ON MATRICES OF ENDMORPHISMS OF ABELIAN VARIETIES

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Abstract. We study endomorphisms of abelian varieties and their action on the ℓ-adic Tate modules. We prove that for every endomorphism one may choose a basis of each \( \mathbb{Q}_\ell \)-Tate module such that the corresponding matrix has rational entries and does not depend on \( \ell \).

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1. Introduction

Let \( X \) be an abelian variety of positive dimension \( g \) over an algebraically closed field \( K \) of arbitrary characteristic. We write \( \text{End}(X) \) for the endomorphism ring of \( X \) and \( \text{End}^0(X) \) for the corresponding \( \mathbb{Q} \)-algebra

\[
\text{End}^0(X) := \text{End}(X) \otimes \mathbb{Q},
\]

which is a finite-dimensional semisimple algebra over \( \mathbb{Q} \). If \( n \) is any integer then we write \( n_X \in \text{End}(X) \) for multiplication by \( n \) in \( X \), which is an isogeny if \( n \neq 0 \). For example, \( 1_X \) is the identity automorphism of \( X \).

One may view \( \text{End}(X) = \text{End}(X) \otimes 1 \) as an order in \( \text{End}^0(X) \). Let \( \ell \neq \text{char}(K) \) be a prime and \( T_\ell(X) \) be the \( \ell \)-adic Tate module of \( X \), which is a free \( \mathbb{Z}_\ell \)-module of rank \( 2g \) \([5]\). We write \( V_\ell(X) \) for the corresponding \( \mathbb{Q}_\ell \)-vector space

\[
V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell
\]

of dimension \( 2g \).

By functoriality, there is the natural injective ring homomorphism \([5]\)

\[
\text{End}(X) \to \text{End}_{\mathbb{Z}_\ell}(T_\ell(X)), \; u \mapsto u_\ell
\]

that extends by \( \mathbb{Z}_\ell \)-linearity to the injective \( \mathbb{Z}_\ell \)-algebra homomorphism

\[
\text{End}(X) \otimes \mathbb{Z}_\ell \hookrightarrow \text{End}_{\mathbb{Z}_\ell}(T_\ell(X)), \; u \mapsto u_\ell
\]

and extends by \( \mathbb{Q}_\ell \)-linearity to the injective \( \mathbb{Q}_\ell \)-algebra homomorphism

\[
\text{End}^0(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \text{End}(X) \otimes \mathbb{Q}_\ell \hookrightarrow \text{End}_{\mathbb{Q}_\ell}(V_\ell(X)), \; u \mapsto u_\ell
\]

(see \([3, 4]\)).

If \( u \in \text{End}^0(X) \) then let us consider the monic degree \( 2g \) characteristic polynomial

\[
P_u(t) := \det(t\text{Id} - u_\ell, V_\ell(X)) \in \mathbb{Q}_\ell[t]
\]

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of \( u_\ell \in \End_{\mathbb{Q}_\ell}(V_\ell(X)) \). (Here \( \Id : V_\ell(X) \to V_\ell(X) \) is the identity map.) A classical result of Weil \[3\] asserts that if \( u \in \End(X) \) then \( \mathcal{P}_u(t) \) lies in \( \mathbb{Z}[t] \) and does not depend on the choice of \( \ell \). It follows readily that if \( u \in \End^0(X) \) then \( \mathcal{P}_u(t) \) lies in \( \mathbb{Q}[t] \) and does not depend on the choice of \( \ell \).

The aim of this note is to prove the following assertion.

**Theorem 1.1.** Let \( X \) be an abelian variety of positive dimension \( g \) over an algebraically closed field \( K \) of arbitrary characteristic. Let \( u \in \End^0(X) \). Then there exists a square matrix \( M(u) \) of size \( 2g \) with entries in \( \mathbb{Q} \) that enjoys the following property.

If \( \ell \) is any prime \( \neq \text{char}(K) \) then there is a basis of the \( 2g \)-dimensional \( \mathbb{Q}_\ell \)-vector space \( V_\ell(X) \) such that the corresponding matrix of \( u_\ell \in \End_{\mathbb{Q}_\ell}(V_\ell(X)) \) coincides with \( M(u) \).

**Remark 1.2.** In the case of characteristic 0 the assertion of Theorem 1.1 reduces to the case of \( K = \mathbb{C} \) where it is obvious; in addition, if \( u \in \End(X) \) then one may chose bases in the free \( \mathbb{Z}_\ell \)-modules \( T_\ell(X) \) in such a way that the corresponding matrices \( M(u) \) have entries in \( \mathbb{Z} \) and do not depend on the choice of \( \ell \) when.

**Examples 1.3.**

(i) Let \( E \) be a subfield of \( \End^0(X) \) that contains \( 1_X \). Then \( E \) is a number field; let \( d \) be its degree over \( \mathbb{Q} \). The natural \( \mathbb{Q}_\ell \)-algebra homomorphism

\[
E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to \End_{\mathbb{Q}_\ell}(V_\ell(X)), \quad u \mapsto u_\ell
\]

is injective \[5\] and provides \( V_\ell(X) \) with the structure of a faithful \( E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \)-module. It is known \[4\] that this module is free and its rank is

\[
h := \frac{2g}{d}
\]

