AMBIENT CONSTRUCTIONS FOR SASAKIAN ETA-EINSTEIN MANIFOLDS

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Abstract. The theory of ambient spaces is useful to define CR invariant objects, such as CR invariant powers of the sub-Laplacian, the $P$-prime operators, and $Q$-prime curvature. However, in general, it is difficult to write down these objects in terms of the Tanaka-Webster connection. In this paper, we give those explicit formulas for CR manifolds satisfying an Einstein condition, called Sasakian $\eta$-Einstein manifolds. As an application, we study properties of the first and the second variation of the total $Q$-prime curvature at Sasakian $\eta$-Einstein manifolds.

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1. Introduction

CR geometry has very similar properties to conformal geometry. As an important example, the theory of ambient spaces gives a powerful tool for both geometries. An ambient space is an asymptotically Ricci-flat space determined for a conformal or CR structure. Many invariants for both geometries have been constructed as obstructions to harmonic extensions, for example, GJMS operators and the $Q$-curvature.

In conformal geometry, a conformal class containing an Einstein metric plays an important role. Several authors have studied relations between invariants constructed by using ambient spaces and Einstein metrics, while there exist few studies...
for corresponding relations in CR geometry. The aim of this paper is to reveal relations between the theory of ambient spaces and CR manifolds satisfying an Einstein condition, called Sasakian $\eta$-Einstein manifolds.

Via Fefferman construction, we can see CR geometry as a special class of even dimensional conformal geometry; we first recall the latter. Let $N$ be a smooth manifold of even dimension $n$, and $[g]$ a conformal class on $N$. The metric bundle $G$ is the principal $\mathbb{R}_+$-bundle over $N$ whose sections are the metrics in the conformal class. An ambient space is the space $G \times \mathbb{R}$ with an asymptotically Ricci-flat metric $\tilde{g}$ homogeneous of degree 2 with respect to the $\mathbb{R}_+$-action. We can define GJMS operators as obstructions to harmonic extensions for the Laplacian of $\tilde{g}$.

Now we give the statements of our results, relations between the theory of ambient spaces and CR manifolds satisfying an Einstein condition, called Sasakian $\eta$-Einstein manifolds.

Now assume that there exists an Einstein metric $g$ in $[g]$. Gover [9] proved that the $k$-th GJMS operator is decomposed into $k$ factors, and each factor is the sum of the Laplacian and a constant depending on $n$ and the Einstein constant. He also showed that the $Q$-curvature with respect to $g$ is a constant depending only on $n$ and the Einstein constant. Moreover, the variation of the total $Q$-curvature under deformations of conformal structures is well-understood; and the second variation is written in terms of the Lichnerowicz Laplacian [18]; see also [13].

Next we recall CR analogs of the above objects. Let $M$ be a strictly pseudoconvex real hypersurface in an $m$-dimensional complex manifold $X$, and $\pi_X: X \to X$ be the canonical bundle of $X$ with the zero section removed, which is a principal $\mathbb{C}^*$-bundle over $X$. Denote by $\mathcal{M}$ the restriction of $\pi_X$ on $M$. From the $\mathbb{C}^*$-action, we define the space of homogeneous functions of degree $(w, w') \in \mathbb{R}^2$ on $\mathcal{M}$ and $\mathcal{M}$, which is written as $\mathcal{E}(w, w')$ and $\mathcal{E}(w, w')$ respectively. A Fefferman defining function is a defining function $\rho \in \mathcal{E}(1, 1)$ of $M$ satisfying the complex Monge-Ampère equation [22]. This defining function induces a Lorentz-Kähler metric $g$ that is Ricci-flat to some order. The CR counterparts of GJMS operators are CR invariant powers of the sub-Laplacian. For $(w, w') \in \mathbb{R}^2$ with $w + w' + m \in \{1, 2, \ldots, m\}$, Gover-Graham [10] defined a CR invariant operator $P_{w, w}$ whose leading part agrees with a power of the sub-Laplacian acting on $\mathcal{E}(w, w')$ by using ambient spaces. The natural CR analog of the $Q$-curvature in conformal geometry is the $Q$-prime curvature $Q'_{\rho}$ [11][13]. This is a homogeneous function defined for a pseudo-Einstein contact form $\theta$. The transformation law of $Q'_{\rho}$ under a conformal change is given in terms of $P_{0, 0}$ and the $P$-prime operator $P'_{0, 0}$. Its integral, the total $Q$-prime curvature, defines a global CR invariant of $M$ under some mild assumptions.

Sasakian $\eta$-Einstein manifolds are CR manifolds satisfying an Einstein condition. Let $(S, T^{1,0}S)$ be a $(2m - 1)$-dimensional strictly pseudoconvex CR manifold and $\eta$ a contact form on $S$. The triple $(S, T^{1,0}S, \eta)$ is called a Sasakian manifold if its Tanaka-Webster torsion vanishes identically. Then the cone $C(S) = \mathbb{R}_+ \times S$ of $S$ has a canonical complex structure, and the level set $\{r = 1\}$, where $r$ is the coordinate of $\mathbb{R}_+$, is canonically isomorphic to $S$ as CR manifolds; in the following, we identify $S$ with this level set. A Sasakian manifold $(S, T^{1,0}S, \eta)$ is called a Sasakian $\eta$-Einstein manifold if its Tanaka-Webster Ricci curvature is a constant multiple of the Levi form; in this case, we call this constant the Einstein constant of $(S, T^{1,0}S, \eta)$.

Now we give the statements of our results, relations between the theory of ambient spaces in CR geometry and Sasakian $\eta$-Einstein manifolds.
First we give formulas of CR invariant powers of the sub-Laplacian and the $P$-prime operator on Sasakian $\eta$-Einstein manifolds. To simplify the formulas, we use unbold differential operators $P_{w,w'}^*_{\eta}$ and $P_{\eta}^*$ acting on densities (Definition 2.3). The operators $P_{w,w'}^*_{\eta}$ and $P_{\eta}^*$ have expressions in terms of the sub-Laplacian $\Delta_b$ and the Reeb vector field $\xi$.

**Theorem 1.1.** Let $(S,T^{1,0}S,\eta)$ be a Sasakian $\eta$-Einstein manifold of dimension $2m - 1$ with Einstein constant $m\lambda$. Then $P_{w,w'}^*_{\eta}$ for $k = w + w' + m$ has the formula

$$P_{w,w'}^*_{\eta} = \prod_{j=0}^{k-1} L_{w'-w+k-2j-1}.$$

Here $L_\mu$ is the differential operator on $S$ defined by

$$L_\mu = \frac{1}{2} \Delta_b + \frac{\sqrt{-1}}{2} \mu \cdot \xi + \frac{1}{4} \lambda(m-1-\mu)(m-1+\mu).$$

**Theorem 1.2.** Let $(S,T^{1,0}S,\eta)$ be as in Theorem 1.1. For any CR pluriharmonic function $\Upsilon$ on $S$,

$$P_{\eta}^* \Upsilon = (m - 1)^{-1} P_{-1,-1}^* (\Delta_b^2 + (m-1)^2 \lambda^2 \Delta_b) \Upsilon.$$ 

In particular, the $P$-prime operator is formally self-adjoint.

In the case of the sphere or the Heisenberg group, Theorems 1.1 and 1.2 were already obtained by Graham [11] and Branson-Fontana-Morpurgo [2], respectively.

We also compute the $Q$-prime curvature of Sasakian $\eta$-Einstein manifolds. Similar to the above, we use unbold $Q'_{\eta}$ instead of $Q''_{\eta}$; see 5.1.

**Theorem 1.3.** Let $(S,T^{1,0}S,\eta)$ be as in Theorem 1.1. Then the $Q$-prime curvature $Q'_{\eta}$ is written as

$$Q'_{\eta} = 2((m-1)!)^2 \lambda^m.$$

As an application of this formula, we will compute the total $Q$-prime curvature for some Sasakian $\eta$-Einstein manifolds in Section 6.

Finally, we consider the variation of the total $Q$-prime curvature under deformations of real hypersurfaces. Let $S$ be a closed Sasakian $\eta$-Einstein manifold of dimension $2m-1$, and $(M_t)_{t\in(-1,1)}$ be a smooth family of closed real hypersurfaces in $C(S)$ such that $M_0 = S$. It can be proved that the total $Q$-prime curvature is a CR invariant of $M_t$ for any sufficiently small $t$, denoted by $\overline{Q}(M_t)$ (Corollary 5.3). Then we have

**Corollary 1.4.** Let $(S,T^{1,0}S,\eta)$ be a closed $(2m-1)$-dimensional Sasakian $\eta$-Einstein manifold, and $(M_t)_{t\in(-1,1)}$ be a smooth family of closed real hypersurfaces in $C(S)$ such that $M_0 = S$. Then the first variation of the total $Q$-prime curvature vanishes, that is,

$$\frac{d}{dt} \big|_{t=0} \overline{Q}(M_t) = 0.$$

Moreover, the second variation is written in terms of $P_{1,1}$, whose formula is obtained in Theorem 1.1. Spectral properties of $\Delta_b$ and $\xi$ give the following

**Theorem 1.5.** Let $(S,T^{1,0}S,\eta)$ and $(M_t)_{t\in(-1,1)}$ be as in Corollary 1.4. Assume that $m = 2$ or the Einstein constant is non-negative. Then, the second variation of the total $Q$-prime curvature is non-positive, that is,

$$\frac{d^2}{dt^2} \big|_{t=0} \overline{Q}(M_t) \leq 0.$$

Moreover, the equality holds if and only if $(M_t)_{t\in(-1,1)}$ is infinitesimally trivial as a deformation of CR structures (see Definition 2.3).
On the other hand, the conclusion of Theorem 1.5 does not hold for Sasaki\-an \(\eta\)-Einstein manifolds of dimension greater than three and with negative Einstein constant.

**Theorem 1.6.** For each integer \(m \geq 3\), there exist a closed Sasaki\-an \(\eta\)-Einstein manifold of dimension \(2m-1\) with negative Einstein constant and an infinitesimally non-trivial smooth deformation such that the second variation of the total \(Q\)-prime curvature along this deformation is equal to zero. If \(m\) is odd, one can also find an example of a Sasaki\-an \(\eta\)-Einstein manifold and a smooth deformation such that the second variation of the total \(Q\)-prime curvature along this deformation is positive.

