Abstract

We prove a formula relating the Hausdorff dimension of a subset of the unit interval and the Hausdorff dimension of the same set with respect to a random path metric on the interval, which is generated using a multiplicative cascade. When the random variables generating the cascade are exponentials of Gaussians, the well known KPZ formula of Knizhnik, Polyakov and Zamolodchikov from quantum gravity [KPZ88] appears. This note was inspired by the recent work of Duplantier and Sheffield [DS08] proving a somewhat different version of the KPZ formula for Liouville gravity. In contrast with the Liouville gravity setting, the one dimensional multiplicative cascade framework facilitates the determination of the Hausdorff dimension, rather than some expected box count dimension.

1 Introduction

There is growing interest in establishing a rigorous theory of two dimensional continuum quantum gravity. Heuristically, quantum gravity is a metric chosen on the sphere uniformly among all possible metrics. Although there are successful discrete mathematical quantum gravity models, we do not yet have a satisfactory continuum definition. A highlight of quantum gravity in the physics literature is the mysterious KPZ formula of Knizhnik, Polyakov and

---

*Microsoft Research and the Weizmann Institute of Science
†Microsoft Research
Zamolodchikov [Pol87, KPZ88] relating the dimensions of fractals in the random geometry to the corresponding dimension in Euclidean geometry. More specifically, the KPZ formula is

$$\Delta - \Delta_0 = \frac{\Delta(1 - \Delta)}{k + 2},$$

(1.1)

where $2 - 2 \Delta$ is the dimension of a set in quantum gravity and $2 - 2 \Delta_0$ is the dimension of the corresponding set in Euclidean geometry.

Recently Duplantier and Sheffield [DS08] were able to prove an expected box count dimension version of the KPZ formula in Liouville gravity. In their setup, they avoid the very difficult issue of defining the random metric and instead define a random measure.

Multiplicative cascades is a well studied object and defines naturally a random metric $\rho$ on $[0, 1]$. (The definition will be recalled in Section 2.) The metric space $([0, 1], \rho)$ is a path metric space, which in this case just means that it is isometric with some interval $[0, \ell]$ with its Euclidean metric $|x - y|$. The length $\ell = \rho(0, 1)$ of $[0, 1]$ in the metric $\rho$ is in general random. In this note we prove a formula relating the Hausdorff dimensions of sets $K \subset [0, 1]$ with respect to the random metric $\rho$ on the one hand and with respect to the Euclidean metric on the other hand. The KPZ relation appears precisely when the random variables defining the cascade are exponentials of Gaussians. Interestingly, this is hinted at in the introduction of [OW00].

One goal of the note is to establish in this simple one-dimensional setup a Hausdorff dimension version of the KPZ relation. We also generalize the discussion to the non-Gaussian setting, mainly to gain perspective on the underpinnings of the KPZ relation.

The proof in [DS08] is based on large deviations arguments, which is not the case for the present paper. While in a very general sense one could say that the ideas are similar, it can reasonably be claimed that our proof is substantially different and is rather elegant. As we have not yet seen a draft of [DS08], this comparison is based on lectures by Scott Sheffield and on conversations with him.

\footnote{The parameter $k$ comes up since it is presumed that there is essentially one free parameter in the construction of quantum gravity. This parameter is intimately related to the central charge, and the various variants of quantum gravity are believed to arise by weighting a uniform measure with the partition function of a statistical physics model.}

\footnote{We use the term “metric” to mean a “distance function”, not a Riemannian metric tensor.}
Early versions of multiplicative cascades were introduced by Kolmogorov already in 1941 \cite{Kol91} and were developed by Yaglom \cite{Yag66} and by Mandelbrot \cite{Man74}. Many fundamental properties of multiplicative cascades were first proved in the remarkable paper \cite{KP76} by Kahane and Peyrière. For further background and references regarding the long history of multiplicative cascades see e.g., \cite{OW00}.

In Section 2 we describe the basic setup of multiplicative cascades and define the random metric $\rho$. Our main result is stated and proved in Section 3. An appendix follows in which we prove some essentially known necessary background facts about multiplicative cascades which we need to use. In some cases, the proofs in the appendix are simpler than the proofs that we were able to find in the literature, and in other cases, the results are slightly stronger.