(in particular, \( d \mid 2g \)). Let us choose an \( h \)-element basis

\[
\{z_1, \ldots, z_h\} \subset V_\ell(X)
\]

of the \( E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \)-module \( V_\ell(X) \). Choose also a \( d \)-element basis

\[
B_E = \{\alpha_1, \ldots, \alpha_d\} \subset E
\]

of the \( \mathbb{Q} \)-vector space \( E \). Then the \( d \)-element set

\[
B_{E,\ell} := \{\alpha_1 \otimes 1, \ldots, \alpha_d \otimes 1\} \subset E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell
\]

is a basis of the \( d \)-dimensional \( \mathbb{Q}_\ell \)-vector space \( E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \). It follows that the \( 2g (=dh) \)-element set

\[
B_{E,\ell,X} := \{(\alpha_i)_{\ell}z_j \mid 1 \leq i \leq d, \ 1 \leq j \leq h\} \subset V_\ell(X)
\]

is a basis of the \( \mathbb{Q}_\ell \)-vector space \( V_\ell(X) \). Now assume that

\[
u \in E = E \otimes \mathbb{Q} \subset E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell,
\]

and let \( M_0(u) \) be the matrix of the \( \mathbb{Q} \)-linear map

\[
\text{mult}_u : E \to E, \ w \mapsto uw \ \forall w \in E
\]

with respect to \( B_E \). Clearly, \( M_0(u) \) coincides with the matrix of the \( \mathbb{Q}_\ell \)-linear map

\[
E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, \ w \mapsto uw
\]

with respect to \( B_{E,\ell} \). It follows that the matrix \( M(u) \) of \( u_\ell \) wrt \( B_{E,\ell,X} \) is the block diagonal matrix, all whose diagonal entries coincide with \( M_0(u) \).
In particular, all the entries of $M(u)$ are rational numbers and $M(u)$ does not depend on the choice of $\ell$. Notice that we use the same basis $B_{E,\ell,X}$ for all $u \in E$.

(ii) Let $m$ be a positive integer and let us consider the abelian variety $Y = X^m$. Then $\text{End}^0(Y) = \text{Mat}_m(\text{End}^0(X))$ and there is the natural embedding

$$\text{Mat}_m(E) \subset \text{Mat}_m(\text{End}^0(X)) = \text{End}^0(Y).$$

We also have

$$V_\ell(Y) = \oplus_{i=1}^m V_\ell(X), \quad \text{End}_{Q_\ell}(V_\ell(Y)) = \text{Mat}_m(\text{End}_{Q_\ell}(V_\ell(X))).$$

The basis $B_{E,\ell,X}$ of $V_\ell(X)$ gives rise to the obvious basis $B_{E,\ell,X}^{(m)}$ of the $Q_\ell$-vector space $\oplus_{i=1}^m V_\ell(X) = V_\ell(Y)$ that enjoys the following properties. If

$$u = (u_{ij})_{i,j=1}^m \in \text{Mat}_m(E) \subset \text{Mat}_m(\text{End}^0(X)) = \text{End}^0(Y)$$

then the matrix of $u_\ell \in \text{End}_{Q_\ell}(V_\ell(Y))$ wrt $B_{E,\ell,X}^{(m)}$ coincides with the block matrix $(M(u_{ij}))_{i,j=1}^m$. In particular, all its entries lie in $Q$ and do not depend on the choice of $\ell$.

The paper is organized as follows. In Section 2 we discuss the plan of the proof of Theorem 1.1. In particular, we obtain that Theorem 1.1 follows from certain auxiliary assertions about endomorphism subalgebras of abelian varieties (Theorems 2.1 and 2.2) and about elements of finite-dimensional semisimple algebras in characteristic 0 (Theorem 2.4). The results about endomorphism subalgebras are proven in Section 3. The assertion about semisimple algebras is proven in Section 4.

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2. Endomorphism subalgebras of abelian varieties: statements

In the course of the proof of Theorem 1.1 we will use the following assertions.

**Theorem 2.1.** Let $D$ be a finite-dimensional $Q$-algebra with identity element $1_D$ and let $W$ be a positive-dimensional abelian variety over $K$ provided with a $Q$-algebra embedding $\tau : D \rightarrow \text{End}^0(W)$ that sends $1_D$ to $1_W$. Suppose that $r \geq 2$ is an integer and $D$ splits into a direct sum $D = \oplus_{i=1}^r D_i$ of $r$ nonzero finite-dimensional $Q$-algebras $D_i$. We will identify $D_i$’s with the corresponding two-sided ideals in $D$. Then for all $i = 1, \ldots, r$ there exist positive-dimensional abelian subvarieties $W_i \subset W$ and $Q$-algebra embeddings $\tau_i : D_i \rightarrow \text{End}^0(W_i)$ that send $1_{D_i}$ to $1_{W_i}$ and enjoy the following properties.

(i) The homomorphism of abelian varieties

$$S : \prod_{i=1}^r W_i \rightarrow W, \quad \{w_i\}_{i=1}^r \mapsto \sum_{i=1}^r w_i$$

is an isogeny.

(ii) For each $u = \sum_{i=1}^r u_i \in D$ with $u_i \in D_i \ \forall i$ we have

$$\{\tau_i(u_i)\}_{i=1}^r \in \oplus_{i=1}^r \text{End}^0(W_i) \subset \text{End}^0(\prod_{i=1}^r W_i), \quad S(\{\tau_i(u_i)\}_{i=1}^r) S^{-1} = \tau(u) \in \text{End}^0(W).$$
Theorem 2.2. Let $E$ be a number field, $m$ a positive integer, $\operatorname{Mat}_m(E)$ the matrix algebra of size $m$ over $E$, and $Z$ an abelian variety of positive dimension over $K$ endowed with a $\mathbb{Q}$-algebra embedding $\bar{\kappa} : \operatorname{Mat}_m(E) \hookrightarrow \operatorname{End}_0^0(Z)$ that sends the identity matrix $\operatorname{Id}_m \in \operatorname{Mat}_m(E)$ to $1_Z$.

Then there is a positive-dimensional abelian variety $X$ over $K$, a $\mathbb{Q}$-algebra embedding $\kappa : E \hookrightarrow \operatorname{End}_0^0(X)$ that sends $1$ to $1_X$, and an isogeny of abelian varieties $\psi : X^m \to Z$ that enjoy the following properties.

