ASYMPTOTIC PROFILES FOR A NONLINEAR KIRCHHOFF EQUATION WITH COMBINED POWERS NONLINEARITY

SHIWANG MA

School of Mathematical Sciences and LPMC, Nankai University
Tianjin 300071, China

VITALY MOROZ

Department of Mathematics, Swansea University
Fabian Way, Swansea SA1 8EN, Wales, UK

Abstract. We study asymptotic behavior of positive ground state solutions of the nonlinear Kirchhoff equation

$$-(a + b \int_{\mathbb{R}^N} |\nabla u|^2) \Delta u + \lambda u = u^{q-1} + u^{p-1} \quad \text{in } \mathbb{R}^N,$$

as \( \lambda \to 0 \) and \( \lambda \to +\infty \), where \( N = 3 \) or \( N = 4 \), \( 2 < q \leq p \leq 2^* \), \( 2^* = \frac{2N}{N-2} \) is the Sobolev critical exponent, \( a > 0 \), \( b \geq 0 \) are constants and \( \lambda > 0 \) is a parameter. In particular, we prove that in the case \( 2 < q < p = 2^* \), as \( \lambda \to 0 \), after a suitable rescaling the ground state solutions of \((P_\lambda)\) converge to the unique positive solution of the equation \(-\Delta u + u = u^{q-1}\) and as \( \lambda \to +\infty \), after another rescaling the ground state solutions of \((P_\lambda)\) converge to a particular solution of the critical Emden-Fowler equation \(-\Delta u = u^{2^*-1}\). We establish a sharp asymptotic characterisation of such rescalings, which depends in a non-trivial way on the space dimension \( N = 3 \) and \( N = 4 \). We also discuss a connection of our results with a mass constrained problem associated to \((P_\lambda)\) with normalization constraint \( \int_{\mathbb{R}^N} |u|^2 = c^2 \).

As a consequence of the main results, we obtain the existence, non-existence and asymptotic behavior of positive normalized solutions of such a problem. In particular, we obtain the exact number and their precise asymptotic expressions of normalized solutions if \( c > 0 \) is sufficiently large or sufficiently small. Our results also show that in the space dimension \( N = 3 \), there is a striking difference between the cases \( b = 0 \) and \( b \neq 0 \). More precisely, if \( b \neq 0 \), then both \( p_0 := 10/3 \) and \( p_0 := 14/3 \) play a role in the existence, non-existence, the exact number and asymptotic behavior of the normalized solutions of the mass constrained problem, which is completely different from those for the corresponding nonlinear Schrödinger equation and which reveals the special influence of the nonlocal term.

E-mail addresses: shiwang@nankai.edu.cn, v.moroz@swansea.ac.uk.

Date: November 29, 2022.

2010 Mathematics Subject Classification. Primary 35J60; Secondary 35B25, 35B40.

Key words and phrases. Nonlinear Kirchhoff equation; mass \( L^2 \)-critical exponent; critical Sobolev exponent; concentration compactness; asymptotic behavior, normalized solutions.
1. Introduction and notations

We consider the following Kirchhoff equation

\[-(a + b \int_{\mathbb{R}^N} |\nabla u|^2) \Delta u + \lambda u = |u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N,\]  

\((P_\lambda)\)

where \(N \geq 3\), \(2 < q \leq p \leq 2^* = \frac{2N}{N-2}\), \(a > 0\), \(b \geq 0\) and \(\lambda > 0\) are parameters. For a fixed \(\lambda > 0\), the corresponding to \((P_\lambda)\) action functional is given by

\[I_\lambda(u) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\lambda}{2} \int_{\mathbb{R}^N} |u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*},\]  

\((1.1)\)

and critical points of \(I_\lambda\) in \(H^1(\mathbb{R}^N)\) correspond to solutions of \((P_\lambda)\). By a ground state solution of \((P_\lambda)\) we understand a solution \(u_\lambda \in H^1(\mathbb{R}^N)\) such that \(I_\lambda(u_\lambda) \leq I_\lambda(u)\) for every nontrivial solution \(u\) of \((P_\lambda)\).

In this paper we are interested in the limit asymptotic profile of the ground states \(u_\lambda\) of the problem \((P_\lambda)\), and in the precise asymptotic behavior of different norms of \(u_\lambda\), as \(\lambda \to 0\) and \(\lambda \to \infty\). Of particular importance is the \(L^2\)-mass of the groundstates

\[M(\lambda) := \|u_\lambda\|_2^2,\]  

\((1.2)\)

which plays a key role in the analysis of stability of the standing wave solution of the time-dependent NLS, cf. Lewin and Nodari\,[14, Section 3.2] for a discussion in the context of the local combined power NLS.

For the local prototype of \((P_\lambda)\) with \(b = 0\) the asymptotic profiles of the ground states were studied in \([1, 2]\), where the authors considered the following nonlinear Schrödinger equation with focusing combined powers nonlinearity:

\[-\Delta u + \lambda u = |u|^{q-2}u + |u|^{p-2}u, \quad \text{in } \mathbb{R}^N,\]  

\((1.3)\)

where \(N \geq 3\) and \(2 < q < p \leq 2^*\). When \(p = 2^*\) and \(q \in (2 + \frac{4}{N}, 2^*)\), Akahori et al. in \([1]\) proved that for small \(\lambda > 0\) the positive ground state state of \((1.3)\) is unique and non-degenerate, and as \(\lambda \to 0\) the unique positive ground state \(u_\lambda\) converges after an explicit rescaling to the unique positive solution of the limit equation \(-\Delta u + u = u^{q-1}\) in \(\mathbb{R}^N\). In \([2]\), after a suitable implicit rescaling the authors establish a uniform decay estimate for the positive ground states \(u_\lambda\), and then prove the uniqueness and nondegeneracy of ground states \(u_\lambda\) for \(N \geq 5\) and large \(\lambda > 0\), and show that for \(N \geq 3\), as \(\lambda \to \infty\), \(u_\lambda\) converges to a particular solution of the critical Emden–Fowler equation. Recently, for \(p = 2^*\), Coles and Gustafson\,[5] proved that the radial ground state \(u_\lambda\) is also unique and non-degenerate for all large \(\lambda > 0\) when \(N = 3\) and \(q \in (4, 2^*)\). In \([19]\), the authors studied a related problem and its connection with a mass constrained problem by using a rescaling argument and the concentration–compactness principle. See also \([20]\) for a nonlinear Choquard type equation.

The techniques in this work (as well as in \([19, 20]\)) is inspired by \([21]\), where the second author and C. Muratov studied the asymptotic properties of ground states for a combined powers Schrödinger equations with a focusing exponent \(p > 2\) and a defocusing exponent \(q > p\),

\[-\Delta u + \lambda u = |u|^{p-2}u - |u|^{q-2}u, \quad \text{in } \mathbb{R}^N,\]  

\((1.4)\)

and obtained a sharp asymptotic characterisation of the limit profiles of positive ground states \(u_\lambda\) of \((1.4)\) as \(\lambda \to 0\). Later, in \([14]\), M. Lewin and S. Rota Nodari proved a general result
about the uniqueness and non-degeneracy of positive radial solutions of (1.4). The non-degeneracy of the unique positive solution allowed them to refine the asymptotic results in [21] and, amongst other things, to establish the exact asymptotic behavior of $M(\lambda) = \|u(\lambda)\|_2^2$. In particular, this implied the uniqueness of normalised energy minimizers at fixed masses in certain regimes. See also [17], where Zeng Liu and the second author extended the results in [21] to a class of Choquard type equation.

In the present paper, we study the limit asymptotic profiles of the ground states $u_\lambda$ of the Kirchhoff problem $(P_\lambda)$ by using a rescaling argument and the concentration–compactness principle, and obtain an explicit asymptotic expression of different norms of ground states for fixed frequency problem $(P_\lambda)$. To do so, we adapt the technique developed in [19]. However, additional difficulties arise since $(P_\lambda)$ contains five terms and as a consequence, the Pohožaev–Nehari algebraic relations can not be resolved, in general. Fortunately, we succeed to overcome this difficulty in the case $p = 2^*$, but the method does not work any more for $p < 2^*$. In the latter case, we shall use a suitable scaling to reduce $(P_\lambda)$ to the local equation (1.3) (cf. (5.2)). The disadvantage of such a rescaling is that for $b \neq 0$, the rescaled family of ground states for $(P_\lambda)$ should not necessarily be a ground state for (1.3). Besides, to obtain a precise estimate of the least energy, the scaling should transform a ground state for $(P_\lambda)$ into a ground state of the local equation (1.3). Generally speaking, this is not the case, which prevents us from deriving a precise energy estimate of the ground state $p < 2^*$, see Section 5 for a discussion.

Alternatively to the study of fixed frequency solutions of $(P_\lambda)$, one can search for solutions to $(P_\lambda)$ with a prescribed mass, that is for a fixed $c > 0$ to search for $u \in H^1(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ that satisfy

$$
\begin{align*}
-(a + b \int_{\mathbb{R}^N} |\nabla u|^2) \Delta u + \lambda u &= \mu |u|^{q-2} u + |u|^{p-2} u \quad \text{in} \quad \mathbb{R}^N, \\
\mu &> 0, \quad \int_{\mathbb{R}^N} |u|^2 = c^2,
\end{align*}
$$

(1.5)

where $\mu > 0$ is a new parameter and the frequency $\lambda \in \mathbb{R}$ becomes a part of the unknown. The solutions of (1.5) are usually denoted by a pair $(u, \lambda)$ and referred to as normalised solutions. Normalised solutions can be obtained by searching critical points of the energy functional

$$
E_\mu(u) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p,
$$

subject to the constraint

$$
S_c := \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 = c^2 \},
$$

where $\lambda \in \mathbb{R}$ appears a posteriori as a Lagrange multiplier.

In the local case $b = 0$, by rescaling we also assume $a = 1$. Then equation $(P_\lambda)$ reduces to the classical non linear Schrödinger equation

$$
-\Delta u + \lambda u = \mu |u|^{q-2} u + |u|^{p-2} u, \quad \text{in} \quad \mathbb{R}^N.
$$

(1.7)

Normalized solutions of (1.7) were studied by T. Cazenave and P.-L. Lions [3], N. Soave [24, 25], L. Jeanjean et al. [10, 12], L. Jeanjean and T. Le [11]. The additional parameter $\mu > 0$ is often introduced to control the unknown Lagrange multipliers $\lambda \in \mathbb{R}$. Some of the results on normalized solutions to (1.7) are summarized in [15]. The asymptotic behavior of normalized solution as $\mu$ varies in its range is studied in [11, 24, 25, 26, 27]. We mention that in the case $N \geq 4, 2 < q < 2 + \frac{4}{N}$ and $p = 2^*$, L. Jeanjean and T. Le [11] obtained a normalized
solution $u_c$ of mountain pass type for small $c > 0$ and proved that $\lim_{c \to 0} \|\nabla u_c\|^2_2 = S^{N/2}$ and $\lim_{c \to 0} E_c(u_c) = \frac{1}{N} S^{N/2}$, where $S$ is the best Sobolev constant.

In the above results, the number $p_0 := 2 + \frac{4}{N}$, called the $L^2$-critical exponent, is crucial for the existence and asymptotic behavior of normalized solutions of (1.5). However, it is showed in [30, 31] that when $b \neq 0$, $p_b := 2 + \frac{8}{N}$ is the $L^2$-critical exponent for the minimization problem

$$E_{c}(c) := \inf_{u \in S_c} E_{c}(u). \quad (1.8)$$

in the sense that for each $c > 0$, $E_{c}(c) > -\infty$ if $2 < p < p_b$ and $E_{c}(c) = -\infty$ if $p_b < p < 2^*$. In particular, when $N = 3$, the $L^2$-critical exponent for the problem (1.5) is given by

$$p_b = \begin{cases} 
10/3, & \text{if } b = 0, \\
14/3, & \text{if } b \neq 0.
\end{cases} \quad (1.9)$$

In [15], Li, Luo and Yang consider the existence and multiplicity of normalized solutions of (1.5) when $N = 3$ and prove that if $2 < q < \frac{10}{3}$ and $\frac{14}{3} < p < 6$, then for small $\mu > 0$, $E_{\mu}(u_c)$ has a local minimizer at a negative energy level $m(\mu) < 0$, and has a second critical point of mountain pass type at a positive energy level $\sigma(\mu) > 0$. If $2 < q < \frac{10}{3} < p < 6$, then for small $\mu > 0$ a ground state solution is obtained. If $\frac{14}{3} < q < p \leq 6$, then for any $\mu > 0$, a critical point of mountain pass type is also obtained. Furthermore, as the parameter $\mu \to 0^+$, the asymptotic behavior of energy $m(\mu)$ and the normalized solution is also investigated. To our best knowledge, the existence of normalized solutions to (1.5) with $\frac{10}{3} < q < p < \frac{14}{3}$ is still unknown.

When $\mu = 0$, then the equation $(P_{\lambda})$ reduces to the following Kirchhoff equation with a homogeneous nonlinearity

$$- (a + b \int_{\mathbb{R}^N} |\nabla u|^2) \Delta u + \lambda u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \quad (1.10)$$

Amongst other things, Li and Ye [31] studied the existence and concentration behavior of minimizers for (1.10) and obtained precise asymptotic behavior of normalized solutions to (1.10) with $2 < p < 2 + \frac{4}{N}$ as $c \to +\infty$. For $2 < p < 2 + \frac{4}{N}$, Zeng and Zhang [33] proved the existence and uniqueness of the minimizer to the minimization problem $E_0(c) := \inf_{u \in S_c} E_0(u)$ for any $c > 0$, while for $2 + \frac{4}{N} \leq p < 2 + \frac{8}{N}$ the authors proved that there exists a threshold mass $c_* > 0$ such that for any $c \in (0, c_*)$ there is no minimizer and for $c > c_*$ there is a unique minimizer. Moreover, a precise formula for the minimizer and the threshold value $c_*$ is given according to the mass $c$. In the case $2 < p < 2^*$, Qi and Zou [23] obtain the exact number and expressions of the positive normalized solutions for (1.10) and then answer an open problem concerning the exact number of positive solutions to the Kirchhoff equation with fixed frequency. Recently, Qihan He et al. [9] studied the existence and asymptotic behavior of normalized solutions of (1.10) with $|u|^{p-2}u$ replaced by a general subcritical nonlinearity $g(u)$ of mass super-critical type. In particular, $g(u)$ contains the nonlinearity in (1.5) with $2 + \frac{8}{N} < q \leq p < 2^*$ as a special case. Under some suitable assumptions, they obtain the existence of ground state normalized solutions for any given $c > 0$. After a detailed analysis via the blow up method, they also described the asymptotic behavior of these solutions as $c \to 0^+$, as well as $c \to +\infty$.

If both $b \neq 0$ and $\mu \neq 0$, then $(P_{\lambda})$ become a nonlocal equation with a non-homogeneous nonlinearity. It is much more challenging and interesting to investigate the existence and qualitative properties of solutions of (1.5). Some existence results concerning the normalized
solutions of (1.5) have been obtained over the past few years, see [15, 32] and reference therein for Kirchhoff equations with combined powers nonlinearity. However, less progress is made on the asymptotic behavior of these solutions whenever the associated parameter varies in a suitable range. The technique used in [23, 31, 33, 34] for the asymptotic study of equation (1.10) is not applicable any more, and any explicit expression of normalized solution in terms of the mass $c$ is not available for the nonlocal problem $(P_b)$.

Our main purpose in this paper is to study the effect of the nonlocal term in the case $b \neq 0$ on the existence, non-existence, multiplicity and properties of normalized solutions of (1.5) and to understand the role of the $L^2$-critical exponent $p_b$ in the existence and asymptotic behavior of normalized solutions of (1.5) as the parameter $c$ varies. As a direct consequence of our main results on the fixed frequency problem $(P_{\lambda})$ in this paper, we are able to obtain an explicit asymptotic expression of different norms of positive normalized solutions, and to give a complete description on the existence, multiplicity and precise asymptotic behavior of positive normalized solutions of (1.5). In particular, we prove that both $p_b = \frac{10}{3}$ and $p_b = \frac{14}{7}$ play a key role in the existence, multiplicity and the asymptotic behavior of normalized solutions of (1.5) if $b \neq 0$.

**Organization of the paper.** In Section 2, we state the main results in this paper. In Section 3, we give a proof of Theorem 2.1 for small $\lambda > 0$. Section 4 is devoted to the proof of Theorems 2.1 for large $\lambda > 0$. In Section 5, we prove Theorem 2.2, and in the last section, as an application of our main results, we present some results concerning the existence, non-existence, and exact number of normalized solutions of the associated mass constrained problem.

**Basic notations.** Throughout this paper, we assume $N \geq 3$.

- $C_c^\infty(\mathbb{R}^N)$ denotes the space of smooth functions with compact support in $\mathbb{R}^N$.
- $L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$ is the Lebesgue space with the norm $\|u\|_p = (\int_{\mathbb{R}^N} |u|^p)^{1/p}$.
- $H^1(\mathbb{R}^N)$ is the usual Sobolev space with the norm $\|u\|_{H^1(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2)^{1/2}$.
- $H^1_0(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u$ is radially symmetric$\}$.
- $D^{1,2}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$.
- For any $q \in (2, 2^*)$ where $2^* = \frac{2N}{N-2}$, we define
  \[
  S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}}, \quad S_q = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2}{\left(\int_{\mathbb{R}^N} |u|^{q}\right)^{2/q}}. 
  \]  

- $B_r$ denotes the ball in $\mathbb{R}^N$ with radius $r > 0$ and centered at the origin, $|B_r|$ and $B_r^c$ denote its Lebesgue measure and its complement in $\mathbb{R}^N$, respectively.
- As usual, $C$, $c$, etc., denote generic positive constants.

**Asymptotic notations.** For $\lambda > 0$ and nonnegative functions $f(\lambda)$ and $g(\lambda)$, we write:

1. $f(\lambda) \lessgtr g(\lambda)$ if there exists a positive constant $C$ independent of $\lambda$ such that $f(\lambda) \leq Cg(\lambda)$.
2. $f(\lambda) \sim g(\lambda)$ if $f(\lambda) \leq g(\lambda)$ and $f(\lambda) \geq g(\lambda)$.

If $|f(\lambda)| \lessgtr |g(\lambda)|$, we write $f(\lambda) = O((g(\lambda)))$. We also denote by $\Theta = \Theta(\lambda)$ a generic positive function satisfying $C_1\lambda \leq \Theta(\lambda) \leq C_2\lambda$ for some positive numbers $C_1, C_2 > 0$, which are
independent of $\lambda$. Finally, if $\lim f(\lambda)/g(\lambda) = 1$ as $\lambda \to \lambda_0$, then we write $f(\lambda) \simeq g(\lambda)$ as $\lambda \to \lambda_0$.

2. Main results

The existence of ground state solutions established in [18] for Kirchhoff equations with a general nonlinearity and in [29, Theorem 2.1] for Kirchhoff equations with critical nonlinearities applies to $(P_\lambda)$ directly or after a suitable scaling. In this paper, we are interested in the asymptotic behavior of ground state solutions of $(P_\lambda)$. Our main results are the following two theorems. In the first one, we consider $(P_\lambda)$ with $p = 2^*$ in dimensions $N = 3, 4$. In the second one, we consider $(P_\lambda)$ in dimension $N = 3$.

