Vector-Field Domain Walls

J.L. Chkareuli\textsuperscript{a,b}, Archil Kobakhidze\textsuperscript{a,c} and Raymond R. Volkas\textsuperscript{c}

\textsuperscript{a}E. Andronikashvili Institute of Physics, 0177 Tbilisi, Georgia
\textsuperscript{b}I. Chavchavadze State University, 0162 Tbilisi, Georgia
\textsuperscript{c}School of Physics, The University of Melbourne, Victoria 3010, Australia

E-mail: j.chkareuli@iliauni.edu.ge, archilk@unimelb.edu.au, raymondv@unimelb.edu.au

Abstract

We argue that spontaneous Lorentz violation may generally lead to metastable domain walls related to the simultaneous violation of some accompanying discrete symmetries. Remarkably, such domain wall solutions exist for space-like Lorentz violation and do not exist for the time-like violation. Because a preferred space direction is spontaneously induced, these domain walls have no planar symmetry and produce a peculiar static gravitational field at small distances, while their long-distance gravity appears the same as for regular scalar-field walls. Some possible applications of vector-field domain walls are briefly discussed.
1 Gauge fields as vector Nambu-Goldstone bosons

Relativistic invariance and local gauge symmetries are cornerstones of our current understanding of elementary particles and their interactions. Elementary particles as quanta of their corresponding fields are classified through irreducible representations of the Poincaré group, while the local gauge symmetries prescribe their dynamics. This theoretical picture is most successfully realized within the celebrated Standard Model of quarks and leptons and their fundamental strong, weak and electromagnetic interactions.

Nevertheless, it is conceivable that local gauge symmetries and the associated masslessness of gauge bosons might have a completely different origin, being in essence dynamical rather than due to a fundamental principle. This point of view is partially motivated by the peculiarities of local gauge symmetries themselves, which, unlike global symmetries, represent redundancies of the description of a theory rather than being “true” symmetries. In addition, the dynamical origin of massless particle excitations is very well understood in terms of spontaneously broken global symmetries. Based on these observations, the origin of massless gauge fields as the vector Nambu-Goldstone bosons appearing due to spontaneous Lorentz invariance violation (SLIV) \[1\] has gained new impetus \[2\]-\[7\] in recent years.\footnote{Independently of the problem of the origin of local symmetries, Lorentz violation in itself has attracted considerable attention as an interesting phenomenological possibility which may be probed in direct Lorentz non-invariant extensions of quantum electrodynamics (QED) and the Standard Model \[8\]-\[10\].}

While the first models realizing the SLIV conjecture were based on the four fermion interaction (where the photon was expected to appear as a fermion-antifermion composite state \[1\]), the simplest model for SLIV is in fact given by a conventional QED type Lagrangian extended by an arbitrary vector field potential energy. For the minimal polynomial containing only the vector field bilinear and quadrilinear terms one comes to the Lagrangian

\[
\mathcal{L} = -\frac{1}{4} F_{ab} F^{ab} - \frac{\lambda}{4} \left( A_a A^a - n^2 v^2 \right)^2 ,
\]

where \(n_a\) \((a = 0, 1, 2, 3)\) is a properly-oriented unit Lorentz vector, \(n^2 = n_a n^a = \pm 1\), while \(\lambda\) and \(v^2\) are, respectively, dimensionless and mass-squared dimensional positive parameters. This potential means in fact that the vector field \(A_a\) develops a constant background value \(\langle A_a \rangle = n_a v\) and Lorentz symmetry \(SO(1, 3)\) breaks down to \(SO(3)\) or \(SO(1, 2)\) depending on whether \(n_a\) is time-like \((n_a^2 > 0)\) or space-like \((n_a^2 < 0)\). Expanding the vector field around this vacuum configuration,

\[
A_a(x) = n_a(v + \phi) + A_a(x) , \quad n_a A^a = 0
\]

