Adaptive MCMC for Generalized Method of Moments with Many Moment Conditions

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Abstract

A generalized method of moments (GMM) estimator is unreliable when the number of moment conditions is large, that is, it is comparable or larger than the sample size. While a number of provisions for this problem is proposed in classical GMM literature, the literature on its Bayesian counterpart (i.e., Bayesian inference using a GMM criterion as a quasi-likelihood) has paid scant attention to this problem. This paper fills this gap by proposing an adaptive Markov Chain Monte Carlo (MCMC) approach to a GMM inference with many moment conditions. Particularly, this paper focuses on the adaptive tuning of a weighting matrix on the fly. Our proposal consists of two elements. The first is the random update of a weighting matrix, which substantially reduces computational cost, while maintaining the accuracy of the estimation. The second is the use of the nonparametric eigenvalue-regularized precision matrix estimator, which contributes to numerical stability. A simulation study and a real data application then are presented to illustrate the performance of the proposed approach in comparison with existing approaches.

Keywords: Bayesian analysis, generalized method of moments, many instruments, adaptive Markov chain Monte Carlo, nonparametric eigenvalue-regularization precision estimator

JEL Codes: C11, C15, C18

1 Introduction

The generalized method of moments (GMM) is a widely used statistical framework (Hansen [1982], Hall [2005]). Under GMM, unknown parameters are estimated via a set of moment conditions. A parameter estimate is obtained by minimizing a GMM criterion constructed as a quadratic form and composed of a vector of the sample mean of the moment conditions and a weighting matrix. While GMM uses only lower-order moments, thus being statistically less efficient than full-information methods such as the maximum likelihood method, it has many advantages, including robustness to model misspecification, nonparametric treatment of heteroskedasticity, and computational simplicity.

This paper concerns the Bayesian version of the GMM. A GMM criterion can be viewed as a quasi-likelihood, being theoretically equivalent to the Laplace approximation of the true likelihood around its mode (Chernozhukov and Hong [2003]). Exploiting this feature, one can conduct
a (quasi-)Bayesian inference by replacing true likelihood by a GMM criterion, as discussed by, for example,\textsuperscript{1} Kim (2002); Yin (2009).\textsuperscript{1} Posterior draws from a quasi-posterior density (product of quasi-likelihood and prior density) can be simulated using standard Bayesian Markov Chain Monte Carlo (MCMC) techniques, such as the Metropolis-Hastings algorithm. In the following, we call this inferential approach Bayesian GMM, in contradistinction to the classical GMM.

For applications, a GMM criterion has many moment conditions, making the estimator considerably unreliable. There are cases where the number of moment conditions can be large, including dynamic panel models (e.g., Arellano and Bond, 1991; Blundell and Bond, 1998; Roberts and Rosenthal, 2009; Veire et al., 2012), instrumental variable methods (e.g., Chernozhukov and Hansen, 2005, 2013), and identification through heteroskedasticity (Lewbel, 2012).

In the literature on classical GMM, many provisions to the problem are proposed such as systematic moment selection (Andrews, 1999; Andrews and Lu, 2001; Hall and Peixe, 2003; Hall et al., 2007; Okui, 2009; Donald et al., 2009; Canay, 2010; DiTraglia, 2016; Chang and DiTraglia, 2018), averaging (Chen et al., 2016), and shrinkage estimation (Liao, 2013; Fan and Liao, 2014; Cheng and Liao, 2015; Caner et al., 2018). On the other hand, the literature on Bayesian GMM has paid scant attention to the problem, although remedies tailored to classical GMM are not straightforwardly applicable to Bayesian GMM. The purpose of this paper is thus to fill this gap by proposing a novel method to deal with Bayesian GMM with many moment conditions.

For both classical and Bayesian GMM, choosing a good weighting matrix is not a trivial problem. It is theoretically optimal to set a weighting matrix to the precision matrix (i.e., the inverse of the covariance matrix) of moment conditions, evaluated based on true parameter values. Since this approach is infeasible in practice, a two-step and continuously updated estimators are commonly used in classical GMM (Hansen, 1982; Hansen et al., 1996). By contrast, the literature on Bayesian GMM has paid less attention to the weighting matrix choice. Chernozhukov and Hong (2003), who use the random-walk Metropolis-Hasting algorithm, suggest recomputing the weighting matrix each time a parameter proposal is drawn. This approach is motivated by setting a weighting matrix to a locally optimal one; a posterior mean estimate of the weighting matrix is supposed to be nearly optimal on average. In this approach, the unknown parameters and a weighting matrix are updated concurrently. Consequently, the surface of the quasi-posterior becomes complicated, making the MCMC algorithm inefficient and unstable. To tackle this problem, Yin et al. (2011) propose an approach they call stochastic GMM, where unknown parameters are updated one by one and the corresponding weighting matrix is also updated accordingly. Their approach improves the numerical stability of the posterior simulator by suppressing changes in the posterior in a single cycle. However, this approach requires so many matrix inversions of the weighting matrix that it is not practical for models with many moment conditions.