**Theorem 2.1.** Let $p = 2^*$ and $\{u_\lambda\}$ be a family ground states of $(P_\lambda)$. If $N = 3, 4$, and $q \in (2, 2^*)$, then for small $\lambda > 0$, $u_\lambda$ satisfies

$$u_\lambda(0) = \lambda^{-\frac{1}{2-q}}(V_0(0) + o(1)), $$

$$\|\nabla u_\lambda\|_2^2 = \lambda^{-\frac{2N-q(N-2)}{2(2-q)}} \left\{ \frac{N(q-2)}{2q} a^\frac{N}{2} \frac{S_q^2}{q} + O\left(\lambda^{\frac{2N-q(N-2)}{2(2-q)}}\right) \right\}, $$

$$\|u_\lambda\|_q^q = \lambda^{-\frac{2N-q(N-2)}{2(2-q)}} \left\{ a^\frac{N}{2} \frac{S_q^2}{q} + o(1) \right\}, $$

$$\|u_\lambda\|_2^2 = \lambda^{-\frac{2N-q(N-2)}{2(2-q)}} \left\{ a^\frac{N}{2} \|V_0\|_2^2 + o(1) \right\}. $$

and the least energy $m_\lambda$ of the ground state satisfies

$$m_\lambda = I_\lambda(u_\lambda) = \lambda^{-\frac{2N-q(N-2)}{2(2-q)}} \left\{ q - 2 a^\frac{N}{2} S_q^2 + O\left(\lambda^{\frac{2N-q(N-2)}{2(2-q)}}\right) \right\}. $$

Moreover, for small $\lambda > 0$, the rescaled family of ground states

$$v_\lambda(x) = \lambda^{-\frac{1}{2-q}} u_\lambda(\lambda^{-\frac{1}{2}} x) $$

satisfies

$$\|\nabla v_\lambda\|_2^2 \sim \|v_\lambda\|_2^2 \sim \|v_\lambda\|_q^q \sim \|v_\lambda\|_2^2 \sim 1, $$

and as $\lambda \to 0$, $v_\lambda$ converges in $H^1(\mathbb{R}^N)$ to $v_0$, where $v_0(x) = V_0(\frac{x}{\sqrt{\lambda}})$ and $V_0$ is the unique positive solution of the equation

$$-\Delta V + V = V^{q-1} \text{ in } \mathbb{R}^N. $$

If $N = 4$, $q \in (2, 4)$ and $bS^2 < 1$ or $N = 3$ and $q \in (4, 6)$, then for large $\lambda > 0$, $u_\lambda$ satisfies

$$u_\lambda(0) \sim \begin{cases} (\lambda \ln \lambda)^{-\frac{1}{2-q}}, & \text{if } N = 4, \\ \lambda^{-\frac{1}{2(q-1)}}, & \text{if } N = 3, \end{cases} $$

$$\frac{aS^2}{1 - bS^2} \sim \|\nabla u_\lambda\|_2^2 \sim (\lambda \ln \lambda)^{-\frac{q-2}{q-1}}, \text{ if } N = 4, $$

$$\frac{bS^3 + S^2 \sqrt{b^2 S^4 + 4a}}{2} \sim \lambda^{-\frac{6-q}{2(3-q)}}, \text{ if } N = 3. $$
Moreover, there exists $\zeta_\lambda \in (0, +\infty)$ verifying

$$\zeta_\lambda \sim \begin{cases} (\ln \lambda)^{-\frac{q}{q-2}}, & \text{if } N = 4, \\ \lambda^{-\frac{6}{6-q}}, & \text{if } N = 3, \end{cases}$$

such that for large $\lambda > 0$, the rescaled family of ground states

$$w_\lambda(x) = \frac{N-2}{4} u_\lambda(\lambda x)$$

satisfies

$$\|
abla w_\lambda\|_2^2 \sim \|w_\lambda\|_2^2 \sim \|w_\lambda\|_q^q \sim 1, \quad \|w_\lambda\|_p^p \sim \begin{cases} \ln \lambda, & \text{if } N = 4, \\ \lambda^{-\frac{6-q}{2(6-q)}}, & \text{if } N = 3, \end{cases}$$

and as $\lambda \to \infty$, $w_\lambda$ converges in $D^{1,2}(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ to $w_\infty$, where $w_\infty(x) = U_1(\gamma_N x)$ with $\gamma_N$ being given by

$$\gamma_N = \begin{cases} \sqrt{1 - bS^2 \frac{6-a}{a}} \lambda, & \text{if } N = 4, \\ \frac{bS^2 + \sqrt{b^2S^4 + 4a}}{2b}, & \text{if } N = 3. \end{cases}$$

Moreover, the least energy $m_\lambda$ of the ground state satisfies

$$m_\infty - m_\lambda \sim \begin{cases} (\ln \lambda)^{-\frac{q}{q-2}}, & \text{if } N = 4, \\ \lambda^{-\frac{6-q}{2(6-q)}}, & \text{if } N = 3, \end{cases}$$

where $m_\infty$ is given by

$$m_\infty = \begin{cases} \frac{a^2S^2}{4(1-bS^2)}, & \text{if } N = 4, \\ \frac{1}{6}b(bS^3 + S^3 \sqrt{b^2S^4 + 4a}) & \text{if } N = 3. \end{cases}$$

**Theorem 2.2.** Let $N = 3$, $b > 0$, $2 < q < p < 2^*$ and $\{u_\lambda\}$ be a family ground states of $(P_{\lambda})$. If $p = q$, then for any $\lambda > 0$, $u_\lambda$ is the unique positive solution of $(P_{\lambda})$ and satisfies

$$u_\lambda(0) = (\lambda/2)^{\frac{1}{p-2}} W_0(0),$$

$$\|
abla u_\lambda\|_2^2 = \begin{cases} \lambda^{\frac{6-p}{2(p-2)}} \frac{3(p-2)}{p} \sqrt{\lambda^{3-p}} (S_p/2)^{\frac{p}{p-2}} & \text{as } \lambda \to \infty, \\ \lambda^{\frac{6-p}{2p}} \left(\frac{bS^3}{2p}\right)^{\frac{p}{p-2}} + \Theta(\lambda^{\frac{6-p}{2p}}) & \text{as } \lambda \to 0, \end{cases}$$

and

$$\|u_\lambda\|_q^q = \begin{cases} \lambda^{\frac{6-p}{2(p-2)}} \frac{3(p-2)}{p} \sqrt{\lambda^{3-p}} (S_p/2)^{\frac{p}{p-2}} & \text{as } \lambda \to \infty, \\ \lambda^{\frac{6-p}{2p}} \left(\frac{bS^3}{2p}\right)^{\frac{p}{p-2}} + \Theta(\lambda^{\frac{6-p}{2p}}) & \text{as } \lambda \to 0, \end{cases}$$

and

$$\|u_\lambda\|_p^p = \begin{cases} \lambda^{\frac{6-p}{2(p-2)}} \frac{3(p-2)}{p} \sqrt{\lambda^{3-p}} (S_p/2)^{\frac{p}{p-2}} & \text{as } \lambda \to \infty, \\ \lambda^{\frac{6-p}{2p}} \left(\frac{bS^3}{2p}\right)^{\frac{p}{p-2}} + \Theta(\lambda^{\frac{6-p}{2p}}) & \text{as } \lambda \to 0. \end{cases}$$
\[ \|u_\lambda\|_2^2 = \lambda^{10-3p} (\sqrt{\varpi \lambda})^3 (S_p/2)^{\frac{p}{p-2}} \]
\[ = \begin{cases} \lambda^{\frac{14}{p-2}} \left( \frac{27b^3(p-2)^2(p-2)}{6p^2} (S_p/2)^{\frac{p}{p-2}} + \Theta(\lambda^{-\frac{6-p}{p-2}}) \right), & \text{as } \lambda \to \infty, \\
\lambda^{\frac{10-3p}{p-2}} \left( \frac{6}{p} a \varpi \lambda \right)^{\frac{2}{p-2}} (S_p/2)^{\frac{p}{p-2}} + \Theta(\lambda^{-\frac{6-p}{p-2}}), & \text{as } \lambda \to 0, \end{cases} \]

Moreover, for any \( \lambda > 0 \), there holds
\[ u_\lambda(x) = (\lambda/2)^{-\frac{1}{p-2}} W_0(\lambda^{-\frac{1}{2}} \varpi^{\frac{1}{2}} x), \]
(10.0)

where
\[ \sqrt{\varpi \lambda} = \frac{3b(p-2)}{4p} \lambda^{\frac{6-p}{2(p-2)}} (S_p/2)^{\frac{p}{p-2}} + \sqrt{\frac{9b^2(p-2)^2}{16p^2} \lambda^{\frac{6-p}{p-2}} (S_p/2)^{\frac{2p}{p-2}} + a}, \]
(11.0)

and \( S_p = \|W_0\|_p^{p-2} \), \( W_0 \in H^1(\mathbb{R}^3) \) is the unique positive solution of the equation
\[ -\Delta W + W = W^{p-1} \text{ in } \mathbb{R}^N. \]

If \( q < p \) and \( \lambda > 0 \) is sufficiently small, then \( u_\lambda \) is the unique positive solution of \((P_\lambda)\) and satisfies
\[ u_\lambda(0) = \lambda^{\frac{1}{p-2}} (V_0(0) + o(1)), \]
(12.0)

\[ ||\nabla u_\lambda||_2^2 = \begin{cases} \lambda^{\frac{6-p}{2(p-2)}} \left( \frac{3(q-2)}{2q} a \varpi S_{q-2}^{\frac{q}{q-2}} - \Theta(\lambda^{\frac{p-q}{p-2}}) \right), & \text{if } q > 2p - 6, \\
\lambda^{\frac{6-p}{2(p-2)}} \left( \frac{3(q-2)}{2q} a \varpi S_{q-2}^{\frac{q}{q-2}} + O(\lambda^{\frac{p-q}{p-2}}) \right), & \text{if } q \leq 2p - 6, \end{cases} \]
(13.0)

\[ ||u_\lambda||_2^2 = \begin{cases} \lambda^{\frac{10-3q}{2(q-2)}} \left( \frac{6-q}{2q} a \varpi S_{q-2}^{\frac{q}{q-2}} - \Theta(\lambda^{\frac{p-q}{p-2}}) \right), & \text{if } q > 2p - 6, \\
\lambda^{\frac{10-3q}{2(q-2)}} \left( \frac{6-q}{2q} a \varpi S_{q-2}^{\frac{q}{q-2}} + O(\lambda^{\frac{p-q}{p-2}}) \right), & \text{if } q \leq 2p - 6, \end{cases} \]
(14.0)

Moreover, as \( \lambda \to 0 \), the rescaled family of groundstates
\[ u_\lambda(x) = \lambda^{-\frac{1}{p-2}} u_\lambda(\lambda^{-1/2} \varpi \lambda x), \]
(15.0)

converge in \( H^1(\mathbb{R}^3) \) to the unique positive solution \( V_0 \) of the equation
\[ -\Delta V + V = V^{p-1} \text{ in } \mathbb{R}^N. \]

If \( q < p \) and \( \lambda > 0 \) is sufficiently large, then \( u_\lambda \) is the unique positive solution of \((P_\lambda)\) and satisfies
\[ u_\lambda(0) = \lambda^{\frac{1}{p-2}} (W_0(0) + o(1)), \]
Moreover, as \( \lambda \to \infty \), the rescaled family of groundstates

\[
   w_\lambda(x) = \lambda^{-\frac{1}{p-2}} u_\lambda(\lambda^{-1/2} \sqrt{\omega_\lambda} x), \quad \omega_\lambda = a + b \int_{\mathbb{R}^N} |\nabla v_\lambda|^2,
\]

converge in \( H^1(\mathbb{R}^3) \) to the unique positive solution \( W_0 \) of the equation

\[
   -\Delta W + W = W^{p-1} \quad \text{in} \mathbb{R}^N.
\]

Assume \( M \in C^1((0, \infty), \mathbb{R}) \), we denote \( M(0) := \lim_{\lambda \to 0} M(\lambda) \) and \( M(\infty) := \lim_{\lambda \to \infty} M(\lambda) \). The following lemma is proved in [20, Lemma 1.1].

**Lemma 2.1.** Let \( \eta > 0 \) is a constant. Then the following statements hold true:

1. If \( M(\lambda) \sim \lambda^0 \) as \( \lambda \to 0 \), then there is \( \lambda_0 > 0 \) such that \( M'(\lambda) > 0 \) for \( \lambda \in (0, \lambda_0) \).
2. If \( M(\lambda) \sim \lambda^{-\eta} \) as \( \lambda \to 0 \), then there is \( \lambda_0 > 0 \) such that \( M'(\lambda) < 0 \) for \( \lambda \in (0, \lambda_0) \).
3. If \( M(\lambda) \sim \lambda^{-\eta} \) as \( \lambda \to \infty \), then there is \( \lambda_\infty > 0 \) such that \( M'(\lambda) < 0 \) for \( \lambda \in (\lambda_\infty, \infty) \).
4. If \( M(\lambda) \sim \lambda^0 \) as \( \lambda \to \infty \), then there is \( \lambda_\infty > 0 \) such that \( M'(\lambda) > 0 \) for \( \lambda \in (\lambda_\infty, \infty) \).

The following corollary is a direct consequence of Theorems 2.1–2.2 and Lemma 2.1.

**Corollary 2.1.** Let \( 2 < q \leq p \leq 6 \), then

\[
   M(0) = \begin{cases} 
   0, & \text{if } q < 10/3 \text{ and } q \leq p < 6, \\
   \frac{6-q}{2q} \frac{3}{2} S_q^\frac{q}{q-2}, & \text{if } q = 10/3 \text{ and } 10/3 < p < 6, \\
   \infty, & \text{if } q > 10/3 \text{ and } q \leq p < 6,
   \end{cases}
\]

and

\[
   M(\infty) = \begin{cases} 
   27b^3(p-2)^3(6-p) S_p^\frac{4p}{p-2}, & \text{if } p = 14/3 \text{ and } q < p, \\
   0, & \text{if } p > 14/3 \text{ and } q \leq p.
   \end{cases}
\]

Moreover, there exists a small \( \lambda_0 > 0 \) such that for any \( \lambda \in (0, \lambda_0) \),

\[
   \begin{cases} 
   M'(\lambda) > 0, & \text{if } q < 10/3 \text{ and } q \leq p < 6, \\
   M'(\lambda) < 0, & \text{if } q = 10/3 \text{ and } 10/3 < p < 14/3, \\
   M'(\lambda) < 0, & \text{if } q > 10/3 \text{ and } q \leq p,
   \end{cases}
\]

and there exists a large \( \lambda_\infty > 0 \) such that for any \( \lambda \in (\lambda_\infty, +\infty) \),

\[
   \begin{cases} 
   M'(\lambda) < 0, & \text{if } p < 14/3 \text{ and } q \leq p, \\
   M'(\lambda) > 0, & \text{if } p > 14/3 \text{ and } q \leq p.
   \end{cases}
\]
Remark 2.1. In the Sobolev critical case \( p = 2^* \), a similar result as above also holds and in particular, we have \( M(0) = \infty \) and \( M(\infty) = 0 \) if \( N = 4 \) and \( q \in (3, 4) \), or \( N = 3 \) and \( q \in (4, 6) \); \( M(0) = M(\infty) = 0 \) if \( N = 4 \) and \( q \in (2, 3) \); \( M(0) = \infty \) if \( N = 3 \) and \( q \in (10/3, 4] \), and \( M(0) = 0 \) if \( N = 3 \) and \( q \in (2, 10/3) \). The sign of \( M'(\lambda) \) can also be determined for small \( \lambda > 0 \) or for large \( \lambda > 0 \), and we omitted the details. We mention that in the subcritical case, the sign of \( M'(\lambda) \) is still open in the following cases:

1. \( q = \frac{10}{3}, p \in \left(\frac{14}{3}, 6\right) \) and \( \lambda > 0 \) small,
2. \( p = \frac{14}{3}, q \in \left(2, \frac{14}{3}\right) \) and \( \lambda > 0 \) large.

According to Corollary 2.1 and results concerning the case \( b = 0 \) given in Section 5, we draw the following figures which reveal the variations of \( M(\lambda) \) for small \( \lambda > 0 \) and large \( \lambda > 0 \) when \( (p, q) \) belongs to different regions in the \((p, q)\) plane. Plainly, there exists a dramatic change between \( b = 0 \) and \( b \neq 0 \). This observation results in a striking different feature in the existence, non-existence and exact number of normalized solutions of (1.5).

3. Proof of Theorem 2.1 as \( \lambda \to 0 \)

Let
\[
v(x) = \lambda^{-\frac{1}{p-2}} u(\lambda^{-\frac{1}{2}} x).
\]
Then the equation \((P_\lambda)\) reduces to
\[
-a \Delta v + v - b \lambda^{2N-q(N-2)} \int_{\mathbb{R}^N} |\nabla v|^2 dx \Delta v = v^{q-1} + \lambda^{\frac{2^*-q}{q-2}} v^{2^*-1}.
\]
(\(Q_\lambda\))

The energy functional for \((Q_\lambda)\) is defined by
\[
J_\lambda(v) = \frac{a}{q} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 + \frac{b}{2} \lambda^{\frac{2N-q(N-2)}{2(q-2)}} \left( \int_{\mathbb{R}^N} |\nabla v|^2 \right)^2
\]
\[- \frac{1}{q} \int_{\mathbb{R}^N} |v|^q - \frac{1}{2} \lambda^{\frac{2^*-q}{q-2}} \int_{\mathbb{R}^N} |v|^{2^*}.
\]
(3.2)
The formal limit equation for \((Q_\lambda)\) as \(\lambda \to 0\) is given by
\[-a\Delta v + v = v^{q-1}\quad \text{in } \mathbb{R}^N.
\] (Q_0)

The energy functional for \((Q_0)\) is given by
\[J_0(v) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 - \frac{1}{q} \int_{\mathbb{R}^N} |v|^q.
\]

**Lemma 3.1.** Let \(\lambda > 0\), \(u \in H^1(\mathbb{R}^N)\) and \(v\) is the rescaling \((3.1)\) of \(u\). Then:
\[(a) \quad \|\nabla u\|^2_2 = \lambda^{2N-q(N-2)} \|\nabla v\|^2_2, \quad \|u\|^q_2 = \lambda^{2\frac{N-q(N-2)}{q-2}} \|v\|^q_2,
\]
\[(b) \quad \|u\|^2_2 = \lambda^{\frac{4-N(q-2)}{2(q-2)}} \|v\|^2_2, \quad \|u\|^2_2^* = \lambda^{\frac{N[2N-q(N-2)]}{2(q-2)(q-4)}} \|v\|^2_2^*,
\]
\[(c) \quad I_\lambda(u) = \lambda^{\frac{2N-q(N-2)}{2(q-2)}} J_\lambda(v).
\]

The above lemma is easily proved and the details will be omitted. In particular, it follows from Lemma 3.1 \((c)\) that the rescaling \(v\) of the ground state \(u\) of \((P_\lambda)\) corresponds to a ground state of \((Q_\lambda)\).

**Lemma 3.2.** The rescaled family of ground states \(\{v_\lambda\}\) is bounded in \(H^1(\mathbb{R}^N)\) for small \(\lambda > 0\).