one finds that the \(A_a\) field components, which are orthogonal to the Lorentz violating direction \(n_a\), describe a massless vector Nambu-Goldstone boson, while the \(\phi\) field corresponds to a Higgs mode.
This minimal polynomial QED extension, Eq. (1), is sometim es referred to as the “bumblebee” model (see paper [5] and references therein). Its nonlinear version [11] with a directly imposed vector field constraint $A_{a}^{2} = n_{a}v_{a}^{2}$ (which appears virtually in the limit $\lambda \to \infty$ from the potential (1)) was also intensively discussed in the literature (see paper [7] and references therein). Actually, both of these models are equivalent in the infrared energy domain, where the Higgs mode is considered infinitely massive, and amount to gauge invariant QED taken in the axial gauge, as was shown in tree [11] and one-loop [6] approximations. This axial gauge, $n_{a}A_{a}^{a} = 0$, singles out the pure Goldstonic modes in the vector field as per Eq. (2).

2 Vector field domain walls in flat spacetime

We show here that systems described by Lagrangians that are extensions of type (1) may possess topologically stable domain wall configurations. Remarkably, such domain wall solutions exist for space-like Lorentz violation and do not exist for time-like violation.

One may at first suspect that the topological stability of such domain wall solutions may stem from the fact that the discrete $Z_{2}$ symmetry $A_{a}(x) \to -A_{a}(x)$ of the Lagrangian (1) lies outside the connected part of the $SO(1,3)$ Lorentz symmetry group and, when minimal couplings to matter fields $\psi(x)$ are included, is nothing but the charge conjugation invariance

$$C: A_{a}(x) \to -A_{a}(x), \psi(x) \to \psi^{c}(x).$$

(3)

The nonzero VEVs $\langle A_{a} \rangle = \pm n_{a}v$ spontaneously break this symmetry. We may thus search for a vector-field domain wall solution which asymptotes to the two $Z_{2}$ degenerate vacua, $+vn_{a}$ and $-vn_{a}$, along some spatial direction.

However, the model, as it stands in Eq.(1), does not yet have topologically stable walls. The point is that the potential terms in the model (1) have an extra accidental symmetry $SO(1,3)'$ which rotates vector fields like the Lorentz symmetry while leaving the space-time coordinates untransformed. This $SO(1,3)'$ is in fact a generic internal symmetry of the potential in the model (1) or in any other QED polynomial extension. As a result, the vacuum state four-vector of type $+vn_{a}$ can be transformed into the vector $-vn_{a}$ by a continuous $SO(1,3)'$ rotation, so the system does not possess disconnected vacua. For pure geometrical reasons, this argument holds just for space-like domain wall solutions ($n^{2} = -1$) and would not

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2 We shall discuss what we mean by “topologically stable” in this context in more detail below. But in short: it is well-known that SLIV theories do not have stable “vacua”, since the Hamiltonians are not bounded from below. Since “vacua”, which are more accurately to be termed “local minima”, are used as the boundary conditions for domain wall solutions, those solutions can never be absolutely stable. Nevertheless, local minima will be metastable. Topological (meta-)stability then means that the local minima used as boundary conditions for domain wall solutions form a disconnected manifold, just as in the usual case of absolutely-stable topological domain walls in scalar field theories. See also [12] for a discussion of topological defects in scalar field theories with explicit Lorentz invariance violation.
be applicable to time-like domain walls \((n^2 = 1)\), if such solutions might appear through spontaneous Lorentz violation. However, as we show below, only the space-like domain wall solutions specifically appear in the model of type (1) so one might think that they all are unstable.

To achieve stability, the model can be extended, as we shall see shortly, by introducing a second vector field \(B_a\) into the model (1). There naturally appears, apart from the discrete \(Z_2\) symmetry \((A_a \rightarrow -A_a, B_a \rightarrow -B_a)\), also the interchange symmetry \(A \leftrightarrow B\) in the properly arranged Lagrangian model of the \(A\) and \(B\) fields. We shall show that the spontaneous breakdown of this interchange discrete symmetry may provide stability for vector-field domain walls.