There are two difficulties in setting a weighting matrix when the number of moment conditions is large. First, it is computationally demanding because the inversion of the sample covariance matrix is repeatedly computed. This problem is peculiar to Bayesian GMM. Second, as in classical GMM, the sample estimate of the covariance matrix of the moment conditions is unreliable, and the inversion of the covariance matrix can amplify estimation errors.

In this paper, we develop an adaptive MCMC approach to deal with the problem of many moment conditions in Bayesian GMM. The proposal consists of two main contributions. First, we propose to update a weighting matrix randomly using the recursive mean of the posterior samples. In our approach, adaptation probabilities are set to be exponentially decreasing, which ensures the validity of the MCMC algorithm, and significantly saves computational cost. Second,
we propose estimating the precision matrix of the moment conditions using the nonparametric eigenvalue-regularized precision matrix estimator developed by Lam (2016). This estimator is more numerically stable than the standard estimator. Through a series of Monte Carlo experiments, we show that the proposed approach outperforms existing ones in terms of both statistical and computational efficiency. Even if the number of moment conditions is significantly smaller than the sample size, a GMM estimator can be ill-posed. While our primary focus is a problem posed by many moment conditions, the proposed approach can be also beneficial to cases where the number of moment conditions is not so many, as shown in the subsequent sections.

The rest of the paper is structured as follows. Section 2 introduces the proposed approach. Section 3 conducts a simulation study. In Section 4, we apply the approach to a real data problem as an example. Section 5 concludes the paper with a discussion.

2 Methodology

2.1 Setup and challenges

We consider the Bayesian inference of a statistical model by means of a set of moment conditions. Assume that a likelihood function can be approximated by a quasi-likelihood based on a generalized method of moments (GMM) criterion (Hansen, 1982). We call this inferential approach Bayesian GMM (Kim, 2002; Yin, 2009). Given data \( D \) and an \( L \)-dimensional parameter \( \theta \), a quasi-likelihood is derived from the GMM criterion:

\[
q(\theta; D) = \left( \frac{2\pi}{N} \right)^{-K/2} \text{det}(W)^{-1/2} \exp \left[ -\frac{N}{2} \bar{m}(\theta)^\top W \bar{m}(\theta) \right],
\]

where \( \bar{m}(\theta) \) contains the sample means of the moment conditions, \( W \) is a symmetric positive definite weighting matrix, and \( N \) is the sample size. A GMM criterion can be seen as the Laplace approximation of the negative true likelihood evaluated around the mode (Chernozhukov and Hong, 2003). Given a prior density \( p(\theta) \), the posterior density \( p(\theta|D) \) is approximated as

\[
p(\theta|D) \approx \frac{q(\theta|D) p(\theta)}{\int q(\theta'|D) p(\theta') d\theta'},
\]

where the denominator is generally unknown but constant. The posterior samples \( \theta_{[j]} = (\theta_{[j],1}, ..., \theta_{[j],L})^\top \) are drawn from this target density (evaluated up to the normalizing constant) using Bayesian simulation techniques. For simplicity, we consider using the Random walk Metropolis-Hastings (RWMH) algorithm as in previous studies (e.g., Chernozhukov and Hong, 2003; Yin, 2009).

As in classical GMM, the statistical efficiency of the Bayesian GMM critically depends on the choice of the weighting matrix \( W \). \( W \) is optimal when it is set to the precision matrix of the moment conditions based on true parameter values \( \theta_0 \): \( W(\theta_0) = E \left[ m_n(\theta_0) m_n(\theta_0)^\top \right]^{-1} \). This choice is optimal in that it minimizes the Kullback-Leibler divergence of the true data generating process to the set of all asymptotically less restrictive distributions:

\[
\lim_{N \to \infty} E \left[ N \bar{m}(\theta_0)^\top W(\theta_0) \bar{m}(\theta_0) \right] K^{-1} = 1.
\]
Let $M(\theta) = (m_1(\theta), ..., m_N(\theta))^\top$ denote an $N$-by-$K$ matrix of the moment conditions. The optimal choice of weighting matrix in finite sample is
\[
W(\theta_0) = \left[ N^{-1}M(\theta_0)^\top M(\theta_0) \right]^{-1}.
\]

In classical GMM, it is common practice to employ the two-step\cite{Hansen1982} or continuously updated estimators\cite{Hansen1996}. While in classical GMM, the choice of $W$ does not affect the consistency of the parameter estimate, Bayesian GMM does not inherit this property due to the use of a prior. A sub-optimal choice of $W$ can decrease the curvature of $q(\theta; D)$, making the inference undesirable from a Bayesian perspective: the more uncertain we are about the true values $\theta_0$, the less the prior is likely to contribute to the posterior. Therefore, for Bayesian GMM, there is an urgent need to choose $W$ efficiently before or along with the posterior simulation. We take the latter route: choosing $W$ on the fly.