**Proof.** It is standard to see that ground states of \((Q_\lambda)\) satisfy the Nehari identity
\[a \int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} |v|^2 + b\lambda^{\frac{2N-q(N-2)}{2(q-2)}} (\int_{\mathbb{R}^N} |\nabla v|^2)^2 = \int_{\mathbb{R}^N} |v|^q + \lambda^{\frac{2-q}{q-2}} \int_{\mathbb{R}^N} |v|^2^*
\]
and the Pohožaev identity
\[a \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 + \frac{b}{2} \lambda^{\frac{2N-q(N-2)}{2(q-2)}} (\int_{\mathbb{R}^N} |\nabla v|^2)^2 = \frac{1}{q} \int_{\mathbb{R}^N} |v|^q + \frac{1}{2} \lambda^{\frac{2-q}{q-2}} \int_{\mathbb{R}^N} |v|^2^*.
\]

Therefore, it follows that
\[
\left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |v|^2 = \left(\frac{1}{q} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |v|^q,
\]
and hence
\[\frac{1}{N} \int_{\mathbb{R}^N} |v|^2 \leq \frac{\lambda^{\frac{2-q}{2(q-2)}}}{\frac{2-q}{2q}} \left(\frac{1}{q} \int_{\mathbb{R}^N} |v|^q \right)^{\frac{2^*-q}{2^*_q}} + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla v|^2 \right)^{\frac{2^*_q}{2^*_q}},
\]
which implies that
\[
\frac{1}{N} \left(\int_{\mathbb{R}^N} |v|^2 \right)^{\frac{2^*-q}{2q}} \leq \frac{\lambda^{\frac{2-q}{2(q-2)}}}{\frac{2-q}{2q}} \left(\frac{1}{q} \int_{\mathbb{R}^N} |v|^q \right)^{\frac{2^*-q}{2^*_q}}.
\]

To prove the boundedness of \(\{v_\lambda\}\) in \(H^1(\mathbb{R}^N)\), it suffices to show that \(\{v_\lambda\}\) is bounded in \(D^{1,2}(\mathbb{R}^N)\).

Let \(v_0\) be the unique positive solution of the equation \((Q_0)\), then by the Pohožaev’s identity, we have
\[\frac{1}{2} \int_{\mathbb{R}^N} |v_0|^2 < \frac{a}{2^*} \int_{\mathbb{R}^N} |\nabla v_0|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |v_0|^2 = \frac{1}{q} \int_{\mathbb{R}^N} |v_0|^q,
\] (3.5)
which implies that for small \(\lambda > 0\), there is a unique \(t_\lambda > 0\) such that
\[
\frac{a}{2} \int_{\mathbb{R}^N} |\nabla v_0|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |v_0|^2 + \frac{b}{2} \lambda^{\frac{2N-q(N-2)}{2(q-2)}} \left(\int_{\mathbb{R}^N} |\nabla v_0|^2 \right)^2
\]
\[= \frac{1}{q} \int_{\mathbb{R}^N} |v_0|^q + \frac{1}{2} \lambda^{\frac{2-q}{q-2}} \int_{\mathbb{R}^N} |v_0|^2^*.
\] (3.6)
If $N = 3$ and $t_\lambda > 1$, then

$$
t_\lambda \leq \frac{1}{2} \lambda^2 \int_\mathbb{R}^N |\nabla v_0|^2 + \frac{b}{2} \lambda \frac{2^{2(N-2)} - 2(N-2)}{2(N-2)} \left( \int_\mathbb{R}^N |\nabla v_0|^2 \right)^2
$$

$$
= \frac{1}{2} \lambda^2 \int_\mathbb{R}^N |v_0|^2 + \frac{b}{2} \lambda \frac{2^{2(N-2)} - 2(N-2)}{2(N-2)} \left( \int_\mathbb{R}^N |\nabla v_0|^2 \right)^2
$$

$$
< C < \infty.
$$

If $N = 4$ and $t_\lambda > 1$, then

$$
t_\lambda^2 = \frac{1}{2} \lambda^2 \int_\mathbb{R}^N |\nabla v_0|^2
$$

$$
= \frac{1}{2} \lambda^2 \int_\mathbb{R}^N |v_0|^2 + \frac{b}{2} \lambda \frac{2^{2(N-2)} - 2(N-2)}{2(N-2)} \left( \int_\mathbb{R}^N |\nabla v_0|^2 \right)^2
$$

$$
< C < \infty.
$$

Therefore, we obtain

$$
m_\lambda \leq \sup_{t > 0} J_\lambda((v_0)_t)
$$

$$
= \sup_{t > 0} \frac{1}{2} t^{N-2} \int_\mathbb{R}^N |\nabla v_0|^2 + \frac{1}{4} t^N \int_\mathbb{R}^N |v_0|^2 + \frac{b}{4} t^{2(N-2)} \lambda \frac{2^{2(N-2)} - 2(N-2)}{2(N-2)} \left( \int_\mathbb{R}^N |\nabla v_0|^2 \right)^2
$$

$$
< \sup_{t > 0} \frac{1}{2} t^{N-2} \int_\mathbb{R}^N |\nabla v_0|^2 + \frac{1}{4} t^N \lambda \frac{2^{2(N-2)} - 2(N-2)}{2(N-2)} \left( \int_\mathbb{R}^N |\nabla v_0|^2 \right)^2
$$

$$
< m_0 + C \lambda \frac{2^{2(N-2)} - 2(N-2)}{2(N-2)}.
$$

This proves that $m_\lambda \leq C < +\infty$ for all small $\lambda > 0$.

On the other hand, we have

$$
m_\lambda := J_\lambda(v_\lambda) = \frac{1}{2} \int_\mathbb{R}^N |\nabla v_\lambda|^2 + \frac{1}{4} \int_\mathbb{R}^N |v_\lambda|^2 + \frac{b}{4} \lambda \frac{2^{2(N-2)} - 2(N-2)}{2(N-2)} \left( \int_\mathbb{R}^N |\nabla v_\lambda|^2 \right)^2
$$

$$
= \frac{1}{2} \int_\mathbb{R}^N |\nabla v_\lambda|^2 + \frac{1}{4} \int_\mathbb{R}^N |v_\lambda|^2 + \frac{b}{4} \lambda \frac{2^{2(N-2)} - 2(N-2)}{2(N-2)} \left( \int_\mathbb{R}^N |\nabla v_\lambda|^2 \right)^2
$$

$$
\geq \frac{1}{4} \lambda \frac{2^{2(N-2)} - 2(N-2)}{2(N-2)} \left( \int_\mathbb{R}^N |\nabla v_\lambda|^2 \right)^2.
$$

This yields the boundedness of $||\nabla v_\lambda||_2$ for small $\lambda > 0$ and completes the proof.

**Lemma 3.3.** Set

$$
v_\lambda(x) = \begin{cases} v(\frac{x}{\lambda}) & \text{if } t > 0, \\ \frac{1}{2} & \text{if } t = 0. \end{cases}
$$

Then for small $\lambda > 0$, there holds

$$
m_\lambda = \inf_{\substack{v \in H^1(\mathbb{R}^N) \setminus \{0\} \\ t \geq 0}} J_\lambda(tv) = \inf_{\substack{v \in H^1(\mathbb{R}^N) \setminus \{0\} \\ t \geq 0}} J_\lambda(v_t).
$$

In particular, we have $m_\lambda = J_\lambda(v_\lambda) = \sup_{t \geq 0} J_\lambda(tv_\lambda) = \sup_{t \geq 0} J_\lambda((v_\lambda)_t)$.

The proof of Lemma 3.3 is similar to that of [20, Lemma 3.2] and is omitted. Next we obtain an estimation of the least energy.
Lemma 3.4. Let \( N = 3 \) or \( 4 \), and \( u_\lambda \) is a ground state solution of \((P_\lambda)\), then
\[
m_\lambda - m_0 = O(\lambda^{\frac{2N-q(N-2)}{4(q-2)}}),
\] as \( \lambda \to 0 \), where \( m_0 := \inf_{v \in \mathcal{H}^1(\mathbb{R}^N) \backslash \{0\}} \sup_{t \geq 0} J_0(tv) \) is the ground state energy for \((Q_0)\).

Proof. By (3.7), we have
\[
m_\lambda \leq m_0 + C\lambda^{\frac{2N-q(N-2)}{4(q-2)}}.
\]
On the other hand, by Lemma 3.3, we have
\[
m_0 \leq \sup_{t \geq 0} J_0(tv_\lambda) = J_0(t_\lambda v_\lambda)
\]
\[
\leq \sup_{t \geq 0} J_\lambda(tv_\lambda) - \frac{b_\lambda}{\lambda} \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + \frac{1}{q} t_\lambda^2 \lambda^{\frac{2^*-q}{q-2}} \int_{\mathbb{R}^N} |v_\lambda|^{2^*}
\]
where
\[
t_\lambda = \frac{a \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + b_\lambda \int_{\mathbb{R}^N} |v_\lambda|^2}{\int_{\mathbb{R}^N} |v_\lambda|^q} \|v_\lambda\|^2
\]
\[
\leq \frac{2N-q(N-2)}{2(q-2)} \left( \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 - \lambda^{\frac{2^*-2}{q-2}} \int_{\mathbb{R}^N} |v_\lambda|^{2^*} \right)
\]
\[
\leq 1 + C\lambda^{\frac{2N-q(N-2)}{4(q-2)}}.
\]
Therefore, we get
\[
m_0 \leq m_\lambda + C\lambda^{\frac{2N-q(N-2)}{4(q-2)}}.
\]
The proof is complete. \( \square \)

Lemma 3.5. Let \( N = 3 \) or \( 4 \), and \( u_\lambda \) is a ground state solution of \((P_\lambda)\), then
\[
\int_{\mathbb{R}^N} |v_\lambda|^q = \int_{\mathbb{R}^N} |v_0|^q + O(\lambda^{\frac{2N-q(N-2)}{4(q-2)}}),
\] (3.10)
\[
\int_{\mathbb{R}^N} |v_\lambda|^2 = \int_{\mathbb{R}^N} |v_0|^2 + O(\lambda^{\frac{2N-q(N-2)}{4(q-2)}}),
\] (3.11)
\[
\int_{\mathbb{R}^N} |\nabla v_\lambda|^2 = \int_{\mathbb{R}^N} |\nabla v_0|^2 + O(\lambda^{\frac{2N-q(N-2)}{4(q-2)}}),
\] (3.12)
as \( \lambda \to 0 \).

Proof. Let \( u_\lambda \) be a ground state solution of \((P_\lambda)\) and
\[
A_\lambda = \int_{\mathbb{R}^N} |\nabla v_\lambda|^2, \quad B_\lambda = \int_{\mathbb{R}^N} |v_\lambda|^2, \quad C_\lambda = \int_{\mathbb{R}^N} |v_\lambda|^q, \quad D_\lambda = \int_{\mathbb{R}^N} |v_\lambda|^{2^*}.
\] (3.13)
Then the Nehari and Pohožaev identities imply that
\[
\begin{aligned}
& A_\lambda + B_\lambda + b_\lambda \lambda^{\frac{2N-q(N-2)}{4(q-2)}} A_\lambda^2 = C_\lambda + \lambda^{\frac{2^*-q}{q-2}} D_\lambda,
& \frac{q}{2^*} A_\lambda + \frac{1}{2} B_\lambda + b_\lambda \lambda^{\frac{2N-q(N-2)}{4(q-2)}} A_\lambda^{2^*} = \frac{1}{q} C_\lambda + \frac{1}{2^*} \lambda^{\frac{2^*-q}{q-2}} D_\lambda.
\end{aligned}
\]
From which, we conclude that
\[
B_\lambda = \frac{N(2^* - q)}{2^*} C_\lambda, \quad a A_\lambda = \frac{N(q - 2)}{2q} C_\lambda + O(\lambda^{\frac{2N-q(N-2)}{2(q-2)}}).
\] (3.14)
Therefore, we obtain
\[ m_\lambda = \frac{1}{q} aA_\lambda + \frac{1}{q} B_\lambda - \frac{1}{q} C_\lambda + O(\lambda^{2 N - q(N - 2) \over 2(q - 2)}) \]
\[ = \left( \frac{N(q-2)}{4q} + \frac{N(2^* - q)}{2^*q} - \frac{1}{q} \right) C_\lambda + O(\lambda^{2 N - q(N - 2) \over 2(q - 2)}) \]
\[ = \frac{q - 2}{2q} C_\lambda + O(\lambda^{2 N - q(N - 2) \over 2(q - 2)}). \]

In a similar way, we can show that
\[ m_0 = \frac{q - 2}{2q} C_0. \]
Thus, we obtain
\[ q - 2 \over 2q (C_\lambda - C_0) = m_\lambda - m_0 + O(\lambda^{2 N - q(N - 2) \over 2(q - 2)}), \]
which together with Lemma 3.4 implies that
\[ C_\lambda - C_0 = \int_{\mathbb{R}^N} |v_\lambda|^q - \int_{\mathbb{R}^N} |v_0|^q = O(\lambda^{2 N - q(N - 2) \over 2(q - 2)}). \]

Since
\[ B_0 = \frac{N(2^* - q)}{2^*q} C_0, \quad aA_0 = \frac{N(q - 2)}{2q} C_0. \quad (3.15) \]

it follows from (3.14) and (3.15) that
\[ A_\lambda - A_0 = \frac{N(q - 2)}{2aq}(C_\lambda - C_0) + O(\lambda^{2 N - q(N - 2) \over 2(q - 2)}) = O(\lambda^{2 N - q(N - 2) \over 2(q - 2)}), \]
\[ B_\lambda - B_0 = \frac{N(2^* - q)}{2^*q}(C_\lambda - C_0) = O(\lambda^{2 N - q(N - 2) \over 2(q - 2)}), \]
from which (3.11) and (3.12) follows. The proof is complete. \( \square \)

Proof of Theorem 2.1 for small \( \lambda \). Observe that \( v_\lambda \to v_0 \) in \( H^1(\mathbb{R}^N) \) with \( v_0 \) being the unique ground state solution of \((Q_0)\). For small \( \lambda > 0 \), Theorem 2.1 follows from Lemmas 3.1-3.5 and the details will be omitted. \( \square \)

4. Proof of Theorem 2.1 as \( \lambda \to \infty \)

4.1. Rescalings. For \( \lambda > 0 \), define the rescaling
\[ v(x) = \lambda^{-\frac{1}{q-2}} u(\lambda^{2^* - q - 2} x). \quad (4.1) \]
Rescaling (4.1) transforms \((P_\lambda)\) into the equivalent equation
\[ -(a + b \int_{\mathbb{R}^N} |\nabla v|^2) \Delta v + \lambda^{2^* - q} v = \lambda^{2^* - q} v^{q-1} + v^{2^* - 1} \text{ in } \mathbb{R}^N. \quad (R_\lambda) \]
The corresponding energy functional is given by
\[ J_\lambda(v) = {a \over 2} \int_{\mathbb{R}^N} |\nabla v|^2 + {\lambda - \sigma \over 2} \int_{\mathbb{R}^N} |v|^2 + {b \over 4} \left( \int_{\mathbb{R}^N} |\nabla v|^2 \right)^2 - {\lambda - \sigma \over q} \int_{\mathbb{R}^N} |v|^q - {1 \over 2} \int_{\mathbb{R}^N} |v|^{2^*}, \quad (4.2) \]
here and in what follows, we set
\[ \sigma := \frac{2^* - q}{q - 2}. \]
The formal limit equation for \((R_\lambda)\) as \(\lambda \to \infty\) is given by the equation
\[
-(a + b \int_{\mathbb{R}^N} |\nabla v|^2) \Delta v = v^{2^*-1} \quad \text{in } \mathbb{R}^N. \tag{R_\infty}
\]

The corresponding functional is given by
\[
J_\infty(v) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + b \left( \int_{\mathbb{R}^N} |\nabla v|^2 \right)^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*}. \tag{4.3}
\]

We denote their corresponding Nehari manifolds as follows:
\[
\mathcal{N}_\lambda := \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \int_{\mathbb{R}^N} a|\nabla v|^2 + \lambda - \sigma |v|^2 + b(\int_{\mathbb{R}^N} |\nabla v|^2)^2 = \int_{\mathbb{R}^N} |v|^{2^*} + \lambda - \sigma |v|^q \};
\]
\[
\mathcal{N}_\infty := \{ v \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} \mid \int_{\mathbb{R}^N} a|\nabla v|^2 + b(\int_{\mathbb{R}^N} |\nabla v|^2)^2 = \int_{\mathbb{R}^N} |v|^{2^*} \}.
\]

Then
\[
m_\lambda := \inf_{v \in \mathcal{N}_\lambda} J_\lambda(v), \quad m_\infty := \inf_{v \in \mathcal{N}_\infty} J_\infty(v)
\]
are well-defined and positive.

Let \(\varpi := \varpi(v) = a + b \int_{\mathbb{R}^N} |\nabla v|^2\), then \(\varpi(v)\) is invariant respect to the rescaling
\[
\tilde{v}(x) = \eta^{\frac{N-2}{2}} v(\eta x). \tag{4.4}
\]

That is, \(\varpi(\tilde{v}) = \varpi(v)\).

Set
\[
w(x) = v(\sqrt{\varpi} x), \tag{4.5}
\]
then the equation \((R_\infty)\) reduces to
\[
- \Delta w = w^{2^*-1}. \tag{4.6}
\]

Moreover, \(\varpi\) satisfies the equation
\[
\varpi = a + \varpi^{\frac{N-2}{2}} b \int_{\mathbb{R}^N} |\nabla w|^2.
\]

If \(N = 3\), then
\[
\sqrt{\varpi} = \frac{b \int_{\mathbb{R}^N} |\nabla w|^2 + \sqrt{b^2(\int_{\mathbb{R}^N} |\nabla w|^2)^2 + 4a}}{2} = \frac{bS^2 + \sqrt{b^2S^3 + 4a}}{2}.
\]

If \(N = 4\) and \(bS^2 < 1\), then
\[
\varpi = \frac{a}{1 - b \int_{\mathbb{R}^N} |\nabla w|^2} = \frac{a}{1 - bS^2}.
\]

It is easy to see that \(m_\infty\) is attained on \(\mathcal{N}_\infty\) by \(v_1(x) := W_1(\frac{x}{\sqrt{\varpi}})\) and the family of its rescalings
\[
v_\rho(x) := \rho^{\frac{N-2}{2}} v_1(x/\rho), \quad \rho > 0, \tag{4.7}
\]
where \(W_1\) is the the Talenti function
\[
W_1(x) := \left[ N(N-2) \right]^{\frac{N-2}{4}} \left( \frac{1}{1 + |x|^2} \right)^{\frac{N-2}{2}}.
\]
Lemma 4.1. Let $\lambda > 0$, $u \in H^1(\mathbb{R}^N)$ and $v$ is the rescaling (4.1) of $u$. Then:

(a) $\|\nabla u\|_2^2 = \|\nabla v\|_2^2$, $\|u\|^{2^*_s}_2 = \|v\|^{2^*_s}_2$,

(b) $\lambda^{1+\sigma}\|u\|_2^2 = \|v\|_2^2$, $\lambda^{\sigma}\|u\|_q^q = \|v\|_q^q$,

(c) $I_\lambda(u) = J_\lambda(v)$.

In particular, if $v_\lambda$ is the rescaling (4.1) of the ground state $u_\lambda$, then

$$J_\lambda(v_\lambda) = I_\lambda(u_\lambda)$$

and hence $v_\lambda$ is the ground state of $(R_\lambda)$. Moreover, $v_\lambda$ satisfies the Pohozaev’s identity [4]:

$$\frac{a}{2^*} \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + \frac{\lambda - \sigma}{2} \int_{\mathbb{R}^N} |v_\lambda|^2 + \frac{b}{2^*} \left( \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 \right)^2 = \frac{\lambda - \sigma}{q} \int_{\mathbb{R}^N} |v_\lambda|^q + \frac{1}{2^*} \int_{\mathbb{R}^N} |v_\lambda|^{2^*}. \quad (4.9)$$

Define the Pohozaev manifold

$$P_\lambda := \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} \mid P_\lambda(v) = 0 \},$$

where

$$P_\lambda(v) := \frac{a}{2^*} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{\lambda - \sigma}{2} \int_{\mathbb{R}^N} |v|^2 + \frac{b}{2^*} \left( \int_{\mathbb{R}^N} |\nabla v|^2 \right)^2 - \frac{\lambda - \sigma}{q} \int_{\mathbb{R}^N} |v|^q - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*}. \quad (4.10)$$

Clearly, $v_\lambda \in P_\lambda$. Moreover, we have the following minimax characterizations for the least energy level $m_\lambda$:

$$m_\lambda = \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} J_\lambda(tv) = \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} J_\lambda(v_t). \quad (4.11)$$

In particular, we have $m_\lambda = J_\lambda(v_\lambda) = \sup_{t>0} J_\lambda(t v_\lambda) = \sup_{t>0} J_\lambda((v_\lambda)_t)$.