We consider first the one-vector field case to illustrate some generic features of vector-field domain walls, and then proceed to the two-vector field model.

### 2.1 One-vector field model

The general domain wall solution in flat spacetime can be searched for using the simple ansatz,

\[
A_a(x) = n_a V(n \cdot x),
\]

where \(n_b\) is another constant unit four-vector, \(n \cdot x = n_b x^b\). The equation of motion following from (1) then reads

\[
\left[n^2 n_a - (n \cdot n) n_a \right] V''(n \cdot x) - \lambda n^2 n_a \left(V^2(n \cdot x) - v^2\right) V(n \cdot x) = 0,
\]

where primes denote the differentiation with respect to the argument of a function. As one can immediately see multiplying the equation (5) by vector \(n^a\) that static domain wall solutions appear only for the pure space-like spontaneous Lorentz violation with orthogonal \(n\) and \(n\) directions,

\[
n^2 = n^2 = -1, \quad n_a n^a = 0.
\]

Then the equation (5) for \(V\) reduces to the familiar equation for the scalar domain wall:

\[
V'' - \lambda V \left(V^2 - v^2\right) = 0.
\]

The solution for the above equation is well known:

\[
V(n \cdot x) = v \tanh \left[ m \ (n \cdot x) \right],
\]

where \(m = \sqrt{\lambda/2v}\).

Consider for definiteness a domain wall extending along the \(z\)-direction, i.e. \(n_a = (0, 0, 0, 1)\), and centered at \(z = 0\) with the Lorentz violating vacua taken in the \(y\)-direction, i.e. \(n_a = (0, 0, 1, 0)\):

\[
A_2 = v \tanh (mz), \quad A_0 = A_1 = A_3 = 0.
\]

Observe that due to the Lorentz non-invariant VEVs, this configuration does not have an \(x - y\) planar symmetry, contrary to the case of the ordinary scalar field.
domain wall \[13], \[14], \[15]. This important difference is reflected in the energy-momentum tensor of our domain wall configuration (9):

\[
T^{ab} \equiv -F_a^c F^{bc} - \lambda A^a A^b (A_c A^c - n^2 v^2) + \frac{1}{4} \eta^{ab} \left[ F_{cd} F^{cd} + \lambda (A_c A^c - n^2 v^2)^2 \right] + \frac{m^2 v^2}{\cosh^4 (m z)} \text{diag} [1, -1, -2 \sinh^2 (m z), 0].
\]

(10)

Note the fact that the preferred Lorentz-violating direction in the plane of the vector domain wall implies that \(T^{11} \neq T^{22}\) in contrast to a standard scalar domain wall, though its surface energy density,

\[
\sigma = \int_{-\infty}^{+\infty} T_0^0 dz = \frac{4}{3} m v^2,
\]

(11)

is the same as for a scalar wall, as directly follows from Eq. (10).

However, as was argued above, this vector domain wall solution is unstable since the vacuum \((0, 0, +v, 0)\) in this model can be continuously changed to the vacuum \((0, 0, -v, 0)\) by a proper rotation in the \(x - y\) plane. So, one has to think about some extensions of the model among which the two-vector field system seems to be the simplest possibility.

### 2.2 Two-vector field model

The most general Lagrangian of two vector fields \(A\) and \(B\) possessing the independent discrete symmetries,

\[
A \rightarrow -A, \quad B \rightarrow B,
\]

\[
A \rightarrow A, \quad B \rightarrow -B,
\]

(12)

together with the interchange symmetry \(I\),

\[
A \leftrightarrow B,
\]

(13)

may be written in the following simple form,

\[
\mathcal{L}_{(A,B)} = -\frac{1}{4} F_A^2 - \frac{1}{4} F_B^2 - \frac{\mu^2}{2} (A^2 + B^2) - \frac{\alpha}{4} (A^4 + B^4)
\]

\[
-\frac{\beta}{2} (A^2 B^2) - \frac{\gamma}{2} (AB)^2,
\]

