to (semi)stability (cf. [1, 3, 1]) are the notions of \( Q \)-subsheaves and quotient \( Q \)-sheaves, as well as indecomposable and simple \( Q \)-sheaves, which are defined as in any abelian category.
2.2. **Equivariant filtrations.** Let $M$ be a complex $G$-manifold. To simplify the notation, throughout this paper a $G$-equivariant coherent sheaf on $M$ will mean a coherent sheaf $\mathcal{F}$ on $M$ together with a holomorphic $G$-equivariant action on $\mathcal{F}$ (cf. §2.3.2, or e.g. [AK] for more details).

**Definition 2.2.** A $G$-equivariant sheaf filtration on $M$ is a finite sequence of $G$-equivariant coherent sheaf $\mathcal{F}$ on $M$,

(2.3) \[ \mathcal{F} : 0 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_m = \mathcal{F}. \]

We say that $\mathcal{F}$ is a holomorphic filtration if the sheaves $\mathcal{F}_i$ are locally free.

When $G$ is the trivial group, a $G$-equivariant coherent sheaf will be referred to as a sheaf filtration.

The $G$-equivariant sheaf filtrations on $M$ form an abelian category, whose morphisms are defined as follows. Let $\mathcal{F}$, given by (2.3), and $\mathcal{F}'$, given by

(2.4) \[ \mathcal{F}' : 0 \hookrightarrow \mathcal{F}'_0 \hookrightarrow \mathcal{F}'_1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}'_m = \mathcal{F}', \]

be $G$-equivariant sheaf filtrations. A $G$-equivariant morphism from $\mathcal{F}'$ to $\mathcal{F}$ is a morphism of coherent $G$-equivariant sheaves $f : \mathcal{F}' \rightarrow \mathcal{F}$, such that $f(\mathcal{F}_i') = \mathcal{F}_i \cap \text{Im}(f)$ for $0 \leq i \leq m$. In particular, a subobject of a $G$-equivariant sheaf filtration (2.3) is a sheaf filtrations (2.4), where $\mathcal{F}'$ is a $G$-invariant coherent subsheaf of $\mathcal{F}$, and $\mathcal{F}_i' = \mathcal{F}_i \cap \mathcal{F}'$, for $0 \leq i \leq m$.

2.3. **Correspondence between equivariant sheaves, equivariant filtrations, and quiver sheaves.**

We now prove the main results of this section. Throughout §2.3, $(Q, K)$ is the quiver with relations associated to $P$.

**Theorem 2.5.** There is an equivalence of categories

\[
\begin{cases}
\text{coherent } G\text{-equivariant sheaves on } X \times G/P \\
\text{coherent } (Q, K)\text{-sheaves on } X 
\end{cases}
\]

\[ \longleftrightarrow \]

\[
\begin{cases}
\text{coherent } G\text{-equivariant sheaves on } X \times G/P \\
\text{coherent } (Q, K)\text{-sheaves on } X 
\end{cases}
\]

The holomorphic $G$-equivariant vector bundles on $X \times G/P$ and the holomorphic $(Q, K)$-bundles on $X$ are in correspondence by this equivalence.

**Proposition 2.6.** Let us fix a total order in the set $Q_0$, as in §1.3.3 Any coherent $G$-equivariant sheaf $\mathcal{F}$ on $X \times G/P$ admits a $G$-equivariant filtration

(2.7) \[ \mathcal{F} : 0 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_m = \mathcal{F}, \]

\[ \mathcal{F}_s/\mathcal{F}_{s-1} \cong p^* E_{\lambda_s} \otimes q^* \mathcal{O}_{\lambda_s}, \quad 0 \leq s \leq m, \]

where $\{\lambda_0, \lambda_1, \ldots, \lambda_m\}$ is a finite subset of $Q_0$, listed in ascending order as $\lambda_0 < \lambda_1 < \cdots < \lambda_m$, and $E_0, \ldots, E_m$ are non-zero coherent sheaves on $X$, with trivial $G$-action. If $\mathcal{F}$ is a holomorphic $G$-equivariant vector bundle, then $E_0, \ldots, E_m$ are holomorphic vector bundles.

The proof of Theorem 2.5 will be given in two steps, by (i) reduction of a coherent $G$-equivariant sheaf $\mathcal{F}$ on $X \times G/P$ to the slice $i : X \cong X \times P/P \hookrightarrow X \times G/P$, obtaining coherent $P$-equivariant sheaf $\mathcal{E} = i^* \mathcal{F}$ on $X$, following §1.1.1; and (ii) given $\mathcal{E}$, the construction of a $(Q, K)$-sheaf $\mathcal{R} = (\mathcal{E}, \phi)$ on $X$, following the proof of Theorem 1.4. The first step is Lemma 2.8. A preliminary result needed in the second step is Lemma 2.9, where the $L$-equivariant constant sheaf on $X$ associated to the $L$-module $M_\lambda$ is also denoted $\mathbb{M}_\lambda$, for each $\lambda \in Q_0$. In Lemma 2.8 (resp. Lemma 2.9), the $P$-action (resp. $L$-action) on $X$ is trivial.

**Lemma 2.8.** There is an equivalence of categories

\[
\begin{cases}
\text{coherent } G\text{-equivariant sheaves on } X \times G/P \\
\text{coherent } P\text{-equivariant sheaves on } X 
\end{cases}
\]

\[ \longleftrightarrow \]

\[
\begin{cases}
\text{coherent } G\text{-equivariant sheaves on } X \times G/P \\
\text{coherent } P\text{-equivariant sheaves on } X 
\end{cases}
\]
Lemma 2.9. A coherent L-equivariant sheaf \( \mathcal{E} \) on \( X \) admits an isotopical decomposition

\[
\mathcal{E} \cong \bigoplus_{\lambda \in Q'_0} \mathcal{E}_\lambda \otimes M_\lambda
\]

where \( Q'_0 \subset Q_0 \) is finite, and \( \mathcal{E}_\lambda \) is a coherent sheaf with trivial \( L \)-action, for each \( \lambda \in Q'_0 \).

Since we are dealing with equivariant sheaves, an ingredient in the proof of the previous lemmas, which did not appear for homogeneous bundles on \( G/P \), will be standard techniques of representation theory of Lie groups on the Fréchet spaces of sections of equivariant coherent sheaves. The necessary preliminaries are explained in §2.3.1-2.3.3. The proofs of Lemmas 2.8 and 2.9 are in §2.3.4-2.3.5, and the proof of Theorem 2.5 is completed in §2.3.6. The proof of Proposition 2.6 is the content of §2.3.7.

2.3.1. Equivariant holomorphic vector bundles. To understand the proof of Theorem 2.5, let us first prove the equivalence between \( G \)-equivariant holomorphic vector bundles on \( X \times G/P \) and holomorphic \( (Q, K) \)-bundles on \( X \). In §1.1.3 we defined an equivalence functor between the category of \( G \)-equivariant holomorphic vector bundles on \( X \times G/P \) and the category of \( P \)-equivariant holomorphic vector bundles on \( X \). We now define an equivalence functor between the category of \( P \)-equivariant holomorphic vector bundles on \( X \) and the category of holomorphic \( (Q, K) \)-bundles on \( X \), generalising Theorem 1.4. Let \( \mathcal{E} \) be a \( P \)-equivariant holomorphic vector bundle on \( X \). The equivalence functor of Theorem 1.4 associates a \( (Q, K) \)-module \( \mathcal{R}_x = (\mathcal{E}_x, \phi_x) \), to the holomorphic representation \( \mathcal{E}_x \) of \( P \), for each \( x \in X \). The point is to show that these \( (Q, K) \)-modules ‘vary holomorphically in \( X \’ , i.e. that they are the fibers of a holomorphic \( (Q, K) \)-bundle on \( X \). To do this, one can use standard techniques in representation theory. First, there is a (unique) holomorphic projection operator \( \Pi : \mathcal{E} \to \mathcal{E} \) onto the \( L \)-invariant part of \( \mathcal{E} \), and its image \( \Pi \mathcal{E} \) and kernel \( (id - \Pi) \mathcal{E} \) have induced structures of holomorphic vector subbundles of \( \mathcal{E} \). A proof is as follows. Let \( \Pi \) be defined by \( \Pi(e) = \int_J g \cdot e \, dg \), where \( dg \) is the Haar measure of \( J \). The map \( \Pi \) is obviously a \( J \)-invariant smooth projection operator onto the \( J \)-invariant part. One can prove that, since \( L \) is the universal complexification of \( J \) (for \( L \) is reductive), \( \Pi \) is actually \( L \)-invariant and, moreover, holomorphic. To prove that \( \Pi \mathcal{E} \) and \( (id - \Pi) \mathcal{E} \) have induced structures of holomorphic vector subbundles of \( \mathcal{E} \), one follows e.g. as in [AB, Lemma (1.4)], where a similar result is proved for a smooth projection operator on a smooth vector bundle —one simply changes the word ‘smooth’ by ‘holomorphic’ in that proof. The image of the projection operator \( \Pi \) picks out the \( L \)-invariant part of \( \mathcal{E} \), so the isotopical decomposition of \( \mathcal{E} \) is obtained as in the familiar case of finite dimensional representations:

\[
\mathcal{E} \cong \bigoplus_{\lambda \in Q'_0} \mathcal{E}_\lambda \otimes M_\lambda, \quad \mathcal{E}_\lambda = (\text{Hom}(M_\lambda, \mathcal{E}))^L,
\]

where \( M_\lambda \) is the \( L \)-equivariant vector bundle \( X \times M_\lambda \), for \( \lambda \in Q_0 \), \( \text{Hom}(M_\lambda, \mathcal{E}) \) is the holomorphic bundle of endomorphisms, \((-)^L := \Pi(-)\), \( \mathcal{E}_\lambda \) is a holomorphic vector bundle with trivial \( L \)-action, and \( Q'_0 \subset Q_0 \) is the subset of weights \( \lambda \) with \( \mathcal{E}_\lambda \neq 0 \). This isotopical decomposition defines the action of \( L \) on \( \mathcal{E} \). The extension of the \( L \)-action to a \( P \)-action is defined by means of holomorphic morphisms \( \phi_a : \mathcal{E}_{ta} \to \mathcal{E}_{ha} \) satisfying the relations in \( K \). The proof follows as in Theorem 1.4. The holomorphic vector bundles \( \mathcal{E}_\lambda \) and the holomorphic morphisms \( \phi_a \) define the holomorphic \((Q, K)\)-bundle \( \mathcal{R} = (\mathcal{E}, \phi) \) associated to \( \mathcal{E} \).

It is worth mentioning that the equivalence proved in §2.3.1 can be stated, and proved, in terms of \( \overline{\partial} \)-operators on smooth equivariant vector bundles, as we shall do in §3.

2.3.2. Holomorphic actions on coherent sheaves. Throughout §2.3.2, \( \Gamma \) is a complex Lie group (later on, it will be either \( G \), \( P \), or \( L \)), and \( M \) is a complex \( \Gamma \)-manifold. As mentioned in §2.2, to simplify the notation, throughout this paper a \( \Gamma \)-equivariant coherent sheaf on \( M \) is a coherent sheaf \( \mathcal{F} \) on \( M \) together with a holomorphic \( \Gamma \)-equivariant action on \( \mathcal{F} \), following the definitions given e.g.
in [Ar]. This means that $\mathcal{F}$ is a coherent sheaf on $M$, hence a covering space with projection map $\pi : \mathcal{F} \to M$, together with a $\Gamma$-action on $\mathcal{F}$, so each $g \in \Gamma$ acts on the stalk $\mathcal{F}_y$, for $y \in M$, by an isomorphism $\rho_y : \mathcal{F}_y \to \mathcal{F}_{y^g}$ of $\mathcal{O}_{g,y}$-modules, such that the $\Gamma$-action commutes with $\pi$, and is holomorphic in the following sense: Since $\mathcal{F}$ is coherent, the spaces of sections $\mathcal{F}(B)$, for $B \subset M$ open, have canonical Fréchet topologies (cf. e.g. [GrR, §V.6]); the $\Gamma$-action on $\mathcal{F}$ is holomorphic if for all open subsets $B, B' \subset X$ and $W \subset \Gamma$ with $W \cdot B \subset B'$, and all sections $s \in \mathcal{F}(B')$, the map $W \to \mathcal{F}(B), g \mapsto g^{-1} \cdot (s|_{g-B})$, is holomorphic with respect to the Fréchet topology on $\mathcal{F}(B)$.

2.3.3. Sheaves of invariant sections. Let $\Gamma$ and $M$ be as in §2.3.2. Let us assume that the $\Gamma$-action on $M$ is such that $M/\Gamma$ is a complex manifold. The structure sheaf of $M/\Gamma$ is given by $\mathcal{O}_{M/\Gamma}(B) = \mathcal{O}_M(\pi^{-1}(B))^\Gamma$, for $B \subset M/\Gamma$ open, where $\pi : M \to M/\Gamma$ is projection. We now define a functor from the category of coherent $\Gamma$-equivariant sheaves on $M$ to that of (not necessarily coherent) sheaves on $M/\Gamma$, by ‘taking invariant sections’. Given a coherent $\Gamma$-equivariant sheaf $\mathcal{F}$ on $M$, the sheaf $\mathcal{F}^\Gamma$ associates to $B \subset M/\Gamma$ open, the subspace $\mathcal{F}^\Gamma(B) = \mathcal{F}(\pi^{-1}(B))^\Gamma$ of $\Gamma$-invariant sections; the restriction maps of $\mathcal{F}^\Gamma$ are those of $\mathcal{F}$ restricted to the spaces of invariant sections. This defines a presheaf, which is in fact a sheaf, called the sheaf of invariant sections of $\mathcal{F}$. Given a $\Gamma$-equivariant morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ of coherent $\Gamma$-equivariant sheaves on $M$, the morphism $f^\Gamma : \pi^*_x \mathcal{F}_1 \to \pi^*_x \mathcal{F}_2$ is defined by $f^\Gamma(B) = f(\pi^{-1}(B))|_{\mathcal{F}(\pi^{-1}(B))^\Gamma}$ for any open set $B \subset M/\Gamma$.

2.3.4. Proof of Lemma 2.8. Let $\pi_X : X \times G \to X, \pi_G : X \times G \to G, \pi : X \times G \to X \times G/P$ be projections. The $P$-action on $G, r : P \times G \to G, (p, g) \mapsto prg^{-1}$, induces a $P$-action on $\mathcal{O}_{G}, \mathcal{O}_G(B) \to \mathcal{O}_G(rp(B)), f \mapsto f \circ r^{-1}$, for $p \in P$ and $B \subset G$ open, which is holomorphic (cf. e.g. [Ar, §4.1]). This action induces another $P$-action on $\mathcal{O}_{X \times G} = \pi_X^* \mathcal{O}_X \otimes \pi_G^* \mathcal{O}_G$, obtained from the trivial $P$-action on the first factor, and the induced $P$-action on the second factor; thus, $\mathcal{O}_{X \times G/P} = \mathcal{O}_{X \times G}^P$. Let $\mathcal{E}$ be a coherent $P$-equivariant sheaf on $X$. The coherent sheaf $\mathcal{H} = \pi_X^* \mathcal{E} \otimes \pi_G^* \mathcal{O}_G$ on $X \times G$ has a holomorphic $P$-equivariant action, obtained from the induced $P$-actions on the first and the second factor. We now prove that the sheaf $\mathcal{F} = \mathcal{H}^P$ on $X \times G/P$ is coherent. The set $S(\mathcal{F})$ of points $y \in X \times G/P$ where $\mathcal{F}$ is not coherent, i.e. where it does not admit a presentation

$$\mathcal{O}_{X \times G/P,y}^P \longrightarrow \mathcal{O}_{X \times G/P,y} \longrightarrow \mathcal{F}_y \longrightarrow 0,$$

is $G$-invariant, i.e. $G \cdot S(\mathcal{F}) = S(\mathcal{F})$, and $G \cdot y = \{ x \} \times G/P$ for $y = (x, gp) \in X \times G/P$, so $S(\mathcal{F}) = S(\mathcal{E}) \times G/P$, where $S(\mathcal{E})$ is the set of points $x \in X$ where $\mathcal{E}$ is not coherent. Since $\mathcal{E}$ is coherent, $\mathcal{F}$ is coherent. Analogously, one shows that $\mathcal{F}$ is locally free if $\mathcal{E}$ is locally free.