Let $\operatorname{Mat}_m(\kappa) : \operatorname{Mat}_m(E) \hookrightarrow \operatorname{Mat}_m(\operatorname{End}_0^0(X)) = \operatorname{End}_0^0(X^m)$ be the natural $\mathbb{Q}$-algebra embedding that sends a matrix $(a_{ij})_{i,j=1}^m \in \operatorname{Mat}_m(E)$ to

$$
(\kappa(a_{ij}))_{i,j=1}^m \in \operatorname{Mat}_m(\operatorname{End}_0^0(X)) = \operatorname{End}_0^0(X^m).
$$

Then

$$
(1) \quad \bar{\kappa}(u) = \psi \circ \operatorname{Mat}_m(\kappa)(u) \circ \psi^{-1} \quad \forall u \in \operatorname{Mat}_m(E).
$$

Remark 2.3. Notice that in the notation of Theorem 2.2 $\psi$ induces (by functoriality of Tate modules) the isomorphism

$$
\psi_{\ell} : V_\ell(X^m) \cong V_\ell(Z)
$$

of $\mathbb{Q}_{\ell}$-vector spaces. Thus $\psi_{\ell}(B_{E,\ell,X}^{(m)})$ is a basis of the $\mathbb{Q}_{\ell}$-vector space $V_\ell(Z)$. It follows from [1] combined with Example 2 that for each $u \in \operatorname{Mat}_m(E)$ there exists a square matrix $M(u)$ of size $2\dim(Z)$ with rational entries that enjoys the following property. For each prime $\ell \neq \text{char}(K)$ the matrix of $\bar{\kappa}(u)_\ell$ with respect to $\psi_{\ell}(B_{E,\ell,X}^{(m)})$ coincides with $M(u)$.

In light of Theorem 2.1, Theorem 2.2 and Remark 2.3, Theorem 1.1 is an immediate corollary of the following assertion applied to semisimple $\operatorname{End}_0^0(X)$.

Theorem 2.4. Let $D$ be a finite-dimensional semisimple algebra over $\mathbb{Q}$. Then every element of $D$ is contained in a subalgebra of $D$ with the same 1 that is isomorphic to a direct sum of matrix algebras over number fields.

We prove Theorem 2.4 in Section 4. Theorems 2.1 and 2.2 will be proven in Section 3.

3. Endomorphism subalgebras of abelian varieties: proofs

Throughout this section, $D$ be a finite-dimensional $\mathbb{Q}$-algebra with identity element $1_D$ and let $W$ be a positive-dimensional abelian variety over $K$ provided with a $\mathbb{Q}$-algebra embedding

$$
\tau : D \hookrightarrow \operatorname{End}_0^0(W)
$$

that sends $1_D$ to $1_W$.

Lemma 3.1. Let $u_1, u_2$ be two conjugate elements of $D$, i.e., there exists $s \in D^*$ such that

$$
u_2 = su_1s^{-1}.
$$

Let $N$ be a positive integer such that

$$
N\tau(u_1) = \tau(Nu_1), \quad N\tau(u_2) = \tau(Nu_2), \quad N\tau(s) = \tau(Ns) \in \operatorname{End}(X).
$$

Let us consider abelian subvarieties

$$
W_1 := \tau(Nu_1)(W) \subset W, \quad W_2 := \tau(Nu_2)(W) \subset W
$$
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of $W$. Then

$$\tau(Ns)(W_1) = W_2.$$  

In addition, the restriction

$$\tau(Ns) \big|_{W_1} : W_1 \to W_2$$

is an isogeny of abelian varieties.

Proof. Replacing $u_1, u_2, s$ by $Nu_1, Nu_2, Ns$ respectively, we may and will assume that

$$\tau(u_1), \tau(u_2), \tau(s) \in \text{End}(X), \ N = 1.$$  

Since $s$ is invertible in $D$, the endomorphism

$$\tau(s) : W \to W$$

is invertible in $\text{End}^0(W)$ and therefore is an isogeny. This means that

$$\tau(s)(W) = W.$$  

and there is a positive integer $r$ such that multiplication by $r$ kills $\ker(\tau(s))$. On the other hand, the equality $u_2 = su_1s^{-1}$ means that $u_2s = su_1$ and therefore

$$\tau(u_2)\tau(s) = \tau(s)\tau(u_1).$$

Combining it with (2) and the definition of $W_1$ and $W_2$, we obtain that

$$W_2 = \tau(u_2)(W) = \tau(u_2)\tau(s)(W) = \tau(s)\tau(u_1)(W) = \tau(s)W_1,$$

i.e.,

$$W_2 = \tau(s)W_1.$$  

This means that the restriction

$$\tau(s) \big|_{W_1} : W_1 \to W_2, \ w_1 \mapsto s(w_1)$$

is a surjective morphism of abelian varieties. Since its kernel $\ker(\tau(s) \big|_{W_1})$ is a group subscheme of $\ker(\tau(s))$, it is also killed by multiplication by $r$. This implies that $\tau(s) \big|_{W_1}$ is an isogeny. □

The following assertion contains Theorem 2.1.

Theorem 3.2. Suppose that $r \geq 2$ is an integer and $D$ splits into a direct sum

$$D = \bigoplus_{i=1}^{r} D_i$$

of $r$ nonzero finite-dimensional $\mathbb{Q}$-algebras $D_i$. We will identify $D_i$’s with the corresponding two-sided ideals in $D$. Let us consider the subrings

$$O_i := \{u_i \in D_i \subset D \mid \tau(u_i) \in \text{End}(W)\}$$

of $D_i$ ($1 \leq i \leq r$).

Let $e_i := 1_{D_i}$ be the identity element of $D_i$ viewed as a nonzero central idempotent in $D$. Let $N$ be a positive integer such that $Ne_i \in O_i$ for all $i$, i.e., all $\tau(Ne_i) = N\tau(e_1)$ lie in $\text{End}(W)$. (Such an $N$ always exists.) Let us consider abelian subvarieties $W_i := \tau(Ne_i)(W) \subset W$ of $W$.

Then:

(i) (a) The natural $\mathbb{Q}$-algebra homomorphisms

$$O_i \otimes \mathbb{Q} \to D_i, \ u_i \otimes r \mapsto r \cdot u_i$$

are isomorphisms for $i = 1, 2$. 
Remark 3.3.  

(i) The rings $\tau(\text{End}(W))$ are isomorphic, which proves (1).