Acknowledgements: We are obliged to Scott Sheffield for numerous discussions explaining his insights to us. Yuval Peres and Ed Waymire have been very helpful in enlightening us with regards to the published work on multiplicative cascades.

2 Setup

We now describe our setup. Let $I_n$ denote the set of dyadic subintervals of $[0,1]$ of length $2^{-n}$; namely,

$$I_n := \left\{ [k2^{-n}, (k+1)2^{-n}] : n \in \mathbb{N}, k \in \{0,1,\ldots,2^n-1\} \right\}.$$  

Then each interval in $I_n$ has precisely two subintervals in $I_{n+1}$, its left half and its right half. Also set $\mathcal{I} := \bigcup_{n \in \mathbb{N}} I_n$.

Let $W$ be some positive random variable with mean 1, and let $W_I$, $I \in \mathcal{I}$, be an independent collection of random variables, each of which has the distribution of $W$. We now define inductively a sequence of random measures on $[0,1]$. Let $\mu_0$ denote Lebesgue measure on $[0,1]$, and let $\mu_1 := W_{[0,1]} \mu_0$. Let $\mu_2$ denote the measure which agrees with $W_{[0,1/2]} \mu_1$ on $[0,1/2]$ and agrees with $W_{[1/2,1]} \mu_1$ on $[1/2,1]$. Inductively, define $\mu_{n+1}$ as the measure that on every $I \in I_n$ agrees with $W_I \mu_n$. Alternatively,

$$\mu_n := w_n \mu_0 , \quad \text{where} \quad w_n(x) := \prod_{j=0}^{n-1} W_{I_j(x)} ,$$
and $I_j(x)$ denotes the interval $I \in \mathcal{I}_j$ that contains $x$ (and if there is more than one, the one whose maximum is $x$, say).

We will need the following general result regarding multiplicative cascades.

**Theorem 2.1.** The weak limit $\mu := \lim_{n \to \infty} \mu_n$ exists almost surely. Moreover, if $\mathbb{E}[W \log_2 W] < 1$ then $\mu[0, 1] > 0$ a.s. and $\mu$ has no atoms a.s.

This theorem is entirely or almost entirely proved in [KP76], but we present in the appendix a different perhaps simpler proof. Since $\mu[0, s]$ is a positive martingale for every $s \in [0, 1]$, the first claim is very easy to verify. In [KP76] they show that $\mu[0, 1] > 0$ a.s. if and only if $\mathbb{E}[W \log_2 W] < 1$. The claim about the non-existence of atoms would follow from the last remark in [KP76], but we were unable to verify the justification of that remark (though it does follow under additional moment assumptions).

Henceforth, we will be assuming that

$$\mathbb{E}[W \log_2 W] < 1. \quad (2.1)$$

By the result of [KP76] mentioned above, this assumption is necessary for the limit $\mu$ to be nonzero.

On $[0, 1]$, define the random metric $\rho$ by

$$\rho(x, y) := \mu[x, y], \quad \text{for all } 0 \leq x \leq y \leq 1.$$ 

If $F(x) := \mu[0, x]$, then $\rho$ is just the pullback of the Euclidean metric on $[0, F(1)]$ under $F$. We also set

$$\ell_n := \mu_n[0, 1], \quad \ell := \mu[0, 1]. \quad (2.2)$$

Clearly, $\mathbb{E}[\ell_n] = 1$ and $\mathbb{E}[\ell] \leq 1$. In fact, $\mathbb{E}[\ell] = 1$ by [KP76], but we do not need this result.

### 3 Hausdorff dimension

**Theorem 3.1.** Suppose that

$$\mathbb{E}[W^{-s}] < \infty \quad \text{for all } s \in [0, 1) \quad (3.1)$$

(in addition to the standard assumptions $\mathbb{E}[W \log_2 W] < 1$, $\mathbb{E}[W] = 1$, and $\mathbb{P}[W > 0] = 1$.) Let $K \subset [0,1]$ be some (deterministic) nonempty set, let $\zeta_0$ denote its Hausdorff dimension with respect to the Euclidean metric, and let $\zeta$ denote its Hausdorff dimension with respect to the random metric $\rho$. Then a.s. $\zeta$ is the unique solution of the equation

$$2^{\zeta_0} = \frac{2^\zeta}{\mathbb{E}[W^\zeta]}$$  \hspace{1cm} (3.2)

in $[0,1]$.