Lemma 4.2. The rescaled family of ground-states $\{v_\lambda\}$ is bounded in $H^1(\mathbb{R}^N)$ for large $\lambda > 0$.

Proof. The Nehari identity

$$a \int_{\mathbb{R}^N} |\nabla v|^2 + \lambda - \sigma \int_{\mathbb{R}^N} |v|^2 + b \left( \int_{\mathbb{R}^N} |\nabla v|^2 \right)^2 = \lambda - \sigma \int_{\mathbb{R}^N} |v|^q + \int_{\mathbb{R}^N} |v|^{2^*}$$

and the Pohozaev identity

$$\frac{a}{2^*} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{\lambda - \sigma}{2} \int_{\mathbb{R}^N} |v|^2 + \frac{b}{2^*} \left( \int_{\mathbb{R}^N} |\nabla v|^2 \right)^2 = \frac{1}{q} \int_{\mathbb{R}^N} |v|^q + \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*}. \quad (4.10)$$

imply that

$$\left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |v|^2 = \left( \frac{1}{q} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |v|^q. \quad (4.12)$$

Hence

$$\frac{1}{N} \int_{\mathbb{R}^N} |v|^2 \leq \frac{2^* - q}{2^* q} \int_{\mathbb{R}^N} |v|^q \leq \frac{2^* - q}{2^* q} \left( \int_{\mathbb{R}^N} |v|^2 \right)^{\frac{2^*}{2^* - q}} \left( \frac{1}{S} \int_{\mathbb{R}^N} |\nabla v|^2 \right)^{\frac{2^*(q-2)}{2^* (2^* - q)}},$$

which implies that

$$\frac{1}{N} \int_{\mathbb{R}^N} |v|^2 \leq \frac{2^* - q}{2^* q} \left( \frac{1}{S} \int_{\mathbb{R}^N} |\nabla v|^2 \right)^{\frac{2^*(q-2)}{2^* (2^* - q)}}.$$
Lemma 4.3. There exists a constant $C = C(q) > 0$ such that for all large $\lambda > 0$,

$$m_\lambda \leq \begin{cases} m_\infty - C(\lambda \ln \lambda)^{-\frac{4}{q-2}} & \text{if } N = 4, \\ m_\infty - C\lambda^{-\frac{6-q}{2(q-1)}} & \text{if } N = 3 \text{ and } q > 4. \end{cases} \tag{4.13}$$

Proof. Let $\rho > 0$, $R \gg 1$ be a large parameter and $\eta_R \in C_0^\infty(\mathbb{R})$ is a cut-off function such that $\eta_R(r) = 1$ for $|r| < R$, $0 < \eta_R(r) < 1$ for $R < |r| < 2R$, $\eta_R(r) = 0$ for $|r| > 2R$, and $|\eta_R'(r)| \leq 2/R$.

For $\ell \gg 1$, a straightforward computation shows that

$$\int_{\mathbb{R}^N} |\nabla (\eta \tilde{W}_1)|^2 = S_1^N + O(\ell^{-N-2}) = \begin{cases} S_1^N + O(\ell^{-2}) & \text{if } N = 4, \\ S_1^N + O(\ell^{-1}) & \text{if } N = 3. \end{cases} \tag{4.14}$$

$$\int_{\mathbb{R}^N} |\eta W_1|^{2^*} = S_1^N + O(\ell^{-N}), \tag{4.15}$$

$$\int_{\mathbb{R}^N} |\eta W_2| \leq \begin{cases} \ln \ell(1 + o(1)) & \text{if } N = 4, \\ \ell(1 + o(1)) & \text{if } N = 3. \end{cases} \tag{4.16}$$

By Lemma 3.4, we find

$$m_\lambda \leq \sup_{\ell \geq 0} J_\lambda((\eta_R \tilde{W}_\rho)_\ell) = J_\lambda((\eta_R \tilde{W}_\rho)_{t_\lambda}) \tag{4.17}$$

$$\leq \sup_{\ell \geq 0} \left( \frac{a t_\lambda^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla (\eta_R \tilde{W}_\rho)|^2 + \frac{b 2^{(N-2)-4} t_\lambda^{N-2}}{4} \left( \int_{\mathbb{R}^N} |\nabla (\eta_R \tilde{W}_\rho)|^2 \right)^2 - \frac{t_\lambda^N}{2} \int_{\mathbb{R}^N} |\eta_R \tilde{W}_\rho|^{2^*} \right)$$

$$- \lambda^{-\sigma} t_\lambda^{-\frac{6-q}{2(q-1)}} \left( \int_{\mathbb{R}^N} \frac{1}{4} |\eta_R \tilde{W}_\rho|^q + \frac{1}{2} |\eta_R \tilde{W}_\rho|^{2^*} \right)$$

$$= (I) - \lambda^{-\sigma} (II),$$

where $t_\lambda > 0$ is the unique solution of the following equation

$$\frac{a}{2 t_\lambda^2} \int_{\mathbb{R}^N} |\nabla (\eta_R \tilde{W}_\rho)|^2 + \frac{b}{2 t_\lambda^2} \left( \int_{\mathbb{R}^N} |\nabla (\eta_R \tilde{W}_\rho)|^2 \right)^2$$

$$= \frac{1}{2} \int_{\mathbb{R}^N} |\eta_R \tilde{W}_\rho|^{2^*} + \lambda^{-\sigma} \left[ \frac{1}{4} \int_{\mathbb{R}^N} |\eta R \tilde{W}_\rho|^q + \frac{1}{2} \int_{\mathbb{R}^N} |\eta R \tilde{W}_\rho|^{2^*} \right].$$

Since

$$\int_{\mathbb{R}^N} |\eta R \tilde{W}_\rho|^q \leq \left( \int_{\mathbb{R}^N} |\eta R \tilde{W}_\rho|^2 \right)^{\frac{q}{2^* - 2}} \left( \int_{\mathbb{R}^N} |\eta R \tilde{W}_\rho|^{2^*} \right)^{\frac{2^* - q}{2^* - 2}},$$

So, to prove the boundedness of $\{v_\lambda\}$ in $H^1(\mathbb{R}^N)$, it suffices to show that $\{v_\lambda\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$. It is easy to see that

$$m_\lambda := J_\lambda(v_\lambda) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + \frac{b}{2} \lambda^{-\sigma} \int_{\mathbb{R}^N} |v_\lambda|^2 + \frac{b}{4} \int_{\mathbb{R}^N} |\nabla v_\lambda|^2$$

$$- \frac{1}{q} \lambda^{-\sigma} \int_{\mathbb{R}^N} |v_\lambda|^q - \frac{1}{2^*} \int_{\mathbb{R}^N} |v_\lambda|^{2^*}$$

$$= (\frac{1}{2} - \frac{1}{2^*}) a \int_{\mathbb{R}^N} |v_\lambda|^2 + (\frac{1}{2} - \frac{1}{2^*}) b \int_{\mathbb{R}^N} |\nabla v_\lambda|^2$$

$$\geq \frac{b}{2^*} \int_{\mathbb{R}^N} |\nabla v_\lambda|^2.$$
We also deduce
\[
\frac{1}{q} \frac{\int_{\mathbb{R}^N} |\nabla \tilde{W}_1|^q}{\int_{\mathbb{R}^N} |\nabla \tilde{W}_1|^{2^*q}} \leq \sup_{x \geq 0} \left[ \frac{1}{q} \frac{2^{q-2} - q}{2^* - q} \right] = \frac{q - 2}{2(2^*-q)} \left( \frac{2(2^* - q)}{q(2^*-2)} \right)^{\frac{2^* - 2}{2^* - q}}.
\]

Set \( \ell = R/\rho \) and \( \tilde{W}_1(x) = W_1(\gamma_N x) \), then
\[
\int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 = \gamma^{-(N-2)} \int_{\mathbb{R}^N} |\nabla (\eta_t W_1)|^2 = \gamma^{-(N-2)} S_N^{\frac{2}{2}} + O(\ell^{-(N-2)}) = \int_{\mathbb{R}^N} |\tilde{W}_1|^2 + O(\ell^{-(N-2)}).
\]

We also deduce
\[
\int_{\mathbb{R}^N} |\eta_t \tilde{W}_1|^2 = \gamma^{-N} \int_{\mathbb{R}^N} |\eta_t W_1|^2 = \gamma^{-N} S_N^{\frac{2}{2}} + O(\ell^{-N}) = \int_{\mathbb{R}^N} |\tilde{W}_1|^2 + O(\ell^{-N}),
\]
\[
\int_{\mathbb{R}^N} |\eta_t \tilde{W}_1|^q = \gamma^{-N} \int_{\mathbb{R}^N} |\eta_t W_1|^q = \gamma^{-N} \int_{\mathbb{R}^N} |W_1|^q + o(1) = \int_{\mathbb{R}^N} |\tilde{W}_1|^q + o(1),
\]
\[
\int_{\mathbb{R}^N} |\eta_t \tilde{W}_1|^2 = \gamma^{-N} \int_{\mathbb{R}^N} |\eta_t W_1|^2 = \left\{ \begin{array}{ll} \ln(\ell) (\gamma^{-4} + o(1)) & \text{if } N = 4, \\ \ell (\gamma^{-2} + o(1)) & \text{if } N = 3.
\end{array} \right.
\]

Therefore
\[
(I) = \sup_{t > 0} \left( \frac{a t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 + \frac{b t^{N-2}}{4} \left( \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 \right)^2 - \frac{q}{2^*} \int_{\mathbb{R}^N} |\eta_t \tilde{W}_1|^2 \right) \]
\[
= a \left( \frac{1}{4} - \frac{1}{2} \right) t^{N-2} \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 + b \left( \frac{1}{4} - \frac{1}{2} \right) t^{2(N-2)} \left( \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 \right)^2,
\]
where \( t_\ell > 0 \) is given by
\[
at_\ell^{N-2} \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 + b t_\ell^{2(N-2)} \left( \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 \right)^2 = t_\ell^{N} \int_{\mathbb{R}^N} |\eta_t \tilde{W}_1|^2.
\]

If \( N = 4 \), then \( t = t_\ell \) satisfies
\[
at^2 \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 + b t^4 \left( \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 \right)^2 = t^4 \int_{\mathbb{R}^N} |\eta_t \tilde{W}_1|^4.
\]
Hence, we have
\[
t_\ell^2 = \frac{a \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2}{\int_{\mathbb{R}^N} |\eta_t \tilde{W}_1|^4 - b \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2} \]
\[
= \frac{a \int_{\mathbb{R}^N} |\nabla \tilde{W}_1|^4}{\int_{\mathbb{R}^N} |\tilde{W}_1|^4 - b \int_{\mathbb{R}^N} |\nabla \tilde{W}_1|^2} + O(\ell^{-2}).
\]
\[
\int_{\mathbb{R}^N} |\eta_t \tilde{W}_1|^4 - b \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2 = \gamma^{-4} S_N^{\frac{2}{2}} + O(\ell^{-4}) - b(\gamma^{-2} S^{\frac{2}{2}} + O(\ell^{-2}))^2 = \gamma^{-4} S^2(1 - b S^2) + O(\ell^{-2}).
\]

Therefore, we obtain
\[
(I) = \frac{a}{4} \frac{t_\ell^2}{\int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2} = \frac{a}{4} \frac{\int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2}{\int_{\mathbb{R}^N} |\eta_t \tilde{W}_1|^4 - b \int_{\mathbb{R}^N} |\nabla (\eta_t \tilde{W}_1)|^2} \]
\[
= \frac{a}{4} \frac{\gamma^{-4} S_N^{\frac{2}{2}}}{\gamma^{-4} S_N^{\frac{2}{2}}(1 - b S^{\frac{2}{2}}) + O(\ell^{-2})} \]
\[
= \frac{a}{4} \frac{\int_{\mathbb{R}^N} |\nabla \tilde{W}_1|^2}{\int_{\mathbb{R}^N} |\tilde{W}_1|^4 - b \int_{\mathbb{R}^N} |\nabla \tilde{W}_1|^2} + O(\ell^{-2}) = m_\infty + O(\ell^{-2}).
\]
If $N = 3$, then $t = t_\ell$ satisfies
\[
t^2 \int_{\mathbb{R}^N} |\eta \bar{W}_1|^6 - bt(\int_{\mathbb{R}^N} |\nabla (\eta \bar{W}_1)|^2)^2 - a \int_{\mathbb{R}^N} |\nabla (\eta \bar{W}_1)|^2 = 0.
\]
Therefore, we get
\[
t_\ell = \frac{b(\int_{\mathbb{R}^N} |\nabla (\eta \bar{W}_1)|^2)^2 + \sqrt{b^2(\int_{\mathbb{R}^N} |\nabla (\eta \bar{W}_1)|^2)^4 + 4a(\int_{\mathbb{R}^N} |\nabla (\eta \bar{W}_1)|^2)^2 + 4a(\int_{\mathbb{R}^N} |\nabla (\eta \bar{W}_1)|^6)}}{2 \int_{\mathbb{R}^N} |\eta \bar{W}_1|^6}
\]
\[
= t_\infty + O(\ell^{-1}).
\]
Thus, we obtain
\[
(I) = \sup_{\rho \geq 0} \frac{\rho}{2} \int_{\mathbb{R}^N} |\nabla (\eta \bar{W}_1)|^2 - \frac{1}{4} t^2 \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2 - \frac{1}{2} t^2 \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2 - \frac{1}{2} t^2 \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2 - \frac{1}{2} t^2 \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2 - \frac{1}{2} t^2 \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2
\]
\[
= \sup_{\rho \geq 0} J_0(W_1) + O(\ell^{-1})
\]
\[
= m_\infty + O(\ell^{-1}).
\]
On the other hand, we have
\[
(II) = t_\lambda^N \left[ \frac{1}{2} \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2 - \frac{1}{2} \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2 \right]
\]
\[
= t_\lambda^N \left[ \frac{1}{2} \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2 - \frac{1}{2} \rho^2 \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2 \right].
\]
Let $h(\rho) := \frac{1}{q} \int_{\mathbb{R}^N} |\eta \bar{W}_1|^q - \frac{1}{2} \rho^2 \int_{\mathbb{R}^N} |\eta \bar{W}_1|^2$. Then $h(\rho)$ take its maximum value $\varphi(\rho_\ell)$ at the unique point $\rho_\ell > 0$, and
\[
\sup_{\rho > 0} h(\rho) = h(\rho_\ell)
\]
\[
= \frac{(q-2)(N-2)}{4q} \left[ \frac{2N-q(N-2)}{2q} \right] 2^{N-q(N-2)} 2^{(q-2)(N-2)} \left( \frac{\|\eta \bar{W}_1\|_2^{(q-2)}}{\|\eta \bar{W}_1\|_2^{(q-2)}} \right)^{\frac{2(q-2)}{q-2}}.
\]
Then we obtain
\[
(II) = t_\lambda^N h(\rho_\ell) \geq 2C_q \|\eta \bar{W}_1\|_2^{2(q^2-4)} \geq \left\{ \begin{array}{ll}
C_q (\ln \ell)^{\frac{4-q}{2}} & \text{if } N = 4, \\
C_q \ell^{-\frac{4-q}{2}} & \text{if } N = 3.
\end{array} \right.
\]
For the rest of the proof we consider separately the cases $N = 4$ and $N = 3$.

**CASE** $N = 4$. In this case, we have
\[
m_\lambda \leq m_\infty + O(\ell^{-2}) - C_q \lambda^{-\frac{4-q}{4-2}} (\ln \ell)^{\frac{4-q}{2}}.
\]
Take $\ell = \lambda^M$. Then
\[
m_\lambda \leq m_\infty + O(\lambda^{-2M}) - C_q M^{-\frac{4-q}{4-2}} (\lambda \ln \lambda)^{-\frac{4-q}{4-2}}.
\]
If $2M > \frac{4-q}{4-2}$, then for large $\lambda > 0$, we have
\[
m_\lambda \leq m_\infty - \frac{1}{2} C_q M^{-\frac{4-q}{4-2}} (\lambda \ln \lambda)^{-\frac{4-q}{4-2}}.
\]
Thus, if $N = 4$, the result of Lemma 4.3 is proved by choosing
\[
\varpi = \frac{1}{2} C_q M^{-\frac{4-q}{4-2}}.
\]
Corollary 4.4. Let \( \tau > 0 \) be such that \( \tau = (1 + r) \frac{6 - q}{q - 2} \), that is, \( \tau = \frac{6 - q}{2(q - 4)} \).

Take \( \ell = \delta^{-1} \lambda \tau \). Then
\[
m_{\lambda} \leq m_{\infty} + O(\ell^{-1}) - C_{q} \lambda^{-\frac{6-q}{2}} \ell^{-\frac{6-q}{2^{*}}}.
\]

Let \( \tau > 0 \) be such that \( \tau = (1 + r) \frac{6 - q}{q - 2} \), that is, \( \tau = \frac{6 - q}{2(q - 4)} \).

Since \( q > 4 \), we have \( \frac{6-q}{q-2} < 1 \), we can choose a small \( \delta > 0 \) such that
\[
m_{\lambda} \leq m_{\infty} - \frac{1}{2} C_{q} \delta^{\frac{6-q}{2}} \lambda^{-\frac{6-q}{2^{*}}}.
\]

and take
\[
(\lambda \ln \lambda)^{-\frac{6-q}{2}} \quad \text{if} \quad N = 4,
\]
\[
\lambda^{-\frac{6-q}{2(q-4)}} \quad \text{if} \quad N = 3 \quad \text{and} \quad q \in (4, 6).
\]

which finished the proof in the case \( N = 3 \).

\[\square\]

Corollary 4.4. Let \( \delta_{\lambda} := m_{\infty} - m_{\lambda} \), then
\[
\lambda^{-\frac{6-q}{q-2}} \gtrsim \delta_{\lambda} \gtrsim \begin{cases} 
(\lambda \ln \lambda)^{-\frac{6-q}{2}} & \text{if} \quad N = 4, \\
\lambda^{-\frac{6-q}{2(q-4)}} & \text{if} \quad N = 3 \quad \text{and} \quad q \in (4, 6).
\end{cases}
\]

Proof. Arguing as in the proof of Lemma 4.3, it is easy to show that
\[
\delta_{\lambda} \lesssim \lambda^{-\frac{6-q}{q-2}},
\]

which together with Lemma 4.3 yields the desired conclusion.

\[\square\]

4.2. Proof of Theorem 2.1 for large \( \lambda \). We recall the P.-L. Lions’ concentration–compactness lemma, which is at the core of our proof of Theorem 2.1 as \( \lambda \to \infty \).

Lemma 4.5 (P.-L. Lions [16]). Let \( r > 0 \) and \( 2 \leq q \leq 2^{*} \). If \( (u_{n}) \) is bounded in \( H^{1}(\mathbb{R}^{N}) \) and if
\[
\sup_{y \in \mathbb{R}^{N}} \int_{B_{r}(y)} |u_{n}|^{q} \to 0 \quad \text{as} \quad n \to \infty,
\]
then \( u_{n} \to 0 \) in \( L^{p}(\mathbb{R}^{N}) \) for \( 2 < p < 2^{*} \). Moreover, if \( q = 2^{*} \), then \( u_{n} \to 0 \) in \( L^{2^{*}}(\mathbb{R}^{N}) \).