(ii) The natural morphisms of abelian varieties

\[ S : \prod_{i=1}^{r} W_i \to W, \quad \{w_i\}_{i=1}^{r} \mapsto \sum_{i=1}^{r} w_i, \quad P : W \to \prod_{i=1}^{r} W_i, \quad w \mapsto \{\tau(\text{End}(W))\}_{i=1}^{r} \]

are isogenies such that

\[ P \circ S = N_W, \quad S \circ P = N_{\hat{W}} \text{ where } \hat{W} := \prod_{i=1}^{r} W_i. \]

In particular, $\sum_{i=1}^{r} W_i = \{\sum_{i=1}^{r} w_i \mid w_i \in W_i \forall i\}$ coincides with $W$.

(iii) For each $u_i \in O_i$ and $u_j \in O_j$ with $i \neq j$

\[ \tau(u_i)(W_i) \subset W_i, \quad \tau(u_j)(W_j) = \{0\}, \quad \tau(u_i)(W_i) = \{0\}, \quad \tau_j(u_j)(W_j) \subset W_j. \]

(iv) There exist $Q$-algebra embeddings

\[ \tau_i : D_i \to \text{End}^0(W_i) \]

that enjoy the following properties.

(c) $\tau_i(1_{D_i}) = 1_{W_i}$.

(d) If $u_i \in O_i \subset D_i$ then $\tau_i(u_i) \in \text{End}(W_i)$ and

\[ \tau_i(u_i)(w_i) = \tau(u_i)(w_i) \in W_i \forall w_i \in W_i. \]

(e) For each

\[ u = \sum_{i=1}^{r} u_i \in D \quad \text{with } u_i \in D_i \forall i \]

we have

\[ \{\tau_i(u_i)\}_{i=1}^{r} \in \otimes_{i=1}^{r} \text{End}^0(W_i) \subset \text{End}^0\left(\prod_{i=1}^{r} W_i\right) = \text{End}^0(\hat{W}) \]

and

\[ S \circ \left(\{\tau_i(u_i)\}_{i=1}^{r}\right) S^{-1} = \tau(u) \in \text{End}^0(W). \]

Remark 3.3.  

(i) The rings $O_i$ do not have to have the identity elements.

(ii) The abelian subvarieties $W_i$ do not depend on the choice of $N$.

Proof of Theorem 3.2. (i). Since $\text{End}(X)$ is a free $Z$-module of finite rank and $\tau$ is a ring embedding, $O_i$ is a also a free $Z$-module of finite rank that generates the finite-dimensional $Q$-vector space $D_i$. This implies that $O_i \otimes Q \to D_i$ is an isomorphism, which proves (1).

Clearly, \[ \sum_{i=1}^{r} e_i = 1_D, \quad e_i^2 = e_i \forall i, \quad e_i e_j = 0 \forall i \neq j. \]

This implies that

\[ \tau(\text{End}(W)) = N \cdot \tau(\text{End}(W)), \quad \sum_{i=1}^{r} \tau(\text{End}(W)) = 1_W, \]

\[ \sum_{i=1}^{r} e_i = 1_D, \quad e_i^2 = e_i \forall i, \quad e_i e_j = 0 \forall i \neq j. \]
This implies that
\[ \tau(\text{Ne}_i)\tau(\text{Ne}_j) = 0 \quad \forall i \neq j. \]

Let
\[ i, j \in \{1, \ldots, r\}, \quad w_i \in W_i, w_j \in W_j. \]
Then there exist
\[ \tilde{w}_i, \tilde{w}_j \in W \] such that
\[ w_i = \tau(\text{Ne}_i)(\tilde{w}_i), w_j = \tau(\text{Ne}_j)(\tilde{w}_j). \]
This implies that
\[ \tau(\text{Ne}_i)(w_i) = \tau(\text{Ne}_i)^2(\tilde{w}_i) = N\tau(\text{Ne}_i)(\tilde{w}_i) = Nw_i. \]
On the other hand, if \( i \neq j \) then
\[ \tau(\text{Ne}_i)(w_j) = \tau(\text{Ne}_i)\tau(\text{Ne}_j)(\tilde{w}_j) = 0 \in W \]
in light of (7).

(ii) Recall that for each \( w \in W \)
\[ \tau(\text{Ne}_i)(w) \in W_i \quad \forall i \in \{1, \ldots, r\}. \]
This implies that
\[ S \circ P(w) = S(\tau(\text{Ne}_1)(w), \tau(\text{Ne}_2)(w), \ldots, \tau(\text{Ne}_r)(w)) = \sum_{i=1}^{r} \tau(\text{Ne}_i)(w) = \tau(N \cdot 1_D)(w) = \tau(N \left( \sum_{i=1}^{r} e_i \right))w = Nw. \]
This proves that
\[ (8) \quad S \circ P = N_W. \]
On the other hand, let \( w_i \in W_i \quad \forall i \in \{1, \ldots, r\}. \) Thanks to (7).

\[ P \circ S(w_1, w_2, \ldots, w_r) = P \left( \sum_{i=1}^{r} w_i \right) = \left( \sum_{i=1}^{r} \tau(\text{Ne}_1)(w_i), \sum_{i=1}^{r} \tau(\text{Ne}_2)(w_i), \ldots, \sum_{i=1}^{r} \tau(\text{Ne}_r)(w_i) \right) = \]
\[ (\tau(Nw_1)(w_1), \tau(Nw_2)(w_2), \ldots, \tau(Nw_r)(w_r)) = (Nw_1, Nw_2, \ldots, Nw_r) = N \cdot (w_1, w_2, \ldots, w_r). \]
This proves that
\[ (9) \quad P \circ S = N_W. \]
Combining (8) and (9), we obtain that both \( S \) and \( P \) are isogenies. This proves (ii).