The $G$-action on $G, l : G \times G \to G, (g, g') \mapsto l_g g' = gg'$, induces a holomorphic $G$-action on $\mathcal{O}_G, \mathcal{O}_G(W) \to \mathcal{O}_G(l_g(W)), f \mapsto f \circ l_g^{-1}$, for $g \in G, W \subset G$ open, which induces a holomorphic $G$-action on $\mathcal{O}_{X \times G}$ and on the tensor product $\mathcal{H} = \pi_X^* \mathcal{E} \otimes \pi_G^* \mathcal{O}_G$: $G$ acts trivially on the first factor and in the induced way in the second factor. Thus, for all open subsets $B, B' \subset X \times G, W \subset G$ with $W \cdot B \subset B'$, and $s \in \mathcal{H}(B')$, the map $W \to \mathcal{H}(B), g \mapsto g^{-1} \cdot (s|_{g-B})$, is holomorphic. Hence, if $B_1, B'_1 \subset X \times G/P$ are open, $W \cdot B_1 \subset B'_1$, and $s \in \mathcal{F}(B'_1) = \mathcal{H}(B'_1)^P$, with $B' = \pi^{-1}(B'_1)$, then the map $W \to \mathcal{F}(B_1) = \mathcal{H}(B)^P, g \mapsto g^{-1} \cdot (s|_{g-B})$, is holomorphic, where $B = \pi^{-1}(B_1)$. Therefore, $\mathcal{F}$ is a coherent $G$-equivariant sheaf on $X \times G/P$.

2.3.5. Proof of Lemma 2.9. Since $L$ acts trivially on $X$, a holomorphic $L$-equivariant action on a coherent sheaf $\mathcal{E}$ on $X$, as defined in §2.3.2, is simply an $L$-action on $\mathcal{E}$ such that for each $B \subset X$ open and each $s \in \mathcal{E}(B)$, the map $L \to \mathcal{E}(B), g \mapsto g \cdot s$, is holomorphic (because the maps $\mathcal{E}(B) \to \mathcal{E}(B')$, $s \mapsto s|_{B'}$, for $B' \subset B \subset X$ open, and $L \to L, g \mapsto g^{-1}$, are holomorphic). To prove Lemma 2.3.3 we shall need the following three lemmas, where it is crucial that $L$ is reductive. The first one is an equivariant analogue of a well-known property of coherent sheaves.
Lemma 2.11. A coherent $L$-equivariant sheaf on $X$ is a sheaf $E$ of $O_X$-modules, together with an $L$-action on the space of sections $E(B)$ by automorphisms of $O(B)$-modules, for each $B \subset X$, with commutative with the restriction maps $\rho_{B'B} : E(B) \to E(B')$ for $B' \subset B \subset X$ open, and such that for each $x \in X$, there is a neighbourhood $B$ of $x$, finite-dimensional complex representations $V,W$ of $L$ and a $L$-equivariant exact sequence

\[
\begin{array}{c}
O_B \otimes W & \overset{g^L}{\longrightarrow} & O_B \otimes V & \overset{f^L}{\longrightarrow} & E^L_B & \longrightarrow & 0,
\end{array}
\]

such that, for any $B' \subset B$ open, each $s \in E(B')$ is $L$-finite, i.e. the $\mathbb{C}$-linear span of the orbit $L \cdot s$ is finite-dimensional.

Proof. The existence of $B, B', W$ and the $L$-equivariant exact sequence is a consequence of a result of Roberts. [R, Proposition 2.1]. Actually, Roberts states a very close result for complex $L$-spaces which can be covered by $L$-stable Stein open sets, and he also proves the existence of a finite set $s_1, \ldots, s_r \in E(B)$ of $L$-finite sections such that the restrictions $s_1|_{B'}, \ldots, s_r|_{B'}$ generate $E(B')$ as an $O_X(B')$-module, for each $B' \subset B$ open. Moreover, the representation $V$ in the lemma is a finite dimensional $L$-invariant complex subspace of $E(B)$ generated by $s_1, \ldots, s_r$ over $\mathbb{C}$. We now show that for $B' \subset B$ open, each $s \in E(B')$ is $L$-finite. Let $B' \subset B$ open and $s'_i = s_i|_{B'}$ for $1 \leq i \leq r$, so $s'_1, \ldots, s'_r$ generate $E(B')$ over $O_X(B')$, and they are also $L$-finite. Let $B' \subset E(B')$ be a finite dimensional $\mathbb{C}$-vector subspace containing $s'_1, \ldots, s'_r$. By adding more generators if necessary, we can assume that the $\mathbb{C}$-linear span of $s'_1, \ldots, s'_r$ is $V'$. Thus, there are functions $f_i^L : L \to \mathbb{C}$ such that $g \cdot s'_j = \sum_{i=1}^n f_i^L(g)s'_i$, for each $g \in L$. To show that each $\xi \in E(B')$ is $L$-finite, we first expand it in the set of $O(B')$-generators $s'_i, \xi = \sum_{i=1}^n \xi^i s'_i$, with $\xi^i \in E(B')$. Let $V'' \subset E(B')$ be the finite-dimensional $\mathbb{C}$-vector subspace generated by the subset $\{\xi^i s'_i | 1 \leq i, j \leq n\}$ of $E(B')$. If $g \in G$, then $g \cdot \xi = \sum_{j=1}^n \xi^i g \cdot s'_j = \sum_{i,j=1} f_i^L(g) (\xi^i s'_i') \in V''$, so $\xi$ is $L$-finite.

In the following two Lemmas we shall use the functor which takes a coherent $L$-equivariant sheaf $E$ into the sheaf $E^L$ of invariant sections, as defined in [2.3.2]. We notice that $E$ is a sheaf on $X$, since $X/L = X$. A projection operator on a coherent $L$-equivariant sheaf $E$ on $X$, i.e. a morphism $\Pi : E \to \mathcal{E}$ with $\Pi^2 = \Pi$, is called $L$-invariant, if $\Pi(g \cdot s) = \Pi(s)$ for $g \in L, s \in E(B), B \subset X$ open. Lemma 2.13 is an immediate consequence of [R, Proposition 2.2].

Lemma 2.13. (a) Let $E$ be a coherent $L$-equivariant sheaf on $X$. There is a unique $L$-invariant projection operator $\Pi : E \to \mathcal{E}$ onto $\mathcal{E}^L$. If $B' \subset B$ are two open subsets of $X$ and $\rho_{B'B} : \mathcal{E}(B) \to \mathcal{E}(B')$ is the restriction map, then $\Pi_B \circ \rho_{B'B} = \rho_{B'B} \circ \Pi$.

(b) The $L$-invariant projection operator commutes with $L$-equivariant homomorphisms, i.e. if $f : \mathcal{E}_1 \to \mathcal{E}_2$ is an $L$-equivariant morphism of coherent $L$-equivariant sheaves on $X$, then $\Pi \circ f = f^L \circ \Pi$.

Lemma 2.14. (a) The functor which takes a coherent $L$-equivariant sheaf $E$ into $\mathcal{E}^L$ is exact.

(b) If $E$ is a coherent (resp. locally free) $L$-equivariant sheaf on $X$, then $\mathcal{E}^L$ is coherent (resp. locally free).

Proof. (a) This functor is obviously left exact. To see that it is right exact, we have to prove that if $f : \mathcal{E}_1 \to \mathcal{E}_2$ is an $L$-equivariant epimorphism then $f^L : \mathcal{E}_1^L \to \mathcal{E}_2^L$ is surjective. Given $x \in X$, $s_{2x} \in \mathcal{E}_{2,x}^L$, there exists $s_{1,x} \in \mathcal{E}_{1,x}$ with $f_x(s_{1,x}) = s_{2x}$. By Lemma 2.13, $f_x^L(\Pi_x(s_{1,x})) = s_{2x}$, so $f^L$ is surjective.

(b) Let $x \in X$. Assume first that $E$ is coherent. By Lemma 2.11, there exists a neighbourhood $B$ of $x$, finite-dimensional complex representations $V,W$ of $L$ and a $L$-equivariant exact sequence (2.12). By part (a), the induced sequence

\[
(\mathcal{O}_B \otimes W)^L \overset{g^L}{\longrightarrow} (\mathcal{O}_B \otimes V)^L \overset{f^L}{\longrightarrow} (\mathcal{E}|_B)^L \longrightarrow 0
\]
is exact. Obviously \((O_B \otimes V)^L = O_B \otimes V^L, (O_B \otimes W)^L = O_B \otimes W^L\) and \((E|_B)^L = (E^L)|_B\). This proves (b) for coherent sheaves. For locally free sheaves the argument is analogous. \(\square\)

We now prove Lemma 2.9. Since \(M_\lambda^\vee \otimes E\) is a coherent (resp. locally free) \(L\)-equivariant sheaf, for \(\lambda \in Q_0\), the ‘multiplicity sheaf’ \(E_\lambda = (M_\lambda^\vee \otimes E)^L\) is coherent (resp. locally free). Then \(E_\lambda(B) = (M_\lambda^\vee \otimes E(B))^L\), for \(B \subset X\) open (for taking invariant sections obviously commutes with restriction). Let \(x \in X\). Let \(B \subset X\) open as in Lemma 2.11. Since all the sections of \(E(B')\) are \(L\)-finite, for \(B' \subset B\) open, and \(L\) is reductive, there is an isomorphism

\[ E(B') \cong \bigoplus_{\lambda \in Q_0'} (M_\lambda^\vee \otimes E(B'))^L \otimes M_\lambda = \bigoplus_{\lambda \in Q_0'} E_\lambda(B') \otimes M_\lambda, \]

where \(Q_0'\) is the set of vertices \(\lambda \in Q_0\) such that \(E_\lambda \neq 0\). Taking direct limits in \(B' \ni x\), we get \(E_x \cong \bigoplus_{\lambda \in Q_0'} E_\lambda(x) \otimes M_\lambda\), which proves (2.10). We now prove that \(Q_0'\) is a finite set. Since \(X\) is compact, it is enough to see that there are isomorphisms \(E|_B \cong \bigoplus_{\lambda \in Q_0'} E_\lambda|_B \otimes M_\lambda\), for the open sets \(B \subset X\) satisfying the conditions in Lemma 2.11, where \(Q_0' \subset Q_0\) is finite. Let \(V\) be defined as in Lemma 2.11, so there is an \(L\)-equivariant epimorphism \(f: O_B \otimes V \to E|_B\). By Lemma 2.14, tensoring by \(M_\lambda^\vee\) and taking the \(L\)-invariant, we get another epimorphism

\[ (2.15) \quad O_B \otimes V_\lambda \xrightarrow{(f \otimes \text{id})^L} E_\lambda|_B \longrightarrow 0, \]

where \((O_B \otimes M_\lambda^\vee \otimes V)^L = O_B \otimes V_\lambda\), with \(V_\lambda := (M_\lambda^\vee \otimes V)^L\), and \((M_\lambda^\vee \otimes E|_B)^L = E_\lambda|_B\). Now \(V\) is a finite-dimensional complex representation of \(L\), so it has an isotopical decomposition \(V \cong \bigoplus_{\lambda \in Q_0'} V_\lambda \otimes M_\lambda\), where \(Q_0' \subset Q_0\) is finite, i.e. \(V_\lambda = 0\) for all but finitely many \(\lambda \in Q_0\). By (2.15), \(E_\lambda = 0\) for all but finitely many \(\lambda \in Q_0\), so \(Q_0'\) is finite. \(\square\)

2.3.6. Proof of Theorem 2.5. Our proof is similar to that of Theorem 1.4, together with the previous lemmas. By Lemma 2.8, to prove the theorem we define an equivalence functor from the category of coherent \(P\)-equivariant sheaves on \(X\), to the category of \((Q, K)\)-sheaves on \(X\). Let \(E\) be a coherent \(P\)-equivariant sheaf on \(X\). By restriction to the Levi subgroup \(L \subset G\), we obtain a coherent \(L\)-equivariant sheaf on \(X\), whose \(L\)-action is given by an isotopical decomposition (2.10). Obviously, the action of the unipotent radical \(U \subset P\) defines an action \(\tau: U \to \text{End}_X(E)\) of its Lie algebra, satisfying conditions similar to (CR1), (CR2) in the proof of Theorem 1.4. Condition (CR1) means that \(\tau\) is a global section of the \(L\)-invariant part of the coherent \(L\)-equivariant sheaf \(W_1 = \mathfrak{u}^* \otimes \text{End}_{O_X}(E)\), i.e. \(s \in W_1^L(X)\). Now, \(E\) has an isotopical decomposition (2.10), so

\[ W_1 \cong \bigoplus_{\lambda, \mu \in Q_0'} A_{\mu \lambda} \otimes \text{Hom}_{O_X}(E_\lambda, E_\mu) \otimes \text{Hom}(M_\mu, M_\lambda), \]

hence Schur’s lemma implies \(W_1^L \cong \bigoplus_{\lambda, \mu \in Q_0'} A_{\mu \lambda} \otimes \text{Hom}_{O_X}(E_\lambda, E_\mu)\). Thus, \(\tau\) is in

\[ W_1^L(X) \cong \bigoplus_{\lambda, \mu \in Q_0'} A_{\mu \lambda} \otimes \text{Hom}_X(E_\lambda, E_\mu). \]

The collection \(E\) of coherent sheaves \(E_\lambda\), together with the collection \(\phi\) of morphisms \(\phi^{(i)}_{\mu \lambda}\), define a \(Q\)-sheaf \(\mathcal{R} = (E, \phi)\). Condition (CR2) means that \(\mathcal{R}\) satisfies the relations \(K\). \(\square\)

2.3.7. Proof of Proposition 2.6. Let \(F\) be a coherent \(G\)-equivariant sheaf on \(X \times G/P\). The corresponding coherent \(P\)-equivariant sheaf \(E\) on \(X\), given by Lemma 2.6, has an isotopical decomposition (2.10), defining the \(L\)-action on \(E\). Let \(Q_0' = \{\lambda_0, \lambda_1, \ldots, \lambda_m\}\) be listed in ascending order as \(\lambda_0 < \lambda_1 < \cdots < \lambda_m\). The coherent subsheaves of \(E\) defined by

\[ E_{(\leq s)} = \bigoplus_{j=0}^{s} E_j \otimes M_{\lambda_j}, \]

with \(E_s := E_{\lambda_s}\), for \(0 \leq s \leq m\),
are $P$-invariant, so there is a $P$-equivariant filtration, and $P$-equivariant isomorphisms
\[
0 \hookrightarrow \mathcal{E}_{\leq 0} \hookrightarrow \mathcal{E}_{\leq 1} \hookrightarrow \cdots \hookrightarrow \mathcal{E}_{\leq m} = \mathcal{E}, \quad \mathcal{E}_{(\leq s)}/(\leq s-1) \cong \mathcal{E}_{s} \otimes M_{\lambda_{s}}, \text{ for } 1 \leq s \leq m
\]
Applying Lemma 2.6 to each $P$-equivariant sheaf $\mathcal{E}_{(\leq s)}$, to each map $\mathcal{E}_{(\leq s-1)} \hookrightarrow \mathcal{E}_{(\leq s)}$, and to each isomorphism $\mathcal{E}_{(\leq s)}/(\leq s-1) \cong \mathcal{E}_{s} \otimes M_{\lambda_{s}}$, gives a $G$-equivariant filtration (2.7). \hfill \Box

3. INVARIANT HOLONOMIC STRUCTURES AND QUIVER BUNDLES

In Theorem 2.5, we proved an equivalence between the category of $G$-equivariant holomorphic vector bundles on $X \times G/P$ and the category of holomorphic $(Q, K)$-bundles on $X$, where $(Q, K)$ is the quiver with relations associated to $P$. Our purpose in this section is to study the same equivalence in terms of invariant $\partial$-operators on smooth equivariant vector bundles on $X \times G/P$. The main result, stated as Proposition 3.4, will be used in the next section to obtain the dimensional reduction of the gauge equations and of the stability criteria for equivariant holomorphic bundles on $X \times G/P$. Throughout this section, $(Q, K)$ is the quiver with relations associated to $P$.

3.1. Invariant holomorphic structures on $G$-manifolds. A $K$-equivariant smooth complex vector bundle on a smooth $K$-manifold $M$ is a smooth complex vector bundle $F$ on $M$ together with a smooth lifting of the $K$-action to an action on $F$. Let $F$ be such a smooth $K$-equivariant vector bundle on a compact $K$-manifold $M$. Since $G$ is the universal complexification of $K$, the $K$-actions on $M$ and $F$ lift to unique smooth $G$-actions, and hence $F$ is a smooth $G$-equivariant bundle over the $G$-manifold $M$. The group $G$ acts naturally on the space $\mathcal{G}$ of $\partial$-operators on $F$, by $\gamma(\partial_{F}) = \gamma \circ \partial_{F} \circ \gamma^{-1}$ for $\gamma \in G$ and $\partial_{F} \in \mathcal{G}$. This action of $G$ leaves invariant the subset $\mathcal{E}$ of $\partial$-operators $\partial_{F}$ with $(\partial_{F})^{2} = 0$, which is in bijection with the space of holomorphic structures on $F$. The group $G$ also acts naturally on the complex gauge group $\mathcal{G}^{c} = \Omega^{0}(\text{Aut}(F))$ of $F$, by $\gamma(g) = \gamma \circ g \circ \gamma^{-1}$ for $\gamma \in G$ and $g \in \mathcal{G}^{c}$. The space of fixed points $\mathcal{E}^{G}$ is in bijection with the space of holomorphic structures on $F$ such that the action of $G$ is holomorphic.