As our proof shows, the assumption (3.1) may be significantly relaxed. See Theorems 3.4 and 3.5.

Now consider the case in which $\log W$ is a Gaussian random variable. Since $\mathbb{E}[W] = 1$, this implies $W = \exp(\sigma Y - \sigma^2/2)$, where $Y$ is a standard Gaussian of zero mean and unit variance and $\sigma \geq 0$. The assumption (2.1) is then equivalent to the requirement $\sigma^2 < \log 4$. In this case, the moments $\mathbb{E}[W^s]$ are easily evaluated and (3.2) gives

$$\zeta_0 - \zeta = \frac{\sigma^2}{\log 4} \zeta (1 - \zeta),$$  \hspace{1cm} (3.3)

in agreement with (1.1).

For comparison, suppose instead that $W = 1 \pm \sigma$, each with probability $1/2$. Then (2.1) becomes $|\sigma| < 1$ and (3.2) transforms to

$$2^{\zeta_0} = \frac{2^\zeta}{\frac{1}{2}(1 - \sigma)^\zeta + \frac{1}{2}(1 + \sigma)^\zeta},$$

or

$$\zeta_0 = 1 + \zeta - \log_2 \left( (1 - \sigma)^\zeta + (1 + \sigma)^\zeta \right).$$

We now proceed to prove Theorem 3.1. Define

$$\phi(s) := s - \log_2 \mathbb{E}[W^s].$$

Then (3.2) reads $\zeta_0 = \phi(\zeta)$. The following lemma implies the uniqueness of the $\zeta$ satisfying (3.2).

---

\footnote{In (1.1), $2(1 - \Delta_0)$ is the dimension and similarly for $\Delta$. The factor of 2 comes from the fact that the ambient space is two dimensional, and that the dimension is defined in terms of the measure, not the distance function. The transition from $\Delta_0$ to $1 - \Delta_0$ is a passage from the dimension to the co-dimension, and does not change the form of (1.1).}
Lemma 3.2. The function $\phi$ is continuous, strictly monotone increasing in $[0,1]$ and maps $[0,1]$ onto $[0,1]$.

Proof. Set $\psi(s) := E[(W/2)^s]$. Continuity of $\psi$ follows from the dominated convergence theorem and convexity of $\psi$ is immediate by the convexity of $(W/2)^s$ in $s$. Since $\psi$ is convex and

$$\psi'(1-)=E[(W/2)\log(W/2)]=\frac{1}{2}E[W\log W]-\frac{1}{2}\log 2 < 0,$$

it is strictly monotone decreasing in $[0,1]$. The lemma follows since $\phi = -\log_2 \psi$, $\phi(0) = 0$ and $\phi(1) = 1$. \hfill $\Box$

The following simple lemma can serve to motivate Theorem 3.1 and is also important in its proof.

Lemma 3.3. Let $x, y \in [0,1]$, and let $s \in (0,1]$. Then

$$E[\rho(x,y)^s] \leq 8|x-y|^\phi(s).$$

Proof. Let $[a,b] \in I_n$. Then by the construction of $\rho$ and the independence of the different variables $W_I, I \in I$,

$$E[\rho(a,b)^s] = 2^{-ns} E[W^s]^n E[\ell^s] = |a-b|^{\phi(s)} E[\ell^s].$$

Now, Jensen’s inequality gives $E[\ell^s] \leq E[\ell]^s \leq 1$. Note that if $|y-x| \in (2^{-n-1}, 2^{-n}]$, then the interval joining $x$ and $y$ can be covered by two consecutive intervals in $I_n$, say $[a,b]$ and $[b,c]$. Then

$$E[\rho(x,y)^s] \leq E[(\rho(a,b)+\rho(b,c))^s] \leq E[(2\rho(a,b))^s+(2\rho(b,c))^s] = 2^{1+s} E[\rho(a,b)^s] \leq 2^{1+s} |a-b|^{\phi(s)} \leq 2^{1+s+\phi(s)} |x-y|^{\phi(s)}.$$

The lemma follows, since $\phi(s) \leq 1$ by Lemma 3.2. \hfill $\Box$

Theorem 3.4. Let $K$, $\zeta_0$ and $\zeta$ be as in Theorem 3.1. Then a.s. $\phi(\zeta) \leq \zeta_0$.