This paper is organized as follows. In Section 2 we recall basic concepts of CR manifolds, ambient spaces, and Sasaki\-an manifolds. Section 3 is devoted to the construction of a Fefferman defining function. In Section 4 we provide the proofs of Theorems 1.1 and 1.2. Section 5 deals with the variation of the total \(Q\)-prime curvature. In Section 6 we compute the total \(Q\)-prime curvature for some examples.

**Notation.** We use Einstein’s summation convention and assume that
- uppercase Latin indices \(A, B, C,\ldots\) run from 0 to \(m\);
- lowercase Latin indices \(a, b, c,\ldots\) run from 1 to \(m\);
- lowercase Greek indices \(\alpha, \beta, \gamma, \ldots\) run from 1 to \(m-1\).

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### 2. Preliminaries

#### 2.1. CR manifolds.

Let \(M\) be a smooth \((2m-1)\)-dimensional manifold without boundary. A CR structure is a complex \((m-1)\)-dimensional subbundle \(T^{1,0}M\) of the complexified tangent bundle \(TM \otimes \mathbb{C}\) such that
- \(T^{1,0}M \cap T^{0,1}M = 0\), where \(T^{0,1}M = T^{1,0}M^*\);
- \([\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M)\).

For example, if \(M\) is a real hypersurface in a complex manifold \(X\), then \(M\) has the natural CR structure
\[
T^{1,0}M = (TM \otimes \mathbb{C}) \cap T^{1,0}X.
\]
Set \(HM = \text{Re} T^{1,0}M\) and let \(J\): \(HM \to HM\) be a unique complex structure on \(HM\) such that
\[
T^{1,0}M = \ker(J - \sqrt{-1}) : HM \otimes \mathbb{C} \to HM \otimes \mathbb{C}.
\]
In the following, assume that there exists a nowhere vanishing 1-form \(\theta\) that annihilates \(HM\). The Levi form \(\mathcal{L}_\theta\) with respect to \(\theta\) is the Hermitian form on \(T^{1,0}M\) defined by
\[
\mathcal{L}_\theta(Z,W) = -\sqrt{-1} d\theta(Z,\overline{W}), \quad Z, W \in T^{1,0}M.
\]
We consider only strictly pseudoconvex CR manifolds, i.e., CR manifolds that has a positive definite Levi form for some \(\theta\); such a \(\theta\) is called a contact form. Denote by \(T\) the Reeb vector field with respect to \(\theta\), that is, a unique vector field satisfying
\[
\theta(T) = 1, \quad \nu_T d\theta = 0.
\]
Then the tangent bundle has the decomposition $TM = HM \oplus RT$. One can define the Riemannian metric $g_\theta$ on $M$ by

$$g_\theta(X, Y) = \frac{1}{2} d\theta(X, JY) + \theta(X) \theta(Y), \quad X, Y \in TM.$$  

Here, we extend $J$ to an endomorphism on $TM$ by $JT = 0$. Take a local frame $(Z_\alpha)$ of $T^{3,0}M$. Then we have a local frame $(T, Z_\alpha, Z_\overline{\beta} = \overline{Z_\beta})$ of $TM \otimes \mathbb{C}$, and take the dual frame $(\theta, \theta^\alpha, \theta^{\overline{\beta}})$. For this frame, the 2-form $d\theta$ is written as follows:

$$d\theta = \sqrt{-1} l_{\alpha \beta} \theta^\alpha \wedge \theta^{\overline{\beta}},$$

where $(l_{\alpha \beta})$ is a positive Hermitian matrix. In the following, we will use $l_{\alpha \beta}$ and its inverse $l^{\alpha \beta}$ to lower and raise indices of various tensors.

A contact form $\theta$ induces a canonical connection $\nabla$, called the Tanaka-Webster connection with respect to $\theta$. This connection is given by

$$\nabla Z_\alpha = \omega^\beta_\alpha Z_\beta, \quad \nabla Z_{\overline{\beta}} = \omega_{\overline{\beta} \alpha} Z_\alpha, \quad \nabla T = 0,$$

and $\omega^\beta_\alpha, \omega_{\overline{\beta} \alpha} = \overline{\omega^\alpha_\beta}$ satisfy the following structure equations:

$$dl_{\alpha \beta} = \omega_{\alpha \beta} + \omega_{\beta \alpha},$$

$$d\theta^\alpha = \theta^{\overline{\beta}} \wedge \omega_{\alpha \beta} + A^\alpha_{\overline{\beta} \gamma} \theta^{\overline{\beta}} \wedge \theta^{\gamma},$$

Here the tensor $A^\alpha_{\overline{\beta} \gamma} = A_{\overline{\beta} \gamma}^\alpha$ is symmetric and called the Tanaka-Webster torsion. We denote the components of successive covariant derivatives of a tensor by subscripts preceded by the comma, for example, $K_{\alpha \beta \gamma}^\delta$; we omit the comma if the derivatives are applied to functions. With this notation, introduce the operator $\overline{\partial}_b$ acting on $C^\infty(M)$ by

$$\overline{\partial}_b f = f_{\alpha} \theta^\alpha.$$

A smooth function $f$ is called a CR holomorphic function if $\overline{\partial}_b f = 0$. A CR pluriharmonic function is a real-valued smooth function that is locally the real part of a CR holomorphic function. We denote by $\mathcal{P}$ the space of CR pluriharmonic functions. If $M$ is a real hypersurface in a complex manifold $X$, then any CR holomorphic function (resp. CR pluriharmonic function) can be extended to a holomorphic function (resp. pluriharmonic function) on the pseudoconvex side.

The curvature form $\Omega^\alpha_{\beta \gamma}$ of $\nabla$ is given by

$$\Omega^\alpha_{\beta \gamma} = R^\alpha_{\beta \gamma \delta} \theta^\delta \wedge \theta^\gamma + A^{\alpha \gamma}_{\beta \delta} \theta^\delta \wedge \theta - A^\alpha_{\overline{\gamma} \delta \gamma} \theta^\delta \wedge \theta + \sqrt{-1} l_{\alpha \delta} A^\beta_{\overline{\gamma} \delta \gamma} \theta^\delta \wedge \theta^{\overline{\gamma}}.$$  

We call the tensor $R^\alpha_{\beta \gamma \delta}$ the Tanaka-Webster curvature. This tensor has the symmetry

$$R^\alpha_{\alpha \beta \gamma} = R^\beta_{\beta \gamma \alpha} = R^\gamma_{\gamma \alpha \beta} = R^\delta_{\delta \alpha \beta}.$$

Define the Tanaka-Webster Ricci curvature $\text{Ric}_{\alpha \beta}$ by

$$\text{Ric}_{\alpha \beta} = R^\gamma_{\alpha \gamma \beta} = R^\gamma_{\beta \gamma \alpha} = R^\gamma_{\alpha \beta \gamma} = R^\gamma_{\beta \gamma \alpha},$$

Commutators of derivatives are important for computation. However, we postpone formulas of commutators until we introduce Sasakian manifolds, since commutation relations are simplified for such CR manifolds.
2.2. Ambient space. Let $X$ be an $m$-dimensional complex manifold and $\pi_X: \mathcal{X} = K_X^\infty \to X$ the total space of the canonical bundle of $X$ with the zero section removed. For $\mu \in \mathbb{C}^\times$, define the dilation $\delta_\mu: \mathcal{X} \to \mathcal{X}$ by the scalar multiplication $\delta_\mu(\xi) = \mu^{m+1} \xi$ for $\xi \in \mathcal{X}$. Denote by $Z_\mu$ the holomorphic vector field generating $\delta_\mu$, that is, $Z_\mu = (d/d\mu)|_{\mu=1}$. A smooth function $f$ on an open set of $\mathcal{X}$ is said to be homogeneous of degree $(w, w')$ for $w, w' \in \mathbb{R}$ if $Z_\mu f = w f$ and $Z_\mu f = w' f$ hold. We write $\tilde{\mathcal{X}}(w, w')$ for the space of smooth homogeneous functions of degree $(w, w')$.

To simplify notation, write $\tilde{\mathcal{X}}(w, w')$ and for each homogeneous function $f$ of degree $(w, w')$, there exists a smooth function $f$ locally on $X$ such that $f = (z^0)^w (z^0)^{w'} f$. Let $M$ be a strictly pseudoconvex real hypersurface in $X$ and $\mathcal{M} = \pi_X^{-1}(M)$. Then $\mathcal{M}$ is a pseudoconvex real hypersurface in $\mathcal{X}$, and set $\mathcal{M}(w, w')$ for $\mathcal{X}(w, w')|_{\mathcal{M}}$.

For a defining function $\rho \in \tilde{\mathcal{X}}(1)$ of $\mathcal{M}$, the $(1, 1)$-form $dd^c \rho$ defines the Lorentz-Kähler metric $g[\rho]$ in a neighborhood of $\mathcal{M}$, where $dd^c = (\sqrt{-1}/2)(\partial - \bar{\partial})$. We normalize $\rho$ by a complex Monge-Ampère equation. Take the tautological $(m, 0)$-form $\zeta$ on $K_X$. Then
\[
\text{vol}_X = (\sqrt{-1})^{(m+1)^2} d\zeta \wedge d\overline{\zeta}
\]
gives a volume form on $\mathcal{X}$.

**Proposition 2.1** (from [15] Proposition 2.2). There exists a defining function $\rho \in \tilde{\mathcal{X}}(1)$ of $\mathcal{M}$ such that
\[
(dd^c \rho)^{m+1} = k_m (1 + O_{m+1}) \text{vol}_X,
\]
where $O \in \tilde{\mathcal{X}}(-m - 1)$ and $k_m = -m!(m + 1)$. Moreover, such a $\rho$ is unique modulo $O(\rho^{m+1})$ and $O$ modulo $O(\rho)$ is independent of the choice of $\rho$.

We call such a $\rho$ a Fefferman defining function and the Lorentz-Kähler metric $g[\rho]$ with respect to $\rho$ an ambient metric. The function $O$ is called the obstruction function.