Using Lemma 4.5, we establish the following.

Lemma 4.6. If \( N = 4 \) or \( N = 3 \) then there exists \( \xi_{\lambda} \in (0, +\infty) \) such that \( \xi_{\lambda} \to 0 \) and
\[
v_{\lambda} - \xi_{\lambda}^{\frac{N+2}{2}} v_{1}(\xi_{\lambda}^{-1}. \to 0
\]
in \( D^{1,2}(\mathbb{R}^{N}) \) and \( L^{2^{*}}(\mathbb{R}^{N}) \) as \( \lambda \to \infty \), where \( v_{1} \) is given by (4.7).

Proof. Note that \( v_{\lambda} \) is a positive radially symmetric function, and by Lemma 4.2, \( \{v_{\lambda}\} \) is bounded in \( H^{1}(\mathbb{R}^{N}) \). Then there exists constant \( A \in \mathbb{R} \) and \( 0 \leq v_{\infty} \in H^{1}(\mathbb{R}^{N}) \) such that as \( \lambda \to \infty \), up to a subsequence, we have
\[
\int_{\mathbb{R}^{N}} |\nabla v_{\lambda}|^{2} \to A^{2},
\]
\[
v_{\lambda} \to v_{\infty} \quad \text{weakly in} \quad H^{1}(\mathbb{R}^{N}), \quad v_{\lambda} \to v_{\infty} \quad \text{in} \quad L^{p}(\mathbb{R}^{N}) \quad \text{for any} \quad p \in (2, 2^{*}),
\]

(4.19)
Therefore, we obtain
\[ v_\lambda(x) \to v_\infty(x) \quad \text{a.e. on } \mathbb{R}^N, \quad v_\lambda \to v_\infty \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N). \] (4.20)
Moreover, \( v_\infty \) verifies the equation
\[ -(a + bA^2)\Delta v = v^{2^* - 1}. \]

Observe that
\[ J_\infty(v_\lambda) = J_\lambda(v_\lambda) + \frac{\lambda^{-\sigma}}{q} \int_{\mathbb{R}^N} |v_\lambda|^q - \frac{\lambda^{-\sigma}}{2} \int_{\mathbb{R}^N} |v_\lambda|^2 = m_\lambda + o(1) = m_\infty + o(1), \]
and
\[ J'_\infty(v_\lambda)v = J'_\lambda(v_\lambda)v + \lambda^{-\sigma} \int_{\mathbb{R}^N} |v_\lambda|^{q^*} v_\lambda - \lambda^{-\sigma} \int_{\mathbb{R}^N} v_\lambda v = o(1). \]

Therefore, \( \{v_\lambda\} \) is a \((PS)\) sequence for \( J_\infty \) at level \( m_\infty \).

By Lemma 4.5 and an argument similar to that in [28], it is standard to show that there exists \( \zeta^{(j)}_\lambda \in (0, +\infty), \ v^{(j)} \in D^{1,2}(\mathbb{R}^N) \) with \( j = 1, 2, \ldots, k \) where \( k \) is a nonnegative integer, such that
\[ v_\lambda = v_\infty + \sum_{j=1}^{k} \left( \zeta^{(j)}_\lambda \right)^{-\frac{N-2}{2}} v^{(j)} \left( \zeta^{(j)}_\lambda \right)^{-1} x + \tilde{v}_\lambda, \] (4.21)
where \( \tilde{v}_\lambda \to 0 \) in \( D^{1,2}(\mathbb{R}^N), \ \zeta^{(j)}_\lambda \to 0 \) as \( \lambda \to \infty \), and \( v^{(j)} \) are nontrivial solutions of the equation
\[ -(a + bA^2)\Delta v = v^{2^* - 1}. \]
Moreover, we have
\[ A^2 = \|v_\infty\|^2_{D^{1,2}(\mathbb{R}^N)} + \sum_{j=1}^{k} \|v^{(j)}\|^2_{D^{1,2}(\mathbb{R}^N)}, \] (4.22)
and
\[ m_\infty = J_\infty^A(v_\infty) + \sum_{j=1}^{k} J_\infty^A(v^{(j)}), \] (4.23)
where
\[ J_\infty^A(v) = \left( \frac{a}{2} + \frac{bA^2}{4} \right) \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*}. \]
For any solution \( v \) of the equation \(- (a + bA^2)\Delta v = v^{2^* - 1}, \) we have
\[ (a + bA^2) \int_{\mathbb{R}^N} |\nabla v|^2 = \int_{\mathbb{R}^N} |v|^{2^*}. \]
Therefore, we obtain
\[ J_\infty^A(v) = \left( \frac{a}{2} + \frac{bA^2}{4} \right) \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} \]
\[ = \left( \frac{a}{2} + \frac{bA^2}{4} \right) \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{1}{2^*} (a + bA^2) \int_{\mathbb{R}^N} |\nabla v|^2 \]
\[ = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + (\frac{1}{4} - \frac{bA^2}{2^*}) \int_{\mathbb{R}^N} |\nabla v|^2 \]
\[ \geq \frac{a}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + (\frac{1}{4} - \frac{bA^2}{2^*}) \int_{\mathbb{R}^N} |\nabla v|^2 \]
\[ = J_\infty(v). \]
Since \( J_\infty(v_\infty) \geq 0 \) and \( J_\infty(v^{(j)}) \geq m_\infty \) for \( j = 1, 2, \ldots, k \), we conclude that \( J_\infty^A(v_\infty) \geq 0 \) and \( J_\infty^A(v^{(j)}) \geq m_\infty \) for all \( j = 1, 2, \ldots, k \).
If $N = 4$ or $3$ then by (4.12) and Fatou’s lemma we have
\[ |v_\infty|^2 \leq \liminf_{\lambda \to \infty} |v_\lambda|^2 = \frac{2(2^* - q)}{q(2^* - 2)} \int_{\mathbb{R}^N} |v_\infty|^q < \infty. \]
Note that $v_\infty \notin L^2(\mathbb{R}^N)$ whenever $v_\infty \neq 0$, therefore, $v_\infty = 0$ and hence $k = 1$. Thus, we obtain $J_\infty(v^{(1)}) = m_\infty$ and hence $v^{(1)} = v_\rho$ for some $\rho \in (0, +\infty)$. Therefore, we conclude that
\[ v_\lambda - \xi_{\lambda}^{-\frac{N-2}{2}} v_1(\xi_{\lambda}^{-1} \cdot) \to 0 \]
in $D^{1,2}(\mathbb{R}^N)$ as $\lambda \to 0$, where $v_1$ is given by (4.7) and $\xi_{\lambda} := \rho_{\xi_1}^{(1)} \in (0, +\infty)$ satisfying $\xi_{\lambda} \to 0$ as $\lambda \to 0$. Moreover, $A^2 = \lim_{\lambda \to 0} \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 = \int_{\mathbb{R}^N} |\nabla v^{(1)}|^2$, we conclude that $v^{(1)}$ is a solution of the equation $-(a + b \int_{\mathbb{R}^N} |\nabla v|^2) \Delta v = v^{2^*-1}$. \[\square\]

We perform an additional rescaling
\[ w(x) = \xi_{\lambda}^{-\frac{N-2}{2}} v(\xi_{\lambda} x), \quad (4.24) \]
where $\xi_{\lambda} \in (0, +\infty)$ is given in Lemma 4.6. This rescaling transforms $(Q_{\lambda})$ into an equivalent equation
\[-(a + b \int_{\mathbb{R}^N} |\nabla w|^2) \Delta w + \lambda^{-\sigma} \xi_{\lambda}^2 w = \lambda^{-\sigma} \xi_{\lambda}^{\frac{2N(2N-4)}{4}} w^{2^*-1} + w^{2^*-1}, \quad \text{in} \ \mathbb{R}^N, \quad (\tilde{R}_\lambda) \]
The corresponding energy functional is given by
\[ \tilde{J}_\lambda(w) := \frac{1}{2} \int_{\mathbb{R}^N} a|\nabla w|^2 + \lambda^{-\sigma} \xi_{\lambda}^2 |w|^2 + \frac{1}{4} (\int_{\mathbb{R}^N} |\nabla w|^2)^2 \underbrace{- \frac{1}{4} \lambda^{-\sigma} \xi_{\lambda}^{\frac{2N(2N-4)}{4}}}_{(4.25)} \int_{\mathbb{R}^N} |w|^q - \frac{1}{2^*} \int_{\mathbb{R}^N} |w|^{2^*}. \]
It is straightforward to verify the following.

**Lemma 4.7.** Let $\lambda > 0$, $v$ is the rescaling of $u \in H^1(\mathbb{R}^N)$ and $w$ is the rescaling of $v$ given in (4.1) and (4.24), respectively. Then:
(a) $\|\nabla w\|^2 = \|\nabla v\|^2$, $\|\nabla u\|^2$, $\|w\|_2^2 = \|w\|_2^2$, $\|v\|_2^2 = \|u\|_2^2$,
(b) $\xi_{\lambda}^2 \|w\|^2 = \|v\|^2 = \lambda^{1+\sigma} \|u\|^2$, $\xi_{\lambda}^{\frac{2N(2N-4)}{4}} \|w\|^q = \|v\|^q = \lambda^q \|u\|^q$,
(c) $\tilde{J}_\lambda(w) = J_\lambda(v) = I_\lambda(u)$.

Let $w_\lambda(x) = \xi_{\lambda}^{-\frac{N-2}{2}} v_\lambda(\xi_{\lambda} x)$ where the $v_\lambda$ is a ground-state of $(R_\lambda)$. Then it follows from Lemma 4.7(c) that $w_\lambda$ is a ground state of $(\tilde{R}_\lambda)$. By Lemma 4.6 we conclude that
\[ \|\nabla(w_\lambda - v_1)\|_2 \to 0, \quad \|w_\lambda - v_1\|_2 \to 0 \quad \text{as} \ \lambda \to \infty. \quad (4.26) \]
Note that the corresponding Nehari and Pohožaev identities read as follows
\[ a \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \lambda^{-\sigma} \xi_{\lambda}^2 \int_{\mathbb{R}^N} |w_\lambda|^2 + b \left( \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 \right)^2 \int_{\mathbb{R}^N} |w_\lambda|^q, \]
and
\[ \frac{\alpha^2}{2} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \frac{1}{2} \lambda^{-\sigma} \xi_{\lambda}^2 \int_{\mathbb{R}^N} |w_\lambda|^2 + \frac{1}{2^*} \left( \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 \right)^2 \int_{\mathbb{R}^N} |w_\lambda|^q. \]
We conclude that
\[
\left(\frac{1}{2} - \frac{1}{2^*}\right) \lambda^{-\sigma} \xi_\lambda^2 \int_{\mathbb{R}^N} |w_\lambda|^2 = \left(\frac{1}{q} - \frac{1}{2^*}\right) \lambda^{-\sigma} \xi_\lambda^{2N-q(N-2)} \int_{\mathbb{R}^N} |w_\lambda|^q.
\]
Thus, we obtain
\[
\xi_\lambda^{\frac{(N-2)(q-2)}{2}} \int_{\mathbb{R}^N} |w_\lambda|^2 = \frac{2(2^* - q)}{q(2^* - 2)} \int_{\mathbb{R}^N} |w_\lambda|^q.
\] (4.27)
To control the norm \(\|w_\lambda\|_2\) from below, we give the following estimate:

**Lemma 4.8.** There exists a constant \(C > 0\) such that
\[
w_\lambda(x) \geq C \varpi_\lambda^{\frac{N-2}{2}} |x|^{-(N-2)} \exp(-\varpi_\lambda^{-\frac{1}{2}} \lambda^{-\frac{2}{3}} \xi_\lambda^2 |x|), \quad |x| \geq 1,
\] (4.28)
where \(\varpi_\lambda = a + b \int_{\mathbb{R}^N} |\nabla w_\lambda|^2\).

**Proof.** It is easy to see that \(\tilde{w}_\lambda(x) = w_\lambda(\sqrt{\varpi_\lambda} x)\) satisfies the following
\[-\Delta \tilde{w}_\lambda + \lambda^{-\sigma} \xi_\lambda^2 \tilde{w}_\lambda = \lambda^{-\sigma} \xi_\lambda^{2N-q(N-2)} \tilde{w}_\lambda^{q-1} + \tilde{w}_\lambda^{2q-1} > 0.
\]
Arguing as in the proof in [21, Lemma 4.8], we show that
\[
\tilde{w}_\lambda(x) \geq C |x|^{-(N-2)} \exp(-\lambda^{-\frac{1}{2}} \xi_\lambda^2 |x|), \quad |x| \geq 1.
\]
Therefore, we obtain
\[
w_\lambda(x) = \tilde{w}_\lambda\left(\frac{x}{\sqrt{\varpi_\lambda}}\right) \geq C \varpi_\lambda^{\frac{N-2}{2}} |x|^{-(N-2)} \exp(-\varpi_\lambda^{-\frac{1}{2}} \lambda^{-\frac{2}{3}} \xi_\lambda^2 |x|), \quad |x| \geq 1.
\]
The proof is complete. \(\square\)

Since \(0 < C_1 < \varpi_\lambda < C_2 < +\infty\) for some constants \(C_1, C_2\) which are independent of \(\lambda > 0\), as consequences of the above lemma, we have the following two lemmas.

**Lemma 4.9.** If \(N = 3\), then \(\|w_\lambda\|_2 \gtrsim \lambda^{\frac{2}{3}} \xi_\lambda^{-1}\).

**Lemma 4.10.** If \(N = 4\), then \(\|w_\lambda\|_2 \gtrsim -\ln(\lambda^{-\sigma} \xi_\lambda^2)\).

To prove our main result, the key point is to show the boundedness of \(\|w_\lambda\|_q\).

**Lemma 4.11.** If \(N = 3, 4\) and \(\frac{N}{N-2} < s < 2^*\), then \(\|w_\lambda\|_s^s \sim 1\) as \(\lambda \to \infty\). Furthermore, \(w_\lambda \to v_1\) in \(L^s(\mathbb{R}^N)\) as \(\lambda \to \infty\).

**Proof.** By (4.26), we have \(w_\lambda \to v_1\) in \(L^q(\mathbb{R}^N)\). Then, as in [21, Lemma 4.6], using the embeddings \(L^{2^*}(B_1) \hookrightarrow L^s(B_1)\) we prove that \(\liminf_{\lambda \to \infty} \|w_\lambda\|_s^s > 0\).

On the other hand, arguing as in [2, Proposition 3.1], we show that there exists a constant \(C > 0\) such that for all large \(\lambda > 0\),
\[
w_\lambda(x) \leq \frac{C}{(1 + |x|)^{N-2}}, \quad \forall x \in \mathbb{R}^N,
\] (4.29)
which together with the fact that \(s > \frac{N}{N-2}\) implies that \(w_\lambda\) is bounded in \(L^s(\mathbb{R}^N)\) uniformly for large \(\lambda > 0\), and by the dominated convergence theorem \(w_\lambda \to v_1\) in \(L^s(\mathbb{R}^N)\) as \(\lambda \to \infty\). \(\square\)
Hence, by Corollary 4.4, we obtain
\[ m_\infty \leq \sup_{t \geq 0} J_\infty ((w_\lambda)_t) = J_\infty ((w_\lambda)_t) \]
\[ \leq \sup_{t \geq 0} \tilde{J}_\lambda ((w_\lambda)_t) + \lambda^{-\sigma} t_\lambda^{N} \left\{ \frac{1}{q} \xi_\lambda^{\frac{2N-q(N-2)}{2}} \int_{\mathbb{R}^N} |w_\lambda|^q - \frac{1}{2} \xi_\lambda^2 \int_{\mathbb{R}^N} |w_\lambda|^2 \right\} \]
\[ \leq m_\lambda + \lambda^{-\sigma} t_\lambda^{N} \frac{1}{q} \xi_\lambda^{\frac{2N-q(N-2)}{2}} \int_{\mathbb{R}^N} |w_\lambda|^q, \]
where \( t_\lambda > 0 \) is such that \((w_\lambda)_t \in \mathcal{N}_\infty\), and more precisely,
\[ t_\lambda^2 = \begin{cases} \frac{b(f_{R,N} |\nabla w_\lambda|^2)^2 + \sqrt{b^2(f_{R,N} |\nabla w_\lambda|^2)^4 + 4a f_{R,N} |w_\lambda|^2 f_{R,N} |\nabla w_\lambda|^2}}{2 f_{R,N} |w_\lambda|^2}, & \text{if } N = 3, \\ \frac{a f_{R,N} |\nabla v_1|^2}{f_{R,N} |v_1|^2 - b f_{R,N} |\nabla v_1|^2} & \text{if } N = 4, \end{cases} \]
and hence by (4.8) and (4.26), as \( \lambda \to \infty \), we have
\[ t_\lambda^2 \to \begin{cases} \frac{b(f_{R,N} |\nabla v_1|^2)^2 + \sqrt{b^2(f_{R,N} |\nabla v_1|^2)^4 + 4a f_{R,N} |v_1|^2 f_{R,N} |\nabla v_1|^2}}{2 f_{R,N} |v_1|^2}, & \text{if } N = 3, \\ \frac{a f_{R,N} |\nabla v_1|^2}{f_{R,N} |v_1|^2 - b f_{R,N} |\nabla v_1|^2} & \text{if } N = 4. \end{cases} \]
The above inequality implies that
\[ \xi_\lambda^{\frac{2N-q(N-2)}{2}} t_\lambda^N \int_{\mathbb{R}^N} |w_\lambda|^q \geq \lambda^\sigma (m_\infty - m_\lambda). \]
Hence, by Corollary 4.4, we obtain
\[ \xi_\lambda^{\frac{2N-q(N-2)}{2}} \int_{\mathbb{R}^N} |w_\lambda|^q \gtrsim \begin{cases} (\ln \lambda)^{-\frac{q-2}{q}} & \text{if } N = 4, \\ \lambda^{-\frac{(6-q)}{2(q-2)(q-4)}} & \text{if } N = 3. \end{cases} \]
(4.31)
Therefore, by Lemma 4.11, we have
\[ \xi_\lambda \gtrsim \begin{cases} (\ln \lambda)^{-\frac{q-2}{q}} & \text{if } N = 4, \\ \lambda^{-\frac{6-q}{2(q-2)(q-4)}} & \text{if } N = 3. \end{cases} \]
(4.32)
On the other hand, if \( N = 3 \), then by (4.27) and Lemma 4.9 and Lemma 4.11, we have
\[ \xi_\lambda^q \lesssim \frac{1}{\|w_\lambda\|^q_2} \lesssim \lambda^{-\frac{q}{2}} \xi_\lambda. \]
Then, observing that \( \sigma = \frac{2^{q+2}}{q-2} = \frac{4}{4-q} \), for \( q \in (4,6) \) we obtain
\[ \xi_\lambda \lesssim \lambda^{-\frac{q-4}{2}} = \lambda^{-\frac{6-q}{(q-2)(q-4)}}. \]
(4.33)
If \( N = 4 \), then by (4.27) and Lemma 4.10 and Lemma 4.11, we have
\[ \xi_\lambda^{q-2} \lesssim \frac{1}{\|w_\lambda\|^q_2} \lesssim \frac{1}{\ln(\lambda^{-\sigma} \xi_\lambda^q)} \lesssim (\ln \lambda)^{-1}, \]
here we have used the fact that \( \xi_\lambda \to 0 \) as \( \lambda \to +\infty \) and for large \( \lambda > 0 \),
\[ -\ln(\lambda^{-\sigma} \xi_\lambda^q) = \sigma \ln \lambda - 2 \ln \xi_\lambda \geq \sigma \ln \lambda. \]
Thus, we obtain
\[ \xi_\lambda \lesssim (\ln \lambda)^{-\frac{1}{q-2}}. \]
Thus, it follows from (4.30), (4.33), (4.34) and Lemma 4.11 that
\[ m_\infty - m_\lambda \lesssim \lambda^{-\sigma} \xi_\lambda^{2N-2(N-2)} \\lesssim \begin{cases} (\lambda \ln \lambda)^{-\frac{4-\sigma}{2}} & \text{if } N = 4, \\ \lambda^{-\frac{6-q}{2(q-4)}} & \text{if } N = 3, \end{cases} \]
and hence Corollary 4.4 yields that
\[ m_\infty - m_\lambda \sim \lambda^{-\sigma} \xi_\lambda^{2N-2(N-2)} \sim \begin{cases} (\lambda \ln \lambda)^{-\frac{4-\sigma}{2}} & \text{if } N = 4, \\ \lambda^{-\frac{6-q}{2(q-4)}} & \text{if } N = 3. \end{cases} \tag{4.35} \]
Since
\[ m_\lambda = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \frac{1}{2} \lambda^{-\sigma} \xi_\lambda^{2} \int_{\mathbb{R}^N} |w_\lambda|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 \right)^2 - \frac{1}{q} \lambda^{-\sigma} \xi_\lambda^{\frac{2N-2q}{2}} \int_{\mathbb{R}^N} |w_\lambda|^q - \frac{1}{2q} \int_{\mathbb{R}^N} |w_\lambda|^{2q}. \]
we get
\[ m_\lambda = a \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + b \left( \frac{1}{4} - \frac{1}{2^*} \right) \left( \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 \right)^2. \]
Similarly, we have
\[ m_\infty = a \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla w_\infty|^2 + b \left( \frac{1}{4} - \frac{1}{2^*} \right) \left( \int_{\mathbb{R}^N} |\nabla w_\infty|^2 \right)^2. \]
Therefore, we obtain
\[ m_\infty - m_\lambda = \left[ \frac{a}{2} + b \left( \frac{2^* - 4}{2^{*4}} \left( \int_{\mathbb{R}^N} |\nabla w_\infty|^2 + \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 \right) \right) \left( \int_{\mathbb{R}^N} |\nabla w_\infty|^2 - \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 \right) \right]. \]
which together with (4.35) implies that
\[ ||\nabla w_\infty||^2_2 - ||\nabla w_\lambda||^2_2 \sim m_\infty - m_\lambda \sim \begin{cases} (\lambda \ln \lambda)^{-\frac{4-\sigma}{2}} & \text{if } N = 4, \\ \lambda^{-\frac{6-q}{2(q-4)}} & \text{if } N = 3. \end{cases} \tag{4.36} \]
Since
\[ \frac{1}{2^*} \left( \int_{\mathbb{R}^N} |w_\infty|^2 - \int_{\mathbb{R}^N} |w_\lambda|^2 \right) = \frac{1}{2} \left( m_\infty - m_\lambda \right) \]
\[ + \frac{a}{2} + \frac{b}{4} \left( \frac{2^* - 4}{2^*4} \left( \int_{\mathbb{R}^N} |\nabla w_\infty|^2 + \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 \right) \right) \left( \int_{\mathbb{R}^N} |\nabla w_\infty|^2 - \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 \right) \]
\[ + \frac{(q-2)(N-2)}{4q} \lambda^{-\sigma} \xi_\lambda^{2N-2q} \int_{\mathbb{R}^N} |w_\lambda|^q. \]
It follows from (4.35) and (4.36) that
\[ ||w_\infty||^2_* - ||w_\lambda||^2_* = \begin{cases} O((\lambda \ln \lambda)^{-\frac{4-\sigma}{2}}) & \text{if } N = 4, \\ O(\lambda^{-\frac{6-q}{2(q-4)}}) & \text{if } N = 3. \end{cases} \]
Finally, by (4.27), (4.35) and Lemma 4.11, we obtain
\[ ||w_\lambda||^2_2 \sim \begin{cases} \frac{\ln \lambda}{\lambda^{-\frac{6-q}{2(q-4)}}} & \text{if } N = 4, \\ \frac{\ln \lambda}{\lambda^{-\frac{6-q}{2(q-4)}}} & \text{if } N = 3. \end{cases} \]
Statements on \( u_\lambda \) follow from the corresponding results on \( v_\lambda \) and \( w_\lambda \). This completes the proof of Theorem 2.1 for large \( \lambda \). \( \square \)
5. Proof of Theorem 2.2