It is worth pointing out that we are not assuming (3.1) here.
Proof. Let $s \in [0, 1]$ and assume that $t := \phi(s) > \zeta_0$. We now show that $s \geq \zeta$ a.s. Let $\epsilon > 0$. Then there is a covering of $K$ by at most countably many intervals $[x_i, y_i]$ such that $\sum_i |x_i - y_i|^t < \epsilon$. By Lemma 3.3 we have

$$E\left[ \sum_i \rho(x_i, y_i) \right] \leq 8 \sum_i |x_i - y_i|^t \leq 8 \epsilon.$$ 

By Markov’s inequality, with probability at least $1 - \sqrt{\epsilon}$ we have a covering of $K$ with balls whose radii in the $\rho$ metric satisfy $\sum r_i^t \leq 8 \sqrt{\epsilon}$. Thus $s \geq \zeta$ a.s. Hence $\zeta \leq \inf \phi^{-1}(\zeta_0, 1]$. By Lemma 3.2, the theorem follows.

Theorem 3.5. Let $K$, $\zeta_0$ and $\zeta$ be as in Theorem 3.1. Then a.s.

$$\zeta \geq \sup \{ s \in (0, 1) : \phi(s) < \zeta_0, E[W^{-s}] < \infty \}.$$ 

Proof. Suppose that $s \in (0, 1)$ satisfies $t := \phi(s) < \zeta_0$ and $E[W^{-s}] < \infty$. We need to prove that $\zeta \geq s$. Since $E[W^s]$ is convex in $s$ and equals to 1 at $s = 0, 1$, we have $t \geq s \geq 0$. Since $\zeta_0 > t$, by Frostman’s lemma [Mat95, Chapter 8] there is a Borel probability measure $\nu_0$ supported on $K$ such that

$$E_t(\nu_0) := \int \int \frac{d\nu_0(x) d\nu_0(y)}{|x - y|^t} < \infty.$$ 

Set $a := E[W^s]$, $Z := W^s/a$ and $Z_I := W^s_I/a$. Define

$$f_n(x) := \prod_{j<n} Z_{I_j(x)}, \quad \nu_n := f_n \nu_0.$$ 

Since for every $a \in [0, 1]$ the sequence $\nu_n[0, a]$ is a non-negative martingale, it easily follows that the weak limit $\nu := \lim_{n \to \infty} \nu_n$ exists. (See, e.g., the proof of Theorem 2.1 in the appendix.) Then the support of $\nu$ is contained in the support of $\nu_0$ and therefore in $K$.

Define

$$\rho_n(x, y) := \rho(x, y) \vee \mu(I_n(x)) \vee \mu(I_n(y)).$$

In order to estimate the expectation of

$$E_s(\nu_n; \rho_n) := \int \int \frac{d\nu_n(x) d\nu_n(y)}{\rho_n(x, y)^s}$$

we fix $x, y \in [0, 1]$, and estimate

$$E[f_n(x) f_n(y) \rho_n(x, y)^{-s}].$$
Let \( k \) be the smallest integer such that \([x, y]\) contains some interval of \( I_k \). Then
\[
|x - y| < 4 \cdot 2^{-k}.
\] (3.6)

Let \( J \in I_k \) satisfy \( J \subset [x, y] \), and let \( J' \in I_{k-1} \) satisfy \( J' \supset J \). Then \( x \in J' \) or \( y \in J' \). By symmetry, assume \( x \in J' \). Let \( \mathcal{G} \) denote the \( \sigma \)-field \( \langle W_{I_j(x)}, W_{I_j(y)} : j < n \rangle \). Assume first that \( k \leq n \). Then
\[
E[\rho(x, y)^{-s} | \mathcal{G}] \leq E[\mu(J)^{-s} | \mathcal{G}] = 2^{ks} E[\ell^{-s}] \prod_{j=0}^{k-1} W_{I_j(x)}^{-s}.
\]