(14)

in a self-evident notation for contractions of \(A\) and \(B\) fields and their field-strength tensors \((A^2 = A_a A^a, AB = A_a B^a, A^4 = (A_a A^a)^2, F_A^2 = F_{Aab} F_A^{ab} \text{ etc.})\) and positive dimensionless coupling constants \(\alpha, \beta\) and \(\gamma\). The sign of the mass parameter, \(\mu^2/2\), has been chosen so as to have just space-like Lorentz violation in the model, as was argued in the above.
Note that, as in the one-field model (1), the potential in $\mathcal{L}_{(A,B)}$ also possesses an accidental symmetry $SO(1,3)^{\prime}_A \times SO(1,3)^{\prime}_B$ which is broken to a “diagonal” subgroup by the $\gamma$-term in Eq. (14). Also, when $\alpha = \beta$, these two terms combine into $(A^2 + B^2)^2$ which possess an internal $SO(2)$ also exhibited by the mass and kinetic terms. But this continuous symmetry is explicitly broken by the $\gamma$-term, and also if $\alpha \neq \beta$. The non-existence of this $SO(2)$ will be important for establishing domain wall topological stability.

One can readily confirm that this potential has the following extrema. The first extremum has both the $A$ and $B$ fields identically equal to zero, but it is arranged to be a local maximum, providing $\mu^2 > 0$. Second, there is the “symmetrical” extremum given by the following expectation values of the fields:

$$\langle A_a \rangle = \langle B_a \rangle = n_a \sqrt{\frac{\mu^2}{\alpha + \beta + \gamma}},$$

where $n_a$ is a spacelike unit vector, $n_a^2 = -1$. If this were the vacuum, it would spontaneously break the discrete symmetries of Eq. (12) while leaving the interchange symmetry $I$ of Eq. (13) exact.

There are also “asymmetric” extrema. One class is given by

$$\langle A_a \rangle = n_a v, \quad \langle B_a \rangle = 0 \quad \text{with} \quad v \equiv \sqrt{\frac{\mu^2}{\alpha}},$$

which is degenerate with the $I$-related configuration

$$\langle A_a \rangle = 0, \quad \langle B_a \rangle = n_a v.$$

A vacuum of this type breaks $I$ and one of the $Z_2$ symmetries of Eq. (12). The other class is given by

$$\langle A_a \rangle = n_a v, \quad \langle B_a \rangle = n_b v \quad \text{with} \quad v \equiv \sqrt{\frac{\mu^2}{\alpha + \beta}},$$

where $n_a^2 = n_b^2 = -1$ and $n_a \cdot n_b = 0$. It is degenerate with its $I$-symmetry correlate.

Note that the vacua of Eqs. (15) and (18) are unsuitable for domain-wall stability since all configurations of those types form connected manifolds. For example, there are two-field analogues of the solution we derived in the one-vector field model. One such configuration is

$$A_a(x) = n_a v \tanh[m(n \cdot x)], \quad B_a(x) = 0$$

and another is its $I$-symmetry partner. However, for the parameter space region where Eqs. (16) and (17) are the vacua, a qualitatively different wall solution exists, related to the spontaneous breaking of the interchange symmetry $I$ rather than the

$^3$Recall that by “vacuum” we really mean metastable local minimum.
reflection symmetries of Eq. (12). We find that the extremum (16) (and (17)) is indeed realized as a local minimum if the coupling constants satisfy the inequalities

\[ 0 < \alpha \leq \frac{\beta + \gamma}{2} , \quad \alpha \leq \beta. \]  

(20)

All the other extrema discussed above are local maxima when the inequalities (20) hold. Note that in this range of parameters the potential in (14) is not bounded from below. The reason is that \( (A^2B^2) \) term is neither positive nor negative definite, and thus for \( 0 < \alpha < \beta \) one always finds a direction in \( (A, B) \) space along which the potential runs to \(-\infty\). However, it is unreasonable to insist on the boundedness of the potential since, as mentioned earlier, the total Hamiltonian is known to be unbounded [17], [18]. The corresponding instabilities are caused by sufficiently strong fluctuations of the Higgs modes of vector fields around a vacuum. For small (linear) perturbations (e.g., in the low energy regime of the theory), however, stability can be maintained [18]. Thus we only demand that vacua are realized as local minima, and they are stable under the linear perturbations. We will confirm shortly below that our domain wall solutions are also stable perturbatively against linear perturbations of the Higgs modes.