Proof of (iii). Let \( u_i \in O_i \subset D_i. \) Then
\[ u_i = e_i u_i = e_i u_i e_i. \]
This implies that
\[ \tau(Nu_i) = N\tau(u_i) = \tau(\text{Ne}_i)\tau(u_i) \in \text{End}(W) \]
and therefore
\[ \tau(Nu_i)(W) = \tau(\text{Ne}_i)\tau(u_i)(W) \subseteq \tau(\text{Ne}_i)(W) = W_i, \]
i.e.,
\[ N\tau(u_i)(W) \subseteq W_i. \]
Clearly, \( \tau(u_i)(W) \) is an abelian subvariety of \( W \) and therefore coincides with \( N\tau(u_i)(W) \).
This implies that
\[ \tau(u_i)(W) = N\tau(u_i)(W) \subseteq W_i. \]
In particular, 
\[ \tau(u_i)(W) \subset W_i \ \forall i. \]

On the other hand, if \( i \neq j \) then 
\[ 0 = u_i e_j, 0 W = \tau(Nu_i e_j) = \tau(u_i) \tau(N e_j) \]
and therefore 
\[ \tau(Nu_i)(W_j) = \{0\}. \]
Again, \( \tau(u_i)(W_j) \) is an abelian subvariety of \( W \) and therefore coincides with 
\[ N \tau(u_i)(W_j) = \{0\}. \]
Hence, \( \tau(u_i)(W_j) = \{0\} \). This ends the proof of (e).

(iv) Now (4) allows us to define the ring homomorphisms 
\[ \tau_i : O_i \to \text{End}(W_i), \ u_i \mapsto \{ w_i \mapsto \tau(u_i)(w_i) \in W_i \ \forall w_i \in W_i \} \ \forall u_i \in O_i. \]
It follows from already proven (i) that \( \tau_i \) extend to \( \mathbb{Q} \)-algebra homomorphisms 
\[ D_i = O_i \otimes \mathbb{Q} \to \text{End}(W_i) \otimes \mathbb{Q} = \text{End}^0(W_i), \]
which we continue to denote by \( \tau_i \). It follows from already proven (i) that \( \tau_i(N e_i) = N W_i \) and therefore 
\[ \tau_i(e_i) = \frac{1}{N} \tau_i(N e_i) = \frac{1}{N} N W_i = 1_{W_i}. \]
This proves (c). The property (d) follows from the very definition of \( \tau_i \). Let us prove that 
\[ \tau_i : D_i \to \text{End}^0(W_i) \]
is an embedding. Suppose that \( u_i \in D_i \) satisfies \( \tau_i(u_i) = 0 \). Replacing \( u_i \) by \( m u_i \) for sufficiently divisible positive integer \( m \), we may and will assume that \( u_i \in O_i \). Then 
\[ \{0\} = \tau_i(u_i)(W_i) = \tau_i(u_i)(W_i). \]
On the other hand, if \( i \neq j \) then \( \tau(u_i)(W_j) = 0 \). It follows that \( \tau(u_i) \) kills 
\[ W_i + \sum_{j \neq i} W_j = \sum_{j=1}^r W_j = W, \]
because the isogeny \( S \) is surjective. This implies that \( \tau(u_i) = 0_{W_i} \). Since \( \tau \) is injective, \( u_i = 0 \), i.e., \( \tau_i \) is an embedding.

Let us prove (e). Replacing 
\[ u = (u_1, \ldots, u_i, \ldots, u_r) = \sum_{i=1}^r u_i \in \bigoplus_{i=1}^r D_i = D \]
by \( m u = (m u_1, \ldots, m u_i, \ldots, m u_r) \) for sufficiently divisible positive integer \( m \), we may and will assume that all \( u_i \in O_i \). Let us check that in \( \text{Hom}(\prod_{i=1}^r W_i, W) = \text{Hom}(\bar{W}, W) \) we have 
\[ S \circ (\tau_1(u_1), \ldots, \tau_i(u_i), \ldots, \tau_r(u_r)) = \tau(u) \circ S. \]
So, let \( (w_1, \ldots, w_i, \ldots, w_r) \in \prod_{i=1}^r W_i \). Then 
\[ (\tau_1(u_1), \ldots, \tau_i(u_i), \ldots, \tau_r(u_r))(w_1, \ldots, w_i, \ldots, w_r) = (\tau_1(u_1)(w_1), \ldots, \tau_i(u_i)(w_i), \ldots, \tau_r(u_r)(w_r)) \in \prod_{i=1}^r W_i, \]
\[ S \circ ((\tau_1(u_1), \ldots, \tau_i(u_i), \ldots, \tau_r(u_r))(w_1, \ldots, w_i, \ldots, w_r)) = S (\tau(u_1)(w_1), \ldots, \tau(u_i)(w_i), \ldots, \tau(u_r)(w_r)) = \]
\[ \sum_{i=1}^{r} \tau(u_i)(w_i). \]

On the other hand, \( S(w_1, \ldots, w_i, \ldots, w_r) = \sum_{i=1}^{r} w_i; \) in addition, thanks to (iii),
\[ \tau(u_i)(w_j) = 0 \forall i \neq j. \]

This implies that
\[ \tau(u) \circ S(w_1, \ldots, w_i, \ldots, w_r) = \tau(u) \left( \sum_{i=1}^{r} w_i \right) = \tau(u) \left( \sum_{i=1}^{r} w_i \right) = \sum_{i=1}^{r} \tau(u_i)(w_i) = S \circ (\tau_1(u_1)w_1, \ldots, \tau_i(u_i)w_i, \ldots, \tau_r(u_r)w_r). \]

This proves (10). Multiplying both sides of (10) by \( S^{-1} \) from the right, we get the desired equality
\[ S \circ (\tau_1(u_1), \ldots, \tau_i(u_i), \ldots, \tau_r(u_r))S^{-1} = \tau(u) \quad \text{in End}^0(W). \]

\textbf{Proof of Theorem 2.2.} We write \( e_{ij} \in \text{Mat}_m(E) \) for the matrix, whose only nonzero entry is 1 at the intersection of \( i \)th row and \( j \)th column, \( i, j \in \{1, \ldots, m\} \).