Next we introduce the pseudo-Einstein condition for a contact form on $M$, which is necessary for the definition of the $Q$-prime curvature. To this end, review a correspondence between Hermitian metrics of $K_X$ and defining functions of $M$. For a Hermitian metric $h$ of $K_X$, the function $\rho \cdot h^{-1/(m+1)} \in \tilde{\mathcal{X}}(0)$ gives a defining function of $M$. Conversely, let $\rho$ be a defining function of $M$. Then $h_\rho = (\rho/\rho)^{m+1} \in \tilde{\mathcal{X}}(m+1)$ defines a Hermitian metric of $K_X$ near $M$. Moreover, if $\rho$ is normalized by a contact form $\theta$, i.e., $\theta = dd^c \rho|_M$, $h_\rho|_M \in \tilde{\mathcal{X}}(m+1)$ depends only on $\theta$, denoted by $h_\theta$. In particular, if we fix a contact form $\theta$, then multiplication by $h_\theta^{-w/(m+1)}$ defines an identification between $\tilde{\mathcal{X}}(w)$ and $C^\infty(M)$.

**Definition 2.2.** A contact form $\theta$ on $M$ is said to be **pseudo-Einstein** if there exists a defining function $\rho$ normalized by $\theta$ such that $h_\rho$ is flat on the pseudoconvex side; see [15] Proposition 2.6] for equivalent conditions. For another contact form $\tilde{\theta} = e^T \theta$, $\tilde{\theta}$ is pseudo-Einstein if and only if $\mathcal{Y} \in \mathcal{P}$.

Before the end of this subsection, note that there exists a canonical bijection between $\tilde{\mathcal{X}}(-m)$ and the space of volume forms on $M$. For $\varphi \in \tilde{\mathcal{X}}(-m)$, the $(2m - 1)$-form $\varphi d^c \rho \wedge (dd^c \rho)^{m-1}$ descends to a volume form on $M$. Hence we denote by $\int_M \varphi$ the integral of the volume form corresponding to $\varphi \in \tilde{\mathcal{X}}(-m)$ for a closed CR manifold $M$. 

2.3. Deformation of CR structures. This subsection deals with deformations of real hypersurfaces in a complex manifold and corresponding deformations of CR structures.

Let $M$ be a closed strictly pseudoconvex real hypersurface in an $m$-dimensional complex manifold $X$, and $(M_t)_{t \in (-1,1)}$ be a smooth family of closed real hypersurfaces in $X$ such that $M_0 = M$. Take a Fefferman defining function $\rho_t$ of $M_t = \pi_X^*(M)$ that is smooth in $t$. Then $(d/dt)|_{t=0} \rho_t|_M \in \mathcal{E}(1)$ is independent of the choice of $\rho_t$. Conversely, for any real-valued function $\varphi \in \mathcal{E}(1)$, there exists a smooth family $(M_t)_{t \in (-1,1)}$ such that $\varphi = (d/dt)|_{t=0} \rho_t|_M$. Thus the space of infinitesimal deformations of real hypersurfaces is naturally parametrized by $\mathcal{E}(1)$. The space of real-valued functions in $\mathcal{E}(1)$, denoted by $\mathfrak{D}(M, T^{1,0}M)$, is a linear subspace of $\Gamma(\operatorname{Hom}(T^{0,1}M, T^{1,0}M))$. Each infinitesimal deformation of real hypersurfaces induces that of CR structures.

Definition 2.3. A smooth family $(M_t)_{t \in (-1,1)}$ of closed real hypersurfaces is said to be infinitesimally trivial as a deformation of CR structures if $(d/dt)|_{t=0} \rho_t|_M \in \text{Reker } D$.

As stated in Section 2.2, a contact form $\theta$ on $M$ gives an identification between $\mathcal{E}(1)$ and $C^\infty(M)$. Thus we obtain a differential operator

$$D_\theta: C^\infty(M) \to \mathfrak{D}(M, T^{1,0}M),$$

written in terms of the Tanaka-Webster connection as follows:

$$2(D_\theta F)_{\bar{\pi} \pi} = F_{\bar{\pi} \pi} - \sqrt{-1} A_{\bar{\pi} \pi} F.$$

2.4. Ambient construction. In this subsection, we recall CR invariant powers of the sub-Laplacian, the $P$-prime operator, and $Q$-prime operator, which are main subjects in this paper.

Let $\Delta$ be the $\bar{\partial}$-Laplacian with respect to an ambient metric $g[\rho]$. This operator maps $\mathcal{E}(w, w')$ to $\mathcal{E}(w - 1, w' - 1)$.

Lemma 2.4 ([10] Theorem 1.1]). Let $(w, w') \in \mathbb{R}^2$ such that $k = w + w' + m$ is a positive integer. Then for $f \in \mathcal{E}(w, w')$,

$$(\Delta^k \tilde{f})|_M \in \mathcal{E}(w - k, w' - k)$$

depends only on $f = \tilde{f}|_M$ and defines a differential operator

$$P_{w, w'}: \mathcal{E}(w, w') \to \mathcal{E}(w - k, w' - k).$$

Moreover, the operator $P_{w, w'}$ is independent of the choice of a Fefferman defining function if $k \leq m$.

When $w = w' = 1$, the operator $P_{1,1}$ depends on the choice of a Fefferman defining function. However, a slight modification gives a new CR invariant differential operator, which is closely related to the variation of the total $Q$-prime curvature.
Lemma 2.5 ([15] Lemma A.2]). Let $\nabla$ be the Levi-Civita connection with respect to the Lorentz-Kähler metric $g(\rho)$. Then for $f \in \mathcal{E}(1)$,
\[
[\text{Re}(\Delta^m \nabla^{AB})f]_{|\mathcal{M}} \in \mathcal{E}(-m - 1)
\]
depends only on $f = \tilde{f}|_{|\mathcal{M}}$ and defines a differential operator
\[
R: \mathcal{E}(1) \to \mathcal{E}(-m - 1).
\]
Moreover, $R$ is independent of the choice of a Fefferman defining function. The operator $R$ coincides with $P_{1,1}$ if we choose $\rho$ such that the obstruction function $\mathcal{O}$ vanishes near $\mathcal{M}$.

To define the $Q$-prime curvature, we need to fix a pseudo-Einstein contact form on $M$.

Definition 2.6 ([14] Definition 5.4]). Let $\theta$ be a pseudo-Einstein contact form on $M$ and $\rho$ a defining function of $M$ normalized by $\theta$ such that $h_\rho$ is flat on the pseudoconvex side. The $Q$-prime curvature $Q'_\theta$ is defined by
\[
Q'_\theta = (m + 1)^{-1} |\Delta^m (\log h_\rho)^2|_{|\mathcal{M}} \in \mathcal{E}(-m).
\]

If we take another pseudo-Einstein contact form $\tilde{\theta} = e^T \theta$ for $\Upsilon \in \mathcal{P}$, the $Q$-prime curvature transforms as follows [14] Proposition 5.5):
\[
Q'_{\theta'} = Q'_\theta + 2P'_{\Upsilon} \Upsilon + P_{0,0}(\Upsilon^2).
\]
Here $P'_{\Upsilon}: \mathcal{P} \to \mathcal{E}(-m)$ is the $P$-prime operator defined by
\[
P'_{\Upsilon} \Upsilon = -(m + 1)^{-1} |\Delta^m (\tilde{\Upsilon} \log h_\rho)|_{|\mathcal{M}},
\]
where $\tilde{\Upsilon}$ is a smooth extension of $\Upsilon$ that is pluriharmonic on the pseudoconvex side [14] Definition 4.2]. Since $P_{0,0}$ is formally self-adjoint and annihilates constant functions [10] Proposition 5.1], we have
\[
\int_M Q'_\theta = \int_M Q'_\theta + 2 \int_M P'_{\Upsilon} \Upsilon.
\]
for a closed CR manifold $M$. Hence the integral of the $Q$-prime curvature, the total $Q$-prime curvature, defines a global CR invariant if the second term on the right hand side is equal to zero for every $\Upsilon \in \mathcal{P}$. A sufficient condition for its CR invariance is that $P_{\Upsilon}'$ is formally self-adjoint; this is because $P_{\Upsilon}' 1 = 0$.

2.5. Sasakian manifolds. This subsection contains a brief summary of Sasakian manifolds from CR point of view. See [1] and [20] for a comprehensive introduction to Sasakian manifolds. Let $(S, T^{1,0} S)$ be a strictly pseudoconvex CR manifold, $\eta$ a contact form on $S$, and $\xi$ the Reeb vector field with respect to $\eta$.

Definition 2.7. The triple $(S, T^{1,0} S, \eta)$ is called a Sasakian manifold if the Tanaka-Webster torsion with respect to $\eta$ vanishes.

For a Sasakian manifold $(S, T^{1,0} S, \eta)$, the almost complex structure $I$ on the cone $C(S) = \mathbb{R}^+ \times S$ of $S$ is defined by
\[
I(V + a\xi + b(r\partial_r)) = JV + b\xi - a(r\partial_r),
\]
where $r$ is the coordinate of $\mathbb{R}^+$, $V \in HS$ and $a, b \in \mathbb{R}$. Then this almost complex structure is integrable, that is, $(C(S), I)$ is a complex manifold. The Riemannian metric $\tilde{g} = dr \otimes dr + r^2 g_0$ is a Kähler metric on $C(S)$ and its Kähler form $\omega$ is equal to $d\phi r^2/2$. Moreover, the level set $\{r = 1\}$ is isomorphic to $S$ as CR manifolds and the 1-form $\eta$ is equal to $d\phi \log r^2$. 
Consider the Tanaka-Webster connection with respect to $\eta$. Note that the index 0 is used for the component $T$ or $\theta$ in our index notation. The commutators of the derivatives for $f \in C^\infty(X)$ are given by

$$f_{[\alpha\beta]} = 0, \quad 2f_{[\alpha\beta \gamma]} = \sqrt{-1}R_{\alpha\beta\gamma}f_0, \quad f_{[\alpha\gamma]} = 0,$$

where $[\cdots]$ means the antisymmetrization over the enclosed indices. Define the Kohn Laplacian $\Box_b$ and sub-Laplacian $\Delta_b$ by

$$\Box_b f = -f^\alpha_{\alpha\beta}, \quad \Delta_b f = -f^\alpha_{\alpha\beta} - f^\alpha_{\alpha} = \Box_b f + \square_b f,$$

respectively. From the above commutation relations, we have

$$\Box_b - \square_b = \sqrt{-1}(m - 1)\xi.$$

The third covariant derivatives of $f$ satisfy

$$f^\alpha_{\alpha[\gamma\delta]} = 0, \quad f^\alpha_{\alpha[\gamma\delta \eta]} = 0,$$

(2.3)

$$2f^\alpha_{\alpha[\gamma\delta \eta]} = \sqrt{-1}R_{\alpha\gamma\delta}f_0 + R_{\alpha\gamma\delta}f_\eta.$$

From these, it follows that the Kohn Laplacian and sub-Laplacian commute with the Reeb vector field $\xi$. See [17, Lemma 2.3] for the proofs of these commutation relations.