A prototype for a domain wall solution has the form

\[ A_a(x) = n_a v (1 + \tanh[m(n \cdot x)])/2 , \quad B_a(x) = n_a v (1 - \tanh[m(n \cdot x)])/2 \]  

(21)

which asymptotes to the two \( I \)-degenerate vacua,

\[ \langle (A_a, B_a) \rangle = (v n_a, 0) \quad \text{and} \quad (0, v n_a) \]  

(22)

along some spatial direction \( n \cdot x \in (-\infty, +\infty) \). One can easily see that these vacua cannot be rotated to each other in principle. The only continuous symmetry that might do it is the \( SO(2) \) acting in the \( (A, B) \) space, but this symmetry is explicitly broken in the Lagrangian, as discussed earlier.

The prototype solution above is the simplest analytic configuration of this type one can write down using the standard hyperbolic tangent function. In theories involving multiple fields, other kinds of kink-like functions can also be solutions, though they can usually only be obtained numerically. It is often the case that the analytic prototype corresponds to a particular relationship holding amongst the parameters in the potential, and if that relationship does not hold then wall solutions still exist but must be computed numerically.

To establish that wall solutions of the type we want exist, it suffices to constrain ourselves to the analytic prototype and show that it indeed can be a solution. Putting the ansatz (21) into the equations of motion of the vector fields \( A_a(x) \) and \( B_a(x) \) one finds that such a solution indeed exists, with \( m = \sqrt{\lambda/2} = \mu/\sqrt{2} \), in the parameter plane

\[ \beta + \gamma = 3 \alpha \]  

(23)

This relation is consistent with the potential stability conditions (20), providing \( \gamma \leq 2 \beta \). As emphasized already, for other values of these constants a numerical integration of the equations of motion is required.
Consider, again as in the one-vector field case, a domain wall extending along the \( z \)-direction, i.e. \( n_a = (0, 0, 0, 1) \), and centered at \( z = 0 \) with the Lorentz violating vacua taken both in the \( y \)-direction, i.e. \( n_a = (0, 0, 1, 0) \):

\[
A_2(z) = v\frac{1 + \tanh(mz)}{2}, \quad B_2(z) = v\frac{1 - \tanh(mz)}{2}
\]

while \( A_0 = A_1 = A_3 = 0 \) and \( B_0 = B_1 = B_3 = 0 \). Equally, they could be taken in the orthogonal \( x \)- and \( y \)-directions, respectively. In any case, due to the preferred space directions in these vacua, our domain wall configuration does not possess an \( x-y \) planar symmetry and has, as it can easily be confirmed, the following energy-momentum tensor

\[
T^{ab} = \frac{m^2 v^2}{2 \cosh^4(mz)} \text{diag} [1, -1, -2 \sinh^2(mz), 0]
\]

being exactly half of the corresponding tensor in the one-vector field case (10).

To conclude, we found vacuum configurations in the model where only one of the two vector fields \( A \) or \( B \) is condensed, thus producing three Goldstone modes collected into a photon-like multiplet, while its counterpart has a mass of order the Lorentz violation scale and decouples from the low-energy physics. Actually, the mass-squared of the non-condensed field being from the outset negative (tachyonic) becomes now positive. If, say, \( A \) is condensed (while \( B \) is not) the mass of the \( B \)-field excitations along \( n_a \) direction is given by

\[
M_{B||} = \mu \sqrt{(\beta + \gamma)/\alpha - 1} = \mu \sqrt{2},
\]

while the excitations orthogonal to \( n_a \) have the mass,

\[
M_{B\perp} = \mu \sqrt{\beta/\alpha - 1}.
\]

So, on one side of the wall one has massless \( A \) photons and massive \( B \) bosons, while on the other side one has massive \( A \) bosons and massless \( B \) photons. In this sense, the model contains in fact only one massless vector field in all possible observational manifestations.