We have
\[ e_{ii}^2 = e_{ii} \sum_{i=1}^{m} e_{ii} = \text{Id}_m, \quad e_{ii}e_{jj} = 0 \forall i \neq j. \]

In addition, if \( i \neq j \), then the monomial matrix
\[ s_{ij} = s_{ji} \in \text{GL}(m, \mathbb{Q}) \subset \text{Mat}_m(E) \]
attached to the transposition \((ij)\) satisfies
\[ s_{ij} = \text{Id}_m - (e_{ii} + e_{jj}) + (e_{ij} + e_{ji}) \in \text{GL}(m, \mathbb{Q}), \quad s_{ij}^2 = \text{Id}_m. \]

(11)
\[ s_{ij}e_{ii}s_{ij}^{-1} = e_{jj}. \]

There is a positive integer \( N \) such that
\[ N\tilde{\kappa}(e_{ij}) \in \text{End}(Z) \forall i, j = 1, \ldots, m. \]

Let us consider the nonzero abelian subvarieties
\[ Z_i := \tilde{\kappa}(Ne_{ii})(Z) \subset Z. \]

It follows from (11) combined with Lemma 3.3 that there are isogenies of abelian varieties
\[ P_{ij} := \tilde{\kappa}(N s_{ij}) |_{Z_i}: Z_i \to Z_j, \quad z_i \mapsto \tilde{\kappa}(N s_{ij})(z_i), \]

Since
\[ s_{ij}^2 = \text{Id}_m, \quad \tilde{\kappa}(s_{ij})^2 = 1_Z, \]

we get
\[ P_{ji} \circ P_{ij} = N^2_{Z_i}. \]

They give rise to the product-isogenies
\[ \mathbf{P} := \prod_{i=1}^{m} P_{i1}: Z_1^m \to \prod_{i=1}^{m} Z_i, \quad \mu := \prod_{i=1}^{m} P_{1i}: \prod_{i=1}^{m} Z_i \to Z_1^m. \]
such that
\[
\mu \circ P = N_{Z_i}^2, \ P \circ \mu = N_{Z_i}^2 \quad \text{where} \quad Z := \prod_{i=1}^m Z_i.
\]
Applying Theorem 3.2 to the subalgebra of diagonal matrices
\[
D = \bigoplus_{i=1}^m E \cdot e_{ii} \subset \text{Mat}_m(E),
\]
and
\[
r = m, D_i = E \cdot e_{ii}, \ e_i := e_{ii}
\]
and
\[
W := Z, \ \tau : D \hookrightarrow \text{End}^0(Z), \ u \mapsto \tilde{\kappa}(u),
\]
we obtain that the morphism of abelian varieties
\[
(12) \quad S : Z = \prod_{i=1}^m Z_i \rightarrow Z, \ \{z_i\}_{i=1}^m \mapsto \sum_{i=1}^r z_i
\]
is an isogeny. In addition, we get the \(\mathbb{Q}\)-algebra embeddings
\[
\tau_i : E \cdot e_{ii} = D_i \hookrightarrow \text{End}^0(Z_i)
\]
that send \(e_{ii}\) to \(1_{Z_i}\) and such that for every collection \(u_i \in D_i \ (1 \leq i \leq m)\)
\[
\sum_{i=1}^r \tau_i(u_i) \in \oplus_{i=1}^m \text{End}^0(Z_i) \subset \text{End}^0 \left( \prod_{i=1}^m Z_i \right) = \text{End}^0(Z),
\]
\[
S \left( \sum_{i=1}^m \tau_i(u_i) \right) S^{-1} = \tilde{\kappa} \left( \sum_{i=1}^m u_i \right) \in \text{End}^0(Z).
\]
This implies that
\[
S^{-1} \circ \tilde{\kappa} \left( \sum_{i=1}^m u_i \right) \circ S = \sum_{i=1}^m \tau_i(u_i) \in \oplus_{i=1}^m \text{End}^0(Z_i) \subset \text{End}^0 \left( \prod_{i=1}^m Z_i \right).
\]
In particular, for each \(j \in \{1, \ldots, m\}\)
\[
S^{-1} \circ \tilde{\kappa}(e_{jj}) \circ S = \tau_j(e_{jj}) = 1_{Z_j} \in \text{End}^0(Z_j) \subset \oplus_{i=1}^m \text{End}^0(Z_i) \subset \text{End}^0 \left( \prod_{i=1}^m Z_i \right).
\]
Let us put
\[
X := Z_1, \ \psi := P \circ S : X^m = Z_1^m \rightarrow \prod_{i=1}^m Z_i \rightarrow Z.
\]
In order to define \(\kappa\) let us consider
\[
O := \{u \in E \mid \tilde{\kappa}(u) \in \text{End}(Z)\}.
\]
Clearly, \(O\) is an order in \(E\) and the natural \(\mathbb{Q}\)-algebra homomorphism
\[
O \otimes \mathbb{Q} \rightarrow E, \ u \otimes a \mapsto au
\]
is an isomorphism. In addition, for each nonzero \(u \in O\) the selfmap of \(Z\)
\[
\tilde{\kappa}(u) : Z \rightarrow Z
\]
is an isogeny. Since \(E\) is the center of \(\text{Mat}_m(E)\), every \(u \in O\) commutes with all \(e_{ij}\) and \(s_{ij}\). In particular, for all \(u \in O\)
\[
\tilde{\kappa}(u)(Z_i) = \tilde{\kappa}(u)\tilde{\kappa}(Ne_{ii})(Z) = \tilde{\kappa}(uNe_{ii})(Z) = \tilde{\kappa}(Ne_{ii}u)(Z) \subset \tilde{\kappa}(Ne_{ii})(Z) = Z_i
\]
and the diagrams

$$
\begin{array}{ccc}
Z_1 & \xrightarrow{\tau_i(u)} & Z_1 \\
\downarrow & & \downarrow \\
Z_i & \xrightarrow{\tau_i(u)} & Z_i
\end{array}
$$

(13)

are commutative. This gives rise to the ring homomorphisms

$$
\tau_i : O \rightarrow \text{End}(Z_i), \ u \mapsto \{ z_i \mapsto \tilde{\kappa}(u)(z_i) \ \forall z_i \in Z_i \} \ \forall u \in O,
$$

which are obviously injective, and extend them by $\mathbb{Q}$-linearity to the injective $\mathbb{Q}$-algebra homomorphisms $E \rightarrow \text{End}^0(Z_i)$, which we continue to denote by $\tau_i$. Let us put

$$
\psi := \tau_1 : E \hookrightarrow \text{End}^0(X = Z_1).
$$

4. Linear Algebra

**Definition 4.1.** Let $k$ be a field and $C$ be a finite-dimensional $k$-algebra with identity element $1_C$. We say that $C$ is splittable over $k$ if it is isomorphic to a direct sum of matrix algebras over fields that are finite algebraic separable extension of $k$.