We next consider an Einstein condition for Sasakian manifolds.

**Definition 2.8.** Let $(S, T^{1,0}S, \eta)$ be a $(2m - 1)$-dimensional Sasakian manifold. It is called a Sasakian $\eta$-Einstein manifold if there exists a constant $\lambda$ such that the Tanaka-Webster Ricci curvature $\text{Ric}_{\alpha\beta}$ of $\eta$ satisfies

$$\text{Ric}_{\alpha\beta} = m\lambda l_{\alpha\beta}.$$

In particular if $\lambda = 1$, it is called a Sasaki-Einstein manifold. In this paper, we call the constant $m\lambda$ the Einstein constant of $(S, T^{1,0}S, \eta)$.

There exist characterizations of Sasakian $\eta$-Einstein manifolds in terms of $g_\eta$ or $\overline{\eta}$.

**Proposition 2.9.** Let $(S, T^{1,0}S, \eta)$ be a $(2m - 1)$-dimensional Sasakian manifold and $\lambda$ a real constant. Then the following are equivalent:

1. $(S, T^{1,0}S, \eta)$ is a Sasakian $\eta$-Einstein manifold with Einstein constant $m\lambda$;
2. the Ricci curvature $\text{Ric}_{\eta}$ of $g_\eta$ satisfies
   $$\text{Ric}_{\eta} = 2(m\lambda - 1)g_\eta + 2m(1 - \lambda)\eta \otimes \eta;$$
3. the Ricci form of $\overline{\eta}$ is equal to $m(\lambda - 1)d\eta$, or, the Ricci curvature $\text{Ric}_{\overline{\eta}}$ of $\overline{\eta}$ satisfies
   $$\text{Ric}_{\overline{\eta}} = 2m(\lambda - 1)(g_\eta - \eta \otimes \eta).$$

**Proof.** First, we show the equivalence between (1) and (2). We denote by $\nabla^{g_\eta}$ the Levi-Civita connection with respect to $g_\eta$. Then for $U, V \in \Gamma(TS)$,

$$\nabla^{g_\eta} U V = \nabla_U V - g_\eta(U,V)\xi + \eta(U)JV + \eta(V)JU.$$ 

This follows from the fact that the Tanaka-Webster connection preserves the metric $g_\eta$ and the torsion $\text{Tor}$ of $\nabla$ satisfies $\text{Tor}(U,V) = 2g_\eta(U,V)\xi$. Hence the curvature $R_{g_\eta}$ of $\nabla^{g_\eta}$ is related with the curvature $R$ of $\nabla$ as follows:

$$R_{g_\eta}(U,V)W = R(U,V)W - g_\eta(JU,W)JU + g_\eta(JV,W)JU$$

(2.4)

$$- 2g_\eta(U,JV)JW - \eta(V)g_\eta(JU,JW)\xi$$

$$+ \eta(U)g_\eta(JV,JW)\xi - \eta(U)\eta(W)V + \eta(V)\eta(W)U$$

for $U, V, W \in \Gamma(TS)$. Taking the trace of (2.4) gives

$$\text{Ric}_{g_\eta} = \text{Ric} - 2g + 2m \eta \otimes \eta,$$
where $\text{Ric}$ is the Ricci curvature of $\nabla$. On the other hand, $\text{Ric}$ is given by

$$\text{Ric} = \text{Ric}_{\overline{\nabla}}(\theta^\alpha \otimes \theta^\beta + \theta^\alpha \otimes \theta^\beta),$$

which follows from (2.4). This proves the equivalence between (2) and (4).

Next, we show the equivalence between (2) and (3). Let $\overline{\nabla}$ be the Levi-Civita connection with respect to $\overline{g}$. Then for $U, V \in \Gamma(TS)$,

$$\nabla^\overline{g}_{r \delta}(r \partial_r) = r \partial_r, \quad \nabla^\overline{g}_{\delta}(U, r \partial_r) = U,$$

$$\nabla^\overline{g}_V U = \nabla^\overline{g}_U V - g_\eta(U, V) r \partial_r.$$

Hence the curvature $R_{\overline{g}}$ of $\overline{\nabla}$ satisfies

$$R_{\overline{g}}(\cdot, r \partial_r) = 0, \quad R_{\overline{g}}(\cdot, r \partial_r) = 0,$$

$$R_{\overline{g}}(U, V)W = R_\eta(U, V)W - g_\eta(V, W)U + g_\eta(U, W)V,$$

where $U, V, W \in \Gamma(TS)$. Taking the trace, we obtain

$$\text{Ric}^\overline{g} = \text{Ric}_\eta - 2(m - 1)g_\eta.$$

Therefore, (2) is equivalent to (3). \hfill \Box

Note that (3) is equivalent to

(2.5) $$- dd^c \log \det (\partial_{ab}r^2) = m(\lambda - 1)dd^c \log r^2$$

for any holomorphic coordinate $(z^1, \ldots, z^m)$ of $C(S)$ since $r^2/2$ is a Kähler potential of $\overline{g}$.

**Example 2.10.** Let $S^{2m-1} \subset \mathbb{C}^m$ be the unit sphere centered at the origin with the canonical CR structure, and $\eta_0$ be the contact form on $S^{2m-1}$ defined by

$$\eta_0 = \frac{\sqrt{-1}}{2} \sum_{i=1}^m (z^i d\overline{z}^i - \overline{z}^i dz^i)|_{S^{2m-1}}.$$

Then the triple $(S^{2m-1}, T^{1,0}S^{2m-1}, \eta_0)$ is a Sasakian manifold. Moreover, the cone $(C(S^{2m-1}), \overline{g})$ is isomorphic to $(\mathbb{C}^m \setminus \{0\}, g_{\text{Eucl}})$ as Kähler manifolds by the map $C(S^{2m-1}) \ni (r, p) \mapsto r^2 p \in \mathbb{C}^m$. Here, $g_{\text{Eucl}}$ is the Euclidean metric on $\mathbb{C}^m$. Hence the metric $\overline{g}$ is Ricci-flat, and $(S^{2m-1}, T^{1,0}S^{2m-1}, \eta_0)$ is a Sasaki-Einstein manifold. Note that there exists a canonical projection $S^{2m-1} \rightarrow \mathbb{CP}^{m-1}$, and the 2-form $\omega_{FS}$ descends to the Fubini-Study form $\omega_{FS}$ on $\mathbb{CP}^{m-1}$.

**Example 2.11.** Let $Y$ be an $(m - 1)$-dimensional complex manifold, $L$ a holomorphic line bundle over $Y$, and $h$ a Hermitian metric of $L$. Assume that the $(1,1)$-form $\omega_h = dd^c \log h$ defines a Kähler-Einstein metric on $Y$ with Einstein constant $m\lambda$. Consider the tube $S = \{v \in L \mid h(v, v) = 1\} \subset L$ and the contact form $\eta = d^c \log h$ on $S$. Then the triple $(S, T^{1,0}S, \eta)$ is a Sasakian $\eta$-Einstein manifold with Einstein constant $m\lambda$. Note that the Reeb vector field $\xi$ with respect to $\eta$ is the generator of the $S^1$-action on $S$ induced from that on $L$.

### 3. Construction of Fefferman defining function

In this section, we construct a Fefferman defining function for Sasakian $\eta$-Einstein manifolds. To this end, we first construct a "good" defining function $\rho_S$ of $S$ in $C(S)$. From this defining function, we obtain a flat Hermitian metric $h_S$ of $K_{C(S)}$, and the desired Fefferman defining function $\rho_S$ is given as the product $\rho_S \cdot h_S^{1/(m+1)}$. 


Let \((S, T^{1,0}_S, \eta)\) be a \((2m - 1)\)-dimensional Sasakian \(\eta\)-Einstein manifold with Einstein constant \(m\lambda\). Regard \(S\) as a real hypersurface in a complex manifold \(X = C(S)\). Define the smooth function \(\psi_\lambda\) on \(\mathbb{R}\) by
\[
\psi_\lambda(x) = \begin{cases} 
\lambda^{-1}(\exp(\lambda x) - 1) & \text{if } \lambda \neq 0, \\
x & \text{if } \lambda = 0.
\end{cases}
\]
It can be seen that
\[
\psi_\lambda' = 1 + \lambda \psi_\lambda, \quad \psi_\lambda'' = \lambda \psi_\lambda',
\]
and
\[
\rho_S = \psi_\lambda(\log r^2) \in C^\infty(X)
\]
is a defining function of \(S\) normalized by \(\eta\).

**Proposition 3.1.** The defining function \(\rho_S\) satisfies the equation
\[
ddf \log J_z[\rho_S] = 0,
\]
where \(z = (z^1, \ldots, z^m)\) is a local coordinate of \(X\) and
\[
J_z[\phi] = -\det \left( \frac{\partial \phi}{\partial z^a} \frac{\partial \phi}{\partial z^b} \right) \frac{\partial^2 \phi}{\partial z^a \partial z^b}.
\]

**Proof.** To simplify notation, we write \(\partial_a = \partial/\partial z^a\) and \(\partial_{\bar{a}} = \partial/\partial z^\bar{a}\).

\[
J_z[\rho_S] = -\det \left( \frac{\rho_S}{\partial_{\bar{a}} \rho_S} \frac{\partial_a \rho_S}{\partial_{\bar{a}} \rho_S} \right)
= -\det \left( \frac{\rho_S}{\partial_{\bar{a}} \log r^2} \frac{1 + \lambda \rho_S}{1 + \lambda \rho_S} \partial_a \log r^2 \right)
= -(1 + \lambda \rho_S)^m \det \left( \frac{\rho_S}{\partial_{\bar{a}} \log r^2} \frac{\partial_a \log r^2}{\partial_{\bar{a}} \log r^2} \right)
= -(1 + \lambda \rho_S)^m \det \left( \frac{\rho_S}{r^2 \partial_{\bar{a}} \log r^2} \frac{r^{-2} \partial_a \log r^2}{r^{-2} \partial_{\bar{a}} \log r^2} \right)
= -(1 + \lambda \rho_S)^m \log(1 + \lambda \rho_S) \det \left( \frac{\partial_a \rho_S}{\partial_{\bar{a}} \rho_S} \right)
\]

Since \(r^2/2\) is a Kähler potential of \(\mathcal{T}\), \(\partial_{\bar{a}} \rho_S = 2\mathcal{T}(\partial_a, \partial_{\bar{a}})\). Thus
\[
J_z[\rho_S] = -(1 + \lambda \rho_S)^m \log(\rho_S - (1 + \rho_S)(2r)^{-2} \|\partial_a \log r^2\|^2) \det(\partial_{\bar{a}} \rho_S)
= (1 + \lambda \rho_S)^m \log(\rho_S - (1 + \rho_S)(2r)^{-2} \|\partial_a \log r^2\|^2),
\]

where the last equality follows from \(\|\partial_a \log r^2\|^2 = 2r^2\). Therefore from \(2.3\),
\[
-ddf \log J_z[\rho_S] = -\log(1 + \lambda \rho_S) + mddf \log r^2 - ddf \log \det(\partial_{\bar{a}} \rho_S)
= -m\lambda \log r^2 + m\lambda \log r^2
= 0.
\]
This proves the statement. \(\square\)

Next, we construct a flat Hermitian metric of \(K_X\) by using \(\rho_S\).