3.2. Preliminaries on smooth equivariant vector bundles on $X \times G/P$. Since $P \subset G$ is a parabolic subgroup, the natural map $K/J \to G/P$ is a diffeomorphism. Since $G/P$ is a projective variety, this map induces the same structure on $K/J$. In particular, $K/J$ is a compact Kähler homogeneous manifold with a symplectic action of $K$.

Let $F$ be a $K$-equivariant smooth vector bundle on $X \times K/J$. The $K$-action on $E$ lifts to a unique smooth $G$-action, giving $F$ the structure of a smooth $G$-equivariant bundle over $X \times K/J \cong X \times G/P$. Now, there is an equivalence, similar to §1.1.1, between smooth $K$-equivariant vector bundles on $X \times K/J$ and smooth $J$-equivariant vector bundles on $X$: a smooth $J$-equivariant bundle $E$ on $X$ induces a smooth $K$-equivariant bundle $F = K \times_{J} E$. Furthermore, any smooth equivariant $J$-action on $E$ extends uniquely to an $L$-action on $E$ (since $L$ is the universal complexification of $J$). Thus, if $\mathcal{E}$ is a holomorphic $P$-equivariant vector bundle on $X$ whose underlying smooth $J$-equivariant vector bundle is $E$, then the induced holomorphic $G$-equivariant vector bundle $\mathcal{F} = G \times_{P} \mathcal{E}$ on $X \times K/J \cong X \times G/P$ has underlying smooth $K$-equivariant $F = K \times_{J} E$. This means that there is a one-to-one correspondence between (i) $G$-invariant holomorphic structures on the smooth $K$-equivariant vector bundle $F = K \times_{J} E$ and (ii) $L$-invariant holomorphic structures on the smooth $J$-equivariant vector bundle $E$ together with extensions of the $L$-action on $E$ to a holomorphic $P$-action on $E$. As a preliminary step to describe this correspondence in terms of $\partial$-operators and quiver bundles, we introduce some notation and state the following lemma, whose proof is standard (see e.g. [52]).

3.2.1. Notation. Given an irreducible representation $M_{\lambda}$ of $J$, corresponding to an integral dominant weight $\lambda$, $H_{\lambda} := K \times_{J} M_{\lambda}$ is the induced irreducible smooth homogeneous vector bundle on $K/J$. It is the smooth $K$-equivariant vector bundle underlying $O_{\lambda} = G \times_{P} M_{\lambda}$ (cf. §1.5.4). The $\partial$-operator
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corresponding to the holomorphic structure $\mathcal{O}_X$ is denoted by $\bar{\partial}_H$. The maps $p : X \times K/J \to X$ and $q : X \times K/J \to K/J$ are the canonical projections.

**Lemma 3.1.** Every smooth $K$-equivariant complex vector bundle $F$ on $X \times K/J$ can be equivariantly decomposed, uniquely up to isomorphism, as

$$F \cong \bigoplus_{\lambda \in Q_0'} F_{\lambda}, \quad F_{\lambda} := p^* E_\lambda \otimes q^* H_\lambda,$$

for some finite collection $E$ of smooth complex vector bundles $E_\lambda$ on $X$, with trivial $K$-action, where $Q_0' \subset Q_0$ is the set of vertices with $E_\lambda \neq 0$, which is of course a finite set.

The following lemma, proved in [3.4.2], is needed before introducing some more notation.

**Lemma 3.3.** There is a natural isomorphism $\Omega^{0,1}(\text{Hom}(H_\lambda, H_\mu))^K \cong A_{\mu \lambda}$.

3.2.2. **Notation.** Let $\{\eta_a | a \in Q_0, ta = \lambda, ha = \mu\}$ be a basis of $\Omega^{0,1}(\text{Hom}(H_\lambda, H_\mu))^K$ corresponding to a basis $\{a^{(i)}_{\mu \lambda} | i = 1, \ldots, n_{\mu \lambda}\}$ of $A_{\mu \lambda}$ by Lemma 3.3, for each $\lambda, \mu \in Q_0$.

3.3. **Spaces of $\bar{\partial}$-operators, complex gauge groups, and quiver bundles.** Let $F$ be a smooth $K$-equivariant vector bundle on $X \times K/J$, with equivariant decomposition (3.2). The $K$-action on $F$ lifts to a unique smooth $G$-action, so $F$ is a smooth $G$-equivariant over $X \times K/J$ (cf. 3.2). Let $\bar{\mathcal{D}}$ (resp. $\mathcal{D}$) be the space of $\bar{\partial}$-operators (resp. $\bar{\partial}$-operators with square zero) on $F$, and let $\mathcal{G}^c$ be the complex gauge group of $F$, with the $G$-actions on these spaces defined in (3.3). Let $Q' = (Q_0', Q_1')$ be the full subquiver of $Q$ whose vertices $\lambda$ are defined by the condition $E_\lambda \neq 0$ (the arrows $\alpha$ are defined by the conditions $E_{\alpha \lambda} \neq 0$ and $E_{\alpha \lambda} \neq 0$). Let $\mathcal{D}_\lambda$ (resp. $\mathcal{G}^c_\lambda$) be the space of $\bar{\partial}$-operators (resp. $\bar{\partial}$-operators with square zero) on $E_\lambda$, and let $\mathcal{G}^c_\lambda$ be the complex gauge group of $E_\lambda$, for each $\lambda \in Q_0'$. The group

$$\mathcal{G}^c = \prod_{\lambda \in Q_0'} \mathcal{G}^c_\lambda$$

acts on the space $\mathcal{D}'$ of $\bar{\partial}$-operators, and on the representation space $\mathcal{R}(Q', E)$, defined by

$$\mathcal{D}' = \bigoplus_{\lambda \in Q_0'} \mathcal{D}_\lambda, \quad \mathcal{R}(Q', E) = \bigoplus_{a \in Q_1'} \Omega^0(\text{Hom}(E_{\alpha \lambda}, E_{\beta \lambda})).$$

An element $g \in \mathcal{G}^c$ is a collection of elements $g_\lambda \in \mathcal{G}^c_\lambda$, for each $\lambda \in Q_0'$, and an element $\bar{\partial}_E \in \mathcal{D}'$ (resp. $\phi \in \mathcal{R}(Q', E)$) is a collection of $\bar{\partial}$-operators $\bar{\partial}_{E_\lambda} \in \mathcal{D}_\lambda$ (resp. smooth morphisms $\phi_a : E_{\alpha \lambda} \to E_{\beta \lambda}$), for each $\lambda \in Q_0'$ (resp. $a \in Q_1'$). The $\mathcal{G}^c$-actions on $\mathcal{D}'$ and $\mathcal{R}(Q', E)$ are given by $(g(\bar{\partial}_E))_\lambda = g_\lambda \circ \bar{\partial}_{E_\lambda} \circ g^{-1}_\lambda$, and $(g \cdot \phi)_a = g_{\alpha \lambda} \circ \phi_a \circ g^{-1}_{\beta \lambda}$, respectively. The induced $\mathcal{G}^c$-action on the product $\mathcal{D}' \times \mathcal{R}(Q', E)$ leaves invariant the subset $\mathcal{N}$ of pairs $(\bar{\partial}_E, \phi)$ such that $\bar{\partial}_{E_\lambda} \in \mathcal{G}^c_\lambda$, for each $\lambda \in Q_0'$, $\phi_a : E_{\alpha \lambda} \to E_{\beta \lambda}$ is holomorphic with respect to $\bar{\partial}_{\alpha \lambda}$ and $\bar{\partial}_{\beta \lambda}$, for each $a \in Q_0$, and the holomorphic $Q'$-bundle $\mathcal{R} = (E, \phi)$, defined by these holomorphic structures and morphisms, satisfies the relations in $\mathcal{K}$.

**Proposition 3.4.** (a) There is a one-to-one correspondence between $\mathcal{D}^G$ and $\mathcal{D}' \times \mathcal{R}(Q', E)$ which, to any $(\bar{\partial}_E, \phi) \in \mathcal{D}' \times \mathcal{R}(Q', E)$, associates, the $\bar{\partial}$-operator $\bar{\partial}_F$ in $\mathcal{D}^G$ given by

$$\bar{\partial}_F = \sum_{\lambda \in Q_0'} \bar{\partial}_{F_\lambda} \circ \pi_\lambda + \sum_{a \in Q_1'} \beta_a \circ \pi_{\alpha \lambda}.$$

Here $\bar{\partial}_{F_\lambda}$ is the $\bar{\partial}$-operator of $F_\lambda$ given by $\bar{\partial}_{F_\lambda} = p^* \bar{\partial}_{E_\lambda} \otimes \text{id} + \text{id} \otimes q^* \bar{\partial}_{H_\lambda}$, for each $\lambda \in Q_0'$, and $\beta_a := p^* \phi_a \otimes q^* \eta_a \in \Omega^{0,1}(\text{Hom}(F_{\alpha \lambda}, F_{\beta \lambda}))$ for each $a \in Q_1'$.
(b) The previous correspondence restricts to a one-to-one correspondence between \( \mathcal{C}_G \) and \( \mathcal{N} \).

(c) There is a one-to-one correspondence between \( (\mathcal{G}_c)^G \) and \( \mathcal{G}_c^\mathcal{N} \) which, to any \( g \in \mathcal{G}_c \), associates \( g = \sum_{\lambda \in \mathcal{G}_0} \delta \circ \tau \in (\mathcal{G}_c)^G \), with \( \delta \lambda = p^\ast g \lambda \in \Omega^0(\Aut(F_\lambda)) \cong \Omega^0(\Aut(p^\ast E_\lambda)) \).

(d) These correspondences are compatible with the actions of the groups of (c) on the sets of (a) and (b), hence there is a one-to-one correspondence between \( \mathcal{C}_G / (\mathcal{G}_c)^G \) and \( \mathcal{N} / \mathcal{G}_c^\mathcal{N} \).

3.4. **Preliminaries on smooth homogeneous vector bundles.** To prove the previous proposition, in this subsection we collect several preliminary results about the homogeneous space \( K/J \) and the homogeneous bundles on \( K/J \). We first recall several standard results (§§3.4.1-3.4.5), which we adapt to the notation used throughout this paper. We then prove other results (§§3.4.6-3.4.8), which are elementary but necessary to relate invariant connections on homogeneous bundles to our definition of the quiver with relations associated to \( P \).

3.4.1. **The canonical complex structure on \( K/J \).** The canonical complex structure on \( K/J \) is the complex structure on \( K/J \) induced by the complex structure on the projective variety \( G/P \) and by the diffeomorphism \( K/J \cong G/P \). Let \( T = H \cap K \) be a maximal torus of \( K \), and let \( \mathfrak{t} \) be its Lie algebra. Let \( \mathfrak{r} \subseteq \mathfrak{g} \) be the Lie subalgebra, which is also a \( J \)-submodule, given by the isomorphisms of \( J \)-modules

\[
\mathfrak{t} = j \oplus \mathfrak{r}, \quad \mathfrak{r} \cong \mathfrak{t}/j.
\]

That is, \( \mathfrak{r} \) is the direct sum of the even-dimensional real subspaces of \( \mathfrak{t} \) on which the spectrum for the action of \( \mathfrak{t} \) is \( \pm \sqrt{-1} \lambda \), for \( \lambda \in \Delta(\mathfrak{t}) \), where \( \Delta(\mathfrak{t}) \) is the set of roots of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) which are not roots of \( \mathfrak{t} \). The complexification of \( \mathfrak{r} \) is \( \mathfrak{r}_c = u \oplus u \). We define a \( J \)-invariant complex structure of the \( J \)-module \( \mathfrak{r} \) by the condition that

\[
\mathfrak{r}^{1,0} = \bar{u}, \quad \mathfrak{r}^{0,1} = u
\]

are the \((1, 0)\)- and \((0, 1)\)-subspaces of \( \mathfrak{r}_c \), respectively. Then there are natural isomorphisms

\[
(3.6) \quad \Lambda^i j T^* (K/J) \cong K \times_J \Lambda^i j \mathfrak{r}.
\]

To prove this, we first note that the holomorphic and anti-holomorphic cotangent bundles on \( G/P \) are isomorphic to \( G \times_P u \) and \( G \times_P u^* \), respectively, as holomorphic \( G \)-equivariant vector bundles. This follows from: (i) the holomorphic tangent bundle on \( G/P \) is isomorphic to \( G \times_P (\mathfrak{g}/\mathfrak{p}) \) as a holomorphic \( G \)-equivariant vector bundle; (ii) the Killing form on \( \mathfrak{g} \) induces isomorphisms \( \mathfrak{g}/\mathfrak{p} \cong u^* \cong u \) of representations of \( P \). Second, by the arguments of §3.2, the underlying smooth homogeneous vector bundles of \( G \times_P u \) and \( G \times_P u^* \) are \( K \times_J \mathfrak{r}^{1,0} \) and \( K \times_J \mathfrak{r}^{0,1} \), respectively. This proves (3.6). It is worth mentioning that the canonical complex structure can be defined directly on \( K/J \), without the use of the diffeomorphism \( K/J \cong G/P \) (see e.g. [Be]); however we shall be interested in the relation between the holomorphic structures on smooth homogeneous vector bundles on \( K/J \) and the representation theory of \( P \), so we shall need the isomorphisms (3.6).

3.4.2. **Spaces of invariant forms on \( K/J \).** Given a smooth \( K \)-equivariant vector bundle \( F = K \times_J W \) on \( K/J \), induced by a representation \( W \) of \( J \), there is a bijection between the space of \( K \)-invariant sections of \( F \) and the \( J \)-invariant subspace \( W^J \) of \( W \). If \( W = \Lambda^i j \mathfrak{r}^* \otimes V \), where \( V \) is another complex representation of \( J \), then (3.6) implies \( \Lambda^i j T^* (K/J) \otimes E \cong K \times_J (\Lambda^i j \mathfrak{r}^* \otimes V) \) as smooth \( K \)-equivariant vector bundles, where \( E := K \times_J V \). Therefore, the spaces of \( K \)-invariant \( E \)-valued \( r \)- and \((i, j)\)-forms on \( K/J \) are

\[
(3.7) \quad \Omega^r (E)^K \cong C^r (V), \quad \text{and} \quad \Omega^{i, j} (E)^K \cong C^{i, j} (V),
\]

respectively, where

\[
(3.8) \quad C^r (V) := (\Lambda^r \mathfrak{r}^* \otimes V)^J, \quad \text{and} \quad C^{i, j} (V) := (\Lambda^{i, j} \mathfrak{r}^* \otimes V)^J.
\]
In particular, if $V = \text{Hom}(M_\lambda, M_\mu)$, for $\lambda, \mu \in Q_0$, and $i = 1, 2$, we get (see §12.1)

$$\Omega^{0,1}(\text{Hom}(H_\lambda, H_\mu))K \cong A_{\mu\lambda}, \quad \Omega^{0,2}(\text{Hom}(H_\lambda, H_\mu))K \cong B_{\mu\lambda}.$$  

Let $V$ be a representation of $J$, and let $E = K \times_J V$ be the induced homogeneous vector bundle on $K/J$. A basic $V$-valued $r$-form on the principal $J$-bundle $K \to K/J$ is an element of $\Omega^r_K(V) := \Omega^r_K \otimes V$ which is $J$-invariant and horizontal (cf. e.g. §1.1 of [BGGV] for definitions). The space $\Omega^r_K(V)$ of basic $V$-valued $r$-forms on $K \to K/J$ is isomorphic to the space $\Omega^r(E)$ of $E$-valued $r$-forms on $K/J$. The space $\Omega^r(E)$ is naturally a $K$-module and, since the principal $J$-bundle $K \to K/J$ is $K$-equivariant (with the canonical left $K$-action), the space $\Omega^r_K(V)$ is a $K$-module as well. Moreover, the previous isomorphism $\Omega^r_K(V) \cong \Omega^r(E)$ is an isomorphism of $K$-modules. Therefore their $K$-invariant parts are isomorphic, so (3.7) gives an isomorphism.

### 3.4.3. Spaces of invariant forms on $K$

There is a natural isomorphism of Lie algebras between $\mathfrak{k}$ and the space $\mathfrak{X}(K)^K$ of (left) $K$-invariant vector fields on $K$. Given $x \in \mathfrak{k}$, let $\bar{x}$ be the corresponding $K$-invariant vector field on $K$ generated by $x$, i.e. $\bar{x}(k) = \frac{d}{dt}(k \exp(tx))|_{t=0}$, for each $k \in K$. Let $V$ be a representation of $J$. The left action of $J$ on $K$ together with the action of $J$ on $V$ define a structure of $J$-module on the space $\Omega^r_K(V) = \Omega^r_K \otimes V$ of $V$-valued $r$-forms on $K$, while the right action of $K$ on itself induces naturally a structure of $K$-module on $\Omega^r_K(V)$. Moreover, the $K$-invariant subspace $\Omega^r_K(V)^K$ is a $J$-submodule of $\Omega^r_K(V)$. Given $a \in \wedge^r \mathfrak{k} \otimes V$, let $\bar{a} \in \Omega^r_K(V)^K$ be defined by

$$\bar{a}(\bar{x}_1(k), \ldots, \bar{x}_r(k)) = a(x_1, \ldots, x_r), \quad \text{for } k \in K, \ x_1, \ldots, x_r \in \mathfrak{k}.$$  

This map defines an isomorphism of representations of $J$:

$$\wedge^r \mathfrak{k} \otimes V \cong \Omega^r_K(V)^K.$$

### 3.4.4. An invariant connection for smooth homogeneous bundles

Let $A' : \mathfrak{k} \to j$ be the canonical projection from $\mathfrak{k} = j \oplus \mathfrak{r}$ onto $j$. Thus $A' \in \mathfrak{k}^* \otimes j$ is $J$-equivariant, so it induces a $K$-invariant $j$-valued one-form $\bar{A}' \in \Omega^1(j)^K$ on $K$, by (3.10).