In this section, we always assume $N = 3$. We divide this section into three subsections, in which we consider the cases (i) $p = q$, (ii) $q < p$, $\lambda > 0$ large, and (iii) $q < p$, $\lambda > 0$ small, respectively. If $2 < q \leq p < 2^*$, then the arguments used in Section 3 and Section 4 do not work any more, so we try to transform the nonlocal equation $(P_\lambda)$ into the local equation (1.3) by using a suitable rescaling which is dependent of the ground state solution. Unfortunately, if $b \neq 0$, such a rescaling does not necessarily transforms a ground state solution into a ground state solution of a new equation, which prevents us from deriving a precise energy estimate of the ground state.

5.1. The case $p = q$. Let $W \in H^1(\mathbb{R}^3)$ is the unique positive solution of the equation

$$-\Delta W + W = W^{p-1},$$

and $S_p = \|W\|^{p-2}$. Let $u_\lambda$ be a ground state solution of $(P_\lambda)$, then for any $\lambda > 0$, there holds

$$u_\lambda(x) = (\lambda/2)^{\frac{1}{p-2}} W(\lambda^{\frac{1}{2}} \omega_\lambda^\frac{1}{2} x),$$

where $\omega_\lambda = a$ if $b = 0$, and if $b \neq 0$, then

$$\sqrt{\omega_\lambda} = \frac{1}{2} \left\{ \frac{3b(p-2)}{2p} \lambda^{\frac{6-p}{2(p-2)}} (S_p/2)^{\frac{p}{p-2}} + \sqrt{\frac{9b^2(p-2)^2}{4p^2} \lambda^{\frac{6-p}{2}} (S_p/2)^{\frac{4p}{p-2}} + 4a} \right\}$$

$$\lambda^{\frac{6-p}{2}} \left( \frac{3b(p-2)}{2p} \right) (S_p/2)^{\frac{p}{p-2}} + \Theta(\lambda^{\frac{6-p}{2}}) \right\}, \quad \text{as } \lambda \to \infty,$$

$$\lambda^{\frac{6-p}{2}} \left( \frac{3b(p-2)}{p} a^\frac{1}{2} (S_p/2)^{\frac{p}{p-2}} + \Theta(\lambda^{\frac{6-p}{2}}) \right), \quad \text{as } \lambda \to 0,$$

Clearly, we have

$$u_\lambda(0) = (\lambda/2)^{\frac{1}{p-2}} W(0),$$

and a direct computation shows that for $b \neq 0$, we have

$$\|\nabla u_\lambda\|_2^2 = \lambda^{\frac{6-p}{2}} \left( \frac{3b(p-2)}{2p} \right) (S_p/2)^{\frac{p}{p-2}}$$

$$\lambda^{\frac{6-p}{2}} \left( \frac{3b(p-2)}{p} a^\frac{1}{2} (S_p/2)^{\frac{p}{p-2}} + \Theta(\lambda^{\frac{6-p}{2}}) \right), \quad \text{as } \lambda \to \infty,$$

$$\lambda^{\frac{6-p}{2}} \left( \frac{3b(p-2)}{p} a^\frac{1}{2} (S_p/2)^{\frac{p}{p-2}} + \Theta(\lambda^{\frac{6-p}{2}}) \right), \quad \text{as } \lambda \to 0,$$

For $b = 0$, we have

$$\|\nabla u_\lambda\|_2^2 = \lambda^{\frac{6-p}{2}} \frac{3(p-2)}{p} \sqrt{\omega_\lambda} (S_p/2)^{\frac{p}{p-2}} = \lambda^{\frac{6-p}{2}} \frac{3(p-2)}{p} a^\frac{1}{2} (S_p/2)^{\frac{p}{p-2}},$$

$$\|u_\lambda\|_2^2 = \lambda^{\frac{6-p}{2}} \frac{6-p}{p} \left( \sqrt{\omega_\lambda} \right)^3 (S_p/2)^{\frac{p}{p-2}} = \lambda^{\frac{6-p}{2}} \frac{6-p}{p} a^\frac{3}{2} (S_p/2)^{\frac{p}{p-2}},$$
\[
\|u_\lambda\|_p^p = \lambda^{\frac{6-p}{p(p-2)}} (\sqrt{a})^3 (S_p/2)^{\frac{p}{p-2}} = \lambda^{\frac{6-p}{p(p-2)}} a^\frac{3}{2} (S_p/2)^{\frac{p}{p-2}}.
\]

5.2. The case \( q < p \) and \( \lambda > 0 \) is sufficiently large. Let \( u_\lambda \) be a ground state solution of \((P_\lambda)\), and

\[
w_\lambda(x) = \lambda^{-\frac{1}{p-2}} u_\lambda(\sqrt{\lambda} x), \quad \lambda_\lambda = a + b \int_{\mathbb{R}^N} |\nabla u_\lambda|^2.
\]

Then \( w = w_\lambda \) satisfies

\[- \Delta w + w = \lambda^{\frac{6-p}{p-2}} w^{q-1} + w^{b-1}, \quad \text{in } \mathbb{R}^N.\]

The corresponding functional is given by

\[
J_\lambda(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + |w|^2 - \frac{1}{q} \lambda^{\frac{6-p}{p-2}} \int_{\mathbb{R}^N} |w|^q + \frac{1}{p} \int_{\mathbb{R}^N} |w|^p.
\]

Observe that

\[
\omega_\lambda = \begin{cases} 
a + b \lambda^{\frac{6-p}{p(p-2)}} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2, & \text{if } b \neq 0, \\
a, & \text{if } b = 0,
\end{cases}
\]

it follows that

\[
I_\lambda(u) = \frac{\|u\|^2}{2} + \frac{\|u\|^2}{2} + \frac{b}{4} \lambda^{-\frac{6-p}{p-2}} \omega_\lambda^{-\frac{1}{2}} \|\nabla u\|^2 - \frac{1}{q} \|u\|^q - \frac{1}{p} \|u\|^p
\]

\[
= \begin{cases} 
\lambda^{\frac{6-p}{p(p-2)}} (\sqrt{a})^3 [J_\lambda(w) + K_\lambda(w)], & \text{if } b \neq 0, \\
\lambda^{\frac{6-p}{p(p-2)}} a^\frac{3}{2} J_\lambda(w), & \text{if } b = 0.
\end{cases}
\]

where

\[
K_\lambda(w) = a \omega_\lambda^{-1} \|\nabla w_\lambda\|^2 + b \lambda^{-\frac{6-p}{p-2}} \omega_\lambda^{-\frac{3}{2}} \|\nabla w_\lambda\|^2 - \frac{1}{2} \|\nabla w_\lambda\|^2.
\]

Clearly, for any \( \varphi \in H^1(\mathbb{R}^N) \), we have

\[
K_\lambda'(w_\lambda) \varphi = (a \omega_\lambda^{-1} + b \lambda^{-\frac{6-p}{p-2}} \omega_\lambda^{-\frac{1}{2}} \|\nabla w_\lambda\|^2 - 1) \int_{\mathbb{R}^N} \nabla w_\lambda \nabla \varphi = 0,
\]

therefore, \( w = w_\lambda \) is a critical point of \( K_\lambda(w) \). Therefore, if \( u_\lambda \) is a critical point of \( I_\lambda \), then \( w_\lambda \) is a critical point of \( J_\lambda \).

On the other hand, assume \( w_\lambda \) is a critical point of \( J_\lambda \) and \( \omega_\lambda \) is given by (5.4), then

\[
u_\lambda(x) = \lambda^{\frac{1}{p-2}} w_\lambda (\lambda^{\frac{1}{2}} (\sqrt{\omega_\lambda})^{-1} x)
\]

is a critical point of \( I_\lambda \). These observation reduces the problem of finding critical point of \( I_\lambda \) to the corresponding problem of finding critical point of \( J_\lambda \).

For general \( \lambda > 0 \), the ground state solutions of (5.3) should not be unique. But for large \( \lambda > 0 \), this is not the case. Arguing as in [13, Theorem 5.1] (see also [1]), by the implicit function theorem, we can show that for large \( \lambda > 0 \), (5.3) admits a unique positive ground state solution, therefore, \((P_\lambda)\) has only one ground state solution for large \( \lambda > 0 \). This also yields that \( w_\lambda \) is a ground state solution of (5.3).

Let \( m_\lambda \) be the least energy of nontrivial solutions of (5.3). Put

\[
A_\lambda = \int_{\mathbb{R}^N} |\nabla w_\lambda|^2, \quad B_\lambda = \int_{\mathbb{R}^N} |w_\lambda|^2, \quad C_\lambda = \int_{\mathbb{R}^N} |w_\lambda|^q, \quad D_\lambda = \int_{\mathbb{R}^N} |w_\lambda|^p,
\]

Then the Nehari and Pohožaev identities hold true:

\[
A_\lambda + B_\lambda = \lambda^{\frac{6-p}{p-2}} C_\lambda + D_\lambda,
\]
\[
\frac{1}{2^*} A_\lambda + \frac{1}{2} B_\lambda = \frac{1}{q} \lambda^{-\frac{p-q}{p-2}} C_\lambda + \frac{1}{p} D_\lambda, \tag{5.9}
\]

As a consequence, it follows that

\[
A_\lambda = \frac{N(q-2)}{2q} \lambda^{-\frac{p-q}{p-2}} C_\lambda + \frac{N(p-2)}{2p} D_\lambda
\]

\[
B_\lambda = \frac{N(2^*-q)}{2^*q} \lambda^{-\frac{p-q}{p-2}} C_\lambda + \frac{N(2^*-p)}{2^*p} D_\lambda
\]

Hence, we get

\[
m_\lambda = \frac{1}{2} A_\lambda + \frac{1}{2} B_\lambda - \frac{1}{q} \lambda^{-\frac{p-q}{p-2}} C_\lambda - \frac{1}{p} D_\lambda
\]

\[
= \frac{N(q-2)}{4q} \lambda^{-\frac{p-q}{p-2}} C_\lambda + \frac{N(p-2)}{4p} D_\lambda + \frac{N(2^*-q)}{22^*q} \lambda^{-\frac{p-q}{p-2}} C_\lambda + \frac{N(2^*-p)}{22^*p} D_\lambda - \frac{1}{q} \lambda^{-\frac{p-q}{p-2}} C_\lambda - \frac{1}{p} D_\lambda
\]

\[
= \frac{q-2}{2} \lambda^{-\frac{p-q}{p-2}} C_\lambda + \frac{p-2}{2} D_\lambda.
\]

In a similar way, we show that

\[
m_\infty = \frac{p-2}{2} D_\infty = \frac{p-2}{2} S_p^{\frac{p}{p-2}}.
\]

Thus, we get

\[
m_\lambda - m_\infty = \frac{p-2}{2} (D_\lambda - D_\infty) + \frac{q-2}{2} \lambda^{-\frac{p-q}{p-2}} C_\lambda.
\tag{5.10}

Arguing as in [19], it is shown that

\[
m_0 - m_\lambda \sim \lambda^{-\frac{p-q}{p-2}}, \quad \text{as } \lambda \to \infty.
\tag{5.11}

Therefore, from (5.11), we obtain

\[
\int_{\mathbb{R}^N} |w_\lambda|^p = D_\infty - \frac{2}{p-2} (m_0 - m_\lambda - \frac{q-2}{p-2} \lambda^{-\frac{p-q}{p-2}} C_\lambda = S_p^{\frac{p}{p-2}} - \Theta(\lambda^{-\frac{p-q}{p-2}}).
\tag{5.12}
\]

\[
\int_{\mathbb{R}^N} |\nabla w_\lambda|^2 = \frac{N(q-2)}{2q} \lambda^{-\frac{p-q}{p-2}} C_\lambda + \frac{N(p-2)}{2p} (D_\infty - \frac{2}{p-2} (m_0 - m_\lambda - \frac{q-2}{p-2} \lambda^{-\frac{p-q}{p-2}} C_\lambda)
\]

\[
= \frac{N(p-2)}{2p} D_\infty - \frac{N}{p} (m_\infty - m_\lambda) - N(q-2) \frac{1}{2} (1 - \frac{1}{q}) \lambda^{-\frac{p-q}{p-2}} C_\lambda
\tag{5.13}
\]

\[
= \frac{N(p-2)}{2p} S_p^{\frac{p}{p-2}} + O(\lambda^{-\frac{p-q}{p-2}}),
\]

\[
\int_{\mathbb{R}^N} |w_\lambda|^2 = \frac{N(2^*-q)}{2^*q} \lambda^{-\frac{p-q}{p-2}} C_\lambda + \frac{N(2^*-p)}{2^*p} (D_\infty - \frac{2}{p-2} (m_\infty - m_\lambda) - \frac{q-2}{p-2} \lambda^{-\frac{p-q}{p-2}} C_\lambda)
\]

\[
= \frac{N(2^*-p)}{2^*p} S_p^{\frac{p}{p-2}} - \frac{2N(2^*-p)}{2^*p(p-2)} (m_\infty - m_\lambda) - N(\frac{2^*-p(q-2)}{2^*p(p-2)} - \frac{2^*-q}{2^*q}) \lambda^{-\frac{p-q}{p-2}} C_\lambda
\]

\[
= \frac{N(2^*-p)}{2^*p} S_p^{\frac{p}{p-2}} + O(\lambda^{-\frac{p-q}{p-2}}).
\tag{5.14}
\]
Thus, by (5.6), (5.12), (5.13) and (5.14), for \( b \neq 0 \), we obtain

\[
\|\nabla u_\lambda\|^2 \frac{2}{b} = \lambda^{6-p-2} \sqrt{\omega} \int_{B_\lambda} |\nabla w_\lambda|^2 \\
= \lambda^{6-p-2} \left( b + O\left( \lambda^{6-q} \right) \right)^3 \int_{B_\lambda} |\nabla w_\lambda|^2 \\
= \lambda^{6-p-2} \left( b + O\left( \lambda^{6-q} \right) \right)^3 \left( \lambda^{6-2} S^p_{p-2} + O\left( \lambda^{6-2} \right) \right) \\
= \left\{ \begin{array}{ll}
\lambda^{6-p-2} \left( \frac{9(3-p)}{4p} S^p_{p-2} + O\left( \lambda^{6-2} \right) \right), & \text{if } q > 2p - 6, \\
\lambda^{6-p-2} \left( \frac{9(3-p)}{4p} S^p_{p-2} + O\left( \lambda^{6-2} \right) \right), & \text{if } q \leq 2p - 6,
\end{array} \right.
\]

For \( b = 0 \), noting that \( \omega = a \), by (5.6), (5.12), (5.13) and (5.14), we have

\[
\|\nabla u_\lambda\|^2 = \lambda^{6-p-2} \sqrt{\omega} \int_{B_\lambda} |\nabla w_\lambda|^2 = \lambda^{6-p-2} \left( \frac{3(p-2)}{2p} S^p_{p-2} + O\left( \lambda^{6-2} \right) \right),
\]

\[
\|u_\lambda\|^p = \lambda^{6-p-2} \sqrt{\omega} \int_{B_\lambda} |w_\lambda|^p = \lambda^{6-p-2} \left( \frac{3(p-2)}{2p} S^p_{p-2} + O\left( \lambda^{6-2} \right) \right),
\]

where \( \omega = a \).
On the other hand, as before, it is easy to show that

\[ \|u_\lambda\|_p^p = \lambda^{\frac{6-p}{2(p-q)}} (\sqrt{\varpi_\lambda})^3 \int_{\mathbb{R}^N} |w_\lambda|^p = \lambda^{\frac{6-p}{2(p-q)}} \left( a^{\frac{3}{2}} S_p^{\frac{p}{2(p-q)}} - \Theta(\lambda^{-\frac{p-q}{2}}) \right). \]