**Lemma 3.2.** For each point \(p \in X\), there exists a local coordinate \(z = (z^1, \ldots, z^m)\) near \(p\) such that \(J_z[\rho_S] = 1\). Moreover, if \(w = F(z)\) is also such a local coordinate, then \(F'\) is a locally constant function with the absolute value one, where \(F'\) is the holomorphic Jacobian of \(F\).

**Definition 3.3.** A local coordinate \(z\) is called a flat local coordinate if \(J_z[\rho_S] = 1\) holds.
Proof. Take a flat local coordinate \( w = (w^1, \ldots, w^m) \) near \( p \). From Proposition 3.1 \( \log J_w[p_S] \) is a pluriharmonic function. We may assume that this is the real part of a holomorphic function \( f \), that is,

\[
J_w[p_S] = e^{\Re f},
\]

if we take a sufficiently small neighborhood of \( p \). Take a holomorphic function \( g \) such that \( \partial g / \partial w^1 = e^{f/2} \). In general, for another coordinate \( w' = G(w) \) of \( X \),

\[
J_{w'}[\phi] = |\det G'|^{-2} J_w[\phi].
\]

holds. Thus the new local coordinate \( z = (z^1 = g(w), z^2 = w^2, \ldots, z^m = w^m) \) satisfies \( J_z[p_S] = 1 \). The second statement follows from (3.1) and the fact that a holomorphic function with its absolute value one is locally constant. \( \square \)

**Corollary 3.4.** There exists a unique flat Hermitian metric \( h_S \) on \( K_X \) such that \( dz^1 \wedge \cdots \wedge dz^m \) is a local orthonormal frame of \( K_X \) for any flat local coordinate \( z \), or equivalently, \( h_S \) is written as \( |z^0|^{2(m+1)} \), where \( z^0 \) is a branched fiber coordinate with respect to \( z \).

**Proposition 3.5.** The defining function \( \rho_S = \rho_S \cdot h_S^{1/(m+1)} \in \tilde{\mathcal{E}}(1) \) of \( S = \pi_X^{-1}(S) \) satisfies

\[
(dd^c \rho_S)^{m+1} = k_m \text{vol}_X.
\]

In particular, the obstruction function \( \mathcal{O} \) vanishes on \( X \).

**Proof.** Take a flat local coordinate \( (z^1, \ldots, z^m) \) and a branched fiber coordinate \( z^0 \) with respect to \( z \). Then the volume form \( \text{vol}_X \) is written as

\[
\text{vol}_X = (\sqrt{-1})^{m+1} (m+1)! |z^0|^{2m} dz^0 \wedge d\bar{z}^0 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m.
\]

On the other hand, the \((m+1,m+1)\)-form \((dd^c \rho_S)^{m+1}\) is of the form

\[
(\sqrt{-1})^{m+1} (m+1)! \det (\partial^2 \rho_S / \partial z^A \partial \bar{z}^B) dz^0 \wedge d\bar{z}^0 \wedge d\bar{z}^0 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m.
\]

Thus it suffices to show that

\[
\det (\partial^2 \rho_S / \partial z^A \partial \bar{z}^B) = -|z^0|^{2m},
\]

which follows from the computation below:

\[
\det (\partial^2 \rho_S / \partial z^A \partial \bar{z}^B) = \det \begin{pmatrix} \rho_S & z^0 \partial_a \rho_S \\ \partial_a \rho_S & |z^0|^2 \partial_a \rho_S \end{pmatrix} = |z^0|^{2m} \det \begin{pmatrix} \rho_S & \partial_a \rho_S \\ \partial_a \rho_S & \partial_a \rho_S \end{pmatrix} = -|z^0|^{2m} J_{z^0}[p_S] = -|z^0|^{2m}.
\]

Note that the last equality is a consequence of the definition of a flat local coordinate. \( \square \)

4. **Proof of factorization theorem**

This section is devoted to the proofs of Theorems 1.1 and 1.2 product formulas for CR invariant powers of the sub-Laplacian and the \( P \)-prime operator. To prove these, introduce operators that change homogeneous degrees.

**Definition 4.1.** Let \( z \) be a flat local coordinate of \( X = C(S) \) and \( z^0 \) a branched fiber coordinate with respect to \( z \). The operator \( M_{v,v'} : \tilde{\mathcal{E}}(w, w') \to \tilde{\mathcal{E}}(w+v, w'+v') \) is defined by the multiplication by \( (z^0)^v(z^0)^{v'} \). Note that \( M_{v,v} \) coincides with the multiplication by \( h_S^{v/(m+1)} \).
These multiplication operators $M_{w,w'}$ induce differential operators on $S$ corresponding to $P_{w,w'}$ and $P'_{\eta}$.

**Definition 4.2.** Let $(w, w') \in \mathbb{R}^2$ such that $k = w + w' + m$ is a positive integer.

The differential operator $P_{w,w'}$ on $C^\infty(S)$ is defined by

$$P_{w,w'} = M_{k-w,k-w'} \circ P_{w,w'} \circ M_{w,w'}.$$ 

Similarly, the operator $P'_{\eta}$: $\mathcal{P} \to C^\infty(S)$ is defined by

$$P'_{\eta} = M_{m,m} \circ P'_{\eta}.$$ 

These are independent of the choice of a flat local coordinate and a branched fiber coordinate.

In the following, we use the ambient metric $g = g[\rho s]$ for $\rho s$ defined in Proposition 4.3. The most important ingredient for the proofs of Theorems 1.1 and 1.2 is the following

**Proposition 4.3.**

\[ \Delta^k = M_{-k-1,0}(M_{2,0}\Delta)^k M_{-k+1,0}. \]  

If $k \geq 2$, then

\[ \Delta^k = M_{-1,-1} \Delta^{k-2} M_{0,k-1} \Delta M_{-k+2} \Delta M_{-k+1,0}. \]

We need additionally the following lemma to obtain the factorization formula for the $P$-prime operator.

**Lemma 4.4.** Let $\Upsilon$ be a CR pluriharmonic function on $S$ and $\tilde{\Upsilon}$ its pluriharmonic extension. Then the function $I\tilde{\Upsilon}$ defined by

$$I\tilde{\Upsilon} = M_{0,m-1} \Delta M_{m,-m+2} \Delta M_{-m+1,0}(\tilde{\Upsilon} \log |\rho|^2)$$

is an element of $\tilde{\mathcal{D}}(-1)$ modulo a term that vanishes to infinite order at $S$.

Computations of the homogeneous degrees in (4.1) and (4.2) give the following

**Corollary 4.5.** The operator $P_{w,w'}$ for $k = w + w' + m$ has the formula

$$P_{w,w'} = M_{-k-1,0} \prod_{j=0}^{k-1} (M_{2,0} L_{w'w+k-2j-1}) M_{-k+1,0},$$

where

$$L_{\mu} = P_{\mu-m+1 \mu-m+1}.$$ 

Similarly, the $P$-prime operator $P'_{\eta}$ is written as

$$P'_{\eta} \Upsilon = - M_{-1,-1} P_{-1,-1} [I\tilde{\Upsilon}] |S|.$$ 

From this observation, it suffices to prove Proposition 4.3 Lemma 4.4

\[ L_{\mu} = M_{\frac{\mu+1}{2},\frac{\mu+1}{2}} \circ L_{\mu} \circ M_{\frac{\mu-1}{2},\frac{\mu-1}{2}}. \]

and

\[ I\tilde{\Upsilon} |S| = -(m-1)^{-1} M_{-1,-1} (\Delta^2 + (m-1)^2 \lambda \Delta) \Upsilon \]

for the proofs of Theorems 1.1 and 1.2.

To show Proposition 4.3 define the differential operator $C$ by

$$C = [\Delta, M_{1,0}].$$

Key commutation relations are the following

**Lemma 4.6.**

$$[\Delta, M_{v,0}] = v M_{v-1,0} C, \quad [C, M_{v,v'}] = \lambda v' M_{v,v'-1}, \quad [\Delta, C] = 0.$$
Proof. Define the \((1,0)\)-vector field \(Z_m\) on \(X\) by
\[
Z_m = \frac{1}{2}(r \partial_r - \sqrt{-1} \xi),
\]
which is shown to be holomorphic. For a local frame \((Z_\alpha)\) of \(T^{1,0}S\), \((Z_0, Z_\alpha, Z_m)\) is that of \(T^{1,0}X\). With this frame, the matrix representations of \(g\) and its inverse are given by
\[
g_{\alpha\beta} = |z^0|^2 \begin{pmatrix}
\rho S & 0 & 1 + \lambda \rho S \\
0 & (1 + \lambda \rho S)l_\alpha & 0 \\
1 + \lambda \rho S & 0 & \lambda(1 + \lambda \rho S)
\end{pmatrix},
\]
and
\[
g^{A\overline{B}} = |z^0|^{-2} \begin{pmatrix}
-\lambda & 0 & 1 \\
0 & (1 + \lambda \rho S)^{-1}l_{\overline{\beta}} & 0 \\
1 & 0 & -\rho S(1 + \lambda \rho S)^{-1}
\end{pmatrix}.
\]
Denote by \(f\) the holomorphic function \(z^0\) for simplicity. Since the \(\partial\)-Laplacian is of the form \(-\nabla A^A\),
\[
C = -f_A \nabla A^A.
\]
Hence
\[
[\Delta, M_{e,0}] = -v f^{-1} f_A \nabla A^A = v M_{e-1,0} C,
\]
and
\[
[C, M_{e,e'}] = -v' f^{-1} f_A \nabla A^A = \lambda v' M_{e,e'-1}.
\]
Here, we use the fact that \(f_A \nabla A^A = -\lambda\), which follows from (4.5). Similarly,
\[
\Delta C = \nabla_B (f_A \nabla A^B)
= f_{AB} \nabla A^B + f_A \nabla_B A^B
= f_{AB} \nabla A^B + C \Delta;
\]
the last equality holds since \(g\) is Ricci-flat. Thus, it is sufficient to show that \(f_{AB} = 0\). From definition,
\[
f_{AB} = Z_B Z_A f - (\nabla Z_B Z_A)f,
\]
and
\[
g(\nabla Z_\alpha Z_A, Z_B) = Z_B (g(Z_A, Z_B)) - g(Z_A, [Z_B, Z_\alpha]).
\]
Since \(f = z^0\), we need only to consider the \(Z_0\)-component of \(\nabla Z_B Z_A\). Thus it is enough to compute the value \(g(\nabla Z_\alpha Z_A, Z_B)\) for \(C = 0\) or \(m\) from the matrix representation of \(g\). In this case,
\[
g(\nabla Z_\alpha Z_A, Z_B) = Z_B (g(Z_A, Z_B)),
\]
since the \((0,1)\)-vector fields \(Z_\alpha\) and \(Z_B\) are anti-holomorphic. Under these observations, a direct calculation shows \(Z_B Z_A f = (\nabla Z_\alpha Z_A) f\). □