Despite its critical importance, the practical choice of $W$ in the context of Bayesian GMM has received rather scant attention. A straightforward approach to choosing $W$, which is employed by, for instance,\cite{ChernozhukovHong2003,Yin2009}, can be described as follows. At the $j$th MCMC iteration, given the current parameters $\theta_{[j-1]}$, a proposal $\theta'$ is simulated for a proposal density $p(\theta'|\theta_{[j-1]})$. The weighting matrix is set to the precision matrix of the moment condition based on $\theta'$, that is, the parameter vector and weighting matrix are concurrently proposed and updated (i.e., accepted or rejected). We call this approach the concurrent GMM. The Metropolis-Hastings (MH) ratio is calculated as
\[
\alpha(\theta', \theta_{[j-1]}) = \frac{q(\theta'|D)p(\theta')p(\theta'|\theta_{[j-1]})}{q(\theta_{[j-1]}|D)p(\theta_{[j-1]}|p(\theta'_{[j-1]}|\theta')}
\]
\[
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad= \frac{\det(W(\theta'))^{-\frac{1}{2}} \exp\left[-\frac{N}{2} \bar{m}(\theta')^\top W(\theta') \bar{m}(\theta')\right] p(\theta')}{\det(W(\theta_{[j-1]}))^{-\frac{1}{2}} \exp\left[-\frac{N}{2} \bar{m}(\theta_{[j-1]})^\top W(\theta_{[j-1]}) \bar{m}(\theta_{[j-1]})\right] p(\theta_{[j-1]})}.
\]

\cite{Yin2011} argue this approach is numerically unstable, because the concurrent updating of $\theta$ and $W$ complicates the surface of the target kernel, resulting in an inefficient move of the MH sampler. They propose an alternative approach, named stochastic GMM, where the elements of $\theta$ are updated one by one, keeping $W$ unchanged. This approach is designed to update $\theta$ and $W$ gradually, suppressing instantaneous changes in the shape of the target kernel. Let $\theta_{[j,l]} = (\theta_{[j,l],1}, ..., \theta_{[j,l],1}, \theta_{[j,l-1],l+1}, ..., \theta_{[j,l-1],L})^\top$ denote a state at the $j$th MCMC iteration after the $l$th parameter was updated. Once a proposed value of $\theta_{[j,l],l}$ is simulated, a proposal is constructed as $\theta'_{[j,l]} = (\theta_{[j,l],1}, ..., \theta_{[j,l],1}, \theta'_{[j,l],l}, \theta_{[j,l-1],l+1}, ..., \theta_{[j,l-1],L})^\top$, and the MH ratio is given by
\[
\alpha(\theta'_{[j,l]}, \theta_{[j,l-1]}) = \frac{\exp\left[-\frac{N}{2} \bar{m}(\theta'_{[j,l]})^\top W(\theta_{[j,l-1]}) \bar{m}(\theta'_{[j,l]})\right] p(\theta'_{[j,l]})}{\exp\left[-\frac{N}{2} \bar{m}(\theta_{[j,l-1]})^\top W(\theta_{[j,l-1]}) \bar{m}(\theta_{[j,l-1]})\right] p(\theta_{[j,l-1]})}.
\]

As previously mentioned, when number of moment conditions is large, this approach is computationally heavy, because it requires many matrix inversions.

\footnote{The two-step estimation method obtains a first-stage estimate using an arbitrary weighting matrix (e.g., an identity matrix), then obtains a second-stage estimate using a weighting matrix to a precision matrix of the moment conditions based on the first-stage estimate. The continuously updating estimation method repeats the two-step estimation for more than one time.}
There are two challenges in regards the choice of the weighting matrix for Bayesian GMM, especially when the number of moment conditions $K$ is large, that is, $K$ is comparable or even larger than the sample $N$. The first challenge is computational cost. The existing approaches require repeated inversion of the sample covariance of the moment conditions, thus imposing severe computational loads. Second, when $K$ is large, the covariance of the moment condition is ill-estimated, and estimation errors are amplified through matrix inversions. As mentioned in Section 1, remedies in the classical GMM literature cannot be straightforwardly imported to Bayesian GMM. Using the Moore-Penrose generalized inverse is a simple solution, but it does not work well, as shown by the simulation study reported in Section 3.