5.3. **The case \( q < p \) and \( \lambda > 0 \) is sufficiently small.** Let \( u_\lambda \) be a ground state solution of \((P_\lambda)\), and

\[ w_\lambda(x) = \lambda^{-\frac{1}{n-2}} u_\lambda(\lambda^{-\frac{1}{n-2}} x), \quad \varpi_\lambda = a + b \int_{\mathbb{R}^N} |\nabla u_\lambda|^2. \quad (5.15) \]

Then \( w = w_\lambda \) satisfies

\[ -\Delta w + w = w^{q-1} + \lambda^{\frac{p-q}{2}} w^{p-1}, \quad \text{in} \ \mathbb{R}^N. \quad (5.16) \]

As before, we show that \( w_\lambda \) is the unique positive solution of \((5.16)\) for small \( \lambda > 0 \). Moreover, we have

\[ \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \int_{\mathbb{R}^N} |w_\lambda|^2 = \int_{\mathbb{R}^N} |w_\lambda|^q + \lambda^{\frac{p-q}{2}} \int_{\mathbb{R}^N} |w_\lambda|^p, \quad (5.17) \]

\[ \frac{1}{2^*} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |w_\lambda|^2 = \int_{\mathbb{R}^N} |w_\lambda|^q + \lambda^{\frac{p-q}{2}} \int_{\mathbb{R}^N} |w_\lambda|^p, \quad (5.18) \]

Put

\[ A_\lambda = \int_{\mathbb{R}^N} |\nabla w_\lambda|^2, \quad B_\lambda = \int_{\mathbb{R}^N} |w_\lambda|^2, \quad C_\lambda = \int_{\mathbb{R}^N} |w_\lambda|^q, \quad D_\lambda = \int_{\mathbb{R}^N} |w_\lambda|^p, \quad (5.19) \]

then

\[ A_\lambda = \frac{N(q-2)}{2q} C_\lambda + \frac{N(p-2)}{2p} \lambda^{\frac{p-q}{2}} D_\lambda, \]

\[ B_\lambda = \frac{N(2^* - q)}{2^* q} C_\lambda + \frac{N(2^* - p)}{2^* p} \lambda^{\frac{p-q}{2}} D_\lambda, \]

and hence

\[ m_\lambda = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |w_\lambda|^2 - \frac{1}{q} \int_{\mathbb{R}^N} |w_\lambda|^q - \frac{1}{p} \lambda^{\frac{p-q}{2}} \int_{\mathbb{R}^N} |w_\lambda|^p \]

\[ = \frac{\varpi_\lambda}{2} C_\lambda + \frac{\varpi_\lambda}{2} \lambda^{\frac{p-q}{2}} D_\lambda, \quad (5.20) \]

In a similar way, we show that

\[ m_0 = \frac{q-2}{2} C_0 = \frac{q-2}{2} S_q^{\frac{q}{q-2}}, \quad C_0 = S_q^{\frac{q}{q-2}}. \]

Thus, we obtain

\[ m_\lambda - m_0 = \frac{q-2}{2} (C_\lambda - C_0) + \frac{p-2}{2} \lambda^{\frac{p-q}{2}} D_\lambda. \quad (5.21) \]

On the other hand, as before, it is easy to show that

\[ m_0 - m_\lambda \sim \lambda^{\frac{p-q}{2}}, \quad \text{as} \ \lambda \to 0. \quad (5.22) \]

Therefore, we obtain

\[ C_0 - C_\lambda = \frac{2}{q-2} (m_0 - m_\lambda) + \frac{p-2}{q-2} \lambda^{\frac{p-q}{2}} D_\lambda \sim \lambda^{\frac{p-q}{2}}, \]

that is,

\[ \int_{\mathbb{R}^N} |w_\lambda|^q = C_\lambda = C_0 - \Theta(\lambda^{\frac{p-q}{2}}) = S_q^{\frac{q}{q-2}} - \Theta(\lambda^{\frac{p-q}{2}}). \quad (5.23) \]
Therefore, we have

\[
\int_{\mathbb{R}^N} |\nabla w_\lambda|^2 = \frac{N(q-2)}{2q} (C_0 - \frac{2}{q-2} (m_0 - m_\lambda) - \frac{p_0}{q-2} \lambda^{\frac{p_0}{q-2}} D_\lambda) + \frac{N(p-2)}{2p} \lambda^{\frac{p-2}{p}} D_\lambda
\]

\[
= \frac{N(q-2)}{2q} C_0 - \frac{N}{q} (m_0 - m_\lambda) - \frac{N(p-2)}{2} (\frac{1}{q} - \frac{1}{p}) \lambda^{\frac{p}{q-2}} D_\lambda
\]

\[
= \frac{N(q-2)}{2q} S_q^{2/3} - \Theta(\lambda^{\frac{p}{q-2}}). \tag{5.24}
\]

\[
\int_{\mathbb{R}^N} |w_\lambda|^2 = \frac{N(2^*-q)}{2^*q} (C_0 - \frac{2}{q-2} (m_0 - m_\lambda) - \frac{p_0}{q-2} \lambda^{\frac{p_0}{q-2}} D_\lambda) + \frac{N(2^*-p)}{2^*p} \lambda^{\frac{p}{q-2}} D_\lambda
\]

\[
= \frac{N(q-2)}{2q} S_q^{2/3} - \frac{2N(2^*-q)}{2^*q(q-2)} (m_0 - m_\lambda) - \left[ \frac{N(2^*-q)(p-2)}{2^*q(q-2)} - \frac{N(2^*-p)}{2^*p} \right] \lambda^{\frac{p}{q-2}} D_\lambda.
\]

Notice that

\[
\frac{(2^* - q)(p-2)}{2^*q(q-2)} - \frac{(2^* - p)}{2^*p} = \frac{p - 2}{q - 2} (\frac{1}{q} - \frac{1}{2^*}) - \left( \frac{1}{p} - \frac{1}{2^*} \right) \geq \frac{p - q}{q - 2} \left( \frac{1}{p} - \frac{1}{2^*} \right) > 0,
\]

we conclude that

\[
\int_{\mathbb{R}^N} |w_\lambda|^2 = \frac{N(2^*-q)}{2^*q} S_q^{2/3} - \Theta(\lambda^{\frac{p}{q-2}}). \tag{5.25}
\]

Since

\[
\varpi_\lambda = a + b \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \begin{cases} 
  a + b \lambda^{\frac{6-q}{4(2^*-q)}} \varpi_\lambda \int_{\mathbb{R}^N} |\nabla w_\lambda|^2, & \text{if } b \neq 0, \\
  a, & \text{if } b = 0,
\end{cases}
\]

by (5.24), for \( b \neq 0 \), we have

\[
\sqrt{\varpi_\lambda} = \frac{1}{a} \left( b \lambda^{\frac{6-q}{4(2^*-q)}} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \sqrt{b^2 \lambda^{\frac{p-2}{2}} (\int_{\mathbb{R}^N} |\nabla w_\lambda|^2)^2 + 4a} \right)
\]

\[
= \frac{1}{a} \left( b \lambda^{\frac{6-q}{2(2^*-q)}} + \sqrt{b^2 \lambda^{\frac{p-2}{2}} + 4a (\int_{\mathbb{R}^N} |\nabla w_\lambda|^2)^2} \right) \int_{\mathbb{R}^N} |\nabla w_\lambda|^2
\]

\[
= a^{\frac{1}{2}} + O(\lambda^{\frac{6-q}{2(2^*-q)}}).
\]

Thus, it follows from (5.15), (5.23), (5.24) and (5.25) that for \( b \neq 0 \),

\[
\|\nabla u_\lambda\|_2^2 = \lambda^{\frac{2N-(p-2)}{2(2^*-q)}} (\sqrt{\varpi_\lambda})^{N-2} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2
\]

\[
= \lambda^{\frac{6-q}{2(2^*-q)}} \varpi_\lambda \int_{\mathbb{R}^N} |\nabla w_\lambda|^2
\]

\[
= \begin{cases} 
  \lambda^{\frac{6-q}{2(2^*-q)}} \left( \frac{3(q-2)}{2q} a^{\frac{1}{2}} S_q^{2/3} - \Theta(\lambda^{\frac{p}{q-2}}) \right), & \text{if } q > 2p - 6, \\
  \lambda^{\frac{6-q}{2(2^*-q)}} \left( \frac{3(q-2)}{2q} a^{\frac{1}{2}} S_q^{2/3} + O(\lambda^{\frac{6-q}{2(2^*-q)}}) \right), & \text{if } q \leq 2p - 6,
\end{cases}
\]

\[
\|u_\lambda\|_2^2 = \lambda^{\frac{4N-(4p-2)}{2(2^*-q)}} (\sqrt{\varpi_\lambda})^N \int_{\mathbb{R}^N} |w_\lambda|^2
\]

\[
= \lambda^{\frac{10-q}{2(2^*-q)}} \left( \frac{6-q}{2q} a^2 S_q^{2/3} - \Theta(\lambda^{\frac{p}{q-2}}) \right), \quad \text{if } q > 2p - 6,
\]

\[
= \lambda^{\frac{10-q}{2(2^*-q)}} \left( \frac{6-q}{2q} a^2 S_q^{2/3} + O(\lambda^{\frac{6-q}{2(2^*-q)}}) \right), \quad \text{if } q \leq 2p - 6,
\]

\[
\|u_\lambda\|_q^q = \lambda^{\frac{2N-(p-2)}{2(2^*-q)}} (\sqrt{\varpi_\lambda})^N \int_{\mathbb{R}^N} |w_\lambda|^q
\]

\[
= \lambda^{\frac{6-q}{2(2^*-q)}} \left( \frac{3}{a} S_q^{2/3} - \Theta(\lambda^{\frac{p}{q-2}}) \right), \quad \text{if } q > 2p - 6,
\]

\[
= \lambda^{\frac{6-q}{2(2^*-q)}} \left( \frac{3}{a} S_q^{2/3} + O(\lambda^{\frac{6-q}{2(2^*-q)}}) \right), \quad \text{if } q \leq 2p - 6.
\]
For $b = 0$, by (5.23), (5.24) and (5.25), we have

$$\|\nabla u_\lambda\|^2 = \lambda^{\frac{6 - q}{2(q - 2)}} \sqrt{\omega_\lambda} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 = \lambda^{\frac{6 - q}{2(q - 2)}} \left( \frac{3(q - 2)}{2q} a^\frac{q}{2} S_q^{-\frac{q}{2}} - \Theta(\lambda^{\frac{p - q}{2}}) \right),$$

$$\|u_\lambda\|^2 = \lambda^{\frac{10 - 3q}{2(q - 2)}} \int_{\mathbb{R}^N} |w_\lambda|^2 = \lambda^{\frac{10 - 3q}{2(q - 2)}} \left( \frac{6 - q}{2q} a^\frac{q}{2} S_q^{-\frac{q}{2}} - \Theta(\lambda^{\frac{p - q}{2}}) \right),$$

$$\|u_\lambda\|_q^q = \lambda^{\frac{6 - q}{2(q - 2)}} \int_{\mathbb{R}^N} |w_\lambda|^q = \lambda^{\frac{6 - q}{2(q - 2)}} \left( a^\frac{q}{2} S_q^{-\frac{q}{2}} - \Theta(\lambda^{\frac{p - q}{2}}) \right).$$

The proof of Theorem 2.2 is complete.

6. A CONNECTION WITH THE MASS CONSTRAINED PROBLEM

It is clear that if $u_\lambda \in H^1(\mathbb{R}^N)$ is a ground state of $(P_\lambda)$, and for some $c > 0$ there holds

$$M(\lambda) = \|u_\lambda\|^2 = c^2,$$

then $u_\lambda$ is a positive normalized solution of (1.5) with $\mu = 1$. We denote this normalized solution by a pair $(u_\lambda, \lambda_c)$. In what follows, we always assume $\mu = 1$ in (1.5). As a consequence of Theorem 2.1, we have the following

**Proposition 6.1.** Let $p = 2^*$, $2 < q < 2^*$ and $bS^2 < 1$ if $N = 4$. Then the following statements hold true:

If $N = 4$ and $q \in (3, 4)$, or $N = 3$ and $q \in (4, 6)$, then for any $c > 0$ the problem (1.5) has at least one positive normalized solution $(u_{\lambda_c}, \lambda_c)$ with $\lambda_c > 0$, $\lim_{c \to 0} \lambda_c = \infty$ and $\lim_{c \to \infty} \lambda_c = 0$. If $N = 3$ and $q \in (10/3, 4]$, then there exists a constant $c_1 > 0$ such that for any $c > c_1$, the problem (1.5) has at least one positive normalized solution $(u_{\lambda_c}, \lambda_c)$ with $\lambda_c > 0$ and $\lim_{c \to \infty} \lambda_c = 0$.

If $N = 4$ and $q \in (2, 3)$, then there exists $c_1 > 0$ such that for any $c \in (0, c_1)$ the problem (1.5) has at least two positive normalized solutions $(u_{\lambda_{c(i)}}, \lambda_{c(i)})$ with $\lambda_{c(i)} > 0$, $i = 1, 2$, $\lim_{c \to 0} \lambda_{c(1)} = 0$ and $\lim_{c \to 0} \lambda_{c(2)} = +\infty$. If $N = 3$ and $q \in (2, 10/3)$, then there exists $c_1 > 0$ such that for any $c \in (0, c_1)$ the problem (1.5) has at least one positive normalized solution $(u_{\lambda_c}, \lambda_c)$ with $\lambda_c > 0$ and $\lim_{c \to 0} \lambda_c = 0$.

Furthermore, if $q \neq 2 + \frac{4}{N}$ and $\lim \lambda_c = 0$, then $\lim c^{\frac{q(N - 2)}{4 - N(q - 2)}} = 0$ and

$$\lambda_c \simeq \left( \frac{2q}{2N - q(N - 2)} a^{\frac{N}{2}} S_q^{-\frac{q}{2}} \right)^{\frac{2(q - 2)}{4 - N(q - 2)}},$$

$$u_{\lambda_c}(0) \simeq V_0(0) \left( \frac{2q}{2N - q(N - 2)} a^{\frac{N}{2}} S_q^{-\frac{q}{2}} \right)^{\frac{2}{4 - N(q - 2)}},$$

$$\|\nabla u_{\lambda_c}\|^2 \simeq \frac{N(q - 2)}{2q} a^{\frac{N - 2}{2}} S_q^{\frac{q}{2}} \left( \frac{2q}{2N - q(N - 2)} a^{\frac{N}{2}} S_q^{-\frac{q}{2}} \right)^{\frac{2N - q(N - 2)}{4 - N(q - 2)}},$$

$$\|u_{\lambda_c}\|_2^2 \simeq a^{\frac{N}{2}} \|V_0\|_2^2 \left( \frac{2q}{2N - q(N - 2)} a^{\frac{N}{2}} S_q^{-\frac{q}{2}} \right)^{\frac{N(2N - 2)(N - 2)}{(N - 2)(4 - N(q - 2))}},$$

$$\|u_{\lambda_c}\|_q^q \simeq a^{\frac{N}{2}} S_q^{-\frac{q}{2}} \left( \frac{2q}{2N - q(N - 2)} a^{\frac{N}{2}} S_q^{-\frac{q}{2}} \right)^{\frac{2N - q(N - 2)}{4 - N(q - 2)}}.$$
If \( N = 4, q \in (2, 4) \) and \( \lim \lambda_c = \infty \), then \( \lim c = 0 \), and
\[
\lambda_c(\ln \lambda_c)^{\frac{4-q}{2}} \sim \frac{1}{c^{q-2}}, \quad u_{\lambda_c}(0) \sim \frac{1}{c^2} \ln \lambda_c,
\]
\[
\| \nabla u_{\lambda_c} \|_2^2 = \frac{a S^2}{1 - b S^2} - \Theta(c^{4-q}(\ln \lambda_c)^{-\frac{4-q}{2}}),
\]
\[
\| u_{\lambda_c} \|_2^2 = \frac{a^2 S^2}{(1 - b S^2)^2} + O(c^{4-q}(\ln \lambda_c)^{-\frac{4-q}{2}}), \quad \| u_{\lambda_c} \|_q^q \sim c^{4-q}(\ln \lambda_c)^{-\frac{4-q}{2}},
\]
if \( N = 3, q \in (4, 6) \) and \( \lim \lambda_c = \infty \), then \( \lim c = 0 \), and
\[
\lambda_c \sim c^{-\frac{2-q}{q}}, \quad u_{\lambda_c}(0) \sim c^{-\frac{1}{2(q-2)}},
\]
\[
\| \nabla u_{\lambda_c} \|_2^2 = \frac{b S^3 + S^3 \sqrt{b^2 S^3 + 4a}}{2} - \Theta(c^{\frac{6-q}{2(q-2)}}),
\]
\[
\| u_{\lambda_c} \|_2^2 = \frac{1}{8}(b S^2 + S^2 \sqrt{b^2 S^3 + 4a})^2 + O(c^{\frac{6-q}{2(q-2)}}), \quad \| u_{\lambda_c} \|_q^q \sim c^{\frac{6-q}{2(q-2)}}.
\]
The same conclusions hold true for \( (u_{\lambda_c}^{(i)}, \lambda_c^{(i)}), i = 1, 2 \).