The following lemma is a consequence of the above commutation relations.

**Lemma 4.7.**
\[
M_{-1,0} \Delta^k M_{1,0} = \Delta^k + k M_{-1,0} \Delta^{k-1} C.
\]
Proof.

\[
M_{-1,0} \Delta^k M_{1,0} = \Delta^k + \sum_{j=0}^{k-1} M_{-1,0} \Delta^{k-j-1} [\Delta, M_{1,0}] \Delta^j \\
= \Delta^k + \sum_{j=0}^{k-1} M_{-1,0} \Delta^{k-j-1} C \Delta^j \\
= \Delta^k + kM_{-1,0} \Delta^{k-1} C.
\]

This proves the statement. \(\square\)

Proof of Proposition 4.3 We prove (4.1) and (4.2) by induction in \(k\). First we prove (4.1). The case \(k=1\) is trivial. Assume that the formula holds for \(k\). Then,

\[
M_{-k-2,0}(M_{2,0}\Delta)^{k+1} M_{-k,0} = M_{-1,0} [M_{-k-1,0}(M_{2,0}\Delta)^k M_{-k+1,0}] [M_{k+1,0}\Delta M_{-k,0}] = M_{-1,0} \Delta^k (M_{1,0} \Delta - kC) = (\Delta^k + kM_{-1,0} \Delta^{k-1} C) \Delta - kM_{-1,0} \Delta^k C = \Delta^{k+1}.
\]

This proves the formula for \(k+1\). Next, consider (4.2). If \(k=2\),

\[
M_{-1,-1} M_{0,1} \Delta M_{2,0} \Delta M_{-1,0} = M_{-1,0} \Delta M_{1,0} \Delta - M_{-1,0} \Delta C = \Delta^2 + M_{-1,0} C \Delta - M_{-1,0} \Delta C = \Delta^2.
\]

Assume that the formula holds for \(k\). Then,

\[
M_{-1,-1} \Delta^{k-1} M_{0,k} \Delta M_{k+1,-k+1} \Delta M_{-k,0} = M_{-1,-1} \Delta^{k-1} [\Delta, M_{0,k-1}] M_{0,1} \Delta M_{k+1,-k+1} \Delta M_{-k,0} + M_{-1,-1} \Delta^{k-2} M_{0,k-1} \Delta M_{0,1} [\Delta, M_{0,-k+1}] M_{k+1,0} \Delta M_{-k,0} + M_{-1,-1} \Delta^{k-2} M_{0,k-1} \Delta M_{0,-k+2} [\Delta, M_{k,0}] M_{1,0} \Delta M_{-k,0} + M_{-1,-1} \Delta^{k-2} M_{0,k-1} \Delta M_{1,0} \Delta M_{-k,0} + M_{-1,-1} \Delta^{k-2} M_{0,k-1} \Delta M_{-k,2} \Delta M_{-k+1,0} \Delta = \Delta^{k+1}.
\]

This proves (4.2) for \(k+1\). \(\square\)

Proof of Theorem 4.4 What is left is to prove (4.3). Since \(\Delta = -g^{\alpha \overline{\beta}} \nabla_{\alpha} \overline{\beta}\), we need only to consider \(\partial \overline{\partial} f(Z_A, Z_{\overline{A}})\) for \(f \in \mathcal{D}(w, w')\) and \((A, \overline{A})\) with \(g^{\alpha \overline{\beta}} \neq 0\). Since \(Z_0\) and \(Z_m\) are holomorphic vector fields,

\[
\partial \overline{\partial} f(Z_0, Z_{\overline{A}}) = w w' f, \quad \partial \overline{\partial} f(Z_0, Z_{\overline{m}}) = w Z_{\overline{m}} f, \\
\partial \overline{\partial} f(Z_m, Z_{\overline{A}}) = w' Z_m f, \quad \partial \overline{\partial} f(Z_m, Z_{\overline{m}}) = \frac{1}{2} (Z_m Z_{\overline{m}} + Z_{\overline{m}} Z_m) f.
\]

On the other hand, the commutator \([Z_\alpha, Z_{\overline{m}}]\) is equal to

\[
\nabla Z_\alpha Z_{\overline{m}} - \nabla Z_{\overline{m}} Z_\alpha + l_\alpha(Z_m - Z_{\overline{m}}).
\]
Thus we have
\[
\partial \bar{\partial} \tilde{f}(Z_\alpha, Z_\beta) = Z_\alpha Z_\beta \tilde{f} - \bar{\partial} \tilde{f}([Z_\alpha, Z_\beta])
\]
\[
= \frac{1}{2} (Z_\alpha Z_\beta + Z_\beta Z_\alpha) \tilde{f} + \frac{1}{2} \partial \bar{\partial} \tilde{f}([Z_\alpha, Z_\beta]) = \frac{1}{2} \tilde{f}(Z_\alpha, Z_\beta).
\]
Hence for \(\tilde{f} \in C^\infty(X)\),
\[
(M_{1-w,1-w} \circ \Delta \circ M_{w,w}) \tilde{f}|_S = \left( \frac{1}{2} \Delta_b + \lambda w' \right) \tilde{f}|_S + \left( -\frac{m-1}{2} - w' \right) (Z_m \tilde{f})|_S
\]
(4.6)
In particular, taking \((w, w') = ((-\mu - m + 1)/2, (\mu - m + 1)/2)\), we have
\[
M_{1-w,1-w} \circ L_\mu \circ M_{w,w} = \frac{1}{2} \Delta_b + \frac{\sqrt{-1}}{2} \mu \xi + \frac{\mathbf{4}}{4} \lambda (m - 1 - \mu)(m + 1 + \mu)
\]
\[
= \lambda|_\mu.
\]
This proves the theorem. \(\square\)

**Proof of Lemma 4.4 and Theorem 1.2.** It is sufficient to show Lemma 4.4 and (1.3). First, note that \(Z_m Z_\alpha \tilde{\eta} \) and \(\mathbf{\Delta} \tilde{\eta} \) vanish to infinite order at the boundary. In particular, (4.6) for \((w, w') = (0, 0)\) gives that
\[
[(Z_m + Z_\alpha) \tilde{\eta}]|_S = (m - 1)^{-1} \Delta_b \tilde{\eta}.
\]
In the following, we compute modulo functions that vanish to infinite order at the boundary.
\[
\Delta M_{-m+1,0}(\tilde{\eta} \log |z|^2)
\]
\[
= -(d\tilde{\eta}, d(z^0)^{-m+1} \log |z|^2),_g + \tilde{\eta} \Delta ((z^0)^{-m+1} \log |z|^2)
\]
\[
= M_{-m+1}[(m - 1)(Z_m \tilde{\eta}) \log |z|^2 - (Z_m + Z_\alpha \tilde{\eta}) - (m - 1)\lambda \tilde{\eta}].
\]
Here \(\langle \cdot, \cdot \rangle_g\) is the inner product on \(T^* \mathcal{X}\) induced from \(g\). Hence
\[
M_{0,m-1} \Delta M_{-m+2} \Delta M_{-m+1,0}(\tilde{\eta} \log |z|^2)
\]
\[
= M_{0,m-1} \Delta M_{-m+1}[(m - 1)(Z_m \tilde{\eta}) \log |z|^2 - (Z_m + Z_\alpha \tilde{\eta}) - (m - 1)\lambda \tilde{\eta}]
\]
\[
= M_{-m+1}[-(m - 1)(Z_m + Z_\alpha \tilde{\eta}) - (m - 1)^2 \lambda (Z_m + Z_\alpha \tilde{\eta})]
\]
\[
= M_{-m+1}[(m - 1)^2 \tilde{\eta} - (m - 1)^2 \lambda (Z_m + Z_\alpha \tilde{\eta})],
\]
which is an element of \(\tilde{E}(-1)\). Moreover, on \(S\),
\[
(I \tilde{\eta})|_S = -(m - 1)^{-1} |z|^2 \Delta_b^2 + (m - 1)^2 \lambda \Delta_b \tilde{\eta}.
\]
Here we use the fact that \(\Delta_b^2 + (m - 1)^2 \xi \) annihilates CR pluriharmonic functions; see \([12\] Proposition 3.2]. \(\square\)

In particular, this expression of the \(P\)-prime operator gives the formal self-adjointness.

**Corollary 4.8.** The total \(Q\)-prime curvature of a Sasakian \(\eta\)-Einstein manifold is a global \(CR\) invariant, that is, independent of the choice of a pseudo-Einstein contact form.
5. Variation of total $Q$-prime curvature

In this section, we consider the first and the second variation of the total $Q$-prime curvature.