### 2.3 Linear stability

Before discussing the gravitational properties of our two-field domain wall solutions we would like to show that the wall solution is stable against linear perturbations in the Higgs modes, \( \phi_A \) and \( \phi_B \): \( A_\mu = n_\mu(V_A + \phi_A) + a_\mu \) and \( B_\mu = n_\mu(V_B + \phi_B) + b_\mu \). Here \( V_A = v(1 + \tanh|m(n \cdot x)|)/2 \) and \( V_B = v(1 - \tanh|m(n \cdot x)|)/2 \) correspond to the background solutions in (25). We plug the above expansion into the equations of motion obtained from the Lagrangian (14). Then, using the equation of motions for the background solutions \( V_A \) and \( V_B \) and keeping only the terms linear in \( \phi_A \) and \( \phi_B \), we obtain the linearized equations of motion for the Higgs modes:

\[
\Box [\phi_A - \mu (V_A + \phi_A) + a_\mu] = 2\alpha \phi_A - 6\alpha V_A V_B \phi_B,
\]

\[
\Box [\phi_B - \mu (V_B + \phi_B) + b_\mu] = 2\alpha \phi_B - 6\alpha V_A V_B \phi_A.
\]
The stability can be easily established by looking at linear combinations: \( \phi_+ = \phi_A + \phi_B \) and \( \phi_- = \phi_A - \phi_B \). Namely, it can be straightforwardly obtained from (27) and (28) that the mode \( \phi_+ \) is simply a massive \( m_+ = \sqrt{2\mu} \) free field with the (Lorentz-violating) dispersion relation \( \omega^2 = 2\mu^2 + \overline{k}^2 - (\overline{n} \cdot \overline{k})^2 > 0 \). The solution for the the mode \( \phi_- \) can be written as \( \phi_- = N_0 e^{-i\omega z+k_xx+i\eta y f(z)} \). For \( f(z) \) we obtain,

\[
 f''(z) + \left[ -\omega^2 + k_x^2 + \mu^2(1 + 3 \tanh^2(mz)) \right] f(z) = 0
\]

This equation is essentially the same as the corresponding equation obtained in the case of the the usual one-field scalar domain wall. Hence, the domain wall solutions are perturbatively stable against linear Higgs mode perturbations, despite the fact that the total Hamiltonian is not bounded from below.

A more generic analysis of the linear perturbations (including those of Goldstonic modes \( a \) and \( b \)) is a complicated problem and can not be handled analytically. However, we expect that the Goldstonic perturbations do not induce instabilities due to the topological reasons. Indeed, due to the topological charge conservation, the domain wall configuration (21) can "decay" by emitting \( a \) and \( b \) quanta only into a configuration with the same topological charge and lower surface energy. However, it is not difficult to check that any domain wall configuration which interpolates between \( I \)-degenerate vacua, \( (n_a,0,0) \) and \( (0,n_a,0) \), have the same energy density \( T^{00} \) (25). Therefore, the only source of the instability of domain walls is the metastability of the vacua (16,17), and this instability shows up at non-linear level. Thus we argue that the wall solutions are indeed metastable, though we acknowledge that this has not been explicitly checked through linear stability analysis.