**Lemma 3.11.**

(a) The one-form $\bar{A}' \in \Omega^1_K(j)^K$ is a $K$-invariant connection one-form on the smooth principal $J$-bundle $K \to K/J$.

(b) Its curvature $\bar{F}' \in \Omega^{1,1}_{K/J}(\text{ad } K)$, with $\text{ad } K := K \times_J j$, is given by means of its isomorphic element $F' \in C^{1,1}(j)$, defined by

$$F'(x, x') = -A'([x, x']) \quad \text{for } x, x' \in \mathfrak{r}.$$

**Proof.** Part (a) is straightforward. By §3.4.2 the curvature is defined by its isomorphic element $\bar{F}' = d\bar{A}' + \frac{1}{2}[[\bar{A}, \bar{A}], \bar{A}], \text{ in } \Omega^2(K)^K \subset \Omega^2_K(j)^K$, which is given by

$$\bar{F}'(\bar{x}(k), \bar{x}'(k)) = d\bar{A}'(\bar{x}(k), \bar{x}'(k)) + [\bar{A}'(\bar{x}(k)), \bar{A}'(\bar{x}'(k))], \quad \text{for } x, x' \in \mathfrak{k} \text{ and } k \in K.$$  

(3.12) Then $d\bar{A}'(\bar{x}, \bar{x}') = \bar{x}(\bar{A}'(\bar{x}') - \bar{A}'(\bar{x})) - \bar{A}'(\bar{x}, \bar{x}')$. But $\bar{A}'(\bar{x}) = A'(x)$ is constant, so $\bar{x}(\bar{A}'(\bar{x})) = 0$, and analogously $\bar{x}(\bar{A}'(\bar{x}')) = 0$. On the other hand $\bar{F}'(\bar{x}, \bar{x}') = [\bar{A}'(\bar{x}), \bar{A}'(\bar{x}')] - \bar{A}'(\bar{x}, \bar{x}') = 0$. Therefore

$$\bar{F}'(\bar{x}, \bar{x}') = [\bar{A}'(\bar{x}), \bar{A}'(\bar{x}')] - \bar{A}'(\bar{x}, \bar{x}') = A'(x, x') - A'(x, x').$$  

This shows that $\bar{F}'(\bar{x}, \bar{x}') = 0$ for $x \in j$ or $x' \in j$, i.e. $\bar{F}'$ is horizontal, as we already knew from part (a), and for $x, x' \in \mathfrak{r}$, it is given by $\bar{F}'(\bar{x}, \bar{x}') = -A'([x, x'])$ which is (3.12). Let us denote by the same symbol the complexification of $F'$. Given $e, e' \in u_\pm$, we have $[e, e'] \in u_\pm$, since $u_\pm$ are Lie algebras, hence its projection by $A' : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{u} \oplus \mathfrak{u} \to \mathfrak{l} = j|_C$ is zero, i.e. $F'([e, e']) = -A'([e, e']) = 0$. Hence $\bar{F}' \in C^{1,1}(j)$. \qed
Let $V$ be a $J$-module. Since the curvature $\tilde{F}'$ is of type $(1, 1)$, the induced connection on any homogeneous vector bundle $E = K \times J V$ defines a holomorphic structure, which is obviously $G$-invariant. Let us define the linear maps $d: C^r(V) \to C^{r+1}(V)$ between the linear spaces $C^r(V) := (\wedge^r r^* \otimes V)^g$ by

\begin{equation}
\label{3.13}
da(x_0, \ldots, x_r) = \sum_{0 \leq i < j \leq r} (-1)^{i+j} a([x_i, x_j], x_0, \ldots, x_i, \ldots, \hat{x}_j, \ldots, x_r), \quad \text{for } x_0, \ldots, x_r \in \mathfrak{r}.
\end{equation}

\textbf{Lemma 3.14.} Let $A$ be the $K$-invariant connection on $E = K \times J V$ induced by the $K$-invariant connection one-form $\tilde{A}'$ on $K \to K/J$. The following diagram is commutative:

$$
\begin{array}{ccc}
C^r(V) & \xrightarrow{d} & C^r(V) \\
\Downarrow & & \Downarrow \\
\Omega^r(E)^K & \xrightarrow{dA} & \Omega^r(E)^K
\end{array}
$$

Here $C^r(V) \cong \Omega^r(E)^K$ and $C^{r+1}(V) \cong \Omega^{r+1}(E)^K$ are the isomorphism appearing in §3.4.2.

\textbf{Proof.} First we assume that $r = 0$, so let $v \in C^0(V) = V^J$, $s \in \Omega^0(E)^K$ and $\tilde{v} \in \Omega^0_b(V)$ be related by the isomorphisms of §3.4.2. By definition $dv = 0$, so we have to see that $d_A s = 0$, or equivalently, that $d_A \tilde{v} = 0$, where $d_A = d + \rho(\tilde{A}') : \Omega^0_b(V) \to \Omega^1_b(V)$. Since $\tilde{v} : K \to V$ is a constant map, $d\tilde{v} = 0$, while $\rho(\tilde{A}') \cdot \tilde{v} \in \Omega^1_b(E)$ is given by $(\rho(\tilde{A}') \cdot \tilde{v}) \cdot e = \rho(\tilde{A}' \cdot e \cdot v$ for $e \in TK$, which is zero, since $v$ is $J$-invariant. This proves $d_A s = 0$. For $r \geq 1$, let $a \in C^r(V)$, and let $\eta \in \Omega^r(E)^K$ and $v \in \Omega^r_b(V)$ be related the isomorphism of §3.4.2. Let $x_0, \ldots, x_r \in \mathfrak{r}$, and let $\tilde{x}_0, \ldots, \tilde{x}_r$ be as in §3.4.3. Then

\begin{equation}
\label{3.14.1}
d_A a(x_0, \ldots, x_r) = \sum_{i=0}^r (-1)^i \iota(x_i) d_A \tilde{a}(\tilde{x}_0, \ldots, \tilde{x}_i, \ldots, \tilde{x}_r)
\end{equation}

\begin{equation}
\label{3.14.2}
+ \sum_{0 \leq i < j \leq r} (-1)^{i+j} \tilde{a}([\tilde{x}_i, \tilde{x}_j], \tilde{x}_0, \ldots, \tilde{x}_i, \ldots, \hat{\tilde{x}}_j, \ldots, \tilde{x}_r),
\end{equation}

where $\iota(x) : \Omega^1 \to \Omega^0$ is contraction with the vector field $\tilde{x}$. But $\tilde{a}(\tilde{x}_0, \ldots, \tilde{x}_i, \ldots, \tilde{x}_r) = a(x_0, \ldots, x_i, \ldots, x_r)$ is constant, so $d_A a(x_0, \ldots, x_i, \ldots, x_r) = 0$ as seen before, while $[\tilde{x}_i, \tilde{x}_j] = [x_i, x_j]$. This proves the assertion. \hfill \Box

\subsection{3.4.5. Invariant forms on $K/J$ and the linear maps $\psi_{\mu \lambda}$, $\psi_{\mu \nu \lambda}$.}

Let $\{\eta^{(i)}_{\mu \lambda} | i = 1, \ldots, m_{\mu \lambda}\}$ and $\{\xi_{p}^{(i)} | p = 1, \ldots, m_{p \mu \lambda}\}$, for fixed $\lambda, \mu \in Q_0$, be the bases of $\Omega^{0,1}(\text{Hom}(H_{\lambda}, H_{\mu}))^K$ and $\Omega^{0,2}(\text{Hom}(H_{\lambda}, H_{\mu}))^K$, that correspond to the basis $\{a_{\mu \lambda}^{(i)} | i = 1, \ldots, n_{\mu \lambda}\}$ of $A_{\mu \lambda}$ and to the basis $\{b_{p \mu \lambda}^{(i)} | p = 1, \ldots, b_{p \mu \lambda}\}$ of $B_{p \mu \lambda}$ by the isomorphisms (3.9), respectively (cf. §3.2.2). Let $c_{\mu \lambda}^{(i,p)}$ and $c_{\mu \nu \lambda}^{(j,i,p)}$ be the coefficients defined in §3.3.2.

\textbf{Lemma 3.15.} Let $A'$ be the unique $K$-invariant connection of $H_{\lambda}$, and let $A'_{\mu \lambda}$ be the connection induced by $A'$ and $A'_{\mu \lambda}$ on the vector bundle $\text{Hom}(H_{\lambda}, H_{\mu})$. Then

\begin{equation}
\label{3.16}
\eta^{(i)}_{\mu \lambda} \wedge \eta^{(i)}_{\nu \lambda} = \sum_{p=1}^{m_{\mu \lambda}} c_{\mu \lambda}^{(i,p)} \xi_{\mu \lambda}^{(p)}, \quad \partial A'_{\mu \lambda} \eta^{(i)}_{\mu \lambda} = \sum_{p=1}^{m_{\mu \lambda}} c_{\mu \lambda}^{(i,p)} \xi_{\mu \lambda}^{(p)}, \quad \text{for each } \lambda, \mu \in Q_0.
\end{equation}

\textbf{Proof.} The first equation follows immediately from the first in (3.3). The second equation is obtained from the second equation in (3.3). Indeed, $d A'_{\mu \lambda} \eta^{(i)}_{\mu \lambda} \in \Omega^2_{\mu \lambda}(\text{Hom}(H_{\lambda}, H_{\mu}))^K$ corresponds, by the
isomorphism (3.9), to \( da^{(i)}_{\mu\lambda} \in B_{\mu\lambda} \) as given by (3.13). To evaluate its \((0,2)\)-part \( \partial a^{(i)}_{\mu\lambda} := (da^{(i)}_{\mu\lambda})^{0,2} \in \wedge^{0,2}r \otimes \text{Hom}(M_{\lambda}, M_{\mu}) \), let \( e, e' \in e^{0,1} = u : \)

\[
da^{(i)}_{\mu\lambda}(e, e') = -a^{(i)}_{\mu\lambda}([e, e']) = \psi_{\mu\lambda}(a)(e, e') = \sum_{p=1}^{m_{\mu\lambda}} c^{(i,p)}_{\mu\lambda} b^{(p)}_{\mu\lambda}.
\]

Using the isomorphism (3.9) once more implies the second equation in (3.16). \( \Box \)

From the point of view of \( \partial \)-operators, the occurrence of the coefficients \( c^{(i,p)}_{\mu\lambda} \) and \( c^{(j,i,p)}_{\mu\lambda} \) in the relations of the quiver will appear, via equation (3.16), when demanding the integrability condition \((\partial_F)^2 = 0\) on the \( \partial \)-operators \( \partial_F \) defined on equivariant bundles over \( X \times K/J \). Thus, the linear terms in the relations \( f^{(p)}_{\mu\lambda} \), corresponding to the coefficients \( c^{(i,p)}_{\mu\lambda} \), can be seen as a consequence of a non-holomorphic phenomenon (i.e. \( \partial a^{(i)}_{\mu\lambda}(\eta^{(i)}_{\mu\lambda}) \) may not be zero). Note that the forms \( \eta_{\mu\lambda} \) are holomorphic precisely when the unipotent radical \( U \) is abelian, e.g. for Grassmann varieties. It is worth remarking that Hille already proved that the relations are quadratic when the flag variety \( G/P \) is a Grassmannian (cf. [H12 Corollary 2.2]), using a different method which involves his level function (cf. [4.3]), and which, therefore, in principle can only be applied when \( P \) is a parabolic subgroup.

3.5. Proof of Proposition 3.4. (a) Let us fix a \( \partial \)-operator \( \partial_{E_{\lambda}} \) on each \( E_{\lambda} \). We define \( \partial \)-operators \( \partial_{F_{\lambda}} \) on each \( F_{\lambda} \) by \( \partial_{F_{\lambda}} = p^{*}\partial_{E_{\lambda}} \otimes \text{id} + \text{id} \otimes q^{*}\partial_{H_{\lambda}} \). They are obviously \( G \)-invariant, so the \( \partial \)-operator \( \partial_{F_{\lambda}} = \sum_{\lambda \in Q_{0}^{*}} \partial_{E_{\lambda}} \circ \pi_{\lambda} \) on \( F \), is also \( G \)-invariant. Thus, any \( G \)-invariant \( \partial \)-operator \( \partial_{F} \) on \( F \) can be written as \( \partial_{F} = \partial_{F_{\lambda}} + \theta \) for \( \theta \in \Omega^{0,1}(\text{End}(F))^{G} \), the \( G \)-invariant subset of \( \Omega^{0,1}(\text{End}(F)) \). What we have to prove is that there is a one-to-one correspondence between \( \Omega^{1}(\text{End}(F))^{G} \) and \( \Theta \times \mathcal{R}(Q', E) \), where \( \Theta := \bigoplus_{\lambda \in Q_{0}^{*}} \Omega^{0,1}(\text{End}(E_{\lambda})) \). It is clear that \( \Omega^{1}(\text{End}(F))^{G} = \bigoplus_{\lambda, \mu \in Q_{0}^{*}} \Omega^{0,1}(\text{Hom}(F_{\lambda}, E_{\mu}))^{G} \), where \( T_{C}^{0,1}(X \times K/J) \cong p^{*}T_{C}^{0,1}(X) \otimes q^{*}T_{C}^{0,1}(K/J) \), so

\[
\Omega^{0,1}(\text{Hom}(F_{\lambda}, E_{\mu}))^{G} \cong \Omega^{0,1}(\text{Hom}(E_{\lambda}, E_{\mu})) \otimes \Omega^{0,1}(\text{Hom}(H_{\lambda}, H_{\mu}))^{G} \\
\cong \Omega^{0,1}(\text{Hom}(E_{\lambda}, E_{\mu})) \otimes \Omega^{0,1}(\text{Hom}(H_{\lambda}, H_{\mu}))^{G}.
\]

By the isomorphisms (3.8) and (3.9) and Schur’s lemma, it follows that

\[
\Omega^{0,1}(\text{End}(F))^{G} \cong \bigoplus_{\lambda \in Q_{0}^{*}} \Omega^{0,1}(\text{End}(E_{\lambda})) \oplus \bigoplus_{\lambda, \mu \in Q_{0}^{*}} A_{\lambda\mu} \otimes \Omega^{0,1}(\text{Hom}(E_{\lambda}, E_{\mu})) \cong \Theta \times \mathcal{R}(Q', E).
\]

(b) Let \( \phi^{(i)}_{\mu\lambda} = \phi_{a}, \beta^{(i)}_{\mu\lambda} = \beta_{a} \) for each \( a = a^{(i)}_{\mu\lambda} \in Q_{1}' \), let \( \partial_{E_{\mu\lambda}} \) be the \( \partial \)-operator induced by \( \partial_{E_{\lambda}} \) and \( E_{\mu\lambda} \) on the vector bundle \( \text{Hom}(E_{\lambda}, E_{\mu}) \), and let \( \partial_{F} \in \mathcal{G}^{G} \) be given by (3.5). Then

\[
(\partial_{F})^{2} = \sum_{\lambda \in Q_{0}^{*}} (\partial_{E_{\lambda}})^{2} + \sum_{\lambda, \mu \in Q_{0}^{*}} \sum_{i=1}^{n_{\mu\lambda}} \beta^{(i)}_{\mu\lambda} \circ \pi_{\lambda} + \sum_{\lambda, \mu, \nu \in Q_{0}^{*}} \sum_{i=1}^{n_{\mu\nu}} \sum_{j=1}^{n_{\mu\nu}} \beta^{(j)}_{\mu\nu} \wedge \beta^{(i)}_{\mu\lambda} \circ \pi_{\lambda},
\]

where \( (\partial_{E_{\lambda}})^{2} = p^{*}(\partial_{E_{\lambda}})^{2} \otimes \text{id} + \text{id} \otimes q^{*}(\partial_{H_{\lambda}})^{2} \). Therefore (3.16) implies

\[
(\partial_{F})^{2} = \sum_{\lambda \in Q_{0}^{*}} (p^{*}(\partial_{A_{\lambda}})^{2} \otimes \text{id}) \circ \pi_{\lambda} + \sum_{\lambda, \mu \in Q_{0}^{*}} \sum_{i=1}^{n_{\mu\lambda}} \sum_{j=1}^{n_{\mu\nu}} (p^{*}\partial_{E_{\lambda\mu}})(\phi^{(i)}_{\mu\lambda} \otimes q^{*}\eta^{(i)}_{\mu\lambda}) \circ \pi_{\lambda}
\]

\[
+ \frac{1}{2} \sum_{\lambda, \mu \in Q_{0}^{*}} \sum_{p=1}^{m_{\mu\lambda}} \sum_{\nu \in Q_{0}^{*}} \sum_{i=1}^{n_{\mu\nu}} \sum_{j=1}^{n_{\mu\nu}} \sum_{p=1}^{m_{\nu\lambda}} \phi^{(j)}_{\mu\nu} \otimes \phi^{(i)}_{\mu\lambda} + \sum_{i=1}^{n_{\mu\lambda}} \sum_{j=1}^{n_{\mu\nu}} \sum_{p=1}^{m_{\mu\lambda}} \phi^{(j)}_{\mu\nu} \otimes \phi^{(i)}_{\mu\lambda} \otimes q^{*}\xi^{(p)}_{\mu\lambda}.
\]
So \((\bar{\partial}_F)^2 = 0\) if and only if \((\bar{\partial}_E)^2 = 0\) for all \(\lambda, \bar{\partial}_{E_{\mu,\lambda}}(\xi_{\mu,\lambda}^i) = 0\) for all \(\lambda, \mu, i\), and the corresponding holomorphic \(Q\)-bundle \(\mathcal{R} = (\mathcal{E}, \phi)\) satisfies the relations in \(\mathcal{K}\).