We recall a variational formula for the total $Q$-prime curvature. Let $(M_t)_{t \in (-1, 1)}$ be a smooth family of closed strictly pseudoconvex real hypersurfaces in a complex manifold $X$. Take a Fefferman defining function $\rho_t$ of $M_t = \pi^{-1}_X(M_t)$ such that it is smooth in the parameter $t \in (-1, 1)$. Assume that there exists a flat Hermitian metric $h$ of $K_X$ near $M = M_0$. In this setting, the function $\rho_t = \rho_1 \cdot h^{-1/(m+1)}$ is a defining function of $M_t$, and the corresponding contact form $\theta_t = \frac{d}{dt}\rho_1|_{M_t}$ is pseudo-Einstein.

**Theorem 5.1 ([15 Theorem 1.2]).** Under the above assumptions, the total $Q$-prime curvature satisfies

\[
\frac{d}{dt}\bigg|_{t=0} \int_{M_t} Q_{\theta_t} = c_m \int_M \varphi \mathcal{O},
\]

where $\mathcal{O}$ is the obstruction function of $M$, $\varphi = (d/dt)|_{t=0} \rho_t|_M \in \mathcal{E}(1)$, and $c_m = 2(m-1)!/(m+1)!$. Moreover if the obstruction function of $M$ vanishes,

\[
\frac{d^2}{dt^2}\bigg|_{t=0} \int_{M_t} Q'_{\theta_t} = c'_m \int_M \varphi(R\varphi),
\]

where $R$ is as in Lemma 2.2 and $c'_m = -2(m(m+1))^{-1}$.

As an application of (5.1), we prove that the invariance of the total $Q$-prime curvature is preserved under deformations in the direction of the pseudoconcave side.

**Proposition 5.2.** Let $M$ be a closed strictly pseudoconvex hypersurface in a complex manifold $X$ and $\rho$ a defining function of $M$. Assume that the total $Q$-prime curvature of $M$ is independent of the choice of a pseudo-Einstein contact form, and there exists a flat Hermitian metric $h$ of $K_X$ near $M$. Then for any sufficiently $C^2$-small positive smooth function $\phi$, the total $Q$-prime curvature of $M_\phi = \{\rho = \phi\}$ is also independent of the choice of a pseudo-Einstein contact form.

**Proof.** We use the fact that a pseudo-Einstein contact form does not appear in the integrand of the right hand side in (5.1). Let $\theta_t$, $t \in [0, 1]$, be the pseudo-Einstein contact form on $M_{\theta_t}$ corresponding to $h$, and $\tilde{\theta}_t = e^t \theta_1$ be another pseudo-Einstein contact form on $M_\phi$, where $\tilde{\theta}$ is a CR pluriharmonic function on $M_\phi$. Since $\phi$ is sufficiently $C^2$-small and positive, $\tilde{\theta}$ extends to a pluriharmonic function near $M$, denoted by $\tilde{\theta}$. Then $e^{(m+1)\tilde{\theta}} h$ is also a flat Hermitian metric of $K_X$ and we obtain another pseudo-Einstein contact form $\tilde{\theta}_t$ on $M_{\theta_t}$. From the assumption and the variational formula (5.1),

\[
\int_{M_\phi} Q'_{\tilde{\theta}_1} = \int_M Q'_{\tilde{\theta}_t} + \int_0^1 \left( \frac{d}{dt} \int_{M_{\theta_t}} Q'_{\tilde{\theta}_t} \right) dt
\]

\[
= \int_M Q'_{\theta_t} + \int_0^1 \left( \frac{d}{dt} \int_{M_{\theta_t}} Q'_{\theta_t} \right) dt
\]

\[
= \int_{M_{\theta_t}} Q'_{\theta_t}.
\]

Hence the total $Q$-prime curvature of $M_\phi$ is invariant under a conformal change. $\square$

**Corollary 5.3.** Let $S$ be a closed Sasakian $\eta$-Einstein manifold. Then the total $Q$-prime curvature defines a CR invariant for the real hypersurface $\{ r = c \}$, where $f$ is a sufficiently $C^2$-small smooth function on $S$. 
Proof. First note that the hypersurface \{r = c\} is CR isomorphic to \(S\) for any \(c > 0\), and in particular the total Q-prime curvature of this hypersurface is a CR invariant. For a sufficiently \(C^2\)-small function \(f\) on \(S\), we can apply Proposition 5.5\(\text{a}\) for \(p = r - 1 + \epsilon\) and \(\phi = e^f - 1 + \epsilon > 0\), where \(\epsilon > 0\) is a sufficiently small constant. \(\Box\)

In the following, we consider a closed Sasakian \(\eta\)-Einstein manifold \(S\) and a smooth family \((M_t)_{t \in (-1, 1)}\) of real hypersurfaces in \(X = C(S)\) such that \(M_0 = S\). From Corollary 5.3, the total Q-prime curvature of \(M_t\) is a CR invariant for sufficiently small \(t\); denote by \(\overline{Q}(M_t)\) for simplicity.

The first variation of the total Q-prime curvature can be computed from (5.1).

Proof of Corollary 7.4. Since \(K_X\) has a flat Hermitian metric \(h_S\), we can apply (5.1). Thus Corollary 7.4 follows from the vanishing of the obstruction function. \(\Box\)

Next, consider the second variation of the total Q-prime curvature. Take a Fefferman defining function \(\rho_t\) of \(M_t\) that is smooth in \(t\) and coincides with \(\rho_S\) constructed in Proposition 8.3 at \(t = 0\). Then the second variation of \(\overline{Q}(M_t)\) satisfies

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \overline{Q}(M_t) = c'_m \int_M \phi(P_{1,1}\varphi) \eta \wedge (dn\eta)^{m-1},
\]

where \(\varphi = (d/dt)|_{t=0}(\rho_t \cdot h_S^{-1/(m+1)})\) is \(C^\infty(S)\). Note that the constant \(c'_m\) is negative. Hence it is enough to study spectral properties of \(P_{1,1}\) for the proofs of properties of the second variation.

Before studying \(P_{1,1}\), we consider a relation between \(D_\eta\) introduced in Section 2.3 and \(L_\mu\)'s.

Lemma 5.4. The operator \(4D_\eta^*D_\eta\) coincides with \(L_{m+1}L_{m-1}\). In particular, the operator \(L_{m+1}L_{m-1}\) is a non-negative operator and its kernel coincides with \(\text{ker} \ D_\eta\) if \(S\) is closed. Similarly, the operator \(L_{m-1}L_{m+1}\) is non-negative and its kernel is equal to that of \(\overline{D_\eta}\).

Proof. Since \(L_{m-1}L_{m+1}\) is the complex conjugate of \(L_{m+1}L_{m-1}\), it suffices to prove the lemma for \(D_\eta\) and \(L_{m+1}L_{m-1}\). Since the Tanaka-Webster torsion for \(\eta\) vanishes, the operator \(D_\eta\) is given by \(2(D_\eta F)_{\overline{\beta}} = F_{\overline{\beta}}\). Hence \(4D_\eta^*D_\eta F = F_{\overline{\beta}}^{\overline{\beta}}\).

From (2.3),

\[
F_{\overline{\gamma}}^{\overline{\beta}} = F_{\overline{\gamma}}^{\overline{\beta}} - \sqrt{-1}F_{\overline{\beta}0}^\beta + R_{\overline{\gamma}}^{\overline{\beta}} F_{\overline{\beta}}^\beta = F_{\overline{\gamma}}^{\overline{\beta}} - \sqrt{-1}F_{\overline{\beta}0}^\beta + m \lambda F_{\overline{\beta}}^\beta;
\]

in the last equality, we use the fact that

\[
R_{\overline{\gamma}}^{\overline{\beta}} = \text{Ric}^{\overline{\beta}} = m \lambda \overline{\beta}^\gamma.
\]

Thus

\[
F_{\overline{\gamma}}^{\overline{\beta}} = F_{\overline{\gamma}}^{\overline{\beta}} - \sqrt{-1}F_{\overline{\beta}0}^\beta + m \lambda F_{\overline{\beta}}^\beta = \Box_{\overline{\gamma}}^\beta F + \sqrt{-1}\Omega_{\overline{\gamma}} \Box_\beta F - m \lambda \Box_\beta F
\]

\[
= L_{m+1}L_{m-1}F.
\]

In particular if \(S\) is closed, \(L_{m+1}L_{m-1}\) is non-negative, and its kernel coincides with \(\text{ker} \ D_\eta\). \(\Box\)

We rewrite Lemma 5.4 by using the spectral theory of the sub-Laplacian and Reeb vector field. Since the sub-Laplacian \(\Delta_b\) is a non-negative subelliptic self-adjoint operator, its spectrum \(\sigma(\Delta_b)\) is a discrete subset of \([0, \infty)\), consists only of eigenvalues, and the eigenspace \(\mathcal{H}_p\) with eigenvalue \(p \in \sigma(\Delta_b)\) is a finite-dimensional
holds if and only if \( \ker D \neq 0 \). Moreover, the equalities hold if and only if \( H \leq 0 \) and equal to zero if and only if \( \phi \geq 0 \). In particular, if \( g \) is Kähler, then \( \mu(g) = 0 \).

Consider the case \( m \geq 3 \). Let \( \Sigma \) be a closed Riemann surface of genus 2, and \( g_{\Sigma} \) a hyperbolic metric on \( \Sigma \). Then there exists a complex structure on \( \Sigma \) such that \( g_{\Sigma} \) is Kähler. Consider the \((m-1)\)-dimensional complex manifold \( Y = \Sigma^{m-1} \). The product metric \( g_Y \) on \( Y \) satisfies \( \text{Ric}_{g_Y} = -g_Y \). Moreover, its anti-canonical line bundle \( K_Y^{-1} \) has the Hermitian metric \( h \) induced by \( g_Y \), and \( \omega_h = dd^c \log h \) coincides with the Kähler form of \( g_Y \). Hence the tube \( S = \{ v \in K_Y^{-1} | h(v, v) = 1 \} \) is a Sasakian \( \eta \)-Einstein manifold with Einstein constant \( -1 \).
with respect to $\eta = df \log h$. Denote by $\pi_1$ the composition of the projection $S \to Y$ and the projection from $Y$ to the first factor $\Sigma$. Let $f \in C^\infty(\Sigma)$ be an eigenfunction of $\Delta_{g_2}$ with eigenvalue $p$. Then $\pi_1^* f$ is an element of $\mathcal{H}_{p,0}$, and

$$L_{m+1-2j}(\pi_1^* f) = \left(\frac{1}{2^p} - \frac{(j-1)(m-j)}{m}\right) \pi_1^* f.$$

**Proposition 5.5.** For any $0 < p < 2$, there exists a hyperbolic metric $g_2$ on $\Sigma$ such that the first positive eigenvalue $\lambda_1(\Delta_{g_2})$ of $\Delta_{g_2}$ is equal to $p$.