In what follows, we consider the problem (1.5) with subcritical nonlinearity. Let
\[
g(w) = w^{p-1} + w^{q-1}, \quad w \geq 0,
\]
then for \( 2 < q < p < 6 \), \( g(u) \) satisfies all the assumptions (G1)-(G3) in [13]. For \( 0 < \Lambda_1 < \Lambda_2 < +\infty \), it follows from [13, Corollary 3.2] that
\[
\mathcal{W}_{\lambda_1} = \{ w \in H^1_{rad}(\mathbb{R}^3) : w \text{ is a nonnegative solution of } (P_\lambda) \text{ with } a = 1, b = 0, \lambda \in [\Lambda_1, \Lambda_2] \}
\]
is compact in \( H^1(\mathbb{R}^3) \), and hence is compact in \( L^2(\mathbb{R}^3) \) and \( D^{1,2}(\mathbb{R}^3) \). Set
\[
\mathcal{S}_0^\Lambda = \{ w \in H^1_{rad}(\mathbb{R}^3) : w \text{ solves } (P_\lambda) \text{ with } a = 1, b = 0 \text{ and } w > 0 \}.
\]
Then the map \( \rho : \cup_{\lambda \in [\Lambda_1, \Lambda_2]} \mathcal{S}_0^\Lambda \to (0, +\infty) \) defined by \( \rho(w) = \| w \|_2^2 \) is compact. Therefore, there exist positive constants \( C_i = C_i(\Lambda_1, \Lambda_2), i = 1, 2 \) and \( D_i = D_i(\Lambda_1, \Lambda_2), i = 1, 2 \) such that
\[
0 < C_1 \leq \| w \|_2^2 \leq C_2 < +\infty, \quad 0 < D_1 \leq \| \nabla w \|_2^2 \leq D_2 < +\infty, \quad \forall w \in \cup_{\lambda \in [\Lambda_1, \Lambda_2]} \mathcal{S}_0^\Lambda.
\]
Set
\[
\mathcal{S}_0^\Lambda = \{ u \in H^1_{rad}(\mathbb{R}^3) : u \text{ solves } (P_\lambda) \text{ and } u > 0 \}.
\]
Then there exists an one-to-one correspondence through the rescaling
\[
w(x) = u(\sqrt{c} x), \quad \varpi = a + b \int_{\mathbb{R}^3} | \nabla u |^2 = a + b \varpi^\frac{1}{2} \int_{\mathbb{R}^3} | \nabla w |^2
\]
(6.2)
between \( \mathcal{S}_0^\Lambda \) and \( \mathcal{S}_b^\Lambda \). Clearly, by (6.2), we have \( \| w \|_2^2 = \varpi^{-\frac{1}{2}} \| u \|_2^2 \) and
\[
\varpi^\frac{1}{2} = \frac{b + \sqrt{b^2 \| \nabla w \|_2^2 + 4a}}{2}.
\]
Therefore, for any \( u \in \cup_{\lambda \in [\Lambda_1, \Lambda_2]} \mathcal{S}_b^\Lambda \), we have
\[
\frac{(b + \sqrt{b^2 D_2^2 + 4a})C_1}{8} \leq \| u \|_2^2 \leq \frac{(b + \sqrt{b^2 D_2^2 + 4a})^3 C_2}{8}.
\]
Thus arguing as in [13], Theorem 2.2 implies the following result concerning the existence, non-existence and exact number of normalized solutions of (1.5), and their precise asymptotic behavior as the parameter \( c \) varies.
Proposition 6.2. Let $2 < q < p < 6$, $b > 0$ and

$$m_1 := \frac{\sqrt{6 - q}}{2q} \left( \frac{a}{4} S_q^{\frac{q}{2(q - 1)}} \right), \quad m_2 := \frac{\sqrt{27b^3(p - 2)^2(6 - p)}}{4p^2}. \quad (6.3)$$

If $q < 10/3$ and $p < 14/3$, then for any $c > 0$ the problem (1.5) has at least one positive normalized solution $(u_{c}, \lambda_{c})$ with $\lambda_{c} > 0$, $\lim_{c \to 0} \lambda_{c} = 0$ and $\lim_{c \to \infty} \lambda_{c} = +\infty$. Moreover, for sufficiently small $c > 0$ and for sufficiently large $c > 0$, the problem (1.5) has exactly one positive normalized solution.

If $10/3 < q < p < 14/3$, then there exists $c_1 > 0$ such that for any $c > c_1$ the problem (1.5) has two positive normalized solutions $(u_{c_{1}}(i), \lambda_{c_{1}}^{(i)})$ with $\lambda_{c_{1}}^{(i)} > 0$, $i = 1, 2$, $\lim_{c \to 0} \lambda_{c_{1}}^{(1)} = 0$ and $\lim_{c \to \infty} \lambda_{c_{1}}^{(2)} = +\infty$. Moreover, if $c > 0$ is sufficiently large, the problem (1.5) has exactly two positive normalized solutions, and if $c > 0$ is sufficiently small, the problem (1.5) has no normalized solution.

If $q < 10/3$ and $p > 14/3$, then there exists $c_1 > 0$ such that for any $c \in (0, c_1)$ the problem (1.5) has two positive normalized solutions $(u_{c_{1}}(i), \lambda_{c_{1}}^{(i)})$ with $\lambda_{c_{1}}^{(i)} > 0$, $i = 1, 2$, $\lim_{c \to 0} \lambda_{c_{1}}^{(1)} = 0$ and $\lim_{c \to \infty} \lambda_{c_{1}}^{(2)} = +\infty$. Moreover, if $c > 0$ is sufficiently small, the problem (1.5) has exactly two positive normalized solutions, and if $c > 0$ is sufficiently large, the problem (1.5) has no normalized solution.

If $q > 10/3$ and $p > 14/3$, then for any $c > 0$ the problem (1.5) has at least one positive normalized solution $(u_{c}, \lambda_{c})$ with $\lambda_{c} > 0$, $\lim_{c \to 0} \lambda_{c} = +\infty$ and $\lim_{c \to \infty} \lambda_{c} = 0$. Moreover, for sufficiently small $c > 0$ and for sufficiently large $c > 0$, the problem (1.5) has exactly one positive normalized solution.

If $q = 10/3$ and $p < 14/3$, then there exists positive number $c_1 \in (0, m_1)$ such that for any $c \in (c_1, m_1)$, the problem (1.5) has at least two normalized solutions $(u_{c_{1}}(i), \lambda_{c_{1}}^{(i)})$ with $\lambda_{c_{1}}^{(i)} > 0$, $\lim_{c \to m_{1}} \lambda_{c}^{(1)} = 0$ and $\lim_{c \to m_{1}} \lambda_{c}^{(2)} > 0$, and for any $c \geq m_1$, the problem (1.5) has at least one positive normalized solution $(u_{c_{1}}, \lambda_{c})$ with $\lambda_{c} > 0$ and $\lim_{c \to \infty} \lambda_{c} = +\infty$. Moreover, if $c > 0$ is sufficiently small, the problem (1.5) has exactly one positive normalized solution, and if $c > 0$ is sufficiently small, the problem (1.5) has no normalized solution.

If $q = 10/3$ and $p > 14/3$, then there exists a positive number $c_1 \geq m_1$ such that for any $c \in (0, c_1)$, the problem (1.5) has at least one positive normalized solution $(u_{c_{1}}, \lambda_{c})$ with $\lambda_{c} > 0$, $\lim_{c \to 0} \lambda_{c} = 0$ and $\lim_{c \to \infty} \lambda_{c} = +\infty$. Moreover, if $c > 0$ is sufficiently small, the problem (1.5) has exactly one positive normalized solution, and if $c > 0$ is sufficiently large, the problem (1.5) has no normalized solution.

If $q < 10/3$ and $p = 14/3$, then there exists $c_1 \geq m_2$ such that for any $c \in (0, c_1)$, the problem (1.5) has at least one positive normalized solution $(u_{c_{1}}, \lambda_{c})$ with $\lambda_{c} > 0$, $\lim_{c \to m_{2}} \lambda_{c} = +\infty$ and $\lim_{c \to \infty} \lambda_{c} = 0$. Moreover, if $c > 0$ is sufficiently large, the problem (1.5) has exactly one positive normalized solution, and if $c > 0$ is sufficiently small, the problem (1.5) has no normalized solution.

If $q > 10/3$ and $p = 14/3$, then there exists a positive number $c_1 \leq m_2$ such that for any $c > c_1$, the problem (1.5) has at least one positive normalized solution $(u_{c_{1}}, \lambda_{c})$ with $\lambda_{c} > 0$, $\lim_{c \to m_{2}} \lambda_{c} = +\infty$ and $\lim_{c \to \infty} \lambda_{c} = 0$. Moreover, if $c > 0$ is sufficiently large, the problem (1.5) has exactly one positive normalized solution, and if $c > 0$ is sufficiently small, the problem (1.5) has no normalized solution.
Furthermore, if $q \neq \frac{10}{3}$ and $\lim \lambda_c = 0$, then $\lim c^{\frac{1}{10-3q}} = 0$, and
\[
\lambda_c \simeq \left(\frac{2q}{6-q}\right) \frac{2(q-2)}{10-3q} a^{\frac{3(q-2)}{10-3q}} S_q^{-\frac{2q}{10-3q}} c^{\frac{2(q-2)}{10-3q}},
\]
\[
\|\nabla u_{\lambda_c}\|_2^2 \simeq \frac{3(q-2)}{2q} \left(\frac{2q}{6-q}\right) \frac{6-q}{10-3q} a^{\frac{3(q-2)}{10-3q}} S_q^{-\frac{2q}{10-3q}} c^{\frac{2(q-2)}{10-3q}},
\]
\[
\|u_{\lambda_c}\|_q^q \simeq \left(\frac{2q}{6-q}\right) \frac{6-q}{10-3q} a^{\frac{3(q-2)}{10-3q}} S_q^{-\frac{2q}{10-3q}} c^{\frac{2(q-2)}{10-3q}}.
\]
If $p \neq \frac{14}{3}$ and $\lim \lambda_c = +\infty$, then $\lim c^{\frac{1}{14-3p}} = +\infty$, and
\[
\lambda_c \simeq \left(\frac{16p^4}{27b^3(p-2)^3(6-p)}\right)^{\frac{p-2}{4p^2}} S_p^{-\frac{4p}{14-3p}} c^{\frac{2(p-2)}{14-3p}},
\]
\[
\|\nabla u_{\lambda_c}\|_2^2 \simeq \frac{9b(p-2)^2}{4p^2} \left(\frac{16p^4}{27b^3(p-2)^3(6-p)}\right)^{\frac{6-p}{14-3p}} S_p^{-\frac{2p}{4p^2}} c^{\frac{2(p-2)}{14-3p}},
\]
\[
\|u_{\lambda_c}\|_p^p \simeq \frac{27b^3(p-2)^3}{8p^3} \left(\frac{16p^4}{27b^3(p-2)^3(6-p)}\right)^{\frac{2(p-2)}{14-3p}} S_p^{-\frac{4p}{14-3p}} c^{\frac{4(p-2)}{14-3p}}.
\]
If $q = \frac{10}{3}$ and $\lim \lambda_c = 0$, then
\[
\lim c = m_1 = \sqrt[2q]{\frac{6-q}{a^2 S_q^{\frac{4}{10-3q}}}},
\]
and
\[
\lim u_{\lambda_c}(0) = \lim \|\nabla u_{\lambda_c}\|_2^2 = \lim \|u_{\lambda_c}\|_q^q = 0.
\]
If $p = \frac{14}{3}$ and $\lim \lambda_c = +\infty$, then
\[
\lim c = m_2 = \sqrt[4p^2]{\frac{27b^3(p-2)^3(6-p)}{4p^2} S_p^{-\frac{4p}{14-3p}} c^{\frac{4(p-2)}{14-3p}}},
\]
and
\[
\lim u_{\lambda_c}(0) = \lim \|\nabla u_{\lambda_c}\|_2^2 = \lim \|u_{\lambda_c}\|_p^p = +\infty.
\]
The same conclusions hold true for $(u_{\lambda_c^{(i)}}, \lambda_c^{(i)})$, $i = 1, 2$.

**Remark 6.1.** In the case that $b = 0$ and $2 < q \leq p < 6$, Jeanjean, Zhang and Zhong [13] obtain the existence, non-existence and multiplicity of positive normalized solutions to $(P_\lambda)$. The authors in [13] also obtain some asymptotic behavior of normalized solutions as the Lagrange multiplier $\lambda \to 0$ or $\lambda \to +\infty$. In fact, by the discussion in Section 5, a direct computation shows that if $b = 0$, then we have
\[
\lambda_c = \left(\frac{p}{6-p}\right)^{\frac{2(p-2)}{10-3p}} a^{-\frac{3(p-2)}{14-3p}} S_p^{\frac{2p}{14-3p}} c^{\frac{4(p-2)}{10-3p}},
\]
\[
\|\nabla u_{\lambda_c}\|_2^2 = \frac{3(p-2)}{p} \left(\frac{p}{6-p}\right)^{\frac{6-p}{10-3p}} a^{-\frac{4}{10-3p}} S_p^{\frac{2p}{10-3p}} c^{\frac{2(p-2)}{10-3p}},
\]
By Proposition 6.2 and Remark 6.1, we see that in the space dimension expression of normalized solutions are not addressed there.

eral subcritical nonlinearity and obtain some results concerning the existence, non-existence

We mention that Zeng et al. [34] extend the results in [13] to a Kirchhoff equation with gen-

Asymptotic behavior of

Remark 6.3.

and nonlinear Schrödinger equations have not been addressed there.

[23]. But the difference between the Kirchhoff equations with combined powers nonlinearity

power nonlinearity and nonlinear Schrödinger equations has also been observed by Qi and Zou

the nonlocal term. We mention that the difference between the Kirchhoff equations with pure

asymptotic behavior of normalized solutions of (1.5), which are completely different from those

then both

For

For

We mention that Zeng et al. [34] extend the results in [13] to a Kirchhoff equation with gen-

eral subcritical nonlinearity and obtain some results concerning the existence, non-existence and multiplicity of normalized solutions, but the exact number and the precise asymptotic expression of normalized solutions are not addressed there.

Remark 6.2. By Proposition 6.2 and Remark 6.1, we see that in the space dimension \( N = 3 \), there is a striking difference between the cases \( b = 0 \) and \( b \neq 0 \) (See also Figure 1 and Figure 2). More precisely, if \( b = 0 \) then \( p_0 = \frac{10}{3} \) plays a key role in the existence, non-existence, multiplicity and asymptotic behavior of normalized solutions of (1.5). However, if \( b \neq 0 \), then both \( p_0 = \frac{10}{3} \) and \( p_0 = \frac{14}{3} \) play a role in the existence, non-existence, multiplicity and asymptotic behavior of normalized solutions of (1.5), which are completely different from those for the corresponding nonlinear Schrödinger equation and which reveal the special influence of the nonlocal term. We mention that the difference between the Kirchhoff equations with pure power nonlinearity and nonlinear Schrödinger equations has also been observed by Qi and Zou [23]. But the difference between the Kirchhoff equations with combined powers nonlinearity and nonlinear Schrödinger equations have not been addressed there.

Remark 6.3. Asymptotic behavior of \( M(\lambda) \) similar to the cases depicted in Figure 3 and Figure 4 have been observed in nonlinear Schrödinger equations with a power nonlinearity, the cases depicted in Figure 3, Figure 4 and Figure 6 have been observed in nonlinear Schrödinger equations with general nonlinearity [13], while the cases depicted in Figure 3, Figure 4 and Figure 5 have been observed in Kirchhoff equations with a pure power nonlinearity [23]. If \( q = \frac{10}{3} \) and \( p < \frac{14}{3} \), then there exists positive number \( c_1 \in (0, m_1) \) such that for any \( c \in (c_1, m_1) \), the problem (1.5) has at least two normalized solutions, if \( c > 0 \) is sufficiently large, the problem (1.5) has exactly one positive normalized solution, and if \( c > 0 \) is sufficiently small, the problem (1.5) has no normalized solution. This is new phenomenon, which is not observed before in the literature and which does not shared by nonlinear Schrödinger equations and Kirchhoff equations with pure power nonlinearity. Some new phenomenon is
also observed in the case $\frac{10}{3} < q < p = \frac{14}{3}$. See the diagrams of $M(\lambda)$ given below in Figure 3-Figure 8, where $m_1$ and $m_2$ are given in (6.3).
We mention that when \( 2 < q < \frac{10}{3}, \frac{14}{3} < p < 6 \), a nontrivial variation of \( M(\lambda) \) can also be observed in Figure 6, which affects the existence, non-existence and multiplicity of normalized solutions of (1.5). This special behavior of \( M(\lambda) \) is mainly caused by the combined nonlinearity and have been observed in nonlinear Schrödinger equations and Kirchhoff equations with general subcritical nonlinearity [13, 9]. We also mention that this type of behavior of \( M(\lambda) \) does not appear in the Kirchhoff equations with a pure power nonlinearity [23]. Typically, the asymptotic behavior of \( M(\lambda) \) depicted in Figure 5 is mainly caused by the appearance of the nonlocal term \( b \int_{\mathbb{R}^N} |\nabla u|^2 \), which has been reported by Qi and Zou [23] as a new phenomenon for Kirchhoff equation with a pure power nonlinearity. While the asymptotic behaviors of \( M(\lambda) \) depicted in Figure 7 and Figure 8 are mainly caused by the combined effect of the nonlocal term and the combined nonlinearity, which have not been reported before in the literature.

Besides, the value of \( b > 0 \) has also an effect on the existence, non-existence and the number of normalized solutions of (1.5), which can be seen from Figure 8 and the following diagrams.

**Acknowledgements.** Part of this research was carried out while S.M. was visiting Swansea University. S.M. thanks the Department of Mathematics for its hospitality. S.M. was supported by National Natural Science Foundation of China (Grant Nos.11571187, 11771182).

**References**

[1] T. Akahori, S. Ibrahim, H. Kikuchi and H. Nawa, *Global dynamics above the ground state energy for the combined power type nonlinear Schrödinger equations with energy critical growth at low frequencies*. Mem. Amer. Math. Soc. **272** (2021), no. 1331, v+130 pp.

[2] T. Akahori, S. Ibrahim, N. Ikoma, H. Kikuchi and H. Nawa, *Uniqueness and nondegeneracy of ground states to nonlinear scalar field equations involving the Sobolev critical exponent in their nonlinearities for high frequencies*. Calc. Var. Partial Differential Equations **58** (2019), Paper No. 120, 32 pp.

[3] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[4] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), 313–345.

[5] M. Coles and S. Gustafson, *Solitary Waves and Dynamics for Subcritical Perturbations of Energy Critical NLS*. Publ. Res. Inst. Math. Sci. **56** (2020), 647–699.
[6] J. Dávila, M. del Pino and I. Guerra, Non-uniqueness of positive ground states of nonlinear Schrödinger equations. Proc. Lond. Math. Soc. (3) 106 (2013), no. 2, 318–344.

[7] A. Ferrero and F. Gazzola, On subcriticality assumptions for the existence of ground states of quasilinear elliptic equations. Adv. Differential Equations 8 (2003), 1081–1106.

[8] R. Fukuizumi, Stability and instability of standing waves for nonlinear Schrödinger equations. Dissertation, Tohoku University, Sendai, 2003. Tohoku Mathematical Publications 25 (2003), vi+68 pp.

[9] Qihan He, Zongyan Lv, Yimin Zhang and Xuexiu Zhong, Positive normalized solution to the Kirchhoff equation with general nonlinearities of mass supercritical, arXiv:2110.12921v2.

[10] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, Nonlinear Anal. 28 (1997), 1633–1659.

[11] L. Jeanjean and T. Le, Multiple normalized solutions for a Sobolev critical Schrödinger equation, Math. Ann. 384 (2022), 101–134.

[12] L. Jeanjean, J. Jendrej, T. Le and N. Visciglia, Orbital stability of ground states for a Sobolev critical Schrödinger equation, arXiv:2112.05869v1.

[13] L. Jeanjean, Jianjun Zhang and Xuexiu Zhong, A global branch approach to normalized solutions for the Schrödinger equation, arXiv:2008.12084v1.

[14] L. Jeanjean, Jianjun Zhang and Xuexiu Zhong, A global branch approach to normalized solutions for the Schrödinger equation, arXiv:2112.05869v1.

[15] L. Jeanjean and S. R. Nodari, The double-power nonlinear Schrödinger equation and its generalizations: uniqueness, non-degeneracy and applications. Calc. Var. Partial Differential Equations 59 (2020), Paper No. 197, 49 pp.

[16] G. Li, X. Luo and T. Yang, Normalized solutions to a class of Kirchhoff equations with Sobolev critical exponent, arXiv:2103.08106v1.

[17] S. Qi and W. Zou, Exact number of positive solutions for the Kirchhoff equation, SIAM J. Math. Anal., 54 (2022), 5424–5446.

[18] N. Soave, Normalized ground state for the NLS equations with combined nonlinearities. J. Functional Analysis, 269 (2020), 6941–6987.

[19] Q. Xie and B.-X. Zhou, A study on the critical Kirchhoff problem in high-dimensional space, Z. Angew. Math. Phy., 2022, 73:4.
[33] X. Zeng and Y. Zhang, *Existence and uniqueness of normalized solutions for the Kirchhoff equation*, Appl. Math. Letters, 74 (2017), 52–59.

[34] X. Zeng, J. Zhang, Y. Zhang and X. Zhong, *Positive normalized solution to the Kirchhoff equation with general nonlinearities*, arXiv:2112.10293v1, 20 Dec 2021.