(c) By Schur’s lemma, \((\mathcal{E}^c)^G\) is included in

\[
\Omega^0(\text{End}(F))^G \cong \bigoplus_{\lambda, \mu \in Q_0'} \Omega^0(\text{Hom}(F_{\lambda}, F_{\mu}))^G \cong \bigoplus_{\lambda \in Q_0} \Omega^0(\text{End}(F_{\lambda}))^G.
\]

The result is now immediate.

(d) This is trivial. □

4. GAUGE EQUATIONS, STABILITY, AND DIMENSIONAL REDUCTION

The goal of this section is to study natural gauge equations and stability criteria for equivariant bundles on \(X \times G/P\), and to investigate their dimensional reduction to \(X\). Proposition 2.6 establishes an equivalence between holomorphic equivariant bundles on \(X \times G/P\) and their \(G\)-equivariant filtrations. As a result, one can consider certain deformed Hermite–Einstein equation and deformed stability criteria on equivariant bundles, which take into account the additional structure encoded in the filtration. As we shall see, these deformations will depend on as many stability criteria on equivariant bundles, which take into account the additional structure encoded in the filtration. We review this correspondence in the next subsection.

In Theorem 2.3, we proved an equivalence between the category of \(G\)-equivariant holomorphic vector bundles on \(X \times G/P\) and the category of holomorphic \((Q, \mathcal{K})\)-bundles on \(X\), where \((Q, \mathcal{K})\) is the quiver with relations associated to \(P\). In fact, in Proposition 3.4 we described this equivalence in terms of \(\bar{\partial}\)-operators. Thus, the deformed Hermite–Einstein equation and deformed stability condition lead, by the so-called process of dimensional reduction, to new gauge equations and stability conditions on the associated quiver bundle on \(X\), while the Hitchin–Kobayashi correspondence for equivariant holomorphic filtrations on \(X \times G/P\) lead to a Hitchin–Kobayashi correspondence for holomorphic quiver bundles on \(X\). By using Proposition 3.4 in Sections 4.2 and 4.3 we describe the resulting equations and stability criteria for quiver bundles on \(X\), which will also depend on certain stability parameters. We relate the stability parameters for the equivariant filtration on \(X \times G/P\) with the stability parameters for the corresponding quiver bundle. In these two subsections we notice that there is some freedom in the relationship between the parameters, that can be traced back to the fact that the \(K\)-invariant symplectic form \(\omega_g\) depends on as many positive parameters \(\varepsilon_{\alpha}\) as simple non-parabolic roots \(\alpha \in \Sigma\) of \(G\) which are not roots of the Levi subgroup \(L\) of \(P\). Another effect of the process of dimensional reduction is the appearance of purely group-theoretic multiplicity factors \(n_\lambda = \dim M_\lambda\), for \(\lambda \in \Lambda^+_P\), integral dominant weights of \(P\), in the relation between the parameters.

4.1. Hitchin–Kobayashi correspondence for equivariant holomorphic filtrations. Given a holomorphic vector bundle \(\mathcal{F}\) on a Kähler manifold \((M, \omega)\), we define the degree and slope of \(\mathcal{F}\) as

\[
\deg(\mathcal{F}) = \frac{1}{\operatorname{Vol}(M)} \int_M \operatorname{tr}(\sqrt{-1} \Lambda F_A) \omega^n, \quad \mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\operatorname{rk}(\mathcal{F})},
\]

respectively, where \(n = \dim_c(M)\), \(\operatorname{Vol}(M)\) is the volume of \(M\), \(F_A\) is the curvature of a connection \(A\) on \(E\), \(\Lambda\) is contraction with \(\omega\), and \(\operatorname{rk}(\mathcal{F})\) is the rank of \(\mathcal{F}\). The degree of a torsion-free coherent sheaf on \(M\) is normalised with these conventions as well. The Hermite–Einstein equation for a hermitian metric \(h\) on a holomorphic vector bundle \(\mathcal{F}\) is \(\sqrt{-1} \Lambda F_h = \mu(\mathcal{F}) I,\) where \(F_h\) is the curvature of the Chern connection associated to the metric \(h\) on \(\mathcal{F}\).
4.1.1. Deformed Hermite–Einstein equation. Throughout this subsection, \((M, \omega)\) is a compact Kähler \(K\)-manifold, where \(K\) is a compact Lie group. We assume then that the \(K\)-action leaves \(\omega\) invariant, and extends to a unique holomorphic action of \(G\) (the complexification of \(K\)) on \(M\). Let \(\mathcal{F}\), given by
\[
\mathcal{F} : 0 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_m = \mathcal{F}
\]
be a \(G\)-equivariant holomorphic filtration over \(M\). The deformed Hermite–Einstein equation involves as many parameters \(\tau_0, \tau_1, \ldots, \tau_m \in \mathbb{R}\) as steps are in the filtration, and has the form
\[
\sqrt{-1} \Lambda F_h = \begin{pmatrix}
\tau_0 I_0 \\
\tau_1 I_1 \\
\vdots \\
\tau_m I_m
\end{pmatrix},
\]
where the RHS is a diagonal matrix, written in blocks corresponding to the splitting which a hermitian metric \(h\) defines in the filtration \(\mathcal{F}\). If \(\tau_0 = \cdots = \tau_m\), (4.3) reduces to the Hermite–Einstein equation. Taking traces in (4.3) and integrating over \(M\), we see that there are only \(m\) independent parameters, since they are constrained by
\[
\sum_{i=0}^{m} \tau_i \text{rk}(\mathcal{F}_i/\mathcal{F}_{i-1}) = \text{deg}(\mathcal{F}).
\]
The group \(K\) acts in the space \(\text{Met}\) of hermitian metrics on \(\mathcal{F}\), in a natural way, by \(K \times \text{Met} \to \text{Met}, (\gamma, h) \mapsto \gamma \cdot h = (\gamma^{-1})^*h\). We are interested in the \(K\)-invariant solutions of (4.3).

**Definition 4.5.** Let \(\tau = (\tau_0, \ldots, \tau_m) \in \mathbb{R}^{m+1}\), and let \(\mathcal{F}\), as in (4.2), be a holomorphic filtration on \(M\). We say that a hermitian metric \(h\) on \(\mathcal{F}\) is a \(K\)-invariant solution of the \(\tau\)-Hermite–Einstein equation on \(\mathcal{F}\) if it is \(K\)-invariant and it satisfies the \(\tau\)-Hermite–Einstein equation (4.3). If such an \(K\)-invariant \(\tau\)-Hermite–Einstein metric \(h\) exists on \(\mathcal{F}\), we say that \(\mathcal{F}\) is a \(K\)-invariantly \(\tau\)-Hermite–Einstein holomorphic filtration.

4.1.2. Deformed stability. As in the ordinary Hermite–Einstein equation, the existence of invariant solutions to the \(\tau\)-Hermite–Einstein equation on an equivariant holomorphic filtration is related to a stability condition for the equivariant holomorphic filtration.

**Definition 4.6.** Let \(\sigma = (\sigma_0, \ldots, \sigma_{m-1}) \in \mathbb{R}^{m-1}\), and let \(\mathcal{F}\), as in (4.2), be a \(G\)-equivariant sheaf filtration on \(M\). We define its \(\sigma\)-degree and \(\sigma\)-slope respectively by
\[
\text{deg}_\sigma(\mathcal{F}) = \text{deg}(\mathcal{F}) + \sum_{i=0}^{m-1} \sigma_i \text{rk}(\mathcal{F}_i), \quad \mu_\sigma(\mathcal{F}) = \frac{\text{deg}_\sigma(\mathcal{F})}{\text{rk}(\mathcal{F})}.
\]
We say that \(\mathcal{F}\) is \(G\)-invariantly \(\sigma\)-(semi)stable if for all \(G\)-invariant proper sheaf subfiltrations \(\mathcal{F}' \hookrightarrow \mathcal{F}\), we have \(\mu_\sigma(\mathcal{F}') < (\leq) \mu_\sigma(\mathcal{F})\). A \(G\)-invariantly \(\sigma\)-polystable sheaf filtration is a direct sum of \(G\)-invariantly \(\sigma\)-stable sheaf filtrations, all of them with the same \(\sigma\)-slope.

4.1.3. Hitchin–Kobayashi correspondence.

**Theorem 4.7.** Let \(\mathcal{F}\) be a \(G\)-equivariant holomorphic filtration on a compact Kähler \(K\)-manifold \(M\). Let \(\tau = (\tau_0, \cdots, \tau_m) \in \mathbb{R}^{m+1}\) be related by (4.4) and let \(\sigma = (\sigma_0, \ldots, \sigma_{m-1}) \in \mathbb{R}^m\) be defined by \(\sigma_i = \tau_{i+1} - \tau_i\), for \(0 \leq i \leq m-1\), so that \(\sigma_i > 0\). Then \(\mathcal{F}\) admits a \(K\)-invariant \(\tau\)-Hermite–Einstein metric if and only if it is \(G\)-invariantly \(\sigma\)-polystable.

**Proof.** This theorem was proved in §2.3 of [AG1] when \(G\) is trivial and also for \(SL(2, \mathbb{C})\)-equivariant holomorphic filtrations on \(X \times \mathbb{P}^1\). Since Proposition 2.6 establishes that the \(G\)-equivariant coherent
sheaves admit similar $G$-equivariant filtrations, the same proof given there can be applied to this more general situation. \hfill \square

4.2. Dimensional reduction and equations. We study now the dimensional reduction of the deformed Hermite–Einstein equation for a $G$-equivariant holomorphic filtration on $X \times G/P$, as defined in §4.1.1. This subsection is organised as follows. In §4.2.1 we study $K$-invariant Kähler form on $G/P$, since the Kähler metric on $X \times G/P$ plays an important role in the Hermite–Einstein equation for the equivariant holomorphic filtration. In §4.2.2 we define natural gauge equations for quiver bundles, which actually make sense for any quiver, not necessarily associated to a Lie group $P\{AG\}$. The main result is then stated in Theorem 4.13, which is proved in §4.2.4 after some preliminaries about connections on homogeneous bundles, which are covered in §4.2.3.

4.2.1. Invariant Kähler structures on $K/J$. The following lemma, which we adapt to the notation used throughout this paper, is standard and can be found e.g. in [Be]. Let $\kappa(\cdot, \cdot)$ be the Killing form on $\mathfrak{g}$, given by $\kappa(e, e') = \text{tr}(\text{ad}(e) \circ \text{ad}(e'))$, $e, e' \in \mathfrak{g}$. Let $\mathcal{S}$ be a system of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, such that all the negative roots of $\mathfrak{g}$ are roots of $\mathfrak{p}$, and let $\Sigma$ be the set of non-parabolic simple roots, defined as in §1.5.1. Further, let $\mathfrak{t}$ be as in §3.4.1. For each $\alpha \in \mathcal{S}$, let $h_\alpha$ be dual to the co-root $\alpha^\vee = 2\alpha/\kappa(\alpha, \alpha)$ w.r.t. the Killing form (so $\{h_\alpha | \alpha \in \mathcal{S}\}$ is a basis of $\mathfrak{h}$). Thus, $\{\sqrt{-1} h_\alpha | \alpha \in \mathcal{S}\}$ is a basis of $\mathfrak{t}$, and $-\kappa(\cdot, \cdot)$ restricted to $\mathfrak{t}$ is an inner product.

**Lemma 4.8.** There is a bijection between the set of $K$-invariant Kähler forms on $K/J$ compatible with the canonical complex structure and the set $\mathbb{R}_{+}\Sigma$ of collections of positive numbers indexed by $\Sigma$. The bijection associates to a collection $\varepsilon$ of positive numbers $\varepsilon_\alpha$, for $\alpha \in \Sigma$, the unique $K$-invariant Kähler form $\omega_\varepsilon$ whose value at the base point $o = J \subset K/J$ of $K/J$ is the $J$-invariant 2-form $\omega_\varepsilon \in \wedge^2 \mathfrak{t}^*$ given by

$$\omega_\varepsilon(x, x') = -\kappa(t, [x, x'])$$

for $x, x' \in \mathfrak{t}$. Here $t \in \mathfrak{t}$ is given in terms of $\varepsilon$ by $-\kappa(t, \sqrt{-1} h_\alpha) = \varepsilon_\alpha$ for $\alpha \in \Sigma$, and $-\kappa(t, \sqrt{-1} h_\alpha) = 0$ for $\alpha \in \mathcal{S} \setminus \Sigma$.

**Proof.** It is well known (see e.g. [Be, Proposition 8.83]) that the set of $K$-invariant Kähler forms on $K/J$ compatible with the canonical complex structure is in bijection with the set of vectors $t \in \mathfrak{t}$ such that $-\kappa(t, \sqrt{-1} h_\alpha) > 0$ for the vectors $\sqrt{-1} h_\alpha$ spanning the center $\mathfrak{z} = \bigoplus_{\alpha \in \Sigma} \mathbb{R} \sqrt{-1} h_\alpha$ of $J$, and $\kappa(t, \sqrt{-1} h_\alpha) = 0$ for the vectors $\sqrt{-1} h_\alpha$ in the semisimple part $[j, j]$ of $J$. In other words, the set of $K$-invariant Kähler forms on $K/J$, compatible with the canonical complex structure, are in one-to-one correspondence with the set of vectors $\varepsilon \in \mathbb{R}_{+}\Sigma^\vee$. Formula (4.9) follows for instance from [Be, Proposition 8.83]; actually, the stabiliser of the vector $t$ for the adjoint action of $K$ on $\mathfrak{t}$ is $J$, so the adjoint orbit $K \cdot t$ is isomorphic to $K/J$, and by identifying $t$ with $t^\vee$ by means of the Killing form, $\omega_\varepsilon$, as given by (4.9), transforms into be the well-known Kirillov–Kostant symplectic form on the coadjoint orbits. \hfill \square

4.2.2. Statement of the main result. Let $\mathcal{S}$ and $\Sigma$ be defined as in §1.5.1, and let $\Delta_+(\mathfrak{t})$ be the set of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\mathcal{S}$ which are not roots of $\mathfrak{l}$ (cf. §3.4.1). Let $\{\lambda_\alpha | \alpha \in \mathcal{S}\}$ be the basis of fundamental weights of $\mathfrak{h}^*$, i.e. it is the dual basis of $\{h_\alpha | \alpha \in \mathcal{S}\}$ given in §4.2.1. Let $\varepsilon$ be a collection of positive real numbers $\varepsilon_\alpha$, for $\alpha \in \Sigma$.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be a collection of real parameters $\tau_\lambda$ and $\tau'_\lambda$ for each $\lambda \in Q_0$, related to $\varepsilon$ by

$$\tau'_\lambda = n_\lambda \tau_\lambda - n_\lambda \sum_{\alpha \in \Delta_+(\mathfrak{t})} \varepsilon_\alpha^{-1} \kappa(\lambda, \alpha^\vee), \quad \text{for } \lambda \in Q_0,$$

The following lemma, which we adapt to the notation used throughout this paper, is standard and can be found e.g. in [Be]. Let $\mathfrak{g}$ be a collection of positive real numbers $\varepsilon_\alpha$, for $\alpha \in \Sigma$, the unique $K$-invariant Kähler form $\omega_\varepsilon$ whose value at the base point $o = J \subset K/J$ of $K/J$ is the $J$-invariant 2-form $\omega_\varepsilon \in \wedge^2 \mathfrak{t}^*$ given by

$$\omega_\varepsilon(x, x') = -\kappa(t, [x, x'])$$

for $x, x' \in \mathfrak{t}$. Here $t \in \mathfrak{t}$ is given in terms of $\varepsilon$ by $-\kappa(t, \sqrt{-1} h_\alpha) = \varepsilon_\alpha$ for $\alpha \in \Sigma$, and $-\kappa(t, \sqrt{-1} h_\alpha) = 0$ for $\alpha \in \mathcal{S} \setminus \Sigma$.