3 Gravitational field of a vector-field domain wall

Next we are interested in the gravitational properties of vector-field domain walls. Let us start by considering the weak gravity approximation,

\[
g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) .
\]

(where the tensors now have “curved” indices \( \mu, \nu \) rather than the previous “flat” indices \( a, b, \ldots \)). In this approximation we can use the flat spacetime solution for the vector domain wall (24) and the associated energy-momentum tensor (25). In the harmonic gauge, \( \partial^\mu (h_{\mu\nu} - 1/2\eta_{\mu\nu} h) = 0 \), the Einstein equations are,

\[
h''_{\mu\nu}(z) = \frac{2}{M_P^2} (T_{\mu\nu} - 1/2\eta_{\mu\nu} T) ,
\]
where $T = T^\mu_\mu$ is the trace of the energy-momentum tensor. The reflection ($z \to -z$) symmetric solution to the above equations is

\[
\begin{align*}
    h_{00} &= -h_{11} = -\frac{v^2}{6M_P^2} \left[ 2 \ln(\cosh(mx)) + \cosh^{-2}(mz) \right], \\
    h_{22} &= \frac{v^2}{3M_P^2} \left[ \ln(\cosh(mz)) - \cosh^{-2}(mz) \right] \\
    h_{33} &= \frac{v^2}{M_P^2} \left[ \ln(\cosh(mz)) \right].
\end{align*}
\]

One can see a peculiar static gravitational field of the vector wall at small distances. Interestingly, one of the $h$ components, namely $h_{33}$, is vanishing at small distances $mz \ll 1$. At the same time all $h$ components grow linearly with $|z|$ away from the wall’s center. Thus, strictly speaking, the weak field approximation is not valid at large $|z|$, so one has to find a solution in the full nonlinear theory. As is well known [14], [15], planar symmetric spacetime metrics (in the case of scalar-field domain walls) in Einstein gravity without a cosmological constant are necessarily time-dependent. We find below that the same appears in our case as well despite the planar symmetry being broken for vector-field domain walls.

In principle, we have to simultaneously solve the Einstein equations and the equations of motion for the vector field in the spacetime given by an a priori unknown metric. Analytic solutions are difficult, if not impossible, to obtain. To proceed further we will thus consider the thin wall approximation which is sufficient for our purpose because we are interested here in how the metric behaves away from the wall (where the above linear approximation is not valid). Indeed, the finite-thickness domain-wall solutions must approach thin-wall-limit solution in regions distant from the wall, where the “microscopic” details of the wall structure are not essential. In the thin-wall limit , $m \to \infty$ such that $mv^2 = \text{const}$., the energy-momentum tensor (25) takes the form,

\[
T^\mu_\nu = \sigma \delta(z) \text{diag} [1, -1, -1, 0],
\]

which coincides with the energy-momentum tensor for the scalar field domain wall. Therefore, we eventually come to the same solution[14], [15] for metric

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (1 - k|z|)^2[dt^2 - e^{2kt}(dx^2 + dy^2) - dz^2].
\]

where $k = 2\pi G_N \sigma$.

We would like to conclude this section with the following remark concerning the thin wall approximation, comparing a limiting procedure from polynomial theory (which causes Lorentz violation) to the case of standard QED. Consider for simplicity the one-vector model given by the Lagrangian (I). As we have pointed out in the end of the Section 1, in the limit $\lambda \to \infty$ such that $v$ remains finite, which is different from the thin-wall limit considered immediately above, the potential in (I) reduces to the gauge field constraint $A_\mu^2 = n^2v^2$, and this model is in fact equivalent to a gauge invariant $U(1)$ theory which may be identified with QED taken in the
nonlinear gauge. This is also a type of thin-wall limit, since $m \to \infty$, but in this case the energy density $\sigma$ diverges and thus the wall, even if it might be stable in this generic model, would be infinitely massive and decoupled from low-energy physics. Therefore, standard QED, though it may be considered as a low-energy approximation for a general polynomial vector field theory, is fundamentally free from domain-wall type solutions.

4 Cosmological evolution of vector-field domain walls

The vector-field domain walls described above can be produced in the early universe through the Kibble mechanism [19]. Ordinary (scalar field) domain walls evolve cosmologically as per $\rho \sim 1/a$ in the non-relativistic limit, where $a$ is the cosmological scale factor. They therefore dominate the energy density of the universe from very early times unless the corresponding discrete symmetry breaking scale $v$ is less than $\sim 100$ MeV. In fact, it must be less even than $\sim 1$ MeV if the anisotropy induced by the walls on the cosmic microwave background radiation is to be below experimental limits.