**Examples 4.2.**

(i) If $C$ is a field that is a finite algebraic separable extension of $k$ then it is splittable over $k$.

(ii) If $C_1$ and $C_2$ are splittable over $k$ then their direct sum $C_1 \oplus C_2$ is also splittable over $k$.

(iii) If $C$ is splittable over $k$ then for all positive integers $d$ the matrix algebra $\text{Mat}_d(C)$ is also splittable over $k$.

(iv) Let $L/k$ be a finite algebraic separable field extension of $k$. If $C$ is splittable over $k$ then

$$
C_L = C \otimes_k L
$$

is splittable over $k$ and over $L$.

(iv) Suppose that $E$ is an overfield of $k$ and $C$ is a a finite-dimensional splittable $E$-algebra. If $L/k$ is a finite separable field extension then $C$ is splittable over $k$ as well.

Clearly, Theorem 2.4 is a special case of the following assertion.

**Theorem 4.3.** Let $k$ be a field of characteristic zero, $A$ a nonzero semisimple finite-dimensional $K$-algebra with identity element $1_A$, and $f$ an element of $A$. Then there exists a $k$-subalgebra $C$ of $A$ that contains $1_A$ and $f$, and is splittable over $k$.

We will need the following lemma that will be proven at the end of this section.

**Lemma 4.4.** Let $A$ be a nonzero finite-dimensional algebra over a field $k$ with identity element $1_A$. Let $V$ be a nonzero finite-dimensional $k$-vector space that is a faithful semisimple $A$-module. Assume additionally that every simple $A$-submodule $M$ of $V$ is absolutely simple, i.e., the centralizer $\text{End}_A(M)$ of $A$ in $\text{End}_k(M)$ coincides with $k \cdot 1_M$ where $1_M : M \rightarrow M$ is the identity map.

Then $A$ is isomorphic to a direct sum of matrix algebras over $k$. 


Proof of Theorem 4.3. Induction by $\dim_k(A)$.

**Step 0.** If $\dim_k(A) = 1$ then $A = k \cdot 1_A \cong k$ is obviously splittable over $k$.

Now assume that $d := \dim_k(A) > 1$ and the assertion of Theorem 4.3 holds true for all semisimple algebras of dimension $< d$ over any field of characteristic 0.

**Step 1.** Suppose that the $k$-algebra $A$ is not simple, i.e., it splits into a direct sum

$$A = A_1 \oplus A_2$$

of two nonzero semisimple $k$-algebras $A_1$ and $A_2$. Clearly, both $\dim_k(A_1)$ and $\dim_k(A_1)$ are strictly less than $d$. There are elements $f_1 \in A_1$, $f_2 \in A_2$ such that $f = f_1 + f_2$. Applying the induction assumption to both $(A_1, f_1)$ and $(A_2, f_2)$, we obtain that there are splittable over $k$ subalgebras

$$C_1 \subset A_1, C_2 \subset A_2$$

such that $1_{A_1}, f_1 \in C_1$, $1_{A_2}, f_2 \in C_2$.

Now the direct sum

$$C = C_1 \oplus C_2 \subset A_1 \oplus A_2 = A$$

is splittable over $k$ and contains $1_{A_1} + 1_{A_2} = 1_A$ and $f_1 + f_2 = f$.

So, in the course of the proof, we may and will assume that $A$ is a simple $k$-algebra.

**Step 2.** Let $E$ be the center of the simple $k$-algebra $A$. Then $E$ is a field that is a finite algebraic extension of $k$. Clearly, $A$ carries the natural structure of central simple $E$-algebra. If $E \neq k$ then

$$\dim_E(A) < \dim_k(A)$$

and the induction assumption implies that there is a splittable over $E$ subalgebra $C \subset A$ that contains both $1_A$ and $f$. Since $E/k$ is finite algebraic, $C$ is splittable over $k$ as well.

So, in the course of the proof, we may and will assume that $E = k$, i.e., $A$ is a central simple $k$-algebra.

**Step 3.** So, $A$ is a *central simple* $k$-algebra of finite $k$-dimension $d > 1$. Recall that $d = m^2$ where $m > 1$ is an integer. Then $A$ carries the natural structure of a $m^2$-dimensional $k$-Lie algebra with brackets

$$[u, v] := uv - vu \forall u, v \in A.$$ 

The center of the $k$-Lie algebra $A$ coincides with $k \cdot 1_A$. Let us consider the $k$-linear *reduced trace map* (see [6])

$$\text{tr}_A : A \to k.$$ 

Recall [6] that

$$\text{tr}_A(uv) = \text{tr}_A(vu) \forall u, v \in A; \text{tr}_A(\alpha) = m\alpha \forall \alpha \in k = k \cdot 1_A.$$ 

This implies that

$$\text{tr}_A([u, v]) = \text{tr}_A(uv) - \text{tr}_A(vu) = 0,$$

and therefore one may view $\text{tr}_A$ as a homomorphism of $k$-Lie algebras (here $k$ is viewed as the Lie algebra with zero brackets operation). It follows that the $k$-Lie algebra $A$ splits into a direct sum

$$A = k \cdot 1_A \oplus \text{sl}(A).$$
of its center $k \cdot 1_A$ and the nonzero $(m^2 - 1)$-dimensional $k$-Lie algebra $sl(A) := \ker(\text{tr}_A)$; in addition, $sl(A)$ contains the derived $k$-Lie subalgebra $[A, A]$ of $A$. It is known that $[A, A]$ is an absolutely simple $k$-Lie algebra of type $A_{m-1}$ over $k$, see [3, Ch. X, Sect. 3].) This implies that
\[ \dim_k([A, A]) = m^2 - 1 = \dim_k(sl(A)). \]
Since $[A, A] \subset sl(A)$, we have $[A, A] = sl(A)$, and therefore
\[ A = k \cdot 1_A \oplus sl(A) = k \cdot 1_A \oplus [A, A] \]
is a reductive $k$-Lie algebra, whose center is $k \cdot 1_A$ and the semisimple part is $sl(A)$.