2.2 Proposed approach

The proposal of this paper is comprised of two elements: random update of weighting matrix and regularized precision matrix estimation.

First, we consider randomly updating a weighting matrix $W$. While the existing methods compute $W$ for each MCMC cycle, we treat $W$ as a tuning parameter, and update it on the fly as in adaptive MCMC algorithms [Haario et al., 2001, Andrieu and Thoms, 2008; Roberts and Rosenthal, 2009]. Our adaptation procedure is motivated by Bhattacharya and Dunson (2011). At the $j$th MCMC iteration, the adaptation of $W$ occurs with probability $s(j) = \exp(\alpha_0 + \alpha_1 j)$, regardless of the previous proposal being accepted or rejected. For example, in the simulation study below, we choose $\alpha_0 = -1$ and $\alpha_1 = -5 \times 10^{-4}$ so that the probability of adaptation is around 0.1 at the beginning of the MCMC, and then decreases exponentially to zero. If an adaptation occurs, $W$ is updated using the means of the hitherto sample obtained; at the $j$th iteration, $\bar{\theta}_{j-1} = (j-1)^{-1} \sum_{j'=1}^{j-1} \theta_{j'}$. After warmup iterations, $W$ is fixed to the end. This adaptation strategy satisfies the convergence condition in Theorem 5 of Roberts and Rosenthal (2007). In our implementation, at every $j$th iteration, a random variable is simulated from a standard uniform distribution, $u_j \sim \mathcal{U}(0, 1)$, and $W$ is updated if $u_j < s(j)$. At the $j$th iteration, given a proposal $\theta'$, the MH ratio is calculated as

$$
\alpha(\theta', \theta_{j-1}) = \frac{\exp\left[\frac{-N}{2} \bar{m}(\theta')^\top W(\bar{\theta}_{j-1}) \bar{m}(\theta')\right] p(\theta')}{\exp\left[\frac{-N}{2} \bar{m}(\theta_{j-1})^\top W(\bar{\theta}_{j-1}) \bar{m}(\theta_{j-1})\right] p(\theta_{j-1})}
$$

This treatment of $W$ does not conflict with the theoretical results of Bayesian GMM in existing papers, since in the theoretical analyses, the weighting matrix is pre-fixed. In a sense, there is a discrepancy between theory and practical computation in how a weighting matrix is treated, and our treatment of $W$ rather accords with the theoretical results than the existing approaches. A serious theoretical investigation on the effects of estimation/tuning of a weighting matrix on the posterior density is an important topic but will be addressed in future studies.

Next, we propose to compute $W$ using the nonparametric eigenvalue-regularized (NER) precision matrix estimator [Lam, 2016]. Given $\theta$, the moment conditions are partitioned as

3See Satchachai and Schmidt (2008) on this point for frequentist GMM.
4$\mathcal{U}(a, b)$ denotes a uniform distribution with support on interval $(a, b)$.
5See, e.g., Kim (2002); Chernozhukov and Hong (2003), Belloni and Chernozhukov (2009); Li and Jiang (2016).
6Abadir et al. (2014) consider a closely related covariance estimator.
7In frequentist GMM, Doran and Schmidt (2006) suggest using principal components of a weighting matrix.
8A strategy using the standard principal component analysis to estimate the weighting matrix does not work for Bayesian GMM, not being considered in the paper. The simulation results are available upon request.

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that when guidance for setting \( c, a \) is not applicable when \( a \rightarrow 0 \). Lam (2016) suggests improving this estimator by averaging many (e.g., 50) estimates using \( m \) randomly permute different sets of partitioned data that are generated via random permutation. We also randomly permute \( m_n (\theta) \), \( n = 1, ..., N \), once a computation of \( W \) for robustness.

The choice of the split location \( N^* \) is non-trivial. Theorem 5 of Lam (2016, p. 941) suggests that when \( K/N \rightarrow c \), it is asymptotically efficient to choose \( N^* = N - aN^{1/2} \), with some constants \( c, a > 0 \). There are two difficulties in this regard. First, this asymptotic property is not applicable when \( N^*/N \) goes to a constant smaller than 1. Second, there is no practical guidance for setting \( a \). Lam (2016) proposes to choose \( N^* \) to minimize the following criterion by means of a grid search:

\[
g (N^*) = \left\| \sum_{m=1}^{M} \tilde{\Sigma}_{N^*}^{(m)} - \tilde{\Sigma}_2^{(m)} \right\|_F^2,
\]

where the superscripts for \( \Sigma \)s denote indices for different permutations, \( M \) is a number of permutations executed, and \( \| \cdot \|_F \) denotes the Frobenius norm. He considers the following grid as a set of candidates for \( N^* \):

\[
\{ 2N^{1/2}, 0.2N, 0.4N, 0.6N, 0.8N, N - 2.5N^{1/2}, N - 1.5N^{1/2} \}.
\]