The cosmological evolution of the vector-field domain walls seems to largely follow to the same scenario as the regular scalar field walls. However, there may be some difference as well. A full analysis of this problem is beyond the scope of this paper, but we do wish to make some simple observations.

One reason for a difference between scalar and vector domain walls is the violation of Lorentz invariance on the domain wall surface. Heuristically, this can be understood as follows. Because of Lorentz symmetry breaking on the surface of a vector-field wall [$SO(1,2) \to SO(1,1)$], tangential motion along the Lorentz-violating direction becomes also relevant for the dynamical evolution of a domain wall, while the motion along the Lorentz-preserving direction is still unobservable. As a result, the vector-field wall may effectively behave in a different way from a scalar-field wall.

Another reason is the generic dynamical instability in polynomial theories [17], [18] already discussed. In general, such an instability can only be removed by removing the Lorentz Higgs mode thus going to pure QED in an axial or non-linear gauge. For a very heavy Higgs, this instability may be vanishingly small. In general, however, this instability will make the vector domain walls unstable, even though the “vacua” serving as boundary conditions are topologically disconnected. This may be welcome phenomenologically: absolutely stable vector-field walls would be as phenomenologically unacceptable as the usual stable scalar-field walls. Long-lived quasi-stable walls (whose lifetime is controlled by the heavy mass of Lorentz Higgs mode) may be well suited to the course of cosmological evolution.

Due to the above reasons, the phenomenological bounds for vector walls should

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4 In the context of the inflationary scenario, the domain wall problem exists if the universe gets reheated enough to allow the production of domain walls after inflation, i.e. $T_{\text{reheat}} > v$. Otherwise, the domain walls are either inflated away from the visible universe, or they simply are not produced.
be different from those for ordinary walls, and may be weaker. Detailed numerical modelling of a network of vector-field domain walls would probably be necessary to compute the precise bounds, including from the generation of microwave background anisotropy.

5 Conclusion and outlook

We have argued that spontaneous Lorentz-invariance violation leads to domain wall solutions related to the simultaneous violation of the accompanying discrete symmetries, which may be charge-conjugation and/or interchange symmetry. Depending on the specific model, such a configuration may be unstable or metastable.

As was illustrated by the example of the simple one-vector field model, such domain wall solutions can exist for space-like Lorentz violation and do not exist for time-like violations. Though the one-vector field model leads to an unstable wall, the two-vector field extension seems to also be of physical interest. There naturally appears, apart from the discrete $Z_2$ symmetries ($A_a \rightarrow -A_a, B_a \rightarrow -B_a$), the interchange symmetry $A \leftrightarrow B$ in an appropriate Lagrangian model of the $A$ and $B$ fields. This additional discrete symmetry provides a mechanism to achieve metastability for vector-field domain walls. We found a metastable vacuum configuration in the model where only one of the two vector fields condenses on each side of the wall, thus producing Goldstone modes collected into a photon-like multiplet while its counterpart acquires the Lorentz violation scale order mass and decouples from the low-energy physics. Because a preferred space direction is spontaneously induced, these domain walls have no planar symmetry and produce a peculiar static gravitational field at small distances, while their long-distance gravity appears the same as for regular scalar-field walls. Cosmological bounds on these domain walls may be weaker than for the usual scalar walls, but a more detailed analysis is required before this can be quantified.

The vector-field domain walls are especially interesting if this QED type model is further extended to the Standard Model (where the Lorentz-violating vector field is then taken to be coupled to the hypercharge current rather than the electromagnetic one), and also to grand unified theories. Because the discrete symmetry that is spontaneously broken may be the charge conjugation, one application could be to baryogenesis, since particles and antiparticles are expected to behave differently due to their interactions with a wall. Another application might concern the extension of the model to higher dimensions and the possibility of trapping of gauge fields (both Goldstonic and non-Goldstonic) to a 4-dimensional vector-field domain wall appearing in the higher dimensional bulk. We may return to these interesting points elsewhere.

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