By Lemma A.3 in the appendix and our assumption that \( E[W^{-s}] < \infty \), we have \( E[\ell^{-s}] < \infty \). Using (3.6), we therefore obtain
\[
E[\rho(x, y)^{-s} | \mathcal{G}] \leq O(1) |x - y|^{-s} \prod_{j=0}^{k-1} W_{I_j(x)}^{-s},
\]
where the implied constant may depend on \( s \) and the law of \( W \). Now,
\[
f_n(x) = a^{-n} \prod_{j=0}^{n-1} W_{I_j(x)}^{s},
\]
and we have a similar expression for \( f_n(y) \). Thus,
\[
E[f_n(x) f_n(y) \rho(x, y)^{-s} | \mathcal{G}] \leq O(1) a^{-2n} |x - y|^{-s} \prod_{j=k}^{n-1} W_{I_j(x)}^{s} \prod_{j=0}^{n-1} W_{I_j(y)}^{s}.
\]

Note that for \( j \geq k \) we have \( I_j(x) \neq I_j(y) \). Taking expectations and using the definition of \( a \) yields
\[
E[f_n(x) f_n(y) \rho_n(x, y)^{-s}] \leq E[f_n(x) f_n(y) \rho(x, y)^{-s}]
\leq O(1) |x - y|^{-s} a^{-k} \leq O(1) |x - y|^{-s + \log_2 a} = O(1) |x - y|^{-t}.
\]

Now, if \( k > n \), we have instead
\[
E[\rho_n(x, y)^{-s} | \mathcal{G}] \leq O(1) E[\mu(I_n(x))^{-s} | \mathcal{G}] \leq O(1) 2^{sn} \prod_{j=0}^{n-1} W_{I_j(x)}^{-s}.
\]
The above argument therefore gives in this case,
\[
\mathbb{E} \left[ f_n(x) f_n(y) \rho_n(x, y)^{-s} \right] \leq O(1) \ 2^{n s} a^{-n} \leq O(1) |x - y|^{-t}.
\]
Thus, we have \( \mathbb{E} \left[ f_n(x) f_n(y) \rho_n(x, y)^{-s} \right] \leq O(1) |x - y|^{-t} \) for every \( x, y \in [0, 1] \). Integrating this with respect to \( d\nu_0(x) \times d\nu_0(y) \) and applying Fubini, one obtains
\[
\mathbb{E} \left[ \mathcal{E}_s(\nu_n; \rho_n) \right] \leq O(1) \mathcal{E}_t(\nu_0). \tag{3.7}
\]
Since \( \rho_n(x, y) \leq \ell \) holds for \( x, y \in [0, 1] \), this estimate gives
\[
\mathbb{E} \left[ (\nu_n[0, 1])^2 \ell^{-s} \right] \leq O(1) \mathcal{E}_t(\nu_0).
\]
Now Hölder’s inequality comes into play:
\[
\mathbb{E} \left[ (\nu_n[0, 1])^{2/(1+s)} \right] \leq \mathbb{E} \left[ (\nu_n[0, 1])^{2} \ell^{-s} \right]^{1/(1+s)} \mathbb{E} \left[ \ell \right]^{s/(1+s)} \leq O(1) \mathcal{E}_t(\nu_0)^{1/(1+s)}.
\]
Thus, the martingale sequence \( \nu_n[0, 1] \) is uniformly bounded in \( L^p \) with \( p = 2/(1 + s) > 1 \). It follows by the corresponding martingale convergence theorem that \( \mathbb{E} [\nu[0, 1]] = \nu_0[0, 1] = 1 \), and in particular, \( \nu[0, 1] > 0 \) with positive probability. The event \( \nu[0, 1] > 0 \) is clearly independent of \( \sigma \)-field generated by any finite number of the random variables \( W_I \), and therefore has probability 0 or 1, and in this case, \( \mathbb{P} [\nu[0, 1] > 0] = 1 \).

Since a.s. \( \rho \) is continuous, \( \rho_n \to \rho \) uniformly as \( n \to \infty \) and \( \nu_n \to \nu \) weakly, we have a.s.
\[
\mathcal{E}_s(\nu; \rho) \leq \lim \inf_{n \to \infty} \mathcal{E}_s(\nu_n; \rho_n) \leq \mathcal{E}_t(\nu_0)\ . \tag{3.7}
\]

The proof is now completed by appealing to Frostman’s criterion \cite[Chapter 8]{Mat1995}, since \( \nu[0, 1] > 0 \) a.s. \( \Box \)

**Proof of Theorem 3.1.** The theorem follows immediately from Lemma 3.2 and Theorems 3.4 and 3.5. \( \Box \)

**A Some multiplicative cascades background**

**Lemma A.1.** Our standing assumption \( \mathbb{E}[W \log_2 W] < 1 \) implies that \( \ell > 0 \) a.s.
Proof. Set \( a := \mathbb{E}[W \log_2 W] < 1 \). We first prove that \( \ell > 0 \) with positive probability. Since \( \ell_n := \mu_n[0,1] \) is a positive martingale, we have a.s. convergence \( \ell_n \to \ell \).