In our framework, one might consider tuning \( N^* \) adaptively based on the above criterion as well. However, we do not adopt such a strategy, because the criterion is not informative enough to pin down the optimal choice of \( N^* \), as shown in the subsequent section. A default choice in this paper is \( N^* = 0.6N \), that is, the median of Lam's (2016) grid. As shown in the next section, simulated posteriors are not sensitive to \( N^* \), as long as \( N^* \) is within a moderate range.

### 3 Simulation Study

We compare the proposed approach with alternatives.\(^8\) We compare the nonparametric eigenvalue-regularization precision matrix estimator given by (2) with the standard estimators specified by

\[
W (\theta) = \begin{cases} 
N^{-1} M (\theta)^\top M (\theta)^{-1}, & K \leq N, \\
N^{-1} M (\theta)^\top M (\theta)^+, & K > N,
\end{cases}
\]

\(^8\)Using a conventional but rather confusing notation, \( \text{Diag} (A) \) is written as \( \text{diag} (\text{Diag} (A)) \).

\(^9\)The programs in this paper are written in Matlab 2016a (64bit), and executed on an Ubuntu Desktop 16.04 LTS (64bit), running on Intel Xeon E5-2607 v3 processors (2.6GHz).
where \( A^+ \) denotes the Moore-Penrose generalized inverse of a matrix \( A \). Six adaptation strategies are considered. The first is fixing the weighting matrix of the moment conditions based on the true parameter value (Oracle), the second is the concurrent Bayesian GMM (Concurrent) (Chernozhukov and Hong [2003], Yin [2009]), and the third is the stochastic GMM (Stochastic) (Yin et al., 2011). The fourth is a MCMC version of the continuously updating GMM estimator (Hansen et al., 1996) (Continuous), that is, \( W \) is updated in each cycle based on the current recursive means of the sampled parameters. The fifth is the random update strategy we propose (Random).

We adopt an instrumental variable (IV) regression as laboratory. A true data generating process is specified by the following two equations, for \( n = 1, \ldots, N \),

\[
x_n = z_n^\top \delta + w_n, \quad w_n \sim \mathcal{N} \left( 0, \sigma^2_x \right), \tag{5}
\]

\[
y_n = \gamma x_n + u_n, \quad u_n \sim \mathcal{N} \left( 0, \sigma^2_y \right), \tag{6}
\]

where \( y_n \) is a response variable, \( x_n \) is an endogenous covariate, \( z_n \) is a \( K \)-dimensional vector of instruments, \( u_n \) and \( w_n \) are normally distributed errors, and \( \mathcal{N} (\mu, \sigma^2) \) denotes a normal distribution with mean \( \mu \) and variance \( \sigma^2 \). \( \gamma = 0.5 \) is a coefficient to be inferred. The instruments and their corresponding coefficients \( \delta \) are generated as follows, for \( n = 1, \ldots, N \),

\[
z_n \sim \mathcal{N} \left( 0_K, BB^\top + \Psi^2 \right),
\]

\[
\delta = A^\top \eta, \quad A = B^\top \left( BB^\top + \Psi^2 \right)^{-1},
\]

\[
\Psi^2 = \text{diag} \left( \psi^2_1, \ldots, \psi^2_K \right), \quad \psi_k \sim \mathcal{U} (2, 4), \quad k = 1, \ldots, K,
\]

\[
B = (b_{k,l}), \quad b_{k,l} \sim \mathcal{U} (0, 1), \quad k = 1, \ldots, K; \quad l = 1, \ldots, L.
\]

The signal-to-noise ratios of equations (5) and (6) are fixed to one. The variances of the errors are chosen as

\[
\sigma^2_x = \delta^\top \left( BB^\top + \Psi^2 \right) \delta, \quad \sigma^2_y = 2\gamma \sigma^2_x.
\]

Unknown parameter \( \gamma \) is inferred through a set of moment conditions,

\[
E \left[ (y_n - \gamma x_n) z_n \right] = 0_K.
\]

We assign a uniform prior on \( \gamma \), \( \gamma \sim \mathcal{U} (-4.5, 5.5) \). The prior is set as centered at and symmetric around the true value to minimize prior-induced bias.