Replacing $f$ by $f - \frac{\text{tr}_A(f)}{m} 1_A$, we may and will assume that $f$ lies in $(semi)simple sl(A)$.

**Step 4.** Since $f$ is an element of the semisimple $k$-Lie algebra $sl(A)$, it can be presented as a sum
\[ f = f_s + f_n \]
of commuting semisimple $f_s$ and nilpotent $f_n$. Let us consider the natural faithful representation of $sl(A)$ in the $k$-vector space $A$
\[ sl(A) \rightarrow \text{End}_k(A), u \mapsto \{z \mapsto uz \in A\} \forall u \in sl(A). \]
Then $f_s$ acts on $A$ as a semisimple operator with zero trace and $f_n$ acts as a nilpotent operator.

**Step 5.** Suppose that $f_s \neq 0$. Then the subalgebra $k[f_s]$ of $A$ generated by $f_s$ and $1_A$ is a semisimple commutative $k$-subalgebra that contains $k \cdot 1_A$ as a proper $k$-subalgebra. The centralizer $Z_{f_s}$ of semisimple $k[f_s]$ in semisimple $A$ is a semisimple $k$-subalgebra ([2], Th. 4.3.2), ([9], Th. 4.1) that contains both $f_s$ and $f_n$ and therefore contains $f_s + f_n = f$. Clearly, $Z_{f_s}$ does not coincide with the whole $A$, since $f_s$ does not belong to the center $k \cdot 1_A$. This implies that
\[ \dim_k(Z_{f_s}) < \dim_k(A) = d. \]
The induction assumption applied to $Z_{f_s}$ proves the desired assertion. So, we may and will assume that $f_s = 0$, i.e., $f = f_n$ is a nonzero nilpotent element.

**Step 6.** So, $f$ is a nonzero nilpotent element. By Jacobson-Morozov theorem [1, Ch. 8, Sect. 11, Prop. 2], there is a Lie $k$-subalgebra $g$ of $sl(A)$ that contains $f$ and is isomorphic to $sl(2, k)$. Let $C \subset A$ be the associative $k$-subalgebra of $A$ generated by $g$ and $1_A$. Clearly,
\[ f \in g \subset C \subset A. \]
Let us consider the natural faithful action of $C$ on the $k$-vector space $V := A$ induced by multiplication in $A$. Clearly, a $k$-vector subspace $W$ of $V$ is a $C$-submodule (resp. a simple $C$-submodule) if and only it is a $g$-submodule (resp. a simple $g$-submodule). In addition, if $W$ is a $C$-submodule then the centralizer
\[ \text{End}_C(W) = \text{End}_g(W). \]
Recall [1, Ch. VII, Sect. 1, Prop. 3 and Th. 1] that every $sl(2, k)$-module of finite $k$-dimension is semisimple; in addition, if $W$ is a simple $sl(2, k)$-module of finite $k$-dimension then it is absolutely simple (ibid). This implies that $V$ is a semisimple $C$-module and its every simple submodule is absolutely simple. It follows from Lemma [4,4] that $C$ is a direct sum of matrix algebras over $k$ and therefore is splittable. This ends the proof of Theorem 4.3. \[ \square \]
Proof of Lemma 4.4. We may and will identify $A$ with its isomorphic image in End$_k(V)$. The semisimplicity and faithfulness of $V$ combined with [4, Ch. XVII, Sect. 4, Prop. 4.7] imply that $A$ is a finite-dimensional semisimple $k$-algebra. Let us consider the centralizer $B := \text{End}_A(V) \subset \text{End}_k(V)$ of $A$ in End$_k(V)$. The semisimplicity of $V$ and absolute simplicity of all its simple $A$-submodules combined with [4, Ch. XVII, Sect. 1, Prop. 1.2] imply that there are a positive integer $s$ and positive integers $d_1, \ldots, d_s$ such that the $k$-algebra
$$B \cong \oplus_{i=1}^s \text{Mat}_{d_i}(k)$$
In particular, $B$ is semisimple and therefore the faithful $B$-module $V$ is semisimple.
Let
$$p_i : \oplus_{i=1}^s \text{Mat}_{d_i}(k) \to \text{Mat}_{d_i}(k)$$
be the (surjective) projection map to the $i$th summand. Recall that the coordinate $k$-vector space $k^{d_i}$ endowed with the natural action of $\text{Mat}_{d_i}(k)$ is the only (up to an isomorphism) simple $\text{Mat}_{d_i}(k)$-module and this module is absolutely simple. Using $p_i$, one may endow $k^{d_i}$ with the natural structure of $\oplus_{i=1}^s \text{Mat}_{d_i}(k)$-module and this module, which we denote by $M_i$, is an absolutely simple $\oplus_{i=1}^s \text{Mat}_{d_i}(k)$-module. Clearly, every simple $\oplus_{i=1}^s \text{Mat}_{d_i}(k)$-module is isomorphic to one of $M_i$; in particular, each simple $\oplus_{i=1}^s \text{Mat}_{d_i}(k)$-module is absolutely simple. This means that each simple $B$-module is absolutely simple. Recall that the $B$-module $V$ is semisimple and therefore is isomorphic to a direct sum of simple $B$-modules, each of which is absolutely simple. It follows from [4, Ch. XVII, Sect. 1, Prop. 1.2] that the centralizer End$_B(V)$ of $B$ in End$_k(V)$ is isomorphic to a direct sum of matrix algebras over $k$. By Jacobson’s density theorem [4, Ch. XVII, Sect. 3, Th. 3.2], our $A$ coincides with End$_B(V)$ and therefore is isomorphic to a direct sum of matrix algebras over $k$. This ends the proof.

\[\square\]

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