**Proof.** It is well known (see e.g. [Be, Proposition 8.83]) that the set of $K$-invariant Kähler forms on $K/J$ compatible with the canonical complex structure is in bijection with the set of vectors $t \in \mathfrak{t}$ such that $-\kappa(t, \sqrt{-1} h_\alpha) > 0$ for the vectors $\sqrt{-1} h_\alpha$ spanning the center $\mathfrak{z} = \bigoplus_{\alpha \in \Sigma} \mathbb{R} \sqrt{-1} h_\alpha$ of $J$, and $\kappa(t, \sqrt{-1} h_\alpha) = 0$ for the vectors $\sqrt{-1} h_\alpha$ in the semisimple part $[j, j]$ of $J$. In other words, the set of $K$-invariant Kähler forms on $K/J$, compatible with the canonical complex structure, are in one-to-one correspondence with the set of vectors $\varepsilon \in \mathbb{R}_{+}\Sigma^\vee$. Formula (4.9) follows for instance from [Be, Proposition 8.83]; actually, the stabiliser of the vector $t$ for the adjoint action of $K$ on $\mathfrak{t}$ is $J$, so the adjoint orbit $K \cdot t$ is isomorphic to $K/J$, and by identifying $t$ with $t^\vee$ by means of the Killing form, $\omega_\varepsilon$, as given by (4.9), transforms into be the well-known Kirillov–Kostant symplectic form on the coadjoint orbits. \hfill \square

4.2.2. Statement of the main result. Let $\mathcal{S}$ and $\Sigma$ be defined as in §1.5.1, and let $\Delta_+(\mathfrak{t})$ be the set of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\mathcal{S}$ which are not roots of $\mathfrak{l}$ (cf. §3.4.1). Let $\{\lambda_\alpha | \alpha \in \mathcal{S}\}$ be the basis of fundamental weights of $\mathfrak{h}^*$, i.e. it is the dual basis of $\{h_\alpha | \alpha \in \mathcal{S}\}$ given in §4.2.1. Let $\varepsilon$ be a collection of positive real numbers $\varepsilon_\alpha$, for $\alpha \in \Sigma$.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be a collection of real parameters $\tau_\lambda$ and $\tau'_\lambda$ for each $\lambda \in Q_0$, related to $\varepsilon$ by

$$\tau'_\lambda = n_\lambda \tau_\lambda - n_\lambda \sum_{\alpha \in \Delta_+(\mathfrak{t})} \varepsilon_\alpha^{-1} \kappa(\lambda, \alpha^\vee), \quad \text{for } \lambda \in Q_0,$$
where \( n_\lambda = \dim \mathbb{C} M_\lambda \), and \( \varepsilon_\alpha \) is defined, for \( \alpha \not\in \Sigma \), by
\[
(4.11) \quad \varepsilon_\alpha := \sum_{\beta \in \Sigma} \varepsilon_\beta \kappa(\lambda\beta, \alpha^\vee).
\]

**Remark 4.12.** The numbers \( \varepsilon_\alpha \) in \((4.11)\) do not depend on the choice of \( \kappa \), i.e. any \( I \)-invariant metric \( \kappa \) on \( I \) gives the same \( \varepsilon_\alpha \). Actually, if we multiply the Killing form \( \kappa \) by a positive constant \( c > 0 \), then we obtain the same \( \varepsilon_\alpha \) in \((4.11)\) \((\text{for } \kappa \mapsto c\kappa \mapsto c^{-1}\alpha^\vee)\), and clearly this transformation can be made separately for the factors of the Killing form corresponding to the different simple factors of \( I \).

**Theorem 4.13.** Let \( \mathcal{F} \) be a \( G \)-equivariant holomorphic vector bundle on \( X \times G/P \). Let \( \mathcal{F} \) be the \( G \)-equivariant holomorphic filtration associated to \( \mathcal{F} \) and \( \mathcal{R} = (\mathcal{E}, \phi) \) be its corresponding holomorphic \((Q, K)\)-bundle on \( X \), where \((Q, K)\) is the quiver with relations associated to \( P \). Then \( \mathcal{F} \) has a \( K \)-invariant \( \tau \)-Hermite–Einstein metric, with respect to the Kähler form \( \rho \omega + q^* \omega_\epsilon \), if and only if the vector bundles \( \mathcal{E}_\lambda \) in \( \mathcal{R} \) admit hermitian metrics \( k_\lambda \) on \( \mathcal{E}_\lambda \), for each \( \lambda \in Q_0 \) with \( \mathcal{E}_\lambda \neq 0 \), satisfying
\[
(4.14) \quad \sqrt{-1} n_\lambda \mathcal{A}_{F_{k_\lambda}} + \sum_{a \in h^{-1}(\lambda)} \phi_a \circ \phi_a^* - \sum_{a \in r^{-1}(\lambda)} \phi_a^* \circ \phi_a = \tau_\lambda \id_{\mathcal{E}_\lambda},
\]
where \( F_{k_\lambda} \) is the curvature of the Chern connection \( A_{k_\lambda} \) associated to the metric \( k_\lambda \) on the holomorphic vector bundle \( \mathcal{E}_\lambda \), for each \( \lambda \in Q_0 \) with \( \mathcal{E}_\lambda \neq 0 \), and \( n_\lambda = \dim \mathbb{C} (M_\lambda) \) is the multiplicity of the irreducible representation \( M_\lambda \), for each \( \lambda \in Q_0 \).

To prove this theorem, we need some preliminaries about connections on irreducible homogeneous vector bundles.

### 4.2.3. Hermite–Einstein connections on irreducible homogeneous vector bundles

In this subsection we evaluate the slope \( \mu_\epsilon(\mathcal{O}_\lambda) \) of any irreducible homogeneous vector bundle, with respect to the invariant Kähler form \( \omega_\epsilon \) defined in Lemma 4.18. To do this, we reprove a well-known fact that these bundles are Hermite–Einstein (a result originally due to Kobayashi [Ko]), hence stable (as originally proved by Ramanan [Ra]; see also Umemura [U]). Let \( A'_{\lambda} \) be the \( K \)-invariant connection induced by the connection one-form \( A' \) on \( H_\lambda \) \((\text{cf. } 3.2.1, 3.5, 3.4.4)\), which is unitary with respect to the unique (up to a constant) \( K \)-invariant hermitian metric \( k'_{\lambda} \) on \( H_\lambda \). Let \( \text{End}(H_\lambda, k'_{\lambda}) \) be the vector bundle of anti-hermitian endomorphisms of \((H_\lambda, k'_{\lambda})\). Contraction with the Kähler form \( \omega_\epsilon \) \((\text{cf. } 4.18)\) is denoted by \( \Lambda_{\epsilon} \).

**Lemma 4.15.** The connection \( A'_{\lambda} \) is the unique \( K \)-invariant connection on \( H_\lambda \). It defines the unique \( G \)-invariant holomorphic structure \( \partial_{H_\lambda} \) on \( H_\lambda \), and it is unitary with respect to the unique (up to scale) \( K \)-invariant hermitian metric \( k'_{\lambda} \) on \( H_\lambda \). Moreover, \( A'_{\lambda} \) is Hermite–Einstein with respect to the Kähler form \( \omega_\epsilon \) \((i.e. \sqrt{-1} \Lambda_{\epsilon} F_{A'_{\lambda}} = \mu_\epsilon(\mathcal{O}_\lambda) \id)\), and the slope of \( \mathcal{O}_\lambda \) with respect to the Kähler form \( \omega_\epsilon \) is
\[
(4.16) \quad \mu_\epsilon(\mathcal{O}_\lambda) = \sum_{\alpha \in \Delta^+(\mathfrak{t})} \varepsilon_\alpha^{-1} \kappa(\lambda, \alpha^\vee).
\]

Here, \( \varepsilon_\alpha \) is defined by \((4.11)\) for \( \alpha \not\in \Sigma \).

**Proof.** By construction, \( A'_{\lambda} \) is \( K \)-invariant and unitary. Its curvature \( F_{A'_{\lambda}} \in \Omega^{1,1}(\text{End}(H_\lambda, k'_{\lambda})) \) is given, in terms of the isomorphic element \( F'_{A_{\lambda}} \in C^{1,1}(\text{End}(M_\lambda), k'_{\lambda}) \), by \( F'_{A_{\lambda}}(x, x') = -\rho_\lambda(A'(x, x')) \), where \( \rho_\lambda : K \to \text{U}(M_\lambda) \) is the unitary representation associated to the dominant integral weight \( \lambda \). Thus, \( A'_{\lambda} \) defines a holomorphic structure on \( H_\lambda \). It is easy to see that \( A'_{\lambda} \) is the only \( K \)-invariant connection on \( H_\lambda \), because any other would be \( dA'_{\lambda} + \theta \) with \( \theta \in \Omega^1(\text{End}(H_\lambda)) \text{End}(H_\lambda) \cong (\mathfrak{t}_{\mathbb{C}} \otimes \text{End}(M_\lambda))^J = A_{\lambda\lambda} + A'_{\lambda\lambda} = 0 \) \((\text{cf. } 3.7 \text{ and } \mathfrak{t}_{\mathbb{C}} = \hat{u} \oplus u)\). Analogously one proves...
that \( \bar{\partial}_{\lambda'} \) is the only \( G \)-invariant \( \partial \)-operator on \( H_\lambda \). Since \( \omega_\xi \) and \( F_{\lambda',\xi} \) are both \( K \)-equivariant, 
\( \sqrt{-1} \Lambda_\xi F_{\lambda',\xi} \) is \( K \)-equivariant as well, so by Schur’s lemma, \( \sqrt{-1} \Lambda_\xi F_{\lambda',\xi} = \mu_\xi(O_\lambda) \text{id} \) for some constant \( \mu_\xi(O_\lambda) \), which of course is the slope \( \mu_\xi(O_\lambda) \) of \( H_\lambda \) w.r.t. \( \omega_\xi \). To evaluate \( \mu_\xi(O_\lambda) \), first we compute (the complexification of) \( \omega_{\xi\epsilon} \in \wedge^2 t_\epsilon \). For every root \( \alpha \) of \( g \), let \( e_\alpha \) be the corresponding Chevalley generator, and for each pair of roots \( \alpha, \beta \), let \( N_{\alpha\beta} \in \mathbb{Z} \) be the coefficients defined by the commutation relations \([e_\alpha, e_\beta] = N_{\alpha\beta}e_{\alpha+\beta} \) if \( \alpha + \beta \) is a root as well, and \( N_{\alpha\beta} = 0 \) otherwise. Note that \( e_\alpha = -e_{-\alpha} \) for \( \alpha \in \Delta_+(t) \). Let \( \alpha, \beta \) be two roots of \( t_\epsilon = u \oplus \bar{u} \); if \( \alpha \neq \beta \), then \( \omega_{\xi\epsilon}(e_\alpha, e_\beta) = -\kappa(t, N_{\alpha\beta}e_{\alpha+\beta}) = 0 \) while the center \( \mathfrak{z} \) is orthogonal to \( e_{\alpha+\beta} \), while if \( \alpha = \beta \), then \( [e_\alpha, e_{-\alpha}] = h_\alpha \) implies \( \omega_{\xi\epsilon}(e_\alpha, e_{\alpha}) = -\kappa(t, h_\alpha) \). To evaluate this number, first we expand \( \alpha^\vee = \sum_{\beta \in S} \kappa(\alpha^\vee, \beta)\beta^\vee \) and take into account that \(-\sqrt{-1}\beta^\vee(t) = -\kappa(t, \sqrt{-1}\beta)\) for any root \( \beta \), and that this is \( \epsilon_\beta \) if \( \beta \in \Sigma \) and zero if \( \beta \in S \setminus \Sigma \). Thus,
\[
\omega_{\xi\epsilon}(e_\alpha, e_{\alpha}) = -\sqrt{-1}\alpha^\vee(t) = -\sqrt{-1}\sum_{\beta \in S} \kappa(\alpha^\vee, \beta)\beta^\vee(t) = -\sqrt{-1}\sum_{\beta \in S} \kappa(\alpha^\vee, \beta)\epsilon_\beta = \epsilon_\alpha,
\]
where \( \epsilon_\alpha \) is as given in Equation (4.11) for \( \alpha \notin \Sigma \). Therefore
\[
\omega_{\xi\epsilon} = \sqrt{-1} \sum_{\alpha \in \Delta_+(t)} g_{\alpha\beta} \epsilon_\alpha \wedge e_\beta \quad \text{with} \quad g_{\alpha\beta} := \delta_{\alpha\beta} \epsilon_\alpha.
\]
If \( \alpha \in \Delta_+(t) \) then \([e_\alpha, e_{-\alpha}] = [e_\alpha, e_{-\alpha}] = -h_\alpha \in \mathfrak{h} \subset \mathfrak{l} \) so \( F'(e_\alpha, e_{-\alpha}) = A'(h_\alpha) = h_\alpha \). Therefore, contraction of \( \omega_{\xi\epsilon} \) with \( F' \in C^{1,1}(j) \) is
\[
\sqrt{-1} \Lambda_{\xi\epsilon} F' = \sum_{\alpha \in \Delta_+(t)} \sum_{\beta \in S} \bar{g}_{\alpha\beta} \epsilon_\alpha \wedge e_\beta = \sum_{\alpha \in \Delta_+(t)} \epsilon_{-\alpha}^{-1} h_\alpha,
\]
so \( \sqrt{-1} \Lambda_{\xi\epsilon} F''_\lambda \) is \( J \)-invariant and \( M_\lambda \) is irreducible, \( \sqrt{-1} \Lambda_{\xi\epsilon} F' = c_\lambda \text{id} \) for some number \( c_\lambda \) which can be computed by evaluating \( \rho_\lambda(h_\alpha) \) at a highest weight vector \( v^+ \) of \( M_\lambda \); \( \rho_\lambda(h_\alpha)v^+ = \lambda(h_\alpha)v^+ \), where \( \lambda(h_\alpha) = \kappa(\lambda, \alpha^\vee) \). Thus, \( \sqrt{-1} \Lambda_{\xi\epsilon} F''_\lambda = \left( \sum_{\alpha \in \Delta_+(t)} \epsilon_{-\alpha}^{-1} \kappa(\lambda, \alpha^\vee) \right) \text{id}_{M_\lambda} \) which implies Equation (4.16).

4.2.4. **Proof of Theorem 4.13**. The smooth vector bundle \( F \) underlying \( F \) has a \( K \)-equivariant decomposition \( (5.2) \). Let \( Q' \) be the subquiver of \( Q \) defined as in \( \S 3.3 \). Proposition 3.4 determines the \( \partial \)-operator \( \bar{\partial}_F \) associated to the holomorphic structure on \( F \) in terms of the \( \partial \)-operators \( \bar{\partial}_{E_\lambda} \) associated to the holomorphic structures on \( E_\lambda \), for \( \lambda \in Q'_0 \), and the maps \( \phi^{(i)}_{\mu} \). Any \( K \)-invariant hermitian metric on \( F \) has a \( K \)-invariant orthogonal decomposition \( k = \otimes_\lambda \bar{k}_\lambda \), where the sum is over \( \lambda \in Q'_0 \). \( \bar{k}_\lambda := p^*k_\lambda \otimes q^*k'_\lambda \) is a \( K \)-invariant metric on \( F_\lambda, k_\lambda \) is a metric on \( E_\lambda \), and \( k'_\lambda \) is defined as in \( \S 4.2.3 \). Let \( A \) (resp. \( A_\lambda \)) be the Chern connection associated to such a \( K \)-invariant hermitian metric \( k \) (resp. \( k_\lambda \)) on the holomorphic vector bundle \( F \) (resp. \( E_\lambda \)). From (5.5),
\[
\bar{\partial}_A = \sum_{\lambda \in Q'_0} \bar{\partial}_{A_\lambda} \circ \pi_\lambda + \sum_{\lambda, \mu \in Q'_0} \sum_{i=1}^{n_{\lambda\mu}} \beta_{\mu}(i) \circ \pi_\lambda, \quad \bar{\partial}_A = \sum_{\lambda \in Q'_0} \bar{\partial}_{A_\lambda} \circ \pi_\lambda - \sum_{\lambda, \mu \in Q'_0} \sum_{j=1}^{n_{\lambda\mu}} \beta_{\lambda\mu}(j) \circ \pi_\lambda.
\]
Therefore \( F_A = F_{A}^{1,1} = \bar{\partial}_A \circ \bar{\partial}_A + \bar{\partial}_A \circ \bar{\partial}_A \) is given by
\[
F_A = \sum_{\lambda} \bar{F}_{A_\lambda} \circ \pi_\lambda + \sum_{a} \bar{\partial}_{A_\lambda}(\beta_a) \circ \pi_{ta} + \sum_{a} \bar{\partial}_{A_\lambda}(\beta_a) \circ \pi_{ha} - \sum_{\lambda, \mu, \nu} (\beta_{\mu\nu}(i) \wedge \beta_{\nu\lambda}(j)) \circ \pi_\lambda.
\]
(4.18)
Here \( \bar{A}_a \) (resp. \( \bar{A}_a' \)) is the connection induced by \( A_\lambda \) and \( \bar{A}_a \) on the bundle \( \text{Hom}(F_{\lambda a}, F_\lambda) \) (resp. \( \text{Hom}(F_{h_a}, F_{\lambda a}) \)). Let \( A_\lambda \) and \( A'_a \) (resp. \( A_a' \) and \( A''_a \)) be the connections induced by \( A_\lambda, A_\lambda \) on
Combining (4.18) and the results for (i)-(v), we can evaluate $\sqrt{-1} \Lambda F_A$ in (4.18), we need:

- (i) $\sqrt{-1} \Lambda F_{\lambda} = p^* \sqrt{-1} \Lambda F_{\lambda} \otimes q^* \text{id} + p^* \text{id} \otimes q^* \sqrt{-1} \Lambda \varepsilon F_{A'}$;
- (ii) $\sqrt{-1} \Lambda \partial d_{\lambda} (\beta_a) = \sqrt{-1} \Lambda (p^* \partial d_{\lambda} (\phi_a) \otimes q^* \eta_a + p^* \phi_a \otimes q^* \partial d_{\lambda} (\eta_a))$;
- (iii) $\sqrt{-1} \Lambda \partial d_{\lambda} (\beta_a) = \sqrt{-1} \Lambda (p^* \partial d_{\lambda} (\phi_a) \otimes q^* \eta_a + p^* \phi_a \otimes q^* \partial d_{\lambda} (\eta_a))$;
- (iv) $\sqrt{-1} \Lambda (\beta_{\nu\mu}^i) + \beta_{\lambda\nu}^j = p^* (\phi_{\nu\mu}^i \circ \phi_{\lambda\nu}^j) \otimes q^* \sqrt{-1} \Lambda (\eta_{\nu\mu}^i) \wedge \eta_{\lambda\nu}^j$;
- (v) $\sqrt{-1} \Lambda (\beta_{\nu\mu}^i) + \beta_{\lambda\nu}^j = p^* (\phi_{\nu\mu}^i \circ \phi_{\lambda\nu}^j) \otimes q^* \sqrt{-1} \Lambda (\eta_{\nu\mu}^i) \wedge \eta_{\lambda\nu}^j$.