The proof will come out of a recurrence relation for the sequence \( b_n := \mathbb{E}[\ell_n \log_2 \ell_n] \). Let \( \ell_n' \) and \( \ell_n'' \) have the law of \( \ell_n \) and be independent and independent from \( W \). Then the law of \( \ell_{n+1} \) is the same as the law of \( W (\ell_n' + \ell_n'')/2 \). Thus,

\[
\begin{align*}
    b_{n+1} &= \mathbb{E}[(W/2) (\ell_n' + \ell_n'') \log_2 W] + \\
    &\quad + \mathbb{E}[(W/2) (\ell_n' + \ell_n'') \log_2 (\ell_n' + \ell_n'')] + \mathbb{E}[(W/2) (\ell_n' + \ell_n'') \log_2 (1/2)].
\end{align*}
\]

We now use independence, \( \mathbb{E}[\ell_n'] = \mathbb{E}[W] = 1 \), the symmetry between \( \ell_n' \) and \( \ell_n'' \), and the definition of \( a \), and simplify the above to

\[
b_{n+1} = a + \mathbb{E}[\ell_n' \log_2 (\ell_n' + \ell_n'')] - 1.\]

Thus,

\[
\begin{align*}
    b_{n+1} - b_n &= a - 1 + \mathbb{E}[\ell_n' \log_2 (\ell_n' + \ell_n'')] - \mathbb{E}[\ell_n' \log_2 \ell_n'] \\
    &= a - 1 + \mathbb{E}[\ell_n' \log_2 ((\ell_n' + \ell_n'')/\ell_n')].
\end{align*}
\]

Since \( \log_2 (1 + x/\ell_n') \) is concave as a function of \( x \), and since \( \ell_n'' \) is independent from \( \ell_n' \), Jensen’s inequality applied to the above gives

\[
b_{n+1} - b_n \leq a - 1 + \mathbb{E}[\ell_n' \log_2 (1 + \mathbb{E}[\ell_n'']/\ell_n')] = a - 1 + \mathbb{E}[\ell_n' \log_2 (1 + 1/\ell_n)]. \quad (A.1)
\]

Because \( \inf\{b_n : n \in \mathbb{N}\} \geq \inf\{x \log_2 x : x > 0\} > -\infty \) and \( a < 1 \), the set \( S := \{n \in \mathbb{N} : b_{n+1} - b_n > (a - 1)/2\} \) is infinite. For \( n \in S \), (A.1) implies

\[
    \mathbb{E}[g(\ell_n)] > (1 - a)/2, \quad \text{where} \quad g(x) := x \log_2 (1 + 1/x).
\]

Note that \( c := \sup\{g(x) : x > 0\} < \infty \). For all \( \epsilon > 0 \) and \( n \in S \), we have

\[
    (1 - a)/2 < \mathbb{E}[g(\ell_n)] \leq c \mathbb{P}[\ell_n \geq \epsilon] + \sup\{g(x) : 0 < x < \epsilon\}.
\]

By taking \( \epsilon \) sufficiently small, we can make sure that the last summand is at most \((1 - a)/4\). Then, \( \mathbb{P}[\ell_n > \epsilon] > (1 - a)/(4c) \). This proves that \( \ell_n \) does not tend to zero in probability, and hence does not tend to zero a.s. Thus \( \mathbb{P}[\ell > 0] > 0 \).
Set \( \ell' := 2 \mu[0, 1/2]/W_{[0,1]} \) and \( \ell'' := 2 \mu[1/2, 1]/W_{[0,1]} \). Then \( \ell', \ell'' \) and \( W_{[0,1]} \) are independent, and each of \( \ell' \) and \( \ell'' \) has the law of \( \ell = (W_{[0,1]}/2)(\ell' + \ell'') \). But \( \ell = 0 \) if and only if \( \ell' = 0 = \ell'' \), since \( W > 0 \) a.s. Thus,

\[
P[\ell = 0] = P[\ell' = 0, \ell'' = 0] = P[\ell = 0]^2.
\]

Since \( P[\ell > 0] > 0 \), we conclude that \( \ell > 0 \) a.s., which completes the proof.