The sample size is fixed at \( N = 200 \). We consider three scenarios with different numbers of instruments \( K = \{50, 150, 250\} \). For posterior sampling, we employ an adaptive MH sampler of Vihola (2012), which automatically tunes the covariance of a proposal density. The tuning parameters of the sampler are chosen as in Vihola (2012). For all experiments, we simulate a total of 70,000 draws: the initial 20,000 draws are used for warmup and the subsequent 50,000 for posterior estimates. We evaluate the results according to three measures. The first is the failure rate (Fail): when the estimated interquantile range of a target posterior density is larger than 1 or smaller than 0.01, we regard the MCMC run as failed. The second is the mean squared error of the posterior mean estimate (MSE), and the third the total computation time measured in seconds (Speed). We conduct 500 experiments.

The upper part of Table 1 reports the results for \( K = 50 \), the middle part for \( K = 150 \), and the lower part for \( K = 250 \). The left half of Table 1 shows the results for the standard precision matrix estimator and the right half those for the NER estimator. There are three points worth
mentioning. First, Concurrent is the obvious loser: high probability of failure, large MSE, and high computational cost. The relative advantage of Stochastic to Concurrent in terms of numerical stability is in line with Yin et al. (2011). Second, in terms of MSE, all Stochastic, Continuous and Random work well and are largely comparable. Third, Random is much faster than Stochastic and Continuous. Figure 1 provides a typical example of recursive posterior mean and occurrence of random adaptation (NER estimator, $K = 150$). From this figure, a posterior mean is fairly fast to converge, which indicates that most updates of the weighting matrix in Continuous are essentially redundant. To conclude, we find Random has a good balance between statistical and computational efficiency.

Next, we compare the results of the alternative precision matrix estimators. When $K > N$, while the posterior simulations using the standard estimator are unsuccessful, the NER estimator always provides reasonable posterior estimates. Therefore, when $K > N$, only the NER estimator is a viable option. In terms of MSE, the NER estimator outperforms the standard estimator overall. Even when the number of moment conditions $K$ is smaller than the sample size $N$, the NER estimator is likely to obtain a more accurate posterior estimate than the standard precision estimator. It is also worth mentioning that, when $K > N$, the posterior simulation using the NER estimator is almost as precise as the cases with $K < N$. A comparison between the results for the Oracle cases with different precision estimators and $K = 50, 150$ reveals that the NER estimator is not better than the standard one, but the gain from the numerical stability of the NER estimator outweighs its efficiency loss. The increased computation cost incurred by the NER estimator can be mitigated by using the Random adaptation method. When the NER estimator is used, Stochastic, Continuous and Random yield virtually the same MSEs. Therefore, a combination of Random and the NER estimator is preferred.

We also investigate the sensitivity of the above results to the choice of split location $N^*$. We conduct Monte Carlo experiments using different $N^*$ and Random adaptation strategy. Following Lam (2016), we consider the grid of (4) (each $N^*$ is rounded to the nearest integer). Table 2 shows that the NER estimator consistently outperforms the standard estimator, irrespective of the split location choice. In our testing environment, as $N^*$ becomes smaller, MSE is likely to be small, regardless of $K$. To investigate how much this result accords with the criteria based on the Frobenius norm (3), we simulate the values of (3) for different random permutations of the moment conditions using the true parameter. Panel (a) of Figure 2 reports the median and 90 percentile intervals of the simulated values for a fine grid $\{0.1N, 0.15N, \ldots, 0.9N\}$. We only report the results for $K = 250$, as those for $K = 50, 150$ are qualitatively similar. As evident from the panel, an extremely high $N^*$ is not preferred, but the criterion is not informative enough to select a good $N^*$ from a considerably large range. The variability of the criterion is not attributable to the small sample size. We conduct the same simulation as in panel (a) but the sample size increases to $N = 5,000$. Panel (b) of Figure 2 shows the results. When $N = 5,000$, the minimum $N^*$ is no longer the best choice. As is the case of $N = 200$, the values of the criterion based on the Frobenius norm are almost indifferent for a large range. As such, we recommend setting $N^*$ to approximately the half the sample size as default.

4 Application

To demonstrate the proposed method, we apply it to a demand analysis for automobiles. Berry et al. (1995) consider an IV regression model of demand for automobiles specified by

$$y_{i,t} = \gamma p_{i,t} + \delta^\top x_{i,t} + u_{i,t},$$

$$y_{i,t} = \log (s_{i,t}) - \log (s_{0,t}).$$
\[ s_{i,t} \] denotes the market share of product \( i \) on market \( t \), with subscript \( 0 \) denoting the outside option. A treatment \( p_{i,t} \) is the product price. \( u_{i,t} \) is an error term, and \( \gamma \) and \( \delta \) are the parameters to be estimated. The primary focus of this application is inference of \( \gamma \).