The expression (i) is given by Lemma 4.15: $\sqrt{-1} \Lambda \varepsilon F_{A'} = \mu_{\varepsilon} (O_\lambda) \text{id}$. Let us prove that (ii) is zero. First, $\sqrt{-1} \Lambda (p^* \partial d_{\lambda} (\phi_a) \otimes q^* \eta_a)$ is zero because the direct sum $T(X \times K/J) = p^* TX \otimes q^* T(K/J)$ is orthogonal with respect to the metric associated to the Kähler form $p^* \omega + q^* \omega_e$. To see that the second term is zero, we note that $\sqrt{-1} \Lambda \partial d_{\lambda} (\eta_a)$ corresponds to $\sqrt{-1} \Lambda \varepsilon \partial a$ by Lemma 3.14 and (4.17) gives

$$\sqrt{-1} \Lambda \varepsilon \partial a = \sum_{\alpha \in \Delta_\lambda (\Omega^1)} \varepsilon^{-1}_\alpha (\partial a) (e, e^{-\alpha}).$$

Let $\lambda = ta$ and $\mu = ha$, so $\partial a$ is in $B_{\mu \lambda}$ which is a subspace of $\wedge^2 \Omega^1 \otimes \text{Hom}(M_{\lambda}, M_{\mu})$ and this space is injected in $\wedge^2 \Omega^1 \otimes \text{Hom}(M_{\lambda}, M_{\mu})$ by the monomorphism induced by the projection $f = j \otimes \pi \rightarrow f/\bar{f} \otimes \pi$. Therefore, $\partial a$ is zero in $j$, so $(\partial a)(e, e^{-\alpha}) = -\partial a (e, e^{-\alpha}) = a([e, e^{-\alpha}]) = a(h_a) = 0$. Thus, (ii) is zero. Similarly, (iii) is zero. To evaluate (iv) (and (v)), we note that $\omega_{\varepsilon}$ is $K$-invariant and $\eta_{\nu\mu}^i \wedge \eta_{\lambda\nu}^j$ is $K$-invariant, so $\sqrt{-1} \Lambda (\eta_{\nu\mu}^i) \wedge \eta_{\lambda\nu}^j$ is $K$-invariant as well. By Schur’s lemma, it is zero if $\lambda \neq \mu$. If $\lambda = \mu$, we choose the basis $\{a_{\nu\mu}^i | i = 1, \ldots, n_{\mu\lambda}\}$ of $A_{\mu\lambda}$ which is orthonormal with respect to the metric induced by the hermitian metric associated to $\omega_{\varepsilon}$ and the canonical complex structure, and by the $J$-invariant hermitian metrics $k_{\nu\mu}^i, l_{\nu\mu}^i$ on $M_{\lambda}$ and $M_{\mu}$, i.e. they are normalised so that $\sqrt{-1} \Lambda \varepsilon \text{tr}(a_{\nu\mu}^i \wedge a_{\nu\mu}^j) = \delta_{i,j}$ (so $\sqrt{-1} \Lambda \varepsilon \text{tr}(a_{\nu\mu}^i \wedge a_{\lambda\nu}^j) = -\delta_{i,j}$). By Schur’s lemma

$$\sqrt{-1} \Lambda \varepsilon (a_{\nu\mu}^i \wedge a_{\nu\mu}^j) = \frac{1}{n_{\lambda}} \delta_{i,j} \text{id}_{M_{\lambda}}, \quad \sqrt{-1} \Lambda \varepsilon (a_{\nu\mu}^i \wedge a_{\lambda\nu}^j) = \frac{1}{n_{\lambda}} \delta_{i,j} \text{id}_{M_{\lambda}};$$

(since $n_{\lambda} = \text{dim}_C (M_{\lambda})$). Therefore (iv) and (v) are given by

$$\sqrt{-1} \Lambda \varepsilon (\eta_{\nu\mu}^i \wedge \eta_{\lambda\nu}^j) = \frac{1}{n_{\lambda}} \delta_{i,j} \text{id}_{M_{\lambda}}, \quad \sqrt{-1} \Lambda \varepsilon (\eta_{\nu\mu}^i \wedge \eta_{\lambda\nu}^j) = \frac{1}{n_{\lambda}} \delta_{i,j} \text{id}_{M_{\lambda}}.$$

Combining (4.18) and the results for (i)-(v), we can evaluate $\sqrt{-1} \Lambda F_A$:

$$\sqrt{-1} \Lambda F_A = \sum_{\lambda} \left( p^* \left( \sqrt{-1} F_{\lambda} + \mu_{\varepsilon} (O_\lambda) \text{id}_{M_{\lambda}} - \frac{1}{n_{\lambda}} \sum_{\mu} \sum_{i} \phi_{\mu\lambda}^{(i)} \circ \phi_{\nu\mu}^{(i)} \right) \otimes q^* \text{id}_{H_{\lambda}} \right) \circ \pi_{\lambda}.$$

The relation between $\tau_{\lambda}$ and $\tau_{\lambda}^\prime$ given in (4.10) is $\tau_{\lambda}^\prime = n_{\lambda} (\tau_{\lambda} - \mu_{\varepsilon} (O_\lambda))$, due to (4.16). The theorem is now straightforward. □
4.3. Dimensional reduction and stability. In this subsection we introduce a notion of stability for quiver sheaves on $X$ and prove that this is precisely the stability criteria which appears by dimensional reduction for $G$-invariant stability of $G$-equivariant sheaf filtrations, by the correspondences in §4. We also compute the relations among the stability parameters for the equivariant filtration and the quiver sheaf, and the way that the $K$-invariant Kähler form on $K/J$ enters in these relations.

4.3.1. Stability for quiver bundles. Let $Q$ be the quiver associated to $P$, and let $\tau$ be collections of real numbers $\tau_\lambda$, for each $\lambda \in Q_0$. Let $\mathcal{R} = (\mathcal{E}, \phi)$ be a $Q$-sheaf on $X$. Let $n_\lambda = \dim_{\mathbb{C}}(M_\lambda)$, for each $\lambda \in Q_0$.

Definition 4.19. The $\tau$-degree and $\tau$-slope of $\mathcal{R}$ are

$$\deg_\tau(\mathcal{R}) = \sum_{\lambda \in Q_0} \left( n_\lambda \deg(\mathcal{E}_\lambda) - \tau_\lambda \rk(\mathcal{E}_\lambda) \right), \quad \mu_\tau(\mathcal{R}) = \frac{\deg_\tau(\mathcal{R})}{\sum_{\lambda \in Q_0} n_\lambda \rk(\mathcal{E}_\lambda)},$$

respectively. The $Q$-sheaf $\mathcal{R}$ is called $\tau$-(semi)stable if for all proper $Q$-subsheaves $\mathcal{R}'$ of $\mathcal{R}$, $\mu_\tau(\mathcal{R}') < (\leq) \mu_\tau(\mathcal{R})$. A $\tau$-polystable $Q$-sheaf is a direct sum of $\tau$-stable $Q$-sheaves, all of them with the same $\tau$-slope.

Remark 4.20. If $K$ is the set of relations of $Q$ and $\mathcal{R}$ is a $(Q, K)$-sheaf, then all its $Q$-subsheaves are $(Q, K)$-sheaves, so the $\tau$-polystable stability criteria does not depend on $K$.

Theorem 4.21. Let $\mathcal{F}$ be a $G$-equivariant sheaf on $X \times G/P$. Let $\mathcal{F}$ be its associated $G$-equivariant sheaf filtration, and $\mathcal{R} = (\mathcal{E}, \phi)$ be its corresponding $(Q, K)$-sheaf on $X$, where $(Q, K)$ is the quiver with relations associated to $P$. Let $Q'_0 = \{\lambda_0, \ldots, \lambda_m\}$ be the set of vertices $\lambda \in Q_0$ such that $E_\lambda \neq 0$, listed in ascending order as $\lambda_0 < \cdots < \lambda_m$. Let $\varepsilon$ be a collection of positive real numbers $\varepsilon_\alpha$, for each $\alpha \in \Sigma$, and let $\varepsilon_\alpha$ be defined, for $\alpha \in \Delta_+(\tau) \setminus \Sigma$, by (4.11). Let $\sigma = (\sigma_0, \ldots, \sigma_{m-1})$ with $\sigma_s > 0$ for each $0 \leq s \leq m - 1$, and let $\tau'_\lambda$, for each $\lambda \in Q'_0$, be given by

$$\tau'_\lambda = n_\lambda \sum_{s = 0}^{s-1} \sigma_{s'} - n_\lambda \sum_{\alpha \in \Delta_+(\tau)} \varepsilon^{-1}_\alpha \kappa(\lambda, \alpha^\vee),$$

for $0 \leq s \leq m - 1$, where $n_\lambda = \dim_{\mathbb{C}}(M_\lambda)$. Then $\mathcal{F}$ is $G$-invariantly $\sigma$-(semi)stable with respect to the Kähler form $p^*\omega + q^*\omega_\varepsilon$ if and only if $\mathcal{R}$ is $\tau'$-(semi)stable.

Proof. To simplify the notation, let us denote $\sigma_s$ by $\sigma_\lambda$ when $\lambda = \lambda_s \in Q'_0$. Using (4.16), we can write (4.22) as

$$\frac{\tau'_\lambda}{n_\lambda} = \sum_{\mu < \lambda} \sigma_\mu - \mu_\varepsilon(\mathcal{O}_\lambda),$$

for all $\lambda \in Q'_0$. Let now $\mathcal{F}'$, given by (2.24), be a $G$-invariant sheaf subfiltration of $\mathcal{F}$, and let $\mathcal{R}' = (\mathcal{E}', \phi')$ be the corresponding $(Q, K)$-subsheaf of $\mathcal{R}$. Then

$$\sum_{s=0}^{m-1} \sigma_\lambda \rk(\mathcal{F}'_s) = \sum_{\lambda} \sum_{\mu \leq \lambda} \sigma_\lambda \rk(p^*\mathcal{E}'_\mu \otimes q^*\mathcal{O}_\mu) = \sum_{\lambda} \sum_{\mu \leq \lambda} \sigma_\lambda \rk(\mathcal{E}'_\mu)$$

$$= - \sum_{\lambda} \sum_{\mu > \lambda} \sigma_\lambda \mu_\varepsilon(\mathcal{O}_\mu) + \sum_{\lambda} \sum_{\mu} \sigma_\lambda \mu_\varepsilon(\mathcal{E}'_\mu)$$

$$= - \sum_{\lambda} n_\lambda \left( \sum_{\mu < \lambda} \sigma_\mu \right) \rk(\mathcal{E}_\lambda) + \left( \sum_{\lambda} \sigma_\lambda \right) \left( \sum_{\lambda} n_\lambda \rk(\mathcal{E}_\lambda) \right),$$
since \( \text{rk}(\mathcal{O}_\lambda) = n_\lambda \) (sums are only over \( \lambda, \mu \) in \( Q_0 \)), while
\[
\deg(\mathcal{F}') = \sum_\lambda \deg(p^* \mathcal{E}_\lambda \otimes q^* \mathcal{O}_\lambda) = \sum_\lambda \left( \text{rk}(\mathcal{O}_\lambda) \deg(\mathcal{E}_\lambda) + \text{rk}(\mathcal{E}_\lambda) \deg_\epsilon(\mathcal{O}_\lambda) \right)
= \sum_\lambda \left( n_\lambda \deg(\mathcal{E}_\lambda) + n_\lambda \mu_\epsilon(\mathcal{O}_\lambda) \text{rk}(\mathcal{E}_\lambda) \right)
\]
and
\[
\text{rk}(\mathcal{F}') = \sum_\lambda \text{rk}(p^* \mathcal{E}_\lambda \otimes q^* \mathcal{O}_\lambda) = \sum_\lambda n_\lambda \text{rk}(\mathcal{E}_\lambda).
\]
Therefore
\[
\deg_\sigma(\mathcal{F}') = \sum_\lambda n_\lambda \left( \deg(\mathcal{E}_\lambda) - \sum_{\mu < \lambda} \sigma_\mu - \mu_\epsilon(\mathcal{O}_\lambda) \right) \text{rk}(\mathcal{E}_\lambda) + \left( \sum_\lambda \sigma_\lambda \right) \left( \sum_\lambda n_\lambda \text{rk}(\mathcal{E}_\lambda) \right)
= \sum_\lambda \left( n_\lambda \deg(\mathcal{E}_\lambda) - \tau'_\lambda \text{rk}(\mathcal{E}_\lambda) \right) + \left( \sum_\lambda \sigma_\lambda \right) \left( \sum_\lambda n_\lambda \text{rk}(\mathcal{E}_\lambda) \right),
\]
so
\[
\mu_\sigma(\mathcal{F}') = \frac{\deg_\sigma(\mathcal{F}')}{\sum_\lambda n_\lambda \text{rk}(\mathcal{E}_\lambda)} = \mu_\tau'(\mathcal{R}') + \sum_\lambda \sigma_\lambda,
\]
and we are done. \( \square \)

4.4. **A Hitchin–Kobayashi correspondence for quiver bundles.** Combining Theorems 4.7, 4.13 and 4.21, we obtain as a corollary a Hitchin–Kobayashi correspondence for the quiver bundles that arise from dimensional reduction. More precisely.

**Theorem 4.24.** Let \((Q, \mathcal{K})\) be the quiver with relations associated to \( P \). Let \( \mathcal{R} = (\mathcal{E}, \phi) \) be a holomorphic \((Q, \mathcal{K})\)-bundle on \( X \) corresponding to a \( G \)-equivariant holomorphic vector bundle on \( X \times G/P \). Let \( \tau' \) be a collection of real numbers \( \tau'_\lambda \), for each \( \lambda \in Q_0 \). Then the vector bundles \( \mathcal{E}_\lambda \) in \( \mathcal{R} \) admit hermitian metrics \( k_\lambda \) on \( \mathcal{E}_\lambda \), for each \( \lambda \in Q_0 \) with \( \mathcal{E}_\lambda \neq 0 \), satisfying
\[
\sqrt{-1} n_\lambda \Lambda F_{k_\lambda} + \sum_{a \in h^{-1}(\lambda)} \phi_a \circ \phi_a^* - \sum_{a \in t^{-1}(\lambda)} \phi_{a^*} \circ \phi_a = \tau'_\lambda \text{id}_{\mathcal{E}_\lambda},
\]
if and only if \( \mathcal{R} = (\mathcal{E}, \phi) \) is \( \tau' \)-polystable.