**Proof of Theorem 2.1.** Since \( E[\mu_n[0,1]] = 1 \), the sequence of measures \( \mu_n \) is tight in the space of Borel measures on \([0,1]\) with the topology of weak convergence. Thus, some subsequence converges to a limit \( \mu \). For every rational \( r \in [0,1] \cap \mathbb{Q} \) the sequence \( \mu_n[0,r] \) is a positive martingale, and hence the limit \( f(r) := \lim_{n \to \infty} \mu_n[0,r] \) exists for all \( r \in \mathbb{Q} \cap [0,1] \) a.s. It is immediate to verify that \( \mu[0,x] = \sup\{f(r) : r \in \mathbb{Q} \cap [0,x]\} \) and \( \mu[0,1] = f(1) \). This implies that the subsequential limit \( \mu \) is unique, and hence is the weak limit of the entire sequence \( \mu_n \). Thus, the theorem follows from Lemma A.1 and the next lemma.

**Lemma A.2.** A.s. \( \mu \) has no atoms.

**Proof.** Let \( Z \) have the distribution of the \( \mu \)-measure of the largest atom in \([0,1] \). Clearly, \( E[Z] \leq 1 \). Let \( Z_1 \) and \( Z_2 \) have the law of \( Z \) with \( W, Z_1 \) and \( Z_2 \) independent. Then \( W \max(Z_1,Z_2)/2 \) has the law of \( Z \). Therefore,

\[
E[Z_1 + Z_2]/2 = E[Z] = E[W \max(Z_1,Z_2)/2] = E[\max(Z_1,Z_2)]/2.
\]

Since \( Z_1 + Z_2 \geq \max(Z_1,Z_2) \), and the expectations are the same, it follows that they are equal a.s. Thus, on the event \( Z_1 > 0 \), we have \( Z_2 = 0 \) a.s. Since \( Z_1 \) and \( Z_2 \) are independent, we get \( Z_1 = 0 \) a.s., as required.

**Lemma A.3.** Suppose that

\[
E[W^{-r}] < \infty \tag{A.2}
\]

for some constant \( r > 0 \). Then also

\[
E[\ell^{-r}] < \infty.
\]

A very slightly weaker form of this lemma can be found in [Liu01], where the assumption \((A.2)\) is the same, but the conclusion is that \( E[\ell^{-s}] < \infty \) for all \( s \in [0,r) \). However, the setup in [Liu01] is more general.
Proof. Let $\ell_1 := 2 \mu [0, 1/2] / W$ and $\ell_2 := 2 \mu [1/2, 1] / W$. By construction $\ell_1, \ell_2$ and $W$ are independent and each of $\ell_1$ and $\ell_2$ has the law of $\ell$. Moreover
\[
\ell = W (\ell_1 + \ell_2) / 2. \tag{A.3}
\]
Assume, for a moment, that there is some $\delta > 0$ such that
\[
\mathbb{E}[\ell^{-\delta}] < \infty. \tag{A.4}
\]
By the means inequality and (A.3), we have
\[
\ell \geq W \sqrt{\ell_1 \ell_2}. \tag{A.5}
\]
Now independence gives for every $s > 0$
\[
\mathbb{E}[\ell^{-s}] \leq \mathbb{E}[W^{-s}] \mathbb{E}[\ell_1^{-s/2}] \mathbb{E}[\ell_2^{-s/2}] = \mathbb{E}[W^{-s}] \mathbb{E}[\ell^{-s/2}]^2. \tag{A.6}
\]
Let $S$ denote the set of $s \in [0, \infty)$ such that $\mathbb{E}[\ell^{-s}] < \infty$. By (A.4), we have $\delta \in S$. Since $\mathbb{E}[W^{-s}]$ is a convex function of $s$, the set $S$ must be an interval. Moreover, $[0, \delta] \subset S$. Similarly, $\mathbb{E}[W^{-s}] < \infty$ for every $s \in [0, r]$. Now, (A.6) shows that $[0, r] \cap (2S) \subset S$, where $2S = \{ 2s : s \in S \}$. This implies that $[0, r] \subset S$, as needed.