We consider two specifications\(^{10}\). The first specification coincides with Berry et al. (1995) as follows. A vector of covariates \( x_n \) includes four covariates, namely, air conditioning dummy, horsepower to weight ratio, miles per dollar, and vehicle size. A set of instruments contains the four covariates and ten variables, namely, sum of each covariate taken across models made by product \( t \)'s firm, sum of each covariate taken across competitor firms’ products, total number of models produced by product \( t \)'s firm, and total number of models produced by the firm’s competitors. The second specification is an extension of the first, which is considered in Chernozhukov et al. (2015). \( x_n \) and \( z_n \) are extended from the first case by incorporating a time trend, quadratic and cubic terms of all continuous covariates, and first-order interaction terms. The numbers of the instruments in the first and second specifications are 10 and 48, respectively. The sample size is 2,217, being larger than the numbers of instruments. Nevertheless, because of ill-posedness of the data set, the covariance of a classical estimator is nearly singular. We use a constant prior; thus, if the relationship between the instruments and the treatment is linear and the distributions of residuals are normal, a posterior estimate coincides with a two-stage least square estimate. The posterior estimate is obtained using different combinations of precision matrix estimators and adaptation of proposal density. A total of 250,000 posterior draws are sampled and the last 200,000 for posterior analysis. We set \( \alpha_0 = -1 \) and \( \alpha_1 = -5 \times 26^{-4} \).

Table 3 summarizes the results of the posterior estimate for the coefficient on price. Although the number of moment conditions is fairly smaller than the sample size, MCMC runs using existing adaptation strategies (Concurrent and Stochastic) and the standard precision estimator fails to converge. By contrast, MCMC runs using the NER estimator obtain sensible posterior samples, irrespective of adaptation strategy. For comparison, Table 3 also includes the estimates obtained using four alternative methods. The first two are conventional: ordinary (OLS) and two-stage least square methods (2SLS). The second two are state-of-the-art: IV with instrument selection based on a least absolute shrinkage and selection operator (Chernozhukov et al., 2015), and Bayesian IV with a factor shrinkage prior (Hahn et al., 2018). Chernozhukov et al. (2015) propose to select fewer relevant instruments, while Hahn et al. (2018) propose to compress observed information into few latent factors. The two methods assume a linear relationship between instruments and the endogenous variable and Gaussianity of the error terms, while our method does not impose such assumptions. These alternative methods obtain larger estimates than the conventional ones, and the estimates considerably depend on a set of (potential) instruments. By contrast, our method estimated the coefficient to be intermediate between OLS and 2SLS, nearly irrespective of the choice of instruments.

5 Discussion

We propose a new adaptive MCMC approach to infer Bayesian GMM with many moment conditions. Our proposal consists of two elements. The first is the use of a nonparametric eigenvalue-regularized precision matrix estimator (Lam, 2016) for estimating the weighting matrix (i.e., the precision matrix of the moment conditions based on the recursive mean of the unknown parameters). This prevents us from ill-estimating the weighting matrix. The second is the use of random adaptation. By setting adaptation probability as exponentially decreasing, it can significantly reduce the computational burden, while retaining statistical efficiency. We show the superior-

\(^{10}\)All data are extracted from R package hdm (version 0.2.3).
ity of the proposed approach over existing approaches through simulation, and demonstrate the approach by applying it to a demand analysis for automobiles.

There are many promising research areas that stem from this study. First, a theoretical investigation of the effects of tuning/estimation of a weighting matrix on the posterior density is needed but absent in the literature. Second, while the proposed approach seems to be fairly robust to \( N^* \), there is room for improvement by finding a better \( N^* \). Third, while this paper addresses only problems caused by many moment conditions, problems caused by many unknown parameters are also important. The proposed method should serve as a stepping stone for the further development of inferential methods for high-dimensional Bayesian GMM. Finally, it is worth conducting a thorough comparison between the proposed approach and existing frequentist and Bayesian approaches tailored to a specific class of models such as IV regressions and dynamic panel models.

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Table 1: Comparison of different approaches