**Proof.** Consider the space $\mathcal{M}_{-1}$ of hyperbolic metrics on $\Sigma$ with $C^\infty$ topology. This space is contractible, and in particular connected [22 Section 3.4]. Moreover, the map $g_2 \mapsto \lambda_1(\Delta_{g_2})$ defines a continuous function on $\mathcal{M}_{-1}$. Hence it is sufficient to show that

$$\inf_{g_2 \in \mathcal{M}_{-1}} \lambda_1(\Delta_{g_2}) = 0, \quad \sup_{g_2 \in \mathcal{M}_{-1}} \lambda_1(\Delta_{g_2}) \geq 2$$

from the intermediate value theorem. The first equality was obtained by Buser [3 Satz 1]. On the other hand, it is known that there exists a hyperbolic metric on $\Sigma$ such that its first eigenvalue is greater than $3.83$ [16 Section 1.2]; see [21 Section 5.3] for a more precise estimate of its value. This finishes the proof. □

**Proof of Theorem 1.6.** Since $0 < 2(m-2)/m < 2$, we can take a hyperbolic metric $g_2$ on $\Sigma$ such that $\lambda_1(\Delta_{g_2}) = 2(m-2)/m$, and a real-valued eigenfunction $0 \neq f$ on $\Sigma$ with eigenvalue $2(m-2)/m$. Then $\pi_1^* f$ is not contained in $\text{Reker} D_n$, but $P_{1,1}(\pi_1^* f) = 0$ since $L_{m-3}(\pi_1^* f) = 0$. Therefore if we take a smooth deformation of $S$ such that $\varphi = f$, the second variation of the total $Q$-prime curvature along this deformation is equal to zero though this deformation is infinitesimally non-trivial as a deformation of CR manifolds.

Assume that $m = 2n + 1$ for an integer $n \geq 1$. Let $0 \neq f \in C^\infty(\Sigma)$ be an eigenfunction of $\Delta_{g_2}$ with eigenvalue $p$.

$$P_{1,1}(\pi_1^* f) = \prod_{j=0}^{2n+2} \left(\frac{1}{2^p} - \frac{(j-1)(2n+1-j)}{2n+1}\right) \pi_1^* f$$

$$= \left(\frac{1}{2^p} - \frac{n^2}{2n+1}\right) \prod_{j=0}^{n} \left(\frac{1}{2^p} - \frac{(j-1)(2n+1-j)}{2n+1}\right)^2 \pi_1^* f.$$

Hence if we choose $g_2$ and $f$ such that $p$ is sufficiently small, $\pi_1^* f$ is an eigenfunction of $P_{1,1}$ with negative eigenvalue. Thus there exists a smooth deformation of $S$ with positive second variation. □

### 6. Computation of Total $Q$-Prime Curvature

In this section, we compute the total $Q$-prime curvature for some examples. To this end, we first compute the $Q$-prime curvature for Sasakian $\eta$-Einstein manifolds. The unbold $Q'_{\eta}$ is defined by

$$Q'_{\eta} = M_{m,m}Q'_{\eta} = Q'_{\eta} \cdot h^m_{S/(m+1)} \in C^\infty(S).$$

**Proof of Theorem 1.6.** It suffices to compute $\Delta^m (\log |z^0|^2)$ for a branched fiber coordinate $z^0$ with respect to a flat local coordinate. It can be seen that

$$\Delta (\log |z^0|^2) = -\langle d\log |z^0|^2, d\log |z^0|^2 \rangle_g = 2\lambda |z^0|^{-2}$$

and

$$\Delta |z^0|^{-2} = -(d(z^0)^{-1}, d(z^0)^{-1})_g = i^2 \lambda |z^0|^{-2(l+1)}.$$

Hence $Q'_{\eta} = 2((m-1)!)^2 \lambda$. □
Let \((S, T^{1,0}S, \eta)\) be a closed Sasakian \(\eta\)-Einstein manifold with Einstein constant \(m\lambda\). Then the total \(Q\)-prime curvature \(\mathcal{Q}(S)\) has the formula
\[
\mathcal{Q}(S) = 2((m-1)!)^2 \lambda^m \int_S \eta \wedge (d\eta)^{m-1} = 2^m((m-1)!)^2 \lambda^m \text{Vol}(S, g_\eta),
\]
where \(\text{Vol}(S, g_\eta)\) is the volume of the Riemannian manifold \((S, g_\eta)\). We apply this formula to some examples of Sasakian \(\eta\)-Einstein manifolds.

### 6.1. Sasaki-Einstein manifolds

If \((S, T^{1,0}S, \eta)\) is a Sasaki-Einstein manifold, the total \(Q\)-prime curvature \(\mathcal{Q}(S)\) is equal to
\[
\mathcal{Q}(S) = 2^m((m-1)!)^3 \text{Vol}(S, g_\eta).
\]
Hence it is enough to compute the volume of \((S, g_\eta)\).

**Example 6.1** \((S^{2m-1})\). Consider the sphere \((S^{2m-1}, T^{1,0}S^{2m-1}, \eta_0)\) as in Example 2.10. Then the total \(Q\)-prime curvature is equal to
\[
\mathcal{Q}(S^{2m-1}) = 2^m((m-1)!)^3 \text{Vol}(S^{2m-1}, g_{\eta_0}) = 2^{m+1}((m-1)!)^2 \pi^{m}.
\]
Here we use the fact that the metric \(g_{\eta_0}\) is equal to that induced from the Euclidean metric on \(\mathbb{C}^m\).

**Example 6.2** \((Y^{p,q})\). In the study of the AdS/CFT correspondence, Gauntlett-Martelli-Spark-Waldram constructed 5-dimensional Sasaki-Einstein manifolds \(Y^{p,q}\) for coprime positive integers \(q < p\), which are diffeomorphic to \(S^2 \times S^3\). The total \(Q\)-prime curvature of \(Y^{p,q}\) is given by
\[
\mathcal{Q}(Y^{p,q}) = 2^3((3-1)!)^3 \text{Vol}(Y^{p,q}) = \frac{2^6 q^2 (2p + (4p^2 - 3q^2)^{1/2})}{3p^2 (3p^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2})} \pi^3.
\]

### 6.2. Links of affine cones over projective varieties

Let \(Y\) be an \((m-1)\)-dimensional smooth projective variety in \(\mathbb{CP}^N\) and \(\text{Aff}(Y) \subset \mathbb{CN+1}\) the affine cone of \(Y\), which may have an isolated singularity at the origin. Assume that the restriction of the Fubini-Study metric on \(Y\) defines a Kähler-Einstein metric with Einstein constant \(m\lambda\). Then the intersection \(S\) of \(\text{Aff}(Y)\) and the unit sphere centered at the origin is a Sasakian \(\eta\)-Einstein manifold with Einstein constant \(m\lambda\) with respect to \(\eta = \eta_0|_S\). In this case, \(S\) is a principal \(S^1\)-bundle over \(Y\) and \(\eta\) is a connection 1-form. Hence the total \(Q\)-prime curvature \(\mathcal{Q}(S)\) of \(S\) is equal to
\[
\mathcal{Q}(S) = 2((m-1)!)^2 \lambda^m \int_S \eta \wedge (d\eta)^{m-1} = 4\pi((m-1)!)^2 \lambda^m \int_Y \omega_{FS}^{m-1}.
\]
Since the Fubini-Study form \(\omega_{FS}\) on \(\mathbb{CP}^N\) is a representative of the cohomology class \(2\pi c_1(O(1))\), we obtain
\[
\mathcal{Q}(S) = 2^{m+1}((m-1)!)^2 \pi^m \lambda^m \cdot \deg Y,
\]
where
\[
\deg Y = \int_Y c_1(O(1)|_Y)^{m-1}.
\]
Hence it is enough to compute the constant \(\lambda\) and the degree \(\deg Y\). Note that \(\lambda\) is determined by the formula \(c_1(Y) = m\lambda c_1(O(1)|_Y)\).

**Example 6.3** (Fermat hypersurface of degree 2). Let \(F \in \mathbb{C}[z^1, \ldots, z^{m+1}]\) be the homogeneous polynomial defined by
\[
F(z^1, \ldots, z^{m+1}) = (z^1)^2 + \cdots + (z^{m+1})^2.
\]
The hypersurface \(Y \subset \mathbb{CP}^m\) defined by \(F\) is called the Fermat hypersurface of degree 2. It is known that \(\omega_{FS}|_Y\) defines a Kähler-Einstein metric on \(Y\) [19] Section 2. Moreover, \(\deg Y = 2\) and \(c_1(Y) = (m-1)c_1(O(1)|_Y)\). Therefore, the intersection
$S$ of its affine cone and the unit sphere has a Sasakian $\eta$-Einstein structure, and the total $Q$-prime curvature is given by
\[
\mathcal{Q}(S) = 2^{m+2}((m-1))^{2}(m-1)m^{m-\frac{m}{2}}.
\]

**Example 6.4 (Grassmannian manifold).** Let $G(k,n)$ be the complex Grassmannian manifold, that is, the space of $k$-dimensional $\mathbb{C}$-linear subspaces in $\mathbb{C}^n$. By the Plücker embedding, identify $G(k,n)$ with its image in $\mathbb{C}P^{\binom{n}{k}-1}$. Note that its affine cone $\text{Aff}(G(k,n))$ is the affine variety defined by the Plücker relations. A direct calculation shows that the Fubini-Study metric on $\mathbb{C}P^{\binom{n}{k}-1}$ induces a Kähler-Einstein metric on $G(k,n)$ with Einstein constant $n$. Hence the intersection $S = \text{Aff}(G(k,n)) \cap S^{2^k(n-1)}$ is a Sasakian $\eta$-Einstein manifold with respect to $\eta_\rho|S$. The degree of $G(k,n)$ can be computed from so-called Schubert calculus. The result is
\[
\deg G(k,n) = (k(n-k))!\prod_{i=1}^{k} \frac{(i-1)!}{(n-k+i-1)!},
\]
see for example [1] Example 14.7.11]. Thus, the total $Q$-prime curvature is given by
\[
\mathcal{Q}(S) = 2^{k(n-k)+2}((k(n-k))!)^{2}\left(\frac{n\pi}{k(n-k)+1}\right)^{k(n-k)+1} \cdot \deg G(k,n).
\]

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