It remains to prove (A.4). From (A.2) we know that
\[
\lim_{x \downarrow 0} x^{-r} \mathbb{P}[W < x] = 0. \tag{A.7}
\]
By (A.3), for every $b, x > 0$, we have that if $\ell < bx/2$ then $W < x$ or $\ell_1 + \ell_2 < b$. Thus,
\[
\mathbb{P}[\ell < bx/2] \leq \mathbb{P}[W < x] + \mathbb{P}[\ell_1 + \ell_2 < b] \\
\leq \mathbb{P}[W < x] + \mathbb{P}[\ell_1 < b, \ell_2 < b] \tag{A.8}
\]
\[
= \mathbb{P}[W < x] + \mathbb{P}[\ell < b]^2.
\]
Set $\epsilon_j := 2^{-2^j-1}$ for $j \in \mathbb{N}$. Then $\epsilon_{j+1} = 2 \epsilon_j^2$. Let $b_0 > 0$ satisfy
\[
\mathbb{P}[\ell < b_0] \leq \epsilon_0
\]
(Theorem 2.1 implies the existence of such a $b_0$). Set
\[
x_j := \sup \left\{ x : \mathbb{P}[W < x] \leq \epsilon_j^2 \right\}.
\]
Then our assumption (A.7) implies that $x_j \geq \epsilon_j^{2/r}$ for all but finitely many $j$. Inductively define

$$b_{j+1} = b_j x_j / 2.$$ 

Then using induction, the relation $\epsilon_{j+1} = 2 \epsilon_j^2$ and the estimate (A.8) give

$$P[\ell < b_j] \leq \epsilon_j.$$  

(A.9)

The definition of $b_j$ gives

$$b_j = 2^{-j} b_0 \prod_{k=0}^{j-1} x_k \geq C' 2^{-j} \prod_{k=0}^{j-1} \epsilon_k^{2/r},$$

where $C' > 0$ is the product of $b_0$ and the finitely many $x_k$ that satisfy $x_k < \epsilon_k^{2/r}$. Taking into account the definition of $\epsilon_j$, we get

$$b_j \geq C' 2^{-j-2(j-1+2^j)/r}.$$ 

Now,

$$E[\ell^{-\delta}] \leq b_0^{-\delta} + \sum_{j=0}^{\infty} b_{j+1}^{-\delta} P[b_{j+1} \leq \ell < b_j].$$

For all but finitely many $j$, the above estimate on $b_j$ gives $b_{j+1}^{-\delta} \leq 2^{5\delta 2^j / r}$, while $P[b_{j+1} \leq \ell < b_j] \leq P[\ell < b_j] \leq \epsilon_j = 2^{-2^j-1}$. Thus, we have (A.4) for every $\delta \in (0, r/5)$, and the proof is thus complete.

\[\square\]

References

[DS08] Bertrand Duplantier and Scott Sheffield, 2008. In preparation.

[Kol91] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. Proc. Roy. Soc. London Ser. A, 434(1890):9–13, 1991. Translated from the Russian by V. Levin, Turbulence and stochastic processes: Kolmogorov’s ideas 50 years on.

[KP76] J.-P. Kahane and J. Peyrière. Sur certaines martingales de Benoît Mandelbrot. Advances in Math., 22(2):131–145, 1976.
[KPZ88] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov. Fractal structure of 2D-quantum gravity. *Modern Phys. Lett. A*, 3(8):819–826, 1988.

[Liu01] Quansheng Liu. Asymptotic properties and absolute continuity of laws stable by random weighted mean. *Stochastic Process. Appl.*, 95(1):83–107, 2001.

[Man74] B. Mandelbrot. Intermittent turbulence in self similar cascades: divergence of high moments and dimension of carrier. *J. Fluid Mech.*, 62:331–333, 1974.

[Mat95] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[OW00] Mina Ossiander and Edward C. Waymire. Statistical estimation for multiplicative cascades. *Ann. Statist.*, 28(6):1533–1560, 2000.

[Pol87] A. M. Polyakov. Quantum gravity in two dimensions. *Modern Phys. Lett. A*, 2(11):893–898, 1987.

[Yag66] A.M. Yaglom. Effect of fluctuations in energy dissipation rate on the form of turbulence characteristics in the inertial subrange. *Dokl. Akad. Nauk SSSR*, 166:49–52, 1966.