**Proof.** First, by Theorem 4.13, the bundles \( \mathcal{E}_\lambda \) have hermitian metrics satisfying (4.25) if and only if the corresponding \( G \)-equivariant holomorphic filtration \( \mathcal{F} \) on \( X \times G/P \) has a \( G \)-invariant \( \tau \)-Hermite–Einstein metric, where \( \tau \) is a collection of parameters \( \tau_\lambda \) related to \( \tau' \) by (4.10), or equivalently, by
\[
\frac{\tau'_\lambda}{n_\lambda} = \tau_\lambda - \mu_\epsilon(\mathcal{O}_\lambda).
\]
Similarly, by Theorem 4.21, the quiver bundle \( \mathcal{R} \) is \( \tau' \)-stable if and only if \( \mathcal{F} \) is \( G \)-invariantly \( \sigma \)-stable, where \( \sigma = (\sigma_0, \ldots, \sigma_{n-1}) \) is related to \( \tau' \) by (4.22), or equivalently, by (4.23) (we are using the notation of the proof of Theorem 4.21). By Theorem 4.7, we only have to check that \( \sigma_\lambda = \tau_{\lambda+1} - \tau_\lambda \).

But this follows from the previous two equations (4.26) and (4.23):
\[
\sigma_\lambda = \sum_{\mu < \lambda} \sigma_\mu - \sum_{\mu < \lambda} \sigma_\mu = \left( \frac{\tau'_{\lambda+1}}{n_{\lambda+1}} + \mu_\epsilon(\mathcal{O}_{\lambda+1}) \right) - \left( \frac{\tau'_{\lambda}}{n_{\lambda}} + \mu_\epsilon(\mathcal{O}_{\lambda}) \right) = \tau_{\lambda+1} - \tau_\lambda.
\]
\( \square \)
Remark 4.27. It is clear that equations (4.25) make also sense for a \((Q, K)\)-bundle, where \((Q, K)\) is not necessarily associated to a parabolic group \(P\). Moreover, the \(n_\lambda\) appearing in the equations could actually be arbitrary positive numbers. All these generalizations, not related \emph{a priori} to dimensional reduction, had been carried out in [AG2].

5. Examples of dimensional reduction

In §§1.6.2 and 1.6.3 we described explicitly the quiver when \(G/P\) is \(\mathbb{P}^1\) or \(\mathbb{P}^2\). We now apply Theorem 4.13 to obtain, in these examples, the dimensional reduction of the deformed \(\tau\)-Hermite–Einstein equation, for an equivariant holomorphic filtration \(\mathcal{F}\), on \(X \times G/P\), i.e. we evaluate the multiplicities \(n_\lambda\) and the stability parameters \(\tau'\), appearing in the quiver vortex equations, in terms of \(\tau\) and the parameters \(\epsilon\) parametrising the invariant Kähler form \(\omega_\epsilon\) on \(G/P\). In particular, we shall recover the vortex equations corresponding to holomorphic triples [BG, GC] and to their generalizations, the holomorphic chains [AGI], which are obtained by dimensional reduction when \(G/P = \mathbb{P}^1\).

5.1. Dimensional reduction for \(\mathbb{P}^2\). We begin with \(G/P = \mathbb{P}^2\). The dimensional reduction of the \(\tau\)-Hermite–Einstein equation for an equivariant holomorphic filtration on \(X \times \mathbb{P}^2\) gives

\[
\sqrt{-1} n_x \Lambda F_{k_x} + \phi_x^{(1)} \circ \phi_x^{(1)*} + \phi_x^{(2)} \circ \phi_x^{(2)*} - \phi_x^{(1)*} \circ \phi_x^{(1)} - \phi_x^{(2)*} \circ \phi_x^{(2)} = \tau_x' \text{id}, \quad \text{for } x \in Q_0,
\]

where we are using the notation in §1.6.3 and \(\phi_x^{(i)} := \phi_{a_x}\), for \(a = a_x^{(i)} \in Q_1\) and \(i = 1, 2\). The relations give \(\phi_{x-3\epsilon} \circ \phi_x^{(1)} = \phi_{x-3\epsilon}^{(1)} \circ \phi_x^{(2)}\) for \(x \in Q_0\). The multiplicities can be found e.g. in [FH, (15.17)]: \(n_\lambda := \dim_{\mathbb{C}}(M_\lambda) = 1 + \lambda_1 - \lambda_2\), or using the notation of §1.6.3

\[
n_x := 1 + \frac{1}{3}(x_1 - x_2)
\]

they take all possible positive integer values, since \((x_1 - x_2)/3 = \lambda_1 - \lambda_2 \geq 0\). Note that any arrow \(a_x^{(1)} : x \to y\) decreases \(n_y = \sigma_x - 1\), for \((y_1, y_2) = (x_1, x_2 + 3)\) (resp. \(a_x^{(2)} : x \to y\) increases \(n_y = \sigma_x + 1\), for \((y_1, y_2) = (x_1 + 3, x_2)\)). There is only one non-parabolic simple root, i.e. \(\Sigma = \{\alpha_2\}\) (cf. §1.6.3), so the family of invariant Kähler forms \(\omega_\epsilon\) on \(\mathbb{P}^2\) only depends on a positive real number \(\epsilon > 0\) (with the notation of Lemma 4.3, this is \(\epsilon_{\alpha_2}\)). According to Theorem 4.13, \(\tau'\) is given by \(\tau_x' = n_\lambda (\tau_\lambda + \mu_\epsilon(O_\lambda))\), where \(\mu_\epsilon(O_\lambda)\) is given by Lemma 4.11. To evaluate \(\mu_\epsilon(O_\lambda)\), we take into account that \(\Delta_+(\tau) = \{-\gamma_1, -\gamma_2\}\) (cf. §4.2.2, so (4.11) gives \(\gamma_k = \epsilon \gamma_k = \epsilon \kappa(\lambda_{\alpha_2}, -\gamma_k)\), and (4.16) gives

\[
\mu_\epsilon(O_\lambda) = \sum_{k=1}^{2} \epsilon_k^{-1} \kappa(\lambda, -\gamma_k) = \epsilon^{-1} \sum_{k=1}^{2} \kappa(\lambda, -\gamma_k) = \epsilon^{-1} \sum_{k=1}^{2} \kappa(\lambda, -\gamma_k).
\]

The Killing form is a multiple of \(\kappa(\lambda, \lambda') = \sum_{i=1}^{3} \lambda_i \lambda'_i - \frac{1}{3} \sum_{j=1}^{3} \lambda_i \lambda'_j\), for \(\lambda = \sum_{i=1}^{3} \lambda_i L_i\), \(\lambda' = \sum_{i=1}^{3} \lambda'_i L_i\) (cf. e.g. [FH, (15.2)]). Now, \(\gamma_k = L_k - L_3\) and \(\lambda_{\alpha_2} = -L_3\), so

\[
\mu_\epsilon(O_\lambda) = \frac{\lambda_1 + \lambda_2}{\epsilon} = -\frac{x_1 + x_2}{\epsilon}.
\]

Thus, the \(\tau'\)-parameters are related to the \(\tau\)-parameters appearing in the \(\tau\)-Hermite–Einstein equation for equivariant holomorphic filtrations by

\[
\tau_x' = n_\lambda (\tau_\lambda + \mu_\epsilon(O_\lambda)) = \left(1 + \frac{1}{3}(x_1 - x_2)\right) \frac{\tau_\lambda - (x_1 + x_2)}{\epsilon}, \quad \text{for } x \in Q_0.
\]
5.2. Dimensional reduction for \((\mathbb{P}^1)^N\). Let us now consider \(G/P = (\mathbb{P}^1)^N\). We make use of the results and notation of \([16, 2]\). The Levi subgroup \(L \cong (\mathbb{C}^*)^N\) is abelian, so its irreducible representations are one-dimensional, i.e. \(n_\lambda := \text{dim}_\mathbb{C}(M_\lambda) = 1\), for \(\lambda \in Q_0\). Thus, the dimensional reduction of the \(\tau\)-Hermite–Einstein equation for an equivariant holomorphic filtration on \(X \times (\mathbb{P}^1)^N\) gives

\[
\sqrt{-1} \Lambda F_{k_\lambda} + \sum_{i=1}^N \phi^{(i)}_{\lambda - 2L_i} \circ \phi^{(i)}_{\lambda - 2L_i} - \sum_{i=1}^N \phi^{(i)}_{\lambda} \circ \phi^{(i)}_{\lambda} = \tau'_\lambda \text{id}_{\xi_\lambda}, \quad \text{for } \lambda \in Q_0.
\]

The relations lead to \(\phi^{(j)}_{\lambda - 2L_i} \circ \phi^{(i)}_{\lambda} = \phi^{(i)}_{\lambda - 2L_i} \circ \phi^{(j)}_{\lambda}\). All the positive roots \(\alpha_i = 2L_i, 1 \leq i \leq N\), are simple, so the invariant Kähler form \(\omega_\epsilon\) on \((\mathbb{P}^1)^N\) depends on \(N\) parameters \(\epsilon_i > 0, 1 \leq i \leq N\). Since the \(\epsilon\)-slope of \(O_\lambda\) is \(\mu_\epsilon(O_\lambda) = \sum_{i=1}^N \lambda_i/\epsilon_i\), for \(\lambda = \sum_{i=1}^N \lambda_i L_i \in Q_0\), \((4.10)\) and \((4.22)\) give

\[
\tau'_\lambda = \tau_\lambda - \sum_{i=1}^N \frac{\lambda_i}{\epsilon_i}, \quad \tau'_{s,i} = \sum_{s=1}^{s-1} \sigma_{s} - \sum_{i=2}^N \frac{\lambda_{s,i}}{\epsilon_i}, \quad \text{for } \lambda \in \mathbb{Z}^N, \text{ and } 0 \leq s \leq m.
\]

Here \(\lambda_0 < \lambda_1 \cdots < \lambda_m\) are the weights appearing in the filtration \((2.7)\) with components \(\lambda_{s,i} \in \mathbb{Z}\) given by \(\lambda_s = \sum_{i=1}^N \lambda_{s,i} L_i\). The arrow \(a^{(i)}_{\lambda}\) takes \(\lambda_i\) into \(\lambda_i - 2L_i\), so if \(F\) is indecomposable, then necessarily \(\lambda_{s+1,i} - \lambda_{s,i} = 2\) for some \(1 \leq i \leq N\). In particular, we can apply the preceding discussion to the complex projective line \(G/P = \mathbb{P}^1\): the quiver has two connected components \(Q^{(b)}\), for \(h = 0, 1\), each one isomorphic to the quiver with vertex set \(\mathbb{Z}\) (the weight of \(i \in \mathbb{Z}\) being \(\lambda = 2i + h\)), and arrows \(a_i : i \rightarrow i - 1\), for \(i \in \mathbb{Z}\). The process of dimensional reduction for any equivariant bundle on \(X \times \mathbb{P}^1\) was already studied in \([AG1]\). Each indecomposable holomorphic quiver bundle on \(X\) is given by a sequence of weights \(\lambda_0, \lambda_1, \ldots, \lambda_m \in Q_0^{(h)}\), with \(\lambda_i - \lambda_{i-1} = 2\), and by a sequence of morphisms \(\phi_i : E_i \rightarrow E_{i-1}\) among holomorphic vector bundles \(E_0, \ldots, E_m\), where \(E_i\) corresponds to the weight \(\lambda_i\). Such a quiver bundle, or holomorphic chain \([AG1]\), \(C = (E, \phi)\), is specified by a diagram

\[
\mathcal{C} : E_m \phi_m E_{m-1} \phi_{m-1} \cdots \phi_1 E_0.
\]

(A sheaf chain is analogously defined, by using coherent sheaves instead of holomorphic bundles (see \([AG1]\) for more details.).) By making a translation \(\lambda \mapsto \lambda - \lambda_0\), we can always assume that the weights are \(\lambda_i = 2i\), for \(0 \leq i \leq m\). The equivariant holomorphic filtration on \(X \times \mathbb{P}^1\) corresponding to \(\mathcal{C}\), by Theorem \([2, 3]\) and Proposition \([2, 6]\), is given by

\[
F_i/F_{i-1} \cong p^*E_i \otimes q^*O(2i), \quad 0 \leq i \leq m.
\]

Translating \(\lambda \mapsto \lambda - \lambda_0\) corresponds to twisting \(F\) by \(q^*O(-\lambda_0)\). The quiver vortex equations, or chain \(\tau\)-vortex equations \([AG1]\), for metrics \(k_i\) on the bundles \(E_i\), are (cf. \((5.3)\))

\[
\sqrt{-1} \Lambda F_{k_i} + \phi_i \circ \phi_i^* = \tau_0 \text{id}_{\xi_0},
\]

\[
\sqrt{-1} \Lambda F_{k_i} + \phi_{i+1} \circ \phi_{i+1}^* - \phi_i \circ \phi_i = \tau_i \text{id}_{\xi_i}, \quad (1 \leq i \leq m - 1),
\]

\[
\sqrt{-1} \Lambda F_{k_m} - \phi_m^* \circ \phi_m = \tau_m \text{id}_{\xi_m}.
\]

The story for stability follows similarly: a coherent sheaf chain \(\mathcal{C}\) is \(\tau'\)-(semi)stable if for all proper sheaf subchains \(\mathcal{C}' \subset \mathcal{C}\), \(\mu_{\tau'}(\mathcal{C}') < (\leq) \mu_{\tau'}(\mathcal{C})\), where \(\mu_{\tau'}(\mathcal{C}) = \deg_{\tau'}(\mathcal{C})/\sum_{i=0}^m \text{rk}(E_i)\), \(\deg_{\tau'}(\mathcal{C}) = \sum_{i=0}^m \deg(E_i) - \sum_{i=0}^m \tau_i^* \text{rk}(E_i)\). The relation among \(\sigma, \tau, \tau'\) is given by \((5.4)\) for \(N = 1\). Thus, if the vertices of \(\mathcal{C}\) are \(\lambda_i = 2i\), for \(0 \leq i \leq 2m\), then \(\tau_i = \tau_i - \mu_\epsilon(O_\lambda) = \tau_i - \frac{1}{2}\), \(\sigma_i = \tau_{i+1} - \tau_i - \frac{2}{3}\). These results were already obtained in \([AG1]\) when \(\epsilon = 1\). The case \(m = 1\) has been deeply studied in \([3, 2, BC]\); then the sheaf filtration \((5.6)\) is just an invariant extension of equivariant holomorphic vector bundles on \(X \times \mathbb{P}^1\),

\[
0 \rightarrow p^*E_0 \rightarrow F \rightarrow p^*E_1 \otimes q^*O(2) \rightarrow 0,
\]
and the holomorphic chain \( \langle 5, 5 \rangle \) is a triple (cf. [BG]), i.e. a representation \( \phi : \mathcal{E}_1 \rightarrow \mathcal{E}_0 \) of the quiver \( \bullet \rightarrow \bullet \). Assume now that \( \alpha_0 = 0 \), i.e. \( \tau_0 = \tau_1 \). Then the \( \tau \)-Hermite–Einstein equation \((4.3)\), for a holomorphic filtration \( \mathcal{F} \), is the usual Hermite–Einstein equation for \( \mathcal{F} \), and the \( \sigma \)-stability condition is the usual Mumford–Takemoto stability condition for a (torsion free) coherent sheaf on \( X \times \mathbb{P}^1 \). So the parameter \( \varepsilon \), which was encoded in the invariant Kähler form \( p^* \omega + q^* \omega_\varepsilon \), allows us to get the moduli space of \((\tau_0', \tau_1')\)-stable triples on \( X \) as the set of fixed points of the moduli space of stable sheaves on \( X \times \mathbb{P}^1 \), under the induced action of \( SL(2, \mathbb{C}) \). The constraint in Remark \( 4.2(\text{c}) \) is 

\[
\deg(\mathcal{E}_0) + \deg(\mathcal{E}_1) = \tau_0' \text{rk}(\mathcal{E}_0) + \tau_1' \text{rk}(\mathcal{E}_1).
\]

Thus, the relation \( \tau_0' - \tau_1' = 2/\varepsilon \), among \( \tau_0', \tau_1', \) and \( \varepsilon \), turns out to be

\[
\frac{2}{\varepsilon} = \frac{\text{rk}(\mathcal{E}_0) + \text{rk}(\mathcal{E}_1) \tau_0 - (\deg(\mathcal{E}_0) + \deg(\mathcal{E}_1))}{\text{rk}(\mathcal{E}_1)},
\]

which is precisely as in [G2]. It provides the value of \( \varepsilon \) in terms of the stability parameter \( \tau_0' \). Thus, the mechanism of dimensional reduction, when applied to the well-known Hitchin–Kobayashi correspondence for the usual Hermite–Einstein equation, proved by Donaldson, Uhlenbeck and Yau, for equivariant bundles on \( X \times \mathbb{P}^1 \), gives a Hitchin–Kobayashi correspondence for the coupled vortex equations for holomorphic triples (cf. [BG]). However, when \( m > 1 \) in \( \langle 5, 5 \rangle \), the parameter \( \varepsilon \), encoded in the polarisation \( \omega_\varepsilon \), is not enough to obtain a Hitchin–Kobayashi correspondence for the chain \( \tau' \)-vortex equations by dimensional reduction of the usual Hermite–Einstein equation — we need the deformed \( \tau \)-Hermite–Einstein equation \((4.3)\). Of course, the same occurs for other flag varieties.

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