| K   | Estimator  | Standard | Fail | MSE   | Time | NER | Fail | MSE   | Time |
|-----|------------|----------|------|-------|------|-----|------|-------|------|
| 50  | Oracle     | 0/500    | 0.0014 | 4.6   | 0/500 | 0.0023 | 4.6 |
|     | Concurrent | 4/500    | 24.0333 | 8.6   | 500/500 | –     | 43.7 |
|     | Stochastic | 0/500    | 0.0034 | 8.0   | 0/500 | 0.0026 | 15.7 |
|     | Continuous | 0/500    | 0.0034 | 8.3   | 0/500 | 0.0028 | 17.1 |
|     | Random     | 0/500    | 0.0034 | 4.7   | 0/500 | 0.0028 | 5.1  |
| 150 | Oracle     | 0/500    | >1e-3  | 18.7  | 0/500 | 0.0019 | 18.6 |
|     | Concurrent | 12/500   | 24.1474 | 53.4  | 421/500 | 19.5521 | 191.1 |
|     | Stochastic | 1/500    | 0.0075 | 40.4  | 0/500 | 0.0023 | 93.6 |
|     | Continuous | 0/500    | 0.0210 | 43.0  | 0/500 | 0.0027 | 103.4 |
|     | Random     | 0/500    | 0.0226 | 19.6  | 0/500 | 0.0026 | 21.8 |
| 250 | Oracle     | 500/500  | –      | 14.6  | 0/500 | 0.0018 | 52.7 |
|     | Concurrent | 500/500  | –      | 304.5 | 498/500 | 16.3000 | 287.0 |
|     | Stochastic | 394/500  | 0.0294 | 315.5 | 0/500 | 0.0026 | 270.4 |
|     | Continuous | 500/500  | –      | 361.8 | 0/500 | 0.0029 | 300.2 |
|     | Random     | 500/500  | –      | 27.9  | 0/500 | 0.0029 | 61.7 |

Notes: The column labeled Fail reports the number of failed runs. Column MSE reports the mean squared errors of posterior mean estimates. Column Time reports averages of computation time measured in seconds.
Table 2: Comparison of different choices of $N^*$

| Estimator | $N^*$ | $K = 50$ | Fail | MSE | $K = 150$ | Fail | MSE | $K = 250$ | Fail | MSE |
|-----------|-------|----------|------|-----|----------|------|-----|----------|------|-----|
| Standard  | –     |          | 0/500| 0.0034 | 0/500 | 0.0226 | 500/500 | –     |
| NER       | 28 ($= 2N^{1/2}$) | 0/500 | 0.0027 | 0/500 | 0.0026 | 0/500 | 0.0028 |
|           | 40 ($= 0.2N$) | 0/500 | 0.0027 | 0/500 | 0.0025 | 0/500 | 0.0028 |
|           | 80 ($= 0.4N$) | 0/500 | 0.0028 | 0/500 | 0.0025 | 0/500 | 0.0028 |
|           | 120 ($= 0.6N$) | 0/500 | 0.0027 | 0/500 | 0.0026 | 0/500 | 0.0030 |
|           | 160 ($= 0.8N$) | 0/500 | 0.0028 | 0/500 | 0.0029 | 0/500 | 0.0034 |
|           | 164 ($= \lceil N - 2.5N^{1/2} \rceil$) | 0/500 | 0.0028 | 0/500 | 0.0028 | 0/500 | 0.0033 |
|           | 178 ($= \lceil N - 1.5N^{1/2} \rceil$) | 0/500 | 0.0031 | 0/500 | 0.0031 | 1/500 | 0.0037 |

Notes: The column labeled Fail reports the number of failed runs. Column MSE reports the mean squared errors of posterior mean estimates.
Table 3: Posterior estimates

| $K$ | Standard Mean | Std | NER Mean | Std |
|-----|---------------|-----|----------|-----|
| Concurrent | – | – | -0.122 | 0.051 |
| Stochastic | – | – | -0.129 | 0.051 |
| Continuous | -0.120 | 0.049 | -0.122 | 0.051 |
| Random | -0.117 | 0.051 | -0.121 | 0.049 |
| OLS | -0.089 | 0.004 |
| 2SLS | -0.142 | 0.012 |
| LASSO-IV | -0.185 | 0.014 |

| $K$ | Standard Mean | Std | NER Mean | Std |
|-----|---------------|-----|----------|-----|
| Concurrent | – | – | -0.117 | 0.010 |
| Stochastic | – | – | -0.126 | 0.010 |
| Continuous | -0.116 | 0.011 | -0.117 | 0.010 |
| Random | -0.120 | 0.014 | -0.113 | 0.011 |
| LASSO-IV | -0.221 | 0.015 |
| HS-IV | -0.275 | 0.018 |

Notes: The column labeled Mean reports mean estimates. Column Std. reports standard errors.
Figure 1: An example of random adaptation

Notes: The x-axis denotes MCMC iterations and the y-axis denotes parameter values. A thin solid vertical line denotes the occurrence of adaptation. A bold solid line denotes a recursive mean of posterior samples.
Figure 2: The Frobenius norm criterion for different permutations

(a) $N = 200$ 

(b) $K = 5000$

Notes: Solid lines denote the median, and dashed lines denote the 90 percentile interval. $K = 250$. Moment conditions are calculated based on the true parameter value.